Stability of the separable solutions for a nonlinear boundary diffusion problem

Tianling Jin*, Jingang Xiong†, Xuzhou Yang

February 7, 2024

Abstract

In this paper, we study a nonlinear boundary diffusion equation of porous medium type arising from a boundary control problem. We give a complete and sharp characterization of the asymptotic behavior of its solutions, and prove the stability of its separable solutions.

Keywords: boundary diffusion, blow up, extinction, asymptotics, rate of convergence
MSC(2020): Primary 35K57; Secondary 35K55, 35B40, 35R11

Declarations of interest: none.

Contents

1 Introduction 2
2 Steady states and some quantitative elliptic estimates 8
3 Existence, uniqueness, and infinite speed of propagation 14
4 Extinction or blow up in finite time, and integral bounds 19
5 Uniform upper and lower bounds 24

*T. Jin is partially supported by Hong Kong RGC grants GRF 16306320 and GRF 16303822, and National Natural Science Foundation of China grant 12122120.
†J. Xiong is partially supported by National Natural Science Foundation of China grants 11922104 and 11631002.
1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded smooth domain and $0 < p < \infty$ be a real number. We consider the nonlinear boundary diffusion problem

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega \times (0, \infty), \\
\partial_t u^p &= -\partial_\nu u - au \quad \text{on } \partial \Omega \times (0, \infty),
\end{align*}
\]

with the initial condition

\[ u(\cdot, 0) = u_0(\cdot) > 0 \quad \text{on } \partial \Omega, \]

where $\Delta$ is the Laplace operator with respect to the spatial variable $x \in \Omega$, $\nu$ is the unit outer normal to $\partial \Omega$, $\partial_\nu$ is the outer normal derivative, $a \in C^\infty(\partial \Omega)$, and $u_0 \in C^\infty(\partial \Omega)$ is a positive function. This nonlinear boundary diffusion problem arises from a boundary control problem; see Duvaut-Lions [31] and Athanasopoulos-Caffarelli [7]. When $n \geq 3$, $p = \frac{n}{n-2}$ and $a = -\frac{n-2}{2(n-1)} H_{\Omega}$ with $H_{\Omega}$ being the mean curvature of $\Omega$, it is the unnormalized boundary Yamabe flow; see Brendle [24] and Almaraz [3]. If we denote $\mathcal{B} : H^{\frac{1}{2}}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega)$ as the Dirichlet to Neumann map, that is, for any $f \in H^{\frac{1}{2}}(\partial \Omega)$,

\[ \mathcal{B} f := \frac{\partial}{\partial \nu} F \big|_{\partial \Omega}, \]

where $F$ satisfies $\Delta F = 0$ in $\Omega$ and $F = f$ on $\partial \Omega$, then the nonlinear boundary diffusion equation (1) can be rewritten as

\[ \partial_t u^p = -\mathcal{B} u - au \quad \text{on } \partial \Omega \times (0, \infty). \]

Therefore, the equation (1) can be viewed as a nonlinear evolution equation with a nonlocal diffusion operator on the manifold $\partial \Omega$ of principal symbol the same as the $1/2$–Laplace operator.

If $\Omega = \mathbb{R}^n_+ = \{ x = (x', x_n) : x_n > 0 \}$ and $a \equiv 0$, then the equation (1) becomes the fractional porous medium equation on the Euclidean space

\[ \partial_t u^p = -(-\Delta)^\sigma u \quad \text{on } \mathbb{R}^{n-1} \times (0, \infty) \]

with $\sigma = 1/2$. This equation (5) with $\sigma = 1/2$ has been systematically studied by de Pablo-Quirós-Rodríguez-Vázquez [29], which was later extended to the general fractional Laplace operator $(-\Delta)^\sigma$ for all $\sigma \in (0, 1)$ by themselves in [30]. Vázquez [57] established the existence, uniqueness and main properties of the fundamental solutions to (5),
and obtained some asymptotics properties. When \( p \) is the critical Sobolev exponent, asymptotic behavior was studied by the first two authors in \([43]\). Bonforte-Vázquez \([21]\) obtained weighted global integral estimates for \((5)\) that allow to establish existence of solutions for classes of large data, and obtain quantitative pointwise lower estimates of the positivity of the solutions, depending only on the norm of the initial data in a certain ball. These estimates were later improved by Vázquez-Volzone \([59]\). Vázquez-de Pablo-Quirós-Rodríguez \([58]\) proved that weak solutions of \((5)\) are classical for all positive times. More general nonlocal porous medium equations were studied in Bonforte-Endal \([10]\). There are also studies on the fractional porous medium equation \((5)\) posed in bounded domains with the Dirichlet condition:

\[
\begin{aligned}
\partial_t u^p &= -(-\Delta)\sigma u & \text{on } \mathcal{V} \times (0, \infty), \\
 u &= 0 & \text{on } (\mathbb{R}^{n-1} \setminus \mathcal{V}) \times (0, \infty),
\end{aligned}
\]

where \( \mathcal{V} \subset \mathbb{R}^{n-1} \) is a bounded domain. The wellposedness and regularity of this equation, as well as their extensions to more general nonlocal porous medium equations, have been obtained by Bonforte-Sire-Vázquez \([18]\), Bonforte-Vázquez \([22]\), Bonforte-Figalli-Ros-Oton \([12]\), Bonforte-Figalli-Vázquez \([13]\), Bonforte-Ibarrondo-Ispizua \([15]\), Brasco-Volzone \([23]\) and Franzina-Volzone \([33]\). Note that \( \mathcal{V} \) in \((6)\) is a Euclidean open set in \( \mathbb{R}^{n-1} \), but not a boundary of a domain \( \Omega \subset \mathbb{R}^n \). Therefore, even though the equations \((1)\) and \((6)\) share some similarities in principles, they are two different equations.

In this paper, we would like to study the large time behavior of the solutions to the nonlinear boundary diffusion equation \((1)\) with Sobolev subcritical exponents:

\[
\begin{aligned}
p &\in (0, 1) \cup (1, +\infty) & \text{if } n = 2, \\
p &\in (0, 1) \cup \left(1, \frac{n}{n-2}\right) & \text{if } n \geq 3.
\end{aligned}
\]

The dynamics of the solutions to \((1)-(2)\) will depend on the range of \( p \) and the sign of the first eigenvalue of the operator \( \mathcal{B} + a \) defined by

\[
\lambda_1 = \inf_{u \in H^1(\Omega)} \left\{ \int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial\Omega} au^2 \, dS : \int_{\partial\Omega} u^2 \, dS = 1 \right\},
\]

where \( dS \) is the area element of \( \partial\Omega \).

**Theorem 1.1.** Let \( n \geq 2, \, \Omega \subset \mathbb{R}^n \) be a bounded smooth domain, \( p \) satisfy \((7)\) and \( a \in C^\infty(\partial\Omega) \). Suppose that \( u_0 \in C^\infty(\partial\Omega) \) and \( u_0 \) is positive. Then \((1)-(2)\) admits a unique positive smooth solution \( u \) on \( \overline{\Omega} \times [0, T^*) \), where \( [0, T^*) \), with \( 0 < T^* \leq \infty \), is the largest interval on which \( u \) is positive. Moreover, there exists \( C > 0 \) depending only on \( n, \, p, \, a, \, \Omega, \max_{\partial\Omega} u_0 \) and \( \min_{\partial\Omega} u_0 \) such that

(i) if \( \lambda_1(p-1) < 0 \), then \( T^* = \infty \) and

\[
\frac{1}{C(t+1)^{\frac{1}{p-1}}} \leq u(x, t) \leq C(t+1)^{\frac{1}{p-1}} \quad \text{on } \overline{\Omega} \times [0, \infty);
\]
(ii) if \( \lambda_1 = 0 \), then \( T^* = \infty \) and
\[
\frac{1}{C} \leq u(x, t) \leq C \quad \text{on } \overline{\Omega} \times [0, \infty);
\]

(iii) if \( \lambda_1 (p - 1) > 0 \), then \( T^* < \infty \) and
\[
\frac{1}{C} (T^* - t)^{\frac{1}{p-1}} \leq u(x, t) \leq C (T^* - t)^{\frac{1}{p-1}} \quad \text{on } \overline{\Omega} \times [0, T^*).
\]

Part (iii) in Theorem 1.1 shows that the solution will be extinct in finite time if \( \lambda_1 > 0 \) and \( p > 1 \), and will blow up in finite time if \( \lambda_1 < 0 \) and \( 0 < p < 1 \). The assumption of \( p \) being Sobolev subcritical is used crucially in Propositions 2.6 and 5.1, which will be used to prove Theorem 1.1.

The quantitative estimates in Theorem 1.1 can be illustrated via the separable solutions
\[
U_c(x, t) = \varphi(x)b_c(t)
\]
of (1), where
\[
b_c(t) = \begin{cases} (c + t)^{\frac{1}{p-1}} & \text{if } \lambda_1 (p - 1) < 0, \\ c^{\frac{1}{p-1}} & \text{if } \lambda_1 = 0, \\ (c - t)^{\frac{1}{p-1}} & \text{if } \lambda_1 (p - 1) > 0 \end{cases}
\]
is the solution of the ODE
\[
\partial_t [b(t)]^p = -\text{sgn}(\lambda_1) \frac{p}{|p-1|} b(t)
\]
with the initial data \( b(0) = c^{\frac{1}{p-1}} \) for some \( c > 0 \), and \( \varphi \) is a positive solution of
\[
-\Delta \varphi = 0 \quad \text{in } \Omega, \quad \partial_\nu \varphi + a \varphi = \text{sgn}(\lambda_1) \frac{p}{|p-1|} \varphi^p \quad \text{on } \partial\Omega
\]
with \( \text{sgn}(\lambda_1) \in \{-1, 0, 1\} \) denoting as the sign of \( \lambda_1 \). The equation (10) is equivalent to
\[
\mathcal{B} \varphi + a \varphi - \text{sgn}(\lambda_1) \frac{p}{|p-1|} \varphi^p = 0 \quad \text{on } \partial\Omega.
\]

Since \( p \) is Sobolev subcritical, the existence of \( \varphi \) can be obtained by the variational method; see Proposition 2.6. If \( \lambda_1 (p - 1) < 0 \), then the solution of (10) is unique; see Proposition 2.7. Following the definition of Adams-Simon [1] (pages 229-230), we call a solution \( \varphi \) of (11) is integrable if for every nonzero \( \phi \in \text{Ker} \mathcal{L}_\varphi := \{ \phi \in H^{1/2} (\partial\Omega) : \mathcal{L}_\varphi \phi = 0 \} \), where
\[
\mathcal{L}_\varphi := \mathcal{B} + a - \text{sgn}(\lambda_1) \frac{p^2}{|p-1|} \varphi^{p-1},
\]

and
\[
\mathcal{B} \varphi + a \varphi - \text{sgn}(\lambda_1) \frac{p}{|p-1|} \varphi^p = 0 \quad \text{on } \partial\Omega.
\]
there exists a family \( \{ \varphi_s \}_{s \in (-1, 1)} \) of solutions to (11) such that \( \varphi_s \to \varphi \) in \( C^2(\partial \Omega) \) and \((\varphi_s - \varphi)/s \to \phi \) in \( L^2(\partial \Omega) \) as \( s \to 0 \).

Furthermore, the above separable solutions characterize the limits of all the solutions in Theorem 1.1 as \( t \to T^* \). To show the stability of these separable solutions, we shall use the following changes of variables:

\[
\begin{aligned}
\begin{cases}
   w(x, \tau) = u(x, t)/b_1(t) \text{ and } t = e^\tau - 1, & \text{if } \lambda_1(p - 1) < 0; \\
   w(x, \tau) = u(x, t)/b_1(t) \text{ and } t = \tau, & \text{if } \lambda_1 = 0; \\
   w(x, \tau) = u(x, t)/b_{T^*}(t) \text{ and } t = T^*(1 - e^{-\tau}), & \text{if } \lambda_1(p - 1) > 0.
\end{cases}
\end{aligned}
\]

(13)

Then

\[
\begin{aligned}
\begin{cases}
   \Delta w = 0 \text{ in } \Omega \times (0, \infty), \\
   \partial_\tau w^p = -\partial_\tau w - aw + \text{sgn}(\lambda_1) \frac{p}{|p - 1|} w^p \text{ on } \partial \Omega \times (0, \infty).
\end{cases}
\end{aligned}
\]

(14)

It follows from Theorem 1.1 that \( 1/C \leq w \leq C \) on \( \overline{\Omega} \times [0, \infty) \) for some \( C \geq 1 \) which depends only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \).

**Theorem 1.2.** Let \( u \) be the one in Theorem 1.1 and \( w \) be defined as in (13). Then \( w(x, \tau) \) converges to a positive solution \( \varphi \) of (10) in \( C^2(\overline{\Omega}) \) as \( \tau \to \infty \). Moreover, there exist \( C > 0, \delta > 0 \) and \( \gamma > 0 \), all of which depend only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \), such that

(i) if \( \lambda_1(p - 1) < 0 \), then

\[
\|w(\cdot, \tau) - \varphi\|_{C^2(\overline{\Omega})} \leq Ce^{-\tau} \quad \forall \tau > 1.
\]

(15)

(ii) if \( \lambda_1 = 0 \), then

\[
\|w(\cdot, \tau) - \varphi\|_{C^2(\overline{\Omega})} \leq Ce^{-\gamma \tau} \quad \forall \tau > 1.
\]

(16)

(iii) if \( \lambda_1(p - 1) > 0 \), then either

\[
C\tau^{-1} \leq \|w(\cdot, \tau) - \varphi\|_{C^2(\overline{\Omega})} \leq C\tau^{-\delta} \quad \forall \tau > 1
\]

(17)

or

\[
\|w(\cdot, \tau) - \varphi\|_{C^2(\overline{\Omega})} \leq Ce^{-\gamma \tau} \quad \forall \tau > 1.
\]

(18)

(iv) if \( \lambda_1(p - 1) > 0 \) and \( \varphi|_{\partial \Omega} \) is integrable, then (18) holds.

**Remark 1.3.** The decay rate \( e^{-\tau} \) in (15) is sharp, since the separable solution \( (2 + t)\frac{1}{p - 1} \varphi(x) \) of (1) indeed satisfies this decay rate. The exponential exponent \( \gamma \) in (16) and (18) is in fact equal to \( \gamma_p \) which is defined in (62); see Theorem 6.4. Moreover, this \( \gamma_p \) is the same as in the linear case, and thus, no better rate shall be expected in (16) and (18) in this degree of generality. Furthermore, we obtain a higher order expansion (76) of the solution in the exponential decay case (in fact, one can expand it to arbitrary orders).
When $\lambda_1 > 0$, then the diffusion operator $\mathcal{B} + a$ is coercive, and our results are the analogues of those for the classical nonlinear diffusion equation in bounded domains:

$$
\begin{align*}
\partial_t u^y = \Delta u & \quad \text{in } \Omega \times (0, \infty), \\
u = 0 & \quad \text{on } \partial \Omega \times [0, \infty).
\end{align*}
$$

(19)

In the case of $0 < q < 1$, the equation (19) is called the porous medium equation. It is a slow diffusion equation, which means that if $u_0$ is compactly supported in $\Omega$, then the solution $u(\cdot, t)$ with such initial data will still be compactly supported in $\Omega$ at least for a short time. Suppose $u_0 \in C^1(\overline{\Omega})$, $u_0 \geq 0$ in $\overline{\Omega}$, $u_0 = 0$ on $\partial \Omega$, and $u_0 \not\equiv 0$ in $\Omega$, then part (i) in Theorem 1.1 and the estimate in Theorem 1.2 under the Lipschitz norm (after a waiting time) were proved by Aronson-Peletier [6]. See Vázquez [56] for another similar stability result of the separable solutions to the porous medium equation. In [48], the first two authors and X. Ros-Oton recently proved the optimal regularity of the solution $u$, improved the convergence from the Lipschitz topology to the $C^{2,q}(\Omega)$ topology and obtained its finer asymptotics. In the case of $1 < q < +\infty$, the equation (19) is called the fast diffusion equation. It has infinite speed of propagation, which means that if the initial data is nonnegative and not identically zero, then the solution will be positive everywhere in $\Omega$ immediately. It is known from the work of Sabinina [51, 52] that solutions of (19) with positive initial data will be extinct after a finite time $T^* > 0$. If $q$ is a Sobolev subcritical exponent, then under the assumption that $\partial_t u, \nabla u, \nabla \partial_t u, \nabla^2 u \in C(\overline{\Omega} \times (0, T^*))$, Berryman-Holland [8] proved the integral estimates in Theorem 1.1 for the solutions of (19), as well as the convergence in Theorem 1.2 but just along a sequence of times in $H^1_0(\Omega)$. In 2000, Feireisl-Simondon [34] proved the uniform convergence without the regularity assumption. Later, Bonforte-Grillo-Vázquez [14] proved the uniform convergence of the relative error, and Bonforte-Figalli [11] quantified the convergence rate of the relative error and obtained a sharp exponential rate in generic domains. See also Akagi [5] for a different proof. Recently, the first two authors proved Berryman-Holland’s regularity assumption and established the optimal regularity in [44, 47]. As applications, they improved the uniform convergence of the relative error to be in the $C^2$ topology in [44], and proved the polynomial decay rate for all smooth domains in [45]. More recently, Choi-McCann-Seis [28] proved that the relative error either decays exponentially with the sharp rate, or else decays algebraically at a rate $1/t$ or slower. Furthermore, they obtained higher order asymptotics, which refines and confirms a conjecture of Berryman-Holland [8]. Besides these results, there are many other works on the quantitative properties of the solutions to the fast diffusion equation (19) in bounded domains, and we refer to DiBenedetto-Kwong-Vespri [32], Galaktionov-King [37], Bonforte-Vázquez [19, 20], Akagi [4], Jin-Xiong [45], Sire-Wei-Zheng [54] and the references therein.

When $\lambda_1 < 0$, then one may look for analogues between (1) and

$$
\begin{align*}
\partial_t u^y = \Delta u + cu & \quad \text{in } \Omega \times (0, \infty), \\
u = 0 & \quad \text{on } \partial \Omega \times [0, \infty),
\end{align*}
$$

(20)
where $c$ is a positive constant larger than the first Dirichlet eigenvalue of the Laplace operator $\Delta$ on $\Omega$. Both the solution of (1) with $0 < p < 1$ and the solution of (20) with $0 < q < 1$ blow up in finite time, where the latter conclusion was proved by Galaktionov [36]. However, there is a notable difference on the uniform lower bound in part (iii) of Theorem 1.1. If $\Omega$ is large, and the support of the initial data $u_0$ is small and sufficiently away from $\partial \Omega$, then Theorem 1 of Galaktionov [35] shows that the blow up for the solution of (20) is localized. That is, the support of the solution of (20) stays uniformly away from $\partial \Omega$ up to the blow up time, so that it is impossible to prove proper uniform lower bounds for them. However, the solution of (1) blows up everywhere uniformly. The main reason why the uniform lower bound holds for the solution of (1) with $0 < p < 1$ is that the solution satisfies a quantitative positivity estimate in Theorem 1.4 in the below. This is different from (20) which fails to be so for $0 < q < 1$. This positive estimate implies the solutions of (1) has infinite speed of propagation on $\partial \Omega$, which is analogous to the phenomenon of infinite speed of propagation observed for porous medium equations with nonlocal operators in Euclidean space by de Pablo-Quirós-Rodríguez-Vázquez [29] and Bonforte-Figalli-Ros-Oton [12].

**Theorem 1.4.** Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p$ satisfy (7), $a \in C^\infty(\partial \Omega)$, $u_0 \in C^\infty(\partial \Omega)$ be a positive function, and $u$ be a smooth positive solution of (1)–(2) on $\Omega \times [0, T]$ for some $T > 0$. Then there exist $t_0 > 0$ and $\varepsilon_0 > 0$, both of which depend only on $n, p, a, \Omega, \max_{\partial \Omega} u_0$ and $\|u_0\|_{L^p(\partial \Omega)}$, but not on $\min_{\partial \Omega} u_0$, such that

$$u(x, t) \geq \varepsilon_0 t^{1/p} \text{ for all } (x, t) \in \overline{\Omega} \times [0, t_0].$$

Since the $\varepsilon_0$ in Theorem 1.4 does not depend on $\min_{\partial \Omega} u_0$, then one can consider the equation (1) with nonnegative initial data by approximations, and its solutions will be immediately positive everywhere. Finite time blow up for the porous medium equations with reactions of type (20) has been very extensively studied in the literature, and we refer the readers to the survey Galaktionov-Vázquez [38] for more references.

The estimates in Theorem 1.1 are sometimes called the global Harnack inequality in the literature (see, for example, Bonforte-Vázquez [22] for the fractional porous medium equations in Euclidean spaces, Bonforte-Vazquez [19] for a result of having a lower and upper bound in terms of Barenblatt profiles for (19) in the global case that $\Omega = \mathbb{R}^n$, and also Bonforte-Simonov [17] for a complete characterization of the maximal set of initial data that produces solutions which are pointwisely trapped between two Barenblatt solutions). Parts (i) and (ii) of Theorem 1.1 follow directly from the comparison principle and a short time existence theorem. Part (iii) is more delicate. From the comparison principle, we first show that $T^* < +\infty$, from which we know that $u$ either blows up or becomes extinct at the time $T^*$ at least at one point on $\partial \Omega$. Secondly, we show elliptic type weak Harnack inequalities and local maximum principles for solutions of (1) on each time slice and then use them to show that $\|u\|_{L^{p+1}(\partial \Omega)}$ either blows up or becomes extinct at the time $T^*$. Harnack type inequalities for solutions of classical fast diffusion equations (19) were obtained in Bonforte-Simonov [16] and Bonforte-Dolbeault-Nazaret-Simonov [9]. In this step, the elliptic type weak Harnack inequalities and local maximum principles for (1) are
obtained by considering an evolution equation of a curvature-like quantity, which was used before by the first two authors in [44] for (19). Then by using a similar argument to that in Berryman-Holand [8], we prove that \( \|u\|_{L^{p+1}(\partial\Omega)} \) is bounded from below and above by two uniform multiples of \( (T^* - t)^{\frac{1}{p-1}} \). Finally, the uniform upper bound of \( u \) is obtained by Moser’s iteration, and the uniform lower bound of \( u \) is obtained by a quantitative Hopf’s lemma (Lemma 2.5). The proof of the uniform lower bound is the place where one can sense a nonlocal nature of the nonlinear boundary diffusion problem (1).

Part (i) of Theorem 1.2 follows directly from the comparison principle again. The convergence, and the rates in part (ii) and part (iv) of Theorem 1.2, as well as the upper bound in (17), are obtained by using Łojasiewicz’s inequality in infinite dimensional spaces developed by Simon [53]. Whether or not the exponent \( \theta \) in the Łojasiewicz inequality (57) can reach \( \frac{1}{2} \) would lead to either exponential convergence rates or algebraic convergence rates. The integrability condition on the solution of (11) introduced by Adams-Simon [1] (see also Allard-Almgren [2] for an earlier integrability hypothesis) implies the Łojasiewicz inequality with exponent \( \frac{1}{2} \), leading to the exponential convergence. If \( \lambda_1(p-1) < 0 \), then the smooth positive solution \( \varphi \) of (10) is unique, and the linearized operator at \( \varphi \) has a trivial kernel; see Proposition 2.8. Thus, \( \varphi \) is integrable. If \( \lambda_1 = 0 \), then \( \text{Ker} \mathcal{L}_{\varphi} = \text{span}\{\varphi\} \), and thus, \( \varphi \) is also integrable, by choosing \( \varphi_s = (1 + s)\varphi \). The dichotomy between (18) and the lower bound in (17) is proved by adapting the arguments of Choi-McCann-Seis [28] for the classical fast diffusion equation (19).

This paper is organized as follows. In Section 2, we collect some elementary inequalities and prove several elliptic results as a preparation. In Section 3, we show the short time existence of the solutions and its infinite speed of propagation. In Section 4, we show the finite time extinction or finite time blow up phenomena, and prove the uniform lower and upper integral bounds for the rescaled solution \( w \) defined in (13). In Section 5, we prove the uniform bounds in Theorem 1.1. In Section 6, we prove the convergence results in Theorem 1.2, including the sharp convergence rate, and show higher order asymptotics.

**Acknowledgement:** This is an improvement of part of the third author’s thesis at Beijing Normal University. We would like to thank the anonymous referee for his/her careful reading of the paper and for invaluable suggestions that greatly improved the presentation of the paper.

## 2 Steady states and some quantitative elliptic estimates

Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a bounded smooth domain. We first recall a Sobolev inequality and a trace inequality, which will make it easier for us to refer later. For \( 0 < q_1 < +\infty \) if \( n = 2 \), and \( 0 < q_1 \leq \frac{n+2}{n-2} \) if \( n \geq 3 \), there exists \( C_1 > 0 \) depending only on \( n, q_1 \) and \( \Omega \) such that

\[
\left( \int_{\Omega} |u|^{q_1+1} \, dx \right)^{\frac{2}{q_1+1}} \leq C_1 \left( \int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial\Omega} u^2 \, dS \right) \quad \text{for all } u \in H^1(\Omega). \tag{21}
\]
For $0 < q_2 < +\infty$ if $n = 2,$ and $0 < q_2 \leq \frac{n}{n-2}$ if $n \geq 3,$ there exists $C_2 > 0$ depending only on $n, q_2$ and $\Omega$ such that

\[
\left( \int_{\partial \Omega} |u|^{q_2+1} \, dS \right)^{\frac{2}{q_2+1}} \leq C \left( \int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial \Omega} u^2 \, dS \right)
\]

Consequently, by a compactness argument, we have the following coercivity.

**Proposition 2.1.** Let $a \in C^\infty(\partial \Omega).$ Assume $\lambda_1 > 0,$ where $\lambda_1$ is defined in (8). Then there exists $C > 0$ depending only on $n, \Omega, \lambda_1$ and $a$ such that

\[
\int_{\Omega} (|\nabla u|^2 + u^2) \, dx \leq C \left( \int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial \Omega} au^2 \, dS \right)
\]

for all $u \in H^1(\Omega).$

We will use the following known local maximum principle and weak Harnack inequality. For reader’s convenience, we sketch their proofs.

**Lemma 2.2 (Local maximum principle).** Suppose $w$ is a positive smooth (in $\Omega$) solution of

\[
\begin{cases}
-\Delta w \leq 0 & \text{in } \Omega \\
\frac{\partial}{\partial \nu} w \leq gw & \text{on } \partial \Omega
\end{cases}
\]

with $g^+ \in L^\infty(\partial \Omega),$ where $g^+(x) = \max(g(x), 0).$ Then there exist $C > 0,$ which depends only on $n, \Omega$ and $\|g^+\|_{L^\infty(\partial \Omega)},$ such that

\[
\sup_{\partial \Omega} w \leq C \|w\|_{L^1(\partial \Omega)}.
\]

**Proof.** For $k \geq 1,$ multiplying $w^k$ on both sides and integrating by parts, we obtain

\[
k \int_{\Omega} w^{k-1} |\nabla w|^2 \, dx \leq \int_{\partial \Omega} w^k \frac{\partial}{\partial \nu} w \, dS \leq \int_{\partial \Omega} gw^{k+1} \, dS.
\]

If we let $v = w^{k+1},$ then this implies that

\[
\int_{\Omega} |\nabla v|^2 \, dx \leq \frac{C(k+1)^2}{4k} \int_{\partial \Omega} v^2 \, dS.
\]

By the trace inequality (22),

\[
\left( \int_{\partial \Omega} |v|^q \, dx \right)^{\frac{2}{q}} \leq \frac{C(k+1)^2}{4k} \int_{\partial \Omega} v^2 \, dS;
\]

where $q = \frac{2(n-1)}{n-2}$ if $n \geq 3,$ and $q = 4$ if $n = 2.$ Then it follows from the standard Moser’s iteration (see, e.g., Chapter 8 of Gilbarg-Trudinger [42]) that

\[
\sup_{\partial \Omega} w \leq C \|w\|_{L^2(\partial \Omega)}.
\]

Then the conclusion follows from the fact that $\|w\|_{L^2(\partial \Omega)} \leq \left( \sup_{\partial \Omega} w \right)^{\frac{n}{2}} \left( \|w\|_{L^1(\partial \Omega)} \right)^{\frac{1}{2}}.$
We need an auxiliary lemma for the weak Harnack inequality in a form that we need.

**Lemma 2.3.** Let \( u \in C^2(\Omega) \) be a positive function such that \(-\Delta u \geq 0 \) in \( \Omega \). Then for every \( 0 < \delta \leq 1 \), there exists \( C > 0 \) depending only on \( n, \Omega \) and \( \delta \) such that

\[
\| u \|_{L^1(\Omega)} \geq C \| u \|_{L^1(\partial \Omega)}.
\]

**Proof.** Let \( \varphi(\xi) = u(\xi) \) for every \( \xi \in \partial \Omega \). Let \( P(x, \xi) : \Omega \times \partial \Omega \to \mathbb{R}^+ \) be the Poisson kernel. Note that we have the following estimates for the Poisson kernel (see, e.g., Theorem 1 in Krantz [49]):

\[
c_1 \frac{\text{dist}(x, \partial \Omega)}{|x - \xi|^n} \leq P(x, \xi) \leq c_2 \frac{\text{dist}(x, \partial \Omega)}{|x - \xi|^n},
\]

(23)

where \( c_1 \) and \( c_2 \) are two positive constants depending only on \( n \) and \( \Omega \). Let

\[
U(x) := \int_{\partial \Omega} P(x, \xi) \varphi(\xi) \, dS(\xi).
\]

Then

\[
\int_{\Omega} \frac{dx}{U(x)} = \int_{\Omega} \frac{dx}{U(x)^{\delta - 1}} \left( \int_{\partial \Omega} P(x, \xi) \varphi(\xi) \, dS(\xi) \right) \, dx
\]

\[
= \int_{\partial \Omega} \varphi(\xi) \left( \int_{\Omega} U(x)^{\delta - 1} P(x, \xi) \, dx \right) \, dS(\xi).
\]

By the reverse Hölder inequality, we have

\[
\int_{\Omega} U(x)^{\delta - 1} P(x, \xi) \, dx \geq \left( \int_{\Omega} U(x)^{\delta} \, dx \right)^{\frac{\delta - 1}{\delta}} \left( \int_{\Omega} P(x, \xi)^{\delta} \, dx \right)^{\frac{1}{\delta}}.
\]

Since \( \overline{\Omega} \) is smooth and compact, it satisfies a uniform interior ball condition, that is, there exists \( r > 0 \) such that for every \( \xi \in \partial \Omega \), there exists \( x_\xi \in \Omega \) such that the ball \( B_r(x_\xi) \subset \Omega \), and \( B_r(x_\xi) \cap \partial \Omega = \{\xi\} \). Hence,

\[
\int_{\Omega} P(x, \xi)^{\delta} \, dx \geq c_1 \int_{B_{r/2}(x_\xi)} \left( \frac{\text{dist}(x, \partial \Omega)}{|x - \xi|^n} \right)^{\delta} \, dx \geq c_1 |B_1| \left( \frac{1}{2n+1r^{n-1}} \right)^{\delta} \left( \frac{r}{2} \right)^n > 0.
\]

Therefore,

\[
\left( \int_{\Omega} U(x)^{\delta} \, dx \right)^{\frac{1}{\delta}} \geq C \int_{\partial \Omega} \varphi(\xi) \, dS(\xi).
\]

Since \( u \geq U \) in \( \Omega \) by the maximum principle, the conclusion follows. \( \square \)

**Lemma 2.4 (Weak Harnack inequality).** Suppose \( w \) is a positive smooth (in \( \overline{\Omega} \) ) solution of

\[
\begin{align*}
-\Delta w &\geq 0 \quad \text{in } \Omega \\
\frac{\partial}{\partial \nu} w &\geq gw \quad \text{on } \partial \Omega
\end{align*}
\]
with \( g^- \in L^\infty(\partial\Omega) \), where \( g^-(x) = -\min(g(x), 0) \). Then there exists \( C > 0 \), which depends only on \( n, \Omega \) and \( \|g^-\|_{L^\infty(\partial\Omega)} \), such that

\[
\inf_{\partial\Omega} w \geq \frac{1}{C}\|w\|_{L^1(\Omega)}.
\]

**Proof.** By the proof of Lemma A.1 of Han-Li [40], there exist \( C > 0 \) and \( \delta \in (0, 1) \), which depend only on \( n, \Omega \) and \( \|g^-\|_{L^\infty(\partial\Omega)} \), such that

\[
\inf_{\partial\Omega} w \geq \frac{1}{C}\|w\|_{L^\delta(\Omega)}.
\]

Then the conclusion follows from Lemma 2.3. \( \square \)

We will also use the following quantitative Hopf’s lemma for nonnegative harmonic functions when proving the uniform lower bound in part (iii) of Theorem 1.1. It should be known in the literature, but we cannot find a reference. Hence, we provide a proof.

**Lemma 2.5** (A quantitative Hopf’s lemma). Let \( u \in C^2(\overline{\Omega}) \) be a nonnegative function such that \( \Delta u = 0 \) in \( \Omega \). If \( u(x_0) = 0 \) for some \( x_0 \in \partial\Omega \), then there exists \( C > 0 \) depending only on \( n, \Omega \) such that

\[
-\partial_n u(x_0) \geq C \int_{\partial\Omega} u \, dS.
\]

**Proof.** Since \( u \) is nonnegative, if \( \int_{\partial\Omega} u \, dS = 0 \), then by the strong maximum principle, \( u \equiv 0 \) in \( \overline{\Omega} \), by which the lemma clearly holds.

If \( \int_{\partial\Omega} u \, dS > 0 \), by scaling, we can assume that \( \int_{\partial\Omega} u \, dS = 1 \). Similar to the proof of Lemma 2.3, we denote \( \varphi(\xi) = u(\xi) \) for every \( \xi \in \partial\Omega \). Let \( P(x, \xi): \Omega \times \partial\Omega \to \mathbb{R}^+ \) be the Poisson kernel. Then we have representation

\[
u(x) := \int_{\partial\Omega} P(x, \xi) \varphi(\xi) \, dS(\xi) \quad x \in \Omega.
\]

Without loss of generality, we assume that \( 0 \in \Omega \) and \( x_0 = (0, \cdots, 0, -1) \). Denote \( r = \frac{1}{2} \text{dist}(0, \partial\Omega) \). Then we know from (23) that

\[
u(x) \geq \tilde{C}(\Omega, n) > 0 \quad \forall \, x \in \overline{\Omega}.
\]

Let

\[
U := \Omega \cap \{x \mid x_n < 0\}, \quad B'_r := \{x : |x| < r, x_n = 0\},
\]

\( \eta \in C^\infty_c(B'_r) \) be nonnegative everywhere and equals to 1 in \( B'_{r/2} \), and \( \psi \) be the solution of

\[
\Delta \psi = 0 \quad \text{in} \, U, \quad 
\psi = \tilde{C} \eta \quad \text{on} \, \{x : |x| < r, x_n = 0\}, \quad 
\psi = 0 \quad \text{on} \, \partial U \setminus \{x : |x| < r, x_n = 0\}.
\]
Then by the maximum principle, it follows that
\[ u \geq \psi > 0 \text{ in } U. \]
Since \( u(x_0) = \psi(x_0) = 0 \), we have
\[ \partial_\nu (u - \psi)(x_0) \leq 0. \]
Finally it gives
\[ -\partial_\nu u(x_0) \geq -\partial_\nu \psi(x_0) > 0 \]
where we used Hopf’s Lemma for harmonic functions in the last inequality. The lemma is proved by choosing \( C = -\partial_\nu \psi(x_0) \) and by recalling the normalization \( \int_{\partial \Omega} u \, dS = 1 \) at the beginning of the proof.

Next, we prove the existence of solutions of (10) or (11).

**Proposition 2.6.** Let \( a \in C^\infty(\partial \Omega) \) and \( p \) satisfy (7). Then there exists a positive smooth solution of (10).

**Proof.** For \( u \in H^1(\Omega), u \not\equiv 0 \) on \( \partial \Omega \), we define
\[
E_p[u] := \frac{\int_\Omega |\nabla u|^2 \, dx + \int_{\partial \Omega} au^2 \, dS}{(\int_{\partial \Omega} |u|^{p+1} \, dS)^{\frac{2}{p+1}}}.
\]
Let
\[
Y_p = \inf_{u \in H^1(\Omega)} E_p[u]. \tag{24}
\]
If \( \lambda_1 > 0 \), then by Proposition 2.1, (22) and Hölder’s inequality, we have \( Y_p > 0 \). If \( \lambda_1 < 0 \), then by noticing that \( E_p[\phi_1] < 0 \), where \( \phi_1 \) is an eigenfunction associated to \( \lambda_1 \), we see that \( Y_p < 0 \). If \( \lambda_1 = 0 \), then on one hand, we have \( E_p[u] \geq 0 \) for all \( u \in H^1(\Omega) \) so that \( Y_p \geq 0 \), and on the other hand, \( E(\phi_1) \leq 0 \) so that \( Y_p \leq 0 \). Therefore, if \( \lambda_1 = 0 \), then \( Y_p = 0 \). Hence,
\[
\text{sgn}(Y_p) = \text{sgn}(\lambda_1).
\]
Furthermore, by an interpolation inequality (when \( 0 < p < 1 \)) or Hölder’s inequality (when \( p > 1 \)), we have
\[
\int_{\partial \Omega} |a|u^2 \, dS \leq \int_\Omega |\nabla u|^2 \, dx + C_0 \left( \int_{\partial \Omega} |u|^{p+1} \, dS \right)^{\frac{2}{p+1}} \quad \text{for all } u \in H^1(\Omega),
\]
where \( C_0 > 0 \) depends only on \( n, \Omega, p \) and \( \|a\|_{L^\infty(\Omega)} \). It follows that
\[
\int_\Omega |\nabla u|^2 \, dx + \int_{\partial \Omega} au^2 \, dS \geq -C_0 \left( \int_{\partial \Omega} |u|^{p+1} \, dS \right)^{\frac{2}{p+1}},
\]
and thus,

\[ Y_p \geq -C_0. \]  \hspace{1cm} (25)

Since \( p \) is subcritical, then by the standard variational method, the compact Sobolev embedding, and the fact that \( E[u] = E[|u|] \), we have that \( Y_p \) is achieved by some nonnegative function \( \varphi \in H^1(\Omega) \) satisfying \( \int_{\partial\Omega} \varphi^{p+1} \, dS = 1 \) and

\[-\Delta \varphi = 0 \quad \text{in} \ \Omega, \quad \partial_\nu \varphi + a \varphi = Y_p \varphi^p \quad \text{on} \ \partial\Omega \]

in the distribution sense. By the regularity result of Cherrier [27] and those for harmonic functions, we have \( \varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\Omega) \) for some \( \alpha > 0 \). Then by Hopf’s Lemma, we see that \( \varphi \) is positive in \( \Omega \), and then is smooth in \( \Omega \). Finally, since \( \text{sgn}(Y_p) = \text{sgn}(\lambda_1) \), we know that if \( \lambda_1 = 0 \), then \( \varphi \) is a solution of (10), and if \( \lambda_1 \neq 0 \), then \( \left( \frac{p}{|p-1|Y_p} \right)^\frac{p}{p-1} \varphi \) is a solution of (10).

The next two propositions are on uniqueness and non-degeneracy of the solutions to the stationary equation (10) for \( \lambda_1(p-1) < 0 \).

**Proposition 2.7.** Let \( a \in C^\infty(\partial\Omega) \) and \( p \) satisfy (7). If \( \lambda_1(p-1) < 0 \), then there exists a unique positive smooth (in \( \Omega \)) solution of

\[-\Delta \varphi = 0 \quad \text{in} \ \Omega, \quad \partial_\nu \varphi + a \varphi = \text{sgn}(\lambda_1) \frac{p}{|p-1|} \varphi^p \quad \text{on} \ \partial\Omega. \]

**Proof.** We only need to prove the uniqueness here.

Suppose \( u, v \in C^2(\overline{\Omega}) \) are two positive solutions. Suppose by contradiction that there exists \( x_0 \in \partial\Omega \) such that \( u(x_0) < v(x_0) \).

For \( \lambda \geq 0 \), define

\[ u_\lambda = \lambda u \]

and

\[ \bar{\lambda} = \inf \{ \lambda \geq 0 : u_\lambda \geq v \text{ on } \partial\Omega \}. \]

Since \( u(x_0) < v(x_0) \), we have \( \bar{\lambda} \geq 1 \), and thus,

\[ \partial_\nu u_\lambda + au_\lambda = \text{sgn}(\lambda_1) \frac{p}{|p-1|} \bar{\lambda}^{1-p} u_\lambda^p \geq \text{sgn}(\lambda_1) \frac{p}{|p-1|} u_\lambda^p \quad \text{on} \ \partial\Omega. \]

Then, we have \( -\Delta (u_\lambda - v) = 0 \) in \( \Omega \), and

\[ \partial_\nu (u_\lambda - v) + a(u_\lambda - v) + \text{sgn}(\lambda_1) \frac{p}{|p-1|} (u_\lambda^p - v^p) \geq 0 \quad \text{on} \ \partial\Omega. \]

By the definition of \( \bar{\lambda} \), \( u_\bar{\lambda} \geq v \) on \( \partial\Omega \), and there exists \( \bar{x} \in \partial\Omega \) such that \( u_\lambda(\bar{x}) = v(\bar{x}) \). It follows from the maximum principle and Hopf’s lemma for harmonic functions that

\[ u_\lambda \equiv v \quad \text{in} \ \Omega. \]
Hence, by the equations of \( v \) and \( u_\lambda \), we have
\[
\text{sgn}(\lambda_1) \left| \frac{p}{p-1} \right| u^p = \partial_\nu v + av = \partial_\nu u_\lambda + au_\lambda = \text{sgn}(\lambda_1) \left| \frac{p}{p-1} \right| \bar{\lambda}^{1-p} u_\lambda^p.
\]
Hence, \( \bar{\lambda} = 1 \), and thus,
\[
u \equiv v.
\]
This is a contradiction to the existence of \( x_0 \). This finishes the proof. \( \square \)

**Proposition 2.8.** Let \( a \in C^\infty(\partial \Omega) \) and \( p \) satisfy (7). Suppose \( \lambda_1(p-1) < 0 \), where \( \lambda_1 \) is defined in (8). Let \( \phi \) be the solution in Proposition 2.7. Then the linearized operator \( \mathcal{L}_\phi \) defined in (12) has a trivial kernel.

**Proof.** Consider the eigenvalue problem:
\[
\mathcal{L}_\phi \phi := B\phi + a\phi - \text{sgn}(\lambda_1) \left| \frac{p}{p-1} \right| \phi^{p-1} \phi = \mu \phi^{p-1} \phi.
\]
Since \( \phi \) is positive and satisfies
\[
\mathcal{L}_\phi \phi = p\phi^{p-1} \phi,
\]
then \( p \) must be the first eigenvalue of \( \mathcal{L}_\phi \) and \( \phi \) must be a corresponding first eigenfunction. Therefore, all the eigenvalues of \( \mathcal{L}_\phi \) are larger than or equal to \( p \). The conclusion follows. \( \square \)

## 3 Existence, uniqueness, and infinite speed of propagation

In this section, we first show that the nonlinear boundary diffusion problem (1)–(2) has a unique solution on a small time interval. The following a priori estimates in Sobolev spaces were proved in Lemma 3.4 of Brendle [24].

**Lemma 3.1.** Let \( 0 < p < \infty \). Let \( \phi \) be a smooth solution of the linear initial boundary value problem
\[
\begin{align*}
\Delta \phi &= 0 \quad \text{in } \Omega \times (0, \infty), \\
\partial_t \phi &= -\frac{1}{p} w^{-(p-1)} \partial_\nu \phi + f \quad \text{on } \partial \Omega \times (0, \infty), \\
\phi &= 0 \quad \text{on } \partial \Omega \times \{t = 0\},
\end{align*}
\]
where \( c_0 \leq u \leq C_0 \) on \( \Omega \times (0, \infty) \) for some positive constants \( c_0 \) and \( C_0 \). In addition, we assume that \( u \) satisfies
\[
\| u \|_{W^{m,2}(\partial \Omega \times [0,T])} \leq C_1
\]
for some nonnegative integer \( m \). Then there exists \( C > 0 \) depending only on \( n, p, \Omega, m, T, c_0, C_0 \) and \( C_1 \) such that
\[
\| \phi \|_{W^{m+1,2}(\partial \Omega \times [0,T])} \leq C \| f \|_{W^{m,2}(\partial \Omega \times [0,T])}.
\]
Then, the short time existence for (1)–(2) follows from Lemma 3.1 and the implicit function theorem. The proof is standard and we omit it here.

**Theorem 3.2.** Let \( n \geq 2, \Omega \subset \mathbb{R}^n \) be a bounded smooth domain, \( a \in C^\infty(\partial\Omega), 0 < p < \infty, \) and \( u_0 \in C^\infty(\partial\Omega) \) be a positive function. Then the initial boundary value problem (1)–(2) has a unique smooth positive solution on a small time interval.

We will use the next comparison principle to show various dynamics of the solutions to (1)–(2).

**Proposition 3.3 (Comparison Principle).** Let \( 0 < p < \infty, \ a \in C^\infty(\partial\Omega), \) and \( b_j \in C^\infty(\overline{\Omega}) \) for \( j = 1, \cdots, n. \) Let \( c \in C^\infty(\partial\Omega) \) be positive everywhere. Suppose \( u_1 \) and \( u_2 \) are two smooth positive functions in \( \overline{\Omega} \times [0, T) \) satisfying \( u_1(\cdot, 0) \leq u_2(\cdot, 0) \) in \( \Omega, \)

\[
\Delta u_1 - \sum_{j=1}^{n} b_j \partial_{x_j} u_1 \geq 0 \quad \text{in} \ \Omega \times (0, T),
\]

\[
\partial_t u_1^p \leq -c \partial_{\nu} u_1 - au_1 \quad \text{on} \ \partial\Omega \times (0, T),
\]

and

\[
\Delta u_2 - \sum_{j=1}^{n} b_j \partial_{x_j} u_2 \leq 0 \quad \text{in} \ \Omega \times (0, T),
\]

\[
\partial_t u_2^p \geq -c \partial_{\nu} u_2 - au_2 \quad \text{on} \ \partial\Omega \times (0, T).
\]

Then

\[
u_1 \leq u_2 \quad \text{on} \ \overline{\Omega} \times [0, T).
\]

**Proof.** The difference \( u_1 - u_2 \) satisfies

\[
- \Delta (u_1 - u_2) + \sum_{j=1}^{n} b_j \partial_{x_j} (u_1 - u_2) \leq 0 \quad \text{in} \ \Omega \times (0, T),
\]

\[
pu_1^{p-1} \partial_t (u_1 - u_2) \leq -c \partial_{\nu} (u_1 - u_2) - g(x, t)(u_1 - u_2) \quad \text{on} \ \partial\Omega \times (0, T),
\]

where

\[
g(x, t) = a(x) + p(p-1)\partial_t u_2(x, t) \cdot \int_0^1 [\lambda u_1(x, t) + (1 - \lambda)u_2(x, t)]^{p-2} d\lambda.
\]

Let \( s \in (0, T). \) Then \( u_1 \) and \( u_2 \) are smooth and positive in \( \overline{\Omega} \times [0, s], \) and \( g \) is bounded on \( \partial\Omega \times [0, s]. \) Choose \( C > 0 \) such that

\[
C pu_1^{p-1} + g > 0 \quad \text{on} \ \partial\Omega \times [0, s].
\]
Let \( v(x, t) = e^{-Ct}(u_1(x, t) - u_2(x, t)) \). Then
\[
-\Delta v + \sum_{j=1}^{n} b_j \partial_{x_j} v \leq 0 \quad \text{in } \Omega \times (0, s],
\]
\[
p u_1^{p-1} \partial_{x_1} v \leq -c \partial_{\nu} v - (C p u_1^{p-1} + g) v \quad \text{on } \partial\Omega \times (0, s],
\]
\[
v(\cdot, 0) \leq 0 \quad \text{in } \Omega.
\]

Then, it follows from the maximum principle and Hopf’s lemma for elliptic equations that
\[
v \leq 0 \quad \text{on } \overline{\Omega} \times [0, s].
\]
That is,
\[
u_1 \leq u_2 \quad \text{on } \overline{\Omega} \times [0, s].
\]
Since \( s \in (0, T) \) is arbitrary, we have
\[
u_1 \leq u_2 \quad \text{on } \overline{\Omega} \times [0, T).
\]

Now, let us go back to the equations (1)–(2) for \( p \) satisfying (7). Using the comparison principle in Proposition 3.3, we will derive the following estimates.

**Proposition 3.4.** Let \( n \geq 2, \Omega \subset \mathbb{R}^n \) be a bounded smooth domain, \( a \in C^\infty(\partial\Omega) \), \( p \) satisfy (7), \( u_0 \in C^\infty(\partial\Omega) \) be a positive function, and \( u \) be a smooth positive solution of (1)–(2) on \( \overline{\Omega} \times [0, T] \) for some \( T > 0 \). Then there exist \( s_1 > 0 \) depending only on \( n, p, a, \Omega \) and \( \min_{\partial\Omega} u_0 \), \( s_2 > 0 \) depending only on \( n, p, a, \Omega \) and \( \max_{\partial\Omega} u_0 \), and \( C > 0 \) depending only on \( n, p, a \) and \( \Omega \), such that

(i). If \( \lambda_1(p-1) < 0 \), then
\[
(t + s_1)^{\frac{1}{p-1}} \varphi(x) \leq u(x, t) \leq (t + s_2)^{\frac{1}{p-1}} \varphi(x) \quad \text{on } \overline{\Omega} \times [0, T],
\]
where \( \varphi \) is the unique positive solution of (10).

(ii). If \( \lambda_1 = 0 \), then
\[
\frac{s_1}{C} \leq u(x, t) \leq C s_2 \quad \text{on } \overline{\Omega} \times [0, T].
\]

(iii). If \( \lambda_1 > 0 \) and \( p > 1 \), then
\[
u(x, t) \geq \frac{1}{C} (s_1 - t)^{\frac{1}{p-1}} \quad \text{on } \overline{\Omega} \times [0, \min(T, s_1)],
\]
\[
u(x, t) \leq C (s_2 - t)^{\frac{1}{p-1}} \quad \text{on } \overline{\Omega} \times [0, T].
\]
(iv). If $\lambda_1 < 0$ and $0 < p < 1$, then
\[
\begin{align*}
    u(x, t) &\geq \frac{1}{C} (s_1 - t)^{\frac{1}{p-1}} & \text{on } \overline{\Omega} \times [0, T], \\
    u(x, t) &\leq C (s_2 - t)^{\frac{1}{p-1}} & \text{on } \overline{\Omega} \times [0, \min(T, s_2)].
\end{align*}
\]

Proof. Let $\varphi$ be a positive solution of (10) and $b_c(t)$ be defined in (9). If $\lambda_1 (p - 1) < 0$, then we know from Proposition 2.7 that $\varphi$ is unique. The function $b_c(t) \varphi(x)$ satisfies the equation (1). Then the conclusion follows from the comparison principle in Corollary 3.3 by choosing proper $c$ such that either $b_c(0) \varphi(x) \leq u_0(x)$ or $b_c(0) \varphi(x) \geq u_0(x)$. \qed

Next, we show that the equation (1) has infinite speed of propagation for all $p$ satisfying (7), including $0 < p < 1$.

Proof of Theorem 1.4. First, it follows from Proposition 3.4 that there exist $T_0 > 0$ and $C_0 > 0$, both of which depend only on $n, p, a, \Omega$ and $\max_{\partial \Omega} u_0$, such that
\[
u \leq C_0 \quad \text{on } (x, t) \in \overline{\Omega} \times [0, T_0]. \tag{26}
\]

Secondly, we would like to show in the below that there exist $t_0 \in (0, T_0]$ and $C_1 > 0$, both of which depend only on $n, p, a, \Omega, \max_{\partial \Omega} u_0$ and $\|u_0\|_{L^p(\partial \Omega)}$, such that
\[
\int_{\partial \Omega} u^p(\cdot, t) \, dS \geq C_1 \quad \text{for all } t \in [0, t_0]. \tag{27}
\]

The proof of (27) will be split into several cases. To start with, let $\phi_1$ be the positive eigenfunction associated to $\lambda_1$ defined in (8) such that $\|\phi_1\|_{L^2(\partial \Omega)} = 1$. Then by multiplying $\phi_1$ to (1) and integrating over $\partial \Omega$, we obtain
\[
\frac{d}{dt} \int_{\partial \Omega} u^p(\cdot, t) \phi_1 \, dS = -\lambda_1 \int_{\partial \Omega} u(\cdot, t) \phi_1 \, dS. \tag{28}
\]

Case 1: $\lambda_1 \leq 0$. Then it follows from (28) that
\[
\frac{d}{dt} \int_{\partial \Omega} u^p(\cdot, t) \phi_1 \, dS \geq 0,
\]
and thus,
\[
\int_{\partial \Omega} u^p(\cdot, t) \phi_1 \, dS \geq \int_{\partial \Omega} u_0^p \phi_1 \, dS \quad \forall \ t \in [0, T_0].
\]

Since $\phi_1$ is bounded from below and above by two positive constants that depend only on $n, p, a$ and $\Omega$, this proves (27).

Case 2: $\lambda_1 > 0$ and $0 < p \leq 1$. Then from (26) and (28), we have
\[
\frac{d}{dt} \int_{\partial \Omega} u^p(\cdot, t) \phi_1 \, dS = -\lambda_1 \int_{\partial \Omega} u(\cdot, t) \phi_1 \, dS \geq -\lambda_1 C_0^{1-p} \int_{\partial \Omega} u^p(\cdot, t) \phi_1 \, dS.
\]

17
Solving this differential inequality, we obtain
\[ \int_{\partial\Omega} u^p(\cdot, t) \phi_1 \, dS \geq e^{-\lambda_1 C_0^{1-p} t} \int_{\partial\Omega} u_0^p \phi_1 \, dS \quad \forall \, t \in [0, T_0]. \]
Hence,
\[ \int_{\partial\Omega} u^p(\cdot, t) \phi_1 \, dS \geq e^{-\lambda_1 C_0^{1-p} T_0} \int_{\partial\Omega} u_0^p \phi_1 \, dS \quad \forall \, t \in [0, T_0]. \]
Therefore, as in case 1, (27) follows.

Case 3: \( \lambda_1 > 0 \) and \( p > 1 \). Then by using Hölder’s inequality, it follows from (28) that
\[
\frac{d}{dt} \int_{\partial\Omega} u^p(\cdot, t) \phi_1 \, dS = -\lambda_1 \int_{\partial\Omega} u(\cdot, t) \phi_1 \, dS \geq -C_2 \left( \int_{\partial\Omega} u^p(\cdot, t) \phi_1 \, dS \right)^{\frac{1}{p}},
\]
where \( C_2 > 0 \) depends only on \( n, p, a \) and \( \Omega \). Since we know from (28) that the integral \( \int_{\partial\Omega} u^p(\cdot, t) \phi_1 \, dS \) is decreasing in \( t \), we obtain
\[
\frac{d}{dt} \int_{\partial\Omega} u^p(\cdot, t) \phi_1 \, dS \geq -C_2 \left( \int_{\partial\Omega} u_0^p \phi_1 \, dS \right)^{\frac{1}{p}}.
\]
Integrating in the time variable, we obtain
\[
\int_{\partial\Omega} u^p(\cdot, t) \phi_1 \, dS \geq \int_{\partial\Omega} u_0^p \phi_1 \, dS - t C_2 \left( \int_{\partial\Omega} u_0^p \phi_1 \, dS \right)^{\frac{1}{p}}.
\]
Hence, one proves (27) by choosing a proper \( t_0 \).

Finally, we claim that for a sufficiently small \( \varepsilon_0 \), which will be fixed in the end, there holds
\[ u(x, t) > \varepsilon_0 t^{1/p} \quad \text{for all} \quad (x, t) \in \overline{\Omega} \times [0, t_0]. \]
Suppose not, then let \( t_1 < t_0 \) be the first time that \( u \) touches the function \( \varepsilon_0 t^{1/p} \) at some point \( x_1 \in \partial\Omega \). Since \( u_0 \) is positive, we have \( t_1 > 0 \). Then
\[ \partial_t (u^p)(x_1, t_1) \leq \varepsilon_0^p \quad \text{and} \quad u(x_1, t_1) = \varepsilon_0 t_1^{1/p}. \]
Using the equation (1), we then have
\[ -\partial_t u(x_1, t_1) \leq \varepsilon_0^p + \varepsilon_0 t_1^{1/p} \|a\|_{L^\infty(\partial\Omega)} \leq \varepsilon_0^p + \varepsilon_0 t_1^{1/p} \|a\|_{L^\infty(\partial\Omega)}. \]
By Lemma 2.5, then we have
\[ \int_{\partial\Omega} u(\cdot, t_1) \, dS \leq C_3 \varepsilon_0^p + C_3 \varepsilon_0 t_1^{1/p}, \]
where \( C_3 > 0 \) depends only on \( n, \Omega \) and \( \|a\|_{L^\infty(\partial\Omega)} \). Then when \( p \geq 1 \), it follows from (26) and (27) that
\[ C_1 \leq \int_{\partial\Omega} u^p(\cdot, t_1) \, dS \leq C_0^{p-1} \int_{\partial\Omega} u(\cdot, t_1) \, dS \leq C_0^{p-1} (C_3 \varepsilon_0^p + C_3 \varepsilon_0 t_1^{1/p}). \]
This would be impossible if we choose \( \varepsilon_0 \) sufficiently small. When \( 0 < p < 1 \), then by Hölder’s inequality and (27), we have

\[
C_4^{1/p} \leq \left( \int_{\partial \Omega} u^p(\cdot, t_1) \, dS \right)^{\frac{1}{p}} \leq C_4 \int_{\partial \Omega} u(\cdot, t_1) \, dS \leq C_4(C_3 \varepsilon_0^p + C_3 \varepsilon_0^{1/p}),
\]

where \( C_4 > 0 \) depends only on \( n \) and \( \Omega \). This would be impossible either, if we choose \( \varepsilon_0 \) sufficiently small.

This theorem is proved. \( \square \)

4 Extinction or blow up in finite time, and integral bounds

Part (iii) of Proposition 3.4 immediately implies that if \( \lambda_1 > 0 \) and \( p > 1 \), then the solution \( u \) cannot be positive forever, that is,

\[
\sup \{ t > 0 : u > 0 \text{ on } \partial \Omega \times [0, t) \} = +\infty.
\]

Similarly, part (iv) of Proposition 3.4 implies that if \( \lambda_1 < 0 \) and \( 0 < p < 1 \), then the solution \( u \) cannot be bounded forever, that is,

\[
\sup \{ t > 0 : \|u\|_{L^\infty(\partial \Omega \times [0, t))} < +\infty \} < +\infty.
\]

**Definition 4.1.** Let \( u \) be as in Proposition 3.4. Suppose \( \lambda_1(p-1) > 0 \). We define

\[
T^* := \begin{cases} 
\sup \{ t > 0 : u > 0 \text{ on } \partial \Omega \times [0, t) \}, & \text{if } \lambda_1 > 0 \text{ and } p > 1, \\
\sup \{ t > 0 : \|u\|_{L^\infty(\partial \Omega \times [0, t))} < +\infty \}, & \text{if } \lambda_1 < 0 \text{ and } 0 < p < 1.
\end{cases}
\]

When \( \lambda_1 > 0 \) and \( p > 1 \), we call \( T^* \) as the extinction time of \( u \). When \( \lambda_1 < 0 \) and \( 0 < p < 1 \), we call \( T^* \) as the blow-up time of \( u \).

Consequently, it follows from Proposition 3.4 that

**Corollary 4.2.** Let \( T^* \) be as in Definition 4.1. Then there exist \( s_1 > 0 \) small, and \( s_2 > 0 \) large, both of which depend only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \), such that

\[
s_1 \leq T^* \leq s_2.
\]

The reason why we call this supremum \( T^* \) as the extinction time of the solution when \( \lambda_1 > 0 \) and \( p > 1 \), is that if \( u \) vanishes at one point at time \( T \), then \( u(\cdot, T) \equiv 0 \), by using the following elliptic type weak Harnack inequality for the problem (1)–(2). This idea of proving such a lower bound was used earlier in Jin-Xiong [46].

**Proposition 4.3.** Let \( u \) be a smooth positive solution of (1)–(2) on \( \Omega \times [0, T) \) with \( p > 1 \) and \( T > 0 \). Suppose \( u \leq M \) on \( \Omega \times [0, T) \). There exists \( C > 0 \) depending only on \( n, p, a, \Omega, M, \min_{\partial \Omega} u_0 \) and \( \|u_0\|_{C^{1,\alpha}(\Omega)} \) (with any \( \alpha > 0 \) fixed) such that

\[
\inf_{x \in \partial \Omega} u(x, t) \geq \frac{1}{C} \int_{\partial \Omega} u^{p+1}(\cdot, t) \, dS \quad \forall \ 0 < t < T.
\]
Proof. Define 
\[ H = -p \frac{\partial_t u}{u}. \]
Then 
\[ \Delta(uH) = 0 \quad \text{in } \Omega \]
and 
\[ H = u^{-p}(\partial_{\nu} u + au) \quad \text{on } \partial\Omega. \]
Then 
\[ \partial_t H = -pu^{-p+1}(\partial_{\nu} u)(\partial_{\nu} u + au) + u^{-p}(\partial_{\nu} + a)\partial_t u 
\[ = -u^{-2p}(\partial_t u^p)(\partial_{\nu} u + au) + \frac{1}{p} u^{-p}(\partial_{\nu} + a)(u^{-p+1}\partial_t u^p) 
\[ = u^{-2p}(\partial_{\nu} u + au)^2 - \frac{1}{p} u^{-p}(\partial_{\nu} + a)(u^{-p+1}(\partial_{\nu} + a)u) 
\[ = H^2 - \frac{1}{p} u^{-p}(\partial_{\nu} + a)(uH) 
\[ = -\frac{1}{p} u^{-p}(H\partial_{\nu} u + u\partial_{\nu} H + auH) + H^2 
\[ = -\frac{1}{p} u^{-p+1}\partial_{\nu} H + \left(1 - \frac{1}{p}\right)H^2 + \left(1 - \frac{1}{p}\right)H^2 \]
\[ \geq -\frac{1}{p} u^{-p+1}\partial_{\nu} H. \quad (30) \]

So we have the linear elliptic equation for \( H \):
\[ \begin{cases} 
\Delta H + 2u^{-1}\nabla u \cdot \nabla H = 0 & \text{in } \Omega \times (0, T), \\
\partial_t H \geq -\frac{1}{p} u^{-p+1}\partial_{\nu} H & \text{on } \partial\Omega \times (0, T). 
\end{cases} \]

By Proposition 3.3, we have
\[ H \geq c_1 := \min \left\{ \min_{\partial\Omega} H(\cdot, 0), 0 \right\} \quad \text{on } \overline{\Omega} \times [0, T). \]

That is
\[ \partial_{\nu} u + au \geq c_1 u^p \quad \text{on } \partial\Omega \quad \text{for all } t \in (0, T). \]

So we have for all \( t \in [0, T) \) that
\[ \begin{cases} 
\Delta u = 0 & \text{in } \Omega, \\
\partial_{\nu} u \geq g(x, t)u & \text{on } \partial\Omega 
\end{cases} \]
with \( g(x, t) = c_1 u^{p-1} - a \) and \( |g(x, t)| \leq |c_1| |M^{p-1} + |a| \) on \( \partial\Omega \). By Lemma 2.4, we have
\[ \inf_{x \in \partial\Omega} u(x, t) \geq \frac{1}{C} \int_{\partial\Omega} u(\cdot, t) \, dS \quad \text{for all } t \in (0, T). \]

Since \( u \leq M \) on \( \overline{\Omega} \times [0, T) \), we have \( \int_{\partial\Omega} u(\cdot, t) \, dS \geq M^{-p} \int_{\partial\Omega} u^{p+1}(\cdot, t) \, dS \), from which the conclusion follows. \( \square \)
Similarly, if $0 < p < 1$, and $\|u(\cdot, t)\|_{L^\infty(\partial\Omega)}$ blows up at time $T^*$, then $\|u(\cdot, t)\|_{L^{p+1}(\partial\Omega)}$ will also blow up at time $T^*$, following from the local maximum principle in Lemma 2.2.

**Proposition 4.4.** Let $u$ be a smooth positive solution of (1)–(2) on $\overline{\Omega} \times [0, T)$ with $0 < p < 1$ and $T > 0$. Suppose $u \geq m$ on $\overline{\Omega} \times [0, T)$ for some $m > 0$. There exists $C > 0$ depending only on $n, p, a, \Omega, m, \max_{\partial\Omega} u_0$ and $\|u_0\|_{C^{1, \alpha}(\partial\Omega)}$ (with any $\alpha > 0$ fixed) such that

$$\sup_{x \in \partial\Omega} u(x, t) \leq C \int_{\partial\Omega} u^{p+1}(\cdot, t) \, dS \quad \forall 0 < t < T.$$ 

**Proof.** Let $H$ be the one defined in Proposition 4.3. Then it follows from (30) that

$$\begin{cases} \Delta H + u^{-1} \nabla u \cdot \nabla H = 0 & \text{in } \Omega \times (0, T), \\ \partial_t H \leq -\frac{1}{p} u^{p+1} \partial_t H & \text{on } \partial\Omega \times (0, T). \end{cases}$$

By Proposition 3.3, we have

$$H \leq c_2 := \max \{ \max_{\partial\Omega} H(\cdot, 0), 0 \} \quad \text{on } \overline{\Omega} \times [0, T).$$

That is

$$\partial_t u + au \leq c_2 u^p \quad \text{on } \partial\Omega \quad \text{for all } t \in (0, T).$$

So we have for all $t \in [0, T)$ that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_t u \leq g(x, t) u & \text{on } \partial\Omega \end{cases}$$

with $g(x, t) = c_1 u^{p-1} - a$ and $|g(x, t)| \leq |c_1| m^{p-1} + |a|$ on $\partial\Omega$. By Lemma 2.2, we have

$$\sup_{x \in \partial\Omega} u(x, t) \leq C \int_{\partial\Omega} u(\cdot, t) \, dS \quad \forall t \in (0, T).$$

Since $u \geq m$ on $\overline{\Omega} \times [0, T)$, we have $\int_{\partial\Omega} u(\cdot, t) \, dS \leq m^{-p} \int_{\partial\Omega} u^{p+1}(\cdot, t) \, dS$, from which the conclusion follows. \hfill \Box

Consequently, we have

**Proposition 4.5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p$ satisfy (7), $a \in C^\infty(\partial\Omega)$ and $u_0 \in C^\infty(\partial\Omega)$ be a positive function. Suppose $\lambda_1 (p - 1) > 0$, and $u$ is the smooth positive solution of (1)–(2) on $\overline{\Omega} \times [0, T^*)$, where $T^* < \infty$ is the one defined in (29).

(i) If $\lambda_1 > 0$ and $p > 1$, then

$$\liminf_{t \to (T^*)^-} \|u(\cdot, t)\|_{L^{p+1}(\partial\Omega)} = 0. \quad (31)$$
(ii). If $\lambda_1 < 0$ and $0 < p < 1$, then

$$\limsup_{t \to (T^*)^{-}} \| u(\cdot, t) \|_{L^{p+1}(\partial \Omega)} = +\infty.$$  \hfill (32)

**Proof.** It follows from Proposition 4.3, Proposition 4.4, and the definition of $T^*$. \hfill \Box

Now we can derive the precise decay rate of $\| u(\cdot, t) \|_{L^{p+1}(\partial \Omega)}$ near the extinction time $T^*$, or the precise blow up rate of $\| u(\cdot, t) \|_{L^{p+1}(\partial \Omega)}$ near the blow up time $T^*$.

**Proposition 4.6.** Assume all the assumptions in Proposition 4.5. Then there exist $C > 0$ and $c > 0$, both of which depend only on $n, p, a, \Omega, \max_{\partial \Omega} u_0$ and $\min_{\partial \Omega} u_0$, such that

$$c(T^* - t)^{\frac{1}{p-1}} \leq \| u(\cdot, t) \|_{L^{p+1}(\partial \Omega)} \leq C(T^* - t)^{\frac{1}{p-1}}.$$  

**Proof.** By (1), we have

$$\frac{d}{dt} \int_{\partial \Omega} u^{p+1}(x, t) \, dS = -\frac{p+1}{p} \left( \int_{\Omega} |\nabla u(x, t)|^2 \, dx + \int_{\partial \Omega} au^2(x, t) \, dS \right),$$  \hfill (33)

and

$$\frac{d}{dt} \left( \int_{\Omega} |\nabla u(x, t)|^2 \, dx + \int_{\partial \Omega} au^2(x, t) \, dS \right) = -\frac{2}{p} \int_{\partial \Omega} \frac{(\partial_{\nu} u + au)^2}{u^{p-1}} \, dS \leq 0.$$  

Let

$$I(t) = \frac{\int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial \Omega} au^2 \, dS}{\left( \int_{\partial \Omega} u^{p+1} \, dS \right)^{\frac{2}{p+1}}}.$$  

Then

$$\frac{d}{dt} I(t) = \frac{2}{p} \left( \int_{\partial \Omega} u^{p+1}(x, t) \, dS \right)^{-\frac{2}{p+1}} \left[ \left( \frac{\int_{\Omega} |\nabla u(x, t)|^2 \, dx + \int_{\partial \Omega} au^2(x, t) \, dS}{\int_{\partial \Omega} u^{p+1}(x, t) \, dS} \right)^2 - \int_{\partial \Omega} \frac{(\partial_{\nu} u + au)^2}{u^{p-1}} \, dS \right].$$

Since $u$ is harmonic in $\Omega$, we have $\int_{\Omega} |\nabla u|^2 \, dx = \int_{\partial \Omega} u \partial_{\nu} u \, dS$. Applying the Cauchy-Schwarz inequality, we obtain:

$$\left( \int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial \Omega} au^2 \, dS \right)^2 \leq \int_{\partial \Omega} u^{p+1} \, dS \int_{\partial \Omega} \frac{(\partial_{\nu} u + au)^2}{u^{p-1}} \, dS.$$
Hence,
\[
\frac{d}{dt} I(t) \leq 0. \quad (34)
\]
Define
\[
Z(t) := \left( \int_{\partial \Omega} u^{p+1}(x, t) \, dS \right)^{\frac{p-1}{p+1}}.
\]
Then it follows from (33) that
\[
Z'(t) = -\frac{p-1}{p} I(t). \quad (35)
\]
Case 1: \( \lambda_1 > 0 \) and \( p > 1 \).
Then it follows from (35) that
\[
Z'(t) \leq -C
\]
for some \( C > 0 \) depending only on \( n, p, a \) and \( \Omega \), where we used \( p > 1 \), the trace inequality (22) and Proposition 2.1 in the last inequality. Then
\[
Z(T) - Z(t) \leq -C(T-t).
\]
Taking \( \lim \inf_{T \to (T^*)^-} \) on both sides and using (31), we obtain
\[
Z(t) \geq C(T^* - t).
\]
Hence, the first inequality follows.
By (35) and (34), we have \( Z''(t) \geq 0 \). Hence, for \( 0 < s < t < T < T^* \), we have
\[
Z(t) \leq Z(T) + \frac{Z(s) - Z(T)}{s-T}(t-T).
\]
Taking \( \lim \inf_{T \to (T^*)^-} \) and \( \lim_{s \to 0^+} \) on both sides, and using (31), we have
\[
Z(t) \leq \frac{Z(0)}{T^*} (T^* - t).
\]
Hence, the second inequality follows, with the help of Corollary 4.2.
Case 2: \( \lambda_1 < 0 \) and \( 0 < p < 1 \).
It follows from (35) and (25) that
\[
Z'(t) = \frac{1-p}{p} I(t) \geq \frac{Y_p(1-p)}{p} \geq -C,
\]
where \( C > 0 \) depending only on \( n, p, \Omega \) and \( a \). Then integrating in \( t \), we have
\[
Z(T) - Z(t) \geq -C(T-t).
\]
Taking \( \limsup_{T \to (T^*)^-} \) on both sides and using (32), we obtain
\[
Z(t) \leq C(T^* - t).
\]

Hence, the second inequality follows.

By (35) and (34), we have \( Z''(t) \leq 0 \). Hence, for \( 0 < s < t < T < T^* \), we have
\[
Z(t) \geq Z(T) + \frac{Z(s) - Z(T)}{s - T}(t - T).
\]

Taking \( \limsup_{T \to (T^*)^-} \) and \( \lim_{s \to 0^+} \) on both sides, and using (32), we have
\[
Z(t) \geq \frac{Z(0)}{T^*}(T^* - t).
\]

Hence, the first inequality follows, with the help of Corollary 4.2. \( \square \)

5 Uniform upper and lower bounds

Let \( \lambda_1(p - 1) > 0, u \) and \( T^* < \infty \) be as in Proposition 4.6. Define
\[
w(x, \tau) = \frac{u(x, t)}{(T^* - t)^{\frac{1}{p-1}}} \quad \text{with} \quad t = T^*(1 - e^{-\tau}). \tag{36}
\]

Then \( w(x, \tau) \) is smooth, positive, locally bounded on \( \Omega \times [0, +\infty) \), and satisfies
\[
\begin{cases}
\Delta w(x, \tau) = 0 & \text{in } \Omega \times (0, \infty), \\
\frac{\partial \tau}{w^p} = -\frac{\partial_n w}{w} - aw + p \frac{w^p}{p - 1} \quad & \text{on } \partial \Omega \times (0, \infty), \\
w(x, 0) = w_0 := \frac{u_0}{T^*^{p-1}} \quad & \text{on } \partial \Omega \times \{t = 0\}. \tag{37}
\end{cases}
\]

It follows from Proposition 4.6 that there exist \( C > 0 \) and \( c > 0 \), both of which depend only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \), such that
\[
c \leq \|w(\cdot, \tau)\|_{L^{p+1}(\partial \Omega)} \leq C \quad \text{for all } \tau > 0. \tag{38}
\]

Once we have the integral bounds (38) of \( w \), we can use Moser’s iteration, similar to that in Bonforte-Vázquez [20] for the classic fast diffusion equation (19), to derive its \( L^\infty \) bound.

**Proposition 5.1.** Assume all the assumptions in Proposition 4.6. Let \( u \) and \( T^* \) be as in Proposition 4.6. Let \( w \) be defined in (36). Then there exists \( C > 0 \) depending only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \) such that
\[
w(x, \tau) \leq C \quad \text{for all } x \in \partial \Omega \text{ and } \tau \geq 0.
\]
Proof. It follows from Proposition 3.4 and Corollary 4.2 that there exists $\tau_0 > 0$ depending only on $n, p, a, \Omega, \max_{\partial \Omega} u_0$ and $\min_{\partial \Omega} u_0$ such that the conclusion holds for $0 \leq \tau \leq \tau_0$. In the following, we shall prove it for $\tau \geq \tau_0$. By rescaling, we can assume $\tau_0 = 1$.

We only provide the proof for $n \geq 3$, while the other case $n = 2$ can be proved in the same way.

Let $0 < S_1 < S_2 < S$, such that $|S_2 - S_1| \leq 1$. Let $\eta(\tau)$ be a smooth cut-off function satisfying $\eta(\tau) = 0$ for $\tau < S_1$, $0 \leq \eta(\tau) \leq 1$ for $S_1 \leq \tau \leq S_2$, $\eta(\tau) = 1$ for $\tau > S_2$, and $\eta'(\tau) \leq \frac{2}{S_2 - S_1}$. Let $\Gamma_1 = \partial \Omega \times [S_1, S]$, $\Gamma_2 = \partial \Omega \times [S_2, S]$, $Q_1 = \Omega \times [S_1, S]$, $Q_2 = \Omega \times [S_2, S]$. Let $\alpha > -1$. In the below, $C$ will be different constants depending only on $n, p, a, \Omega, \max_{\partial \Omega} u_0$ and $\min_{\partial \Omega} u_0$, but may change from lines to lines.

Multiplying $\eta^2 w^{1+\alpha}$ to the boundary equation of (37) and integrating over $\Gamma_1$, we have

$$\int_{\Gamma_1} \eta^2 w^{1+\alpha} \frac{\partial}{\partial \tau} w^p + \int_{\Gamma_1} \eta^2 w^{1+\alpha} \frac{\partial}{\partial \nu} w \leq - \int_{\Gamma_1} a \eta^2 w^{2+\alpha} + \frac{p}{|p-1|} \int_{\Gamma_1} \eta^2 w^{p+1+\alpha}.$$  
Then integration by parts gives

$$\int_{\Gamma_1} \eta^2 w^{1+\alpha} \frac{\partial}{\partial \tau} w^p + \int_{\Gamma_1} \eta^2 \nabla (w^{1+\alpha}) \nabla w \leq C \int_{\Gamma_1} \eta^2 w^{2+\alpha} + C \int_{\Gamma_1} \eta^2 w^{p+1+\alpha}.$$  

On the left hand side, for the first term, we have

$$\int_{\Gamma_1} \eta^2 w^{1+\alpha} \frac{\partial}{\partial \tau} w^p = \frac{p}{p+1+\alpha} \int_{\Gamma_1} \eta^2 \frac{\partial}{\partial \tau} w^{p+1+\alpha}$$
$$= \frac{p}{p+1+\alpha} \int_{\partial \Omega} (w^{p+1+\alpha})(S) - \frac{p}{p+1+\alpha} \int_{\Gamma_1} 2 \eta w^{p+1+\alpha}$$
$$\geq \frac{p}{p+1+\alpha} \int_{\partial \Omega} (w^{p+1+\alpha})(S) - \frac{C}{S_2 - S_1} \int_{\Gamma_1} w^{p+1+\alpha}.$$  

For the second term, we obtain

$$\int_{Q_1} \eta^2 \nabla (w^{1+\alpha}) \nabla w = \frac{4(1+\alpha)}{(2+\alpha)^2} \int_{Q_1} \eta^2 |\nabla (w^{\frac{2+\alpha}{2}})|^2.$$  

Using above three estimates, we obtain

$$\int_{\partial \Omega} (w^{p+1+\alpha})(S) + \int_{Q_2} |\nabla (w^{\frac{2+\alpha}{2}})|^2 \leq \frac{C(1+\alpha)}{S_2 - S_1} \int_{\Gamma_1} w^\alpha (w^{p+1} + w^2). \quad (39)$$  

We can find a $s_0 \in [S_2, S]$ such that

$$\int_{\partial \Omega} (w^{p+1+\alpha})(s_0) \geq \frac{1}{2} \sup_{\tau \in [S_2, S]} \int_{\partial \Omega} (w^{p+1+\alpha})(\tau).$$  

We replace $S$ by $s_0$ in (39) to obtain

$$\sup_{\tau \in [S_2, S]} \int_{\partial \Omega} (w^{p+1+\alpha})(\tau) + \int_{Q_2} |\nabla (w^{\frac{2+\alpha}{2}})|^2 \leq \frac{C(1+\alpha)}{S_2 - S_1} \int_{\Gamma_1} w^\alpha (w^{p+1} + w^2). \quad (40)$$  

25
Case 1: \( p > 1 \). By (38) and Hölder’s inequality we get

\[
C_{p+1} \leq \int_{\partial \Omega} w^{p+1} \leq |\partial \Omega| \left( \int_{\partial \Omega} w^{p+1+\alpha} \right)^{\frac{p+1}{p+1+\alpha}} \leq C \left( \int_{\partial \Omega} w^{p+1+\alpha} \right)^{\frac{p+1}{p+1+\alpha}}.
\]

So we have

\[
\frac{1}{C} \leq \left( \int_{\partial \Omega} w^{p+1+\alpha} \right)^{\frac{1}{p+1+\alpha}}.
\]

The constant \( C \) in (41) is independent of \( \alpha \), because \( |\partial \Omega|^{\frac{\alpha}{p+1+\alpha}} \) is bounded by a constant independent of all \( \alpha \geq 0 \). By \( p > 1 \), we have

\[
\int_{\partial \Omega} w^{\alpha + 2} \leq |\partial \Omega|^{\frac{\alpha}{p+1+\alpha}} \left( \int_{\partial \Omega} w^{p+1+\alpha} \right)^{\frac{\alpha+2}{p+1+\alpha}} \leq C \int_{\partial \Omega} w^{p+1+\alpha}.
\]

We obtain from (40) that

\[
\sup_{\tau \in [S_2, S]} \int_{\partial \Omega} (w^{p+1+\alpha})(\tau) + \int_{Q_2} |\nabla (w^{2+\alpha})|^2 \leq C \left( \frac{1 + \alpha}{S_2 - S_1} \right) \int_{\Gamma_1} w^{\alpha + p + 1}.
\]

For each \( \sigma > 1 \), using (22) and (42), we have

\[
\int_{\partial \Omega} w^{(\alpha+2)\sigma} = \int_{\partial \Omega} w^{\alpha+2} w^{(\alpha+2)(\sigma-1)}
\leq \left( \int_{\partial \Omega} w^{\frac{(\alpha+2)(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}} \left( \int_{\partial \Omega} w^{(\alpha+2)(\sigma-1)(n-1)} \right)^{\frac{1}{n-1}}
\leq C \left( \int_{\Omega} |\nabla (w^{\frac{\alpha+2}{n-2}})|^2 + \int_{\partial \Omega} w^{\alpha + 2} \right) \left( \int_{\partial \Omega} w^{(\alpha+2)(\sigma-1)(n-1)} \right)^{\frac{1}{n-1}}
\leq C \left( \int_{\Omega} |\nabla (w^{\frac{\alpha+2}{n-2}})|^2 + \int_{\partial \Omega} w^{\alpha + p + 1} \right) \left( \int_{\partial \Omega} w^{(\alpha+2)(\sigma-1)(n-1)} \right)^{\frac{1}{n-1}}.
\]

Integrating the above inequality over time interval \([S_2, S]\), and using (42) and (43), we obtain

\[
\int_{\Gamma_2} w^{(\alpha+2)\sigma}
\leq C \left( \int_{Q_2} |\nabla (w^{\frac{\alpha+2}{n-2}})|^2 + \int_{\Gamma_2} w^{\alpha + p + 1} \right) \left( \sup_{\tau \in [S_2, S]} \int_{\partial \Omega} w^{(\alpha+2)(\sigma-1)(n-1)}(\tau) \right)^{\frac{1}{n-1}}
\leq C \frac{(1 + \alpha)}{S_2 - S_1} \left( \int_{\Gamma_1} w^{\alpha + p + 1} \right) \left( \sup_{\tau \in [S_2, S]} \int_{\partial \Omega} w^{(\alpha+2)(\sigma-1)(n-1)}(\tau) \right)^{\frac{1}{n-1}}.
\]

Let

\[
\alpha + p + 1 = (\alpha + 2)(\sigma - 1)(n - 1).
\]

Then

\[
\sigma = 1 + \frac{\alpha + p + 1}{(n - 1)(\alpha + 2)}.
\]
For \( p > 1 \), i.e. \( 2 < p + 1 < \frac{2(n-1)}{n-2} \), we know for all \( \alpha \geq 0 \) that
\[
\frac{n}{n - 1} < \sigma < \frac{n}{n - 2}.
\]
Then by (43) and (44), we have
\[
\int_{\Gamma_2} w_{(\alpha+2)\sigma} \leq C \left( \frac{1 + \alpha}{S_2 - S_1} \right)^{\frac{1}{n - 1}} \left( \int_{\Gamma_1} w^{\alpha + p + 1} \right)^{\frac{1}{n - 1}}.
\]  \( \text{(45)} \)

For \( k = 0, 1, 2, 3, \ldots \), define \( \alpha_0 = 0 \),
\[
\sigma_k = 1 + \frac{\alpha_k + p + 1}{(n - 1)(\alpha_k + 2)} \in \left( \frac{n}{n - 1}, \frac{n}{n - 2} \right) \text{ for all } \alpha_k \geq 0,
\]
\[
\alpha_{k+1} = (\alpha_k + 2)\sigma_k - (p + 1),
\]
\[
q_k = \alpha_k + p + 1.
\]

Then
\[
\alpha_k = \left( 1 + \frac{1}{n - 1} \right)^k - 1 \left( p + 1 - (n - 1)(p - 1) \right),
\]
\[
q_k = \left( 1 + \frac{1}{n - 1} \right)^k \left( p + 1 - (n - 1)(p - 1) \right) + (n - 1)(p - 1).
\]

Since \( p < \frac{n}{n-2} \), one knows that \( p + 1 - (n - 1)(p - 1) > 0 \). So \( \alpha_k \) and \( q_k \) are strictly increasing and tend to \(+\infty\) as \( k \to \infty \). Let \( s_0 = \frac{1}{10}, s_1 = c_0, s_{k+1} - s_k = c_0k^{-4} \), where \( c_0 = \frac{1}{8} (\sum_{k=0}^{\infty} k^{-4})^{-1} \geq \frac{3}{16} \). Then \( s_k = c_0(\sum_{j=0}^{k-1} l^{-4}) \), \( \lim_{k \to \infty} s_k = \frac{1}{2} \). Let \( \Gamma_k = \partial \Omega \times [s_k, 1] \). By (45) we get
\[
\int_{\Gamma_{k+1}} w^{q_{k+1}} = \int_{\Gamma_{k+1}} w^{(\alpha_k+2)\sigma_k} \leq C \left( \frac{1 + \alpha_k}{c_0} k^4 \right)^{\frac{1}{n-1}} \left( \int_{\Gamma_k} w^{q_k} \right)^{\frac{1}{n-1}}.
\]

Then
\[
\left( \int_{\Gamma_{k+1}} w^{q_{k+1}} \right)^{\frac{1}{q_{k+1}}}
\leq C \frac{1}{q_{k+1}} \left( 1 + \frac{1}{n - 1} \right)^{\frac{2k}{q_{k+1}} \left( 1 + \frac{1}{n - 1} \right)} \left( \int_{\Gamma_k} w^{q_k} \right)^{\frac{1}{q_k} \left( 1 + \frac{1}{n - 1} \right) \frac{q_k}{q_{k+1}}}
\leq C \frac{1}{q_{k+1}} \sum_{j=0}^{k} \left( 1 + \frac{1}{n - 1} \right)^j D \frac{1}{q_{k+1}} \sum_{j=0}^{k} (k-j)(1 + \frac{1}{n - 1})^j \left( \int_{\Gamma_0} w^{q_0} \right)^{\frac{1}{q_0} \left( 1 + \frac{1}{n - 1} \right)^{k+1}},
\]
where \( D = (1 + \frac{1}{n - 1})^{2(k+1)} \). We send \( k \to \infty \) to get
\[
\|w\|_{L^\infty(\partial \Omega \times [\frac{1}{2}, 1])} \leq C \left( \int_{\Gamma_0} w^{p+1} \right)^{\left( p+1-(n-1)(p-1) \right)^{-1}} \leq C.
\]
Since the equation is translation invariant in the time variable, this proposition for \( p > 1 \) is proved.

Case 2: \( 0 < p < 1 \). By (38) and H"older’s inequality we get

\[
C^\frac{p+1}{p} \leq \int_{\partial \Omega} w^{p+1} \leq |\partial \Omega|^{\frac{1-p+\alpha}{2p+\alpha}} \left( \int_{\partial \Omega} w^{2+\alpha} \right)^{\frac{p+1}{2p+\alpha}} \leq C \left( \int_{\partial \Omega} w^{2+\alpha} \right)^{\frac{p+1}{2p+\alpha}}.
\]

So we have

\[
\frac{1}{C} \leq \left( \int_{\partial \Omega} w^{2+\alpha} \right)^{\frac{1}{2p+\alpha}}.
\]  

(46)

The constant \( C \) in (46) is independent of \( \alpha \), because \( |\partial \Omega|^{\frac{1-p+\alpha}{2p+\alpha}} \) is bounded by a constant independent of all \( \alpha \geq p - 1 \). By \( 0 < p < 1 \), we have

\[
\int_{\partial \Omega} w^{p+1+\alpha} \leq |\partial \Omega|^{\frac{1-p+\alpha}{2p+\alpha}} \left( \int_{\partial \Omega} w^{2+\alpha} \right)^{\frac{p+1}{2p+\alpha}} \leq C \int_{\partial \Omega} w^{2+\alpha}.
\]  

(47)

We obtain from (40) that

\[
\sup_{\tau \in [S_2, S]} \int_{\partial \Omega} (w^{p+1+\alpha})^\sigma (\tau) + \int_{Q_2} |\nabla (w^{2+\alpha})|^2 \leq \frac{C(1 + \alpha)}{S_2 - S_1} \int_{\Gamma_1} w^{\alpha+\sigma}.
\]  

(48)

For each \( \sigma \in (1, \frac{n-1}{n-2}) \), i.e. \( 2\sigma \in (2, \frac{2(n-1)}{n-2}) \), using (22) and (47), we have

\[
\int_{\partial \Omega} w^{(\alpha+2)\sigma} = \int_{\partial \Omega} w^{\alpha+2} w^{(\alpha+2)(\sigma-1)} \\
\leq \left( \int_{\partial \Omega} w^{(\alpha+2)(\alpha-1)} \right)^{\frac{n-2}{n-1}} \left( \int_{\partial \Omega} w^{(\alpha+2)(\sigma-1)(n-1)} \right)^{\frac{1}{n-1}} \\
\leq C \left( \int_{\Omega} |\nabla (w^{\alpha+2})|^2 + \int_{\partial \Omega} w^{\alpha+2} \right) \left( \int_{\partial \Omega} w^{(\alpha+2)(\sigma-1)(n-1)} \right)^{\frac{1}{n-1}}.
\]

Integrating the above inequality over time interval \( [S_2, S] \), and using (47) and (48), we obtain

\[
\int_{\Gamma_2} w^{(\alpha+2)\sigma} \\
\leq C \left( \int_{Q_2} |\nabla (w^{\alpha+2})|^2 + \int_{\Gamma_2} w^{\alpha+2} \right) \left( \sup_{\tau \in [S_2, S]} \int_{\partial \Omega} w^{(\alpha+2)(\sigma-1)(n-1)} (\tau) \right)^{\frac{1}{n-1}} \\
\leq \frac{C(1 + \alpha)}{S_2 - S_1} \left( \int_{\Gamma_1} w^{\alpha+2} \right) \left( \sup_{\tau \in [S_2, S]} \int_{\partial \Omega} w^{(\alpha+2)(\sigma-1)(n-1)} (\tau) \right)^{\frac{1}{n-1}}.
\]  

(49)

Let

\[
\alpha + p + 1 = (\alpha + 2)(\sigma - 1)(n - 1).
\]

28
Then
\[ \sigma = 1 + \frac{\alpha + p + 1}{(n-1)(\alpha + 2)}. \]

For \(0 < p < 1\), we know that for all \(\alpha \geq p - 1\),
\[ 1 + \frac{2p}{(n-1)(p+1)} < \sigma < \frac{n}{n-1}. \]

Then by \((48)\) and \((49)\), we have
\[
\int \int_{\Gamma_2} w^{(\alpha + 2)\sigma} \leq C \left( \frac{1 + \alpha}{S_2 - S_1} \right)^{1 + \frac{1}{n-1}} \left( \int \int_{\Gamma_1} w^{\alpha + 2} \right)^{1 + \frac{1}{n-1}}. \tag{50}
\]

For \(k = 0, 1, 2, 3, \ldots\), define \(\alpha_0 = p - 1\),
\[ \sigma_k = 1 + \frac{\alpha_k + p + 1}{(n-1)(\alpha_k + 2)}, \]
\[ \alpha_{k+1} = (\alpha_k + 2)\sigma_k - 2, \]
\[ q_k = \alpha_k + 2. \]

Then
\[ \alpha_k = 2p \left( 1 + \frac{1}{n-1} \right)^k - p - 1, \]
\[ q_k = 2p \left( 1 + \frac{1}{n-1} \right)^k - p + 1. \]

So \(q_k\) is strictly increasing and tend to \(+\infty\) as \(k \to \infty\). Let \(s_0 = \frac{1}{16}, s_1 = c_0, s_{k+1} - s_k = c_0 k^{-4}\), where \(c_0 = \frac{1}{2} (\sum_{k=0}^{\infty} k^{-4})^{-1} > \frac{3}{16}\). Then \(s_k = c_0 (\sum_{l=0}^{k-1} l^{-4})\), \(\lim_{k \to \infty} s_k = \frac{1}{2}\). Let \(\Gamma_k = \partial \Omega \times [s_k, 1]\). By \((50)\) we get
\[
\int \int_{\Gamma_{k+1}} w^{q_{k+1}} = \int \int_{\Gamma_{k+1}} w^{(\alpha_k + 2)\sigma_k} \leq C \left( \frac{1 + \alpha_k}{c_0 k^4} \right)^{1 + \frac{1}{n-1}} \left( \int \int_{\Gamma_k} w^{q_k} \right)^{1 + \frac{1}{n-1}}.
\]

Then by the similar iteration to that in Case 1, we obtain
\[ \|w\|_{L^\infty(\partial \Omega \times [\frac{1}{2}, 1])} \leq C \left( \int \int_{\Gamma_0} w^{p+1} \right)^{\frac{1}{p}} \leq C. \]

Since the equation is translation invariant in the time variable, this proposition for \(0 < p < 1\) is proved. \(\square\)

Using the uniform upper bound in Proposition 5.1 and the integral lower bound \((38)\), we can obtain the uniform lower bound of \(w\), with the help of Lemma 2.5.
Proposition 5.2. Let $w$ be as in Proposition 5.1. Then there exists $C > 0$, which depends only on $n, p, a, \Omega, \max_{\partial \Omega} u_0$ and $\min_{\partial \Omega} u_0$, such that

$$
w(x, \tau) \geq \frac{1}{C} \quad \text{for all } x \in \partial \Omega \text{ and } \tau > 0.
$$

Proof. It follows from Proposition 3.4 and Corollary 4.2 that there exists $\tau_0 > 0$ depending only on $n, p, a, \Omega, \max_{\partial \Omega} u_0$ and $\min_{\partial \Omega} u_0$ such that the conclusion holds for $0 < \tau \leq \tau_0$.

We claim that there exists some $\varepsilon > 0$ sufficiently small, which will be determined in the end, such that $w > \varepsilon$ on $\partial \Omega$ for all time. If not, we denote $\tau_1$ as the first time that $w$ touches $\varepsilon$ at some point $x_1 \in \partial \Omega$. We can choose $\varepsilon$ small such that $\tau_1 > \tau_0$.

Then it follows from the equation (37) that

$$
-\partial_\nu w \big|_{(x_1, \tau_1)} \leq C(\varepsilon + \varepsilon^p),
$$

where $C > 0$ depends only on $p$ and $\|a\|_{L^\infty(\partial \Omega)}$. By applying Lemma 2.5 to $w(\cdot, \tau_1) - \varepsilon$, we obtain

$$
-\partial_\nu w \big|_{(x_1, \tau_1)} \geq C \int_{\partial \Omega} (w(\cdot, \tau_1) - \varepsilon) \, dS.
$$

Combining above two inequalities, we have

$$
\int_{\partial \Omega} w(\cdot, \tau_1) \, dS \leq C(\varepsilon + \varepsilon^p).
$$

By the uniform upper bound in Proposition 5.1 and the integral lower bounds (38), we have

$$
\int_{\partial \Omega} w(\cdot, \tau_0) \, dS \geq c,
$$

where $c > 0$ depends only on $n, p, a, \Omega, \max_{\partial \Omega} u_0$ and $\min_{\partial \Omega} u_0$. This will reach a contradiction if $\varepsilon$ is sufficiently small.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. The conclusions for the cases of $\lambda_1(p-1) \leq 0$ follow from Theorem 3.2, Proposition 3.4 (i) and (ii). The conclusion for the case $\lambda_1(p-1) > 0$ follows from Theorem 3.2, Proposition 4.6, Proposition 5.1 and Proposition 5.2.

6 Convergence

Throughout this section, we always let $u$ and $T^n$ be as in Theorem 1.1, and $w$ be defined in (13). We will show the asymptotic behavior of $w$ in this section.
6.1 Convergence to a steady state with rates

It follows from Theorem 1.1 that \( 1/C \leq w \leq C \) on \( \Omega \times [0, +\infty) \) for some \( C \geq 1 \) which depends only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \). Then we can obtain higher order regularity estimates for \( w \).

**Theorem 6.1.** For every positive integer \( k \), there exists a constant \( C > 0 \) depending only on \( k, n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \) such that

\[
\| w \|_{C^k(\Omega \times [1, +\infty))} \leq C.
\]

**Proof.** Let \( \tau_0 \geq 1 \) be arbitrarily fixed. Let \( \eta \) be a smooth cut-off such that \( \eta = 0 \) in \( [\tau_0 - 1, \tau_0 - \frac{2}{3}] \), \( \eta = 1 \) in \( [\tau_0 - \frac{1}{2}, +\infty) \) and \( |\eta^{(k)}| \leq c_k \) for every positive integer \( k \), where \( c_k \) is a positive constant depending only on \( k \). Let \( \tilde{w}(x, \tau) = \eta(\tau)w(x, \tau) \). Then

\[
\begin{align*}
\Delta \tilde{w} &= 0 \quad \text{in} \ \Omega \times (0, \infty), \\
pw^{p-1}\partial_\tau \tilde{w} &= -\partial_\nu \tilde{w} - aw + \text{sgn}(\lambda_1)\frac{p}{|p-1|}w^{p-1}\tilde{w} + pw^{p}\partial_\nu \eta \quad \text{on} \ \partial \Omega \times (0, \infty).
\end{align*}
\]

Applying Lemma 3.1 finitely many times and using the Sobolev embedding, we have

\[
\| w \|_{C^k(\Omega \times [\tau_0, \tau_0+1])} \leq C.
\]

Since \( \tau_0 \) is arbitrary, then the conclusion follows. \( \square \)

We need the following two error estimates for the proof of convergence.

**Lemma 6.2.** Let \( \varphi \) be a positive solution of (10), and let \( h(x, \tau) = w(x, \tau) - \varphi(x) \). Then there exists \( C > 0 \) depending only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \) such that

\[
\| h(\cdot, \tau + t) \|_{L^2(\partial \Omega)} \leq Ce^{Ct}\| h(\cdot, \tau) \|_{L^2(\partial \Omega)} \quad \forall \tau > 1, \ t > 0.
\]

**Proof.** Consider the parabolic equation of \( w - \varphi \):

\[
pw^{p-1}\partial_\tau h = -\mathcal{B} h - ah + \text{sgn}(\lambda_1)\frac{p}{|p-1|}gh,
\]

where

\[
g = \int_0^1 p((1 - \lambda)\varphi + \lambda w)^{p-1} d\lambda.
\]

Note that \( g(x, \tau) \) is a bounded smooth function on \( \partial \Omega \times [0, +\infty) \). Multiplying \( h \) to both sides of (52) and integrating by parts, and using \( 1/C \leq w \leq C \) and \( \| \partial_\nu w \| \leq C \) with constant \( C > 0 \) by means of Theorem 6.1, we obtain

\[
\frac{d}{d\tau} \int_{\partial \Omega} h^2(\cdot, \tau)w^{p-1}(\cdot, \tau) dS \leq C \int_{\partial \Omega} h^2(\cdot, \tau)w^{p-1}(\cdot, \tau) dS.
\]

Using Gronwall’s inequality and the fact that \( 1/C \leq w \leq C \) again, we obtain (51). \( \square \)
Lemma 6.3. Let \( \varphi \) be a positive solution of \((10)\), and let \( h(x, \tau) = w(x, \tau) - \varphi(x) \). Then there exists \( C > 0 \) depending only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \) such that

\[
\| \partial_\tau h(\cdot, \tau) \|_{C^3(\partial \Omega)} + \| h(\cdot, \tau) \|_{C^3(\partial \Omega)} \leq C \| h(\cdot, \tau - 1) \|_{L^2(\partial \Omega)} \quad \forall \tau > 1. \tag{53}
\]

Proof. Let \( \tau_0 > 1 \) be arbitrarily fixed. Let \( \eta \) be a smooth cut-off such that \( \eta = 0 \) in \([\tau_0 - 1, \tau_0 - \frac{2}{3}]\), \( \eta = 1 \) in \([\tau_0 - \frac{1}{2}, \infty)\) and \( |\eta^{(k)}| \leq c_k \) for every positive integer \( k \), where \( c_k \) is a positive constant depending only on \( k \). Let \( \tilde{h}(x, \tau) = \eta(\tau) h(x, \tau) \). Then

\[
pw^{p-1} \partial_\tau \tilde{h} = -\partial_\tau \tilde{h} - \frac{\partial_\tau g}{p} \tilde{h} + pw^{p-1} \partial_\tau \eta \quad \text{on } \partial \Omega \times [\tau_0 - 1, \infty).
\]

Using Theorem 6.1, applying Lemma 3.1 finitely many times and using the Sobolev embedding, we have

\[
\| \partial_\tau h(\cdot, \tau_0) \|_{C^3(\partial \Omega)} + \| h(\cdot, \tau_0) \|_{C^3(\partial \Omega)} \leq C \| h \|_{L^2(\partial \Omega \times [\tau_0 - \frac{2}{3}, \tau_0])}.
\]

By (51), we have

\[
\| h(\cdot, \tau) \|_{L^2(\partial \Omega)}^2 \leq C \| h(\cdot, \tau - 1) \|_{L^2(\partial \Omega)}^2 \quad \text{for all } \tau \in [\tau_0 - 1, \tau_0].
\]

Combining these two estimates, we obtain

\[
\| \partial_\tau h(\cdot, \tau_0) \|_{C^3(\partial \Omega)} + \| h(\cdot, \tau_0) \|_{C^3(\partial \Omega)} \leq C \| h(\cdot, \tau_0 - 1) \|_{L^2(\partial \Omega)}.
\]

Since \( \tau_0 \) is arbitrary, then the conclusion follows. \( \square \)

For \( v \in H^\frac{1}{2}(\partial \Omega) \) and \( v \geq 0 \), let

\[
G(v) = \int_{\partial \Omega} \left[ \frac{1}{2} (v \mathcal{B} v + av^2) - \frac{\text{sgn}(\lambda_1)p}{|p-1|} v^{p+1} \right] \, dS,
\]

where \( \mathcal{B} \) is the Dirichlet to Neumann map defined in \((3)\). Then

\[
\langle G'(v), \phi \rangle = \int_{\partial \Omega} \left( \mathcal{B} v + av - \frac{\text{sgn}(\lambda_1)p}{|p-1|} v^p \right) \phi \, dS \quad \text{for all } \phi \in H^\frac{1}{2}(\partial \Omega).
\]

Then we have

\[
\frac{d}{d\tau} G(w(\cdot, \tau)) = \langle G'(w(\cdot, \tau)), w_{\tau}(\cdot, \tau) \rangle
\]

\[
= \int_{\partial \Omega} -\frac{\partial}{\partial \tau} (pw^p) w_{\tau} \, dS
\]

\[
= -\int_{\partial \Omega} pw^{p-1} w_{\tau}^2 \, dS = -\frac{4p}{(p+1)^2} \int_{\partial \Omega} \left| \frac{\partial}{\partial \tau} (w^{\frac{p+1}{2}}) \right|^2 \, dS \leq 0.
\]
That is, \( G(w(\cdot, \tau)) \) is decreasing in \( \tau \). Moreover, it follows from Theorem 1.1 that
\[
G(w(\cdot, \tau)) \geq -M \quad \text{for all } \tau \geq 0,
\]
where \( M > 0 \) depends only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \). Consequently,
\[
\lim_{\tau \to +\infty} G(w(\cdot, \tau)) \text{ exists and is finite.} \tag{56}
\]

Now we give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** By Theorem 6.1, there exists a sequence of times \( \tau_j, \tau_j \to \infty \) as \( j \to \infty \), such that
\[
w(\cdot, \tau_j) \to \varphi \quad \text{as } j \to \infty \quad \text{in } C^2(\overline{\Omega}).
\]

First, we will show that \( \varphi \) is a solution to (10). Integrating (55) from \( \tau_j \) to \( \tau_j + t \), and using the Cauchy-Schwarz inequality, we obtain
\[
\int_{\partial \Omega} |w^{\frac{p+1}{2}}(\cdot, \tau_j + t) - w^{\frac{p+1}{2}}(\cdot, \tau_j)|^2 \, dS = \int_{\partial \Omega} \left( \int_{\tau_j}^{\tau_j + t} \frac{\partial}{\partial \tau} \left( w^{\frac{p+1}{2}}(\cdot, \tau) \right) \, d\tau \right)^2 \, dS
\leq \int_{\partial \Omega} t \int_{\tau_j}^{\tau_j + t} \left| \frac{\partial}{\partial \tau} \left( w^{\frac{p+1}{2}}(\cdot, \tau) \right) \right|^2 \, d\tau dS
= \frac{(p + 1)^2 t}{4p} \left( G(w(\cdot, \tau_j)) - G(w(\cdot, \tau_j + t)) \right).
\]

Then by (56), we have
\[
\int_{\partial \Omega} |w^{\frac{p+1}{2}}(\cdot, \tau_j + t) - \varphi^{\frac{p+1}{2}}|^2 \, dS \to 0
\]
locally uniformly in \( t \) as \( j \to \infty \). Since \( 1/C \leq w \leq C \), we have \( 1/C \leq \varphi \leq C \). Then by using the inequality
\[
|w - \varphi|^{(2,p+1)} \leq C|w^{\frac{p+1}{2}} - \varphi^{\frac{p+1}{2}}|^2,
\]
we have
\[
w(\cdot, \tau_j + t) \to \varphi \quad \text{in } L^{p+1}(\partial \Omega)
\]
locally uniformly in \( t \) as \( j \to \infty \). Since they are all uniformly bounded in the \( C^3 \) norm, then by interpolation inequalities, we have
\[
w(\cdot, \tau_j + t) \to \varphi \quad \text{in } C^2(\overline{\Omega})
\]
locally uniformly in \( t \) as \( j \to \infty \). By integrating the equation (14) on \( \partial \Omega \times [\tau_j, \tau_j + 1] \), we have for \( \forall \phi \in C^\infty(\partial \Omega) \) that
\[
\int_{\partial \Omega} \left( w^p(\cdot, \tau_j + 1) - w^p(\cdot, \tau_j) \right) \phi \, dS
= \int_0^1 \int_{\partial \Omega} \left( -\mathcal{D}w(\cdot, \tau_j + t) - aw(\cdot, \tau_j + t) + \frac{\text{sgn}(\lambda_1)p}{|p-1|} w^p(\cdot, \tau_j + t) \right) \phi \, dS \, dt.
\]
Let $j \to \infty$ we have

\[
\int_{\partial \Omega} \left( -B \varphi - a \varphi + \frac{\text{sgn}(\lambda_1)p}{|p-1|} \varphi^p \right) \phi \, dS = 0.
\]

Therefore, $\varphi$ is a stationary solution to (10).

Secondly, we are going to prove that

\[ w(\cdot, \tau) \to \varphi \quad \text{as} \quad \tau \to \infty \quad \text{in} \quad C^2(\overline{\Omega}). \]

This follows from the same idea as the uniqueness result of Simon \cite{53}, and the details are given in the below. Denote

\[
\nabla G(w) = Bw + aw - \frac{\text{sgn}(\lambda_1)p}{|p-1|}w^p.
\]

Hence,

\[
\nabla G(\varphi) = 0.
\]

We also know that $\varphi \in S_C$ for some $C > 0$, where $S_C = \{ u \in H^{1/2}(\partial \Omega) : \frac{1}{C} \leq u \leq C \}$, and there exists a neighborhood $U$ of $\varphi$, $U \subset S_{2C}$ such that $G$ is analytic in $U$. Then by Theorem 3 in Simon \cite{53} (or Corollary 3.11 in Chill \cite{25} or Theorem 4.1 of Haraux-Jendoubi \cite{41} for an abstract setting), we have the following Łojasiewicz type inequality: There exist $C, \delta_0 > 0$ and $\theta \in (0, 1/2]$ such that for every $w \in C^{2,\alpha}(\partial \Omega)$ with $\| w - \varphi \|_{C^{2,\alpha}(\partial \Omega)} < \delta_0$, the following inequality holds.

\[
\| \nabla G(w) \|_{L^2(\partial \Omega)} \geq C|G(w) - G(\varphi)|^{1-\theta}. \tag{57}
\]

From the equality (55) and uniform bounds of $w$, we find a constant $c_0 > 0$ such that

\[
-\frac{d}{d\tau} G(w(\cdot, \tau)) \geq c_0 \| (w^p)_{\tau}(\cdot, \tau) \|_{L^2(\partial \Omega)}^2
\]

\[
\geq c_0 \| w_{\tau}(\cdot, \tau) \|_{L^2(\partial \Omega)} \| \nabla G(w(\cdot, \tau)) \|_{L^2(\partial \Omega)}. \tag{58}
\]

For any $\varepsilon \in (0, \delta_0)$, we shall find a $\tilde{\tau}(\varepsilon)$ such that for any $\tau > \tilde{\tau}(\varepsilon)$,

\[
\| w(\cdot, \tau) - \varphi \|_{C^{2,\alpha}(\partial \Omega)} < \varepsilon.
\]

Let $\tau_0$ be large enough such that $\| w(\cdot, \tau_0) - \varphi \|_{L^2(\partial \Omega)} < \delta$, where $\delta$ will be chosen later. Then by Lemma 6.2,

\[
\| w(\cdot, \tau) - \varphi \|_{L^2(\partial \Omega)} < C_1 \delta \quad \forall \tau \in [\tau_0, \tau_0 + 1].
\]

Then by Lemma 6.3, we have

\[
\| w(\cdot, \tau) - \varphi \|_{C^{2,\alpha}(\partial \Omega)} < C_1 C_2 \delta \quad \forall \tau \in [\tau_0 + 1, \tau_0 + 2]. \tag{59}
\]
Choose $\delta$ small so that $C_1C_2\delta < \varepsilon$, and define

$$T = \sup \{ t : \| w(\cdot, s) - \varphi \|_{C^{2,\alpha}(\partial\Omega)} < \varepsilon \ \forall \ s \in [\tau_0 + 1, t] \}.$$  

Consequently,

$$T \geq \tau_0 + 2.$$  

We are going to prove that $T$ must be $\infty$. If $T < \infty$, then for any $s \in [\tau_0 + 1, T]$, we have

$$-\frac{d}{d\tau} \left( G(w(\cdot, \tau)) - G(\varphi) \right)^\theta = -\theta \left( G(w(\cdot, \tau)) - G(\varphi) \right)^{\theta-1} \frac{d}{d\tau} G(w(\cdot, \tau))$$

$$\geq c_0 \theta \left( G(w(\cdot, \tau)) - G(\varphi) \right)^{\theta-1} \| w_\tau(\cdot, \tau) \|_{L^2(\partial\Omega)} \| \nabla G(w(\cdot, \tau)) \|_{L^2(\partial\Omega)}$$

$$\geq c_0 \theta \| w_\tau(\cdot, \tau) \|_{L^2(\partial\Omega)},$$

where we used (57) and (58). By integrating the above inequality from $\tau_0 + 1$ to $s$, we obtain

$$\int_{\tau_0 + 1}^{s} c_0 \theta \| w_\tau(\cdot, \tau) \|_{L^2(\partial\Omega)} \ d\tau \leq \left( G(w(\cdot, \tau_0 + 1)) - G(\varphi) \right)^\theta - \left( G(w(\cdot, s)) - G(\varphi) \right)^\theta$$

$$\leq \left( G(w(\cdot, \tau_0 + 1)) - G(\varphi) \right)^\theta.$$  

By the Minkovski integral inequality, we obtain

$$\| w(\cdot, s) - w(\cdot, \tau_0 + 1) \|_{L^2(\partial\Omega)} = \left\| \int_{\tau_0 + 1}^{s} w_\tau(\cdot, \tau) \ d\tau \right\|_{L^2(\partial\Omega)}$$

$$\leq \int_{\tau_0 + 1}^{s} \| w_\tau(\cdot, \tau) \|_{L^2(\partial\Omega)} \ d\tau$$

$$\leq \frac{1}{c_0 \theta} \left( G(w(\cdot, \tau_0 + 1)) - G(\varphi) \right)^\theta.$$  

Then

$$\| w(\cdot, s) - \varphi \|_{L^2(\partial\Omega)} \leq \| w(\cdot, s) - w(\cdot, \tau_0 + 1) \|_{L^2(\partial\Omega)} + \| \varphi - w(\cdot, \tau_0 + 1) \|_{L^2(\partial\Omega)}$$

$$\leq \frac{1}{c_0 \theta} \left( G(w(\cdot, \tau_0 + 1)) - G(\varphi) \right)^\theta + \| \varphi - w(\cdot, \tau_0 + 1) \|_{L^2(\partial\Omega)}$$

$$\leq C \left( \| \varphi - w(\cdot, \tau_0 + 1) \|_{C^{2,\alpha}(\partial\Omega)}^\theta + \| \varphi - w(\cdot, \tau_0 + 1) \|_{C^{2,\alpha}(\partial\Omega)} \right)$$

$$\leq C_3 \theta^\theta.$$  

35
for \( \forall s \in [\tau_0 + 1, T] \). If we choose \( \delta \) smaller such that \( C_2 C_3 \delta^\theta < \varepsilon \), then it follows from (59) that

\[
\|w(\cdot, s) - \varphi\|_{C^2(\partial \Omega)} < \varepsilon \quad \forall s \in [\tau_0 + 2, T + 1].
\]

This is a contradiction to the definition of \( T \). Thus \( T = \infty \). Therefore,

\[
\|w(\cdot, \tau) - \varphi\|_{C^2(\partial \Omega)} \to 0 \quad \text{as} \quad \tau \to \infty.
\]

Finally, we can use the decay rate of \( G \) to obtain the decay rate of \( w \) in \( C^{2,\alpha}(\partial \Omega) \). By (55), (57) and the uniform bounds of \( w \), we have

\[
\frac{d}{d\tau} \left( G(w(\cdot, \tau)) - G(\varphi) \right) = -\int_{\partial \Omega} \frac{\partial}{\partial \tau} (w^p) w_{\tau} dS
\]

\[
\leq -C \int_{\partial \Omega} \left( \frac{\partial}{\partial \tau} (w^p(\cdot, \tau)) \right)^2 dS
\]

\[
= -C \| \nabla G(w(\cdot, \tau)) \|_{L^2(\partial \Omega)}^2
\]

\[
\leq -C |G(w(\cdot, \tau)) - G(\varphi)|^{2-2\theta}
\]

for \( \tau \geq \tau_0 \). For \( \theta \in (0, 1/2) \), we know

\[
\frac{d}{d\tau} \left( G(w(\cdot, \tau)) - G(\varphi) \right)^{2\theta-1} \geq C(1 - 2\theta),
\]

and thus,

\[
G(w(\cdot, \tau)) - G(\varphi) \leq C \tau^{\frac{1}{2-\theta}}.
\]

Then for \( r \) very large,

\[
\int_r^{2r} \|w_{\tau}(\cdot, \tau)\|_{L^2(\partial \Omega)} d\tau \leq r^{1/2} \left( \int_r^{2r} \|w_{\tau}(\cdot, \tau)\|_{L^2(\partial \Omega)}^2 \right)^{1/2}
\]

\[
\leq C r^{1/2} \left( \int_r^{2r} \int_{\partial \Omega} p w^{p-1} w_{\tau}^2 dS \right)^{1/2}
\]

\[
= C r^{1/2} \left( G(w(\cdot, r)) - G(w(\cdot, 2r)) \right)^{1/2}
\]

\[
\leq C r^{1/2} \left( G(w(\cdot, r)) - G(\varphi) \right)^{1/2}
\]

\[
\leq C r^{\frac{1}{2} - \frac{1}{4\theta - 2}}
\]

\[
\leq C r^{\frac{\theta}{2\theta - 1}},
\]

\[
\text{36}.
\]
where we used (55) in the first equality. Then
\[
\int_r^\infty \| w_r(\cdot, \tau) \|_{L^2(\partial\Omega)} \, d\tau = \sum_{k=0}^\infty \int_r^{r \cdot 2^{k+1}} \| w_r(\cdot, \tau) \|_{L^2(\partial\Omega)} \, d\tau \\
\leq C' \sum_{k=0}^\infty (r \cdot 2^k)^{\frac{\theta}{2^k+1}} \\
= C' \sum_{k=0}^\infty (2^{\theta r-\frac{1}{2^k}})^k r^{\frac{\theta}{2^k}}.
\]

Since \( \theta \in (0, 1/2) \), we know \( 2^{\theta r-\frac{1}{2^k}} < 1 \), and thus, \( \sum_{k=0}^\infty (2^{\theta r-\frac{1}{2^k}})^k \) is finite. So
\[
\int_r^\infty \| w_r(\cdot, \tau) \|_{L^2(\partial\Omega)} \, d\tau \leq C' r^{\frac{\theta}{2^k}}.
\]

Finally, for \( t \) large,
\[
\| w(\cdot, t) - \varphi \|_{L^2(\partial\Omega)} \leq \left\| \int_t^\infty w_r(\cdot, \tau) \, d\tau \right\|_{L^2(\partial\Omega)} \leq \int_t^\infty \| w_r(\cdot, \tau) \|_{L^2(\partial\Omega)} \, d\tau \leq C t^{\frac{\theta}{2^k}}.
\]

Then it follows from Lemma 6.3 that
\[
\| w(\cdot, \tau) - \varphi \|_{C^{2,\alpha}(\partial\Omega)} \leq C \tau^{-\gamma},
\]
where \( \gamma = \frac{\theta}{1 - 2\theta} > 0 \).

If \( \varphi \) is integrable, then by Lemma 1 of Adams-Simon [1] (see also Corollary 3.12 of Chill [25]), the Łojasiewicz inequality (57) holds for \( \theta = \frac{1}{2} \). That is,
\[
G(w(\cdot, \tau)) - G(\varphi) \leq C \left\| -\partial_\tau w(\cdot, \varphi) - aw(\cdot, \tau) + \text{sgn}(\lambda_1) \frac{p}{p-1} w((\cdot, \tau)) \right\|_{L^2(\partial\Omega)}^2 \\
= C \| \partial_\tau (w^p) \|_{L^2(\partial\Omega)}^2.
\]

Therefore,
\[
\frac{d}{d\tau} \left[ G(w(\cdot, \tau)) - G(\varphi) \right] = -\int_{\partial\Omega} (w^p)_\tau \partial_\tau w \, dS \\
\leq -C \| \partial_\tau (w^p) \|_{L^2(\partial\Omega)}^2 \\
\leq -C(G(w(\cdot, \tau)) - G(\varphi)).
\]

So by Gronwall’s inequality, we have
\[
G(w(\cdot, \tau)) - G(\varphi) \leq Ce^{-4\gamma \tau}
\]
for some $C > 0$ and $\gamma > 0$. Similar to (60), we have
\[
\int_r^{2r} \| w_\tau(\cdot, \tau) \|_{L^2(\partial\Omega)} \, d\tau \leq C r^{1/2} \left( G(w_\cdot, r) - G(\varphi) \right)^{1/2} \leq C r^{1/2} e^{-2\gamma r} \leq C e^{-\gamma r}
\]
for $r$ sufficiently large. Then
\[
\int_r^\infty \| w_\tau(\cdot, \tau) \|_{L^2(\partial\Omega)} \, d\tau = \sum_{k=0}^\infty \int_{r-2^k}^{r-2^{k+1}} \| w_\tau(\cdot, \tau) \|_{L^2(\partial\Omega)} \, d\tau \leq C \sum_{k=0}^\infty e^{-\gamma r - 2^k} \leq C e^{-\gamma r}
\]
for $r$ sufficiently large. Finally, for $t$ large
\[
\| w(\cdot, t) - \varphi \|_{L^2(\partial\Omega)} \leq \left\| \int_t^\infty w_\tau(\cdot, \tau) \, d\tau \right\|_{L^2(\partial\Omega)} \leq \int_t^\infty \| w_\tau(\cdot, \tau) \|_{L^2(\partial\Omega)} \, d\tau \leq C e^{-\gamma t}.
\]
Then it follows from Lemma 6.3 that
\[
\| w(\cdot, \tau) - \varphi \|_{C^1(\partial\Omega)} \leq C e^{-\gamma \tau} \quad \text{for all } \tau > 1.
\]
This finishes the proof of parts (ii), (iv) of Theorem 1.2, as well as the upper bound in (17).

For part (i) of Theorem 1.2, we know from part (i) of Proposition 3.4 that
\[
\| w(\cdot, \tau) - \varphi \|_{L^\infty(\partial\Omega)} \leq C e^{-\tau} \quad \text{for all } \tau > 1.
\]
It follows from Lemma 6.3 that
\[
\| w(\cdot, \tau) - \varphi \|_{C^1(\partial\Omega)} \leq C e^{-\tau} \quad \text{for all } \tau > 1.
\]
This finishes the proof of part (i) of Theorem 1.2.

The dichotomy between (18) and the lower bound in (17) follows from Theorem 6.4 in the next subsection.

\[\square\]

### 6.2 Sharp rates and higher order asymptotics

In this subsection, we adapt the argument of Choi-McCann-Seis [28] to show that the solutions of (14) will converge to steady states either exponentially with the sharp rate or algebraically slow with the rate $\tau^{-1}$, and also to obtain higher order asymptotics.

Let us assume all the assumptions in Theorem 1.2, and $w(\cdot, \tau) \to \varphi$ in $C^2(\partial\Omega)$, where $\varphi$ is a positive smooth solution of (10). Let $\mathcal{L}_\varphi$ be defined in (12). Then the weighted eigenvalue problem
\[
\mathcal{L}_\varphi(e) = \mu \varphi^{p-1} e \quad \text{on } \partial\Omega
\]
(61)
admits eigenpairs $\{ (\mu_j, e_j) \}_{j=1}^\infty$ such that
the eigenvalues with multiplicities can be listed as \( \mu_1 < \mu_2 \leq \cdots \leq \mu_j \rightarrow +\infty \) as \( j \rightarrow +\infty \).

- the eigenfunctions \( \{e_j\}_{j=1}^{\infty} \) forms a complete orthonormal basis of \( L^2(\partial \Omega; \varphi^{p-1} \, dS) \), that is, \( \int_{\partial \Omega} e_i e_j \varphi^{p-1} \, dS = \delta_{ij} \) for \( i, j \in \mathbb{N} \).

Since \( \varphi \) is a positive solution to (10), then

\[
\mu_1 = -\text{sgn}(\lambda_1(p-1))p \quad \text{and} \quad e_1 = \varphi/\|\varphi\|_{L^2(\partial \Omega; \varphi^{p-1} \, dS)}.
\]

We suppose \( I \) is the dimension of the eigenspace corresponding to all the negative eigenvalues, if negative eigenvalues exist, and we denote \( \mu_I \) as the largest negative eigenvalue. If \( \mu_1 \geq 0 \), then there are no negative eigenvalues, and we just let \( I = 0 \). Let \( K \) be the multiplicity of the zero eigenvalue. Let \( k = I + K + 1 \) so that \( \mu_k \) is the smallest positive eigenvalue. For example, if \( \mu_1 < 0 \), then we can list the eigenvalues with multiplicities as

\[-\text{sgn}(\lambda_1(p-1))p = \mu_1 < \mu_2 \leq \cdots \leq \mu_I < 0 = \mu_{I+1} = \cdots = \mu_{I+K} < \mu_k \leq \cdots .\]

Denote

\[
\gamma_p = \frac{\mu_k}{p}.
\]

In particular, if \( \lambda_1(p-1) < 0 \), then \( k = 1, \mu_1 = p, e_1 = \varphi \), and thus, \( \gamma_p = 1 \).

As in [28], we call the eigenfunctions corresponding to the negative eigenvalues the unstable modes, those corresponding to \( \mu_{I+1} \) to \( \mu_{I+K} \) the central modes and the remaining eigenfunctions the stable modes. Their corresponding eigenspaces are denoted as \( E_u, E_c \), and \( E_s \). Then we have \( L^2(\partial \Omega; \varphi^{p-1} \, dS) = E_u \oplus E_c \oplus E_s \).

As before, we let \( h = w - \varphi \). Then by (11) and (14), \( h \) satisfies

\[
p\varphi^{p-1}\partial_\tau h + \mathcal{L}_\varphi h = N(h),
\]

where \( \mathcal{L}_\varphi \) is the linearized operator defined in (12) and \( N(h) \) is the nonlinearity

\[
N(h) = \text{sgn}(\lambda_1) \frac{p}{|p-1|} \left( (h + \varphi)^p - \varphi^p - ph\varphi^{p-1} \right) + p \left( \varphi^{p-1} - (h + \varphi)^{p-1} \right) \partial_\tau h.
\]

**Theorem 6.4.** Assume the assumptions in Theorem 1.2. Suppose \( w(x, \tau) \) converges to a positive solution \( \varphi \) of (10) in \( C^2(\overline{\Omega}) \) as \( \tau \rightarrow \infty \). Then \( h := w - \varphi \) satisfies the following dichotomy for some \( C > 0 \) depending only on \( n, p, a, \Omega, \max_{\partial \Omega} u_0 \) and \( \min_{\partial \Omega} u_0 \):

(a) either \( h \) decays algebraically or slower, that is,

\[
\|h(\cdot, \tau)\|_{L^2(\partial \Omega)} \geq C \tau^{-1} \quad \forall \tau > 1;
\]

(b) or \( h \) decays exponentially or faster, that is,

\[
\|h(\cdot, \tau)\|_{C^2(\partial \Omega)} \leq Ce^{-\gamma_p \tau} \quad \forall \tau > 1,
\]

where \( \gamma_p > 0 \) is defined in (62).
Proof. Let $P_u$, $P_c$ and $P_s$ be the orthogonal projections of $L^2(\partial \Omega; \varphi^{p-1} \, dS)$ onto the eigen-spaces $E_u$, $E_c$ and $E_s$, respectively. From now on, for the sake of convenience we use

$$\|f\| := \|f\|_{L^2(\partial \Omega; \varphi^{p-1} \, dS)} , \quad \langle f, g \rangle := \int_{\partial \Omega} fg \varphi^{p-1} \, dS,$$

and $h_u, h_c, h_s$ denote the projected solutions $P_u h, P_c h, P_s h$, respectively. Then we have

$$\frac{d}{d\tau} \|h_u\| \geq -\frac{\mu_I}{p} \|h_u\| - \frac{1}{p} \|N(h)\|, \quad (65)$$
$$\left| \frac{d}{d\tau} \|h_c\| \right| \leq \frac{1}{p} \|N(h)\|, \quad (66)$$
$$\frac{d}{d\tau} \|h_s\| \leq -\frac{\mu_k}{p} \|h_s\| + \frac{1}{p} \|N(h)\|, \quad (67)$$

which are obtained by multiplying corresponding eigenfunctions to (63) and integrating.

Next, we recall from Theorem 1.2 that

$$\|h(\cdot, \tau)\|_{C^{2}(\partial \Omega)} = o(1) \text{ as } \tau \to \infty.$$ 

Hence, it follows from (53) that for any small $\epsilon > 0$, there exists a time $\tau_0(\epsilon)$ such that $|h|, |\partial_\tau h| \leq \epsilon$ for all $\tau \geq \tau_0(\epsilon)$. By the Taylor expansion, we derive

$$|N(h)| \leq C|h| (|h| + |\partial_\tau h|). \quad (68)$$

Therefore,

$$\|N(h(\cdot, \tau))\| \leq \epsilon \|h(\cdot, \tau)\| \text{ for } \tau \geq \tau_0(\epsilon). \quad (69)$$

Plugging (69) into (65), (66) and (67), and using the change of variables $s = \lambda \tau$, we obtain

$$\frac{d}{ds} \|h_u\| - \frac{\mu_I}{p\lambda} \|h_u\| \geq -\frac{\epsilon}{p\lambda} \|h\|, \quad (70)$$
$$\left| \frac{d}{ds} \|h_c\| \right| \leq \frac{\epsilon}{p\lambda} \|h\|,$$
$$\frac{d}{ds} \|h_s\| + \frac{\mu_k}{p\lambda} \|h_s\| \leq \frac{\epsilon}{p\lambda} \|h\|,$$

where

$$\lambda = \begin{cases} \frac{\mu_k}{2p} & \text{if } I = 0 \text{ (i.e., there are no negative eigenvalues),} \\ \frac{1}{2p} \min\{\mu_I, \mu_k\} & \text{if } I \neq 0. \end{cases}$$

Then $h_u, h_c, h_s$ satisfy the hypothesis in Lemma 4.6 of Choi-Haslhofer-Hershkovits [26] (which is a slight refinement of Lemma A.1 of Merle-Zaag [50]). Therefore, either

$$\|h_u\| + \|h_s\| = o(\|h_c\|) \text{ as } \tau \to \infty \quad (71)$$
or
\[ \|h_u\| + \|h_c\| \leq \frac{100\epsilon}{\lambda p} \|h_s\| \text{ for } \tau \geq \tau_0(\epsilon) \] (72)
except that \( h = 0 \). We investigate these two alternatives separately.

**Case 1.** If (71) holds, we can assume that
\[ \|h_u\| + \|h_s\| \leq \frac{1}{2} \|h_c\| \text{ for } \tau \gg 1. \]
Then by (69) and (66), we have
\[ \frac{d}{d\tau} \|h_c\| \geq -\frac{3\epsilon}{2p} \|h_c\|. \]
Hence,
\[ \|h(\cdot, \tau)\| \leq 2\|h_c(\cdot, \tau)\| \leq \|h(\cdot, \tau + 1)\| \leq C\|h(\cdot, \tau + 1)\|. \]
Then by (53) and (68), we drive
\[ \|N(h(\cdot, \tau))\| \leq C\|h(\cdot, \tau)\| (\|h(\cdot, \tau)\| + \|\partial_\tau h(\cdot, \tau)\|) \leq C\|h(\cdot, \tau)\| \|h(\cdot, \tau + 1)\| \leq C\|h(\cdot, \tau)\|^2 \leq C\|h_c(\cdot, \tau)\|^2 \text{ for } \tau \gg 1. \]
Consequently, (66) becomes
\[ \frac{d}{d\tau} \|h_c\| \geq -C\|h_c\|^2, \]
which leads to a lower bound
\[ \|h(\cdot, \tau)\| \geq \|h_c(\cdot, \tau)\| \geq (C\tau)^{-1} \text{ for } \tau \gg 1. \]
This proves the first alternative in Theorem 6.4.

**Case 2.** If (72) holds, then we can assume that
\[ \|h_u\| + \|h_s\| \leq \frac{1}{2} \|h_s\| \text{ for } \tau \gg 1. \]
Then by (69) we obtain
\[ \|N(h(\cdot, \tau))\| \leq 2\epsilon\|h_s(\cdot, \tau)\| \text{ for } \tau \geq \tau_0(\epsilon) \gg 1. \]
Consequently, (67) becomes
\[ \frac{d}{d\tau} \|h_s\| \leq -\frac{1}{p} (\mu_k - 2\epsilon) \|h_s\|, \]
which, by Gronwall’s inequality, leads to an exponential decay
\[ \|h(\cdot, \tau)\| \leq 2\|h_s(\cdot, \tau)\| \leq Ce^{-\frac{1}{\tau}(\mu_k - 2\epsilon)(\tau - \tau_0)} \|h_s(\tau_0)\| \text{ for } \tau \gg 1. \]
By (53) and (68), we have
\[
\| N(h(\cdot, \tau)) \| \leq C\| h(\cdot, \tau) \| (\| h(\cdot, \tau) \| + \| \partial_\tau h(\cdot, \tau) \|)
\]
\[
\leq C\| h(\cdot, \tau) \| (\| h(\cdot, \tau) \| + \| h(\cdot, \tau - 1) \|)
\]
\[
\leq Ce^{\frac{1}{\beta} (\mu_k - 2\epsilon)(\tau - \tau_0)}\| h(\cdot, \tau_0) \|^2 \text{ for } \tau \gg 1.
\]

Plugging this into (67) and using Gronwall’s argument again, it yields
\[
\| h(\cdot, \tau) \| \leq Ce^{-\frac{\mu_k}{p} \tau} \| h(\cdot, \tau_0) \| \text{ for } \tau \gg 1.
\]

Then part (b) in Theorem 6.4 follows from (53).

This finishes the proof of Theorem 6.4.

In fact, if part (b) in Theorem 6.4 happens, then one can obtain higher order expansions of the solution as follows. First of all, the estimate of the nonlinearity improves as
\[
\| N(h(\cdot, \tau)) \| \leq Ce^{-2\gamma p \tau} \| h(\cdot, \tau_0) \|^2 \text{ for all } \tau \gg 1.
\]

For \( i \in \mathbb{N} \), let \( y_i(\tau) := \langle h(\cdot, \tau), e_i \rangle \). Then these projections satisfy
\[
\left| \frac{d}{d\tau} y_i + \frac{\mu_i}{p} y_i \right| \leq \| N(h) \|.
\]

Using (73) we get
\[
\left| \frac{d}{d\tau} \left( e^{\frac{\mu_i}{p} \tau} y_i \right) \right| \leq Ce^{-(2\gamma p - \frac{\mu_i}{p}) \tau}.
\]

Let \( J \in \mathbb{N} \) is chosen such that \( \frac{\mu_{I+J}}{p} < 2\gamma p \leq \frac{\mu_{I+J+1}}{p} \). Then \( J \geq K + 1 \). For \( i \in \{ I + K + 1, \ldots, I + J \} \), by integrating the above inequality in the time variable, we obtain
\[
\left| e^{\frac{\mu_i}{p} \tau} y_i(\tau) - e^{\frac{\mu_i}{p} T} y_i(T) \right| \leq Ce^{-(2\gamma p - \frac{\mu_i}{p}) \tau}
\]
for any \( T \geq \tau \gg 1 \). This inequality tells us that \( \tau \mapsto e^{\frac{\mu_i}{p} \tau} y_i(\tau) \) is a Cauchy sequence, and therefore,
\[
e^{\frac{\mu_i}{p} \tau} y_i(\tau) \to C_i \in \mathbb{R} \text{ as } \tau \to \infty
\]
for \( i \in \{ I + K + 1, \ldots, I + J \} \). We rewrite this condition as
\[
y_i(\tau) = C_i e^{-\frac{\mu_i}{p} \tau} + O(e^{-2\gamma p \tau}).
\]

Next, with this decay in mind, we have
\[
\left\| h - \sum_{i=I+K+1}^{I+J} C_i e^{-\frac{\mu_i}{p} \tau} e_i \right\|
\]

42
Finally, plugging the above estimates into (75) with the help of the alternative (b)

\[ \tau \leq h_s - \sum_{i=I+K+1}^{I+J} C_i e^{-\mu_i \tau} e_i + \| h_u \| + \| h_c \| \]

then an integration from \( \tau \) to \( \infty \)

\[ \begin{aligned}
    &\| h_s - \sum_{i=I+K+1}^{I+J} y_i e_i \| + \| h_u \| + \| h_c \| \leq \sum_{i=I+J+1}^{\infty} y_i e_i \\
    &\leq \sum_{i=I+K+1}^{I+J} \| y_i - C_i e^{-\mu_i \tau} \| + \| h_s - \sum_{i=I+K+1}^{I+J} y_i e_i \| + \| h_u \| + \| h_c \|. \\
    &\leq \sum_{i=I+K+1}^{I+J} \| y_i - C_i e^{-\mu_i \tau} \| + \| h_s - \sum_{i=I+K+1}^{I+J} y_i e_i \| + \| h_u \| + \| h_c \|. \\
\end{aligned} \] 

(75)

The first term on the right hand side of (75) is bounded by \( Ce^{-2\gamma_p \tau} \) as we showed in (74).

For the second term on the right hand side of (75), we let

\[ z := \| h_s - \sum_{i=I+K+1}^{I+J} y_i e_i \| = \| \sum_{i=I+J+1}^{\infty} y_i e_i \|. \]

Then it satisfies

\[ \frac{d}{d\tau} z + \frac{\mu_{I+J+1}}{p} z \leq C \| N(h) \| \leq C e^{-2\gamma_p \tau}, \]

which can be obtained in a similar way to that of (67). Then it becomes

\[ \frac{d}{d\tau} \left( e^{\frac{\mu_{I+J+1}}{p} \tau} z \right) \leq C e^{-\left(2\gamma_p - \frac{\mu_{I+J+1}}{p}\right)\tau}. \]

By Gronwall’s argument again, we have

\[ z(\tau) \leq C \| h(\tau_0) \| \times \begin{cases} e^{-2\gamma_p \tau} & \text{if } \mu_{I+J+1} > 2\gamma_p p, \\
                          \tau e^{-2\gamma_p \tau} & \text{if } \mu_{I+J+1} = 2\gamma_p p, \quad \tau \gg 1. \end{cases} \]

For the remaining two terms on the right hand side of (75), by using (65) and (66), we obtain a differential inequality

\[ \frac{d}{d\tau} (\| h_u \| + \| h_c \|) \geq -Ce^{-2\gamma_p \tau} \text{ for } \tau \gg 1. \]

Then an integration from \( \tau \) to \( \infty \) leads to

\[ \| h_u \| + \| h_c \| \leq Ce^{-2\gamma_p \tau}, \]

with the help of the alternative (b) that

\[ \| h_u \| + \| h_c \| \to 0 \text{ as } \tau \to \infty. \]

Finally, plugging the above estimates into (75), we obtain

\[ \begin{aligned}
    \| h(\cdot, \tau) - \sum_{i=I+K+1}^{I+J} C_i e^{-\mu_i \tau} e_i \| \leq & \begin{cases} Ce^{-2\gamma_p \tau} & \text{if } \mu_{I+J+1} > 2\gamma_p p, \\
                          Ce^{-2\gamma_p \tau} & \text{if } \mu_{I+J+1} = 2\gamma_p p, \end{cases} \\
    &\text{for all } \tau \gg 1 \text{ and some } C = C(n, p, a, \Omega, \varphi). \end{aligned} \] 

Finally, we can keep expanding the solution up to an arbitrary order in the same way as that Han-Li-Li [39] did for the singular Yamabe equation. In the final expansion, the exponential exponents are not only the \{\mu_j/p\} but also some of their linear combinations.
References

[1] D. Adams and L. Simon, *Rates of asymptotic convergence near isolated singularities of geometric extrema*. Indiana Univ. Math. J. **37** (1988), no. 2, 225–254.

[2] W. Allard and F. Almgren, *On the radial behaviour of minimal surfaces and the uniqueness of their tangent cones*. Ann. of Math. (2) **113** (1981), 215–265.

[3] S. Almaraz, *Convergence of scalar-flat metrics on manifolds with boundary under a Yamabe-type flow*. J. Differential Equations **259** (2015), 2626–2694.

[4] G. Akagi, *Stability of non-isolated asymptotic profiles for fast diffusion*. Comm. Math. Phys. **345** (2016), no. 1, 77–100.

[5] G. Akagi, *Rates of convergence to non-degenerate asymptotic profiles for fast diffusion via energy methods*. Arch. Ration. Mech. Anal. **247** (2023), no. 2, Article No. 23.

[6] D. G. Aronson and L. A. Peletier, *Large time behaviour of solutions of the porous medium equation in bounded domains*. J. Differential Equations **39** (1981), no. 3, 378–412.

[7] I. Athanasopoulos and L. A. Caffarelli, *Continuity of the temperature in boundary heat control problems*. Adv. Math. **224** (2010), no. 1, 293–315.

[8] J. B. Berryman and C. J. Holland, *Stability of the separable solution for fast diffusion*. Arch. Rat. Mech. Anal. **74** (1980), 379–388.

[9] M. Bonforte, J. Dolbeault, B. Nazaret, N. Simonov, *Stability in Gagliardo-Nirenberg- Sobolev inequalities. Flows, regularity and the entropy method*. arXiv: 2007.03674.

[10] M. Bonforte and J. Endal, *Nonlocal nonlinear diffusion equations. Smoothing effects, Green functions, and functional inequalities*. J. Funct. Anal. **284** (2023), no. 6, Paper No. 109831, 104 pp.

[11] M. Bonforte and A. Figalli, *Sharp extinction rates for fast diffusion equations on generic bounded domains*. Comm. Pure Appl. Math. **74** (2021), no. 4, 744–789.

[12] M. Bonforte, A. Figalli and X. Ros-Oton, *Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains*. Comm. Pure Appl. Math. **70** (2017), no. 8, 1472–1508.

[13] M. Bonforte, A. Figalli, J. L. Vázquez, *Sharp global estimates for local and nonlocal porous medium-type equations in bounded domains*. Anal. PDE **11** (2018), 945–982.

[14] M. Bonforte, G. Grillo and J. L. Vázquez, *Behaviour near extinction for the fast diffusion equation on bounded domains*. J. Math. Pures Appl. (9) **97** (2012), no. 1, 1–38.

[15] M. Bonforte, P. Ibarraondo and M. Isipuzia, *The Cauchy-Dirichlet problem for singular nonlocal diffusions on bounded domains*. Discrete Contin. Dyn. Syst. **43** (2023), no. 3-4, 1090–1142.

[16] M. Bonforte and N. Simonov, *Quantitative a priori estimates for fast diffusion equations with Caffarelli-Kohn-Nirenberg weights. Harnack inequalities and Hölder continuity*. Adv. Math. **345** (2019), 1075–1161.

[17] M. Bonforte and N. Simonov, *Fine properties of solutions to the Cauchy problem for a Fast Diffusion Equation with Caffarelli-Kohn-Nirenberg weights*. Ann. Inst. H. Poincaré Anal. Non Linéaire **40** (2023), no. 1, pp. 1–59.

[18] M. Bonforte, Y. Sire, J. L. Vázquez, *Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains*. Discrete Contin. Dyn. Syst. **35** (2015), no. 12, 5725–5767.

[19] M. Bonforte and J. L. Vázquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation*. J. Funct. Anal. **240** (2006), no. 2, 399–428.

44
[20] M. Bonforte and J. L. Vázquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations*. Adv. Math. 223 (2010), no. 2, 529–578.

[21] M. Bonforte and J. L. Vázquez, *Quantitative local and global a priori estimates for fractional nonlinear diffusion equations*. Adv. Math. 250 (2014), 242–284.

[22] M. Bonforte and J. L. Vázquez, *A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains*. Arch. Ration. Mech. Anal. 218 (2015), no. 1, 317–362.

[23] L. Brasco and B. Volzone, *Long-time behavior for the porous medium equation with small initial energy*. Adv. Math. 394 (2022), Paper No. 108029, 57 pp.

[24] S. Brendle, *A generalization of the Yamabe flow for manifolds with boundary*. Asian J. Math. 6 (2002), no. 4, 625–644.

[25] R. Chill, *On the Łojasiewicz-Simon gradient inequality*. J. Funct. Anal. 201 (2003), no. 2, 572–601.

[26] K. Choi, R. Haslhofer and O. Hershkovits, *Ancient low entropy flows, mean convex neighborhoods, and uniqueness*. Acta Math. 228 (2022), no. 2, 217–301.

[27] P. Cherrier, *Problèmes de Neumann non linéaires sur les variétés riemanniennes*. J. Funct. Anal. 57 (1984), 154–206.

[28] B. Choi, R. J. McCann and C. Seis, *Asymptotics near extinction for nonlinear fast diffusion on a bounded domain*. Arch. Ration. Mech. Anal. 247 (2023), no. 2, Article No. 16.

[29] A. de Pablo, F. Quirós, A. Rodríguez and J. L. Vázquez, *A fractional porous medium equation*. Adv. Math. 226 (2011), no. 2, 1378–1409.

[30] A. de Pablo, F. Quirós, A. Rodríguez and J. L. Vázquez, *A general fractional porous medium equation*. Comm. Pure Appl. Math. 65 (2012), no. 9, 1242–1284.

[31] G. Duvaut and J.-L. Lions, *Les Inéquations en mécanique et en physique*. Travaux et Recherches Mathématiques, No. 21. Dunod, Paris, 1972.

[32] E. DiBenedetto, Y. C. Kwong and V. Vespri, *Local space-analyticity of solutions of certain singular parabolic equations*. Indiana Univ. Math. J. 40 (2) (1991), 741–765.

[33] G. Franzina and B. Volzone, *Large time behavior of fractional porous media equation*. arXiv:2302.04266.

[34] E. Feireisl and F. Simondon, *Convergence for semilinear degenerate parabolic equations in several space dimension*. J. Dynam. Differential Equations 12 (2000), 647–673.

[35] V. A. Galaktionov, *A proof of the localization of unbounded solutions of the nonlinear parabolic equation* $u_t = (u^\sigma u_x)_x + u^\beta$. Differentsial’nye Uravneniya 21 (1985), no. 1, 15–23.

[36] V. A. Galaktionov, *A boundary value problem for the nonlinear parabolic equation* $u_t = \Delta u^{\sigma+1} + u^\beta$. Differentsial’nye Uravneniya 17 (1981), no. 5, 836–842.

[37] V. A. Galaktionov and J. R. King, *Fast diffusion equation with critical Sobolev exponent in a ball*. Nonlinearity 15 (2002), no. 1, 173–188.

[38] V. A. Galaktionov and J. L. Vázquez, *The problem of blow-up in nonlinear parabolic equations*. Discrete Contin. Dyn. Syst. 8 (2002), no. 2, 399–433.

[39] Q. Han, X. Li and Y. Li, *Asymptotic expansions of solutions of the Yamabe equation and the $\sigma_k$-Yamabe equation near isolated singular points*. Comm. Pure Appl. Math. 74 (2021), no. 9, 1915–1970.

[40] Z.-C. Han and Y. Y. Li, *The Yamabe problem on manifolds with boundary: existence and compactness results*. Duke Math. J. 99 (1999), no. 3, 489–542.

[41] A. Haraux and M. J. Jendoubi, *The Łojasiewicz gradient inequality in the infinite-dimensional Hilbert space framework*. J. Funct. Anal. 260 (2011), no. 9, 2826–2842.
[42] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[43] T. Jin and J. Xiong, A fractional Yamabe flow and some applications. J. Reine Angew. Math. 696 (2014), 187–223.

[44] T. Jin and J. Xiong, Optimal boundary regularity for fast diffusion equations in bounded domains. Amer. J. Math. 145 (2023), no. 1, 151–219.

[45] T. Jin and J. Xiong, Bubbling and extinction for some fast diffusion equations in bounded domains, arXiv:2008.01311.

[46] T. Jin and J. Xiong, Singular extinction profiles of solutions to some fast diffusion equations, J. Funct. Anal. 283 (2022), no. 7, Paper No. 109595, 29 pp.

[47] T. Jin and J. Xiong, Regularity of solutions to the Dirichlet problem for fast diffusion equations, arXiv:2201.10091.

[48] T. Jin, X. Ros-Oton and J. Xiong, Optimal regularity and fine asymptotics for the porous medium equation in bounded domains, arXiv:2211.06124.

[49] S. G. Krantz, Calculation and estimation of the Poisson kernel. J. Math. Anal. Appl. 302 (2005), no. 1, 143–148.

[50] F. Merle and H. Zaag, Optimal estimates for blowup rate and behavior for nonlinear heat equations. Comm. Pure Appl. Math. 51 (1998), no. 2, 139–196.

[51] E.S. Sabinina, A class of nonlinear degenerating parabolic equations. Sov. Math. Doklady 143 (1962), 495–498.

[52] E.S. Sabinina, On a class of quasilinear parabolic equations, not solvable for the time derivative. Sibirski Mat. Z. 6 (1965), 1074–1100.

[53] L. Simon, Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems. Ann. of Math. (2) 118 (1983), no. 3, 525–571.

[54] Y. Sire, J. Wei and Y. Zheng, Extinction behavior for the fast diffusion equations with critical exponent and Dirichlet boundary conditions. J. Lond. Math. Soc. (2) 106 (2022), no. 2, 855–898.

[55] M. E. Taylor, Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials. Mathematical Surveys and Monographs, 81. American Mathematical Society, Providence, RI, 2000.

[56] J. L. Vázquez, The Dirichlet problem for the porous medium equation in bounded domains. Asymptotic behavior. Monatsh. Math. 142 (2004), no. 1–2, 81–111.

[57] J. L. Vázquez, Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type. J. Eur. Math. Soc. (JEMS) 16 (2014), no. 4, 769–803.

[58] J. L. Vázquez, A. de Pablo, F. Quirós and A. Rodríguez, Classical solutions and higher regularity for nonlinear fractional diffusion equations. J. Eur. Math. Soc. (JEMS) 19 (2017), no. 7, 1949–1975.

[59] J. L. Vázquez and B. Volzone, Optimal estimates for fractional fast diffusion equations. J. Math. Pures Appl. (9) 103 (2015), no. 2, 535–556.

T. Jin
Department of Mathematics, The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong
Email: tianlingjin@ust.hk
J. Xiong
School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, MOE
Beijing Normal University, Beijing 100875, China
Email: jx@bnu.edu.cn

X. Yang
Department of Mathematics, The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong
Email: maxzyang@ust.hk