AN INFINITE SELF DUAL RAMSEY THEOREM

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ABSTRACT. In a recent paper [So] S. Solecki proves a finite self dual Ramsey theorem that in a natural way gives simultaneously the classical finite Ramsey theorem [Ra] and the Graham-Rothschild theorem [Gr-Ro]. In this paper we prove the corresponding infinite dimensional self dual theorem, giving similarly as a consequence the infinite classical Ramsey theorem [Ra] and the Carlson-Simpson theorem [Ca-Si].

1. INTRODUCTION

Recall that the classical Ramsey theorem [Ra] states that given any finite coloring of the set of all $K$ elements subsets of $\omega$ there exists an infinite subset $A \subseteq \omega$ where the restriction of the coloring is constant.

The dual form of Ramsey theorem, the Carlson-Simpson theorem [Ca-Si], states that given any finite Borel coloring of the set of all partitions of $\omega$ into $K$ many classes, there exists a partition $r$ of $\omega$ into $\omega$ many classes such that the set of all $K$ partitions of $\omega$ resulting by identifying classes of $r$ is monochromatic.

There are also the corresponding finite versions of these results, the finite Ramsey theorem and the Graham-Rothschild theorem [Gr-Ro], respectively. S. Solecki recently proved in [So] a self dual theorem that implies simultaneously the finite version of the Ramsey theorem and the Graham-Rothschild theorem. He achieved that by introducing the notion of a connection, which roughly speaking is a labelled partition of $L$ into $K$ many classes, for $K$ and $L$ integers. By a labelled partition we mean a partition of $L = \{0, \ldots, L - 1\}$ onto $K$ many classes, a $K$ partition of $L$, and a choice function with domain $K$ such that for each of the $K$ classes picks a representative. Solecki then proved that given any positive integers $K, L$ and $M$ there exists $N$ such that for any $L$ coloring of all labelled partitions of $N$ into $K$ many pieces, there exists a labelled partition of $M$ into $K$ pieces, such that the set of all labelled partitions of $N$ into $M$ composed with the particular labelled partition of $M$ into $K$ is monochromatic.

The composition is defined in the most natural way by composing partitions, namely that partition $N$ into $M$ pieces and then $M$ into $K$ pieces, so we finally partition $N$ into $K$. The composition of the choice functions is done in the reverse order.

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We extend canonically his notion of connection to labelled partitions of $\omega$, with finite or infinitely many classes and we prove the following (see Theorem 2):

**Theorem.** For any finite Borel coloring of all labelled $K$-partitions of $\omega$ there is a fixed labelled $\omega$-partition of $\omega$ such that the set of all of its reductions, i.e. labelled $K$-partitions of $\omega$ which result from putting pieces of the fixed partition together, is monochromatic.

The proof is done by induction on $K$ and the use of the left variable Hales-Jewett theorem. In the final section of this paper we extend this result by building the corresponding topological Ramsey space $F_{\omega,\omega}$.

2. Basic facts and definitions

We follow the terminology introduced in [So]. Let $K$ and $L$ be finite linear orders, for now. By a rigid surjection $t : L \to K$ we mean a surjection with the additional property that images of initial segments of $L$ are also initial segments of $K$. We denote its image $K$, by $im(t)$ and its domain $L$ by $dom(t)$. We call a pair $(t, i)$ a connection between $L$ and $K$ if $t : L \to K$, $i : K \to L$ such that for all $x \in K$, $y \in L$:

$t(i(x)) = x$ and $\forall y \leq i(x) \ t(y) \leq x$.

It is easy to see that if $(t, i)$ is a connection then $t$ is a rigid surjection and $i$ is an increasing injection. If now we consider countably infinite linear orders and we allow $L = \omega$ we get a connection $(s, j)$ between $\omega$ and $K$. Once again, we have that if $(s, j)$ is a connection then $s$ is a rigid surjection and $j$ is an increasing injection. Finally if we allow both $L = K = \omega$ we get a connection $(r, c)$ between $\omega$ and itself.

With this terminology the classical Ramsey Theorem can be stated as follows: Let $l, K$ be natural numbers. For any $l$-coloring of all increasing injections $j : K \to \omega$ there exists an increasing injection $j_0 : \omega \to \omega$ such that the set $\{ j_0 \circ j : j : K \to \omega \}$ is monochromatic.

Similarly, the Carlson-Simpson Theorem can be stated as follows: Let $l$ a natural number. For any Borel $l$-coloring of all rigid surjections $s : \omega \to K$, there exists a rigid surjection $s_0 : \omega \to \omega$ such that the set $\{ s \circ s_0 : s : \omega \to K \}$ is monochromatic.

Given a finite, possibly empty alphabet $A = \{ \alpha_0, \ldots, \alpha_{|A|-1} \}$, written in an increasing order, and given $K \leq L \leq \omega$ we consider the corresponding spaces $F_{L, K}^A = \{ (r, c) \ | \ r : A \cup L \to A \cup K, \ c : K \to L, \ r \upharpoonright A = id_A \text{ and } c \text{ is an increasing injection such that } r(c(x)) = x, y \leq c(x) \Rightarrow r(y) \leq x, \text{ for all } x \in K, y \in L \}$.

We adapt the convention that on $A \cup L$ every element of $A$ is less than any element of $L$. Notice that $A$ is not in the domain of $c$. 
We introduce the following notational convention: If \( K = L = \omega \) then an arbitrary element of \( F_{\omega, \omega}^A \) will be denoted by \((r, c)\). If \( L = \omega, K \in \omega \), then \((s, j)\) denotes an arbitrary element of \( F_{\omega, K}^A \) and finally if \( L, K \in \omega \), then \((t, i)\) denotes an element of \( F_{L, K}^\omega \).

In the case that the finite alphabet \( A \) is empty then the above spaces become:

\[
F_{L, K} = \{ (r, c) \mid r : L \rightarrow K, \, c : K \rightarrow L \text{ and } c \text{ is an increasing injection such that } r(c(x)) = x, y \leq c(x) \Rightarrow r(y) \leq x \}.
\]

In other words, \( A \) disappears from the domain and the range of the rigid surjection. Similarly for \( F_{\omega, \omega} \) and \( F_{\omega, K}^\omega \).

An arbitrary element \((r, c) \in F_{\omega, \omega}^A\) is essentially an equivalence relation on \( A \cup \omega \) onto \( \omega \) many equivalent classes, where each equivalent class is permitted to contain at most one element of the finite alphabet \( A \). For \( n \in \omega \), let \( X_n = r^{-1}(\{n\}) \) and let \( E_n = \min X_n \). That gives us \((E_n)_{n \in \omega}\). If \((r, c) \in F_{\omega, \omega}\) then \( E_0 = 0 \). We define

\[
(r \upharpoonright E_n, c \upharpoonright n) = (r, c)[n] = (r, c)[E_n] \in F_{E_n, n}^A
\]

Having adapted the convention that on \( A \cup L \) every element of \( A \) is less than any element of \( L \) we remark here that \( r \upharpoonright E_n \) includes \( A \) in its domain.

Identical definition applies in the case that \( A = \emptyset \). In that case, for \((r, c) \in F_{\omega, \omega}\), we still get \((E_n)_{n \in \omega}\) and we consider \((r, c)[n] = (r, c)[E_n] \) for all \( n \in \omega \). Similarly for \((t, i) \in F_{M, K}\) we get \((E_n)_{n \in K - 1}\) and

\[
(t \upharpoonright E_n, i \upharpoonright n) = (t, i)[E_n] = (t, i)[n].
\]

Therefore we have a way to restrict our connections to certain places, namely the minimum elements of each piece of the partition. That gives us our notion of an initial segment of a connection.

**Definition 1.** We define \((t, i)\) to be an initial segment of \((r, c)\) denoted by \((t, i) \preceq (r, c)\) if \((t, i) = (r, c)[n] = (r \upharpoonright E_n, c \upharpoonright n) \) for some \( n \in \omega \).

Similarly for \((t, i) \in F_{M, K}\) and \((t', i') \in F_{N, L}\), with \( K < L \) and \( M < N \), we define \((t, i)\) to be an initial segment of \((t', i')\), denoted by \((t, i) \preceq (t', i')\) if \((t', i') = (t, i)[n] \), for some \( n \in K \).

Finally we define \((t, i) \preceq (s, j)\), for \((t, i) \in F_{M, K}, \, (s, j) \in F_{\omega, L}, \, K \leq L, \) if and only if \( s \upharpoonright M = t \) and \( j \upharpoonright K = i \).

We notice here that to get initial segments of \((r, c) \in F_{\omega, \omega}\) we restrict in certain places, namely the minimal elements of each piece of the partition that \( r \) defines, similarly for \((t, i) \in F_{M, K}\). To get initial segments in the case of \((s, j) \in F_{\omega, K}\) though, we simply restrict to any place where we can get a connection.

Let \((s, j)\) be an arbitrary element of \( F_{\omega, K}\). Then \((E_n)_{n \in K - 1}\) is defined as above and \( E_{K-1} = \min \{ m : s(m) = K - 1, \, s(m + 1) < K - 1 \} \).
We define

\[(s \upharpoonright E_n, j \upharpoonright n) = (s, j)[n] \in F_{E_n,n} \text{ for } n \leq K - 1.\]

For \(n = K\) we define \((t, i) = (s \upharpoonright E_K, j \upharpoonright K) = (s, j)[K]\)

where the domain of \(t\) is the least natural number \(E_K\) with the property that \(K - 1 = t(E_K - 1) \neq t(E_K), t(E_K) < K - 1.\)

The mapping \(c\) in \((r, c)\) is a choice function that from each free class, the ones that get mapped by \(r\) to an \(n \in \omega\), chooses \(c(k)\) such that

\[\min r^{-1}(k) \leq c(k) < \min r^{-1}(k + 1).\]

Similarly for all other variations of the above spaces, we think of \(j\) in \((s, j)\) and of \(i\) in \((t, i)\) as choice functions that respect the above equation.

Given \((s, j), (r, c)\) we define the multiplication as follows

\[(s, j) \cdot (r, c) = (s \circ r, c \circ j) \in F^A_{\omega,K}.\]

Notice that the order of composition in the two coordinates is not the same. Similarly we can define the above multiplication between any two members of \(F^A_{\omega,\omega}\) as well as between \((t_0, i_0) \in F^A_{M,L}\) and \((t_1, i_1) \in F^A_{L,K}\), where \(K \leq L \leq M\), to get \((t_1, i_1) \cdot (t_0, i_0) \in F^A_{M,K}\).

Notice also that the multiplication the way defined above is nothing more than a way to put equivalence classes together and accordingly adjusting the choice functions.

In \([So]\) Solecki proved the following:

**Theorem 1.** \((Solecki)[So]\) For any \(K, l\) and \(M\) positive integers, there exists \(N\) such that for any finite coloring \(c : F_{N,K} \to l\) there exists a connection \((s_0, j_0) \in F_{N,M}\) such that the set

\[F_{M,K} \cdot (s_0, j_0) = \{ (s, j) : (s, j) \in F_{N,M} \} \]

is monochromatic.

In order to proceed we need some new concepts.

**Definition 2.** For \((t, i) \in F^A_{L,K}\) we define its length \(lh(t, i)\) to be equal to the domain \(L\) of \(t\).

In other words the length of a \((t, i)\) is the initial segment \(L\) of \(\omega\) that gets partitioned by \(t\) onto \(A \cup K\). If \((t, i) \in F^A_{L,K}\) then its length is equal to \(L\) as well. Sometimes we will talk of the length of the rigid surjection \(t\) itself, which is identical with the length of \((t, i)\), the domain of \(t\).

As we noted above the way that the multiplication is defined tells us how to put equivalence classes together and to get a new connection. In the next definition we formalize that procedure.

**Definition 3.** We introduce the notion of a reduct.
(1) Let \((r_0, c_0), (r_1, c_1) \in F^A_{\omega, \omega}\). We say that \((r_0, c_0)\) is a reduct of \((r_1, c_1)\), denoted by \((r_0, c_0) \leq (r_1, c_1)\), if there exists \((r, c) \in F^A_{\omega, \omega}\) such that 
\[
(r_0, c_0) = (r, c) \cdot (r_1, c_1).
\]

(2) For \((t_0, i_0) \in F^A_{M,K}, (t_1, i_1) \in F^A_{M,L}\), with \(L \geq K\), we have already defined the length of \((t_0, i_0)\) to be equal to \(M = \ell h(t_0, i_0)\). We say that \((t_0, i_0)\) is a reduct of \((t_1, i_1)\) denoted by \((t_0, i_0) \leq (t_1, i_1)\) if both have the same length, where here it is the case, and there exists \((t, i) \in F_{L,K}\) such that 
\[
(t_0, i_0) = (t, i) \cdot (t_1, i_1).
\]

Then \(F^A_{\omega, \omega}\) and \(F^A_{\omega, K}\) become topological spaces, by defining basic open sets as follows:
\[
\{(t, i), (r, c)\} = \{(r', c') \in F^A_{\omega, \omega} : (t, i) \preceq (r', c')\} \text{ and } (r', c') \leq (r, c)\}
\]
So, a basic open set contains all possible reducts of \((r, c)\) that have \((t, i)\) as an initial segment. Similarly for the set \(F^A_{\omega, K}\),
\[
\{(t, i), (r, c)\} = \{(s', j') \in F^A_{\omega, K} : (s', j') \leq (r, c)\}
\]
where \((t, i) \in F^A_{M,L}\) with \(L \leq K\). In other words basic open sets are like above except that we only consider \((t, i) \in F^A_{M,L}\) with \(L \leq K\). Identical in the case that \(A = \emptyset\), namely
\[
\{(t, i), (r, c)\} = \{(r', c') \in F_{\omega, \omega} : (t, i) \preceq (r', c')\} \text{ and } (r', c') \leq (r, c)\}
\]
and
\[
\{(t, i), (r, c)\} = \{(s', j') \in F_{\omega, K} : (s', j') \leq (r, c)\}
\]
where \((t, i) \in F_{M,L}\) with \(L \leq K\).

In the above definition by taking \((t, i) = (\emptyset, \emptyset)\) we get the following set
\[
\{(r, c]\}^A = \{(\emptyset, \emptyset), (r, c)\} = \{(r', c') \in F^A_{\omega, \omega} : (r', c') \leq (r, c)\}.
\]
Equivalently \((r, c)\) is the set of all possible reducts of \((r, c)\). Similarly
\[
\{(r, c)\}^K = \{(s, j) \in F_{\omega, K} : (s, j) \leq (r, c)\}.
\]

Once more if the fine alphabet \(A\) is empty then
\[
\{(r, c)\}_K = \{(\emptyset, \emptyset), (r, c)\} = \{(r', c') \in F_{\omega, \omega} : (r', c') \leq (r, c)\}
\]
and similarly for
\[
\{(r, c)\}_K^K = \{(s, j) \in F_{\omega, K} : (s, j) \leq (r, c)\}.
\]

For any \((r, c)\) and \((t, i) \in F^A_{M,K}\), we define \(t^{-r}(r, c) \in F^A_{\omega, \omega}\) the translation of \((r, c)\) by \((t, i)\) defined as follows: for all \(n \in M\), \((t^{-r})(n) = t(n)\), and for \(n \geq M\), \((t^{-r})(n) = r(n - M)\). Similarly for the choice functions, i.e. for all \(m \in K\), \((c^{-i})(m) = i(m)\) and for all \(m \geq K\), \((c^{-i})(m) = c(m) + M\). Identical definition
applies in the case that $A = \emptyset$.

Let now $(r, c)^0_A = \{ (t, i) : (t, i) = (r', c')[0], (r', c') \in [(r, c)]^ω_A \}$. Notice that $(t, i) = (t, \emptyset)$, where by $\emptyset$ we emphasize that the increasing injections in the second coordinate do not have $A$ in their domain. Essentially $(r, c)^0_A$ contains all initial segments of all possible reducts of $(r, c)$ whose surjection has image equal to $A$. As a result $(r, c)^0_A$ is not defined in the case that $(r, c) \in F_{\omega, \omega}$. Similarly let

$$(r, c)^L_A = \{ (t, i) : (t, i) = (r', c')[L], (r', c') \in [(r, c)]^ω_A \}.$$  

So, $(r, c)^L_A$ is the set of all initial segments of all possible reducts of $(r, c)$ whose choice function has domain $L$. If $A = \emptyset$ then

$$(r, c)^L = \{ (t, i) : (t, i) = (r', c')[L], (r', c') \in [(r, c)]^ω \}.$$  

Finally, let $(t, i) \preceq (r, c), (t, i) \in (r, c)^L_A$, with length of $(t, i)$ $M$ and with domain of $i$ equal to $L$, i.e. $(t, i) \in F^A_M L$. By $(t, i)^* \in (r, c)^{L+1}_A$ we mean the initial segment of $(r, c)$ that extends $(t, i)$ as follows:

$$(t, i)^* = (r, c)[L + 1].$$  

Notice that given $(t, i)$ to consider $(t, i)^*$ the connection $(r, c)$ whose initial segment is $(t, i)$, needs to be specified. Once more identical definition applies in the case that $A = \emptyset$.

3. Main theorem

The main result of this chapter is the following theorem:

**Theorem 2.** Let $K, l > 0$ be a natural numbers. For each Borel $l$ coloring of all connections between $\omega$ and $K$, there exists $(r_0, c_0) \in F_{\omega, \omega}$ such that the set

$$F_{\omega, K} \cdot (r_0, c_0) = [(r_0, c_0)]^K = \{ (s, j) \cdot (r_0, c_0) : (s, j) \in F_{\omega, K} \}$$

is monochromatic.

Note that for colorings that do not depend on the first coordinate, the above theorem reduces to the classical Ramsey theorem and for colorings not depending on the second coordinate it reduces to the Carlson-Simpson theorem.

We shall prove the following more general result:

**Theorem 3.** If $F^A_{\omega, K} = C_0 \cup \cdots \cup C_l$ where each $C_i$ is Borel, then there exists $(r_0, c_0) \in F^A_{\omega, \omega}$ and $k \in l$ such that $[(r_0, c_0)]^K_A \subseteq C_k$.

Toward this end, we prove the following three results:

**Lemma 1.** If $(r, c) \in F^A_{\omega, \omega}$ and $[(r, c)]^0_A = C_0 \cup \cdots \cup C_l$ where each $C_k$ is Borel, then there exists $(r', c') \in [(r, c)]^ω_A$ and $k \in l$ such that $[(r', c')]^0_A \subseteq C_k$.

**Proof.** Since the coloring does not depend on the second coordinate this result readily follows from the Carlson-Simpson theorem.
Lemma 2. If \((r, c) \in F^{A}_{\omega, \omega}\) and \((r, c)^0_A = C_0 \cup \cdots \cup C_{l-1}\), then there exists \((r', c') \in [(r, c)]^\omega A\) and \(k \in l\) such that \((r', c')^0_A \subseteq C_k\).

We postpone the proof of this lemma to the end of this section. Assuming that we have this result, the next step is to prove the following:

**Lemma 3.** Assume the Theorem 3 holds for \(F^{A+1}_{\omega, K}\), where \(A \subset A+1\) and \(|A+1| = |A| + 1\). Then Theorem 3 also holds for \(F^{A}_{\omega, K+1}\).

*Proof.* Let

\[ A + 1 = \{ \alpha_0, \ldots, \alpha_{|A|-1}, \alpha_A \} \]

At first notice that given \((r, c) \in F^A_{\omega, \omega}\), \((t, i) \in (r, c)^0_A\), then by the definition of \((r, c)^0_A\), \((t, i)\) is an initial segment of \((\bar{r}, \bar{c})\), a reduct of \((r, c)\), that can be written as \((t, i) \preceq (\bar{r}, \bar{c}) = (t, i)^*\sim (\bar{r}, \bar{c})\). We remind here that as defined in the above section \((t, i)^*\) is the initial segment of \((\bar{r}, \bar{c})\) that extends \((t, i)\).

**Claim 1.** There exists a natural homeomorphism \(\theta\) between \(F^{A+1}_{\omega, K}\) and \([[(t, i)^*, (\bar{r}, \bar{c})]^K+1_A]\).

Where

\[
[[t, i]^*, (\bar{r}, \bar{c})]^K+1_A = \{ (s, j) \in F^A_{\omega, K+1} : (s, j) \preceq (r', c') \text{ for } (r', c') \in [(t, i)^*, (\bar{r}, \bar{c})],
(t, i)^* \preceq (s, j) \}
\]

is the set of all possible reducts of the members of \([[(t, i)^*, (\bar{r}, \bar{c})]\) into \(K + 1\) many free classes, that also have \((t, i)^*\) as an initial segment.

*Proof.* Given \((s, j) \in F^{A+1}_{\omega, K}\) by \((\bar{s}, \bar{j}) \in F^A_{\omega, K+1}\) we denote the connection that results from \((s, j)\) that takes the equivalence class which contains the letter \(\alpha_{|A|}\) and removes it, so it becomes a new free class, the first free class of \((\bar{s}, \bar{j})\). The choice function \(\bar{j}\) has now domain \(K + 1\) and \(\bar{j}(0) \in [\min \bar{s}^{-1}(0), \min \bar{s}^{-1}(1)]\).

The homomorphism is defined by \(\theta(s, j) = (t, i)^*\sim (\bar{s}, \bar{j}) \cdot (\bar{r}, \bar{c})\). Conversely given \((t, i)^*\sim (\bar{r}, \bar{c}) \in [[[t, i)^*, (\bar{r}, \bar{c})]^K+1_A\), there exists \((\bar{s}, \bar{j}) \in F^A_{\omega, K+1}\) so that

\[(t, i)^*\sim (\bar{s}, \bar{j}) \cdot (\bar{r}, \bar{c}) = (t, i)^*\sim (\bar{r}, \bar{c})\).

Define \(\theta^{-1}((t, i)^*\sim (\bar{r}, \bar{c})) = \theta^{-1}((t, i)^*\sim (\bar{s}, \bar{j}) \cdot (\bar{r}, \bar{c})) = (s, j) \in F^{A+1}_{\omega, K}\), where \((s, j)\) results from \((\bar{s}, \bar{j})\) by taking its first free class and mapping it to \(\alpha_{|A|}\) and the increasing injection \(j\) has domain equal to \(K\) now.

It is left to the reader to verify that \(\theta\) is a homeomorphism.

\[\square\]

Let \(F^{A}_{\omega, K+1} = C_0 \cup \cdots \cup C_{l-1}\) be a Borel coloring. Consider now the corresponding partition \(F^{A+1}_{\omega, K} = C'_0 \cup \cdots \cup C'_{l'}\) defined by \((s, j) \in C'_i\) if \(\theta(s, j) \in C_i\). We are always in the context of a given \((r, c) \in F^A_{\omega, \omega}\) and \((t, i) \in (r, c)^0_A\).

Theorem 3 holds for \(F^{A+1}_{\omega, K}\), so there exists \((\bar{r}, \bar{c}) \in F^A_{\omega, \omega}\) such that

\[\[(\bar{r}, \bar{c})]^A+1_K \subseteq C'_{l'}\]

for some fixed \(i\). Let \((r'', c'') \in F^A_{\omega, \omega}\) resulting from \((\bar{r}, \bar{c})\), with \(\bar{r}^{-1}(\alpha_{|A|})\) becoming a free class, by removing the letter \(\alpha_{|A|}\) from the alphabet. Let now

\[(\bar{r}, \bar{c}) = (t, i)^*\sim (r'', c'') \cdot (\bar{r}, \bar{c})\]
it is obvious that \((\hat{r}, \hat{c}) \leq (r, c)\), in fact \((\hat{r}, \hat{c}) \in [(t, i)^*, (r, c)]\), since \((\hat{r}, \hat{c}) = (t, i)^* \sim (\hat{r}, \hat{c})\) and \((\hat{r}, \hat{c})\) is a reduct of \((r, c)\).

**Claim 2.** \([(t, i)^*, (\hat{r}, \hat{c})]\)_{\mathcal{A}}^{K+1} \subseteq C_i \text{ for the above fixed } i.

**Proof.** Let \((t, i)^* \sim (s, j) \cdot (r'', c'') \cdot (\hat{r}, \hat{c}) \in [(t, i)^*, (\hat{r}, \hat{c})]\)_{\mathcal{A}}^{K+1}. By the above definition of \(\theta^{-1}\) we get
\[
\theta^{-1}((t, i)^* \sim (s, j) \cdot (r'', c'') \cdot (\hat{r}, \hat{c})) = (s, j) \cdot (\hat{r}, \hat{c}) \in [(\hat{r}, \hat{c})\]_{\mathcal{K}}^{A+1} \subseteq C'_i.
\]

**Remark 1.** Up to this point, we have shown that given any \((r, c) \in F_{\omega, \omega}^A\), a \((t, i) \in (r, c)^A_0\) and a partition \(F_{\omega, \omega+1}^A = C_0 \cup \cdots \cup C_{l-1}\), there exists an \((\hat{r}, \hat{c}) \in [(t, i)^*, (r, c)]\) such that \([(t, i)^*, (\hat{r}, \hat{c})]\)_{\mathcal{A}}^{K+1} \subseteq C_i\) for a fixed \(i\). Notice that \((t, i)^*\) extends \((t, i)\) in both \((r, c)\) as well as \((\hat{r}, \hat{c})\), since \((\hat{r}, \hat{c}) \in [(t, i)^*, (r, c)]\).

Keeping that remark in mind we proceed in the proof of our lemma.

Let \((r_0, c_0) \in F_{\omega, \omega}^A\) arbitrary and \((t_0, i_0) \in (r_0, c_0)^A_0\). By Remark 1 there exists \((r_1, c_1) \in [(t_0, i_0]^*, (r_0, c_0)]\) such that \([(t_0, i_0)^*, (r_1, c_1)]\)_{\mathcal{A}}^{K+1} \subseteq C_k\) for some \(k \in l\). Set \((t_1, i_1) = (t_0, i_0)^* \leq (r_1, c_1)\), notice that the domain of \(i_1\) is equal to one. By Remark 1, once more, there exists \((r_2, c_2) \in [(t_1, i_1)^*, (r_1, c_1)]\) such that for every \((t, i) \leq (t_1, i_1)\) with \((t, i) \in (r_1, c_1)^A_0\) then \([(t, i)^*, (r_2, c_2)] \subseteq C_k\), \((t, i)^* \leq (t_1, i_1)^*\)

for some fixed \(k \in l\). Set \((t_2, i_2) = (t_1, i_1)^* \leq (r_2, c_2)\).

Suppose we have constructed \((r_n, c_n) \in F_{\omega, \omega}^A\), \((t_n, i_n) = (t_{n-1}, i_{n-1})^\ast\), with the property that the domain of \(i_n\) is equal to \(n\) and for all \((t, i) \leq (t_{n-1}, i_{n-1})\), where \((t, i) \in (r_{n-1}, c_{n-1})^A_0\), we have \([(t, i)^*, (r_n, c_n)]\)_{\mathcal{A}}^{K+1} \subseteq C_k\) for some \(k\) depending on \((t, i)\). Notice that \((t, i)^*\) has the same length as \((t_n, i_n)\), in fact \((t, i)^* \leq (t_n, i_n)\).

**Claim 3.** There exists \((r_{n+1}, c_{n+1})\) such that \([(t, i)^*, (r_{n+1}, c_{n+1})]\)_{\mathcal{A}}^{K+1} \subseteq C_k\) for some \(k\) depending on \((t, i)\), for all \((t, i) \leq (t_n, i_n)\), \((t, i) \in (r_n, c_n)^A_0\).

**Proof.** Let \{\((t_{nj}, i_{nj}) : j \leq m\}\} be an enumeration of all \((t, i) \leq (t_n, i_n)\), where the range of \(t\) is equal to \(A\) and \(i = \emptyset\). In other words we enumerate all reducts \((t, i)\) of \((t_n, i_n)\) such that \((t, i) \in (r_n, c_n)^A_0\). Set \((r_{n, c_n}^0) = (r_n, c_n)\) and \((r_{n, c_n}^{ni})\) be given by the Remark 1 so that \(\forall j \leq i[(t_{nj}, i_{nj}^\ast)^*, (r_{n, c_n}^{ni})] \subseteq C_k\) for some \(k \in l\). Set \((r_{n+1}, c_{n+1}) = (r_{n+1}^{ni}, c_{n+1}^{ni})\) and \((t_{n+1}, i_{n+1}) = (t_{n+1}, i_{n+1})^\ast \leq (r_{n+1}, c_{n+1})\).

Let now \((r, c) = \lim_{n \to \infty} (r_n, c_n)\).

Consider the finite coloring \((r, c)^A_0 = C_0^\prime \cup \cdots \cup C_{l-1}^\prime\) defined by: for \((t, i) \in (r, c)^A_0\) if \([(t, i)^*, (r, c)]\)_{\mathcal{A}}^{K+1} \subseteq C_i\) then we put it in the class of \(C_i^\prime\). By Lemma 2 we get
an \((r', c') \in [(r, c)]_A^\infty\) such that \((r', c')_A^i \subseteq C_i\) for some fixed \(i\), which it means that 
\([t, i]^*, (r', c')]_A^{K+1} \subseteq C_i\) for all \((t, i) \in (r', c')_A^0\). This implies that 
\([r', c')]_A^{K+1} \subseteq C_i\). 

\(\square\)

Theorem 3 follows by induction on \(K\). The case \(K = 0\) reduces to Lemma 1, 
while the inductive step is a consequence of Lemma 3.

**Proof of Lemma 2**

The proof of the Lemma 2 uses the Infinite Hales-Jewett theorem for left-variable 
words, so we need to introduce some definitions related to it.

Fix a finite alphabet \(A\). By \(W_A\) we denote the set of all words over \(A\) of finite 
length i.e. all finite strings of elements of \(A\), and by \(W_{Av}\) the set of all variable 
words over \(A\), i.e. all finite strings of elements of \(A \cup \{v\}\) in which \(v\) occurs at least 
once. For \(w \in W_A\) or \(w \in W_{Av}\) we define its length \(|w|\) to be the length of the finite 
string of members of \(A\) or \(A \cup \{v\}\) respectively. We distinguish a certain subset of 
the variable words, the left variable words, the ones that the variable \(v\) occurs at least 
on the first position of the word. For an infinite sequence \(X = (x_n)_{n \in \omega}\) of 
elements of \(W_{Av}\), by \([X]_A\) we denote the partial subsemigroup of \(W_A\) generated by 
\(X\) as follows:

\([X]_A = \{ x_{n_0}(\alpha_0) \ldots x_{n_k}(\alpha_k) : n_0 < \ldots < n_k \text{ and } \alpha_i \in A(i \leq k) \}\).

We shall need the following variation of the infinite Hales-Jewett theorem see ([To], 
chapter 3).

**Theorem 4.** (Infinite Hales -Jewett theorem for Left-variable words) Let \(A\) be a 
finite alphabet, then for any finite coloring of \(W_A\) there is an infinite sequence \(X = 
(x_n)_{n \in \omega}\) of left variable-words and a variable free word \(w_0\) such that the translate 
\(w_0^\omega [X]_A\) of the partial semigroup of \(W_A\) generated by \(X\) is monochromatic.

We can now proceed to the proof of the Lemma 2

**Proof.** Let \((r, c)_A^0 = C_0 \cup \ldots C_{l-1}\) be an \(l\) coloring. We introduce a mapping 
\(\sigma : W_A \rightarrow (r, c)_A^0\)

defined as follows: given \(w \in W_A\) of length \(K\) let \(\sigma(w) = (t, i) \in (r, c)_A^0\) for 
\((t, i) = (r_0, c_0)[0]\), where \((r_0, c_0) = (r_1, c_1) \cdot (r, c)\). The definition of \((r_1, c_1)\) is as 
follows: for \(k \in K\), \(r_1(k) = w(k)\) and for \(k \geq K\), \(r_1(k) = k - K\). Any choice of \(c_1\) 
will do here since we are interested for \((t, i) = (r_0, c_0)[0]\) so \(i = 0\).

Conversely we define \(\sigma^{-1} : (r, c)_A^0 \rightarrow W_A\) in the following way: given \((t, i) \in 
(r, c)_A^0\), with \(K = lh(t, i)\), there exists \((r_1, c_1)\) such that \((t, i) = (r_1, c_1) \cdot (r, c)[0]\), 
then \(\sigma^{-1}(t, i) = r_1 \upharpoonright K \in W_A\).
Consider the partition \( W_A = C'_0 \cup \cdots \cup C'_{l-1} \) defined by \( w \in C'_i \) iff \( \sigma(w) \in C_i \).

By the left variable Hales-Jewett Theorem, there exists an infinite sequence \( X = (x_n)_n \in \omega \) of left variable words and a variable free word \( w_0 \) such that the translate \( w_0^{-}[X]_A \) of the partial subsemigroup of \( W_A \) generated by \( X \) is contained in \( C'_k \) for some fixed \( k \in l \).

Consider now the infinite variable word

\[
w' = w_0^{-} x_0 \cdots \hat{x}_n \cdots
\]

We use it to define an \( (r_2, c_2) \in F^A_{\omega, \omega} \) such that \( (r', c') = (r_2, c_2) \cdot (r, c) \) has the property that \( (r', c')_A \subseteq C_k \) for the same fixed \( k \in l \) as above.

The rigid surjection \( r_2 : A \cup \omega \to A \cup \omega \) is defined as follows:

\[
r_2(n) = \begin{cases} 
    w'(n) & \text{if } w'(n) \in A, \\
    m-1 & \text{if } w'(n) = v, n \in [l_{m-1}, l_m),
\end{cases}
\]

where \( l_0 = |w_0^{-}|, l_1 = |w_0^{-} x_0|, l_{m-1} = |w_0^{-} \cdots \hat{x}_{m-2}| \) and \( l_m = |w_0^{-} \cdots \hat{x}_{m-1}| \).

We defined \( c_2 \) for the sake of completeness. Since we are working with \( (r, c)_A \) the increasing injection \( c_2 \) plays no role here.

\( c_2 \) is defined by \( c_2(0) = |w_0| \) and for \( m > 0 \) \( c_2(m) = |w| \), where

\[
w = w_0^{-} x_0 \cdots \hat{x}_{m-1}.
\]

Obviously \( (r', c') = (r_2, c_2) \cdot (r, c) \in [(r, c)]^A_1 \). We would like to demonstrate that \( (r', c')_A \subseteq C_k \). Given \( (t, i) \in (r', c')_A \) it suffices to show that \( \sigma^{-1}(t, i) \in w_0^{-}[X]_A \).

Notice that \( (t, i) = ((r_3, c_3) \cdot (r_2, c_2))(0) = ((r_3, c_3) \cdot (r_2, c_2) \cdot (r, c))(0) \) for some connection \( (r_3, c_3) \in F^A_{\omega, \omega} \), with \( \text{ht}((r_3, c_3)(0)) = M \). In other words \( r_3 \upharpoonright M \subseteq A \) and also \( r_3(M) = E_0 \). Then by the definition of multiplication it follows that

\[
\sigma^{-1}(t, i) = ((r_3, c_3) \cdot (r_2, c_2))(0) = r_3 \circ r_2 \upharpoonright M
\]

\[
= w_0^{-} x_0 (r_3(0)) \cdots x_1 (r_3(1)) \cdots \hat{x}_{M-1} (r_3(M-1))
\]

is an element of \( w_0^{-}[X]_A \).

\[ \square \]

4. The spaces \( F_{\omega, \omega} \) and \( F_{\omega, K} \)

In this section we obtain the corresponding version of Theorem 2 for the space of all connections that partition \( \omega \) onto \( \omega \) many pieces, in other words \( F_{\omega, \omega} \). The reader at this point is assumed to be familiar with the theory of Ramsey spaces as introduced and developed by S. Todorcevic in [16]. We have already introduced a topology on \( F_{\omega, \omega} \), the one with basic open sets \( [(t, i), (r, c)] \). We recall that a subset \( \mathcal{X} \) of \( F_{\omega, \omega} \) is *Ramsey* if for every \( [(t, i), (r, c)] \neq \emptyset \) there is a \( (r', c') \in [(t, i), (r, c)] \) such that either \( [(t, i), (r', c')] \subseteq \mathcal{X} \) or \( [(t, i), (r', c')] \subseteq \mathcal{X}^c \), and \( \mathcal{X} \) is *Ramsey null* if for every \( [(t, i), (r, c)] \neq \emptyset \), there is a \( (r', c') \) such that \( [(t, i), (r', c')] \cap \mathcal{X} = \emptyset \). We are going to see that Ramsey subsets of \( F_{\omega, \omega} \) are exactly those with the Baire property, and Ramsey null are the meager ones. Once this is established, the desired
partition property i.e. the corresponding version of Theorem 2 for $F_{\omega, \omega}$ follows immediately as a result of the general theory of Ramsey spaces in [To]. In order to see the equivalence of those topological and Ramsey notions we use Theorem 5.4 from [To], where it is proved that the list of properties A.1- A.4, listed just below, gives that equivalence between Ramsey subsets and those that have the property of Baire.

As we defined above for $(t, i) \in F_{M, L}$ and $(r, c) \in F_{\omega, \omega}$ let

$$[(t, i), (r, c)] = \{ (r', c') : (r', c') \leq (r, c), (t, i) \leq (r', c') \}$$

be the basic open sets in the topology on $F_{\omega, \omega}$ with respect to which we consider the property of Baire.

In order to use Theorem 5.4 we have to introduce a notion of finite approximation of a connection. Define $u : F_{\omega, \omega} \times \omega \to AF_{\omega, \omega}$ by $u((r, c), n) = u_n((r, c)) = (r, c)[n]$. In other words each $u_n$, a finite approximation, of the connection $(r, c)$, is the initial segment $(r, c)[n]$ introduced in the second section. The image of $u_n$ is denoted by $(AF_{\omega, \omega})_n$ and the union $\bigcup_{n \in \omega}(AF_{\omega, \omega})_n = AF_{\omega, \omega}$ forms the set of all finite approximations of all elements of the space $F_{\omega, \omega}$.

Similarly we can define finite approximations of elements in $F_{\omega, K}$ by

$$u'(s, j), n) = u'_n(s, j) = (s \upharpoonright n, j \upharpoonright \text{im}(s \upharpoonright n))$$

where $(s \upharpoonright n, j \upharpoonright \text{im}(s \upharpoonright n))$ is a connection. In other words $u'_n(s, j)$ is an initial segment of $(s, j)$ as defined in Definition 1.

We pass now to verify the properties A.1–A.4. The first three are immediate consequence of the definitions:

**A.1.**

1. $u_0((r, c)) = \emptyset$ for all $(r, c) \in F_{\omega, \omega}$.
2. $(r', c') \neq (r, c)$ implies $u_n((r, c)) \neq u_n((r', c'))$ for some $n \in \omega$.
3. $u_n((r, c)) = u_m((r', c'))$ implies $n = m$ and $u_k((r, c)) = u_k((r', c'))$ for all $k < n$.

**A.2.** Given $(t, i), (t', i')$ we define $(t', i') \leq_{fin} (t, i)$ if they have the same length, i.e., $lh(t, i) = lh(t', i')$, also denoted by $[t, i] = [t', i']$ and $(t', i') \leq (t, i)$. In other words $(t', i') \leq_{fin} (t, i)$ iff $(t', i')$ is a reduct of $(t, i)$ as defined in section 2, Definition 3.

1. For any $(t, i) \in F_{M, L}$ the set
   $$\{(t', i') \in F_{M, K} : K < L, (t', i') \leq_{fin} (t, i) \}$$
   is finite.
2. $(r', c') \leq (r, c)$ iff $(\forall n)(\exists m)u_n((r', c')) \leq_{fin} u_m((r, c))$. 
(3) For all \((t, i), (t', i')\), \((t', i') \preceq (t, i) \land (t, i) \preceq_{\text{fin}} (t'', i'') \rightarrow \exists (\tilde{t}, \tilde{i}) \preceq (t'', i'') : (t', i') \preceq_{\text{fin}} (\tilde{t}, \tilde{i})\).

A.3

(1) If \([[(t, i), (r, c)] \neq \emptyset\) then \([[(t, i), (r', c')] \neq \emptyset\) for all \((r', c') \in \[(t, i), (r, c)\]\).

(2) \((r', c') \leq (r, c)\) and \([[(t, i), (r', c')] \neq \emptyset\) imply that there is \((r'', c'') \in \[(t, i), (r, c)\]\)

\[\begin{align*}
\text{such that } \emptyset \neq \[(t, i), (r'', c'')\] \subseteq \[(t, i), (r', c')\].
\end{align*}\]

The last property A.4 is less obvious.

A.4

Let \(\mathcal{O} \subseteq (AF_{\omega, \omega})_{L+1}\), i.e. \(\mathcal{O}\) is a subset of the set of all finite approximations of all elements of \(F_{\omega, \omega}\) of length \(L + 1\), \((t, i) \in F_{M, L}\), for some \(M\), and \([[(t, i), (r, c)] \neq \emptyset\), then there exists \((r', c') \in \[(t, i), (r, c)\]\) such that:

\[u_{L+1}([[(t, i), (r', c')] \subseteq \mathcal{O}\text{ or } u_{L+1}([[(t, i), (r', c')] \subseteq \mathcal{O}^c,\]

where \(u_{L+1}([[(t, i), (r', c')] = \{ (t', i') \in (r', c')^{L+1} : (t, i) \preceq (t', i') \}.\]

Proof. Consider the open set \([[(t, i), (r, c)] \neq \emptyset\). By A.3(1) we can assume that \((r, c) = (t, i) \sim (\tilde{r}, \tilde{c}).\)

Let \(A\) be a finite alphabet of cardinality \(L\) and consider the space \(F^A_{\omega, 2}\). There is a natural homeomorphism \(\sigma : F^A_{\omega, 2} \to F_{\omega, L+2}\) defined by \(\sigma((s, j)) = (\hat{s}, \hat{j})\), where \((\hat{s}, \hat{j})\) results from \((s, j)\) by removing the elements of the alphabet \(A\), so the first \(L\) classes become free. Conversely \(\sigma^{-1}((\hat{s}, \hat{j})) = (s, j)\), where \((s, j)\) results from \((\hat{s}, \hat{j})\) by adding to the first \(L\) free classes of \((\hat{s}, \hat{j})\) an element of the alphabet \(A\). Now we proceed as in the Lemma 3 above.

There is an homeomorphism \(\theta : F^A_{\omega, 2} \to [[(t, i), (r, c)]^{L+2}\) where

\[[(t, i), (r, c)]^{L+2} = \{ (s', j') \in F_{\omega, L+2} : (t, i) \preceq (s', j') \leq (r, c) \}.\]

For \((s, j) \in F^A_{\omega, 2}\) let

\[\theta((s, j)) = (t, i) - \sigma((s, j) \cdot (r', c'))\]

where, given \((\tilde{r}, \tilde{c}) \in F_{\omega, \omega}\) by \((\tilde{r}', \tilde{c}') \in F^A_{\omega, \omega}\) we mean the connection that is the identity on \(A\) and maps \(\omega\) onto \(\tilde{\omega}\) as \(\tilde{r}\) does, i.e. \(\tilde{r}' | \omega = \tilde{r} | \omega = \omega\), the equivalence class of the connection \((\tilde{r}', \tilde{c}')\) that contains an element of the alphabet \(A\) has only one element namely the letter from \(A\). Conversely given an arbitrary \((s', j') \in [(t, i), (r, c)]^{L+2}\) then \((s', j') = (\hat{s}, \hat{j}) \cdot ((t, i) \sim (\tilde{r}, \tilde{c}))\) for some \((\hat{s}, \hat{j}) \in F_{\omega, L+2}\)

\[\theta^{-1}((s', j')) = \theta^{-1}((\hat{s}, \hat{j}) \cdot ((t, i) \sim (\tilde{r}, \tilde{c}))) = \sigma^{-1}((\hat{s}, \hat{j})).\]

Let \(c : [(t, i), (r, c)]^{L+2} \to 2\) be a two coloring, defined by

\[c((s', j')) = \chi_{\mathcal{O}}(\langle s', j' \rangle | [L + 1]),\]
where $X_\emptyset$ is the characteristic function of the set $\mathcal{O} \subseteq (A \mathcal{F}_\omega, \omega)_{L+1}$. This is obviously a Borel coloring so Theorem 3 applies to give us $(r', c') \leq (r, c)$ such that $[(t, i), (r', c')]^L_{L+2}$ is monochromatic with respect to $c$. Then the open set $[(t, i), (r', c')]$ is such that $u_{L+1}[(t, i), (r', c')] \subseteq \mathcal{O}$ or $\mathcal{O}^c$. □

As a result of the properties A.1 to A.4 being satisfied, we have that

$$\langle \mathcal{F}_{\omega, \omega}, u, \leq \rangle$$

forms a Ramsey topological space as introduced and discussed in [To]. Equivalently one has that every property of Baire subset of $\mathcal{F}_{\omega, \omega}$ is Ramsey and every meager subset is Ramsey null. Moreover the field of Ramsey subsets, that coincides with the field of Baire measurable subsets, is closed under the Suslin operation. Recall that a mapping $f : X \to Y$ between two topological spaces is Suslin measurable, if the preimage $f^{-1}(U)$ of every open subset $U$ of $Y$ belong to the minimal $\sigma -$field of subsets of $Y$ containing its closed sets and it is closed under the Suslin operation [Ke].

Now the above Ramsey and topological equivalence in any Ramsey topological space gives us as a corollary a Ramsey theorem that in the case of our Ramsey topological space $\langle \mathcal{F}_{\omega, \omega}, u, \leq \rangle$ is the following theorem:

**Theorem 5.** Given a finite Suslin measurable coloring $g : \mathcal{F}_{\omega, \omega} \to l$, there exists a $(r, c) \in \mathcal{F}_{\omega, \omega}$ such that $[(r, c)]^\omega$ is monochromatic.

Given that $\langle \mathcal{F}_{\omega, \omega}, u, \leq \rangle$ forms a Ramsey topological space where its field of Baire measurable subsets coincides with that of Ramsey and is closed under the Suslin operation, then for any finite coloring, where each color is Suslin measurable, the assertion of the above theorem follows immediately.

We can show that $\langle \mathcal{F}_{\omega, \omega}, \mathcal{F}_{\omega, \omega}, \leq, \leq, u', u \rangle$ forms a Ramsey space, which would give us the corresponding extension of the main Theorem of this paper to the Suslin measurable colorings. Instead we give a direct proof.

**Theorem 6.** Let $g : \mathcal{F}_{\omega, K} \to l$ be a Suslin measurable coloring. Then there exists $(r, c) \in \mathcal{F}_{\omega, \omega}$ such that $[(r, c)]^K$ is monochromatic.

**Proof.** Let $\pi : \mathcal{F}_{\omega, \omega} \to \mathcal{F}_{\omega, K}$ be defined by $\pi((r, c)) = (s_0, j_0) \cdot (r, c)$, where $(s_0, j_0) \in \mathcal{F}_{\omega, K}$ is defined by $s_0 \upharpoonright K$ being the identity and $s_0 \upharpoonright [K, \omega] = 0$. Consider now the composition $g \circ \pi : \mathcal{F}_{\omega, \omega} \to l$ which is also Suslin measurable. By Theorem 5 there exists $(r, c) \in \mathcal{F}_{\omega, \omega}$ such that $[(r, c)]^\omega$ is monochromatic with respect the above composition.

Let $(r', c') = (r_1, c_1) \cdot (r, c)$, where $(r_1, c_1)$ has the property that for all $n \in \omega$ set $r_1^{-1}(n)$ is of infinite cardinality. Notice that any $(s, j) \in [(r', c')]^K$ can be written as $\pi(r'', c'')$ for some $(r'', c'') \in \mathcal{F}_{\omega, \omega}$ and $(r'', c'') \in [(r', c')]^\omega$. Therefore $(r', c')$ is such that $[(r', c')]^K$ monochromatic with respect to the coloring $g$. □
We shall show that Theorem 6 does extend to the realm of Baire measurable colorings relative to an appropriate topology on $F_{\omega,K}$. Any element $(s,j) \in F_{\omega,K}$ is consisted by a rigid surjection $s : \omega \to K$ that can be seen as an element of the space $\{0, \ldots, K - 1\}^\omega$, as well as an increasing surjection $j : K \to \omega$. We can consider the metric topology on the space $F_{\omega,K}$ having as basic open sets, sets of the form:

$$[(t,i)] = \{ (s,j) \in F_{\omega,K} : (t,i) \preceq (s,j) \text{ and domain of } i \text{ is equal to } K \}.$$  

Notice that since the domain of $i$ is $K$ we have that $j = i$. To indicate that $(s,j) \in F_{\omega,K}$ we will still write $(s,j)$ instead of $(i,s)$.

We could think of a rigid surjection defined on a finite interval, in a set theoretic way, as a set of ordered pairs, so given two rigid surjections defined on two disjoint intervals we can consider their union, namely the rigid surjection defined on both intervals and is equal to each of its components when restricted to one of them.

The extension of Theorem 6 to the Baire measurable colorings relative to the new topology defined just above, is as follows:

**Theorem 7.** Let $g : F_{\omega,K} \to l$ is a finite coloring that is Baire measurable relative to the metrizable topology defined just above. Then there exists an $(r,c) \in F_{\omega,\omega}$ such that the family $[(r,c)]^K = \{ (s,j) \in F_{\omega,K} : (s,j) \leq (r,c) \}$ is $g$-monochromatic.

**Proof.** The coloring $g$ is Baire measurable, then there is a dense $G_\delta$ subset $G$ of $F_{\omega,\omega}$ that the coloring is actually continuous.

We call a sequence $(D_p)_{p \in \omega}$ of finite subsets of $\mathbb{N}$ a block sequence, if for any two $p_0, p_1$ with $p_0 < p_1$ we have $\max D_{p_0} < \min D_{p_1}$.

To continue we need the following:

**Lemma 4.** Given a dense $G_\delta$ subset $G$ of $F_{\omega,K}$ and $(t,i) \in F_{\omega,K}$, there is an infinite block sequence $(D_p)_{p \in \omega}$ of finite subsets of $\mathbb{N}$ and for each $p$ a surjective mapping $f_p : D_p \to K$ such that for every $(s,j) \in F_{\omega,K}$ if $(t,i) \preceq (s,j)$ and $s$ extends infinitely many of the $f_p$, then $(s,j) \in G$.

**Proof.** $G$ is a dense $G_\delta$ subset of $F_{\omega,K}$, so its complement is a countable union of closed nowhere dense sets namely $G^c = \cup_{n \in \omega} N_n$. Start with the open set $[(t,i)]$ and $N_0$, there is an extension $t_0$ of $t$ such that $[(t_0,i)] \cap N_0 = \emptyset$ set $d_0 = \text{dom}(t_0 \setminus t)$ and $g_0 : d_0 \to K$ defined by $g_0 = t_0 \upharpoonright d_0$. We remark here that $i$ is fixed so no matter how we extend $t$ has no impact on $i$.

Let $\{ t^n_h : h \in [0,m) \}$ where $t_0^n = t_0$, be an enumeration of all possible extensions of $t$ with length equal to $lh(t_0) = lh(t_0,i)$. Consider the open set $[(t^n_1,i)]$ and let $t_1$ be an extension of the rigid surjection $t^n_1$ such that $[(t_1,i)] \cap N_0 = \emptyset$. Set now $d_1 = \text{dom}(t_1 \setminus t^n_1)$ and $g_1 : d_1 \to K$ defined by $g_1 = t_1 \upharpoonright d_1$. Then consider the open set $[(t^n_2 \cup g_1,i)]$ and let $t_2$ be an extension of the rigid surjection $t^n_1 \cup g_1$ such that $[(t_2,i)] \cap N_0 = \emptyset$. Once more set $d_2 = \text{dom}(t_2 \setminus t^n_2 \cup g_1)$ and $g_2 : d_2 \to K$ defined by $g_2 = t_2 \upharpoonright d_2$. After $m$-steps we have defined $(d_i)_{i \in m}$ and similarly $(g_i)_{i \in m}$. Set
now $D_0 = d_1 \cup \cdots \cup d_{m-1}$ and $f_0 = g_1 \cup \cdots \cup g_{m-1}$. They have the property that for all $t^h_r$ with $h < m$

$$[(t^h_r \cup f_0, i)] \cap N_0 = \emptyset.$$

Consider now the open set $[(t \cup g_0 \cup f_0, i)]$ and let $t_2$ be an extension of the rigid surjection $t' = t \cup g_0 \cup f_0$ such that $[(t_2, i)] \cap N_1 = \emptyset$. Notice that $[(t_2, i)] \cap N_0 = \emptyset$ as well, since $t_2$ extends $t \cup g_0 = t_0$. Set $d_m = dom(t_2 \setminus t')$ and $g_m : d_m \to K$ defined by $g_m(t_2) = g_m$. Let $t^h_r = t_2 | t^h_r$, be an enumeration of all possible extensions of $t$ with length equal to $lb(t_2) = lb(t_2, i)$. As above start off with $t^1_r$ and consider the open set $[(t^1_r, i)]$. Let $g_{m+1} : d_{m+1} \to K$ be a such that $[(t^1_r \cup g_{m+1}, i)] \cap (N_0 \cup N_1) = \emptyset$. After $n$ steps we have created $(d_h)_{h \in [m, m+n]}$ and their corresponding mappings $(g_h)$. Set $D_1 = d_{m+1} \cup \cdots \cup d_{(m+n)-1}$ and $f_1 = g_{m+1} \cup \cdots \cup g_{(m+n)-1}$. Then for all $h \in n$, $D_1$ and $f_1$ have the property:

$$[(t^h_r \cup f_1, i)] \cap (N_0 \cup N_1) = \emptyset.$$

We proceed in that manner to get $(D_p)_{p \in \omega}$ and $f_p : D_p \to K$ their corresponding surjections. Now suppose that $(s, j) \in F_{\omega, K}$ is such that $(t, i) \preceq (s, j)$, $s$ extends infinitely many of $f_p$ and is in $G^\omega$. Then $(s, j) \in N_n$ for some $n$. But $s$ extends in particular some $f_m$ where $n \leq m$ which implies that $(s, j)$ does not belong to the finite union $N_0 \cup \cdots \cup N_n$ a contradiction.

We continue now the proof of Theorem 7. Let $(b_l)_{l \in \omega}$ be an enumeration of all $(t, i) \in F_{M, K}$, $M \in \omega$, i.e. all finite approximations of all elements of $F_{\omega, K}$. For each $l$, let $(f^l_p)_{p \in \omega}$ be the sequence of block mappings given by Lemma 4 when applied to $G$ and $b_l$. We will create $(r, c) \in F_{\omega, \omega}$ with the property that $[(r, c)]^K \subseteq G$.

**Claim 4.** There exists $(r, c) \in F_{\omega, \omega}$ such that $[(r, c)]^K \subseteq G$.

from this claim, we see that $g \mid [(r, c)]^K$ is continuous so we apply Theorem 6 to get the desired result.

It rests to prove the claim.

**Proof.** We build $(r, c) \in F_{\omega, \omega}$ recursively by deciding the restrictions $(r, c)[n]$ for some strictly increasing sequence $(n_i)$ of positive integers, as follows: That strictly increasing sequence $(n_i)$ of positive integers will be a subset of $(E_n)_{n \in \omega}$ that was defined at section 2. At odd stages we choose $(r, c)[n2l-1]$ in a way that guarantees that $(r, c)$ has infinitely many infinite classes. We define the increasing injection arbitrary. At even stages, consider the finite set

$$A = \{(b_q)_{l \in \omega} : b_q \leq (r, c)[n] \mid b_q \mid = lb(b_q) \leq n_{2l-1}\}.$$

Start off with $b_0$ choose a finite partial mapping $f_0 : D_0 \to K$ associated with $b_0$ such that $D_0$ lies above $n_{2l-1}$ i.e. $\max D_0 < n_{2l-1}$. Let $n_{2l}^0 = \max(D_0) + 1$ and define the extension $(r, c)[n_{2l}^0]$ of $(r, c)[n_{2l-1}]$ such that each point of the preimage $f_0^{-1}(m)$, $(m \in K)$ is equivalent to the minimal point of the preimage of $b_0^{-1}(m)$ and makes no other commitments beyond $(r, c)[b_0]$ being an equivalence relation.
Now consider $b_1$ and repeat the above step to get $n^1_{2l} > n^0_{2l}$ and $f_1$, with $D_1$ lying above of $D_0$. After $m$ steps we finally define $n^m_{2l} = n^{m-1}_{2l}$ and the extension of $(r,c)[n_{2l-1}]$ to $(r,c)[n_{2l}]$ as the union of each of the above extensions. Notice that at the even stages we add no new pieces on the partition, the equivalence relation that $r$ defines. As a result we make no choices for $c$.

Consider now an arbitrary $(s,j) \in [(r,c)]^K$. Let $(t,i) = (s,j)[K]$, see section 2 page 4, for the definition. Therefore the map $s: \omega \to \omega$ extends $t: lh(t,i) \to K$ and infinitely many of the $f^t_i$ with $b_t = (t,i)$, so $(s,j) \in G$. This is since $b_t$ is contained in the above set $A$ at all odd stages $n_{2l-1}$ with $lh(b_t) \leq l \leq n_{2l-1}$. □

Therefore our Theorem 6 holds in the realm of Baire measurable colorings as well.

REFERENCES

[Ca-Si] T.J. Carlson, S.G. Simpson, A Dual from of Ramsey’s Theorem, Adv. Math.,53 (1984),265-290.

[Gr-Ro] R.L. Graham, B.L. Rothschild, Ramsey’s theorem for n-parameter sets. Trans. Amer. Math. Soc. 159 1971 257-292.

[Ke] A.S. Kechris, Classical Descriptive Set Theory, Springer-Verlag.

[Ra] F. P. Ramsey, On a problem of Formal Logic, Proc. London Math Society Ser. 230 (1929), pp. 264-286.

[So] S. Solecki Abstract approach to finite Ramsey theory and a self-dual Ramsey theorem. Preprint 2011

[To] S. Todorcevic, Introduction to Ramsey Spaces, Annals of Mathematics Studies, No.174, Princeton Univ. Press, 2010.