Harnack inequalities, Kobayashi distances and holomorphic motions

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We prove some generalizations and analogies of Harnack inequalities for pluriharmonic, holomorphic and “almost holomorphic” functions. The results are applied to the proving of smoothness properties of holomorphic motions over almost complex manifolds.

The Harnack inequalities in this paper are considered only from the point of view of Complex Analysis, in application to real parts of holomorphic functions. In this context they can be written in invariant form independent of biholomorphic transformations and thus can be generalized onto arbitrary almost complex manifolds what we show in sec.1.

First of all we are interesting in holomorphic functions which do not take the values 0 and 1 because just such functions constitute in the main the normalized holomorphic motions (see sec. 3). The main estimates of such functions give the theorems of Landau and Schottky which were specified many times and are presented in the final form in [H]. The estimates of J.Hempel can also be written in invariant form and thus they are evidently extendable onto general almost complex manifolds (sec. 2).

Most likely, these results are already known but the proofs are so simple that it seemed for me more complicated to look for precise references, and I do not affirm that these results are new.

The general Landau theorem is applied in sec. 3 in the proving of an analogy of Harnack inequalities for holomorphic functions with values in $\mathbb{C} \setminus \{0, 1\}$ (sec. 2) and (following [GJW]) in the proof of Hölder conditions for arbitrary holomorphic motions (sec. 3).

In sec. 4 we establish analogies of Harnack inequalities for functions in disk or on the plane which satisfy special estimates of derivatives in $\bar{z}$ what
has direct relation with almost holomorphic motions, in particular, with holomorphic motions in nonstandard complex structures.

1. Pluriharmonic functions. Let us start with base Harnack inequalities for positive harmonic functions \( u(z) \) in the unit disk \( \mathbb{D} : |z| < 1 \) on the complex plane \( \mathbb{C} \):

\[
\frac{1 - |z|}{1 + |z|} \leq \frac{u(z)}{u(0)} \leq \frac{1 + |z|}{1 - |z|}
\]

(see e.g. [HK]). Noticing that the Poincaré distance \( \rho_\mathbb{D} \) between the points 0 and \( z \) in \( \mathbb{D} \) is equal to \( \log \frac{1 + |z|}{1 - |z|} \) (assuming that Poincaré metric has curvature \( \equiv -1 \)) we can rewrite these inequalities in invariant form independent of Möbius transformations of \( \mathbb{D} \):

\[
e^{-\rho_\mathbb{D}(z,z_0)} \leq \frac{u(z)}{u(z_0)} \leq e^{\rho_\mathbb{D}(z,z_0)}
\]

for any \( z, z_0 \in \mathbb{D} \).

But the metric of curvature \( \equiv -1 \) exists on arbitrary hyperbolic Riemann surface, and it is natural to ask if the correspondent inequalities are valid on such surfaces. And if one notices that harmonic functions in the disk are partial case of pluriharmonic functions on a complex manifold then it is natural to interest in analogy of the last inequalities for such functions also.

A generalization of Poincaré distance on arbitrary complex manifold is the Kobayashi distance (see [K]) which coincides in the disk with \( \rho_\mathbb{D} \). Consider it more detaily in very general setting.

Let \( B \) be arbitrary path connected complex Banach manifold (in particular, finite dimensional one). For any \( z, z' \) placed in one coordinate ball of the manifold \( B \) there exists evidently a holomorphic disk \( h : \mathbb{D} \rightarrow B \) such that \( h(0) = z', h(\zeta) = z \) for some \( \zeta \in \mathbb{D} \). As \( B \) is path connected and any path \([0, 1] \rightarrow B \) is covered by a finite number of coordinate balls, then for any \( z, z' \in B \) there exists a chain of holomorphic disks \( h_j : \mathbb{D} \rightarrow B \) and points \( \zeta_j \in \mathbb{D} \) such that \( h_1(0) = z', h_{j+1}(0) = h_j(\zeta_j), j = 1, ..., N - 1, \) and \( h_N(\zeta_N) = z \). The amount

\[
\kappa_B(z, z') = \inf \sum_{1}^{N} \log \frac{1 + |\zeta_j|}{1 - |\zeta_j|}
\]
where infimum is taken by all described chains of holomorphic disks ($N$ can be arbitrary) is called Kobayashi distance between the points $z, z'$ on the manifold $B$. As $\log \frac{1 + |\zeta|}{1 - |\zeta|} = 2 \text{arcth} |\zeta|$ is the Poincaré distance in $\mathbb{D}$ between 0, $\zeta$ and this distance is invariant with respect to Möbius automorphisms of $\mathbb{D}$ then $\kappa_B(z, z') = \kappa_B(z', z)$.

In general case $\kappa_B$ is only a pseudometric, the distances between some different points can be equal to zero but the triangle inequality follows easily from the definition.

The most important property of Kobayashi distance is its evident non-increasing by holomorphic mappings: if $f : B \to X$ is such a map then

$$\kappa_X(f(z), f(z')) \leq \kappa_B(z, z') \quad (2)$$

(it is one of abstract variants of Schwarz lemma).

Kobayashi distance can be defined as above on arbitrary almost complex Banach manifold $B$. As the existence of finite chains of holomorphic disks connecting given $z, z' \in B$ on such a manifold is not evident at all, then we set for definitness that $\kappa_B(z, z') = \infty$ if there is no such a chain. For marked (base) point $z_0$, the ball $\{z \in B : \kappa_B(z, z_0) < R\}$ will be denoted by $B_R$.

Let us remind that a continuous function $u : B \to \mathbb{R}$ on almost complex Banach manifold is called pluriharmonic if for any holomorphic map $f : \mathbb{D} \to B$ the function $u \circ f$ is harmonic in the unit disk.

**Proposition 1.** Let $B$ be an almost complex Banach manifold and $u$ be a positive pluriharmonic function on $B$. Then, for any $z, z_0 \in B$,

$$e^{-\kappa_B(z, z_0)} \leq \frac{u(z)}{u(z_0)} \leq e^{\kappa_B(z, z_0)}. \quad (3)$$

If $\kappa_B(z, z_0) = \infty$ then there is nothing to prove, thus we assume further that $\kappa_B(z, z_0) < \infty$.

Let us fix $\varepsilon > 0$ and choose a chain of holomorphic disks $h_j : \mathbb{D} \to B$ and $\zeta_j \in \mathbb{D}$ such that $h_1(0) = z_0, h_{j+1}(0) = h_j(\zeta_j), j = 1, ..., N - 1, h_N(\zeta_N) = z$ and $\sum_1^N \log \frac{1 + |\zeta_j|}{1 - |\zeta_j|} \leq \kappa_B(z, z_0) + \varepsilon$.

By the classical Harnack inequality

$$\frac{1 - |\zeta_j|}{1 + |\zeta_j|} \leq \frac{u \circ h_j(\zeta_j)}{u \circ h_j(0)} \leq \frac{1 + |\zeta_j|}{1 - |\zeta_j|}.$$
As
\[
\frac{u(z)}{u(z_0)} = \frac{u \circ h_N(\zeta_N)}{u \circ h_N(0)} \cdot \frac{u \circ h_1(\zeta_1)}{u \circ h_1(0)},
\]
it follows that
\[
e^{-\kappa_B(z,z_0)-\varepsilon} \leq \frac{u(z)}{u(z_0)} \leq e^{\kappa_B(z,z_0)+\varepsilon}
\]
and the inequalities are proved due to arbitrary \(\varepsilon\).

The inequalities (3) are applicable to the functions of type \(\log M/|f|\) where \(f\) is a holomorphic function with \(0 < |f| < M\). Consider as an example a “pointwise” analogy of two constant theorem for such functions in which the point \(z_0\) plays the role of a set of positive harmonic measure.

**Corollary.** If \(f\) is a holomorphic function on \(B\) and \(0 < |f| < M\) then
\[
|f(z)| < |f(z_0)|^{\alpha(z)}M^{1-\alpha(z)}, \quad z \in B
\]
where \(\alpha(z) = e^{-\kappa_B(z,z_0)}\).

The function \(\log M/|f|\) is pluriharmonic and positive. According (3), \(M/|f(z)| \geq (M/|f(z_0)|)^{\alpha(z)}\) what is claimed.

2. Functions with values in \(\mathbb{C} \setminus \{0, 1\}\). The hyperbolic domain \(\mathbb{C} \setminus \{0, 1\}\) with complete hyperbolic Poincaré metric \(\rho_{0,1}\) plays big role in different problems of Complex Analysis. By the universal covering \(\mathbb{D} \to \mathbb{C} \setminus \{0, 1\}\) this \(\rho_{0,1}\) is lifted to Poincaré metric \(\rho_{\mathbb{D}}\). As the lifting to \(\mathbb{D}\) of Kobayashi metric on \(\mathbb{C} \setminus \{0, 1\}\) is the same then \(\rho_{0,1}\) coincides with Kobayashi metric in this domain.

Infinitesimal Poincaré metric in \(\mathbb{C} \setminus \{0, 1\}\) has the form \(\rho_{0,1}(z)|dz|\) (i.e. \(\rho_{0,1}(z_1,z_2) = \inf \gamma \int_{\gamma} \rho_{0,1}(z)|dz|\) for any \(z_1, z_2\) where infimum is taken by all smooth paths with ends \(z_1, z_2\)). We will need in the following estimates of this metric.

**Lemma 1.** \(\rho_{0,1}(z) \geq \rho_{0,1}(-|z|) \geq \left(|z|(C_{0,1} + \log \frac{1}{|z|})\right)^{-1}\) in \(\mathbb{D} \setminus 0\) with the constant \(C_{0,1} = 1/\rho_{0,1}(-1) > 1\).

The first inequality is proved in the paper of Lehto and Virtanen [LV], p.6 (see also [Ag]).

The second inequality is proved in the paper of Hempel [H], p.443, but we present here a simpler proof.
It is wellknown (see e.g. [A1], 1-8) that
\[ \log \rho_{0,1}(z) + \log \frac{1}{|z|} + \log (\log \frac{1}{|z|}) \to 0 \]
as \( z \to 0 \). (More detailed asymptotics see in [H].)

As \( \min_{\partial D} \rho_{0,1} = \rho_{0,1}(-1) \) then \( \rho_{0,1}(z) \geq \rho_{0,1}(-1) = 1/(|z| \log C) \) when \( |z| = 1 \) with the constant \( C = \exp 1/\rho_{0,1}(-1) \). Setting \( \rho_0(z) = 1/(|z| \log C) \) we obtain that \( \log \rho_0(z) - \log (1/(|z| \log C)) \to 0 \) as \( z \to 0 \), hence, due to the asymptotic obtained above, \( \log \rho_{0,1}(z) - \log \rho_0(z) \to 0 \) as \( z \to 0 \).

As the metrics \( \rho_{0,1}(z) \) and \( \rho_0(z) \) have the same Gauss curvature \( \Delta \log \rho \equiv \rho^2 \) then \( \Delta (\log \rho_{0,1} - \log \rho_0) = \rho_{0,1}^2 - \rho_0^2 \). The function \( \rho_{0,1}/\rho_0 \geq 1 \) on \( \partial \mathbb{D} \) by the definition of \( C \), and the same at 0 in sense of limit as it is proved above. If it takes minimum at some point \( z_0 \in \mathbb{D} \setminus 0 \) then there will be also the minimum of \( \log \rho_{0,1} - \log \rho_0 \). At the minimum point of a smooth function its laplasian is non-negative, hence \( \rho_{0,1}(z_0) \geq \rho_0^2(z_0) \) and thus \( \rho_{0,1} \geq \rho_0 \) everywhere in \( \mathbb{D} \setminus 0 \).

Comparing \( \rho_{0,1} \) with Poincaré metric for half-plane \( \{ \text{Re} z < 0 \} \subset \mathbb{C} \setminus \{0,1\} \) we obtain that \( C_{0,1} > 1 \).

Another proof of this lemma (with a bigger constant \( C \)) is contained in [GJW] where we have taken the idea of using Gauss curvatures going back to Ahlfors (see [A1], 1-5).

By Agard formula [Ag]
\[ \frac{1}{\rho_{0,1}(z)} = \frac{1}{2\pi} \int_{\mathbb{C}} \left| \frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)} \right| dS_\zeta \]
and thus the precise value of the constant \( C_{0,1} \) is equal to \( \frac{1}{\pi} \int_{\mathbb{C}} |\zeta(\zeta^2-1)|^{-1} dS_\zeta \).

A simpler expression
\[ C_{0,1} = \Gamma(\frac{1}{4})^4/4\pi^2 = 4.3768796... \]
is given in [H], a proof can be found in [N], Ch.6, sec.6.

Kobayashi metric is invariant with respect to holomorphic automorphisms, in particular, the metric \( \rho_{0,1}(z)|dz| \) is invariant with respect to the transformation \( z \to 1/z \). From this and infinitesimal Schwarz lemma (see (5) below) for imbeddings of \( \mathbb{D} \setminus 0 \) and \( \mathbb{C} \setminus \bar{\mathbb{D}} \) into \( \mathbb{C} \setminus \{0,1\} \) we obtain
Corollary 1. For all $z \in \mathbb{C} \setminus \{0, 1\}$,
\[
|z| |\log|z|| \leq \frac{1}{\rho_{0,1}(z)} \leq |z|(C_{0,1} + |\log|z||). \tag{4}
\]

The same estimates are valid also for Poincaré metrics $\rho_{0,a}$ in $\mathbb{C} \setminus \{0, a\}$ with arbitrary $a$, $|a| = 1$, because $\rho_{0,a}(z) = \rho_{0,1}(z/a)$ and $C_{0,1} = 1/\rho_{0,a}(-a)$.

Infinitesimal form of Kobayashi distance on arbitrary finite dimensional complex manifold $B$ is the Royden metric (or Kobayashi – Royden) which measures the lengths of tangent vectors $V \in T_zB$ to $B$ by the formula
\[
|V|_\kappa \equiv |V|_{\kappa(B)} = \inf \left\{ \frac{2}{R} : \exists \text{ holomorphic disk } h : \mathbb{D} \rightarrow B, \zeta = \xi + i\eta, \text{ such that } h(0) = z \text{ and } h_*(\frac{\partial}{\partial \xi}|0) = RV \right\},
\]
where $h_* : T\mathbb{D} \rightarrow TB$ is the tangent map to $h$. (If Poincaré metric in the disk is normalized in another way, as $|d\zeta|_1 = 1 - |\zeta|^2$ with curvature $\equiv -4$, then instead of $2/R$ in the definition one has to put $1/R$). In particular, in the disk $\mathbb{D}$ the Kobayashi – Royden metric coincides with the Poincaré metric,
\[
|V|_{\kappa(\mathbb{D})} = 2 |V|/(1 - |z|^2) \text{ for all } V \in T_0\mathbb{D}.
\]

It is, as a rule, not Riemannian but Finsler (semi)metric (there can be nonzero vectors of zero length) and $\kappa_B(z, z') = \inf \int_0^1 |\gamma'(t)|_\kappa dt$ where infimum is taken by all smooth paths $\gamma : [0, 1] \rightarrow B$ with ends $z, z'$ (see [R1,R2]).

The definition of Kobayashi – Royden metric given above suits surely for arbitrary almost complex Banach manifolds but I do not know if the integral representation given above is valid for almost complex and infinite dimensional manifolds. Nevertheless, also in general case we have infinitesimal Schwarz lemma:
\[
|f_*(V)|_{\kappa(Y)} \leq |V|_{\kappa(X)}, \quad V \in TX, \tag{5}
\]
for any holomorphic map $f : X \rightarrow Y$ of almost complex Banach manifolds. The proof follows evidently from the definition of Kobayashi – Royden metric.

For example, if $B$ is the ball $\|z\| < R$ in complex Banach space and $h : \mathbb{D} \rightarrow B$, $h(0) = 0$, is a holomorphic disk then $\|h'(|\zeta|)\| \leq R |\zeta|$ by Schwarz lemma, hence $\|h_*(0)\| \leq R (h_*(0) : TB \rightarrow X$ because $TB = B \times X$). If $h_*(\frac{\partial}{\partial \xi}|0) = R'V$ then it follows that $R'\|V\| \leq R, 2/R' \geq 2\|V\|/R$, hence $|V|_{\kappa(B)} \geq 2 \|V\|/R$. The extremal disk $h : \mathbb{D} \rightarrow B$ realising the infimum is the linear map $\zeta \mapsto R\zeta V/\|V\|; \text{ therefore } |V|_{\kappa} = 2 \|V\|/R$ for all $V \in T_0B$. 

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We apply these notions to holomorphic functions with values in \( \mathbb{C} \setminus \{ 0, 1 \} \).

**Proposition 2.** Let \( f \) be a holomorphic function on almost complex Banach manifold \( B \) with values in \( \mathbb{C} \setminus \{ 0, 1 \} \) and \( V \in T_z B \). Then

\[
|(Vf)(z)| \leq |V|_\kappa \cdot |f(z)| \left( C_{0,1} + |\log |f(z)|| \right)
\]

where \( C_{0,1} = 1/\rho_{0,1}(-1) > 1 \).

Let us fix arbitrary \( \varepsilon > 0 \). Let \( h : \mathbb{D} \to B \) be a holomorphic disk such that \( h(0) = z, h_*(\partial/\partial \xi|_0) = 2V/(|V|_\kappa + \delta) \) with \( 0 \leq \delta \leq \varepsilon \). Then \( g := f \circ h : \mathbb{D} \to \mathbb{C} \setminus \{ 0, 1 \} \). By infinitesimal Schwarz lemma, for any such map \( g \) the following inequality is valid:

\[
\rho_{0,1}(g(\zeta)) |dg(\zeta)| \leq 2 |d\zeta|/(1 - |\zeta|^2),
\]

in particular, \( |g'(0)| \leq 2/\rho_{0,1}(g(0)) \). As \( g(0) = f(z) \) then by Lemma 1

\[
|g'(0)| \leq 2 |g(0)| \left( C_{0,1} + |\log |g(0)|| \right) = 2 |f(z)| \left( C_{0,1} + |\log |f(z)|| \right).
\]

By the choice of \( h \),

\[
g'(0) = \frac{\partial}{\partial \zeta} \bigg|_0 (f \circ h)(\zeta) = \frac{\partial}{\partial \zeta} \bigg|_0 (f \circ h)(\zeta) = \frac{2(Vf)(z)}{|V|_\kappa + \delta},
\]

hence \( |(Vf)(z)| \leq (|V|_\kappa + \varepsilon) \cdot |f(z)| \left( C_{0,1} + |\log |f(z)|| \right) \) and the statement is proved due to arbitrary \( \varepsilon \).

If \( B = \mathbb{D} \) and \( V = \frac{\partial}{\partial \zeta}|_0 \) then \( |V|_\kappa = \frac{2}{1 - |z|^2} \), hence

\[
|f'(z)| \leq \frac{2}{1 - |z|^2} \cdot |f(z)| \left( C_{0,1} + |\log |f(z)|| \right).
\]

By \( z = 0 \) it is classical Landau theorem:

**Corollary 2.** Let \( f(z) = \sum_0^\infty a_n z^n \) be a function holomorphic in \( \mathbb{D} \) which does not take the values \( 0 \) and \( 1 \). Then

\[
|a_1| \leq 2 |a_0| \left( C_{0,1} + |\log |a_0|| \right).
\]

The exact constant \( C_{0,1} = 1/\rho_{0,1}(-1) \) in this theorem is established by J.Hempel [H]; the equality is attained by the universal covering \( S : \mathbb{D} \to \mathbb{C} \setminus \{ 0, 1 \}, S(0) = -1 \).
Due to the estimate of Kobayashi – Royden metric in a ball we obtain similar estimate.

**Corollary 3.** Let $f$ be a holomorphic function in the unit ball of a Banach space $(X, \| \cdot \|)$ which does not take the values $0, 1$. Then

$$\|f'(0)\| \leq 2 |f(0)| \left( C_{0,1} + |\log |f(0)|| \right) .$$

Thus, Proposition 2 is a generalization of Landau theorem onto arbitrary almost complex manifolds and, as we see, the proof is practically the same as for $B = \mathbb{D}$ and is a simple corollary of infinitesimal Schwarz lemma (5) and the estimate (4) of Poincaré metric $\rho_{0,1}$.

The following are analogies of Harnack inequalities (3).

**Proposition 3.** Let $f$ be a holomorphic function on almost complex Banach manifold $B$ with values in $\mathbb{C} \setminus \{0, 1\}$. Then

$$e^{-\kappa_B(z,z_0)} \leq \frac{C_{0,1} + |\log |f(z)||}{C_{0,1} + |\log |f(z_0)||} \leq e^{\kappa_B(z,z_0)}$$

(6)

for any $z, z_0 \in B$. Furthermore, if there exists continuous $\log f$ on $B$ then

$$\frac{C_{0,1} + |\log |f(z)||}{C_{0,1} + |\log |f(z_0)||} \leq e^{\kappa_B(z,z_0)}$$

(7)

Let us consider first the crucial case $B = \mathbb{D} \subset \mathbb{C}_\zeta$ with $\zeta = re^{it}$; here continuous logarithm exists.

Fix an arbitrary point $a = |a|e^{i\alpha} \in \mathbb{D}$ and denote $u(r) := |\log |f(re^{i\alpha})||$, $C := C_{0,1}$. If $u$ is not constant (what we assume further) then there can be on the interval $(0, a)$ only finite set of points where $u = 0$, hence the function $u$ is piecewise smooth. As

$$\left| \frac{\partial}{\partial r} \log f \right| \geq \left| \frac{\partial}{\partial r} \log |f| \right| \geq \frac{\partial}{\partial r} |\log |f||$$

almost everywhere on $(0, a)$ then by Prop. 2

$$\frac{u'}{C + u} \leq \frac{2}{1 - r^2} = \frac{1}{1 + r} + \frac{1}{1 - r} .$$
Integrating this inequality by \((0, |a|)\) we obtain that

\[
\log \frac{C + u(|a|)}{C + u(0)} \leq \log \frac{1 + |a|}{1 - |a|}.
\]

The substitution \(\zeta = \frac{a - \eta}{1 - \bar{a}\eta}\) changes 0, \(a\) in places, hence

\[
\frac{1 - |a|}{1 + |a|} \leq \frac{C + |\log|f(a)||}{C + |\log |f(0)||} \leq \frac{1 + |a|}{1 - |a|}.
\]

For obtaining inequalities (7) let us notice that there exists continuous \(\log (\log f)\) on \([0, a]\). As \(\frac{\partial}{\partial r} \log (\log f) \geq \frac{\partial}{\partial r} \log |\log f|\) then \(\frac{\partial}{\partial r} \log f \geq \frac{\partial}{\partial r} |\log f|\).

Thus setting \(v(r) = |\log f(re^{i\alpha})|\) we obtain, again by Prop. 2, that \(v'(C + v) \leq 2/(1 - r^2)\). Integrating this as above we obtain the inequalities (7) in \(\mathbb{D}\).

In the case of arbitrary base \(B\) we repeat the argument from the proof of Prop. 1. If \(\kappa_B(z, z_0) = \infty\) then there is nothing to prove, therefore we assume further that \(\kappa_B(z, z_0) < \infty\).

Fix \(\varepsilon > 0\), choose a chain of holomorphic disks \(h_j : \mathbb{D} \to B\) and points \(\zeta_j \in \mathbb{D}, z_0, z_1, ..., z_N = z\) such that \(h_j(0) = z_{j-1}, h_j(\zeta_j) = z_j, j = 1, ..., N\), and \(\kappa_B(z, z_0) \geq \sum_1^N \log \frac{1 + |\zeta_j|}{1 - |\zeta_j|} - \varepsilon\). As it is proved above,

\[
\frac{1 - |\zeta_j|}{1 + |\zeta_j|} \leq \frac{C + |\log |f(z_j)||}{C + |\log |f(z_{j-1})||} \leq \frac{1 + |\zeta_j|}{1 - |\zeta_j|}
\]

and corresponding inequalities for \(|\log f|\) if there exists on \(B\) a continuous logarithm of \(f\).

It follows that

\[
e^{-\kappa_B(z, z_0) - \varepsilon} \leq \frac{C + |\log |f(z)||}{C + |\log |f(z_0)||} = \prod_1^N \frac{C + |\log |f(z_j)||}{C + |\log |f(z_{j-1})||} \leq e^{\kappa_B(z, z_0) + \varepsilon}
\]

with arbitrary \(\varepsilon > 0\), and corresponding inequalities for \(|\log f|\) if a continuous logarithm of \(f\) exists. ♦

Using for metrics \(\rho_{0, a}\) with \(|a| = 1\) the same estimates (4) as for \(\rho_{0, 1}\) (see above) we obtain similarly the inequalities

\[
e^{-\kappa_B(z, z_0)} \leq \frac{C_{0, 1} + |\log e^{-i \arg f(z_0)}f(z)||}{C_{0, 1} + |\log |f(z_0)||} \leq e^{\kappa_B(z, z_0)}
\]
which give more precise estimate of the argument of the function $f$.

The constant $C_{0,1}$ in the estimates (6) and the last inequalities is exact. Indeed, let $S : \mathbb{D} \to \mathbb{C} \setminus \{0,a\}$, $|a| = 1$, be the universal covering normalized by conditions $S(0) = -a$, $S'(0) > 0$. Then $\rho_{0,a}(S(z))|S'(z)| = 2/(1 - |z|^2)$, hence $\rho_{0,a}(-a)S'(0) = 2$. As $C_{0,1} = 1/\rho_{0,a}(-a) = S'(0)/2$ then the right inequality can be rewritten in the form

$$1 + |\log (1 - S'(0)z (1 + o(1)))|/C_{0,1} = 1 + 2|z| (1 + o(1)) \leq \frac{1 + |z|}{1 - |z|},$$

and it is arbitrary close to equality for small $|z|$.

**Corollary 4.** Let $h$ be a holomorphic function on $B$ which does not take values from $2\pi i \mathbb{Z}$. Then

$$|h(z)| \leq (C_{0,1} + |Re h(z_0)|) \cdot e^{\kappa_B(z,z_0)} + |Im h(z_0)| - C_{0,1}.$$

\(\blacktriangleleft\) Let $f = e^h$. Then $h = \log f$, $\log |f| = Re h$ and $arg f = Im h$. Put this in the last right inequality. \(\blacktriangledown\)

Similar estimates are valid surely for functions which do not take values in an arithmetic progression $a + b\mathbb{Z}$, $b \neq 0$. Therefore the growth of such a function $f(z)$, say in the unit ball, does not exceed $C/(1 - \|z\|)$ with corresponding constant $C = C(f,a,b)$.

Proposition 3 permits to estimate the growth of a function $B \to \mathbb{C} \setminus \{0,1\}$ by its value in a fixed point and the distance to this point in Kobayashi metric. In the case $B = \mathbb{D}$ the classical Schottky’ theorem states that, for such functions, $|f(z)|$ is estimated by a quantity depending only on $|f(0)|$ and $|z|$. In general, let us denote by $M(R,R')$ the supremum of numbers $|f(z)|$ by all holomorphic functions $f : B \to \mathbb{C} \setminus \{0,1\}$ such that $|f(z_0)| \leq R'$ and by all $z \in B_R$.

**Corollary 5.** If $|f(z_0)| \leq R'$ and $z \in B_R$ then

$$|f(z)| \leq M(R,R') \leq e^{-C_{0,1} \log e^{C_{0,1} (\max(1,R'))} e^R}.$$

\(\blacktriangleleft\) If $|f(z_0)| \geq 1$ then by Prop. 2

$$|\log |f(z)|| \leq e^{R} (C_{0,1} + \log R') - C_{0,1},$$

hence

$$|f(z)| \leq e^{\log ((C_{0,1} + \log R') e^R - C_{0,1})} = e^{-C_{0,1} (e^{C_{0,1} R'} - e^R)} e^R.$$
It remains to notice that \( M(R, R'') \leq M(R, R') \) if \( R'' < R' \) and thus \( M(R, 1) \leq e^{C_0,1(R-1)} \). ▷

This estimate is far from being exact even for the case \( f(z_0) = -1 \). An exact but implicit estimate for \( B = \mathbb{D} \) is pointed by Hempel [H]:

\[
\left| \int_{|f(z)|}^{\left| f(z_0) \right|} \rho_{0,1}(-r) \, dr \right| \leq \frac{1 + |z|}{1 - |z|}.
\]

It follows evidently from the inequality \( \rho_{0,1}(z) \geq \rho_{0,1}(-|z|) \) (see [LV],[Ag]) and Schwarz lemma (2) and thus it is generalized to holomorphic functions \( f : B \to \mathbb{C} \setminus \{0, 1\} \) on arbitrary almost complex Banach manifold:

\[
\left| \int_{|f(z)|}^{\left| f(z_0) \right|} \rho_{0,1}(-r) \, dr \right| \leq \kappa_B(z, z_0).
\]

The equality here with \( B = \mathbb{D} \) is attained by any universal covering \( F : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\} \) such that \( F(0) < 0 \) and \( F(z) < 0 \).

To estimate \( M(R, R') \) by this formula seems rather difficult. Several simple explicit estimates for \( B = \mathbb{D} \), such as that one in Corollary 5, are contained in the paper of Jenkins [J] but I do not see how to extend them to the case of arbitrary \( B \).

At the end of the section, several words on functions with values in \( \mathbb{D} \setminus 0 \). Instead of estimates (4) we have here precise formula for Poincaré metric, \( \rho_{\mathbb{D}\setminus 0}(z) = 1/|z| \log 1/|z| \). Repeating the proof of Prop. 2 we obtain inequalities

\[
e^{-\kappa_B(z, z_0)} \leq \frac{\log 1/|f(z_0)|}{\log 1/|f(z_0)|} \leq e^{\kappa_B(z, z_0)}
\]

for holomorphic functions \( f : B \to \mathbb{D} \setminus 0 \) on almost complex Banach manifold \( B \) as well as other similar inequalities when there exists a continuous logarithm of \( f \).

3. **On the smoothness of holomorphic motions.** *Holomorphic motion* of a set \( E \subset \hat{\mathbb{C}} \) over almost complex Banach manifold \( B \) with a base point \( z_0 \) is an arbitrary map \( \phi : B \times E \to \hat{\mathbb{C}} \) with the following properties:

1. \( \phi(z_0, w) \equiv w \),
2. \( \phi(z, \cdot) : E \to \hat{\mathbb{C}} \) is injective for every fixed \( z \in B \),
(3) $\phi(\cdot, w)$ is holomorphic in $B$ for every fixed $w \in E$.
A motion $\phi$ is called normalized if $w = 0, 1, \infty \in E$ and $\phi(\lambda, w) \equiv w$ for these $w$.

The most important and basic for the proofs of properties of holomorphic motions is the case $B = \mathbb{D}$, $z_0 = 0$. Just in this form holomorphic motions have appeared for the first time in the paper of R.Mañé, P.Sad and D.Sullivan [MSS] where it was proved so called $\lambda$-lemma (the term is already excepted) which states that holomorphic motions are continuous by joint variables and quasiconformal for every fixed $z \in B$.

Any quasiconformal mapping, say, of the plane $\mathbb{C}$ onto itself can be included into some holomorphic motion of the set $E = \mathbb{C}$ as a map $\phi(z, \cdot) : \mathbb{C} \to \mathbb{C}$ with suitable $z \in \mathbb{D}$. As quasiconformal mappings do not satisfy in general Lipschitz condition (even locally) one can not wait from holomorphic motions a smoothness in $w$ better than Hölder one. At the same time it was noticed in the paper [GJW] that the estimates as in Landau theorem imply the Hölder conditions for normalized holomorphic motions. We expose this below in more general context, with some specifications.

If $f_1, f_2$ are different functions from correspondent normalized holomorphic motion then $f_1 - f_2$ in general can take value 1 and by this reason the estimate of “derivatives” of the function $f_1 - f_2$ comparing with $f_j$ itself is a little more complicated.

**Lemma 2.** Let $f_1, f_2$ be holomorphic functions on almost complex Banach manifold with values in $\mathbb{C} \setminus \{0, 1\}$ which are different at every point of $B$. Then for any vector field $V$ on $B$ the following inequality is valid

$$\frac{|V(f_1 - f_2)|}{|f_1 - f_2|} \leq |V|_{\kappa} \cdot (2 C_{0,1} + 2 \min (|\log |f_1||, |\log |1 - f_1||) + |\log |f_1 - f_2||).$$

The function $1 - f_2/f_1$ maps $B$ to $\mathbb{C} \setminus \{0, 1\}$, hence Prop. 2 is applicable to it. As

$$V \left( \frac{f_1 - f_2}{f_1} \right) \cdot \frac{f_1}{f_1 - f_2} = \frac{f_2 V f_1 - f_1 V f_2}{f_1(f_1 - f_2)} = \frac{V f_1 - V f_2}{f_1 - f_2} - \frac{V f_1}{f_1},$$

and $|V f_1|/|f_1| \leq |V|_{\kappa}(C_{0,1} + |\log |f_1||)$ by Prop. 2 then

$$\left| \frac{V f_1 - V f_2}{f_1 - f_2} \right| \leq |V|_{\kappa} \left( 2 C_{0,1} + |\log \left| \frac{f_1 - f_2}{f_1} \right| \right) + |\log |f_1||.$$

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inequality (8) is invariant with respect to the change of \( \phi \)

and it remains to notice that this inequality is valid also for functions \( 1 - f_1, 1 - f_2 \) with the difference \( f_2 - f_1 \).

Lemma 2 shows that, for any normalized holomorphic motion \( \phi : B \times E \to \mathbb{C} \) and any Kobayashi bounded vector field \( V \) on \( B \), the function \( V\phi \) has almost the same modulus of continuity in \( w \) as the motion \( \phi \) itself. This permits to establish the following estimate which can not be improved in general.

**Proposition 4.** Let \( \phi : B \times E \to \mathbb{C} \) be a normalized holomorphic motion of a set \( E \subset \mathbb{C} \) over almost complex Banach manifold \( (B, z_0) \). Then for any \( w_1, w_2 \in E \), any \( R > 0 \) and \( z \in B_R \) the following inequality is valid (Hölder inequality with respect to spherical metric on \( \mathbb{C} \)):

\[
\frac{|\phi(z, w_1) - \phi(z, w_2)|}{\sqrt{1 + |\phi(z, w_1)|^2} \sqrt{1 + |\phi(z, w_2)|^2}} \leq C_R \left( \frac{|w_1 - w_2|}{\sqrt{1 + |w_1|^2} \sqrt{1 + |w_2|^2}} \right)^e^{-R} \tag{8}
\]

with constant \( C_R \) depending only on \( R \) (and not depending on \( \phi \)). Moreover, for any \( R' > 0 \) there exists a constant \( C(R, R') > 1 \) such that

\[
(|w_1 - w_2|/C(R, R'))^e^{R} \leq |\phi(z, w_1) - \phi(z, w_2)| \leq C(R, R') |w_1 - w_2|e^{-R} \tag{9}
\]

by the condition that \( |w_1|, |w_2| < R' \).

Let us note at once that for the unit ball \( B \) in a complex Banach space and \( z_0 = 0 \) the exponent \( e^{-R} \) over the ball \( ||z|| < r \) is equal to \( \frac{1}{1+r} \) and this estimate can not be improved in general.

Fix \( R > 0 \), denote \( e^{-R} = \alpha \), \( \phi(\cdot, w_j) = f_j, f = f_1 - f_2 \) and prove at first the inequality (8).

If \( |w_1| \leq 1/2 \) and \( |w_2| > 2 \) or vice versa then \( |w_1 - w_2|/\sqrt{1 + |w_1|^2} \sqrt{1 + |w_2|^2} > 1/2 \) and inequality (8) is valid with any constant \( C_R > 2 \) as its left side \( \leq 1 \). Therefore we can assume that both \( |w_j| \leq 2 \) or both \( |w_j| \geq 1/2 \). As the inequality (8) is invariant with respect to the change of \( \phi \) onto \( 1/\phi \) then one can assume further that both \( |w_j| \leq 2 \).

By Schottky’ theorem (Cor. 5 sec.2), \( |f_j(z)| \leq M(R, 2), j = 1, 2 \), for \( z \in B_R \). By lemma 2 \( |V f| \leq |V|_k \cdot |f|(C + |\log \ |f|) \) on \( B_R \) for any vector field \( V \) on \( B \) with the constant \( C = 2C_{0,1} + 2\log M \), \( M := M(R, 3) \), if \( w_1, w_2 \in E \setminus \{0, 1\} \). The same is true if one of \( w_j \) is equal to 0 or 1 (see
Prop. 2). Repeating literally the proof of Prop. 2 (with this new constant $C$) we obtain Harnack type inequalities

$$\alpha \leq \frac{C + |\log |f(z)||}{C + |\log |f(z_0)||} \leq 1/\alpha, \quad z \in B_R.$$ 

If $|w_1 - w_2| > 1$ then there is nothing to prove, the inequality (8) is valid with any $C_R \geq 5$ and therefore we assume further that $|w_1 - w_2| \leq 1$ (and $|w_j| \leq 2$). If $|w_1 - w_2| \geq e^C/(2 e^C M)^{1/\alpha}$ then inequality (8) is fulfilled with $C_R \geq 5^\alpha e^{C(1-\alpha)} 2 M$. If $|w_1 - w_2| < e^C/(2 e^C M)^{1/\alpha}$ then it follows from the inequality obtained above that $|f(z)| \leq 1$ (otherwise $\alpha \cdot (C + \log 1/|w_1 - w_2|) \leq \log e^C |f(z)| \leq \log (2 e^C M)$ in contradiction with the condition on $|w_1 - w_2|$). Therefore $\alpha \log (e^C/|w_1 - w_2|) \leq \log (e^C/|f(z)|)$, hence

$$\frac{|(f_1 - f_2)(z)|}{\sqrt{1 + |f_1(z)|^2} \sqrt{1 + |f_2(z)|^2}} < |f(z)| \leq e^{C(1-\alpha)} |w_1 - w_2|^\alpha$$

$$\leq 5^\alpha e^{C(1-\alpha)} \left( \frac{|w_1 - w_2|}{\sqrt{1 + |w_1|^2} \sqrt{1 + |w_2|^2}} \right)^\alpha$$

and the inequality (8) is proved.

At last, if $|w_j| \leq R'$ then, due to the Cor. 4 sec. 2, $|f_j(z)| \leq M(R, R')$ for $z \in B_R$, hence

$$|(f_1 - f_2)(z)| \leq C_R (1 + M(R, R')^2) |w_1 - w_2|^\alpha.$$ 

As the choice of $z_0$ on $B$ is arbitrary and $z_0 \in B_R(z)$ if $z \in B_R$ then, as it is proved,

$$|w_1 - w_2| \leq C(R, R') |(f_1 - f_2)(z)|^\alpha,$$

and this is the left inequality in (9). ▶

Lemma 2 together with Prop. 4 show that for any normalized holomorphic motion $\phi : B \times E \to \mathbb{C}$ and any vector field $V$ bounded on $B_R$ the function $V\phi$ satisfies on $B_R$ Hölder condition in $w$ with any exponent $\beta < \alpha = e^{-R}$.

4. Other generalizations. Harnack type inequalities are valid not only for holomorphic functions but also for solutions of other elliptic equations and inequalities. Let us start with evident generalizations for solutions of Beltrami equation.
Let a function $f$ in $\mathbb{D}$ be a (generalized) solution of the equation $f_\bar{z} = \mu f_z$ where $\mu \in L^\infty(\mathbb{D})$, $\|\mu\|_\infty < 1$. Then there exists a quasiconformal homeomorphism $\psi : \mathbb{D}_z \to \mathbb{D}_\zeta$, $\psi(0) = 0$, such that $\psi_\bar{z} = \mu \psi_z$ in sense of distributions (see e.g. [A2]). The function $h(\zeta) = f(\psi^{-1}(\zeta))$ is holomorphic in $\mathbb{D}$, $h(0) = f(0)$. If $0 < |f| < M$ then by the corollary in sec.1 $|f(z)| \leq |f(0)|^{\beta(z)} M^{1-\beta(z)}$ where $\beta(z) = \frac{1-|\psi(z)|}{1+|\psi(z)|}$. Whereas if $f$ does not assume in $\mathbb{D}$ the values 0 and 1 then by Prop. 3 the values 0 and 1 then by Prop. 3

\[
1 - |\psi(z)| \leq C_{0,1} + |\log f(z)| \leq \frac{1 + |\psi(z)|}{1 - |\psi(z)|},
\]
and corresponding inequalities for $|\log f(z)|$.

But these are evident generalizations. More substantial is the following statement related with holomorphic motions with a nonstandard complex structure in $B \times \mathbb{C}$.

**Proposition 5.** Let $f$ be a continuous function without zeros in the disk $\mathbb{D}$ such that $|f| < 1/e$ and the partial derivative of $f_\bar{z}$ by $\bar{z}$ in sense of distributions is locally integrable and satisfies almost everywhere in $\mathbb{D}$ the inequality

\[
|f_\bar{z}| \leq A |f| \log 1/|f|
\]
where $A$ is a function from $L^p(\mathbb{D})$ with some $p > 2$. Then there exists a constant $c$ depending only on $p$ and $\|A\|_{L^p}$ such that, for $z \in \mathbb{D}$,

\[
e^{-c/(1-|z|)^{2-2/p}} \leq \frac{\log 1/|f(z)|}{\log 1/|f(0)|} \leq e^{c/(1-|z|)}.
\]

\[\clubsuit\] One can assume that $f(0) > 0$.

Denote by $\log 1/f$ the continuous logarithm of the function $1/f$ defined by the condition $\log 1/f(0) > 0$ and by $g$ denote the continuous logarithm of the function $\log 1/f$ also defined by the condition $g(0) > 0$. As $Re \log 1/f > 1$ then $|Im g| < \pi/2$. As $\log 1/|f| \leq |\log 1/f|$ then the function $g$ satisfies the inequality $|g_\zeta| \leq A$, hence $g_\zeta \in L^p(\mathbb{D})$.

The function

\[a(z) = -\frac{1}{\pi} \int_{\mathbb{D}} \left( \frac{1}{\zeta - z} - \frac{1}{\zeta} \right) g_{\zeta}(\zeta) dS_\zeta\]
is continuous on whole the plane (because $g_\zeta \in L^p$), is bounded, $|a| \leq c_1(p)\|A\|_{L^p}$, $a(0) = 0$ and the generalized derivative of $a$ by $\bar{z}$ is equal to $g_\bar{z}$ (see [V] Ch.1).
By Weil lemma the function \( g - a - g(0) =: h \) is holomorphic in \( \mathbb{D} \) and equal to 0 at 0. As \( |Im g| < \pi/2 \) and \( |a| < c_0(p)\|A\|_p \) then \( |Im h| < c_0(p)\|A\|_p + \pi/2 \) and therefore the function \( h \) has angular limit values almost everywhere on \( \partial\mathbb{D} \). By Schwarz formula

\[
h(z) = \frac{i}{2\pi} \int_{\partial\mathbb{D}} Imh(\zeta) \cdot \frac{\zeta + z}{\zeta - z} |d\zeta|
\]

what follows that \( |Re h| \leq (\pi + 2c_0(p)\|A\|_p)/(1 - |z|) \), and the right inequality (10) follows from the equality \( |log 1/|f\| = e^{Re(a+h)} log 1/|f(0)| \).

The left inequality (10) is obtained from the right one using automorphisms of \( \mathbb{D} \). Let us fix an arbitrary point \( z' \in \mathbb{D} \) and set \( z = \psi(\zeta) := \frac{\zeta + z'}{1 + \bar{z}'\zeta} \). As

\[
(f \circ \psi)\zeta = f_{z} \circ \psi \cdot \frac{1 - |z'|^2}{(1 + \bar{z}'\zeta)^2}
\]

then

\[
|(f \circ \psi)\zeta| \leq A' \|f \circ \psi\| log 1/|f \circ \psi| \quad \text{with} \quad A'(\zeta) = \frac{1 - |z'|^2}{|1 + \bar{z}'\zeta|^2} A(\psi(\zeta)) \in L^p(\mathbb{D}).
\]

As \( \zeta = \frac{z - z'}{1 - z'\bar{z}} \) then

\[
\|A'\|^p_{L^p} = \int_{\mathbb{D}} \frac{|1 - \bar{z}'z|^{2p-4}}{(1 - |z'|^2)^{p-2}} |A(z)|^p dS_z \leq \frac{2^{p-2}\|A\|^p_{L^p}}{(1 - |z'|)^{p-2}}.
\]

Let \( \alpha = arg f(z') \in (-\pi, \pi] \) and \( log e^{i\alpha}/f, \ log log e^{i\alpha}/f \) are continuous branches of logarithms which are positive at \( \zeta = 0 \) (corresponding to \( z = z' \)). As it is proved above

\[
\frac{log 1/|f \circ \psi(\zeta)|}{log 1/|f \circ \psi(0)|} \leq \exp [(\pi + 3c_0(p)\|A\|_{L^p})/(1 - |\zeta|)]
\]

As \( \psi(0) = z' \) and \( \psi(-z') = 0 \) it follows at \( \zeta = -z' \) that

\[
log \frac{1}{|f(0)|} \leq e^{c/(1 - |z'|)^{2-2/p}} log \frac{1}{|f(z')|}
\]

and the left inequality (10) is proved. \( \blacksquare \)

More symmetric inequalities would be obtained by \( p = 2 \) but \( c_0(p) \to \infty \) as \( p \downarrow 2 \).

Similar estimates are valid also for functions on the whole plane but they depend not on geometry (Kobayashi distance on the plane vanishes identically) but on estimates of \( f_{z} \).
Proposition 6. Let $f$ be a continuous function without zeros on the plane $\mathbb{C}_z$ such that $|f| < 1$ and the partial derivative $f_\bar{z}$ by $\bar{z}$ in sense of distributions is locally integrable and satisfies almost everywhere on $\mathbb{C}$ the inequality

$$|f_\bar{z}| \leq A |f| \cdot |\log M/f|$$

where $A$ is a function from $(L^p \cap L^{p'}) (\mathbb{C})$ with some $p > 2$, $p' < 2$, $M \geq 1$ is a constant and $\log M/f$ is a continuous branch of logarithm. Then $f$ extends continuously onto $\hat{\mathbb{C}}$ and there exists a constant $C > 0$ depending only on $p, p'$ and $\|A\|_{L^p} + \|A\|_{L^{p'}}$ such that

$$C^{-1} \leq \frac{|\log M/f(z)|}{|\log M/f(0)|} \leq C. \quad (11)$$

Set $g = \log M/f$. From the estimate of $f_\bar{z}$ we obtain that $|g_\bar{z}| \leq A |g|$ and $Re g > 0$.

Set $a = g_\bar{z}/g$. Then $a \in (L^p \cap L^{p'}) (\mathbb{C})$ and $g_\bar{z} = a g$. The function

$$\hat{a}(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \left( \frac{1}{\zeta - z} - \frac{1}{\zeta} \right) a(\zeta) dS_\zeta$$

is continuous (as $a \in L^p$) and bounded (as $a \in L^{p'}$) on whole the plane $\mathbb{C}$ and the generalized derivative of $\hat{a}$ by $\bar{z}$ equals $a$ (see [V] Ch.1), besides $a(0) = 0$ and $|\hat{a}| \leq C_1 (\|A\|_{L^p} + \|A\|_{L^{p'}})$ with a constant $C_1$.

For generalized derivative by $\bar{z}$ of the function $ge^{-\hat{a}}$ the Leibnitz rule is applicable: if $g^\varepsilon, a^\varepsilon$ is a standard mollification with a smooth “hat” then $g^\varepsilon e^{-a^\varepsilon} \rightarrow ge^{-\hat{a}}$ in sense of distributions, hence

$$(ge^{-\hat{a}})_\bar{z} = \lim_{\varepsilon \rightarrow 0} (g^\varepsilon e^{-a^\varepsilon} - g^\varepsilon a^\varepsilon e^{-a^\varepsilon}) = (g_\bar{z} - a g) e^{-\hat{a}}.$$ 

As $g_\bar{z} = a g$ it follows from this and Weil lemma that $ge^{-\hat{a}}$ is a holomorphic function on whole the plane and it does not have zeros. And then the function $\log g - \hat{a} =: h$ is also holomorphic on whole the plane, has uniformly bounded imaginary part, hence is constant. As $\hat{a}(0) = 0$ then $h \equiv \log g(0)$, hence $g = g(0) e^{\hat{a}}$.

It follows that $\log M/f = e^{\hat{a}} \log M/f(0)$ and the right inequality (11) is fulfilled with the constant $C = sup e^{Re \hat{a}}$. As the conditions of the proposition do not change by shifts $z \mapsto z + const$ then the left inequality follows also. 

Corollary 1. If $\lim_{|z| \rightarrow \infty} |f| = 0$ then $f \equiv 0$.

Apply the right inequality (11) to $f(z + R)$ with arbitrary big $R$. }
Corollary 2. \( \sup |f| \leq (\inf |f|)^{1/C} \).

\( \blacklozenge \) We can assume that \( \inf |f| = f(0) > 0 \) (shift by \( z \) and multiply \( f \) on a constant equal to one by modulus). \( \blacklozenge \)

Corollary 3. If \( |f| \leq A|f| \) with a function \( A \) as in proposition 6 then

\[ \sup |f| \leq C(A) \cdot \inf |f| \]

with similar constant \( C(A) > 1 \).

\( \blacklozenge \) It follows from given Lipschitz condition that \( f/e \) satisfies the log-Lipschitz condition from the proposition and therefore \( \sup |f|^{1/C} \leq e^{1-1/C} \cdot \inf |f|^{1/C} \). \( \blacklozenge \)

Propositions 5, 6 and corollaries are valid surely also for functions satisfying the inequalities from these propositions in which the Cauchy – Riemann operator \( \partial/\partial \bar{z} \) is substituted by the Beltrami operator \( \partial/\partial \bar{z} - \mu \partial/\partial z \) (the proofs are reduced to Prop. 5, 6 by evident quasiconformal transformations of \( \mathbb{D} \) or \( \mathbb{C} \)).

\[ *** *** \]

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