GAUSS-TYPE FORMULAS FOR LINK MAP INVARIANTS

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We find that Koschorke’s $\beta$-invariant and the triple $\mu$-invariant of link maps in the critical dimension can be computed as degrees of certain maps of configuration spaces — just like the linking number. Both formulas admit geometric interpretations in terms of Vassiliev’s ornaments via new operations akin to the Jin suspension, and both were unexpected for the author, because the only known direct ways to extract $\mu$ and $\beta$ from invariants of maps between configuration spaces involved some homotopy theory (Whitehead products and the stable Hopf invariant, respectively).

INTRODUCTION

The linking number of a link $S^1 \sqcup S^1 \subset \mathbb{R}^3$ is the degree of the Gauss map $S^1 \times S^1 \to \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta \simeq S^2$. A considerable progress has been made in finding similar expressions for other invariants of knots and links and higher-dimensional embeddings, and the question now arises, do there ever exist two embeddings of the same manifold into $\mathbb{R}^m$ that cannot be distinguished by a so expressible invariant? We shall now see that this question, to be made more precise in a second, is a natural generalization of the Vassiliev Conjecture on completeness of finite type invariants of classical knots.

Configuration space integrals. It is known that all rational Vassiliev invariants of knots can be computed from “configuration space integrals” (see [Vo2] and references there). These are roughly the degrees of the maps between the Fulton–MacPherson compactification $(S^1)^{(r)}$ (see definition at the end of the introduction) of the configuration space $(S^1)^{(r)}$ of $r$-tuples of distinct points of $S^1$ and manifolds homotopy equivalent to $(\mathbb{R}^3)^{(r)}$ with some diagonals filled in, for $r = 2, 3, \ldots$. More precisely, since each $(S^1)^{(r)}$ is a manifold with nonempty boundary, the integrals are only well-defined when summed up with “correction terms”, which involve “joint” configuration spaces, where some points are constrained to be on the knot yet are not allowed to collide with additional points running over the 3-space. Necessity of correction terms is readily clear from the fact that the Fulton–MacPherson compactification is a PL invariant (see its definition below) — but every PL knot in $\mathbb{R}^3$ is PL isotopic (i.e. non-locally-flat PL homotopic through embeddings) to the unknot.

Another approach to extracting Vassiliev invariants from configuration spaces is found in [BCSS]. It follows from results of D. Sinha [Si] and I. Volić [Vo1] that all rational type $n$ Vassiliev invariants of a string knot $(I, \partial) \hookrightarrow (I^3, \partial)$ are contained in the aligned $S_{2n}$-equivariant stratum preserving homotopy class of the induced map between the Fulton–MacPherson compactifications $I^{[2n]}$ and $(I^3)^{[2n]}$. (Technically, here $(I^3)^{[2n]}$ must be endowed with tangent vectors, but this can be remedied

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by increasing $2n$ to $4n$.) As indicated above, this result couldn’t have been true without the word “aligned”, so let us recall its meaning from [Si]. A point $x$ of $(I^3)^r$ represents an $r$-tuple $(x_1, \ldots, x_r)$ of (not necessarily distinct) points of $I^3$ along with some additional data for each substring $s_j$ of coincident points $x_{i_1} = \cdots = x_{i_{k_j}}$. Thinking of $x$ as the limit of a curve of points $x(t)$, $t \to \infty$, in the configuration space, these data can be derived from the relative velocities of the moving points $x_{i_1}(t), \ldots, x_{i_{k_j}}(t)$ (the details can be found in [FM]). If $v_{i_k}$ is the tangent vector to the curve $x_{i_k}(t)$ at $t = \infty$ (which must exist for $x$ to be defined), $x$ is called aligned if $v_{i_1}, \ldots, v_{i_{k_j}}$ are all collinear for each $j$. In particular, all points in $(I^3)^r$ are aligned since each $k_j = 1$. A map $I^r \to (I^3)^r$ is termed aligned if all points in its image are aligned. Every map $I^r \to (I^3)^r$ yielded by a genuine knot $I \hookrightarrow I^3$ is aligned simply because $I$ is one-dimensional.

On the other hand, there is a direct geometric argument due to J. Conant showing that knots that are $C_n$-equivalent ($\Leftrightarrow$ indistinguishable by invariants of type $< n$) yield maps $I^n \to (I^3)^n$ that are connected by an aligned $S_n$-equivariant stratum preserving homotopy. It also works to show that this homotopy $I^r \to (I^3)^r$ can be chosen to agree with analogous homotopies $I^i \to (I^3)^i$, (a precise definition of such agreement is given at the end of the introduction).

**Polynomial compactification.** Motivated by entirely unrelated considerations (resolution of singularities of smooth maps), the present author has defined a new compactification $M^{(r)}$ of the configuration space $M^r$ of a compact smooth manifold $M$, which can be viewed as a polynomial analogue of the “linear” Fulton–MacPherson compactification, and which is obtained by successfully blowing it up along submanifolds of increasingly degenerate configurations, as measured by coranks of the collections of vectors $v_{i_1}, \ldots, v_{i_{k_j}}$ [M*]. In particular, aligned maps of [BCSS] are precisely those $(S_r$-equivariant stratum preserving) maps between the Fulton–MacPherson compactifications that lift to the polynomial compactifications.

Thus, #1 below in the case of classical knots is nearly equivalent to (follows from the rational version of and implies) the Vassiliev Conjecture.

**Conjecture.** If $m - n \geq 3$, let $r$ be the maximal number such that $m > \frac{(r+1)(n+1)}{r}$, i.e. there are no $r$-tuple points in a generic $r$-parametric homotopy $I^n \to I^m$. If $m - n = 2$, let $r$ stand for an arbitrarily large finite number depending on the given embeddings.

1. Smooth embeddings between compact smooth manifolds $N^n \hookrightarrow M^m$ are classified up to smooth isotopy by homotopy of the induced maps $N^{(r)} \to M^{(r)}$ through $S_r$-equivariant stratum preserving maps agreeing with those at all levels $r' < r$.

2. PL embeddings $X^n \hookrightarrow M^m$ of a compact polyhedron into a compact PL manifold are classified up to PL isotopy by homotopy of the induced maps $N^{[r]} \to M^{[r]}$ through $S_r$-equivariant stratum preserving maps agreeing with those at all levels $r' < r$.

3. Topological embeddings between compact manifolds $N^n \hookrightarrow M^m$ are classified up to topological isotopy by homotopy of the power maps $N^r \to M^r$ through $S_r$-equivariant maps whose restriction to each diagonal $\Delta^S_N$, $S \subset \{1, \ldots, r\}$ is $S[S]$-equivariant and sending the complement to every $\Delta^S_N$ into the complement to $\Delta^S_M$.

4. Link maps $S^{n(1)} \sqcup \cdots \sqcup S^{n(k)} \to \mathbb{R}^m$ are classified up to link homotopy by homotopy classes of the induced maps $S^{n(i_1)} \times \cdots \times S^{n(i_k)} \to (\mathbb{R}^m)^s \setminus \bigcup_{p,q} \Delta_{i_p=1q}$, where $s \leq r$ and $\Delta^{i=1}$ stands for $\emptyset$. 


By the results of Haefliger and Weber–Harris [Har] the answer is certainly yes to #1 and #2 in the metastable range \( m > \frac{3(n+1)}{2} \), \( n > 1 \). By results of Habegger–Kaiser and Koschorke/Massey, #4 is known to hold for \( r \leq 3 \). Validity of #3 for non-triangulable manifolds in \( r = 2 \) follows from [MS], where the Haefliger–Weber criterion is generalized to arbitrary compacta.

A homotopy theoretic classification of embeddings with \( r = 3 \) was given in [GKW] using Goodwillie’s Calculus of Functors. It may possibly follow from this (cf. [BCSS]) that the answer to #1 is affirmative for \( r = 3 \) as well. However, since the classification of [GKW] involves taking limits of diagrams indexed by all (uncountably many) points of the manifold, it should be more valuable, from a geometric standpoint, if #1 is proved directly rather than deduced from such results.

As for #2, by an argument from [M3] it implies the “links modulo knots” version of the Vassiliev Conjecture [M3].

In codimension two, there is a natural weaker version of #1 and #2, involving simultaneous homotopies on all levels \( r = 1, 2, \ldots \), agreeing for different levels, or, equivalently, a homotopy between the infinite telescopes. The difference between this and the original version may be captured by a lim \(^1\) obstruction. A similar weaker version can be stated for #3.

For embeddings into \( \mathbb{R}^m \) rather than other manifolds, the thinnest diagonal \( \Delta^1=\cdots=r \) in #3 and the corresponding strata of the compactifications in #1 and #2 can be deleted both in the domain and range. Other diagonals cannot be deleted:

**Quasi-isotopy and Whitehead link.** Every embedding obstruction from the deleted cube in fact obstructs quasi-embeddability, i.e. existence for each \( \varepsilon > 0 \) of a map \( g_\varepsilon: X \to \mathbb{R}^m \) with point-inverses of diameter \( < \varepsilon \) (such a map is called an \( \varepsilon \)-map). Indeed, if \( \varepsilon \) is sufficiently small, \( g_\varepsilon \) yields an \( S_3 \)-equivariant map between the simplicial deleted cubes, which are \( S_3 \)-equivariantly homotopy equivalent to the ordinary deleted cubes. Similarly, every isotopy invariant from the deleted cube is actually invariant under quasi-isotopy, i.e. existence for each \( \varepsilon > 0 \) of a homotopy through \( \varepsilon \)-maps between the given embeddings.

Haefliger’s Whitehead link \( W: S^{2k-1} \sqcup S^{2k-1} \hookrightarrow \mathbb{R}^{3k} \) can be obtained from the Borromean rings \( B: S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \hookrightarrow \mathbb{R}^{3k} \) (see §2) by connecting any two components by a thin tube. Assuming \( k > 1 \), it does not matter which tube is chosen, since the complement is simply-connected. Since \( B \) is Brunnian, the tubed component is null-homotopic in the complement to the unmodified one. Hence \( B \) is link homotopic and so quasi-isotopic to the unlink. On the other hand, \( W \) is not isotopic to the unlink if \( n \neq 1, 3, 7 \). Indeed, the homotopy class of the untubed component in the complement to the tubed component is the Whitehead product \([i, i] \in \pi_{2k-1}(S^k)\) (see [Hae]), which is nonzero when \( k \neq 1, 3, 7 \). So \( W \) is not ambient isotopic to the unlink for such \( k \), hence not PL isotopic to the unlink by Zeeman’s theorem and hence not topologically isotopic to the unlink by Edwards’ theorem (see references e.g. in [M2]). See [SkA] for further examples of this type.

The version of #3 omitting the condition that the restrictions to the diagonals be equivariant was stated in [M2]. However, this condition, as well the conditions of agreement with levels \( < r \) in #1 and #2, turn out to be necessary for \( r = 3 \) by the following example.

**Semi-contractible Whitehead links.** In the cases \( n = 3, 7 \) each component of
the Whitehead link $W: S^{2k-1} \sqcup S^{2k-1} \hookrightarrow \mathbb{R}^{3k}$ is null-homotopic in the complement to the other one (see above). Combining the tracks of such two null-homotopies yields a link map $S^{2k} \sqcup S^{2k} \to \mathbb{R}^{3k+1}$ with a non-zero $\beta$-invariant. Since the $\beta$-invariant vanishes on such link maps with one component embedded (see §3), it does not depend on the choice of the homotopies and so is an isotopy invariant of $W$. Thus $W$ is non-trivial for $k = 3, 7$ as well.

Now to get an isovariant homotopy between the power maps $(S_1^{2k-1} \sqcup S_2^{2k-1})^3 \to (\mathbb{R}^{3k})^3$ of the Whitehead link and a trivial link we only need to construct it separately for each connected component of the domain. It is easy to define it on $(S_1^{2k-1})^3$ and $(S_2^{2k-1})^3$. Now using a null-homotopy of $S_1^{2k-1}$ in the complement to $S_2^{2k-1}$, we can also define it on $S_1^{2k-1} \times (S_2^{2k-2})^2$, and vice versa.

The possibility of coincidence of indices in #4 is essential: without allowing it, we would be dealing with Koschorke’s $\kappa$-invariant, that is, the homotopy class of the map $S_0^{(1)} \times \cdots \times S_0^{(k)} \to (\mathbb{R}^m)^{(k)}$. But for link maps $S^2 \sqcup S^2 \to \mathbb{R}^4$ it is strictly weaker than Kirk’s $\sigma$-invariant. Correspondingly, a notable special case of #4 concerns Koschorke’s $\beta$-invariant, and in particular Kirk’s $\sigma$-invariant of link maps $S^2 \sqcup S^2 \to \mathbb{R}^4$.

Remark. In the last paragraph of the paper [Ko4], Koschorke mentions a “possibly new additive invariant” of link maps $S^2 \sqcup S^2 \to \mathbb{R}^4$ taking values in “a cyclic group”, which “might detect nontrivial elements in the kernel of $\sigma$” (the possible source of this invariant can be inferred from [Ko4]). The present author has shown in [MR; Theorem 2.6(b)], but failed to state explicitly there, that the “cyclic group” in question is trivial after all.

Outline of the paper and discussion. Problems 1–4 are concerned with a homotopy-theoretic classification of embeddings and link maps. In view of the Bott–Taubes integrals, one may also wonder whether a classification in terms of (generalized equivariant) cohomology invariants of maps between configuration spaces is possible. As observed in [M5], this is indeed the case when $r = 2$ (for smooth and PL embeddings and link maps). In this paper it is proved that such a classification is also possible for link maps with $r = 3$, by expressing the triple $\mu$-invariant and the $\beta$-invariant of link maps as (generalized equivariant) degrees of certain maps between configuration spaces. The only previously known ways of extracting $\mu$ and $\beta$ from configuration spaces involved a Whitehead product (in the case of $\mu$) and the stable Hopf invariant (in the case of $\beta$), even in those dimensions where our formulae involve only ordinary (equivariant) cohomology.

These formulas for $\mu$ and $\beta$ are geometrically well understood. Geometrically, the link map is converted in both cases into Vassiliev’s “ornament”, which is a map $S^i \sqcup S^j \sqcup S^k \to \mathbb{R}^m$ with no triple points involving all three components (any intersections and self-intersections of any pair of components are allowed). In the range $r = 3$, ornaments are classified (up to homotopy through ornaments in the case of $\mu$, and up to homotopy through “equivariant ornaments” in the case of $\beta$) by a cohomological invariant directly analogous to the linking number.

These two geometric constructions producing ornaments from link maps are in fact simplest instances of a general method, allowing to “abelianize” a lower-dimensional, “non-linear” situation into a higher-dimensional “linear” one. A third instance, producing link maps from semi-contractible links, was known as “Jin suspension” since mid 80s, and is used e.g. in the author’s new proof of Kirk–
Livingston and Nakanishi–Ohyama theorems [M6]. It is expected that this method, further instances of which are now known, will be a promising substitute for the Whitney tower/grope/Ck-move techniques, which nobody managed to make work adequately in the multi-component case as yet.

**Fulton–MacPherson compactification.** Let $X$ be any compact polyhedron. What we call the *Fulton–MacPherson compactification* $X^{(2)}$ of the two-point configuration space $X^{(2)} = X \times X \setminus \Delta$ is PL homeomorphic to $X \times X \setminus \text{Int}N$, where $N$ is an equivariant regular neighborhood of $\Delta$. Using an equivariant collapse of $N$ onto $\Delta$, one can define a map $\pi : X^{(2)} \to X \times X$ such that $\pi^{-1}(\Delta) = \text{Fr}N$ and $\pi$ sends $\text{Int}X^{(2)}$ homeomorphically onto $X^{(2)}$ — thus indeed making $X^{(2)}$ a compactification of $X^{(2)}$. In contrast to the usual definition of $X^{(2)}$ in the smooth case, our definition does not uniquely describe $X^{(2)}$ up to a homeomorphism commuting with $\pi$, nor even up to a homeomorphism fixed on $X^{(2)}$. But it does uniquely define $\pi$ up to PL “block equivalence”, which is a notion from the same category as PL block bundle isomorphism. That is, given an equivariant triangulation of $X \times X$, any two instances $X^{(2)}_\alpha$, $X^{(2)}_\beta$ of $X^{(2)}$ are related by an equivariant PL homeomorphism taking the preimage in $\pi$ with $X$ space the cylinders of all $\pi$ sends $\text{Int}X^{(2)}$ into its preimage in $X^{(2)}$ of each dual cone of the triangulation of $X \times X$ into its preimage in $X^{(2)}$; moreover the homeomorphism strictly commutes with $\pi$ over dual cones disjoint from $\Delta$.

For $r = 2, 3$ the Fulton–MacPherson compactification of the $r$-point configuration space $X^{(r)}$ coincides with its simplified version $X^{(r)}$, defined as follows. Let us consider an $S_r$-regular neighborhood $N_1$ of the thinnest diagonal $\Delta^1 = \cdots = r$ of $X^r = X \times \cdots \times X$; an $S_r$-equivariant regular neighborhood $N_2$ of union of the next-thick diagonals $\Delta^i_1 = \cdots = i_{i-1}$ and $\Delta^i_1 = \cdots = i_i; i_{i+1} = \cdots = i_r$ relative to $N_1$; and so on, ending up with an $S_r$-equivariant regular neighborhood $N_{r-1}$ of union of the thickest diagonals $\Delta^i = j$ relative to $N_1 \cup \cdots \cup N_{r-2}$. (The diagonals are indexed by all partitions of $\{1, \ldots, r\}$, except for the partition into singletons. The thickness of a diagonal is the number of blocks in the partition.) Now $X^{(r)}$ is PL homeomorphic to $X^r \setminus \text{Int}(N_1 \cup \cdots \cup N_{r-1})$, and is endowed with the collection of codimension zero subpolyhedra $\partial X^{(r)} = \text{Fr}N_1 \cap \partial X^{(r)}$ of the corona $\partial X^{(r)} = \text{Fr}(N_1 \cup \cdots \cup N_{r-1})$. Connected (normally) components $\partial_{\Pi}X^{(r)}$ of each $\partial X^{(r)}$ are indexed by partitions $\Pi$ with $|\Pi| = i$ blocks. If $X$ is a closed manifold, $X^{(r)}$ is a manifold with boundary $\partial X^{(r)}$ and with corners $\partial_{\Pi}X^{(r)} \cap \cdots \cap \partial_{\Pi_{i+1}}X^{(r)}$, where each $\Pi_i$ refines $\Pi_{i+1}$. The projection $\pi : X^{(r)} \to X^r$ is well-defined up to block equivalence, and its restriction to each $\partial_{\Pi}X^{(r)}$ factors through a map $\pi_{\Pi} : \partial_{\Pi}X^{(r)} \to X^{(|\Pi|)}$, well-defined up to block equivalence, and the projection $\pi : X^{(|\Pi|)} \to X^{[\Pi]}$. Gluing together $X^{(r)}$ and the cylinders of all $\pi_{\Pi}$’s, we recover the product $X^r$.

Restricting our attention to the diagonals indexed by subsets (which can be identified with partitions into one subset and singletons) in the above definition, we get the *Fulton–MacPherson compactification* $X^{[r]}$. The role of $\partial_{\Pi}X^{(r)}$’s is played by the images $\partial_SX^{[r]}$ under the natural projection $X^{(r)} \to X^{[r]}$ of $\partial_{\{S\}}X^{(r)}$ for all subsets $S \subset \{1, \ldots, r\}$. A map $f : X^{[r]} \to Y^{[r]}$ will be called stratum preserving if $f(\partial SX^{[r]}) = \partial SY^{[r]}$ for each subset $S$. (It would perhaps be more accurate to call $f$ preserving closures of strata, but “stratum preserving” appears to be a standard terminology [Si].) One can define the maps $\pi_S : \partial SX^{[r]} \to X^{[r+1]-|S|}$ similarly to the above. We shall say that an $S_r$-equivariant stratum preserving map $f_r : X^{[r]} \to Y^{[r]}$ agrees with similar lower level maps if there exist $S_r$-equivariant
stratum preserving maps \( f_i : X^{[i]} \to Y^{[i]} \) for all \( i < r \) such that \( \pi_S f_i = f_j \pi_S \) for all subsets \( S \) of \( \{1, \ldots, i\} \) with \( j = i + 1 - |S| \).

If \( X \) is a smooth manifold or \( r = 2 \), each \( \partial S X^{[r]} \) and \( \partial_i X^{(r)} \) admits a canonical involution, and these can be used to make \( X^{[r]} \) and \( X^{(r)} \) into closed manifolds — the original projective versions of the Fulton–MacPherson and the Fulton–MacPherson–Ulyanov compactifications [FM], [U].

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**Note added in 2017.** This preprint was privately circulated and presented at conferences and seminars by the author in 2006-07 (Alexandroff Readings, Moscow, May 2006; UF-FSU Topology Conference, Gainesville, December 2006; Dartmouth College topology seminar, April 2007; etc.) I hesitated to publish it at that time as I hoped to get more progress on the conjectures stated in the introduction; but numerous other projects have been distracting me from this task so far.

In recent years, a construction pretty similar to (but not exactly same as) the triple-point Whitney trick in Theorem 1.1 below has been independently discovered by I. Mabillard and U. Wagner, who successfully employed it to obtain nice results. (They call theirs the “triple Whitney trick”, but I prefer to reserve this title for a more elaborate construction, involving the triple-point Whitney trick as one of several steps; it can be used, in particular, to obtain a geometric proof of the Habegger–Kaiser classification of link maps in the 3/4 range — as presented in my talk at the Postnikov Memorial Conference in Bedlewo, June 2007, — and will hopefully appear elsewhere.) As the triple-point Whitney trick has attracted considerable interest, I have added several footnotes clarifying the proof of Theorem 1.1 without altering its original (admittedly, somewhat sketchy) text from 2006.

**Notation.** The following notation will be used throughout the paper. Let \( X \) be a compact \( n \)-polyhedron. \( \bar X \) will denote the quotient of the deleted product \( \tilde X := X \times X \setminus \Delta X \) by \( \mathbb{Z}/2 \), acting by the factor exchanging involution.

If \( X \) is a polyhedron, let \( \tilde X \) denote the deleted cube \( X \times X \times X \setminus (\Delta^{1=2} \cup \Delta^{2=3} \cup \Delta^{3=1}) \), where each \( \Delta^{i=j} \) denotes the thick diagonal \( \{(x_1, x_2, x_3) \mid x_i = x_j\} \). Also let \( \tilde X_{\Delta} \) denote \( X \times X \times X \setminus \Delta^{1=2=3} \), where \( \Delta^{1=2=3} \) denotes the thin diagonal \( \{(x, x, x) \mid x \in X\} \).

1. **Ornaments** \((2k - 1 \text{ in } 3k - 1)\)

Doodles were originally defined by Fenn and Taylor [FT] (see also [Fe]) as triple point free maps of a collection of circles into the plane that embed each circle. In our terminology we follow Khovanov [Kh], who redefined them by dropping the condition that each component be embedded. It has been established that Khovanov’s doodles are classified up to *doodle homotopy*, i.e. homotopy through doodles, by their finite type invariants [Mer2].

**Invariant of ornaments.** If \( f = f_1 \sqcup f_2 \sqcup f_3 : X_1 \sqcup X_2 \sqcup X_3 \to \mathbb{R}^m \) is an ornament\(^1\),

\(^1\)Ornaments were introduced by Vassiliev [Va1] as a modification of Fenn and Taylor’s doodles [FT]; see also [Mer1] (especially §6 and Appendix D) and [Va2; Chapter 6].
that is \( f(X_1) \cap f(X_2) \cap f(X_3) = \emptyset \) (but \( f \) may have triple points), let

\[
\tilde{\mu}(f) \in H^{2m-1}(X_1 \times X_2 \times X_3)
\]

be the image of a fixed generator of \( H^{2m-1}(S^{2m-1}) \) under the composition

\[
X_1 \times X_2 \times X_3 \xrightarrow{f_1 \times f_2 \times f_3} \mathbb{R}^m_\Delta \xrightarrow{\tilde{\mu}} S^{2m-1}.
\]

Then \( \tilde{\mu}(f) \) is invariant under ornament homotopy (i.e. homotopy through ornaments). We note that \( \tilde{\mu} \) depends on the ordering of \( X_i \). Since \( S_3 \) acts on \( H^{2m-1}(\mathbb{R}^m_\Delta) \) by the sign homomorphism, \( \tilde{\mu} \) remains unchanged when \( X_1, X_2, X_3 \) are permuted cyclically, and reverses the sign when any two are interchanged.

If each \( X_i \) is an oriented connected closed \((2k-1)\)-manifold and \( m = 3k-1 \), then \( \tilde{\mu}(f) \) is an integer, and it clearly equals the algebraic number of \( 1 = 2 = 3 \) points, i.e. triple points of intersection between \( X_1, X_2 \) and \( X_3 \), in a generic homotopy from \( f \) to \( t \). In particular, in the case of ornaments \( f: S^1 \sqcup S^1 \sqcup S^1 \to \mathbb{R}^2 \) it is an extension of the \( \mu \)-invariant of [Fe]. Note that in this case it is incomplete up to ornament homotopy (see e.g. [Va1]) or ornament concordance, but is complete up to ornament cobordism (see [Fe]).

**Borromean ornament.** Let us think of \( S^{3k-1} \) as the unit sphere in \( \mathbb{R}^{3k} \). Let \( b \) be a triple point free map \( S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to S^{3k-1} \) embedding the components onto \( S^{2k-1} \cap \mathbb{R}^k \times \mathbb{R}^k \times 0, S^{2k-1} \cap \mathbb{R}^k \times 0 \times \mathbb{R}^k \) and \( S^{2k-1} \cap 0 \times \mathbb{R}^k \times \mathbb{R}^k \). (This was called the Borromean doodle in [Fe] when \( k = 1 \).) The obvious null-homotopy of \( b \) has one \( 1 = 2 = 3 \) point, so \( \tilde{\mu}(b) = \pm 1 \).

**Theorem 1.1.** \( \tilde{\mu}(f) \) is a complete invariant of ornament homotopy if each \( X_i \) is an orientable \((2k-1)\)-manifold and \( m = 3k-1, k > 2 \).

**Proof.** Let us assume for simplicity that each \( X_i \) is connected and closed. If \( \tilde{\mu}(f) = \tilde{\mu}(g) \), there is a homotopy \( h: X \times I \to \mathbb{R}^m \times I \) between the ornaments \( f \) and \( g \) whose \( 1 = 2 = 3 \)-points can be paired up with opposite signs. Each pair \((p^+, p^-)\) can then be cancelled by a triple Whitney trick. In more detail, let \( p^\pm_i \) be the preimage of \( p^\pm \) in \( X_i \times I \). We first arrange that \((p^+_1, p^+_2, p^-_2)\) and \((p^-_1, p^-_2, p^+_2)\) be in the same component of the double point set \( \{(x, y) | h(x) = h(y)\} \) in \( X_1 \times I \times X_2 \times I \) (in case that initially they are not). To this end we pick points \((q^+_1, q^+_2)\) in the same components with \((p^+_1, p^+_2)\) and such that the double points \( f(q^+_1) = f(q^+_2) \) and \( f(q^-_1) = f(q^-_2) \) are not triple points. Now connect \( q^+_1 \) and \( q^-_1 \) by a generic path in \( X_1 \times I \), disjoint from preimages of any double points (using that \( k > 1 \)) and attach a thin 1-handle (i.e. remove \( B^0 \times S^0 \) and paste in \( S^{n-1} \times I \)) to \( h(X_2 \times I) \) along the image of this path.

To restore the topology of \( X_2 \times I \), we cancel the 1-handle geometrically by attaching a 2-handle along an embedded 2-disk, meeting \( h(X \times I) \) only in the boundary circle (such a disk exists since \( k > 2 \)).

\(^2\) In more detail, it may be assumed that the double point set is an oriented manifold that is immersed in \( X_1 \times I \). Let us take an oriented connected sum of its components along the original path in \( X_1 \times I \). This yields an embedded tube \( S^{2k-1} \times I \) in \( h(X_1 \times I) \), and its fiberwise join with the spherical normal bundle of \( h(X_1 \times I) \) over the image of the path is the desired embedded tube \( S^{2k-1} \times I \) in \( \mathbb{R}^m \times I \).

\(^3\) Moreover, since \( k > 2 \), the boundary circle may be assumed to be disjoint from \( h(X_1 \times I) \), so that the double point set is unaffected by this step. Since the normal bundle of the boundary circle in \( h(X_2 \times I) \) is trivial, it extends over the disk; the associated sphere bundle is the desired 2-handle.
the 1 = 2 = 3 points, let us connect \((p_1^+, p_2^+\) and \((p_1^-, p_2^-)\) by a path\(^4\), and attach a thin 1-handle to \(h(X_3 \times I)\) along the image of this path.\(^5\) The topology of \(X_3 \times I\) can be restored using another 2-disk like before.\(^6\)

Finally, we must either make sure that this construction can be done in a level-preserving fashion, or alternatively apply the “ornament concordance implies ornament homotopy in codimension three” theorem [M1]. □

We note that \(\hat{\mu}(g) = 0\) if \(g\) is a link map, i.e. \(g(X_i) \cap f(X_j) = \emptyset\) whenever \(i \neq j\). Indeed, by first moving \(X_1\) away while keeping \(f\) fixed on \(X_2 \cup X_3\), and subsequently moving \(X_2\) away from \(X_3\), we get an ornament homotopy from \(g\) to a trivial link map.

This is not a coincidence:

**Theorem 1.2.** Consider link maps \(f: S^{n_1} \sqcup \cdots \sqcup S^{n_k} \to \mathbb{R}^m, n_i < m - 1\), and let \(\omega(f)\) be the image in the cohomology of \(S^{n_1} \times \cdots \times S^{n_k}\) of some cohomology class of \((\mathbb{R}^m)^k \setminus \bigcup \Delta^{i=j}\). Suppose that \(\omega\) vanishes on any \(f\) with one component contained in an \(m\)-ball disjoint from the other components. Then \(\omega\) vanishes on all \(f\).

**Proof.** Since spherical link maps in codimension two up to link homotopy (i.e. homotopy through link maps) form a group [BT], [M1], and \(\omega\) is clearly invariant under link homotopy and additive under connected sum, it suffices to consider the case where \(f\) is homotopically Brunnian. Now by an analysis of Koschorke [K1], [K7], \(f_1 \times \cdots \times f_n\) factors through a Whitehead product, hence is trivial on cohomology. □

\(\beta_i\)-invariants of animated ornaments. An animated ornament is an ornament \(f: X_1 \sqcup X_2 \sqcup X_3 \to \mathbb{R}^m\) along with three ornament homotopies starting with \(f\), where the \(i\)th homotopy has support in the \(i\)th component and shrinks it to a point.

An animated ornament \((F, f) = \bigsqcup(F_i, f_i); \bigsqcup(CX_i, X_i) \to \mathbb{R}^m\) yields a map

\[
\Sigma_*(X_1 \times X_2 \times X_3) \xrightarrow{\bigsqcup f_i \times f_j \times F_k} \mathbb{R}_\Delta^m \xrightarrow{\varepsilon} S^{2m-1},
\]

which suspends to \(S^1 \ast (X_1 \times X_2 \times X_3) \to \Sigma_* S^{2m-1}.\) A \(\mathbb{Z}[\mathbb{Z}/3]\)-module isomorphism \(H^{2m}(\Sigma_* S^{2m-1}) \cong \mathbb{Z}[\zeta]\) can be chosen so that the homotopy between \(f_i \times f_j \times f_k\) and \(f_i \times f_j \times F_k\) only meets the support of \(\zeta^{-1}\). Let us denote the image of \(\zeta^{-1}\) in \(H^{2m}(S^1 \ast (X_1 \times X_2 \times X_3)) \cong H^{2m-2}(X_1 \times X_2 \times X_3)\) by

\[
\beta_i(F) \in H^{2m-2}(X_1 \times X_2 \times X_3).
\]

By construction, \(\beta_i\) is invariant under homotopy through animated ornaments, remains unchanged if \(X_j\) and \(X_k\) are interchanged, and does not depend on \(F_i\). Also \(\beta_i + \beta_j + \beta_k = 0\) since \(1 + \zeta + \zeta^2 = 0\).

In fact, it follows from the definition that \(\beta_i\) is simply the \(\hat{\mu}\)-invariant of the ornament \(\Sigma X_1 \sqcup \Sigma X_2 \sqcup \Sigma X_3 \to \mathbb{R}^{m+1}\) given by \(F_j\) and \((f_i \sqcup f_k) \times \text{id}_{[0,1/2]}\) plus some null-homotopies of \(f_i\) and \(f_k\) on the upper half of the suspensions, and by \(F_k\) and \((f_i \sqcup f_j) \times \text{id}_{[-1/2,0]}\) plus some null-homotopies of \(f_i\) and \(f_j\) on the lower half of the suspensions.

\(^4\)within the double point set

\(^5\)This 1-handle is the spherical normal bundle of \(h(X_1 \times I) \cap h(X_2 \times I)\) over the path. It is attached orientably since the two \(1 = 2 = 3\) points have opposite signs.

\(^6\)In particular, this 2-disk is disjoint from \(h(X_1 \times I \sqcup X_2 \times I)\), so no new \(1 = 2 = 3\) points arise.
$\beta_i$-invariants of $\vec{v}$-animated ornaments. An ornament $f: X_1 \sqcup X_2 \sqcup X_3 \to \mathbb{R}^m$ is called $\vec{v}$-animated if no vector of the type $f(x) - q$ is a positive scalar multiple of $\vec{v}$, where $q = f(y) = f(z)$ and each $X_i$ contains precisely one of $x, y, z$. Clearly, every $\vec{v}$-animated ornament is animable.

A $\vec{v}$-animated ornament $f = \bigsqcup f_i: \bigsqcup X_i \to \mathbb{R}^m$ yields a map

$$X_1 \times X_2 \times X_3 \xrightarrow{\bigsqcup f_i \times f_j \times f_k} (\mathbb{R}^m)^3 \setminus \Pi_1 \vec{v} \cup \Pi_2 \vec{v} \cup \Pi_3 \vec{v} \xrightarrow{\sim} \Sigma_3 S^{2m-3}.$$  

Then with an appropriately chosen isomorphism $H^{2m-2}(\Sigma_3 S^{2m-3}) \simeq \mathbb{Z}[\zeta]$, the image of $\zeta^{i-1}$ in $H^{2m-2}(X_1 \times X_2 \times X_3)$ equals $\beta_i$.

$\beta_0$-invariant for $\pm \neq 0$ maps. Let $f: X_+ \sqcup X_- \sqcup X_0 \to \mathbb{R}^{3k-2}$ be a $\pm \neq 0$ map of a $(2k-2)$-polyhedron (see §3). Then any null-homotopy of $f|_{X_1}$, any null-homotopy of $f|_{X_2}$ and (using the hypothesis on dimensions) any generic null-homotopy of $f|_{X_3}$ combine into an animated ornament. Moreover, since $\beta_0$ does not depend on the choice of the latter null-homotopy, it only depends on $f$ and is invariant under $\pm \neq 0$ homotopy of $f$. Specifically, $\beta_0(f)$ coincides with $\beta(f)$ from §3 since both equal the $\mu$-invariant of the ornament $\hat{f}$ from the proof of Theorem 3.1. Alternatively, choosing the animated ornament to be $\vec{v}$-animated for some $\vec{v}$, we see that $\beta_0(f)$ as defined in the previous paragraph (making use of $\vec{v}$) equals $\beta(f)$ as defined in §3 using the composition $X_1 \times X_2 \times X_2 \to (\mathbb{R}^m)^3 \setminus (\Pi_2 \vec{v} \cup \Pi_3 \vec{v}) \simeq S^{2m-1}$.

2. $\mu$-INVARIENT OF LINK MAPS ($2k - 3$ IN $3k - 3$)

The first dimension beyond the metastable range is $n = 2k - 2, m = 3k - 2$ for embeddability and $n = 2k - 3, m = 3k - 3$ for isotopy. For convenience, let us substitute $k - 1$ for $k$ and write $n = 2k - 1, m = 3k$ from now on.

Borromean rings. The Borromean rings $B: S^{2k-1}_+ \sqcup S^{2k-1}_- \sqcup S^{2k-1}_0 \hookrightarrow \mathbb{R}^{3k}$ can be obtained from concentrically embedded spheres $S^{k-1}_+, S^{k-1}_- \subset \mathbb{R}^k$ by forming the joins

$$S^{k-1}_+ \times 0 \times 0 \sqcup (0 \times S^{k-1}_+ \times 0) \sqcup (0 \times 0 × S^{k-1}_-) \subseteq \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k.$$  

The non-triviality of $B$ cannot be detected by the deleted product, since every two-component sublink is easily seen to be trivial. It is well-known, however, that it can be detected by the homotopy class of one component in the complement to the other two, which is a Whitehead product [Ha2] and by the triple Massey product of the Alexander duals to the fundamental homology classes of the components [Ma1].

As noticed by Koschorke [Ko1], the non-triviality of $B$ up to link homotopy can also be detected from the homotopy class of $F: S^{2k-1} \times S^{2k-1} \times S^{2k-1} \to \mathbb{R}^{3k}$, which turns out to factor through the Whitehead product $S^{6k-1} \to S^{3k-1} \vee S^{3k-1}$. Indeed, since $B$ restricted to the first two components is link homotopically trivial, $F$ composed with the projection $\pi: \mathbb{R}^{3k} \to \mathbb{R}^{3k}$ that forgets the third point of the configuration is null-homotopic. Therefore $F$ is homotopic to a map into a fiber of $\pi$, i.e. $\mathbb{R}^m$ minus two points, which collapses onto $S^{3k-1} \vee S^{3k-1}$.

In light of this observation it may look like no triple analogue of the van Kampen obstruction to isotopy, defined by inducing some cohomology class of $S_3$, can detect the non-triviality of the Borromean rings (compare Theorem 1.2). Nevertheless, we
will show in Theorem 2.5 that the triple \( \mu \)-invariant, which can be defined using either of the three mentioned approaches to prove the non-triviality of \( B \) (see [Ko7], [Ko5; 3.11]), in fact equals a certain 3-parametric version of the triple van Kampen obstruction. This agrees with what was apparently expected in [Kr; \S 5].

\( \mu \)-invariant of link maps. The following definition of the \( \mu \)-invariant is essentially found in [Ah; \S 4] (in the classical dimension and mod2); Akhmetiev informed the author that he mentioned it as early as 1991 in his talk at a conference in Kiev. It may also be viewed as a specific instance of Koschorke’s geometric definition [Ko6], essentially rediscovered in [MM] (in the classical dimension), although neither [Ko6] nor [MM] singles out this specific instance; compare [Ko5; 3.10].

Let \( f: S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to \mathbb{R}^{3k} \) be a Brunnian link, i.e. one where each proper sublink is isotopically trivial. We define \( \mu(f) \) to be the algebraic number of triple points of intersection between the tracks of a generic null-homotopy of the first component in the complement to the second; a generic null-homotopy of the second component in the complement to the third; and a generic null-homotopy of the third component in the complement to the first. We will refer to such a triple of null-homotopies as a \((123)\)-null-homotopy of \( f \); also recall that we abbreviate a triple point of intersection between three distinct components as a “\( 1 = 2 = 3 \) point”. The three obvious flat disks bounded by the Borromean rings \( B \), where \( S^{k-1}_- \) is taken in the bounded complementary domain of \( S^{k-1}_+ \), yield a \((123)\)-null-homotopy of \( B \) with precisely one \( 1 = 2 = 3 \) point, thus \( \mu(B) = \pm 1 \).

The following lemma allows to extend the definition to all \( h \)-Brunnian link maps, i.e. ones where every proper sub-link-map is link homotopic to an unlink.

**Lemma 2.1.** Every \( h \)-Brunnian link map \( f: 3S^{2k-1} \to \mathbb{R}^{3k} \) is link homotopic to a link admitting a \((123)\)-null-homotopy as well as a \((321)\)-null-homotopy (in fact, to a Brunnian link).

**Proof.** If \( k > 1 \), we first link homotop \( f \) to an embedded link using the Penrose–Whitehead–Zeeman trick (this is the lowest dimension where it works). We then make it Brunnian by adding split links resulting from the reflected 2-component sublinks of \( f \) (using that concordance implies isotopy in codimension three).

For \( k = 1 \) we could resort to Milnor’s classification to get the assertion. (It might be undesirable to use Milnor’s work on \( \mu \)-invariants in defining a \( \mu \)-invariant. To this end, we can simply unknot the components by a link homotopy, so that each component becomes null-homotopic in the complement to every other. This suffices to get the weaker conclusion.) \( \square \)

**Theorem 2.2.** \( \mu(f) \) is well-defined up to link homotopy.

**Proof.** Let \( F_1, F_2, F_3: S^{2k-1} \times I \to \mathbb{R}^{3k} \times I \) be a generic link homotopy between \( f \) and \( g \), and let \( h_0 \) be a \((123)\)-null-homotopy of \( f \) and \( h_1 \) a \((321)\)-null-homotopy of \( g \). We can extend \( h_0 \) and \( h_1 \) to some triple of null-homotopies \( H_1, H_2, H_3: D^{2k+1} \to \mathbb{R}^{3k} \times I \) of \( F_1, F_2, F_3 \). This triple yields an oriented bordism between the \( 1 = 2 = 3 \) points of \( h_0 \) and those of \( h_1 \) as well as the points of intersection between \( H_i, H_j \) and \( F_k \) whenever \((ijk) = (123)\). The latter are the \( F_k \)-images of the double points between \( M_{ki} := F_k^{-1}(H_i(D^{2k+1})) \) and \( M_{kj} := F_k^{-1}(H_j(D^{2k+1})) \), which by transversality may be assumed to be manifolds. Since \( h_0 \) is a \((123)\)-null-homotopy, \( \partial M_{ki} \subset S^{2k-1} \times \{1\} \), and since \( h_1 \) is a \((321)\)-null-homotopy, \( \partial M_{kj} \subset S^{2k-1} \times \{0\} \).
Pushing $M_{ki}$ upwards and $M_{kj}$ downwards in $S^{2k-1} \times I$ until they become disjoint yields an oriented bordism between the double points in question. □

It also follows from Lemma 2.1 and the proof of Theorem 2.2 that $\mu(f)$ can be evaluated using any (123)-null-homotopy or (321)-null-homotopy that $f$ admits.

**From ornaments to link maps.** An ornament $f : S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to \mathbb{R}^{3k-1}$ gives rise to a link map $\tilde{f} : S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \hookrightarrow \mathbb{R}^{3k}$ by lifting the first sphere to overweight the second, the second to overweight the third, and the third to overweight the first wherever they intersected in $\mathbb{R}^{2k-1}$. Now a (123)-null-homotopy of $\tilde{f}$ can be found within the track (in the upper half-space of $\mathbb{R}^{3k}$) of any generic homotopy between $f$ and a trivial link map $t$, constant on each component. So $\tilde{\mu}(f) = \mu(\tilde{f})$. This formula is well-known in the classical dimension [Fe].

Another way to see that $\tilde{\mu}(f) = \mu(\tilde{f})$ is by noticing that every $1 = 2 = 3$ point in a generic homotopy between $f$ and $f_0$ corresponds under $\sim$ to connected summation with the Borromean rings. Indeed, $\sim$ when applied to the Borromean ornament $b$ yields the (slightly smoothed) Borromean rings $B$ — if instead of the vertical decomposition $\mathbb{R}^{3k} = \mathbb{R}^{3k-1} \times \mathbb{R}$ we think of the radial decomposition $\mathbb{R}^{3k} \setminus \{0\} = S^{3k-1} \times \mathbb{R}$.

**Reading off the $\mu$-invariant.** There are two ways to generalize the above definition of $\mu(f)$, without resorting to Lemma 2.1, to arbitrary link maps $f : S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to \mathbb{R}^{3k}$ when $k > 1$, and arbitrary h-Brunnian links when $k = 1$.

In the first approach, $\mu(f)$ can be defined as above, except that each null-homotopy is replaced by a null-homology (i.e. a map of a punctured $2k$-pseudo manifold). Such null-homologies exist, since each $H^{2k-1}(\mathbb{R}^{3k} \setminus f(S^{2k-1})) = 0$ because the double point locus of $f$ has dimension $\leq k - 2$ and so $H_k(f(S^{2k-1})) = 0$ unless $k = 1$. This definition generalizes the well-known formula [Co; §5] for the $\mu$-invariant of a classical h-Brunnian link as the algebraic number of $1 = 2 = 3$ points for Seifert surfaces, bounded by each component in the complement to the other two. This formula can be proved for h-Brunnian link maps $3S^{2k-1} \to \mathbb{R}^{3k}$ with arbitrary $k$ along the lines of [Ma1; §4].

In the other approach ([Ko6], [MM]), one considers an arbitrary triple of null-homotopies of the components, and adds three “correction terms” to the algebraic number of $1 = 2 = 3$ points using that the preimage of each null-homotopy in every other component is null-bordant, since the two-component sublink is link homotopically trivial. An advantage of this definition is a straightforward generalization to all dimensions and a straightforward proof of invariance under link homotopy [Ko6]. (The above proof of Theorem 2.2 is a special case of Koschorke’s argument; yet it has the advantage of not breaking the symmetry, as it has already been broken in our definition.) It leads to the same $\mu$-invariant as defined in [Ko1], [Ko7] for h-Brunnian link maps [Ko6], and, as is easily seen, to the same $\mu$-invariant as defined in the previous paragraph.

---

7 Invariance under link homotopy can also be proved in this fashion. If $F$ is a link homotopy, its double point locus $D$ is a $(k - 1)$-pseudomanifold, so that $H^{2k}(\mathbb{R}^{3k} \setminus F(S^{2k-1} \times I)) \simeq H_k(F(S^{2k-1} \times I))$ may be nonzero. However its generators can be represented by “characteristic tori” $S^k \times S^k$ in a neighborhood of $D$, see [HK; §2]. So, although we cannot avoid inadmissible intersections between one immersed cylinder $F(S^{2k-1} \times I)$ and a null-homology for another, they can be confined to a neighborhood of $D$, which is disjoint from the null-homology of the third cylinder by general position.
Inducing $\mu$ from a cohomology class of $\mathbb{Z}/3$. We want to find a Gauss style description for the $\mu$-invariant of h-Brunnian link maps. Here is the basic idea. From the viewpoint of configuration spaces, we should not think of the complement of a component in $\mathbb{R}^m$ — we may only think of two components being disjoint. Hence each null-homotopy from the initial definition of $\mu$ (for Brunnian links) becomes a homotopy of the entire link, possibly moving all components. Now in order to catch the $1 = 2 = 3$ point between the null-homotopies as a generic event in their eyes, we have to treat each homotopy as living in its own time, and combine them together into a 3-parametric family (just like in [Ko6; (4.3)], albeit for a different purpose). Thus, we need to rewrite the $(123)$-condition in terms of the new data to start with.

Let $X = X_1 \sqcup X_2 \sqcup X_3$ be a polyhedron. Consider a $\partial I^3$-homotopy (parametrized by the boundary of a cube) $h_T: X \to \mathbb{R}^m$, $T \in \partial I^3$. We shall call it triangular if $h_T(X_i \cap X_j) = \emptyset$ whenever $T$ lies on either of the faces $t_j = 0$, $t_i = 1$ of the cube $I^3 = \{0 \leq t_1, t_2, t_3 \leq 1\}$, where $(ijk) = (123)$.

Lemma 2.3. If $f: S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to \mathbb{R}^{3k}$ is an h-Brunnian link map, there exists a generic cube of homotopies $H_T$, $T \in I^3$, such that $H_{0,0,0} = f$, $H_{1,1,1}$ is a trivial link map, and the boundary $\partial I^3$-homotopy is triangular. Moreover, the algebraic number of $1 = 2 = 3$ points in the $I^3$-homotopy equals $\mu(f)$.

Proof. By Lemma 2.1, there exists a link homotopy $h_t$ from $f$ to a link $l$ whose components bound an admissible triple of null-homotopies. Consider this triple as a homotopy $g_t$ from $l$ to a trivial link map $t$, sending each component to a generic point in $\mathbb{R}^{3k}$. Define $G_{t_1,t_2,t_3}$ by $g_{t_i}$ on the $i$th component $S^{2k-1}$. This gives the required $I^3$-homotopy except that $G_{0,0,0} = l$ instead of $f$. So it remains to modify $G$ near $(0,0,0)$. To this end let us first extend it via $h_t$ to an $I \vee I^3$-homotopy, where the whisker $I$ is attached to $I^3$ at $(0,0,0)$. Now set $H_T = G_{\varphi(T)}$, where $\varphi: I^3 \to I \vee I^3$ is some continuous surjection whose composition with the collapse of the whisker $I \vee I^3 \to I^3$ sends each face of the cube to itself. $\square$
We shall call a \( \partial I^3 \)-homotopy \( h_T: Q \rightarrow (\mathbb{R}^m)^3 \) hexagonal if \( h_T(Q) \cap \Delta^{i=j} = \emptyset \) whenever \( T \) lies on either of the faces \( t_k = 0, t_k = 1 \) of the cube \( I^3 = \{0 \leq t_1, t_2, t_3 \leq 1\} \) whenever \( \{i, j, k\} = \{1, 2, 3\} \), and additionally \( h_T(Q) \subset \tilde{\mathbb{R}}^m \) whenever \( T \) lies on either edge of the form \( t_i = t_j = 0 \) or \( t_i = t_j = 1 \). Note that if \( h_T \) is hexagonal, its image lies in \( \tilde{\mathbb{R}}^m \).

**Lemma 2.4.** Let \( G: Q \rightarrow \tilde{\mathbb{R}}^m \) be such that its composition with each projection \( p_i: \tilde{\mathbb{R}}^m \rightarrow \tilde{\mathbb{R}}^m \) forgetting the \( i \)th point is null-homotopic. Then there exists a hexagonal \( \partial I^3 \)-homotopy \( H_T: Q \rightarrow \tilde{\mathbb{R}}^m \) such that \( H_{0,0,0} = G \) and \( H_{1,1,1} \) is a constant map into \( \tilde{\mathbb{R}}^m \).

**Proof.** We first construct a null-homotopy \( H_{T_{ij}(t)}: Q \rightarrow (\mathbb{R}^m)^3 \setminus (\Delta^{k=j} \cup \Delta^{k=i}) \) of \( G \) corresponding to each of the six shortest paths \( T_{ij} \) from \((0,0,0)\) to \((1,1,1)\) in the 1-skeleton of the cube.

**First edge.** The fiber \( F_i := p_i^{-1}(x_j, x_k) \) is homeomorphic to \( \mathbb{R}^m \setminus \{x_j, x_k\} \); we may assume that \( x_j - x_k \) is a fixed vector \( \vec{c} \in \mathbb{R}^m \) with \( |\vec{c}| > 1 \). Since \( p_iG \) is null-homotopic, \( G \) is homotopic to a map \( G_i \) into \( F_i \). Writing \( \vec{e}_i \) for the unit vector in the direction of the \( t_i \)-axis, let \( H_{t\vec{e}_i}: Q \rightarrow \tilde{\mathbb{R}}^m \) be this homotopy.

**Second edge.** Let \( \pi: \tilde{\mathbb{R}}^m \rightarrow \mathbb{R}^m \setminus pt \) be the projection along the diagonal, \( \pi(x, y) = x - y \). Let us extend \( p_i \) to the projection \( p_{ij}: (\mathbb{R}^m)^3 \setminus (\Delta^{k=j} \cup \Delta^{k=i}) \rightarrow \tilde{\mathbb{R}}^m \). The fiber \( F_{ij} = p_{ij}^{-1}\pi^{-1}(\vec{c}) \) of \( \pi p_{ij} \) contains \( F_i \) and coincides with the image of the section \( s_{ji} \) of \( p_{ji} \), given by \( s_{ji}(x_i, x_k) = (x_i, x_k, x_k + \vec{c}) \). Hence \( G_i = s_{ji}p_{ji}G \) is homotopic to \( s_{ji}p_{ji}G \) with values in \( (\mathbb{R}^m)^3 \setminus (\Delta^{k=j} \cup \Delta^{k=i}) \) via the homotopy \( f_{t}^{ji} := s_{ji}p_{ji}H_{(1-t)\vec{e}_i} \). 

**Figure 1: Triangular \( \partial I^3 \)-homotopy**
Let \( s_j : \mathbb{R}^m \to \mathbb{R}^m \) be the section of \( p_j \) given by \( s_j(x_i, x_k) = (x_i, x_k, (|x_i| + |x_k|)\overline{c}) \). It is homotopic to \( s_{ij} \) by a homotopy through sections of \( p_{ji} \), given by combining in order the two homotopies

\[
\begin{align*}
g_t^{ji}(x_i, x_k) &= (x_i, x_k, x_k + [t(|x_i| + |x_k|) + (1 - t)]\overline{c}), \\
h_t^{ji}(x_i, x_k) &= (x_i, x_k, (1 - t)x_k + (|x_i| + |x_k|)\overline{c}).
\end{align*}
\]

Here \( g_t^{ji} \) does not touch \( \Delta^{k=j} \) since \( t(|x_i| + |x_k|) + (1 - t) \neq 0 \) for each \( t \), and \( h_t^{ji} \) since \( |tx_k| \leq |x_i| + |x_k| \) for each \( t \), and \( |\overline{c}| > 1 \) (and both homotopies do not touch \( \Delta^{k=i} \) since \( x_i \neq x_k \) in the domain). We define \( H_{t\epsilon_i+\epsilon_k} : Q \to (\mathbb{R}^m)^3 \setminus (\Delta^{k=i} \cup \Delta^{k=j}) \) by combining in order the homotopies \( f_t^{ji} \) and \( g_t^{ji}p_{ji}G \) and \( h_t^{ji}p_{ji}G \).

**Third edge.** Finally, \( s_jp_jG \) is null-homotopic via \( s_jk_t^{ji} \), where \( k_t^{ji} \) is the given null-homotopy of \( p_jG \). Let \( H_{t\epsilon_i+\epsilon_k+\epsilon_j} : Q \to \mathbb{R}^m \) be this null-homotopy.

**Late face.** We first observe that \( H_{t\epsilon_i+\epsilon_k+\epsilon_j} \) followed by \( H_{t\epsilon_i+\epsilon_k+\epsilon_j} \), followed through homotopies of \( G_1 \) with values in \( \mathbb{R}^m \setminus \Delta^{k=j} \). Indeed, the former (resp. the latter) is homotopic to a null-homotopy of \( G_1 \) within \( F_1 \cup \{x_j\} \) (resp. within \( F_1 \cup \{x_k\} \)), using a deformation retraction of \( F_{ij} \) (resp. \( F_{ik} \)) onto \( F_1 \). But \( F_1 \cup \{x_j, x_k\} \) is a Euclidean space disjoint from \( \Delta^{k=j} \).

**Early face.** It remains to check that \( H_{t\epsilon_i} \), followed by \( H_{t\epsilon_i+\epsilon_k} \), followed by \( H_{t\epsilon_k} \), and \( s_jp_jG \) with values in \( \mathbb{R}^m \setminus \Delta^{k=i} \). Let us extend \( p_{ji} \) and \( p_{jk} \) to the projection \( P_j : (\mathbb{R}^m)^3 \setminus \Delta^{k=j} \to \mathbb{R}^m \).

Using the homotopy \( \varphi_t^{ji} \) between the identity map and \( s_jp_j \), given by a deformation retraction of \( (\mathbb{R}^m)^3 \setminus \Delta^{k=j} \) onto \( F_{ij} \), we find that \( H_{t\epsilon_i} \), followed by \( f_t^{ji} \), is homotopic to \( \varphi_t^{ji}G \) through homotopies between \( G \) and \( s_jp_jG \). Finally, \( \varphi_t^{ji} \), followed by \( g_t^{ji}p_j \) and \( h_t^{ji}p_j \) and \( g_t^{ji}p_j \) and \( h_t^{ji}p_j \), is a self-homotopy of the identity map. It is null-homotopic since when post-composed with the homotopy equivalence \( P_j \), it equals identically \( P_j \) for each value of \( t \in [0, 6] \).

**Gauss style description of \( \mu(f) \).** Consider a polyhedron \( X_1 \sqcup X_2 \sqcup X_3 \) and let \( Q = X_1 \times X_2 \times X_3 \). Given an h-Brunnian link map \( f : X_1 \times X_2 \times X_3 \to \mathbb{R}^m \), Lemma 2.4 yields a map \( F : Q \times \partial I^3 \to \mathbb{R}^m_\Delta \simeq S^{2m-1} \) such that \( F_{0,0,0} \) is the restriction of \( f_\Delta \) and \( F_{1,1,1} \) is constant. Now \((Q \times S^2)/(Q \times pt) \) is homeomorphic to \((Q \sqcup pt) \ast S^1 \), so we have precisely a well-defined map \( \varphi : (Q \sqcup pt) \ast S^1 \to S^{2m-1} \). Writing \( \xi \) for a generator of \( H^{2m-1}(S^{2m-1}) \), we denote the image of \( \varphi^*(\xi) \) under the suspension isomorphism \( H^{2m-1}((Q \sqcup pt) \ast S^1) \simeq H^{2m-3}(Q) \) by \( \mu^*(f) \in H^{2m-3}(X_1 \times X_2 \times X_3) \).

**Theorem 2.5.** \( \mu^*(f) \) is well-defined up to link homotopy and equals \( 2\mu(f) \) when the latter is defined.

**Proof.** Let \( f, g : X_1 \sqcup X_2 \sqcup X_3 \to \mathbb{R}^m \) be link homotopic h-Brunnian link maps, and let \( F_T, G_T : X_1 \times X_2 \times X_3 \to \mathbb{R}^m_\Delta \) be hexagonal \( \partial I^3 \)-homotopies with \( F_{0,0,0} = f, G_{0,0,0} = g \) and \( F_{1,1,1} = G_{1,1,1} = \) a constant map. We need to show that \( F \) and \( G \) are homotopic as maps \( X_1 \times X_2 \times X_3 \times \partial I^3 \to \mathbb{R}^m_\Delta \).

Let \( R \) be the restriction to \( \partial I^3 \) of a cyclic permutation of the coordinate axes in \( \mathbb{R}^3 \). Since \( R \) is isotopic to the identity, \( F_T \) is homotopic to \( F_{R(T)} \) with values in \( \mathbb{R}^m_\Delta \). Let \( H_T : X_1 \times X_2 \times X_3 \times \partial I^3 \to (\mathbb{R}^m)^3 \) be a generic smooth homotopy between
Similarly to the proof of Theorem 2.2, and using the polyhedral Pontryagin-Thom construction [BRS], we can amend \( H_T \) so that its restriction to \( X_1 \times X_2 \times X_3 \times \partial I^3 \) has values in \( \tilde{\mathbb{R}}_\Delta^m \).

The equation \( \mu^*(f) = 2\mu(f) \) now follows from Lemma 2.3. \( \square \)

**Figure 2: Hexagonal \( \partial I^3 \)-homotopy**

**From link maps to ornaments.** The new algebraic definition of \( \mu \) contained essentially in the proof of Lemma 2.4 admits a purely geometric interpretation. Given an h-Brunnian a link map \( f : 3S^{2k-1} \to \mathbb{R}^3 \), we shall construct geometrically an ornament \( J(f) : 3S^{2k+1} \to \mathbb{R}^{3k+2} \) such that \( 2\mu(f) = \tilde{\mu}(J(f)) \). Let us put \( k = 1 \) for simplicity of notation.

A circle of null-homotopies of \( S^1 \) in \( S^3 \) is a map \( S^1 \ast S^1 \to S^3 \ast S^1 \), i.e. \( S^3 \to S^5 \). Prolonging all null-homotopies slightly in time (by bringing their final point to the basepoint say), we also get a map \( S^3 \to S^5 \setminus N(S^1) \), where \( N(S^1) \) is a small regular neighborhood of the distinguished \( S^1 \). Thus a triple of circles of null-homotopies for the components of \( f : S^1 \sqcup S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3 \subset S^3 \) yields a map \( S^3 \sqcup S^3 \sqcup S^3 \to S^5 \setminus N(S^1) \subset \mathbb{R}^5 \). We shall construct just such a triple, or, equivalently, a circle of homotopies from \( f \) to a trivial link map \( t \), constant on each component.

A circle of homotopies between \( f \) and \( t \) is conveniently parametrized by the boundary of a cube \( I^3 = \{(t_1,t_2,t_3) \mid 0 \leq t_i \leq 1\} \), with \( f \) corresponding to the vertex \( 0 = (0,0,0) \) and \( g \) to \( \vec{e}_1 + \vec{e}_2 + \vec{e}_3 = (1,1,1) \), where \( \vec{e}_i \) are the coordinate unit vectors in the parameter space. We shall thus write the homotopy as \( H_T \) with \( T \in \partial I \). Let also \( \vec{v}_i \) be the coordinate unit vectors in the “physical” space \( \mathbb{R}^3 \), and
let $B_i$ be the balls of radius $1/3$ centered at (the endpoints of) $v_i$. We may assume that the image of $f$ is contained in the ball $B_0$ of radius $1/3$ centered at the origin, and that $t$ sends the $i$th component to the point $v_i$. Also let $B_{ij}, B_{ijk}$ be the convex hulls of $B_i$ and $B_j$, resp. $B_i, B_j$ and $B_k$.

First edge. The homotopy along each edge $t\vec{v}_i$, $t \in I$, is a link homotopy carrying the $j$th component into $B_j$ and the $k$th component into $B_k$, where $\{i,j,k\} = \{1,2,3\}$. Its restriction to the $j$th and $k$th components is provided by the hypothesis (that $f$ is h-Brunnian) and it can be extended to the $i$th component by dragging it past the movement of the other components, which is possible since it never gets close to their double point loci by general position (this also works in higher dimensions). We may assume without loss of generality that the homotopy first "untangles" the $j$th and $k$th components within $B_0$ and then brings them into their respective balls, so that the $j$th component always stays within $B_{0i}$ and the $k$th within $B_{0k}$. Also without loss of generality, the final map of the homotopy is disjoint from the points $\vec{v}_j$ and $\vec{v}_k$.

Second edge. The homotopy along each edge $\vec{v}_i + t\vec{v}_k$ first shrinks the $j$th component to the point $\vec{v}_j$ by a radial null-homotopy and then brings the $i$th and $k$th components back to their original position via the restriction of $H_{(1-t)\vec{v}_i}$. Thus the $j$th component may arbitrarily intersect the $i$th in this homotopy, but remains disjoint from the $k$th since they stay within disjoint convex sets $B_j, B_{0k}$. Also the $i$th component is disjoint from the $k$th since they are immobile in the first stage and move by a link homotopy thereafter.

Third edge. The homotopy along each edge $\vec{v}_i + \vec{v}_k + t\vec{v}_j$ is a link homotopy keeping the $j$th component fixed (it has been shrunk to $\vec{v}_j$ already) and shrinking the $i$th to $\vec{v}_i$ and the $k$th to $\vec{v}_k$ within $B_{0ik}$. It is provided by the hypothesis.

Late face. Now the combination of the latter two homotopies is fiberwise linearly homotopic (through homotopies between $H_{\vec{v}_i}$ and $H_{\vec{v}_i + \vec{v}_k + \vec{v}_j}$) to the linear null-homotopy, shrinking each $l$th component to $v_l$. Note that the $j$th and $k$th component remain disjoint in this fiberwise homotopy since they are always contained in disjoint convex sets $B_j, B_{0ik}$. Along with the symmetric homotopy (with $j$ and $k$ interchanged) this defines our homotopy along each face $\vec{v}_i + t\vec{v}_k + s\vec{v}_j$.

Early face. Finally, to define it along each face $t\vec{v}_i + s\vec{v}_k$, we similarly note that the combination of $H_{t\vec{v}_i}$ and $H_{\vec{v}_i + t\vec{v}_k}$ is fiberwise homotopic (through homotopies between $H_0 = f$ and $H_{\vec{v}_i + s\vec{v}_k}$) to the homotopy keeping the $i$th and $k$th components fixed and shrinking the $j$th radially to $\vec{v}_j$. Such a fiberwise homotopy can be defined on the $i$th and $k$th components by $H_{t\vec{v}_i + s\vec{v}_k} = H_{(t-s)\vec{v}_i}$ for $t \geq s$. Since the $i$th and $k$th components remain disjoint under $H_{t\vec{v}_i}$, so they do under this fiberwise homotopy. The homotopy can be extended to the entire face by defining is symmetrically (with $i$ and $k$ interchanged) on the other side from the diagonal.

The procedure $J$ is not entirely canonical, as it slightly depends on general position arguments. However, we have

**Theorem 2.6.** $2\mu(f) = \tilde{\mu}(J(f))$, and $J$ descends to a well-defined map

$$\{\text{link maps } S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to \mathbb{R}^{3k}\}/\text{link homotopy}$$

$$\downarrow$$

$$\{\text{ornaments } S^{2k+1} \sqcup S^{2k+1} \sqcup S^{2k+1} \to \mathbb{R}^{3k+2}\}/\text{ornament homotopy}.$$
Proof. Since $J(f)$ is essentially a hexagonal $\partial I^3$-homotopy between $f$ and $t$, the first assertion follows from Theorem 2.5. Now the second assertion follows from Theorem 1.1. □

Combining $J$ with $\gamma$, we get

**Corollary 2.7.** Let $\text{TLM}_k$ be the group of $h$-Brunnian link maps $S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \rightarrow \mathbb{R}^{3k}$ up to link homotopy. There is a geometrically defined suspension homomorphism $\sigma: \text{TLM}_k \rightarrow \text{TLM}_{k+1}$ such that $\mu \sigma = 2 \mu$.

Another homomorphism $\text{TLM}_k \rightarrow \text{TLM}_{k+1}$ is a sequence of three Nezhinskij suspensions. By [Ko7; Theorem 7.2], it commutes with the $\mu$-invariant.

It would be interesting to know if $J$ can be decomposed into two operations, each raising dimension by one. Note that the $\hat{\beta}$-invariant from §3 below vanishes on the $\pm \neq 0$ map given by two null-homotopies of the Borromean rings $B$ along an appropriate choice of two three-edge paths on the cube.

### 3. $\beta$-Invariant of Link Maps ($2k - 2$ in $3k - 2$)

Given a link map $f: S^{2k-2} \sqcup S^{2k-2} \rightarrow \mathbb{R}^{3k-2}$, one may study obstructions to embedding a component by a link homotopy. One way to get such an obstruction is to double the component and consider an obstruction to removing the intersection between the two copies while keeping them disjoint from the other original component.

**$\hat{\beta}$-invariant of $\pm \neq 0$ maps.** Let $f = f_+ \sqcup f_- \sqcup f_0: S^{2k-2} \sqcup S^{2k-2} \sqcup S^{2k-2} \rightarrow \mathbb{R}^{3k-2}$ be a $\pm \neq 0$ map, i.e. a map such that the image of $f_0$ is disjoint from those of $f_+$ and $f_-$. Let $h_+ \sqcup h_- \sqcup h_0: 3D^{2k-1} \rightarrow \mathbb{R}^{3k-2}$ be a generic homotopy from $f$ to a map $t$ sending the components to 3 distinct points. We set $\hat{\beta}(f)$ to be the algebraic number of $1 = 2 = 3$ points between $h_+, h_-$ and $f_0$. This is clearly an invariant of $\pm \neq 0$ homotopy, i.e. homotopy through $\pm \neq 0$ maps.

**Theorem 3.1.** $\hat{\beta}(f) = \deg(\varphi)$, where $\varphi$ denotes the composition

$$S^{2k-2} \times S^{2k-2} \times S^{2k-2} \xrightarrow{f_+ \times f_- \times f_0} (\mathbb{R}^{3k-2})^3 \setminus (\Delta^+ = 0 \cup \Delta^- = 0) \xrightarrow{\hat{\beta}} S^{3k-3} \times S^{3k-3}.$$ 

Proof. We will relate $\hat{\beta}(f)$ to $\deg(\varphi)$ through a sequence of auxiliary invariants (which might be useful in their own right). By definition, $\hat{\beta}(f)$ equals the linking number of $f_0$ and $\partial(h_+ \cap h_-) = (f_+ \cap h_-) \cup (f_- \cap h_+)$. This linking number equals the number of $1 = 2 = 3$ points between $f_+, h_-$ and $h_0$ plus the number of $1 = 2 = 3$ points between $f_-, h_+$ and $h_0$. This is the same as $\tilde{\mu}(f)$, where the ornament $\hat{f}: 3S^{2k-1} \rightarrow \mathbb{R}^{3k-2} \times [-2, 2]$ is the track of the following self-homotopy $\hat{f}(t)$ of $t = t_+ \sqcup t_- \sqcup t_0$:

$$
t \xleftarrow{h_- \sqcup h_0} f_- \sqcup t_- \sqcup f_0 \xleftarrow{h_+} f \xrightarrow{h_-} f_+ \sqcup t_- \sqcup f_0 \xleftarrow{h_+ \sqcup h_0} t.
$$

Indeed, writing $\hat{f} = \hat{f}_+ \sqcup \hat{f}_- \sqcup \hat{f}_0$, we can extend $\hat{f}$ to a generic map $F: 3D^{2k} \rightarrow \mathbb{R}^{3k-2} \times [-2, 2] \times [0, \infty)$ such that $F^{-1}(\mathbb{R}^{3k-2} \times [-2, 2] \times [1, \infty))$ consists of two 2k-balls $D_+, \ D_-$ with $D_+ \cap \partial(3D^{2k}) = \hat{f}_-^{-1}(\mathbb{R}^{3k-2} \times [\pm 1, \pm 2])$ and such that $F^{-1}(\mathbb{R}^{3k-2} \times [0, \pm 2] \times [0, 1])$ is sent by $F$ composed with the projection onto $\mathbb{R}^{3k-2}$.
onto the image of $f_+ \sqcup h_+ \sqcup h_0$. Then the $1 = 2 = 3$ points of $F$ project onto those between $f_\pm$, $h_+$ and $h_0$.

Now $\hat{\mu}(f)$ equals the degree of the composition

$$S^{2k-1} \times S^{2k-1} \times S^{2k-1} \xrightarrow{\hat{\mu}} \mathbb{R}_\Delta^{3k-1} \simeq S^{6k-3}.$$ 

This factors through the double suspension of the composition

$$\Sigma(S^{2k-2} \times S^{2k-2} \times S^{2k-2}) \xrightarrow{\hat{\mu}} \mathbb{R}_\Delta^{3k-2} \simeq S^{6k-5}.$$ 

Thus $\hat{\beta}(f)$ equals the degree of the latter composition $\Phi$. Let us take the homotopies $h_+$ and $h_-$ to be given by translations (in distinct directions) sufficiently far apart from a cube $I^{3k-2}$ containing the image of $f$, followed by radial null-homotopies. Then $\Phi$ is the following self-homotopy of a constant map $0$:

$$0 \xleftarrow{\rho_+} \varphi_+ \xleftarrow{\tau_+} \varphi \xrightarrow{\tau} \varphi_- \xrightarrow{\rho_-} 0,$$

where $\tau_\pm$ is given by a translation of $(I^{3k-2})_+ \setminus \Delta^{\pm=0}$ within $(\mathbb{R}^{3k-2})_+ \setminus \Delta^{\pm=0}$ along the $\mp$-coordinate, until it is disjoint from $\Delta^{\mp=0}$; and $\rho_\pm$ is given by a null-homotopy of the translated $(I^{3k-2})_+ \setminus \Delta^{\pm=0}$ within $(\mathbb{R}^{3k-2})_+ \setminus (\Delta^{\mp=0} \cup \Delta^{\mp=\mp})$. It follows that $\Phi$ is homotopic to the suspension of $\varphi$ composed with the degree one map $\Sigma(S^{3k-3} \times S^{3k-3}) \to S^{6k-5}$ given by a self-homotopy of a constant map $S^{3k-3} \times S^{3k-3} \to S^{6k-5}$, whose image is contained in the following subsets of $S^{6k-5} = S^{3k-3} \ast S^{3k-3}$:

$$\ast \xleftarrow{D^{3k-3} \times \ast} S^{3k-3} \times \ast \xleftarrow{S^{3k-3} \times D^{3k-2}} S^{3k-3} \times S^{3k-3} \xrightarrow{D^{3k-2} \times S^{3k-3}} \ast \times S^{3k-3} \ast \xrightarrow{D^{3k-3}} \ast.$$ 

Thus $\deg(\varphi) = \deg(\Phi)$. □

**$\beta$-invariant of link maps.** Let $g: S^{2k-2}_0 \sqcup S^{2k-2}_0 \to \mathbb{R}^{3k-2}$ be a generic link map. Consider a generic null-homotopy $h: D^{2k-1}_* \simeq \mathbb{R}^{3k-2}$ of the first component. If $k$ is even, $\beta(g) \in \mathbb{Z}/2$ is defined to be the parity of the number of double points between $g(S^{2k-2}_0)$ and the double point manifold $\Delta(h)$. If $k$ is odd, the central symmetry of $\mathbb{R}^{3k-2}$ is orientation reversing, hence the factor exchanging involution of $\mathbb{R}^{3k-2} \times \mathbb{R}^{3k-2}$ reverses co-orientation of the diagonal, so the factor exchanging involution of $D^{2k-1}_* \times D^{2k-1}_*$ reverses the co-orientation of $\Delta(h) := \{(x, y) \mid h(x) = h(y), x \neq y\}$ and therefore preserves its orientation. Thus $\Delta(h)$, which is the quotient of $\Delta(h)$ by the involution, is orientable, and we may set $\beta(g) \in \mathbb{Z}$ to be the algebraic number of double points between $g(S^{2k-2}_0)$ and the double point manifold $\Delta(h)$. This is a special case of Koschorke’s definition, see [Ko2; proof of Theorem 4.8]. (We note that this definition of $\beta$ also applies to link maps of any two oriented $(2k - 2)$-manifolds in $\mathbb{R}^{3k-2}$; the proof of invariance is straightforward.)

$\beta$ and $\hat{\beta}$. Consider now the projection $\pi: S^{2k-2}_+ \sqcup S^{2k-2}_- \to S^{2k-2}_*$ and let $f = g \pi$. If $k$ is odd, $\hat{\beta}(f)$ coincides by definition with $2\beta(g)$. Unfortunately, $\beta$ is identically zero for odd $k$ — in fact, it is only nonzero for $k = 2, 4, 8$ [Ko2; Theorem 4.8]. (The extension of $\beta$ to manifold link maps may be nonzero and is similarly related with the extension of $\hat{\beta}$ for odd $k$.) If $k$ is even, $\beta(f)$ is identically zero since it counts every double point between $g(S^{2k-2}_0)$ and $\Delta(h)$ twice — with opposite signs.
Gauss style description of $\beta$. Let $X_1 \sqcup X_2$ be a polyhedron and let $f = f_1 \sqcup f_2: X_1 \sqcup X_2 \to \mathbb{R}^{3k-2}$ be a link map. The composition

$$
\varphi_f: X_1 \times X_2 \times X_2 \xrightarrow{f_1 \times f_2 \times f_2} (\mathbb{R}^{3k-2})^3 \setminus (\Delta^{1=2} \cup \Delta^{1=3}) \xrightarrow{\sim} S^{3k-3} \times S^{3k-3}
$$

is equivariant with respect to the involution exchanging the second and third factors of $X_1 \times X_2 \times X_2$ and the factor exchanging involution of $S^{3k-3} \times S^{3k-3}$. The latter is orientation-preserving iff $k$ is odd, so $H_{S/2}^{6k-6}(S^{3k-3} \times S^{3k-3}; \mathbb{Z}_r \circ (k-1)) \simeq \mathbb{Z}$. Let $\xi^k$ be a generator of this group, and let us denote $\varphi_f^*(\xi^k)$ by

$$
\beta^*(f) \in H_{S/2}^{6k-6}(X_1 \times X_2 \times X_2; \mathbb{Z}_r \circ (k-1)).
$$

Theorem 3.2. If $X_1 = X_2 = S^{2k-2}$, then $\beta^*(f) = \beta(f)$.

Proof. The factor exchanging involution of $S^{2k-2} \times S^{2k-2}$ is orientation-preserving, so $H_{S/2}^{6k-6}(S^{2k-2} \times S^{2k-2} \times S^{2k-2}; \mathbb{Z}_r \circ (k-1)) \simeq \mathbb{Z}$ when $k$ is odd and $\mathbb{Z}/2$ when $k$ is even. An examination of the proof of Theorem 3.1 shows that it carries over to the equivariant setting, thus establishing the assertion. □

Corollary 3.3. $\beta(f) = 0$ if $f: S^{2k-2}_* \sqcup S^{2k-2}_0 \to \mathbb{R}^{3k-2}$ PL embeds $S^{2k-2}_0$.

If $f$ PL embeds $S^{2k-2}_*$, this embedding is PL isotopic to the boundary of an embedded $D^{2k-1}_*$, and it follows from the definition that $\beta(f) = 0$.

A standard proof of Corollary 3.3 is that $\alpha$ is symmetric with respect to interchange of the components, hence so is $\beta = h(\alpha)$. The point of the following proof is that it does not use homotopy theory.

Proof. The embedding of $S^{2k-2}_0$ is PL isotopic to the standard embedding. Hence $f$ is link homotopic to the composition of some map $S^{2k-2}_* \to S^{k-1}_* \sqcup S^{k-1}_{0*} \to \mathbb{R}^{3k-2}$. Hence the map $\varphi_f: S^{2k-2}_* \times S^{2k-2}_* \times S^{2k-2}_0 \to S^{3k-3} \times S^{3k-3}$ factors up to homotopy through $S^{k-1}_* \times S^{k-1}_* \times S^{2k-2}_0$ and so has zero equivariant degree. □

Sphere instead of the torus. Let $\vec{v}$ be a nonzero vector in $\mathbb{R}^m$ and let $\Pi_{\vec{v}}^\alpha$, resp. $\Pi_{\vec{v}}^\alpha_\ast$ be the subsets of $(\mathbb{R}^m)^3$ consisting of all triples $(x, x+\alpha \vec{v}, x)$, resp. $(x, x, x+\alpha \vec{v})$ with $\alpha \geq 0$. Then $(\mathbb{R}^m)^3 \setminus (\Pi_{\vec{v}}^{\alpha} \cup \Pi_{\vec{v}}^{\alpha})$ collapses onto $S^{2m-2}$ equivariantly with respect to the involution exchanging the second and third factors of $(\mathbb{R}^m)^3$ and the join of the identity involution on $S^{m-1}$ and the antipodal involution on $S^{m-3}$. Moreover, the preimage of $S^{m-1}$ under the collapse is $\Delta^{2=3}$, and the collapse factors up to equivariant homotopy through the collapse of $(\mathbb{R}^m)^3 \setminus (\Delta^{1=2} \cup \Delta^{1=3})$ onto $S^{m-1} \times S^{m-1}$ and a degree one map $S^{m-1} \times S^{m-1} \to S^{2m-2}$. (Note that $\Delta^{2=3}$ cannot be the preimage of the diagonal of the torus under the latter collapse.)

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