On the Expected Complexity of Random Convex Hulls

Sariel Har-Peled†

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Abstract

In this paper we present several results on the expected complexity of a convex hull of \(n\) points chosen uniformly and independently from a convex shape.

(i) We show that the expected number of vertices of the convex hull of \(n\) points, chosen uniformly and independently from a disk is \(O(n^{1/3})\), and \(O(k \log n)\) for the case a convex polygon with \(k\) sides. Those results are well known (see [RS63, Ray70, PS85]), but we believe that the elementary proof given here are simpler and more intuitive.

(ii) Let \(D\) be a set of directions in the plane, we define a generalized notion of convexity induced by \(D\), which extends both rectilinear convexity and standard convexity.

We prove that the expected complexity of the \(D\)-convex hull of a set of \(n\) points, chosen uniformly and independently from a disk, is \(O\left(n^{1/3} + \sqrt{n\alpha(D)}\right)\), where \(\alpha(D)\) is the largest angle between two consecutive vectors in \(D\). This result extends the known bounds for the cases of rectilinear and standard convexity.

(iii) Let \(B\) be an axis parallel hypercube in \(\mathbb{R}^d\). We prove that the expected number of points on the boundary of the quadrant hull of a set \(S\) of \(n\) points, chosen uniformly and independently from \(B\) is \(O(\log^{d-1} n)\). Quadrant hull of a set of points is an extension of rectilinear convexity to higher dimensions. In particular, this number is larger than the number of maxima in \(S\), and is also larger than the number of points of \(S\) that are vertices of the convex hull of \(S\).

Those bounds are known [BKST78], but we believe the new proof is simpler.

1 Introduction

Let \(C\) be a fixed compact convex shape, and let \(X_n\) be a random sample of \(n\) points chosen uniformly and independently from \(C\). Let \(Z_n\) denote the number of vertices of the convex hull of \(X_n\). Rényi and Sulanke [RS63] showed that \(E[Z_n] = O(k \log n)\), when \(C\) is a convex polygon with \(k\) vertices in the plane. Raymond [Ray70] showed that expected number of facets of the convex hull is \(O(n^{(d-1)/(d+1)})\), where \(C\) is a ball in \(\mathbb{R}^d\), so \(E[Z_n] = O(n^{1/3})\) when

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†Department of Computer Science; University of Illinois; 201 N. Goodwin Avenue; Urbana, IL, 61801, USA; sariel@uiuc.edu; http://www.uiuc.edu/~sariel/.
$C$ is a disk in the plane. Raynaud [Ray70] showed that the expected number of facets of $\text{CH}(X_n) = \text{ConvexHull}(X_n)$ is $O((\log(n))^{(d-1)/2})$, where the points are chosen from $\mathbb{R}^d$ by a $d$-dimensional normal distribution. See [WW93] for a survey of related results.

All these bounds are essentially derived by computing or estimating integrals that quantify the probability of two specific points of $X_n$ to form an edge of the convex hull (multiplying this probability by $\binom{n}{2}$ gives $E[Z_n]$). Those integrals are fairly complicated to analyze, and the resulting proofs are rather long, counter-intuitive and not elementary.

Efron [Efr65] showed that instead of arguing about the expected number of vertices directly, one can argue about the expected area/volume of the convex hull, and this in turn implies a bound on the expected number of vertices of the convex hull. In this paper, we present a new argument on the expected area/volume of the convex hull (this method can be interpreted as a discrete approximation to the integral methods). The argument goes as follows: Decompose $C$ the into smaller shapes (called tiles). Using the topology of the tiling and the underlining type of convexity, we argue about the expected number of tiles that are exposed by the random convex hull, where a tile is exposed if it does not lie completely in the interior of the random convex hull. Resulting in a lower bound on the area/volume of the random convex hull. We apply this technique to the standard case, and also for more exotic types of convexity.

In Section 2, we give a rather simple and elementary proofs of the aforementioned bounds $E[Z_n] = O(n^{1/3})$ for $C$ a disk, and $E[Z_n] = O(k \log n)$ for $C$ a convex $k$-gon. We believe that these new elementary proofs are indeed simpler and more intuitive than the previous integral-based proofs.

The question on the expected complexity of the convex hull remains valid, even if we change our type of convexity. In Section 3 we define a generalized notion of convexity induced by $D$, a given set of directions. This extends both rectilinear convexity, and standard convexity. We prove that the expected complexity of the $D$-convex hull of a set of $n$ points, chosen uniformly and independently from a disk, is $O \left( n^{1/3} + \sqrt{n\alpha(D)} \right)$, where $\alpha(D)$ is the largest angle between two consecutive vectors in $D$. This result extends the known bounds for the cases of rectilinear and standard convexity.

Finally, in Section 4 we deal with another type convexity, which is an extension of the generalized convexity mentioned above for the higher dimensions, where the set of the directions is the standard orthonormal basis of $\mathbb{R}^d$. We prove that the expected number of points that lie on the boundary of the quadrant hull of $n$ points, chosen uniformly and independently from the axis-parallel unit hypercube in $\mathbb{R}^d$, is $O(\log^{d-1} n)$. This readily imply $O(\log^{d-1} n)$ bound on the expected number of maxima and the expected number of vertices of the convex hull of such a point set. Those bounds are known [BKST78], but we believe the new proof is simpler and more intuitive.

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1Preparata and Shamos [PSS85] pp. 152 comment on the older proof for the case of a disk: “Because the circle has no corners, the expected number of hull vertices is comparatively high, although we know of no elementary explanation of the $n^{1/3}$ phenomenon in the planar case.” It is the author’s belief that the proof given here remedies this situation.


2 On the Complexity of the Convex Hull of a Random Point Set

In this section, we show that the expected number of vertices of the convex hull of \( n \) points, chosen uniformly and independently from a disk, is \( O(n^{1/3}) \). Applying the same technique to a convex polygon with \( k \) sides, we prove that the expected number of vertices of the convex hull is \( O(k \log n)^2 \). The following lemma, shows that the larger the expected area outside the random convex hull, the larger is the expected number of vertices of the convex hull.

**Lemma 2.1** Let \( C \) be a bounded convex set in the plane, such that the expected area of the convex hull of \( n \) points, chosen uniformly and independently from \( C \), is at least \((1 - f(n)) \text{Area}(C)\), where \( 1 \geq f(n) \geq 0 \), for \( n \geq 0 \). Then the expected number of vertices of the convex hull is \( \leq nf(n/2) \).

**Proof:** Let \( N \) be a random sample of \( n \) points, chosen uniformly and independently from \( C \). Let \( N_1 \) (resp. \( N_2 \)) denote the set of the first (resp. last) \( n/2 \) points of \( N \). Let \( V_1 \) (resp. \( V_2 \)) denote the number of vertices of \( H = \text{CH}(N_1 \cup N_2) \) that belong to \( N_1 \) (resp. \( N_2 \)), where \( \text{CH}(N_1 \cup N_2) = \text{ConvexHull}(N_1 \cup N_2) \).

Clearly, the expected number of vertices of \( C \) is \( E[V_1] + E[V_2] \). On the other hand,

\[
E[V_1 \mid N_2] \leq \frac{n}{2} \left( \frac{\text{Area}(C) - \text{Area}(\text{CH}(N_2))}{\text{Area}(C)} \right),
\]

since \( V_1 \) is bounded by the expected number of points of \( N_1 \) falling outside \( \text{CH}(N_2) \).

We have

\[
E[V_1] = E_{N_2} E[V_1 \mid N_2] \leq E \left[ \frac{n}{2} \left( \frac{\text{Area}(C) - \text{Area}(\text{CH}(N_2))}{\text{Area}(C)} \right) \right]
\]

\[
\leq \frac{n}{2} f(n/2),
\]

since \( E[X] = E_Y E[X \mid Y] \) for any two random variables \( X, Y \). Thus, the expected number of vertices of \( H \) is \( E[V_1] + E[V_2] \leq nf(n/2) \).

**Remark 2.2** Lemma 2.1 is known as Efron’s Theorem. See [Efr65].

**Theorem 2.3** The expected number of vertices of the convex hull of \( n \) points, chosen uniformly and independently from the unit disk, is \( O(n^{1/3}) \).

**Proof:** We claim that the expected area of the convex hull of \( n \) points, chosen uniformly and independently from the unit disk, is at least \( \pi - O(n^{-2/3}) \).

Indeed, let \( D \) denote the unit disk, and assume without loss of generality, that \( n = m^3 \), where \( m \) is a positive integer. Partition \( D \) into \( m \) sectors, \( S_1, \ldots, S_m \), by placing \( m \) equally spaced points on the boundary of \( D \) and connecting them to the origin. Let \( D_1, \ldots, D_m \)

\[\text{As already noted, these results are well known (RS63, Ray70, PS85), but we believe that the elementary proofs given here are simpler and more intuitive.}\]
denote the $m^2$ disks centered at the origin, such that (i) $D_1 = D$, and (ii) $\text{Area}(D_i) = \pi/m^2$, for $i = 2, \ldots, m^2$. Let $r_i$ denote the radius of $D_i$, for $i = 1, \ldots, m^2$.

Let $S_{i,j} = (D_i \setminus D_{i+1}) \cap S_j$, and $S_m, j = D_m \cap S_j$, for $i = 1, \ldots, m^2 - 1$, $j = 1, \ldots, m$. The set $S_{i,j}$ is called the $i$-th tile of the sector $S_j$, and its area is $\pi/n$, for $i = 1, \ldots, m^2$, $j = 1, \ldots, m$.

Let $N$ be a random sample of $n$ points chosen uniformly and independently from $D$. Let $X_j$ denote the first index $i$ such that $N \cap S_{i,j} \neq \emptyset$, for $j = 1, \ldots, m$. For a fixed $j \in \{1, \ldots, m\}$, the probability that $X_j = k$ is upper-bounded by the probability that the tiles $S_{1,j}, \ldots, S_{(k-1),j}$ do not contain any point of $N$; namely, by $(1 - k^{-1})^n$. Thus, $P[X_j = k] \leq (1 - k^{-1})^n \leq e^{-(k-1)}$, since $1 - x \leq e^{-x}$, for $x \geq 0$. Thus,

$$E[X_j] = \sum_{k=1}^{m^2} k P[X_j = k] \leq \sum_{k=1}^{m^2} k e^{-(k-1)} = O(1),$$

for $j = 1, \ldots, m$.

Let $K_o$ denote the convex hull of $N \cup \{o\}$, where $o$ is the origin. The tile $S_{i,j}$ is exposed by a set $K$, if $S_{i,j} \setminus K \neq \emptyset$. We claim that at most $X_{j-1} + X_{j+1} + O(1)$ tiles are exposed by $K_o$ in the sector $S_j$, for $j = 1, \ldots, m$ (where we put $X_0 = X_m$, $X_{m+1} = X_1$).

Indeed, let $w = w(N, j) = \max(X_{j-1}, X_{j+1})$, and let $p, q$ be the two points in $S_{j-1}, S_{j+1}$, respectively, such that the number of sets exposed by the triangle $T = \triangle opq$, in the sector $S_j$, is maximal. Both $p$ and $q$ lie on $\partial D_{w+1}$ and on the external radii bounding $S_{j-1}$ and $S_{j+1}$, as shown in Figure 1. Clearly, any tile which is exposed in $S_j$ by $K_o$ is also exposed by $T$. Let $s$ denote the segment connecting the middle of the base of $T$ to its closest point on $\partial D_w$. The number of tiles in $S_j$ exposed by $T$ is bounded by $\max(X_{j-1}, X_{j+1})$, plus the number of tiles intersecting the segment $s$. The length of $s$ is

$$|op| - |op| \cos \left( \frac{3}{2} \cdot \frac{2\pi}{m} \right) \leq 1 - \cos \left( \frac{3}{2} \cdot \frac{2\pi}{m} \right) \leq \frac{1}{2} \left( \frac{3\pi}{m} \right)^2 = \frac{4.5\pi^2}{m^2},$$

since $\cos(x) \geq 1 - x^2/2$, for $x \geq 0$.

On the other hand, $r_{i+1} - r_i \geq r_i - r_{i-1} \geq 1/(2m^2)$, for $i = 2, \ldots, m^2$. Thus, the segment $s$ intersects at most $\lceil |s|/(1/(2m^2)) \rceil = \lceil 9\pi^2 \rceil = 89$ tiles, and we have that the number of

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Figure 1: Illustrating the proof that bounds the number of tiles exposed by $T$ inside $S_j$.
tiles exposed in the sector $S_i$ by $K_o$ is at most \( \max(X_{j-1}, X_{j+1}) + 89 \leq X_{j-1} + X_{j+1} + 89 \), for \( j = 1, \ldots, m \).

Thus, the expected number of tiles exposed by $K_o$ is at most

\[
E \left[ \sum_{i=1}^{m} (X_{j-1} + X_{j+1} + 89) \right] = O(m).
\]

The area of $K = CH(N)$ is bounded from below by the area of tiles which are not exposed by $K$. The probability that $K \subsetneq K_o$ (namely, the origin is not inside $K$, or, equivalently, all points of $N$ lie in some semidisk) is at most $2\pi/2n$, as easily verified. Hence,

\[
E[Area(K)] \geq E[Area(C)] - P \left[ C \neq K \right] \pi = \pi - O(m) \frac{\pi}{n} - \frac{2\pi}{2n} \pi = \pi - O \left( n^{-2/3} \right).
\]

The assertion of the theorem now follows from Lemma 2.1. 

**Lemma 2.4** The expected number of vertices of the convex hull of $n$ points, chosen uniformly and independently from the unit square, is $O(\log n)$.

**Proof:** We claim that the expected area of the convex hull of $n$ points, chosen uniformly and independently from the unit square, is at least $1 - O(\log(n)/n)$.

Let $S$ denote the unit square. Partition $S$ into $n$ rows and $n$ columns, such that $S$ is partitioned into $n^2$ identical squares. Let $S_{i,j} = [(i-1)/n, i/n] \times [(j-1)/n, j/n]$ denote the $j$-th square in the $i$-th column, for $1 \leq i, j \leq n$. Let $S_i = \cup_{j=1}^{n} S_{i,j}$ denote the $i$-th column of $S$, for $i = 1, \ldots, n$, and let $S(l, k) = \cup_{i=l}^{k} S_i$, for $1 \leq l \leq k \leq n$.

Let $N$ be a random sample of $n$ points chosen uniformly and independently from $S$. Let $X_j$ denote the first index $i$ such that $N \cap (\cup_{i=1}^{j-1} S_{i,j}) \neq \emptyset$, for $j = 2, \ldots, n-1$; namely, $X_j$ is the index of the first row in $S(1, j-1)$ that contains a point from $N$. Symmetrically,
let $X'_j$ be the index of the first row in $S(j + 1, n)$ that contains a point of $N$. Clearly, $E[X_j] = E[X'_{n-j+1}]$, for $j = 2, \ldots, n - 1$.

Let $Z_j$ denote the number of squares $S_{i,j}$ in the bottom of the $j$-th column that are exposed by $CH(N)$, for $j = 2, \ldots, n - 1$. Arguing as in the proof of Theorem 2.3, we have that $Z_j \leq \max(X_j, X'_j) \leq X_j + X'_j$. Thus, in order to bound $E[Z_j]$, we first bound $E[X_j]$ by covering the strips $S(1, j - 1), S(j + 1, n)$ by tiles of area $\geq 1/n$. In particular, let $h(l) = \lceil n/(l - 1) \rceil$, and let $R_j(m) = [0, (j - 1)/n] \times [h(n - j + 1)(m - 1)/n, h(j)m/n]$, and let $R'_j(m) = [(j + 1)/n, 1] \times [h(j)(m - 1)/n, h(j)m/n]$, for $j = 2, \ldots, n - 1$. See Figure 2.

Let $Y_j$ denote the minimal index $i$ such that $R_j(i) \cap N \neq \emptyset$. The area of $R_j(i)$ is at least $1/n$, for any $i$ and $j$. Arguing as in the proof of Theorem 2.3, it follows that $E[Y_j] = O(1)$. On the other hand, $E[X_j] \leq h(j)E[Y_j] = O(n/(j - 1))$. Symmetrically, $E[X'_j] = O(n/(n - j))$.

Thus, by applying the above argument to the four directions (top, bottom, left, right), we have that the expected number of squares $S_{i,j}$ exposed by $CH(N)$ is bounded by

$$4n - 4 + 4 \sum_{j=2}^{n-1} E[Z_j] < 4n + 4 \sum_{j=2}^{n-1} (E[X_j] + E[X'_j]) = 4n + 8 \sum_{j=2}^{n-1} O\left(\frac{n}{j-1}\right) = O(n \log n),$$

where $4n - 4$ is the number of squares adjacent to the boundary of $S$.

Since the area of each square is $1/n^2$, it follows that the expected area of $CH(N)$ is at least $1 - O(\log(n)/n)$.

By Lemma 2.1 the expected number of vertices of the convex hull is $O(\log n)$. ■

**Lemma 2.5** The expected number of vertices of the convex hull of $n$ points, chosen uniformly and independently from a triangle, is $O(\log n)$.

**Proof:** We claim that the expected area of the convex hull of $n$ points, chosen uniformly and independently from a triangle $T$, is at least $(1 - O(\log(n)/n))\text{Area}(T)$. We adapt the tiling used in Lemma 2.4 to a triangle. Namely, we partition $T$ into $n$ equal-area triangles, by segments emanating from a fixed vertex, each of which is then partitioned into $n$ equal-area trapezoids by segments parallel to the opposite side, such that each resulting trapezoid has area $1/n^2$. See Figure 3.

Notice that this tiling has identical topology to the tiling used in Lemma 2.4. Thus, the proof of Lemma 2.4 can be applied directly to this case, repeating the tiling process three times, once for each vertex of $T$. This readily implies the asserted bound. ■

**Theorem 2.6** The expected number of vertices of the convex hull of $n$ points, chosen uniformly and independently from a polygon $P$ having $k$ sides, is $O(k \log n)$.

**Proof:** We triangulate $P$ in an arbitrary manner into $k$ triangles $T_1, \ldots, T_k$. Let $N$ be a random sample of $n$ points, chosen uniformly and independently from $P$. Let $Y_i = |T_i \cap N|$, $N_i = T_i \cap N$, and $Z_i = |CH(N_i)|$, for $i = 1, \ldots, k$. Notice that the distribution of the points of $N_i$ inside $T_i$ is identical to the distribution of $Y_i$ points chosen uniformly and independently from $T_i$. In particular, $E[Z_i|Y_i] = O(\log Y_i)$, by Lemma 2.5 and $E[Z_i] = E[Y_iE[Z_i|Y_i]] = O(\log n)$, for $i = 1, \ldots, k$.

Thus, $E[|CH(N)|] \leq E\left[\sum_{i=1}^{k} |CH(N_i)|\right] \leq \sum_{i=1}^{k} E[Z_i] = O(k \log n)$. ■
3 On the Expected Complexity of a Generalized Convex Hull Inside a Disk

In this section, we derive a bound on the expected complexity on a generalized convex hull of a set of points, chosen uniformly and independently for the unit disk. The new bound matches the known bounds, for the case of standard convexity and maxima. The bound follows by extending the proof of Theorem 2.3.

We begin with some terminology and some initial observations, most of them taken or adapted from [MP97]. A set \( D \) of vectors in the plane is a set of directions, if the length of all the vectors in \( D \) is 1, and if \( v \in D \) then \( -v \in D \). Let \( D_{IR} \) denote the set of all possible directions. A set \( C \) is \( D \)-convex if the intersection of \( C \) with any line with a direction in \( D \) is connected. By definition, a set \( C \) is convex (in the standard sense), if and only if it is \( D_{IR} \)-convex.

For a set \( C \) in the plane, we denote by \( CH_D(C) \) the \( D \)-convex hull of \( C \); that is, the smallest \( D \)-convex set that contains \( C \). While this seems like a reasonable extension of the regular notion of convexity, its behavior is counterintuitive. For example, let \( D_Q \) denote the set of all rational directions (the slopes of the directions are rational numbers). Since \( D_Q \) is dense in \( D_{IR} \), one would expect that \( CH_{D_Q}(C) = CH_{D_{IR}}(C) = CH(C) \). However, if \( C \) is a set of points such that the slope of any line connecting a pair of points of \( C \) is irrational, then \( CH_{D_Q}(C) = C \). See [OSSW85, RW88, RW87] for further discussion of this type of convexity.

**Definition 3.1** Let \( f \) be a real function defined on a \( D \)-convex set \( C \). We say that \( f \) is \( D \)-convex if, for any \( x \in C \) and any \( v \in D \), the function \( g(t) = f(x + tv) \) is a convex function of the real variable \( t \). (The domain of \( g \) is an interval in \( IR \), as \( C \) is assumed to be \( D \)-convex.)

Clearly, any convex function, in the standard sense, defined over the whole plane satisfies this condition.

**Definition 3.2** Let \( C \subseteq IR^2 \). The set \( CH^D(C) \), called the functional \( D \)-convex hull of \( C \), is
Figure 4: (a) A set of directions $\mathcal{D}$, (b) the set of quadrants $\mathcal{Q}(\mathcal{D})$ induced by $\mathcal{D}$, and (c) the $\mathcal{D}$-CH of three points.

A set $C$ is functionally $\mathcal{D}$-convex if $C = \mathcal{CH}^\mathcal{D}(C)$.

**Definition 3.3** Let $\mathcal{D}$ be a set of directions. A pair of vectors $v_1, v_2 \in \mathcal{D}$, is a $\mathcal{D}$-pair, if $v_2$ is counterclockwise from $v_1$, and there is no vector in $\mathcal{D}$ between $v_1$ and $v_2$. Let $\mathcal{D}_{\text{pairs}}$ denote the set of all $\mathcal{D}$-pairs. Let $\text{pspan}(u_1, u_2)$ denote the portion of the plane that can be represented as a positive linear combination of $u_1, u_2 \in \mathcal{D}$. Thus $\text{pspan}(u_1, u_2)$ is the open wedge bounded by the rays emanating from the origin in directions $u_1, u_2$. We define by $(v_1, v_2)_L = \text{pspan}(-v_1, v_2)$ and $(v_1, v_2)_R = \text{pspan}(v_1, -v_2)$: these are two of the four quadrants of the plane induced by the lines containing $v_1$ and $v_2$. Similarly, for $v \in \mathcal{D}$ we denote by $v_L$ and $v_R$ the two open half-planes defined by the line passing through $v$. Let

$$
\mathcal{Q}(\mathcal{D}) = \left\{ v_L, v_R \mid v \in \mathcal{D} \right\} \cup \left\{ (v_1, v_2)_R, (v_1, v_2)_L \mid (v_1, v_2) \in D_{\text{pairs}} \right\}.
$$

**Definition 3.4** For a set $S \subseteq \mathbb{R}^2$ we denote by $T(S)$ the set of translations of $S$ in the plane, that is $T(S) = \left\{ S + p \mid p \in \mathbb{R}^2 \right\}$. Given a set of directions $\mathcal{D}$, let $T(\mathcal{D}) = \bigcup_{Q \in \mathcal{Q}(\mathcal{D})} T(Q)$.

For $\mathcal{D}_{\mathbb{R}}$, the set $T(\mathcal{D}_{\mathbb{R}})$ is the set of all open half-planes. The standard convex hull of a planar point set $S$ can be defined as follows: start from the whole plane, and remove from it all the open half-planes $H^+$ such that $H^+ \cap S = \emptyset$. We extend this definition to handle $\mathcal{D}$-convexity for an arbitrary set of directions $\mathcal{D}$, as follows:

$$
\mathcal{D}\text{-CH}(S) = \mathbb{R}^2 \setminus \bigcup_{\mathcal{I} \in T(\mathcal{D}), \mathcal{I} \cap S = \emptyset} \mathcal{I};
$$

that is, we remove from the plane all the translations of quadrants and halfplanes in $\mathcal{Q}(\mathcal{D})$ that do not contain a point of $S$. See Figures 4 and 5.

For the case $\mathcal{D}_{xy} = \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$, Matoušek and Plecháč [MP97] showed that if $\mathcal{D}_{xy}\text{-CH}(S)$ is connected, then $\mathcal{CH}^{\mathcal{D}_{xy}}(S) = \mathcal{D}_{xy}\text{-CH}(S)$.
Figure 5: (a) A set of directions $\mathcal{D}$, such that $\alpha(\mathcal{D}) > \pi/2$, (b) the set of quadrants $\mathcal{Q}(\mathcal{D})$ induced by $\mathcal{D}$, and (c) the $\mathcal{D}$-\textit{CH} of a set of points which is not connected.

**Definition 3.5** For a set of directions $\mathcal{D}$, we define the density of $\mathcal{D}$ to be

$$\alpha(\mathcal{D}) = \max_{(v_1, v_2) \in \mathcal{D}} \alpha(v_1, v_2),$$

where $\alpha(v_1, v_2)$ denotes the counterclockwise angle from $v_1$ to $v_2$.

See Figure 5 for an example of a set of directions with density larger than $\pi/2$.

**Corollary 3.6** Let $\mathcal{D}$ be a set of directions in the plane. Then:

- The set $\mathcal{D}$-\textit{CH}(A) is $\mathcal{D}$-convex, for any $A \subseteq \mathbb{R}^2$.
- For any $A \subseteq B \subseteq \mathbb{R}^2$, one has $\mathcal{D}$-\textit{CH}(A) $\subseteq$ $\mathcal{D}$-\textit{CH}(B).
- For two sets of directions $\mathcal{D}_1 \subseteq \mathcal{D}_2$ we have $\mathcal{D}_1$-\textit{CH}(S) $\subseteq$ $\mathcal{D}_2$-\textit{CH}(S), for any $S \subseteq \mathbb{R}^2$.
- Let $S$ be a bounded set in the plane, and let $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathcal{D}_3 \cdots$ be a sequence of sets of directions, such that $\lim_{i \to \infty} \alpha(\mathcal{D}_i) = 0$. Then, $\text{int CH}(S) \subseteq \lim_{i \to \infty} \mathcal{D}_i$-\textit{CH}(S) $\subseteq$ \text{CH}(S).

**Lemma 3.7** Let $\mathcal{D}$ a set of directions, and let $S$ be a finite set of points in the plane. Then $C = \mathcal{D}$-\textit{CH}(S) is a polygonal set whose complexity is $O(|S \cap \partial C|)$.

**Proof:** It is easy to show that $C$ is polygonal. We charge each vertex of $C$ to some point of $S' = S \cap \partial C$. Let $C'$ be a connected component of $C$. If $C'$ is a single point, then this is a point of $S'$. Otherwise, let $e$ be an edge of $C'$, and let $I$ be a set in $\mathcal{T}(\mathcal{D})$ such that $e \subseteq \partial I$, and $I \cap S = \emptyset$.

Since $e$ is an edge of $C'$, there is no $q \in \mathbb{R}^2$ such that $e \subseteq q + I$, and $(q + I) \cap S = \emptyset$. This implies that there must be a point $p$ of $S$ on $\partial I \cap l_e$, where $l_e$ is the line passing through $e$. However, $C$ is a $\mathcal{D}$-convex set, and the direction of $e$ belongs to $\mathcal{D}$. It follows that $l_e$ intersects $C$ along a connected set (i.e., the segment $e$), and $p \in l_e \cap C = e$. We charge the edge $e$ to $p$. We claim that a point $p$ of $S'$ can be charged at most 4 times. Indeed, for each edge $e'$ of $C$ incident to $p$, there is a supporting set in $\mathcal{T}(\mathcal{D})$, such that $p$ and $e'$ lie on
its boundary. Only two of those sets can have angle less than $\pi/2$ at $p$ (because such a set corresponds to a $\mathcal{D}$-pair $(v_1, v_2)$ with $\alpha(v_1, v_2) > \pi/2$). Thus, a point of $S'$ is charged at most \( \max(2\pi/(\pi/2), \pi/(\pi/2) + 2) = 4 \) times.

**Lemma 3.8** Let $\mathcal{D}$ be a set of directions, and let $K$ be a bounded convex body in the plane, such that the expected area of $\mathcal{D}\cdot\text{CH}(N)$ of a set $N$ of $n$ points, chosen uniformly and independently from $K$, is at least $(1 - f(n))\text{Area}(K)$, where $1 \geq f(n) \geq 0$, for $n \geq 1$. Then, the expected number of vertices of $C = \mathcal{D}\cdot\text{CH}(N)$ is $O(nf(n/2))$.

**Proof:** By Lemma 3.7 the complexity of $C$ is proportional to the number of points of $N$ on the boundary of $C$. Using this observation, it is easy to verify that the proof of Lemma 2.1 can be extended to this case.

We would like to apply the proof of Theorem 2.3 to bound the expected complexity of a random $\mathcal{D}$-convex hull inside a disk. Unfortunately, if we try to concentrate only on three consecutive sectors (as in Figure 1) it might be that there is a quadrant $I$ of $\mathcal{T}(\mathcal{D})$ that intersects the middle the middle sector from the side (i.e. through the two adjacent sectors). This, of course, can not happen when working with the regular convexity. Thus, we first would like to decompose the unit disk into “safe” regions, where we can apply a similar analysis as the regular case, and the “unsafe” areas. To do so, we will first show that, with high probability, the $\mathcal{D}\cdot\text{CH}$ of a random point set inside a disk, contains a “large” disk in its interior. Next, we argue that this implies that the random $\mathcal{D}\cdot\text{CH}$ covers almost the whole disk, and the desired bound will readily follows from the above Lemma.

**Definition 3.9** For $r \geq 0$, let $B_r$ denote the disk of radius of $r$ centered at the origin.

**Lemma 3.10** Let $\mathcal{D}$ be a set of directions, such that $0 \leq \alpha(\mathcal{D}) \leq \pi/2$. Let $N$ be a set of $n$ points chosen uniformly and independently from the unit disk. Then, with probability $1 - n^{-10}$ the set $\mathcal{D}\cdot\text{CH}(N)$ contains $B_r$ in its interior, where $r = 1 - c\sqrt{\log n/n}$, for an appropriate constant $c$.

**Proof:** Let $r' = 1 - c\sqrt{\log n/n}$, where $c$ is a constant to be specified shortly. Let $q$ be any point of $B_{r'}$. We bound the probability that $q$ lies outside $C = \mathcal{D}\cdot\text{CH}(N)$ as follows: Draw 8 rays around $q$, such that the angle between any two consecutive rays is $\pi/4$. This partitions $q + B_{r''}$, where $r'' = c\sqrt{\log n/n}$, into eight portions $R_1, \ldots, R_8$, each having area $\pi c^2 \log n/(8n)$. Moreover, $R_i \subseteq q + B_{r''} \subseteq B_1$, for $i = 1, \ldots, 8$. The probability of a point of $N$ to lie outside $R_i$ is $1 - c^2 \log n/(8n)$. Thus, the probability that all the points of $N$ lie outside $R_i$ is

$$P \left[ N \cap R_i = \emptyset \right] \leq \left( 1 - \frac{c^2 \log n}{8n} \right)^n \leq e^{-(c^2 \log n)/8} = n^{-c^2/8},$$

since $1 - x \leq e^{-x}$, for $x \geq 0$. Thus, the probability that one of the $R_i$’s does not contain a point of $N$ is bounded by $8n^{-c^2/8}$. We claim that if $R_i \cap N \neq \emptyset$, for every $i = 1, \ldots, 8$, then $q \in C$. Indeed, if $q \notin C$ then there exists a set $Q \subseteq \mathcal{Q}(\mathcal{D})$, such that $(q + Q) \cap N = \emptyset$. Since $\alpha(\mathcal{D}) \leq \pi/2$ there exists an $i$, $1 \leq i \leq 8$, such that $R_i \subseteq q + Q$; see Figure 6. This is a contradiction, since $R_i \cap N \neq \emptyset$. Thus, the probability that $q$ lies outside $C$ is $\leq 8n^{-c^2/8}$.
Figure 6: Since $\alpha(D) \leq \pi/2$, any quadrant $Q \in Q(D)$, when translated by $q$, must contain one of the $R_i$’s.

Figure 7: The dark areas are the unsafe areas for a consecutive pairs of directions $v_1, v_2 \in D$.

Let $N'$ denote a set of $n^{10}$ points spread uniformly on the boundary of $B_{r'}$. By the above analysis, all the points of $N'$ lie inside $C$ with probability at least $1 - 8n^{10-c^2/8}$. Furthermore, arguing as above, we conclude that $B_r \subseteq D-\mathcal{CH}(N')$, where $r = 1 - 2c\sqrt{(\log n)/n}$. Hence, with probability at least $1 - 8n^{10-c^2/8}$, $D-\mathcal{CH}(C)$ contains $B_r$. The lemma now follows by setting $c = 20$, say.

Since the set of directions may contain large gaps, there are points in $B_1 \setminus B_r$ that are “unsafe”, in the following sense:

**Definition 3.11** Let $D$ be a set of directions, and let $0 \leq r \leq 1$ be a prescribed constant, such that $0 \leq \alpha(D) \leq \pi/2$. A point $p$ in $B_1$ is safe, relative to $B_r$, if $op \subseteq D-\mathcal{CH}(B_r \cup \{p\})$.

See Figure 7 for an example how the unsafe area looks like. The behavior of the $D-\mathcal{CH}$ inside the unsafe areas is somewhat unpredictable. Fortunately, those areas are relatively small.

**Lemma 3.12** Let $D$ be a set of directions, such that $0 \leq \alpha(D) \leq \pi/2$, and let $r = 1 - O\left(\sqrt{(\log n)/n}\right)$. The unsafe area in $B_1$, relative to $B_r$, can be covered by a union of $O(1)$ caps. Furthermore, the length of the base of such a cap is $O((\log n)/n)^{1/4}$, and its height is $O\left(\sqrt{(\log n)/n}\right)$.

**Proof:** Let $p$ be an unsafe point of $B_1$. Let $\vec{v}_1, \vec{v}_2$ be the consecutive pair of vectors in $D$, such that the vector $\overrightarrow{po}$ lies between them. If $\text{ray}(p, \vec{v}_1) \cap B_r \neq \emptyset$, and $\text{ray}(p, \vec{v}_2) \cap B_r \neq \emptyset$ then
The set $S$ that contains any point of $p_0$ that is $r$ away. By Lemma 3.12, the number of sectors in $U$ is at most $8$. Let $D$ be a set of directions, such that $0 \leq \alpha(D) \leq \pi/2$. The expected number of vertices of $D$-CH($N$), where $N$ is a set of $n$ points, chosen uniformly and independently from the unit disk, is $O\left(n^{1/3} + \sqrt{n\alpha(D)}\right)$.

Proof: We claim that the expected area $A(D)$ is at least $\pi - O\left(n^{-2/3} + \sqrt{\alpha/n}\right)$, where $\alpha = \alpha(D)$. The theorem will then follow from Lemma 3.12.

Indeed, let $m$ be an integer to be specified later, and assume, without loss of generality, that $m$ divides $n$. Partition $B$ into $m$ congruent sectors, $S_1, \ldots, S_m$. Let $B^1, \ldots, B^m$ denote the $\mu = n/m$ disks centered at the origin, such that (i) $B^1 = B_1$, and (ii) $Area(B^{i-1}) - Area(B^i) = \pi/\mu$, for $i = 2, \ldots, \mu$. Let $r_i$ denote the radius of $B^i$, for $i = 1, \ldots, \mu$. Note that $r_i - r_{i+1} \geq r_{i-1} - r_i \geq 1/(2\mu)$, for $i = 2, \ldots, \mu - 1$.

Let $r = 1 - O\left(\sqrt{(\log n)/n}\right)$, and let $U$ be the set of sectors that either intersect an unsafe area of $B$ relative to $B_r$, or their neighboring sectors intersect the unsafe area of $B$. By Lemma 3.12, the number of sectors in $U$ is $O(1) \cdot O\left((\log n/n)^{1/4}\right) = O(m((\log n)/n)^{1/4})$.

Let $S_{i,j} = (B^i \setminus B^{i+1}) \cap S_j$, and $S_{i,j} = B^i \cap S_j$, for $i = 1, \ldots, \mu - 1$, and $j = 1, \ldots, m$. The set $S_{i,j}$ is called the $i$-th tile of the sector $S_j$, and its area is $\pi/n$, for $i = 1, \ldots, \mu$, and $j = 1, \ldots, m$.

Let $X_j$ denote the first index $i$ such that $N \cap S_{i,j} \neq \emptyset$, for $j = 1, \ldots, m$. The probability that $X_j = k$ is upper-bounded by the probability that the tiles $S_{1,j}, \ldots, S_{(k-1),j}$ do not contain any point of $N$; namely, by $(1 - k^{-1})^n$. Thus, $P[X_j = k] \leq (1 - k^{-1})^n \leq e^{-(k-1)}$. Thus,

$$E[X_j] = \sum_{k=1}^{\mu} kP[X_j = k] \leq \sum_{k=1}^{\mu} ke^{-(k-1)} = O(1),$$

for $j = 1, \ldots, m$.

---

3 Area($B^1$) - Area($B^2$) = $\pi(1 - r_2^2) = \pi/\mu$, thus $r_2^2 = 1 - 1/\mu$. We have $r_2 \leq 1 - 1/(2\mu)$, and $r_1 - r_2 \geq 1 - (1 - 1/(2\mu)) = 1/(2\mu)$. 
Let $C$ denote the set $\mathcal{D}\cdot\mathcal{CH}(N \cup B_r)$. The tile $S_{i,j}$ is exposed by a set $K$, if $S_{i,j} \setminus K \neq \emptyset$. We claim that the expected number of tiles exposed by $C$ in a section $S_j \notin U$ is at most $X_j + X_{j+1} + O(\mu/m^2 + \alpha \mu/m)$, for $j = 1, \ldots, m$ (where we put $X_0 = X_m, X_{m+1} = X_1$).

Indeed, let $w = \max(X_{j-1}, X_{j+1})$, and let $p, q$ be the two points in $S_{j-1,w}, S_{j+1,w}$, respectively, such that the number of sets exposed by the triangle $T = \triangle opq$, in the sector $S_j$, is maximal. Both $p$ and $q$ lie on $\partial B^w$ and on the external radii bounding $S_{j-1}$ and $S_{j+1}$, as shown in Figure 8. Let $s$ denote the segment connecting the midpoint $p$ of the base of $T$ to its closest point on $\partial B^w$. The number of tiles in $S_j$ exposed by $T$ is bounded by $w$, plus the number of tiles intersecting the segment $s$. The length of $s$ is

$$|oq| - |pq| \cos \left( \frac{3}{2} \cdot \frac{2\pi}{m} \right) \leq 1 - \cos \left( \frac{3\pi}{m} \right) \leq \frac{1}{2} \left( \frac{3\pi}{m} \right)^2 \leq \frac{4.5\pi^2}{m^2},$$

since $\cos x \geq 1 - x^2/2$, for $x \geq 0$.

On the other hand, the segment $s$ intersects at most $\lfloor ||s||/(2\mu) \rfloor = O(\mu/m^2)$ tiles, and we have that the number of tiles exposed in the sector $S_j$ by $T$ is at most $w + O(\mu/m^2)$, for $j = 1, \ldots, m$.

Since $S_j \notin U$, the points $p, q$ are safe, and $op, oq \subseteq C$. This implies that the only additional tiles that might be exposed in $S_j$ by $C$, are exposed by the portion of the boundary of $C$ between $p$ and $q$ that lie inside $T$. Let $V$ be the circular cap consisting of the points in $T$ lying between $pq$ and a circular arc $\gamma \subseteq T$, connecting $p$ to $q$, such that for any point $p' \in \gamma$ one has $\angle pp'q = \pi - \alpha$. See Figure 8.

Let $Q \in \mathcal{T}(D)$ be any quadrant of the plane induce by $D$, such that $Q \cap N = \emptyset$ (i.e. $C \cap Q = \emptyset$), and $Q \cap T \neq \emptyset$. Then, $Q \cap op = \emptyset, Q \cap oq = \emptyset$ since $p$ and $q$ are safe. Moreover, the angle of $Q$ is at least $\pi - \alpha$, which implies that $Q \cap T \subseteq V$. See Figure 8.

Let $s'$ be the segment $op \cap V$, where $\rho$ is as above, the midpoint of $pq$. The length of $s''$ is

$$|s'| \leq \sin \left( \frac{3}{2} \cdot \frac{2\pi}{m} \right) \tan \frac{\alpha}{2} \leq \frac{3\pi \sqrt{2} \alpha}{2m} \leq \frac{3\pi \alpha}{m},$$

since $\sin x \leq x$, for $x \geq 0$, and $1/\sqrt{2} \leq \cos (\alpha/2)$ (because $0 \leq \alpha \leq \pi/2$).
Thus, the expected number of tiles exposed by $C$, in a sector $S_j \not\in U$, is bounded by
\[
X_{j-1} + X_{j+1} + O \left( \frac{\mu}{m^2} \right) + O \left( \frac{3\pi \alpha / m}{1/(2\mu)} \right) = X_{j-1} + X_{j+1} + O \left( \frac{\mu}{m^2} \right) + O \left( \frac{\alpha \mu}{m} \right).
\]

Thus, the expected number of tiles exposed by $C$, in sectors that do not belong to $U$, is at most
\[
E \left[ \sum_{j=1}^{m} \left( X_{j-1} + X_{j+1} + O \left( \frac{\mu}{m^2} \right) + O \left( \frac{\alpha \mu}{m} \right) \right) \right] = O \left( m + \frac{\mu}{m} + \alpha \mu \right).
\]

Adding all the tiles that lie outside $B_r$ in the sectors that belong to $U$, it follows that the expected number of tiles exposed by $C$ is at most
\[
O \left( m + \frac{\mu}{m} + \alpha \mu + |U| \cdot \frac{1 - r}{1/(2\mu)} \right) = O \left( m + \frac{\mu}{m} + \alpha \mu + m \left( \log n \right) \right)^{1/4} \cdot \mu \sqrt{\frac{\log n}{n}}
\]
\[
= O \left( m + \frac{n}{m^2} + \frac{\alpha n}{m} + n \left( \frac{\log n}{n} \right)^{3/4} \right) = O \left( m + \frac{n}{m^2} + \frac{\alpha n}{m} + n^{1/4} \log^{3/4} n \right).
\]

Setting $m = \max \left( n^{1/3}, \sqrt{n \alpha} \right)$, we conclude that the expected number of tiles exposed by $C$ is $O \left( n^{1/3} + \sqrt{n \alpha} \right)$.

The area of $C' = \mathcal{D} \cdot \mathcal{CH}(N)$ is bounded from below by the area of the tiles which are not exposed by $C'$. The probability that $C' \neq C$ (namely, that the disk $B_r$ is not inside $C'$) is at most $n^{-10}$, by Lemma 3.10. Hence the expected area of $C'$ is at least
\[
E[\text{Area}(C)] - \text{Prob} \left[ C \neq C' \right] = \pi - O \left( n^{1/3} + \sqrt{n \alpha} \right) \frac{\pi}{n} - n^{-10} \pi = \pi - O \left( n^{-2/3} + \sqrt{\frac{\alpha}{n}} \right).
\]

The assertion of the theorem now follows from Lemma 3.8.

Corollary 3.14 Let $\mathcal{D}'_{xy}$ be the set of directions resulting from rotating $\mathcal{D}_{xy}$ by 45 degrees. Let $N$ be a set of $n$ points, chosen independently and uniformly from the unit square $S'$. The expected complexity of $\mathcal{D}'_{xy} \cdot \mathcal{CH}(N)$ is $\Omega \left( \sqrt{n} \right)$.

Proof: Without loss of generality, assume that $n = m^2$ for some integer $m$. Tile $S'$ with $n$ translated copies of a square of area $1/n$. Let $S_1, \ldots, S_m$ denote the squares in the top row of this tiling, from left to right. Let $A_j$ denote the event that $S_j$ contains a point of $N$, and neither of the two adjacent squares $S_{j-1}, S_{j+1}$ contains a point of $N$, for $j = 2, \ldots, m-1$.

We have
\[
\text{Prob} \left[ A_j \right] = \text{Prob} \left[ S_{j-1} \cap N = \emptyset \text{ and } S_{j+1} \cap N = \emptyset \right] - \text{Prob} \left[ (S_{j-1} \cup S_j \cup S_{j+1}) \cap N = \emptyset \right],
\]
Figure 9: If \( A_j \) happens, then the squares \( S_{j-1}, S_{j+1} \) do not contain a point of \( N \). Thus, if \( q \) is the highest point in \( S_j \), then \( q + Q_{\text{top}} \) can not contain a point of \( N \), and \( q \) is a vertex of \( D'_{\text{xy}}-\mathcal{H}(N) \).

for \( j = 2, \ldots, m - 1 \). Hence,

\[
\lim_{n \to \infty} \Pr[ A_j ] = \lim_{n \to \infty} \left( \left( 1 - \frac{2}{n} \right)^n - \left( 1 - \frac{3}{n} \right)^n \right) = e^{-2} - e^{-3} \approx 0.0855
\]

This implies, that for \( n \) large enough, \( \Pr[ A_j ] \geq 0.01 \). Thus, the expected value of \( Y \) is \( \Omega(m) = \Omega(\sqrt{n}) \), where \( Y \) is the number of \( A_j \)'s that have occurred, for \( j = 2, \ldots, m - 1 \). However, if \( A_j \) occurs, then \( C = D'_{\text{xy}}-\mathcal{H}(N) \) must have a vertex at \( S_j \). Indeed, let \( Q_{\text{top}} \) denote the quadrant of \( Q(D'_{\text{xy}}) \) that contains the positive \( y \)-axis. If we translate \( Q_{\text{top}} \) to the highest point in \( S_j \cap N \), then it does not contain a point of \( N \) in its interior, implying that this point is a vertex of \( C \), see Figure 9.

This implies that the expected complexity of \( D'_{\text{xy}}-\mathcal{H}(N) \) is \( \Omega(\sqrt{n}) \).

4 On the Expected Number of Points on the Boundary of the Quadrant Hull Inside a Hypercube

In this section, we show that the expected number of points on the boundary of the quadrant hull of a set \( S \) of \( n \) points, chosen uniformly and independently from the unit cube is \( O(\log^{d-1} n) \). Those bounds are known [BKST78], but we believe the new proof is simpler.

Definition 4.1 ([MP97]) Let \( Q \) be a family of subsets of \( \mathbb{R}^d \). For a set \( A \subseteq \mathbb{R}^d \), we define the \( Q \)-hull of \( A \) as

\[
Q-\text{co}(A) = \bigcap \left\{ Q \in Q \mid A \subseteq Q \right\}.
\]

Definition 4.2 ([MP97]) For a sign vector \( s \in \{-1,+1\}^d \), define

\[
q_s = \left\{ x \in \mathbb{R}^d \mid \text{sign}(x_i) = s_i, \text{ for } i = 1, \ldots, d \right\},
\]

and for \( a \in \mathbb{R}^d \), let \( q_s(a) = q_s + a \). We set \( Q_{\text{sc}} = \left\{ \mathbb{R}^d \setminus q_s(a) \mid a \in \mathbb{R}^d, s \in \{-1,+1\}^d \right\} \). We shall refer to \( Q_{\text{sc}}-\text{co}(A) \) as the quadrant hull of \( A \). These are all points which cannot be separated from \( A \) by any open orthant in space (i.e., quadrant in the plane).
Definition 4.3 Given a set of points \( S \subseteq \mathbb{R}^d \), a point \( p \in \mathbb{R}^d \) is \( Q_{sc} \)-exposed, if there is \( s \in \{-1, +1 \}^d \), such that \( q_s(p) \cap S = \emptyset \). A set \( C \) is \( Q_{sc} \)-exposed, if there exists a point \( p \in C \) which is \( Q_{sc} \)-exposed.

Definition 4.4 For a set \( S \subseteq \mathbb{R}^d \), let \( n_{sc}(S) \) denote the number of points of \( S \) on the boundary of \( Q_{sc} \)-co\( (S) \).

Theorem 4.5 Let \( C \) be a unit axis parallel hypercube in \( \mathbb{R}^d \), and let \( S \) be a set of \( n \) points chosen uniformly and independently from \( C \). Then, the expected number of points of \( S \) on the boundary of \( H = Q_{sc} \)-co\( (S) \) is \( O(\log^{d-1}(n)) \).

Proof: We partition \( C \) into equal size tiles, of volume \( 1/n^d \); that is \( C(i_1, i_2, \ldots, i_d) = [(i_1 - 1)/n, i_1/n) \times \cdots \times [(i_d - 1)/n, i_d/n] \), for \( 1 \leq i_1, i_2, \ldots, i_d \leq n \).

We claim that the expect number of tiles in our partition of \( C \) which are exposed by \( S \) is \( O(n^{d-1} \log^{d-1} n) \).

Indeed, let \( q = q(-1, -1, \ldots, -1) \) be the “negative” quadrant of \( \mathbb{R}^d \). Let \( X(i_2, \ldots, i_d) \) be the maximal integer \( k \), for which \( C(k, i_2, \ldots, i_d) \) is exposed by \( q \). The probability that \( X(i_2, \ldots, i_d) \geq k \) is bounded by the probability that the cubes \( C(l_1, l_2, \ldots, l_d) \) does not contain a point of \( S \), where \( l_1 < k, l_2 < i_2, \ldots, l_d < i_d \). Thus,

\[
\Pr \left[ X(i_2, \ldots, i_d) \geq k \right] \leq \left( 1 - \frac{(k - 1)(i_2 - 1)\cdots(i_d - 1)}{n^d} \right)^n \leq \exp \left( - \frac{(k - 1)(i_2 - 1)\cdots(i_d - 1)}{n^{d-1}} \right),
\]

since \( 1 - x \leq e^{-x} \), for \( x \geq 0 \).

Hence, the probability that \( \Pr \left[ X(i_2, \ldots, i_d) \geq i \cdot m + 1 \right] \leq e^{-i} \), where

\[
m = \left\lceil \frac{n^{d-1}}{(i_2 - 1)\cdots(i_d - 1)} \right\rceil.
\]

Thus,

\[
E \left[ X(i_2, \ldots, i_d) \right] = \sum_{i=1}^{\infty} i \Pr \left[ X(i_2, \ldots, i_d) = i \right] = \sum_{i=0}^{\infty} \sum_{j=im+1}^{\infty} j \Pr \left[ X(i_2, \ldots, i_d) = j \right] \leq \sum_{i=0}^{\infty}(i + 1)m \Pr \left[ X(i_2, \ldots, i_d) \geq im + 1 \right] \leq \sum_{i=0}^{\infty}(i + 1)me^{-i} = O(m).
\]

Let \( r \) denote the expected number of tiles exposed by \( q \) in \( C \). If \( C(i_1, \ldots, i_d) \) is exposed by \( q \), then \( X(i_2, \ldots, i_d) \geq i_1 \). Thus, one can bound \( r \) by the number of tiles on the boundary of \( C \), plus the sum of the expectations of the variables \( X(i_2, \ldots, i_d) \). We have

\[
r = O(n^{d-1}) + \sum_{i_2=2}^{n-1} \sum_{i_3=2}^{n-1} \cdots \sum_{i_d=2}^{n-1} O \left( \frac{n^{d-1}}{(i_2 - 1)(i_3 - 1)\cdots(i_d - 1)} \right)
\]

\[
= O \left( n^{d-1} \sum_{i_2=2}^{n-1} \frac{1}{i_2 - 1} \sum_{i_3=2}^{n-1} \frac{1}{i_3 - 1} \cdots \sum_{i_d=2}^{n-1} \frac{1}{i_d - 1} \right) = O \left( n^{d-1} \log^{d-1} n \right).
\]
The set $Q_{sc}$ contains translation of $2^d$ different quadrants. This implies, by symmetry, that the expected number of tiles exposed in $C$ by $S$ is $O \left( 2^d n^{d-1} \log^{d-1} n \right) = O \left( n^{d-1} \log^{d-1} n \right)$. However, if a tile is not exposed by any $q_s$, for $s \in \{-1, +1\}^d$, then it lies in the interior of $H$. Implied that the expected volume of $H$ is at least

$$\frac{n^d - O \left( n^{d-1} \log^{d-1} n \right)}{n^d} = 1 - O \left( \frac{\log^{d-1} n}{n} \right).$$

We now apply an argument similar to the one used in Lemma 2.1 (Efron’s Theorem), and the theorem follows.

\begin{remark}
A point $p$ of $S$ is a \textit{maxima}, if there is no point $p'$ in $S$, such that $p_i \leq p'_i$, for $i = 1, \ldots, d$. Clearly, a point which is a maxima, is also on the boundary of $Q_{sc}$-co($S$). By Theorem 4.5, the expected number of maxima in a set of $n$ points chosen independently and uniformly from the unit hypercube in $\mathbb{R}^d$ is $O(\log^{d-1} n)$. This was also proved in [BKST78], but we believe that our new proof is simpler.

Also, as noted in [BKST78], a vertex of the convex hull of $S$ is a point of $S$ lying on the boundary of the $Q_{sc}$-co($S$). Hence, the expected number of vertices of the convex hull of a set of $n$ points chosen uniformly and independently from a hypercube in $\mathbb{R}^d$ is $O(\log^{d-1} n)$.
\end{remark}

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