Analysis of synchronized behaviour of two metronomes: experimental verification

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ABSTRACT
In this paper, we experimentally evaluate the performance of a method for theoretically analysing the synchronized behaviour of two metronomes on a hanging plate using the describing function approach, proposed in our previous work. This is because the performance evaluation for real systems is important for practical application, but it was not conducted previously. By applying the analysis method to an experimental system with real metronomes, we demonstrate that the analysis results similar to experimental ones are obtained even though there are some approximations associated with the use of the describing function approach. This indicates that our analysis method is applicable to the synchronized behaviour of real metronomes.

ARTICLE HISTORY
Received 31 July 2021
Revised 8 October 2021
Accepted 26 October 2021

KEYWORDS
Synchronization; metronomes; hanging plate; describing function approach; experimental verification

1. Introduction
Synchronization is a phenomenon wherein the rhythms of oscillating objects are adjusted by their interaction. This phenomenon can be found in various fields such as biology [1,2], physics [3], engineering [4], and social life [5]. In addition, there are practical applications, including radio-controlled clocks and cardiac pacemakers [6]. For these reasons, synchronization has become an important research topic.

A typical example of synchronization is that of pendulum clocks or metronomes. In the synchronization of metronomes, the oscillation frequencies of the pendulums of multiple metronomes placed on an object converge to a common value by their interaction through the object. For example, consider the system illustrated in Figure 1, which is composed of a hanging plate and two metronomes placed on it. In this system, the vibration of one metronome is transmitted to the other metronome through the plate, which leads to their synchronization.

The synchronization of metronomes has been actively studied by many researchers. For example, Sato et al. [7] investigated the synchronized behaviour of two metronomes on a hanging plate through experiments and simulations. Czolczynski et al. [8,9] characterized the types of synchronization for metronomes on a horizontally moving base and those on a vertically moving base. Boda et al. [10] studied the synchronization of metronomes on a freely rotating disk in experimental and numerical ways. Zhang et al. [11] conducted experiments and simulations on metronomes in two layers, wherein two metronomes were placed on each of the two horizontally moving bases that were coupled by placing one base on top of the other base. More recently, Ikeguchi and Shimada [12] discussed how to obtain a mathematical model to describe the dynamical behaviour of metronomes on a hanging plate and performed simulations of the model to investigate their synchronized behaviour. Furthermore, as another related work, Peña-Ramírez et al. [13] developed an experimental system with two controllable mass-spring-damper oscillators coupled through an elastically supported rigid bar in order to address the problem of synchronizing two arbitrary nonlinear oscillators with a certain coupling.

In [14–16], our group has addressed problems of theoretically analysing the synchronized behaviour of metronomes (i.e., the oscillation amplitudes and frequencies of their pendulums and the phase differences between the oscillations at synchronization). Compared to the above-mentioned existing works, a feature of our analysis problems is that in order to investigate the types of synchronization in detail, we must derive not only the phase differences between the oscillations but also their amplitudes and frequencies (without using simulations). Although the analysis problems are difficult because of the presence of a discontinuous function representing a torque generated in each metronome, we addressed them by approximating the discontinuous function using the describing function approach [17,18]. However, the analysis methods presented in these works were not verified through
experiments, and their accuracy may be low for real systems. In general, there exist gaps between theory and practice, caused by, for example, the mathematical modelling of target systems. In addition, our analysis methods are based on some approximations, including those inherent in the describing function approach.

Thus, this paper focuses on the analysis method presented in [14], which is the most basic one in our framework, as the first step of the research, and aims to verify its effectiveness through experiments. To this end, we apply the analysis method to an experimental system and evaluate the analysis results by comparison with experimental results. The evaluation demonstrates that some of the analysis results are similar to the experimental ones, even in the presence of the above-mentioned gaps and approximations. This means that our analysis method is applicable to the synchronized behaviour of real metronomes.

Finally, we stress the differences between this paper and our related papers [19,20]. This paper is based on the conference version [19]. However, to study the performance of the analysis method in detail, we conduct a few tens of experiments by changing the frequencies of metronomes and the initial angles of both pendulums. In addition, we provide a discussion on the differences between the analysis and experimental results. Meanwhile, this paper uses the experimental system developed in [20], but its purpose is different from that of [20]. The purpose of [20] was to obtain a mathematical model describing the behaviour of the experimental system, whereas this paper aims to evaluate the performance of our analysis method.

**Notation.** Let \( \mathbb{R}, \mathbb{R}_+, \) and \( \mathbb{C} \) be the real number field, the set of positive real numbers, and the complex number field, respectively. We use \( j \) to denote the imaginary unit (i.e., \( j := \sqrt{-1} \)). For the complex number \( z \in \mathbb{C}, |z| \) and \( \angle z \) denote its absolute value and argument, respectively. Both the zero scalar and the zero vector are represented by \( 0 \). The identity matrix is represented by \( I \). Finally, the Laplace transform of the function \( f(t) \) is denoted as \( \mathcal{L}[f(t)] \).

### 2. Model description

This section describes a mathematical model of the system discussed in this paper.

Consider the model illustrated in Figure 2. This model is a model of the system shown in Figure 1, where the pendulums correspond to the metronomes. We assume that the two pendulums are identical. The dynamics of the model is described by

\[
\begin{align*}
(m_0 + 2m)\ell_i^2\ddot{\phi}_i(t) + \mu_0\dot{\phi}_i(t) + m\ell_i\ell_0^2\ddot{\phi}_0(t) \\
+ m\ell_i\ell_0\dot{\phi}_0(t)\cos(\phi_i(t) - \phi_0(t)) \\
+ m\ell_i\ell_0\dot{\phi}_0(t)\sin(\phi_i(t) - \phi_0(t)) \\
+ (m_0 + 2m)g\ell_0\sin(\phi_0(t)) = 0,
\end{align*}
\]

\[
\begin{align*}
\dot{\phi}_i(t) + m\ell_i\ell_0\dot{\phi}_0(t)\cos(\phi_i(t) - \phi_0(t)) \\
+ m\ell_i\ell_0\dot{\phi}_0(t)\sin(\phi_i(t) - \phi_0(t)) = \tau_i(\phi_i(t), \dot{\phi}_i(t)) \quad \forall i \in \{1, 2\},
\end{align*}
\]

where \( \phi_0(t) \in \mathbb{R} \) is the swing angle of the pendulum, \( \phi_i(t) \in \mathbb{R} \) is the angle of pendulum \( i \) (\( i \in \{1, 2\} \)), \( m_0 \in \mathbb{R}_+ \) is the mass of the pendulum, \( m \in \mathbb{R}_+ \) is the mass of the bob of each pendulum, \( \ell_0 \in \mathbb{R}_+ \) is the length of the rope from which the plate is hung, \( \ell \in \mathbb{R}_+ \) is the length of each pendulum, \( \mu_0 \in \mathbb{R}_+ \) is the damping coefficient for the swinging motion of the plate, \( \mu \in \mathbb{R}_+ \) is the damping coefficient for each pendulum, and \( g \in \mathbb{R}_+ \) is the acceleration due to gravity. The variable \( \tau_i(t) \in \mathbb{R} \) is a torque acting on pendulum \( i \) and is defined as the following discontinuous function [8]:

\[
\tau_i(\phi_i(t), \dot{\phi}_i(t)) = \begin{cases} 
\delta & \text{if } 0 < \phi_i(t) < \gamma, \phi_i(t) > 0, \\
-\delta & \text{if } -\gamma < \phi_i(t) < 0, \phi_i(t) < 0, \\
0 & \text{otherwise,}
\end{cases}
\]

where \( \delta, \gamma \in \mathbb{R}_+ \) are positive numbers. Equation (2) is based on a component of metronomes called the escapement mechanism, which means that a torque of magnitude \( \delta \) is applied for accelerating the motion of pendulum \( i \) after it passes through the downward position (i.e., \( \phi_i = 0 \)). This prevents the damping of the oscillation of each pendulum, producing a sustained oscillation. Moreover, for simplicity of notation, let \( \varphi(t) := [\phi_0(t) \phi_1(t) \phi_2(t)]^\top \) and \( \tau(t) := [\tau_1(t) \tau_2(t)]^\top \).

The behaviour of the above-mentioned model is now illustrated through an example. The values of the model parameters are set as \( m_0 := 0.514 \text{ kg}, m := 4.40 \times 10^{-3} \text{ kg}, \ell_0 := 0.174 \text{ m}, \ell := 0.102 \text{ m}, \mu_0 := 1.77 \times 10^{-3} \text{ Nms/rad}, \mu := 2.45 \times 10^{-3} \text{ Nms/rad}, g := 9.81 \text{ m/s}^2, \delta := 1.19 \times 10^{-3} \text{ Nm}, \) and \( \gamma := 0.150 \text{ rad}. \) These values (except for \( g \)) are based on our experimental system detailed in Section 4. Figure 3 shows the time evolution of \( \tau(t) \) and \( \varphi(t) \) for \( t \in [0, 5] \) and \( t \in [55, 60], \)
where $\psi(0) := [-0.04, 0.9, -0.4]^{T}$ and $\dot{\psi}(0) := 0$. By applying the intermittent torque $\tau_i(t)$, each pendulum $i$ continues to oscillate even in the presence of the damping. In addition, through the interaction between the pendulums, which manifests as the motion of the plate, the behaviour of the two pendulums is synchronized.

3. Analysis of synchronized behaviour [14]

For the model provided in Section 2, Muraoka et al. [14] presented a method for theoretically analysing the behaviour of the pendulums at synchronization. This section briefly introduces the analysis method.

3.1. Problem formulation

For the model shown in Figure 2, we say that synchronization occurs if the oscillation frequencies of the two pendulums converge to a common value. Then, assuming that the oscillation amplitudes of the pendulums are the same at synchronization, we address the problem of finding the common amplitude and frequency of the oscillations and the phase difference between them. More precisely, we assume that the behaviour of pendulum $i$ ($i \in \{1, 2\}$) at synchronization is described by

$$\psi_i(t) = a \sin(\omega t + \theta_i),$$  

(3)

for $i \in \{1, 2\}$.

where $a, \omega \in \mathbb{R}_+$ and $\theta_i \in \mathbb{R}$ are the amplitude, frequency, and phase shift of the oscillation, respectively; we theoretically derive $a, \omega,$ and $\theta_d := \theta_2 - \theta_1$ from (1) and (2).

3.2. Analysis method using describing function approach

The difficulty of the analysis problem stated above is that the torque $\tau_i(t)$ is defined as a discontinuous function, as indicated in (2). To overcome this difficulty, Muraoka et al. [14] used the describing function approach [17,18], in which a nonlinear function is approximated by regarding the input and output signals as a sinusoidal signal and the first harmonic of the Fourier expansion of the resulting output signal, respectively. By approximating $\tau_i(t)$ via the describing function approach, they developed a method to solve the analysis problem.
of the form (3) is approximated by the downward equilibrium point \( \phi \). Besides, for the input \( \varphi_-(t) \) shown in Figure 4, where \( L(\tau(t)) \) gives \( \tau \in \{1,2\} \), let \( P(s) \in \mathbb{C}^{2 \times 2} \) denote the transfer function matrix from \( \mathcal{L}[\tau(t)] \) to \( \mathcal{L}[\varphi_-(t)] \). Linearizing the dynamics (1) around the downward equilibrium point \( [\phi^\top \varphi^-^\top]^\top = 0 \) gives

\[
P(s) := \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{12}(s) & P_{11}(s) \end{bmatrix},
\]

where

\[
P_{11}(s) := \frac{c_4s^4 + c_3s^3 + c_2s^2 + c_1s + c_0}{d_5s^6 + d_3s^5 + d_4s^4 + d_3s^3 + d_2s^2 + d_1s + d_0},
\]

\[
P_{12}(s) := \frac{m^2\ell_2^2s^4}{d_5s^6 + d_3s^5 + d_4s^4 + d_3s^3 + d_2s^2 + d_1s + d_0},
\]

for

\[
c_0 := (m_0 + 2m)m\ell_0\ell_2g^2,
\]

\[
c_1 := (m_0 + 2m)\ell_0\mu g + m\ell_2\mu g,
\]

\[
c_2 := (m_0 + 2m)(\ell_0 + \ell)m\ell_0\ell_2g + \mu_0\mu g,
\]

\[
c_3 := (m_0 + 2m)\ell_0^2\mu + m\ell_2^2\mu_0,
\]

\[
c_4 := (m_0 + m)\ell_2^2\ell_0^2g^2,
\]

\[
d_0 := (m_0 + 2m)m^2\ell_0\ell_2^3g^3,
\]

\[
d_1 := 2(m_0 + 2m)m\ell_0\ell_2\mu g^2 + m^2\ell_2^2\mu_0g^2,
\]

\[
d_2 := (m_0 + 2m)(m^2\ell_0^2g + 2m^2\ell_3^3g + \mu_0^2\ell_0g + 2m\ell_0\mu g
\]

\[
+ 2m^2\ell_3^2\mu g + \mu_0\mu_2^2,
\]

\[
d_3 := 2(m_0 + 2m)(\ell_0 + \ell)m\ell_0\ell_2\mu g
\]

\[
+ 2m^2\ell_3^2\mu_0 g + \mu_0\mu_2,
\]

\[
d_4 := (m_0 + 2m)(m^2\ell_0^2g + \ell_0\mu_2^2\ell_0)
\]

\[
+ 2(m_0 + m)m^2\ell_2^3\mu g + 2m^2\ell_0^2\mu_0,
\]

\[
d_5 := 2(m_0 + m)m\ell_0^2\ell_2^2\mu + m^2\ell_4^3\mu_0,
\]

\[
d_6 := m_0^2\ell_0^2\ell_2^4.
\]

Using these notations, we introduce the feedback system shown in Figure 4, where \( \hat{\tau}, \hat{\varphi}_- \in \mathbb{C}^2 \) are the phasor representations of \( \tau(t) \) and \( \varphi_-(t) \), respectively (i.e., they are the complex vectors in which each element contains the absolute value and the argument, specified by the amplitude and phase shift of the oscillation, respectively). This feedback system approximates the model shown in Figure 2 in the frequency domain under (3), (7), and the linearization of (1). Therefore, if an appropriate \( (a, \omega, \theta_d) \) exists as a solution to the system, it will provide a solution to the analysis problem.

To find such a tuple \( (a, \omega, \theta_d) \), we focus on the input–output relation of each component of the feedback system shown in Figure 4 and then obtain

\[
\hat{\varphi}_- = P(j\omega)\hat{\tau},
\]

\[
\hat{\tau} = f(a)\hat{\varphi}_-.
\]

Combining these equalities and using the fact that \( f(a) \) is a scalar function yield

\[
(I - f(a)P(j\omega))\hat{\varphi}_- = 0.
\]

By solving (13), the following theorem [14] is obtained.

**Theorem 3.1**: Consider the feedback system shown in Figure 4. The synchronized behaviour of the form (3) is characterized as follows.

(i) **Anti-phase synchronization**: In this case, the amplitude \( a \) and frequency \( \omega \) of the oscillations of the two pendulums are given as a solution to

\[
P_{11}(j\omega) - P_{12}(j\omega) = \frac{1}{f(a)},
\]

and the phase difference \( \theta_d \) is equal to \( \pi \) (in the sense of modulo \( 2\pi \)).

(ii) **In-phase synchronization**: In this case, the amplitude \( a \) and frequency \( \omega \) of the oscillations of the two pendulums are given as a solution to

\[
P_{11}(j\omega) + P_{12}(j\omega) = \frac{1}{f(a)},
\]

and the phase difference \( \theta_d \) is equal to zero.

Anti-phase synchronization and in-phase synchronization are illustrated in Figure 5. In the former, the two pendulums oscillate in opposite directions. In the latter, they oscillate in the same direction.

Theorem 3.1 indicates that anti-phase synchronization and in-phase synchronization are possible in the
feedback system shown in Figure 4 and that the amplitude $a$ and frequency $\omega$ at synchronization are determined by (14) and (15). Thus, by solving (14) and (15), we can analyse the behaviour of the two pendulums at synchronization.

However, the accuracy of this analysis method may be low for real systems because of the approximations of (3) and (7) and the linearization of the dynamics (1). In fact, these factors cause the following effects.

(E1) The errors between the values of $a$, $\omega$, and $\theta_d$ provided by the analysis method and those obtained by experiments may occur.

(E2) The analysed behaviour may not exist in real systems.

(E3) The analysis method may not estimate the synchronized behaviour that exists in real systems.

Remark 3.1: Although we can consider $f(a)$ as a controller for $P(j\omega)$ from Figure 4, the problem considered in this subsection is not a controller design problem. In fact, the describing function $f(a)$ is given by (6) and is not a design parameter. The problem considered here is to find an appropriate $(a, \omega, \theta_d)$ as a solution to the feedback system in Figure 4 for investigating the behaviour of the two pendulums at synchronization.

Remark 3.2: Theorem 3.1 does not guarantee the stability of the resulting synchronized behaviour. Therefore, unstable behaviour may be analysed.

4. Experimental verification

In this section, we verify the effectiveness of the analysis method explained in Section 3 through experiments.

Figure 6 shows a photo of the experimental system. This system was developed in [20] and corresponds to the system shown in Figure 1. The metronomes are the type 331 of Nikko Seiki Co., Ltd. [21], and the distance between them is 183 mm. The plate is a wooden one of size $450 \times 200$ mm and hangs by strings. These settings are chosen to realize and observe the synchronization of metronomes, and they are not special. In fact, in the above metronomes, the motion of the entire pendulums can be directly observed (see Figure 6). In addition, the distance between the metronomes, which corresponds to the size of the plate, and the material of the plate are chosen to be small and light, respectively, because for a large and heavy plate, it is difficult to transmit the vibration of the metronomes for the interaction between them.

Figure 7 shows the experimental setup, which consists of the system shown in Figure 6, four motion capture cameras, and a desktop computer. The cameras are the type Flex 3 in the motion capture system OptiTrack [22]. The desktop computer calculates the angles of the pendulums of the metronomes from the camera images and provides their time evolution as experimental data, where the sampling period is 10 ms. We acquire the experimental data of more than 70 s until the metronomes reach their steady states and a certain period of time elapses, and use the data from 60 s to 70 s for evaluating the performance of the analysis method.

For the experimental system shown in Figure 6, we analyse the synchronized behaviour of the metronomes using Theorem 3.1. To this end, consider the model shown in Figure 2. The values of the model parameters are the same as those given in Section 2. These values are determined using the method given in [20], where the frequencies of the metronomes are 184 BPM (beats per minute). We first consider (i) in Theorem 3.1. Figure 8 (a) shows the plots of $P_{11}(j\omega) - P_{12}(j\omega)$ and $1/f(a)$ in (14) in the complex plane, where the case of $a \geq \gamma$ is chosen for $f(a)$ in (6) because $a < \gamma$ is impractical according to (2). Investigating the intersection $p_1$ of the two curves provides a solution to (14) as $(a, \omega) := (0.689, 9.78)$. Therefore, it follows from Theorem 3.1 that one of the synchronization types is
given by \((a, \omega, \theta_d) = (0.689, 9.78, \pi)\). We then consider (ii) in Theorem 3.1. Figure 8(b) shows the plots of \(P_{11}(j\omega) + P_{12}(j\omega)\) and \(1/f(a)\) in (15) in the same way as in Figure 8(a). From the intersections \(p_2, p_3,\) and \(p_4\) of the two curves and Theorem 3.1, we obtain \((a, \omega, \theta_d) = (0.276, 7.45, 0), (0.193, 7.50, 0), (0.676, 9.97, 0)\), respectively, as the other synchronization types.

Table 1 summarizes these analysis results and the corresponding simulation and experimental results, where \(p_1, p_2, p_3,\) and \(p_4\) correspond to the points shown in Figure 8 and \(\omega_i, \omega_j, (i \in \{1, 2\})\) are the oscillation amplitude and frequency of pendulum \(i\), respectively. The simulation results are based on the data from 60 s to 70 s similar to the experimental results, and synchronization types that were not observed are indicated as N/A. Table 1 also shows the stability of each synchronization type that is analysed by applying the Nyquist stability criterion to Figure 8. Figure 9 shows the time evolution of \(\varphi_1(t)\) and \(\varphi_2(t)\) from 60 to 65 s for the analysis and experimental results in \(p_4\) in Table 1, where Figure 9(a) is plotted using (3). These results indicate that although the analysis results contain synchronization types that are not observed in practice, the analysis method can estimate the synchronized behaviour of real metronomes. In addition, regarding \(p_4\) in Table 1, we conduct 10 experiments by changing the initial angles \(\varphi_1(0)\) and \(\varphi_2(0)\), where \(\varphi_1(0)\) and \(\varphi_2(0)\) are chosen as 10 combinations from approximately \(\varphi_1(0), \varphi_2(0) \in [-\pi/3, -\pi/6, \pi/6, \pi/3]\) and the other initial values are set as zero. Table 2 presents the maximum, average, and minimum of the absolute values of the errors \(\Delta a_i, \Delta \omega_j, (i \in \{1, 2\})\) and \(\Delta \theta_d, i.e. the differences between the values of \(\varphi_1(0)\) and \(\varphi_2(0)\) that are shown in Table 1 as the analysis result and those obtained for each of the 10 experiments. This demonstrates that results similar to \(p_4\) in Table 1 are obtained even for the different values of \(\varphi_1(0)\) and \(\varphi_2(0)\).

Next, we change the frequencies of the metronomes from 184 BPM to 144 BPM. For this setting, we analyse the synchronized behaviour of the metronomes using Theorem 3.1. The change in the frequencies of the metronomes results in \(m := 1.20 \times 10^{-3} \text{kg}\) and \(\ell := 0.166 \text{m}, \text{where the other parameter values are the same as before.}\) Figure 10 shows the plots of \(P_{11}(j\omega) + P_{12}(j\omega)\) and \(1/f(a)\) in the same way as in Figure 8. The intersection \(p_5\) of the curves implies that \((a, \omega) := (0.779, 7.65)\) is a solution to (14), and \(p_6, p_7,\) and \(p_8\) imply that \((a, \omega) := (0.707, 7.33), (0.388, 7.52), (0.735, 7.85)\) are solutions to (15). Hence, from Theorem 3.1, \((a, \omega, \theta_d) := (0.779, 7.65, \pi), (0.707, 7.33, 0), (0.388, 7.52, 0), (0.735, 7.85, 0)\) are obtained as the synchronization types.

Table 3 summarizes these analysis results and the corresponding simulation and experimental results in
Table 1. Analysis, simulation, and experimental results (184 BPM).

|                  | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
|------------------|-------|-------|-------|-------|
| Anal. Sim. Exp.  |       |       |       |       |
| $a_1$ [rad]      | 0.689 | 0.672 | N/A   | 0.276 |
| $a_2$ [rad]      | 0.689 | 0.672 | N/A   | 0.276 |
| $\omega_1$ [rad/s] | 9.78  | 9.50  | N/A   | 7.45  |
| $\omega_2$ [rad/s] | 9.78  | 9.50  | N/A   | 7.45  |
| $\theta_d$ [rad] | 3.14  | 3.14  | N/A   | 0     |
| Stability        | Stable | Unstable | Unstable | Stable |

Figure 8

(a) Analysis.

Figure 9.

Time evolution of $\varphi_1(t)$ and $\varphi_2(t)$ for the analysis and experimental results in $p_4$ in Table 1.

Table 2. Errors between the analysis and experimental results for 10 experiments (184 BPM).

|                  | Maximum | Average | Minimum |
|------------------|---------|---------|---------|
| $|\Delta a_1|$ [rad] | 0.0146  | 0.00589 | 0.000200 |
| $|\Delta a_2|$ [rad] | 0.0682  | 0.0566  | 0.0387  |
| $|\Delta \omega_1|$ [rad/s] | 0.342   | 0.327   | 0.317   |
| $|\Delta \omega_2|$ [rad/s] | 0.342   | 0.328   | 0.317   |
| $|\Delta \theta_d|$ [rad] | 0.341   | 0.286   | 0.223   |

Figure 10.

(a) $P_{11}(j\omega) - P_{12}(j\omega)$ and $1/f(a)$.

(b) $P_{11}(j\omega) + P_{12}(j\omega)$ and $1/f(a)$.

Remark 4.1: We comment on the synchronization types that were analysed but not observed in the experiments. We first consider the synchronization types indicated in $p_2$ and $p_3$ in Table 1 and $p_6$ and $p_7$ in Table 3. It can be considered that these synchronization types are unstable as shown in Tables 1 and 3 and thus were not observed in the experiments. Next, we discuss the synchronization types indicated in $p_5$ in Table 1 and $p_8$ in Table 3. From Tables 1 and 3, these synchronization types are stable, and their existence was confirmed in the simulation but not in the experiments. The reason for this gap can be explained as follows. According to
Table 3. Analysis, simulation, and experimental results (144 BPM).

|       | p5       |       | p6       |       | p7       |       | p8       |       |
|-------|----------|-------|----------|-------|----------|-------|----------|-------|
| Anal. | Sim.     | Exp.  | Anal.    | Sim.  | Exp.     | Anal. | Sim.     | Exp.  |
| α1 [rad] | 0.779 | 0.756 | 0.667 | 0.707 | N/A | N/A | 0.388 | N/A | N/A | 0.735 | 0.736 | N/A |
| α2 [rad] | 0.779 | 0.756 | 0.780 | 0.707 | N/A | N/A | 0.388 | N/A | N/A | 0.735 | 0.736 | N/A |
| ω1 [rad/s] | 7.65 | 7.37 | 7.48 | 7.33 | N/A | N/A | 7.52 | N/A | N/A | 7.85 | 7.23 | N/A |
| ω2 [rad/s] | 7.65 | 7.37 | 7.48 | 7.33 | N/A | N/A | 7.52 | N/A | N/A | 7.85 | 7.23 | N/A |
| θd [rad] | 3.14 | 3.15 | 2.92 | 0 | N/A | N/A | 0 | N/A | N/A | 0 | 0.00 | N/A |
| Stability | Stable | Unstable | Unstable | Stable |

Figure 10

(a) Analysis.

(b) Experiment.

Table 4. Errors between the analysis and experimental results for 10 experiments (144 BPM).

|       | Maximum | Average | Minimum |
|-------|---------|---------|---------|
| | | | |
| Δα1 [rad] | 0.116 | 0.0971 | 0.0735 |
| | | | |
| Δω1 [rad/s] | 0.0260 | 0.0116 | 0.0004 |
| | | | |
| Δθd [rad] | 0.192 | 0.176 | 0.162 |
| | | | |
| ΔΔθd [rad] | 0.182 | 0.174 | 0.162 |
| | | | |
| | | | |

Remark 4.2: We discuss the errors Δα1, Δω1 (i ∈ {1, 2}) and Δθd between the analysis and experimental results in Tables 2 and 4. It can be considered that these errors occur due to the approximations of (3) and (7) and the linearization of the dynamics (1), as described in effect (E1) in Section 3.2. In addition, the magnitudes of Δα1 and Δω1 on the oscillation amplitudes are not uniform. The reason is that the metronomes used in the experiments are not completely identical, and thus the values of α1 and ω1 in the experimental results are different from each other unlike those in the analysis results, as shown in Tables 1 and 3.

Remark 4.3: Clarifying the application range of the analysis method is difficult. However, as the first step of the application, it is important to demonstrate that the analysis method can estimate the synchronized behaviour of real metronomes even though there are concerns about effects (E1)–(E3) in Section 3.2. In this respect, this paper makes a contribution.

5. Conclusion

In this paper, we experimentally evaluated the performance of our analysis method for the synchronized behaviour of two metronomes placed on a hanging plate. For the two cases where the frequencies of the metronomes are 184 BPM and 144 BPM, we analysed their synchronized behaviour and compared the analysis results with the experimental results. The comparison showed that the analysis results contained synchronization types that were not observed in practice, but some of the analysis results were similar to the experimental results. This demonstrates that our analysis method allows us to estimate the synchronized behaviour of real metronomes only from their model parameters.

Although the analysis method considered in this paper assumes that the number of metronomes is two and the oscillation amplitudes of their pendulums are the same at synchronization, these assumptions were removed in [16]. In the future, we plan to evaluate the performance of the extended method developed in [16] through experiments by introducing additional metronomes to the experimental system. Another future work is to investigate the effects of the experimental conditions (e.g. the distance between the two metronomes, the material of the plate, and the choice of the metronomes) on the accuracy of the analysis method.

Disclosure statement

No potential conflict of interest was reported by the author(s).
Funding

This work was supported in part by Japan Society for the Promotion of Science (JSPS) KAKENHI [grant number 19K15016].

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