A COMPARISON BETWEEN PRETRIANGULATED $A_{\infty}$-CATEGORIES AND $\infty$-STABLE CATEGORIES

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Abstract. In this paper we will prove that the $A_{\infty}$-nerve of two quasi-equivalent $A_{\infty}$-categories are weak-equivalent in the Joyal model structure, a consequence of this fact is that the $A_{\infty}$-nerve of a pretriangulated $A_{\infty}$-category is $\infty$-stable. Moreover we give a comparison between the notions of pretriangulated $A_{\infty}$-categories, pretriangulated dg-categories and $\infty$-stable categories.

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Introduction

In 1963 Jean-Louis Verdier and Alexandre Grothendieck developed the notion of triangulated category in order to capture the additional structure on the derived category of an abelian category. Triangulated categories played an important role in algebraic geometry even if they have some problems, for example the non-functoriality of the mapping cone or the non-existence of homotopy colimits and homotopy limits.

For this purpose, in the 90’s it was developed the notion of pretriangulated envelope of a differential graded category and of a $A_{\infty}$-categories. Roughly speaking, pretriangulated dg-categories (resp. $A_{\infty}$-categories) are dg-categories (resp. $A_{\infty}$-categories) whose homotopy category is "canonically" triangulated so they can be viewed as enhanced triangulated categories.

Finally, more recently, it was exploited the notion of stable $\infty$-category that is a
pointed, complete and stable under loop space $\infty$-category whose homotopy category is a triangulated category.

It is a folklore belief that over a field $\mathbb{K}$ of characteristic zero the notions of ($\mathbb{K}$-linear) pretriangulated $A_\infty$-categories, pretriangulated dg-categories and $\infty$-stable categories are equivalent, under suitable localization. Unfortunately we cannot find any satisfying reference in the existing literature.

Regarding the category of pretriangulated dg-categories, in 2013 Lee Cohn proves that the nerve of the category of dg-categories localized on Morita equivalences is $\infty$-equivalent to the $\infty$-category of stable idempotent complete $\infty$-categories enriched over the Eilenberg-MacLane spectra $H\mathbb{K}$ ([Coh]). This fact proves that the categorical nerve of the category of dg-categories (localizing on Morita equivalence) is equivalent to an idempotent complete $\infty$-stable category. The problem is that the strategy used by Cohn does not extend to pretriangulated $A_\infty$-categories. However in 2015 Giovanni Faonte proved that, in general, the dg-nerve of a pretriangulated dg-category (in the sense of [BoKa]) is an $\infty$-stable category.

The aim of the present work is to extend the same result to pretriangulated $A_\infty$-categories, to clarify the relationship between pretriangulated dg-categories and pretriangulated $A_\infty$-categories. At the same time, we investigate some new possibilities offered by the $A_\infty$-nerve recently defined by Giovanni Faonte and Jacob Lurie. In particular we will prove that the $A_\infty$-nerve sends quasi-equivalences of unital $A_\infty$-categories in weak-equivalences of $\infty$-categories (Theorem 4.4) then, using classical theorems about $A_\infty$-categories due to Kenji Fukaya and Paul Seidel, we will prove the following:

**Theorem 4.4.** Let $\mathcal{A}$ be a pretriangulated $A_\infty$-category then $N_{A_\infty}(\mathcal{A})$ is an $\infty$-stable category. The functor induced in the homotopy categories is an equivalence of triangulated categories. Moreover $\mathcal{A}$ is idempotent complete if and only if $N_{A_\infty}(\mathcal{A})$ is an idempotent complete $\infty$-stable category.

This means that the $A_\infty$-nerve of a (unital) pretriangulated $A_\infty$-category is $\infty$-stable and the nerve induces a triangulated functor at homotopy categories level. Unfortunately, using the $A_\infty$-nerve, we do not have an equivalence of $\infty$-categories between the nerve of the category of the $A_\infty$-categories and a $\infty$-stable category as in the case of the category of dg-categories (localized over Morita equivalences). For this reason in the last part of the paper we will describe better the correspondence given by the $A_\infty$-nerve, in particular we will prove that a quasi-equivalence between the $A_\infty$-nerves of pretriangulated $A_\infty$-categories in $\text{Cat}_{\infty}^{\text{Ex}}$ (the category of $\infty$-stable categories with the exact functors) induces a weak equivalences of $A_\infty$-categories.

The plan of the paper is as follows. In Section 1 we survey some definitions and important properties of $A_\infty$-categories, $A_\infty$-Yoneda Lemma, modules over an $A_\infty$-category, quasi-equivalences and pretriangulated $A_\infty$-categories. In Section 2 we recall the construction of the $A_\infty$-nerve due to Giovanni Faonte. Section 3 is devoted to prove that the $A_\infty$-nerves of two quasi equivalent $A_\infty$-categories are weak equivalent. In Section 4 we show some consequences of the results proved in the previous section. In particular we analyze the aforementioned correspondence between pretriangulated $A_\infty$-categories and $\infty$-stable categories.

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1. \(A_\infty\)-modules, quasi-equivalences and pretriangulated \(A_\infty\)-categories

This section is divided in four parts: in the first one we will recall some basic definitions about the theory of \(A_\infty\)-categories, in the second we will give an \(A_\infty\)-version of Yoneda Lemma, in the third we will define the \(A_\infty\)-equivalences. In the last part we will discuss the pretriangulated envelopment of the \(A_\infty\)-categories.

1.1. Brief background on \(A_\infty\)-categories. First of all we give some information about \(A_\infty\)-categories well known to the experts, a good reference for the theory of the \(A_\infty\)-categories is [Sei]. We omit the notion of dg-category that will further be useful, cf. [Kel] for a survey about this topic.

Let \(K\) be a commutative ring.

**Definition 1.1** (\(A_\infty\)-category). We define an \(A_\infty\)-category \(\mathcal{A}\) to be a set of objects, a \(K\)-linear graded vector space \(\text{Hom}_{\mathcal{A}}(x_0, x_1)\) for any pair of objects, and \(K\)-linear maps

\[
m_{d}^n : \text{Hom}_{\mathcal{A}}(x_{d-1}, x_d) \otimes \ldots \otimes \text{Hom}_{\mathcal{A}}(x_0, x_1) \to \text{Hom}_{\mathcal{A}}(x_0, x_d)[2 - d],
\]

for every \(d > 0\). Moreover the maps above must verifying the followings:

\[
(1) \sum_{m,n} (-1)^{m+n} m_{d-m+1}^n (a_d, \ldots, a_{n+m+1}, m_{d}^n (a_{n+m}, \ldots, a_1), a_n, \ldots, a_1) = 0.
\]

where \(1 \leq m \leq d, 0 \leq n \leq d - m\) and \(\delta_n = \deg(a_1) + \ldots + \deg(a_n) - n\).

**Example 1.1.** The category of differential graded chains \(\text{Ch}_k\), whose morphisms are given by \(\text{Hom}^k_{\text{Ch}_k}(X, Y) := \bigoplus_{l \in \mathbb{Z}} \text{Hom}(X^l, Y^{l+k})\), equipped by the maps:

- \(m^1_{\text{Ch}_k}(f) := df + (-1)^{\deg f} f d\);
- \(m^2_{\text{Ch}_k}(f, g) := (-1)^{\deg f} g (\delta g + 1) g f\);
- \(m^m_{\text{Ch}_k} = 0\), for all \(n \geq 2\);

is an \(A_\infty\)-category.

From now on we consider only \(A_\infty\)-unitary categories i.e. \(A_\infty\)-categories with unit, according to the following definition:

**Definition 1.2** (Unit). Given an object \(x\) in \(\mathcal{A}\). We define the unit of \(x\) to be a morphism of degree 0 such that:

\[
(u1) m^2_{\mathcal{A}}(f, 1_x) = f;
\]

\[
(u2) m^2_{\mathcal{A}}(1_x, g) = (-1)^{\deg g} g;
\]

\[
(u3) m^n_{\mathcal{A}}(\ldots, 1_x, \ldots) = 0, \text{ for all } n > 2.
\]

We denote by \(1_x\) such a morphism.

**Definition 1.3** (\(A_\infty\)-unitary functor). We define an \(A_\infty\)-functor \(\mathcal{F} : \mathcal{A} \to \mathcal{A}'\) to be a map \(\mathcal{F}_0\) between the objects of \(\mathcal{A}\) and \(\mathcal{A}'\) and a collection of \(K\)-linear maps (for all \(n \geq 1\)):

\[
\mathcal{F}_n : \text{Hom}_{\mathcal{A}}(x_{n-1}, x_n) \otimes \ldots \otimes \text{Hom}_{\mathcal{A}}(x_0, x_1) \to \text{Hom}_{\mathcal{A}'}(\mathcal{F}_0(x_0), \mathcal{F}_0(x_n))[1 - n]
\]

such that for every \(0 < m \leq n\):

\[\text{where } [n] \text{ denotes the shift of a vector space down by an integer } n.\]
\[ \sum_{0 \leq j, k, l \leq m} (-1)^{j+k+l} F(1^{\otimes j} \otimes m^k_{\mathcal{A}} \otimes 1^{\otimes l}) = \sum_{1 \leq r \leq n} (-1)^{s} m^r_{\mathcal{A}}(F_1 \otimes \ldots \otimes F_n) \]

where \( j + k + l = m, n = j + 1 + l, i_1 + \ldots + i_v = n \) and

\[ s = \sum_{u=2}^{n} ((1 - i_u) \sum_{v=1}^{u} i_v). \]

Moreover we require that if \( \mathcal{A} \) and \( \mathcal{A}' \) are unitary the unit have to be preserved by \( F \) and \( F(\ldots, 1_x, \ldots) = 0 \) for all \( n \geq 2 \).

**Definition 1.4 (A\(_\infty\)-opposite category).** We define the opposite category of \( \mathcal{A} \) (denoted by \( \mathcal{A}'^{op} \)) to be the category defined by:

1. \( \text{Obj}(\mathcal{A}'^{op}) = \text{Obj}(\mathcal{A}) \);
2. \( \forall x, y \in \mathcal{A}'^{op} \) we have \( \text{Hom}_{\mathcal{A}'^{op}}(x, y) = \text{Hom}_{\mathcal{A}}(y, x) \);
3. \( \forall n > 1 \) we have \( m_{\mathcal{A}'^{op}}^n(f_1, \ldots, f_n) = (-1)^{\sum f_i} m^n_{\mathcal{A}}(f_1, \ldots, f_i) \), where \( \epsilon(f_n, \ldots, f_1) = \sum_{1 \leq i < j \leq k} (\text{deg} f_i + 1)(\text{deg} f_j + 1) + 1 \).

**Definition 1.5 (Coderivations).** We define the coderivations to be the \( K \)-linear maps:

\[ \hat{d}_k : \text{Hom}_{\mathcal{A}}(x_{n-1}, x_n) \otimes \ldots \otimes \text{Hom}_{\mathcal{A}}(x_0, x_1) \to \text{Hom}_{\mathcal{A}}(x_0, x_n)[2-k] \]

such that

\[ \hat{d}_k(f_n, \ldots, f_1) = \sum_{i=1}^{n-k+1} (-1)^{\hat{k} + 1} f_n \otimes \ldots \otimes m_k(f_{i+k-1}, \ldots, f_i) \otimes \ldots \otimes f_1, \]

where \( \hat{k} = (\text{deg} f_{i+1}) + \ldots + (\text{deg} f_{i-1} + 1) \). We denote by \( \hat{d} \), the following:

\[ \hat{d} = \sum_{k=1}^{n} \hat{d}_k \]

**Definition 1.6 (Opposite coderivations).** We define the opposite coderivations to be the \( K \)-linear maps:

\[ \hat{d}_{\mathcal{A}'^{op}}^n : \text{Hom}_{\mathcal{A}'^{op}}(x_{n-1}, x_n) \otimes \ldots \otimes \text{Hom}_{\mathcal{A}'^{op}}(x_0, x_1) \to \text{Hom}_{\mathcal{A}'^{op}}(x_0, x_n)[2-k] \]

such that

\[ \hat{d}_{\mathcal{A}'^{op}}^n(f_n, \ldots, f_1) = \sum_{i=1}^{n-k+1} (-1)^{\hat{k} + 1} f_n \otimes \ldots \otimes m_{\mathcal{A}'^{op}}^k(f_{i+k-1}, \ldots, f_i) \otimes \ldots \otimes f_1, \]

where \( \hat{k} = (\text{deg} f_{i+1}) + \ldots + (\text{deg} f_{i-1} + 1) \). We denote by \( \hat{d} \), the following:

\[ \hat{d}^{op} = \sum_{k=1}^{n} \hat{d}_{\mathcal{A}'^{op}}^k \]

If \( f = f_1 \otimes \ldots \otimes f_k \), we set

\[ \text{deg} f := \text{deg} f_1 + \ldots + \text{deg} f_n \]

and

\[ \text{deg} f := \text{deg} f_1 + \ldots + \text{deg} f_n + k \]

**Definition 1.7 (Right module over an A\(_\infty\)-category).** We define a right module \( \mathcal{M} \) over \( \mathcal{A} \) to be an \( A_{\infty} \)-functor \( \mathcal{M} : \mathcal{A}'^{op} \to \text{Ch}_K \).
Given two right $\mathcal{A}$-modules $\mathcal{F}_0$ and $\mathcal{F}_1$. We recall the definition of pre-natural transformation between them:

**Definition 1.8** (Pre-natural transformation). We define a pre-natural transformation $T$ of degree $g$ to be a sequence $(T^0, T^1, ..., T^d, ...)$ of $\mathbb{K}$-multilinear maps, such that, for all $x_0 \in \mathcal{A}$, we have:

$$T^d(x_0) \in \text{Hom}_{\text{Ch}_{\mathbb{K}}}^{\mathbb{K}}(\mathcal{F}_0(x_0), \mathcal{F}_1(x_0)),$$

and for every $d > 0$:

$$T^d : \text{Hom}_{\mathcal{A}}(x_{d-1}, x_d) \otimes ... \otimes \text{Hom}_{\mathcal{A}}(x_0, x_1) \to \text{Hom}_{\text{Ch}_{\mathbb{K}}}(\mathcal{F}_0(x_0), \mathcal{F}_1(x_0))[g-d].$$

We denote by $\text{Hom}_{\mathcal{A}_{\infty}}(\mathcal{F}_0, \mathcal{F}_1)^{\mathbb{K}}$ of pre-natural transformations between $\mathcal{F}_0$ and $\mathcal{F}_1$ of degree $g$. On the other hand, $\text{Hom}_{\mathcal{A}_{\infty}}(\mathcal{F}_0, \mathcal{F}_1)$ of all pre-natural transformations (in any degree) between $\mathcal{F}_0$ and $\mathcal{F}_1$.

**Definition 1.9** (Boundary operation). Given $T \in \text{Hom}_{\mathcal{A}_{\infty}}(\mathcal{F}_0, \mathcal{F}_1)^{\mathbb{K}}$, we define the differential $\partial$ of $T$ to be the pre-natural transformation $\partial T$ such that, for every $d > 0$:

$$\partial(T)^d(a_d, ..., a_1) = m^1_{\text{Ch}_{\mathbb{K}}} T^d(a_d, ..., a_1)$$

$$+ \sum_{s_1=1}^{d} m^2_{\text{Ch}_{\mathbb{K}}}(T^{s_1}(a_{s_1}, ..., a_1), \mathcal{F}_1^{d-s_1}(a_{d}, ..., a_{s_1+1}))$$

$$+ \sum_{s_1=1}^{d} (-1)^{(\deg T - 1) \deg a_{s_1} + ... + \deg a_{d}} m^2_{\text{Ch}_{\mathbb{K}}}(\mathcal{F}_0^{s_1}(a_{s_1}, ..., a_1)),$$

$$T^d(a_d, ..., a_{s_1+1}) + (-1)^{\deg T} d^{\mathbb{K}}(a).$$

**Definition 1.10** (Product of pre-natural transformations). Given two prenatural transformations $T_1 \in \text{Hom}_{\mathcal{A}_{\infty}}(\mathcal{F}_0, \mathcal{F}_1)$ and $T_2 \in \text{Hom}_{\mathcal{A}_{\infty}}(\mathcal{F}_1, \mathcal{F}_2)$, we define the product $T_2 \circ T_1$ to be the pre-natural transformation in $\text{Hom}_{\mathcal{A}_{\infty}}(\mathcal{F}_0, \mathcal{F}_2)$ given by:

$$(T_2 \circ T_1)^d(a_d, ..., a_1)(b) = \sum_{n=1}^{d} m^2_{\text{Ch}_{\mathbb{K}}}(T_2^{d-n}(a_d, ..., a_{n+1}), T_1^n(a_n, ..., a_1))(b).$$

**Theorem 1.1.** The category of right modules over an $\mathcal{A}_{\infty}$-category together with pre-natural transformations is a dg-category.

**Proof.** See §7 of [Fuk] or [Sei; pp. 19-20].

1.2. $\mathcal{A}_{\infty}$-Yoneda Lemma. There exists a version of the Yoneda Lemma for $\mathcal{A}_{\infty}$-categories. This implies that we can embed any $\mathcal{A}_{\infty}$-category into the category of modules. In this subsection we just want to recall how to associate a right module to an $\mathcal{A}_{\infty}$-category. We refer to §7 and §9 of [Fuk] for a more exhaustive exposition.

Let $\mathcal{A}$ be an $\mathcal{A}_{\infty}$-category. For every object $x \in \mathcal{A}$ we have an $\mathcal{A}_{\infty}$-right module $\text{Rep}(x) : \mathcal{A}_{\infty}^{op} \to \text{Ch}_{\mathbb{K}}$ in the following way:

If $n = 0$:

$$\text{Rep}(x)_0 := \text{Hom}_{\mathcal{A}}(-, x),$$

if $n > 0$:

$$\text{Rep}(x)_n : \text{Hom}_{\mathcal{A}_{\infty}}(x_{n-1}, x_n) \otimes ... \otimes \text{Hom}_{\mathcal{A}_{\infty}}(x_0, x_1)$$

$$\to \text{Hom}_{\text{Ch}_{\mathbb{K}}}(\text{Rep}(x)_0 x_0, \text{Rep}(x)_0 x_n)[1-n]$$

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is defined to be:

$$\text{Rep}(x_0, \ldots, f_1)(z) = (-1)^{\epsilon(f_n, \ldots, f_1)} m_{n+1}^1(z, f_1, \ldots, f_n)$$

where

$$\epsilon(f_n, \ldots, f_1) = \sum_{1 \leq i < j \leq k} (\deg f_i + 1)(\deg f_j + 1) + 1,$$

with $f_i \in \text{Hom}_{\mathcal{A}^{\text{op}}}(x_{i-1}, x_i)$ and $z \in \text{Hom}_\mathcal{A}(x_0, x)$.

We will denote by $\text{Rep}(\mathcal{A}^{\text{op}})$ the set of all modules $\text{Rep}(x)$ where $x \in \mathcal{A}$.

Remark 1. After easy calculations, we can show that the set $\text{Rep}(\mathcal{A}^{\text{op}})$ together with pre-natural transformations forms a dg-category with differentials $\partial$ and composition $\circ$. We call such a category category of right representable modules.

Make it functorial! Given an $A_\infty$-category, we built a bijection from $\text{Obj}(\mathcal{A})$ to $\text{Obj}(\text{Rep}(\mathcal{A}^{\text{op}}))$. Now we want to make this construction functorial.

Definition 1.11. We define $\text{Rep}$ from $\mathcal{A}$ to $\text{Rep}(\mathcal{A}^{\text{op}})$ such that:

$$\text{Rep} : \mathcal{A} \to \text{Hom}_{A_\infty\text{-Cat}}(\mathcal{A}^{\text{op}}, \mathcal{C}_{\mathbb{K}})$$

$x \mapsto \text{Rep}(x)$.

to be:

- If $n = 0$ we set $\text{Rep}_0(x) := \text{Rep}(x)$;
- If $n > 1$ we want a $\mathbb{K}$-linear map:

$$\text{Rep}_n : \text{Hom}_{\mathcal{A}^{\text{op}}}(x_{n-1}, x_n) \otimes \cdots \otimes \text{Hom}_{\mathcal{A}^{\text{op}}}(x_0, x_1)$$

$$\to \text{Hom}_{A_\infty\text{-Fun}}(\text{Rep}(x_0)(-), \text{Rep}(x_n)(-)).$$

$\text{Rep}_n(f_n, \ldots, f_1)$ is a prenatural transformation. So for every $l > 1$ we have:

$$\text{Rep}_n(f_n, \ldots, f_1)_l : \text{Hom}_{\mathcal{A}^{\text{op}}}(y_{l-1}, y_l) \otimes \cdots \otimes \text{Hom}_{\mathcal{A}^{\text{op}}}(y_0, y_1)$$

$$\to \text{Hom}_{\mathcal{C}_{\mathbb{K}}}(\text{Rep}(x_0)y_0, \text{Rep}(x_n)y_l)[1 - (n + l)].$$

If we take $c_j \in \text{Hom}_{\mathcal{A}^{\text{op}}}(y_{j-1}, y_j)$, we have that,

$$\text{Rep}_n(f_n, \ldots, f_1)_l(c_q, \ldots, c_0) : \text{Rep}(x_0)y_0 \to \text{Rep}(x_n)y_l$$

is defined as

$$\text{Rep}_n(f_n, \ldots, f_1)_l(c_q, \ldots, c_0)(z) := (-1)^{\epsilon(c)} m_{n+l+1}(f_n, \ldots, f_1, z, c_1, \ldots, c_l)$$

where $\epsilon(c) = (\deg f)(\deg c + \deg z)$, $f = (f_n, \ldots, f_1)$, $z \in \text{Hom}_{\mathcal{A}^{\text{op}}}(y_0, x_0)$ and $c = (c_1, \ldots, c_l)$.

Theorem 1.2. The functor $\text{Rep}$ is an $A_\infty$-functor.

Proof. [Fuk; Lemma 9.8]
1.3. $A_\infty$ quasi-equivalences.

**Definition 1.12** (Homotopy category). Let $\mathcal{A}$ be an $A_\infty$-category, we define the *homotopy category* $\text{Ho}(\mathcal{A})$ of $\mathcal{A}$, as the category whose objects are the objects of $\mathcal{A}$ and whose morphisms, for $x$ and $y \in \text{Obj}(\mathcal{A})$, are given by the quotients

$$\text{Hom}_{\text{Ho}(\mathcal{A})}(x,y) := \frac{Z^0(\text{Hom}_{\mathcal{A}}(x,y))}{B^0(\text{Hom}_{\mathcal{A}}(x,y))} = H^0(\text{Hom}_{\mathcal{A}}(x,y)),$$

where $Z^0(\text{Hom}_{\mathcal{A}}(x,y)) := \ker(m^1_{\mathcal{A}} : \text{Hom}^0_{\mathcal{A}}(x,y) \to \text{Hom}^1_{\mathcal{A}}(x,y))$ and $B^0(\text{Hom}_{\mathcal{A}}(x,y)) := \text{Im}(m^1_{\mathcal{A}} : \text{Hom}^1_{\mathcal{A}}(x,y) \to \text{Hom}^0_{\mathcal{A}}(x,y))$.

**Definition 1.13** (Quasi-equivalence). Let $\mathcal{A}$, $\mathcal{A}'$ be $A_\infty$-categories (unitary), we say that an $A_\infty$-functor $\{F^n\} : \mathcal{A} \to \mathcal{A}'$ is a *quasi-equivalence* if:

1. ($\text{we1}$) $\text{Ho}(F) : \text{Ho}(\mathcal{A}) \to \text{Ho}(\mathcal{A}')$ is an equivalence of categories.
2. ($\text{we2}$) $F^1 : \text{Hom}_{\mathcal{A}}(x,y) \to \text{Hom}_{\mathcal{A}'}(F^0x,F^0y)$ is a quasi-isomorphism.

**Example 1.2.** Two dg-categories which are weak-equivalent are quasi-equivalent as $A_\infty$-categories.

**Remark 2.** Given an $A_\infty$-category $\mathcal{A}$ the $A_\infty$-functor $\text{Rep}$ is a quasi-equivalence, ($\text{we1}$) is trivial and ($\text{we2}$) follows immediately from [Fuk, Lemma 9.22]. It means that every $A_\infty$-category is quasi-equivalent to a dg-category.

1.4. **Pretriangulated $A_\infty$-categories.** The next definition is probably due to Kontsevich, we refer to [LyM2] for the proofs. Let $\mathcal{A}$ be a $K$-linear $A_\infty$-category.

**Definition 1.14** (Shift category and shift functor). We define the category $\Sigma(\mathcal{A})$ to be the $A_\infty$-category such that $\text{Obj}(\Sigma \mathcal{A}) = (\text{Obj}(\mathcal{A})) \times \mathbb{Z}$, and morphisms are defined as follows

$$\text{Hom}_{\Sigma(\mathcal{A})}(x[n],y[m]) := \text{Hom}_{\mathcal{A}}(x,y)[m-n],$$

where $x, y \in \mathcal{A}$. The endofunctor sending $x[n]$ to $x[n+1]$ is called shift functor.

**Definition 1.15** (Closed under shift). We say that $\mathcal{A}$ is closed under shift if $\mathcal{A} \hookrightarrow \Sigma(\mathcal{A})$ is a quasi-equivalence.

**Definition 1.16** ($A_\infty$-twisted complexes). A *twisted complex* in $\mathcal{A}$ is a finite set of objects of $\Sigma(\mathcal{A})$ $(E_i[n_i])_{i \in \mathbb{Z}}$ together with maps $\alpha_{ij} \in \text{Hom}_{\mathcal{A}}(E_i,E_j)^{n_j-n_i+1}$ if $i < j$ such that:

$$\sum_{k=1}^{+\infty} m_k(\alpha,...,\alpha) = 0$$

**Remark 3.** The set of $A_\infty$-twisted complexes defined above has the structure of an $A_\infty$-category [Sei], [LyM2], we denote such an $A_\infty$-category by $\text{pretr}(\mathcal{A})$. Moreover we have an $A_\infty$-functor $i_{A_\infty} : \mathcal{A} \hookrightarrow \text{pretr}(\mathcal{A})$ and it was proven that the construction is functorial (cf. §3 of [Sei]). Given an $A_\infty$-morphism $\mathcal{F}$ we denote by $\text{pretr}_{A_\infty}(\mathcal{F})$ the induced functor.

**Definition 1.17** (Pretriangulated $A_\infty$-categories). We say that an $A_\infty$-category $\mathcal{A}$ is *pretriangulated* if $\mathcal{A}$ is closed under shift and the functor $i_{A_\infty} : \mathcal{A} \hookrightarrow \text{pretr}(\mathcal{A})$ is a quasi-equivalence.

**Remark 4.** If $\mathcal{C}$ is a dg-category $\text{pretr}(\mathcal{C}) = \text{pretr}(\mathcal{C})$. Where $\text{pretr}(\mathcal{C})$ denotes the pretriangulated envelope of the dg-category $\mathcal{C}$ according to the notation of [Kel].

We recall the fundamental proposition:

**Proposition 1.3.** Let $\mathcal{C}$ be a pretriangulated dg-category (pretriangulated $A_\infty$-category) then the homotopy category $\text{Ho}(\mathcal{C})$ is a triangulated category.
We have the following [Sei; Lemma 3.25.]:

**Theorem 1.4.** Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a quasi-equivalence between two $A_\infty$-categories then $\text{pretr}_{A_\infty}(\mathcal{F}) : \text{pretr}_{A_\infty}(\mathcal{A}) \to \text{pretr}_{A_\infty}(\mathcal{B})$ is a quasi-equivalence.

By the following diagram we deduce that an $A_\infty$-category $\mathcal{A}$ is $A_\infty$-pretriangulated if and only if $\text{Rep}(\mathcal{A})$ is (dg) pretriangulated.

\[ \begin{array}{ccc}
\mathcal{A} & \sim & \text{Rep}(\mathcal{A}) \\
\text{pretr}_{A_\infty}(\mathcal{A}) & \sim & \text{pretr}(\text{Rep}(\mathcal{A}))
\end{array} \]

**Definition 1.18** (Idempotent complete). We say that an additive category $\mathcal{X}$ is *idempotent complete* if any endomorphism $E : k \to k$ such that $E^2 = E$ (idempotent) is such that $k = \text{Im}(E) \oplus \ker(E)$.

According to [BaSc], in general, we can always embed an additive category in an idempotent complete category (we denote by $(-)^{ic}$ such an embedding) moreover if $\mathcal{X}$ is a triangulated category we have the following [BaSc; Thm.1.5.]:

**Proposition 1.5.** If $\mathcal{X}$ is a triangulated category, its idempotent completion $(\mathcal{X})^{ic}$ admits a unique triangulated structure such that the canonical functor $(-)^{ic}$ is exact.

**Definition 1.19** (Idempotent complete). We say that a pretriangulated dg-category $\mathcal{F}$ (or $A_\infty$-category) is *idempotent complete* if the homotopy category $\text{Ho}(\mathcal{F})$ is idempotent complete.

2. $A_\infty$-**Nerve**

### 2.1. Brief background on $\infty$-categories.

We briefly recall the basics about $\infty$-categories. The non-expert reader can have a look at Chapter 1 and 2 of [Lur1].

**Definition 2.1** (Minimal $\mathbb{K}$-linear category). Let $n$ be a positive integer (or zero). We define the *minimal $\mathbb{K}$-linear category* $[n]_{\mathbb{K}}$ to be the category such that the objects are the positive integers $\{0, 1, 2, \ldots, n\}$ and the maps are defined by

\[ \text{Hom}_{[n]_{\mathbb{K}}}(i, k) = \begin{cases} 
0_{\mathbb{K}}, & \text{if } i > k \\
\langle j_{ik}\rangle_{\mathbb{K}}, & \text{if } i < k \\
1_{\mathbb{K}}, & \text{if } i = k.
\end{cases} \]

where $\langle j_{ik}\rangle_{\mathbb{K}}$ is the $\mathbb{K}$-vectorial space generated by the element $j_{ik}$. The composition is defined as follow; let $i_1 < i_2 < i_3$ be positive integers. Then:

\[ \cdot := \text{Hom}_{[n]_{\mathbb{K}}}(i_2, i_3) \otimes_{\mathbb{K}} \text{Hom}_{[n]_{\mathbb{K}}}(i_1, i_2) \to \text{Hom}_{[n]_{\mathbb{K}}}(i_1, i_3) \]

is such that

\[ j_{i_2i_3} \cdot j_{i_1i_2} = j_{i_1i_3}, \]

where $j_{i_1i_3}$ is the unique morphism in $\text{Hom}_{[n]_{\mathbb{K}}}(i_1, i_3)$.

**Remark 2.** The definition above works even without the $\mathbb{K}$-linear enrichment.

**Definition 2.2** (Simplex category). We define the *simplex category* to be the category whose objects are the minimal $\mathbb{K}$-linear categories $[n]$ and whose morphisms are the functions $f$ such that $f(i) \leq i$ and $f(i_1) \leq f(i_2)$ if $i_1 \leq i_2$. We denote by $\Delta$ such a category.

**Definition 2.3** (Simplicial set). We define a *simplicial set* to be a contravariant functor from the simplex category $\Delta$ to the category of sets.
We will denote by sSet the category of simplicial sets.

**Example 2.1.** Given a positive integer \( n \), the functor \( \Delta^n \) defined as \( \text{Hom}_{\Delta}(-,[n]) : \Delta^{\text{op}} \to \text{Sets} \) is a simplicial set. Moreover for each \( 0 \leq i \leq n \) the functor generated by all the maps \( d^i : [n-1] \to [n] \) (which are the injective maps not having \( j \) in the image), with \( i \neq j \), is a subsimplicial set of \( \Delta^n \). We call such a simplicial set \((n,i)\)-horn and we denote it by \( \Lambda^i_n \).

**Definition 2.4** (\(\infty\)-category). We define an \(\infty\)-category to be a simplicial set \( X \) such that, for every positive integer \( n \) and every natural transformation \( \phi : \Lambda^k_n \to X \), with \( 0 < k < n \), there exists (at least) one map \( \tilde{\phi} \) such that the following diagram:

\[
\begin{array}{ccc}
\Lambda^k_n & \xrightarrow{\phi} & X \\
\downarrow \quad & & \downarrow \quad \tilde{\phi} \\
\Delta^n & \xrightarrow{\tilde{\phi}} & X \\
\end{array}
\]

commutes.

Let \( X \) be an \(\infty\)-category, the objects of \( X \) are given by the elements of the set \( X_0 \) and the set of morphisms from \( x \) to \( y \), denoted by \( \text{Map}_X(x,y) \), is given as the pullback of the following diagram:

\[
\begin{array}{ccc}
\text{Map}_X(x,y) & \to & X_1 \\
\downarrow & & \downarrow (d,c) \\
\bullet & \to & X_0 \times X_0 \\
(x,y) & \quad & (x,y) \\
\end{array}
\]

where \( d = X(d_1) : X_1 \to X_0 \), and \( c = X(d_0) : X_1 \to X_0 \).

**Example 2.2.** Let \( X \) be an \(\infty\)-category. Fixing two elements \( x \) and \( y \in X_0 \), we get a simplicial set, denoted by \( \text{Hom}^n_X(x,y) \), whose 0-simplices are 1-simplices in \( X \) from \( x \) to \( y \), whose 1-simplices are 2-simplices of the form:

\[
\begin{array}{ccc}
x & \xrightarrow{1_x} & x \\
\downarrow & & \downarrow \\
x & \quad & y \\
\end{array}
\]

and whose \( n \)-simplices are \((n+1)\)-simplices whose target is \( y \) and whose \((n+1)\)th-face degenerates at \( \Lambda^k_n \).

**Example 2.3.** Let \( \mathcal{C} \) be a category, the simplicial set defined as the set of the compositions of \( n \)-arrows of \( \mathcal{C} \), for every \( n > 0 \), and as the set of objects of \( \mathcal{C} \), if \( n = 0 \), is an \(\infty\)-category. We call such a simplicial set the nerve of \( \mathcal{C} \) and we denote it by \( \text{N}_\text{cat}(\mathcal{C}) \).

Given an \(\infty\)-category \( X \) and two morphisms \( f, g \in \text{Map}_X(x,y) \) we say that \( f \) is homotopic to \( g \) if there exists a natural transformation \( \sigma : \Delta^2 \to X \) of the form:

\[
\begin{array}{ccc}
x & \xrightarrow{1_x} & x \\
\downarrow & & \downarrow f \\
x & \xrightarrow{g} & y \\
\end{array}
\]

The homotopy relation is an equivalence relation.

\(^2\text{cf. Definition 3.5}\)
Definition 2.5 (Homotopy category). We define the homotopy category of an $\infty$-category $X$ to be the category whose objects are the elements of $X_0$ and whose morphisms, fixed two objects $x$ and $y$, are given by the quotient of $\text{Map}_X(x, y)$ by the homotopy relation defined above. We denote such a category by $\text{Ho}(X)$.

2.2. $\infty$-stable categories.

Definition 2.6 (Zero object in $\infty$-category). Let $X$ be an $\infty$-category, we define the zero object $0$ to be an object of $X$ that is both initial and final, i.e.

$$\text{Map}_X(c, 0) \simeq \text{Map}_X(0, c) \simeq *$$

for all $c \in X_0$.

Remark 6. The zero object is unique up to equivalence.

Definition 2.7 (Pointed $\infty$-category). We define a pointed $\infty$-category to be an $\infty$-category equipped with a zero object.

Definition 2.8 (Fiber (cofiber) sequence). Let $X$ be a pointed $\infty$-category, we consider the functor of simplicial sets $T : \Delta^1 \times \Delta^1 \to X$ of the form:

$$x \xrightarrow{f} y \xleftarrow{g} z$$

We call $T$ a triangle in $X$. If $T$ is a pullback square we call it fiber sequence (fiber of $g$), if $T$ is a pushout square we call it cofiber sequence (cofiber of $f$).

Remark 7. It easy to check that a triangle $T$ is the datum of:

- Two morphisms $f, g \in X_1$.
- Two 2-simplices in $X_2$ of the form:

$$x \downarrow \downarrow h \xleftarrow{f} y \xrightarrow{g} z$$

We will indicate the triangle $T$ by

$$x \xrightarrow{f} y \xleftarrow{g} z.$$

Definition 2.9 ($\infty$-stable). We say that $X$ is a $\infty$-stable category if

(S1) $X$ is an $\infty$-category equipped with zero object (pointed $\infty$-category).
(S2) Every morphism has fibers and cofibers.
(S3) Every triangle in $X$ is a fiber sequence if and only if it is a cofiber sequence.

Given an $\infty$-stable category $X$, we have an auto-equivalence $\Sigma : X \to X$ called suspension functor, with inverse $\Omega$ called loop functor, obtained via the category of subfunctors of $\text{Fun}(\Delta^1 \times \Delta^1, X)$ generated by the following pullbacks and pushouts in $\Delta^1 \times \Delta^1 \to X$:

$$\begin{array}{ccc}
0 & \xrightarrow{f} & x \\
\Sigma z & \leftarrow & z \\
0' & \xrightarrow{g} & x
\end{array}$$

where $0$ and $0'$ are zero objects in $X$ (cf. Chapter 1 of [Lur2] for a precise definition).

If $n > 0$ we will denote by $x[n]$ the $\Sigma$ functor applied $n$-times to $x \in X$, if $n < 0$ we will denote by $x[n]$ the $\Omega$ functor applied $n$-times to $x$.

We have the following fundamental theorem:
Theorem 2.1. If $X$ is an $\infty$-stable category then the homotopy category $\text{Ho}(X)$ is a triangulated category with $\Sigma$ the suspension functor as shift functor and distinguished triangles given by the following $\Delta^2 \times \Delta^1 \to X$ diagram:

\[
\begin{array}{ccc}
x & \rightarrow & 0 \\
y & \downarrow & \downarrow \\
z & \rightarrow & w \\
\end{array}
\]

We denote by $\text{Cat}_\infty^{St}$ the category of $\infty$-stable categories whose objects are the $\infty$-stable categories and whose morphisms are the functors of $\infty$-categories.

A functor between $\infty$-categories "a priori" does not give information about the zero object and the fiber sequences, so in the case of $\infty$-stable categories we prefer use the following definition of functors.

Definition 2.10 (Exact functor). Let $F : X \to X'$ be a functor between $\infty$-stable categories. We say that $F$ is exact if the following are satisfy:

1. $F(0_X) = 0_{X'}$.
2. $F$ carries fiber sequences to fiber sequences.
3. $F$ carries two exact functors.

Remark 8. If (E1) holds true, than $F$ carries triangles to triangles. Moreover $F$ satisfies (E2) if and only if $F$ satisfies (E2').

Example 2.4. The identity functor of an $\infty$-stable category and the composition of two exact functors are exact functors.

We denote by $\text{Cat}_\infty^{Ex}$ the exact $\infty$-stable category whose objects are the $\infty$-stable categories and whose morphisms are the exact functors.

2.3. $A_\infty$-nerve. The nerves are a useful tool to pass from a category to an $\infty$-category, in this section we will define the $A_\infty$-nerve, originally defined in [Fao], which is a generalization of the dg-nerve of Lurie.

Proposition 2.2. Let $n$ be a positive integer and $\mathcal{C}$ be an $A_\infty$-category (unitary). Every maps $\{\mathcal{F}^n\} \in \text{Hom}_{A_\infty\text{-Cat}}([n], \mathcal{C})$ is uniquely determined by:

1. $n + 1$-objects $\{X_i\}_{0 \leq i \leq n}$ of $\mathcal{C}$,
2. A set of morphisms $f_I$ for all set of integers $I = \{i_0 < i_1 < ... < i_m < i_{m+1}\}$ where $0 \leq i_0 < i_{m+1} \leq n$ that satisfying the following:

$$m_\mathcal{C}^1(f_I) = \sum_{1 \leq j \leq m} (-1)^{j-1} f_{I-j} + \sum_{1 \leq j \leq m} (-1)^{1+(m+1)(j-1)} m_\mathcal{C}^2(f_{i_j...i_{m+1}}, f_{i_0...i_{j}})$$

$$+ \sum_{r \geq 2} \sum_{1 \leq j_1 < ... < j_r} (-1)^{1+r} m_\mathcal{C}^r(f_{i_{m+1-j_r}...i_{m+1}, ..., f_{i_0...i_{j_r}}}),$$

where

$$\mathcal{T}_r = \{s_1, ..., s_r \in \mathbb{N} | \sum_{j=1}^r s_j = m + 1\}$$

$$\epsilon_r(i_1, ..., i_r) = \sum_{2 \leq k \leq r} (1 - i_k + i_{k-1})i_{k-1}.$$
Proof. Given an $\Lambda_\infty$-unitary functor $F = \{F_m\}_{m \geq 0} : [n]_k \to \mathcal{C}$ the image of the map $F_0$ is uniquely determined by $n + 1$ objects $\{X_i\}_{0 \leq i \leq n}$ in $\mathcal{C}$ because $[n]_k$ has exactly $n + 1$ objects. Moreover fixed two integers $i_-$ and $i_+$ such that $i_- < i_+$, for every $0 \leq m \leq n$ we consider the map:

$$F_m : \text{Hom}_{[n]_k}(m_{i_-, i_+}) \to \text{Hom}_{\mathcal{C}}(F_{X_{i_-}}, F_{X_{i_+}})[1 - m]$$

the unique non-trivial ones are those such that $i_- < i_1 < i_2 < \ldots < i_{m-1} < i_m < i_+$. So the image of $F_m$ is non-zero if and only if we have a set $I$ of $m + 1$-elements in $[n]$ such that $I = \{i_- < i_1 < i_2 < \ldots < i_{m-1} < i_m < i_+\}$. Then $F_m$ is uniquely determined by the image $f_I = F_m(j_{i_{m-1} i_+}, \ldots, j_{i_1 i_+})$ where $j_{il}$ denotes the only non trivial map in $\text{Hom}_{[n]_k}(i_k, i_l)$, and clearly they satisfy (3) because they are the image of the $\Lambda_\infty$-functor $F$.

$\square$

**Proposition 2.3.** Given a map $\alpha : [m] \to [n]$ in $\Delta$, we have an induced map $\text{Hom}_{\Lambda_\infty, \text{cat}}(\alpha, \mathcal{C})$ given by:

$$\text{Hom}_{\Lambda_\infty, \text{cat}}(\alpha, \mathcal{C}) : \text{Hom}_{\Lambda_\infty, \text{cat}}([m]_k, \mathcal{C}) \to \text{Hom}_{\Lambda_\infty, \text{cat}}([n]_k, \mathcal{C})$$

$$(\{X_i\}_{0 \leq i \leq n}, \{f_I\}) \mapsto (\{X_{\alpha(j)}\}_{0 \leq j \leq m}, \{g_J\}).$$

where $g_J$ is:

$$g_J = \begin{cases} f_{\alpha(j)}, & \text{if } \alpha_j \text{ is injective} \\ 1_{X_i}, & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') = X_i \\ 0, & \text{otherwise}, \end{cases}$$

such that, given $\alpha : [m] \to [n]$ and $\beta : [n] \to [l]$, then

$$\text{Hom}_{\Lambda_\infty, \text{cat}}(\beta \cdot \alpha, \mathcal{C}) = \text{Hom}_{\Lambda_\infty, \text{cat}}(\alpha, \mathcal{C}) \cdot \text{Hom}_{\Lambda_\infty, \text{cat}}(\beta, \mathcal{C}).$$

Moreover given $Id : [n] \to [n]$ then

$$\text{Hom}_{\Lambda_\infty, \text{cat}}(Id, \mathcal{C}) = Id_{\text{Hom}_{\Lambda_\infty, \text{cat}}([n], \mathcal{C})}.$$

**Proof.** First of all, we want to associate to $\alpha$ an $\Lambda_\infty$-unitary functor (denoted by $\{\alpha\}$) between the minimal categories $[m]_k \to [n]_k$. We define the $\Lambda_\infty$-functor $\{\alpha_n\}_{n \geq 0} : [m]_k \to [n]_k$ in the following way:

- if $k = 0$, $\alpha_k = \alpha$,
- if $k = 1$,
  $$\alpha_1 : \text{Hom}_{[m]_k}(l, s) \to \text{Hom}_{[n]_k}(\alpha(l), \alpha(s))$$
  $$j_{ls} \mapsto \alpha_1(j_{ls}) = \begin{cases} 0, & \text{if } l > s \\ 1, & \text{if } l = s \\ j_{\alpha(l) \alpha(s)}, & \text{if } l < s \end{cases}$$

- if $k > 1$, $\alpha_k = 0$.

The induced map $\text{Hom}_{\Lambda_\infty, \text{cat}}(\alpha, \mathcal{C})$ is given by the composition with the $\Lambda_\infty$-functor $\{\alpha_n\}_{n \geq 0}$. Let $F \in \text{Hom}_{\Lambda_\infty, \text{cat}}([n]_k, \mathcal{C})$. For all $t \geq 1$ we have:

$$(F \alpha)_t = \sum_{r=1}^{t} \sum_{1 \leq i_1 + \ldots + i_t = t} F_r(\alpha_{i_1}, \ldots, \alpha_{i_t}).$$

Since only $\alpha_1$ is non-trivial. We have $r = t$, $i_1 = i_2 = \ldots = i_t = 1$ and $(F \alpha)_t$ becomes:

$$(F \alpha)_t = F_t(\alpha_1, \ldots, \alpha_1).$$

Therefore

$$F_1(\alpha_1(j_{i_0 i_1})) = F_1(j_{\alpha_0 \alpha_1}(\alpha_{i_1})),
F_2(\alpha_1(j_{i_0 i_1}), \alpha_1(j_{i_1 i_2})) = F_2(j_{\alpha_0 \alpha_1}(\alpha_{i_1}), J_{\alpha_1 \alpha_1}(\alpha_{i_2})).$$
... \( F_n(\alpha_1(j_{i_1}), \ldots, \alpha_1(j_{i_m})) = F_n(j_{\alpha(i_1)}, \ldots, j_{\alpha(i_l)}) \)

Of course, \( i_l \) are positive integers smaller than \( m \) (because \( \alpha : [m] \to [n] \)), so if we take an element in \( \text{Hom}_{A_\infty}\text{-cat}([n]_\mathbb{K}, \mathcal{C}) \) denoted by \( \{X_i\}_{0 \leq i \leq n}, \{f_i\} \) this is sent to \( \{X_{\alpha(j)}\}_{0 \leq j \leq m}, \{g_j\} \) where \( g_j \) is:

\[
g_j = \begin{cases} 
  f_{\alpha(j)}, & \text{if } \alpha(j) \text{ is injective} \\
  1_{X_i}, & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') = X_i \\
  0, & \text{otherwise.}
\end{cases}
\]

This is what we want to prove. \( \square \)

**Definition 3.11** (\( A_\infty \)-nerve). Let \( \mathcal{C} \) be a unitary \( A_\infty \)-category. We define the \( A_\infty \)-nerve of \( \mathcal{C} \) to be the simplicial set (denoted by \( N_{A_\infty}(\mathcal{C}) \)) such that for all positive integers \( n \)

\[
N_{A_\infty}(\mathcal{C})_n := \text{Hom}_{A_\infty}\text{-cat}([n]_\mathbb{K}, \mathcal{C}).
\]

And for every \( \alpha : [m] \to [n] \in \Delta \) the element \( \{X_i\}_{0 \leq i \leq n}, \{f_i\} \) in \( N_{A_\infty}(\mathcal{C})_n \) is sent to \( \{X_{\alpha(j)}\}_{0 \leq j \leq m}, \{g_j\} \) where \( g_j \) is:

\[
g_j = \begin{cases} 
  f_{\alpha(j)}, & \text{if } \alpha(j) \text{ is injective} \\
  1_{X_i}, & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') = X_i \\
  0, & \text{otherwise.}
\end{cases}
\]

**Remark 9.** Note that if \( \mathcal{C} \) is a dg-category then \( N_{A_\infty}(i(\mathcal{C})) = N_{\text{dg}}(\mathcal{C}) \) where \( N_{\text{dg}} \) is the dg-nerve defined in [Lur1; 1.3.1.6].

**Theorem 2.4.** Let \( \mathcal{C} \) be an \( A_\infty \)-category, then \( N_{A_\infty}(\mathcal{C}) \) is an \( \infty \)-category.

**Proof.** [Fao; Prop. 2.2.12]. \( \square \)

### 3. Properties of the \( A_\infty \)-Nerves

This section is divided in three parts: in the first one we will give a useful characterization of the mapping space of the \( A_\infty \)-nerve, in the second we will recall some classical result about model categories, finally we will prove the main theorem of the paper that will be the fundamental tool to give a comparison between the \( A_\infty \)-categories and the \( \infty \)-stable categories.

#### 3.1. Simplicial Objects and DK-correspondence

Let \( \mathcal{A} \) be an abelian category, we denote by \( \text{Ch}^0_\mathbb{K} \) the category of chain complexes bounded above. In particular if \( \mathcal{A} \) is the category of \( \mathbb{K} \)-modules, we denote by \( \text{Ch}^\geq_\mathbb{K} \) the category of chain complexes of \( \mathbb{K} \)-modules bounded above.

**Definition 3.1** (Simplicial Object). A simplicial object \( A \) in \( \mathcal{A} \) is a functor \( X : \Delta^{op} \to \mathcal{A} \).

We have a functor \( N_* : \text{Fun}(\Delta^{op}, \mathcal{A}) \to \text{Ch}^\geq_\mathbb{K} \) that associates to each simplicial object \( A \) the chain:

\[
... \to N_2(A) \overset{A(d_0)}{\to} N_1(A) \overset{A(d_0)}{\to} N_0(A) \to 0 \to ...
\]

where:

\[
N_n(A) := \bigcap_{1 \leq i \leq n} \ker(A(d_i))
\]

and \( d_j : [n-1] \to [n] \) is the natural injective map such that \( j \not\in \text{Im}(d_j) \).
We have also a functor $D K_\bullet : Ch_{\geq 0}^\infty \to \text{Fun}(\Delta^{op}, A)$ that associates to each chain $C^\bullet$ the simplicial object $D K_\bullet(C) : \Delta^{op} \to A$ defined, for every $n$, to be:

$$D K_\alpha(C) := \bigoplus_{\alpha : [n] \to [k]} C_k,$$

where $\alpha$ is a surjective map.

Moreover, given a map $\beta : [n'] \to [n]$, we define $D K_\bullet(\beta)$ to be the matrix with $(\alpha, \alpha')$ entries:

$$(f_{\alpha, \alpha'}) : \bigoplus_{\alpha} C_k \to \bigoplus_{\alpha'} C_{k'},$$

such that:

$$f_{\alpha, \alpha'} = \begin{cases} 
1_{C_k}, & \text{if } \alpha \text{ and } \alpha' \text{ are fit in a diagram} \\
d_k, & \text{if } \alpha \text{ and } \alpha' \text{ are fit in a diagram} \\
0, & \text{otherwise.}
\end{cases}$$

**Theorem 3.1.** The functors $D K_\bullet$, $N_\ast$ are adjoint in both directions (i.e. $D K_\bullet \dashv N_\ast$ and $N_\ast \dashv D K_\bullet$):

**Proof.** [DoPu; Satz 3.6]. □

Let $Z \Delta^n$ denote the free abelian group generated by $\Delta^n[j]$, for every $j$. Let us build the chain associated $N_\ast(Z \Delta^n)$.

**Example 3.1.** We take $\Delta^0 = \text{Hom}_\Delta(-, [0])$, if $n = 0$ then $N_0(Z \Delta^0) = \ker(Z \Delta_0 \to 0) = Z \Delta_0^0 = \{1 \text{ generator } g_0\}$. If $n = 1$, by definition, $N_1(Z \Delta^0) = \ker(d^1 : Z \Delta_0^1 \to Z \Delta_0^0) = 0$, because $Z \Delta_0^0$ is generated by $g_0$ and $d^1(g_0) = g_0 \neq 0$. We can proceed in the same way for all the other $n \geq 1$. Hence the chain associated to $Z(\Delta^0)$ is given by:

$$\ldots \to 0 \xrightarrow{d_0} 0 \xrightarrow{d_0} < g_0 > \to \ldots$$

**Example 3.2.** We take $\Delta^1 = \text{Hom}_\Delta(-, [1])$, if $n = 0$ we have $N_0(Z \Delta^1) = \ker(Z \Delta_0^1 \to 0) = Z \Delta_0^1 = \{2 \text{ generators } g_0 \text{ and } g_1\}$. If $n = 1$ we have $N_1(Z \Delta^1) = \ker(d^1 : Z \Delta_1^1 \to Z \Delta_1^0)$. In $Z \Delta_1^1$ we have three generators $g_{00}$, $g_{01}$ and $g_{11}$ given by the following maps:

$$0 \xrightarrow{g_{00}} 0 \xrightarrow{g_{01}} 0 \quad 0 \xrightarrow{g_{11}} 0$$

$N_1(Z \Delta^1)$ is given by the elements $Z \Delta_1^1$ of the form $\alpha_{00}g_{00} \oplus \alpha_{01}g_{01} \oplus \alpha_{11}g_{11}$ such that $d^1 = 0$, where $\alpha_{ij} \in \mathbb{K}$. By definition:

$$d^1(\alpha_{00}g_{00} \oplus \alpha_{01}g_{01} \oplus \alpha_{11}g_{11}) = \alpha_{00}g_{00} \oplus \alpha_{01}g_{01} \oplus \alpha_{11}g_{11}$$

$$= (\alpha_{00} + \alpha_{01})g_{00} \oplus \alpha_{11}g_{11}. \quad (4)$$
and it is zero only if \( \alpha_{00} + \alpha_{01} = 0 \) and \( \alpha_{11} = 0 \).

Hence \( \ker(\mathbb{Z}\Delta^1 \to \mathbb{Z}\Delta^0) = \langle g_{00} - g_{01} \rangle \).

Then the associated chain \( \mathbb{Z}(\Delta^1) \) is given by:

\[
\begin{array}{c}
\ldots \\
0 \\
\Delta_1 \\
\ldots \\
\end{array} \quad 0 < g_{00} - g_{01} > \quad d_0 < g_0 > \oplus < g_1 > \quad 0 \quad \ldots
\]

such that \( d^0 < g_{00} - g_{01} >= g_0 - g_1 \).

Let \( \mathcal{C} \) be a dg-category and \( x, y \) two fixed objects in \( \mathcal{C} \). By Example [3.1] we can identify the homomorphisms of complexes \( f : \mathbf{N}_*(\mathbb{Z}\Delta^0) \to \text{Map}_\mathcal{C}(x, y) \) with the maps \( f : x \to y \) of degree zero such that \( df = 0 \). By Example [3.2] we can identify the homomorphisms of complexes \( f : \mathbf{N}_*(\mathbb{Z}\Delta^1) \to \text{Map}_\mathcal{C}(x, y) \) with the set of the maps \( f_{02}, f_{12}, f_{012} : x \to y \) such that \( \deg f_{02} = \deg f_{12} = 0, \deg f_{012} = -1, \ d f_{012} = f_{02} - f_{12} \) and \( df_{02} = df_{12} = 0. \)

More generally let us discuss an important lemma (implicitly assumed by Lurie [Lur2; pg. 66]) which characterizes the maps between \( \mathbf{N}_*(\mathbb{Z}\Delta^n) \) and \( \text{Map}_\mathcal{C}(x, y). \)

**Lemma 3.2.** We can identify \( f : \mathbf{N}_*(\mathbb{Z}\Delta^n) \to \text{Map}_\mathcal{C}(x, y) \) of degree \( |l| - 2 \) for all subset \( l = \{ 0 \leq i_0 < \ldots < i_j < j + 1 \leq n \} \) such that:

\[
(f)
\]

\[
d_f = \sum_{0 \leq k \leq j} (-1)^k f_{l-k}.
\]

**Proof.** We denote by \( g_{i_0 \ldots i_j} \) the free generator associated to the map \( [j] \to [n] \) which sends the integer \( k \in [j] \) to \( i_k \in [n] \). It follows immediately that

\[
\{ \bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} g_{i_0 \ldots i_j} \} = \mathbb{Z}\Delta^n.
\]

By definition, an element \( \alpha_{i_0 \ldots i_j} g_{i_0 \ldots i_j} \) is in \( \mathbb{N}_j(\mathbb{Z}\Delta^n) \) if and only if

\[
\begin{cases}
 d^j( \bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} g_{i_0 \ldots i_j} ) = 0 \\
 \ldots \\
 d^j( \bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} g_{i_0 \ldots i_j} ) = 0
\end{cases}
\]

(5)

Now, if we focus on the first row in (5), we have that

\[
\begin{align*}
& d^j( \bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} g_{i_0 \ldots i_j} ) = 0 \\
& \quad \text{if and only if}
\end{align*}
\]

(6)

\[
\sum_{i_j = i_{j-1} + 1}^n \alpha_{i_0 \ldots i_j} = -\alpha_{i_0 \ldots i_{j-1} i_{j-1} i_{j-1}}.
\]

So we can rewrite (5) in terms of the following system of \( j - 1 \) equations

\[
\begin{cases}
 d^{j-1}( \bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} (g_{i_0 i_1 \ldots i_j} - g_{i_0 \ldots i_{j-1} i_{j-1}} ) ) = 0 \\
 \ldots \\
 d^j( \bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} (g_{i_0 i_1 \ldots i_j} - g_{i_0 \ldots i_{j-1} i_{j-1}} ) ) = 0
\end{cases}
\]

(7)

Proceeding as for the first row, we obtain the following system of \( j - 2 \) equations equivalent to (7)
\begin{align*}
\sum_{0 \leq k_1^0 \ldots k_j^0 \leq 1} (-1)^{k_1^0 + \ldots + k_j^0} \delta_{i_1 \ldots i_j} \left( (g_{i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_j - 2 i_j}) \right) = 0 \\
\sum_{0 \leq k_1^0 \ldots k_j^0 \leq 1} (-1)^{k_1^0 + \ldots + k_j^0} \delta_{i_1 \ldots i_j} \left( (g_{i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_j - 2 i_j}) \right) = 0.
\end{align*}

We can go on as before by removing one by one the equations from the system. In the end we have that $\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \delta_{i_0 \ldots i_j} \alpha_{i_0 \ldots i_j}$ is in $N_j(Z \Delta^n)$ if it is of the form

\begin{align*}
\sum_{0 \leq k_1^0 \ldots k_j^0 \leq 1} (-1)^{k_1^0 + \ldots + k_j^0} \delta_{i_1 \ldots i_j} \left( (g_{i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_j - 2 i_j}) \right) = 0.
\end{align*}

where

\begin{align*}
k_{i_1}^{j_1} = \begin{cases}
  i_{12}, & \text{if } k_{i_1}^{j_1} = 0 \\
  i_{11}, & \text{if } k_{i_1}^{j_1} = 1
\end{cases}
\end{align*}

and

\begin{align*}
\Delta_{i_1 \ldots i_j}^{j_1 \ldots j_1} = k_1^0 + \ldots + k_{j}^{j-1}.
\end{align*}

We note that, if there exists $p$ such that $i_p = i_{p-1}$, then

\begin{align*}
\sum_{0 \leq k_1^0 \ldots k_j^0 \leq 1} (-1)^{k_1^0 + \ldots + k_j^0} \delta_{i_1 \ldots i_j} \left( (g_{i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_j - 2 i_j}) \right) = 0.
\end{align*}

This means that $N_j(Z \Delta^n) = 0$ if $j > n$. Otherwise $N_j(Z \Delta^n)$ is generated by

\begin{align*}
\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \sum_{0 \leq k_1^0 \ldots k_j^0 \leq 1} (-1)^{k_1^0 + \ldots + k_j^0} \delta_{i_1 \ldots i_j} \left( (g_{i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_j - 2 i_j}) \right).
\end{align*}

Now, every map of complexes $f : N_\ast(Z \Delta^n) \to \text{Map}_\ast(x, y)$ is uniquely determined, for every integer $j$, by the image of the generators in (8). We will denote by $f_{i_0 \ldots i_j(j+1)}$ such images. Moreover $f$ is a chain of complexes. So

\begin{align*}
d^j(f_{i_0 \ldots i_j(j+1)}) &= f_{j-1}(\sum_{0 \leq k_1^0 \ldots k_j^0 \leq 1} (-1)^{k_1^0 + \ldots + k_j^0} \delta_{i_1 \ldots i_j} \left( (g_{i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_j - 2 i_j}) \right)) \\
&= f_{j-1}(\sum_{0 \leq k_1^0 \ldots k_j^0 \leq 1} (-1)^{k_1^0 + \ldots + k_j^0} \delta_{i_1 \ldots i_j} \left( (g_{i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_j - 2 i_j}) \right)) \\
&= f_{i_1 \ldots i_j(j+1)} - f_{j-1}(\sum_{0 \leq k_1^0 \ldots k_j^0 \leq 1} (-1)^{k_1^0 + \ldots + k_j^0} \delta_{i_1 \ldots i_j} \left( (g_{i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_j - 2 i_j}) \right)).
\end{align*}
Note that, for every $t$, we have
\[
g_{i_1^{k_1}i_2^{k_2}\ldots i_t^{k_t}} = g_{i_1^{k_1}i_2^{k_2}\ldots i_t^{k_t}} - g_{i_1^{k_1}i_2^{k_2}\ldots i_t^{k_t}i_{t+1}^{k_{t+1}}\ldots i_j^{k_j}}.
\]
This means that equation (10) gives precisely the condition (†). □

**Remark 10.** By Theorem 3.3 we have that
\[
\text{Hom}(\mathbb{Z}\Delta^n, \text{DK}_*(\tau_{\geq 0}\text{Map}_\mathcal{E}(x,y))) \simeq \text{Hom}_{\text{ch}}(\mathbb{N}_*(\mathbb{Z}\Delta^n), \tau_{\geq 0}\text{Map}_\mathcal{E}(x,y))
\]

Using the characterization in Lemma 3.2 we have that the morphisms $f_j$ with the property (†) are in bijection with $\text{DK}_*(\tau_{\geq 0}\text{Map}_\mathcal{E}(x,y))$.

### 3.2. Model structures.

We briefly recall some classical notions about model structures on categories. A good reference about model structures for the beginners is [Hov].

**Example 3.3.** The category of (small) dg-categories has two canonically model structures due to Tabuada [Tab1] [Tab2]: the first one has as weak-equivalences the "classical" quasi-equivalences and the second one has as weak-equivalences the Morita equivalences. We recall that $F : \mathcal{C} \to \mathcal{C}'$ is a Morita equivalence if:

- (Me1) $F$ induces an equivalence on perfect-complexes
  \[
  \text{Ho}(F) : \text{Ho}(\text{pretr}(\mathcal{C}))^? \to \text{Ho}(\text{pretr}(\mathcal{C}'))^?
  \]
  \[
  \text{Hom}_\mathcal{E}(x, y) \to \text{Hom}_\mathcal{E}'(F(x), F(y))
  \]
  is a quasi-isomorphism for all $x, y \in \mathcal{C}$. Clearly every weak equivalence in the first model structure is a Morita equivalence.

- (Me2) $\text{Hom}_\mathcal{E}(x, y) \to \text{Hom}_\mathcal{E}'(F(x), F(y))$ is a quasi-isomorphism for all $x, y \in \mathcal{C}$.

**Remark 11.** A functor between pretriangulated idempotent complete dg-categories is a weak-equivalence if and only if it is a Morita equivalence.

**Definition 3.2 (Weak equivalence [Joy]).** Let $X, Y$ be ∞-categories, $F : X \to Y$ is a weak equivalence if:

- $\text{Ho}(X) \simeq \text{Ho}(Y)$ (as categories),
- $\forall x, y \in X$ the geometric realization of the morphism $\text{Map}_\mathcal{X}(x, y) \to \text{Map}_\mathcal{Y}(F_0(x), F_0(y))$

  is a weak homotopy equivalence of topological spaces.

Weak equivalences together with monomorphisms (i.e. $F_n : X_n \to Y_n$ monomorphisms for all $n > 0$) as cofibrations and fibrations, defined by the right left property (cf. Definition 1.1.2. [Hov]), forms a model structure over $\text{sSet}$ called Joyal model structure.

**Remark 12.** We can see a simplicial object as a simplicial set by applying the forgetful functor.

Using [Qui: Thm. 4] we can endow the category of simplicial objects with a model structure defining weak equivalences (resp. fibrations) as the morphisms of simplicial objects where the underlying functor is a weak equivalence (resp. Kan fibrations) of simplicial sets.

**Remark 13.** Let $x, y \in \mathcal{C}$ where $\mathcal{C}$ is a unitary $\Lambda_\infty$-category. The simplicial set $\text{Map}_{\Lambda_\infty}^R(x, y)$ can be naturally enriched over the monoidal category of modules over the commutative ring $\mathbb{K}$. So $\text{Map}_{\Lambda_\infty}^R(x, y) \in \text{Fun}(\Delta^{op}, \mathbb{K}-\text{Mod})$ and the identification $\text{Map}_{\Lambda_\infty}^R(x, y) \simeq \text{DK}_{\mathcal{E}}(\tau_{\geq 0}\text{Map}_\mathcal{E}(x, y))$ makes sense.
Remark 14. The functors $\text{DK}_\bullet$ and $\text{N}_\bullet$ match cofibrations, fibrations and weak equivalences in the model structures on $\text{Ch}_R^0$ (where weak equivalences are quasi-isomorphisms, fibrations are degreewise epimorphisms and cofibrations are degreewise monomorphisms with degreewise projective cokernels) in the above model structure over the simplicial objects $\text{Fun}(\Delta^{op}, [\mathcal{K}, \text{Mod}])$ [SeSh: 4.1].

3.3. Main results. Now we are ready to prove some new results about $\Lambda_\infty$-nerves that will be useful to give a comparison between pretriangulated $\Lambda_\infty$-categories and $\infty$-stable categories in the last section. Let $X$ be a simplicial set and let $x, y$ be two elements in $X_0$.

Definition 3.3 (Degenerate simplex). We define the degenerate $n$-simplex on $x$ to be the image of $x$ via $X(\sigma)$, where $\sigma : [n] \to [0]$.

Example 3.4. A degenerate 2-simplex in $\mathcal{C}$ is given by the following diagram:

```
      1_x
      |   \\
1_x |   | 1_x
-----|---|---
   0   |   |
```

Definition 3.4 (Mapping space). For every couple of elements of $\mathcal{C}$, we define the mapping space $\text{Hom}_X(x, y)$ to be the $\infty$-category whose $n$-simplices are the $n+1$-simplices of $X_{n+1}$ such that $X_{\{n+1\}} = y$ and $X_{\{0, \ldots, n\}}$ is the degenerate $n$-simplex on $x$.

Lemma 3.3. Let $\mathcal{C}$ be an $\Lambda_\infty$-category. The mapping space $\text{Hom}_{N_{\Lambda_\infty}(\mathcal{C})}(x, y)$ is equivalent to $\text{DK}_\bullet(\tau_{\geq 0}\text{Map}_\mathcal{C}(x, y))$.

Proof. First of all we calculate the degenerate $n$-simplex in $N_{\Lambda_\infty}(\mathcal{C})$. Let us consider the degenerate map $\sigma : [n] \to [0]$. Using Theorem 2.3, the image of $x$ in $N_{\Lambda_\infty}(\mathcal{C})_n$ via $N_{\Lambda_\infty}(\sigma)$ is given by:

- $n + 1$-copies of $x$, because $\alpha(i_0) = \ldots = \alpha(i_n) = 0$;
- identity maps between $x$, because $\alpha(j_{0, i_1}) = 1_{X_0}$;
- all the higher maps $f_{i_0, i_1, \ldots}$ are zeroes, because $[0]$ has only one object.

By definition we have that, for every integer $n$, $\text{Hom}_{N_{\Lambda_\infty}(\mathcal{C})}(x, y)_n \subset N_{\Lambda_\infty}(\mathcal{C})_{n+1}$.

Then an element of $\text{Hom}_{N_{\Lambda_\infty}(\mathcal{C})}(x, y)_n$ is a set of elements satisfying (3) for all sets $I = \{0 \leq i_0 < i_1 < \ldots < i_m < i_{m+1} \leq n + 1\}$.

Now, using the previous calculation on degenerate $n$-simplex, we have that every $f_{i_0, i_q}$ with $i_q \neq n + 1$ is the identity and every $f_{i_0 \ldots i_q}$, with $q \neq n + 1$, is 0.

Then we can say that every element in $\text{Hom}_{N_{\Lambda_\infty}(\mathcal{C})}(x, y)_n$ is given by the identity maps on the vertex $x$ and, for all subsets $I = \{0 \leq i_0 < i_1 < \ldots < i_m < i_{m+1} = n + 1\}$, the maps $f_I$ (i.e. the maps with target $y$) satisfy:

$$m_1^0(f_I) = \sum_{1 \leq j \leq m} (-1)^{j-1}(f_{I-i_j}) - (-1)^0m_2^0(f_{i_1 \ldots i_{m+1}}, f_{i_0 i_1}) + \sum_{r \geq 2} \sum_{i_1 \ldots i_r} (-1)^{1+r}0.$$  

This means that

$$m_1^0(f_I) = \sum_{1 \leq j \leq m} (-1)^{j-1}(f_{I-i_j}) - (-1)^0m_2^0(f_{i_1 \ldots i_{m+1}}, f_{i_0 i_1}) + \sum_{r \geq 2} \sum_{i_1 \ldots i_r} (-1)^{1+r}0$$

$$= -f_{i_1 \ldots i_{m+1}} + \sum_{1 \leq j \leq m} (-1)^{j-1}(f_{I-i_j})$$

$$= \sum_{0 \leq j \leq m} (-1)^{j+1}(f_{I-i_j})$$
Remark of a right Quillen functor because of lack of limits. The correspondence between equivalences and fibrations do not guarantee the existence of a model structure preserves weak equivalences and fibrations in such a structure. Obviously this is what we wanted to prove.

Theorem 3.4. Let $\mathcal{C}$, $\mathcal{D}$ be $A_\infty$-categories (unitary) and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a quasi-equivalence of $A_\infty$-categories. Then $N_{A_\infty} \mathcal{F} : N_{A_\infty}(\mathcal{C}) \to N_{A_\infty}(\mathcal{D})$ is an weak-equivalence in the Joyal model structure.

Proof. If $\{\mathcal{F}^n\}$ is a quasi-equivalence then, by definition, the functor induced between the homotopy category $\text{Ho}(\mathcal{C})$ and $\text{Ho}(\mathcal{D})$ is an equivalence (we1). We observe that the homotopy category of an $\infty$-category $X$ is given by the category having as objects the elements of $X_0$ and as morphisms the elements of $X_1$ that are quotient by the homotopy relation. So $\text{Ho}(N_{A_\infty}(\mathcal{C}))$ has the same objects as $\mathcal{C}$ and as morphisms the set $Z^0(\text{Hom}_\mathcal{C}(x,y))$ such that $f \sim g$ if and only if there exists $h \in \text{Hom}_\mathcal{C}(x,y)^{-1}$ such that $dh = f - g$. It follows that $N_{A_\infty}(\mathcal{F})$ induces an equivalence between the homotopy categories of $N_{A_\infty}(\mathcal{C})$ and $N_{A_\infty}(\mathcal{D})$.

Now we have to prove that, given two objects $x, y \in \mathcal{C}$, the map

$$\text{Hom}_{N_{A_\infty}(\mathcal{C})}(x, y) \to \text{Hom}_{N_{A_\infty}(\mathcal{C})}(\mathcal{F}_0(x), \mathcal{F}_0(y))$$

is an homotopy equivalence between the Kan complex corresponding. Using Lemma 3.3 we have that it is enough to prove that

$$\text{DK}_*(\tau_{\geq 0}\text{Map}_\mathcal{C}(x, y)) \to \text{DK}_*(\tau_{\geq 0}\text{Map}_\mathcal{D}(\mathcal{F}_0(x), \mathcal{F}_0(y)))$$

is a weak equivalence, and this is true because the functor $\text{DK}_*$ preserves weak equivalences and the map of complexes $\text{Map}_\mathcal{C}(x, y) \to \text{Map}_\mathcal{D}(\mathcal{F}_0(x), \mathcal{F}_0(y))$, induced by $\mathcal{F}$, is a quasi-isomorphism by (we2).

Corollary 3.5. Given a unitary $A_\infty$-category $\mathcal{C}$, we have that the following $\infty$-categories are weak-equivalent:

$$N_{A_\infty}(\mathcal{C}) \simeq N_{A_\infty}(\text{Rep}(\mathcal{C})) \simeq N_{A_\infty}(\text{Rep}(\mathcal{C})).$$

Proof. The first weak-equivalence is a consequence of Theorem 3.4 using the fact that $\text{Rep}$ is a weak-equivalence of $A_\infty$-categories, the second weak-equivalence is a straightforward consequence of Remark 9 and the last equivalence follows from [Lur2; Prop 1.3.1.17].

Remark 15. In the case of dg-categories, Lurie proved in [Lur2; prop. 1.3.1.20] that the dg-nerve induces a right Quillen functor from the classical model structure on the category of (small) dg-categories (the first one in Example 3.3) to the Joyal model structure over sSet. Unfortunately in the case of the category of $A_\infty$-categories there is no model structure, we will analyze this in a future work. So for every $A_\infty$-category we prefer using the quasi equivalent dg-nerve on representable modules. However there exists a canonical model structure (without limits) on the categories of $A_\infty$-algebras due to Lefèvre [Le1] and Le Grignou proves in [LeG] that $N_{A_\infty}$ preserves weak equivalences and fibrations in such a structure. Obviously this correspondence between equivalences and fibrations do not guarantee the existence of a right Quillen functor because of lack of limits.

Remark 16. Given a weak equivalence $F : N_{A_\infty}(\mathcal{C}) \to N_{A_\infty}(\mathcal{C}')$, we have that $F$ induces an equivalence between the homotopy categories $\text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{D})$, moreover given two objects $x$ and $y \in \mathcal{C}$ we have a quasi-isomorphism $\mathcal{F}_1$ between $f$ and $g$ in $\mathcal{C}$, and $f$ and $g$ in $\mathcal{D}$.

Hence, after a change of signs, all the maps in $\text{Hom}^R_{N_{A_\infty}(\mathcal{C})}\{x, y\}$ satisfy (†) so,

$$\text{Hom}^R_{N_{A_\infty}(\mathcal{C})}\{x, y\} \simeq \text{DK}_*(\tau_{\geq 0}\text{Map}_\mathcal{C}(x, y))$$

This is what we wanted to prove.

□
\[ \tau_{\geq 0} \text{Map}_{\mathcal{E}}(x, y) \] and \[ \tau_{\geq 0} \text{Map}_{\mathcal{E}'}(F_0(x), F_0(y)) \] given by the following diagram:

\[
\begin{array}{ccc}
N_{A_{\infty}}(\mathcal{E}) & \xrightarrow{F} & N_{A_{\infty}}(\mathcal{E}') \\
\text{Map}_X(x, y) & \xrightarrow{F} & \text{Map}_Y(F_0(x), F_0(y)) \\
\text{DK}_* (\tau_{\geq 0} \text{Map}_{\mathcal{E}}(x, y)) & \xrightarrow{\sim} & \text{DK}_* (\tau_{\geq 0} \text{Map}_{\mathcal{E}'}(F_0(x), F_0(y))) \\
\tau_{\geq 0} \text{Map}_{\mathcal{E}}(x, y) & \xrightarrow{\sim} & \tau_{\geq 0} \text{Map}_{\mathcal{E}'}(F_0(x), F_0(y))
\end{array}
\]

More explicitly we set \( \mathcal{F}_1 = \text{DK}_* \circ F \circ \mathcal{A} \).

Unfortunately in general it is not true that, given a weak equivalence \( F : N_{A_{\infty}}(\mathcal{E}) \to N_{A_{\infty}}(\mathcal{E}') \), then \( \mathcal{E} \) and \( \mathcal{E}' \) are quasi-equivalent as \( A_{\infty} \). For example if we take the category \( \mathcal{X} \) with two objects \( x \) and \( y \) and a morphism \( g : x \to y \) of degree \(-1\) such that \( dg = 0 \) and the category \( \mathcal{X}' \) with two objects without nontrivial morphisms then \( N_{A_{\infty}}(\mathcal{X}) = N_{A_{\infty}}(\mathcal{X}') \) but \( \mathcal{X} \not\equiv \mathcal{X}' \). In the last section we will see, that under specific hypothesis Theorem 3.4 has a viceversa.

4. \( \infty \)-stable categories vs pretriangulated \( A_{\infty} \)-categories

In this section we will prove that the pretriangulated \( A_{\infty} \)-categories identified to the \( \infty \)-stable categories, via the \( A_{\infty} \)-nerve.

**Theorem 4.1.** Let \( \mathcal{A} \) be a pretriangulated \( A_{\infty} \)-category. Then \( N_{A_{\infty}}(\mathcal{A}) \) is an \( \infty \)-stable category. The functor induced between the homotopy categories is an equivalence of triangulated categories. Moreover \( \mathcal{A} \) is idempotent complete if and only if \( N_{A_{\infty}}(\mathcal{A}) \) is an idempotent complete \( \infty \)-stable category.

**Proof.** If \( \mathcal{A} \) is pretriangulated, then \( \text{Rep}(\mathcal{A}) \) is a pretriangulated dg-category. By [Fao; Thm. 4.3.1.] we have that the dg nerve \( N_{dg}(\text{Rep}(\mathcal{A})) \) is a stable \( \infty \)-category. By Corollary 3.5, we have that \( N_{dg}(\text{Rep}(\mathcal{A})) \) is weak-equivalent to \( N_{A_{\infty}}(\mathcal{A}) \) hence is an \( \infty \)-stable category. Moreover, by Lemma 1.2.4.6 in [Lur2], an \( \infty \)-stable category is idempotent complete if and only if the homotopy category is idempotent complete, so \( \mathcal{A} \) is idempotent complete if and only if \( N_{A_{\infty}}(\mathcal{A}) \) is idempotent complete. □

**Lemma 4.2.** Let \( F : N_{A_{\infty}}(\mathcal{A}) \to N_{A_{\infty}}(\mathcal{A}') \) be an exact functor between \( A_{\infty} \)-nerves then, for every object \( x \), \( F_0(\Sigma(x)) \simeq \Sigma(F_0(x)) \).

**Proof.** If \( \mathcal{A} \) is pretriangulated \( A_{\infty} \)-category, then \( N_{A_{\infty}}(\mathcal{A}) \) is a \( \infty \)-stable category. Moreover, in [Fao], it is proven that, given a morphism \( g \in \mathcal{A} \) of degree 0 with trivial differential (i.e. \( g \in N_{A_{\infty}}(\mathcal{A})_1 \)), the diagram

\[
\begin{array}{ccc}
x & \xrightarrow{g} & y \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Cone}(g)
\end{array}
\]
is a cofiber sequence. In particular, if we take \( g = 0 \) and \( Y = 0 \), then we have that
\[
\begin{array}{ccc}
x & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Cone}(0)
\end{array}
\]
is a cofiber sequence. Using the axioms \( \text{TR1} \) and \( \text{TR2} \) of triangulated categories in \( \text{Ho}(\mathcal{A}) \) (see Definition 1.1.2. [Nee]), we have that \( \text{Cone}(0) \simeq \Sigma(x) \). Hence the diagram
\[
\begin{array}{ccc}
x & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Sigma(x)
\end{array}
\]
is a cofiber sequence. By definition, \( F \) carries cofiber sequences to cofiber sequences. In particular, the diagram above will be carried to a cofiber sequence,
\[
\begin{array}{ccc}
F_0(x) & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & F_0(\Sigma(x))
\end{array}
\]
in \( N_{\mathcal{A}_{\infty}}(\mathcal{A}') \). Therefore we have that \( F_0(\Sigma(x)) \simeq \Sigma(F_0(x)) \), for every object \( x \in \mathcal{A} \). \( \square \)

Now we are ready to give a viceversa to Theorem 3.4.

**Theorem 4.3.** Let \( \mathcal{A} \), \( \mathcal{A}' \) be two pretriangulated \( A_{\infty} \)-categories. A weak equivalence \( F : N_{\mathcal{A}_{\infty}}(\mathcal{A}) \rightarrow N_{\mathcal{A}_{\infty}}(\mathcal{A}') \) in \( \text{Cat}_{\text{Ex}}^{\infty} \) induces a quasi-equivalence between \( \mathcal{A} \) and \( \mathcal{A}' \).

**Proof.** A weak equivalence \( F \) induces an equivalence between the categories \( \text{Ho}(\mathcal{A}) \) and \( \text{Ho}(\mathcal{A}') \) (see Remark 16) so, for all \( x, y \in \mathcal{A} \), we have the following equivalence of categories
\[
H^0(\text{Hom}_{\mathcal{A}}(x, y)) \xrightarrow{\sim} H^0(\text{Hom}_{\mathcal{A}'}(F_0(x), F_0(y)))
\]
Moreover, for all \( n \in \mathbb{Z} \), we have
\[
H^n(\text{Hom}_{\mathcal{A}}(x, y)) \simeq H^n(\text{Hom}_{\mathcal{A}'}(x, y)) \simeq H^n(\text{Hom}_{\mathcal{A}}(x[n], y)),
\]
because \( \mathcal{A} \) is pretriangulated. By the previous equivalence, we have
\[
H^0(\text{Hom}_{\mathcal{A}}(x[n], y)) \simeq H^0(\text{Hom}_{\mathcal{A}'}(F_0(x[n]), F_0(y))).
\]
Now, by Lemma 4.2 we have that \( \text{Hom}_{\mathcal{A}'}(F_0(x[n]), F_0(y)) \simeq \text{Hom}_{\mathcal{A}'}(F_0(x[n]), F_0(y)). \) Then \( H^n(\text{Hom}_{\mathcal{A}'}(F_0(x[n]), F_0(y))) \simeq H^n(\text{Hom}_{\mathcal{A}'}(F_0(x), F_0(y))) \) and we are done. \( \square \)

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