Cosmological Evolution for Dark Energy Models in $f(T)$ Gravity

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Abstract

In this paper, we investigate the behavior of equation of state parameter and energy density for dark energy in the framework of $f(T)$ gravity. For this purpose, we use anisotropic LRS Bianchi type I universe model. The behavior of accelerating universe is discussed for some well-known $f(T)$ models. It is found that the universe takes a transition between phantom and non-phantom phases for $f(T)$ models except exponential and logarithmic models. We conclude that our results are relativity analogous to the results of FRW universe.

Keywords: $f(T)$ gravity; LRS Bianchi type I universe; Equation of state; Dark energy.

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1 Introduction

The extensions of general relativity seem attractive to explain the late time acceleration of the universe and dark energy (DE). High redshift type Supernova Ia experiments show that the universe is experiencing accelerated expansion in every direction, while the other observations of anisotropies give indirect evidence (Perlmutter et al. 1999; Knop et al. 2003; Riess et
al. 1998). The mysterious anti-gravity DE material is smoothly filled in the universe. Dark energy with negative pressure and positive energy density depends on the equation of state (EoS), \( p = \omega \rho \), where \( \omega \) is the function of cosmic time called EoS parameter (Sharif and Zubair 2010a). There are different forms of dynamically varying DE phases related with negative behavior of EoS parameter. The range, \(-1 < \omega < -1/3\), corresponds to the non-phantom phase of the universe while the phantom phase occurs for \( \omega < -1 \) (Yadav 2011; Sahni and Starobinsky 2008). Quintom is such a DE model which can cross the phantom divide line \( \omega = -1 \) from both sides (Khatua et al. 2011; Feng et al. 2005).

The dynamical nature of DE can originate from a variable cosmological constant, phantom as well as scalar field, tachyon, Chaplygin gas and modified gravities (Martinelli and Melchiorri 2009). In spite of all observational evidences, the expanding universe is still a challenging issue in modern physics (Yang et al. 2010). An alternative approach to accommodate the current accelerating expansion of the universe is to modify the GR on large scale, such as, the scalar-tensor theories, \( f(R) \) theory, \( f(T) \) theory and DGP braneworld theory (Yang 2011a; Zheng and Huang 2011) etc.

Among these theories, the generalized teleparallel theory of gravity has recently gained a lot of interest due to its possible explanation about DE. In this generalization, the torsion scalar, \( T \), is replaced by its general function \( f(T) \) in the Lagrangian of teleparallel gravity (Sharif and Rani 2011a). The \( f(T) \) gravity models use the Weitzenböck connection which inherits only torsion and responsible for the accelerating expansion of the universe (Myrzakulov 2011). This approach (theory) originally developed by Einstein in 1928 under the name “Fern-Parallelismus” or “distant Parallelism” or “teleparallelism” (Linder 2010; Unzicker and Case 2005). The \( f(T) \) theory has significant advantage of its second order field equations as compared to \( f(R) \) theory with fourth order field equations (Sharif and Shamir 2009; Sharif and Kausar 2010, 2011a, 2011b; Jamil et al. 2012a, 2012b).

Bengochea and Ferraro (2009) described the recently detected acceleration of the universe without DE and performed observational viability tests by using recent SNIa data for some \( f(T) \) models. Linder (2010) explored that the cosmological constant is not only the component for the acceleration of the universe. This was also observed through a generalization of GR to other gravity theories. Zhang et al. (2011) discussed the dynamical analysis for the logarithmic and power form of \( f(T) \) models. They investigated phase-space analysis of models, their stability of the critical points and evolution of
EoS parameter. In a recent paper, Sharif and Shamaila (2011b) constructed some $f(T)$ models by using Bianchi type I (BI) universe. They also derived EoS parameter for two modified teleparallel models. Yang (2011b) investigated three new $f(T)$ models and described their physical implications and cosmological behavior.

Recently, Bamba et al. (2011) discussed different $f(T)$ models to investigate the cosmological evolutions of EoS parameter for DE and their observational constraints. Wu and Yu (2011) proposed two new $f(T)$ models and analyzed that the crossing of the phantom divide line is consistent with recent cosmological observational data. Chen et al. (2011) investigated this problem by extending modified gravity at the background and perturbed level. They also explored this theory for quintessence scenario.

One of the predictions from inflation is that the observed universe should nearly isotropic on large scales. Spatially homogenous and isotropic universe can be well described by FRW model which can not be explained the early universe. The existence of anisotropy at early times is a natural phenomenon to investigate the local anisotropies that we observe today in galaxies and cluster. Recent measurements on the cosmic microwave background radiation (CMBR) through WMAP suggest that the anisotropic inflationary models should be considered (Vielva 2004; Costa 2004). In early epoch of the big bang, the universe does not maintain its isotropic behavior at very small scales (Kumar and Singh 2008). In order to get a realistic model representing an expanding, homogenous and anistropic universe, Bianchi type cosmological models are considered. The isotropic behavior of today universe makes BI model a prime candidate for describing the possible anisotropic effects of the early universe on modern day data observations.

Many people worked on simplest anisotropic model called BI universe to discuss the effects of anisotropy in several contexts. Kumar and Singh (2007) developed mechanism for solution of the field equations by considering BI universe model. The same authors (Kumar and Singh 2008) investigated the exact BI solutions which give the constant value of the deceleration parameter in scalar-tensor theory. Recently, Sharif and Waheed (2012) studied exact solutions for anisotropic fluids by taking the locally rotationally symmetric Bianchi Type I (LRS BI) universe model which generalizes the flat FRW universe in the modified theory (BD). There are many papers available in literature where this model has been used widely (Bali and Kumawat 2008; Amirhashchi 2011; Yadav and Saha 2012).

In this paper, we explore cosmological evolution for some well-known
\( f(T) \) models by using BI universe. For this purpose, we evaluate the EoS parameter and energy density of DE and display graphically. The paper is organized as follows: In the next section, we review teleparallel theory of gravity. We formulate the field equations of \( f(T) \) gravity and some cosmological parameters for BI universe in section 3. Section 4 is devoted to examine the cosmological behavior of some \( f(T) \) models to check whether the crossing of phantom divide line occurs or not. We summarize and conclude the results in the last section 5.

2 The \( f(T) \) Formalism

Here we briefly review the modified teleparallel theory of gravity. In this theory, the vierbein field \( h_i(x^\mu) \) at each point \( x^\mu \) of the manifold are orthonormal basis, where the Latin \( (i, j, \ldots = 0, 1, 2, 3) \) and Greek alphabets \( (\mu, \nu, \ldots = 0, 1, 2, 3) \) are used for the tangent space and spacetime indices respectively. The spacetime metric tensor is defined from the dual vierbein as

\[
g_{\mu\nu} = \eta_{ij} h^i_\mu h^j_\nu, \tag{1}\]

where \( \eta_{ij} = diag(1, -1, -1, -1) \) is the Minkowski spacetime for the tangent space. For a given metric, there exists infinite tetrad fields \( h^i_\mu \) which satisfy the following properties (Ferraro and Fiorini 2007; Hayashi and Shirafuji 1979)

\[
h^i_\mu h^\mu_j = \delta^i_j, \quad h^i_\mu h^\nu_i = \delta^\nu_\mu. \tag{2}\]

The action for \( f(T) \) theory with matter is given by (Bamba et al. 2011)

\[
I = \frac{1}{16\pi G} \int d^4x e(T + f(T) + L_m). \tag{3}\]

Here \( f(T) \) is a differentiable function of the torsion scalar \( T \), \( L_m \) is the matter Lagrangian and \( e = \sqrt{-g} \). The torsion scalar has the form

\[
T = S^\rho \, ^{\mu\nu} T^\rho \, _{\mu\nu}, \tag{4}\]

here the torsion and the antisymmetric tensors are defined, respectively by

\[
T^\rho \, _{\mu\nu} = \Gamma^\rho \, _{\nu\mu} - \Gamma^\rho \, _{\mu\nu} = h^i_\rho (\partial_\mu h^i_\nu - \partial_\nu h^i_\mu), \tag{5}\]

\[
S^\rho \, ^{\mu\nu} = \frac{1}{2} (K^{\mu\nu} \, _\rho + \delta^\rho_\mu T^{\theta\nu} \, _\theta - \delta^\rho_\nu T^{\theta\mu} \, _\theta), \tag{6}\]
where $\Gamma^\rho_{\nu\mu}$ is the Weitzenböck connection. The difference between Weitzenböck and Levi-Civita connections is called contorsion tensor given by

$$K^\mu_\rho^\nu = \frac{1}{2}(T^\mu_\rho^\nu - T^\nu_\rho^\mu - T^\mu_\rho^\nu).$$

The modified field equations of the teleparallel theory of gravity (Sharif and Jawad 2011) are obtained by varying the action with respect to the vierbien $h^i_\mu$ as (Bengochea and Ferraro 2009)

$$[\epsilon^{-1}\partial_\mu(\epsilon S_\mu) - h^\rho_\mu T^\rho_\mu S_\rho]S_\mu = (1 + f_T)S_\mu \partial^a(T)h^\rho_\mu,$$

where $S_\mu = h^\rho_\mu S_\rho$, $\kappa^2 = 8\pi G$, $f_T \equiv \frac{df}{dT}$.

3 Bianchi I Universe and Some Cosmological Parameters

The spatially homogenous and anisotropic LRS BI universe which has one transverse direction $x$ and two equivalent longitudinal directions $y$ and $z$, responsible for the anisotropic behavior is (Sharif and Zubair 2010b)

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)(dy^2 + dz^2),$$

where $A$ and $B$ are the cosmic scale factors. Using Eqs. (1) and (3), the tetrad components are obtained as

$$h^i_\mu = diag(1, A, B, B), \quad h^\mu_i = diag(1, A^{-1}, B^{-1}, B^{-1}).$$

Substituting Eqs. (1) and (3) in (3), the torsion tensor for BI becomes

$$T = -2\left(2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}\right).$$

The corresponding average scale factor $R$, the Hubble parameter $H$ and the anisotropy parameter $\Delta$ will become

$$R = (AB^2)^{\frac{1}{3}}, \quad H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right), \quad \Delta = \frac{1}{3} \sum_{i=1}^{3} \left(\frac{H_i - H}{H}\right)^2.$$
where \( H_1 = \frac{\dot{A}}{A}, \ H_2 = \frac{\dot{B}}{B} = H_3 \) are directional Hubble parameters along \( x, \ y \) and \( z \)-axes respectively. The energy-momentum tensor \( T_{\mu\nu} \) for perfect fluid is

\[
T_{\nu}^{\rho} = \text{diag}(\rho_M, -P_M, -P_M, -P_M), 
\]

where \( \rho_M \) and \( P_M \) are the energy density and pressure of matter inside the universe.

For \( i = 0 = \nu \) and \( i = 1 = \nu \) in Eq. (7), we obtain the following field equations

\[
T + f(T) - 4 \left( 2 \frac{\dot{A}B}{AB} + \frac{\dot{B}^2}{B^2} \right) (1 + f_T) = 2 \kappa^2 \rho_M, \tag{13}
\]

\[
4 \left( \frac{\dot{A}B}{AB} + \frac{\dot{B}^2}{B^2} + \frac{\dot{B}}{B} \right) (1 + f_T) - 16 \frac{\dot{B}}{B} \left[ \frac{\dot{B}}{B} - \frac{\dot{A}^2}{A^2} \right] f_T \left[ (T + f) = 2 \kappa^2 P_M. \right] \tag{14}
\]

For a spatially homogeneous metric, the normal congruence to homogeneous expansion implies that the expansion scalar \( \Theta \) is proportional to the shear scalar \( \sigma \), i.e,

\[
\Theta \propto \sigma. \tag{15}
\]

The expansion scalar and shear scalar for BI universe is given by

\[
\Theta = \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B}, \quad \sigma = \frac{1}{\sqrt{3}} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) \tag{16}
\]

This leads to the condition (Sharif and Waheed 2012; Bali and Kumawat 2008; Amirhashchi 2011)

\[
A(t) = B^m(t), \quad m \geq 2. \tag{17}
\]

By using above condition, the anisotropy parameter of the expansion is found to be

\[
\Delta = 2 \left( \frac{m - 1}{m + 2} \right)^2. \tag{18}
\]

It is mentioned here that the isotropic behavior of the expanding universe is obtained for \( \Delta = 0 \). Under this supplementary condition, the above field
equations reduce to the following form

\[ H^2 = -\frac{(m+2)^2}{9(2m+1)} \left( \frac{8\pi G}{3} \rho_M - \frac{f}{6} - \frac{T f_T}{3} \right), \]  

\[ (H^2)' = \frac{m+2}{3} \left( \frac{2\kappa^2 P_M + \frac{2(m+2)}{2m+1} T f_T + \frac{4m+5}{2m+1} T + f}{2 + 2f_T + 4T f_{TT}} \right), \]  

where prime denotes derivative with respect to \( \ln(B^{\frac{m+2}{3}}) \). Notice that the above equations reduce to the usual field equations for \( T + f(T) = T \) i.e.

\[ H^2 = \frac{(m+2)^2}{9(2m+1)} \left[ -\frac{8\pi G}{3} (\rho_M + \rho_{DE}) \right], \]  

\[ (H^2)' = \left( \frac{m+2}{3} \right) \frac{8\pi G}{3} \left[ P_M + P_{DE} + \frac{4m+5}{3(2m+1)} (\rho_M + \rho_{DE}) \right]. \]

We assume only non-relativistic matter in which pressure is zero i.e., \( P_M = 0 \). Comparing Eqs. (19) with (21) and (20) with (22), the energy density and pressure of the effective DE become

\[ \rho_{DE} = \frac{-1}{16\pi G} (f + 2T f_T), \]  

\[ P_{DE} = \frac{1}{16\pi G} \left( \frac{f - T f_T - \frac{2(4m+5)}{2m+1} T f_{TT}}{1 + f_T + 2T f_{TT}} \right). \]

Dividing Eq. (24) by (23), the EoS parameter for DE turns out to be

\[ \omega_{DE} = -\frac{\frac{f}{T} - f_T - \frac{2(4m+5)}{2m+1} T f_{TT}}{(1 + f_T + 2T f_{TT})(\frac{f}{T} + 2f_T)}. \]

The conservation laws of BI universe filled with pressureless matter and DE take the following form

\[ \dot{\rho}_M + 3H \rho_M = 0, \]

\[ \dot{\rho}_{DE} + 3H \rho_{DE}(1 + \omega_{DE}) = 0. \]

### 4 Cosmological Evolution in \( f(T) \) Models

In this section, we discuss the crossing of the phantom divide line for the EoS parameter (25) by using some particular \( f(T) \) models. An appropriate
scale factor should satisfy two requirements from observations: one is the past transition from acceleration to deceleration expansion phase and other is the currently slow variation of the DE density. For the radiation, matter and DE epoch, the universe can be characterized by a power law scale factor with constant exponent. Due to the complexity of the field equations, it is very difficult to evaluate the explicit analytical form of the scale factor. So, we choose the scalar factor of the form (Sharif and Rani 2011a; Radinschi 2000) to describe the possible supper accelerated transition.

\[ B(t) = (nst)^{1/n}, \]  

(26)

where \( n \) and \( s \) are positive real constants. For the sake of convenience, \( s = 1 \).

### 4.1 Model I

Consider the exponential \( f(T) \) model (Linder 2010; Jamil et al. 2011; Bamba et al. 2011)

\[ f(T) = \alpha T (1 - e^{pT_0/T}), \]  

(27)

where

\[ \alpha = -\frac{1 - \Omega_M^{(0)}}{1 - (1 - 2p)e^{p}}, \quad p = \text{constant}. \]

Here \( T_0 = T(z = 0) \) is the current value of the torsion and the redshift is

\[ z \equiv 1/B^{\frac{m+2}{3}} - 1. \]  

(28)

The current value of the fractional density \( \Omega_M \) of non-relativistic matter is defined by (Komatsu et al. 2011)

\[ \Omega_m^{(0)} \equiv \rho_M^{(0)}/\rho_{\text{crit}} = 0.26, \]

where \( \rho_M^{(0)} \) is the current energy density and \( \rho_{\text{crit}}^{(0)} = 3H_0^2/8\pi G \) is the critical density (Bamba et al. 2010a) with current Hubble parameter \( H_0 = 1 \) (Sharif and Jawad 2012). Notice that Eq. (27) has only one parameter \( p \) if the value of \( \Omega_M^{(0)} \) is known.

The EoS parameter in terms of \( |T/T_0| \) can be written as

\[ \omega_{DE} = -\frac{e^{pT_0/T}}{E} \left( \frac{pT_0}{T} \right) \left[ -1 + \frac{2(4m + 5)}{2m + 1} \left( \frac{pT_0}{T} \right) \right], \]  

(29)
The graphical behavior of $\omega_{DE}$ as a function of $|T/T_0|$ is shown in Figure 1. For both $p > 0$ and $p < 0$, $\omega_{DE}$ reaches to $-1$ but does not cross the phantom divide line ($\omega_{DE} = -1$). Thus the universe stays in DE era as $|T/T_0|$ approaches to infinity.

Inserting Eq. (17) in (10) and using (26) and (28), we get a torsion scalar as a function of redshift $z$

$$T = -2(2m + 1)(1 + z)^{\frac{6n}{m+2}}, \quad T_0 = -2(2m + 1). \quad (30)$$

Substituting these values in Eq. (29), the corresponding EoS parameter in terms of $z$ takes the following form as

$$\omega_{DE} = \frac{-pe^{p(1+z)^{\frac{6n}{m+2}}} - 2(4m + 5)p}{D(1 + z)^{\frac{6n}{m+2}} \left[ -1 + \frac{2(4m + 5)p}{(2m + 1)(1 + z)^{\frac{6n}{m+2}}} \right]}, \quad (31)$$

where

$$D = \left[ 3 \left(1 - e^{p(1+z)^{\frac{6n}{m+2}}} + 2p(1 + z)^{-\frac{6n}{m+2}}e^{p(1+z)^{\frac{6n}{m+2}}} \right) \right] \times \left[ 1 + \alpha \left( 1 - e^{p(1+z)^{\frac{6n}{m+2}}} \right) + \alpha p(1 + z)^{-\frac{6n}{m+2}}e^{p(1+z)^{\frac{6n}{m+2}}} \times \left(1 - 2p(1 + z)^{-\frac{6n}{m+2}} \right) \right].$$
The graphical representation of $\omega_{DE}$ versus redshift is shown in Figure 2. We see from the left graph that for $p > 0$, $\omega_{DE}$ increases towards negative and attains $-1$ but does not cross the phantom divide line for $z \to \infty$. Thus the universe stays in non-phantom phase (quintessence). For $p < 0$, $\omega_{DE}$ is less than $-1$ without crossing the phantom divide line and gives the phantom phase of the universe initially. As $z \to \infty$, it reaches to $-1$ which is shown in the right graph of Figure 2. Thus the universe always stays in DE era for both cases.

The parameter $\rho_{DE}^{(s)} = \rho_{DE}/\rho_{DE}^{(0)}$ in terms of $z$ is obtained by putting Eq.(30) in (23) as

$$\rho_{DE}^{(s)} = \frac{(2m+1)(1+z)^{\frac{6n}{m+2}}}{8\pi G \rho_{DE}^{(0)}} \left[ 3 \left( 1 - e^{p/(1+z)}^{\frac{6n}{m+2}} \right) + \frac{2pe^{p/(1+z)}}{(1+z)^{\frac{6n}{m+2}}} \right], \tag{32}$$

where $\rho_{DE}^{(0)} = 0.74\rho_{\text{crit}}^{(0)}$ (Bamba et al. 2011). The evolution of $\rho_{DE}^{(s)}$ in terms of redshift $z$ for $p > 0$ and $p < 0$ is shown in Figure 3. For $p > 0$, this indicates a slight increment in $\rho_{DE}^{(s)}$ for smaller values of $z$ and becomes constant for larger values of $z$. When $p < 0$, it decreases initially with respect to $z$ and approaches to constant value as $z \to \infty$. We would like to mention here that $\rho_{DE}^{(s)}$ attains different values at $z = 0$ for both the cases of $p$ for BI universe. On the other hand, $\rho_{DE}^{(s)}(z = 0)$ coincides at the same point for several values of $p$ for FRW universe (Bamba et al. 2011).

Now we discuss the viability of the exponential $f(T)$ model for phantom and non-phantom phases by using an approximate method. From Eq.(27),
we obtain

\[ f_T = \alpha \left( 1 - e^{pT_0/T} + \frac{pT_0}{T} e^{pT_0/T} \right), \quad f_{TT} = -\alpha \left( \frac{pT_0}{T} \right)^2 \frac{1}{T} e^{pT_0/T}. \] (33)

Assuming \( X = pT_0/T, \ T_0/T \lesssim 1 \) in Eqs. (27) and (33), it follows that (Bamba et al. 2010b)

\[ \frac{f}{T} \approx -\alpha \left( X + \frac{X^2}{2} \right), \quad f_T \approx \frac{\alpha X^2}{2}, \quad T f_{TT} \approx -\alpha X^2. \] (34)

Inserting these values in Eq. (25), \( \omega_{DE} \) takes the form

\[ \omega_{DE} \approx -1 + \frac{14m + 19}{2(2m + 1)} X. \] (35)

Here, we have take \( \alpha \sim O(1) \). This implies that the behavior of EoS parameter depends on the sign of \( p \) and correspondingly on \( X \). For \( X > 0 \), the universe always stays in non-phantom phase as \( \omega_{DE} > -1 \). The universe rests in the phantom region for \( X < 0 \) as \( \omega_{DE} < -1 \). This is consistent with the graphical results shown in Figure 2.

### 4.2 Model II

Assume the logarithmic \( f(T) \) model as (Bamba et al. 2011)

\[ f(T) = \beta T_0 \left( \frac{qT_0}{T} \right)^{-1/2} \ln \left( \frac{qT_0}{T} \right), \] (36)
Figure 4: Plot of $\omega_{DE}$ versus $|T/T_0|$ in left side and $\omega_{DE}$ versus $z$ in right side for $q = 1$, $m = 2 = n$ and $\Omega_{M}^{(0)} = 0.26$ for model II.

where

$$\beta \equiv \frac{1 - \Omega_{M}^{(0)}}{2q^{-1/2}}, \quad q > 0.$$ 

If the value of $\Omega_{M}^{(0)}$ is given, then the only parameter $q$ is involved in the logarithmic $f(T)$ model same as the exponential $f(T)$ model. The corresponding EoS parameter is given by

$$\omega_{DE} = -\frac{[1 + \frac{3(m+1)}{2m+1} \ln(T_0/T)]}{[2 - (1 - \Omega_{M}^{(0)})(T_0/T)^{1/2}][\ln(T_0/T) - 1]}$$  \hspace{1cm} (37)

which is independent of $q$. Using Eq.(30) in above equation, $\omega_{DE}$ in terms of redshift $z$ becomes

$$\omega_{DE} = -\frac{[1 + \frac{3(m+1)}{2m+1} \ln(1/(1 + z)\frac{6n}{m+2})]}{[2 - (1 - \Omega_{M}^{(0)})(1/(1 + z)\frac{6n}{m+2})^{1/2}][\ln(1/(1 + z)\frac{6n}{m+2}) - 1]}.$$  \hspace{1cm} (38)

Its behavior is shown in Figure 4. The left graph shows that $\omega_{DE}$ becomes negative as $|T/T_0| \to \infty$. In the beginning, the model represents a universe having both properties of matter and radiation. After a very short interval, the universe enters in DE phase. It is mentioned here that the universe remains in non-phantom phase as the time elapses. The right graph has the same behavior as that of the exponential $f(T)$ model and the universe stays in the non-phantom phase ($\omega_{DE} > -1$).
4.3 Model III

Here we take the combination of both exponential and logarithmic $f(T)$ models which has the following form (Bamba et al. 2011)

$$f(T) = \gamma \left[ T_0 \left( \frac{uT_0}{T} \right)^{-1/2} \ln \left( \frac{uT_0}{T} \right) - T(1 - e^{uT_0/T}) \right],$$  \hspace{1cm} (39)

where

$$\gamma \equiv \frac{1 - \Omega^{(0)}_{M}}{2u^{-1/2} + [1 - (1 - 2u)e^u]}.\hspace{1cm} (40)$$

The positive constant $u$ is the only parameter in this model. The EoS parameter for DE in terms of $|T/T_0|$ is given by

$$\omega_{DE} = -\frac{1}{I} \left[ \frac{1}{u} \sqrt{\frac{uT_0}{T}} \left\{ 1 + 3m \frac{m+1}{2m+1} \ln \left( \frac{uT_0}{T} \right) \right\} \right. \left.+ e^{uT_0/T} \left( \frac{uT_0}{T} \right) \left\{ 1 - 2(4m+5) \frac{m+1}{2m+1} \left( \frac{uT_0}{T} \right) \right\} \right],$$  \hspace{1cm} (41)

where

$$I = \left[ \frac{2}{u} \sqrt{\frac{uT_0}{T}} \left\{ \ln \left( \frac{uT_0}{T} \right) - 1 \right\} + e^{uT_0/T} \left( \frac{uT_0}{T} \right) \right. \left( 3 - \frac{2uT_0}{T} \right) - 3 \right] \left. \times \left[ 1 - \gamma(1 - e^{uT_0/T}) - \gamma \sqrt{\frac{uT_0}{T}} \left\{ \frac{1}{u} + \sqrt{\frac{uT_0}{T}} e^{uT_0/T} \left( 1 - \frac{2uT_0}{T} \right) \right\} \right].\hspace{1cm} (42)$$

Initially, the expanding universe is lying in non-phantom phase with $\omega_{DE}$ as a function of $|T/T_0|$ as shown in left graph of Figure 5. The EoS parameter decreases with increase in $|T/T_0|$ by crossing the phantom divide line at $|T/T_0| = 0.27, 0.36$ and $0.42$ for $u = 0.5, 0.8$ and 1 respectively evolving the phantom phase. After a short interval, $\omega_{DE}$ crosses the phantom divide line again at $|T/T_0| = 0.65, 1.2$ and 1.7 which turns out to be constant as $|T/T_0|$ increases.

Inserting Eq.(30) in (41), $\omega_{DE}$ in terms of $z$ turns to be

$$\omega_{DE} = -\frac{1}{F} \left[ \frac{1}{u} \sqrt{u(1+z)^{m/2}} \left\{ 1 + 3m \frac{m+1}{2m+1} \ln \left( u(1+z)^{m/2} \right) \right\} \right. \left.+ u(1+z)^{m/2} e^{u(1+z)^{m/2}} \left\{ 1 - 2u(4m+5)(1+z)^{m/2} \left( \frac{m+1}{2m+1} \right) \right\} \right],$$  \hspace{1cm} (43)
Figure 5: Plot of $\omega_{DE}$ versus $|T/T_0|$ in left graph and $\omega_{DE}$ versus $z$ in right graph with $m = 2 = n$ and $u = 1$ (black line), 0.8 (red line), 0.5 (green) for model III.

where

$$F = \left(\frac{2}{u} \sqrt{u(1+z)} \cdot \frac{6n}{m+2} \left( \ln \left( u(1+z) \cdot \frac{6n}{m+2} \right) - 1 \right) + e^{u(1+z) \cdot \frac{6n}{m+2}} \times \left\{ 3 - 2u(1+z) \cdot \frac{6n}{m+2} \right\} - 3 \right) \left[ 1 - \gamma \left( 1 - e^{u(1+z) \cdot \frac{6n}{m+2}} \right) - \gamma \sqrt{u(1+z) \cdot \frac{6n}{m+2}} \right] \times \left\{ \frac{1}{u} + e^{u(1+z) \cdot \frac{6n}{m+2}} \left( 1 - 2u(1+z) \cdot \frac{6n}{m+2} \right) \sqrt{u(1+z) \cdot \frac{6n}{m+2}} \right\}.$$

The right graph of Figure 5 shows that for $u = 0.8$ and $u = 1$, the universe is in the phantom phase ($\omega_{DE} < -1$) at initial epoch. As $z$ increases, it crosses the phantom divide line ($\omega_{DE} = -1$) at $z = 0.067$ and 0.18 respectively. Thus $\omega_{DE}$ enters in the non-phantom phase and converges to constant value with increment in $z$. It is interesting to note that for $u \leq 0.6$, the universe always rests in non-phantom phase. Notice that the combined $f(T)$ model behaves as the quintom model (Khatua et al. 2011).

4.4 Model IV

Now we take the $f(T)$ model of the type (Yang 2011)

$$f(T) = T + \eta T_0 \frac{(T^2/T_0^2)^\lambda}{1 + (T^2/T_0^2)^\lambda},$$

(42)
where \( \eta \) and \( \lambda \) are positive constants. The corresponding EoS parameter is

\[
\omega_{DE} = -\frac{\eta(T/T_0)^{2\lambda-1}}{K(1+(T^2/T_0^2)^\lambda)} \left[ 1 - 2\lambda \left( 1 - \frac{(T^2/T_0^2)^\lambda}{1+(T^2/T_0^2)^\lambda} \right) \right] + \left( 2\lambda - 1 - \frac{(T^2/T_0^2)^\lambda}{1+(T^2/T_0^2)^\lambda} \left( 6\lambda - 1 - \frac{4\lambda(T^2/T_0^2)^\lambda}{1+(T^2/T_0^2)^\lambda} \right) \right) \times \frac{2(4m+5)}{2m+1} \right],
\]

(43)

where

\[
K = \left[ 3 + \frac{\eta(T/T_0)^{2\lambda-1}}{1+(T^2/T_0^2)^\lambda} \left( 1 + 4\lambda \left( 1 - \frac{(T^2/T_0^2)^\lambda}{1+(T^2/T_0^2)^\lambda} \right) \right) \right] \times \left[ 1 + \frac{2\eta\lambda(T/T_0)^{2\lambda-1}}{1+(T^2/T_0^2)^\lambda} \left( 4\lambda - 1 + \frac{(T^2/T_0^2)^\lambda}{1+(T^2/T_0^2)^\lambda} \right) \right] \times \left( 12\lambda - 1 + \frac{8\lambda(T^2/T_0^2)^\lambda}{1+(T^2/T_0^2)^\lambda} \right). \]

The left graph of \( \omega_{DE} \) as a function of \(| T/T_0 |\) is shown in Figure 6. Initially, the behavior of \( \omega_{DE} \) is positive and stays in radiation era for very small region. After a slight variation in \(| T/T_0 |\), the crossing of phantom divide line occurs and results a phantom era. There are two singularities at \(| T/T_0 | = 0.95 \) and \( 2.5 \) with positive behavior. After these singularities, \( \omega_{DE} \) evolves from phantom to non-phantom phase by crossing phantom divide line and remains in non-phantom phase for \(| T/T_0 | \to \infty \) causing accelerated
expansion of the universe. In term of $z$, the parameter $\omega_{DE}$ is obtained by using Eq.(30) in (43) as

$$\omega_{DE} = -\frac{\eta(1+z)^{\frac{6n(2\lambda-1)}{m+2}}}{J(1+(1+z)^{\frac{12n\lambda}{m+2}})} \left[ 1 - 2\lambda \left\{ 1 - \frac{(1+z)^{\frac{12n\lambda}{m+2}}}{1+(1+z)^{\frac{12n\lambda}{m+2}}} + \frac{2(4m+5)}{2m+1} \right\} \right] \times \left( 2\lambda - 1 - \frac{(1+z)^{\frac{12n\lambda}{m+2}}}{1+(1+z)^{\frac{12n\lambda}{m+2}}} \right) \left( 6\lambda - 1 - 4\lambda \frac{(1+z)^{\frac{12n\lambda}{m+2}}}{1+(1+z)^{\frac{12n\lambda}{m+2}}} \right),$$

(44)

where

$$J = \left[ 3 + \eta(1+z)^{\frac{6n(2\lambda-1)}{m+2}} \left\{ 1 + 4\lambda \left( 1 - \frac{(1+z)^{\frac{12n\lambda}{m+2}}}{1+(1+z)^{\frac{12n\lambda}{m+2}}} \right) \right\} \right] \times \left[ 1 + 2\eta\lambda(1+z)^{\frac{6n(2\lambda-1)}{m+2}} \left\{ 4\lambda - 1 + \frac{(1+z)^{\frac{12n\lambda}{m+2}}}{1+(1+z)^{\frac{12n\lambda}{m+2}}} \right\} \times \left( 12\lambda - 1 + \frac{8\lambda(1+z)^{\frac{12n\lambda}{m+2}}}{1+(1+z)^{\frac{12n\lambda}{m+2}}} \right) \right].$$

For small values of $z$, $\omega_{DE}$ is found to be the non-phantom phase and converges to zero with the increment in the value of $z$. This shows that matter becomes dominant over the DE as shown in right side of Figure 6. Inserting Eq.(30) in (23), the corresponding parameter $\rho_{DE}^{(s)}(z)$ can be written as

$$\rho_{DE}^{(s)} = \frac{(2m+1)(1+z)^{\frac{6n}{m+2}}}{8\pi G \rho_{DE}^{(0)}} \left[ 3 + \frac{\eta(1+z)^{\frac{6n(2\lambda-1)}{m+2}}}{1+(1+z)^{\frac{12n\lambda}{m+2}}} \right] \times \left\{ 1 + 4\lambda \left( 1 - \frac{(1+z)^{\frac{12n\lambda}{m+2}}}{1+(1+z)^{\frac{12n\lambda}{m+2}}} \right) \right\}. \quad (45)$$

The graphical representation (Figure 7) shows the increasing behavior of $\rho_{DE}^{(s)}$ for $z \to \infty$.

4.5 Model V

Finally, we take the $f(T)$ model in the form (Yang 2011)

$$f(T) = T - \eta T_0 \left[ \left( 1 + \frac{T^2}{T_0^2} \right)^{-\lambda} - 1 \right]. \quad (46)$$
The expression for $\omega_{DE}$ takes the form

$$\omega_{DE} = \frac{1}{L} \left[ \eta \left( \frac{T}{T_0} \right) \left\{ 1 - \left( 1 + \frac{T^2}{T_0^2} \right)^{-\lambda} \right\} - 2 \eta \lambda \left( \frac{T}{T_0} \right) \left( 1 + \frac{T^2}{T_0^2} \right)^{-\lambda-1} \right]$$

$$\times \left\{ 1 + \frac{2(4m+5)}{2m+1} \left( 1 - 2(\lambda + 1) \left( \frac{T}{T_0} \right)^2 \left( 1 + \frac{T^2}{T_0^2} \right)^{-1} \right) \right\}, \quad (47)$$

where

$$L = \left[ 2 + 2 \eta \lambda \left( \frac{T}{T_0} \right) \left( 1 + \frac{T^2}{T_0^2} \right)^{-\lambda-1} \left\{ 3 - 4(\lambda + 1) \left( \frac{T}{T_0} \right)^2 \left( 1 + \frac{T^2}{T_0^2} \right)^{-1} \right\} \right]$$

$$\times \left[ 3 - \eta \left( \frac{T_0}{T} \right) \left( 1 + \frac{T^2}{T_0^2} \right)^{-\lambda} \left\{ 1 - 4\lambda \left( \frac{T}{T_0} \right)^2 \left( 1 + \frac{T^2}{T_0^2} \right)^{-1} \right\} + \eta \left( \frac{T_0}{T} \right) \right].$$

The left graph of Figure 8 represents the cosmological evolution of $\omega_{DE}$ in terms of $|T/T_0|$ which shows the same behavior as model IV. However, the singularities appear at $|T/T_0| = 0.7, 1.7$ and the universe stays in non-phantom phase for a short interval. Here the universe becomes matter dominated for higher values of $|T/T_0|$.

Substituting Eq. (30) in (47), the corresponding $\omega_{DE}$ as a function of $z$
Figure 8: Plot of $\omega_{DE}$ versus $|T/T_0|$ in the left and $\omega_{DE}$ versus $z$ in the right for $\eta = 3$, $\lambda = 3$ and $m = 2 = n$ for model V.

turns out to be

$$\omega_{DE} = \frac{1}{M} \left[ -2\eta\lambda(1+z)^{6n_{m+2}} \left( 1 + (1+z)^{\frac{12n}{m+2}} \right)^{-\lambda-1} \right.$$  
$$\times \left\{ 1 + \frac{2(4m+5)}{2m+1} \left( 1 - 2(\lambda + 1)(1+z)^{\frac{12n}{m+2}} \left( 1 + (1+z)^{\frac{12n}{m+2}} \right)^{-1} \right) \right\} + \eta \left( 1 + z \right)^{\frac{6n}{m+2}} \left\{ 1 - \left( 1 + (1+z)^{\frac{12n}{m+2}} \right)^{-\lambda} \right\} \right], \quad (48)$$

where

$$M = \left[ 1 + \eta\lambda(1+z)^{\frac{6n}{m+2}} \left( 1 + (1+z)^{\frac{12n}{m+2}} \right)^{-\lambda-1} \right.$$  
$$\times \left\{ 3 - 4(\lambda + 1)(1+z)^{\frac{12n}{m+2}} \left( 1 + (1+z)^{\frac{12n}{m+2}} \right)^{-1} \right\} \right]$$  
$$\times \left[ 3 - \frac{\eta}{(1+z)^{\frac{6n}{m+2}}} \left( 1 + (1+z)^{\frac{12n}{m+2}} \right)^{-\lambda} \right.$$  
$$\times \left\{ 1 - 4\lambda(1+z)^{\frac{12n}{m+2}} \left( 1 + (1+z)^{\frac{12n}{m+2}} \right)^{-1} \right\} + \frac{\eta}{(1+z)^{\frac{6n}{m+2}}} \right].$$

The parameter $\omega_{DE}$ in terms of $z$ enters from phantom phase to non-phantom phase and approaches to matter dominated era similar to $\omega_{DE}(T/T_0)$ shown
in the right graph of Figure 8. Using Eq. (30) in (23), it follows that

$$\rho_{DE}^{(s)}(z) = \frac{(2m+1)(1+z)^{\frac{6n}{m+2}}}{8\pi G \rho_{DE}^{(0)}} \left[ 3 - \frac{\eta}{(1+z)^{\frac{6n}{m+2}}} \left( 1 + (1+z)^{\frac{12n}{m+2}} \right)^{-\lambda} \right] + \frac{\eta}{(1+z)^{\frac{6n}{m+2}}}$$

Figure 9 shows that $\rho_{DE}^{(s)}$ is positive and increases with $z \to \infty$.

5 Summary and Conclusion

The main purpose of this paper is to discuss the well-known phenomenon of the universe expansion in the context of $f(T)$ gravity for BI universe model. We have investigated the cosmological evolution of EoS parameter $\omega_{DE}$ and energy density $\rho_{DE}^{(s)}$ for some well-known $f(T)$ models. These parameters are evaluated in terms of $| T/T_0 |$ and redshift $z$. The graphical representation of the phantom and non-phantom phases of the expanding universe is examined. The behavior of these models can be summarized as follows:

- Model I is the exponential $f(T)$ model in which the phase of the universe changes with the sign of parameter $p$. For $p > 0$, $\omega_{DE}(z)$ indicates a universe dominated by the non-phantom era while the phantom phase is obtained for $p < 0$. The EoS parameter $\omega_{DE}(| T/T_0 |)$ corresponds to the DE phase. The energy density shows positive behavior both for $p > 0$ and $p < 0$. 

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Figure 9: Plot of $\rho_{DE}^{(s)}$ versus $z$ for $\eta = 3$ and $\lambda = 2$
• For the logarithmic model II, $\omega_{DE}$ as a function of $|T/T_0|$ and $z$ remains in non-phantom phase which shows the accelerated expansion of the universe.

• In the combined model III, $\omega_{DE}(|T/T_0|)$ shows that the universe initially transits from the non-phantom to phantom phase. After that, the universe again crosses the phantom divide line and retains its initial phase. On the other hand, $\omega_{DE}(z)$ crosses the phantom divide line from phantom phase to non-phantom phase only once.

• For model IV, the parameter $\omega_{DE}(|T/T_0|)$ enters from phantom to non-phantom phase and stays in it while the universe becomes matter dominated as EoS parameter becomes zero for $z \to \infty$. The parameter $\rho_{DE}^{(s)}$ represents the positive increment in its value as a function of $z$.

• The behavior of model V remains the same as that of the model IV. However, the parameter $\omega_{DE}$ indicates a matter dominated universe for both cases.

The comparison of our results with Bamba et al. (2011) is as follows: For model I, $\omega_{DE}(z)$ stays in the phantom phase for $p > 0$ and non-phantom phase for $p < 0$ in FRW universe, whereas in BI universe, it shows opposite behavior. $\omega_{DE}(|T/T_0|)$ does not cross phantom divide line in BI universe while it crosses phantom divide line in FRW universe for models I and II. $\omega_{DE}(z)$ gives the same behavior for both BI and FRW in model II. The crossing of phantom divide line occurs in combined model for both BI and FRW. In models IV and V, the universe turns out to be matter dominated era in BI universe while these models behave like the cosmological constant in FRW universe. We can conclude from the above discussion that the DE component is responsible for the accelerating expansion of the universe. This is consistent with recent observations like SNIa and WMAP data (Perlmutter et al. 1999; Knop et al. 2003; Riess et al. 1998).

References

Amirhashchi, H.: Phys. Lett. B697, 429(2011)
Bamba, K., et al.: JCAP 01, 021(2011)
Bamba, K. et al.: JCAP 11, 001(2010a)
Bamba, K., Geng, C.Q. and Lee, C.C.: arXiv1008.4036
Bali, R. and Kumawat, P.: Phys. Lett. B665, 332(2008)
Bengochea, G.R. and Ferraro, R.: Phys. Rev. D79, 124019(2009)
Chen, S.H. et al.: Phys. Rev. D 83(2011)023508.
Costa, D.O.: Phys. Rev. D69, 063516(2004)
Feng, B., Wang, X. and Zhang, X.: Phys. Lett. B 607, 35(2005)
Ferraro, R. and Fiorini, F.: Phys. Rev. D75, 084031(2007)
Hayashi, K. and Shirafuji, T.: Phys. Rev. D19, 3524(1979)
Jamil, M., Momeni, D. and Myrzakulov, M.: Gen. Relativ. Gravit. 45, 263(2011)
Jamil, M., Momeni, D. and Myrzakulov, M.: Eur. Phys. J. C 72, 2075(2012a)
Jamil, M., Momeni, D. and Myrzakulov, M.: Eur. Phys. J. C 72, 2137(2012b)
Khatua, P.B., Chakraborty, S. and Debnath, U.: arXiv1105.3393
Knop, R.A., et al.: Astrophys. J. 598, 102(2003)
Komatsu, E., et al., Astrophys. J. Suppl. 192, 18(2011)
Kumar, S. and Singh, C.P.: Int. J. Theor. Phys. 47, 1722(2008)
Kumar, S. and Singh, C.P.: Astrophys. Space Sci. 312, 57(2007)
Linder, E.V.: Phys. Rev. D81, 127301(2010)
Martinelli, M. and Melchiorri, A.: Phys. Rev. D79, 123516(2009)
Myrzakulov, R.: Eur. Phys. J. C71, 1752(2011)
Perlmutter, S., et al.: Astrophys. J. 517, 565(1999)
Radinschi, I.: Acta Phys. Slov. 50, 609(2000)
Riess, A.G., et al.: Astrophys. J. 116, 1009(1998)
Sahni, V., Shaifeloo, A. and Starobinsky, A.A.: Phys. Rev. D78, 103502(2008)
Sharif, M. and Jawad, A.: Astrophys. Space Sci. 337, 789(2012)
Sharif, M. and Jawad, A.: Astrophys. Space Sci. 331, 257(2011)
Sharif, M. and Kausar, H.R.: Mod. Phys. Lett. A253299(2010)
Sharif, M. and Kausar, H.R.: Astrophys. Space Sci. 331, 281(2011a)
Sharif, M. and Kausar, H.R.: Astrophys. Space Sci. 332, 463(2011b)
Sharif, M. and Rani, S.: Mod. Phys. Lett. A26, 1657(2011a)
Sharif, M. and Rani, S.: Phys. Scr. 84, 055005(2011b)
Sharif, M. and Shamir, M.F.: Class. Quantum Grav. 26, 235020(2009)
Sharif, M. and Waheed, S.: Eur. Phys. J. C72, 1876(2012)
Sharif, M. and Zubair, M.: Int. J. Mod. Phys. D19, 1957(2010a)
Sharif, M. and Zubair, M.: Astrophys. Space Sci. 330, 399(2010b)
Unzicker, A. and Case, T.: arXiv0503046
Vielva, P., at el.: Astrophys. J. 609, 22(2004)
Wu, P. and Yu, H.: Eur. Phys. J. C71, 1552(2011)
Yadav, A.K.: Astrophys. Space Sci. 335, 565(2011)
Yadav, A.K. and Saha, B.: Astrophys. Space Sci. 337, 759(2012)
Yang, L., et al.: Phys. Rev. D82, 103515(2010)
Yang, R.J.: Eur. Phys. Lett. 93, 60001(2011a)
Yang, R.J.: Eur. Phys. J. C71, 1797(2011b)
Zheng, R. and Huang, Q.G.: JCAP 03, 002(2011)
Zhang, Y., et al.: JCAP 07, 015(2011)