Twistors, Holomorphic Disks, and Riemann Surfaces with Boundary

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Abstract

Moduli spaces of holomorphic disks in a complex manifold \( Z \), with boundaries constrained to lie in a maximal totally real submanifold \( P \), have recently been found to underlie a number of geometrically rich twistor correspondences. The purpose of this paper is to develop a general Fredholm regularity criterion for holomorphic curves-with-boundary \( (\Sigma, \partial \Sigma) \subset (Z, P) \), and then show how this applies, in particular, to various moduli problems of twistor-theoretic interest.

1 Introduction

Many interesting differential-geometric structures can best be understood by means of twistor correspondences. Here the main lesson is that moduli spaces of compact complex curves \( \Sigma \) in a complex manifold \( Z \) tend to carry tautological differential-geometric structures. Moreover, the structures arising in this way often actually represent the general solution of some natural system of partial differential equations.

The prototypical construction of this type was first discovered by Penrose, who called it the *nonlinear graviton* [22]. Suppose that \( Z \) is a complex 3-manifold, and let \( \mathcal{M} \) denote the moduli space of those embedded \( \mathbb{C}\mathbb{P}_1 \)'s in \( Z \) which have the same normal bundle as does a projective line in \( \mathbb{C}\mathbb{P}_3 \). Penrose

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discovered that $\mathcal{M}$ is then a complex 4-manifold, and naturally carries the holomorphic analog of a conformal class of anti-self-dual metrics.

In order to get real geometries out of this fundamentally complex picture, one traditionally supposes that $Z$ is also equipped with an anti-holomorphic involution $\sigma : Z \to Z$, and then focuses on the moduli space $M \subset \mathcal{M}$ of $\sigma$-invariant curves. The geometries that one obtains in this manner are necessarily real-analytic, but in many problems this is an expected feature of the general solution, anyway — e.g. for reasons of elliptic regularity. For example, by considering complex 3-folds $Z$ equipped with free anti-holomorphic involutions $\sigma$, Atiyah, Hitchin, and Singer [1] showed that every self-dual Riemannian 4-manifold arises from the Penrose construction. Moreover, this conclusion holds not only locally, but also globally.

Now one can also construct real-analytic self-dual manifolds of metric signature $(++--)$ by instead equipping a complex 3-fold $Z$ with an anti-holomorphic involution $\sigma$ with fixed-point set $P \neq \emptyset$. However, the relevant differential-geometric problem now corresponds to an ultra-hyperbolic system of PDE rather than an elliptic one, and we must therefore expect most solutions to be of low regularity. Moreover, the focusing of bicharacteristics makes it exceedingly difficult to apply the Penrose approach to understand the global structure of solutions, even in the real-analytic context.

However, a new paradigm has recently emerged that substantially sweeps away these obstacles. First, forget about the involution $\sigma$; instead, focus on a totally real submanifold $P \subset Z$, previously thought to merely be the fixed-point set of $\sigma$. Second, forget about compact holomorphic curves; instead, look for holomorphic curves-with-boundary, where the boundary is constrained to lie on $P$. Smooth solutions will arise from smooth $P$, while rougher solutions will arise from rougher $P$.

Lionel Mason and I found this approach to be remarkably fruitful in our joint work on Zoll surfaces [16] and split-signature 4-manifolds [17]. In this article, I will present a new way of determining when a complex curve-with-boundary is stable under small deformations of $(Z,P)$, and then show how this tool can be used in the context of some interesting examples. My primary goal here is to indicate the wide applicability of these ideas to topics that now seem ripe for further exploration.

It is perhaps worth mentioning that quite different considerations have recently led physicists to intensively study both closed [25] and open [3] strings in twistor spaces. The distinction between closed and open strings precisely parallels the contrast between traditional twistor geometry and the
new kind of twistor correspondence explored here, with the submanifold $P$ playing the rôle of an $m$-brane in open string theory. I thus hope to convince you that it’s time we twistor geometers opened up and used our branes!

2 Reflections on Kodaira

The systematic study of moduli spaces of complex submanifolds was largely instigated by Kodaira, who proved the following paradigmatic result [13]:

**Theorem 1 (Kodaira)** Suppose that $X$ is a compact complex submanifold of a complex manifold $Z$, and let $N = \left[ \left( T_{1,0}Z \right) \right] / T_{1,0}X$ be the normal bundle of $X$. If $H^1(X, \mathcal{O}(N)) = 0$, then the moduli space $\mathcal{M}$ of all compact complex submanifolds of $Z$ near $X$ is a complex manifold. Moreover, if $x \in \mathcal{M}$ is the the base-point representing $X$, then there is a natural isomorphism $T_{1,0}x \mathcal{M} \cong H^0(X, \mathcal{O}(N))$.

In this picture, $X$ is assumed to be embedded in $Z$, but a moment’s thought immediately gives one a result in the non-embedded case, too:

**Corollary 1** Let $X$ be a compact complex manifold, and let $f : X \to Y$ be a holomorphic map. If $H^1(X, \mathcal{O}(f^*T_{1,0}Y)) = 0$, then there is a universal deformation for $f$ which is parameterized by a neighborhood of the origin in $H^0(X, \mathcal{O}(f^*T_{1,0}Y))$.

**Proof.** Set $Z = X \times Y$, and embed $X$ in $Z$ as the graph of $f$. The normal bundle of this embedding is exactly $N = f^*T_{1,0}Y$. Now apply Theorem 1.

When $X = \mathbb{CP}_1$, this corollary is frequently used in Mori theory [15]. Indeed, little harm is done by even invoking it to deform embedded rational curves $\mathbb{CP}_1 \subset Z$. However, this corollary is ill suited to the study of embedded complex curves $X$ of higher genus, since in this setting one often has $H^1(X, \mathcal{O}(T_{1,0}Z)) \neq 0$ even when $H^1(X, \mathcal{O}(N)) = 0$.

Many applications of twistor ideas depend on the persistence of families of complex submanifolds after deformation of the ambient complex manifold. Fortunately, a minor modification [14] of Theorem 1 provides a criterion for guaranteeing the survival of complex submanifolds in this context:
Theorem 2 (Kodaira) Suppose that \(X \subset Z\) is a compact complex submanifold whose normal bundle satisfies \(H^1(X, \mathcal{O}(N)) = 0\). Then any small deformation \(Z'\) of \(Z\) contains an \(h^0(X, \mathcal{O}(N))\)-complex-dimensional family of compact complex submanifolds, obtained by deforming \(X\).

Now these theorems do not depend at all on the dimension of \(X\). Nonetheless, the special case in which \(X\) is a Riemann surface enjoys a somewhat privileged status. For example, the same framework then works even for pseudo-holomorphic curves in almost-complex manifolds \(Z\), and Gromov [10] was able to make systematic use of this observation to prove a family of truly revolutionary results on the structure of symplectic manifolds. However, it is yet another special feature of the 1-dimensional case which will concern us here; namely, as will be explained below, these ideas can also be naturally generalized so as to handle Riemann surfaces with non-empty boundary [7, 9, 18, 20].

Let \(Z\) be a complex \(n\)-manifold, and let \(J\) denote its complex structure tensor. Suppose that \(P\) is a differentiable submanifold of \(Z\) of real dimension \(n\). We will then say that \(P\) is a (maximal) totally real submanifold if 

\[T_yP \cap J(T_yP) = 0\]

for all \(y \in P\), so that 

\[TZ|_P = TP \oplus J(TP)\].

Now suppose that \(\Sigma\) is a compact complex curve-with-boundary, and that \(\Sigma \hookrightarrow Z\) is a holomorphic embedding that is differentiable up to the boundary. Also assume that \(\partial \Sigma \subset P\), where \(P\) is a maximal totally real submanifold of differentiability class \(C^{k+4}\), \(k \geq 1\). A regularity theorem of Chirka [6] then asserts that \(\Sigma \hookrightarrow Z\) is actually \(C^{k+3}\). Our goal here is to understand the space of nearby \(C^1\) holomorphic curves \(\Sigma'\) with \(\partial \Sigma' \subset P\). However, Chirka’s regularity result tells us that this is equivalent, for example, to studying holomorphic curves of class \(C^{k+1,\alpha}\) for any chosen \(\alpha \in (0, 1)\). Needless to say, we could now elect to set \(k = 1\), once and for all, but I will leave the choice of \(k\) up to the reader, as doing so may actually clarify certain aspects of the argument.

Let us next choose a \(C^{k+3}\) open surface \(S \subset Z\) containing \(\Sigma\) and a \(C^{k+3}\) Riemannian metric \(g\) on \(Z\) with respect to which \(P\) is totally geodesic. Let \(E \subset TP|_{\partial \Sigma}\) be the orthogonal complement of \(T\partial \Sigma\) relative to \(P\), and, after shrinking \(S\) if necessary, let \(\tilde{N} \subset T^{1,0}Z|_S\) be a \(C^{k+2}\) complex sub-bundle whose real part \(\text{Re}\tilde{N}\) is complementary to \(T\Sigma\) and agrees with \(E \oplus J(E)\) along \(\partial \Sigma\). Applying the geodesic spray of \(g\) to \(\text{Re}\tilde{N}\) and invoking the inverse function theorem, we thus obtain a \(C^{k+2}\) diffeomorphism \(\Phi\) between some neighborhood \(\mathcal{U} \subset Z\) of \(\Sigma \subset 0_S\) and an open subset \(\mathcal{U} \subset Z\). Notice,
moreover, that we have arranged that

\[ P \cap U = \Phi(E \cap V). \]

Now notice that \( N = \tilde{N}|_\Sigma \) can canonically be identified with the normal bundle \( T^{1,0}Z/T^{1,0}\Sigma \) of \( \Sigma \). Choose some inner product and connection on \( N \), and let \( C^{k+1,\alpha}(N, E) \) denote the Banach space of \( C^{k+1,\alpha} \) sections of \( N \) whose boundary values are sections of \( E \to \partial \Sigma \); let \( C^{k+1,\alpha}(N, E)_\varepsilon \subset C^{k+1,\alpha}(N, E) \) be the \( \varepsilon \) ball about 0 in this Banach space. If \( \varepsilon \) is sufficiently small, the graph of any \( f \in C^{k+1,\alpha}(N, E)_\varepsilon \) is contained in \( V \), and so is sent by \( \Phi \) to a \( C^{k+1,\alpha} \) surface \( \Sigma' \subset Z \) with \( \partial \Sigma' \subset P \). Now let \( V^1 \subset T_z N \) denote the vertical vectors of type \((1,0)\), and notice that this is naturally isomorphic to the pull-back of \( N \) to its own total space. By possibly shrinking our neighborhood \( V \) of \( \Sigma \), we then have

\[ T^{0,1}Z \cap \Phi^*V^1 = 0. \]

On the other hand, by shrinking \( \varepsilon \) if necessary, we may also arrange that

\[ T^{0,1}Z \cap (\Phi \circ f)^*[T^{1,0}\Sigma] = 0 \]

for all \( f \in C^{k+1,\alpha}(N, E)_\varepsilon \). Hence

\[ T\Sigma = T^{0,1}Z + \Phi^*V^1 + (\Phi \circ f)^*[T^{1,0}\Sigma] \]

at each point of the image of any \( f \in C^{k+1,\alpha}(N, E)_\varepsilon \). Composing \((\Phi \circ f)^*\) with the projection \( T\Sigma \to \Phi^*V^1 \) thus defines a linear map \( T\Sigma \to N \) for every such \( f \); and since this linear map kills \( T^{1,0}\Sigma \) by construction, it may be viewed as a \((0,1)\)-form \( \mathcal{D}f \) with values in \( N \). It is now easy to see that

\[ \mathcal{D} : C^{k+1,\alpha}(N, E)_\varepsilon \to C^{k,\alpha}(\Lambda^{0,1} \otimes N) \]

is a differentiable map of Banach manifolds, and that the linearization of \( \mathcal{D} \) at 0 is exactly the canonical operator

\[ \overline{\mathcal{D}} : C^{k+1,\alpha}(N, E) \to C^{k,\alpha}(\Lambda^{0,1} \otimes N), \]

obtained by remembering that \( N = T^{1,0}Z/T^{1,0}\Sigma \) is a holomorphic vector bundle over \( \Sigma \). Notice, moreover, that \( \mathcal{D}^{-1}(0) \) exactly consists of those holomorphic curves \((\Sigma', \partial \Sigma') \hookrightarrow (Z, P)\) which are sufficiently near \( \Sigma \).
I now want to explain a simple geometric trick that not only proves that this linearized operator is Fredholm, but actually provides a practical method of precisely calculating its kernel and cokernel. The key idea is to first construct the abstract double of our Riemann surface. That is, we begin with our Riemann-surface-with-boundary $\Sigma$

and then attach a mirror-image copy $\overline{\Sigma}$ to $\Sigma$ along $\partial\Sigma$:

Let $\mathbb{X} = \Sigma \cup_{\partial\Sigma} \overline{\Sigma}$ denote this double, and notice that it comes equipped with an anti-holomorphic involution

$$\rho : \mathbb{X} \rightarrow \mathbb{X}, \quad \rho^2 = \text{id}_\mathbb{X},$$

obtained by interchanging $\Sigma$ and $\overline{\Sigma}$. This is true because $\overline{\Sigma}$ is by definition simply $\Sigma$, equipped with its conjugate complex structure.

Now notice that $\overline{N}$ is a a holomorphic vector bundle on $\overline{\Sigma}$. On the other hand, both $N$ and $\overline{N}$ restrict to $\partial\Sigma$ as complexifications of the real vector bundle $E = TP/T(\partial\Sigma)$. We can therefore construct a complex vector bundle $\mathcal{N} \rightarrow \mathbb{X}$ by attaching $\overline{N} \rightarrow \overline{\Sigma}$ to $N \rightarrow \Sigma$ in such a manner that $E$ is sent to itself by the identity:

$$\mathcal{N} = N \cup_{E \otimes \mathbb{C}} \overline{N}$$

$$\downarrow \quad \downarrow$$

$$\mathbb{X} = \Sigma \cup_{\partial\Sigma} \overline{\Sigma}$$

We can make this into a holomorphic vector bundle by taking the complex structure tensor on its total space to be that of $N$ over $\Sigma$ and that of $\overline{N}$ over $\overline{\Sigma}$. Of course, we still need to check that this gives us a locally trivial structure in the vicinity of $\partial\Sigma$. To see this, first recall that we arranged for $\Sigma \hookrightarrow \mathbb{Z}$
to be at least $C^2$ up to the boundary, so that, even near $\partial \Sigma$, the bundle $N$ has $C^1$ local holomorphic trivializations induced by local holomorphic trivializations of $T^{1,0} \mathcal{Z}$. The integrable almost-complex structure on total space of $N$ is therefore $C^1$ up to the boundary. By reflection, the conjugate complex structure of $\overline{N}$ is therefore $C^1$ up to the boundary, too. Now, the manner in which we glue $N$ and $\overline{N}$ together to make a $C^1$ manifold is exactly chosen so that these two almost complex structures agree along the interface. Hence the total space of $\mathcal{N}$ carries an induced almost-complex structure which is at least Lipschitz. However, a result of Nijenhuis and Woolf [19] asserts that a Lipschitz almost-complex manifold contains pseudo-holomorphic curves tangent to any given complex tangent line in its tangent space. But in our case the generic such pseudo-holomorphic curve is necessarily the graph of a local section of $\mathcal{N}$, and we thus obtain enough local holomorphic sections of $\mathcal{N}$ to generate local holomorphic trivializations, even near points of $\partial \Sigma$. Thus $\mathcal{N} \to \mathcal{X}$ really is a holomorphic vector bundle, as claimed.

Now notice that, by construction, the total space of $\mathcal{N}$ carries a tautological involution $\rho$ which covers $\rho$:

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\rho} & \mathcal{N} \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\rho} & \mathcal{X}
\end{array}
$$

The fixed-point set of this involution is exactly the given sub-bundle $E \subset \mathcal{N}|_{\partial \Sigma}$. A somewhat surprising consequence of this is that, no matter how rough $E$ may have appeared in our original picture, it is actually real-analytic as a subspace of $\mathcal{N}$. Indeed, we can even find holomorphic local trivializations of $\mathcal{N}$ near any point of $\partial \Sigma$ in which $E$ becomes the trivial bundle with fiber $\mathbb{R}^n$ as a sub-bundle of the trivial bundle with fiber $\mathbb{C}^n$. Such special local trivializations will play a prominent rôle in what follows.

For this reason, it is important that we now check that these special local trivializations are actually of class $C^{k+1,\alpha}$ relative to the naïve local trivializations of $\mathcal{N}$. To see this, let $\{h_j\}$ be a local holomorphic frame for $\mathcal{N}$ whose real span along $\partial \Sigma$ is $E$. Working instead with respect to a local trivialization of $N \to \Sigma$ induced by some local holomorphic vector fields on $\mathcal{Z}$, the assumed regularity of $P$ allow us to choose a $C^{k+2}$ local frame $\{e_k\}$ for $N \to \Sigma$ whose real span along $\partial \Sigma$ is $E$. Let $\psi$ be a smooth bump function
supported in the common domain of \{e_j\} and \{h_k\}. We then have
\[ \psi e_j = \sum_k c_{jk} h_k \]
where \(c_{jk}\) is real along \(\partial \Sigma\), and where \(\overline{\partial} c_{jk}\) is \(C^{k+1}\). By taking local coordinates on \(\Sigma\), we can then view each \(c_{jk}\) as a compactly supported function on the upper half-plane, and then convert this into a smooth function on the 2-disk \(D^2\) by applying a Möbius transformation. Expressing \(c_{jk} = a_{jk} + ib_{jk}\) in terms of its real and imaginary parts, \(b_{jk}\) then vanishes along the \(\partial D^2\), while \(\Delta b_{jk}\) is of class \(C^{k-1}\) on \(D^2\). Elliptic regularity for the Dirichlet problem \[8\] thus predicts that \(b_{jk}\) is of class \(C^{k+1, \alpha}\) for any \(\alpha \in (0,1)\), and the fact that \(da_{jk} + J(d b_{jk})\) is \(C^{k+1}\) then implies that the \(a_{jk}\) must be of class \(C^{k+1, \alpha}\), too. It follows that the \(\{h_j\}\) are also \(C^{k+1, \alpha}\), as claimed.

Next, notice that \(\varrho\) induces complex-anti-linear involutions
\[ \varrho^*: H^j(\mathbb{X}, \mathcal{O}(\mathcal{N})) \to H^j(\mathbb{X}, \mathcal{O}(\mathcal{N})), \quad j = 0, 1. \]
Let \(H^j(\mathbb{X}, \mathcal{O}(\mathcal{N}))\) denote the \((+1)\)-real-eigenspace of \(\varrho^*\), so that
\[ H^j(\mathbb{X}, \mathcal{O}(\mathcal{N})) = H^j(\mathbb{X}, \mathcal{O}(\mathcal{N})) \oplus iH^j(\mathbb{X}, \mathcal{O}(\mathcal{N})) \]
as a real vector space. With this notation in hand, we can now formulate our key technical result:

**Lemma 1** For any integer \(k \geq 1\) and any \(\alpha \in (0,1)\), the linear operator
\[ \overline{\partial}: C^{k+1, \alpha}(\Sigma; N, E) \to C^{k, \alpha}(\Sigma; \Lambda^{0,1} \otimes N) \]
is Fredholm, with kernel canonically isomorphic to \(H^0(\mathbb{X}, \mathcal{O}(\mathcal{N}))\) and cokernel canonically isomorphic to \(H^1(\mathbb{X}, \mathcal{O}(\mathcal{N}))\). In particular, \(\ker \overline{\partial}\) has real dimension \(h^0(\mathbb{X}, \mathcal{O}(\mathcal{N}))\), while \(\text{coker } \overline{\partial}\) has real dimension \(h^1(\mathbb{X}, \mathcal{O}(\mathcal{N}))\).

**Proof.** First, let us compute the kernel of \(\overline{\partial}\). If \(f \in C^{k+1, \alpha}(\Sigma; N, E)\), define a continuous section \(\mathfrak{f}\) of \(\mathcal{N} \to \mathbb{X}\) by
\[ \mathfrak{f} = \begin{cases} f & \text{on } \Sigma \\ \varrho^* f & \text{on } \bar{\Sigma} \end{cases} \]
This is well defined and continuous because, by assumption, \(f = \varrho^* f\) along \(\partial \Sigma\). Now if \(f \in \ker \overline{\partial}\), \(\mathfrak{f}\) is continuous up to \(\partial \Sigma\) and holomorphic on its
complement, and so is holomorphic on all of $\mathbb{X}$ by the reflection principle. Moreover, $f$ is invariant under the action of $\rho^*$ by construction. Hence $f \in H^0(\mathbb{X}, \mathcal{O}(\mathcal{N}))$. Since any element of $H^0(\mathbb{X}, \mathcal{O}(\mathcal{N}))$ conversely restricts to $\Sigma$ as an element of $C^{k+1,\alpha}(\Sigma; N, E)$ which is killed by $\bar{\partial}$, we thus conclude that $H^0(\mathbb{X}, \mathcal{O}(\mathcal{N}))$ can naturally be identified with the kernel of the operator.

Now, what is the image of the operator? Given $\phi \in C^{k,\alpha}(\Sigma; \Lambda^{0,1} \otimes N)$, define a section of $\Lambda^{0,1} \otimes \mathcal{N} \to \mathbb{X}$ by

$$\varphi = \begin{cases} \phi & \text{on } \Sigma \\ \rho^*\phi & \text{on } \mathbb{X} - \Sigma \end{cases}$$

Of course, this $\varphi$ may not be continuous, but at any rate it is certainly $L^\infty$, and in particular may be considered as a twisted distribution-valued $(0,1)$-form. Since Dolbeault cohomology can be computed using currents instead of smooth forms, it follows that there is a well defined cohomology class $[\varphi] \in H^1(\mathbb{X}, \mathcal{O}(\mathcal{N}))$ which precisely measures the obstruction to writing $\varphi$ as

$$\varphi = \overline{\partial} f$$

for some distributional section $f$ of $\mathcal{N} \to \mathbb{X}$; moreover, $[\varphi] \in H^1(\mathbb{X}, \mathcal{O}(\mathcal{N}))$, since, by construction, $\varphi$ is $\rho$-invariant almost everywhere. We thus have a continuous linear map

$$\Pi : C^{k,\alpha}(\Sigma; \Lambda^{0,1} \otimes N) \to H^1(\mathbb{X}, \mathcal{O}(\mathcal{N}))$$

$$\phi \mapsto [\varphi]$$

Moreover, this map is a surjection, since $\mathbb{X}$ has a Stein cover consisting of any small neighborhood $U$ of $\Sigma$ and its conjugate $\overline{U}$, and every element of $H^1(\mathbb{X}, \mathcal{O}(\mathcal{N}))$ can be expressed as $[\varphi]$ for a smooth section $\phi$ of $\Lambda^{0,1} \otimes N \to \Sigma$ obtained by cutting off a Čech representative $\in \Gamma(U \cap \overline{U}, \mathcal{O}(\mathcal{N}))$ with a bump function and then restricting to $\Sigma$.

Now notice that $\overline{\partial} C^{k+1,\alpha}(\Sigma; N, E) \subset \ker \Pi$. Indeed, if $\phi = \overline{\partial} f$ for some $f \in C^{k+1,\alpha}(\Sigma; N, E)$, the continuous section

$$f = \begin{cases} f & \text{on } \Sigma \\ \rho^*f & \text{on } \mathbb{X} - \Sigma \end{cases}$$

of $\mathcal{N}$ then satisfies $\overline{\partial} f = \varphi$ in the distributional sense, and $[\varphi] = \Pi(\phi)$ therefore vanishes.
To finish the proof, it therefore suffices to show that \( \ker \Pi \subset \mathcal{O}_{k+1,\alpha}(\Sigma; N, E) \). Thus, suppose that we are given some \( \phi \in C^{k,\alpha}(\Sigma; \Lambda^0,1 \otimes N) \) for which \( \hat{\phi} = \mathcal{O}(\mathcal{O}_{k,\alpha}(\Sigma)) \). It then follows that \( \phi = \mathcal{O}_{\hat{\phi}} \) for some distributional \( \hat{f}_0 \). Since \( \phi \) is \( L^p \) for any \( p \), elliptic regularity then tells us that \( \hat{f}_0 \in L^p(\mathcal{X}, N) \) for any \( p \). Taking \( p > 2 \), thus have \( \hat{f}_0 \in C^0(\mathcal{X}, N) \) by the Sobolev embedding theorem. Setting \( \hat{f} = (\hat{f}_0 + \hat{\phi}_0)/2 \) we then have \( \phi = \mathcal{O}_{\hat{f}} \), where \( \hat{f} \) is a \( \hat{\phi} \)-invariant continuous section of \( N \), and so takes values in \( E \) along \( \partial \Sigma \) by \( \hat{\phi} \) invariance. Letting \( f \) denote \( \hat{f}|_{\Sigma} \), we then have \( f \in C^0(\Sigma; N, E) \), and the point worth emphasizing is that \( f|_{\partial \Sigma} \) is a section of \( E \). Moreover, Schauder theory tells us that \( f \) is of class \( C^{k+1,\alpha} \) on the interior of \( \Sigma \). It therefore only remains to show that \( f \) is \( C^{k+1,\alpha} \) in the vicinity of any boundary point.

Since this last issue is completely local, we can multiply \( f \) by a smooth bump function supported in the domain of a special local trivialization near a given boundary point, and so obtain a weak solution of the equation

\[
\nabla \hat{f} = \hat{\phi}
\]

where \( \hat{f} \) is a compactly supported \( \mathbb{C}^n \)-valued continuous function on the upper half-plane which is \( \mathbb{R}^n \)-valued along the real axis, and where \( \hat{\phi} \) is a \( \mathbb{C}^n \)-valued \( (0,1) \)-form of class \( C^{k,\alpha} \). We now identify the upper half-plane with the 2-disk \( D \) via a Möbius transformation. Expressing the \( j \)th component of \( \hat{f} \) in terms of its real and imaginary parts

\[
\hat{f}_j = u + iv,
\]

and correspondingly expressing the \( j \)th component of \( \hat{\phi} \) as

\[
\hat{\phi}_j = \frac{\alpha + i\beta}{2} d\xi,
\]

it then follows that \( v \) is a weak solution of the Dirichlet problem

\[
\Delta v = h \quad \text{on } D, \quad v = 0 \quad \text{on } \partial D,
\]

where

\[
h = \frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial x}
\]

is of class \( C^{k-1,\alpha} \). It follows that \( v \) is of class \( C^{k+1,\alpha} \), as desired. But we also have

\[
du = -Jdv + \alpha \, dx + \beta \, dy
\]
so this implies that $u$ is of class $C^{k+1,\alpha}$, too. Hence $f$ is everywhere of class $C^{k+1,\alpha}$, as claimed.

Notice that the same reasoning actually applies to any holomorphic vector bundle on any $\Sigma$ and any maximal real sub-bundle of its restriction to $\partial \Sigma$. Of course, the precise computations of the kernel and cokernel are delicate in nature, but the fact that the map is Fredholm is stable under perturbation by compact operators. Thus the Fredholm property holds for quite general first-order operators of Cauchy-Riemann type with these boundary conditions. Since \[18\] the linearization of $\mathcal{D}$ always falls under this heading, we therefore have the following:

**Proposition 1** For any integer $k > 1$ and any real number $\alpha \in (0, 1)$,

$$
\mathcal{D} : C^{k+1,\alpha}(N, E) \rightarrow C^{k,\alpha}(\Lambda^{0,1} \otimes N)
$$

is a Fredholm map of Banach manifolds, and $\mathcal{M} = \mathcal{D}^{-1}(0)$ exactly parameterizes the holomorphic curves $(\Sigma', \partial \Sigma') \subset (Z, P)$ which are sufficiently close to $\Sigma$.

The implicit function theorem \[24\] thus implies an analog of Theorem \[11\]

**Theorem 3** Let $\Sigma$ be a compact holomorphic curve-with-boundary in a complex manifold $Z$, and suppose that the boundary $\partial \Sigma$ of this curve lies on a maximal totally real $C^5$ submanifold $P \subset Z$. If the double $X$ of $\Sigma$ satisfies $H^1(X, \mathcal{O}(\mathcal{N})) = 0$, then the moduli space $\mathcal{M}$ of nearby holomorphic curves $(\Sigma', \partial \Sigma') \subset (Z, P)$ is a manifold. Moreover, the tangent space of this manifold at the base-point $x$ representing $(\Sigma, \partial \Sigma)$ is canonically given by

$$
T_x \mathcal{M} = H^0_G(X, \mathcal{O}(\mathcal{N})).
$$

Indeed, the entire point of Lemma \[11\] is that $\Sigma$ is a regular point of $\mathcal{D}$ iff $H^1(X, \mathcal{O}(\mathcal{N})) = 0$. When this happens, one then says that the holomorphic curve $(\Sigma, \partial \Sigma) \subset (Z, P)$ is Fredholm regular.

If we now wish to consider a 1-parameter family of deformations of either $P$ or of the complex structure of $Z$, we may do so by simply multiplying both $C^{k+1,\alpha}(\Sigma; N, E)$ and $C^{k,\alpha}(\Sigma; \Lambda^{0,1} \otimes N)$ by an interval, and augmenting the parameterized form of $\mathcal{D}$ with the identity map on the second factor. This new map is still Fredholm, and has exactly the same kernel and cokernel at the origin. The implicit function theorem therefore also yields the following analog of Theorem \[24\]
Theorem 4 Suppose \((\Sigma, \partial \Sigma) \subset (Z, P)\), as above. If the double \(X\) of \(\Sigma\) satisfies \(H^1(X, \mathcal{O}(\mathcal{N})) = 0\), then any small deformation \((Z', P')\) of \((Z, P)\) contains an \(h^0(X, \mathcal{O}(\mathcal{N}))\)-dimensional family of holomorphic curves-with-boundary obtained by deforming \(\Sigma\).

In order to make good use of this result, one must be able to concretely identify both \(X\) and \(\mathcal{N} \to X\). However, we shall now see that when \(P\) is the fixed point set of an anti-holomorphic involution of \(Z\), \(X\) can be identified with a holomorphic curve immersed in \(Z\), and \(\mathcal{N}\) then precisely becomes the normal bundle of \(X\), in the usual sense.

3 Plane Curves

The general theory developed in §2 turns out to have some rather surprising consequences for algebraic curves in the projective plane.

Let \(X \subset \mathbb{C}\mathbb{P}^2\) be a complex algebraic curve

\[ P(z_1, z_2, z_3) = 0 \]

where \(P\) is a homogeneous polynomial of degree \(d\) with real coefficients. Assume that \(dP \neq 0\) along \(X\), so that \(X\) is smooth. Also assume that the real locus \(C = X \cap \mathbb{R}\mathbb{P}^2\) is non-empty. Let \(\sigma : X \to X\) be the anti-holomorphic involution induced by complex conjugation in \(\mathbb{C}\mathbb{P}^2\). It is not hard to see, by an elementary covering-space argument, that \(X - C\) has either one or two components, depending on whether the connected surface-with-boundary \(X/\sigma\) is non-orientable or orientable, respectively. Both possibilities really do occur. For example, a real cubic can have either one or two components, and in this \(d = 3\) case one can check that \(X - C\) has the same number of components as the real locus.

Let us first consider the case in which \(X - C\) has two connected components. Let \(\Sigma\) be the closure of one of these two components. Then the corresponding double \(\overline{X} = \Sigma \cup \Sigma\) can be identified with \(X\), and \(\sigma : X \to X\) can be identified with \(\rho : \overline{X} \to \overline{X}\), and the virtual normal bundle \(\mathcal{N} \to \overline{X}\) can be identified with the usual normal normal bundle \(\mathcal{O}(\mathcal{N})\) of \(X\). Since the canonical line bundle of \(X\) is \(\mathcal{O}(d - 3)\) by the adjunction formula, Serre
duality tells us that
\[
H^1(\mathbb{X}, \mathcal{O}(\mathcal{N})) = H^1(X, \mathcal{O}(d)) = [H^0(X, \Omega^1(-d))]^* \\
= [H^0(X, \mathcal{O}([d - 3] - d))]^* \\
= [H^0(X, \mathcal{O}(-3))]^* \\
= 0
\]
since \( \mathcal{O}(-3) \) has negative degree. The hypotheses of Theorems 3 and 4 are therefore fulfilled.

In fact, Theorem 3 tells us essentially nothing new in this case, because the predicted family of dimension
\[
h^0(\mathbb{X}, \mathcal{O}(\mathcal{N})) = \frac{d(d + 3)}{2} = \left(\frac{d + 2}{2}\right) - 1
\]
simply arises by varying the real coefficients of the homogeneous polynomial \( P \). However, the prediction made by Theorem 4 is, by contrast, rather surprising. If we wiggle the embedding of \( P = \mathbb{RP}^2 \) in \( Z = \mathbb{CP}^2 \), leftover halves of algebraic curves continue to cling to it, and the collections of ovals in \( P' \approx \mathbb{RP}^2 \) which are their boundaries give us some strange sort of deformation of the algebraic geometry of the real projective plane.

When \( X - C \) is connected, the story is basically similar, although a few modest changes are necessary. In this case, we instead take \( \Sigma \) to be a surface-with-boundary diffeomorphic to \( X \) minus an annular neighborhood of \( C \), obtained from \( X \) by formally replacing \( C \) with two disjoint copies of itself. The double \( \tilde{X} \) of \( \Sigma \) then becomes the double cover \( \pi : \tilde{X} \to X \) given by the element of \( H^1(X, \mathbb{Z}_2) \) Poincaré dual to \([C] \in H_1(X, \mathbb{Z}_2)\). In this case, we have \( \mathcal{N} = \pi^*\mathcal{O}(d) \), while the canonical line bundle of \( \mathbb{X} = \tilde{X} \) is \( \pi^*\mathcal{O}(d - 3) \), so Serre duality tells us that
\[
H^1(\mathbb{X}, \mathcal{O}(\mathcal{N})) = [H^0(\tilde{X}, \pi^*\mathcal{O}(-3))]^* = 0
\]
because the degree of \( \pi^*\mathcal{O}(-3) \) is once again negative. Thus the hypotheses of Theorems 3 and 4 are once again fulfilled. Here, even Theorem 3 predicts something interesting, as
\[
h^0(\mathbb{X}, \mathcal{O}(\mathcal{N})) = d(d + 3)
\]
is twice as large as before, so we get deformations of real algebraic curves which do not simply arise from varying the coefficients of $P$, but instead arise from real deformations of the map $\tilde{X} \to \mathbb{CP}_2$. The observed doubling of parameters is analogous to what happens in the previous case if we simultaneously keep track of deformations $\Sigma$ and $\tilde{\Sigma}$, without requiring that their boundaries match up in any way. Indeed, this point makes it obvious that when we consider the deformed analogs of real algebraic geometry arising from the replacement of $P = \mathbb{RP}^2$ with some nearby totally real submanifold $P' \subset \mathbb{CP}_2$, we must remember that the entire story has to do with \textit{oriented} curves in $\mathbb{RP}^2$. In this context, an algebraic curve with one orientation must be viewed as a completely different object from its orientation-reversed twin. After deformation, oppositely oriented versions of any given algebraic curve will generally go their own separate ways!

Of course, the methods described here can of also be used to study real projective spaces curves of higher codimension, but the hypotheses of Theorems 3 and 4 are unfortunately no longer hold in general, even for complete intersections. Details are left to the interested reader.

## 4 Partial Indices of Holomorphic Disks

Many interesting applications of Theorem 3 already arise when $\Sigma$ is a disk. Under these circumstances, the double of $\Sigma$ is $X = \mathbb{CP}_1$, so the Grothendieck splitting theorem \cite{21} guarantees that

$$\mathcal{N} = \mathcal{O}(j_1) \oplus \cdots \oplus \mathcal{O}(j_m).$$

If we want to consider holomorphic maps $f : (D, S^1) \to (Y, M)$ which are not embeddings, the theory continues to work extremely well provided we take $Z = Y \times \mathbb{C}$, $P = M \times S^1$, and let $\Sigma$ be the graph of $f$.

In this setting, the numbers $j_1, \ldots, j_m$ are called the \textit{partial indices} of the disk \cite{9}. Notice that Theorem 3 tells us \cite{20} that such a disk is Fredholm regular iff all the partial indices are $\geq -1$. The sum of the partial indices is called the \textit{Maslov index}, and corresponds to the Chern class or \textit{degree} of the double of the normal bundle. Notice that the Maslov index is a topological invariant, whereas the partial indices depend quite sensitively on the complex-analytic structure.

If an embedded holomorphic disk is real-analytic up to its boundary, it is necessarily contained in a complex coordinate domain. Thus, for many
purposes it is quite sufficient to thoroughly understand the special case of $Z = \mathbb{C}^n$. In this setting, however, the problem is amenable to a more elementary treatment, using Fourier series or the Riemann-Hilbert transform. In fact, many of the phenomena under discussion were originally discovered [7, 9, 20] from this perspective, making them seem at first sight to be completely unrelated to Kodaira’s results. It is hoped that the present article may play a useful rôle in bringing together these disparate strands of thought.

5 Twistor Geometry

I will now quickly describe some twistor correspondences involving holomorphic disks.

Let’s first return to the setting of §3, and reconsider the pair consisting of the complex manifold $Z = \mathbb{C}P^2$ and the totally real submanifold $P = \mathbb{R}P^2 \subset \mathbb{C}P^2$. If $C \subset \mathbb{R}P^2$ is any real projective line $\mathbb{R}P^1$, the corresponding complex projective line $X \cong \mathbb{C}P^1$ is divided into two hemispheres by $C$, and we may single out one of these holomorphic disks $(D, \partial D) \subset (\mathbb{C}P^2, \mathbb{R}P^2)$ by choosing an orientation of $C$. Thus the space $M$ of all these disks may be identified with the Grassmannian of oriented projective lines in $\mathbb{R}P^2$, or in other words the Grassmannian $\tilde{Gr}_2(\mathbb{R}^3)$ of oriented 2-planes in $\mathbb{R}^3$. This is of course just a fancy way of saying $S^2$, and we can clarify this point by observing that each of the disks in question meets the conic

$$z_1^2 + z_2^2 + z_3^2 = 0$$

in a unique point. This conic is of course diffeomorphic to $S^2$, and provides a serviceable model for $M$. Now, as a special case of the discussion in §3 the family $M$ of holomorphic disks is stable under deformations of $P$. Thus, if we wiggle the embedding $\mathbb{R}P^2 \hookrightarrow \mathbb{C}P^2$ to produce a nearby totally real submanifold $P' \subset \mathbb{C}P^2$, there is an associated $S^2$-family of holomorphic disks $D'$ with boundaries on $P'$. Let $M' \approx S^2$ be the moduli space of these disks. Then $M'$ contains a tautological family of closed curves. Indeed, for each $y \in P'$, one can consider the set $L_y \subset M$ consisting of all of the holomorphic disks $D'$ passing through $y$. Remarkably, the $L_y$ turn out to be exactly the unparameterized geodesics of an affine connection $\nabla$ on $M'$. Moreover, this construction can be shown to give rise to every connection on a compact surface for which every geodesic is a simple closed curve [16]. The special case in which $\nabla$ is the Levi-Civita connection of some Riemannian metric
g can also be thoroughly analyzed from this point of view, leading to an entirely new understanding \[16\] of the classical theory of Zoll surfaces \[4\].

Next, let us consider what happens if we instead take $Z = \mathbb{CP}^3$ and $P = \mathbb{RP}^3 \subset \mathbb{CP}^3$. Again, every real projective line bounds two holomorphic disks, and the moduli space of these disks is now the Grassmannian $\tilde{Gr}_2(\mathbb{R}^4)$ of oriented 2-planes in $\mathbb{R}^4$. Each such disk meets the quadric

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$$

in a unique point, so this oriented Grassmannian can be identified, if we like, with a complex 2-quadric $Q_2 \approx S^2 \times S^2$. Because the double $X$ of any such disk is a projective line $\mathbb{CP}_1$ with normal bundle $\mathcal{N} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$, these disks all satisfy $H^1(X, \mathcal{O}(\mathcal{N})) = 0$. Theorem \[4\] thus predicts that if we perturb $\mathbb{RP}^3 \hookrightarrow \mathbb{CP}^3$ to obtain a nearby totally real submanifold $P' \subset \mathbb{CP}^3$, there is an analogous $(S^2 \times S^2)$-family of holomorphic disks $D'$ with boundaries in $P'$. Let $M' \approx S^2 \times S^2$ denote this moduli space. Then $M'$ comes equipped with a natural family of embedded 2-spheres $S_y \subset M'$, where, for each $y \in P'$, $S_y$ consists of all the disks in the family passing through $y$. One can then show that there is a pseudo-Riemannian metric $g$ on $M$ with respect to which the $S_y$ are all null surfaces. This $g$ is unique up to conformal rescaling, and is self-dual, in the sense that its Weyl curvature $W$ satisfies $\star W = W$ as a bundle-valued 2-form. Moreover, every self-dual conformal metric near the standard one arises from this construction \[17\]. Notice that the geometries that arise from this construction, unlike the previous one, now satisfy a local curvature condition. On the other hand, global conditions on the periodicity of geodesics do not have to be explicitly stipulated in this case, as they turn out to automatically hold for any solution which is $C^2$ close to the standard one.

It is now only natural to ask what happens if we instead take $Z = \mathbb{CP}_{m+1}$ and $P = \mathbb{RP}^{m+1} \subset \mathbb{CP}_{m+1}$ for some $m \geq 3$. While this is a story which has never properly been set down in detail, many of the broad outlines are certainly similar to what we have already seen. Every real projective line in $\mathbb{RP}^{m+1}$ bounds two holomorphic disks in $\mathbb{CP}_{m+1}$, and the moduli space of these disks is the Grassmannian $\tilde{Gr}_2(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. Each such disk meets the quadric

$$z_1^2 + z_2^2 + \cdots + z_{m+2}^2 = 0$$

in a unique point, so this oriented Grassmannian can be identified with the complex $m$-quadric $Q_m$. Because the double $X$ of any such disk is a projective
line with normal bundle $\mathcal{N} \cong [\mathcal{O}(1)]^\oplus m$, these disks are all Fredholm regular, and Theorem 4 again tells us that each totally real submanifold $P'$ near the standard $\mathbb{RP}^{m+1} \subset \mathbb{CP}_{m+1}$ has an associated family of holomorphic disks $D'$ with boundaries in $P'$. The moduli space $M' \approx Q_m$ of these disks contains a tautological family of embedded $m$-spheres $S_y \subset M'$, $y \in P'$, given by sub-families of those disks passing through any given $y$. This time, however, the associated geometry of $M'$ is, in the terminology of [2], exactly a right-flat $(m, 2)$-paraconformal structure; when $m$ is even, this sort of structure may be thought of as a Wick-rotated version of a quaternionic structure [23].

Given any volume form on $M' \approx Q_m$, there is a unique torsion-free affine connection of holonomy $\subset [SL(2, \mathbb{R}) \times SL(m, \mathbb{R})]/\mathbb{Z}_2$ which is compatible with the paraconformal structure and the volume form; the submanifolds $S_y$ are then totally geodesic with respect to this connection. The arguments in [17] strongly indicate that every such structure on $Q_m$ sufficiently near the standard one should arise from this construction. In particular, the general such structure on $Q_m$ should depend on $(m + 1)$ real functions of $(m + 1)$ real variables. Details are left to the interested reader.

These are but a few simple examples of the manner in which moduli of holomorphic curves-with-boundary can naturally give rise to geometrically rich twistor correspondences. Of course, we have in each case simply taken $Z$ to be $\mathbb{CP}_n$, and taken $\Sigma$ to be half a projective line. There are certainly many, many more correspondences of this same flavor, just waiting to be developed. For example, by taking $\Sigma$ to be a disk in a different rational complex surface, one would encounter versions of the Hitchin correspondence for Einstein-Weyl spaces [12] or Bryant’s connections with exotic affine holonomy [5]. But what if we take $\Sigma$ to be something other than a disk? Such moduli spaces must certainly carry fascinating geometries whose secrets are simply waiting to be unlocked. I can only hope that some interested reader will take up the challenge, and try to chart a bit of this terra incognita.

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