Filon-Clenshaw-Curtis rules for a class of highly-oscillatory integrals with logarithmic singularities

V. Domínguez∗

June 12, 2013

Abstract

In this work we propose and analyse a numerical method for computing a family of highly oscillatory integrals with logarithmic singularities. For these quadrature rules we derive error estimates in terms of \( N \), the number of nodes, \( k \) the rate of oscillations and a Sobolev-like regularity of the function. We prove that that the method is not only robust but the error even decreases, for fixed \( N \), as \( k \) increases. Practical issues about the implementation of the rule are also covered in this paper by: (a) writing down ready-to-implement algorithms; (b) analysing the numerical stability of the computations and (c) estimating the overall computational cost. We finish by showing some numerical experiments which illustrate the theoretical results presented in this paper.

Keywords numerical integration; highly oscillatory integrals; Clenshaw-Curtis rules, Chebyshev polynomials, logarithmic singularities

MSC 65D30, 42A15, 65Y20.

1 Introduction

This paper concerns itself with the approximation of

\[
I_{k,N}(f) := \int_{-1}^{1} f(x) \log((x - \alpha)^2) \exp(ikx) \, dx \tag{1.1}
\]

where \( \alpha \in [-1,1] \). For the sake of simplicity, we will assume throughout this paper that \( k \geq 0 \), although the algorithm and the theoretical results can be straightforwardly adapted for \( k \leq 0 \).

Our aim is to design numerical methods whose rates of convergence do not depend on \( k \) but only on \( f \) and the number of nodes of the quadrature rules. No information about the derivatives, which is very common in the approximation of oscillatory integrals (see [3] or [12] and references therein), will be used, which results in a simpler and less restrictive method. At first sight, \( \alpha \in \{-1,0,1\} \) could be the more common cases but since the analysis we develop here is actually valid for any \( \alpha \in [-1,1] \), we cover the general case in this paper.

We choose in this work the Clenshaw-Curtis approach:

\[
I_{k,N}^\alpha(f) := \int_{-1}^{1} Q_N f(x) \log((x - \alpha)^2) \exp(ikx) \, dx \approx I_{k}^\alpha(f), \tag{1.2}
\]

*Dep. Ingeniería Matemática e Informática, E.T.S.I.I.T. Universidad Pública de Navarra. Campus de Tudela 31500 - Tudela (SPAIN), email:victor.dominguez@unavarra.es
where
\[ \mathbb{P}_N \ni Q_N f, \quad \text{s.t.} \quad (Q_N f)(\cos(n\pi/N)) = f(\cos(n\pi/N)), \quad n = 0, \ldots, N. \] (1.3)

In other words, \( Q_N f \) is the polynomial of degree \( N \) which interpolates \( f \) at Chebyshev nodes.

Classical, and modified, Clenshaw-Curtis rules (1.2) enjoy very good properties which have made them very popular in the scientific literature cf. [5, 20, 21, 24, 25] and have been considered competitive with respect to Gaussian rules even for smooth integrands (we refer to [26] for an interesting discussion about this fact). First of all, the error of the rule is, in the worst case, like the error of the interpolating polynomial in \( L_2(-1,1) \). Thus, the rule is robust respect to \( k \) and it inherits the excellent approximation properties of the interpolant. On the other hand, and from a more practical view, nested grids can be used in the computations. Hence, if \( I_{k,N}(f) \) has been already computed, \( I_{k,2N}(f) \) only requires \( N \) new evaluations of \( f \), i.e. previous calculations can be reused. Moreover, by comparing both approximations, a-posteriori error estimate is at our disposal almost for free. Finally, \( Q_N f \) can be expressed in the Chebyshev basis very fast, in about \( O(n^2) \) operations, using FFT techniques.

If \( k = 0 \), or if \( k \) is small enough (\( k \leq 2 \) has been used throughout this paper), the complex exponential can be incorporated to the definition of \( f \). This leads us to consider, in the same spirit, the following integral and numerical approximation,
\[ I_0^a(f) := \int_{-1}^{1} f(x) \log((x-\alpha)^2) \, dx \approx \int_{-1}^{1} Q_N f(x) \log((x-\alpha)^2) \, dx =: I_{0,N}^a(f). \] (1.4)

This problem is also dealt with in this work since the combination of both algorithms gives rise to a method which can be applied to non-, mildly and highly oscillatory integrals.

For these rule we will show that the rule converges superalgebraically for smooth functions \( f \). Moreover, the error is not only not deteriorated as \( k \) increases but it even decreases as \( k^{-1} \) as \( k \to \infty \). Furthermore, for some particular values of \( \alpha \), which include the more common choices \( \alpha \in \{-1,0,1\} \), the error decay faster, as \( k^{-2} \), which means that both, the absolute and relative error of the rule decreases )cf. Theorem 2.4).

The implementation of the rule hinges on finding a way to compute, fast and accurately, the weights
\[ \xi_n^\alpha(k) := \int_{-1}^{1} T_n(x) \log((x-\alpha)^2) \exp(ikx) \, dx, \quad k > 2, \] (1.5)
\[ \xi_n^\alpha := \xi_n^\alpha(0) = \int_{-1}^{1} T_n(x) \log((x-\alpha)^2) \, dx \] (1.6)

(\( T_n(x) := \cos(n \arccos x) \) is the Chebyshev polynomial of the first kind) for \( n = 0, 1, \ldots, N \). The second set of coefficients \( (\xi_n^\alpha)_n \) is computed by using a three-term recurrence relation which we show to be stable. For the first set, \( (\xi_n^\alpha(k))_n \), the situation is more delicate. First we derive a new three-term linear recurrence which can be used to evaluate \( \xi_n^\alpha(k) \). The calculations, however, turn out to be stable only for \( n \leq k \). This could be understood, somehow, as consequence of potentially handling two different sources of oscillations in \( \xi_n^\alpha(k) \). The most obvious is that coming from the complex exponential, which is fixed independent of \( n \). However, when \( n \) is large, the Chebyshev polynomials, like the classical orthogonal polynomials, have all their roots in \([-1,1]\). This results in a increasing oscillatory behaviour of the polynomial as \( n \to \infty \). As long as the first oscillations source dominates the second one, i.e. as \( k > n \), the recurrence is stable: any perturbation introduced in the computation is amplified very little. However, when \( n > k \) increases, such perturbations are hugely magnified, which makes this approach completely useless. Of course, if \( k \) is large, so should be \( N \) to find these instabilities. Hence, this only causes difficulties for practical computations in the middle range, that is, when \( k \) is not yet very large but we need to use a relatively large number of points to evaluate the integral within the prescribed precision.
This phenomenon is not new: It has been already observed, among other examples, when computing the simpler integral
\[ \int_{-1}^{1} T_n(x) \exp(ikx) \, dx. \]
(See [6] and references therein). Actually, the problem is circumvented using the same idea, the so-called Oliver method (cf. [18]) which consists in rewriting appropriately the difference equation used before now as a tridiagonal linear system whose (unique) solution gives the sought coefficients except the last one which is part now of the right-hand-side. Therefore, the evaluation of this last coefficient has to be carried out in a different way. Thus, we make use of an asymptotic argument, namely the Jacobi-Anger expansion, which expresses \( \xi_\alpha^N(k) \) as a series whose terms are a product of Bessel functions and integrals as in (1.6). Despite the fact that it could seem at first sight, the series can also be summed in about \( O(N) \) operations. The resulting algorithm has a cost \( O(N \log N) \), cost which is lead by the FFT method used in the construction of the interpolant \( Q_N f \).

Let us point out that the case of \( \alpha = 0 \), for both the oscillatory and non-oscillatory case, has been previously considered in [4] using a different strategy. Roughly speaking, it relies on using the asymptotic Jacobi-Anger expansion for all the coefficients, no matter how large \( k \) is respect to \( n \). Our approach is, in our opinion, more optimal since the algorithm is simpler to implement and the computational cost is smaller.

The interest in designing efficient methods for approximating oscillatory integrals has been increased in the last years, fueled by new problems like high frequency scattering simulations cf. [4, 9, 6]. For instance, in the boundary integral method, the assembly of the matrix of the systems requires computing highly oscillatory integrals which are smooth except on the diagonal. Hence, after appropriate change of variables, we can reduce the problem to evaluate
\[ \int_{0}^{1} f(s) \exp(iks) \, ds. \]
Typically, \( f \) is smooth except at the end-points where an integrable singularity, which could be either in the original integral or introduced in the change of variables, occurs. Actually, the log-singularity is very common since one can find it in the fundamental solutions for many differential operators in 2D, for instance, in the Helmholtz equation.

Different strategies have been suggested for computing oscillatory integrals. For instance steepest descent methods, based on analytic continuation in the complex plane [8] or Levin methods which reduces the problem to solving ODE by collocation methods [15, 19]. On the other hand, we find Filon rules which consists in interpolating the function by a (piecewise) polynomial. Therefore, our method can be characterised as a Filon rule. The general case for smooth functions has been considered eg. in [10, 11, 12, 16, 29, 28]. Provided that the new integral with the interpolating polynomial replacing the original function can be computed exactly, a robust method is obtained in the sense that it converges as the size of the subintervals shrink to zero. Depending on the choice of the nodes we have Filon-Cloenshaw-Curtis rules, Filon-Gaussian rules or, if the derivatives, usually at the end points, are also interpolated, Filon-Hermite rules. Oscillatory integrals with algebraic singularities in the integrand, and more general oscillators, have been considered in [17, 13]. A different approach was considered in [7] where the use of graded meshes toward the singularities has shown to be also efficient. Let us point out that this last example gives another example of the importance of having robust methods, which covers all possible values of \( k \), since graded meshes can easily have very small subintervals so that the oscillations are reduced or even disappear.

This paper is structured as it follows: In section 2 we derive the error estimates for the quadrature rule. In section 3 we deduce the algorithms to evaluate the coefficients (1.5) and (1.6). The stability
of such evaluations is analysed in detail in section 4. Some numerical experiments are presented in section 5, demonstrating the results proven in this work. In the appendix we collect those properties of Chebyshev polynomials used in this paper.

2 Error estimates for the Product Clenshaw-Curtis rule

The aim of this section is to derive convergence estimates for the error of the quadrature rule (1.2)-(1.4). Obviously,

\[ T^a_{k,N} - T^a_k(f) = \int_{-1}^{1} E_N(x) \log((x-\alpha)^2) \exp(ikx) \, dx \]

where

\[ E_N := Q_N f - f, \quad (2.1) \]

and \( Q_N f \) is the interpolating polynomial at Chebyshev nodes cf. (1.3).

A very popular technique when working with Chebyshev polynomial approximations is to perform the change of variable \( x = \cos \theta \). This transfers the problem to the frame of even periodic functions and their approximations by trigonometric polynomials. Hence, if we denote \( f_c(\theta) := f(\cos \theta) \) (note that \( f_c \) is now even and \( 2\pi \)-periodic) we have that

\[ \text{span} \{ \cos n \theta : n = 0, \ldots, N \} \ni (Q_N f)_c, \quad (Q_N f)(n\pi/N) = f_c(n\pi/N), \quad n = 0, \ldots, N. \]

Let us denote by \( H^r(I) \) the classical Sobolev space of order \( r \) on an interval \( I \subset \mathbb{R} \) and define

\[ H^r_{\#} := \{ \varphi \in H^r_{\text{loc}}(\mathbb{R}) \mid \varphi = \varphi(2\pi + \cdot) \}. \]

The norm of these spaces can be characterised in terms of the Fourier coefficients of the elements as follows

\[ \| \varphi \|_{H^r_{\#}}^2 := |\hat{\varphi}(0)|^2 + \sum_{n \neq 0} |n|^{2r} |\hat{\varphi}(n)|^2, \quad \hat{\varphi}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\theta) \exp(-in\theta) \, d\theta. \]

If \( r = 0 \), we just have the \( L_2(-\pi, \pi) \) norm, whereas for positive integers an equivalent norm is given by

\[ \left( \int_{-\pi}^{\pi} |\varphi(\theta)|^2 \, d\theta + \int_{-\pi}^{\pi} |\varphi^{(r)}(\theta)|^2 \, d\theta \right)^{1/2}. \]

The convergence estimates for the trigonometric interpolant in Sobolev spaces (see for instance [23 §8.3]) can be straightforwardly adapted to prove that

\[ \|(E_N)_c\|_{H^r(0,\pi)} \leq \|(E_N)_c\|_{H^r_{\#}} \leq C_{s,r_0} N^{s-r} \|f_c\|_{H^r_{\#}} \quad (2.2) \]

where \( r \geq s \geq 0 \) with \( r \geq r_0 > 1/2 \) and \( C_{s,r_0} \) independent of \( f, N \) and \( r \).

Set \( w(x) := (1-x^2)^{1/2} \) and define for \( \beta \in \{-1,1\} \)

\[ \|f\|_{\beta,w} := \|fw^{\beta/2}\|_{L_2(-1,1)} = \left[ \int_{-1}^{1} |f(x)|^2 (1-x^2)^{\beta/2} \, dx \right]^{1/2}. \]

Notice in pass

\[ \left| \int f(x)g(x) \, dx \right| \leq \|f\|_{-1,w} \|g\|_{1,w}, \quad \|g\|_{1,w} \leq \sqrt{\pi} \|g\|_{L_\infty(-1,1)}. \quad (2.3) \]

From the relations

\[ \|f\|_{-1,w} = \|f_c\|_{L_2(0,\pi)}, \quad \|f'\|_{1,w} = \|(f_c)'\|_{L_2(0,\pi)}, \]

4
estimates \( \text{[2.2]} \), and the Sobolev embedding theorem \( \text{[23, Lemma 5.3.3]} \), we can easily derive the following estimate: For any \( r \geq 1 \),

\[
\| E_N \|_{-1,w} + N^{-1}\| E'_N \|_{1,w} + N^{-1/2-\varepsilon}\| E_N \|_{L\infty(-1,1)} \leq C_\varepsilon N^{-r}\| f_c \|_{H^r_w}, \tag{2.4}
\]

with \( C_\varepsilon \) depending only on \( \varepsilon > 0 \).

To prove the main result of this section we previously need some technical results we collect in the next three Lemmas. The first result concerns the asymptotics of

\[
\xi^\alpha_0(k) = \int_{-1}^{1} \exp(ikx) \log((x - \alpha)^2) \, dx
\]

as \( k \to \infty \).

**Lemma 2.1** For all \( \alpha \in (-1,1) \) there exists \( C_\alpha > 0 \) so that for all \( k \geq 2 \)

\[
|\xi^\alpha_0(k)| \leq C_\alpha(1 + \log(1 - \alpha^2))k^{-1}.
\]

Moreover, for \( \alpha = \pm 1 \),

\[
|\xi^{\pm 1}_0(k)| \leq C_1 k^{-1} \log k,
\]

with \( C_1 > 0 \) independent of \( k \geq 2 \).

**Proof.** The result follows from working on the explicit expression for \( \xi^\alpha_0(k) \), see \( \text{[3.18]} - \text{[3.19]} \), and from using the limit of the functions involved as \( k \to \infty \). We omit the proof for the sake of brevity. \( \square \)

The next Lemma complements the estimates given in (2.4).

**Lemma 2.2** Let \( E_N \) be given in (2.1) and

\[
e^\alpha_N(x) := \frac{E_N(x) - E_N(\alpha)}{x - \alpha}. \tag{2.5}
\]

Then for all \( r \geq s_0 > 5/2 \), there exists \( C_{s_0} \) independent of \( f, N \) and \( r \) so that

\[
\| E'_N \|_{L\infty(-1,1)} + \| e^\alpha_N \|_{L\infty(-1,1)} \leq C_{s_0} N^{s_0 - r}\| f_c \|_{H^r_w}. \tag{2.6}
\]

Moreover,

\[
\| wE''_N \|_{L\infty(-1,1)} + \| w(e^\alpha_N)' \|_{L\infty(-1,1)} \leq C_{s_1} N^{s_1 - r}\| f_c \|_{H^r_w}, \tag{2.7}
\]

for \( r \geq s_1 > 7/2 \), with \( C_{s_1} \) independent also of \( f, N \) and \( r \).

**Proof.** Recall cf. \( \text{[A.3]} - \text{[A.4]} \)

\[
\| T_n' \|_{L\infty(-1,1)}^2 = n^2, \quad \| wT_n' \|_{L\infty(-1,1)} = n, \quad \| wT_n'' \|_{L\infty(-1,1)} \leq C n^3 \tag{2.8}
\]

where \( C > 0 \) is independent of \( n \). Define now,

\[
p^\alpha_n(x) := \frac{T_n(x) - T_n(\alpha)}{x - \alpha} \in \mathbb{P}_{n-1}.
\]

Obviously

\[
\| p^\alpha_n \|_{L\infty(-1,1)} \leq \| T'_n \|_{L\infty(-1,1)} \leq n^2. \tag{2.9}
\]
Besides, from (A.10) (note that in the notation used there, \( U_j = T_{j+1}/(j + 1) \)), we derive

\[
(p_n^0)'(x) = 2 \sum_{j=0}^{n-2} (j + 1)^{-1} T_{j+1}^\prime(\alpha) T_{n-j-1}(x).
\]

Then, using (2.8)

\[
\|w(p_n^0)\|_{L_\infty(-1,1)} \leq 2 \sum_{j=0}^{n-2} (j + 1)^{-1} T_{j+1}^\prime(\alpha) \|wT_{n-j-1}\|_{L_\infty(-1,1)} = 2 \sum_{j=0}^{n-2} (j + 1)(n - 1 - j) = \frac{1}{3}(n - 1)n(n + 1) < \frac{n^3}{3}.
\]

On the other hand, for all \( f \) smooth enough,

\[
f = \tilde{f_c}(0) + 2 \sum_{n=1}^\infty \tilde{f_c}(n)T_n, \quad \tilde{f_c}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f_c}(\theta) \exp(i\pi \theta) \, d\theta = \frac{1}{\pi} \int_{-1}^{1} f(x)T_n(x) \, dx.
\]

To prove (2.9), we recall first the definition of \( \varepsilon_N^0 \) in (2.5), and combine (2.11), (2.10) and (2.8) to obtain

\[
\|\varepsilon_N^0\|_{L_\infty(-1,1)} + \|E_N'\|_{L_\infty(-1,1)} = \left\| \sum_{n=0}^{\infty} \widehat{(E_N)c} (n)p_n^0 \right\|_{L_\infty(-1,1)} + \left\| \sum_{n=0}^{\infty} \widehat{(E_N)c} (n)T_n' \right\|_{L_\infty(-1,1)} \\
\leq \sum_{n=1}^{\infty} \|\widehat{(E_N)c} (n)\| \left( \|p_n^0\|_{L_\infty(-1,1)} + \|T_n'\|_{L_\infty(-1,1)} \right) \\
\leq 2 \left[ \sum_{n=1}^{\infty} n^{-1-2\epsilon} \right]^{1/2} \left[ \sum_{n=1}^{\infty} \|\widehat{(E_N)c} (n)\|^2 n^{5+2\epsilon} \right]^{1/2} =: C_\varepsilon \|E_N\|_{H^{5/2+\epsilon}}.
\]

Estimate (2.2) proves (2.6). Proceeding similarly, but using the last bound in (2.8) and (2.10) instead, we prove (2.7).

\[
\text{Lemma 2.3} \quad \text{There exists } C > 0 \text{ such that for any } \alpha \in [-1, 1] \text{ and for all } g \in H^1(-1,1) \text{ with } g(\alpha) = 0 \text{ and for all } x \in [-1, 1],
\]

\[
|g(x)| \leq C|x - \alpha|^{1/4}||g'||_{1,w},
\]

(2.12)

\[
\int_{-1}^{1} \frac{|g(x)|}{|x - \alpha|} \, dx = C||g'||_{1,w}.
\]

\[
\text{Proof.} \quad \text{Clearly, (2.13) follows from (2.12).}
\]

Note first

\[
C := \max_{\alpha \in [-1, 1]} \left\| \arcsin(\cdot) - \arcsin \alpha \right\|_{L_\infty(-1,1)} < \infty.
\]

Since \( g(\alpha) = 0 \), it follows

\[
|g(x)| = \left| \int_\alpha^x g'(s) \, ds \right| \leq \left[ \int_\alpha^x \frac{ds}{\sqrt{1 - s^2}} \right]^{1/2} \left[ \int_\alpha^x |g'(s)|^2 (1 - s^2)^{1/2} \, ds \right]^{1/2} \\
\leq |\arcsin x - \arcsin \alpha|^{1/2} \left[ \int_{-1}^{1} |g'(s)|^2 w(s) \, ds \right]^{1/2} \leq C|x - \alpha|^{1/4}||g'||_{1,w}.
\]

6
The result is then proven. \qed

We are ready to give the main result of this section which summarises the convergence property of $\mathcal{I}_{k,N}(f)$ in terms of $N$, $k$ and the regularity of $f$.

**Theorem 2.4** For all $\alpha \in [-1, 1]$ there exists $C_\alpha > 0$ so that for $\delta \in \{0, 1\}$

$$|\mathcal{I}_k^\alpha(f) - \mathcal{I}_{k,N}^\alpha(f)| \leq C_\alpha (1 + k)^{-\delta} N^{\delta - r} \|f_c\|_{H_r^\alpha}$$  \hspace{1cm} (2.14)

for all $r \geq 1$ and $k \geq 0$.

Furthermore, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that if $\alpha = \pm 1$ or $\alpha = 0$ and $N$ is even it holds

$$|\mathcal{I}_k^\alpha(f) - \mathcal{I}_{k,N}^\alpha(f)| \leq C_\varepsilon (1 + k)^{-2(1 + \alpha^2 \log k)} N^{7/2 + \varepsilon - r} \|f_c\|_{H_r^\alpha},$$  \hspace{1cm} (2.15)

for all $f \in H_r^\alpha$ with $r > 7/2 + \varepsilon$.

**Proof.** Note first

$$|\mathcal{I}_k^\alpha(f) - \mathcal{I}_{k,N}^\alpha(f)| = \left| \int_{-1}^{1} E_N(x) \log ((x - \alpha)^2) \exp(ikx) \, dx \right| \leq \|\log (\cdot - \alpha^2)\|_{1,w} \|E_N\|_{1,w}$$

where we have used \ref{2.1} (see also \ref{2.1}). This proves \ref{2.1} for $\delta = 0$. Observe that from now on we can assume, without loss of generality, that $k \geq 1$.

To obtain \ref{2.14} for $\delta = 1$ we write

$$|\mathcal{I}_k^\alpha(f) - \mathcal{I}_{k,N}^\alpha(f)| = \left| \int_{-1}^{1} \left( E_N(x) - E_N(\alpha) \right) \log ((x - \alpha)^2) \exp(ikx) \, dx + E_N(\alpha)\xi_0^\alpha(k) \right|$$

where we have used \ref{2.12} in Lemma 2.3 with $g = E_N - E_N(\alpha)$:

$$R_N^{(1)}(k) \leq C \|E_N\|_{1,w} \left( \|h_\alpha(-1)\| + \|h_\alpha(1)\| \right), \quad h_\alpha(x) := \|x - \alpha\|^{1/4} \log ((x - \alpha)^2) \leq C' \|E_N\|_{1,w}. \hspace{1cm} (2.18)$$

where we have used that $h_\alpha \in C^0[-1, 1]$ for any $\alpha \in [-1, 1]$.

For the second term, notice that \ref{2.13} and \ref{2.14} imply

$$R_N^{(2)}(k) \leq \|\log (\cdot - \alpha^2)\|_{-1,w} \|E_N'\|_{1,w}. \hspace{1cm} (2.19)$$

On the other hand, \ref{2.13} of Lemma 2.3 yields

$$R_N^{(3)}(k) \leq C \|E_N'\|_{1,w}. \hspace{1cm} (2.20)$$
Finally, \( E_N(\pm 1) = 0 \) and therefore \( R_N^{(4)}(k) \) vanishes for \( \alpha = \pm 1 \). Otherwise, Lemma 2.1 implies
\[
R_N^{(4)}(\alpha) \leq C_\alpha |E_N(\alpha)| \leq C_\alpha \|E_N\|_{L_\infty(-1,1)}. \tag{2.21}
\]
Bounding (2.18)–(2.21) with (2.21), we derive (2.14) for \( \delta = 1 \).

To prove (2.15) we have to perform another step of integration by parts. First, we note, that, by hypothesis \( \alpha \) is dealt with similarly.

Similarly, one can prove easily
\[
R_N^{(3)}(k) \leq \frac{1}{k} \left[ |e_N^\alpha(-1)| + |e_N^\alpha(1)| + C\|e_N^\alpha\|_{L_\infty(-1,1)} \right] \leq C_\varepsilon k^{-1} N^{7/2+\varepsilon-r} \|f_c\|_{H^r} \tag{2.24}
\]
for \( \varepsilon > 0 \), \( r \geq 7/2 + \varepsilon \) with \( C_\varepsilon \) independent of \( N, k \) and \( f \).

Similarly, one can prove easily
\[
R_N^{(3)}(k) \leq \frac{1}{k} \left[ |e_N^\alpha(-1)| + |e_N^\alpha(1)| + C\|e_N^\alpha\|_{L_\infty(-1,1)} \right] \leq C_\varepsilon k^{-1} N^{7/2+\varepsilon-r} \|f_c\|_{H^r} \tag{2.25}
\]
for all \( r \geq 7/2 + \varepsilon \), and \( C_\varepsilon \) depending only on \( \varepsilon > 0 \).

Plugging (2.24) and (2.25) into (2.17) and recalling (2.22), we have completed the proof of (2.15).
Remark 2.5 We will show in the last section (see Experiment 5) that the restriction of \(N\) to be even if \(\alpha = 0\) for achieving the \(k^{-2}\)-decay of the error is really needed. In the same experiment, we can check that the error, specially for high values of \(k\), is smaller for \(\alpha = 0\) than for that obtained if \(\alpha = 1\). This supports empirically the fact that in the second case the \(\log k\) term is certainly part of the error term and therefore it affects, although very slightly, the convergence of the rule.

3 Stable computation of the weights

When the practical implementation of the quadrature rule is considered, we face that it essentially reduces to find a way of evaluating \(\xi_\alpha^N(k)\) cf. (1.5)–(1.6) fast and accurately. In this section we present the algorithms to carry out this evaluation and we leave for the next one the proofs of the results concerning the stability of such computations.

For both, the oscillatory and non-oscillatory case, what we will actually compute is

\[
\eta_\alpha^N(k) := \int_{-1}^{1} \log((x - \alpha)^2)U_n(x) \exp(ikx)dx \tag{3.1}
\]

where

\[
U_n := \frac{1}{n+1} T_{n+1}'
\]

is the Chebyshev polynomial of the second kind and degree \(n\). Notice that, from this definition, we have

\[
U_{-1} = 0
\]

which simplifies some forthcoming expressions. From (A.9) we have

\[
\xi_0^\alpha(k) = \eta_0^\alpha(k), \quad \xi_\alpha^N(k) = \frac{1}{2}(\eta_\alpha^N(k) + \eta_{\alpha-2}^N(k)), \quad n = 1, 2 \ldots \tag{3.3}
\]

Observe that by (A.3)–(A.5), there exists \(C > 0\) such that for any \(\alpha \in [-1, 1]\) and \(n\)

\[
|\xi_\alpha^N(k)| \leq \int_{-1}^{1} |\log((x - \alpha)^2)| \, dx \leq C, \tag{3.4}
\]

\[
|\eta_\alpha^N(k)| \leq \left[ \int_{-1}^{1} |U_n(x)|^2 \sqrt{1 - x^2} \, dx \right]^{1/2} \left[ \int_{-1}^{1} \left(\log((x - \alpha)^2)\right)^2 \frac{dx}{\sqrt{1 - x^2}} \right]^{1/2} \leq C. \tag{3.5}
\]

That is, these coefficients are bounded independent of \(n\), \(\alpha\) and \(k\).

3.1 The non-oscillatory case

Recall that for \(k = 0\), we have denoted \(\xi_\alpha^N\) and \(\eta_\alpha^N\) instead of \(\xi_\alpha^N(0)\) and \(\eta_\alpha^N(0)\) to lighten the notation.

Assume that \(\alpha \neq \pm 1\). Using the recurrence relation for Chebyshev polynomials cf. (A.2) and (3.2), we deduce for \(n \geq 1\)

\[
\eta_\alpha^N = \int_{-1}^{1} U_n(x) \log((x - \alpha)^2) \, dx
\]

\[
= \int_{-1}^{1} 2xU_{n-1}(x) \log((x - \alpha)^2) \, dx - \int_{-1}^{1} U_{n-2}(x) \log((x - \alpha)^2) \, dx
\]

\[
= \frac{2}{n} \int_{-1}^{1} (x - \alpha)T_n'(x) \log((x - \alpha)^2) \, dx + 2\alpha \eta_{n-1}^\alpha - \eta_{n-2}^\alpha. \tag{3.6}
\]
Integrating by parts in the first integral, we easily see that

\[
\int_{-1}^{1} (x - \alpha) T_n(x) \log((x - \alpha)^2) \, dx =
\]

\[
= (x - \alpha) T_n(x) \log((x - \alpha)^2)|_{x=-1}^{x=1} - \int_{-1}^{1} T_n(x) \log((x - \alpha)^2) \, dx - 2 \int_{-1}^{1} T_n(x) \, dx
\]

\[
= (1 - \alpha) \log((1 - \alpha)^2) + (-1)^n (1 + \alpha) \log((1 + \alpha)^2)
\]

\[- \frac{1}{2} [\eta_n^\alpha - \eta_{n-2}^\alpha] + \begin{cases} 
\frac{4}{n^2 - 1}, & \text{if } n \text{ is even}, \\
0, & \text{otherwise}, 
\end{cases} \tag{3.7}
\]

where we have used that \(T_n(1) = 1 = (-1)^n T_n(-1)\), \(\text{(3.3)}\) (see also \(\text{(A.9)}\)) and \(\text{(A.6)}\). Hence, inserting \(\text{(3.7)}\) in \(\text{(3.6)}\) and resorting appropriately the elements above, we arrive to the following two-terms linear recurrence

\[
\eta_n^\alpha = \frac{2\alpha n}{n+1} \eta_{n-1}^\alpha - \frac{n-1}{n+1} \eta_{n-2}^\alpha + \gamma_n^\alpha, \quad n = 1, 2, \ldots \tag{3.8}
\]

with

\[
\gamma_n^\alpha := \frac{4}{n+1} \begin{cases} 
(1 - \alpha) \log(1 - \alpha) + (1 + \alpha) \log(1 + \alpha) + \frac{2}{n^2 - 1}, & \text{for even } n, \\
(1 - \alpha) \log(1 - \alpha) - (1 + \alpha) \log(1 + \alpha), & \text{for odd } n. 
\end{cases} \tag{3.9}
\]

For \(\alpha = \pm 1\), \(\text{(3.8)}\) remains valid with

\[
\gamma_{\pm 1}^\alpha := \frac{8}{n+1} \begin{cases} 
\log 2 + \frac{1}{n^2 - 1}, & \text{for even } n, \\
\mp \log 2, & \text{for odd } n, 
\end{cases} \tag{3.10}
\]

which corresponds to take the limit as \(\alpha \to \pm 1\) in \(\text{(3.9)}\).

Straightforward calculations show, in addition, that

\[
\eta_0^\alpha := \begin{cases} 
-(\alpha - 1) \log ((\alpha - 1)^2) + (\alpha + 1) \log ((\alpha + 1)^2) - 4, & \text{if } \alpha \neq \pm 1, \\
4 \log 2 - 4, & \text{if } \alpha = \pm 1. 
\end{cases} \tag{3.11}
\]

Recalling that \(\eta_{-1}^\alpha = 0\), we are ready to write down the first algorithm.

**Algorithm I: compute \(\xi_n^\alpha\) for \(n = 0, 1, \ldots, N\)**

1. Set \(\eta_{-1}^\alpha = 0\) and compute \(\eta_0^\alpha\) according to \(\text{(3.11)}\).
2. For \(n = 1, \ldots, N\)

\[
\eta_n^\alpha = \frac{2\alpha n}{n+1} \eta_{n-1}^\alpha - \frac{n-1}{n+1} \eta_{n-2}^\alpha + \gamma_n^\alpha,
\]

with \(\gamma_n^\alpha\) defined in \(\text{(3.9)}-\text{(3.10)}\).
3. Set

\[
\xi_0^\alpha = \eta_0^\alpha, \quad \xi_n^\alpha = \frac{1}{2} [\eta_n^\alpha - \eta_{n-2}^\alpha], \quad n = 1, 2, \ldots, N.
\]

**Remark 3.1** For \(\alpha = 0\) the algorithm is even simpler since by parity \(\eta_0^0 = \xi_s^0 = 0\) and step 2 of the algorithm becomes

\[
\eta_{2n}^0 = -\frac{2n - 1}{2n + 1} \eta_{2n-2}^0 + \frac{8}{(2n + 1)(4n^2 - 1)}.
\]
3.2 The oscillatory case

Because of (3.3),

\[
\frac{1}{2}(\eta_n^\alpha(k) + \eta_{n-2}^\alpha(k)) = \xi_n^\alpha(k) = \int_{-1}^{1} (T_n(x) - T_n(\alpha)) \log((x - \alpha)^2) \exp(ikx) \, dx
\]

\[+ T_n(\alpha) \int_{-1}^{1} \log((x - \alpha)^2) \exp(ikx) \, dx. \tag{3.12} \]

Assume now that \( \alpha \neq \pm 1 \). Integrating by parts we derive

\[
\int_{-1}^{1} (T_n(x) - T_n(\alpha)) \log((x - \alpha)^2) \exp(ikx) \, dx
\]

\[= \frac{1}{ik} \left[ (T_n(x) - T_n(\alpha)) \log((x - \alpha)^2) \exp(ikx) \right]_{x=-1}^{x=1}
- \int_{-1}^{1} T_n'(x) \log((x - \alpha)^2) \exp(ikx) \, dx - 2 \int_{-1}^{1} \frac{T_n(x) - T_n(\alpha)}{x - \alpha} \exp(ikx) \, dx \tag{3.13} \]

\[= \frac{1}{ik} \left[ (1 - T_n(\alpha)) \log((1 - \alpha)^2) \exp(ik) + ((-1)^n + T_n(\alpha)) \log((1 + \alpha)^2) \exp(-ik) \right.
- n \int_{-1}^{1} U_{n-1}(x) \log((x - \alpha)^2) \exp(ikx) \, dx
- 2 \int_{-1}^{1} U_{n-1}(x) \exp(ikx) \, dx - 4 \sum_{j=0}^{n-2} T_{n-1-j}(\alpha) \int_{-1}^{1} U_j(x) \exp(ikx) \, dx \right]. \tag{3.14} \]

(We have applied (A.10) to write the last integral in (3.13) as the sum in the right-hand-side of (3.14)). Inserting (3.14) in (3.12) and using (3.2) we derive the following recurrence equation

\[\eta_n^\alpha(k) + \frac{2n}{ik} \eta_{n-1}^\alpha(k) + \eta_{n-2}^\alpha(k) = \gamma_n^\alpha(k) \tag{3.15} \]

where

\[\gamma_n^\alpha(k) := \frac{2}{ik} \left[ (1 - T_n(\alpha)) \log((1 - \alpha)^2) \exp(ik) + ((-1)^n + T_n(\alpha)) \log((1 + \alpha)^2) \exp(-ik) \right]
- 4 \sum_{j=0}^{n-2} T_{n-1-j}(\alpha) \rho_j(k) + \rho_{N-1}(k) \right] + 2T_n(\alpha) \eta_0^\alpha(k), \tag{3.16} \]

with

\[\rho_j(k) := \int_{-1}^{1} U_j(x) \exp(ikx) \, dx, \quad j = 0, \ldots, n - 1. \]

Let us point out that \((\rho_j(k))_{j=0}^{N}\) can be computed in \(O(N)\) operations (see [6]).

For \( \alpha = \pm 1 \) we obtain the same recurrence (3.15) with

\[\gamma_n^{\pm 1}(k) = \frac{4}{ik} \left\{ \begin{array}{ll}
\log(4) \exp(\mp ik), & \text{if } n \text{ is odd} \\
0, & \text{otherwise}
\end{array} \right.
- 2 \sum_{j=0}^{n-2} (\pm 1)^{n-j+1} \int_{-1}^{1} U_j(x) \exp(ikx) \, dx - \int_{-1}^{1} U_{n-1}(x) \exp(ikx) \, dx \right] + 2(\pm 1)^n \eta_0^{\pm 1}(k). \tag{3.17} \]

11
It just remains to compute $\eta_0^\alpha(k)$ for setting up the algorithm. For this purpose we introduce the sine and cosine integral functions

$$
\text{Si}(t) := \int_0^t \frac{\sin x}{x} \, dx, \quad \text{Ci}(t) := \gamma + \log(t) + \int_0^t \frac{\cos x - 1}{x} \, dx,
$$

with $\gamma \approx 0.57721$ the Euler-Mascheroni constant. Straightforward calculations show that

$$
\eta_0^\alpha(k) = \xi_0^\alpha(k) = \frac{2}{k} \left[ \log(1 - \alpha^2) \sin k + \sin(\alpha k) \left( \text{Ci}((\alpha + 1)k) - \text{Ci}((1 - \alpha)k) \right) - \cos(\alpha k) \left( \text{Si}((\alpha + 1)k) + \text{Si}((1 - \alpha)k) \right) \right]
$$

$$
+ \frac{2i}{k} \left[ \log \left( \frac{1 + \alpha}{1 - \alpha} \right) \cos k + \cos(\alpha k) \left( \text{Ci}((1 - \alpha)k) - \text{Ci}((1 + \alpha)k) \right) - \sin(\alpha k) \left( \text{Si}((1 - \alpha)k) + \text{Si}((1 + \alpha)k) \right) \right],
$$

(3.18)

for $\alpha \neq \pm 1$, and

$$
\eta_{0}^{\pm 1}(k) = \xi_{0}^{\pm 1}(k) = \frac{2}{k} \left[ - (\gamma - \text{Ci}(2k) + \log(k/2)) \sin k - \text{Si}(2k) \cos k \right]
$$

$$
\pm \frac{2i}{k} \left[ (\gamma - \text{Ci}(2k) + \log(2k)) \cos k - \text{Si}(2k) \sin k \right].
$$

(3.19)

From now on, we will denote by $\lfloor x \rfloor$ the floor function, i.e., the largest integer smaller than $x$.

**Algorithm II: computation of $\xi_n^\alpha(k)$ for $n = 0, \ldots, N$ with $N \leq |k| - 1$**

1. Set $\eta_{-1}^\alpha(k) = 0$ and evaluate $\eta_0^\alpha(k)$ according to (3.18)–(3.19).

2. Compute $\gamma_n^\alpha(k)$ for $n = 1, \ldots, N$ using (3.16)–(3.17).

3. For $n = 1, 2, \ldots, N$, define

$$
\eta_n^\alpha(k) = \gamma_n^\alpha(k) - \frac{2n}{ik} \eta_{n-1}^\alpha(k) + \eta_{n-2}^\alpha(k).
$$

(3.20)

4. Set

$$
\xi_0^\alpha(k) = \eta_0^\alpha(k), \quad \xi_n^\alpha(k) = \frac{1}{2} \left[ \eta_n^\alpha(k) - \eta_{n-2}^\alpha(k) \right], \quad n = 1, \ldots, N.
$$

Observe that we have restricted the range for which this algorithm can be used to $N \leq k - 1$. This is because the recurrence relation (3.20), as it will be shown in the next sections, is not longer stable for $n \geq k$. Thus, we have to explore different ways to compute $\eta_n^\alpha(k)$ when $n \geq k$.

Then assume that $N > |k| - 1$. Note that Algorithm II returns $\eta_0^\alpha(k), \ldots, \eta_{|k|-1}^\alpha$. In order to compute the remaining weights we still use (3.20) but rewriting it in a different way, namely as a tridiagonal system. (This is the so-called Oliver method cf. [18].) Hence, let

$$
A_N^\alpha(k) := \begin{bmatrix}
\frac{2(|k|+1)}{ik} & 1 & & \\
-1 & \frac{2(|k|+2)}{ik} & 1 & \\
& \ddots & \ddots & \\
& & -1 & \frac{2N-2}{ik}
\end{bmatrix}
$$

$$
b_N^\alpha(k) := \begin{bmatrix}
\eta_{|k|-1}^\alpha + \gamma_{|k|+1}^\alpha \\
\gamma_{|k|+2}^\alpha \\
\vdots \\
\gamma_{N-1}^\alpha - \eta_N^\alpha(k)
\end{bmatrix}.
$$

(3.21)
Note that \( A_N^\alpha(k) \) is row dominant, which implies first that the system is uniquely solvable and next suggests that all the calculations become stable. This will be rigorously proven in next section.

In short, if \( \eta \) solves
\[
A_N^\alpha(k)\eta = b_N^\alpha(k), \tag{3.22}
\]
ecessarily
\[
\eta = \begin{bmatrix}
\eta_{[k]}(k) & \eta_{[k]+1}(k) & \cdots & \eta_{N-1}(k)
\end{bmatrix}^T.
\]
In the definition of the right-hand-side we find \( \eta_N^\alpha(k) \) and \( \eta_{[k]-1}^\alpha \). The latter is already known. Thus, only the problem of finding \( \eta_N^\alpha(k) \) remains open. For these purposes, as in \cite{4}, we will use the Jacobi-Anger expansion cf. \cite{27} §2.2, \cite{2} (9.1.44-45):
\[
\exp(ikx) = J_0(k) + 2 \sum_{n=1}^{\infty} i^n J_n(k) T_n(x),
\]
where \( J_n \) is the Bessel function of the first kind and order \( n \). Hence, using (A.7), we derive
\[
\eta_N^\alpha(k) = \int_{-1}^{1} U_N(x) \exp(ikx) \log((x-\alpha)^2) \, dx \\
= J_0(k) \int_{-1}^{1} U_N(x) \log((x-\alpha)^2) \, dx + 2 \sum_{m=1}^{\infty} i^m J_m(k) \int_{-1}^{1} U_N(x) T_n(x) \log((x-\alpha)^2) \, dx \\
= J_0(k) \eta_N^0 + \sum_{m=1}^{N} i^m J_m(k) (\eta_N^0 - \eta_{N-m}^0) + \sum_{m=N+1}^{\infty} i^m J_m(k) (\eta_N^0 - \eta_{N-m}^0).
\]
Observe that the coefficients \( \eta_N^0 \) can be obtained from Algorithm I. By (3.5), in order to estimate how many terms are needed to evaluate this coefficient, we need to estimate how fast \( J_M(k) \) decays as \( M \to \infty \). We point out cf. \cite{2} (9.1.10), (9.3.1),
\[
J_M(k) \approx \frac{1}{M!} \left( \frac{k}{2} \right)^M \approx \frac{1}{\sqrt{2\pi M}} \left( \frac{ek}{2M} \right)^M \tag{3.23}
\]
which shows that \( J_M(k) \) decreases very fast as \( n \to \infty \). In addition, it suggests that taking \( \approx k \) terms in the series (3.23), should be enough to approximate \( \eta_N(k) \) within the machine precision.

In our implementation we have taken
\[
\eta_N^\alpha(k) \approx J_0(k) \eta_N^0(k) + \sum_{n=1}^{N} i^n J_n(k) (\eta_N^0 - \eta_{N-n}^0) + \sum_{n=N+1}^{M(k)} i^n J_n(k) (\eta_N^0 - \eta_{N-n}^0). \tag{3.24}
\]
with \( M(k) = 25 + \lceil ek/2 \rceil \) which has demonstrated to be sufficient for our purposes.

**Algorithm III: compute** \( \xi_n^\alpha(k) \) for \( n = [k], \ldots, N \)

1. Construct \( b_N^\alpha(k) \) using
   (a) \( \eta_{[k]-1}^\alpha(k) \) returned in Algorithm II
   (b) \( \gamma_n^k(k) \) for \( n = [k]+1, \ldots, N \) defined in (3.16)-(3.17).
   (c) \( \eta_N^\alpha(k) \) evaluated with the sum (3.24).
2. Construct the tridiagonal matrix $A^\alpha_N(k)$ defined in (3.21) and solve

$$A^\alpha_N(k)\eta = b^\alpha_N(k).$$

Set

$$\eta^\alpha_{[k]−1+\ell}(k) = (\eta)^\ell, \quad \ell = 1, \ldots, N−[k].$$

3. Set

$$\xi^\alpha_n(k) = \frac{1}{2} [\eta^\alpha_n(k) − \eta^\alpha_{n−2}(k)], \quad n = [k], \ldots, N.$$

**On the computational cost**

Certainly, one could use (3.24) for computing all the coefficients $(\eta^\alpha_n(k))_n$, as it was proposed in [4] (for $\alpha = 0$). However, this choice results in a more expensive algorithm. By restricting this approach to the last coefficient, and only if $N > k$, we can speed up the algorithm since all the terms but the last one, are computed by solving a tridiagonal system which can be done in $O(N − [k])$ operations by Thomas algorithm.

The vector $(\gamma^\alpha_n(k))^N_{n=1}$ can be also constructed very fast. Hence, note that the bulk part in (3.9) is the convolution of the vectors

$$(T_n(\alpha))^{N−1}_{n=0}, \quad (\rho_n)^{N−1}_{n=0}$$

which can be done in $O(N \log(N))$ operations by using FFT. (For $\alpha \in \{-1, 0, 1\}$ this could be achieved even faster from (3.9), since $T_n(\pm1) = (\pm1)^n$ and $T_n(0) = 1$ if $n$ is even and 0 otherwise).

Another possible bottleneck of the algorithm could be found in the evaluation of the Bessel functions $J_n(k)$. Let us show how it can be overcome. We recall that the Bessel functions obey the recurrence relation

$$J_{n+1}(k) − \frac{2n}{k}J_n(k) + J_{n−1}(k) = 0. \tag{3.25}$$

Notice in pass that it is very similar to that obtained in (3.15) for evaluating our coefficients. Thus, we can exploit these similarities to get a faster evaluation of these functions: Once $J_0(k)$ and $J_1(k)$ are evaluated by usual methods, (3.25) can be safely used for evaluating $J_n(k)$ for $n \leq [k]$. For the remainder values, i.e. for $n \geq [k] + 1$, we make use of the Oliver approach and solve

$$\begin{bmatrix} \frac{-2(k+1)}{k} & 1 & -2(k+2) \\ 1 & -2k & 1 \\ \vdots & \vdots & \vdots \\ 1 & -2M(k) & 1 \end{bmatrix} \begin{bmatrix} J_{[k]+1}(k) \\ J_{[k]+2}(k) \\ \vdots \\ J_{M(k)}(k) \end{bmatrix} = \begin{bmatrix} −J_{[k]−1}(k) \\ 0 \\ \vdots \\ −J_{M(k)+1}(k) \end{bmatrix}.$$

The asymptotics (3.23) can be used to approximate $J_{M(k)+1}(k)$, which gives even better results that setting simply $J_{M(k)+1}(k) \approx 0$. The evaluation turns out to be stable just for the same reasons that ensure the stability of Algorithms II and III (see next section).

**4 Numerical stability**

We analyse the stability of the algorithms separately in three propositions and collect the stability results for Algorithms II and III, when they work together, in a theorem which ends this section.

The (usually small) parameter $\varepsilon_j > 0$ will be used in this section to represent any possible perturbation occurring in the evaluation such as round-off errors or errors coming from previous computations.
Theorem 4.1 Let \( \varepsilon_N := (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N) \in \mathbb{R}^{N+1} \) with \( \| \varepsilon \|_{\infty} \leq \varepsilon \) and define the sequence

\[
\eta_{N-1}^{\alpha, \varepsilon} = \eta_{-1}^{\alpha} = 0, \quad \eta_0^{\alpha, \varepsilon} = \eta_0^{\alpha} + \varepsilon_0, \\
\eta_n^{\alpha, \varepsilon} = \gamma_n + \frac{2\alpha n}{n+1} \eta_{n-1}^{\alpha, \varepsilon} - \frac{n-1}{n+1} \eta_{n-2}^{\alpha, \varepsilon} + \varepsilon_n, \quad \text{for } n = 1, 2, \ldots.
\]

Then for all \( N > 0 \)

\[
|\eta_N^{\alpha} - \eta_N^{\alpha, \varepsilon}| \leq \left[ \frac{1}{N+1} \sum_{j=0}^{N} (j+1)|U_{N-j}(\alpha)| \right] \varepsilon \leq \frac{1}{6} (N+2)(N+3)\varepsilon.
\]

Proof. Clearly,

\[
\eta_N^{\alpha} - \eta_N^{\alpha, \varepsilon} = \sum_{j=0}^{N} \delta_N^{(j)},
\]

where \( \delta_N^{(j)} \) is given by

\[
\delta_{j-1}^{(j)} := 0, \quad \delta_j^{(j)} := \varepsilon_j, \quad \delta_n^{(j)} := \frac{2\alpha n}{n+1} \delta_{n-1}^{(j)} - \frac{n-1}{n+1} \delta_{n-2}^{(j)}, \quad n = j+1, j+2, \ldots
\] (4.1)

It is easy to check, using (A.2), that the solution of the problem above is given by

\[
\delta_n^{(j)} = \frac{j+1}{n+1} U_{n-j}(\alpha) \varepsilon_j.
\]

Therefore, using (A.3)

\[
|\eta_N^{\alpha} - \eta_N^{\alpha, \varepsilon}| \leq \frac{1}{N+1} \sum_{j=0}^{N} (j+1)|U_{N-j}(\alpha)| \varepsilon \leq \frac{1}{N+1} \left[ \sum_{j=0}^{N} (j+1)(N-j+1) \right] \varepsilon
\]

\[= \frac{1}{6} (N+2)(N+3)\varepsilon.
\]

The proof is now finished. \( \square \)

Remark 4.2 In view of this result, we conclude that theoretically \( \alpha = 0 \) turns out to be the most stable case. Indeed, since \( U_{2j}(0) = (-1)^j \),

\[
|\eta_{2N}^{0} - \eta_{2N}^{0, \varepsilon}| \leq \frac{\varepsilon}{2N+1} \sum_{j=0}^{N} (2j+1) = \frac{(N+1)^2 \varepsilon}{2N+1}.
\]

(Note that \( \eta_{2j+1}^{0} = 0 \) and therefore only \( \eta_{2j}^{0} \) has to be considered).

On the other hand, \( \alpha = \pm 1 \) are precisely the most unstable cases, since \( |U_j(\pm 1)| = j+1 \). We point out, however, that in practical computation the algorithm has demonstrated, see section 5, that: (a) the computation is stable for \( \alpha \in [-1, 1] \), much better than that theory predicts; (b) the error observed for \( \alpha = 0 \) is a little smaller than that for \( \alpha = \pm 1 \).

Next we consider the stability of Algorithms II and III, i.e., the oscillatory case.
Proposition 4.3 Let $N \leq k - 1$ and set $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_N) \in \mathbb{C}^{N+1}$ with $\|\varepsilon\|_\infty \leq \varepsilon$ and consider the sequence
\[
\begin{align*}
\eta_{-1}^{\alpha, \varepsilon}(k) &= \eta_{-1}^\alpha(k) = 0, \\
\eta_0^{\alpha, \varepsilon}(k) &= \eta_0^\alpha(k) + \varepsilon_0, \\
\eta_n^{\alpha, \varepsilon}(k) &= \gamma_n^\alpha(k) - \frac{2n}{ik} \eta_{n-1}^{\alpha, \varepsilon}(k) + \eta_{n-2}^{\alpha, \varepsilon}(k) + \varepsilon_n, \quad n = 1, 2, \ldots, N.
\end{align*}
\]
Then, for all $0 \leq n < k - 1$,
\[
|\eta_n^{\alpha, \varepsilon}(k) - \eta_n^\alpha(k)| \leq \left[ 1 + \frac{4}{3} \frac{(n + 1)^{1/2}}{(k^2 - (n + 1)^2)^{1/4}} \right] \varepsilon. \tag{4.2}
\]
Therefore, for all $n \leq k - 2$,
\[
|\eta_n^{\alpha, \varepsilon}(k) - \eta_n^\alpha(k)| \leq \left[ 1 + \frac{2^{3/4}}{3} (n + 1)^{3/4} k^{1/2} \right] \varepsilon, \tag{4.3}
\]
whereas for $k - 2 < n \leq k - 1$, i.e., for $n = |k| - 1$,
\[
|\eta_{|k|-1}^{\alpha, \varepsilon}(k) - \eta_{|k|-1}^\alpha(k)| \leq [4 + 2^{7/4} k^{5/4}] \varepsilon. \tag{4.4}
\]
Proof. As before, it suffices to study the sequence
\[
\begin{align*}
\delta_{-1} &= 0, \quad \delta_0 = \varepsilon_0, \\
\delta_n &= -\frac{2n}{ik} \delta_{n-1} + \delta_{n-2} + \varepsilon_n, \quad n = 1, 2, \ldots, N.
\end{align*}
\]
We refer now to [6] Th. 5.1 where the stability of this sequence is analysed and whose proof can be straightforwardly adapted to derive (4.2).

To prove (4.3), we observe that (4.2) implies that for $n \leq k - 2$,
\[
\begin{align*}
|\eta_n^{\alpha, \varepsilon}(k) - \eta_n^\alpha(k)| &\leq \left[ 1 + \frac{4}{3} \frac{(n + 1)^{1/2}}{(n + 2)^2 - (n + 1)^2)^{1/4}} \right] \varepsilon \leq \left[ 1 + \frac{4}{3} \frac{(n + 1)^{1/2}}{2n + 3} \right] \varepsilon \\
&\leq \left[ 1 + \frac{2^{3/4}}{3} (n + 1)^{3/4} k^{1/2} \right] \varepsilon.
\end{align*}
\]
If $n = |k| - 1$, we can use (4.3) as follows
\[
\begin{align*}
|\eta_n^{\alpha, \varepsilon}(k) - \eta_n^\alpha(k)| &\leq \varepsilon + \frac{2n}{k} |\eta_{n-1}^{\alpha, \varepsilon}(k) - \eta_{n-1}^\alpha(k)| + |\eta_{n-2}^{\alpha, \varepsilon}(k) - \eta_{n-2}^\alpha(k)| \\
&\leq \left[ 4 + 2 \cdot \frac{2^{7/4}}{3} (n - 1)^{3/4} k^{1/2} \right] \varepsilon \leq \left[ 4 + 2^{7/4} n^{3/4} k^{1/2} \right] \varepsilon \\
&\leq \left[ 4 + 2^{7/4} k^{5/4} \right] \varepsilon.
\end{align*}
\]
Bound (4.4) is now proven. \hfill \Box

The stability of Algorithm III is consequence of the next result.

Proposition 4.4 Let $N > k$ and consider the solutions of the original and perturbed systems
\[
A_N^\alpha(k) \eta = b_N^\alpha(k), \quad (A_N^\alpha(k) + \Delta A_N^\alpha(k)) \eta^\varepsilon = b_N^\alpha(k) + \Delta b_N^\alpha(k).
\]
Then, if $(k + 2) \|\Delta A_N^\alpha(k)\|_\infty < 2$, it holds
\[
\|\eta^\varepsilon - \eta\|_\infty \leq \frac{k + 2}{2 - (k + 2) \|\Delta A_N^\alpha(k)\|_\infty} \left[ \|\Delta b_N^\alpha(k)\|_\infty + \|\Delta A_N^\alpha(k)\|_\infty \|\eta\|_\infty \right].
\]

16
Proof. A classical result in stability theory for systems of linear equations (see for instance [3, Th. 8.4]) states that
\[
\| \eta^\varepsilon - \eta \|_\infty \leq \frac{\| (A_N^\alpha(k))^{-1} \|_\infty}{1 - \| \Delta A_N^\alpha(k) \|_\infty \| (A_N^\alpha(k))^{-1} \|_\infty} \left[ \| \Delta b_N^\alpha(k) \|_\infty + \| \Delta A_N^\alpha(k) \|_\infty \| \eta \|_\infty \right].
\] (4.5)

Thus, we just have to estimate \( \| (A_N^\alpha(k))^{-1} \|_\infty \). Let
\[
D_N(k) = \begin{bmatrix}
[k] + 1 & 0 & \cdots & 0 \\
0 & [k] + 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & [N - 1]
\end{bmatrix},
\]
\[
K_N(k) = \begin{bmatrix}
0 & \frac{ik}{2([k] + 1)} & 0 & \cdots & 0 \\
0 & 0 & \frac{ik}{2([k] + 2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -\frac{ik}{2(N - 2)}
\end{bmatrix}.
\]

Let I denote the identity matrix. Clearly, it holds
\[
A_N^\alpha(k) = (I + K_N(k))D_N(k) \Rightarrow (A_N^\alpha(k))^{-1} = (D_N(k))^{-1}(I + K_N(k))^{-1}.
\]

Notice also
\[
\| (D_N(k))^{-1} \|_\infty = \frac{k}{2([k] + 1)} < \frac{1}{2},
\]
\[
\| K_N(k) \|_\infty = \frac{k}{2([k] + 1)} + \frac{k}{2([k] + 3)} < \frac{k}{2} + \frac{k}{2(k + 2)} = \frac{k + 1}{k + 2}.
\]

Collecting these inequalities, we conclude
\[
\| (A_N^\alpha(k))^{-1} \|_\infty \leq \frac{\| (D_N(k))^{-1} \|_\infty}{1 - \| K_N(k) \|_\infty} < \frac{k + 2}{2}.
\] (4.6)

Inserting (4.6) in (4.5) the result is proven. \(\Box\)

The perturbation \( \Delta A_N^\alpha(k) \) in the matrix is essentially round-off errors. Since \( A_N^\alpha(k) \) is a tridiagonal matrix we can safely expect \((k + 2)\| \Delta A_N^\alpha(k) \|_\infty \ll 1\).

The last result of this section states the numerical stability of the algorithm in the oscillatory case and is result of combining appropriately propositions 4.3 and 4.4.

**Theorem 4.5** With the notations of Propositions 4.3–4.4, it holds

(a) For any \( \nu \in (0, 1) \) there exists \( C_\nu \), depending only on \( \nu \), so that for \( N < \nu k \),
\[
\max_{n=0,\ldots,N} | \eta_n^\alpha(\varepsilon(k)) - \eta_n^\alpha(k) | \leq C_\nu N \varepsilon.
\] (4.7)

with \( C_\nu \) independent of \( k \) and \( N \).
| \( n \) | \( e^0_{\text{abs}} \) | \( e^0_{\text{rel}} \) |
|---|---|---|
| 10 | 1.11E-16 | 4.33E-16 |
| 200 | 1.11E-16 | 1.07E-15 |
| 400 | 1.11E-16 | 1.31E-15 |

Table 1: Absolute and relative errors in \( \xi^0_n \) for different values of \( n \)

| \( n \) | \( e^1_{\text{abs}} \) | \( e^1_{\text{rel}} \) |
|---|---|---|
| 10 | 5.83E-16 | 3.70E-14 |
| 200 | 5.83E-16 | 7.92E-14 |
| 400 | 5.83E-16 | 3.90E-13 |

Table 2: Absolute and relative errors \( \xi^1_n \) for different values of \( n \)

(b) There exists \( C > 0 \) independent of \( k \) and \( \alpha \) so that

\[
\max_{n=0,\ldots,[k]-1} |\eta^\alpha_n(k) - \eta^\alpha_n(k)| \leq C k^{5/4} \varepsilon. \tag{4.8}
\]

(c) For \( N > k \), if the following conditions are, in addition, satisfied

\[
\|A^\alpha_N(k)\|_\infty \leq \varepsilon, \quad \|A^\alpha_N(k)\|_\infty \leq \varepsilon + |\eta^\alpha_{[k]-1}(k) - \eta^\alpha_{[k]-1}(k)|,
\]

and \( \varepsilon < (k + 1)^{-1} \), then it holds

\[
\max_{j=[k]-1,\ldots,N} |\eta^\alpha_j(k) - \eta^\alpha_j(k)| \leq (k + 2)(1 + |\eta^\alpha_{[k]-1}(k) - \eta^\alpha_{[k]-1}(k)|) + \|\eta\|_\infty \varepsilon \tag{4.9}
\]

\[
\leq C' k^{9/4} \varepsilon \tag{4.10}
\]

where \( C \) is independent of \( k \) and \( N \).

Proof. Estimates (4.7)-(4.8) follow from Proposition 4.3. For item (c) (4.9) is a consequence of Proposition 4.4. Finally, (4.10) is obtained by applying (4.8), which bounds the last term in (4.9), and using that \( \|\eta\|_\infty \) is uniformly bounded independent of \( N \) and \( \alpha \) cf. (3.5). \( \square \)

Let us emphasise that, in our computations, (4.10) has been demonstrated to be very pessimistic.

5 Numerical Experiments

We collect in this section some numerical experiments to illustrate the theoretical results presented in this paper. The implementation of the rule, for \( \alpha = 0, -1 \) is available in [1].

5.1 Stability for \( \xi^\alpha_n \)

We have computed here \( \xi^\alpha_n \) for \( n = 0, \ldots, 100 \) using the implementation of our method in Matlab. Next, we compare the numerical results with that obtained using symbolic calculations in Mathematica, which will be denoted by \( \xi^\alpha_n^{\text{Symb}} \). The evaluation of these expressions is done using (very) high arithmetic
precision to keep the round-off errors well below the significant digits returned in our implementation in Matlab.

We present

\[
e_{\text{abs}}^\alpha(N) = \max_{j=0,\ldots,N} |\xi_j^\alpha - S_j|, \quad e_{\text{rel}}^\alpha(N) = \max_{j=0,\ldots,N} \frac{|\xi_j^\alpha - S_j|}{|\xi_j^\alpha|},
\]

in Tables 1 (for \(\alpha = 0\)) and 2 (for \(\alpha = 1\)) for different values of \(n\). We clearly see that for all \(n\) the error is very close to the machine’s unit round off and that the results returned for \(\alpha = 0\) are slightly better than that for \(\alpha = 1\). This should indicate that Theorem 4.1 is sharp (see also Remark 4.2).

### 5.2 Stability for \(\xi_n^\alpha(k)\)

As before, we compare here the values of \(\eta_n^\alpha(k)\) computed by our code with that returned by Mathematica. The results are shown for \(\alpha = 0\) in Table 3 and for \(\alpha = 1\) in Table 4.

It is worth mentioning that in our implementation in Matlab we face an annoying bug. Algorithm II (and therefore indirectly Algorithm III) makes use of the sine and cosine integral functions (Si and Ci in our notation) just for evaluating the first coefficient \(\eta_n^\alpha(k)\). These functions are only included in Matlab as part of the symbolic toolbox, and therefore it is not presented in all distributions. Moreover, any call to these functions consumes a significant CPU time because of the own nature of the symbolic toolbox. Hence, in some of our experiments we observed that when using the built-in functions almost half of the CPU time was consumed in performing these two evaluations.

Thus we wrote our own implementation for sine and cosine integral functions. The evaluation is accomplished by a combination of asymptotic expansion for large arguments [2, (5.2.34)-(5.2.35)] and a sum of Bessel functions of fractional order for small and moderate arguments [2, (5.2.15)]. Despite our efforts, our code introduces a very small error in the last or in the last but one significant digits. However, such errors only affect the first few coefficients very slightly and do not propagate to the rest of coefficients. Hence, it gives us an unwanted proof of the stability of the algorithm.

| \(N \setminus k\) | 10 | 20 | 40 | 80 | 160 |
|-----------------|----|----|----|----|-----|
| 1               | 1.39E-17 | 1.04E-17 | 1.30E-18 | 4.34E-19 | 2.71E-020 |
| 10              | 1.33E-15 | 2.22E-16 | 2.78E-17 | 2.78E-17 | 0.00 |
| 20              | 1.67E-16 | 6.61E-16 | 0.00 | 2.78 \times 10^{-17} | 6.94E-18 |
| 40              | 2.78E-17 | 1.39E-16 | 1.11E-15 | 4.16E-17 | 0.00 |
| 80              | 2.78E-17 | 1.39E-17 | 5.55E-17 | 1.11E-15 | 2.08E-17 |
| 160             | 0.00 | 1.39E-17 | 2.78E-17 | 7.63 \times 10^{-17} | 1.55E-15 |

Table 3: Absolute (top) and relative (below) error in computing \(\xi_0^\alpha(k)\)
Table 4: Absolute (top) and relative (below) error in computing $\xi_n^\alpha(k)$.

| $N \setminus k$ | 10     | 20     | 40     | 80     | 160    |
|---------------|--------|--------|--------|--------|--------|
| 1             | 3.86E-16 | 1.39E-17 | 2.95E-16 | 2.78E-17 | 1.39E-17 |
| 10            | 2.24E-15 | 1.24E-16 | 4.79E-16 | 2.86E-17 | 1.55E-17 |
| 20            | 2.22E-17 | 1.56E-15 | 1.25E-15 | 2.08E-17 | 4.39E-17 |
| 40            | 1.03E-16 | 3.71E-17 | 4.10E-15 | 1.67E-16 | 1.12E-16 |
| 80            | 1.81E-17 | 6.35E-17 | 2.70E-17 | 1.60E-15 | 1.31E-16 |
| 160           | 1.32E-17 | 1.86E-17 | 1.00E-16 | 5.02E-16 | 8.68E-16 |

| $N \setminus k$ | 10     | 20     | 40     | 80     | 160    |
|---------------|--------|--------|--------|--------|--------|
| 1             | 5.47E-16 | 3.05E-17 | 1.35E-15 | 1.89E-16 | 2.15E-16 |
| 10            | 4.55E-15 | 5.19E-16 | 2.00E-15 | 1.87E-16 | 1.95E-16 |
| 20            | 7.89E-16 | 4.42E-15 | 3.57E-15 | 1.04E-16 | 4.97E-16 |
| 40            | 1.18E-14 | 3.84E-15 | 1.89E-14 | 4.12E-16 | 5.95E-16 |
| 80            | 6.94E-15 | 2.23E-14 | 9.93E-15 | 1.21E-14 | 6.56E-16 |
| 160           | 1.73E-14 | 2.28E-14 | 1.27E-13 | 6.02E-13 | 1.01E-14 |

5.3 Experiments for an oscillatory integral

Let

$$I_\alpha(k) := \int_{-1}^{1} \frac{\cos(4x)}{x^2 + x + 1} \log((x - \alpha)^2) \exp(ikx) \, dx$$

We have computed the errors returned by our numerical method for different values of $k$, $N$ and for $\alpha = 0$ (Table 5) and $\alpha = 1$ (Table 6). As exact value we just have used that returned by the rule when a huge number of points is used.

Several facts can be observed right from the beginning. First, the convergence is very fast: with modest values of $N$ we get approximations with an error of the same order as the round-off unity. Second, if we read the Table by rows, we clearly see that in both cases the error decrands as $k^{-2}$ only for even $N$, whereas for odd $N$ and $\alpha = 0$ the error decreases only as $k^{-1}$ (and therefore, the relative error keeps bounded independent of $k$).

Such phenomenon does not occur when $\alpha = 1$, i.e., when the logarithmic singularity occurs at the end of the interval (See Table 6).

5.4 Non-smooth functions

In this last experiment we run our code to compute

$$I_1(k, \alpha, \beta) := \int_{-1}^{1} (1 + x)^{\beta} \log((x - \alpha)^2) \exp(ikx) \, dx,$$

$$I_0(k, \alpha, \beta) := \int_{-1}^{1} |1/2 + x|^{\beta} \log((x - \alpha)^2) \exp(ikx) \, dx,$$

for $\alpha \in \{-1, 0\}$ and $\beta \in \{1/2, 3/2\}$ to analyse how precise are the regularity assumptions in the hypothesis Theorem 2.2.1.

We expect the convergence of the rule to be faster for the first integral since, after performing the cosine change of variables, $|1 + \cos \theta|^{\beta} \in H^{\beta + 1/2 - \varepsilon} \#$ whereas $|1/2 + \cos \theta|^{\beta} \in H^{\beta + 1/2 - \varepsilon} \#$. The regularity
| $N \setminus k$ | 0    | 10   | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
|--------------|------|------|--------|--------|--------|--------|
| 11           | 1.71E−03 | 4.00E−03 | 1.75E−04 | 1.82E−05 | 1.83E−06 | 1.83E−07 |
| 12           | 4.56E−05 | 3.28E−04 | 1.44E−06 | 1.37E−08 | 1.37E−10 | 1.37E−12 |
| 23           | 1.65E−08 | 2.56E−08 | 4.80E−09 | 3.89E−10 | 3.80E−11 | 3.80E−12 |
| 24           | 2.96E−10 | 8.24E−09 | 9.93E−10 | 9.09E−12 | 9.09E−14 | 9.08E−16 |
| 47           | 6.66E−16 | 1.11E−16 | 8.97E−17 | 1.29E−17 | 1.08E−19 | 1.36E−20 |
| 48           | 6.66E−16 | 2.73E−16 | 8.85E−17 | 1.26E−17 | 1.08E−19 | 2.71E−20 |

| $N \setminus k$ | 0    | 10   | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
|--------------|------|------|--------|--------|--------|--------|
| 11           | 9.38E−04 | 5.49E−03 | 2.78E−03 | 2.89E−03 | 2.91E−03 | 2.91E−03 |
| 12           | 2.50E−05 | 4.50E−04 | 2.28E−05 | 2.18E−06 | 2.17E−07 | 2.18E−08 |
| 23           | 9.04E−09 | 3.51E−08 | 7.61E−08 | 6.20E−08 | 6.05E−08 | 6.04E−08 |
| 24           | 1.63E−10 | 1.13E−08 | 1.58E−08 | 1.45E−09 | 1.45E−10 | 1.45E−11 |
| 47           | 3.66E−16 | 1.52E−16 | 1.42E−15 | 2.05E−15 | 1.73E−16 | 2.17E−16 |
| 48           | 3.66E−16 | 3.75E−16 | 1.40E−15 | 2.01E−15 | 1.73E−16 | 4.32E−16 |

Table 5: Absolute (top) and relative (bottom) errors for integral (5.1) with $\alpha = 0$

| $N \setminus k$ | 0    | 10   | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
|--------------|------|------|--------|--------|--------|--------|
| 11           | 1.81E−05 | 8.89E−04 | 3.04E−05 | 5.04E−07 | 6.33E−09 | 7.90E−11 |
| 12           | 2.43E−06 | 7.72E−05 | 8.94E−06 | 1.74E−07 | 1.77E−09 | 2.15E−11 |
| 13           | 4.21E−11 | 2.60E−11 | 1.50E−09 | 5.51E−12 | 1.25E−13 | 1.48E−15 |
| 24           | 5.25E−11 | 4.91E−11 | 1.89E−09 | 1.84E−11 | 2.81E−13 | 3.55E−15 |
| 47           | 1.04E−18 | 7.85E−17 | 9.22E−17 | 2.47E−17 | 2.09E−18 | 1.10E−19 |
| 48           | 7.31E−17 | 8.89E−17 | 9.17E−17 | 2.17E−17 | 1.89E−18 | 1.12E−19 |

| $N \setminus k$ | 0    | 10   | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
|--------------|------|------|--------|--------|--------|--------|
| 11           | 8.09E−04 | 3.07E−03 | 1.00E−03 | 1.71E−04 | 1.26E−05 | 1.27E−06 |
| 12           | 1.09E−04 | 2.67E−04 | 2.94E−04 | 5.92E−05 | 3.53E−06 | 3.45E−07 |
| 23           | 1.89E−09 | 9.00E−11 | 4.92E−08 | 1.87E−09 | 2.49E−10 | 2.39E−11 |
| 24           | 2.30E−09 | 1.70E−10 | 6.22E−08 | 6.26E−09 | 5.61E−10 | 5.72E−11 |
| 47           | 4.67E−17 | 2.71E−16 | 3.03E−15 | 8.38E−15 | 4.17E−15 | 1.77E−15 |
| 48           | 3.27E−15 | 3.07E−16 | 3.02E−15 | 7.37E−15 | 3.78E−15 | 1.80E−15 |

Table 6: Absolute (top) and relative (bottom) errors for integral (5.1) with $\alpha = 1$
of the transformed function is precisely what appears in the estimate of Theorem 2.4 (function $f_c$ in
the right-hand-side).

We show in Tables 7-10 the error of the rule for different values of $k$ and $N$. (The exact integral
was computed by using the Clenshaw-Curtis rule on graded meshes towards the singular points, cf. [14]. Clearly, the errors are in almost all cases smaller for (5.2a) than for (5.2b).

It is difficult to estimate the order of convergence of the rule because it becomes chaotic as $k$
increases in such a way that the larger is $k$, the bigger has to be $N$ to make the error decay steady to
zero. This is specially noticeable in Tables 7, 9, 10. One can observe, however, that the error in the
first columns of Table 7 decreases as $N^{-3-2\beta}$, in Table 9 the rule converges approximately as $N^{-2.5-\beta}$ and
as $N^{-1-\beta}$ for Tables 9, 10.

If we read the table by rows, we can detect that the $O(k^{-2})$ decay of the error occurs only in Table 7 and in Table 9 for $\beta = 3/2$. Only for the first integral (5.2a) with $\beta = 3/2$, this property has been rigorously
proved since in the notation of Theorem 2.4 $f_c \in H^{7/2-\varepsilon}$. There is however no theoretically justification
for the other cases and it certainly deserves more attention to study if the regularity assumptions can be relaxed for $\alpha = 0$.

On the other hand, the error does not behave as $O(k^{-2})$ in Table 8 although for $\beta = 3/2$ Theorem
2.4 should imply such decay of the error. We think that the very irregular convergence of the rule in
this case could force $N$ and $k$ to be larger to observe it.

A Some relevant properties for Chebyshev polynomials

For the sake of completeness we present in this section those properties of Chebyshev polynomials we
have used in this work. These results can be found in many classical text books on special functions
or Chebyshev polynomials (see for instance [2, Ch. 22] or [22]).

From the definitions of the Chebyshev polynomials of first and second kind we have the relations

\[ T_n(\cos \theta) = \cos n\theta, \quad U_n(\cos \theta) = \frac{1}{n+1} T'_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}, \]  

(A.1)

As a byproduct, one can deduce that if $n$ is even (respectively odd), so are $T_n$ and $U_n$. Note that
as usual in this work, we have taken $U_{-1} = 0$, which is also consistent with (A.1). Both families of
polynomials obey the recurrence relation

\[ P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x) \]  

(A.2)

but with, obviously, different starting values, simply $T_0(x) = 1$, $T_1(x) = x$ and $U_{-1}(x) = 0$, $U_0(x) = 1$
respectively.

From (A.1) we easily deduce

\[ T'_n(\cos \theta) = n \frac{\sin n\theta}{\sin \theta}, \quad \sin \theta T''_n(\cos \theta) = n \frac{d}{d\theta} \left( \frac{\sin n\theta}{\sin \theta} \right). \]

Therefore,

\[ \|T_n\|_{L_\infty(0,1)} \leq T_n(1) = 1, \quad \|U_n\|_{L_\infty(0,1)} = \frac{1}{n+1} \|T'_{n+1}\|_{L_\infty(-1,1)} = n + 1 \]  

(A.3)

and (recall that $w(x) = \sqrt{1 - x^2}$)

\[ \|wT'_{n+1}\|_{L_\infty(0,1)} = (n + 1), \quad \|wT''_{n+1}\|_{L_\infty(0,1)} \leq C(n + 1)^3 \]  

(A.4)
| $N \setminus k$ | 0          | 1          | 10         | $10^2$      | $10^3$      | $10^4$      | $10^5$      |
|-----------------|------------|------------|------------|-------------|-------------|-------------|-------------|
| 11              | 2.59E−03   | 2.59E−03   | 6.90E−03   | 2.80E−04    | 2.81E−05    | 2.83E−06    | 2.83E−07    |
| 12              | 1.72E−04   | 1.72E−04   | 4.82E−03   | 3.38E−05    | 3.23E−07    | 2.97E−09    | 2.92E−11    |
| 23              | 2.86E−04   | 2.86E−04   | 2.96E−04   | 4.80E−05    | 6.48E−06    | 6.56E−07    | 6.57E−08    |
| 24              | 1.07E−05   | 1.07E−05   | 1.29E−04   | 2.24E−05    | 1.90E−07    | 1.53E−09    | 1.50E−11    |
| 47              | 3.36E−05   | 3.36E−05   | 3.18E−05   | 3.04E−05    | 1.50E−06    | 1.57E−07    | 1.58E−08    |
| 48              | 6.64E−07   | 6.64E−07   | 6.91E−06   | 1.20E−05    | 1.05E−07    | 7.61E−10    | 7.78E−12    |
| 95              | 4.07E−06   | 4.07E−06   | 3.91E−06   | 5.62E−05    | 4.14E−07    | 3.84E−08    | 3.86E−09    |
| 96              | 4.13E−08   | 4.13E−08   | 4.17E−07   | 9.36E−05    | 5.52E−08    | 3.03E−10    | 4.05E−12    |

Table 7: Errors of the quadrature rule for integral (5.2a) with $\alpha = 0$, $\beta = 1/2$ (top) and $\beta = 3/2$ (bottom)

| $N \setminus k$ | 0          | 1          | 10         | $10^2$      | $10^3$      | $10^4$      | $10^5$      |
|-----------------|------------|------------|------------|-------------|-------------|-------------|-------------|
| 11              | 5.07E−05   | 5.05E−05   | 1.34E−04   | 5.33E−06    | 5.51E−07    | 5.56E−08    | 5.56E−09    |
| 12              | 2.66E−06   | 2.65E−06   | 8.00E−05   | 4.74E−07    | 4.72E−09    | 4.78E−11    | 4.76E−13    |
| 23              | 1.32E−06   | 1.32E−06   | 1.37E−06   | 2.34E−07    | 2.99E−08    | 3.03E−09    | 3.03E−10    |
| 24              | 4.41E−08   | 4.41E−08   | 5.47E−07   | 9.01E−08    | 6.59E−10    | 6.27E−12    | 6.17E−14    |
| 47              | 3.74E−08   | 3.74E−08   | 3.55E−08   | 3.35E−08    | 1.69E−09    | 1.76E−10    | 1.76E−11    |
| 48              | 7.01E−10   | 7.01E−10   | 7.38E−09   | 1.25E−08    | 1.01E−10    | 8.11E−13    | 7.83E−15    |
| 95              | 1.11E−09   | 1.11E−09   | 1.07E−09   | 1.54E−08    | 1.12E−10    | 1.05E−11    | 1.06E−12    |
| 96              | 1.10E−11   | 1.10E−11   | 1.12E−10   | 2.50E−08    | 1.36E−11    | 9.53E−14    | 9.99E−16    |

Table 8: Errors of the quadrature rule for integral (5.2a) with $\alpha = -1$, $\beta = 1/2$ (top) and $\beta = 3/2$ (bottom)
\[ N \setminus k \quad 0 \quad 1 \quad 10 \quad 10^2 \quad 10^3 \quad 10^4 \quad 10^5 \]
\[ \begin{array}{cccccccc}
11 & 2.13E-02 & 2.15E-02 & 5.10E-02 & 2.96E-03 & 8.24E-05 & 1.31E-05 & 1.27E-06 \\
12 & 5.71E-02 & 5.72E-02 & 1.21E-01 & 1.63E-03 & 5.31E-05 & 1.69E-06 & 5.45E-08 \\
23 & 5.85E-03 & 5.86E-03 & 6.49E-03 & 2.10E-03 & 2.81E-05 & 4.38E-06 & 3.79E-07 \\
24 & 2.15E-02 & 2.15E-02 & 2.49E-02 & 1.66E-03 & 5.36E-05 & 1.71E-06 & 5.46E-08 \\
47 & 1.83E-03 & 1.83E-03 & 1.78E-03 & 1.87E-03 & 4.51E-05 & 2.24E-06 & 1.28E-07 \\
48 & 7.73E-03 & 7.73E-03 & 8.01E-03 & 1.59E-03 & 5.39E-05 & 1.72E-06 & 5.47E-08 \\
95 & 6.11E-04 & 6.11E-04 & 5.92E-04 & 1.98E-03 & 5.17E-05 & 1.84E-06 & 6.65E-08 \\
96 & 2.75E-03 & 2.75E-03 & 2.77E-03 & 6.56E-03 & 5.42E-05 & 1.72E-06 & 5.48E-08 \\
\end{array} \]

Table 9: Errors of the quadrature rule for integral (5.20) with \( \alpha = 0, \beta = 1/2 \) (top) and \( \beta = 3/2 \) (bottom)

\[ N \setminus k \quad 0 \quad 1 \quad 10 \quad 10^2 \quad 10^3 \quad 10^4 \quad 10^5 \]
\[ \begin{array}{cccccccc}
11 & 1.74E-02 & 1.75E-02 & 2.53E-02 & 1.63E-03 & 5.63E-05 & 1.73E-05 & 5.49E-08 \\
12 & 6.28E-02 & 6.29E-02 & 5.65E-02 & 1.41E-03 & 6.94E-05 & 1.67E-06 & 5.46E-08 \\
23 & 5.34E-03 & 5.34E-03 & 6.02E-03 & 1.92E-03 & 5.60E-05 & 1.73E-06 & 5.49E-08 \\
24 & 2.20E-02 & 2.20E-02 & 2.26E-02 & 2.79E-03 & 6.28E-05 & 1.68E-06 & 5.47E-08 \\
47 & 1.76E-03 & 1.76E-03 & 1.81E-03 & 1.59E-03 & 5.40E-05 & 1.73E-06 & 5.49E-08 \\
48 & 7.78E-03 & 7.78E-03 & 7.83E-03 & 8.18E-04 & 5.07E-05 & 1.68E-06 & 5.47E-08 \\
95 & 6.01E-04 & 6.01E-04 & 6.06E-04 & 1.30E-03 & 5.68E-05 & 1.72E-06 & 5.49E-08 \\
96 & 2.75E-03 & 2.75E-03 & 2.75E-03 & 3.19E-03 & 6.43E-05 & 1.66E-06 & 5.47E-08 \\
\end{array} \]

Table 10: Errors of the quadrature rule for integral (5.2b) with \( \alpha = -1, \beta = 1/2 \) (top) and \( \beta = 3/2 \) (bottom)
where $C$ is independent of $n$.

Unlike $T_n$, $U_n$ is not uniformly bounded in $n$ and $x \in [-1, 1]$. However,

\[
\|U_n\|_{1,\omega}^2 = \int_{-1}^{1} |U_n(x)|^2 \sqrt{1-x^2} \, dx = \int_0^\pi \sin^2 n\theta \, d\theta = \frac{\pi}{2}. \quad (A.5)
\]

On the other hand,

\[
\int_{-1}^{1} T_n(x) \, dx = \int_0^\pi \cos n\theta \sin \theta \, d\theta = \begin{cases} 
\frac{-2}{n^2-1}, & \text{if } n \text{ is even}, \\
0, & \text{otherwise}.
\end{cases} \quad (A.6)
\]

The trigonometric identity

\[
\cos n\theta \sin (m+1)\theta = \frac{1}{2} (\sin (m+n+1)\theta + \sin (m+1-n)\theta)
\]

implies

\[
T_nU_m = \begin{cases} 
\frac{1}{2}(U_{m+n} + U_{m-n}), & \text{if } m \geq n-1, \\
\frac{1}{2}(U_{m+n} - U_{n-m-2}), & \text{if } m \leq n-2.
\end{cases} \quad (A.7)
\]

In particular, we obtain for $n \geq 1$

\[
2xT'_n(x) = 2nT_1(x)U_{n-1}(x) = n[U_n(x) + U_{n-2}(x)],
\]

\[
T_n(x) = T_n(x)U_0(x) = \frac{1}{2}[U_n(x) - U_{n-2}(x)]. \quad (A.8)
\]

Finally, it holds

\[
\frac{T_n(x) - T_n(y)}{x-y} = 2 \sum_{j=0}^{n-2} U_j(x)T_{n-1-j}(y) + U_{n-1}(x) = 2 \sum_{j=0}^{n-2} U_j(y)T_{n-1-j}(x) + U_{n-1}(y) \quad (A.10)
\]

which can be easily proven by induction on $n$.

**Acknowledgements**

The author is supported partially by Project MTM2010-21037. The author wants to thank Prof. Ivan Graham for several useful discussions which help to improve both the quality and readability of this paper.

**References**

[1] Clenshaw-Curtis rules for highly oscillatory integrals. available in. http://www.unavarra.es/personal/victor_dominguez/clenshawcurtisrule.

[2] M. Abramowitz and I. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[3] K. Atkinson. *An introduction to numerical analysis*. John Wiley & Sons Inc., New York, second edition, 1989.
[4] O. Bruno and M. Haslam. Efficient high-order evaluation of scattering by periodic surfaces: deep gratings, high frequencies, and glancing incidences. *J. Opt. Soc. Am. A*, 26(3):658–668, 2009.

[5] C. W. Clenshaw and A. R. Curtis. A method for numerical integration on an automatic computer. *Numer. Math.*, 2:197–205, 1960.

[6] V. Domínguez, I. Graham, and V. Smyshlyaev. Stability and error estimates for Filon-Clenshaw-Curtis rules for highly-oscillatory integrals. *IMA Journal of Numerical Analysis*, 31(4):1253–1280, 2011.

[7] V. Domínguez, I. G. Graham, and T. Kim. Filon-Clenshaw-Curtis rules for highly-oscillatory integrals with algebraic singularities and stationary points. *SIAM J. Numer. Anal.*, 2013. To appear. Preprint available in arXiv:1207.2283.

[8] D. Huybrechs and S. Vandewalle. On the evaluation of highly oscillatory integrals by analytic continuation. *SIAM J. Numer. Anal.*, 44(3):1026–1048 (electronic), 2006.

[9] D. Huybrechs and S. Vandewalle. A sparse discretization for integral equation formulations of high frequency scattering problems. *SIAM J. Sci. Comput.*, 29(6):2305–2328 (electronic), 2007.

[10] A. Iserles. On the numerical quadrature of highly-oscillating integrals. I. Fourier transforms. *IMA J. Numer. Anal.*, 24(3):365–391, 2004.

[11] A. Iserles. On the numerical quadrature of highly-oscillating integrals. II. Irregular oscillators. *IMA J. Numer. Anal.*, 25(1):25–44, 2005.

[12] A. Iserles and S. Nørsett. Efficient quadrature of highly oscillatory integrals using derivatives. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461(2057):1383–1399, 2005.

[13] H. Kang and S. Xiang. Efficient quadrature of highly oscillatory integrals with algebraic singularities. *J. Comput. Appl. Math.*, 237(1):576–588, 2013.

[14] T. Kim, V. Domínguez, I. Graham, and V. Smyshlyaev. Recent progress on hybrid numerical-asymptotic boundary integral methods for high-frequency scattering. Preprint, 2009.

[15] D. Levin. Analysis of a collocation method for integrating rapidly oscillatory functions. *J. Comput. Appl. Maths*, 78:131–138, 1997.

[16] J. M. Melenk. On the convergence of Filon quadrature. *J. Comput. Appl. Math.*, 234(6):1692–1701, 2010.

[17] H. Mo and S. Xiang. On the asymptotic order of Filon-type methods for highly oscillatory integrals with an algebraic singularity. *Appl. Math. Comput.*, 217(22):9105–9110, 2011.

[18] J. Oliver. Relative error propagation in the recursive solution of linear recurrence relations. *Numer. Math.*, 9:323–340, 1966/1967.

[19] S. Olver. Shifted GMRES for oscillatory integrals. *Numer. Math.*, 114(4):607–628, 2010.

[20] R. Piessens. Computing integral transforms and solving integral equations using Chebyshev polynomial approximations. *J. Comput. Appl. Math.*, 121(1-2):113–124, 2000. Numerical analysis in the 20th century, Vol. I, Approximation theory.
[21] R. Piessens, E. de Doncker-Kapenga, C. Überhuber, and D. Kahaner. *QUADPACK*, volume 1 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1983. A subroutine package for automatic integration.

[22] T. J. Rivlin. *Chebyshev polynomials*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1990. From approximation theory to algebra and number theory.

[23] J. Saranen and G. Vainikko. *Periodic Integral and Pseudodifferential Equations with Numerical Approximation*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

[24] I. Sloan and W. Smith. Product integration with the Clenshaw-Curtis points: implementation and error estimates. *Numer. Math.*, 34(4):387–401, 1980.

[25] E. Tadmor. The exponential accuracy of Fourier and Chebyshev differencing methods. *SIAM J. Numer. Anal.*, 23(1):1–10, 1986.

[26] L. N. Trefethen. Is Gauss quadrature better than Clenshaw-Curtis? *SIAM Rev.*, 50(1):67–87, 2008.

[27] G. N. Watson. *A treatise on the theory of Bessel functions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995. Reprint of the second (1944) edition.

[28] S. Xiang. Efficient Filon-type methods for \( \int_a^b f(x)e^{i\omega g(x)}dx \). *Numer. Math.*, 105(4):633–658, 2007.

[29] S. Xiang. On the Filon and Levin methods for highly oscillatory integral \( \int_a^b f(x)e^{i\omega g(x)}dx \). *J. Comput. Appl. Math.*, 208(2):434–439, 2007.