Star sampling with and without replacement

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Abstract—Star sampling (SS) is a random sampling procedure on a graph wherein each sample consists of a randomly selected vertex (the star center) and its (one-hop) neighbors (the star points). We consider the use of SS to find any member of a target set of vertices in a graph, where the figure of merit (cost) is either the expected number of samples (unit cost) or the expected number of star centers plus star points (linear cost) until a vertex in the target set is encountered, either as a star center or as a star point. We analyze these two performance measures on three related star sampling paradigms: SS with replacement (SSR), SS without center replacement (SSC), and SS without star replacement (SSS). Exact and approximate expressions are derived for the expected unit and linear costs of SSR, SSC, and SSS on Erdős-Rényi (ER) random graphs. The approximations seen to be accurate. SSC/SSS are notably better than SSR under unit cost for low-density ER graphs, while SSS is notably better than SSR/SSC under linear cost for low- to moderate-density ER graphs. Simulations on twelve “real-world” graphs show the cost approximations to be of variable quality: the SSR and SSC approximations are uniformly accurate, while the SSS approximation, derived for an ER graph, is of variable accuracy.

Index Terms—graph sampling; sampling with replacement; graph search; Erdős-Rényi random graph.

I. INTRODUCTION

Let \( G = (V, E) \) denote a simple undirected graph with vertex set \( V \) and edge set \( E \). Suppose each vertex has a property value, defined via a function \( f : V \to P \), for \( P \) the set of property values. Then \( V^* \equiv f^{-1}(P^*) \), for \( P^* \subset P \), is the subset of vertices holding property values of interest (i.e., vertices with properties in \( P^* \)). This paper evaluates the performance of three related random sampling approaches to finding a vertex in \( V^* \), called star sampling, described below, that differ in terms of which part of the sample is replaced.

Star sampling. Using random sampling to search for a vertex of interest is often suitable for large and/or dynamic graphs, where either the order/size and/or the rapid evolution of the graph precludes holding the graph in local memory. In such cases the searcher may be required to query the graph, by requesting a random vertex. Star sampling (SS) is a variant on vertex sampling in which each sample returns not only the property value \( f(v) \) of the selected vertex \( v \in V \), termed the star center, but also the property values \( f(N_G(v)) \) of its (one-hop) neighbors \( N_G(v) \) (where \( N_G(v) \equiv \{ u \in V : uv \in E \} \) is the neighborhood of \( v \), and \( uv \in E \) denotes vertices \( u \) and \( v \) are joined by an edge), termed the star points. Random star sampling selects a vertex uniformly at random, then checks whether either it or any of its neighbors hold the property of interest; this is repeated until a vertex in \( V^* \) is found.

Cost. Although large graphs are encoded in a variety of ways, it is often the case that the data structure corresponding to each vertex (e.g., the profile of a particular member in a social network) includes the list of neighbors of that vertex (e.g., the social connections of that member in the network). Star sampling is a practical sampling paradigm whenever such neighbor information is available. The property may be readily available if it is easily computable, or if it is not easily computable, but has been precomputed and stored in the data structure for the vertex. To address this issue we consider two natural cost models (unit and linear), where cost is measured either as the number of star samples (unit cost) or as the number of vertices (linear cost) for which property values are queried / computed. Unit cost is most natural for the case when the primary cost incurred is the query of the star itself and property values of neighbors are readily available, while linear cost is most natural for the case where the primary cost incurred is the computation of the property value.

Target set. We focus on the case where \( P^* \) is independent of \( G \), i.e., the target set is a subset \( V^* \), chosen uniformly at random from \( V \), with prescribed cardinality \( |V^*| = n^* \). As such, the expected search cost to find a member of the target set depends upon the set only through \( n^* \).

Variants. Three SS variants are considered:

- **SS with replacement (SSR)**: the star center is selected uniformly at random from the set of vertices;
- **SS without center replacement (SSC)**: the star center is selected uniformly at random from the set of remaining vertices; the star center (along with its adjacent edges) is removed from the graph after the query;
- **SS without star replacement (SSS)**: the star center is selected uniformly at random from the set of remaining vertices; the entire star (center, points, and all adjacent edges) is removed from the graph after the query.

Urn sampling. One motivation in considering these variants is to understand their relative performance, in a manner similar to the elementary case of sampling balls from an urn. When seeking any one of \( n^* \) marked balls out of a total of \( n \geq n^* \) balls in an urn, sampling with replacement requires on average \( n/n^* \) samples; this follows immediately from the observation that the number of draws until the first success, say \( c^R \), is a geometric random variable (RV) with success probability \( n^*/n \), and expectation \( E[c^R] = n/n^* \). In contrast, sampling without replacement requires a random number of draws, \( c^{NR} \), with expectation given by the following Fact.

**Fact 1.** \( E[c^{NR}] = (n + 1)/(n^* + 1) \).

The proof is found in App. A.
improves the mean search time by at most two, relative to sampling with replacement, with equality for \( n^* = 1 \).

**Contributions.** The three primary contributions are:

- Derivation of exact, bound, and approximate expected unit and linear cost expressions, for both arbitrary graphs and Erdős-Rényi (ER) random graphs, for the three star sampling variants; these results are summarized in Table I.

- Numerical assessment of quality, in which the bounds are shown to be tight, and the approximations are seen to be accurate, relative to simulation results of ER random graphs; these results are seen in §IV-C (Fig. 3) and §V-C (Fig. 4).

- Numerical assessment on “real-world” graphs, in which the relative error of the approximations are compared with simulation results; c.f. §VI (Table IV and Table V).

Three points of context for the results bear mention.

- The accuracy of the cost approximations is notable, particularly for SSS, given the complexity of the dynamics under star sampling without star replacement. The conditional probability of hitting the star at time \( t \), conditioned on not hitting it before, is approximated by tracking both the expected reduction in the overall order of the graph and the expected reduction in the number of neighbors of the target set in the previous \( t-1 \) samples. This yields an approximate unconditional probability of first hitting the target set at \( t \), which in turn yields an approximate expected cost.

- The (mostly) small relative error of the cost approximations for “real-world” graphs is notable, particularly for SSS, given the approximations are derived for ER graphs, which are quite distinct from the “real-world” graphs.

- It is notable that all cost approximations require knowing only three quantities: the graph order \( (n) \), the target set cardinality \( (n_0^*) \), and the graph edge density \( s \).

**Outline.** The paper is organized as follows. §II discusses related work. §III provides basic notation and definitions. §IV and §V study performance under the unit and linear cost models, respectively. §VI assesses the accuracy and relative performance of the cost estimates on “real-world” graphs. §VII gives a brief conclusion. Most proofs are in the Appendices.

## II. RELATED WORK

There is an extensive literature on graph sampling, too vast to credibly review here. Classic graph exploration strategies include: random sampling of vertices or edges, random walks, and random jump sampling, which alternates between a random walk and random sampling. These graph exploration strategies are general in the sense that they may be applied towards a variety of objectives. Two notable classes of objectives include i) sampling to derive an unbiased estimate of some graph property, and ii) sampling to search for vertices with particular properties; this paper falls into the latter category.

**Snowball sampling** [4] was introduced in [5] (see also [6], [7], [8], [9]); **star sampling**, a special case of snowball sampling, appears in [10]. Ref. [11] argues that Metropolis-Hastings sampling algorithms should avoid backtracking. Ref. [12] proposed a rejection controlled Metropolis-Hastings algorithm and a non-backtracking generalized maximum-degree sampling algorithm. Ref. [13] found that a Metropolis-Hasting random walk and a re-weighted random walk both outperform a simple random walk in returning a uniform sample of Facebook users. Ref. [14] has shown that weighted random walks can be used to carry out stratified sampling on graphs. Ref. [15] bounds the number of steps required to return a uniform sample of a network using rejection sampling, maximum-degree sampling, and Metropolis-Hastings sampling.

Our pertinent prior and related work includes [1], [2], [3]. Ref. [1] compares the cost of using random walk vs. random sampling to find a target vertex. Ref. [2] analyzes the cost of finding a degree \( j \) vertex or an edge with vertices of degrees \( j, k \) via star sampling without replacement. Ref. [3] holds preliminary results comparing star sampling with and without replacement. To our knowledge, ours is the first work to study the performance impact of replacement on star sampling.

## III. NOTATION, SAMPLING MODEL, BACKGROUND

Notation is defined in §III-A, ER graphs and their properties are described in §III-B, and the three sampling models and two cost models are stated in §III-C.

**A. Notation**

Let \( a \equiv b \) denote \( a \) and \( b \) are equal by definition. Let \( [n] \) denote \( \{1, \ldots , n\} \), \( n \in \mathbb{N} \). Scalar random variables are denoted in a lowercase sans-serif font, e.g., \( x, k \), while graph- and set-valued random variables are denoted with uppercase sans-serif font, e.g., \( G, V, E \). Expectation is denoted \( \mathbb{E}[\cdot] \), probability is denoted \( \mathbb{P}(\cdot) \), and \( \text{IID} \) refers to independent and identically distributed. Four probability distributions are used: i) \( \text{uni}(U) \) denotes a RV taking value uniformly over \( U \), ii) \( \text{Ber}(p) \) denotes a Bernoulli RV with parameter \( p \), iii) \( \text{bin}(n, p) \) denotes a binomial RV with parameters \( n, p \), and iv) \( \text{geo}(p) \) denotes a geometric RV with parameter \( p \).

The following graph and sampling notation is used:

- **Order, size, edges.** An undirected and simple graph of order \( n \) is denoted \( G = (V, E) \), with vertex set \( V = [n] \) and edge set \( E \); size is denoted by \( m \equiv |E| \). An edge is denoted \( uv \).

- **Neighborhoods.** Let \( \Gamma_G(v) = \{ u \in V : uv \in E \} \) denote the (direct) neighbors of \( v \), \( \Gamma_G(v) = \Gamma_G(v) \cup \{ v \} \) the extended neighborhood of \( v \), \( \Gamma_G(v) = \{ uv \in E \} \) the edge neighborhood of \( v \), i.e., the edges adjacent to \( v \), and

\[
\Gamma^*_G(v) = \bigcup_{u \in \Gamma_G(v)} \Gamma_G(u),
\]

the extended edge neighborhood of \( v \), i.e., all edges adjacent to \( v \) or any of \( v \)'s neighbors. Observe \( \Gamma^*_G(v) \) is a star sample with star center \( v \) and star points \( \Gamma_G(v) \). For \( V^* \subseteq V \), let

\[
\Gamma_G(V^*) = (\bigcup_{v \in V^*} \Gamma_G(v)) \setminus V^*
\]

denote neighbors of \( V^* \)
not including $V^*$, and let $N^*_G(v)$ denote $V^*$ and its neighbors.

- Degrees. Let $d_G(v) = |N_G(v)|$ denote the degree of $v$, and $d^*_G(v) = |N^*_G(v)|$ the “extended degree”, i.e., $d^*_G(v) = d_G(v) + 1$. Let $D_G = \bigcup_{v \in V} d_G(v)$ denote the set of degrees found in $G$. Partition $V$ by degree into subsets $(V_G(k), k \in D)$, with $V_G(k) = \{v \in V : d_G(v) = k\}$ the set of vertices with degree $k$ and $w_G(k) = |V_G(k)|/n$ the fraction of vertices with degree $k$, to obtain the degree distribution of $G$, denoted $(w_G(k), k \in D)$ (with $\sum_{k \in D} w_G(k) = 1$). The expected degree of a randomly selected vertex $\sim \text{uni}(V)$ is

$$d_G = \mathbb{E}[d_G(v)] = \sum_{k \in D_G} kw_G(k) = \frac{1}{n} \sum_{v \in V} d_G(v). \quad (2)$$

The notation in this subsection is summarized in Table III-A.

### B. Erdős–Rényi (ER) graph properties

An Erdős–Rényi (ER) random graph $G = (V, E)$ has parameters $(n, s)$, where $n \in \mathbb{N}$ denotes the order, $V = [n]$, and $s \in (0, 1)$ denotes the edge probability. A realization of an ER random graph, i.e., of the random edge set $E$, is obtained by including each of the $\binom{n}{2}$ possible edges independently at random with probability $s$. The random size of $G$, denoted $m = |E|$, is a binomial RV, i.e., $m \sim \text{bin}(\binom{n}{2}, s)$, as is the degree $d_G(v)$ of a randomly selected vertex $v \sim \text{uni}(V)$, namely, $d_G(v) \sim \text{bin}(n - 1, s)$. The RV $s \equiv m/\binom{n}{2}$ is the edge density (with $\mathbb{E}[s] = s$), and the RV $d \equiv (n - 1)s = 2m/n$ is the average degree of $G$. Given an ER graph $G$, the RV $d$ is the expected degree of a vertex selected uniformly at random.

We focus our analysis of star sampling on ER graphs since they are closed under both SSC and SSS, i.e., if either a star-center (SSC) or a star (SSS) is removed from an ER graph the resulting graph is still an ER graph, albeit one with a different (reduced) order. This follows intuitively from the fact that the presence or absence of an edge is independent across edges, but is established formally in Lem. 1 below.

**Lemma 1.** Given an ER random graph, the graph that remains after an SSS or SSC random sample is an ER random graph.

The proof is found in App. B.

### C. Sampling variant definitions and cost models

Random star sampling from a given graph $G = (V, E)$ produces a sequence of random graphs, denoted $(G_t, t \in \mathbb{N})$, where $G_t = (V_t, E_t)$ is the random graph after sample $t$. Both the vertex set and edge set are (in general) random variables. It is convenient to denote the given graph as $G_0 = (V_0, E_0)$, i.e., the initial member of the list of random graphs, corresponding to $t = 0$. The target set is denoted by $V^*$ or $V^*_t$. Define the sequence of random vertex sets $(V_t, t \in \mathbb{N})$, where $V_t = V_t \cap V^*$ holds the members of the initial target set still “alive” after $t$ samples. The sampling construction ensures the random sets are nested: $V_{t+1} \subseteq V_t$, $E_{t+1} \subseteq E_t$, and $V^*_{t+1} \subseteq V^*_t$. Recalling $|\cdot|$, the three SS variants are as follows.

**Definition 1.** SS with replacement (SSR). Generate the IID random sequence of star centers $(v_t, t \in \mathbb{N})$, with $v_t \sim \text{uni}(V_0)$. As SSR uses replacement, $G_t = G_0$ for all $t$.

**Definition 2.** SS without center replacement (SSC). Generate the random sequence of star centers $(v_t, t \in \mathbb{N})$, with $v_t \sim \text{uni}(V_{t-1})$, and update the graph by removing the star center, i.e., $E_t = E_{t-1} \setminus \{v_t\}$, and the edges in the edge neighborhood of the star center, i.e., $E_t = E_{t-1} \setminus \Gamma_{G_{t-1}}(v_t)$.

**Definition 3.** SS without replacement (SSS). Generate the random sequence of star centers $(v_t, t \in \mathbb{N})$, with $v_t \sim \text{uni}(V_{t-1})$, and update the graph by removing the star, i.e., $V_t = V_{t-1} \setminus N^*_G(v_t)$, and the edges in the extended edge neighborhood of the star center, i.e., $E_t = E_{t-1} \setminus \Gamma^*_{G_{t-1}}(v_t)$.

Lem. 2 gives the expected number of edges removed from an ER graph under SSS. Consider a star sample with star center $v \sim \text{uni}(V)$ of an ER graph $G$ with parameters $(n, s)$; recall $v$ has degree distribution $d \sim \text{bin}(n - 1, s)$.

**Lemma 2.** Given an ER random graph $G$ with parameters $(n, s)$, a randomly selected vertex $v$ as star center, and conditioned on the degree $d$ of $v$, the random number of edges in the extended edge neighborhood of $v$, denoted $g \equiv |\Gamma^*_G(v)|$, has a binomial distribution

$$g|d \sim d + \binom{d}{2} + d(n - d - 1), s; \quad (3)$$

with (unconditional) expectation

$$\mathbb{E}[g] = (n - 1)s(1 + (n/2 - 1)(2 - s)) = (n - 1)s(2 - s). \quad (4)$$

The asymptotic (in $n$) ratio of $\mathbb{E}[g]$ to the expected total number of edges in the graph $(\binom{n}{2})s$ in the graph is

$$\lim_{n \to \infty} \mathbb{E}[g]/(\binom{n}{2}s) = (2 - s)s.$$
is the expectation of \( c_u(G, V^*) \), taken with respect to the distribution over all possible realizations of graph sequences induced by SS that begin with \( G_0 = G \).

**Definiton 5.** The linear cost of a SS is the random sum of the extended degrees of the randomly selected vertex centers from each star sample until a star, either the star center or one of the star points, intersects the target set \( V^* \), i.e.,

\[
    c_l(G, V^*) = \sum_{t=1}^{c_u(G, V^*)} d_{G_{t-1}}(v_t).
\]

This cost is a function of the sequence of random graphs \( (G_t, t \in \mathbb{N}) \). The expected linear cost of a SS, denoted

\[
    c_l(G, V^*) = \mathbb{E}[c_l(G, V^*)],
\]

is the expectation of \( c_l(G, V^*) \), taken with respect to the distribution over all possible realizations of graph sequences induced by SS that begin with \( G_0 = G \).

**IV. UNIT COST MODEL**

Expected unit costs of SSR and SSC are given in §IV-A and the expected unit cost of SSS is given in §IV-B. Exact results in §IV-A are given for an arbitrary graph and as bounds for ER random graphs, while approximate results in §IV-B are only given for ER random graphs. §IV-C gives numerical results.

**A. Unit cost analysis for SSR and SSC**

Let \( G = (V, E) \) be an arbitrary graph of order \( n \), and let \( V^* \subseteq V \) be an arbitrary target set. Recall from §III-A that \( N^c_G(V^*) \) contains the target set \( V^* \) and its neighbors in \( G \). Observe the equivalence: a star sample \( N^c_G(v) \) intersects \( V^* \) if and only if \( v \in N^c_G(V^*) \). This observation yields the expected unit cost of SSR and SSC. Set \( n^* = |N^c_G(V^*)| \).

**Fact 2 (Unit cost of SSR).** Under SSR, for any graph \( G \) and any target set \( V^* \), the unit cost \( c_u \) in Def. 4 is a geometric RV with success probability \( n^* \), i.e., \( \mathbb{E}[c_u] = \operatorname{geo}(n^*) \), and the expected unit cost is \( c_u = n / n^c \).

**Proof:** SSR unit cost is the random number of independent Bernoulli trials until the first “success”, i.e., the random star intersects the target set (equivalently, the random star center intersects the target set extended neighborhood).

**Fact 3 (Unit cost of SSC).** Under SSC, for any graph \( G \) and any target set \( V^* \), the expected unit cost \( c^{SSC}_u \) in Def. 4 is

\[
    c^{SSC}_u = (n + 1)(n^c + 1).
\]

**Proof:** SSC with target set \( V^* \) on a graph \( G \) with extended neighborhood \( N^c_G(V^*) \) is equivalent to sampling without replacement from an urn with \( n \) balls, of which \( n^c \) are marked, until a marked ball is drawn. The expected number of samples is \( (n + 1)(n^c + 1) \) by Fact 1.

**Fact 2 and Fact 3 show the expected number of SSR samples exceeds the expected number of SSC samples:** \( c_u > c^{SSC}_u \).

We now adapt the previous two facts to the case where the initial graph is an ER random graph \( G_0 = (V_0, E_0) \) with parameters \((n, s)\), and where the expectation is with respect to both the graph and sampling distributions. Define the RVs:

- \( n_t = |V_t| \): order of graph \( G_t \) (note \( n_t = n \));
- \( n^c_t = |V^*_t| \): number of vertices from the target set in \( G_t \);
- \( n^c_0 \equiv |V^*_0| \) (where \( V^*_0 \equiv N^c_0(V^*_0) \)); number of vertices in extended neighborhood of target set in \( G_0 \).

**Fact 4.** The random order of the extended neighborhood of the target set in the ER random graph \( G_0 \) is a binomial RV:

\[
    n^c_0 \sim \binomial(n, n^*_0, 1 - \bar{s}^n_0).
\]

Define \( n^{c, *}_0 \equiv \mathbb{E}[n^c_0] \), where

\[
    \mathbb{E}[n^c_0] = n^*_0 + (n - n^*_0)(1 - \bar{s}^n_0),
\]

\[
    \text{var}(n^c_0) = (n - n^*_0)(\bar{s}^n_0)(1 - \bar{s}^n_0).
\]

**Proof:** Each \( v \in V_0 \setminus V^*_0 \) is connected to \( V^*_0 \) (independently of other vertices) if there exists an edge (or edges) from \( v \) to \( V^*_0 \), which happens with probability \( 1 - \bar{s}^n_0 \).

Fig. 1 (right) shows \( \bar{s}^n_0 / n \), the expected fraction of vertices in the extended target set (Fact 4), vs. the edge probability \( s \).

The next result, from [16], is leveraged in Prop. 1 below.

**Fact 5 (Bounds on inverse moments, [16] (Eq. 3.1, p. 729)).** Let a random variable \( x \) have mean \( \mu \), variance \( \sigma^2 \), and minimum support \( x_{\min} > 0 \) (i.e., \( P(x \geq x_{\min}) = 1 \)). Then

\[
    \frac{1}{\mu} \leq \mathbb{E} \left[ \frac{1}{x} \right] \leq \frac{1}{x_{\min}} \frac{\sigma^2 + (\mu - x_{\min})x_{\min}}{\sigma^2 + (\mu - x_{\min})\mu}.
\]

Specializing (11) to a binomial RV \( x \sim \binomial(m, p) \), for \( a > 0 \):

\[
    \frac{1}{a + mp} \leq \mathbb{E} \left[ \frac{1}{a + x} \right] \leq \frac{a + 1 - p}{a(a + 1 + (m - 1)p)}.
\]

**Proposition 1 (Unit cost of SSR and SSC for ER graph).** Fix the initial graph as an ER random graph \( G_0 \) with parameters \((n, s)\). The expected unit cost under SSR and SSC has bounds

\[
    c_u \leq c^{SSR}_u \leq c^{SSC}_u \leq c^{SSC}_u \leq c^{SSC}_u.
\]
Let the target set \( V_0^* \subseteq V_0 \) have cardinality \( n_0^* = |V_0^*| \):

\[
\begin{align*}
\ell_{SSR}^u &= \frac{n}{n - \hat{n}_0^*(n - n_0^*)}, \\
\ell_{SSR}^v &= \frac{n(n_0^* + n_0^*)}{n_0^*(n - \hat{n}_0^*(n - 1 - n_0^*))}, \\
\ell_{SSC}^u &= \frac{n + 1}{n + 1 - \hat{n}_0^*(n - n_0^*)}, \\
\ell_{SSC}^v &= \frac{(n + 1)(\hat{n}_0^* + 1 + n_0^*)}{(n_0^* + 1)(n + 1 - \hat{n}_0^*(n - 1 - n_0^*)}. 
\end{align*}
\]  

(14)

**Proof:** Using Fact 2, the SSR bounds are derived by applying (12) in Fact 5 to \( nE[1/n_0^*] \), for \( n_0^* \) in (9) of Fact 4, i.e., with \( a = n_0^*, \quad m = n - n_0^*, \quad p = 1 - \hat{n}_0^* \). Using Fact 3, the SSC bounds are derived by applying (12) in Fact 5 to \((n + 1)E[1/(n_0^* + 1)]\), i.e., with \( a = n_0^* + 1, \quad m = n - n_0^*, \quad p = 1 - \hat{n}_0^* \). \[\blacksquare\]

**B. Unit cost analysis for SSS**

Let \((y, z)\) be a pair of continuous RVs, with \( z \neq 0 \) almost surely, means \((\mu_y, \mu_z)\), variances \((\sigma_y^2, \sigma_z^2)\), and covariance \(\text{cov}(y, z)\). Consider the expectation of their ratio, i.e., \(E[\frac{y}{z}]\). The following result is found in [17] and [18].

**Proposition 2** ([17], [18]). The second order Taylor series approximation of \( E[\frac{y}{z}] \) around \((\mu_y, \mu_z)\) is

\[
E\left[\frac{y}{z}\right] \approx \frac{\mu_y}{\mu_z} + \frac{\sigma_y^2 \mu_y}{\mu_z^2} - \frac{\text{cov}(y, z)}{\mu_z^2}. \tag{15}
\]

In particular, the error associated with a first-order Taylor series approximation is, to second order,

\[
\epsilon_1 = \left| E\left[\frac{y}{z}\right] - \frac{\mu_y}{\mu_z} \right| \leq \frac{\sigma_y^2 \mu_y}{\mu_z^2} - \frac{\text{cov}(y, z)}{\mu_z^2}. \tag{16}
\]

This fact will be leveraged in Thm. 1 below.

Let the initial graph be an ER random graph \( G_0 = (V_0, E_0) \) with parameters \((n, s)\). Recall a star will hit the target set if its star center is in the extended neighborhood of the target. The (random) probability that a star \( t + 1 \) hits the target set is

\[
\tilde{p}_{t+1}^{SSS} = \frac{n_{t}^{c,*}}{n_t}. \tag{17}
\]

Define events \( (\xi_t, t \in \mathbb{Z}^+) \), with \( \xi_0 \) trivial, and \( \xi_t = \{v_t \notin V_0^*\} \) the event that star \( t \) misses the target set. Next, define events \( (\tilde{\xi}_t, t \in \mathbb{Z}^+) \), with \( \xi_0 \) trivial, and

\[
\tilde{\xi}_t \equiv \bigcap_{t' \in [t]} \xi_{t'} = \{v_{t'} \notin V_0^* \text{ for all } t' \in [t]\}. \tag{18}
\]

Thus, \( \tilde{\xi}_t \) is the event that the stars of the first \( t \) samples have each missed the target set. Observe that, conditioned on \( \xi_t \), the target set on \( \xi_t \), the target set in graph \( G_t \) is identical to the same set in the initial graph \( G_0 \), i.e., \( V_0^* (\xi_t) = V_0^* \), although the degrees of vertices in \( V_0^* \) may have decreased due to sampling.

The expected probability of hitting the target with star \( t + 1 \), conditioned on missing the target set in the first \( t \) samples, is:

\[
E[\tilde{p}_{t+1}^{SSS} | \tilde{\xi}_t] = E\left[\frac{n_{t}^{c,*}}{n_t} | \tilde{\xi}_t\right]. \tag{19}
\]

Leveraging (16), the error in approximating this expected conditional probability by its ratio of expectations, i.e.,

\[
\tilde{p}_{t+1}^{SSS} = \frac{E[n_{t}^{c,*} | \tilde{\xi}_t]}{E[n_t | \tilde{\xi}_t]}, \tag{20}
\]

is approximately

\[
\epsilon_{n,t} = \left| \frac{\text{var}(n_{t}^{c,*} | \tilde{\xi}_t)}{E[n_t | \tilde{\xi}_t]^3} - \frac{\text{cov}(n_{t}^{c,*}, n_t | \tilde{\xi}_t)}{E[n_t | \tilde{\xi}_t]^2}\right|. \tag{21}
\]

This approximate conditional hitting probability and its approximation error are expressed in terms of the three fundamental parameters \((n, n_0^*, s)\) and the sample index \( t \) in Thm. 1.

**Theorem 1**. The approximate probability of hitting the target set in sample \( t + 1 \) under SSS, conditioned on missing it in the first \( t \) samples, is \((n_0^* = E[n_0^*] \text{ in Fact 4 (10))})

\[
\hat{p}_{t+1}^{SSS} = \frac{(n_0^* - n_0^*) s^t + n_0^*}{n^t s^t - \frac{2}{5}(1 - s^t)}. \tag{22}
\]

Viewing \( \hat{p}_{t+1}^{SSS} \) as a continuous function of \( t \), \( \hat{p}_{t+1}^{SSS} \) is convex increasing in \( t \) over \( (0, t_p^{(2)}) \), where

\[
t_p^{(2)} = \frac{\log(n_s^t + 1)}{\log(1/s)}. \tag{23}
\]

Moreover, \( \hat{p}_{t}^{SSS} \) (hitting in the first sample) equals the exact value \( \hat{p}_{t}^{SSS} \equiv n_{0}^{c,*}/n \), and \( \hat{p}_{t+1}^{SSS} = 1 \) at

\[
t_p^{(1)} = \frac{\log(s(n - E[n_0^*]) + n_0 + s)}{\log(1/s)} < t_p^{(2)}. \tag{24}
\]

The error in this approximation, \( \epsilon_{n,t} \), has upper bound

\[
\begin{align*}
\left(n(1 - s^t) + \frac{\bar{s}(1 + \bar{s}(1 - s^t))}{s(1 + \bar{s})}\right) & \left(s^t((n_0^* - n_0^*) s^t + n_0^*) \right) \\
& \left(n^t s^t - \frac{2}{5}(1 - s^t)^3\right)
\end{align*} \tag{25}
\]

The approximation error is asymptotically negligible in \( n \):

\[
\lim_{n \to \infty} \epsilon_{n,t} = 0, \quad \forall t \in \mathbb{N}, s \in (0, 1). \tag{26}
\]

and in fact \( \epsilon_{n,t} = O(n_0^*/n^2) \).

The proof is found in App. E. Fig. 2 (top left) shows \( \hat{p}_{t}^{SSS} \) from (22) vs. the sample index \( t \). Note the curves are convex increasing even on a logarithmic scale, i.e., \( \hat{p}_{t}^{SSS} \) is log-convex.

The approximate unconditional hitting probability \( \hat{q}_{t}^{SSS} \) and the corresponding approximate expected unit cost under SSS for an ER graph \( c_{SSS}^{u} \) are expressed in terms of the three fundamental parameters \((n, n_0^*, s)\) in Thm. 2.

Let the RV \( c_{SSS}^{u} \) have distribution \( c_{SSS}^{u} \equiv c_{SSS}^{u} \), \( t \in [t_{p}^{(1)}] \), so that the approximate expected unit cost is the expectation of this RV with respect to that distribution. \( c_{SSS}^{u} \equiv E[c_{SSS}^{u}] \).

**Theorem 2**. The approximate unconditional probability of first hitting the target set in sample \( t \) under SSS is

\[
\hat{q}_{t}^{SSS} \equiv \hat{p}_{t}^{SSS} \prod_{u \in [t-1]} (1 - \tilde{p}_{u}^{SSS}), \tag{27}
\]

for \( t \in [[t_{p}^{(1)}]], \quad \text{with } t_{p}^{(1)} \text{ defined in (24)}. \) The associated approximate expected unit cost under SSS is

\[
c_{SSS}^{u} \equiv \sum_{t \in [t_{p}^{(1)}]} \prod_{u \in [t]} (1 - \tilde{p}_{u}^{SSS}). \tag{28}
\]
Fig. 2. Top left: Thm. 1: approximate conditional probability of first hitting the target in an ER graph with parameters \((n = 100, s)\) at sample \(t\), conditioned on not yet hitting it, under SSR, i.e., \(p_{t|t}^{SSR}\) from (22), vs. \(t\), for \(n_0^* = 2\) and \(s \in \{0.0001, 0.0005, 0.001, 0.005, 0.01\}\). Horizontal lines show \(p_{t|t}^{SSR} \equiv n_{0|t}^{c_e}/n\) and vertical lines show \(t_{\bar{V}}^{(1)}\) from (24). Top right: Thm. 2: approximate unconditional probability of first hitting the target in an ER graph with parameters \((n = 100, s)\) at sample \(t\) under SSR, i.e., \(p_{t|t}^{SSR}\) from (27), vs. \(t\), for \(n_0^* = 2\) and \(s \in \{0.0001, 0.0005, 0.001, 0.005, 0.01\}\). Vertical lines show \(t_{\bar{V}}^{(1)}\) from (24). Bottom: Prop. 3: three conditional probability ratios vs. graph order \(n\), for \(t \in \{2, 8\}\) \((n_0^* = 2, s(n) = 1/n)\).

The proof is found in App. F. Fig. 2 (top right) shows \(\bar{q}_{t|t}^{SSS}\) from (27) vs. the sample index \(t\). As \(s\) increases the distribution support decreases and the distribution slope decreases.

The last result in this section, Prop. 3, considers the parameter regime where the ER edge probability \(s\) is \(O(1/n)\). Let

\[ p_{t|t}^{SSR} = \frac{n_{0|t}^{c_e}}{n}, \quad p_{t|t}^{SSC} = \frac{n_{0|t}^{c_e}}{n - t + 1} \]  

be the conditional probabilities of hitting the target set for the first time on sample \(t\) under SSR and SSC, respectively.

**Proposition 3.** Set the ER edge probability \(s(n) = c/n\) (with \(c > 0\)), and fix \(t\) and \(n_0^*\). The conditional probability of hitting the target set in sample \(t\) (conditioned on not yet having hit it) is, asymptotically as \(n \uparrow \infty\), equal for all three SS variants:

\[ \frac{p_{t|t}^{SSR}}{p_{t|t}^{SSC}} \to 1, \quad \frac{p_{t|t}^{SSR}}{p_{t|t}^{SSC}} \to 1, \quad \frac{p_{t|t}^{SSS}}{p_{t|t}^{SSC}} \to 1. \]  

The proof is found in App. G. Fig. 2 (bottom) shows all three conditional probability ratios in Prop. 3 vs. the graph order \(n\), for \(n_0^* = 2\) and \(s(n) = c/n\) with \(c = 1\) for \(t \in \{2, 8\}\). The convergence of all three ratios to 1 as \(n \uparrow \infty\) is evident.

**C. Numerical results**

The four plots in Fig. 3 show numerical and simulation results for the expected unit cost to find \(v \in V^*\) (with \(n_0^* = 2\)) under SSR, SSC, and SSS on ER random graphs with order \(n = 1,000\) vs. the edge density \(s\) (on a log scale). Three conclusions are evident from the plots. First, the bounds on SSR (Prop. 1, top left plot) and SSC (Prop. 1, top right plot) are “tight” and the approximation for SSS (Thm. 1, bottom left plot) is “accurate”. Second, the average unit cost ordering on ER graphs is \(c_{SSR} \geq c_{SSC} \geq c_{SSS}\) (bottom right plot). Third, there is a notable difference in unit cost between SSR and SSC/SSS for “small” \(s\), but no appreciable difference among the three sampling schemes for “large” \(s\) (bottom right plot).

**V. LINEAR COST MODEL**

The expected linear cost (Def. 5) of SSR on an arbitrary graph is given in §V-A. Approximate expected linear costs of SSR, SSC, and SSS on an ER random graph are given in §V-B. Numerical results are presented in §V-C.

**A. Arbitrary graph**

Let \(G = (V, E)\) be an arbitrary graph with order \(|V| = n\), and let \(V^* \subseteq V\) be an arbitrary target set. Fact 6 below is the linear cost analog of the unit cost result for SSR in Fact 2.

We first develop some necessary notation, extending that introduced in §III-A. Let \(V^*_G = N_2(V^*)\) denote the extended neighborhood of \(V^*\) (with order \(n_0^* = |V^*_G|\), and let \(V^c_G = V \setminus V^*_G\) denote its complement. Define \(V^c_G(v) = \{v \in V^c_G : d_G(v) = k\}\) and \(V^c_G(k) = \{v \in V^c_G : d_G(v) = k\}\) as the subsets of vertices in \(V^c_G\), respectively, of degree \(k\). The conditional degree distributions in \(V^c_G\) are denoted \(w_{G}^{c,e}(k), \tilde{w}_{G}^{c,e}(k)\), respectively, with components

\[ w_{G}^{c,e}(k) = \frac{|V^c_G(v)|}{|V^c_G|}, \quad \tilde{w}_{G}^{c,e}(k) = \frac{|V^c_G(k)|}{|V^c_G|}. \]
Finally, the average degrees in \( V^{c_{\ast}}_{G} \), \( \bar{c}_{G}^{c_{\ast}} \), respectively, are
\[
d_{G}^{c_{\ast}} = \sum_{k \in D_{G}^{c_{\ast}}} k u_{G}^{c_{\ast}}(k), \quad \bar{c}_{G}^{c_{\ast}} = \sum_{k \in D_{G}^{c_{\ast}}} k u_{G}^{c_{\ast}}(k). \tag{32}
\]

**Fact 6** (Linear cost of SSR). Under SSR, for any graph \( G \) and any target set \( V^{\ast} \), the expected linear cost \( c_{l} \) is
\[
c_{l}^{SSR}(G, V^{\ast}) = \bar{c}_{G}^{c_{\ast}} \left( \frac{n}{\bar{c}_{G}^{c_{\ast}}} - 1 \right) + d_{G}^{c_{\ast}}. \tag{33}
\]

The proof is found in App. H.

Despite the success in deriving the previous result for SSR, we are of the impression that a simple exact expression for the expected linear cost under SSC on an arbitrary graph is not available. In contrast to the unit cost case, for which the SSC expected unit cost is given by Fact 3, the linear cost case appears to be much more difficult to analyze, as it requires a representation of the evolution under SSC of the (effectively arbitrary) degree distribution \( u_{G} \) of the initial graph \( G \). This evolution does not appear to be sufficiently tractable to yield “closed-form” results like those in §IV-A. Given this difficulty, we turn to approximations for the case of an ER random graph.

**B. ER random graph**

Let the initial graph be an ER random graph \( G_{0} = (V_{0}, E_{0}) \) with parameters \((n, s)\). Expectations below are taken with respect to the graph and sampling distributions. The starting point for all three sampling paradigms is to express the expected linear cost in terms of the conditional expectations of the extended degrees of each sampled vertex, conditioned on the unit cost, and to approximate the per-sample extended degree as independent of the unit cost:
\[
\begin{aligned}
c_{l} &= \mathbb{E}[c_{l}] = \mathbb{E} \left[ \sum_{t \in [c_{\max}]} d_{t}^{c_{\ast}} \mathbf{1}(t \leq c_{u}) \right] \tag{34} \\
&= \sum_{t \in [c_{\max}]} \mathbb{E}[d_{t}^{c_{\ast}}] \mathbb{P}(t \leq c_{u}).
\end{aligned}
\]

Here, \( d_{t}^{c_{\ast}} \) is the extended degree of sample \( t \), \( c_{u} \) is the unit cost, i.e., the random number of samples required for the star center to hit the extended target set, and \( c_{\max} \) is an upper bound, possibly infinite, on \( c_{u} \). The extended degree is not independent of the number of samples, as the target set in \( G_{t} \) will have a distinct degree distribution from its complement, conditioned on the previous \( t - 1 \) samples missing the target. Nonetheless, the approximation is useful in that it facilitates analysis and our numerical investigations support our claim that the approximation is accurate over a wide array of parameter values. Below, we adapt the above approximation to give the approximate expected linear cost conditioned on the random extended target set, \( n_{0}^{c_{\ast}} \), i.e.,
\[
\mathbb{E}[c_{l} | n_{0}^{c_{\ast}}] \approx \mathbb{E}[c_{l} | n_{0}^{c_{\ast}}] = \sum_{t \in [c_{\max}(n_{0}^{c_{\ast}})]} \mathbb{E}[d_{t}^{c_{\ast}}] \mathbb{P}(t \leq c_{u} | n_{0}^{c_{\ast}}). \tag{35}
\]

The additional approximation in (35), beyond the one in (34), is that \( d_{t}^{c_{\ast}} \) is independent of \( n_{0}^{c_{\ast}} \), i.e., \( \mathbb{E}[d_{t}^{c_{\ast}} | n_{0}^{c_{\ast}}] = \mathbb{E}[d_{t}^{c_{\ast}}] \).

1) **SSR**: In the case of SSR, the approximation in (35) leads to a particularly simple expression on account of the fact that \( (d_{t}^{c_{\ast}}) \) are IID, i.e., \( \mathbb{E}[d_{t}^{c_{\ast}}] \) does not depend upon \( t \), and as such Wald’s identity for the expected value of the sum of a random number of IID RVs applies:
\[
\hat{\mathbb{E}}[c_{l}^{SSR} | n_{0}^{c_{\ast}}] = \mathbb{E}[d_{t}^{c_{\ast}}] \sum_{t \in [c_{\max}(n_{0}^{c_{\ast}})]} \mathbb{P}(t \leq c_{u} | n_{0}^{c_{\ast}}) \tag{36}
\]
\[
= \mathbb{E}[d_{t}^{c_{\ast}} | n_{0}^{c_{\ast}}] = (1 + (n - 1)s) \frac{n}{n_{0}^{c_{\ast}}}.
\]

Here, \( \mathbb{E}[d_{t}^{c_{\ast}}] = 1 + (n - 1)s \) is the extended expected degree of a random vertex in an ER random graph, and \( \mathbb{E}[d_{t}^{c_{\ast}} | n_{0}^{c_{\ast}}] = n/n_{0}^{c_{\ast}} \) is the conditional expected unit cost of SSR. Taking expectation with respect to \( n_{0}^{c_{\ast}} \) yields the following.

**Proposition 4** (SSR linear cost for ER graph). Fix the initial graph as an ER random graph \( G_{0} \) with parameters \((n, s)\), and the extended target set cardinality \( n_{0}^{c_{\ast}} \). The approximate expected linear cost under SSR, \( c_{l}^{SSR} \), is
\[
c_{l}^{SSR} = \mathbb{E}[c_{l}^{SSR} | n_{0}^{c_{\ast}}] = (1 + (n - 1)s)c_{u}^{SSR}, \tag{37}
\]
and the SSR expected unit cost, \( c_{u}^{SSR} \), has bounds in Prop. 1.

2) **SSC**: Leveraging the approximation (35) for SSC requires the expected extended degree of the random star center selected in sample \( t \) and the hitting probability for sample \( t \).

**Proposition 5** (SSC linear cost for ER graph). The approximate expected linear cost under SSC for an ER graph, conditioned on the random extended target set order \( n_{0}^{c_{\ast}} \), is
\[
\mathbb{E}[c_{l}^{SSC} | n_{0}^{c_{\ast}}] = \sum_{t=1}^{n-n_{0}^{c_{\ast}}+1} \left( 1 + (n - t)s \right) \prod_{u=1}^{t-1} \left( 1 - \frac{n_{0}^{c_{\ast}}}{n - u + 1} \right). \tag{38}
\]

The proof is found in App. I.

3) **SSS**: As with SSC, leveraging the approximation (35) for SSS requires the expected extended degree of the random star center selected in sample \( t \) and the approximate conditional hitting probability for sample \( t \).

**Proposition 6** (SSS linear cost for ER graph). The approximate expected linear cost under SSS for an ER graph, conditioned on the random extended target set order \( n_{0}^{c_{\ast}} \), is
\[
\mathbb{E}[c_{l}^{SSS} | n_{0}^{c_{\ast}}] = \sum_{t=1}^{(n-1)s+1} \sum_{u=1}^{t} \left( 1 - \frac{n_{0}^{c_{\ast}}}{n - u + 1} \right). \tag{39}
\]

where \( \rho_{(n-1)s+1}^{SSS} \) (c.f. (24)) and \( p_{(n-1)s+1}^{SSS} \) (c.f. (22)) are defined in Thm. 1.

The proof is found in App. J.

**C. Numerical results**

The four plots in Fig. 4 show numerical and simulation results for the expected linear cost to find \( v \in V^{\ast} \) (with \( n_{0} = 2 \)) under SSR, SSC, and SSS on ER random graphs with order \( n = 1,000 \) vs. the edge density \( s \) (on a log scale). Three conclusions are evident from the plots. First, the bounds on SSR (Prop. 4, top left plot) are not as tight as for the unit cost case, but both bounds are tight for separate ranges of values of \( s \), and the approximation for SSS (Prop. 5, top
right plot) and SSS (Prop. 6, bottom left plot) are “accurate.” Second, the average linear cost ordering on ER graphs is $c_{SSR}^0 \geq c_{SSC}^0 \geq c_{SSS}^0$ (bottom right plot), as was true for unit cost. Third, there is a notable difference in linear cost between SSR and SSC/SSS for “small” $s$, and a notable difference between SSR/SSC and SSS for “large” $s$ (bottom right plot).

Fig. 4. Numerical and simulation results for expected linear cost on ER random graphs with order $n = 1,000$ and target size $n_0^* = 2$ vs. edge density $s$ (on log scale). Top left: SSR bounds and simulation. Top right: SSC approximation and simulation. Bottom left: SSS approximation and simulation. Bottom right: Simulation results for SSR, SSC, SSS.

VI. RESULTS ON “REAL-WORLD” GRAPHS

This section assesses the accuracy when the SSR, SSC, and SSS unit and linear cost estimates, derived for ER graphs in §IV and §V, are applied to twelve “real-world” graphs. These graphs come from a variety of sources: power-network is from [19]; condMat, gnutella08, gnutella04, web-google, astroPh, epinions, fb-combined, oregon1, and brightkite are from the SNAP repository [20]; while the web-edu and tech-routers are from the network data repository [21].

Table III gives statistics for these graphs, including order $n = |V|$, size $m = |E|$, edge density $s = m/(n(n-1)/2)$, assortativity $\alpha$ (the correlation of the degrees of a randomly selected edge), and maximum degree $\delta_{max} \equiv \max D$.

Table IV and Table V present empirical and estimated costs, unit and linear respectively, on these graphs under SSR, SSC, and SSS, assuming $n_0^* = 4$, and assuming $n_0^{**}$ is known. Simulation results are presented as a 95% confidence interval after 1,000 independent trials. The percent relative error between the simulation mean cost and the estimated expected cost are given for all three sampling variants. The relative error column entries are bolded when the estimate does not lie in the 95% confidence interval from simulation.

Several comments bear mention. First, the SSR and SSC approximations are remarkably accurate, all within 5% relative error. Second, the SSS approximations, while less accurate than SSR and SSC, are still remarkably accurate given i) these graphs are quite different from one another and quite different from an ER graph, and ii) the approximate costs require so few summary statistics about the graph, i.e., $(n, n_0^*, s)$. Both unit and linear SSS relative errors are all under 30%, several having much smaller relative error, aside from the 60% error for epinions (unit) and oregon1 (linear).

Fig. 5 shows the unit (top) and linear (bottom) costs for the twelve “real-world” graphs vs. the sampling variant, i.e., SSR, SSC, or SSS. The cost for most graphs is relatively constant across the three variants, with several notable exceptions. Namely, the unit costs for epinions1 and britekite are notably higher for SSS than for SSR/SSC, and the linear costs for britekite and astroPh are notably lower for SSS than for SSR/SSC.

Fig. 5. Simulation results for the twelve “real-world” graphs.
VII. Conclusion

Star sampling (SS) is a natural means by which to randomly search for an element of a target set in a large / dynamic graph that is not available for deterministic search. Assessing the performance of the three SS variants (SSR, SSC, and SSS) is an important question to answer in order to assess whether the complexity incurred in avoiding sample replacement will yield a notable reduction in cost. ER random graphs are a useful class of graphs to analyze given their tractability; the bounds are seen to be tight, and the approximations are seen to be accurate. Moreover, applying these approximations to “real-world” graphs shows them to have, in most but not all cases, reasonably small relative errors, despite the approximations, particularly for SSS, having been derived for an ER graph. Finally, the results are easy to apply in that they require knowledge of only three basic facts: the graph order, the cardinality of the target set, and the edge density.

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Appendix A

Proof of Fact 1

Proof: Define: (i) \((n)_k \equiv (n-0)(n-1) \cdots (n-k+1)\) as the falling factorial (recall \(\frac{n!}{k!} = (n)_k\)); (ii) \(p_t \equiv n^*/(n-t+1)\) for \(t \in [n-n^*+1]\) as the probability of success on trial \(t\), conditioned on failure in the first \(t-1\) trials (observe \(p_1 = n^*/n\) and \(p_n = n^*+1\)); (iii) \(q_t \equiv p_t \prod_{s=1}^{t-1}(1-p_s)\) as the unconditioned probability of a first success on trial \(t\); (iv) \(x\) as the RV for the number of required trials, where \(x\) has distribution \((q_t, t \in [n-n^*+1])\). It is straightforward but tedious to show that \((q_t, t \in [n-n^*+1])\) is correctly normalized. Substituting into the definition \(E[x] \equiv \sum_{t=1}^{n-n^*} t q_t\) yields

\[
E[x] = \frac{n^*}{n} + n^* \sum_{t=2}^{n-n^*} t \frac{n+1-n^*}{n+1-s} \frac{n+1-n^*}{n+1-s} = (n)_t (n)_t = \frac{n^* (n^*)}{n^* (n^*)} = \frac{n^*}{n^*} \tag{40}
\]

Using the falling factorial definition yields:

\[
E[x] = \frac{n^*}{n} + n^* \sum_{t=1}^{n-n^*} \binom{n}{t} \binom{n}{t} = \frac{n^*}{n} \tag{41}
\]

It is straightforward but tedious to verify that

\[
\sum_{s=0}^{n-n^*} \binom{n}{s} \binom{n}{s} = \frac{(n^* + n^*)!}{n^*! (n^* - 1)!} = \frac{n^* + n^* + 1}{n^* - 1} \tag{42}
\]

Substitution and simple algebra yields the result.

Appendix B

Proof of Lem. 1

Lem. 1 holds by Lem. 3 and Lem. 4, both proved below.

Lemma 3. Take \(t' \in [n]\) samples from an ER random graph \(G\) with parameters \((n, s)\) using SSS, yielding a sequence of random graphs \((G_t, t \in [t'])\). Then \(G_t = (V_t, E_t)\) is an ER random graph with parameters \((n-t, s)\) for each \(t \in [t']\).

Proof: Let \(G_0 = G\) denote the initial ER random graph, with \(G_0 = (V_0, E_0)\), where \(V_0 = [n] = [0] = E_0\) may be considered as a sequence of \(\binom{n}{2}\) IID Bernoulli RVs, say \(E_0 = (x_e, e \in \binom{n}{2})\), with \(x_e \sim \text{Ber}(s)\) indicating the random inclusion or exclusion of an edge at the “site” at (unordered) vertex pair \(e\).

Define the sequence of random vertex and set triples \((V_t, V_{t-1}, E_t), t \in [t']\) with \(V_t \sim \text{uni}(V_{t-1})\) the randomly selected star center in sample \(t\), \(V_t = V_{t-1} \backslash \{v_t\}\) the set of vertices that remain after removal of star center \(v_t\), and \(E_t = E_{t-1} \backslash \{E_t\}\), for \(E_t \equiv E_{t-1} \backslash \{E_t\}\). Then \(|V_t| = n - t\) and, as \(E_0, \ldots, E_t\) are IID Ber(s), the RVs in \(E_t\) are IID Ber(s), so that \(G_t\) is an ER random graph (with vertex random labels) with parameters \((n-t, s)\).

Lemma 4. Take \(t' \in [n]\) samples from an ER random graph \(G\) with parameters \((n, s)\) using SSS, yielding a sequence of random graphs \((G_t, t \in [t'])\). Then \(G_t = (V_t, E_t)\) is an ER random graph with random order parameter \(n_t\) and edge probability \(s\), for each \(t \in [t']\). The random order parameter, \(n_t \equiv |V_t|\), obeys the recursion \(n_t = n_{t-1} - d_t^i\), for \(d_t^i \sim 1 + \text{bin}(n_{t-1} - 1, s), t \in [t']\).

Proof: The first two paragraphs in the proof of Lem. 3 hold here as well; we adopt the same notation below. Define the sequence of random vertex and set triples \((v_t, V_t, E_t), t \in [t']\), with \(v_t \sim \text{uni}(V_{t-1})\) the randomly selected star center in sample \(t\), \(V_t = V_{t-1} \backslash \{v_t\}\) the set of vertices that remain after removal of star \(N_{G_{t-1}}^c(v_t)\), and \(E_t = E_{t-1} \backslash \{E_t\}\) the edges that remain after removal of the extended edge neighborhood of \(v_t\) from \(G_{t-1}\). The sequence of RV pairs \((n_t, d_t^i), t \in [t']\), where \(n_t = |V_t|\) is the random order of \(G_t\) and \(d_t^i = |N_{G_{t-1}}^c(v_t)|\) the size of the extended neighborhood of \(v_t\), obeys (by construction) the recursion \(n_t = n_{t-1} - d_t^i\), with \(n_0 = n\). Proof by induction in \(t\) will show \(G_t\) is ER, and \(d_t^i \sim 1 + \text{bin}(n_{t-1} - 1, s), t \in [t']\).

Base case. It must be shown i) \(G_0 = (V_0, E_0)\) is an ER random graph with parameters \(n_0 = n - d_0^i\) and \(s\), and ii) \(d_0^i \sim 1 + \text{bin}(n-1, s)\). Recall \(V_1 = V_0 \backslash N_{G_0}^c(v_1)\) and \(E_1 = E_0 \backslash \{v_1\}\). First, the edges in \(E_1\) are a (randomly selected) subset of the IID Ber(s) RVs in \(E_0\), and as such the edges in \(E_1\) are also IID Ber(s) RVs; this shows i). Second, \(d_1^i = |N_{G_0}^c(v_1)|\) counts both the star center \(v_1\) and its neighbors \(N_{G_0}^c(v_1)\), where \(d_1^i \equiv |N_{G_0}^c(v_1)|\) has distribution Ber\((n-1, s)\) by construction; this shows ii).

Induction hypothesis. Fix \(t \in [t']\) and assume the induction hypothesis for \(t - 1, i.e., assume i) \(G_{t-1}\) is an ER random graph with parameters \(n_{t-1} - d_{t-1}^i\) and \(s\), and ii) \(d_{t-1}^i \sim 1 + \text{bin}(n_{t-2} - 1, s)\). It follows that this implies i) the random graph \(G_t\) obtained by removing a random star from \(G_{t-1}\) is an ER graph with parameters \(n_t = n_{t-1} - d_{t-1}^i\) and \(s\), and ii) \(d_t^i \sim 1 + \text{bin}(n_{t-1} - 1, s)\). First, the edges in \(E_t\) are a (randomly selected) subset of the IID Ber(s) RVs in \(E_{t-1}\), and as such the edges in \(E_t\) are also IID Ber(s) RVs; this shows i). Second, \(d_t^i = |N_{G_{t-1}}^c(v_t)|\) counts both the star center \(v_t\) and its neighbors \(N_{G_{t-1}}^c(v_t)\), where \(d_t^i \equiv |N_{G_{t-1}}^c(v_t)|\) has distribution Ber\((n_{t-1} - 1, s)\) by construction; this shows ii).

Appendix C

Proof of Lem. 2

Proof: A star sample has two “types” of vertices (the star center \(v\) and its \(d\) neighbors) and three “types” of edges, namely, i) “neighbor” edges connecting \(v\) with \(N_G(v), ii) “internal” edges with both vertices in \(N_G(v), and iii) “external” edges with one vertex in \(N_G(v) and the other outside \(N_G(v).\)
edges with one vertex in \( N_G(v) \) and the other in \( V \setminus N_G(v) \). There are \( d \) neighbor edges by assumption, \( (\frac{\beta}{2}) \) potential internal edges, and \( d(n-d-1) \) potential external edges, where (due to the ER random graph properties) all potential edges are present or absent independently with probability \( s \).

As \( d \sim \binom{n-1}{s} \), we have \( E[d] = (n-1)s \) and \( E[d^2] = (n-1)s + 2\left(\frac{n-1}{2}\right)s^2 \), and with these (4) is obtained by conditional expectation and simple algebra:

\[
E[g] = E[E[g|d]] = E\left[ d + \left(\frac{d}{2} + d(n-d-1)\right) s \right] = E\left[ 1 + \left(\frac{n-3}{2}\right)s \right] E[d] - \frac{1}{2} sd^2 = \left(1 + \frac{n-3}{2}s\right) E[d] - \frac{1}{2} sE[d^2] (43)
\]

The limit of \( E[g]/\binom{n}{2}s \) as \( n \to \infty \) follows from (4).

**APPENDIX D**

**PROOF OF LEM. 5**

Let \( (G_t, t \in \mathbb{Z}^+) \) be a sequence of ER random graphs induced by SSS, with \( G_t = (V_t, E_t) \) the graph following the removal of the random star selected in sample \( t \), starting from an initial ER random graph \( G_0 = (V_0, E_0) \) with parameters \((n,s)\). Let \( s \equiv 1 - s \).

Fix “watch” \( W_0 \subseteq V_0 \) and “draw” \( D_0 \subseteq V_0 \) subsets, and define an “immune” subset \( Z_0 \subseteq W_0 \) comprised of vertices in \( W_0 \) with no neighbors in \( D_0 \). Define the random set-valued sequences \((W_t, t \in \mathbb{Z}^+)\) and \((D_t, t \in \mathbb{Z}^+)\), with \( W_t \equiv W_0 \cap V_t \) and \( D_t \equiv D_0 \cap V_t \) the watch and draw sets surviving the first \( t \) samples. Consider two cases for star centers: \( i) \) they are never drawn from the watch set, i.e., \( D_0 \cap W_0 = \emptyset \), and \( ii) \) they are always drawn from the watch set, i.e., \( D_0 \subseteq W_0 \).

Let \((v_t, t \in \mathbb{N})\), with \( v_t \sim \text{uni}(V_{t-1})\), denote the sequence of random star centers associated with each random sample, and define the event sequence \((C_t, t \in \mathbb{N})\) that star center \( t \) is drawn from the surviving draw set, i.e., \( C_t \equiv \{v_t \in D_{t-1}\} \). Let \( \bar{C}_t \equiv \bigcap_{v \in [t]} C_t \), be the event that the first \( t \) star centers are each drawn from the surviving draw set.

Define the random sequence \((n_t, t \in \mathbb{Z}^+)\), with \( n_t \equiv |W_t| \) the watch set order, and define the sequences of conditional means \( (\mu_{W|D,t}, t \in \mathbb{Z}^+) \) and conditional variances \( (\sigma_{W|D,t}^2, t \in \mathbb{Z}^+) \), with \( \mu_{W|D,t} \equiv E[n_t|C_t] \) and \( \sigma_{W,t}^2 \equiv \text{Var}(n_t|C_t) \) associated with \( n_t \) and \( C_t \). Let \( n_{Z,0} = |Z_0| \).

**Lemma 5.** The mean and variance in the size of the watch set \( W_t \), conditioned on drawing from the draw set \( D_t \) and removing no vertices from the immune set \( Z_0 \) after \( t \) samples, where \( n_{W,0} = |W_0| \) and \( n_{Z,0} = |Z_0| \), are: \( i) \) if \( D_0 \cap W_0 = \emptyset \),

\[
\mu_{W|D,t} = \binom{n_{W,0} - n_{Z,0}}{s}^2 + n_{Z,0} \quad (44)
\]

\[
\sigma_{W|D,t}^2 = \binom{n_{W,0} - n_{Z,0}}{s}^2(1 - s^2) \quad (45)
\]

or, \( ii) \) if \( D_0 \subseteq W_0 \) and \( Z_0 = \emptyset \),

\[
\mu_{W|D,t} = \mu_{W|D,t}^{(i)} = \frac{s}{s}(1 - s^t) \quad (46)
\]

\[
\sigma_{W|D,t}^2 = \sigma_{W|D,t}^{(s)} = \frac{s}{s(1 + s)}(1 + s)(1 - s^t))s^t \quad (47)
\]

**Proof:** Let \( r_t \) be the RV denoting the number of vertices in the watch set \( W_t \), removed by sample \( t \). The random recurrence induced by the sampling is \( n_t = n_{t-1} - r_t \).

First:

\[
\mu_{W|D,t} = E[n_t|C_t] = E[E[n_t|n_{t-1}][C_t]] = E[E[n_{t-1} - r_t[n_{t-1}][C_t]] = E[n_{t-1} - E[r_t[n_{t-1}][C_t]].
\]

Second: \( \sigma_{W|D,t}^2 \)

\[
\equiv \text{Var}(n_t|C_t) = \text{Var}(E[n_t|n_{t-1}][C_t]) + \text{Var}(E[n_{t-1} - r_t[n_{t-1}][C_t]) + \text{Var}(E[r_t[n_{t-1}][C_t]) \quad (49)
\]

Case \( i) \) \( (D_0 \cap W_0 = \emptyset \). As the star center is never drawn from the watch set \( r_t(n_{t-1}, C_t) \sim \binom{n_{t-1} - n_{Z,0}}{s} \), and therefore \( i) E[r_t[n_{t-1}, C_t] = (n_{t-1} - n_{Z,0})s \) and \( ii \) \( \text{Var}(r_t[n_{t-1}, C_t] = (n_{t-1} - n_{Z,0}^2)s^2 \).

First:

\[
\mu_{W|D,t} - n_{Z,0} = E[(n_{t-1} - n_{Z,0})] = E[n_{t-1} - E[E[r_t[n_{t-1}][C_t]] = E[n_{t-1} - E[(n_{t-1} - n_{Z,0})s] = \bar{s}(n_{t-1} - n_{Z,0}). \quad (50)
\]

The recurrence \( \mu_{W|D,t} - n_{Z,0} = \bar{s}(\mu_{W|D,t-1} - n_{Z,0}) \) with initial condition \( \mu_{W|D,0} = n_{W,0} \) has solution (44).

\[
\sigma_{W|D,t}^2 = \text{Var}(\mu_{W|D,t}) + s^2(\text{Var}(n_{W,0} - n_{Z,0})) \quad (51)
\]

The recurrence \( \sigma_{W|D,t}^2 = \sigma_{W|D,t+1}^2 + s^2(\text{Var}(n_{W,0} - n_{Z,0})) \) with initial condition \( \sigma_{W|D,0} = 0 \) has solution (45).

Case \( ii) \) \( (D_0 \subseteq W_0, Z_0 = \emptyset \). As the star center is always drawn from the watch set and \( n_{Z,0} = 0 \), \( r_t(n_{t-1}, C_t) \sim 1 + \binom{n_{t-1} - 1, s} {1} \), and therefore \( i) E[r_t[n_{t-1}] = 1 + (n_{t-1} - 1)s \) and \( ii) \( \text{Var}(r_t[n_{t-1}] = (n_{t-1} - 1)s^2 \).

First:

\[
\mu_{W|D,t} = E[n_{t-1} - 1 + (n_{t-1} - 1)s] = \bar{s}E[n_{t-1}] - \bar{s} = \bar{s}E[n_{t-1}] - \bar{s} = \bar{s}(\mu_{W|D,t-1} - 1). \quad (52)
\]

The recurrence \( \mu_{W|D,t} = \bar{s}(\mu_{W|D,t-1} - 1) \) with initial condition \( \mu_{W|D,0} = n_{W,0} \) has solution (46).

Second:

\[
\sigma_{W|D,t}^2 = \text{Var}(\mu_{W|D,t}) + s^2(\text{Var}(n_{W,0} - n_{Z,0})) \quad (53)
\]

\[
\sigma_{W|D,t}^2 = \sigma_{W|D,t+1}^2 + s^2(\text{Var}(n_{W,0} - n_{Z,0})) \quad (54)
\]
The recurrence
\[ \sigma_{W|D,t}^2 = \hat{s}^2 \sigma_{W|D,t-1}^2 + ((n_{W,0} - 1)s + 1)\hat{s}^2 - \hat{s}, \]
with initial condition \( \sigma_{W|D,0}^2 = 0 \) has solution (47).

\section*{Appendix E}

\section*{Proof of Thm. 1}

\textit{Proof:} The proof relies upon Lem. 5 in App. D. (22) through (24). Apply Lem. 5 to each of the numerator and denominator in (20). Specifically, \( \mathbb{E}[n_i^{c,t}|\hat{E}_i] \) follows from the expectation expression for case \( i \) with watch set \( \mathcal{N}_{\mathcal{C}_t}(V^*), \) (disjoint) draw set \( V_0 \setminus \mathcal{N}_{\mathcal{C}_0}(V^*) \), conditioned on \( \hat{E}_t \) and hence \( V^* \) being unsampled, while \( \mathbb{E}[n_i|\hat{E}_i] \) follows from the expectation expression for case \( ii \) with watch set \( V_0 \) and (subset) draw set \( V_0 \setminus \mathcal{N}_{\mathcal{C}_0}(V^*) \): \( \mathbb{E}[n_i|\hat{E}_i] = (\mathbb{E}[n_i^{c,t}] - n_0^t)\hat{s}^4 \quad n_0^t \) \( \mathbb{E}[n_i|\hat{E}_i] = n s^2 - \frac{\hat{s}}{1 - \hat{s}} \). \( \mathbb{E}[n_i|\hat{E}_i] = \frac{n}{s^2} - \frac{\hat{s}}{1 - \hat{s}}. \) (54) (55) The ratio of (54) and (55) gives (22). The first two derivatives of \( \hat{p}_t^{SSS} \) with respect to \( t \) are:
\[ \hat{p}_t^{SSS'} = \frac{s\hat{s}(n_0^t n_s + \mathbb{E}[n_i^{c,t}]\hat{s})\log(\frac{1}{\hat{s}})}{(\hat{s} - \hat{s}(n_s + \hat{s}))^2} > 0 \]
\[ \hat{p}_t^{SSS''} = \frac{s\hat{s}(\mathbb{E}[n_i^{c,t}](\hat{s} + \hat{s}(n_s + \hat{s})) - \hat{s}^3)}{(\hat{s}(n_s + \hat{s}) - \hat{s})^3} > 0. \]
This establishes that \( \hat{p}_t^{SSS} \) is convex increasing. The value \( \hat{p}_t^{SSS} = 1 \) is found by equating the denominator in \( \hat{p}_t^{SSS} \) to zero and solving for \( t \), and \( \hat{p}_t^{SSS} = 1 \) is found by solving \( \hat{p}_t^{SSS} = 1 \) for \( t \).

\textit{Proof of (25).} Apply the variance expression in case \( ii \) of Lem. 5 with watch set \( V_0 \) and (subset) draw set \( V_0 \setminus \mathcal{N}_{\mathcal{C}_0}(V^*) \):
\[ \text{var}(n_i|\hat{E}_i) = \left( n(1 - \hat{s}^3) + \frac{\hat{s}}{s(1 + \hat{s})}(1 + \hat{s}(1 - \hat{s}^3)) \right)\hat{s}^4. \] (56)
Next, observe \( \text{cov}(n_i^{c,t}, n_i|\hat{E}_i) \) in (21) is nonnegative. To see this, consider a general scenario where each member of a population is given a property value, and two subsets of the population are defined as holding those members with property values in a given target set, with the second target set a subset of the first. Let the property values be random, and consider the two random variables denoting the cardinalities of the two random population subsets. These random variables are by construction positively correlated. This general scenario applies here to the population \( V_t \), with the first subset equal to \( V_t \) and the second subset equal to \( \mathcal{N}_{\mathcal{C}_0}(V^*_t) \).
By the above argument, (21) may be upper bounded as:
\[ \epsilon_{n,t} \leq \frac{n\text{var}(n_i|\hat{E}_i)\mathbb{E}[n_i^{c,t}|\hat{E}_t]}{\mathbb{E}[n_i|\hat{E}_t]^3}. \] (57)
Substitution of (54), (55), and (56) into (57) gives (25).

\textit{Proof of (26).} This follows by observing the numerator is \( O(n_0^{c,t}) = O(n^2) \) while the denominator is \( O(n^3). \)

\section*{Appendix F}

\section*{Proof of Thm. 2}

\textit{Proof:} The proofs of (27) and (28) are given in turn.
\textit{Proof of (27).} Set \( C_t \) as the event that star \( t \) misses the target, \( \hat{C}_t = C_t \cap \cdot \cdot \cdot \cap C_{t-1} \) as the event that the first \( t \) stars each miss the target, and \( C^*_{t+1} \) as the complement of \( C_t \), i.e., the event that star \( t \) hits the target. Set \( p_{t+1} \equiv \mathbb{P}(C^*_{t+1} | \hat{C}_t) \) as the conditional probability that star \( t+1 \) hits the target given that the first \( t \) stars miss the target. Note \( 1 - p_{t+1} = \mathbb{P}(C_{t+1} | \hat{C}_t) \). Set \( q_{t+1} \equiv \mathbb{P}(C^*_{t+1} \cap \hat{C}_t) \) as the unconditioned probability that star \( t+1 \) hits the target \( and \) that the first \( t \) stars miss the target. Thus \( q_{t+1} \) is the unconditioned probability that the first star to hit the target is star \( t+1 \). Set \( r_t \equiv \mathbb{P}(\hat{C}_t) \) as the probability that the first \( t \) stars each miss the target. As
\[ q_{t+1} = \mathbb{P}(C^*_{t+1} | \hat{C}_t) \mathbb{P}(\hat{C}_t) = p_{t+1} r_t. \]
and
\[ r_t = \mathbb{P}(C_1)\mathbb{P}(C_2|C_1)\mathbb{P}(C_3|C_2) \cdots \mathbb{P}(C_t|C_{t-1}) = \prod_{s=1}^{t}(1 - p_s). \]
It follows that \( q_{t+1} = p_{t+1} \prod_{s=1}^{t-1}(1 - p_s) \). Given approximation \( \hat{p}_{t+1} \approx p_{t+1} \) of the conditional distribution, (27) is a corresponding approximation of the unconditioned distribution.

\textit{Proof of (28).} Consider a sequence \( \hat{p} \equiv (\hat{p}_t, t \in \mathbb{N}) \) with each \( \hat{p}_t \in (0,1) \). For \( T \in \mathbb{N} \) set \( \hat{p}^{(T)}(t) \equiv (\hat{p}_t^{(T)}, t \in [T]) \), with
\[ \hat{p}^{(T)}(T) = \begin{cases} \hat{p}_t, & t \in [T-1] \ni, \\ 1, & t = T \end{cases} \] (60)
Define the following quantities:
\begin{itemize}
  \item \( \hat{p}^{(T)}_T \equiv 1 - \hat{p}^{(T)}_T; \)
  \item \( \pi^{(T)}_T \equiv \pi_1^{(T)}, t \in [T] \), with \( \pi_1^{(T)} \equiv 1; \)
  \item \( q^{(T)}_T \equiv q_1^{(T)}, t \in [T-1] \), with \( q_1^{(T)} \equiv (\hat{p}_1^{(T)} | \hat{p}^{(T)}_T); \) and
  \item \( \mu^{(T)}_T \equiv \sum_{t \in [T]} \pi_1^{(T)} \hat{p}_t^{(T)} \).
\end{itemize}
Then:
\begin{itemize}
  \item \( q^{(T)}_T \) is a probability distribution, \( b \) if \( x^{(T)} \sim q^{(T)}_T \) then \( \mathbb{P}(x^{(T)} > t) = \pi_1^{(T)} \), \( c \) and \( \mathbb{E}[x^{(T)}] = \mu^{(T)}_T \).
  \item \( q^{(T)}_T = \sum_{t \in [T]} \pi_1^{(T)} \hat{p}_t^{(T)} \). It must be shown that \( \Sigma_q^{(T)} = 1 \).
\end{itemize}

The proof is by induction in \( T \). The base case \( T = 1 \) is trivial. Suppose it is true for \( T \), i.e., suppose \( \Sigma_q^{(T)} = 1 \), so that \( \Sigma_q^{(T)} = \Sigma_q^{(T-1)} + \pi_1^{(T)} = 1. \) For induction hypothesis case \( T+1 \):
\[ \Sigma_q^{(T+1)} = \sum_{t \in [T-1]} \hat{p}^{(T+1)}_t + \hat{p}^{(T+1)}_T + \hat{p}^{(T+1)}_T \]
\[ = \sum_{t \in [T-1]} \hat{p}^{(T+1)}_t + \hat{p}^{(T+1)}_T + \hat{p}^{(T+1)}_T \]
\[ = \sum_{t \in [T-1]} q^{(T-1)}_t + \pi_1^{(T-1)} = 1 + \pi_1^{(T+1)} \]
\[ = \sum_{t \in [T-1]} q^{(T+1)}_t + \pi_1^{(T)} = 1 \] (61)
This proves the induction step.
\textbf{APPENDIX G}

\textbf{PROOF OF Prop. 3.}

\textit{Proof:} For finite \((c, x, y)\),
\begin{equation}
\lim_{n \to \infty} n \left[ \left(1 - \frac{c}{n}\right)^x - \left(1 - \frac{c}{n}\right)^{x+y} \right] = yc. 
\end{equation}

The three claims from Prop. 3 are established in turn.

\textit{Claim i): SSR and SSs.} By Fact 2 and Thm. 1:
\begin{equation}
\frac{\tilde{p}_{\text{SSS}^*}}{\tilde{p}_{\text{SSR}}} = \frac{(n_0^c + n_0^{\pi})s^t + n_0^s}{s^t - (1 - s^t)} \frac{n_0^c}{n} 
= \frac{n_s^t n_0^c + n_s^t n_0^s}{n_s^t n_0^c + n_s^t n_0^s + n_s^0 c^s + n_s^0 c^r}.
\end{equation}

Recall from (10) that \(n_0^c = n - s^t u(n - n_0^c)\). Substitution into (65) and rearranging gives \(\tilde{p}_{\text{SSS}^*}/\tilde{p}_{\text{SSR}} = P(n)/(Q(n) + R(n))\) where, using \(s(n) = c/n\),
\begin{align*}
P(n) &= n \left[ \left(1 - \frac{c}{n}\right)^t - \left(1 - \frac{c}{n}\right)^{n_0^c + t} \right] 
+ n_0^c \left[ \left(1 - \frac{c}{n}\right)^{n_0^c + t} - \left(1 - \frac{c}{n}\right)^t \right] + n_0^s, \\
Q(n) &= n \left[ \left(1 - \frac{c}{n}\right)^t - \left(1 - \frac{c}{n}\right)^{n_0^c + t} + \frac{c}{n} \left[ \left(1 - \frac{c}{n}\right)^{t+1} 
- \left(1 - \frac{c}{n}\right)^{n_0^c + 1} \right] 
- \left(1 - \frac{c}{n}\right) + \left(1 - \frac{c}{n}\right)^{n_0^c + 1} \right] 
\end{align*}
\begin{align*}
R(n) &= n_0^c \left[ \left(1 - \frac{c}{n}\right)^{t+n_0^c} \right] 
+ \frac{c}{n} \left[ \left(1 - \frac{c}{n}\right)^{t+n_0^c} - \left(1 - \frac{c}{n}\right)^{n_0^c + 1} \right] \right]. 
\end{align*}

Given (63) it follows that
\begin{equation}
\lim_{n \to \infty} P(n) = (c + 1)n_0^c, \quad \lim_{n \to \infty} Q(n) = cn_0^c, \quad \lim_{n \to \infty} R(n) = n_0^c. 
\end{equation}

It follows that \(\lim_{n \to \infty} \tilde{p}_{\text{SSS}^*}/\tilde{p}_{\text{SSR}} = \lim_{n \to \infty} P(n)/(Q(n) + R(n)) = 1\). This establishes Claim i).

\textit{Claim ii): SSR and SSs.} Since each SSs sample removes a vertex from \(V_0 \setminus V_{c^r}^e\), it follows that \(p_{\text{SSS}^*} = n_0^c + (n - t - 1)\), and as such, Claim ii) follows directly from Fact 2.

\textit{Claim iii): SSC and SSs.} This follows immediately from Claims i) and ii).

\textbf{APPENDIX H}

\textbf{PROOF OF FACT 6}

\textit{Proof:} Let \(p_t^{\text{SSR}} = n_0^c + n/\) denote the probability of success under SSR, with \(n_0^c = |V_{c^r}^e|\). Recall from Def. 4 that \(c_u = c_u^{\text{SSR}}(G, V^*)\), the random unit cost, denotes the random number of samples until success, and recall from Fact 2 that this quantity has distribution \(\text{geo}(p_t^{\text{SSR}})\). The random linear cost under SSR, denoted \(c_l^{\text{SSR}}(G, V^*)\), equals \(c_l = x_1 + \cdots + x_{e_u}\), where \(x_i\) is the random extended degree of the star center in sample \(t\). The expected linear cost is
\begin{align*}
c_l^{\text{SSR}}(G, V^*) &= \mathbb{E}[c_l^{\text{SSR}}(G, V^*)] 
= \mathbb{E}\mathbb{E}[c_l|c_u] + \mathbb{E}[x_i|c_u] 
= \mathbb{E}[c_l - 1] + \mathbb{E}[x_i|c_u].
\end{align*}
The first term represents the expected linear cost up until but not including the cost of the final sample, while the second term is the expected cost of the final sample. The unsuccessful samples are identically distributed, due to replacement. The expected cost of an unsuccessful sample is \(\bar{c}_l^{\text{SSR}}\) while the expected cost of a successful sample is \(c_l^{\text{SSR}}\). The distribution of the cost of a sample is conditionally independent of the number of samples, conditioned on whether or not the sample is successful or not. As such, \(\mathbb{E}[x_i|c_u] = \bar{c}_l^{\text{SSR}}\), \(\mathbb{E}[x_i|c_u] = c_l^{\text{SSR}}\), and \(\mathbb{E}[c_u] = n/\bar{c}_l^{\text{SSR}}\), yielding (33).

\textbf{APPENDIX I}

\textbf{PROOF OF Prop. 5.}

\textit{Proof:} Consider an urn with \(n\) balls of which \(k\) are marked. Draw \(m\) balls uniformly at random and let \(x\) be the random number of marked balls drawn, with support \(S = \{0, \ldots, n - k\}\). The distribution \(\mathbb{P}(x = t) = \binom{n - k}{t} \binom{m}{t} / \binom{n}{m}\) for \(t \leq S\). The expectation of \(x\) is \(\mathbb{E}[x] = \sum_{t \leq S} \mathbb{P}(x = t) = \frac{km}{n}\). Note this is the same as if the \(m\) balls were drawn with replacement, where \(x \sim \binom{m}{k/n}\).

This expectation is pertinent in the derivation below of the expected degree when sampling an ER random graph using SSC. Pick uniformly at random any vertex in \(G_t\), i.e., a vertex that was not selected in the first \(t\) draws. This vertex has a random initial degree in \(G_0\) of \(d_0 \sim \binom{n - 1, s}\). Let \(x_{t-1}\) be the random number of neighbors that are removed in the first \(t - 1\) samples, so that its random degree in \(G_t\) is \(d_t = d_0 - x_{t-1}\). Then:
\begin{align*}
\mathbb{E}[d_t] &= \mathbb{E}[\mathbb{E}[d_t|d_0]] 
= \mathbb{E}[d_0 - \mathbb{E}[x_{t-1}|d_0]] 
= \mathbb{E}[d_0 - \mathbb{E}[d_0|t - 1]] 
= \left(1 - \frac{t - 1}{n - 1}\right) \mathbb{E}[d_0] 
= \left(1 - \frac{t - 1}{n - 1}\right) (n - 1) s = \frac{(n - 1)s}{n - 1}. \quad (68)
\end{align*}

The proof that \(\mathbb{E}[x_{t-1}|d_0] = \frac{d_0(t-1)}{n-1}\) comes from the discussion above, where the \(n - 1\) “balls” are the potential neighbors.
in $G_0$ of the randomly selected vertex, of which the $d_0$ marked “balls” are the actual neighbors, and $t−1$ “balls” are drawn.

The event $\{c_u \geq t\}$, conditioned on $n_0^{c_u^*}$, means the first $t−1$ samples fail to find the target. As graph $G_u−1$ has order $n−u+1$, conditioned on the failure of each previous sample, sample $u \in [t−1]$ fails with probability $1−n_0^{c_u^*}/(n−u+1)$. As these random outcomes are independent:

$$\mathbb{P}(c_u \geq t|n_0^{c_u^*}) = \prod_{u=1}^{t−1} \left(1 - \frac{n_0^{c_u^*}}{n−u+1}\right).$$

(69)

The maximum number of samples possible under SSC, $c_{SSC}^{max}(n_0^{c_u^*})$, is $n−n_0^{c_u^*}+1$. Substitution into (35) yields (38).

**APPENDIX J**

**PROOF OF PROP. 6.**

Proof: The proof follows the same lines as that of Prop. 5, i.e., we establish $i)$ $\mathbb{E}[d_t]$, $ii)$ $c_{SSS}^{max}(n_0^{c_u^*})$, and $iii)$ $\mathbb{P}(c_u \geq t|n_0^{c_u^*})$.

$i)$ $\mathbb{E}[d_t]$. The expected degree of the star center in sample $t$ is obtained as follows. Consider any vertex not yet removed by the first $t−1$ samples. Conditioned on $n_{t−1}$, this vertex has a random degree $d_t|n_{t−1} \sim \text{bin}(n_{t−1}−1,s)$, and as such

$$\mathbb{E}[d_t] = \mathbb{E}[\mathbb{E}[d_t|n_{t−1}]] = s\mathbb{E}[n_{t−1}−1],$$

(70)

By case $ii)$ (46) in Lem. 5:

$$\mathbb{E}[d_t] = s(\mathbb{E}[n_{t−1}]−1) = s\left(ns^{t−1}−\bar{s}(1−\bar{s}^{t−1})−1\right)$$

$$= ns^{t−1}−\bar{s}(1−\bar{s}^{t−1})−s$$

(71)

and as such

$$\mathbb{E}[d_t^2] = ns^{t−1}−\bar{s}(1−\bar{s}^{t−1})+\bar{s}$$

$$= ns^{t−1}+\bar{s} = ((n−1)s+1)\bar{s}^{t−1}$$

(72)

$ii)$ $c_{SSS}^{max}(n_0^{c_u^*})$. The maximum number of samples before the target set is reached may be upper bounded by $t_p^{(1)}$ the time at which $p_{t−1} = 1$ given in (24) of Thm. 1, i.e. $c_{SSS}^{max}(n_0^{c_u^*}) \leq t_p^{(1)}$.

$iii)$ $\mathbb{P}(c_u \geq t|n_0^{c_u^*})$. The probability of requiring $t$ or more samples under SSS is approximated by leveraging the results in Thm. 2, namely,

$$\mathbb{P}(c_u \geq t|n_0^{c_u^*}) \approx \prod_{u=1}^{t−1}(1−\hat{p}_u^{SSS}),$$

(73)

where $\hat{p}_u^{SSS}$ is defined in (22) in Thm. 1.

As in the case of Prop. 5, (39) follows by substitution. ■