Zero Modes for the Boundary Giant Magnons

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ABSTRACT

We study the fermion zero-mode dynamics for open strings ending on the giant graviton branes. For the open string ending on the $Z = 0$ brane, the quantization of the fermion zero-modes of boundary giant magnons reproduces the 256 states of the boundary degrees with the precise realization of the $SU(2|2) \times SU(2|2)$ symmetry algebra. Also for the open string ending on the $Y = 0$ brane, we reproduce the unique vacuum state from the fermion zero-modes.
1 Introduction

There have been great advances in our understanding of the correspondence between the planar \( \mathcal{N} = 4 \) super-Yang-Mills (SYM) theory and the type IIB strings on the AdS\(_5 \times S^5\) background\([1]\). The integrability plays a crucial role for the check of the correspondence of the string sigma model and the SYM spin chain dynamics\([2, 3, 4, 5, 6]\).

The elementary excitation of the SYM spin chains, which is called a magnon, is composed of 16 states that are organized by \( SU(2|2) \times SU(2|2) \) symmetry. The \( S \) matrix describing the scattering of two magnons is fully determined and becomes the basis of solving the asymptotic spectrum of the SYM spin chain operators\([7, 8, 9]\). In the string sigma model side, the giant magnon solution of the spinning string describes the corresponding elementary excitation\([10]\). In Ref. \([11]\), the dynamics of the fermion zero modes around the giant magnon solution was studied. It was shown that the quantization of these zero modes is precisely reproducing the 16 states of the magnon of the SYM spin chain side.

In Ref. \([12]\), the correspondence between open spin chain in the \( \mathcal{N} = 4 \) SYM theory and the open strings ending on the giant gravitons was proposed. As we shall explain details later on, there are two classes of open SYM spin chain operators: One is the so called open spin chain of the \( Z = 0 \) brane and the other is open spin chain of the \( Y = 0 \) brane. The \( Z = 0 \) brane vacuum involves two boundary degrees, which are localized at the left and the right boundaries respectively. It was shown that each boundary magnon is again organized by the \( SU(2|2) \times SU(2|2) \) symmetry carrying 16 states. Hence there are 256 states in total. On the other hand, the ground state for the open spin chain of the \( Y = 0 \) brane does not involve any boundary states and is characterized by a unique vacuum state.

The reflection amplitudes of bulk magnon on the boundary were constructed up to overall phases in Ref. \([12]\). The overall dressing phases for the \( Y = 0 \) brane and the \( Z = 0 \) brane were later determined respectively in Ref. \([13]\) and Ref. \([14]\). For the related aspects of the open spin chain correspondence, see Refs. \([15]\).

In this note, we shall first construct the finite-size boundary giant graviton solution of the strings ending on the \( Z = 0 \) giant graviton brane. We study the fermion zero mode dynamics of the string sigma model and identify the 256 states that are organized by the \( SU(2|2) \times SU(2|2) \) symmetry. For the strings ending on the \( Y = 0 \) brane, we also construct the finite-size vacuum solution and show that the quantization of the fermion zero mode leads to the unique vacuum state of no boundary degrees.
2 Boundary states

In Ref. [12], it was proposed that a magnon in a class of open spin chain in $\mathcal{N} = 4$ super Yang-Mills theory is corresponding to a configuration of open string ending on a giant graviton. We shall briefly review the relevant part of this proposal here.

The proposal is an open spin chain and open string version of the the closed string giant magnon dynamics [10]. There are two types of giant gravitons that allow BPS ground-state configuration. If the giant graviton is located at $Y = 0$ or $Z = 0$ hyper surfaces inside $S^5$, they are called $Y = 0$ and $Z = 0$ branes, respectively. We choose the open string vacuum oriented along $Z$-direction. We see that the open string can end on the $Y = 0$ brane with Neumann boundary condition. The open string can also end on the $Z = 0$ brane with Dirichlet boundary condition; An additional localized boundary degree is necessary at each boundary of $Z = 0$.

In the $\mathcal{N} = 4$ SYM theory side, the $Y = 0$ brane open spin chain is represented by composite operators containing a determinant factor $\text{det}(Y)$:

$$O_Y = \varepsilon^{j_1\ldots j_{N-1}A}_{i_1\ldots i_{N-1}B} Y_{j_1}^i \ldots Y_{j_{N-1}}^i (Z \ldots Z \chi_1 Z \ldots Z \chi_2 Z \ldots Z)^B_A,$$

where $\chi_1, \chi_2, \ldots$ represent other SYM fields. The other is the $Z = 0$ brane open spin chain, represented by composite SYM operators containing a determinant factor $\text{det}(Z)$:

$$O_Z = \varepsilon^{j_1\ldots j_{N-1}A}_{i_1\ldots i_{N-1}B} Z_{j_1}^i \ldots Z_{j_{N-1}}^i (\chi_L Z \ldots Z \chi_1 Z \ldots Z \chi_2 Z \ldots \chi_R)^B_A. \tag{2.2}$$

An important difference of the $Z = 0$ brane from the $Y = 0$ brane is that the open SYM spin chain is connected to the giant graviton through boundary impurities $\chi_L$ and $\chi_R$. In this note, we are mainly interested in the ground states where the bulk magnon excitation $\chi_1, \chi_2, \ldots$ are absent.

It is clear that the ground state for the $Y = 0$ brane is described by a unique state because there are no boundary degrees. On the other hand, the $Z = 0$ brane involves the multiplet of left and right boundary states due to the presence of the boundary excitations. Each boundary state is organized by the $SU(2|2)^2$ representation. The elementary boundary magnon involves 16 degenerate states with the energy spectrum,

$$E_B = \sqrt{1 + 4g^2}, \tag{2.3}$$

where $g$ is related to the t’Hooft coupling by $g = \sqrt{\lambda}/(4\pi)$. Each $SU(2|2)$ algebra consists of the $SU(2) \times SU(2)$ rotation generators $\mathfrak{R}_a^b$, $\mathfrak{L}^\alpha_\beta$, the supersymmetry generators $\Omega_\alpha^a$ and $\mathcal{S}_a^\alpha$ and the central charge $\mathcal{C}[8]$.

Their commutators are given by[8]

$$[\mathfrak{R}_a^b, \mathfrak{J}^c] = \delta_a^b \mathfrak{J}^c - \frac{1}{2} \delta_a^c \mathfrak{J}^c, \quad [\mathfrak{L}^\alpha_\beta, \mathfrak{J}^\gamma] = \delta_\beta^\gamma \mathfrak{J}^\alpha - \frac{1}{2} \delta_\beta^\alpha \mathfrak{J}^\gamma$$

$$\{\Omega_\alpha^a, \mathcal{S}_b^\alpha\} = \delta_a^b \Omega_\alpha^a + \delta_\beta^a \Omega_\beta^b + \delta_\alpha^b \mathcal{S}_a^\alpha \mathcal{C}. \tag{2.4}$$
The central element $\mathcal{C}$ is related to the energy by $E_B = 2\mathcal{C}$. There are further central extensions,

$$\{\Omega^\alpha_a, \Omega^\beta_b\} = \varepsilon^{ab} \varepsilon_{ab} \frac{k}{2}, \quad \{\mathcal{S}^a_a, \mathcal{S}^b_b\} = \varepsilon_{ab} \varepsilon^{ab} k^*,$$

(2.5)

For the construction of the boundary states [12], we first represent the $SU(2|2)$ acting on the $2\mid 2$ space. We label the bosonic states by $|\phi^a\rangle$ and the fermionic states by $|\psi^\alpha\rangle$. Then the generators are acting on the states by

$$\mathcal{R}^a_a|\phi^c\rangle = \delta^a_c|\phi^a\rangle - \frac{1}{2} \delta^a_b|\phi^c\rangle, \quad \mathcal{S}^a_a|\phi^c\rangle = \delta^a_b|\phi^a\rangle - \frac{1}{2} \delta^a_c|\phi^b\rangle$$

(2.6)

and by

$$\mathcal{S}^a_a|\phi^b\rangle = a_B \delta^b_a|\psi^\alpha\rangle, \quad \mathcal{R}^a_a|\psi^b\rangle = b_B \varepsilon^{ab} \varepsilon_{ab} |\phi^b\rangle$$

$$\mathcal{S}^a_a|\psi^b\rangle = c_B \varepsilon_{ab} \varepsilon^{ab} |\psi^b\rangle, \quad \mathcal{S}^a_a|\psi^b\rangle = d_B \delta^b_a|\phi^a\rangle.$$  

(2.7)

The condition

$$a_B d_B - b_B c_B = 1$$

(2.8)

is necessary for the closure of the algebra. One finds also that $k/2 = a_B b_B, k^*/2 = c_B d_B$ and the energy $E_B = a_B d_B + b_B c_B$. From the string theory picture explained in [12], we assume that

$$|k|^2 = 4g^2.$$  

(2.9)

Then $a_B, b_B, c_B$ and $d_B$ are in general parametrized by

$$a_B = \sqrt{g} \eta_B, \quad b_B = \frac{\sqrt{g} f_B}{\eta_B}$$

$$c_B = i \frac{\sqrt{g} x_B}{f_B}, \quad d_B = \frac{\sqrt{g} x_B}{\eta_B}.$$  

(2.10)

The unitarity demands that $f_B$ should be a pure phase with $|\eta_B|^2 = -ix_B$. The shortening condition, $a_B d_B - b_B c_B = 1$, implies

$$x_B + \frac{1}{x_B} = \frac{i}{g}, \quad x_B = \frac{i}{2g} \left(1 + \sqrt{1 + 4g^2}\right).$$  

(2.11)

and we recover (2.3) with $E_B = \frac{g}{4}(x_B - x_B^{-1})$.

Let us denote a representation of one $SU(2|2)$ by $|q_L\rangle = (|\phi^a\rangle, |\psi^\alpha\rangle)$ with $L = 1, 2, 3, 4$ and $a, \alpha = 1, 2$. Then the representation of $SU(2|2) \times SU(2|2)$ is given by the tensor product $|q_L\rangle \otimes |q_M\rangle$ corresponding to the sixteen states. Therefore there are total 256 states if the both boundary magnons are elementary.
3 Open string description of boundary giant magnons

The main purpose of this note is to reproduce the above ground state degeneracy by studying the open string zero mode dynamics.

For this purpose, we shall first describe the finite-size string ending on the $Z = 0$ or the $Y = 0$ giant gravitons. Since we are interested in the boundary degrees, we focus on the strings without turning on bulk magnon excitations.

To get the string configuration, we first find the classical solution of boundary magnons with finite size $J$. We begin with the bosonic part of the string action in the conformal gauge,

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau \int_0^{2\pi} d\sigma \, \partial_a X_I \partial^a X_I$$

with the constraint $X_I X_I = 1$ ($I = 1, 2, \cdots, 6$). We have set the AdS radial coordinate to zero since we are interested in the string moving in $S^5$ while staying at the center of the AdS$_5$. We shall work in the gauge, $T = \tau$, where $T$ is the global AdS time. In this set-up, the Virasoro constraints

$$(X_I \pm X'_J)(X_I \pm X'_J) = 1,$$  

have to be imposed in addition.

The energy density is uniform in the static gauge and the string energy is proportional to the spatial coordinate size:

$$E = \frac{\sqrt{\lambda}}{2\pi} 2r.$$  

For the description of the $Z = 0$ or the $Y = 0$ boundary states, we turn on only $X_1, X_2$ and $X_3$ and use the coordinates $Z = X_1 + iX_2 = \sqrt{1 - z^2} \, e^{i\phi}$ and $X_3 = z$.

3.1 String ending on the $Z = 0$ brane

For the $Z = 0$ brane, we begin with an ansatz ($\omega \geq 1$),

$$z = z(\sigma - v\omega \tau), \quad \phi = \omega \tau + \varphi(\sigma - v\omega \tau).$$

The equations of motion are reduced to

$$(z')^2 = \frac{\omega^2}{(1 - v^2 \omega^2)^2} \left( z^2 - 1 + \frac{1}{\omega^2} \right) \left( 1 - v^2 - z^2 \right)$$

$$\varphi' = \frac{v \omega^2}{(1 - v^2 \omega^2)} \frac{z^2 - 1 + \frac{1}{\omega^2}}{1 - z^2}.$$  

\footnote{This solution is first found in the Ref. \cite{16}. The following is the review of the solution.}
The general solution can be found as \[17\]

\[
z = \frac{\sqrt{1 - v^2}}{\omega \sqrt{\eta}} \text{dn} \left( \frac{\sigma - \nu \tau}{\sqrt{\eta \sqrt{1 - v^2}}}, \eta \right)
\] (3.6)

where \(\text{dn}(x, k^2)\) is the Jacobi elliptic function and we introduce the parameter \(\eta\) by

\[
\eta = \frac{1 - \omega^2 \nu^2}{\omega^2 (1 - v^2)}.
\] (3.7)

Since we are only interested in the boundary degrees which are not in motion, we set \(\nu = 0\). The solution then becomes

\[
z = \text{dn} \left( \omega (\sigma - \sigma_0), \frac{1}{\omega^2} \right),
\]

\[
\phi = \omega \tau.
\] (3.8)

Or in terms of \(\sin \theta \equiv \sqrt{1 - z^2}\), the solution is

\[
\sin \theta = \frac{1}{\omega} \text{sn} \left( \omega (\sigma - \sigma_0), \frac{1}{\omega^2} \right).
\] (3.9)

For the string ending on the \(Z = 0\) branes, the Dirichlet boundary condition \(\dot{Z} = 0\) at \(Z = 0\) will be imposed for the open string boundaries, \(\sigma = 0\) and \(\sigma = 2r\). The \(\sigma = 0\) boundary condition can be satisfied by setting \(\sigma_0 = 0\). The remaining boundary condition at \(\sigma = 2r\) is satisfied by the choice \(\omega r = K(k)\) with \(k = \frac{1}{\omega}\) where the complete elliptic integrals \(K(k)\) and \(E(k)\) are defined by

\[
K(k) = \int_0^1 dx \frac{1}{\sqrt{1 - x^2} \sqrt{1 - k^2 x^2}}
\]

\[
E(k) = \int_0^1 dx \sqrt{1 - k^2 x^2} \sqrt{1 - x^2}.
\] (3.10)

The other choice \(\omega r = mK(k)\) (\(m \in \mathbb{Z}\)) with \(m \geq 2\) is possible but it simply describes the multiple open strings.

The angular momentum on the \(1 - 2\) plane is given by

\[
J = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2r} d\sigma (1 - z^2) \dot{\phi}.
\] (3.11)

Hence the combination \(E - J\) can be expanded to the leading correction for the large \(J\) by

\[
E - J = \frac{\sqrt{\lambda}}{\pi} \left( 1 - 4 e^2 e^{-\frac{2\nu}{\sqrt{\lambda}}} + \cdots \right) = 4g \left( 1 - 4 e^2 e^{-\frac{J}{2g}} + \cdots \right).
\] (3.12)
The energy and correction is doubled here because we add up the energy of the left and the right boundary together. Thus one boundary energy and correction is just one half of the above $E - J$, which precisely reproduces the classical part of the energy of (2.3). From the viewpoint of classical string, the construction of the boundary states requires the study of fermion zero modes. This will lead to the 16 boundary states for each boundary, so there are 256 combinations of boundary states in total as we shall see later on. Since this construction is independent of the above finite size correction, we conclude that, for any combination of boundary states of the left and right boundaries, the finite energy correction remains the same.

3.2 Strings ending on the $Y = 0$ brane

In this section, we describe the strings ending on the $Y = 0$ brane.

We are interested in the open strings moving on the $Z$ space. Hence at the open string boundaries, one has to satisfy the Neumann boundary condition $Z' = 0$ since $Z$ is now parallel to the worldvolume of the brane. The action and the equation of motion are the same as (3.1) and (3.5). The trivial solution,

$$z = \sqrt{1 - \frac{1}{\omega^2}}, \quad \phi = \omega \tau,$$

(3.13)

satisfies the necessary boundary condition. There is no restriction of $r$ and $\omega \geq 1$. The energy and angular momentum are given by

$$E = 4gr, \quad J = \frac{4gr}{\omega}.$$  

(3.14)

When $\omega = 1$, one has

$$E_B = E - J = 0,$$

(3.15)

which corresponds to the ground state. It describes a point-like open string carrying finite angular momentum $J$ moving along the equator.\footnote{The size in the target space is point-like here. The finite-size means a finite $J$ corresponding to the finite R-charge of SYM spin chain.} It is clear that there is no finite size correction to the energy at least classically. This is also quite consistent with the fact the $Y = 0$ open string does not involve any boundary degrees.

4 Fermion zero modes

In this section we consider the fermion zero modes of the strings around the solutions constructed in the previous section. For this we begin with the fermionic part of the string action to
the quadratic order\[18],

\[ I_F = 2g \int d\tau d\sigma \, L_F , \]  

(4.1)

with

\[ L_F = i(\eta^{ab}\delta_{IJ} - \epsilon^{ab} s_{IJ})\hat{\theta}^I \rho_a D_b \theta^J. \]  

(4.2)

Here \( I \) and \( J \) run over 1, 2 and \( s_{IJ} \) is diagonal with \( s_{11} = -s_{22} = 1 \). \( \rho_a \) is the world sheet gamma matrix defined by

\[ \rho_a = \Gamma_A e^A_a = \Gamma_A E^A_{\mu} \partial_a X^\mu \]  

(4.3)

where \( \Gamma_A \) and \( E^A_{\mu} \) are respectively the 10d gamma matrices, which are taken to be real, and the einbein. \( \theta^I \) denotes 16 component Majorana spinor. The covariant derivative is defined as

\[ D_a \theta^I = (\delta^I_J D_a - \frac{i}{2} \epsilon^{IJK} \Gamma_a \rho_a)\theta^J \]  

(4.4)

where

\[ D_a = \partial_a + \frac{1}{4} \omega_{AB}^a \partial_a X^\mu \Gamma_A B , \quad \Gamma_* = i \Gamma_{01234} . \]  

(4.5)

The equations of motion take the form,

\[ (\rho_0 - \rho_1)(D_0 + D_1)\theta^1 = 0 , \]

\[ (\rho_0 + \rho_1)(D_0 - D_1)\theta^2 = 0 . \]  

(4.6)

Let us now work out how the equations look in the background we consider. We note that we turn on only \( \theta \) and \( \phi \) components. The relevant nonvanishing component of the spin connection \( \omega^{AB}_\mu \) is

\[ \omega^{\delta \theta} = \cos \theta , \]  

(4.7)

and

\[ \rho_0 = \Gamma_0 + \omega \sin \theta \Gamma_\phi , \quad \rho_1 = \phi' \sin \theta \Gamma_\phi + \theta' \Gamma_\theta = \theta' \Gamma_\theta , \]  

(4.8)

where we have used \( \dot{\theta} = 0 , \dot{\phi} = \omega \) and \( \phi' = 0 \). Therefore the equations become

\[ (\rho_0 - \rho_1) \left[ \partial_t \theta^1 + D \theta^1 - \frac{i}{2} \Gamma_* (\rho_0 + \rho_1) \theta^2 \right] = 0 \]

\[ (\rho_0 + \rho_1) \left[ - \partial_t \theta^2 + \bar{D} \theta^2 - \frac{i}{2} \Gamma_* (\rho_0 - \rho_1) \theta^1 \right] = 0 , \]  

(4.9)
where we introduce
\[ D = \partial_\sigma + \frac{\omega \cos \theta}{2} \Gamma_{\phi \theta}, \quad \bar{D} = \partial_\sigma - \frac{\omega \cos \theta}{2} \Gamma_{\phi \theta}. \tag{4.10} \]

These equations are further rewritten as
\[
\begin{align*}
(\partial_t + D)\psi^1 - \frac{i}{2} [\rho_0, \Gamma_{\phi \theta}]\psi^2 &= (\partial_t + D)\psi^1 + i \omega \sin \theta \Gamma_{\phi \theta} \psi^2 = 0, \\
(-\partial_t + \bar{D})\psi^2 - \frac{i}{2} [\rho_0, \Gamma_{\phi \theta}]\psi^1 &= (-\partial_t + \bar{D})\psi^2 + i \omega \sin \theta \Gamma_{\phi \theta} \psi^1 = 0, 
\end{align*} \tag{4.11} \]

where we introduced new spinors \( \psi^I \) defined by
\[
\psi^1 = i(\rho_0 - \rho_1) \theta^1, \quad \psi^2 = i(\rho_0 + \rho_1) \theta^2, \tag{4.12} \]
and used the relations
\[
[\rho_0, \Gamma_{\phi \theta}] = -2 \omega \sin \theta \Gamma_{\phi \theta}, \tag{4.13} \]
\[
[\rho_0 - \rho_1, D] = [\rho_0 + \rho_1, \bar{D}] = 0. \tag{4.14} \]

The boundary contributions of the variation of the action should vanish, which leads to the condition,
\[
\left[ \bar{\theta}^1 (\rho_0 - \rho_1) \delta \theta^1 - \bar{\theta}^2 (\rho_0 + \rho_1) \delta \theta^2 \right]_{\text{boundary}} = 0. \tag{4.15} \]

Finally for the zero mode, the equations are reduced to
\[
\begin{align*}
D\psi^1 + i \omega \sin \theta \Gamma_{\phi \theta} \psi^2 &= 0, \\
\bar{D}\psi^2 + i \omega \sin \theta \Gamma_{\phi \theta} \psi^1 &= 0. \tag{4.16} \end{align*} \]

By eliminating \( \psi^2 \), one gets
\[
\left[ \left( \frac{1}{\omega \sin \theta} D \right)^2 - 1 \right] \psi^1 = 0, \tag{4.17} \]
which is equivalent to two first-order equations,
\[
D\psi^1 = \mp \omega \sin \theta \psi^1. \tag{4.18} \]

Then \( \psi^2 \) is given by
\[
\psi^2 = \pm i \Gamma_{\phi \theta} \psi^1. \tag{4.19} \]

One comment is that the boundary condition \((4.15)\) is satisfied automatically since the boundary term vanishes with the relations \((4.19)\). These equations will be the starting point of our analysis of the fermion zero modes.
4.1 Zero modes for the $Z = 0$ boundary

For the $Z = 0$ boundary, we use $\cos \theta = \text{dn}(\omega \sigma)$ and $\sin \theta = k \text{sn}(\omega \sigma)$. The solution of (4.18) can be found as

$$
\psi_{\pm}^1 = iN(k)(\rho_0 - \rho_1)[\text{dn}(\omega \sigma) \pm k \text{cn}(\omega \sigma)][\text{sn}(\omega \sigma) + \text{cn}(\omega \sigma)\Gamma_{\theta}]^\frac{1}{2} U_{\pm}
$$

$$
= iN(k)[\text{dn}(\omega \sigma) \pm k \text{cn}(\omega \sigma)][\text{sn}(\omega \sigma) + \text{cn}(\omega \sigma)\Gamma_{\theta}]^\frac{1}{2}[\Gamma_0 + \Gamma_{\phi}] U_{\pm},
$$

(4.20)

where $U_{\pm}$ is a constant Majorana spinor and the normalization factor $N(k)$ defined by

$$
N^2(k) = \frac{1}{2k(1+k)E\left(\frac{2\sqrt{k}}{1+k}\right)}
$$

(4.21)

is introduced for the normalization. Note that $D\psi_{\pm}^1|_{\text{boundary}} = \bar{D}\psi_{\pm}^2|_{\text{boundary}} = 0$ and we shall require these as extra boundary conditions for the $Z = 0$ and the $Y = 0$ branes.

The solutions $\psi_{\pm}^I$ and $\psi_{\mp}^I$ have a maximum at $\sigma = 0$ and $\sigma = 2kK(k)$ respectively. Hence $\psi_{+}^I$ is concentrated on the left boundary $\sigma = 0$ while $\psi_{-}^I$ is concentrated on the right boundary $\sigma = 2kK(k)$. Therefore $\psi_{+}^I$ and $\psi_{-}^I$ can be viewed as describing the left and the right boundary degrees respectively. This becomes clear if the string length $2kK(k)$ becomes infinite as $k \to 1$.

For $k \to 1$, the Jacobi elliptic functions become

$$
\text{sn}(\omega \sigma) \to \tanh \sigma
$$

$$
\text{dn}(\omega \sigma), \text{cn}(\omega \sigma) \to 1/\cosh \sigma.
$$

(4.22)

Also for $k \to 1$, one has

$$
\text{sn}(\omega \sigma) \to \tanh(2r - \sigma)
$$

$$
\text{dn}(\omega \sigma), -\text{cn}(\omega \sigma) \to 1/\cosh(2r - \sigma),
$$

(4.23)

when $2r - \sigma$ is finite. In this limit, one can check that the overlap of $\psi_{+}^I$ and $\psi_{-}^I$ disappear completely.

The full solution is given by

$$
\psi^1 = \psi_{+}^1 + \psi_{-}^1, \quad \psi^2 = i\Gamma_\ast \Gamma_{\phi}(\psi_{+}^1 - \psi_{-}^1).
$$

(4.24)

The effective action for the zero mode dynamics can be obtained by giving the time dependence to the zero mode fermion coordinates as $U_{\pm}(\tau)$; This leads to

$$
I_{\text{zero}} = 2g \int d\tau \left[ iU_{+}^T(\Gamma_0 + \Gamma_{\phi})^\dagger(\Gamma_0 + \Gamma_{\phi}) U_{+} + iU_{-}^T(\Gamma_0 + \Gamma_{\phi})^\dagger(\Gamma_0 + \Gamma_{\phi}) U_{-} \right],
$$

(4.25)
where the cross terms from $\psi^1$ and $\psi^2$ are cancelling with each other. With further definitions,

$$U_L = (\Gamma_0 + \Gamma_\phi) U_+, \quad U_R = (\Gamma_0 + \Gamma_\phi) U_-,$$

the action for the zero mode dynamics becomes

$$I_{\text{zero}} = 2g \int d\tau \left[ i U_L^T \dot{U}_L + i U_R^T \dot{U}_R \right].$$

It is clear that the left and the right degrees behave independently. Let us consider the dynamics of the left boundary first. Among the 16 real components $U_L$, the light-con condition $(\Gamma_0 + \Gamma_\phi) U_L = 0$ projects down by half and only 8 real degrees remain. We organize this in terms of the bispinor components $U_{\alpha a}$ and $\bar{U}_{\dot{\alpha} \dot{a}}$ where $\alpha$ and $\dot{\alpha}$ are the spinor indices for $SO(4) \simeq SU(2) \times SU(2)$ isometry in the transverse part of $S^5$\cite{11}. The quantization leads to the anticommutation relations

$$\{U_{\alpha a}, U_{\beta b}\} = \frac{1}{2g} \epsilon_{\alpha\beta} \epsilon_{ab}$$

$$\{\bar{U}_{\dot{\alpha} \dot{a}}, \bar{U}_{\dot{\beta} \dot{b}}\} = \frac{1}{2g} \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{a}\dot{b}}$$

$$\{U_{\alpha a}, \bar{U}_{\dot{\beta} \dot{b}}\} = 0.$$  \hspace{1cm} (4.28)

Then the Hilbert space and operators are realized as

$$U_{\alpha a} | \phi_b \rangle = \frac{1}{\sqrt{2g}} \epsilon_{ba} | \psi_\alpha \rangle$$

$$U_{\alpha a} | \psi_\beta \rangle = \frac{1}{\sqrt{2g}} \epsilon_{\alpha\beta} | \phi_a \rangle$$

up to the freedom of the usual unitary transformations. Then the operators $\Omega_{\alpha a}$ and $\mathcal{S}_{\alpha a}$ are realized as

$$\Omega_{\alpha a} = \sqrt{\frac{g}{2}} \left[ (a_B + b_B) - (a_B - b_B)(-)^F \right] U_{\alpha a}$$

$$\mathcal{S}_{\alpha a} = \sqrt{\frac{g}{2}} \left[ (c_B + d_B) - (c_B - d_B)(-)^F \right] U_{\alpha a}$$

(4.30)

where $F$ is the fermion number operator with $(-)^F U_{\alpha a} + U_{\alpha a} (-)^F = 0$.

For the $\bar{U}$, one has the same construction and there are consequently altogether 16 states for the left boundary. For the right boundary, one has 16 states too. Therefore the 256 states with the precise algebra are constructed, which is matching with the Yang-Mills theory side.
4.2 Zero modes for the $Y = 0$ boundary

For the $Y = 0$ boundary, we use the vacuum solution, $\cos \theta = 1$ and $\phi = \tau$. The equations in (4.18) become

$$\frac{d}{d\sigma} \psi^1 = \pm \psi^1.$$  \hspace{1cm} (4.31)

Then the most general solution is

$$\psi^1 = i (\Gamma_0 + \Gamma_\phi) \left[ e^{-\sigma} U_+ + e^{-(2r-\sigma)} U_- \right].$$  \hspace{1cm} (4.32)

However there is no way to satisfy the boundary conditions $D\psi^1\big|_{\text{boundary}} = \bar{D}\psi^2\big|_{\text{boundary}} = 0$. Hence there are no fermion zero modes. Therefore the corresponding state is simply describing the unique vacuum, which is again consistent with the SYM theory construction of the state.

5 Discussions

In this paper, we studied the fermion zero mode problem for the open strings ending on the $Z = 0$ or the $Y = 0$ branes. For the open string ending on the $Z = 0$ brane, the fermion zero mode dynamics reproduces the 256 states of the boundary degrees with the precise realization of the $SU(2|2) \times SU(2|2)$ symmetry algebra. Also for the open string ending on the $Y = 0$ brane, we reproduce the unique vacuum state by studying its fermion zero-mode dynamics.

Recently there appeared the proposal of the correspondence between the $\mathcal{N} = 6$ super Chern-Simons theory and the strings on the $\text{AdS}_4 \times \mathbb{CP}^3$ [19]. The integrability of the planar two loop integrability is checked in Refs. [20, 21] with further developments [22]. The string sigma model, its giant magnon solutions and some related issue are also studied in Refs. [23, 24, 25]. However we still lack some direct information on the states of the elementary magnon of this Chern-Simons spin chains. In this sense, the identification of the states with symmetry algebra arising from the fermion zero-mode dynamics is of much interest.

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3 The Dirichlet type boundary condition $\psi^i(0) = \psi^i(2r) = 0$ does not allow any zero modes either. One needs a further clarification of the required boundary conditions. This is an interesting problem but beyond the scope of this paper. The author would like to thank A. Tseytlin for his comment on this point.
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