Parameterized Norm and Parameterized Fixed-Point Theorem by Using Fuzzy Soft Set Theory

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Abstract: From last decade, when Molodtsov introduced the theory of soft set as a new approach to deal with uncertainties, until now this theory was considered sharply by a fair number of researchers. Combination of fuzzy set theory and soft set theory, called fuzzy soft set theory, by Maji et.al opened a new way for researchers whose framework of study is soft sets and fuzzy sets. Although published papers in this area have considered both application and theoretical aspects of fuzzy soft set theory, the concept of norm for a fuzzy soft set has not been studied yet. We begin this paper by introducing fuzzy soft real numbers, which are needed for study fuzzy soft norm, and then continued by considering fuzzy soft norm. Fixed-point theorem is also investigated for fuzzy soft normed spaces.

Key–Words: soft set, fuzzy soft set, fuzzy soft topology, fuzzy soft point, fuzzy soft real number, fuzzy soft norm, fuzzy soft continuous map, fuzzy soft fixed-point theorem

1 Introduction

We live in the world of uncertainties, where most of the problems which we face are vague rather than precise. These vague concepts are due to limited knowledge or incomplete information. During past decades different mathematical theories like probability theory, fuzzy set theory [15], and rough set theory [27] were introduced to deal with various types of uncertainties. Although a wide range of problems can be solved by these methods, some difficulties have been remained.

Lack of parameterization tools in all previous theories, led to introduce soft set theory by Molodtsove [1] in 1999. A soft set is in fact a set-valued map which gives an approximation description of objects under consideration based on some parameters. Hence the set of all soft sets over a universal set can be considered as a function space. Since then, Maji et.al [2, 3] discussed theoretical aspects and practical applications of soft sets in decision making problems. For more details about soft set theory see [2, 5, 6, 7, 8]. But since almost all the time in real life situations we have inexact information about considered objects, in 2002, Maji et.al [4] studied the combination of fuzzy set theory and soft set theory and gave a new concept called fuzzy soft set. This new notion expanded the concept of soft set from crisp cases to fuzzy cases and then in [5], they applied this new method to solve decision making problems. Kharal and Ahmad in [25] studied some properties of fuzzy soft set and then in [26] defined the concept of mapping on fuzzy soft classes. Topological studies of fuzzy soft sets was started by Tanay and Kandemir in [10]. Mahanta and Das [11] continued working on fuzzy soft topology and studied separation axioms and connectedness of fuzzy soft topological spaces. Simsekler and Yuksel [12] modified the definition of fuzzy soft topology and also gave the concept of soft quasi neighborhood of a fuzzy soft point. Roy and Samanta [9] gave the new definition of fuzzy soft topology and proposed the concept of base and subbase for this new space.

Although different aspects of fuzzy soft set theory has been studied by several authors, fuzzy soft number and fuzzy soft norm have not been investigated yet. In real life situations, we usually use some phrases or sentences like "distance between A and B is about 10 kilometers" or "distance between A and B is around 20 minutes". Such vague expressions are used when we deal with quantities which we do not know their values precisely. Moreover, these inexact values are usually depend on some parameters. For instance, distance between A and B can be changed based on selected measure. In addition, it can be presented longer or shorter with regards to chosen path or time of movement. Whereas in classic mathematics distance between two objects in a normed space is supposed constant, in real life it is not a fixed concept. To solve such problems a parameterization version of numbers and norm shall be needed.

In this paper, we first recall the definition of fuzzy soft topology which has been established in [9] and then answer to the natural question "is there any relation between point-set topology, fuzzy topology and fuzzy soft topology over a common universe?". We continue our work by introducing the concept of fuzzy soft real numbers and then initiate...
fuzzy soft norm over a set. We also consider the relationship between fuzzy soft norm and fuzzy norm over a common set. Finally, we will consider fuzzy soft fixed-point theorem in fuzzy soft normed spaces.

2 Preliminaries

Let \( X \) be the set of all objects and \( E \) be the set of all parameters. Let \( A \subseteq E \), and \( 2^X \) and \( I^X \), where \( I = [0, 1] \), denote the set of all subsets and fuzzy subsets of \( X \), respectively. Molodtsov [1] introduced the concept of soft set as follow:

**Definition 2.1**: (1) A pair \((F, A)\) is called a soft set over \( X \) if \( F \) is a mapping given by \( F : A \rightarrow 2^X \) such that for all \( e \in A \), \( F(e) \subseteq X \).

The soft set \((F, A)\) can be denoted by \( F_A \).

Maji et. al [4] gave the concept of fuzzy soft set as be-

**Definition 2.2**: (2) A pair \((f, A)\) is called a fuzzy soft set over \( X \), FS set briefly, if \( f \) is a mapping given by \( f : A \rightarrow I^X \). So \( \forall e \in A \), \( f(e) \) is a fuzzy subset of \( X \), with membership function

\[ f_e : X \rightarrow [0, 1] \]

In fact, the membership function \( f_e \) indicates degree of belongingness of each element of \( X \) in \( f(e) \) or shows how much each member of \( X \) has the parameter \( e \in E \). So the soft set \((f, A)\) can be represented by the set of triplet ordered

\[ \{(e, x, f_e(x)) : e \in A, x \in X, f(e) \in I^X \} \]

where \( f_e(x) \) is degree of membership \( x \) in fuzzy set \( f(e) \).

The fuzzy soft set \((f, A)\) can be denoted by \( F_A \).

**Example 2.1** In geography science, definition of a forest is expressed based on some parameters such as domain and density of vegetation, type of vegetation, soil types, amount of annual rain, and species of plant and animal. Let \( X = \{A, B, C\} \) be a set of regions under consideration. Suppose that \( E = \{e_1, e_2, e_3, e_4\} \) is a set of parameters where \( e_i \) \( (i = 1, \ldots, 4) \) stand for the parameters: wide, dense vegetation, rainy, variety of plant and animal, respectively. The available information of these regions can be presented by the fuzzy soft set \( F_E \) as follow:

\[
F_E = \{(e_1, \left\{\frac{0.8}{A}, \frac{0.3}{B}, \frac{0.5}{C}\right\}), (e_2, \left\{\frac{0.1}{A}, \frac{0.5}{B}, \frac{0.7}{C}\right\}), \\
(e_3, \left\{\frac{0.2}{A}, \frac{0.3}{B}, \frac{0.8}{C}\right\}), (e_4, \left\{\frac{0.1}{A}, \frac{0.3}{B}, \frac{0.5}{C}\right\}) \}
\]

which means that \( A \) is the largest zone among these three areas, while it has the lowest amount of vegetation, annual rain, and diversity of animal and plant species.

**Definition 2.3**: (3) (Rules of fuzzy soft set) For two fuzzy soft sets \( F_A \) and \( g_B \) over the common universe \( X \) with respect to parameter set \( E \) where \( A, B \subseteq E \) we have,

i. \( F_A \) is a fuzzy soft subset of \( g_B \) shown by \( F_A \preceq g_B \) if:

1. \( A \subseteq B \),
2. For all \( e \in A \), \( f_e(x) \leq g_e(x), \forall x \in X \).

ii. \( F_A \preceq g_B \) if \( F_A \preceq g_B \) and \( g_B \preceq f_A \).

iii. The complement of fuzzy soft set \( F_A \) is denoted by \( f_A^c \) where \( f_A^c : A \rightarrow I^X \) and \( f_A^c(e) \) is the complement of fuzzy set \( f(e) \), i.e. \( f_A^c(e) = (f(e))^c \), with membership function \( f_A^c(e) = 1 - f_e(e), \forall e \in A \).

iv. \( F_A = \Phi_A \) (null fuzzy soft set with respect to \( A \)), if for each \( e \in A \), \( f_e(x) = 0, \forall x \in X \).

v. \( F_A = \tilde{X}_A \) (absolute fuzzy soft set with respect to \( A \)), if \( \forall e \in A \), \( f_e(x) = 1, \forall x \in X \).

If \( A = E \), the null and absolute fuzzy soft set is denoted by \( \Phi \) and \( \tilde{X} \), respectively.

vi. The union of two fuzzy soft sets \( F_A \) and \( g_B \), denoted by \( F_A \vee g_B \), is the fuzzy soft set \( (f \vee g)_C \) where \( C = A \cup B \) and \( \forall e \in C \), we have \((f \vee g)(e) = f(e) \vee g(e)\) where

\[
(f \vee g)_e(x) = \begin{cases} 
  f_e(x) & \text{if } e \in A - B \\
  g_e(x) & \text{if } e \in B - A \\
  \max\{f_e(x), g_e(x)\} & \text{if } e \in A \cap B 
\end{cases}
\]

for all \( x \in X \).

vii. The intersection of two fuzzy soft sets \( F_A \) and \( g_B \), denoted by \( F_A \wedge g_B \), is the fuzzy soft set \( (f \wedge g)_C \) where \( C = A \cap B \) and \( \forall e \in C \), we have \((f \wedge g)(e) = f(e) \wedge g(e)\) where \((f \wedge g)_e(x) = \min\{f_e(x), g_e(x)\}\) for all \( x \in X \).

In (vii), \( A \cap B \) must be nonempty to avoid the degenerate case.

Note that during this paper, \( X \) and \( E \) are used to show the universal sets of objects and parameters, respectively. \( F_A \) denotes a fuzzy soft set over \( X \) where \( A \subseteq E \) and \( X_E \) denotes the set of all fuzzy soft sets over \( X \) with regards to parameter set \( E \).

**Proposition 2.1** (3) (De Morgan Laws) Let \( F_E \) and \( g_E \) be FS sets over \( X \) with respect to \( E \). Then we have

1. \( F_E \vee g_E \) \( \preceq \) \( F_E \preceq g_E \)
2. \( F_E \wedge g_E \) \( \preceq \) \( F_E \preceq g_E \)

**Proof.** See [3].
Definition 2.4: Let $X$ and $Y$ be universal sets, and $E$ and $E'$ be corresponding parameter sets, respectively. Suppose that $f_A$ be a F.S set on $X$ and $g_B$ be a F.S set on $Y$. Let $u : X \to Y$ and $p : E \to E'$ be ordinary functions.

The map $h_{up} : X_E \to Y_{E'}$ is called a F.S map from $X$ to $Y$ mapping $f_A$ to F.S set $h_{up}(f_A)$ and for all $y \in Y$ and $e' \in p(E)$ it is defined as below:

$$[h_{up}(f)]_{e'}(y) = \sup_{x \in u^{-1}(y)} \sup_{e \in p^{-1}(e') \cap A} f(e)(x)$$

if $p^{-1}(e') \cap A \neq \emptyset$, $u^{-1}(y) \neq \emptyset$ and otherwise $[h_{up}(f)]_{e'}(y) = 0$.

Let $h_{up} : X_E \to Y_{E'}$ be a F.S map from $X$ to $Y$. Then the inverse image of F.S set $g_B$, denoting by $h_{up}^{-1}(g_B)$, is a F.S set on $X$ and for all $x \in X$ and $e \in E$ is defined as below:

$$[h_{up}^{-1}(g)]_e(x) = \begin{cases} g_p(e)(u(x)) & \text{if } p(e) \in B \\ 0 & \text{otherwise} \end{cases}$$

if $p^{-1}(e') \cap A \neq \emptyset$, $u^{-1}(y) \neq \emptyset$ and otherwise $0$. This is shown in the following Diagram

| $E$ | $\longrightarrow^p$ | $E'$ |
|-----|----------------|-----|
| $u(f)$ | $\downarrow^g$ | $Y$ |
| $f(e)$ | $\triangleright$ | $g(e')ou^{-1}$ |
| $[0, 1]$ |

Hence $(uf)(e') = f(p^{-1}(e'))ou^{-1}$.

Let $g_B$ be a F.S set in $Y$. The inverse image of $g_B$ is a F.S set in $X$ whose membership function is given by:

$$[u^{-1}(g)]_e(x) = \begin{cases} g_p(e)(u(x)) & \text{if } p(e) \in B \\ 0 & \text{otherwise} \end{cases}$$

This is shown in the following Diagram

| $E$ | $\longrightarrow^p$ | $E'$ |
|-----|----------------|-----|
| $u^{-1}g$ | $\downarrow^g$ | $Y$ |
| $g(e')ou^{-1}$ | $\triangleright$ | $g(e')$ |
| $[0, 1]$ |

So $g_p(e)ou = (u^{-1}g)(e)$.

Note that, If $p : E \to E'$ is a bijective mapping, then the former definition can be written as below:

$$[u(f)]_e(y) = \begin{cases} \sup_{x \in u^{-1}(y)} f_e(x) & e = p^{-1}(e') \in A, u^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

since there exists only one element in the parameter set $E$ such that $p(e) = e'$. If $A = E$, then we have

$$[u(f)]_e(y) = \begin{cases} \sup_{x \in u^{-1}(y)} f_e(x) & u^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Proof. See [26].

Here we apply Definition 2.4 and Fuzzy Extension Principle for an ordinary function (see [16]) to give a parameterized extension of an ordinary function to a F.S map.

Definition 2.5: Let $u : X \to Y$ be a function. Let $f_A$ be a F.S set in $X$. Then the image of $f_A$ is a F.S set in $Y$ whose membership function is given by:

$$[u(f)]_{e'}(y) = \sup_{x \in u^{-1}(y)} \sup_{e \in p^{-1}(e') \cap A} f(e)(x)$$

if $p^{-1}(e') \cap A \neq \emptyset$, $u^{-1}(y) \neq \emptyset$ and otherwise $0$.

3 Fuzzy Soft Topology and its Relation with Point-Set and Fuzzy Topologies

Topological studies of fuzzy soft sets was begun by Tanay and Kandemir in [10]. The concept of fuzzy soft topology introduced by them is in fact, a topological structure over a fuzzy soft set. So it can be seen as a collection of fuzzy soft subsets of an arbitrary fuzzy soft set. This means that parameter set is not fixed everywhere (see Definition 2.3 part (i)). But since in this case De Morgan law’s are not hold in general (see [14]), in [9], Roy and Samanta initiated the concept of fuzzy soft topology over a universal set where...
The parameter set is supposed fixed all over the universe. Here we recall definition of fuzzy soft topology introduced in [9], and consider the relationship between this new topology with two previous topologies, point-set topology and fuzzy topology, over a common universe.

**Definition 3.1**: ([9]) A fuzzy soft topology over \( X \) denoted by \( \tau \), is a collection of fuzzy soft subsets of \( X \) such that:

i. \( \bar{X} \) and \( \Phi \in \tau \).

ii. The union of any number of fuzzy soft sets in \( \tau \) belongs to \( \tau \).

iii. The intersection of any two fuzzy soft sets in \( \tau \) belongs to \( \tau \).

The triplet \( (X, \tau, E) \) is called a fuzzy soft topological space, F.S topological space in brief, and each element of \( \tau \) is called a fuzzy soft open set, say F.S open set, in \( X \). The complement of a F.S open set is called F.S closed set.

**Example 3.1**: Let \( X \) be a universal set including objects under consideration and \( E \) be the set of parameters. Then the family of all F.S sets over \( X \), denoting by \( X_E \), forms a F.S topology over \( X \) which is called discrete F.S topology, while indiscrete or trivial F.S topology on \( X \) contains only \( \Phi, X \).

**Example 3.2**: Let \( (X, \tau) \) be a topological space. Then the family

\[ \{ V_E : V \in \tau, E = (0, 1) \} \]

forms a F.S topology over \( X \) denoting by \( \tau_{F.S} \) where \( V_E \) is a F.S set over \( X \) with respect to \( \tau \)-open set \( V \) defined as below:

\[ V : E = (0, 1) \rightarrow I^X \]

where for each \( \alpha \in (0, 1) \), \( V(\alpha) \) is defined by characteristic function of \( \tau \)-open set \( V \) i.e.,

\[ V(\alpha) = \chi_V \]

So \( V_\alpha(x) = \begin{cases} 1 & x \in V \\ 0 & x \notin V \end{cases} \)

**Example 3.3**: Let \( (X, \tau) \) be a topological space and \( E \) be a parameter set. The collection \( \tau_{F.S} = \{ f : E \rightarrow I^X : \forall \alpha \in E, (f_\alpha)^{-1}(0, 1) \in \tau \} \) is a fuzzy soft topology over \( X \) where for each \( \alpha \in E \), \( (f_\alpha)^{-1}(0, 1) \) denotes the support of fuzzy set \( f(\alpha) \). Moreover \( \tau \subset \tau_{F.S} \).

**Example 3.4**: Let \( (X, \gamma) \) be a fuzzy topological space. Let \( \mu \) be a fuzzy subset of \( X \) and \( \chi \) denotes the characteristic function. We define the characteristic function of \( \alpha \)-cut sets of \( \mu \), denoting by \( \chi_{\mu, \alpha} \), as below:

\[ \chi_{\mu, \alpha} : X \rightarrow [0, 1] \]

\[ (\chi_{\mu, \alpha})(x) = \begin{cases} 1 & \mu(x) \geq \alpha \\ 0 & \mu(x) < \alpha \end{cases} \]

Now we construct the F.S map \( \chi_{\mu, \alpha} \) by using \( \chi_{\mu, \alpha} \)'s as below:

\[ \chi_{\mu} : E = (0, 1) \rightarrow I^X \]

\[ (\chi_{\mu})(\alpha) = \chi_{\mu, \alpha} \]

for each \( \alpha \in E \). Then the collection

\[ \tau_{F.S} = \{ \chi_{\mu, \alpha} : \chi_{\mu} : E \rightarrow I^X, \forall \alpha \in E, (\chi_{\mu})(\alpha) = \chi_{\mu, \alpha} \} \]

is a fuzzy soft topology over \( X \).

**Theorem 3.1**: ([13]) Let \( (X, E, \tau) \) be a fuzzy soft topological space. Then corresponding to each \( e \in E \), \( \tau_e = \{ f_e : f_E \in \tau, e \in E \} \) forms a fuzzy topology over \( X \) (in the sense of Chang [70]).

**Proof**.

i. \( \Phi, \bar{X} \in \tau \) where \( \emptyset \) denotes the empty fuzzy set and \( \underline{1} \) shows the fuzzification of \( X \), i.e.

\[ \emptyset = \chi_\emptyset, \underline{1} = \chi_X \]

and \( \chi \) denotes the characteristic function.

ii. Let \( \Lambda \) be an index set and \( \{ f_E^\lambda \}_{\lambda \in \Lambda} \subseteq \tau \). Let \( e \in E \), then for all \( \lambda \in \Lambda \), \( f^\lambda(e) \in \tau_e \). Since \( \tau \) is a F.S topology over \( X \), then by applying Definition 2.3 part (vi) we have \( \bigvee_{\lambda \in \Lambda} f_E^\lambda \in \tau \Rightarrow (\bigvee_{\lambda \in \Lambda} f^\lambda)_E \in \tau \Rightarrow (\bigvee_{\lambda \in \Lambda} f^\lambda)(e) \in \tau_e \).

iii. Suppose \( f_E^1, f_E^2 \in \tau \) and \( e \in E \). Then \( f^1(e), f^2(e) \in \tau_e \). Since \( \tau \) is a F.S topology, then by applying Definition 2.3 part (vii) we have \( f_E^1 \wedge f_E^2 \in \tau \Rightarrow (f^1 \wedge f^2)_E \in \tau \Rightarrow f^1(e) \wedge f^2(e) \in \tau_e \).

So \( \tau_e \) is a fuzzy topology over \( X \).

\[ \square \]

Now we define the concept of fuzzy soft point over set \( X \). In the literature, the concept of fuzzy soft point was introduced as bellows:

1. \( \{11\} \) A fuzzy soft set \( f_E \) over \( X \) is called a fuzzy soft point if for the element \( e^* \in E \) we have

\[ f_e(x) = \begin{cases} \lambda_x & \text{if } x = e^* \\ 0 & \text{otherwise} \end{cases} \]

\forall e \in E, \forall x \in X \text{ where } \lambda_x \in (0, 1].

2. \( \{12\} \) A fuzzy soft set \( f_E \) over \( X \) is called a fuzzy soft point if for the element \( x^* \in X \) we have

\[ f_e(x) = \begin{cases} \lambda & \text{if } x = x^* \\ 0 & \text{otherwise} \end{cases} \]

\forall e \in E, \forall x \in X \text{ where } \lambda \in (0, 1].
But these definitions are not free of difficulties. First one investigates the cases which are related to only one parameter. Second one, although is more general than the former, considers specific cases in which the membership degree is supposed fixed for all parameters. In fact, it can be seen as a parameterized version of a fuzzy point, whereas fuzzy soft set theory is a new method to give a fuzzy extension of soft set theory.

In classic set theory, each member of a non-empty set is defined as an object which has the same property of the set. On the other hand, fuzzy soft set theory is a method used to represent the imprecise information about a set based on some parameters. So to define the concept of fuzzy soft point, we restrict the universal set $X$ to the single set $\{x\} \subset X$ and define the fuzzy soft point $x$ as a fuzzy description of element $x \in X$ with regards to some parameters.

**Definition 3.2** 

1. Let $x^\lambda$ be a fuzzy point with support $x \in X$ and membership degree $\lambda \in (0, 1]$ (see [28]). The fuzzy soft set $P_x$ is called fuzzy soft point, say F.S point, whenever $P_x : E \rightarrow 2^X$ is a map such that for each $e \in E$ and $\forall z \in X$ 

$$
(P_x)_e(z) = \begin{cases} 
\lambda & \text{if } z = x \\
0 & \text{otherwise}
\end{cases}
$$

So for each $e \in E$, $(P_x)(e) = x^\lambda$, or $(P_x)(e) = \lambda_\chi_{\{x\}}$ where $\chi_{\{x\}}$ is the characteristic function of $\{x\}$. In other words, the F.S point $P_x$, is a fuzzy description of $x \in X$ based on parameter set $E$.

2. The F.S point $P_x$ belongs to F.S set $f_E$ denoting by $P_x \in f_E$, whenever for all $e \in E$ we have $0 < \lambda \leq f_e(x)$.

3. The restriction of F.S point $P_x$ to an element $e \in E$, denoting by $P_x|_e$, is called fuzzy soft single point over $E$, say F.S single point, whenever for all $\alpha \in E$ and $\forall z \in X$ 

$$
(P_x|_e)_z = \begin{cases} 
(P_x)_e(x) = \lambda & \text{if } \alpha = \epsilon, z = x \\
0 & \text{otherwise}
\end{cases}
$$

The F.S single point $P_x|_e$ belongs to F.S set $f_E$, denoting by $P_x|_e \in f_E$, whenever for $\epsilon \in E$ we have $0 < \lambda \leq f_e(x)$.

It is clear that Definition 3.2 is an extension of both crisp and fuzzy points. In addition, the concept of fuzzy soft point introduced in the literature [11][12] can be seen as a specific case of our definition.

**New Notation.**

- The F.S point $P_x$ will be denoted by $\tilde{x}_E$ where for all $e \in E$, $(P_x)(e) = \tilde{x}(e) = x^\lambda$ i.e., the image of each parameter under map $P_x$ is a fuzzy point. Consequently $(\tilde{x}_E)^\alpha$ can be applied to show the complement of F.S point $\tilde{x}_E$ such that for all $e \in E$ and $\forall z \in X$, we have 

$$
(\tilde{x}_E)^\alpha(z) = \begin{cases} 
1 - \lambda & z = x \\
1 & z \neq x
\end{cases}
$$

- The crisp F.S point $P_x$ will be denoted by $x^\lambda_E$ or $\tilde{x}_E$.

From now, notation $\tilde{x}_E$ means F.S point $x$ in $X$ where $\tilde{x}(e) = x^\lambda$, $\forall e \in E$.

Note that crisp point $x$ and fuzzy point $x^\lambda$ can be viewed as a fuzzy soft point $\tilde{x}_{(0,1)}$ and $\tilde{x}_{(1)}$ where $\forall \alpha \in E = (0, 1]$, $\tilde{x}(\alpha) = x^\lambda$ and $\tilde{x}(\alpha) = x^\lambda$, respectively

**Definition 3.3** ([13][24]) Let $(X, E, \tau)$ be a fuzzy soft topological space. Two fuzzy soft points $\tilde{x}_E$ and $\tilde{y}_E$ where for all $e \in E$, $\tilde{x}(e) = x^\lambda$ and $\tilde{y}(e) = y^\tau$ are said to be

- different if and only if
  1. $x \neq y$ or
  2. Whenever $x = y$, we have $\lambda \neq \gamma$ for some $\epsilon \in E$.

- distinct if and only if $\tilde{x}_E \tilde{y}_E = \Phi$.

**Definition 3.4** ([13]) Let $\tilde{x}_E$ be a F.S point over $X$. Fuzzy soft set $g_E$ is called fuzzy soft neighborhood, F.S - $N$ in brief, of F.S point $\tilde{x}_E$ whenever there exists a F.S open set $f_E$ such that $\tilde{x}_E \in f_E \subseteq g_E$.

**Example 3.5** : Take the set of all real numbers $\mathbb{R}$ with usual topology $\tau_w$. Let $V = (a, b) \subseteq \mathbb{R}$ be an open neighborhood of $x \in \mathbb{R}$. Define fuzzy soft set $V_E$ as the following:

$$
V : (0, 1] \rightarrow 2^\mathbb{R} \\
\alpha \mapsto V(\alpha) = \chi_V
$$

Then $V_E$ is an F.S-open-N of F.S point $\tilde{x}_E$ in F.S topological space $(X, [0, 1], \tau_{FS})$ where $\tau_{FS}$ is the F.S topology mentioned in Example 3.2. In fact, $V_E$ is a parametrization version of open interval $V$ in $\mathbb{R}$.

**Definition 3.5** ([13])

1. Let $\tilde{x}_E$ be a F.S point in $X$. We say that $\tilde{x}_E$ is soft quasi-coincident with F.S set $f_E$, denoting by $\tilde{x}_E \tilde{q}_f f_E$, if there exists $e \in E$, such that $\lambda_e + f_e(x) > 1$.

2. The fuzzy soft set $g_E$ is called soft quasi-coincident with fuzzy soft set $f_E$ at $x \in X$, denoting by $g_E \tilde{q}_f f_E$, if there exist $e \in E$ such that $g_e(x) + f_e(x) > 1$. If not we say that $g_E$ is not soft quasi-coincident with $f_E$ denoting by $f_E \tilde{q}_f g_E$. 


3. The fuzzy soft single point \( x_e \) is called soft quasi-coincident with fuzzy soft set \( f_E \) at \( x \), denoting by \( x_e \tilde{\in} f_E \), if for \( e \in E \) we have \( \lambda_e + f_e(x) > 1 \).

**Definition 3.6** : (13) The fuzzy soft set \( g_E \) is called a soft quasi neighborhood of \( F .S \) point \( x \), say soft \( Q - N \), if there exists the \( F .S \) open set \( f_E \), such that \( x \tilde{\in} f_E \hat{\in} g_E \).

Let \( x \tilde{\in} E \) be a \( F .S \) point in \( X \). Let \( \Lambda \) be an index set and \( (f_E)_{\alpha \in \Lambda} \) be a family of fuzzy soft sets over \( X \) with respect to parameter set \( E \).

**Proposition 3.1** If \( x \tilde{\in} E \tilde{\in} \bigwedge_{\alpha \in \Lambda} (f_E)_{\alpha} \Rightarrow \forall \alpha \in \Lambda : x \tilde{\in} E \tilde{\in} (f_E)_{\alpha} \).

**Proof.** See [13].

**Proposition 3.2** If \( x \tilde{\in} E \tilde{\in} \bigvee_{\alpha \in \Lambda} (f_E)_{\alpha} \Rightarrow \exists \alpha \in \Lambda : x \tilde{\in} E \tilde{\in} (f_E)_{\alpha} \).

**Proof.** See [13].

**Definition 3.7** (13) Let \( (X, E, \tau) \) be a fuzzy soft topological space. We say that \( X \) is a fuzzy soft topological space if \( X \) is a fuzzy soft set.

1. \( T_0 \), say \( F .S \) \( T_0 \), if and only if for every two distinct \( F .S \) points in \( X \), at least one of them has a \( F .S \) open - \( N \) which is not intersected with the other.

2. \( T_1 \) space, say \( F .S \) \( T_1 \), if and only if for every two distinct \( F .S \) points in \( X \) such as \( x_E \) and \( y_E \), there exist two \( F .S \) open - \( N \) of \( x_E \), \( y_E \) like \( f_E \) and \( g_E \) respectively, such that \( y_E \hat{\in} f_E = \Phi \), and \( x_E \hat{\in} g_E = \Phi \).

3. Hausdorff space, say \( F .S \) \( T_2 \) or \( F .S \) Hausdorff, if and only if for every two distinct \( F .S \) points in \( X \) such as \( x_E \) and \( y_E \), there exist two \( F .S \) open - \( N \) like \( f_E \) and \( g_E \), such that \( f_E \hat{\in} g_E = \Phi \).

Now we introduce a way to construct a fuzzy soft topology over a set associate with the given point-set topology or fuzzy topology over it.

We know that for each ordinary set like \( A \) if \( A \) is an uncountable set, then we have \( A \sim \mathbb{R} \sim (0, 1) \), and if \( A \) is a finite or countable set, then \( A \sim \mathbb{N}_k \) or \( A \sim \mathbb{N} \), respectively where \( k \) is the number of elements of \( A \).

Now consider the parameter set \( E \). In the following remark we discuss how the parameter set \( E \) can be replaced with a suitable subset of \((0, 1] \).

**Remark 3.1** : (13) Let \( X \) be the set of objects under consideration, and \( E \) be the set of parameters.

1. If \( E \) is an uncountable set, then we replace \( E \) with \((0, 1] \). So associate with each \( e \in E \) we have an \( \alpha \in (0, 1] \).

2. If \( E \) is an infinite but countable set, we replace \( E \) with the set of all rational numbers in \((0, 1] \). Thus in this case, each \( e_i \in E \) can be replaced with \( \frac{m_i}{n} \in (0, 1] \) such that \( m_1, \ldots, n = 1, 2, \ldots, \text{and } m \leq n \).

3. If \( E \) is a finite set where \(|E| = k \), we replace \( E \) with \( \{1, \ldots, k\} \). So corresponding to each \( e_i \in E \), we have \( \frac{1}{k} \in (0, 1] \), where \( i = 1, \ldots, k - 1, k \).

From now, \((0, 1]_E \) is used to show the subset of \((0, 1] \) corresponding to \( E \).

Now we are ready to introduce a fuzzy soft topology on \( X \) associate with the initial topology on \( X \). Put \( T(X) \) the set of all topologies on \( X \), \( T_F(X) \) the set of all fuzzy topologies on \( X \), and \( T_{F.S}(X) \) the set of all fuzzy soft topologies on \( X \) and consider the following mappings

\[
i'_e : T_{F.S}(X) \rightarrow T_F(X), i'_e : T_F(X) \rightarrow T(X)
\]

\[
w : T(X) \rightarrow T_F(X), w' : T_F(X) \rightarrow T_{F.S}(X)
\]

**Definition 3.8** : (13)

1. Let \((X, E, \delta)\) be a fuzzy soft topological space, then corresponding to each parameter \( e \in E \) we can define a fuzzy topology on \( X \) by

\[
i'_e(\delta) = \{ f(e) : f_E \in \delta \}
\]

(see Theorem 5.1) and a topology over \( X \) by

\[
i_e(i'_e(\delta)) = \{(f(e))^{-1}(t, 1) : t \in [0, 1], f_E \in \delta \}
\]

as a topology over \( X \) where \( t \in (0, 1]_E \).

2. Suppose \((X, \tau)\) be a topological space and \( E \) be the set of parameters. We can define a fuzzy soft topology on \( X \) with regards to \( \tau \) by

\[
w'(w(\tau)) = \{ f : X \rightarrow I^X; f(e) \in w(\tau); \forall e \in E \}
\]

where

\[
w(\tau) = \{ \mu : X \rightarrow [0, 1] : \mu^{-1}(t, 1) \in \tau; \forall t \in [0, 1] \}
\]

is a fuzzy topology over \( X \) (see [17]).

It is easy to check that \( w'(w(\tau)) \) and \( i_e(i'_e(\delta)) \) are indeed \( F .S \) topology and point-set topology over \( X \), respectively (see [13]).

## 4 Fuzzy Soft Real Number

A fuzzy soft number is a fuzzy soft set on real numbers \( \mathbb{R} \) associated with some parameters. This notion is a mathematical tool to represent terms such as "about \( m \) with respect to \( n". Before giving the concept of fuzzy soft real number, we recall some definitions connected with this subject which are considered in the literature (see [15, 19, 20, 18]).


4.1 Fuzzy Real number

**Definition 4.1** : (20 78) The fuzzy real number \( \mu \) is a fuzzy subset of real number set \( \mathbb{R} \) which satisfies the below conditions:

- \( \mu \) is normal, i.e., \( \exists r \in \mathbb{R}; \mu(r) = 1 \).
- \( \mu \) is convex, i.e., \( \forall \alpha \in (0,1], \alpha \)-level’s of \( \mu \) are convex sets in \( \mathbb{R} \).
- \( \mu \) is upper semi-continuous, i.e., for all \( t \in (0,1] \) and \( \varepsilon > 0 \), \( \mu^{-1}((0,t)) \) is an open set in \( \mathbb{R} \) when topology over \( \mathbb{R} \) is supposed the usual topology \( \tau_\alpha \).

The set of all fuzzy real numbers is denoted by \( \mathcal{F}(\mathbb{R}) \). It can be proved that \( \alpha \)-level sets of an upper semi-continuous convex normal fuzzy set for each \( \alpha \in (0,1] \) is a closed interval in \( \mathbb{R} \). So we have

\[
[\mu]_\alpha = \{ t \in \mathbb{R} : \mu(t) \geq \alpha \} = [\mu_1, \mu_2]
\]

where \( \mu_1, \mu_2 \in \mathbb{R} \).

A fuzzy number \( \mu \) is called non-negative if \( \mu(t) = 0 \) for \( t < 0 \). The set of all non-negative fuzzy real numbers is denoted by \( \mathcal{F}(\mathbb{R}^+) \).

**Definition 4.2** : (20 78) The operations \( \oplus, \ominus, \otimes \) are defined on \( \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \) as below:

\[
(\mu \oplus \delta)(t) = \sup \{ \mu(s), \delta(t-s) \}, t, s \in \mathbb{R}
\]

\[
(\mu \ominus \delta)(t) = \sup \{ \mu(s), \delta(s-t) \}, t, s \in \mathbb{R}
\]

\[
(\mu \otimes \delta)(t) = \sup \{ \mu(s), \delta(t/s) \}, t, s \in \mathbb{R}, s \neq 0
\]

\[
(\mu \otimes \delta)(t) = \sup \{ \mu(s), \delta(s) \}, t, s \in \mathbb{R}
\]

\[
[\mu]_\alpha = \{ t \in \mathbb{R} : \mu(t) \geq \alpha \} = [\mu_1, \mu_2]
\]

where \( \mu_1, \mu_2 \in \mathbb{R} \).

**Definition 4.3** (20 78) We can also define fuzzy arithmetic operations by \( \alpha \)-level sets as below:

\[
[\mu \oplus \delta]_\alpha = [\mu_1^\alpha \delta_\alpha, \mu_2^\alpha \delta_\alpha]
\]

\[
[\mu \ominus \delta]_\alpha = \left[ \min\{\mu_1^\alpha - \delta_\alpha, \mu_2^\alpha \delta_\alpha\}, \max\{\mu_1^\alpha - \delta_\alpha, \mu_2^\alpha - \delta_\alpha\} \right]
\]

\[
[\mu \otimes \delta]_\alpha = \left[ \min\{\mu_1^\alpha \delta_\alpha, \mu_2^\alpha \delta_\alpha\}, \max\{\mu_1^\alpha \delta_\alpha, \mu_2^\alpha \delta_\alpha\} \right]
\]

\[
[\mu \otimes \delta]_\alpha = \left[ \min\{\mu_1^\alpha \delta_\alpha, \mu_2^\alpha \delta_\alpha\}, \max\{\mu_1^\alpha \delta_\alpha, \mu_2^\alpha \delta_\alpha\} \right]
\]

\[
[\mu]_\alpha = \left[ \min\{0, \mu_1^\alpha - \mu_2^\alpha\}, \max\{[\mu_1^\alpha], [\mu_2^\alpha]\} \right]
\]

**Definition 4.4** (20 78) Let \( \mu, \delta \in \mathcal{F}(\mathbb{R}) \) and \( [\mu]_\alpha = [\mu_1^\alpha, \mu_2^\alpha] \) and \( [\delta]_\alpha = [\delta_1^\alpha, \delta_2^\alpha] \).

The partial order \( \leq \) in \( \mathcal{F}(\mathbb{R}) \) is defined as below:

\[
\mu \leq \delta \text{ if and only if } \mu_1^\alpha \leq \delta_1^\alpha \text{ and } \mu_2^\alpha \leq \delta_2^\alpha \text{ for all } \alpha \in (0,1].
\]

The equality of fuzzy numbers \( \mu \) and \( \delta \) is defined as below:

\[
\mu = \delta \leftrightarrow [\mu]_\alpha = [\delta]_\alpha \text{ for all } \alpha \in (0,1].
\]

4.2 Fuzzy Soft Real Number

**Definition 4.5** : Let \( \mathbb{R} \) be the set of all real numbers, and \( E \) be the parameter set. The set of all fuzzy soft real numbers, say F.S real numbers, is denoted by \( \mathbb{R}_E = \{ f : E \rightarrow \mathbb{R} \} \), where for all \( e \in E \), \( f(e)’s \) are fuzzy real numbers.

The real number \( r \in \mathbb{R} \) can be seen as a F.S real number \( r_E \) if for all \( e \in E \), \( r(e) \) is defined by characteristic function of \( r \). So we have

\[
\tilde{r} : E \rightarrow \mathbb{I}_\mathbb{R}
\]

\[
e \mapsto \tilde{r}(e) = \chi_e
\]

where \( \tilde{r}(e) \) is defined by

\[
\tilde{r}_e(t) = \begin{cases} 
1 & \text{if } t = r \\
0 & \text{if } t \neq r 
\end{cases}
\]

\[
\begin{array}{c}
\tilde{0}_e(t) = \begin{cases} 
1 & \text{if } t = 0 \\
0 & \text{if } t 
eq 0 
\end{cases} \\
\tilde{1}_e(t) = \begin{cases} 
1 & \text{if } t = 1 \\
0 & \text{if } t \neq 1 
\end{cases}
\end{array}
\]

During this study, we will show the F.S real numbers by \( \tilde{r}_E \), where \( r \in \mathbb{R} \) and \( \tilde{r} : E \rightarrow \mathbb{I}_\mathbb{R} \).

**Remark 4.1** : In daily-life situations, saying like " weather is pretty warm, although it is winter " are used usually. So a mathematical concept is needed for modelling these kinds of term.

**Remark 4.2** : In application, F.S real numbers are used to display numbers which are approximately equal to a real number or being approximately between two real numbers with respect to some parameters.

**Definition 4.6** : [\( \bar{F}_{E\alpha} \)] is called the \( \alpha \)-level set of F.S real number \( \bar{r}_E \) corresponding to the parameter \( e \in E \) and defined as below

\[
[\bar{F}_{E\alpha}] = \{ t : \tilde{r}(t) \geq \alpha \}
\]

So it can be considered as the \( \alpha \)-level set of fuzzy real number \( \bar{r}(e) \), i.e. \( \bar{F}_{E\alpha} = [\bar{r}(e)]_\alpha \).

**Definition 4.7** : Let \( \bar{r}_E \) and \( \bar{r}’_E \) be two F.S real numbers. We say that

\[ \bar{r}_E \leq \bar{r}’_E \text{ if and only if } \bar{r}_E \leq \bar{r}’_E \text{ for all } e \in E \]
2. \( \tilde{r}_E = \tilde{r}_E' \iff \tilde{r}(e) = \tilde{r}'(e) \), for all \( e \in E \).

So if

\[ [\tilde{r}_E]_{e,\alpha} = [\tilde{r}(e)]_{\alpha} = [r^1_e, r^2_e] \]

and

\[ [\tilde{r}'_E]_{e,\alpha} = [\tilde{r}'(e)]_{\alpha} = [r^1_{e,\alpha}, r^2_{e,\alpha}] \]

then we have

1. \( \tilde{r}_E \tilde{r}_E' \Rightarrow r^1_e \leq r^1_{e,\alpha} \) and \( r^2_e \leq r^2_{e,\alpha} \) \( \forall \alpha \in (0, 1] \) and \( \forall e \in E \).

2. \( \tilde{r}_E = \tilde{r}_E' \Rightarrow r^1_e = r^1_{e,\alpha} \) and \( r^2_e = r^2_{e,\alpha} \), \( \forall \alpha \in (0, 1] \) and \( \forall e \in E \).

4.3 Arithmetical Operations of Fuzzy Soft Real Number

**Definition 4.8**: Let \( \tilde{r}_E \) and \( \tilde{r}_E' \) be two F.S real numbers. We define arithmetical operations \( \oplus, \otimes, \ominus, \odot \) by applying Definition 4.4 as below:

\( \tilde{r}_E \oplus \tilde{r}_E' \) is defined by map \( \tilde{r} \oplus \tilde{r}' : E \rightarrow \tilde{R} \) such that for each \( e \in E \) we have \( (\tilde{r} \oplus \tilde{r}')(e) = \tilde{r}(e) + \tilde{r}'(e) \) where \( \forall t \in \mathbb{R} \),

\[ [\tilde{r}(e) + \tilde{r}'(e)](t) = [r^1_e, r^2_e] \]

(\( \tilde{r}_E \oplus \tilde{r}'_E \)) is \( \tilde{r}(e) + \tilde{r}'(e) \), for each \( e \in E \),

(\( \tilde{r}_E \oplus \tilde{r}'_E \)) is \( \tilde{r}(e) + \tilde{r}'(e) \), for each \( e \in E \),

(\( \tilde{r}_E \oplus \tilde{r}'_E \)) is \( \tilde{r}(e) + \tilde{r}'(e) \), for each \( e \in E \).

If \( [\tilde{r}_E]_{e,\alpha} = [\tilde{r}(e)]_{\alpha} = [r^1_e, r^2_e] \), and \( [\tilde{r}'_E]_{e,\alpha} = [\tilde{r}'(e)]_{\alpha} = [r^1_{e,\alpha}, r^2_{e,\alpha}] \) be non-negative F.S real numbers, then by applying Definition 4.4, we have

\[ [\tilde{r}_E \oplus \tilde{r}'_E]_{e,\alpha} = [\tilde{r}(e) + \tilde{r}'(e)]_{\alpha} \]

\[ [\tilde{r}_E \oplus \tilde{r}'_E]_{e,\alpha} = [\tilde{r}(e) + \tilde{r}'(e)]_{\alpha} \]

\[ [\tilde{r}_E \oplus \tilde{r}'_E]_{e,\alpha} = [\tilde{r}(e) + \tilde{r}'(e)]_{\alpha} \]

\[ [\tilde{r}_E \oplus \tilde{r}'_E]_{e,\alpha} = [\tilde{r}(e) + \tilde{r}'(e)]_{\alpha} \]

\[ [\tilde{r}_E \oplus \tilde{r}'_E]_{e,\alpha} = [\tilde{r}(e) + \tilde{r}'(e)]_{\alpha} \]

\[ \forall e \in E \text{ and } \forall \alpha \in (0, 1] \].

**Proposition 4.1**: The additive and the multiplicative identities in \( \mathbb{R}_E \) are \( 0_E \) and \( 1_E \), respectively.

**Proof**: Let \( r^* \) be a F.S real number and \( 0_E \) be crisp F.S zero element. It is clear that \( [0_E]_{e,\alpha} = [0]_{\alpha} = [0] = [0, 0] \). Suppose that \( [r^*]_{e,\alpha} = [r^*]_{\alpha} = [r^1_{e,\alpha}, r^2_{e,\alpha}] \). Then for each \( \alpha \in (0, 1] \),

\[ [r^*]_{e,\alpha} = [r^*]_{\alpha} = [r^1_{e,\alpha}, r^2_{e,\alpha}] \]

Thus \( \tilde{r}_E \oplus 0_E = \tilde{r}_E \).

Similarly, it can be shown that \( \forall e \in E \) and \( \forall \alpha \in (0, 1] \),

\[ [r^*]_{e,\alpha} = [r^*]_{\alpha} \]

5 Fuzzy Soft Normed Spaces

5.1 Fuzzy Soft Norm

Due to introduce a norm for a fuzzy soft point, extension of some operations such as addition and scalar multiplication are needed. In this section we firstly suggest a parametrization version of these well-known concepts, and then introduce the notion of fuzzy soft norm.

**Definition 5.1**: Let \( f_E, g_E \in X_E \). Then we define \( f_E \odot g_E \in X_E \times X_E \) as the F.S multiplication by the map

\[ f \odot g : E \rightarrow I^{x \times x} \]

\[ e \rightarrow (f \odot g)(e) = f(e) \odot g(e) \]

where

\[ (f \odot g)(e)(x_1, x_2) = (f(e) \odot g(e))(x_1, x_2) = \min \{f_e(x_1), g_e(x_2)\} \]

**Definition 5.2**: Let \( X \) be a vector space over the field \( F \) (\( \mathbb{R} \) or \( \mathbb{C} \)). Let \( h^*_1 : X \times X \rightarrow X \) be the ordinary addition function on \( X \), i.e. \( h^*_1(t, x) = x + t \) for all \( x, t \in X \).

Let \( f_E, g_E \in X_E \). If F.S addition is denoted by \( \oplus \), then by applying Definition 2.5 we have \( h^*_1(f_E \times g_E) = f_E \odot g_E \) where

\[ (f \oplus g)(e)(x) = \sup_{(x_1, x_2) \in h^*_1(e)} ([f \times g](e))(x_1, x_2) = \sup_{(x_1, x_2) \in h^*_1(e)} ([f \times g](e))(x_1, x_2) = \sup_{(x_1, x_2) \in h^*_1(e)} \min \{f_e(x_1), g_e(x_2)\} \]

where \( x_1 + x_2 = x \).

**Definition 5.3**: Let \( X \) be a vector space over the field \( F \) (\( \mathbb{R} \) or \( \mathbb{C} \)). Let \( h^*_2 : F \times X \rightarrow X \) be the ordinary scalar multiplication function on \( X \), i.e. \( h^*_2(t, x) = tx \) for \( t \in F \) and \( x \in X \).

Let \( f_E \in X_E \) and \( r \in \mathbb{R} \) such that \( \tilde{r}_E \in \mathbb{R}_E \). If F.S scalar multiplication is denoted by \( \odot \), by applying Definition 2.5 we have \( h^*_2(r_E \times f_E) = r_E \odot f_E \) where

\[ (r \odot f)(e)(z) = h^*_2(\tilde{r} \times f)(e)(z) = \sup_{(t, x) \in h^*_1(e)} ([\tilde{r} \times f](e))(t, x) = \sup_{(t, x) \in h^*_1(e)} ([\tilde{r} \times f](e))(t, x) \]

\[ = \sup_{(t, x) \in h^*_1(e)} \min \{\tilde{r}_e(t), f_e(x)\} \]

where \( tx \).
Note that if $\tilde{r}_E$ be the crisp F.S real number $\tilde{r}_E$, then the F.S scalar multiplication is defined by

$$(\tilde{r} \otimes f)(e)(z) = \begin{cases} f_e(\tilde{z}) & \text{if } r \neq 0 \\ \sup_{x \in X} f_e(x) & \text{if } r = 0, z = 0 \\ 0 & \text{if } r = 0, z \neq 0 \end{cases}$$

We denote $\tilde{r}_E \otimes f_E$ by $r f_E$.

Next, let $f_E$ be the F.S point $\tilde{x}_E$, then

$$(\tilde{x} \otimes x)(e)(z) = \begin{cases} 1 & \text{if } z = r x \\ 0 & \text{otherwise} \end{cases}$$

So $\tilde{x}_E \otimes x_E = \tilde{x} E$, crisp F.S real number $r x$.

**Definition 5.4**: Let $X$ be a vector space over field $F$ ($\mathbb{R}$ or $\mathbb{C}$) and let $E$ be the set of parameters. We define fuzzy soft norm $\| | |$, say F.S norm, over $X$ by map $\| | | : X E \rightarrow R^+ E$ which satisfies the below conditions

1. $\tilde{x}_E = 0_E \iff \| \tilde{x}_E \| = 0_E$
2. $\| r \tilde{x}_E \| = \| \tilde{r}_E \otimes \| \tilde{x}_E \|$
3. $\| \tilde{x}_E \otimes \tilde{y}_E \| \leq \| \tilde{x}_E \| \otimes \| \tilde{y}_E \|$

We denote the fuzzy soft normed space $X$, say F.S normed space $X$, by $(X, E, |||)$.

Note that F.S norm of F.S point $\tilde{x}_E \in X_E$ is denoted by map $\| \tilde{x}_E \| : E \rightarrow I^R$ which is a non-negative F.S real number. It means that for each $e \in E$, $\| \tilde{x}_E \| (e) : R \rightarrow I$ where $\| \tilde{x}_E \| (e)(t) = 0$ for all $t < 0$ (see Definition 4.1).

We can also define $\alpha$-level sets of F.S real number $\| \tilde{x}_E \|$ as below

$$\| \tilde{x}_E \|_{e, \alpha} = \| \tilde{x}_E \| (e)(t) \geq \alpha$$

where $\| \tilde{x}_E \|_{e, \alpha}, \| \tilde{x}_E \|_{e, \alpha}^2$ are real numbers and $\| \tilde{x}_E \| (e)(t)$ is a closed interval in real line (see Definition 4.1).

**Lemma 5.1**: Let $r, x \in \mathbb{R}$. Suppose $\tilde{x}_E$ and $\tilde{r}_E$ be crisp F.S real numbers corresponding to $x$ and $r$, respectively. Then

1. $\| \tilde{r}_E \| = |r|$ 
2. $\tilde{r}_E \otimes \tilde{x}_E$

**Proof.**

1. Let $r \in \mathbb{R}$. Definition 4.7 implies $\| \tilde{r}_E \|_{e, \alpha} = [r, r]$ and $\| \tilde{r}_E \|_{e, \alpha} = [r, r]$. By applying Definition 4.7 we have

$$\| \tilde{r}_E \|_{e, \alpha} = \| \tilde{r}_E \|_{e, \alpha} = \{ \max \{0, r, -r\}, \max \{r, |r|\} \} = |r|, |r|$$

So $\| \tilde{r}_E \| = |r|$. 

2. It is clear by Definition 4.7.

**Theorem 5.1**: Let $(X, E, |||)$ be a F.S normed space. Then for each $e \in E$,

1. $|||_e$ is a fuzzy norm, introduced in [20, 22, 23], such that $|||_e : X \rightarrow \mathcal{F}(\mathbb{R}^*)$ is defined as $|||_e = |||_E|||_e$ where $\mathcal{F}(\mathbb{R}^*)$ denotes the set of all non-negative fuzzy real numbers.
2. $(X, |||_e)$ is normed space where $i = 1, 2, e \in E$ and $\alpha \in (0, 1]$ as below $|||_e, \alpha : X \rightarrow \mathbb{R}^*$ is defined by $\alpha$-level set of fuzzy norm $|||_e$ as $|||_e, \alpha = \| | |_e, | |_\alpha = \{ |||_e, | |_\alpha \}$

**Proof.** Let $(X, E, |||)$ be a F.S normed space.

1. Take $e \in E$.

(1) Let $r = 0$ and $0$ denotes crisp fuzzy number zero. Then by applying Definitions 5.1, we have $\tilde{x}_E = 0_E \iff \| \tilde{x}_E \| = 0_E \iff \forall e \in E, \forall t \in \mathbb{R} : |||_e(t) = \tilde{x}_E = \tilde{0}_E \iff \| \tilde{x}_E \| = 0_E.$

(2) Lemma 5.1 implies

$$\{ |||_e, | |_\alpha \} = \{ t : |||_e, | |_\alpha \geq \alpha \}$$

On the other hand

$$\{ t : |||_e, | |_\alpha \geq \alpha \}$$

(3) For any pair $\tilde{x}_E, \tilde{y}_E \in \tilde{x}$,

$$|||_e \tilde{x}_E \otimes \tilde{y}_E|||_e \leq |||_e \tilde{x}_E|||_e \otimes |||_e \tilde{y}_E|||_e$$

Then for each $e \in E$ and $\alpha \in (0, 1],$

$$\{ |||_e \tilde{x}_E \otimes \tilde{y}_E|||_e \} \leq \{ |||_e \tilde{x}_E|||_e \} + \{ |||_e \tilde{y}_E|||_e \}$$

when $i = 1, 2$. So

$$\{ ||x + y|||_e \} \leq \{ ||x|||_e \} + \{ ||y|||_e \}$$

where $x + y = \text{supp} (\tilde{x}_E \otimes \tilde{y}_E)$. This implies that $||x + y|||_e \leq ||x|||_e \oplus ||y|||_e$. Thus $|||_e$'s are fuzzy norm on $X$ for all $e \in E$. 

$\square$
2. It is similar to (1).

**Theorem 5.2** : Let \((X, ||.||)\) be a normed space. Then

1. \(||.||^F\) is a fuzzy norm on \(X\), where \(||.||^F : X \rightarrow \mathcal{F}()\) and for each \(x \in X\), \(||x||^F = \chi_{|x|}\) where \(\chi\) is a characteristic function. So for all \(t \in \mathbb{R}^+\)
   \[
   ||x||^F(t) = \begin{cases} 
   1 & \text{if } t = ||x|| \\
   0 & \text{otherwise}
   \end{cases}
   \]

2. \(||.||^F\) is a F.S norm on \(X\) with respect to the parameter set \(E\). \(||.||^F\) is defined by the mapping \(||.||^F : X_E \rightarrow \mathbb{R}^*_E\) such as for all \(\tilde{x} \in X_E\) and \(e \in E\), \(\forall t \in \mathbb{R}^*_E\)
   \[
   (||\tilde{x}|^F_E\rangle(t) = \begin{cases} 
   1 & \text{if } t = ||x|| \\
   0 & \text{otherwise}
   \end{cases}
   \]
   where \(x = \text{support } \tilde{x}(e)\) for all \(e \in E\).

**Proof.** Part (1) is clear. We only prove (2).

(1) Let \(\tilde{x}_E = 0_E\). Suppose \(x = \text{support } \tilde{x}(e)\) for each \(e \in E\), then for \(e \in E\) we have
   \[
   \tilde{x}_E = 0_E \iff \tilde{x}(e) = 0
   \]
   \[
   \iff x = 0
   \]
   \[
   \iff ||x|| = 0
   \]
   \[
   \iff (||\tilde{x}_E||^F_E\rangle(t) = 1, (||\tilde{x}_E||^F_E\rangle(t = 0) \neq 0
   \]
   \[
   \iff ||\tilde{x}_E||^F_E = 0_E
   \]

(2) It is clear that \(\text{supp}(\tilde{x}_E \otimes \tilde{x}_E)\rangle(t) = rx\), then
   \[
   ||\tilde{x}_E \otimes \tilde{x}_E||^F_E\rangle\rangle = \begin{pmatrix} ||\tilde{x}_E \otimes \tilde{x}_E||^F_E\rangle\rangle\rangle \\
   = \begin{cases} 
   \{ t : ||\tilde{x}_E \otimes \tilde{x}_E||^F_E\rangle\rangle(t) \geq \alpha \} \\
   \{ t : t = ||\text{supp}(\tilde{x}_E \otimes \tilde{x}_E)\rangle(t) || \} \\
   \{ t : t = ||rx|| \} = ||r|| = || |x|| |r|| |x||
   \end{cases}
   \]
   \[
   \begin{aligned}
   ||\tilde{x}_E \otimes \tilde{x}_E||^F_E\rangle\rangle\rangle &= \begin{pmatrix} r ||x|| \} || |x|| = ||r|| |x||
   \end{aligned}
   \]

(3) Let \(\tilde{x}_E\) and \(\tilde{y}_E\) belong to \(\tilde{X}\). Then
   \[
   ||\tilde{x}_E \otimes \tilde{y}_E||^F_E\rangle\rangle\rangle = \begin{pmatrix} ||\tilde{x}_E \otimes \tilde{y}_E||^F_E\rangle\rangle\rangle\rangle \\
   = \begin{cases} 
   \{ t : |||\tilde{x}_E \otimes \tilde{y}_E||^F_E\rangle\rangle\rangle(t) \geq \alpha \} \\
   \{ t : t = ||\text{supp}(\tilde{x}_E \otimes \tilde{y}_E)\rangle\rangle\rangle(t) \} \\
   \{ t : t = ||x + y|| \} = ||x + y||
   \end{cases}
   \]
   \[
   \begin{aligned}
   ||\tilde{x}_E \otimes \tilde{y}_E||^F_E\rangle\rangle\rangle\rangle &= \begin{pmatrix} r ||x|| \} || |x|| = ||r|| |x||
   \end{aligned}
   \]

**5.2 Fuzzy Soft Topology Generated by F.S Norm**

If \((X, E, ||.||)\) is a F.S normed space, then the F.S norm \(||.||\) induces a F.S topology over \(X\) as below.

**Definition 5.5** : Let \((X, E, ||.||)\) be a F.S normed space. The F.S topological space \((X, E, w[w(\tau_{||.||}^{E, \alpha})])\) is called F.S normed topology over \(X\) generated by F.S norm \(||.||\).

**Theorem 5.3** : Let \((X, E, ||.||)\) be a F.S normed space. The topological space \((X, E, w[w(\tau_{||.||}^{E, \alpha})])\) is a F.S Hausdorff space.

**Proof.** Let \(\tilde{x}_E\) and \(\tilde{y}_E\) be two F.S disjoint points in F.S normed space \((X, E, ||.||)\), where for all \(e \in E\), \(\tilde{x}(e) = x^e\) and \(\tilde{y}(e) = y^e\). Hence \(x\) and \(y\) are disjoint points in normed space \((X, ||.||^{E, \alpha})\). Since every normed space is Hausdorff, then there exist open sets \(U\) and \(V\) in \(\tau_{||.||}^{E, \alpha}\), such that \(U \cap V = \emptyset\). Consider F.S open sets \(U_E\) and \(V_E\) in \(w[w(\tau_{||.||}^{E, \alpha})])\) such that for all \(e \in E\), \(U(e) = \chi_U\) and \(V(e) = \chi_V\). It is clear that \(\tilde{x}_E \in U_E\) and \(\tilde{y}_E \in V_E\) and moreover \(U_E \land V_E = \emptyset\).

**5.2.1 Convergency in Fuzzy Soft Normed Spaces**

Here we suggest the concept of "approach" and consequently "limit" in the fuzzy soft set theory. In point-set topology, \(\lim_{x \rightarrow c} f(x) = L\) means that when \(x\) approaches \(c\) sufficiently, \(f(x)\) becomes arbitrarily close to \(L\). By extending this idea in fuzzy soft set theory, where a typical fuzzy soft set is in fact a map whose image is a function from \(X\) into \([0,1]\), the concept of "approximately near with respect to some parameters" can be presented as the following.

Let \(\tilde{x}_E\) and \(\tilde{y}_E\) be F.S points in \(X\). \(\tilde{x}_E\) approaches \(\tilde{y}_E\) means that, for all \(e \in E\) the real function \(\tilde{x}(e)\) becomes close to the real function \(\tilde{y}(e)\) whenever \(x\) approaches \(y\) in normed space \((X, ||.||^{E, \alpha})\). In other words, behavior of objects \(x\) and \(y\) on the basis of some parameters are similar.

**Definition 5.6** : Let \((X, E, ||.||)\) be a F.S normed space. Let \(\tilde{x}_E\) and \(\tilde{y}_E\) be two F.S points in \(X\) such that for each \(e \in E\)
   \[
   \tilde{x}_E(z) = \begin{cases} 
   \lambda_e & \text{if } z = x \\
   0 & \text{if } z \neq x
   \end{cases}
   \]
   \[
   \tilde{y}_E(z) = \begin{cases} 
   \gamma_e & \text{if } z = y \\
   0 & \text{if } z \neq y
   \end{cases}
   \]
   where \(\lambda_e, \gamma_e \in (0,1]\). We say that \(\tilde{x}_E\) approaches \(\tilde{y}_E\), denoting by
   \[
   \tilde{x}_E \rightarrow \tilde{y}_E
   \]
   whenever for all \(e \in E\) and \(\forall e > 0\) there exists \(\delta > 0\) such that
   \[
   ||x - y||^{E, \alpha}_e < \delta \Rightarrow |\lambda_e - \gamma_e| < \epsilon
   \]
   for \(i = 1, 2, \ldots, 2\) and \(\forall e \in (0,1]\).
Definition 5.7: Let $(X, E, ||.||)$ be a F.S normed space. Let $\{x_n\}$ be a sequence in $X$ where $n = 1, 2, 3, \ldots$. For each $n$, let $(\tilde{x}_E)_n$ be a F.S point in $X$ such that $\forall e \in E$ and $\forall z \in X$
$$
(\tilde{x}_E)_n(z) = \begin{cases} \lambda_{e,n} & \text{if } z = x_n \\ 0 & \text{if } z \neq x_n \end{cases}
$$
where $\lambda_{e,n} \in (0, 1]$. Then $\{(\tilde{x}_E)_n\}$ includes F.S points $(\tilde{x}_E)_n$ is called a F.S sequence in $X$.

Definition 5.8: Let $(X, E, ||.||)$ be a F.S normed space. Let $\{(\tilde{x}_E)_n\}$ be a F.S sequence in $X$ mentioned in Definition 5.7. Suppose that $\tilde{x}_E$ is a F.S point in $X$, such that $\forall e \in E$ and $\forall z \in X$
$$
\tilde{x}_E(z) = \begin{cases} \gamma_e & \text{if } z = x \\ 0 & \text{if } z \neq x \end{cases}
$$
where $\gamma_e \in (0, 1]$. By using Definition 5.8, we say that the sequence $\{(\tilde{x}_E)_n\}$ converges to $\tilde{x}_E$ denoting by
$$
(\tilde{x}_E)_n \rightarrow \tilde{x}_E
$$
if and only if $\forall \varepsilon > 0$ there exist positive integer $N$ and $\delta > 0$ such that for all $e \in E$,
$$
\forall n \geq N: ||x_n - x||_{e,\delta} < \delta \Rightarrow |\lambda_{e,n} - \gamma_e| < \varepsilon
$$
or $\lim_{n \to \infty} \lambda_{e,n} = \gamma_e$ whenever in normed spaces $(X, E, ||.||_{e,\delta})$ we have $\lim_{n \to \infty} x_n = x$. We call the sequence $\{(\tilde{x}_E)_n\}$ a F.S convergent sequence in $X$.

Theorem 5.4: Let $(X, E, ||.||)$ be a F.S normed space. If F.S sequence $\{(\tilde{x}_E)_n\}$ converges to $\tilde{x}_E$ and $\tilde{y}_E$, then $\tilde{x}_E = \tilde{y}_E$.

Proof. Let $\{(\tilde{x}_E)_n\}$ be a F.S sequence in $X$ mentioned in Definition 5.7. Let $(\tilde{x}_E)_n \rightarrow \tilde{x}_E$ and $(\tilde{x}_E)_n \rightarrow \tilde{y}_E$ where $\tilde{x}_E$ and $\tilde{y}_E$ are two different F.S points in $X$ such that $\forall e \in E$, are defined as below
$$
\tilde{x}_E(z) = \begin{cases} \xi_e & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}
$$
and
$$
\tilde{y}_E(z) = \begin{cases} \gamma_e & \text{if } z = y \\ 0 & \text{otherwise} \end{cases}
$$
where $z \in X$ and $\xi_e, \gamma_e \in (0, 1]$. Since limit is unique in every normed (or metric) spaces, Definition 5.8 implies that $x = y$ and $\xi_e = \gamma_e, \forall e \in E$. This means that $\tilde{x}_E = \tilde{y}_E$. □

Definition 5.9: Let $(X, E, ||.||)$ be a F.S normed space. Let $\{(\tilde{x}_E)_n\}$ be a F.S sequence in $X$ mentioned in Definition 5.7. We say that the F.S sequence $\{(\tilde{x}_E)_n\}$ is a fuzzy soft Cauchy sequence, say F.S Cauchy sequence, in $X$ if F.S points $\{(\tilde{x}_E)_n\}$ become arbitrarily close to each other as the sequence progress. So we say $\{(\tilde{x}_E)_n\}$ is a F.S Cauchy sequence in $X$ if and only if $\forall \varepsilon > 0$ there exists a positive integer $N$ such that $\forall n, m \geq N$
$$
|\lambda_{e,n} - \lambda_{e,m}| \leq \varepsilon
$$
while $||x_n - x_m||_{e,\alpha} \leq \varepsilon$.

Definition 5.10: Let $(X, E, ||.||)$ be a F.S normed space and $\{(\tilde{x}_E)_n\}$ be a F.S sequence in $X$. Let $\{n_k\}$ be a sequence of positive integers such that $n_1 < n_2 < \ldots$. Then the F.S sequence $\{(\tilde{x}_E)_{n_k}\}$ is called a subsequence of $\{(\tilde{x}_E)_n\}$.

Theorem 5.5: Let $(X, E, ||.||)$ be a F.S normed space and $\{(\tilde{x}_E)_n\}$ be a F.S sequence in $X$. The sequence $\{(\tilde{x}_E)_n\}$ is converges to $\tilde{y}_E$ in $X$ if and only if every subsequence of it is convergent to $\tilde{y}_E$.

Proof. Let $\{(\tilde{x}_E)_n\}$ be a F.S sequence mentioned in Definition 5.7 and $\tilde{y}_E$ be a F.S point in $X$ such that $\forall e \in E$ and $\forall z \in X$,
$$
\tilde{y}_E(z) = \begin{cases} \gamma_e & \text{if } z = y \\ 0 & \text{otherwise} \end{cases}
$$
where $\gamma_e \in (0, 1]$. ⇒ Let the sequence $\{(\tilde{x}_E)_n\}$ converges to $\tilde{y}_E$. Definition 5.8 implies that for each $e \in E$, sequence $\{\lambda_{e,n}\}$ of real numbers converges to $\gamma_e$ while sequence $\{x_n\}$ converges to $y$ in normed spaces $(X, ||.||_{e,\alpha})$. So for every subsequence $\{n_k\}$ of $\{n\}$, we have
$$
\lim_{n_k \to \infty} \lambda_{e,n_k} = \gamma_e
$$
whenever
$$
\lim_{n_k \to \infty} x_{n_k} = y
$$
This implies that $\{(\tilde{x}_E)_n\} \rightarrow \tilde{y}_E$.

⇐ Let $\{n_k\}$ be an arbitrary subsequence of $\{n\}$. So the subsequence $\{(\tilde{x}_E)_{n_k}\}$ of $\{(\tilde{x}_E)_n\}$ is convergent to $\tilde{y}_E$. Definition 5.8 implies $\lambda_{e,n_k} \rightarrow \gamma_e$ whenever $x_{n_k} \rightarrow y$ in normed spaces $\{X, ||.||_{e,\alpha}\}$. Since $\{n_k\}$ is an arbitrary subsequence of $\{n\}$, then $\lambda_{e,n} \rightarrow \gamma_e$ while $x_n \rightarrow y$. Hence $(\tilde{x}_E)_n \rightarrow \tilde{y}_E$. This complete the proof.

Theorem 5.6: Let $(X, E, ||.||)$ be a F.S normed space. Then every F.S convergent sequence is a F.S Cauchy sequence.

Proof. It is implied from Definitions 5.8 and 5.9 □

Definition 5.11: Let $(X, E, ||.||)$ be a F.S normed space. If every F.S Cauchy sequence in $X$ be a F.S convergent sequence, the F.S normed space $X$ is called a F.S Banach space.
5.2.2 Continuity in Fuzzy Soft Normed Spaces

Definition 5.12: Let $X$ and $Y$ be universal sets and $E$ and $E'$ are the parameter sets. Let $(X, E, ||.||)$ and $(Y, E', ||.||)$ be F.S normed spaces. Then the map

$$T : (X, E, ||.||) \rightarrow (Y, E', ||.||)$$

is called a fuzzy soft operator, say F.S operator in brief.

Definition 5.13: Let $(X, E, ||.||)$ and $(Y, E', ||.||)$ be F.S normed spaces where $E$ and $E'$ are the parameter sets and $X$ and $Y$ are universal sets. Then $T : (X, E, ||.||) \rightarrow (Y, E', ||.||)$ is continuous at F.S point $\tilde{x}_E \in X$, called F.S continuous, if for every $\tilde{z}_E \in X$, such that $\tilde{z}_E \rightarrow \tilde{x}_E$ we have $T\tilde{z}_E \rightarrow T\tilde{x}_E$.

We say that $T$ is F.S continuous over $X$ whenever $T$ is continuous at every F.S point of $X$.

Lemma 5.2: Let $(X, E, ||.||)$ and $(Y, E', ||.||)$ be F.S normed spaces. Let $T : (X, ||.||_E^{\epsilon, \alpha}) \rightarrow (Y, ||.||_{E'}^{\epsilon, \alpha})$ be an operator from $X$ into $Y$. Then if there is a bijection between $E$ and $E'$, then for every F.S point $\tilde{x}_E$ in $X$ $T\tilde{x}_E = \tilde{T}\tilde{x}_E$.

Proof: Definition 5.13 implies that

$$(T\tilde{x})_{e'}(y) = \begin{cases} \tilde{x}_e(x) & \text{if } y = Tx \\ 0 & \text{otherwise} \end{cases}$$

This means that $T\tilde{x}_E$ is a F.S point in $Y$ corresponding to $Tx \in Y$. This complete the proof.

Theorem 5.7: Let $(X, E, ||.||)$ and $(Y, E', ||.||)$ be two F.S normed spaces. Let $T : (X, ||.||_E^{\epsilon, \alpha}) \rightarrow (Y, ||.||_{E'}^{\epsilon, \alpha})$ be an operator from $X$ into $Y$. Let $\tilde{x}_E$ and $\tilde{z}_E$ are used to denote F.S points in $X$ corresponding to $x, z \in X$, respectively. Then $T$ is continuous at $\tilde{x}_E \in X$ if and only if $T$ is continuous at $x \in X$ and moreover $\lim_{z \rightarrow x} |\tilde{x}_e(x) - \tilde{z}_e(x)| = 0$ for all $z \in X$.

Proof: It is clear by Lemma 5.2 and Definitions 5.8 and 6.13.

Theorem 5.8: Let $(X, E, ||.||)$ and $(Y, E', ||.||)$ be two F.S normed spaces and $T : (X, ||.||_E^{\epsilon, \alpha}) \rightarrow (Y, ||.||_{E'}^{\epsilon, \alpha})$ be an operator from $X$ into $Y$. Let $g_{E'}$ be a F.S open subset of $Y$ with respect to norm topology $w'(w(\eta||.||_{E'}^{\epsilon, \alpha}))$ induced by F.S norm $||.||$ where $i = 1, 2$ and $\alpha \in (0, 1]$. Then $T$ is F.S continuous over $X$ if and only if $T^{-1}(g_{E'})$ is F.S open subset of $X$ with respect to norm topology $w'(w(\eta||.||_{E'}^{\epsilon, \alpha})).$

Proof: It is clear by applying Theorem 5.7.

6 Fuzzy Soft Fixed-Point Theorem

In this section we give a parameterized extension of fixed-point theorem in fuzzy soft set normed spaces.

Definition 6.1: Let $X$ and $E$ are sets of objects and parameters, respectively. Let $T : (X, ||.||_E^{\epsilon, \alpha}) \rightarrow (X, ||.||_E^{\epsilon, \alpha})$ be an operator. The F.S point $\tilde{x}_E$ is called a F.S fixed point of $T$ if and only if $T\tilde{x}_E = \tilde{x}_E$.

Theorem 6.1: Let $(X, E, ||.||)$ be a F.S Banach spaces. Let $T : (X, ||.||_E^{\epsilon, \alpha}) \rightarrow (X, ||.||_E^{\epsilon, \alpha})$ be an contraction operator, i.e., for some real number like $k$ such that $0 < k < 1$ we have

$$||T x - Ty||_E^{\epsilon, \alpha} \leq k||x - y||_E^{\epsilon, \alpha}$$

where $i = 1, 2, \epsilon, \alpha \in (0, 1]$ and $x, y \in X$. Then $T$ is a F.S continuous map on $X$ and moreover

$$||T\tilde{x}_E - T\tilde{y}_E||_E^{\epsilon, \alpha} \leq k||\tilde{x}_E - \tilde{y}_E||_E^{\epsilon, \alpha}$$

Proof: It is clear by applying Theorem 5.7 and Definition 6.1.

Theorem 6.2: Let $(X, E, ||.||)$ be a F.S Banach spaces. Let $T : (X, ||.||_E^{\epsilon, \alpha}) \rightarrow (X, ||.||_E^{\epsilon, \alpha})$ be an contraction operator, i.e., for some real number like $k$ such that $0 < k < 1$

$$||T x - Ty||_E^{\epsilon, \alpha} \leq k||x - y||_E^{\epsilon, \alpha}$$

where $i = 1, 2, \epsilon, \alpha \in (0, 1]$. Then there exists a unique F.S point in $X$ like $\tilde{x}_E$ such that $T\tilde{x}_E = \tilde{x}_E$. Moreover for any $\tilde{z}_E$ in $X$, $T(T(...(T\tilde{z}_E)...)) \rightarrow \tilde{x}_E$.

Proof: Let $X$ and $E$ be the sets of objects and parameters, respectively. Let $\tilde{z}_E$ be a F.S point in $X$ such that

$$\tilde{z}_e(x) = \begin{cases} \xi_e & \text{if } x = z \\ 0 & \text{otherwise} \end{cases}$$

where for all $e \in E$, $\xi_e \in (0, 1]$. Define the F.S sequence $\{\tilde{x}_E\}$ in $X$ as below:

$T\tilde{x}_E = (\tilde{x}_E)_{1}
\begin{cases} 
T(\tilde{x}_E)_{1} = \tilde{x}_E
\end{cases}$


$T(\tilde{x}_n) = (\tilde{x}_n)_{n=1}^\infty$

where $\tilde{x}_n : E \to I^X$ such that

$$(\tilde{x}_n)_e(x) = \begin{cases} 
\lambda_{e,n} & \text{if } x = x_n \\
0 & \text{otherwise}
\end{cases}$$

Since $\tilde{x}_E$ and $(\tilde{x}_n)_e$ are F.S points in $X$, for all $e \in E$, we can define the sequence $\{x_n\}$ in $X$ as below:

$x_1 = \text{supp} \tilde{x}_1 = T z$

$x_2 = \text{supp} \tilde{x}_2 = T x_1 = T (T z) = T^2 z$

$\vdots$

$x_n = \text{supp} \tilde{x}_n = T x_{n-1} = T (T (\ldots (T z) \ldots ) \ldots ) = T^n z$

where for every $i = 1, 2, \ldots$, $\text{supp} \tilde{x}_i = [\tilde{x}_i]^{-1}(0, 1]$ indicates the support of fuzzy set $\tilde{x}_i$ and $T^n$ shows the composition of $T$ with itself $n$ times.

On the other hand, Lemma 5.1 implies that the membership functions of $(\tilde{x}_n)_e$ are defined as below:

$$(\tilde{x}_1)_e(x) = (T \tilde{z})_e(x) = \begin{cases} 
\xi_e & \text{if } x = x_1 = z \\
0 & \text{otherwise}
\end{cases}$$

$$(\tilde{x}_2)_e(x) = (T \tilde{x}_1)_e(x) = \begin{cases} 
(\tilde{x}_1)_e(x) = \xi_e & \text{if } x = x_2 = x_1 = z \\
0 & \text{otherwise}
\end{cases}$$

$\vdots$

$$(\tilde{x}_n)_e(x) = (T \tilde{x}_{n-1})_e(x) = \begin{cases} 
\xi_e & \text{if } x = x_n = \ldots = x_1 = z \\
0 & \text{otherwise}
\end{cases}$$

So for all $e \in E$ we have $\lambda_{e,n} = \xi_e$, $\forall n \in \mathbb{N}$.

Since $T$ is a contraction map, we can show that

$$\|x_{n+1} - x_n\|_e \leq k^n \|x_1 - z\|_e$$

and moreover, for any $\varepsilon > 0$ since $0 < k < 1$ we can find a large number $N \in \mathbb{N}$ such that $k^N < \frac{\varepsilon}{\|x_1 - z\|_e}$. So for all $m, n \in \mathbb{N}$ such that $m \geq N$ we have

$$\|x_m - x_n\|_e \leq \frac{\varepsilon}{1 - k}$$

Hence we have $\|x_m - x_n\|_e \leq \varepsilon$ and $|\lambda_{e,m} - \lambda_{e,n}| = |e(n) - e(\lambda)| = 0 < \varepsilon$. So Definition 5.1 implies that $(\tilde{x}_E)_n$ is a F.S Cauchy sequence in $X$ and consequently since $X$ is a F.S Banach space, it is a F.S convergent sequence.

Let $(\tilde{x}_E)_n \to \tilde{x}_E$. Therefore Theorems 5.9 and 6.1 imply that $T(\tilde{x}_E)_n \to T\tilde{x}_E$.

On the other hand, $(\tilde{x}_E)_n = T(\tilde{x}_E)_{n-1}$. So $T(\tilde{x}_E)_n \to \tilde{x}_E$. Theorem 6.1 implies that $\tilde{x}_E = T\tilde{x}_E$ i.e., $\tilde{x}_E$ is a F.S fixed point of $T$ in $X$.

Now we show that it is unique. Suppose that $T \tilde{y}_E = \tilde{y}_E$ then Definition 5.9 implies that $0_E \leq \|\tilde{y}_E - \tilde{x}_E\| = \|T \tilde{x}_E - T \tilde{y}_E\| \leq k \|\tilde{x}_E - \tilde{y}_E\|$. Therefore Theorems 5.9 and 6.1 imply that $\tilde{x}_E = \tilde{y}_E = 0_E$.

It means that for all $e \in E$ and $z \in X$ we have

$$(\tilde{x}_E)(z) = \begin{cases} 
1 & \text{if } z = \tilde{x}_E = \tilde{y}_E = 0_E \\
0 & \text{otherwise}
\end{cases}$$

So $(\tilde{x}_E)(e) = (\tilde{y}_E)(e)$.

Hence $\tilde{x}_E$ is the only F.S fixed point of $T$ in $X$. □

7 Discussion and Conclusion

The language used in our daily life situations is usually full of imprecise phrases. Furthermore, measurement applied in the areas related to economics, social science, environmental science, and etc., is not crisp and depends on some parameters such as our measurement tool, time and place of measurement, individual observer, and etc. On the other hand, the collected data from such surveys are not accurate and have degree of uncertainty. Theory of fuzzy sets can model these imprecise data while soft set theory provides a useful method to deal with information related to some parameters. So fuzzy soft set theory can be used as a framework to approach and model these kind of occasions.

Due to determine how much behavior of some objects in an information system are close to each other, the concepts of norm and limit for fuzzy soft sets are needed. To consider the similarity between some geographical regions based on amount of annual rain or behavior of several shareholders with respect to some decision parameters related to stock market are examples of application of fuzzy soft norm and fuzzy soft convergency in real-life situations. In this work, we introduce a norm over the fuzzy soft classes, called F.S norm, to indicate how much the elements of a universal set are close to each other with regards to some parameters. In fact the fuzzy soft norm may help us to construct the equivalence classes over a set based on degree of having some parameters.

We firstly introduce the concept of fuzzy soft number for the first time and then consider some arithmetic operations over them. Then we present fuzzy soft norm and give the relation classes over a set based on degree of having some parameters.
tionship between the concept of norm in the sense of classic and F.S norm. The concept of F.S sequence and F.S convergent sequence are given. We also introduce a parametrization extension of fixed-point theorem in F.S normed spaces. This paper may be the beginning for future research on fuzzy soft inner product, Banach spaces, fixed-point theorem and etc.

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