ON AVERAGED TRACING
OF PERIODIC AVERAGE PSEUDO ORBITS

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Abstract. We propose a definition of average tracing of finite pseudo-orbits
and show that in the case of this definition measure center has the same prop-
erty as nonwandering set for the classical shadowing property. We also show
that the average shadowing property trivializes in the case of mean equicon-
tinuous systems, and that it implies distributional chaos when measure center
is nondegenerate.

1. Introduction. A dynamical system is a pair \((X, T)\), where \(X\) is a compact met-
ric space with a metric \(d\) and \(T: X \to X\) is a continuous map. The notion of the
average shadowing property was introduced in 1988 (see [2, 3]) as a generalization
of the shadowing property. The main motivation was a property suitable for dy-
amical systems obtained by random perturbations, where we cannot control error
of pseudo-orbit in each iterate, but on average the error can be controlled. First ex-
amples of dynamical systems with the average shadowing property were obtained on
manifolds in the class of Axiom A diffeomorphisms (see [2, 14]) and their appropri-
ately chosen random perturbations. Presently it is known that dynamical systems
with the average shadowing property are much more common, since this property is

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a consequence of the specification property or its averaged variant called the almost specification property (e.g. see [2] and [13]). Since the average shadowing property in the system with a fully supported measure implies weak mixing (e.g. see [6] Theorem 4.3), it is natural to expect that such systems will share some properties of systems with shadowing. One of standard properties in theory of shadowing (see [1]) is that shadowing of finite pseudo-orbits is equivalent to shadowing of infinite ones. It is also well known that when a dynamical system \((X, T)\) is that shadowing of finite pseudo-orbits is equivalent to shadowing of infinite systems with shadowing. One of standard properties in theory of shadowing (see Theorem 4.3], it is natural to expect that such systems will share some properties in the system with a fully supported measure implies weak mixing (e.g. see [6, specification property (e.g. see [6] and [18]). Since the average shadowing property is a consequence of the specification property or its averaged variant called the almost\(t\).

Preliminaries.

2. The set of real numbers, integers, natural numbers and nonnegative integers are denoted, respectively, by \(\mathbb{R}, \mathbb{Z}, \mathbb{N} = \mathbb{Z} \cap (0, +\infty)\) and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). If \(A\) is a set, then its complement is denoted \(A^c\) and its closure \(\overline{A}\). The cardinality of a set \(A\) is denoted \(|A|\).

Let \(2^\mathcal{B}(X)\) denote the space of all Borel probability measures on \(X\). A measure \(\mu \in 2^\mathcal{B}(X)\) is invariant for \(T: X \to X\) if \(\mu(A) = \mu(T^{-1}(A))\) for any Borel set \(A \subset X\). The classical Krylov-Bogolyubov theorem implies that every compact dynamical system \((X, T)\) has at least one such measure.

A subset \(A \subset X\) is measure saturated if for every open set \(U\) satisfying \(U \cap A \neq \emptyset\), there exists an invariant measure \(\mu\) such that \(\mu(U) > 0\). The measure center of \(T\) which is denoted \(\text{supp}(X, T)\) is the largest measure saturated subset. It is not difficult to verify that \(\text{supp}(X \times X, T \times T) = \text{supp}(X, T) \times \text{supp}(X, T)\).

2.1. Average tracing of approximate trajectories. Let \(\{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty} \subset X\) and fix any \(\varepsilon > 0\). We define

\[
\Lambda(\{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty}, T, \varepsilon) := \{i \in \mathbb{N}_0 : d(x_i, y_i) < \varepsilon\},
\]

\[
\Lambda^c(\{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty}, T, \varepsilon) := \mathbb{N}_0 \setminus \Lambda(\{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty}, T, \varepsilon)
= \{i \in \mathbb{N}_0 : d(x_i, y_i) \geq \varepsilon\}.
\]

When the map \(T\) is clear from the context, we simply write \(\Lambda(\{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty}, \varepsilon)\) and \(\Lambda^c(\{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty}, \varepsilon)\). Similarly, we use the following simplified notation (for both \(\Lambda\) and \(\Lambda^c\)):

\[
\Lambda(\{x_i\}_{i=0}^{\infty}, T, \varepsilon) := \Lambda(\{x_{i+1}\}_{i=0}^{\infty}, \{T(x_i)\}_{i=0}^{\infty}, T, \varepsilon)
= \{i \in \mathbb{N}_0 : d(T(x_i), x_{i+1}) < \varepsilon\},
\]

\[
\Lambda^c(\{x_i\}_{i=0}^{\infty}, T, \varepsilon) := \mathbb{N}_0 \setminus \Lambda(\{x_{i+1}\}_{i=0}^{\infty}, \{T(x_i)\}_{i=0}^{\infty}, T, \varepsilon)
= \{i \in \mathbb{N}_0 : d(T(x_i), x_{i+1}) \geq \varepsilon\}.
\]
\[ \Lambda^\varepsilon(\{x_i\}_{i=0}^\infty, T, \varepsilon) := \mathbb{N}_0 \setminus \Lambda(\{x_{i+1}\}_{i=0}^\infty, T, \varepsilon) = \{ i \in \mathbb{N}_0 : d(T(x_i), x_{i+1}) \geq \varepsilon \} , \]
\[ \Lambda(\{x_i\}_{i=0}^\infty, T, \varepsilon) := \Lambda(\{f^i(z)\}_{i=0}^\infty, \{x_i\}_{i=0}^\infty, T, \varepsilon) = \{ i \in \mathbb{N}_0 : d(T^i(z), x_i) < \varepsilon \} , \]
\[ \Lambda^\varepsilon(z, \{x_i\}_{i=0}^\infty, T, \varepsilon) := \mathbb{N}_0 \setminus \Lambda(z, \{x_i\}_{i=0}^\infty, T, \varepsilon) = \{ i \in \mathbb{N}_0 : d(T^i(z), x_i) \geq \varepsilon \} . \]

Finally, we will denote the above sets restricted on a subset \( A \) of \( \mathbb{N}_0 \) by
\[ \Lambda_A(\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty, T, \varepsilon) := A \cap \Lambda(\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty, T, \varepsilon) , \]
\[ \Lambda_A^\varepsilon(\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty, T, \varepsilon) := A \cap \Lambda^\varepsilon(\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty, T, \varepsilon) . \]

**Definition 2.1.** Let \( \delta > 0 \) and let \( \xi = \{x_i\}_{i=0}^\infty \subset X \). We say that \( \xi \) is
1. a \( \delta \)-ergodic pseudo-orbit (of \( T \)) if
\[ \lim_{n \to \infty} \frac{1}{n} |\Lambda^\varepsilon_{[0,n]}(\xi, T, \delta)| = 0 ; \]
2. a \( \delta \)-average-pseudo-orbit (of \( T \)) if there exists \( N > 0 \) such that for all \( n \geq N \) and \( k \in \mathbb{N}_0 \),
\[ \frac{1}{n} \sum_{i=0}^{n-1} d(T(x_{i+k}), x_{i+k+1}) < \delta ; \]
3. a \( \delta \)-asymptotic-average-pseudo-orbit (of \( T \)) if
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T(x_i), x_{i+1}) < \delta ; \]
4. an asymptotic average pseudo-orbit (of \( T \)) if
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T(x_i), x_{i+1}) = 0 . \]

A \( \delta \)-average-pseudo-orbit is periodic if it is formed by a periodic sequence, i.e. \( x_i = x_{i+s} \) for some integer \( s > 0 \) and all \( i \in \mathbb{N}_0 \).

We use the above notions of approximate trajectories to define three main shadowing properties of the paper.

**Definition 2.2.** A dynamical system \((X, T)\) has
1. the average shadowing property (abbrev. ASP) if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that every \( \delta \)-average-pseudo-orbit \( \{x_i\}_{i=0}^\infty \) is \( \varepsilon \)-shadowed on average by a point \( z \in X \), i.e.
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(z), x_i) < \varepsilon ; \]
2. the average shadowing of periodic (or finite) pseudo-orbits property (abbrev. FinASP) if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that every periodic \( \delta \)-average pseudo-orbit is \( \varepsilon \)-shadowed on average by a point in \( X \);
3. the asymptotic average shadowing property (abbrev. AASP) if every asymptotic average pseudo-orbit \( \{x_i\}_{i=0}^\infty \) is asymptotically shadowed on average by a point \( z \in X \), i.e.
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(z), x_i) = 0 . \]
The following simple fact shows that there is no much difference between finite and periodic average $\delta$-pseudo orbits. The proof is simple and left to the reader.

**Proposition 2.1.** Let $\delta > 0$. If a finite sequence $w_0, \ldots, w_{s-1}$ satisfies

$$\frac{1}{s-1} \sum_{i=0}^{s-2} d(T(w_i), w_{i+1}) < \frac{\delta}{2}$$

and $\text{diam}(X)/s < \delta/2$, then periodic sequence given by $x_{j+i} = w_i$ for all $j \in \mathbb{N}_0$ and $0 \leq i < s$ is a $\delta$-average pseudo-orbit.

A $\delta$-chain from $x$ to $y$ is a finite $\delta$-pseudo-orbit between these points, that is, a sequence $x_1, \ldots, x_{n+1}$ such that $d(T(x_i), x_{i+1}) < \delta$ for all $i = 1, \ldots, n$, and $x_1 = x$, $x_{n+1} = y$. A map is chain transitive if for any $\delta > 0$ and any two points $x, y \in X$ there is a $\delta$-chain from $x$ to $y$. Chain transitivity is a natural generalization of transitivity. It is clear that if a map is chain transitive then it must be surjective as well. There is a surprising result [13] which shows that chains do not distinguish between total transitivity and mixing. Precisely speaking, if $(X, T^n)$ is chain transitive for all $n > 0$ then it is chain mixing, that is, for any $x, y \in X$ and $\delta > 0$ there is $N > 0$ such that there is a $\delta$-chain from $x$ to $y$ consisting of exactly $n$ elements for every $n > N$. Recently, Wu et al. [19] proved that this also holds for iterated function systems. A point $x \in X$ is chain recurrent if for every $\delta > 0$ there exists a $\delta$-chain from $x$ to $x$. The set of all chain recurrent points is denoted $\text{CR}(T)$. By compactness, it is not difficult to check that $\text{CR}(T)$ is a closed set and $T(\text{CR}(T)) = \text{CR}(T)$.

### 2.2. Furstenberg families and tracing.

A (Furstenberg) family $\mathcal{F}$ is a collection of subsets of $\mathbb{N}_0$ which is upwards hereditary, that is

\[ F_1 \in \mathcal{F} \text{ and } F_1 \subset F_2 \implies F_2 \in \mathcal{F}. \]

The dual family of $\mathcal{F}$ is

\[ \mathcal{F}^* := \{ A \subset \mathbb{N}_0 : \forall F \in \mathcal{F}, A \cap F \neq \emptyset \}. \]

A set $A \subset \mathbb{N}_0$ is syndetic if it has bounded gaps, i.e. there is $k > 0$ such that $A \cap [i, i+k) \neq \emptyset$ for all $i \geq 0$ and thick if it belongs to the dual family $\mathcal{F}^* = \mathcal{F}_s^*$, where $\mathcal{F}_s$ denotes the family of syndetic sets.

For any $A \subset \mathbb{N}_0$, the upper density of $A$ is defined by

\[ d(A) := \limsup_{n \to \infty} \frac{1}{n} |A \cap \{0, 1, \ldots, n-1\}|. \]  

(1)

Replacing $\lim sup$ with $\lim inf$ in (1) we obtain the definition of $d(A)$, the lower density of $A$. If there exists a number $d(A)$ such that $d(A) = d(A) = d(A)$ then we say that the set $A$ has density $d(A)$. Fix any $\alpha \in (0, 1)$ and denote by $\mathcal{M}_\alpha$ (resp. $\mathcal{M}^\alpha$) the family consisting of sets $A \subset \mathbb{N}_0$ with $d(A) > \alpha$ (resp. $d(A) > \alpha$). We denote by $\mathcal{M}_\alpha$ the family of sets with $d(A) \geq \alpha$. Clearly $\mathcal{M}_1$ consists of sets $A$ with $d(A) = 1$.

**Definition 2.3.** A dynamical system $(X, T)$ has (ergodic) $\mathcal{F}$-shadowing property if, for any $\varepsilon > 0$ there is $\delta > 0$ such that every $\delta$-ergodic pseudo-orbit $\xi$ is $\mathcal{F}$-$\varepsilon$-shadowed by some point $z \in X$, i.e.

\[ \Lambda(z, \xi, \varepsilon) \in \mathcal{F}. \]
In the special case of $\mathcal{F} = \mathcal{M}_1$ (resp., $\mathcal{F} = \mathcal{M}_0$ and $\mathcal{M}^{1/2}$), we say that $(X, f)$ has the ergodic shadowing property (resp., $d$-shadowing property and $\bar{d}$-shadowing property).

2.3. The (almost) specification property. The specification property was first introduced by Bowen [4]. It is one of the strongest mixing properties that can be expected from a dynamical system. Recently, Pfister and Sullivan introduced in [12] a property called the $g$-almost product property, which generalizes Bowen’s specification in terms of average tracing. Inspired by [12], Thompson in [16] modified slightly this definition and proposed to call it the almost specification property, which in turn generalizes the notion of specification. In this paper, we adopt the concepts of [16]. First, we introduce some auxiliary notation.

Let $\epsilon_0 > 0$. A function $g: \mathbb{N}_0 \times (0, \epsilon_0] \to \mathbb{N}$ is called a mistake function if, for all $\epsilon \in (0, \epsilon_0]$ and all $n \in \mathbb{N}_0$, we have $g(n, \epsilon) \leq g(n + 1, \epsilon)$ and

$$\lim_{n \to \infty} \frac{g(n, \epsilon)}{n} = 0.$$ 

Given a mistake function $g$, if $\epsilon > \epsilon_0$, then we define $g(n, \epsilon) = g(n, \epsilon_0)$.

For $n$ sufficiently large satisfying $g(n, \epsilon) < n$, we define the set of $(g; n, \epsilon)$ almost full subsets of $\{0, \ldots, n - 1\}$ as the family $I(g; n, \epsilon)$ consisting of subsets of $\{0, 1, \ldots, n - 1\}$ with at least $n - g(n, \epsilon)$ elements, that is,

$$I(g; n, \epsilon) := \{A \subset \{0, 1, \ldots, n - 1\} : |A| \geq n - g(n, \epsilon)\}.$$

For a finite set of indexes $A \subset \{0, 1, \ldots, n - 1\}$, we define the Bowen distance between $x, y \in X$ along $A$ by $d_A(x, y) = \max\{d(f^j(x), f^j(y)) : j \in A\}$ and the Bowen ball (of radius $\epsilon$ centered at $x \in X$) along $A$ by $B_A(x, \epsilon) = \{y \in X : d_A(x, y) < \epsilon\}$. When $g$ is a mistake function and $(n, \epsilon)$ is such that $g(n, \epsilon) < n$, we define $x$ to be a $g(\cdot; n, \cdot)$-Bowen ball of radius $\epsilon$, center $x$, and length $n$ by

$$B_n(g; x, \epsilon) := \left\{y \in X : y \in B_A(x, \epsilon) \text{ for some } A \in I(g; n, \epsilon) \right\} = \bigcup_{A \in I(g; n, \epsilon)} B_A(x, \epsilon).$$

Using the above notation, we are able to present the definition of the almost specification property.

**Definition 2.4.** A dynamical system $(X, T)$ has the almost specification property if there exists a mistake function $g$ and a function $k_g: (0, \infty) \to \mathbb{N}$ such that for any $m \geq 1$, any $\epsilon_1, \ldots, \epsilon_m > 0$, any points $x_1, \ldots, x_m \in X$, and any integers $n_1 \geq k_g(\epsilon_1), \ldots, n_m \geq k_g(\epsilon_m)$ setting $n_0 = 0$ and

$$l_j = \sum_{s=0}^{j-1} n_s, \text{ for } j = 1, \ldots, m,$$

one can find a point $z \in X$ such that for every $j = 1, \ldots, m$,

$$T_{l_j}(z) \in B_{n_j}(g; x_j, \epsilon_j).$$

In other words, the appropriate part of the orbit of $z$, $\epsilon_j$-traces with at most $g(\epsilon_j, n_j)$, mistakes the orbit of $x_j$, $j = 1, \ldots, m$. 
2.4. Distributional chaos. A very important generalization of the concept of Li-Yorke chaos is distributional chaos which was introduced by Schweizer and Smítal in [15]. Let \((X, T)\) be a dynamical system. For any pair \(x, y \in X\) and any \(n \in \mathbb{N}\), let
\[
\Phi_{x,y}^{(n)}(t) := \left| \{0 \leq i < n : g(T^i(x), T^i(y)) < t\} \right|.
\]
Define the lower and upper distributional functions, \(\mathbb{R} \rightarrow [0, 1]\) generated by \(T, x\) and \(y\), as
\[
\Phi_{x,y}(t, T) = \liminf_{n \to \infty} \frac{1}{n} \Phi_{x,y}^{(n)}(t),
\]
and
\[
\Phi_{x,y}^*(t, T) = \limsup_{n \to \infty} \frac{1}{n} \Phi_{x,y}^{(n)}(t),
\]
respectively. Both functions \(\Phi_{x,y}\) and \(\Phi_{x,y}^*\) are nondecreasing. For any pair \(x, y \in X\), if \(\Phi_{x,y} \equiv 1\) and \(\Phi_{x,y}(\eta, T) = 0\) for some \(\eta > 0\), then \((x, y)\) is called a distributionally \(\eta\)-chaotic pair of \(T\). A subset \(D \subset X\) containing at least two points is called a distributionally \(\eta\)-scrambled set for some \(\eta > 0\) if any pair of its distinct points is distributionally \(\eta\)-chaotic. A dynamical system is said to be distributionally \(\eta\)-chaotic if there exists an uncountable distributionally \(\eta\)-scrambled set.

3. FinASP. In the case of standard definition of shadowing property, it is not much different situation if we know that every finite \(\delta\)-pseudo orbit can be traced or we know that tracing is possible for infinite \(\delta\)-pseudo orbits. Namely, using compactness it is easy to see that both approaches are equivalent. This argument does not work, however, in the case of averaged versions of shadowing. It is also not completely clear how finite pseudo-orbits and their tracing should be defined. A possible approach is introduced below. It is easy to see that this new notion is not stronger than ASP, however presently we still do not know how whether both definitions are equivalent.

3.1. FinASP and measure center. It is known that if \((X, T)\) has the shadowing property, then also its restriction to the nonwandering set \((\Omega(T), T)\) has the shadowing property. A natural candidate to replace \(\Omega(T)\) in the case of average shadowing is the measure center. We prove that this intuition is true in the case of FinASP. Unfortunately our argument is not sufficient to work for ASP.

**Theorem 3.1.** A dynamical system \((X, T)\) has FinASP if and only if \((\text{supp}(X, T), T)\) has FinASP.

**Proof.** \((\Leftarrow).\) Without loss of generality, assume that \(\text{diam}(X) = 1\). For any \(\varepsilon > 0\), let \(0 < \delta < \varepsilon/4\) be such that every periodic \(\delta\)-average pseudo-orbit contained in \(\text{supp}(X, T)\) is \(\varepsilon/4\)-traced on average by a point in \(\text{supp}(X, T)\). Applying [6] Lemma 5.4 implies that there is \(0 < \delta_1 < \delta/4\) such that for every \(\delta_1\)-average pseudo-orbit \(\{x_i\}_{i=0}^\infty\), there is a \(\delta/4\)-average pseudo-orbit \(\{y_i\}_{i=0}^\infty \subset \text{supp}(X, T)\) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \|\{0 \leq i < n : d(x_i, y_i) \geq \delta/4\}\| < \frac{\delta}{4}.
\]

For any periodic \(\delta_1\)-average pseudo-orbit \(\{x_i\}_{i=0}^\infty \subset X\), take a \(\delta/4\)-average pseudo-orbit \(\{y_i\} \subset \text{supp}(X, T)\) satisfying condition \([2]\) and let \(p\) be a period of \(\{x_i\}\) such that at least \(p > 12/\delta\). There is an integer \(K > p\) such that for any \(k \geq K\),
\[
\frac{1}{k} \sum_{i=0}^{k-1} d(T(y_i), y_{i+1}) < \frac{\delta}{4}.
\]
and
\[ \frac{1}{k} \{ 0 \leq i < k : d(x_i, y_i) \geq \delta/4 \} < \frac{\delta}{4}. \] 

(4)

Take a sequence
\[ \xi = \xi_0 \xi_1 \xi_2 \cdots = y_0 y_1 \cdots y_{Kp-1} y_0 y_1 \cdots y_{Kp-1} \cdots. \]

For any \( n \geq Kp \) and any \( j \in \mathbb{N}_0 \), applying (3), it can be verified that
\[ \frac{1}{n} \sum_{i=0}^{n-1} d(T(\xi_{i+j}), \xi_{i+j+1}) \leq \frac{\delta}{4} + \frac{n/Kp + 2 + 2K}{n} \text{diam}(X) \leq \frac{\delta}{4} + \frac{5}{p} < \delta. \]

This shows that \( \xi \) is a periodic \( \delta \)-average pseudo-orbit contained in \( \text{supp}(X, T) \).

Then there is \( z \in \text{supp}(X, T) \) such that
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} d(T^i(z), \xi_i) < \frac{\varepsilon}{4}. \]

Combining this with [3] and (4), it follows that for any \( n \geq Kp \),
\[ \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(z), x_i) \leq \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(z), \xi_i) + \frac{1}{n} \sum_{i=0}^{n-1} d(\xi_i, x_i) \]
\[ \leq \frac{1}{n-1} \sum_{i=0}^{n} d(T^i(z), \xi_i) + \frac{\delta}{4} + \frac{1}{n} \sum_{i \in \Lambda_{[0,n)}(\{\xi_i, x_i\}, \delta/4)} d(\xi_i, x_i) \]
\[ \leq \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(z), \xi_i) + \frac{\delta}{4} + \frac{\delta}{2} = \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(z), \xi_i) + \frac{3\varepsilon}{16} \]

and therefore
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(z), x_i) < \varepsilon. \]

\((\implies)\). Suppose that there is \( \varepsilon > 0 \) such that for every \( \delta > 0 \) there is a periodic \( \delta \)-average pseudo-orbit contained in \( \text{supp}(X, T) \) which cannot be \( \varepsilon \)-traced in average by any point in \( \text{supp}(X, T) \). Let \( \delta > 0 \) be provided to \( \varepsilon/2 > 0 \) by FinASP and let \( \{x_i\}_{i=0}^{\infty} \subset \text{supp}(X, T) \) be a periodic \( \delta \)-average pseudo-orbit such that for every \( x \in \text{supp}(X, T) \) we have \( \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} d(T^j(x), x_j) \geq \varepsilon \). Let \( z \) be a point which \( \varepsilon/2 \)-traces \( \{x_i\}_{i=0}^{\infty} \) in average, that is
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} d(T^j(z), x_j) < \varepsilon/2. \]

By our assumption, \( z \notin \text{supp}(X, T) \).

Take \( \gamma < \varepsilon/8 \) and let \( p \) be the least period of \( \{x_i\}_{i=0}^{\infty} \). There is an integer \( N > 0 \) such that \( p \text{diam}(X)/N < \gamma \). Since \( \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} d(T^j(x), x_j) > \varepsilon - \gamma \) for every \( x \in \text{supp}(X, T) \), there is an open cover \( U_1, \ldots, U_s \) of \( \text{supp}(X, T) \) and numbers \( n_i > N \) such that, if \( x \in U_i \) then
\[ \frac{1}{n_i} \sum_{j=0}^{n_i-1} d(T^j(x), x_j) > \varepsilon - \gamma. \]
Let $M = \max_{i=1,\ldots,s} n_i$ and let $\eta > 0$ be such that if $d(x, y) < \eta$ then $d(T^j(x), T^j(y)) < \gamma$ for $j = 0, \ldots, M$. Note that $T|_{\text{supp}(X, T)}$ is onto, hence the set $\bigcap_{k=0}^p T^{-k}(\bigcup_{i=1}^s U_i)$ is a neighborhood of $\text{supp}(X, T)$. We assume that $\eta$ is sufficiently small, so that

$$
\overline{B}_\eta(\text{supp}(X, T)) \subset \bigcap_{k=0}^p T^{-k}(\bigcup_{i=1}^s U_i),
$$

where $B_\eta(A)$ is the $\eta$-neighborhood of a subset $A$ of $X$, i.e., $B_\eta(A) = \bigcup_{y \in A} \{y \in X : d(x, y) < \eta\}$. Let $\xi < \eta/4$ be such that $2\xi$ is a Lebesgue number for each of the covers $T^{-k}(\{U_1, \ldots, U_s\})$, $k = 0, \ldots, p$ of the set $\overline{B}_\eta(\text{supp}(X, T))$. Observe that $A = X \setminus B_\xi(\text{supp}(X, T))$ is a universally null set such that $\text{dist}(A, \text{supp}(X, T)) > 0$, hence there is a set $K$ of density $d(K) = 1$ such that $T^j(z) \in B_\xi(\text{supp}(X, T))$ for every $j \in K$.

Define a sequence $\{t_j\}_{j=0}^\infty$ of integers in the following way. Put $t_0 = 0$ and, next, if $T^j(z) \in A$ then we put $t_{j+1} = t_j + 1$. If $T^j(z) \notin A$ then there is $y_j \in \text{supp}(X, T)$ such that $d(y_j, T^j(z)) < \xi$. By the definition of Lebesgue number and condition $2\xi < \eta$, there is $1 \leq i = i(j) \leq s$ such that

$$
B_\xi(T^j(y_j)) \subset B_\xi(y_j) \subset T^{-k}(U_i) \cap B_\eta(\text{supp}(X, T)),
$$

where $0 \leq k < p$ is a number such that $x_{t_j + i + k} = x_i$ for every $i \geq 0$. Then we put $t_{j+1} = t_j + n_i$. Since sequence $\{t_j\}_{j=0}^\infty$ is syndetic, there is a set $K \subset \mathbb{N}$ such that $d(K) = 1$ and $\{t_j : j \in K\} \subset K$. Note that there exists $j \in K$ such that

$$
\frac{1}{t_j} \sum_{i=t_j}^{t_{j+1}-1} d(T^i(z), x_i) < \frac{\varepsilon}{2} (t_{j+1} - t_j),
$$

because otherwise

$$
\frac{1}{t_j} \sum_{i=0}^{t_{j-1}-1} d(T^i(z), x_i) = \frac{1}{t_j} \sum_{r=0}^{j-1} (t_{r+1} - t_r) \frac{1}{t_{r+1} - t_r} \sum_{i=t_r}^{t_{r+1}-1} d(T^i(z), x_i)
\geq \frac{1}{t_j} \sum_{r \in K \cap [0,j)} (t_{r+1} - t_r) \frac{1}{t_{r+1} - t_r} \sum_{i=t_r}^{t_{r+1}-1} d(T^i(z), x_i)
\geq \frac{1}{t_j} \sum_{r \in K \cap [0,j)} (t_{r+1} - t_r) \varepsilon/2
\geq \frac{\varepsilon}{2t_j} \left( \sum_{r \in K \cap [0,j]} d(\hat{K}) - M \left( \left| 0, t_j \right| \setminus \hat{K} \right) \right)
\to \frac{\varepsilon}{2} \left( d(\hat{K}) - M(1 - d(\hat{K})) \right) = \frac{\varepsilon}{2},
$$

which is a contradiction. Let $j \in K$ be an integer such that

$$
\frac{1}{t_{j+1}-t_j} \sum_{i=t_j}^{t_{j+1}-1} d(T^i(z), x_i) < \frac{\varepsilon}{2} (t_{j+1} - t_j).
$$

There is a unique $0 \leq k < p$ such that $\{x_i\}_{i=0}^\infty = \{x_i\}_{i=t_j+k}^\infty$. Since $T^i(z) \notin A$, by the definition of $t_j$ there are $y_j \in \text{supp}(X, T)$ and $1 \leq m \leq s$ such that

$$
T^i(z) \in B_\xi(y) \subset T^{-k}(U_m).
$$
Then for \( r = 0, \ldots, n_m - 1 \) we have \( d(T^{i+r}(z), T^r(y)) < \gamma \) and hence
\[
\sum_{i=t_j}^{t_{j+1}-1} d(x_i, T^{i-t_j}(y)) \leq \sum_{i=t_j}^{t_{j+1}-1} d(x_i, T^i(z)) + \sum_{i=t_j}^{t_{j+1}-1} d(T^i(z), T^{i-t_j}(y)) \\
< (\varepsilon/2 + \gamma)(t_{j+1} - t_j).
\]
Denote \( x = T^k(y) \in \text{supp}(X,T) \cap U_m \) and observe that \( \{x_i\}_{i=0}^{\infty} = \{x_i\}_{i=t_j+k}^{\infty} \) and \( t_{j+1} = t_j + n_m \) which gives
\[
\sum_{i=0}^{n_m-1} d(T^i(x), x_i) \leq \sum_{i=0}^{n_m+k-1} d(T^i(y), x_{t_j+i}) \\
\leq (\varepsilon/2 + \gamma)n_m + \sum_{i=n_m}^{n_m+k-1} d(x_{t_j+i}, T^i(y)) \\
\leq (\varepsilon/2 + \gamma)n_m + p\text{diam}(X).
\]
Since \( x \in U_m \), this implies that
\[
\varepsilon - \gamma < \frac{1}{n_m} \sum_{i=0}^{n_m-1} d(x_i, T^i(x)) < \frac{\varepsilon}{2} + 2\gamma < \varepsilon - \gamma
\]
which is a contradiction. The proof is completed. \( \Box \)

By the fact that \( (\text{supp}(X,T), T) \) is a surjection and statement of Theorem 3.1 a slight change in the proof of [6, Lemma 3.1] leads to the following result. We leave details to the reader.

**Corollary 3.2.** If a dynamical system \((X,T)\) has FinASP, then \((\text{supp}(X,T), T)\) is weakly mixing. In particular, \((\text{supp}(X,T), T)\) is chain mixing.

3.2. **A remark on the ergodic shadowing property.** In [5, Theorem A], Fakhari and Gane proved that for a surjective dynamical system, ergodic shadowing is equivalent to pseudo-orbital specification. Now, we shall show that ergodic shadowing implies almost specification. Before we are able to present a simple proof, we need the following facts from [18].

**Theorem 3.3.** [18, Corollary 5.6] If a dynamical system \((X,T)\) has the ergodic shadowing property, then it also has the average shadowing property.

**Theorem 3.4.** [18, Theorem 6.7] A dynamical system \((X,T)\) has the almost specification property if and only if \((\text{supp}(X,T), T)\) has the almost specification property.

Now we are ready to prove the following.

**Theorem 3.5.** If a dynamical system \((X,T)\) has the ergodic shadowing property, then \((X,T)\) has the almost specification property.

Proof. Clearly, the ergodic shadowing property implies the thick shadowing property, i.e., \(\mathcal{F}_T\)-shadowing property. According to the proof of [10, Theorem 4.5], it follows that every point in \(\text{CR}(T)\) can be presented as a limit of minimal points in \(X\). This, together with [10, Theorem 4.5 (1)], implies that the measure center of \((X,T)\) is \(\text{CR}(T)\), i.e., \(\text{supp}(X,T) = \text{CR}(T)\) and that \((\text{CR}(T), T)\) has the shadowing property. It follows from Theorem 3.3 and Corollary 5.2 that \((\text{CR}(T), T)\) is chain mixing. Applying [5, Theorem A] yields that \((\text{CR}(T), T)\) has the specification property as \((\text{CR}(T), T)\) is surjective. Hence the result follows by Theorem 3.4.
As a direct consequence of the proof of Theorem 3.5, we have

**Corollary 3.6.** If a dynamical system \((X, T)\) has the ergodic shadowing property, then \((\text{supp}(X, T), T)\) has the specification property.

**Corollary 3.7.** If a dynamical system \((X, T)\) has the ergodic shadowing property, then \((X, T)\) has AASP.

*Proof.* It follows immediately from Theorem 3.5 and [18, Corollary 6.9].

Note that if \((X, T)\) is chain transitive on \(\text{CR}(T)\) and \(x\) has well defined backward extension, i.e. there exists a sequence \(\{x_i\}_{i=0}^{\infty}\) such that \(x = x_0\) and \(T(x_{i-1}) = x_i\) for every \(i \leq 0\) then \(x \in \text{CR}(T)\). Then the only situation when \((X, T)\) with the ergodic shadowing property has the almost specification property but does not have the specification property occurs when there are starting points in \(X\), that is points \(x \in X\) with \(T^{-1}(\{x\}) = \emptyset\). An example of such a map is \(T: [0, 2] \to [0, 2]\) defined by \(T(x) = 4x(1-x)\) for \(x \leq 1\) and \(T(x) = x - 1\) for \(x > 1\).

### 3.3. Mean equicontinuity and FinASP

In [7], the authors introduced the notion of mean equicontinuity which is equivalent to mean-L-stability and proved that every ergodic invariant measure of a mean equicontinuous system has discrete spectrum. According to them, a dynamical system \((X, T)\) is called *mean equicontinuous* if for every \(\varepsilon > 0\), there exists \(\delta > 0\) such that whenever \(x, y \in X\) with \(d(x,y) < \delta\),

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(x), T^i(y)) < \varepsilon.
\]

A pair \((x, y) \in X \times X\) is *proximal* if \(\lim \inf_{n \to \infty} d(T^n(x), T^n(y)) = 0\). A dynamical system \((X, T)\) is called *proximal* if any pair of two points in \(X\) is proximal. If \((x, y) \in X \times X\) is not proximal, then it is said to be *distal*.

Clearly each equicontinuous system is mean equicontinuous system. For example it can be periodic orbit or an odometer (systems having classical shadowing property) or irrational rotation of the circle (which does not have shadowing property). Another example of mean equicontinuous map (which is not equicontinuous) can be the circle map induced from \([0, 1]\) by \(x \to x^2\) with both endpoints \(0, 1\) identified. Clearly this circle map has the unique invariant measure concentrated on the fixed point \(0\), hence its measure center is trivial.

Here, we shall show that for a mean equicontinuous dynamical system, almost specification, AASP, ASP, FinASP, \(d\)-shadowing and \(\bar{d}\)-shadowing are all equivalent. Strictly speaking, we prove that if any of the above properties is present in mean equicontinuous dynamical system, then it is trivial from measure theoretic point of view.

**Theorem 3.8.** If a dynamical system \((X, T)\) is mean equicontinuous and has FinASP, \(\bar{d}\)-shadowing or \(d\)-shadowing, then \((X, T)\) has trivial measure center. In particular, \((X, T)\) is proximal.

*Proof.* As a preliminary step we prove that \((X, T)\) is proximal. First, we consider the case of FinASP. Assume on the contrary that \((X, T)\) is not proximal. Let \(x, y\) be a distal pair and put \(\varepsilon = \frac{1}{3} \inf \{d(T^n(x), T^n(y)) : n \in \mathbb{N}_0\} > 0\). Let \(\delta > 0\) be such that if \(d(p, q) < \delta\) then

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(p), T^i(q)) < \varepsilon. \tag{5}
\]
Let \( \eta > 0 \) be such that every periodic \( \eta \)-average pseudo-orbit is \( \delta/3 \)-traced on average by some point. Take \( N \in \mathbb{N} \) sufficiently large, so that \( 3 \text{diam}(X)/N < \eta \). Then the periodic sequence

\[
\xi = x, T(x), \ldots, T^{N-1}(x), T^N(y), T^{N+1}(y), \ldots, T^{2N-1}(y), x, \ldots
\]

is a periodic \( \eta \)-average pseudo-orbit because if we fix any \( n \geq 2N \) and \( k \in \mathbb{N}_0 \), then there is \( j \) such that \( jN \leq n < (j+1)N \) and then

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(T(x_{i+k}), x_{i+k+1}) < \frac{1}{jN} (j+2)N \text{diam}(X) \leq \frac{2 \text{diam}(X)}{N} < \eta.
\]

Let \( z \in X \) be a point which \( \delta/3 \)-traces \( \xi \) on average. Suppose that for every \( k \in \mathbb{N}_0 \) and we either have \( d(T^{KN+j}(z), T^j(x)) \geq \delta \) for every \( j = 0, \ldots, N-1 \) or \( d(T^{KN+j+N}(z), T^{j+N}(y)) \geq \delta \) for every \( j = 0, \ldots, N-1 \). Then for each \( s > 2 \) and each \( 2sN \leq n \leq 2(s+1)N \) we have

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(T^i(z), \xi_i) \geq \frac{\delta s N}{2(s+1)N} \geq \frac{\delta}{3},
\]

which is impossible. Therefore there exist \( K \geq 0 \) and \( 0 \leq i, j < N \) such that

\[
d(T^{KN+i}(z), T^i(x)) < \delta \quad \text{and} \quad d(T^{KN+N+j}(z), T^{N+j}(y)) < \delta.
\]

This, together with [5], implies that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^{KN+i}(z), T^i(x)) < \varepsilon, \quad \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^{KN+i}(z), T^i(y)) < \varepsilon,
\]

and therefore

\[
2\varepsilon \geq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} (d(T^{KN+i}(z), T^i(x)) + d(T^{KN+i}(z), T^i(y)))
\]

\[
\geq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(x), T^i(y)) \geq 3\varepsilon,
\]

which is a contradiction. Indeed \((X, T)\) is proximal.

Next assume that \( \text{supp}(X, T) \) is nontrivial. By Theorem [3.1] without loss of generality we may assume that \( X = \text{supp}(X, T) \). Let \( p \) be the unique fixed point in \( X \) (as \((X, T)\) is proximal) and let \( U \) be a nonempty open set such that \( \text{dist}(p, \overline{U}) := \inf \{d(p, y) : y \in \overline{U}\} = \eta > 0 \). By ergodic decomposition theorem there is an ergodic measure \( \mu \) such that \( \mu(U) = \tau > 0 \). Let \( x \) be a generic point for \( \mu \). Then there exists \( N > 0 \) such that for every \( n \geq N \), we have \( \frac{1}{n} \left| \{0 \leq i < n : T^i(x) \in U\} \right| > \tau/2 \). Take any \( \varepsilon > 0 \) such that \( \varepsilon < \tau \eta/6 \).

Increasing \( N \) if necessary and repeating previous arguments, we can find a point \( z, k \geq 0 \) and \( 0 \leq i, j < N \) such that

\[
d(T^{kN+i}(z), T^i(x)) < \varepsilon \quad \text{and} \quad d(T^{kN+N+j}(z), T^{N+j}(p)) < \varepsilon.
\]

Similarly as before we obtain that

\[
2\varepsilon \geq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(x), T^i(p))
\]

\[
\geq \liminf_{n \to \infty} \frac{1}{n} \left| \{0 \leq i < n : T^i(x) \in U\} \right| \text{dist}(p, \overline{U}) > 3\varepsilon,
\]
which is a contradiction.

The proof of the case of $d$-shadowing or $d$-shadowing is analogous, with the main difference in definition of pseudo-orbit $\xi$. Let $m_0 = m'_0 = 0$, $m_1 = 2$, $m'_1 = 1$, $m_n = 2^{m_1 + \cdots + m_{n-1}}$ and $m'_n = m_{n-1} + n$ for $n \geq 2$ and take $M_1 = \bigcup_{i=0}^{\infty}[m_{2i}, m_{2i+1})$, $M_2 = \bigcup_{i=0}^{\infty}[m_{2i+1}, m_{2i+2})$ and $M'_1 = \bigcup_{i=0}^{\infty}[m'_{2i}, m'_{2i+1})$, $M'_2 = \bigcup_{i=0}^{\infty}[m'_{2i+1}, m'_{2i+2})$. It is easy to see that $d(M_1) = d(M_2) = 1$ and $d(M'_1) = d(M'_2) = 1/2$. For the cases of $d$-shadowing and $d$-shadowing, we replace $\xi$, respectively, by the ergodic pseudo-orbits
\[
\xi = x, \ldots, T^{m_1-1}(x), T^{m_1}(y), \ldots, T^{m_2-1}(y),
T^{m_2}(x), \ldots, T^{m_3-1}(x)T^{m_3}(y), \ldots, T^{m_4-1}(y), \ldots
\]
and
\[
\xi = x, \ldots, T^{m'_1-1}(x), T^{m'_1}(y), \ldots, T^{m'_2-1}(y),
T^{m'_2}(x), \ldots, T^{m'_3-1}(x)T^{m'_3}(y), \ldots, T^{m'_4-1}(y), \ldots
\]
We leave verification of the details to the reader.

\[\square\]

**Corollary 3.9.** Let $(X, T)$ be a mean equicontinuous dynamical system. Then, the following statements are equivalent:

1. $(X, T)$ has the almost specification property;
2. $(X, T)$ has AASP;
3. $(X, T)$ has ASP;
4. $(X, T)$ has FinASP;
5. $(X, T)$ has the $d$-shadowing property;
6. $(X, T)$ has the $d$-shadowing property.
7. $(X, T)$ has trivial measure center.

**Proof.** By [18] Theorem 4.2, Corollary 6.9] we obtain $1 \implies 2 \implies 3 \implies 4$ + $5$ + $6$. When $4$, $5$ or $6$ holds, we obtain by Theorem 3.8 that $(X, T)$ has trivial measure center, so $(7)$ holds. Then both $(supp(X, T), T)$ has the almost specification property, and so $(X, T)$ has the almost specification property by [18] Theorem 6.7].

4. **FinASP and distributional chaos.** In [11], Oprocha and Štefánková proved that a dynamical system with the specification property and with a pair of distal points is distributionally chaotic. Very recently, Wang et al. [17] proved that a dynamical system with AASP having a distal pair is distributionally chaotic. Here, we shall show that this also holds for FinASP which is weaker than AASP. By [6] Lemma 8.2], it follows that there exists a proximal dynamical system with the average shadowing property, hence we need an additional assumption of distal pair to have a chance to prove distributional chaos.

**Theorem 4.1.** Let $(X, T)$ be a dynamical system having FinASP. Then, the following statements are equivalent:

1. $(X, T)$ is distributionally $\varepsilon$-chaotic for some $\varepsilon > 0$;
2. there exists a distal pair $(p,q)$;
3. there exists a Cantor distributionally $\varepsilon$-scrambled subset for some $\varepsilon > 0$.

**Proof.** By [8] Corollary 6.9, we obtain $1 \iff 3$. It is proved in [9] that proximal system does not have distributionally chaotic pairs. Therefore, it suffices to check
that \( \{2\} \implies \{1\} \). Let \((p, q)\) be a distal pair. Then \(\omega(T \times T, (p, q)) \cap \Delta = \emptyset\), therefore there is a minimal set \(M\) for \(T \times T\) such that \(M \cap \Delta = \emptyset\). Taking any \((u, v) \in M\) we see that \((u, v)\) is a distal pair, and since \((u, v)\) is uniformly recurrent for \(T \times T\), both \(u, v\) are uniformly recurrent. But each minimal set is contained in measure center, hence \(u, v \in \text{supp}(X, T)\), that is \(\text{supp}(X, T)\) contains a distal pair.

Fix a distal pair \((p', q') \in \text{supp}(X, T)\) and let \(\eta = \frac{1}{3} \inf \{d(f^n(p'), f^n(q')) : n \in \mathbb{N}_0\}\). For \(M = 1, 2, \ldots\), we define sets \(P_M, S_M \subset \text{supp}(X, T) \times \text{supp}(X, T)\) in the following way: \((x, y) \in S_M\) (resp. \((x, y) \in P_M\)) if there exist \(n > M\) and \(a > 0\) such that \(\Phi_{x,y}^{(n)}(\eta + a) < \frac{1}{3}\) (resp. \(\Phi_{x,y}^{(n)}(\frac{1}{3}) > 1 - \frac{1}{3}\)). It is not difficult to verify that each set \(P_M\) and \(S_M\) is open in \(\text{supp}(X, T) \times \text{supp}(X, T)\). We are going to show that each set \(S_M\) and \(P_M\) is also dense in \(\text{supp}(X, T) \times \text{supp}(X, T)\).

Let \(\mu\) be an invariant measure such that \(\text{supp}(\mu) = \text{supp}(X, T)\) and let \(\mu^2 = \mu \times \mu\) be the product measure on \(X \times X\). Fix any positive integer \(M\), take any two nonempty open set \(U, V \subset X\) with \(U \cap \text{supp}(X, T) \neq \emptyset \neq V \cap \text{supp}(X, T)\) and let \(U', V'\) be nonempty open sets such that \(U' \subset U\), \(V' \subset V\), and \(U' \cap \text{supp}(X, T) \neq \emptyset\), \(V' \cap \text{supp}(X, T) \neq \emptyset\). Clearly \(\mu^2(U' \times V') > 0\), hence by the ergodic decomposition theorem, there is an ergodic measure \(\lambda\) positive on \(U' \times V'\). But then, by Birkhoff ergodic theorem there exists a pair \((u, v) \in (U' \times V') \cap \text{supp}(\lambda) \subset (U' \times V') \cap (\text{supp}(X, T) \times \text{supp}(X, T))\) such that

\[
\lim_{n \to \infty} \frac{1}{n} |N_{T \times T}((u, v), U' \times V') \cap [0, n]| = \gamma > 0.
\]

Take \(\varepsilon \in (0, \min\{\gamma/(8M), \eta/2, 1\})\) such that \(B_{\varepsilon}(U') \subset U\) and \(B_{\varepsilon}(V') \subset V\). Noting that \((X \times X, T \times T)\) has the average shadowing property, applying Theorem 3.1 yields that there is \(\delta > 0\) such that every periodic \(\delta\)-average pseudo-orbit contained in \(\text{supp}(X, T) \times \text{supp}(X, T)\) is \(\varepsilon^2\)-traced on average by a point in \(\text{supp}(X, T) \times \text{supp}(X, T)\). For any \(n \in \mathbb{N}\), define a sequence \(\xi\) in \(\text{supp}(X, T) \times \text{supp}(X, T)\) by putting

\[
\xi_i = \begin{cases} 
(T \times T)^i(u, v), & i \in [0, n), \\
(T \times T)^i(u, u), & i \in [n, (3M + 1)n),
\end{cases}
\]

and take

\[
\xi^{(n)} = \{\xi_i^{(n)}\} = \xi \xi \cdots \xi \cdots .
\]

Clearly, there is a sufficiently large integer \(N\) such that for any \(n \geq N\), \(\xi^{(N)}\) is a periodic \(\delta\)-average pseudo-orbit contained in \(\text{supp}(X, T) \times \text{supp}(X, T)\) and

\[
\frac{1}{n} |N_{T \times T}((u, v), U' \times V') \cap [0, n)| \geq \frac{3}{4} \gamma.
\]

Then, there is a point \(z = (z_1, z_2) \in \text{supp}(X, T) \times \text{supp}(X, T)\) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d((T \times T)^i(z), \xi_i^{(N)}) < \varepsilon^2 < \varepsilon.
\]

This implies that there is a positive integer \(R\) such that

\[
\varepsilon \left[\frac{\sum_{i=0}^{(3M+1)NR-1} d((T \times T)^i(z), \xi_i^{(N)})}{(3M + 1)NR} < \varepsilon^2\right].
\]
This implies that there is 0 ≤ j < R such that
\[ \frac{1}{(3M + 1)N} |A_{[(3M + 1)N^2]}\{\{(T \times T)^i(z)\}, \{\xi_i^{(N)}\}, \varepsilon/(4M)| > 1 - \frac{\gamma}{8M}. \]

Denote \( \beta = \|\{(3M + 1)N_j \leq i < (3M + 1)N_j + N : d((T \times T)^i(z), \xi_i^{(N)}) \leq \varepsilon/(4M)\|/N. \) Clearly, \( \frac{N\beta + 3M^2N}{(3M + 1)N} > 1 - \frac{\gamma}{8M}. \) Then, \( \beta \geq 1 - \frac{3M + 1}{8M} \gamma > 1 - \frac{\gamma}{8M}. \) Combining this with [7], it follows that there is (3M+1)Nj ≤ k < (3M+1)Nj+N such that \( d((T \times T)^k(z), \xi_k^{(N)}) \leq \varepsilon/(4M) \) and \( T^{k-(3M+1)N_j(u)} \in U', T^{k-(3M+1)N_j(v)} \in V'. \) This, together with [7], implies that \( T^k(z_1) \in U \) and \( T^k(z_2) \in V. \)

On the other hand we have
\[ \frac{1}{(3M + 1)N} |A_{[(3M + 1)N^2]}\{\{(T^i(z_1)), \{(T^i(z_2)), 1/(2M)\)| \geq \frac{1}{(3M + 1)N} |A_{[(3M + 1)N^2]}\{\{(T \times T)^i(z)\}, \{\xi_i^{(N)}\}, \varepsilon/(4M)| \]
\[ \geq \frac{1 - \gamma/(8M)}{(3M + 1)N - N} = 1 - \left( \frac{\gamma}{8M} + \frac{1}{3M + 1} \right) > 1 - \frac{1}{M}. \]

Noting that
\[ ||\{k \leq i < (3M + 1)N(j + 1) : d(T^i(z_1), T^i(z_2)) < 1/M\}| \geq ||\{(3M + 1)N_j + N \leq i < (3M + 1)N(j + 1) : d((T \times T)^i(z), \xi_i^{(N)}) \leq \varepsilon/(4M)\}|, \]
it is easy to see that \( (T^k(z_1), T^k(z_2)) \in P^M \) which implies
\[ (U \times V \cap (\text{supp}(X, T) \times \text{supp}(X, T))) \in P^M \neq \emptyset. \]

Next, replacing definition of \( \xi \) in [8] by
\[ \xi_i = \left\{ \begin{array}{ll} (T \times T)^i(u, v), & i \in [0, n), \\ (T \times T)^i(p', q'), & i \in [n, (3M + 1)n), \end{array} \right. \]
and repeating previous argument, we easily obtain that there is a positive integer \( k' \) such that \( (T^{k'}(z_1), T^{k'}(z_2)) \in S^M, i.e., (U \times V \cap (\text{supp}(X, T) \times \text{supp}(X, T))) \cap S^M \neq \emptyset. \) Since all the sets \( P^M, S^M \) are open, we obtain that the set
\[ R = \bigcap_{M=1}^{\infty} (P^M \cap S^M) \]
is residual in \( \text{supp}(X, T) \times \text{supp}(X, T) \). Note that every pair contained in \( R \) is distributionally \( \eta \)-chaotic, hence the result follows by application of Mycielski theorem. □

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