THE BOUNDED HEIGHT CONJECTURE FOR SEMIABELIAN VARIETIES

LARS KÜHNE

Abstract. The Bounded Height Conjecture of Bombieri, Masser, and Zannier states that for any sufficiently generic algebraic subvariety of a semiabelian \( \overline{Q} \)-variety \( G \) there is an upper bound on the Weil height of the points contained in its intersection with the union of all algebraic subgroups having (at most) complementary dimension in \( G \). In this article, we demonstrate the conjecture for general semiabelian varieties.

A generalization of the classical Manin-Mumford conjecture is the following theorem, proven by Hindry \cite{hindry98} in the late eighties.

**Theorem 1.** Let \( G \) be a semiabelian variety over \( \overline{Q} \) with torsion points \( \text{Tor}(G) \subseteq G(\overline{Q}) \). For any algebraic subvariety \( X \) of \( G \), there are finitely many connected algebraic subgroups \( G_i \) of \( G \) and finitely many torsion points \( x_i \in \text{Tor}(G) \) such that \( \bigcup_{i=1}^{n}(G_i + x_i) \) is the Zariski closure of \( X \cap \text{Tor}(G) \).

More recently, another type of intersections in semiabelian varieties has been widely studied. These intersections are with algebraic subgroups instead of torsion points. Of course, investigating the intersection of \( X \) with a single such subgroup is a dreary task. However, very interesting phenomena appear when intersecting \( X \) with the countable union \( G^{[s]} \) of all algebraic subgroups having codimension \( \geq s \) for some fixed integer \( s \).

Since the pioneering work of Bombieri, Masser, and Zannier \cite{bombieri90} in this direction, two choices of \( s \) are of paramount importance. If \( s = \dim(X) + 1 \), an algebraic subgroup \( H \subseteq G \) of codimension \( \geq s \) usually does not meet \( X \) at all. The intersection \( X \cap G^{[s]} \) may nevertheless be dense in the Zariski topology – even in generic cases. If \( X \) is not contained in a proper algebraic subgroup of \( G \), conjectures of Pink \cite{pink09} and Zilber \cite{zilber06} imply that this never happens. Such statements about “unlikely intersections” are still unsettled problems, on which the reader finds a comprehensive overview in \cite{guevara13}. This article treats the other important and related case where \( s = \dim(X) \). In this case, a generic subgroup \( H \subseteq G \) of codimension \( \geq s \) intersects \( X \) already in finitely many points, and \( X \cap G^{[\dim(X)]} \) can be dense with respect to the Zariski topology of \( X \). The gist of the Bounded Height Conjecture (BHC) stated below is that the Weil height of the \( \overline{Q} \)-points in \( X \cap G^{[\dim(X)]} \) should be nevertheless bounded from above.

In order to state this conjecture, we have to introduce some additional notions to tackle also non-generic cases. A closed irreducible subvariety \( Y \) of \( X \) is called \( s \)-anomalous if there exists a connected algebraic subgroup \( H \subseteq G \) satisfying

\[
\max\{0, s - \text{codim}_G(H)\} < \dim(Y)
\]

and a point \( y \in Y(\overline{Q}) \) such that \( Y \subseteq yH \) (i.e., \( Y \) is contained in a translate of \( H \)). In this situation, we say that \( Y \) is associated with \( H \). By \( X^{(s)} \) we mean the union of all positive dimensional closed irreducible \( s \)-anomalous subvarieties of \( X \). It is a corollary of Kirby’s work \cite{kirby95} that \( X^{(s)} \) (resp. \( X \setminus X^{(s)} \)) is a Zariski closed (resp. Zariski open) subset of \( X \) (cf. \cite{szamuely07} Proposition 2.6). In addition, a proof allowing to determine \( X^{(s)} \) effectively was given by Bombieri, Masser and Zannier \cite{bombieri90} for tori and carried over to abelian varieties by Rémont \cite{remont08}.

Let now \( G \) be a semiabelian variety over \( \overline{Q} \) and \( X \) a closed irreducible subvariety of \( G \). To be able to work with heights, we choose a compactification \( \overline{G} \) of \( G \) (i.e., an open embedding \( G \hookrightarrow \overline{G} \) such that \( \overline{G} \) is proper). Let \( L \) be a line bundle on \( \overline{G} \) of \( G \). Finally, let \( h_L : \overline{G}(\overline{Q}) \to \mathbb{R} \) be a Weil height associated with \( L \). We can now state the

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Bounded Height Conjecture (BHC). The height $h_L$ is bounded from above on the set $(X \setminus X^{(\dim(X))}(\overline{\mathbb{Q}})) \cap G^{(\dim(X))}(\overline{\mathbb{Q}})$.

This conjecture was first proposed by Bombieri, Masser, and Zannier [6] in the case where $G$ is a torus. Even before this, they had provided a proof if $G$ is torus and $X$ is a curve [5]. The extension of their conjecture from tori to semiabelian varieties is merely formal and can be found in Habegger’s article [22], where a proof of the BHC for abelian varieties is given. This extension is in fact natural as semiabelian varieties have proven to be the right object for many standard conjectures in diophantine geometry (e.g., Manin-Mumford, Mordell-Lang, Bogomolov). In addition, they appear naturally as Jacobians of semistable curves (cf. Example 9.2.8) like abelian varieties do for smooth curves so that they still retain a connection with the original study of rational points on curves. We also mention intermediate results in the direction of the BHC given by Bombieri, Masser, and Zannier [7], Maurin [38, 39], Viada [51], and Zannier [61].

In parallel to his work on the BHC for abelian varieties, Habegger [23] obtained a complete proof of the conjecture for tori. Regarding the general case of the BHC, no further progress was made since his two breakthrough articles [22, 23]. In fact, several additional problems precluded further generalizations up to now. These problems originate from the “mixed” nature of semiabelian varieties (i.e., the additional structures induced by the non-triviality of the extension constituting the semiabelian variety). The aim of this article is to solve these problems. Its main result, Theorem 2 below, yields the BHC in general. In line with [22], we actually prove a stronger version of the BHC here. To announce it, we introduce certain “height cones”; for each subset $\Sigma \subset G(\overline{\mathbb{Q}})$ and each real number $\varepsilon > 0$, we define such a height cone by setting

$$C(\Sigma, h_L, \varepsilon) = \{ x \in G(\overline{\mathbb{Q}}) | \exists a \in \Sigma, b \in G(\overline{\mathbb{Q}}): x = a + b \text{ and } h_L(b) \leq \varepsilon (1 + h_L(a)) \}.$$  

Theorem 2. Let $G$ be a semiabelian variety and $\overline{G}$ a compactification endowed with an ample line bundle $L$. Furthermore, let $X$ be a closed subvariety of $G$. In addition, assume that $G$, $\overline{G}$, $L$, and $X$ are defined over $\overline{\mathbb{Q}}$. Let $h_L$ be a Weil height associated with $L$. For each integer $s$, there exists some $\varepsilon > 0$ such that $h_L$ is bounded from above on $(X \setminus X^{(s)}(\overline{\mathbb{Q}})) \cap C(G^{(s)}(\overline{\mathbb{Q}}), h_L, \varepsilon)$.

The above Theorem 2 is proven in Section 4. Let us sketch our proof and compare our approach with the one of Habegger from [22, 23]. For expository reasons, we frequently use the notion of $\mathbb{Q}$-line bundles in this sketch, though we always clear denominators in the proof itself and hence have no use for $\mathbb{Q}$-line bundles there. Ignoring some preliminary reductions (Section 7.1), the proof consists of three major steps, which we outline successively in the following.

In the first step (Section 7.2 and Lemma 25), we pass from algebraic subgroups to a well-behaved relatively compact family of $\mathbb{Q}$-line bundles on $G$ and their associated heights. Subsequently, we use compactness to approximate the members of this family by finitely many $\mathbb{Q}$-line bundles. To start with, we replace subgroups $H \subseteq G$ by their associated quotients $\pi_H : G \rightarrow G' = G/H$. A point $x \in G(\overline{\mathbb{Q}})$ lies on a subgroup $H$ if and only if it is contained in the kernel of $\pi_H$. For an ample (or any) line bundle $L'$ on $G'$, this implies

$$\widehat{h}_{\pi_H^* L'}(x) = 0$$

for the normalized Weil height $\widehat{h}_{\pi_H^* L'} : G(\mathbb{Q}) \rightarrow \mathbb{R}$ (see Section 3 for this notion). To wind up with a relatively compact family, we have to minimize the number of line bundles $\pi_H^* L'$ appearing in this procedure. For abelian varieties, the quotients $G/H$ fall into finitely many isogeny classes when $H$ runs through the algebraic subgroups of $H$. Indeed, this is a direct consequence of Poincaré’s complete reducibility theorem. For tori, the same assertion is trivially true but we temporarily restrict to abelian varieties so as to avoid compactifications. For these, we choose representatives $G'_1, \ldots, G'_n$ for the isogeny classes of quotients and endow each $G'_i$ with an ample line bundle $L'_i$. In this way, we can restrict our attention to line bundles of the form $\varphi^* L'_i$, $\varphi \in \text{Hom}(G, G'_i)$. By homogeneity, it is possible to define $\mathbb{Q}$-line bundles $\phi^* L'_i$ for each $\phi \in \text{Hom}_{\mathbb{Q}}(G, G'_i)$. Some manipulation of homomorphisms (cf. Lemma
allows to further restrict to the $\mathbb{Q}$-line bundles $\phi^* L'_i$, $\phi \in K_i \cap \text{Hom}_{\mathbb{R}}(G, G'_i)$, for some compact subset $K_i \subset \text{Hom}_{\mathbb{R}}(G, G'_i)$. Using compactness, we can eventually arrange for the following assertion, which corresponds to our Lemmas 24 and 25: For each $\delta > 0$, there exist finitely many “surjective” $\phi_{i,\delta}^{(t)}$, $\phi_{i,\delta}^{(c)} \in \text{Hom}_{\mathbb{Q}}(G, G'_i)$ and a constant $c(\delta) > 0$ such that for every $x \in G^{[\delta]}(\mathbb{Q})$ we have

$$
\widehat{h}_{(\phi_{i,\delta}^{(t)})^* L'_i}(x) \leq \delta \widehat{h}_L(x) + c(\delta)
$$

for some $1 \leq i \leq n$ and $1 \leq k \leq k_i, \delta$.

Comparing this inequality with (3), we notice that passing to a finite family of $\mathbb{Q}$-line bundles worsens the bound but that the dependence on $\widehat{h}_L(x)$ can be curbed by choosing $\delta$ sufficiently small. So far, this is just Habegger’s argument for abelian varieties as in [22], although the focus there is more on the quasi-homomorphisms in $\text{Hom}_{\mathbb{Q}}(G, G'_i)$ than on the associated $\mathbb{Q}$-line bundles $\phi^* L'_i$ on $G$.

In general, however, a shift to $\mathbb{Q}$-line bundles on $G$ instead of quasi-homomorphism becomes essential. Indeed, the quotients of a semiabelian variety $G$ regularly fall into infinitely many different isogeny classes (cf. the footnote on p. 27). This makes it difficult to work with quasi-homomorphisms along the lines sketched above. In fact, repeating the above procedure does not lead to finitely many line bundles. Consequently, it does not yield an inequality like (4) with some uniform constant $c(\delta)$. To circumvent this problem, we define suitable $\mathbb{Q}$-line bundles directly on $G$. These should generalize pullbacks of line bundles along quasi-homomorphisms. There are some indications on how to write down such line bundles. First, it is a well-known fact that a homomorphism $\varphi$ between semiabelian varieties is describable in terms of the induced homomorphism $\varphi_{\text{tor}}$ between their maximal subtori and the induced homomorphism $\varphi_{\text{ab}}$ between their underlying abelian varieties (see Lemma 1). If $G$ is a semiabelian variety with split maximal subtorus $G'_m$ and underlying abelian variety $A$, it is hence natural to consider families of $\mathbb{Q}$-line bundles parameterized by the $\mathbb{Q}$-vector spaces

$$
V_{Q}(t, j) = \text{Hom}_{\mathbb{Q}}(G'_m, G'_m) \times \text{Hom}_{\mathbb{Q}}(A, A_j), \ t \in \{0, \ldots, t\}, \ j \in \{1, \ldots, j_0\},
$$

where $\{A_1, \ldots, A_{j_0}\}$ is a set of representatives for the isogeny classes of the quotients of $A$. Second, a result of Knop and Lange [31] Theorem 2.1] states that linearized line bundles on compactifications of $G$ retain a “product-like” shape even if $G$ is a non-trivial extension and there is no projection to the maximal subtorus.

To discuss what these facts mean in our particular situation, we have to introduce compactifications for semiabelian varieties. In Section 2 we recall the standard method to associate with any semiabelian variety $G$ having a split maximal subtorus $G'_m$ and any $G'_m$-equivariant compactification $\overline{T}$ of $G'_m$ a compactification $\overline{G}$ of $G$. In addition, we associate with any $G'_m$-linearized line bundle $(M, \varrho)$ on $\overline{T}$ a line bundle $G(M, \varrho)$ on $G$. We employ two basic types of compactifications in this article. A $G'_m$-equivariant embedding $G'_m \rightarrow (\mathbb{P}^1)^t$ yields a compactification $\overline{G} = G(\mathbb{P}^1)^t$ (Construction 5). In addition, the Zariski closure $\overline{\Gamma(\varphi_{\text{tor}})} \subset (\mathbb{P}^1)^t \times (\mathbb{P}^1)^t$ of the graph of any homomorphism $\varphi_{\text{tor}} : G'_m \rightarrow G'_m$ provides a compactification $G_{\overline{\Gamma(\varphi_{\text{tor}})}}$ (Construction 4). In both cases, the homomorphism $\varpi : G \rightarrow A$ to the underlying abelian variety extends to maps $\varpi : \overline{G} \rightarrow A$ and $\varpi_{\overline{\Gamma(\varphi_{\text{tor}})}} : G_{\overline{\Gamma(\varphi_{\text{tor}})}} \rightarrow A$, respectively.

Assume now given a homomorphism $\varphi : G \rightarrow G'$ to a second semiabelian variety $G'$ with maximal subtorus $G'_m$ and maximal abelian quotient $\pi' : G' \rightarrow A'_j$. Let $(M'_{\varrho}, \varrho'_{\varrho})$ be a $G'_m$-linearized line bundle on $(\mathbb{P}^1)^t$ and $N'_{j_0}$ an ample line bundle on $A'_j$. Then,

$$
G' (M'_{\varrho}, \varrho'_{\varrho}) \otimes (\varpi')^* N'_{j_0}
$$

is an ample line bundle on $\overline{G}'$ (Lemma 4). Additionally, $\varphi$ extends to a map $p_{\varphi} : G_{\overline{\Gamma(\varphi_{\text{tor}})}} \rightarrow \overline{G}'$ (see Construction 7 for details). The pullback of (5) along $p_{\varphi}$ is the line bundle

$$
G((\text{pr}_2|_{\overline{\Gamma(\varphi_{\text{tor}})}})^* (M'_{\varrho}, \varrho'_{\varrho})) \otimes \varpi_{\overline{\Gamma(\varphi_{\text{tor}})}}^* \varphi_{\text{ab}}^* N'_{j_0}
$$
where $\text{pr}_2 : (\mathbb{P}^1)^t \times (\mathbb{P}^1)^{t'} \rightarrow (\mathbb{P}^1)^{t'}$ denotes projection to the second factor. Surprisingly, the line bundle in (6) can be defined solely in terms of $(\varphi_{\text{tor}}, \varphi_{\text{ab}}) \in V_Q^{(t', 3)}$. In other words, there is neither need for a homomorphism $\varphi : G \rightarrow G'$ nor for a semiabelian variety $G'$. Line bundles of the form (6) play a prominent role in our proof, generalizing the line bundles $\phi^*L'_i$ from the abelian case above. Naturally, some checking is necessary to guarantee that they simulate pullbacks along homomorphisms sufficiently well (see e.g. Lemmas 9, 11 and 11).

In the second step of the proof (Section 7.3), we establish two concurring height bounds similar to [22, 23]. However, the non-homogeneity of the canonical height on semiabelian varieties, which decomposes into a linear and a quadratic part, is yet another problem. A clever choice of line bundles is needed to counterbalance this in the height estimates (cf. the proof of Lemma 25). The first of the two bounds considered is similar to (4). The second opposing height bound is a consequence of Siu’s numerical bigness criterion ([53, Corollary 1.2]). To apply Siu’s criterion, we need to estimate two types of intersection numbers related to the line bundles in (6) and the Zariski closure of $X$ in $G_{\Gamma(\varphi_{\text{tor}})}$ (Lemma 27). Subject to sufficiently strong estimates on these intersection numbers (as stated in Lemmas 28 and 29), we already finish the proof of Theorem 2 at this point by combining the two opposite height bounds.

In the third and last step of our proof (Section 7.4), we estimate these intersection numbers. Homogeneity, or the lack hereof, is once again an issue. Serious difficulties seem to arise when trying to obtain lower bounds on intersection numbers by counting torsion points as in [22, 23]. Although some technical tools such as [22, Proposition 3] were already written up more generally than strictly necessary in order to foster future generalizations, it is not clear whether this can be done at all. Therefore, we provide an alternative to this argument (Lemma 28) based on hermitian differential geometry (see also Sections 4 and 6 for details). In fact, this alternative is strikingly simple in the special case of abelian varieties treated in [22]. We obtain the sought-after lower bounds on intersection numbers by integrating appropriately chosen $(1, 1)$-forms. These $(1, 1)$-forms are defined in Section 5 as real interpolations of Chern forms associated with specific hermitian metrics on the line bundles appearing in (6). On the level of $(1, 1)$-forms, balancing the different homogeneities of the “toric” and “abelian” contributions is an easy task (see e.g. our definition (70)).

Whereas the definition of the used $(1, 1)$-forms and the verification of their basic properties is almost trivial for abelian varieties (Section 5.2), tori and hence general semiabelian varieties demand considerably more work (Section 5.1). The reason for this is that any invariant hermitian metric on the line bundles under consideration is merely continuous and leads to a singular Chern current supported on the maximal compact subgroup $K_G \subseteq G(\mathbb{C})$ (see e.g. [10, Lemme 6.3]). A singular Chern current being detrimental for the application of Ax’s Theorem [1] in Section 5 we have to work with a less natural non-invariant hermitian metric instead. For the Chern forms associated to such a metric, establishing some natural properties is a non-trivial task; the reader may compare the proof of Lemma 16 with the evident relation (42).

It should be mentioned that Chern forms were also used by Maurin [39], Rémond [47], and Vojta [57] to control intersections numbers appearing in diophantine geometry. In particular, both [39] – in the case of tori – and [57] – in the case of semiabelian varieties – endow line bundles with non-invariant hermitian metrics. Apart from this, it seems that the overlap of their work with our Sections 5 and 6 is rather narrow. It is nevertheless noteworthy that Ax’s Theorem plays an essential role here as it does in the work of Habegger [22, 23] and Rémond [47]. In contrast to Lemma 28, our proof of the supplementary upper bounds on intersection numbers in Lemma 29 uses algebraic intersection theory [16] to avoid problems stemming from the non-compactness of $G$.

The above outline already suggests that our approach is able to deal with potentially much larger sets than $G^{[s]}(\mathbb{Q})$. In fact, to each connected algebraic subgroup of a semiabelian variety $G$ with maximal torus $T$ and maximal abelian quotient $\pi : G \rightarrow A$ is associated a unique pair $(S, B)$ consisting of a subtorus $S \subseteq T$ and a connected algebraic subgroup $B \subseteq A$ (cf. Lemma 1). Whereas all these pairs $(S, B)$ are realized by connected algebraic subgroups for a product
G = T × A, this is not the case for general semiabelian varieties. However, these pairs are still relatable to line bundles as in (2). Denote by \((V' \otimes (t', j) \subset V_\mathbb{Q}'(t', j)\) the subset of pairs \((\varphi_{\text{tor}}, \varphi_{\text{ab}})\) such that both \(\varphi_{\text{tor}}\) and \(\varphi_{\text{ab}}\) are surjective homomorphisms. We leave it to the interested reader to modify our proof in order to replace \(G^s\) in Theorem 2 by

\[
G^s.[s](\mathcal{H}) = \bigcup \left\{ x \in G(\mathcal{H}) \mid \exists (\varphi_{\text{tor}}, \varphi_{\text{ab}}) \in (V' \otimes (t', j)) : \hat{h}_{M_{\Gamma(\varphi_{\text{tor}})}}(x) = 0 \text{ and } \pi(x) \in \ker(\varphi_{\text{ab}}) \right\};
\]

the union is over all pairs \((t', A'_j)\) such that \(t' + \dim(A'_j) \geq s\). Here, the line bundle \(M_{\Gamma(\varphi_{\text{tor}})}\) on \(G_{\Gamma(\varphi_{\text{tor}})}\) is as defined in Construction 3 below and the definition of its normalized height \(\hat{h}_{M_{\Gamma(\varphi_{\text{tor}})}}\) can be found in Section 3. It may well be that the general framework of our method (i.e., the use of relatively compact families of \(\mathbb{Q}\)-line bundles in combination with real interpolations of Chern forms) gives also some leeway in problems beyond the BHC with no group structure.

Finally, it should be mentioned that a previous announcement [33] stated a non-optimal version of the first step in the proof of Theorem 2. This included a non-effective compactness argument ([33, Lemma 2]), which is replaced here by the simpler Lemma 24. The improvement of the first step in the proof of Theorem 2. This included a non-effective compactness argument ([17, Corollaire VI.5.2]) and of finite type over \(\mathbb{Q}\). Note that \(H\) is necessarily Zariski closed in \(G\) ([17, Corollaire VI.5.2]) and of finite type over \(\text{Spec}(k)\). Left-multiplication by an element \(g \in G(k)\) is written \(l_g : G \to G\). More generally, we use the same notation \(l_g\) for the left-multiplication with respect to an action \(G \times X \to X\).

A split \(k\)-torus is an algebraic \(k\)-group that is isomorphic to some direct product of copies of multiplicative groups \(\mathbb{G}_m\). A \(k\)-torus is an algebraic group \(G\) such that its base change \(G_{\text{sep}}\) to the separable closure \(k_{\text{sep}}\) of \(k\) is a split torus. A linear \(k\)-algebraic group is an algebraic \(k\)-group whose underlying scheme is both affine and connected.

For us, the quotient of a \(k\)-scheme under the action of an algebraic \(k\)-group is a sheaf on the big fppf site of \(\text{Spec}(k)\). If such a fppf quotient sheaf is represented by a \(k\)-scheme both
are identified. This is in particular the case for the quotient $G/H$ of an algebraic group $G$ by a normal subgroup $H$ (cf. [17 Théorème VI$_A$.3.2]). In this way, the commutative algebraic $k$-groups form an abelian subcategory of the fppf sheaves in abelian groups (cf. [17 Théorème VI$_A$.5.4.2]). A sequence

$$(7) \quad 0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

of commutative algebraic $k$-groups is called exact if the associated sequence of fppf sheaves in abelian groups is exact. For fixed $G_1$ and $G_3$, the isomorphism classes of Yoneda extensions as in [33] form an abelian group $\text{Ext}^1_k(G_1, G_3)$ with respect to Baer summation (cf. [13 Section I.3]).

We write $[n]_G$ for the multiplication-by-$n$ map on any commutative algebraic group $G$. The notation $\cdot_G: G \times G \to G$ is used for the group law of $G$ and $e_G: k \to G$ denotes the identity of $G$. We omit the reference to $G$ in these notations when this group can be inferred from context.

We write $A^\vee$ for the dual abelian variety associated with an abelian variety $A$. Pulling back line bundles along a homomorphism $\varphi: A \to B$ induces a homomorphism $\varphi^\vee: B^\vee \to A^\vee$ of the associated dual abelian varieties.

Line bundles are denoted by capital italic letters $L, M, \ldots$ whereas the corresponding calligraphic letters $L, M, \ldots$ are reserved for the invertible sheaves formed by their sections. The line bundle dual to $L$ is written $L^\vee$. In the situation where we have an algebraic group $G$ acting on a scheme $X$, we use Mumford’s definition of $G$-linearization ([43 Definition 1.6]) for general $\mathcal{O}_X$-modules. For an invertible sheaf $\mathcal{L}$ on $X$, a $G$-linearization corresponds to an action $\varrho: G \times L \to L$ such that the projection $L \to X$ is $G$-equivariant. We refer to [43 Section 1.3] for details. Given a $G$-linearized line bundle $(L, \varrho)$ we write $(L, \varrho)^{\otimes n}$ for the line bundle $L^{\otimes n}$ with the $T$-linearization induced by $\varrho$. If $\varphi: H \to G$ is a homomorphism from another algebraic group $H$, $Y$ a scheme with $H$-action and $f: Y \to X$ a $\varphi$-equivariant algebraic map, we write $f^*(L, \varrho)$ for the line bundle $f^*L$ with the induced $H$-linearization. For a $G$-equivariant closed immersion $i: Y \hookrightarrow X$, we also write $(L, \varrho)|_Y$ instead of $i^*(L, \varrho)$.

With a line bundle $L$ on a projective variety, we associate a first Chern class $c_1(L)$ in the sense of [16]: we refer the reader to there for an exposition on the basic properties of Chern classes and the basic intersection theory we are using. We denote by $|X|$ the $k$-cycle class associated with an irreducible algebraic variety $X$ of dimension $k$ (in some ambient projective variety).

1. Preliminaries on Semiabelian Varieties

1.1. Basics. A semiabelian variety $G$ over $k$ is a connected smooth algebraic $k$-group that is the extension of an abelian variety $A$ over $k$ by a $k$-torus $T$. In other words, we have an exact sequence

$$(8) \quad 1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

of fppf sheaves in abelian groups. Any homomorphism from a smooth linear algebraic group to an abelian variety is the zero homomorphism (see e.g. [12 Lemma 2.3]). Therefore, any smooth linear algebraic $k$-subgroup of $G$ must be contained in $T$. It follows that $T$ is the maximal smooth linear algebraic subgroup of $G$. We hence call $T$ the toric part of $G$ and $G \to G/T = A$ (or just $A$) the abelian quotient of $G$. For a semiabelian variety $G$, we write $\eta_G$ for the Yoneda extension class in $\text{Ext}^1(A, T)$ described by [33]. Each homomorphism $\varphi: B \to A$ (resp. $\varphi: T \to S$) of abelian varieties (resp. tori) induces a pullback $\varphi^*: \text{Ext}^1_k(A, T) \to \text{Ext}^1_k(B, T)$ (resp. a pullforward $\varphi_*: \text{Ext}^1_k(A, T) \to \text{Ext}^1_k(A, S)$).

The Weil-Barsotti formula (see [15 Section III.18] or the appendix to [12]) gives a canonical identification $\text{Ext}^1_k(A, G_m) = A^\vee(k)$. If $T$ is split (i.e., $T = G^t_m$) we make frequent use of the identify $\text{Ext}^1_k(A, G^t_m) = \text{Ext}^1_k(A, G^t_m)^t = (A^\vee)^t(k)$. The pullback

$$(9) \quad \varphi^*: \text{Ext}^1_k(A, G^t_m) \longrightarrow \text{Ext}^1_k(B, G^t_m)$$
along a homomorphism $\varphi : B \rightarrow A$ corresponds to the $t$-fold product $\varphi^\vee \times \cdots \times \varphi^\vee$ of the dual morphism $\varphi^\vee : A^\vee \rightarrow B^\vee$. Pushforwards also allow a simple description. Indeed, let $\varphi : G_m \rightarrow G_m'$ be the homomorphism described by $\varphi^*(Y_v) = \prod_{u=1}^t X_u^{a_{uv}}$ in standard coordinates $X_1, \ldots, X_t$ (resp. $Y_1, \ldots, Y_t$) on $G_m'$ (resp. $G_m$). Then, the pushforward
\begin{equation}
\varphi_* : \operatorname{Ext}^1_k(A, G_m') \rightarrow \operatorname{Ext}^1_k(A, G_m')
\end{equation}
corresponds to the homomorphism $(A^\vee)^t \rightarrow (A^\vee)^t$ sending $(\eta_1, \ldots, \eta_t)$ to $((\sum_{u=1}^t a_{uv}\eta_u)_{1 \leq v \leq t}$.

Finally, we note that quotients as well as smooth algebraic subgroups of a semiabelian variety are themselves semiabelian varieties (cf. [51, Chapter VII], yet described in a pre-schematic language). All of this is contained in [51, Corollary 5.4.6]). In particular, the algebraic subgroups appearing in Theorem 2 are all semiabelian varieties because a well-known result of Cartier ([17, Corollaire VI_B.1.6.1]) states that all algebraic $k$-groups are smooth if $k$ has characteristic 0.

1.2. Homomorphisms and quasi-homomorphisms. We recall the fundamental result on homomorphisms between semiabelian varieties.

**Lemma 1.** Let $G$ (resp. $G'$) be a semiabelian variety such that $A$ (resp. $A'$) is the abelian quotient and $T$ (resp. $T'$) is the toric part of $G$ (resp. $G'$). For any homomorphism $\varphi : G \rightarrow G'$ there exist unique homomorphisms $\varphi_{\text{tor}} : T \rightarrow T'$ and $\varphi_{\text{ab}} : A \rightarrow A'$ such that
\begin{equation}
\begin{array}{ccc}
1 & \longrightarrow & T \\
\varphi_{\text{tor}} & \downarrow & \varphi \\
1 & \longrightarrow & T' \\
\end{array}
\end{equation}
\begin{equation}
\begin{array}{ccc}
1 & \longrightarrow & G \\
\varphi & \downarrow & \varphi_{\text{ab}} \\
1 & \longrightarrow & A' \\
\end{array}
\end{equation}
is a homomorphism of exact sequences. Furthermore, the induced map
\begin{equation}
\operatorname{Hom}(G, G') \rightarrow \operatorname{Hom}(T, T') \times \operatorname{Hom}(A, A'), \quad \varphi \mapsto (\varphi_{\text{tor}}, \varphi_{\text{ab}}),
\end{equation}
is an injective homomorphism with image
\begin{equation}
\{(\varphi_{\text{tor}}, \varphi_{\text{ab}}) \in \operatorname{Hom}(T, T') \times \operatorname{Hom}(A, A') \mid (\varphi_{\text{tor}})_*\eta_G = (\varphi_{\text{ab}})^*\eta_{G'} \text{ in } \operatorname{Ext}^1_k(A, T')\}.
\end{equation}

This lemma is well-known in the literature (see e.g. [2] or [55]). In fact, the existence of a pair $(\varphi_{\text{ab}}, \varphi_{\text{tor}})$ follows directly from the fact that any map from a smooth linear algebraic group to an abelian variety is zero ([12, Lemma 2.3]) and its uniqueness is obvious. The remaining assertions can be shown by abstract homological algebra in the abelian category of commutative algebraic groups (or fppf sheaves in abelian groups). All of this is contained in [51, Chapter VII], yet described in a pre-schematic language.

It should be also noted that $\varphi$ is surjective (resp. an isogeny) if and only if both $\varphi_{\text{tor}}$ and $\varphi_{\text{ab}}$ are surjective (resp. isogenies). In the situation of Lemma 1 we call $\varphi_{\text{tor}}$ (resp. $\varphi_{\text{ab}}$) the toric (resp. abelian) component of $\varphi$. In addition, we say that $\varphi$ is represented by the pair $(\varphi_{\text{tor}}, \varphi_{\text{ab}})$ and, conversely, that $(\varphi_{\text{tor}}, \varphi_{\text{ab}})$ represents $\varphi$. We state an immediate consequence of Lemma 1 for later reference as a separate lemma.

**Lemma 2.** Assume that $k$ is algebraically closed. Let $G$ be a semiabelian variety with abelian quotient $A$ and toric part $G_m'$. For every homomorphism $\varphi_{\text{tor}} : G_m' \rightarrow G_m'$ and every isogeny $\varphi_{\text{ab}} : A \rightarrow B$ there exists a semiabelian variety $G'$ and a homomorphism $\varphi : G \rightarrow G'$ represented by $(\varphi_{\text{tor}}, \varphi_{\text{ab}})$.

Recall that an isogeny is a homomorphism of algebraic groups that is surjective and has finite kernel.

**Proof.** Write $(\varphi_{\text{tor}})_*\eta_G = (\eta_1', \ldots, \eta_t') \in (A^\vee)^t(k)$. Since $\varphi_{\text{ab}}^\vee : B^\vee \rightarrow A^\vee$ is an isogeny (cf. [44, Remark (3) on p. 81]), there exist $\eta_i' \in B_i^\vee(k)$ such that $\eta_i' = \varphi_{\text{ab}}^\vee(\eta_i)$. Let $G'$ be the semiabelian variety described by $\eta_{G'} = (\eta_1, \ldots, \eta_t) \in (B^\vee)^t(k) = \operatorname{Ext}^1_k(B, G_m')$. As $(\varphi_{\text{tor}})_*\eta_G = (\varphi_{\text{ab}})^*\eta_{G'}$, there exists a homomorphism $\varphi : G \rightarrow G'$ representing $(\varphi_{\text{tor}}, \varphi_{\text{ab}})$ by Lemma 1. \qed
As for abelian varieties, one calls two semiabelian varieties $G, G'$ isogeneous if there exists an isogeny $G \to G'$. Evidently, the multiplication-by-$n$ homomorphism $[n]$ of a semiabelian variety is an isogeny. As for abelian varieties, isogeny is an equivalence relation on semiabelian varieties.

We need to work also with quasi-homomorphisms of semiabelian varieties. First of all, note that for any semiabelian varieties $G$ and $G'$ the $\mathbb{Z}$-module $\text{Hom}(G, G')$ of homomorphisms is torsion-free. Indeed, this is true for both tori and abelian varieties so that we may infer the general case from Lemma\footnote{Using descent along a finite Galois extension of $k'/k$ (compare [8] Example 6.2.B) such that $T_{k'}$ splits, one can get rid of the splitting assumption a posteriori but we do not need this generality.}. By quasi-homomorphisms we mean the elements of $\text{Hom}_\mathbb{Q}(G, G') = \text{Hom}(G, G') \otimes \mathbb{Q}$. In analogy to actual homomorphisms, each quasi-homomorphism is denoted in the form $\phi : G \to G'$. By tensoring (12) with $\mathbb{Q}$, we can also associate with each quasi-homomorphism $\phi : G \to G'$ uniquely a toric component $\phi_{\text{tor}} : T \to T'$ and an abelian component $\phi_{\text{ab}} : A \to A'$. With each quasi-homomorphism $\phi : G \to G'$ we can associate a “kernel up to torsion” $\ker(\phi) + \text{Tors}(G)$ in the following way: Let $n$ be a denominator of $\phi$ (i.e., $n \cdot \phi \in \text{Hom}(G, G')$) and set $\ker(\phi) + \text{Tors}(G) = \ker(n \cdot \phi) + \text{Tors}(G)$. Additionally, we may say that $\phi$ is surjective if $n \cdot \phi$ is. These definitions are clearly independent of the chosen denominator $n$. Albeit a quasi-homomorphism $\phi_{\text{ab}} \in \text{Hom}_\mathbb{Q}(A', A)$ does not induce a pullback as in (9), it gives rise to a homomorphism

$$(\phi_{\text{ab}})^*: \text{Ext}^1_k(A, G'_{\text{m}})_\mathbb{Q} = (A^\vee(k) \otimes \mathbb{Q})^t \to ((A')^\vee(k) \otimes \mathbb{Q})^t = \text{Ext}^1_k(A', G'_{\text{m}})_\mathbb{Q}.$$  

Similarly, a quasi-homomorphism $\phi_t \in \text{Hom}_\mathbb{Q}(G'_{\text{m}}, G'_{\text{m}})$ induces a homomorphism

$$(\phi_t)^*: \text{Ext}^1_k(A, G'_{\text{m}})_\mathbb{Q} = (A^\vee(k) \otimes \mathbb{Q})^t \to (A^\vee(k) \otimes \mathbb{Q})^t = \text{Ext}^1_k(A, G'_{\text{m}})_\mathbb{Q}$$

in place of (10).

2. Compactifications

To compactify semiabelian varieties we use a well-known construction proposed by Serre (cf. [52] Section 3.2] and Serre’s appendix in [59]). Let $G$ be a semiabelian variety over $k$ with split toric part $T = G'_{\text{m}}$ and abelian quotient $A_{\text{m}}$. Furthermore, let $T$ be a $T$-equivariant compactification of $T$. This means that we are given a dense open embedding $T \to T$ with $T$ a proper $k$-variety and that there is an extension $\tau: T \times T \to \tau(T)$ of the group law $\tau: T \times T \to T$.

We endow $G \times_k T$ with the $T$-action given by

$$(g, t) \cdot (g', t') = (g \cdot g', t^{-1} \tau t'), \quad t \in T(S), t' \in T(S), g \in G(S),$$

on $S$-points. It is well-known that $G\tau = G \times_k T/T$ is a proper $k$-scheme (see e.g. [15, 30]) but we briefly indicate the argument here for lack of a reference. By definition, $G$ is a $T$-torsor over $A$ in the fppf topology. Hilbert’s Theorem 90 ([10] Proposition III.4.9]) implies that $G$ is even a $T$-torsor in the Zariski topology. In other words, there exist finitely many Zariski opens $U_i$ of $A$ together with compatible $T$-equivariant trivializations $\phi_i : U_i \times_A G \to U_i \times_k T$ over each $U_i$. The isomorphisms

$$\phi_i \circ \phi_i^{-1}|_{(U_i \cap U_j) \times_k T} : (U_i \cap U_j) \times_k T \to (U_i \cap U_j) \times_k T$$

determine sections $t_{ij} \in \Gamma(U_i \cap U_j, T)$. The variety $G$ can be described as a gluing of these trivial $T$-torsors by means of the $\check{C}$ech cocycle $\{t_{ij}\} \in H^1(\{U_i\}, T)$. (In fact, $\{t_{ij}\}$ is also the cocycle describing $\eta_G = (A^\vee)^t(k) = \text{Ext}^1_k(A, T)$ in the Barsotti-Weil formula.) Via the extension $\tau$ of the group law $\tau$, the same $\check{C}$ech cocycle $\{t_{ij}\}$ determines also a gluing of the $k$-varieties $U_i \times_k T$, yielding a proper $k$-variety $X$ and a projection $\pi: X \to A$. There is a canonical map $p: G \times_k T \to X$ over $A$ such that its base change

$$p \times_X (U_i \times_k T): (U_i \times_A G) \times_k T \to U_i \times_k T$$
coincides with the action

\[ U_i \times_k \mathcal{T} : U_i \times_k (T \times_k \mathcal{T}) \rightarrow U_i \times_k \mathcal{T} \]

under the identification \( U_i \times_A G = U_i \times_k T \) described by \( \phi_i \). Neither \( X \) nor \( p \) depends on the Zariski covering \( \{ U_i \} \) as the definitions are compatible with any refinement. It is easy to conclude now from \([14]\) that \((X, p)\) represents indeed the fppf sheaf \( G_{\mathcal{T}} \), using e.g. \([17]\) Théorème V.10.1.1. In addition, the \( G \)-action given by the group law \( +_G : G \times G \rightarrow G \) extends uniquely to an action \( G \times G_{\mathcal{T}} \rightarrow G_{\mathcal{T}} \).

If \((M, \varrho : T \times_k M \rightarrow M)\) is a \( T \)-linearized line bundle on \( \mathcal{T} \), we endow \( G \times_k M \) with a \( T \)-action in a way similar to \([14]\) and form the quotient \( G(M, \varrho) = G \times_k M/T \). Repeating the above procedure, it is easy to infer that \( G(M, \varrho) \) is a line bundle over \( G_{\mathcal{T}} \). One checks also a compatibility \( G(M \otimes M', \varrho \otimes \varrho') \cong G(M, \varrho) \otimes G(M', \varrho') \) with tensor products.

**Lemma 3.** Let \((M, \varrho)\) be an ample \( T \)-linearized line bundle on \( \mathcal{T} \) and \( N \) an ample line bundle on \( A \). Then, the line bundle \( G(M, \varrho) \otimes \pi^* N \) (resp. \( G(M, \varrho) \)) is ample (resp. nef).

**Proof.** The ampleness part is educed easily from \([19]\) Théorème 4.6.13 (ii) (cf. \([30]\) Lemma 1.7]). For nefness, let \( C \) be a proper curve in \( G_{\mathcal{T}} \). We already know that \( G(M, \varrho) \otimes_k \pi^* N \) is ample for any integer \( k \geq 1 \). Hence, the degree of the \( 0 \)-cycle class

\[ kc_1(G(M, \varrho)) \cap [C] + c_1(\pi^* N) \cap [C] \]

is positive for any \( k \) (see e.g. \([16]\) Lemma 12.1)). Dividing by \( k \) and taking the limit \( k \rightarrow \infty \), we obtain

\[ \deg(c_1(G(M, \varrho)) \cap [C]) \geq 0. \]

We are interested in the behavior of the above constructions with regard to homomorphisms. For this, let \( \varphi : G \rightarrow G' \) be a homomorphism of semiabelian varieties with toric component \( \varphi_{\text{tor}} : T \rightarrow T' \) as in \([11]\). In addition, let \( \mathcal{T} \) (resp. \( \mathcal{T}' \)) be a \( T \)-equivariant (resp. \( T' \)-equivariant) compactification of \( T \) (resp. \( T' \)) so that \( \varphi_{\text{tor}} \) extends to a \( \varphi_{\text{tor}} \)-equivariant map \( \varphi_{\text{tor}} : \mathcal{T} \rightarrow \mathcal{T}' \).

Endowing \( G \times_k \mathcal{T} \) (resp. \( G' \times_k \mathcal{T}' \)) with a \( T \)-action (resp. \( T' \)-action) as in \([11]\), the \( \varphi_{\text{tor}} \)-equivariant map \( \varphi \times_k \varphi_{\text{tor}} : G \times_k \mathcal{T} \rightarrow G' \times_k \mathcal{T}' \) induces a map \( \varphi : G_{\mathcal{T}} \rightarrow G'_{\mathcal{T}'} \). Let now \((M, \varrho)\) be a \( T' \)-linearized line bundle on \( \mathcal{T}' \). We have \( \varphi^* G'(M, \varrho) \approx G(\varphi_{\text{tor}}(M, \varrho)) \); for the line bundle \( G \times_k \varphi_{\text{tor}} M \) is the pullback of \( G' \times_k M \) along \( \varphi \times_k \varphi_{\text{tor}} \) and the induced map \( G \times_k \varphi_{\text{tor}} M \rightarrow G' \times_k M \) is \( \varphi_{\text{tor}} \)-equivariant.

In these considerations, the case where \( \varphi \) is the multiplication-by-\( n \) homomorphism \( [n]_G \) for a semiabelian variety \( G \) with toric part \( T \) is of particular importance. To avoid pathologies, some further technical requirements on both the \( T \)-equivariant compactification \( \mathcal{T} \) and the \( T \)-linearizable line bundle \( M \) should be met. First, an extension of \([n]_T \) to a map \([n]_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T} \) should exist. (Such an extension is unique by separatedness.) Under this condition, there is an extension \( \varphi = [n]_{\mathcal{T}} : G_{\mathcal{T}} \rightarrow G_{\mathcal{T}} \) of \([n]_G \) by the last paragraph. Second, there should be a \( T \)-equivariant isomorphism \([n]_{\mathcal{T}}^* M \approx M^{\otimes n} \). If this is satisfied, the last assertion of the preceding paragraph specializes to \([n]_{\mathcal{T}}^* G(M, \varrho) \approx G([n]_{\mathcal{T}}(M, \varrho)) \approx G((M, \varrho)^{\otimes n}) \approx G(M, \varrho)^{\otimes n} \).

Before introducing the two types of compactifications to be employed in our proof of Theorem \([2]\) we recall a further notion. Let \( T \) be a torus with \( T \)-equivariant compactification \( \mathcal{T} \). Pulling meromorphic functions back fabricates a \( T \)-linearization of \( K_{\mathcal{T}} \). Denote by \( \text{pr}_2 : T \times T \rightarrow \mathcal{T} \) the projection and by \( \sigma : T \times \mathcal{T} \rightarrow \mathcal{T} \) the \( T \)-action on \( \mathcal{T} \). A Cartier divisor \( D \) on \( \mathcal{T} \) is called \( T \)-invariant if the pullbacks \( \text{pr}_2^* D \) and \( \sigma^* D \) are equal. In this case, \( D \) gives rise to a \( T \)-invariant invertible subsheaf \( O(D) \) of \( K_{\mathcal{T}} \). Hence, there is an induced \( T \)-linearization on \( O(D) \). We always mean this linearization when associating a \( T \)-linearized line bundle \((L(D), g_0)\) with a \( T \)-invariant Cartier divisor \( D \). Note that this \( T \)-linearization on \( O(D) \) is uniquely characterized by the fact that its rational section \( 1 \in K_{\mathcal{T}}(\mathcal{T}) \) is \( T \)-invariant.

Any \( T \)-invariant Cartier divisor \( D \) on \( \mathcal{T} \) yields naturally a Cartier divisor on \( G_{\mathcal{T}} \). Indeed, assume that \( D \) is represented by \((V_j, f_j)\) with Zariski opens \( V_j \subset \mathcal{T} \). For each Zariski open \( U_i \subset \mathcal{T} \)
A this gives a Cartier divisor on $U_i \times_{\mathcal{A}} G_{T^\ast} = U_i \times_k \mathcal{T}$ that is represented by $(U_i \times_k V_i, f_j \circ pr_2)$. By $T$-invariance, these Cartier divisors glue together to a Cartier divisor $G(D)$ on $G_{T^\ast}$. Furthermore, it is easy to see that $L(G(D))$ is isomorphic to $G(L(D), \varrho_D)$.

**Construction 4** $(D_t, (M_t, \varrho_t))$. The torus $G_m = \text{Spec}(k[X, X^{-1}])$ has a $G_m$-equivariant compactification $\tau_t : G_m \hookrightarrow \mathbb{P}^1 = \text{Proj}(k[Z_1, Z_2])$ with $\tau_t^* (Z_2/Z_1) = X$. There is an extension $[n_i] : \mathbb{P}^1 \to \mathbb{P}^1$ of $[n_i]_{G_m} : G_m \to G_m$. Let $E_0$ (resp. $E_\infty$) be the $G_m$-invariant Cartier divisor on $\mathbb{P}^1$ represented by

$$(D(Z_1), Z_2/Z_1) \text{ and } (D(Z_2), 1) \text{ (resp. } (D(Z_1), 1) \text{ and } (D(Z_2), Z_1/Z_2)).$$

For the torus $T = G^\ast_m$, the map $\iota_t = \tau_1 \times \cdots \times \tau_i : G^\ast_i \to \mathcal{T} = (\mathbb{P}^1)^\ast$ gives a $T$-equivariant compactification. Denoting by $pr_i : \mathcal{T} = (\mathbb{P}^1)^\ast \to \mathbb{P}^1$ the projection to the $i$-th component, we set $D_i = \sum_{1 \leq i \leq t} pr_i^*(E_0 + E_\infty)$ and $M_i = \mathcal{O}(D_i)$. By the above, there is a natural $T$-linearization $\varrho_t = \varrho_{D_t}$ on the associated line bundle $M_t$ that acts trivially on its global section $1 \in \mathcal{M}_t(\mathcal{T})$.

Furthermore, from the evident identity $[n_i]_{G}^\ast M_t = \mathcal{M}_t^{\otimes n}$ of $\mathcal{O}_X$-submodules of $\mathcal{K}_T$ so that $[n_i]_{G}^\ast (M_t, \varrho_t) = (M_t, \varrho_t)^{\otimes n}$.

**Construction 5** $(\mathcal{G}, M_{\mathcal{T}}, G(D_t))$. Given a semialgebraic variety $G$ having split toric part $T = G^\ast_g$ and abelian quotient $\pi : G \to A$, we use the $T$-equivariant compactification $\iota_t : G^\ast_i \to (\mathbb{P}^1)^\ast = \mathcal{T}$ constructed above to obtain a smooth compactification $\mathcal{G} = G_{\mathcal{T}}$ with abelian quotient $\pi : \mathcal{G} \to A$. The $T$-invariant line bundle $(M_t, \varrho_t)$ yields further a line bundle $M_{\mathcal{T}} = G(M_t, \varrho_t)$ on $\mathcal{G}$, which satisfies $[n_i]_{\mathcal{G}}^\ast M_{\mathcal{T}} \approx M_{\mathcal{T}}^{\otimes n}$. In addition, the line bundle $M_{\mathcal{T}}$ is associated with the Cartier divisor $G(D_t)$.

**Construction 6** $(G_{(\varphi_{\text{tor}}^\ast)}, M_{(\varphi_{\text{tor}}^\ast)}, \pi_{(\varphi_{\text{tor}}^\ast)}).$ Assume given a semialgebraic variety $G$ with split toric part $G^\ast_m$ and abelian quotient $\pi : G \to A$ as well as a homomorphism $\varphi_{\text{tor}} \in \text{Hom}(G^\ast_m, G^\ast_g)$. Let $G(\varphi_{\text{tor}}) \subset G^\ast_m \times G^\ast_g$ be the graph of $\varphi_{\text{tor}}$ and $\overline{G(\varphi_{\text{tor}})}$ its Zariski closure in the $(G^\ast_m \times G^\ast_g)$-equivariant compactification $(\mathbb{P}^1)^\ast \times (\mathbb{P}^1)^\ast$. The projection to $G^\ast_m$ induces an identification $G(\varphi_{\text{tor}}) = G^\ast_m$. In this way, $G(\varphi_{\text{tor}})$ can be considered as a $G^\ast_m$-equivariant compactification of $G^\ast_m$. As $[n_i]_{G(\varphi_{\text{tor}})}$ is just the restriction of $[n_i]_{G^\ast_m \times G^\ast_g}$, it clearly extends to $\overline{G(\varphi_{\text{tor}})}$ because $[n_i]_{G^\ast_m \times G^\ast_g}$ extends to $(\mathbb{P}^1)^\ast \times (\mathbb{P}^1)^\ast$. Therefore, there is an extension of $[n_i]_G$ to the “graph compactification” $G_{(\varphi_{\text{tor}}^\ast)}$. To fix notations, we record a self-explanatory commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
(\mathbb{P}^1)^\ast \\
\downarrow \pi \\
G \\
\downarrow \pi_{(\varphi_{\text{tor}})} \\
\overline{G(\varphi_{\text{tor}})} \\
\downarrow \pi_{(\varphi_{\text{tor}})} \\
A \\
\downarrow \text{id}_A \\
A
\end{array}
\end{array}
\]

(15)

Construction 4 gives a $G^\ast_m$-linearized line bundle $(M_t, \varrho_t)$ on $(\mathbb{P}^1)^\ast$. Its $(G^\ast_m \times G^\ast_m)$-linearized pullback $pr_2^*(M_t, \varrho_t)$ yields a line bundle $M_{(\varphi_{\text{tor}}^\ast)} = G_{(\varphi_{\text{tor}}^\ast)}(pr_2^*(M_t, \varrho_t)|_{\overline{G(\varphi_{\text{tor}})}})$ on $G_{(\varphi_{\text{tor}}^\ast)}$. Setting $\varphi_{\text{tor}} = \text{id}_{G^\ast_m}$, this construction specializes to Construction 5 above (i.e., $\mathcal{G} \approx G_{(\text{id}_{G^\ast_m})}$) with compatible $M_{\mathcal{T}} \approx M_{(\text{id}_{G^\ast_m})}$.

For any positive integer $n$, we can relate $(G_{(n \cdot \varphi_{\text{tor}}^\ast)}, M_{(n \cdot \varphi_{\text{tor}}^\ast)})$ with $(G_{(\varphi_{\text{tor}}^\ast)}, M_{(\varphi_{\text{tor}}^\ast)})$ for this, we define $G'$ and $G''$ to be the semialgebraic varieties such that $\eta_{G'} = (\eta_G, (\varphi_{\text{tor}}^\ast), \eta_G)$ and $\eta_{G''} = (\eta_G, (n \cdot \varphi_{\text{tor}}^\ast), \eta_G)$ in $\text{Ext}^1(A, G^\ast_m \times G^\ast_m)$. The equivariant closed embeddings $\overline{G(\varphi_{\text{tor}}^\ast)}, \overline{G(n \cdot \varphi_{\text{tor}}^\ast)} \subset (\mathbb{P}^1)^\ast \times (\mathbb{P}^1)^\ast$ yield closed embeddings $G_{(\varphi_{\text{tor}}^\ast)} \subset \overline{G'}$ and $G_{(n \cdot \varphi_{\text{tor}}^\ast)} \subset \overline{G''}$. In addition, the finite morphism $[1]_{(\mathbb{P}^1)^\ast} \times [n_i]_{(\mathbb{P}^1)^\ast}$ yields a map $\vartheta_n : \overline{G'} \to \overline{G''}$. As $[1]_{(\mathbb{P}^1)^\ast} \times [n_i]_{(\mathbb{P}^1)^\ast}$
restricts to a \( \mathbb{G}_m \)-equivariant morphism \( \varphi(n_{\text{tor}}) \to \varphi(n_{\text{tor}} \cdot T) \). Furthermore,
\[
\varphi(n_{\text{tor}} \cdot T) \cong \varphi(n_{\text{tor}})^* T_{\text{tor}}(\varphi)^{-1} \cong G_{\varphi(n_{\text{tor}})} \cong G_{\varphi(n_{\text{tor}} \cdot T)}.
\]
In addition, there are the evident relations \( \pi_{\varphi} \) and \( \pi_{\varphi(n_{\text{tor}})} \).

**Construction 7 \((\varphi^*)\).** We describe a subcase of Construction 6 for later reference. In this case, we start with a homomorphism \( \varphi : G \to G' \) of semiabelian varieties with split toric parts \( T = \mathbb{G}_m \) and \( T' = \mathbb{G}_m'. \) We obtain a compactification \( \overline{G'} = G_{\varphi(n_{\text{tor}})} \) from Construction 6. Furthermore, the homomorphism \( \varphi \) induces now an even larger commutative diagram
\[
\begin{array}{ccc}
\mathbb{P}^1 \rightarrow T & \xrightarrow{\varphi} & \overline{G'} \\
\downarrow & & \downarrow \\
G & \xrightarrow{\varphi} & G' \\
\downarrow & & \downarrow \\
A & \xrightarrow{\varphi_{\text{ab}}} & A'
\end{array}
\]

such that there is a decomposition \( \varphi = \varphi_{\text{ab}} \circ \varphi_{\text{tor}} \).

This construction also allows us to show that each line bundle \( M_{\varphi(n_{\text{tor}})} \) is a pullback \( p_\varphi^* M_{\varphi} \) for some homomorphism \( \varphi : G \to G' \) of semiabelian varieties. In fact, we can take \( \eta_G = (\varphi_{\text{tor}})_* \eta \in \text{Ext}^1(A, \mathbb{G}_m) \) and the homomorphism \( \varphi : G \to G' \) represented by \( (\varphi_{\text{tor}}, \text{id}_A) \).

## 3. Heights

We consistently work with (logarithmic) Weil heights and refer to [26] Theorem B.3.6] for the main features of Weil’s height machinery. In short, it provides for each line bundle \( L \) on a projective \( \overline{\mathbb{Q}} \)-variety \( X \) a class of height functions \( h_L : X(\overline{\mathbb{Q}}) \to \mathbb{R} \) such that any two such height functions differ by a globally bounded function on \( X(\overline{\mathbb{Q}}) \).

Let \( G \) be a semiabelian variety over \( \overline{\mathbb{Q}} \) with toric part \( T \) and abelian quotient \( \pi : G \to A \). Assume also given a \( T \)-equivariant compactification \( \overline{T} \) of the torus \( T \) and a \( T \)-linearized line bundle \( (M, g) \) on \( \overline{T} \) such that \([n]_T \) extends to \([n]_{\overline{T}} : \overline{T} \to \overline{T}\) and that there is an isomorphism \([n]_{\overline{T}}(M, g) \cong (M, g)^{\otimes n}\). For our purposes, these conditions on \( \overline{T} \) and \((M, g)\) are always satisfied. Additionally, we choose a symmetric line bundle \( N \) on \( A \). We furnish \( G_{\overline{T}} \) with the line bundle \( L = G(M, g) \otimes \overline{T} N \), which is ample by the above Lemma [3] and Weil’s height machinery supplies us with some height function \( h_L : G_{\overline{T}}(\overline{\mathbb{Q}}) \to \mathbb{R} \). The function \( h_L \) is neither unique nor does it enjoy homogeneity properties like the Weil height of a symmetric line bundle on an abelian variety. However, the following lemma remedies this partially. We call a \( T \)-linearized line bundle \( T \)-effective if it has a \( T \)-invariant non-zero global section.

**Lemma 8.** For any \((M, g)\) (resp. \(N\)) as above, there exists a function \( \widehat{h}_{G(M, g)} : G_{\overline{T}}(\overline{\mathbb{Q}}) \to \mathbb{R} \) (resp. \( \widehat{h}_{\overline{T}}(\overline{\mathbb{Q}}) \to \mathbb{R} \)) such that

(a) \[ |\widehat{h}_{G(M, g)}(n_{\overline{T}}) - \widehat{h}_{G(M, g)}| (\text{resp. } |\widehat{h}_{\overline{T}}(\overline{\mathbb{Q}}) - \widehat{h}_{\overline{T}}(\overline{\mathbb{Q}})|) \text{ is globally bounded on } G(\overline{\mathbb{Q}}), \]

(b) \[ \widehat{h}_{G(M, g)}([n]_g) = n \widehat{h}_{G(M, g)}(x) \text{ (resp. } \widehat{h}_{\overline{T}}([n]_g) = n^2 \widehat{h}_{\overline{T}}(x) \text{) for any } x \in G(\overline{\mathbb{Q}}) \text{ and any integer } n. \]

(c) Given a second \( T \)-linearized line bundle \((M', g')\) (resp. a symmetric ample line bundle \( N' \)) as above, we have the additivity relations
\[
\widehat{h}_{G(M \otimes M', g \otimes g')} = \widehat{h}_{G(M, g)} + \widehat{h}_{G(M', g')}, \text{ and } \widehat{h}_{\overline{T}(N \otimes N')} = \widehat{h}_{\overline{T}}(N) + \widehat{h}_{\overline{T}}(N').
\]
(d) If \((M, \varphi)\) (resp. \(N\)) is \(T\)-effective (resp. ample), then \(\hat{h}_{G(M, \varphi)}|_{G(T)}\) (resp. \(\hat{h}_{\varphi, N}\)) is non-negative.

Furthermore, \(\hat{h}_{G(M, \varphi)}\) (resp. \(\hat{h}_{\varphi, N}\)) is uniquely characterized by (a) and (b).

It is natural to work with the unique \(\hat{h}_L = \hat{h}_{G(M, \varphi)} + \hat{h}_{\varphi, N}\) instead of a non-canonical Weil height \(h_L\). By (a) of the above theorem, their difference is globally bounded on \(G(\overline{\mathbb{Q}})\). As for abelian varieties, the zero set of \(\hat{h}_L\) coincides with the torsion points of \(G\) if both \(M\) and \(N\) are ample.

The assumption of \(T\)-effectivity in (d) cannot be relaxed to mere effectivity. In fact, assume that \(T = \mathbb{G}_m^r\) and let \(Q_i, 1 \leq i \leq t\), be the line bundles on \(A\) such that \(\eta_0 = (Q_1, \ldots, Q_t) \in A^r(k)^t = \operatorname{Ext}^1_{k}(A, T)\). For \(g\) running through all possible \(T\)-linearizations of the trivial line bundle \(\mathbb{A}^1_T\), the line bundle \(G(A^t, g)\) runs through \(\pi^*(Q^{k_1}_1 \otimes \cdots \otimes Q^{k_t}_t)\) for arbitrary integers \(k_i\), as a comparison of \(\check{\mathrm{C}}\)ech cocycles shows. Except for this caveat, we do not need this and leave the verification to the interested reader.

**Proof.** (a), (b): The first two assertions of the lemma as well as uniqueness can be inferred directly from [26, Theorem B.4.1] applied to \((M, \varphi)\) (resp. \(N\)). Indeed, \([n]^\ast_{\overline{\mathbb{Q}}}(M, \varphi) \approx G(M, \varphi)^{\otimes n}\) by our assumption (compare Section 2) and \([n]^\ast_{\overline{\mathbb{Q}}} N \approx \pi^\ast [n]^\ast_{\overline{\mathbb{Q}}} N \approx \pi^\ast N^\otimes n^2\) since \(N\) is symmetric. (The result in [26] is stated for divisor classes on smooth varieties but it is also true for line bundles on general varieties with exactly the same proof.)

(c): Inspecting the proof of [26, Theorem B.4.1], one sees that the third assertion follows from the global boundedness of \(|h_{G(M \otimes M', \varphi))} - h_{G(M, \varphi)} - h_{G(M', \varphi')}|\) and \(|h_{\varphi, N \otimes \varphi, N'} - h_{\varphi, N} - h_{\varphi, N'}|\). This is a standard property of the Weil height.

(d): Similarly, one observes that \(\hat{h}_{G(M, \varphi)}\) (resp. \(\hat{h}_{\varphi, N}\)) is non-negative if \(h_{G(M, \varphi)}\) (resp. \(h_{\varphi, N}\)) is bounded from below on \(G(\overline{\mathbb{Q}})\) (resp. \(G(\overline{\mathbb{Q}})\)). For the height \(h_{\varphi, N}\), this is true because the ampleness of \(N\) implies that \(N\) and hence \(\pi^\ast N\) has empty base locus. By assumption, we have a \(T\)-invariant non-zero global section \(s : \overline{\mathbb{T}} \to M\). This gives rise to local sections \(s_i^t = U_i \times_k s : U_i \times_A G_T = U_i \times_k \overline{\mathbb{T}} \to U_i \times_k M = U_i \times_A G(M, \varphi)\). Due to the \(T\)-invariance of \(s\), the sections \(s_i^t\) glue together to a non-zero global section \(s^t\) of \(G(M, \varphi)\). Furthermore, \(T\)-invariance guarantees that \(s^t\) generates \(M_x\) for every \(x \in T(\overline{\mathbb{T}})\). We infer that \(s_i^t\) generates \(G(M, \varphi)\) for every \(x \in G(\overline{\mathbb{Q}})\). Therefore the base locus of \(G(M, \varphi)\) is contained in \(G(\overline{\mathbb{Q}})\) and \(h_{G(M, \varphi)}|_{G(\overline{\mathbb{Q}})}\) is bounded from below.

In addition, we have a good functorial behavior of the heights \(\hat{h}_{G(M, \varphi)}\) and \(\hat{h}_{\varphi, N}\). To state precisely what this means, let \(G\) (resp. \(G'\)) be a semiabelian variety over \(\overline{\mathbb{Q}}\) with torus \(T\) (resp. \(T'\)) and abelian quotient \(A\) (resp. \(A'\)). Take furthermore equivariant compactifications \(\overline{T}\) and \(\overline{T}'\) so that \(\varphi_{\text{tor}} : T \to T'\) extends to a \(\varphi_{\text{tor}}\)-equivariant map \(\overline{\varphi}_{\text{tor}} : \overline{T} \to \overline{T}'\). In this situation, we consider a \(T'\)-linearized line bundle \((M, \varphi)\) on \(\overline{T}'\) such that there is a \(T'\)-equivariant isomorphism \([n]^\ast_{\overline{\mathbb{Q}}}(M, \varphi) \approx (M, \varphi)^{\otimes n}\). We also take a symmetric ample line bundle \(N\) on \(A'\). Then, for any homomorphism \(\varphi : G \to G'\) with toric (resp. abelian) component \(\varphi_{\text{tor}}\) (resp. \(\varphi_{\text{ab}}\)) we have equalities

\[
\tag{17} \hat{h}_{G(M, \varphi)}(\varphi(x)) = \hat{h}_{G(\varphi_{\text{tor}}(M, \varphi))}(x) \quad \text{and} \quad \hat{h}_{\varphi, N}(\varphi(x)) = \hat{h}_{\varphi_{\text{ab}}, N}(x) \quad \text{for any} \ x \in G(\overline{\mathbb{Q}}).
\]

In fact, this follows directly from the functorial behavior of the Weil height under pullback and the uniqueness assertion of Lemma 8.

We note an addendum to Lemma 8 specifically related to the line bundles \(M_T\) and \(M_{\Gamma(\varphi_{\text{tor}})}\).

**Lemma 9.** Let \(G\) a semiabelian variety with split torus \(\mathbb{G}_m^r\) and abelian quotient \(\pi : G \to A\). For any \(\varphi_{\text{tor}} \in \operatorname{Hom}(\mathbb{G}_m^r, \mathbb{G}_m^r)\), the restriction of \(\hat{h}_{M_{\Gamma(\varphi_{\text{tor}})}} : G(\overline{\mathbb{Q}}) \to \mathbb{R}\) to \(G(\overline{\mathbb{Q}})\) is non-negative. In particular, the restriction of \(\hat{h}_{M_T}\) to \(G(\overline{\mathbb{Q}})\) is non-negative.
Proof. In Construction 7 it is shown that each $M_{\Gamma(\varphi_{tor})}$ is isomorphic to the pullback of a line bundle of the form $M_{\Gamma}$. Using functoriality (17), it hence suffices to prove the non-negativity of $\hat{h}_{M_{\Gamma}}$. For this, we inspect Construction 11. The Cartier divisor $D_{t}$ on $G_{m}^{t}$ is effective and $G_{m}^{t}$-invariant so that the constant function $1 \in K_{\psi}\xi((\mathbb{P}^{1})')$ gives rise to a $G_{m}^{t}$-invariant global section of its associated line bundle $(M_{t}, \varrho_{t})$. In other words, $(M_{t}, \varrho_{t})$ is $G_{m}^{t}$-effective and we can use Lemma 8 (d).

Fix again a semiabelian variety $G$ over $\overline{\mathbb{Q}}$ with toric part $G_{m}^{t}$ and abelian quotient $\pi : G \to A$. Furthermore, let $\overline{G}$ be a compactification of $G$ and $\overline{\pi} : \overline{G} \to A$ its abelian quotient as in Construction 5. We want to estimate the difference between $\hat{h}_{M_{\Gamma(\varphi_{tor})}}$ and $\hat{h}_{M_{\Gamma(\varphi_{tor})}}$ for two “close” homomorphisms $\varphi_{tor}, \tilde{\varphi}_{tor} \in \text{Hom}(G_{m}^{t}, G_{m}^{t})$. Simultaneously, we examine the corresponding “abelian” analogue. For this purpose, let $A'$ be a second abelian variety and $N$ (resp. $N'$) an ample symmetric line bundle on $A$ (resp. $A'$). We choose some linear norms $\cdot \cdot$ on $\text{Hom}(G_{m}^{t}, G_{m}^{t})$ and Hom($A, A'$) for quantification.

Lemma 10. In the above situation, there exist constants $c_{1}$ and $c_{2}$ depending only on $G$, $N$, $N'$ and the linear norms $\cdot \cdot$ on $\text{Hom}(G_{m}^{t}, G_{m}^{t})$ and Hom($A, A'$) such that the following assertions are true: For any pair $(\varphi_{tor}, \tilde{\varphi}_{tor}) \in \text{Hom}(G_{m}^{t}, G_{m}^{t})^{2}$ and any $x \in G(\mathbb{Q})$, we have

\begin{equation}
|\hat{h}_{M_{\Gamma(\varphi_{tor})}}(x) - \hat{h}_{M_{\Gamma(\tilde{\varphi}_{tor})}}(x)| \leq c_{1}|\varphi_{tor} - \tilde{\varphi}_{tor}| \cdot \hat{h}_{\overline{G}}(x).
\end{equation}

Similarly, we have

\begin{equation}
|\hat{h}_{\overline{\pi} - \varphi_{ab}, N'}(x) - \hat{h}_{\overline{\pi} - (\varphi_{ab}), N'}(x)| \leq c_{2}|\varphi_{ab} - \tilde{\varphi}_{ab}| \cdot \hat{h}_{\overline{\pi} - N}(x)
\end{equation}

for any pair $(\varphi_{ab}, \tilde{\varphi}_{ab}) \in \text{Hom}(A, A')^{2}$.

Proof. We prove first the inequality (18). The proof takes place on the “graph compactification” $G_{\Gamma}$ with $\Gamma = \Gamma(\varphi_{tor} \times \varphi_{tor}) \subset (\mathbb{P}^{1})' \times (\mathbb{P}^{1})', (\mathbb{P}^{1})''$. We denote the projections corresponding to these three factors by $pr_{i}$ ($i = 1, 2, 3$). The projections $(pr_{1} \times pr_{2})|_{\Gamma} : \Gamma \to \Gamma(\varphi_{tor})$ and $(pr_{1} \times pr_{3})|_{\Gamma} : \Gamma \to \Gamma(\tilde{\varphi}_{tor})$ are $G_{m}^{t}$-equivariant and hence induce maps $G_{\Gamma} \to G_{\Gamma(\varphi_{tor})}$ and $G_{\Gamma} \to G_{\Gamma(\tilde{\varphi}_{tor})}$. Applying (17) to these, we obtain

\begin{equation}
\hat{h}_{M_{\Gamma(\varphi_{tor})}}(x) = \hat{h}_{pr_{3}(M_{t, \varrho_{t}})}(x) \quad \text{and} \quad \hat{h}_{M_{\Gamma(\tilde{\varphi}_{tor})}}(x) = \hat{h}_{pr_{3}(M_{t, \varrho_{t}})}(x)
\end{equation}

for every $x \in G(\mathbb{Q})$. Similarly, we have

\begin{equation}
\hat{h}_{\overline{\pi} - \varphi_{ab}, N'}(x) = \hat{h}_{pr_{3}(M_{t, \varrho_{t}})}(x)
\end{equation}

for every $x \in G(\mathbb{Q})$.

Using standard coordinates $X_{u}$, $1 \leq u \leq t$, (resp. $Y_{v}$, $1 \leq v \leq t'$) on $G_{m}^{t}$ (resp. $G_{m}^{t}$), we write

\begin{equation}
\varphi_{tor}(Y_{v}) = X_{1}^{a_{1,v}}X_{2}^{a_{2,v}} \cdots X_{t}^{a_{t,v}} \quad \text{(resp. } \varphi_{tor}'(Y_{v}) = X_{1}^{a_{1,v}}X_{2}^{a_{2,v}} \cdots X_{t}^{a_{t,v}}), 1 \leq v \leq t',
\end{equation}

with integers $a_{uv}$ (resp. $a_{uv}'$). Our strategy is to compare the restriction of the line bundles $pr_{i}(M_{t, \varrho_{t}})^{\otimes l}$, $l$ sufficiently large, and $pr_{i}(M_{t, \varrho_{t}}) \otimes pr_{3}(M_{t, \varrho_{t}})^{-1}$ on $\Gamma$. In fact, we claim that both

\begin{equation}
(pr_{1}^{t}M_{t}^{\otimes l} \otimes pr_{2}^{t}M_{t} \otimes pr_{3}^{t}M_{t}^{-1})|_{\Gamma} \quad \text{and} \quad (pr_{1}^{t}M_{t}^{\otimes l} \otimes pr_{2}^{t}M_{t} \otimes pr_{3}^{t}M_{t})|_{\Gamma}
\end{equation}

are $G_{m}^{t}$-effective with respect to the induced linearizations. In this case, Lemma 8 (c, d) implies that

\begin{equation}
|\hat{h}_{pr_{3}(M_{t, \varrho_{t}})}|_{\Gamma}(x) - \hat{h}_{pr_{3}(M_{t, \varrho_{t}})}|_{\Gamma}(x)| \leq l \cdot \hat{h}_{pr_{1}(M_{t, \varrho_{t}})}|_{\Gamma}(x).
\end{equation}

Using the equalities (20) and (21), the inequality (18) can be derived from (23) if we have adequate control on $l$. For this, we note that $pr_{1}^{t}(L_{t, \varrho_{t}})|_{\Gamma}$ (resp. $pr_{2}^{t}(L_{t, \varrho_{t}})|_{\Gamma}$, $pr_{3}^{t}(L_{t, \varrho_{t}})|_{\Gamma}$) can be defined by means of the $G_{m}^{t}$-invariant Cartier divisor $pr_{1}^{t}D_{i}|_{\Gamma}$ (resp. $pr_{2}^{t}D_{i}|_{\Gamma}$, $pr_{3}^{t}D_{i}|_{\Gamma}$).
We next describe these divisors explicitly and start with giving a covering of $\Gamma$. With each $(t + 2t')$-tuple $\underline{m}$ of numbers $m_r \in \{-1, 1\}$, $1 \leq r \leq t + 2t'$, we associate a Zariski open

$$U_{\underline{m}} = \Gamma \cap \bigcap_{1 \leq u \leq t} D(\text{pr}_1^* X_u^{m_u}) \cap \bigcap_{1 \leq v \leq t'} D(\text{pr}_2^* Y_v^{m_{v+t}}) \cap \bigcap_{1 \leq v \leq t'} D(\text{pr}_3^* Y_v^{m_{v+t+t'}}).$$

Evidently, $\text{pr}_1^* D_1|\Gamma$ is represented by $(U_{\underline{m}}, f_{\underline{m}})$ with

$$f_{\underline{m}} = \text{pr}_1^*(X_1^{-m_1} X_2^{-m_2} \cdots X_t^{-m_t})$$

and $\text{pr}_2^* D_2|\Gamma$ (resp. $\text{pr}_3^* D_2|\Gamma$) is represented by $(U_{\underline{m}}, g_{\underline{m}})$ (resp. $(U_{\underline{m}}, g'_{\underline{m}})$) with

$$g_{\underline{m}} = \prod_{1 \leq v \leq t'} \text{pr}_2^*(Y_v^{-m_{v+t}}) \quad (\text{resp. } g'_{\underline{m}} = \prod_{1 \leq v \leq t'} \text{pr}_3^*(Y_v^{-m_{v+t+t'}})).$$

The meromorphic function $1 \in K_2(\Gamma)$ gives a $G^t_\underline{m}$-invariant rational section of $O(\text{pr}_1^* D_1|\Gamma)$, $O(\text{pr}_2^* D_2|\Gamma)$ and $O(\text{pr}_3^* D_2|\Gamma)$ by our choice of linearizations. It thus also gives a $G^t_\underline{m}$-invariant rational section of $O(l \cdot \text{pr}_1^* D_1|\Gamma + \text{pr}_2^* D_2|\Gamma - \text{pr}_3^* D_2|\Gamma)$ and $O(l \cdot \text{pr}_1^* D_1|\Gamma - \text{pr}_2^* D_2|\Gamma + \text{pr}_3^* D_2|\Gamma)$, to which the line bundles in (22) are associated. For $G^t_\underline{m}$-effectivity, we may hence prove that it is actually a global section. In other words, we have to prove that both $f_{\underline{m}}^l \cdot g_{\underline{m}} \cdot (g'_{\underline{m}})^{-1}$ and $f_{\underline{m}}^l \cdot (g_{\underline{m}})^{-1} \cdot g'_{\underline{m}}$ are regular on $U_{\underline{m}}$. Let us remark first that for $l_v = \max_{1 \leq u \leq t} |a_{uv} - a'_{uv}|$ the meromorphic function

$$f_{\underline{m}}^l \cdot \text{pr}_2^*(Y_v) \text{pr}_3^*(Y_v) = \prod_{1 \leq u \leq t} \text{pr}_1^*(X_u^{s_u}) \quad (s_u = -m_u l_v + (a_{uv} - a'_{uv}),$$

is regular on $U_{\underline{m}}$; for $\text{pr}_1^*(X_u^{-m_u})$ is regular on $U_{\underline{m}} \subset D(\text{pr}_1^*(X_u^{-m_u}))$. Similarly, the meromorphic function $f_{\underline{m}}^l \cdot \text{pr}_2^*(Y_v)^{-1} \text{pr}_3^*(Y_v)$ is regular on $U_{\underline{m}}$. We write $g_{\underline{m}} \cdot (g'_{\underline{m}})^{-1} = \prod_{l_v = 1} \text{h}_{\underline{m}}$ with

$$h_{\underline{m}} = \text{pr}_2^*(Y_v^{-m_{v+t}}) \text{pr}_3^*(Y_v^{-m_{v+t+t'}})$$

and claim that $f_{\underline{m}}^l \cdot h_{\underline{m}}$ is regular on $U_{\underline{m}}$. If $m_{v+t} = m_{v+t+t'} = 1$ or $m_{v+t} = m_{v+t+t'} = -1$, this follows directly from our previous remark. In case $m_{v+t} = -1$ and $m_{v+t+t'} = 1$, the function

$$f_{\underline{m}}^l \cdot h_{\underline{m}} = \text{pr}_2^*(Y_v)^2 \cdot (f_{\underline{m}}^l \cdot \text{pr}_3^*(Y_v)^{-1} \text{pr}_3^*(Y_v))$$

is regular by our remark and the fact that $\text{pr}_3^*(Y_v)$ is regular on $U_{\underline{m}} \subset D(\text{pr}_2^*(Y_v)^{-1})$. The case $m_{v+t} = 1$ and $m_{v+t+t'} = -1$ can be handled in the same way, establishing our claim. In conclusion, the condition

$$l \geq \sum_{1 \leq u \leq t} l_u = \sum_{1 \leq v \leq t'} \left( \max_{1 \leq u \leq t} |a_{uv} - a'_{uv}| \right)$$

suffices to ensure the regularity of $f_{\underline{m}}^l \cdot g_{\underline{m}} \cdot (g'_{\underline{m}})^{-1}$. The same argument shows that each $f_{\underline{m}}^l \cdot (g_{\underline{m}})^{-1} \cdot g'_{\underline{m}}$ is regular on $U_{\underline{m}}$. Combining (23) and (24), we obtain (18).

The inequality (19) boils down to

$$|\widehat{h}_{N'}(\varphi_{ab}(y)) - \widehat{h}_{N'}(\varphi'_{ab}(y))| \leq c_2 |\varphi_{ab} - \varphi'_{ab}| \cdot \widehat{h}_N(y), \quad y = \pi(x),$$

where $\widehat{h}_N$ and $\widehat{h}_{N'}$ are now just the Néron-Tate heights on the abelian varieties $A$ and $A'$. This follows straightforwardly from the fact that the map

$$\text{Hom}(A, A') \to \text{Pic}(A), \varphi_{ab} \to \varphi_{ab} N',$$

is quadratic, which is a direct consequence of the Theorem of the Cube ([44 Corollary II.6.2]). The reader may refer to [22, p. 417] for details.

Finally, we state a lemma on the behavior of the heights $\widehat{h}_{\Gamma_{\varphi_{tor}}}$ with respect to the group law. Again, there is an “abelian” analogue and we mention this also for later reference.

**Lemma 11.** For any $\varphi_{tor} \in \text{Hom}(G_{m}, G^t_{m})$ and any points $x, y \in G(\overline{\mathbb{Q}})$, we have $\widehat{h}_{\Gamma_{\varphi_{tor}}}(xy) \leq \widehat{h}_{\Gamma_{\varphi_{tor}}}(x) + \widehat{h}_{\Gamma_{\varphi_{tor}}}(y)$. Similarly, we have $\widehat{h}_{N}(xy) \leq 2\widehat{h}_{N}(x) + 2\widehat{h}_{N}(y)$. \qed
Note that this statement includes the fact that
\begin{equation}
\hat{h}_{M_\Gamma}(xy) \leq \hat{h}_{M_\Gamma}(x) + \hat{h}_{M_\Gamma}(y)
\end{equation}
for all \(x, y \in G(\mathbb{Q})\) (set \(\varphi_{\text{tor}} = \text{id}_{G_m}\)).

**Proof.** For the first assertion, it suffices to prove (25). In fact, each \(M_{\Gamma([\text{tor}])}\) is isomorphic to some pullback \(p^*_\varphi M_{\Gamma}\) along a homomorphism \(\varphi : G \to G'\) to another semiabelian variety \(G'\) (see Construction [7]). In order to prove (25), we use the same strategy as for Lemma 10. This means we consider the Zariski closure \(\overline{\Gamma} \subset (\mathbb{P}^1)^t \times (\mathbb{P}^1)^t\) of the group of the graph law \(\pi : G_m \times G_m' \to G_m\). Again, we denote the projection to the \(i\)-th component by \(\text{pr}_i\) \((i = 1, 2, 3)\). For this, we use standard coordinates \(X_u, u \leq t\), on \(G_m(t)\) (and on its extension to \((\mathbb{P}^1)^t\)).

Note that \((\text{pr}_3^*X_u) = (\text{pr}_1^*X_u)(\text{pr}_2^*X_u)\) on \(\overline{\Gamma}\).

With each \((3t)\)-tuple \(m \in \{-1, 1\}^{3t}\) of numbers \(m_r \in \{-1, 1\}, 1 \leq r \leq 3t\), we associate a Zariski open. To wit, we define
\[
U_m = \overline{\Gamma} \cap \bigcap_{1 \leq u \leq t} D(\text{pr}_1^*X_u^{m_u}) \cap \bigcap_{1 \leq u \leq t} D(\text{pr}_2^*X_u^{m_u+t}) \cap \bigcap_{1 \leq u \leq t} D(\text{pr}_3^*X_u^{m_u+2t}).
\]

It is easy to see that each \(\text{pr}_1^*D_t(\overline{\Gamma})\) \((i = 1, 2, 3)\) is represented by \((U_m, f_m^{(i)})\) where
\[
f_m^{(i)} = \text{pr}_i^*(X_1^{-m_{1+i-(i-1)t}})X_2^{-m_{2+i-(i-1)t}} \cdots X_t^{-m_{t+i-(i-1)t}).
\]

Consequently, the restriction of \(\text{pr}_1^*D_t + \text{pr}_2^*D_t - \text{pr}_3^*D_t\) to \(\overline{\Gamma}\) is represented by \((U_m, f_m^{(1)} \cdot f_m^{(2)} \cdot (f_m^{(3)})^{-1})\). The meromorphic function \(f_m^{(1)} \cdot f_m^{(2)} \cdot (f_m^{(3)})^{-1}\) equals
\[
\text{pr}_1^*(X_1)^{-m_1+m_{2t+1}} \cdots \text{pr}_2^*(X_t)^{-m_{t+1}+m_{2t+1}} \text{pr}_3^*(X_1)^{-m_{t+1}+m_{2t+1}} \cdots \text{pr}_2^*(X_t)^{-m_{2t+m_2}}.
\]

By definition, each \(\text{pr}_i^*(X_u)^{-m_{u+i-(i-1)t}}, u \in \{1, 2, 3\}, 1 \leq u \leq t\), is regular on \(U_m\). Since \(-m_u + m_{2t+u} \in \{0, -2m_u\}\) and \(-m_{t+u} + m_{2t+u} \in \{0, -2m_{t+u}\}\), we infer the regularity of \(f_m^{(1)} \cdot f_m^{(2)} \cdot (f_m^{(3)})^{-1}\) on \(U_m\). As in the proof of Lemma 10, we see that this implies that \(1 \in K_*(\overline{\Gamma})\) is a \(T\)-invariant global section of \(\mathcal{O}(\text{pr}_1^*D_t + \text{pr}_2^*D_t - \text{pr}_3^*D_t)\). Thus, the first assertion follows from Lemma 3 (d). For the second assertion, it suffices to note the equivalence of the assertion with
\[
\hat{h}_{\pi N}(\pi(x) + \pi(y)) \leq 2\hat{h}_{\pi N}(\pi(x)) + 2\hat{h}_{\pi N}(\pi(y)).
\]

Indeed, this inequality follows directly from the fact that the Néron-Tate height \(\hat{h}_N\) is related to a scalar product (cf. [3] Theorem 9.3.5)).

\[
\square
\]

4. Hermitian Differential Geometry

In the next two sections, we make extensive use of hermitian differential geometry at the level of rather explicit computations on semiabelian varieties. To avoid potential interruptions in these, we recall here the necessary abstract framework separately. The reader is referred to [56] Section 3.1] as well as [18] Section 0.2] and [27] Section 1.2] for details.

Let \(Y\) be a complex manifold (e.g., \(X^{sm}(\mathbb{C})\) for a complex algebraic variety \(X\)). To \(Y\) is associated its real tangent bundle \(T_\mathbb{R}Y\), its holomorphic tangent bundle \(T^{1,0}_\mathbb{C}Y\) (e.g., \(T_xX(\mathbb{C})\) for a smooth complex algebraic variety \(X\)) and its anti-holomorphic tangent bundle \(T^{0,1}_\mathbb{C}Y\). As real vector bundles, all three can be canonically identified (cf. [18] p. 17)] and we do so from now on. In this way, we obtain an almost complex structure \(I : T_\mathbb{R}Y \to T_\mathbb{R}Y\) (i.e., a linear map \(I : T_\mathbb{R}Y \to T_\mathbb{R}Y\) such that \(I^2 = -\text{id}_{T_\mathbb{R}Y}\)) from the multiplication-by-\(-i\) (resp. multiplication-by-\((i)\)) on the complex vector bundle \(T^{1,0}_\mathbb{C}Y\) (resp. \(T^{0,1}_\mathbb{C}Y\)). A \((1, 1)\)-form of real type on \(Y\) is an alternating \(\mathbb{R}\)-bilinear pairing
\[
\omega : T^{1,0}_\mathbb{C}Y \times Y \to \mathbb{R} \times Y \text{ (resp. } \omega : T^{1,0}_\mathbb{C}Y \times Y \to \mathbb{R} \times Y).\]
such that $\omega(I(\cdot), I(\cdot)) = \omega(\cdot, \cdot)$ (resp. $\omega(i(\cdot), i(\cdot)) = -\omega(\cdot, \cdot)$). The Chern forms of hermitian line bundles are the basic examples of such $(1, 1)$-forms. More generally, for any smooth function $\lambda : Y \to \mathbb{R}$ the $(1, 1)$-form $dd^c\lambda$ $(d^c = i/4\pi(\overline{\partial} - \partial))$ is always of real type. To such a $(1, 1)$-form $\omega$ is associated a symmetric $\mathbb{R}$-bilinear pairing

$$g_\omega : T_\mathbb{R}Y \times Y T_\mathbb{R}Y \longrightarrow \mathbb{R} \times Y, (v, w) \longmapsto \omega(v, Iw).$$

In fact, this establishes a one-to-one correspondence between $(1, 1)$-forms of real type and symmetric $\mathbb{R}$-bilinear forms on $T_\mathbb{R}Y$. Using our identification of $T_\mathbb{R}Y$ with $T^* Y$ and $T^0 Y$, the $(1, 1)$-form $\omega$ is positive (resp. semipositive) in the ordinary sense (e.g. [27, Definition 4.3.14]) if and only if $g_\omega$ is positive definite (resp. positive semidefinite). We note that for a smooth function $\lambda : Y \to \mathbb{R}$, the $(1, 1)$-form $dd^c\lambda$ is semipositive if and only if $\lambda$ is plurisubharmonic (cf. [20, Theorem K.8]).

For later reference, we remark that for any smooth function $f : Y \to \mathbb{R}$ the $(1, 1)$-form $\omega = i(\partial f \wedge \overline{\partial} f)$ is of real type and

$$\omega = \frac{1}{2}(\partial f \otimes \overline{\partial} f + \overline{\partial} f \otimes \partial f);$$

for this is a local assertion that reduces by linearity to the fact that the $(1, 1)$-form

$$\omega = i \cdot (\alpha dz_i \wedge d\overline{z}_j + \overline{\alpha} dz_j \wedge d\overline{z}_i), \quad \alpha \in \mathbb{C},$$

on $\mathbb{C}^n$ is of real type and the fact that

$$idz_i \wedge d\overline{z}_j(v, Iw) = \frac{1}{2}((dz_i(v)dz_j(w) + d\overline{z}_j(v)d\overline{z}_i(w)), \quad v, w \in T_{\mathbb{R}, x}\mathbb{C}^n, x \in \mathbb{C}^n.$$

To a $(1, 1)$-form $\omega$ of real type is also associated a hermitian form (with respect to $I$)

$$H_\omega : T_\mathbb{R}Y \times Y T_\mathbb{R}Y \longrightarrow \mathbb{C} \times Y, (v, w) \longmapsto g_\omega(v, w) - i \cdot \omega(v, w),$$

and this can also be seen to be a one-to-one correspondence. Indeed, $\omega = -\text{Im}(H_\omega)$.

Let $Z$ be a complex submanifold of a complex manifold $Y$ and $\omega$ a $(1, 1)$-form of real type on $Y$. Restricting and taking exterior products, we obtain an alternating $\mathbb{R}$-multilinear map

$$(\omega|_Z)^{\wedge \dim(Z)} : (T_{\mathbb{R}, x}Z)^{2\dim(Z)} \longrightarrow \mathbb{R}$$

for each $x \in Z$. If the restriction of the $\mathbb{R}$-bilinear form $g_{\omega, x}$ to $T_{\mathbb{R}, x}Z$ is moreover positive definite, we have a non-zero Riemannian volume form ([24, pp. 361–362])

$$\text{vol}(g_{\omega, x}) : (T_{\mathbb{R}, x}Z)^{2\dim(Z)} \longrightarrow \mathbb{R}.$$

By [56, Lemma 3.8], $\text{vol}(g_{\omega, x})$ agrees with $\dim(Z)!^{-1}(\omega|_Z)^{\wedge \dim(Z)}$. If $\omega$ is continuous, this implies immediately that there is a euclidean neighborhood $U$ of $x$ in $Z$ such that $\int_U (\omega|_Z)^{\wedge \dim(Z)} > 0$.

To use this argument effectively, we need a criterion to check whether the restriction of $g_{\omega, x}$ to $T_{\mathbb{R}, x}Z$ is positive definite. For an arbitrary $\mathbb{R}$-bilinear form $g$ on a real vector space $V$, we define its kernel by

$$\ker(g) = \{v \in V : \forall w \in V : g(v, w) = 0\}.$$

In our applications, $\omega$ is always semipositive so that $g_\omega$ is positive semidefinite. For a positive semidefinite bilinear form $g$, it is a straightforward consequence of Sylvester’s law of inertia ([18, Theorem 11.25]) that

$$\ker(g) = \{v \in V : g(v, v) = 0\}$$

and hence that $\ker(g|_W) = \ker(g) \cap W$ for any $\mathbb{R}$-linear subspace $W \subset V$. Consequently, the restriction of $g_{\omega, x}$ to $T_{\mathbb{R}, x}Z$ is positive definite if and only if $\ker(g_{\omega, x}) \cap T_{\mathbb{R}, x}Z = \{0\}$. Finally,

One frequently identifies a $(1, 1)$-form $\omega$ of real type with its scalar extension $\omega_\mathbb{C} : T_{\mathbb{C}}Y \times Y T_{\mathbb{C}}Y \rightarrow \mathbb{C} \times Y$, $T_{\mathbb{C}}Y = \mathbb{R}Y \otimes \mathbb{C}$. Since the restriction to $T_{\mathbb{R}}Y \times Y T_{\mathbb{R}}Y$ or $T^{1,0}_\mathbb{C}Y \times Y T^{0,1}_\mathbb{C}Y$ retains all information, we allow ourselves to switch tacitly between $\omega$ and $\omega_\mathbb{C}$.
let us note that for any positive semidefinite \( \mathbb{R} \)-bilinear forms \( g_1, g_2 \) on \( V \) their sum \( g_1 + g_2 \) is also a positive semidefinite \( \mathbb{R} \)-bilinear form and (28) implies that

\[
\ker(g_1 + g_2) = \ker(g_1) \cap \ker(g_2).
\]

5. Weil Functions, Hermitian Metrics and Chern Forms

We provide here the necessary tools for Section 7.4, in which bounds on certain intersections numbers are established. Our approach is to endow all line bundles under consideration with smooth hermitian metrics so that intersection numbers become integrals of the associated Chern forms. Throughout this section, we hence take \( k = \mathbb{C} \) as our base field. A major issue is to interpolate between the Chern forms of different line bundles. For this purpose, we introduce certain explicit smooth \((1,1)\)-forms of real type, namely the “toric” \((1,1)\)-forms \( \omega(\phi_{tor}) \) in Subsection 5.1 and the “abelian” \((1,1)\)-forms \( \omega(N, \phi_{ab}) \) in Subsection 5.2.

5.1. “Toric” \((1,1)\)-forms. Our first aim is to endow the line bundles \( M_X \) from Construction 5 with a hermitian metric and to compute the associated Chern forms. Functoriality allows us to endow additionally the line bundles \( M_X(\phi_{tor}) \) from Construction 6 with a hermitian metric. A closer look at the associated Chern forms leads us to introduce the “toric” \((1,1)\)-forms \( \omega(\phi_{tor}) \).

Our main instrument are Weil functions, on which the reader may find details in [34, Chapter 10] and [35, Chapter I]. Let \( X \) be a complex algebraic variety and \( D \) a Cartier divisor on \( X \). In this situation, a function \( \lambda : (X \setminus \text{supp}(D)) \rightarrow \mathbb{R} \) is called a Weil function for \( D(\mathbb{C}) \) if every point \( x \in X(\mathbb{C}) \) has an open neighborhood \( U \) (in the euclidean topology) such that

\[
\lambda = -\log |f| + \alpha \text{ on } U \setminus \text{supp}(D)(\mathbb{C})
\]

with \( f \) a meromorphic function on \( U \) such that \( \text{div}(f) = D \) (as formal sums of irreducible analytic varieties on \( U \)) and \( \alpha \) a continuous function on \( U \). Furthermore, \( \lambda \) is called a smooth Weil function if \( \alpha \) can be even assumed smooth on \( U \). Every (smooth) Weil function \( \lambda : (X \setminus \text{supp}(D))(\mathbb{C}) \rightarrow \mathbb{R} \) associated with \( D \) yields a (smooth) hermitian metric \( g \) on the associated line bundle \( L(D) \). In fact, its sections \( \mathcal{O}(D) \) form a \( \mathcal{O}_X \)-submodule of \( K_X \) and we can just set \( g_x(f) = e^{-\lambda(x)}|f(x)| \) for any meromorphic function \( f \) on \( U \) and any \( x \in X(\mathbb{C}) \) in its domain of definition. To a smooth Weil function \( \lambda \) for \( D \) is associated a smooth closed \((1,1)\)-form of real type, namely the Chern form \( c_1(L(D), g) \) of the associated smooth metric on \( L(D) \). On an open euclidean neighborhood \( U \) such that (30) is true, we have \( c_1(L, g) = dd^c \alpha \). Additionally, \( dd^c \alpha = dd^c \lambda \) outside \( \text{supp}(D)(\mathbb{C}) \).

We now record a standard result on Weil functions. Let \( D \) be a Cartier divisor on a complex projective variety \( X \) and \( \lambda \) a Weil function for \( D \). Assume that \( D \) is the difference \( D_1 - D_2 \) of two effective Cartier divisors \( D_1, D_2 \) with disjoint supports. From [34] Propositions 10.2.1 and 10.3.2, we infer that \( \sup \{ \lambda, 0 \} \) (resp. \( \inf \{ \lambda, 0 \} \)) is a Weil function for \( D_1 \) (resp. \( D_2 \)). The next lemma provides a smooth variant of this observation in the same situation.

**Lemma 12.** In the situation described above, assume additionally that \( \lambda \) is a smooth Weil function. Then, \( \log(1 + e^{2\lambda})/2 \) (resp. \( \log(1 + e^{-2\lambda})/2 \)) is a smooth Weil function for \( D_1 \) (resp. \( D_2 \)).

**Proof.** By assumption, we know that for each \( x \in X(\mathbb{C}) \) there exists an open euclidean neighborhood \( U \), a meromorphic function \( f \) representing \( D \) on \( U \) and a smooth function \( \alpha \) satisfying (30). Since \( D_1 \) and \( D_2 \) have disjoint supports, we may shrink \( U \) to guarantee that it is relatively compact and that its topological closure \( \overline{U} \) does not intersect \( \text{supp}(D_1)(\mathbb{C}) \) or \( \text{supp}(D_2)(\mathbb{C}) \). Suppose \( \overline{U} \cap D_1 = \emptyset \) (resp. \( \overline{U} \cap D_2 = \emptyset \)). Then, \(|f| \geq \varepsilon > 0 \) (resp. \(|f| \leq \varepsilon^{-1} \)) for some sufficiently small \( \varepsilon > 0 \). Furthermore, 1 (resp. \( f \)) is a local equation for \( D_1 \). Note that
\[ \beta = \log(1 + |f|^{-2}e^{2\lambda})/2 \] (resp. \( \beta = \log(|f|^2 + e^{2\lambda})/2 \)) is a smooth function on \( U \). In addition, \[ \frac{1}{2} \log(1 + e^{2\lambda}) = -\log |l| + \beta \] (resp. \( \frac{1}{2} \log(1 + e^{2\lambda}) = -\log |f| + \beta \)).

This demonstrates that \( \frac{1}{2} \log(1 + e^{2\lambda}) \) is a smooth Weil function for \( D_1 \). Similarly, \( \frac{1}{2} \log(1 + e^{-2\lambda}) \) can be shown to be a smooth Weil function for \( D_2 \).

Let us next recollect a fundamental result of Vojta \cite{57}. Let \( G \) be a semialgebraic variety with split toric part \( T = G_{m}^r \) and abelian quotient \( A \). Recall from Construction \( 3 \) its compactification \( \overline{G} \) as well as the Cartier divisor \( G(D_t) \) on \( \overline{G} \). With \( \text{pr}_u : (\mathbb{P}^1)^t \to \mathbb{P}^1 \) being the projection to the \( u \)-th component as in Construction \( 4 \) we set \( D_{u,0} = G(\text{pr}_u^* E_0) \) and \( D_{u,\infty} = G(\text{pr}_u^* E_\infty) \) so that \( G(D_t) = \sum_{u=1}^{t} (D_{u,0} + D_{u,\infty}) \).

**Lemma 13.** For each divisor \( D_{u,0} - D_{u,\infty}, 1 \leq u \leq t \), there exists a unique smooth Weil function
\[ \lambda_u : \overline{G}(\mathbb{C}) \setminus \text{supp}(D_{u,0} - D_{u,\infty})(\mathbb{C}) \to \mathbb{R} \]
that satisfies
\[ \lambda_u(xy) = \lambda_u(x) + \lambda_u(y) \]
for all \( x, y \in G(\mathbb{C}) \). In addition, \( e^{\lambda_u} \) is locally the absolute value of a meromorphic function.

Outside \( \text{supp}(D_{u,0} - D_{u,\infty})(\mathbb{C}) \), we have locally \( \lambda_u = \log |f| \) for some holomorphic function \( f \). This implies \( dd^c \lambda_u = 0 \) on \( G(\mathbb{C}) \).

**Proof.** This is stated in \cite{57}, Proposition 2.6] except for the assertion about \( e^{\lambda_u} \). Inspecting (2.6.3) in the proof of the said proposition, one sees that it suffices to prove the same assertion for the Néron function \( \lambda(s) \) (cf. \cite{34}, Theorem 11.1.1] attached to the divisor \( (s) \) on \( A \). As \( s \) is a rational section of an algebraically trivial line bundle by construction, the divisor \( (s) \) is algebraically equivalent to the zero divisor. The explicit formula for \( \lambda(s) \) in terms of a normalized theta function (\cite{34} Theorem 13.1.1] directly yields the assertion in this case; the hermitian form \( H \) in the formula is zero because \( (s) \sim_{\text{alg}} 0 \) (cf. \cite{3} Proposition 2.2.2]).

Using the Weil functions \( \lambda_u \) we can define a subgroup
\[ \{ x \in G(\mathbb{C}) \mid \lambda_1(x) = \lambda_2(x) = \cdots = \lambda_t(x) = 0 \} \subset G(\mathbb{C}) \]
This coincides with the maximal compact subgroup \( K_G \) of \( G(\mathbb{C}) \). Indeed, any homomorphism \( K_G \to \mathbb{R} \) vanishes by compactness so that \( \lambda_u|_{K_G} = 0 \). By uniqueness, the restriction of \( \lambda_u \) to the maximal torus \( T(\mathbb{C}) \) equals \( -\log |X_u| \) (in standard coordinates \( X_1, \ldots, X_t \)). Hence, the subgroup in (31) is topologically a fiber bundle with compact fiber \( (S^1)^t \), \( S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \), over the compact base \( A(\mathbb{C}) \). Therefore, it is compact itself and hence contained in \( K_G \). From this explicit presentation, it is easy to see that \( K_G \) is an abstract closed subgroup of \( G(\mathbb{C}) \). Using \cite{24} Theorem II.2.3] and counting dimensions, we see that \( K_G \) is a real Lie subgroup of (real) dimension \( 2 \dim (A) + t \).

Recall that \( M/\overline{\Delta} \) is the line bundle associated to the \( T \)-invariant Weil divisor \( G(D_t) \). By Lemmas \cite{12} and \cite{13} the function
\[ \lambda = \frac{1}{2} \sum_{u=1}^{t} (\log(1 + e^{2\lambda_u}) + \log(1 + e^{-2\lambda_u})) \]
is a smooth Weil function for \( G(D_t) \). For the associated smooth hermitian line bundle, which is denoted \( \overline{\Delta} \) in the sequel, we have
\[ c_1(\overline{\Delta}) = \frac{1}{2} \sum_{u=1}^{t} (dd^c \log(1 + e^{2\lambda_u}) + dd^c \log(1 + e^{-2\lambda_u})) \] on \( G(\mathbb{C}) \).

---

\(^3\)Note that \( z \mapsto |z|^2 = x^2 + y^2 \) is smooth at \( z = 0 \) in contrast to \( z \mapsto |z| = \sqrt{x^2 + y^2} \). This rules out the straightforward choice \( \log(1 + e^{\lambda}) \) (resp. \( \log(1 + e^{-\lambda}) \)).
The Weil functions of Lemma 13 also satisfy some functoriality. To be precise, let \( \varphi : G \to G' \) be a homomorphism of semiabelian varieties with toric component \( \varphi_{\text{tor}} : T = G_m \to T' = G'_m \). Let \( X_i \) (resp. \( Y_j \)) be the standard algebraic coordinates on \( G'_m \) (resp. \( G'_m \)) and write \( \varphi_{\text{tor}}(Y_v) = X^{a_1}_{i_1} \cdots X^{a_{t_v}}_{i_{t_v}} \) with integers \( a_{uv} \). Lemma 13 supplies Weil functions \( \lambda_v' \), \( 1 \leq u \leq t' \), on \( G' \) and there is an identity

\[
\varphi^* \lambda_v' = \lambda_v' \circ \varphi = a_{1v} \lambda_1 + a_{2v} \lambda_2 + \cdots + a_{tv} \lambda_t \text{ on } G(\mathbb{C}).
\]

Indeed, the equality is valid on \( T_0 \) since the restriction of \( \lambda_u \) (resp. \( \lambda_v' \)) to the maximal torus \( T_0(\mathbb{C}) \approx (\mathbb{C}^*)^t \) (resp. \( t(\mathbb{C}) \approx (\mathbb{C}^*)^t \)) is \( -\log |X_u| \) (resp. \( -\log |Y_v| \)) as we noted above. It is also true on \( K_G \) because \( \varphi(K_G) \subset K_{G'} \). As \( K_G \) and \( T_0(\mathbb{C}) \) span \( G(\mathbb{C}) \) as an abstract group, \( \lambda_v' \) is true for all of \( G(\mathbb{C}) \). Note that \( \varphi^* \lambda_v' \) is independent of the abelian component of \( \varphi \). Abusing notation, we therefore write \( \varphi_{\text{tor}} \lambda_v' \) instead of \( \varphi^* \lambda_v' \). Even more, we can use (33) to formally define \( \phi_{\text{tor}} \lambda_v' : G(\mathbb{C}) \to \mathbb{R}, v = 1, \ldots, t' \), for an arbitrary \( \phi_{\text{tor}} \in \text{Hom}_\mathbb{R}(G'_m, G'_m) \).

We now apply the results of the previous paragraph to endow the line bundles \( M_{(\varphi_{\text{tor}})} \), \( \varphi_{\text{tor}} \in \text{Hom}(G'_m, G'_m) \), with hermitian metrics. For this, we use the homomorphism \( \varphi : G \to G' \) from Construction 7 with \( p_{\varphi}^* M_{\varphi} \approx M_{(\varphi_{\text{tor}})} \). We may endow \( M_{(\varphi_{\text{tor}})} \) with a hermitian metric such that \( p_{\varphi}^* M_{\varphi} \approx M_{(\varphi_{\text{tor}})} \). Since the isomorphism between \( p_{\varphi}^* M_{\varphi} \) and \( M_{(\varphi_{\text{tor}})} \) is unique up to multiplication with a non-zero constant, this singles out a hermitian metric on \( M_{(\varphi_{\text{tor}})} \) up to a non-zero constant scaling factor. Regardless of this indeterminate scaling factor, we have an identity of Chern forms \( c_1(M_{(\varphi_{\text{tor}})}) = p_{\varphi}^* c_1(M_{\varphi}) \). Thus,

\[
c_1(M_{(\varphi_{\text{tor}})}) = \frac{1}{2} \sum_{v=1}^{t'} \left( dd^c \log(1 + e^{2\phi_{\text{tor}} \lambda_v'}) + dd^c \log(1 + e^{-2\phi_{\text{tor}} \lambda_v'}) \right) \text{ on } G(\mathbb{C}).
\]

Since the indeterminacy in the metric is negligible for our purposes, we suppress it in writing \( M_{(\varphi_{\text{tor}})} \) for any hermitian line bundle as constructed above. Again, the right hand side of (34) depends only on \( \varphi_{\text{tor}} \) and is moreover well-defined for any \( \phi_{\text{tor}} \in \text{Hom}_\mathbb{R}(G'_m, G'_m) \). In other words, we can associate with each \( \phi_{\text{tor}} \in \text{Hom}_\mathbb{R}(G'_m, G'_m) \) a \((1,1)\)-form

\[
\omega(\phi_{\text{tor}}) = \frac{1}{2} \sum_{v=1}^{t'} \left( dd^c \log(1 + e^{2\phi_{\text{tor}} \lambda_v'}) + dd^c \log(1 + e^{-2\phi_{\text{tor}} \lambda_v'}) \right)
\]

on \( G(\mathbb{C}) \). In the remainder of this section, we establish basic properties of this \((1,1)\)-form.

**Lemma 14.** Each \( dd^c \log(1 + e^{\pm 2\phi_{\text{tor}} \lambda_v'}) \), \( 1 \leq v \leq t' \), is a semipositive \((1,1)\)-form of real type on \( G(\mathbb{C}) \). Consequently, \( \omega(\phi_{\text{tor}}) \) is a semipositive \((1,1)\)-form of real type.

**Proof.** It suffices to prove that \( \log(1 + e^{\pm 2\phi_{\text{tor}} \lambda_v'}) \) is a plursubharmonic function. This follows directly from \( dd^c \lambda_u = 0 \) (i.e., both \( \lambda_u \) and \( -\lambda_u \) are plursubharmonic on \( G(\mathbb{C}) \)) and the fact that \( \log(1 + e^x) \) is a convex monotonically increasing function (cf. [20] Theorem K.5 (d)]). \( \square \)

Furthermore, the map \( \phi_{\text{tor}} \mapsto \omega(\phi_{\text{tor}}) \) is continuous with respect to the euclidean topology on \( \text{Hom}_\mathbb{R}(G'_m, G'_m) \approx \mathbb{R}^{t \times t'} \) and the usual topology on smooth \((1,1)\)-forms (cf. [13] Section I.2 and [19] Section 1.46]). By Lemma 13 there exists locally a holomorphic function \( \kappa_u \) such that \( e^{\lambda_u} = |\kappa_u| \). From \( \lambda_u = \log(|\kappa_u|^2)/2 \), we deduce

\[
\partial \lambda_u = \frac{\kappa_u \overline{\partial \kappa_u}}{2|\kappa_u|^2} = \frac{\partial \kappa_u}{2\kappa_u} \text{ and } \bar{\partial} \lambda_u = \frac{\kappa_u \bar{\partial} \kappa_u}{2|\kappa_u|^2} = \left( \frac{\partial \kappa_u}{2\kappa_u} \right).
\]

Using Lemma 15 below, we hence obtain

\[
dd^c \log(1 + e^{2\phi_{\text{tor}} \lambda_v'}) + dd^c \log(1 + e^{-2\phi_{\text{tor}} \lambda_v'}) = \frac{i(\partial \phi_{\text{tor}}^{-1} \lambda_v' \wedge \bar{\partial} \phi_{\text{tor}} \lambda_v')}{\pi(1 + e^{2\phi_{\text{tor}} \lambda_v'})(1 + e^{-2\phi_{\text{tor}} \lambda_v'})}.
\]
Lemma 15. Let $\kappa_1, \ldots, \kappa_m$ be zero-free holomorphic functions on an open subset $U \subset \mathbb{C}^n$. Then,

$$dd^c \log(1 + |\kappa_1|^{a_1} \cdots |\kappa_m|^{a_m}) = \frac{i}{8\pi} \frac{|\kappa_1|^{a_1} \cdots |\kappa_m|^{a_m}}{(1 + |\kappa_1|^{a_1} \cdots |\kappa_m|^{a_m})^2} \left( \sum_{u=1}^m a_u \frac{\partial \kappa_u}{\kappa_u} \wedge \sum_{u=1}^m a_u \frac{\partial \kappa_u}{\kappa_u} \right)$$

for any real numbers $a_1, \ldots, a_m$.

Proof. Without loss of generality, we assume that all $a_1, \ldots, a_m$ are non-zero. To simplify our notation, we set $f(z) = |\kappa_1|^{a_1} \cdots |\kappa_m|^{a_m}$. First, we note that $\partial |\kappa|^q = \partial (\kappa |\kappa|^{q/2}) = \frac{q}{2} |\kappa|^{q-2} \kappa \partial \kappa = \frac{q}{2} |\kappa|^q \left( \frac{\kappa}{|\kappa|} \right)$ (resp. $\overline{\partial} |\kappa|^q = \frac{q}{2} |\kappa|^q \left( \frac{\kappa}{|\kappa|} \right)$) implies that

$$\partial f(z) = f(z) \cdot \sum_{u=1}^m \left( a_u \frac{\partial \kappa_u}{2\kappa_u} \right) \quad \text{(resp. $\overline{\partial} f(z) = f(z) \cdot \sum_{u=1}^m \left( a_u \frac{\partial \kappa_u}{2\kappa_u} \right)$)}.

Since both $\kappa_u$ and $\kappa_u^{-1}$ are holomorphic, we have

$$\partial \kappa_u^{-1} \partial \kappa_u = \overline{\partial} (\kappa_u^{-1} \partial \kappa_u) = \overline{\partial} (\kappa_u^{-1}) \wedge \partial \kappa_u + \kappa_u^{-1} \overline{\partial} \partial (\kappa_u) = 0$$

and hence

$$\partial \overline{\partial} f(z) = \partial f(z) \wedge \left( \sum_{u=1}^m \left( a_u \frac{\partial \kappa_u}{2\kappa_u} \right) \right) + f(z) \cdot \partial \left( \sum_{u=1}^m \left( a_u \frac{\partial \kappa_u}{2\kappa_u} \right) \right)$$

$$= f(z) \left( \sum_{u=1}^m \left( a_u \frac{\partial \kappa_u}{2\kappa_u} \right) \right) \wedge \left( \sum_{u=1}^m \left( a_u \frac{\partial \kappa_u}{2\kappa_u} \right) \right).$$

We compute

$$\partial \overline{\partial} \log(1 + f(z)) = \partial \left( \frac{1}{1 + f(z)} \overline{\partial} f(z) \right)$$

$$= \frac{-1}{(1 + f(z))^2} \partial f(z) \wedge \overline{\partial} f(z) + \frac{1}{1 + f(z)} \overline{\partial} \partial f(z)$$

$$= \left( \frac{-f(z)^2}{(1 + f(z))^2} + \frac{f(z)}{1 + f(z)} \right) \left( \sum_{u=1}^m a_u \frac{\partial \kappa_u}{2\kappa_u} \wedge \sum_{u=1}^m \left( a_u \frac{\partial \kappa_u}{2\kappa_u} \right) \right)$$

$$= \frac{f(z)}{4(1 + f(z))^2} \left( \sum_{u=1}^m a_u \frac{\partial \kappa_u}{\kappa_u} \wedge \sum_{u=1}^m \left( a_u \frac{\partial \kappa_u}{\kappa_u} \right) \right).$$

The assertion follows directly as $dd^c = (i/4\pi) (\partial + \overline{\partial})(\overline{\partial} - \partial) = (i/2\pi) \partial \overline{\partial}$. \qed

The next lemma establishes an essential homogeneity property for $\omega(\phi_{\text{tor}})$.

Lemma 16. Let $G$ be a semiabelian variety with abelian quotient $\pi : G \to A$ and toric part $G^t_m$. In addition, let $t'$ be a non-negative integer and $\omega$ a smooth closed $(1,1)$-form on $A(\mathbb{C})$. Then,

$$\int_{X(\mathbb{C})} \omega(n \cdot \phi_{\text{tor}})^{s_1} \wedge (\pi^* \omega)^{s_2} = n^{s_1} \cdot \int_{X(\mathbb{C})} \omega(\phi_{\text{tor}})^{s_1} \wedge (\pi^* \omega)^{s_2}$$

for any $\phi_{\text{tor}} \in \text{Hom}_Q(G^t_m, \mathbb{G}_m^t)$, any algebraic subvariety $X \subset G$ and any non-negative integers $n$, $s_1$, $s_2$ satisfying $s_1 + s_2 = \dim(X)$.

A priori, the improper integrals in (36) need not be well-defined. They are well-defined, however, if there is a compactification $\overline{X}$ of $X$ and the integrand extends to a smooth form on $\overline{X}(\mathbb{C})$. Up to an étale covering of $X$, this is a by-product of our proof below and should best be understood as part of the assertion.
Proof. Let us first prove the lemma assuming that $\phi_{\text{tor}} = \varphi_{\text{tor}} \in \text{Hom}(G'_{m}, G''_{r})$. In this situation, both $\omega(\varphi_{\text{tor}})$ and $\omega(n \cdot \varphi_{\text{tor}})$ extend (uniquely) to smooth closed $(1,1)$-forms $\tilde{\omega}(\varphi_{\text{tor}})$ and $\tilde{\omega}(n \cdot \varphi_{\text{tor}})$ on $G_{T(\varphi_{\text{tor}})}(C)$. In fact, $\omega(\varphi_{\text{tor}})$ (resp. $\omega(n \cdot \varphi_{\text{tor}})$) is precisely defined to agree with the restriction of $c_1(M_{T(\varphi_{\text{tor}})})$ (resp. $c_1(M_{T(n \cdot \varphi_{\text{tor}})})$) to $G(C) \subset G_{T(\varphi_{\text{tor}})}(C)$ (resp. $G(C) \subset G_{T(n \cdot \varphi_{\text{tor}})}(C)$). Thus, we have found an extension of $\omega(\varphi_{\text{tor}})$ to $G_{T(\varphi_{\text{tor}})}(C)$. Construction 6 supplies us with a map $\vartheta_{\varphi_{\text{tor}}, n} : G_{T(\varphi_{\text{tor}})} \rightarrow G_{T(n \cdot \varphi_{\text{tor}})}$, which is the identity on $G$. Therefore, $\vartheta_{\varphi_{\text{tor}}, n} \cdot c_1(M_{T(n \cdot \varphi_{\text{tor}})})$ is the sought-after extension $\tilde{\omega}(n \cdot \varphi_{\text{tor}})$.

Denote by $\overline{X}$ the Zariski closure of $X$ in $G_{T(\varphi_{\text{tor}})}(C)$ and by $\overline{\pi} : G_{T(\varphi_{\text{tor}})} \rightarrow A$ the abelian quotient. Since the boundary $(\overline{X} \setminus X)/(C)$ has measure zero, (36) would follow now from the equality

$$\int_{\overline{X}(C)} \tilde{\omega}(m \cdot \varphi_{\text{tor}})^{\lambda_{s_{1}}} \wedge (\pi^{*} \varpi)^{\lambda_{s_{2}}} = m^{s_{1}} \cdot \int_{\overline{X}(C)} \tilde{\omega}(\varphi_{\text{tor}})^{\lambda_{s_{1}}} \wedge (\pi^{*} \varpi)^{\lambda_{s_{2}}}$$

of proper integrals. For this, we claim that any $\rho_{\varphi}^{\pm} : G(C) \rightarrow \mathbb{R}^{\geq 0}$, $x \mapsto (1 + e^{\pm 2m \varphi_{m}})^{m}$, $1 \leq v \leq t'$, extends smoothly to $G_{T(\varphi_{\text{tor}})}(C)$. It suffices to prove that each $x \in (G_{T(\varphi_{\text{tor}})} \setminus G(C))$ has a (euclidean) neighborhood on which $\rho_{\varphi}^{\pm}$ extends to a smooth function. For this, we let $\varphi : G \rightarrow G'$ be again the homomorphism from Construction 7 so that $G_{T(\varphi_{\text{tor}})} = \overline{G}^{\varphi}$. As before, Lemma 13 affords a Weil function $\lambda_{\varphi}$, $1 \leq v \leq t'$, for each divisor $D_{v,0} - D_{v,\infty}$ on $\overline{G}$. Its pullback $p_{\varphi}^{*} \lambda_{\varphi}$ along $p_{\varphi} : \overline{G}^{\varphi} \rightarrow \overline{G}$ restricts to the function $\varphi_{\varphi}^{*} \lambda_{\varphi} : G(C) \rightarrow \mathbb{R}$ formally defined by (33). There exists a euclidean neighborhood $U$ of $p_{\varphi}(x)$ and a meromorphic function $f$ on $U$ with $\text{div}(f) = (D_{v,0} - D_{v,\infty})|_{U}$ such that $\lambda_{\varphi} + \log |f|$ extends to a smooth function $\alpha$ on $\overline{U}$. On $\overline{G}(C) \cap p_{\varphi}^{-1}(U) \subset G_{T(\varphi_{\text{tor}})}(C)$, we have

$$\varphi_{\text{tor}}^{*} \lambda_{\varphi} = \lambda_{\varphi} \circ p_{\varphi} = -\log |f \circ p_{\varphi}| + (\alpha \circ p_{\varphi})$$

and thus

$$\rho_{\varphi}^{\pm}(x) = \frac{1 + |f \circ p_{\varphi}|^{\pm 2m \varphi_{m}}}{1 + |f \circ p_{\varphi}|^{\pm 2m \varphi_{m}}^{m}} = \frac{|f \circ p_{\varphi}|^{\pm 2m \varphi_{m}}}{|f \circ p_{\varphi}|^{\pm 2m \varphi_{m}}^{m}} \cdot e^{\pm 2m \varphi_{m}}.$$

Since $\text{supp}(D_{v,0}) \cap \text{supp}(D_{v,\infty}) = \emptyset$, we have $x \notin \text{supp}(D_{v,0})$ or $x \notin \text{supp}(D_{v,\infty})$. Shrinking $U$ if necessary, we may hence assume that $f$ or $f^{-1}$ is holomorphic on $U$. In either case, (38) yields a smooth extension of $\rho_{\varphi}^{\pm}$ on $p_{\varphi}^{-1}(U)$. By uniqueness, these extensions glue together to a smooth function $p_{\varphi}^{\pm} : G_{T(\varphi_{\text{tor}})}(C) \rightarrow \mathbb{R}^{\geq 0}$. (In fact, $p_{\varphi}^{\pm}(x) = 1$ for all $x \in \text{supp}(D_{v,0})(C) \cup \text{supp}(D_{v,\infty})(C)$ because $(1 + e^{m})/(1 + e^{m})^{m} \rightarrow 1$ as $x \rightarrow \pm \infty$.) In addition,

$$\tilde{\omega}(m \cdot \varphi_{\text{tor}}) - m \tilde{\omega}(\varphi_{\text{tor}}) = \frac{1}{2} \sum_{v=1}^{t'} \left( d\lambda^{\alpha} \log(\rho_{\varphi}^{\pm}) + d\lambda^{B} \log(\rho_{\varphi}^{\pm}) \right);$$

indeed, this equality is obvious on $G(C)$ and any $(1,1)$-form on $G(C)$ has at most one smooth extension to the compactification $G_{T(\varphi_{\text{tor}})}(C)$. We deduce from (39) that $\tilde{\omega}(m \cdot \varphi_{\text{tor}}) - m \tilde{\omega}(\varphi_{\text{tor}})$ is exact and (37) follows.

Now, let us consider a general $\phi_{\text{tor}} \in \text{Hom}_{Q}(G'_{m}, G''_{r})$ and a positive integer $n$ that is a denominator for $\phi_{\text{tor}}$ (i.e., $n \cdot \phi_{\text{tor}} \in \text{Hom}(G'_{m}, G''_{r})$). Setting $Y = [n]^{-1}(X)$, the restriction $[n]G_{0}|_{Y} : Y \rightarrow X$ is finite étale of degree $n^{2 \text{dim}(A)+t}$. By functoriality (33), we have $[n]G_{0} \omega(\phi_{\text{tor}}) = \omega(n \cdot \phi_{\text{tor}})$ and $[n]G_{0} \omega(m \cdot \phi_{\text{tor}}) = \omega(m \cdot n \cdot \phi_{\text{tor}})$. We infer that

$$n^{2 \text{dim}(A)+t} \int_{X(C)} \omega(m \cdot \phi_{\text{tor}})^{\lambda_{s_{1}}} \wedge (\pi^{*} \varpi)^{\lambda_{s_{2}}} = \int_{Y(C)} \omega(m \cdot n \cdot \phi_{\text{tor}})^{\lambda_{s_{1}}} \wedge (\pi^{*} [n]G_{0} \varpi)^{\lambda_{s_{2}}}$$

and

$$n^{2 \text{dim}(A)+t} \int_{X(C)} \omega(\phi_{\text{tor}})^{\lambda_{s_{1}}} \wedge (\pi^{*} \varpi)^{\lambda_{s_{2}}} = \int_{Y(C)} \omega(n \cdot \phi_{\text{tor}})^{\lambda_{s_{1}}} \wedge (\pi^{*} [n]G_{0} \varpi)^{\lambda_{s_{2}}}.$$
Since \( n \cdot \phi_{tor} \in \text{Hom}(\mathbb{G}_m^t, \mathbb{G}_m^t) \), this reduces (36) to the already proven special case.

5.2. “Abelian” (1,1)-forms. This subsection is the “abelian” equivalent of the last one and we introduce here (1,1)-forms \( \omega(N; \phi_{ab}) \) analogous to the (1,1)-forms \( \omega(\phi_{tor}) \). Having pullbacks from abelian quotients at our disposal, it suffices to define these forms on abelian varieties and their definition is technically less demanding.

Let \( \bar{\varphi}_{ab} : A \to A' \) be a homomorphism of abelian varieties. We choose lattices \( \Lambda \subseteq \mathbb{C}^g \), \( g = \dim(A) \), and \( \Lambda' \subseteq \mathbb{C}^{g'} \), \( g' = \dim(A') \), such that \( \mathbb{C}^g \to \mathbb{C}^{g'/\Lambda} = A(\mathbb{C}) \) and \( \mathbb{C}^{g'} \to \mathbb{C}^{g'/\Lambda'} = A'(\mathbb{C}) \) are universal coverings. In the sequel, each holomorphic tangent space \( T_x A(\mathbb{C}), x \in A(\mathbb{C}) \), (resp. \( T_x A'(\mathbb{C}) \), \( x' \in A'(\mathbb{C}) \)), is identified with \( \mathbb{C}^g \) (resp. \( \mathbb{C}^{g'} \)) by virtue of this quotient map. We write \( \bar{\varphi}_{ab} : \mathbb{C}^g \to \mathbb{C}^{g'} \) for the lifting of \( \bar{\varphi}_{ab} \), along the universal coverings.

Let \( N \) be an ample line bundle on \( A' \). The Appell-Humbert Theorem (see e.g. [3, Section 2.2]) allows us to describe \( N \) in terms of a pair \( (H, \chi) \) consisting of a hermitian form \( H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C} \) such that \( \text{Im}(H(N', N')) \subseteq \mathbb{Z} \) and a semicharacter \( \chi : A' \to S^1 \) for \( H \). It is well-known (cf. [3, Exercise 2.6.2] and [56, Theorem 7.10]) that \( N \) can be endowed with a metric \( g \) such that the Chern form \( c_1(N) \) of the hermitian line bundle \( N = (N, g) \) is given by

\[
(40) \quad c_1(N)_x : T_{\mathbb{R}, x} A'(\mathbb{C}) \times T_{\mathbb{R}, x} A'(\mathbb{C}) = \mathbb{C}^{g'} \times \mathbb{C}^{g'} \to \mathbb{C}, \quad (v, w) \mapsto -\text{Im}(H)(v, w), \quad x \in A'(\mathbb{C}).
\]

Ampleness of \( N \) is equivalent to \( H \) being positive definite ([3, Proposition 4.5.2]), which is equivalent to \( c_1(N) \) being a positive (1,1)-form. The pullback of \( c_1(N) \) along \( \bar{\varphi}_{ab} \) is given by

\[
(41) \quad \varphi_{ab}^* c_1(N)_x : T_{\mathbb{R}, x} A(\mathbb{C}) \times T_{\mathbb{R}, x} A(\mathbb{C}) = \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}, \quad (v, w) \mapsto -\text{Im}(H)(\bar{\varphi}_{ab}(v), \bar{\varphi}_{ab}(w)),
\]

for each \( x \in A(\mathbb{C}) \). Lifting homomorphisms \( A \to A' \) to homomorphisms \( \mathbb{C}^g \to \mathbb{C}^{g'} \) of the universal coverings induces an injection \( \text{Hom}_{\mathbb{R}}(A, A') \to \text{Hom}_{\mathbb{R}}(\mathbb{C}^g, \mathbb{C}^{g'}) \), \( \phi_{ab} \to \bar{\varphi}_{ab} \). As in (33), the right hand side of (41) is well-defined for any \( \bar{\varphi}_{ab} \in \text{Hom}_{\mathbb{R}}(\mathbb{C}^g, \mathbb{C}^{g'}) \). For an element \( \phi_{ab} \in \text{Hom}_{\mathbb{R}}(A, A') \), we hence define the (1,1)-form \( \omega(N; \phi_{ab}) \) on \( A \) by demanding

\[
(42) \quad \omega(N; \phi_{ab}) : T_{\mathbb{R}, x} A(\mathbb{C}) \times T_{\mathbb{R}, x} A(\mathbb{C}) = \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}, \quad (v, w) \mapsto -\text{Im}(H)(\bar{\varphi}_{ab}(v), \bar{\varphi}_{ab}(w)),
\]

for each \( x \in A(\mathbb{C}) \). Since \( c_1(N) \) is positive and of real type, \( \omega(N; \phi_{ab}) \) is semipositive and of real type as well. In addition, \( \omega(N; \phi_{ab}) \) only depends on \( \phi_{ab} \) and the hermitian form \( H \) associated with \( N \) (i.e., the Neron-Severi class of \( N \)) but we have no use for this fact in the following. Yet again, the assignment \( \phi_{ab} \mapsto \omega(N; \phi_{ab}) \) is continuous with respect to the usual topologies. Finally, there is the obvious homogeneity relation

\[
(43) \quad \omega(N; n \cdot \phi_{ab}) = n^2 \cdot \omega(N; \phi_{ab}).
\]

6. Distributions, analytic subgroups, and Ax’s Theorem

In general, the (1,1)-forms \( \omega(\phi_{tor}) \) and \( \omega(N; \phi_{ab}) \) introduced in Section 5 have no realization as Chern forms of hermitian line bundles. As we show in this section, they nevertheless convey geometric information and are closely connected to the group structure of the semiabelian variety.

Once again, we consider a semiabelian variety \( G \) with abelian quotient \( \pi : G \to A \) and toric part \( T = \mathbb{G}_m^t \). Let \( t' \) be a non-negative integer, \( A' \) an abelian variety and \( N \) an ample line bundle on \( A' \). For each \( \phi_{tor} \in \text{Hom}_{\mathbb{R}}(\mathbb{G}_m^t, \mathbb{G}_m^t) \) (resp. \( \phi_{ab} \in \text{Hom}_{\mathbb{R}}(A, A') \)), we have a semipositive (1,1)-form \( \omega(\phi_{tor}) \) (resp. \( \pi^* \omega(\phi_{ab}) \)) of real type on \( G(\mathbb{C}) \). Set

\[
\omega = c \cdot \omega(\phi_{tor}) + \pi^* \omega(N; \phi_{ab})
\]

for some arbitrary positive constant \( c > 0 \). Since \( g_\omega(\phi_{tor}) \) and \( g_\pi \omega(N; \phi_{ab}) \) are positive semidefinite, we infer from (29) that

\[
(43) \quad \ker(g_\omega,x) = \ker(g_\omega(\phi_{tor}),x) \cap \ker(g_\pi \omega(N; \phi_{ab}),x) \subseteq T_{\mathbb{R}, x} G(\mathbb{C})
\]

for each \( x \in G(\mathbb{C}) \). In addition, \( \omega(I(\cdot), I(\cdot)) = \omega(\cdot, \cdot) \) implies \( g_\omega(I(\cdot), I(\cdot)) = g_\omega(\cdot, \cdot) \) and thus \( \ker(g_\omega, x) \) is invariant under \( I \). In fact, both \( \ker(g_\omega(\phi_{tor}),x) \) and \( \ker(g_\omega(N; \phi_{ab}),x) \) are \( I \)-invariant for
the same reason. Under our standing identification of $T_Z G(\mathbb{C})$ and $T^{1,0}_Z G(\mathbb{C})$, this means that ker$(g_{\omega,x})$ is a $\mathbb{C}$-linear subspace of $T^{1,0}_Z G(\mathbb{C})$. Our next observation is that this yields a left-invariant holomorphic distribution (i.e., a holomorphic vector subbundle) ker$(g_{\omega}) \subset T^{1,0}_Z G(\mathbb{C})$, which is a straightforward consequence of the lemma below.

**Lemma 17.** For every $x, y \in G(\mathbb{C})$, we have $(dl_y)_x \ker(g_{\omega,x}) = \ker(g_{\omega,y,x})$.

**Proof.** Because of (13), it suffices to prove that

$$(dl_y)_x \ker(g_{\omega(\phi),x}) = \ker(g_{\omega(\phi),y,x})$$

for all $x, y \in G(\mathbb{C})$. The latter equality is a direct consequence of the fact that the Chern form $c_1(N)$ on $A'(\mathbb{C})$ is translation-invariant, which can be read off from (10). Using (26), we see that (35) implies

$$g_{\omega(\phi)} = \sum_{v=1}^{t'} \partial \phi_{\text{tor}}^* \lambda_v' \otimes \overline{\partial \phi_{\text{tor}}^* \lambda_v'} + \partial \phi_{\text{tor}}^* \lambda_v' \otimes \partial \phi_{\text{tor}}^* \lambda_v'.
$$

By Lemma 14 each $\partial \phi_{\text{tor}}^* \lambda_v' \otimes \overline{\partial \phi_{\text{tor}}^* \lambda_v'} + \partial \phi_{\text{tor}}^* \lambda_v' \otimes \partial \phi_{\text{tor}}^* \lambda_v'$ is a positive semidefinite bilinear form on $T_Z G(\mathbb{C})$ and it follows by (29) that

$$\ker(g_{\omega(\phi)}) = \bigcap_{v=1}^{t'} \ker(\partial \phi_{\text{tor}}^* \lambda_v' \otimes \overline{\partial \phi_{\text{tor}}^* \lambda_v'} + \partial \phi_{\text{tor}}^* \lambda_v' \otimes \partial \phi_{\text{tor}}^* \lambda_v').$$

In addition, (28) implies that

$$\ker(\partial \phi_{\text{tor}}^* \lambda_v' \otimes \overline{\partial \phi_{\text{tor}}^* \lambda_v'} + \partial \phi_{\text{tor}}^* \lambda_v' \otimes \partial \phi_{\text{tor}}^* \lambda_v')_x = \{ w \in T_{R,x} G(\mathbb{C}) \mid \partial \phi_{\text{tor}}^* \lambda_v'(w) \cdot \overline{\partial \phi_{\text{tor}}^* \lambda_v'}(w) = 0 \}.$$

Since each $\phi_{\text{tor}}^* \lambda_v'$ is real-valued, we have $\overline{\partial \phi_{\text{tor}}^* \lambda_v'}(v) = \overline{\partial \phi_{\text{tor}}^* \lambda_v'}(v)$ and thus

$$\ker(\partial \phi_{\text{tor}}^* \lambda_v' \otimes \overline{\partial \phi_{\text{tor}}^* \lambda_v'} + \partial \phi_{\text{tor}}^* \lambda_v' \otimes \partial \phi_{\text{tor}}^* \lambda_v')_x = \{ w \in T_{R,x} G_0(\mathbb{C}) \mid \partial \phi_{\text{tor}}^* \lambda_v'(w) = 0 \} = \ker(\partial \phi_{\text{tor}}^* \lambda_v').$$

Note that $(\partial \phi_{\text{tor}}^* \lambda_v')_x$ is a $\mathbb{R}$-linear map $T_{R,x} G(\mathbb{C}) \to \mathbb{C}$. As each $\lambda_v'$ is a homomorphism, we have $(\partial \phi_{\text{tor}}^* \lambda_v' \circ l_y)(\cdot) = \phi_{\text{tor}}^* \lambda_v'(\cdot) + \phi_{\text{tor}}^* \lambda_v'(y)$ and therefore $\partial(\phi_{\text{tor}}^* \lambda_v' \circ l_y) = \partial \phi_{\text{tor}}^* \lambda_v'$. We infer

$$(dl_y)_x \ker(\partial \phi_{\text{tor}}^* \lambda_v')_x = \ker(\partial(\phi_{\text{tor}}^* \lambda_v' \circ l_y))_{yz} = \ker(\partial \phi_{\text{tor}}^* \lambda_v')_{yz},$$

thus establishing the first equation of (14). \hfill \Box

Having proven translation-invariance, we can easily determine the rank of ker$(g_{\omega})$ by determining the dimension of ker$(g_{\omega,e})$. We do this next under some surjectivity assumption on $\phi_{\text{tor}}$ and $\phi_{\text{ab}}$. To describe this assumption, we recall that the complex exponential map gives a universal covering $\mathbb{C} \to G_m(\mathbb{C})$. Taking products, we obtain universal coverings $\mathbb{C}' \to G'_{m'}(\mathbb{C})$ and $\mathbb{C}' \to G'_{m'}(\mathbb{C})$. Each homomorphism $\varphi_{\text{tor}} \in \text{Hom}(G'_m, G'_{m'})$ lifts to a linear map $\tilde{\varphi}_{\text{tor}} : \mathbb{C}' \to \mathbb{C}'$ (cf. [60] Theorems 3.25 and 3.27). Tensoring with $\mathbb{R}$, we obtain an injection $\text{Hom}_R(G'_m, G'_{m'}) \hookrightarrow \text{Hom}_R(\mathbb{C}', \mathbb{C}')$, $\phi_{\text{tor}} \mapsto \tilde{\phi}_{\text{tor}}$, and set

$$\text{Hom}_{R}^{\circ}(G'_m, G'_{m'}) = \{ \phi_{\text{tor}} \in \text{Hom}_R(G'_m, G'_{m'}) \mid \tilde{\phi}_{\text{tor}} \text{ is surjective} \}.$$ 

In Subsection 5.2, we have associated with each $\phi_{\text{ab}} \in \text{Hom}(A, A')$ a linear map $\tilde{\phi}_{\text{ab}} : \mathbb{C}^g \to \mathbb{C}^{g'}$ and we define similarly

$$\text{Hom}_{R}^{\circ}(A, A') = \{ \phi_{\text{ab}} \in \text{Hom}_R(A, A') \mid \tilde{\phi}_{\text{ab}} \text{ is surjective} \}.$$ 

**Lemma 18.** If $\phi_{\text{tor}} \in \text{Hom}_{R}^{\circ}(G'_m, G'_{m'})$ and $\phi_{\text{ab}} \in \text{Hom}_{R}^{\circ}(A, A')$, then ker$(\omega)$ has rank $(t - t') + (\dim(A) - \dim(A'))$ (as a complex vector bundle).
Proof. First, we claim that there is a commutative exact diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & \ker(g_{\omega(\phi_{\text{tor}}),e}|_{T_{R,e}T(C)}) & \longrightarrow & \ker(g_{\omega,e}) & \longrightarrow & \ker(g_{\omega(N;\phi_{ab}),e}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{R,e}T(C) & \longrightarrow & T_{R,e}G(C) & \longrightarrow & (d\pi)_{e} & \longrightarrow & T_{R,e}A(C) & \longrightarrow & 0.
\end{array}
\]
(45)

Except for the surjectivity of \(\ker(g_{\omega,e}) \to \ker(g_{\omega(N;\phi_{ab}),e})\), this is a direct consequence of semi-positivity and (28). For surjectivity, it suffices to prove that there exists an \(I\)-invariant subspace \(V \subset \ker(g_{\omega(\phi_{\text{tor}}),e})\) such that
\[
T_{R,e}T(C) \oplus V = T_{R,e}G(C).
\]
(46)

Given such a decomposition, we can find for any \(v \in \ker(g_{\omega(N;\phi_{ab}),e})\) a \((d\pi)_{e}\)-preimage \(w \in V\). Furthermore, we have
\[
g_{\omega,e}(w, w) = g_{\omega(\phi_{\text{tor}}),e}(w, w) + g_{\omega,N;\phi_{ab},e}(w, w) = g_{\omega,N;\phi_{ab},e}(v, v) = 0
\]
since \(w \in \ker(g_{\omega(\phi_{\text{tor}}),e})\) and \((d\pi)_{e}(w) = v \in \ker(g_{\omega(N;\phi_{ab}),e})\). Recall that the maximal compact subgroup \(K_{G} \subset G(C)\) is a real Lie subgroup such that \(\dim_{R}(T_{R,e}K_{G}) = 2 \dim(A) + t\).

We now claim that there is a commutative exact diagram
\[
\begin{array}{cccc}
\ker(\partial\lambda_{u}) & \longrightarrow & \ker(\partial\lambda_{u}|_{T_{R,e}T(C)}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ker(\partial\lambda_{u}) & \longrightarrow & \ker(\partial\lambda_{u}|_{T_{R,e}T(C)}) & \longrightarrow & 0.
\end{array}
\]
which shows that the restrictions of \(\partial\lambda_{1}, \ldots, \partial\lambda_{t}, \partial\lambda_{1}', \ldots, \partial\lambda_{t}'\) to \(T_{R,e}T(C)\) form a \(C\)-basis of \(\text{Hom}_{R}(T_{R,e}T(C), C)\). Since each of these forms vanishes on \(V\) (see (31)), we have \(T_{R,e}T(C) \cap V = \{0\}\). As \(\dim_{R}(V) = \dim_{R}(T_{R,e}K_{G} \cap I(T_{R,e}K_{G})) = 2 \dim(A)\), we obtain the direct sum decomposition (40).

Using (15), it remains to compute the dimensions of the \(I\)-invariant \(R\)-linear subspaces \(\ker(g_{\omega(\phi_{\text{tor}}),e}|_{T_{R,e}T_{0}(C)})\) and \(\ker(g_{\omega(M;\phi_{ab}),e})\). For the former one, let us represent \(\phi_{\text{tor}} \in \text{Hom}_{R}(C', C')\) as a matrix \((a_{uv})_{1 \leq u \leq t, 1 \leq v \leq v' \in \mathbb{R}^{t \times t'}\). As in the proof of Lemma 17 above, we have
\[
\ker(g_{\omega(\phi_{\text{tor}}),e}|_{T_{R,e}T(C)}) = \bigcap_{v=1}^{t'} \ker((a_{1v}\partial\lambda_{1} + \cdots + a_{tv}\partial\lambda_{t})|_{T_{R,e}T(C)})
\]
\[
= \bigcap_{v=1}^{t'} \ker(a_{1v}dx_{1} + \cdots + a_{tv}dx_{t} : T_{R,e}T(C) \to C).
\]
Setting \(dz_{v} = dx_{i} + idy_{i}\) with \(dx_{i}, dy_{i} : T_{R,e}T(C) \to \mathbb{R}\), we can rewrite this as
\[
\ker(g_{\omega(\phi_{\text{tor}}),e}|_{T_{R,e}T(C)}) = \bigcap_{v=1}^{t'} \ker((a_{1v}dx_{1} + \cdots + a_{tv}dx_{t}) \cap \bigcap_{v=1}^{t'} \ker(a_{1v}dy_{1} + \cdots + a_{tv}dy_{t}).
\]

The condition \(\phi_{\text{tor}} \in \text{Hom}_{R}(C'_{m}, C'_{m})\) is equivalent to the matrix \((a_{uv})\) having maximal rank \(t'\). This implies that the real-valued functionals

\[a_{1v}dx_{1} + \cdots + a_{tv}dx_{t}, a_{1v}dy_{1} + \cdots + a_{tv}dy_{t} (1 \leq v \leq t')\]
on \(T_{R,e}T(C)\) are \(R\)-linearly independent. From this, we infer
\[\dim_{R} \ker(g_{\omega(\phi_{\text{tor}}),e}|_{T_{R,e}T(C)}) = 2(t - t').\]
For \(\ker(g_{\omega(N;\phi_{ab}),e})\), it follows directly from (11) that
\[
g_{\omega(N;\phi_{ab}),e} : T_{R,e}A(C) \times T_{R,e}A(C) = C^{g} \times C^{g} \to C, (v, w) \mapsto \text{Re}(H)(\tilde{\phi}_{ab}(v), \tilde{\phi}_{ab}(w))
\]
(compare with (27)). Since $H$ is positive definite, so is its real part $\text{Re}(H)$. Using (28), we deduce that
\[ \ker(g_\omega(N;\phi_{ab}),x) = \tilde{\phi}_{ab}^{-1}(\ker(\text{Re}(H))) = \tilde{\phi}_{ab}^{-1}(\{0\}) = \ker(\tilde{\phi}_{ab}). \]
Finally, $\phi_{ab} \in \text{Hom}_{\mathbb{R}}^0(A,A')$ implies that $\dim_{\mathbb{R}} \ker(g_\omega(N;\phi_{ab}),x) = 2(\dim(A) - \dim(A'))$. 

In summary, we have proven that $\ker(\omega) \subset T_{c,0}^1 G(\mathbb{C})$ is a left-invariant holomorphic distribution on $G(\mathbb{C})$. By the holomorphic Frobenius theorem ([24, Theorem II.1.7]), the distribution $\ker(\omega)$ is (holomorphically) integrable since the Lie bracket on $T_{c,0}^1 G(\mathbb{C})$ vanishes. In fact, the integral manifold of $\ker(\omega)$ through a given point $x \in G(\mathbb{C})$ coincides with the analytic subgroup $x \cdot \exp_{G(\mathbb{C})}(\ker(g_\omega,e))$ with $\exp_{G(\mathbb{C})} : T_{c,0}^1 G(\mathbb{C}) \to G(\mathbb{C})$ being the Lie group exponential (cf. [24, Theorem II.1.7]).

As indicated in Section 4, we are interested in determining when a submanifold $Y \subset G(\mathbb{C})$ and a point $x \in Y(\mathbb{C})$ are such that $g_\omega|_{T_{x,Y}}$ is positive definite. For this purpose, we introduce an elementary lemma about integrable (holomorphic) distributions.

**Lemma 19.** Let $M$ be a complex manifold of dimension $n$, $D \subset T_{c,0}^1 M$ an integrable holomorphic distribution of rank $m$ on $M$, $Z \subset M$ a $k$-dimensional analytic subvariety and $x$ a point on $Z$. Assume that there exists an open neighborhood $U \subset M$ of $x$ such that $\dim_{\mathbb{C}}(D \cap T_{c,0}^1 Z) \geq l$ for any $y \in Z \cap U$. Then, the integral submanifold $L \subset U$ of $D$ through $x$ satisfies $\dim_x(L \cap Z) \geq l$. 

**Proof.** By shrinking $U$ if necessary, we can assume that there exists a holomorphic flat chart $f : U \to \mathbb{C}^{n-m}$ for $D|_U$. Recall that this means that $f$ is a submersion and that each non-empty fiber of $f$ is an integral submanifold for $D|_U$. By our assumption, the differential $d(f|_Z) : T_{c,0}^1 Z \to \mathbb{C}^{n-m}$ has rank $\leq k - l$ for every $y \in Z$. By [21, Lemma L.6], the local dimension of any fiber $f|_Z^{-1}(f(y)) = f^{-1}(f(y)) \cap Z$, $y \in Z \cap U$, is $\geq l$ everywhere. In particular, $\dim_x(f^{-1}(f(x)) \cap Z) \geq l$. 

Finally, we are ready to use Ax’s Theorem to show non-degeneracy in all cases of interest.

**Lemma 20.** Let $X \subset G$ be an algebraic subvariety such that $X^{(s)} \neq X$ for some non-negative integer $s$. Then $(\omega|_X)^{\dim(X)} \neq 0$ for every $(\phi_{\text{tor}}, \phi_{ab}) \in \text{Hom}_{\mathbb{R}}^0(G'_m, G'_m) \times \text{Hom}_{\mathbb{R}}^0(A,A')$ with $t' + \dim(A') \geq s$.

**Proof.** Assume $(\omega|_X)^{\dim(X)} = 0$, which means that $\dim_{\mathbb{C}}(\ker(g_\omega,x) \cap T_x X) \geq 1$ for any $x \in X(\mathbb{C})$. For each $x \in X(\mathbb{C})$, let $L_x = x \cdot \exp_{G(\mathbb{C})}(\ker(g_\omega,e))$ be the integral manifold of $\ker(g_\omega)$ through $x$. By Lemma 18, the holomorphic distribution $\ker(g_\omega)$ has rank $\leq \dim(G) - s$ and this is also the dimension of $L_x$. From Lemma 19, we know that $\dim_x(L_x \cap X(\mathbb{C})) \geq 1$. This is an intersection of an algebraic subvariety with an analytic subgroup in $G(\mathbb{C})$. Applying Ax’s Theorem ([11, Corollary 1]), we obtain for each $x \in X(\mathbb{C})$ an algebraic subgroup $H \subset G$ such that $X \subset xH$ and
\[ \dim(H) \leq \dim(X) + \dim(L_x) - \dim_x(L_x \cap X(\mathbb{C})) < \dim(X) + \dim(G) - s. \]
A comparison with [11] shows that this implies that $X$ is itself an $s$-anomalous variety, associated with $H$, and hence $X = X^{(s)}$. 

7. **Proof of Theorem 2**

In this section, all algebraic groups are over $\text{Spec}(\mathbb{C})$ without further mention. As usual, $T$ denotes the toric part of $G$ and $A$ the underlying abelian variety. Since our base field is $\mathbb{C}$, the torus $T$ is split and we keep fixed a splitting throughout this section (i.e., assume $T = G'_m$).
7.1. Reductions. We start with an elementary observation related to the “height cones” introduced in (2). Let \( h, h' : G(\overline{Q}) \to \mathbb{R} \) be functions satisfying
\[
c_3 h'(x) - c_4 \leq h(x) \leq c_5 h(x) + c_6 \quad \text{and} \quad h'(x_1 + x_2) \leq c_7 (h'(x_1) + h'(x_2)) + c_8
\]
for all \( x, x_1, x_2 \in G(\overline{Q}) \) with constants \( c_i > 0 (i \in \{3, \ldots, 8\}) \). For any subset \( \Sigma \subseteq G(\overline{Q}) \) and any \( \varepsilon > 0 \), there is an inclusion
\[
C(\Sigma, h', \varepsilon') \subseteq \{ x \in G(\overline{Q}) \mid h'(x) < c_9 \} \cup C(\Sigma, h, \varepsilon)
\]
with \( \varepsilon' = \varepsilon \max \{ c_3 c_1^{-1}, c_5 c_1^{-1} \} \) and \( c_9 = c_7 (2 \varepsilon^{-1} (1 + \varepsilon') (c_5 \varepsilon' + c_6) + c_1^{-1} c_8 + \varepsilon') \). The straightforward computation (or the deduction of a similar assertion) is left to the reader.

Let \( \overline{G} \) be the compactification of \( G \) and \( M_{\overline{G}} \) the line bundle as in Construction 5. Furthermore, let \( N \) be a ample symmetric line bundle on \( A \). By Lemma 3 \( L = M_{\overline{G}} \otimes \pi^* N \) is an ample line bundle on \( \overline{G} \). For Theorem 2 it is sufficient to prove the boundedness of \( h_L \) on \( (X \setminus X^{(s)}) (\overline{Q}) \cap C(G^{[s]}(\overline{Q}), h_L, \varepsilon) \). In fact, let \( L' \) be an arbitrary ample line bundle on an arbitrary compactification \( \overline{G}' \) of \( G \) and \( h_{L'} \), an associated Weil height. Applying [58, Proposition 2.3] to the identity map \( \text{id}_G \), which gives a birational map \( \overline{G} \dashrightarrow \overline{G}' \), and the line bundles \( L \) and \( L' \) we obtain the first two inequalities in [18]. The third inequality follows from applying the same proposition to the group law \( +_G \), understood as a rational map \( \overline{G}' \times \overline{G}' \dashrightarrow \overline{G}' \). We may thus use our above observation to ensure the asserted reduction. Considering also Lemma 8 (a), we see that it even suffices to prove that \( \hat{h}_L = \hat{h}_{M_{\overline{G}}} + \hat{h}_N : G(\overline{Q}) \to \mathbb{R} \geq 0 \) is bounded from above on \( (X \setminus X^{(s)}) (\overline{Q}) \cap C(G^{[s]}(\overline{Q}), \hat{h}_L, \varepsilon) \).

Our last reduction step is to note that Theorem 2 is easily inferred from the following proposition, which is shown in the remaining parts of Section 7.

Proposition 21. Let \( X \subseteq G \) be an irreducible Zariski closed subset of positive dimension such that \( X^{(s)} \neq X \). Then, there exists a non-empty Zariski open subset \( U \subseteq X \) and some \( \varepsilon > 0 \) such that \( \hat{h}_L \) is bounded on \( U \cap C(G^{[s]}(\overline{Q}), \hat{h}_L, \varepsilon) \).

Proof of Theorem 2 (using Proposition 21). We perform an induction on \( \dim(X) \). Theorem 2 is clearly trivial if \( X \) has dimension zero, which starts our induction. Assume now that \( X \) is positive dimensional and that the assertion of the theorem, with \( h_L \) replaced by \( \hat{h}_L \), is already known for any \( X' \) with \( \dim(X') < \dim(X) \). Without loss of generality, we can additionally assume that \( X \) is irreducible and that \( X^{(s)} \neq X \). Applying Proposition 21 to \( X \), we obtain a non-empty Zariski open subset \( U \subseteq X \) and a real number \( \varepsilon > 0 \) such that \( \hat{h}_L \) is bounded on \( U(\overline{Q}) \cap C(G^{[s]}(\overline{Q}), \hat{h}_L, \varepsilon) \). Now, \( X' = X \setminus U \) has dimension strictly less than \( \dim(X) \) so that we may apply our inductive hypothesis to \( X' \). We obtain that \( \hat{h}_L \) is bounded on \( (X' \setminus (X')^{(s)}) (\overline{Q}) \cap C(G^{[s]}(\overline{Q}), \hat{h}_L, \varepsilon') \) for some \( \varepsilon' > 0 \). In conclusion, we know that \( \hat{h}_L \) is bounded on
\[
(X \setminus (X')^{(s)}) (\overline{Q}) \cap C(G^{[s]}(\overline{Q}), \hat{h}_L, \min\{\varepsilon, \varepsilon'\})
\]
As \( (X')^{(s)} \subseteq X^{(s)} \) by (1), this yields the assertion of Theorem 2 for \( X \).

\[ \square \]

7.2. Approximating homomorphisms. The following lemma is useful for reducing the proof of the main theorem to a manageable situation.

Lemma 22. There exist finitely many abelian varieties \( A'_1, \ldots, A'_{j_0} \) (depending on \( s \)) such that each \( x \in G^{[s]}(\overline{Q}) \) is contained in the kernel of some surjective homomorphism \( \varphi : G \to G' \), \( \dim(G') \geq s \), that is represented (as in Lemma 1) by a diagram
\[
\begin{array}{cccc}
1 & \longrightarrow & G'_{m} & \longrightarrow & G' & \longrightarrow & A'_{j} & \longrightarrow & 0 \\
& & \varphi_{\text{tor}} & \downarrow \varphi & & & & \varphi_{\text{sur}} & \downarrow \\
1 & \longrightarrow & G_{m} & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0.
\end{array}
\]
As \( \varphi_{\text{tor}} \) is surjective, we clearly have \( t' \leq t \).

Proof. Evidently, if \( x \in H(\mathbb{Q}) \) with \( \text{codim}_G(H) \geq s \) then \( x \) is in the kernel of the quotient \( \pi: G \to G/H \). The toric part of \( G/H \) can be identified with \( G'_{\text{tor}} \). The abelian component \( \pi_{\text{ab}}: A \to B \) of \( \pi \) is surjective. By Poincaré’s complete reducibility theorem ([14, Theorem 1 on p. 173]), there exist only finitely many quotients \( A \to A_j' \), \( 1 \leq j \leq j_0 \), up to isogeny. In particular, \( B \) is isogeneous to one of these abelian varieties, which we denote by \( A_j' \).

By Lemma 23 there exists a unique homomorphism \( G' \to G/H \) with toric component \( id_{G'_{\text{tor}}} \) and abelian component \( \psi \). This is an isogeny and hence there exists also an isogeny \( \psi': G/H \to G' \). As \( G' \) has abelian quotient \( A_j' \) by construction, we can take \( \varphi = \psi' \circ \pi \).

If \( G \) is an abelian variety, Poincaré’s complete reducibility theorem yields immediately the existence of finitely many quotients \( \varphi_i: G \to G_i \), \( \text{dim}(G_i) \geq s \), such that each \( x \in G[i](\mathbb{Q}) \) is contained in the kernel of some \( \varphi_i \). In addition, if \( G \) is a torus a similar statement is true for more trivial reasons. Nevertheless, the analogous statement is false for general semiabelian varieties as simple examples show. Our lemma is optimal in the general case.

By Lemma 1 we may associate with each \( \varphi \in \text{Hom}(G, G') \) as in Lemma 22 a pair

\[
(\varphi_{\text{tor}}, \varphi_{\text{ab}}) \in \text{Hom}(G'_{\text{tor}}, G'_{\text{m}}) \times \text{Hom}(A, A_j'), \ t' \in \{0, \ldots, t\}, \ j \in \{1, \ldots, j_0\}.
\]

This allows us to concentrate on a finite number of fixed finite rank \( \mathbb{Z} \)-modules

\[
(49) \quad V(t', j) = \text{Hom}(G'_{\text{tor}}, G'_{\text{m}}) \times \text{Hom}(A, A_j'), \ t' \in \{0, \ldots, t\}, \ j \in \{1, \ldots, j_0\}.
\]

instead of infinitely many different \( \text{Hom}(G, G') \). We study now one of these modules separately and drop the superscripts, writing \( V \) instead of \( V(t', j) \). As \( V \) is a free \( \mathbb{Z} \)-module, it embeds into \( V_\mathbb{Q} = V \otimes \mathbb{Q} \) and \( V_\mathbb{R} = V \otimes \mathbb{R} \). Furthermore, a quasi-homomorphism \( \phi \in \text{Hom}_\mathbb{Q}(G, G') \) determines a pair \( (\phi_{\text{tor}}, \phi_{\text{ab}}) \in V_\mathbb{Q} \). However, the relation between elements \( (\phi_{\text{tor}}, \phi_{\text{ab}}) \in V_\mathbb{Q} \) and actual quasi-homomorphisms \( \phi: G \to G' \) of semiabelian varieties is quite intricate. The reader is referred to Appendix 8 for details. As witnessed by the results of Section 4, we have a special interest in pairs that are contained in

\[
V_{\mathbb{R}} := \text{Hom}_\mathbb{R}(G'_{\text{tor}}, G'_{\text{m}}) \times \text{Hom}_\mathbb{R}(A, A_j') \subset V_{\mathbb{R}}.
\]

For this reason, we also define \( V_{\mathbb{Q}} = V_\mathbb{Q} \cap V_{\mathbb{R}} \). It is easy to see that a quasi-homomorphism \( \phi_{\text{tor}} \in \text{Hom}_\mathbb{Q}(G'_{\text{tor}}, G'_{\text{m}}) \) (resp. \( \phi_{\text{ab}} \in \text{Hom}_\mathbb{Q}(A, A_j') \)) is contained in \( \text{Hom}_\mathbb{R}(G'_{\text{tor}}, G'_{\text{m}}) \) (resp. \( \text{Hom}_\mathbb{R}(A, A_j') \)) if and only if it is surjective in the sense of Section 12.

With these preparations, we can state our first approximation result. The proof is a simple reduction to the abelian and toric cases treated in [22, 23].

Lemma 23. There exists a compact subset \( K = K_{\text{tor}} \times K_{\text{ab}} \subset V_{\mathbb{R}} \) such that the following assertion is true: Let \( x \in G(\mathbb{Q}) \) be contained in the kernel of a surjective homomorphism \( \varphi: G \to G' \) of semiabelian varieties that is represented by some \( (\varphi_{\text{tor}}, \varphi_{\text{ab}}) \in V \). Then, there exists a semiabelian variety \( G'' \) and a surjective quasi-homomorphism \( \tilde{\phi}: G \to G'' \) such that \( x \in \ker(\tilde{\phi}) + \text{Tors}(G) \) and \( \phi \) is represented by some \( (\tilde{\phi}_{\text{tor}}, \tilde{\phi}_{\text{ab}}) \in V_{\mathbb{Q}} \cap K \).

The reader may be reminded that \( \text{dim}(G') = \text{dim}(G'') \) as well as the fact that \( G'_{\text{m}} \) (resp. \( A_j' \)) is the toric part (resp. the abelian quotient) of both \( G' \) and \( G'' \) is automatic.

---

4 In fact, consider the semiabelian variety \( G \) that is the \( G'_{\text{m}} \)-extension of a non-CM elliptic curve \( E \) represented by \( (\eta_1, \eta_2) \in E'(\mathbb{Q})^2 \). Assume also that \( \Xi_{\eta_1} + \Xi_{\eta_2} \) is a free \( \mathbb{Z} \)-module of rank 2. For each integer \( n \), we consider the \( G'_{\text{m}} \)-extension \( G(n) \) of \( G \) by \( n \eta_1 + n \eta_2 \in E'(\mathbb{Q}) \) and the homomorphism \( \varphi(n): G \to G(n) \) described by \( \varphi(n)(\eta_1) = X_{\eta_1}X_2 \) and \( \varphi(n)(\eta_2) = id_{E} \). There exists a point \( x \in \ker(\varphi(n))(\mathbb{Q}) \subset G[2](\mathbb{Q}) \) that is not contained in any other algebraic subgroup of codimension 2. Therefore any surjective homomorphism \( \varphi: G \to G', \text{dim}(G') = 2, \) with \( p \in \ker(\varphi)(\mathbb{Q}) \) factors through \( \varphi(n) \). However, Lemma 4 implies that \( G(n) \) and \( G'(m) \) are not isogeneous if \( n \neq m \). Indeed, all \( G'_{\text{m}} \)-extensions isogeneous to \( G(n) \) are represented by "rational multiples" of \( n \eta_1 + n \eta_2 \in E'(\mathbb{Q}) \).
Proof. Using again Lemma 1, we obtain a commutative diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & G^t_m & \rightarrow & G & \rightarrow & 0 \\
\downarrow \varphi_{tor} & & \downarrow \varphi & & \downarrow \varphi_{ab} & \\
1 & \rightarrow & G^t_m & \rightarrow & G' & \rightarrow & 0.
\end{array}
\]

By [22] Lemma 2, there exists some compact subset \(K_{ab} \subset \text{Hom}_R(A, A'_j)\) such that for every surjective \(\psi \in \text{Hom}_Q(A, A'_j)\) there exists a surjective \(\psi' \in \text{Hom}_Q(A'_j, A'_j)\) with \(\psi' \circ \psi \in K_{ab}\). As \(\varphi\) is surjective, the same is true for its abelian component \(\varphi_{ab}\). Hence, we may apply the lemma with \(\psi = \varphi_{ab}\) and obtain a quasi-homomorphism \(\psi'_2 : A'_j \rightarrow \mathbb{Q} A'_j\) such that \(\psi'_2 \circ \varphi_{ab} \in K_{ab}\). Similarly, we can extract from the proof of [23] Lemma 4.2] that there exists a compact set \(K_{tor} \subset \text{Hom}_R(G^t_m, G^t_m)\) such that there always exists a surjective quasi-homomorphism \(\psi'_3 : G^t_m \rightarrow \mathbb{Q} G^t_m\) with \(\psi'_3 \circ \varphi_{tor} \in K_{tor}\). We claim that \(K = K_{tor} \times K_{ab} \subset V_R^0\) satisfies the assertion of the lemma.

Let \(n \) be a positive integer such that \(n \cdot \psi'_3 \in \text{Hom}(A'_j, A'_j)\) and \(n \cdot \psi'_2 \in \text{Hom}(G^t_m, G^t_m)\). By Lemma 3 there exists a semiabelian variety \(G''\) and a homomorphism \(\varphi' : G' \rightarrow G''\) such that

\[
\begin{array}{cccccc}
1 & \rightarrow & G^t_m & \rightarrow & G' & \rightarrow & A'_j & \rightarrow & 0 \\
\downarrow n \cdot \psi'_2 & & \downarrow n \cdot \psi'_3 & & \downarrow \varphi' & & \downarrow n \cdot \psi_{ab} & & \downarrow n \cdot \psi_{tor} & \\
1 & \rightarrow & G^t_m & \rightarrow & G'' & \rightarrow & A'_j & \rightarrow & 0.
\end{array}
\]

is a commutative diagram with exact rows. The homomorphism \(\varphi' \circ \varphi : G \rightarrow G''\) is represented by

\[
n \cdot (\psi'_3 \circ \varphi_{tor}, \psi'_3 \circ \varphi_{ab}) \in V_Z \cap n \cdot K.
\]

Multiplying with \(n^{-1}\), we get a quasi-homomorphism \(\phi : G \rightarrow Q G''\) that is represented by

\[
(\psi'_3 \circ \varphi_{tor}, \psi'_3 \circ \varphi_{ab}) \in V_Q \cap K.
\]

This is evidently the quasi-homomorphism we are searching for. \(\square\)

For the next lemma, we endow \(\text{Hom}_R(G^t_m, G^t_m)\) and \(\text{Hom}_R(A, A'_j)\) with linear norms. As all norms on a finite-dimensional \(\mathbb{R}\)-vector space are equivalent, the precise choice is irrelevant for our purposes. Therefore, we just fix an arbitrary norm \(|\cdot|\) on \(\text{Hom}_R(G^t_m, G^t_m)\) and \(\text{Hom}_R(A, A'_j)\) for the sequel. We slightly abuse notation in denoting both norms by \(|\cdot|\). For each real \(r > 0\), we denote by \(B_r(\phi_{tor})\) (resp. \(B_{r,1/2}(\phi_{ab})\)) the open ball with radius \(r\) (resp. \(r^{1/2}\)) around \(\phi_{tor} \in \text{Hom}_R(G^t_m, G^t_m)\) (resp. \(\phi_{ab} \in \text{Hom}_R(A, A'_j)\)). In addition, we set

\[
B_r(\phi_{tor}, \phi_{ab}) = B_r(\phi_{tor}) \times B_{r,1/2}(\phi_{ab}), \quad (\phi_{tor}, \phi_{ab}) \in V_R.
\]

Lemma 24. Let \(\delta > 0\) be arbitrary. Then, there exists an integer \(n_{\delta} \geq 1\) and a finite set

\[
\{(\phi_{1,tor}, \phi_{1,ab}), \ldots, (\phi_{k_{\delta},tor}, \phi_{k_{\delta},ab})\} \subset n_{\delta}^{-1} V
\]

such that for each \((\phi_{tor}, \phi_{ab}) \in K\) we have \((\phi_{tor}, \phi_{ab}) \in B_{\delta}(\phi_{tor}, \phi_{k,ab})\) for some \(1 \leq k \leq k_{\delta}\).

Proof. For sufficiently large \(n_{\delta}\), the open sets

\[
B_{\delta}(\phi_{tor}, \phi_{ab}) = B_{\delta}(\phi_{tor}) \times B_{\delta,1/2}(\phi_{ab}), \quad (\phi_{tor}, \phi_{ab}) \in n_{\delta}^{-1} V,
\]

cover all of \(V_R\). By compactness, finitely many of these open sets suffice to cover all of \(K\). \(\square\)

In both [22] and [23], a step analogous to Lemma 24 is performed quite explicitly with a quantitatively much better result, using diophantine approximation. The above weaker estimate is however sufficient for our proof.
7.3. Height bounds. In this section, we derive two competing height bounds. The first one (Lemma 25) is valid for any \( x \in C(G^{[s]}, \hat{h}_L, \varepsilon) \), whereas the second one (65) is valid for almost all \( x \in (X \setminus X^{(s)})/\bar{\Q} \). In combination, they imply the desired Proposition 21.

Throughout this section, we keep fixed some sufficiently small \( \delta \); the precise conditions on \( \delta \) can be found in (50) and (66). For the constants to be introduced in the sequel, we have to distinguish between those depending only on \( G \) and \( X \) and those that depend additionally on \( \delta \). For this purpose, the former are written plainly \( c_0 \), whereas the latter are written \( c_i(\delta) \). None of these constants depends on the point \( x \in G(\bar{\Q}) \) under consideration.

As in Lemma 22 we fix finitely many abelian varieties \( A_1, \ldots, A_{n_0} \). We endow each \( A'_j \) with an ample symmetric line bundle \( N_j, 1 \leq j \leq j_0 \). Furthermore, we consider the \( \mathbb{Z} \)-modules \( V^{(t,j)} \) defined in (19) and a linear form \( \delta \) on each \( \text{Hom}_{\mathbb{Z}}(G_m^{t}, G_m^{t}) \), \( 0 \leq t \leq t_0 \), as well as on each \( \text{Hom}_{\mathbb{Z}}(A, A'_j) \), \( 1 \leq j \leq j_0 \). We choose a compact subset \( \mathcal{K}^{(t,j)} = \mathcal{K}^{(t,j)}_{\text{tor}} \times \mathcal{K}^{(t,j)}_{ab} \) in each \( (V^{(t,j)})^\mathbb{R} \) such that the assertion of Lemma 23 is satisfied with \( \mathcal{K} = \mathcal{K}^{(t,j)} \). Since \( \mathcal{K}^{(t,j)}_{\text{tor}} \) (resp. \( \mathcal{K}^{(t,j)}_{ab} \)) is compact and contained in the open subset \( \text{Hom}_{\mathbb{R}}^\circ(G_m^{t}, G_m^{t}) \) (resp. \( \text{Hom}_{\mathbb{R}}^\circ(A, A'_j) \)), the distance between \( \mathcal{K}^{(t,j)}_{\text{tor}} \) (resp. \( \mathcal{K}^{(t,j)}_{ab} \)) and the complement \( \mathcal{K}^{(t,j)}_{\text{tor}} = \text{Hom}_{\mathbb{R}}(G_m^{t}, G_m^{t}) \setminus \text{Hom}_{\mathbb{R}}^\circ(G_m^{t}, G_m^{t}) \) (resp. \( \mathcal{K}^{(t,j)}_{ab} = \text{Hom}_{\mathbb{R}}(A, A'_j) \setminus \text{Hom}_{\mathbb{R}}^\circ(A, A'_j) \)) is strictly positive. We assume that

\[
\delta < \{ \text{dist}(\mathcal{K}^{(t,j)}_{\text{tor}}, \mathcal{K}^{(t,j)}_{\text{tor}}), \text{dist}(\mathcal{K}^{(t,j)}_{ab}, \mathcal{K}^{(t,j)}_{ab}) \}. 
\]

By the triangle inequality, this implies that the distance between \( \mathcal{K}^{(t,j)}_{\delta} = \mathcal{K}^{(t,j)} + B_\delta(0, 0) \) and \( V_\mathbb{R} \setminus V_\mathbb{R}^\circ \) is strictly positive. Consequently, \( \mathcal{K}^{(t,j)}_{\delta} \) is a relatively compact subset of \( V_\mathbb{R}^\circ \).

For any \( t' \in \{ 0, \ldots, t_0 \} \) and \( j \in \{ 1, \ldots, j_0 \} \), we choose pairs

\[
(\phi^{(t,j)}, \phi^{(t,j)}_{ab}) \in (n_\delta^{(t,j)})^{-1} V^{(t,j)}, 1 \leq k \leq k^{(t,j)}(\delta),
\]

such that the conclusion of Lemma 24 is true for these and \( \mathcal{K}^{(t,j)} = \mathcal{K}^{(t,j)}_{\delta} \). Discarding pairs if necessary, we may assume that \( (\phi^{(t,j)}_{\text{tor}}, \phi^{(t,j)}_{ab}) \in \mathcal{K}^{(t,j)}_{\delta} \) and hence that \( (\phi^{(t,j)}_{\text{tor}}, \phi^{(t,j)}_{ab}) \in \mathcal{K}^{(t,j)}_\delta \). Let \( n_\delta \) be the least common multiple of all \( n_\delta^{(t,j)} \) (0 \leq t' \leq t, 1 \leq j \leq j_0). In addition, set \( \Gamma^{(t,j)}_k = \Gamma(n_\delta \cdot \phi^{(t,j)}_{ab}) \subset (\mathbb{P}^1)^t \times (\mathbb{P}^1)^{t_0} \). From Construction 6 we obtain a compactification \( G^{(t,j)} \) endowed with a line bundle \( M^{(t,j)} \) on \( G^{(t,j)} \). Denoting by \( \pi^{(t,j)}_k : G^{(t,j)} \to A \) the projection to the abelian quotient, the line bundle

\[
L^{(t,j)} = M^{\otimes n_\delta} \cdot (\pi^{(t,j)}_k)^* (n_\delta \cdot \phi^{(t,j)}_{ab})^* N_j
\]

is nef by Lemma 8.

With this rearrangement, we now consider a point \( x \in (X \setminus X^{(s)})/\bar{\Q} \cap C(G^{[s]}(\bar{\Q}), \hat{h}_L, \varepsilon) \). Write \( x = y + z \) with \( y \in G^{[s]}(\bar{\Q}) \) and \( \hat{h}_L(z) \leq \varepsilon(1 + \hat{h}_L(y)) \). Assuming \( \varepsilon < 1/4 \), we obtain

\[
\hat{h}_L(y) = \hat{h}_L(x - z) \leq 2\hat{h}_L(x) + 2\hat{h}_L(z) = 2\hat{h}_L(x) + \frac{\hat{h}_L(y)}{2} + \frac{1}{2}
\]

by using Lemma 8 (b) and Lemma 11 for the first inequality. Hence, we have that

\[
\hat{h}_L(y) \leq 4\hat{h}_L(x) + 1 \quad \text{and} \quad \hat{h}_L(z) \leq \varepsilon(4\hat{h}_L(x) + 2).
\]

By Lemma 23 there exists a quasi-homomorphism \( \phi_0 : G \to \hat{G}_0 \) represented by some \( (\phi_{0,\text{tor}}, \phi_{0,ab}) \in \mathcal{K}^{(t,j)}_{\delta} \) such that \( y \in \ker(\phi_0) + \text{Tors}(G) \). We compactify \( G_0 \) by \( \overline{G}_0 \) as in Construction 5 and endow \( \overline{G}_0 \) with the ample line bundle (cf. Lemma 8)

\[
L_0 = M^{\otimes n_\delta} \cdot (\pi^{(t,j)}_0)^* N_j,
\]

where \( \pi^{(t,j)}_0 : \overline{G}_0 \to A'_j \) denotes the usual projection. Our choice of the pairs (51) allows us to pick a pair \( (\phi^{(t,j)}_{\text{tor}}, \phi^{(t,j)}_{ab}) \) with \( (\phi_{\text{tor}}, \phi_{ab}) \in B_\delta(\phi^{(t,j)}_{\text{tor}}, \phi^{(t,j)}_{ab}) \).

To avoid amassing sub- and superscripts in the sequel, we write \( L_1 \) (resp. \( M_1, \pi_1, \phi_{1,\text{tor}}, \phi_{1,ab}, V, \mathcal{K} \)) for \( L^{(t,j)} \) (resp. \( M^{(t,j)}_k, \pi^{(t,j)}_k, \phi^{(t,j)}_{\text{tor}}, \phi^{(t,j)}_{ab}, V^{(t,j)}, \mathcal{K}^{(t,j)} \)). Let \( n \) be a denominator of...
\(\phi \) (i.e., \( n \) is an integer such that \( \psi_0 = n \cdot \phi \) \( \in \text{Hom}(G, G_0) \)). We also write \((\psi_{1,\text{tor}}, \psi_{1,\text{ab}})\) for \( n_{\delta} \cdot (\phi_{1,\text{tor}}, \phi_{1,\text{ab}}) \in V \).

We can now demonstrate the first of the two announced height bounds.

**Lemma 25.** There is some constant \( c_{10} > 0 \) such that
\[
(53) \quad \hat{h}_{L_1}(x) \leq c_{10} n_{\delta}^2 (\delta + \varepsilon) (\hat{h}_L(x) + 1).
\]

**Proof.** From Lemma 14 we know the estimate
\[
(54) \quad \hat{h}_{L_1}(x) \leq 2\hat{h}_{L_1}(y) + 2\hat{h}_{L_1}(z).
\]

We may hence bound \( \hat{h}_{L_1}(y) \) and \( \hat{h}_{L_1}(z) \) separately. Recall that
\[
\hat{h}_L(y) = \hat{h}_{M_{\psi}^n}(y) + \hat{h}_{M_{\psi}^{\psi_{\text{tor}}}}(y)
\]
and note that
\[
\hat{h}_{L_1}(y) = n_{\delta} \hat{h}_{M_{\psi}^n}(y) + \hat{h}_{M_{\psi}^{\psi_{\text{tor}}}}(y)
\]
by Lemma \( \Box \) (b, c). Let \( c_1(t, j) \) and \( c_2(t, j) \) be the constants \( c_1 \) and \( c_2 \) of Lemma 14 if applied to \( G = G, N_0 = N, t = G_m, A_1 = A', N_1 = N_j \) and our fixed linear norms \( \cdot \) \( \cdot \) \( \text{Hom}_A(G_m, G_m') \) and \( \text{Hom}_G(A, A') \). Set \( c_1 = \max_{t, j} \{ c_1(t, j) \} \) and \( c_2 = \max_{t, j} \{ c_2(t, j) \} \). Comparing Constructions \( \Box \) and \( \Box \) we infer \( \psi_0 \cdot M_{G_0} \approx M_{\psi_{\text{tor}} N_j} \). From Construction \( \Box \) we also know the homogeneities
\[
M_{G_0}^n = M_{W_1}^{\psi_{\text{tor}}} \cdot \gamma \psi_{\text{tor}} = M_{F(\psi\text{tor})}^n \quad \text{and} \quad \psi_0 M_{G_0}^n \approx M_{W_1}^{\psi_{\text{tor}}} \cdot \gamma \psi_{\text{tor}} = M_{F(\psi\text{tor})}^n,
\]

implying
\[
(55) \quad n \hat{h}_{M_1}(y) = \hat{h}_{M_{\psi_{\text{tor}}} (\psi_0 \cdot n \cdot \psi_{\text{tor}})}(y)
\]
and
\[
(56) \quad |n^2 \hat{h}_{\psi_{\text{tor}}}^{\psi_{\text{tor}}} N_j(y) - n^2 \hat{h}_{\psi_{\text{tor}}}^{\psi_{\text{tor}}} N_j(y)| < c_2|n \cdot \psi_{\text{tor}} - n_{\delta} \cdot \psi_{\text{tor}}| \hat{h}_{M_{\psi_{\text{tor}}} N_j}(y).
\]

Invoking Lemma 14 for \( n \cdot (\psi_{\text{tor}}, \psi_{\text{ab}}) \) and \( n_{\delta} \cdot (\psi_{\text{tor}}, \psi_{\text{ab}}) \) yields
\[
(57) \quad \hat{h}_{L_0}(\psi_0(y)) = \hat{h}_{M_{\psi_{\text{tor}}} N_j}(\psi_0(y)) = 0.
\]

With Lemma \( \Box \) we obtain \( \hat{h}_{\psi_{\text{tor}}}^{\psi_{\text{tor}}} N_j(y) = 0 \) and \( \hat{h}_{\psi_{\text{tor}}}^{\psi_{\text{tor}}} N_j(y) = \hat{h}_{\psi_{\text{tor}}}^{\psi_{\text{tor}}} N_j(y) = 0 \) from \( \Box \). Since
\[
(58) \quad n \cdot (\psi_{\text{tor}}, \psi_{\text{tor}}) - n_{\delta} \cdot (\psi_{\text{tor}}, \psi_{\text{ab}}) = n_{\delta} \cdot ((\phi_{\text{tor}}, \phi_{\text{tor}}) - (\phi_{\text{tor}}, \phi_{\text{ab}}));
\]
we may cancel \( n \) (resp. \( n^2 \)) in \( \Box \) (resp. \( \Box \)) and obtain
\[
(59) \quad \hat{h}_{L_1}(y) < c_{11} n_{\delta} \hat{h}_L(y) \leq c_{11} n_{\delta}^2 (4\hat{h}_L(x) + 1)
\]
for some constant \( c_{11} > 0 \). Applying Lemma 14 to \( (\psi_{\text{tor}}, \psi_{\text{tor}}) \) and \( (0, 0) \) \( \in V \), we obtain similarly
\[
(60) \quad \hat{h}_{L_1}(z) < c_{12} (n_{\delta} \cdot (\psi_{\text{tor}})^{\psi_{\text{tor}}} + |\psi_{\text{tor}}|^2) \hat{h}_L(z)
\]
for some constant \( c_{12} > 0 \). As \( K_{\delta} \subset V_{\psi}^2 \) is relatively compact, there exists a constant \( c_{13} > 1 \) such that
\[
(61) \quad c_{13} \hat{h}_{L_1}(z) < 2c_{12} n_{\delta}^2 \hat{h}_L(z) \leq 4\varepsilon c_{12} n_{\delta}^2 (2\hat{h}_L(x) + 1).
\]
Finally, (53) follows from combining (54), (59) and (62).

Our second height bound is a consequence of Siu’s numerical bigness criterion (Corollary 1.2). Recall from (15) the maps \( t = t_{T_k^P} : G \to G_{T_k^P} \) and \( q = q_{T_k^P} : G_{T_k^P} \to T \). The idea is to compare the line bundles \( L_1 \) and \( q^*L \) on the Zariski closure \( X \) of \( i(X) \subset G_{T_k^P} \). Set \( r = \dim(X) \geq 1 \) and

\[
\alpha = \frac{\deg(c_1(L_1)^r \cap [X])}{\max\{1, 2r \deg(c_1(L_1)^{r-1}(q^*L) \cap [X])\}}.
\]

We note that both \( L_1 \) and \( q^*L \) are nef.

**Lemma 26.** There exists a non-empty Zariski open subset \( U_\delta \subseteq X \) and a constant \( c_{14}(\delta) \), both depending on \( \delta \), such that

\[
\alpha \hat{h}_L(x) - c_{14}(\delta) \leq \hat{h}_{L_1}(x)
\]

if \( x \in U_\delta(Q) \).

**Proof.** If \( \deg(c_1(L_1)^r \cap [X]) = 0 \), then \( \alpha = 0 \) and there is nothing left to prove because \( \hat{h}_{L_1}(x) \) is non-negative by Lemma 9. Hence, we may and do assume \( \deg(c_1(L_1)^r \cap [X]) \neq 0 \). As \( L_1 \) is nef, this actually means \( \deg(c_1(L_1)^r \cap [X]) > 0 \). Set

\[
u = \deg(c_1(L_1)^r \cap [X]) \text{ and } v = \max\{1, 2r \deg(c_1(L_1)^{r-1}(q^*L) \cap [X])\}.
\]

This is arranged so that

\[
\deg(c_1(L_i^{\otimes \nu})^r \cap [X]) = \nu^r \deg(c_1(L_1)^r \cap [X]) \geq 2r \nu^{r-1} u \deg(c_1(L_1)^{r-1}(q^*L) \cap [X]) = 2r \deg(c_1(L_i^{\otimes \nu})^{r-1}c_1(q^*L^{\otimes u}) \cap [X]).
\]

Thus, Siu’s criterion as stated in [36] Theorem 2.2.15] implies that \( L_2 = (L_i^{\otimes \nu} \otimes q^*L^{\otimes(u)})^{\otimes 1}(X) \) is big. In particular, some power of \( L_2 \) is effective. By [26] Theorem B.3.6, there exists a non-empty Zariski-open set \( U_\delta \subseteq X \) and a constant \( c_{15}(L_2) > 0 \) such that

\[-c_{15}(L_2) \leq \hat{h}_{L_1^{\otimes \nu} \otimes q^*L^{\otimes(u)}}(x) \]

for all \( x \in U_\delta(Q) \). For a fixed \( \delta > 0 \), we wind up here with finitely many choices for \( X \subset G_{T_k^P} \) and the line bundles \( L_1 \) and \( q^*L \) on \( G_{T_k^P} \). As \( L_2 \) can be determined from this data, we can hence replace \( c_{15}(L_i^{\otimes \nu}) \) by some constant depending only on \( \delta \). Combining this fact with Lemma 3 (a), we conclude the existence of some constant \( c_{16}(\delta) > 0 \) such that

\[-c_{16}(\delta) \leq \hat{h}_{L_1^{\otimes \nu} \otimes q^*L^{\otimes(u)}}(x) \]

whenever \( x \in U_\delta(Q) \). Inequality (64) follows immediately by using Lemma 3 (c), (17) and \( \alpha = u/v \).

It remains to bound the quantity \( \alpha \) from below.

**Lemma 27.** There exists a constant \( c_{17} > 0 \) such that \( \alpha \geq c_{17}n_\delta^2 \).

**Proof.** We first define auxiliary functions \( \beta_i, 0 \leq i \leq r \), and \( \gamma_i, 0 \leq i \leq r-1 \), on \( V_Q \). Once again, we use that Construction 6 gives for each \( \varphi_{\text{tor}} \in \text{Hom}(G_{m}\Gamma_{l}, G_{m}\Gamma_{l}') \) a compactification \( G_{\varphi_{\text{tor}}} \) of \( G \) with abelian quotient \( \pi_{\Gamma_{l}^{\varphi_{\text{tor}}}} : G_{\Gamma_{l}^{\varphi_{\text{tor}}}} \to A_j' \) and a line bundle \( M_{\Gamma_{l}^{\varphi_{\text{tor}}}} \) on \( G_{\Gamma_{l}^{\varphi_{\text{tor}}}} \). We use the notations introduced in (15). In addition, we let \( X_{\Gamma_{l}^{\varphi_{\text{tor}}}} \) be the Zariski closure of \( i_{\Gamma_{l}^{\varphi_{\text{tor}}}}(X) \) in \( G_{\Gamma_{l}^{\varphi_{\text{tor}}}} \). For any \( \varphi_{\text{tor}}, \varphi_{ab} \in V_Q \), we can now define

\[
\beta_i(\varphi_{\text{tor}}, \varphi_{ab}) = (|\varphi_{\text{tor}}| + |\varphi_{ab}|)^i \deg\left(c_1(M_{\Gamma_{l}^{\varphi_{\text{tor}}}})^{\otimes i}\left(c_1((\varphi_{ab} \circ \pi_{\Gamma_{l}^{\varphi_{\text{tor}}}})^*N_j)^{\otimes i} \cap [X_{\Gamma_{l}^{\varphi_{\text{tor}}}}]\right)\right)
\]

and

\[
\gamma_i(\varphi_{\text{tor}}, \varphi_{ab}) = (|\varphi_{\text{tor}}| + |\varphi_{ab}|)^i \deg\left(c_1(M_{\Gamma_{l}^{\varphi_{\text{tor}}}})^{\otimes i}\left(c_1((\varphi_{ab} \circ \pi_{\Gamma_{l}^{\varphi_{\text{tor}}}})^*N_j)^{\otimes i} \cap [X_{\Gamma_{l}^{\varphi_{\text{tor}}}}]\right)\right).
\]
This defines a \( \mathbb{Z} \)-homogeneous function \( \beta_i \) (resp. \( \gamma_i \)) of degree 2r (resp. 2r - 2) on \( V \). To prove this, we recall that Construction 6 provides a finite morphism \( \theta_{n, \varphi_{\tor}} : G_{\Gamma(\varphi_{\tor})} \to G_{\Gamma(n \varphi_{\tor})} \) such that \( \varphi^*_{\varphi_{\tor}, n} M_{\Gamma(\varphi_{\tor})} \approx M_{\Gamma(n \varphi_{\tor})}^{\otimes n} \). The homogeneity relation follows from the projection formula (cf. [16, Proposition 2.5 (c)]) by using the straightforward relations (18), which gives the desired upper bound on \( \hat{h}_{\varphi_{\tor}}(x) \).

Lemma 27 allows us to make (64) precise: There exists a non-empty Zariski open \( U_\delta \subset X \) such that

\[
c_{17}n^2_3 \hat{h}_L(x) - c_{14}(\delta) \leq \hat{h}_{L_1}(x)
\]

whenever \( x \in U_\delta(\mathbb{Q}) \). Combining this with (63), we obtain

\[
c_{17}n^2_3 \hat{h}_L(x) - c_{14}(\delta) \leq c_{10}n^2_3(\delta + \varepsilon)(\hat{h}_L(x) + 1).
\]

Canceling \( n^2_3 \), this can be rewritten as

\[
(c_{17} - c_{10}(\delta + \varepsilon))\hat{h}_L(x) \leq n^2_3 c_{14}(\delta) + c_{10}(\delta + \varepsilon).
\]

This inequality gives the desired upper bound on \( \hat{h}_L(x) \) if

\[
\max\{\delta, \varepsilon\} < \frac{1}{2}c_{10}^{-1}c_{17}.
\]

Consequently, Proposition 21 is proven up to Lemmas 28 and 29, whose proofs are provided next in Section 7.4.

7.4. Bounds on intersection numbers. The reader may profitably compare our derivation of Lemma 28 with the lengthy one of [23] Proposition 4] to appreciate the technical advantage provided by using Chern forms. In fact, our argument is particularly simple if \( G \) is an abelian variety because most of Section 5 is not needed in this case and only the functions \( \beta_0 \) and \( \gamma_0 \) are non-zero.

Lemma 28. Assume \( X^{(s)} \neq X \). There exists a constant \( c_{18} > 0 \) such that

\[
\max_{0 \leq i \leq r}\{\beta_i(\phi_{\tor}, \phi_{\ab})\} \geq c_{18}
\]

for all \( (\phi_{\tor}, \phi_{\ab}) \in \mathcal{K}_\delta \cap V_\mathbb{Q} \).
Before starting the proof, let us recall the compatibility between algebraic Chern classes and analytic Chern forms on proper complex algebraic varieties. By [16, Proposition 19.1.2], there is the expectable compatibility between algebraic Chern classes and their topological counterparts acting on singular homology. Furthermore, the topological Chern class of a line bundle is given by its Chern form (see e.g. [18, Proposition on p. 141]).

**Proof.** Since \( K_\delta \) is a relatively compact subset of \( V^c_\mathbb{R} \), it suffices to prove the following claim: For each \((\phi_\text{tor}, \phi_\text{ab}) \in V^c_\mathbb{R}\), there exists a euclidean neighborhood \( U \subset V^c_\mathbb{R} \) of \((\phi_\text{tor}, \phi_\text{ab})\) and a constant \( c_{20}(\phi_\text{tor}, \phi_\text{ab}) > 0 \) such that

\[
\max_{0 \leq i \leq r} \{ \beta_i(\phi_\text{tor}, \phi_\text{ab}) \} \geq c_{20}(\phi_\text{tor}, \phi_\text{ab})
\]

for all \((\phi_\text{tor}, \phi_\text{ab}) \in U \cap V_Q\).

In order to prove this claim, let \((\phi_\text{tor}, \phi_\text{ab}) \in V_Q\) and let \( n \) denote a denominator for \((\phi_\text{tor}, \phi_\text{ab})\). In Section [3] the line bundle \( M_{\Gamma(n, \phi_\text{tor})} \) is endowed with a hermitian metric such that

\[
c_1(M_{\Gamma(n, \phi_\text{tor})}) = \omega(n \cdot \phi_\text{tor}).
\]

Similarly, the line bundle \( N_j \) is endowed with a hermitian metric such that \( c_1(N_j) = \omega(N_j; n \cdot \phi_\text{ab}) \). These hermitian line bundles can be used to express \( \beta_i(\phi_\text{tor}, \phi_\text{ab}) \) analytically; to wit, \( \beta_i(\phi_\text{tor}, \phi_\text{ab}) = n^{-2r} \beta_i(n \cdot \phi_\text{tor}, n \cdot \phi_\text{ab}) \) and

\[
\beta_i(n \cdot \phi_\text{tor}, n \cdot \phi_\text{ab}) = \left( |n \cdot \phi_\text{tor}| + |n \cdot \phi_\text{ab}| \right)^r \int_{X_{\Gamma(n, \phi_\text{tor})}(\mathbb{C})} \omega(n \cdot \phi_\text{tor})^l \wedge (\bar{\omega}(N_j; n \cdot \phi_\text{ab})^{r-1}.
\]

Since each \( \beta_i \) is a non-negative function, it suffices to prove that there exists a positive constant \( c_{21}(\phi_\text{tor}, \phi_\text{ab}) \) and a neighborhood \( U \) of \((\phi_\text{tor}, \phi_\text{ab})\) such that

\[
n^{2r} \sum_{s=0}^{r} \binom{r}{s} \beta_i(\phi_\text{tor}, \phi_\text{ab})
\]

\[
= \int_{X_{\Gamma(n, \phi_\text{tor})}(\mathbb{C})} \left( |n \cdot \phi_\text{tor}| + |n \cdot \phi_\text{ab}| \right) \omega(n \cdot \phi_\text{tor}) + (\bar{\omega}(N_j; n \cdot \phi_\text{ab})^{r-1}
\]

exceeds \( n^{2r} c_{21}(\phi_\text{tor}, \phi_\text{ab}) \) for any \((\phi_\text{tor}, \phi_\text{ab}) \in U \cap V_Q\) with denominator \( n \). As the boundary \( X_{\Gamma(n, \phi_\text{tor})}(\mathbb{C}) \setminus \pi_{\Gamma(n, \phi_\text{tor})}^{-1}(X)(\mathbb{C}) \) has measure zero, the integral in (68) equals

\[
\int_{X(\mathbb{C})} \left( |n \cdot \phi_\text{tor}| + |n \cdot \phi_\text{ab}| \right) \omega(n \cdot \phi_\text{tor}) + (\bar{\omega}(N_j; n \cdot \phi_\text{ab})^{r-1},
\]

which by Lemma [16] and (42) simplifies to

\[
n^{2r} \int_{X(\mathbb{C})} \left( |\phi_\text{tor}| + |\phi_\text{ab}| \right) \omega(\phi_\text{tor}) + (\bar{\omega}(N_j; \phi_\text{ab})^{r-1}.
\]

It remains to show that the integral

\[
\int_{X(\mathbb{C})} \left( |\phi_\text{tor}| + |\phi_\text{ab}| \right) \omega(\phi_\text{tor}) + \pi^* \omega(N_j; \phi_\text{ab})^{r-1}
\]

is bounded from below by a positive constant \( c_{21}(\phi_\text{tor}, \phi_\text{ab}) \) for all \((\phi_\text{tor}, \phi_\text{ab}) \in V_Q\) in a neighborhood \( U \) of \((\phi_\text{tor}, \phi_\text{ab})\). In the sequel, we write

\[
\omega(\phi_\text{tor}, \phi_\text{ab}) = \left( |\phi_\text{tor}| + |\phi_\text{ab}| \right) \omega(\phi_\text{tor}) + \pi^* \omega(N_j; \phi_\text{ab})
\]

for any \((\phi_\text{tor}, \phi_\text{ab}) \in V_\mathbb{R}\). From Section [3] we know that each \( \omega(\phi_\text{tor}, \phi_\text{ab}) \) is a semipositive \((1, 1)\)-form of real type. Furthermore, our assumption \( X \neq X^{(s)} \) implies \( (\omega(\phi_\text{tor}, \phi_\text{ab})|_X)^{\dim(X) \neq 0} \) by Lemma [20] (with \( c = |\phi_\text{tor}| + |\phi_\text{ab}| \)). We infer from this the existence of a non-empty relatively compact open set \( K \) such that \( (\omega(\phi_\text{tor}, \phi_\text{ab})|_X)^{\dim(X) \neq 0} \) is a positive volume form for each \( y \in K \). By continuity of \( \omega(\phi_\text{tor}, \phi_\text{ab}) \) with respect to \((\phi_\text{tor}, \phi_\text{ab})\) and compactness, there exists an open neighborhood \( U \subset V_\mathbb{R} \) such that

\[
\omega(\phi_\text{tor}, \phi_\text{ab})^{\dim(X)} - \frac{1}{2} \omega(\phi_\text{tor}, \phi_\text{ab})^{\dim(X)}
\]
restricts to a positive volume form on each $T_{x,y}X^{sm}(\mathbb{C})$, $y \in K$. Using the semipositivity of $\omega(\phi_{tor}, \phi_{ab})$, we obtain that (69) is bounded from below by

$$\int_K \omega(\phi_{tor}, \phi_{ab})^{\dim(X)} \geq \frac{1}{2} \int_K \omega(\phi'_{tor}, \phi'_{ab})^{\dim(X)} = c_21(\phi'_{tor}, \phi'_{ab}) > 0.$$

This proves our claim.

Lemma 29. There exists a constant $c_{22} > 0$ such that

$$\max_{0 \leq i \leq r-1} \{ \gamma_i(\phi_{tor}, \phi_{ab}) \} \leq c_{22}$$

for all $(\phi_{tor}, \phi_{ab}) \in K_\delta \cap V_Q$.

It is tempting to provide a proof resembling the one of Lemma 28. In fact, we can reduce the statement of the lemma to bounds on certain integrals of volume forms on $X(\mathbb{C})$ that vary continuously with $(\phi_{tor}, \phi_{ab})$. If $X(\mathbb{C})$ were compact (e.g. because $G = A$ is an abelian variety), the above lemma could be immediately inferred from this continuity. However, non-compactness of $X(\mathbb{C})$ precludes such a direct argument in the general case. We circumvent these problems by using algebraic intersection theory [16] instead. This resembles the proof of [23, Lemma 3.3] by a multiprojective version of Bézout’s Theorem. We use the standard notation from [16] freely.

Proof. Consider a fixed $(\phi_{tor}, \phi_{ab}) \in K_\delta \cap V_Q$ with denominator $n$. By compactness, $(|\phi_{tor}| + |\phi_{ab}|)$ is bounded on $K_\delta$. It suffices to bound

$$\deg(c_1(M_{G(\phi_{tor})})^i c_1((|n \cdot \phi_{ab}| \cap \mathbb{P}(\phi_{tor}))^* N_j)^{r-1-i} c_1(q_{G(\phi_{tor})}^*(\phi'_{tor}) L \cap [X_{G(\phi_{tor})}])$$

by $n^{2r-2}c_{22}$ because $\gamma_i(\phi_{tor}, \phi_{ab})$ is homogeneous of degree $2r - 2$. As in the proof of Lemma 28, it is enough to demonstrate that

$$(71) \quad \deg(c_1(M_{G(\phi_{tor})} \otimes (|n \cdot \phi_{ab}| \cap \mathbb{P}(\phi_{tor}))^* N_j)^{r-1-i} c_1(q_{G(\phi_{tor})}^*(\phi'_{tor}) L \cap [X_{G(\phi_{tor})}]))$$

is bounded by $n^{2r-2}c_{22}$.

Let $G'$ be the semiabelian variety described by $\eta_{G'} = (n \cdot \phi_{tor}), \eta_G \in \text{Ext}_G^1(A, G_{m}^n)$. From Construction 4 we recall the compactification $\overline{G}$ (resp. $G'$) of $G$ (resp. $G'$) with its abelian quotient $\pi : \overline{G} \to A$ (resp. $\pi' : \overline{G'} \to A$) and the line bundle $M_{G(\pi)}$ (resp. $M_{G'(\pi)}$) on $\overline{G}$ (resp. $\overline{G'}$). The Zariski closure of $X$ in $\overline{G}$ is denoted $\overline{X}$. Then, $L = M_{\overline{G}} \otimes \mathbb{P}^i N$ and we also set

$$L' = M_{\overline{G}'} \otimes (n \cdot \phi_{ab}) \cap \mathbb{P}^{i} N.$$

The homomorphism $(\text{id}_{G_{m}^n}, n \cdot \phi_{tor}) : G_{m}^n \to G_{m}^n \times G_{m}^n$ extends to a $(\text{id}_{G_{m}^n}, n \cdot \phi_{tor})$-equivariant map $G(n \cdot \phi_{tor}) \to (\mathbb{P}^1)^i \times (\mathbb{P}^1)^i$, yielding a closed embedding $\iota : G_{G(\phi_{tor})} \to \overline{G} \times \overline{G}$ by means of the constructions in Section 2. Furthermore, the line bundle $q_{G(\phi_{tor})}^*(\phi'_{tor}) L$ (resp. $M_{G(\phi_{tor})}$) on $G_{G(\phi_{tor})}$ coincides with the pullback $\iota^{*}p_{1}^{*} L$ (resp. $\iota^{*}p_{2}^{*} M_{G}$). Using the projection formula ([16], Proposition 2.5.c]), we infer that (71) equals the degree of

$$c_1(p_{1}^{*} L)c_1(p_{2}^{*} L')^{r-1} \cap [\iota(X_{G(\phi_{tor})})] \in A_{0}(\overline{G} \times \overline{G}).$$

To estimate this degree, we use suitable projective embeddings $\overline{G} \hookrightarrow \mathbb{P}^{1}$ and $\overline{G}' \hookrightarrow \mathbb{P}^{2}$. By Lemma 3, the line bundles $L$ and $L' = L' \otimes (\mathbb{P})^* N$ are ample. Consequently, there exists an integer $l_1$ such that $L^{\otimes l_1}$ is very ample. Since $L$ is independent of $(\phi_{tor}, \phi_{ab})$, we can choose $l_1$ less than some constant $c_{22}$ that only depends on $G$ and $X$. The line bundle $(L')^{\otimes l_2}$ is very ample if $l_2$ is sufficiently large; in contrast to $l_1$, there is an implicit dependence on $(\phi_{tor}, \phi_{ab})$ here. These very ample line bundles determine projective embeddings $\iota_1 : G \hookrightarrow \mathbb{P}^{1}$ and $\iota_2 : G' \hookrightarrow \mathbb{P}^{2}$ such that $\iota_1^{*}O_{\mathbb{P}^{1}}(1) \approx L^{\otimes l_1}$ and $\iota_2^{*}O_{\mathbb{P}^{2}}(1) \approx (L')^{\otimes l_2}$. Setting $\kappa = (\iota_1 \times \iota_2) \circ \iota$, we continue by estimating the degree of

$$c_1(p_{1}^{*} O_{\mathbb{P}^{1}}(1))c_1(p_{2}^{*} O_{\mathbb{P}^{2}}(1))^{r-1} \cap [\kappa(X_{G(\phi_{tor})})] \in A_{0}(\mathbb{P}^{1} \times \mathbb{P}^{2}).$$
If it is shown that the degree of (73) is less than \( l_1 l_2^{r-1} n^{2r-2} c_{23} \), the desired degree bound on (72) follows immediately. In fact, the degree of (73) equals the degree of

\[
l_1 l_2^{r-1} c_1 (pr_1^* L) (c_1 (pr_2^* L') + c_1 (pr_2^* (\pi^*)^* N))^{r-1} \cap \mathcal{I}(X_{\Gamma (n \cdot \phi_{\text{tor}})}) \in A_0 (G \times G)
\]

by the projection formula. By Lemma 5 the line bundles \( pr_1^* L \), \( pr_2^* L' \) and \( pr_2^* (\pi^*)^* N \) are nef so that this can be expanded into a sum of zero-cycle classes with non-negative degrees (see 29 Theorem III.2.1). Since one of the summands is a \((l_1 l_2^{r-1})\)-multiple of (72), the reduction is clear.

The variety \( \kappa (X_{\Gamma (n \cdot \phi_{\text{tor}})}) \) is an irreducible component of \( \kappa (G_{\Gamma (n \cdot \phi_{\text{tor}})}) \cap (\iota_1 (X) \times \mathbb{P}^r) \subset \mathbb{P}^r \times \mathbb{P}^r \).

In fact, both are subvarieties of \( \iota_1 (G) \times \iota_2 (G') \) whose restrictions to the open dense subset \( \iota_1 (G) \times \iota_2 (G') \) coincide with \( \kappa (X) \). Choose hypersurfaces \( S_1, S_2, \ldots, S_k \subset \mathbb{P}^n \) such that \( \iota_1 (X) = S_1 \cap S_2 \cap \ldots \cap S_k \) as varieties (i.e., set-theoretically). As \( X \) is irreducible, we can select a subset \( \{ S_{k_1}, \ldots, S_{k_{\dim (G) - r}} \} \) of these hypersurfaces such that \( \kappa (X_{\Gamma (n \cdot \phi_{\text{tor}})}) \) is an irreducible component of

\[
\kappa (G_{\Gamma (n \cdot \phi_{\text{tor}})}) \cap (S_{k_1} \times \mathbb{P}^r) \cap \ldots \cap (S_{k_{\dim (G) - r}} \times \mathbb{P}^r).
\]

For reasons of dimension (cf. 16 Lemma 7.1 (a) and 16 Example 8.2.1), we have

\[
\iota (\kappa (X_{\Gamma (n \cdot \phi_{\text{tor}})})), \kappa (G_{\Gamma (n \cdot \phi_{\text{tor}})}) \cdot (S_{k_1} \times \mathbb{P}^r) \cdots (S_{k_{\dim (G) - r}} \times \mathbb{P}^r) ; \mathbb{P}^r \times \mathbb{P}^r \geq 1.
\]

It is well-known (compare 16 Section 12.3) that the tangent vector bundle \( T^{\mathbb{P}^r \times \mathbb{P}^r} = pr_1^* (T^{\mathbb{P}^r}) \oplus pr_2^* (T^{\mathbb{P}^r}) \) is ample and hence globally generated. By 16 Corollary 12.2 (a), every distinguished subvariety contributes a non-negative cycle to the intersection product in (74). The degree of the 0-cycle class (73) is hence majorized by the degree of the 0-cycle class

\[
c_1 (pr_1^* O_{\mathbb{P}^r (1)}) \cap [H_1^i \times H_2^j] \mathcal{I}(pr_2^* (\pi^*)^* N),
\]

on \( \mathbb{P}^r \times \mathbb{P}^r \). The Chow ring \( A_4 (\mathbb{P}^r \times \mathbb{P}^r) \) is of the form

\[
\mathbb{Z}[H_1]/([H_1]^{r+1}) \otimes \mathbb{Z}[H_2]/([H_2]^{r+1})
\]

for any two hyperplanes \( H_1 \subset \mathbb{P}^r \) and \( H_2 \subset \mathbb{P}^r \) (see 16 Example 8.3.7). Thus, we may write \( [S_{r_1 + r_2}] = d_{r_1} [H_1^i] \times [H_2^j] \) and

\[
[\kappa (G_{\Gamma (n \cdot \phi_{\text{tor}})})] = \sum_{i_1 + i_2 = r_1 + r_2 - \dim (G)} e_{i_1, i_2} [H_1^{i_1} \times H_2^{i_2}].
\]

Furthermore, the definition of the first Chern class immediately implies that

\[
c_1 (pr_1^* O_{\mathbb{P}^r (1)}) \cap [H_1^i \times H_2^j] = \begin{cases} [H_1^{i+1} \times H_2^{j}] & \text{if } i = 1 \\ [H_1^{i} \times H_2^{j+1}] & \text{if } i = 2. \end{cases}
\]

With these notations, the degree of (73) is

\[
d_{k_1} \cdots d_{k_{\dim (G) - r}} \epsilon_{r_1 + r - \dim (G) - 1, r_2 - r + 1} \leq \max_{K \subseteq \{1, \ldots, k\}} \left( \prod_{k \in K} d_k \right) \cdot \epsilon_{r_1 + r - \dim (G) - 1, r_2 - r + 1}.
\]

Additionally, we have

\[
\epsilon_{r_1 + r - \dim (G) - 1, r_2 - r + 1} = \deg (c_1 (pr_1^* O_{\mathbb{P}^r (1)})^{\dim (G) + 1 - r} c_1 (pr_2^* O_{\mathbb{P}^r (1)})^{r-1} \cap [\kappa (G_{\Gamma (n \cdot \phi_{\text{tor}})})],
\]

which is less than

\[
c_{24}^{\dim (G) + 1 - r} l_2^{r-1} \deg (c_1 (pr_1^* L)^{\dim (G) + 1 - r} c_1 (pr_2^* L')^{r-1} \cap \mathcal{I}(G_{\Gamma (n \cdot \phi_{\text{tor}})})]
\]

by the projection formula. To ease our exposition notationally, we write \( \alpha_1 = c_1 (pr_1^* M_{\Gamma}) \), \( \alpha_2 = c_1 (pr_2^* M_{\Gamma}) \), \( \beta_1 = c_1 (pr_1^* (\pi^*)^* N) \), \( \beta_2 = c_1 (pr_2^* ((n \cdot \phi_{\text{ab}}) \circ (\pi^*)^* N_j) \), \( \beta_3 = c_1 (pr_2^* (\pi^*)^* N) \) and
\[ r' = \dim(G) + 1 - r \text{ in the following computations. Then,} \]
\[ c_1(\pr_1^r L)^r c_1(\pr_2^r L'_r)^{-1} = (\alpha_1 + \beta_1)^r (\alpha_2 + \beta_2 + \beta_3)^{-1} \]
\[ = \left( \sum_{s_1=0}^{r} \binom{r'}{s_1} \alpha_1^s \beta_1^{r'-s} \right) \left( \sum_{s_2=0}^{r-1} \binom{r-1}{s_2} \alpha_2^s \beta_2 \beta_3^{r-1-s} \right). \]

For each positive integer \( n \), the isogeny \([n]_G : G \rightarrow G\) of degree \( n^{t+2 \dim(A)} \) extends to a proper map \([n]_G : G \rightarrow G\) such that \([n]_G [\Gamma(n \cdot \phi_{tor})]_G = n^{t+2 \dim(A)} [\Gamma(n \cdot \phi_{tor})]_G\).

Furthermore, pulling back the line bundles \( \pr_1^p M_{\sigma} \) and \( \pr_2^s M_{\sigma} \) (resp. \( \pr_1^p (\Pi) \cdot N_j \) and \( \pr_2^s (\Pi) \cdot N_j \)) along \([n]_G\) amounts to rising them to the \( n \)-th (resp. \( n^2 \)-th) power. Therefore, the projection formula (applied to \([n]_G\)) yields that
\[ \text{deg}(\alpha_1^s \alpha_2^s \beta_1^{r'-s} - \beta_2 \beta_3^{r-1-s}) \cap [\Gamma(n \cdot \phi_{tor})]_G = \text{deg}(\alpha_1^s \alpha_2^s \beta_1^{r'-s} - \beta_2 \beta_3^{r-1-s}) \cap [\Gamma(n \cdot \phi_{tor})]_G. \]

It follows that (77) is zero whenever \( s_1 + s_2 \neq t \). Hence, the quantity (76) can be rewritten as
\[ \text{deg}(\alpha_1^s \alpha_2^s \beta_1^{r'-s} \beta_2 \beta_3^{r-1-s}) \cap [\Gamma(n \cdot \phi_{tor})]_G \]

is bounded from above by \( c_{25} n^{2r-2} \) for some constant \( c_{25} \).

As \( \pi_1^{\phi \lor} \) is flat of relative dimension \( t \), we can pull back cycle classes on \( A \) to cycle classes on \( \Gamma(n \cdot \phi_{tor}) \). In particular, we have \( \pi_1^{\phi \lor}(A) = [\Gamma(n \cdot \phi_{tor})]_G \) and \( \pi_2^s (\phi \lor)(\bar{p}) = [\pi_1^{\phi \lor}(\bar{p})]_G \)
for any point \( p \in A \). Setting
\[ \sigma_s = c_1([N]) \cdot c_1((n \cdot \phi_{tor}) \cdot N_j) + c_1([N]) \]
we know from (76) Proposition 2.5 (d)] that there exist points \( p_1, ..., p_{\deg(\alpha_1^{m})}, q_1, ..., q_m \in A \) such that
\[ \text{deg}(\alpha_1^s \alpha_2^s \beta_1^{r'-s} \beta_2 \beta_3^{r-1-s}) \cap [\Gamma(n \cdot \phi_{tor})]_G = \sum_{l=1}^{\deg(\alpha_2^{m})} [\pi_1^{\phi \lor}(\bar{p}_l)]_G - \sum_{l=1}^{\deg(\alpha_2^{m})} [\pi_1^{\phi \lor}(\bar{q}_l)]_G. \]

By construction, there exists a non-canonical isomorphism between each fiber \( \pi_1^{\phi \lor}(x), \]
\( x \in A, \) and \( \Gamma(n \cdot \phi_{tor}) \subset (\mathbb{P}^1)^t \times (\mathbb{P}^1)^t \) such that the restrictions of \( \pr_1^p M_{\sigma} \) and \( \pr_2^s M_{\sigma} \) to \( \Gamma(n \cdot \phi_{tor}) \) correspond to the line bundles \( \pr_1^p M_{\sigma} \Gamma(n \cdot \phi_{tor}) \) and \( \pr_2^s M_{\sigma} \Gamma(n \cdot \phi_{tor}) \). Once again, we apply the projection formula to obtain
\[ \text{deg}(\alpha_1^s \alpha_2^s \beta_1^{r'-s} \beta_2 \beta_3^{r-1-s}) \cap [\Gamma(n \cdot \phi_{tor})]_G = \text{deg}(\alpha_1^s \alpha_2^s \beta_1^{r'-s} \beta_2 \beta_3^{r-1-s}) \cap [\Gamma(n \cdot \phi_{tor})]_G. \]

In particular, (78) is bounded by
\[ \text{deg}(\alpha_1^s \alpha_2^s \beta_1^{r'-s} \beta_2 \beta_3^{r-1-s}) \cap [\Gamma(n \cdot \phi_{tor})]_G = 0. \]

We first show that \( \alpha_1^s \alpha_2^s \beta_1^{r'-s} \beta_2 \beta_3^{r-1-s} \cap [\Gamma(n \cdot \phi_{tor})]_G \) has degree less than \( c_{26} n^{t-s} \) for some constant \( c_{26} \).

Using standard coordinates \( X_1, ..., X_t \) (resp. \( Y_1, ..., Y_t \)) on \( \mathbb{G}_m^t \) (resp. \( \mathbb{G}_m^t \)), let us write
\[ \cdot \]
where integers \( a_{uv} \) (\( 1 \leq u \leq t, 1 \leq v \leq t' \)). By dimension, we have again
\[ \text{dim}(n \cdot \phi_{tor}) \geq (1 \leq u \leq t, 1 \leq v \leq t'). \]

With \( M_{\sigma} \in \mathbb{G}_m^t \) and \( M_{\sigma} \in \mathbb{G}_m^t \), we obtain
\[ \text{deg}(\alpha_1^s \alpha_2^s \beta_1^{r'-s} \beta_2 \beta_3^{r-1-s}) \cap [\Gamma(n \cdot \phi_{tor})]_G = 0. \]

In particular, (78) is bounded by
\[ \text{deg}(\alpha_1^s \alpha_2^s \beta_1^{r'-s} \beta_2 \beta_3^{r-1-s}) \cap [\Gamma(n \cdot \phi_{tor})]_G = 0. \]
We determine next the intersection product
\[ (Y_1 = X_{a_1} \cdots X_{a_{t'}}) \cdots (Y_{t'} = X_{a_{t'}}') \in A_t((\mathbb{P}^1)^t \times (\mathbb{P}^1)^{t'}). \]
From [16, Example 8.3.7], we deduce an identification
\[ A_\phi G = \mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_t, \varepsilon'_1, \ldots, \varepsilon'_{t'}]/((\varepsilon_1)^2, \ldots, (\varepsilon_t)^2, (\varepsilon'_1)^2, \ldots, (\varepsilon'_{t'})^2) \]
such that \( \varepsilon_i \) (resp. \( \varepsilon'_i \)) corresponds to the flat pullback of the cycle class associated with an arbitrary point in the \( i \)-th factor of \((\mathbb{P}^1)^t\) (resp. \((\mathbb{P}^1)^{t'}\)). Considering appropriate intersections, it is easy to verify
\[ [Y_v = X_{a_{1v}} \cdots X_{a_{tv}}] = |a_{1v}|\varepsilon_1 + |a_{2v}|\varepsilon_2 + \cdots + |a_{tv}|\varepsilon_t + \varepsilon'_v. \]
Thus, \( (80) \) is simply the product
\[ \prod_{1 \leq v \leq t'} (|a_{1v}|\varepsilon_1 + |a_{2v}|\varepsilon_2 + \cdots + |a_{tv}|\varepsilon_t + \varepsilon'_v). \]
Inspecting the definition of \( M_t \) (resp. \( M_{t'} \)) in Construction [4], we note that intersecting a cycle class on \((\mathbb{P}^1)^t \times (\mathbb{P}^1)^{t'}\) with \( c_1(\text{pr}_1^* M_t) \) (resp. \( c_1(\text{pr}_2^* M_{t'}) \)) amounts to multiplication with \( 2(\varepsilon_1 + \cdots + \varepsilon_t) \) (resp. \( 2(\varepsilon'_1 + \cdots + \varepsilon'_{t'}) \)) in the Chow ring. We infer that the degree of \( (79) \) is majorized by the degree of
\[ 2^s(\varepsilon_1 + \cdots + \varepsilon_t)^s(\varepsilon'_1 + \cdots + \varepsilon'_{t'})^{t-s} \prod_{1 \leq v \leq t'} (|a_{1v}|\varepsilon_1 + \cdots + |a_{tv}|\varepsilon_t + \varepsilon'_v). \]
Exploiting cancellations, this can be simplified to
\[ 2^s!t!(t-s)! \cdot \sum_{1 \leq u_1, \ldots, u_{t-s} \leq t \text{ distinct}} \prod_{1 \leq v \leq t'} (|a_{u_1v_1}a_{u_2v_2} \cdots a_{u_{t-s}v_{t-s}}|(\varepsilon_1 \cdots \varepsilon_t \varepsilon'_1 \cdots \varepsilon'_{t'}) \]
Since \((\phi_{tor}, \phi_{ab}) \in \mathcal{K}_d, (79) \) can be consequently bounded from above by \( c_{26}n^{t-s} \) as claimed.

We finally demonstrate that \( \deg(\sigma_s) \) is bounded from above by \( c_{27}n^{2\dim(A) - (r'-s)} \) for some constant \( c_{27} \). For this, it suffices to note that \( \text{Hom}(A, A'_j) \) is a finitely generated \( \mathbb{Z} \)-module and that
\[ \text{Hom}(A, A'_j) \longrightarrow \text{Pic}(A), \varphi_{ab} \longmapsto \varphi_{ab}^* N_j, \]
is quadratic by the Theorem of the Cube ([14, Corollary II.6.2]) because \( N_j \) is symmetric (see [22, p. 417] for details). Combining this with the previous estimate, we immediately obtain the bound
\[ c_{26}c_{27}n^{t-s+2\dim(A) - (r'-s)} \leq c_{26}c_{27}n^{2(t-s+\dim(A) - r'+s)} = c_{26}c_{27}n^{2r'-2} \]
on (78). Taking our previous reductions into account, this completes the proof of the lemma. \( \square \)

8. Quotients of Semiabelian Varieties

In this section, we elucidate the set of quotients belonging to a fixed semiabelian variety. Let \( G \) be a semiabelian variety over \( \overline{\mathbb{Q}} \) with split toric part \( G'_m \) and abelian quotient \( \pi : G \to A. \) For a fixed torus \( G'_m \) and a fixed abelian variety \( A', \) we ask which elements \((\phi_{tor}, \phi_{ab})\) of
\[ V_Q = \text{Hom}_Q(G'_m \times G'_m, \times \text{Hom}_Q(A, A')) \]
are such that there exists a quasi-homomorphism \( \phi : G \to G' \) represented by \((\phi_{tor}, \phi_{ab})\) in the sense of Section [12]. Let \( Z(Q) \subset V_Q \) denote the subset consisting of these elements. For a fixed semiabelian variety \( G' \) with toric part \( G'_m \) and abelian quotient \( A', \) we know from Lemma [1] that the surjective quasi-homomorphisms \( \phi : G \to G' \) are parameterized by a linear subspace of \( V_Q. \) The set \( Z(Q) \) is the union of all these linear subspaces for varying \( G'. \) It is, however, not a union of finitely many linear subspaces in general. Nevertheless, we can interpret \( V_Q \) as the \( \mathbb{Q} \)-points of an additive algebraic group, which we abusively denote also by \( V_Q, \) and ask whether there is an algebraic subvariety \( Z \subset V_Q \) with \( Z(Q) \) as its set of \( \mathbb{Q} \)-points. This would also motivate our notation \( Z(Q) \) retroactively. In the next theorem, a cone \( Z \subset V_Q \) is a (not
Theorem 3. There exists a cone $Z \subset V_Q$ such that its $\mathbb{Q}$-points are precisely the pairs $(\phi_{\text{tor}}, \phi_{ab}) \in V_Q$ representing quasi-homomorphisms.

In the following, pairs $(\phi_{\text{tor}}, \phi_{ab}) \in V_Q$ representing quasi-homomorphisms are called realizable.

Proof. Write $\eta_G = (\eta_G^{(1)}, \ldots, \eta_G^{(t)}) \in A^t(\mathbb{Q})$. By Lemma 1 a pair $(\phi_{\text{tor}}, \phi_{ab}) \in V_Q$ is realizable if and only if for one of its multiples $n \cdot (\phi_{\text{tor}}, \phi_{ab}) \in V$ there exists some $\mu = (\mu_1, \ldots, \mu_t) \in (A')^\vee(\mathbb{Q})^t$ such that

$$
(\eta_G^{(1)}, \ldots, \eta_G^{(t)}) = (n \cdot \phi_{\text{tor}}) \cdot \eta_G = (n \cdot \phi_{ab})^\vee \mu = (\mu_1, \ldots, \mu_t) \cdot \eta_G
$$

in $A^\vee(\mathbb{Q})$. Write

$$
\begin{pmatrix}
  a_{11} & \cdots & a_{1t}
  \\
  \vdots & & \vdots
  \\
  a_{t1} & \cdots & a_{tt}
\end{pmatrix}, \quad a_{uv} \in n^{-1} \mathbb{Z},
$$

for the matrix representing $\phi_{\text{tor}} \in \text{Hom}_\mathbb{Q}(G_{m}, G_m^{t})$ and let $\phi_{v,\text{tor}} : G_m^t \rightarrow \mathbb{Q} G_m, 1 \leq v \leq t'$, be the quasi-homomorphism described by the column vector $(a_{1v}, \ldots, a_{tv})^t$. Then, (81) is equivalent to the equations

$$
(n \cdot \phi_{\text{tor}}) \cdot \eta_G = n a_{tv} \cdot \eta_G^{(1)} + \cdots + n a_{tv} \cdot \eta_G^{(t)} = (n \cdot \phi_{ab})^\vee \mu_v, 1 \leq v \leq t',
$$

having solutions $\mu_v \in A^\vee(\mathbb{Q})$. Hence, the pair $(\phi_{\text{tor}}, \phi_{ab})$ is realizable if and only if each $(\phi_{v,\text{tor}}, \phi_{ab}), 1 \leq v \leq t'$, is representing a quasi-homomorphism $G \rightarrow G_m$. Assume that there are cones $Z_v \subset \text{Hom}(G_m^t, G_m) \times \text{Hom}(A, A') = V_{v,\mathbb{Q}}, 1 \leq v \leq t'$, with $Z_v(\mathbb{Q})$ consisting of the pairs in $V_{v,\mathbb{Q}}$ representing quasi-homomorphisms. Denoting by $p_v : V_Q \rightarrow V_{v,\mathbb{Q}}$ the standard projection, the cone $Z = \bigcap_{v=1}^{t'} p^{-1}(Z_v) \subset V_Q$ is as wanted. In conclusion, it suffices to show the assertion for $t' = 1$.

Choose pairwise non-isogeneous simple abelian varieties $B_1, \ldots, B_k$ such that there exist isogenies

$$
\chi : A \rightarrow B_1^{r_1} \times \cdots \times B_k^{r_k}
$$

and set

$$
W = \text{Hom}(G_m^t, G_m) \times \text{Hom}(B_1^{r_1} \times \cdots \times B_k^{r_k}, B_1^{r_1} \times \cdots \times B_k^{r_k}).
$$

Let further $\psi$ (resp. $\psi'$) be isogenies such that $\psi \circ \chi = \chi \circ \psi = [m]_A$ (resp. $\psi' \circ \chi' = \chi' \circ \psi' = [m]_{A'}$) for some integer $m \geq 1$. We have $\mathbb{Q}$-linear maps

$$
f : V_Q \rightarrow W_Q, \quad (\phi_{\text{tor}}, \phi_{ab}) \mapsto (\phi_{\text{tor}}, \chi \circ \phi_{ab} \circ \psi)
$$

and

$$
g : W_Q \rightarrow V_Q, \quad (\phi_{\text{tor}}, \phi_{ab}) \mapsto (\phi_{\text{tor}}, \psi' \circ \phi_{ab} \circ \chi)
$$

such that both $g \circ f : V_Q \rightarrow V_Q$ and $f \circ g : W_Q \rightarrow W_Q$ send $(\phi_{\text{tor}}, \phi_{ab})$ to $(\phi_{\text{tor}}, m^2 \cdot \phi_{ab})$. Hence, $f$ and $g$ are bijections between $V_Q$ and $W_Q$. Using Lemma 2 we additionally deduce that both $f$ and $g$ preserve realizable pairs. Consequently, we may assume that $A = B_1^{r_1} \times \cdots \times B_k^{r_k}$ and $A' = B_1^{r_1} \times \cdots \times B_k^{r_k}$ in proving the theorem. In this case, we can also identify

$$
V_Q = \text{Hom}_\mathbb{Q}(G_m^t, G_m) \times \prod_{i=1}^{k} \text{Hom}_\mathbb{Q}(B_i^{r_i}, B_i^{r_i}).
$$

By Lemma 1 an element $(\phi_{\text{tor}}, \phi_{ab}^{(1)}, \ldots, \phi_{ab}^{(k)}) \in V_Q, \phi_{ab}^{(i)} \in \text{Hom}_\mathbb{Q}(B_i^{r_i}, B_i^{r_i})$, is realized by a quasi-homomorphism if and only if for one of its multiples $n \cdot (\phi_{\text{tor}}, \phi_{ab}^{(1)}, \ldots, \phi_{ab}^{(k)}) \in V$ there exists some tuple $(\eta^{(1)}, \ldots, \eta^{(k)}) \in B_1^\vee(\mathbb{Q})^{r_1} \times \cdots \times B_k^\vee(\mathbb{Q})^{r_k}$ such that

$$
(n \cdot \phi_{\text{tor}}) \cdot \eta_G = (n \cdot \phi_{ab}^{(1)})^\vee (\eta^{(1)}), \ldots, (n \cdot \phi_{ab}^{(k)})^\vee (\eta^{(k)}) = ((n \cdot \phi_{ab}^{(1)})^\vee(\eta^{(1)}), \ldots, (n \cdot \phi_{ab}^{(k)})^\vee(\eta^{(k)})).
$$
Arguing as above, we deduce that it suffices to prove the theorem under the additional assumption that \( k = 1 \) (i.e., \( A = B' \) and \( A' = B'' \) with a simple abelian variety \( B' \)).

Let us write \( \eta_G = (\eta_1, \ldots, \eta_t) \in (B')^\vee(\mathbb{Q})^t = \text{Ext}_B^1(B', \mathbb{G}_m) \) and \( \eta_j = (\eta_{ij}, \ldots, \eta_{ir})^t \in (B')^\vee(\mathbb{Q})^r = (B')^\vee(\mathbb{Q})^r \). Again, \((\phi_{tor}, \phi_{ab}) \in V_Q\) is realizable if and only if there exists some multiple \( n \cdot (\phi_{tor}, \phi_{ab}) \in V \) such that

\[
(n \cdot \phi_{tor}) \cdot \eta_G = (n \cdot \phi_{ab})^* \mu
\]

has a solution \( \mu = (\mu_1, \ldots, \mu_r) \in B'(\mathbb{Q})^r = \text{Ext}_B^1(A', \mathbb{G}_m) \). This condition can be translated into linear algebra over the \( \mathbb{Q} \)-division algebra \( D = \text{End}(B')_\mathbb{Q} \) (cf. [44, Corollary 2 on p. 174]). For this, we denote by \( \Gamma \) the left \( \text{End}(B') \)-submodule of \( B'(\mathbb{Q}) \) generated by

\[
\eta_{ij}, \quad 1 \leq i \leq r, \quad 1 \leq j \leq t.
\]

The tensor product \( \Gamma_Q = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \) is a left \( D \)-submodule of \( B'(\mathbb{Q}) \otimes \mathbb{Q} \). For any \( \gamma \in B'(\mathbb{Q}) \), we let \([\gamma]\) denote \( \gamma \otimes 1 \in \Gamma_Q \). This \( \mathbb{Q} \)-div ring, \( \Gamma_Q \), is a free left \( D \)-module so that we may choose \( \text{End}(B') \)-linearly independent elements \( \gamma_1, \ldots, \gamma_t \in B'(\mathbb{Q}) \) satisfying

\[
\Gamma = \text{End}(B') \cdot \gamma_1 \oplus \cdots \oplus \text{End}(B') \cdot \gamma_t \oplus \text{Tors}(\Gamma);
\]

here \( \text{Tors}(\Gamma) \) denotes the \( \mathbb{Z} \)-torsion elements of \( \Gamma \). If \( \eta_G \) has a solution \( \mu = (\mu_1, \ldots, \mu_r) \in B'(\mathbb{Q})^r \) for some \( n \), then it also has a solution \( \mu \in \Gamma^r \subset B'(\mathbb{Q})^r \) for a possibly larger \( n \). In fact, one may take any image under a \( D \)-linear projection from \( \Gamma_Q + D \cdot [\mu_1] + \cdots + D \cdot [\mu_r] \) to \( \Gamma_Q \). Since we can always arrange for \( n \) to annihilate the finite group \( \text{Tors}(\Gamma) \), we infer that \((\phi_{tor}, \phi_{ab})\) is realizable if and only if, in the notation from Section 1.2,

\[\eta_{ij} \in \alpha \cdot \eta_G = (\phi_{ab})^* \cdot \alpha(\mu)\]

has a solution \( \mu \in \Gamma^r \subset B'(\mathbb{Q}) \otimes \mathbb{Q} \). Both \( \phi_{tor} \in \text{Hom}_\mathbb{Q}(\mathbb{G}_m^t, \mathbb{G}_m) \) and \( \phi_{ab}^\vee \in \text{Hom}_\mathbb{Q}( (A')^\vee, A^\vee) \) can be represented by matrices

\[
\begin{pmatrix}
(a_1, a_2, \cdots, a_t)^t, a_i \in \mathbb{Q},
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
(b_{11} & \cdots & b_{1r} \\
\vdots & \ddots & \vdots \\
b_{r1} & \cdots & b_{rr}
\end{pmatrix}, b_{ij} \in D.
\]

Using this notation, we are searching for \( n \cdot \phi_{ab} \) such that

\[
(n \cdot \phi_{tor}) \cdot \eta_G = (n \cdot \phi_{ab})^* \mu
\]

has a solution \((\mu_1, \ldots, [\mu_r]) \in \Gamma_Q \). Using the decomposition \( \eta_G \), we expand

\[
[\eta_{ij}] = c_{ij}^{(1)} [\gamma_1] + \cdots + c_{ij}^{(t)} [\gamma_t], c_{ij}^{(t)} \in D.
\]

Then, \( \eta_G \) has a solution \((\mu_1, \ldots, \mu_r)\) if and only if each of the \( l \) linear equations

\[
a_1 \cdot \begin{pmatrix}
c_{11}^{(1)} \\
\vdots \\
c_{t1}^{(t)}
\end{pmatrix} + \cdots + a_t \cdot \begin{pmatrix}
c_{1t}^{(1)} \\
\vdots \\
c_{tt}^{(t)}
\end{pmatrix} = \begin{pmatrix}
b_{11} & \cdots & b_{1r} \\
\vdots & \ddots & \vdots \\
b_{r1} & \cdots & b_{rr}
\end{pmatrix} \begin{pmatrix}
[\mu_1] \\
\vdots \\
[\mu_t]
\end{pmatrix}
\]

has a solution \((\delta_1^{(1)}, \ldots, \delta_r^{(t)}) \in D^r \). By Lemma 30 below, the corresponding condition on \( \eta_G \) and \( \mu \) is described by a subcone of \( \mathbb{Q}^l \setminus D^r \times D^r \). The intersection \( Z^\vee \) of these \( l \) cones is almost what we are searching for. In fact, a pair \((\phi_t, \phi_a) \in V_Q \) is realizable if and only if \((\phi_t, \phi_a^\vee) \in \mathbb{Q}^l \setminus D^r \times D^r \) is in \( Z^\vee(\mathbb{Q}) \). The theorem follows now from the \( \mathbb{Q} \)-linearity (cf. [44, (ii) on p. 75]) of

\[\text{Hom}_\mathbb{Q}(A, A') \longrightarrow \text{Hom}_\mathbb{Q}( (A^\vee)' , A^\vee), f \mapsto f^\vee.\]
The following lemma is certainly standard (for \( t = 1 \) and \( a_1 = 1 \) at least) but I have found no trace of it in the literature so that a complete proof is given.

**Lemma 30.** Let \( D \) be a finite-dimensional \( \mathbb{Q} \)-algebra and \( y_1, \ldots, y_t \in D^r \) column vectors. Then, the pairs \((a, M) \in \mathbb{Q}^t \times D^r \times r'\), \( a = (a_1, \ldots, a_t) \), such that

\[
a_1 y_1 + \cdots + a_t y_t = M \cdot x
\]

has a solution \( x \in D^{r'} \), are the \( \mathbb{Q} \)-points of a cone \( Z \subset \mathbb{Q}^t \times D^{r \times r'} \).

Here, \( \mathbb{Q}^t \times D^{r \times r'} \) is given its canonical structure as an affine linear space over \( \mathbb{Q} \). We also remark that \( Z \) is generally not a closed subvariety.

**Proof.** Choosing a \( \mathbb{Q} \)-linear isomorphism \( \varphi : D \to \mathbb{Q}^n \), we obtain a map \( l : D \to \mathbb{Q}^{n \times n} \) such that \( l(d_1)\varphi(d_2) = \varphi(d_1 d_2) \). This realizes \( D \) as a \( n \)-dimensional subspace \( l(D) \) of \( \mathbb{Q}^{n \times n} \). With these identifications, the equation (88) can be written as

\[
a_1 y'_1 + \cdots + a_t y'_t = M' \cdot x'
\]

with \( M' \in \mathbb{Q}^{n \times n} \), \( x' \in \mathbb{Q}^{n} \) and \( y'_1, \ldots, y'_t \in \mathbb{Q}^{nr} \). We then search for \( M' \in \mathbb{Q}^{n \times nr} \) such that a solution \( x' \) exists under the additional restraint that \( M' \) comes from a matrix \( M \in D^{r \times r'} \) by applying \( l \) to each entry. Since this restraint can be evidently expressed as \( M' \) being contained in a \( \mathbb{Q} \)-subcone of \( \mathbb{Q}^{nr \times nr} \), we can restrict to the case \( D = \mathbb{Q} \).

To deal with this special case, we make the following elementary observation: Write \( M = (m_1, \ldots, m_n) \) with column vectors \( m_i \in \mathbb{Q}^{nr} \). For any \( y \neq 0 \), we have \( y \in \text{im}(M) \) if and only if there exists a subset \( I \subset \{1, \ldots, r\} \) such that \( \bigwedge_{i \in I} m_i \neq 0 \in \bigwedge^{|I|} \mathbb{Q}^{r'} \) and \( \bigwedge_{i \in I} m_i \wedge y = 0 \in \bigwedge^{|I|+1} \mathbb{Q}^{r'} \). From this, we straightforwardly obtain equations for the sought-after \( \mathbb{Q} \)-cone \( Z \subset \mathbb{Q}^t \times D^{r \times r'} \).

The proof of Theorem 34 gives evidently a procedure to determine \( Z \) via linear algebra so that one may hope that its rational points \( Z(\mathbb{Q}) \) are equally easy to describe. However, \( Z(\mathbb{Q}) \) can be rather complicated if \( G \) is neither an abelian variety nor a torus. For example, \( Z \) is not even rational in general, although it is in these two special cases. With respect to the proof of Theorem 22, this means in particular that (Dirichlet) approximation arguments as in [22 Section 4] and [23 Section 4] break down if one insists on the use of surjective quasi-homomorphisms \( G \to G' \). This makes it necessary to work with explicit line bundles on \( G \) as we do in this article.

**Example 31.** Let \( E \) be an elliptic curve without complex multiplication (i.e., \( \text{End}(E) = \mathbb{Z} \)). Furthermore, let \( \gamma_1, \gamma_2, \gamma_3 \in E^\vee(\mathbb{Q}) \) be such that \( \Gamma = \sum_{i=1}^3 \mathbb{Z} \cdot \gamma_i \) is a free \( \mathbb{Z} \)-module of rank 3. For an arbitrary tuple \((n_1, n_2, n_3) \in \mathbb{Z}^3 \), we define

\[
\eta_1 = \begin{pmatrix} n_1 \cdot \gamma_1 \\ \gamma_3 \\ \gamma_2 \end{pmatrix}, \eta_2 = \begin{pmatrix} n_2 \cdot \gamma_2 \\ \gamma_1 \\ \gamma_3 \end{pmatrix}, \eta_3 = \begin{pmatrix} n_3 \cdot \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{pmatrix},
\]

considering these column vectors as elements of \( (E^3)^\vee(\mathbb{Q}) \). Let \( G \) be the semiabelian variety determined by

\[
(\eta_1, \eta_2, \eta_3) \in ((E^3)^\vee)^3 = \text{Ext}^1(E^3, \mathbb{G}_m)^3 = \text{Ext}^1(E^3, \mathbb{G}_m^3).
\]

From Theorem 13, we know that the realizable pairs in

\[
\text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \times \text{Hom}(E, E^2) = V_Q = \text{Hom}(\mathbb{G}_m^3, \mathbb{G}_m^3) \times \text{Hom}(E^3, E^2)
\]

are the \( \mathbb{Q} \)-rational points of an algebraic subvariety \( Z \subset V_Q \). Consider the projection \( \pi : V_Q \to \text{Hom}(\mathbb{G}_m^3, \mathbb{G}_m^3) \). An inspection of the three linear equations given by [37] tells us that the
image $\pi(Z)$ is described by
\[
\det \begin{pmatrix}
  n_1 a_1 & n_2 a_2 & n_3 a_3 \\
  a_2 & a_3 & a_1 \\
  a_3 & a_1 & a_2
\end{pmatrix}
= (n_1 + n_2 + n_3)a_1 a_2 a_3 - n_1 a_1^3 - n_2 a_2^3 - n_3 a_3^3.
\]

It is easy to check (cf. [11] Chapter 10 or [14] Section 3.1) that
\[
n_1 X^3 + n_2 Y^3 + n_3 Z^3 - (n_1 + n_2 + n_3)XYZ = 0
\]
is the projective equation of an elliptic curve $E'_{n_1,n_2,n_3}$ for generic tuples $(n_1,n_2,n_3) \in \mathbb{Z}^3$. In these cases, $\pi(Z)$ is birationally equivalent to $\mathbb{P}^1 \times E'_{n_1,n_2,n_3}$. The existence of a global non-zero one-form (i.e., the pull-back of the invariant differential form on $E'_{n_1,n_2,n_3}$) precludes unirationality of $\mathbb{P}^1 \times E'_{n_1,n_2,n_3}$ (cf. [32] Theorem 1.52). Therefore, $Z$ itself cannot be a rational variety. In addition, the set $Z(\mathbb{Q})$ surjects onto the Mordell-Weil group of the $\mathbb{Q}$-elliptic curve $E'_{n_1,n_2,n_3}$. Given that no known algorithm produces the Mordell-Weil rank, this should demonstrate that the “mixed structure” of a semiabelian variety can lead to an intricate set of quotients and subgroups.

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**References**

[1] James Ax. Some topics in differential algebraic geometry. I. Analytic subgroups of algebraic groups. Amer. J. Math., 94:1195–1204, 1972.

[2] D. Bertrand. Endomorphismes de groupes algébriques; applications arithmétiques. In *Diophantine approximations and transcendental numbers* (Luminy, 1982), volume 31 of Progr. Math., pages 1–45. Birkhäuser Boston, Boston, MA, 1983.

[3] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 2004.

[4] Enrico Bombieri and Walter Gubler. *Heights in Diophantine geometry*, volume 4 of New Mathematical Monographs. Cambridge University Press, Cambridge, 2006.

[5] Enrico Bombieri, David Masser, and Umberto Zannier. Intersecting a curve with algebraic subgroups of multiplicative groups. Internat. Math. Res. Notices, (20):1119–1140, 1999.

[6] Enrico Bombieri, David Masser, and Umberto Zannier. Anomalous subvarieties – structure theorems and applications. Int. Math. Res. Not. IMRN, (19):Art. ID rnm057, 33, 2007.

[7] Enrico Bombieri, David Masser, and Umberto Zannier. Intersecting a plane with algebraic subgroups of multiplicative groups. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 7(1):51–80, 2008.

[8] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990.

[9] Michel Brion. Some structure theorems for algebraic groups. In *Algebraic Groups: Structure and Actions*, Proc. Sympos. Pure Math., 94, pages 53–125. Amer. Math. Soc., 2017.

[10] Antoine Chambert-Loir. Points de petite hauteur sur les variétés semi-abéliennes. Ann. Sci. École Norm. Sup. (4), 33(6):789–821, 2000.

[11] Antoine Chambert-Loir. Relations de dépendance et intersections exceptionnelles. Astérisque, (348):Exp. No. 1032, viii, 149–188, 2012. Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042.

[12] Brian Conrad. A modern proof of Chevalley’s theorem on algebraic groups. J. Ramanujan Math. Soc., 17(1):1–18, 2002.

[13] Demaill. Complex analytic and differential geometry. available from [http://www-fourier.ujf-grenoble.fr/~demailly/](http://www-fourier.ujf-grenoble.fr/~demailly/) 09 2012.
[42] L. Moret-Bailly. Familles de courbes et de variétés abéliennes sur $\mathbb{P}^1$. I. In L. Szpiro, editor, Séminaire sur les Pinceaux de Courbes de Genre au Moins Deux, volume 86 of Astérisque. Société Mathématique de France, Paris, 1981.

[43] D. Mumford, J. Fogarty, and F. Kirwan. Geometric invariant theory, volume 34 of Ergebnisse der Mathematik und ihrer Grenzbereiche (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, third edition, 1994.

[44] David Mumford. Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[45] F. Oort. Commutative group schemes, volume 15 of Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1966.

[46] Richard Pink. A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang. available from [http://www.math.ethz.ch/~pink] 04 2005.

[47] Gaël Rémond. Intersection de sous-groupes et de sous-variétés. III. Comment. Math. Helv., 84(4):835–863, 2009.

[48] Steven Roman. Advanced linear algebra, volume 135 of Graduate Texts in Mathematics. Springer, New York, second edition, 2005.

[49] Walter Rudin. Functional analysis. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[50] A. Schinzel. Polynomials with special regard to reducibility, volume 77 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2000. With an appendix by Umberto Zannier.

[51] Jean-Pierre Serre. Algebraic groups and class fields, volume 117 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1988. Translated from the French.

[52] Jean-Pierre Serre. Espaces fibrés algébriques. In Exposés de séminaires (1950-1999). Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 1, pages viii+259. Société Mathématique de France, Paris, 2001.

[53] Yum Tong Siu. An effective Matsusaka big theorem. Ann. Inst. Fourier (Grenoble), 43(5):1387–1405, 1993.

[54] Evelina Viada. The intersection of a curve with algebraic subgroups in a product of elliptic curves. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2(1):47–75, 2003.

[55] Oscar G. Villareal. Moving subvarieties by endomorphisms. Manuscripta Math., 125(1):81–93, 2008.

[56] Claire Voisin. Hodge theory and complex algebraic geometry. I, volume 76 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.

[57] Paul Vojta. Integral points on subvarieties of semiabelian varieties. I. Invent. Math., 126(1):133–181, 1996.

[58] Paul Vojta. Integral points on subvarieties of semiabelian varieties. II. Amer. J. Math., 121(2):283–313, 1999.

[59] Michel Waldschmidt. Nombres transcendants et groupes algébriques, volume 69 of Astérisque. Société Mathématique de France, Paris, 1979. With appendices by Daniel Bertrand and Jean-Pierre Serre, With an English summary.

[60] Frank W. Warner. Foundations of differentiable manifolds and Lie groups, volume 94 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[61] Umberto Zannier. Proof of Conjecture 1, appendix in [50, pp. 517–539].

[62] Umberto Zannier. Some problems of unlikely intersections in arithmetic and geometry, volume 181 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2012. With appendices by David Masser.

[63] Boris Zilber. Exponential sums equations and the Schanuel conjecture. J. London Math. Soc. (2), 65(1):27–44, 2002.

E-mail address: lkuehne@fields.utoronto.ca

Fields Institute, 222 College Street, Toronto ON M5T 3J1, Canada