DEFORMATIONS OF COALGEBRA MORPHISMS

DONALD YAU

ABSTRACT. An algebraic deformation theory of coalgebra morphisms is constructed.

1. Introduction

Algebraic deformation theory, as first described by Gerstenhaber [4], studies perturbations of algebraic structures using cohomology and obstruction theory. Gerstenhaber’s work has been extended in various directions. For example, Balavoine [1] describes deformations of any algebra over a quadratic operad. In the same direction, Hinich [9] studies deformations of algebras over a differential graded operad. It is a natural problem to try to extend deformation theory to morphisms and, more generally, diagrams.

Deformations of morphisms are much harder to describe than that of the algebraic objects themselves. Specifically, the difficulty arises when one tries to show that certain obstruction classes (obstructions to integration) are cocycles in the deformation complex. Even in the case of an associative algebra morphism, proving this relies on a powerful result called the Cohomology Comparison Theorem (CCT) [5, 6, 7]. This Theorem says that the cohomology of a morphism, or a certain diagram, of associative algebras is isomorphic to the Hochschild cohomology of an auxiliary associative algebra. This allows one to bypass the obstruction class issue by reducing the problem to the case of a single associative algebra.

The purpose of this paper is to establish the dual picture: A deformation theory of coalgebra morphisms. Deformations of coalgebras (not their morphisms) have been described by Gerstenhaber and Schack [8]. Their theory is essentially the same as that of the original theory [4], with Hochschild coalgebra cohomology in place of Hochschild cohomology for associative algebras. We will build upon their work, regarding it as the absolute case, ours being the relative case.

There are two aspects of deformations of coalgebra morphisms that are different from the associative case. First, the CCT requires a rather involved argument and has only been established for associative algebras. Since we do not have a coalgebra version of the CCT, we deal with the obstruction
class issue mentioned above differently, using instead an elementary, computational approach. Second, working with coalgebras and their morphism seems to be even more conceptual and transparent than in the associative case. In fact, much of our coalgebra morphism deformation theory is element-free. In other words, although the Hochschild coalgebra cochain modules are of the form $\text{Hom}(M, A^\otimes n)$, we never need to pick elements in $M$ in our arguments.

It should be noted that deformations of Lie algebra morphisms have been studied by Nijenhuis-Richardson [10] and Fréiger [3]. Also, recent work of Borisov [2] sheds new light into the structure of the deformation complex of a morphism of associative algebras, showing that it is an $L_\infty$-algebra (“strongly homotopy Lie algebra”).

**Organization.** The next section is a preliminary one, in which Hochschild coalgebra cohomology is recalled. Section 3 introduces deformations of a coalgebra morphism and identifies infinitesimals as 2-cocycles in the deformation complex $C^*_c(f)$ of a morphism of coalgebras (Theorem 3.4). It ends with the Rigidity Theorem, which states that a morphism is rigid, provided that $H^2(C^*_c(f))$ is trivial (Theorem 3.6). In section 4 the obstructions to extending a 2-cocycle in $C^*_c(f)$ to a deformation of $f$ are identified. They are shown to be 3-cocycles in $C^*_c(f)$ (Theorem 4.2). In particular, such extensions automatically exist if $H^3(C^*_c(f))$ is trivial (Corollary 4.3). Most of the arguments in this paper are contained in that section.

2. Hochschild coalgebra cohomology

Throughout the rest of this paper, $K$ will denote a fixed ground field. The following discussion of coalgebra cohomology is exactly dual to that of Hochschild cohomology of associative algebras. See, for example, [8].

2.1. Bicomodule. A $K$-coalgebra is a pair $(A, \Delta)$ (or just $A$) in which $A$ is a vector space over $K$ and $\Delta: A \to A \otimes A$ is a linear map, called comultiplication, that is coassociative, in the sense that $(\text{Id} \otimes \Delta)\Delta = (\Delta \otimes \text{Id})\Delta$. When the ground field is understood, we will omit the reference to $K$. If $A$ and $B$ are coalgebras, a morphism $f: A \to B$ is a linear map that is compatible with the comultiplications, in the sense that $\Delta_B \circ f = (f \otimes f) \circ \Delta_A$.

An $A$-bicomodule is a vector space $M$ together with a left action map, $\psi_l: M \to A \otimes M$, and a right action map, $\psi_r: M \to M \otimes A$, that make the usual diagrams commute. For example, if $f: A \to B$ is a coalgebra morphism, then (the underlying vector space of) $A$ becomes a $B$-bicomodule.
via the structure maps:

$$(f \otimes \text{Id}_A) \circ \Delta_A: A \to B \otimes A,$$

$$(\text{Id}_A \otimes f) \circ \Delta_A: A \to A \otimes B.$$ 

In this case, we say that $A$ is a $B$-bicomodule via $f$. One can also consider $A$ as an $A$-bicomodule with the structure maps $\psi_l = \psi_r = \Delta_A$.

2.2. Coalgebra cohomology. The Hochschild coalgebra cohomology of a coalgebra $A$ with coefficients in an $A$-bicomodule $M$ is defined as follows. For $n \geq 1$, the module of $n$-cochains is defined to be

$$C^*_n(M, A) := \text{Hom}_K(M, A \otimes A^{\otimes n}),$$

with differential

$$\delta_c \sigma = (\text{Id}_A \otimes \sigma) \circ \psi_l + \sum_{i=1}^n (-1)^i (\text{Id}_A \otimes (i-1) \otimes \Delta \otimes \text{Id}_A \otimes (n-i)) \circ \sigma + (-1)^{n+1} (\sigma \otimes \text{Id}_A) \circ \psi_r$$

for $\sigma \in C^n(A, M)$, $\psi_l = \psi_r = \Delta_A$. The cohomology of the cochain complex $(C^*_n(M, A), \delta_c)$ is denoted by $H^*_c(M, A)$.

3. Coalgebra morphism deformations

Fix a coalgebra morphism $f: A \to B$ once and for all. Consider $A$ as a $B$-bicomodule via $f$ wherever appropriate.

The purpose of this section is to introduce deformations of a coalgebra morphism and discuss infinitesimals and rigidity. All the assertions in this section are proved by essentially the same arguments as the ones in the associative case \[4, 5, 6, 7\]. Therefore, we can safely omit the proofs.

3.1. Deformation complex. For $n \geq 1$, define the module of $n$-cochains to be

$$C^n_c(f) := C^*_n(A, A) \times C^*_n(B, B) \times C^{n-1}_c(A, B)$$

and the differential

$$d_c: C^n_c(f) \to C^{n+1}_c(f)$$

by

$$d_c(\xi; \pi; \varphi) = (\delta_c \xi; \delta_c \pi; \pi \circ f - f^\otimes \circ \xi - \delta_c \varphi).$$

It is straightforward to check that $d_c d_c = 0$ \[7, p. 155\]. The cochain complex $(C^*_c(f), d_c)$ (or just $C^*_c(f)$) is called the deformation complex of $f$. Its cohomology is denoted by $H^*_c(f)$.

Note that the signs in front of the terms $\pi \circ f$ and $f^\otimes \circ \xi$ are different from their counterparts in the associative case \[7, p. 155, line 4\]. This change of
signs is needed in order to correctly identify infinitesimals as 2-cocycles in the deformation complex (Theorem 3.4).

3.2. Deformation. First, recall from [8] that a deformation of a coalgebra $A$ is a power series $\Delta_t = \sum_{n=0}^{\infty} \Delta_n t^n$ in which each $\Delta_n \in C^2_c(A, A)$ with $\Delta_0 = \Delta$, such that $\Delta_t$ is coassociative: $(\text{Id} \otimes \Delta_t) \Delta_t = (\Delta_t \otimes \text{Id}) \Delta_t$. In particular, $\Delta_t$ gives a $K[[t]]$-coalgebra structure on the module of power series $A[[t]]$ with coefficients in $A$ that restricts to the original coalgebra structure on $A$ when setting $t = 0$.

With this in mind, we define a deformation of $f$ to be a power series $\Omega_t = \sum_{n=0}^{\infty} \omega_n t^n$, with each $\omega_n = (\Delta_{A,n}; \Delta_{B,n}; f_n) \in C^2_c(f)$, satisfying the following three statements:

1. $\Delta_{A,t} = \sum_{n=0}^{\infty} \Delta_{A,n} t^n$ is a deformation of $A$.
2. $\Delta_{B,t} = \sum_{n=0}^{\infty} \Delta_{B,n} t^n$ is a deformation of $B$.
3. $F_t = \sum_{n=0}^{\infty} f_n t^n : (A[[t]], \Delta_{A,t}) \to (B[[t]], \Delta_{B,t})$ is a $K[[t]]$-coalgebra morphism with $f_0 = f$.

A deformation $\Omega_t$ will also be denoted by the triple $(\Delta_{A,t}; \Delta_{B,t}; F_t)$.

A formal isomorphism of $f$ is a power series $\Phi_t = \sum_{n=0}^{\infty} \phi_n t^n$ with each $\phi_n = (\phi_{A,n}; \phi_{B,n}) \in C^1_c(f)$ and $\phi_0 = (\text{Id}_A; \text{Id}_B)$.

Suppose that $\Omega_t = (\Delta_{A,t}; \Delta_{B,t}; F_t)$ is also a deformation of $f$. Then $\Omega_t$ and $\Omega_t$ are said to be equivalent if and only if there exists a formal isomorphism $\Phi_t$ such that

1. $\Delta_{A,t} = (\Phi_{A,t} \otimes \Phi_{A,t}) \circ \Delta_{A,t} \circ \Phi_{A,t}^{-1}$,
2. $\Delta_{B,t} = (\Phi_{B,t} \otimes \Phi_{B,t}) \circ \Delta_{B,t} \circ \Phi_{B,t}^{-1}$, and
3. $F_t = \Phi_{B,t} \circ F_t \circ \Phi_{A,t}^{-1}$,

where $\Phi_{*,t} = \sum_{n=0}^{\infty} \phi_{*,n} t^n$ for $* = A, B$.

3.3. Infinitesimal. The linear coefficient $\omega_1 = (\Delta_{A,1}; \Delta_{B,1}; f_1) \in C^2_c(f)$ of a deformation $\Omega_t$ of $f$ is called the infinitesimal of $\Omega_t$. This element is more than just a 2-cocycle.

Theorem 3.4. Let $\Omega_t = \sum_{n=0}^{\infty} \omega_n t^n$ be a deformation of $f$. Then $\omega_1$ is a 2-cocycle in $C^2_c(f)$ whose cohomology class is determined by the equivalence class of $\Omega_t$. Moreover, if $\Omega_t = 0$ for $1 \leq i \leq l$, then $\omega_{l+1}$ is a 2-cocycle in $C^2_c(f)$.

3.5. Rigidity. The trivial deformation of $f$ is the deformation $\Omega_t = \omega_0 = (\Delta_A; \Delta_B; f)$. The morphism $f$ is said to be rigid if and only if every one of its deformations is equivalent to the trivial deformation. The following cohomological criterion for rigidity is standard.
Theorem 3.6. If $H^2_c(f)$ is trivial, then $f$ is rigid.

4. Extending 2-cocycles to deformations

In view of Theorem 3.4, a natural question is: Given a 2-cocycle $\omega$ in $C^*_c(f)$, is there a deformation of $f$ with $\omega$ as its infinitesimal? The purpose of this section is to identify the obstructions for the existence of such a deformation. Following [4], if such a deformation exists, then $\omega$ is said to be integrable.

Fix a positive integer $N$. By a deformation of $f$ of order $N$, we mean a polynomial $\Omega_t = \sum_{n=0}^{N} \omega_n t^n$ with each $\omega_n \in C^2_c(f)$ and $\omega_0 = (\Delta_A; \Delta_B; f)$, satisfying the definition of a deformation of $f$ modulo $t^{N+1}$. In other words, for $X \in \{A,B\}$, $\Delta_{X,t} = \sum_{n=0}^{N} \Delta_{X,n} t^n$ defines a $K[t]/(t^{N+1})$-coalgebra structure on $X[t]/(t^{N+1})$ and $F_t = \sum_{n=0}^{N} f_n t^n$ is a $K[t]/(t^{N+1})$-coalgebra morphism.

To answer the integrability question, it suffices to consider the obstruction to extending $\Omega_t$ to a deformation of $f$ of order $N+1$. So let $\omega_{N+1} = (\Delta_{A,N+1}; \Delta_{B,N+1}; f_{N+1}) \in C^2_c(f)$ be a 2-cochain and set

$$\tilde{\Omega}_t := \Omega_t + \omega_{N+1} t^{N+1}. \quad (4.0.1)$$

Is $\tilde{\Omega}_t$ a deformation of $f$ of order $N+1$? Since $\tilde{\Omega}_t \equiv \Omega_t \pmod{t^{N+1}}$, it suffices to consider the coefficients of $t^{N+1}$ in the definition of a deformation of $f$.

To this end, consider the following cochains ($X \in \{A,B\}$):

$$\text{Ob}_X = \sum_{i=1}^{N} \left( (\Delta_{X,i} \otimes \text{Id}_X) \circ \Delta_{X,N+1-i} - (\text{Id}_X \otimes \Delta_{X,i}) \circ \Delta_{X,N+1-i} \right),$$

$$\text{Ob}_F = \left( \sum' (f_j \otimes f_k) \circ \Delta_{A,i} \right) - \sum_{i=1}^{N} \Delta_{B,N+1-i} \circ f_i, \quad (4.0.2a)$$

where

$$\sum' = \sum_{\begin{array}{c} i+j+k=N+1 \\
0 \leq i,j,k \leq N
\end{array}} + \sum_{\begin{array}{c} i+j=N+1 \\
k=0
\end{array}} + \sum_{\begin{array}{c} i+k=N+1 \\
j=0
\end{array}} + \sum_{\begin{array}{c} j+k=N+1 \\
i=0
\end{array}} + \sum_{i+j+k=N+1} \quad (4.0.2b)$$

From now on, integer indexes appearing in a summation are assumed non-negative, unless otherwise specified. Let $\text{Ob}_\Omega \in C^3_c(f)$ be the element

$$\text{Ob}_\Omega = (\text{Ob}_A; \text{Ob}_B; \text{Ob}_F). \quad (4.0.3)$$
Note that
\[ \text{Ob}_X = \sum_{i=1}^{N} \Delta_{X,i} \circ \Delta_{X,N+1-i} \in C^3_c(X,X) \]
for \( X \in \{A,B\} \), where \( \bar{\circ} \) is the “comp” in [8, p. 63]. A standard deformation theory argument [8, p. 60] shows that \( \text{Ob}_X \) is a 3-coboundary if and only if \( \Delta_{X,t} \) extends to a \( K[t]/(t^{N+2}) \)-coalgebra structure on \( X[t]/(t^{N+2}) \). In this case, any 2-cochain whose coboundary is \( \text{Ob}_X \) gives an extension. An analogous argument applied to our setting yields the following result.

**Theorem 4.1.** The polynomial \( \bar{\Omega}_t \) is a deformation of \( f \) of order \( N+1 \) if and only if \( \text{Ob}_\Omega = d_c \omega_{N+1} \).

It is also known [8, Theorem 3] that \( \text{Ob}_X \) is a 3-cocycle. We extend this statement to \( \text{Ob}_\Omega \).

**Theorem 4.2.** The element \( \text{Ob}_\Omega \in C^3_c(f) \) is a 3-cocycle.

Before giving the proof of Theorem 4.2, let us record the following immediate consequence of the previous two Theorems.

**Corollary 4.3.** If \( H^3_c(f) \) is trivial, then every 2-cocycle in \( C^*_{c}(f) \) is integrable.

**Proof of Theorem 4.2.** Since \( \text{Ob}_A \) and \( \text{Ob}_B \) are 3-cocycles, to show that \( \text{Ob}_\Omega \) is a 3-cocycle, it suffices to show that
\[ \delta_c \text{Ob}_F = \text{Ob}_B \circ f + f \otimes^3 \circ \text{Ob}_A = 0. \]

To do this, first note that for a 2-cochain \( \psi \in C^2_c(A,B) \), \( \delta_c \psi \) is given by
\[ \delta_c \psi = (\text{Id}_B \otimes \psi) \circ (f \otimes \text{Id}_A) \circ \Delta_A - (\Delta_B \otimes \text{Id}_B) \circ \psi \]
\[ + (\text{Id}_B \otimes \Delta_B) \circ \psi - (\psi \otimes \text{Id}_B) \circ (\text{Id}_A \otimes f) \circ \Delta_A \]
\[ = (f \otimes \psi) \circ \Delta_A - (\Delta_B \otimes \text{Id}_B) \circ \psi + (\text{Id}_B \otimes \Delta_B) \circ \psi - (\psi \otimes f) \circ \Delta_A. \]

Applying this to \( \text{Ob}_F \), we have
\[ \delta_c \text{Ob}_F = \]
\[ \left( \sum' f \otimes [(f_j \otimes f_k) \circ \Delta_{A,i}] \right) \circ \Delta_A \]  
(4.3.2a)
\[ - \sum_{i=1}^{N} (f \otimes (\Delta_{B,N+1-i} \circ f_i)) \circ \Delta_A \]  
(4.3.2b)
\[ - \sum' (\Delta_B \otimes \text{Id}_B) \circ (f_j \otimes f_k) \circ \Delta_{A,i} \]  
(4.3.2c)
\[ + \sum_{i=1}^{N} (\Delta_B \otimes \text{Id}_B) \circ \Delta_{B,N+1-i} \circ f_i \]  
(4.3.2d)
\[ + \sum' (\text{Id}_B \otimes \Delta_B) \circ (f_j \otimes f_k) \circ \Delta_{A,i} \]  
(4.3.2e)
\[ - \sum_{i=1}^{N} \left( \text{Id}_B \otimes \Delta_B \right) \circ \Delta_{B,N+1-i} \circ f_i \] (4.3.2f)

\[ - \left( \sum' \left( f_j \otimes f_k \right) \circ \Delta_{A,i} \otimes f \right) \circ \Delta_A \] (4.3.2g)

\[ + \sum_{i=1}^{N} \left( \left( \Delta_{B,N+1-i} \circ f_i \right) \otimes f \right) \circ \Delta_A. \] (4.3.2h)

The terms \(4.3.2c\), \(4.3.2d\), and \(4.3.2e\) need to be expanded.

For \(4.3.2c\), first note that, since \(F_t = \sum_{n=0}^{N} f_n t^n\) is a \(K[t]/(t^{N+1})\)-coalgebra morphism, we have that

\[ \sum_{i+j+k=n} (f_j \otimes f_k) \circ \Delta_{A,i} = \sum_{i=0}^{N} \Delta_{B,i} \circ f_{n-i} \] (4.3.3)

for \(0 \leq n \leq N\). In particular, it follows that

\[ \Delta_B \circ f_j = \sum_{\alpha+\beta+\gamma=j} (f_{\beta} \otimes f_{\gamma}) \circ \Delta_{A,\alpha} - \sum_{\lambda+\mu=j}^{\lambda+\mu=j} \Delta_{B,\mu} \circ f_\lambda. \] (4.3.4)

Putting \(4.3.4\) into \(4.3.2c\), we have that

\[ \sum' = \sum_{i+j+k=n} \left( \Delta_B \circ f_j \right) \circ f_k \circ \Delta_A,i = \sum_{i=0}^{N} \Delta_{B,i} \circ f_{n-i} \] (4.3.5a)

and

\[ \sum' = \sum_{\lambda+\mu=j}^{\lambda+\mu=j} \left( \Delta_B \circ f_j \right) \circ f_k \circ \Delta_A,i. \] (4.3.5b)

The last two summations are given by

\[ \sum_{\alpha+\beta+\gamma=j}^{\alpha+\beta+\gamma=j} + \sum_{i+k=N+1}^{i+k=N+1} + \sum_{i+k=N+1}^{i+k=N+1} + \sum_{i+k=N+1}^{i+k=N+1} \] (4.3.5a)

and

\[ \sum_{\lambda+\mu=j}^{\lambda+\mu=j} + \sum_{i+k=N+1}^{i+k=N+1} + \sum_{i+k=N+1}^{i+k=N+1} \] (4.3.5b)

Therefore, \(4.3.2c\) can be written as

\[ - \sum' \left[ \left( \Delta_B \circ f_j \right) \circ f_k \right] \circ \Delta_A,i = \] (4.3.6a)
\[ \sum_{i+\alpha+\beta+\gamma=N+1} \left[ (f_{i} \otimes f_{\alpha}) \circ \Delta_{A,i} \otimes f \right] \circ \Delta A, i \]  

(4.3.6b)

\[ \sum_{i+k=N+1} \left[ (f \otimes f) \circ \Delta_{A,i} \right] \Delta_{A,i} \]  

(4.3.6c)

\[ \sum_{\alpha+\beta+\gamma+k=N+1} \left[ (f_{\beta} \otimes f_{\gamma}) \circ \Delta_{A,i} \right] \otimes f_{k} \circ \Delta_{A} \]  

(4.3.6d)

\[ \sum_{i+\alpha+\beta+\gamma=k=N+1} \left[ (f_{\beta} \otimes f_{\gamma}) \circ \Delta_{A,i} \right] \otimes f_{k} \circ \Delta_{A,i} \]  

(4.3.6e)

\[ \sum'_{\lambda+\mu=j} \left[ (\Delta_{B,\mu} \circ f_{\lambda}) \otimes f_{k} \right] \circ \Delta_{A,i}. \]  

(4.3.6f)

A similar argument can be applied to (4.3.2d), which yields

\[ \sum_{i=1}^{N} (\Delta_{B} \otimes \text{Id}_{B}) \circ \Delta_{B,N+1-i} \circ f_{i} \]  

(4.3.2d)

\[ = \sum_{i=1}^{N} (\text{Id}_{B} \otimes \Delta_{B}) \circ \Delta_{B,N+1-i} \circ f_{i} \]  

(4.3.7a)

\[ + \sum_{i+j+k=N+1} (\text{Id}_{B} \otimes \Delta_{B,k}) \circ \Delta_{B,j} \circ f_{i} \]  

(4.3.7b)

\[ - \sum_{i+j+k=N+1} (\Delta_{B,k} \otimes \text{Id}_{B}) \circ \Delta_{B,j} \circ f_{i} \]  

(4.3.7c)

\[ + \sum'_{\lambda+\mu=k} \left[ f_{j} \otimes (\Delta_{B,\mu} \circ f_{\lambda}) \right] \circ \Delta_{A,i} \]  

(4.3.7d)

\[ - \sum_{j+k=N+1} (\text{Id}_{B} \otimes \Delta_{B,k}) \circ \Delta_{B,j} \circ f \]  

(4.3.7e)

\[ - \sum_{i=1}^{N} \left[ (\Delta_{B,N+1-i} \circ f_{i}) \otimes f \right] \circ \Delta A \]  

(4.3.7f)

\[ - \sum'_{\lambda+\mu=j} \left[ (\Delta_{B,\mu} \circ f_{\lambda}) \otimes f_{k} \right] \circ \Delta_{A,i} \]  

(4.3.7g)
+ \sum_{j+k=N+1 \atop j,k>0} (\Delta_{B,k} \otimes \text{Id}_B) \circ \Delta_{B,j} \circ f. \quad (4.3.7g)

A similar argument when applied to (4.3.2e) gives

\begin{align*}
\text{(4.3.2e)} &= \sum' (f_j \otimes (\Delta_B \circ f_k)) \circ \Delta_{A,i} \\
&= \sum_{i+j=N+1 \atop i,j>0} [f_j \otimes ((f \otimes f) \circ \Delta_A)] \circ \Delta_{A,i} \quad (4.3.8a) \\
&+ \sum_{i+\alpha=N+1 \atop \lambda,\mu \geq 0} [f \otimes ((f \otimes f) \circ \Delta_{A,\alpha})] \circ \Delta_{A,i} \quad (4.3.8b) \\
&+ \sum_{j+\beta+\gamma=N+1 \atop \lambda,\mu \geq 0} [f_j \otimes ((f \beta \otimes f \gamma) \circ \Delta_{A,\alpha})] \circ \Delta_{A,i} \quad (4.3.8c) \\
&+ \sum_{i+j+\alpha+\beta+\gamma=N+1 \atop \lambda,\mu \geq 0} [f_j \otimes ((f \beta \otimes f \gamma) \circ \Delta_{A,\alpha})] \circ \Delta_{A,i} \quad (4.3.8d) \\
&- \sum' [f_j \otimes (\Delta_{B,\mu} \circ f_\lambda)] \circ \Delta_{A,i}, \quad (4.3.8f)
\end{align*}

where the last sum is interpreted as in (4.3.5b) with the roles of \( j \) and \( k \) interchanged.

Now observe that each of the following sums is equal to 0: \( -\text{Ob}_B \circ f + (4.3.7d) + (4.3.7g), f^\otimes 3 \circ \text{Ob}_A + (4.3.6a) + (4.3.8b), (4.3.7a) + (4.3.2d), (4.3.7b) + (4.3.2d), (4.3.7c) + (4.3.8f), (4.3.7d) + (4.3.6f). \)

It follows that

\[ \delta_c \text{Ob}_F - \text{Ob}_B \circ f + f^\otimes 3 \circ \text{Ob}_A \]

\[ = \sum \{ [f_\alpha \otimes ((f_\beta \otimes f_\gamma) \circ \Delta_{A,\mu})] \circ \Delta_{A,\lambda} \]

\[ - \{ ((f_\alpha \otimes f_\beta) \circ \Delta_{A,\mu}) \otimes f_\lambda \} \circ \Delta_{A,\lambda}. \]

The sum on the right-hand side is taken over all \( \alpha, \beta, \gamma, \lambda, \mu \geq 0 \) such that:

1. \( \alpha + \beta + \gamma + \lambda + \mu = N + 1 \) with \( 1 \leq \alpha + \beta + \gamma \leq N \), or
2. \( \alpha + \beta = N + 1 \) with \( \alpha, \beta > 0 \) and \( \gamma = \lambda = \mu = 0 \), or
3. \( \alpha + \gamma = N + 1 \) with \( \alpha, \gamma > 0 \) and \( \beta = \lambda = \mu = 0 \), or
4. \( \beta + \gamma = N + 1 \) with \( \beta, \gamma > 0 \) and \( \alpha = \lambda = \mu = 0 \), or
5. \( \alpha + \beta + \gamma = N + 1 \) with \( \alpha, \beta, \gamma > 0 \) and \( \lambda = \mu = 0 \).

This sum is equal to 0, since \( \Delta_{A,t} = \sum_{i=0}^{N} \Delta_{A;it}t^i \) gives a \( K[t]/(t^{N+1}) \)-coalgebra structure on \( A[t]/(t^{N+1}) \).

This finishes the proof of Theorem 4.2 \( \square \)
ACKNOWLEDGMENT

The author thanks the referee for reading an earlier version of this paper and for pointing out the reference [9].

REFERENCES

[1] D. Balavoine, Deformations of algebras over a quadratic operad, Contemp. Math. 202 (1997), 207-234.
[2] D. V. Borisov, Formal deformations of morphisms of associative algebras, Int. Math. Res. Notices 41 (2005), 2499-2523.
[3] Y. Fréiger, A new cohomology theory associated to deformations of Lie algebra morphisms, Lett. Math. Phys. 70 (2004), 97-107.
[4] M. Gerstenhaber, On the deformation of rings and algebras, Ann. Math. 79 (1964), 59-103.
[5] M. Gerstenhaber and S. D. Schack, On the deformation of algebra morphisms and diagrams, Trans. Amer. Math. Soc. 279 (1983), 1-50.
[6] M. Gerstenhaber and S. D. Schack, On the cohomology of an algebra morphism, J. Algebra 95 (1985), 245-262.
[7] M. Gerstenhaber and S. D. Schack, Sometimes $H^1$ is $H^2$ and discrete groups deform, Contemp. Math. 74 (1988), 149-168.
[8] M. Gerstenhaber and S. D. Schack, Algebras, bialgebras, quantum groups, and algebraic deformations, Contemp. Math. 134 (1992), 51-92.
[9] V. Hinich, Deformations of homotopy algebras, Comm. Alg. 32 (2004), 473-494.
[10] A. Nijenhuis and R. W. Richardson, Deformations of homomorphisms of Lie algebras, Bull. Amer. Math. Soc. 73 (1967), 175-179.

Department of Mathematics, The Ohio State University at Newark, 1179 University Drive, Newark, OH 43055, USA

E-mail address: dyau@math.ohio-state.edu