Viscous Flows and Conditions for Existence of
Shocks in Relativistic Magnetic
Hydrodynamics

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Abstract

We present a criterion for a shock wave existence in relativistic magnetic hydrodynamics with an arbitrary (possibly non-convex) equation of state. The criterion has the form of algebraic inequality that involves equation of state of the fluid; it singles out the physical solutions and it can be easily checked for any discontinuity satisfying conservation laws. The method of proof uses introduction of small viscosity into the coupled set of equations of motion of ideal relativistic fluid with infinite conductivity and Maxwell equations.

1 Introduction

It is well known that not every discontinuous solution that can be formally obtained from equations of hydrodynamics is physically allowed. The most well known example is prohibition of rarefaction shock waves in media with a convex equation of state as ones through which entropy is decreasing. But from the point of view of hydrodynamic equations only, such shock waves have the same “rights” as usual compression shock waves. This example says that for a shock wave to be physical some additional (non-hydrodynamic) restrictions must be applied.

In case of usual hydrodynamics (without magnetic field) there are three famous types of additional criteria that all should hold.

• Entropy condition. This criterion means that entropy of a fluid must increase as it is crossing a shock front.
Figure 1: Shock wave viscous profile. Dotted line: limiting case of zero viscosity

- Evolutionarity conditions: \( v_{\text{ahead}} > c_s \), \( v_{\text{behind}} < c_s \). This criterion follows from condition that small perturbations must be uniquely defined [1].

- Existence of viscous profile. This condition takes into account that shock wave is not a sharp step but, due to dissipative processes, some continuously smeared out region that we call a profile or structure (see Fig.1). From mathematical viewpoint the shock is a generalized solution that may be represented as a certain limit of viscous flows in zero viscosity limit [2].

If all these conditions hold then one can be sure that a shock wave exists. In case of ”normal” media these criteria are equivalent. Conditions for normal media are regulated by the set of Bethe-Weyl conditions [2]; and among them the most important is the requirement of convexity of the equation of state (EOS), i.e. convexity of the Poisson adiabats.

For a normal EOS all compression shock waves exist and all rarefaction shock waves are prohibited. However, in case of a non-convex (anomalous) EOS the question of existence becomes much more complicated and the criteria 1-3 are not equivalent. Rarefaction shocks, split shocks, compression simple waves and complex wave configurations become possible. H.Bethe and H.Weyl have first studied this situation for shock waves in arbitrary fluids in classical hydrodynamics (see, e.g., [2]). How the conditions 1-3 are ordered relatively to each other is discussed in [3]; the relativistic case is considered in [4]. Note that criterion 3 appears to be the most effective one and it imposes the most stringent requirements in case of a general EOS.

In relativistic hydrodynamics the anomalous equation of state arises in super-dense media particularly with phase transitions; this has applications in elementary particle physics (see, e.g., [5, 6]) and in relativistic astrophysics (super-dense matter in neutron and exotic stars, gravitational collapse and
supernova explosions, models of gamma ray bursts). The relativistic version of the convexity condition is
\[ \frac{\partial^2 p}{\partial \tau^2} \bigg|_S > 0, \quad \tau = (\varepsilon + p)/n^2 \] 
where \( n \) is the baryon number density, \( p \) is the pressure; \( \varepsilon \) is the energy density, \( S \) is the entropy per baryon. In a neighbourhood of phase transitions in super-dense matter the convexity condition can be violated. The conditions for existence of relativistic hydrodynamic shocks viscous profile in case of a general EOS have been investigated in \[5, 6\].

The situation in magnetohydrodynamics is more complicated, because of additional degrees of freedom due to magnetic field. In present paper we review the main points of investigation of existence criteria for magnetohydrodynamic shocks. We introduce a viscosity term into equations of relativistic magnetohydrodynamics in order to study shock structure and determine conditions for a viscous profile existence. We derive these conditions for arbitrary EOS on basis of Landau-Lifshits viscosity tensor with one of viscosity coefficients put equal to zero. We show that there is a domain of avoidance that makes our conditions more stringent than evolutionarity conditions. Then we extend this analysis to both non-zero viscosity coefficients. Our consideration relies upon the results of \[7]-[9].

2 Basic Equations

We consider ideal relativistic fluid with infinite conductivity in magnetic field with energy-momentum tensor
\[ T^{\mu\nu} = (p^* + \varepsilon^*)u^\mu u^\nu - p^* g^{\mu\nu} - \frac{\mu}{4\pi} h^\mu h^\nu, \] 
where \( u^\mu \) – four velocity, \( F_{\mu\nu} \) – electromagnetic field tensor, \( g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1,-1,-1,-1) \), \( h^\mu = -\frac{1}{2} e^{\mu\alpha\beta\gamma} F_{\alpha\beta} u_\gamma \) (4- vector of magnetic field), \( e^{\alpha\beta\gamma\delta} \) – absolutely anti-symmetric symbol, \( p^* = p + \frac{\mu}{8\pi} |h|^2 \), \( \varepsilon^* = \varepsilon + \frac{\mu}{8\pi} |h|^2 \), \( |h|^2 = -h^\alpha h_\alpha > 0 \), \( \mu = \text{const} \) – magnetic permeability. The tensor (2) is the sum of corresponding tensors of hydrodynamics and electrodynamics.

Then we introduce the Landau-Lifshits viscosity tensor \[1\]:
\[ \tau_{\mu\nu} = \eta (u_{\mu,\nu} + u_{\nu,\mu} - u_\mu u^\alpha u_{\nu,\alpha} - u_\nu u^\alpha u_{\mu,\alpha}) + (\xi - 2\eta/3) u^\alpha_{,\alpha} (g_{\mu\nu} - u_{\mu} u_{\nu}), \]
into equations of motion

\[ \partial_{\mu}(T^{\mu\nu} + \tau^{\mu\nu}) = 0, \] (4)

\[ \partial_{\mu}(nu^{\mu}) = 0, \] (5)

\[ \partial_{\mu}(u^{\mu}h^{\nu} - u^{\nu}h^{\mu}) = 0. \] (6)

Here Eq. (4) describes energy-momentum conservation, Eq. (5) – baryon number conservation; Eq. (6) follows from Maxwell equations for the electromagnetic field.

In order to obtain shock structure in the rest frame of the shock we have to take a one-dimensional stationary solution of (4)–(6) depending upon the only variable \( x \) corresponding to the direction perpendicular to the shock front. This yields

\[ T^{1\nu} + \tau^{1\nu} = T^{1\nu}_{(0)}, \] (7)

\[ u^{1}h^{\nu} - h^{1}u^{\nu} = H^{\nu} \equiv u^{1}_{(0)}h^{\nu}_{(0)} - h^{1}_{(0)}u^{\nu}_{(0)} \] (8)

\[ nu^{1} = n_{(0)}u^{1}_{(0)} \] (9)

If we denote the state ahead of the shock transition by index ”0” then relation between states on both sides of the shock (with state ”1” being the state behind a shock) must be as follows

\[ T^{1\nu}_{(2)} = T^{1\nu}_{(0)}, \] (10)

\[ u^{1}_{(2)}h^{\nu}_{(2)} - h^{1}_{(2)}u^{\nu}_{(2)} = H^{\nu} \equiv u^{1}_{(0)}h^{\nu}_{(0)} - h^{1}_{(0)}u^{\nu}_{(0)}, \] (11)

\[ n_{(2)}u^{1}_{(2)} = n_{(0)}u^{1}_{(0)}. \] (12)

By definition we say that shock wave ”0”→”1” has a viscous profile if there is a continuous solution of (7)–(9) having corresponding asymptotics for \( x \rightarrow \infty \) and \( x \rightarrow -\infty \).
3 The case of one viscosity coefficient $\eta=0$.

One can choose the reference frame such as $u^3 \equiv 0$ and $h^3$. Then one can obtain

$$\varepsilon^* u^1 = T_{(0)}^{1\mu} u^\mu$$  \hspace{1cm} (13)

from (7) and

$$h^\mu = \frac{1}{u^1} [H^\mu - u^\mu H^\alpha u_\alpha]$$  \hspace{1cm} (14)

from (8).

From continuity equation

$$n = u_{(0)}^1 n_0 / u^1,$$  \hspace{1cm} (15)

then Eqs. (13)-(16) allow us to express all the variables in terms of $u^1$ and $u^2$.

Using this one can get then relationship between $u^1$ and $u^2$

$$\left( H^2 u^2 - H^0 u^0 \right) \left( H^2 u^0 - H^0 u^2 \right) = 4\pi u^1 \left( T_{(0)}^{10} u^2 - T_{(0)}^{12} u^0 \right),$$  \hspace{1cm} (16)

This allows us to relate three-dimensional velocity components $v_1 = u^1 / u^0$ and $v_2 = u^2 / u^0$ with each other. Note that we do not consider here the trivial case, when the magnetic field is zero. We have an explicit dependence $v_1(v_2)$ from (16) that leaves us the only ordinary differential equation for $u^1$

$$\zeta \left( 1 + (u^1)^2 \right) \frac{du^1}{dx} = T^{11} - T_{(0)}^{11},$$  \hspace{1cm} (17)

that describes the viscous structure of profile. Our problem is reduced to find a necessary condition for a regular solution of Eq. (17) to exist. We assume also that hydrodynamic parameters in the shock structure are monotonous functions. If we suppose that such solution exists then we have finite $dv_1/dv_2$ and for monotonous structures this cannot change its sign. Then the function $v_2(v_1)$ is well defined during the transition ”0”$\rightarrow$”1”. In this case the right hand side of (17) must not change its sign; otherwise the hydrodynamic parameters could not reach the point ”1”. This yields necessary condition we are looking for. For details see [7, 8].

In view of (9) we can express $v_1$ and $v_2$ in terms of the specific volume $V = 1/n$. 

5
If introduce the function
\[ \tilde{p}(V) = \left\{ 1 + (u_1^1)^2 \right\}^{-1} \left\{ T_{(0)}^{11} + \frac{1}{4\pi} (H^\alpha u_\alpha)^2 - T_{(0)}^{1\mu} u_\mu u_1^1 \right\} \]
\[- \frac{1}{8\pi} (u_1^1)^2 \left\{ (H^\alpha u_\alpha)^2 - H^\alpha H\right\} ;
\]
where \( \bar{\varepsilon}(V) = T_{(0)}^{1\mu} u_\mu / u_1^1 - \frac{1}{8\pi} |h|^2 \), then we obtain following criterion of admissibility of stationary shock transition:
\[ (V_1 - V_0) (p(V, \bar{\varepsilon}(V)) - \tilde{p}(V)) \geq 0 \]  \hspace{1cm} (18)
for all \( V \) between \( V_0 \) and \( V_1 \).

This criterion has the following consequences [7,8]:

- If the shock satisfies the criterion, then it also satisfies the entropy criterion.
- The criterion can be written in terms of the shock adiabat \( p_H(V) \) as follows
\[ (V_1 - V_0) (p_H(V) - \tilde{p}(V)) \geq 0 \ . \]  \hspace{1cm} (19)
In case of a non-single valued shock adiabat the criterion can be formulated in terms of absence of intersections of \( p_H(V) \) and \( \tilde{p}(V) \) everywhere except \( V_0 \) and \( V_1 \).

- In the neighbourhoods of initial and final points we get that the possible fast shock transitions are found to satisfy the following relations between the speed of the shock and the characteristic speeds at initial state “0” (ahead of the shock)
\[ v_{sh(0)} > v_f(0) > v_{N(0)} > v_{A(0)} > v_{sl(0)} \]  \hspace{1cm} (20)
and at the final state “1” (behind the shock).
\[ v_f(2) > v_{sh(2)} > v_{N(2)} > v_{A(2)} > v_{sl(2)} \]  \hspace{1cm} (21)

Here we have introduced a new characteristic velocity that satisfies equation
\[ (p + \varepsilon) (u_N^1)^2 [(u_N^1)^2 + 1] = \frac{1}{4\pi} (h^1)^2 \]  \hspace{1cm} (22)
It is distinct from Alfven velocity that enters usual evolutionarity conditions.
For slow shocks we have the usual evolutionarity conditions.

\[ v_{A(0)} > v_{sh(0)} > u_{sl(0)} ; \quad v_{f(2)} > v_{N(2)} > v_{A(2)} > v_{sl(2)} > v_{sh(2)}. \]  

(23)

So we have obtained a domain of avoidance behind the fast shock \([v_N, v_A]\) that satisfies evolutionarity conditions but nevertheless must be considered as non-physical one as according to our criterion there is no shock wave with viscous profile there. Note that the existence of \(v_N\) is due to monotonicity of the dependence \(v_1(v_2)\).

This relations for the final state appear to be more restrictive than the standard evolutionary criterion that does not involve \(V_N\). However in the nonrelativistic limit \(V_N \rightarrow V_A\) and these inequalities tend to the standard evolutionarity conditions.

### 4 Two nonzero viscosity coefficients

In this case when \(\eta \neq 0\) instead of algebraic equation (15) we have the second ordinary differential equation (for \(u^2\)). It is convenient to introduce a new variable \(v = u^2/\sqrt{1 + (u^1)^2}\).

After this we have the following dynamical system regarding to \(u^1\) and \(v\)

\[
(\xi + 4\eta/3) \frac{du^1}{dx} = F_1(u^1, v),
\]

\[
\frac{dv}{dx} = F_2(u^1, v)
\]

(24)

where

\[
F_1(u^1, v) = p - (1 + (u^1)^2)^{-1} \left[ T_0^{11} + \frac{\mu}{4\pi}(H^\alpha u_\alpha)^2 - T_0^{1\mu}u_\mu u^1 \right] + \frac{\mu}{8\pi}(u^1)^{-2}[(H^\alpha u_\alpha)^2 - H^\alpha H_\alpha]
\]

\[
F_2(u^1, v) = \frac{(T_0^{10} u^0 - T_0^{12} u^2) u^2 u^1 - (\mu/4\pi) u^2 (H^\alpha u_\alpha)^2}{4\pi u^1 [1 + (u^1)^2]^{3/2}} + \frac{(\mu/4\pi) H^2 (H^\alpha u_\alpha) - T_0^{12} u^1}{[1 + (u^1)^2]^{3/2} u^1}
\]

Let the state parameters \(u^\mu_{(0)}, h^\mu_{(0)}, n_0, p_0\) ahead of the shock and \(u^\mu_{(1)}, h^\mu_{(1)}, n_1, p_1\) behind the shock satisfy the conservation laws (10)–(12) that relate hydrodynamic quantities on both sides of the shock. We denote \(y = u^1, y_0 = u^1_{(0)}\),
Figure 2: Fast shock. The arrows show direction of the phase trajectories when crossing the curves $V_1$ and $V_2$. Red dashed line corresponds to viscous profile of the shock.

$y_1 = u\left(\frac{y_1}{2}\right)$. In this section we deal with the state variables $u^\mu$, $h^\mu$, $n$, $p$ ahead of the shock unless otherwise stated, and we omit further the index "0" for these variables.

Now we rewrite the conditions of viscous profile existence in terms of the right hand sides of the system (24) and the curves $V_1, V_2$:

$v = V_1(y): F_1(y, V_1(y)) = 0$
$v = V_2(y): F_2(y, V_2(y)) = 0$
$v_1 = V_1(y_1) = V_2(y_1); v_0 = V_1(y_0) = V_2(y_0), V_1(y) \neq V_2(y) \quad \forall y \in (y_1, y_0)$

The conditions can be formulated as follows:

A The function $v = V_2(y)$ is monotonous on $(y_1, y_0)$.

B For $y \in (y_1, y_0)$ the following inequality is valid:

$$(y_0 - y_1)F_1(y, V_2(y)) < 0$$

C We suppose that $h^1 h^2 \neq 0$ at the point "0" (this is a technical requirement).

D (D1): $u^1 > u_f$ ahead of the shock at the point "0" and $u_N < u^1 < u_f$ behind the shock at the point "1", or (D2): $u_A > u^1 > u_{sl}$ ahead of the shock at the point "0" and $u_{sl} > u^1$ behind the shock at the point "1".

The last inequalities are the famous relations between the velocities ahead of the shock. The first inequality (D1) corresponds to the fast shock and the second one (D2) – to the slow shock. In case $\eta = 0$ inequalities (D1) and (D2) follow from criterion of viscous profile existence of the section (5).
Figure 3: Slow shock. The arrows show direction of the phase trajectories when crossing the curves $V_1$ and $V_2$. Red dashed line corresponds to viscous profile of the shock.

Under these conditions one can show, e.g., that in case D1 the rest point "1" of the system (17) is a saddle point. In case D2 the rest point "0" is a saddle point. This enables us to restore the qualitative behaviour of solutions to the system (22) inside the domain of the phase plane between the curves $V_1$ and $V_2$.

Typical situation is shown on Figs.2 and 3. In case of the fast shock (D1) the solutions leaving "0" go out of the domain, except the only solution representing the separatrix of the saddle point "1"; just this solution represents the shock viscous structure.

In case of the slow shock (D2) there is the only solution leaving "0" that tends to "1"; just this solution represents the shock viscous structure. The other solutions enter the domain, crossing $V_2$ from right to left and $V_1$ bottom-up.

As the first result we have obtained following sufficiency of conditions for $\eta > 0$, $\xi > 0$.

Let the states "0" ahead of the shock and "1" behind the shock satisfy the conservation equations. If the conditions (A–D) are satisfied, then the MHD shock transition "0"→"1" has a viscous profile satisfying the equations (7)–(12).

More detailed proof will be published elsewhere [9].

If additional limitations on EOS (e.g., convexity) are absent, the criterion (B) dealing with EOS for the whole interval between the states "0" and "1"
is evidently more restrictive than, e.g., evolutionarity conditions [9] or any other conditions that involve characteristics of the fluid only at initial and final states. On the other hand, the evolutionarity conditions can be derived from the viscous profile existence [8, 9].

Our criteria can be applied to an arbitrary smooth EOS. However, we must note that the requirement for $V_1(y)$ to be a continuous (single-valued) function is not trivial and can not be fulfilled in case of a certain equations of state (see, e.g., [2, 3]). Though consideration of a viscous profile seems to be rather effective for investigation of shock existence and stability, this method can not work in case of complicated EOS (see, e.g., remarks in [3] in case of relativistic hydrodynamics) that require either modification of the equations of motion or using additional physical information about solutions.

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