String Spectrum of 1+1-Dimensional Large $N$ QCD with Adjoint Matter

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ABSTRACT

We propose gauging matrix models of string theory to eliminate unwanted non-singlet states. To this end we perform a discretised light-cone quantisation of large $N$ gauge theory in 1+1 dimensions, with scalar or fermionic matter fields transforming in the adjoint representation of $SU(N)$. The entire spectrum consists of bosonic and fermionic closed-string excitations, which are free as $N \to \infty$. We analyze the general features of such states as a function of the cut-off and the gauge coupling, obtaining good convergence for the case of adjoint fermions. We discuss possible extensions of the model and the search for new non-critical string theories.
1. Introduction

The construction of string theories in non-critical dimensions is an interesting problem which remains largely unsolved. A fruitful idea is to study large-$N$ field theories, where the Feynman diagrams are grouped according to their two-dimensional topology and, after some adjustment of parameters, can be identified with string worldsheets. Recently this idea has been extensively developed to sum over all worldsheets embedded in dimensions $c \leq 1$ [1-4]. It is of great interest to extend this progress to higher-dimensional embeddings.

In a previous paper [5] we considered a two-dimensional interacting scalar matrix field theory with a global $SU(N)$ symmetry, where the Feynman graphs (without tadpoles) are finite. In such a theory the sum over the planar graphs is expected to possess a critical point similar to those found for $c \leq 1$. At this critical point the model should describe two-dimensional quantum gravity coupled to $c = 2$ matter. Our light-cone analysis indeed suggested that the critical point exists. The large-$N$ scalar field theory is a straightforward extension to higher dimension of the technique that has been so successful for $c \leq 1$. Is this a viable candidate for non-critical string theory in any dimension? We believe that the answer is negative. The main reason is the lack of decoupling of the $SU(N)$ non-singlet states, which have infinite degeneracies and cannot be regarded as closed-string states. The non-singlets are like strings that have disintegrated into separate bits. It is not hard to show [5] that the non-singlets have finite energy and, therefore, do not decouple. This should be contrasted with the $c = 1$ case, where the non-singlets are pushed to infinite energy in the continuum limit [4, 6, 7]. The basic physical reason for this difference is perhaps that, in the $0 + 1$-dimensional theory ($c = 1$) the space consists of only a point and a string cannot fall apart into its elementary constituents. In $1 + 1$ dimensions, however, the question of string stability already becomes crucial. For a large-$N$ model to be a good string theory, it is necessary that the field quanta (string bits) are absolutely bound into $SU(N)$ invariant states (the strings). In a scalar theory we may find some attraction of the quanta, as suggested by our results in [5], but the binding is not absolute. Therefore at sufficiently high energies the strings fall apart and the model does not make sense as a theory of elementary string excitations.

In [5] we suggested an obvious cure for this problem: introduction of a gauge field. The resulting confinement pushes the non-singlet states to infinite energy, so that the string
bits become absolutely bound into strings. Let us note that all the $c \leq 1$ models can be reformulated as gauge theories [8]. For instance, the gauged matrix quantum mechanics ($c = 1$) has the lagrangian

$$L = \text{Tr} \left\{ \frac{1}{2}(\dot{\phi} + i[A, \phi])^2 - V(\phi) \right\}.$$  \hspace{1cm} (1)

The corresponding hamiltonian is

$$H = \text{Tr} \left\{ \frac{1}{2}\Pi^2 + V(\phi) + i[\Pi, \phi]A \right\}.$$  \hspace{1cm} (2)

Thus, $A$ is simply a Lagrange multiplier which imposes the condition $i[\Pi, \phi]|\Psi >= 0$ on the physical states. Only the $SU(N)$ singlet wave functions satisfy this constraint, and the non-singlets are discarded. In the ungauged models the energies of the non-singlets diverge at the critical point of continuous worldsheets [4, 6, 7], resulting in equivalence with the gauged models at this point.

In this paper we consider the gauged matrix models in 1+1 dimensions, i.e. $SU(N)$ gauge theories coupled to matter fields in the adjoint representation. Here the effect of gauge fields is absolutely crucial because they produce confinement which otherwise would not exist. Also the two-dimensional nature of the problem means that there are no physical gluons and so strings are still built from the matter quanta alone. We carry out light-cone quantization of these models as $N \rightarrow \infty$ and study them as a function of the gauge coupling strength, this being of some interest in itself apart from the string interpretation. Our approach is similar to ‘t Hooft’s classic solution [11] of the two-dimensional large-$N$ gauge theory coupled to fermions in the fundamental representation. In that case quark and anti-quark pairs are confined into mesons whose masses are found from a simple bound state integral equation. This theory can be thought of as an open string model, the spectrum forming a single rising “Regge trajectory”. We will argue that the coupling to adjoint matter fields produces a kind of closed string analogue of the ‘t Hooft model. This is much more complicated than ‘t Hooft’s open string because our closed string can contain any number of matter quanta.

* The case of compact $c = 1$ is somewhat more subtle because there the non-singlets implement the effects of Kosterlitz-Thouless vortices [4, 6, 7, 8].

† The infinite coupling limit of such models has recently been studied on a lattice with a rather different motivation [9, 10].
(string bits), the light-cone hamiltonian connecting different number sectors. Therefore the bound state equation is hard to write down explicitly. However, if we introduce a cut-off in the form of discretized longitudinal momentum \([12, 13]\), the bound state equation can be set up easily and reduces to a matrix eigenvalue problem, with the size of the matrix diverging as the cut-off is removed. Even for relatively low values of the cut-off, some qualitative and sometimes quantitative conclusions can be drawn from this method.

The adjoint matter field can be taken to be either a scalar or a fermion, and we analyze both cases. The scalar case gives a kind of closed bosonic string theory with rising “Regge trajectories”. The adjoint fermion theory is perhaps more intriguing. It can be interpreted as a fermionic string whose spectrum contains both bosons (strings with even numbers of bits) and fermions (strings with odd numbers of bits). The discretized free string eigenvalue equation can be easily set up for both types of states. Low-lying states are found to be spectacularly pure in the sense that the mass eigenstates are almost exactly eigenstates of string length (number of matter quanta). This encourages us to go to much higher cutoffs by diagonalising in the subspace of states of a given length.

In section 2 we introduce the models and carry out light-cone quantization. In section 3 we introduce the discretization of longitudinal momentum and reduce the free string theory to a matrix eigenvalue problem. We give some of our numerical results on the spectrum as a function of the gauge coupling and cut-off. In section 4 we discuss possible extensions of this work to higher dimensions, supersymmetric spectrum, etc. and speculate on the search for new non-critical string theories.

2. Light-cone quantisation

In this section we consider the light-cone quantization of 1+1-dimensional SU\((N)\) gauge theory coupled to matter in the adjoint representation. Strings will arise as collective excitations and, depending on whether the matter field is a scalar or a fermion, will be of bosonic or bosonic and fermionic type.

**Scalar Matter**

Here the Minkowski space action is taken to be

\[
S_{sc} = \int d^2x \text{Tr} \left[ \frac{1}{2} D_\alpha \phi D^\alpha \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4g^2} F_{\alpha\beta} F^{\alpha\beta} \right],
\]

(3)
where \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i[A_\alpha, A_\beta] \) and the covariant derivative is \( D_\alpha \phi = \partial_\alpha \phi + i[A_\alpha, \phi] \).

The scalar \( \phi \) and the gauge potential \( A_\alpha \) are traceless \( N \times N \) hermitian matrix fields. To leading order in \( N \) it is not necessary to impose the tracelessness condition from the beginning. Instead, as we will show later, this condition is implemented by a simple restriction on the physical states. A similar theory, with \( \phi \) taken as a general complex matrix, was considered by Bardeen, Pearson and Rabinovici \[14\] as part of their attempt to solve higher-dimensional pure gauge theory. We have not included explicit self-interactions of the matter fields in (3) but we shall return to this question in section 4 where we discuss the possibility of finding new critical behaviour.

An interesting feature of the theory in eq. (3) is that it corresponds to dimensional reduction of the 2+1-dimensional pure gauge theory. If the 3-d gauge field \( A_\mu = (A_\alpha, A_3) \) is taken to be independent of \( x^3 \), then the action reduces to eq. (3) with \( m = 0 \) and \( \phi \sim A_3 \). Such a reduction is valid when \( x^3 \) is compact and small, for example at high temperature in the euclidean version. In fact polynomials of \( \phi \) can also be obtained in the reduced action by adding appropriate combinations of Wilson lines winding around the \( x^3 \) direction. The reduction generalises to higher numbers of compact and non-compact dimensions also, involving gauged multi-matrix models etc..

To perform light-cone quantisation of (3) we introduce variables \( x^\pm = (x^0 \pm x^1)/\sqrt{2} \) and treat \( x^+ \) as the time variable. Our conventions will be such that \( g^{+-} = g^{-+} = 1 \) so for any 2-vector \( k^\alpha \) we have \( k^+ = k^- \). As is standard we choose the convenient light-cone gauge \( A_- = 0 \); there are no Fadeev-Popov ghosts, and no need for a gauge fixing term since there are no dynamical degrees of freedom from the gauge potential to be quantized. The action reduces to

\[
S_{sc} = \int dx^+ dx^- \text{Tr} \left[ \partial_+ \phi \partial_- \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2g^2} (\partial_- A_+)^2 + A_+ J^+ \right] \tag{4}
\]

where the longitudinal momentum current \( J^+_ij = i[\phi, \partial_- \phi]_{ij} \). The action does not depend on \( x^+ \) ‘time’ derivatives of the gauge potential \( A_+ \). Therefore \( A_+ \) is not dynamical and may be eliminated by the constraint of its Euler-Lagrange equations. It is convenient to split the gauge field into its zero and non-zero modes as \( A_+ = A_{+,0} + \tilde{A}_+ \). Then the constraints are

\[
\int dx^- J^+ = 0, \quad \partial_-^2 \tilde{A}_+ - g^2 J^+ = 0. \tag{5}
\]
The first of these equations is enforced by $A_{+,0}$ which acts as a lagrange multiplier.

We can now proceed to quantise the remaining matter degrees of freedom. The light-cone momentum and energy $P^\pm = \int dx^- T^\pm$ are found to be

\begin{align*}
P^+ &= \int dx^- \text{Tr}[(\partial_- \phi)^2], \\
P^- &= \int dx^- \text{Tr} \left[ \frac{1}{2} m^2 \phi^2 - \frac{1}{2} g^2 J^+ \frac{1}{\partial^2} J^+ \right].
\end{align*}

(6)

Given an initial field configuration on the line $x^+ = 0$ we obtain the field at later times by Hamiltonian development. We choose the free field representation for $\phi(x^+ = 0)$, with mode decomposition:

\begin{equation}
\phi_{ij} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left( a_{ij}(k^+) e^{-ik^+ x^-} + a^\dagger_{ji}(k^+) e^{ik^+ x^-} \right). 
\end{equation}

(7)

As usual in light-cone quantisation, positive energy quanta have positive light-cone momentum $k^+$; this fact leads in large part to the successful employment of the discretised approach we formulate later. The modes $a_{ij}$ are normalised so that when we impose the canonical commutation relation at $x^+ = 0$;

\begin{equation}
[\phi_{ij}(x^-), \partial_- \phi_{kl}(\bar{x}^-)] = \frac{1}{2} i \delta(x^- - \bar{x}^-) \delta_{il} \delta_{jk},
\end{equation}

(8)

we obtain

\begin{equation}
[a_{ij}(k^+), a^\dagger_{lk}(\bar{k}^+)] = \delta(k^+ - \bar{k}^+) \delta_{il} \delta_{jk}.
\end{equation}

(9)

Substituting the mode expansion (7) into eq. (6) and normal ordering, we can derive explicit oscillator expressions. Thus, we find

\begin{equation}
P^+ = \int_0^\infty dk \; k a^\dagger_{ij}(k) a_{ij}(k)
\end{equation}

(10)

The hamiltonian $P^-$ of the $m \to 0$ (strong coupling) limit of the theory can be neatly written

* The symbol $\dagger$ is from here on always understood as the quantum version of complex conjugation and does not act on indices.
as (repeated indices are summed over);

\[ P^-(m \rightarrow 0) = g^2 \int_0^{\infty} \frac{dk^+}{(k^+)^2} \tilde{J}^+_{ik}(k^+) \tilde{J}^+_{ki}(-k^+) \]  \hspace{1cm} (11)

where we introduced the current Fourier transform

\[ \tilde{J}^+(k^+) = \frac{1}{\sqrt{2\pi}} \int dx^+ J^+(x^+) \exp(-ik^+x^+) . \]

We will often drop the superscript on \( k^+ \) for brevity. Explicitly we find, for \( q > 0 \),

\[ 2\sqrt{2\pi} \tilde{J}^+_{ki}(-q) = \int_0^{\infty} dp \frac{2p + q}{\sqrt{p(p + q)}} (a^\dagger_{jk}(p)a_{ji}(p + q) - a^\dagger_{ij}(p)a_{kj}(p + q)) \]

\[ + \int_0^q dp \frac{q - 2p}{\sqrt{p(q - p)}} a_{kj}(p)a_{ji}(q - p) \]  \hspace{1cm} (12)

and \( \tilde{J}^+_{ik}(q) = (\tilde{J}^+_{ki}(-q))^\dagger \). Implicit in eq. (11) is a choice of normal ordering that preserves positivity and ensures that the light-cone oscillator vacuum \( |0\rangle \) is the ground state of \( P^- \) with eigenvalue zero.

In general, a basis for the Fock space states can be taken of the form;

\[ a^\dagger_{i_1j_1}(k^+_1) \cdots a^\dagger_{i_bj_b}(k^+_b)|0\rangle ; \hspace{1cm} a_{ij}(k^+)|0\rangle = 0 . \]  \hspace{1cm} (13)

In a gauge theory however, there is a dramatic reduction in the number of physical states due to the gauge invariance. Namely, the quantum version of the first constraint in eq. (5) acts as the condition on physical states

\[ \tilde{J}^+_{ki}(q = 0)|\Psi\rangle = 0 . \]  \hspace{1cm} (14)

It follows that the \( k^+ \)-integral in eq. (11) has to be defined in a principal value sense, and that the energies of physical states stay finite. It is not hard to check that the only states that satisfy condition (14) are the singlets under the residual global \( SU(N) \) symmetry;

\[ N^{-b/2} \text{Tr}[a^\dagger(k^+_1) \cdots a^\dagger(k^+_b)]|0\rangle \]  \hspace{1cm} (15)

As we discussed in [5], these are precisely the states that have a nice closed string interpretation. For instance, (15) can be thought of as a single closed string consisting of \( b \) bits, each
bit carrying a proportion of the total longitudinal momentum of the string \( P^+ = \sum_{i=1}^{b} k_i^+ \). To leading order in \( N \), the only effect of the condition \( \text{Tr} \phi = 0 \) on our quantization procedure is to exclude the one-bit state \( \text{Tr}[a^+(P^+)]|0> \).

For comparison, recall that in the scalar theory with self-interaction but no gauge fields it was found in [5] that the non-singlet states, which have no simple closed string interpretation, were always of finite energy. After gauging the theory, the trivial confinement in two dimensions pushes these states to infinite energy. This apparently removes the main obstacle to building a connection between string theories and large-\( N \) models in dimensions greater than one.

In the limit \( N \to \infty \) the light-cone hamiltonian \( P^- \) propagates single closed string states to single closed string states. Terms that convert one closed string into two closed strings (two oscillator traces acting on the vacuum) are of order \( 1/N \). Thus, as expected, the string coupling constant is \( \sim 1/N \) and sending it to zero will allow us to study the spectrum of free closed strings. Practically speaking this limit means that the hamiltonian \( P^- \) acts locally on the string, splitting bits and joining adjacent bits in the trace. Explicitly, we find;

\[
P^- = \frac{1}{2} m^2 \int_0^\infty \frac{dk}{k} a^\dagger_{ij}(k)a_{ij}(k) + \frac{g^2 N}{4\pi} \int_0^\infty \frac{dk}{k} Ca_{ij}(k)a_{ij}(k)
+ \frac{g^2}{8\pi} \int_0^\infty \frac{dk_1dk_2dk_3dk_4}{\sqrt{k_1k_2k_3k_4}} \left\{ A\delta(k_1 + k_2 - k_3 - k_4)a^\dagger_{kj}(k_3)a^\dagger_{ji}(k_4)a_{ki}(k_1)a_{li}(k_2)
+ B\delta(k_1 + k_2 + k_3 - k_4)(a^\dagger_{kj}(k_1)a^\dagger_{ji}(k_2)a^\dagger_{ti}(k_3)a_{ki}(k_4) + a^\dagger_{kj}(k_4)a_{ki}(k_1)a_{ti}(k_2)a_{ij}(k_3)) \right\}
\]

(16)

The coefficients \( A, B \) and \( C \) are;

\[
A = \frac{(k_2 - k_1)(k_4 - k_3)}{(k_1 + k_2)^2} - \frac{(k_3 + k_1)(k_4 + k_2)}{(k_4 - k_2)^2}, \\
B = \frac{(k_1 + k_4)(k_3 - k_2)}{(k_3 + k_2)^2} + \frac{(k_3 + k_4)(k_1 - k_2)}{(k_1 + k_2)^2}, \\
C = \int_0^k dp \frac{(k + p)^2}{p(k - p)^2}.
\]

(17)

As we argued previously, those momentum integrals in (16) which have singularities are understood in the principal value sense. We shall not specify this more precisely since
in the following section we will discretise the momentum integrals explicitly for numerical computation, in which case the finiteness of the spectrum will be easy to demonstrate. Also, the ‘self-induced inertia’ \( C \) has been kept explicit and not absorbed in a renormalisation of the mass since it really removes a divergence present in the quartic terms. Though the dynamics governed by \( P^- \) are quite complicated, each term in (16) has a simple interpretation. The mass term represents a potential which measures the tensional energy of a string without changing its state. The kinetic terms either take three neighboring bits into one or one bit into three, or redistribute the momentum in two adjacent bits. (Obviously this does not mix states with odd and even numbers of bits, and these two sectors can be treated separately). These dynamics are to be understood by analogy with the fluctuations of the Liouville degree of freedom in non-critical string theory. The light-cone formalism for the \( c = 2 \) case, which leads to a string hamiltonian similar to (16), was developed in [5]. However, as we explained, there are significant physical differences between the pure scalar model and the gauge theory which manifest themselves in the absence of the non-singlet states in the latter. We will also find that the gauged model has no phase transition as a function of the only dimensionless parameter \( g/m^2 \). Perhaps, as we suggest in section 4, critical behaviour can be found in a gauged model with the quartic scalar potential.

**Fermionic Matter**

In this case the action is taken to be

\[
S_f = \int d^2x \text{Tr} \left[ i\gamma^T \gamma^0 \partial_{\alpha} \Psi - m\gamma^0 \Psi - \frac{1}{4g^2} F_{\alpha\beta} F^{\alpha\beta} \right]
\]  

(18)

where \( \Psi_{ij} = 2^{-1/4}(\psi_{ij}) \) is a two-component spinor, and the transposition acts only on the Dirac indices, \( \Psi^T_{ij} = 2^{-1/4}(\psi_{ij} \chi_{ij}) \). \( \chi \) and \( \psi \) are traceless hermitian \( N \times N \) matrices of grassmann variables,

\[
\chi^{*}_{ij} = \chi_{ji}, \quad \psi^{*}_{ij} = \psi_{ji},
\]

and the covariant derivative is defined by \( D_{\alpha} \Psi = \partial_{\alpha} \Psi + i[A_{\alpha}, \Psi] \). Choosing the light-cone gauge \( A_- = 0 \), and the ‘chiral’ representation \( \gamma^0 = \sigma_2, \gamma^1 = i\sigma_1 \), we find the action

\[
S_f = \int dx^+ dx^- \text{Tr} \left[ i\psi \partial^+ \psi + i\chi \partial^- \chi - i\sqrt{2}m\chi \psi + \frac{1}{2g^2}(\partial^- A_+)^2 + A_+ J^+ \right]
\]  

(19)

where the longitudinal momentum current is now of the form \( J^+_{ij} = 2\psi_{ik}\psi_{kj} \). The gauge
potential $A_+$ and the left-moving fermion $\chi$ are non-dynamical degrees of freedom and can be eliminated by their constraint equations. Two of the constraints are identical to those in the scalar case, eq. (5), while $\chi$ is determined by

$$\sqrt{2}\partial_-\chi - m\psi = 0 .$$ (20)

Using the constraints, we find that eq. (6) of the scalar case is replaced by

$$P^+ = \int dx^- \text{Tr}[i\psi\partial_-\psi] ,$$

$$P^- = \int dx^- \text{Tr} \left[ -\frac{im^2}{2}\psi \frac{1}{\partial^-}\psi - \frac{1}{2}g^2 J^+ \frac{1}{\partial^-} J^+ \right] .$$ (21)

Now we introduce the mode expansion,

$$\psi_{ij} = \frac{1}{2\sqrt{\pi}} \int_0^\infty dk^+ \left( b_{ij}(k^+) e^{-ik^+x^-} + b_{ji}^\dagger(k^+) e^{ik^+x^-} \right) .$$ (22)

From the canonical commutation relations

$$\{ \psi_{ij}(x^-), \psi_{kl}(\tilde{x}^-) \} = \frac{1}{2} \delta(x^- - \tilde{x}^-) \delta_{ij}\delta_{jk}$$ (23)

it follows that

$$\{ b_{ij}(k^+), b_{lk}^\dagger(\tilde{k}^+) \} = \delta(k^+ - \tilde{k}^+) \delta_{il}\delta_{jk}$$ (24)

In terms of the oscillators, eq. (21) assumes the form

$$P^+ = \int_0^\infty dk \ k b_{ij}^\dagger(k) b_{ij}(k) ,$$ (25)

$$P^- = \frac{1}{2}m^2 \int_0^\infty \frac{dk}{k} b_{ij}^\dagger(k) b_{ij}(k) + \frac{g^2 N}{\pi} \int_0^\infty \frac{dk}{k} C b_{ij}^\dagger(k) b_{ij}(k)$$

$$+ \frac{g^2}{2\pi} \int_0^\infty dk_1 dk_2 dk_3 dk_4 \left\{ A\delta(k_1 + k_2 - k_3 - k_4)b_{k_3j}(k_3)b_{j1}(k_4)b_{kl}(k_1)b_{li}(k_2) + B\delta(k_1 + k_2 + k_3 - k_4)(b_{k_3j}(k_4)b_{kl}(k_1)b_{li}(k_2)b_{ij}(k_3) - b_{k_3j}(k_1)b_{jl}(k_2)b_{li}(k_3)b_{kl}(k_4)) \right\}$$ (26)
where the coefficients are now given by

\[
A = \frac{1}{(k_4 - k_2)^2} - \frac{1}{(k_1 + k_2)^2}, \\
B = \frac{1}{(k_2 + k_3)^2} - \frac{1}{(k_1 + k_2)^2}, \\
C = \int_0^k dp \frac{p}{(p - k)^2}.
\] (27)

The discussion of the space of states is similar to that in the scalar case. Again, the confinement requires the physical states to be SU(\(N\)) singlets. One important new feature is that the strings with odd numbers of bits are fermions, while the strings with even numbers of bits are bosons. To leading order in \(1/N\) (free string approximation) these two sectors do not mix. They will be discussed separately in section 3.

3. Discretized formulation and numerical results

In order to study the theories of section 2, we will introduce a cut-off in the form of discretized longitudinal momentum \([12, 13]\). Now \(k^+\) is only allowed to take values \(nP^+/K\) where the integer \(n \leq K\) and \(K \to \infty\) in the continuum limit. This can be arranged by making \(x^-\) compact and adopting periodic boundary conditions on the matter fields \([13]\). Moreover in a massive theory quanta with \(k^+ = 0\) are excluded because they carry infinite energy. Therefore \(n\) is restricted to be a positive integer and the mode expansions become

\[
\phi_{ij} = \frac{1}{\sqrt{4\pi}} \sum_{n=1}^\infty \frac{1}{\sqrt{n}} \left( A_{ij}(n)e^{-iP^+nx^-/K} + A_{ji}^\dagger(n)e^{iP^+nx^-/K} \right), \\
\psi_{ij} = \frac{1}{\sqrt{4\pi}} \sum_{n=1}^\infty \left( B_{ij}(n)e^{-iP^+nx^-/K} + B_{ji}^\dagger(n)e^{iP^+nx^-/K} \right),
\] (28)

with the oscillator algebra

\[
[A_{ij}(n), A_{lk}^\dagger(n')] = \{ B_{ij}(n), B_{lk}^\dagger(n') \} = \delta_{nm}\delta_{ij}\delta_{jk}.
\] (29)
In the case of the scalar matter, a normalized state of $b$ bits is of the form

$$\frac{1}{N^{b/2}\sqrt{s}} \text{Tr}[A^\dagger(n_1) \cdots A^\dagger(n_b)]|0 >.$$ \hspace{1cm} (30)

The states are defined by ordered partitions of $K$ into $b$ positive integers, modulo cyclic permutations. Therefore the closed strings are oriented. If $(n_1, n_2, \ldots, n_b)$ is taken into itself by $s$ out of $b$ possible cyclic permutations, then the corresponding state receives a normalization factor $1/\sqrt{s}$. In the absence of special symmetries, $s = 1$.

For the fermionic matter, if $b$ is odd (fermionic states) the discussion above goes through after replacing $A_{ij}$ by $B_{ij}$. Thus the normalized states

$$\frac{1}{N^{b/2}\sqrt{s}} \text{Tr}[B^\dagger(n_1) \cdots B^\dagger(n_b)]|0 >$$ \hspace{1cm} (31)

are again in one-to-one correspondence with ordered partitions of $K$ into $b$ positive integers, modulo cyclic permutations. For even $b$ however, this is no longer the case since some states vanish due to the fermionic statistics of the oscillators. A simple example is

$$\text{Tr}[B^\dagger(n_1)B^\dagger(n_1)]|0 >= 0.$$ \hspace{1cm} (32)

In general, all partitions of $K$ where $b/s$ is odd do not give rise to states. For even $b/s$, the normalized states are constructed in the same way as before and are given by eq. (31).

In all cases the restriction $\sum_{i=1}^b n_i = K$ on the positive integers $n_i$ makes the total number of states finite. This allows for an explicit diagonalization of the $(\text{mass})^2$ operator $2P^+P^-$. The goal is to study the mass spectrum as a function of the increasing cut-off $K$, looking for convergence of the low-lying masses. For the scalar matter, we obtain

$$2P^+P^- = \frac{g^2 N}{4\pi} K(yV_{sc} + T_{sc}) ,$$ \hspace{1cm} (33)

where

$$V_{sc} = \sum_{n=1}^\infty \frac{1}{n} A^\dagger_{ij}(n)A_{ij}(n)$$ \hspace{1cm} (34)
is the mass term, and

\[
T_{sc} = 2 \sum_{n=1}^{\infty} \frac{1}{n} A^\dagger_{ij}(n) A_{ij}(n) \sum_{m=1}^{n-1} \frac{(n + m)^2}{(n - m)^2m} + \\
\frac{1}{N} \sum_{n_i=1}^{\infty} \left\{ \frac{\delta_{n_1+n_2,n_3+n_4}}{\sqrt{n_1n_2n_3n_4}} \left[ \frac{(n_2 - n_1)(n_4 - n_3)}{(n_1 + n_2)^2} - \frac{(n_3 + n_1)(n_4 + n_2)}{(n_4 - n_2)^2} \right] A^\dagger_{kj}(n_3) A^\dagger_{ji}(n_4) A_{kl}(n_1) A_{li}(n_2) \\
+ \frac{\delta_{n_1+n_2+n_3,n_4}}{\sqrt{n_1n_2n_3n_4}} \left[ \frac{(n_1 + n_4)(n_3 - n_2)}{(n_3 + n_2)^2} + \frac{(n_3 + n_4)(n_1 - n_2)}{(n_1 + n_2)^2} \right] \\
(B^\dagger_{kj}(n_4) B_{kl}(n_1) B_{li}(n_2) B_{ij}(n_3) - B^\dagger_{kj}(n_1) B^\dagger_{ji}(n_2) B^\dagger_{li}(n_3) B_{kl}(n_4)) \right\} 
\]

(35)
is the term generated by the gauge current-current interaction. The cases where the denominator is zero are understood to be excluded from the sum; this follows from the discretized analogue of the principal value prescription. The only dimensionless parameter in the problem is \( y = \frac{4\pi m^2}{g^2N} \), while \( g^2N \) sets the scale of the “string tension”. Large \( y \) corresponds to weak coupling, while \( y \to 0 \) is the strong coupling limit.

For the fermionic matter, we find

\[
2P^+P^- = \frac{g^2N}{\pi} K (xV_f + T_f) , 
\]

(36)
where \( x = \frac{\pi m^2}{g^2N} \) is the dimensionless parameter. Here the mass term \( V_f \) is given by eq. (34) with the \( A \)'s replaced by the \( B \)'s. The term generated by the gauge interaction is now

\[
T_f = 2 \sum_{n=1}^{\infty} B^\dagger_{ij}(n) B_{ij}(n) \sum_{m=1}^{n-1} \frac{1}{(n - m)^2m} + \\
\frac{1}{N} \sum_{n_i=1}^{\infty} \left\{ \frac{\delta_{n_1+n_2,n_3+n_4}}{\sqrt{n_1n_2n_3n_4}} \left[ \frac{1}{(n_4 - n_2)^2} - \frac{1}{(n_1 + n_2)^2} \right] B^\dagger_{kj}(n_3) B^\dagger_{ji}(n_4) B_{kl}(n_1) B_{li}(n_2) \\
+ \frac{\delta_{n_1+n_2+n_3,n_4}}{\sqrt{n_1n_2n_3n_4}} \left[ \frac{1}{(n_3 + n_2)^2} - \frac{1}{(n_1 + n_2)^2} \right] \\
(B^\dagger_{kj}(n_4) B_{kl}(n_1) B_{li}(n_2) B_{ij}(n_3) - B^\dagger_{kj}(n_1) B^\dagger_{ji}(n_2) B^\dagger_{li}(n_3) B_{kl}(n_4)) \right\} 
\]

(37)
To illustrate the calculation of the \((mass)^2\) matrix with a simple example, consider the theory with the adjoint fermions. We will set \( K = 6 \) and study the even-bit (boson) and
the odd-bit (fermion) sectors separately: working to leading order in $1/N$ the hamiltonian preserves the number of bits modulo 2. For the even-bit sector the normalized states are

$$|1> = \frac{1}{N} \text{Tr}[B^\dagger(3)B^\dagger(1)B^\dagger(1)B^\dagger(1)]|0>,$$

$$|2> = \frac{1}{N^2} \text{Tr}[B^\dagger(2)B^\dagger(2)B^\dagger(1)B^\dagger(1)]|0>,$$

$$|3> = \frac{1}{N^2\sqrt{2}} \text{Tr}[B^\dagger(2)B^\dagger(1)B^\dagger(2)B^\dagger(1)]|0>,$$

$$|4> = \frac{1}{N} \text{Tr}[B^\dagger(5)B^\dagger(1)]|0>,$$

$$|5> = \frac{1}{N} \text{Tr}[B^\dagger(4)B^\dagger(2)]|0>.$$  

(38)

Our calculation gives

$$xV_f + T_f = \begin{pmatrix}
\frac{10x}{3} + \frac{7}{2} & 0 & 0 & 0 & 0 \\
0 & 3x + \frac{653}{144} & 0 & -\frac{7}{72} & \frac{5}{18} \\
0 & 0 & 3x + \frac{40}{9} & 0 & 0 \\
0 & -\frac{7}{72} & 0 & \frac{6x}{5} + \frac{107}{36} & -\frac{16}{9} \\
0 & \frac{5}{18} & 0 & -\frac{16}{9} & \frac{3x}{4} + \frac{47}{9}
\end{pmatrix}.$$  

(39)

Now for the odd-bit sector the normalized states are

$$|1> = \frac{1}{N^{5/2}} \text{Tr}[B^\dagger(2)B^\dagger(1)B^\dagger(1)B^\dagger(1)B^\dagger(1)]|0>,$$

$$|2> = \frac{1}{N^{3/2}} \text{Tr}[B^\dagger(4)B^\dagger(1)B^\dagger(1)]|0>,$$

$$|3> = \frac{1}{N^{3/2}} \text{Tr}[B^\dagger(3)B^\dagger(2)B^\dagger(1)]|0>,$$

$$|4> = \frac{1}{N^{3/2}} \text{Tr}[B^\dagger(3)B^\dagger(1)B^\dagger(2)]|0>,$$

$$|5> = \frac{1}{N^{3/2}} \text{Tr}[B^\dagger(2)B^\dagger(2)B^\dagger(1)]|0>.$$  

(40)

Here we find

$$xV_f + T_f = \begin{pmatrix}
\frac{9x}{2} + \frac{43}{36} & 0 & 0 & 0 & 0 \\
0 & \frac{9x}{4} + \frac{291}{100} & -\frac{117}{100} & -\frac{117}{100} & 0 \\
0 & -\frac{117}{100} & \frac{11x}{6} + \frac{16969}{3600} & -\frac{7331}{3600} & -\frac{15\sqrt{3}}{16} \\
0 & -\frac{117}{100} & -\frac{7331}{3600} & \frac{11x}{6} + \frac{16969}{3600} & -\frac{15\sqrt{3}}{16} \\
0 & 0 & -\frac{15\sqrt{3}}{16} & \frac{15\sqrt{3}}{16} & \frac{3x}{2} + \frac{99}{16}
\end{pmatrix}.$$  

(41)

For larger values of $K$ we used Mathematica to generate the states, calculate and diagonalize the (mass)$^2$ matrices for a range of values of $x, y$ and $K$. At the values of $K$ up to which we were able to work, we found very poor convergence of the spectrum for the
scalar matter. The convergence improves as $y$ increases toward weak coupling, since the $y \to \infty$ limit yields the free particle spectrum, but no reliable data were obtainable in the strong coupling region. To illustrate this we plot in Figure 1 the (mass)$^2$ at $y = 0$ for a low-lying state of scalar quanta in the even-bit sector as a function of $K$. In this and subsequent convergence plots the full matrix was used up to $K = 12$. In order to obtain some idea of how convergence continues, we used the finding that low-lying mass eigenstates are composed almost entirely of strings of some given length (number of bits) to extrapolate to higher $K$ by diagonalizing in the sub-space of bits of that length. For the state in Figure 1 we continued beyond $K = 12$ by diagonalizing in the subspace of 2-bit strings, since for $K = 12$ this state has probability of only $\sim 10^{-6}$ of containing other than 2-bits. This is a judicious choice since the 2-bit space of states is especially simple and we could diagonalise up to $K \sim 400$, at which point the mass finally began to converge. Such a cutoff on the full problem is way beyond the bounds of attainability. Although quantitative conclusions are difficult to obtain in the strong coupling region, we can say with some confidence that there are no massless states in the $y \to 0$ limit. If we track each state as a function of the cut-off $K$, we find the mass monotonically increasing with $K$.

The masses of the bound states of fermion quanta converge extremely well by comparison. Figures 2 and 3 show the full spectrum of eigenvalues at $K = 9$ as a function of $x$ for the even-bit (bosonic) and odd-bit (fermionic) sectors. (Close inspection reveals significant bending of the paths.) The model still seems to be interacting fairly strongly past $x = 1$. Again there are many states of exceptional purity of length. Figures 4 and 5 illustrate the convergence for the lowest eigenstates at $x = 0$ and $x = 0.8$ including both sectors, extrapolating past $K = 12$ by identifying the dominant length component of the eigenfunction and diagonalizing in the subspace of states of that length. The reader can identify these levels on Figures 2 and 3. We conclude that the latter are a fairly accurate reflection of the very low-lying spectrum. One should keep in mind though, that for a cut-off $K$ the results for states of length close to $K$ are dominated by lattice artifacts, while states longer than $K$ do not appear at all. Indeed for $K = 12$ the masses of the smallest length states are already close to converging, while as the length increases the calculation gets less reliable. As $K$ grows, more and more states are included in the scheme, and more masses are obtained reliably.

To better quantify the length-purity of these string eigenfunctions we plot in Figure 6(i) the low-lying spectrum in the fermionic sector at $K = 13$ and $x = 0$ vs. average length
of the bound states. It is tempting to identify “Regge”-like trajectories in this diagram; horizontally we have excitations of the length degree of freedom while vertically we have glue excitations. Note, in particular, the rising line that can be drawn through the purest states, whose average lengths are very close to 3, 5, 7, 9, ... This can be thought of as a “trajectory” connecting the length excitations of the ground state. This trajectory appears to repeat for the excited states, and in the continuum limit we may see an infinite number of such trajectories. In Figure 6(ii) we repeat the plots for $x = 0.8$. This demonstrates the effect of $x$ on the relation between the characteristic energies for the glue and length excitations.

5. Discussion

Our discretized light-cone analysis of the large-$N$ two-dimensional gauge theory coupled to adjoint matter can certainly be improved by extending the calculation to higher values of the cut-off $K$, which requires a more efficient computer program. We will however, attempt to draw some physical conclusions at this stage.

1. For all values of $g/m$ the spectrum of $(\text{mass})^2$ is positive and appears to be discrete. There is no sign of a phase transition for any value of the dimensionless parameter. This should be contrasted with the behavior of the scalar matrix model with only global $SU(N)$ symmetry. There we found evidence [5] that the theory is tachyonic, and that there is a phase transition, which is associated with divergence of the sum of planar diagrams.

2. Keeping $g$ fixed and sending $m$ to zero, we find no massless states either for the scalar or for the fermion matter. It follows that if $m$ is kept fixed while the gauge coupling $g$ is sent to infinity, then the entire spectrum is pushed to infinite mass and the theory becomes trivial. This behavior differs from what is found in two-dimensional QCD with fundamental fermions, where there is a massless state for $m = 0$ [11].

3. The low-lying states are surprisingly close to being eigenstates of the number of bits. For the fermionic matter this is certain to be true in the continuum limit, since the convergence rate is good, while for the scalar matter this property may also hold for $K \to \infty$. This result could hardly be expected because the matrix elements that change
the number of bits are not small. A qualitative picture that seems to emerge is that the low-lying excitations can be roughly divided into two types: pure glue excitations that do not alter the number of bits, and excitations of the “number of bits” degree of freedom. The relation between the two characteristic energy scales is, of course, a function of $m/g$ (see Figures 6). This separation is only approximate and breaks down for high enough mass. The structure of states is therefore quite intricate: they can be roughly separated into “Regge trajectories”, which group the glue excitations of the approximately constant “bit state”. This structure certainly requires further analysis.

It is clearly more complex than in QCD with fundamental fermions, where all the states consist of a quark and an antiquark so that there is only one Regge trajectory [11]. Coupling to the adjoint matter gives rise to an additional “number of bits” degree of freedom, which allows many new possibilities.

4. The rate of convergence of the discretized light-cone method is much better for the fermions than for the scalars. This rate appears to be quite sensitive to which theory is under study. Poor convergence in discretised light-cone quantisation, especially at small mass, is often associated with the fact that low momentum quanta are not accurately accounted for at a given $K$. Perhaps the models with the adjoint matter will be of use as a testing ground for the power of the method in higher-dimensional gauge theories. It is also of great physical interest to extend our studies of the longitudinal dynamics to gauge theories in dimensions greater than two, where it becomes intertwined with transverse dynamics. Theories of this type, with latticized transverse dimensions, were studied in ref. [14-16], and it is desirable to extend this analysis.

The models that we have analyzed can be regarded as close string theories in the sense that they possess a light-cone string description. However, the absence of a phase transition makes it unlikely that these theories have a continuum string formulation. We should, in this case, look for gauged matrix models which do have critical behavior. As we suggested in [5], a good candidate model is (in euclidean space)

$$
S_{gauged} = \int d^2x \text{Tr} \left( \frac{1}{4g^2} F_{ab}^2 + \frac{1}{2} (\partial_a \phi + i[A_a, \phi])^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4N} \phi^4 \right)
$$

which has a scalar self-interaction in addition to the gauge field. Now there are two dimen-

* At strictly zero mass the zero momentum quanta are allowed and can even alter the vacuum from the naive Fock one [13].
sionless parameters, $g/m$ and $\lambda/m^2$. Criticality may occur for $\lambda < 0$, where the number of scalar propagators in a typical planar graph can begin to diverge. We have studied the model of eq. (42) for $g = 0$ and found results similar to those in the $M^3$ model [5], suggesting the possibility of a phase transition at $\lambda = \lambda_c < 0$. Although the $g = 0$ theory has problems due to the non-singlet states, they can be eliminated by turning on arbitrary $g$, however small. The effect of this modification on the singlet states vanishes in the $g \to 0$ limit. Thus, eq. (42) with $g = 0^+$ appears to define a model with a critical point where the string theory has an extra continuous dimension. Its origin is that the “number of bits” degree of freedom is tuned to criticality and becomes a massless field on the string. Unfortunately, this string theory appears to be tachyonic. Could it be that there is an entire critical line, extending through non-zero values of $g$, that in some range passes through the non-tachyonic region of the $g - \lambda$ plane? If we do find a string theory of a new type, it would be extremely interesting to find any world sheet formulation.

A nice feature of the model with the fermion matter is that it gives rise to strings that can be either space-time bosons or fermions. An obvious extension is to formulate the supersymmetric matrix theory in 1+1 dimensions, generalising the $c = 1$ model of Marinari and Parisi [17]. This question and the other issues mentioned above are under investigation.

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FIGURE CAPTIONS

**Fig.1.** A low-lying energy-level in the even-bit sector of the scalar matter theory as a function of $K$. For $K \geq 13$ extrapolation is performed by diagonalising in the 2-bit sector only.

**Fig.2.** The full spectrum of 29 states at $K = 9$ for the odd-bit (fermionic) sector of the fermionic matter theory as a function of $x$.

**Fig.3.** The full spectrum of 29 states at $K = 9$ for the even-bit (bosonic) sector of the fermionic matter theory as a function of $x$.

**Fig.4.** The lowest few energy-levels for fermionic matter (both even (b) and odd (f) sectors) as a function of $K$ at $x = 0$. From the bottom, the sequence of statistics is \{f, f, b, f, b, \ldots\}, with extrapolation for $K \geq 13$ using sectors of \{3, 5, 2, 7, 4, \ldots\} bits.

**Fig.5.** The lowest few energy-levels for fermionic matter (both even (b) and odd (f) sectors) as a function of $K$ at $x = 0.8$. From the bottom, the sequence of statistics is \{f, b, f, f, b, \ldots\}, with extrapolation for $K \geq 13$ using sectors of \{3, 2, 5, 3, 4, \ldots\} bits.

**Figs.6.** The lowest 15 energy-levels plotted against their average length (number of quanta) for the odd-bit sector of fermionic matter at $K = 13$ and (i) $x = 0$ (ii) $x = 0.8$. 
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