Another look at $e$

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Abstract

This note describes a way of obtaining $e$ that differs from the standard one. It could be used as an alternate way of showing how the value of $e$ is obtained. No attempt is made to show the existence of the limit in the definition of $e$ that appears in the final equation.

1. Introduction.

Traditionally the value of $e$ has been obtained, for instance, by taking the limit of ever-decreasing interest intervals in the compound interest formula (see Greenleaf [1]) or linear interpolation (see Flanders and Price [2]). We describe an alternate technique of obtaining $e$ that should have pedagogic value. In this section we give an approximation of $e$ using this technique and generalize it in the next section.

If $f'(x)$, the derivative of $f(x)$, exists at point $x$, and you start at point $x$ and move a distance $\Delta x$, the value at the point $x + \Delta$ is given by

$$ f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x \quad (1) $$

We want to find a constant, let’s call it $e$, such that when it’s raised to the power $x$ obtaining the function $e^x$, the function’s derivative is also $e^x$.‡

Since $f'(x)$ equals $f(x)$, we rewrite equation (1) as

$$ f(x + \Delta x) \approx f(x)(1 + \Delta x) \quad (2) $$

We will analyse this in the interval $[1,2]$. Let’s take $x = 1$ and $\Delta x = 0.1$. So $x + \Delta x$ is 1.1. Equation (2) gives

$$ f(1.1) \approx f(1)(1 + 0.1) \quad (3) $$

or

$$ e^{1.1} \approx 1.1e \quad (4) $$

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‡ Our analysis also holds if $f(x) = Ce^x$ where $C$ is a constant.
Now take \( x = 1.1 \) and use the same value of \( \Delta x \), i.e., 0.1. We will be using the same increment in \( x \) in this and all subsequent steps since eventually we will let \( \Delta x \) approach zero. Continuing in this way

\[
f(1.1 + 0.1) \approx 1.1 f(1.1)
\]

So \( f(1.2) \approx 1.1e^{1.1} \). Or

\[
e^{1.2} \approx (1.1)^2 e
\]

Eventually we will get \( e^2 \) on the left side of the equation, so we can solve for \( e \). So let’s compute \( e^{1.3} \). We get \( e^{1.3} \approx 1.1e^{1.2} \). But this equals \((1.1)^3 e\). If we extrapolate to \( x = 1.8 \), we see that

\[
e^{1.9} \approx (1.1)^9 e
\]

and finally that

\[
e^2 \approx (1.1)^{10} e
\]

Solving for \( e \) we get \( e \approx (1.1)^{10} \) or \( e \) equals 2.59 to three digits, where the 10 corresponds to dividing 1 by 0.1. Equation (1) presupposes that \( \Delta x \) approaches zero. If we let \( \Delta x = .00000001 \), or \( 10^{-8} \), we raise \((1 + .00000001)\) to \( 10^8 \). The answer for \( e \) is 2.71828 to five significant figures.

2. Generalization.

We now sketch the steps that describe the preceding method in general. Using equation (2), and setting \( x = 1 \), we write

\[
e^{1 + \Delta x} \approx e(1 + \Delta x)
\]

We continue, letting \( x = x + \Delta x \) and keeping \( \Delta x \) the same, and write

\[
e^{1 + \Delta x + \Delta x} \approx e^{1 + \Delta x}(1 + \Delta x)
\]

or

\[
e^{1 + 2\Delta x} \approx e(1 + \Delta x)^2
\]

We have to add \( \Delta x \) to \( x \) \( 1/\Delta x \) times to get \( e^2 \) on the left side of these equations. So we get

\[
e^{1 + (1/\Delta x) \cdot \Delta x} \approx e(1 + \Delta x)^{1/\Delta x}
\]

or

\[
e^2 = e(1 + \Delta x)^{1/\Delta x}
\]
Solve for $e$ and since the definition of the derivative in equation (1) lets $\Delta x \to 0$, take the same limit here. We get

$$e = \lim \limits_{\Delta x \to 0} (1 + \Delta x)^{1/\Delta x}$$

which is one of the definitions of $e$.

3. References.

[1] Greenleaf, Fredrick E.. *Quantitative Reasoning, 3/e*, McGraw Hill (2006).
[2] Flanders, Harley and Price, Justin J., *Calculus with Analytic Geometry*, Academic Press (1978).