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WEAK ERROR EXPANSION OF THE IMPLICIT EULER SCHEME

OMAR ABOURA

Abstract. In this paper, we extend the Talay Tubaro theorem to the implicit Euler scheme.

1. Introduction

Let \((\Omega, \mathcal{F}, P)\) a probability space and \(T > 0\) a fixed time. \(W\) will be a Brownian motion in \(\mathbb{R}\) with respect to its own filtration \(\mathcal{F}_t\). We will consider the following stochastic differential equation

\[
X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \tag{1.1}
\]

where \(x \in \mathbb{R}\), \(b\) and \(\sigma\) are real functions defined on \(\mathbb{R}\). It is well know that, under Lipschitz conditions on \(b\) and \(\sigma\), this equation admits a unique strong solution.

For various reasons, including mathematical finance or partial differential equations, the approximation of \(\mathbb{E}f(X_T)\) is of importance. One way to do this is to use an Euler scheme and to study the speed of convergence. There is a vast literature on this subject and one of the pioneering work is the paper of D. Talay and L. Tubaro [7].

Let \(N \in \mathbb{N}^*\) and \(h := T/N\). Consider \((t_k)_{0 \leq k \leq N}\) the uniform subdivision of \([0, T]\) defined by \(t_k := kh\). In their paper [7] the authors deal with the explicit Euler scheme \((\bar{X}_t^k)_{0 \leq k \leq N}\) defined as: \(\bar{X}_0 = x\) and for \(k = 0, \ldots, N - 1,\)

\[
\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(\bar{X}_{t_k})h + \sigma(\bar{X}_{t_k})\Delta W_{t_{k+1}}. \tag{1.2}
\]

Despite the fact that this implicit scheme cannot be implemented in most cases, it has been studied in [5] but, to the best of our knowledge, its weak error expansion has not been given. The main reason of this study is that we believe it would be a step in order to study a weak convergence error for SPDEs. So far in that framework only few cases have been studied in [2]-[4] for the stochastic heat or Schrödinger equation.

Notations. Let \(n \in \mathbb{N}\) and \(v, w : [0, T] \times \mathbb{R} \to \mathbb{R}\) be smooth functions. We will denote by \(\partial^n v(t, x)\) the \(n^{th}\) derivative of \(v\) with respect to the space variable \(x\), except for the second derivative denoted \(\Delta v(t, x)\) as usual. Moreover, by an abuse of notation, for a function \(v : \mathbb{R} \to \mathbb{R}\) and \(w : [0, T] \times \mathbb{R} \to \mathbb{R}\), we will write \((vw)(t, x) := v(x)w(t, x)\).

Given \(p \in \mathbb{N}\), \(C_p\) will denote a constant that depends on \(p, T\) and the coefficients \(b\) and \(\sigma\), but does not depend on \(N\). As usual, \(C_p\) may change from line to line.

For \(h\) small enough, we denote by \(S_h\) the functions defined on \(\mathbb{R}\) by

\[
S_h(x) := 1/(1 - hb'(x)). \tag{1.4}
\]

It is similar to the map used by Debussche in [3].
2. The main result

Let $u$ the (classical) solution of the following pde, called the Kolmogorov equation:

$$
\begin{cases}
\frac{\partial u}{\partial t}(t, x) + b(x)\partial u(t, x) + \frac{1}{2}\sigma^2(x)\Delta u(t, x) = 0, \\
u(T, x) = f(x).
\end{cases}
$$

The properties of $u$ will be given in the next section. Let us mention that for $b$ and $\sigma$ smooth enough, $u$ is smooth too. We define the function $\psi_i : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, where $i$ stands for implicit, as follows for a smooth enough function $u$:

$$
\psi_i := \frac{1}{2} b \partial (b \partial u) + \frac{1}{4} \sigma^2 \Delta (b \partial u) - \frac{1}{2} b^2 \Delta u + \frac{1}{8} \sigma^4 \partial^4 u - \frac{1}{4} b \partial (\sigma^2 \Delta u) - \frac{1}{8} \sigma^2 \Delta (\sigma^2 \Delta u).
$$

We are now in position to state the main result of this paper.

**Theorem 2.1.** Let $b, \sigma, f$ be $C^\infty$-functions with bounded derivatives.

(i) The implicit Euler scheme (1.3) is of weak order 1, that is, there exists a constant $C$, such that for $h$ small enough $|Ef(X_T^N) - Ef(X_T)| \leq Ch$.

(ii) The weak error can be expanded as

$$
Ef(X_T^N) - Ef(X_T) = hE \int_0^T \psi_i(t, X_t)dt + O(h^2).
$$

We have not given the minimal hypothesis; indeed we want to focus on the ideas and not on the best set of assumptions. The proof of this theorem is quite long; it uses intensively the Kolmogorov equation (2.1), the Itô and Clark-Ocone formulas. It will be proved in the next section. We at first compare our result with that of Talay Tubaro. In their paper [7], the authors introduce the following function

$$
\psi_e = \frac{1}{2} b^2 \Delta u + \frac{1}{2} b \sigma^2 \partial^3 u + \frac{1}{8} \sigma^4 \partial^4 u + \frac{1}{4} \partial^2 \partial u + b \partial u + \frac{1}{2} \sigma^2 \partial \Delta u,
$$

and prove the following result (see [7] page 489).

**Theorem 2.2.** Let $(X_{t_k})_{k=0, \ldots, N}$ denote the explicit Euler scheme defined by (1.2). Then weak error has the following expansion

$$
Ef(X_T) - Ef(X_T) = hE \int_0^T \psi_e(t, X_t)dt + O(h^2).
$$

Applying $\frac{\partial}{\partial t}$, $b \partial$ and finally $\frac{1}{2}\sigma^2 \Delta$ to (2.1) and summing these equations we have

$$
\frac{\partial^2}{\partial t^2} u + 2b \partial \partial t u + \sigma^2 \partial \partial t u = -b \partial (b \partial u) - \frac{1}{2} b \partial (\sigma^2 \Delta u) - \frac{1}{2} \sigma^2 \Delta (b \partial u) - \frac{1}{8} \sigma^2 \Delta (\sigma^2 \Delta u)
$$

So we can rewrite the function $\psi_e$ as

$$
\psi_e = \frac{1}{2} b^2 \Delta u + \frac{1}{2} b \sigma^2 \partial^3 u + \frac{1}{8} \sigma^4 \partial^4 u - \frac{1}{2} b \partial (b \partial u) - \frac{1}{4} b \partial (\sigma^2 \Delta u) - \frac{1}{4} \sigma^2 \Delta (b \partial u) - \frac{1}{8} \sigma^2 \Delta (\sigma^2 \Delta u)
$$

For $b = 0$, we have $\psi_e = \psi_i = \frac{1}{8} \sigma^4 \partial^4 u - \frac{1}{4} \sigma^2 \Delta (\sigma^2 \Delta u)$ as expected since in this case the explicit and the implicit Euler scheme coincide. We can notice that $\psi_i = \psi_e - b^2 \Delta u + \frac{1}{2} \sigma^2 \Delta (b \partial u) + b \partial (b \partial u) - \frac{1}{2} b \sigma^2 \partial^3 u$.

3. Proof Theorem 2.1

Here is a sketch of the proof: After proving some property of the scheme, we introduce a continuous interpolation of this scheme. Finally, after decomposing the weak error, we study a remainder term.
3.1. Some tools.

**Proposition 3.1** (Property of \(u\)). Let \((X_{s}^{t,x})_{s \in [t,T]}\) denote the stochastic flow, that is the solution of (1.1) starting from \(x\) at time \(t\) and let \(u(t,x) = Ef(X_{t}^{x})\). Then \(u\) belongs to \(C^{\infty,\infty}([0,T] \times \mathbb{R})\) and satisfies the Kolmogorov equation (2.1). Moreover, for any \(n,p \in \mathbb{N}\), there exists constants \(C\) and \(k\) such that

\[
|\frac{\partial^{n}}{\partial t^{n}}\partial^{p}u(t,x)| \leq C\left(1 + |x|^{k}\right).
\]

See for example [7] page 486 Lemma 2.

Now we recall several results from Malliavin Calculus that will be used in the sequel. For a detailed introduction, we send the reader to D.Nualart’s book [6].

**Proposition 3.2** (Clark-Ocone formula). Let \(t \in [0,T]\) and \(F \in L^{2}(\mathcal{F}_{t}) \cap D^{1,2}\); then we have for all \(s \in [0,t]\)

\[
F = E(F|\mathcal{F}_{s}) + \int_{s}^{t} E(D_{r}F|\mathcal{F}_{r}) \, dW_{r}.
\]

**Lemma 3.3.** Let \(F, G \in D^{1,2}\).

(i) If \(F\) and \(DF\) are bounded, then \(FG \in D^{1,2}\) and \(D(FG) = FDG + GDF\).

(ii) Let \(f \in C^{1}\) with a bounded derivative; then \(f(F) \in D^{1,2}\) and \(Df(F) = f'(F)DF\).

(iii) Let \((s,t) \in [0,T]^{2}\) such that \(s < t\) and let \(F \in D^{1,2} \cap L^{2}(\mathcal{F}_{s})\). Then \(F(W_{t} - W_{s}) \in D^{1,2}\) and

\[
D_{r}[F(W_{t} - W_{s})] = D_{r}F(W_{t} - W_{s}) + F1_{\{s \leq r \leq t\}}.
\]

(iv) Let \(\{H_{n}, n \geq 1\}\) be a sequence of random variables in \(D^{1,2}\) that converges to \(H\) in \(L^{2} (\Omega)\) and such that \(\sup_{n} E\left(\|DH_{n}\|_{L^{2}(0,T)}^{2}\right) < \infty\). Then \(H\) belongs to \(D^{1,2}\).

For a proof of (iii), see [6] Lemma 1.3.4. Now we state some technical lemmas that will be useful in the sequel. The following discrete Gronwall lemma is classical.

**Lemma 3.4** (Gronwall’s lemma). For any nonnegative sequences \((a_{k})_{0 \leq k \leq N}\) and \((b_{k})_{0 \leq k \leq N}\), satisfying \(a_{k+1} \leq (1 + Ch)a_{k} + b_{k+1}\), with \(C > 0\). Then we have \(a_{k} \leq e^{C(T-t_{k})} \left(a_{0} + \sum_{i=1}^{k} b_{i}\right)\).

**Lemma 3.5.** Let \(L > 0;\) then for \(h^{*}\) small enough (more precisely \(Lh^{*} < 1\)) there exists \(\Gamma := \frac{L}{1-Lh^{*}} > 0\) such that for all \(h \in (0, h^{*})\) we have \(\frac{1}{1-Lh} < 1 + \Gamma h\).

**Proof.** Let \(h \in (0, h^{*})\); then we have \(1 - Lh > 1 - Lh^{*} > 0\). Hence \(\frac{L}{1-Lh} < \frac{L}{1-Lh^{*}} = \Gamma\), so that \(Lh < \Gamma h(1-Lh)\), which yields \(1 + \Gamma h - Lh - \Gamma Lh^{2} = (1 + \Gamma h)(1-Lh) > 1\). This concludes the proof. \(\square\)

**Lemma 3.6** (Generalization of Young’s lemma). For an integer \(p \geq 1\) and for \(\epsilon > 0\), we have

\[
(a + b)^{2p} \leq (1 + \epsilon)^{2p-1}a^{2p} + \left(1 + \frac{1}{\epsilon}\right)^{2p-1}b^{2p}.
\]

**Proof.** We use an induction argument. The inequality is true for \(p = 1\), that is \((a + b)^{2} \leq (1 + \epsilon)a^{2} + (1 + \frac{1}{2}b)^{2}\). Now, suppose that it is true until \(p\) and will prove it for \(p + 1\); indeed the induction hypothesis yields

\[
(a + b)^{2p+1} \leq \left|(1 + \epsilon)^{2p-1}a^{2p} + \left(1 + \frac{1}{\epsilon}\right)^{2p-1}b^{2p}\right|^{2}\n\leq (1 + \epsilon)|(1 + \epsilon)^{2p-1}|^{2}a^{2p} + \left(1 + \frac{1}{\epsilon}\right)^{2p-1}2\left|b^{2p}\right|^{2}.
\]
This concludes the proof.

\section{Property of the implicit Euler scheme.}

\begin{lemma}[Existence of the scheme] For small $h$, the implicit Euler scheme \eqref{eq:1.3} is well defined. Moreover, for all $k = 0, \ldots, N$, we have $X^N_{t_k} \in L^2(\mathcal{F}_{t_k})$.

We will denote by $N_0$ the smallest integer such that the scheme is well defined.

\begin{proof}
 For $k = 0$, we have $X^N_{t_0} = x \in L^2(\mathcal{F}_{t_0})$. Suppose that for all $j = 0, \ldots, k$, $X^N_{t_j}$ is well defined and belongs to $L^2(\mathcal{F}_{t_j})$; we prove this for $j = k + 1$. We define $\xi_{k+1} := X^N_{t_k} + \sigma (X^N_{t_k}) \Delta W_{k+1}$. By independence of $\Delta W_{k+1}$ and $\mathcal{F}_{t_k}$ and the linear growth of $\sigma$, we have that $\xi_{k+1} \in L^2(\Omega)$. Let $F_{k+1}: L^2(\Omega) \to L^2(\Omega)$ be defined by

\begin{equation}
F_{k+1}(X) := \xi_{k+1} + b(X) h,
\end{equation}

for all $X \in L^2(\Omega)$. Using the Lipschitz property of $b$ we have $E |F_{k+1}(X) - F_{k+1}(Y)|^2 \leq \|b\|_\infty h^2 E |X - Y|^2$. So by the fixed point theorem, if $\|b\|_\infty h < 1$ there exist an unique element of $L^2(\Omega)$, noted $X^N_{t_{k+1}}$, such that $X^N_{t_k} = F_{k+1}(X^N_{t_{k+1}})$. The measurability of $X^N_{t_k}$ with respect to $\mathcal{F}_{t_{k+1}}$ is obvious.

\begin{lemma}[Malliavin derivability] Let $h > 0$ small enough; then for all $k = 0, \ldots, N$, we have $X^N_{t_k} \in D^{1,2}$. Moreover, for all $t \in [t_k, t_{k+1})$, we have $D_t X^N_{t_{k+1}} = S_h (X^N_{t_{k+1}}) \sigma (X^N_{t_k})$, where $S_h$ is defined by \eqref{eq:1.4}.

\begin{proof}
 It is true for $k = 0$, since $X^N_{t_0} = x$. Now suppose that for all $j = 1, \ldots, k$, $X^N_{t_j} \in D^{1,2}$ and prove that $X^N_{t_{k+1}} \in D^{1,2}$. First, we define the following sequence in $L^2(\Omega)$:

\begin{equation}
X^N_{t_{k+1}}(0) = 0 \text{ and for } i \geq 0, X^N_{t_{k+1}}(i + 1) = F_{k+1}(X^N_{t_{k+1}}(i)) \text{ where } F_{k+1} \text{ is defined by } \eqref{eq:3.1}.
\end{equation}

Using the Lipschitz property of $F_{k+1}$, since $X^N_{t_{k+1}}$ is a fixed point of $F_{k+1}$, we have

\begin{equation}
E \left| X^N_{t_{k+1}} - X^N_{t_{k+1}}(i + 1) \right|^2 \leq \|b\|_\infty h^2 E \left| X^N_{t_{k+1}} - X^N_{t_{k+1}}(i) \right|^2 \leq \left( \|b\|_\infty h^2 \right)^{i+1} E \left| X^N_{t_{k+1}} \right|^2.
\end{equation}

So $X^N_{t_{k+1}}(i)$ converge to $X^N_{t_{k+1}}$ in $L^2(\Omega)$ if $\|b\|_\infty h < 1$. Using the induction hypothesis, the assumptions on $\sigma$ and Lemma 3.3 (ii) and (iii), we deduce that $\xi_{k+1} = X^N_{t_k} + \sigma (X^N_{t_k}) \Delta W_{k+1}$ belongs to $D^{1,2}$. Finally, since $b$ is Lipschitz, we deduce by induction that for all $i \geq 0$ $X^N_k(i) \in D^{1,2}$. Moreover we have $D X^N_k(i + 1) = D \xi_{k+1} + b h (X^N_k(i)) D X^N_k(i)$ and $D X^N_{t_{k+1}}(0) = 0$, so that

\begin{equation}
\| D X^N_k(i + 1) \|^2_{L^2(0,T)} \leq 2 \| D \xi_{k+1} \|^2_{L^2(0,T)} + 2 h^2 \| b \|^2_{\infty \sigma} \| D X^N_k(i) \|^2_{L^2(0,T)}.
\end{equation}

An induction argument yields for $i \geq 1$ and $2 h^2 \| b \|^2_{\infty \sigma} < 1,$

\begin{equation}
\| D X^N_k(i) \|^2_{L^2(0,T)} \leq \frac{2 \| D \xi_{k+1} \|^2_{L^2(0,T)}}{1 - 2 h^2 \| b \|^2_{\infty \sigma}}.
\end{equation}

Finally, we have $\sup_i \| D X^N_k(i) \| < \infty$. Lemma 3.3 (iv) proves that $X^N_{t_{k+1}} \in D^{1,2}$.

Finally, let $t \in [t_k, t_{k+1})$; applying the Malliavin derivative $D$ to \eqref{eq:1.3} and using Lemma 3.3 we have

\begin{equation}
D_t X^N_{t_{k+1}} = h b (X^N_{t_{k+1}}) D_t X^N_{t_{k+1}} + \sigma (X^N_{t_k});
\end{equation}

which concludes the proof.
\end{proof}

The following result gives a bound of $p$th moments of the implicit scheme.
Lemma 3.10. For all \( k \geq 1 \); then for \( N_0 \) large enough, there exists a constant \( C(p) > 0 \) such that

\[
\sup_{N \geq N_0} \max_{k=0, \ldots, N} E \left| X_{t_k}^N \right|^p \leq C(p). \tag{3.2}
\]

Proof. Holder’s inequality shows that it suffices to consider moments which are power of 2, that is to check \( \sup_{N \geq N_0} \max_{k=0, \ldots, N} E \left| X_{t_k}^N \right|^{2p} \leq C_p \), for every integer \( p \geq 1 \). Using the generalized Young Lemma 3.6 the independence between \( \Delta W_{k+1} \) and \( F_{t_k} \), and the fact that for all \( j \in \mathbb{N} \), \( E (\Delta W_{k+1})^{2j+1} = 0 \), we have for \( h \in (0, h^*) \) and some constant \( C_p \) depending on \( h^* \)

\[
E \left| X_{t_{k+1}}^N \right|^{2p} \leq (1 + h)^{2p-1} E \left| X_{t_k}^N \sigma (X_{t_k}^N) \Delta W_{k+1} \right|^{2p} + \left( 1 + \frac{1}{h} \right) \left( \frac{2p-1}{2} \right) h^{2p} E \left| b \left( X_{t_k}^N \right) \right|^{2p}
\]

\[
\leq (1 + C_p h) E \left| X_{t_k}^N \right|^{2p} + (1 + C_p h) \sum_{j=1}^{2p-1} \left( \frac{2p}{2j} \right) E \left( \left| \sigma (X_{t_k}^N) \right|^{2j-2} \left( 1 + \left| X_{t_k}^N \right|^{2j} \right) \right) E \left| \Delta W_{k+1} \right|^{2j} + C_p h \left( 1 + E \left| X_{t_k}^N \right|^{2p} \right)
\]

Using the identity \( E |\Delta W_{k+1}|^{2j} = C(2j) h^j \) and the linear growth of \( \sigma \) we deduce for \( h < 1 \)

\[
E \left| X_{t_{k+1}}^N \right|^{2p} \leq (1 + C_p h) E \left| X_{t_k}^N \right|^{2p} + C_p h + C_p h E \left| X_{t_k}^N \right|^{2p} + (1 + C_p h) C_p h \sum_{j=1}^{2p-1} E \left( \left| \sigma (X_{t_k}^N) \right|^{2j} \left( 1 + \left| X_{t_k}^N \right|^{2j} \right) \right)
\]

Using the inequality: \( a^{2p+1-2j} \leq a^{2p+1} + 1 \) valid for any \( a > 0 \), we get for some constant \( C_p > 0 \) and \( h < 1 \)

\[
E \left| X_{t_{k+1}}^N \right|^{2p} \leq (1 + C_p h) E \left| X_{t_k}^N \right|^{2p} + C_p h + C_p h E \left| X_{t_k}^N \right|^{2p}
\]

Provided that \( h \) is small enough, the Gronwall Lemma 3.4 and Lemma 3.5 conclude the proof. \( \square \)

3.3. Some martingales and related process: \( \beta_t^{k,N}, z_t^{k,N}, \gamma_t^{k,N} \) and \( \eta_t^{k,N} \). Let \( k \in \{0, \ldots, N-1\} \) be fixed; in the sequel, we will use the following processes defined for \( t \in [t_k,t_{k+1}] \)

\[
\beta_t^{k,N} := E \left( b \left( X_{t_k}^N \right) \bigg| F_t \right), \quad z_t^{k,N} := E \left( D_t b \left( X_{t_k}^N \right) \bigg| F_t \right), \tag{3.3}
\]

\[
\gamma_t^{k,N} := \sigma (X_{t_k}^N) + (t-t_k) z_t^{k,N}, \quad \gamma_t^{k,N} := \sigma (X_{t_k}^N) E \left( D_t \left( S_h b' \right) \left( X_{t_k}^N \bigg| F_t \right) \right).
\]

The following lemma describes the time evolution of these processes.

Lemma 3.10. For all \( k = 0, \ldots, N-1 \), and for \( t \in [t_k,t_{k+1}] \), we have the following relation

\[
d\beta_t^{k,N} = z_t^{k,N} dW_t, \quad d\gamma_t^{k,N} = \eta_t^{k,N} dW_t, \quad d\gamma_t^{k,N} = z_t^{k,N} dt + \eta_t^{k,N} (t-t_k) dW_t,
\]

\[
d\eta_t^{k,N} = \left| \sigma (X_{t_k}^N) \right|^2 E \left( D_t \left( S_h b'' \right) \left( X_{t_k}^N \bigg| F_t \right) \right) dW_t.
\]
Proof. Let $k = 0, \ldots, N-1$ and let $t \in [t_k, t_{k+1}]$. Lemmas 3.3 (ii), 3.7 and 3.8 and the bounds of $\|b''\|_\infty$ imply that $b \left( X_{t_{k+1}}^N \right) \in L^2 \left( \mathcal{F}_{t_{k+1}} \right) \cap D^{1,2}$ and

$$D_t b \left( X_{t_{k+1}}^N \right) = b' \left( X_{t_{k+1}}^N \right) D_t X_{t_{k+1}}^N = (S_h b') \left( X_{t_{k+1}}^N \right) \sigma \left( X_{t_{k+1}}^N \right).$$

(3.4)

So the Clark-Ocone formula in Proposition 3.2 yields $\beta_{t}^{k,N} = b \left( X_{t_{k+1}}^N \right) - \int_{t}^{t_{k+1}} z_{k,N}^s dW_s$ and hence $d\beta_{t}^{k,N} = z_{k,N}^s dW_t$; where $z_{k,N}^s = E \left( D_u b \left( X_{t_{k+1}}^N \right) \big| \mathcal{F}_s \right)$; using (3.4)

$$z_{k,N}^s = \sigma \left( X_{t_{k+1}}^N \right) E \left( \left( S_h b' \right) \left( X_{t_{k+1}}^N \right) \big| \mathcal{F}_s \right).$$

(3.5)

So taking conditional expectation with respect to $\mathcal{F}_t$, we have (3.5). Since $b''$ is bounded and $b'$ Lipschitz we have that for $h$ small enough, $S_h b' = \frac{b'}{1-hb'} \in C_b^1$. So we can conclude that $(S_h b') \left( X_{t_{k+1}}^N \right) \in D^{1,2}$ and using the Clark-Ocone formula we deduce that for $s \in [t_k, t_{k+1})$,

$$\left( S_h b' \right) \left( X_{t_{k+1}}^N \right) = E \left( S_h b' \left( X_{t_{k+1}}^N \right) \big| \mathcal{F}_s \right) + \int_{s}^{t_{k+1}} E \left( D_u \left[ S_h b' \left( X_{t_{k+1}}^N \right) \right] \big| \mathcal{F}_u \right) dW_u,$$

and hence

$$dz_{k,N}^s = \sigma \left( X_{t_{k+1}}^N \right) E \left( D_s \left[ \frac{b'}{1-hb'} \left( X_{t_{k+1}}^N \right) \big| \mathcal{F}_s \right) dW_s = \eta_{k,N}^s dW_s.$$}

The differential of $\gamma_{t}^{k,N}$ is a consequence of the previous result and Itô’s formula. Finally, since $S_h b' \in C_b^1$ and $(S_h b')' = S_h^2 b''$, Lemma 3.8 implies that $D_t(S_h b') \left( X_{t_{k+1}}^N \right) = \sigma \left( X_{t_{k+1}}^N \right) \left( S_h^2 b'' \right) \left( X_{t_{k+1}}^N \right)$ and then

$$\eta_{k,N}^s = \left| \sigma \left( X_{t_{k+1}}^N \right) \right|^2 E \left( \left( S_h^2 b'' \right) \left( X_{t_{k+1}}^N \right) \big| \mathcal{F}_t \right).$$

(3.6)

Applying once more the Clark-Ocone formula in Proposition 3.2, we deduce

$$E \left( S_h^2 b'' \left( X_{t_{k+1}}^N \right) \big| \mathcal{F}_t \right) = \left( S_h^2 b'' \right) \left( X_{t_{k+1}}^N \right) - \int_{t}^{t_{k+1}} E \left( D_s \left( S_h^2 b'' \right) \left( X_{t_{k+1}}^N \right) \big| \mathcal{F}_s \right) dW_s.$$}

Multiplying this by $\left| \sigma \left( X_{t_{k+1}}^N \right) \right|^2$ and using (3.6), we conclude the proof. \[ \square \]

The next lemma provides uniform moment estimates of the above processes.

**Lemma 3.11.** Let $p \in \mathbb{N}$; then for $N$ large enough there exists a constant $C_p$ such that for $N \geq N_0$,

$$\max_{k=0, \ldots, N-1} \sup_{t_k \leq t \leq t_{k+1}} E \left\{ \left| \beta_{t}^{k,N} \right|^p + \left| z_{k,N}^s \right|^p + \left| \gamma_{t}^{k,N} \right|^p + \left| \eta_{k,N}^s \right|^p \right\} \leq C_p.$$

**Proof.** Using Jensen’s inequality, the Lipschitz property of $b$ and Lemma 3.9 we have

$$E \left| \beta_{t}^{k,N} \right|^p \leq E \left| b \left( X_{t_{k+1}}^N \right) \right|^p \leq C_p \left( 1 + E \left| X_{t_{k+1}}^N \right|^p \right) \leq C_p.$$

The identity (3.5), Jensen’s inequality, the growth property of $\sigma$ and the upper estimate $b'/(1-hb') \geq C$ for small $h$, Schwarz’s inequality and Lemma 3.9 yield

$$E \left| z_{k,N}^s \right|^p \leq E \left| \sigma \left( X_{t_{k+1}}^N \right) \left( S_h b' \right) \left( X_{t_{k+1}}^N \right) \right|^p \leq C_p.$$

Using the definition of $\gamma_{t}^{k,N}$ in (3.3) and the previous upper estimates we deduce

$$E \left| \gamma_{t}^{k,N} \right|^p \leq C_p E \left| \sigma \left( X_{t_{k+1}}^N \right) \right|^p + C_p h^p E \left| z_{k,N}^s \right|^p \leq C_p.$$
Finally (3.6), the Jensen inequality, the growth condition on $\sigma$, the upper bounds of $b'$ and $b''$, Lemma 3.9 and Schwarz’s inequality yield

$$E \left| \eta_t^{k,N} \right|^p \leq E \left| \sigma \left( X_{t_k}^N \right) \right|^{2p} \left| \left( S_t^h b'' \right) \left( X_{t_k+1}^N \right) \right|^p \leq C_p$$

This concludes the proof. \hfill \square

3.4. **Continuous interpolation.** As usual we need to introduce a continuous process that interpolates the implicit Euler scheme (1.3). With an abuse of notation, let $(X_t^N)_{t \in [0,T]}$ be the process defined as follow: $X_0^N = x_0$ and for $k = 0, \ldots, N - 1$ and $t_k \leq t \leq t_{k+1}$

$$X_t^N := X_{t_k}^N + E \left( b \left( X_{t_k+1}^N \right) \right) \mathcal{F}_t, t - t_k) + \sigma \left( X_{t_k}^N \right) \left( W_t - W_{t_k} \right).$$

(3.7)

This process satisfies the following

**Lemma 3.12.** The process $(X_t^N)_{t \in [0,T]}$ is continuous $\mathcal{F}_t$-adapted and is an interpolation of the scheme (1.3). Moreover, for $k \in \{0, \ldots, N - 1\}$ and $t \in [t_k, t_{k+1}]$ we have $dX_t^N = \beta_t^{k,N} dt + \gamma_t^{k,N} dW_t$, where the process $(\beta_t^{k,N})$ and $(\gamma_t^{k,N})$ are defined by (3.3).

**Remark 3.13.** (1) If $b = 0$, (3.7) corresponds to the classical interpolation given by Talay Tubaro [7], since the explicit and implicit Euler scheme are the same.

(2) If $b$ is linear, this continuous process differs from that used by Debussche in [3]. Indeed, the finite dimensional analog of the interpolation corresponding to the process $dX_t = -\beta X_t dt + \sigma \left( X_t \right) dW_t$, is defined by

$$X_t^D = X_{t_k}^N + \int_{t_k}^t -\beta X_s^N \frac{dX_s^N}{1 + h\beta} ds + \int_{t_k}^t \sigma \left( X_s^N \right) \frac{dW_s}{1 + h\beta};$$

for $t \in [t_k, t_{k+1}]$ (see [3] page 96 equation (3.2)). In this particular case, our interpolation is given by

$$X_t^N = X_{t_k}^N + \int_{t_k}^t E \left( -\beta X_{t_{k+1}}^N \mathcal{F}_s \right) ds + \int_{t_k}^t \left\{ \sigma \left( X_{t_k}^N \right) \mathcal{F}_t \left( s - t_k \right) E \left( -D_s \beta X_{t_{k+1}}^N \mathcal{F}_s \right) \right\} dW_s.$$

**Proof of Lemma 3.12.** The fact that $(X_t^N)$ is an $(\mathcal{F}_t)$-adapted process which interpolates the scheme (1.3) is a consequence of (3.7). The continuity is a consequence of the fact that the map $(t \to E(X|\mathcal{F}_t))$ has a continuous modification. So, applying Itô’s formula and Lemma 3.10, we obtain $d(\beta_t^{k,N} (t - t_k)) = (t - t_k) \beta_t^{k,N} dW_t + \beta_t^{k,N} dt$, and hence

$$dX_t^N = \beta_t^{k,N} dt + \left( \sigma \left( X_{t_k}^N \right) + (t - t_k) \beta_t^{k,N} \right) dW_t.$$

This concludes the proof. \hfill \square

We next give moment estimates of the interpolation process $X_t^N$.

**Lemma 3.14.** Let $p \geq 1$ and $h^* > 0$ be small enough. There exits a constant $C_p > 0$ depending on $h^*$ such that

$$\sup_{N \geq N_0} \sup_{t \in [0,T]} E \left| X_t^N \right|^p < C_p.$$

**Proof.** Using Lemma 3.9, Jensen’s inequality and the independence of $W_t - W_{t_k}$ and $X_{t_k}^N$, we have for $t \in [t_k, t_{k+1}]$: 

$$E \left| X_t^N \right|^p \leq C_p E \left| X_{t_k}^N \right|^p + C_p E \left| b \left( X_{t_{k+1}}^N \right) \mathcal{F}_t \right|^p \left| t - t_k \right|^p + C_p E \left| \sigma \left( X_{t_k}^N \right) \right|^p \left| W_t - W_{t_k} \right|^p \leq C_p + C_p h^p E \left| b \left( X_{t_{k+1}}^N \right) \mathcal{F}_t \right|^p + E \left| \sigma \left( X_{t_k}^N \right) \right|^p E \left| W_t - W_{t_k} \right|^p.$$

Using the growth condition on $b$ and $\sigma$, moments of the normal law and Lemma 3.9, we deduce the result. \hfill \square
The following is a straightforward consequence of Lemmas 3.11 and 3.14

**Corollary 3.15.** Let \( v : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a function with polynomial growth, and let \( n_1, \ldots, n_6 \) non negative integers. Then there exists a constant \( C \) independent of \( h^* \) such that for \( r \in [t_k, t_{k+1}] \) and \( h \in (0, h^*) \)
\[
E \left( |\beta_{r,N}^{k,N}|^n_3 |\gamma_{r,N}^{k,N}|^n_2 |\eta_{r,N}^{k,N}|^n_4 |\nu (r, X_r^{N}) |^{n_6} \right) \leq C.
\]

3.5. **Local decomposition.** Now we return to the proof of the main theorem. Let \( u \) be the solution to the Kolmogorov equation (2.1). Using (2.1), we decompose the weak error into a sum of local errors. Let \( \delta_k^N := E u \left( t_{k+1}, X_{t_{k+1}}^N \right) - E u \left( t_k, X_{t_k}^N \right) \); we deduce
\[
E f \ (X_r^N) - E f \ (X_T) = E u \ (T, X_T^N) - E u \ (0, x) = \sum_{k=0}^{N-1} \delta_k^N. \tag{3.8}
\]

We introduce, for \( t_k \leq t \leq t_{k+1} \),
\[
\mathcal{I}_k^N \ (t) := E \left[ \left( \beta_t^{k,N} - b \ (X_t^N) \right) \partial u \ (t, X_t^N) \right], \quad \mathcal{J}_k^N \ (t) := E \left[ \left( \left| \gamma_t^{k,N} \right|^2 - \sigma^2 \ (X_t^N) \right) \Delta u \ (t, X_t^N) \right].
\]

Since \( u \in C^{1,2} \), using Itô’s formula, Lemma 3.11 and the Kolmogorov equation (2.1) at the point \((t, X_t^N)\), we obtain
\[
\delta_k^N = E \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} u + \beta_t^{k,N} \partial u + \frac{1}{2} \left| \beta_t^{k,N} \right|^2 \Delta u \right\} \ (t, X_t^N) \ dt \tag{3.9}
\]
\[
= E \int_{t_k}^{t_{k+1}} \left\{ \mathcal{I}_k^N \ (t) + \frac{1}{2} \mathcal{J}_k^N \ (t) \right\} \ dt. \tag{3.10}
\]

Now for \( k = 0, \ldots, N-1 \), we introduce the following quantities for \( s \in [t_k, t_{k+1}] \):
\[
i_k^N \ (s) := \frac{\partial}{\partial s} \ (b \partial u) \ (s, X_s^N) + \beta_s^{k,N} \partial \ (b \partial u) \ (s, X_s^N) \tag{3.11}
\]
\[
\quad + \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta \ (b \partial u) \ (s, X_s^N) - \beta_s^{k,N} \frac{\partial}{\partial s} \partial u \ (s, X_s^N) - \left( \left| \beta_s^{k,N} \right|^2 + z_s^{k,N} \gamma_s^{k,N} \right) \Delta u \ (s, X_s^N) - \frac{1}{2} \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^2 u \ (s, X_s^N),
\]
\[
j_k^N \ (s) := \left| \gamma_s^{k,N} \right|^2 \frac{\partial}{\partial s} \Delta u \ (s, X_s^N) + \beta_s^{k,N} \gamma_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^3 u \ (s, X_s^N) \tag{3.12}
\]
\[
\quad + \frac{1}{2} \left| \gamma_s^{k,N} \right|^4 \partial^4 u \ (s, X_s^N) + 2 \gamma_s^{k,N} z_s^{k,N} \Delta u \ (s, X_s^N) + |s - t_k|^2 \left| \gamma_s^{k,N} \right|^2 \Delta u \ (s, X_s^N) + 2(s - t_k) \left| \gamma_s^{k,N} \right|^2 \eta_s^{k,N} \partial^2 u \ (s, X_s^N) - \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta \ (\sigma^2 \Delta u) \ (s, X_s^N). \]

The next two lemmas explain that, up to some sign, \( \mathcal{I}_k^N \) (resp. \( \mathcal{J}_k^N \)) can be viewed as an antiderivative of \( i_k^N \) (resp. \( j_k^N \)).

**Lemma 3.16.** For all \( k = 0, \ldots, N-1 \), we have \( \mathcal{I}_k^N \ (t) = E \int_{t_k}^{t_{k+1}} i_k^N \ (s) \ ds \) for \( t \in [t_k, t_{k+1}] \).

**Proof.** If we denote by \( A := E \left[ \beta_t^{k,N} \partial u \ (t, X_t^N) \right] - E \left[ b \left( X_{t_{k+1}}^N \right) \partial u \left( t_{k+1}, X_{t_{k+1}}^N \right) \right] \) and by \( B := E \left[ b \left( X_{t_{k+1}}^N \right) \partial u \left( t_k, X_{t_k}^N \right) \right] - E \left[ b \left( X_{t_k}^N \right) \partial u \ (t, X_t^N) \right] \) we can write \( \mathcal{I}_k^N \ (t) = A + B \).
Lemma 3.11 enables us to apply Itô’s formula: Let \( v : [0, T] \times \mathbb{R} \to \mathbb{R} \) be of class \( C^{1,2} \); Itô’s formula yields

\[
dv(t, X^N_t) = \left\{ \frac{\partial}{\partial t} v + \beta^k_N \partial v + \frac{1}{2} |\gamma^k_N|^2 \Delta v \right\} (t, X^N_t) \, dt + \gamma^k_N \partial v (t, X^N_t) \, dW_t. \tag{3.13}
\]

Using this equation with Lemma 3.10 we have for \( v \in C^{1,2} \)

\[
d \left[ \beta^k_N v (r, X^N_r) \right] = \left\{ \beta^k_N \frac{\partial}{\partial r} v + \frac{1}{2} \beta^k_N |\gamma^k_N|^2 \Delta v + z^k_N \gamma^k_N \partial v \right\} (r, X^N_r) \, dr + \left\{ \beta^k_N \gamma^k_N \partial v + z^k_N v \right\} (r, X^N_r) \, dW_r. \tag{3.14}
\]

The function \( \Delta u \) has polynomial growth; hence corollary 3.15 implies that

\[
E \int_t^{t_{k+1}} \left\{ \beta^k_N \gamma^k_N \Delta u + z^k_N \gamma^k_N \partial u \right\} (s, X^N_s) \, dW_s = 0.
\]

Using equation (3.14) with \( v = \partial u \), integrating between \( t \) and \( t_{k+1} \), using the fact that \( \beta^k_{k+1} = b \left( X^N_{t_{k+1}} \right) \) and taking expectation we obtain

\[
A = -E \int_t^{t_{k+1}} \left\{ \beta^k_N \frac{\partial}{\partial s} u + \frac{1}{2} \beta^k_N |\gamma^k_N|^2 \Delta u + \frac{1}{2} \beta^k_N |\gamma^k_N|^2 \partial^3 u + z^k_N \gamma^k_N \Delta u \right\} (s, X^N_s) \, ds. \tag{3.15}
\]

Similarly, Corollary 3.15 implies that

\[
E \int_t^{t_{k+1}} \beta^k_N \partial (b \partial u) (s, X^N_s) \, dW_s = 0.
\]

Using (3.13) with \( v = b \partial u \), integrating between \( t \) and \( t_{k+1} \) and taking expectation yields

\[
B = E \int_t^{t_{k+1}} \left\{ \frac{\partial}{\partial s} (b \partial u) + \frac{1}{2} \left| \gamma^k_N \right|^2 \Delta (b \partial u) \right\} (s, X^N_s) \, ds.
\]

The stochastic integral is centered by Corollary 3.15. This identity combined with (3.15) concludes the proof.

**Lemma 3.17.** For all \( k = 0, \ldots, N - 1 \), we have \( \mathcal{J}^N_k (t) = E \int_{t_k}^{t} j^N_j (s) \, ds \) for \( t \in [t_k, t_{k+1}] \).

**Proof.** Using (3.3) we clearly deduce that \( \mathcal{J}^N_k (t) = C + D \) where

\[
C := E \left[ \sigma \left( X^N_{t_k} \right) + (t - t_k) \beta^k_N \right] \Delta u (t, X^N_t) - E \left[ \sigma^2 (X^N_{t_k}) \Delta u (t_k, X^N_{t_k}) \right],
\]

\[
D := E \left[ \sigma^2 (X^N_{t_k}) \Delta u (t_k, X^N_{t_k}) \right] - E \left[ \sigma^2 (X^N_{t}) \Delta u (t, X^N_t) \right].
\]

We at first rewrite the term \( D \): using (3.13) with \( v = \sigma^2 \Delta u \), integrating between \( t_k \) and \( t \) and taking expectation, we obtain:

\[
D = -E \int_{t_k}^{t} \left\{ \frac{\partial}{\partial t} (\sigma^2 \Delta u) + \beta^k_N \partial (\sigma^2 \Delta u) + \frac{1}{2} \left| \gamma^k_N \right|^2 \Delta (\sigma^2 \Delta u) \right\} (s, X^N_s) \, ds,
\]

since \( \sigma^2 \Delta u \) has polynomial growth which implies that the stochastic integral is centered using Corollary 3.15. Itô’s formula and Lemma 3.10 yield for \( r \in [t_k, t_{k+1}] \)

\[
d \left| \gamma^k_r \right|^2 = \left\{ 2 \gamma^k_r \beta^k_r \frac{\partial}{\partial r} + \frac{1}{2} \left| \gamma^k_N \right|^2 \right\} (r - t_k) \, dr + 2 \gamma^k_r \eta^k_r \eta^k_r (r - t_k) \, dW_r. \tag{3.16}
\]

Using this equation with Lemma 3.10 we obtain for \( t \in [t_k, t_{k+1}] \):

\[
dv(t, X^N_t) = \left\{ \frac{\partial}{\partial t} v + \beta^k_N \partial v + \frac{1}{2} |\gamma^k_N|^2 \Delta v \right\} (t, X^N_t) \, dt + \gamma^k_N \partial v (t, X^N_t) \, dW_t. \tag{3.13}
\]
Using this equation and (3.13), we have for \(v\) of class \(C^{1,2}\) and \(r \in [t_k, t_{k+1}]\)

\[
d | \gamma^{k,N}_r |^2 v (r, X^N_r) = \left\{ \begin{array}{l}
| \gamma^{k,N}_r |^2 \frac{\partial}{\partial r} v + \beta^{k,N}_r | \gamma^{k,N}_r |^2 \partial v + \frac{1}{2} | \gamma^{k,N}_r |^4 \Delta v \\
+ 2 \gamma^{k,N}_r z^{k,N}_r v + | \eta^{k,N}_r |^2 |r - t_k|^2 v + 2 | \gamma^{k,N}_r | \eta^{k,N}_r (r - t_k) \partial v \\
+ \left\{ \left(2 \gamma^{k,N}_r \right)^3 \partial v + 2 \gamma^{k,N}_r \eta^{k,N}_r (r - t_k) v \right\} (r, X^N_r) dr
\end{array} \right.
\]  

(3.17)

Using equation (3.17) with \(v = \Delta u\), integrating between \(t_k\) and \(t\), using the identity \(\gamma^{k,N}_{t_k} = \sigma (X^N_{t_k})\) and taking expectation, we deduce

\[
C = E \int_{t_k}^t \left\{ \begin{array}{l}
| \gamma^{k,N}_s |^2 \frac{\partial}{\partial s} \Delta u + \beta^{k,N}_s | \gamma^{k,N}_s |^2 \partial \beta u + \frac{1}{2} | \gamma^{k,N}_s |^4 \partial^4 u \\
+ 2 \gamma^{k,N}_s z^{k,N}_s \Delta u + | s - t_k |^2 | \eta^{k,N}_s |^2 \Delta u + 2 (s - t_k) \gamma^{k,N}_s \beta | \eta^{k,N}_s \partial^3 u \right\} (s, X^N_s) ds.
\]

Indeed, once more Corollary 3.15 and the polynomial growth of \(\partial \Delta u\) and \(\partial u\) implies that the corresponding stochastic integral is centered. This concludes the proof.

\[
\]

Plugging the results of Lemmas 3.16 and 3.17 into (3.10) we obtain

\[
Ef (X^N_T) - Ef (X_T) = \sum_{k=0}^{N-1} E \int_{t_k}^{t_{k+1}} \left\{ \int_{t_k}^{t_{k+1}} i^N_i (s) ds + \frac{1}{2} \int_{t_k}^{t_{k+1}} j^N_i (s) ds \right\} dt.
\]

Note: Thanks to Corollary 3.15 and the assumptions growth or boundness on the coefficients, all the stochastic integrals appearing in the next section, are centered.

3.6. Upper estimate of \(I^N_k(t)\). We next upper estimate the difference \(\phi_1(s) - \phi_1(t_{k+1})\), where \(\phi_1\) is one of the seven terms in the right hand side of (3.11)

3.6.1. The term \(\phi_1(s) = \frac{\partial}{\partial s} (b \partial u) (s, X^N_s)\). Using (3.13) with \(v = \frac{\partial}{\partial s} (b \partial u)\), integrating from \(s\) to \(t_{k+1}\) and taking expected value we deduce

\[
E \frac{\partial}{\partial s} (b \partial u) (s, X^N_s) = E \frac{\partial}{\partial s} (b \partial u) (t_{k+1}, X^N_{t_{k+1}}) + R_1(s),
\]

where

\[
R_1(s) := - E \int_s^{t_{k+1}} \left\{ \beta^{k,N}_r \frac{\partial^2}{\partial s^2} (b \partial u) + \beta^{k,N}_r \frac{\partial}{\partial s} (b \partial u) + \frac{1}{2} | \gamma^{k,N}_r |^2 \Delta (b \partial u) \right\} (r, X^N_r) dr.
\]

Furthermore, Lemmas 3.14 and 3.11 and the polynomial growth of the functions involved imply that \(|R_1(s)| \leq Ch\).

3.6.2. The term \(\phi_2(s) = \beta^{k,N}_s \partial (b \partial u) (s, X^N_s)\). Using (3.14) with \(v = \partial (b \partial u)\), integrating between \(s\) and \(t_{k+1}\) and taking expectation we obtain

\[
E \left[ \beta^{k,N}_s \partial (b \partial u) (s, X^N_s) \right] = E \left[ b \left( X^N_{t_{k+1}} \right) \partial (b \partial u) (t_{k+1}, X^N_{t_{k+1}}) \right] + R_2(s),
\]

where

\[
R_2(s) := - E \int_s^{t_{k+1}} \left\{ \beta^{k,N}_r \frac{\partial}{\partial s} (b \partial u) + \beta^{k,N}_r \Delta (b \partial u) + \frac{1}{2} | \gamma^{k,N}_r |^2 \partial^3 (b \partial u) \right\} + \gamma^{k,N}_r \frac{\partial}{\partial r} (b \partial u) \Delta (b \partial u) \right\} (r, X^N_r) dr.
\]

The polynomial growth of the functions and Lemmas 3.14 and 3.11 imply that \(|R_2(s)| \leq Ch|\).
3.6.3. The term $\phi_3(s) = \frac{1}{2} \left| \gamma^{k,N}_s \right|^2 \Delta (b \partial u) (s, X^N_s)$. Let
\[
R_3(s) := -\frac{1}{2} E \int_s^{t_{k+1}} \left\{ \left| \gamma^{k,N}_r \right|^2 \frac{\partial}{\partial t} \Delta (b \partial u) + \beta^{k,N}_r \left| \gamma^{k,N}_r \right|^2 \frac{\partial^3}{\partial t^3} (b \partial u) + \frac{1}{2} \left| \gamma^{k,N}_r \right|^4 4 b \partial^4 (b \partial u) \\
+ 2 \Delta (b \partial u) \right\} \left( r, X^N_r \right) dr.
\]
Using (3.17) with $v = \frac{1}{2} \Delta (b \partial u)$, integrating between $s$ and $t_{k+1}$, and taking expectation give us
\[
\frac{1}{2} E \left[ \left| \gamma^{k,N}_s \right|^2 \Delta (b \partial u) (s, X^N_s) \right] = \frac{1}{2} E \left[ \left| \gamma^{k,N}_{t_{k+1}} \right|^2 \Delta (b \partial u) \left( t_{k+1}, X^N_{t_{k+1}} \right) \right] + R_3(s),
\]
with $|R_3(s)| \leq Ch$.

3.6.4. The term $\phi_4(s) = \frac{\beta^{k,N}_s}{s} \frac{\partial}{\partial s} u (s, X^N_s)$. Let
\[
R_4(s) := E \int_s^{t_{k+1}} \left[ \beta^{k,N}_r \frac{\partial^2}{\partial s^2} \frac{\partial}{\partial t} u + \beta^{k,N}_r \frac{\partial^2}{\partial s \partial t} u + \frac{1}{2} \beta^{k,N}_r \left| \gamma^{k,N}_r \right|^2 \Delta \frac{\partial}{\partial s} u \\
+ \frac{\gamma^{k,N}_r \partial}{\partial s} u \right] \left( r, X^N_r \right) dr.
\]
Using (3.14) for $v = \frac{\partial}{\partial s} u$ and integrating between $s$ and $t_{k+1}$, we obtain
\[
- E \left[ \beta^{k,N}_s \frac{\partial}{\partial t} u (s, X^N_s) \right] = - E \left[ b \left( X^N_{t_{k+1}} \right) \frac{\partial}{\partial t} u \left( t_{k+1}, X^N_{t_{k+1}} \right) \right] + R_4(s),
\]
with $|R_4(s)| \leq Ch$.

3.6.5. The term $\phi_5(s) = \left| \beta^{k,N}_s \right|^2 \Delta u (s, X^N_s)$. Using Itô’s formula and Lemma 3.10 we have
\[
d \left| \beta^{k,N}_r \right|^2 = \left| \beta^{k,N}_r \right|^2 dr + 2 \beta^{k,N}_r \frac{\partial}{\partial s} \left( \beta^{k,N}_r \right) dW_r.
\]
Using this equation, (3.13) and Itô’s formula we obtain
\[
d \left[ \beta^{k,N}_r \right]^2 \Delta u (r, X^N_r) = \left[ \beta^{k,N}_r \right]^2 \frac{\partial}{\partial t} \Delta u + \left( \beta^{k,N}_r \right)^3 \frac{\partial^3}{\partial t^3} u + \frac{1}{2} \left| \beta^{k,N}_r \right|^2 \left| \gamma^{k,N}_r \right|^2 \frac{\partial^4}{\partial t^4} u \\
+ \left| \beta^{k,N}_r \right|^2 \Delta u + 2 \beta^{k,N}_r \frac{\partial}{\partial s} \left( \beta^{k,N}_r \right) \left( r, X^N_r \right) dr + dM_r,
\]
where $dM_r = \left( 2 \beta^{k,N}_r \frac{\partial}{\partial s} \left( \beta^{k,N}_r \right) \right) \left( r, X^N_r \right) dW_r$ and $M_t$ is a square integrable martingale. Let
\[
R_5(s) := E \int_s^{t_{k+1}} \left[ \beta^{k,N}_r \frac{\partial}{\partial t} \Delta u + \left( \beta^{k,N}_r \right)^3 \frac{\partial^3}{\partial t^3} u + \frac{1}{2} \left| \beta^{k,N}_r \right|^2 \left| \gamma^{k,N}_r \right|^2 \frac{\partial^4}{\partial t^4} u \\
+ \frac{\beta^{k,N}_r \frac{\partial}{\partial s} \left( \beta^{k,N}_r \right)}{r, X^N_r} \right] \left( r, X^N_r \right) dr.
\]
Integrating between $s$ and $t_{k+1}$ and taking expectation we have
\[
- E \left[ \beta^{k,N}_s \right]^2 \Delta u (s, X^N_s) = - E \left[ \beta^{k,N}_{t_{k+1}} \right]^2 \Delta u \left( t_{k+1}, X^N_{t_{k+1}} \right) + R_5(s),
\]
with $|R_5(s)| \leq Ch$. 

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3.6.6. The term \( \phi_{6}(s) = z_{r}^{k,N} \gamma_{s}^{k,N} \Delta u (s, X_{s}^{N}) \). Applying Itô's formula to the product of \( z_{r}^{k,N} \gamma_{s}^{k,N} \) and (3.13), and Lemma 3.10, we obtain for \( r \in [t_{k}, t_{k+1}] \)

\[
d d \left[ z_{r}^{k,N} \gamma_{r}^{k,N} u (r, X_{r}^{N}) \right] = \left\{ \frac{z_{r}^{k,N} \gamma_{r}^{k,N} \partial u}{\partial t} + \frac{z_{r}^{k,N} \gamma_{r}^{k,N} \partial u}{\partial v} + \frac{1}{2} \left[ z_{r}^{k,N} \right]^{2} \Delta u + \left[ \gamma_{r}^{k,N} \right]^{2} v + \left| \eta_{r}^{k,N} \right|^{2} \right. \\
+ \left. \left( r - t_{k} \right) v + z_{r}^{k,N} \eta_{r}^{k,N} \left( r - t_{k} \right) \gamma_{r}^{k,N} \partial v + \left| \gamma_{r}^{k,N} \right|^{2} \eta_{r}^{k,N} \partial v \right\} (r, X_{r}^{N}) \, dr + dM_{r}
\]

(3.19)

where \( dM_{r} = \left\{ \frac{z_{r}^{k,N} \gamma_{r}^{k,N} \partial u}{\partial v} + \frac{z_{r}^{k,N} \gamma_{r}^{k,N} \partial v}{\partial v} + \frac{1}{2} \left[ \gamma_{r}^{k,N} \right]^{2} \Delta u + \left[ \gamma_{r}^{k,N} \right]^{2} v \right\} (r, X_{r}^{N}) \, dW_{r} \) and \( M_{r} \) is a square integrable martingale. Using equation (3.19) with \( v = \Delta u \), integrating between \( s \) and \( t_{k+1} \) and taking expectation give us

\[
- E \left[ z_{s}^{k,N} \gamma_{s}^{k,N} \Delta u (s, X_{s}^{N}) \right] = - E \left[ z_{t_{k+1}}^{k,N} \Delta u \left( t_{k+1}, X_{t_{k+1}}^{N} \right) \right] + R_{6}(s),
\]

with \( |R_{6}(s)| \leq Ch \).

3.6.7. The term \( \phi_{7}(s) = \frac{1}{2} \Delta \gamma_{s}^{k,N} \left| \gamma_{s}^{k,N} \right|^{2} \partial^{3} u (s, X_{s}^{N}) \). Using Lemma 3.10 and equation (3.17), Itô's formula give us for \( v \) of class \( C^{1,2} \)

\[
d \left[ \frac{\partial^{3} u}{\partial r^{3}} \right]^{2} (r, X_{r}^{N}) = \left\{ \frac{\partial^{3} u}{\partial r^{3}} \right\}^{2} \frac{\partial u}{\partial t} + \left[ \frac{\partial^{3} u}{\partial r^{3}} \right]^{2} \frac{\partial u}{\partial v} + \frac{1}{2} \left[ \frac{\partial^{3} u}{\partial r^{3}} \right]^{2} \Delta u
\]

(3.20)

\[
+ 2 \frac{\partial^{3} u}{\partial r^{3}} \left( r_{r}^{k,N} \right) \gamma_{r}^{k,N} \left( r_{r}^{k,N} \right) \frac{\partial u}{\partial v} + \frac{1}{2} \left[ \frac{\partial^{3} u}{\partial r^{3}} \right]^{2} \eta_{r}^{k,N} \left( r_{r}^{k,N} \right) \frac{\partial v}{\partial v} + \frac{1}{2} \left[ \frac{\partial^{3} u}{\partial r^{3}} \right]^{2} \gamma_{r}^{k,N} \left( r_{r}^{k,N} \right) \frac{\partial v}{\partial v} \right\} (r, X_{r}^{N}) \, dr
\]

Using this equation with \( v = \frac{1}{2} \partial v^{3} \), integrating between \( s \) and \( t_{k+1} \) and taking expectation we have

\[
- \frac{1}{2} E \left[ \frac{\partial^{3} u}{\partial r^{3}} \left( r_{r}^{k,N} \right) \right]^{2} \partial^{3} u (s, X_{s}^{N}) = - \frac{1}{2} E \left[ \frac{\partial^{3} u}{\partial r^{3}} \left( t_{k+1} \right) \right]^{2} \partial^{3} u \left( t_{k+1}, X_{t_{k+1}}^{N} \right) \right) + R_{7}(s),
\]

with \( |R_{7}(s)| \leq Ch \).

3.7. Upper estimate of \( \mathcal{J}_{k}^{N} (t) \). We upper estimate the error \( \tilde{\phi}_{i}(s) - \phi_{i}(t_{k}) \) where \( \tilde{\phi}_{i} \) is one of the nine terms in the right hand side of (3.12)

3.7.1. The term \( \tilde{\phi}_{1}(s) = \left| \gamma_{s}^{k,N} \right|^{2} \frac{\partial}{\partial r} \Delta u (s, X_{s}^{N}) \). Using (3.17) with \( v = \frac{\partial}{\partial r} \Delta u \), integrating between \( t_{k} \) and \( s \), taking expectation and using the fact that \( \gamma_{t_{k}}^{k,N} = \sigma (X_{t_{k}}^{N}) \) we have

\[
E \left[ \left| \gamma_{s}^{k,N} \right|^{2} \frac{\partial}{\partial t} \Delta u (s, X_{s}^{N}) \right] = E \left[ \left| \sigma (X_{t_{k}}^{N}) \right|^{2} \frac{\partial}{\partial t} \Delta u (t_{k}, X_{t_{k}}^{N}) \right] + \tilde{R}_{1}(s),
\]

with

\[
\tilde{R}_{1}(s) = \int_{t_{k}}^{s} \left[ \left| \gamma_{r}^{k,N} \right|^{2} \frac{\partial^{2}}{\partial r^{2}} \Delta u + \left( 2 \gamma_{r}^{k,N} \gamma_{r}^{k,N} + \left| \eta_{r}^{k,N} \right|^{2} \right) r - t_{k} \frac{\partial}{\partial r} \Delta u \right.
\]

\[
+ \left. \left[ \frac{\partial^{3} u}{\partial r^{3}} + 2 \frac{\partial^{3} u}{\partial r^{3}} \left( r_{r}^{k,N} \right) \right] \gamma_{r}^{k,N} \frac{\partial}{\partial v} + \frac{1}{2} \left| \gamma_{r}^{k,N} \right|^{4} \frac{\partial}{\partial v} \right] \left( r, X_{r}^{N} \right) \, dr.
\]

Corollary 3.15 implies that \( |\tilde{R}_{1}(s)| \leq Ch \).
3.7.2. The term \( \tilde{\phi}_2(s) = \beta^{k,N}_s \left| \gamma^{k,N}_s \right|^2 \partial^3 u (s, X^N_s) \). For an \( F_s \)-measurable random variable \( Z \), we have \( E \left( Z \gamma^{k,N}_s \right) = E \left( Z b \left( X^N_{t_{k+1}} \right) \right) \). Using (3.20) with \( v = \partial^3 u \), integrating between \( t_k \) and \( s \) and taking expectation we have

\[
E \left[ \beta^{k,N}_s \left| \gamma^{k,N}_s \right|^2 \partial^3 u (s, X^N_s) \right] = E \left[ b \left( X^N_{t_{k+1}} \right) \left| \sigma \left( X^N_{t_{k}} \right) \right|^2 \partial^3 u \left( t_k, X^N_{t_k} \right) \right] + \tilde{R}_2(s),
\]

where

\[
\tilde{R}_2(s) := E \int_{t_k}^{s} \left\{ 2 \gamma^{k,N}_r \gamma^{k,N}_s + \beta^{k,N}_r \left| \gamma^{k,N}_r \right|^2 \left| r - t_k \right|^2 + 2 \gamma^{k,N}_r \gamma^{k,N}_s \gamma^{k,N}_s \left( r - t_k \right) \right\} \partial^3 u + \beta^{k,N}_r \left| \gamma^{k,N}_r \right|^2 \partial^3 u + \frac{1}{2} \beta^{k,N}_r \left| \gamma^{k,N}_r \right|^4 \partial^3 u + \left( \beta^{k,N}_r \right)^2 \left( 2 \beta^{k,N}_r \left| \gamma^{k,N}_r \right| \partial^3 u + \frac{1}{2} \beta^{k,N}_r \left| \gamma^{k,N}_r \right|^4 \partial^3 u \right) \left( r, X^N_r \right) dr.
\]

Corollary 3.15 implies that \( |\tilde{R}_2(s)| \leq Ch \).

3.7.3. The term \( \tilde{\phi}_3(s) = \frac{1}{2} \left| \gamma^{k,N}_s \right|^4 \partial^4 u (s, X^N_s) \). Using Lemma 3.10 and Itô’s formula we deduce

\[
\frac{1}{2} E \left[ \gamma^{k,N}_s \right]^4 \partial^4 u (s, X^N_s) = \frac{1}{2} E \left[ \left| \gamma^{k,N}_s \right|^4 \partial^4 u (t_k, X^N_{t_k}) \right] + \tilde{R}_3(s),
\]

where \( \tilde{R}_3(s) \leq Ch \) by Corollary 3.15.

3.7.4. The term \( \tilde{\phi}_4(s) = 2 \gamma^{k,N}_s z^{k,N}_s \Delta u (s, X^N_s) \). Using (3.19) with \( v = \Delta u \) we have

\[
E \left[ 2 \gamma^{k,N}_s z^{k,N}_s \Delta u (s, X^N_s) \right] = E \left[ 2 \gamma^{k,N}_s z^{k,N}_s \Delta u \left( t_k, X^N_{t_k} \right) \right] + \tilde{R}_4(s),
\]

and Corollary 3.15 implies \( \left| \tilde{R}_4(s) \right| \leq Ch \).

3.7.5. The term \( \tilde{\phi}_5(s) := \tilde{R}_5(s) := |s - t_k|^2 \left| \eta^{k,N}_s \right|^2 \Delta u (s, X^N_s) + 2 \left| s - t_k \right| \left| \gamma^{k,N}_s \right|^2 \left| \eta^{k,N}_s \right| \partial^3 u (s, X^N_s) \). Using Corollary 3.15, we have \( \left| \tilde{R}_5(s) \right| \leq Ch \).

3.7.6. The term \( \tilde{\phi}_6(s) = \beta^{k,N}_s \partial \left( \sigma^2 \Delta u \right) (s, X^N_s) \). Using (3.13) with \( v = \partial \left( \sigma^2 \Delta u \right) \), integrating between \( t_k \) and \( s \) and taking expectation, we have

\[
- E \left[ \frac{\partial}{\partial t} \left( \sigma^2 \Delta u \right) (s, X^N_s) \right] = - E \left[ \frac{\partial}{\partial t} \left( \sigma^2 \Delta u \right) \left( t_k, X^N_{t_k} \right) \right] + \tilde{R}_6(s),
\]

with \( \left| \tilde{R}_6(s) \right| \leq Ch \) by Corollary 3.15.

3.7.7. The term \( \tilde{\phi}_7(s) = \beta^{k,N}_s \partial \left( \sigma^2 \Delta u \right) (s, X^N_s) \). Using (3.14) with \( v = \partial \left( \sigma^2 \Delta u \right) \), integrating between \( t_k \) and \( s \), taking expectation we have

\[
- E \left[ \beta^{k,N}_s \partial \left( \sigma^2 \Delta u \right) (s, X^N_s) \right] = - E \left[ b \left( X^N_{t_k} \right) \partial \left( \sigma^2 \Delta u \right) \left( t_k, X^N_{t_k} \right) \right] + \tilde{R}_7(s),
\]

with \( \left| \tilde{R}_7(s) \right| \leq Ch \) by Corollary 3.15.
3.7.8. The term $\phi_s(s) = \frac{1}{2} \left| \gamma_{s,N}^k \right|^2 \Delta (\sigma^2 \Delta u)(s, X_s^N)$. Using (3.17) with $v = \frac{1}{2} \Delta (\sigma^2 \Delta u)$, integrating between $t_k$ and $s$, and finally taking expectation we have

$$-\frac{1}{2} E \left[ \left| \gamma_{s,N}^k \right|^2 \Delta (\sigma^2 \Delta u)(s, X_s^N) \right] = -\frac{1}{2} E \left[ \left| \sigma (X_{t_k}^N) \right|^2 \Delta (\sigma^2 \Delta u)(t_k, X_{t_k}^N) \right] + \tilde{R}_s(s),$$

with $|\tilde{R}_s(s)| \leq C h$ by Corollary 3.15.

3.8. **Proof Theorem 2.1 (i).** The identity (3.18) and the upper estimate in section 3.6 and 3.7 imply that

$$Ef(X_T^N) - Ef(X_T) = \frac{1}{2} \sum_{k=0}^{N-1} E \int_{t_k}^{t_{k+1}} \left\{ \int_t^{t_{k+1}} i_k^N(t_{k+1}) ds + \frac{1}{2} \int_t^{t_{k+1}} j_k^N(t_{k+1}) ds \right\} dt + R \tag{3.21}$$

where

$$R := \sum_{k=0}^{N-1} \sum_{j=1}^{7} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} R_j(s) ds dt + \sum_{k=0}^{N-1} \sum_{j=1}^{8} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \tilde{R}_j(s) ds dt.$$ 

Hence $|R| \leq C h^2$. Note that $\beta_{t_{k+1}} = b(X_{t_{k+1}}^N)$. Using (3.3) and (3.5) we deduce that

$$ \tilde{z}_{t_{k+1}}^{k,N} = \sigma (X_{t_k}^N) (S_h b') (X_{t_{k+1}}^N) + \gamma_{t_{k+1}}^{k,N} = \sigma (X_{t_k}^N) \left[ 1 + h(S_h b')(X_{t_{k+1}}^N) \right] = \sigma (X_{t_k}^N) S_h (X_{t_{k+1}}^N).$$

Therefore, we deduce that

$$i_k^N(t_{k+1}) = \frac{\partial}{\partial t} (b \partial u)(t_{k+1}, X_{t_{k+1}}^N) + \left[ b \partial (b \partial u) \right] (t_{k+1}, X_{t_{k+1}}^N) + \frac{1}{2} \sigma^2 (X_{t_k}^N) \left[ \sigma^2 \Delta (b \partial u) \right] (t_{k+1}, X_{t_{k+1}}^N)$$

$$- \left[ \frac{\partial}{\partial t} \partial u \right] (t_{k+1}, X_{t_{k+1}}^N) - \left[ \partial^2 \Delta u \right] (t_{k+1}, X_{t_{k+1}}^N) - \frac{1}{2} \sigma^2 (X_{t_k}^N) \left[ \sigma^2 \partial^2 \Delta u \right] (t_{k+1}, X_{t_{k+1}}^N).$$

Similarly, $\gamma_{t_k}^{k,N} = \sigma (X_{t_k}^N)$. So we have

$$E j_k^N(t_k) = \frac{\partial}{\partial t} \Delta u(t_k, X_{t_k}^N) + \frac{1}{2} \left( \frac{\partial}{\partial t} \sigma^2 \Delta u(t_k, X_{t_k}^N) \right) + \frac{1}{2} E \sigma^4 \Delta u(t_k, X_{t_k}^N)$$

$$+ 2 E S_h b' (X_{t_{k+1}}^N) \sigma^2 \Delta u(t_k, X_{t_{k+1}}^N) - \frac{\partial}{\partial t} \left( \sigma^2 \Delta u(t_k, X_{t_{k+1}}^N) \right)$$

$$- \frac{1}{2} \sigma^2 \Delta \left( \sigma^2 \Delta u(t_k, X_{t_{k+1}}^N) \right).$$

Notice that $b$ and $\sigma$ do not depend upon $t$; hence after simplification we have

$$i_k^N(t_{k+1}) = b \partial (b \partial u)(t_{k+1}, X_{t_{k+1}}^N) - \frac{1}{2} \sigma^2 \Delta u(t_{k+1}, X_{t_{k+1}}^N) + \frac{1}{2} \sigma^2 (X_{t_k}^N) \left( S_h ^2 b' \partial u \right) (t_{k+1}, X_{t_{k+1}}^N), \tag{3.22}$$

$$E j_k^N(t_k) = \frac{1}{2} \sigma^2 \Delta \left( \sigma^2 \Delta u(t_k, X_{t_{k+1}}^N) \right) + \frac{1}{2} \sigma^4 \Delta u(t_k, X_{t_{k+1}}^N) - \frac{1}{2} \sigma^2 \Delta (\sigma^2 \Delta u(t_k, X_{t_{k+1}}^N)). \tag{3.23}$$

Corollary 3.15 implies the existence of a constant $C$, such that for all $k = 0, \ldots, N - 1$:

$$|E (i_k^N(t_{k+1}) + j_k^N(t_k))| \leq C.$$ 

Using this bound with (3.21) proves the first part of Theorem 2.1.
3.9. **Proof Theorem 2.1 (ii).** We at first prove the following lemma, which upper estimates the error in the approximation of an integral by a Riemann sum.

**Lemma 3.18.** Let \( v \) and \( w \) in \( C^\infty_b([0,T] \times \mathbb{R}) \). Then there exists a constant \( C \) independent of \( h \) such that

\[
\left| h \sum_{k=0}^{N-1} E v(t_{k+1}, X^N_{k+1}) w(t_k, X^N_{tk}) - E \int_0^T vw(t, X_t)dt \right| \leq Ch.
\]

**Proof.** Using (3.13) multiply by \( w(t_k, X^N_{tk}) \) and taking expected value, we deduce for \( v \in C^{1,2} \),

\[
E v(t_{k+1}, X^N_{k+1}) w(t_k, X^N_{tk}) = E vw(t_k, X^N_{tk}) + A_k,
\]

where

\[
A_k := Ew(t_k, X^N_{tk}) \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} v + \beta^k u \partial v + \frac{1}{2} \left| \gamma^k \right|^2 \Delta v \right\} (t, X^N_t) dt.
\]

This yields

\[
h \sum_{k=0}^{N-1} E v(t_{k+1}, X^N_{k+1}) w(t_k, X^N_{tk}) - E \int_0^T vw(t, X_t)dt = \sum_{k=0}^{N-1} (hA_k + hB_k + C_k).
\]

Using the Cauchy-Schwarz inequality, the fact that \( \frac{\partial}{\partial t} v, \partial v \) and \( \Delta v \) have polynomial growth so that Corollary 3.15 can be applied, we deduce

\[
|A_k|^2 \leq E \left[ \left| w(t_k, X^N_{tk}) \right|^2 \right] E \left[ \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} v + \beta^k u \partial v + \frac{1}{2} \left| \gamma^k \right|^2 \Delta v \right\} (t, X^N_t) dt \right]^2 \leq ChE \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} v + \beta^k u \partial v + \frac{1}{2} \left| \gamma^k \right|^2 \Delta v \right\} (t, X^N_t) dt \leq Ch^2,
\]

Hence, \( |A_k| \leq Ch \) which implies \( h \sum_{0 \leq k \leq N-1} |A_k| \leq Ch \).

Since \( (vw)(t_k, \cdot) \) is in \( C^\infty_b \), we use Theorem 2.1 (i), changing \( T \) by \( t_k \), which yields \( |B_k| \leq Ch \) and then \( h \sum_{0 \leq k \leq N-1} |B_k| \leq Ch \). Finally, Itô’s formula implies

\[
C_k = E \int_{t_k}^{t_{k+1}} (vw)(t_k, X_k) - (vw)(t, X_t) dt = -E \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \left\{ \frac{\partial}{\partial t} (vw) + b\partial(vw) + \frac{1}{2} \sigma^2 \Delta (vw) \right\} (s, X_s)dsdt.
\]

Once more the polynomial growth imposed on \( v, w \) and their partial derivatives implies that \( |C_k| \leq Ch^2 \) and then \( \sum_{0 \leq k \leq N-1} |C_k| \leq Ch \). This concludes the proof. \( \square \)

Now we introduce the function \( \psi_{ih} : [0,T] \times \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
\psi_{ih}(t, x) := \frac{1}{2} b\partial(b\partial u)(t, x) - \frac{1}{2} b^2 \Delta u(t, x) + \frac{1}{4} \sigma^2 \sigma_h^2 b\partial u(t, x) + \frac{1}{4} b^2 \partial^4 u(t, x) + \frac{1}{8} \sigma^4 \partial^4 u(t, x) + \frac{1}{2} b^2 \sigma^2 \Delta u(t, x) - \frac{1}{4} b\partial(\sigma^2 \Delta u)(t, x) - \frac{1}{8} \sigma^2 \Delta(\sigma^2 \Delta u)(t, x) \tag{3.24}
\]
Using the expression of $i^N_k$ in (3.22) (resp. $j^N_k$ in (3.23)) and the previous lemma, we deduce

$$ \left| \frac{1}{2h} \sum_{k=0}^{N-1} Ei^N_k(t_{k+1}) + \frac{1}{4h} \sum_{k=0}^{N_1} Ej^N_k(t_k) - \int_0^T E\psi_{ih}(t,X_t)dt \right| \leq Ch. \quad (3.25) $$

Using the definitions of $\psi_i$ and $\psi_{ih}$ given in (2.2) and (3.24) respectively, we have

$$ \psi_{ih}(t,x) - \psi_i(t,x) = \frac{1}{4} \{ \sigma^2 S_h^2 b'' \partial u + b \sigma^2 \partial^3 u + 2b'(S_h - 1) \sigma^2 \Delta u - \sigma^2 \Delta (b \partial u) \} (t,x) $$

$$ = \frac{1}{4} \sigma^2 (S_h^2 - 1) b'' \partial u(t,x) + \frac{1}{2} b'(S_h - 1) \sigma^2 \Delta u(t,x). $$

Since $(S_h - 1)(x) = \frac{b'}{1+bh}(x)$ and $|S_h(x) + 1| \leq C$ for $h \in (0,h^*)$, we have $|(S_h - 1)(x)| + |(S_h^2 - 1)(x)| \leq Ch$, where as usually $C$ does not depend on $N$ and $h$. This yields

$$ \left| \int_0^T E \{ \psi_{ih}(t,X_t) - \psi_i(t,X_t) \} dt \right| \leq Ch. $$

This last equation with (3.21) and (3.25) concludes the proof.

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