Geodesic complexity for non-geodesic spaces

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Abstract
We define the notion of near-geodesic between points of a metric space when no geodesic exists, and use this to extend Recio-Mitter’s notion of geodesic complexity to non-geodesic spaces. This has potential application to topological robotics. We determine explicit near-geodesics and geodesic complexity in a variety of cases.

Keywords Geodesic · Configuration space · Topological robotics

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1 Introduction
In [6], Recio-Mitter defined the geodesic complexity GC(X) of a metric space X to be the smallest number k such that X × X can be partitioned into k + 1 locally compact sets E_i, 0 ≤ i ≤ k, such that on each E_i, there is a continuous map s_i : E_i → PX, called a geodesic motion planning rule (GMPR) on E_i, such that, for all (x_0, x_1) ∈ E_i, s_i(x_0, x_1) is a (minimal) geodesic from x_0 to x_1. This was an analogue of Farber’s notion of topological complexity TC(X) ([3]), which applied to any topological space X and did not require that s_i(x_0, x_1) be a geodesic. Clearly, TC(X) ≤ GC(X) for any metric space X. These notions are of particular interest if X is a space of configurations of one or more robots.

A geodesic space is one in which for all pairs (x_0, x_1) of points, there is a geodesic from x_0 to x_1. According to the definition in [6], GC(X) = ∞ if X is not geodesic. In [1] and [2], some non-geodesic spaces X were replaced by

1 By “geodesic,” we always mean a minimal geodesic as defined in [6, Definition 1.3].
homotopically equivalent geodesic spaces, whose GC was computed and interpreted as representing GC(\(X\)). This seems reasonable, since TC is a homotopy invariant.

Let \(F(X, 2)\) denote the space of ordered pairs of distinct points of \(X\), with the induced metric from \(X \times X\). In [1], we replaced the non-geodesic space \(F(\mathbb{R}^n, 2)\) by the homotopically equivalent geodesic space \(F_\varepsilon(\mathbb{R}^n, 2)\), consisting of points \((x_0, x_1)\) satisfying \(d(x_0, x_1) \geq \varepsilon\). We determined explicit geodesics in \(F_\varepsilon(\mathbb{R}^n, 2)\), but the work was quite complicated.

For a topological space \(Y\), \(P(Y) = Y'\) denotes the free path space of continuous maps \(I \to Y\), with the compact-open topology, and \(P(Y; y_0, y_1)\) the subspace consisting of paths from \(y_0\) to \(y_1\).

**Definition 1.1** Let \(X\) be a metric space whose completion \(\overline{X}\) is geodesic. For \(x_0, x_1 \in X\), a near-geodesic from \(x_0\) to \(x_1\) is a continuous map \(\phi : I \to P(\overline{X}; x_0, x_1)\) satisfying

i. \(\phi(0)\) is a geodesic in \(\overline{X}\) from \(x_0\) to \(x_1\);
ii. \(\phi([0, 1]) \subset P(X; x_0, x_1)\);
iii. length(\(\phi(s)\)) is finite for all \(s\), and if \(s_n \to 0\), then length(\(\phi(s_n)\)) \to length(\(\phi(0)\)).

Here, length of a path is defined (as in [6]) to be the supremum of sums of distances between successive points on the path. Item (iii) of the above definition guarantees that for values of \(s\) close enough to \(0\), the paths \(\phi(s)\) in \(X\) are good approximations to the geodesic in \(\overline{X}\). If there is a geodesic in \(X\) from \(x_0\) to \(x_1\), one can use the constant homotopy, but we still call it a near-geodesic for uniformity.

**Definition 1.2** For \(E \subset X \times X\), a near-geodesic motion planning rule (NGMPR) on \(E\) is a continuous map \(\Phi\) from \(E\) to \(P(\overline{X})\), such that, for all \((x_0, x_1) \in E\), \(\Phi(x_0, x_1)\) is a near-geodesic from \(x_0\) to \(x_1\), with the additional proviso that \(\Phi(x_0, x_1)(0)\) is a geodesic in \(X\) if one exists. The geodesic complexity \(\text{GC}(X)\) is defined as the smallest \(k\), such that \(X \times X\) can be partitioned into locally compact sets \(E_0, \ldots, E_k\), such that each \(E_j\) has an NGMPR.

The following elementary proposition suggests that this is a good definition.

**Proposition 1.3** If \(X\) is geodesic, then the definition of \(\text{GC}(X)\) in Definition 1.2 agrees with that in [6].

**Proof** If \(s\) is a GMPR on \(E\), then using the constant homotopy yields an NGMPR on \(E\). If \(\Phi\) is an NGMPR on \(E\), then \(\Phi|E \times 0\) is a GMPR. \(\square\)

One might prefer that the “additional proviso” in Definition 1.2 be omitted. We have included it so that Proposition 1.3 is true. It is conceivable that this proposition might still be true if that requirement were omitted, but we have not been able to prove this. Note that \(\text{TC}(X) \leq \text{GC}(X)\), since if \(\Phi\) is an NGMPR on \(E\), then the map \(E \to P(X)\) defined by \((x_0, x_1) \mapsto \Phi(x_0, x_1)(\frac{1}{2})\) is a motion planning rule on \(E\).

In this paper, we show that \(\text{GC}(X) = \text{TC}(X)\) for the following non-geodesic spaces \(X\) by constructing explicit NGMPRs.
\begin{itemize}
  \item $\mathbb{R}^n - Q$ with $n \geq 2$ and $Q$ a finite subset,
  \item $F(\mathbb{R}^n, 2)$ with $n \geq 2$,
  \item $F(\mathbb{R}^n - \{x_0\}, 2)$ with $n \geq 2$,
  \item the unordered configuration space $C(\mathbb{R}^2 - \{x_0\}, 2)$,
  \item $F(Y, 2)$, where $Y$ is a graph with exactly one essential vertex, of order 3.
\end{itemize}

In Sect. 3, we study $GC(X)$ for $X = F(\mathbb{R}^n - Q, 2)$ if $Q$ is a finite subset with at least two points. This might be a case in which $GC(X) \neq TC(X)$.

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### 2 Some examples with $GC = TC$

In this section, we determine explicit NGMPRs in the first four of the five examples with $GC(X) = TC(X)$ listed above. In each of these examples, we find it convenient to let $g : I \to I$ denote a continuous function, such as $g(t) = \sin(\pi t)$ or $g(t) = 1 - |2t - 1|$, satisfying $g^{-1}(0) = \{0, 1\}$.

**Definition 2.1** The nogeo set of $X$ is the set of $(x_0, x_1) \in X \times X$ for which there is no (minimal) geodesic from $x_0$ to $x_1$. Its complement is the geoset of $X$.

**Theorem 2.2** For $n \geq 2$ and $x_0 \in \mathbb{R}^n$

\[
GC(\mathbb{R}^n - \{x_0\}) = TC(\mathbb{R}^n - \{x_0\}) = \begin{cases} 
1 & n \text{ even} \\
2 & n \text{ odd}
\end{cases}
\]

**Proof** Here, $\overline{X} = \mathbb{R}^n$. By a linear homeomorphism of $\mathbb{R}^n$, we may assume that $x_0 = 0$. The nogeo set $E$ is $\{(a, b) : a = \lambda b, \lambda < 0\}$. We use linear paths as a GMPR on the geoset. In such cases, the constant homotopy gives an NGMPR. We will not usually mention this. In this case, we have $\Phi(a, b)(s)(t) = (1 - t)a + tb$.

If $n$ is even, let $v$ be a unit vector field on $S^{n-1}$. Throughout Sects. 2 and 3, we will use $v$ for a unit vector field on $S^m$ when $m$ is odd or on $S^m - \{e_1\}$ when $m$ is even, and $e_1 \in S^m$. An NGMPR on $E$ is given by

\[
\Phi(a, b)(s)(t) = (1 - t)a + tb + s \cdot g(t) \cdot v \left( \frac{b - a}{\|b - a\|} \right). \tag{2.3}
\]

If $n$ is odd, let $v$ be a unit vector field on $S^{n-1} - \{e_1\}$ and let

\[E' = \{(a, b) : b - a = ke_1, k > 0\} \text{ and let}
\]

\[E' = \{(a, b) : b - a = ke_1, k > 0\} \text{ and let}
\]

Then, (2.3) is an NGMPR on $E - E'$, while $\Phi(a, b)(s)(t) = (1 - t)a + tb + s \cdot g(t) \cdot e_2$ is an NGMPR on $E'$. Thus, $GC(\mathbb{R}^n - \{0\}) \leq 1$ (resp. 2) if $n$ is even (resp. odd). Equality follows from the well-known value \cite{3} of $TC(S^{n-1})$, since $\mathbb{R}^n - \{0\} \simeq S^{n-1}$.
Recall that two subsets are topologically disjoint if the closure of each is disjoint from the other. Then, continuous maps on each can be combined on the union. This notion will be used in the next and subsequent proofs.

**Theorem 2.4** For $n \geq 2$, let $X = \mathbb{R}^n - Q$, where $Q$ is a finite set with at least two points. Then, $G(X) = T(X) = 2$.

**Proof** Again, $X = \mathbb{R}^n$. First, let $n$ be even. We split the nogeo set as $E_1 \cup E_2$, where $E_1$ is the set of $(a, b)$ for which exactly one point of $Q$ lies on the segment $ab$, while $E_2$ is those for which two or more points of $Q$ lie on the segment. An NGMPR on $E_1$ is given by

$$
\Phi(a, b)(s)(t) = (1 - t)a + tb + \delta \cdot s \cdot g(t) \cdot v\left(\frac{b - a}{\|b - a\|}\right),
$$

where $\delta = \delta(a, b) = \frac{1}{2}\min(1, d(ab, Q - (Q \cap ab)))$, a continuous function on $X \times X$. If it is not the case that $Q$ is collinear, then the function (2.5) works on $E_2$, as well. Note that this function would not be continuous on $E_1 \cup E_2$ because of the role of $\delta$. [[If $(a_n, b_n) \in E_1$, with all segments $a_n b_n$ passing through the same point $x \in Q$, have the property that $a_n \rightarrow a$, $b_n \rightarrow b$, and $d(a_n b_n, x') \rightarrow 0$ for some $x' \in Q \cap ab$ with $x' \neq x$, then $\delta(a_n, b_n) \rightarrow 0$ but $(a, b) \in E_2$ has $\delta(a, b) \neq 0$.]]

If $Q$ is collinear, and $q_1$ and $q_2$ are two of the points in $Q$, then we can use

$$
\Phi(a, b)(s)(t) = (1 - t)a + tb + s \cdot g(t) \cdot v\left(\frac{q_2 - q_1}{\|q_2 - q_1\|}\right)
$$
on $E_2$.

If $n$ is odd, let $V$ be a unit vector which is not realizable as $(x' - x)/\|x' - x\|$ for any $x, x' \in Q$, and let $v$ be a unit vector field on $S^{n-1} - \{V\}$. Let $E_1$ be the set of $(a, b)$ for which exactly one point of $Q$ lies on the segment $ab$, and $(b - a)/\|b - a\| \neq V$, and define $\Phi$ on $E_1$ using (2.5). Let $E_2$ be the set of $(a, b)$ for which two or more points of $Q$ lie on $ab$ or exactly one point of $Q$ lies on $ab$ and $(b - a)/\|b - a\| = V$. These two portions of $E_2$ are topologically disjoint. Define $\Phi(a, b)$ on $E_2$ using (2.5), interpreting $v(V)$ to be any particular vector orthogonal to $V$. Modify as above if $Q$ is collinear.

Using linear geodesics on the geoset, we obtain $G(X) \leq 2$, and it must equal 2, since $T(X) = 2$ ( [4, Lemma 10.2]).

**Theorem 2.6** For $n \geq 2$, $G(F(\mathbb{R}^n, 2)) = T(F(\mathbb{R}^n, 2)) = \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$.

**Proof** We have $X = \mathbb{R}^n \times \mathbb{R}^n$. Since $F(\mathbb{R}^n, 2) \simeq S^{n-1}$, it suffices to prove the upper bound. As noted in [1], the nogeo set $E$ is

$$
\{(a, a'), (b, b') : b - b' = \lambda(a - a'), \lambda < 0\},
$$
and we use linear geodesics on its complement.

If $n$ is even, an NGMPR on $E$ is given by
\[
\Phi((a, a'), (b, b'))(s)(t) = \left((1 - t)a + tb, (1 - t)a' + tb' + s \cdot g(t) \cdot v \left( \frac{b' - b}{\|b' - b\|} \right) \right). \tag{2.7}
\]

For a point on a path in the homotopy of (2.7) to have both components equal would require that \(v(\frac{b' - b}{\|b' - b\|})\) is a scalar multiple of \(b' - b\), which cannot happen.

If \(n\) is odd, decompose the nogeo set into subsets determined by whether or not \((b' - b)/\|b' - b\| = e_1\). We use a vector field on \(S^{n-1} - \{e_1\}\) in (2.7) for one, and can replace the \(v(-)\) expression by \(e_2\) for the other.

\[\square\]

**Theorem 2.8** If \(X = F(\mathbb{R}^n - \{x_0\}, 2)\), then \(\text{GC}(X) = \text{TC}(X) = \begin{cases} 3 & n \text{ even} \\ 4 & n \text{ odd}. \end{cases}\)

**Proof** Again, \(\overline{X} = \mathbb{R}^n \times \mathbb{R}^n\). We say that paths \(\gamma\) and \(\gamma'\) collide if \(\gamma(t) = \gamma'(t)\) for some \(t \in I\). As noted in [1], segments \(ab\) and \(a'b'\) collide if and only if \(b' - b = \lambda(a' - a)\) for some \(\lambda < 0\).

Let \(n\) be even. We partition \(X \times X\) into seven sets, on each of which we will define a GMPR or NGMPR. Sets \(C_0, C_1,\) and \(C_{x_0}\) consist of those \(((a, a'), (b, b'))\) for which segments \(ab\) and \(a'b'\) collide and

- \(C_0:\) neither segment contains \(x_0\),
- \(C_1:\) one of the segments contains \(x_0\) and the other segment has positive length, and
- \(C_{x_0}\): they collide at \(x_0\).

Sets \(E_j, j \in \{0, 1, 2\}\) consist of those \(((a, a'), (b, b'))\) which do not collide and \(j\) of the segments \(ab\) and \(a'b'\) contain \(x_0\). The set \(L_1\) is those \(((a, a'), (b, b'))\), such that \(x_0\) lies on \(a'b'\) as does \(a = b\), or \(x_0\) lies on \(ab\) as does \(a' = b'\).

Note that \(C_0\) and \(E_1\) are topologically disjoint, as are \(C_1\) and \(E_2\), and also \(C_{x_0}\) and \(L_1\). Indeed, each set has a property, preserved under limits, which is not true of any element of the paired set. Once we have noted the GMPR and NGMPR sets on each of the seven regions when \(n\) is even, the domains \(E_0, C_0 \cup E_1, C_1 \cup E_2,\) and \(C_{x_0} \cup L_1\) imply \(\text{GC}(X) \leq 3\) when \(n\) is even. Since \(\text{TC}(X) = 3\) when \(n\) is even ( [5]), we obtain the result in this case.

We use the linear GMPR on \(E_0\). For \(E_1\) and \(E_2\), let \(\delta = \min(1, d(ab, a'b'))\), where \(d(ab, a'b')\) is the minimum distance for corresponding values of \(t\). On \(E_2\), we use

\[
\Phi((a, a'), (b, b'))(s)(t) = \left((1 - t)a + tb + \frac{1}{3} \delta \cdot s \cdot g(t) \cdot v \left( \frac{b - a}{\|b - a\|} \right), \right. \\
(1 - t)a' + tb' + \frac{1}{3} \delta \cdot s \cdot g(t) \cdot v \left( \frac{b' - a'}{\|b' - a'\|} \right) \tag{2.9}
\]

Because both \(ab\) and \(a'b'\) pass through \(x_0\) and do not collide, both \(b - a\) and \(b' - a'\) must be nonzero. On \(E_1\), we modify this formula by removing the \(\frac{1}{3} \delta s g v\) term in the component which did not pass through \(x_0\).

On \(C_0\), we use the NGMPR of (2.7) with the \(s \cdot g(t)\) multiplied by an additional factor \(\frac{1}{2} \min(1, d(ab, x_0))\). On \(C_1\), we use
\[
\Phi((a, a'), (b, b'))(s)(t) = \left((1 - t)a + tb, (1 - t)a' + tb' + s \cdot g(t) \cdot \frac{b - a}{||b - a||}\right),
\]

(2.10)

when \(a'b'\) passes through \(x_0\), and a similar formula when \(ab\) passes through \(x_0\). These curves do not pass through \(x_0\), since \(x_0\) cannot be written as the sum of a point on \(a'b'\) plus a nonzero multiple of \(b - a\). Since \(b - b'\) is a scalar multiple of \(a - a'\), if the components of (2.10) were to collide, \(b - a\) would be a nonzero multiple of \(a - a'\), which it is not.

On \(L_1\), we use (2.7) when \(x_0\) lies on \(a'b'\), as does \(a = b\), with an obvious modification if \(ab\) and \(a'b'\) play opposite roles. On \(C_{x_0}\), we use similarly to \(C_1\)

\[
\Phi((a, a'), (b, b'))(s)(t) = \left((1 - t)a + tb + s \cdot g(t) \cdot \frac{b' - b}{||b' - b||}\right),
\]

\[
(1 - t)a' + tb' + s \cdot g(t) \cdot \frac{a - b}{||a - b||};
\]

(2.11)

For these to collide, we would need \(ab\) to be parallel to \(aa'\), which it is not. Also, note that \(a - b\) is nonzero, since otherwise we would have \(a = b = x_0\), which cannot happen.

When \(n\) is odd, we no longer have a vector field on \(S^{n-1}\). For the cases that used such a vector field, we use a vector field on \(S^{n-1} - \{e_1\}\). By choosing the excluded point differently in different cases, we can arrange it so that all the excluded cases are topologically disjoint, and so can be combined into one additional domain, again agreeing with the known result for \(TC\). Indeed, choose vectors \(V_i, 1 \leq i \leq 4\), in \(S^{n-1}\), such that if \(i \neq j\), then \(V_i \neq \pm V_j\). Our fifth domain is

\[
(C_0 \cap \{b' - b \in \langle V_1 \rangle\}) \cup (E_1 \cap \{b - a \text{ or } b' - a' \in \langle V_2 \rangle\})
\]

\[
\cup (E_2 \cap \{b - a \text{ or } b' - a' \in \langle V_3 \rangle\}) \cup (L_1 \cap \{b' - b \in \langle V_4 \rangle\}),
\]

where \(\langle V \rangle\) denotes the span of a vector \(V\). 

\[ \square \]

**Theorem 2.12** For the unordered configuration space \(X = C(\mathbb{R}^2 - x_0, 2)\)

\[ \text{GC}(X) = \text{TC}(X) = 2. \]

**Proof** We begin by looking at \(C(\mathbb{R}^n - x_0, 2)\), and then will specialize to \(n = 2\). The metric on \(C(\mathbb{R}^n, 2)\) is as in [1]

\[ d(\{a, a'\}, \{b, b'\}) = \min(d((a, a'), (b, b')), d((a, a'), (b', b))), \]

and we use this on the subspace \(C(\mathbb{R}^n - x_0, 2)\). Here, we use results from [1, Prop 4.3, (4.1)] that in \(\mathbb{R}^n \times \mathbb{R}^n\), \(d((a, a'), (b, b')) = d((a, a'), (b', b))\) if and only if \(aa' \perp bb'\), and if \(d((a, a'), (b, b')) < d((a, a'), (b', b))\), then \(ab\) and \(a'b'\) do not collide. Also, \(X = \mathbb{R}^n \times \mathbb{R}^n / (a, a') \sim (a', a)\).

We first consider pairs \(\{(a, a'), \{b, b'\}\}\) with \(d((a, a'), (b, b')) \neq d((a, a'), (b', b))\),

\[ \square \]
and label them so that \(d((a, a'), (b, b')) < d((a, a'), (b', b))\). Let \(E_0\) denote the set of those for which neither \(ab\) nor \(a'b'\) passes through \(x_0\) and use the linear GMPR on \(E_0\).

Let \(E_1\) denote the set of those for which exactly one of \(ab\) and \(a'b'\) passes through \(x_0\), and, if \(x_0 \in ab\), assuming now that \(n\) is even, so that there is a unit vector field \(v\) on \(S^{n-1}\), use

\[
\Phi((a, a'), \{b, b'\}) (s)(t) = \left\{ (1-t)a + tb + \delta \cdot s \cdot g(t) \cdot v\left(\frac{a-b}{\|a-b\|}\right), (1-t)a' + tb' \right\}
\]

(2.13)

with \(\delta = \frac{1}{2}\min(1, d(ab, a'b'))\), with obvious reversal if, instead, \(x_0 \in a'b'\). Let \(E_2\) denote those for which \(ab\) and \(a'b'\) both pass through \(x_0\) (but not for the same \(t\)), and use (2.9) as the NGMPR.

Now, we consider pairs \(((a, a'), \{b, b'\})\) for which \(aa' \perp bb'\), so \(d((a, a'), (b, b')) = d((a, a'), (b', b))\). Note that \(ab\) and \(a'b'\) do not collide, nor do \(ab'\) and \(a'b\). We now specialize to \(n = 2\). Choose \(ab\) so that the oriented segment \(bb'\) is a 90-degree counterclockwise rotation from \(aa'\). Let \(Y_0\) denote the set of those for which none of the segments \(ab, a'b, ab',\) and \(a'b'\) pass through \(x_0\). A GMPR on \(Y_0\) can be obtained using the linear path from \((a, a')\) to \((b, b')\). Let \(Y_1\) denote the set of those such that exactly one of the pairs \(\{ab, a'b'\}\) and \(\{ab', a'b\}\) has a segment passing through \(x_0\). Use the linear path on that pair as a GMPR. For example, if \(x_0\) lies on \(ab'\), use the linear path from \((a, a')\) to \((b, b')\). Let \(Y_2\) denote the set of those for which the pairs \(\{ab, a'b'\}\) and \(\{ab', a'b\}\) have one segment each passing through \(x_0\). The situation will be like that in Fig. 1, or with \(a\) and \(b\) reversed, or with primed and unprimed reversed. These different situations are topologically disjoint.

We can use (2.13) as an NGMPR on diagrams in \(Y_2\) of the form in Fig. 1, with obvious modifications for its variants.

We can use \(E_0, E_1 \cup Y_0, E_2 \cup Y_2,\) and \(Y_1\) as our four domains, since \(E_1\) and \(Y_0\) are topologically disjoint, as are \(E_2\) and \(Y_2\).

\(\Box\)

### 3 \(F(\mathbb{R}^n - Q, 2)\) with \(2 \leq |Q| < \infty\)

**Theorem 3.1** Let \(n\) be even, and let \(Q\) be a finite subset of \(\mathbb{R}^n\) with at least two points, and \(X = F(\mathbb{R}^n - Q, 2)\). Then, \(\text{GC}(X) \leq 5\).

![Fig. 1 A typical \(Y_2\) configuration](image-url)
By [5], $\text{TC}(F(\mathbb{R}^n - Q, 2)) = 4$. We think that is likely that, at least for $n = 2$, our result for $\text{GC}(X)$ is sharp, which would give a nice example of $\text{TC} < \text{GC}$.

**Proof** We partition $X$ into 18 subsets, on each of which there is a GMPR or NGMPR. Then, we will group them into six collections of topologically disjoint subsets.

We use the word ‘collide’ as in the proof of Theorem 2.8. Recall that $ab$ and $a'b'$ collide if and only if $b - b'$ is a negative multiple of $a - a'$. There are sets $E_0$, $E_1$, $E_2$, $E_{1,1}$, $E_{1,2}$, and $E_{2,2}$, in which the segments $ab$ and $a'b'$ do not collide, and the subscripts indicate how many points of $Q$ lie on each segment, with “2” referring to “2 or more.” For example, $E_2$ consists of those $((a, a'), (b, b'))$ for which $ab$ and $a'b'$ do not collide and one of these segments contains two or more points of $Q$, while the other has none. If the segments intersect at a point of $Q$ (for differing values of the parameter $t$), then that point counts for both lines. Therefore, for example, $E_{1,1}$ consists both of noncolliding elements where the segments do not meet at a point of $Q$ and each contains a point of $Q$, and those where the two segments meet at a point of $Q$, and neither contains other points of $Q$. If $Q$ has only two or three points, some of these sets can be empty.

There are also sets $C_0$, $C_1$, $C_2$, $C_{1,1}$, $C_{1,2}$, and $C_{2,2}$, in which the segments collide, but not at a point of $Q$, and the subscripts have the same meaning as before. For these, there is not the issue of classifying what happens if segments intersect at a point of $Q$. For these $C$-sets, we exclude colliding elements in which $a$, $a'$, $b$, and $b'$ are collinear.

Next, we have sets $Y_j$, $j = 0, 1, 2$, in which the segments collide at a point of $Q$, and $j$ of the segments contain one or more additional points of $Q$. Again, we exclude the case in which $a$, $a'$, $b$, and $b'$ are collinear. Finally, we have the linear cases $L_j$, $j = 0, 1, 2$, in which the four points are collinear and $aa'$ and $bb'$ have opposite directions, and $j$ points of $Q$ lie on $ab \cup a'b'$.

On $E_0$, we use the linear geodesic. On the other $E$ sets, we use formulas like (2.9), using just the linear part when a segment does not contain any points of $Q$, and modifying $\delta$ to equal $\min(1, d_1, d_2)$, where $d_1$ is the distance between the parametrized segments, and $d_2$ is the distance from the segment to the nearest point of $Q$ not on it.

Formulas for near-geodesics on the $C$ and $Y$ sets are similar to those that worked for the $C$ sets in the proof of Theorem 2.8. We use a factor $\delta_1 = \frac{1}{2} \min(1, d(ab, Q - (Q \cap ab)))$ on the first component, and an analogue on the second. Incorporating that, we use an analogue of (2.7) on $C_0$, of (2.10) on $C_1$ and $C_2$, and of (2.11) on $C_{1,1}$, $C_{1,2}$, $C_{2,2}$, and each $Y_j$. On each set $L_j$, we can use

$$
\Phi((a, a'), (b, b'))(s)(t) = \left( (1 - t)a + tb + \delta_2 \cdot s \cdot g(t) \cdot v\left(\frac{b' - b}{\|b' - b\|}\right),
\right.

\left. (1 - t)a' + tb' - \delta_2 \cdot s \cdot g(t) \cdot v\left(\frac{b' - b}{\|b' - b\|}\right) \right),
$$

We can group these into six collections of topologically disjoint subsets as follows:
To show two sets are topologically disjoint, we usually show that each has a property which is preserved under limits of sequences and is not shared by any element of the other set. For example, to see that \( L_1 \) and \( Y_1 \) are topologically disjoint, we note that the limit of collinear configurations is collinear and hence is not in \( Y_1 \), while if the limit of \( Y_1 \) configurations is collinear, then it will contain more than one point of \( Q \), and hence not be in \( L_1 \). As another example, if the limit of a sequence of \( E_{2,2} \) configurations collides, the limiting configuration would be either in \( C_{2,2} \) or \( Y_2 \). Similarly, the limit of a sequence of \( C_{1,2} \) configurations could be in \( Y_2 \), but not in \( Y_1 \). Since two points determine a line and \( Q \) is finite, a convergent sequence of \( C_{2,2} \) sets will have the point of convergence eventually constant, and so the limiting configuration cannot be in \( Y_2 \).

4 Configuration spaces of graphs

Configuration spaces \( F(G, 2) \) of graphs \( G \), as studied in \([2]\), are handled quite differently than the cases considered above. In \([2]\), \( F(G, 2) \) was given the subspace metric from \( G \times G \), where \( G \times G \) often had the Euclidean metric, using distance in the graph \( G \). Thus, 

\[
d((a, a'), (b, b')) = \sqrt{d(a, b)^2 + d(a', b')^2}.
\]

When \( X = F(G, 2) \) in the above metric, the completion \( \overline{X} \) equals \( G \times G \) and is geodesic. However, certain geodesics in \( \overline{X} \) cannot be approximated by paths in \( X \). For example, let \( G \) be the \( Y \)-graph \( Y \) with essential vertex \( v \), and suppose \( a, a', b', b \) are on the same edge at distance 1, \( 1 + \delta \), \( 1 + 2\delta \), and \( 1 + 3\delta \), respectively, from the vertex \( v \).

The geodesic in \( \overline{X} \) from \((a, a')\) to \((b, b')\) is the linear path of length \( \delta \sqrt{10} \). However, since direct motion will involve a collision, there is no path in \( X \) whose length is close to this. A short path in \( X \) from \((a, a')\) to \((b, b')\) is one that moves from \((a, a')\) back just beyond \( v \) onto the two empty arms, and from there to \((b, b')\), with length slightly greater than \( \sqrt{1 + (1 + 3\delta)^2 + (1 + \delta)^2 + (1 + 2\delta)^2} \). Thus, there is no near-geodesic from \((a, a')\) to \((b, b')\).

Since the definition of \( GC \) in \([6]\) only applies well to geodesic spaces, to consider \( GC(F(Y, 2)) \) in \([2]\), we replaced \( F(Y, 2) \) by the homotopically equivalent subspace \( F_\epsilon(Y, 2) \) which consisted of points \((a, a')\) satisfying \( d(a, a') \geq \epsilon \) for some fixed positive number \( \epsilon \). For example, in the diagram at the left in Fig. 2, there is no geodesic in \( F(Y, 2) \) from \((a, a')\) to \((b, b')\), because in the linear motion from \((a, a')\) to \((b, b')\), the first particle would overtake the second, which is not allowed. In \([2]\), we represented paths in the graph by paths in the \( xy \) plane, where, in this case, the \( x \)-axis corresponds to motion of the first particle on the two upper arms on the graph, with the vertex at 0, and the \( y \)-axis, similarly, corresponds to motion of the second particle on the arms on the left. This is shown on the right side of Fig. 2, in which
the interior of the shaded region is excluded, as those points do not satisfy
\(d(a, a') \geq \epsilon\). The representation of the geodesic in \(F_\epsilon(Y, 2)\) is indicated. It

corresponds to the path in the graph which goes from \((a, a')\) to the point \((v, \epsilon)\),

with the \(\epsilon\) on the bottom arm, and from there to \((b, b')\). Here, \(\epsilon < \min(d(a, a'), d(v, b'))\).

For our new approach, we use the intrinsic metric \(d_I\) on \(X = F(G, 2)\).

**Definition 4.2** If \((X, d)\) is a metric space, the intrinsic metric \(d_I\) on the set \(X\) is
defined by \(d_I(x, y)\) is the infimum of the \(d\)-lengths of paths in \(X\) from \(x\) to \(y\).

For the spaces considered here, this metric induces the same topology as does the
original metric, since if \(\epsilon < d(a, a')/\sqrt{2}\), the \(\epsilon\)-balls around \((a, a')\) in the two
topologies are equal. This is true because linear motion between points \((b, b')\) in the
\(\epsilon\)-ball and \((a, a')\) avoids collision. This also implies that lengths of paths using the \(d_I\)
metric equal the \(d\)-length, because length of a path, being the supremum of sums of
distances between successive points, is determined from arbitrarily small segments.

Since the intrinsic and Euclidean topologies are the same, we again have

\(\text{TC}(F(G, 2)) \leq \text{GC}(F(G, 2))\), where \(\text{GC}(F(G, 2))\) is defined as in Definition 1.2,
using the metric \(d_I\).

Let \(F_I(Y, 2)\) denote \(F(Y, 2)\) in the intrinsic metric.

**Lemma 4.3** The completion of \(F_I(Y, 2)\) is formed by adjoining to \(F_I(Y, 2)\) points \((x, x)\) and
\((x, x)_2\) for all \(x \in Y \setminus \{v\}\), and also \((v, v)\). Here, \((x, x)_1\) (resp. \((x, x)_2)\) is the limit of Cauchy
sequences of points \((a_n, a'_n)\) with \(a_n, a'_n \to x\) in \(Y\) and \(d(a_n, v) > d(a'_n, v)\)
(resp. \(d(a_n, v) < d(a'_n, v)\)). We have \(d_I((x, x)_1, (x, x)_2) = 2\sqrt{2}d(x, v)\).

**Proof** The reason for having both \((x, x)_1\) and \((x, x)_2\) is that, as noted above, if points
\(a, a', b,\) and \(b'\) are all very close to \(x\) and \(d(a, v) < d(a', v)\) and \(d(b', v) < d(b, v)\), then
\(d_I((a, a'), (b, b'))\) is not small. The formula for \(d_I((x, x)_1, (x, x)_2)\) is true, since it
equals \(\lim_{n \to \infty} d_I((a_n, a'_n), (b_n, b'_n))\) when all four points approach \(x\), with
\(d(a_n, v) > d(a'_n, v)\) and \(d(b_n, v) < d(b'_n, v)\). This issue is not present for sequences
approaching \((v, v)\). If \(a, a', b, b'\) are on the same arm at distance \(\delta, 2\delta, 3\delta, 4\delta\),
respectively, from \(v\), then \(d_I((a, a'), (b, b')) = \delta(\sqrt{1^2 + 2^2} + \sqrt{3^2 + 4^2})\), so a
Cauchy sequence of such points can approach \((v, v)\). 

\[\square\]
If you tried to follow the linear path from \((a, a')\) to \((b, b')\) in the situation on the left side of Fig. 2, \((a, a')\) will quickly get to a point \((x, x)\). If the linear motion is attempted beyond that, then \(a\) is now closer to \(v\) than is \(a'\), so the path is not continuous in the \(d_I\)-metric. The geodesic in \(F_I(Y, 2)\) is the path from \((a, a')\) to \((v, v)\) to \((b, b')\). Our near-geodesic is a homotopy which at parameter \(s = \epsilon\) could be chosen to be represented by the path on the right side of Fig. 2.

The remainder of the paper is devoted to the proof of the following result, which is an analogue of [2, Theorem 1(a)].

**Theorem 4.4** If \(Y\) denotes the \(Y\) graph, then \(GC(F_I(Y, 2)) = TC(F(Y, 2)) = 1\).

**Proof** Let \(F_I = F_I(Y, 2)\). Our analysis of \(GC(F_I)\) is patterned after the analysis of \(GC(F_c(Y, 2))\) in [2]. The three arms of the \(Y\)-graph are referred to as closed or open depending on whether or not the vertex \(v\) is included. We will assume that the arms have length \(\geq 2\). If the four points \(a, a', b,\) and \(b'\) are all on one or two arms, then their orientation (same or opposite) refers to the direction of \(aa'\) compared with \(bb'\).

We partition \(F_I \times F_I\) into the following six sets.

- **\(C_{1,1,2}\):** Three open arms occupied, and direct motion involves a collision. (The fourth point cannot be at \(v\).)
- **\(X_{1,1,2}\):** Three open arms occupied, direct motion does not involve a collision. (The fourth point possibly at \(v\).)
- **\(C_4\):** All four points are on a single closed arm, with opposite orientation.
- **\(C_{2,2}\):** Two points on each of two open arms, with opposite orientation.
- **\(C_{3,1}\):** Three points on one closed arm, the other on a different (open) arm, with opposite orientation.
- **\(L\):** All are on one or two arms, with the same orientation.

Our two domains are

\[ E_0 = C_{1,1,2} \cup C_4 \cup C_{2,2}, \quad E_1 = X_{1,1,2} \cup C_{3,1} \cup L. \]

We number the three arms cyclically (mod 3) in clockwise order. We first describe the NGMPR on \(E_0\).

Here, we introduce the notation that \(s_i\) denotes the point at distance \(s\) from \(v\) on arm \(i\), and 0 denotes \(v\). Arrows denote uniform linear motion along the graph, and the overall path is parametrized by arc length in \(Y \times Y\) (in the \(d\)-metric).

The NGMPR on \(C_4\) is: if they are all on arm \(i\) with \(d(a, v) < d(a', v)\), then

\[(a, a') \rightarrow (s_{i+1}, 0) \rightarrow (0, s_{i+2}) \rightarrow (b, b'),\]

while if \(d(a', v) < d(a, v)\), then

\[(a, a') \rightarrow (0, s_{i+1}) \rightarrow (s_{i+2}, 0) \rightarrow (b, b').\]

This is a very convenient way of describing complicated formulas. Here, \(s\) is the homotopy parameter. Let \(d_a = d(a, v)\), etc.
where $D_1 = \sqrt{(d_a + s)^2 + d_a'^2}$, $D_2 = \sqrt{s^2 + s^2}$, $D_3 = \sqrt{d_b^2 + (s + d_b')^2}$,

\[ r_1 = \frac{d_a - d_a'}{D_1}, \quad \text{and} \quad r_2 = \frac{d_a'}{d_a} r_1. \]

Then, the first part of the paths in the homotopy is given by

\[ \Phi((a, a'), (b, b'))(s)(t) = \begin{cases} 
\left( \frac{r_2 - t}{r_2}, \frac{r_1 - t}{d_a} \right) & 0 \leq t \leq r_2 \\
\left( \frac{s - r_2}{r_1 - r_2}, \frac{r_1 - t}{d_a'} \right) & r_2 \leq t \leq r_1, 
\end{cases} \]

with the first component of the last part being on arm $i + 1$.

On $C_{1,1,2}$, the NGMPR is as follows. If $a$ is alone on arm $i$ and $a'$ is alone on a different arm, and $d(b', v) > d(b, v)$, or if $b$ is alone on arm $i$ and $b'$ is alone on a different arm, and $d(a', v) > d(a, v)$, we use

\[ (a, a') \rightarrow (s, 0) \rightarrow (b, b'). \]

Such configurations cannot approach an element of $C_4$. Otherwise, if $a'$ is alone on an open arm, so $a$ and $b'$ are on another arm $i$, we use

\[ (a, a') \rightarrow (0, s_{i+1}) \rightarrow (s_{i+2}, 0) \rightarrow (b, b'), \]

while if $a$ is alone on an open arm, so $a'$ and $b$ are on another arm $i$, we use

\[ (a, a') \rightarrow (s_{i+1}, 0) \rightarrow (0, s_{i+2}) \rightarrow (b, b'). \]

This NGMPR agrees with that of the limit in $C_4$ of such configurations as $a'$ and $b$ approach $v$. If $a$ and $a'$ are on the same open arm $i$ with $d(a, v) > d(a', v)$, we can use $(a, a') \rightarrow (s, 0) \rightarrow (b, b')$.

On $C_{2,2}$, we consider first the case in which $a$ and $a'$ lie on one arm and $b$ and $b'$ on another, with $a'$ and $b'$ closer to $v$. A sequence of such diagrams cannot converge to an element of $C_{1,1,2} \cup C_4$. If $i$ is the free arm, we can use $(a, a') \rightarrow (0, s) \rightarrow (b, b')$. The other type of $C_{2,2}$ diagram is a bit more complicated. We just consider the case when $a$ and $b'$ are on open arm 1 and $a'$ and $b$ are on open arm 2. Our rule must satisfy that the limit as $a$ and $b'$ approach $v$ should be

\[ (a, a') \rightarrow (s_3, 0) \rightarrow (0, s_1) \rightarrow (b, b'), \]

while the limit as $a'$ and $b$ approach $v$ should be

\[ (a, a') \rightarrow (0, s_2) \rightarrow (s_3, 0) \rightarrow (b, b'). \]

We consider the portion from $(a, a')$ to $(s_3, 0)$. The portion from $(s_3, 0)$ to $(b, b')$ can be treated in an analogous way, looking backward from $(b, b')$.

Let $d = d(a, v)$ and $d' = d(a', v)$. We use $(a, a') \rightarrow (s_3, 0)$ if $d > s$ or $d' > d$, and we use $(a, a') \rightarrow (0, s_2) \rightarrow (s_3, 0)$ if $d' < s/2$ and $d' < d/2$. In Fig. 3, the $x$ (resp. $y$)-axis represents the motion of the first (resp. second) particle for a fixed value of the homotopy parameter $s$, somewhat similarly to Fig. 2. The starting position $(d, d')$ is a point in the first quadrant, and we wish to represent the path to the point $(-s, 0)$.
corresponding to the point \((s_3, v)\) on the \(Y\)-graph on the left. For starting positions above the intermediate region \(R\), we move along the straight segment from \((d, d')\) to \((-s, 0)\). For starting positions below \(R\), we move from \((d, d')\) to \((0, s)\) and then to \((-s, 0)\). This satisfies the requirement regarding the behavior as \(a\) or \(a'\) approaches \(v\). Note that we need not be concerned with the limit as both \(a\) and \(a'\) approach 0, as the point \((v, v)\) is not in \(F(Y, 2)\).

For a point in the intermediate region \(R\), we move from \((d, d')\) to \((0, y)\) and then to \((0, \frac{s}{2})\), where \(y\) is defined as follows. Let \(y_0\) (resp. \(y_1\)) be the smallest (resp. largest) \(y\) value of the points in \(R\) with \(x = d\). Let \((0, y_2)\) be the intersection of the segment from \((d, y_1)\) to \((-s, 0)\) with the \(y\)-axis. If \(d' = (1 - t)y_0 + ty_1\), then \(y = (1 - t)s + ty_2\). This choice of paths varies continuously with nonnegative values of \(d, d'\), and \(s\), provided that \(d + d' > 0\).

This completes the description of the NGMPR on \(E_0\). We next describe the NGMPR on \(E_1\). On \(L\), we use linear motion \((a, a') \rightarrow (b, b')\) and the constant homotopy.

The NGMPR on \(C_{3,1}\) is, at homotopy parameter \(s\), given as follows. If \(a, a'\), and \(b\) are on arm \(i\) with \(d(a, v) < d(a', v)\), and \(b'\) is on arm \(j\), and arm \(k\) is free, or if \(b, b'\), and \(a\) are on arm \(j\) with \(d(b, v) < d(b', v)\), and \(a'\) is on arm \(i\), and arm \(k\) is free, then

\[(a, a') \rightarrow (0, s_i) \rightarrow (s_k, 0) \rightarrow (0, s_j) \rightarrow (b, b').\]

Reversing primed and unprimed reverses the coordinates of the parts involving 0 and \(s\). This has the property that if \(a\) and \(b\) are at the vertex, with \(a'\) and \(b'\) on distinct open arms, the same motion is obtained if it is thought of as \(a, a'\), and \(b\) on an arm or as \(b, b'\), and \(a\) on an arm.

On \(X_{1,1,2}\), we would like to use linear motion and the constant homotopy. However, for noncolliding diagrams of any of the three types on the left side of
Fig. 4, this homotopy would not agree, in the limit, with that in the $C_{3,1}$ diagram on the right side, as $a'$ and $b'$ approach $v$. In the first two diagrams, the two letters on an arm may appear in either order.

We will allow $a, a', b,$ and $b'$ to denote distance of the corresponding points from the vertex. The NGMPR for the first of the diagrams of Fig. 4 has $\Phi((a, a'), (b, b'))(s) = \Phi((a, v), (b, v))(s)$ equal to the path described as follows. Let $M = \frac{\max(a, b)}{\min(a, b)}$. For $s \leq M$, it is the linear path $(a, a') \to (b, b')$, which can also be written as

$$
(a, a') \to \left(0, \left(\frac{b}{a + b} a' + \frac{a}{a + b} b'\right)\right) \to (b, b').
$$

For $s = 2M$, it is the path $(a, a') \to (0, (2M)_{i+1}) \to (b, b')$, and for $M \leq s \leq 2M$, it is the linear interpolation between these two paths. Note that $s \leq 1$, so what this really means is that for $s \in [0, 1] \cap [M, 2M]$, it is the path $(a, a') \to (0, y_i) \to (b, b')$, where

$$
y = \left(1 - \frac{s - M}{2M}\right) \left(a \left(\frac{b}{a + b} a' + \frac{a}{a + b} b'\right) + \frac{s - M}{M} \cdot 2M\right)
$$

For $4M \leq s \leq 1$, it is the path

$$
(a, a') \to (s_{i+1}, 0) \to (0, s_i) \to (s_{i+2}, 0) \to (b, b'),
$$

and for $s \in [0, 1] \cap [2M, 4M]$, it is a linear interpolation between the paths for $2M$ and $4M$, similar to what was done in Fig. 3.

This satisfies for all $s \in [0, 1]$

$$
\lim_{a', b' \to v} \Phi((a, a'), (b, b'))(s) = \Phi((a, v), (b, v))(s),
$$

where the latter is from the $\Phi$ formula on $C_{3,1}$. It also satisfies

$$
\lim_{a \to v} \Phi((a, a'), (b, b'))(s) = \Phi((a, v), (b, v))(s),
$$

where the latter is the constant linear homotopy from the $\Phi$ formula on $L$, and similarly as $b \to v$.

The same formulation works for the third diagram, using $M = a'/a$. This will approach the $C_{3,1}$ motion as $a' \to v$, and will approach the $L$ motion as $a \to v$. The second diagram of Fig. 4 is slightly more complicated, because we have to also worry about the $C_{3,1}$ motion in the limit as $a$ and $b$ approach $v$.

Let $M = \max(a, b)$ and $M' = \max(a', b')$, and $\overline{M} = \min(M, M')$. For $s \leq \overline{M}$, we use the linear path, which can also be written as

$$
(a, a') \to \left(\left(\frac{b'}{a' + b'} a + \frac{a'}{a' + b'} b\right)_{i+1}, 0\right) \to \left(0, \left(\frac{b}{a + b} a' + \frac{a}{a + b} b'\right)\right) \to (b, b').
$$

For $s = 2\overline{M}$, use $(a, a') \to ((2\overline{M})_{i+1}, 0) \to (0, (2\overline{M})_i) \to (b, b')$, and for $\overline{M} \leq s \leq 2\overline{M}$, interpolate linearly between these, in the sense described above for the first diagram of Fig. 4. For $4\overline{M} \leq s \leq 1$, the $(a, a') \to (s_{i+1}, 0)$ part of the path is replaced by $(a, a') \to (0, s_{i+2}) \to (s_{i+1}, 0)$, and for $s \in [0, 1] \cap [2\overline{M}, 4\overline{M}]$, this part of
the path is interpolated. Similarly, on the other end, for $4M' \leq s \leq 1$, the $(0, s_i) \rightarrow (b, b')$ part has $(s_{i+2}, 0)$ inserted in the middle, and we interpolate on this side for $s \in [0, 1] \cap [2\bar{M}, 4M']$. Only one or the other of $M$ and $M'$ can approach 0, and each will approach the appropriate $C_{3,1}$ motion in the limit.

This completes our NGMPR on $E_1$. □

We see that for these graph configuration spaces, the analysis of GC is closely related to the analysis of $GC(F_c(G, 2))$ in [2], but the perspective is quite different.

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