COGARCH: SYMBOL, GENERATOR AND CHARACTERISTICS

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Abstract. We describe the technique how to use the symbol in order to calculate the generator and the characteristics of an Itô process. As an example we analyze the COGARCH process which is used to model financial data.

1 Introduction

The COGARCH process was introduced by Klüppelberg et al. in [13] in order to model financial data. It is a continuous time analog of the classic GARCH process (in discrete time) and it is based on a single background driving Lévy process in contrast to the well known model by Barndorff-Nielsen and Shephard [1]. Lévy processes are càdlàg universal Markov processes which are homogeneous in time and space. Our main reference for this class of processes is [16]. For the Lévy triplet we write $(\ell, Q, N)$.

In the present paper we calculate the so called symbol of the COGARCH process (and its volatility process). The origins of the symbols are in the theory of partial differential equations, namely they appear in the Fourier representation of certain operators. The symbol found its way into probability theory for the following reason: suppose we are given a Feller process $X$ with associated semigroup $(T_t)_{t \geq 0}$ and generator $(A, D(A))$. Suppose further that the test functions $C_0^\infty(\mathbb{R}^d)$ are contained in the domain $D(A)$. In this case $A$ is a pseudo-differential operator with symbol $-q(x, \xi)$. For every $x \in \mathbb{R}^d$ $q(x, \cdot)$ is a continuous negative definite function in the sense of Schoenberg (cf. [2] Chapter 2).

For a detailed, self contained treatment on the interplay between the process and its symbol cf. the monograph [9]. In this context the following four questions are of interest:

I) Given a process, (say as the solution of an SDE) what is its symbol? (E.g. [19])
II) Given a symbol, does there exist a corresponding process? (6 [7] [11])
III) Which properties of the process can be characterized via the symbol? (17 [18])
IV) For which bigger classes of processes is it possible (and useful) to define a symbol? (20 [21])
All four questions are a vital part of ongoing research. In the present paper we emphasize, how one can calculate the symbol of a given process using a probabilistic formula and derive directly the generator as well as the semimartingale characteristics.

The notation we are using is (more or less) standard. Vectors are meant to be column vectors and the transposed of a vector $v$ or a matrix $Q$ is denoted by $v'$ respectively $Q'$.

Let us recall how the COGARCH process is defined: we start with a Lévy process $Z = (Z_t)_t$ with triplet $(\ell, Q, N)$. Fix $0 < \delta < 1, \beta > 0, \lambda \geq 0$. Then the volatility process $(\sigma_t)_{t \geq 0}$ is the solution of the SDE

$$d\sigma_t^2 = \beta \, dt + \sigma_t^2 \left( \log \delta \, dt + \frac{\lambda}{\delta} \, d[Z, Z]_t^{disc} \right)$$

where $\sigma_0 = S > 0$ and

$$[Z, Z]_t^{disc} = \sum_{0 < s \leq t} (\Delta Z_s)^2.$$

It turns out, that $(\sigma_t)_{t \geq 0}$ is a time homogeneous Markov process.

**Definition:** The process

$$G_t := g + \int_0^t \sigma_s \, dZ_s, \quad g \in \mathbb{R},$$

is called **COGARCH process** (starting in $g$).

We allow the process to start everywhere in order to bring our methods into account. The pair $(G_t, \sigma_t^2)$ is a (normal) Markov process which is homogeneous in time. It is homogeneous in space in the first component. Furthermore $(G_t, \sigma_t^2)$ is an Itô process, which follows from Theorem 3.33 of [4] which characterizes Itô processes as solutions of certain stochastic differential equations and Proposition IX.5.2. of [10] giving a representation of the semimartingale characteristics of a stochastic integral.

To avoid problems which might arise for processes defined on $\mathbb{R} \times \mathbb{R}_+$ we consider in the following: $(G_t, V_t) = (G_t, \log(\sigma_t^2))$, i.e. $V$ is the logarithmic squared volatility.

2 The Symbol of a Stochastic Process

**Definition:** Let $X$ be an $\mathbb{R}^d$-valued universal Markov process, which is conservative and normal. Fix a starting point $x$ and define $T = T^x_X$ to be the first exit time from
the ball of radius $R > 0$:

$$T := T^x_R := \inf\{t \geq 0 : \|X_t - x\| > R\} \text{ under } \mathbb{P}^x(x \in \mathbb{R}^d).$$

(1)

We call the function $p : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$, given by

$$p(x, \xi) := -\lim_{t \downarrow 0} \mathbb{E}^x \frac{e^{i(X^T_t - x)\xi} - 1}{t},$$

(2)

the (probabilistic) symbol of the process, if the limit exists for every $x, \xi$ and $R$ and is independent of the choice of $R$.

In [21] Theorem 4.4. we have shown that for Itô processes in the sense of Cinlar, Jacod, Protter and Sharpe (cf. [5]) having differential characteristics which are finely continuous (cf. [3]) and locally bounded the above limit exists and coincides for every choice of $R$. For the reader’s convenience we recall the definition of Itô processes, as it is used here:

**Definition:** A Markov semimartingale $X = (X_t)_{t \geq 0}$, i.e. a universal Markov process which is a semimartingale w.r.t. every initial probability $\mathbb{P}^x (x \in \mathbb{R})$, is called **Itô process** if it has characteristics of the form:

$$B^j_t(\omega) = \int_0^t \ell^j(X_s(\omega)) \, ds \quad j = 1, \ldots, d$$

$$C^j_k(t, \omega) = \int_0^t Q^j_k(X_s(\omega)) \, ds \quad j, k = 1, \ldots, d$$

$$\nu(\omega; ds, dy) = N(X_s(\omega), dy) \, ds$$

where $\ell^j, Q^j_k : \mathbb{R}^d \rightarrow \mathbb{R}$ are measurable functions, $Q(x) = (Q^j_k(x))_{1 \leq j, k \leq d}$ is a positive semidefinite matrix for every $x \in \mathbb{R}^d$, and $N(x, \cdot)$ is a Borel transition kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. $\ell, Q$ and $\int_{y \neq 0} (1 \wedge y^2) N(\cdot, dy)$ are called **differential characteristics**.

Example 1: Let $X$ be a $d$-dimensional Lévy process. It is a well known fact that the characteristic function of $X_t$ ($t \geq 0$) can be written as

$$\mathbb{E}^0 \exp(iX^j_t\xi) = \exp(-t\psi(\xi)).$$

The function $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$ is called characteristic exponent. By an elementary calculation one obtains $p(x, \cdot) = \psi(\cdot)$ for every $x \in \mathbb{R}^d$.

Example 2: Let $X$ be a rich Feller process, i.e. the test functions $C^\infty_c(\mathbb{R}^d)$ are contained in the domain $D(A)$ of the generator $A$. In this case the generator restricted to $C^\infty_c(\mathbb{R}^d)$ is a pseudo-differential operator with (functional analytic) symbol $-q(x, \xi)$. In [21] we have shown that $X$ is an Itô process and $p(x, \xi) = q(x, \xi)$ for every $x, \xi \in \mathbb{R}^d$. 
Example 3: Let \((Z_t)_{t \geq 0}\) be an \(\mathbb{R}^n\)-valued Lévy process. The solution of the stochastic differential equation \((x \in \mathbb{R}^d)\),
\[
dX_t^x = \Phi(X_{t-}^x) dZ_t \\
X_0^x = x,
\]
where \(\Phi : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^n\) is Lipschitz continuous admits the symbol
\[
p(x, \xi) = \psi(\Phi(x)\xi).
\]
This was shown in [19].

3 Symbol, Generator and Characteristics

In the present section we calculate the symbol of the COGARCH process. Using the close relationship between the symbol, the extended generator and the semimartingale characteristics we are able to write down the latter two objects directly. Let us emphasize that the symbol does not depend on \(g\), since the process is homogeneous in the first component.

**Theorem:** The stochastic process \((G_t, V_t) = (G_t, \log(\sigma_t^2))\) admits the symbol \(p : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}\) given by
\[
p \left( \left( \frac{g}{v} \right), \xi \right) = \\
- i \xi_1 \left( e^{v/2} + e^{-v/2} \int_{\mathbb{R} \setminus \{0\}} y \cdot \left( 1_{\{|e^{v/2}y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta) y^2)|<1\}} - 1_{\{|y|<1\}} \right) N(dy) \right) \\
- i \xi_2 \left( \frac{\beta}{e^v} + \log \delta + \int_{\mathbb{R} \setminus \{0\}} \log(1 + \lambda/\delta) y^2 \cdot \left( 1_{\{|e^{v/2}y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta) y^2)|<1\}} \right) N(dy) \right) \\
+ \frac{1}{2} \xi_1^2 e^v Q \\
- \int_{\mathbb{R}^2 \setminus \{0\}} \left( e^{i(z_1^1 z_2^2) \xi} - 1 - i z^1 \cdot (1_{\{|z_1|<1\}} \cdot 1_{\{|z_2|<1\}} ) \right) \tilde{N} \left( \left( \frac{g}{v} \right), dz \right),
\]
where \(\tilde{N}\) is the image measure
\[
\tilde{N} \left( \left( \frac{g}{v} \right), dz \right) = N(f_v \in dz)
\]
under \(f : \mathbb{R} \to \mathbb{R}^2\) given by
\[
f_v(w) = \left( \frac{e^{v/2} w}{\log(1 + (\lambda/\delta) w^2)} \right).
\]
Remark: It is not surprising, that the transformation of the jump measure depends only on $v$ since the process is space homogeneous in the first component.

**Proof:** Let $T$ be the stopping time defined in (1). At first we use Itô’s formula:

$$
\mathbb{E} g,v e^{i(G^T - g, V^T - v)} \xi - 1 = \mathbb{E}^{0,v} e^{i(G^T, V^T - v)} \xi - 1
$$

(1)

$$
= \frac{1}{t} \mathbb{E}^{0,v} \int_{0+}^{t} i \xi_1 e^{i(G^T_{s-}, V^T_{s-} - v)} \xi \ dG^T_s
$$

(II)

$$
= \frac{1}{2t} \mathbb{E}^{0,v} \int_{0+}^{t} \xi_1^2 e^{i(G^T_{s-}, V^T_{s-} - v)} \xi \ d[G^T, G^T]^c_s
$$

(III)

$$
= \frac{1}{t} \mathbb{E}^{0,v} \int_{0+}^{t} \xi_1 \xi_2 e^{i(G^T_{s-}, V^T_{s-} - v)} \xi \ d[G^T, V^T]^c_s
$$

(IV)

$$
= \frac{1}{2t} \mathbb{E}^{0,v} \int_{0+}^{t} \xi_2^2 e^{i(G^T_{s-}, V^T_{s-} - v)} \xi \ d[V^T, V^T]^c_s
$$

(V)

$$
+ \frac{1}{t} \mathbb{E}^{0,v} \sum_{0 < s \leq t} e^{i(G^T_{s-}, V^T_{s-} - v)} \xi \left( e^{i \Delta(G^T_s, V^T_s)} \xi - 1 - (i \xi_1 \Delta G^T_s + i \xi_2 \Delta V^T_s) \right)
$$

(VI)

We deal with this formula term-by-term. In the calculation of the first term we use:

$$
dG^T_s = \sigma_s - 1_{\{s \in [0,T]\}} \ dZ_s.
$$

Recall that the integrand is bounded and for the Lévy process $Z$ we have the Lévy-Itô-decomposition:

$$
Z_t = \ell t + \sqrt{Q} W_t + \left\{ \int_{[0,t] \times \{|y|<1\}} y \left( \mu^Z(ds, dy) - dsN(dy) \right) \right\} + \sum_{0 < s \leq t} \Delta Z_s 1_{\{\Delta Z_s \geq 1\}},
$$

where $\mu^Z$ denotes the jump measure of the process (cf. [10] Proposition II.1.16).

The integrals with respect to the martingale parts are again $L^2$-martingales and the respective terms disappear. What remains from the first term is:

$$
\frac{1}{t} \mathbb{E}^{0,v} \int_{0+}^{t} i \xi_1 e^{i(G^T_{s-}, V^T_{s-} - v)} \xi \sigma_s - 1_{\{s \in [0,T]\}} \ d\left( \ell s + \sum_{0 < r \leq s} \Delta Z_r \cdot 1_{\{\Delta Z_r \geq 1\}} \right)
$$

(3)
For the first part of this integrand we get:

\[
\frac{1}{t} \mathbb{E}^{0,v} \int_{0+}^{t} i \xi_1 e^{i(G_{s}^T, V_{s}^T - v)} \xi \sigma_s^{-1} \{ s \in [0,T] \} \, d(\ell s)
\]

\[
= \mathbb{E}^{0,v} \frac{1}{t} \int_{0}^{t} i \xi_1 \ell e^{i(G_{s}^T, V_{s}^T - v)} \xi^1 \{ s \in [0,T] \} \sigma_s \, ds
\]

\[
= i \xi_1 \ell \mathbb{E}^{0,v} \int_{0}^{1} e^{i(G_{s}^T, V_{s}^T - v)} \xi^1 \{ s \in [0,T] \} \sigma_{s \rightarrow 1} \, ds \rightarrow S
\]

\[
\longrightarrow i \xi_1 \ell S.
\]

In the first equation we used the fact that we are integrating with respect to Lebesgue measure. For this the countable number of jump times is a nullset. In the last step we used Lebesgue’s theorem twice. A similar argumentation is used in the consideration of the second and the third term. The jump term of (3) above will be compared to the sixth term.

Using Itô’s formula we obtain for the second term

\[
\frac{1}{t} \mathbb{E}^{0,v} \int_{0+}^{t} i \xi_2 e^{i(G_{s}^T, V_{s}^T - v)} \xi \left\{ \frac{1}{\sigma_s^{-2}} \, d(\sigma_s^T)^2 + d \left( \sum_{0 < r \leq s} \log \sigma_r^2 - \log \sigma_{r-}^2 - \frac{1}{\sigma_{r-}^2} \Delta(\sigma_r^2) \right) \right\}
\]

and by plugging in the defining SDE for \((\sigma^2)\):

\[
\frac{1}{t} \mathbb{E}^{0,v} \int_{0+}^{t} i \xi_2 e^{i(G_{s}^T, V_{s}^T - v)} \xi^1 \{ s \in [0,T] \} \left\{ \left( \frac{\beta}{\sigma_s^2} - \frac{\sigma_s^2}{\sigma_{s-}^2} \log \delta \right) \, ds + \frac{\sigma_s^2}{\sigma_{s-}^2} \log \delta \, ds \right\}
\]

\[
+ \frac{\lambda}{\delta} \, d \left( \sum_{0 < r \leq s} (\Delta Z_r)^2 \right) + d \left( \sum_{0 < r \leq s} \Delta(\log \sigma_r^2) - \frac{1}{\sigma_{r-}^2} \Delta(\sigma_r^2) \right) \right\}
\]

We postpone the jump parts and for the remainder term we get in the limit, using a similar argumentation as for the first term,

\[
\longrightarrow i \xi_2 \beta / S^2 + i \xi_2 \log \delta.
\]
For the third term we obtain in an analogous manner to the first one
\[-\frac{1}{2t} E^{0,v} \int_{0+}^{t} i \xi_1 e^{i(G_{s-}^{T},V_{s-}^{T}-v)\xi} d[G_{s-}^{T},G_{s}^{T}]_{s} \]
\[= -\frac{1}{2t} E^{0,v} \int_{0+}^{t} i \xi_1 e^{i(G_{s-}^{T},V_{s-}^{T}-v)\xi} \sum_{s \in \{0,T\}} \sigma_{s-}^2 d[Z,Z]_{s} \]
\[\longrightarrow t \downarrow 0 \quad -\frac{1}{2} \xi_1^2 S^2 Q. \]

The terms four and five are constant zero: since \((t)_t\) and \((|Z,Z|)_t\) are both of finite variation on compacts, the process \((\sigma^2)_t\) has this property as well, by its very definition. Therefore it is a quadratic pure jump process (see [14], Section II.6).

Using Itô’s formula we obtain that \(V = \log(\sigma^2)\) is again a quadratic pure jump process and therefore
\([V^T, V^T]_s^c = 0\) and \([V^T, G^T]_s^c = 0\).

The only thing that remains to do is dealing with the various ‘jump parts’. From the first term we left the following behind
\[\frac{1}{t} E^{0,v} \int_{0+}^{t} i \xi_1 e^{i(G_{s-}^{T},V_{s-}^{T}-v)\xi} \sum_{s \in \{0,T\}} \Delta Z_r \cdot 1_{\{\Delta Z_r \geq 1\}} \cdot \sigma_{s-}1_{\{s \in \{0,T\}\}} d \left( \sum_{0 < r \leq s} (\Delta Z_r)^2 \right) \]
and from the second one
\[\frac{1}{t} E^{0,v} \int_{0+}^{t} i \xi_2 e^{i(G_{s-}^{T},V_{s-}^{T}-v)\xi} \sum_{s \in \{0,T\}} \Delta Z_r \cdot 1_{\{s \in \{0,T\}\}} d \left( \sum_{0 < r \leq s} (\Delta Z_r)^2 \right) \]
\[+ \frac{1}{t} E^{0,v} \int_{0+}^{t} i \xi_2 e^{i(G_{s-}^{T},V_{s-}^{T}-v)\xi} \sum_{s \in \{0,T\}} \Delta V_r \cdot \frac{1}{\sigma_{r-}^2} \Delta(\sigma_{r-}^2) \cdot \sigma_{s-}1_{\{s \in \{0,T\}\}} d \left( \sum_{0 < r \leq s} (\Delta Z_r)^2 \right) \]
\[\longrightarrow t \downarrow 0 \quad \frac{1}{t} \sum_{0 < s \leq t} i \xi_2 e^{i(G_{s-}^{T},V_{s-}^{T}-v)\xi} \sum_{s \in \{0,T\}} \frac{1}{\sigma_{s-}^2} \Delta(\sigma_{s-}^2). \]
Adding these terms to term number six and using the equalities
\[ \Delta G^T_s = (\sigma_s - 1_{\{s \in [0,T]\}}) \Delta Z_s \] and
\[ (\Delta \sigma_s^T)^2 = \frac{\lambda}{\delta} (\sigma_s^2 - 1_{\{s \in [0,T]\}}) (\Delta Z_s)^2 \]
as well as
\[ \Delta \log(\sigma_s^2)^T = \log \left( \frac{(\sigma_s^2)^T + \Delta(\sigma_s^2)^T}{(\sigma_s^2)^T} \right) = \log \left( 1 + \frac{\Delta(\sigma_s^2)^T}{(\sigma_s^2)^T} \right) \]
we obtain
\[
\frac{1}{t} \mathbb{E}^{0,v} \sum_{0 < s \leq t} e^{i(G^{T}_{s-}, V^{T}_{s-} - v)} \xi 1_{\{s \in [0,T]\}} \times \left( e^{i\sigma_s - \Delta Z_s \xi_1 + i \log(1 + (\lambda/\delta) \Delta(Z_s)^2)} \xi_1 - 1 - i \xi_1 \sigma_s - \Delta Z_s \cdot 1_{\{|\Delta Z_s| < 1\}} \right)
\]
\[
= \frac{1}{t} \mathbb{E}^{0,v} \int_{[0,t] \times \{ y \neq 0 \}} e^{i(G^{T}_{s-}, V^{T}_{s-} - v)} \xi 1_{\{s \in [0,T]\}} \times \left( e^{i\sigma_s - y \xi_1 + i \log(1 + (\lambda/\delta) y^2)} \xi_1 - 1 - i \xi_1 \sigma_s - y \cdot 1_{\{|y| < 1\}} \right) \mu^Z (\cdot; ds, dy)
\]
\[
= \frac{1}{t} \mathbb{E}^{0,v} \int_{[0,t] \times \{ y \neq 0 \}} e^{i(G^{T}_{s-}, V^{T}_{s-} - v)} \xi 1_{\{s \in [0,T]\}} \times \left( e^{i\sigma_s - y \xi_1 + i \log(1 + (\lambda/\delta) y^2)} \xi_1 - 1 - i \left( \frac{\sigma_s - y}{\log(1 + \frac{\lambda}{\delta} y^2)} \right)' \cdot 1_{\{|y| < 1\}} \cdot 1_{\{|\log(1 + \frac{\lambda}{\delta} y^2)| < 1\}} \right) + \left( i \xi_2 \log(1 + \frac{\lambda}{\delta} y^2) \cdot 1_{\{|y| < 1\}} \cdot 1_{\{|\log(1 + \frac{\lambda}{\delta} y^2)| < 1\}} \right) \mu^Z (\cdot; ds, dy).
\]
It is possible to calculate the integral with respect to the compensator \( \nu(\cdot; ds, dy) = N(dy)\) \( ds \) instead of the measure itself ‘under the expectation’, since the integrands are of class \( F_p^2 \) of Ikeda-Watanabe (IW):
\[ F_p^2 = \left\{ f(s, y, \omega) : f \text{ is predictable, } \mathbb{E} \int_0^t \int_{[0,t]} |f(s, y, \cdot)|^2 N(dy)ds \text{ for every } t > 0 \right\} . \]
One obtains this, because \( 1_{\{|y| < 1\}} \cdot 1_{\{|\log(1 + (\lambda/\delta) y^2)| < 1\}} - 1_{\{|y| < 1\}} \) is zero near the origin and bounded and \( \log(1 + \frac{\lambda}{\delta} y^2) \leq (\lambda/\delta) \cdot y^2 \) for \( |(\lambda/\delta) \cdot y^2| < 1 \).
For $t$ tending to zero (and multiplying with $-1$) we obtain by using Lebesgue’s theorem again twice
\[
p\left(\left(\frac{g}{v}, \left(\xi_1, \xi_2\right)\right) = -i\xi_1 \left(\ell S + S \int_{\mathbb{R}\setminus\{0\}} y \cdot (1_{\{|Sy|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta) y^2)|<1\}} - 1_{\{|y|<1\}}) \, N(dy)\right)
\]
\[
- i\xi_2 \left(\frac{\beta}{S^2} + \log \delta + \int_{\mathbb{R}\setminus\{0\}} \log(1 + \frac{\lambda}{\delta} y^2) \cdot (1_{\{|S y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta) y^2)|<1\}}) \, N(dy)\right)
+ \frac{1}{2} \xi_1^2 S^2 Q
\]
\[- \int_{\mathbb{R}^2\setminus\{0\}} \left(e^{i(z_1, z_2)\xi} - 1 - i z \cdot 1_{\{|z_1|<1\}} \cdot 1_{\{|z_2|<1\}}\right) \tilde{N} \left(\left(\frac{g}{S}, dz\right)\right),
\]
where $\tilde{N}$ is the image measure
\[
\tilde{N} \left(\left(\frac{g}{S}, dz\right)\right) = N \left(\left(\frac{S}{\log(1+(\lambda/\delta) y^2)}\right) \in dz\right).
\]

And by writing the starting point as $S = \exp(v/2)$ we obtain the result. \qed

It is an advantage of our approach that, having calculated the symbol, one can write down the (extended) generator and the semimartingale characteristics at once.

For the reader’s convenience we recall the definition of the extended generator (cf. Definition (7.1) of [5]):

**Definition:** An operator $G$ with domain $\mathcal{D}_G$ is called **extended generator** of a Markov semimartingale $X$ if $\mathcal{D}_G$ consists of those functions $f \in \mathcal{B}(\mathbb{R}^d)$ for which there exists a function $Gf \in \mathcal{B}(\mathbb{R}^d)$ such that the process
\[
C_t^f := f(X_t) - f(X_0) - \int_0^t Gf(X_s) \, ds
\]
is well defined and a local martingale.

Combining Theorem 4.4 of [21] and Theorem 7.16 of [5] we obtain:

**Corollary 1:** The extended generator $G$ on $\mathcal{C}^2_0(\mathbb{R}^2)$ of the process
\((X^{(1)}, X^{(2)})' = (G, \log(\sigma^2))'\) can be written as

\[
G u(x) = \\
\partial_1 u(x) \left( t e^{x^2/2} + e^{x^2/2} \int_{\mathbb{R} \setminus \{0\}} y \cdot \left( 1 \{ e^{x^2/2} y < 1 \} \cdot 1 \{ 1/\log(1 + \lambda/\delta) \cdot y^2 < 1 \} - 1 \{ y < 1 \} \right) N(dy) \right) \\
+ \partial_2 u(x) \left( \frac{\beta}{e^{x^2}} + \log \delta + \int_{\mathbb{R} \setminus \{0\}} \log(1 + \lambda \delta y^2) \cdot 1 \{ e^{x^2/2} y < 1 \} \cdot 1 \{ 1/\log(1 + \lambda/\delta) \cdot y^2 < 1 \} N(dy) \right) \\
+ \partial_1 \partial_2 u(x)e^{x^2} Q \\
+ \int_{\mathbb{R}^2 \setminus \{0\}} \left( u(x - y) - u(x) + y' \nabla u(x) \cdot \left( 1 \{ y_1 < 1 \} \cdot 1 \{ y_2 < 1 \} \right) \right) \tilde{N}(x, dy)
\]

with the \(\tilde{N}\) from above.

Writing \(D(A)\) for the domain of the generator \(A\) of the process we have \(D(A) \subseteq D_G\) and the operators \(A\) and \(G\) coincide on \(D(A)\).

**Corollary 2:** The semimartingale characteristics \((B, C, \nu)\) of the process \((X^{(1)}, X^{(2)})' = (G, \log(\sigma^2))'\) are

\[
B^{(1)}_t(x) = \int_0^t \left( t e^{x^2/2} + e^{x^2/2} \int_{\mathbb{R} \setminus \{0\}} y \cdot \left( 1 \{ e^{x^2/2} y < 1 \} \cdot 1 \{ 1/\log(1 + \lambda/\delta) \cdot y^2 < 1 \} - 1 \{ y < 1 \} \right) N(dy) \right) ds \\
B^{(2)}_t(x) = \int_0^t \left( \frac{\beta}{e^{x^2}} + \log \delta + \int_{\mathbb{R} \setminus \{0\}} \log(1 + \lambda \delta y^2) \cdot 1 \{ e^{x^2/2} y < 1 \} \cdot 1 \{ 1/\log(1 + \lambda/\delta) \cdot y^2 < 1 \} N(dy) \right) ds \\
C_t(x) = \int_0^t \left( e^{x^2} Q 0 \\
0 0 \right) ds \\
\nu(\cdot, ds, dy) = \tilde{N}(X_s(\cdot), dy) ds
\]

with the \(\tilde{N}\) from above.

Remark: A different approach to calculate the characteristics of the COGARCH process is described in [12]. Furthermore our results are related to earlier work of B. Rajput and J. Rosinski. In their interesting article [15] they derive under certain restrictions a representation of the characteristic function of processes of the form \(X_t = \int_0^t f(t, s) dZ_s\) where \(f\) is a deterministic function and \(Z\) is a Lévy process.

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References

1. O. E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Statist. Soc B*, 63 (2001), Part 2, 167–241.
2. C. Berg and G. Forst. *Potential Theory on Locally Compact Abelian Groups*, volume 87 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1975.
3. R. M. Blumenthal and R. K. Getoor. *Markov Processes and Potential Theory*. Academic Press, New York 1968.
4. E. Cinlar and J. Jacod. Representation of Semimartingale Markov Processes in Terms of Wiener Processes and Poisson Random Measures. *Seminar on Stochastic Processes*, pages 159–242, 1981.
5. E. Cinlar, J. Jacod, P. Protter, and M. J. Sharpe. Semimartingales and Markov Processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 54 (1980): 161–219.
6. W. Hoh. Pseudo differential operators with negative definite symbols of variable order. *Rev. Mat. Iberoam*, 16 (2000): 219–241.
7. W. Hoh. Perturbations of pseudodifferential operators with negative definite symbol. *Applied Mathematics and Optimization*, 45 (2002)(3): 269–281.
8. N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. North-Holland Math. Library vol. 24, North Holland, Tokyo 1981.
9. N. Jacob. *Pseudo-Differential Operators and Markov Processes I-III*. Imperial College Press, London 2001-2005.
10. J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, Grundlehren math. Wiss. vol. 288, Berlin 1987.
11. N. Jacob and R. L. Schilling. Lévy-type processes and pseudo differential operators. In *Lévy Process-Theory and Applications*, pages 139–168. Birkhäuser, 2001.
12. J. Kallsen and J. Vesenmayer. COGARCH as a continuous-time limit of GARCH(1,1). *Stoch. Proc. Appl.* 119 (2009): 74–98
13. C. Klüppelberg, A. Lindner and R. Maller. A Continuous-Time GARCH Process Driven by a Lévy Process: Stationarity and Second-Order Behaviour. *J. Appl. Prob.*, 41 (2004): 601–622.
14. P. Protter. *Stochastic Integration and Differential Equations*. Springer Appl. Math. vol. 21, Berlin 2005.
15. B. Rajput and J. Rosinski. Spectral Representations of Infinitely Divisible Processes. *Probab. Theory Rel. Fields*, **82** (1989): 451–487.

16. K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Studies in Advanced Mathematics, vol. 68. Cambridge University Press, Cambridge 1999.

17. R. L. Schilling. Feller Processes Generated by Pseudo-Differential Operators: On the Hausdorff Dimension of Their Sample Paths. *J. Theor. Probab.*, **11** (1998): 303–330.

18. R. L. Schilling. Growth and Hölder conditions for the sample paths of Feller processes. *Probab. Theory Rel. Fields*, **112** (1998): 565–611.

19. R.L. Schilling and A. Schnurr. The Symbol Associated with the Solution of a Stochastic Differential Equation. *Electr. J. Probab.*, **15** (2010): 1369–1393.

20. A. Schnurr. A Generalization of the Blumenthal-Getoor Index to the Class of Homogeneous Diffusions with Jumps and some Applications. Preprint 2011.

21. A. Schnurr. *The Symbol of a Markov Semimartingale*. PhD thesis, TU Dresden 2009.

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