PAPER

Pececi-Quinn-like symmetries for nonabelian axions

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Abstract

Axions were first introduced in connection with chiral symmetry but are now being looked for mainly as dark matter. In this paper we introduce a nonabelian analogue of axions which can also be potential candidates for dark matter. Their nonabelian symmetries, which are generalizations of the Pececi-Quinn symmetry, are interesting in their own right. Detailed analysis, using fermion measure and zeta function approaches, shows that these symmetries are not anomalous.

1. Introduction

The chiral symmetry which holds in classical Dirac theory with massless fermions interacting with gauge fields is broken by what is called an anomaly [1]: the transformation

\[ \psi \rightarrow e^{i\alpha \gamma_5} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\alpha \gamma_5}, \]

is a symmetry of the kinetic term \( \bar{\psi} [i \gamma^\mu] \psi \) and also of the interaction term \( \bar{\psi} [i A^\mu] \psi \) with the gauge field \( A^\mu_i \), but the axial current \( \bar{\psi} \gamma^\mu \gamma^5 \gamma^i \psi \), which is classically conserved, is found to violate this conservation when the fermion triangle diagram is regularized and evaluated. The divergence of the axial current is finite and proportional to \( \text{tr} FF \) [2].

In general, a classical symmetry may or may not survive quantization. The simple phase symmetry, whereby \( \psi \) is multiplied by a phase factor, does survive quantization for all masses \( m \). To see whether a symmetry survives quantization, the action has to be regularized. If the regularized action still has the symmetry, the symmetry obviously has no anomaly. If the regularized action does not possess the symmetry, one tends to think that the symmetry has an anomaly, but there are different ways of regularizing fermion field theories and one must check whether a different regularization can preserve the symmetry. Familiarity with the chiral anomaly may make one suspect all classical symmetries involving chiral rotations in any manner to be anomalous. But whether a symmetry is anomalous or not has to be checked individually for each symmetry.

Below we review the example of the Pececi-Quinn symmetry which occurs in the presence of the hypothetical field called the axion and was introduced by these authors. It has been shown to survive quantization [3]. We point out that the symmetry can even be made local. In section 2, a new nonabelian analogue of axions is introduced: the new Pececi-Quinn symmetry too has no anomaly, as shown by a measure analysis and a zeta function approach.

1.1. Pececi-Quinn symmetry and the axion

Chiral symmetry is explicitly broken by the mass term \( m \bar{\psi} \psi \) and also by quantum effects, i.e. the anomaly. However, an artificial chiral symmetry for massive fermions works by letting a new field \( \varphi \) absorb the chiral transformation. The mass term is replaced by [4]

\[ \bar{\psi} m e^{i\alpha \gamma_5} \psi, \]

which is invariant if the field \( \varphi \) transforms under

\[ \psi \rightarrow e^{i\alpha \gamma_5} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\alpha \gamma_5}, \quad \varphi \rightarrow \varphi - 2\alpha. \]
This transformation leaves the action invariant provided the new field \( \varphi \) is massless. This is the Peccci-Quinn symmetry. The particle [3] corresponding to the new field \( \varphi \) introduced by them is called the axion, but it has not been seen in any experiment [6]. It is being studied extensively because it is expected to contribute to the elusive dark matter. For a discussion of strong CP symmetry in the absence of axions, one may look at [7].

Careful regularization has been shown to respect the Peccci-Quinn symmetry, which accordingly is not anomalous but survives quantization [3].

In spite of this subtlety about the Peccci-Quinn symmetry, the axion can still be used to remove any \( FF \) term in the action dynamically by coupling the axion directly to \( FF \) so that its vacuum expectation value cancels the coefficient of the \( FF \) term.

Observe that the axial symmetry can be made local by introducing an extra gauge field \( B_\mu \) for this purpose:

\[
\bar{\psi}i\not{\partial} + B_\gamma \gamma_5 - me^{i\varphi/\varphi_0}\psi + \frac{1}{2}F^2(\partial_\mu \varphi + 2B_\mu)(\partial^\mu \varphi + 2B^\mu),
\]

where

\[ B_\mu \rightarrow B_\mu + \partial_\mu \alpha \]  

under a local chiral transformation. Here \( F \) is a constant of mass dimension such that the axion kinetic term is \( \frac{1}{4}F^2e^{i\varphi/\varphi_0}\varphi \partial^\mu \varphi \) and there are additional kinetic terms of the gauge fields. It is to be noted that this provides a formulation of a chiral gauge theory similar to but different from the Wess-Zumino formulation suggested in [8]. The similarity is that in both cases there is an extra degree of freedom. The difference is that gauge invariance occurs only at the quantum level

\[ \text{invariance occurs only at the quantum level} \]  

The question now is whether this nonabelian symmetry survives quantization. Anomalies arise when regularizations break some symmetries of classical actions. In the functional integral approach, it is said that the action has a symmetry which is broken by the measure [10].

2. Nonabelian chiral symmetry and nonabelian axions

The usual chiral symmetry is under a transformation of the fermion in spinor space. If the fermion is an \( \text{SU}(N) \) multiplet, there exist nonabelian chiral symmetries. The kinetic piece

\[
\bar{\psi}i\not{\partial}\psi = \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R
\]

is invariant under the chiral transformations

\[ \psi_L \rightarrow U_L \psi_L, \quad \psi_R \rightarrow U_R \psi_R, \]

where \( U_L, U_R \) are spacetime independent \( \text{SU}(N) \) matrices acting on the two chiral projections of \( \psi \). The gauge interactions will also be invariant under these provided the matrix \( A_\mu \) commutes with \( U_L, U_R \). For instance, the \( \text{SU}(N) \) could be a flavour group and the colour \( \text{SU}(3) \) or the \( \text{U}(1) \) could be gauged.

The usual mass term \( m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \) is not invariant under (7) unless \( U_L = U_R \), in which case of course the transformation is not a chiral transformation. An analogue of the Peccci-Quinn mass term can be introduced: \( m(\bar{\psi}_L W \psi_R + \bar{\psi}_R W^\dagger \psi_L) \). Here \( W \) is a hypothetical \( \text{SU}(N) \) matrix field analogous to the axion. Considering that the original axion has not been detected, we must be cautious about such an object. However, just as the usual axion is expected to be a kind of dark matter, this nonabelian object too could be relevant as dark matter. Note that it is visualized as a new degree of freedom and not as mesonic matter. The mathematical construction may in any case be useful for calculations because of the symmetry. This term is invariant under (7) if \( W \) transforms as

\[ W \rightarrow U_L W U_R^\dagger, \]

As an \( \text{SU}(N) \) matrix it involves \( N^2 - 1 \) parameters which become fields. The kinetic term for this matrix field has to be of the form \( \text{Tr}[\partial_\mu W \partial^\mu W^\dagger] \), familiar from chiral models. This is invariant under (8). Thus the full action is invariant under the generalized Peccci-Quinn symmetry

\[ \psi_L \rightarrow U_L \psi_L, \quad \psi_R \rightarrow U_R \psi_R, \quad W \rightarrow U_L W U_R^\dagger. \]

The question now is whether this nonabelian symmetry survives quantization. Anomalies arise when regularizations break some symmetries of classical actions. In the functional integral approach, it is said that the action has a symmetry which is broken by the measure [10].

2.1. Fermion measure approach

To formulate the fermion measure, it is customary to expand the fermion field in eigenfunctions of some operator. To maintain gauge invariance, the covariant Dirac operator is considered. The eigenvalue equation is

\[ i\not{D}f_n = \lambda_n f_n, \]
where the subscript labels the eigenvalue and the eigenfunction. Under a gauge transformation,

$$D_\mu \rightarrow U D_\mu U^{-1}, \quad \psi \rightarrow U \psi, \quad \bar{\psi} \rightarrow \bar{\psi} U^{-1},$$  \hspace{1cm} (11)

so that

$$f \rightarrow Uf.$$ \hspace{1cm} (12)

The field is expanded as

$$\psi = \sum_n a_n f_n, \quad \bar{\psi} = \sum_n \bar{a}_n f_n^\dagger.$$ \hspace{1cm} (13)

Each $a$, $\bar{a}$ is gauge invariant because $\psi$ and $f$ transform the same way under gauge transformations and $\bar{\psi}$ and $f^\dagger$ also transform like each other. The gauge invariant measure $\prod_n da_n \bar{d}a_n$ is used for the fermion integration. It is well known that the measure is not chirally invariant; chiral transformations alter $a$, $\bar{a}$ and the change of the measure is a Jacobian which can be evaluated after some regularization and yields the chiral anomaly. One needs measures for other fields too, but these do not break symmetries.

Given this situation, it would appear that the Peccei-Quinn transformation would also alter the measure. The above measure would certainly be altered, but remembering that the requirement of gauge invariance led to the use of a fermion measure involving the eigenfunctions of the Dirac operator which contains the gauge field, we can involve the axion field now. First for the abelian axion, we consider the new expansion

$$\psi = e^{-i\chi} \sum_n b_n f_n, \quad \bar{\psi} = \sum_n \bar{b}_n f_n^\dagger e^{-i\chi}.$$ \hspace{1cm} (14)

Although the fermion field changes under the Peccei-Quinn transformation, the exponential factor too changes and cancels it, because of (3), leaving $b$, $\bar{b}$ invariant. Hence the measure $\prod_n db_n \bar{d}b_n$ is invariant under the transformation. In other words, although the naïve fermion measure is altered by the Peccei-Quinn transformation, there does exist a fermion measure which is left invariant. This is very similar to what happens with regularizations. It is also similar to the alteration occurring in the fermion measure in the presence of a twisted fermion mass term [11]. It may be added that the measure for $\phi$ is translation invariant.

For the $SU(N)$ version of axions, the construction of the measure is a bit complicated. First, note that eigenvalues and eigenfunctions of $i \partial \phi$ come in pairs:

$$i \partial f_n = \lambda_n f_n, \quad i \partial \gamma_5 f_n = -\lambda_n \gamma_5 f_n.$$ \hspace{1cm} (15)

Hence it is possible to consider expansions in $f_n^L, f_{nR}$ which are chiral combinations of the $f_n$, $\gamma_5 f_n$, though they are not eigenfunctions of $i \partial \phi$. We expand

$$\psi_L = \sum_n a_n^L f_n^L, \quad \bar{\psi}_L = \sum_n \bar{a}_n^L f_n^L, \quad \psi_R = \sum_n a_n^R f_n^R, \quad \bar{\psi}_R = \sum_n \bar{a}_n^R f_n^R.$$ \hspace{1cm} (16)

Of course, the range of $n$ is implicitly altered here. The measure $\prod_n da_n^L \bar{d}a_n^L da_n^R \bar{d}a_n^R$ is not invariant under an $SU(N)$ chiral transformation (7) because $a$, $\bar{a}$ have to change unless $U_L = U_R$, in which case the common vector transformation may be absorbed in $f$.

However, a new measure can be constructed using the generalized axion field. Consider the expansions

$$W \psi_L = \sum_n b_n^L f_n^L, \quad \psi_R = \sum_n b_n^R f_n^R,$$

$$\bar{\psi}_L W = \sum_n \bar{b}_n^L f_n^L, \quad \bar{\psi}_R = \sum_n \bar{b}_n^R f_n^R.$$ \hspace{1cm} (17)

As $W$ is invertible, it may also be transferred to the right if desired. This construction is not unique, but serves the purpose. Note the asymmetric use of $W$ here. Because of this asymmetry, the left hand sides of both equations in the first line acquire $U_L$ under (9) and the left hand sides in the second line acquire $U_R^\dagger$, so that the common vector factor $U_R$ may be absorbed in $f$:

$$f_n \rightarrow U_R f_n,$$ \hspace{1cm} (18)

leaving the $b$, $\bar{b}$ invariant. This means that there exists a measure $\prod_n db_n^L db_n^R d\bar{b}_n^L d\bar{b}_n^R$ invariant under the $SU(N)$ version of the Peccei-Quinn transformation, exactly as before. As regards the measure for $W$, it can be chosen to be $SU(N)$ invariant. So the measure respects the symmetry and the new nonabelian Peccei-Quinn symmetry is not anomalous.

### 2.2. Zeta function approach

Instead of considering a regularized action, one may also look at the fermion determinant which then has to be regularized. The most convenient way to do this in this context is the zeta function regularization [12]. The determinant is that of the Dirac operator, which in the simple case of a singlet axion is
\[ i\mathcal{J} - me^{i\gamma_5}. \tag{19} \]

The zeta function regularization works for a hermitian, positive definite operator, which has to be obtained by constructing the Laplacian,

\[ \Delta = [-i\mathcal{J} - me^{-i\gamma_5}][i\mathcal{J} - me^{i\gamma_5}]. \tag{20} \]

Here the gamma matrices have been taken to be antihermitean in euclidean spacetime. The determinant of the Dirac operator is defined as the square root of the determinant of \( \Delta \). The anomaly has been checked in this framework \( \cite{13} \).

Now one can write

\[ \Delta = [-i\mathcal{J} - me^{-i\gamma_5}]e^{i\phi/2}e^{-i\phi/2}[i\mathcal{J} - me^{i\gamma_5}]. \tag{21} \]

This can be written after some formal manipulations as

\[ \Delta = e^{-i\phi/2}[i\mathcal{J} + \not{\phi}\gamma_5/2 - m][i\mathcal{J} + \not{\phi}\gamma_5/2 - m]e^{i\phi/2}. \tag{22} \]

Apart from the initial and final exponential factors, this depends on the axion field \( \phi \) only through its derivative. When the determinant is calculated, those exponential factors cancel. Hence, its determinant will also involve only derivatives of this field and will be invariant under translations thereof. But after the fermion is integrated out, the Peccei-Quinn transformation is just a constant translation of the axion field, so the determinant is invariant under Peccei-Quinn transformations and it has no anomaly when the product of its eigenvalues is regularized through the zeta function.

The case of the nonabelian Peccei-Quinn symmetry is more complicated. Here, the Dirac operator is

\[ i\mathcal{J} - m(W_R + W^\dagger_L) \tag{23} \]

So one needs the Laplacian

\[ \Delta = [-i\mathcal{J} - m(W_R + W^\dagger_L)][i\mathcal{J} - m(W_R + W^\dagger_L)]. \tag{24} \]

This can be recast as

\[ \Delta = [-i\mathcal{J} - m(W_R + W^\dagger_L)] \]
\[ [V^\dagger_R + Y_P][V_P + Y^\dagger_L][i\mathcal{J} - m(W_R + W^\dagger_L)], \tag{25} \]

where \( V, Y \) are some \( SU(N) \) matrices to be constrained later. This becomes

\[ \Delta = [V^\dagger_P + Y_P][-i\mathcal{J} - m(V^\dagger_P + Y^\dagger_R)](V^\dagger_P + Y^\dagger_R) - m] \]
\[ [i\mathcal{J} - m(V^\dagger_P + Y^\dagger_R)(V^\dagger_P + Y^\dagger_R) - m][V^\dagger_P + Y^\dagger_R], \tag{26} \]

if

\[ V^\dagger = WY, Y = W^\dagger V^\dagger, VW = Y^\dagger, Y^\dagger W^\dagger = V, \tag{27} \]

which can all be satisfied by requiring \( V, Y \) to obey the single relation

\[ VWY = 1. \tag{28} \]

When the determinant is calculated, the initial and final factors cancel out. The remaining expression

\[ \text{det}(-i\mathcal{J} - imV^\dagger \not{\phi}Y^\dagger - iY^\dagger P_R \not{\phi}Y - m)[i\mathcal{J} - i\not{\phi}V^\dagger P_L - i\not{\phi}Y^\dagger Y^\dagger P_R - m]) \]

involves \( V, Y \) only in the combinations \( V \partial V^\dagger, Y \partial Y^\dagger, \partial V V^\dagger, \partial Y Y^\dagger \). These are invariant under the Peccei-Quinn-like transformations of \( W \) whose \( U_R \) actions are taken as left actions on \( Y \) and \( U_L \) actions as right actions on \( V \) because of the constraint on \( VWY \). Thus the determinant is invariant under Peccei-Quinn-like transformations. This persists upon zeta function regularization of the product of its eigenvalues. Hence it is seen once again that these symmetries are not anomalous.

As there is no anomaly in the Peccei-Quinn-like symmetries, gauge fields can again be used to extend these global chiral symmetries to local ones. For example, for the left handed chiral symmetry, one needs an \( SU(N) \) gauge field matrix \( B_{\mu} \):

\[ \tilde{\psi}[i\mathcal{J} + \not{\phi}P_L - m(W_R + W^\dagger_L)]\psi \]
\[ + \frac{1}{2} F^2 Tr[(\partial_{\mu} - iB_{\mu}) W (\partial_{\nu} W^\dagger + iW^\dagger B^{\nu})]. \tag{29} \]

Here the transformation of the new gauge field is given by

\[ (\partial_{\mu} - iB_{\mu}) \rightarrow U_L (\partial_{\mu} - iB_{\mu}) U_L^{-1}. \tag{30} \]

Gauge field kinetic terms have to be added. The right handed chiral symmetry too can be gauged if desired in a similar way. This reformulation of a chiral gauge theory in a gauge invariant way is again reminiscent of \( \cite{8} \).
3. Conclusion

Symmetries which appear to be anomalous because they are not consistent with obvious regularizations may turn out to be consistent if regularized with care and therefore may be non-anomalous. The first known case of such a faux anomaly is the Peccei-Quinn symmetry which arises in QCD in the presence of axions, which are now being looked for as dark matter but have not so far been found. Our new $SU(N)$ version of this symmetry, which holds if $SU(N)$ analogues of axions are introduced, provides the second example: a fermion measure invariant under this symmetry has been explicitly constructed and the determinant in the zeta function approach is also invariant under nonabelian transformations of $W$. We hope these mathematical observations will be of as much interest as ordinary axions in the context of dark matter. While axions have been expected to be useful in the context of the strong CP issue, there is no such possibility with the nonabelian analogues discussed here because they cannot be coupled in a natural way to the gluon $\tilde{F} F$ term. However, as they couple to quarks, they may also transform into other gauge bosons in the same way as the ordinary axions get feebly converted to photons. They could be detected for example as decaying $Z$ bosons just as ordinary axions are sought to be caught in the form of photons. Apart from this, nonabelian axions can be of use in chiral gauge theories where they are unphysical and get swallowed up while providing mass to gauge bosons [9].

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References

[1] Adler S 1969 Phys. Rev. 177 2426
   Bell J S and Jackiw R 1969 Nuovo Cim. 60 47
[2] Brown L, Carlitz R and Lee C 1977 Phys Rev. D16 417
[3] Mitra P 2017 Journ. Phys. Communications 1 015002
[4] Peccei R and Quinn H 1977 Phys. Rev. Letters 38 1440
   Peccei R and Quinn H 1977 Phys. Rev. D16 1791
[5] Weinberg S 1978 Phys. Rev. Letters 40 223
   Wilczek F 1978 Phys. Rev. Letters 40 279
[6] Zyla P A et al 2020 The Review of Particle Physics, Prog. Theor. Exp. Phys. 2020 083C01
[7] Banerjee H, Chatterjee D and Mitra P 2003 Phys. Letters B573 109
[8] Faddeev L D and Shatashvili S L 1986 Phys. Letters B167 225
[9] Chiral gauge theories and nonabelian analogues of axions, in preparation
[10] Fujikawa K 1980 Phys. Rev. D21 2848
   Fujikawa K 2001 Int. J. Mod. Phys. A16 331
[11] Mitra P 2014 Symmetries and Symmetry Breaking in Field Theory (Florida: CRC Press)
[12] Elizalde E, Odintsov S D, Romeo A, Bytsenko A A and Zerbini S 1994 Zeta Regularization Techniques with Applications (Singapore: World Scientific)
[13] Reuter M 1985 Chiral anomalies and zeta–function regularization Phys. Rev. D31 1374