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Simplicity of tangent bundles of smooth horospherical varieties of Picard number one

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Abstract. Recently, Kanemitsu has discovered a counterexample to the long-standing conjecture that the tangent bundle of a Fano manifold of Picard number one is (semi)stable. His counterexample is a smooth horospherical variety. There is a weaker conjecture that the tangent bundle of a Fano manifold of Picard number one is simple.

We prove that this weaker conjecture is valid for smooth horospherical varieties of Picard number one. Our proof follows from the existence of an irreducible family of unbendable rational curves whose tangent vectors span the tangent spaces of the horospherical variety at general points.

1. Introduction

Tangent bundles of Fano manifolds have been studied from various points of view, in relation with the existence of a Kähler–Einstein metric. For example, if $X$ admits a Kähler–Einstein metric, then the Lie algebra of holomorphic vector fields on $X$ is reductive. A weaker condition, the existence of a Hermitian–Einstein metric on the tangent bundle $TX$ is equivalent to the polystability of $TX$, and is equivalent to the stability of $TX$ when $X$ has Picard number one.

In this regard, when $X$ has Picard number one, the tangent bundle $TX$ of a Fano manifold $X$ has been expected to be semistable for a long time.

Conjecture 1 ([9, Conjecture 0.1]). The tangent bundle of a Fano manifold of Picard number one is semistable.
This conjecture has been verified in many cases (see e.g., [4, 13]). Recently, however, a counterexample has been discovered by Kanemitsu ([9]): The tangent bundle of a smooth horospherical variety $X$ of Picard number one is not semistable if $X$ is of type $(B_n, \omega_{n-1}, \omega_n)$, where $n \geq 4$ or of type $(F_4, \omega_3, \omega_2)$, and is stable, otherwise. For the types of horospherical varieties of Picard number one, see Proposition 10.

Only scalar multiplications are endomorphisms of a stable vector bundle, so that stable vector bundles are simple. Instead of stability, we consider a weaker conjecture.

**Conjecture 2.** The tangent bundle of a Fano manifold of Picard number one is simple.

In this paper, we prove that Conjecture 2 is valid for any smooth horospherical variety of Picard number one.

**Theorem 3.** The tangent bundle of any smooth horospherical varieties of Picard number one is simple.

To prove Theorem 3 we use unbendable rational curves. A rational curve $f : \mathbb{P}^1 \to X$ in a uniruled projective manifold $X$ is said to be *unbendable* if the pull-back $f^* TX$ of the tangent bundle $TX$ of $X$ is decomposed as $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some nonnegative integers $p, q$. Minimal rational curves are unbendable but the converse is not true (Remark 8). Here, by a minimal rational curve we mean a rational curve $f : \mathbb{P}^1 \to X$ with $f^* TX$ being nonnegative, and the degree of $f$ with respect to a fixed ample line bundle is minimal among all such rational curves.

We expect unbendable rational components will play an important role in understanding the geometry of a uniruled projective manifold $X$ as minimal rational components did (see e.g., [6] and [10] and references therein). When $X$ has Picard number one, we get the simplicity of the tangent bundle $TX$ if there is an irreducible family of unbendable rational curves whose tangent vectors span $\mathbb{P} T_x X$ at a general point $x$ in $X$ (Proposition 5).

We prove that for a rational homogeneous variety $G/P$ with $G$ simple or a smooth horospherical variety of Picard number one, such an unbendable rational component exists (Proposition 6 and Proposition 12). Then Theorem 3 follows from Proposition 5 and the condition that $X$ has Picard number one.

In relation with Conjecture 2 we ask whether there is such an unbendable rational component for any Fano manifold of Picard number one.

**Conjecture 4.** A Fano manifold of Picard number one admits an irreducible family of unbendable rational curves whose tangent vectors span the tangent spaces of the Fano manifold at general points.

If Conjecture 4 holds, then so does Conjecture 2 by Proposition 5. Minimal rational curves are unbendable curves, but their tangent vectors do not always generate the tangent space at a general point. Since a Fano manifold of Picard number one is rationally connected, there exists an irreducible family of rational curves whose tangent vectors span the tangent space at a general point, but the restriction of the tangent bundle to rational curves in this family may have more than one $\mathcal{O}(2)$-factors, or more generally, may have $\mathcal{O}(a)$-factors with $a \geq 3$. The question is whether we have an irreducible family of rational curves having both properties, rational curves in the family are unbendable and their tangent vectors span the tangent space at a general point.

2. **Unbendable rational curves**

Let $X$ be a uniruled projective manifold. We say that a rational curve $f : \mathbb{P}^1 \to X$ in $X$ is

- *unbendable* if the pull-back $f^* TX$ of the tangent bundle $TX$ of $X$ is decomposed as $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some nonnegative integers $p, q$;
• *minimal* if the pull-back $f^*TX$ of the tangent bundle $TX$ of $X$ is nonnegative, and the degree of $f$ with respect to a fixed ample line bundle is minimal among all such rational curves.

An irreducible component $\mathcal{H}$ of the Hilbert scheme of rational curves in $X$ containing an unbendable (minimal, respectively) rational curve is called an *unbendable (minimal, respectively) rational component* of $X$.

Fix an unbendable rational component $\mathcal{H}$ of $X$. For a point $x \in X$ denote by $\mathcal{H}_x$ the subscheme of $\mathcal{H}$ consisting of members passing through $x$, and define a rational map

$$\tau_x : \mathcal{H}_x \rightarrow \mathbb{P}T_x X$$

by sending a rational curve $[C] \in \mathcal{H}_x$ smooth at $x$ to its tangent direction $[T_x C] \in \mathbb{P}T_x X$. The closure $\mathcal{E}_x$ of the image of $\tau_x$ is called the *variety of tangents* at $x$ of the family $\mathcal{H}$.

**Proposition 5 ([5, Theorem 2]).** If $X$ has an unbendable rational component $\mathcal{H}$ such that

1. the variety of tangents at $x$ of $\mathcal{H}$ is nondegenerate in the projective tangent space $\mathbb{P}T_x X$ for general $x \in X$;
2. a general point of $X$ is joined to a point in $X$ by a connected chain of curves in $\mathcal{H}$.

then $T X$ is simple, that is, any endomorphism of $T X$ is a scalar multiplication.

When $X$ has Picard number one, the condition (2) of Proposition 5 is satisfied automatically: There is a sequence of locally closed submanifolds $\mathcal{X}^0 = \{x\} \subset \mathcal{X}^1 \subset \cdots \subset \mathcal{X}^m$, where $\dim \mathcal{X}^m = \dim X$, such that any point in $\mathcal{X}^k$ can be connected to a point in $\mathcal{X}^{k-1}$ by a rational curve in $\mathcal{H}$ for any $1 \leq k \leq m$ (see [7, Section 4.3] or [10, Section 3]).

For example, the moduli of semistable vector bundles of rank $r$ and with a fixed determinant ([5]) or a wonderful group compactification ([3]) have an unbendable rational component whose variety of tangents is nondegenerate in $\mathbb{P}(T_x X)$ at a general point $x \in X$. In the first case, in fact, a minimal rational component satisfies the desired property. However, this is not the case in general (Remark 8), and we need to consider an unbendable rational component which is not minimal to get the desired property.

### 3. Existence of unbendable rational curves

#### 3.1. Rational homogeneous varieties

Let $\mathfrak{g}$ be a complex simple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a system of simple roots. For each root $\alpha$, there is an element $H_\alpha$ in $\mathfrak{h}$ satisfying that $\alpha(H) = (H_\alpha, H)$ for any $H \in \mathfrak{h}$, where $(\cdot, \cdot)$ is the Killing form of $\mathfrak{g}$. Then the Killing form induces a symmetric bilinear form $(\cdot, \cdot)$ on the set $\Phi$ of roots defined by $(\alpha, \beta) := (H_\alpha, H_\beta)$ for $\alpha, \beta \in \Phi$. Put $h_\alpha := 2H_\alpha/\langle \alpha, \alpha \rangle$ and $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ for $\alpha \in \Phi$. Then $\beta(h_\alpha) = (\beta, \alpha^\vee)$.

For a subset $\Delta_1 \subset \Delta$, define a function $n_{\Delta_1} : \Phi \rightarrow \mathbb{Z}$ by

$$n_{\Delta_1}(\alpha) = \sum_{\alpha_i \in \Delta_1} n_i$$

for $\alpha = \sum_{i=1}^l n_i \alpha_i \in \Phi$. Define $\Phi_{\Delta_1}^\pm$ and $\Phi_{\Delta_1}^0$ by

$$\Phi_{\Delta_1}^\pm := \{\alpha \in \Phi : \Phi : n_{\Delta_1}(\alpha) \in \mathbb{Z}_\pm\} \quad \text{and} \quad \Phi_{\Delta_1}^0 := \{\alpha \in \Phi : n_{\Delta_1}(\alpha) = 0\}.$$

and put $p_{\Delta_1} = \mathfrak{h} \oplus (\Phi_{\Delta_1}^0 \cup \Phi_{\Delta_1}^\pm)$ and $m_{\Delta_1} = \Phi_{\Delta_1}^0 \cup \Phi_{\Delta_1}^\pm$ and $\mathfrak{g} = p_{\Delta_1} \oplus m_{\Delta_1}$. Fix $p = p_{\Delta_1}$ from now on. Note that $p$ contains the negative Borel subalgebra $\mathfrak{b} := p_{\Delta}.$

Let $G$ be a simply connected algebraic group with Lie algebra $\mathfrak{g}$ and $P$ be the subgroup of $G$ with Lie algebra $\mathfrak{p}$. For each root $\alpha$, take $E_\alpha \in \mathfrak{g}_\alpha, E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $(E_\alpha, E_{-\alpha}) = 1$. Then
Let \( [E_a, E_{-a}] = H_a \in \mathfrak{h} \) and \( E_a, E_{-a}, H_a \) generate a subalgebra \( \mathfrak{sl}_a \) of \( \mathfrak{g} \) which is isomorphic to \( \mathfrak{sl}_2 \). Let \( S_a \) be the subgroup of \( G \) with Lie algebra \( \mathfrak{sl}_a \). The \( S_a \)-orbit of \( o \in G/P \) is a rational curve in \( G/P \), denoted by \( C_a \).

The tangent bundle \( T(G/P) \) is the homogeneous vector bundle on \( G/P \) associated to the representation \( P \to GL(\mathfrak{g}/p) \). The tangent bundle \( T(G/P) \) restricted to \( C_a \) is decomposed as \( \oplus_{\beta \in \Phi^+_a} \mathcal{O}(\beta(h_a)) = \oplus_{\beta \in \Phi^+_a} \mathcal{O}(\beta, \alpha^\vee) \) (See [8, proof of Proposition 2]).

**Proposition 6.** Let \( G/P \) be a rational homogeneous variety. Assume that \( G \) is simple. Then there is an unbendable rational component \( \mathcal{H} \) whose variety \( \mathfrak{c}_y \) of tangents at any \( y \in G/P \) is nondegenerate in \( \mathbb{P}T_y(G/P) \).

**Proof.** Let \( \theta \) be the maximal positive root of \( \mathfrak{g} \). Since \( \theta \) has nonzero coefficient in \( \alpha_i \) for any \( i = 1, \ldots, \ell \), \( S_\theta \) is not contained in \( P_{\Delta_i} \) for any subset \( \Delta_i \) of \( \Delta \), and thus \( C_\theta \) is a (nonconstant) rational curve in \( G/P \) for any \( P = P_{\Delta_i} \). We will show that \( C_\theta \) is an unbendable rational curve in \( G/P \). It suffices to show that \( (\beta, \theta^\vee) \leq 1 \) for all \( \beta \in \Phi^+_{\Delta_i} \setminus \{ \theta \} \). To do this we will use the following two facts:

(i) If \( G \) is not of type \( A \), then there is \( i_0 \in \{ 1, \ldots, \ell \} \) such that \( (\alpha_i, \theta) = 0 \) for all \( i \neq i_0 \) in \( \{ 1, \ldots, \ell \} \) and \( (\alpha_{i_0}, \theta^\vee) = 1 \).

(ii) If \( G \) is of type \( A \), then \( (\alpha_j, \theta) = 0 \) for all \( j \in \{ 2, \ldots, \ell - 1 \} \) and \( (\alpha_1, \theta^\vee) = (\alpha_\ell, \theta^\vee) = 1 \).

Following the conventions in [11] for the indices of simple roots, we have \( i_0 = 2 \) if \( G \) is of type \( B_\ell, D_\ell \), and \( i_0 = 1 \) if \( G \) is of type \( C_\ell \), and \( i_0 = 6 \) (6, 1, 4, 2, respectively) if \( G \) is of type \( E_6, E_7, E_8, F_4, G_2 \), respectively. See the affine Dynkin diagram of \( \Delta^{(1)} \) in [11, Table 6], which is the diagram of the union \( \Delta \cup \{ -\theta \} \) of \( \Delta \) in [11, Table 1] and \( \{ -\theta \} \).

If \( G \) is not of type \( A \), then, by (i), the maximal root among roots in \( \Phi \setminus \{ \theta \} \) is \( \theta - \alpha_i \). Since \( (\theta - \alpha_i, \theta^\vee) = 2 - 1 = 1 \), we have \( (\beta, \theta^\vee) \leq 1 \) for all \( \beta \in \Phi^+_{\Delta_i} \setminus \{ \theta \} \). If \( G \) is of type \( A \), then, by (ii), among roots in \( \Phi \setminus \{ \theta \} \), maximal ones are either \( \theta - \alpha_1 \) or \( \theta - \alpha_\ell \). From \( (\theta - \alpha_1, \theta^\vee) = (\theta - \alpha_\ell, \theta^\vee) = 1 \) it follows that \( (\beta, \theta^\vee) \leq 1 \) for all \( \beta \in \Phi^+_{\Delta_i} \setminus \{ \theta \} \).

The curve \( C_\theta \) is tangent to \( \mathbb{C} E_\theta \subset T_\theta G/P \). The variety \( \mathfrak{c}_o \) of tangents at the base point \( o \) is the closure of the \( P \)-orbit \( P[E_\theta] \) in \( \mathbb{P}(m) \). We claim that \( \mathfrak{c}_o \) is nondegenerate in \( \mathbb{P}(g/p) \). Consider the projection \( g \to g/p \). The closure of the \( P \)-orbit \( P[E_\theta] \) in \( \mathbb{P}(g) \) is the \( G \)-orbit \( G[E_\theta] \), which is nondegenerate in \( \mathbb{P}(g) \) because it is the highest weight orbit. Therefore, the closure of the \( P \)-orbit \( P[E_\theta] \) in \( \mathbb{P}(g/p) \) is nondegenerate in \( \mathbb{P}(g/p) \).

**Remark 7.** The inequality \( (\beta, \theta^\vee) \leq 1 \) for any \( \beta \in \Phi^+ \setminus \{ \theta \} \) also follows from [2, Proposition 25 in VI.1.8].

**Remark 8.** Take a simple root \( \alpha_i \) which is not short and consider the rational homogenous variety \( G/P \) associated with \( \Delta_1 = \Delta - \{ \alpha_i \} \). Then the rational curve \( C_{\alpha_i} \) is a minimal rational curve and the variety of tangents of the minimal rational component containing \( [C_{\alpha_i}] \) at a point \( x \in G/P \) spans a proper subspace of \( \mathbb{P}T_x(G/P) \) if the coefficient of \( \alpha_i \) in \( \theta \) is \( > 1 \) ([8, Proposition 1]). Therefore, an unbendable rational curve is not necessarily a minimal rational curve.

Together with Proposition 5, Proposition 6 reproves the following result.

**Proposition 9 (11, Theorem 2.1).** Let \( X = G/P \) be a rational homogeneous variety with \( G \) being simple. Then the tangent bundle of \( X \) is simple.

**Proof.** By Proposition 5 and Proposition 6, it suffices to show that a general point \( x \) in \( G/P \) is joined to the base point \( o \) by a connected chain of curves in the unbendable rational component \( \mathcal{H} \) constructed in Proposition 6.

Let \( \Sigma \) denote the subset of \( G/P \) consisting of \( x \in G/P \) which are joined to \( o \) by a connected chain of curves in \( \mathcal{H} \). Then the stabilizer \( Q \) of \( \Sigma \) in \( G \) contains the subgroup of \( G \) generated by \( P \) and \( S_\theta \) because the action of \( S_\theta \) moves the point \( o \) to a point in \( C_\theta \). Thus \( Q \) contains \( P \) properly.
Let \( \{\varnothing_1, \ldots, \varnothing_\ell\} \) denote the system of fundamental weights corresponding the system \( \Delta \) of simple roots. Then each \( \varnothing_j \) can be written as a linear combination of \( \alpha_i \)'s with positive coefficients (See [11, Table 2]). By (i) and (ii) in the proof of Proposition 6, \( (\varnothing_j, \theta^V) \) is positive for any \( j = 1, \ldots, \ell \). Therefore, \( C_\varnothing \) cannot be contained in a fiber of a nontrivial projection \( G/P \to G/Q \). Since \( Q/P \) is not a point, \( G/Q \) is a point, that is, \( G \) is \( Q \) and \( \Sigma \) is \( G/P \).

\[ \square \]

3.2. \textit{Smooth horospherical varieties of Picard number one}

For \( i = 1, \ldots, \ell \), let \( V_{\varnothing_i} \) be the irreducible representation of \( G \) of highest weight \( \varnothing_i \) and let \( v_{\varnothing_i} \) be a highest weight vector of \( V_{\varnothing_i} \). Let \( (G, \varnothing_i, \varnothing_j) \) denote the closure of the \( G \)-orbit of the sum \( [v_{\varnothing_i} + v_{\varnothing_j}] \) in \( \mathbb{P}(V_{\varnothing_i} \oplus V_{\varnothing_j}) \).

**Proposition 10 ([12]).** \( \text{Let } X \text{ be a smooth horospherical variety of Picard number one. Then } X \text{ is either a rational homogeneous variety or one of the following:} \)

\begin{itemize}
  \item[(1)] \( (B_n, \varnothing_{n-1}, \varnothing_n) \) \( (n \geq 3) \)
  \item[(2)] \( (B_3, \varnothing_1, \varnothing_3) \)
  \item[(3)] \( (C_n, \varnothing_k, \varnothing_{k-1}) \) \( (n \geq 2, 2 \leq k \leq n) \)
  \item[(4)] \( (F_4, \varnothing_3, \varnothing_2) \)
  \item[(5)] \( (G_2, \varnothing_2, \varnothing_1) \).
\end{itemize}

In the latter case, the automorphism group \( \text{Aut}^0(X) \) is given by \( \bar{G} \ltimes H^0(G/P, G \times P V) \), where \( \bar{G} \) is a reductive group with \( G \) a maximal semisimple subgroup, and the open \( \text{Aut}^0(X) \)-orbit \( \Theta \) in \( X \) is \( G \)-equivariantly isomorphic to the total space of a homogeneous vector bundle \( G \times P V \) on \( G/P \), where \( P \) is the maximal parabolic subgroup associated to

\[ \varnothing_{n-1}, \varnothing_1, \varnothing_{k-1}, \varnothing_3, \varnothing_2, \text{ respectively} \]

and \( V \) is the simple \( P \)-module of highest weight \( \lambda_V \) given by

\[ \varnothing_{n-1} - \varnothing_n, \varnothing_1 - \varnothing_3, \varnothing_k - \varnothing_{k-1}, \varnothing_3 - \varnothing_2, \varnothing_2 - \varnothing_1, \text{ respectively} \].

**Remark 11.** We remark that our convention is different from that of [12]. The way of indexing simple roots and fundamental weights is the same as in [11] in this paper while [12] follows the convention in [2]. The isotropy subgroup \( P \) contains the negative Borel subgroup in this paper while it contains the positive Borel subgroup in [12].

**Proposition 12.** \( \text{Let } X \text{ be a smooth horospherical variety of Picard number one. Then there is an unbendable rational component } \mathscr{H} \text{ whose variety } \mathcal{E}_x \text{ of tangents at any } x \text{ in the open } \text{Aut}^0(X) \text{-orbit } \Theta \text{ is nondegenerate in } \mathbb{P}T_x X. \)

**Proof.** We will show that there is an unbendable rational component \( \mathscr{H} \) such that for any point \( x \) in the open \( \text{Aut}^0(X) \)-orbit \( \Theta \), the variety of tangents of \( \mathscr{H} \) at \( x \) is nondegenerate in \( \mathbb{P}T_x X. \) Since \( G \times H^0(G/P, G \times P V) \) acts on \( \Theta = G \times P V \) transitively, we may assume that \( x = [0,0] \), where \( 0 \) is the base point in \( G/P \). Let \( \mathcal{C}_0 \) be the rational curve in \( G/P \cong Y \) constructed in Proposition 6. We claim that \( \mathcal{C}_0 \) is an unbendable curve in \( \Theta \).

Let \( \lambda_V \) be the highest weight of \( V \) listed in Proposition 10. By the facts (i) and (ii) in the proof of Proposition 6, \( (\lambda_V, \theta^V) \) is the coefficient of \( \alpha_{i_0} \) in the expression of \( \lambda_V \) via simple roots, \( \alpha_1, \ldots, \alpha_\ell \). For the description of \( \varnothing_i \) as a linear combination of simple roots, see [11, Table 2]: The \( i \)-th column of the matrix \( (A^t)^{-1} \) inverse to the transposed Cartan matrix \( A \) is the list of coefficients of simple roots in \( \varnothing_i \). Let \( b_{i,j} \) denote the \((i, j)\)-th element of the matrix \( (A^t)^{-1} \). Then \( (\lambda_V, \theta^V) \) is given by

\[ b_{2,n-1} - b_{2,n}, \ b_{2,1} - b_{2,3}, \ b_{1,k} - b_{1,k-1}, \ b_{4,3} - b_{4,2}, \ b_{2,2} - b_{2,1}, \text{ respectively}. \]
Using the description of the matrix \((A^t)^{-1} = (b_{i,j})\) in [11, Table 2] we get that \((\lambda_V, \theta^V)\) is given by
\[
\frac{1}{2}(4-2), \quad \frac{1}{2}(2-2), \quad \frac{1}{2}(4-4), \quad 3-2, \quad 2-1,
\]
respectively.

Therefore, \((\lambda_V, \theta^V)\) is 1 or 0, and thus the vector bundle \(G \times_P V\) restricted to \(C_\theta \subset G/P\) splits as a \(\bigoplus_{i=1}^m \theta(a_i)\), where
\[
1 \geq a_1 \geq \cdots \geq a_m \geq 0.
\]
From this property and the nonnegativity of \(TY|_{C_\theta}\), it follows that the short exact sequence
\[
0 \to TY \to TX|_V \to G \times_P V \to 0
\]
restricted to \(C_\theta\) splits. Since \(TY|_{C_\theta}\) has only one \(\theta(2)\)-factor by Proposition 6, we get that \(TX|_{C_\theta}\) is also nonnegative and has only one \(\theta(2)\)-factor.

Now let \(V\) be the vector space spanned by
\[
\{T_{[a,0]}(aC_\theta) : [a,0] \in aC_\theta, a \in G \times H^0(G/P, G \times_P V)\}.
\]
Then the projection of \(V\) to \(V\) is a nonzero \(P\)-stable subspace of \(V\), and thus is the whole \(V\) because \(V\) is an irreducible \(P\)-module. Hence the tangent directions to translates of \(C_\theta\) passing through \([a,0]\) span \(T_{[a,0]}\).

**Proof of Theorem 3.** By Proposition 12, there is an unbendable rational component \(\mathcal{H}\) whose variety of tangents is nondegenerate in \(\mathbb{P}(T_x X)\) for any \(x \in \mathcal{O}\). Since \(X\) has Picard number one, by Proposition 5, any endomorphism of the tangent bundle of \(X\) is a scalar multiple.

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**References**

[1] H. Azad, I. Biswas, "A note on the tangent bundle of \(G/P\)", *Proc. Indian Acad. Sci., Math. Sci.* 120 (2010), no. 1, p. 69-71.

[2] N. Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie*, Springer, 2006.

[3] M. Brion, B. Fu, "Minimal rational curves on wonderful group compactifications", *J. Éc. Polytech., Math.* 2 (2015), p. 153-170.

[4] J.-M. Hwang, "Stability of tangent bundles of low-dimensional Fano manifolds with Picard number 1", *Math. Ann.* 312 (1998), no. 4, p. 599-606.

[5] ———, "Hecke curves on the moduli space of vector bundles over an algebraic curve", in *Proceedings of the symposium on Algebraic Geometry in East Asia (Kyoto, 2001)*, World Scientific, 2001, p. 155-164.

[6] ———, "Geometry of varieties of minimal rational tangents", in *Current developments in algebraic geometry*, Mathematical Sciences Research Institute Publications, vol. 59, Cambridge University Press, 2012, p. 197-226.

[7] J.-M. Hwang, N. Mok, "Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation", *Invent. Math.* 131 (1998), no. 2, p. 393-418.

[8] ———, "Deformation rigidity of the rational homogeneous space associated to a long root", *Ann. Sci. Éc. Norm. Supér.* 35 (2002), no. 2, p. 173-184.

[9] A. Kanemitsu, "Fano manifolds and stability of tangent bundles", *J. Reine Angew. Math.* 774 (2021), p. 163-183.

[10] N. Mok, "Geometric structures on uniruled projective manifolds defined by their varieties of minimal rational tangents", in *Differential geometry, mathematical physics, mathematics and society (II)*, Astérisque, vol. 322, Société Mathématique de France, 2008, p. 151-205.

[11] A. L. Onishchik, È. B. Vinberg, *Lie groups and algebraic groups*, Springer Series in Soviet Mathematics, Springer, 1990.

[12] B. Pasquier, "On some smooth projective two-orbit varieties with Picard number 1", *Math. Ann.* 344 (2009), no. 4, p. 963-987.

[13] T. Peternell, J. A. Wiśniewski, "On stability of tangent bundles of Fano manifolds with \(b_2 = 1\)", *J. Algebr. Geom.* 4 (1995), no. 2, p. 362-384.