Higher Order Derivatives in Costa’s Entropy Power Inequality

Fan Cheng, Member, IEEE and Yanlin Geng, Member, IEEE

Abstract

Let \( X \) be an arbitrary continuous random variable and \( Z \) be an independent Gaussian random variable with zero mean and unit variance. For \( t > 0 \), Costa proved that \( e^{2h(X + \sqrt{t}Z)} \) is concave in \( t \), where the proof hinged on the first and second order derivatives of \( h(X + \sqrt{t}Z) \). Specifically, these two derivatives are signed, i.e., \( \frac{\partial}{\partial t} h(X + \sqrt{t}Z) \geq 0 \) and \( \frac{\partial^2}{\partial t^2} h(X + \sqrt{t}Z) \leq 0 \). In this paper, we show that the third order derivative of \( h(X + \sqrt{t}Z) \) is nonnegative, which implies that the Fisher information \( J(X + \sqrt{t}Z) \) is convex in \( t \). We further show that the fourth order derivative of \( h(X + \sqrt{t}Z) \) is nonpositive. Following these results, we make two conjectures on \( h(X + \sqrt{t}Z) \): the first is that \( \frac{\partial}{\partial n} h(X + \sqrt{t}Z) \) is nonnegative in \( t \) if \( n \) is odd, and nonpositive otherwise; the second is that \( \log J(X + \sqrt{t}Z) \) is convex in \( t \). The concavity of \( h(\sqrt{tX + \sqrt{t}Z}) \) is studied, revealing its connection with Costa’s EPI.

Index Terms

Costa’s EPI, Differential entropy, Entropy power inequality, Fisher information, Heat equation.

I. INTRODUCTION

For a continuous random variable \( X \) with density \( g(x) \), the differential entropy is defined as

\[
h(X) := -\int_{-\infty}^{+\infty} g(x) \log g(x) dx,
\]

where \( \log \) is the natural logarithm. The Fisher information (e.g., p. 671, Cover [13]) is defined as

\[
J(X) := \int_{-\infty}^{+\infty} g(x) \left[ \frac{\partial}{\partial x} \log g(x) \right]^2 dx.
\]

The entropy power inequality (EPI) introduced by Shannon [1] states that for any two independent continuous random variables \( X \) and \( Y \),

\[
e^{2h(X+Y)} \geq e^{2h(X)} + e^{2h(Y)},
\]

where the equality holds if and only if both \( X \) and \( Y \) are Gaussian.

Shannon didn’t give a proof and there was a gap in his argument. The first rigorous proof was made by Stam in [2], where he applied an equality that connected the Fisher information and the differential entropy and the so-called Fisher information inequality (FII) was proved; i.e.,

\[
\frac{1}{J(X+Y)} \geq \frac{1}{J(X)} + \frac{1}{J(Y)}.
\]

Later, Stam’s proof was simplified by Blachman [3]. Zamir [4] proved the FII via a data processing argument in Fisher information. Lieb [5] showed an equivalent form of EPI and proved the equivalent form via Young’s inequality. Lieb’s argument has been widely used as a common step in the subsequent proofs of EPI. Recently, Verdú and Guo [6] gave a proof by invoking an equality that related the minimum mean square error estimation and the differential entropy. Rioul [7] devised a Markov chain on \( X, Y \) and the additive Gaussian noise, from which EPI can be proved via the data processing inequality and properties of mutual information.

There are several generalizations of EPI. Costa [8] proved that the entropy power \( e^{2h(X + \sqrt{t}Z)} \) is concave in \( t \), where the first and second order derivatives of \( h(X + \sqrt{t}Z) \) were obtained. Moreover, these two derivatives are signed, i.e., positive or negative. Villani [14] simplified the proof in [8] by using some advanced techniques as well as the heat equation noticed by [2], which is instrumental in our work. The generalization of EPI in matrix form was obtained in Zamir and Feder [9]. Liu and Viswanath [10] generalized EPI by considering a covariance-constrained optimization problem which was motivated by multi-terminal coding problems. Wang and Madiman [11] discussed EPI from the perspective of rearrangement.

F. Cheng is with the Institute of Network Coding, The Chinese University of Hong Kong, N.T., Hong Kong. Email: fcheng@inc.cuhk.edu.hk
Y. Geng is with the Department of Information Engineering, The Chinese University of Hong Kong, N.T., Hong Kong. Email: gengyanlin@gmail.com

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As one of the most important information inequalities, EPI has numerous proofs, generalizations and applications. The literature is so vast that instead of trying to be complete, we only mention the results that are most relevant to our discussion. A comprehensive survey can be found in [4], and the book by El Gamal and Kim [12] also serves as a very good repository.

In this paper, inspired by [8], we make some progress and introduce related conjectures which reveal even more fundamental facts about Gaussian random variables in the view of information theory. By harnessing the power of the techniques in [8], i.e., heat equation and integration by parts, we obtain the third and fourth order derivatives of $h(X + \sqrt{t}Z)$, which are also signed. Summarizing all the derivatives of $h(X + \sqrt{t}Z)$, we conjecture that $\frac{\partial^n}{\partial t^n} h(X + \sqrt{t}Z)$ is signed for any $n$. Corresponding to Costa’s EPI, we further conjecture that $\log J(X + \sqrt{t}Z)$ is convex in $t$. We investigate the concavity of $h(\sqrt{t}X + \sqrt{1-t}Z)$, showing that it is concave in $t$ and is equivalent to Costa’s EPI. The connection between the convexities of $J(\sqrt{t}X + \sqrt{1-t}Z)$ and $\log J(X + \sqrt{t}Z)$ is also revealed.

The paper is organized as follows. In Section II we introduce the background and the main result on derivatives. In Section III some preliminaries are stated. In Section IV and V the derivatives are verified. We discuss the uniqueness of the signed form in Section VI. The conjecture is introduced in Section VII. In Section VIII we prove an inequality which is equivalent to Costa’s EPI. We conclude the paper in Section IX.

II. THE HIGH-ORDER DERIVATIVES

Consider a random variable $X$ with density $g(x)$, and an independent standard Gaussian random variable $Z$, denoted as $Z \sim \mathcal{N}(0, 1)$. For $t \geq 0$, let

$$Y_t := X + \sqrt{t}Z.$$  

The density of $Y_t$ is

$$f(y, t) = \int_{-\infty}^{+\infty} g(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dx.$$  

**Notation:** For the derivatives, in addition to the usages of $f_{yy}$, $f_y$ and $\frac{\partial^2}{\partial y^2} f$, by $f^{(n)}$ we always mean

$$f^{(n)} := \frac{\partial^n}{\partial y^n} f.$$  

Sometimes, for ease of notation we also denote

$$f_n := f^{(n)} = \frac{\partial^n}{\partial y^n} f.$$  

The integration interval, usually $(-\infty, +\infty)$, will be omitted, unless it is not clear from the context.

In this paper, the main result is the following two theorems.

Theorem 1. For $t > 0$,

$$\frac{\partial^3}{\partial t^3} h(Y_t) = \frac{1}{2} \int f \left( \frac{f_3}{f} - \frac{f_1 f_2}{f^2} + \frac{1}{3} \frac{f_1^3}{f^3} \right)^2 + \frac{f_1^6}{45 f^5} dy. \tag{4}$$  

This implies that $J(Y_t)$ is convex in $t$.

Theorem 2. For $t > 0$,

$$\frac{\partial^4}{\partial t^4} h(Y_t)$$  

$$= -\frac{1}{2} \int f \left( \frac{f_4}{f} - \frac{6}{5} \frac{f_1 f_3}{f^2} - \frac{7}{10} \frac{f_2^2}{f^2} + \frac{8}{5} \frac{f_1 f_2}{f^3} - \frac{1}{2} \frac{f_1^4}{f^4} \right)^2$$  

$$+ f \left( \frac{2}{5} \frac{f_1 f_3}{f^2} - \frac{1}{3} \frac{f_2^2}{f^3} + \frac{9}{10} \frac{f_1^4}{f^4} \right)^2$$  

$$+ f \left( \frac{4}{100} \frac{f_1 f_2}{f^3} + \frac{4}{100} \frac{f_1^4}{f^4} \right)^2$$  

$$+ \frac{1}{300} \frac{f_2^2}{f^3} + \frac{56}{90000} \frac{f_1^4 f_2^2}{f^5} + \frac{13}{70000} \frac{f_1^8}{f^7} dy. \tag{5}$$  

This implies that $\frac{\partial^4}{\partial t^4} h(Y_t) \leq 0$.

Our work is highly related to the following theorem.

Theorem 3 (Costa’s EPI [8]). $e^{2h(Y_t)}$ is concave in $t$, where $t > 0$.  

There are several methods to prove Costa’s EPI, and a straightforward way is to calculate the first and second order derivatives of \( h(Y_t) \) and show some inequality holds. The expressions on \( \frac{\partial}{\partial t} h(Y_t) \) and \( \frac{\partial^2}{\partial t^2} h(Y_t) \) are already obtained in Lemma 1.

**Lemma 1.**

\[
\frac{\partial}{\partial t} h(Y_t) = \frac{1}{2} J(Y_t); \\
\frac{\partial^2}{\partial t^2} h(Y_t) = -\frac{1}{2} \int \left( \frac{f_{yy}}{f} - \frac{f_y^2}{f^2} \right)^2 \, dy. 
\]

The proof can be found in [8], [14]. The first equation is called de Bruijn’s identity in literature and is due to de Bruijn. Using Lemma 1 one can readily show that \( \frac{\partial}{\partial t} h(Y_t) \geq 0 \) and \( \frac{\partial^2}{\partial t^2} h(Y_t) \leq 0 \). In Theorem 1 [8] we have presented the expressions of \( \frac{\partial}{\partial t} h(Y_t) \) and showed that \( \frac{\partial^2}{\partial t^2} h(Y_t) \geq 0 \). A much more complicated result on \( \frac{\partial^2}{\partial t^2} h(Y_t) \) is stated in Theorem 2. We notice that in Guo et al. [16], the third and fourth order derivatives of \( h(\sqrt{t} X + Z) \) are obtained (Proposition 9), which are not signed. However, for \( h(X + \sqrt{t} Z) \), they are determined by Theorem 1 and 2.

In the next section, we introduce the necessary tools to prove Theorem 1 and 2.

### III. Preliminaries

The differential entropy and Fisher information may not be well defined due to the integration issue. In literature, there is no simple and general conditions which can guarantee their existence (c.f. [7]). In general, the behavior of the differential entropy and Fisher information may be unpredictable as shown by Wu and Verdù [15]. However, this work studies the higher order derivatives of \( h(Y_t) \), where \( t > 0 \) is imposed. Under this assumption, \( Y_t \) has some good properties; e.g., in [8], \( Y_t \) is proved to be infinitely differentiable everywhere.

#### A. Properties of \( f(y, t) \)

The following property is well known (e.g., Lemma 1, [7]).

**Proposition 1.** For any fixed \( t > 0 \) and any \( n \in \mathbb{Z}_+ \), all the derivatives \( f^{(n)}(y, t) \) exist, are bounded, and satisfy

\[
\lim_{|y| \to \infty} f^{(n)}(y, t) = 0.
\]

The following property is used repeatedly in the rest of the paper, for dealing with integration by parts. The proof is presented in Appendix A.

**Proposition 2.** For any \( r, m_i, k_i \in \mathbb{Z}_+ \), the following integral exists

\[
\int f \prod_{i=1}^{r} \left( \frac{f^{(m_i)}(y)}{f^{k_i}} \right) \, dy.
\]

In particular, this implies that

\[
\lim_{|y| \to \infty} \int f \prod_{i=1}^{r} \left( \frac{f^{(m_i)}(y)}{f^{k_i}} \right) \, dy = 0. 
\]

#### B. The “heat equation”

For a Gaussian random variable \( \hat{X} \sim \mathcal{N}(\mu, \sigma^2) \) with density function \( \hat{f}(x) \), one can show that the following “heat equation” holds

\[
\frac{\partial}{\partial (\sigma^2)^t} \hat{f} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \hat{f}.
\]

The “heat equation” also holds for \( Y_t \) [2], and was used by [14] to simplify Costa’s proof.

**Lemma 2.**

\[
\frac{\partial}{\partial t} f(y, t) = \frac{1}{2} \frac{\partial^2}{\partial y^2} f(y, t). 
\]

**Proof:** The proof is known in literature and we present it here for completeness. By some calculus,

\[
f_t = \int g(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \left( \frac{1}{2} \frac{1}{t^2} - \frac{1}{2t} \right) \, dx,
\]

\[
f_y = \int g(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \left( -\frac{1}{t} (y-x) \right) \, dx,
\]

\[
f_{yy} = \int g(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \left[ \left( \frac{1}{t} (y-x) \right)^2 - \frac{1}{t} \right] \, dx.
\]

By comparing \( f_{yy} \) with \( f_t \), the lemma can be proved.
C. Proof to Lemma 1

The proof to Lemma 1 is known in literature. Here we slightly modify the proof, so that the idea carries over to the proof of Theorem 1 and even the cases with higher-order derivatives.

Proof: For the first order derivative we have

$$\frac{\partial}{\partial t} h(Y_t) = \frac{\partial}{\partial t} \left[ - \int f(y, t) \log f(y, t) dy \right]$$

$$= - \int f_t (1 + \log f) dy$$

$$= - \int \frac{1}{2} f_{yy} (1 + \log f) dy$$

$$= - \frac{1}{2} \int (1 + \log f) dy$$

$$\overset{(a)}{=} - \frac{1}{2} f_y (1 + \log f) \bigg|_{y = -\infty}^{+\infty} + \frac{1}{2} \int \frac{f_y^2}{f} dy$$

$$\overset{(b)}{=} 0 + \frac{1}{2} \int \frac{f_y^2}{f} dy$$

$$= \frac{1}{2} J(Y_t).$$

In (a) we apply integration by parts. In (b) the limits are zero, because $f_y (1 + \log f) = \frac{f_y}{\sqrt{f}} (\sqrt{f} + \sqrt{f} \log f)$, where $\frac{f_y^2}{f} \to 0$ from Proposition 2, $\sqrt{f} \to 0$ as $|y| \to \infty$, and $\sqrt{f} \log f \to 0$ because $x \log x \to 0$ as $x \to 0$.

For the second order derivative, similarly

$$2 \frac{\partial^2}{\partial t^2} h(Y_t) = \int \frac{f_y f_{yy} f - f_y^2 f_y}{f^2} dy$$

$$\overset{(a)}{=} \int \frac{f_y f_{yy} f}{f^2} - \frac{f_y^2 f_y}{2f^2} dy.$$

For the second term

$$\int \frac{f_y^2 f_{yy}}{f^2} dy = \int \frac{f_y^2}{f^2} df_y$$

$$= - \frac{f_y^3}{f^2} \bigg|_{y = -\infty}^{+\infty} + \int \frac{f_y^2 f_y}{f} \frac{f_y}{f} f - f_y f_y}{f^2} dy$$

$$\overset{(a)}{=} 0 - 2 \int \frac{f_y^2 f_{yy}}{f^2} dy + 2 \int \frac{f_y^4}{f^3} dy.$$

Hence

$$\int \frac{f_y^2 f_{yy}}{f^2} dy = \int \frac{2f_y^4}{f^3} dy.$$ (10)

For the first term

$$\int \frac{f_y f_{yy} f}{f} dy = \int \frac{f_y}{f} df_y$$

$$= \frac{f_y f_{yy}}{f} \bigg|_{y = -\infty}^{+\infty} - \int \frac{f_y f_{yy}}{f^2} f^2 dy$$

$$\overset{(a)}{=} 0 + \int \frac{-f_{yy}^2}{f} + \frac{f_y^2 f_{yy}}{f^2} dy$$

$$\overset{(b)}{=} \int \frac{-f_{yy}^2}{f} + \frac{2f_y^4}{3f^3} dy.$$ (11)

Combining these two terms we have

$$2 \frac{\partial^2}{\partial t^2} h(Y_t) = \int \frac{f_y f_{yy} f - f_y^2 f_y}{f^2} dy$$

$$\overset{(11)(10)}{=} \int - \frac{f_{yy}^2}{f} + \frac{f_y^4}{3f^3} dy.$$ (12)
Now it suffices to show that the right-hand side term in (7) has the same form:

\[- \int f \left( \frac{f_y}{f} \right)^2 \, dy = \int -\frac{f_y^2}{f} + \frac{2f_y^2 f_{yy}}{f^2} - \frac{f_y^4}{f^3} \, dy\]

\[\int -\frac{f_y^2}{f} + \frac{4f_y^4}{3f^3} - \frac{f_y^4}{f^3} \, dy = (12).\]

Thus the proof is finished.

One may notice that we first use the heat equation to deal with \( f_t \), then apply integration by parts to eliminate those terms whose highest-order derivatives have power one. Equation (11) explains this elimination, as one can see that in the final expression the highest-order derivatives are \( f_{yy}^2 \) and \( f_y^2y \), whose powers are bigger than one.

IV. PROOF TO THEOREM 1

The following lemma is instrumental in proving Theorem 1.

Lemma 3.

\[\int \frac{f_x^4 f_y}{f^4} \, dy = \int \frac{4f_x^6}{5f^5} \, dy\]  
(13)

\[\int \frac{f_x^3 f_y^3}{f^4} \, dy = \int -\frac{3f_x^2 f_y^2}{f^3} + \frac{12f_x^6}{5f^5} \, dy\]  
(14)

\[\int \frac{f_x f_y^3}{f^2} \, dy = \int -\frac{f_x^2}{2f^2} + \frac{f_x^2 f_x^2}{f^3} \, dy\]  
(15)

\[\int \frac{f_x^3 f_y^3}{f^4} \, dy = \int -\frac{f_x^2}{2f^2} - \frac{f_x^2 f_x^2}{f^3} \, dy\]  
(16)

\[\int \frac{f_x^4 f_y}{f^4} \, dy = \int \frac{4f_x^6}{1f^5} \, dy\]  
(9)

\[\int \frac{f_x^3 f_y^3}{f^4} \, dy = \int \frac{f_x^1 f_y^3}{f^2} \, dy\]  
(12)

**Proof:** See Appendix C.

This lemma is similar to what we did in equations (10) and (11); for the terms on the left-hand side, the highest-order derivatives have power one; while for the right-hand side, they are bigger than one.

Next, we prove Theorem 1.

**Proof:** From (12)

\[2 \frac{\partial^2 h(Y_t)}{\partial t^2} = \int -\frac{f_y^2}{f} + \frac{f_y^4}{3f^3} \, dy = \int -\frac{f_y^2}{f} + \frac{f_y^4}{3f^3} \, dy.\]

Thus

\[2 \frac{\partial^3 h(Y_t)}{\partial t^3} = \int \left( -\frac{f_y^2}{f} + \frac{f_y^4}{3f^3} \right) \, dy.\]

By repeatedly applying the heat equation,

\[\int \left( \frac{f_y^2}{f} \right)_t \, dy = \int \frac{2f_y f_y f - f_y^2 f_t}{f^2} \, dy\]

\[\int \frac{2f_y f_y f - f_y^2 f_t}{f^2} \, dy = \int \frac{f_y f_y}{f} - \frac{f_y^2}{2 f^2} \, dy\]

\[\int \left( \frac{f_y^3}{3f^3} \right)_t \, dy = \int \frac{4f_y^3 f_y f^3 - f_y^4 f_y^2 f_t}{3f^6} \, dy\]

\[\int \frac{4f_y^3 f_y f^3 - f_y^4 f_y^2 f_t}{3f^6} \, dy = \int \frac{2f_y^3 f_y}{3f^3} - \frac{f_y^4 f_y}{2f^4} \, dy.\]
Substitute these terms and use Lemma 3:

\[
2 \frac{\partial^3 h(Y_t)}{\partial t^3} = \int \left( \frac{-f_2 f_4 + f_3^2}{f} \right) + \left( \frac{2 f_1^3 f_3 - f_1^4 f_2}{3 f^3} \right) \, dy
\]

Lemma 3:

\[
\int - \left( \frac{f_2^3}{f} - \frac{f_3}{2 f_s^2} + \frac{f_2 f_2^2}{f^3} \right) \, dy
\]

\[
+ \frac{2}{3} \left( -3 f_1^2 f_2^2 + 12 f_1^6 \right) - \frac{1}{2} \left( \frac{4 f_1^6}{5 f^5} \right) \, dy
\]

\[
= \int \frac{f_2^3}{f} + \frac{f_2^3}{f^2} - \frac{3 f_1^2 f_2}{f^3} + \frac{6 f_1^6}{5 f^5} \, dy
\]

(17)

Then we do the same manipulations to \(2 \frac{\partial^3 h(Y_t)}{\partial t^3} \) in Theorem 1. That is, applying Lemma 3 to the corresponding terms and we have:

\[
\int f \left( \frac{f_3}{f} - \frac{f_1 f_2}{f^2} + \frac{1 f_1^3}{3 f^3} \right)^2 + \frac{f_1^6}{45 f^5} \, dy
\]

\[
= \int \frac{f_2^3}{f} + \frac{f_1^2 f_2^2}{f^3} + \frac{f_1^6}{9 f^5} - \frac{2 f_1 f_2 f_3}{f^2} + \frac{2 f_1^3 f_3}{3 f^3}
\]

\[
+ \frac{2 f_1^6}{3 f^4} - \frac{2 f_1^3 f_2}{3 f^4} + \frac{f_1^6}{45 f^5} \, dy
\]

Lemma 3:

\[
\int \frac{f_2^3}{f} + \frac{f_1^2 f_2^2}{f^3} + \frac{6 f_1^6}{45 f^5} - 2 \left( \frac{f_2^3}{2 f^2} + \frac{f_2 f_2^3}{f^3} \right)
\]

\[
+ \frac{2}{3} \left( -3 f_1^2 f_2^2 + 12 f_1^6 \right) - \frac{2}{3} \left( \frac{4 f_1^6}{5 f^5} \right) \, dy
\]

\[
= \int \frac{f_2^3}{f} - \frac{3 f_1 f_2^2}{f^3} + \frac{54 f_1^6}{45 f^5} + \frac{f_2^3}{f^2} \, dy
\]

(17)

Thus the expression is proved.

Finally,

\[
\frac{\partial^2}{\partial t^2} J(Y_t) = 2 \frac{\partial^3}{\partial t^3} h(Y_t) \geq 0,
\]

which means \( J(Y_t) \) is convex in \( t \).

V. PROOF TO THEOREM 2

The proof is the same as that to Theorem 1 except there are more manipulations. The following lemma is instrumental in proving Theorem 2.
Lemma 4.

\[ \int \frac{f_1^6 f_2}{f_0} dy = \int \frac{6 f_1^8}{7 f_1^2} dy \]  
(18)

\[ \int \frac{f_1^5 f_3}{f_5} dy = \int -\frac{5 f_1^4 f_2^2}{f_5} + \frac{30 f_1^8}{7 f_1^2} dy \]  
(19)

\[ \int \frac{f_1 f_2 f_3}{f_4} dy = \int \frac{2 f_1^2 f_2}{2 f_4} + \frac{2 f_1^2 f_2^2}{f_4} dy \]  
(20)

\[ \int \frac{f_1 f_2^3 f_3}{f_3} dy = \int -\frac{f_1^3 f_2^3}{3 f_3} + \frac{f_1^3 f_2^3}{f_3} dy \]  
(21)

\[ \int \frac{f_1 f_2 f_4}{f_4} dy = \int \frac{6 f_1^2 f_3^2}{f_4} - \frac{28 f_1^2 f_2^2}{f_5^2} + \frac{120 f_1^8}{7 f_1^2} dy \]  
(22)

\[ \int \frac{f_1 f_2 f_4}{f_3} dy = \int \frac{2 f_1^4 f_3^2}{3 f_3^3} - \frac{13 f_1^2 f_2^3}{2 f_4} \]  
\[ - \frac{f_1^3 f_2^3}{f_3^2} + \frac{6 f_1^3 f_2^3}{f_5^2} dy \]  
(23)

\[ \int \frac{f_2 f_4}{f_2} dy = \int -\frac{2 f_2 f_3^2}{f_2} - \frac{2 f_2^4}{3 f_3} + \frac{2 f_1^2 f_2^2}{f_4} dy \]  
(24)

\[ \int \frac{f_1 f_3 f_4}{f_2} dy = \int \frac{f_2 f_3^2}{2 f_2} + \frac{f_1^3 f_3^2}{f_3} dy \]  
(25)

\[ \int \frac{f_3 f_5}{f} dy = \int -\frac{f_2 f_3^2}{2 f^2} + \frac{f_1^3 f_3^2}{f_3} dy \]  
(26)

\[ \frac{2 \partial^4 h(Y_i)}{\partial t^4} = \int \left( \frac{f_1^2}{f} + \frac{f_2^3}{f^2} - \frac{3 f_1^2 f_2^2}{f_3} + \frac{6 f_1^6}{5 f_5} \right) dy \]

Proof: We first apply the heat equation:

\[ \int \left( \frac{f_2^2}{f} \right)_t dy = \int \frac{2 f_2 f_3 f_4 f - f_2^2 f_1}{f^2} dy \]
\[ \int \left( \frac{f_2 f_3}{f^2} \right)_t dy = \int \frac{3 f_2^2 f_4 f^2 - f_2^3 f f_4}{f^4} dy \]
\[ \int \left( \frac{f_2}{f} \right)_t dy = \int \frac{f_3 f_4}{f^3} \]

\[ \int \left( \frac{f_2^2}{f_3} \right)_t dy = \int 6 f_1 f_2 (f_1 f_2 + f_1 f_2) f^3 - 3 f_1^2 f_2^2 s f_1 dy \]
\[ \int \frac{3 f_1 f_2 (f_3 f_2 + f_1 f_4) f^3 - 9 f_1^2 f_2^2 f_2}{f^6} dy \]
\[ = \int \frac{3 f_1 f_2 f_3 f_4}{f^3} + \frac{3 f_1^2 f_2 f_4}{f^3} - \frac{9 f_1^2 f_2^2}{2 f^2} dy \]
\[
\int \left( \frac{6f^6}{5f^5} \right) \, dy = \int \frac{36f^5 f_1 f_5 - 6f^6 5f^4 f_t}{5f^{10}} \, dy
\]
\[
= \int \frac{18f^5 f_3 f^5 - 15f^6 f_1 f_5}{5f^{10}} \, dy
\]
\[
= \int \frac{18f^5 f_3}{5f^5} = \frac{3f^6 f_2}{f^6} \, dy
\]

Substitute these terms and use Lemma 4:

\[
2 \frac{\partial^4 h(Y_t)}{\partial t^4}
\]
\[
= \int \left( \frac{f_3 f_5}{f} - \frac{f_2 f_3}{2f^2} \right) + \left( \frac{3f_2 f_4}{2f^2} - \frac{f_2}{f^2} \right)
\]
\[
- \left( \frac{3f_1 f_2^2 f_3}{f^3} + \frac{3f_1 f_2^2 f_3}{f^3} - \frac{9f_1 f_2^3}{2f^4} \right)
\]
\[
+ \left( \frac{18f_3^4 f_5}{5f^5} - \frac{3f_1 f_2}{f^6} \right) \, dy
\]

Lemma 4
\[
\int \left( \frac{f_2^4}{f^2} - \frac{f_2 f_3}{2f^2} + \frac{f_1 f_3^3}{f^4} \right) - \frac{f_2 f_3}{2f^2}
\]
\[
+ \frac{3}{2} \left( - \frac{2f_2 f_3^2}{f^2} - \frac{2f_4^2}{3f^3} + \frac{2f_1 f_3^3}{f^4} \right) - \frac{f_2}{f^3}
\]
\[
- 3 \left( \frac{f_2^4}{3f^3} + \frac{f_2 f_3^3}{f^4} \right)
\]
\[
- 3 \left( \frac{f_1 f_3^3}{2f^3} - \frac{13f_1 f_3}{2f^4} - \frac{6f_1 f_2^2}{f^5} \right)
\]
\[
+ \frac{9f_2^4 f_3^2}{2f^4} + 18 \left( - \frac{5f_1 f_2^2}{f^5} + \frac{30f_1^4}{7f^7} \right) - 3 \left( \frac{6f_1^8}{7f^7} \right) \, dy
\]
\[
= \int \frac{f_2^4}{f} + \left( - \frac{1}{2} - 1 - 3 \right) \frac{f_2 f_3}{f^2} + \left( 1 + 3 \right) \frac{f_2 f_3^3}{f^4}
\]
\[
+ \left( -1 - 1 + 1 - 2 \right) \frac{f_2^4}{f^3} + \left( 3 - 3 + \frac{39}{2} + \frac{9}{2} \right) \frac{f_2 f_3^3}{f^4}
\]
\[
+ \left( -18 - 18 \right) \frac{f_1 f_2 f_3^3}{f^5} + \left( -18 - 18 \right) \frac{f_3}{f^2} \, dy
\]
\[
= \int \frac{f_2^4}{f} - \frac{4f_2 f_3^3}{f^2} + \frac{4f_2 f_3^3}{f^3} - \frac{3f_2^4}{f^3} + \frac{24f_2^4 f_3^3}{f^5}
\]
\[
- \frac{36f_1 f_2^2}{f^5} + \frac{90f_1^4}{7f^7} \, dy
\]  

(27)

Then we do the same manipulations to \( 2 \frac{\partial^4 h(Y_t)}{\partial t^4} \) in Theorem 2. That is, applying Lemma 4 to the corresponding terms.
To simplify the calculation, we first consider the following general expression

\[
\int f \left( x_0 f_4 + x_1 f_1 f_3 + x_2 f_2 f_3 + x_3 f_3^2 + x_4 f_4^2 \right)^2 \, dy
\]

\[
= \int x_0^2 f_1^2 f_3^2 + x_1^2 f_1 f_3 f_4 + x_2^2 f_2 f_4 + x_3 f_3^2 f_4 + x_4 f_4^4 \, dy
\]

\[
+ 2x_0 x_1 f_1 f_3 f_4 + 2x_0 x_2 f_2 f_4 + 2x_0 x_3 f_3^2 f_4 + 2x_0 x_4 f_4^3
\]

\[
+ 2x_1 x_2 f_1 f_3 f_4 + 2x_1 x_3 f_1 f_2 f_3 + 2x_1 x_4 f_1 f_2 f_3 + 2x_2 x_3 f_2 f_3 + 2x_2 x_4 f_2 f_3 + 2x_3 x_4 f_3 f_4 + 2x_4 f_4^4 \, dy
\]

(28)
With this general simplification, we have

\[
\int f \left( \frac{f_4}{f} - \frac{6f_1f_3}{f^2} - \frac{7f_2^2}{10f^2} + \frac{8f_1^2f_2}{5f^3} - \frac{1f_1^4}{2f^4} \right)^2 dy
\]

\[
= \int \frac{f_4^2}{f} + \left( (-\frac{6}{5})^2 + 2(-\frac{6}{5}) - 2(\frac{8}{5}) \right) f_2^2 f_3 f_4^3 + \left( (-\frac{7}{10})^2 - 4(-\frac{7}{10}) + \frac{4}{3}(\frac{8}{5}) \right) f_2^2 f_3^2 f_4^3 - \frac{2}{3}(-\frac{6}{5})(-\frac{7}{10}) f_2^2 f_3^3 + \left( \frac{8}{5})^2 + 12(\frac{8}{5}) - 56(-\frac{1}{2}) + 4(-\frac{6}{5})(\frac{8}{5}) \right) f_2^2 f_3^3 f_4^3 - 10(-\frac{6}{5})(-\frac{1}{2}) + 2(-\frac{7}{10})(-\frac{1}{2}) f_2^2 f_3^3 f_4^3 + \left( (-\frac{1}{2})^2 + \frac{240}{7}(-\frac{1}{2}) + \frac{60}{7}(-\frac{6}{5})(-\frac{1}{2}) \right) + \frac{12}{7}(-\frac{1}{2}) f_3 f_4^3 + \left( (-\frac{6}{5})^2 - 4(-\frac{7}{10}) \right) f_2^2 f_3^3 f_4^3 + \left( (-\frac{7}{10})^2 - 4(-\frac{7}{10}) + \frac{4}{3}(\frac{8}{5}) \right) f_3^2 f_4^3 f_5 + \left( 4(-\frac{7}{10}) - 13(\frac{8}{5}) + 12(-\frac{1}{2}) + 2(-\frac{6}{5})(-\frac{7}{10}) \right) f_2^2 f_3^2 f_4^3 f_5 - 3(-\frac{6}{5})(\frac{8}{5}) + 2(-\frac{7}{10})(\frac{8}{5}) f_2^2 f_3^2 f_4^3 f_5 \cdot dy
\]

\[
= \int \frac{f_4^2}{f} + \frac{104f_1f_3}{25f^3} + \frac{899f_2^2}{300f^3} + \frac{1839f_1^2f_2}{50f^5} - \frac{1837f_3^3}{140f^7} + \frac{4f_2^2f_3^2}{100f^4} - \frac{122f_2f_3^2}{5f^4} \cdot dy
\]

\[
(29)
\]

\[
\int f \left( \frac{2}{5} \frac{f_1f_3}{f^2} - \frac{1}{3} \frac{f_2^2}{f^3} + \frac{9}{100} \frac{f_1^4}{f^4} \right)^2 dy
\]

\[
= \int \frac{2}{5} \frac{f_1f_3}{f^2} + \left( \frac{1}{3} \frac{f_2^2}{f^3} \right) \frac{f_1f_3}{f^2} + \left( \frac{9}{100} \frac{f_1^4}{f^4} \right) \frac{f_1f_3}{f^2} + \left( \frac{9}{100} \frac{f_1^4}{f^4} \right) \frac{f_2^2}{f^3} + \left( \frac{9}{100} \frac{f_1^4}{f^4} \right) \frac{f_2^2}{f^3} \cdot dy
\]

\[
= \int \frac{4f_1f_3^2}{25f^3} - \frac{1}{25f^3} + \frac{1}{25f^3} + \frac{1}{25f^3} + \frac{1}{25f^3} \cdot dy
\]

\[
= \int \frac{4f_1f_3^2}{25f^3} - \frac{1}{25f^3} + \frac{1}{25f^3} + \frac{1}{25f^3} + \frac{1}{25f^3} \cdot dy
\]

\[
(30)
\]

\[
\int f \left( -\frac{4}{100} \frac{f_2f_4}{f^3} + \frac{4}{100} \frac{f_1^4}{f^4} \right)^2 dy
\]

\[
= \int \left( -\frac{4}{100} \frac{f_2f_4}{f^3} \right) f_1f_3 \left( -\frac{4}{100} \frac{f_2f_4}{f^3} \right) \frac{f_1f_3}{f^2} + \left( -\frac{4}{100} \frac{f_2f_4}{f^3} \right) f_2^2 \left( -\frac{4}{100} \frac{f_2f_4}{f^3} \right) \frac{f_1f_3}{f^2} \cdot dy
\]

\[
= \int \frac{16f_1f_2^3}{10000f^5} - \frac{80f_1^8}{70000f^7} \cdot dy
\]

\[
(31)
\]
By (29), (30) and (31)

\[- \int f \left( \frac{f_4}{f} - \frac{6 f_1 f_3}{5 f^2} - \frac{7 f_2^2}{10 f^2} + \frac{8 f_1 f_2}{5 f^3} - \frac{1 f_4^2}{2 f^4} \right)^2
+ f \left( \frac{2 f_1 f_3}{5 f^2} - \frac{1 f_1 f_2}{3 f^3} + \frac{9 f_4}{100 f^4} \right)^2
+ f \left( - \frac{4}{100} f_1 f_2 + \frac{4 f_4^2}{100 f^4} \right)^2
+ \frac{1}{300} \frac{f_2}{f^3} + \frac{56}{90000} \frac{f_1 f_2}{f^5} + \frac{13 f_4^2}{70000 f^7} \right) dy
+ \int \left( \frac{f_2}{f} - \frac{104 f_2^2}{25 f^3} + \frac{389 f_2^4}{300 f^5} + \frac{1839 f_2^4 f_4^2}{50 f^7} \right)
+ \frac{1 f_2^4}{300 f^3} + \frac{56}{90000} \frac{f_4}{f^5} + \frac{13 f_4^2}{70000 f^7} \right) dy
+ \left( \frac{16 f_2^4 f_4^2}{10000 f^5} - \frac{80 f_4^2}{70000 f^7} \right)
+ \frac{1 f_2^4}{300 f^3} + \frac{56}{90000} \frac{f_4}{f^5} + \frac{13 f_4^2}{70000 f^7} \right) dy
+ \left( \frac{1839 f_2^4}{50} - \frac{704}{900} + \frac{16}{10000} \frac{f_4}{f^5} \right)
+ \left( \frac{1837}{140} + \frac{18567}{70000} - \frac{80}{70000} + \frac{13 f_4^2}{70000} \right)
+ \frac{4 f_2 f_4^2}{f^2} + \left( \frac{122}{5} + \frac{2 f_2^3 f_4^2}{f^4} \right) \right) dy
+ \frac{90 f_4^2}{7 f^2} + \frac{4 f_2^3 f_4^2}{f^4} - \frac{24 f_2^3 f_4^2}{f^4} \right) dy
= (27),
\]
which completes the proof of Theorem 2

VI. ALTERNATIVE SIGNED REPRESENTATIONS

In this section, we discuss alternative signed representations of \( \frac{\partial^2}{\partial t^2} h(Y_t) \) in Lemma 1, Theorem 1 and Theorem 2. For the first order derivative, the representation is unique due to its simplicity. For the second and third order derivatives, we have the following alternative representations stated in Corollary 1 and 2. The proof of Corollary 1 though simple, contains the idea of how we obtain the formulae in Theorem 1 and 2.

Corollary 1.

\[
\frac{\partial^2}{\partial t^2} h(Y_t) = -\frac{1}{2} \int f \left( \alpha \frac{f_2}{f} + \beta \frac{f_1^2}{f^2} \right)^2 + f \left( \gamma \frac{f_1^3}{f^2} \right)^2
+ (1 - \alpha^2) \frac{f_2}{f} + (-\beta^2 - \gamma^2 - 4 \alpha \beta - \frac{1}{3} \frac{f_4}{f^3}) \right) dy
\]
where

\[
1 - \alpha^2 \geq 0
- \beta^2 - \gamma^2 - 4 \alpha \beta - \frac{1}{3} \geq 0
\]

(32)
One set of solution is

$$\alpha = 1, \quad \gamma = 0, \quad -1 \leq \beta \leq -\frac{1}{3},$$

where the case $\beta = -1$ corresponds to the result in Lemma 1.

**Proof:** After applying the heat equation, the orders of derivatives in each term of $\frac{\partial^2}{\partial t^2} h(Y_t)$ have sum equals four. Thus we consider expressing the second derivative as

$$2 \frac{\partial^2}{\partial t^2} h(Y_t) = - \sum_i \int f \left( \alpha_i \frac{f_i}{f^2} + \beta_i \frac{f_i^2}{f^2} \right)^2 dy,$$

where $\alpha_i$ and $\beta_i$ are coefficients. Since for the reals $A, B, C, a, b$, the following equality holds

$$(aA + B)^2 + (bA + C)^2 = \left( \sqrt{a^2 + b^2}A + \frac{a}{\sqrt{a^2 + b^2}}B + \frac{b}{\sqrt{a^2 + b^2}}C \right)^2 + \left( \frac{b}{\sqrt{a^2 + b^2}}B - \frac{a}{\sqrt{a^2 + b^2}}C \right)^2,$$

it suffices to consider the following expression

$$2 \frac{\partial^2}{\partial t^2} h(Y_t) = - \int f \left( \alpha \frac{f_2}{f} + \beta \frac{f_1^2}{f^2} \right)^2 + f \left( \gamma \frac{f_1}{f^2} \right)^2 dy.$$

Now similar to the proof to Theorem 1

$$- \int f \left( \alpha \frac{f_2}{f} + \beta \frac{f_1^2}{f^2} \right)^2 + f \left( \gamma \frac{f_1}{f^2} \right)^2 dy = - \int \alpha^2 \frac{f_2^2}{f} + 2\alpha\beta \frac{f_2 f_1}{f^2} + (\beta^2 + \gamma^2) \frac{f_1^4}{f^3} dy \quad \text{[10]} - \int \alpha^2 \frac{f_2^2}{f} + (\beta^2 + \gamma^2 + \frac{4}{3} \alpha \beta) \frac{f_1^4}{f^3} dy.$$

Comparing with (12), one obtains

$$2 \frac{\partial^2}{\partial t^2} h(Y_t) = - \int f \left( \alpha \frac{f_2}{f} + \beta \frac{f_1^2}{f^2} \right)^2 + f \left( \gamma \frac{f_1}{f^2} \right)^2 + (1 - \alpha^2) \frac{f_2^2}{f} + (-\beta^2 - \gamma^2 - \frac{4}{3} \alpha \beta - \frac{1}{3}) \frac{f_1^4}{f^3} dy.$$

To show that the second derivative is negative, one requires

$$1 - \alpha^2 \geq 0$$

$$-\beta^2 - \gamma^2 - \frac{4}{3} \alpha \beta - \frac{1}{3} \geq 0$$

And it is easy to verify the set of solution

$$\alpha = 1, \quad \gamma = 0, \quad -1 \leq \beta \leq -\frac{1}{3}.$$

For the third derivative, similar to Corollary 1 one could determine the coefficients $c_i$ in the following

$$2 \frac{\partial^3}{\partial t^3} h(Y_t) = \int f \left( c_0 \frac{f_3}{f} + c_1 \frac{f_1 f_2}{f^2} + c_2 \frac{f_1^3}{f^3} \right)^2 + f \left( c_3 \frac{f_1 f_2}{f^2} + c_4 \frac{f_1^3}{f^3} \right)^2 + f \left( c_5 \frac{f_1}{f^2} \right)^2 dy.$$

Since there is no essential difference, we would not present the general expression for the third derivative, but just prove the following corollary.
Corollary 2.

\[ \frac{\partial^3}{\partial t^3} h(Y_t) = \frac{1}{2} \int f \left( \frac{f_3}{f} - \frac{f_1 f_2}{f^2} + \frac{\beta f_3^3}{f^3} \right)^2 \]
\[ + (6\beta - 2) \frac{f_1^2 f_2^2}{f^3} + (\frac{6}{5} - \frac{16}{5}\beta - \beta^2) \frac{f_1^6}{f^3} \text{d}y, \]

where \( \frac{1}{3} \leq \beta \leq \frac{-8 + \sqrt{74}}{5}. \)

Proof: We have

\[ \int f \left( \frac{f_3}{f} - \frac{f_1 f_2}{f^2} + \frac{\beta f_3^3}{f^3} \right)^2 \]
\[ + (\frac{6}{5} - \frac{16}{5}\beta - \beta^2) \frac{f_1^6}{f^3} \text{d}y \]
\[ = \int \frac{f_3^2}{f} + (6\beta - 1) \frac{f_1^2 f_2^2}{f^3} + (\frac{6}{5} - \frac{16}{5}\beta) \frac{f_1^6}{f^5} \]
\[ - 2 \frac{f_1 f_2 f_3}{f^2} + 2\beta \frac{f_3^3}{f^3} \text{d}y \]
\[ = \int \frac{f_3^2}{f} + (6\beta - 1) \frac{f_1^2 f_2^2}{f^3} + (\frac{6}{5} - \frac{16}{5}\beta) \frac{f_1^6}{f^5} \]
\[ - 2 \frac{f_1 f_2 f_3}{f^2} + 2\beta \left( 4 \frac{f_1^6}{f^5} \right) \text{d}y \]
\[ = \int \frac{f_3^2}{f} - 3 \frac{f_1^2 f_2^2}{f^3} + \frac{6}{5} \frac{f_1^6}{f^5} + \frac{f_3^3}{f^2} \text{d}y \]
\[ = (17). \]

The interval of \( \beta \) ensures that the coefficients are positive.

For the second and third order derivatives of \( h(Y_t) \), the representations can be obtained by hand. For the fourth order derivative, we consider the following representation

\[ 2 \frac{\partial^4}{\partial t^4} h(Y_t) \]
\[ = - \int f \left( c_0 \frac{f_4}{f} + c_1 \frac{f_1 f_3}{f^2} + c_2 \frac{f_2^2}{f^2} + c_3 \frac{f_3 f_2}{f^3} + c_4 \frac{f_1^4}{f^4} \right)^2 \]
\[ + f \left( c_5 \frac{f_1 f_3}{f^2} + c_6 \frac{f_2^2}{f^2} + c_7 \frac{f_3 f_2}{f^3} + c_8 \frac{f_1^4}{f^4} \right)^2 \]
\[ + f \left( c_9 \frac{f_2^2}{f^2} + c_{10} \frac{f_1 f_2}{f^2} + c_{11} \frac{f_1^4}{f^4} \right)^2 \]
\[ + f \left( c_{12} \frac{f_1 f_3}{f^2} + c_{13} \frac{f_1^4}{f^4} \right)^2 + f \left( c_{14} \frac{f_1^4}{f^4} \right)^2 \text{d}y. \]

By (28), we can obtain some constraints similar to (32), and finally find the feasible set of coefficients in Theorem 2 by numerical methods. The process is much more complicated, and we would not present it here.

The general pattern for the signed form of the \( n \)-th order derivative is that, first we need to find all the partitions of \( n \), and then each partition is an item in the squares. But the exact coefficients are hard to obtain. One can reuse the same idea to deal with the fifth-order derivative, or even higher. However, the manipulation is huge and hence it is prohibitive in computational cost, unless one can find some patterns for the coefficients in the signed representations.

VII. conjectures

Motivated by Theorem 1–3, we would like to introduce the following conjectures.

Conjecture 1. The \( n \)-th order derivative of \( h(Y_t) \) satisfies

1. \( \frac{\partial^n}{\partial t^n} h(Y_t) \leq 0 \) when \( n \) is even;
2. \( \frac{\partial^n}{\partial t^n} h(Y_t) \geq 0 \) when \( n \) is odd;
Fig. 1. $\frac{1}{J(Y_t)}$ is neither concave nor convex.

i.e., $\frac{\partial^n}{\partial t^n} h(Y_t)$ is either convex or concave in $t$ for a fixed $n$.

It is easy to see that when $X$ is Gaussian, the above conjectures hold. Conjecture 1 speculates that for a fixed $n$, the convexity or concavity of $\frac{\partial^n}{\partial t^n} h(Y_t)$ remains as if $X$ is Gaussian. Conjecture 1 has been verified when $n \leq 2$ in literature (Lemma 1), and for $n = 3, 4$ by Theorem 1 and 2.

The second conjecture is on the log-convexity of Fisher information. From the grand picture of differential entropy and Fisher information, nearly every result on different entropy has a counterpart in Fisher information, e.g., Shannon EPI and FII, the concavity of $h(Y_t)$ and the convexity of $J(Y_t)$ as well as de Bruijn’ identity. Corresponding to Costa’s EPI, there may be a strengthened convexity of $J(Y_t)$.

**Conjecture 2 (log-convex).** $\log J(Y_t)$ is convex in $t$.

When $X$ is standard white Gaussian, $J(Y_t) = \frac{1}{t+1}$. We may speculate $\frac{1}{J(Y_t)}$ or $\log J(Y_t)$ is convex in $t$. Simulations show that $\frac{1}{J(Y_t)}$ is neither convex nor concave. Fig. 1 illustrates an example of $\frac{1}{J(Y_t)}$, where $X$ is mixed Gaussian with p.d.f.

\[ g(x) = 0.5f_G(0, 0.1) + 0.5f_G(10, 0.1) \]

and $f_G(\mu, \sigma^2)$ is the p.d.f. of Gaussian $N(\mu, \sigma^2)$. Limited simulations show that $\log J(Y_t)$ is convex.

**VIII. CONCAVITY OF $h(\sqrt{t}X + \sqrt{1-t}Z)$**

For $0 < t < 1$, let $W_t := \sqrt{t}X + \sqrt{1-t}Z$, (33) where $Z \sim N(0, 1)$ is independent of $X$. In this section, we study the concavity and convexity of $h(W_t)$ and $J(W_t)$, respectively.

Lieb showed that Shannon EPI (3) is equivalent to

\[ h(\sqrt{\lambda}X_1 + \sqrt{1-\lambda}X_2) \geq \lambda h(X_1) + (1 - \lambda)h(X_2) \]

(34) for any $0 \leq \lambda \leq 1$. Here we use $X_1$ and $X_2$ in lieu of $X$ and $Y$ as the independent random variables.

In literature, $(X_1, X_2) \rightarrow \sqrt{\lambda}X_1 + \sqrt{1-\lambda}X_2$ is referred to as the covariance-preserving transformation, which can be found in many generalizations of Shannon EPI (7). The original proof of Lieb is a little tricky. Next, we give a geometrical interpretation of this transformation which can help us to have a better appreciation on $\sqrt{\lambda}X_1 + \sqrt{1-\lambda}X_2$.

**A. Covariance-preserving Transformation**

Recall that a convex function has the following three equivalent statements.

1. $f(x)$ is convex in $x$.
2. The Hessian matrix of $f(x)$ is positive semi-definite; i.e.,
\[ \nabla^2 f \succeq 0. \]

(35)
3. For any fixed point \( x_0 \),
\[
f(x) \geq f(x_0) + (x - x_0)^T \nabla f(x_0).
\] (36)

Furthermore, \( y = f(x_0) + (x - x_0)^T \nabla f(x_0) \) can be viewed as the tangent plane at point \( (x_0, f(x_0)) \) for function \( y = f(x) \). In the following, we shall apply the above argument on convex functions to study the so-called covariance-preserving transformation.

Shannon EPI [3] can be equivalently transformed to
\[
h(X_1 + X_2) \geq \frac{1}{2} \log \left( e^{2h(X_1)} + e^{2h(X_2)} \right).
\] (37)

Let's study function \( f(x_1, x_2) = \frac{1}{2} \log \left( e^{2x_1} + e^{2x_2} \right) \). By some manipulations,
\[
\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) = \left( \frac{e^{2x_1}}{e^{2x_1} + e^{2x_2}}, \frac{e^{2x_2}}{e^{2x_1} + e^{2x_2}} \right),
\]
\[
\nabla^2 f = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{ij} = \frac{2e^{2x_1}e^{2x_2}}{e^{2x_1} + e^{2x_2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
\]

It is easy to see that \( f(x_1, x_2) \) is convex since \( \nabla^2 f \succeq 0 \). By (36), the tangent plane of \( f(x_1, x_2) \) at point \( (x_1, x_2) = \left( \frac{1}{2} \log(\sigma_1^2), \frac{1}{2} \log(\sigma_2^2) \right) \) is
\[
y = \frac{1}{2} \log \left( \sigma_1^2 + \sigma_2^2 \right) + \left( x_1 - \frac{1}{2} \log(\sigma_1^2) \right) \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}
+ \left( x_2 - \frac{1}{2} \log(\sigma_2^2) \right) \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.
\] (38)

Hence, (3) is equivalent to
\[
h(X_1 + X_2) \geq \frac{1}{2} \log \left( \sigma_1^2 + \sigma_2^2 \right) + \left( h(X_1) - \frac{1}{2} \log(\sigma_1^2) \right) \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}
+ \left( h(X_2) - \frac{1}{2} \log(\sigma_2^2) \right) \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.
\] (39)

Let
\[
\lambda = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}.
\]

Notice that \( h(aX) = h(X) + \log |a| \), we have
\[
h(X_1 + X_2) \geq \lambda h(X_1/\sqrt{\lambda}) + (1 - \lambda) h(X_2/\sqrt{1 - \lambda}).
\] (40)

Substitute \( (X_1, X_2) \) with \( (\sqrt{\lambda}X_1, \sqrt{1 - \lambda}X_2) \),
\[
h(\sqrt{\lambda}X_1 + \sqrt{1 - \lambda}X_2) \geq \lambda h(X_1) + (1 - \lambda) h(X_2),
\] (41)

which is exactly the inequality (34).

In the above proof, the points share the same tangent plane (40) as long as they admit the same \( \lambda \). In fact, all the results (see [7]) that applied covariance-preserving transformation can be proved in this manner.

B. The concavity of \( h(W_t) \)

**Theorem 4.** \( h(W_t) \) is concave in \( t \), \( 0 < t < 1 \).

**Proof:** Since
\[
h(W_t) = h(X + \sqrt{1/t - 1}Z) + \frac{1}{2} \log t,
\]
by some algebra, we obtain
\[
\frac{\partial}{\partial t} h(W_t) = \frac{1}{2} J(X + \sqrt{1/t - 1}Z) \left( -\frac{1}{t^2} \right) + \frac{1}{2t}
\] (42)

and
\[
\frac{\partial^2}{\partial t^2} h(W_t) = \frac{1}{2} J'(X + \sqrt{1/t - 1}Z) \left( -\frac{1}{t^2} \right)^2
+ \frac{1}{2} J(X + \sqrt{1/t - 1}Z) \left( \frac{2}{t^3} \right) - \frac{1}{2t^2}.
\] (43)
To show
\[ \frac{\partial^2}{\partial t^2} h(W_t) \leq 0, \]
we need to prove
\[ \frac{1}{2} J'(X + \sqrt{1/t - 1} Z) \left( -\frac{1}{t^2} \right)^2 + \frac{1}{2t^2} \]
\[ \geq \frac{1}{2} J(X + \sqrt{1/t - 1} Z) \left( \frac{2}{t^3} \right). \]
That is
\[ - J'(X + \sqrt{1/t - 1} Z) + t^2 \geq 2t J(X + \sqrt{1/t - 1} Z). \] (44)
By (61), Costa's EPI is equivalent to
\[ - J'(X + \sqrt{s} Z) \geq J(X + \sqrt{s} Z)^2, \]
for any \( s > 0 \). Therefore,
\[ - J'(X + \sqrt{1/t - 1} Z) + t^2 \]
\[ \geq J(X + \sqrt{1/t - 1} Z)^2 + t^2 \]
\[ \geq 2t J(X + \sqrt{1/t - 1} Z), \]
which is (44).

In all the results above, as \( t > 0 \), \( X + \sqrt{t} Z \) can be replaced by \( X' + \sqrt{s} Z |_{s=0} \), where \( X' = X + \sqrt{\hat{t} Z} \) and \( \hat{Z} \) is the standard Gaussian and is independent of \( X \) and \( Z \). In this manner, we only need to prove that the result holds for any such \( X' \) at point \( s = 0 \). In light of the smoothness introduced by \( \sqrt{t} Z \) where \( t > 0 \), without weakening our result, we can just assume that when \( t \to 0 \), the \( n \)-th order derivative of \( h(X + \sqrt{t} Z) \) exists in the sequel.

Next, we show that Theorem 4 can imply Costa’s EPI. In the above proof, if \( J(X) \) and \( J'(X) \) are well defined, then let \( t \to 1 \) in (44).

\[ - J'(X) + 1 \geq 2 J(X). \] (45)
Let \( \hat{X} = X / \sqrt{J(X)} \), then
\[ J'(\hat{X}) = \frac{J'(X)}{J(X)^2}, \text{ and } J(\hat{X}) = 1. \] (46)
Substitute \( X \) with \( \hat{X} \) in (45),
\[ - \frac{J'(X)}{J(X)^2} \geq 1, \]
which is just Costa’s EPI by (61).

C. The convexity of \( J(W_t) \)

In this section, we study the convexity of \( J(W_t) \) via the relations among the convexities of \( J(W_t) \), \( \frac{1}{J(Y_t)} \) and \( \log J(Y_t) \).

Claim 1. \( J(W_t) \) is not convex.

By some algebra, \( \log J(Y_t) \) is convex in \( t \) if and only if
\[ J''(Y_t) J(Y_t) \geq (J'(Y_t))^2. \] (48)
\( \frac{1}{J(Y_t)} \) is convex in \( t \) if and only if
\[ J''(Y_t) J(Y_t) \leq 2 (J'(Y_t))^2, \] (49)
and concave if and only if
\[ J''(Y_t) J(Y_t) \geq 2 (J'(Y_t))^2. \] (50)

The first and second order derivatives of \( J(W_t) \) are
\[ \frac{\partial}{\partial t} J(W_t) \]
\[ = \frac{\partial}{\partial t} \frac{1}{t} J(X + \sqrt{1/t - 1} Z) \]
\[ = - \frac{1}{t^2} J(X + \sqrt{1/t - 1} Z) - \frac{1}{t^3} J'(X + \sqrt{1/t - 1} Z) \] (51)
and

\[
\frac{\partial^2}{\partial t^2} J(W_t) \\
= \frac{2}{t^3} J(X + \sqrt{1/t - 1} Z) + \frac{1}{t^4} J'(X + \sqrt{1/t - 1} Z) \\
+ \frac{3}{t^5} J'(X + \sqrt{1/t - 1} Z) + \frac{1}{t^6} J''(X + \sqrt{1/t - 1} Z) \\
= \frac{2}{t^3} J(X + \sqrt{1/t - 1} Z) + \frac{4}{t^4} J'(X + \sqrt{1/t - 1} Z) \\
+ \frac{1}{t^5} J''(X + \sqrt{1/t - 1} Z).
\]

(52)

If we can show that

\[
J''(X + \sqrt{s} Z) J(X + \sqrt{s} Z) \geq 2(J'(X + \sqrt{s} Z))^2
\]

(53)
holds for any \(s > 0\), then (48) holds and

\[
\frac{\partial^2}{\partial t^2} J(W_t) \\
= \frac{2}{t^3} J(X + \sqrt{1/t - 1} Z) + \frac{4}{t^4} J'(X + \sqrt{1/t - 1} Z) \\
+ \frac{1}{t^5} J''(X + \sqrt{1/t - 1} Z) \\
\geq 2\sqrt{\frac{2}{t^3} J \times \frac{1}{t^5} J''} + \frac{4}{t^4} J' \\
\geq 2\sqrt{\frac{2}{t^3} 2(J')^2 + \frac{4}{t^4} J'} \\
\geq 0.
\]

(54)

Conversely, if \( \frac{\partial^2}{\partial t^2} J(W_t) \geq 0 \) holds, we can show that also holds. In (52), let \( t \to 1 \), we obtain that

\[
2J(X) + 4J'(X) + J''(X) \geq 0.
\]

Substitute \( X' \) by \( X' = aX \), where \( a > 0 \),

\[
2 \frac{J(X)}{a^2} + 4 \frac{J'(X)}{a^4} + \frac{J''(X)}{a^6} \geq 0.
\]

(55)

Note that \( J \geq 0 \) and \( J'' \geq 0 \). Choose proper \( a \) such that

\[
2 \frac{J(X)}{a^2} = \frac{J''(X)}{a^6}.
\]

Hence

\[
2 \frac{J(X)}{a^2} = \frac{J''(X)}{a^6} \\
= \sqrt{2 \frac{J(X)}{a^2} \times \frac{J''(X)}{a^6}} = \frac{1}{a^4} \sqrt{2J(X)J''(X)}.
\]

Therefore, (55) becomes

\[
2 \frac{J(X)}{a^2} + 4 \frac{J'(X)}{a^4} + \frac{J''(X)}{a^6} \\
= \frac{2}{a^4} \sqrt{2J(X)J''(X)} + 4 \frac{J'(X)}{a^4} \geq 0,
\]

which is

\[
J(X)J''(X) \geq 2(J'(X))^2.
\]

Hence, \( \frac{\partial^2}{\partial t^2} J(W_t) \geq 0 \) if and only if (53) holds for arbitrary \( X \). Because \( \frac{1}{J(W_t)} \) is neither convex nor concave for arbitrary \( X \), which means neither (49) nor (50) holds always, thus \( J(W_t) \) is neither convex nor concave.
IX. Conclusion

The Gaussian random variables have many fascinating properties. In this paper, we have obtained the third and the fourth order derivatives of \( h(X + \sqrt{t}Z) \). The signed representations have a very interesting form. We wish to show that, though we cannot obtain a closed-form expression on \( h(X + \sqrt{t}Z) \) when \( X \) is arbitrary, we can still obtain its convexity or concavity for any order derivative. Our progress verifies a small part of the conjectures and has nearly exhausted the power of fundamental calculus. A new approach may be needed towards solving these conjectures.

In literature, the approach that employed heat equation and integration by parts is merely one of many different approaches to prove Costa’s EPI. For the approaches like data processing argument in [4] [7], and the advanced tools in [14], it is unknown whether they can go further than what we have done. However, if these conjectures are correct, a rather fundamental fact about the Gaussian random variable will be revealed in the language of differential entropy.

REFERENCES

[1] C. E. Shannon, “A mathematical theory of communication,” Bell System Tech. J., vol. 27, pp. 623-656, 1948.
[2] A. Stam, “Some inequalities satisfied by the quantities of information of Fisher and Shannon,” Information and Control, vol. 2, pp. 101-112, 1959.
[3] N. Blachman, “The convolution inequality for entropy powers,” IEEE Trans. Inform. Theory, vol. 11, no. 2, pp. 267-271, Apr. 1965.
[4] R. Zamir, “A Proof of the Fisher Information Inequality via a Data Processing Argument,” IEEE Trans. Inform. Theory, vol. 44, no. 3, pp. 1246-1250, May 1998.
[5] E. H. Lieb, “Proof of an entropy conjecture of Wehrl,” Comm. Math. Phys., vol. 62, no. 1, pp. 35-41, 1978.
[6] S. Verdù and D. Guo, “A simple proof of the entropy-power inequality,” IEEE Trans. Inform. Theory, vol. 52, no. 5, pp. 2165-2166, May 2006.
[7] O. Rioul, “Information theoretic proofs of entropy power inequalities,” IEEE Trans. Inform. Theory, vol. 57, no. 1, pp. 33-55, 2011.
[8] M. H. M. Costa, “A new entropy power inequality,” IEEE Trans. Inform. Theory, vol. 31, no. 6, pp. 751-760, Nov. 1985.
[9] R. Zamir and M. Feder, “A Generalization of the Entropy Power Inequality with Applications,” IEEE Trans. Inform. Theory, vol. 39, no. 5, pp. 1723-1728, Sep. 1993.
[10] T. Liu and P. Viswanath, “An extremal inequality motivated by multiterminal information theoretic problems,” IEEE Trans. Inform. Theory, vol. 53, no. 5, pp. 1839-1851, May 2007.
[11] L. Wang and M. Madiman, “A New Approach to the Entropy Power Inequality, via Rearrangements,” IEEE International Symposium on Information Theory, 2013.
[12] A. El Gamal and Y.-H. Kim, Network Information Theory, Cambridge Univ. Press, 2011.
[13] T. M. Cover and J. A. Thomas, Elements of Information Theory, Second Edition, Wiley.
[14] C. Villani, “A short proof of the concavity of entropy power,” IEEE Trans. Inform. Theory, vol. 46, no. 4, pp. 1695-1696, 2000.
[15] Y. Wu and S. Verdú, “MMSE dimension,” IEEE Trans. Inform. Theory, vol. 57, no. 8, 2011.
[16] D. Guo, S. Shamai (Shitz), and S. Verdú, “Estimation in Gaussian noise: properties of the minimum mean-square error,” IEEE Trans. Inform. Theory, vol. 57, no. 4, pp. 2371-2385, Apr. 2011.
[17] F. Cheng and Y. Geng, “Convexity of Fisher Information with Respect to Gaussian Perturbation,” IEEE Iran Workshop on Communication and Information Theory, 2014.

APPENDIX

A. Proof to Proposition [2]

The technique used in this proof is essentially the same as that by Costa. One may refer to [3] for more details.

Proof: One can obtain the formulae for the derivatives as

\[
 f^{(n)}(y, t) = \int g(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} H_n(y-x) dx,
\]

where \( H_0 = 1 \) and \( H_n \) satisfies the recursion formula

\[
 H_n(y-x) = \frac{y-x}{t} H_{n-1} + \frac{\partial}{\partial y} H_{n-1}.
\]

In general, \( H_n \) can be expressed as

\[
 H_n(y-x) = \sum_{j=0}^{n} \alpha_{n,j} (y-x)^{n-j}
\]

where \( \alpha_{n,j} \)'s are some constants that also depend on \( t \) (and actually these constants are zeroes for odd \( j \)).

Notice that

\[
 \frac{f^{(n)}}{f} = \int g(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} H_n(y-x) \frac{1}{f(y,t)} dx
\]

\[
 = E[H_n(Y_t - X)Y_t = y]
\]

\[
 = \sum_{j=0}^{n} \alpha_{n,j} E[(Y_t - X)^{n-j}|Y_t = y].
\]
Let $\alpha_n := \sum_l |\alpha_{n,l}|$. We prove Proposition 2 by induction on $r$. When $r = 1$,
\[
\int f \frac{f^{(n)}}{f}^k \, dy = E \left[ \frac{f^{(n)}}{f} \right]^k
\]
\[
= E \left[ \sum_{j=0}^n \alpha_{n,j} E \left[ (Y_t - X)^{n-j} | Y_t = y \right] \right]^k
\]
\[
= \alpha_n^k E \left[ \sum_{j=0}^n \frac{\alpha_{n,j}}{\alpha_n} E \left[ (Y_t - X)^{n-j} | Y_t = y \right] \right]^k
\]
\[
\leq \alpha_n^k \sum_{j=0}^n \frac{\alpha_{n,j}}{\alpha_n} E \left[ |(Y_t - X)^{n-j} | Y_t = y \right] \right]^k
\]
(56)
\[
\leq \alpha_n^k \sum_{j=0}^n \frac{\alpha_{n,j}}{\alpha_n} E \left[ |Y_t - X|^{k(n-j)} | Y_t = y \right] \right]
\]
(57)
\[
= \alpha_n^k \sum_{j=0}^n \frac{\alpha_{n,j}}{\alpha_n} E \left[ Y_t - X \right]^{k(n-j)} \right]
\]
\[
= \alpha_n^k \sum_{j=0}^n \frac{\alpha_{n,j}}{\alpha_n} E \left[ \sqrt{\mathcal{I}}Z \right]^{k(n-j)} \right]
\]
< +\infty,
\]
where (56) and (57) are due to Jensen’s inequality.
When $r \geq 2$, by induction,
\[
\int f \prod_{i=1}^r \left[ \frac{f^{(m_i)}}{f^{(k_i)}} \right] \, dy = E \left[ \prod_{i=1}^r \left[ \frac{f^{(m_i)}}{f^{(k_i)}} \right] \right]^k
\]
\[
\leq \left( E \left[ \prod_{i=1}^{r-1} \left[ \frac{f^{(m_i)}}{f^{(k_i)}} \right] \right] \cdot E \left[ \left[ \frac{f^{(m_r)}}{f^{(k_r)}} \right]^2 \right] \right)^{\frac{1}{2}}
\]
(58)
\[
< +\infty,
\]
where (58) is by the Cauchy-Schwartz inequality.
The fact that $\int f \prod_{i=1}^r \left[ \frac{f^{(m_i)}}{f^{(k_i)}} \right] \, dy$ vanishes as $|y| \to \infty$ can be obtained from the existence of integral.

B. Proof to Costa’s EPI

Proof: Costa’s EPI is equivalent to
\[
\frac{\partial^2}{\partial t^2} e^{2h(Y_t)} \leq 0.
\]
(59)
By some algebra, one needs to show
\[
2 \left( \frac{\partial}{\partial t} h(Y_t) \right)^2 \leq -\frac{\partial^2}{\partial t^2} h(Y_t),
\]
(60)
or
\[
J(Y_t)^2 \leq -J(Y_t),
\]
(61)
i.e.,
\[
\int f \left( \frac{f_y y}{f} - \frac{f_y^2}{f^2} \right)^2 \, dy \geq \left( \int \frac{f_y^2}{f} \, dy \right)^2,
\]
(62)
which can be proved by the inequality of arithmetic and geometric means:

\[
\int f \left( \frac{f_{yy}}{f} - \frac{f_y^2}{f^2} \right)^2 dy \geq \left( \int f \left( \frac{f_{yy}}{f} - \frac{f_y^2}{f^2} \right) dy \right)^2
\]

\[
= \left( \int f_{yy} dy - \int \frac{f_y^2}{f} dy \right)^2
\]

\[
= \left( \int \frac{f^2}{f} dy \right)^2.
\]

In (63), \( \int f_{yy} dy = \int 2f_1 dy = \frac{\alpha}{\pi} \int 2f dy = \frac{\alpha}{\pi} 2 = 0.\)

C. Proof to Lemma 3

Proof: We use integration by parts to eliminate the high-order terms:

\[
\int \frac{f_1^4 f_2}{f^4} dy = \int \frac{f_1^4}{f^4} df_1
\]

\[
= \int \frac{1}{5f^4} df_1^5
\]

\[
= \frac{f_1^5}{5f^4} \bigg|^{+\infty}_y - \int \frac{f_1^5}{5} \left( \frac{1}{f^4} \right) dy
\]

\[
= 0 + \int \frac{4f_1^6}{5f^4} dy.
\]

\[
\int \frac{f_1^3 f_2}{f^3} dy = \int \frac{f_1^3}{f^3} df_2
\]

\[
= \frac{f_1^3 f_2}{f^3} \bigg|^{+\infty}_y - \int \frac{f_2}{f^3} \frac{f_1}{f^2} df_2 dy
\]

\[
= 0 - \int f_2 \frac{3f_1^2 f_2}{f^2} \frac{f_2 f - f_1 f_1}{f^2} dy
\]

\[
= \int - \frac{3f_1^2 f_2^2}{f^3} + \frac{3f_1^2 f_2}{f^4} dy
\]

\[
= \int - \frac{3f_1^2 f_2^2}{f^3} + \frac{12f_1^4 f_2}{5f^4} dy.
\]

\[
\int \frac{f_1 f_2 f_3}{f^2} dy = \int \frac{f_1 f_2}{f^2} df_2
\]

\[
= \int \frac{f_1 f_2}{2f^2} df_2^2
\]

\[
= \frac{f_1 f_2^2}{2f^2} \bigg|^{+\infty}_y - \int \frac{f_2^2}{2} \left( \frac{f_1}{f^2} \right) dy
\]

\[
= 0 - \int \frac{f_1 f_2 f_3 f_2^3}{2f^2} - f_1 f_2 f_1 \frac{f_2^2}{f^4} dy
\]

\[
= \int - \frac{f_2^3}{2f^2} + \frac{f_2^2 f_2}{f^3} dy.
\]
\[
\int \frac{f_2 f_4}{f} \, dy = \int \frac{f_2}{f} \, df_3 = \frac{f_2 f_3}{f} \bigg|_{y=-\infty}^{+\infty} - \int f_3 \left( \frac{f_2}{f} \right)_y \, dy
\]
\[
= 0 - \int f_3 \frac{f_3 f - f_2 f_1}{f^2} \, dy
\]
\[
= \int -\frac{f_3^2}{f} + f_1 f_2 f_3 \, dy
\]
\[
= \int -\frac{f_3^2}{f} - \frac{f_2^3}{2f^2} + \frac{f_1^2 f_2^2}{f^3} \, dy.
\]

In the above, the limits are zero due to Proposition 2. Since all the integrals and limits exist, all the steps which use integration by parts are valid.

\[\text{D. Proof to Lemma 4}\]

Proof: We use integration by parts to eliminate the high-order terms:

\[
\int \frac{f_1^3 f_2}{f^6} \, dy = \int \frac{f_1^6}{f^6} \, df_1
\]
\[
= \int \frac{1}{f^6} \, df_1
\]
\[
= \frac{f_1^7}{7f^6} \bigg|_{-\infty}^{+\infty} - \int \frac{f_1^6}{7f^6} \, dy
\]
\[
= 0 + \int \frac{6f_1^8}{7f^7} \, dy
\]
\[
(66)
\]

\[
\int \frac{f_1^5 f_3}{f^5} \, dy = \int \frac{f_1^5}{f^5} \, df_2
\]
\[
= \frac{f_1^5 f_2}{f^5} \bigg|_{-\infty}^{+\infty} - \int f_2 \left( \frac{f_1^5}{f^5} \right)_y \, dy
\]
\[
= 0 - \int f_2 \frac{5f_1^4 f_2 f - f_1^2}{f^4} \, df_2
\]
\[
= \int -5f_1^4 f_2^2 + \frac{5f_1^6 f_2}{f^6} \, dy
\]
\[
= \int -5f_1^4 f_2^2 + 30f_1^3 f_2 \, dy
\]
\[
(67)
\]

\[
\int \frac{f_1^3 f_2 f_3}{f^4} \, dy = \int \frac{f_1^3 f_2}{f^4} \, df_3
\]
\[
= \int \frac{f_1^3}{2f^4} \, df_2
\]
\[
= \frac{f_1^3 f_2^2}{2f^4} \bigg|_{-\infty}^{+\infty} - \int \frac{f_2^2}{2} \left( \frac{f_1^3}{f^4} \right)_y \, dy
\]
\[
= 0 - \int \frac{f_2^2}{2} \frac{3f_1^3 f_2 f^4 - f_1^3 f^3 f_1}{f^8} \, dy
\]
\[
= \int -\frac{3f_1^3 f_2^2}{2f^4} + \frac{2f_1^4 f_2^2}{f^5} \, dy
\]
\[
(68)
\]
\[
\int \frac{f_1 f_2^2 f_3}{f^3} \, dy = \int \frac{f_1 f_2^2}{f^3} \, df_2
\]
\[
= \int \frac{f_1}{3f^3} \, df_2^3
\]
\[
= \frac{f_1 f_2^3}{3f^3} \bigg|_{-\infty}^{+\infty} - \int \frac{f_2^3}{3f^3} \left( \frac{f_3}{f^3} \right)_y \, dy
\]
\[
= 0 - \int \frac{f_3^3}{3} f_2 f_3 - f_1 3 f_2^3 f_1 \, df_1
\]
\[
= \int -\frac{f_2^4}{3f^3} + \frac{f_2^2 f_1^2}{f^4} \, dy
\]
\[
(69)
\]

\[
\int \frac{f_1^4 f_2}{f^4} \, dy = \int \frac{f_1^4}{f^4} \, df_2
\]
\[
= \frac{f_1^4 f_2^3}{f^4} \bigg|_{-\infty}^{+\infty} - \int \frac{f_3}{f^3} \left( \frac{f_1^4}{f^4} \right)_y \, dy
\]
\[
= 0 - \int \frac{f_3^4}{f^6} f_2 f_3 - f_1^3 f_2 f_3 \, df_1
\]
\[
= \int -\frac{4 f_3^4 f_2 f_3}{f^4} + \frac{4 f_3^5 f_2}{f^5} \, dy
\]
\[
= \int -4 \left( \frac{3 f_2^2 f_1^2}{2 f^4} + \frac{2 f_1^4 f_2^2}{f^5} \right)
\]
\[
+ 4 \left( -\frac{5 f_1^4 f_2^2}{f^5} + \frac{30 f_3^8}{7 f^7} \right) \, dy
\]
\[
= \int \frac{6 f_2^4 f_1^2}{f^4} - \frac{28 f_1^4 f_2^2}{f^5} + \frac{120 f_3^8}{7 f^7} \, dy
\]
\[
(70)
\]

\[
\int \frac{f_1 f_2 f_3}{f^3} \, dy
\]
\[
= \int \frac{f_1 f_2 f_3}{f^3} \, df_3
\]
\[
= \frac{f_1 f_2 f_3}{f^3} \bigg|_{-\infty}^{+\infty} - \int f_3 \left( \frac{f_1 f_2}{f^3} \right)_y \, dy
\]
\[
= 0 - \int \frac{f_3^3}{f^6} (f_2 f_1) f_3^3 - f_2^3 f_2 (f_1^3) \, df_1
\]
\[
= \int -2 f_1 f_2^2 f_3 - \frac{f_3^3}{f^3} f_3^3 + \frac{3 f_3 f_2 f_3}{f^4} \, dy
\]
\[
= \int -2 \left( \frac{f_2^2}{3f^3} + \frac{f_3 f_2}{f^4} \right) \, dy
\]
\[
+ 3 \left( -\frac{3 f_2^2 f_3}{2 f^4} + \frac{2 f_1^4 f_2^2}{f^5} \right) \, dy
\]
\[
= \int \frac{2 f_2^2}{3f^3} - \frac{13 f_2^2 f_3}{2 f^4} - \frac{f_2^2 f_1^2}{f^3} + \frac{6 f_1^4 f_2^2}{f^5} \, dy
\]
\[
(71)
\]
\[ \int \frac{f_3 f_4}{f^2} dy = \int \frac{f_3 f_4}{f^2} df_3 \]
\[ = \frac{f_3 f_4}{f^2} \bigg|_{-\infty}^{+\infty} - \int f_3 \left( \frac{f_4}{f^2} \right) dy \]
\[ = 0 - \int f_3 \frac{2f_2 f_3 f - f_2 f_1}{f^2} dy \]
\[ = \int -\frac{2f_2 f_3^2}{f^2} + \frac{2f_1 f_2 f_3}{f^3} dy \]
\[ = \int \left[ -\frac{2f_2 f_3^2}{f^2} + 2 \left( -\frac{f_2^4}{3f^3} + \frac{f_1^2 f_2^2}{f^4} \right) \right] dy \]
\[ = \int \left[ -\frac{2f_2 f_3^2}{f^2} - \frac{2f_1^2 f_3}{3f^3} + \frac{2f_1^2 f_2^2}{f^4} \right] dy \]  
(72)

\[ \int \frac{f_1 f_3 f_4}{f^2} dy = \int \frac{f_1 f_3}{f^2} df_3 \]
\[ = \int \frac{f_1 f_3}{2f^2} df_3 \]
\[ = \frac{f_1 f_3}{2f^2} \bigg|_{-\infty}^{+\infty} - \int \frac{f_1^2}{2} \left( \frac{f_3}{f^2} \right) dy \]
\[ = 0 - \int \frac{f_2^3}{2} f_2 f^2 - f_1 2f_3 f_3 dy \]
\[ = \int -\frac{2f_2 f_3^2}{2f^2} + \frac{2f_2^2 f_3}{f^3} dy \]  
(73)

\[ \int \frac{f_3 f_5}{f} dy = \int \frac{f_3}{f} df_4 \]
\[ = \frac{f_3 f_4}{f} \bigg|_{-\infty}^{+\infty} - \int f_4 \left( \frac{f_3}{f} \right) dy \]
\[ = 0 - \int f_4 f_1 f - f_3 f_1 dy \]
\[ = \int -\frac{f_2^3}{f} + \frac{f_1 f_3 f_4}{f^2} dy \]
\[ = \int \left[ -\frac{f_2^3}{f} - \frac{2f_2 f_3^2}{2f^2} + \frac{f_1^2 f_2^2}{f^3} \right] dy \]  
(74)