Fourier and Zak transforms of multiplicative characters

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Abstract

In this paper we derive formulas for the $N$-point discrete Fourier transform and the $R_1 \times R_2$ finite Zak transform of multiplicative characters on $\mathbb{Z}/N$, where $N$ is an odd integer, and $R_1$ and $R_2$ are co-prime factors of $N$. In one special case this permits computation of the discrete Fourier transform and the finite Zak transform of the Jacobi symbol, the modified Jacobi sequence, and the Golomb sequence. In other cases, not addressed here, this permits computation of the discrete Fourier transform and the finite Zak transform of certain complex-valued sequences. These results constitute, to our knowledge, the first unified treatment of key Fourier and Zak space properties of multiplicative characters. These results also provide a convenient framework for the design of new character-based sequences.

Index terms: Chinese Remainder Theorem, Björck sequences, discrete Fourier transform, finite Zak transform, Gauss sum, Golomb sequences, Good-Thomas Prime Factor algorithm, ideal sequences, Jacobi symbol, Legendre symbol, modified Jacobi sequences, multiplicative characters, Quadratic Reciprocity Law, twin-prime sequences.

1 Introduction

The focus of this work is on Fourier properties of multiplicative characters. A multiplicative character on $\mathbb{Z}/N$ is a group homomorphism from its group of units, $U(N)$, to the multiplicative group of non-zero complex numbers, $\mathbb{C}^*$ [20]. Multiplicative characters are of theoretical interest, as they link concepts in group theory, linear algebra, harmonic analysis, and number theory. Among multiplicative characters a class of multiplicative characters, the primitive multiplicative characters, is especially important, for two main reasons. First, the discrete Fourier transform (DFT) of primitive multiplicative characters has a simple structure explicitly involving the Gauss sums that can be used to re-derive certain classical number-theoretical results. Second, and of more direct interest in this work, primitive multiplicative characters are closely associated with sequences with favorable pseudo-random and correlation properties, including
difference sets and quadratic residue sequences. These sequences are often used in cryptography [25], [28], communications [16], [23] and radar [16], [23], [27-28]. The association between characters and sequences has been explored previously, among others by Turyn in [32] (in the construction of Hadamard difference sets) and by Scholtz and Welch in [27] (in investigation of sequences with good correlation properties).

The principal results of this work include:

1. The formulas for the \( N \)-point DFT and the \( R_1 \times R_2 \) finite Zak transform (FZT) of primitive multiplicative characters on \( \mathbb{Z}/N \), where \( N = R_1R_2 \) is an arbitrary odd integer and \( (R_1, R_2) = 1 \). While the DFT formula was given previously in [4], the presentation given here is more pedagogical, as it builds on detailed analysis of special cases, \( \mathbb{Z}/p \) and \( \mathbb{Z}/p^m \), and as it provides a link to a subsequent time-frequency analysis of sequences.

2. The use of the formulas for the DFT and the FZT of primitive characters leads to the derivation of the DFT and the FZT of three special sequences: the Jacobi symbol, the modified Jacobi sequence and the Golomb sequence. The FZT result appears especially interesting, as it suggests a link between the constructions of complex-valued and binary sequences.

3. The use of an intermediate result in the derivation of the DFT of the Jacobi symbol that links the Jacobi symbol with Gauss sums (Theorem 6) and a prior result for the trace of the DFT matrix leads to re-derivation of the Quadratic Reciprocity Law (QRL).

The DFT of primitive multiplicative characters is derived twice, in two different contexts and using different tools. The first derivation relies, in part, on prior results, summarized in Section 2, for the primitive multiplicative characters on \( \mathbb{Z}/p \) and for the primitive multiplicative characters on \( \mathbb{Z}/p^m \), \( m \geq 2 \), and, in part, on the Chinese Remainder Theorem (CRT). The CRT defines a ring isomorphism between \( \mathbb{Z}/N \) and \( \mathbb{Z}/R_1 \times \mathbb{Z}/R_2 \). This ring isomorphism establishes a relationship between the multiplicative characters on \( \mathbb{Z}/N \), and those on \( \mathbb{Z}/R_1 \) and \( \mathbb{Z}/R_2 \),

\[
\chi(a_1e_1 + a_2e_2) = \chi_1(a_1)\chi_2(a_2), \quad a_1 \in U(R_1), \quad a_2 \in U(R_2),
\]

where \( e_1 \) and \( e_2 \) are idempotents for \( \mathbb{Z}/N \) corresponding to the factorization \( N = R_1R_2 \). This relationship permits construction of the primitive multiplicative characters on \( \mathbb{Z}/N \) from the primitive multiplicative characters on \( \mathbb{Z}/R_1 \) and \( \mathbb{Z}/R_2 \) (Theorem 3), and computation of the DFT of primitive multiplicative characters on \( \mathbb{Z}/N \) (Theorem 4). The last result is given by the formula

\[
\hat{\chi}(a) = G(\chi)\chi^*(a),
\]

where

\[
G(\chi) = G(\chi_1)G(\chi_2)\chi_1^*(e_1/R_2)\chi_2^*(e_2/R_1).
\]

In Section 4 we present a second proof of Theorem 4, based on the Good-Thomas Prime Factor (GTPF) algorithm. This derivation is more explicit, as it relies on the relationship between the DFT matrix \( F(N) \) and the tensor product of the associated DFT matrices \( F(R_1) \) and \( F(R_2) \), perturbed by permutation matrices.

The second main theoretical result of this work (Theorem 9) is the \( R_1 \times R_2 \) FZT of a primitive multiplicative character on \( \mathbb{Z}/N \),

\[
X_{R_2}(j,k) = G(\chi_2)\chi_2(R_1)e^{-2\pi i \frac{j-1}{R_2} R_2} \chi_2^*(j)\chi_1(k), \quad 0 \leq j < R_2, \quad 0 \leq k < R_1.
\]
This result can be viewed as a generalization of the result for the DFT. The formula for the FZT is more broadly applicable than the formula for the DFT, in the sense that it does not require both multiplicative characters on $\mathbb{Z}/R_1$ and $\mathbb{Z}/R_2$ to be primitive. The use of the FZT in the analysis of multiplicative characters is new; it complements and sometimes is preferable to the use of the DFT, especially in sequence design applications. We elaborate on this in Section 7.

One consequence of these results is that they avail convenient tools for investigating special cases that are important in sequence design applications. We focus here, in particular, on three sequences: the Jacobi symbol, the modified Jacobi sequence [22], and the Golomb sequence [15]. The Jacobi symbol is the second order primitive multiplicative character and a well-known ternary sequence. The last two sequences are not multiplicative characters, but they can be obtained from multiplicative characters by simple modifications. The importance of the modified Jacobi sequence stems from the fact that while, like the Jacobi symbol, it has good aperiodic correlation properties and large linear complexity, unlike the Jacobi symbol, it is binary. The importance of the Golomb sequence stems from the fact that, unlike the Jacobi symbol and the modified Jacobi sequence, it is a constant magnitude ideal sequence, that is a sequence with an ideal autocorrelation.

We derive the DFT (Corollary 4 and Theorems 7 and 8) and the FZT (Corollary 5 and Theorems 10 and 11) of all three sequences. The DFT results are not entirely new; partial results for all three cases have appeared before. The DFT of the Jacobi sequence was used by Jensen et al in [22], to facilitate computation of merit factor of binary sequences, but the formula is not stated there explicitly in a complete, general form. The DFT of the modified Jacobi sequence was given in [22] as well, but the formula is more complex than necessary, and it depends on certain auxiliary parameters. The DFT of the Golomb sequence for the special case of prime length (referred there as the Björck sequence) was given by Popovic in [24]. Our results generalize and unify these results. In further research the general result for the DFT of multiplicative characters can be used to identify new sequences and to compute their DFTs. The FZT results are entirely new. Among others, these results show that the FZT of the Golomb sequence, while, unlike the FZT of a chirp [10], is neither unimodular nor semi-unimodular, it is highly structured, suggesting Zak space sequence design approaches, similar to those developed for chirps [2], [6-11], might be viable.

This work is a sequel to a sequence of papers and books. In particular, it is a sequel to [10], which, in part, synthesizes our principal results for chirps. The DFT formula in Theorem 4 extends the result in [31] for the DFT of multiplicative characters on $\mathbb{Z}/p^m$ to the general case, and is an analogue of the formulas for the DFT of a discrete chirp given by Theorem 1 in [6] and Theorem 1 in [10]. Associated with the DFT formula in Theorem 4, the FZT formula in Theorem 7 is an analogue of the formulas for the FZT of a discrete chirp given by Theorem 5 in [7] and Theorems 4 and 5 in [10].

The content of this paper is as follows. In Section 2 we review prior results on the DFT of multiplicative characters on $\mathbb{Z}/N$ for $N = p^m$, $m \geq 1$. In Section 3 we use the CRT to extend the results of Section 2 to the case of $N = R_1R_2$, $N$ odd, $(R_1, R_2) = 1$. In Section 4 we re-derive this result using the GTPF algorithm. In Section 5 we derive a formula for the DFT of the Jacobi symbol. In Section 6 we derive formulas for the DFT of the modified Jacobi sequence and the Golomb sequence. In Section 7 we derive formulas for the FZT of multiplicative characters, the Jacobi symbol, the modified Jacobi sequence, and the Golomb sequence. In Section 8 we discuss further work. In Appendix 1 we summarize basic properties of the Legendre and Jacobi symbols. In Appendix 2 we re-derive the QRL using Theorem 6 and a prior result for the trace of the DFT matrix.
2 DFT of multiplicative characters in a special case

We will start by introducing basic notation. $N$ is an odd positive integer with distinct prime power factorization $N = p_1^{c_1}...p_r^{c_r}$, $\mathbb{Z}/N$ is the ring of integers modulo $N$ and $U(N)$ is its multiplicative group of units. $U(N)$ consists of all $a \in \mathbb{Z}/N$ such that $(a, N) = 1$ and has order

$$t = p_1^{c_1-1}...p_r^{c_r-1}(p_1 - 1)...(p_r - 1).$$

(1)

It follows that the order of $U(N)$ is even.

Suppose $x$ is an arbitrary $N$-periodic sequence. The $N$-point Fourier transform of $x$, $\hat{x}$, is given by

$$\hat{x}(a) = \sum_{b \in \mathbb{Z}/N} x(b)e^{2\pi i \frac{ab}{N}}, \quad a \in \mathbb{Z}/N.$$  

(2)

A multiplicative character on $\mathbb{Z}/N$ is a group homomorphism, $\chi : U(N) \rightarrow \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers, by definition,

$$\chi(ab) = \chi(a)\chi(b), \quad a, b \in U(N).$$  

(3)

Multiplicative characters take their values on the $t$-th root of unity, $t$ the order of $U(N)$. The set of multiplicative characters forms a group of order $t$ under the rule

$$\chi_1\chi_2(a) = \chi_1(a)\chi_2(a), \quad a \in U(N), \quad \chi_1, \chi_2 \text{ multiplicative characters on } \mathbb{Z}/N.$$  

(4)

In particular

$$\chi(a^{-1}) = \chi^*(a) = \chi^{-1}(a), \quad a \in U(N).$$  

(5)

We set the values of multiplicative characters equal to zero off of $U(N)$ and assume that they are defined on $\mathbb{Z}$ by periodicity mod $N$,

$$\chi(a) = \chi(a_1), \quad a \in \mathbb{Z}, \quad a_1 \equiv a \text{ mod } N.$$  

(6)

An important role in this work is played by the properties of $U(p)$ and its multiplicative characters, $p$ an odd prime. $U(p)$ is a cyclic group of order $t = p - 1$. Denote a generator of $U(p)$ by $g$, so that $U(p)$ consists of the powers

$$g^a, \quad 0 \leq a < t.$$  

(7)

There are $t$ distinct multiplicative characters on $\mathbb{Z}/p$ defined by

$$\chi_j(g^a) = e^{2\pi i \frac{ja}{p}}, \quad 0 \leq j < t, \quad 0 \leq a < t.$$  

(8)

By well-known results (see, for example, [20])

$$\hat{x}_0(a) = \begin{cases} p - 1, & a = 0, \\ -1, & \text{otherwise}, \end{cases}$$  

(9)

and

$$\hat{x}_j(a) = \hat{x}_j(1)\chi_j^*(a), \quad j > 0, \quad a \in \mathbb{Z}/p,$$  

(10)
where

$$\hat{\chi}_j(1) \neq 0. \quad (11)$$

In fact, for $j > 0$,

$$|\hat{\chi}_j(1)|^2 = p. \quad (12)$$

Suppose $\chi$ is an arbitrary multiplicative character on $U(N)$ and $\hat{\chi}$ is its $N$-point Fourier transform

$$\hat{\chi}(a) = \sum_{b \in U(N)} \chi(b) e^{2\pi i \frac{ab}{N}}, \quad a \in \mathbb{Z}/N. \quad (13)$$

For $a \in U(N)$

$$\chi(a)\hat{\chi}(a) = \sum_{b \in U(N)} \chi(ab) e^{2\pi i \frac{ab}{N}} = \sum_{c \in U(N)} \chi(c) e^{2\pi i \frac{c}{N}} = \hat{\chi}(1). \quad (14)$$

This leads to the following well-known result [20].

**Theorem 1** If $\chi$ is a multiplicative character on $\mathbb{Z}/N$, then

$$\hat{\chi}(a) = \hat{\chi}(1)\chi^*(a), \quad a \in U(N). \quad (15)$$

There are two mutually exclusive cases. If

$$\hat{\chi}(1) = 0, \quad (16)$$

then $\hat{\chi}$ vanishes on $U(N)$. Otherwise $\hat{\chi}$ is a non-zero scalar multiple of $\chi^*$ on $U(N)$. In this case we set

$$G(\chi) = \hat{\chi}(1), \quad (17)$$

and call $G(\chi)$ the Gauss sum of the multiplicative character $\chi$.

We call a multiplicative character $\chi$ on $\mathbb{Z}/N$ primitive if

$$\hat{\chi}(a) = G(\chi)\chi^*(a), \quad a \in \mathbb{Z}/N. \quad (18)$$

By necessity, if $\chi$ is primitive, then $\hat{\chi}(1) \neq 0$, since otherwise $\hat{\chi}$ vanishes everywhere, and therefore $\chi$ must vanish everywhere.

If $N = p$, the multiplicative characters $\chi_j$, $0 < j < p - 1$, are primitive, while $\chi_0$ is not, since $\hat{\chi}_0(0) \neq 0$. In the next section we construct the set of all primitive multiplicative characters on $\mathbb{Z}/N$ for arbitrary $N$. We remark that Theorem 1 does not address the values of $\hat{\chi}(a)$, $a \notin U(N)$, a problem that has been studied in [31].

Next, we will describe the primitive multiplicative characters on $\mathbb{Z}/p^m$, $m \geq 2$. Suppose $\chi$ is a multiplicative character on $\mathbb{Z}/p^m$, $m \geq 2$, satisfying $\hat{\chi}(1) \neq 0$. By Theorem 1,

$$\hat{\chi}(a) = G(\chi)\chi^*(a), \quad a \in U(p^m). \quad (19)$$
Below we will show that \( \hat{\chi} \) vanishes outside \( U(p^m) \). Take \( a \notin U(p^m) \) and write
\[
a = pa', \quad 0 \leq a' < p^{m-1}.
\]
(20)

Consider the subgroup \( \Delta \) of \( U(p^m) \),
\[
\Delta = 1 + p^{m-1}\mathbb{Z}/p^m.
\]
(21)

If \( \chi(\Delta) = 1 \), then for \( v \in U(p^m) \)
\[
\chi(v + p^{m-1}\mathbb{Z}) = \chi(v)\chi(1 + v^{-1}p^{m-1}\mathbb{Z}) = \chi(v)
\]
and \( \chi \) is periodic mod \( p^{m-1}\mathbb{Z} \). It follows that \( \hat{\chi} \) is decimated mod \( p\mathbb{Z} \), contradicting the assumption \( \hat{\chi}(1) \neq 0 \). It follows that we can find \( c \in \Delta \), such that \( \chi(c) \neq 1 \). Since \( pc \equiv p \mod p^m \), we have
\[
w^{ab} = wp^{a'b} = wp^{aca'b} = w^{abc}, \quad w = e^{2\pi i \frac{l}{t}}, \quad b \in U(p^m),
\]
(23)

\[
\hat{\chi}(a) = \sum_{b \in U(p^m)} \chi(b)w^{abc} = \chi(c)\hat{\chi}(a).
\]
(24)

Since \( \chi(c) \neq 1 \), \( \hat{\chi}(a) = 0 \), proving (18) holds for \( a \notin U(p^m) \). This yields the next result [31].

**Theorem 2** If \( \chi \) is a multiplicative character on \( \mathbb{Z}/p^m \), \( m \geq 2 \), satisfying the condition \( \hat{\chi}(1) \neq 0 \), then \( \chi \) is primitive.

We can explicitly describe the primitive multiplicative characters on \( \mathbb{Z}/p^m \), \( m \geq 2 \) [31]. For an odd prime \( p \), \( U(p^m) \) is a cyclic group of order \( t = p^{m-1}(p-1) \). Suppose \( z \) is a generator of \( U(p^m) \). The multiplicative characters on \( \mathbb{Z}/p^m \) are completely determined by their values on generator \( z \) and are given by
\[
\chi_l(z) = e^{2\pi i \frac{l}{t}}, \quad 0 \leq l < t.
\]
(25)

The primitive multiplicative characters are given by
\[
\chi_l, \quad 0 < l < t, \quad (l, p^m) = 1.
\]
(26)

The non-primitive multiplicative characters are \( \chi_0 \) and
\[
\chi_l, \quad 0 < l < t, \quad (l, p^m) > 1.
\]
(27)

### 3 DFT of multiplicative characters and the CRT

In this section we use the CRT to extend the results of Section 2 to an arbitrary integer \( N = R_1R_2 \), \((R_1, R_2) = 1\).

Suppose \( N = R_1R_2 \). The mapping
\[
\Phi(a) = (a_1, a_2), \quad a_1 \equiv a \mod R_1, \quad a_2 \equiv a \mod R_2,
\]
(28)
is a ring homomorphism of $\mathbb{Z}/N$ into $\mathbb{Z}/R_1 \times \mathbb{Z}/R_2$. The CRT states that if $(R_1, R_2) = 1$, then $\Phi$ is a ring isomorphism of $\mathbb{Z}/N$ onto $\mathbb{Z}/R_1 \times \mathbb{Z}/R_2$.

Throughout $(R_1, R_2) = 1$. By one form of the CRT there exist

$$e_1, e_2 \in \mathbb{Z}/N$$

such that

$$e_1 \equiv 1 \mod R_1, \quad e_1 \equiv 0 \mod R_2$$

and

$$e_2 \equiv 0 \mod R_1, \quad e_2 \equiv 1 \mod R_2.$$  

$e_1$ and $e_2$ satisfy the idempotent conditions

$$e_1^2 \equiv e_1 \mod N, \quad e_2^2 \equiv e_2 \mod N, \quad e_1 e_2 \equiv 0 \mod N, \quad e_1 + e_2 \equiv 1 \mod N.$$  

The mapping

$$\Phi(a_1, a_2) \equiv a_1 e_1 + a_2 e_2 \mod N, \quad a_1 \in \mathbb{Z}/R_1, \quad a_2 \in \mathbb{Z}/R_2$$

is a ring isomorphism from $\mathbb{Z}/R_1 \times \mathbb{Z}/R_2$ onto $\mathbb{Z}/N$, providing the inverse to $\Phi$. In particular, every $a$ in $\mathbb{Z}/N$ can be uniquely written as

$$a = a_1 e_1 + a_2 e_2 \mod N, \quad a_1 \in \mathbb{Z}/R_1, \quad a_2 \in \mathbb{Z}/R_2.$$  

$U(N)$ consists of the points

$$a_1 e_1 + a_2 e_2 \mod N, \quad a_1 \in U(R_1), \quad a_2 \in U(R_2).$$

Suppose $\chi_1$ and $\chi_2$ are multiplicative characters on $\mathbb{Z}/R_1$ and on $\mathbb{Z}/R_2$, respectively, then

$$\chi(a_1 e_1 + a_2 e_2) = \chi_1(a_1) \chi_2(a_2), \quad a_1 \in U(R_1), \quad a_2 \in U(R_2),$$

is a multiplicative character on $\mathbb{Z}/N$. In the inverse direction, if $\chi$ is a multiplicative character on $\mathbb{Z}/N$, then

$$\chi_1(a_1) = \chi(a_1 e_1 + e_2) \quad \text{and} \quad \chi_2(a_2) = \chi(e_1 + e_2 a_2),$$

are multiplicative characters on $\mathbb{Z}/R_1$ and on $\mathbb{Z}/R_2$.

Also,

$$\chi_1(a_1) \chi_2(a_2) = \chi(a_1 e_1 + e_2) \chi(e_1 + e_2 a_2) = \chi(a_1 e_1 + e_2 a_2).$$  

In this way we can identify multiplicative characters on $\mathbb{Z}/N$ with pairs of multiplicative characters on $\mathbb{Z}/R_1$ and on $\mathbb{Z}/R_2$. We can also write

$$\chi(a) = \chi_1(a_1) \chi_2(a_2), \quad a_1 \equiv a \mod R_1, \quad a_2 \equiv a \mod R_2.$$  


Setting
\[ a \equiv a_1 e_1 + a_2 e_2 \mod N \quad \text{and} \quad b \equiv b_1 e_1 + b_2 e_2 \mod N. \] (40)
in
\[ \hat{\chi}(a) = \sum_{b \in U(N)} \chi(b) e^{2\pi i \frac{ab}{N}}, \] (41)
we obtain
\[ \hat{\chi}(a_1 e_1 + a_2 e_2) = \sum_{b_1 \in U(R_1)} \chi_1(b_1) e^{2\pi i \frac{a_1 e_1}{R_2}} \sum_{b_2 \in U(R_2)} \chi_2(b_2) e^{2\pi i \frac{a_2 e_2}{R_1}} = \hat{\chi}_1 \left( a_1 \frac{e_1}{R_2} \right) \hat{\chi}_2 \left( a_2 \frac{e_2}{R_1} \right). \] (42)

Since \( e_1 \equiv 0 \mod R_2 \) and \( e_1 \equiv 1 \mod R_1 \), we have that \( e_1 / R_2 \) is an integer relatively prime to \( R_1 \). In the same way \( e_2 / R_1 \) is an integer relatively prime to \( R_2 \), leading to the next result.

**Theorem 3** If \( \chi_1 \) and \( \chi_2 \) are multiplicative characters on \( \mathbb{Z}/R_1 \) and on \( \mathbb{Z}/R_2 \), and \( \chi \) is the corresponding multiplicative character on \( \mathbb{Z}/N \), then
\[ \hat{\chi}(a_1 e_1 + a_2 e_2) = \hat{\chi}_1 \left( a_1 \frac{e_1}{R_2} \right) \hat{\chi}_2 \left( a_2 \frac{e_2}{R_1} \right), \quad a_1 \in \mathbb{Z}/R_1, \quad a_2 \in \mathbb{Z}/R_2, \] (43)
where \( \frac{e_1}{R_2} \in U(R_1) \) and \( \frac{e_2}{R_1} \in U(R_2) \).

Since \( e_1 + e_2 \equiv 1 \mod N \),
\[ \hat{\chi}(1) = \hat{\chi}_1 \left( \frac{e_1}{R_2} \right) \hat{\chi}_2 \left( \frac{e_2}{R_1} \right), \] (44)
proving the following.

**Corollary 1** \( \hat{\chi}(1) \neq 0 \) if and only if \( \hat{\chi}_1(1) \neq 0 \) and \( \hat{\chi}_2(1) \neq 0 \).

Moreover, since \( \hat{\chi}(a) = 0 \) for \( a \notin U(N) \) if and only if \( \hat{\chi}_1(a_1) = 0 \) for \( a_1 \notin U(R_1) \), and \( \hat{\chi}_2(a_2) = 0 \) for \( a_2 \notin U(R_2) \), we have the next result.

**Corollary 2** \( \chi \) is primitive if and only if \( \chi_1 \) and \( \chi_2 \) are primitive.

We can now construct the primitive multiplicative characters on \( \mathbb{Z}/N \),
\[ N = p_1^{e_1} \ldots p_r^{e_r}. \] (45)
Choose primitive multiplicative characters \( \chi_j \) in \( \mathbb{Z}/p_j^{e_j} \), \( 0 \leq j \leq r \). If \( c_j = 1 \), we can choose any non-trivial character on \( \mathbb{Z}/p \) for \( \chi_j \). Otherwise \( c_j > 1 \), and by Theorem 2 we can choose any multiplicative character on \( \mathbb{Z}/p_j^{c_j} \) whose DFT does not vanish at 1. By Theorem 3
\[ \chi(a) = \chi_1(a_1) \ldots \chi_r(a_r), \quad a_j \equiv a \mod p_j^{e_j}, \quad 0 \leq j \leq r, \] (46)
is a primitive multiplicative character on \( \mathbb{Z}/N \), and every primitive multiplicative character on \( \mathbb{Z}/N \) has this form. Usually the definition of primitive multiplicative character is given in terms of induced characters [4]. A primitive multiplicative character is a multiplicative character not induced from a lower order.
character. Our definition follows the program of using the CRT to bootstrap general results from results on factors.

Suppose $\chi_1$ and $\chi_2$ are primitive multiplicative characters on $\mathbb{Z}/R_1$ and on $\mathbb{Z}/R_2$, and $\chi$ is the corresponding primitive multiplicative character on $\mathbb{Z}/N$, $N = R_1 R_2$, $(R_1, R_2) = 1$. Set

$$f_1 = \frac{e_1}{R_2} \quad \text{and} \quad f_2 = \frac{e_2}{R_1}.$$  

(47)

By primitivity

$$\hat{\chi}_1(a_1 f_1) = G(\chi_1)\chi_1^*(a_1)\chi_1^*(f_1),$$  

(48)

$$\hat{\chi}_2(a_2 f_2) = G(\chi_2)\chi_2^*(a_2)\chi_2^*(f_2),$$  

(49)

and

$$\hat{\chi}(a_1 e_1 + a_2 e_2) = G(\chi)\chi_1^*(a_1)\chi_2^*(a_2). \quad a_1 \in \mathbb{Z}/R_1, \quad a_2 \in \mathbb{Z}/R_2.$$  

(50)

This leads to the next result.

**Theorem 4** If $\chi_1$ and $\chi_2$ are primitive multiplicative characters on $\mathbb{Z}/R_1$ and on $\mathbb{Z}/R_2$, and $\chi$ is the corresponding primitive multiplicative character on $\mathbb{Z}/N$, $N = R_1 R_2$, $(R_1, R_2) = 1$, then

$$\hat{\chi}(a_1 e_1 + a_2 e_2) = G(\chi)\chi_1^*(a_1)\chi_2^*(a_2), \quad a_1 \in \mathbb{Z}/R_1, \quad a_2 \in \mathbb{Z}/R_2,$$

where

$$G(\chi) = G(\chi_1)G(\chi_2)\chi_1^*(f_1)\chi_2^*(f_2).$$  

(52)

## 4 DFT of multiplicative characters and the GTPF algorithm

In this section we derive the main result of Section 3, Theorem 4, in a slightly different way, using the GTPF algorithm. The GTPF algorithm was first described in [18] and [30]. It is an analogue of the Cooley-Tukey (CT) algorithm for computing the DFT. While the CT algorithm addresses the case when $N = 2^n$, the GTPF algorithm addresses the case when $N$ is an odd prime. The key components of the GTPF algorithm are the DFTs of the factors and permutation matrices. The details of the algebraic structure of the algorithm are given in [31].

Suppose $\chi_1$ and $\chi_2$ are primitive multiplicative characters on $\mathbb{Z}/R_1$ and $\mathbb{Z}/R_2$, and $\chi$ is the corresponding primitive multiplicative characters on $\mathbb{Z}/N$. Then

$$\hat{\chi}_1 = G(\chi_1)\chi_1^*, \quad \hat{\chi}_2 = G(\chi_2)\chi_2^*,$$

and

$$\hat{\chi} = G(\chi)\chi^*.$$  

(54)

We will now use the GTPF algorithm to relate the constants $G(\chi_1), G(\chi_2)$ and $G(\chi)$.  

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Every $a \in \mathbb{Z}/N$ can be uniquely written as
\[ a = a_2 + a_1 R_2, \quad 0 \leq a_1 < R_1, \quad 0 \leq a_2 < R_2. \tag{55} \]
with $a_2$ the fastest running variable for consistency with the tensor product. Define the permutation $\pi$ of $\mathbb{Z}/N$ by
\[ \pi(a) \equiv a_1 e_1 + a_2 e_2 \mod N, \tag{56} \]
and set $Q_\pi$ equal to the corresponding permutation matrix. Then \[ F(N) = Q_\pi^{-1} F_\pi Q_\pi, \tag{57} \]
where
\[ F_\pi = \begin{bmatrix} w^{\pi(a)\pi(b)} \end{bmatrix}_{0 \leq a, b < N}, \quad w = e^{2\pi i \frac{1}{N}}. \tag{58} \]
We describe the structure of $F_\pi$ below.

Applying $Q_\pi$ to $\chi(a)$ we have
\[ Q_\pi \chi(a) = \chi(\pi(a)) = \chi(a_1 e_1 + a_2 e_2) = \chi_1(a_1) \otimes \chi_2(a_2). \tag{59} \]
Set
\[ u_1 = e^{2\pi i \frac{1}{R_1}}, \quad u_2 = e^{2\pi i \frac{1}{R_2}}, \tag{60} \]
and
\[ f_1 = \frac{e_1}{R_2}, \quad f_2 = \frac{e_2}{R_1}. \tag{61} \]
$f_1$ and $f_2$ are integers with $f_1$ relatively prime to $R_1$, and $f_2$ relatively prime to $R_2$. Then
\[ f_1 t, \quad 0 \leq t < R_1, \tag{62} \]
is a permutation of $\mathbb{Z}/R_1$ and
\[ f_2 t, \quad 0 \leq t < R_2 \tag{63} \]
is a permutation of $\mathbb{Z}/R_2$.

We can now describe $F_\pi$. From
\[ w^{\pi(a)\pi(b)} = u_1^{a_1 b_1} u_2^{a_2 b_2}, \tag{64} \]
we have
\[ F_\pi = F_{R_1} \otimes F_{R_2} \tag{65} \]
where
\[ F_{R_1} = \begin{bmatrix} u_1^{f_1 a_1 b_1} \end{bmatrix}_{0 \leq a_1, b_1 < R_1}. \tag{66} \]
and

\[ F_{R_2} = [u_2^{f_2a_2b_2}]_{0 \leq a_2, b_2 < R_2}. \]  (67)

\( F_{R_1} \) is found by permuting the rows of \( F(R_1) \) by \( f_1t, \ 0 \leq t < R_1 \), with a similar statement for \( F_{R_2} \). Then

\[ F_\pi Q_\pi \chi(a) = F_{R_1}(\chi_1(a_1)) \otimes F_{R_2}(\chi_2(a_2)) = G(\chi_1)G(\chi_2) \chi_1^*(f_1)\chi_2^*(f_2) \chi_1^*(a_1) \otimes \chi_2^*(a_2) \]  (68)

and

\[ \hat{\chi}(a) = Q_\pi^{-1} F_\pi Q_\pi \chi(a) = G(\chi_1)G(\chi_2) \chi_1^*(f_1)\chi_2^*(f_2) \chi^*(a). \]  (69)

This leads to the next result.

**Theorem 5** Suppose \( \chi_1 \) and \( \chi_2 \) are primitive multiplicative characters on \( \mathbb{Z}/R_1 \) and on \( \mathbb{Z}/R_2 \), and

\[ \chi(a) = \chi_1(a_1)\chi_2(a_2), \ a_1 \equiv a \mod R_1 \text{ and } a_2 \equiv a \mod R_2, \]  (70)

the corresponding primitive multiplicative characters on \( \mathbb{Z}/N \). Then

\[ \hat{\chi}(a) = G(\chi_1)G(\chi_2)\chi_1^*(f_1)\chi_2^*(f_2)\chi^*(a), \ a \in \mathbb{Z}/N. \]  (71)

In the next two sections we use results from Sections 3 and 4 to derive the DFTs of the Jacobi symbol, the modified Jacobi sequence and the Golomb sequence.

## 5 Application of the DFT formula to the Jacobi symbol

Let \( N = R_1R_2 \), \( (R_1, R_2) = 1 \), and \( \chi_{R_1} \) and \( \chi_{R_2} \) be the Jacobi symbols on \( \mathbb{Z}/R_1 \) and \( \mathbb{Z}/R_2 \), viewed as primitive multiplicative characters on \( \mathbb{Z}/R_1 \) and \( \mathbb{Z}/R_2 \) (Appendix A1),

\[ \chi_{R_1}(a_1) = \left( \frac{a_1}{R_1} \right) \text{ and } \chi_{R_2}(a_2) = \left( \frac{a_2}{R_2} \right). \]  (72)

Then

\[ \left( \frac{R_2}{R_1} \right) \chi_{R_1}(f_1) = \left( \frac{R_2}{R_1} \right) \left( \frac{e_1/R_2}{R_1} \right) = \left( \frac{e_1}{R_1} \right) = 1 \]  (73)

and

\[ \left( \frac{R_1}{R_2} \right) \chi_{R_2}(f_2) = \left( \frac{R_1}{R_2} \right) \left( \frac{e_2/R_1}{R_2} \right) = \left( \frac{e_2}{R_2} \right) = 1. \]  (74)

By Theorem 4 we have the next result.

**Theorem 6** If \( \chi_{R_1} \) and \( \chi_{R_2} \) are the Jacobi symbols on \( \mathbb{Z}/R_1 \) and \( \mathbb{Z}/R_2 \), and \( \chi_N, N = R_1R_2 \), \( (R_1, R_2) = 1 \), is the corresponding Jacobi symbol on \( \mathbb{Z}/N \), then

\[ G(\chi_N) = G(\chi_{R_1})G(\chi_{R_2}) \left( \frac{R_1}{R_2} \right) \left( \frac{R_2}{R_1} \right). \]  (75)

Applying QRL in Appendix 1, Theorem 6 implies the following result.
Corollary 3

\[ G(\chi_N) = G(\chi_{R_1}) G(\chi_{R_2}) (-1)^{\frac{R_1-1}{2} \frac{R_2-1}{2}}. \] (76)

The Legendre symbol

\[ \chi_p(a) = \left( \frac{a}{p} \right), \] (77)

where \( p \) is an odd prime, is a primitive multiplicative character on \( \mathbb{Z}/p \). We can identify \( \chi_p \) with the unique real valued multiplicative character

\[ \chi_{t/2}, \quad t = p - 1 \] (78)

in \( \mathbb{Z}/p \). \( \chi_p \) is an eigenvector of the \( p \)-point DFT,

\[ \hat{\chi}_p = c_p \chi_p, \quad c_p = G(\chi_p). \] (79)

By Theorem 4, the Jacobi symbol

\[ \chi_N(a) = \left( \frac{a}{N} \right) = \left( \frac{a}{p_1 \cdots p_r} \right) = \left( \frac{a}{p_1} \right) \cdots \left( \frac{a}{p_r} \right) \] (80)

is a primitive multiplicative character on \( \mathbb{Z}/N \),

\[ \hat{\chi}_N(a) = c_N \chi_N(a). \] (81)

Applying Corollary 3 permits computing an explicit formula for \( c_N \), which leads to the next result.

**Corollary 4** \( \chi_N \) is an eigenvector of the \( N \)-point DFT, i.e.,

\[ \hat{\chi}_N(a) = c_N \chi_N(a), \] (82)

where

\[ c_N = c_{p_1 \cdots p_r} = c_{p_1} \cdots c_{p_r} (-1)^{\frac{1}{2} \sum_{i=1}^{r-1} (p_1 \cdots p_i - 1)(p_{i+1} - 1)}, \] (83)

and

\[ c_{p_i} = G(\chi_{p_i}) = \begin{cases} \sqrt{p_i}, & p_i \equiv 1 \mod 4, \\ i\sqrt{p_i}, & p_i \equiv 3 \mod 4. \end{cases} \] (84)

In particular, when \( N = p_1 p_2 \),

\[ c_{p_1 p_2} = c_{p_1} c_{p_2} (-1)^{\frac{p_1-1}{2} \frac{p_2-1}{2}} \] (85)

and, when \( N = p_1 p_2 p_3 \),

\[ c_{p_1 p_2 p_3} = c_{p_1} c_{p_2} c_{p_3} (-1)^{\frac{p_1-1}{2} \frac{p_2-1}{2} \frac{p_3-1}{2}} (-1)^{\frac{p_1 p_2 - 1}{2} \frac{p_2 p_3 - 1}{2} \frac{p_1 p_3 - 1}{2}}. \] (86)
6 Application of the DFT formula to the modified Jacobi sequence and the Golomb sequence

Suppose \( N = pq \), \( p \) and \( q \) are distinct odd primes, \( p < q \). Define the modified Legendre and Jacobi symbols, \( x_p \) and \( x_{pq} \), by

\[
x_p(n) := \begin{cases} 0, & n = 0, \\ \rho \left( \left( \frac{n}{p} \right) \right), & \text{otherwise.} \end{cases} \tag{87}
\]

and

\[
x_{pq}(n) := \begin{cases} 0, & n = 0, \\ 1, & n \equiv 0 \mod p, n \not\equiv 0 \mod q, \\ 0, & n \equiv 0 \mod q, n \not\equiv 0 \mod p, \\ \rho \left( \left( \frac{n}{p} \right) \left( \frac{n}{q} \right) \right), & (n, pq) = 1, \end{cases} \tag{88}
\]

where \( \rho(-1) = 1 \) and \( \rho(1) = 0 \). The modified Legendre and Jacobi sequences are of interest in sequence design applications, as they have large linear complexity and large merit factor [19], [22], [33]. They are also linked to the Golomb sequences [15], a well-known unimodular two-valued ideal sequence. The DFT of the Golomb sequence, which we obtain by first computing the DFT of the modified Jacobi sequence, is the main result of this section.

The modified Jacobi sequence in (88) can be re-written as

\[
x_{pq}(n) = \frac{1}{2} \left( \left( \frac{n}{pq} \right) + 1_{pq} - \text{comb}_p + \text{comb}_q + \delta_0 \right), \tag{89}
\]

where \( 1_{pq}, \text{comb}_p \) and \( \delta_0 \) are the length-\( pq \) vectors

\[
1_{pq} = [1, 1, \ldots, 1], \tag{90}
\]

\[
\text{comb}_p = \begin{cases} 1, & n \equiv 0 \mod p, \\ 0, & \text{otherwise}, \end{cases} \tag{91}
\]

and

\[
\delta_0 = [1, 0, \ldots, 0]. \tag{92}
\]

Since

\[
\hat{1}_{pq} = pq\delta_0, \tag{93}
\]

\[
\hat{\text{comb}}_p = \sum_{r=0}^{q-1} e^{2\pi i \frac{mr}{q}} = \begin{cases} q, & m \equiv 0 \mod q, \\ 0, & \text{otherwise}, \end{cases} \tag{94}
\]

\[
\hat{\delta}_0 = 1_{pq}, \tag{95}
\]

\[\text{In sequence design literature, including in [22], } x_p(0) \text{ and } x_{pq}(0) \text{ are usually set to } 1. \text{ We choose here to follow the standard mathematical convention.}\]
and
\[
\left( \frac{n}{pq} \right) = c_{pq} \left( \frac{m}{pq} \right) = \begin{cases} 
0, & (m, pq) > 1, \\
c_{pq} \left( \frac{m}{pq} \right), & \text{otherwise},
\end{cases}
\]  
(96)

where
\[
c_{pq} = \begin{cases} 
i\sqrt{pq}, & q = p + 2, \\
\sqrt{pq}, & \text{otherwise},
\end{cases}
\]  
(97)

which leads to the next result.

**Theorem 7**

\[
\hat{x}_{pq}(m) = \begin{cases} 
pq - q + p + 1, & m = 0, \\
-q + 1, & m \equiv 0 \mod q, m \neq 0, \\
p + 1, & m \equiv 0 \mod p, m \neq 0, \\
c_{pq} \left( \frac{m}{pq} \right) + 1, & \text{otherwise},
\end{cases}
\]  
(98)

It follows that \(|\hat{x}_{pq}(m)|\) is constant for all \(m > 0\) iff \(q = p + 2\).

Define the Golomb sequence\(^2\) as the following modification of the modified Jacobi sequence\(^3\)

\[
y_{pq}(n) := \begin{cases} 
1, & x_{pq}(n) = 1, \\
\alpha, & x_{pq}(n) = 0,
\end{cases}
\]  
(99)

where \(\alpha = e^{i\Phi}, \Phi = \cos^{-1}\left(-\frac{pq-1}{pq+1}\right)\) and \(q = p + 2\).

The equation (99) can be re-written as

\[
y_{pq} = (1 - \alpha)x_{pq} + \alpha 1_{pq}.
\]  
(100)

Then

\[
\hat{y}_{pq}(m) = \frac{1}{2}(1 - \alpha) \left( \left( \frac{n}{pq} \right) - \tilde{comb}_p + \tilde{comb}_q + \delta_0 \right) + \frac{1}{2}(1 + \alpha) \tilde{1}_{pq}
\]  
(101)

This leads to the next result.

**Theorem 8**

\[
\hat{y}_{pq}(m) = \frac{1}{2} \begin{cases} 
(1 + \alpha)pq + (1 - \alpha)(-q + p + 1), & m = 0, \\
(1 - \alpha)(-q + 1), & m \equiv 0 \mod q, m \neq 0, \\
(1 - \alpha)(p + 1), & m \equiv 0 \mod p, m \neq 0, \\
(1 - \alpha)(c_{pq} \left( \frac{m}{pq} \right) + 1), & \text{otherwise},
\end{cases}
\]  
(102)

---

\(^2\)The Golomb sequence is sometimes referred to as the Björck sequence [26], as it was discovered independently by Björck [5].

\(^3\)Golomb defines his sequence with respect to the Paley-Hadamard difference set, i.e., a cyclic difference set with the parameters \((n, k, \lambda) = (4t - 1, 2t - 1, t - 1)\). The construction of the cyclic Paley-Hadamard difference set is known for three cases: (1) \(N = p\), (2) \(p(p + 2)\), and (3) \(N = 2^m - 1\). For \(q = p + 2\) the modified Jacobi sequence is identical with the second case (also known as the twin-prime sequence [29]), and for \(N = p\) the modified Legendre sequence is identical with the first case.
When \( q = p + 2 \), \( \hat{y}_{pq} \) has a constant magnitude and the Golomb sequence is ideal.

When \( N = p \), then
\[
x_p(n) = \frac{1}{2} \left( \left( \frac{n}{p} \right) + 1_p - \delta_0 \right),
\]
(103)
\[
y_p = (1 - \alpha)x_p + \alpha 1_p,
\]
(104)
\[
\hat{x}_p(m) = \frac{1}{2} \left( \left( \frac{n}{p} \right) + \hat{1}_p - \hat{\delta}_0 \right) = \frac{1}{2} \left( \frac{p - 1}{c_p \left( \frac{m}{p} \right)} - 1 \right), \quad m = 0,
\]
(105)
and
\[
\hat{y}_p(m) = (1 - \alpha)\hat{x}_p + \alpha \hat{1}_p = \frac{1}{2} \left\{ \begin{array}{ll}
(1 + \alpha)p + \alpha - 1, & m = 0, \\
(1 - \alpha)(c_p \left( \frac{m}{p} \right) - 1), & \text{otherwise}
\end{array} \right.
\]
(106)

When \( p \equiv 3 \mod 4 \), then \( c_p = i\sqrt{p} \), \( \hat{y}_p \) has a constant magnitude and the Golomb sequence is ideal.

### 7 FZT of multiplicative characters

Suppose \( N = LM \). Define the FZT of an arbitrary \( N \)-periodic sequence \( x \) by
\[
X_L(j, k) = \sum_{r=0}^{L-1} x(k + rM)e^{2\pi i \frac{jr}{L}}, \quad 0 \leq j < L, \quad 0 \leq k < M.
\]
(107)

FZT has several applications in mathematics [12], quantum mechanics [34] and signal analysis [21]. In particular, it plays a major role in the analysis of time-frequency representations, including ambiguity functions [3] and Weyl-Heisenberg expansions [1], and in polyphase sequence design [2], [6-11].

The FZT is a time-frequency representation that is closely related to the DFT [21]. For \( L = 1 \) \( X_L = x \). For \( M = 1 \) \( X_L = \hat{x} \). In general, \( X_L \) can be viewed as a collection of DFTs of appropriate decimations of \( x \) taken at various values of \( k \). In particular, when \( x \) is a discrete periodic chirp, the FZT of \( x \) can be linked with the DFT of the decimation of an appropriate component of \( x \) [10]. In effect, the condition for polyphase/semi-polyphase support of the DFT of a discrete periodic chirp can be replaced by the condition of polyphase/semi-polyphase support of the FZT of a discrete periodic chirp, and the Fourier space design setting can be replaced with the Zak space design setting. In this section we set the stage for future investigations of existence of similar relationships for binary sequences.

Suppose \( N = R_1R_2 \), \( (R_1, R_2) = 1 \), \( \chi_1 \) and \( \chi_2 \) are multiplicative characters on \( \mathbb{Z}/R_1 \) and \( \mathbb{Z}/R_2 \), and \( \chi \) is the corresponding multiplicative character on \( \mathbb{Z}/N \). In this section we describe the \( R_1 \times R_2 \) FZT of \( \chi \),
\[
X_{R_2}(j, k) = \sum_{r=0}^{R_2-1} \chi(k + rR_1)e^{2\pi i \frac{jr}{R_2}}.
\]
(108)

Suppose
\[
\chi(k + rR_1) = \chi_1(k)\chi_2(k + rR_1).
\]
(109)
We have
\[ X_{R_2}(j,k) = \chi_1(k) \sum_{r=0}^{R_2-1} \chi_2(k + rR_1) e^{2\pi i \frac{jr}{R_2}}, \quad 0 \leq j < R_2, \quad 0 \leq k < R_1. \]  
(110)

Consider
\[ \hat{\chi}_2(j) = \sum_{r=0}^{R_2-1} \chi_2(r) e^{2\pi i \frac{jr}{R_2}}. \]  
(111)

Multiplying by \( \chi_2(R_1) \)
\[ \chi_2(R_1) \hat{\chi}_2(j) = \sum_{r=0}^{R_2-1} \chi_2(rR_1) e^{2\pi i \frac{jr}{R_2}}. \]  
(112)

Changing variables by \( rR_1 = r'R_1 + k \),
\[ \chi_2(R_1) \hat{\chi}_2(j) = e^{2\pi i \frac{jR_1}{R_2}} \sum_{r'=0}^{R_2-1} \chi_2(k + r'R_1) e^{2\pi i \frac{j'r'}{R_2}}. \]  
(113)

Plugging this result into the formula for \( X_{R_2}(j,k) \), we get
\[ X_{R_2}(j,k) = \chi_1(k) \chi_2(R_1) e^{-2\pi i \frac{jR_1}{R_2}} \hat{\chi}_2(j). \]  
(114)

Suppose now that \( \chi_2 \) is primitive, then
\[ \hat{\chi}_2(j) = G(\chi_2) \chi_2^*(j), \quad 0 \leq j < R_2, \]  
(115)

and we have the following result.

**Theorem 9** If \( \chi_1 \) and \( \chi_2 \) are multiplicative characters on \( \mathbb{Z}/R_1 \) and \( \mathbb{Z}/R_2 \), with \( \chi_2 \) primitive, then
\[ X_{R_2}(j,k) = G(\chi_2) \chi_2^*(j) \chi_1(k), \quad 0 \leq j < R_2, \quad 0 \leq k < R_1. \]  
(116)

Theorem 9 permits, in particular, an assessment of the DFT of the Jacobi symbol and related sequences. Set \( \chi(k + rR_1) = \left( \frac{k + rR_1}{R_1R_2} \right) \). We have the following result.

**Corollary 5**
\[ X_{R_1}(j,k) = c_{R_2} \left( \frac{R_1}{R_2} \right) e^{-2\pi i \frac{jR_1}{R_2}} \left( \frac{j}{R_2} \right) \left( \frac{k}{R_1} \right), \quad 0 \leq j < R_2, \quad 0 \leq k < R_1, \]  
(117)

where \( c_{R_2} \) is computed via formula (82).

The FZT of the modified Jacobi sequence can be written as
\[ X_q\{x_{pq}\}(j,k) = \frac{1}{2} X_q \left\{ \begin{array}{c} n \quad \text{if } p - q, \\ 1 \quad \text{if } q \end{array} \right\} + 1_{pq} - comb_p + comb_q + \delta_0 \].
(118)

Since
\[ X_q\{1_{pq}\}(j,k) = \sum_{r=0}^{q-1} e^{2\pi i \frac{rj}{q}} = \begin{cases} q, & j = 0, \quad 0 \leq k < p, \\ 0, & \text{else,} \end{cases} \]  
(119)
\[ X_q\{\text{comb}_p\}(j, k) = \begin{cases} \sum_{r=0}^{q-1} e^{2\pi i \frac{rj}{q}}, & k = 0, \\
0, & \text{else,} \end{cases} \quad j = k = 0, \quad (120) \]

\[ X_q\{\text{comb}_q\}(j, k) = \sum_{k+rp \equiv 0 \mod q} e^{2\pi i \frac{rj}{q}} = e^{-2\pi i \frac{1-kj}{q}}, \quad \text{for all } j, k, \quad (121) \]

\[ X_q\{\delta_0\}(j, k) = \begin{cases} 1, & k = 0, \quad 0 \leq j < q, \\
0, & \text{else}, \end{cases} \quad (122) \]

and

\[ X_q\left\{\left(\frac{n}{pq}\right)\right\}(j, k) = c_q \left(\frac{p}{q}\right) \left(\frac{k}{p}\right) \left(\frac{j}{q}\right) e^{-2\pi i \frac{j-k}{q}}, \quad \text{for all } j, k, \quad (123) \]

we have the next result.

**Theorem 10**

\[ X_q\{x_{pq}\}(j, k) = \begin{cases} 2, & k = 0, \\
q + 1, & j = 0, \quad k \neq 0, \\
Ae^{-2\pi i \frac{1-kj}{q}}, & \text{otherwise}, \end{cases} \quad (124) \]

where

\[ A = 1 + c_q \left(\frac{p}{q}\right) \left(\frac{k}{p}\right) \left(\frac{j}{q}\right) \quad (125) \]

and

\[ c_q = \sqrt{q} \begin{cases} 1, & q \equiv 1 \mod 4, \\
i, & q \equiv 3 \mod 4. \end{cases} \quad (126) \]

Similarly, since the FZT of the Golomb sequence can be written as

\[ X_q\{y_{pq}\} = (1 - \alpha)X_q\{x_{pq}\} + \alpha X_q\{1_{pq}\}, \quad (127) \]

we have the next result.

**Theorem 11**

\[ X_q\{y_{pq}\} = \begin{cases} 1 - \alpha + \alpha q, & j = k = 0, \\
\frac{(1-\alpha)(q+1)}{2} + \alpha q, & j = 0, \quad k \neq 0, \\
1 - \alpha, & k = 0, \quad j \neq 0, \\
\frac{1-\alpha}{2} Ae^{-2\pi i \frac{1-kj}{q}}, & \text{otherwise}. \end{cases} \quad (128) \]

The array \(X_q\{x_{pq}\}(j > 0, k > 0)\) is unimodular when \(q \equiv 3 \mod 4\), and semi-unimodular when \(q \equiv 1 \mod 4\). The support of \(X_q\{y_{pq}\}\) is the same as the support of \(X_q\{x_{pq}\}\).
8 Further work

In this work we derived formulas for the \( N \)-point DFT and the \( R_1 \times R_2 \) FZT of multiplicative characters on \( \mathbb{Z}/N \), where \( N = R_1 R_2 \) an odd integer and \( (R_1, R_2) = 1 \), and of three special sequences: the Jacobi symbol, the modified Jacobi sequence and the Golomb sequence.

The importance of the results of this work stems from their potential utility in sequence design. DFT and FZT enter sequence design at several stages in fundamental ways [2]. One of the main tasks in this field is identification of unimodular sequences with ideal periodic autocorrelation. The ideal autocorrelation condition is equivalent to unimodularity of the DFT of a sequence. Sequences satisfying this condition are called bi-unimodular or ideal [5], [26]. A corresponding condition exists for the FZT of a sequence [2], [11]. Other sequences of interest that are not ideal include Golay sequences [13] and zero autocorrelation zone sequences [17]. These sequences are often semi-unimodular, that is, their DFTs and FZTs have constant magnitude on a subset of points and are zero otherwise. The structure of the supports of the DFTs and the FZTs, have been previously used to design new families of chirp-like sequences [2], [6-11]. We anticipate the results for the DFT and FZT of multiplicative characters will be of equal utility.

In particular, these results avails convenient tools for investigation of two topics. First, in the special case of Theorem 4, explored in Section 5, the DFT of the second order primitive multiplicative characters was used to derive the DFT of the modified Jacobi sequence and the Golomb sequence. In further research it might be of interest to consider the DFT of higher order primitive multiplicative characters, which might lead to identification of new sequences and description of their Fourier properties. Second, since, in general, there are many choices for the factors of \( N, R_1 \) and \( R_2 \), there are many distinct, but equivalent factorizations of primitive multiplicative characters on \( \mathbb{Z}/N \), and hence many equivalent factorizations of the associated sequences. The choice of a factorization or a subset of factorizations can be used in schemes that require distributed processing or sharing of information between multiple users.

Apart from sequence design applications, results of this work raise some broader questions:

1. Primitive multiplicative characters are defined by the eigenvector property. What is the relationship between eigenvectors and eigenvalues of different multiplicative characters?

2. Modification of Jacobi sequences trades the eigenvector property for biunimodularity. What is the mathematical relationship between the eigenvector property and biunimodularity? Are there sequences other than ideal chirps, described in [6], that share these two properties?

Appendix A1 Synopsis of Legendre and Jacobi symbols

The Legendre symbol is defined by

\[
\left( \frac{a}{p} \right) = \begin{cases} 
0, & a = 0, \\
1, & a \text{ is a quadratic residue mod } p, \\
-1, & a \text{ is not a quadratic residue mod } p.
\end{cases}
\]

(129)

The Legendre symbol is a primitive multiplicative character of order two on \( \mathbb{Z}/p \) having the following properties:

\[
\left( \frac{ab}{p} \right) = \left( \frac{a}{p} \right) \left( \frac{b}{p} \right)
\]

(130)
if \( a \equiv b \mod p \) then \( \left( \frac{a}{p} \right) = \left( \frac{b}{p} \right) \) \hspace{1cm} (131)

\[ \left( \frac{a^2}{p} \right) = 1 \] \hspace{1cm} (132)

In addition,

\[ a^{\frac{p-1}{2}} \equiv \left( \frac{a}{p} \right) \mod p. \] \hspace{1cm} (133)

If \( p \) and \( q \) are odd primes, the Legendre symbols satisfy the QRL,

\[ \left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}}. \] \hspace{1cm} (134)

The Jacobi symbol is defined by

\[ \left( \frac{a}{N} \right) = \left( \frac{a}{p_1 \cdots p_r} \right) = \left( \frac{a}{p_1} \right) \cdots \left( \frac{a}{p_r} \right). \] \hspace{1cm} (135)

Since the Legendre symbol has order two, we can assume that the primes \( p_1, \ldots, p_r \) are distinct. By Theorem 4 the Jacobi symbol is a primitive multiplicative character of order two on \( \mathbb{Z}/N \) having the following properties:

\[ \left( \frac{ab}{N} \right) = \left( \frac{a}{N} \right) \left( \frac{b}{N} \right) \] \hspace{1cm} (136)

If \( a \equiv b \mod N \) then \( \left( \frac{a}{N} \right) = \left( \frac{b}{N} \right) \) \hspace{1cm} (137)

In addition,

\[ \left( \frac{a}{N_1 N_2} \right) = \left( \frac{a}{N_1} \right) \left( \frac{a}{N_2} \right). \] \hspace{1cm} (138)

If \( a \) and \( b \) are odd positive integers, the Jacobi symbols satisfy the QRL,

\[ \left( \frac{a}{b} \right) \left( \frac{b}{a} \right) = (-1)^{\frac{a-1}{2} \frac{b-1}{2}}. \] \hspace{1cm} (139)

**Appendix A2 Proof of the QRL**

The QRL and the quadratic Gauss sum computation for the Jacobi symbol, first derived by Gauss [20], are fundamental results in classical number theory. These Gauss sums are critical to the determination of the class number of quadratic number fields, as well as many other problems. Gauss also observed the connection between these sums and the QRL. Since Gauss there have been many derivations of these results.

In this appendix we describe another approach towards relating these results based on combining two independent formulas. One, given in Theorem 6, uses the CRT to relate quadratic Gauss sums to the Jacobi symbol. The second, derived in [13], uses the algebra of null-theta functions in the 3 dimensional
real Heisenberg group to relate the quadratic Gauss sum to the trace of the DFT matrix.

We start by retracing the principal steps of the second approach.

A basis for $L(\mathbb{Z}/p)$, the complex valued functions on $\mathbb{Z}/p$, can be chosen, so that in this basis

$$\frac{1}{\sqrt{p}} F(p)$$

is a block diagonal matrix consisting of $2 \times 2$ blocks having zero trace and the single eigenvalue

$$\frac{1}{\sqrt{p}} G(\chi_p)$$

(141)

corresponding to the eigenvector $\chi_p$. Then

$$\frac{1}{\sqrt{p}} G(\chi_p) = Tr \left( \frac{1}{\sqrt{p}} F(p) \right) = \frac{1}{\sqrt{p}} \sum_{x=0}^{p-1} e^{2\pi i x^2/p}.$$  

(142)

Details can be found in [31].

In the same way

$$\frac{1}{\sqrt{N}} G(\chi_N) = Tr \left( \frac{1}{\sqrt{N}} F(N) \right) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{2\pi i x^2/N}.$$  

(143)

In [13] we derived a formula for $Tr(\frac{1}{\sqrt{N}} F(N))$, independent of the QRL, based on the algebra of nil-theta functions on the 3 dimensional Heisenberg group,

$$\frac{1}{\sqrt{N}} G(\chi_N) = Tr \left( \frac{1}{\sqrt{N}} F(N) \right) = \begin{cases} 1 + i, & N \equiv 0 \mod 4, \\ 1, & N \equiv 1 \mod 4, \\ 0, & N \equiv 2 \mod 4, \\ i, & N \equiv 3 \mod 4. \end{cases}$$

(144)

In particular, when $N$ is odd,

$$\frac{1}{\sqrt{N}} G(\chi_N) = i \frac{N-1}{2}.$$  

(145)

Since, by (145),

$$\frac{G(\chi_N)}{G(\chi_{R_1}) G(\chi_{R_2})} = i \frac{N-1}{2} \frac{R_{\frac{R_1}{R_2}}^{R_{\frac{R_2}{R_1}}} - 1}{R_{\frac{R_1}{R_2}}},$$

we have

$$G(\chi_N) = G(\chi_{R_1}) G(\chi_{R_2}) (-1)^{\frac{R_1-1}{2} \frac{R_2-1}{2}}.$$  

(146)

Combining (147) with Theorem 6,

$$G(\chi_N) = G(\chi_{R_1}) G(\chi_{R_2}) \left( \frac{R_1}{R_2} \right) \left( \frac{R_2}{R_1} \right),$$

we have the QRL for the Jacobi symbol,

$$\left( \frac{R_1}{R_2} \right) \left( \frac{R_2}{R_1} \right) = (-1)^{\frac{R_1-1}{2} \frac{R_2-1}{2}}.$$  

(148)
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