REACHING GENERALIZED CRITICAL VALUES OF A POLYNOMIAL

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Abstract. Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We give an algorithm to compute the set of generalized critical values. The algorithm uses a finite dimensional space of rational arcs along which we can reach all generalized critical values of $f$.

1. Introduction.

Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Over forty years ago R. Thom proved that $f$ is a $C^\infty$–fibration outside a finite set, the smallest such a set is called the bifurcation set of $f$, we denote it by $B(f)$. In a natural way appears a fundamental question: how to determine the set $B(f)$.

Let us recall that in general the set $B(f)$ is bigger than $K_0(f)$ - the set of critical values of $f$. It contains also the set $B_\infty(f)$ of bifurcations points at infinity. Briefly speaking the set $B_\infty(f)$ consists of points at which $f$ is not a locally trivial fibration at infinity (i.e., outside a large ball). To control the set $B_\infty(f)$ one can use the set of asymptotic critical values of $f$

$$K_\infty(f) = \{ y \in \mathbb{K} : \exists x_l \in \mathbb{K}^n, x_l \to \infty \text{ s.t. } f(x_l) \to y \text{ and } \|x_l\| \|d f(x_l)\| \to 0 \}.$$ 

If $c \notin K_\infty(f)$, then it is usual to say that $f$ satisfies Malgrange’s condition at $c$. It is proved ([16], [17]), that $B_\infty(f) \subset K_\infty(f)$. We call $K(f) = K_0(f) \cup K_\infty(f)$ the set of generalized critical values of $f$. Thus we have that in general $B(f) \subset K(f)$.

In the case $\mathbb{K} = \mathbb{C}$ we gave in [9] an algorithm to compute the set $K(f)$.

In the real case, that is for a given real polynomial $f : \mathbb{R}^n \to \mathbb{R}$ we can compute $K(f_\mathbb{C})$ the set of generalized critical values of $f_\mathbb{C}$ which stands for the complexification of $f$. However in general the set $K_\infty(f)$ of asymptotic critical values of $f$ may be smaller than $\mathbb{R} \cap K_\infty(f_\mathbb{C})$. Precisely, it is possible (c.f. Example 4.1) that there exists a sequence $x_l \in \mathbb{C}^n \setminus \mathbb{R}^n, \|x_l\| \to \infty$ such that $f(x_l) \to y \in \mathbb{R}$ and $\|x_l\| \|d f(x_l)\| \to 0$, but there is no sequence $x_l \in \mathbb{R}^n$ with this property.

To our best knowledge no method was known to detect algorithmically this situation.

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In the paper we propose another approach to the computation of generalized critical values which works both in the complex and in the real case. The main new idea is to use a finite dimensional space of rational arcs along which we can reach all asymptotic critical values.

Asymptotic and generalized critical values appear for instance in the problem of optimization of real polynomials, see e.g. papers of Hà and Pham [13], [14]. In fact, as they observed, if a polynomial \( f \) is bounded from below then \( \inf f \in K(f) \). Numerical and complexity aspects of this approach to the optimization of polynomials were studied recently by M. Safey El Din, eg. [18], [19].

2. The complex case

We start with the following variant of the Puiseux Theorem:

Lemma 2.1. Let \( C \subset \mathbb{C}^n \) be a curve of degree \( d \). Assume that \( a \in (\mathbb{P}^n \setminus \mathbb{C}^n) \cap C \) is a point at infinity of \( C \). Let \( \Gamma \) be an irreducible component of the germ \( C_a \). Then there is an integer \( s \leq d \) and a real number \( R > 0 \), such that the \( \Gamma \) has a holomorphic parametrization of the type

\[
x = \sum_{-\infty \leq i \leq s} a_i t^i, \quad |t| > R,
\]

where \( t \in \mathbb{C}, a_i \in \mathbb{C}^n \) and \( \sum_{i>0} |a_i| > 0 \).

Proof. Let \( \overline{C} \setminus C = \{a, b_1, ..., b_r\} \). First choose the affine system of coordinates in \( \mathbb{C}^n \) in a generic way. Let \( a = (0 : a_1 : ... : a_n), b_j = (0 : b_{j1} : ... b_{jn}) \in \mathbb{P}^n \). Since our system of coordinates was generic we can assume that \( a_i \neq 0 \) for \( i > 0 \) and \( b_{ji} \neq 0 \) for \( i, j > 0 \). Choose a new projective system of coordinates, at which the new hyperplane at infinity is a hyperplane \( H = \{x : x_1 = 0\} \). Take \( y_1 = x_0/x_1 \) and \( y_i = x_i/x_0 \) for \( i = 2, ..., n \). Put \( L = \{y \in \mathbb{C}^n : y_2 = y_3 = ... = y_n = 0\} \). By our construction we have \( L \cap \overline{C} = \emptyset \). In particular the projection \( \pi_L : \overline{C} \setminus H \to \mathbb{C} \) is finite. This means that there is a punctured disc \( U = \{z \in \mathbb{C} : 0 < |z| < \delta\} \) such that the mapping

\[
\rho : \Gamma \cap \pi_L^{-1}(U) \ni x \to \pi_L(x) \in U
\]

is proper. We can also assume that the set \( \Gamma' := \Gamma \cap \pi_L^{-1}(U) \) is smooth and \( \rho \) has no critical values on \( U \). In particular \( \rho \) is a holomorphic covering of degree \( s \leq d \).

In particular the function \( \rho^{-1} : U \ni z \to (z, h_2(z), ..., h_n(z)) \in \Gamma' \) is an \( s \)-valued holomorphic function. If we compose it with the mapping \( z \to z^s \) we obtain a holomorphic function. Consequently the mapping \( t \to (t^s, h_2(t^s), ..., h_n(t^s)) = (t^s, g_2(t), ..., g_n(t)) \) is holomorphic. If we go back to the old coordinates we have the following parametrization of \( \Gamma' \):

\[
t \to (1/t^s, g_2(t)/t^s, ..., g_n(t)/t^s),
\]

where \( 0 < |t| < \delta \). Now exchange \( t \) by \( 1/t \) and put \( R = 1/\delta \). □

Let \( X \subset \mathbb{C}^m \) be a variety, recall that a mapping \( F : X \to \mathbb{C}^m \) is not proper at a point \( y \in \mathbb{C}^m \) if there is no neighborhood \( U \) of \( y \) such that \( F^{-1}(U) \) is compact. In other words, \( F \) is not proper at \( y \) if there is a sequence \( x_l \to \infty \) such that \( F(x_l) \to y \).
Let $S_F$ denote the set of points at which the mapping $F$ is not proper. The set $S_F$ has the following properties (see [6], [7], [8]):

**Theorem 2.2.** Let $X \subset \mathbb{C}^m$ be an irreducible variety of dimension $n$ and let $F = (F_1, ..., F_m) : X \to \mathbb{C}^m$ be a generically-finite polynomial mapping. Then the set $S_F$ is an algebraic subset of $\mathbb{C}^m$ and it is either empty or it has pure dimension $n - 1$. Moreover, if $n = m$ then

$$\deg S_F \leq \frac{D(\prod_{i=1}^n \deg F_i) - \mu(F)}{\min_{1 \leq i \leq n} \deg F_i},$$

where $D = \deg X$ and $\mu(F)$ denotes the geometric degree of $F$ (i.e., it is a number of points in a generic fiber of $F$).

The following elementary lemma will be useful in the sequel.

**Lemma 2.3.** Assume that a holomorphic curve has parametrization of the type

$$x(t) = \sum_{-\infty \leq i \leq s} a_i t^i, \quad |t| > R,$$

where $t \in \mathbb{C}$, $a_i \in \mathbb{C}^n$, $\sum_{i > 0} |a_i| > 0$ and $s \geq 0$ is an integer. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d$. Set

$$\tilde{x}(t) = \sum_{-(d-1)s \leq i \leq s} a_i t^i, \quad |t| > R.$$

Assume that $\lim_{t \to \infty} f(x(t)) = b \in \mathbb{C}$, then

$$\lim_{t \to \infty} f(\tilde{x}(t)) = \lim_{t \to \infty} f(x(t)).$$

The same statement holds in the real case.

We also need the following obvious lemma:

**Lemma 2.4.** Let $K$ be an infinite field. Let $X \subset K^m$ be an affine variety of dimension $n$. Then a generic linear map $\pi : K^m \to K^n$ is finite on $X$.

We we state now an effective variant of the curve selection lemma:

**Theorem 2.5.** Let $F : \mathbb{C}^n \ni x \to (f_1(x), ..., f_m(x)) \in \mathbb{C}^m$ be a generically finite polynomial mapping. Assume that $\deg f_i = d_i$ and $d_1 \geq d_2 \geq ... \geq d_m$. Let $b \in \mathbb{C}^m$ be a point at which the mapping $F$ is not proper. Then there exists a rational curve with a parametrization of the form

$$x(t) = \sum_{-(d-1)d-1 \leq i \leq d} a_i t^i, \quad t \in \mathbb{C}^*,$$

where $a_i \in \mathbb{C}^n$, $\sum_{i > 0} |a_i| > 0$ and $D = \prod_{i=2}^n d_i$, $d = d_1$, such that

$$\lim_{t \to \infty} F(x(t)) = b.$$
Proof. By Lemma 2.4 we can assume that \( m = n \). Again by this lemma we can assume that the system of coordinates is sufficiently general. In particular we can assume that the line \( l = \{ x : x_2 = b_2, x_3 = b_3, \ldots, x_n = b_n \} \), where \( b = (b_1, \ldots, b_n) \), is not contained neither in the set \( S_F \) nor in the set of critical values of \( F \). Let \( C = F^{-1}(l) \). Then \( b \) is a non proper point of the mapping \( F|_C \). In particular there exists a holomorphic branch \( \Gamma \) of \( C \) such that \( \lim_{x \in \Gamma, x \to \infty} F(x) = b \). Note that \( \deg C \leq D \). By Lemma 2.4 we can assume that the branch \( \Gamma \) has a parametrization of the form:

\[
x = \sum_{-\infty \leq i \leq a} a_i t^i, \ |t| > R, \text{ and } \sum_{i>0} |a_i| > 0.
\]

From Lemma 2.3 follows that

\[
F(\sum_{-(d-1)D-1\leq i \leq D} a_i t^i) = b + \sum_{i=1}^{\infty} c_i / t^i,
\]

which proves the theorem. \( \square \)

Definition 2.6. By a rational arc we mean a curve \( \Gamma \subset \mathbb{C}^n \) which has a parametrization \( x(t) = \sum_{D_2 \leq i \leq D_1} a_i t^i, \ t \in \mathbb{C}^*, \) where \( a_i \in \mathbb{C}^n \). By a bidegree of the parametrization \( x(t) \) we mean a pair of integers \((D_1, D_2)\).

Definition 2.7. Let \( F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m \) be a generically finite polynomial mapping, which is not proper. Assume that deg \( f_i = d_i \), where \( d_1 \geq d_2 \geq \ldots \geq d_m \). By asymptotic variety of rational arcs of the mapping \( F \) we mean the variety \( AV(F) \subset \mathbb{C}^{n(2+\prod_{i=1}^m d_i)} \), which consists of those rational arcs \( x(t) \) of bidegree \((D_1, D_2)\) where \( D_1 = \prod_{i=2}^m d_i \), \( D_2 = 1 + (d_1 - 1) \prod_{i=2}^m d_i \), that

\[\begin{align*}
&a) \ F(x(t)) = b + \sum_{i=1}^{\infty} c_i / t^i, \\
&b) \ \sum_{i>0} \sum_{j=1}^n a_{ij} = 1, \text{ where } a_i = (a_{i1}, \ldots, a_{in}).
\end{align*}\]

By generalized asymptotic variety of \( F \) we mean the variety \( GAV(F) \subset \mathbb{C}^{n(2+\prod_{i=1}^m d_i)} \) defined only by the condition a).

Remark 2.8. The condition b) assures that the arc \( x(t) \) ”goes to infinity”.

Let us note that \( AV(F) \) and \( GAV(F) \) are algebraic subsets of \( \mathbb{C}^{n(2+\prod_{i=1}^m d_i)} \). We identify an arc \( x(t) \) with its coefficients \( a_{ij} \in \mathbb{C}^{n(2+\prod_{i=1}^m d_i)} \).

Moreover, if \( x(t) \in AV(F) \), respectively \( x(t) \in GAV(F) \), then \( F(x(t)) = \sum c_i(a) t^i \).

Note that the function \( c_0 : AV(F) \to \mathbb{C}^m \) plays important role:

Proposition 2.9. Let \( c_0 : AV(F) \ni a \mapsto c_0(a) \in \mathbb{C}^m \) be as above. Then

\[ c_0(AV(F)) = S_F. \]

Proof. Let \( x(t) = \sum a_i t^i \in AV(F) \). Then \( F(x(t)) = c_0(a) + \sum_{i=1}^{\infty} c_i(a) / t^i \), this implies that \( c_0(a) \in S_F \). Conversely, let \( b \in S_F \). By Theorem 2.3 we can find a rational arc \( x(t) = \sum_{i=-(d-1)D-1}^D a_i t^i \) such that \( \lim_{t \to \infty} F(x(t)) = b \). Now change the parametrization of \( x(t) \), \( t \to \lambda t \) in this way that \( \sum_{i>0} \sum_{j=1}^n \lambda^i a_{ij} = 1 \). The new arc \( x'(t) := x(\lambda t) \) belongs to \( AV(F) \) and \( c_0(x'(t)) = b \). \( \square \)
Now let \( f \in \mathbb{C}[x_1, ..., x_n] \) be a polynomial. Let us define a polynomial mapping \( \Phi : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^n \) by

\[
\Phi = (f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, ..., h_{nn}),
\]
where \( h_{ij} = x_i \frac{\partial f}{\partial x_j}, \ i = 1, ..., n, \ j = 1, ..., n. \)

**Definition 2.10.** Let \( \Phi \) be as above. Consider the mapping \( c_0 : AV(\Phi) \to \mathbb{C}^n \) and the line \( L := \mathbb{C} \times \{(0, ..., 0)\} \subset \mathbb{C} \times \mathbb{C}^n \). By \( a \) we mean a variety

\[
BV(f) = \{x(t) \in AV(F) : x(t) \in c_0^{-1}(L)\}.
\]

Similarly, we define a generalized bifurcation variety of rational arcs of the polynomial \( f \):

\[
GBV(f) = \{x(t) \in GAV(F) : x(t) \in c_0^{-1}(L)\}.
\]

As an immediate consequence of [9] we have:

**Proposition 2.11.** Let \( K(f) = K_0(f) \cup K_\infty(f) \) denote the set of generalized critical values of \( f \). If we identify the line \( L = \mathbb{C} \times \{(0, ..., 0)\} \subset \mathbb{C} \times \mathbb{C}^n \) with \( \mathbb{C} \), then we have \( c_0(BV(\Phi)) = K_\infty(f) \) and \( c_0(GBV(\Phi)) = K(f) \).

3. Algorithm

In this section we give an algorithm to compute the set \( K_\infty(f) \) of asymptotic critical values as well as the set \( K(f) \) of generalized critical values of a complex polynomial \( f \). Let \( \deg f = d \) and \( D_1 = d^{n-1}, D_2 = d^n - d^{n-1} + 1 \).

**Algorithm for the set \( K_\infty(f) \).**

1) Compute equations for the variety \( BV(f) \):
   a) consider the arc \( x(t) = \sum_{t=0}^{D_1} a_it^i \in \mathbb{C}^{n(D_1+D_2+1)} \)
   b) compute \( f(x(t)) = \sum c_i(a)t^i \),
   c) compute \( \frac{\partial f}{\partial x_i}(x(t)) = \sum d_{ik}(a)t^i, i = 1, ..., n, \)
   d) compute \( \frac{\partial f}{\partial x_i}(x(t))x_j(t) = \sum e_{ijk}(a)t^i, i, j = 1, ..., n \)
   e) equations for \( BV(f) \) are \( c_i = 0 \) for \( i > 0, d_{ik} = 0 \) for \( k \geq 0, i = 1, ..., n, e_{ijk} = 0 \) for \( k \geq 0, i, j = 1, ..., n \) and \( \sum_{i > 0} \sum_{j=1}^{n} a_{ij} = 1, \) where \( a_{ij} = (a_{i1}, ..., a_{in}) \).

2) Find equations for irreducible components of \( BV(f) = \bigcup_{i} P_i \). It can be done by standard method of computational algebra. We can use e.g., the MAGMA system and radical decomposition of ideal in this system (see also [1]).

3) Find a point \( x_i \in P_i \). It can be also done by standard methods. We can use e.g. the MAGMA system and use several time the elimination procedure in this system. Indeed Let \( P_i = V(f) \). Compute \( I_k = \mathbb{C}[x_1, ..., x_k] \cap I \) for \( k = n, n - 1, ..., 1 \) until \( I_k = (0) \). Then take randomly chosen integer point \((a_1, ..., a_k)\) find a zero \((a_1, ..., a_k, b_i)\) of ideal \( I_{k+1} \) and so on.

4) \( K_\infty(f) = \{c_0(x_i), i = 1, ..., r\} \).
If we replace above the variety $BV(f)$ by the variety $GAV(f)$ we get an algorithm for computing $K(f)$. Indeed, it is enough to delete at point 1 e) the equation $\sum_{i>0} \sum_{j=1}^{n} a_{ij} = 1$.

4. The real case

We begin with a simple example.

**Example 4.1.** For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ consider $f_{\mathbb{K}} : \mathbb{K}^2 \to \mathbb{K}, f(x, y) = x(x^2 + 1)^2$. Observe that $K_{\infty}(f_{\mathbb{R}}) = K_0(f_{\mathbb{R}}) = \emptyset$. But $0 \in K_{\infty}(f_{\mathbb{C}}) = K_0(f_{\mathbb{C}})$. So in general $K_{\infty}(f_{\mathbb{R}}) \neq \mathbb{R} \cap K_{\infty}(f_{\mathbb{C}})$.

It shows that the computation of the asymptotic critical values of a real polynomial can not be reduced to the computation of the asymptotic critical values of its complexification.

4.1. Effective curve selection lemma at infinity. First we give a construction of a curve selection in a special affine case.

Let $X \subset \mathbb{R}^{2n}$ be an algebraic set described by a system of polynomial equations $p_i = 0$, $\deg p_i \leq d$, where $i = 1, \ldots, n$. Denote by $H$ the hyperplane $\{x_1 = 0\}$. Assume that on $Y := X \setminus H$ the system is non degenerate i.e. $P = (p_1, \ldots, p_n) : \mathbb{R}^{2n} \to \mathbb{R}^n$ is a submersion at each point of $Y$. Thus $Y$ is a smooth manifold of dimension $n$.

**Proposition 4.2.** Let $a \in H \cap \overline{Y}$ then there exists an algebraic curve $C \subset \mathbb{R}^{2n}$ of degree $D \leq d^n((d - 1)^n + 2)^{n-1}$ such that $a \in C \cap Y$.

**Proof.** For simplicity we assume that $a = 0$. Denote $\rho(x) = (\sum_{i=1}^{2n} x_i^2)^{\frac{1}{2}}$ and by $S(r)$ the sphere centered at 0 of radius $r$ and finally $Y(r) := Y \cap S(r)$. With our hypothesis we have.

**Lemma 4.3.** There exists $\varepsilon > 0$ such that for any $r \in (0, \varepsilon)$ the set $Y(r)$ is a smooth manifold of dimension $n - 1$, in particular is nonempty.

Indeed, the function $\rho$ restricted to the manifold $Y$ is smooth and semialgebraic, so it has finitely many critical values (see e.g. [3]). Let $\varepsilon_1 > 0$ be the smallest critical value (or $\varepsilon_1 = 1$ if there are no critical values).

On the other hand $\rho|_Y : Y \to \mathbb{R}_+$ is locally trivial (by Hardt’s trivialization theorem cf. [3] or [4]), so there is $\varepsilon_2 > 0$ such that for any $r, r' \in (0, \varepsilon_2)$ the sets $Y(r)$ and $Y(r')$ are homeomorphic. Since $a \in H \cap \overline{Y}$, then there is $r' \in (0, \varepsilon_2)$ such that $Y(r')$ is nonempty. Hence $Y(r)$ is nonempty for any $r \in (0, \varepsilon_2)$. Finally we put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

Let us now consider a family of functions $g_\alpha : \mathbb{R}^{2n} \to \mathbb{R}$ of the form $g_\alpha(x) := x_1 \alpha(x)$, where $\alpha \in (\mathbb{R}^{2n})^*$ is a linear function on $\mathbb{R}^{2n}$.

**Lemma 4.4.** For any $r \in (0, \varepsilon)$ there exists an algebraic set $A_r \subset (\mathbb{R}^{2n})^*$ such that $g_\alpha$ is a Morse function on $Y(r)$, for any $\alpha \notin A_r$. Moreover the set $\bigcup_{\varepsilon \in (a, \varepsilon)} \{r\} \times A_r$ is contained in a proper algebraic set $A \subset \mathbb{R} \times (\mathbb{R}^{2n})^*$.
The proof of the lemma uses standard arguments in Morse theory, see e.g. [11], [12]. Consider a map

$$\Phi : Y(r) \times (\mathbb{R}^{2n})^* \to (\mathbb{R}^{2n})^*$$

given by $$\Phi(x, \alpha) = d_x g_\alpha$$. It is enough to show that $$\Phi$$ is a submersion. Indeed the Jacobian matrix of $$\Phi$$ (with respect to variables in $$(\mathbb{R}^{2n})^*$$) is triangular with $$x_1$$ on the diagonal except the entry in the left superior corner where it is $$2x_1$$. So this matrix is invertible since $$x_1 \neq 0$$ for $$x \in Y(r)$$. The second statement follows from the fact that set $$A_r$$ is defined by polynomial equations with $$r$$ as a variable parameter.

**Lemma 4.5.** There exists $$\alpha \in (\mathbb{R}^{2n})^*$$ and $$0 < \varepsilon' \leq \varepsilon$$ such that $$g_\alpha$$ is a Morse function on each $$Y(r) \cap S(r)$$, for any $$r \in (0, \varepsilon')$$.

Indeed, let $$A \subset \mathbb{R} \times (\mathbb{R}^{2n})^*$$ by the proper algebraic set in Lemma 4.4. Thus there exists an affine line $$\mathbb{R} \times \alpha$$ which meets the set $$A$$ only in finitely many points. So $$(0, \varepsilon') \times \{\alpha\}$$ is disjoint with $$A$$, for some $$\varepsilon' > 0$$ small enough.

Since $$Y(r)$$ is not compact so a priori it is not obvious that $$g_\alpha$$ has a critical point on $$Y(r)$$. However we have.

**Lemma 4.6.** Assume that $$Y(r) \neq \emptyset$$ and that $$g_\alpha$$ is Morse on $$Y(r)$$. Then $$g_\alpha$$ has a critical point on $$Y(r)$$.

Note that image of $$Y(r)$$ by $$g_\alpha$$ consists of finitely many nontrivial intervals. Since $$\overline{Y(r)}$$ is compact and

$$\overline{(Y(r) \setminus Y(r))} \subset \{x_1 = 0\},$$

Hence at least one of the endpoints of those intervals belongs to $$g_\alpha(Y(r))$$. Thus $$g_\alpha$$ achieves a minimum or a maximum in $$Y(r) \cap S(r)$$.

**Proof of Proposition 4.2.** Let us fixe a linear form $$\alpha$$ which satisfies Lemma 4.5. Let $$\Xi$$ be the locus of critical points of $$g_\alpha$$ on $$Y(r)$$ for $$r \in (0, \varepsilon')$$. The Zariski closure of $$\Xi$$ is contained in the algebraic set given by the following equations

$$p_1 = \cdots = p_n = 0$$

and

$$dp_1 \wedge \cdots \wedge dp_n \wedge d\rho^2 \wedge dg_\alpha = 0.$$  

Let us fix $$\Xi_1$$ a smooth connected component of $$\Xi$$ such that $$a \in \overline{\Xi_1}$$. Then locally $$\Xi_1$$ is given by a non degenerate system (4.1) of $$n$$ equation of degree at most $$d$$ and $$n - 1$$ equations of degree at most $$(d - 1)^n + 2$$, which are $$(n + 2) \times (n + 2)$$ minors of the matrix corresponding to the system (4.2).

Let $$C$$ be the Zariski closure of $$\Xi_1$$. Hence by the general Bezout’s formula (cf. e.g. [3] Thm.2.2.5) degree of the curve $$C$$ is at most $$d^n((d - 1)^n + 2)^{n-1}$$, note that for $$d \geq 3$$ we have $$d^n((d - 1)^n + 2)^{n-1} \leq d^{n^2}$$.

$$\square$$

We can state now a real version of Theorem 2.5.

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Theorem 4.7. Let $F : \mathbb{R}^n \ni x \to (f_1(x), \ldots, f_m(x)) \in \mathbb{R}^m$ be a polynomial mapping. Assume that $\deg f_i \leq d$. Let $b \in \mathbb{R}^m$ be a point at which the mapping $F$ is not proper. Then there exists a rational curve with a parametrization of the form

$$x(t) = \sum_{-(d-1)D-1 \leq i \leq D} a_i t^i, \ t \in \mathbb{R}^n,$$

where $a_i \in \mathbb{R}^n, \sum_{i \geq 0} |a_i| > 0$ and $D = (d + 1)^n (d^n + 2)^{n-1}$ such that

$$\lim_{t \to \infty} F(x(t)) = b.$$

Remark 4.8. Note that contrary to the complex case we do not assume that $F$ is generically finite, but the bidegree of the real rational curve is much higher, namely $D = O(d^n^2)$.

Proof. By Lemma 2.14 we can assume that $m = n$. Let $\gamma(t) \in \mathbb{R}^n$ be a semi-algebraic curve such that $\lim_{t \to \infty} |\gamma(t)| = +\infty$ and $\lim_{t \to \infty} F(\gamma(t)) = b$. Let $\tilde{\gamma}(t)$ be the image of $\gamma(t)$ by the canonical imbedding $\mathbb{R}^n \ni (x_1, \ldots, x_n) \to (1 : x_1 : \cdots : x_n) \in \mathbb{P}^n$. Since $\gamma$ is semi-algebraic there exists $a := \lim_{t \to \infty} \tilde{\gamma}(t) \in H_0$, where $H_0$ stands for the hyperplane at infinity. Let us denote by $Y$ the graph of $F$, embedded in $\mathbb{P}^n \times \mathbb{R}^n$ and by $X$ its Zariski closure in $\mathbb{P}^n \times \mathbb{R}^n$. Note that the point $(a, b)$ belongs to the closure (in the strong topology) of $Y$. Let $\bar{f}_i$ stands for the homogenization of the polynomial $f_i$, that is

$$\bar{f}_i(x_0, x_1, \ldots, x_n) = x_0^d f_i\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Hence $X \subset \mathbb{P}^n \times \mathbb{R}^n$ is defined by the equations

$$x_0^d y_i = \bar{f}_i(x_0, x_1, \ldots, x_n), \ i = 1, \ldots, n.$$

Assume that $a_1 = 1$, so $\mathbb{R}^n \ni (x_0, x_2, \ldots, x_n) \mapsto (x_0, 1, x_2, \ldots, x_n) \in \mathbb{P}^n$ is an affine chart around the point $a$. In this chart $X$ is given by the equations

$$p_i(x_0, x_2, \ldots, x_n, y_1, \ldots, y_n) := x_0^d y_i - \bar{f}_i(x_0, 1, x_2, \ldots, x_n) = 0,$$

$i = 1, \ldots, n$. Clearly $\deg p_i = 1 + d$, and $X = Y \setminus \{x_0 = 0\}$ in this chart. So we may apply Proposition 4.12 at the point $(a, b)$. Hence there exists an algebraic curve $C_1 \subset \mathbb{R}^{2n}$ of degree $d_* \leq (d + 1)^n (d^n + 2)^{n-1}$ such that $(a, b) \in C_1 \cap Y$. Now take $C$ the projection of $C_1$ on $\mathbb{P}^n$. Note that degree of $C$ is less or equal than $d_*$. Now we can argue as in the proof of Theorem 2.5 to conclude that there is a real rational arc $x(t)$ of bidegree $D_2 = (d - 1)D + 1$, $D_1 = D$ where $D = (d + 1)^n (d^n + 2)^{n-1}$, such that $\lim_{t \to \infty} F(x(t)) = b$.

Definition 4.9. By a real rational arc we mean a curve $\Gamma \subset \mathbb{R}^n$ which has a parametrization $x(t) = \sum_{-D_2 \leq i \leq D_2} a_i t^i, \ t \in \mathbb{R}^n$, where $a_i \in \mathbb{C}^n$. By bidegree of $\Gamma$ we mean a pair of integers $(D_1, D_2)$.

Definition 4.10. Let $F = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a generically finite polynomial mapping, which is not proper. Assume that $\deg f_i \leq d$ and $D = (d + 1)^n (d^n + 2)^{n-1}$ By asymptotic variety of real rational arcs of the mapping $F$ we mean the
variation $AV_{\mathbb{R}}(F) \subset \mathbb{R}^{n(dD+2)}$, which consists of those real rational arcs $x(t)$ of bidegree $(D, (d-1)D + 1)$ that

a) $F(x(t)) = b + \sum_{i=1}^{\infty} c_i/t^i$,

b) $\sum_{i>0} \sum_{j=1}^{n} a_{ij}^2 = 1$, where $a_i = (a_{i1}, ..., a_{in})$.

If we omit the condition b) we get definition of a generalized asymptotic variety of real arcs $GAV_{\mathbb{R}}(F)$.

**Remark 4.11.** The condition b) assures that the arc $x(t)$ "goes to infinity".

As before we see that $AV_{\mathbb{R}}(F), GAV_{\mathbb{R}}$ are algebraic subsets of $\mathbb{R}^{n(dD+2)}$. Moreover, for $x(t) \in AV_{\mathbb{R}}(F)(GAV_{\mathbb{R}}(F))$ we have $F(x(t)) = \sum c_i(a)t^i$. Note that again the function $c_0 : AV_{\mathbb{R}}(F) \to \mathbb{R}^m$ plays important role:

**Proposition 4.12.** Let $c_0 : AV_{\mathbb{R}}(F) \in a \mapsto c_0(a) \in \mathbb{R}^m$ be as above. Then

$c_0(AV_{\mathbb{R}}(F)) = S_F(\mathbb{R})$.

**Proof.** Let $x(t) = \sum_{i=1}^{\infty} a_i t^i \in AV_{\mathbb{R}}(F)$. Then $F(x(t)) = c_0(a) + \sum_{i=1}^{\infty} c_i(a)/t^i$, this implies that $c_0(a) \in S_F$. Conversely, let $b \in S_F$. By Theorem 4.7 we can find a rational arc $x(t) = \sum_{i=-1}^{dD-1} a_i t^i$ such that $\lim_{t \to \infty} F(x(t)) = b$. There exist a $\lambda \in \mathbb{R}$ such that $\sum_{i>0} \sum_{j=1}^{n} a_{ij}^2 = 1$. Change a parametrization by $t \to \lambda t$. The new arc $x'(t) := x(\lambda t)$ belongs to $AV_{\mathbb{R}}(F)$ and $c_0(x'(t)) = b$. □

Now let $f \in \mathbb{R}[x_1, ..., x_n]$ be a polynomial. Let us define a polynomial mapping $\Phi : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^N$ by

$\Phi = (f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, ..., h_{nn})$,

where $h_{ij} = x_i \frac{\partial f}{\partial x_j}$, $i = 1, 2, ..., n, \quad j = 1, ..., n$.

**Definition 4.13.** Let $\Phi$ be as above. Consider the mapping $c_0 : AV_{\mathbb{R}}(\Phi) \to \mathbb{R}^N$ and the line $L := \mathbb{R} \times \{(0, ..., 0)\} \subset \mathbb{R} \times \mathbb{R}^N$. By a bifurcation variety of real rational arcs of the polynomial $f$ we mean a variety

$BV_{\mathbb{R}}(f) = \{x(t) \in AV_{\mathbb{R}}(\Phi) : x(t) \in c_0^{-1}(L)\}$.

Similarly we define

$GBV_{\mathbb{R}}(f) = \{x(t) \in GAV_{\mathbb{R}}(\Phi) : x(t) \in c_0^{-1}(L)\}$.

As an immediate consequence of [9] we have:

**Proposition 4.14.** Let $K(f)(\mathbb{R}) = K_0(f) \cup K_\infty(f)$ denote the set of generalized critical values of real polynomial $f$. If we identify the line $L = \mathbb{R} \times \{(0, ..., 0)\} \subset \mathbb{R} \times \mathbb{R}^N$ with $\mathbb{R}$, then we have $c_0(BV_{\mathbb{R}}(f)) = K_\infty(f)$ and $c_0(GBV_{\mathbb{R}}(f)) = K(f)$. 
5. Real algorithm

In this section we describe an algorithm to compute the set \( K_\infty(f) \) of asymptotic critical values as well as the set \( K(f) \) of generalized critical values of a real polynomial \( f \in \mathbb{R}[x_1, \ldots, x_n] \). Let \( \deg f = d \) and \( D_1 = (d+1)^n(d^n+2)^{n-1}, D_2 = (d-1)D_1+1. \)

**Algorithm for the set \( K_\infty(f) \).**

1) Compute equations \( g_\alpha \) for the variety \( BV_\mathbb{R}(f) \):
   a) consider the arc \( x(t) = \sum_{i=0}^{D_1} a_i t^i \in \mathbb{R}^{n(D_1+D_2+1)} \)
   b) compute \( f(x(t)) = \sum c_i(a)t^i \),
   c) compute \( \frac{\partial f}{\partial x_i}(x(t)) = \sum d_{ik}(a)t^k, i = 1, 2 \),
   d) compute \( \frac{\partial f}{\partial x_i}(x(t))x_j(t) = \sum e_{ijk}(a)t^k, i, j = 1, 2 \),
   e) equations for \( BV_\mathbb{R}(f) \) are \( c_i = 0 \) for \( i > 0 \), \( d_{ik} = 0 \) for \( k \geq 0, i = 1, \ldots, n \), \( e_{ijk} = 0 \) for \( k \geq 0, i, j = 1, \ldots, n \) and \( \sum_{i>0} \sum_{j=1}^n a_{ij}^2 = 1 \), where \( a_i = (a_{1i}, \ldots, a_{ni}) \).

2) Form a polynomial \( G = \sum a_\alpha g_\alpha \) where \( g_\alpha \) are \( c_i \) for \( i > 0 \), or \( d_{ik} \) for \( k \geq 0, i = 1, \ldots, n \), or \( e_{ijk} \) for \( k \geq 0, i, j = 1, \ldots, n \) or \( \sum_{i>0} \sum_{j=1}^n a_{ij}^2 = 1 \).

3) In each connected component \( S_i \) of the set \( G = 0 \) find a point \( x_i \in S_i, i = 1, \ldots, r \). It can be done by standard method of computational algebra, e.g. Theorem 15.13 p. 585 in [2].

4) \( K(f) = \{c_0(x_i), i = 1, \ldots, r \} \).

If we replace above the variety \( BV_\mathbb{R}(f) \) by the variety \( GAV_\mathbb{R}(f) \) we get an algorithm for computing \( K(f) \). Actually it is enough to delete at points 1 e) and 2) the equation \( \sum_{i>0} \sum_{j=1}^n a_{ij}^2 = 1 \).

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REACHING GENERALIZED CRITICAL VALUES OF A POLYNOMIAL

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Abstract. Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$. We give an algorithm to compute the set of generalized critical values. The algorithm uses a finite dimensional space of rational arcs along which we can reach all generalized critical values of $f$.

1. Introduction.

Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Over forty years ago R. Thom [27] proved that $f$ is a $C^\infty$-fibration outside a finite subset of the target; the smallest such a set is called the bifurcation set of $f$, we denote it by $B(f)$. In a natural way appears a fundamental question: how to determine the set $B(f)$?

Let us recall that in general the set $B(f)$ is bigger than $K_0(f)$, the set of critical values of $f$. It also contains the set $B_\infty(f)$ of bifurcations points at infinity. Roughly speaking, $B_\infty(f)$ consists of points at which $f$ is not a locally trivial fibration at infinity (i.e., outside a large ball). For $n = 2$ in the complex case the set $B_\infty(f)$ can be effectively computed (see e.g., [15], [26]). In the real case and $n = 2$ the set $B_\infty(f)$ has an explicit description, see [4]. For $n \geq 3$ the computation of $B_\infty(f)$ is a challenging open problem even for $\mathbb{K} = \mathbb{C}$. To control the set $B_\infty(f)$ one can use the set of asymptotic critical values of $f$

$$K_\infty(f) = \{y \in \mathbb{K} : \exists x_\nu \in \mathbb{K}^n, x_\nu \to \infty \text{ s.t. } f(x_\nu) \to y, \|x_\nu\|\|df(x_\nu)\| \to 0\}.$$ 

If $c \notin K_\infty(f)$, then it is usually said that $f$ satisfies Malgrange’s condition at $c$. It is known (20, 21) that $B_\infty(f) \subset K_\infty(f)$. We call $K(f) = K_0(f) \cup K_\infty(f)$ the set of generalized critical values of $f$. Thus in general $B(f) \subset K(f)$. In the case $\mathbb{K} = \mathbb{C}$ we gave in [10] an algorithm to compute the set $K(f)$.

In the real case, that is, for a given real polynomial $f : \mathbb{R}^n \to \mathbb{R}$ we can compute $K(f_\mathbb{C})$, the set of generalized critical values of the complexification $f_\mathbb{C}$ of $f$. However in general the set $K_\infty(f)$ of asymptotic critical values of $f$ may be smaller than $\mathbb{R} \cap K_\infty(f_\mathbb{C})$. Precisely, it is possible (see Example 6.1) that there exist a sequence $x_\nu \in \mathbb{C}^n \setminus \mathbb{R}^n$ with $\|x_\nu\| \to \infty$ such that $f(x_\nu) \to y \in \mathbb{R}$ and $\|x_\nu\|\|df(x_\nu)\| \to 0$, but

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there is no sequence $x_\nu \in \mathbb{R}^n$ with this property. To our best knowledge no method was known to detect algorithmically this situation.

In the paper we propose another approach to the computation of generalized critical values which works both in the complex and in the real case. The main new idea is to use a finite dimensional space of rational arcs along which we can reach all asymptotic critical values.

Asymptotic and generalized critical values appear for instance in the problem of optimization of real polynomials (see e.g., papers of Hà and Pham [16], [17]). In fact, as they observed, if a polynomial $f$ is bounded from below then $\inf f \in K(f)$. Numerical and complexity aspects of this approach to the optimization of polynomials were studied recently by M. Safey El Din [23], [24].

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2. Preliminaries

Here we assume that $\mathbb{K} = \mathbb{C}$. If $f : X \to Y$ is a dominant, generically finite polynomial map of smooth affine varieties and $\{x\}$ is an isolated component of the fiber $f^{-1}(f(x))$, then we denote the multiplicity of $f$ at $x$ by $\text{mult}_x(f)$ and the number of points in a generic fiber of $f$ by $\mu(f)$.

Let $X, Y$ be affine varieties. Recall that a mapping $F : X \to Y$ is not proper at a point $y \in Y$ if there is no neighborhood $U$ of $y$ such that $F^{-1}(U)$ is compact. In other words, $F$ is not proper at $y$ if there is a sequence $x_\nu \to \infty$ such that $F(x_\nu) \to y$. Let $S_F$ denote the set of points at which the mapping $F$ is not proper. The set $S_F$ has the following properties (see [7], [8], [9]):

**Theorem 2.1.** Let $X \subset \mathbb{C}^k$ be an irreducible variety of dimension $n$ and let $F = (F_1, \ldots, F_m) : X \to \mathbb{C}^m$ be a generically finite polynomial mapping. Then the set $S_F$ is an algebraic subset of $\mathbb{C}^m$ and it is either empty or of pure dimension $n - 1$. Moreover, if $n = m$ then

$$\deg S_F \leq \frac{D(\prod_{i=1}^m \deg F_i) - \mu(F)}{\min_{1 \leq i \leq n} \deg F_i},$$

where $D = \deg X$ and $\mu(F)$ denotes the geometric degree of $F$ (i.e., the number of points in a generic fiber of $F$).

In the case of a polynomial map of normal affine varieties it is easy to show the following.

**Proposition 2.2.** Let $f : X \to Y$ be a dominant polynomial map of normal affine varieties. Then $f$ is proper at $y \in Y$ if and only if $f^{-1}(y) = \{x_1, \ldots, x_r\}$ is a finite set and $\sum_{i=1}^r \text{mult}_{x_i}(f) = \mu(f)$.

Actually we prove a stronger result.

**Proposition 2.3.** Let $f : X \to Y$ be a dominant polynomial map of normal affine varieties. Assume that points $x_1, \ldots, x_r$ are isolated components of the fiber $f^{-1}(y)$. If $\sum_{i=1}^r \text{mult}_{x_i}(f) = \mu(f)$, then $f$ is proper at $y$ and $f^{-1}(y) = \{x_1, \ldots, x_r\}$. 
**Proof.** Let $\overline{X}$ be the projective completion of $X$. Let $V$ be the normalization of the closure of the graph $f$ in $\overline{X} \times Y$. Then $X \subset V$ and there is a proper mapping $\overline{f} : V \rightarrow Y$ such that $\overline{f}|_X = f$. By the Stein Factorization Theorem (see [6], pp. 141-142), there exists a normal variety $W$, a finite morphism $h : W \rightarrow Y$, and a surjective morphism $g : V \rightarrow W$ with connected fibers, such that $\overline{f} = h \circ g$. In particular the mapping $g$ is birational. Let $w_i = g(x_i)$. By the Zariski Main Theorem the mapping $g$ is a local biholomorphism near $x_i$. This means that $\text{mult}_{x_i} \overline{f} = \text{mult}_{x_i} f = \text{mult}_{w_i} h$. Since the mapping $h$ is finite and $\mu(h) = \mu(f)$, we have $h^{-1}(y) = \{w_1, \ldots, w_r\}$. Consequently, $\overline{f}^{-1}(y) = \{x_1, \ldots, x_r\}$, which implies that $f$ is proper at $y$. \qed

The next fact will be useful.

**Proposition 2.4.** Let $X$ be a normal affine variety of dimension $n$ and let $f : X \rightarrow \mathbb{C}^n$ be a dominant mapping. Let $b \in \mathbb{C}^n$. If there exists a relatively compact set $K \subset X$ and a sequence of points $b_i \in \mathbb{C}^n$ such that

1) $f^{-1}(b) \subset K$ and $f^{-1}(b_i) \subset K$ for $i \in \mathbb{N}$,

2) $\# f^{-1}(b_i) = \mu(f)$,

3) $\lim_{i \rightarrow \infty} b_i = b$,

then $f$ is proper at $b$.

**Proof.** Since $X$ is locally compact we can assume that the set $K$ is open. Let $U_n \subset Y$ be a ball of radius $1/n$ with center at $b$. Assume that for any $n$ the mapping $f : K \cap f^{-1}(U_n) \rightarrow U_n$ is not proper. Since the mapping $f : K \cap f^{-1}(U_n) \rightarrow U_n$ is proper, we have $K \cap f^{-1}(U_n) \neq K \cap f^{-1}(U_n)$. Take $x_n \in K \cap f^{-1}(U_n) \setminus K \cap f^{-1}(U_n)$. Since $K$ is compact, the sequence $x_n$ has an accumulation point $x \in K \setminus K$. But $f(x) = y$, which contradicts the fact that $f^{-1}(y) \subset K$.

Consequently, the mapping $f : K \cap f^{-1}(U_n) \rightarrow U_n$ is proper for some $n$. Thus it is an analytic cover (see Theorem 21, p. 108 in [12]). Moreover, it is easily seen that this cover is $\mu(f)$-sheeted. In particular for any $y \in U_n$ we have $\sum_{f(x) = y, x \in K} \text{mult}_x(f) = \mu(f)$. To conclude we apply Proposition 2.3. \qed

### 3. Parametrizing branches at infinity

We begin with a variant of the Puiseux Theorem.

**Lemma 3.1.** Let $C \subset \mathbb{C}^n$ be an algebraic curve of degree $d$. Assume that $a \in (\mathbb{P}^n \setminus \mathbb{C}^n) \cap \overline{C}$ is a point at infinity of $C$. Let $\Gamma$ be an irreducible component of the germ $\overline{C}_a$. Then there exist an integer $s \leq d$ and a real number $R > 0$ such that $\Gamma$ has a holomorphic parametrization of the type

$$x = \sum_{-\infty \leq \ell \leq s} a_i t^\ell, \quad |t| > R,$$

where $t \in \mathbb{C}$, $a_i \in \mathbb{C}^n$ and $\sum_{i > 0} |a_i| > 0$.

Note that $\overline{C}$, the closure of $C$ in $\mathbb{P}^n$, is the same for the strong topology and the Zariski topology, since we work over $\mathbb{C}$. So in fact $\overline{C}$ is an algebraic set in $\mathbb{P}^n$. 
Proof. First choose an affine system of coordinates in $\mathbb{C}^n$ in a generic way. Let $a = (0 : a_1 : \cdots : a_n) \in \mathbb{P}^n$. We can assume that $a_1 \neq 0$. Now choose coordinates in the affine chart $U_1 = \mathbb{P}^n \setminus H$, where $H = \{ x : x_1 = 0 \}$. Take $y_1 = x_0/x_1$ and $y_i = x_i/x_1$ for $i = 2, \ldots, n$. Put $L = \{ x \in H : x_0 = 0 \}$. By our assumption we have $L \cap C = \emptyset$. In particular the projection $\pi_L : C \setminus H \ni (y_1, \ldots, y_n) \mapsto y_1 \in \mathbb{C}$ is finite. So there is a punctured disc $U_\delta^* = \{ z \in \mathbb{C} : 0 < |z| < \delta \}$ such that the mapping

$$
\rho : \Gamma \cap \pi_L^{-1}(U_\delta^*) \ni x \mapsto \pi_L(x) \in U_\delta^*
$$

is proper. We can also assume that the set $\Gamma' := \Gamma \cap \pi_L^{-1}(U_\delta^*)$ is connected smooth and $\rho$ has no critical values on $U_\delta^*$. In particular $\rho$ is a holomorphic covering of degree $s \leq d$.

In particular $\rho^{-1} : U_\delta^* \ni z \mapsto (z, h_2(z), \ldots, h_n(z)) \in \Gamma'$ is an $s$-valued holomorphic function. If we compose it with $z \mapsto z^s$ we obtain a holomorphic bounded function on $U_{\delta^{1/s}}$. By Riemann’s theorem on removable singularities this function extends to a holomorphic function on the disc $U_{\delta^{1/s}} = \{ z \in \mathbb{C} : |z| < \delta^{1/s}\}$. Consequently, the mapping

$$
z \mapsto (z^s, h_2(z^s), \ldots, h_n(z^s)) = (z^s, g_2(z), \ldots, g_n(z))
$$

is holomorphic in $U_{\delta^{1/s}}$. (Precisely we have $s$ such functions which are of the form $(z^s, g_2(\xi z), \ldots, g_n(\xi z))$, where $\xi^s = 1$.) Hence in the original coordinates we have

$$
z \mapsto (1/z^s, g_2(z)/z^s, \ldots, g_n(z)/z^s).
$$

Now put $z = t^{-1}$, where $|t| > R := 1/\delta$. So

$$
t \mapsto (t^s, t^s g_2(t^{-1}), \ldots, t^s g_n(t^{-1}))
$$

is the desired parametrization of $\Gamma$. 

To state the real version let us recall some well known facts (see e.g., [18], Preliminaries). Consider the standard embedding of $\mathbb{R}^n$ in its projective closure denoted by $\mathbb{P}^n(\mathbb{R})$. Let $C \subset \mathbb{R}^n$ be an algebraic curve, denote by $\overline{C}^Z$ the Zariski closure of $C$ in $\mathbb{P}^n(\mathbb{R})$ and by $\overline{C}$ the closure of $C$ in $\mathbb{P}(\mathbb{R}^n)$ for the strong topology. Clearly $\overline{C} \subset \overline{C}^Z$ and $\overline{C}^Z \setminus \overline{C}$ is a finite (possibly empty) set. Hence, if $a \in (\mathbb{P}^n(\mathbb{R}) \setminus \mathbb{R}^n) \cap \overline{C}$, then we have the equality of germs $\overline{C}_a = \overline{C}^Z_a$. Finally, recall that $\Gamma$ is an irreducible component of the germ of a real algebraic curve $A$ at $a \in \mathbb{P}^n$, which is a non-isolated point of $A$, if and only if there exists a real analytic injective parametrization $\gamma : (-\delta, \delta) \ni a \mapsto A \subset \mathbb{P}^n(\mathbb{R})$ of $\Gamma$. By the degree of a real algebraic curve we mean the degree of its complexification.

**Lemma 3.2.** Let $C \subset \mathbb{R}^n$ be a real algebraic curve of degree $d$. Assume that $a \in (\mathbb{P}^n(\mathbb{R}) \setminus \mathbb{R}^n) \cap \overline{C}$ is a point at infinity of $C$. Let $\Gamma$ be an irreducible component of the germ $\overline{C}_a$. Then there is an integer $s \leq d$ and a real number $R > 0$ such that $\Gamma$ has a real analytic parametrization of the type

$$
x = \sum_{-\infty \leq i \leq s} a_i t^i, \quad |t| > R,
$$


where \( t \in \mathbb{R} \), \( a_i \in \mathbb{R}^n \) and \( \sum_{i>0} |a_i| > 0 \).

**Proof.** We shall explain how to adapt the proof of Lemma 3.1 to the real case. Denote by \( C' \subset \mathbb{P}^n \) the complexification of the real curve \( C_\mathbb{R} \). Hence the complex conjugation \( \sigma \) acts as an involution on \( C' \), moreover \( C_\mathbb{R} \) is the set of fixed points of \( \sigma \). Note that in the proof of Lemma 3.1 we can choose real coordinates, so the action of \( \sigma \) will be preserved. We shall use the action of \( \sigma \) on the germ \( C'_{\mathbb{R}} \). By the assumption there exists a branch of \( C'_{\mathbb{R}} \) which contains the real branch \( \Gamma \); we call this branch \( \Gamma' \). So \( \sigma(\Gamma') = \Gamma' \), which means that the graph of the multivalued function \( h = (h_2, \ldots, h_n) \) is invariant under the action of \( \sigma \). The set \( \Gamma_s := \{ (z^n, h(z^n)) : z \in U_{\delta/2/s}^* \} \) is a disjoint union of graphs of \( s \) holomorphic functions in the punctured disc \( U_{\delta/2/s}^* \).

Note that \( \sigma(\Gamma_s) = \Gamma_s \). Since one of those graphs contains real points (coming from the real branch \( \Gamma \)), this graph is stable under \( \sigma \). We call this function \( g = (g_2, \ldots, g_n) \). Therefore \( g_j(\bar{z}) = g_j(z) \) for \( j = 2, \ldots, n \), which means that all coefficients in the power series expansion (at \( 0 \in \mathbb{C} \)) of \( g_j \) are real. Hence the lemma follows.

\[ \square \]

The following elementary but useful lemma can be checked by direct computations.

**Lemma 3.3.** Assume that a holomorphic curve has a parametrization of the type

\[ x(t) = \sum_{-\infty \leq i \leq s} a_i t^i, \quad |t| > R, \]

where \( t \in \mathbb{C} \), \( a_i \in \mathbb{C}^n \), \( \sum_{i>0} |a_i| > 0 \) and \( s \geq 0 \) is an integer. Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a polynomial of degree \( d \). Set

\[ \tilde{x}(t) = \sum_{-(d-1)s \leq i \leq s} a_i t^i, \quad |t| > R. \]

Assume that \( \lim_{t \to \infty} f(x(t)) = b \in \mathbb{C} \). Then

\[ \lim_{t \to \infty} f(\tilde{x}(t)) = \lim_{t \to \infty} f(x(t)). \]

The same statement holds in the real case.

4. The complex case

Let us recall a well-known fact.

**Lemma 4.1.** For sufficiently general numbers \( a_{ij} \in \mathbb{C} \) the mapping

\[ \pi : X \ni (x_1, \ldots, x_m) \mapsto \left( \sum_{j=1}^m a_{1j}x_j, \sum_{j=2}^m a_{2j}x_j, \ldots, \sum_{j=n}^m a_{nj}x_j \right) \in \mathbb{C}^n \]

is finite.

We can now state an effective variant of the curve selection lemma.
Theorem 4.2. Let $F : \mathbb{C}^n \ni x \mapsto (f_1(x), \ldots, f_m(x)) \in \mathbb{C}^m$ be a generically finite polynomial mapping. Assume that $\deg f_i = d_i$ and $d_1 \geq \cdots \geq d_m$. Let $b \in \mathbb{C}^m$ be a point at which $F$ is not proper. Then there exists a rational curve with a parametrization of the form

$$x(t) = \sum_{-(d-1)D-1 \leq i \leq D} a_i t^i, \quad t \in \mathbb{C}^*, \quad \text{where } a_i \in \mathbb{C}^n, \sum_{i>0} |a_i| > 0 \text{ and } D = \prod_{i=2}^n d_i, \quad d = d_1, \text{ such that}$$

$$\lim_{t \to \infty} F(x(t)) = b.$$

Proof. First we consider the case $m = n$. We have two possibilities:

1) the fiber $F^{-1}(b)$ is finite,
2) the fiber $F^{-1}(b)$ is infinite.

1) We can assume that the system of coordinates in the target is sufficiently general. In particular we can assume that the line $L = \{ x : x_2 = b_2, x_3 = b_3, \ldots, x_n = b_n \}$, where $b = (b_1, \ldots, b_n)$, is contained neither in the set $S_F$ nor in the set of critical values of $F$ and it omits the (constructible) set over which the mapping $F$ has infinite fibers.

Let $C = F^{-1}(L)$. By our assumptions $C$ is a curve. By Proposition 2.4 we see that the preimage under $F|_C$ of any neighborhood of $b$ on $L$ cannot be relatively compact, i.e., the point $b$ is a non-proper point of the mapping $F|_C$. In particular there exists a holomorphic branch $\Gamma$ of $C$ such that $\lim_{x \in \Gamma, x \to \infty} F(x) = b$. Note that $\deg C \leq D$. By Lemma 3.1 we can assume that the branch $\Gamma$ has a parametrization of the form

$$x = \sum_{-\infty \leq i \leq D} a_i t^i, \quad |t| > R, \quad \text{and } \sum_{i>0} |a_i| > 0.$$

From Lemma 3.3 it follows that

$$F\left( \sum_{-(d-1)D-1 \leq i \leq D} a_i t^i \right) = b + \sum_{i=1}^\infty c_i/t^i,$$

which proves the theorem in case 1).

2) Let $W$ be an irreducible component of $F^{-1}(b)$ of positive dimension. By the general Bezout’s formula [5] Thm. 2.2.5] we see that $\deg W \leq D$. Using a hyperplane section we easily obtain a curve $C \subset W$ with $\deg C \leq D$. Now we conclude as in the in the previous case.

In the general case when $m > n$, we set $X = F(\mathbb{C}^n)$. Let $\pi : \mathbb{C}^m \to \mathbb{C}^n$ be a generic projection as in Lemma 4.1. We can assume that $\pi^{-1}(\pi(b)) \cap S_F = \{b\}$ since $S_F$ is of dimension less than $n$. Take $G = \pi \circ F$. Then $G = (g_1, \ldots, g_n)$ and $\deg g_i \leq d_i$.

By the first part of our proof there exists a holomorphic branch $\Gamma$ of a curve $C$ such that $\lim_{x \in \Gamma, x \to \infty} G(x) = \pi(b)$ and $\deg C \leq D$. Since the mapping $\pi|_X$ is proper
we have
\[ b' := \lim_{x \in \Gamma, x \to \infty} F(x) \in \pi^{-1}(\pi(b)) \cap X. \]
So \( b' \in S_F \), hence \( b' = b \).

**Definition 4.3.** By a rational arc we mean a function of the form
\[ x(t) = \sum_{-D_2 \leq i \leq D_1} a_i t^i, \quad t \in \mathbb{C}^*, \]
where \( a_i \in \mathbb{C}^n \). If \( a_i \neq 0 \) for \( i = D_1, -D_2 \) we say that \( x(t) \) is of bidegree \( (D_1, D_2) \).

We identify an arc \( x(t) \) with its coefficients. Clearly the space of all rational arcs of bidegree at most \( (D_1, D_2) \) is isomorphic to \( \mathbb{C}^{n(1+D_1+D_2)} \).

**Definition 4.4.** Let \( F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m \) be a generically finite polynomial mapping which is not proper. Assume that \( \deg f_i = d_i \), where \( d_1 \geq \cdots \geq d_m \).

By the asymptotic variety of rational arcs of the mapping \( F \) we mean the variety \( \text{AV}(F) \subset \mathbb{C}^{n(2+\Pi_{i=1}^n d_i)} \) which consists of those rational arcs \( x(t) \) of bidegree at most \( (D_1, D_2) \), where \( D_1 = \Pi_{i=2}^n d_i, D_2 = 1 + (d_1 - 1) \Pi_{i=2}^n d_i \) such that

a) \( F(x(t)) = c_0 + \sum_{i=1}^{\infty} c_i t^i \),

b) \( \sum_{i>0} \sum_{j=1}^n a_{ij} = 1 \), where \( a_i = (a_{i_1}, \ldots, a_{in}) \).

The generalized asymptotic variety of \( F \) is the variety \( \text{GAV}(F) \subset \mathbb{C}^{n(2+\Pi_{i=1}^n d_i)} \) defined only by condition a).

**Remark 4.5.** Condition b) ensures that \( \lim_{|t|\to\infty} \|x(t)\| = \infty \).

Notice that \( \text{AV}(F) \) and \( \text{GAV}(F) \) are algebraic subsets of \( \mathbb{C}^{n(2+\Pi_{i=1}^n d_i)} \). Recall that we identify an arc \( x(t) \) with its coefficients \( a = (a_{ij}) \in \mathbb{C}^{n(2+\Pi_{i=1}^n d_i)} \). If \( x(t) \in \text{GAV}(F) \) then \( F(x(t)) = \sum_{i=0}^{\infty} c_i(a)/t^i \). Clearly each \( c_i \) is a polynomial in \( a \). The function \( c_0 : \text{AV}(F) \to \mathbb{C}^m \) plays an important role.

**Proposition 4.6.** \( c_0(\text{AV}(F)) = S_F \).

**Proof.** Let \( x(t) = \sum a_i t^i \in \text{AV}(F) \). Then \( F(x(t)) = c_0(a) + \sum_{i=1}^{\infty} c_i(a)/t^i \), which implies that \( c_0(a) \in S_F \). Conversely, let \( b \in S_F \). By Theorem 1.2 we can find a rational arc \( x(t) = \sum_{i=-(d-1)}^{D-1} a_i t^i \) such that \( \lim_{t \to \infty} F(x(t)) = b \). Now change the parametrization of \( x(t) \), \( t \mapsto \lambda t \), so that \( \sum_{i>0} \sum_{j=1}^n \lambda^i a_{ij} = 1 \). The new arc \( x'(t) := x(\lambda t) \) belongs to \( \text{AV}(F) \) and \( c_0(x'(t)) = b \). \( \square \)

Now let \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial. Let us define a polynomial mapping \( \Phi : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^N \) by
\[
\Phi = \left( f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, \ldots, h_{mn} \right),
\]
where \( h_{ij} = x_i \frac{\partial f}{\partial x_j}, i = 1, \ldots, n, j = 1, \ldots, n \).
Definition 4.7. Let $\Phi$ be as above. Consider the mapping $c_0 : AV(\Phi) \to \mathbb{C}^N$ and the line $L := \mathbb{C} \times \{(0, \ldots, 0)\} \subset \mathbb{C} \times \mathbb{C}^N$. By the bifurcation variety we mean the variety

$$BV(f) = \{x(t) \in AV(\Phi) : x(t) \in c_0^{-1}(L)\}.$$  

Similarly, we define the generalized bifurcation variety of rational arcs of the polynomial $f$:

$$GBV(f) = \{x(t) \in GAV(\Phi) : x(t) \in c_0^{-1}(L)\}.$$  

As an immediate consequence of [10] we have:

Proposition 4.8. Let $K(f) = K_0(f) \cup K_\infty(f)$ denote the set of generalized critical values of $f$. If we identify the line $L = \mathbb{C} \times \{(0, \ldots, 0)\} \subset \mathbb{C} \times \mathbb{C}^N$ with $\mathbb{C}$, then we have $c_0(BV(\Phi)) = K_\infty(f)$ and $c_0(GBV(\Phi)) = K(f)$.

5. Algorithm

In this section we give an algorithm to compute the set $K_\infty(f)$ of asymptotic critical values as well as the set $K(f)$ of generalized critical values of a complex polynomial $f$. Let $\deg f = d$ and $D_1 = d^{n-1}$, $D_2 = d^n - d^{n-1} + 1$.

**Algorithm for the set $K_\infty(f)$**.
1) Compute equations for the variety $BV(f)$:
   a) consider an arc $x(t) = \sum_{i=1}^{D_1-D_2} a_it^i \in \mathbb{C}^{n(D_1+D_2+1)}$,
   b) compute $f(x(t)) = \sum c_i(a)t^i$,
   c) compute $\frac{\partial f}{\partial x_i}(x(t)) = \sum d_{ik}(a)t^k$, $i = 1, \ldots, n$,
   d) compute $\frac{\partial f}{\partial x_i}(x(t))x_j(t) = \sum e_{ijk}(a)t^k$, $i, j = 1, \ldots, n$
   e) equations for $BV(f)$ are $c_i = 0$ for $i > 0$, $d_{ik} = 0$ for $k \geq 0$, $i = 1, \ldots, n$, $e_{ijk} = 0$ for $k \geq 0$, $i, j = 1, \ldots, n$ and $\sum_{i>0} \sum_{j=1}^n a_{ij} = 1$, where $a_i = (a_{i1}, \ldots, a_{in})$.

2) Find equations for irreducible components of $BV(f) = \bigcup_{j=1}^r P_j$. This can be done by standard methods of computational algebra. We can use the MAGMA system and a radical decomposition of an ideal (see also [25]).

3) Find a point $x_i \in P_j$. This can also be done by standard methods. Again we can use the MAGMA system and the elimination procedure. Indeed, let $P_i = V(I)$. Compute $I_k = \mathbb{C}[x_1, \ldots, x_k] \cap I$ for $k = n, n-1, \ldots$ until $I_k = (0)$. Then take a randomly chosen integer point, $(a_1, \ldots, a_k)$ find a zero $(a_1, \ldots, a_k, b_1)$ of ideal $I_{k+1}$ and so on.

4) $K_\infty(f) = \{c_0(x_i) : i = 1, \ldots, r\}$.

If we replace above the variety $BV(f)$ by the variety $GAV(f)$ we get an algorithm for computing $K(f)$. Indeed, it is enough to delete the equation $\sum_{i>0} \sum_{j=1}^n a_{ij} = 1$ in item 1e).
6. The real case

We begin with a simple example.

**Example 6.1.** For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ consider $f_\mathbb{K}: \mathbb{K}^2 \to \mathbb{K}$, $f(x, y) = x(x^2 + 1)^2$. Observe that $K_\infty(f_\mathbb{K}) = \emptyset$. But $0 \in K_\infty(f_{\mathbb{C}})$. So in general

$$K_\infty(f_\mathbb{R}) \neq \mathbb{R} \cap K_\infty(f_{\mathbb{C}}).$$

This shows that the computation of the asymptotic critical values of a real polynomial cannot be reduced to the computation of the asymptotic critical values of its complexification.

6.1. Effective curve selection lemma at infinity. First we give a construction of a curve selection in a special affine case.

Let $X \subset \mathbb{R}^{2n}$ be an algebraic set described by a system of polynomial equations $p_i = 0$, $\deg p_i \leq d$, where $i = 1, \ldots, n$. Denote by $H$ the hyperplane $\{x_1 = 0\}$. Assume that on $Y := X \setminus H$ the system is nondegenerate, i.e., $P = (p_1, \ldots, p_n) : \mathbb{R}^{2n} \to \mathbb{R}^n$ is a submersion at each point of $Y$. Thus $Y$ is a smooth manifold of dimension $n$.

**Proposition 6.2.** Let $a \in H \cap Y$. Then there exists an algebraic curve $C \subset \mathbb{R}^{2n}$ of degree $D \leq d^n((d - 1)^n + 2)^{n-1}$ such that $a \in C \cap Y$.

For simplicity we assume that $a = 0$. Denote $\rho(x) = (\sum_{i=1}^{2n} x_i^2)^{\frac{1}{2}}$, let $S(r)$ be the sphere centered at 0 of radius $r$, and finally $Y(r) := Y \cap S(r)$. The proof is a based on the following lemmas.

**Lemma 6.3.** There exists $\varepsilon > 0$ such that for any $r \in (0, \varepsilon)$ the set $Y(r)$ is a smooth manifold of dimension $n - 1$, in particular is nonempty.

**Proof.** Indeed, the function $\rho$ restricted to the manifold $Y$ is smooth and semialgebraic, so it has finitely many critical values (see [2], p. 82 or [3], p. 235). Let $\varepsilon_1 > 0$ be the smallest critical value (or $\varepsilon_1 = 1$ if there are no critical values).

Since $\rho|_Y : Y \to \mathbb{R}_+$ is locally trivial, by Hardt’s trivialization theorem (see [2], p. 54 or [3], p. 232) there is $\varepsilon_2 > 0$ such that for any $r, r' \in (0, \varepsilon_2)$ the sets $Y(r)$ and $Y(r')$ are homeomorphic. Since $a \in H \cap Y$, there is $r' \in (0, \varepsilon_2)$ such that $Y(r')$ is nonempty. Hence $Y(r)$ is nonempty for any $r \in (0, \varepsilon_2)$. Finally we put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. \hfill $\square$

Let us now consider the family of functions $g_\alpha : \mathbb{R}^{2n} \to \mathbb{R}$ of the form $g_\alpha(x) := x_1 \alpha(x)$, where $\alpha \in (\mathbb{R}^{2n})^*$ is a linear function on $\mathbb{R}^{2n}$.

**Lemma 6.4.** For any $r \in (0, \varepsilon)$ there exists a proper algebraic set $A_r \subset (\mathbb{R}^{2n})^*$ such that $g_\alpha$ is a Morse function on $Y(r)$, for any $\alpha \notin A_r$. Moreover the set $\bigcup_{r \in (0, \varepsilon)} \{r\} \times A_r$ is contained in a proper algebraic set $A \subset \mathbb{R} \times (\mathbb{R}^{2n})^*$.

**Proof.** The proof of the lemma uses standard arguments in Morse theory (see [13], [14]). Recall that a function $g : Y(r) \to \mathbb{R}$ is Morse if the map $j g : Y(r) \to T^* Y(r)$,
$jg(x) = (x, d_xg)$, is transverse to the zero-section $\Theta \subset T^*Y(r)$. We have a natural projection $\pi : Y(r) \times (\mathbb{R}^{2n})^* \to T^*Y(r)$, $\pi(x, \beta) = (x, \beta|_{T_xY(r)})$, where $\beta|_{T_xY(r)}$ is the restriction of a linear form $\beta \in (\mathbb{R}^{2n})^*$ to the subspace $T_xY(r)$. Consider the map 

$$\Phi : Y(r) \times (\mathbb{R}^{2n})^* \to Y(r) \times (\mathbb{R}^{2n})^*$$

given by $\Phi(x, \alpha) = (x, d_xg_\alpha)$. Note that $\Phi$ is a submersion. Indeed, the Jacobian matrix of $\Phi$ (with respect to variables in $(\mathbb{R}^{2n})^*$) is triangular, with $x_1$ on the diagonal except the entry in the left upper corner which is $2x_1$. So this matrix is invertible since $x_1 \neq 0$ for $x \in Y(r)$.

Hence $\pi \circ \Phi : Y(r) \times (\mathbb{R}^{2n})^* \to T^*Y(r)$ is also a submersion. So it is transverse to any submanifold of $T^*Y(r)$, in particular to the zero-section $\Theta \subset T^*Y(r)$. Note that $g_\alpha(x) = \pi \circ \Phi(x, \alpha)$.

By the Transversality Theorem (see [14], p. 68 or [13] Theorem 2.2.3, p. 53) the set $A_r$ of $\alpha \in (\mathbb{R}^{2n})^*$ such that $g_\alpha$ is not transverse to $\Theta$, i.e., is not a Morse function on $Y(r)$, is nowhere dense. Since $A_r$ is semialgebraic, its Zariski closure $A_\ast$ is a proper algebraic subset of $\alpha \in (\mathbb{R}^{2n})^*$.

The second statement follows from the fact that set $A_r$ is defined by polynomial equations with $r$ as a variable parameter. \hfill $\square$

**Lemma 6.5.** There exist $\alpha \in (\mathbb{R}^{2n})^*$ and $0 < \varepsilon' \leq \varepsilon$ such that $g_\alpha$ is a Morse function on $Y \cap S(r)$ for any $r \in (0, \varepsilon')$.

**Proof.** Indeed, let $A \subset \mathbb{R} \times (\mathbb{R}^{2n})^*$ by the proper algebraic set in Lemma 6.4. Thus there exists an affine line $\mathbb{R} \times \alpha$ which meets the set $A$ only in finitely many points. So $(0, \varepsilon') \times \{\alpha\}$ is disjoint with $A$, for some $\varepsilon' > 0$ small enough. \hfill $\square$

Since $Y(r)$ is not compact, a priori it is not obvious that $g_\alpha$ has a critical point on $Y(r)$. However we have

**Lemma 6.6.** Assume that $Y(r) \neq \emptyset$ and that $g_\alpha$ is Morse on $Y(r)$. Then $g_\alpha$ has a critical point on $Y(r)$.

**Proof.** Note that the image of $Y(r)$ under $g_\alpha$ consists of finitely many nontrivial intervals. Since $\overline{Y(r)}$ is compact and

$$\overline{Y(r)} \setminus Y(r) \subset \{x_1 = 0\},$$

at least one of the endpoints of each of those intervals belongs to $g_\alpha(Y(r))$. Thus $g_\alpha$ achieves a minimum or a maximum in $Y \cap S(r)$.

**Proof of Proposition 6.4** Let us fix a linear form $\alpha$ which satisfies the requirements of Lemma 6.5. Let $\Xi$ be the locus of critical points of $g_\alpha$ on $Y(r)$ for $r \in (0, \varepsilon')$. Since $g_\alpha$ is a Morse function on $Y(r)$, for each $r \in (0, \varepsilon')$ the set $\Xi \cap \{|x| = r\}$ is finite and nonempty by Lemma 6.6. Hence $\Xi$ is a semialgebraic curve.

The Zariski closure of $\Xi$ is contained in the algebraic set given by the equations

$$p_1 = \cdots = p_n = 0$$

\hfill (6.1)
and

\begin{equation}
(6.2) \quad dp_1 \wedge \cdots \wedge dp_n \wedge dp^2 \wedge dg_\alpha = 0.
\end{equation}

Let \( \Xi_1 \) be a smooth connected component of \( \Xi \) such that \( a \in \Xi_1 \). Then locally \( \Xi_1 \) is given by a nondegenerate system \((6.1)\) of \( n \) equations of degree at most \( d \) and \( n-1 \) equations of degree at most \( (d-1)^n + 2 \), which are \((n+2) \times (n+2)\) minors of the matrix corresponding to the system \((6.2)\). Note that the system \((6.1)\) with \((6.2)\) is actually nondegenerate at each point of \( \Xi_1 \), because critical points of a Morse function are described by a transversality condition.

Let \( C \) be the Zariski closure of \( \Xi_1 \). Hence by the general formula of Bezout (see [5 Thm. 2.2.5]) the degree of the curve \( C \) is at most \( d^n((d-1)^n + 2)^{n-1} \); note that for \( d \geq 3 \) we have \( d^n((d-1)^n + 2)^{n-1} \leq d^{n^2} \). Hence Proposition 6.2 follows.

We can now state a real version of Theorem 4.2.

**Theorem 6.7.** Let \( F: \mathbb{R}^n \ni x \mapsto (f_1(x), \ldots, f_m(x)) \in \mathbb{R}^m \) be a polynomial mapping. Assume that \( \deg f_i \leq d \). Let \( b \in \mathbb{R}^m \) be a point at which the mapping \( F \) is not proper. Then there exists a rational curve with a parametrization of the form

\[ x(t) = \sum_{-(d-1)d-1 \leq i \leq D} a_i t^i, \quad t \in \mathbb{R}, \]

where \( a_i \in \mathbb{R}^n \), \( \sum_{i > 0} |a_i| > 0 \) and \( D = (d+1)^n(d^n + 2)^{n-1} \) such that

\[ \lim_{t \to \infty} F(x(t)) = b. \]

**Remark 6.8.** Note that in contrast to the complex case we do not assume that \( F \) is generically finite, but the bidegree of the real rational curve is much higher, namely \( D = O(d^{n^2}) \).

**Proof.** By Lemma 4.1 we can assume that \( m = n \). Let \( \gamma(t) \in \mathbb{R}^n \) be a semialgebraic curve such that \( \lim_{t \to \infty} |\gamma(t)| = +\infty \) and \( \lim_{t \to \infty} F(\gamma(t)) = b \). Let \( \bar{\gamma}(t) \) be the image of \( \gamma(t) \) under the canonical embedding \( \mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto (1 : x_1 : \cdots : x_n) \in \mathbb{P}^n \).

Since \( \gamma \) is semi-algebraic there exists a \( a := \lim_{t \to \infty} \bar{\gamma}(t) \in H_0 \), where \( H_0 \) stands for the hyperplane at infinity. Let us denote by \( Y \) the graph of \( F \), embedded in \( \mathbb{P}^n \times \mathbb{R}^n \), and by \( X \) its Zariski closure in \( \mathbb{P}^n \times \mathbb{R}^n \). Note that the point \((a, b)\) belongs to the closure (in the strong topology) of \( Y \). Let \( \tilde{f}_i \) stand for the homogenization of the polynomial \( f_i \), that is,

\[ \tilde{f}_i(x_0, x_1, \ldots, x_n) = x_0^{d_i} f_i \left( \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right). \]

Hence \( X \subset \mathbb{P}^n \times \mathbb{R}^n \) is defined by the equations

\[ x_0^{d_i} y_i = \tilde{f}_i(x_0, x_1, \ldots, x_n), \quad i = 1, \ldots, n. \]

Assume that \( a_1 = 1 \), so \( \mathbb{R}^n \ni (x_0, x_2, \ldots, x_n) \mapsto (x_0, 1, x_2, \ldots, x_n) \in \mathbb{P}^n \) is an affine chart around the point \( a \). In this chart \( X \) is given by the equations

\[ p_i(x_0, x_2, \ldots, x_n, y_1, \ldots, y_n) := x_0^{d_i} y_i - \tilde{f}_i(x_0, 1, x_2, \ldots, x_n) = 0, \]
Remark 6.11. Let $\Phi : \mathbb{R} \to \mathbb{R}^n$ be a generically finite polynomial mapping which is not proper. Assume that $(a, b)$. Hence there exists an algebraic curve $C_1 \subset \mathbb{R}^2$ of degree $d_e \leq (d+1)^n(d^n+2)^{n-1}$ such that $(a, b) \in C_1 \cap \overline{Y}$. Now let $C$ be the projection of $C_1$ on $\mathbb{P}^n$. Note that the degree of $C$ is less than or equal to $d_e$. Now we can argue as in the proof of Theorem 6.4 to conclude that there is a real rational arc $x(t)$ of bidegree $D_2 = (d-1) + 1$, $D_1 = D$ where $D = (d+1)^n(d^n+2)^{n-1}$, such that $\lim_{t \to \infty} F(x(t)) = b$.

\[ \Phi = \left( f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, \ldots, h_{nn} \right), \]

Definition 6.9. By a real rational arc we mean a function of the form

\[ x(t) = \sum_{-D_2 \leq i \leq D_1} a_i t^i, \quad t \in \mathbb{R}, \]

where $a_i \in \mathbb{R}$. If $a_i \neq 0$ for $i = D_1, -D_2$ we say that $x(t)$ is of bidegree $(D_1, D_2)$.

We identify an arc $x(t)$ with its coefficients. Clearly the space of all real rational arcs of bidegree at most $(D_1, D_2)$ is isomorphic to $\mathbb{R}^{n(D_1+D_2)}$.

Definition 6.10. Let $F = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a generically finite polynomial mapping which is not proper. Assume that $\deg f_i \leq d$ and $D = (d+1)^n(d^n+2)^{n-1}$. By the asymptotic variety of real rational arcs of the mapping $F$ we mean the variety $AV_{\mathbb{R}}(F) \subset \mathbb{R}^{n(dD+2)}$ which consists of those real rational arcs $x(t)$ of bidegree at most $(D, (d-1)D + 1)$ such that

a) $F(x(t)) = c_0 + \sum_{i=1}^{\infty} c_i/t^i$,

b) $\sum_{i>0} \sum_{j=1}^{n} a_{ij}^2 = 1$, where $a_i = (a_{i1}, \ldots, a_{in})$.

If we omit condition b) we get the definition of the generalized asymptotic variety of real arcs, $GAV_{\mathbb{R}}(F)$.

Remark 6.11. Condition b) ensures that the arc $x(t)$ "goes to infinity".

As before we see that $AV_{\mathbb{R}}(F), GAV_{\mathbb{R}}(F)$ are algebraic subsets of $\mathbb{R}^{n(dD+2)}$. Recall that we identify an arc $x(t)$ with its coefficients $a = (a_{ij}) \in \mathbb{R}^{n(2+D_1+D_2)}$. Moreover, for $x(t) \in AV_{\mathbb{R}}(F)$, respectively $x(t) \in GAV_{\mathbb{R}}(F)$, we have $F(x(t)) = \sum_{i=0}^{\infty} c_i(t)/t^i$. Note that again the function $c_0 : AV_{\mathbb{R}}(F) \to \mathbb{R}^m$ plays an important role.

Proposition 6.12. $c_0(AV_{\mathbb{R}}(F)) = S_F(\mathbb{R})$.

Proof. Let $x(t) = \sum a_it^i \in AV_{\mathbb{R}}(F)$. Then $F(x(t)) = c_0(a) + \sum_{i=1}^{\infty} c_i(t)/t^i$; this implies that $c_0(a) \in S_F$. Conversely, let $b \in S_F$. By Theorem 6.4 we can find a rational arc $x(t) = \sum_{i=-d-1}^{D} a_it^i$ such that $\lim_{t \to \infty} F(x(t)) = b$. There exists a $\lambda \in \mathbb{R}$ such that $\sum_{i>0} \sum_{j=1}^{n} \lambda^2 a_{ij}^2 = 1$. Change a parametrization by $t \mapsto \lambda t$. The new arc $x'(t) := x(\lambda t)$ belongs to $AV_{\mathbb{R}}(F)$ and $c_0(x'(t)) = b$.

Now let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial. Let us define a polynomial mapping $\Phi : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^N$ by

\[ \Phi = \left( f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, \ldots, h_{nn} \right), \]
where \( h_{ij} = x_i \frac{\partial f}{\partial x_j}, \ i, j = 1, \ldots, n. \)

**Definition 6.13.** Let \( \Phi \) be as above. Consider the mapping \( c_0 : AV_{\mathbb{R}}(\Phi) \to \mathbb{R}^N \) and the line \( L := \mathbb{R} \times \{(0, \ldots, 0)\} \subset \mathbb{R} \times \mathbb{R}^N \). By the bifurcation variety of real rational arcs of the polynomial \( f \) we mean the variety
\[
BV_{\mathbb{R}}(f) = \{ x(t) \in AV_{\mathbb{R}}(\Phi) : x(t) \in c_0^{-1}(L) \}.
\]
Similarly we define
\[
GBV_{\mathbb{R}}(f) = \{ x(t) \in GAV_{\mathbb{R}}(\Phi) : x(t) \in c_0^{-1}(L) \}.
\]

As an immediate consequence of [10] we have:

**Proposition 6.14.** Let \( K(f) = K_0(f) \cup K_\infty(f) \) denote the set of generalized critical values of a real polynomial \( f \). If we identify the line \( L = \mathbb{R} \times \{(0, \ldots, 0)\} \subset \mathbb{R} \times \mathbb{R}^N \) with \( \mathbb{R} \), then we have \( c_0(BV_{\mathbb{R}}(f)) = K_\infty(f) \) and \( c_0(GBV_{\mathbb{R}}(f)) = K(f) \).

7. Real algorithm

In this section we describe an algorithm to compute the set \( K_\infty(f) \) of asymptotic critical values as well as the set \( K(f) \) of generalized critical values of a real polynomial \( f \in \mathbb{R}[x_1, \ldots, x_n] \). Let \( \text{deg} f = d \) and \( D_1 = (d+1)^n(d^n+2)^n-1, D_2 = (d-1)D_1+1 \).

**Algorithm for the set \( K_\infty(f) \).**
1) Compute equations \( g_a \) for the variety \( BV_{\mathbb{R}}(f) \):
   a) consider an arc \( x(t) = \sum a_{ijk} t^j \in \mathbb{R}^{n(D_1+D_2+1)}, \)
   b) compute \( f(x(t)) = \sum c_i(a) t^j \),
   c) compute \( \frac{\partial f}{\partial x_i}(x(t)) x_j(t) = \sum e_{ij}(a) t^j, i, j = 1, \ldots, n, \)
   e) equations for \( BV_{\mathbb{R}}(f) \) are \( c_i = 0 \) for \( i > 0, d_{ik} = 0 \) for \( k \geq 0, i = 1, \ldots, n, \)
   \( e_{ijk} = 0 \) for \( k \geq 0, i, j = 1, \ldots, n \) and \( \sum_{j=1}^n a_{ijk}^2 = 1 \), where \( a_i = (a_{i1}, \ldots, a_{in}) \).
2) Form a polynomial \( G = \sum_{a} g_a^2 \), where \( g_a \) are \( c_i \) for \( i > 0, \) or \( d_{ik} \) for \( k \geq 0, i = 1, \ldots, n, \) or \( e_{ijk} \) for \( k \geq 0, i, j = 1, \ldots, n \) or \( \sum_{j=1}^n a_{ijk}^2 - 1 \).
3) Using Algebraic Sampling [11 Theorem 11.61 p. 412]) we can find a finite set \( A \subset \{ G = 0 \} \) which meets every connected component of \( \{ G = 0 \} \).
4) \( K_\infty(f) = \{ c_0(a) : a \in A \} \).

If we replace above the variety \( BV_{\mathbb{R}}(f) \) by the variety \( GAV_{\mathbb{R}}(f) \) we get an algorithm for computing \( K(f) \). Actually it is enough to delete the equation \( \sum_{i>0} \sum_{j=1}^n a_{ij}^2 = 1 \) in items 1e) and 2).

**Conclusion.** In the paper we have developed only the general geometric aspect of our method. In a forthcoming work we plan to discuss effectiveness of the algorithms and possibly technical improvements. Recently M. Raibaut [22] introduced the
motivic bifurcation set of a complex polynomial. It seems that our approach can give a more geometric explanation of his work.

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