Multi-index filtrations and motivic Poincaré series.

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Abstract

The Poincaré series of a multi-index filtration on the ring of germs of functions can be written as a certain integral with respect to the Euler characteristic over the projectivization of the ring. Here this integral is considered with respect to the generalized Euler characteristic with values in the Grothendieck ring of varieties. For the filtration defined by orders of functions on the components of a plane curve singularity C and for the so called divisorial filtration for a modification of (C², 0) by a sequence of blowing-ups there are given formulae for this integral in terms of an embedded resolution of the germ C or in terms of the modification respectively. The generalized Euler characteristic of the extended semigroup corresponding to the divisorial filtration is computed.

Introduction

Let (C, 0) ⊂ (Cⁿ, 0) be a germ of a reduced analytic curve, let C = \bigcup_{k=1}^{r} C_k be its decomposition into irreducible components, and let \( \varphi_k : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0) \), \( k = 1, \ldots, r \), be parametrizations of the components \( C_k \) of the curve \( C \); i.e. germs of analytic maps such that \( \text{Im} \varphi_k = C_k \) and \( \varphi_k \) is an isomorphism between \( \mathbb{C} \) and \( C_k \) outside of the origin (in a neighbourhood of it). For a germ \( g \in \mathcal{O}_{\mathbb{C}^n,0} \) let \( v_k = v_k(g) \) and \( a_k = a_k(g) \) be the power of the leading term and the
coefficient at it in the power series decomposition of the germ $g \circ \varphi_k : (\mathbb{C}, 0) \to \mathbb{C}, \ g \circ \varphi_k(\tau) = a_k \tau^{v_k} + \text{terms of higher degree}, \ a_k \neq 0$. If $g \circ \varphi_k(\tau) \equiv 0$, $v_k(g)$ is assumed to be equal to $\infty$ and $a_k(g)$ is not defined. Let $v(g) := (v_1(g), \ldots, v_r(g)) \in \mathbb{Z}^r_{\geq 0}, \ a(g) := (a_1(g), \ldots, a_r(g)) \in (\mathbb{C}^*)^r$.

The functions (valuations) $v_k$ define a multi-index filtration on the ring $\mathcal{O}_{\mathbb{C}^n, 0}$: for $v \in \mathbb{Z}^r$, the corresponding subspace $J(v)$ is defined as $\{g \in \mathcal{O}_{\mathbb{C}^n, 0} : v(g) \geq v\}$. In [2] there was computed the (appropriately defined) Poincaré series of the described multi-index filtration on the ring $\mathcal{O}_{\mathbb{C}^2, 0}$ for a plane curve singularity $C$ (i.e., for $n = 2$). It appeared to be equal to the Alexander polynomial of the algebraic link $C \cap S^3_\epsilon \subset S^3_\epsilon$ corresponding to the curve $C$ ($S^3_\epsilon$ is the sphere of radius $\epsilon$ centred at the origin of $\mathbb{C}^2$ with positive $\epsilon$ small enough).

Inspired by the notion of motivic integration (see, e.g., [7], [12]) there was defined the notion of the Euler characteristic of (some) subsets of the ring $\mathcal{O}_{V, 0}$ of functions on a germ $(V, 0)$ of an analytic variety or of its projectivization $\mathbb{P}\mathcal{O}_{V, 0}$ and the corresponding notion of the integration with respect to the Euler characteristic (see, e.g., [10]). Here the Euler characteristic can be considered both as the usual one $\chi$ with values in $\mathbb{Z}$ and the generalized one $\chi_g$ with values in the Grothendieck ring of complex algebraic varieties localized by the class $L$ of the complex affine line: $K_0(V_{\mathbb{C}}(L))$.

It was shown that the Poincaré series $P(t_1, \ldots, t_r)$ of a multi-index filtration $\{J(v)\}$ on the ring $\mathcal{O}_{V, 0}$ (finitely determined: see the definition below) is equal to the integral with respect to the Euler characteristic over the projectivization $\mathbb{P}\mathcal{O}_{V, 0}$ of the ring of functions:

$$P(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}_{V, 0}} \frac{t^{v(g)}}{1 - t^{v(g)}} d\chi$$

($t = (t_1, \ldots, t_r), \ t^{v(g)} = t_1^{v_1} \cdots t_r^{v_r}$). Moreover, this representation permitted to give a considerably shorter proof of the mentioned formula for the Poincaré series of the filtration defined by orders $v_i(g)$ of a function on the components of the plane curve singularity $C$ (see [3]) and to compute Poincaré series of some other multi-index filtrations: [8], [4], ... .

It is natural to consider the integral like the one in the equation (1) with respect to the generalized Euler characteristic $\chi_g$. This integral is a series in $t$ and $q = L^{-1}$. Generally speaking it is a more fine invariant than the Poincaré series itself (for non-plane curves). We give two formulae for it. The first one is in terms of the Hilbert function $h(v) = \dim \mathcal{O}_{V, 0}/J(v)$. The second one, for the described filtration defined by the orders of functions on the components of a plane curve singularity (and also for the so called divisorial filtration for a modification of $(\mathbb{C}^2, 0)$ by a series of blowing-ups) is in terms of an embedded
resolution of the curve \( C \) (or in terms of the modification respectively). We also give a formula for the generalized Euler characteristic of the extended semigroup corresponding to the divisorial filtration.

1 Multi-index filtrations and integrals with respect to the Euler characteristic.

Let \((V, 0)\) be a germ of an analytic variety. A one index filtration

\[
O_{V,0} = J_0 \supset J_1 \supset \ldots \supset J_n \supset \ldots
\]  

(2)
on the ring \( O_{V,0} \) of germs of functions on \((V, 0)\) by vector subspaces can be defined by a function \( v : O_{V,0} \to \mathbb{Z}_{\geq 0} \cup \{ \infty \} \) (\( v(g) = \sup \{ i : g \in J_i \} \)) with the properties

\[
v(\lambda g) = v(g) \quad \text{for } \lambda \in \mathbb{C}^*, \]
\[
v(g_1 + g_2) \geq \min \{ v(g_1), v(g_2) \}.
\]  

(3)

Sometimes the additional property

\[
v(g_1 g_2) = v(g_1) + v(g_2)
\]  

(4)
is required (i.e., \( v \) is a valuation). It guaranties that \( \{2\} \) is a filtration by ideals and that \( J_i \cdot J_j \subset J_{i+j} \).

The Poincaré series of the filtration \( \{2\} \) is the series

\[
P(t) = \sum_{i=0}^{\infty} \dim(J_i/J_{i+1}) \cdot t^i
\]  

(5)

(it makes sense if the dimensions of all the factors \( J_i/J_{i+1} \) are finite).

A multi-index filtration on the ring \( O_{V,0} \) is defined by a system \( \{v_1, \ldots, v_r\} \) of functions \( O_{V,0} \to \mathbb{Z}_{\geq 0} \cup \{ \infty \} \) satisfying the properties \( \{3\} \): the corresponding system of subspaces \( J(\underline{v}) \), \( \underline{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r \), is defined by \( J(\underline{v}) = \{ g \in O_{V,0} : v(g) \geq \underline{v} \} \). (Here \( \underline{v}' \geq \underline{v}'' \) (\( \underline{v}', \underline{v}'' \in \mathbb{Z}^r \)) if and only if \( v'_k \geq v''_k \) for all \( k = 1, \ldots, r \).)

Remark. It is sufficient to define the subspaces \( J(\underline{v}) \) only for nonnegative \( \underline{v} \), i.e for \( \underline{v} \in \mathbb{Z}^r_{\geq 0} \); however, it will be convenient if \( J(\underline{v}) \) is defined for all \( \underline{v} \in \mathbb{Z}^r \).

We say that the filtration \( \{ J(\underline{v}) \} \) is finitely determined if, for any \( \underline{v} \in \mathbb{Z}^r \), there exists \( N \) such that \( J(\underline{v}) \supset \mathfrak{m}^N \), where \( \mathfrak{m} \) is the maximal ideal in the ring \( O_{V,0} \). If the multi-index filtration \( \{ J(\underline{v}) \} \) is finitely determined, all subspaces \( J(\underline{v}) \) have finite codimension. In what follows all filtrations are assumed to

3
be finitely determined. Let \( h(\mathfrak{u}) = \dim \mathcal{O}_{V,0}/J(\mathfrak{u}) \). We shall call \( h(\mathfrak{u}) \) (as a function of \( \mathfrak{u} \)) the Hilbert function of the filtration \( \{J(\mathfrak{u})\} \).

Before describing a multi-index version of the Poincaré series it is convenient to define the notion of integration with respect to the Euler characteristic over the projectivization \( \mathbb{P}\mathcal{O}_{V,0} \) of the ring (space) \( \mathcal{O}_{V,0} \) and to interpret the Poincaré series \( P(t) \) as such an integral (similar to the equation \( \mathbb{I} \)).

Let \( K_0(\mathcal{V}_C) \) be the Grothendieck ring of quasi-projective varieties. It is generated by classes \([X]\) of such varieties subject to the relations:

1) if \( X_1 \cong X_2 \), then \([X_1] = [X_2]\);
2) if \( Y \) is Zariski closed in \( X \), then \([X] = [Y] + [X \setminus Y]\) (the multiplication is defined by the Cartesian product). Let \( \mathbb{L} \) be the class \([\mathbb{A}^1_C]\) of the complex affine line. The class \( \mathbb{L} \) is not equal to zero in the ring \( K_0(\mathcal{V}_C) \).

Moreover the natural ring homomorphism \( \mathbb{Z}[x] \to K_0(\mathcal{V}_C) \) which sends \( x \) to \( \mathbb{L} \) is an inclusion. Let \( K_0(\mathcal{V}_C)(\mathbb{L}) \) be the localization of the Grothendieck ring \( K_0(\mathcal{V}_C) \) by the class \( \mathbb{L} \). The natural homomorphism \( \mathbb{Z}[x]_{(x)} \to K_0(\mathcal{V}_C)(\mathbb{L}) \) is an inclusion as well.

Let \( J^p_{V,0}, p \geq 0 \), be the space of \( p \)-jets of functions at the origin in \( (V,0) \), i.e. \( J^p_{V,0} = \mathcal{O}_{V,0}/\mathfrak{m}^{p+1} \), where \( \mathfrak{m} \) is the maximal ideal in the ring \( \mathcal{O}_{V,0} \). It is a finite dimensional space. Let \( d(p) = \dim J^p_{V,0} \). For \( (V,0) = (\mathbb{C}^n,0) \) one has \( d(p) = \binom{n+p}{p} \). For a complex vector space \( L \) (finite or infinite dimensional), let \( \mathbb{P}L = (L \setminus \{0\})/\mathbb{C}^* \) be its projectivization, let \( \mathbb{P}^*L \) be the disjoint union of \( \mathbb{P}L \) with a point (in some sense \( \mathbb{P}^*L = L/\mathbb{C}^* \)). One has natural maps \( \pi_p : \mathbb{P}\mathcal{O}_{V,0} \to \mathbb{P}^*J^p_{V,0} \) and \( \pi_{p,q} : \mathbb{P}^*J^p_{V,0} \to \mathbb{P}^*J^q_{V,0} \) for \( p \geq q \geq 0 \). Over \( \mathbb{P}J^0_{V,0} \subset \mathbb{P}^*J^0_{V,0} \) the map \( \pi_{p,q} \) is a locally trivial fibration, the fibre of which is a complex affine space of dimension \( d(p) - d(q) \).

**Definition:** A subset \( X \subset \mathbb{P}\mathcal{O}_{V,0} \) is said to be cylindric if \( X = \pi_p^{-1}(Y) \) for a constructible subset \( Y \subset \mathbb{P}J^p_{V,0} \subset \mathbb{P}^*J^p_{V,0} \).

This definition means that the condition for a function \( g \in \mathcal{O}_{V,0} \) to belong to the set \( X \) is a constructible (and \( \mathbb{C}^* \)-invariant) condition on the \( p \)-jet of the germ \( g \).

**Definition:** For a cylindric subset \( X \subset \mathbb{P}\mathcal{O}_{V,0} \) (\( X = \pi_k^{-1}(Y), Y \subset \mathbb{P}J^k_{V,0}, Y \) is constructible), its Euler characteristic \( \chi(X) \) is defined as the Euler characteristic \( \chi(Y) \) of the set \( Y \). The generalized Euler characteristic \( \chi_\theta(X) \) of the cylindric subset \( X \) is the element \([Y] \cdot \mathbb{L}^{d(p)} \) of the ring \( K_0(\mathcal{V}_C)(\mathbb{L}) \).

The generalized Euler characteristic \( \chi_\theta(X) \) is well defined since, if \( X = \pi_q^{-1}(Y'), Y' \subset \mathbb{P}J^q_{V,0} \), \( p \geq q \), then \( Y \) is a locally trivial fibration over \( Y' \) with the fibre \( \mathbb{C}^{d(p)-d(q)} \) and therefore \([Y] = [Y'] \cdot \mathbb{L}^{d(p)-d(q)} \).

Let \( \psi : \mathbb{P}\mathcal{O}_{V,0} \to A \) be a function with values in an Abelian group \( A \) with countably many values.
Definition: We say that the function \( \psi \) is cylindric if, for each \( a \neq 0 \), the set \( \psi^{-1}(a) \subset \mathbb{P}O_{V,0} \) is cylindric.

Definition: The integral of a cylindric function \( \psi \) over the space \( \mathbb{P}O_{V,0} \) with respect to the Euler characteristic (respectively with respect to the generalized Euler characteristic) is

\[
\int_{\mathbb{P}O_{V,0}} \psi \, d\chi := \sum_{a \in A, a \neq 0} \chi(\psi^{-1}(a)) \cdot a,
\]

where \( \chi \) is the usual Euler characteristic \( \chi \) or the generalized Euler characteristic \( \chi_g \) respectively, if this sum makes sense in \( A \) (respectively in \( A \otimes \mathbb{Z} K_0(V_C)_{(L)} \)). If the integral exists (makes sense) the function \( \psi \) is said to be integrable.

One can easily see that the Poincaré series \([5]\) of a one index filtration is equal to the following integral with respect to the Euler characteristic:

\[
P(t) = \int_{\mathbb{P}O_{V,0}} t^{v(g)} \, d\chi.
\]

Here \( t^{v(g)} \) is considered as a function on \( O_{V,0} \) (or \( \mathbb{P}O_{V,0} \)) with values in the Abelian group \( \mathbb{Z}[[t]] \) of power series in \( t \) with integral coefficients; \( t^\infty \) is supposed to be equal to 0. The above equality follows immediately from the fact that, for a complex vector space \( L \) of finite dimension, one has \( \dim L = \chi(\mathbb{P}L) \).

Let us use the following definition inspired by \([6]\).

Definition: The Poincaré series of a multi-index filtration \( \{J(v)\} \), \( v \in \mathbb{Z}^r \), on the ring \( O_{V,0} \) is the power series in \( r \) variables \( t_1, \ldots, t_r \) defined as the following integral with respect to the Euler characteristic:

\[
P(t) = \int_{\mathbb{P}O_{V,0}} t^{v(g)} \, d\chi.
\]

In \([2]\) and \([8]\) it was essentially shown than the Poincaré series \( P(t) \) also has the following description which in fact was its initial definition in \([5]\) (formally speaking, in the cited papers this statement is proved for a particular filtration, however, there is no difference; see also Proposition 2 below). Let \( \mathbf{1} = (1,1,\ldots,1) \in \mathbb{Z}^r \). Let

\[
L(t) = \sum_{v \in \mathbb{Z}^r} \dim(J(v)/J(v+\mathbf{1})) \cdot t^v.
\]

The series \( L(t) \) contains negative powers of variables \( t_i \), however, it does not contain monomials \( t^v \) with purely negative \( v \) (i.e. all components of which
are negative). It is convenient to consider \( L(t) \) as an element of the set \( L \) of formal Laurent series in the variables \( t_1, \ldots, t_r \) with integer coefficients without purely negative exponents; i.e. of expressions of the form \( \sum_{\nu \in \mathbb{Z}^r \setminus \mathbb{Z}_{<1}^r} d(\nu) \cdot t^{\nu} \) with 
\( d(\nu) \in \mathbb{Z} \).

**Proposition 1**

\[
P(t) = \frac{L(t) \cdot \prod_{k=1}^{r} (t_k - 1)}{t_1 \cdot \ldots \cdot t_r - 1}.
\]

In analogy to [9], [8], [4], one can define a notion of the extended semigroup of the multi-index filtration \( \{J(\nu)\} \). For \( K \subset K_0 = \{1, 2, \ldots, r\} \), let \( \#K \) be the number of elements in \( K \), let \( 1_K \) be the element of \( \mathbb{Z}^r_{\geq 0} \) whose \( i \)th component is equal to 1 or 0 if \( i \in K \) or \( i \notin K \) respectively (one has \( 1 = 1_{K_0} \)). For \( \nu \in \mathbb{Z}^r_{\geq 0} \), let
\[
F_{\nu} := \frac{J(\nu)}{J(\nu + 1)} \setminus \bigcup_{k=1}^{r} \frac{J(\nu + 1_{(k)})}{J(\nu + 1)}.
\]

**Definition:** The extended semigroup \( \hat{S} \) is the union of the spaces \( F_{\nu} \) for \( \nu \in \mathbb{Z}^r_{\geq 0} \).

The spaces \( F_{\nu} \) are called fibres of the extended semigroup \( \hat{S} \). The space \( \hat{S} \) is a graded space in the sense of [2] (i.e. its components are numbered by elements of \( \mathbb{Z}^r_{\geq 0} \); in other words there is a function \( \nu \) on \( \hat{S} \) with values in \( \mathbb{Z}^r_{\geq 0} \) constant on connected components \( F_{\nu} \) of \( \hat{S} \): it has value \( \nu \) on \( F_{\nu} \)). It is really a semigroup if the functions \( v_i \) which define the filtration are valuations, i.e. satisfy the property [4]. In this case the semigroup operation in \( \hat{S} \) is defined by multiplication of functions. For the filtration defined by orders of functions on components \( C_k \) of a curve \( C = \bigcup_{k=1}^{r} C_k \subset (\mathbb{C}^n, 0) \), the extended semigroup \( \hat{S} \) is the set \( \{(\nu(g), a(g)) \mid g \in \mathbb{C}^n, 0 \} \) for all \( g \in \mathbb{C}^n, 0 \) with \( v_k(g) < \infty \) for \( k = 1, \ldots, r \). The fibre \( F_{\nu} \) is the complement to an arrangement of linear subspaces in a linear space. Let \( \mathbb{P}F_{\nu} = F_{\nu}/\mathbb{C}^* \) be its projectivization. The union \( \mathbb{P}\hat{S} \) of the spaces \( \mathbb{P}F_{\nu} \) is (called) the projectivization of the extended semigroup \( \hat{S} \). (The projectivization \( \mathbb{P}\hat{S} \) is a semigroup itself if the functions \( v_i \) are valuations.) From [7] one can easily see that the Poincaré series \( P(t) \) is equal to the integral of the function \( t^{\nu} \) over the projectivization \( \mathbb{P}\hat{S} \) of the extended semigroup with respect to the Euler characteristic:

\[
P(t) = \int_{\mathbb{P}\hat{S}} t^{\nu} d\chi.
\]
2 Generalized Poincaré series of multi-index filtrations.

Using integration with respect to the generalized Euler characteristic one can define the following two generalizations of the Poincaré series \( P(t_1, \ldots, t_r) \).

**Definition**: The **generalized Poincaré series** \( P_g(t) \) of a filtration \( J(v) \) defined by \( v(g) = (v_1(g), \ldots, v_r(g)) \) is the integral

\[
P_g(t) = \int_{\mathbb{P}\mathcal{O}_{V,0}} t^{v(g)} d\chi_g \in K_0(\mathcal{O}_{V,C})[[t_1, \ldots, t_r]]
\]

over the projectivization \( \mathbb{P}\mathcal{O}_{V,0} \) of the ring \( \mathcal{O}_{V,0} \) with respect to the generalized Euler characteristic \( \chi_g \).

The subset of the projectivization \( \mathbb{P}\mathcal{O}_{V,0} \) where \( t^{v(g)} \) is equal to \( t^v \) (i.e., where \( v(g) = v \)) is the projectivization of the complement \( J(v) \setminus \bigcup_{k=1}^r J(v+1_{\{k\}}) \).

Because of that all the coefficients of the series \( P_g(t) \) are polynomials in \( L^{-1} \). Therefore we shall write \( P_g(t_1, \ldots, t_r) \) as a series

\[
P_g(t_1, \ldots, t_r, q) \in \mathbb{Z}[[t_1, \ldots, t_r, q]]
\]

in \( t_1, \ldots, t_r \), and \( q = L^{-1} \). One has \( P(t_1, \ldots, t_r) = P_g(t_1, \ldots, t_r, 1) \).

To give "a motivic version" of Proposition 1, let us define the corresponding version of the series \( L(t) : L_g(t_1, \ldots, t_r, q) \in \mathcal{L}[q] \). We have indicated that the reason for Proposition 1 is the following formula for the dimension. Let \( A \) and \( A' \) be subspaces of \( \mathcal{O}_{V,0} \) of finite codimension. Then \( \dim(A/A') = \chi(\mathbb{P}(A) \setminus \mathbb{P}(A')) \). Substituting the usual Euler characteristic \( \chi \) by the generalized one \( \chi_g \), one gets the following "motivic version" of the dimension: "\( \dim_g(A/A') = \chi_g(\mathbb{P}(A) \setminus \mathbb{P}(A')) \). Let \( \operatorname{codim} A = a \), \( \operatorname{codim} A' = a' \) (\( a' > a \)). Then \( \chi_g(\mathbb{P}(A) \setminus \mathbb{P}(A')) = q^a + q^{a+1} + \ldots + q^{a'-1} = q^a \cdot \frac{1-q^{a'-a}}{1-q} \). Therefore let \( L_g(t, q) \) ("a motivic version" of \( L(t) \)) be the series

\[
L_g(t, q) := \sum_{v \in \mathbb{Z}^r} q^{h(v)} \cdot \frac{1 - q^{h(v+1_k)-h(v)}}{1-q} \cdot t^v.
\]

**Proposition 2**

\[
P_g(t, q) = \frac{L_g(t, q) \cdot \prod_{k=1}^r (t_k - 1)}{t_1 \cdot \ldots \cdot t_r - 1}.
\]
Proof. One has

\[
P_g(t, q) = \int_{\mathbb{P} \mathcal{O}_{V, 0}} t^v d\chi_g = \sum_{u \in \mathbb{Z}^r} \chi_g(\{ g \in \mathbb{P} \mathcal{O}_{V, 0} : \mathcal{U}(g) = u \}) \cdot t^u = \\
= \sum_{u \in \mathbb{Z}^r} [\mathbb{P} \mathcal{J}(\mathcal{U}) \setminus \bigcup_{k=1}^r \mathbb{P} \mathcal{J}(\mathcal{U} + 1_{(k)})] \cdot t^u = \\
= \sum_{u \in \mathbb{Z}^r} \left( \sum_{K \subset K_0} (-1)^#K [\mathbb{P} \mathcal{J}(\mathcal{U} - 1_{K}) \setminus \mathbb{P} \mathcal{J}(\mathcal{U} + 1)] \right) \cdot t^u.
\]

Therefore

\[
(t_1 \cdot \ldots \cdot t_r - 1) \cdot P_g(t, q) = \\
= \sum_{u \in \mathbb{Z}^r} \left( \sum_{K \subset K_0} (-1)^#K [\mathbb{P} \mathcal{J}(\mathcal{U} - 1_{K}) \setminus \mathbb{P} \mathcal{J}(\mathcal{U} + 1)] \right) \cdot t^u - \\
- \sum_{u \in \mathbb{Z}^r} \left( \sum_{K \subset K_0} (-1)^#K [\mathbb{P} \mathcal{J}(\mathcal{U} + 1_{K}) \setminus \mathbb{P} \mathcal{J}(\mathcal{U} + 1)] \right) \cdot t^u = \\
= \sum_{u \in \mathbb{Z}^r} \sum_{K \subset K_0} (-1)^#K q^{h(\mathcal{U} - 1_{K})} \cdot \frac{1 - q^{h(\mathcal{U} + 1_{K}) - h(\mathcal{U} + 1_{K})}}{1 - q} \cdot t^u = \\
= \left( \sum_{u \in \mathbb{Z}^r} q^{h(\mathcal{U})} \cdot \frac{1 - q^{h(\mathcal{U} + 1) - h(\mathcal{U})}}{1 - q} \right) \cdot \prod_{k=1}^r (t_k - 1) = L_g(t, q) \cdot \prod_{k=1}^r (t_k - 1).
\]

\[
\square
\]

Generally speaking, the generalized Poincaré series \( P_g(t, q) \) is a more fine invariant of a filtration than the (usual) Poincaré series \( P(t) \). Moreover this already holds for filtrations defined by orders of functions on components of curve singularities. Theorem 1 below shows that this cannot happen for plane curve singularities: two plane curve singularities with the same Poincaré series \( P(t) \) are topologically equivalent, have topologically equivalent resolutions and therefore equal generalized Poincaré series \( P_g(t, q) \). However this can happen for non plane curve singularities what can be seen from the following example taken from [5].

Example. Let \( C = C_1 \cup C_2 \subset (\mathbb{C}^5, 0) \) (respectively \( C' = C_1' \cup C_2' \subset (\mathbb{C}^6, 0) \)) be the germ of a curve whose branches \( C_1 \) and \( C_2 \) (resp. \( C_1' \) and \( C_2' \)) are defined...
by the parametrizations \((t \in \mathbb{C}, u \in \mathbb{C})\):

\[
C_1 = \{(t^2, t^3, t^2, t^4, t^5)\}, \quad C_2 = \{(u^2, u^3, u^4, u^5, u^6)\}
\]

(resp. \(C'_1 = \{(t^3, t^4, t^5, t^4, t^5, t^6)\}, \quad C'_2 = \{(u^3, u^4, u^5, u^6, u^7)\})\).

The local rings \(\mathcal{O}_{C,0}\) and \(\mathcal{O}_{C',0}\) of the curves \(C\) and \(C'\), as subrings of the normalizations \(\mathbb{C}\{t\} \times \mathbb{C}\{u\}\), are:

\[
\mathcal{O}_{C,0} = \mathbb{C}\{(t^2, u^2), (t^3, u^3), (t^4, u^4), (t^5, u^6)\},
\]

\[
\mathcal{O}_{C',0} = \mathbb{C}\{(t^3, u^3), (t^4, u^4), (t^5, u^5), (t^4, u^5), (t^6, u^6)\}.
\]

The corresponding semigroups of values are, respectively,

\[
S = \{(0, 0), (3, 3), (2, k), (\ell, 2), (r, s) : k \geq 2, \ell \geq 3, r, s \geq 4\},
\]

\[
S' = \{(0, 0), (3, 3), (r, s) : r, s \geq 4\}.
\]

An easy computation shows that in both cases the Poincaré series is the polynomial \(P(t) = 1 + t^3t^3\). However one has \(h(3,3) = 1\) for the curve \(C'\) and \(h(3,3) = 3\) for \(C\) (note that a basis for this case consists of the classes of 1, \((t^2, u^4)\) and \((t^4, u^2)\) with the linear dependence \((t^2, u^4) + (t^4, u^2) - (t^2, u^2) = (t^4, u^2)\) in \(J(3,3)\)).

**Remark.** The example shows that, generally speaking, the Poincaré series \(P(t_1, \ldots, t_r)\) of the multi-index filtration \(\{J(v)\}\) defined by a system of functions \(\{v_k\}, k \in K_0 = \{1,2, \ldots, r\}\), does not determine the Hilbert function \(h(v)\). However, the set of the Poincaré series of the filtrations defined by all subsystems of \(\{v_k\}\) does determine the Hilbert function. For \(K \subseteq K_0\), let \(P_K(t)\) be the Poincaré series corresponding to the system \(\{v_k : k \in K\}\) (it really depends on variables \(t_k\) only with \(k \in K\); \(P_{K_0}(t) = P(t)\)). Let

\[
H(t) = \sum_{\mathbb{Z}_{\geq 0}, k \in K_0} h(u)t^k.
\]

The same arguments as in [8] or [4] show that

\[
H(t) = \frac{t}{1 - t} \cdot \frac{\sum_{K \subseteq K_0} (-1)^{r - \#K} P_K(t) \cdot (t_1 \cdot \ldots \cdot t_r - 1) \mid_{t_k = 1 \text{ for } k \notin K}}{\prod_{k=1}^r (t_k - 1)}
\]

(in [8] there is a misprint: the factor \(\frac{1}{t}\) is forgotten; it is corrected in [4]). The fact that, in general, the Poincaré series \(P(t)\) does not determine the Hilbert function \(h(t)\) implies that it does not determine the Poincaré series \(P_K(t)\) for \(K \subseteq K_0\). This can be explained by the fact that both \(P(t)\) and \(P_K(t)\) can be computed as integrals of the type (7). However, those \(g \in \mathcal{O}_{V,0}\), for which \(v_k(g) < \infty\) for all \(k \in K\) but \(v_k(g) = \infty\) for some \(k \notin K\), participate in the integral for \(P_K(t)\), but do not participate in the integral for \(P(t)\): in this case
$t^{v(g)} = 0$. This gives the following fact. Suppose that the set of functions $v_k$ which defines the filtration is such that if $v_k(g) = \infty$ for certain $k \in K_0$ then $v_k(g) = \infty$ for all $k \in K_0$. Then $P_K(t) = P(t) \mid_{t_k=1}$ for $k \notin K$. This holds, in particular, for the divisorial valuations discussed below.

One can also consider a generalization of the Poincaré series $P(t_1, \ldots, t_r)$ which emerges from the equation (9).

**Definition:** The generalized semigroup Poincaré series $\hat{P}_g(t)$ of a filtration $\{J(v)\}$ defined by $v(g) = (v_1(g), \ldots, v_r(g))$ is the integral

$$\hat{P}_g(t) = \int_{\mathbb{P}^S} t^{\mathcal{V}(g)} d\chi_g \in K_0(\mathcal{V})[[t_1, \ldots, t_r]]$$

over the projectivization of the extended semigroup $\hat{S}$ with respect to the generalized Euler characteristic $\chi_g$.

All connected components of $\mathbb{P}\hat{S}$ (i.e. projectivizations $\mathbb{P}F_v$ of the fibres $F_v$) are complements to arrangements of projective subspaces in finite dimensional (!) projective spaces. Because of that all the coefficients of the series $\hat{P}_g(t)$ are polynomials in $L$. Therefore we shall write $\hat{P}_g(t_1, \ldots, t_r)$ as a series $\hat{P}_g(t_1, \ldots, t_r, L) \in \mathbb{Z}[[t_1, \ldots, t_r, L]]$ in $t_1, \ldots, t_r, L$. One has $P(t_1, \ldots, t_r) = \hat{P}_g(t_1, \ldots, t_r, 1)$.

Repeating the arguments of Proposition 2 one gets the following equation:

$$\hat{P}_g(t) = \frac{\left( \sum_{v \in \mathbb{Z}^r} [P(J(v))/J(v + 1))] \cdot t^{v} \right)}{t_1 \cdots t_r - 1} \cdot \prod_{k=1}^r (t_k - 1).$$

### 3 The Poincaré series $P_g(t, q)$ in terms of an embedded resolution.

Let $C \subset (\mathbb{C}^2, 0)$ be a (reduced) plane curve singularity, $C = \bigcup_{k=1}^r C_k$, $C_k = \{f_k = 0\}$, $f_k \in \mathcal{O}_{\mathbb{C}^2,0}$, and let $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ be an embedded resolution of it. Let the exceptional divisor $\mathcal{D}$ of the resolution $\pi$ be the union of irreducible components $E_i$, $i = 1, \ldots, s$, each of them is isomorphic to the projective line $\mathbb{CP}^1$. Let $\hat{E}_i$ be the “nonsingular part” of the component $E_i$, i.e., $E_i$ minus intersection points with all other components of the total transform $(f \circ \pi)^{-1}(0)$ of the curve $C$. We shall denote by $\hat{E}_i$ the “nonsingular part” of the component $E_i$ in the space $\mathcal{X}$ of the resolution, i.e., $E_i$ minus intersection points
with all other components $E_i$ of the exceptional divisor $\mathcal{D}$. For $i = 1, \ldots, s$ and $g \in \mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\}$, let $w_i(g)$ be the multiplicity of the lifting $\tilde{g} = g \circ \pi$ of the function $g$ to the space $\mathcal{X}$ of the resolution along the component $E_i$ of the exceptional divisor $\mathcal{D}$. The valuations $w_i$ define a multi-index filtration (called divisorial) $\{J^D(w)\}$ on the ring $\mathcal{O}_{\mathbb{C}^2,0}$: $J^D(w) = \{g : w(g) \geq w\}$.

For $k = 1, \ldots, r$, let $i(k)$ be the number of the component $E_i$ ($i = i(k)$) of the exceptional divisor $\mathcal{D}$ which intersects the strict transform $\widetilde{C}_k$ of the component $C_k$ of the curve $C$. Let $A = (E_i \circ E_j)$ be the intersection matrix of the components $E_i$. (For each $i$ the self-intersection number $(E_i \circ E_i)$ is negative, for $i \neq j$ the intersection number $(E_i \circ E_j)$ is either 0 or 1; let $A = \pm 1$.) Let $M = -A^{-1}$. The entries $m_{ij}$ of the matrix $M$ are positive and have the following meaning. Let $\tilde{L}_i$ be a germ of a smooth curve on $\mathcal{X}$ transversal to the component $E_i$ of the exceptional divisor $\mathcal{D}$ at a smooth point (i.e., at a point of $\tilde{E}_j$) and let the projection $L_i = \pi(\tilde{L}_i) \subset (\mathbb{C}^2,0)$ of the curve $\tilde{L}_i$ be given by an equation $g_i = 0$, $g_i \in \mathcal{O}_{\mathbb{C}^2,0}$. Then $m_{ij} = w_j(g_i) = w_i(g_j) = L_i \circ L_j$.

Let $I_0 = \{(i,j) : i < j, E_i \cap E_j = pt\}$, $K_0 = \{1, 2, \ldots, r\}$. For $\sigma \in I_0$, $\sigma = (i,j)$, let $P_\sigma = E_i \cap E_j$; for $k \in K_0$, let $P_k = E_{i(k)} \cap \widetilde{C}_k$. For $\sigma = (i,j) \in I_0$, let $i(\sigma) := i$, $j(\sigma) := j$. For $I \subset I_0$, $K \subset K_0$, let

$$N_{I,K} := \{(n) = (n_i, n'_\sigma, n''_{\sigma}, \tilde{n}'_k, \tilde{n}''_k) : n_i \geq 0, i = 1, \ldots, s; n'_\sigma > 0, n''_{\sigma} > 0, \sigma \in I; \tilde{n}'_k > 0, \tilde{n}''_k > 0, k \in K\}.$$

For $n \in N_{I,K}$, $i = 1, \ldots, s$, let

$$\hat{n}_i := n_i + \sum_{\sigma \in I: i(\sigma) = i} n'_\sigma + \sum_{\sigma \in I: j(\sigma) = i} n''_{\sigma} + \sum_{k \in K: i(k) = i} \tilde{n}'_k.$$

Let

$$F(n) := \frac{1}{2} \left( \sum_{i,j=1}^{s} m_{ij} \hat{n}_i \hat{n}_j + \sum_{i=1}^{s} \hat{n}_i \cdot \left( \sum_{j=1}^{s} m_{ij} \chi(E_j)^* + 1 \right) \right) + \sum_{k \in K} \tilde{n}'_k, \quad (10)$$

and $w(n) := \sum_{i=1}^{s} \hat{n}_i m_i$, where $m_i = (m_{i1}, \ldots, m_{is}) \in \mathbb{Z}_{\geq 0}^s$, $v_k(n) := w_{i(k)}(n) + \tilde{n}''_k$.

Theorem 1

$$P_g(t_1, \ldots, t_r, q) = \sum_{I \subset I_0, K \subset K_0} \sum_{n \in N_{I,K}} q^{F(n)} - \sum_{i=1}^{s} n_i - \#I - \#K \cdot (1 - q)^{\#I + \#K} \times$$

$$\times \prod_{i=1}^{s} \left( \min_{\{n_i, 1-\chi(E_i)\}} \sum_{j=0}^{\min_\circ} (-1)^j \left( 1 - \chi(E_i) \right)^j \right) \cdot \tilde{L}_i^{w(n)}.$$
Remark. One can see that the generalized Poincaré series \( P_\sigma(t, q) \) of the discussed filtration represents a rational function in the variables \( t_1, \ldots, t_r \) and \( q \) (in fact with the denominator \( \prod_{k=1}^r (1 - qt_k) \)). This is not immediately clear from the equation of Theorem 1.

Proof. In a neighbourhood of the point \( P_\sigma, \sigma \in I_0 \), (respectively of the point \( P_k, k \in K_0 \)) choose local coordinates \( x_\sigma, y_\sigma \) (respectively \( x_k, y_k \)) so that the components of the total transform \( \pi^{-1}(C) \) of the curve \( C \) are the coordinate lines: \( E_i(\sigma) = \{ y_\sigma = 0 \} \), \( E_j(\sigma) = \{ x_\sigma = 0 \} \) (respectively \( E_i(k) = \{ y_k = 0 \} \), \( \tilde{C}_k = \{ x_k = 0 \} \)).

For a space \( X \), let \( S^n X = X^n / S_n \) be the \( n \)th symmetric power of the space \( X \). Let

\[
Y := \bigcup_{I \subset I_0, K \subset K_0} \bigcup_{\mathbf{n} \in N_{I,K}} \left( \prod_{i=1}^s \mathbb{C}^{n_i} \times \prod_{\sigma \in I} \mathbb{C}_\sigma^* \times \prod_{k \in K} \mathbb{C}_k^* \right),
\]

where \( \mathbb{C}_\sigma^* \) and \( \mathbb{C}_k^* \) are copies of the punctured line \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) numbered by \( \sigma \in I_0 \) and \( k \in K_0 \) respectively. Let \( Y_\mathbf{n} = \prod_{i=1}^s \mathbb{C}^{n_i} \times \prod_{\sigma \in I} \mathbb{C}_\sigma^* \times \prod_{k \in K} \mathbb{C}_k^* \) be the corresponding connected component of the space \( Y \). The space \( Y \) can be considered as a semigroup with respect to the following semigroup operation. Let \( y_1 \) and \( y_2 \) be two points of \( Y \) which belong to the components \( Y_{\mathbf{n}_1} \) and \( Y_{\mathbf{n}_2} \) respectively, where \( 1_{\mathbf{n}} = (n_1, n_\sigma', n_\sigma'', \tilde{n}_k', \tilde{n}_k'' \in N_{I, K} \) and \( 1_{\mathbf{n}} = (n_i, n_\sigma', n_\sigma'', \tilde{n}_k', \tilde{n}_k'') \in N_{I, K} \). Then the product \( y_1 y_2 \) belongs to the component \( Y_{\mathbf{n}_3} \) with \( \mathbf{n}_3 = (n_i, n_\sigma', n_\sigma'', \tilde{n}_k', \tilde{n}_k'') \in N_{I, K} \) such that \( I = I_1 \cup I_2, K = K_1 \cup K_2 \), each of \( n_i, n_\sigma', n_\sigma'', \tilde{n}_k', \tilde{n}_k'' \) is equal to the sum of the corresponding components in \( 1_{\mathbf{n}} \) and \( 2_{\mathbf{n}} \) where, if a component is absent (say, if \( \sigma \in I \), but \( \sigma \not\in I_1 \)), it is assumed to be equal to zero. Moreover, in the factor \( S^n E_i \) of the component \( Y_{\mathbf{n}} \) the product \( y_1 y_2 \) is represented by the union of the corresponding tuples of points of \( E_i^* \) in \( y_1 \) and in \( y_2 \) (if a component is absent, the corresponding tuple is assumed to be empty); in the factor \( \mathbb{C}_\sigma^* \) or \( \mathbb{C}_k^* \) the product \( y_1 y_2 \) is represented by the product of the corresponding components (nonzero numbers) in \( y_1 \) and in \( y_2 \) (if a component is absent, it is assumed to be equal to 1).

Let \( O_{\mathbb{C}^2, 0} = \{ g \in O_{\mathbb{C}^2, 0} : v_i(g) < \infty \text{ for } i = 1, \ldots, r \} \) and let \( \text{Init} \) be the map \( \mathbb{P}O_{\mathbb{C}^2, 0} \to Y \) from the projectivization of \( O_{\mathbb{C}^2, 0} \) defined in the following way. For \( g \in O_{\mathbb{C}^2, 0} \), let \( \Gamma_g \subset X \) be the strict transform of the curve \( \{ g = 0 \} \subset (\mathbb{C}^2, 0) \). Let \( I(g) := \{ \sigma \in I_0 : P_\sigma \subset \Gamma_g \}, K(g) := \{ k \in K_0 : P_k \subset \Gamma_g \} \). Let \( n_i = n_i(g) \), \( i = 1, \ldots, s \), be the number of intersection points of the curve \( \Gamma_g \) with the smooth part \( E_i \) of the component \( E_i \).
counted with multiplicities. For $\sigma \in I(g)$, let $n'_\sigma = n'_\sigma(g) := (\Gamma_g \circ E_i(\sigma))_{P_\sigma}$, $n''_\sigma = n''_\sigma(g) = (\Gamma_g \circ E_j(\sigma))_{P_\sigma}$; for $k \in K(g)$, let $\bar{n}'_k = \bar{n}'_k(g) := (\Gamma_g \circ E_i(k))_{P_\sigma}$, $\bar{n}''_k = \bar{n}''_k(g) := (\Gamma_g \circ C_k)_{P_\sigma}$. For $\sigma \in I(g)$ (respectively for $k \in K(g)$), in a neighbourhood of the point $P_\sigma$ (respectively of the point $P_k$), the germ of the curve $\Gamma_g$ is given by an equation $\varphi_\sigma(x_\sigma, y_\sigma) = 0$, where $\varphi_\sigma(x_\sigma, y_\sigma) = a_\sigma(g)x_\sigma^{n'_\sigma} + b_\sigma(g)y_\sigma^{n''_\sigma}$ + terms with monomials $x_\sigma^\alpha y_\sigma^\beta$ with either $\alpha > 0$, $\beta > 0$, or $\alpha > n'_\sigma$, or $\beta > n''_\sigma$, $a_\sigma(g) \neq 0$, $b_\sigma(g) \neq 0$ (respectively by an equation $\varphi_k(x_k, y_k) = 0$, where $\varphi_k(x_k, y_k) = a_k(g)x_k^{\bar{n}'_k} + b_k(g)y_k^{\bar{n}''_k}$ + ... $a_k(g) \neq 0$, $b_k(g) \neq 0$). Now $\text{Init}(g)$ is an element of the summand $Y_n$ corresponding to $I(g)$, $K(g)$ and $n = (n_1, n'_\sigma, n''_\sigma, \bar{n}'_k, \bar{n}''_k)$ the components (factors) of which are the following ones. In $S^n_i$, $i = 1, \ldots, s$, it is represented by the set of the intersection points of the curve $\Gamma_g$ and $E_i$ counted with multiplicities; in $C^*_\sigma$, $\sigma \in I(g)$, (respectively in $C^*_k$, $k \in K(g)$) it is represented by the element $a_\sigma(g) : b_\sigma(g)$ (respectively by $a_k(g) : b_k(g)$). One can see that the map $\text{Init}$ is a semigroup homomorphism $\mathbb{P}O^*_C \to Y$, where the semigroup structure on $\mathbb{P}O^*_{C^2, 0}$ is defined by multiplication of functions.

**Lemma 1** For any point $y \in Y$, the preimage $\text{Init}^{-1}(y)$ is an affine space in the projectivization $\mathbb{P}O_{C^2, 0}$ of a finite codimension. Moreover, over each connected component $Y_n$ of the space $Y$ the map $\text{Init}$ is a locally trivial fibration.

This statement follows from the next one.

**Lemma 2** For $g$ and $g'$ from $O^*_C$, $\text{Init}(g) = \text{Init}(g')$ if and only if $g' = \alpha g + h$, where $\alpha \in \mathbb{C}^*$, $w_i(h) > w_i(g)(= w_i(g'))$ for all $i = 1, \ldots, s$, $v_k(h) > v_k(g)(= v_k(g'))$ for all $k = 1, \ldots, r$.

**Proof.** Let $\tilde{g} = g \circ \pi$ and $\tilde{g}' = g' \circ \pi$ be the liftings of the functions $g$ and $g'$ to the space $X$ of the resolution. Their ratio $\Psi = \tilde{g}' / \tilde{g}$ is a meromorphic function on $X$. If $\text{Init}(g) = \text{Init}(g')$, then zeroes and poles of the function $\Psi$ on each component of the exceptional divisor $D$ and of the strict transform $\tilde{C}$ of the curve $C$ cancel each other and therefore $\Psi$ is a regular function on each of them (in particular, it is constant on each component $E_i$). The conditions on the coefficients in the power series decompositions of the functions $\tilde{g}$ and $\tilde{g}'$ at the intersection points of the total transform $\pi^{-1}(C)$ of the curve $C$ imply that at all these points the values of the function $\Psi$ on the both components of the total transform coincide. Therefore $\Psi$ is a regular function on $\pi^{-1}(C)$ equal to a constant (say, $\alpha$) on the exceptional divisor $D$. Therefore $w_i(g' - \alpha g) > w_i(g)(= w_i(g'))$, $i = 1, \ldots, s$, $v_k(g' - \alpha g) > v_k(g)(= v_k(g'))$, $k = 1, \ldots, r$. 


In the other direction, if \( q' = \alpha g + h \) with \( \alpha \in \mathbb{C}^* \), \( w_i(g' - \alpha g) > w_i(g) \), \( i = 1, \ldots, s \), \( v_k(g' - \alpha g) > v_k(g) \), \( k = 1, \ldots, r \), then \( \Psi \) is a regular function on the total transform \( \pi^{-1}(C) \) and \( \Psi|_\mathcal{D} \equiv \alpha \). Therefore zeroes and poles of the function \( \Psi \) cancel each other on each component of the exceptional divisor \( \mathcal{D} \) and therefore the intersection points of the strict transform of the curves \( \{ g = 0 \} \) and \( \{ g' = 0 \} \) with each component \( E_i \) of the exceptional divisor \( \mathcal{D} \) coincide (with their multiplicities). The intersection number of the strict transform of the curve \( \{ g = 0 \} \) with the component \( \tilde{C}_k \) of the strict transform of the curve \( C \) is equal to \( v_k(g) - w_i(k)(g) \) and therefore coincides with that of \( \{ g' = 0 \} \). The fact that the ratios of the coefficients in local equations of the strict transforms of the curves \( \{ g = 0 \} \) and \( \{ g' = 0 \} \) at the intersection points of the components of \( \pi^{-1}(C) \) are equal is obvious. \( \Box \)

The number \( w_i(n) \) defined before Theorem 1 is just the multiplicity along the component \( E_i \) of the exceptional divisor \( \mathcal{D} \) of the lifting of a function \( g \) such that \( \text{Init}(g) \in Y_n \). The number \( v_k(n) \) is the order of such a function on the component \( C_k \) of the curve.

Lemma 2 says that the preimage \( \text{Init}^{-1}(y) \) of a point \( y \in Y_n \) is an affine space whose codimension \( F(n) \) in \( \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0} \) is one less than the codimension of the ideal \( I_n \) in \( \mathcal{O}_{\mathbb{C}^2,0} \) which consists of functions \( h \) with \( w_i(h) > w_i(n) \), \( i = 1, \ldots, s \) and \( v_k(h) > v_k(n) \), \( k = 1, \ldots, r \). Applying the Fubini formula to the map \( \text{Init} \) one gets

\[
P_g(l, q) = \int_Y q^{F(n)} L \cdot \overline{w}(n) d\chi_y = \sum_{l \subset I_0, K \subset K_0} \sum_{n \in N_{l,K}} q^{F(n)} \left[ Y_n \right] \cdot \overline{w}(n).
\]

Here \( \left[ Y_n \right] = [\mathbb{C}^*]^{#I + \#K} \cdot \prod_{i=1}^{s} [S^{n_i} E_i] \), \( [\mathbb{C}^*] = q^{-1} - 1 = q^{-1}(1 - q) \). The class \( [S^{n_i} E_i] \) as a polynomial in \( L = q^{-1} \) can be obtained from the formula

\[
\sum_{n=0}^{\infty} [S^n E_i] t^n = (1 - t)^{-|1 - \chi(E_i)|}
\]

\[
= (1 - t)^{-L}(1 - t)^{1 - \chi(E_i)} = \sum_{n=0}^{\infty} L^n t^n (1 - t)^{1 - \chi(E_i)}
\]

(see, e.g., [11]). Therefore

\[
[S^n E_i] = \sum_{j=0}^{\min(n,1-\chi(E_i))} (-1)^j L^n t^j (1 - \chi(E_i))
\]

\[
= q^{-n} \sum_{j=0}^{\min(n,1-\chi(E_i))} (-1)^j \left( \frac{1 - \chi(E_i)}{j} \right) q^j.
\]
Now to prove Theorem 4 it remains to show that the codimension $F(n)$ of the ideals $I_n$ is given by the formula (10). Making additional blowing-ups at intersection points of the strict transforms $\widetilde{C}_k$ of the curve $C$ with the exceptional divisor $D$ one reduces this problem to the case when $K = \emptyset$. In this case the codimension we are interested in is equal to $h^D(w(n) + 1) - 1$ where $h^D(w) = \dim \mathcal{O}_{\mathbb{C}^2,0}/\mathcal{J}^D(w)$ is the Hilbert function of the divisorial filtration $\{\mathcal{J}^D(w)\}$. The codimension $h^D(w(n))$ can be computed by the formula 
\[ h^D(w(n)) = -\frac{1}{2}(\mathcal{D} \circ \mathcal{D} + \mathcal{D} \circ \mathcal{K}) \] (it can be also obtained from the Hoskin–Deligne formula: see, e.g., [12]), where $\mathcal{D} = \sum_{i=1}^{s} w_i(n) E_i = - \sum_{i=1}^{s} \tilde{n}_i E_i^*$, $\mathcal{K} = - \sum_{i=1}^{s} (2 + E_i \circ E_i) E_i^*$ is the canonical divisor of $X$ and $E_1^*, \ldots, E_s^*$ is the basis of divisors on $X$ dual (with respect to the intersection form) to $E_1, \ldots, E_s$. Finally the formula for $h^D(w(n) + 1)$ follows from the fact that an open part of $\mathbb{P}J^D(w(n))$ is fibred over the $\sum_{i=1}^{s} \tilde{n}_i$ dimensional space $\prod_{r=1}^{s} S_{\tilde{n}_r} \hat{E}_r$ with the fibre $J^D(w(n) + 1)$ (Lemma 2 applied to a function $g$ with $I(g) = 0$).

**Remark.** If $w(n) + 1$ itself belongs to the semigroup of values of the set of divisorial valuations $\{w\}$ (i.e., if $w(n) + 1 = w(n')$, the last argument can be substituted by the direct computation of $h^D(w(n) + 1)$ with the use of the formula indicated above.

\[ \square \]

4 The Poincaré series $P_g(t, q)$ for divisorial valuations and the Euler characteristic of its extended semigroup.

Let $\pi : (X, D) \to (\mathbb{C}^2, 0)$ be a modification of the complex plane by a sequence of blowing-ups at preimages of the origin in $\mathbb{C}^2$. Let the exceptional divisor $D$ of the modification $\pi$ be the union of irreducible components $E_i$, $i = 1, \ldots, s$, each of them is isomorphic to the projective line $\mathbb{P}^1$. As above, let $E_i$ be the “nonsingular part” of the component $E_i$, i.e., $E_i$ minus intersection points with all other components $E_j$ of the exceptional divisor $D$. For $i = 1, \ldots, s$ and $g \in \mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\}$, let $w_i(g)$ be the multiplicity of the lifting $\overline{g} = g \circ \pi$ of the function $g$ to the space $X$ of the resolution along the component $E_i$ of the exceptional divisor $D$. Let $A = (E_i \circ E_j)$ be the intersection matrix of the components $E_i$ and let $M = (m_{ij}) = -A^{-1}$. Let $\{J^D(w)\}$ be the divisorial filtration defined by the valuations $w_i(g)$, $i = 1, \ldots, s$: $J^D(w) := \{g \in \mathcal{O}_{\mathbb{C}^2,0} : w_i(g) \geq w\}$.

Let $I_0 = \{(i, j) : i < j, E_i \cap E_j = pt\}$. For $\sigma \in I_0$, $\sigma = (i, j)$, let $P_{\sigma}$, $i(\sigma)$,
and \( j(\sigma) \) be defined as in section 3. For \( I \subset I_0 \), let

\[
N^D_I := \{ n = (n_i, n'_\sigma, n''_\sigma) : n_i \geq 0, i = 1, \ldots, s; n'_\sigma > 0, n''_\sigma > 0, \sigma \in I \}.
\]

For \( n \in N^D_I \), \( i = 1, \ldots, s \), let

\[
\hat{n}_i := n_i + \sum_{\sigma \in I : i(\sigma) = i} n'_\sigma + \sum_{\sigma \in I : j(\sigma) = i} n''_\sigma.
\]

Let

\[
F^D(n) := \frac{1}{2} \left( \sum_{i,j=1}^{s} m_{ij}\hat{n}_i\hat{n}_j + \sum_{i=1}^{s} \hat{n}_i \cdot \left( \sum_{j=1}^{s} m_{ij} \chi(E_j) + 1 \right) \right),
\]

\[
w(n) := \sum_{i=1}^{s} \hat{n}_i \cdot m_i.
\]

**Theorem 2** The generalized Poincaré series \( P^D_g(t_1, \ldots, t_r, q) \) of the divisorial filtration \( \{ J^D(w) \} \) is equal to

\[
P^D_g(t_1, \ldots, t_r, q) = \sum_{I \subset I_0} \sum_{n \in N^D_I} q^{F^D(n) - \sum_{i=1}^{s} n_i - \# I} \cdot (1 - q)^{\# I} \times
\]

\[
\times \prod_{i=1}^{s} \left( \sum_{j=0}^{\min\{n_i, 1-\chi(E_i)\}} (-1)^j \left( 1 - \chi(E_i) \right)^{q^j} \cdot t^{w(n)} \right).
\]

**Proof.** The arguments essentially repeat those of Theorem 1 with the use of the space

\[
Y^D := \bigcup_{I \subset I_0} \bigcup_{n \in N^D_I} \left( \prod_{i=1}^{s} S^{n_i} E_i^* \times \prod_{\sigma \in I} \mathbb{C}_\sigma^* \right).
\]

\( \square \)

In [8] it was shown that

\[
P^D(t_1, \ldots, t_s) = \chi(\mathbb{P}\hat{S}_D) = \prod_{i=1}^{s} (1 - t^{m_i})^{-\chi(E_i)}
\]

(it follows easily from the formula of Theorem 2). Since \( \chi(E_i) = 2 - \#\{ \sigma \in I_0 : i(\sigma) = i \text{ or } j(\sigma) = i \} \), the equation (11) can be rewritten as

\[
\chi(\mathbb{P}\hat{S}_D) = \frac{\prod_{\sigma \in I_0} (1 - t^{m_i(\sigma)})(1 - t^{m_j(\sigma)})}{\prod_{i=1}^{s} (1 - t^{m_i})^2}.
\]
Let us write a version of the equation (12) for the generalized Euler characteristic.

Using the construction described in [2] one can define a map (in fact a semigroup homomorphism) \( \Pi : Y^D \to \mathbb{P}\hat{S}_D \) in the following way. For \( y \in Y \), let \( \Gamma \) be an arbitrary (germ of a) curve on the space \((\mathcal{X}, \mathcal{D})\) of the resolution which represents the point \( y \); see the description of the point of the space \( Y \) corresponding to the curve \( \Gamma \) in the proof of Theorem 1. Let the projection \( \pi(\Gamma) \) of the curve \( \Gamma \) be given by an equation \( \{ g = 0 \} \). If there is another curve \( \Gamma' \) like that and \( \pi(\Gamma') = \{ g' = 0 \} \), then the meromorphic function \( \Psi = g' \circ \pi / g \circ \pi \) on the space \( \mathcal{X} \) of the resolution is constant on the exceptional divisor \( \mathcal{D} \). This implies that the points of the projectivization \( \mathbb{P}\hat{S}_D \) of the extended semigroup corresponding to the functions \( g \) and \( g' \) coincide. By definition this point is the image \( \Pi(y) \) of the point \( y \).

**Proposition 3** \( \Pi \) is a semigroup isomorphism.

**Proof.** This is a direct consequence of Lemma 2. \( \square \)

**Theorem 3**

\[
\hat{P}_g^D (t, \mathbb{L}) = \chi_g (\mathbb{P}\hat{S}_D) = \int_{\mathbb{P}\hat{S}_D} t^w \, d\chi_g = \prod_{\sigma \in I_0} \frac{(1 - t^{m_{i(\sigma)}} - t^{m_{j(\sigma)}} + \mathbb{L} m_{i(\sigma)} t^{m_{j(\sigma)}})}{\prod_{i=1}^s (1 - t^{m_i})(1 - \mathbb{L} t^{m_i})}.
\]

**Proof.** One has the following equalities:

\[
\int_{\mathbb{P}\hat{S}_D} t^w \, d\chi_g = \int_{Y^D} t^w \, d\chi_g = \sum_{I \subseteq I_0} \sum_{n \in \mathcal{N}^D} [Y]\cdot t^v(n) =
\]

\[
= \sum_{I \subseteq I_0} \sum_{n \in \mathcal{N}^D} (\mathbb{C}^*) \# I \prod_{i=1}^s [S^{n_i} E_i] \cdot \mathbb{L} \sum_{i=1}^{n_{i+}} \sum_{\sigma \in I, i(\sigma) = i} n_{i+} + \sum_{\sigma \in I, j(\sigma) = i} n_{i+} m_{j(\sigma)} =
\]

\[
= \left( \sum_{n_{i+} \geq 0, i=1, \ldots, r} \prod_{i=1}^s [S^{n_i} E_i] \cdot \mathbb{L} \sum_{i=1}^{n_{i+}} m_{i(\sigma)} \right) \cdot \left( \sum_{n'_{+} > 0, n''_{+} > 0} \prod_{\sigma \in I} t^{n'_{+} m_{i(\sigma)}} \right).
\]

For the first factor one has

\[
\sum_{n_{i+} \geq 0, i=1, \ldots, r} \prod_{i=1}^s [S^{n_i} E_i] \cdot \mathbb{L} \sum_{i=1}^{n_{i+}} m_{i(\sigma)} = \prod_{i=1}^s \left( \sum_{n=0}^{\infty} [S^n E_i] \cdot t^{n m_i} \right) = \prod_{i=1}^s (1 - t^{m_i})^{-[E_i]} =
\]

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(the last expression is in the sense of [11]; it looks like a motivic version of the formula of A’Campo [1])

\[
= \prod_{i=1}^{s} \left(1 - \frac{t^{m_i}}{1 - L t^{m_i}}\right)^{-\left[C\mathbb{P}^1\right] + \#\{\sigma \in I_0: i(\sigma) = i\} + \#\{\sigma \in I_0: j(\sigma) = i\}} =
\]

\[
= \prod_{i=1}^{s} \left(1 - \frac{t^{m_i}}{1 - L t^{m_i}}\right) - \prod_{\sigma \in I_0} (1 - \frac{t^{m_i(\sigma)}}{1 - t^{m_j(\sigma)}}) (1 - \frac{t^{m_j(\sigma)}}{1 - t^{m_i(\sigma)}})
\]

\[
= \prod_{i=1}^{s} \left(1 - \frac{t^{m_i}}{1 - L t^{m_i}}\right) \prod_{\sigma \in I_0} (1 - \frac{t^{m_i(\sigma)}}{1 - \frac{t^{m_i(\sigma)}}{1 - t^{m_j(\sigma)}}}) \prod_{\sigma \in I_0} \left(1 + \left[C^*\right] \frac{t^{m_i(\sigma)}}{1 - t^{m_i(\sigma)}} \cdot \frac{t^{m_j(\sigma)}}{1 - t^{m_j(\sigma)}}\right)
\]  \hspace{1cm} (13)

(the last equality follows from the fact that \((1 - t)^{-L} = 1/(1 - L t))

For the second factor one has

\[
\sum_{I \subseteq I_0} (C^*)^I \sum_{n_\sigma > 0, n_\sigma' > 0: \sigma \in I} t^{n_\sigma m_i(\sigma)} t^{n_\sigma' m_j(\sigma)} =
\]

\[
= \sum_{I \subseteq I_0} (C^*)^I \cdot \prod_{\sigma \in I} \left(1 - \frac{t^{m_i(\sigma)}}{1 - t^{m_i(\sigma)}} \cdot \frac{t^{m_i(\sigma)}}{1 - t^{m_j(\sigma)}}\right)
\]

\[
= \prod_{\sigma \in I_0} \left(1 + \left[C^*\right] \frac{t^{m_i(\sigma)}}{1 - t^{m_i(\sigma)}} \cdot \frac{t^{m_j(\sigma)}}{1 - t^{m_j(\sigma)}}\right)
\]  \hspace{1cm} (14)

Combining (13) and (14) one gets

\[
\int \mathfrak{P} dX_9 = \prod_{i=1}^{s} \left(1 - \frac{t^{m_i}}{1 - L t^{m_i}}\right) \prod_{\sigma \in I_0} \left[(1 - \frac{t^{m_i(\sigma)}}{1 - \frac{t^{m_i(\sigma)}}{1 - t^{m_j(\sigma)}}}) + \left[C^*\right] \frac{t^{m_i(\sigma)}}{1 - t^{m_i(\sigma)}} t^{m_j(\sigma)}\right] =
\]

\[
= \prod_{\sigma \in I_0} \left(1 - \frac{t^{m_i(\sigma)}}{1 - L t^{m_i}} + \frac{t^{m_i(\sigma)}}{1 - \frac{t^{m_i(\sigma)}}{1 - t^{m_j(\sigma)}}}\right) \prod_{i=1}^{s} \left(1 - \frac{t^{m_i}}{1 - L t^{m_i}}\right)
\]

\[
\square
\]

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