PSEUDOHOLOMORPHIC QUILTS WITH FIGURE EIGHT SINGULARITY

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Abstract. We show that the novel figure eight singularity in a pseudoholomorphic quilt can be continuously removed when composition of Lagrangian correspondences is cleanly immersed. The proof of this result requires a collection of width-independent elliptic estimates that allow for nonstandard complex structures on the domain.

1. Introduction

We consider compact Lagrangian correspondences $L_{01} \subset M_0^− \times M_1$ and $L_{12} \subset M_1^− \times M_2$, where $M_0, M_1, M_2$ are closed symplectic manifolds, and where $M_i^− := (M_i, −ω_M_i)$. The geometric composition of the Lagrangian correspondences is $L_{01} \circ L_{12} := π_{02}(L_{01} ×_{M_1} L_{12})$, the image under the projection $π_{02}: M_0^− \times M_1 \times M_1^− \times M_2 \to M_0^− \times M_2$ of the fiber product

$$L_{01} \times_{M_1} L_{12} := (L_{01} \times L_{12}) \cap (M_0^− \times \Delta_{M_1} \times M_2).$$

Here $Δ_{M_i} \subset M_1 \times M_1^−$ is the diagonal. If $L_{01} \times L_{12}$ intersects $M_0^− \times Δ_1 \times M_2$ transversely then $π_{02}: L_{01} \times_{M_1} L_{12} \to M_0^− \times M_2$ is a Lagrangian immersion (see [GS, WW2]), in which case we call $L_{01} \circ L_{12}$ an immersed composition. In the case of embedded composition, where the projection is injective and hence a Lagrangian embedding, monotonicity and Maslov index assumptions allowed Wehrheim–Woodward [WW1] to establish an isomorphism of quilted Floer cohomologies (as defined in [WW2])

$$HF(…, L_{01}, L_{12}, …) ≅ HF(…, L_{01} \circ L_{12}, …).$$

The analytic core of the proof was a strip-shrinking degeneration, in which a triple of pseudoholomorphic strips coupled by Lagrangian seam conditions degenerates to a pair of strips, via the width of the middle strip shrinking to zero. The monotonicity and embeddedness assumptions allowed for an implicit exclusion of all bubbling, which was conjectured to include a novel figure eight bubbling that (unlike disk or sphere bubbling) could be an algebraic obstruction to [1].

1.1. Removal of singularity. The current author and Katrin Wehrheim prove in [BW] that a blowup of the gradient in a sequence of pseudoholomorphic quilts with an annulus or strip of shrinking width gives rise to one of the standard bubbling phenomena (pseudoholomorphic spheres and disks) or a nontrivial figure eight bubble, as depicted in Figure 1. The latter is a tuple of finite energy pseudoholomorphic maps

$$w_0: \mathbb{R} × (−∞, −\frac{1}{2}] \to M_0, \quad w_1: \mathbb{R} × [−\frac{1}{2}, \frac{1}{2}] \to M_1, \quad w_2: \mathbb{R} × [\frac{1}{2}, ∞) \to M_2$$

satisfying the seam conditions

$$(w_0(s, −\frac{1}{2}), w_1(s, −\frac{1}{2})) \in L_{01}, \quad (w_1(s, \frac{1}{2}), w_2(s, \frac{1}{2})) \in L_{12} \quad ∀ s ∈ \mathbb{R}.$$

In the current paper we apply this Gromov Compactness Theorem to show that the figure eight singularity can be removed, as [WW1] conjectured:

Removal of Singularity Theorem 2.2. If the composition $L_{01} \circ L_{12}$ is cleanly immersed (immersed, and in addition the local branches of $L_{01} \circ L_{12}$ intersect one another cleanly), then $w_0$ resp. $w_2$ extend to continuous maps on $D^2 ≅ (\mathbb{R} × (−∞, 0]) \cup \{∞\}$ resp. $D^2 ≅ (\mathbb{R} × [0, ∞)) \cup \{∞\}$,
Figure 1. The left figure illustrates a figure eight bubble, the middle figure illustrates its reparametrization as a pseudoholomorphic quilt whose domain is the punctured sphere, and the right figure illustrates an inverted figure eight (defined in §2 and equivalent to the left figure via $z \mapsto -1/z$). The domain of the left and right figures is $\mathbb{C}$, and the point at infinity in the left figure corresponds to the punctures in the middle and right figures.

and $w_1(s, -)$ converges to constant paths as $s \to \pm \infty$. If $L_{01} \circ L_{12}$ is embedded, then the latter limits are equal.

This theorem is the first step in the program outlined in [B], which proposes a collection of composition operations amongst Fukaya categories of different symplectic manifolds.

In support of [B], Appendix A also proves the analogous removal of singularities for pseudoholomorphic disks with a type of immersed boundary values in $L_{01} \circ L_{12}$, under the assumption that the latter is cleanly-immersed resp. immersed. These results are not necessarily new, see Appendix A for citations, but provided for the sake of completeness. It is also conceptually useful to recast the (possibly singular) disk bubbles with boundary on $L_{01} \circ L_{12}$ as squashed eight bubbles, that is as triples of finite energy pseudoholomorphic maps

$$w_0: \mathbb{R} \times (-\infty, 0] \to M_0, \quad w_1: \mathbb{R} \to M_1, \quad w_2: \mathbb{R} \times [0, \infty) \to M_2$$

satisfying the generalized seam condition

$$(w_0(s, 0), w_1(s), w_1(s), w_2(s, 0)) \in L_{01} \times_{M_1} L_{12} \quad \forall s \in \mathbb{R}.$$

1.2. Uniform elliptic estimates for varying widths and complex structures. There is a further logical dependence between [BW] and the current paper: In Lemma 3.8 we substantially strengthen the strip-shrinking estimates in [WW1] — in particular, from embedded to immersed geometric composition. These strengthened estimates form the analytic core of Theorem 3.1, which is the analytic core of the Gromov Compactness Theorem in [BW]. One of the ingredients in Lemma 3.8 is a special connection that allows us to obtain estimates without boundary terms for quilted Cauchy–Riemann operators, with uniform constants for all small widths of a strip. This allows us to strengthen the previously-established uniform $H^2 \cap W^{1,4}$ estimates to $H^3$ and thus $C^1$, which is e.g. needed to deduce nontriviality of bubbles with generalized boundary condition in $L_{01} \circ L_{12}$.

Our estimates allow for nonstandard complex structures on the shrinking strip. This is necessitated by the following analytic formulation for the figure eight singularity: In cylindrical coordinates for a neighborhood of infinity in (2), the two seams become two pairs of curves approaching each other asymptotically (see the right figure in Figure 1). On finite cylinders, the standard complex structure on this quilted surface can be pulled back to a quilted surface in which the width of the strips is constant and the complex structures are nonstandard, but converge in $C^0$ and stay within a controlled $C^k$-distance from the standard structure for any $k \geq 1$. 

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The hypothesis that \( M_0, M_1, M_2 \) are closed is not essential: As explained in [BW], it is enough for the symplectic manifolds to be geometrically bounded and to have a priori \( C^0 \)-bounds on the various pseudoholomorphic curves. In a future paper we will treat the noncompact setting in a more systematic way.

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2. **Removal of Singularity for the Figure Eight Bubble**

In this section and the next we will be working with symplectic manifolds \( M_0, M_1, M_2 \), almost complex structures \( J_0, J_1, J_2 \), and pseudoholomorphic curves with seam conditions defined by compact Lagrangian correspondences

\[
L_{01} \subset M_0^+ \times M_1, \quad L_{12} \subset M_1^- \times M_2,
\]

with \( L_{01} \circ L_{12} \) either immersed or cleanly immersed:

- \( L_{01} \) and \( L_{12} \) have **immersed composition** if the intersection
  \[
  L_{01} \times_{M_1} L_{12} = (L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)
  \]
  is transverse. This implies that \( \pi_{02} : L_{01} \times_{M_1} L_{12} \to M_0^- \times M_2 \) is a Lagrangian immersion, e.g. by [WW2] Lemma 2.0.5, and in this situation we will denote the image by \( L_{01} \circ L_{12} := \pi_{02}(L_{01} \times_{M_1} L_{12}) \).
- If \( L_{01}, L_{12} \) have immersed composition and furthermore any two local branches of \( L_{01} \circ L_{12} \) intersect cleanly — i.e. at any intersection of two local branches there is a chart for \( M_0^- \times M_2 \) (as a smooth manifold) in which each of those two branches is identified with an open subset of a vector subspace of \( \mathbb{R}^n \) — then the composition \( L_{01} \circ L_{12} \) is **cleanly immersed**.

The purpose of [2] is to prove a removal of singularity theorem for inverted figure eight bubbles.

**Definition 2.1.** An inverted figure eight bubble between \( L_{01} \) and \( L_{12} \) is a triple of smooth maps

\[
\begin{align*}
  w_0 : B_1(-i) \setminus \{0\} &\to M_0 \\
  w_1 : \mathbb{C}^+ \setminus (B_1(i) \cup B_1(-i)) &\to M_1 \\
  w_2 : \overline{B_1}(i) \setminus \{0\} &\to M_2
\end{align*}
\]

satisfying the Cauchy–Riemann equations \( \partial_s w_\ell + J_\ell(w_\ell) \partial_t w_\ell = 0 \) for \( \ell \in \{0, 1, 2\} \) and the seam conditions

\[
(w_0(-i + e^{i\theta}), w_1(-i + e^{i\theta})) \in L_{01} \quad \forall \, \theta \neq \frac{\pi}{2}, \quad (w_1(i + e^{i\theta}), w_2(i + e^{i\theta})) \in L_{12} \quad \forall \, \theta \neq \frac{3\pi}{2},
\]

and which have finite energy

\[
\int w_0^*\omega_0 + \int w_1^*\omega_1 + \int w_2^*\omega_2 = \frac{1}{2}(\int |dw_0|^2 + \int |dw_1|^2 + \int |dw_2|^2) < \infty,
\]

where we have endowed \( M_\ell \) with the metric

\[
ge_\ell := \omega_\ell(-, J_\ell-).
\]

Throughout [2] the norm of a tangent vector on \( M_\ell \) will always be defined using \( g_\ell \).
Fix for \( \mathbb{R}^2 \) closed symplectic manifolds \( M_0, M_1, M_2 \), compatible almost complex structures \( J_\ell \in \mathcal{J}(M_\ell, \omega_\ell), \ \ell \in \{0, 1, 2\} \), compact Lagrangians \( L_{01}, L_{12} \) as in \( \mathbb{3} \) with cleanly-immersed composition, and an inverted figure eight bubble \( w \) between \( L_{01} \) and \( L_{12} \).

In fact, only the arguments in \( \mathbb{2} \) require the composition \( L_{01} \circ L_{12} \) to be clean immersed, rather than just immersed, but we assume the stronger hypothesis throughout \( \mathbb{2} \) for cohesiveness.

The following theorem says that the singularity at 0 of a figure eight bubble can be continuously removed, under the hypothesis of cleanly-immersed composition.

**Theorem 2.2.** The maps \( w_0, w_2 \) continuously extend to 0, and the limits \( \lim_{z \to 0, \Re(z) > 0} w_1(z) \) and \( \lim_{z \to 0, \Re(z) < 0} w_1(z) \) both exist. If moreover the immersion \( \pi_{02} : L_{01} \times M_1 L_{12} \to M_0^1 \times M_2^1 \) is an embedding, then the latter limits are equal so that \( w_1 \) also extends continuously to 0.

The proof of this theorem draws on the removal of singularity strategies in \([\mathbb{AH}], \mathbb{7}, \mathbb{3}\] and in \([\mathbb{MS}], \mathbb{4}, \mathbb{5}\]. First, we follow \([\mathbb{AH}]\) and establish a uniform gradient bound in cylindrical coordinates near the puncture (Lemma 2.4), which we use to show that the lengths of the paths \( \theta \mapsto w_\ell(\epsilon e^{i\theta}) \) converge to zero as \( \epsilon \to 0 \) (Lemma 2.3). The substantial modification to the argument of \([\mathbb{AH}]\) is that we must use the Gromov Compactness Theorem \([\mathbb{BW}]\) in order to prove uniform gradient bounds in Lemma 2.4. Once we have proven that lengths go to zero, we follow \([\mathbb{MS}]\) and prove an isoperimetric inequality for the energy (Lemma 2.9), which we use to show that the energy on disks around the puncture decays exponentially with respect to the logarithm of the radius. Here the nontrivial modification is in the quilted nature of our isoperimetric inequality. Finally, an argument from \([\mathbb{AH}]\) allows us to conclude that \( w_0 \) and \( w_2 \) extend continuously to the puncture. The continuous extension of \( w_1 \) follows from the gradient bound in cylindrical coordinates and the immersed composition of \( L_{01} \) and \( L_{12} \). The formal proof of Theorem 2.2 is given in \( \mathbb{2} \).

2.1. **Lengths tend to zero.** The first step toward the Removal of Singularity Theorem 2.2 is to show that the lengths of the paths \( \theta \mapsto w_\ell(\epsilon e^{i\theta}) \) converge to zero as \( \epsilon \to 0 \). This is nontrivial since the conformal structure of the quilted surface near the singularity does not allow us to apply mean value inequalities effectively, as in previous removal of singularity results for pseudoholomorphic curves. Hence the finiteness of energy only provides a sequence \( \epsilon^\nu \to 0 \) along which the lengths tend to zero. This allowed Bottman–Wehrheim to deduce a weak removal of singularity in \([\mathbb{BW}]\), but the stronger Theorem 2.2 will require the full strength of the generalized strip-shrinking analysis developed in \([\mathbb{3}]\) and the resulting Gromov Compactness Theorem in \([\mathbb{BW}]\). We record a consequence of the latter as Corollary 2.7 below.

In this subsection we will work in cylindrical coordinates centered at the singularity, hence we define the reparametrized maps

\[
\begin{align*}
v_\ell(s, t) := w_\ell(e^{2\pi(s + it)}) \quad &\text{for} \quad \ell \in \{0, 1, 2\},
\end{align*}
\]

whose domains \( V_0, V_1, V_2 \subset (-\infty, 0] \times \mathbb{R}/\mathbb{Z} \) are given by

\[
V_0 := \{(s, t) \mid s \leq 0, |t - \frac{3}{4}| \leq \frac{1}{4} - \theta(s)\}, \quad V_2 := \{(s, t) \mid s \leq 0, |t - \frac{1}{4}| \leq \frac{1}{4} - \theta(s)\},
\]

\[
V_1 := \{(s, t) \mid s \leq 0, |t - \frac{1}{2}| \leq \theta(s) \lor |t - 1| \leq \theta(s)\},
\]

with

\[
\theta(s) := \frac{1}{2\pi} \arcsin(\frac{1}{2} e^{2\pi s}).
\]

Now the paths \( w_\ell(\epsilon e^{i\theta}) \) for fixed \( \epsilon \in (0, 1) \) correspond to the following paths for fixed \( s = \frac{\log \epsilon}{2\pi} \leq 0: \)

\[
\begin{align*}
\gamma_s^0 := v_0(s, -) : [\frac{1}{2} + \theta(s), 1 - \theta(s)] &\to M_0, \quad \gamma_s^2 := v_2(s, -) : [\theta(s), \frac{1}{2} - \theta(s)] \to M_2, \\
\gamma_s^1 := v_1(s, -) : [\frac{1}{2} - \theta(s), \frac{1}{2} + \theta(s)] \cup [1 - \theta(s), 1 + \theta(s)] &\to M_1.
\end{align*}
\]
The length of $\gamma_s^\ell$ is given by the integral $\ell(\gamma_s^\ell) := \int |\frac{d}{dt} \gamma_s^\ell| dt$ over the respective domain, and will be followed by the main result of this subsection.

**Lemma 2.3.** The $L^2$-lengths of the paths $\gamma_s^0, \gamma_s^1, \gamma_s^2$ defined in (7) converge to zero as $s \to -\infty$:

$$\int_{1/2+\theta(s)}^{1-\theta(s)} |\frac{d}{dt} \gamma_s^0|^2 dt + \left( \int_{1/2-\theta(s)}^{1+\theta(s)} + \int_{1-\theta(s)}^{1+\theta(s)} \right) |\frac{d}{dt} \gamma_s^1|^2 dt + \int_{\theta(s)}^{1/2-\theta(s)} |\frac{d}{dt} \gamma_s^2|^2 dt \to 0 \quad \text{as} \quad s \to -\infty.$$  

In particular, the length $\ell(\gamma_s) := \ell(\gamma_s^0) + \ell(\gamma_s^1) + \ell(\gamma_s^2)$ tends to zero as $s \to -\infty$.

The proof of Lemma 2.3 will use ideas from [AH]. The novel difficulty — due to the conformal structure — is to establish the following uniform gradient bound on $|dv_\ell|$, the upper semicontinuous function defined by

$$|dv_\ell| : (-\infty, 0] \times \mathbb{R}/\mathbb{Z} \to [0, \infty), \quad |dv_\ell(s, t)|^2 := |dv_0(s, t)|^2 + |dv_1(s, t)|^2 + |dv_2(s, t)|^2,$$

where the functions $|dv_\ell(s, t)|$ are set to zero where they are not defined.

**Lemma 2.4.** The gradient $|dv_\ell|$ defined in (8) is uniformly bounded.

We prove Lemma 2.4 by contradiction: if $|dv_\ell|$ is not bounded for some $\ell$, then there is a sequence of points $(s^\nu, t^\nu)$ (necessarily with $s^\nu \to -\infty$) at which $|dv_\ell|$ diverges. Rescaling at these points produces a nonconstant quilted map, as illustrated in Figure 2, but this contradicts the finite-energy hypothesis on $\nu$. The technical input is the Gromov Compactness Theorem in [BW], a consequence of which we record as Theorem 2.7. This theorem is needed to deduce that the rescaled maps actually converge. In order to state it, we need to define the domains of the maps and a controlled fashion in which the strip-width can tend to zero.

The following definition is the only instance in [2] where we allow the almost complex structures to be domain-dependent, so that the notion of a squiggly strip quilt is flexible enough to be used in [3].

**Definition 2.5.** Fix $\rho > 0$, a real-analytic function $f : [-1, 1] \to (0, \frac{1}{2}]$, domain-dependent compatible almost complex structures $J_\ell : [-\rho, \rho]^2 \to \mathcal{J}(M_\ell, \omega_\ell), \ell \in \{0, 1, 2\}$, and a complex structure $j$ on $[-\rho, \rho]^2$.

• A $(J_0, J_1, J_2, j)$-holomorphic size-$(f, \rho)$ squiggly strip quilt for $(L_{01}, L_{12})$ is a triple of smooth maps

$$v = \left( \begin{array}{c} v_0 : \{(s, t) \in (-\rho, \rho)^2 \mid t \leq -f(s)\} \to M_0 \\ v_1 : \{(s, t) \in (-\rho, \rho)^2 \mid |t| \leq f(s)\} \to M_1 \\ v_2 : \{(s, t) \in (-\rho, \rho)^2 \mid t \geq f(s)\} \to M_2 \end{array} \right)$$  

Figure 2. To prove Lemma 2.4 we assume that the cylindrical reparametrizations $\nu_\ell$ do not have uniformly bounded gradient, then bubble off a nonconstant quilted map. In this illustration, the bubbled-off map is a figure eight bubble.
that fulfill the seam conditions

\[(10) \quad (v_0(s, -f(s)), v_1(s, -f(s))) \in L_{01}, \quad (v_1(s, f(s)), v_2(s, f(s))) \in L_{12} \quad \forall s \in (-\rho, \rho),\]
satisfy the Cauchy–Riemann equations

\[(11) \quad dv_\ell(s, t) \circ j(s, t) - J_\ell(s, t, v_\ell(s, t)) \circ dv_\ell(s, t) = 0 \quad \forall \ell \in \{0, 1, 2\}\]
for \((s, t)\) in the relevant domains, and have finite energy

\[E(\nu) := \int v_0^* \omega_0 + \int v_1^* \omega_1 + \int v_2^* \omega_2 < \infty.\]

\[\bullet A (J_0, J_2, j)\)-holomorphic size-\(\rho\) degenerate strip quilt for \(L_{01} \times M_1 L_{12}\) with singularities is a triple of smooth maps

\[(12) \quad \nu = \begin{pmatrix} v_0: (-\rho, \rho) \times (-\rho, 0] \setminus S \times \{0\} \to M_0 \\ v_1: (-\rho, \rho) \setminus S \to M_1 \\ v_2: (-\rho, \rho) \times [0, \rho) \setminus S \times \{0\} \to M_2 \end{pmatrix}\]
defined on the complement of a finite set \(S \subset \mathbb{R}\) that fulfill the lifted seam condition

\[(13) \quad (v_0(s, 0), v_1(s), v_2(s, 0)) \in L_{01} \times M_1 L_{12} \quad \forall s \in (-\rho, \rho) \setminus S,\]
satisfy the Cauchy–Riemann equation \([11]\) for \(\ell \in \{0, 2\}\) and \((s, t)\) in the relevant domains, and have finite energy

\[E(\nu) := \int v_0^* \omega_0 + \int v_2^* \omega_2 < \infty.\]
When \(j\) is the standard complex structure \(i: \partial_s \mapsto \partial_t, \partial_t \mapsto -\partial_s\), the Cauchy–Riemann equation \([11]\) can be expressed in coordinates as:

\[\partial_t v_\ell(s, t) - J_\ell(s, t, v_\ell(s, t)) \partial_s v_\ell(s, t) = 0.\]

The novel hypothesis necessary for a sequence of squiggly strip quilts of widths \((f^\nu)_{\nu \in \mathbb{N}}\) to converge \(C^\infty_{\text{loc}}\) away from the gradient blow-up points is that the widths “obediently shrink to zero”:

**Definition 2.6.** Fix \(\rho > 0\). A sequence \((f^\nu)_{\nu \in \mathbb{N}}\) of real-analytic functions \(f^\nu: [-\rho, \rho] \to (0, \frac{\rho}{2}]\) obediently shrinks to zero, \(f^\nu \Rightarrow 0\), if \(\max_{s \in [-\rho, \rho]} f^\nu(s) \xrightarrow[\nu \to \infty]{} 0\) and and in addition there are holomorphic extensions \(F^\nu: [-\rho, \rho]^2 \to \mathbb{C}\) of \(f^\nu(s) = F^\nu(s, 0)\) such that \((F^\nu)\) converges \(C^\infty\) to zero.

The key to the following special case of the Gromov Compactness Theorem from [BW] is a collection of width-independent elliptic estimates proven in [3] for the linearized Cauchy–Riemann operator. Those elliptic estimates allow for a nonstandard domain complex structure, which is necessary in order to allow widths \(f^\nu\) that are not constant in \(s\).

**Corollary 2.7** (consequence of Gromov Compactness Theorem, [BW]). Fix \(\rho > 0\), a sequence \((f^\nu: [-\rho, \rho] \to (0, \frac{\rho}{2}]\) of real-analytic functions shrinking obediently to zero, and a sequence \((\nu^\nu)_{\nu \in \mathbb{N}}\) of \((J_0, J_1, J_2, i)\)-holomorphic size-\(f^\nu, \rho\) squiggly strip quilts for \((L_{01}, L_{12})\) of bounded energy \(E := \sup_{\nu \in \mathbb{N}} E(\nu^\nu) < \infty\).

If \((s^\nu, t^\nu) \to (s^\infty, t^\infty) \in (-\rho, \rho)^2\) is a sequence of points where the gradient blows up, i.e.

\[\limsup_{\nu \to \infty} |f^\nu|'(s^\nu, t^\nu) = \infty,
\]
then there must be a concentration of energy \( h > 0 \) at \((s^\infty, t^\infty)\), in the sense that there is a sequence of radii \( r^\nu \to 0 \) such that:

\[
\liminf_{\nu \to \infty} \int_{B_{r^\nu}(s^\infty, t^\infty)} \frac{1}{2}|d\nu^\nu|^2 > 0.
\]

We are finally in the position to bound the gradients of the reparametrized maps \( v_\ell \) from [5].

**Proof of Lemma** [2.4] We will prove the equivalent statement that the “folded maps”

\[
u_\ell: U_\ell \to M_\ell \times M_\ell^-,
\quad \nu_\ell(s, t) := (v_\ell(s, t), v_\ell(s, 1/2 - t)) \quad \text{for} \quad \ell = 0, 1, 2
\]

have uniformly-bounded gradients, where the domains \( U_\ell \) are given by

\[
U_0 := \{(s, t) \mid s \leq 0, -\frac{1}{4} \leq t \leq -\theta(s)\},
U_2 := \{(s, t) \mid s \leq 0, \theta(s) \leq t \leq \frac{1}{4}\},
U_1 := \{(s, t) \mid s \leq 0, -\theta(s) \leq t \leq \theta(s)\}.
\]

These maps are pseudoholomorphic with respect to the almost complex structures \( \hat{J}_\ell := J_\ell \oplus (-J_\ell) \) and satisfy the following boundary and seam conditions for \( s \leq 0 \):

\[
u_0(s, -\frac{1}{4}) \in \Delta_{M_0},
\quad (u_0(s, -\theta(s)), u_1(s, -\theta(s))) \in (L_{01} \times L_{01})^T,
\quad u_2(s, \frac{1}{4}) \in \Delta_{M_2},
\quad (u_1(s, \theta(s)), u_2(s, \theta(s))) \in (L_{12} \times L_{12})^T.
\]

(Here \( \theta(s) = \frac{1}{2\pi} \arcsin(\frac{1}{2} e^{2s}) \) as in [6], and \((L_{ij} \times L_{ij})^T\) is the image of \( L_{ij} \times L_{ij} \) under the permutation \((x_i, x_j, y_i, y_j) \mapsto (x_i, y_i, x_j, y_j)\).) Finiteness of the energy of the inverted figure eight \( \nu_\ell \) translates into convergence of the integral \( \lim_{s \to -\infty} \int_{(s,0) \times [-1/4,1/4]} \frac{1}{2}|du|^2 < \infty \) of the energy density

\[
|du|: (-\infty, 0) \times [-\frac{1}{4}, \frac{1}{4}] \to [0, \infty),
|du(s, t)|^2 := |du_0(s, t)|^2 + |du_1(s, t)|^2 + |du_2(s, t)|^2,
\]

where the functions \( |du_\ell(s, t)| \) are set to zero where they are not already defined (so \( |du| \) is upper semi-continuous). This convergence in particular implies

\[
\int_{(\infty, s) \times [-1/4,1/4]} \frac{1}{2}|d\nu|^2 \to 0\quad \text{as} \quad s \to -\infty.
\]

Now assume for a contradiction that there exists a sequence \((s^\nu, t^\nu) \in (-\infty, 0] \times [-1/4, 1/4]\) such that \(|du^\nu(s^\nu, t^\nu)| \to \infty \). Since the \( \nu_\ell \) are smooth, this is possible only for \( s^\nu \rightarrow -\infty \); passing to a further subsequence, we may in fact assume \( s^{\nu+1} \leq s^\nu - 1 \) and \( s^1 \leq 1/4 \). Depending on whether \( t^\infty \) is \( \pm 1/4 \) or is contained in \((-1/4, 1/4)\), we derive a contradiction to (14):

\( t^\infty = \pm 1/4 \). Assume \( t^\infty = -1/4 \); the \( t^\infty = 1/4 \) case can be treated in analogous fashion. Define a sequence \((u^\nu_0)\) by:

\[
u_0^\nu: B_{1/8}(0) \cap \mathbb{H} \to M_0 \times M_0^-,
\quad \nu_0^\nu(s, t) := u_0(s + s^\nu, t - 1/4).
\]

The map \( \nu_0^\nu \) is \( \hat{J}_0\)-holomorphic and satisfies the Lagrangian boundary condition \( u_0(s, 0) \in \Delta_{M_0} \) for \( s \in (-1/8, 1/8) \). Furthermore, \(|du_0^\nu(0, t^\nu + 1/4)| \to \infty \), \( t^\nu + 1/4 \to 0 \) by assumption, and the energy of \( u_0^\nu \) is bounded by the energy of \( \nu_\ell \), so [MS, Lemma 4.6.5] implies the inequality

\[
\liminf_{\nu \to \infty} \int_{B_{1/8}(0)} \frac{1}{2}|du_0^\nu|^2 > 0,
\]

which contradicts (14).

\( t^\infty \in (-1/4, 1/4) \). Define a sequence \((u^\nu_0, u^\nu_1, u^\nu_2)\) of \((\hat{J}_0, \hat{J}_1, \hat{J}_2, i)\)-holomorphic size-\((1/4, \theta^\nu)\) squiggly strip quilts, with

\[
\theta^\nu: [-\frac{1}{4}, \frac{1}{4}] \to (0, \frac{1}{2}],
\quad \theta^\nu(s) := \frac{1}{2\pi} \arcsin(\frac{1}{2} e^{2\pi(s+s^\nu)}),
\]

by:

\[
u^\nu_\ell(s, t) := \nu_\ell(s + s^\nu, t).
\]
The energy \( \int_{B_{1/s}(0)} \frac{1}{2} |du^{\nu}|^2 \) is bounded by the energy of \( u \), and by assumption, the gradient |\( du^{\nu}(0, t^{\nu}) \)| tends to \( \infty \). In the following sublemma we establish the last hypothesis needed to apply Corollary 2.7.

**Sublemma 2.8.** The functions \( \theta^{\nu}(s) = \frac{1}{2\pi} \arcsin(\frac{1}{2} e^{2\pi(s+s^{\nu})}) \) obediently shrink to zero as \( \nu \to \infty \).

**Proof of Sublemma 2.8.** The convergence \( s^{\nu} \to -\infty \) implies \( \frac{1}{2} e^{2\pi(s+s^{\nu})} \to 0 \) in \( C^0 \), so the equality \( \arcsin(0) = 0 \) implies the \( \kappa \) for any \( m \gamma L \)

\[ \begin{aligned} &\text{To check the second condition for obedient shrinking, fix } k \geq 1 \text{ and note that } \frac{\partial \theta^{\nu}}{\partial s}(s) = \frac{d^2 \theta}{d\gamma}(s + s^{\nu}), \text{ with } \theta(s) = \frac{1}{2\pi} \arcsin(\frac{1}{2} e^{2\pi s}) \text{ as above. The derivative } \frac{d^2 \theta}{d\gamma}(s) \text{ is a linear combination of the functions } f_\ell(s) := (4 - e^{4\pi s})^{-\ell(0)/2 \ell e^{4\pi(\ell - 1/2)s}} \text{ for } \ell = 1, \ldots, m. \text{ This can be seen by induction starting from } \frac{d \theta(s)}{ds} = (4 - e^{4\pi s})^{-1/2 e^{2\pi s}}. \text{ This decomposition, the inequality } \arcsin(z) \leq \max_{s \in [-1/4, 1/4]} |f_\ell(s)| \leq \sup_{\nu \in \mathbb{N}} \exp(4\pi(\ell - 1/2)(s^{\nu} + 1/4)) \leq 4\pi \exp(\pi). \end{aligned} \]

The arcsine function extends to a holomorphic function \( \arcsin : B_1(0) \to \mathbb{C} \) by the power series

\[ \arcsin(z) := \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (2k+1)!} z^{2k+1}, \text{ so } f^{\nu} \text{ extends to a holomorphic function } F^{\nu} \text{ from } [-1/4, 1/4]^2 \to \mathbb{C}. \]

Since the functions \( \frac{1}{2} e^{2\pi(z^{\nu} + s^{\nu})} \) tend \( C^\infty \) to zero and since \( \arcsin(0) = 0 \), the extensions \( F^{\nu} \) also tend \( C^\infty \) to zero. \( \square \)

Part (2) of Corollary 2.7 now implies the inequality \( \liminf_{\nu \to \infty} \int_{B_{1/s}(0)} \frac{1}{2} |du^{\nu}|^2 > 0 \), which contradicts [14]. \( \square \)

**Proof of Lemma 2.3.** First, note that the domain \([1/2 - \theta(s), 1/2 + \theta(s)] \cup [1 - \theta(s), 1 + \theta(s)] \) of \( \gamma^1_s \) has total length \( \theta^1(s) = \frac{2}{\pi} \arcsin(\frac{1}{2} e^{2\pi s}) \), which converges to 0 as \( s \to -\infty \). Hence the gradient bounds of Lemma 2.4 immediately imply that the \( L^2 \)-length of \( \gamma^1_s \) converges to zero as \( s \to -\infty \). Moreover, these gradient bounds imply that to show the \( L^2 \)-lengths of \( \gamma^0_s, \gamma^2_s \) converge to zero, it suffices to fix an arbitrary \( \epsilon > 0 \) and show that the \( L^2 \)-lengths of \( \gamma^0_s[1/2 + \epsilon, 1 - \epsilon], \gamma^2_s[1/2 + \epsilon, 1 - \epsilon] \) converge to zero as \( s \to -\infty \).

Fix \( \epsilon > 0 \). We will show that the \( L^2 \)-length of \( \gamma^0_s[1/2 + \epsilon, 1 - \epsilon] \) converges to zero as \( s \to -\infty \); the proof for \( \gamma^2 \) is similar. Choose \( s_0 \) so that the domain of \( \gamma^0_s \) contains \([1/2 + \epsilon/2, 1 - \epsilon/2]\) for all \( s \leq s_0 \). Now the \( C^0 \)-bound on \( |dv_0| \) from Lemma 2.4 induces a \( C^m \)-bound on \( v_0[0, -\infty, s_0 - 1][1/2 + \epsilon, \epsilon - 1] \) for any \( m \geq 0 \). Indeed, we can apply the interior elliptic estimates (e.g. [AHL] [6.3]) on each of the precompactly-nested domains

\[ [s_0 - k - 1, s_0 - k] \times [1/2 + \epsilon, 1 - \epsilon] \subset [s_0 - k - 2, s_0 - k + 1] \times [1/2 + \epsilon/2, 1 - \epsilon/2] \]

for \( k \in \mathbb{N} \). Since for different \( k \) these domains are translations of one other, the constants in the elliptic estimates are independent of \( k \), and thus yield the desired \( C^m \)-bounds.

For \( s \leq s_0 \), define

\[ \Phi(s) := \frac{1}{2} \int_{-\infty}^{s} \int_{1/2 + \epsilon}^{1-\epsilon} (\partial_s v_0)^2 + (\partial_t v_0)^2. \]
Then $\Phi: (-\infty, s_0] \to [0, \infty)$ is nondecreasing with $\lim_{s \to -\infty} \Phi(s) = 0$ and

$$
\Phi'(s) = \frac{1}{2} \int_{1/2+\varepsilon}^{1-\varepsilon} (|\partial_s v_0(s, \tau)|^2 + |\partial_v v_0(s, \tau)|^2) \, d\tau = \int_{1/2+\varepsilon}^{1-\varepsilon} |\partial_v v_0(s, \tau)|^2 \, d\tau,
$$

$$
\Phi''(s) = 2 \int_{1/2+\varepsilon}^{1-\varepsilon} (\partial_s v_0(s, \tau), \nabla^2_{LC} v_0(s, \tau)) \, d\tau,
$$

where in the last quantity we are using the Levi-Civita connection with respect to the metric $g_0$ defined in [4]. By the previous paragraph, there exists a constant $c > 0$ so that $\Phi''(s) \leq c$ for all $s \leq s_0 - 1$. Now for any fixed $\delta > 0$ we can choose $s_1 \leq s_0 - 1$ such that $\Phi(s_1) \leq \delta^2/4c$. For $s \leq s_1$, we obtain:

$$
\frac{\delta^2}{4c} \geq \Phi(s_1) \geq \Phi(s) - \Phi(s - \frac{\delta}{2c}) = \int_{s - \delta/2c}^s \Phi'(\sigma) \, d\sigma \geq \frac{\delta}{2c} (\Phi'(s) - \frac{\delta}{2c}),
$$

where the last step uses the bound on $\Phi''$ to deduce $\Phi'(\sigma) \geq \Phi'(s) - c|s - \sigma|$. This inequality can be rearranged to yield $\Phi'(s) \leq \delta$ for all $s \leq s_1$, and thus proves $\lim_{s \to -\infty} \Phi'(s) = 0$. Since $\Phi'(s)$ is equal to $\|\frac{d}{ds} \gamma_s^0\|_{L^2([1/2+c-1, 1])}$ and since $|\frac{d}{ds} \gamma_s^0|$ is uniformly bounded, we have now shown that the $L^2$-norm of $\gamma_s^0$ converges to zero as $s \to -\infty$.

The Cauchy–Schwarz inequality implies that the $L^1$-norm of $\frac{d}{ds} \gamma_s^0$ — i.e. the length $\ell(\gamma_s^0)$ — also tends to zero as $s \to -\infty$.

2.2. An isoperimetric inequality and the proof of removal of singularity. In this subsection, we prove Theorem 2.2. The crucial inputs will be Lemma 2.3 from [2.1] together with the following isoperimetric inequality for the energy on $(-\infty, s_0] \times \mathbb{R}/\mathbb{Z}$,

$$
E(v; s_0) := \int_{(-\infty, s_0] \times \mathbb{R}/\mathbb{Z}} \frac{1}{2} |dv|^2 \, dsdt.
$$

**Lemma 2.9.** There exists $C > 0$ such that the following inequality holds for all $s \leq 0$:

$$
E(v; s) \leq C \sum_{i \in \{0, 1, 2\}} \ell(\gamma_i^s)^2.
$$

We defer the proof to later in [2.2]; now, we turn to the proof of removal of singularity. Throughout this subsection we denote

$$
M_{0112} := M_0^- \times M_1 \times M_2 \times M_2, \quad M_{02} := M_0^- \times M_2.
$$

**Proof of Theorem 2.2**

**Step 1.** There exist $C_1, C_2 > 0$ such that the inequality $E(v; s) \leq C_1 \exp(C_2 s)$ holds for all $s \leq 0$.

Fix $s \leq 0$. The following inequality follows from Lemma 2.9

$$
E(v; s) \leq C \sum_{i \in \{0, 1, 2\}} \ell(v_i(s, -))^2 \leq C \left( \int_0^1 |dv(s, t)| \, dt \right)^2 \leq C \int_0^1 |dv(s, t)|^2 \, dt = C \frac{d}{ds} (E(v; s)).
$$

Manipulating this inequality and integrating from $s$ to 0, we obtain $E(v; s) \leq E(v; 0) \exp(s/C)$.

**Step 2.** The limit $\lim_{s \to -\infty} v_0(s, -)$ exists in $C^0([5/8, 7/8], M_0)$.

Fix a $C^1$ embedding $i: M_0 \to \mathbb{R}^N$; we will show that $\Lambda := \lim_{s \to -\infty} (i \circ v_0|_{[5/8, 7/8]})$ exists in $C^0$. 

We begin by showing that $\Lambda$ exists in $L^2$, where $L^2([5/8, 7/8], \mathbb{R}^N)$ is defined using the Euclidean metric on $\mathbb{R}^N$. Fix $s_2 \leq s_1 \leq 0$. Cauchy–Schwarz implies the following inequality:

(15)

$$
\| (i \circ v_0)(s_1, -) - (i \circ v_0)(s_2, -) \|^2_{L^2([5/8, 7/8])} = \left( \int_{s_2}^{s_1} \left( \int_{s_2}^{s_1} \partial_s (i \circ v_0) ds \right)^2 dt \right)^{1/2} \leq (s_1 - s_2)^{1/2} \left( \int_{s_2}^{s_1} \| \partial_s (i \circ v_0) \|^2_{g_{M_0}} ds dt \right)^{1/2}.
$$

Since $M_0$ is compact, there exists a constant of equivalence $\mu > 0$ for the norms induced by $g_{M_0}$ and $\overline{v}^*g_{\text{euc}}$, so (15) yields the following:

| (i \circ v_0)(s_1, -) - (i \circ v_0)(s_2, -) \|^2_{L^2([5/8, 7/8])} \leq \mu (s_1 - s_2)^{1/2} \left( \int_{s_2}^{s_1} \| \partial_s v_0 \|^2_{g_{M_0}} ds dt \right)^{1/2}.

(16)

Define $f(s) := \left| \frac{d}{dt} (i \circ v_0)(s, -) \right|_{L^2([5/8, 7/8])}$. This quantity tends to zero as $s \to -\infty$:

$$
\limsup_{s \to -\infty} f(s) \leq \limsup_{s \to -\infty} \mu | \frac{d}{dt} v_0(s, -) |_{L^2([5/8, 7/8])} = 0.
$$

We can now show that $\Lambda$ exists in $W^{1,2}$: We have

$$
| (i \circ v_0)(s_1, -) - (i \circ v_0)(s_2, -) |_{W^{1,2}([5/8, 7/8])} \leq | (i \circ v_0)(s_1, -) - (i \circ v_0)(s_2, -) |_{L^2([5/8, 7/8])} + f(s_1) + f(s_2)
$$

$$
\leq \frac{C_3 | s_1 |^{1/2} \exp(C_2 s_1 / 2)}{1 - \exp(C_2 s_1 / 2)} + f(s_1) + f(s_2),
$$

which implies the equality

$$
\limsup_{s_1 \to -\infty} \sup_{s_2 \in (-\infty, s_1]} | (i \circ v_0)(s_1, -) - (i \circ v_0)(s_2, -) |_{W^{1,2}([5/8, 7/8])} = 0.
$$

Since $W^{1,2}([5/8, 7/8], \mathbb{R}^N)$ is complete, $\Lambda$ exists in $W^{1,2}$. The Sobolev embedding $W^{1,2} \hookrightarrow C^0$ for one-dimensional domains now implies that $\Lambda$ exists in $C^0$.

**Step 3. We prove Theorem 2.2**
By Lemma 2.3 the first claim of Theorem 2.2 would follow from the existence of the limits

\[ \Lambda_0 := \lim_{s \to -\infty} v_0(s, \frac{3}{4}), \quad \Lambda_1 := \lim_{s \to -\infty} v_1(s, \frac{1}{2}), \quad \Lambda_1' := \lim_{s \to -\infty} v_1(s, 1), \quad \Lambda_2 := \lim_{s \to -\infty} v_2(s, \frac{1}{4}). \]

It follows from Step 2 that \( \Lambda_0 \) exists, and an analogous argument shows that \( \Lambda_2 \) exists. It remains to show that \( \Lambda_1, \Lambda_1' \) exist.

To show that \( \Lambda_1 \) exists, we will show convergence of the path

\[ \gamma : s \mapsto (v_0(s, \frac{1}{2} + \theta(s)), v_1(s, \frac{1}{2}), v_1(s, \frac{1}{2}), v_2(s, \frac{1}{2} - \theta(s))) \]

as \( s \to -\infty \). This path takes values in \( M_0 \times \Delta M_1 \times M_2 \) and \( \lim_{s \to -\infty} d_{M_{0112}}(\gamma(s), L_{01} \times L_{12}) = 0 \) (by Lemma 2.4), so the distances \( d_{M_{0112}}(\gamma(s), L_{01} \times M_1 \times L_{12}) \) converge to zero. Hence there exists a path \( \beta : (-\infty, 0] \to L_{01} \times M_1 \times L_{12} \) satisfying the equality

\[ \lim_{s \to -\infty} d_{M_{0112}}(\gamma(s), \beta(s)) = 0. \]  

(Indeed, define \( \beta \) by choosing a tubular neighborhood \( U \) of \( L_{01} \times M_1 \times L_{12} \), and compose \( \gamma \) with the projection \( U \to L_{01} \times M_1 \times L_{12} \).) We will show that \( \lim_{s \to -\infty} \gamma(s) \) exists by showing that \( \lim_{s \to -\infty} \beta(s) \) exists.

Lemma 2.3, the existence of \( \Lambda_0 \) and \( \Lambda_2 \), and \( \{18\} \) imply that \( x_{02} := \lim_{s \to -\infty} \pi_{02}(\beta(s)) \) exists. Since \( \pi_{02} \) restricts to an immersion of \( L_{01} \times M_1 \times L_{12} \) into \( M_{02} \), there exist finitely many preimages \( x_{0112}^1, \ldots, x_{0112}^k \) of \( x_{02} \) in \( L_{01} \times M_1 \times M_2 \). Choose \( \epsilon > 0 \) small enough that the preimage of \( B_\epsilon (x_{02}) \) under \( \pi_{02}|_{L_{01} \times M_1 \times L_{12}} \) consists of \( k \) connected components \( U^1, \ldots, U^k \), with \( x_{0112}^j \) contained in \( U^j \). Now choose \( s_0 \in (-\infty, 0] \) such that \( \pi_{02}(\beta((-\infty, s_0])) \) is contained in \( B_\epsilon (x_{02}) \). The image \( \beta((-\infty, s_2]) \) must then be contained in a single \( U_j \). If \((s_\nu), (s'_{\nu'})\) are sequences with limit \(-\infty \) such that \( x_{0112}^{j1} := \lim_{\nu \to -\infty} \beta(s_\nu) \) and \( x_{0112}^{j2} := \lim_{\nu \to -\infty} \beta(s'_{\nu'}) \) exist, then \( j_1 \) and \( j_2 \) must be equal; since \( L_{01} \times M_1 \times L_{12} \) is compact, this is enough to conclude that \( \lim_{s \to -\infty} \beta(s) \) exists. As noted above, this is enough to conclude the first statement of Theorem 2.2.

The points \((\Lambda_0, \Lambda_1, \Lambda_2)\) and \((\Lambda_0, \Lambda_1', \Lambda_2)\) are lifts in \( L_{01} \times M_1 \times L_{12} \) of \((\Lambda_0, \Lambda_2)\), so if the projection from \( L_{01} \times M_1 \times L_{12} \) to \( M_{02} \) is injective, then \( \Lambda_1, \Lambda_1' \) are the same point. \( \square \)

Our proof of Lemma 2.9 is an adaptation to the quilted setting of [MS, Lemma 4.5.1], which is an isoperimetric inequality for the energy near an interior point of a \( J \)-holomorphic curve. Their argument went like this: restricting the map to an annulus, then reparametrizing, yields a map
Proof of Lemma 2.9.

The cleanly-immersed hypothesis implies that any two branches of a
subspace.

We will give only a brief sketch, since a formal proof is no more enlightening. The key is that
Lemma 2.10. There exist \( C > 0, \epsilon > 0 \) such that:

(i) If \( x_{02}, y_{02} \in L_{01} \circ L_{12} \) have lifts

\[
x, x' \in \pi_{02}^{-1}\{x_{02}\} \cap (L_{01} \times M_{1} L_{12}), \quad y, y' \in \pi_{02}^{-1}\{y_{02}\} \cap (L_{01} \times M_{1} L_{12})
\]

with small distances

\[
\max\{d_{M_{0112}}(x, y), d_{M_{0112}}(x', y')\} \leq \epsilon,
\]

then there exists a smooth path \( \gamma_{02} : [0, 1] \to M_{02} \) with image in \( L_{01} \circ L_{12} \) and smooth lifts

\[
\ell(\gamma_{02}) + \ell(\gamma) + \ell(\gamma') \leq C d_{M_{02}}(x_{02}, y_{02})
\]

and satisfy \( \gamma(0) = x, \gamma(1) = y, \gamma'(0) = x', \) and \( \gamma'(1) = y' \).

(ii) For \( x, x' \in L_{01} \times M_{1} L_{12} \) with \( d_{M_{02}}(\pi_{02}(x), \pi_{02}(x')) \leq \epsilon \), there exists a point \( y_{02} \in L_{01} \circ L_{12} \) and preimages \( y, y' \in \pi_{02}^{-1}(y_{02}) \cap L_{01} \times M_{1} L_{12} \) such that the following inequality holds:

\[
d_{M_{02}}(\pi_{02}(x'), y_{02}) + d_{M_{02}}(\pi_{02}(x), y_{02}) + d_{M_{0112}}(x, y) + d_{M_{0112}}(x', y') \leq C d_{M_{02}}(\pi_{02}(x), \pi_{02}(x')).
\]

We will give only a brief sketch, since a formal proof is no more enlightening. The key is that the cleanly-immersed hypothesis implies that any two branches of \( L_{01} \circ L_{12} \) meet like two vector subspaces.

(i) If \( x, x', y, y' \) lie in the same local branch of \( L_{01} \circ L_{12} \), then the conclusion is immediate. Otherwise, \( x \) and \( y \) lie in one branch, and \( x' \) and \( y' \) lie in another. Represent these branches as open subsets of vector subspaces \( V, V' \subset \mathbb{R}^{N} \). Then \( x_{02}, y_{02} \) lie in \( V \cap V' \), and we may define \( \gamma_{02} \) to be a path in \( V \cap V' \) from \( x_{02} \) to \( y_{02} \) and \( \gamma \) (resp. \( \gamma' \)) to be the lift to the portion of \( L_{01} \times M_{1} L_{12} \) corresponding to \( V \) (resp. to \( V' \)).

(ii) If \( x, x' \) lie in the same local branch of \( L_{01} \circ L_{12} \), the conclusion is again immediate. Otherwise, represent the branches containing \( x, x' \) as open subsets of \( V, V' \subset \mathbb{R}^{N} \). Set \( y_{02} \) to be the nearest point in \( V \cap V' \) to \( x \), and let \( y \) (resp. \( y' \)) be the lift to the portion of \( L_{01} \times M_{1} L_{12} \) corresponding to \( V \) (resp. to \( V' \)).

Proof of Lemma 2.9.
Step 1. We prove Lemma 2.9 up to an extension result, which we defer to Steps 2 and 3.

It suffices to prove the lemma for \( s \leq s_0 \leq 0 \), where \( s_0 \) is chosen so that \( \sup_{s \leq s_0} \| \gamma_i^s \|, \ i \in \{0,1,2\} \) is bounded by a constant \( \delta > 0 \) to be determined later. As illustrated in Figure 4, partition the unit circle \( S_1(0) \) into four segments by

\[
A_0 := \{(x,y) \in S_1(0) \mid y \leq x, \ y \leq -x\}, \quad A_1 := \{(x,y) \in S_1(0) \mid x \geq y, \ x \geq -y\}, \quad A_2 := \{(x,y) \in S_1(0) \mid y \geq x, \ y \geq -x\}, \quad A_3 := \{(x,y) \in S_1(0) \mid x \leq y, \ x \leq -y\}
\]

and set \( p_{i(i+1)} := A_i \cap A_{i+1} \) for \( i \in \mathbb{Z}/4\mathbb{Z} \). Given \( s_1, s_2 \) with \( s_2 < s_1 \leq s_0 \), define maps \( \sigma_i : A_i \times [s_2, s_1] \to M_{i} \), \( i \in \{0,1,2,3\} \) (where we set \( M_3 := M_1 \)) like so:

\[
\sigma_0(\exp(2\pi it), s) := v_0(s, \frac{1}{2} + \theta(s) + 4(\frac{1}{2} - 2\theta(s))(t - \frac{\delta}{2})), \quad \sigma_1(\exp(2\pi it), s) := v_1(s, 8\theta(st)), \quad \sigma_2(\exp(2\pi it), s) := v_2(s, \theta(s) + 4(\frac{1}{2} - 2\theta(s))(t - \frac{1}{2})), \quad \sigma_3(\exp(2\pi it), s) := v_3(s, \frac{1}{2} + 8\theta(st) - 2t \frac{1}{2})
\]

where we take \( t \in [-1/8, 7/8] \). These maps satisfy the seam condition

\[
(\sigma_i(p_{i(i+1)}), \sigma_{i+1}(p_{i(i+1)})) \in L_{i(i+1)}, \quad i \in \mathbb{Z}/4\mathbb{Z},
\]

where we set \( L_{23} := L_{12}^T, \ L_{30} := L_{01}^T \).

In order to apply Stokes’ theorem, we will extend the maps \( \sigma_i \) to the following four quadrants of the closed unit disk:

\[
U_0 := \{(x,y) \in \overline{B}(0,1) \mid y \leq x, \ y \leq -x\}, \quad U_1 := \{(x,y) \in \overline{B}(0,1) \mid x \geq y, \ x \geq -y\},
\]

\[
U_2 := \{(x,y) \in \overline{B}(0,1) \mid y \geq x, \ y \geq -x\}, \quad U_3 := \{(x,y) \in \overline{B}(0,1) \mid x \leq y, \ x \leq -y\}.
\]

Choose \( s_2 = t_0 < t_1 < \cdots < t_k = s_1 \) such that for every \( j \), the diameters of the images \( \sigma_i(A_i \times [t_j, t_{j+1}]) \) are bounded by \( \tilde{\delta} \). As long as \( \tilde{\delta} \) is small enough, Steps 2 and 3 below allow us to extend \( \sigma_i \) to a continuous map \( \tilde{\sigma}_i : U_i \times [s_2, s_1] \to M_i \) that is smooth on \( U_i \times [t_j, t_{j+1}] \), such that the extended maps satisfy the Lagrangian seam conditions

\[
(\tilde{\sigma}_i(p, s), \tilde{\sigma}_{i+1}(p, s)) \in L_{i(i+1)} \quad \forall \ p \in U_i \cap U_{i+1}, \ s \in [s_2, s_1]
\]

Indeed, use Step 2 to define the maps \( \tilde{\sigma}_i \) on the slices \( U_i \times \{t_j\} \), then use Step 3 to extend \( \tilde{\sigma}_i \) to all of \( U_i \times [s_2, s_1] \).

Since \( \omega_0, \omega_1, \omega_2 \) are closed, Stokes’ theorem yields the following:

\[
E(\nu; [s_2, s_1] \times \mathbb{R}/\mathbb{Z}) \leq \sum_{i \in \{1,2\}} \sum_{j \in \{0,1,2,3\}} \int_{U_i \times \{s_j\}} \sigma_j^* \omega_j \leq C \sum_{i \in \{1,2\}} \sum_{j \in \{1,2,3\}} \ell(\gamma_i^j)^2;
\]

where in the first inequality we have used the seam conditions (19), and in the second inequality we have used the isoperimetric inequality for the symplectic area [MS, Theorem 4.4.1]. Taking the limit as \( s_2 \) goes to \( -\infty \) and applying Lemma 2.3 yields the conclusion of the lemma.

Throughout the final two steps, the constants \( C_i \) will depend only on the geometry of \( L_{01}, L_{12} \).

Step 2. There exist \( C > 0, \ k_0 > 0 \) so that if \( \sigma_0, \sigma_1, \sigma_2, \sigma_3 \) are smooth maps with

\[
\sigma_i : A_i \to M_i, \quad (\sigma_i(p_{i(i+1)}), \sigma_{i+1}(p_{i(i+1)})) \in L_{i(i+1)}, \quad \kappa := \max_{i \in \{0,1,2,3\}} \text{diam} \sigma_i(A_i) \leq k_0,
\]

then there exist extensions \( \tilde{\sigma}_i : U_i \to M_i \) of \( \sigma_i \) such that:

\[
(\tilde{\sigma}_i(p), \tilde{\sigma}_{i+1}(p)) \in L_{i(i+1)} \quad \forall \ p \in U_i \cap U_{i+1}, \quad \max_{i \in \{0,2,3\}} \ell(\tilde{\sigma}_i|\partial U_i) + \max_{i \in \{0,2,3\}} \text{diam} \tilde{\sigma}_i(U_i) \leq C \kappa.
\]

The points

\[
z := (\sigma_0(p_0), \sigma_1(p_0), \sigma_1(p_1), \sigma_2(p_1), \sigma_2(p_1)), \quad z' := (\sigma_0(p_30), \sigma_3(p_30), \sigma_3(p_23), \sigma_2(p_23))
\]

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lie in $L_{01} \times L_{12}$. Since the intersection $(L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)$ defining $L_{01} \times M_1 L_{12}$ is transverse, there are points $x, x' \in L_{01} \times M_1 L_{12}$ that are close to $z$ resp. $z'$,

\begin{equation}
(20) \
 d_{M_{0112}}(x, z) \leq C_1 \kappa, \quad d_{M_{0112}}(x', z') \leq C_1 \kappa,
\end{equation}

for a uniform constant $C_1 > 0$. The triangle inequality bounds the distance between the projections of $z, z'$:

\[
d_{M_{02}}(\pi_{02}(x), \pi_{02}(x')) \leq d_{M_{02}}(\pi_{02}(x), \pi_{02}(z)) + d_{M_{02}}(\pi_{02}(z), \pi_{02}(z')) + d_{M_{02}}(\pi_{02}(z'), \pi_{02}(x')) \leq 2(C_1 + 1)\kappa.
\]

As long as $\kappa_0$ is chosen to be small enough, it follows from Lemma 2.10(ii) that there exist lifts $y, y' \in L_{01} \times M_1 L_{12}$ of a single point $y_{02} \in L_{01} \circ L_{12}$ with small distances to $z$ resp. $z'$:

\begin{equation}
(21) \
 d_{M_{0112}}(x, y) \leq C_2 \kappa, \quad d_{M_{0112}}(x', y') \leq C_2 \kappa,
\end{equation}

where $C_2 > 0$ is another constant. We can now define the extensions $\tilde{\sigma}_i$ at the origin:

\[
(\tilde{\sigma}_0(0), \tilde{\sigma}_1(0), \tilde{\sigma}_1(0), \tilde{\sigma}_2(0)) := y, \quad (\tilde{\sigma}_0(0), \tilde{\sigma}_3(0), \tilde{\sigma}_3(0), \tilde{\sigma}_2(0)) := y'.
\]

Inequalities (20) and (21) and the triangle inequality yield:

\[
d_{M_{0112}}(y, z) \leq (C_1 + C_2)\kappa, \quad d_{M_{0112}}(y', z') \leq (C_1 + C_2)\kappa.
\]

The local triviality of smooth submanifolds implies that there exists a constant $C_3 > 0$ such that after redefining $\kappa_0$ if necessary, we may extend the maps $\tilde{\sigma}_i$ to the set $\{(a, b) \in B(0, 1) \mid b = \pm a\}$ such that the seam conditions (19) hold and the length of the loop $\tilde{\sigma}_i|\partial U_i$ is bounded by $C_3\kappa$. Once more redefining $\kappa_0$ if necessary, we may extend each map $\tilde{\sigma}_i$ to $U_i$ in such a way that the diameter of $\tilde{\sigma}_i(U_i)$ is bounded by $C_4\kappa$ for $C_4 > 0$ another constant.

**Step 3.** There exists $\lambda > 0$ such that the following holds. Assume that $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are smooth maps and $a < b$ are real numbers with:

\[
\sigma_i: A_i \times [a, b] \cup U_i \times \{a, b\} \to M_i, \quad \max_{i \in \{0, 1, 2, 3\}} \text{diam} \, \sigma_i \leq \lambda,
\]

\[
(\sigma_i(q), \sigma_{i+1}(q)) \in L_{i(i+1)} \quad \forall q \in \left(p_{i(i+1)} \times [a, b]\right) \cup \left((U_i \cap U_{i+1}) \times \{a, b\}\right).
\]

Then each $\sigma_i$ can be extended to a smooth map $\tilde{\sigma}_i: U_i \times [a, b] \to M_i$ such that the following seam conditions hold:

\[
(\tilde{\sigma}_i(q), \tilde{\sigma}_{i+1}(q)) \in L_{i(i+1)} \quad \forall q \in (U_0 \cap U_1) \times [a, b].
\]

Define $x, x', y, y' \in L_{01} \times M_1 L_{12}$ like so:

\[
x := (\sigma_0, \sigma_1, \sigma_1, \sigma_2)(0, a), \quad x' := (\sigma_0, \sigma_3, \sigma_3, \sigma_2)(0, a),
\]

\[
y := (\sigma_0, \sigma_1, \sigma_1, \sigma_2)(0, b), \quad y' := (\sigma_0, \sigma_3, \sigma_3, \sigma_2)(0, b).
\]

Then $\pi_{02}(x) = \pi_{02}(x')$ and $\pi_{02}(y) = \pi_{02}(y')$, and $x$ resp. $x'$ are close to $y$ resp. $y'$:

\[
 d_{M_{0112}}(x, y) \leq 4\lambda, \quad d_{M_{0112}}(x', y') \leq 4\lambda.
\]

It follows from Lemma 2.10(ii) that as long as $\lambda$ is chosen to be small enough, there exists a path $\gamma_{02}: [a, b] \to L_{01} \circ L_{12}$ and lifts $\gamma, \gamma': [a, b] \to L_{01} \times M_1 L_{12}$ from $x$ to $y$ resp. from $x'$ to $y'$ of small lengths:

\[
\ell(\gamma) + \ell(\gamma') \leq C_5 \lambda
\]

for $C_5 > 0$ a constant. Define $\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ on $(0) \times [a, b]$ like so:

\[
(\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_1, \tilde{\sigma}_2)(0, t) := \gamma(t), \quad (\tilde{\sigma}_0, \tilde{\sigma}_3, \tilde{\sigma}_3, \tilde{\sigma}_2)(0, t) := \gamma'(t).
\]
Theorem 3.1, deferring the \(\delta\) rather than embedded. The purpose of this section is to establish convergence mod bubbling in \(L^\infty\) if necessary, we may extend \((\bar{\sigma}_0, \bar{\sigma}_1)\) to a map \((U_0 \cap U_1) \times [a, b] \to M_0^- \times M_1\) with small diameter:

\[
diam((\bar{\sigma}_0, \bar{\sigma}_1)((U_0 \cap U_1) \times [a, b])) \leq C_0 \lambda
\]

for \(C_0 > 0\) a constant. Extend \((\bar{\sigma}_1, \bar{\sigma}_2), (\bar{\sigma}_2, \bar{\sigma}_3), (\bar{\sigma}_3, \bar{\sigma}_0)\) to \((U_1 \cap U_2) \times [a, b], (U_2 \cap U_3) \times [a, b], (U_3 \cap U_0) \times [a, b]\) in the same fashion. Finally, \(\bar{\sigma}_i|_{\partial(U_i \times [a, b])}\) is a map to \(M_i\) from a domain homeomorphic to \(S^2\), and its diameter is small:

\[
diam(\bar{\sigma}_i(\partial(U_i \times [a, b]))) \leq (2C_0 + 1)\lambda.
\]

Redefining \(\lambda\) if necessary, we may extend \(\bar{\sigma}_i\) to all of \(U_i \times [a, b]\).

\[\square\]

3. CONVERGENCE MODULO BUBBLING FOR STRIP-SHRINKING

The proof of the Gromov Compactness Theorem in [BW] relies on \(C^k\)-compactness in the presence of a uniform gradient bound. This result is based on a strengthening of the strip-shrinking analysis of [WW1] from \(H^2 \cap W^{3,4}\)-convergence to \(C^k\)-convergence; we also allow the domain to be equipped with nonstandard complex structures and the geometric composition \(L_{01} \circ L_{12}\) to be immersed, rather than embedded. The purpose of this section is to establish convergence mod bubbling in Theorem 3.1 deferring the \(\delta\)-independent Sobolev estimate Lemma 3.8 to \[3.2\]

Fix for \(\square\) closed symplectic manifolds \(M_0, M_1, M_2\) and compact Lagrangians \(L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2\) with immersed composition as defined in the beginning of [2]

For convenience, we will denote by \((M_{02}, \omega_{02}), (M_{0211}, \omega_{0211})\) the symplectic manifolds

\[
(M_{0211}, \omega_{0211}) := M_0 \times M_2^- \times M_1^\perp \times M_1 = (M_0 \times M_2 \times M_1 \times M_1, \omega_0 \oplus (-\omega_2) \oplus (-\omega_1) \oplus \omega_1),
\]

\[
(M_{02}, \omega_{02}) := M_0^- \times M_2 = (M_0 \times M_2, (-\omega_0) \oplus \omega_2)
\]

and by \((L_{01} \times L_{12})^T \subset M_{0211}\) the transposed Lagrangian gotten by permuting the factors of \(M_{0211}\) by \((x_0, x_1, y_1, x_2) \mapsto (x_0, x_2, x_1, y_1)\).

The notion of “symmetric complex structure” in the following theorem will be defined in \[3.1\]

**Theorem 3.1.** There exists \(\epsilon > 0\) such that the following holds: Fix \(k \in \mathbb{N}_{\geq 1}\), positive reals \(\delta^\nu \to 0\) and \(\rho > 0\), symmetric complex structures \(J^\nu\) on \([-\rho, \rho]^2\) that converge \(C^\infty\) to \(J^\infty\) with \(\|J^\nu - i\mathbb{C}\|_{C^0} \leq \epsilon\), and \(C^k_{\text{loc}}\)-bounded sequences of domain-dependent compatible almost complex structures \(J^\nu_{\ell} : [-\rho, \rho]^2 \to J^\nu_{\ell}(M, \omega_{\ell}), \ell \in \{0, 1, 2\}\) such that the \(C^{k+1}\)-limit of each \((J^\nu_{\ell})\) is a compatible \(C^\infty\) almost complex structures \(J^\infty_{\ell} : [-\rho, \rho]^2 \to \mathcal{J}(M, \omega_{\ell})\).

Then if \((v^\nu_0, v^\nu_1, v^\nu_2)\) is a sequence of size-\((\delta^\nu, \rho)\) \((J^\nu_{01}, J^\nu_{02}, J^\nu_{12})\)-holomorphic squiggly strip quilts for \((L_{01}, L_{12})\) with uniformly bounded gradients,

\[
\sup_{\nu \in \mathbb{N}, (s, t) \in [-\rho, \rho]^2} |dv^\nu|(s, t) < \infty,
\]

then there is a subsequence in which \((v^\nu_0(t - \delta^\nu)), (v^\nu_1|_{t=0}), (v^\nu_2(t + \delta^\nu))\) converge \(C^k_{\text{loc}}\) to a \((J^\infty_{01}, J^\infty_{02}, J^\infty_{12})\)-holomorphic size-\(\rho\) degenerate strip quilt \((v^\infty_0, v^\infty_1, v^\infty_2)\) for \(L_{01} \times M_1, L_{12}\).

If the inequality \(\liminf_{\nu \to \infty, (s, t) \in [-\rho, \rho]^2} |dv^\nu|(s, t) > 0\) holds, then \(v^\infty_0, v^\infty_2\) are not both constant.

The analysis in our proof of Theorem 3.1 will be phrased in terms of pairs of smooth maps \((w_{02}, \tilde{w}) = ((w_0, w_2), (w'_0, w'_2, w'_1, w_1)):\n
\begin{align*}
(22) & 
(w_{02}, \tilde{w}) : (-\rho, \rho) \times [0, \rho - 2\delta) \to M_{02}, \quad \tilde{w} : (-\rho, \rho) \times [0, \delta) \to M_{0211}, \\
& (w_{02}, \tilde{w})(s, 0) \in \Delta_{M_{02}} \times \Delta_{M_1}, \quad \tilde{w}(s, \delta) \in (L_{01} \times L_{12})^T \quad \forall s \in (-\rho, \rho),
\end{align*}
where $\delta$ is nonnegative. From now on we denote the domains of $w_{02}$ and $\hat{w}$ by

$$Q_{02,\rho} := (-\rho, \rho) \times [0, \rho - 2\delta], \quad \hat{Q}_{\delta, \rho} := (-\rho, \rho) \times [0, \delta],$$

and combine them into the notation $Q := (Q_{02,\rho}, \hat{Q}_{\delta, \rho})$. We denote the closures in $\mathbb{R}^2$ by

$$\overline{Q}_{02,\rho} := [-\rho, \rho] \times [0, \rho - 2\delta], \quad \overline{\hat{Q}}_{\delta, \rho} := [-\rho, \rho] \times [0, \delta].$$

For $\delta > 0$, $\rho > 0$ (resp. $\delta = 0$, $\rho > 0$), the setup $Q_{02,\rho}$ is equivalent to a triple of smooth maps $(v_0, v_1, v_2)$ with the same domain and targets as a size-$(\delta, \rho)$ squiggly strip quilt for $(L_{01}, L_{12})$ (resp. as a size-$\rho$ degenerate strip quilt for $L_0 \times M_1, L_2$) and that fulfill the seam conditions $f = \delta$ (resp. $f = 0$) but are not necessarily pseudoholomorphic or of finite energy. Indeed, given such $(v_0, v_1, v_2)$, define $(w_{02}, \hat{w})$ like so:

$$w_{02}(s,t) := (v_0(s, -t - 2\delta), v_2(s, t + 2\delta)), \quad \hat{w}(s,t) := (v_0(t - 2\delta), v_2(s, -t + 2\delta), v_1(s, -t), v_1(s, t)).$$

Conversely, for $\delta \geq 0$ and $(w_{02}, \hat{w})$ satisfying $Q_{02,\rho}$, define $(v_0, v_1, v_2)$ satisfying $f = \delta$, $f = \delta$ (for $\delta > 0$) or $f = 0$, $f = 0$ (for $\delta = 0$) like so:

$$v_0(s,t) := \begin{cases} w_0'(s,t + 2\delta), & -2\delta \leq t \leq -\delta, \\ w_0(s, -t - 2\delta), & t \leq -2\delta, \end{cases} \quad v_2(s,t) := \begin{cases} w_2'(s, t + 2\delta), & \delta \leq t \leq 2\delta, \\ w_2(s, t - 2\delta), & 2\delta \leq t, \end{cases}$$

$$v_1(s,t) := \begin{cases} w_1'(s, -t), & -\delta \leq t \leq 0, \\ w_1(s, t), & 0 \leq t \leq \delta. \end{cases}$$

The transformations (23), (24) are inverse to one another.

The following proof of Theorem 3.1 uses several notions that will be defined in §3.1–3.2.

**Proof of Theorem 3.1** We divide the proof into steps: in Step 1, we show that the squiggly strip quilts converge $C^0_{loc}$ in a subsequence. In Step 2, we upgrade this convergence to $C^k_{loc}$. Finally, we prove in Step 3 that if the gradient satisfies a lower bound at a sequence of points with limit on the boundary, then at least one of $v_0^\infty, v_2^\infty$ is nonconstant. Throughout this proof, $C_1$ will be a constant that may change from line to line.

**Step 1. After passing to a subsequence, $(v_0^\nu(t - \delta^n))$, $(v_1^\nu|_{t=0})$, $(v_2^\nu(t + \delta^n))$ converge $C^0_{loc}$ to a $(J^0_0, J^\infty_2, t)$-holomorphic size-$\rho$ degenerate strip quilt $(v_0^\infty, v_1^\infty, v_2^\infty)$ for $L_{01} \times M_1, L_{12}$.

The Arzelà–Ascoli theorem implies that there exist continuous maps

$$v_0^\infty : (-\rho, \rho) \times (-\rho, 0] \to M_0, \quad v_1^\infty : (-\rho, \rho) \to M_1, \quad v_2^\infty : (-\rho, \rho) \times [0, \rho) \to M_2$$

such that after passing to a subsequence, $(v_0^\nu(s, t - \delta^n))$, $(v_1^\nu|_{t=0})$, $(v_2^\nu(s, t + \delta^n))$ converge $C^0_{loc}$ to $v_0^\infty, v_1^\infty, v_2^\infty$. Standard compactness for pseudoholomorphic curves (e.g. [MS Theorem B.4.2]) implies that this convergence takes place in $C^k_{loc}$ on the interior (i.e. away from the line $t = 0$); in particular, $v_0^\infty$ resp. $v_2^\infty$ are $J^0_0$ resp. $J^\infty_2$-holomorphic on the interior, hence $C^\infty$ by [MS Theorem B.4.1]. In fact, we claim that $v_0^\infty$ and $v_2^\infty$ are $C^\infty$ on their full domains, and that they satisfy a generalized Lagrangian boundary condition in $L_{01} \times M_1, L_{12}$ at $t = 0$.

Denote by $\sigma$ the map

$$\sigma : (v_0^\infty(-, 0), v_1^\infty(-), v_2^\infty(-), 0)) : (-\rho, \rho) \to M_0 \times M_1 \times M_1^{-} \times M_2.$$
converge to zero. This follows from the uniform gradient bound on \( (v_i^\nu) \) and the convergence of \( \delta^\nu \) to zero.

Let us show that \( v_0^\infty \) and \( v_2^\infty \) are \( C^\infty \). We have already concluded that these maps are \( C^\infty \) on the interior, so it only remains to show that they are \( C^\infty \) at the boundary points, w.l.o.g. at \((0,0)\). For that purpose we choose a neighborhood \( U \subset L_{01} \times M_L \cup L_{12} \) of \( \pi(0) \) such that \( \pi_02|_U: U \to M_{02} \) is a smooth embedding. Then \( \pi_02(U) \subset M_{02} \) is a noncompact embedded Lagrangian, and since \( v_0^\infty \) and \( v_2^\infty \) are continuous we find \( \epsilon > 0 \) such that \( \pi((-\epsilon,\epsilon) \times \{0\}) \) is contained in \( U \). Hence \( (v_0^\infty, v_2^\infty)((-\epsilon,\epsilon) \times \{0\}) \) is contained in \( \pi_02(U) \), so standard elliptic regularity (e.g. [MS] Theorem B.4.1) applied to the map \( (v_0^\infty(s,-t), v_2^\infty(s,t)) \) shows that \( v_0^\infty \) and \( v_2^\infty \) are \( C^\infty \) at \((0,0)\). Since \( \pi_02|_U \) is a diffeomorphism onto its image, \( \pi \) is \( C^\infty \) at 0 and thus we have shown that \( v_0^\infty, v_1^\infty, v_2^\infty \) are \( C^\infty \).

**Step 2.** After passing to a further subsequence, the convergence of \( (v_0^\nu(s,t-\delta^\nu)), (v_1^\nu|_{t=0}), (v_2^\nu(s,t+\delta^\nu)) \) takes place in \( C^k_{\text{loc}} \).

In order to establish \( C^k_{\text{loc}} \) convergence near \((\rho,\rho) \times \{0\}\), we cannot rely on [MS] Theorem B.4.2. Rather, we will establish uniform Sobolev bounds for all three sequences of maps. The compact Sobolev embeddings \( H^{k+2} \hookrightarrow C^k \) resp. \( H^{k+1} \hookrightarrow C^k \) for two-dimensional resp. one-dimensional domains will then provide \( C^k_{\text{loc}} \)-convergent subsequences.

Set \( J^\nu \) resp. \( J^\nu \) to be the coherent pair of almost complex structures resp. coherent collection of complex structures resulting from the transformations \( (32) \) resp. \( (31) \) applied to \( J^\nu_0, J^\nu_1, J^\nu_2 \) resp. \( J^\nu_0, J^\nu_1, J^\nu_2 \). Set \( (\hat{u}_0^\nu, \hat{w}_0^\nu) \) to be the \( (J^\nu, J^\nu) \)-holomorphic size-\( (d^\nu,\rho) \) folded strip quilt resulting from the transformation \( (23) \) applied to \( (v_0^\nu, v_1^\nu, v_2^\nu) \). Then \( u_0^\nu \) resp. \( \hat{w}_0^\nu|_{t=0} \) converge \( C^k_{\text{loc}} \) to \( u_0^\infty \) resp. \( \hat{w}_0^\nu|_{t=0} \) to \( u^\infty \) and \( \hat{w}_0^\nu \) to \( \hat{w}_0^\nu \) as \( \rho \to \rho_1 \), \( \rho_2 \to \rho_1 \), \( \rho \to \rho_1 \), and \( \rho \to \rho_1 \) uniformly \( C^1 \)-bounds on \( (v_0^\nu, v_1^\nu, v_2^\nu) \). Set \( (\hat{J}_0^\nu, J^\nu_1, J^\nu_2) \) resp. \( (\hat{J}_0^\nu, J^\nu_1, J^\nu_2) \) to be the \( (J^\nu_0, J^\nu_1, J^\nu_2) \)-holomorphic size-\( (d^\nu,\rho) \) folded strip quilt resulting from the transformation \( (23) \) applied to \( (\hat{v}_0^\nu, \hat{v}_1^\nu, \hat{v}_2^\nu) \). Then \( w_0^\nu \) resp. \( \hat{w}_0^\nu \) converge \( C^k_{\text{loc}} \) to \( w_0^\infty \) resp. \( \hat{w}_0^\nu \) as \( \rho \to \rho_1 \), \( \rho_2 \to \rho_1 \), \( \rho \to \rho_1 \), and \( \rho \to \rho_1 \) uniformly \( C^1 \)-bounds on \( \hat{w}_0^\nu \) to \( \hat{w}_0^\nu \) for sufficiently large \( \nu \) in terms of the corrected exponential maps \( e_{u_0^\nu, d^\nu} \) resp. \( e_{\hat{u}_0^\nu, \hat{d}^\nu} \) and sections \( (s_0^\nu, \xi^\nu) \in \Gamma^{k+1}_{\text{loc}} \) as introduced in \( (3.2) \).

\[
w_0^\nu = e_{u_0^\nu, d^\nu}(\zeta^\nu), \quad \hat{w}_0^\nu = e_{\hat{u}_0^\nu, \hat{d}^\nu}(\hat{\zeta}^\nu).
\]

The sections \( s_0^\nu, \xi^\nu \) converge to zero in \( C^0 \) as \( \nu \to \infty \), are uniformly bounded in \( C^1 \), and satisfy boundary conditions \( (36) \) in the linearizations of \( (L_{01} \times L_{12})^T \) and \( M_0 \times \Delta M_1 \times M_2 \).

**Iteration claim.** We bound \( \|D^\nu \zeta^\nu\|_{\tilde{H}(Q_{\text{df}, \nu})} \) and \( \|\zeta^\nu\|_{\tilde{H}(Q_{\text{df}, \nu})} \) for \( l \in [1, k+2] \) by induction on \( l \), where \( \tilde{H} \) and \( D^\nu \) are the modified Sobolev space and the linear d-bar operator defined in \( (3.2) \) using \( J^\nu, J^\nu \), and the pair of connections \( \nabla = (\nabla_0, \nabla) \) constructed in Lemma \( 3.4 \).

The first key fact for this claim is the formula

\[
D^\nu \zeta^\nu = e_{u_0, d^\nu}(\zeta^\nu) - (\hat{J}_0^\nu, J^\nu_1, J^\nu_2)(e_{u_0, d^\nu}(\zeta^\nu)) =: G^\nu(\zeta^\nu).
\]

---

1 The hypothesis of [MS] that the Lagrangian submanifold is closed can be removed.
Lemma 3.8 yields:
\[ H^2(27) \]

Since the convergence of \( \nu \) is uniformly bounded in \( \kappa \), the \( \delta \)-independent elliptic estimates.

Since \( \zeta^\nu \) is uniformly bounded in \( C^1 \), \( \|\zeta^\nu\|_{H^1(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}})} \) and \( \|\mathcal{D}^\nu_{\zeta^\nu} \zeta^\nu\|_{H^1(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}})} = \|G^\nu(\zeta^\nu)\|_{H^1(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}})} \) are uniformly bounded. This establishes the base case of the iteration.

Next, say that \( \zeta^\nu \) and \( \mathcal{D}^\nu_{\zeta^\nu} \zeta^\nu \) are uniformly bounded in \( \tilde{H}^l(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}}) \) for some \( l \in [1, k + 1] \). Lemma 3.8 yields:
\[ (27) \]

It remains to bound \( \|\mathcal{D}^\nu_{\zeta^\nu} \zeta^\nu\|_{\tilde{H}^{l+1}(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}})} \). Since \( \zeta^\nu \) is uniformly bounded in \( \tilde{H}^{l+1}(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}}) \), it is uniformly bounded in \( C^{l-1}(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}}) \) by Lemma 3.9 which allows us to bound \( \|\mathcal{D}^\nu_{\zeta^\nu} \zeta^\nu\|_{\tilde{H}^{l+1}(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}})} \):
\[ (26) \]

This, together with \( (27) \), establishes the iteration step and completes the Iteration Claim.

The uniform bounds on \( \|\zeta^\nu\|_{\tilde{H}^{k+1}(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}})} \) and the \( C^k \)-bounds that result from Lemma 3.9 yield uniform bounds on \( \|u^\nu_{\rho_{\delta^\nu}}\|_{H^{k+1}(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}})} \), \( \|\tilde{u}^\nu\|_{H^{k+1}(\mathcal{Q}^\nu_{\delta^\nu,\rho_{\delta^\nu}})} \), and \( \|\tilde{u}^\nu\|_{|t|=0} = \|H_{k+1}((\rho_{\delta^\nu,\rho_{\delta^\nu}})^2) \). These bounds induce uniform bounds on the \( H^{k+2} \)-norms of \( v_0^\nu \) and \( v_2^\nu \) on the relevant subdomains of \( (-\rho_{\delta^\nu,\rho_{\delta^\nu}})^2 \) and on the \( H^{k+1} \)-norms of \( v_0^\nu \) on \( (-\rho_{\delta^\nu,\rho_{\delta^\nu}})^2 \times \{0\} \). The compact embeddings \( H^{k+2} \rightarrow C^k \) resp. \( H^{k+1} \rightarrow C^k \) for two-dimensional resp. one-dimensional domains implies the desired \( C^k \)-convergence of \( v_0^\nu(s, t - \delta^\nu) \) resp. \( v_2^\nu(s, t + \delta^\nu) \) to \( v_0^\infty \) resp. \( v_2^\infty \).

**Step 3.** We show that if for some \( \ell \in \{0, 1, 2\} \) and \( \kappa > 0 \) the gradient satisfies a lower bound \( |\nabla v_0^\nu(s, t)\| \geq \kappa \) for some \( \tau^\nu \rightarrow \tau^\infty \in (-\rho, \rho) \), then at least one of \( v_0^\infty \) or \( v_2^\infty \) is nonconstant.

In the notation of Step 2, it suffices to show that if for some \( \tau^\nu \rightarrow \tau^\infty \in [0, \rho) \) and \( \kappa < 0 \) the inequality \( |\nabla v^\nu(0, \tau^\nu) - w^\nu(0, \tau^\nu)| = |\nabla v_0^\nu(0, \tau^\nu) + |\tilde{w}^\nu(0, \tau^\nu)| \geq \kappa \) is satisfied, then \( u_{\rho_{\delta^\nu,\rho_{\delta^\nu}}} \) is not constant. We prove the contrapositive of this statement: assuming that \( u_{\rho_{\delta^\nu,\rho_{\delta^\nu}}} \) is constant, we will show that the quantities \( \lim_{\nu \rightarrow 0} \sup_{t \in [0, \rho]} |\nabla v_0^\nu(0, t)| \) and \( \lim_{\nu \rightarrow 0} \sup_{t \in [0, \delta^\nu]} |\tilde{w}^\nu(0, t)| \) are both zero.

Since the convergence of \( u_{\rho_{\delta^\nu,\rho_{\delta^\nu}}} \) to \( u_{0_{\rho_{\delta^\nu,\rho_{\delta^\nu}}} \) takes place in \( C^1 \) loc, the quantity \( \lim_{\nu \rightarrow 0} \sup_{t \in [0, \rho]} |\nabla v_0^\nu(0, t)| \) is zero. To see that the quantity \( \lim_{\nu \rightarrow 0} \sup_{t \in [0, \delta^\nu]} |\tilde{w}^\nu(0, t)| \) is also zero, note that by the last paragraph of Step 1, the limit \( \overline{w} \) of \( \tilde{w}^\nu \) is also constant, which implies the formula \( \overline{w}^\nu = \overline{w} \).
de g_\tilde{w}(\zeta')(\nabla \tilde{w}')$. It follows that to prove the equality $\lim_{\nu \to \infty} \sup_{t \in [0, \delta']} |\nabla \tilde{w}'(0, t)| = 0$, it suffices to prove the equality $\lim_{\nu \to \infty} \sup_{t \in [0, \delta']} |\nabla \zeta'(0, t)| = 0$. We can now estimate, using the Sobolev inequality $\| - \|_{C^0} \leq C_1 \| - \|_{H^1}$ for one-dimensional domains whose lengths are bounded away from zero:

$$\limsup_{\nu \to \infty} \sup_{t \in [0, \delta']} |\nabla \zeta'(0, t)| \leq \limsup_{\nu \to \infty} \sup_{t \in [0, \delta']} |\nabla \zeta'(0, 0)| + \limsup_{\nu \to \infty} \sup_{t \in [0, \delta']} |\nabla \zeta(0, t) - \nabla \zeta(0, 0)|$$

This completes the contrapositive of Step 3, which concludes our proof of Theorem 3.1. □

3.1. **Complex and almost complex structures in the folded and unfolded setups.** The Gromov Compactness Theorem in [BW] is proved by “straightening” the seams of a squiggly strip quilt. Pushing forward the standard complex structure from the squiggly strip quilt to the new quilt with horizontal seams produces a nonstandard complex structure, which is symmetric under conjugation. We axiomatize this property in the following definition.

**Definition 3.2.** Fix $\rho > 0$. A **symmetric complex structure** on $[-\rho, \rho]^2$ is a complex structure $\jmath$ such that the equality

$$\jmath(s, t) = -\sigma \circ \jmath(s, -t) \circ \sigma$$

holds for any $(s, t) \in [-\rho, \rho]^2$, where $\sigma$ is the conjugation $\alpha \partial_s + \beta \partial_t \mapsto \alpha \partial_s - \beta \partial_t$.

When a symmetric complex structure, almost complex structures, and a pseudoholomorphic squiggly strip quilt are “pushed forward” by the folding operation (23), the result is a “coherent system of complex structures”, a “coherent pair of almost complex structures”, and a “pseudoholomorphic folded strip quilt”, defined as follows.

**Definition 3.3.** Fix $\delta > 0$ and $\rho > 0$.

- **A coherent collection of complex structures** $\jmath$ on $\bar{\mathcal{Q}}_{\delta, \rho}$ is a pair $\jmath = (\jmath_{\delta, \tilde{j}}) = ((\jmath_0, \jmath_2), (\jmath'_0, \jmath'_2, \jmath'_1, \jmath_1))$, where $\jmath_0, \jmath_2$ (resp. $\jmath'_0, \jmath'_2, \jmath'_1, \jmath_1$) are complex structures on $\bar{\mathcal{Q}}_{\delta, \rho}$ (resp. on $\bar{\mathcal{Q}}_{\delta, \rho}$) such that the following equalities hold for all $s \in (-\rho, \rho)$:

  $$\jmath(s, 0) = -\sigma \circ \jmath'(s, 0) \circ \sigma,$$

  $$\jmath'(s, 0) = \jmath_0'(s, \delta), \quad \jmath'_1(s, \delta) = \jmath_1(s, \delta), \quad \jmath'_2(s, \delta) = \jmath_2(s, \delta) \circ \sigma.$$ 

- **A coherent pair of almost complex structures** $\mathcal{J}$ on $\bar{\mathcal{Q}}_{\delta, \rho}$ is a pair $\mathcal{J} = (\mathcal{J}_{\delta, \tilde{\mathcal{J}}})$, where $\mathcal{J}_{\delta, \tilde{\mathcal{J}}}$ are almost complex structures

  $$\mathcal{J}_{\delta, \tilde{\mathcal{J}}} : \bar{\mathcal{Q}}_{\delta, \rho} \to \tilde{\mathcal{J}}(M_{\delta, \omega_0}), \quad \tilde{\mathcal{J}} : \bar{\mathcal{Q}}_{\delta, \rho} \to \tilde{\mathcal{J}}(M_{\delta, \omega_0}, \omega_{0211})$$

satisfying the following compatibility condition: Set $\iota : M_{\delta, \omega_0} \to M_{\delta, \omega_{0211}}$ resp. $\pi : M_{\delta, \omega_{0211}} \to M_{\delta, \omega_0}$ to be the inclusion resp. projection. Then for any $s \in (-\rho, \rho)$, the following equality must hold:

  $$\mathcal{J}(s, 0) = -d\pi \circ \tilde{\mathcal{J}}(s, 0) \circ d\iota.$$
- Fix a coherent collection \( j \) of complex structures and a coherent pair \( J \) of almost complex structures on \( Q_{\delta,\rho} \). A \((J, j)\)-holomorphic size-\((\delta, \rho)\) folded strip quilt is a collection of smooth maps \( w = (w_{02}, \tilde{w}) = ((w_0, w_2), (w'_0, w'_2, w'_1, w_1)) \) satisfying (22) that have finite energy,

\[
\int_{Q_{02,\delta,\rho}} w_{02}^* \omega_{02} < \infty, \quad \int_{\hat{Q}_{\delta,\rho}} \hat{w}^* \omega_{0211} < \infty,
\]

and satisfy the Cauchy–Riemann equations

\[
\partial_{J,j} w = (\partial_{Q_{02,\delta,\rho} w_{02}}, \hat{\partial}_{J,j} \tilde{w}) = 0,
\]

where \( \partial_{J,j} = (\partial_{Q_{02,\delta,\rho} w_{02}}, \hat{\partial}_{J,j} \tilde{w}) \) is the pair of operators defined by:

\[
\partial_{Q_{02,\delta,\rho} w_{02}} := (dw_0, dw_2) \circ (j_0, j_2) (\partial_s) - J_0 (\cdot, w_0) \circ (\partial_s w_0, \partial_s w_2),
\]

\[
\hat{\partial}_{J,j} \tilde{w} := (dw'_0, dw'_2, dw'_1, dw_1) \circ (j'_0, j'_2, j'_1, j_1) (\partial_s) - \hat{J} (\cdot, \tilde{w}) \circ (\partial_s w'_0, \partial_s w'_2, \partial_s w'_1, \partial_s w_1).
\]

Given a \((J_0, J_1, J_2, j)\)-holomorphic squiggly strip quilt \((v_0, v_1, v_2)\) with \( j \) symmetric, we can produce a folded strip quilt like this: Define a coherent collection \( j \) of complex structures by

\[
\begin{align*}
 j_{02}(s, t) &= (j_0, j_2)(s, t) := (-\sigma \circ j(s, -t - 2\delta) \circ \sigma, j(s, t + 2\delta)), \\
 \hat{J}(s, t) &= (j'_0, j'_2, j'_1, j_1)(s, t) := (j(s, t - 2\delta), -\sigma \circ j(s, -t + 2\delta) \circ \sigma, -\sigma \circ j(s, t) \circ \sigma, \sigma(s, t))
\end{align*}
\]

and a coherent pair \( J \) of almost complex structures by

\[
\begin{align*}
 J_{02}(s, t) &= (-J_0(s, -t - 2\delta)) \oplus J_2(s, t + 2\delta), \\
 \hat{J} &= J_0(t - 2\delta) \oplus (-J_2(t + 2\delta)) \oplus (-J_1(s, -t)) \oplus J_1(s, t).
\end{align*}
\]

If \((w_{02}, \tilde{w})\) is defined by applying (22) to \((v_0, v_1, v_2)\), then \((w_{02}, \tilde{w})\) is a \((J, j)\)-holomorphic size-\((\delta, \rho)\) folded strip quilt. Indeed, \((w_{02}, \tilde{w})\) have the correct domains and codomains and satisfy the seam conditions, as discussed earlier, and the finite-energy hypothesis on \((v_0, v_1, v_2)\) implies that \((w_{02}, \tilde{w})\) has finite energy. The Cauchy–Riemann equation (11) for \( v_0 \) on \((-\rho, \rho) \times (-\rho, -2\delta)\) can be rewritten as

\[
dw_0(s, t) \circ (-\sigma \circ j(s, -t - 2\delta) \circ \sigma) - (-\sigma \circ j(s, -t - 2\delta), w_0(s, t)) \circ dw_0(s, t) = 0
\]

for \( w_0(s, t) := v_0(s, -t - 2\delta) \) as in (23), so \( w_0 \) is \((-J_0(s, -t - 2\delta), J_0(s, t))\)-holomorphic on \( Q_{02,\delta,\rho} \). Five similar calculations complete the check that \((w_{02}, \tilde{w})\) is \((J, j)\)-holomorphic.

Finally, we consider the coordinate representation of a coherent collection of complex structures. Fix a coherent collection \( j = ((j_0, j_2), (j'_0, j'_2, j'_1, j_1)) \) of complex structures on \( Q_{\delta,\rho} \). Define \( a_0(s, t), c_0(s, t) \in \mathbb{R} \) by

\[
\begin{align*}
 j_0(s, t)(\partial_s) &=: a_0(s, t) \partial_s + c_0(s, t) \partial_t,
\end{align*}
\]

and define \( a_j(s, t), c_j(s, t) \) for \( j \in \{1, 2\} \) and \( a'_k(s, t), c'_k(s, t) \) for \( k \in \{0, 1, 2\} \) in the same way. Then (28) and (29) translate into the following conditions on these coefficients:

\[
\begin{align*}
 a_j(s, 0) &= -a'_j(s, 0), \quad c_j(s, 0) = c'_j(s, 0) \quad \forall j \in \{0, 1, 2\}, \\
 a_0(s, \delta) &= a_2(s, \delta), \quad a'_1(s, \delta) = a_1(s, \delta), \quad a_0(s, \delta) = -a'_1(s, \delta), \\
 c_0(s, \delta) &= c_2(s, \delta), \quad c'_1(s, \delta) = c_1(s, \delta), \quad c_0(s, \delta) = c'_1(s, \delta).
\end{align*}
\]

We will use this coordinate representation in [3.2].
3.2. A collection of δ-independent elliptic estimates. This subsection is devoted to proving Lemma 3.8, which is the crucial δ-independent elliptic estimate needed for the proof of Theorem 3.1. In addition to the data fixed at the beginning of §3, fix for $\rho > 0$ and a pair of maps $u = (u_{02}, \pi)$ satisfying (22) $\delta_0 = 0, \rho$.

Furthermore, we continue to denote by $i$ the standard coherent collection of complex structures defined in (25), and for any $\delta \in (0, \rho/4]$ we define a pair $u_\delta = (u_{02, \delta}, \hat{u}_\delta)$ of smooth maps satisfying (22) $\delta_\rho$ by:

$$u_{02, \delta} := u_{02}|_{Q_{02, \delta, \rho}} , \quad \hat{u}_\delta(s, t) := \bar{u}(s).$$

Our approach is inspired by [WW1], but we deviate from that approach by working with a special connection which allows us to drop boundary terms from the $H^2$-estimate [WW1 Lemma 3.2.1(b)]. This special connection is constructed in the following lemma, which is a generalization to the immersed case of a connection constructed in [W2].

**Lemma 3.4.** There is an assignment $\delta \mapsto \nabla_\delta = (\nabla_{02, \delta}, \hat{\nabla}_\delta)$ that sends $\delta \in (0, \rho/4]$ to a pair of connections $\nabla_{02, \delta}$ resp. $\hat{\nabla}_\delta$ on $u_{02, \delta}^* T M_{02} \to Q_{02, \delta, \rho}$ resp. $\hat{u}_\delta^* T M_{0211} \to \hat{Q}_{\delta, \rho}$ such that the following hold:

- Parallel transport under $\hat{\nabla}_\delta$ preserves $\hat{u}_\delta^* T (L_{01} \times L_{12}) \cap$ and $\hat{u}_\delta^* T (M_{02} \times \Delta_{M_1})$;
- For a section $\hat{\zeta} \in \Gamma(\hat{u}_\delta^* T (M_{02} \times \Delta_{M_1}))$ we have $\nabla_{02, \delta, \rho}(p \circ \hat{\zeta}) = p \circ \nabla_{\delta, \rho} \hat{\zeta}$, where $p: \hat{u}_\delta^* T (M_{02} \times \Delta_{M_1}) \to u_{02, \delta}^* T M_{0212}$ is the projection;
- For $\delta_1 < \delta_2$, the restrictions of $\nabla_{\delta_1}, \nabla_{\delta_2}$ agree:

$$\nabla_{02, \delta_1}|_{Q_{02, \delta_1, \rho}} = \nabla_{02, \delta_2}, \quad \hat{\nabla}_{\delta_1} = \hat{\nabla}_{\delta_2}.$$

**Proof.** Fix metrics on $u_{02}^* T M_{02}$ and $\pi^* T M_{0211}$ so that given a smooth subbundle, we may form its orthogonal complement. For any fixed $s \in (-\rho, \rho)$ we denote:

$$\Lambda_{0211} := T_{\pi(s)}(L_{01} \times L_{12})^\perp, \quad \Delta := T_{\pi(s)}(M_{02} \times \Delta_{M_1}), \quad \Lambda_{02} := \Lambda_{0211} \cap \Delta, \quad \Lambda_0 := T_{\pi_{02}, \pi(s)}(\Lambda_{02}).$$

The transversality of $L_{01} \times L_{12} \cap M_{02} \times \Delta_{M_1} \times M_2$ implies $\Lambda_{02} = T_{\pi(s)} \hat{L}_{02}$, so the projection from $\hat{L}_{02}$ to $\Lambda_{02}$ is injective (see e.g. [WW2 Lemma 2.0.5]). Hence the intersection of $\Lambda_{02}$ and $\{0\} \times T(\pi(s), \pi(s)) \Delta_{M_1}$ is trivial. It follows that if we let $C_1$ denote the complement of $\Lambda_{02} + \{0\} \times T(\pi(s), \pi(s)) \Delta_{M_1}$ in $\Delta$, the diagonal decomposes as $\Delta = \Lambda_{02} \oplus C_1 \oplus \{0\} \times T(\pi(s), \pi(s)) \Delta_{M_1}$.

Let $C_2$ be the complement of $\Lambda_{02}$ in $\Lambda_{0211}$. Transversality implies $T(\pi(s))_{0211} = \Lambda_{0211} + \Delta$, so we have deduced the following decomposition:

$$T(\pi(s))_{0211} = C_2 \oplus \Lambda_{02} \oplus C_1 \oplus \{0\} \times T(\pi(s), \pi(s)) \Delta_{M_1}.$$

The subspace $\Lambda_{0211}$ (resp. $\Delta$) is given by the sum of the first two factors (resp. the sum of the last three factors) in this decomposition. Therefore, if we choose connections on each of these four subbundles and set $\nabla$ to be the product connection, then extend $\nabla$ to a connection $\nabla_{\delta}$ on $u_{02}^* T M_{0212} \to \hat{Q}_{\delta, \rho}$ by defining $\nabla_{\delta, s}(s, t) \mapsto \nabla_s(s) \mapsto \hat{\zeta}(s, t)$ and defining $\nabla_{\delta, \rho}(s, t) \mapsto \hat{\zeta}(s, t)$ in terms of the Levi-Civita connection $\nabla_{\delta}, \hat{\nabla}_{\delta}$ satisfies the first bullet.

Denote by $p: \pi^* T (M_{02} \times \Delta_{M_1}) \to u_{02}^* T M_{0212}$ projection and by $i: u_{02}^* T M_{0212} \to u_{02}^* T M_{02} \times \Delta_{M_1}$ the inclusion defined by sending $v \in T_{02}(s, 0) M_{0212}$ to $(v, 0) \in T_{\pi(s)}(M_{02} \times \Delta_{M_1})$. Define a connection $p^* \nabla$ on $u_{02}^* T M_{0212}$ by $p^* \nabla(0, 0) := p \circ \nabla(i \circ \zeta_{02})$. Extend $p^* \nabla$ in any way to a connection $\nabla_{02}$ on $u_{02}^* T M_{0212}$; for $\delta \in (0, \rho/4]$, define $\nabla_{02, \delta} := \nabla_{02, \delta, \rho}$.

The second bullet now follows from a computation, in which $(\zeta_{02}, \hat{\zeta}_{01}, \hat{\zeta}_1)$ is an arbitrary section of $u_{\delta}^* T (M_{02} \times \Delta_{M_1})$:

$$p \circ \nabla_{\delta, s} \hat{\zeta} = p \circ \nabla_{\delta, s}(\zeta_{02}, \hat{\zeta}_{01}, \hat{\zeta}_1) = p \circ \nabla_{\delta, s}(i \circ \zeta) + p \circ \nabla_{\delta, s}(0, \hat{\zeta}_{01}, \hat{\zeta}_1) = \nabla_{02, \delta, s}(p \circ \hat{\zeta}).$$
The term \( p \circ \nabla_{\delta,s}(0, \hat{\zeta}_1, \hat{\zeta}_1) \) in the third quantity vanishes since the subbundle \( \{0\} \times T(\bar{w}_{\delta,1}, w_{\delta,1}) \Delta_{M_1} \) is preserved under parallel transport by \( \hat{\nabla}_{\delta,s} \).

We will use the connections \( \nabla_{\delta} \) just constructed throughout the rest of [3.2]. Due to the third property in Lemma 3.4, it is unambiguous to drop the subscript and refer to \( \nabla_{\delta} \) simply as \( \nabla \). Note that this pair of connections induce connections on the pullbacks by \( u_{02,\delta} \) or \( \hat{u}_{\delta} \) of any tensor bundle of \( TM_{02} \) or \( TM_{0211} \) in a canonical way.

Before we state the elliptic estimate Lemma 3.8, we need to define our function spaces and delbar operators.

**Definition 3.5.** Fix \( r \in (0, \rho) \), \( \delta > 0 \), and \( k \geq 2 \). Define the space of sections \( \Gamma_{u_{\delta}}^k(Q_{\delta,r}) \) and the norms \( \| - \|_{H^k(Q_{\delta,r})}, \| - \|_{\hat{H}^k(Q_{\delta,r})} \) as follows.

- Define \( \Gamma_{u_{\delta}}^k(Q_{\delta,r}) \) by:
  \[
  \Gamma_{u_{\delta}}^k(Q_{\delta,r}) := \left\{ \left( \begin{array}{c}
  \xi_{02} \in H^k(Q_{02,\delta,r}, u_{02,\delta}^* TM_{02}), \\
  \hat{\zeta} \in H^k(\hat{Q}_{\delta,r}, \hat{u}_{\delta}^* TM_{0211})
  \end{array} \right) \right\},
  \]
  where (36) denotes the following linearized boundary conditions:

  \[
  (\xi_{02}(s,0), \hat{\zeta}(s,0)) \in T\Delta_{M_{02}} \times T\Delta_{M_1}, \quad \hat{\zeta}(s,\delta) \in T(L_{01} \times L_{12})^T \quad \forall \ s \in (-r, r).
  \]

- Define two norms \( \| - \|_{H^k(Q_{\delta,r})}, \| - \|_{\hat{H}^k(Q_{\delta,r})} \) on \( \Gamma_{u_{\delta}}^k \) by:

  \[
  \| (\xi_{02}, \hat{\zeta}) \|_{H^k(Q_{\delta,r})}^2 := \| \xi_{02} \|_{H^k(Q_{02,\delta,r}, u_{02,\delta}^* TM_{02})}^2 + \| \hat{\zeta} \|_{H^k(\hat{Q}_{\delta,r}, \hat{u}_{\delta}^* TM_{0211})}^2,
  \]

  \[
  \| (\xi_{02}, \hat{\zeta}) \|_{\hat{H}^k(Q_{\delta,r})}^2 := \| (\xi_{02}, \hat{\zeta}) \|_{H^k(Q_{\delta,r})}^2 + \sum_{l=0}^{k-2} \| (\nabla^l \xi_{02}, \nabla^l \hat{\zeta}) \|_{C^0 H^1(Q_{\delta,r})}^2,
  \]

  \[
  := \| (\xi_{02}, \hat{\zeta}) \|_{H^k(Q_{\delta,r})}^2 + \sum_{l=0}^{k-2} \sup_{t \in [0, r] \cap (0, \delta)} \| \nabla^l \xi_{02}(r, -t) \|_{H^1((-r, r), u_{02,\delta}(-t)^* TM_{02})}^2 + \sup_{t \in [0, \delta]} \| \nabla^l \hat{\zeta}(r, -t) \|_{H^1((-r, r), \hat{u}_{\delta}(-t)^* TM_{0211})}^2.
  \]

Note that \( \| - \|_{\hat{H}^k(Q_{\delta,r})} \) is a well-defined norm on \( \Gamma_{u_{\delta}}^k(Q_{\delta,r}) \) due to the embedding \( H^1 \hookrightarrow C^0 \) for one-dimensional domains. However, the constant in the bound \( \| - \|_{\hat{H}^k(Q_{\delta,r})} \leq C(\delta, r) \) \( \| - \|_{H^k(Q_{\delta,r})} \) is \( \delta \)-dependent.

In [WW1], Wehrheim–Woodward introduced an exponential map with quadratic corrections, which allowed them to treat the Lagrangian boundary conditions as totally geodesic. Wehrheim–Woodward assumed the composition \( L_{01} \circ L_{12} \) to be embedded, but their construction of the corrected exponential map only used the immersedness of that composition. We may therefore import their corrected exponential map into our setting:

**Definition 3.6.** Given \( r > 0 \) and \( \delta > 0 \), define the **corrected exponential map** \( e_{u_{\delta}} \) and its linearization \( de_{u_{\delta}} \) and \( s \)- and \( t \)-derivatives as follows.

- Let \( e_{u_{\delta}} = (e_{u_{02,\delta}}, e_{\hat{u}_{\delta}}) \) be the pair of maps defined in [WW1] Lemma 3.1.2; \( e_{u_{\delta}} \) sends \( \zeta \in \Gamma_{u_{\delta}}^2(Q_{\delta,r}) \) with \( \| \zeta \|_{C^0(Q_{\delta,r})} \) sufficiently small to a pair of maps \( e_{u_{\delta}}(\zeta) = (e_{u_{02,\delta}}(\xi_{02}), e_{\hat{u}_{\delta}}(\hat{\zeta})) \) satisfying (22).

- For \( p_{02} \in u_{02,\delta}^* TM_{02}(s,t) \), \( de_{u_{02,\delta}}(p_{02}) : u_{02,\delta}^* TM_{02}(s,t) \to T e_{u_{02,\delta}}(p_{02}) M_{02} \) is defined by including the fiber \( u_{02,\delta}^* TM_{02}(s,t) \) into \( T p_{02} u_{02,\delta}^* TM_{02} \) as the vertical vectors, then postcomposing.
with the tangent map $T(e_{u_{02},\delta})_{p_02} : T_{p_02}u_{02,\delta}^*TM_{02} \to T_{e_{u_{02},\delta}(p_{02})}M_{02}$. The linearization $de_{\hat{\alpha}}(\hat{p})$ is defined analogously.

- For $p_{02} \in u_{02,\delta}^*TM_{02}(s,t)$, define $D_{e}e_{\sigma_{02}}(p_{02}) \in T_{e_{\sigma_{02}}(p_{02})}M_{02}$ to be the vector gotten by choosing a flat section $\sigma$ of $u_{02}^*TM_{02}((s-\epsilon,s+\epsilon) \times \{t\})$ for $\epsilon$ small, then setting $D_{e}e_{\sigma_{02}}(p_{02}) := T_{e_{\sigma_{02}}(p_{02})}(D_{e}\sigma_{02}(\sigma))$. The derivatives $D_{e}e_{\sigma_{02}}(p_{02})$, $D_{e}e_{\sigma_{02}}(\hat{p})$, $D_{e}e_{\sigma_{02}}(\hat{p})$ are defined analogously, and each of these derivatives depends smoothly on the argument $p_{02}$ or $\hat{p}$.

This exponential map will allow us to define fiberwise complex structures in the following, which are parametrized by vector fields rather than by maps.

In the following definition of the linear delbar operator, we must go into coordinates. Fix $\delta > 0$ and a coherent collection $j = ((j_0,j_2), (j_0',j_2',j_1'))$ of complex structures on $Q_{\delta,\rho}$. Then $j$ induces via (33) two pairs of endomorphisms $A = (A_{02}, \hat{A})$, $C = (C_{02}, \hat{C})$ of $u_{02,\delta}^*TM_{02}, \hat{u}_{02}^*TM_{0211}$, with $C_{02}, \hat{C}$ defined as follows and $A_{02}, \hat{A}$ defined in analogous fashion:

$$C_{02}(s,t) : T_{u_{02,\delta}(s,t)}M_{02} \to T_{u_{02,\delta}(s,t)}M_{02}, \quad (v_0,v_2) \mapsto (c_0(s,t)v_0, c_2(s,t)v_2),$$
$$\hat{C}(s,t) : T_{\hat{u}_{02,\delta}(s,t)}M_{0211} \to T_{\hat{u}_{02,\delta}(s,t)}M_{0211}, \quad (v_0',v_2') \mapsto (c'_0(s,t)v_0', c'_2(s,t)v_2').$$

Note that the conditions (34) (which are equivalent to the coherence conditions (28), (29)) imply that for any $s \in [-\rho, \rho]$, the endomorphisms

$$\hat{C}(s,\delta), \quad C_{02}(s,0) \times (\hat{C}|_{(u_{02},u_{02}')}^*TM_{02})(s,0), \quad (\hat{C}|_{(u_{02},u_{02})^*TM_{02}})(s,0)$$

are scalar multiples of the identity; we will use this fact later in \S 3.2.

**Definition 3.7.** For $\delta > 0$, $r > 0$, $k \geq 2$, a coherent collection $j$ of complex structures and a coherent pair of almost complex structures $J$ on $Q_{\delta,\rho}$, and $\xi \in \Gamma_{u_{02}}^2(Q_{\delta,\rho})$, define the linear delbar operator $D_\xi$ to be the following map from $H^1(Q_{02,\delta,r}, u_{02,\delta}^*TM_{02}) \times H^1(\hat{Q}_{\delta,\rho}, \hat{u}_{02}^*TM_{0211})$ to $H^0(u_{02,\delta}^*TM_{02}) \times H^0(\hat{u}_{02}^*TM_{0211})$:

$$D_\xi := A\nabla_\xi + C\nabla_\xi - J(\xi)\nabla_\xi$$

$$:= (A_{02}\nabla_{u_{02,\delta}\xi_0} + C_0\nabla_{\xi_0} - J_0(\xi_0)\nabla_{u_{02,\delta}\xi_0}, \hat{A}\nabla_\hat{\xi} + \hat{C}\nabla_\hat{\xi} - \hat{J}(\hat{\xi})\nabla_\hat{\xi}),$$

where $J(\xi)$ is the pulled-back complex structure

$$J(\xi)(s,t) := de_{u_{\xi}}(\xi(s,t))^{-1}J(s,t,e_{\xi}(\xi(s,t)))de_{u_{\xi}}(\xi(s,t))$$
$$:= (de_{u_{02,\delta}}(\xi_0(s,t))^{-1}J_0(s,t,e_{\xi_0}(\xi_0(s,t)))de_{u_{02,\delta}}(\xi_0(s,t)),$$
$$de_{\hat{\xi}}(\xi(s,t))^{-1}\hat{J}(s,t,e_{\hat{\xi}}(\hat{\xi}(s,t)))de_{\hat{\xi}}(\hat{\xi}(s,t))).$$

If $\xi = (\xi_0, \hat{\xi})$ is a pair of sections in $\Gamma_{u_{\xi}}^2(Q_{\delta,\rho})$, we can write $\partial_s(e_{\xi}(\xi))$ and $\partial_t(e_{\xi}(\xi))$ in terms of $de_{u_{\xi}, s}e_{u_{\xi}}, de_{u_{\xi}, t}e_{u_{\xi}}$:

$$\partial_s(e_{\xi}(\xi)) := (\partial_s(e_{u_{02,\delta}}(\xi_0)) + D_{s}e_{u_{02,\delta}}(\xi_0) \nabla_{u_{02,\delta}\xi_0},$$
$$\partial_t(e_{\xi}(\xi)) := (\partial_t(e_{u_{02,\delta}}(\xi_0)), D_{t}e_{u_{02,\delta}}(\xi_0)),$$

$$\partial_s(e_{\hat{\xi}}(\hat{\xi})) := (\partial_s(e_{\hat{\xi}}(\xi)), D_{s}e_{\hat{\xi}}(\xi) \nabla_{\xi} + D_{s}e_{\hat{\xi}}(\hat{\xi})),$$
$$\partial_t(e_{\hat{\xi}}(\hat{\xi})) := (\partial_t(e_{\hat{\xi}}(\xi)), D_{t}e_{\hat{\xi}}(\xi) \nabla_{\xi} + D_{t}e_{\hat{\xi}}(\hat{\xi})).$$
This decomposition allows us to relate the delbar operator $\overline{\partial}_{j,j}$ from (30) with the linear delbar operator $D_\zeta$ just defined:

$$\overline{\partial}_{j,j}(e_{u_s}(\zeta)) = A\partial_s(e_{u_s}(\zeta)) + C\partial_t(e_{u_s}(\zeta)) - J(s,t,e_{u_s}(\zeta))\partial_s(e_{u_s}(\zeta))$$

$$= de_{u_s}(\zeta)(A\nabla_s\zeta + C\nabla_t\zeta - de_{u_s}(\zeta)^{-1}J(s,t,e_{u_s}(\zeta))de_{u_s}(\zeta)\nabla_s\zeta)$$

$$+ (AD_se_{u_s}(\zeta) + CD_te_{u_s}(\zeta) - J(s,t,e_{u_s}(\zeta))D_se_{u_s}(\zeta))$$

$$=: de_{u_s}(\zeta)D_\zeta\zeta + F(\zeta).$$

The inhomogeneous term $F$ depends smoothly on $\zeta$, which is crucial for the proof of Theorem 3.1.

The following is the main result of 3.2. It generalizes [WW1, Lemma 3.2.1], which bounds the $H^1$-norm of $\zeta$ when the domain complex structure is standard.

**Lemma 3.8.** There is a constant $\epsilon > 0$ and for every $C_0 > 0$, $k \geq 0$, and $r_1, r_2$ with $0 < r_1 < r_2 < \rho$ there is a constant $C_1$ such that the inequality

$$\|\zeta\|_{\tilde{H}^{k+1}(Q_{\delta,r_1})} \leq C_1 (\|D_\zeta\zeta\|_{\tilde{H}^k(Q_{\delta,r_2})} + \|\zeta\|_{H^0(Q_{\delta,r_2})})$$

holds for any choice of $\delta \in (0, r_1/4]$, a coherent collection $j$ of complex structures on $\overline{Q}_{\delta,\rho}$ with $\|j-i\|_{C_0} \leq \epsilon$ and $\|j-i\|_{C^{max}(k,1)} \leq C_0$, a coherent pair $J$ of almost complex structures on $\overline{Q}_{\delta,\rho}$ which are contained in a $C^{max(k,1)}$-ball of radius $C_0$ and which induce by (1) metrics whose pairwise constants of equivalence are bounded above by $C_0$, and a pair of sections $\zeta \in \Gamma_{u_s}^{k+2}(Q_{\delta,r_2})$ with $\|\zeta\|_{C^0} \leq \epsilon$, $\|\zeta\|_{C^1} \leq C_0$, and $\|\zeta\|_{\tilde{H}^k(Q_{\delta,r_2})} \leq C_0$.

We begin by establishing $\delta$-independent Sobolev estimates for elements of $\Gamma_{u_s}^k(Q_{\delta,r})$.

**Lemma 3.9.** Fix $C_0 > 0$, $k \geq 0$, and $r_1, r_2$ with $0 < r_1 < r_2 < \rho$. Then there is a constant $C_1$ and a polynomial $P$ such that the inequality

$$\|\nabla^k\zeta\|_{C^0H^1(Q_{\delta,r})} \leq C_1 (\|\zeta\|_{H^{k+2}(Q_{\delta,r})} + \|\nabla^{k-1}D_\zeta\zeta\|_{C^0H^1(Q_{\delta,r})})$$

$$+ P \left( \sum_{l=1}^{k-1} \|\nabla^l\xi\|_{C^0H^1(Q_{\delta,r})} \right) \left( \|\zeta\|_{H^{k+1}(Q_{\delta,r})} + \sum_{l=0}^{k-2} \|\nabla^l\xi\zeta\|_{C^0H^1(Q_{\delta,r})} \right)$$

(41)

(where the term $\|\nabla^{k-1}D_\zeta\zeta\|_{C^0H^1(Q_{\delta,r})}$ is to be omitted when $k = 0$) holds for any choice of $\delta \in (0, r_1/4]$, $r \in [r_1, r_2]$, a coherent collection $j$ of complex structures on $\overline{Q}_{\delta,\rho}$ with $\|j-i\|_{C_0} \leq C_0$, a coherent pair $J$ of compatible almost complex structures on $Q_{\delta,\rho}$ which are contained in a $C^k$-ball of radius $C_0$ and which induce by (1) metrics whose pairwise constants of equivalence are bounded above by $C_0$, and pairs of sections $\zeta, \xi \in \Gamma_{u_s}^{k+2}(Q_{\delta,r})$ with $\|\xi\|_{C^1} \leq C_0$.

Here is the idea of the proof: [WW1, Lemma 3.1.4] is a uniform Sobolev inequality for sections $\zeta$ satisfying the linearized boundary conditions. Since the special connection constructed in Lemma 3.4 preserves the linearized boundary conditions, [WW1, Lemma 3.1.4] immediately gives a bound on $\|\nabla^k\zeta\|_{C^{k+1}(Q_{\delta,r})}$. To derive a bound on $\|\nabla^\alpha\zeta\|_{C^{k+1}(Q_{\delta,r})}$ for $\alpha \in \{s, t\}^k$, we trade indices using the operator $D_\zeta$.

**Proof.** We prove this lemma in two steps: first, we prove a slightly different inequality, which has terms of the form $\|\nabla^l\zeta\|_{C^0H^1}$ on the right-hand side. Then, we prove the desired inequality by inductively removing these unwanted terms.

Throughout this proof, $C_1$ and $P$ will denote a $\delta$-independent constant and $\delta$-independent polynomial that may change from line to line.
Step 1. We prove the following inequality:

$$(42) \quad \|\nabla^k \zeta\|_{c^0 H^1} \leq C_1 \left( \|\zeta\|_{H^{k+2}} + \|\nabla^{k-1} D \zeta\|_{c^0 H^1} + P \left( \sum_{l=1}^{k-1} \|\nabla^l \zeta\|_{c^0 H^1} \right) \cdot \sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{c^0 H^1} \right).$$

We begin by proving the $k = 0$ case of (42), which is essentially a consequence of \[WW]\ Lemma 3.1.4. The modification must be made to that lemma: we must relax the hypothesis that the composition $L_1 \circ L_2$ is embedded to the hypothesis that this composition is immersed. To make this modification, change the proof of \[WW]\ Lemma 3.1.4 like so: instead of using \[WW]\ Lemma 3.1.3(c), use the fact that for $\hat{\zeta} = (\zeta'_{02}, \zeta_1') \in C^\infty((-r, r), \pi^* TM_{0211})$,

$$\|\hat{\zeta}\|_{H^1((-r, r))} \leq C_1 \left( \|\zeta_{02}\|_{H^1((-r, r))} + \|\zeta'_1 - \xi_1\|_{H^1((-r, r))} + \|\pi_{0211}^\perp \|_{H^1((-r, r))} \right),$$

where $\pi_{0211}^\perp$ is the projection onto the orthogonal complement of the tangent space of $(L_1 \times L_2)^T$. This inequality follows from the pointwise estimate $|\hat{\zeta}| \leq C(|\zeta'_{02}| + |\zeta'_1 - \xi_1| + |\pi_{0211}^\perp|)$, which can be proved like \[WW]\ Lemma 3.1.3b.

Next, fix $k \geq 1$; let us prove (42) for this $k$. Let $\zeta, \xi$ be sections in $\Gamma_{a_k}^{k+2}$, and assume that the other hypotheses of the lemma are satisfied. We will show that for every tuple $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{s, t\}^k$, there is a polynomial $P_{\alpha}$ so that the following inequality holds:

$$\|\nabla_\alpha \zeta\|_{c^0 H^1} \leq C_1 \left( \|\zeta\|_{H^{k+2}} + \|\nabla^{k-1} D \zeta\|_{c^0 H^1} + P_{\alpha} \left( \sum_{l=1}^{k-1} \|\nabla^l \zeta\|_{c^0 H^1} \right) \cdot \sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{c^0 H^1} \right).$$

We prove this by induction on $n_t(\alpha) := \#\{m \in [1, k] \mid \alpha_m = t\}$.

$n_t(\alpha) = 0$. If $\alpha = (s, \ldots, s)$, then since the special connection we have constructed preserves the boundary conditions of $\Gamma_{a_k}^{k+2}$, the desired inequality follows immediately from the $k = 0$ case of the current lemma: $\|\nabla^k \zeta\|_{c^0 H^1} \leq C_1 \|\nabla^k \zeta\|_{H^2}$.

$n_t(\alpha) \in [1, k]$. Let us prove the inductive step for some $n_t(\alpha) \in [1, k]$. Write $\alpha = (\alpha', \alpha_m = t, s, \ldots, s)$. Using the assumed bound on $j$, we estimate:

$$\|\nabla_\alpha \zeta\|_{c^0 H^1} = \|\nabla_{\alpha'} (C^{-1}(D \xi (\nabla^k \zeta) - (A - J(\xi)) \nabla^k \zeta))\|_{c^0 H^1}$$

$$\leq C_1 \left( \|\nabla_{\alpha'} D \xi \nabla^k \zeta\|_{c^0 H^1} + \|\nabla_{\alpha'} \nabla^k \zeta\|_{c^0 H^1} + \|\nabla_{\alpha'} (J(\xi) \nabla^k \zeta)\|_{c^0 H^1}\right)$$

$$+ \sum_{l=0}^{m-2} \|\nabla^k \zeta\|_{c^0 H^1} + \sum_{l=0}^{m-2} \|\nabla^l (J(\xi) \nabla^k \zeta)\|_{c^0 H^1}$$

$$+ \sum_{l=0}^{k-1} \|\nabla^l (J(\xi) \nabla^k \zeta)\|_{c^0 H^1}.$$

Let us bound separately the five terms in the last expression.

$\|\nabla_{\alpha'} D \xi (\nabla^k \zeta)\|_{c^0 H^1}$. We estimate:

$$\|\nabla_{\alpha'} D \xi (\nabla^k \zeta)\|_{c^0 H^1} \leq \|\nabla_{\alpha'} \nabla^k \zeta\|_{c^0 H^1} + \sum_{l=0}^{k-1} \|\nabla_{\alpha'} \nabla^l (\partial_s A \nabla^k \zeta + \partial_s C \nabla^k \zeta + \nabla^l \zeta)\|_{c^0 H^1}$$

$$+ \sum_{l=1}^{k-1} \|\nabla_{\alpha'} (C \nabla^l \nabla^k \zeta)\|_{c^0 H^1}.$$
By the inductive hypothesis, this term is bounded appropriately: 

$$\sum_{l=1}^{k-m} \|\nabla_{\alpha'} \nabla_s^l (J(\xi)) \nabla_s^{k-m-l+1} \zeta\|_{C^0H^1} \leq \sum_{\beta, \gamma \geq 0, \beta + \gamma = k-2} \|\nabla^{\beta+1}(J(\xi)) \nabla^{\gamma+1} \zeta\|_{C^0H^1} \leq P \left( \sum_{l=1}^{k-1} \|\nabla^l \zeta\|_{C^0H^1} \right)^2 \cdot \sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{C^0H^1}. $$

(In the last inequality we have used the Banach algebra property of $C^0H^1$.) Finally, the curvature of $\nabla$ is a tensor, so the term $\sum_{l=0}^{k-m-1} \|\nabla_{\alpha'}(C\nabla_s^l [\nabla_s, \nabla_t] \nabla_s^{k-m-l-1} \zeta)\|_{C^0H^1}$ can be bounded by a constant times $\sum_{l=0}^{k-2} \|\nabla^l \zeta\|_{C^0H^1}$.

$\|\nabla_{\alpha'} \nabla_s^{k-m+1} \zeta\|_{C^0H^1}$ By the inductive hypothesis, this term is bounded appropriately:

$$\|\nabla_{\alpha'} \nabla_s^{k-m+1} \zeta\|_{C^0H^1} \leq C_1 \left( \|\zeta\|_{H^{k+2}} + \|\nabla^{k-1} D \zeta\|_{C^0H^1} + P(\alpha', s, \ldots, s) \sum_{l=1}^{k-1} \|\nabla^l \zeta\|_{C^0H^1} \right) \times \sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{C^0H^1}.$$

$\|\nabla_{\alpha'}(J(\xi) \nabla_s^{k-m+1} \zeta)\|_{C^0H^1}$ To bound this term, it suffices to bound $\|J(\xi) \nabla_{\alpha'} \nabla_s^{k-m+1} \zeta\|_{C^0H^1}$ and $\|\nabla^{\beta+1}(J(\xi)) \nabla^{\gamma+1} \zeta\|_{C^0H^1}$ separately, where in the second term $\beta$ and $\gamma$ are nonnegative integers with $\beta + \gamma = k - 2$. The quantity $\|J(\xi) \nabla_{\alpha'} \nabla_s^{k-m+1} \zeta\|_{C^0H^1}$ can be bounded using the Banach algebra property of $C^0H^1$, the assumed $C^1$-bounds on $\xi$, and the inductive hypothesis. Using the Banach algebra property of $C^0H^1$, the quantity $\|\nabla^{\beta+1}(J(\xi)) \nabla^{\gamma+1} \zeta\|_{C^0H^1}$ can be bounded by $P \left( \sum_{l=1}^{k-1} \|\nabla^l \zeta\|_{C^0H^1} \right) \cdot \sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{C^0H^1}$.

$\sum_{l=0}^{k-2} \|\nabla_s^{k-m+l+1} \zeta\|_{C^0H^1}$ This term is already bounded appropriately.

$\sum_{l=0}^{m-2} \|\nabla_s^{k-m+l+1} \zeta\|_{C^0H^1}$ By the Banach algebra property of $C^0H^1$, this term is bounded by

$$P \left( \sum_{l=1}^{k-2} \|\nabla^l \zeta\|_{C^0H^1} \right) \cdot \sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{C^0H^1}. $$

This establishes the inductive step, so we have proven (42) for all $k \geq 0$.

**Step 2.** We prove (41) by induction on $k$.

As in Step 1, the $k = 0$ case follows from [WW1] Lemma 3.14. Next, say that (41) holds up to, but not including, some $k \geq 1$. By (42), we have:

$$\|\nabla^k \zeta\|_{C^0H^1} \leq C_1 \left( \|\zeta\|_{H^{k+2}} + \|\nabla^{k-1} D \zeta\|_{C^0H^1} + P \left( \sum_{l=1}^{k-1} \|\nabla^l \zeta\|_{C^0H^1} \right) \cdot \sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{C^0H^1} \right).$$

Replacing the sum $\sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{C^0H^1}$ appearing in the last term using the inductive hypothesis finishes the inductive step. 

We now turn to the proof of Lemma 3.8. Here is our strategy: in Lemma 3.10, we bound $\|\zeta\|_{H^1}$ in terms of $\|\zeta\|_{H^0}$ and $\|D \zeta\|_{H^0}$, for $\zeta$ supported in $Q_{\delta,r}$. In Lemma 3.11, we use Lemma 3.10
to bound $\|\eta \nabla^k \zeta\|_{H^1}$ in terms of $\|\zeta\|_{H^k}$ and $\|D\zeta\|_{H^k}$, where $\eta$ is supported in $Q_{02,\delta, r}$ and $\zeta$ has arbitrary support. Finally, we use Lemma 3.11 to prove Lemma 3.8.

**Lemma 3.10** (elliptic estimate for $k = 0$ and $\zeta$ compactly supported). There is a constant $\epsilon > 0$ and for every $C_0 > 0$, $k \geq 0$, and $r_1, r_2$ with $0 < r_1 < r_2 < \rho$ there is a constant $C_1$ such that the inequality

$$\|\nabla \zeta\|_{H^0(Q_{\delta, r})} \leq C_1 (\|D\zeta\|_{H^0(Q_{\delta, r})} + \|\zeta\|_{H^0(Q_{\delta, r})})$$

holds for any choice of $\delta \in (0, r_1/4]$, $r \in [r_1, r_2]$, a coherent collection $\mathbf{j}$ of complex structures on $\overline{Q}_{\delta, r}$ with $\|\mathbf{j} - i\|_{C^0} \leq \epsilon$ and $\|\mathbf{j} - i\|_{C^1} \leq C_0$, a coherent pair $\mathbf{J}$ of almost complex structures on $\overline{Q}_{\delta, r}$ which are contained in a $C^1$-ball of radius $C_0$ and which induce by (4) metrics whose pairwise constants of equivalence are bounded above by $C_0$, and sections $\zeta, \xi \in \Gamma^2_{u_\delta}(Q_{\delta, r})$ with $\|\xi\|_{C^0} \leq \epsilon$, $\|\xi\|_{C^1} \leq C_0$, and supp $\zeta_{02}$, supp $\zeta_{\delta, r}$ compact subsets of $Q_{02, \delta, r}$, $\overline{Q}_{\delta, r}$.

**Proof.** Throughout this proof, $C_1$ will denote a $\delta$-independent constant that may change from line to line, and $A = (A_{02, A}, C = (C_{02, \hat{C}})$ will be the endomorphisms of $u_{02, \delta}^* TM_2$ and $\hat{u}_{\delta}^* TM_2$ defined in (37).

We begin by fixing convenient metrics on $M_{02}$ and $M_{0211}$ that will be used for the pointwise norms in the definition of the Sobolev norms. Via (1), $\mathbf{J}$ induces fiberwise metrics $g_{02, \hat{g}}$ on $u_{02, \delta}^* TM_2$ and $\hat{u}_{\delta}^* TM_{0211}$. In this proof, however, we will use the pullback metrics $g_{\xi} = (g_{02, \xi}, \hat{g}_{\xi})$ of $g_{02, \hat{g}}$ under $d_{u_{02, \delta}}(\xi_0)$, $d_{\hat{u}_{\delta}}(\xi)$; note that $g_{\xi}$ is $J(\xi)$-invariant. If we pick $\epsilon > 0$ to be sufficiently small, then $d_{u_{\delta}}(\xi)$ is $C^0$-close to the identity, and hence the induced norms $\|\cdot\|_{H^k}$ on $\Gamma^k_{u_{\delta}}(Q_{\delta, r})$ are equivalent to the standard norms $\|\cdot\|_{H^k} = \|\cdot\|_{0, H^k}$.

With these metrics we calculate for $\zeta \in \Gamma^2_{u_{\delta}}$ compactly supported and $\xi \in \Gamma^2_{u_{\delta}}$ satisfying $\|\xi\|_{C^0(Q_{\delta, r})} \leq \epsilon$ and $\|\nabla \xi\|_{C^0(Q_{\delta, r})} \leq C_0$:

$$\|D\zeta\|_{H^0(Q_{\delta, r})}^2 = \int_{Q_{\delta, r}} (\|\nabla \zeta\|_{\xi}^2 + |A \nabla \zeta|_{\xi}^2) + 2g_{\xi}(A \nabla \zeta, C \nabla \zeta) + |C \nabla \zeta|_{\xi}^2) \,ds \,dt$$

(44)

$$+ \int_{Q_{\delta, r}} (g_{\xi}(C \nabla \zeta, J(\xi) \nabla \zeta) - g_{\xi}(C \nabla \zeta, J(\xi) \nabla \zeta)) \,ds \,dt.$$

Let us estimate the two integrals on the right-hand side separately. We begin with the first integral:

$$\int_{Q_{\delta, r}} (\|\nabla \zeta\|_{\xi}^2 + |A \nabla \zeta|_{\xi}^2) + 2g_{\xi}(A \nabla \zeta, C \nabla \zeta) + |C \nabla \zeta|_{\xi}^2) \,ds \,dt$$

(45)

$$\geq \int_{Q_{\delta, r}} (\|\nabla \zeta\|_{\xi}^2 - 3|A \nabla \zeta|_{\xi}^2 + \frac{3}{4}|C \nabla \zeta|_{\xi}^2) \,ds \,dt \geq \frac{5}{8} \|\nabla \zeta\|_{H^0(Q_{\delta, r})}^2,$$

where the last inequality follows from the hypothesis $\|\mathbf{j} - i\| \leq \epsilon$ as long as $\epsilon$ is chosen small enough.

To bound the second integral on the right-hand side of (44), we first derive a convenient formula for its integrand:

$$g_{\xi}(C \nabla \zeta, J(\xi) \nabla \zeta) - g_{\xi}(C \nabla \zeta, J(\xi) \nabla \zeta)$$

(46)

$$= (\partial_t g_{\xi}(C \zeta, J(\xi) \nabla \zeta) - (\nabla_s g_{\xi})(C \zeta, J(\xi) \nabla \zeta) - g_{\xi}((\nabla_s C) \zeta, J(\xi) \nabla \zeta) - g_{\xi}( C \zeta, J(\xi) \nabla \zeta)$$

$$- g_{\xi}(C \zeta, \nabla_s (J(\xi)) \nabla \zeta) - g_{\xi}(C \zeta, J(\xi) \nabla \zeta))$$

$$- (\partial_t (g_{\xi}(C \zeta, J(\xi) \nabla \zeta)) - \nabla_t g_{\xi}(C \zeta, J(\xi) \nabla \zeta) - g_{\xi}((\nabla_t C) \zeta, J(\xi) \nabla \zeta)$$

$$- g_{\xi}(C \zeta, \nabla_t (J(\xi)) \nabla \zeta) + g_{\xi}(C \zeta, J(\xi) \nabla \zeta) - g_{\xi}(C \zeta, J(\xi) \nabla \zeta))$$

$$= (\partial_s (g_{\xi}(C \zeta, J(\xi) \nabla \zeta)) - \partial_t (g_{\xi}(C \zeta, J(\xi) \nabla \zeta)) - \nabla_s g_{\xi}(C \zeta, J(\xi) \nabla \zeta) - \nabla_t g_{\xi}(C \zeta, J(\xi) \nabla \zeta)$$

$$- g_{\xi}((\nabla_s C) \zeta, J(\xi) \nabla \zeta) - g_{\xi}(C \zeta, \nabla_s (J(\xi)) \nabla \zeta) - \nabla_t (J(\xi)) \nabla \zeta)$$

$$- g_{\xi}(C \zeta, J(\xi) \nabla \zeta).$$

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We can now use Green’s formula and the assumed $C^1$-bounds on $j$, $J$, and $\xi$ to bound the second integral on the right-hand side of (45):

$$
\int_{Q_{r,0}} (g_\xi(C\nabla_s \zeta, J(\xi)\nabla_t \zeta) - g_\xi(C\nabla_t \zeta, J(\xi)\nabla_s \zeta)) \, ds dt
$$

(47)

$$
\int_{(-r, r) \times (0)} g_\xi(C\zeta, J(\xi)\nabla_s \zeta) \, ds dt - \int_{(-r, r) \times \{0\}} g_\xi(C\zeta, J(\xi)\nabla_s \zeta) \, ds dt
$$

$$
- \int_{Q_{r,0}} (\nabla_s g_\xi)(C\zeta, J(\xi)\nabla_t \zeta) - (\nabla_t g_\xi)(C\zeta, J(\xi)\nabla_s \zeta) \, ds dt
$$

$$
- \int_{Q_{r,0}} g_\xi((\nabla_s C)\zeta, J(\xi)\nabla_t \zeta) - g_\xi((\nabla_t C)\zeta, J(\xi)\nabla_s \zeta) \, ds dt
$$

$$
- \int_{Q_{r,0}} g_\xi(C\zeta, \nabla_s J(\xi)) \nabla_t \zeta - \nabla_t (J(\xi))\nabla_s \zeta) \, ds dt - \int_{Q_{r,0}} g_\xi(C\zeta, J(\xi)|\nabla_s, \nabla_t|) \zeta) \, ds dt
$$

$$
\geq \int_{Q_{r,0}} C_1 |\xi|_\xi (|\xi| + |\nabla_\xi|) \, ds dt \geq -\frac{1}{2} ||\nabla_\xi||_{\xi, H^0}^2 - C_1 ||\xi||_{\xi, H^0}^2,
$$

where in the first inequality we have eliminated the integrals over the $t = 0$ and $t = \delta$ boundary via the coherence condition on $j$ and the fact that $g_\xi(\zeta, J(\xi)|\nabla_s, \nabla_t|) = 0$ and $\nabla_\xi|_{t=\delta} \zeta$ vanish. Indeed, $\nabla_\xi|_{t=\delta} \zeta$ vanishes by the Lagrangian boundary condition:

$$
\langle \zeta, J(\xi)\nabla_s \zeta \rangle_{t=\delta} = \omega_{0211}(\text{de}_{\zeta}(\xi)\zeta, \text{de}_{\zeta}(\xi))\text{de}_{\zeta}(\xi)\nabla_\xi|_{t=\delta} \zeta = 0,
$$

where we crucially used the fact that both the exponential map $\text{de}_{\zeta}(\xi)$ and the connection $\nabla_\xi$ preserve $T(L_{01} \times L_{12})^T$. The boundary term $g_\xi(\zeta, J(\xi)|\nabla_s, \nabla_t|) = 0$ vanishes due to the facts that $\text{de}_{\zeta}(\xi)$ preserves $T\Delta_{M_{02}} \times T\Delta_{M_1}$, $\nabla$ satisfies $\nabla_{02, \delta}(\xi) = 0 = p \circ \nabla_\xi|_{t=0}$ for $p: M_{02} \rightarrow M_0$, and $\omega_{02, \omega_{02}}$ satisfy $\omega_{02, \omega_{02}}(T\Delta_{M_{02}} \times T\Delta_{M_1}) = -\omega_2 \omega_{02}$:

$$
\langle \zeta, J(\xi)\nabla_s \zeta \rangle_{t=0} = -\omega_{02}(\text{de}_{\zeta}(\xi)\text{de}_{\zeta}(\xi)\text{de}_{\zeta}(\xi)\nabla_\xi|_{t=0} \zeta = 0
$$

$$
= -\omega_{02}(\text{de}_{\zeta}(\xi)\text{de}_{\zeta}(\xi)\text{de}_{\zeta}(\xi)\nabla_\xi|_{t=0} \zeta = 0
$$

Combining (44), (45), and (47) yields the following inequality:

$$
\|D\xi\xi_{\xi, H^0} \geq \frac{1}{\pi} \|\nabla_\xi\xi_{\xi, H^0}^2 - C_1 \|\xi\xi_{\xi, H^0}^2.
$$

Adding $C_1 \|\xi\xi_{\xi, H^0}^2$ to both sides of this inequality and taking the square root of the result, we obtain:

$$
\|\nabla_\xi\xi_{\xi, H^0} \leq C_1 (\|D\xi\xi_{\xi, H^0}^2 + \|\xi\xi_{\xi, H^0}^2)^1/2 \leq C_1 (\|D\xi\xi_{\xi, H^0} + \|\xi\xi_{\xi, H^0}^2).
$$

In this estimate, we may replace $\| - \xi, H^0$ with $\| - \xi, H^0$ by using the $\delta$-independent uniform equivalence of these norms, which yields (43).

\[\square\]

**Lemma 3.11** (elliptic estimate for $k \geq 0$). There is a constant $\epsilon > 0$ and for every $C_0 > 0$, $k \geq 0$, and $0 < r_1 < r_2 < \rho$ there is a constant $C_1$ such that the inequality

$$
\|\eta \nabla^k \xi\|_{H^k(Q_{r_2, r_2})} \leq C_1 (\|D\xi\xi_{H^k(Q_{r_2, r_2})} + \|\xi\|_{H^k(Q_{r_2, r_2})})
$$

(48)

holds for any choice of $\delta \in (0, r_1/4)$, $r \in (r_1, r_2)$, a coherent collection $j$ of complex structures on $Q_{r_2, r_2}$ with $\|j - i\|_{C^0} \leq \epsilon$ and $\|j - i\|_{\max(k, 1)} \leq C_0$, a pair $J$ of compatible almost complex structures on $Q_{r_2, r_2}$ which are contained in a $C^\max(k, 1)$-ball of radius $C_0$ and which induce $\|j - i\|_{C^0}$ metrics whose pairwise constants of equivalence are bounded above by $C_0$, a pair of sections $\zeta \in T^{k+2}(Q_{r_2, r_2})$ with $\|\zeta\xi_{C^0} \leq \epsilon$, $\|\zeta\|_{C^1} \leq C_0$, and $\|\zeta\|_{H^k(Q_{r_2, r_2})} \leq C_0$, and a smooth function $\eta: Q_{r_2, r_2} \rightarrow \mathbb{R}$ with $\|\eta\|_{C^{k+1}} \leq C_0$ and $\supp \eta \subset Q_{r_2, r_2}$. 28
Proof. Throughout this proof, $C_1$ will denote a $\delta$-independent constant and $P$ will denote a $\delta$-independent polynomial, and both may change from line to line.

We break down the proof into several steps: in Step 1, we establish (48), but with an extra term on the right-hand side. In Step 2, we bound this extra term, using different arguments in the $k \neq 3$ and $k = 3$ cases. In Step 3, we establish (48).

**Step 1a.** We prove the following inequality:

$$
\|\eta \nabla^k \zeta\|_{H^1} \leq C_1 \left( \|D \zeta\|_{H^k} + \|\zeta\|_{H^k} + \sum_{\beta \geq 1, \gamma \geq 0, \beta + \gamma = k} \|\eta \nabla^\beta (J(\zeta)) \nabla^\gamma \nabla_s \zeta\|_{H^0} \right)
$$

for $\alpha = \underbrace{s, \ldots, s}_k$.

Since the connection $\nabla$ preserves the linearized boundary conditions and $\eta$ is supported in $Q_{02,\delta,r}$, we may estimate $\|\eta \nabla^k \zeta\|_{H^1}$ using Lemma 3.10:

$$
\|\eta \nabla^k \zeta\|_{H^1} \leq C_1(\|D \zeta(\eta \nabla^k \zeta)\|_{H^0} + \|\nabla^k \zeta\|_{H^0})
$$

$$
= C_1 \left( \|\eta \nabla^k \zeta\|_{H^0} + \left\|\eta \nabla^k D \zeta - \sum_{l=1}^k \binom{k}{l} \eta((\partial^l_s A \nabla^k \zeta + \partial^l_s C \nabla^k \zeta) - \sum_{l=1}^k C \eta \nabla^l \zeta[D_s, \nabla_l] \nabla^k \zeta - \left( \partial_s \eta (A - J(\zeta)) + C \partial_t \eta \right) \nabla^k \zeta \right\|_{H^0} \right)
$$

$$
\leq C_1 \left( \|D \zeta\|_{H^k} + \|\zeta\|_{H^k} + \sum_{\beta \geq 1, \gamma \geq 0, \beta + \gamma = k} \|\eta \nabla^\beta (J(\zeta)) \nabla^\gamma \nabla_s \zeta\|_{H^0} \right).
$$

**Step 1b.** We prove (49) for a general multiindex $\alpha$ of length $k$.

We establish Step 1b by induction on $n_t(\alpha) := \{ \# m \in [1, k] \mid \alpha_m = t \}$. Step 1a is the base case for this induction. For the inductive step, fix $\alpha$ with $n_t(\alpha) \geq 1$, and write $\alpha = (\alpha', \alpha_m = t, s, \ldots, s)_{k-m}$.

We estimate:

$$
\|\eta \nabla^\alpha \zeta\|_{H^1} = \|\eta \nabla^{\alpha'} (C^{-1}(D \zeta - (A - J(\zeta)) \nabla^{k-m+1} \zeta))\|_{H^1}
$$

$$
\leq C_1(\|\zeta\|_{H^k} + \|\eta \nabla^\alpha' D \zeta\|_{H^1} + \|\eta \nabla^\alpha' \nabla^{k-m+1} \zeta\|_{H^1} + \|\eta \nabla^\alpha' (J(\zeta) \nabla^{k-m+1} \zeta)\|_{H^1})
$$

$$
= C_1 \left( \|\zeta\|_{H^k} + \left\|\eta \nabla^{\alpha'} \left( \nabla^{k-m} D \zeta - \sum_{l=1}^{k-m} \binom{k-m}{l} (\partial^l_s A \nabla^{k-m-l+1} \zeta + \partial^l_s C \nabla^{k-m-l} \zeta) \right) \right\|_{H^1}
$$

$$
+ \left\|\eta \nabla^{\alpha'} \nabla^k \zeta\|_{H^1} + \left\|\eta \nabla^\alpha' (J(\zeta) \nabla^{k-m+1} \zeta)\|_{H^1} \right\|_{H^1}
$$

$$
\leq C_1 \left( \|D \zeta\|_{H^k} + \|\zeta\|_{H^k} + \sum_{\beta \geq 1, \gamma \geq 0, \beta + \gamma = k} \|\eta \nabla^\beta (J(\zeta)) \nabla^\gamma \nabla_s \zeta\|_{H^0} \right),
$$

where in the last inequality we have used the inductive hypothesis to bound $\|\eta \nabla^\alpha' \nabla^{k-m+1} \zeta\|_{H^1}$. 


Step 2a. In the \( k \neq 3 \) case, we prove the following inequality:

\[
\sum_{\beta \geq 1, \gamma \geq 0, \beta + \gamma = k} \| \eta \nabla^\beta (J( \zeta )) \nabla^\gamma \nabla s \zeta \|_{H^0} \leq C_1 \| \zeta \|_{H^k}.
\]

It follows from the assumption \( k \neq 3 \) that if \( \beta, \gamma \geq 1 \) satisfy \( \beta + \gamma = k + 1 \), then \( \min\{\beta, \gamma\} \leq \max\{k - 2, 1\} \). Furthermore, the assumption \( \| \zeta \|_{H^k} \leq C_0 \) implies the inequality \( \| \zeta \|_{C^{k-2}} \leq C_1 \) by the embedding of \( H^1 \hookrightarrow C^0 \) for one-dimensional domains whose lengths are bounded away from zero. This, along with the assumed \( C^1 \)-bound on \( \zeta \), yields (50) in the \( k \neq 3 \) case.

Step 2b. In the \( k = 3 \) case, we prove the following inequality:

\[
\sum_{\beta \geq 1, \gamma \geq 0, \beta + \gamma = 3} \| \eta \nabla^\beta (J( \zeta )) \nabla^\gamma \nabla s \zeta \|_{H^0} \leq C_1 (\| D_\zeta \zeta \|_{H^3} + \| \zeta \|_{H^3} + \delta^{1/2} \| \eta \nabla^3 \zeta \|_{H^3}).
\]

The assumed \( C^1 \)-bound on \( \zeta \) implies that the only term in the left-hand side of (51) that is not immediately bounded by \( C_1 \| \zeta \|_{H^3} \) is \( \| \eta \nabla^2 (J( \zeta )) \nabla s \zeta \|_{H^0} \).

Choose smooth maps

\[ S, U : \tilde{u}^* T M_{0211} \to \tilde{u}^* \text{ hom}( (T M_{0211})^{\otimes 2}, T M_{0211} ) \]
\[ T : \tilde{u}^* T M_{0211} \to \tilde{u}^* \text{ hom}( (T M_{0211})^{\otimes 3}, T M_{0211} ) \]
\[ V : \tilde{u}^* T M_{0211} \to u^* \text{ hom}( T M_{0211}, T M_{0211} ) \]

so that the formula

\[
\hat{\nabla}^2 (\hat{J}( \hat{\zeta} )) = S( \hat{\zeta} ) (\hat{\nabla}^2 \hat{\zeta} ) + T( \hat{\zeta} ) (\hat{\nabla} \hat{\zeta} , \hat{\nabla} \hat{\zeta} ) + U( \hat{\zeta} ) (\hat{\nabla} \hat{\zeta} ) + V( \hat{\zeta} )
\]

holds, where the maps \( S, T, U, V \) preserve fibers but may not respect their linear structure. Since \( J \) is bounded in \( C^3 \), \( S, T, U, V \) must be bounded in \( C^1 \). We may now use (52) to bound the hat-part of \( \| \eta \nabla^2 (J( \zeta )) \nabla s \zeta \|_{H^0} \):

\[
\| \eta \hat{\nabla}^2 (\hat{J}( \hat{\zeta} )) \hat{\nabla} \hat{\zeta} \|_{H^0} \leq C_1 (\| \hat{\zeta} \|_{H^2} + \| S( \hat{\zeta} ) (\hat{\nabla}^2 \hat{\zeta} ) \hat{\nabla} \hat{\zeta} \|_{H^0})
\]

\[
= C_1 (\| \hat{\zeta} \|_{H^2} + \| \hat{\nabla} s (S( \hat{\zeta} )) (\eta \nabla^2 \hat{\zeta} ) \hat{\nabla} \hat{\zeta} - \hat{\nabla} s (S( \hat{\zeta} )) (\eta \nabla^2 \hat{\zeta} ) \hat{\nabla} \hat{\zeta} + S( \hat{\zeta} ) (\eta \nabla^2 \hat{\zeta} ) \hat{\nabla} \hat{\zeta})
\]

\[
\leq C_1 (\| \hat{\zeta} \|_{H^3} + \delta^{1/2} \| S( \hat{\zeta} ) (\eta \nabla^2 \hat{\zeta} ) \hat{\nabla} \hat{\zeta} \|_{C^0 H^1})
\]

\[
\leq C_1 (\| \hat{\zeta} \|_{H^3} + \delta^{1/2} \| S( \hat{\zeta} ) (\eta \nabla^2 \hat{\zeta} ) \|_{C^0 H^1} \| \hat{\nabla} \hat{\zeta} \|_{C^0 H^1}),
\]

where in the last inequality we have used the \( \delta \)-independent Banach algebra property of \( C^0 H^1 \).

By Lemma 3.9, \( \| \hat{\nabla} \hat{\zeta} \|_{C^0 H^1} \) is bounded by \( C_1 (\| \hat{\nabla} \hat{\zeta} \|_{H^2} + \| \zeta \|_{H^3}) \) and therefore by \( C_1 \| \hat{\zeta} \|_{H^3} \); on the other hand, the \( C^1 \)-bound on \( S \) and the \( C^1 \)-bound on \( \zeta \) implies the inequality \( \| S( \zeta ) (\eta \nabla^2 \hat{\zeta} ) \|_{C^0 H^1} \leq C_1 \| \eta \nabla^2 \hat{\zeta} \|_{C^0 H^1} \). Substituting these inequalities into (53), we obtain:

\[
\| \eta \nabla^2 (J( \zeta )) \nabla s \zeta \|_{H^0} \leq C_1 (\| \zeta \|_{H^3} + \delta^{1/2} \| \zeta \|_{H^3} \| \eta \nabla^2 \zeta \|_{C^0 H^1}) \leq C_1 (\| \zeta \|_{H^3} + \delta^{1/2} \| \eta \nabla^2 \zeta \|_{C^0 H^1}).
\]

Next, we use Lemma 3.9 to bound \( \| \eta \nabla^2 \zeta \|_{C^0 H^1} \):

\[
\| \eta \nabla^2 \zeta \|_{C^0 H^1} \leq C_1 (\| \zeta \|_{H^3} + \| \nabla^2 (\zeta) \|_{C^0 H^1})
\]

\[
\leq C_1 (\| \zeta \|_{H^3} + \| \nabla D \zeta (\eta \zeta) \|_{C^0 H^1} + \| \zeta \|_{H^3}) + P(\| \varnothing^2 \zeta \|_{C^0 H^1}(\| \zeta \|_{H^3} + \| \zeta \|_{H^3} + \| \nabla^3 \zeta \|_{H^1} + \| \varnothing \|_{H^3} + \| \nabla^3 \zeta \|_{H^1} + \| \varnothing \|_{H^3} + \| \nabla^3 \zeta \|_{H^1})).
\]
where the last inequality follows from the assumed bound on $\|\zeta\|_{\tilde{H}^3}$. Substituting (55) into (54), we obtain:

\begin{equation}
\|\eta \tilde{\nabla}^2 (J(\zeta)) \tilde{\nabla} \tilde{s} \zeta\|_{H^0} \leq C_1(\|\zeta\|_{H^3} + \delta^{1/2}(\|D_\zeta \zeta\|_{\tilde{H}^3} + \|\zeta\|_{\tilde{H}^3} + \|\eta \nabla^3 \zeta\|_{H^1}))
\end{equation}

$$\leq C_1(\|D_\zeta \zeta\|_{\tilde{H}^3} + \|\zeta\|_{\tilde{H}^3} + \delta^{1/2}\|\eta \nabla^3 \zeta\|_{H^1}).$$

To bound the 02-part of $\|\eta \nabla^2 (J(\zeta)) \nabla \nabla s \zeta\|_{H^0}$, we use the theorem that the domains $Q_{02,\delta,r}$ satisfy a uniform cone condition:

\begin{equation}
\|\eta \nabla^2_{02}(\zeta_0(\zeta_{02}))(\nabla \nabla s \zeta_{02})\|_{H^0} \leq C_1(\|\nabla^2_{02}(\zeta_0(\zeta_{02}))(L^4)\|\nabla^2_{02}(\zeta_{02})\|_{L^4})
\end{equation}

$$\leq C_1((1 + \|\zeta\|_{H^3})\|\zeta\|_{H^3}),$$

where the second inequality follows from the Sobolev embedding $H^1 \hookrightarrow L^4$ for two-dimensional domains satisfying a cone condition. Combining (56) and (57) and using the assumed bound on $\|\zeta\|_{\tilde{H}^3}$ yields the desired bound:

$$\|\eta \nabla^2 (J(\zeta)) \nabla \nabla s \zeta\|_{H^0} \leq C_1(\|D_\zeta \zeta\|_{\tilde{H}^3} + \|\zeta\|_{\tilde{H}^3} + \delta^{1/2}\|\eta \nabla^3 \zeta\|_{H^1}).$$

**Step 3.** We prove Lemma 3.11.

The $k \neq 3$ case of Lemma 3.11 is an immediate consequence of Steps 1b and 2a. Toward the $k = 3$ case of Lemma 3.11, let us show that there exists $\delta_0 \in (0, r_1]$ such that (48) holds for $\delta \in (0, \delta_0]$. Combining (49) and (51) yields the following inequality:

\begin{equation}
\|\eta \nabla^3 \zeta\|_{H^1} \leq C_1(\|D_\zeta \zeta\|_{\tilde{H}^3} + \|\zeta\|_{\tilde{H}^3} + \delta^{1/2}\|\eta \nabla^3 \zeta\|_{H^1}).
\end{equation}

If we set $\delta_0 := \min\{(2C_1)^{-1}, r_1\}$, where $C_1$ is the constant appearing in (58), then (58) yields the uniform inequality $\|\eta \nabla^3 \zeta\|_{H^1} \leq C_1(\|D_\zeta \zeta\|_{\tilde{H}^3} + \|\zeta\|_{\tilde{H}^3})$ for all $\delta \in (0, \delta_0]$. It remains to establish the $k = 3$ case of (48) for $\delta \in [\delta_0, r_1]$. To do so, we begin by bounding $\|\nabla^2 (J(\zeta)) \nabla^2 \zeta\|_{H^0}$, using the fact that the domains $Q_{\delta,r}$ satisfy a uniform cone condition for $\delta \in [\delta_0, r_1/4]$:

\begin{equation}
\|\eta \nabla^2 (J(\zeta)) \nabla^2 \zeta\|_{H^0} \leq C_1(\|\nabla^2 (J(\zeta))\|_{L^4} \|\nabla^2 \zeta\|_{L^4}) \leq C_1((1 + \|\zeta\|_{H^2,4})\|\zeta\|_{H^2,4})
\end{equation}

$$\leq C_1((1 + \|\zeta\|_{H^3})\|\zeta\|_{H^3} \leq C_1\|\zeta\|_{H^3}.$$

Substituting (59) into (49) yields the $k = 3$ case of (48) for $\delta \in [\delta_0, r_1/4]$: $\|\eta \nabla^3 \zeta\|_{H^1} \leq C_1(\|D_\zeta \zeta\|_{H^3} + \|\zeta\|_{H^3} + \sum_{\beta \geq 1, \gamma \geq 0, \beta + \gamma = 3} \|\eta \nabla^3 (J(\zeta)) \nabla^2 \nabla s \zeta\|_{H^0}) \leq C_1(\|D_\zeta \zeta\|_{H^3} + \|\zeta\|_{H^3}).$

\[\square\]

**Proof of Lemma 3.8** Lemma 3.8 follows immediately from Lemmata 3.9 and 3.11. Indeed, choose $\eta: \mathcal{Q}_{02,\delta,r_2} \to \mathbb{R}$ to be a smooth function with $\eta|_{\mathcal{Q}_{02,\delta,r_1}} \equiv 1$ and supp $\eta \subset Q_{02,\delta,r_2}$. $C_1$ and $P$ will denote a $\delta$-independent constant and a $\delta$-independent polynomial that may change from line to line. Lemma 3.11 yields a bound on $\|\zeta\|_{H^{k+1}(Q_{\delta,r_1})}$:

\begin{equation}
\|\zeta\|_{H^{k+1}(Q_{\delta,r_1})} \leq \|\eta \zeta\|_{H^{k+1}(Q_{\delta,r_2})} \leq C_1(\|\zeta\|_{\tilde{H}^k(Q_{\delta,r_2})} + \|D_\zeta \zeta\|_{\tilde{H}^k(Q_{\delta,r_2})}).
\end{equation}
Lemma 3.9 yields a bound on \( \sum_{l=0}^{k-1} \| \nabla^l \zeta \|_{C^0 H^1(\mathcal{Q}_{\delta, r_1})} \):

\[
(61) \quad \sum_{l=0}^{k-1} \| \nabla^l \zeta \|_{C^0 H^1(\mathcal{Q}_{\delta, r_1})} \leq C_1 \left( \| \zeta \|_{H^{k+1}(\mathcal{Q}_{\delta, r_1})} + \| D \zeta \|_{H^k(\mathcal{Q}_{\delta, r_1})} \right) + P \left( \| \zeta \|_{H^{k+1}(\mathcal{Q}_{\delta, r_1})} \right) \cdot \left( \| \zeta \|_{H^k(\mathcal{Q}_{\delta, r_1})} + \| D \zeta \|_{H^{k-1}(\mathcal{Q}_{\delta, r_1})} \right)
\]

where in the second inequality we have used the assumed bound on \( \| \zeta \|_{H^{k+1}(\mathcal{Q}_{\delta, r_1})} \). Combining (60) and (61) yields \( \| \zeta \|_{H^{k+1}(\mathcal{Q}_{\delta, r_1})} \leq C_1 \left( \| D \zeta \|_{H^k(\mathcal{Q}_{\delta, r_2})} + \| \zeta \|_{H^k(\mathcal{Q}_{\delta, r_2})} \right) \), which can be used to inductively prove the desired inequality (40).

We will not use the following proposition in this paper. However, it will be used in [B] to show that the linearized Cauchy–Riemann operator defines a Fredholm section.

**Proposition 3.12** (linear elliptic estimate for \( k = 2 \)). There is a constant \( \epsilon > 0 \) and for every \( C_0 > 0, k \geq 0, \) and \( 0 < r_1 < r_2 < \rho \) there is a constant \( C_1 \) such that the inequality

\[
\| \zeta \|_{H^{k+1}(\mathcal{Q}_{\delta, r_1})} \leq C_1 \left( \| D \zeta \|_{H^k(\mathcal{Q}_{\delta, r_2})} + \| \zeta \|_{H^k(\mathcal{Q}_{\delta, r_2})} \right)
\]

holds for any choice of \( \delta \in (0, r_1/4) \), a coherent collection \( j \) of complex structures on \( \mathcal{Q}_{\delta, \rho} \), with \( \| j - i \|_{C^0} \leq \epsilon \) and \( \| j - i \|_{C^2} \leq C_0 \), a pair \( J \) of compatible almost complex structures on \( \mathcal{Q}_{\delta, \rho} \) which are contained in a \( C^2 \)-ball of radius \( C_0 \) and which induce by (4) whose pairwise constants of equivalence are bounded above by \( C_0 \), and two pairs of sections \( \zeta, \zeta \in \Gamma_{a+2}(\mathcal{Q}_{\delta, r_2}) \) with \( \| \zeta \|_{C^0} \leq \epsilon \) and \( \| \zeta \|_{C^1} \leq C_0 \).

The proof is an easier version of the proof of Lemma 3.8

**Appendix A. Removal of singularity for cleanly intersecting Lagrangians**

In this appendix, we sketch a proof of removal of singularity for a holomorphic curve satisfying a generalized Lagrangian boundary condition in an immersed Lagrangian with locally-clean self-intersection. We emphasize that this is not a new result, see e.g. [AB, CEL, H, IS, Sc]. We have included the following proposition in this paper because our methods allow us to give a short proof.

This removal of singularity will be stated for maps \( u \) with Lagrangian boundary conditions lifting to paths \( \gamma, \gamma' \):

\[
(63) \quad u : (B(0, 1) \cap \mathbb{H}) \setminus \{0\} \to M, \quad \gamma : (0, 1) \to L, \quad \gamma' : (0, 1) \to L',
\]

\[
\varphi' \gamma' (s') = u(s', 0), \quad \varphi \gamma (s) = u(s, 0) \quad \forall s' \in (-1, 0), \ s \in (0, 1),
\]

\[
\partial_s u + J(s, t, u) \partial_t u = 0, \quad E(u) := \int u^* \omega < \infty,
\]

where \( (M, \omega) \) is a closed symplectic manifold, \( \varphi : L \to M \) and \( \varphi' : L' \to M' \) are Lagrangian immersions with \( L, L' \) closed, and \( J \) is an almost complex structure \( J : B(0, 1) \cap \mathbb{H} \to J(M, \omega) \). We will assume that \( \varphi(L), \varphi'(L') \) intersect locally cleanly, which means that there are finite covers \( L = \bigcup_{i=1}^k U_i, L' = \bigcup_{j=1}^l U'_j \) such that \( \varphi \) resp. \( \varphi' \) restrict to an embedding on each \( U_i \) resp. \( U'_j \), and \( \varphi(U_i), \varphi'(U'_j) \) intersect cleanly for all \( i, j \).

**Proposition A.1.** If \( u, \gamma, \gamma' \) satisfy (63), then \( u \) extends continuously to 0.

**Sketch proof of Proposition A.1.** The first part of the proof of [AH, Theorem 7.3.1] yields a uniform gradient bound on \( u \) in cylindrical coordinates near the puncture. We must make a minor modification due to the fact that the Lagrangians defining our boundary conditions are immersed, not
embedded: Recall that the uniform gradient bound in cylindrical coordinates is established in [AH] by assuming that there is a sequence \((s_k, t_k)\) \(\subset (-\infty, 0] \times [0, \frac{1}{2}]\) so that \(\lim_{k \to \infty} |du(s_k, t_k)| = \infty\), which necessarily has \(s_k \to -\infty\). Rescaling at the points \((s_k, t_k)\) yields a sequence of maps that converges in \(C^\infty_{\text{loc}}\) to a nonconstant map on either \(\mathbb{R}^2\) or \(\pm \mathbb{H}\), which contradicts the finiteness of the energy. To adapt this proof to our situation, let \(\delta\) be a Lebesgue number for \(L = \bigcup_{i=1}^k U_i\) and \(L' = \bigcup_{i=1}^l U'_i\). That is, if \(A\) is a subset of \(L\) (resp. of \(L'\)) with \(\text{diam} A \leq \delta\), then \(A \subset U_i\) (resp. \(A \subset U'_i\)) for some \(i\). Now rescale at the points \((s_k, t_k)\) as in [AH], but restrict the resulting maps to the intersection of \(B(0, \frac{1}{4}\delta)\) with their domain. The gradient bound on these rescaled maps and our choice of \(\delta\) allows us to pass to a subsequence so that for some \(i, j\), all the rescaled maps have boundary values in \(\pi(U_i)\) or \(\pi'(U'_j)\). A further subsequence converges in \(C^\infty_{\text{loc}}\), so we get a contradiction and therefore a uniform bound on \(|\nabla u|\) in cylindrical coordinates.

The analogue of Lemma 2.3 holds in this setting; the proof is the same as for Lemma 2.3 but simpler. As in the first paragraph, some care must be taken with the immersed Lagrangians. The analogue of Lemma 2.9 holds in this setting, though the proof must be modified. Specifically, the domains \(U_0, U_1, U_2, U_3\) used in the proof of that lemma must be replaced by the domain \(\overline{B}(0, 1) \cap \mathbb{H}\).

A slight modification of the proof of Theorem 2.2 establishes Proposition A.1. □

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