EXPANDING SPEED OF THE HABITAT FOR A SPECIES IN AN ADVECTIVE ENVIRONMENT

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Abstract. Recently, Gu et al. [7, 8] studied a reaction-diffusion-advection equation $u_t = u_{xx} - \beta u_x + f(u)$ in $(g(t), h(t))$, where $g(t)$ and $h(t)$ are two free boundaries satisfying Stefan conditions, $f(u)$ is a Fisher-KPP type of non-linearity. When $\beta \in [0, c_0)$, where $c_0 := 2\sqrt{f'(0)}$, they found that for a spreading solution $(u, g, h)$, $h(t)/t \to c^*_r(\beta)$ and $g(t)/t \to c^*_l(\beta)$ as $t \to \infty$, and $c^*_r(\beta) > c^*_l(0) > -c^*_l(\beta) > 0$. In this paper we study the expanding speed $C^*_*(\beta) := c^*_r(\beta) - c^*_l(\beta)$ of the habitat $(g(t), h(t))$, and show that $C^*_*(\beta)$ is strictly increasing in $\beta \in [0, c_0)$. When $\beta \in [c_0, \beta^*)$ for some $\beta^* > c_0$, [8] also found a virtual spreading phenomena: $h(t)/t \to c^*_r(\beta)$ as $t \to \infty$, and a back forms in the solution which moves rightward with a speed $\beta - c_0$. Hence the expanding speed of the main habitat for such a solution is $C^*_*(\beta) := c^*_r(\beta) - [\beta - c_0]$. In this paper we show that $C^*_*(\beta)$ is strictly decreasing in $\beta \in [c_0, \beta^*)$ with $C^*_*(\beta^* - 0) = 0$, and so there exists a unique $\beta_0 \in (c_0, \beta^*)$ such that the advection is favorable to the expanding speed of the habitat if and only if $\beta \in (0, \beta_0)$.

1. Introduction. Consider a reaction-diffusion-advection equation with free boundaries:

$$
\begin{cases}
    u_t = u_{xx} - \beta u_x + f(u), & t > 0, \ g(t) < x < h(t), \\
    u(t, g(t)) = 0, & g'(t) = -\mu u_x(t, g(t)), & t > 0, \\
    u(t, h(t)) = 0, & h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
    -g(0) = h(0) = h_0, & u(0, x) = u_0(x), & -h_0 \leq x \leq h_0,
\end{cases}
$$

(P)

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where $\mu$ and $h_0$ are positive constants, $\beta \geq 0$ is a parameter representing the strength of the advection, $u_0$ is a $C^2$ function with support $[-h_0, h_0]$, and $f \in C^1([0, \infty))$ is a Fisher-KPP type of nonlinearity which satisfies

$$
\begin{cases}
  f(0) = f(1) = 0, & (1-u)f(u) > 0 \text{ for } u > 0 \text{ and } u \neq 1, \\
  f'(0) > 0, f'(1) < 0, & \text{ and } f(u) \leq f'(0)u \text{ for } u \geq 0.
\end{cases}
$$

(T)

Typical examples for such functions include the logistic nonlinearity $f(u) = u(1-u)$.

When $\beta = 0$ (that is, there is no advection in the environment) and $f(u) = u(1-u)$, the problem (T) was studied by Du and Lin [2]. They used such a problem to model the spreading of a new or invasive species with population density $u(t, x)$ over a one dimensional habitat, with two free boundaries $x = g(t), h(t)$ representing the expanding fronts. Among others, they obtained a dichotomy result: either vanishing happens (that is, $u(t, \cdot) \to 0$ as $t \to \infty$ uniformly in $\mathbb{R}$, and the limiting interval $(g_\infty, h_\infty)$ is a bounded one, where

$$
g_\infty := \lim_{t \to \infty} g(t) \quad \text{and} \quad h_\infty := \lim_{t \to \infty} h(t),
$$

or spreading happens (that is, $u(t, \cdot) \to 1$ as $t \to \infty$, locally uniformly in $\mathbb{R}$, and the limiting interval $(g_\infty, h_\infty) = \mathbb{R}$). Furthermore, when spreading happens, they obtained the existence of the asymptotic spreading speed ([2, Theorem 4.2]):

$$
c_* := \lim_{t \to \infty} \frac{-g(t)}{t} = \lim_{t \to \infty} \frac{h(t)}{t} > 0.
$$

(1)

In the last few years, the paper [2] has brought a small boom on the study of reaction diffusion equations with free boundaries. Among others, [3] also obtained the formula (1) for general equations and [4] improved it into the form $h(t) = -g(t) = c_* t + O(1)$.

In the field of ecology, organisms can often sense and respond to local environmental cues by moving towards favorable habitats, and these movements usually depend upon a combination of local biotic and abiotic factors such as stream, climate, food and predators. For example, in studying the propagation of West Nile virus in North America, it was observed in [9] that West Nile virus appeared for the first time in New York city in the summer of 1999. In the second year the wave front travels 187 km to the North and 1100 km to the South, till 2002, it has been spread across almost the whole America continent. Therefore, the propagation of WNv from New York city to California state is a consequence of the diffusion and advection movements of birds. Especially, bird advection becomes an important factor for lower mosquito biting rates. Another example is that Averill [1] considered the effect of intermediate advection on the dynamics of two-species competition system, and provided a concrete range of advection strength for the coexistence of two competing species. Moreover, three different kinds of transitions from small advection to large advection were illustrated theoretically and numerically. Many other examples involving advection were also found in the field of ecology.

From a mathematical point of view, to involve the influence of advection, one of the simplest but probably still realistic approaches is to assume that species can move up along the gradient of the density. The equation $u_t = u_{xx} - \beta u_x + f(u)$ is such an example. Recently, this equation (more precisely, the problem (P) with $\beta > 0$) was studied by [6-11], where the problem is used to model the spreading of a new species in an environment with advection. In [5], the authors investigated a SIS model like (P) (with advections and free boundaries). The impact of the spatial
heterogeneity and the advection on the persistence and eradication of an infectious disease was studied.

In this paper, we focus on the impact of the advection strength $\beta$ on the expanding speed of the habitat, which is the further investigation to the details of [8]. Gu, Lou and Zhou [8] gave a rather complete description for the long time behavior of the solutions. More precisely, there exists $\beta^* > c_0$ (where $c_0 := 2\sqrt{f'(0)}$ is the minimal speed of the traveling waves of the Fisher-KPP equation $u_t = u_{xx} + f(u)$) such that

(i) in small advection case $\beta \in [0, c_0)$, there is a dichotomy result (cf. [8] Theorem 2.1): either vanishing happens, or spreading happens for each solution;

(ii) in medium-sized advection case $\beta \in [c_0, \beta^*)$, there is a trichotomy result (cf. [8] Theorem 2.2)): either vanishing happens, or virtual spreading happens (which means that $-\infty < g_\infty < h_\infty = +\infty$, $u(t, \cdot) \to 0$ locally uniformly in $(g_\infty, \infty)$, $u(t, \cdot + ct) \to 1$ locally uniformly in $\mathbb{R}$ for some $c > 0$), or the solution is a transition one in the sense that neither (virtual) spreading nor vanishing happens.

(iii) in large advection case $\beta \geq \beta^*$, vanishing happens for all the solutions of [F] (cf. [8] Theorem 2.4]).

In addition, [6, 8] also studied the asymptotic spreading speeds and asymptotic profiles for (virtual) spreading solutions, which are of special importance from the ecological point of view. More precisely, when $\beta \in (0, c_0)$ and when spreading happens for a solution $u$, [8] Theorem 2.5(i)] showed that $h(t) - c^*_t(\beta)t \to H_\infty$ and $g(t) - c^*_t(\beta)t \to G_\infty$ as $t \to \infty$ for some real numbers $c^*_t(\beta), c^*_t(\beta), H_\infty$ and $G_\infty$ satisfying $c^*_t(\beta) > c_\ast > -c^*_t(\beta) > 0$. Moreover, the solution $u$ converges as $t \to \infty$ to the traveling semi-wave $U^*_t(x - c^*_t(\beta)t - H_\infty)$ near the right boundary $x = h(t)$ (resp. to the semi-wave $U^*_t(x - c^*_t(\beta)t - G_\infty)$ near the left boundary $x = g(t)$), where $(c, q) = (c^*_t(\beta), U^*_t)$ is the unique solution of

$$
\begin{cases}
q''(z) + (c - \beta)q'(z) + f(q) = 0, \quad z \in (-\infty, 0), \\
q(0) = 0, \quad q(-\infty) = 1, \quad -\mu q'(0) = c, \quad q'(z) < 0 \text{ for } z \in (-\infty, 0],
\end{cases}
$$

and $(c, q) = (c^*_t(\beta), U^*_t)$ (with $c^*_t(\beta) < 0$) is the unique solution of

$$
\begin{cases}
q''(z) + (c - \beta)q'(z) + f(q) = 0, \quad z \in (0, \infty), \\
q(0) = 0, \quad q(+\infty) = 1, \quad -\mu q'(0) = c, \quad q'(z) > 0 \text{ for } z \in [0, +\infty).
\end{cases}
$$

From these conclusions we see that the habitat domain $(g(t), h(t))$ expands with an asymptotic speed $C^*_t(\beta) := c^*_t(\beta) - c^*_t(\beta)$.

When $\beta \in [c_0, \beta^*)$ and when virtual spreading happens for a solution $u$, [8] Theorem 2.5(ii)] proved that $h(t) - c^*_t(\beta)t \to H_\infty$ as $t \to \infty$ for some positive number $c^*_t(\beta)$, and the solution $u$ converges as $t \to \infty$ to the traveling semi-wave $U^*_t(x - c^*_t(\beta)t - H_\infty)$ near the right boundary $x = h(t)$. On the left side, however, $g_\infty > -\infty$ and a back (that is, a sharp increasing part on the graph of the solution) emerges and moves rightward. This back converges to a function $Q(x - (\beta - c_0)t + o(t))$, with $Q(x - (\beta - c_0)t)$ being the traveling wave of the equation [F]$_1$, that is, $q = Q$ is the unique solution of

$$
\begin{cases}
q''(z) - c_0 q'(z) + f(q) = 0, \quad z \in \mathbb{R}, \\
q(-\infty) = 0, \quad q(+\infty) = 1, \quad q(0) = 1/2, \quad q'(z) > 0 \text{ for } z \in \mathbb{R}.
\end{cases}
$$

Therefore, the main part of the habitat domain is $[(\beta - c_0)t + o(t), h(t)]$, which expands with an asymptotic speed $C^*_2(\beta) := c^*_2(\beta) - [\beta - c_0]$.
Define
\[ C^*(\beta) = \begin{cases} 
C_1^*(\beta), & \text{when } \beta \in [0, c_0) \\
C_2^*(\beta), & \text{when } \beta \in [c_0, \beta^*). 
\end{cases} \quad (5) \]

A natural question is:

\textbf{Q: How does } C^*(\beta) \text{ depend on } \beta \text{? Namely, how does the advection strength } \beta \text{ affect the expanding speed of the habitat?}

In [8, Lemma 3.4], the authors proved that 
\( C_2^*(\beta) > 0 \) (resp. = 0, or < 0) if and only if \( \beta < \beta^* \) (resp. \( \beta = \beta^* \), or \( \beta > \beta^* \)). As a consequence of such a result we see that the advection is not always favorable to the spreading. Especially, a large advection may cause vanishing for the species. The following result gives a complete answer to the question \textbf{Q}.

\textbf{Theorem 1.1.} Assume \( \beta \in [0, \beta^*) \) and \( C^*(\beta) \) is defined as in (5). Then

(i) \( C^*(\cdot) \in C([0, \beta^*)) \cap C^1((0, \beta^*) \setminus \{c_0\}) \);

(ii) \( C^*(0) = 2c_\star, C^*(\beta^* - 0) = 0, \)

\[ (C^*)'(0 + 0) = 0, \quad (C^*)'(c_0 - 0) > 0, \quad (C^*)'(c_0 + 0) < 0, \quad (C^*)'(\beta^* - 0) < 0, \]

and

\[ (C^*)'(\beta) > 0 \text{ for } \beta \in (0, c_0), \quad (C^*)'(\beta) < 0 \text{ for } \beta \in (c_0, \beta^*). \]

From this theorem we immediately obtain the following corollary (see Figure 1).

\textbf{Corollary 1.} Under the assumption of Theorem 1.1, there exists \( \beta_0 \in (c_0, \beta^*) \) such that

\[ C^*(\beta) > 2c_\star \text{ for } \beta \in (0, \beta_0), \quad 0 < C^*(\beta) < 2c_\star \text{ for } \beta \in (\beta_0, \beta^*). \]

This corollary indicates that when the advection strength is not strong (that is, when \( \beta \in (0, \beta_0) \)), the advection environment is more favorable to the spreading of the species than an environment without advection.

![Figure 1. Graph of \( C^*(\beta) \): a simulation result when \( f(u) = u(1 - u) \) and \( \mu = 1. \)]
2. Proof of the main theorem. In this section, we give the proof for Theorem 1.1. First, we use difference method to prove the following lemma.

**Lemma 2.1.** $c^*_\alpha(t) \in C^1([0, \infty))$, $\frac{d}{dt} c^*_\alpha(\beta) < k := \frac{\mu}{1+k}$ for all $\beta \geq 0$.

**Proof.** For any $\beta \geq 0$, by [8, Lemma 3.4] the problem [2] has a unique solution $(c, q) = (c^*_\alpha(\beta), U^*_\alpha)$. Note that the equation in [2] is equivalent to a system

$$q' = p, \quad p' = (\beta - c)p - f(q),$$

and so in the range where $p \neq 0$, [2] is converted into the following problem

$$\begin{align*}
\frac{dp}{dq} &= \beta - c - \frac{f(q)}{p}, \quad 0 < q < 1, \\
p(0) &= -\frac{\mu}{\beta}, \quad p(1) = 0, \quad p(q) < 0 \text{ for } 0 < q < 1.
\end{align*}$$

(6)

For any given $\beta_1 \geq 0$, set $\beta_2 := \beta_1 + \Delta \beta$, where $\Delta \beta \neq 0$ and $|\Delta \beta|$ is small when $\beta_1 > 0$, and $\Delta \beta > 0$ is small when $\beta_1 = 0$. We use $(c_i, p_i)$ to denote the unique solution of [2] with $\beta = \beta_i$ ($i = 1, 2$), and so $c_i = c^*_\alpha(\beta_i)$. Denote $\eta := p_2 - p_1$, $\Delta \beta := \beta_2 - \beta_1$ and $\Delta c := c_2 - c_1$. Then

$$\begin{align*}
\frac{d\eta}{dq} &= \Delta \beta - \Delta c + \frac{\beta}{p_1 p_2} f(q), \quad 0 < q < 1, \\
\eta(0) &= -\frac{\Delta \beta}{\beta}, \quad \eta(1) = 0.
\end{align*}$$

(7)

Solving the equation we obtain

$$\eta(q) = e^{\int_0^q \frac{f(s)}{p_1(s)p_2(s)} ds} \left[ \eta(0) + (\Delta \beta - \Delta c) \int_0^q e^{-\int_0^y \frac{f(s)}{p_1(s)p_2(s)} ds} dy \right].$$

It follows from the boundary conditions of $\eta$ at 0 and 1 we have

$$\Delta c = \frac{A \Delta \beta}{\beta} + A, \quad \text{with } A := \int_0^1 e^{-f(y)} \frac{f(s)}{p_1(s)p_2(s)} ds dy.$$

(8)

1. Continuity of $c^*_\alpha(\beta)$. Since $\frac{f(s)}{p_1(s)p_2(s)} > 0$ for $s \in (0, 1)$, we see that the integral in the definition of $A$ is convergent, that is, $A$ is a real number in $(0, 1)$. Hence $\Delta c \to 0$ as $\Delta \beta \to 0$. This proves the continuity of $c^*_\alpha(\beta)$ at $\beta = \beta_1$.

2. Differentiability of $c^*_\alpha(\beta)$. Note that the differentiability of $c^*_\alpha(\beta)$ is equivalent to the existence of $\lim_{\Delta \beta \to 0} A$ by [8]. Hence we only need to prove

$$\lim_{\Delta \beta \to 0} A = \lim_{\beta_2 \to \beta_1} \int_0^1 e^{-\int_0^y \frac{f(s)}{p_1(s)p_2(s)} ds} dy = \int_0^1 e^{-\int_0^y \frac{f(s)}{p_1(s)p_2(s)} ds} dy,$$

(9)

since the last integral is convergent.

By taking limit as $q \to 1 - 0$ in the equation

$$\frac{dp_i}{dq} = \beta_i - c_i - \frac{f(q)}{p_i}, \quad (i = 1, 2),$$

we have

$$\lambda_i + (c_i - \beta_i) \lambda_i + f'(1) = 0, \quad \text{with } \lambda_i := \frac{dp_i}{dq} (1 - 0).$$

Due to the fact $p_i(q) < 0$ we see that $\lambda_i > 0$. Hence $\lambda_i = \frac{1}{2} [\beta_i - c_i + \sqrt{(\beta_i - c_i)^2 - 4f'(1)}]$. Therefore, if we choose $\delta \in (0, 1)$ sufficiently small (depending on $\beta_1$), then in the interval $(1 - \delta, 1)$ we have

$$2 \lambda_i (q - 1) < p_i(q) < \frac{1}{2} \lambda_i (q - 1) \quad (i = 1, 2), \quad \frac{1}{2} f'(1)(q - 1) < f(q) < 2 f'(1)(q - 1).$$
Thus,
\[
K_1 \leq \frac{f(q)}{p_1(q)p_2(q)} \leq K_2 \quad \text{and} \quad L_1 \leq \frac{f(q)}{p_1(q)} \leq L_2 \quad \text{in } [1 - \delta, 1),
\]
where
\[
K_1 = -f'(1), \quad K_2 = -8f'(1), \quad L_1 = -f'(1), \quad L_2 = -8f'(1).
\]
This implies that
\[
\left| \int_{1-\delta}^1 e^{-\int_0^y \frac{f(s)}{p_1(s)p_2(s)}} \, ds \right| \leq \left| \int_{1-\delta}^1 e^{-\int_0^y \frac{f(s)}{p_1(s)}} \, ds \right| \label{eq:10}
\]
Consequently, for any given \( \epsilon > 0 \), we can choose \( \delta \) sufficiently small such that
\[
\left| \int_{1-\delta}^1 e^{-\int_0^y \frac{f(s)}{p_1(s)p_2(s)}} \, ds \right| - \left| \int_{1-\delta}^1 e^{-\int_0^y \frac{f(s)}{p_1(s)}} \, ds \right| \leq \frac{\epsilon}{2}.
\]

On the other hand, for the \( \delta \) chosen just as above, we consider the following problem
\[
\begin{align*}
\frac{dp_i}{dx} &= \beta_i - c_i - \frac{f(q)}{p_c}, \quad q \in [0, 1 - \delta], \quad i = 1, 2. \\
p_i(0) &= -\frac{\alpha_i}{p},
\end{align*}
\]
Since \( c_2 = c^*_r(\beta_2) \rightarrow c_1 = c^*_r(\beta_1) \) as \( \beta_2 \rightarrow \beta_1 \). By the standard ODE theory we have
\[
\lim_{\beta_2 \rightarrow \beta_1} \left\| p_2 - p_1 \right\|_{C([0, 1 - \delta])} = 0.
\]
In the interval \([0, 1 - \delta]\), both \( p_1 \) and \( p_2 \) are continuous and bounded from above by a negative number \( -\delta_1 \). Hence
\[
\lim_{\beta_2 \rightarrow \beta_1} \int_0^{1-\delta} e^{-\int_0^y \frac{f(s)}{p_1(s)p_2(s)}} \, ds = \int_0^{1-\delta} e^{-\int_0^y \frac{f(s)}{p_1(s)}} \, ds.
\]
Combining with \eqref{eq:10} we prove \eqref{eq:9}. In addition, using the first equality in \eqref{eq:8} we have
\[
\frac{d}{d\beta} c^*_r(\beta_1) = -\frac{B}{\rho + B}, \quad \text{with } B := \lim_{A \rightarrow 0} A = \int_0^1 e^{-\int_0^y \frac{f(s)}{p_1(s)}} \, ds. \label{eq:12}
\]
Since \( f(s) > 0 \) for \( s \in (0, 1) \) we see that \( B \) is a number in \((0, 1)\), and so
\[
\frac{d}{d\beta} c^*_r(\beta_1) = -\frac{B}{\rho + B} < k := \frac{\mu}{1 + \mu}.
\]
This also implies that the function \( \beta \rightarrow c^*_r(\beta) \) is strictly increasing in \( \beta \).

3. Continuity of \( \frac{d}{d\beta} c^*_r(\beta) \). Using a similar argument as above, it is not difficult to show that \( B = B(\beta_1) \) depends on \( \beta_1 \geq 0 \) continuously. This proves \( c^*_r(\beta) \in C^1([0, \infty)) \).

4. Monotonicity of \( \frac{d}{d\beta} c^*_r(\beta) \). Finally, we prove that \( \frac{d}{d\beta} c^*_r(\beta) \) is increasing in \( \beta \). For this purpose, we choose \( \Delta \beta > 0 \) in the beginning of the proof. Then \( \beta_2 := \beta_1 + \Delta \beta > \beta_1 \),
\[
c^*_r(\beta_2) > c^*_r(\beta_1), \quad \beta_2 - c^*_r(\beta_2) > \beta_1 - c^*_r(\beta_1),
\]
(13)
by the monotonicity of \( c_1^*(\beta) \) and \( \beta - c_1^*(\beta) \) in \( \beta \). We only need to verify \( B(\beta_2) > B(\beta_1) \), or equivalently, to show that \( p_2(q) < p_1(q) < 0 \) for all \( q \in [0, 1) \).

Using the first inequality in (13) and noting the problem (10) we see that \( p_2(q) < p_1(q) \) when \( q > 0 \) is small. This relation is also true when \( q \) is near 1 since

\[
p_2'(1 - 0) = \frac{1}{2}((\beta_2 - c_2) + \sqrt{(\beta_2 - c_2)^2 - 4f'(1)} ) > p_1'(1 - 0) = \frac{1}{2}((\beta_1 - c_1) + \sqrt{(\beta_1 - c_1)^2 - 4f'(1)} )
\]

by the second inequality in (13). If \( p_2(q) < p_1(q) \) is not always true, then the graph of \( p_2 \) crosses that of \( p_1 \) at least twice. This clearly is impossible. This proves \( p_2(q) < p_1(q) \).

The proof of the lemma is completed.

**Lemma 2.2.** \( c_1^*(\cdot) \in C^1((0, c_0)) \), \( \frac{d}{d\beta} c_1^*(\beta) \) is strictly increasing in \( \beta \) and \( 0 < \frac{d}{d\beta} c_1^*(\beta) < k := \frac{\mu}{\mu + B} \) for all \( \beta \in [0, c_0) \).

The proof of this lemma is similar as above, so we omit the detail. Note that we also obtain an analogue of (12):

\[
\frac{d}{d\beta} c_1^*(\beta) = \frac{B_l}{1 + \frac{B_l}{B_l}} = \int_0^1 e^{-f(s)} \frac{d}{d(s)} ds dy,
\]

where \( p = p_l(q) \) is the unique solution of the following problem

\[
\begin{align*}
\frac{dp}{dq} &= \beta - c - \frac{f(q)}{p}, \quad 0 < q < 1, \\
p(1) &= 0, \quad p(q) > 0 \text{ for } 0 < q < 1.
\end{align*}
\]

with \( e = c_1^*(\beta) \).

**Proof of Theorem 1.1.** Consider the unique solution \((c_1^*(\beta), p_l(q; \beta))\) of the problem (10) and the unique solution \((c_1^*(\beta), p_l(q; \beta))\) of the problem (15). By the phase plane analysis as in [8] it is easy to know that

(i) when \( \beta = 0 \), \( c_1^*(0) = -c_1^*(0) = c_* \), and \( p_l(q; 0) = -p_l(q; 0) \);

(ii) when \( \beta \in (0, c_0], 0 \leq p_l(q; \beta) < -p_l(q; \beta) \).

Thanks to the formulas (12) and (14), when \( \beta = 0 \) we have \( C_*(0) = 2c_* \) and \( (C_*)'(0 + 0) = 0 \) by (i); when \( \beta \in (0, c_0] \) we have \( (C_*)'(\beta) > 0 \) by (ii). Other conclusions in Theorem 1.1 follow easily from [8 Lemma 3.4], Lemmas 2.1 and 2.2.

This proves Theorem 1.1.

**Remark 1.** Summarizing the above results we see that, there is a critical value \( \beta_0 \in (c_0, \beta^*) \) for the advection strength \( \beta \) such that advection enhances the expanding speed of the habitat for a species when \( \beta \in (0, \beta_0) \), and obstructs the expanding speed when \( \beta \in (\beta_0, \beta^*) \). In other words, small advection is favorable to the expanding of the habitat once (virtual) spreading happens, but large advection is unfavorable. This (more precisely, our main results and Figure 1) answers the question in section 1 completely.

3. **Numerical simulations.** In this section we present some numerical simulations for \( c_1^*(\beta) \), \( -c_1^*(\beta) \) and \( C^*(\beta) \). For simplicity, we choose \( f(u) = u(1-u) \) and \( \mu = 1 \). In this case, \( c_0 = 2 \). Our computation results show that \( c_1^*(0) = -c_1^*(0) = c_* \approx 0.3640, \beta^* = 4.2000 \) and \( \beta_0 = 2.8300 \). In Figure 2, we plot the graph of \( c_1^*(\beta) \) for \( \beta \geq 0 \) (which has a positive and increasing slope), and the graph of \( -c_1^*(\beta) \) for \( \beta \in [0, 2] \) (which decreases monotonically to 0 as \( \beta \to 2 - 0 \)). In Figure 1, we plot the graph
of $C^\ast(\beta)$ for $\beta \geq 0$, which increases in the interval $[0, 2]$, decreases in the interval $[2, \beta^\ast]$. Moreover, $C^\ast(\beta) > C^\ast(0) = 2c_\ast\ast$ if and only if $\beta \in (0, \beta^\ast)$.

**Figure 2.** Graphs of $c^\ast_r(\beta)$ and $-c^\ast_l(\beta)$ when $f(u) = u(1 - u)$ and $\mu = 1$.

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