SIGNED A-POLYNOMIALS OF GRAPHS AND POINCARÉ POLYNOMIALS OF REAL TORIC MANIFOLDS

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Abstract. Recently, Choi and Park introduced an invariant of a finite simple graph, called signed a-number, arising from computing certain topological invariants of some specific kinds of real toric manifolds. They also found the signed a-numbers of path graphs, cycle graphs, complete graphs, and star graphs.

We introduce a signed a-polynomial which is a generalization of the signed a-number and gives a-, b-, and c-numbers. The signed a-polynomial of a graph G is related to the Poincaré polynomial \( P_{M(G)}(z) \), which is the generating function for the Betti numbers of the real toric manifold \( M(G) \). We give the generating functions for the signed a-polynomials of not only path graphs, cycle graphs, complete graphs, and star graphs, but also complete bipartite graphs and complete multipartite graphs. As a consequence, we find the Euler characteristic number and the Betti numbers of the real toric manifold \( M(G) \) for complete multipartite graphs \( G \).

1. Introduction

In algebraic topology, Choi and Park \[CP12\] recently introduced a graph invariant, called signed a-number of finite simple graphs \( G \), denote by \( sa(G) \) as follows:

- \( sa(\emptyset) = 1 \).
- \( sa(G) \) is the product of signed a-numbers of connected components of \( G \).
- \( sa(G) = 0 \) if \( G \) is a connected graph with odd vertices.
- If \( G \) is connected with even vertices, then \( sa(G) \) is given by the negative of the sum of signed a-numbers of all induced subgraphs \( G' \) of \( G \) except itself \( G \).

Let the a-number \( a(G) \) be the absolute value of the signed a-number of \( G \), the b-number \( b(G) \) the sum of signed a-numbers induced subgraphs of \( G \), and the c-numbers \( c_i(G) \) the sum of a-numbers of induced subgraphs of \( G \) with \( i \) vertices.

These numbers arise from computing certain topological invariants of some specific kinds of real toric manifolds which are important objects in toric topology. For a finite simple graph \( G \), the real toric manifold \( M(G) \) is the set of real points in the toric manifold associated to the graph associahedron \( P_{B(G)} \) which is the nestohedron as the Minkowski sum of simplices obtained from connected induced subgraphs of \( G \). For further information, see \[De88, DJ91, Pos09, PRW08\].

Recently, Choi and Park \[CP12, Theorem 1.1\] showed that the Euler characteristic \( \chi(M(G)) \) is equal to \( b(G) \) and the (rational) Betti number \( \beta_i(M(G)) \) is equal to \( c_2(G) \). We remark that \( c_2(G) \) is the same with \( a_2(G) \) in \[CP12\]. They also computed these numbers of path graphs \( P_{2n} \), cycle graphs \( C_{2n} \), complete graphs \( K_{2n} \), and star graphs \( K_{1,2n-1} \).

In this paper, we introduce a signed a-polynomial which is a generalization of the signed a-number and gives a-, b-, and c-numbers. The signed a-polynomial of a graph \( G \) is related
to the Poincaré polynomial $P_{M(G)}(z)$, which is the generating function for the Betti numbers of the real toric manifold $M(G)$. The relation will be shown in the equation (7). We give the signed a-polynomials of not only path graphs, cycle graphs, complete graphs, and star graphs, but also complete bipartite graphs $K_{p,q}$ and complete multipartite graphs $K_{p_1,\ldots,p_m}$. As a consequence, we find $\chi(M(G))$ and $\beta_i(M(G))$ for $G = K_{p,q}$ and $G = K_{p_1,\ldots,p_m}$.

2. Preliminaries

From now on, we assume that a graph is finite, undirected, and simple. We rewrite a formal definition of a signed a-number $sa(G)$ of a graph $G = (V, E)$ in the previous section by

$$sa(G) = \begin{cases} 
1 & \text{if } G \text{ is the empty graph;} \\
0 & \text{if } G \text{ is connected and } |V| \text{ is odd;} \\
-\sum_{V' \subseteq V} sa(G|_{V'}) & \text{if } G \text{ is connected and } |V| \text{ is even } \geq 2; \\
\prod_{G' \in \text{comp}(G)} sa(G') & \text{if } G \text{ is disconnected}, 
\end{cases}$$

where $G|_{V'}$ is the induced subgraph of $G$ by a vertex subset $V'$ and comp($G$) is the set of connected components of $G$. From above definition, it is easy to check $sa(G) = 0$ for every graph $G$ with at least one connected component on odd vertices; and $\sum_{V' \subseteq V} sa(G|_{V'}) = 0$ for every nonempty graph $G$ on $V$ with every connected component on even vertices. Thus, we find a simpler equivalent definition of a signed a-number as follows.

**Definition 1.** A signed a-number $sa(G)$ of a graph $G = (V, E)$ is defined by

$$sa(G) = \begin{cases} 
1, & \text{if } G \text{ is the empty graph;} \\
0, & \text{if } G \text{ has a connected component on odd vertices;} \\
-\sum_{V' \subseteq V} sa(G|_{V'}), & \text{otherwise}. 
\end{cases} \tag{1}$$

Consequently, we define a-, b-, and c-numbers of a graph with the signed a-numbers.

**Definition 2.** The a-, b-, and c-numbers of a graph $G$, denoted by $a(G)$, $b(G)$, and $c_i(G)$, are defined by

$$a(G) = (-1)^{|V|/2} sa(G), \tag{2}$$
$$b(G) = \sum_{V' \subseteq V} sa(G|_{V'}), \tag{3}$$
$$c_i(G) = \sum_{V' \subseteq V} a(G|_{V'}) = (-1)^{|V'|/2} \sum_{V' \subseteq V} sa(G|_{V'}). \tag{4}$$

By definition, for any graph $G$, it hold that $c_i(G) = 0$ if $i$ is odd, and $c_n(G) = a(G)$ if $n$ is the number of vertices of $G$. In topological viewpoint [CP12, Remark 2.2], it is obvious that $a(G)$ and $c_i(G)$ are nonnegative integers.

3. On signed a-polynomials

Now, we introduce a generalization of a-, b-, and c-numbers of graphs.
**Definition 3** (Signed a-polynomial). The signed a-polynomial $sa(G; t)$ of a graph $G$ is defined by

$$ sa(G; t) = \sum_{V' \subseteq V(G)} sa(G|_{V'}) t^{|V'\setminus V'|}. $$

(5)

From the equations (11) – (13), for $|V(G)| = n$, it holds that

$$ sa(G) = sa(G; 0), \quad a(G) = (-1)^{n/2} sa(G; 0), \quad b(G) = sa(G; 1), \quad c_1(G) = (-1)^{i/2} t^{n-1} sa(G; t). $$

Thus, $sa(G; t)$ is represented as the sum of $c_i(G)$’s by

$$ sa(G; t) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j c_{2j}(G) t^{n-2j}. $$

(6)

For example, if $G = \{ \{A, B, C, D\}, \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}\}\}$, then

$$ sa(G) = a(G) = 4, \quad b(G) = 0, \quad \text{and} \quad \{c_i(G)\}_{i=0}^{4} = 1, 0, 5, 0, 4. $$

**Remark.** The Poincaré polynomial $P_{M(G)}(z) = \sum_{i \geq 0} \beta_i(M(G)) z^i$ is the generating function for the Betti numbers $\beta_i(M(G))$ of the real toric manifold $M(G)$. Since $\beta_i(M(G)) = c_{2i}(G)$ in [CP12, Theorem 1.1], it holds that

$$ P_{M(G)}(z) = (\sqrt{-z})^{|V(G)|} \sum_{i=0}^{n} \frac{1}{\sqrt{1-x}}. $$

(7)

In the rest of the section, we compute the generating functions for signed a-polynomials of path graphs, cycle graphs, complete graphs, and star graphs.

**Theorem 1.** Let $P_n$ be the path graph with $n$ vertices, which is a tree with exactly $n-2$ vertices of degree 2. Then the generating function for signed a-polynomials of $P_n$ is given by

$$ \sum_{n \geq 0} sa(P_n; t) x^n = \frac{-1 + 2tx + \sqrt{1 + 4x^2}}{2tx - 2(t^2 - 1)x^2}. $$

(8)

**Proof.** From Theorem 2.5 in [CP12], it is known that

$$ c_{2i}(P_n) = \binom{n}{i} - \binom{n}{i-1} = \text{Cat}_{n-i,i}, $$

with Catalan triangle numbers $\text{Cat}_{n,k} = \binom{n+k}{k} - \binom{n+k}{k-1}$. Using the formula (6), we have

$$ sa(P_n; t) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \text{Cat}_{n-j,j} t^{n-2j}. $$

Thus, we obtain

$$ \sum_{n \geq 0} sa(P_n; t) x^n = \sum_{n \geq 0} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \text{Cat}_{n-j,j} t^{n-2j} x^n = \sum_{k \geq 0} \sum_{j \geq 0} \text{Cat}_{k,j} (-x/t)^j (tx)^k. $$

(9)
Since the generating function for Catalan triangle numbers is

$$\sum_{n \geq 0} \sum_{i \geq 0} Cat_{n,i} w^i z^n = \frac{Cat(wz)}{1 - z Cat(wz)},$$

where $Cat(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$, therefore the formula (9) becomes the formula (8).

Remark. For two given sequences $\sigma = (s_0, s_1, s_2, \ldots)$ and $\tau = (t_1, t_2, t_3, \ldots)$, define the generalized Catalan number $B_n$ by the sum of weighted Motzkin paths from $(0,0)$ to $(n,0)$ with up steps $(1,1)$, horizontal steps $(1,0)$, and down steps $(1,-1)$ where we associate weight $1$ to each up step, weight $s_k$ to each horizontal step on the line $y = k$, and weight $t_k$ to each down step between two lines $y = k - 1$ and $y = k$. For example, if $\sigma \equiv 0$ and $\tau \equiv 1$, then $B_{2n} = Cat_n$. In Section 7.4 in [Aig07], the generating function $B(z) = \sum_{n \geq 0} B_n z^n$ of the generalized Catalan number $B_n$ with $\sigma = (a,s,s,\ldots)$ and $\tau = (b,u,u,\ldots)$ is equal to

$$B(z) = \frac{(2u-b)+(bs-2au)z-b\sqrt{1-2sx+(s^2-4u)z^2}}{2(u-b)+2(bs-2au+ab)z+2(a^2u-2abs+b^2)z^2}. \tag{10}$$

For $(a,s,b,u,z) = (t,0,-1,-1,x)$, the formula (10) gives a combinatorial interpretation of the following formula

$$\sum_{n \geq 0} sa(P_n;t)x^n = \frac{-1+2tx+\sqrt{1+4x^2}}{2tx-2(t^2-1)x^2},$$

and for $(a,s,b,u,z) = (0,0,-1,t^2-1,x)$, the formula (10) gives a combinatorial interpretation of the following formula

$$\sum_{n \geq 0} sa(P_{2n};t)x^{2n} = \frac{-(t^2+1)-(t^2-1)\sqrt{1+4x^2}}{-2t^2+2(t^2-1)x^2}.$$  

Theorem 2. Let $C_n$ be the cycle graph with $n$ vertices, which is a connected graph with all vertices of degree 2. Then the generating function for signed $a$-polynomials of $C_n$ is given by

$$\sum_{n \geq 0} sa(C_n;t)x^n = \frac{1}{2} + \frac{1}{2\sqrt{1+4x^2}} \cdot \frac{(t^2+1)x + t\sqrt{1+4x^2}}{t - (t^2-1)x}. \tag{11}$$
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\[
\begin{array}{c|ccc}
G & C_0 & C_{2n} & C_{2n+1} \\
\hline
\text{sa}(G) & 1 & (-1)^n \binom{2n}{n} & 0 \\
a(G) & 1 & \frac{1}{2} \binom{2n}{n} & 0 \\
b(G) & 1 & 0 & (-1)^n \binom{2n}{n} \\
c_{2i}(G) & \delta_{i,0} \left\{ \begin{array}{cl}
\frac{1}{2} \binom{n}{i}, & \text{if } i = n \\
\binom{2n}{i}, & \text{if } i < n
\end{array} \right.
\end{array}
\]

g.f. for $\text{sa}(G; t)$

\[
\sum_{n \geq 0} \text{sa}(C_n; t) x^n = \frac{1}{2} + \frac{1}{2\sqrt{1 + 4x^2}} \cdot \frac{(t^2 + 1)x + t \sqrt{1 + 4x^2}}{t - (t^2 - 1)x}
\]

Table 2. Numbers for cycle graphs $C_n$.

**Proof.** From Theorem 2.6 in [CP12], it is known that

\[
c_{2i}(C_n) = \left\{ \begin{array}{cl}
1, & \text{if } i = n = 0, \\
\frac{1}{2} \binom{n}{i/2}, & \text{if } 2i = n > 0, \\
\binom{2n}{i}, & \text{if } 2i < n.
\end{array} \right.
\]

Using the formula (6), we obtain

\[
\sum_{n \geq 0} \text{sa}(C_n; t) x^n = \frac{1}{2} - \frac{1}{2} \sum_{j \geq 0} \binom{2j}{j} (-x^2)^j + \sum_{k \geq 0} \sum_{j \geq 0} \binom{2j + k}{j} (tx)^k (-x^2)^j.
\]

(12)

From $\sum_{n \geq 0} \binom{2n}{n} z^n = \frac{1}{1 - \sqrt{1 - 4z}}$, we have two generating functions:

\[
\sum_{n \geq 0} \binom{2n}{n} z^n = \frac{1}{1 - \sqrt{1 - 4z}},
\]

\[
\sum_{n \geq 0} \sum_{k \geq 0} \binom{2n + k}{n} w^k z^n = \frac{1}{1 - \sqrt{1 - 4z}} \cdot \frac{1}{1 - w \left(1 - \sqrt{1 - 4z}\right)}.
\]

Using above two generating functions, the formula (12) becomes the formula (11). \qed

Let $A_n$ be the $n$-th Euler zigzag number for which the exponential generating function is

\[
\sum_{n \geq 0} A_n \frac{z^n}{n!} = \sec z + \tan z.
\]

(13)

**Theorem 3.** Let $K_n$ be the complete graph with $n$ vertices. Then the exponential generating function for signed a-polynomials of $K_n$ is given by

\[
\sum_{n \geq 0} \text{sa}(K_n; t) \frac{z^n}{n!} = e^{tx} \text{sech } x.
\]

(14)
\[
G \quad \quad \quad K_0 \quad K_{2n} \quad K_{2n+1}
\]
\[
\begin{array}{c|c|c|c}
\text{sa}(G) & 1 & (-1)^n A_{2n} & 0 \\
a(G) & 1 & A_{2n} & 0 \\
b(G) & 1 & 0 & (-1)^n A_{2n+1} \\
c_{2i}(G) & \delta_{i,0} & \left(\frac{2n}{2i}\right) A_{2i} & \left(\frac{2n+1}{2i}\right) A_{2i}
\end{array}
\]

\[
e.g.f. \text{ for } sa(G; t) \quad \sum_{n=0}^{\infty} \text{sa}(K_n; t) \frac{x^n}{n!} = e^{tx} \text{sech}(x)
\]

Table 3. Numbers for complete graphs \(K_n\), where \(\sum_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sec z + \tan z\).

\textbf{Proof.} From Theorem 2.8 in [CP12], it is known that
\[
\text{sa}(K_{2n}) = (-1)^n A_{2n}.
\]

Using the formula (15), we obtain
\[
\sum_{n=0}^{\infty} \text{sa}(K_n; t) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \text{sa}(K_{2j}) t^{n-2j} \frac{x^n}{n!} \\
= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{k+2j}{2j} (-1)^j A_{2j} t^k \frac{x^{k+2j}}{(k+2j)!} \\
= \left( \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} \right) \left( \sum_{j=0}^{\infty} A_{2j} \frac{(ix)^{2j}}{(2j)!} \right)
\]

By (13), the formula (15) becomes the formula (14). \(\Box\)

\textbf{Remark.} The Euler polynomials \(E_n(t)\) is defined by the exponential generating function
\[
\sum_{n=0}^{\infty} E_n(t) \frac{x^n}{n!} = \left( \frac{2}{e^{2x}+1} \right) e^{tx}. \text{ See [Com74, pp. 48]. Then it follows}
\]
\[
\text{sa}(K_n; t) = E_n(t + \frac{1}{2}) 2^n
\]

from \(\sum_{n=0}^{\infty} \text{sa}(K_n; t) \frac{x^n}{n!} = e^{tx} \text{sech} x = \left( \frac{2}{e^{2x}+1} \right) e^{2x} \left( \frac{ix}{2} \right)^n = \sum_{n=0}^{\infty} E_n \left( \frac{t+1}{2} \right) \frac{(2x)^n}{n!} \).

\textbf{Theorem 4.} Let \(K_{1,n}\) be the star graph with \(n+1\) vertices, which is a tree with at least one vertex of degree \(n\). Then the exponential generating function for signed \(a\)-polynomials of \(K_{1,n}\) is given by
\[
\sum_{n=0}^{\infty} \text{sa}(K_{1,n}; t) \frac{x^n}{n!} = e^{tx} (t - \tanh x).
\]

\textbf{Proof.} From Theorem 2.9 in [CP12], it is known that
\[
\text{sa}(K_{1,2n+1}) = (-1)^{n+1} A_{2n+1}.
\]
Using the formula (5), we obtain

\[ y = \sum_{n \geq 0} \frac{sa(K_{1,n};t)}{n!} x^n = e^{tx}(t - \tanh x) \]

Table 4. Numbers for star graphs \( K_{1,n} \), where \( \sum_{n \geq 0} A_n \frac{x^n}{n!} = \sec z + \tan z. \)

Using the formula (6), we obtain

\[
\sum_{n \geq 0} \frac{sa(K_{1,n};t)}{n!} x^n = \sum_{n \geq 0} \left( sa(0)t^{n+1} + \sum_{j=0}^{[n/2]} \binom{n}{2j+1} sa(K_{1,2j+1}) t^{n-(2j+1)} \right) \frac{x^n}{n!}
\]

\[
= \sum_{n \geq 0} t^{n+1} \frac{x^n}{n!} + \sum_{k \geq 0} \sum_{j \geq 0} \binom{k+2j+1}{2j+1} (-1)^{j+1} A_{2j+1} t^k \frac{x^{k+2j+1}}{(k+2j+1)!}
\]

\[
= t \left( \sum_{n \geq 0} \frac{(tx)^n}{n!} \right) + \left( \sum_{k \geq 0} \frac{(tx)^k}{k!} \right) \left( t \sum_{j \geq 0} A_{2j+1} \frac{(tx)^{2j+1}}{(2j+1)!} \right)
\]

(17)

By (13), it follows

\[
\sum_{j \geq 0} A_{2j+1} \frac{(tx)^{2j+1}}{(2j+1)!} = \tan(tx) = t \tanh x
\]

and the formula (17) becomes the formula (16). □

Since \( sa(G) = sa(G; 0) \), putting \( t = 0 \) in the generating functions \( 8, 11, 14 \), and (16) yields the generating functions for signed a-numbers of path graphs, cycle graphs, complete graphs, and star graphs as follows:

\[
\sum_{n \geq 0} \frac{sa(P_n)x^n}{n!} = -1 + \sqrt{1 + 4x^2} = \sum_{m \geq 0} (-1)^m \text{Cat}_m x^{2m},
\]

\[
\sum_{n \geq 0} \frac{sa(C_n)x^n}{n!} = \frac{1}{2} + \frac{1}{2\sqrt{1 + 4x^2}} = 1 + \sum_{m \geq 1} \frac{(-1)^m}{2} \binom{2m}{m} x^{2m},
\]

\[
\sum_{n \geq 0} \frac{sa(K_n)x^n}{n!} = \text{sech} x = \sum_{m \geq 0} (-1)^m A_{2m} \frac{x^{2m}}{(2m)!},
\]

\[
\sum_{n \geq 0} \frac{sa(K_{1,n})x^n}{n!} = -\tanh x = \sum_{m \geq 1} (-1)^m A_{2m-1} \frac{x^{2m-1}}{(2m-1)!},
\]

Similarly, since \( b(G) = sa(G; 1) \), putting \( t = 1 \) in the generating functions \( 8, 11, 14 \), and (16) yields the generating functions for b-numbers of path graphs, cycle graphs, complete
graphs, and star graphs as follows:

\[
\sum_{n \geq 0} b(P_n) x^n = 1 + \frac{-1 + \sqrt{1 + 4x^2}}{2x} = 1 + \sum_{m \geq 0} (-1)^m \text{Cat}_m x^{2m+1},
\]

\[
\sum_{n \geq 0} b(C_n) x^n = 1 + \frac{x}{\sqrt{1 + 4x^2}} = 1 + \sum_{m \geq 0} (-1)^m \binom{2m}{m} x^{2m+1},
\]

\[
\sum_{n \geq 0} \frac{b(K_n)}{n!} = 1 + \tanh x = 1 + \sum_{m \geq 0} (-1)^m A_{2m+1} \frac{x^{2m+1}}{(2m+1)!},
\]

\[
\sum_{n \geq 0} \frac{b(K_{1,n})}{n!} = \text{sech } x = \sum_{m \geq 0} (-1)^m A_{2m} \frac{x^{2m}}{(2m)!}.
\]

According to (7), putting \( t \leftarrow \frac{1}{\sqrt{1 - z}} \) and \( x \leftarrow x \sqrt{1 - z} \) in the generating functions (8), (11), (14), and (16) yields the next result.

**Corollary 5.** Let \( P_M(G)(z) \) denote the Poincaré polynomials of the real toric manifolds \( M(G) \) associated to the graph \( G \). Then the generating functions for Poincaré polynomials of the real toric manifolds associated to path graphs \( P_n \), cycle graphs \( C_n \), complete graphs \( K_n \), and star graphs \( K_{1,n} \) are as follows:

\[
\sum_{n \geq 0} P_M(P_n)(z) x^n = \frac{-1 + 2x + \sqrt{1 - 4zx^2}}{2x - 2(1 + z)x^2},
\]

\[
\sum_{n \geq 0} P_M(C_n)(z) x^n = \frac{1}{2} + \frac{1}{2\sqrt{1 - 4zx^2}} \cdot \frac{(1 - z)x + \sqrt{1 - 4zx^2}}{1 - (1 + z)x},
\]

\[
\sum_{n \geq 0} \frac{P_M(K_n)(z)}{n!} x^n = e^x \sec(x\sqrt{z})
\]

\[
\sum_{n \geq 0} \frac{P_M(K_{1,n})(z)}{n!} x^n = e^x \left( 1 + \sqrt{z} \tan(x\sqrt{z}) \right).
\]

4. **Signed a-number of complete multipartite graphs**

Firstly, we consider the exponential generating function for signed a-numbers of complete bipartite graphs. Denote by \( K_{p,q} \) the complete bipartite graph with \( p \)-set and \( q \)-set.

**Theorem 6.** The exponential generating function for signed a-numbers of complete bipartite graphs is

\[
\sum_{p \geq 0} \sum_{q \geq 0} \text{sa}(K_{p,q}) \frac{x^p y^q}{p! q!} = \frac{\cosh x + \cosh y - 1}{\cosh(x + y)}.
\]

**Proof.** For two nonnegative integers \( p \) and \( q \) whose sum is even, there is the recurrence

\[
\sum_{i,j \geq 0} \binom{p}{i} \binom{q}{j} \text{sa}(K_{i,j}) = \begin{cases} 
0 & \text{if } p \text{ and } q \text{ are positive}, \\
1 & \text{if } p \text{ or } q \text{ is zero}.
\end{cases}
\]
The exponential generating function for right-hand side of (19) is
\[
\sum_{p, q \geq 0} (RHS) \frac{x^p y^q}{p! q!} = 1 + (\cosh x - 1) + (\cosh y - 1).
\] (20)

The exponential generating function for left-hand side of (19) is
\[
\sum_{p, q \geq 0} (LHS) \frac{x^p y^q}{p! q!} = \sum_{p, q \geq 0} \sum_{0 \leq i \leq p, 0 \leq j \leq q, i + j \text{ even}} \left( \frac{x^{p-i} y^{q-j}}{(p-i)! (q-j)!} \right)
\]
\[
= \left( \sum_{i, j \geq 0} \text{sa}(i, j) \frac{x^i y^j}{i! j!} \right) \left( \sum_{i, j \geq 0} \frac{x^i y^j}{i! j!} \right)
\]
\[
= \left( \sum_{p, q \geq 0} \text{sa}(K_{p, q}) \frac{x^p y^q}{p! q!} \right) \cosh(x + y).
\] (21)

Thus, by (20) and (21), we are done. \qed

The generating function \( S_A_q(x) \) is defined by \( S_A_q(x) = \sum_{p \geq 0} \text{sa}(K_{p, q}) \frac{x^p}{p!} \), which is the coefficient of \( y^q/q! \) in \( \frac{\cosh(x + \cosh y - 1)}{\cosh(x + y)} \). Given a fixed nonnegative \( q \), we can induce the detailed formula \( S_A_q(x) \) by
\[
S_A_q(x) = \frac{\partial^q}{\partial y^q} \left( \frac{\cosh(x + \cosh y - 1)}{\cosh(x + y)} \right) \bigg|_{y=0}.
\]

For example, the initial generating functions \( A_q(x) \) are listed as follows:
\begin{align*}
S_A_0(x) &= 1, \\
S_A_1(x) &= -\tanh x, \\
S_A_2(x) &= -2 \sech^2 x + \sech x + 1, \\
S_A_3(x) &= (6 \sech^2 x - 3 \sech x - 1) \tanh x, \\
S_A_4(x) &= 24 \sech^4 x - 12 \sech^3 x - 20 \sech^2 x + 7 \sech x + 1.
\end{align*}

Next, we generalize the generating function (18) for complete multipartite graphs. Denote by \( K_{p_1, \ldots, p_m} \) the complete \( m \)-partite graph with \( p_1 \)-set, \ldots, \( p_m \)-set.

**Theorem 7.** The exponential generating function for signed \( a \)-numbers of complete \( m \)-partite graphs is
\[
\sum_{p_1, \ldots, p_m \geq 0} \text{sa}(K_{p_1, \ldots, p_m}) \frac{x_1^{p_1}}{p_1!} \cdots \frac{x_m^{p_m}}{p_m!} = \frac{(1 - m) + \cosh x_1 + \cdots + \cosh x_m}{\cosh(x_1 + \cdots + x_m)}.
\] (22)

**Proof.** For \( m \) nonnegative integers \( p_1, \ldots, p_m \) whose sum is even, there is the recurrence
\[
\sum_{i_1, \ldots, i_m \geq 0} \binom{p_1}{i_1} \cdots \binom{p_m}{i_m} \text{sa}(K_{i_1, \ldots, i_m}) = \begin{cases} 0 & \text{if at least two } p_i \text{'s are positive}, \\ 1 & \text{if all } p_i \text{'s are zeros, but at most one}. \end{cases}
\] (23)
Using both sides of (23), we have the generalized formulae of (20) and (21) as follows:

\[
\sum_{p_1 \geq 0} \cdots \frac{x_1^{p_1}}{p_1!} \cdots \frac{x_m^{p_m}}{p_m!} = 1 + (\cosh x_1 - 1) + \cdots + (\cosh x_m - 1)
\]

and

\[
\sum_{p_1 \geq 0} \cdots \frac{x_1^{p_1}}{p_1!} \cdots \frac{x_m^{p_m}}{p_m!} = \left( \sum_{p_1, \ldots, p_m \geq 0} sa(K_{p_1, \ldots, p_m}) \frac{x_1^{p_1}}{p_1!} \cdots \frac{x_m^{p_m}}{p_m!} \right) \cosh(x_1 + \cdots + x_m),
\]

which completes the proof.

\[\square\]

Remark. Obviously, the exponential generating functions for a-numbers of complete bipartite graphs and complete m-partite graphs are equal to

\[
\sum_{p \geq 0} \sum_{q \geq 0} a(K_{p, q}) \frac{x^p y^q}{p! q!} = \frac{\cos x + \cos y - 1}{\cos(x + y)},
\]

\[
\sum_{p_1, \ldots, p_m \geq 0} a(K_{p_1, \ldots, p_m}) \frac{x_1^{p_1}}{p_1!} \cdots \frac{x_m^{p_m}}{p_m!} = \frac{(1 - m) + \cos x_1 + \cdots + \cos x_m}{\cos(x_1 + \cdots + x_m)}.
\]

5. Signed a-polynomial of complete multipartite graphs

Firstly, we consider the exponential generating function for signed a-polynomials of complete bipartite graphs.

Theorem 8. Let \( K_{p, q} \) be the complete bipartite graph with \( p \)-set and \( q \)-set. Then the exponential generating function for signed a-polynomials of \( K_{p, q} \) is given by

\[
\sum_{p \geq 0} \sum_{q \geq 0} sa(K_{p, q}) \frac{x^p y^q}{p! q!} = e^{(x+y)} \left( \frac{\cosh x + \cosh y - 1}{\cosh(x + y)} \right),
\]

(24)

Proof. By definition, we have

\[
\sum_{p \geq 0} \sum_{q \geq 0} sa(K_{p, q}) \frac{x^p y^q}{p! q!} = \sum_{p \geq 0} \sum_{q \geq 0} \left( \sum_{0 \leq p' \leq p, 0 \leq q' \leq q} \binom{p}{p'} \binom{q}{q'} sa(K_{p', q'}) t^{p' - p + q' - q} \right) \frac{x^p y^q}{p! q!}.
\]

(25)

Substituting \( p'' = p - p' \) and \( q'' = q - q' \), the right-hand side of (25) becomes

\[
\sum_{p'' \geq 0} \sum_{q'' \geq 0} \left( \sum_{p' \geq 0} \sum_{q' \geq 0} \binom{p'}{p''} \binom{q'}{q''} sa(K_{p', q'}) t^{p' - p'' + q''} \right) \frac{x^{p'} y^{q'}}{(p' + p'')!(q' + q'')!},
\]

\[= \left( \sum_{p' \geq 0} \sum_{q' \geq 0} sa(K_{p', q'}) \frac{x^{p'} y^{q'}}{p'! q'!} \right) \left( \sum_{p'' \geq 0} \frac{(tx)^{p''}}{p''!} \right) \left( \sum_{q'' \geq 0} \frac{(ty)^{q''}}{q''!} \right).
\]

The formula (18) completes the proof. \[\square\]
Remark. Since the coefficient of \( \frac{y^q}{q!} \) in the formula (24) is equal to \( \sum_{n \geq 0} sa(K_{q,n}; t) \frac{x^n}{n!} \), it holds that

\[
\sum_{n \geq 0} sa(K_{q,n}; t) \frac{x^n}{n!} = \frac{\partial^q}{\partial y^q} e^{t(x+y)} \left( \frac{\cosh x + \cosh y - 1}{\cosh(x + y)} \right) \bigg|_{y=0} .
\]

In case of \( q = 1 \), we have the exponential generating function (16) for signed a-polynomials of star graphs again.

Similarity, we can deduce the next theorem by the same above method.

**Theorem 9.** Let \( K_{p_1, \ldots, p_m} \) be the complete \( m \)-partite graph with \( p_1 \)-set, \( \ldots, p_m \)-set. Then the exponential generating function for signed a-polynomials of \( K_{p_1, \ldots, p_m} \) is given by

\[
\sum_{\sum p_i \geq 0} \frac{b(K_{p_1, \ldots, p_m})}{p_1! \cdots p_m!} \left( \frac{x_1^{p_1}}{p_1!} \cdots \frac{x_m^{p_m}}{p_m!} \right) = e^{(x_1 + \cdots + x_m)} \left( \frac{(1 - m) + \cosh x_1 + \cdots + \cosh x_m}{\cosh(x_1 + \cdots + x_m)} \right).
\]

Since \( sa(G) = sa(G; 0) \), putting \( t = 0 \) in the generating functions (24) and (26) gives the two formula (13) and (22), respectively. Also, since \( b(G) = sa(G; 1) \), putting \( t = 1 \) in the generating functions (24) and (26) yields the generating functions for b-numbers of complete bipartite graphs and complete multipartite graphs as follows:

\[
\sum_{p \geq 0} \sum_{q \geq 0} b(K_{p,q}) \frac{x^p y^q}{p! q!} = e^{x+y} \left( \frac{\cosh x + \cosh y - 1}{\cosh(x + y)} \right),
\]

\[
\sum_{p_1, \ldots, p_m \geq 0} \frac{b(K_{p_1, \ldots, p_m})}{p_1! \cdots p_m!} \left( \frac{x_1^{p_1}}{p_1!} \cdots \frac{x_m^{p_m}}{p_m!} \right) = e^{x_1 + \cdots + x_m} \left( \frac{(1 - m) + \cosh x_1 + \cdots + \cosh x_m}{\cosh(x_1 + \cdots + x_m)} \right).
\]

The next result follows from two generating functions (24) and (26) by plugging in (7).

**Corollary 10.** Let \( P_{M(K_{p,q})}(z) \) and \( P_{M(K_{p_1, \ldots, p_m})}(z) \) denote the Poincaré polynomials of the real toric manifolds associated to the complete bipartite graph \( K_{p,q} \) and the complete \( m \)-partite graph \( K_{p_1, \ldots, p_m} \). Then the generating functions for Poincaré polynomials \( P_{M(K_{p,q})}(z) \) and \( P_{M(K_{p_1, \ldots, p_m})}(z) \) are equal to

\[
\sum_{n \geq 0} P_{M(K_{p,q})}(z) \frac{x^n}{n!} = e^{x+y} \left( \frac{\cos(x \sqrt{z}) + \cos(y \sqrt{z}) - 1}{\cos(x \sqrt{z} + y \sqrt{z})} \right),
\]

\[
\sum_{n \geq 0} P_{M(K_{p_1, \ldots, p_m})}(z) \frac{x_1^{p_1}}{p_1!} \cdots \frac{x_m^{p_m}}{p_m!} = e^{x_1 + \cdots + x_m} \left( \frac{(1 - m) + \cos(x_1 \sqrt{z}) + \cdots + \cos(x_m \sqrt{z})}{\cos(x_1 \sqrt{z} + \cdots + x_m \sqrt{z})} \right).
\]

Table 5 shows the Poincaré polynomials \( P_{M(K_{p,q})}(z) \) for \( p \leq 6 \) and \( q \leq 3 \).

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\[ \begin{array}{c|ccc}
 p \backslash q & 0 & 1 & 2 \\
 \hline
 0 & 1 & 1 & 1 \\
 1 & 1 & 1 + z & 1 + 2z \\
 2 & 1 & 1 + 2z & 1 + 4z + 3z^2 \\
 3 & 1 & 1 + 3z + 2z^2 & 1 + 6z + 13z^2 \\
 4 & 1 & 1 + 4z + 8z^2 & 1 + 8z + 34z^2 + 27z^3 \\
 5 & 1 & 1 + 5z + 20z^2 + 16z^3 & 1 + 10z + 70z^2 + 167z^3 \\
 6 & 1 & 1 + 6z + 40z^2 + 96z^3 & 1 + 12z + 125z^2 + 597z^3 + 483z^4 \\
 \end{array} \]

Table 5. Table for \( P_{M(K_{p,q})}(z) \)

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