0-Pierced Triangles within a Poisson Overlay

STEVEN FINCH

April 4, 2018

Abstract. Let the Euclidean plane be simultaneously and independently endowed with a Poisson point process and a Poisson line process, each of unit intensity. Consider a triangle $T$ whose vertices all belong to the point process. The triangle is 0-pierced if no member of the line process intersects any side of $T$. Our starting point is Ambartzumian’s 1982 joint density for angles of $T$; our exposition is elementary and raises several unanswered questions.

A triangle with angles $\alpha, \beta, \gamma$ is acute if $\max\{\alpha, \beta, \gamma\} < \pi/2$ and well-conditioned if $\min\{\alpha, \beta, \gamma\} > \pi/6$. Given a random mechanism for generating triangles in the plane, we dutifully calculate corresponding probabilities out of sheer habit and for the sake of numerical concreteness.

Beginning with a Poisson point process of unit intensity, let us form a triangle by taking the convex hull of three particles (members of the process). The triangle is 0-filled if no other particles are contained in the convex hull. Study of such configurations is complicated by the prevalence of long, narrow triangles with angles typically $\approx 0$ or $\approx \pi$. We defer discussion of these until later. $\phi(x) = 4 (3x - \pi + 3 \cot(x) \ln(2 \cos(x)))$ for $\pi/3 < x < \pi/2$.

Beginning with a Poisson point process and a Poisson line process, also of unit intensity and independent, let us form a triangle as before. The triangle is 0-pierced if the intersection of each line with the convex hull is always empty. Nothing is presumed about the existence or number of other interior particles; there may be 0 or 1 or 2 or many more. Since the angles satisfy $\alpha + \beta + \gamma = \pi$, we can eliminate $\gamma$ from consideration and write the joint density for $\alpha, \beta$ as

$$\frac{42}{\pi} \frac{\sin(x) \sin(y) \sin(x + y)}{[\sin(x) + \sin(y) + \sin(x + y)]^4}$$

where $x > 0, y > 0, x + y < \pi$. This is a remarkable result, owing to the scattered complexity of particles overlaid with lines. Integrating out $y$, we obtain the marginal density for $\alpha$:

$$f(x) = \frac{21}{2\pi} \frac{(7 - 5 \cos(x))(1 + \cos(x)) + 4(5 - \cos(x))(1 - \cos(x)) \ln(\sin(\frac{x}{2}))}{(1 + \cos(x))^4}$$

Copyright © 2018 by Steven R. Finch. All rights reserved.
and

\[ \mathbb{E}(\alpha) = \frac{\pi}{3} = 1.0471975511..., \quad \mathbb{E}(\alpha^2) = \frac{13}{10} - 2 \ln(2) + 4 \ln(2)^2 = 1.8355176945.... \]

Corresponding to the density for \( \max\{\alpha, \beta, \gamma\} \), the expression \( 3f(x) \) holds when \( \pi/2 < x < \pi \); the expression when \( \pi/3 < x < \pi/2 \) is \( 3\tilde{f}(x) \) where

\[ \tilde{f}(x) = \frac{21}{2\pi} \frac{(7 - 4 \cos(x))(1 - 2 \cos(x)) - 4(5 - \cos(x))(1 - \cos(x))(\ln(2\sin(x/2)))}{(1 + \cos(x))^4}. \]

It thus follows that

\[ \mathbb{P}(\text{a typical 0-pierced triangle is acute}) = \frac{96 - 132 \ln(2) - \pi}{2\pi} = 0.2169249267.... \]

Corresponding to the density for \( \min\{\alpha, \beta, \gamma\} \), the expression \( -3\tilde{f}(x) \) holds when \( 0 < x < \pi/3 \) and hence

\[ \mathbb{P}(\text{a typical 0-pierced triangle is well-conditioned}) = \frac{-3144 + 1584\sqrt{3} + (-2190 + 1338\sqrt{3})\ln(2) + (4380 - 2676\sqrt{3})\ln(-1 + \sqrt{3}) + (71 + 41\sqrt{3})\pi}{2(71 + 41\sqrt{3})\pi} = 0.2393922701.... \]

From \( \nabla(\alpha + \beta + \gamma) = 0 \), we deduce that \( \mathbb{E}(\alpha\beta) - \mathbb{E}(\alpha)\mathbb{E}(\beta) = -(1/2)\nabla(\alpha) \) and therefore

\[ \mathbb{E}(\alpha\beta) = -\frac{13}{10} + \ln(2) - 2 \ln(2)^2 + \frac{\pi^2}{6} = 0.7271752195.... \]

Simulation provides compelling evidence that Ambartzumian’s \([1, 2]\) joint density is valid – see Figure 1 – although it does not provide insight leading to an actual proof.

### 1. Related Expressions

We turn attention to the bivariate densities

\[ C_j \frac{\sin(x)\sin(y)\sin(x+y)}{[\sin(x) + \sin(y) + \sin(x+y)]^3} \]

where \( x > 0, y > 0, x + y < \pi \) and

\[ C_1 = \frac{4}{12 - \pi^2}, \quad C_2 = \frac{1}{(-2 + 3\ln(2))\pi}, \quad C_3 = 8. \]

The case \( j = 1 \) appears in \([5, 6]\) with regard to cells of a Goudsmit-Miles tessellation (sampled until a triangle emerges); the case \( j = 3 \) appears in \([7, 8]\) with regard to
triangles created via breaking a line segment (in two places at random). For \( j = 2 \),
we obtain the univariate density for \( \alpha \):

\[
g(x) = \frac{1}{2 \left(-2 + 3 \ln(2)\right) \pi} \frac{(\pi - x) \sin(x) + 4 \left(1 - \cos(x)\right) \ln \left(\sin\left(\frac{x}{2}\right)\right)}{1 + \cos(x)}
\]

and

\[
\mathbb{E}(\alpha^2) = \frac{4 \left(\pi^2 - 12 \ln(2)\right) \ln(2) - 3 \zeta(3)}{6 \left(-2 + 3 \ln(2)\right)} = 1.4611131303...
\]

where \( \zeta(3) \) is Apéry’s constant [9].

Corresponding to the density for \( \max\{\alpha, \beta, \gamma\} \), the expression \( 3g(x) \) holds when \( \pi/2 < x < \pi \); the expression when \( \pi/3 < x < \pi/2 \) is \( 3\tilde{g}(x) \) where

\[
\tilde{g}(x) = \frac{1}{2 \left(-2 + 3 \ln(2)\right) \pi} \frac{(3x - \pi) \sin(x) - 4 \left(1 - \cos(x)\right) \ln \left(2 \sin\left(\frac{x}{2}\right)\right)}{1 + \cos(x)}.
\]

It thus follows that

\[
\mathbb{P}(\text{a typical such triangle is acute}) = \frac{-24 \ln(2) + (4 - 3 \ln(2)) \pi + 12G}{4 \left(-2 + 3 \ln(2)\right) \pi} = 0.3903338870...
\]
and $G$ is Catalan’s constant $[10]$. Corresponding to the density for $\min\{\alpha, \beta, \gamma\}$, the expression $-3\tilde{g}(x)$ holds when $0 < x < \pi/3$ and hence

$$\mathbb{P}(\text{a typical such triangle is well-conditioned}) = 0.4190489201...$$

(exact expression omitted for reasons of length). As before, we deduce that

$$\mathbb{E}(\alpha \beta) = \frac{2(\pi^2 + 24 \ln(2)) \ln(2) - 4\pi^2 + 3\zeta(3)}{12(-2 + 3\ln(2))} = 0.9143775016....$$

What’s missing, of course, is a natural procedure for generating (not necessarily planar) triangles whose angles $\alpha$, $\beta$, $\gamma$ obey the distributional law prescribed by $j = 2$.

2. 0-Filled Triangles

The phrase “0-filled” first appeared in $[11, 12]$. Let us initially discuss the simulation underlying 0-pierced triangles. Given a parameter value $\lambda > 0$, we generated data $(\alpha_1, \beta_1)$, $(\alpha_2, \beta_2)$, $\ldots$, $(\alpha_n, \beta_n)$ via Poisson overlays in the planar disk of radius $\lambda$ centered at the origin. Our goal was to verify a probability theoretic expression:

$$\mathbb{P}(x < \alpha \leq x + dx, y < \beta \leq y + dy) = \frac{42}{\pi} \frac{\sin(x) \sin(y) \sin(x + y)}{[\sin(x) + \sin(y) + \sin(x + y)]^2} dx dy$$

as $\lambda \to \infty$. This was done simply by histogramming the data, given large enough $n$ and $\lambda$.

For 0-filled triangles, however, we face a situation where the goal is less tangible. Ambartzumian’s measure theoretic expression $[2]$:

$$\mathbb{M}(x < \alpha \leq x + dx, y < \beta \leq y + dy) = \frac{2}{\sin(x) \sin(y) \sin(x + y)} dx dy$$

cannot be normalized to give a probability density (that is, encompassing unit area). It follows that $[13, 14]$:

$$\mathbb{M}(x < \max\{\alpha, \beta, \gamma\} \leq x + dx) = \begin{cases} -12 \csc(x)^2 \ln(2 \cos(x)) & \text{if } \pi/3 \leq x < \pi/2, \\ \infty & \text{if } \pi/2 \leq x < \pi \end{cases}$$

and, for $0 < x < \pi/3$,

$$\mathbb{M}(x < \min\{\alpha, \beta, \gamma\} \leq x + dx) = 12 \csc(x)^2 \ln(2 \cos(x))$$

– see Figures 2 and 3 – but verification is problematic. As before, we can generate
Figure 2: $\phi(x) = 4(3x - \pi + 3 \cot(x) \ln(2 \cos(x)))$ for $\pi/3 < x < \pi/2$.

Figure 3: $\phi(x) = 4(3x - \pi + 3 \cot(x) \ln(2 \cos(x)))$ for $0 < x < \pi/3$. 
Figure 4: Histograms for arbitrary angle in 0-filled triangles, for increasing $\lambda$.

data over disks of increasing radius $\lambda$. Figure 4 provides histograms of $\alpha$ for $\lambda = 2, 3, 4, 5$; Figures 5 and 6 do likewise for $\max\{\alpha, \beta, \gamma\}$ and $\min\{\alpha, \beta, \gamma\}$. Clearly

$$\lim_{\lambda \to \infty} P(\text{a typical 0-filled triangle is acute}) = 0,$$

$$\lim_{\lambda \to \infty} P(\text{a typical 0-filled triangle is well-conditioned}) = 0$$

on empirical grounds. Unfortunately we do not know how to confirm theoretical predictions stemming from [13, 14]:

$$\mathbb{M}(\text{a typical 0-filled triangle is acute}) = 2\pi \approx 6.283,$$

$$\mathbb{M}(\text{a typical 0-filled triangle is well-conditioned}) = 2 \left(3\sqrt{3}\ln(3) - \pi\right) \approx 5.134$$

via our experimental simulation. A procedure to adjust the histogramming of the data, in order to demonstrate an improved fit as $\lambda \to \infty$, would be welcome.

3. Process Intensities

We report here on work in [15]. Given a Poisson overlay $\Omega$, define a $T_0$ process to be the set of all 0-pierced triangles within $\Omega$. Let the intensity $i$ of the process be the mean number of triangles per unit area. It is known that

$$i(T_0) = \frac{2\pi^2}{21} = 0.9399623239... = \frac{1}{6}(5.6397739434...).$$
Figure 5: Histograms for maximum angle in 0-filled triangles, for increasing $\lambda$.

Figure 6: Histograms for minimum angle in 0-filled triangles, for increasing $\lambda$. 
The factor of $1/6$ arises because the three vertices were (apparently) ordered in [15], thus every triangle was counted $3! = 6$ times. We may similarly examine the set of all 0-filled triangles; it is not surprising that $i(T_0) = \infty$. Most interesting, however, is the set of all triangles that are both 0-filled and 0-pierced:

$$i(T_{00}) = 0.6554010386... = \frac{1}{6}(3.9324062319...)$$

$$= \frac{\pi}{18\sqrt{3}} \int_0^{\pi} \frac{\xi(x)\eta(x)}{\sqrt{b(x)(a(x) - c(x))}} \sin \left( \frac{x}{2} \right) dx$$

where

$$a(x) = \frac{2}{3} \left( \cos \left( \frac{x}{3} \right) + 1 \right), \quad b(x) = \frac{2}{3} \left( \cos \left( \frac{x - 2\pi}{3} \right) + 1 \right), \quad c(x) = \frac{2}{3} \left( \cos \left( \frac{x + 2\pi}{3} \right) + 1 \right),$$

$$q(x) = \frac{a(x)(b(x) - c(x))}{b(x)(a(x) - c(x))}, \quad h(x) = 2(27)^{1/4} \cos \left( \frac{x}{2} \right)^{-1/2},$$

$$\xi(x) = 2 \left( 4 + h(x)^2 \right) - \sqrt{\pi} \left( 6 + h(x)^2 \right) h(x) \exp \left( \frac{h(x)^2}{4} \right) \text{erfc} \left( \frac{h(x)}{2} \right),$$

$$\eta(x) = \left( \frac{3}{c(x)} - \frac{3}{a(x)} \right) E(q(x)) + \left( \frac{3}{a(x)} - 1 \right) K(q(x));$$

$$K(y) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - y^2 \sin^2(\theta)^2}} d\theta, \quad E(y) = \int_0^{\pi/2} \sqrt{1 - y^2 \sin^2(\theta)^2} d\theta$$

are complete elliptic integrals of the first and second kind; and

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\tau^2) d\tau = 1 - \text{erfc}(z)$$

is the error function. Formulas (7) and (8) in [15], devoted to a more general scenario $T_{k\ell}$ than our $T_{00}$, specialize to

$$\int_0^\infty s \exp \left( -s - t\sqrt{s} \right) ds = \frac{1}{8} \left[ 2 \left( 4 + t^2 \right) - \sqrt{\pi} \left( 6 + t^2 \right) t \exp \left( \frac{t^2}{4} \right) \text{erfc} \left( \frac{t}{2} \right) \right]$$

for $t > 0$ (avoiding use of a parabolic cylinder function $D_{-4} \left( t/\sqrt{2} \right)$ which is less familiar).

Theory fails for $T_{00}$ – we do not possess density predictions for the histograms in Figure 7 – nor do we know exact probabilities that a such a triangle is acute or well-conditioned.
Figure 7: Histograms for angles, maximum angle and minimum angle in $T_{00}$ triangles.

References

[1] R. V. Ambartzumian, Random shapes by factorization, *Statistics in Theory and Practise, Essays in Honour of Bertil Matérn*, ed. B. Ranneby, Swedish University of Agricultural Sciences, Section of Forest Biometry, 1982, pp. 35-41; MR0688997 (84c:62004).

[2] R. V. Ambartzumian, Factorization in integral and stochastic geometry, *Stochastic Geometry, Geometric Statistics, Stereology*, Proc. 1983 Oberwolfach conf., ed. R. Ambartzumian and W. Weil, Teubner, 1984, pp. 14–33; MR0794864.

[3] D. G. Kendall, Shape manifolds, Procrustean metrics, and complex projective spaces, *Bull. London Math. Soc.* 16 (1984) 81–121; MR0737237 (86g:52010).

[4] V. K. Oganyan, On triangle shapes formed by points of a Poisson process in the plane (in Russian), *Akad. Nauk Armyan. SSR Dokl.*, v. 81 (1985) n. 2, 59–63; MR0826342 (87h:60032).

[5] R. E. Miles, The various aggregates of random polygons determined by random lines in a plane, *Adv. Math.* 10 (1973) 256–290; MR0319232 (47 #7777).

[6] S. R. Finch, Random triangles V, unpublished note (2010).

[7] S. R. Finch, Uniform triangles with equality constraints, [arXiv:1411.5216](https://arxiv.org/abs/1411.5216).
[8] S. R. Finch, Random triangles VI, unpublished note (2011).

[9] S. R. Finch, Apéry’s constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 40–53; MR2003519 (2004i:00001).

[10] S. R. Finch, Catalan’s constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 53–59; MR2003519 (2004i:00001).

[11] R. E. Miles, On the homogeneous planar Poisson point process, *Math. Biosci.* 6 (1970) 85–127; MR0279853 (43 #5574).

[12] R. Cowan, A more comprehensive complementary theorem for the analysis of Poisson point processes, *Adv. Appl. Probab.* 38 (2006) 581–601; MR2256870 (2007k:60138).

[13] H. S. Sukiasian, Two results on triangle shapes, *Stochastic Geometry, Geometric Statistics, Stereology*, Proc. 1983 Oberwolfach conf., ed. R. Ambartzumian and W. Weil, Teubner, 1984, pp. 210–221; MR0794883.

[14] R. V. Ambartzumian, *Factorization Calculus and Geometric Probability*, Cambridge Univ. Press, 1990, pp. 61–67; 158–160; MR1075011 (92b:60013).

[15] V. R. Fatalov, Intensities of thinned processes of triangles that are generated by a Poisson point process on the plane (in Russian), *Izv. Akad. Nauk Armyan. SSR Ser. Mat.*, v. 25 (1990) n. 4, 344–352, 413; Engl. transl. in *Soviet J. Contemp. Math. Anal.*, v. 25 (1990) n. 4, 32–40; MR1115778 (92h:60018).

Steven Finch
MIT Sloan School of Management
Cambridge, MA, USA
steven_finch@harvard.edu