Deterministic Parallel Fixpoint Computation

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Abstract interpretation is a general framework for expressing static program analyses. It reduces the problem of extracting properties of a program to computing a fixpoint of a system of equations. The de facto approach for computing this fixpoint uses a sequential algorithm based on weak topological order (WTO). This paper presents a deterministic parallel algorithm for fixpoint computation by introducing the notion of weak partial order (WPO). We present an algorithm for constructing a WPO in almost-linear time. Finally, we describe our deterministic parallel abstract interpreter Pikos. We evaluate the performance of Pikos on a suite of 207 C programs. We observe a maximum speedup of 9.38x when using 16 cores compared to the existing WTO-based sequential abstract interpreter.

1 INTRODUCTION

Program analysis is a well established and widely adopted approach for automatically extracting properties of the dynamic behavior of programs [Balakrishnan et al. 2010; Ball et al. 2004; Bessey et al. 2010; Brat and Venet 2005; Delmas and Souyris 2007; Jetley et al. 2008]. Program analyses are used, for instance, for program optimization, bug finding, and program verification. To be effective and useful, a program analysis needs to be efficient, precise, as well as deterministic (each run of the analysis should compute the same output) [Bessey et al. 2010]. This paper tackles the problem of improving the efficiency of program analysis without sacrificing precision or determinism.

Abstract interpretation [Cousot and Cousot 1977] is a general framework for expressing static program analyses. A typical use of abstract interpretation to determine program invariants involves the following components:

C1 An abstract domain \( \mathcal{A} \) that captures the relevant program properties. Abstract domains have been developed to perform, for instance, numerical analysis [Cousot and Cousot 1977; Cousot and Halbwachs 1978; Miné 2002, 2004, 2006; Oulamara and Venet 2015; Singh et al. 2017; Venet 2012], heap analysis [Rinetzky et al. 2005; Wilhelm et al. 2000], escape analysis [Blanchet 2003; Deutsch 1997; Park and Goldberg 1992], information flow [Giacobazzi and Mastroeni 2004], and pointer analysis [Deutsch 1995].

C2 Translating the program to an equation system \( \mathcal{X} = F(\mathcal{X}) \) over \( \mathcal{A} \) that capture the abstract program behavior:

\[
\begin{align*}
X_1 &= F_1(X_1, \ldots, X_n) \\
& \vdots \\
X_n &= F_n(X_1, \ldots, X_n)
\end{align*}
\]

Each index \( i \in \{1, \ldots, n\} \) correspond to control points of the program, the unknowns \( X_i \) of the system correspond to the invariants to be computed for these control points, and each
$F_i$ is a monotone operator incorporating the abstract transformers and control flow of the program.

**Computing the fixpoint of Eq. 1.** The exact least solution of the system can be computed using Kleene iteration starting from the least element of $\mathcal{A}^n$ provided $\mathcal{A}$ is Noetherian. However, most interesting abstract domains do not satisfy this condition, and the termination of the analysis requires the use of *widening* [Cousot and Cousot 1977]. While enforcing termination, widening may result in a crude over-approximation of the invariants of the program. A subsequent *narrowing* iteration tries to improve the post solution by means of a downward fixpoint iteration.

The *iteration strategy* specifies the order in which the equations in Eq. 1 are applied during fixpoint computation and where widening is applied. For a given abstraction, the efficiency, precision, and determinism of an abstract interpreter depends on the iteration strategy. The iteration strategy is determined by the dependencies between the individual equations in Eq. 1. If this dependency graph is acyclic, then the optimal iteration strategy is any topological order of the vertices in the graph. This is not true when the dependency graph contains cycles. Furthermore, each cycle in the dependency graph needs to be cut by at least one widening point.

Since its publication in 1993, Bourdoncle’s algorithm [Bourdoncle 1993] has become the de facto approach for computing an efficient iteration strategy for abstract interpretation. Bourdoncle’s algorithm determines the iteration strategy from the *weak topological order* (WTO) of the vertices in the dependency graph corresponding to the equation system. There are certain disadvantages to Bourdoncle’s algorithm: (i) the iteration strategy computed by Bourdoncle’s algorithm is inherently sequential: WTO gives a total order of the vertices in the control-flow graph; (ii) computing WTO using Bourdoncle’s algorithm has a worst-case cubic time complexity; (iii) the mutually-recursive nature of Bourdoncle’s algorithm makes it difficult to understand (even for seasoned practitioners of abstract interpretation); and (iv) applying Bourdoncle’s algorithm, as is, to deep control-flow graphs can result in a stack overflow in practice.\(^1\,^2\)

This paper addresses the above disadvantages of Bourdoncle’s algorithm by presenting a concurrent iteration strategy for fixpoint computation in an abstract interpreter (§5). This concurrent fixpoint computation can be efficiently executed on modern multi-core hardware. The algorithm for computing our iteration strategy has a worst-case almost-linear time complexity, and lends itself to a simple iterative implementation (§6). The resulting parallel abstract interpreter, however, remains deterministic: for the same program, all possible executions of the parallel fixpoint computation give the same result. In fact, the fixpoint computed by our parallel algorithm is the same as that computed by Bourdoncle’s sequential algorithm (§7).

To determine the concurrent iteration strategy, this paper introduces the notion of a *weak partial order* (WPO) for the dependency graph of the equation system (§4). WPO generalizes the notion of WTO: a WTO is a linear extension of a WPO (§7). Consequently, the almost-linear time algorithm for WPO can also be used to compute WTO. The key insight behind our approach is to adapt algorithms for computing *loop nesting forests* [Ramalingam 1999, 2002] to the problem of computing a concurrent iteration strategy for abstract interpretation.

We have implemented our concurrent fixpoint iteration strategy in a tool called Pikos (§8). Using a suite of C programs, we compared the performance of Pikos against the state-of-the-art IKOS abstract interpreter [Brat et al. 2014], which uses Bourdoncle’s algorithm. We see that when using 16 cores we get a maximum speedup of 9.38x when performing a standard interval analysis (§9).

\(^1\)https://github.com/facebook/redex/commit/6bbf8a5ddbaae0b282e9fd7183a21764db7f0f39
\(^2\)https://github.com/seahorn/crab/issues/18

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The contributions of the paper are as follows:

- We introduce the notion of a weak partial order (WPO) for a directed graph (§4), and show how this generalizes the existing notion of weak topological order (WTO) (§7).
- We present a concurrent algorithm for computing the fixpoint of a set of equations (§5).
- We present an algorithm for WPO and WTO construction that has almost-linear time complexity (§6).
- We describe our deterministic parallel abstract interpreter PiKOS (§8), and evaluate its performance on a suite of C programs (§9).

§2 presents an overview of the technique; §3 presents mathematical preliminaries; §10 describes related work; §11 concludes.

## 2 Overview

Abstract interpretation is a general framework that captures most existing approaches for static program analyses and reduces extracting properties of programs to approximating their semantics [Cousot and Cousot 1977; Cousot et al. 2019]. Consequently, this section is not meant to capture all possible approaches to implementing abstract interpretation or describe all the complex optimizations involved in a modern implementation of an abstract interpreter. Instead it is only meant to set the appropriate context for the rest of the paper, and attempt to capture only the relevant high-level structure of abstract-interpretation implementations such as IKOS [Brat et al. 2014].

### Fixpoint equations.

Consider the simple program $P$ represented by its control flow graph (CFG) in Figure 1(a). We will illustrate how an abstract interpreter would compute the set of values that variable $x$ might contain at each program point $i$ in $P$. In this example, we will use the standard integer interval domain [Cousot and Cousot 1976, 1977] represented by the complete lattice $\langle \text{Int}, \subseteq, \top, \cup, \bot, \cap \rangle$ with $\text{Int} \overset{\text{def}}{=} \{\bot\} \cup \{(l, u) \mid l, u \in \mathbb{Z} \land l \leq u\} \cup \{(-\infty, u) \mid u \in \mathbb{Z}\} \cup \{(l, \infty) \mid l \in \mathbb{Z}\} \cup \{(-\infty, \infty]\}$. The partial order $\subseteq$ on Int is interval inclusion with the empty interval $\bot = \emptyset$ encoded as $[\infty, -\infty) \land \top = [\infty, \infty]$. Figure 1(b) shows the equation system $X = F(X)$ corresponding to program $P$, where $X = (X_0, X_1, X_2, \ldots, X_8)$. Each equation in this equation system is of the form $X_i = F_i(X_0, X_1, X_2, \ldots, X_8)$, where the variable $X_i \in \text{Int}$ represents the interval value at program point $i$ in $P$ and $F_i$ is monotone. The operator $+$ represents the (standard) addition operator over Int. As is common (but not necessary), the dependencies among the equations reflect the CFG of program $P$.

The exact least solution $X^{lfp}$ of the equation system $X = F(X)$ would give the required set of values for variable $x$ at program point $i$. Let $X^0 = (\bot, \bot, \ldots, \bot)$ and $X^{i+1} = F(X^i), i \geq 0$ represent the standard Kleene iterates, which converge to $X^{lfp}$.

### Chaotic iteration and widening.

Instead of applying the function $F$ during Kleene iteration, one can use chaotic iterations [Cousot 1977; Cousot and Cousot 1977] and apply the individual equations $F_i$. The order in which the individual equations are applied is determined by the chaotic iteration strategy.

For non-Noetherian abstract domains, such as the interval abstract domain, termination of this Kleene iteration sequence requires the use of a widening operator $\mathcal{W}$ [Cousot 2015; Cousot and Cousot 1977]. A set of widening points $W$ is chosen and the equation for $i \in W$ is replaced by $X_i = X_i \lor F_i(X_0, \ldots, X_n)$. An admissible set of widening points “cuts” each cycle in the dependency graph of the equation system by the use of a widening operator to ensure termination [Cousot and Cousot 1977]. Finding a minimal admissible set of widening points is an NP-complete problem [Garey and Johnson 2002]. A possible widening operator for the interval abstract domain is defined by: $\bot \lor i = \bot \lor l = i \in \text{Int}$ and $[i, j] \lor [k, l] = [\text{if } k < i \text{ then } -\infty \text{ else } i, \text{if } l > j \text{ then } \infty \text{ else } j]$. This widening operator is non-monotone. The application of a widening operator may result in a
Fig. 1. (a) A simple program P that updates x; (b) Corresponding equation system for interval domain; (c) Corresponding equation system with vertices 2 and 4 as widening points.

crude over-approximation of the least fixpoint; more sophisticated widening operators as well as techniques such as narrowing can be used to ensure precision [Amato and Scozzari 2013; Amato et al. 2016; Cousot and Cousot 1977; Gopan and Reps 2006; Halbwachs and Henry 2012; Kim et al. 2016]. Although the discussion of our fixpoint algorithm will use a simple widening strategy (§5), our implementation incorporates more sophisticated widening and narrowing strategies implemented in IKOS (§8).

Bourdoncle’s approach. Bourdoncle [1993] introduces the notion of hierarchical total order (HTO) of a set and weak topological order (WTO) of a directed graph. From the WTO of the dependency graph of an equation system, Bourdoncle [1993] describes a way to find the set of widening points and a chaotic iteration strategy called the recursive strategy. A WTO for the equation system in Figure 1(b), is $T \equiv 0 \ 8 \ 1 \ (2 \ 3) \ (4 \ 5 \ 6) \ 7$. The set of elements between two matching parentheses are called a component of the WTO, and the first element of a component is called the head of the component. Notice that components are non-trivial strongly connected components (“loops”) in the directed graph of Figure 1(a). Bourdoncle [1993] proves that the set of component heads is an admissible set of widening points. For Figure 1(b), the set of heads {2, 4} is an admissible set of widening points. Figure 1(c) shows the corresponding equation system that uses widening.

The iteration strategy generated using WTO $T$ is $S_1 \equiv 0 \ 8 \ 1 \ [2 \ 3]^* \ [4 \ 5 \ 6]^* \ 7$, where occurrence of $i$ in the sequence represents applying the equation for $X_i$ and [...]" is the “iterate until stabilization” operator. A component is stabilized if iterating it once more does not change the values of elements inside the component. 2 and 4 are chosen as the widening points. The iteration sequence $S_1$ should be interpreted as “apply equation for $X_0$, then apply the equation for $X_8$, then apply the equation for $X_1$, repeatedly apply equations for $X_2$ and $X_3$ until stabilization” and so on. Furthermore, Bourdoncle [1993] showed that stabilization of a component can be detected by the stabilization of its head. For instance, stabilization of component {2, 3} can be detected by the stabilization of its head 2. This property minimizes the number of (potentially expensive) comparisons between abstract values during fixpoint computation. For the equation system of Figure 1(c), the use of Bourdoncle’s recursive iteration strategy would give us $X_7^{\text{fp}} = [0, \infty]$.

Asynchronous iterations. The iteration strategy produced by Bourdoncle’s approach is necessarily sequential, because the iteration sequence is generated from a total order. One could alternatively implement a parallel fixpoint computation using asynchronous iterations [Cousot 1977]: each processor $i$ computes the new value of $X_i$ accessing the shared state consisting of $X$
using the appropriate equation from Figure 1(c). However, the parallel fixpoint computation using asynchronous iterations is non-deterministic; that is, the fixpoint computed might differ based on the order in which the equations are applied (as noted by Monniaux [2005]). The reason for this non-determinism is due to the non-monotonicity of widening. For example, if the iteration sequence $S_2 \overset{\text{def}}{=} 0 \ [4 \ 5 \ 6]^* \ [1 \ 2 \ 3]^* \ [4 \ 5 \ 6]^* \ 7$ is used to compute the fixpoint for the equations in Figure 1(c), then $X_3^{fp} = [\mathcal{v}, \mathcal{v}]$, which differs from the fixpoint value computed using iteration sequence $S_1$.

**Our deterministic parallel fixpoint computation.** In this paper, we present a parallel fixpoint computation that is deterministic, and, in fact, gives the same result as Bourdoncle’s sequential fixpoint computation (§7). Our approach generalizes Bourdoncle’s hierarchical total order and weak topological order to hierarchical partial order (HPO) and weak partial order WPO (§4). The iteration strategy is then based on the WPO of the dependency graph of the equation system. The use of partial orders, instead of total orders, enables us to generate an iteration strategy that is concurrent (§5). For the equation system in Figure 1(c), our approach would produce the iteration sequence represented as

$$S_3 \overset{\text{def}}{=} 0 \ [(1 \ 2 \ 3)]^* \ 8 \ [4 \ 5 \ 6]^* \ 7,$$

where $|$ represents concurrent execution. Thus, the iteration (sub)sequences $1 \ [2 \ 3]^*$ and $8$ can be computed in parallel, as well as the subsequences $5$ and $6$. However, unlike iteration sequence $S_2$, the value for $X_4$ will not be computed until the component $\{2, 3\}$ stabilizes. Intuitively, determinism is achieved by ensuring that no element outside the component will read the value of an element in the component until the component stabilizes. In our algorithm, the value of $2$ is read by elements outside of the component $\{2, 3\}$ only after the component stabilizes. Similarly for the value of $4$, elements outside of the component $\{4, 5, 6\}$ wait for the component to stabilize. Parallel fixpoint computation using WPO results in the same fixpoint as the sequential computation using WTO (§7).

### 3 Mathematical Preliminaries

A binary relation $R$ on set $S$ is a subset of the Cartesian product of $S$ and $S$; that is, $R \subseteq S \times S$. Given $S' \subseteq S$, let $R_{|S'} = R \cap (S' \times S')$. A relation $R$ on set $S$ is said to be one-to-one iff for all $w, x, y, z \in S$, $(x, z) \in R$ and $(y, z) \in R$ implies $x = y$, and $(w, x) \in R$ and $(w, y) \in R$ implies $x = y$. A transitive closure of a binary relation $R$, denoted by $R^+$, is the smallest transitive binary relation that contains $R$. A reflexive transitive closure of a binary relation $R$, denoted by $R^*$, is the smallest reflexive transitive binary relation that contains $R$.

A preorder $(S, R)$ is a set $S$ and a binary relation $R$ over $S$ that is reflexive and transitive. A partial order $(S, R)$ is a preorder where $R$ is antisymmetric. Two elements $u, v \in S$ are comparable in a partial order $(S, R)$ if $(u, v) \in R$ or $(v, u) \in R$. A linear (total) order or chain is a partial order in which every pair of its elements are comparable. A partial order $(S, R')$ is an extension of a partial order $(S, R)$ if $R \subseteq R'$; an extension that is a linear order is called a linear extension. There exists a linear extension for every partial order [Szlpijn 1930].

Given a partial order $(S, R)$, define $\| x \|^R \overset{\text{def}}{=} \{ y \in S \mid (x, y) \in R \}$, and $\| x \|^R \overset{\text{def}}{=} \{ v \in S \mid (v, x) \in R \}$, and $\| x, y \|^R \overset{\text{def}}{=} \| x \|^R \cap \| y \|^R$. A partial order $(S, R)$ is a forest if for all $x \in S$, $\| x \|^R \cap R$ is a chain.

**Example 3.1.** Let $(Y, T)$ be a partial order with $Y = \{y_1, y_2, y_3, y_4\}$ and $T = \{(y_1, y_2), (y_2, y_3), (y_2, y_4)\}$. Let $Y' = \{y_1, y_2\} \subseteq Y$, then $T|_{Y'} = \{(y_1, y_1), (y_1, y_2), (y_2, y_2)\}$.

$$\| y_1 \|^T = \{y_1, y_2, y_3, y_4\} \quad \| y_2 \|^T = \{y_2, y_3, y_4\} \quad \| y_3 \|^T = \{y_3\} \quad \| y_4 \|^T = \{y_4\}$$

$$\| y_1 \|^T = \{y_1\} \quad \| y_2 \|^T = \{y_1, y_2\} \quad \| y_3 \|^T = \{y_1, y_2, y_3\} \quad \| y_4 \|^T = \{y_1, y_2, y_4\}$$

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We see that the partial order \((Y, T)\) is a forest because for all \(y \in Y\), \((\{y\} \uparrow \land T)\) is a chain.

\[
\begin{align*}
\ll y_1, y_1 \uparrow \land T &= \{y_1\} & \ll y_1, y_1 \uparrow \land T &= \{y_1, y_2\} & \ll y_1, y_1 \uparrow \land T &= \{y_1, y_2, y_3\} & \ll y_1, y_1 \uparrow \land T &= \{y_1, y_2, y_4\} \\
\ll y_4, y_1 \uparrow \land T &= \emptyset & \ll y_4, y_2 \uparrow \land T &= \emptyset & \ll y_4, y_3 \uparrow \land T &= \emptyset & \ll y_4, y_4 \uparrow \land T &= \{y_4\}
\end{align*}
\]

A directed graph \(G(V, \rightarrow)\) is defined by a set of vertices \(V\) and a binary relation \(\rightarrow\) over \(V\). \(\rightarrow^*\) is a preorder capturing reachability among vertices: \(u \rightarrow^* v\) iff there is a path from vertex \(u\) to vertex \(v\) in \(G\). \(G\) is a directed acyclic graph (DAG) iff \(\rightarrow^*\) is a partial order, and a topological order of a DAG \(G\) corresponds to a linear extension of the partial order \(\rightarrow^*\).

A depth-first numbering (DFN) of a directed graph is the order in which vertices are discovered during a depth-first search (DFS) of the graph. A post depth-first numbering (post-DFN) of a directed graph is the order in which vertices are finished, that is all outgoing edges are explored, during a DFS of the graph. A depth-first search tree (DFST) of a directed graph is a tree of vertices whose edges are the ones in the graph used to discover the vertices during the DFS. These edges are called tree edges. An edge in the graph is a back edge if it directs the ancestor of the source in the DFST, and a forward edge if it directs the descendant of the source in the DFST. All other edges are cross edges [Cormen et al. 2009]. The lowest common ancestor (LCA) of two vertices in a tree is a vertex with both of the vertices as its descendants and whose distance from the root is the longest among such vertices. It is unique for all pairs of vertices.

A strongly connected component (SCC) of a directed graph \(G(V, \rightarrow)\) is a subgraph of \(G\) such that \(u \rightarrow^* v\) for all \(u, v\) in the subgraph. An SCC is trivial if it only consists of a single vertex without any edges. A feedback edge set \(B\) of a graph \(G(V, \rightarrow)\) is a subset of \(\rightarrow\) such that \((\rightarrow \setminus B)^*\) is a partial order; that is, the directed graph \(G(V, \rightarrow \setminus B)\) is a DAG. The problem of finding the minimum feedback edge set is NP-complete [Karp 1972].

Example 3.2. Let \(G(V, \rightarrow)\) be directed graph shown in Figure 1(a). The ids used to label the vertices \(V\) of \(G\) correspond to a depth-first numbering (DFN) of the directed graph \(G\). The following lists the vertices in increasing post-DFN numbering: 3, 5, 6, 7, 4, 2, 1, 8, 0. Edges (3, 2), (5, 4), and (6, 4) are back edges for the DFST that is assumed by the DFN, edge (8, 4) is a cross edge, and the rest are tree edges. The lowest common ancestor (LCA) of 5 and 7 in this DFST is 2. The subgraphs induced by the vertex sets \{2, 3\}, \{4, 5\}, \{4, 6\}, and \{4, 5, 6\} are all non-trivial SCCs. The minimum feedback edge set of \(G\) is \(F = \{(3, 2), (5, 4), (6, 4)\}\). We see that the graph \(G(V, \rightarrow \setminus F)\) is a DAG.

4 AXIOMATIC CHARACTERIZATION OF WEAK PARTIAL ORDER

This section introduces the notion of Weak Partial Order (WPO) presenting its axiomatic characterization and proving relevant properties. A constructive characterization is deferred to §6. The notion of WPO is built upon the notion of a hierarchical partial order, which we define first.

A Hierarchical Partial Order (HPO) is a partial order \((S, \leq)\) overlaid with a nesting relation \(N \subseteq S \times S\) that structures the elements of \(S\) into well-nested hierarchical components. As we will see in §5, the elements in each HPO component are iterated over until stabilization in a fixpoint iteration strategy, and the partial order enables concurrent execution.

Definition 4.1. A hierarchical partial order \(\mathcal{H}\) is a 3-tuple \((S, \leq, N)\) such that for all \(h, x, u, v \in S\):

H1. \((S, \leq)\) is a partial order.
H2. \(N \subseteq S \times S\) is one-to-one.
H3. \((x, h) \in N\) implies \(h < x\).
H4. Partial order \((C_{\mathcal{H}}, \subseteq)\) is a forest, where \(C_{\mathcal{H}} \overset{\text{def}}{=} \{ \ll h, x \leq \mid (x, h) \in N \}\) is the set of components.
H5. \(h \leq u \leq x\) and \((x, h) \in N\) and \(u \leq v\) implies either \(x < v\) or \(v \leq x\).
For each \((x, h) \in \mathbb{N}\), the set \(\| h, x \| \triangleq \{ u \in S \mid h \leq u \leq x\}\) defines a component of the HPO, with \(x\) and \(h\) referred to as the exit and head of the component. A component can be identified using either its head or its exit due to condition H2; we use \(C_h\) or \(C_x\) to denote a component with head \(h\) and exit \(x\). Condition H3 states that the nesting relation \(N\) is in the opposite direction of the partial order \(\leq\). The reason for this convention will be clearer when we introduce the notion of \(\text{WPO}\) (Definition 4.3), where we show that the nesting relation \(N\) has a connection to the feedback edge set of the directed graph. Condition H4 implies that the set of components \(C_H\) is well-nested; that is, two components should be either mutually disjoint or one must be a subset of the other.

Condition H5 states that if an element \(v\) depends upon an element \(u\) in a component \(C_x\), then either \(v\) depends on the exit \(x\) or \(v\) is in the component \(C_x\). Recall that \((x, h) \in \mathbb{N}\) and \(h \leq u \leq x\) implies \(u \in C_x\) by definition. Furthermore, \(v \leq x\) and \(u \leq v\) implies \(v \in C_x\). Condition H5 ensures determinism of the concurrent iteration strategy (§5); this condition ensures that the value of \(u\) does not "leak" from \(C_x\) during fixpoint computation until the component \(C_x\) stabilizes.

**Example 4.2.** Consider the partial order \((Y, T)\) defined in Example 3.1. Let \(N_1 = \{(y_3, y_1), (y_4, y_2)\}\). \((Y, T, N_1)\) violates condition H4 and is not an HPO. The components \(C_{y_3} = C_{y_1} = \{y_1, y_2, y_3\}\) and \(C_{y_2} = \{y_2, y_4\}\) are neither disjoint nor is one a subset of the other.

Let \(N_2 = \{(y_3, y_1)\}\). \((Y, T, N_2)\) violates condition H5 and is not an HPO. In particular, \(y_2 \in C_{y_3}\) and \((y_2, y_4) \in T\), but we do not have \(y_4 < y_2\) or \(y_4 \leq y_3\).

Let \(N_3 = \{(y_2, y_1)\}\). \((Y, T, N_3)\) is an HPO satisfying all conditions H1–H5.

Building upon the notion of an HPO, we now define a *Weak Partial Order (WPO)* for a directed graph \(G(V, \rightarrow)\). In the context of fixpoint computation, \(G\) represents the dependency graph of the fixpoint equation system. To find an effective iteration strategy, the cyclic dependencies in \(G\) need to be broken. In effect, a WPO partitions the preorder \(\rightarrow^*\) into some partial order \(\rightarrow\) and a feedback edge set \(B_W\) defined in terms of a nesting relation \(\hookrightarrow\).

**Definition 4.3.** A weak partial order \(\mathcal{W}\) for a directed graph \(G(V, \rightarrow)\) is a 4-tuple \((V, X, \rightarrow, \hookrightarrow)\) such that:

W1. \(V \cap X = \emptyset\).

W2. \(\rightarrow^* \subseteq X \times V\), and for all \(x \in X\), there exists \(v \in V\) such that \(x \rightarrow^* v\).

W3. \(\hookrightarrow \subseteq (V \cup X) \times (V \cup X)\).

W4. \(\mathcal{H}(V \cup X, \rightarrow^*, \hookrightarrow)\) is a hierarchical partial order (HPO).

W5. For all \(u \rightarrow v\), either (i) \(u \rightarrow^* v\), or (ii) \(u \in \| v, x \|_{\rightarrow}\) and \(x \rightarrow v\) for some \(x \in X\).

Condition W4 states that \(\mathcal{H}(V \cup X, \rightarrow^*, \hookrightarrow)\) is an HPO. Consequently, \((V \cup X, \rightarrow^*)\) is a partial order and \(\hookrightarrow\) plays the role of the nesting relation in Definition 4.1. We refer to the relation \(\rightarrow\) in WPO \(\mathcal{W}\) as the scheduling constraints, the relation \(\hookrightarrow\) as stabilization constraints, and the set \(X\) as the exits. Furthermore, the notion of components \(C_H\) of an HPO \(\mathcal{H}\) as defined in Definition 4.1 can be lifted to components of WPO \(C_W \triangleq \{ \| h, x \|_{\rightarrow^*} \mid x \rightarrow^* h \}\). Condition W4 ensures that the concurrent iteration strategy for the WPO is deterministic (§5). Condition W1 states that exits are always new. Condition W2 states that \(X\) does not contain any unnecessary elements.

Condition W5 connects the relation \(\rightarrow\) of the directed graph \(G\) with relations \(\rightarrow^*\) and \(\hookrightarrow\) used in the HPO \(\mathcal{H}(V \cup X, \rightarrow^*, \hookrightarrow)\) in condition W4. Condition W5 ensures that all dependencies \(u \rightarrow v\) in \(G\) are captured by the HPO \(\mathcal{H}\) either via a relation in the partial order \(u \rightarrow^* v\) or indirectly via the component corresponding to \(x \rightarrow v\). Consequently, W5 ensures that the resulting fixpoint is correct, as formalized by the following theorem:

**Theorem 4.4.** For graph \(G(V, \rightarrow)\) and its WPO \(\mathcal{W}(V, X, \rightarrow, \hookrightarrow), \rightarrow^* \subseteq (\rightarrow \cup \hookrightarrow)^*\).
Fig. 2. (a) Directed graph $G_1$. Vertices $V$ are labeled using depth-first numbering (DFN); (b) WPO $W_1$ for $G_1$ with exits $X = \{x_2, x_3, x_6\}$. 

**Proof.** By property $W_5$, for each $u \rightarrow v$, either $u \rightarrow^+ v$, or $u \in \llbracket v, x \rrbracket_{\rightarrow^*}$ and $x \rightarrow v$ for some $x \in X$. For the latter, $u \in \llbracket v, x \rrbracket_{\rightarrow^*}$, implies that $u \rightarrow^* x \rightarrow v$. Thus, $\rightarrow^* \subseteq (\rightarrow \cup \rightsquigarrow)\ast$. 

The next two theorems capture the fact that the WPO $W(V, X, \rightarrow, \rightsquigarrow)$ partitions the preorder $\rightarrow^*$ of $G(V, \rightarrow)$ into a partial order $\rightarrow^*$ and a feedback edge set $B_W$ defined using the nesting relation $\rightarrow$.

**Theorem 4.5.** For graph $G(V, \rightarrow)$ and its WPO $W(V, X, \rightarrow, \rightsquigarrow)$, $B_W$ is a feedback edge set for $G$, where $B_W \overset{\text{def}}{=} \{(u, v) \in \rightarrow \mid u \in \llbracket v, x \rrbracket_{\rightarrow^*} \text{ and } x \rightarrow v \text{ for some } x \in X\}$. 

**Proof.** Let $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n \rightarrow v_1$, be a cycle of $n$ distinct vertices in $G$. There exists $i \in [1, n)$ such that $v_i \in \llbracket v_{i+1}, x \rrbracket_{\rightarrow^*}$ and $x \rightarrow v_{i+1}$ for some $x \in X$. Otherwise, $v_1 \rightarrow^* \cdots \rightarrow^* v_n \rightarrow^* v_1$ (using $W_5$), and therefore, $v_1 \rightarrow^* v_n$ and $v_n \rightarrow^* v_1$. For a self loop, this is simply $v_1 \rightarrow^* v_1$. Both cases contradict that $\rightarrow^*$ is a partial order. Thus, $B_W$ cuts all cycles at least once and is a feedback edge set. 

**Theorem 4.6.** For graph $G(V, \rightarrow)$ and its WPO $W(V, X, \rightarrow, \rightsquigarrow)$, $(\rightarrow \setminus B_W)^+ \subseteq \rightarrow^*$, where $B_W \overset{\text{def}}{=} \{(u, v) \in \rightarrow \mid u \in \llbracket v, x \rrbracket_{\rightarrow^*} \text{ and } x \rightarrow v \text{ for some } x \in X\}$. 

**Proof.** Each edge $(u, v) \in B_W$ satisfies $W_5$-(ii) by Theorem 4.5. Therefore, all edges in $(\rightarrow \setminus B_W)$ must satisfy $W_5$-(i). Thus, $u \rightarrow^* v$ for all edges $(u, v) \in (\rightarrow \setminus B_W)$, and $(\rightarrow \setminus B_W)^+ \subseteq \rightarrow^*$. 

**Example 4.7.** Consider the directed graph $G_1(V, \rightarrow)$ in Figure 2(a). Figure 2(b) shows a WPO $W_1(V, X, \rightarrow, \rightsquigarrow)$ for $G_1$, where $X = \{x_2, x_3, x_6\}$. One can verify that $(V \cup X, \rightarrow^*, \rightsquigarrow)$ satisfies all conditions in Definition 4.1 and $(V, X, \rightarrow, \rightsquigarrow)$ satisfies all conditions in Definition 4.3. Furthermore, $B_{W_1} = \{(4, 3), (8, 6), (5, 2)\}$ is a feedback edge set for $G_1$.

Suppose we were to remove $x_6 \rightarrow 5$ and instead add $6 \rightarrow 5$ to $W_1$ (to more closely match the edges in $G_1$), then this change would violate condition $H_5$, and hence condition $W_4$.

If we were to only remove $x_6 \rightarrow 5$ from $W_1$, then it would still satisfy condition $W_4$. However, this change would violate condition $W_5$.

Given an HPO $H(S, \preceq, N)$ and its component $C \in C_H$, we now show that the tuple $(C, \preceq|_C, N|_C)$ is also an HPO, where $\preceq|_C = \preceq \cap (C \times C)$ and $N|_C = N \cap (C \times C)$ as defined in §3. This property enables us to use structural induction when proving properties of HPOs.

**Lemma 4.8.** For HPO $H(S, \preceq, N)$, $H|_C(C, \preceq|_C, N|_C)$, is an HPO for all $C \in C_H$. 

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Proof. We show that \( \mathcal{H}_C(C, \leq_C, N_C) \) satisfies all conditions H2–H5 in Definition 4.1. 

[H1] \( (C, \leq_C) \) is still a partial order.

[H2] \( N_C \) is still one-to-one.

[H3] Because components are well-nested, for all \((x', h') \in N_C, h', x' \) are in \( C \). Thus, \( h' \leq_C x' \).

[H4] Components in \( C \) are still well-nested.

[H5] If \( v \notin C \), H5 is vacuously true because \((u, v) \notin \leq_C \). Else, the order is maintained in \( \leq_C \). □

Similarly, given a graph \( G(V, \rightarrow) \) with its WPO \( \mathcal{W}(V, X, \rightarrow^+, \leftarrow) \) and a component \( C \in G_W \), the tuple \( (V', X', \rightarrow^+_C, \leftarrow^+_C) \) is a WPO for the subgraph \( G_C(V', \rightarrow^+_C) \), where \( V' \overset{\text{def}}{=} V \cap C \) and \( X' \overset{\text{def}}{=} X \cap C \). This property enables us to use structural induction when proving properties of WPOs.

**Theorem 4.9.** For graph \( G(V, \rightarrow) \) and its WPO \( \mathcal{W}(V, X, \rightarrow, \leftarrow) \), a sub-WPO \( \mathcal{W}_C(V', X', \rightarrow^+_C, \leftarrow^+_C) \) is a WPO for subgraph \( G_C(V', \rightarrow^+_C) \) for all \( C \in G_W \), where \( V' \overset{\text{def}}{=} V \cap C \) and \( X' \overset{\text{def}}{=} X \cap C \).

**Proof.** We show that \( \mathcal{W}_C(V', X', \rightarrow^+_C, \leftarrow^+_C) \) satisfies all conditions W1–W5 in Definition 4.3.

[W1] \( V \cap X = \emptyset \) implies \( V' \cap X' = \emptyset \).

[W2] \( \rightarrow^+_C \subseteq X' \times X' \) and because components are well-nested, [W2] still holds.

[W3] \( \rightarrow^+_C \subseteq (V' \cup X') \times (V' \cup X') \).

[W4] This property follows from Lemma 4.8.

[W5] We show that \( u \rightarrow^+_v \) implies \( u \rightarrow^+_C v \) if \( u \) and \( v \) is in \( C \). Let \( C = [h, x]_{\rightarrow^+} \) with \( x \rightarrow h \). If it is not true that \( u \rightarrow^+_C v \), then there exists \( w \in [u]_{\rightarrow^+} \cap [v]_{\rightarrow^+} \) such that \( w \notin C \). However, \( u \rightarrow^+ w \) and \( w \notin C \) implies \( x \rightarrow^+ w \) (using H5). This contradicts that \( \rightarrow^+ \) is a partial order, because \( w \in [v]_{\rightarrow^+} \) and \( v \in [h, x]_{\rightarrow^+} \) implies \( w \rightarrow^+ x \). Thus, condition W5 is satisfied. □

**Corollary 4.10.** For graph \( G(V, \rightarrow) \) and its WPO \( \mathcal{W}(V, X, \rightarrow, \leftarrow) \), a sub-WPO \( \mathcal{W}_C(V', X', \rightarrow^+_C, \leftarrow^+_C) \) is a WPO of subgraph \( G_C(V', \rightarrow^+_C) \) without its feedback edges directing the head of \( C \) for all \( C \in G_W \), where \( C' \) is \( C \) without its exit.

**Example 4.11.** Consider the graph \( G_1 \) and its WPO \( \mathcal{W}_1 \) (Figure 2). \( C_6 = \{6, 7, 8, 9, 9_6\} \) is a component of \( \mathcal{W}_1 \). Sub-WPO \( \mathcal{W}_{1|C_6} \), \( \overset{6}{\xrightarrow{7}} \overset{8}{\xrightarrow{9}} \overset{9}{\xrightarrow{8}} \), is a WPO of subgraph \( G_{1|C_6} \), \( \overset{6}{\xrightarrow{7}} \overset{8}{\xrightarrow{9}} \overset{9}{\xrightarrow{8}} \).

Let \( C'_6 = \{6, 7, 8, 9\} \) be \( C_6 \) without its exit \( x_6 \). Sub-WPO \( \mathcal{W}_{1|C'_6} \), \( \overset{6}{\xrightarrow{7}} \overset{8}{\xrightarrow{9}} \), is a WPO of subgraph \( G_{1|C'_6} \) without its feedback edge \((8, 6)\), \( \overset{6}{\xrightarrow{7}} \overset{8}{\xrightarrow{9}} \). □

## 5 DETERMINISTIC CONCURRENT FIXPOINT ALGORITHM

This section describes a deterministic concurrent fixpoint algorithm that computes the fixpoint of an equation system. Given the equation system \( X = F(X) \) with dependency graph \( G(V, \rightarrow) \), we first construct a WPO \( \mathcal{W}(V, X, \rightarrow, \leftarrow) \). The algorithm described in Figure 3 uses the WPO to compute the fixpoint of \( X = F(X) \). It defines a concurrent iteration strategy for a WPO: equations are applied concurrently by following the scheduling constraints \( \rightarrow \), while stabilization constraints \( \leftarrow \) act as “iterate until stabilization” operators, checking the stabilization at the exits and iterating the components.

**Example 5.1.** Consider the WPO \( \mathcal{W}_1 \) in Figure 2(b). The iteration sequence starts with 1, then 2, followed by the concurrent execution of 3 and 6. The subsequence 3, 4, and \( x_3 \) is iterated until component \( C_3 \) stabilizes at \( x_3 \). Similarly, component \( C_5 = \{6, 7, 8, 9, x_6\} \) is iterated until it stabilizes at \( x_6 \). 5 waits for components \( C_3 \) and \( C_6 \) to be stabilized. After 5, stabilization of component \( C_2 \) is checked, and depending on the result, \( C_2 \) may be iterated. Using the notation in §2, concurrent...
\[
\begin{align*}
\text{forall } v \in V, X[v] &\mapsto (v \text{ is entry of } \top) \\
\text{forall } v \in V \cup X, N[v] &\mapsto 0
\end{align*}
\]

\[
\begin{align*}
v \in V &\quad N[v] = \text{NumSchedPreds}(v) \quad \text{INIT} \\
\text{ApplyF}(v) &\quad \text{forall } v \mapsto w, N[w] \mapsto (N[w] + 1) \quad N[v] \mapsto 0 \quad \text{NONEXIT}
\end{align*}
\]

\[
\begin{align*}
x \in X &\quad N[x] = \text{NumSchedPreds}(x) \quad \text{ComponentStabilized}(x) \quad \text{COMPSTABILIZED} \\
\text{forall } x \mapsto w, N[w] \mapsto (N[w] + 1) &\quad N[x] \mapsto 0
\end{align*}
\]

\[
\begin{align*}
x \in X &\quad N[x] = \text{NumSchedPreds}(x) \quad \neg\text{ComponentStabilized}(x) \quad \text{COMPNOTSTABILIZED} \\
\text{SetNForComponent}(x) &\quad N[x] \mapsto 0
\end{align*}
\]

\[
\begin{align*}
\text{ApplyF}(v) &\quad \text{def} = X[v] \mapsto (v \in \text{image of } \rightarrow ? X(v) \varphi F_U(X) : F_U(X)) \\
\text{ComponentStabilized}(x) &\quad \text{def} = \text{there exists } h, x \rightarrow h \land F_R(h) \subseteq X[h] \\
\text{SetNForComponent}(x) &\quad \text{def} = \text{forall } v \in C_X, N[v] \mapsto \text{NumOuterSchedPreds}(v, x) \\
\text{NumSchedPreds}(v) &\quad \text{def} = \{|u \in V \cup X | u \rightarrow v\} \\
\text{NumOuterSchedPreds}(v, x) &\quad \text{def} = \{|u \in V \cup X | u \rightarrow v, u \notin C_X, v \in C_X\}
\end{align*}
\]

Fig. 3. Deterministic concurrent fixpoint algorithm for WPO. Rule \text{INIT} is applied once initially, and rules \text{NONEXIT, COMPSTABILIZED, COMPNOTSTABILIZED} are applied concurrently whenever their conditions are met. \(X\) maps an element in \(V\) to its value. \(N\) maps an element in \(V \cup X\) to its count of executed predecessors. Operations on \(N\) are atomic.

The concurrent fixpoint algorithm in Figure 3 consists of rules that define the operations to be applied when their conditions are met. Except for the initialization rule \text{INIT}, which is applied once at the beginning, rules are applied concurrently whenever some element in \(V \cup X\) satisfies the conditions. The algorithm uses a value map \(X\), which maps an element in \(V\) to its abstract value, and a count map \(N\), which maps an element in \(V \cup X\) to its counts of executed predecessors. Access to the value map \(X\) is synchronized by scheduling constraints, and operations on \(N\) are assumed to be atomic. Rule \text{INIT} initializes values for elements in \(V\) to \(\bot\) except for the entry of the graph, whose value is initialized to \(\top\). The counts for elements in \(V \cup X\) are all initialized to 0.

Rule \text{NONEXIT} applies to a non-exit element \(v \in V\) whose predecessors in \(\rightarrow\) are all executed \((N[v] = \text{NumSchedPreds}(v))\). This rule applies the equation \(F_U\) to update the value \(X_U(\text{ApplyF}(v))\). Definition of the function \text{ApplyF} shows that the widening is applied at the image of \(\rightarrow\) (see Theorem 5.5). The rule then notifies the scheduling successors of \(v\) that \(v\) has executed by incrementing their counts. Because elements can be iterated multiple times when inside a component, the count of an element is reset after its execution. If there are no components in the WPO, only this \text{NONEXIT} rule gets applied, and the algorithm reduces to a DAG scheduling algorithm.

Example 5.2. Consider WPO \(W_1\) in Figure 2(b) and its iteration sequence in Table 1. The initial value of \(N[8]\) is 0. Applying \text{NONEXIT} to 7 and 9 increments \(N[8]\) to 2. \(N[8]\) now equals \(\text{NumSchedPreds}(8)\), and \text{NONEXIT} is applied to 8. Applying \text{NONEXIT} to 8 updates \(X_8\) by applying the equation \(F_8\), increments \(N[x_8]\), and resets \(N[8]\) to 0. Due to the reset, same thing happens when \(C_8\) is iterated once more.
Rules CompStabilized and CompNotStabilized are applied to an exit $x$ ($x \in X$) whose predecessors are all executed ($N[x] = \text{NumSchedPreds}(x)$). If the component $C_x$ is stabilized, CompStabilized is applied, and CompNotStabilized otherwise. A component is stabilized if it once more does not change the values of elements inside the component. Boolean function ComponentStabilized checks the stabilization of $C_x$ by checking the stabilization of its head (see Theorem 5.7). In case of stabilization, rule CompStabilized simply notifies the successors of $x$ and resets the $x$’s count.

Example 5.3. Consider WPO $\mathcal{W}_1$ (Figure 2(b)) and its iteration sequence in Table 1. The initial value of $N[x_3]$ is 0. Applying NonExit to 8 increments $N[x_6]$ to 1, which equals NumSchedPreds($x_6$). Stabilization check for component $C_6$ is run at $x_6$, and if it is stabilized, CompStabilized is applied to $x_6$. Applying CompStabilized to $x_6$ increments $N[5]$ and resets $N[x_3]$ to 0.

If the component $C_x$ is not stabilized, rule CompNotStabilized is applied instead. This rule does not notify the successors of $x$, blocking further advancement until the component stabilizes. To drive the iteration of $C_x$, each count for an element in $C_x$ is set to SetNForComponent($x$), which is the number of its scheduling predecessors not in $C_x$. In particular, the count for the head of $C_x$, whose scheduling predecessors are all not in $C_x$, is set to the number of all scheduling predecessors, allowing the rule NonExit to be applied to the head. The map NumOuterSchedPreds($v$, $x$), which returns the number of outer scheduling predecessors of $v$ w.r.t. component $C_x$, can be computed by running the WPO construction twice: in the first run, compute NumSchedPreds; in the second run, initialize NumOuterSchedPreds to 0s, and set NumOuterSchedPreds($v'$, exit[$v$]) to NumSchedPreds($v'$) minus the number of scheduling predecessors of $v'$ found so far, if scheduling constraint targeting $v'$ with $v \neq \text{exit}[v]$ is found in Line 28 and Line 49 of Algorithm 3 in §6. The rule also resets the count for $x$.

Example 5.4. Let $G_2(V, \rightarrow)$ be and its WPO $\mathcal{W}_2$ be .

An iteration sequence generated by the concurrent fixpoint algorithm for WPO $\mathcal{W}_2$ is:

| Scheduled element $u \in V \cup X$ | Time step in $N$ |
|----------------------------------|------------------|
| 1 2 3 4 $x_3$ 3 4 $x_3$ 3 4 $x_3$ 5 $x_2$ 10 |

Consider the element 4, whose scheduling predecessors in the WPO are 3, 5, and 6. Furthermore, 4 \(\in C_3\) and 4 \(\in C_2\) with $C_3 \subseteq C_2$. After NonExit is applied to 4, $N[4]$ is reset to 0. Then, if the stabilization check of $C_3$ fails at $x_3$, CompNotStabilized sets $N[4]$ to 2, which is NumOuterSchedPreds(4, $x_3$). If it is not set to 2, then the fact that elements 5 and 6 are executed will not be reflected in $N[4]$, and the iteration of $C_3$ will be blocked at element 4. Similarly, if the stabilization check of $C_2$ fails at $x_2$, CompNotStabilized sets $N[4]$ to NumOuterSchedPreds(4, $x_2$) = 1.
Notice that in ApplyF, the image of \( \rightarrow \) is chosen as the set of widening points. These are heads of the components. As we saw in Theorem 4.5, all feedback edges target one of these heads. Therefore, all the cycles in the program contain at least one head, and choosing heads as the widening points guarantee the termination of the fixpoint computation:

**Theorem 5.5.** Given a dependency graph \( G(V, \rightarrow) \) and its WPO \( W(V, X, \rightarrow, \rightarrow^+) \), the image of \( \rightarrow \), or equivalently the set of heads of the components, is an admissible set of widening points.

**Proof.** Theorem 4.5 states that the set \( B_W = \{(u, v) \in \rightarrow \mid u \in \text{dom } X, x \rightarrow v \text{ for some } x \in X\} \) is a feedback edge set. Consequently, image of \( \rightarrow \), or \( \{h \mid x \rightarrow^+ h \text{ for some } x \in X\} \), forms a feedback vertex set. That is, without these vertices, cycles in the graph are removed. Therefore, the set \( W \) is an admissible set of widening points.

**Example 5.6.** The image of \( \rightarrow \), \( \{2, 3, 6\} \), is an admissible set of widening points for the WPO \( W_1 \) in Figure 2(b).

The following theorem justifies our definition of ComponentStabilized; viz., checking the stabilization of \( X_h \) is sufficient for checking the stabilization of the component \( C_h \), where \( h \) is the head of \( C_h \).

**Theorem 5.7.** During the execution of concurrent fixpoint algorithm with WPO \( W(V, X, \rightarrow, \rightarrow^+) \), stabilization of the head \( h \) implies the stabilization of the component \( C_h \) at its exit for all \( C_h \in C_W \).

**Proof.** Suppose there exists an element \( v \in C_h \) that is not stabilized despite the stabilization of \( h \). That is, \( X_v \) changes if \( C_h \) is iterated once more. For this to be possible, there must exist \( u \) such that \( u \rightarrow v \) and whose value \( X_u \) changed after the last application of equation \( F_v \). By \( W5 \) and \( u \rightarrow v \), it’s either (i) \( u \rightarrow^+ v \) or (ii) \( u \in \text{dom } X \) and \( x \rightarrow v \) for some \( x \in X \). It cannot be (i) because if it were true, equation \( F_v \) cannot be applied after \( F_v \). Even if \( u \) were in some other component \( C_{v'} \), due to \( H5 \), \( v' \rightarrow^+ v \), resulting in the same conclusion. Therefore, (ii) should be true. By \( H4 \), \( C_v \subseteq C_h \). However, because \( u \in C_v \), our algorithm checks the stabilization of \( v \) at the exit of \( C_v \) after the last application of equation \( F_v \). This contradicts the assumption that \( X_u \) changed after the last application of equation \( F_v \).

While the actual order that the rules are applied might differ in each run, the concurrent fixpoint algorithm in Figure 3 always computes the same fixpoint.

**Theorem 5.8.** Given a WPO \( W(V, X, \rightarrow, \rightarrow^+) \), concurrent fixpoint algorithm in Figure 3 is deterministic, computing the same fixpoint for all possible iteration sequences.

**Proof.** We use structural induction on the WPO \( W \) to show this.

**Base case (\( \rightarrow = \emptyset \)):** By Theorem 4.5, feedback edge set \( B_W \) is empty. This means that the graph is acyclic. That is, \((\rightarrow \setminus B_W) = \rightarrow \). By Theorem 4.6, \( \rightarrow^+ \subseteq \rightarrow^+ \). Therefore, the algorithm is deterministic.

**Inductive case (\( \rightarrow \neq \emptyset \)):** Let \( C_W^0 \) be the set of maximal components of the WPO \( W \). Assume that the theorem holds for the sub-WPOs defined as in Corollary 4.10 for all maximal components in \( C_W^0 \). By Theorem 4.5 and \( H4 \), source and target of a feedback edge in \( B_W \) is always contained in the same maximal component. In other words, if there exists no \( C \in C_W^0 \) that contains both \( u, v \) for \( u \rightarrow v \), \((u, v) \notin B_W \). Hence by Theorem 4.6, \((u, v) \in \rightarrow^+ \). Also by \( H5 \), intermediate value of \( u \) does not leak to \( v \). That is, \( v \) always uses stabilized value of \( u \). Due to the inductive hypothesis on the sub-WPO \( W|_{C_h^0} \) for \( C_h \in C_W^0 \), each iteration on \( C_h \) is deterministic. The only dependencies that the sub-WPO does not cover are the dependencies corresponding to feedback edges targeting the head \( h \). Because the equation \( F_h \) is applied only at the beginning of each iteration, the feedback edges do not affect the determinism in each iteration. Therefore, the algorithm is deterministic by induction. \( \square \)
ConstructWPO

Algorithm 1: ConstructWPOTD(G)

Input: Directed graph G(V, \rightarrow)
Output: WPO W(V, X, \rightarrow, SD)
1 C1, C2, ..., Ck := SCC(G)  \quad \triangleright \text{Maximal SCCs.}
2 foreach i \in [1, k] do
3 (Vi, Xi, \rightarrow_i, -\rightarrow_i), hi, xi := ConstructWPOTD(SCC(Ci))  \quad \triangleright \text{WPOs for SCCs.}
4 V, X, \rightarrow, SD := \bigcup_{i=1}^{k} Vi, \bigcup_{i=1}^{k} Xi, \bigcup_{i=1}^{k} i \rightarrow_i, \bigcup_{i=1}^{k} i \rightarrow_i  
5 foreach u \rightarrow v \text{ s.t. } u \in C_i, v \in C_j \text{ and } i \neq j do
6   C_j = \text{maximal SCC containing } u \text{ but not } v.
7 return (V, X, \rightarrow, SD)

Algorithm 2: ConstructWPOTD_SCC(G)

Input: SCC G(V, \rightarrow)
Output: WPO W(V, X, \rightarrow, SD), head h, exit xh
1 h := minargv(DFN(v) \mid v \in V)  \quad \triangleright \text{Minimum DFN.}
2 B := \{(v, h) \mid v \in V \text{ and } v \rightarrow h\}
3 if |B| = 0 then
4   return ((h), \emptyset, \emptyset, h, h)  \quad \triangleright \text{Trivial SCC.}
5 x_h := \text{new exit}
6 V := V \cup \{x_h\}
7 \rightarrow := (\rightarrow \setminus B) \cup \{(v, x_h) \mid (v, h) \in B\}
8 V, X, \rightarrow := ConstructWPOTD((V, \rightarrow))
9 V, X := V \setminus \{x_h\}, X \cup \{x_h\}
10 \rightarrow := \rightarrow \cup \{(x_h, h)\}  \quad \triangleright \text{For removed } B.
11 return (V, X, \rightarrow, SD), h, x_h

6 ALGORITHMS FOR WPO CONSTRUCTION

This section presents two algorithms for constructing a WPO for a graph G(V, \rightarrow). The first algorithm, ConstructWPOTD, is a top-down recursive algorithm that is inefficient but intuitive (§6.1). The second one, ConstructWPOBU, is an efficient bottom-up iterative algorithm that has almost-linear time complexity (§6.2). Both algorithms do not introduce superfluous scheduling constraints that could restrict the parallelism during the fixpoint computation.

6.1 Top-down Recursive Construction

Algorithm 1 presents a top-down recursive algorithm ConstructWPOTD, which acts as a proxy between the axiomatic characterization of WPO in §4 and the efficient construction algorithm ConstructWPOBU in §6.2.

ConstructWPOTD begins with the identification of the maximal strongly connected components (SCCs) of the input graph G on Line 1. The WPO for each SCC is constructed by calling ConstructWPOTD_SCC (Algorithm 2) on Line 3. This call returns the WPO (Vi, Xi, \rightarrow_i, -\rightarrow_i), head hi, and exit xi for each SCC Ci, and ensures that (xi, hi) \in \rightarrow_i, and \[\hat{h}_i, x_i\] = Vi \cup Xi. Line 4 initializes the WPO for graph G to the union of the WPOs constructed for each SCC Ci, utilizing the inductive structure mentioned in Theorem 4.9. On Line 6, scheduling constraints are added to cover the dependencies crossing the SCCs: x \rightarrow v is added for each u \rightarrow v such that x is the exit for the maximal SCC that contains u but not v. These added scheduling constraints ensure that W5 and H5 are satisfied by the output WPO for each edge u \rightarrow v.

ConstructWPOTD_SCC in Algorithm 2 takes as input an SCC and returns a WPO, a head, and an exit for this SCC. ConstructWPOTD_SCC constructs this by removing the feedback edges that target the head to break the SCC, and using ConstructWPOTD on the modified graph to construct a WPO for it. This directly exposes the inductive structure mentioned in Corollary 4.10. With new exit and stabilization constraint, the final output satisfies W5 for the removed feedback edges. Line 1 chooses a vertex with minimum DFS numbering (DFN) as the head. This assumes that DFS was already performed once on the graph first given to ConstructWPOTD. Feedback edges that target the head h are identified on Line 2. If there are no such edges, the SCC is trivial, and a WPO with single
element is returned on Line 4. For non-trivial SCCs, the identified feedback edges are removed, and new exit \( x_h \) is added as a unique sink in the graph on Lines 5–7. The call to \( \text{ConstructWPO}^G \) on Line 8 returns a WPO for the modified graph, and the added exit is moved from \( V \) to \( X \) on Line 9. Finally, \( x_h \rightarrow h \) is added on Line 10 to satisfy W5 for the removed feedback edges.

**Example 6.1.** Consider the graph \( G_3 \) in Figure 4(a). \( \text{SCC}(G_3) \) on Line 1 of \( \text{ConstructWPO}^G \) returns a trivial SCC with vertex 1 and three non-trivial SCCs with vertex sets \( \{5, 6, 7, 8\} \), \( \{2, 3\} \), and \( \{4\} \). For the trivial SCC, the call to \( \text{ConstructWPO}^G \) on Line 3 returns \( (1, 1, 1) \). For the non-trivial SCCs, \( \text{ConstructWPO}^G \) returns \( (5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 5, x_5), (2 \rightarrow 3 \rightarrow 2, x_2), \) and \( (4 \rightarrow 5, 4, x_4) \). On Line 6, the scheduling constraints \( 1 \rightarrow 2, 1 \rightarrow 5, x_5 \rightarrow 3, x_3 \rightarrow 4, \) and \( x_2 \rightarrow 4 \) are added for the edges \( 1 \rightarrow 2, 1 \rightarrow 5, 7 \rightarrow 3, 8 \rightarrow 4, \) and \( 3 \rightarrow 4 \), respectively. The result is identical to \( W_3 \) in Figure 4(c).

Consider the execution of \( \text{ConstructWPO}^G \) when given the SCC with vertex set \( V = \{5, 6, 7, 8\} \) as input. On Line 1, the vertex 5 is chosen as the head \( h \), because it has the minimum DFN among \( V \). The set \( B \) on Line 2 is \( B = \{8, 5\} \). The input SCC is modified on Lines 5–7, and the call to \( \text{ConstructWPO}^G \) on Line 8 returns \( 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 5 \). Moving \( x_5 \) to \( X \) on Line 9 and adding \( x_5 \rightarrow 5 \) on Line 10 yields the WPO for the SCC.

We now prove that \( \text{ConstructWPO}^G \) in Algorithm 1 does actually construct a WPO.

**Lemma 6.2.** Let \( (V, X, \rightarrow, \rightarrow, h, x_h) \) be the output of \( \text{ConstructWPO}^G \). \( h \rightarrow v \) and \( v \rightarrow x_h \) for all \( v \in V \cup X \). That is, \( [h, x_h]_{\rightarrow, \rightarrow} = V \cup X \).

**Proof.** We use structural induction on the output of the call to \( \text{ConstructWPO}^G \) to prove this theorem. On Line 8 of \( \text{ConstructWPO}^G \), \( \text{ConstructWPO}^G \) is called with feedback edges targeting \( h \) removed. Without these feedback edges, calls to \( \text{ConstructWPO}^G \) on Line 3 of \( \text{ConstructWPO}^G \) get strictly smaller SCCs, which are the nested SCCs of the original input SCC, as the inputs. Trivial SCCs are the base cases and are handled on Line 4 of \( \text{ConstructWPO}^G \).

- **Base case (\( B = \emptyset \)):** This is the case where the input is a trivial SCC. On Line 4 of \( \text{ConstructWPO}^G \), \( [h, x_h]_{\rightarrow, \rightarrow} = \{h\} = V \cup X \).

- **Inductive case (\( B \neq \emptyset \)):** The input is a non-trivial SCC. Assume that the theorem holds for the outputs of the calls to \( \text{ConstructWPO}^G \) on Line 3 of \( \text{ConstructWPO}^G \), where \( C_i \) is a nested SCC.
of $G$. Because $h_i$ has the minimum DFN in $C_i$, if there exists a vertex in $C_i$ with a predecessor outside of $C_i$ in the graph, $h_i$ must also have a predecessor, and this will create scheduling constraint targeting $h_i$ on Line 6 of ConstructWPO$^TD$. Also, if there exists a vertex in $C_i$ with a successor outside of $C_i$ in the graph, scheduling constraint with source $x_i$ will be created on Line 6 of ConstructWPO$^TD$. Because the input is originally an SCC, every super-node (an SCC contracted to a single vertex) is reachable from $h$ in the DAG of super-nodes. Also, due to the edges added on Line 7 of ConstructWPO$^TD_{SCC}$, $x_h$ is reachable from every super-node. By induction, the theorem holds for the final output. □

**Theorem 6.3.** The structure output by ConstructWPO$^TD$ in Algorithm 1 satisfies the axiomatic characterization of WPO in Definition 4.3.

**Proof.** We show that the constructed structure satisfies all conditions W1–W5 in Definition 4.3. We use structural induction on the output of the call to ConstructWPO$^TD$ to prove some of the properties. on Line 3 of ConstructWPO$^TD$, ConstructWPO$^TD_{SCC}$ is called for the maximal SCCs. ConstructWPO$^TD_{SCC}$ then removes the feedback edges targeting the head and calls ConstructWPO$^TD$ on the modified graph on Line 8 of ConstructWPO$^TD_{SCC}$. The call receives graph with strictly lesser maximum nesting depth of SCCs. DAGs are the base cases.

- [W1] $V$ equals the vertex set of the input graph, and $X$ consists of new exits on Line 5 of ConstructWPO$^TD_{SCC}$ only.
- [W2] For all exits, $x_h \rightarrow h$ is added on Line 10 of ConstructWPO$^TD_{SCC}$. This is the only place stabilization constraints are created.
- [W3] All scheduling constraints are created on Line 6 of ConstructWPO$^TD$.
- [W4-H1] Assume that this holds true for the output of call to ConstructWPO$^TD$ on Line 8 of ConstructWPO$^TD_{SCC}$. That is, $(\rightarrow^*, V_i \cup X_i)$ is partial order for all $i$ on Line 3 of ConstructWPO$^TD$. Because the graph with maximal SCCs contracted to single vertices (super-nodes) is acyclic, $(\rightarrow^*, V \cup X)$ is antisymmetric after Line 6 of ConstructWPO$^TD$. By induction, $(\rightarrow^*, V \cup X)$ is a partial order.
- [W4-H2] Only one stabilization constraint is created per exit on Line 10 of ConstructWPO$^TD_{SCC}$. Because feedback edges to $h$ are removed, no non-trivial SCC contains $h$ afterwards, so $h$ does not become target of another $\rightarrow^*$.
- [W4-H3] By Lemma 6.2, $x_h \rightarrow h$ implies $h \rightarrow^* x_h$.
- [W4-H4] Outputs of ConstructWPO$^TD_{SCC}$ on Line 3 of ConstructWPO$^TD$ are disjoint because the SCCs are disjoint. Also, by Lemma 6.2, output of call to ConstructWPO$^TD_{SCC}$ on Line 3 of ConstructWPO$^TD$ is nested.
- [W4-H5] Assume that this property holds for the output of call to ConstructWPO$^TD$ on Line 8 of ConstructWPO$^TD_{SCC}$. All additional scheduling constraints have their sources from exits on Line 6 of ConstructWPO$^TD$. By induction, nothing leaks in the final output.
- [W5] For $u \rightarrow v$, either (i) scheduling constraint is added in Line 6 of ConstructWPO$^TD$, or (ii) stabilization constraint is added in Line 10 of ConstructWPO$^TD_{SCC}$. In the case of (i), $u \rightarrow^* x_i$ by Lemma 6.2, and with added $x_i \rightarrow v$, $u \rightarrow^* v$. In the case of (ii), $u \in \{h, x_h\} \rightarrow^*$ by Lemma 6.2 where $v = h$. □

WPO constructed by ConstructWPO$^TD$ does not include superfluous scheduling constraints, which could reduce concurrency during the fixpoint computation.

**Theorem 6.4.** For a graph $G(V, \rightarrow)$, WPO $\mathcal{W}(V, X, \rightarrow^*)$ constructed by ConstructWPO$^TD$ in Algorithm 1 has the smallest $\rightarrow^*$ among the WPOs for $G$ with the same set of $\rightarrow^*$.

**Proof.** Again, we use structural induction on the output of the call to ConstructWPO$^TD$ on Line 8 of ConstructWPO$^TD_{SCC}$ to show this.
[Base case ($\rightarrow^* = \emptyset$):] By Theorem 4.5, $B_W = \emptyset$. That is, input graph is acyclic. In this case, $\text{ConstructWPO}^{TD}$ creates WPO with $\rightarrow = \rightarrow^*$. This has the smallest $\rightarrow^*$.

[Inductive case ($\rightarrow^* \neq \emptyset$):] Let $C_W^0$ be the set of maximal components of the constructed WPO $W$. Assume that the theorem holds for the sub-WPOs defined as in Corollary 4.10 for all maximal components in $C_W^0$. To satisfy both W5 and H5 for $u \rightarrow v$ that crosses the maximal SCCs, all WPOs for $G$ with same set of $\rightarrow^*$ must have scheduling constraint from the exit of the maximal component that contains $u$ to some $w$ outside of the component that will satisfy $w \rightarrow^* v$. $\rightarrow^*$ is the smallest when $w = v$ as on Line 6 of $\text{ConstructWPO}^{TD}$. By induction, the theorem holds.

### 6.2 Bottom-up Iterative Construction

We now present an efficient, almost-linear time iterative construction algorithm $\text{ConstructWPO}^{BU}$ in Algorithm 3. The parameter lift is relevant only in §7, and it is assumed to be false throughout this section. $\text{ConstructWPO}^{BU}$ and $\text{ConstructWPO}^{TD}$ construct the same WPO.

$\text{ConstructWPO}^{BU}$ uses a disjoint-set data structure to track the partition of vertices. Each subset represents maximal set of vertices that is currently known to be strongly connected during the execution of the algorithm. It is used to find all set of SCCs in the graph. If some subsets are found to be connected during the execution, they will be merged into a single subset to reflect this. rep and merge are two operations for the disjoint-set data structure. $\text{rep}(v)$ returns the representative of the subset that contains vertex $v$. For our purposes, we make it so that vertex with minimum DFN in the subset is the representative of the subset. Initially, $\text{rep}(v) = v$ for all vertices $v \in V$. $\text{merge}(v, h)$ merges the subsets containing $v$ and $h$, and assigns $h$ to be the representative for the combined subset.

**Example 6.5.** Consider the graph $G_3$ in Figure 4(a). Let $1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8$ represent the current partition, where the underlined vertices are the representatives of the subsets. For instance, $\text{rep}(1) = 1$, $\text{rep}(6) = \text{rep}(7) = 6$. If it is found that vertices 5, 6, 7, 8 are strongly connected, then $\text{merge}(6, 5)$ and $\text{merge}(8, 5)$ are called to update the partition to $1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8$. We now have $\text{rep}(5) = \text{rep}(7) = \text{rep}(8) = \text{rep}(6) = 5$.

$\text{ConstructWPO}^{BU}$ starts with a depth-first search on the input graph. A depth-first search tree (DFST) for the graph is created on Line 2, and its back edges and cross/forward edges are identified on Lines 3 and 4. Then, data structures used in the algorithm are initialized on Line 6. Map exit is used to map an SCC to its corresponding exit. That is, it returns $x_h$ for $h$. Because $h = x_h$ for trivial SCCs, exit[$v$] is initialized to $v$ itself. It is then updated on Line 34 when non-trivial SCCs are found.

Map R is used when restoring the removed cross and forward edges. Function $\text{FindNestedSCCs}$ relies on the assumption that the graph is reducible [Hecht and Ullman 1972; Tarjan 1973]. It follows the edges backwards to find nested SCCs using $\text{rep}(p)$ instead of predecessor $p$, as on Lines 38 and 45, to skip the search inside the nested SCCs. $\text{rep}(p)$ is the unique entry to the nested SCC that contains the predecessor $p$ if the graph is reducible. To make this algorithm work for irreducible graphs as well, cross and forward edges are removed from the graph initially by function $\text{RemoveAllCrossForwardEdges}$ (called on Line 7) to make it reducible. Removed edges are then restored by function $\text{RestoreCrossForwardEdges}$ (called on Line 11), right before they are used. The graph is guaranteed to be reducible when restoring a removed edge $u \rightarrow v$ as $u \rightarrow \text{rep}(v)$ when $h$ is the lowest common ancestor (LCA) [Tarjan 1979] of $u, v$ in the DFST. Cross and forward edges are removed on Line 17 and are stored at their LCAs in $T$ on Line 18. Then, as $h$ hits the LCAs in the loop, the removed edges are restored on Line 21. Because the graph edges are modified, map 0 is used to track the original edges. 0[$v$] returns set of original non-back edges that now targets $v$ after the modification. It is initialized and updated on Lines 9 and 22, respectively.
A call to function ConstructWPOForSCC on Line 12 constructs a WPO for maximal SCC whose vertex with minimum DFN is $h$. For example, ConstructWPOForSCC(5) constructs WPO for SCC with vertex set $\{5, 6, 7, 8\}$. Because the loop traverses the vertices in descending DFN, it is guaranteed to construct all WPOs for nested SCCs before constructing WPO for the enclosing SCC. For example, ConstructWPOForSCC(6) and ConstructWPOForSCC(8) are called before ConstructWPOForSCC(5), creating WPOs for nested SCCs with vertex sets $\{6, 7\}$ and $\{8\}$. Therefore, ConstructWPOForSCC can reuse the constructed sub-WPOs.

Function FindNestedSCCs is called on Line 24 to find the representatives of the nested SCCs. $N_h$ contains these representatives, and $P_h$ contains $h$'s predecessors of back edges. If $P_h$ is empty,
the SCC is trivial, and the function immediately returns as on Line 26. Line 28 adds scheduling constraints for the dependencies crossing the nested SCCs. As in \texttt{ConstructWPO}^{TD}, this must be from the exit of maximal SCC that contains \( u \) but not \( v' \) for \( u \to v' \). Because \( u \to v' \) is now \( u \to r\text{ep}(v') \), 

0[\text{v}'], where \( v = r\text{ep}(v') \), is looked up to find \( u \to v' \). exit is used to find the exit, where \( r\text{ep}(u) \) is the representative of maximal SCC that contains \( u \) but not \( v \). If the parameter lift is \textit{true}, scheduling constraint targeting \( v \) is also added, forcing all scheduling predecessors outside of a component to be visited before the component’s head. Similarly, function \texttt{ConnectWPOsOfMaximalSCCs} is called after the loop on Line 49 to connect WPOs of maximal SCCs. The exit, scheduling constraints to the the exit, and stabilization constraints are added on Lines 31–33. After the WPO is constructed, Line 34 updates the map exit, and Line 35 updates the partition.

\textbf{Example 6.6.} Table 2 describes the steps of \texttt{ConstructWPO}^{BU} for the graph \( G_3 \) in Figure 4. The \( \rightarrow \) column shows the modifications in the graph edges, the Partition column shows the changes in the disjoint-set data structure, and the WPO column shows the WPO constructed so far.

Each row show the changes made in each step of the algorithm. Row ‘Init’ shows the initialization step on Lines 6–7. Row ‘\( h = k' \)’ shows the \( k \)-th iteration of the loop on Lines 10–12. The loop iterates over the vertices in descending DFN: 8, 7, 6, 5, 4, 3, 2, 1. Row ‘Final’ shows the final step after the loop on Line 13.

In the initialization step, the cross and forward edges \{\( (7, 3), (8, 4) \}\} are removed, making \( G \) reducible. The removed edges are added back as \{\( (7, \text{rep}(3)), (8, \text{rep}(4)) \)\} in \( h = 1 \), where 1 is the LCA of both \( (7, 3) \) and \( (8, 4) \). Note that the removed edges play no role in the previous steps. Also, \( G \) remains reducible after restoration.

In step \( h = 5 \), WPOs for the nested SCCs are connected with \( 5 \to 6 \) and \( x_8 \to 8 \). The new exit \( x_5 \) is created and connected to the WPO via \( 8 \to x_5 \). A stabilization constraint \( x_3 \to x_5 \) is also added.

In the final step, the scheduling constraints \( 1 \to 2, 1 \to 5, x_5 \to 3, x_3 \to 4, \) and \( x_2 \to 4 \) are added, connecting the WPOs for maximal SCCs. Furthermore, if lift is \textit{true}, scheduling constraint \( x_5 \to 2 \) is added.

\texttt{ConstructWPO}^{BU} adapts Tarjan-Havlak-Ramaligam’s almost-linear time algorithm [Ramalingam 1999] for constructing Havlak’s loop nesting forest (LNF) [Havlak 1997]. A LNF represents the nesting relationship among the SCCs; for instance, Figure 4(b) shows the LNF corresponding to the graph \( G_3 \). While the LNF only represents with the nesting relationship among SCCs, the WPO contains additional information about the scheduling constraints. These scheduling constraints are used to generate the concurrent iteration strategy for fixpoint computation. Lines in \texttt{ConstructWPO}^{BU} indicated by \( \star \) were added to the algorithm in Ramalingam [1999].

We now prove the correctness and runtime efficiency of \texttt{ConstructWPO}^{BU}.

\textbf{Theorem 6.7.} Given a graph \( G(V, \rightarrow) \), \texttt{ConstructWPO}^{TD} in Algorithm 3 and \texttt{ConstructWPO}^{BU} in Algorithm 1 construct the same WPO.

\textbf{Proof.} Constructive characterization of loop nesting forest (LNF) [Ramalingam 2002, Definition 3] states that an LNF of a graph is obtained by identifying the maximal SCCs, choosing headers for the maximal SCCs, removing feedback edges directing the headers to break the maximal SCCs, and repeating this on the subgraphs. With choice of minimum DFN in SCC as the header, we get a Havlak’s LNF [Ramalingam 2002, Definition 6]. This is identical to how the maximal SCCs are identified on Line 1 of \texttt{ConstructWPO}^{TD} and how \texttt{ConstructWPO}^{TD} is called recursively on Line 8 of \texttt{ConstructWPO}^{TD}_{SCC}. Because \texttt{ConstructWPO}^{BU} is based on the LNF construction algorithm, both algorithms identify the same set of SCCs, resulting in the same X and -\( \rightarrow \).

Now consider \( u \to v' \) in the original graph that is not a feedback edge. Let \( v \) be \( r\text{ep}(v') \) when restoring the edge if the edge is cross/forward, or \( v' \) otherwise. If both \( u, v' \) is in some non-trivial
### Table 2. Steps of ConstructWPO in Algorithm 3 for Figure 4.

| Step | → | Partition | WPO |
|------|---|-----------|-----|
| Init | {\(\{7, 3\}, \{8, 4\}\)} removed | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |
| \(h=8\) | - | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |
| \(h=7\) | - | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |
| \(h=6\) | - | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |
| \(h=5\) | - | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |
| \(h=4\) | - | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |
| \(h=3\) | - | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |
| \(h=2\) | - | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |
| \(h=1\) | \(\{\{7, 2\}, \{8, 4\}\}\) added | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |
| Final | - | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] | \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8\] |

SCC, \(v \in N_h\) on Line 27 for some \(h\). In this case, \(\text{rep}(u)\) must be in \(N_h\). If not, \(u \rightarrow v\) creates an entry to the SCC other than \(h\). We know that \(h\) must be the entry because it has minimum DFN. This contradicts that the modified graph is reducible over the whole run [Ramalingam 1999, Section 5]. Because all nested SCCs have been already identified, \(\text{rep}(u)\) is the representative of the maximal SCC that contains \(u\) but not \(v\). Also, if no SCC contains both \(u, v'\), \(v = \text{rep}(v)\) on Line 48. Because all maximal SCCs are found, \(\text{rep}(u)\) returns the representative of the maximal SCC that contains \(u\) but not \(v\). Therefore, Line 28 and Line 49 construct the same \(\rightarrow\) as ConstructWPO\(^\text{TD}\). □
Fig. 5. A WTO of $G_1$ in Figure 2(a) in two different representations. The upper representation demonstrates that WTO is a linearly ordered WPO. It is obtained by extending the partial order of WPO $\mathcal{W}_1$ in Figure 2(b) to a total order. The scheduling constraints added to extend $\mathcal{W}_1$ are shown in bold.

\textbf{Theorem 6.8.} The running time of ConstructWPO$_{BU}$ in Algorithm 3 is almost-linear.

\textbf{Proof.} This directly follows from the fact that the running time of Tarjan-Havlik-Ramalingam’s algorithm is almost-linear [Ramalingam 1999, Section 5]. The starred lines in ConstructWPO$_{BU}$ only add constant factors to the algorithm. \hfill $\square$

\section{Connection to Weak Topological Order}

The weak partial order defined in this paper is a strict generalization of weak topological order (WTO) defined by Bourdoncle [1993]. Let us first recall the definitions by Bourdoncle.

\textit{Definition 7.1.} A \textbf{hierarchical total order} of a set $S$ is a well-parenthesized permutation of this set without two consecutive "("

A hierarchical total order is a string over the alphabet $S$ augmented with left and right parenthesis. A hierarchical total order of $S$ induces a total order $\leq$ over the elements of $S$. The elements between two matching parentheses are called a \textit{component}, and the first element of a component is called the \textit{head}. The set of heads of the components containing the element $l$ is denoted by $\omega(l)$.

\textit{Definition 7.2.} A \textbf{weak topological order (WTO)} of a directed graph is a hierarchical total order of its vertices such that for every edge $u \rightarrow v$, either $u \preceq v$ or $v \preceq u$ and $v \in \omega(u)$.

A WTO factors out a feedback edge set from the graph using the matching parentheses and topologically sorts the rest of the graph to obtain a total order of vertices. A feedback edge set defined by a WTO is $\{(u, v) \mid v \preceq u \text{ and } v \in \omega(u)\}$.

\textit{Example 7.3.} $1 (2 (3 4) (6 7 9 8) 5) 10$ is a WTO of the graph $G_1$ in Figure 2(a). The feedback edge set defined by this WTO is $\{(4, 3), (8, 6), (5, 2)\}$. It is the same feedback edge set defined by the WPO $\mathcal{W}_1$ in Figure 2(b).

Algorithm 4 presents a top-down recursive algorithm for constructing a WTO of a graph $G$. Notice the use of increasing post DFN order when merging the results on Line 5. In general, an increasing post DFN order of a graph is also a topological order. Therefore, ConstructWTO$_{TD}$, in effect, topologically sorts the DAG of SCCs recursively. Because it is recursive, it preserves the components and their nesting relationship. Furthermore, by observing the correspondence between ConstructWTO$_{TD}$ and ConstructWPO$_{TD}$, we see that ConstructWTO$_{TD}(G)$ and ConstructWPO$_{TD}(G, *)$ construct the same components with same heads and nesting relationship.

The definition of HPO (Definition 4.1) generalizes the definition of hierarchical total order (Definition 7.1) to partial orders, while the definition of WPO (Definition 4.3) generalizes the definition of WTO (Definition 7.2) to partial orders. In other words, the two definitions define the same structure if we strengthen H1 to a total order. If we view the set of exits $X$ as closing parenthesis ")", and $x \rightarrow h$ as matching parentheses ($h \ldots$), the correspondence between the two
The visiting order is in topological order because an element is added from inside of the component to outside, satisfying H5. Satisfying all the requirements, preserving the axiomatic characterization of WPO. Because only scheduling constraints will be added, we only have to be careful about H1, H4, and H5.

Definition 7.4 hints at how a WTO for a graph can be constructed from a WPO. The key is to extend a partial order \((V \cup X, \rightarrow^*)\) to a total order by adding more scheduling constraints, while preserving the axiomatic characterization of WPO. Because only scheduling constraints will be added, we only have to be careful about H1, H4, and H5.

ConstructWTO\textsuperscript{BU} constructs a WTO by extending a WPO. ConstructWPO\textsuperscript{BU} begins with a call to ConstructWPO\textsuperscript{BU}, constructing a WPO for \(G\). Then, it extends it to a WTO while visiting the elements in topologically order of \(\rightarrow\). The visiting order is in topological order because an element can be visited only after all its predecessors are visited on Line 12. Because the elements are visited in this order, extensions on Line 8 does not violate H1 and lead to a total order \((V \cup X, \rightarrow^*)\). With lift set to true in ConstructWPO\textsuperscript{BU}, all scheduling predecessors of a component are visited before the component’s head. Also, because a stack is used as the worklist and because of H5, once a head of component is visited, no element outside the component is visited until all elements in the component are visited. Therefore, it preserves the components and their nesting relationship, satisfying H4. Because the exit is the last element visited in the component, no scheduling constraint is added from inside of the component to outside, satisfying H5. Satisfying all the requirements, ConstructWTO\textsuperscript{BU} constructs a WTO for \(G\):

**Theorem 7.5.** Given a directed graph \(G\), ConstructWTO\textsuperscript{BU} constructs a WTO for \(G\). □

**Example 7.6.** Figure 5 shows a WTO for the graph \(G_1\) in Figure 2(a) that is constructed by ConstructWTO\textsuperscript{BU}. It first constructs WPO \(\mathcal{W}_1\) in Figure 2(b) using ConstructWPO\textsuperscript{BU} and extends its partial order to a total order. The added \(x_5 \rightarrow 6\) and \(7 \rightarrow 9\) make \(\rightarrow^*\) a total order. The elements are placed linearly in the order they are visited in ConstructWTO\textsuperscript{BU}, which equals the topological order. The components and their nesting relationship in \(\mathcal{W}_1\) are preserved. □

Because the elements and scheduling constraints are visited only once after the WPO construction, ConstructWTO\textsuperscript{BU} have the same time complexity as ConstructWPO\textsuperscript{BU}:

**Theorem 7.7.** Running time of ConstructWTO\textsuperscript{BU} is almost-linear. □
Algorithm 6: ConstructWTO\textsuperscript{BU}(G)

Input: Directed graph G(V, \rightarrow)
Output: WTO W(V, X, \rightarrow, \rightarrow)
1 V, X, \rightarrow, \rightarrow := ConstructWPO\textsuperscript{BU}(G, true)
2 \textbf{foreach} v \in V \cup X \textbf{do}
3 \hspace{1em} count[v] := 0
4 \hspace{1em} \textbf{if} G.predecessors(v) = \emptyset \textbf{then} stack.push(v)
5 \hspace{1em} prev := nil
6 \textbf{while} stack \neq \emptyset \textbf{do}
7 \hspace{2em} v := stack.pop()
8 \hspace{2em} \textbf{if} \text{prev} \neq \text{nil} \textbf{then} \rightarrow := \rightarrow \cup \{(\text{prev}, v)\}
9 \hspace{2em} \text{prev} := v
10 \hspace{2em} \textbf{foreach} v \rightarrow w \textbf{do}
11 \hspace{3em} count[w] := count[w] + 1
12 \hspace{3em} \textbf{if} count[w] = |W.predecessors(w)| \textbf{then} stack.push(w)
13 \textbf{return} (V, X, \rightarrow, \rightarrow)

Bourdoncle \cite{Bourdoncle1993} presents a more efficient version of the ConstructWTO\textsuperscript{TD} for WTO construction; however, it has worst-case cubic time complexity. Thus, Theorem 7.7 improves upon the previously known algorithm for WTO construction.

We now compare ConstructWTO\textsuperscript{BU} with ConstructWTO\textsuperscript{TD}. We have already discussed that ConstructWTO\textsuperscript{TD}(G) and ConstructWPO\textsuperscript{TD}(G) construct the same components with same heads and nesting relationship. Therefore, using Theorem 6.7, we can conclude that ConstructWTO\textsuperscript{BU}(G) and ConstructWTO\textsuperscript{TD}(G) construct the same WTO:

\textbf{Theorem 7.8.} Given a directed graph G, ConstructWTO\textsuperscript{BU} and ConstructWTO\textsuperscript{TD} constructs the same WTO for G. \hfill \Box

With both stabilization constraint, \rightarrow, and matching parentheses, (\ldots), interpreted as the “iteration until stabilization” operator, our concurrent iteration strategy for ConstructWTO\textsuperscript{BU}(G) computes the same fixpoint as Bourdoncle’s recursive iteration strategy for ConstructWTO\textsuperscript{TD}(G). Furthermore, because the only change we make to a WPO in ConstructWTO\textsuperscript{BU} is adding more scheduling constraints, our concurrent iteration strategy, which is deterministic (Theorem 5.8), computes the same fixpoint for both ConstructWTO\textsuperscript{BU}(G) and ConstructWPO\textsuperscript{BU}(G, false). Therefore, our concurrent fixpoint algorithm in Figure 3 computes the same fixpoint as Bourdoncle’s sequential fixpoint algorithm:

\textbf{Theorem 7.9.} The fixpoint computed by the concurrent fixpoint algorithm in Figure 3 is the same as one computed by the sequential Bourdoncle’s algorithm that uses recursive iteration strategy. Therefore, concurrent fixpoint algorithm described in Figure 3 does not reduce the precision of the results. \hfill \Box

8 IMPLEMENTATION

Our deterministic parallel abstract interpreter, which we called Pikos, was built using IKOS \cite{Brat2014}, an abstract-interpretation framework for C/C++ based on LLVM.

Sequential baseline IKOS. IKOS performs interprocedural analysis to compute invariants for all programs points, and can detect and prove the absence of runtime errors in programs. To compute the fixpoint for a function, IKOS constructs the WTO of the CFG of the function and uses Bourdoncle’s recursive iteration strategy. Context sensitivity during interprocedural analysis
is achieved by performing dynamic inlining during fixpoint: formal and actual parameters are
matched, the callee is analyzed, and the return value at the call site is updated after the callee
returns. This inlining also supports function pointers by resolving the set of possible callees and
joining the results.

**Pikos.** We modified IKOS to implement our deterministic parallel abstract interpreter and used
Intel’s Threading Building Blocks (TBB) library [Intel 2019]. We have implemented the almost-
linear time algorithm for WPO construction (§6). We implemented the deterministic parallel
fixpoint iterator (§5) using TBB’s parallel_do. Multiple callees at an indirect call site are analyzed
in parallel using TBB’s parallel_reduce. We refer to this extension of IKOS as Pikos; we use
Pikos[k] to refer to the instantiation of Pikos that uses up to k threads.

**Path-based task spawning in Pikos.** Pikos relies on TBB’s tasks to implement the parallel
fixpoint iterator. Our initial implementation would spawn for each WPO element when it became
ready to be scheduled. Such a naive approach resulted in Pikos being slower than IKOS; there were
10 benchmarks where speedup of Pikos[2] was below 0.90x compared to IKOS, with a minimum
speedup of 0.74x.

To counter such behavior, we implemented a simple path-based heuristic for spawning tasks
during fixpoint computation. We assign ids to each element in the WPO W as follows: assign id 1
to the elements along the longest path in W, remove these elements from W and assign id 2 to the
elements along the longest path in the resulting graph, and so on. The length of the path is based
on the number of instructions as well as the size of the callee functions occurring along the path.
During the fixpoint computation, a new task is spawned only if the id of the current element differs
from that of the successor that is ready to be scheduled. Consequently, elements along critical paths
are executed in the same task.

**Memory Allocator for Concurrency.** In addition to the heuristic above, we used optimized
memory allocator for concurrency. Tcmalloc [Google 2019], Jemalloc [Evans 2019], and Tbbmalloc
[Intel 2019] were considered, and we eventually settled on Tcmalloc because it outperformed the
rest in our settings for both Pikos and IKOS.

**Abstract Domain.** Our fixpoint computation algorithm is orthogonal to the abstract domain
in use. Pikos[k] worked for all abstract domains provided by IKOS as long as the domain was
thread-safe. These abstract domains include interval, congruence, gauge [Venet 2012], DBM, etc.
Variable-packaging domains could not be used because their implementation were not thread-safe.
We intend to explore thread-safe implementations for these domains in the future.

9 EXPERIMENTAL EVALUATION

In this section, we study the runtime performance of Pikos in §8 on a large set of C programs using
IKOS as the baseline. Because we are only concerned with the time taken to perform fixpoint
computation, we disabled program checks, such as buffer-overflow detection and division-by-zero
checks, in both IKOS and Pikos.

The experiments were designed to answer the following questions:

**RQ0 [Determinism]** Is Pikos deterministic as we designed? Is the computed fixpoint the same
as IKOS’s?

**RQ1 [Performance]** What is the performance of Pikos[4] compared to IKOS?

**RQ2 [Scalability]** How does the performance Pikos[k] scale as we increase the number of
threads k?

**Platform.** All of the experiments were run on Amazon EC2 C5, the compute optimized instances.
These use 3.00 GHz Intel Xeon Platinum 8124M CPUs. IKOS and Pikos[k] with 1 ≤ k ≤ 4 were run
on c5.2xlarge (8 vCPUs, 4 physical cores, 16GB memory), Pikos$[k]$ with $5 \leq k \leq 8$ on c5.4xlarge (16 vCPUs, 8 physical cores, 32GB memory), and Pikos$[k]$ with $9 \leq k$ on c5.9xlarge (36 vCPUs, 18 physical cores, 72GB memory). Dedicated EC2 instances and BenchExec [Beyer et al. 2019] were used to ensure the reliability of timing results. The Linux kernel version was 4.4, and gcc 8.1.0 was used to compile Pikos$[k]$ and IKOS.

**Abstract Domain.** We experimented with both interval and gauge domain, and the analysis precision was set to track immediate values, pointers, and memory. The results were similar for both interval and gauge domain. We show the results using the interval domain.

**Benchmarks.** Our final set of benchmarks consisted of 207 benchmarks selected from the following two sources:

**SVC** We considered 2689 of the SV-COMP 2019 benchmarks [Beyer 2019]. In particular, we considered the Linux device drivers, control flows, and loops benchmark suites. Programs from these suites contain indirect call sites with multiple callees, large switch statements, nested loops, and irreducible CFGs.

**OSS** We considered 1605 open-source programs obtained through Arch Linux package manager [Pacman 2019], and compiled to LLVM bitcode by gllvm [gllvm 2019]. These programs include apache, coreutils, cscope, curl, dhcp, fvm, gawk, genius, ghostscript, gnupg, inetutils, iproute, nmap, openssl, postfix, r, socat, sysstat, vim, wget, etc.

Of the total 4284 SVC and OSS benchmarks, we dropped those for which IKOS took less than 5 minutes to analyze. This left us with 210 benchmarks. Among these 210 benchmarks, there were 41 for which IKOS did not terminate in 4 hours. To ensure that IKOS was able to analyze these long running benchmarks in 4 hours, we modified the dynamic inliner implementing the context sensitivity in the interprocedural analysis in both IKOS and Pikos$[k]$. In particular, if a stack depth during dynamic inlining exceeded a limit, then the analysis merely returns $\top$ for that callee. This change enabled us to include these 41 benchmarks in our experiments. Because our experiments are designed to understand the performance improvement in fixpoint computation, we felt this was a reasonable thing to do. We excluded 3 benchmarks for which Pikos$[1]$ outperformed IKOS to ensure that the performance speedup are coming only from the parallelism and not any idiosyncrasies of the Pikos implementation. This resulted in a total of 207 benchmarks being selected, consisting of 106 SVC and 101 OSS benchmarks.

**9.1 RQ0: Determinism of Pikos**

We experimentally validated Theorem 5.8 that states that Pikos computes the same fixpoint regardless of the underlying schedule. In particular, for various abstract domains, we ran Pikos multiple times with varying number of threads and checked that the final fixpoint was indeed the same each time. Furthermore, we also experimentally validated Theorem 7.9 by checking that the fixpoint computed by Pikos was the same as that computed by IKOS. Recall that IKOS uses Bourdoncle’s recursive iteration strategy computing using WTOs. This research question is more of a sanity check for our theoretical results.

**9.2 RQ1: Performance of Pikos$[4]$ compared to IKOS**

Figure 6(a) shows a log-log scatter plot comparing the analysis runtime of Pikos$[4]$ with IKOS for each of the 207 benchmarks. Speedup is defined as the analysis time of IKOS divided by the analysis time of Pikos$[4]$. The maximum speedup was 3.44x, which is close to the maximum speedup of 4.00x. Arithmetic, geometric, and harmonic mean of the speedup were 2.24x, 2.14x, 2.03x respectively. Total speedup of all the benchmarks was 2.34x. As we can see in Figure 6(a), benchmarks for which IKOS took longer to analyze tend to have greater speedup in Pikos$[4]$. Top 25% in terms of the
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(a) Log-log scatter plot of time taken by IKOS and Pikos\cite{4}. 1.00x, 2.00x, and 4.00x speedup lines are shown. Benchmarks for which IKOS takes longer to analyze tend to have higher speedup.

(b) Histogram of speedup of Pikos\cite{4}. 67.1\% of benchmarks had speedup over 2.00x. Only one benchmark had a slowdown of 0.95x speedup.

Fig. 6. Performance comparison of Pikos\cite{4} with IKOS on 207 benchmarks. Speedup is defined as analysis time of IKOS divided by analysis time of Pikos\cite{4}.

| Benchmark            | Src. | IKOS (s) | Pikos\cite{4} Time (s) | Speedup |
|----------------------|------|----------|------------------------|---------|
| ratpoison-1.4.9/ratpoison | OSS  | 1229.23  | 356.82                 | 3.44x   |
| audit-2.8.4/aureport | OSS  | 655.90   | 191.61                 | 3.42x   |
| feh-3.1.3/feh       | OSS  | 8548.71  | 2532.56                | 3.38x   |
| hydra-8.9.1/hydra   | OSS  | 341.73   | 102.14                 | 3.35x   |
| coreutils-8.31/mv   | OSS  | 373.12   | 111.66                 | 3.34x   |
| vice-3.3/x128       | OSS  | 12407.31 | 4199.00                | 2.95x   |
| minicom-2.7.1/minicom | OSS  | 12272.09 | 4513.05                | 2.72x   |
| genius-1.0.24/gnome-genius | OSS  | 10405.26 | 3347.14                | 3.11x   |
| vice-3.3/x64sc      | OSS  | 8720.33  | 4890.95                | 1.78x   |
| feh-3.1.3/feh       | OSS  | 8548.71  | 2532.56                | 3.38x   |

Table 3. A sample of the 207 results in Figure 6. The first 5 rows list benchmarks with the highest speedup, and the remaining 5 rows list benchmarks with the longest analysis time in IKOS. (The feh benchmark is common to both lists.)

The analysis time in IKOS had higher means than the total with arithmetic, geometric, and harmonic mean being 2.61x, 2.56x, 2.48x respectively. Table 3 shows speedups for the 5 benchmarks with longest analysis time.

Figure 6(b) provides details about the distribution of speedups. The frequency on the y-axis represents the number of benchmarks that had the speedup represented by the bucket on the x-axis. A bucket size of 0.25 was used, ranging from 0.75 to 3.75. The most frequent speedup was between 2.00x and 2.25x with frequency of 39 out of total 207 (18.8\%). The second most frequent speedup was between 2.50x and 2.75x with frequency of 32 (15.5\%). The third most frequent speedup was between 1.25x and 1.50x with frequency of 27 (13.0\%). 67.1\% of benchmarks had speedup over 2.00x. The ones with more than 3.00x speedup were 29 out of 207 (14.0\%).
Benchmarks with high speedup contained code with large switch statements nested inside loops. For example, *ratpoison-1.4.9/ratpoison*, a tiling window manager, has an event handling loop that dispatches the events using the switch statement with 15 cases. Each switch case calls an event handler that contain further branches, leading to more parallelism. Most of the analysis time for this benchmarks is spent in this loop. On the other hand, the programs with low speedup usually have a dominant single execution path. An example of such a benchmark is *xlockmore-5.56/xlock*, a program that locks the local X display until a password is entered.

Maximum speedup gained by parallelism in *Pikos*[4] was 3.44x, where 4.00x is the maximum possible speedup. Arithmetic, geometric, and harmonic mean of the speedup were 2.24x, 2.14x, 2.03x respectively. The performance were generally better for the benchmarks for which *IKOS* took longer to analyze.

### 9.3 RQ2: Scalability of *Pikos*

In addition to the performance benefit of *Pikos*[k], we want to know how the speedup scales in proportion to the number of threads. To answer this question, the same measurements as in **RQ1** were made on *Pikos*[k] with *k* ∈ {2, 4, 6, 8}. The results are given in Figure 7 and in Figure 8.

Figure 7 shows the box and violin plots for speedup obtained by *Pikos*[k], *k* ∈ {2, 4, 6, 8}. Box plots show the quartiles and the outliers, and violin plots show the estimated distribution of the observed speedups. The box plot [Tukey 1977] on the left summarizes the distribution of the results for each *k* using lower inner fence (\(Q_1 - 1.5 \times (Q_3 - Q_1)\)), first quartile (\(Q_1\)), median, third quartile (\(Q_3\)), and upper inner fence (\(Q_3 + 1.5 \times (Q_3 - Q_1)\)). Data beyond the inner fences (outliers) are plotted as individual points. Box plot reveals that while the benchmarks above the median (middle line in the box) scaled, speedups for the ones below median saturated. Violin plot [Hintze and Nelson 1998] on the right supplements the box plot by plotting the probability density of the results between minimum and maximum. For example, violin shaped curves of 4 threads in Figure 7 roughly corresponds to the histogram in Figure 6(b) when rotated 90 degrees clockwise. In the best case, speedup scaled from 1.68x to 3.42x, 4.73x, and 6.06x. For each *k*, the arithmetic means were 1.59x to 2.24x, 2.50x, and 2.67x. The geometric means were 1.58x to 2.14x, 2.31x, and 2.42x. The harmonic means were 1.56x to 2.03x, 2.14x, and 2.20x.

To better measure the scalability of *Pikos*[k] for individual benchmarks, we define a *scalability coefficient* as the slope of the linear regression of the number of threads and the speedups. The maximum scalability coefficient is 1, meaning that the speedup increases linearly as the number of threads increase. If the scalability coefficient is 0, the speedup is the same regardless of the
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(a) Speedup of \( \text{Pikos}[k] \) for 3 benchmarks with different scalability coefficients. The lines show the linear regressions of these benchmarks.

(b) Distribution of scalability coefficients for 207 benchmarks. \((x, y)\) in the plot means that \(y\) number of benchmarks have scalability coefficient at least \(x\).

Fig. 8. Scalability coefficient for \( \text{Pikos}[k] \)

Table 4. 5 benchmarks with the highest scalability out of 207 benchmarks

| Benchmark (5 out of 207. Criteria: Scalability) | Src. | IKOS (s) | \( k = 4 \) | \( k = 8 \) | \( k = 12 \) | \( k = 16 \) |
|-----------------------------------------------|------|---------|-------------|-------------|-------------|-------------|
| audit-2.8.4/aureport                         | OSS  | 655.90  | 3.42x       | 6.06x       | 8.03x       | 9.38x       |
| ratpoison-1.4.9/ratpoison                    | OSS  | 1229.23 | 3.44x       | 5.88x       | 7.45x       | 8.30x       |
| sdparm-1.10/sdparm.bc                        | OSS  | 322.63  | 3.16x       | 5.18x       | 6.59x       | 7.26x       |
| feh-3.1.3/feh.bc                             | OSS  | 8548.71 | 3.38x       | 5.20x       | 6.51x       | 7.18x       |
| ldv-3.14/alloc-spinlock_drivers-net-fddi-skfp | SVC  | 6335.42 | 3.25x       | 5.29x       | 6.19x       | 6.49x       |

number of threads used. If it is negative, the speedup goes down with increase in number of threads. The measured scalability coefficients are shown in Figure 8. Figure 8(a) illustrates benchmarks exhibiting different scalability coefficients. For the benchmark with coefficient 0.72, the speedup of PIKOS roughly increases by 4, from 2x to 6x, with 6 more threads. For benchmark with coefficient 0, the speedup does not increase with more threads. Figure 8(b) shows the distribution of scalability coefficients for all benchmarks. From this plot we can infer, for instance, that 28 benchmarks have at least 0.4 scalability coefficient. For these benchmarks, speedups increases by at least 2 when 5 more threads are given.

Table 4 shows the speedup of \( \text{Pikos}[k] \) for \( k > 8 \) for a selection of 5 benchmarks that had the highest scalability coefficient in the prior experiment. In particular, we wanted to explore the limits of scalability of \( \text{Pikos}[k] \) for this smaller selection of benchmarks. With scalability coefficient 0.72, the speedup of audit-2.8.4/aureport reaches 9.38x using 16 threads. This program is a tool that produces summary reports of the audit system logs. Like ratpoison, it contains an event-handler loop containing a large switch statement. A control flow graph of function that contains this switch statement is shown in Figure 9.

In the best case, the speedup of \( \text{Pikos}[k] \) scaled from 1.68x to 3.42x, 4.73x, and 6.06x with \( k = 2, 4, 6 \) and 8. With this benchmark, the speedup reached 9.38x with 16 threads. The scalability varies on the structure of the analyzed programs, and programs with multiple paths of similar lengths exhibit high scalability.
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Fig. 9. Control flow graph of function per_event_detailed in aureport. The benchmark aureport had the maximum scalability. Given an event type, a switch statement in the function calls appropriate handlers. This function is called inside a loop.

10 RELATED WORK

Since its publication in 1993, Bourdoncle's algorithm [Bourdoncle 1993] has become the de facto approach to solving equations in abstract interpretation. Many advances have been developed since, but they rely on Bourdoncle's algorithm; in particular, different ways of intertwining widening and narrowing during fixpoint computation with an aim to improve precision [Amato and Scozzari 2013; Amato et al. 2016; Halbwachs and Henry 2012].

C Global Surveyor (CGS) [Venet and Brat 2004] that performed array bounds checking was the first attempt at distributed abstract interpretation. It performed distributed batch processing, and a relational database was used for both storage and communication between processes. As a consequence, the communication costs were too high, and the analysis did not scale beyond four CPUs.

Monniaux [2005] describes a parallel implementation of the ASTRÉE analyzer [Cousot et al. 2005]. It relies on dispatch points that divide the control flow between long executions, which are found in embedded applications; the analyzer analyzes these two control paths in parallel. Unlike our approach, this parallelization technique would not work for programs with irreducible CFGs. The particular parallelization strategy can also lead to a loss in precision. The experimental evaluation found that the analysis does not scale beyond 4 processors.

Dewey et al. [2015] present a parallel static analysis for Javascript by dividing the analysis into an embarrassingly parallel reachability computation on a state transition system, and a strategy for selectively merging states during that reachability computation.

Prior work has explored the use of parallelism for specific program analysis. BOLT [Albarghouthi et al. 2012] uses a map-reduce framework to parallelize a top-down analysis for use in verification and software model checking. Graspan [Wang et al. 2017] implements a single-machine disk-based graph system to solve graph reachability problems for interprocedural static analysis. Graspan is not a generic abstract interpreter, and solves data-flow analyses in the IFDS framework [Reps et al. 1995]. Su et al. [2014] describe a parallel points-to analysis via CFL-reachability. Garbervetsky et al. [2017] use an actor-model to implement distributed call-graph analysis.
McPeak et al. [2013] parallelize the Coverity Static Analyzer [Bessey et al. 2010] to run on an 8-core machine by mapping each function to its own work unit. Tricorder [Sadowski et al. 2015] is a cloud-based static-analysis platform used at Google. It supports only simple, intraprocedural analyses (such as code linters), and is not designed for distributed whole-program analysis.

Sparse analysis [Oh et al. 2014, 2012] and database-backed analysis [Weiss et al. 2015] are orthogonal approaches that improve the memory cost of static analysis. Newtonian program analysis [Reps 2018; Reps et al. 2017] provides an alternative to Kleene iteration used in this paper.

11 CONCLUSION

We presented a generic, parallel, and deterministic algorithm for computing fixpoints of equation systems for abstract interpretation. The iteration strategy used for fixpoint computation is constructed from a weak partial order (WPO) of the dependency graph of the equation system. We described an axiomatic and constructive characterization of WPOs, as well as an efficient almost-linear time algorithm for constructing a WPO. This new notion of WPO generalizes Bourdoncle’s weak topological order (WTO). We presented a linear-time algorithm to construct a WTO from a WPO, which results in an almost-linear algorithm for WTO construction given a directed graph. The previously known algorithm for WTO construction had a worst-case cubic time-complexity. We also showed that the fixpoint computed using the WPO-based parallel fixpoint algorithm is the same as that computed using the WTO-based sequential fixpoint algorithm. We presented our implementation of the algorithms in PIKOS and observed a maximum speedup of 9.38x with 16 cores.

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