Power counting for three-body decays of a near-threshold state

Mohammad H. Alhakami* and Michael C. Birse
Theoretical Physics Division, School of Physics and Astronomy,
The University of Manchester, Manchester M13 9PL, UK
(Dated: February 24, 2015)

We propose a new power counting for the effective field theory describing a near-threshold state with unstable constituents, such as the \( X(3872) \) meson. In this counting, the momenta of the heavy particles, the pion mass and the excitation energy of the unstable constituent—the \( D^* \) in the case of the \( X \)—are treated as small scales, of order \( Q \). The difference \( \delta \) between the excitation energy of the \( D^* \) and the pion mass is smaller than either by a factor \( \sim 20 \). We therefore assign \( \delta \) an order \( Q^2 \) in our counting. This provides a consistent framework for a double expansion in both \( \delta/m_\pi \) and the ratio of \( m_\pi \) to the high-energy scales in this system. It ensures that amplitudes have the correct behaviour at the three-body threshold. It allows us to derive, within an effective theory, various results which have previously been obtained using physically-motivated approximations.

The \( X(3872) \) meson has provided a puzzle since it was discovered by the Belle collaboration [1]. Its closeness to the \( D^0\bar{D}^{*0} \) threshold suggests that it may not be a standard charmonium, but rather a “molecular” bound state of those mesons, of the type predicted by Tornqvist [2]. (A review of experimental developments and theoretical questions can be found in Ref. [3].) Clues to the nature of the \( X(3872) \) have been sought in its decay modes, one of the most important of which is \( X \rightarrow D^0\bar{D}^{*0}\pi^0 \) (see, for example, Refs. [4–6]), but no definitive conclusion has yet been reached.

The fact that the \( X(3872) \) lies within 1 MeV of a threshold makes it a suitable candidate for study using an effective field theory (EFT), similar to the ones that describe nucleon-nucleon scattering [7, 8]. In particular, a nonrelativistic EFT including pion degrees of freedom, XEFT, was proposed by Fleming et al. [9]. This theory is applicable to near-threshold states with unstable constituents. A key ingredient is the hyperfine splitting between the \( D^0 \) and \( D^{*0} \) which is \( \Delta = M_{D^*} - M_D \approx 142 \text{ MeV} \). This means that the \( D^* \) sits close to the \( D\pi \) threshold, and hence has a very small width for strong decays. Fleming et al. therefore introduce the low-energy scale

\[ \delta = \Delta - m_\pi \approx 7 \text{ MeV} \]  

and take \( \delta/m_\pi \) as an expansion parameter for their EFT.

In this version of XEFT, \( m_\pi \) and \( \Delta \) are treated as high energy scales, corresponding to physics that has been integrated out. This provides no systematic justification for a further expansion in powers of the ratios of \( m_\pi \) to the high-energy scales in the system, for example \( m_D \) or the chiral scale \( 4\pi f_\pi \). In particular, as discussed at the end of Section II of Ref. [9], it does not justify an expansion in \( m_\pi/M_D \), which is a natural one to make and which simplifies the calculations.

A further issue arises in diagrams with a \( D\bar{D}\pi \) intermediate state when these are evaluated at lowest order in this second expansion. At this order, the kinetic energies of the \( D \) mesons are suppressed by \( m_\pi/M_D \) compared to the kinetic energy of the pion. However neglecting these energies removes the constraint on the momenta of the \( D \) mesons, giving a \( D\bar{D}\pi \) threshold with a two-body structure, rather than the correct three-body form. To avoid this, Fleming et al. and subsequent authors [10, 11] retain the \( D \)-meson kinetic energies when they evaluate the contributions from real pions. Although doing so requires terms beyond leading order in their expansion, those authors justified this on physical grounds, noting that the imaginary part of the self-energy is then consistent with the partial width for \( X \rightarrow D\bar{D}\pi \) obtained by Voloshin from effective-range theory [12].

In this note, we present a modified power counting for XEFT that avoids these problems. In contrast to Ref. [9] we treat the pion mass and the hyperfine splitting \( \Delta \) as of order \( Q \), as in heavy-hadron chiral perturbation theory [13]. We treat their difference \( \delta \) as of order \( Q^2 \) in our counting. This reflects the near coincidence of the hyperfine splitting of the \( D \) mesons and the pion mass, which forms the basis for the original expansion of XEFT. In this new scheme, both the ratios \( \delta/m_\pi \) and \( m_\pi/M_D \) are of order \( Q \), providing a common framework for both expansions. As we show below, it also ensures the correct behaviour at the \( D\bar{D}\pi \) threshold.

*Permanent address: KACST, PO Box 6086, Riyadh 11442, Saudi Arabia
Our starting point is the leading-order Lagrangian for the neutral mesons, as given in Eq. (A.7) of Ref. [9]:

\[
\mathcal{L} = H^\dagger \left( i \partial^0 + \frac{\nabla^2}{2 M_D} \right) H + H^\dagger \cdot \left( i \partial^0 + \frac{\nabla^2}{2 M_D} - \Delta \right) H
\]

\[
+ \frac{1}{2} \partial^\mu \pi^0 \partial^\mu \pi^0 - \frac{g}{2 f_\pi} \left( H^\dagger \cdot \nabla \pi^0 H + H^\dagger H \cdot \nabla \pi^0 \right),
\]

where the scalar field \( H \) and the vector field \( H \) describe the \( D^0 \) and \( D^{*0} \) mesons, and the pion decay constant is \( f_\pi = 92.4 \text{ MeV} \). This is supplemented by similar terms for the \( \bar{D}^0 \) and \( \bar{D}^{*0} \) and a contact interaction acting in the \( C \)-even combination of \( D^0 \bar{D}^{*0} \) and \( D^{*0} \bar{D}^0 \) channels. For generality, we treat the pion field as relativistic.

The coupling strength of the pions to the charmed mesons is \( g \approx 0.6 \) [13]. The square of this is significantly smaller that for the corresponding coupling in the nuclear case (where \( g_A = 1.27 \)) and it suggests that treating pion exchange perturbatively may be a good approximation [9]. We therefore focus on the two diagrams shown in Fig. 1.

The imaginary parts of these correspond to decay of the \( X \) arising from coupling to a real or virtual \( D \bar{D}^* \) (or \( \bar{D} D^* \)), followed by decay of the \( D^* \) (or \( \bar{D}^* \)). The second diagram can be viewed as the contribution from interference between the \( D \bar{D}^* \) and \( \bar{D} D^* \) components of the \( X \).

![Diagrams](image)

**Fig. 1:** The lowest-order pionic contributions to the self-energy of the \( X \): (a) the self-energy of the \( \bar{D}^* \), (b) pion exchange or, equivalently, interference between \( D \bar{D}^* \) and \( \bar{D} D^* \) components of the wave function. Solid lines represent \( D \) mesons, double lines \( D^* \) and dashed lines pions.

Using rotational invariance, the contribution of diagram 1(a) can be written in the form

\[
I^{(a)}_{ij} = \delta_{ij} \left( \frac{g}{2 f_\pi} \right)^2 I^{(a)},
\]

where the loop integral is

\[
I^{(a)} = \int \frac{d^4 q}{(2 \pi)^4} \int \frac{d^4 k}{(2 \pi)^4} \frac{q^2}{q_0^2 - q^2 - m_\pi^2 + i \epsilon} \left( \frac{1}{k_0 + \frac{E}{2} - \Delta - \frac{k^2}{2 M_D} + i \epsilon} \right)^2 \times \frac{1}{-k_0 + \frac{E}{2} - \frac{k^2}{2 M_D} + i \epsilon} \left( \frac{1}{k_0 + q_0 + \frac{E}{2} - \frac{(k + q)^2}{2 M_D} + i \epsilon} \right).
\]

After integration over the energies \( k_0 \) and \( q_0 \), this becomes

\[
I^{(a)} = \frac{1}{2} \int \frac{d^3 k}{(2 \pi)^3} \int \frac{d^3 q}{(2 \pi)^3} \frac{q^2}{\sqrt{q^2 + m_\pi^2}} \left( \frac{1}{E' - \frac{k^2}{2 M_D}} \right)^2 \times \frac{1}{E' + \Delta - \frac{k^2}{2 M_D} - \frac{(k + q)^2}{2 M_D} - \sqrt{q^2 + m_\pi^2} + i \epsilon},
\]

where \( M_r \approx M_D/2 \) is the reduced mass of the \( D \bar{D}^* \) system and we have defined \( E' = E - \Delta \), the total energy relative to the \( D \bar{D}^* \) threshold.

We consider first the case where the \( X \) lies below the \( D \bar{D}^* \) threshold, \( E' < 0 \), and we expand the self-energy in powers of small scales. We treat the heavy particle momenta \( (k) \) as of order \( Q \), as in nuclear EFTs [7, 8]. It follows that the energies \( (k^2/2M_r \text{ and } E') \) are of order \( Q^2 \). The leading contribution to this integral comes from regions where the pion momentum \( q \) is also of order \( Q \). In this regime, the \( D \bar{D}^* \) energy denominator,

\[
D(E') = E' + \Delta - \frac{k^2}{2 M_D} - \frac{(k + q)^2}{2 M_D} - \sqrt{q^2 + m_\pi^2},
\]
is of order $Q$ and so it can be approximated by

$$D(E') \simeq \Delta - \sqrt{q^2 + m_\pi^2}. \quad (7)$$

The fact that the kinetic energies do not appear in the denominator at this order shows that this is the contribution of “potential” pions in the terminology of Mehen and Stewart [15]. The resulting contribution to the integral is of order $Q^2$, as expected for a leading two-loop diagram in a nonrelativistic EFT [7, 8].

The near cancellation between and $\Delta$ and $m_\pi$ leads to an enhanced contribution to the integral from the region where $q^2 \lesssim \Delta^2 - m_\pi^2$. In our proposed power counting, where $\Delta - m_\pi$ is taken to be of order $Q^2$, this implies that the pion momentum is

$$q \lesssim \sqrt{\delta(\Delta + m_\pi)} \sim \mathcal{O}(Q^{3/2}). \quad (8)$$

The $D\bar{D}\pi$ energy denominator is thus of order $Q^2$ and can be approximated here by

$$D(E') \simeq E' + \delta - \frac{k^2}{2M_r} - \frac{q^2}{2m_\pi}, \quad (9)$$

showing that the pion is nonrelativistic in this regime. However, in contrast to the original expansion of XEFT [9], both the pion and $D$-meson kinetic energies are of the same order in our counting. This ensures that the nonanalytic behaviour at the $D\bar{D}\pi$ threshold has the correct three-body form.

At leading order in this counting, the imaginary part of the contribution of diagram 1(a) to the $X$ self-energy is proportional to

$$\text{Im}[I^{(a)}] \simeq -\frac{\pi}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{E' - k^2/2M_r} \right]^2 \int \frac{d^3q}{(2\pi)^3} \frac{q^2}{m_\pi} \delta \left( \frac{q^2}{2m_\pi} - E' - \delta + \frac{k^2}{2M_r} \right)$$

$$= -\frac{1}{8\pi^3} \left( \frac{m_\pi}{M_r} \right)^{3/2} \int_0^{k_{\text{max}}} k^2 dk \left[ \frac{2M_r(E' + \delta) - k^2}{{(E' - k^2/2M_r)}^2} \right]^{3/2}, \quad (10)$$

where

$$k_{\text{max}} = \sqrt{2M_r(E' + \delta)}. \quad (11)$$

The resulting imaginary part of the self energy is of order $Q^{7/2}$.

The region around the $D\bar{D}\pi$ threshold also makes a contribution of this same order, $Q^{7/2}$, to the real part of the self-energy. This may be of higher order than the contribution of the potential pions but it governs the nonanalytic behaviour at the threshold. This is relevant to studies of the quark-mass dependence of the $X(3872)$ [11, 16, 17] for analyses of lattice simulations of this state [18].

The situation is more complicated if the state lies above the $D\bar{D}^*$ threshold and so $E' > 0$. This also applies to calculations of the line shape for processes such as $B^+ \rightarrow K^+ + X$ above this threshold [4, 6]. For these energies there is an additional enhancement to the integrals from the region around the $D\bar{D}^*$ threshold. In fact the expression in Eq. (10) for the imaginary part diverges as a result of the double pole at $k^2 = 2M_r E'$. This is because, for energies close to the resonance, we cannot ignore the width of the $D^*$. Similar issues arise in the single-baryon system at energies close to the pole of the $\Delta$ resonance. As in the “$\delta$-counting” developed there [19], we include the imaginary part of the self-energy to all orders in the $D^*$ propagator.

To leading order, the width of the $D^*$ is [20]

$$\Gamma_{D^*} = \sqrt{\frac{\sqrt{2}}{3\pi}} \left( \frac{g}{2f_\pi} \right)^2 [m_\pi(\Delta - m_\pi)]^{3/2}, \quad (12)$$

This is proportional to $(m_\pi\delta)^{3/2}$ and hence is of order $Q^{9/2}$ in our counting. The width is thus much smaller than the typical values of the energy above the $DD^*$ threshold, which are of order $Q^2$. The $D^*$ propagator is enhanced for energies within $\sim \Gamma_{D^*}$ of the pole. As discussed by Hanhart et al. [6], the narrowness of this region means that we can neglect any energy dependence, and just replace the imaginary part of the self-energy by the on-shell width of the $D^*$.

This leads to the following expression for the imaginary part:

$$\text{Im}[I^{(a)}] \simeq -\frac{\pi}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(E' - k^2/2M_r)^2} + \frac{1}{4\Gamma_{D^*}} \int \frac{d^3q}{(2\pi)^3} \frac{q^2}{m_\pi} \delta \left( \frac{q^2}{2m_\pi} - E' - \delta + \frac{k^2}{2M_r} \right)$$
\[ = -\frac{1}{8\pi^3} \left( \frac{m_\pi}{M_r} \right)^{3/2} \int_0^{k_{\text{max}}} k^2 \, dk \frac{[2M_r(E' + \delta) - k^2]^{3/2}}{(E' - k^2/2M_r)^2 + \frac{1}{4}\Gamma_r^2}. \]  

(13)

The large contribution comes from a narrow region around \( k^2 = 2M_rE' \) of width \( \sim M_r\Gamma_r \) and it gives a result of order \( Q \).

This result can be simplified by noting that, in the region that gives the dominant contribution, we can approximate the numerator of the integrand using

\[ 2M_r(E' + \delta) - k^2 \simeq 2M_r\delta + \mathcal{O}(Q^{3/2}). \]

(14)

This leads to

\[ \text{Im}[I^{(a)}] \simeq -\frac{\sqrt{2}}{4\pi^3} \left( \frac{m_\pi\delta}{M_r} \right)^{3/2} \int_0^{k_{\text{max}}} k^2 \, dk \frac{1}{(E' - k^2/2M_r)^2 + \frac{1}{4}\Gamma_r^2}. \]

(15)

The combination of low-energy scales multiplying the integral here is the same as appears in the width of the \( D^* \), Eq. (12). When we multiply by the coupling constants to get the full contribution from diagram 1(a), Eq. (3), we find that the result can expressed in the form

\[ I_{ij}^{(a)} \simeq \delta_{ij} \text{Im} \left[ \frac{1}{2\pi^2} \int k^2 \, dk \frac{1}{E' - k^2/2M_r + \frac{1}{2}\Gamma_r} \right]. \]

(16)

This can be seen to be just the imaginary part of the self-energy in a theory without explicit pions but with an unstable \( D^* \). Such a result should not be surprising, as the width of the \( D^* \) is much smaller than that of the \( X \), at least in the case that the \( X \) lies above threshold and so can decay to the \( D\bar{D}^* \) channel. The decay \( D^* \to D\pi \) therefore occurs on a much longer timescale than that for \( X \to D\bar{D}^* \). The width of the \( X \) is thus independent of the details of the subsequent decay of the \( D^* \). This explains the observations of Ref. 6 that the result of the full calculation for \( X \to D\bar{D}^* \) can be very well approximated by that for \( X \to D\bar{D}^* \), provided the energy is far enough above the threshold, \( E' \gg \Gamma_{D^*} \).

The pion-exchange or interference diagram, Fig. 1(b), can be treated in a very similar way. Integrating its contribution over \( k_0 \) and \( q_0 \) leads to two terms. One term has the energy denominator for the \( D\bar{D} \) intermediate state, Eq. (9). It therefore shows the same threshold enhancements as we have discussed above for the self-energy diagram. In particular, for an \( X \) below the \( D\bar{D}^* \) threshold, the threshold region contributes at order \( Q^{7/2} \). In fact, at leading order in our counting, both diagrams lead to the same integral, Eq. (10), and so make equal contributions to the decay of the \( X \). The second term arises from a virtual \( D^*\bar{D}^* \) state, which has a much higher threshold. For the energies considered here, this term just contributes to the “potential” pion part of the self-energy.

For energies above the \( D\bar{D}^* \) threshold, the additional enhancements just discussed come into play. The charge symmetry of the \( X \) wavefunction might suggest that both the \( D^* \) self-energy and pion exchange should be included to all orders, requiring a full three-body treatment of the \( D\bar{D} \) system as in Ref. 21. However there is an important difference between the diagrams. The momentum \( q \) transferred by the pion means that one of the \( D\bar{D}^* \) energy denominators in Eq. (3) should be replaced by \( E' - (k+q)^2/2M_r \). This separates the poles of the two \( D\bar{D}^* \) denominators by \( \mathbf{k} \cdot \mathbf{q}/M_r \), which is of order \( Q^{5/2} \). For this diagram, there is no potential double pole in the \( k \) integral and so the enhancement in the \( D^* \) pole region is not as strong. This is consistent with the numerical estimates of the interference term by Hanhart et al. 6, who found it made relatively small contributions above the \( D\bar{D}^* \) threshold. All of this indicates that pion exchange can still be treated perturbatively in this region.

In summary: we have proposed a new power counting for XEFT, the effective field theory for a near-threshold state with unstable constituents. Like the counting originally proposed by Fleming et al. 9, this leads to an expansion in \( \delta \), the difference between the hyperfine splitting of the \( D \) mesons and the pion mass. However by counting this difference as order \( Q^2 \), we are able to combine this in a single framework with an expansion in \( m_\pi \), which we count in the usual way as of order \( Q \). In this approach, both the pion and heavy-meson kinetic energies are of the same order and so amplitudes at the \( D\bar{D} \) threshold show the correct three-body behaviour. This expansion allows us to recover within an EFT a number of results that were previously obtained using physically-motivated approximations 6, 9, 12. It can provide a systematic framework for extending them to higher orders.

Acknowledgments

We are grateful to C. Hanhart, V. Lensky and J. McGovern for helpful discussions. MA was supported by a scholarship from the King Abdulaziz City for Science and Technology, Saudi Arabia. MCB was supported in part by
the UK STFC under grant ST/J000159/1 and by the EU Integrated Infrastructure Initiative HadronPhysics3.