Seiberg-Witten map for noncommutative super Yang-Mills theory

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Abstract. In this letter we derive the Seiberg-Witten map for noncommutative super Yang-Mills theory in Wess-Zumino gauge. Following (and using results of) hep-th/0108045 we split the observer Lorentz transformations into a covariant particle Lorentz transformation and a remainder which gives directly the Seiberg-Witten differential equations. These differential equations lead to a $\theta$-expansion of the noncommutative super Yang-Mills action which is invariant under commutative gauge transformations and commutative observer Lorentz transformation, but not invariant under commutative supersymmetry transformations: The $\theta$-expansion of noncommutative supersymmetry leads to a $\theta$-dependent symmetry transformation. For this reason the Seiberg-Witten map of super Yang-Mills theory cannot be expressed in terms of superfields.
1 Introduction

The simplest model for noncommutative space-time is the so-called noncommutative $\mathbb{R}^4$ characterised by a $\star$-product involving a constant antisymmetric tensor $\theta$. Field theories on such a deformed space-time became recently very popular, mainly due to their relation to string theory and the possibility to perform similar calculations of Feynman graphs as on usual commutative space-time. It turned out that field theories which are renormalisable in the commutative world are in general not renormalisable (at any loop order) on noncommutative $\mathbb{R}^4$.

One may ask then whether expanding the $\star$-product in $\theta$ improves the renormalisability. If the action of the field theory has a symmetry on noncommutative level, the $\theta$-expanded symmetry transformation will, in general, mix the orders of the $\theta$-expanded action. A remarkable result due to Seiberg and Witten was that for gauge theories on noncommutative $\mathbb{R}^4$ there exists a change of variables such that the $\theta$-expansion of the gauge transformation in the new variables preserves the order in $\theta$. In other words, each $\theta$-order of the expanded action is individually gauge-invariant in the new variables, and one effectively obtains a commutative gauge theory coupled to a constant external field $\theta$.

This change of variables can be traced back to a deeper discussion of Lorentz transformations. In presence of $\theta$ one has to distinguish between ‘observer Lorentz transformations’, which transform $\theta$ as a Lorentz two-tensor, and ‘particle Lorentz transformations’, which leave $\theta$ invariant. It turns out that observer Lorentz transformations are symmetries of the theory whereas particle Lorentz symmetry is broken. Being (in principle) an observable, the breaking of particle Lorentz symmetry must be gauge-invariant. This is not automatically the case and demands a covariant redefinition of the splitting of the observer Lorentz transformation into particle Lorentz transformation plus $\theta$-transformation, which is governed by the Seiberg-Witten differential equations.

It is clear from the construction that the change of variables is tailored to gauge symmetry. If the action has a second symmetry (apart from Lorentz symmetries), it is interesting to know whether this symmetry is $\theta$-diagonalised at the same time with the gauge symmetry or not. As we show in this letter, supersymmetry (regarded as a transformation of the components of a noncommutative super vector field in Wess-Zumino gauge) is not diagonalised at the same time with gauge symmetry. This does not mean that supersymmetry is lost after the Seiberg-Witten map, it is merely not diagonal in the $\theta$-order. It would be interesting to search for a change of variables which $\theta$-diagonalises the supersymmetry transformations and in turn produces $\theta$-expanded gauge transformations which mix the $\theta$-orders. This will be done elsewhere. We stress that the change of variables is unphysical anyway. The fields become dummy integration variables in the path integral, and the change of variables merely changes the measure of integration, which at first order in $\theta$ is a field redefinition also on quantum level.

The letter is organised as follows. We derive in Sec. 3 the Seiberg-Witten differential equations (which govern the change of variables) of super Yang-Mills theory via a covariant splitting of the observer Lorentz transformations (recalled in Sec. 2) into particle Lorentz transformations and a remainder, using the splitting for the gauge field as the starting point. The Seiberg-Witten differential equations lead to a $\theta$-expansion of the noncommutative su-

\footnote{In spite of encouraging preliminary results, it was shown that $\theta$-expansion does not help.}
per Yang-Mills action in terms of fields living on commutative space-time, see Sec. 4. This \( \theta \)-expanded action is automatically invariant under commutative gauge transformations and commutative Lorentz transformations. It is however not invariant under commutative supersymmetry transformations. Instead, the \( \theta \)-expansion of the noncommutative supersymmetry transformation yields a symmetry transformation of the \( \theta \)-expanded action which extends the usual supersymmetry transformations by terms of order \( n \geq 1 \) in \( \theta \). This result implies that the Seiberg-Witten map for super Yang-Mills theory cannot be expressed in terms of superfields. Some comments on superfields are given in Sec. 5.

2 The noncommutative super Yang-Mills action and its symmetries

The noncommutative \( \mathcal{N}=1 \) super Yang-Mills action is in the component formulation defined by

\[
\Gamma = \int d^4 x \text{tr} \left( -\frac{1}{4} \hat{F}^{\mu\nu} \hat{F}_{\mu\nu} + i \hat{\lambda}^a \sigma^{\mu}_{\dot{a}a} \hat{D}_\mu \hat{\lambda}^\dot{a} + \frac{1}{2} \hat{D}^2 \right),
\]

where

\[
\hat{F}_{\mu\nu} := \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i [\hat{A}_\mu, \hat{A}_\nu],
\]

\[
\hat{D}_\mu \hat{\lambda}^\dot{a} := \partial_\mu \hat{\lambda}^\dot{a} - i [\hat{A}_\mu, \hat{\lambda}^\dot{a}].
\]

Some useful properties of objects carrying spinor indices \( a, \dot{a} \in \{1,2\} \) are listed in the appendix. The \( * \)-(anti)commutators of matrix-valued Schwartz class functions \( f, g \) are defined by

\[
[f, g]_* = g * f - f * g , \quad \{f, g\}_* = g * f + f * g ,
\]

where the \( * \)-product is defined by

\[
(f * g)(x) = \int d^4 y \int \frac{d^4 k}{(2\pi)^4} f(x+y) g(x+y) e^{i k \cdot y},
\]

with \( (\theta \cdot k)\mu := \theta^{\mu\nu} k_\nu \), \( k \cdot y := k_\mu y^\mu \) and \( \theta^{\mu\nu} = -\theta^{\nu\mu} \in M_4(\mathbb{R}) \). We consider \( \theta^{\mu\nu} \) as the components of a translation-invariant tensor field.

The action (1) is invariant under gauge transformations

\[
W^G_\omega = \int d^4 x \text{tr} \left( \delta \hat{D}_\mu \hat{\omega} \frac{\delta}{\delta \hat{A}_\mu} - i [\hat{\lambda}^\dot{a}, \hat{\omega}]_* \frac{\delta}{\delta \hat{\lambda}^\dot{a}} - i [\hat{\lambda}^a, \hat{\omega}]_* \frac{\delta}{\delta \hat{\lambda}^a} - i [\hat{D}, \hat{\omega}]_* \frac{\delta}{\delta \hat{D}} \right),
\]
observer Lorentz transformations
\[
W^T = \int d^4 x \, \text{tr} \left( \partial_\tau \dot{A}_\mu \frac{\delta}{\delta A_\mu} + \partial_\tau \dot{\lambda}^a \frac{\delta}{\delta \lambda^a} + \partial_\tau \dot{\lambda}^a \frac{\delta}{\delta \lambda^\alpha} + \partial_\tau \dot{D} \frac{\delta}{\delta D} \right),
\]
(7)
\[
W^R_{\alpha\beta} := \int d^4 x \, \text{tr} \left( \left( \frac{1}{2} \{ x_\alpha, \partial_\beta \dot{A}_\mu \} \right)_\ast + \frac{1}{2} \{ g_{\mu\alpha} \dot{A}_\beta - g_{\mu\beta} \dot{A}_\alpha \} \right) \frac{\delta}{\delta \dot{A}_\mu} \\
+ \left( \frac{1}{2} \{ x_\alpha, \partial_\beta \dot{\lambda}^a \} \right)_\ast - \frac{1}{2} \{ x_\beta, \partial_\alpha \dot{\lambda}^a \} \right)_\ast + \frac{i}{2} \lambda^b \sigma_{\alpha\beta}^a b^a \right) \frac{\delta}{\delta \dot{A}_\mu} \\
+ \left( \frac{1}{2} \{ x_\alpha, \partial_\beta \dot{D} \} \right)_\ast - \frac{1}{2} \{ x_\beta, \partial_\alpha \dot{D} \} \right)_\ast \frac{\delta}{\delta \dot{D}} \\
+ \left( \delta^\mu_{\alpha\beta} \theta^\nu_{\beta} - \delta^\mu_{\beta\alpha} \theta^\nu_{\alpha} + \delta^\nu_{\alpha\beta} \theta^\mu_{\beta} - \delta^\nu_{\beta\alpha} \theta^\mu_{\alpha} \right) \frac{\partial}{\partial \theta^{\mu\nu}} \right),
\]
(8)
\[
W^D := \int d^4 x \, \text{tr} \left( \left( \frac{1}{2} \{ x_\delta, \partial_\beta \dot{A}_\mu \} \right)_\ast + \dot{A}_\mu \right) \frac{\delta}{\delta A_\mu} + \left( 2 \dot{D} + \frac{1}{2} \{ x_\delta, \partial_\beta \dot{D} \} \right)_\ast \frac{\delta}{\delta D} \\
+ \left( \frac{3}{2} \lambda^\alpha + \frac{1}{2} \{ x_\delta, \partial_\beta \dot{\lambda}^a \} \right)_\ast \frac{\delta}{\delta \lambda^a} + \left( \frac{3}{2} \dot{\lambda}^\alpha + \frac{1}{2} \{ x_\delta, \partial_\beta \dot{\lambda}^a \} \right)_\ast \frac{\delta}{\delta \lambda^a} \right) \\
- 2 \theta^{\mu\nu} \frac{\partial}{\partial \theta^{\mu\nu}},
\]
(9)
and supersymmetry transformations
\[
W^S_a = \int d^4 x \, \text{tr} \left( \sigma_{\mu a} \dot{\lambda}^a \frac{\delta}{\delta A_\mu} + (\delta^b_{\alpha} \dot{D} + \frac{1}{2} \sigma^a_{\mu
u} b \dot{F}^a_{\mu\nu}) \frac{\delta}{\delta \dot{A}_\mu} \right),
\]
(10)
\[
W^S_\dot{a} = \int d^4 x \, \text{tr} \left( \dot{\lambda}^a \sigma_{\mu a} \frac{\delta}{\delta A_\mu} + (\delta^b_{\alpha} \dot{D} - \frac{1}{2} \sigma^a_{\mu
u} a \dot{F}^a_{\mu\nu}) \frac{\delta}{\delta \dot{A}_\mu} \right). \]

(11)

The partial derivative with respect to \( \theta^{\mu\nu} \) has the property
\[
\frac{\partial (\hat{U} \ast \hat{V})}{\partial \theta^{\mu\nu}} = \frac{\partial \hat{U}}{\partial \theta^{\mu\nu}} \ast \hat{V} + \hat{U} \ast \frac{\partial \hat{V}}{\partial \theta^{\mu\nu}} + \frac{i}{2} (\partial_\mu \hat{U}) \ast (\partial_\nu \hat{V}),
\]
(12)
where the fields \( \dot{A}_\mu, \dot{\lambda}^a, \dot{\lambda}^\alpha, \dot{D} \) must be assumed to be independent of \( \theta \).

3. Seiberg-Witten differential equations

As in (non-supersymmetric) noncommutative Yang-Mills theory\[\text{we derive the Seiberg-Witten differential equations via a splitting of the observer Lorentz transformation } W^R_{\alpha\beta} \text{ into the covariant particle Lorentz transformation } \tilde{W}^R_{\phi;\alpha\beta} \text{ and a remaining piece } \bar{W}^R_{\theta;\alpha\beta} \text{ involving the Seiberg-Witten differential equation:}\]
\[
W^R_{\alpha\beta} \equiv \tilde{W}^R_{\phi;\alpha\beta} + \bar{W}^R_{\theta;\alpha\beta},
\]
(13)
\[
\tilde{W}^R_{\phi;\alpha\beta}(\theta^{\mu\nu}) = 0,
\]
(14)
\[
[\tilde{W}^R_{\phi;\alpha\beta}, W^C_\omega] = W^C_{\omega_\alpha\beta}, \quad [\bar{W}^R_{\theta;\alpha\beta}, W^C_\omega] = W^C_{\omega_\alpha\beta}.
\]
(15)
The motivation for this ansatz is the following. The commutator of an observer Lorentz rotation \([3]\) with a gauge transformation \([8]\) is again a gauge transformation,

\[
[W^R_{\alpha\beta}, W^G_\omega] = W^G_{\omega_{\alpha\beta}},
\]

for some infinitesimal gauge parameter \(\hat{\omega}_{\alpha\beta}[\hat{\omega}]\). A particle Lorentz transformation is defined as the part of an observer Lorentz transformation which does not transform the field \(\theta^{\mu\nu}\), see \([4]\). However, one should require that a particle Lorentz transformation transforms a gauge-invariant quantity into another gauge-invariant quantity, otherwise the particle Lorentz transformation cannot be considered as well-defined \([6]\). It is sufficient to demand \([13]\) in order to achieve this property.

To find the sought for splitting we first apply the ansatz of ref. \([5]\) for the Yang-Mills field \(\hat{A}_\mu\):

\[
\hat{W}^R_{\phi\alpha\beta} \hat{A}_\mu = \hat{D}_\mu \hat{\chi}_{\alpha\beta} + \left( \frac{1}{2} \{ \hat{X}_\alpha, \hat{F}_{\beta\mu} \}_* - \frac{1}{2} \{ \hat{X}_\beta, \hat{F}_{\alpha\mu} \}_* - W^R_{\alpha\beta}(\theta^{\rho\sigma}) \hat{\Omega}_{\rho\sigma\mu} \right),
\]

where \(\hat{X}^\mu = x^\mu + \theta^{\mu\nu} \hat{A}_\nu\) are the covariant coordinates \([1\,\bar{2}]\) and \(\hat{\Omega}_{\rho\sigma\mu}\) is a polynomial in covariant quantities such as \(\theta^{\alpha\beta}, \hat{F}_{\alpha\lambda}, \hat{D}_{\mu_1} \ldots \hat{D}_{\mu_n} \hat{F}_{\kappa\lambda}\), antisymmetric in \(\rho, \sigma\), of power-counting dimension 3, and expresses the freedom in the splitting. In the following we set \(\hat{\Omega}_{\rho\sigma\mu} = 0\). The parameter \(\hat{\chi}_{\alpha\beta}\) is unchanged and given by \([8]\)

\[
\hat{\chi}_{\alpha\beta} = \frac{1}{4} \{ 2x_\alpha + \theta^\rho \hat{\hat{A}}_\rho, \hat{A}_\beta \}_* - \frac{1}{4} \{ 2x_\beta + \theta^\rho \hat{\hat{A}}_\rho, \hat{A}_\alpha \}_* .
\]

Comparing \([17]\) with the \(\hat{A}_\mu\)-part of \([8]\) and extending this covariantisation to the remaining fields \(\hat{\lambda}^a, \hat{\hat{\lambda}}^\alpha, D\) we obtain from \([8]\)

\[
\hat{W}^R_{\phi\alpha\beta} = W^G_{\hat{\chi}_{\alpha\beta}} + \int d^4x \text{tr} \left( \left( \frac{1}{2} \{ \hat{X}_\alpha, \hat{F}_{\beta\mu} \}_* - \frac{1}{2} \{ \hat{X}_\beta, \hat{F}_{\alpha\mu} \}_* \right) \frac{\delta}{\delta \hat{A}_\mu} \right.
\]

\[
+ \left( \frac{1}{2} \{ \hat{X}_\alpha, \hat{D}_\beta \hat{\lambda}^a \}_* - \frac{1}{2} \{ \hat{X}_\beta, \hat{D}_\alpha \hat{\lambda}^a \}_* + \frac{i}{2} \hat{\lambda}^b \sigma_{\alpha\beta b} \right) \frac{\delta}{\delta \hat{\lambda}^a} \right.
\]

\[
+ \left( \frac{1}{2} \{ \hat{X}_\alpha, \hat{D}_\beta \hat{\hat{\lambda}}^\alpha \}_* - \frac{1}{2} \{ \hat{X}_\beta, \hat{D}_\alpha \hat{\hat{\lambda}}^\alpha \}_* + \frac{i}{2} \sigma_{\alpha\beta \hat{\hat{\alpha}}} \right) \frac{\delta}{\delta \hat{\hat{\lambda}}^\alpha} \right.
\]

\[
+ \left( \frac{1}{2} \{ \hat{X}_\alpha, \hat{D}_\beta \hat{D} \}_* - \hat{D}_\alpha \hat{D} \}_* \right) \frac{\delta}{\delta \hat{\hat{\lambda}}^\alpha} \right) .
\]

Now it is straightforward to evaluate

\[
\hat{W}^R_{\theta\alpha\beta} = W^R_{\alpha\beta} - \hat{W}^R_{\phi\alpha\beta} = W^R_{\alpha\beta}(\theta^{\rho\sigma}) \frac{d}{d\theta^{\rho\sigma}} ,
\]

with

\[
\frac{d}{d\theta^{\rho\sigma}} = \frac{\partial}{\partial \theta^{\rho\sigma}} + \int d^4x \text{tr} \left( \frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} \frac{\delta}{\delta \hat{A}_\mu} + \frac{d\hat{\lambda}^a}{d\theta^{\rho\sigma}} \frac{\delta}{\delta \hat{\lambda}^a} + \frac{d\hat{\hat{\lambda}}^\alpha}{d\theta^{\rho\sigma}} \frac{\delta}{\delta \hat{\hat{\lambda}}^\alpha} + \frac{d\hat{D}}{d\theta^{\rho\sigma}} \frac{\delta}{\delta \hat{\hat{\lambda}}^\alpha} \right) ,
\]
which yields the Seiberg-Witten differential equations

\[ \frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} = -\frac{1}{8}\{\hat{A}_\rho, \partial_\sigma \hat{A}_\mu + \hat{F}_{\rho\sigma}\}_*, + \frac{1}{8}\{\hat{A}_\sigma, \partial_\rho \hat{A}_\mu + \hat{F}_{\rho\mu}\}_*, \] (22)

\[ \frac{d\hat{\chi}^a}{d\theta^{\rho\sigma}} = -\frac{1}{8}\{\hat{A}_\rho, \partial_\sigma \hat{\chi}^a + \hat{D}_\sigma \hat{\chi}^a\}_* + \frac{1}{8}\{\hat{A}_\sigma, \partial_\rho \hat{\chi}^a + \hat{D}_\rho \hat{\chi}^a\}_*, \] (23)

\[ \frac{d\hat{\lambda}^a}{d\theta^{\rho\sigma}} = -\frac{1}{8}\{\hat{A}_\rho, \partial_\sigma \hat{\lambda}^a + \hat{D}_\sigma \hat{\lambda}^a\}_* + \frac{1}{8}\{\hat{A}_\sigma, \partial_\rho \hat{\lambda}^a + \hat{D}_\rho \hat{\lambda}^a\}_*, \] (24)

\[ \frac{d\hat{D}}{d\theta^{\rho\sigma}} = -\frac{1}{8}\{\hat{A}_\rho, \partial_\sigma \hat{D} + \hat{D}_\sigma \hat{D}\}_* + \frac{1}{8}\{\hat{A}_\sigma, \partial_\rho \hat{D} + \hat{D}_\rho \hat{D}\}_*. \] (25)

The differential equation (23) was first found in ref. [1].

4 \(\theta\)-expansion of the action

The differential equations (22)–(25) are now taken as the starting point for a \(\theta\)-expansion of the action,

\[ \Gamma^{(n)} := \sum_{j=0}^{n} \frac{1}{j!} \theta^{\rho_1 \sigma_1} \cdots \theta^{\rho_j \sigma_j} \left( \frac{d\Gamma}{d\theta^{\rho_1 \sigma_1} \cdots d\theta^{\rho_j \sigma_j}} \right)_{\theta=0}. \] (26)

It follows from the the second identity in (13) that the \(\theta\)-expansion (26) of the action (1) is invariant under commutative gauge transformations. Using (7)–(11) one also checks the identity

\[ \left[ W^{(T,R,D)}, \theta^{\rho\sigma} \frac{d}{d\theta^{\rho\sigma}} \right] = 0 \] (27)

for super Yang-Mills theory, which means that the \(\theta\)-expansion of the fields leads to a commutative action invariant under commutative rotations and translations and with commutative dilatational symmetry. The identity (27) is a consequence of the fact that \(\theta^{\rho\sigma} \frac{d}{d\theta^{\rho\sigma}}\) is a Lorentz scalar with respect to observer Lorentz transformations.

The \(\theta\)-expansion of (1) yields an action which is not invariant under commutative supersymmetry transformations. Indeed, the commutator of a supersymmetry transformation (11) and a \(\theta\)-differentiation is given by

\[ \left[ \frac{d}{d\theta^{\rho\sigma}}, W^S_a \right] = \frac{d}{d\theta^{\rho\sigma}} W^S_a - W^S_a \frac{d}{d\theta^{\rho\sigma}}, \]

\[ + \int d^4x \text{tr}\left( \left( \frac{1}{4} \sigma_{\rho a\hat{a}} \{\hat{A}_\rho, \hat{\lambda}^a\}_* - \frac{1}{4} \sigma_{\sigma a\hat{a}} \{\hat{F}_{\rho\sigma}, \hat{\lambda}^a\}_* \right) \frac{\delta}{\delta A_{\mu}} \right. \]

\[ + \left( \frac{1}{4} \sigma_a^{\mu
u} {\hat{F}_{\mu\nu}, \hat{F}_{\rho\sigma}} \right) * + \frac{1}{4} \sigma_{\rho a\hat{a}} \{\hat{\lambda}^a, \hat{D}_\sigma \hat{\lambda}^b\}_* - \frac{1}{4} \sigma_{\sigma a\hat{a}} \{\hat{\lambda}^a, \hat{D}_\rho \hat{\lambda}^b\}_*, \]

\[ \left. + \frac{1}{4} \sigma_{\rho a\hat{a}} \{\hat{\lambda}^a, \hat{D}_\sigma \hat{\lambda}^b\}_* - \frac{1}{4} \sigma_{\sigma a\hat{a}} \{\hat{\lambda}^a, \hat{D}_\rho \hat{\lambda}^b\}_* \right) \frac{\delta}{\delta \lambda^b} \right. \]

\[ b\text{There is of course a freedom in the differential equations (22)–(23) given by the } \Omega\text{-terms in (11) and similarly for the other fields. This freedom is not sufficient to obtain a vanishing right hand side of (23).} \]
\[ + \left( \frac{1}{4} \sigma^{\mu}_{aa} \{ F_{\sigma \mu}, \hat{D}_\rho \hat{\lambda}^a \}, - \frac{1}{4} \sigma^{\mu}_{aa} \{ \hat{F}_{\sigma \mu}, \hat{D}_\rho \hat{\lambda}^a \}\right) \delta_{\delta \hat{D}}. \]  

(28)

where the gauge transformation with respect to a fermionic parameter \( \hat{\omega} \) is defined by

\[ \hat{W}^G_{\hat{\omega}} = \int d^4x \text{tr} \left( \hat{D}_\mu \hat{\omega} \frac{\delta}{\delta A_\mu} + i \{ \hat{\lambda}^\alpha, \hat{\omega} \}\right) \delta_{\delta \hat{D}} + i \{ \hat{\lambda}^\alpha, \hat{\omega} \}\right) \delta_{\delta \hat{\lambda}^\alpha} + i[\hat{D}, \hat{\omega}] \delta_{\delta \hat{D}}. \]  

(29)

The action (\( W^S \)) is invariant under the transformation (\( \theta^a \)). It follows now from (26) that the \( \theta \)-expansion of (\( W^S \)) is invariant under the transformation

\[ W^{S, \text{comm}}_a = (W^S_a)_{\theta=0} + \sum_{n=1}^{\infty} \frac{1}{n!} \theta^{\rho_1 \sigma_1} \cdots \theta^{\rho_n \sigma_n} \left( \left[ \frac{d}{d\theta^{\rho_1 \sigma_1}}, \left[ \cdots \left[ \frac{d}{d\theta^{\rho_n \sigma_n}}, W^S_a \right] \cdots \right] \right] \right)_{\theta=0}, \]  

(30)

which due to \([\frac{d}{d\theta^a}, W^S_a] \neq 0 \) is different from the commutative supersymmetry transformation \( (W^S_a)_{\theta=0} \). In other words, the Seiberg-Witten map does not diagonalise the \( \theta \)-expansion of the noncommutative supersymmetry transformation, see also the remarks in the Introduction. The first terms of (30) read

\[ W^{S, \text{comm}}_a = \int d^4x \text{tr} \left( \sigma_{\mu \alpha} \hat{\lambda}^\alpha + \frac{1}{2} \theta^{\rho \sigma} \sigma_{\rho \alpha a} \{ F_{\sigma \mu}, \hat{\lambda}^a \right) \delta_{\delta A_\mu} + \left( \frac{1}{2} \theta^{\rho \sigma} \sigma_{\rho \alpha a} \{ \hat{\lambda}^\alpha, D_\sigma \hat{\lambda}^b \right) \delta_{\delta \hat{\lambda}^b} \right.

\[ + \left( \delta^{\hat{\lambda}}_{\hat{\lambda}} \hat{D} + \frac{1}{2} \sigma^{\mu \nu b} F_{\mu \nu} + \frac{1}{4} \theta^{\rho \sigma} \sigma^{\mu \nu b}_{\alpha a} \{ F_{\rho \nu}, F_{\sigma \mu a} \right) + \left( \frac{1}{2} \theta^{\rho \sigma} \sigma_{\rho \alpha a} \{ F_{\sigma \mu}, \hat{\lambda}^a \right) \delta_{\delta \hat{\lambda}^a} \right.

\[ + \left( - i \sigma^{\mu}_{aa} D_\mu \hat{\lambda}^a + \frac{1}{2} \theta^{\rho \sigma} \sigma^{\mu}_{aa} \{ F_{\sigma \mu}, D_\rho \hat{\lambda}^a \right) + \left( \frac{1}{2} \theta^{\rho \sigma} \sigma_{\rho \alpha a} \{ \hat{\lambda}^\alpha, D_\sigma \hat{\lambda}^b \right) \delta_{\delta \hat{\lambda}^b} \right) + O(\theta^2). \]  

(31)

Similar formulae exist for the anti-supersymmetry transformation \( W^S_\alpha \). An analogous result for \( U(1) \)-theory with general \( \theta^{\mu \nu} \) has also been obtained.

At order \( n = 0 \) in \( \theta \) the expansion of (\( W^S \)) is obviously the standard super Yang-Mills action

\[ \Gamma^{(0)} = \int d^4x \text{tr} \left( - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} + i \lambda^a \sigma^{\mu}_{aa} D_\mu \hat{\lambda}^a + \frac{1}{2} D^2 \right), \]  

(32)

where \( \phi = \hat{\phi}_{\theta=0} \) for \( \phi \in \{ A_\mu, \lambda^a, \hat{\lambda}^a, D \}. \) At first order in \( \theta \) one finds

\[ \Gamma^{(1)} = \Gamma^{(0)} - \frac{1}{2} \int d^4x \text{tr} \left( \theta^{\rho \sigma} F_{\rho \sigma} \left( - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} \hat{\lambda}^a \sigma^{\mu}_{aa} D_\mu \lambda^a + \frac{1}{2} \lambda^a \sigma^{\mu}_{aa} D_\mu \hat{\lambda}^a + \frac{1}{2} D^2 \right) \right. \]

\[ + \theta^{\rho \sigma} F_{\rho \nu} F_{\sigma \mu} + \theta^{\rho \sigma} F_{\rho \mu} \left( i \lambda^a \sigma^{\mu}_{aa} D_\sigma \lambda^a + i \lambda^a \sigma^{\mu}_{aa} D_\sigma \hat{\lambda}^a \right) \]  

(33)

The \( \theta \)-expanded action (33) could be further analysed, for instance with respect to new decay channels of supersymmetric particles—\( \text{in a similar manner as investigations of models without supersymmetry} \).
5 Remarks on the superspace formalism

The most compact way to formulate supersymmetric theories is to use the superfield formalism. The above considered fields $\hat{A}_\mu$, $\hat{\lambda}^a$, $\hat{\lambda}^b$, $\hat{D}$ of super Yang-Mills theory can be regarded as components of the superfield

$$\hat{\phi} = \hat{C} + \hat{\chi}^a \theta_a + \hat{\theta}_{\dot{a}} \hat{\chi}^{\dot{a}} + \theta^a \theta_{\dot{a}} \hat{M} + \hat{\theta}_{\dot{a}} \hat{\theta}^{\dot{a}} \hat{\dot{M}}$$

$$-2 \theta^a \sigma_{a\dot{a}} \hat{\theta}^{\dot{a}} \hat{A}_\mu - 2 \hat{\theta}_{\dot{a}} \hat{\lambda}^a \theta_a - 2 \hat{\lambda}^{a \dot{a}} \theta_{\dot{a}} \theta_a - \theta^a \theta_{\dot{a}} \theta_a \theta_{\dot{a}} \hat{D}.$$  \hspace{1cm} (34)

The anticommuting variables $\theta^a, \theta^{\dot{a}}$ should not be confused with the noncommutativity parameter $\theta^{\mu\nu}$. The Wess-Zumino gauge consists in setting the components $\hat{\chi}^a, \hat{\lambda}^a, \hat{M}, \hat{\dot{M}}$ equal to zero. One has $\hat{\phi} \star \hat{\phi} \star \hat{\phi} = 0$ in this gauge. For details about the superfield formalism we refer to ref. [2].

Due to $[\frac{d}{d\theta^a}, W^S_a] \neq 0$, see (28), a Seiberg-Witten map in terms of superfields cannot exit. All one can do is to write the previous formulae in a more compact form, in which the super vector field is understood to be in Wess-Zumino gauge. The gauge transformations and observer Lorentz transformations can be written in the compact form

$$W^G_\omega = \int d^4x \left( -2 \theta^a \sigma_{a\dot{a}} \hat{\theta}^{\dot{a}} \partial_\mu \omega - i [\hat{\phi}, \hat{\omega}]_* \right) \frac{\delta}{\delta \hat{\phi}}$$, \hspace{1cm} (35)

$$W^T_\tau := \int d^4x \text{tr} \left( \partial_\tau \hat{\phi} \frac{\delta}{\delta \hat{\phi}} \right),$$ \hspace{1cm} (36)

$$W^R_{a\dot{b}} := \int d^4x \text{tr} \left( \left( \frac{1}{2} \{ x_a, \partial_\beta \hat{\phi} \}_* - \frac{1}{2} \{ x_{\dot{b}}, \partial_\dot{a} \hat{\phi} \}_* + \Sigma_{a\dot{b}} \hat{\phi} \right) \frac{\delta}{\delta \hat{\phi}} \right)$$

$$+ \left( \delta^a_{a'} \theta^\nu_{\beta'} - \delta^\mu_{\beta'} \delta^a_{\nu} + \delta^a_{\nu} \theta^\mu_{\beta} - \delta^a_{\nu} \theta^\mu_{\beta} \right) \frac{\partial}{\partial \theta^{\mu\nu}}.$$ \hspace{1cm} (37)

$$W^D = \int d^4x \text{tr} \left( \left( \frac{1}{2} \{ x^\delta, \partial_\beta \hat{\phi} \}_* \frac{\delta}{\delta \hat{\phi}} \right) - 2 \theta^a \theta^\dot{a} \frac{\partial}{\partial \theta^{a\dot{a}}} \right).$$ \hspace{1cm} (38)

Here $\Sigma_{a\dot{b}} = -\frac{1}{2} \theta^a \sigma_{a\dot{a}} \theta^{\dot{a}}_{\beta} - \frac{1}{2} \hat{\theta}_{\dot{a}} \hat{\sigma}_{\dot{a}b} \hat{\omega}^b$ is the spin operator for the superfield. The covariant particle Lorentz rotation reads

$$\hat{W}^R_{\varphi a\beta} := W^G_{\hat{\chi}_{a\beta}} + \int d^4x \text{tr} \left( \left( \frac{1}{2} \{ \hat{X}_a, \hat{F}_\beta \}_* - \frac{1}{2} \{ \hat{X}_{\dot{b}}, \hat{F}_a \}_* + \Sigma_{a\beta} (\hat{\phi} + 2 \theta^a \sigma_{a\dot{a}} \hat{\theta}^{\dot{a}} \hat{A}_\mu) \right) \frac{\delta}{\delta \hat{\phi}} \right),$$ \hspace{1cm} (39)

where $\hat{X}_{\hat{a}}$ is given by (18) and

$$\hat{F}_\sigma := \partial_\sigma \hat{\phi} + 2 \theta^a \sigma_{a\dot{a}} \hat{\theta}^{\dot{a}} \partial_\mu \hat{A}_\sigma - i [\hat{A}_\sigma, \hat{\phi}]_*.$$ \hspace{1cm} (40)

This object, resembling the usual field strength tensor $F_{\mu\nu}$, transforms covariantly under supergauge transformations (19). The calculation of the Seiberg-Witten expansion is straightforward and yields

$$\frac{d\hat{\phi}}{d\theta^{a\sigma}} = -\frac{1}{8} \{ \hat{A}_\mu, \partial_\sigma \hat{\phi} + \hat{F}_\sigma \}_* + \frac{1}{8} \{ \hat{A}_\sigma, \partial_\mu \hat{\phi} + \hat{F}_\mu \}_*.$$ \hspace{1cm} (41)
6 Conclusion

Following previous ideas we have derived the Seiberg-Witten map for noncommutative super Yang-Mills theory in Wess-Zumino gauge via the splitting of the observer Lorentz transformation into a covariant particle Lorentz transformation and a remainder, which directly leads to the Seiberg-Witten differential equations. We have also computed the \( \theta \)-expansion of the noncommutative super Yang-Mills action, up to first order in \( \theta \). Each \( \theta \)-order of the action is individually invariant under commutative gauge transformations. In contrast, the \( \theta \)-expansion of the supersymmetry transformation differs from the commutative supersymmetry transformations by terms of order \( n \geq 1 \) in \( \theta \). For this reason the Seiberg-Witten map cannot be expressed in terms of superfields.

A Useful formulae

Spinor indices \( a, \dot{a} \in \{1, 2\} \) are shifted by the antisymmetric metric \( \varepsilon^{ab} = -\varepsilon^{ba}, \varepsilon^{\dot{a} \dot{b}} = -\varepsilon^{\dot{b} \dot{a}} \) according to

\[
\chi_a = \varepsilon_{ab} \chi^b, \quad \bar{\chi}^\dot{a} = \varepsilon^{\dot{b}} \bar{\chi}_{\dot{b}}. \tag{A.1}
\]

Note that spinors are anticommuting,

\[
\chi^a \eta_a = -\chi_a \eta^a = \eta^a \chi_a = -\eta_a \chi^a, \quad \bar{\chi}^\dot{a} \bar{\eta}_\dot{a} = -\bar{\chi}_\dot{a} \bar{\eta}^\dot{a} = \bar{\eta}_\dot{a} \bar{\chi}^\dot{a} = -\bar{\eta}^\dot{a} \bar{\chi}_\dot{a}. \tag{A.2}
\]

The \( 2 \times 2 \) \( \sigma \)-matrices are given by

\[
\sigma^\mu_{aa} = (1, \tilde{\sigma})_{a\dot{a}}, \quad \tilde{\sigma}^\mu_{\dot{a}a} = (1, -\tilde{\sigma})^{\dot{a}a}, \quad \sigma^\mu_{\dot{a}\dot{a}} = \tilde{\sigma}^\mu_{a\dot{a}}, \tag{A.3}
\]

where \( \tilde{\sigma} \) denotes the three Pauli matrices. The \( \sigma \)-matrices satisfy

\[
\sigma^\mu_{aa} \sigma^{\nu \dot{a} \dot{b}} = g^{\mu \nu} \delta^b_a - i \sigma^\mu_{ab}, \quad \tag{A.4}
\]

\[
\tilde{\sigma}^\mu_{a\dot{a}} \sigma^{\nu \dot{a} \dot{b}} = g^{\mu \nu} \delta^\dot{b}_\dot{a} - i \tilde{\sigma}^{\mu \dot{a} \dot{b}} \tag{A.5}
\]

\[
\sigma^\mu_{a\dot{a}} \sigma^{\nu \dot{a} \dot{b}} \sigma^\rho_{bb} = g^{\mu \nu} \sigma^\rho_{ab} + g^{\mu \rho} \sigma^\nu_{ab} - g^{\rho \mu} \sigma^\nu_{ab} - i \epsilon^{\mu \rho \lambda} \sigma_{ab} \lambda, \tag{A.6}
\]

\[
\sigma^{\dot{a} \dot{a}} \sigma^{\nu \dot{a} \dot{b}} \sigma^\rho_{bb} = g^{\nu \rho} \tilde{\sigma}^\dot{b} \dot{a} + g^{\nu \dot{a}} \tilde{\sigma}^\rho \dot{b} - g^{\rho \dot{a}} \tilde{\sigma}^\nu \dot{b} + i \epsilon^{\mu \rho \lambda} \tilde{\sigma}^\dot{b} \dot{a} \lambda, \tag{A.7}
\]

\[
\sigma^\mu_{a\dot{a}} \sigma^{\nu \dot{a} \dot{b}} = 2 \varepsilon_{ab} \varepsilon_{\dot{a} \dot{b}} \tag{A.8}
\]

with \( \sigma^{\mu \nu}_{\dot{a}} = -\sigma^{\mu \nu}_{a} \) and \( \tilde{\sigma}^{\mu \nu \dot{a}} = -\tilde{\sigma}^{\mu \nu \dot{a}} \).

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