A penalty scheme for monotone systems with interconnected obstacles: convergence and error estimates

Christoph Reisinger∗ Yufei Zhang†

Abstract. We present a novel penalty approach for a class of quasi-variational inequalities (QVIs) involving monotone systems and interconnected obstacles. We show that for any given positive switching cost, the solutions of the penalized equations converge monotonically to those of the QVIs. We estimate the penalization errors and are able to deduce that the optimal switching regions are constructed exactly. We further demonstrate that as the switching cost tends to zero, the QVI degenerates into an equation of HJB type, which is approximated by the penalized equation at the same order (up to a log factor) as that for positive switching cost. Numerical experiments on optimal switching problems are presented to illustrate the theoretical results and to demonstrate the effectiveness of the method.

Key words. Quasi-variational inequalities, monotone systems, penalty methods, monotone convergence, error estimate.

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1 Introduction

In this article, we consider the following discrete quasi-variational inequality (QVI):

Problem 1.1. Find $u = (u^1, \ldots, u^d) \in \mathbb{R}^{N \times d}$, such that

$$G_i(u) = \min(F_i(u), u^i - M_i u) = 0, \quad i \in \mathcal{I} := \{1, \ldots, d\}, \quad d \geq 2,$$

(1.1)

where $M_i u := \max_{j \neq i} (u^j - c^i j)$ with $c^i j > 0$ for all $j \neq i$, and $F = (F_i)_{i \in \mathcal{I}} : \mathbb{R}^{N \times d} \to \mathbb{R}^{N \times d}$ is a continuous function satisfying the following condition:

- There exists a constant $\gamma > 0$ such that for any $u, v \in \mathbb{R}^{N \times d}$ and $i \in \mathcal{I}$ with $u^i_l - v^i_l = \max_{j \in \mathcal{I}, k=1,\ldots,N} u^j_k - v^j_k \geq 0$, we have

$$F_i(u)_l - F_i(v)_l \geq \gamma (u^i_l - v^i_l).$$

(1.2)

Following [10], we will refer to $Mu$ as an interconnected obstacle, where $M$ is a special form of the intervention operator in [2]. Similar to [8] in the continuous context, we shall also refer to a continuous function $F = (F_i)_{i \in \mathcal{I}}$ with condition (1.2) as a monotone system. Such discrete systems arise naturally from a sensible discretization (i.e., stable and convergent) of elliptic and parabolic QVIs associated to hybrid optimal control problems with switching controls (see e.g. [19, 2, 10, 12]). Since each $F_i$ could depend on all components of the solution $u$, the system $F$ in Problem 1.1

∗Mathematical Institute, University of Oxford, United Kingdom (christoph.reisinger@maths.ox.ac.uk, yufei.zhang@maths.ox.ac.uk)
not only includes systems of Isaacs equations [17] or possibly nonlocal Hamilton-Jacobi-Bellman (HJB) variational inequalities [23, 20], but also includes coupled systems stemming from regime switching models [3] and non-cooperative games [11]. To simplify the presentation, we shall focus on the case with constant switching cost $c_{i,i} = 0$ and $c_{i,j} \equiv c > 0$ for all $j \neq i$, $i \in \mathcal{I}$, but the results and analysis extend naturally to the cases with general switching costs $c_{i,j} > 0$ for all $j \neq i$ satisfying the following triangle inequality:

\[ c_{i,j} + c_{j,k} \geq c_{i,k}, \quad \forall i, j, k \in \mathcal{I}, \quad l \in \mathcal{N} := \{1, \ldots, N\}. \]

By rewriting the interconnected obstacle $u^i - M_i u$ as $\min_{j \neq i} (P_j^i u - c)$, one can easily see that the matrices $(P_j^i)_{j \neq i}$ involved are neither $M$-matrices nor weakly chained diagonally dominant matrices ([2]). Hence a direct application of policy iteration could fail due to the possible singularity of the matrix iterates. However, as we shall see later, penalty schemes are always applicable (even for zero switching cost) and it is straightforward to derive convergent iterative solvers for the penalized equations, which make penalty schemes more appealing for solving the QVI (1.1). Thus it is important to design efficient penalty schemes for general monotone systems with interconnected obstacles.

Moreover, we point out that the implementation of penalty schemes, in particular the choice of penalty parameters, depends greatly on the accuracy of the penalty approximation with a given penalty parameter, hence it is practically important to quantify the penalty errors. However, the non-diagonal dominance of the matrices $(P_j^i)_{j \neq i}$ poses a significant challenge for estimating the penalization errors. In fact, an essential step in estimating the penalty error for standard obstacle problems is to show that if $u^\rho$ solves the penalized equation with the penalty parameter $\rho \geq 0$, then $u^\rho + C/\rho$ is a feasible solution to the obstacle problem (i.e., it satisfies the constraint) for a large enough constant $C$ independent of the penalty parameter (see e.g. [20]). But this is clearly false in the current setting since the interconnected obstacles remain invariant under any vertical shift of the solutions. We shall overcome these difficulties by introducing certain regularization procedures, which consist of approximating the switching problem by a sequence of obstacle problems involving diagonally dominant matrices, and recover the same convergence rates (up to a log factor) as those for conventional obstacle problems.

Finally, we remark that the monotonicity condition (1.2) is essentially different from the monotonicity in the Euclidean norm discussed in [14], which enables us to consider penalty methods for the QVIs of fully nonlinear degenerate equations including Isaacs equations. To the best of our knowledge, there is no published penalty scheme with rigorous error estimates covering such general QVIs. However, there is a vast literature on penalty methods for variational inequalities (see e.g. [13, 15, 14]). For works covering specific extensions, we refer the reader to [22] for penalty approximations to HJB equations, to [23, 20] for applying policy iteration together with penalization to solve HJB variational inequalities, and to [4, 2, 1] for an application of penalty schemes to classical HJBQVIs (without error estimates).

The main contributions of our paper are:

- We propose penalty schemes for discrete monotone systems with interconnected obstacles. Contrary to [2, 1], the penalty is applied to each component of the system, which enables us to derive easily implementable and efficient iterative schemes for penalized equations without taking the pointwise maximum over all switching components.
- We present a novel analysis technique for the well-posedness of the penalized equations with a general class of penalty terms by smoothing the monotone systems. We further demonstrate that the solution of the penalized equation converges to the solution of (1.1) monotonically.

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from below, which subsequently gives a constructive proof for the existence of solutions to Problem 1.1.

- Based on regularizations of the interconnected obstacles, we estimate the penalization error for monotone systems with concave nonlinearity, which include HJBQVIs as special cases (see the discussions below Assumption 1). We introduce two iterative regularization procedures, namely the iterated optimal stopping approximation and the time-marching iteration, which enable us to demonstrate that for any given positive switching cost, the penalty approximation using a penalty term with degree \( \sigma > 0 \) enjoys convergence of order \( O(\rho^{-\sigma} \ln \rho) \) as the penalty parameter \( \rho \to \infty \), independent of the number of switching regimes \( d \). We emphasize that, unlike the error estimates for HJBQVIs in [7, 12], our analysis does not require the running reward functions or the solutions to have a unique sign. Moreover, our error estimate also enables us to exactly construct the switching regions of Problem 1.1, which to the best of our knowledge is new, even for classical HJBQVIs.

- We further investigate the limiting case with zero switching cost, where Problem 1.1 degenerates to an equation of HJB type, i.e., Problem 4.1 below. In this case, the penalty scheme of (1.1) leads to a novel penalty scheme for such equations, which admits the same convergence rates (up to a log factor) as those for fixed positive switching cost when the penalty parameter tends to infinity. We remark that this error estimate applies to non-convex/non-concave systems, such as systems of Isaacs equations.

- Numerical examples for infinite-horizon optimal switching problems in the two-regime case and the three-regime case are included to illustrate the theoretical results for the asymptotic behaviours of the penalty errors with respect to the penalty parameter and switching cost.

We now summarize some of our main results in the following diagram. Suppose the function \( F \) in Problem 1.1 is concave and the penalty function in Problem 2.1 below is given by \( \pi(y) = y^+ \). Then the following error estimates hold:

\[
\begin{align*}
& u^{c,\rho} \text{ solves Problem 2.1 with } c, \rho > 0 \\
\Rightarrow & u^{c,\rho} \nearrow u^c \text{ at rate } O(\ln \rho/(c\rho)) \text{ as } \rho \to \infty \\
& u^c \text{ solves Problem 1.1 with } c > 0 \\
\Rightarrow & u^c \nearrow u \text{ at rate } O(c) \text{ as } c \to 0 \\
& u^c \text{ solves Problem 1.1 with } c > 0 \\
\Rightarrow & u^c \nearrow u \text{ at rate } O(1/\rho) \text{ as } \rho \to \infty \\
& u^c \text{ solves Problem 1.1 with } c > 0 \\
\Rightarrow & u^c \nearrow u \text{ at rate } O(c) \text{ as } c \to 0 \\
& u^c \text{ solves Problem 1.1 with } c > 0 \\
\Rightarrow & u^c \nearrow u \text{ at rate } O(1/\rho) \text{ as } \rho \to \infty \\
& u^c \text{ solves Problem 4.1}
\end{align*}
\]

The remainder of this paper is organized as follows. We shall propose a class of penalty approximations to Problem 1.1 in Section 2, and demonstrate its well-posedness and monotone convergence. Then we construct two regularization procedures for Problem 1.1 in Section 3.1, which enable us to obtain an error estimate of the penalization error for QVIs with positive switching cost in Section 3.2. We then proceed to estimate the penalization errors for QVIs with vanishing switching cost in Section 4. Numerical examples for two-regime and three-regime optimal switching problems are presented in Section 5 to illustrate the effectiveness of our algorithms.
2 Penalty approximations of QVIs

In this section, we discuss how Problem 1.1 can be approximated by a sequence of penalized equations. The well-posedness of the penalized equations and their monotone convergence shall be established, which subsequently lead to a constructive proof for the well-posedness of (1.1).

We start by collecting some useful notation. For any given matrix $A, B \in \mathbb{R}^{d_1 \times d_2}$, we denote by

$$A \geq B$$

the relation $A_{ij} \geq B_{ij}$ for all indices $i, j$, by $\min(A, B)$ the matrix of elements $\min(A_{ij}, B_{ij})$, by $A^+ = \max(A, 0)$ (resp. $A^- = \max(-A, 0)$) the (element-wise) positive (resp. negative) part of $A$, and by $\|A\|$ the usual sup-norm $\|A\| = \max_{i,j} |A_{ij}|$.

Before introducing the penalty equations, we first adapt the non-loop arguments in [17] to the current discrete setting and establish a comparison theorem of the QVI (1.1).

**Proposition 2.1.** Suppose $c > 0$ and $u = (u^i)_{i \in I}$ (resp. $v = (v^i)_{i \in I}$) satisfies

$$\min(F_i(u), u^i - M_i u) \leq 0 \quad (\text{resp.} \geq 0), \quad i \in I,$$

then we have $u \leq v$.

**Proof.** Let $u^i_l - v^i_l = \max_{j,k} u^i_k - v^i_k := M$. Suppose that $M > 0$, we shall show by contradiction that there exists $i_0 \in I$ such that $u^i_{0} - v^0 = M$ and $u^i_0 > (M_{i_0} u)_l = \max_{j \neq i_0} (u^i_l - c)$.

Suppose $u^i_l \leq \max_{j \neq i} (u^i_j - c)$ for all $i \in \Gamma$, where $\Gamma = \{i \mid u^i_l - v^i_l = M\} \neq \emptyset$. Then we can pick $i \in \Gamma$ such that $u^i_l \leq (M_i u)_l = u^i_l - c$ for some $j \neq i$. By using $v^i_j \geq (M_i v)_l \geq u^i_j - c$, we have

$$u^i_l - u^j_l \leq -c \leq v^i_l - v^j_l,$$

which implies $u^i_l - v^i_l \leq u^j_l - v^j_l$, and hence $j \in \Gamma$ and the above inequality is in fact an equality.

Since $\Gamma$ is a finite set, inductively we can find $(i_k)_{k=1}^n \in \Gamma$, $u^{i_k} - u^{i_{k+1}} \leq -c < 0$, $k = 1 \ldots, n - 1$, and $i_n = i_1$. Thus $0 = \sum_{k=1}^{n-1} (u^{i_k} - u^{i_{k+1}}) \leq \sum_{k=1}^{n-1} -c < 0$, which is a contradiction.

Consequently we have $F_{i_0}(u)_l \leq 0 \leq F_{i_0}(v)_l$, from which by using the monotonicity (1.2) of $F$, we have $M \leq 0$ and obtain a contradiction. \hfill $\Box$

A direct consequence of Proposition 2.1 is the uniqueness of solutions to (1.1). The existence of solutions to Problem 1.1 shall be established constructively via penalty approximations below (see Remark 2.2).

Now we are ready to propose the penalty approximation of the QVI (1.1), which is an extension of the ideas used for HJB obstacle problems in [23, 20]. For any given parameter $\rho \geq 0$, we shall consider the following penalized problem:

**Problem 2.1.** Find $u^\rho = (u^\rho, i)_{i \in I} \in \mathbb{R}^{N \times d}$ such that

$$G^\rho_i(u^\rho)_l := F_i(u^\rho)_l - \rho \sum_{j \neq i} \pi(u^\rho_{ij} - c - u^\rho_{ij}) = 0, \quad i \in I, \ l \in N,$$

where the penalty term $\pi : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function satisfying $\pi|_{(-\infty, 0]} = 0$ and $\pi|_{[0, \infty)} > 0$.

**Remark 2.1.** In (2.1), contrary to [1, 2], the penalty is applied to each component of the switching system. Thus efficient iterative schemes for penalized equations can be implemented without taking the pointwise maximum over all switching components at each index $l \in N$. However, all the statements below can be shown to hold for penalized problems with penalty terms involving the maximum of all switching components, such as $\max_{j \neq i} \pi(u^\rho_{ij} - c - u^\rho_{ij})$ or $\pi(M_i u^\rho - u^\rho_{ij})$. 


The following result shows a comparison principle for the penalized equation with a fixed penalty parameter $\rho$, which not only implies the uniqueness of solutions to the penalized equations, but also plays a crucial role in the convergence analysis of the penalty approximations.

**Proposition 2.2.** For any given penalty parameter $\rho \geq 0$ and switching cost $c \geq 0$, suppose $u^\rho = (u_i^{\rho,i})_{i \in I}$ (resp. $v^\rho = (v_i^{\rho,i})_{i \in I}$) satisfies

$$F_i(u^\rho) - \rho \sum_{j \neq i} \pi(u_i^{\rho,j} - c - u_i^{\rho,i}) \leq 0 \quad \text{(resp.} \geq 0), \quad i \in I,$$

then we have $u^\rho \leq v^\rho$.

**Proof.** Let $u_i^{\rho,j} - u_i^{\rho,i} = \max_{j,k} u_k^{\rho,j} - u_k^{\rho,i} := M$. Suppose that $M > 0$, then we have $u_i^{\rho,j} - c - u_i^{\rho,i} \leq v_i^{\rho,j} - c - v_i^{\rho,i}$ for all $j \neq i$, and hence

$$\sum_{j \neq i} \pi(u_i^{\rho,j} - c - u_i^{\rho,i}) \leq \sum_{j \neq i} \pi(v_i^{\rho,j} - c - v_i^{\rho,i}).$$

Then by using the monotonicity of $F$ and the fact that

$$F_i(u^\rho) - F_i(v^\rho) - \rho \left( \sum_{j \neq i} \pi(u_i^{\rho,j} - c - u_i^{\rho,i}) - \sum_{j \neq i} \pi(v_i^{\rho,j} - c - v_i^{\rho,i}) \right) \leq 0,$$

we are able to conclude that $M \leq 0$, which contradicts the hypothesis that $M > 0$. \qed

The next result presents an a priori estimate of the solution to the penalized equations, independent of the penalty parameter $\rho$ and switching cost $c$.

**Lemma 2.3.** Suppose $u^\rho$ solves Problem 2.1 with given penalty parameter $\rho \geq 0$ and switching cost $c \geq 0$, then $\|u^\rho\| \leq \frac{1}{\gamma} \|F(0)\|$. 

**Proof.** Let $\|u_i^{\rho,j}\| = \|u_i^\rho\| > 0$. Suppose that $u_i^{\rho,i} > 0$, then $u_i^{\rho,j} - c - u_i^{\rho,i} \leq 0$ for all $j \neq i$, hence we deduce from (1.2) that

$$\gamma(u_i^\rho - 0) \leq F_i(u^\rho)_l - F_i(0)_l = \rho \sum_{j \neq i} \pi(u_i^{\rho,j} - c - u_i^{\rho,i}) - F_i(0)_l = -F_i(0)_l,$$

thus $\|u^\rho\| \leq \frac{1}{\gamma} \|(F(0))^-\|$. On the other hand, suppose that $u_i^{\rho,i} < 0$, we can obtain directly from (1.2) and the non-negativity of $\pi$ that

$$\gamma(0 - u_i^\rho) \leq F_i(0)_l - F_i(u^\rho)_l \leq (F_i(0)_l)^+,\text{ hence }\|u^\rho\| \leq \frac{1}{\gamma} \|(F(0))^+\|,$$

which leads us to the desired estimate. \qed

Now we are ready to conclude the well-posedness of the penalized problem (2.1).

**Theorem 2.4.** For any given penalty parameter $\rho \geq 0$ and switching cost $c \geq 0$, Problem 2.1 admits a unique solution $u^\rho$ satisfying $\|u^\rho\| \leq \|F(0)\|/\gamma$.

**Proof.** The uniqueness and the a priori bound have been established in Proposition 2.2 and Lemma 2.3 respectively. Now we shall prove the existence of solutions by approximating the penalized equation $G^\rho$ with a sequence of smooth equations.

Consider a family of smooth functions $\delta_m : \mathbb{R}^{N \times d} \to (0, \infty)$ supported in $B(0, 1/m)$ with unit mass, we define the smooth functions $G^{\rho,m} := G^\rho * \delta_m$, where the convolution is applied elementwise. The continuity of $G^\rho$ implies that $G^{\rho,m}$ converges to $G^\rho$ uniformly on compact sets as $m \to \infty$. Moreover, one can easily deduce from (2.2) and the properties of mollifiers $(\delta_m)_{m \in \mathbb{N}}$ that both $G^\rho$ and $(G^{\rho,m})_{m \in \mathbb{N}}$ satisfy the monotonicity condition (1.2) with the same constant $\gamma$. 

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For any given \( m \in \mathbb{N} \), we shall now apply [18, Theorem 5.3.9] to establish that \( G^{\rho,m} \) is a homeomorphism from \( \mathbb{R}^{Nd} \) onto \( \mathbb{R}^{Nd} \), which implies the equation \( G^{\rho,m} = 0 \) has a solution. More precisely, we shall show (1) the Jacobian matrix \( \nabla G^{\rho,m}(u) \) is nonsingular for any given \( u \in \mathbb{R}^{Nd} \) and (2) \( \lim_{|x| \to \infty} \|G^{\rho,m}(u)\| = \infty \). To prove (1), suppose \( \nabla G^{\rho,m}(u)x = 0 \) for some \( u, x \in \mathbb{R}^{Nd} \) and let \( |x| = \|x\| \geq 0 \) for some \( i \in I \) and \( l \in \mathcal{N} \). If \( x_i^l \geq 0 \), we can deduce from the differentiability and monotonicity of \( G^{\rho,m} \)

\[
\gamma(hx_i^l - 0) \leq G_i^{\rho,m}(u + hx) - G_i^{\rho,m}(u) - h(\nabla G_i^{\rho,m}(u)x)_l = O(h^2), \quad \text{as } h \to 0,
\]

which implies \( \|x\| = 0 \). The same conclusion can be drawn for the case with \( x_i^l \leq 0 \), which implies \( x = 0 \) and consequently the non-singularity of \( \nabla G^{\rho,m}(u) \). To prove the coerciveness of \( G^{\rho,m} \), let \( u \in \mathbb{R}^{Nd} \) and \( \|u_i^l\| = \|u\| \geq 0 \) for some \( i \in I \) and \( l \in \mathcal{N} \). If \( u_i^l \geq 0 \), we can obtain from the monotonicity of \( G^{\rho,m} \)

\[
\|G^{\rho,m}(u)\| \geq G_i^{\rho,m}(u) \geq \gamma(u_i^l) + G_i^{\rho,m}(0)_l \geq \gamma\|u\| - \|G^{\rho,m}(0)\|,
\]

where the same estimate can be derived similarly for the case \( u_i^l \leq 0 \). Therefore, we can conclude the existence of a solution \( u^{\rho,m} \) to \( G^{\rho,m} = 0 \). Since \( G^{\rho,m} \) satisfies (1.2) with the same constant \( \gamma \), one can deduce from Lemma 2.3 and the continuity of \( G^{\rho,m} \) that its solution is uniformly bounded, i.e., \( \|u^{\rho,m}\| \leq \frac{1}{\gamma}\|G^{\rho,m}(0)\| \leq L \) independent of \( m \in \mathbb{N} \).

This uniform boundedness of \( (u^{\rho,m})_{m \in \mathbb{N}} \) enables us to extract a convergent subsequence, which along with the uniform convergence of \( G^{\rho,m} \) to \( G^\rho \) (on compact sets) implies the existence of a solution to the penalized equation (2.1).

We end this section with the following monotone convergence result of the penalty approximations.

**Theorem 2.5.** For any fixed switching cost \( c \geq 0 \), the solution to Problem 2.1 converges monotonically from below to a function \( u \in \mathbb{R}^{N \times d} \) as the penalty parameter \( \rho \to \infty \). Moreover, \( u \) solves Problem 1.1 if the switching cost \( c \) is positive.

**Proof.** It is straightforward to verify that if \( u^{\rho_1} \) satisfies (2.1) with the parameter \( \rho_1 \) and \( \rho_2 \geq 0 \), then \( G^{\rho_2}(u^{\rho_1}) \geq 0 \). Hence one can deduce from Proposition 2.2 that \( u^{\rho_1} \geq u^{\rho_2} \), which together with Lemma 2.3 implies \( u^\rho \) converges to some function \( u \in \mathbb{R}^{N \times d} \) as \( \rho \to \infty \). Owing to the fact that the solution of (1.1) is unique for positive switching cost, it suffices to show \( u \) solves Problem 1.1.

Let \( i \in I \) be fixed. Since \( \|u^\rho\| \leq \frac{1}{\gamma}\|F(0)\| \), we see from the continuity of \( F \) that \( \sum_{j \neq i} \pi(u_t^{i,j} - c - u_t^{i,i}) \leq C/\rho \) for all \( l \in \mathcal{N} \), where the constant \( C \) is defined as:

\[
C = \sup_{\|u\| \leq \|F(0)\|/\gamma} \|F(u)\|.
\]

Hence the limiting function \( u \) satisfies \( u^l \geq \max_{j \neq i}(u^i - c) \) and \( F_i(u) = \lim_{\rho \to \infty} F_i(u^\rho) \geq 0 \). Moreover, suppose \( u_t^{l} - (M_t u)_l > 0 \) at the index \( l \in \mathcal{N} \), we can deduce that \( F_l(u^\rho) = 0 \) for all large enough \( \rho \), which further implies \( F_i(u_l) = 0 \) and completes our proof.

**Remark 2.2.** We point out that, unlike Problem 1.1, the well-posedness of Problem 2.1, and the monotone convergence of their solutions hold for any non-negative switching cost \( c \), which enables us to study penalty schemes with zero switching cost (see Section 4).

Moreover, based on the penalized equations, Theorems 2.4 and 2.5 explicitly construct the solution to Problem 1.1, which is uniformly bounded by \( \|F(0)\|/\gamma \) for all positive switching costs.
3 Penalization errors for positive switching cost

In this section, we shall proceed to analyze the convergence rate of the penalty approximation for Problem 1.1 with a fixed positive switching cost. As discussed in Section 1, it is not easy to construct a supersolution of Problem 1.1 from the solution of Problem 2.1 due to the non-diagonal dominance of the interconnected obstacles. We shall overcome this difficulty by regularizing the obstacles and establish the convergence rates of the penalty approximations with respect to the penalty parameter.

In order to obtain error estimates of the regularization procedures, we impose the following concavity condition on the monotone system:

**Assumption 1.** The function $F$ in Problem 1.1 is concave in the sense that: for any given $i \in I$, $u, v \in \mathbb{R}^{N \times d}$, $\theta \in [0, 1]$, we have $F_i(\theta u + (1 - \theta)v) \geq \theta F_i(u) + (1 - \theta)F_i(v)$.

Assumption 1 will only be used in Section 3 to quantify the regularization errors (not for the well-posedness or the monotone convergence of the regularization procedures). It is well-known that a concave function can be equivalently represented as the infimum of a family of affine functions, i.e., $F_i(u) = \inf_{\alpha \in A_i} B_i(\alpha)u - b_i(\alpha)$ for some set $A_i$ and coefficients $B : A_i \to \mathbb{R}^{N \times N d}$ and $b : A_i \to \mathbb{R}^{N d}$, hence our error estimates apply to the HJBQVIs studied in [7, 21, 2, 12, 1]. However, our setting significantly extends the classical HJBQVIs in the following important aspects: (1) $F_i$ can depend on all components of the solutions to the switching systems, (2) the control set $\mathcal{A}_i$ can be non-compact and coefficients $B, b$ can be discontinuous, (3) $b$ does not necessarily have a unique sign.

3.1 Regularizations of the QVIs

In this section we discuss how to approximate Problem 1.1 by variational inequalities with diagonally dominant obstacle terms. We shall propose two regularization procedures, namely the iterated optimal stopping approximation and the time-marching iteration, and estimate the regularization errors, which are essential for analyzing the penalization error of Problem 2.1.

We start with the iterated optimal stopping approximation (see [21, 12] for its applications to the classical HJBQVIs), which approximates Problem 1.1 as follows: find $u^0 \in \mathbb{R}^{N \times d}$ satisfying $F_i(u^0) = 0$, $i \in I$, and for each $n \geq 1$, given $u^{n-1} \in \mathbb{R}^{N \times d}$, find $u^n \in \mathbb{R}^{N \times d}$ such that $u^n = Q u^{n-1}$, where $Q$ is the solution operator to the following obstacle problem:

$$\min(F_i(u^n), u^{n,i} - M_i u^{n-1}) = 0, \quad i \in I. \quad (3.1)$$

It is not difficult to extend the arguments in Section 2 to show the above procedure is well-defined and also the comparison principle of (3.1). The next result presents some important properties of the solution operator $Q$.

**Lemma 3.1.** The operator $Q$ is monotone, i.e., $Qu \geq Qv$ provided that $u \geq v$, and satisfies the a priori estimate:

$$\|Qu\| \leq \max(\|F(0)\|/\gamma, \|u\|) \quad \forall u \in \mathbb{R}^{N \times d}.$$ 

If we further suppose Assumption 1 holds, then the operator $Q$ is convex.

**Proof.** The monotonicity of $Q$ is implied by the comparison principle of (3.1). For the a priori estimate, suppose $\|Q u\| = \|Qu\|$, if $\|Qu\| \geq 0$ and $\|Q u\| \leq \|M_i u\|$, then we have $\|Q u\| = (Q u)^T \leq \|u\|$. Otherwise, we can adapt the arguments of Lemma 2.3 to show $\|Q u\| \leq \|F(0)\|/\gamma$. Finally, for any given $u, v \in \mathbb{R}^{N \times d}$ and $\theta \in [0, 1]$, one can deduce from the concavity of $F$ and $-M$ that $\theta Q u + (1 - \theta)Q v$ is a supersolution to (3.1) with the obstacle $M(\theta u + (1 - \theta)v)$, hence the comparison principle leads us to the desired result. \[\square\]
The above lemma directly implies the monotone convergence of the iterates \((u^n)_{n \in \mathbb{N}}\).

**Proposition 3.2.** For any given positive switching cost \(c > 0\), the iterates \((u^n)_{n \in \mathbb{N}}\) satisfy \(\|u^n\| \leq \|F(0)\|/\gamma\) for all \(n \in \mathbb{N}\), and converge monotonically from below to the solution \(u\) of Problem 1.1 as \(n \to \infty\).

**Proof.** The uniform bound of \((u^n)_{n \in \mathbb{N}}\) follows from Lemmas 2.3 (with \(\rho = 0\)) and 3.1. Moreover, since the comparison principle of the equation \(F = (F_i)_{i \in \mathcal{I}} = 0\) implies \(u^1 \geq u^0\), by using an inductive argument and the monotonicity of the operator \(Q\), we can show \((u^n)_{n \in \mathbb{N}}\) monotonically increases to some vector \(u\), which solves Problem 1.1 due to the continuity of (3.1).

Now we proceed to estimate the difference \(u - u^n\), where \(u\) and \(u^n\) solve Problem 1.1 and the equation (3.1), respectively. We shall first introduce the concept of strict supersolution, which was used in [21] to study impulse control problems.

**Definition 3.1.** A vector \(w \in \mathbb{R}^{N \times d}\) is said to be a strict supersolution of Problem 1.1 if there exists a constant \(\kappa > 0\), such that \(G_i(w) = \kappa\) for all \(i \in \mathcal{I}\).

We obtain from Theorem 2.4 that for any given \(0 < \kappa < c\), Problem 1.1 admits a unique strict supersolution satisfying the bound \(\|w\| \leq (\|F(0)\| + \kappa)/\gamma\). For convenience, we shall assume without loss of generality that \(\|F(0)\| > 0\) in the remaining part of this paper, which excludes the trivial case where 0 is the unique solution to Problems 1.1 and 2.1.

The next lemma shows a contractive property of the solution operator \(Q\). A similar result has been shown in [21] for a classical (continuous in time and space) HJBQVI via a control-theoretic approach. Here we shall present a simpler proof for our discrete setting based on the comparison principle, which can be easily extended to other regularization methods.

**Lemma 3.3.** Suppose Assumption 1 holds. Let \(w\) be the strict supersolution to Problem 1.1 with \(\kappa \in (0,c)\). If \(u^n - u^{n-1} \leq \lambda(w - u^{n-1})\) for some \(\lambda \in [0,1]\) and \(n \in \mathbb{N}\), then we have \(u^{n+1} - u^n \leq \lambda(1 - \mu)(w - u^n)\) with

\[
\mu = \min\left(1, \frac{\gamma \kappa}{2\|F(0)\| + \kappa}\right) \in (0,1].
\]

**Proof.** One can deduce from the convexity and monotonicity of the solution operator \(Q\) that

\[
Qw^n \leq Q(\lambda w + (1 - \lambda)u^{n-1}) \leq \lambda Qw + (1 - \lambda)Qu^{n-1} = \lambda Qw + (1 - \lambda)u^n,
\]

hence it suffices to show \(Qw \leq (1 - \mu)w + \mu u^n\) for some \(\mu \in (0,1]\). Note that for any given \(i \in \mathcal{I}\), we can obtain from the concavity of \(F_i\) that \(F_i(1 - \mu)w + \mu u^n) \geq 0\), and also for \(\mu \in [0,1]\),

\[
(1 - \mu)w^i + \mu u_i^n - M_i w \geq \kappa - \mu(w^i + (u_n^i)^-) \geq \kappa - \mu(\|w\| + \|(u_n^-)\|) \geq 0
\]

provided that \(\mu \leq \frac{\kappa}{\|w\| + \|(u_n^-)\|}\). Since we have

\[
\|w\| + \|(u_n^-)\| \leq \frac{1}{\gamma}(2\|F(0)\| + \kappa) := L \quad \forall n \in \mathbb{N},
\]

setting \(\mu = \min(1, \frac{\kappa}{L})\) enables us to conclude the desired result from the comparison principle.

Now we are ready to present the error estimate of the iterated optimal stopping approximation.
Theorem 3.4. Suppose Assumption 1 holds, and \( u \) and \( u^n \) solve Problem 1.1 and the equation (3.1), respectively. Then we have for any given \( \kappa \in (0, c) \) that

\[
0 \leq u - u^n \leq L(1 - \mu)^n/\mu, \quad \forall n \geq 0,
\]

where \( \mu \) and \( L \) are defined as in (3.2) and (3.3) respectively.

Proof. The monotone convergence of the iterates \((u^n)_{n \in \mathbb{N}}\) (Proposition 3.2) implies that \( u^n \leq w \) for all \( n \in \mathbb{N} \). Hence \( u^1 - u^0 \leq \lambda(w - u^0) \) with \( \lambda = 1 \), which along with Lemma 3.3 gives that \( u^2 - u^1 \leq \lambda(1 - \mu)(w - u^1) \). Inductively, we have

\[
0 \leq u^{n+1} - u^n \leq (1 - \mu)^n(w - u^n) \leq (1 - \mu)^n(w - u^0), \quad n \geq 0.
\]

Then by summing over \( n \) and using the a priori bounds, we arrive at the desired conclusion. \( \Box \)

Remark 3.1. Similar geometric convergence rates have been establish in [7, 12] for classical HJBQVIs, i.e., \( F_i(u) = \inf_{\alpha \in \mathcal{A}_i} B(\alpha)u^i - b(\alpha) \) for all \( i \in \mathcal{I} \), under the assumptions that \( \mathcal{A}_i \) is a compact (or finite) set and \( b(\alpha) \geq 0 \) for all \( \alpha \in \mathcal{A}_i \). Here we remove these restrictions.

We point out that Theorem 3.4 implies the iterated optimal stopping only gives a good approximation to Problem 1.1 for sufficiently large switching cost \( c \). For small enough switching cost \( c \), we have \( 1 - \mu \approx 1 - \gamma c/(2\|F(0)\|) \), which converges to 1 as \( c \to 0 \).

Due to the slow convergence rate of the iterated optimal stopping approximation for small switching cost, let us now discuss another regularization method, called the time-marching iteration (see [16]), which introduces an additional pseudo-time parameter \( \varepsilon \) to the interconnected obstacle, and gives an accurate approximation to Problem 1.1 even for small switching cost.

For any given parameter \( \varepsilon > 0 \), the time-marching iteration is given as follows: find \( u^0 \in \mathbb{R}^{N \times d} \) satisfying \( F_i(u^0) = 0 \), \( i \in \mathcal{I} \), and for each \( n \geq 1 \), given \( u^{n-1} \in \mathbb{R}^{N \times d} \), find \( u^n \in \mathbb{R}^{N \times d} \) such that \( u^{n+1} = Tu^n \), where \( T \) is the solution operator to the following obstacle problem:

\[
\min(F_i(u^n), \quad u^{n,i} - \mathcal{M}_iw^n + \varepsilon(u^{n,i} - u^{n-1,i})) = 0, \quad i \in \mathcal{I}.
\]

The solution operator \( T \) enjoys analogue properties as the operator \( Q \), i.e., Lemma 3.1 and Proposition 3.2, whose proofs are similar and details are omitted. The next theorem presents the convergence rate of the time-marching iteration.

Theorem 3.5. Suppose Assumption 1 holds, and \( u \) and \( u^n \) solve Problem 1.1 and the equation (3.4) with the pseudo-time parameter \( \varepsilon > 0 \), respectively. Then we have for any \( \kappa \in (0, c) \) that

\[
0 \leq u - u^n \leq L(1 - \mu)^n/\mu, \quad n \geq 0,
\]

where \( L \) is defined as in (3.3) and \( \mu = \frac{\kappa}{\kappa + \varepsilon L} \).

Proof. Following the proofs of Lemma 3.3 and Theorem 3.4, we see it is essential to obtain \( \mu \in (0, 1] \) such that \( Tw \leq (1 - \mu)w + \mu u^n \) for all \( n \in \mathbb{N} \), which leaves us to show that for suitable \( u \) we have

\[
(1 - \mu)w^i + \mu u^{n,i} - \mathcal{M}_i[(1 - \mu)w + \mu u^n] + \varepsilon((1 - \mu)w^i + \mu u^{n,i} - w) \\
\geq (1 - \mu)(w^i - \mathcal{M}_iw^n) + \mu u^{n,i} - \mathcal{M}_i w^n + \varepsilon(u^{n,i} - u^{n-1,i}) - \varepsilon u^i - u^{n-1,i} \\
\geq (1 - \mu)\kappa - \varepsilon \mu(\|w\| + \|(u^n - u^{n-1})\|) \geq (1 - \mu)\kappa - \varepsilon \mu L \geq 0,
\]

for \( L \) defined as in (3.3). Thus by setting \( \mu = \frac{\kappa}{\kappa + \varepsilon L} \), we see the above inequality holds, and one can deduce the desired result following similar arguments as the proof of Theorem 3.4. \( \Box \)
Remark 3.2. Through the choice of the pseudo-time parameter $\varepsilon$, the time-marching iteration gives a more accurate approximation to Problem 1.1 than the iterated optimal stopping approximation for small switching cost $c$. In fact, by choosing $c/\varepsilon \to \infty$, one has $1 - \mu \to 0$ as $c \to 0$. However, as we shall see in Section 3.2, the error of the penalty approximation to (3.4) grows proportionally to $1/\varepsilon$, hence after minimizing over $\varepsilon$, both the iterated optimal stopping approximation and the time-marching iteration lead to the same error estimate for Problem 2.1.

3.2 Convergence order of penalty methods

In this section, we shall use the regularization procedures proposed in Section 3.1 to demonstrate that for fixed positive switching cost and a penalty function with degree $\sigma > 0$, the approximation error of Problem 2.1 is bounded above by the quantity $C_0\rho^{-\sigma} \ln \rho$ for some constant $C_0$, which depends only on the function $F$ and is independent of the number of switching regimes $d$. Since both regularization procedures lead to the same error estimate (see Remark 3.2), we shall focus on the regularization by the iterated optimal stopping, and only outline the essential results for the time-marching iteration.

To quantify the penalty error of Problem 2.1 with a fixed parameter $\rho \geq 0$, we introduce the following sequence of auxiliary problems: find $u^{\rho,0} \in \mathbb{R}^{N \times d}$ satisfying $F_i(u^0) = 0$, $i \in I$, and for each $n \geq 1$, given $u^{\rho,n-1} \in \mathbb{R}^{N \times d}$, find $u^{\rho,n} \in \mathbb{R}^{N \times d}$ such that $u^{\rho,n} = Q\rho u^{\rho,n-1}$, where $Q\rho$ is the solution operator to the following penalized equation:

$$F_i(u^{\rho,n}) - \rho \sum_{j \neq i} \pi(u^{\rho,n-1,j} - c - u^{\rho,n,j}) = 0, \quad i \in I.$$  \hspace{1cm} (3.5)

Note that the above auxiliary problem has the same initialization as the iterated optimal stopping approximation to Problem 1.1, i.e., $u^{\rho,0} = u^0$. Moreover, one can easily establish the comparison principle and well-posedness of (3.5) by adapting the proofs of Proposition 2.2 and Theorem 2.4. The following result summarizes the essential properties of the operator $Q\rho$ and the iterates $(u^{\rho,n})_{n \in \mathbb{N}}$.

Proposition 3.6. The operator $Q\rho$ is monotone, non-expansive and satisfies the a priori estimate:

$$\|Q\rho u\| \leq \max(\|F(0)/\gamma\|, \|u\|) \quad \forall u \in \mathbb{R}^{N \times d}.\]$$

Consequently, for any given penalty parameter $\rho \geq 0$ and switching cost $c \geq 0$, the iterates $(u^{\rho,n})_{n \in \mathbb{N}}$ converge monotonically from below to the solution $u^\rho$ of Problem 2.1 as $n \to \infty$.

Proof. The a priori bound can be obtain exactly as Lemma 3.1. For the monotonicity and non-expansiveness of $Q\rho$, it suffices to show for any given $u, v \in \mathbb{R}^{N \times d}$, we have $Q\rho u - Q\rho v \leq \|(u - v)^+\|$.

We shall show $\hat{u} := Q\rho v + \|(u - v)^+\|$ is a supersolution to (3.5) with the input $u$. It is important to observe that for any given $L \geq 0$ and $u \in \mathbb{R}^{N \times d}$, the monotonicity (1.2) of $F$ implies

$$F_i(u + L) - F_i(u) \geq \gamma L \quad \forall i \in I,$$  \hspace{1cm} (3.6)

which along with the fact that

$$\pi(u^j - c - \hat{u}^i) = \pi(u^j - \|(u - v)^+\| - c - (Q\rho v)^i) \leq \pi(u^j - c - (Q\rho v)^i), \quad j \neq i,$$

enables us to conclude the desired result through the following estimate: for any $i \in I$,

$$F_i(\hat{u}) - \rho \sum_{j \neq i} \pi(u^j - c - \hat{u}^i) \geq F_i(Qv) + \gamma \|(u - v)^+\| - \rho \sum_{j \neq i} \pi(u^j - c - (Q\rho v)^i) \geq 0.$$

Then the monotone convergence of $(u^{\rho,n})_{n \in \mathbb{N}}$ follows from similar arguments as those in Proposition 3.2. \hfill $\Box$
The next result provides an upper bound of the term \( u^n - u^{\rho,n} \), where \( u^n \) and \( u^{\rho,n} \) solve the equations (3.1) and (3.5), respectively.

**Proposition 3.7.** For any given penalty parameter \( \rho \geq 0 \), switching cost \( c > 0 \) and \( n \in \mathbb{N} \), let \( u^n \) and \( u^{\rho,n} \) solve the equations (3.1) and (3.5), respectively. Suppose the penalty function \( \pi(y) \geq \tau y^{1/\sigma} \) for all \( y \in [0,2\|F(0)\|/\gamma] \) with some constants \( \tau, \sigma > 0 \). Then we have

\[
\|u^n - u^{\rho,n}\| \leq \left(\frac{C}{\tau \rho}\right)^\sigma n, \quad n \geq 0,
\]

where \( C = \sup\|u\|\leq\|F(0)\|/\gamma \|F(u)\| \).

*Proof.* The non-expansiveness of \( Q^\rho \) implies that for any \( n \in \mathbb{N} \),

\[
\|u^n - u^{\rho,n}\| = \|u^n - Q^\rho u^{n-1}\| + \|Q^\rho u^{n-1} - Q^\rho u^{\rho,n-1}\| \leq \|u^n - Q^\rho u^{n-1}\| + \|u^{n-1} - u^{\rho,n-1}\|,
\]

where \( Q^\rho u^{n-1} \) satisfies the following equation

\[
F_i(Q^\rho u^{n-1}) - \rho \sum_{j \neq i} \pi(u^{n-1,j} - c - (Q^\rho u^{n-1})^i) = 0, \quad i \in I.
\]

(3.8)

Now we bound \( u^n - Q^\rho u^{n-1} \) for any given \( n \in \mathbb{N} \). From the comparison principle of (3.8), the a priori bounds of \( u^n \) (Proposition 3.2) and \( Q^\rho \) (Proposition 3.6), we know for any given \( n \in \mathbb{N} \), \( \|Q^\rho u^{n-1}\| \leq \|F(0)\|/\gamma \) for all \( \rho \geq 0 \), and \( (Q^\rho u^{n-1})_{\rho>0} \) converges monotonically from below to \( u^n \) as \( \rho \to \infty \). This implies that \( \|\rho \pi(u^{n-1,j} - c - (Q^\rho u^{n-1})^i)\| \leq C \) for all \( j \neq i, i \in I \), where \( C \) is defined as in (2.3). Therefore, we have for \( C_1 = C/\tau \) that

\[
(Q^\rho u^{n-1})^i + (C_1/\rho)^\sigma - M_i u^{n-1} = \min_{j \neq i} \left(C_1^\sigma - \rho^\sigma(u^{n-1,j} - c - (Q^\rho u^{n-1})^i)/\rho^\sigma \right) \geq \frac{1}{(\rho \tau)^\sigma} \min_{j \neq i} \left((\tau C_1)^\sigma - \|\rho \pi(u^{n-1,j} - c - (Q^\rho u^{n-1})^i)\|^\sigma \right) \geq 0,
\]

which implies \( Q^\rho u^{n-1} + (C_1/\rho)^\sigma \) is a supersolution to (3.1). Consequently, we obtain \( 0 \leq u^n - Q^\rho u^{n-1} \leq (C_1/\rho)^\sigma \) for all \( n \in \mathbb{N} \), and conclude the desired result from \( w^{\rho,0} = w^0 \) and (3.7).

**Remark 3.3.** This proposition greatly extends the results in [23] (even for the case with \( \sigma = 1 \)) by removing the continuous differentiability assumption of the penalty function \( \pi \). In practice, one can choose \( \pi(y) = (y^+)^{1/\sigma} \) as the penalty function. Since \( \pi \) is semismooth if \( \sigma = 1 \), a direct application of semismooth Newton methods allows us to solve Problem 2.1 efficiently (see [23, 20]). A penalty term with \( \sigma > 1 \) needs an additional smoothing for the application of Newton methods and usually requires a larger number of Newton iterations to solve the penalized equation [14]. Though the higher convergence rate allows us to use a relatively small value of \( \rho \) to achieve the desired accuracy, which could avoid the numerical instability caused by the usage of a large penalty parameter according to [14], we do not discover any problem using the penalty function \( \pi(y) = y^+ \) in our numerical experiments.

Now we are ready to conclude the penalty error of Problem 2.1 to Problem 1.1.

**Theorem 3.8.** For any given switching cost \( c > 0 \), let \( u \) and \( u^{\rho} \) solve Problem 1.1 and 2.1, respectively. If in addition to the possibilities in Proposition 3.7, also Assumption 1 holds, then there exists a constant \( C > 0 \) (possibly depending on the switching cost \( c \)), such that for any large enough penalty parameter \( \rho > 0 \), we have \( \|u - u^{\rho}\| \leq C \rho^{-\sigma} \ln \rho \).
Proof. Since \( u^0 \leq u \) for all \( \rho \geq 0 \), it remains to derive an upper bound of \( u - u^0 \). Note that

\[
u - u^0 \leq u - u^n + u^n - u^\rho_n + u^\rho_n - u^0,
\]

where \( u^n \) and \( u^\rho_n \) solve the equations (3.1) and (3.5), respectively. Hence we deduce from Theorem 3.4, Propositions 3.6 and 3.7 that for any \( \kappa \in (0, c) \),

\[
u - u^0 \leq L(1 - \mu)^n + \left( \frac{C}{\tau \rho} \right)^\sigma \quad \forall n \in \mathbb{N} \cup \{0\},
\]

where \( C \) and \( L \) are defined as in (2.3) and (3.3) respectively, and \( \mu = \min(1, \kappa/L) \). We can assume without loss of generality that \( c \leq L \), which implies \( \mu < 1 \), since otherwise by taking \( n = 0 \), we get \( u - u^0 = 0 \) for all \( \rho \geq 0 \). Now we minimize the right-hand side of the above inequality over \( n \) by applying [6, Lemma 6.1] with \( \nu = L/\mu \), \( a = 1 - \mu \) and \( b = [C/(\tau \rho)]^\sigma \), and obtain that

\[
u - u^0 \leq \begin{cases} \frac{-[C/(\tau \rho)]^\sigma}{\ln(1-\mu)}, & \frac{-[C/(\tau \rho)]^\sigma}{\ln(1-\mu)} \geq 1, \\ \frac{-[C/(\tau \rho)]^\sigma}{\ln(1-\mu)} + [C/(\tau \rho)]^\sigma \left( \log_{1-\mu} \left( 1 - \frac{[C/(\tau \rho)]^\sigma}{\ln(1-\mu)} \right) + 1 \right), & \text{otherwise.} \end{cases}
\]

Then for large enough \( \rho \), we obtain an upper bound from the second case, i.e.,

\[
u - u^0 \leq O \left( \rho^{-\sigma} \frac{\ln \rho + \ln \left( \frac{1}{\ln(1-\mu)} \right)}{-\ln(1-\mu)} \right) = O \left( \frac{\rho^{-\sigma} \ln \rho}{-\ln(1-\mu)} \right), \quad \text{as } \rho \to \infty.
\]

Remark 3.4. Recall that \( \mu = O(c) \) as \( c \to 0 \), hence the upper bound behaves as \( \rho^{-\sigma} \ln \rho/c \) for small switching cost \( c \). Unfortunately, we are not sure whether this dependence on \( c \) is optimal since the possible blow-up of the penalization error for small enough \( c \) could be due to the fact that the iterated optimal stopping approximation does not provide an accurate approximation to Problem 1.1 with small switching cost. Our numerical experiments show that as the switching cost tends to zero, the penalization error with a fixed penalty parameter indeed grows at a rate \( O(c^{-1/2}) \) for certain ranges of switching costs, but then stabilizes to a limiting value (see Section 5). As we shall see in Section 4, Problem 1.1 degenerates into an HJB equation as the switching cost \( c \to 0 \), and Problem 2.1 with \( c = 0 \) provides a penalty approximation to such equation, with an asymptotical error \( O(1/\rho^\sigma) \) as the penalty parameter \( \rho \to \infty \). Thus the penalization error with sufficiently small positive switching cost is dominated by this limiting error (see (4.2)).

We proceed to outline the key results of the convergence analysis by using the time-marching iteration, and demonstrate that even if the time-marching iteration could improve the regularization error for small switching cost by adjusting the pseudo-time parameter \( \varepsilon \), it reduces the accuracy of penalty approximations at each iterate. Hence it leads to the same error estimate of Problem 2.1 as the iterated optimal stopping approximation for small switching cost.

For any given parameters \( \rho \geq 0 \) and \( \varepsilon \geq 0 \), we introduce the following sequence of auxiliary problems: find \( u^{0,0} \in \mathbb{R}^{N \times d} \) satisfying \( F_i(u^0) = 0 \), \( i \in \mathcal{I} \), and for each \( n \geq 1 \), given \( u^{\rho, n-1} \in \mathbb{R}^{N \times d} \), find \( u^{\rho,n} \in \mathbb{R}^{N \times d} \) such that \( u^{\rho,n} = T^\rho u^{\rho,n-1} \), where \( T^\rho \) is the solution operator to the following penalized equation:

\[
F_i(u^{\rho,n}) - \rho \sum_{j \neq i} \pi(u^{\rho,n,j} - c - u^{\rho,n,i} - \varepsilon(u^{\rho,n,i} - u^{\rho,n-1,i})) = 0, \quad i \in \mathcal{I}.
\]
One can establish analogue results of Proposition 3.6 for the operator $T^\rho$, and demonstrate that $T^\rho u^{n-1} + (C_1/\rho)^\sigma/\varepsilon$ is a supersolution to (3.4) under the same assumptions of Proposition 3.7. Therefore, following the proof of Theorem 3.8, we deduce from Theorem 3.5 that for any $\kappa \in (0, c)$,\\
$$
\|u - u^\rho\| \leq L \left( \frac{1 - \mu}{\mu} + \frac{1}{\varepsilon} \left( \frac{C}{\tau \rho} \right)^\sigma \right) n \quad \forall n \in \mathbb{N}, \varepsilon > 0,
$$
where $C$ and $L$ are defined as in (2.3) and (3.3) respectively, and $\mu = \kappa/(\kappa + \varepsilon L)$. Minimizing over $n$, we obtain $\|u - u^\rho\| = \mathcal{O}(\frac{-\rho^{-\sigma} \ln \rho}{\varepsilon \ln(1 + \frac{\rho}{\varepsilon})})$ for all $\varepsilon > 0$ and large enough $\rho$. Therefore, by discussing the cases $\varepsilon = \mathcal{O}(c)$ and $\varepsilon/c \to \infty$ separately, we arrive at the same error estimate as that in Theorem 3.8.

Finally we end this section with an exact construction of the optimal switching regions

$$
\Gamma_i := \{ l \in \mathcal{N} \mid u^i_l = (M_i u)_l \}, \quad i \in \mathcal{I},
$$
of Problem 1.1 with a given switching cost $c > 0$ using the solution of Problem 2.1. Suppose the estimate $0 \leq u - u^\rho \leq C_0 \rho^{-\sigma} \ln \rho$ holds for some constants $C_0, \sigma > 0$, where $\sigma$ is the degree of the penalty function $\pi$ and $C_0$ in practice can be estimated using numerical results. Then we shall define the sets

$$
\Gamma_{\rho,i} := \{ l \in \mathcal{N} \mid |u^i_l - (M_i u^\rho)_l| \leq C_0 \rho^{-\sigma} \ln \rho \}, \quad i \in \mathcal{I}.
$$

One can easily show $\Gamma_i \subset \Gamma_{\rho,i}$ for all $\rho > 0$ and $i \in \mathcal{I}$. The next result demonstrates that $\Gamma_{\rho,i}$ in fact coincides with $\Gamma_i$ for large enough $\rho$.

**Theorem 3.9.** For any given switching cost $c > 0$, there exists $\rho_0 > 0$ such that we have for all $\rho \geq \rho_0$ that $\Gamma_i = \Gamma_{\rho,i}$ for all $i \in \mathcal{I}$.

**Proof.** Suppose the statement does not hold, then there exists a sequence $\{\rho_n\}$ such that $\rho_n \to \infty$ as $n \to \infty$, and an index $l \in \Gamma_{\rho_n,i} \setminus \Gamma_i$ for all $n$. However, the definition of $\Gamma_{\rho_n,i}$ implies that

$$
\begin{align*}
&u^i_l - (M_i u)_l = u^i_l - (M_i u^\rho)_l + (M_i u^\rho)_l - (M_i u^\rho_n)_l + (M_i u^\rho_n)_l - (M_i u)_l \\
&\leq C_0 \rho_n^{-\sigma} \ln \rho_n + C_0 \rho_n^{-\sigma} \ln \rho_n + 0 \to 0,
\end{align*}
$$

which together with $u^i_l \geq (M_i u)_l$ leads to $l \in \Gamma_i$, and hence a contradiction. \qed

### 4 Penalties for vanishing switching cost

In this section, we investigate the asymptotic behaviours of Problems 1.1 and 2.1 as the switching cost $c$ tends to zero. We shall show that the system in Problem 1.1 degenerates into a single equation of HJB type, and establish that the penalty error of Problem 2.1 with zero switching cost is of the same order (up to a log factor) as that in Theorem 3.8.

Throughout this section, to emphasize the dependence on $c$, we shall denote by $u^c$ and $u^{c,\rho}$ the solutions to Problems 1.1 and 2.1 with a given positive switching cost $c$, respectively, and by $u^\rho$ the solution to Problem 2.1 with $c = 0$. Moreover, to identify the limiting behaviour of Problems 1.1 and 2.1, we introduce the following regularity condition on the monotone system $F$:

**Assumption 2.** The function $F$ in Problem 1.1 is locally Lipschitz continuous.

We emphasize that even though Assumption 1 is a sufficient condition for Assumption 2, it will not be used in this section. In particular, Assumption 2 is general enough to cover non-convex/non-concave equations, such as Isaacs equations.

We first introduce the degenerate problem for zero switching cost.
**Problem 4.1.** Find \( u \in \mathbb{R}^N \) such that the vector \( u = (u, \ldots, u) \in \mathbb{R}^{N \times d} \) satisfies
\[
\min_{i \in I} F_i(u) = 0. \tag{4.1}
\]

For the classical HJBQVIs where \( F_i(u) = \inf_{\alpha \in \mathcal{A}} B(\alpha)u^i - b(\alpha) \) for all \( i \in I \), Problem 4.1 can be equivalently written as an HJB equation as studied in [22, 23]: find \( u \in \mathbb{R}^N \) satisfying \( \inf_{\alpha \in \mathcal{A}} B(\alpha)u - b(\alpha) = 0 \) with \( \mathcal{A} = \cup_{i \in I} \mathcal{A}_i \). However, we reiterate that in this work, \( F_i \) can depend on all components of the system, and is not assumed to be concave in this section. Moreover, even for concave equations, both the penalty scheme, i.e., Problem 2.1 with \( c = 0 \), and its error analysis are essentially different from those in [22, 23].

The uniqueness of solutions to Problem 4.1 is implied by the comparison principle, which can be easily established by using the monotonicity condition (1.2). We shall now demonstrate that the solution to Problem 4.1 can be identified as the limit of the solutions to Problem 1.1 with vanishing switching cost.

**Proposition 4.1.** Let \( u^c \) solve Problem 1.1 with a switching cost \( c > 0 \). Then \( (u^c)_{c>0} \) converges monotonically from below to the solution \( u \) of Problem 4.1 as \( c \to 0 \).

**Proof.** It is easy to check that if \( c_1 > c_2 > 0 \), then \( u^{c_2} \) is a supersolution to Problem 1.1 with a switching cost \( c_1 \). Hence the comparison principle and the a priori bound (see Remark 2.2) imply that for each \( i \in I \), \( (u^{c^i})_{c>0} \) converges monotonically from below to some vector \( \bar{u}^i \in \mathbb{R}^N \) as \( c \to 0 \). Moreover, since \( u^{c^i} \geq M_i u^{\bar{c}} \geq u^{c^j} - c \) for all \( j \neq i \), \( c > 0 \), we have \( \bar{u}^i \equiv \bar{u} \) for all \( i \in I \).

We now show \( u := (\bar{u}, \ldots, \bar{u}) \) solves Problem 4.1. For any given \( i \in I \), using the supersolution property of \( u^c \), we have \( F_i(u^c) \geq 0 \) for all \( c > 0 \). Hence the continuity of \( F \) implies that
\[
\min_{i \in I} F_i(u) \geq 0.
\]

On the other hand, let \( l \in \mathcal{N} \) be a fixed index. For any given \( c > 0 \), we consider the component \( i_{l,c} \) where
\[
u_{i_{l,c},c} = \max_{j \in I} u_{i_{l,c},c}^j > (M_i u^{\bar{c}})_l.
\]

As \( c \to 0 \), since \( I \) is a finite set, by passing to a subsequence, we can assume there exists \( \{c_n\} \to 0 \) as \( n \to \infty \), and a component \( i_l \in I \) such that \( F_i(u^{c_n})_l = 0 \) for all \( n \in \mathbb{N} \). Thus letting \( c_n \to 0 \), we have \( \min_{i \in I} F_i(u)_l = F_i(u)_l = 0 \). Since \( l \) is an arbitrary index, we conclude \( \min_{i \in I} F_i(u) = 0 \).

Because Problem 4.1 is the limiting equation of Problem 1.1 as \( c \to 0 \), we now analyze the approximation error of Problem 2.1 with \( c = 0 \) to Problem 4.1, which indicates the asymptotic behaviour of the penalization error of Problem 2.1 for small enough switching cost.

**Theorem 4.2.** The solution \( u^\rho \) of Problem 2.1 (with \( c = 0 \)) converges monotonically from below to the solution \( u \) of Problem 4.1 as \( \rho \to \infty \). Moreover, if we further assume the penalty function \( \pi(y) \geq \tau y^{1/\sigma} \) for all \( y \in [0, \|F(0)\|/\gamma] \) with some constants \( \tau, \sigma > 0 \) and Assumption 2 holds, then we have the following error estimate:
\[
0 \leq u - u^\rho \leq C_1/\rho^\sigma,
\]
for some constant \( C_1 > 0 \), independent of the penalty parameter \( \rho \).

**Proof.** The monotone convergence of \( (u^\rho)_{\rho \geq 0} \) to \( u \) as \( \rho \to \infty \) follows from Theorem 2.5 and similar arguments as those for Proposition 4.1. Hence it suffices to show that there exists a constant \( C_1 \), such that for each \( i \in I \) and \( \rho > 0 \), \( (u^{\rho,j}, \ldots, u^{\rho,i}) + C_1/\rho^\sigma \in \mathbb{R}^{N \times d} \) is a supersolution to (4.1).

Note that \( (u^\rho)_{\rho \geq 0} \) are bounded by \( \|F(0)\|/\gamma \) (see Lemma 2.3), hence we have
\[
\tau[(u^{\rho,j} - u^{\rho,i})^+]^{1/\sigma} \leq \sum_{j \neq i} \pi(u^{\rho,j} - u^{\rho,i}) \leq C/\rho, \quad \forall i \in I, j \neq i,
\]
where $C$ is defined as in (2.3). Then using the local Lipschitz continuity of $F$, we obtain

$$\|F_j(u^\rho) - F_j(\hat{u}^i)\| \leq L_{\text{lip}}\|u^\rho - \hat{u}^i\| \leq L_{\text{lip}} \left( \frac{C}{\gamma \rho} \right)^\sigma,$$

where $\hat{u}^i := (u^{\rho,i}, \ldots, u^{\rho,i})$. Therefore, by using the inequality (3.6) and setting $\gamma C_1 = L_{\text{lip}} (C/\tau)^\sigma$, one can conclude $\hat{u}^i + C_1/\rho^\sigma$ is a supersolution to (4.1) through the following estimate:

$$F_j(\hat{u}^i + C_1/\rho^\sigma) \geq F_j(\hat{u}^i) + \gamma C_1/\rho^\sigma \geq F_j(u^\rho) \geq 0 \quad \forall j \in I,$$

which implies that $\hat{u}^i + C_1/\rho^\sigma \geq u$ for all $\rho > 0$ and $i \in I$.

**Remark 4.1.** Under Assumption 2, one can also show the rate of convergence for Problem 1.1 to Problem 4.1 is of first order in the switching cost. In fact, let $u^c$ solve Problem 1.1 with a switching cost $c > 0$. Then we have $\|u^{c,i} - u^{c,j}\| \leq c$ for all $i, j \in I$. Then following the proof of Theorem 4.2, we see it holds for some constant $K > 0$ that $0 \leq u - u^c \leq Kc$.

Summarizing the above discussions, we can derive another upper bound of the penalization error for Problem 2.1 with the penalty function $\pi(y) = y^+$ and positive switching cost, which enables us to explain the asymptotic behaviours of the penalty errors observed in Section 5.

**Theorem 4.3.** For any given switching cost $c > 0$ and penalty parameter $\rho > 0$, let $u^c$ and $u^{c,\rho}$ be the solutions to Problems 1.1 and 2.1, respectively. Suppose Assumption 2 holds and the penalty function is given by $\pi(y) = y^+$. Then we have $0 \leq u^c - u^{c,\rho} \leq C_1(1/\rho + cp)$, for some constant $C_1 > 0$, independent of $c$ and $\rho$.

**Proof.** The Lipschitz continuity of the penalty function $\pi$ implies that $u^{c,\rho} + (d - 1)cp/\gamma$ is a supersolution to (2.1) with $c = 0$. Hence one can deduce from the comparison principle of the penalized equation (2.1) the estimate $0 \leq u^\rho - u^{c,\rho} \leq (d - 1)cp/\gamma$. Then using Proposition 4.1 and Theorem 4.2, we conclude for some constant $C_1$, independent of $\rho$ and $c$, that

$$u^c - u^{c,\rho} \leq u^c - u + u^\rho + u^\rho - u^{c,\rho} \leq C_1(1/\rho + cp).$$

A direct consequence of Theorems 3.8 and 4.3 is that for the penalty function $\pi(y) = y^+$ and a fixed large enough penalty parameter, the following estimate of the penalization error holds under Assumption 1 as $c \to 0$:

$$0 \leq u^c - u^{c,\rho} \leq C_1 \min \left( -\frac{\ln \rho}{\ln(1-c)\rho}, \frac{1}{\rho} + cp \right),$$

(4.2)

for some constant $C_1$, independent of $\rho$ and $c$. Thus for a sufficiently small switching cost, the penalty error is dominated by the term $C_1/\rho$, i.e., the penalization error with $c = 0$.

## 5 Numerical experiments

In this section, we illustrate the theoretical findings and demonstrate the effectiveness of the penalty schemes through numerical experiments. We present an infinite-horizon optimal switching problem and investigate the convergence of Problem 2.1 with respect to the penalty parameter. We shall also examine the dependence of the penalization errors on the switching cost.
Motivated by Remark 3.3, we shall focus on the penalty function with degree 1, i.e., \( \pi(y) = y^+ \). Due to the semismoothness of the chosen penalty function, one can easily construct convergent iterative methods for solving Problem 2.1 with a fixed penalty parameter (see e.g. [5, 22, 23, 20] for details). Roughly speaking, starting with an initial guess \( u^{(0)} \) of the solution to Problem 2.1, for each \( k \geq 0 \), we compute the next iterate \( u^{(k+1)} \) by solving

\[
C^\alpha[u^{(k)}] + \mathcal{L}^{(k+1)}[u^{(k)}](u^{(k+1)} - u^{(k)}) = 0,
\]

where \( \mathcal{L}^{(k+1)}[u^{(k)}] \) is a generalized derivative of (2.1) at the iterate \( u^{(k)} \). In practice, such a generalized derivative can be computed by policy iteration if \( F_i(u) = \inf_{\alpha \in \mathcal{A}_i} \sup_{\beta \in \mathcal{B}_i} B(\alpha, \beta)u - b(\alpha, \beta) \) for some sets \( \mathcal{A}_i, \mathcal{B}_i \) and some coefficients \( B \) and \( b \), or more generally by a slanting function of \( F = (F_i)_{i \in \mathcal{I}} \) if it exists. One can further show that the iterates \( (u^k)_{k \geq 0} \) are locally superlinearly convergent to the solution of Problem 2.1 or even globally convergent if \( F \) is concave.

To motivate the discrete QVIs solved in our numerical experiments, we introduce the following infinite-horizon optimal switching problem. Let \( (\Omega, \mathcal{F}_t, \mathbb{P}) \) be a filtered probability space and \( \alpha = (\alpha_t)_{t \geq 0} \) be a control process such that \( \alpha_t = \sum_{k \geq 0} i_k 1_{[\tau_k, \tau_{k+1})}(t) \), where \( (i_k)_{k \geq 0} \) is a regime process valued in the discrete space \( \mathcal{I} = \{1, \ldots, d\} \), \( d \geq 2 \), and \( (\tau_k)_{k \geq 0} \) is a non-decreasing sequence of stopping times. That is, the decision maker chooses regime \( i_k \) at the time \( \tau_k \) for all \( k \geq 0 \).

For any given control strategy \( \alpha \), we consider the following controlled state equation:

\[
dX_t^\alpha = (r + \nu(\alpha_t)(\mu - r))X_t^\alpha dt + \sigma \nu(\alpha_t)X_t^\alpha dB_t, \quad t > 0; \quad X_0^\alpha = x,
\]

where \( r, \mu, \sigma, x > 0 \) are given constants, \( (B_t)_{t \geq 0} \) is a one-dimensional Brownian motion defined on \( (\Omega, \mathcal{F}_t, \mathbb{P}) \), and \( \nu(i) = (i - 1)/(d - 1), i \in \mathcal{I} \). Then the objective function associated with the control strategy \( \alpha \) is given by:

\[
J(x, \alpha) = \mathbb{E}\left[ \int_0^\infty e^{-rt} \ell(X_t^\alpha) dt - \sum_{k \geq 0} e^{-r\tau_{k+1}} c^{i_k,i_{k+1}} \right],
\]

where \( \ell \) represents the running reward function and \( c^{i,j} \) represents the switching cost from regime \( i \) to \( j \), \( \forall i, j \in \mathcal{I} \). For each \( i \in \mathcal{I} \), let \( \mathcal{A}^i \) be all admissible control strategies starting with regime \( i \), i.e., \( i_0 = i \) and \( \tau_0 = 0 \). Then the decision maker has the following value functions:

\[
V^i(x) = \sup_{\alpha \in \mathcal{A}^i} J(x, \alpha), \quad i \in \mathcal{I}.
\]

Suppose the switching costs are positive, i.e., \( c^{i,j} > 0 \) for \( i \neq j \), then one can show by using the dynamic programming principle (see [19]) that the value function \( V = (V^i)_{i \in \mathcal{I}} \) satisfies the following system of quasi-variational inequalities: for each \( i \in \mathcal{I} \),

\[
\min \left[ -\frac{1}{2} \sigma^2 \nu(i)^2 x^2 V_{xx}^i - (r + \nu(i)(\mu - r))x V_x^i + r V^i - \ell(x), V^i - (\mathcal{M}_i V) \right] = 0, \quad x \in (0, \infty),
\]

where \( \mathcal{M}_i V = \max_{j \neq i} (V^i - c^{i,j}) \). For our numerical tests, we assume \( c^{i,j} \equiv c \) for \( i \neq j \), and set other parameters as \( \sigma = 0.2, \mu = 0.06, r = 0.02 \).

Now we derive the finite-dimensional QVIs by discretizing (5.1). Note that in this work, we focus on examining the performance of penalty methods for solving discrete QVIs resulting from discretizing (5.1) with a fixed mesh size, instead of the convergence of the discretization to (5.1) as the mesh size tends to zero. Therefore, for simplicity, we shall localize (5.1) on the computational domain \( (0, 2) \) with homogenous Dirichlet boundary condition \( u = 0 \) at \( x = 2 \), and solve the localized equation on a uniform grid \( \{x_l\} = \{lh\}_{l=0}^{N-1} \) with \( h = 2/N \). We further derive a
monotone discretization of (5.1), which uses forward differences for the first derivates and central difference for all second derivates. It is easy to verify that the resulting discrete system (1.1) satisfies the monotonicity condition (1.2) with \( \gamma = 0.02 \) and also Assumption 1.

We proceed to discuss implementation details for solving Problem 2.1 with semismooth Newton methods. The initial guess \( u^{(0)} \) shall be taken as the solution to (2.1) with \( \rho = 0 \), i.e., \( F_i(u) = 0 \) for all \( i \in I \), and the iterations will be terminated once the desired tolerance is achieved:

\[
\max_{l = 0, \ldots, N-1} \left\{ \frac{|u_i^{(k)} - u_i^{(k-1)}|}{\max(|u_i^{(k)}|, \text{scale})} \right\} < \text{tol},
\]

where the scale parameter is chosen to guarantee that no unrealistic level of accuracy will be imposed if the solution is close to zero. We take \( \text{tol} = 10^{-9} \) and \( \text{scale} = 1 \) for all experiments. Computations are performed using MATLAB R2016b on a laptop with 2.2 GHz Intel Core i7 and 16 GB memory.

We first study the performance of the penalty approximation for the discrete system corresponding to the two-regime case, i.e., \( d = 2 \), and the running reward \( \ell(x) = 2(1 - x)1_{[0,75,1]}(x) \).

Note that the discontinuity of \( \ell \) at \( x = 0.75 \) should not affect our convergence analysis since we are solving a finite-dimensional nonlinear system resulting from a fixed discretization of (5.1).

Table 1 contains, for different switching costs and penalty parameters, the numerical solutions of Problem 2.1 with a fixed mesh size \( h = 0.01 \) (hence the total number of unknowns is \( 2N = 400 \)). Line (a) shows that for a fixed switching cost \( c \), regardless of whether \( c \) is positive or not, the numerical solutions converge monotonically from below to the exact solution as the penalty parameter \( \rho \) can be deduced from line (b), which confirms the theoretical results (the log factor has not been observed, c.f. Theorems 3.8 and 4.2). Moreover, by fixing the penalty parameter \( \rho \) and comparing the increment \( \|u^{c,\rho} - u^{c,\rho/2}\| \) columnwise, one can observe that the penalty errors first grow at a rate 1/2 when the switching cost decreases from 1/2 to 1/128, and then stabilize to the penalty errors of the limiting case (with \( c = 0 \)) when \( c \) tends to 0, as asserted by (4.2).

The lines (c) and (d) clearly indicate the efficiency of the iterative solver. We remark that compared with parabolic QVIs, elliptic QVIs are more challenging to solve due to the fact we cannot take the solution at the previous timestep as an accurate initial guess [13, 2]. In fact, Figure 1 (left) illustrates a large disagreement in the shape and magnitude between the initial guess \( u^{(0)} \) and the final solution \( u^{c,\rho} \) of Problem 2.1 with \( \rho = 10^3 \) and \( c = 1/8 \). However, we can see that the iterative method solves Problem 2.1 at the accuracy \( 10^{-9} \) using only a small number of iterations within several milliseconds, which seems to be independent of the size of the penalty parameter \( \rho \).

We now turn to analyze the convergence of the penalty methods for the nonlinear system resulting from a three-regime problem, i.e., \( d = 3 \). The running reward function is chosen as

\[
\ell(x) = \begin{cases} 
   x - 0.5, & x \in (0.5, 1], \\
   -(x - 1.5), & x \in (1, 1.5], \\
   x - 1.5, & x \in (1.5, 1.75], \\
   -5(x - 1.8), & x \in (1.75, 1.8], \\
   0, & \text{otherwise},
\end{cases}
\]

which admits a mixed convexity. Table 2 presents the numerical solutions of the three-regime penalized equations with a fixed mesh \( h = 0.01 \) (hence the total number of unknowns is \( 3N = 600 \),
Table 1: Numerical results for the two-regime optimal switching problem with different switching costs and penalty parameters. Shown are: (a) the numerical solutions $u^{c,\rho,1}$ at $x = 0.5$; (b) the increments $\|u^{c,\rho} - u^{c,\rho/2}\|$; (c) the number of iterations; (d) the overall runtime in seconds.
Figure 1: Differences between the last components of the initial guess $u^{(0)}$ and the solution $u^{c,\rho}$ for the penalized switching system. Shown are: two-regime problem with $\rho = 10^3$ and $c = 1/8$ (left), and three-regime problem with $\rho = 4 \times 10^3$ and $c = 1/1024$ (right).

and different penalty parameter $\rho$ and switching cost $c$. Lines (a) and (b) indicate that for a fixed switching cost, the numerical solutions converge monotonically with a first-order accuracy, as the penalty parameter tends to infinity. Moreover, similar to the two-regime problem, we can observe that as the switching cost tends to zero, the penalty errors corresponding to a fixed penalty parameter $\rho$ first increase at a rate $O(c^{-1/2})$ (for $c \in [1/4, 1/1024]$) and then approach to the penalization errors with $c = 0$. Lines (c) and (d) summarize the number of required iterations and the computational time, which illustrate the efficiency of the iterative solvers for the penalized problems. Despite the relatively poor initial guess (c.f. Figure 1 (right)), the desired accuracy $10^{-9}$ is in general obtained within 0.015 seconds using a reasonable amount of iterations, which does not depend on the magnitude of the penalty parameter.

6 Conclusions

In this paper, we have shown that the penalty method is a powerful tool to solve a large class of discrete quasi-variational inequalities arising from hybrid control problems involving switching controls. We have established a monotone convergence for the solutions of the penalized equations related to a general class of penalty functions, and rigorously analyzed the penalization errors for both positive switching cost and zero switching cost. These error estimates further led to an exact construction of the optimal switching regions. Numerical examples for infinite-horizon optimal switching problems have been presented to illustrate the theoretical findings.

To the best of our knowledge, this is the first paper which proposes penalty approximations for QVIs in such a generality and presents rigorous error estimates for the penalization errors. Natural next steps would be to extend the penalty approach to interconnected obstacles with negative switching costs as in [19], to more general intervention operators as in [2], and to monotone systems with interconnected bilateral obstacles as in [10].

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| $c \backslash \rho$ | $4 \times 10^3$ | $8 \times 10^3$ | $16 \times 10^3$ | $32 \times 10^3$ | $64 \times 10^3$ | $128 \times 10^3$ |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $1/4$          | (a) 6.849917   | 6.849942       | 6.849954       | 6.849960       | 6.849962       | 6.849964       |
|                | (b) 0.00208    | 0.00104        | 0.0111         | 0.012         | 0.012         | 0.0113         |
|                | (c) 12         | 12             | 12             | 12            | 12            | 12             |
|                | (d) 0.0112     | 0.0113         | 0.0111         | 0.0112        | 0.0113        | 0.0113         |
| $1/16$         | (a) 7.405239   | 7.405407       | 7.405641       | 7.405708       | 7.405742       | 7.405758       |
|                | (b) 0.000451   | 0.000226       | 0.000113       | 0.000056       | 0.000028       | 0.000028       |
|                | (c) 12         | 12             | 12             | 12            | 12            | 12             |
|                | (d) 0.0115     | 0.0115         | 0.0115         | 0.0111        | 0.0115        | 0.0117         |
| $1/64$         | (a) 7.791271   | 7.792091       | 7.792499       | 7.792703       | 7.792805       | 7.792856       |
|                | (b) 0.001003   | 0.000501       | 0.000250       | 0.000125       | 0.000062       | 0.000062       |
|                | (c) 13         | 13             | 13             | 13            | 13            | 13             |
|                | (d) 0.0121     | 0.0122         | 0.0115         | 0.0130        | 0.0127        | 0.0127         |
| $1/256$        | (a) 8.00977    | 8.011300       | 8.012258       | 8.012722       | 8.012955       | 8.013071       |
|                | (b) 0.002016   | 0.001010       | 0.000505       | 0.000253       | 0.000126       | 0.000126       |
|                | (c) 14         | 14             | 14             | 14            | 14            | 14             |
|                | (d) 0.0131     | 0.0130         | 0.0132         | 0.0129        | 0.0129        | 0.0130         |
| $1/1024$       | (a) 8.10854    | 8.112341       | 8.114262       | 8.115229       | 8.115715       | 8.115958       |
|                | (b) 0.003854   | 0.002018       | 0.001017       | 0.000510       | 0.000256       | 0.000256       |
|                | (c) 15         | 15             | 14             | 15            | 15            | 15             |
|                | (d) 0.0144     | 0.0145         | 0.0130         | 0.0145        | 0.0144        | 0.0143         |
| $1/4096$       | (a) 8.135298   | 8.139588       | 8.141012       | 8.142047       | 8.142567       | 8.142828       |
|                | (b) 0.003854   | 0.002156       | 0.001087       | 0.000546       | 0.000273       | 0.000273       |
|                | (c) 14         | 14             | 14             | 14            | 14            | 14             |
|                | (d) 0.0135     | 0.0130         | 0.0131         | 0.0132        | 0.0133        | 0.0132         |
| $1/16384$      | (a) 8.143553   | 8.146389       | 8.147286       | 8.148752       | 8.149280       | 8.149545       |
|                | (b) 0.002975   | 0.001508       | 0.000974       | 0.000554       | 0.000278       | 0.000278       |
|                | (c) 12         | 12             | 14             | 14            | 14            | 14             |
|                | (d) 0.0115     | 0.0115         | 0.0134         | 0.0134        | 0.0134        | 0.0133         |
| $0$            | (a) 8.146318   | 8.149164       | 8.150603       | 8.151326       | 8.151688       | 8.151869       |
|                | (b) 0.002990   | 0.001509       | 0.000758       | 0.000380       | 0.000190       | 0.000190       |
|                | (c) 12         | 12             | 12             | 12            | 12            | 11             |
|                | (d) 0.0112     | 0.0112         | 0.0111         | 0.0111        | 0.0111        | 0.0103         |

Table 2: Numerical results for the three-regime optimal switching problem with different switching costs and penalty parameters. Shown are: (a) the numerical solutions $u^{c,\rho,1}$ at $x = 1$; (b) the increments $\|u^{c,\rho} - u^{c,\rho/2}\|$; (c) the number of iterations; (d) the overall runtime in seconds.
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