THE PSEUDOVARIEITY OF ALL NILPOTENT GROUPS IS TAME

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Abstract. It has been shown that for every prime number \( p \), the pseudovariety \( G_p \) of all finite \( p \)-groups is tame with respect to an implicit signature containing the canonical implicit signature. In this paper we generalize this result and we show that the pseudovariety of all finite nilpotent groups is tame but it is not completely tame.

Keywords. relatively free profinite semigroup, pseudovariety of semigroups, system of equations, implicit signature, completely tame, completely reducible, rational constraint, \( \sigma \)-full, weakly reducible.

1. Introduction

By a pseudovariety we mean a class of semigroups which is closed under taking subsemigroups, finite direct products, and homomorphic images. A pseudovariety is said to be decidable if there is an algorithm to test membership of a finite semigroup; otherwise, the pseudovariety is said to be undecidable. Eilenberg [14] established a correspondence between varieties of rational languages and pseudovarieties of finite semigroups which translates problems in language theory into the decidability of pseudovarieties of semigroups. In general the decidability of pseudovarieties is not preserved by many operations on pseudovarieties such as semidirect product, join and Mal’cev product ([19, 1]). Almeida and Steinberg introduced a refined version of decidability called tameness [11]. The tameness property requires the reducibility property which is a generalization of the notion of inevitability that Ash introduced to prove the type II conjecture of Rhodes [12].

There are various results using tameness of pseudovarieties to establish the decidability of pseudovarieties obtained by application of the operations of semidirect product, Mal’cev product and join [5, 4, 6].

Also there are connections between tameness and geometry and model theory [16, 15, 17, 8, 9]. So, it is worth finding more examples of tame pseudovarieties.
It has been established that for every prime numbers $p$, the pseudovariety $G_p$ of all finite $p$-groups is tame [3]. Using this result, in this paper we show that the pseudovariety $G_{nil}$ of all finite nilpotent groups is tame with respect to an enlarged implicit signature $\sigma$. Since the free $\sigma$-subalgebra generated by a finite alphabet $A$ is not any more the free group, in section 3, we prove the word problem is decidable in this free $\sigma$-subalgebra, meaning that there is an algorithm to decide whether two elements of this $\sigma$-algebra represent the same element.

In the last section, we show that the pseudovariety $G_{nil}$ is $\sigma$-reducible if and only if for all prime number $p$, the pseudovariety $G_p$ is $\sigma$-reducible. This theorem yields as a corollary that the pseudovariety $G_{nil}$ is tame with respect to the systems of equations associated to finite directed graphs but is not completely tame.

2. Preliminaries

A topological semigroup is a semigroup $S$ endowed with a topology such that the basic semigroup multiplication $S \times S \to S$ is continuous. We say that a topological semigroup $S$ is $A$-generated if there is a mapping $\varphi : A \to S$ such that $\varphi(A)$ generates a dense subsemigroup of $S$.

A pseudovariety of semigroups is a class of finite semigroups closed under taking subsemigroups, homomorphic images, and direct products. Given a pseudovariety $V$ of semigroups, by a pro-$V$ semigroup $S$ we mean a compact, zero-dimensional semigroup which is residually in $V$, that is for every two distinct points $s, t \in S$, there exists a continuous homomorphism $\varphi : S \to T$ into some member $T \in V$ such that $\varphi(s) \neq \varphi(t)$.

For a finite set $A$ in the variety generated by $V$, we denote by $\Omega_A V$ the free pro-$V$ semigroup. The free pro-$V$ semigroup has the universal property in variety of pro-$V$ semigroups in the sense that for every mapping $\varphi : A \to S$ into a pro-$V$ semigroup $S$, there exists a unique continuous homomorphism $\hat{\varphi} : \Omega_A V \to S$ such that the following diagram commutes:

$$
\begin{array}{c}
A \\
\varphi \downarrow \\
S \\
\hat{\varphi} \\
\Omega_A V
\end{array}
$$

For an $A$-generated pro-$V$ semigroup $S$, we view $S^A$ both as a direct power of $S$ and as the set of all functions from the set $A$ to $S$. To each element $w \in \Omega_A V$, we may associate an $A$-ary operation $w_S : S^A \to S$: for every $\varphi \in S^A$, by the universal property of $\Omega_A V$, there is a unique extension $\hat{\varphi} : \Omega_A V \to S$. Define $w_S(\varphi) = \hat{\varphi}(w)$. It is easy to see that for every continuous homomorphism $f : S \to T$ between pro-$V$ semigroups, the following diagram

$$
\begin{array}{c}
A \\
\varphi \downarrow \\
S \\
\hat{\varphi} \\
\Omega_A V
\end{array}
$$

...
commutes:

\[ \begin{array}{ccc}
S^A & \xrightarrow{wS} & S \\
\downarrow{f} & & \downarrow{f} \\
T^A & \xrightarrow{wT} & T
\end{array} \]

Operations with that property are called \( A \)-ary implicit operations. The element \( w \in \varpi_A V \) is completely determined by the implicit operation \((w_S)_S \in V \) \([3]\). Note that, the elements of \( A \) correspond to the component projects.

We say that an implicit operation \( w \in \varpi_A V \) is computable if there is an algorithm which given \( S \in V \) and \( \varphi \in S^A \), output the value \( w_S(\varphi) \).

An \( A \)-ary implicit operator on a pro-V semigroup \( S \) is a transformation \( f : S^A \to S^A \) of \( S^A \) to itself whose components \( f_i : S^A \to S \) determined an \( A \)-ary implicit operation. The set of all \( A \)-ary implicit operators on a pro-V semigroup \( S \) is denoted by \( \mathcal{O}_A(S) \). The set \( \mathcal{O}_A(S) \) is a monoid under composition.

**Proposition 2.1.** \([3]\) Proposition 2.2] There is a natural topology on \( \mathcal{O}_A(S) \) such that the correspondence

\[ S \to \mathcal{O}_A(S) \]

defines a functor from the category of pro-V semigroups with onto continuous homomorphisms as morphisms into the category of profinite monoids.

For \( n \)-ary implicit operations \( w_1, \ldots, w_n \in \varpi_n V \) and a pro-V semigroup \( S \), denote by \((w_1, \ldots, w_n)\) the implicit operator

\[ S^n \to S^n 
( (s_1, \ldots, s_n) ) \mapsto ( (w_1)_S(s_1, \ldots, s_n), \ldots, (w_n)_S(s_1, \ldots, s_n) ). \]

Denote composition of operators by concatenation: \((v_1, \ldots, v_n)(w_1, \ldots, w_n)\) has component \( i \) determined by the operation \( v_i(w_1, \ldots, w_n) \).

Recall that, for an element \( v \) of a finite semigroup \( V \), \( v^\omega \) denotes the unique idempotent power of \( v \). This defines a unary implicit operation \( x \mapsto x^\omega \) on finite semigroups (and similarly on finite monoids) which therefore has a natural interpretation on each profinite monoid of the form \( \mathcal{O}_n(S) \). Note that \( x^\omega \) is the limit of the sequence \( \{x^{\omega^j}\}_n \). We denote by \( a_j \circ (w_1, \ldots, w_n)^\omega \), the \( j \)-th component of the \( \omega \)-power of the operator \((w_1, \ldots, w_n)\).

**Lemma 2.2.** \([3]\) Corollary 2.5] Let \( w_1, \ldots, w_n \in \varpi_n V \). Then each component of \((w_1, \ldots, w_n)^\omega \) is also a member of \( \varpi_n V \). Moreover, if \( w_i \) are computable operations, then so is each \( a_j \circ (w_1, \ldots, w_n)^\omega \).

Let \( S \) be the pseudovariety of all finite semigroups. The elements of \((\varpi_A S)^1\), over arbitrary finite alphabets \( A \), are called pseudowords. A pseudoidentity is a formal equality of the form \( u = v \) with \( u, v \in \varpi_A S \) for some finite alphabet \( A \). For a pseudovariety \( V \) of semigroups, we denote by \( \psi_V \) the unique continuous homomorphism \( \varpi_A S \to \varpi_A V \) which restricts to the identity on \( A \). We say that \( V \) satisfies the pseudoidentity \( u = v \) with \( u, v \in \varpi_A S \) if \( \psi_V(u) = \psi_V(v) \).
By an *implicit signature* we mean a set of pseudowords including multiplication. An important example is given by the canonical signature \( \kappa \) consisting of the multiplication and the unary operation \( x \mapsto x^{\omega-1} \) which, to an element \( s \) of a finite semigroup with \( n \) elements, associates the inverse of \( s^{1+n} \) in the cyclic subgroup generated by this power.

Let \( \sigma \) be an implicit signature. Under the natural interpretation of the elements of \( \sigma \), every profinite semigroup may be viewed as a \( \sigma \)-algebra in the sense of universal algebra. The \( \sigma \)-subalgebra of \( \Omega_A^V \) generated by \( A \) is denoted by \( \Omega_A^\sigma V \) and it is freely generated by \( A \). We say that \( \Omega_A^\sigma V \) has *decidable word problem* if there is an algorithm to decide whether two pseudowords \( u, v \in \Omega_A^\sigma \) with \( \psi_V(u), \psi_V(v) \in \Omega_A^\sigma V \) represent the same implicit operation in \( \Omega_A^\sigma V \).

For every subset \( L \) of \( \Omega_A^\sigma \) and \( V \) be a pseudovariety, we denote by \( \text{Cl}(L), Cl_V(L), \) and \( Cl_{\sigma,V}(L) \), the closure of \( L \) in \( \Omega_A^\sigma \), \( \Omega_A^V \), and \( \Omega_A^\sigma V \), respectively.

As it is mentioned in the introduction, the property tameness requires the property reducibility. To define reducibility we need a system of equations. Let \( X \) and \( P \) be disjoint finite sets, whose elements will be the *variables* and the *parameters* of the system, respectively. Consider the following system of equations:

\[
(2.1) \quad u_i = v_i \quad (i = 1 \ldots, m),
\]

where \( u_i \) and \( v_i \) are pseudowords of \( \Omega_{X \cup P}^\sigma \). We also fix a finite set \( A \) and for every \( x \in X \), we choose a rational subset \( L_x \subseteq A^* \). For every parameter \( p \in P \), we associate an element \( w_p \in \Omega_A^\sigma \). A *solution* of the system (2.1) modulo \( V \) satisfying the constraints is a function \( \delta : X \cup P \to \Omega_A^\sigma \) satisfying the following conditions:

1. \( \delta(x) \in \text{Cl}(L_x) \).
2. \( \delta(p) = w_p \).
3. \( V \) satisfies the pseudoidentities \( \delta(u_i) = \delta(v_i) \) \( (i = 1 \ldots, m) \).

**Theorem 2.3.** [2, Theorem 5.6.1] *Let \( V \) be a pseudovariety of finite semigroups. The following conditions are equivalent for a finite system \( \Sigma \) of equations with rational constraints over the finite alphabet \( A \):*

- \( \Sigma \) has a solution modulo every \( A \)-generated semigroup in \( V \)
- \( \Sigma \) has a solution modulo \( \Omega_A^V \).

Let \( \sigma \) be an implicit signature. Consider a system of the form (2.1), with constraints \( L_x \subseteq A^* \) and \( w_p \in \Omega_A^\sigma \) \( (x \in X \text{ and } p \in P) \) where \( u_i \) and \( v_i \) are \( \sigma \)-terms in \( \Omega_{X \cup P}^\sigma \). Assume that this system has a solution modulo \( V \). A pseudovariety \( V \) is said to be *\( \sigma \)-reducible* for this system if it has a solution \( \delta : X \cup P \to \Omega_A^\sigma \) modulo \( V \). We say that \( V \) is *completely \( \sigma \)-reducible* if it is \( \sigma \)-reducible for every such system.

**Proposition 2.4.** [2, Proposition 3.1] *Let \( V \) be a pseudovariety. If \( V \) is \( \sigma \)-reducible respect to the systems of equations of \( \sigma \)-terms without parameters, then \( V \) is completely \( \sigma \)-reducible.*
We say that a pseudovariety $V$ is $\sigma$-tame with respect to a class $\mathcal{C}$ of systems of equations if the following conditions hold:

- for every system of equations in $\mathcal{C}$, the pseudovariety $V$ is $\sigma$-reducible;
- the word problem is decidable in $\Omega^\sigma_A V$;
- $V$ is recursively enumerable, in the sense that there is some Turing machine which outputs successively representatives of all the isomorphism classes of members of $V$.

We say that $V$ is completely $\sigma$-tame if $V$ is completely $\sigma$-reducible. Some important tameness results are as follows:

- The pseudovariety $G$ of all finite groups is $\kappa$-tame with respect to systems of equations associated with finite directed graphs \cite{11, 12}. It follows from results of Coulbois and Khelif that $G$ is not completely $\kappa$-tame \cite{13}.
- For a prime number $p$, the pseudovariety $G_p$ of all finite $p$-groups is tame with respect to the systems of equations associated with finite directed graphs, but not $\kappa$-tame \cite{3}.
- The pseudovariety $Ab$ of all finite abelian groups is completely $\kappa$-tame \cite{10}.

Consider a system of equations (2.1) with constraints $L_x \subseteq A^*$ and $w_p \in \Omega^\sigma_A S$ ($x \in X$ and $p \in P$) where $u_i$ and $v_i$ are $\sigma$-terms in $\Omega^\sigma_{X \cup P} S$. A pseudovariety $V$ is said to be weakly $\sigma$-reducible with respect to this system if in the case it has a solution modulo $V$, then there is a solution $\delta: X \cup P \to \Omega_A S$ modulo $V$ which satisfies the conditions $\psi_V(\delta(x)) \in \Omega^\sigma_A V$ ($x \in X$). It is obvious that if a pseudovariety $V$ is $\sigma$-reducible, then it is weakly $\sigma$-reducible but the converse is not true. For a prime number $p$, the pseudovariety $G_p$ of all finite $p$ groups is weakly $\kappa$-reducible but it is not $\kappa$-reducible \cite{21, 11}.

We say that a pseudovariety $V$ is $\sigma$-full if for every rational language $L \subseteq A^*$, the set $\psi_V(Cl_{\sigma}(L))$ is closed in $\Omega^\sigma_A V$.

**Proposition 2.5.** \cite{11, Proposition 4.5] Every $\sigma$-full weakly $\sigma$-reducible pseudovariety is $\sigma$-reducible.

Let $x \in \Omega_A S$ and fix $n \in \mathbb{N}$. We denote by $x^{n^r}$, the pseudoword

$$\lim_{k \to \infty} x^{n^k}.$$

Let $w_1, \ldots, w_k$ be group words. We denote by $M(w_1, \ldots, w_k)$ the $k \times |A|$ matrix whose $(i, j)$-entry is the sum of the exponents in the occurrences of the letter $a_j \in A$ in the word $w_i$.

**Theorem 2.6.** \cite{3} Let $p$ be a prime number and $\sigma_p$ be the set of all implicit signature obtained by adding to the canonical signature $\kappa$ all implicit operations of the form

$$a_{j} \circ (w_1, \ldots, w_n)^{\omega}$$

with $j = 1, \ldots, n$, $w_i$ are $\kappa$-terms such that the subgroup $\langle w_1, \ldots, w_n \rangle$ is $G_p$-dense. Then $G_p$ is $\sigma_p$-tame.
We use the implicit operations in the preceding theorem and we prove the following theorem:

**Theorem 2.7.** Let \( \sigma \) be the set of all implicit signature obtained by adding to the canonical signature \( \kappa \) all implicit operations of the form

\[
(a_j \circ (w_1, \ldots, w_n)\omega)^m \omega \quad (m \in \mathbb{N})
\]

with \( j = 1, \ldots, n \), \( w_i \) are \( \kappa \)-terms such that \( \det M(w_1, \ldots, w_n) \neq 0 \) and every prime number \( p \) dividing \( \det M(w_1, \ldots, w_n) \) also divides \( m \). Then \( G_{nil} \) is \( \sigma \)-tame.

Note that by [18], for a prime number \( p \) and \( \kappa \)-words \( w_1, \ldots, w_n \), the subgroup \( \langle w_1, \ldots, w_n \rangle \) is \( G_p \)-dense if and only if \( \det M(w_1, \ldots, w_n) \equiv 0 \) (mod \( p \)).

3. The pseudovariety \( G_{nil} \) is \( \sigma \)-tame

3.1. The word problem is decidable in \( \Omega^\sigma_A G_{nil} \). We use the following lemmas to reduced the word problem in \( \Omega^\sigma_A G_{nil} \) to the word problem in the free group generated by \( A \).

**Lemma 3.1.** Let \( u, v \in \Omega_A \). Then the pseudoidentity \( u = v \) is valid in \( G_{nil} \) if and only if, for every prime number \( p \), the pseudoidentity \( u = v \) is valid in \( G_p \).

**Proof.** Since every \( p \)-group is a nilpotent group, if \( G_{nil} \) satisfies the pseudoidentity \( u = v \), then for every prime number \( p \), \( G_p \) satisfies the pseudoidentity \( u = v \).

The converse follows from fact that every finite nilpotent group is isomorphic to the direct product of its \( p \)-Sylow subgroups. \( \square \)

We denote by \( \mathbb{P} \), the set of all prime numbers.

**Lemma 3.2.** Fix \( p \in \mathbb{P} \). The pseudoidentity \( x^{n\omega} = x \) is valid in \( G_p \) if \( n \) is not divisible by \( p \) and the pseudoidentity \( x^{n\omega} = 1 \) is valid in \( G_p \) otherwise.

**Proof.** The result follows from the elementary Euler congruence theorem. \( \square \)

**Theorem 3.3.** For every pseudoword \( u \in \Omega^\sigma_A S \), there is a computable cofinite subset \( S(u) \) of \( \mathbb{P} \) such that the following properties hold:

- there is a computable \( \kappa \)-word \( w_0 \) such that for every \( p \in S(u) \), the pseudoidentity \( u = w_0 \) is valid in \( G_p \);
- for every \( p_i \in \{ p_1, \ldots, p_r \} = \mathbb{P} \setminus S(u) \), there is a computable \( \kappa \)-word \( w_i \) such that the pseudoidentity \( u = w_i \) is valid in \( G_{p_i} \) \( (1 \leq i \leq r) \).

In particular, for every prime number \( p \), we have \( \Omega^\sigma_A G_p = \Omega^\sigma_A G_p \).

**Proof.** Since every \( u \in \Omega^\sigma_A S \) is constructed from the letters in \( A \) using a finite number of times the operations in \( \sigma \) and the intersection of a finite number of cofinite sets is cofinite, it is enough to show that the statement of the theorem holds for every pseudoword \( u \in \sigma \setminus \kappa \).
Consider the following pseudoword in $\sigma \setminus \kappa$

$$u = (a_j \circ (w_1, \ldots, w_n)^\omega)^{m^\omega} \quad (m \in \mathbb{N})$$

where $j = 1, \ldots, n$, $w_i$ are $\kappa$-terms such that $\det M(w_1, \ldots, w_n) \neq 0$ and every prime number $p$ dividing $\det M(w_1, \ldots, w_n)$ also divides $m$. Let

$$S(u) = \mathbb{P} \setminus \{p \mid p \text{ divides } m\}.$$ 

If $p$ lies in $S(u)$, then by [18, Corollary 3.3], the subgroup $\langle w_1, \ldots, w_n \rangle$ is $G_p$-dense in $\Omega_A G_p$. Hence, by [3, Lemma 6.5], the following pseudoidentity holds in $G_p$

$$a_j \circ (w_1, \ldots, w_n)^\omega = a_j.$$ 

Consider $w_0 = a_j$.

Otherwise, $p$ divides $m$ and therefore by Lemma 3.2, the pseudoidentity

$$(a_j \circ (w_1, \ldots, w_n)^\omega)^{m^\omega} = 1$$

holds in $G_p$. □

**Corollary 3.4.** The word problem is decidable in $\Omega_A^\sigma G_{\text{nild}}$.

**Proof.** Let $u, v \in \Omega_A^\sigma S$. Then by Lemma 3.1, the pseudoidentity $u = v$ holds in $G_{\text{nild}}$ if and only if for every prime number $p$, the pseudoidentity $u = v$ holds in $G_p$.

Since $G_p$ is a pseudovariety of finite groups, the pseudoidentity $u = v$ is valid in $G_p$ if and only if the pseudoidentity $u v^{\omega - 1} = 1$ is valid in $G_p$. Now the result follows from the preceding lemma and the fact that the word problem is decidable in the free group. □

3.2. **The pseudovariety $G_{\text{nild}}$ is $\sigma$-reducible.** We denote by $FG(A)$, the free group over $A$ and for a finitely generated (f.g.) subgroup $H$ of a free group, denote by $P(H)$, the set of all prime numbers $p$ such that $H$ is $G_p$-closed.

**Proposition 3.5.** [18, proposition 4.3] Let $H$ be a f.g. subgroup of $FG(A)$. The set $P(H)$ is a finite or a cofinite subset of $\mathbb{P}$, and it is effectively computable.

**Lemma 3.6.** Let $H$ be a f.g. subgroup of $FG(A)$. Then there is a cofinite subset $S(H)$ of $\mathbb{P}$ and a f.g. subgroup $K$ of $FG(A)$ such that for every $p \in S(H)$, $\text{Cl}_{G_p, K}(H) = K$.

**Proof.** Let $K_1, \ldots, K_m$ be the set of all overgroups of the subgroup $H$ (i.e., the automaton of $K_i$ is a quotient of the automaton of $H$). For every prime number $p$, there is $i$ ($1 \leq i \leq m$) such that $\text{Cl}_{G_p, K_i}(H) = K_i$ [18]. Since the number of overgroups of $H$ is a finite set, there is $K_j$ and an infinite subset $S$ of $\mathbb{P}$ such that for every $p \in S$, we have $\text{Cl}_{G_p, K}(H) = K_j$ ($1 \leq j \leq m$). If $S$ is cofinite, then we are done. For every $p \in S$, we have $K_j \subseteq \text{Cl}_{G_p, K}(K_j) \subseteq \text{Cl}_{G_p, K}(\text{Cl}_{G_p, K}(H)) = \text{Cl}_{G_p, K}(H) = K_j$ and hence, $S \subseteq P(K_j)$. Therefore, by the preceding proposition, $P(K_j)$ is a cofinite subset of $\mathbb{P}$. 


Suppose that there is an overgroup $L$ of $H$ properly contained in $K_j$ such that $P(L)$ is a cofinite subset of $P$. Then for every $p \in P(L)$, we have
\[ Cl_{G_p, \kappa}(H) \subseteq Cl_{G_p, \kappa}(L) = L \not\subseteq K_j. \]
Hence, the set of all prime numbers $q$ such that $Cl_{G_q, \kappa}(H) = K_j$ is contained in the set $P \setminus P(L)$. Since we assume that $P(L)$ is a cofinite subset of $P$, the set $P \setminus P(L)$ is a finite set which contradicts the choice of $K_j$.

So, for every overgroup $L$ of $H$ properly contained in $K_j$, $P(L)$ is a finite set. Let $K = K_j$ and
\[ S(H) = P(K) \setminus \bigcup_{L \subseteq K} P(L). \]
As the number of overgroups of $H$ is finite, $S(H)$ is a cofinite subset of $P$. For every $p \in S(H)$, we have
\[ Cl_{G_p, \kappa}(H) \subseteq Cl_{G_p, \kappa}(K) = K, \]
hence, $Cl_{G_p, \kappa}(H)$ is a $G_p$-closed overgroup of $H$ contained in $K$, by the choice of $p$, it follows that $Cl_{G_p, \kappa}(H) = K$. \hfill \square

The following two propositions are the main tools to show that the pseudovariety $G_{nil}$ is $\sigma$-reducible.

**Proposition 3.7.** Let $H_1, \ldots, H_t$ be f.g. subgroups of the free group and fix the cofinite sets $S(H_i)$ and the subgroups $K_i$ as in the preceding lemma. Let $w \in K_1 \ldots K_t$. Then there are $u_i \in \Omega \cap Cl(H_i)$ and cofinite subsets $S_i(H_i)$ of $P$ contained in $S(H_i)$ such that, if $p \in \bigcap_{i=1}^t S_i(H_i)$, then the pseudoidentity $w = u_1 \ldots u_t$ is valid in $G_p$ and the pseudoidentity $u_i = 1$ is valid in $G_p$ otherwise.

**Proof.** Since, for every prime number $p$, $G_p$ is an extension-closed pseudovariety, by [18 Proposition 2.10], there are $\kappa$-words $w_{1,i}, \ldots, w_{s_i,i}$ such that $K_i = \langle w_{1,i}, \ldots, w_{s_i,i} \rangle$ and $rk(K_i) \leq rk(H_i)$ ($1 \leq i \leq t$).

Fix $q \in \bigcap_{i=1}^t S(H_i)$. As $K_i$ contains $H_i$, we can rewrite the generator $h$ of $H_i$ as a reduced word $h'$ in terms of generators of $K_i$. Since $Cl_{G_q, \kappa}(H_i) = K_i$, by [18 Proposition 2.9] $H_i$ is $G_q$-dense in the pro-$G_q$ topology on $K_i$. Hence, by [3 Proposition 5.2], there is a subset $\{h'_{1,i}, \ldots, h'_{s_i,i}\}$ of generators of $H_i$ such that the subgroup $H'_i$ generated by $\{h'_{1,i}, \ldots, h'_{s_i,i}\}$ is $G_q$-dense in the pro-$G_q$ topology on $K_i$. Let $M(h'_{1,i}, \ldots, h'_{s_i,i})$ be the $s_i \times s_i$ matrix whose $(k,j)$-entry is the number of occurrences of $w_{j,i}$ in $h'_{k,i}$. Then by [18 Corollary 3.3], we have $\det M(h'_{1,i}, \ldots, h'_{s_i,i}) \equiv 0 \pmod{q}$ and also for every prime number $p$ not dividing $\det M(h'_{1,i}, \ldots, h'_{s_i,i})$, the subgroup $H'_i$ is $p$-dense in the pro-$G_p$ topology on $K_i$. Since for every $p \in S(H_i)$, $K_i$ is $G_p$-closed, by [18 Proposition 2.9] for every $p \in S(H_i)$ not dividing $\det M(h'_{1,i}, \ldots, h'_{s_i,i})$ we have $Cl_{G_p, \kappa}(H'_i) = Cl_{G_p, \kappa}(H_i) = K_i$, where $Cl_{G_p, \kappa}(H'_i)$ is the $p$-closure of $H'_i$ in the pro-$G_p$ topology on $K_i$. Hence, by the proof of [3] Theorem...
Let $S_1(H_i) = S(H_i) \setminus \{p \mid \det M(h_{1,i}^t, \ldots, h_{s_i,i}^t) \equiv 0 \mod p\}$.

As $\det M(h_{1,i}^t, \ldots, h_{s_i,i}^t) \neq 0$, the sets $S_1(H_i)$ are cofinite subsets of $P$.

Let $P_0 = \{p_1, \ldots, p_r\} = P \cap \bigcap_{i=1}^{t} S_1(H_i)$ and $n_0 = p_1 \ldots p_r$. Consider the following pseudowords:

$$u_{j,i} = (a_j \circ (h_{1,i}^t, \ldots, h_{s_i,i}^t)^\omega)^{n_0} \in \Omega_A S \cap Cl(H_i).$$

If $p \in P \setminus P_0$, by Lemma 3.2, the following pseudoidentities are valid in $G_p$:

$$u_{j,i} = (a_j \circ (h_{1,i}^t, \ldots, h_{s_i,i}^t)^\omega)^{n_0} = a_j \circ (h_{1,i}^t, \ldots, h_{s_i,i}^t)^\omega \ (3.1) \ u_{j,i}.$$

Otherwise, the following pseudoidentities are valid in $G_p$ ($p \in P_0$):

$$u_{j,i} = (a_j \circ (h_{1,i}^t, \ldots, h_{s_i,i}^t)^\omega)^{n_0} = 1.$$

So, we showed that for every generator $w_{k,i}$ of $K_i$, there are $u_{k,i} \in \Omega_A S \cap Cl(H_i)$ such that, if $p \in \bigcap_{i=1}^{t} S_1(H_i)$, then the pseudoidentity $u_{k,i} = w_{k,i}$ is valid in $G_p$ and the pseudoidentity $u_{k,i} = 1$ is valid in $G_p$ otherwise. Hence, every $w \in K_i$ has this property ($1 \leq i \leq t$).

Let $w \in K_1 \cup \cdots \cup K_t$. Then there are $w_i \in K_i$ such that $w = w_1 \cdot \cdots \cdot w_t$. By the preceding paragraph, there are $u_i \in \Omega_A S \cap Cl(H_i)$ such that if $p \in \bigcap_{i=1}^{t} S_1(H_i)$, then the pseudoidentity $u_i = w_i$ is valid in $G_p$ and the pseudoidentity $u_i = 1$ is valid in $G_p$ otherwise. Let $u = u_1 \cdot \cdots \cdot u_t \in \Omega_A S \cap Cl(H_1 \cdot \cdots \cdot H_t)$.

If $p \in \bigcap_{i=1}^{t} S_1(H_i)$, then the pseudoidentity $v = w$ is valid in $G_p$ and the pseudoidentity $v = 1$ is valid in $G_p$ otherwise.

**Proposition 3.8.** Let $H_1, \ldots, H_t$ be f.g. subgroups of the free group and fix $S_1(H_i)$ as in the preceding proposition. Then for every $p$ in the set

$$P \setminus \bigcap_{i=1}^{t} S_1(H_i) = \{p_1, \ldots, p_r\}$$

and every $w \in Cl_{G_{p,r}}(H_1 \cdot \cdots \cdot H_t)$, there are $u_i \in \Omega_A S \cap Cl(H_i)$ such that the pseudoidentity $u_i = 1$ is valid in $G_q$ for every $q \in P \setminus \{p\}$ and the pseudoidentity $u_1 \cdot \cdots \cdot u_t = w$ is valid in $G_p$.

**Proof.** By Lemma 5.2, for every prime number $p$, we have

$$Cl_{G_{p,r}}(H_1 \cdot \cdots \cdot H_t) = Cl_{G_{p,r}}(H_1) \cdots Cl_{G_{p,r}}(H_t).$$

Let $P_0 = \{p_1, \ldots, p_r\}$ and $K_{p_i,j} = Cl_{G_{p_i,r}}(H_j)$ ($1 \leq i \leq r$ and $1 \leq j \leq t$). There are $\kappa$-words $w_1, p_{i,j}, \ldots, w_{\kappa, p_{i,j}, i,j}$ such that

$$K_{p_i,j} = \{w_1, p_{i,j}, \ldots, w_{\kappa, p_{i,j}, i,j}\}.$$
By the proof of [3, Theorem 6.1], there is a subset \( \{ h_{1,p_{i,j}}, \ldots, h_{s_{p_{i,j}},p_{i,j}} \} \) of the generators of \( H_j \) such that
\[
Cl_{\mathbb{P}_{p_{i,j}}}(\{ h_{1,p_{i,j}}, \ldots, h_{s_{p_{i,j}},p_{i,j}} \}) = Cl_{\mathbb{P}_{p_{i,j}}}(H_j) = K_{p_{i,j}},
\]
and the pseudoidentity
\[
w_{k,p_{i,j}} = a_k \circ (h_{1,p_{i,j}}, \ldots, h_{s_{p_{i,j}},p_{i,j}})\omega \quad (1 \leq k \leq s_{p_{i,j}})
\]
is valid in \( G_{p_{i,j}} \). We consider the following finite subsets of \( \mathbb{P} \):
\[
P_i = \bigcup_{j=1}^{r} \{ p \mid p \text{ divides } \det M(h_{1,p_{i,j}}, \ldots, h_{s_{p_{i,j}},p_{i,j}}) \} \quad (1 \leq i \leq r)
\]
where \( h'_{k,p_{i,j}} \) is generator \( h_{k,p_{i,j}} \) written in terms of generators of \( K_{p_{i,j}} \) \((1 \leq k \leq s_{p_{i,j}})\). Note that, since \( K_{p_{i,j}} \) is a \( p_{i,j} \)-closed subgroup and
\[
Cl_{\mathbb{P}_{p_{i,j}}}(\{ h_{1,p_{i,j}}, \ldots, h_{s_{p_{i,j}},p_{i,j}} \}) = K_{p_{i,j}},
\]
by [13, Proposition 2.9] the subgroup \( \{ h'_{1,p_{i,j}}, \ldots, h'_{s_{p_{i,j}},p_{i,j}} \} \) is \( p_{i,j} \)-dense in the pro-\( G_{p_{i,j}} \) topology on \( K_{p_{i,j}} \) and, therefore, \( p_{i,j} \) does not belong to \( P_i \). Consider the following natural numbers:
\[
n_0 = p_1 \cdots p_r,
\]
\[
n_i = \prod_{j=0}^{r} \prod_{p \in P_j \setminus \{ p_{i,j} \}} p \quad (1 \leq i \leq r).
\]
The natural numbers \( n_i \) \((1 \leq i \leq r)\) satisfy the following properties:
\(\)
1. The prime number \( p_{i,j} \) does not divide \( n_i \).
2. Fix \( p_{i,j} \in P_0 \). For every \( p_{j} \in P_0 \) \((j \neq i)\), \( p_{j} \) divides \( n_i \), because \( p_{j} \) belongs to the set \( P_0 \setminus \{ p_{i,j} \} \).
3. Since we have \( p_{i,j} \notin P_0 \), every prime number \( p \in P_i \) divides \( n_i \).
4. For every prime number \( p \) in \( (P_1 \cup \ldots \cup P_r) \setminus P_0 \), \( p \) divides \( n_i \) \((1 \leq i \leq r)\), because there is \( j \) such that \( p \in P_j \) and since \( p \) does not belong to \( P_0 \), \( p \) is in \( P_j \setminus \{ p_{i,j} \} \).
\(\)
For every \( i, j \), and \( k \) \((1 \leq j \leq s_{p_{i,j}}, 1 \leq i \leq r, \text{ and } 1 \leq j \leq t)\), we let
\[
u_{k,p_{i,j}} = (a_k \circ (h_{1,p_{i,j}}, \ldots, h_{s_{p_{i,j}},p_{i,j}})\omega)^{n_i\omega}
\]
\[
(\omega-1)(p_{i,j})\omega^{n_i\omega}.
\]
Note that \( u_{k,p_{i,j}} \) belongs to \( \Omega_{\mathbb{P}_{p_{i,j}}}(\mathbb{P} \setminus Cl(H_j)) \). We claim that the pseudoidentity
\[
u_{k,p_{i,j}} = w_{k,p_{i,j}} \quad \text{valid in } G_{p_{i,j}} \quad \text{for every } p \in \mathbb{P} \setminus \{ p_{i,j} \} \quad \text{and the pseudoidentity}
\]
\[
u_{k,p_{i,j}} = 1 \quad \text{valid in } G_{p_{i,j}} \quad \text{for every } 1 \leq k \leq s_{p_{i,j}} \quad \text{and } 1 \leq j \leq t.
\]
It remains to establish the claim. Consider the following cases:
\(\)
- Let \( p = p_{i,j} \). By the property (1) of \( n_i \), \( p_{i,j} \) does not divides \( n_i \). Hence, we have the following pseudoidentities in \( G_{p_{i,j}} \):
\[
u_{k,p_{i,j}} = (a_k \circ (h_{1,p_{i,j}}, \ldots, h_{s_{p_{i,j}},p_{i,j}})\omega)^{n_i\omega}
\]
\[
(\omega-1)(p_{i,j})\omega^{n_i\omega}
\]
\[
= (a_k \circ (h_{1,p_{i,j}}, \ldots, h_{s_{p_{i,j}},p_{i,j}})\omega)^{1 \textbf{3.3} w_{k,p_{i,j}}}.
\]
• Consider either \( p \in P_0 \setminus \{ p_i \} \) or \( p \in (P_1 \cup \ldots \cup P_r) \setminus P_0 \). By the property (2) and (4) of \( n_i \), \( p \) divides \( n_i \). Hence, we have the following pseudoidentities in \( G_p \)

\[
\left( a_k \circ (h_{1,p_i,j}, \ldots, h_{s_{p_i,j},p_i,j})^\omega \right)^{n_i \omega} = 1,
\]

\[
\left( a_k \circ (h_{1,p_i,j}, \ldots, h_{s_{p_i,j},p_i,j})^\omega \right)^{(\omega-1)(p_i n_i) \omega} = 1.
\]

So, the pseudoidentity \( u_{k,p_i,j} = 1 \) is valid in \( G_p \).

• Consider \( p \in \mathcal{P} \setminus (P_0 \cup \ldots \cup P_r) \). By the choice of \( n_i \), \( p \) does not divide \( p_i n_i \) and, therefore, does not divide \( n_i \). Hence, we have the following pseudoidentities in \( G_p \)

\[
u_{k,p_i,j} = \left( a_k \circ (h_{1,p_i,j}, \ldots, h_{s_{p_i,j},p_i,j})^\omega \right)^{n_i \omega} = \left( a_k \circ (h_{1,p_i,j}, \ldots, h_{s_{p_i,j},p_i,j})^\omega \right)^{(\omega-1)(p_i n_i) \omega} = 1.
\]

This completes the proof of the claim.

We showed for every generator \( u_{k,p_i,j} \) of \( Cl_{G_{p_i,\kappa}}(H_j) = K_{p_i,j} \), there are \( u_{k,p_i,j} \in \omega_A^j \mathcal{S} \cap Cl(H_j) \) such that the pseudoidentity \( u_{k,p_i,j} = u_{k,p_i,j} \) is valid in \( G_{p_i} \) and the pseudoidentity \( u_{k,p_i,j} = 1 \) is valid in \( G_p \) (\( p \in \mathcal{P} \setminus \{ p_i \} \)). Hence, every \( w \in Cl_{G_{p_i,\kappa}}(H_j) \) has this property.

Fix \( p \in P_0 \) and let \( w \in Cl_{G_{p,\kappa}}(H_1 \ldots H_t) \). By (3.2), there are \( w_j \in Cl_{G_{p,\kappa}}(H_j) \) such that \( w = w_1 \ldots w_t \). By the preceding paragraph, there are \( u_j \in \omega_A^j \mathcal{S} \cap Cl(H_j) \) such that the pseudoidentity \( u_j = w_j \) is valid in \( G_p \) and the pseudoidentity \( u_j = 1 \) is valid in \( G_q \) (\( q \in \mathcal{P} \setminus \{ p \} \)). Let

\[
v = u_1 \ldots u_t \in \omega_A^\mathcal{S} \cap Cl(H_1 \ldots H_t).
\]

Then the equality \( v = w \) is valid in \( G_p \) and the equality \( v = 1 \) is valid in \( G_q \) (\( q \in \mathcal{P} \setminus \{ p \} \)).

**Corollary 3.9.** For every prime number \( p \), the pseudovariety \( G_p \) is \( \sigma \)-reducible with respect to the systems of equations associated with finite directed graphs.

**Proof.** By Theorem [3.3] for a rational subset \( L \) of \( A^* \) we have

\[
Cl_{G_p,\sigma}(\psi_{G_p}(L)) = Cl_{G_{p,\kappa}}(\psi_{G_{p}}(L)).
\]

Since the pseudovariety \( G_p \) is weakly \( \kappa \)-reducible with respect to the systems of equations associated with finite directed graphs [21], by Theorem 2.5 we just need to show that \( G_p \) is \( \sigma \)-full.

By definition of \( \sigma \)-full pseudovariety, it is enough to show that for a rational subset of \( A^* \):

\[
(3.4) \quad \psi_{G_p}(Cl(\sigma(L))) \subseteq Cl_{G_{p,\kappa}}(\psi_{G_{p}}(L)),
\]
or equivalently, it is enough to show that

\[(3.5) \quad w \in \Cl_{G_p,\kappa}(\psi_{G_p}(L)) \Rightarrow \Cl(L) \cap \Omega_p^* S \cap (\psi^{-1}(w)) \neq \emptyset\]

The proof is similar to the proof of [3, Theorem 6.1]. Since any rational subset of the free semigroup can be obtained by taking a finite number of finite subsets of the free semigroup and applying the union, product and the plus operation \( L \rightarrow L^+ \) a finite number of times, it is enough to show that these operations preserve this property. As it is mentioned in the proof of [3, Theorem 6.2], for the rational subsets \( L \) and \( K \), we have

1. \( \Cl_{G_p,\kappa}(\psi_{G_p}(L)) = \psi_{G_p}(L) \) if \( L \) is finite;
2. \( \Cl_{G_p,\kappa}(\psi_{G_p}(LK)) = \Cl_{G_p,\kappa}(\psi_{G_p}(L))\Cl_{G_p,\kappa}(\psi_{G_p}(K)) \);
3. \( \Cl_{G_p,\kappa}(\psi_{G_p}(L \cup K)) = \Cl_{G_p,\kappa}(\psi_{G_p}(L)) \cup \Cl_{G_p,\kappa}(\psi_{G_p}(L)) \)
4. \( \Cl_{G_p,\kappa}(\psi_{G_p}(L)^+) = \Cl_{G_p,\kappa}(\{\psi_{G_p}(L)\}) \)

If a language \( L \) is finite, then any \( w \) in \( \Cl_{G_p,\kappa}(\psi_{G_p}(L)) \) is also an element of \( \Cl(L) \cap \Omega_p^* S \cap (\psi^{-1}(w)) \) and so finite languages satisfy \( (3.5) \). Suppose that \( L_1 \) and \( L_2 \) are rational subsets of \( \Omega_p^* S \) satisfy Property \( (3.4) \). By property \( (3) \), at least one of the sets \( \Cl(L_1) \cap \Omega_p^* S \cap (\psi^{-1}(w)) \) and \( \Cl(L_2) \cap \Omega_p^* S \cap (\psi^{-1}(w)) \) is nonempty and, therefore, so is their union.

Let \( u \in \psi_{G_p}(\Cl(L_1L_2)) \cap \Omega_p^* G_p \). By property \( (2) \), there are

\[ u_i \in \Cl_{G_p,\kappa}(\psi_{G_p}(L_i)) \quad (i = 1, 2) \]

such that \( u = u_1u_2 \). By the induction hypotheses, there are \( w_i \in \psi_{G_p}^{-1}(u_i) \cap \Omega_p^* S \cap \Cl(L_i) \). Let \( w = w_1w_2 \in \Omega_p^* S \cap \Cl(L_1) \Cl(L_2) \subseteq \Omega_p^* S \cap \Cl(L_1L_2) \). Then we have

\[ \psi_{G_p}(w) = \psi_{G_p}(w_1w_2) = \psi_{G_p}(w_1)\psi_{G_p}(w_2) = u_1u_2 = u. \]

Thus, \( w \) belongs to \( \Omega_p^* S \cap \Cl(L_1L_2) \cap \psi_{G_p}^{-1}(u) \).

Let \( L \) be a rational subset satisfies the property \( [3, 5] \). By [3, Lemma 6.6], the is a finite subset \( Y \) of \( \kappa \) words, such that \( (L) = (Y) \) and \( \Cl(Y) \subseteq \Cl(L) \). Hence, to show that a rational language of the form \( L^+ \) satisfies \( [3, 5] \), it is enough to establish that, for any finite subset \( Y \) of \( \kappa \) words the following property holds:

\[ w \in \Cl_{G_p,\kappa}((Y)) \Rightarrow \Cl(Y^+) \cap \Omega_p^* S \cap (\psi^{-1}(w)) \neq \emptyset \]

The result follows from the Propositions \( 3.7 \) and \( 3.8 \) by \( t = 1 \).

**Theorem 3.10.** Let \( C \) be a class of systems of equations. For every prime number \( p \), the pseudovariety \( G_p \) is \( \sigma \)-reducible with respect to the systems of equations in \( C \) if and only if the pseudovariety \( G_{nil} \) is \( \sigma \)-reducible with respect to the systems of equations in \( C \).

**Proof.** Let \( V \) and \( W \) be pseudovarieties of semigroups such that \( V \subseteq W \). If a pseudoidentity is valid in \( W \), then it is valid in \( V \). Hence, if an equation has solution \( \delta \) modulo \( V \), then \( \delta \) is a solution modulo \( V \).

Conversely, let \( X \) be a finite set of variable and let \( k \) be the cardinality of \( X \). Consider a system of equations in \( C \) of the form

\[(3.6) \quad u_i = v_i \quad (i = 1, \ldots, m), \]
with rational constraints $L_x \subseteq (\Omega_A S)^1$ and $u_i, v_i \in \Omega^*_X S$. Suppose that this system has the solution $\delta : X \rightarrow \Omega_A S$ modulo $G_{nil}$. Since $L_x$ is a rational subset of $(\Omega_A S)^1$, $L_x$ is a finite union of sets of the form $R^*_0 w_0 R^*_1 \ldots w_t R^*_t$, where $R_i$ are rational subsets of $(\Omega_A S)^1$ and $w_i \in (\Omega_A S)^1$. Hence, for every $x \in X$, there is a simple rational subset

$$L'_x = R^*_0 u_0 R^*_1 \ldots u_{t_x-1} R^*_t \subseteq L_x$$

such that $\delta(x) \in Cl_{G_{nil}}(L'_{x})$. So, the system (3.6) with constraints $L'_x \subseteq \Omega_A S$ has the solution $\delta$ modulo $G_{nil}$ and, therefore, for every prime number $p$, the system (3.6) with constraints $L'_x \subseteq \Omega_A S$ has a solution modulo $G_p$. We show that the system (3.6) with constraints $L'_x (x \in X)$ has a solution $\delta' : X \rightarrow \Omega^*_A S$ modulo $G_{nil}$.

By [3, Lemma 6.6], there are finite subsets $Y_{i,x}$ of $\Omega^*_A S$ such that

$$Cl(Y^*_{i,x}) \subseteq Cl(R^*_{i,x}),$$

and the subgroup generated by $Y_{i,x}$ is equal to the subgroup generated by $R_{i,x}$ in the free group $(x \in X$ and $1 \leq i \leq t_x)$. Hence, we have

$$Cl_{G_p}(R^*_0 u_0 R^*_1 \ldots u_{t_x-1} R^*_t) = Cl_{G_p}(Y^*_0 u_0 Y^*_1 \ldots u_{t_x-1} Y^*_t).$$

Let $S_1(\langle Y_{i,x} \rangle)$ be as in Proposition 3.7 and $S = \bigcap_{x \in X=1}^{t_x} S_1(\langle Y_{i,x} \rangle)$. Fix $q \in S$. Since the system (3.6) with constraints $L'_x \subseteq \Omega_A S$ has a solution modulo $G_q$ and $G_q$ is $\sigma$-reducible, there is a solution $\delta_0 : X \rightarrow \Omega^*_A S$ modulo $G_q$. Hence, for every $x \in X$, $\delta_0(x)$ lies in $Cl_{G_q,n}(L'_{x}) = Cl_{G_q,n}(L'_{x})$. For every $p \in S$, we have

$$Cl_{G_p,n}(L'_{x}) = Cl_{G_p,n}(R^*_0 u_0 R^*_1 \ldots u_{t_x-1} R^*_t)$$

$$= Cl_{G_p,n}(R^*_0 u_0) Cl_{G_p,n}(R^*_1 \ldots u_{t_x-1} R^*_t)$$

$$= Cl_{G_p,n}(R^*_0 u_0) Cl_{G_p,n}(R^*_1) \ldots u_{t_x-1} Cl_{G_p,n}(R^*_t)$$

$$= Cl_{G_p,n}(R^*_0 u_0) Cl_{G_p,n}(R^*_1) \ldots u_{t_x-1} Cl_{G_p,n}(R^*_t)$$

$$= Cl_{G_p,n}(R^*_0 u_0) Cl_{G_p,n}(R^*_1) \ldots u_{t_x-1} Cl_{G_p,n}(R^*_t)$$

$$= Cl_{G_p,n}(R^*_0 u_0) Cl_{G_p,n}(R^*_1) \ldots u_{t_x-1} Cl_{G_p,n}(R^*_t)$$

$$= Cl_{G_p,n}(R^*_0 u_0) Cl_{G_p,n}(R^*_1) \ldots u_{t_x-1} Cl_{G_p,n}(R^*_t)$$

So, for every $p \in S$, $\delta_0(x)$ lies in $Cl_{G_p,n}(L'_{x})$.

By Proposition 3.7, there are $v_{i,x} \in Cl(Y^*_{i,x}) \cap \Omega^*_A S$ such that if $p \in \mathbb{P} \setminus S$, then the pseudoidentity $v_{i,x} = 1$ is valid in $G_p$ and the pseudoidentity

$$v_{0,0} u_{0,0} v_{1,0} \ldots u_{t_x-1,0} v_{t_x,0} = \delta_0(x).$$

is valid in $G_p$ otherwise $(x \in X$ and $1 \leq i \leq t_x)$. Note that since $Cl(Y^*_{i,x}) \subseteq Cl(R^*_i)$, the pseudowords $v_{0,0} u_{0,0} v_{1,0} \ldots u_{t_x-1,0} v_{t_x,0}$ lie in $Cl(L_t)$.

Let $P = \{p_1, \ldots, p_r\} = \mathbb{P} \setminus S$. For every $p_j \in P$, there is a solution $\delta_j : X \rightarrow \Omega^*_A S$ modulo $G_{p_j}$ such that for every $x \in X$, $\delta_j(x)$ lies in $Cl_{G_{p_j},n}(L'_{x})$. 


By Proposition 3.7, there are \( v_{i,x,p_j} \in \text{Cl}(Y^*_{i,x}) \cap \Omega_A^p \mathcal{S} \) such that if \( p = p_j \), then the equality
\[
(3.9) \quad v_{0,i,p_j} u_{0,x} v_{1,x,p_j} \cdots u_{t_x-1,x} v_{t_x,x,p_j} = \delta_j(x)
\]
holds in \( G_p \) and the equality \( v_{i,x,p_j} = 1 \) is valid in \( G_p \) otherwise (\( x \in X \), \( 1 \leq i \leq t_x, 1 \leq j \leq r \)). Note that since \( \text{Cl}(Y^*_{i,x}) \subseteq \text{Cl}(R^*_{i,x}) \), the pseudowords
\[
v_{0,x,p_j} u_{0,x} v_{1,x,p_j} \cdots u_{t_x-1,x} v_{t_x,x,p_j}
\]
lie in \( \text{Cl}(L'_x) \).

We define the function \( \delta': X \to \Omega_A^p \mathcal{S} \) with
\[
\delta'(x) = (v_{0,x} v_{0,x,p_1} \cdots v_{0,x,p_r}) u_{0,x} (v_{1,x} v_{1,x,p_1} \cdots v_{1,x,p_r}) u_{t_x-1,x} (v_{t_x,x} v_{t_x,x,p_1} \cdots v_{t_x,x,p_r}).
\]
We claim that \( \delta' \) is a solution of the system (3.6) modulo \( G_{\text{nil}} \).

It remains to establish the claim. Note that since a closure of a subsemigroup is again a subsemigroup, the pseudoword \( v_{i,x} v_{i,x,p_1} \cdots v_{i,x,p_r} \) lies in \( \text{Cl}(Y^*_{i,x}) \) and so, \( \delta'(x) \) lies in \( \text{Cl}(L'_x) \). We show that the pseudovariety \( G_{\text{nil}} \) satisfies the pseudoidentities \( \delta'(u_i) = \delta'(v_i) \) (1 ≤ \( i \) ≤ \( m \)). By Lemma 3.1, it is enough to show that for every prime number \( p \), the pseudovariety \( G_p \) satisfies the pseudoidentities \( \delta'(u_i) = \delta'(v_i) \) (1 ≤ \( i \) ≤ \( m \)).

First we show that

1. if \( p \in S \), then for every \( x \in X \) the pseudoidentity
\[
(3.10) \quad \delta'(x) = \delta_0(x)
\]
is valid in \( G_p \);
2. if \( p = p_m \in P \setminus S \), then for every \( x \in X \) the pseudoidentity
\[
(3.11) \quad \delta'(x) = \delta_m(x)
\]
is valid in \( G_{p_m} \).

We consider the following cases:

- Let \( p \in S \). Then for every \( x \in X \), the following pseudoidentities hold in \( G_p \):
\[
v_{i,x,p_j} = 1 \quad (1 \leq i \leq t_x \text{ and } 1 \leq j \leq r),
\]
Hence, the following pseudoidentities hold in \( G_p \):
\[
\delta'(x) = v_{0,x} u_{0,x} v_{1,x} u_{t_x-1,x} v_{t_x,x} \delta_0(x).
\]
- Let \( p = p_m \in P \setminus S \). Then the following pseudoidentities hold in \( G_p \):
\[
v_{i,x,p_j} = v_{s,x} = 1 \quad (1 \leq i, s \leq t_x \text{ and } j \neq m).
\]
Hence, the following pseudoidentities hold in \( G_{p_m} \):
\[
\delta'(x) = v_{0,x} u_{0,x} v_{1,x} u_{t_x-1,x} v_{t_x,x} \delta_m(x).
\]
Now we show that for every prime number \( p \), the pseudovariety \( G_p \) satisfies the pseudoidentities \( \delta'(u_i) = \delta'(v_i) \) (1 ≤ \( i \) ≤ \( m \)). We consider the two following cases:
Let $p \in S$. Since $\delta_0$ is the solution of the system (3.6) modulo $G_p$, $\delta_0(x) = \delta'(x)$ ($x \in X$), and $u_i, v_i \in \Omega^X_S$, the pseudovariety $G_p$ satisfies the pseudoidentities $\delta'(u_i) = \delta'(v_i)$ ($1 \leq i \leq m$).

Consider $p_j \in P \setminus S$. Since $\delta_j$ is the solution of the system (3.6) modulo $G_{p_j}$, $\delta_j(x) = \delta'(x)$ ($x \in X$), and $u_i, v_i \in \Omega^X_S$, the pseudovariety $G_{p_j}$ satisfies the pseudoidentities $\delta'(u_i) = \delta'(v_i)$ ($1 \leq i \leq m$).

This proves the theorem. \hfill \Box

Corollary 3.11. The pseudovariety $G_{\text{nil}}$ is $\sigma$-reducible with respect to the systems of equations associated with finite directed graphs.

Proof. The result follows from Corollary 3.9 and the preceding theorem. \hfill \Box

Corollary 3.12. The pseudovariety $G_{\text{nil}}$ is not completely $\sigma$-reducible.

Proof. By the preceding Theorem, it is enough to show that for some prime number $p$, the pseudovariety $G_p$ is not completely $\sigma$-reducible. Let $p$ be an odd prime number, $A = \{a, b\}$ and consider the following equation

$$[x^2a, y^{-1}z^2by] = 1$$

It has been shown that the equation (3.12) does not have solution in the free group [13]. We consider the following constraints:

- $L_x = \{a\}^*$;
- $L_z = \{b\}^*$;
- $L_y = A^*$.

Let $p$ be an odd prime number. We find a solution of the equation (3.12) with the above constraints modulo $G_p$. The pseudowords $x = (a^{w-1})^{2^w}$, $z = (b^{w-1})^{2^w}$ and $y = 1$ are a solutions of this equation modulo $G_p$ because the pseudoidentities $(a^{w-1})^{2^w} = a^{w-1}$ and $(b^{w-1})^{2^w} = b^{w-1}$ hold in $G_p$.

Since $G_p$ is $\sigma$-full, $G_p$ is $\sigma$-reducible for the equation (3.12) if and only if it has a solution $\delta : \overline{\Omega}^X_S \to \overline{\Omega}^A_S$ such that $\psi_{G_p}(\delta(u)) = 1$ and $\psi_{G_p}(\delta(x)) \in \Omega^A_{G_p}$. But as $\Omega^X_{G_p} = \Omega^A_{G_p}$, the equation (3.12) must have a solution in the free group which is contradiction. \hfill \Box

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