ON ABEL CONVERGENT SERIES OF FUNCTIONS
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ABSTRACT
In this paper, we are concerned with Abel uniform convergence and Abel point-wise convergence of series of real functions where a series of functions \( \sum f_n \) is called Abel uniformly convergent to a function \( f \) if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that
\[
|f_x(t) - f(t)| < \varepsilon
\]
For \( 1 - \delta < x < 1 \) and \( \forall t \in X \), and a series of functions \( \sum f_n \) is called Abel point-wise convergent to \( f \) if for each \( t \in X \) and \( \forall \varepsilon > 0 \) there is a \( \delta(\varepsilon, t) \) such that for \( 1 - \delta < x < 1 \)
\[
|f_x(t) - f(t)| < \varepsilon.
\]

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1 INTRODUCTION

Firstly, we give some notations and definitions in the following. Throughout this paper, \( N \) will denote the set of all positive integers. We will use boldface \( p, r, w, \ldots \) for sequences \( p = (p_n), \ r = (r_n), \ w = (w_n), \ldots \) of terms in \( R \), the set of all real numbers. Also, \( s \) and \( c \) will denote the set of all sequences of points in \( R \) and the set of all convergent sequences of points in \( R \), respectively.

A sequences \( (p_n) \) of real numbers is called Abel convergent (or Abel summable), (See [1,3]), to \( \ell \) if for \( 0 \leq x < 1 \) the series \( \sum_{k=0}^{\infty} p_k x^k \) is convergent and

\[
\lim_{x \to 1^-} (1 - x) \sum_{k=0}^{\infty} p_k x^k = \ell
\]

Abel proved that if \( \lim_{n \to \infty} p_n = \ell \), then \( \text{Abel} \lim_{n \to \infty} p_n = \ell \) (Abel).

A series \( \sum_{n=0}^{\infty} p_n \) of real numbers is called Abel convergent series (See [1,3]), (or Abel summable) to \( \ell \) if for \( 0 \leq x < 1 \) the series \( \sum_{k=0}^{\infty} p_k x^k \) is convergent and

\[
\lim_{x \to 1^-} (1 - x) \sum_{k=0}^{\infty} S_k x^k = \ell, \text{ where } S_n = \sum_{k=0}^{n} p_k
\]

In this case we write \( \text{Abel} \sum_{n=0}^{\infty} p_n = \ell \). Abel proved that if \( \lim_{n \to \infty} \sum_{k=0}^{n} = \ell \), then \( \text{Abel} \sum_{n=0}^{\infty} p_n = \ell \) (Abel), i.e. every convergent series is Abel summable. As we know the converse is false in general, e.g. \( \text{Abel} \sum_{n=0}^{\infty} (-1)^n = \frac{1}{2} \) (Abel), but \( \sum_{n=0}^{\infty} (-1)^n \neq \frac{1}{2} \).

2 RESULTS

We are concerned with Abel convergence of sequences of functions defined on a subset \( X \) of the set of real numbers. Particularly, we introduce the concepts of Abel uniform convergence and Abel point-wise convergence of series of real functions and observe that Abel uniform convergence inherits the basic properties of uniform convergence.

Let \( (f_n) \) be a sequences of real functions on \( X \) and for all \( t \in X \) let \( f_n(t) = (1 - x) \sum_{n=0}^{\infty} S_n(t) x^n \), where \( S_n(t) = \sum_{k=0}^{n} f_k(t) \).

**Definition 2.1** A series of functions \( \sum f_n \) called Abel point-wise convergent to a function \( f \) if for each \( t \in X \) and \( \forall \varepsilon > 0 \) there is a \( \delta(\varepsilon, t) \) such that for \( 1 - \delta < x < 1 \)

\[
|f_n(t) - f(t)| < \varepsilon.
\]

In this case we write \( \sum f_n \to f \) (Abel) on \( X \).

It is easy to see that any point-wise convergent sequence is also Abel point-wise convergent. But the converse is not always true as being seen in the following example.

**Example 2.1** Define \( f_n : [0,1] \to R \) by

\[
f_n(t) = (-1)^n = \begin{cases} 1, & n \in N \text{ and } n \text{ odd;} \\ -1, & n \in N \text{ and } n \text{ even} \end{cases}
\]

and

\[
S_n(t) = \begin{cases} 0, & n \text{ odd;} \\ 1, & n \text{ even} \end{cases}
\]

Then, for every \( \varepsilon > 0 \),

\[
\left| (1 - x) \sum_{n=0}^{\infty} (S_n(t) - \frac{1}{2}) x^n \right| < \varepsilon.
\]

Hence

\[
\lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} S_n(t) x^n = \frac{1}{2}
\]

So \( \sum f_n \) is Abel point-wise convergent to \( \frac{1}{2} \) on \([0,1]\). But observe that \( \sum f_n \) is not point-wise on \([0,1]\).
**Definition 2.2** A series of functions $\sum f_n$ is called Abel uniform convergent to a function $f$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f_x(t) - f(t)| < \varepsilon$$

for $1 - \delta < x < 1$ and $\forall t \in X$.

In this case we write $\sum f_n \rightarrow f$ (Abel) on $X$.

The sequence is equicontinuous if for every $\varepsilon > 0$ and every $x \in X$, there exists a $\delta > 0$, such that for all $n$ and all $x' \in X$ with $|x' - x| < \delta$ we have

$$|f_n(x') - f_n(x)| < \varepsilon.$$

The next result is a Abel analogue of a well-known result.

**Theorem 2.1** Let $(f_n)$ be equicontinuous on $X$. If a series of functions $\sum f_n$ converges Abel uniform to a function $f$ on $X$, then $f$ is continuous on $X$.

**Proof.** Let $t_0$ be an arbitrary point of $X$. By hypothesis $\sum f_n \rightarrow f$ (Abel) on $X$. Then, for every $\varepsilon > 0$, there is a $\delta_1 > 0$ such that $1 - \delta_1 < x < 1$ implies $|f_x(t) - f(t)| < \frac{\varepsilon}{3}$ and $|f_x(t_0) - f(t_0)| < \varepsilon$ for each $t \in X$. Since $f_n$ is quicontinuous at $t_0 \in X$, there is a $\delta_2 > 0$ and $n \in N$ such that $|t - t_0| < \delta_2$ implies $|f_n(t) - f_k(t_0)| < \frac{\varepsilon}{3n}$ for each $t \in X$, so

$$|f_x(t) - f_x(t_0)| = |(1 - x)\sum_{n=0}^{\infty} S_n(t) x^n - (1 - x)\sum_{n=0}^{\infty} S_n(t_0) x^n|$$

$$= |(1 - x)\sum_{n=0}^{\infty} (S_n(t) - S_n(t_0)) x^n|$$

$$\leq (1 - x)\sum_{n=0}^{\infty} |S_n(t) - S_n(t_0)| x^n$$

$$\leq (1 - x)\sum_{n=0}^{\infty} \frac{\varepsilon}{3} x^n = \frac{\varepsilon}{3}.$$

Now for all $0 < x < 1$, for $\delta = \min(\delta_1, \delta_2)$ and for all $t \in X$ for which $|t - t_0| < \delta$, we have

$$|f(t) - f(t_0)| = |f(t) - f_x(t) + f_x(t) - f_x(t_0) + f_x(t_0) - f(t_0)|$$

$$\leq |f(t) - f_x(t)| + |f_x(t) - f_x(t_0)| + |f_x(t_0) - f(t_0)| < \varepsilon.$$

Since $t_0 \in X$ is arbitrary, $f$ is continuous on $X$.

The next example shows that neither of the converse of Theorem 2.1 is true.

**Example 2.2** Define $f_n : [0, 1] \rightarrow R$ by

$$f_n(t) = n^2 t (1 - t)^n$$

Then we have $\sum f_n : [0, 1] \rightarrow f = 0$ (Abel) on $[0, 1]$. Though all $f_n$ and $f$ are continuous on $[0, 1]$, it follows from Definition 2.2 that the Abel point-wise convergence of $(f_n)$ is not uniform, since

$$c_n = \max_{0 \leq s \leq 1} |\sum_{k=0}^{n} f_k(s) - f(s)| = \infty$$

and Abel-$\lim c_n = \infty \neq 0$.

The following result is a different form of Dini’s theorem.

**Theorem 2.2** Let $X$ be compact subset of $R$, $(f_n)$ be a sequence of continuous functions on $X$. Assume that $f$ is continuous and $\sum f_n \rightarrow f$ (Abel) on $X$. Also let $\sum_{k=0}^{n} f_k$ be monotonic decreasing on $X$ ; $\sum_{k=0}^{n} f_k(t) \geq \sum_{k=0}^{n+1} f_k(t)$ for all $t \in X$. Then

$$\sum_{k=0}^{\infty} f_k(t) \rightarrow f(t)$$

for all $t \in X$. If in addition $\sum_{k=0}^{n} f_k(t)$ is continuous for all $n$ then so is $f(t)$, and the converse is also true.
(n = 1,2,3, ... ) for every t ∈ X. Then ∑f_n ⇒ f (Abel) on X.

Proof. Put h_n(t) = ∑_{k=0}^{n} (f_k(t) - f(t)). By hypothesis, each h_n is continuous and h_n → 0 (Abel) on X, also h_n is a monotonic decreasing sequence on X. Since continuous functions h_n on set compact X, it is bounded on X. As all a series of functions h_n is bound and monotonic decreasing, it is pointwise convergence for all a t ∈ X. Since h_n is Abel pointwise to zero for all a t ∈ X, it find pointwise converge to zero for all a t ∈ X. Hence for every ε > 0 and each t ∈ X there exists a number n(t) := n(ε,t) ∈ N such that 0 ≤ h_n(t) < ε for all n ≥ n(t).

Since h_n(t) is continuous a t ∈ X for every ε > 0, there is an open set V(t) which contains t such that |h_n(t)(t) - h_n(t)(t)| < ε/2 for all ℓ ∈ V(t). Hence for given ε > 0, by monotonicity we have

0 ≤ h_n(ℓ) ≤ h_n(t)(ℓ) = h_n(t)(t) - h_n(t)(t) + h_n(t)(t) < ε

for every ℓ ∈ V(t) and for all n ≥ n(t). Since X ⊂ U_{t∈X} V(t) and it is compact set, by the the Heine Borel theorem it has a finite open covering as

X ∈ V(t_1) ∪ V(t_2) ... U V(t_m).

Now, let N = max{n(t_1), n(t_2), n(t_3), ..., n(t_m)}. Then 0 ≤ h_n(ℓ) < ε for every t ∈ X and for all n ≥ N. So ∑f_n ⇒ f (Abel) on X.

Using Abel uniform convergence, we can also get some applications. We merely state the following theorems and omit the proofs.

**Theorem 2.3** If a series function sequence ∑f_n converges Abel uniformly on [a, b] to a function f on [a, b] and each f_n is integral on [a, b], then f is integral on [a, b]. Moreover,

\[ \lim_{x \to a^+} f(x) = f(ax) dt = f(b) dt \]

**Theorem 2.4** Suppose that ∑f_n is a function series such that each (f_n) has a continuous derivative on [a, b]. If ∑f_n → f on [a, b] and ∑f_n ⇒ g (Abel) on [a, b], then ∑f_n ⇒ f (Abel) on [a, b], where f is differentiable and f' = g.

3 FUNCTIONS SERIES THAT PRESERVE ABEL CONVERGENCE

Recall that a function sequence (f_n) is called convergence-preserving (or conservative) on X ⊂ R if the transformed sequence (f_n(p_n)) converges for each convergent sequence p = (p_n) from X (see [4]). In this section, analogously, we describe the function sequences which preserve the Abel convergence of sequences. Our arguments also give a sequential characterization of the continuity of Abel limit functions of Abel uniformly convergent function series. First we introduce the following definition.

**Definition 3.1** Let X ⊂ R and let ∑f_n be a series of real functions, and f a real function on X. Then series of functions ∑f_n is called Abel preserving Abel convergence (or Abel conservative) on X, if it transforms Abel convergent sequences to Abel convergent sequences, i.e. series of functions ∑f_n = (p_n) is Abel convergent to f(ℓ) whenever (p_n) is Abel convergent to f. If series of functions ∑f_n is Abel conservative and preserves the limits of all Abel convergent sequences from X, then series of functions ∑f_n is called Abel regular on X.

Hence, if series of functions ∑f_n is conservative on X, then series of functions ∑f_n is Abel conservative on X. But the following example shows that the converse of this result is not true.

**Example 3.1** Let f_n: [0,1] → R defined by

\[ f_n(t) = (-1)^n n = \begin{cases} 
-1/n, & n \text{ odd}; \\
1/n, & n \text{ even} 
\end{cases} \]

and

\[ S_n(t) = \begin{cases} 
-n + 1/2, & n \in N \text{ and } n \text{ odd}; \\
2/n, & n \in N \text{ and } n \text{ even} 
\end{cases} \]

Suppose that (w_n ) is an arbitrary sequence in [0,1] such that \( \lim_{x \to 1^{-}}(1 - x)\sum_{n=0}^{\infty} w_n(t)x^n = L \). Then, for every ε > 0, \( (1 - x)\sum_{n=0}^{\infty} (S_n(w_n) - (-1/4^2)x^n) < \epsilon \). Hence \( \lim_{x \to 1^{-}}(1 - x)\sum_{n=0}^{\infty} S_n(w_n) = -1/4 \). So \( \sum f_n \) is Abel conservative on [0,1]. But observe that \( \sum f_n \) is not conservative on [0,1]. The next well-known theorem plays an important role in the proof of Theorem 3.2.
Theorem 3.1 If the series \( \sum_{n=0}^{\infty} f_n \) is Abel pointwise convergent to \( f \) on \( X \) and \( f_n(t) \geq 0 \) for \( n \) sufficiently large for all \( t \in X \) then \( \sum_{n=0}^{\infty} f_n \) converges to \( f \) for all \( t \in X \).

Proof. There exists \( n_0 \) such that if \( n > n_0 \) then \( f_n(t) > 0 \) for all \( t \in X \). Thus the \( (S_n)_{n=0}^{\infty} \) is an increasing sequence if \( S_n \) is bounded then \( \sum_{n=0}^{\infty} f_n(t) = f(t) \) for all \( t \in X \). So for all \( t \in X \)

\[
\lim_{x \to 1^{-1}} (1-x) \sum_{k=0}^{\infty} f_k(t) x^k = \sum_{k=0}^{\infty} f_k(t)
\]

If \( S_n \) is not bounded \( \lim_{n \to \infty} S_n = \infty \), so \( \sum_{n=0}^{\infty} f_n(t) \) is not Abel point-wise convergent for all \( t \in X \) (which contradicts the hypothesis).

Now we are ready to prove the following theorem.

Theorem 3.2 Let \( (f_n) \) be a sequence of nonnegative functions defined on a closed interval \([a, b] \subseteq R\), \( a, b > 0 \). Then a series of nonnegative functions \( \sum f_n \) is Abel conservative on \([a, b]\) if and only if a series of nonnegative functions \( \sum f_k \) converges Abel uniformly on \([a, b]\) to a continuous function.

Proof. Necessity. Assume that a series of nonnegative functions \( \sum f_n \) is Abel conservative on \([a, b]\). Choose the sequence \((r_n) = (r, r, \ldots)\) for each \( r \in [a, b] \). Since \( A - \lim_{n \to \infty} r_n = r \), \( A = \lim_{n \to \infty} S_n(r_n) \) exists, hence \( A - \lim_{n \to \infty} S_n(r) = f(r) \) for all \( r \in [a, b] \). We claim that \( f \) is continuous on \([a, b]\). To prove this we suppose that \( f \) is not continuous at a point \( p_0 \in [a, b] \). Then there exists a sequence \((p_k)\) in \([a, b]\) such that \( \lim_{k \to \infty} p_k = p_0 \), but \( \lim f(p_k) \) exists and \( \lim f(p_k) = L \neq f(p_0) \). Since a series of nonnegative functions \( \sum f_k \) is Abel pointwise convergent to \( f \) on \([a, b] \), we obtain \( \sum f_n \to f (Abel) \) on \([a, b]\) from Theorem 3.1. Hence we write,

\[
\begin{align*}
\text{for } k = 1 & \Rightarrow \lim_{x \to 1^{-1}} (1-x) \sum_{n=0}^{\infty} (S_n(p_1) - f(p_1) x^n) = 0 \Rightarrow \lim_{n \to \infty} S_n(p_1) = f(p_1) \\
\text{for } k = 2 & \Rightarrow \lim_{x \to 1^{-1}} (1-x) \sum_{n=0}^{\infty} (S_n(p_2) - f(p_2) x^n) = 0 \Rightarrow \lim_{n \to \infty} S_n(p_2) = f(p_2) \\
\text{for } k = 3 & \Rightarrow \lim_{x \to 1^{-1}} (1-x) \sum_{n=0}^{\infty} (S_n(p_3) - f(p_3) x^n) = 0 \Rightarrow \lim_{n \to \infty} S_n(p_3) = f(p_3) \\
& \quad \quad \quad \quad \vdots \\
\text{for } k = j & \Rightarrow \lim_{x \to 1^{-1}} (1-x) \sum_{n=0}^{\infty} (S_n(p_j) - f(p_j) x^n) = 0 \Rightarrow \lim_{n \to \infty} S_n(p_j) = f(p_j).
\end{align*}
\]

Now, by the “diagonal process” as in [5] and [6]

\[
|1-x| \sum_{n=0}^{\infty} (S_n(p_n) - f(p_n)) x^n | \leq |(1-x) \sum_{n=0}^{\infty} (S_n(p_n) - f(p_n)) x^n |
\]

So we have

\[
\lim_{x \to 1^{-1}} (1-x) \sum_{n=0}^{\infty} (S_n(p_n) - f(p_n)) x^n = 0
\]

(3.1)

Then,

\[
\sum_{n=0}^{\infty} S_n(p_n)x^n = \sum_{n=0}^{\infty} (S_n(p_n) - f(p_n)) x^n + \sum_{n=0}^{\infty} f(p_n) x^n
\]

and hence from (3.1) one obtains

\[
\lim_{x \to 1^{-1}} (1-x) \sum_{n=0}^{\infty} S_n(p_n) x^n = \lim_{x \to 1^{-1}} (1-x) \sum_{n=0}^{\infty} f(p_n) x^n
\]

If \( \lim f(p_n) = L \), then

\[
\lim_{x \to 1^{-1}} (1-x) \sum_{n=0}^{\infty} f(p_n) x^n = L
\]

So we find that

\[
\lim_{x \to 1^{-1}} (1-x) \sum_{n=0}^{\infty} S_n(p_n) x^n = L.
\]

(3.2)

Hence series of nonnegative functions \( \sum_{n=0}^{\infty} f_n(p_n) \) is not Abel convergent since the series of functions \( \sum_{n=0}^{\infty} f_n(p_n) \) has two different limit value. So, the series of nonnegative functions \( \sum f_n(p_n) \) is not Abel convergent.
convergent, which contradicts the hypothesis. Thus \( f \) must be continuous on \([a, b]\). It remains to prove that series of nonnegative functions \( \sum f_n \) converges Abel uniformly on \([a, b]\) to \( f \). Assume that a series of functions \( \sum f_n \) is not Abel uniformly convergent to \( f \) on \([a, b]\). Hence there exists a number \( \epsilon_0 > 0 \) and numbers \( r_n \in [a, b] \) such that
\[
\left| (1 - x) \sum_{n=0}^{m} (S_n(r_n) - f(r_n))x^n \right| > 2\epsilon_0.
\]
We obtain from Theorem 3.1 that \( |S_n(r_n) - f(r_n)| \geq 2\epsilon_0 \). The bounded sequence \( r = (r_n) \) contains a convergent subsequence \( (r_{n_i}) \), \( \lim_{i \to \infty} (1 - x) \sum_{n=0}^{m} r_{n_i} x^n = \alpha \), say. By the continuity of \( f \),
\[
\lim f(r_{n_i}) = f(\alpha).
\]
So there is an index \( i_0 \) such that \( |f(r_{n_i}) - f(\alpha)| < \epsilon_0 \), \( i \geq i_0 \). For the same \( i \)'s, we have
\[
\left| (1 - x) \sum_{i=0}^{n} (S_i(r_{n_i}) - f(\alpha))x^n \right| \geq \left| (1 - x) \sum_{i=0}^{n} (S_i(r_{n_i}) - f(r_{n_i}))x^n \right| - \left| (1 - x) \sum_{i=0}^{n} (f(r_{n_i}) - f(\alpha))x^n \right| \geq \epsilon_0.
\]
Hence a series of nonnegative functions \( \sum f_n(r_{n_i}) \) is not Abel convergent, which contradicts the hypothesis. Thus a series of nonnegative functions \( \sum f_n \) must be Abel uniformly convergent to \( f \) on \([a, b]\).

**Sufficiency.** Assume that \( \sum f_n = f \) (Abel) on \([a, b]\) and \( f \) is continuous. Let \( p = (p_n) \) be a Abel convergent sequence in \([a, b]\) with \( \Lambda \lim \sum f_n = p_0 \). Since Theorem 3.1 and \( \sum f_n = f \) (Abel) on \([a, b]\) and, we obtain that \( \lim p_n = p_0 \). Since \( \lim p_n = p_0 \) and \( f \) is continuous, we obtain that there is \( A = \lim f(p_n) \) and let
\[
\Lambda \lim f(p_n) = f(p_0).
\]
Let \( \epsilon > 0 \) be given. We write \( \left| (1 - x) \sum_{n=0}^{m} (f_n(t) - f(t))x^n \right| < \frac{\epsilon}{2} \) for every \( t \in [a, b] \). Hence taking \( t = (p_n) \) we have
\[
\left| (1 - x) \sum_{n=0}^{m} (f_n(p_n) - f(p_0))x^n \right| \leq \left| (1 - x) \sum_{n=0}^{m} (f_n(p_n) - f(p_0))x^n \right| + \left| (1 - x) \sum_{n=0}^{m} (f_n(p_n) - f(p_0))x^n \right| < \epsilon.
\]
This shows that \( \sum f_n(p_n) = f(p_0) \) (Abel), whence the proof follows.

Theorem 3.2 contains the following necessary and sufficient condition for the continuity of Abel limit functions of function series that converge Abel uniformly on a closed interval.

**Theorem 3.3** Let \( \sum f_k \) be a series of nonnegative functions that converges Abel uniformly on a closed interval \([a, b] \), \( a, b \geq 0 \) to a function \( f \). The A-limit function \( f \) is continuous on \([a, b] \) if and only if the series of nonnegative functions \( \sum f_k \) is Abel conservative on \([a, b] \).

Now, we study the Abel regularity of function series. If series of nonnegative functions \( \sum f_k \) is Abel regular on \([a, b] \), then obviously \( \Lambda \lim \sum f_k(t) = t \) for all \( t \in [a, b] \), \( a, b > 0 \). So, taking \( f(t) = t \) in Theorem 3.2, we immediately get the following result.

**Theorem 3.4** Let \( \sum f_k \) be a series of nonnegative functions on \([a, b] \), \( a, b > 0 \). Then series of nonnegative functions \( (f_k) \) is Abel regular on \([a, b] \) if and only if series of nonnegative functions \( \sum f_k \) is Abel uniformly convergent on \([a, b] \) to the function defined by \( f(t) = t \).

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