Chern-Weil theory for Haefliger-singular foliations

Lachlan E. MacDonald  
Australian Institute for Machine Learning  
The University of Adelaide  
Adelaide, SA, 5000

Benjamin McMillan  
School of Mathematical Sciences  
The University of Adelaide  
Adelaide, SA, 5000

March 2022

Abstract

We give a Chern-Weil map for the Gel’fand-Fuks characteristic classes of Haefliger-singular foliations, those foliations defined by smooth Haefliger structures with dense regular set. Our characteristic map constructs, out of singular geometric structures adapted to singularities, explicit forms representing characteristic classes in de Rham cohomology. The forms are functorial under foliation morphisms. We prove that the theory applies, up to homotopy, to general smooth Haefliger structures: subject only to obvious necessary dimension constraints, every smooth Haefliger structure is homotopic to a Haefliger-singular foliation, and any morphism of Haefliger structures is homotopic to a morphism of Haefliger-singular foliations. As an application, we provide a generalisation to the singular setting of the classical construction of forms representing the Godbillon-Vey invariant.

1 Introduction

In this paper, we give a Chern-Weil construction of the Gel’fand-Fuks characteristic classes of certain singular foliations in terms of singular metrics and connections. Our methods apply specifically to singular foliations arising from Haefliger structures [24], which are the closest singular cousins of regular foliations, as their leaves are still locally determined by complete families of first integrals. We show that this class is sufficiently general for our theory to apply (up to homotopy) to all smooth Haefliger structures on sufficiently high-dimensional manifolds. Our work opens the way for further study of the topology of Haefliger’s classifying space with all the flexibility afforded by singularities (cf. [47]), and opens up the potential advancement of the study of singular foliations via noncommutative geometry and index theory, which has made great strides in recent years [2, 3, 4, 5].

Let us briefly recall some of the history of foliation characteristic classes most relevant to our work. A regular foliation of codimension \( q \) on an \( n \)-manifold \( M \) is given by an involutive subbundle \( TF \subset TM \) of rank \( n-q \). By the Frobenius theorem, the data \( TF \) is equivalent to a decomposition \( \mathcal{F} \) of \( M \) as a union of non-intersecting, immersed submanifolds of dimension \( n-q \). Assuming the foliation is transversely orientable (i.e. that the normal bundle \( \nu \mathcal{F} := TM/TF \) is an orientable vector bundle), the subbundle \( TF \) can alternatively be regarded as the kernel of a nowhere vanishing, decomposable \( q \)-form \( \omega = \omega^1 \wedge \ldots \wedge \omega^q \) on \( M \) that is integrable, in the sense that there exists a 1-form \( \eta \) for which \( d\omega = \eta \wedge \omega \).
The starting point for the tremendous advances made in understanding regular foliations via their characteristic classes was the discovery of Godbillon and Vey [22] that, for codimension 1 foliations, the associated 3-form $\eta \wedge d\eta$ is closed, and its class in de Rham cohomology (now called the Godbillon-Vey class) depends only on $\mathcal{F}$, and not on the choices of $\eta$ and $\omega$. Nontriviality of the Godbillon-Vey class was first shown by Roussarie (also in [22]), while in [47] Thurston gave a construction exhibiting continuous variation of the Godbillon-Vey class for a family of foliations of the 3-sphere.

It was discovered by Bott [9] that the Godbillon-Vey class of codimension 1 foliations is the simplest example of a large family of so-called secondary characteristic classes associated to regular foliations of arbitrary codimension, transversely orientable or not. More precisely, Bott showed in [8] that the normal bundle of any regular codimension $q$ foliation admits connections that are flat along leaves (now called Bott connections), and proved as an easy consequence that any Pontryagin polynomial of degree more than $2q$ vanishes when evaluated on the curvature of any Bott connection. Bott deduced that certain Chern-Simons transgression forms of such polynomials-in-curvature therefore define de Rham cohomology classes; in particular, for a codimension 1 foliation, the Godbillon-Vey class arises as a transgression of the square of the first Pontryagin form. These characteristic classes, being associated to the normal bundle of the foliation, can be regarded as characteristic classes for the leaf space of the foliation.

Bott’s Chern-Weil account of the characteristic classes of foliations coincided with deep work by Gel’fand and Fuks studying the (continuous) cohomology of infinite-dimensional Lie algebras of vector fields [18, 19, 20, 21]. Gel’fand and Fuks in particular computed the O$(q)$-basic cohomology $H^*(A(a_q), \mathcal{O}(q))$ of the Lie algebra $a_q$ of $\infty$-jets at zero of vector fields on $\mathbb{R}^q$, which is now frequently referred to simply as the “Gel’fand-Fuks cohomology”. They discovered that, in addition to encoding the usual Pontryagin classes of real vector bundles, this Lie algebra cohomology automatically encodes Bott’s vanishing theorem and the consequent secondary characteristic classes, and is isomorphic to the cohomology of a well-understood finite-dimensional subalgebra $WO_q \subset A^*(a_q)$. The subalgebra $WO_q$ furthermore factors through the 2-jets of vector fields. The relationship between Gel’fand-Fuks cohomology and the characteristic classes of foliations was formalised by Bott and Haefliger [11, 10], drawing on work of Kobayashi regarding higher order frame bundles [33].

One of the key properties of vector bundle characteristic classes is functoriality under pullbacks. One must expect the same to hold for the characteristic classes of foliations, but immediately the problem arises that a smooth map $\phi : N \to M$ into a regularly foliated manifold $M$ does not in general pull back a regular foliation. More precisely, by the Frobenius theorem, a regular foliation of $M$ is determined by an open covering $U := \{U_\alpha\}_{\alpha \in A}$ together with submersive first integrals $f_\alpha : U_\alpha \to \mathbb{R}^q$ whose level sets define the leaves of the foliation in $U_\alpha$. The “pullback foliation” of $N$, defined by the open covering $\{\phi^{-1}(U_\alpha)\}_{\alpha \in A}$ and functions $f_\alpha \circ \phi : \phi^{-1}(U_\alpha) \to \mathbb{R}^q$, admits singularities wherever $\phi$ is not transverse to leaves. So, that the characteristic classes of foliations be functorial necessitates their definition for singular foliations of this nature.

In an abstract sense, this lack of functoriality was solved by Haefliger, whose brilliant insight in [24] was to categorify codimension $q$ foliations, regular or not, by considering the transition
functions between local first integrals as fundamental. Since Haefliger’s insight is essential to our work, we record his definition here.

Definition 1.1. Let $X$ be a topological space. A Haefliger cocycle of codimension $q$ on $X$ consists of an open covering $\{U_\alpha\}_{\alpha \in A}$ of $X$ together with continuous functions $f_\alpha : U_\alpha \to \mathbb{R}^q$, called Haefliger charts, and for each $x \in U_\alpha \cap U_\beta$, the transition function $h^x_{\alpha\beta}$, a local diffeomorphism of $\mathbb{R}^q$ defined on a neighbourhood of $f_\beta(x)$, such that for all indices $\alpha, \beta$ and $x \in U_\alpha \cap U_\beta$,

1. the assignment $x \mapsto h^x_{\alpha\beta}$ is continuous, in the sense that the map $(y, \tilde{s}) \mapsto h^y_{\alpha\beta}(\tilde{s})$ is continuous for $(y, \tilde{s})$ near $(x, f_\beta(x)) \in X \times \mathbb{R}^q$,
2. $f_\alpha = h^x_{\alpha\beta} \circ f_\beta$ on some open neighbourhood of $x$ in $U_\alpha \cap U_\beta$, and
3. the transition functions satisfy the following cocycle condition: for all $x \in U_\alpha \cap U_\beta \cap U_\delta$, one has $h^x_{\alpha\beta} \circ h^x_{\beta\delta} = h^x_{\alpha\delta}$ as germs.

Two Haefliger cocycles are equivalent if there exists a third Haefliger cocycle refining them, and a Haefliger structure is an equivalence class of Haefliger cocycles.

Of particular concern to Haefliger were the numerable Haefliger structures, those which may be represented by a cocycle defined over an open cover admitting a subordinate partition of unity. This class includes all Haefliger structures on smooth manifolds and, in particular, all regular foliations. Haefliger showed that the category $\mathcal{H}^q$, whose objects are numerable Haefliger structures of codimension $q$ and whose morphisms are continuous maps pulling back the structure on the codomain to that on the domain, admits a terminal object $(B\Gamma_q, \gamma)$ [25, Theorem 7], the classifying space of codimension $q$ Haefliger structures (here $\Gamma_q$ is the groupoid of germs of local diffeomorphisms of $\mathbb{R}^q$). As an application of this fact, he showed that there exists a universal characteristic map $H^*(A(a_q), O(q)) \to H^*(B\Gamma_q)$ from Gel’fand-Fuks cohomology to the cohomology of $(B\Gamma_q, \gamma)$, thereby furnishing a characteristic map for all numerable Haefliger structures of codimension $q$ [26, p. 53], which is fully functorial.

Haefliger’s categorical lens enables us to systematically expose the limits of the existing characteristic class theory of foliations as follows. Let $\mathcal{T}$ denote the category whose objects are pairs $(X, c)$, where $X$ is a topological space and $c$ is a cohomology class on $X$, and whose morphisms are continuous maps pulling back the class associated to the codomain to that associated to the domain. Then Haefliger’s characteristic map furnishes an assignment $A^*(a_q)O(q) \times \mathcal{H}^q \to \mathcal{T}$, which is functorial in the second argument and homomorphic (with respect to the ring structures of $A^*(a_q)O(q)$ and of cohomology) in the first. Let us denote by $\mathcal{F}_{reg}$ the category of regular foliations of codimension $q$ on smooth manifolds, whose morphisms are transverse maps pulling back the foliation of the codomain to that of the domain. Then the Frobenius theorem induces an inclusion $\mathcal{F}_{reg} \hookrightarrow \mathcal{H}^q$ of categories, and Bott’s Chern-Weil homomorphism amounts to a similarly homomorphic-functorial assignment $WO_q \times \mathcal{F}_{reg} \to \mathcal{T}$. Bott proves in [9, Theorem 10.16] that his Chern-Weil characteristic map is compatible with that of Haefliger, in the sense that it recovers the characteristic classes coming from Haefliger’s classifying space. Put another
way, Bott’s Chern-Weil characteristic map makes the diagram

$$WO_q \times \mathcal{F}_{\text{reg}}^q \to A^*(a_q)_{O(q)} \times H^q$$

commute. We note that, by the homotopy-invariance of cohomology, both characteristic maps factor through the category $H^q_{\text{htpy}}$, whose objects and morphisms are the homotopy classes of those of $H^q$.

With this diagram in mind, the limits of the existing theory are clear. On the one hand, the category $\mathcal{F}_{\text{reg}}^q$ is not sufficiently large to accurately model the full subcategory of smooth manifolds in $H^q$, even up to homotopy, since there exist manifolds with Haefliger structure not homotopic to a regular foliation. This phenomenon can be seen already by choosing a codimension 1 Haefliger structure on $S^2$, which cannot be homotoped to a regular structure, because $S^2$ admits no regular codimension 1 foliation. For such examples, Bott’s Chern-Weil theory offers no way to probe topology. On the other hand, while Haefliger’s characteristic map works for all Haefliger structures, and the abstract definition allows quick proofs of general properties, this very abstraction makes it difficult to use differential geometric methods for the concrete study of topology, as is typically necessary for finer-grained results [22, 47]. A middle ground is needed, one permitting the construction of characteristic classes for all Haefliger structures on manifolds, yet in terms of differential geometry. This gap in the literature has stood for almost five decades.

Our primary result is the provision of such a middle ground. Our solution is an extension of Bott’s Chern-Weil theory to a natural class of singular foliations, which we now define, restricting ourselves to smooth Haefliger cocycles on smooth manifolds (those whose Haefliger charts and transition functions are smooth).

**Definition 1.2.** The singular set $\Sigma$ of a smooth Haefliger cocycle $q$ on a manifold $M$ of dimension at least $q$ is the closed set of all critical points of each $f_\alpha$; the singular set depends only on the associated Haefliger structure. The regular set is the complement of $\Sigma$, which we often denote by $\tilde{M} := M - \Sigma$. If $\tilde{M}$ is dense in $M$, we say that the Haefliger cocycle determines a Haefliger-singular foliation $(M, F)$ of codimension $q$, whose regular subfoliation is the regular foliation of $M$ determined by the restriction of the Haefliger cocycle to $\tilde{M}$.

We note that the term “singular foliation” is often used to refer to Stefan-Sussmann singular foliations, namely those defined by integrable families of vector fields [45, 46]. We show in the appendix that every Haefliger-singular foliation (in fact, every smooth Haefliger structure) does indeed define a Stefan-Sussmann singular foliation. However, the converse is not true; for instance, the foliation of $\mathbb{R}^2$ by the integrals of $x\partial_x + y\partial_y$ is Stefan-Sussmann, but admits no nontrivial first integrals in any neighbourhood of the origin, so cannot be associated to a Haefliger structure.

We show that the category $\mathcal{F}_{\text{sing}}^q$ of codimension $q$ Haefliger-singular foliations, whose morphisms are smooth maps pulling back the singular foliation of the codomain to that of the
domain, fits snugly in between $\mathcal{F}^q_{\text{reg}}$ and $\mathcal{H}^q$. In addition, we describe a Chern-Weil map extending that of Bott to Haefliger-singular foliations, expanding the commuting diagram \((\mathbf{1})\) to
\[
\begin{array}{ccc}
WO_q \times \mathcal{F}^q_{\text{reg}} & \xrightarrow{\cong} & WO_q \times \mathcal{F}^q_{\text{sing}} \\
\downarrow & & \downarrow \\
\mathcal{F}^q_{\text{sing}} & \xrightarrow{\cong} & A^*(a_q)_{\Omega(M)} \times \mathcal{H}^q
\end{array}
\]
\[(\mathbf{2})\]

Importantly, we prove that $\mathcal{F}^q_{\text{sing}}$ is large enough so that every object in $\mathcal{H}^q$ whose underlying space is a manifold of dimension at least $q$ is homotopic to a Haefliger-singular foliation, and that every morphism in $\mathcal{H}^q$ between two such objects is homotopic to a morphism in $\mathcal{F}^q_{\text{sing}}$. In this way, our theory provides a complete, Chern-Weil solution to the problem of accessing the Gel’fand-Fuks characteristic classes of Haefliger structures on manifolds in terms of geometric data.

As a further, geometric application of our theory, we describe how the classical algorithm for the geometric construction of the Godbillon-Vey invariant, used extensively for the study of regular foliations \([22, 47]\), extends cleanly to the singular setting.

Section 2 consists of a review of the necessary prerequisites, on Gel’fand-Fuks cohomology and on the Chern-Weil approach to characteristic classes for regular foliations. In Section 3 we prove a smooth (diffeological) generalisation of Haefliger’s classifying theorem, and an associated de Rham-theoretic universal characteristic map. This is the most complete and explicit account of the universal characteristic map that we are aware of, enabling a clean proof that our Chern-Weil construction recovers the correct characteristic classes.

Section 4 gives a new, de Rham-theoretic proof that the Chern-Weil approach to regular foliations recovers the characteristic classes coming from Haefliger’s classifying space. Although the result, originally due to Bott \([9]\), has been known for some decades, our presentation is novel in that it invokes an under-utilised family of fibre bundles, that we call the Haefliger bundles, which are defined over both regular and singular foliations. The Haefliger bundles act as an intermediary between the universal characteristic map and the Chern-Weil characteristic map for regular foliations.

Section 5 consists of an extension of these ideas to singular foliations. We characterise those metric-Bott-connection pairs on the normal bundle of the regular subfoliation of a Haefliger-singular foliation which are adapted to the singularities via the jets of their exponential maps. Such pairs we term adapted geometries. We then prove our Chern-Weil homomorphism for singular foliations.

**Theorem 1.3.** Let $M$ be a Haefliger-singular foliation of codimension $q$. Any adapted geometry on $M$ specifies a unique Chern-Weil homomorphism from $WO_q \subset A^*(a_q)$ to $\Omega^*(M)$. The Chern-Weil homomorphism descends to cohomology to agree with the Haefliger characteristic map from Gel’fand-Fuks cohomology.

We prove in addition that our Chern-Weil homomorphism is functorial under Haefliger-singular maps, namely those smooth maps which pull back a Haefliger-singular foliation of the codomain to a Haefliger-singular foliation of the domain. As an application of our geometric theory, we generalise the classical algorithm for the construction of the Godbillon-Vey invariant,
famously used by Thurston to study the topology of $B\Gamma_q$ [AS], to Haefliger-singular foliations.

Section 5 is concluded by showing that Haefliger-singular foliations with adapted geometries suffice to recover the characteristic classes of all Haefliger structures on manifolds of sufficiently high dimension. That is:

**Theorem 1.4.** All Haefliger-singular foliations admit adapted geometries. Moreover, the category $\mathcal{F}^q_{\text{sing}}$ of codimension $q$ Haefliger-singular foliations with adapted geometries is homotopy-equivalent to the category $\mathcal{H}^q_{\text{man}}$ consisting of codimension $q$ Haefliger structures on smooth manifolds of dimension at least $q$.

1.1 Acknowledgements

LM was supported by the Australian Research Council Discovery Project grant DP200100729. BM was supported by the Australian Research Council Discovery Project grant DP190102360. LM wishes to thank Iakovos Androulidakis for encouraging him to think about the characteristic classes of singular foliations, and Adam Rennie, Alan Carey, Mathai Varghese and David Roberts for helpful discussions. BM wishes to thank Mike Eastwood and Thomas Leistner for helpful discussions.

2 Background

We work under the convention that the set of natural numbers $\mathbb{N}$ includes zero. All manifolds are assumed to be connected, paracompact, Hausdorff, and without boundary, unless otherwise stated.

2.1 Gel’fand-Fuks cohomology

Given a Lie group $G$ with Lie algebra $\mathfrak{g}$, a $G$-differential graded algebra, or $G$-DGA, is a differential graded $\mathbb{R}$-algebra $(A^\bullet, d)$, where $A^\bullet$ carries a smooth, degree 0 action of $G$, with associated Lie derivative defined by

$$L_\xi(a) := \left. \frac{d}{dt} \right|_0 \exp(t\xi) \cdot a, \quad \xi \in \mathfrak{g}, \quad a \in A,$$

and a contraction operator $\iota_\xi : A^\bullet \to A^{\bullet-1}$ for each $\xi \in \mathfrak{g}$ such that $d$ and $\iota$ anticommute up to the Lie derivative,

$$L_\xi = d\iota_\xi + \iota_\xi d.$$

Given any Lie subgroup $K$ of $G$, with Lie algebra $\mathfrak{k}$, the $K$-basic subalgebra of $A^\bullet$ is the differential graded subalgebra

$$A^\bullet_K := \{ a \in A^\bullet : k \cdot a = a \text{ for all } k \in K, \iota_\xi a = 0 \text{ for all } \xi \in \mathfrak{k} \}.$$

We denote the cohomology of $A^\bullet_K$ by $H^* (A, K)$. Our primary example of such will be the Gel’fand-Fuks cohomology, which we turn to presently.
For $0 \leq q, k < \infty$, fix the Lie group $G^k_q$, the $k^{th}$-order jet group of $\mathbb{R}^q$, comprising the $k$-jets at zero of diffeomorphisms of $\mathbb{R}^q$ that fix zero. Let $g^k_q$ denote the Lie algebra of $G^k_q$. As $k$ ranges from 0 to $\infty$, the natural maps between the $G^k_q$ form a projective system of Lie groups. The projective limit $G^\infty_q$ of this system is the infinite order jet group of $\mathbb{R}^q$, which inherits a natural smooth structure [33] Section 7.1. With this smooth structure, $G^\infty_q$ is an infinite-dimensional Lie group, with Lie algebra $\mathfrak{g}^\infty_q$ equal to the projective limit of the projective system of Lie algebras determined by the $g^k_q$.

For $q \geq 1$, denote by $a_q$ the Lie algebra of $\infty$-jets at zero of smooth vector fields on $\mathbb{R}^q$. Endow $a_q$ with a projective limit topology by identifying it as the projective limit of the system $\{a^k_q, \pi_k\}$ of finite-dimensional manifolds of $k$-jets $\alpha^k_q$ at zero of vector fields on $\mathbb{R}^q$, with $\pi_k : a^k_q \to a^{k-1}_q$ the canonical projection. This, plus the Lie bracket on $a_q$ determined by that of vector fields, determines a topological Lie algebra structure on $a_q$ [21].

Any $X \in a_q$ can be realised as the time derivative of a path of $\infty$-jets,

$$X = \left. \frac{d}{dt} \right|_0 j^\infty_k(\varphi_t)$$

where $t \mapsto \varphi_t$, $\varphi_0 = \text{id}$ is the flow of any vector field representing $X$. (There is no issue of completeness of flows here, because the flow is only required in an infinitesimal neighbourhood of 0). The construction also exhibits a natural inclusion $\mathfrak{g}^\infty_q \hookrightarrow a_q$ of Lie algebras, with image characterised as the jets of those vector fields that vanish at 0.

For $k \geq 1$, denote by $A^k(a_q)$ the space of continuous, alternating, multi-linear functionals $\wedge^k a_q \to \mathbb{R}$. The usual Chevalley-Eilenberg formula defines a differential $d : A^k(a_q) \to A^{k+1}(a_q)$, given for $c \in A^k(a_q)$ and $X_0, \ldots, X_k \in a_q$ by

$$dc(X_0, \ldots, X_k) := \sum_{i<j} (-1)^{i+j} c([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

Now, $A^*(a_q)$ is a $G^\infty_q$-differential graded algebra. Indeed, the inclusion $\mathfrak{g}^\infty_q \hookrightarrow a_q$ defines a contraction operator $\iota_\xi : A^*(a_q) \to A^{*-1}(a_q)$ for all $\xi \in \mathfrak{g}^\infty_q$, and the right action of $G^\infty_q$ on $a_q$ defined by

$$\left( \left. \frac{d}{dt} \right|_0 j^\infty_k(\varphi_t) \right) \cdot j^\infty_k(g) = \left. \frac{d}{dt} \right|_0 j^\infty_k(g^{-1} \circ \varphi_t \circ g), \quad j^\infty_k(g) \in G^\infty_q, \quad \left. \frac{d}{dt} \right|_0 j^\infty_k(\varphi_t) \in a_q,$$

induces an action of $G^\infty_q$ on $A^*(a_q)$ compatible with the contraction operator. The $O(q)$-basic cohomology $H^*(A(a_q), O(q))$ of $A^*(a_q)$ is frequently referred to as simply the Gel’fand-Fuks cohomology, and was computed by Gel’fand and Fuks in [21].

We now review the method outlined by Bott [10] for the computation of Gel’fand-Fuks cohomology $H^*(A(a_q), O(q))$, and in the next subsection, how the $O(q)$-basic $A^*(a_q)$-cocycles define characteristic classes. Fix the standard linear coordinates $s^i$ on $\mathbb{R}^q$. For any multi-index $\alpha \in \mathbb{N}^q$ and for $1 \leq i \leq q$, the Dirac-derivative functional on $a_q$ defined by

$$\delta_\alpha^i(X) := (-1)^{|\alpha|} \left. \frac{\partial |\alpha|}{\partial s^\alpha} X^i \right|_0,$$
is an element of $A^1(a_q)$, and by elementary distribution theory, the collection of all such functionals generates $A^1(a_q)$ linearly, hence generates $A^*(a_q)$ as an algebra. The following structure equations then follow from a routine calculation:

\[ d\delta^i + \delta^j_k \wedge \delta^j = 0, \quad d\delta^j_k + \delta^j_k \wedge \delta^k_k = 0, \]

with the Einstein summation convention assumed. Notice that the first of these equations resembles the structure equation for torsion-free affine connections (with the $\delta^i$ playing the role of the components of the solder form, and the $\delta^i_j$ the components of the connection form), while the second resembles the equation defining the curvature $\Delta_i^j := -\delta^j_k \delta^k + \delta^j_k \delta^k_k$ of the connection in terms of the components of the connection form. We will see in the next subsection that this is not a coincidence—these structure equations are the universal structure equations for torsion-free affine connections on manifolds.

Define the $q \times q$ matrices of 1- and 2-forms

\[ \delta := (\delta^i_j), \quad \Delta := (\Delta^i_j), \quad i, j = 1, \ldots, q \]

(5)

corresponding to the functionals $\delta^i_j$ and $\Delta^i_j$ respectively, and denote by $\delta := \delta_s + \delta_o$ and $\Delta := \Delta_s + \Delta_o$ their respective decompositions into symmetric and antisymmetric components.

Let $\mathbb{R}[c_1, \ldots, c_q]_q$ denote the quotient of the polynomial algebra in symbols $c_i$ of degree $2i$ by the ideal of elements consisting of total degree greater than $2q$, and let $\wedge(h_1, h_3, \ldots, h_l)$ denote the exterior algebra generated by symbols $h_i$ of degree $2i - 1$, with $l$ the largest odd integer that is less than or equal to $q$. Equip the graded-commutative algebra

\[ WO_q := \mathbb{R}[c_1, \ldots, c_q]_q \otimes \wedge(h_1, h_3, \ldots, h_l) \]

with the differential $d$ defined by $dc_i = 0$ for all $i$ and $dh_j = c_j$ for all $j$ odd. Then by the results of Ref. 23, $WO_q$ embeds as a differential graded subalgebra of $A^*(a_q)_{O(q)}$ according to the formulæ

\[ c_i := \text{Tr}(\Delta^i), \quad 1 \leq i \leq q, \]

(6)

\[ h_j := j \text{Tr} \left( \int_0^1 \delta_s(t \Delta_s + \Delta_o + (t^2 - 1)\delta_s^2) \wedge (t^{-1}) dt \right), \quad 1 \leq j \leq q, j \text{ odd}. \]

(7)

In fact, the $c_i$ are $GL(\mathbb{R}^q)$-basic, and correspond to the Pontryagin classes of tangent bundles, as we will see in the next subsection. One has the following theorem, which computes the cohomology of the infinite-dimensional algebra $A^*(a_q)_{O(q)}$ in terms of the finite-dimensional subalgebra $WO_q$.

**Theorem 2.1.** [11, Theorem 2] The algebra inclusion $WO_q \hookrightarrow A^*(a_q)_{O(q)}$ induces an isomorphism on cohomology, $H^*(WO_q) \cong H^*(A(a_q), O(q))$. \hfill \Box

### 2.2 Frame bundles and tautological forms

Let $(M, F)$ be a regular foliation of codimension $q$. For $x \in M$, a transverse embedding through $x$ is an embedding $u: \mathbb{R}^q \to M$ such that $u(0) = x$ and for each $\bar{s} \in \mathbb{R}^q$ one has $T_u(\bar{s})M = \mathbb{R}^q$. For $s \in \mathbb{R}$, let $x_s$ denote the point $u(s)$. Then $\{x_s : s \in \mathbb{R}^q\}$ is a locally trivial principal frame bundle 

\[ \mathbb{R}^q 

\]
Two $k$-jets $j^k_0(u_1)$ and $j^k_0(u_2)$ of transverse embeddings through $x$ are said to be leaf space equivalent if for some (hence any) Haefliger chart $f: U \to \mathbb{R}^q$ defined around $x$ one has $j^k_0(f \circ u_1) = j^k_0(f \circ u_2)$. The leaf space equivalence class of a $k$-jet $j^k_0(u)$ will be denoted $j^k_{\Omega}(u)$ (the $\Omega$ being used to denote transverse).

**Definition 2.2.**\[41] Let $(M, \mathcal{F})$ be a regular foliation of codimension $q$, and let $0 \leq k \leq \infty$. The transverse $k$-frame bundle of $(M, \mathcal{F})$ is the principal $G^k_q$-bundle $\operatorname{Fr}_k(M/\mathcal{F}) \to M$ whose fibre over $x \in M$ is the set of leaf space equivalence classes of $k$-jets at $\hat{0}$ of transverse embeddings through $x$.

**Example 2.3.** If $q = \dim(M)$, then $\mathcal{F}$ is a regular foliation of $M$ by points. Haefliger charts are coordinate charts, and this definition recovers the standard definition of the $k$-frame bundle $\operatorname{Fr}_k(M)$ of $M$. In this case, the diffeomorphism group $\operatorname{Diff}(M)$ acts from the left on $\operatorname{Fr}_k(M)$ by postcomposition, and this action commutes with the principal right action of $G^k_q$. Denote by $\Omega^*(\operatorname{Fr}_k(M))^{\operatorname{Diff}(M)}$ the $G^k_q$-DGA of $\operatorname{Diff}(M)$-invariant forms on $\operatorname{Fr}_k(M)$. Using Equation \([9]\), the tautological $a^k_q$-valued 1-form $\omega^k$ defined by the formula

\[
\omega^k_{j^k_0(\hat{u})}(\frac{d}{dt} j^k_0(\hat{u})) := \frac{d}{dt} j^k_{\Omega}(u^{-1} \circ \hat{u}) \quad \text{for} \quad \frac{d}{dt} j^k_0(\hat{u}) \in T_{j^k_0(\hat{u})} \operatorname{Fr}_k(M)
\]

is easily seen to be an element of $\Omega^1(\operatorname{Fr}_k(M))^{\operatorname{Diff}(M)} \otimes a^k_q$. The tautological 1-forms $\omega^k$ were introduced by Kobayashi \[33\].

The $a_q$-valued 1-form $\omega := \omega^\infty$ on $\operatorname{Fr}_\infty(M)$ satisfies the Maurer-Cartan identity $d\omega^\infty + \frac{1}{2}[\omega^\infty, \omega^\infty] = 0$ \[41\] p. 113 (the identities in Equation \([41]\) are low-order manifestations of this fact), has trivial kernel and defines a canonical trivialisation $T\operatorname{Fr}_\infty(M) \cong \operatorname{Fr}_\infty(M) \times a_q$ of the tangent bundle of $\operatorname{Fr}_\infty(M)$. As such, any $c \in A^*(a_q)$ naturally defines a form $c(\omega \wedge \cdots \wedge \omega) \in \Omega^*(\operatorname{Fr}_\infty(M))^{\operatorname{Diff}(M)}$, and this assignment gives rise to a canonical isomorphism

\[
A^*(a_q) \xrightarrow{\omega} \Omega^*(\operatorname{Fr}_\infty(M))^{\operatorname{Diff}(M)}
\]

of $G^\infty_q$-differential graded algebras (cf. \[32\] Proposition 3.5)].

It is known \[31\] p. 131, Proposition that for each finite $k \geq 1$, the natural projection $G^k_q \to \operatorname{GL}(\mathbb{R}^q)$ is a principal fibre bundle, with typical fibre a contractible nilpotent Lie group whose Lie exponential map is a global diffeomorphism. The same is therefore also true of the fibration $G^\infty_q \to \operatorname{GL}(\mathbb{R}^q)$. It follows that $\operatorname{Fr}_\infty(M) \to \operatorname{Fr}_1(M)$ has contractible fibres, and always admits sections.

In particular, letting $\nabla$ be an affine connection on $M$, with exponential map $\exp \nabla$, the formula

\[
\sigma_\nabla(\tilde{e}_x) := j^\infty_0(\tilde{s} \mapsto \exp^\nabla_x (\tilde{s} \cdot \tilde{e}_x)) \quad \tilde{e}_x \in \operatorname{Fr}_1(M)
\]

defines a $G^1_q$-equivariant section $\sigma_\nabla$ of $\operatorname{Fr}_\infty(M) \to \operatorname{Fr}_1(M)$. Recall now the functional $\delta$ and $\Delta$ introduced in Equation \([4]\).

**Proposition 2.4.**\[66\] Lemma 18] Let $\nabla$ be an affine connection on a manifold $M$, and let $\sigma_\nabla: \operatorname{Fr}_1(M) \to \operatorname{Fr}_\infty(M)$ be the section given in Equation \([9]\). Then $(\sigma_\nabla)^*\delta(\omega) \in \Omega^1(\operatorname{Fr}_1(M); \mathfrak{gl}(\mathbb{R}^q))$.
is the connection form associated to $\nabla$ and $(\sigma_\nabla)^* \Delta(\omega) \in \Omega^2(\Fr_1(M) ; \gl(\mathbb{R}^q))$ is its curvature.

It follows immediately from Proposition 2.4 that the forms $(\sigma_\nabla)^* c_i(\omega) \in \Omega^2(\Fr_1(M))_{\GL(\mathbb{R}^q)}$ from Equation (5) are the Pontryagin forms of $M$ defined with respect to the curvature tensor of $\nabla$, which displays the fundamental relationship between Chern-Weil theory and Gel’fand-Fuks cohomology. Moreover, since the $c_i$ and $h_i$ only depend on 2-jets, and $WO_q \rightarrow A(a_q)$ is a quasi-isomorphism, one sees that in cohomology it suffices to work with $\omega^2 \in \Omega^1(\Fr_2(M))_{\Diff(M) \otimes a_1}$.

The facts elucidated in Example 2.3 generalise to the transverse frame bundles of foliations more generally. Specifically, the transverse frame bundles of a regular foliation admit tautological forms, which model geometric structures on the normal bundle of the foliation. This is folklore, but will be made precise after our introduction of Haefliger bundles in Section 3.

2.3 Review of Chern-Weil for regular foliations

In the early nineteen-seventies, R. Bott showed that characteristic classes for a regular foliation $(M, \mathcal{F})$ of codimension $q$ could be obtained in the following manner [9]. Let the normal bundle $\nu \mathcal{F} := T M / T \mathcal{F}$ of $\mathcal{F}$ be regarded as a subbundle of $T M$ that is complementary to $T \mathcal{F}$. (For instance, the orthogonal complement of $T \mathcal{F}$ with respect to some Riemannian metric on $M$.) For a vector field $Z$ on $M$, denote by $Z = Z_\mathcal{F} + Z_\nu$ its decomposition into leafwise and normal components respectively.

Bott showed [8] that there exist connections $\nabla$ on $\nu \mathcal{F}$ satisfying the equation

$$\nabla_X Y = \nabla_{X_\nu} Y + [X_\mathcal{F}, Y]_\nu, \quad X \in \mathfrak{X}(M), \ Y \in \Gamma(\nu \mathcal{F}) \subset \mathfrak{X}(M).$$

Such connections, now called Bott connections, are flat along leaves, in that their curvature forms $R_\nabla$ vanish on restriction to the tangent distribution of leaves. As a consequence, one has Bott’s vanishing theorem: for any Bott connection $\nabla$ on $\nu \mathcal{F}$, the associated Chern-Weil characteristic map $\mathbb{R}[c_1, \ldots, c_q] \rightarrow \Omega^*(M)$ encoding the Pontryagin classes of $\nu \mathcal{F}$, defined on generators by

$$c_i \mapsto \text{Tr}(R_\nabla^i),$$

vanishes on all monomials (in the $c_i$) of degree greater than $2q$. Bott’s vanishing theorem has the following consequence.

Let $\nabla^1$ be a Bott connection on $\nu \mathcal{F}$ and let $\nabla^0$ be a connection on $\nu \mathcal{F}$ that is compatible with some Euclidean metric on $\nu \mathcal{F}$. For $t \in [0, 1]$, denote by $\nabla^t := t\nabla^1 + (1 - t)\nabla^0$ the affine combination of $\nabla^0$ and $\nabla^1$.

**Theorem 2.5** (Bott-Chern-Weil characteristic map). [9, p. 67-69] The map $\lambda_{\nabla, 0} : WO_q \rightarrow \Omega^*(M)$ defined on generators by the formulae

$$\lambda_{\nabla^1, \nabla^0}(c_i) := \text{Tr}(R_{\nabla^1}^i) \in \Omega^{2i}(M),$$

$$\lambda_{\nabla^1, \nabla^0}(h_i) := i \int_0^1 \text{Tr} ((\nabla^1 - \nabla^0) \wedge R_{\nabla^1}^{i-1}) \, dt \in \Omega^{2i-1}(M)$$
is a homomorphism of differential graded algebras whose descent to cohomology does not depend on the Bott connection or metric connection chosen.

Continuing with a regular foliation \((M, \mathcal{F})\), denote projection to the normal bundle \(\nu \mathcal{F}\) by \(p : TM \to \nu \mathcal{F}\). The torsion of a Bott connection \(\nabla\) on \(\nu \mathcal{F}\) is the tensor \(T \in \Gamma^\infty(M; T^* M \otimes T^* M \otimes \nu \mathcal{F})\) defined by

\[
T(X, Y) := \nabla_X (pY) - \nabla_Y (pX) - p[X, Y]
\]

for all \(X, Y \in \mathfrak{X}(M)\). The connection \(\nabla\) is said to be torsion-free if \(T(X, Y) = 0\) for all \(X, Y \in \mathfrak{X}(M)\). Torsion-free Bott connections will play an essential role in our theory.

One way of constructing torsion-free Bott connections is via a choice of Riemannian metric \(g\) on \(M\). Such a choice gives rise to three canonical objects:

1. an identification of \(\nu \mathcal{F}\) with the orthogonal complement of \(T \mathcal{F}\),
2. a Euclidean structure \(\epsilon\) induced by \(g\) on \(\nu \mathcal{F} \subset TM\), defining a section \(M \to Fr(\nu \mathcal{F})/O(q)\), and
3. a torsion-free Bott connection, the Bott Levi-Civita connection \(\nabla\), defined by

\[
\nabla_X Y := [X_F, Y]_\nu + (\nabla^{LC}_X Y)_\nu,
\]

where \(\nabla^{LC}\) is the Levi-Civita connection associated to \(g\).

These geometric structures are used by Guelorget in the definition of an alternative characteristic map to that of Bott [23], to which our own map bears substantial resemblance. Moreover, as will be clear in Section 5, the geometric structures arising from appropriately singular Riemannian metrics can be used in our Chern-Weil map for Haefliger-singular foliations.

In Section 4, we will describe how Bott’s Chern-Weil map for a regular foliation relates to the higher transverse frame bundles of the foliation. This will furnish a new proof of the fact that Bott’s Chern-Weil map recovers the characteristic classes arising from the classifying space \(B\Gamma_q\) of codimension \(q\) Haefliger structures. Following this, in Section 5, we will use the same techniques to give our Chern-Weil characteristic map for singular Haefliger foliations.

3 Classifying spaces and the universal characteristic map

In this section, following a review of some well-known definitions and constructions concerning classifying spaces, we prove a generalisation of Haefliger’s classifying theorem [25, Theorem 7] to the smooth setting. Our proof factors through Mostow’s smooth structure for classifying spaces [42], which we identify with a diffeology that we call the Mostow diffeology. It is via diffeological methods that we are able to view our classifying theorem as an honest generalisation of that of Haefliger, and by which we can give a neat, explicit, de Rham-theoretic, universal characteristic map from Gel’fand-Fuks cohomology to the cohomology of the smooth Haefliger classifying space. To the best of our knowledge, ours is the most explicit (and the only smooth) construction of a universal characteristic map available in the literature. Ultimately, this de
3.1 Classifying spaces

We begin by briefly recalling the framework of diffeology, referring to the book [31] for details. A diffeological space is a set \( X \) with a smooth structure, determined by a collection of plots \( U \to X \), as \( U \) ranges over the open subsets of finite dimensional Euclidean spaces. This collection of plots, called a diffeology, must satisfy three axioms: constant maps are plots, all maps that are locally given by plots are plots, and closure under precomposition by smooth functions between open subsets of Euclidean space. (The map from the empty set is vacuously locally a plot, so a plot). A diffeology on \( X \) determines a topology on \( X \), the D-topology, which is the finest topology for which all plots are continuous—a subset \( U \subset X \) is D-open if and only if \( P^{-1}(U) \) is open for all plots \( P \) of \( X \). Unless otherwise stated, diffeological spaces will always be assumed to carry the D-topology. Conversely, any topological space carries a natural diffeology, called the continuous diffeology, whose plots are the continuous maps.

A map \( f: X_1 \to X_2 \) of diffeological spaces is smooth if \( f \circ P \) is a plot of \( X_2 \) for each plot \( P \) of \( X_1 \). By definition, smooth maps between diffeological spaces are continuous with respect to the underlying D-topologies. Given a subset \( U \subset X_1 \), a map \( f: U \to X_2 \) is locally smooth if \( P^{-1}(U) \) is open and the composition \( P^{-1}(U) \to U \to X_2 \) is a plot of \( X_2 \), for all plots \( P \) of \( X_1 \). To check that \( f: U \to X_2 \) is locally smooth, it suffices to check first that \( U \) is D-open, and then that \( f \) sends the plots of \( X_1 \) with image in \( U \) to plots of \( X_2 \).

Subsets and quotients of diffeological spaces carry natural diffeologies (the subspace and quotient diffeologies), and the category of diffeological spaces and smooth maps is both complete and cocomplete with respect to limits, and contains the category of manifolds and smooth maps as a full, faithful subcategory. Finally, any diffeological space \( X \) has an associated de Rham complex \( (\Omega^\ast(X), d) \), which is contravariantly functorial in the manner familiar from manifold theory, and which coincides with the usual de Rham complex when \( X \) is a manifold.

Having discussed diffeology, recall now that a groupoid is a small category with inverses. The range (or target) and source maps of a groupoid will always be denoted \( r \) and \( s \) respectively. A groupoid is said to be diffeological if its morphism set is a diffeological space, with object set equipped, via the unit map, with the subspace diffeology, for which the range, source, composition and inversion maps are all smooth maps. A diffeological groupoid is étale if the source and target maps are local homeomorphisms with respect to the D-topology. The most important examples for our purposes are the following.

**Example 3.1** (Čech groupoid). Let \( M \) be a diffeological space and \( \mathcal{U} = \{U_\alpha\}_{\alpha \in \mathbb{A}} \) an open cover of \( M \). The Čech groupoid \( \tilde{\mathcal{U}} \) of \( \mathcal{U} \) is the groupoid whose morphism space is the disjoint union

\[
\tilde{\mathcal{U}}^{(1)} := \bigsqcup_{(\alpha, \beta) \in \mathbb{A}^2} U_\alpha \cap U_\beta.
\]

An element of \( \tilde{\mathcal{U}} \) is an ordered triple \( (x, \alpha, \beta) \) with \( x \in U_\alpha \cap U_\beta \), so that the source and target maps are defined by the rules \( s(x, \alpha, \beta) = (x, \beta, \beta) \) and \( r(x, \alpha, \beta) = (x, \alpha, \alpha) \) respectively. In
particular, the unit space $\mathcal{U}$ is the disjoint union of the degenerate intersections $U_\alpha \cap U_\beta$. Composition is given by

$$(x, \alpha, \beta) \cdot (x, \beta, \delta) := (x, \alpha, \delta).$$

(Note the convention that morphisms act right to left.) As a disjoint union of diffeological spaces, $\mathcal{U}$ inherits a canonical diffeology (the sum diffeology \cite[Section 1.39]{31}), with respect to which it is an étale diffeological groupoid.

**Example 3.2 (Haefliger groupoid).** The *Haefliger groupoid* $\Gamma_q$ is the groupoid of germs of local diffeomorphisms of $\mathbb{R}^q$ \cite{24}. For $\gamma \in \Gamma_q^{(1)}$ with source $x \in \mathbb{R}^q$, representative local diffeomorphism $g : \text{dom}(g) \to \mathbb{R}^q$ and any open neighbourhood $U$ of $x$ contained in $\text{dom}(g)$, define an open neighbourhood of $\gamma$ by

$$\mathcal{N}(\gamma, g, U) := \{\text{germ}_{x'}(g) : x' \in U\}.$$ 

The collection of these for all $(\gamma, g, U)$ determine the basis for a topology on $\Gamma_q$. This topology induces the standard topology on the unit space $\mathbb{R}^q$, with respect to which the range and source are local homeomorphisms. As such, the manifold structure of $\mathbb{R}^q$ induces a (non-Hausdorff) manifold structure on $\Gamma_q$, which is thus an étale diffeological groupoid.

The following definition is the diffeological analogue of Haefliger’s \cite[Definition 2]{25].

**Definition 3.3.** Let $X$ be a diffeological space and $\Gamma$ a diffeological groupoid. A $\Gamma$-cocycle on $X$ defined over a $D$-open cover $\mathcal{U}$ of $X$ is a smooth morphism $h : \mathcal{U} \to \Gamma$ of diffeological groupoids. Two $\Gamma$-cocycles over open covers $\mathcal{U}$ and $\mathcal{V}$ are said to be equivalent if they are restrictions to $\mathcal{U}$ and $\mathcal{V}$ respectively of a smooth $\Gamma$-cocycle over $\mathcal{U} \cup \mathcal{V}$. An equivalence class of $\Gamma$-cocycles is called a $\Gamma$-structure. Two $\Gamma$-structures are homotopic if there exists a $\Gamma$-structure on $X \times [0,1]$ inducing the given structures on $X \cong X \times \{0\}$ and $X \cong X \times \{1\}$ respectively. In particular we call $\Gamma_q$-structures Haefliger structures of codimension $q$.

For the rest of the paper, we will freely use the abbreviation $h_{\alpha\beta}$ for the restriction $h|_{U_\alpha \cap U_\beta}$ of a Haefliger cocycle $h$ to a single intersection $U_\alpha \cap U_\beta$.

It is clear that Definition 1.1 can be reformulated as a special case of Definition 3.3 for the Haefliger groupoid $\Gamma_q$. In particular, a $\Gamma_q$-cocycle $h : \mathcal{U} \to \Gamma_q$ over an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of some manifold $M$ induces Haefliger charts $f_\alpha := s \circ h_{\alpha\alpha} : U_\alpha \to \mathbb{R}^q$, and if these chart maps are submersions, one obtains a regular foliation of $M$.

Ultimately we will show that certain groupoid structures on diffeological spaces are classified by smooth homotopy classes of maps into some diffeological classifying space associated to the groupoid. These classifying spaces will be constructed semi-simplicially. A semi-simplicial object in a category $\mathcal{C}$ is a sequence $X^\bullet = \{X^{(n)}\}_{n \in \mathbb{N}}$ of objects in $\mathcal{C}$ together with morphisms $\partial_i : X^{(n+1)} \to X^{(n)}$ defined for $i = 0, \ldots, n + 1$, called face maps, satisfying the relations

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i, \quad \text{for } i < j.$$ 

A morphism of semi-simplicial objects $X^\bullet$ and $Y^\bullet$ in $\mathcal{C}$ is a natural transformation, a family $\phi^\bullet = \{\phi^{(n)} : X^{(n)} \to Y^{(n)}\}$ of morphisms that commute with respective face maps. Moreover, if $\mathcal{C}$ is a Cartesian category, and $X^\bullet$ and $Y^\bullet$ are two semi-simplicial objects in $\mathcal{C}$, then their
The sets $U$ topology to follow). Given a composable tuple $(\gamma_1, \ldots, \gamma_n) \in \Gamma^{(n)}$, define elements of $\Gamma$ by $\gamma_\alpha = \alpha_0 < \cdots < \alpha_k$ of natural numbers. The face maps $\partial_j : \mathbb{N}^{(k)} \to \mathbb{N}^{(k-1)}$ are given by omission: $(\alpha_0, \ldots, \alpha_k) \mapsto (\alpha_0, \ldots, \alpha_j, \ldots, \alpha_k)$.

Associated to any semi-simplicial diffeological space $X^\bullet$ is a diffeological space, the fat realisation $\|X^\bullet\|$. For each $n \in \mathbb{N}$, let $\Delta_n$ denote the standard $n$-simplex, thought of as a diffeological subspace of Euclidean space. Let $(t_0, \ldots, t_n)$ denote the barycentric coordinates on $\Delta_n$, and denote by $d_i : \Delta_n \to \Delta_{n+1}$ the $i$th face inclusion (which inserts a 0 at position $i$). The fat realisation is the quotient

$$\|X^\bullet\| := \left( \bigsqcup_{n \in \mathbb{N}} \Delta_n \times X^{(n)} \right) / \sim$$

by the relation $(d_i(\tilde{t}), x) \sim (\tilde{t}, \partial_i(x))$ for $(\tilde{t}, x) \in \Delta_n \times X^{(n+1)}$, $n \geq 0$ and $i = 0, \ldots, n+1$. The fat realisation may be equipped with the quotient diffeology, but for our purposes it is necessary to consider a coarser (that is, larger [31, Section 1.18]) diffeology, as we now describe, following Mostow [42].

Let $\Gamma$ be a diffeological groupoid, and consider the fat realisation $\|\mathbb{N}^\bullet \times \Gamma^\bullet\|$ as a set. For each $\alpha \in \mathbb{N}$ there is a natural barycentric coordinate map $u_\alpha : \|\mathbb{N}^\bullet \times \Gamma^\bullet\| \to [0, 1]$, given by the (well-defined) quotient of the maps

$$\Delta_n \times (\mathbb{N}^n \times \Gamma^{(n)}) \ni (t_0, \ldots, t_n; \alpha_0, \ldots, \alpha_n; \gamma_1, \ldots, \gamma_n) \mapsto \begin{cases} t_j & \text{if } \alpha = \alpha_j \\ 0 & \text{otherwise.} \end{cases}$$

The sets $U_\alpha := u_\alpha^{-1}(0, 1]$ define a cover of $\|\mathbb{N}^\bullet \times \Gamma^\bullet\|$ (which will be an open cover in the topology to follow). Given a composable tuple $(\gamma_1, \ldots, \gamma_n) \in \Gamma^{(n)}$, define elements of $\Gamma$ by $g_0 := r(\gamma_1)$ and $g_i := \gamma_1 \cdots \gamma_i$ for $i = 1, \ldots, n$. Then there are functions $\gamma_{\alpha \beta} : U_\alpha \cap U_\beta \to \Gamma$ given

\[ \begin{align*}
\partial_0(\gamma_1, \ldots, \gamma_{n+1}) := & \begin{cases} (\gamma_2, \ldots, \gamma_{n+1}) & \text{if } i = 0, \\
(\gamma_1, \gamma_\alpha \gamma_{i+1}, \ldots, \gamma_{n+1}) & \text{if } 1 \leq i \leq n \\
(\gamma_1, \ldots, \gamma_n) & \text{if } i = n + 1
\end{cases} \\
\partial_i(\gamma_1, \ldots, \gamma_{n+1}) := & \begin{cases} (\gamma_2, \ldots, \gamma_{n+1}) & \text{if } i = 0, \\
(\gamma_1, \gamma_\alpha \gamma_{i+1}, \ldots, \gamma_{n+1}) & \text{if } 1 \leq \alpha < i \leq n \\
(\gamma_1, \ldots, \gamma_n) & \text{if } i = n + 1
\end{cases}
\end{align*} \]
by
\[ \gamma_{\alpha\beta}(t_0, \ldots, t_n; \alpha_0, \ldots, \alpha_n; \gamma_1, \ldots, \gamma_n) := g_i^{-1}g_j \quad \text{where } \alpha_i = \alpha \text{ and } \alpha_j = \beta. \]

**Definition 3.6.** Let \( \Gamma \) be a diffeological groupoid. The **classifying space** \( B\Gamma \) of \( \Gamma \) is the diffeological space whose underlying set is \( \| N^* \times \Gamma(\bullet) \| \), equipped with the **Mostow diffeology**, which is the largest diffeology for which the maps \( u_\alpha : \| N^* \times \Gamma(\bullet) \| \to [0,1] \) are smooth and the maps \( \gamma_{\alpha\beta} : U_\alpha \cap U_\beta \to \Gamma \) are locally smooth. With respect to the Mostow diffeology, \( \mathcal{U} := \{ U_\alpha \}_{\alpha \in \mathbb{N}} \) is an open cover, the **canonical open cover**, and \( \gamma : \mathcal{U} \to \Gamma \) is a smooth \( \Gamma \)-cocycle, the **canonical cocycle**. The Haefliger structure determined by \( \gamma \) is the **canonical \( \Gamma \)-structure** on \( B\Gamma \).

To be explicit, \( P : U \to B\Gamma \) is a plot of \( B\Gamma \) whenever the \( u_\alpha \circ P \) are smooth, the preimages \( P^{-1}(U_\alpha \cap U_\beta) \) are open, and the compositions \( P^{-1}(U_\alpha \cap U_\beta) \xrightarrow{P} B\Gamma \xrightarrow{\gamma_{\alpha\beta}} \Gamma \) are plots of \( \Gamma \).

Note that while Mostow never invokes diffeology explicitly in [12], the Mostow diffeology defined here is implicit in his constructions, via the definition of differential forms [42, Section 2].

It follows immediately from the definitions that classifying spaces are functorial: for any morphism \( \phi : \Gamma_1 \to \Gamma_2 \) of diffeological groupoids, there is a smooth map \( B\phi : B\Gamma_1 \to B\Gamma_2 \), which furthermore preserves the canonical cocycles: \( u_\alpha^i \circ B\phi = u_\alpha^1 \) and \( \phi \circ \gamma_{\alpha\beta}^1 = \gamma_{\alpha\beta}^2 \circ B\phi \), with \( u_\alpha^i \) and \( \gamma_{\alpha\beta}^j \) the maps defining the canonical cocycle of \( B\Gamma_1 \). Just as in the topological setting, Definition 3.6 is a special case of a more general construction, which assigns to any semi-simplicial diffeological \( X^\bullet \) its **unwound geometric realisation** \( \mu(X^\bullet) \) (cf. [41, 49, 42]).

In [24], Haefliger uses Milnor’s infinite join construction to construct the classifying space \( B\Gamma \) for any topological groupoid \( \Gamma \), so that homotopy classes of continuous, numerable \( \Gamma \)-structures on a topological space \( X \) are in bijective correspondence with homotopy classes of continuous maps \( X \to B\Gamma \). Haefliger’s construction can easily be identified with the topological version of Definition 3.6 [12, p. 278], provided one equips \( B\Gamma \) with the so-called **strong topology**, which is the weakest topology making the maps \( u_\alpha \) and \( \gamma_{\alpha\beta} \) continuous. For a diffeological groupoid \( \Gamma \), the strong topology on \( B\Gamma \) is coarser than the D-topology induced by the Mostow diffeology, since smooth maps are always continuous with respect to the D-topology. As such, any smooth map of a diffeological \( X \) into \( B\Gamma \) is automatically continuous with respect to the D-topology on \( X \) and the strong topology on \( B\Gamma \).

The next theorem generalises Haefliger’s [25, Theorem 7] to the diffeological setting, and justifies the nomenclature of “classifying space” for the diffeological space \( B\Gamma \). Say that a smooth, countable partition of unity \( \{ \lambda_\alpha \}_{\alpha \in \mathbb{N}} \) on a diffeological space is **locally finite** if the covering \( \lambda_\alpha^{-1}(0,1) \) of \( X \) by D-closures is locally finite, and **subordinate** to an open cover \( \{ U_\alpha \}_{\alpha \in \mathbb{N}} \) of \( X \) if \( \lambda_\alpha^{-1}(0,1) \subset U_\alpha \) for all \( \alpha \). A countable open cover of a diffeological space \( X \) is **smoothly numerable** if it admits a subordinate, locally finite, smooth partition of unity. A smooth \( \Gamma \)-structure on \( X \) is **smoothly numerable** if it admits a representative cocycle over a countable, smoothly numerable open cover, and two smooth, numerable \( \Gamma \)-structures on \( X \) are said to be **smoothly, numerably homotopic** if there exists a smoothly numerable homotopy between them.

Our next theorem generalises Haefliger’s classifying space theorem to the diffeological category (that our theorem really is a generalisation can be seen by equipping any topological
groupoid, as considered by Haefliger, with the continuous diffeology; see Appendix [13]. Although the diffeology for $B\Gamma$ is inspired by Mostow [22], Mostow does not prove a classifying theorem. An alternative diffeological approach to classifying spaces, which is insufficient for our purposes, is presented in [38].

**Theorem 3.7.** Let $\Gamma$ be a diffeological groupoid.

1. The canonical $\Gamma$-structure $\gamma$ on $B\Gamma$ is smoothly numerable.

2. For any smoothly numerable $\Gamma$-structure $h$ on a diffeological space $X$, there is a smooth map $\eta : X \to B\Gamma_{\eta}$ such that $h = \eta^*\gamma$.

3. If $\eta_0, \eta_1 : X \to B\Gamma$ are smooth maps, then the $\Gamma$-structures $\eta_0^*\gamma$ and $\eta_1^*\gamma$ are smoothly, numerably homotopic if and only if $\eta_0$ and $\eta_1$ are smoothly homotopic.

**Proof.** (1) We show first smooth numerability of the canonical cocycle on the classifying space $B\{\varepsilon\}$ of the trivial groupoid. Note that it follows from the definition that $B\{\varepsilon\}$ may be identified with the infinite simplex $\Delta_\infty$, which comprises the infinite sequences $\{t_\alpha\}_{\alpha \in \mathbb{N}}$ for which only finitely many elements are nonzero and for which $\sum_\alpha t_\alpha = 1$. Under this identification, the barycentric coordinate maps $t_\alpha : \Delta_\infty \to [0, 1]$ are smooth maps, and the canonical open cover is given by $\{t_\alpha^{-1}(0, 1]\}_{\alpha \in \mathbb{N}}$.

The barycentric coordinates $\{t_\alpha\}_{\alpha \in \mathbb{N}}$ define a pointwise-finite, smooth partition of unity on $\Delta_\infty$. Mostow [22 p. 273] constructs from this a locally finite partition of unity $\{s_\alpha\}_{\alpha \in \mathbb{N}}$ that remains subordinate to the canonical open cover with respect to the strong topology (the weakest topology for which the $t_\alpha$ are continuous). But the D-topology on $\Delta_\infty$ is finer than the strong topology, so $\{s_\alpha\}_{\alpha \in \mathbb{N}}$ is also subordinate in the D-topology.

To see the claim for general $\Gamma$, consider the smooth map $u_\infty : B\Gamma \to \Delta_\infty = B\{\varepsilon\}$ obtained by applying the classifying space functor to $\gamma$. It pulls back the canonical open cover of $B\{\varepsilon\}$ to the same on $B\Gamma$, so the pullback functions $v_\alpha := s_\alpha \circ u_\infty$ define a smooth, locally finite partition of unity subordinate to the canonical open cover of $B\Gamma$.

(2) Let $h : \tilde{V} \to \Gamma$ be a $\Gamma$-cocycle for $X$, defined over a countable open cover $\mathcal{V} = \{V_\alpha\}_{\alpha \in \mathbb{N}}$ with smooth, subordinate, locally finite partition of unity $\{\lambda_\alpha\}_{\alpha \in \mathbb{N}}$. For $x \in X$, choose a D-open neighbourhood $W$ of $x$ such that $W$ has nonempty intersection with $\text{supp} \lambda_\alpha$ only for $\alpha$ contained in some finite list $\tilde{\alpha} := (\alpha_0, \ldots, \alpha_n)$ of indices. Then the formula

$$
\eta(y) := [(\lambda_{\alpha_0}(y), \ldots, \lambda_{\alpha_n}(y); \alpha_0, \ldots, \alpha_n; h_{\alpha_0\alpha_1}(y), \ldots, h_{\alpha_{n-1}\alpha_n}(y))], \quad y \in W
$$

(12)
gives rise to a well-defined map $\eta : X \to B\Gamma$. To see this, let us denote $(\lambda_{\alpha_0}(y), \ldots, \lambda_{\alpha_n}(y))$ by $\tilde{\lambda}_{\tilde{\alpha}}(y)$, and $(h_{\alpha_0\alpha_1}(y), \ldots, h_{\alpha_{n-1}\alpha_n}(y))$ by $\tilde{h}_{\tilde{\alpha}}(y)$. Suppose then that $W'$ were some other open neighbourhood of $x$ intersecting $\text{supp} \lambda_\alpha$ only for $\alpha$ contained in some other finite list $\tilde{\alpha}' := (\alpha'_0, \ldots, \alpha'_m)$, and that $y \in W \cap W'$. Since both $W$ and $W'$ contain $x$, the intersection $\tilde{\alpha} \cap \tilde{\alpha}' = (\alpha_i, \ldots, \alpha_k)$ is nonempty, and there are sequences $\tilde{\partial}$ and $\tilde{\partial}'$ of composites of face maps on $\mathbb{N}^*$ such that $\tilde{\partial} \tilde{\alpha} = \tilde{\alpha} \cap \tilde{\alpha}' = \tilde{\partial}' \tilde{\alpha}'$. Moreover, only those functions $\lambda_\alpha$ for which $\alpha \in \tilde{\alpha} \cap \tilde{\alpha}'$ will take nonzero values at $y$, so that $\tilde{\lambda}_{\tilde{\alpha}}(y) = \tilde{d}(\tilde{\lambda}_{\tilde{\alpha} \cap \tilde{\alpha}'}(y))$ and $\tilde{\lambda}_{\tilde{\alpha}'}(y) = \tilde{d}'(\tilde{\lambda}_{\tilde{\alpha} \cap \tilde{\alpha}'}(y))$, where $\tilde{d}$ and $\tilde{d}'$ denote the composites of face maps for the standard simplices corresponding to $\tilde{\partial}$ and
\( \tilde{\varphi} \) respectively, with order of composition reversed. Therefore, by definition of the equivalence relation defining \( B\Gamma \), we then have

\[
(d(\lambda_{\alpha \gamma \alpha'}(y)); \bar{\alpha}; \bar{h}_{\alpha}(y)) \sim (\lambda_{\alpha \gamma \alpha'}(y)); \tilde{\varphi}(\bar{\alpha}; \bar{h}_{\alpha}(y)))
\]

and

\[
(\lambda_{\alpha \gamma \alpha'}(y)); \tilde{\varphi}(\bar{\alpha'}; \bar{h}_{\alpha'}(y))) \sim (\lambda'(\lambda_{\alpha \gamma \alpha'}(y)); \alpha'; \bar{h}_{\alpha'}(y))
\]

so that

\[ (\lambda_{\alpha}(y); \bar{\alpha}; \bar{h}_{\alpha}(y)) \sim (\lambda_{\alpha}(y); \alpha'; \bar{h}_{\alpha'}(y)) \]

as claimed. To see that \( \eta \) is smooth, let \( P : U \to X \) be a plot. For \( u \in U \), choose an open neighbourhood \( W \) of \( P(u) \) in \( X \) such that \( W \) intersects only those \( \text{supp} \lambda_{\alpha} \) for \( \alpha \) contained in some sequence \( \tilde{\alpha} \). Then \( W := P^{-1}(W) \) is open, and for any \( \alpha \in \mathbb{N} \) the composite

\[
u_{\alpha} \circ \eta \circ P|_W = \begin{cases} 
\lambda_{\alpha}, \circ P \text{ if } \alpha = \alpha_i \in \tilde{\alpha} \\
0 \text{ otherwise }
\end{cases}
\]

is smooth. It follows that \( u_{\alpha} \circ \eta \circ P \) is smooth, hence that \( \eta \) is smooth with respect to the largest diffeology for which the \( u_{\alpha} \) are smooth. Therefore for any pair \( \alpha, \beta \in \mathbb{N} \), \( \eta^{-1}(U_{\alpha} \cap U_{\beta}) \) is open in \( X \), and to complete the proof that \( \eta \) is smooth with respect to the Mostow diffeology on \( B\Gamma \), it remains only to show that \( \gamma_{\alpha \beta} \circ \eta \circ P|_{P^{-1}(U_{\alpha} \cap U_{\beta})} \) is a plot for \( \Gamma \). If \( P^{-1}(U_{\alpha} \cap U_{\beta}) \) is empty then we are done. Otherwise, for \( u \in P^{-1}(U_{\alpha} \cap U_{\beta}) \), let us again take \( W \) to be an open neighbourhood of \( P(u) \) which intersects only those \( \text{supp} \lambda_{\alpha'} \) for which \( \alpha' \in \tilde{\alpha} \). We then have \( \alpha = \alpha_i \in \tilde{\alpha} \) and \( \beta = \beta_j \in \tilde{\alpha} \), and setting \( W := P^{-1}(W \cap \eta^{-1}(U_{\alpha} \cap U_{\beta})) \), the composite

\[
\gamma_{\alpha \beta} \circ \eta \circ P|_W = ((h_{\alpha_i \alpha_1} \circ P) \cdots (h_{\alpha_i \alpha_1 \alpha_1} \circ P))^{-1}((h_{\alpha_i \alpha_1} \circ P) \cdots (h_{\alpha_i \alpha_1 \alpha_1} \circ P))
\]

is smooth by the smoothness of \( h \), and therefore \( \gamma_{\alpha \beta} \circ \eta \circ P|_{P^{-1}(U_{\alpha} \cap U_{\beta})} \) is smooth. It follows that \( \eta : X \to B\Gamma \) is smooth. To complete the proof of the second item, one need only show that \( h \) and \( \eta^* \gamma \) are cocyclic. This follows from the definition of \( \eta \).

(3) Suppose that \( \eta_0, \eta_1 : X \to B\Gamma \) are smooth maps. It is clear that if \( \eta_0 \) and \( \eta_1 \) are smoothly homotopic then \( \eta_0^* \gamma \) and \( \eta_1^* \gamma \) are smoothly, numerably homotopic. Suppose conversely that \( \eta_0^* \gamma \) and \( \eta_1^* \gamma \) are smoothly, numerably homotopic through a \( \Gamma \) structure on \( X \times [0, 1] \) which by item (2) may be assumed to be of the form \( \eta^* \gamma \) for some smooth map \( \eta : X \times [0, 1] \to B\Gamma \). We then need only prove that \( \eta_1 \) is smoothly homotopic to \( \eta|_{X \times \{i\}} \). Thus it suffices to prove item (3) assuming that \( \eta_0^* \gamma = \eta_1^* \gamma \).

Our argument is inspired by those found in [30] p.57-58 and in [38] Proposition 3.16, however it is sufficiently different from both that we feel obliged to spell it out in some detail for the reader. Let \( B\Gamma^0 \) and \( B\Gamma^1 \) denote the subsets of \( B\Gamma \) consisting of tuples \([\bar{\gamma}, \bar{\alpha}, \bar{\gamma}]\) for which \( \bar{\alpha} \) consists entirely of even or odd numbers respectively. Replacing the linear functions \( \alpha_n : I_n := [1 - 2^{-n}, 1 - 2^{-n+1}] \to [0, 1] \) of [30] p. 57 with smooth functions \( b_n : I_n \to [0, 1] \) which are everywhere nondecreasing, and constant on a small neighbourhood of each endpoint, the arguments of [30] p.57-58 can be used to show that the maps \( h^1 : B\Gamma \to B\Gamma \) and \( h^0 : B\Gamma \to B\Gamma \)
Moreover that by the Diff(R) over singularities, Haefliger bundles are defined globally even in the singular case. However, while transverse frame bundles do not make sense Haefliger bundles of a regular Haefliger structure are isomorphic to the transverse frame bundles. Although previously identified (at least for regular foliations [41, Section 3.4.1]), Haefliger bundles appear to have been underutilised in the literature. As we will see in the next section, the equivalence relation
\[ x, \alpha, \varphi \sim y, \beta, \varphi(x, \beta, h_{\beta\alpha}(x) \cdot \varphi) \text{ for all } x \in U_\alpha \cap U_\beta. \]

Now \( \eta_0^*\gamma \) is defined over the cover \( \{ V_{2n} := \eta_0^{-1}(U_{2n}) \}_{n \in \mathbb{N}} \) while \( \eta_1^*\gamma \) is defined over the cover \( \{ V_{2n+1} := \eta_1^{-1}(U_{2n+1}) \}_{n \in \mathbb{N}} \). By hypothesis \( \eta_0^*\gamma \) and \( \eta_1^*\gamma \) are cocyclic, and we denote by \( V = \{ V_{\alpha} \}_{\alpha \in \mathcal{A}} \) the corresponding cover and \( h : V \to \Gamma \) the corresponding cocycle. Now fix \( x \in X \). Since the maps \( \eta_i \) are smooth, we can find an open neighbourhood \( W \) of \( x \), indices \( \alpha^0 := (\alpha_0^0, \ldots, \alpha_{n_0}^0) \) and \( \alpha^1 := (\alpha_0^1, \ldots, \alpha_{n_1}^1) \), and smooth functions \( \{ t_{i,j} : W \to (0, 1) \}_{j=0, \ldots, n_i} \) such that
\[
\eta_i(y) = [t_0^i(y), \ldots, t_{n_i}^i(y); \alpha_0^i, \ldots, \alpha_{n_i}^i; h_{\alpha_0^i\alpha_1^i}(y), \ldots, h_{\alpha_{n_i-1}\alpha_n^i}(y)]
\]
for all \( y \in W \), \( i = 0, 1 \). The indices \( \alpha^0 \) and \( \alpha^1 \) may be formed uniquely into a new list \( \alpha \in \mathbb{N}^* \) of length \( n = n_0 + n_1 \). For \( i = 0, \ldots, n \), define \( t_i : W \times [0, 1] \to [0, 1] \) by
\[
t_i(y, s) := \begin{cases} (1 - s) t_{i,j}^0(y) & \text{if } \alpha_i = \alpha_j^0 \\ s t_{i,j}^1(y) & \text{if } \alpha_i = \alpha_j^1. \end{cases}
\]
Then the formula
\[
\eta(y, s) = [t_0(y, s), \ldots, t_n(y, s); \alpha_0, \ldots, \alpha_n; h_{\alpha_0\alpha_1}(y), \ldots, h_{\alpha_{n-1}\alpha_n}(y)], \quad y \in W.
\]
defines a homotopy \( \eta : X \times [0, 1] \to B\Gamma \) between \( \eta_0 \) and \( \eta_1 \) over \( W \).

### 3.2 Universal characteristic map via Haefliger bundles

In this section we define the **Haefliger bundles** of a smoothly numerable Haefliger structure on a diffeological space, which are natural, locally trivial principal \( G^\alpha_\beta \)-bundles on the space. Although previously identified (at least for regular foliations [111, Section 3.4.1]), Haefliger bundles appear to have been underutilised in the literature. As we will see in the next section, the Haefliger bundles of a regular Haefliger structure are isomorphic to the transverse frame bundles of the associated regular foliation. However, while transverse frame bundles do not make sense over singularities, Haefliger bundles are defined globally even in the singular case.

Consider a smoothly numerable Haefliger cocycle \( h : \mathcal{U} \to \Gamma_q \) on a diffeological space \( X \). For \( k \in \mathbb{N} \cup \{ \infty \} \), the **Haefliger k-frame bundle of h**, or simply the **k-Haefliger bundle**, is the smooth quotient
\[
\text{Fr}_k(h) := \left( \bigcup_{\alpha \in \mathbb{N}} (s \circ h_{\alpha})^* \text{Fr}_k(\mathbb{R}^q) \right) / \sim,
\]
the equivalence relation \( \sim \) given by \( (x, \alpha, \varphi) \sim (x, \beta, h_{\beta\alpha}(x) \cdot \varphi) \) for all \( x \in U_\alpha \cap U_\beta \). Note moreover that by the Diff(\( \mathbb{R}^q \))-invariance of the tautological forms \( \omega^h \) on \( \text{Fr}_k(\mathbb{R}^q) \), the pullbacks
\[1\text{In the diffeological setting, principal does not imply locally trivial, but only locally trivial along plots [111, Section 8.13].} \]
\((s \circ h_{\alpha})^* \omega^k\) glue to give an \(a^k_q\)-valued 1-form \(\omega^k_h\) on \(\text{Fr}_k(h)\). We have the following result.

**Proposition 3.8.** Let \(X\) be a diffeological space and let \(k \in \mathbb{N} \cup \{\infty\}\).

1. If \(U\) is an open cover of \(X\) and \(h : \tilde{U} \to \Gamma_q\) represents a smoothly numerable Haefliger structure on \(X\), then \(\text{Fr}_k(h)\) is a smooth, principal \(G^k_q\)-bundle over \(X\), locally trivial over \(U\).

2. If \(\eta : X \to B\Gamma_q\) is a smooth map, then \(\text{Fr}_k(\eta^* \gamma)\) and \(\eta^* \text{Fr}_k(\gamma)\) are canonically isomorphic as principal \(G^k_q\)-bundles over \(X\), and under this isomorphism \(\eta^* \omega^k_\gamma\) identifies with \(\omega^k_{\eta^* \gamma}\).

3. If \(h_0\) and \(h_1\) are smoothly, numerably homotopic Haefliger structures on \(X\), then \(\text{Fr}_k(h_0)\) and \(\text{Fr}_k(h_1)\) are isomorphic as principal \(G^k_q\)-bundles over \(X\).

As a consequence, isomorphism classes of Haefliger bundles are functorial for diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & \text{Fr}_k(h) \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\eta} & B\Gamma_q
\end{array}
\]

of smooth maps that commute up to homotopy.

**Proof.** Item (1) follows from the triviality of \(\text{Fr}_k(\mathbb{R}^q) \to \mathbb{R}^q\), while item (2) follows from the naturality of pullbacks. The final item follows from [38, Lemma 3.11]. Note that the hypothesis there that \(X\) be Hausdorff, second-countable and smoothly paracompact is required only for the existence of a smoothly numerable open cover over which the bundle is locally trivial. In our setting, this follows from the hypothesis that \(h_0\) and \(h_1\) are smoothly numerable Haefliger structures.

The tautological form \(\omega^\infty_h\) on the Haefliger bundle \(\text{Fr}_\infty(h) \to X\) associated to a smoothly numerable Haefliger structure on a diffeological space \(X\) induces a homomorphism

\[
A^*(a^q_q)_{O(q)} \to \Omega^*(\text{Fr}_\infty(h)\big/O(q)).
\]

Thus, given a smooth section \(X \to \text{Fr}_\infty(h)\big/O(q)\), one obtains a homomorphism \(A^*(a^q_q)_{O(q)} \to \Omega^*(X)\) of DGAs. Our next theorem shows that, despite the potentially pathological topology of \(X\), such sections are in abundance, and their induced maps on cohomology unambiguous.

**Theorem 3.9.** Let \(h\) be a smoothly numerable Haefliger structure on a diffeological space \(X\). Then \(\text{Fr}_\infty(h)\big/O(q) \to X\) admits smooth sections, and any two such sections are smoothly homotopic.

**Proof.** The theorem is essentially a consequence of the fact that the fibre \(G^\infty_q\big/O(q)\) of the bundle is contractible. When taken with the strong topology, \(B\Gamma_q\) has the homotopy type of a CW complex [24, Theorem 1.6], and the existence of sections is then classical. A full, constructive proof in our diffeological case is given in Appendix [C].

By the homotopy-invariance of de Rham cohomology, Theorem 3.9 enables the following definition of de Rham characteristic maps for smoothly numerable Haefliger structures.
Definition 3.10. If $h$ is a smoothly numerable Haefliger structure on a diffeological space $X$, the characteristic map associated to $h$ is the homomorphism $H^*(A(a_q); O(q)) \to H^*(\Omega(X))$ induced by the composite

$$A^*(a_q)_{O(q)} \xrightarrow{\omega^\infty} \Omega^*(Fr_\infty(h)/ O(q)) \xrightarrow{\sigma^*} \Omega^*(X),$$

where $\sigma : X \to Fr_\infty(h)/ O(q)$ is any smooth section. In particular, the characteristic map $u : H^*(A(a_q); O(q)) \to H^*(\Omega(B\Gamma_q))$ associated to the canonical Haefliger structure $\gamma$ on $B\Gamma_q$ is called the universal characteristic map.

It follows easily from Proposition 3.8 that characteristic maps for smoothly numerable Haefliger structures are functorial under smooth maps. In particular if $h = \eta^*\gamma$ is the smoothly numerable Haefliger structure defined by a smooth map $\eta : X \to B\Gamma_q$, then the characteristic map associated to $h$ is equal to $\eta^* \circ u : H^*(A(a_q); O(q)) \to H^*(\Omega(B\Gamma_q))) \to H^*(\Omega(X))$. Our final result in this section is a corollary of Theorem 3.9 which describes the characteristic map in terms of the quasi-isomorphic, finite-dimensional subalgebra $WO_q$ of $A^*(a_q)_{O(q)}$, and will be the means by which we relate our Chern-Weil theory to classifying space theory.

Corollary 3.11. Let $X$ be a diffeological space and $h = \eta^*\gamma$ a smoothly numerable Haefliger structure on $X$. If $\sigma : X \to Fr_2(h)/ O(q)$ is any smooth section, then the map induced on cohomology by the homomorphism

$$WO_q \xrightarrow{\omega^2} \Omega^*(Fr_2(h)/ O(q)) \xrightarrow{\sigma^*} \Omega^*(X)$$

of DGAs is equal to the characteristic map $\eta^* \circ u : H^*(WO_q) \to H^*(\Omega(X))$.

We now turn to describing how sections of $Fr_2(h)/ O(q) \to X$ may be induced via geometric structures (connections and curvatures). We first consider the regular case, and then the singular case.

4 Regular foliations and Haefliger bundles

We begin by establishing notation. Given a regular foliation $(M, F)$ of codimension $q$, we will assume it to be defined by a smooth map $\eta_F : M \to B\Gamma_q$. Abusing notation, we will assume that the pullback $h_F := \eta^*_F \gamma$ of the canonical Haefliger cocycle $\gamma$ on $B\Gamma_q$ is defined over an open cover $\{U_a\}_{a \in \mathbb{N}}$ of $M$, with submersive Haefliger charts $f_a := s \circ \gamma_a \circ \eta_F : U_a \to \mathbb{R}^q$. The following will allow us to give a new proof of Bott’s result that the Chern-Weil map for $(M, F)$ recovers the characteristic classes arising from the map $\eta_F : M \to B\Gamma_q$. Ultimately, it is this approach that will allow us to extend Chern-Weil theory to singular foliations.

Proposition 4.1. Let $(M, F)$ be a regular foliation of codimension $q$. For all $1 \leq k \leq \infty$, there are canonical isomorphisms $i_k : Fr_k(M/ F) \to Fr_k(h_F)$ of principal $G^k_q$-bundles. These commute with the natural projection maps.

Proof. The local maps $i_{k,\alpha} : Fr_k(M/ F)|_{U_a} \to Fr_k(h_F)|_{U_a}$ defined by

$$i_{k,\alpha}(j^k_{a0}(u)) \mapsto [(u(0), \alpha, j^k_{a0}(f_a \circ u))]_{\sim}$$

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glue to give the isomorphism $i_k$. Indeed, since each $f_\alpha$ is a submersion and each $u$ a frame, the composition $f_\alpha \circ u$ is a local diffeomorphism of $\mathbb{R}^q$. The maps $i_{k,\alpha}$ are manifestly equivariant for the respective right actions of $G_q^k$, and on each overlap $U_\alpha \cap U_\beta$, one has

$$\left(x, \alpha, j_{01}^k(f_\alpha \circ u)\right) = \left(x, \alpha, j_{01}^k(h_{\alpha,\beta}^\omega \circ f_\beta \circ u)\right) \sim \left(x, \beta, j_{01}^k(f_\beta \circ u)\right)$$

for each $j_{01}^k(u) \in \text{Fr}_k(M/F)|_{U_\alpha \cap U_\beta}$, so that $i_{k,\alpha}(x, j_{01}^k(u)) = i_{k,\beta}(x, j_{01}^k(u))$. \hfill $\Box$

Note that $\text{Fr}_k(h_F)$ has a natural bundle foliation: the trivial bundle $\text{Fr}_k(\mathbb{R}^q) \cong \mathbb{R}^q \times G_q^k$ has horizontal foliation from constant sections, and each chart $(U_\alpha, f_\alpha)$ of $h_F$ pulls this foliation back to a bundle foliation of $\text{Fr}_k(h_F)$ over $U_\alpha$, compatible on overlaps. The well-known fact that the transverse frame bundles $\text{Fr}_k(M/F) \to M$ of a regular foliation $(M, F)$ are foliated bundles then follows easily the isomorphism of Proposition 4.1. Moreover, Proposition 4.1 allows us to extend Proposition 2.4 to the transverse frame bundles of regular foliations. The next theorem is folklore, and elements of its proof appear in a less precise form in [36, Section 5.2]. We give a precise proof here, which makes the role of Haefliger bundles clear.

**Theorem 4.2.** Let $(M, F)$ be a regular foliation of codimension $q$. Torsion-free Bott connections on $\nu F$ are in bijective correspondence with $\text{GL}(q, \mathbb{R})$-equivariant sections $\text{Fr}_1(M/F) \to \text{Fr}_2(M/F)$.

**Proof.** The isomorphism $i_2 : \text{Fr}_2(M/F) \to \text{Fr}_2(h_F)$ pulls back the tautological matrix-valued form $\omega_h^2$ on $\text{Fr}_2(h_F)$ to a matrix-valued form $i_2^*\omega_h^2$ on $\text{Fr}_2(M/F)$. The pullback of this form by any $\text{GL}(q, \mathbb{R})$-equivariant section $\sigma : \text{Fr}_1(M/F) \to \text{Fr}_2(M/F)$ is then easily verified, by the properties of $\omega_h^2$ and the equivariance of $\sigma$, to define a torsion-free connection form on $\text{Fr}_1(M/F)$. To see that $\sigma^*i_2^*\omega_h^2$ is a Bott connection, it suffices [35, Definition 2.3] to show that $T_F \subset \ker(\sigma^*\omega_h^2)$, with $F_1$ the natural bundle foliation on $\text{Fr}_1(M/F)$. Suppose therefore that $\varepsilon : t \mapsto \varepsilon(t) \in \text{Fr}_1(M/F)$ is a smooth path contained in a leaf of $F_1$, whose projection to $M$ is contained in the domain $U_\alpha$ of a Haefliger chart map $f_\alpha$. It follows that $Df_\alpha(\varepsilon(t))$ is constant in $t$. Representing $\sigma(\varepsilon(t))$ by a transversal embedding $u_t : \mathbb{R}^q \to M$, we compute

$$\sigma^*i_2^*\omega_h^2 \left( \frac{d}{dt} \bigg|_0 \varepsilon(t) \right) = \omega_h^2 \left( \frac{d}{dt} \bigg|_0 i_2(f_\alpha \circ u_t) \right)$$

$$= \frac{d}{dt} \bigg|_0 D(f_\alpha \circ u_0)^{-1}(f_\alpha(u_0(0))) Df_\alpha(\varepsilon(t)) = 0,$$

yielding $T_F \subset \ker(\sigma^*i_2^*\omega_h^2)$ as claimed.

Conversely, let $\nabla^\nu$ be a torsion-free Bott connection on $\nu F$. The aim is to construct from $\nabla^\nu$ an equivariant lift $\sigma_{\nabla^\nu} : \text{Fr}_1(M/F) \to \text{Fr}_2(M/F)$, which will be done via local construction and then shown to be independent of choices made. To this end, fix index sets $i, j, k, \ldots = 1, \ldots, q$ and $a, b, c, \ldots = 1, \ldots, \dim(M) - q$, as well as $A, B, C, \ldots = 1, \ldots, \dim(M)$. Recall the notation $p : TM \to \nu F$ for projection to the normal bundle. For a point of $\text{Fr}_1(M/F)$ over $x$—a frame $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_q)$ of $\nu F_x$—there exist local foliation coordinates $x^i, y^a$ about $x$ so that $p(\partial/\partial x^i)_x = \varepsilon_i$. Denote by $e_i := \partial/\partial x^i$, $f_a := \partial/\partial y_a$ the induced local framing of $TM$. 21
We call any such local framing an \textit{adapted lift} of $\varepsilon$, and remark that the vector fields $e_i$ have horizontally parallel projection, in the sense that

$$\nabla_X^\nu (e_i) = [X, e_i]_\nu = 0 \quad \text{for all local vector fields} \quad X \in T_\nu F$$

in the neighborhood on which they are defined.

Now, choose a connection $\nabla^F$ on $T_\nu F$. Any adapted lift determines a local embedding $\iota: \nu F \to T M$ that splits the projection $T M \to \nu F$; equivalently, a choice of splitting $T M \cong T F \oplus \nu F$. As such, $\nabla^F$ plus an adapted lift determine an affine connection $\nabla = \nabla^\nu \oplus \nabla^F$ on $T M$ near $x$. To be precise, the connection $\nabla$ is given by

$$\nabla_X Y = \nabla_X (\iota(v) + w) := \iota \nabla_X^v v + \nabla^F_X w$$

for any vectors $X, Y \in \mathfrak{X}$ and the associated splitting $Y = \iota(v) + w$ into image-of-$\iota$ and $T F$ components.

All this granted, we define the (clearly equivariant) lift $\sigma_{\nabla^\nu}: \operatorname{Fr}_1(M/\mathcal{F}) \to \operatorname{Fr}_2(M/\mathcal{F})$ by the rule that to a frame $\varepsilon$ of $\nu F$ at $x$ assigns the transverse 2-jet of the exponential through any adapted lift $(\vec{e}, \vec{f})$ of $\varepsilon$:

$$\sigma_{\nabla^\nu} (\varepsilon) := j_{\vec{0}}^2 (\vec{s} \mapsto \exp^\nu (\vec{s} \cdot \vec{e})) \quad \text{with argument} \quad \vec{s} \in \mathbb{R}^q. \quad (14)$$

It remains only to check that the transverse component of this two-jet does not depend on the choice of adapted lift, nor the choice of tangential connection $\nabla^F$. This follows by writing the geodesic equation in local foliation coordinates.

First suppose that the foliation coordinates, and consequent adapted lift $(\vec{e}, \vec{f})$, are fixed. Relative $e_i, f_a$, the Christoffel symbols are given by

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k + \Gamma^a_{ij} f_a, \quad \nabla_{f_a} e_j = \Gamma^k_{aj} e_k + \Gamma^b_{aj} f_b, \quad \ldots$$

and the two other obvious permutations of the $e_i$ and $f_a$. The exponential map is determined by geodesics; a path $\gamma(t)$ through $x \in M$, with components $\gamma = (\gamma^A)$ in the foliation coordinates, is geodesic if and only if it satisfies the equation

$$\ddot{\gamma}^A (t) = -\Gamma^A_{BC} (\gamma(t)) \dot{\gamma}^B (t) \dot{\gamma}^C (t)$$

for $t$ in the domain of $\gamma$. But for any geodesic through $x$ and tangent to $\iota(\nu F)$ at $x$, this reduces to $\dot{\gamma}^A (0) = -\Gamma^A_{jk} (x) \dot{\gamma}^j (0) \dot{\gamma}^k (0)$, and the normal component of this reduces further to

$$\dot{\gamma}^i (0) = -\Gamma^i_{jk} (x) \dot{\gamma}^j (0) \dot{\gamma}^k (0).$$

The point here is that only ‘normal’ indices $i, j, k$ appear, but these particular components of the Christoffel symbol depend only on the Bott connection. So, the normal component of the two-jet of the exponential map is independent of choice of $\nabla^F$.

Now, suppose given two different choices of foliation coordinates lifting a fixed frame $\varepsilon$ of $\nu F_x$, with associated adapted lifts $(\vec{e}, \vec{f})$ and $(\vec{e}', \vec{f}')$ near $x \in M$. Denoting all data associated to
the ‘primed’ lift with a prime, note that there exists a matrix function $A^q_3$ so that $e'_i = e_i + A^q_3 f_3$. Furthermore, we have $e_i = i\langle p(e_i) \rangle$ and $e'_i = i\langle p(e'_i) \rangle$ and that $p(e_i)_x = e_i = p(e'_i)_x$. Computing, at $x$ we have

$$\nabla'_e e'_j = i\nabla'_e p(e'_j) = i\nabla'_e p(e_j) + i\nabla'_{A_3 f_3} p(e_j) = i\nabla'_e p(e_j), \quad \text{while} \quad \nabla e_i e_j = i\nabla'_e p(e_j).$$

These have equal projection to the normal bundle, and it follows that the normal component of the ‘bare’ and ‘primed’ Christoffel symbols at $x$ does not depend on the choice of foliations coordinates made. These determine the transverse two jet of the exponential map \cite{14}, which is seen to be independent of choices, as required. \hfill \Box

With Theorem \ref{thm:4.2} in hand, we can now reformulate the Chern-Weil theorem for regular foliations. Using this reformulation as a starting point, it will be relatively clear how to proceed in the singular case. To state the theorem, recall that if $\epsilon$ is a Euclidean structure on a vector bundle, then any connection $\nabla$ on the bundle induces an $\epsilon$-compatible connection $\nabla_\epsilon$, whose connection form on the orthogonal frame bundle is simply the antisymmetric component of the connection form of $\nabla$.

**Theorem 4.3** (Chern-Weil theorem for regular foliations). Let $(M, F)$ be a regular foliation of codimension $q$, and suppose $\nu F$ is equipped with a Euclidean structure $\epsilon$ and a torsion-free Bott connection $\nabla$. Denote by $\nabla_\epsilon$ the $\epsilon$-compatible connection on $\nu F$ induced by $\nabla$, and for $t \in [0, 1]$ denote by $\nabla^t := t\nabla + (1 - t)\nabla_\epsilon$ the affine combination on $M \times I$. The map of algebras $\lambda_{\epsilon, \nabla} : WO_q \to \Omega^*(M)$ defined on generators of $WO_q$ by

$$\lambda_{\epsilon, \nabla}(c_i) := \text{Tr}(R^i_\epsilon) \in \Omega^{2i}(M)$$

$$\lambda_{\epsilon, \nabla}(h_i) := i \int_0^1 \text{Tr} \left( \left( \nabla - \nabla_\epsilon \right) \wedge R^{\alpha(i-1)}_{\nabla^t} \right) dt \in \Omega^{2i-1}(M)$$

is a DGA map. Descending to cohomology, the diagram

$$\begin{align*}
H^*(WO_q) & \xrightarrow{\lambda_{\epsilon, \nabla}} H^*(\Omega(\Gamma q)) \\
\text{Descend to cohomology} & \xrightarrow{\psi} H^*(\Omega(M))
\end{align*}$$

commutes.

**Proof.** Let $\sigma_\epsilon : M \to Fr_1(M/F)/O(q)$ be the section defined by the Euclidean structure, locally determined by any choice of orthonormal framing. Per Theorem \ref{thm:4.2} the Bott connection $\nabla$ determines an equivariant section $\sigma_\nabla : Fr_1(M/F) \to Fr_2(M/F)$, which thus descends to a section $\sigma_\nabla : Fr_1(M/F)/O(q) \to Fr_2(M/F)/O(q)$. It is a routine calculation on generators to verify that for $a \in WO_q$, the image $\lambda_{\epsilon, \nabla}(a)$ is the pullback by $\sigma_\nabla \sigma_\epsilon : M \to Fr_2(M/F)/O(q)$ of the form $a(i^*_\infty \omega^\infty_{h_\nabla}) \in \Omega^*(Fr_2(M/F)/O(q))$. But this latter is defined by evaluation of the Gel’fand-Fuks cocycle associated to $a \in WO_q \subset A(a_q)O(q)$ on the $a_q$-valued form $i^*_\infty \omega^\infty_{h_\nabla}$ on $Fr_\infty(M/F)$. Since $\omega^\infty$ satisfies the Maurer-Cartan identity, $a \mapsto a(i^*_\infty \omega^\infty_{h_\nabla})$ is a homomorphism of DGAs (cf. Equation \cite{8}).
To see that the diagram commutes, recall that the universal characteristic map \( u \) is induced by the cochain map

\[
WO_q \hookrightarrow A^*(a_q)O(q) \xrightarrow{\omega_{\gamma}} \Omega^*(Fr_\infty(\gamma)/ O(q)) \xrightarrow{\sigma^*} \Omega^*(B\Gamma_q),
\]

defined for some choice of smooth section \( \sigma \) of \( Fr_\infty(\gamma)/ O(q) \to B\Gamma_q \). Using contractibility of the fibre of \( Fr_2(M/ F)/ O(q) \to M \), commutativity of the diagram follows from the commutativity up to homotopy of the following diagram.

\[
\begin{array}{cccccc}
\text{Fr}_2(M/ F)/ O(q) & \xrightarrow{\simeq} & \text{Fr}_2(h_F)/ O(q) & \xrightarrow{\simeq} & \eta_F^* \text{Fr}_2(\gamma)/ O(q) & \xrightarrow{\sigma} \text{Fr}_2(\gamma)/ O(q) \\
\sigma \circ \sigma & & & & & \sigma \circ \sigma \\
M & \xrightarrow{id} & M & \xrightarrow{id} & M & \xrightarrow{\eta_F} B\Gamma_q
\end{array}
\]

\[\square\]

5 Chern-Weil for Haefliger-singular foliations

In this section we extend the ideas developed in the previous section to yield a Chern-Weil map for Haefliger-singular foliations admitting adapted geometries, which are pairs consisting of a Bott connection and Euclidean structure over the normal bundle of the regular subfoliation. For such a pair to be adapted means essentially that the Euclidean structure blows up appropriately towards singularities. As an immediate geometric application, we describe how the classical algorithm for the construction of the Godbillon-Vey class, using differential forms, may be extended to the Haefliger-singular setting.

As a final application of our theory, we describe how any smooth Haefliger structure on a manifold is homotopic to a Haefliger-singular foliation. This is fortuitous, because all Haefliger-singular foliations admit adapted geometries. In this way, the theory realises a Chern-Weil map for all homotopy classes of smooth Haefliger structures. Furthermore, the theory is fully functorial (up to homotopy) for maps between Haefliger structures, because any map into a Haefliger-singular foliation may be slightly perturbed to a map pulling back a Haefliger-singular foliation.

Extending the notational convention of the previous section, assume any Haefliger foliation \((M, F)\) of codimension \(q\) to be associated to a smooth map \(\eta_F : M \to B\Gamma_q\), with accompanying Haefliger cocycle \(h_F := \eta_F^* \gamma\).

5.1 Adapted geometries and the Chern-Weil map

Chern-Weil theory provides for the construction of explicit forms that represent the characteristic classes of regular foliations, taking as input certain geometric data—a Bott connection and a Euclidean structure on the normal bundle. The results of the previous section allow the same construction on Haefliger-singular foliations, provided one allows geometric data that are singular but appropriately adapted.

**Definition 5.1.** Let \((M, F)\) be a Haefliger-singular foliation of codimension \(q\). A pair \((\epsilon, \nabla)\) consisting of a Euclidean metric \(\epsilon\) and a torsion-free Bott connection \(\nabla\) on \(\nu F \to \tilde{M}\) is an
adapted geometry for \((M, F)\) if the composition \(i_2\sigma \nabla \epsilon : \tilde{M} \to Fr_2(h_F)/O(q)\) of the associated sections \(\sigma : \tilde{M} \to Fr_1(\tilde{M}/F)/O(q)\) and \(\sigma \nabla : Fr_1(\tilde{M}/F)/O(q) \to Fr_2(\tilde{M}/F)/O(q)\) with \(i_2 : Fr_2(\tilde{M}/F)/O(q) \to Fr_2(h_F)/O(q)\) extends smoothly to a section of \(Fr_2(h_F)/O(q) \to M\).

It will follow from functoriality, to be discussed in the next subsection, that all Haefliger-singular foliations admit adapted geometries (Corollary 5.7). We first describe how an adapted geometry allows for the construction of a Chern-Weil map.

**Theorem 5.2** (Chern-Weil for Haefliger-singular foliations). Let \((M, F)\) be a Haefliger-singular foliation of codimension \(q\), equipped with an adapted geometry \((\epsilon, \nabla)\). Denote by \(\nabla_\epsilon\) the \(\epsilon\)-compatible connection on \(\nu F\) induced by \(\nabla\), and by \(\nabla_t := t\nabla + (1-t)\nabla_\epsilon\) the affine-interpolating family of connections between \(\nabla\) and \(\nabla_\epsilon\). Then the forms

\[
\lambda_{\epsilon, \nabla}(c_i) := Tr(R_{\nabla_t}^i) \in \Omega^{2i}(\tilde{M})
\]

\[
\lambda_{\epsilon, \nabla}(h_i) := i \int_0^1 Tr((\nabla - \nabla_\epsilon) \wedge R_{\nabla_t}^{i(i-1)}) \, dt \in \Omega^{2i-1}(\tilde{M})
\]

extend to globally-defined smooth forms on \(M\). Furthermore, the resulting homomorphism \(\lambda_{\epsilon, \nabla} : WO_q \to \Omega^*(M)\) of DGAs makes the following diagram commute.

\[
\begin{array}{ccc}
H^*(WO_q) & \xrightarrow{u} & H^*(\Omega(B\Gamma_q)) \\
\downarrow^{\lambda_{\epsilon, \nabla}} & & \downarrow^{\eta_F} \\
H^*(\Omega(M)) & &
\end{array}
\]

**Proof.** Let \(\sigma : \tilde{M} \to Fr_1(\tilde{M}/F)/O(q)\) and \(\sigma \nabla : Fr_1(\tilde{M}/F) \to Fr_2(\tilde{M}/F)\) be the sections associated to \(\epsilon\) and \(\nabla\). Then \(\lambda_{\epsilon, \nabla}(c_i) = (i_2\sigma \nabla \epsilon)^*c_i(\omega_{h_F}^2)\) and \(\lambda_{\epsilon, \nabla}(h_i) = (i_2\sigma \nabla \epsilon)^*h_i(\omega_{h_F}^2)\), where \(\omega_{h_F}^2\) is the tautological \(gl(q, \mathbb{R})\)-valued 1-form on \(Fr_2(h_F)\). The extension of \(\lambda_{\epsilon, \nabla}(c_i)\) and \(\lambda_{\epsilon, \nabla}(h_i)\) to globally smooth forms then follows from the fact that \((\epsilon, \nabla)\) is an adapted geometry. Commutativity of the diagram follows from essentially the same argument as in Theorem 4.3. \(\square\)

### 5.2 Functoriality

It is a well-established fact that regular foliations are functorial under transverse maps \([12, 13]\), and that the characteristic classes of regular foliations are similarly functorial. Here we extend this functoriality to smooth maps which are transverse only on a dense set.

**Definition 5.3.** Let \((N, F')\) be a Haefliger foliation, and \(f : M \to N\) a smooth map. Say that \(f\) is **regular at** \(x \in M\) if for some (and so any) Haefliger chart \(h_\alpha : U_\alpha \to \mathbb{R}^q\) with \(f(x) \in U_\alpha\), it holds that \(h_\alpha f\) is a submersion at \(x\). Say that \(f\) is **Haefliger-singular** if it is regular on a dense open set.

It follows immediately from Definition 5.3 that Haefliger-singular foliations are functorial under pullbacks by Haefliger-singular maps. Similarly, one has functoriality of adapted geometries.
Theorem 5.4. Let $(N, F)$ be a Haefliger-singular foliation of codimension $q$, and let $f : M \to (N, F)$ be a Haefliger-singular map. Let $f^* F$ be the induced Haefliger-singular foliation of $M$. If $(\epsilon, \nabla)$ is an adapted geometry for $(N, F)$, then $(f^* \epsilon, f^* \nabla)$ is an adapted geometry for $(M, f^* F)$, and the diagram

$$
\begin{array}{ccc}
\lambda_\epsilon \sigma & \Omega^* (N) \\
WO_q & \downarrow f^* \\
\lambda_{f^* \epsilon, f^* \nabla} & \Omega^* (M)
\end{array}
$$

commutes.

**Proof.** The pullback data $f^* \epsilon$ and $f^* \nabla$ are associated to the pullbacks by $f$ of the corresponding sections $\sigma_\epsilon : \tilde{N} \to \text{Fr}_1(\tilde{N} / F) / O(q)$ and $\sigma_\nabla : \text{Fr}_1(\tilde{N} / F) \to \text{Fr}_2(\tilde{N} / F)$. Consider the commuting diagram, with $\sigma$ the unique smooth section extending $i_2 \sigma_\nabla \sigma_\epsilon$.

$$
\begin{array}{ccc}
\tilde{N} & \sigma \sigma_\epsilon & N \\
\downarrow f & \downarrow \sigma & \downarrow f \\
\text{Fr}_2(\tilde{N} / F) / O(q) & \text{Fr}_2(h_F) / O(q) & N \\
\downarrow i_2 & \downarrow f_2 & \downarrow f \\
\text{Fr}_2(\tilde{M} / F) / O(q) & \text{Fr}_2(h_{f^* F}) / O(q) & f^* \sigma
\end{array}
$$

Functoriality of the Haefliger bundle affords the identification $\text{Fr}_2(h_{f^* F}) \equiv f^* \text{Fr}_2(h_F)$, and thus a well defined, smooth pullback section $f^* \sigma$. At any regular point $x \in \tilde{M}$, one sees that $i_2 \sigma_\nabla \sigma_\epsilon (x) = f^* \sigma (x)$ by chasing around the top of the diagram, and $\tilde{M}$ is dense, so the section $f^* \sigma$ is the unique extension of $i_2 \sigma_\nabla \sigma_\epsilon$. As such, $(f^* \epsilon, f^* \nabla)$ is adapted.

The commutativity of Diagram (15) follows also from commutativity of Diagram (16), because the Chern-Weil morphism factors through the forms on the Haefliger bundle. □

Theorem 5.4 greatly enlarges the class of Haefliger-singular foliations admitting adapted geometries. Indeed, all geometries $(\epsilon, \nabla)$ on a regular foliation $(N, F)$ are adapted, so every Haefliger-singular map $f : M \to (N, F)$ induces an adapted geometry on $(M, f^* F)$. In the final subsection, we will show that in fact every Haefliger-singular foliation admits adapted geometries. Moreover, in the primary application of our theory, we will show that the category of Haefliger-singular foliations with adapted geometries, whose morphisms are Haefliger-singular maps pulling back the foliation and geometry of the codomain to that of the domain, is homotopy-equivalent to the category of all smooth Haefliger structures on manifolds. Before turning to this, we describe an immediate geometric application.

### 5.3 Application: a singular Godbillon-Vey algorithm

The classical Godbillon-Vey algorithm [22] for a regular, transversely orientable foliation $(M, F)$ of codimension 1 proceeds by choosing a nonvanishing 1-form $\omega$ defining $F$. Then by the classical
Frobenius theorem, there exists a 1-form $\eta$ such that

$$d\omega = \eta \wedge \omega.$$ 

The 3-form $\eta \wedge d\eta$ is closed, and its class in de Rham cohomology, which is independent of the choices of $\eta$ and $\omega$, is the Godbillon-Vey class of the foliation. This algorithm is a key tool in the constructions of Roussarie [22] and Thurston [47] of foliations with nonvanishing Godbillon-Vey invariant.

In the recent paper [37], an attempt is made to employ the Godbillon-Vey algorithm in the context of codimension 1 singular foliations, in an application to fluid mechanics. However, its use in this singular context requires more care. For instance, if a 1-form $\omega$ is singular in that is allowed to have zeroes, then it is no longer true that the identity $\omega \wedge d\omega = 0$ suffices to guarantee integrability in the sense that $\omega$ is locally of the form $f \, dg$ for some functions $f$ and $g$ [39]. More generally, integrability of singular $q$-forms is an extremely subtle problem [40], and the relationship between integrable singular $q$-forms and codimension $q$ Haefliger structures (and therefore to characteristic classes) is far from clear.

Our next theorem supplies a singular generalisation of the Godbillon-Vey algorithm using our Chern-Weil theorem. We will say that a Haefliger-singular foliation is transversely orientable if each connected component of its regular subfoliation is transversely orientable.

**Theorem 5.5** (Singular Godbillon-Vey algorithm). Let $(M, \mathcal{F})$ be a transversely orientable, Haefliger-singular foliation of codimension $q$, and suppose that $(\varepsilon, \nabla)$ is an adapted geometry for $(M, \mathcal{F})$. Then there is a singular $q$-form $\omega$ on $M$ for which:

1. $\omega|_{\tilde{M}}$ is nonvanishing and defines the regular subfoliation,
2. $d\omega = \eta \wedge \omega$ for some smooth 1-form $\eta$ on $M$, and
3. $(-1)^{q+1} \eta \wedge (d\eta)^q \in \Omega^{2q+1}(M)$ is a closed form representing the Godbillon-Vey class of the foliation.

**Proof.** Let $h$ denote the Haefliger structure defining the foliation. We begin with a computation using the tautological forms $\omega^k_h$ on the Haefliger bundles. For notational convenience, denote the tautological $\mathbb{R}^q$-valued form $\omega^i_h$ on the Haefliger bundle $\text{Fr}_2(h)$ associated to $h$ by the tuple $(\omega^i)_i^q$, and similarly denote the tautological $\mathfrak{g}(q, \mathbb{R})$-valued form $\omega^j_h$ by the tuple $(\omega^j)^j_{j=1}^q$. Denote by $\bar{\omega}$ the $q$-form $\omega^1 \wedge \cdots \wedge \omega^q$. Using the first structure equation of Equation (4) one has

$$d\bar{\omega} = \sum_{k=1}^q (-1)^{k+1} \omega^1 \wedge \cdots \wedge \omega^{k-1} \wedge d\omega^k \wedge \omega^{k+1} \wedge \cdots \wedge \omega^q$$

$$= \sum_{k=1}^q (-1)^k \omega^1 \wedge \cdots \wedge \omega^{k-1} \wedge \omega^j_k \wedge \omega^j \wedge \omega^{k+1} \wedge \cdots \wedge \omega^q$$

$$= \sum_{k=1}^q -\omega^k \wedge \bar{\omega} = -\text{Tr}(\omega^2_h) \wedge \bar{\omega}.$$ 

Being $O(q)$-basic, the form $\bar{\omega}$ descends to a form on $\text{Fr}_2(h)/O(q)$, hence so too does $\text{Tr}(\omega^2_h)$.
Now, the section $\sigma, \sigma_\nabla$ pulls back $\omega^1 \wedge \cdots \wedge \omega^q$ to a smooth $q$-form $\omega$ on $M$ which, since $(M, \mathcal{F})$ is transversely orientable, defines the regular subfoliation on each connected component. The calculation above shows, moreover, taking $\eta$ to be the pullback of $-\text{Tr}(\omega^2_h)$ along $\sigma, \sigma_\nabla$, that one has $d\omega = \eta \wedge \omega$. Finally, by Theorem 5.2 we see that the Godbillon-Vey invariant of $(M, \mathcal{F})$ is represented by

$$\lambda_{e, \nabla}(h_1 c_1^q) = \text{Tr}(\omega_h^2) \wedge d\text{Tr}(\omega_h^2) = (-1)^{q+1} \eta \wedge (d\eta)^q$$

as claimed.

In the next and final subsection, we prove one of the main results of the paper - that smooth Haefliger structures on manifolds of sufficiently high dimension are categorically homotopy-equivalent to Haefliger-singular foliations. This result furnishes the primary application of our theory in giving a complete geometric solution to the problem of representing the characteristic classes of Haefliger structures.

### 5.4 Application to Haefliger structures

This section is devoted to proving the general applicability of our theory; specifically that the class of Haefliger-singular foliations is homotopically identical to that of smooth Haefliger structures on manifolds.

The first result pertains to the graph of a Haefliger structure, which allows embeddings of an arbitrary Haefliger foliation into a regular foliation. The construction is due to Haefliger, [24, p. 188], but we recall it for the convenience of the reader. Fix a representative Haefliger cocycle over some locally finite open cover $\{U_\alpha\}_{\alpha \in \mathbb{N}}$ of a manifold $M$, with Haefliger charts $f_\alpha : U_\alpha \to \mathbb{R}^q$. Form an open manifold $G(h)$ by gluing appropriate neighborhoods $V_\alpha$ of $\Gamma(f_\alpha) = \{(x, f_\alpha(x)) : x \in U_\alpha\}$ in $U_\alpha \times \mathbb{R}^q$ along the change of chart maps associated to the cocycle. One obtains a regular foliation on $G(h)$ by gluing the foliations given on each $V_\alpha$ by the level sets of $V_\alpha \hookrightarrow U_\alpha \times \mathbb{R}^q \rightarrow \mathbb{R}^q$. The graph of $h$ is the map $i : M \to G(h)$ obtained by gluing the graphs $U_\alpha \to \Gamma(f_\alpha)$, well-defined by construction of $G(h)$. The following is easy from the construction.

**Proposition 5.6.** Every Haefliger structure $(M, h)$ on a manifold $M$ admits a Haefliger embedding into a regular foliation, namely, $G(h)$ with its regular foliation. In particular, $h$ is the pullback of the regular foliation on $G(h)$, and the embedding is regular wherever $h$ is regular.

Since regular foliations trivially admit adapted geometries, one has the following immediate corollary of Proposition 5.6 and Theorem 5.4.

**Corollary 5.7.** Every Haefliger-singular foliation admits adapted geometries.

Recall that a Haefliger-singular map $M \to (N, \mathcal{F}')$ is one whose regular set is dense. A map $M \to N$ is Haefliger-singular if and only if the induced pullback foliation on $M$ is Haefliger-singular.

**Proposition 5.8.** Given a codimension $q$ Haefliger-singular foliation $(N, \mathcal{F}')$, every smooth map $f : M \to N$ from a manifold $M$ of dimension at least $q$, is homotopic to a Haefliger-singular map.
Proof. Up to a small perturbation, we may assume $f$ smooth. We will show the stronger statement that the subset of Haefliger-singular maps is dense (in fact, residual) in $\mathcal{C}^\infty(M,N)$ equipped with the strong topology. Since $\mathcal{C}^\infty(M,N)$ is locally path connected in this topology, there exists a continuous path to some Haefliger-singular map, which induces a homotopy by uncurrying.

First suppose that the foliation on $N$ is regular. Associated to $(N,\mathcal{F}')$ is a regular distribution $D \subset TN$, of codimension $q$. The argument proceeds through the Thom jet-transversality theorem, to show that the set of smooth maps $M \to N$ transverse to $D$ on a dense subset of $M$ is itself dense in $\mathcal{C}^\infty(M,N)$. (Note that the integrability of $D$ is not used in the following, and the argument works for any regular smooth distribution.) Fix $m = \dim M$ and $n = \dim N$.

In the 1-jet bundle $J^1(M,N)$, let $\mathcal{V}$ denote the subset of $J^1(M,N)$ comprising 1-jets that are not transverse to $D$. Although $\mathcal{V}$ is not generally a manifold, it is a finite union of manifolds, each of positive codimension. To see this, note that the bundle map $J^1(M,N) \to M \times N$ restricts to a bundle map $\mathcal{V} \to M \times N$, with fiber over each $(x,y) \in M \times N$ a certain variety $\mathcal{V}_0$, and $\mathcal{V}_0$ is a finite union of manifolds, each of positive codimension in the fiber $J^1(M,N)(x,y) \cong \text{Hom}(\mathbb{R}^m,\mathbb{R}^n)$. More precisely, $\mathcal{V}_0$ is, up to isomorphism, the variety of elements in $\text{Hom}(\mathbb{R}^m,\mathbb{R}^n)$ that pull back to zero the $q$-form $dy_1 \wedge \cdots \wedge dy^n$ on $\mathbb{R}^n$; this condition determines polynomial equations in the matrix coefficients, which can be written out explicitly if required. One might also compare the proof of Theorem 3.2.6 in [29].

Write $\mathcal{V} = \bigcup W_a$ for a finite collection of manifolds $W_a$. Given a map $g: M \to N$, the set of points where $g$ is not transverse to $D$ is exactly $j^1(g)^{-1}(\mathcal{V})$. But each $W_a$ has positive codimension, so if $j^1(g)$ is transverse to each $W_a$, the set $j^1(g)^{-1}(\mathcal{V})$ is contained in a finite union of positive-codimension submanifolds, so has dense complement. On the other hand, by Theorem 3.2.8 of [29], the set of elements in $\mathcal{C}^\infty(M,N)$ that have 1-jet lift transverse to $W_a$ is a residual set in $\mathcal{C}^\infty(M,N)$. Since $\mathcal{C}^\infty(M,N)$ is Baire ([29] Theorem 2.4.2), residual sets are dense.

Now suppose $(N,\mathcal{F}')$ is Haefliger-singular, so that the regular set $\tilde{N}$ is a dense open submanifold of $N$. Let $D$ be the (regular) distribution over $\tilde{N}$, and $\mathcal{V}$ the set of 1-jets with target in $\tilde{N}$ and not transverse to $D$. As in the regular case, $\mathcal{V}$ is a $\mathcal{V}_0$ bundle over $\tilde{N}$, a finite union of positive-codimension submanifolds in $J^1(M,N)$. Theorem 3.2.8 of [29] still applies, and the set of maps densely transverse to $D$ remains residual in $\mathcal{C}^\infty(M,N)$. On the other hand, the following holds, with proof deferred momentarily.

Lemma 5.9. Given an open dense set $\tilde{N} \subset N$, let $A$ be the set of smooth maps $g: M \to N$ such that $g^{-1}(\tilde{N})$ is dense in $M$. The set $A$ is residual in $\mathcal{C}^\infty(M,N)$.

The singular set of a map $g: M \to N$ is contained in $g^{-1}(\Sigma') \cup j^1(g)^{-1}(\mathcal{V})$, with $\Sigma'$ the singular set of $N$. As such, any map in the intersection of the set of maps $g: M \to N$ such that $g^{-1}(\Sigma')$ has dense complement, and the set of maps $g$ so that $j^1(g)$ is transverse to $\mathcal{V}$, is a Haefliger-singular map. Both sets are residual, so their intersection is residual, so is dense, as required.

Proof of Lemma 5.9. Consider the evaluation map $ev: \mathcal{C}^\infty(M,N) \times M \to N$. Since $ev$ is continuous, open, and surjective, the pullback $D = ev^{-1}(\tilde{N})$ is open and dense in $\mathcal{C}^\infty(M,N) \times M$. 29
The evaluation map is open because it is open on each product open $U \times V$, which is a union of $U \times \{x\}$ over $x$ in $V$. Each $ev(U \times \{x\})$ is already open, sufficing to take $U$ basic in $C^\infty (M, N)$, say the set of all maps $M \to N$ whose graph lies in some open neighborhood $W$ of the graph of a fixed $g_0: M \to N$. The projection of the set $W \cap (\{x\} \times N)$ is open in $N$, and equals $ev(U \times \{x\})$.

That the set $A$ is residual follows from the more general following claim: given topological spaces $X, Y$ such that $Y$ is a metric space admitting countable dense sequence $y_n$, and a subset $D \subset X \times Y$, denote by $D_x := \pi_Y (\{x\} \times Y) \cap D \subset Y$ the $x$-slice of $D$ in $Y$, and by $X$ the set of points $x \in X$ so that $D_x$ is dense in $Y$. The sets $D_x$ are open because $D$ is open. The (residual) countable intersection

$$X' = \bigcap_{n \in \mathbb{N}} D^{B_n}$$

is contained in $X$ by construction, which is to say that for each $x \in X'$, we have $D_x$ dense in $Y$. Indeed, fix $x \in X'$, and for any $y \in Y$, take a subsequence so that $y_n \to y$. There is for each $n$ some $y'_n \in B_n \cap D_x$, so that $(x, y'_n) \in D$. The new sequence $y'_n$ also converges to $y$, so $D_x$ contains a sequence converging to $y$ for all $y \in Y$, i.e. $D_x$ is dense in $Y$.

Putting together the results of this subsection, we obtain one of the primary theorems of our paper, enabling the application of our Chern-Weil theorem to all smooth Haefliger structures on manifolds of sufficiently high dimension. Let $\mathcal{F}^q_{\text{sing}}$ denote the category whose objects are Haefliger-singular foliations of codimension $q$ with adapted geometries, and whose morphisms are Haefliger-singular maps pulling back the foliation and geometry on the codomain to that on the domain. Let $\mathcal{H}^q_{\text{man}}$ denote the category whose objects are smooth Haefliger structures on manifolds of dimension at least $q$, and whose morphisms are smooth functions pulling back the Haefliger structure on the codomain to that on the domain.

**Theorem 5.10.** The inclusion of categories $\mathcal{F}^q_{\text{sing}} \hookrightarrow \mathcal{H}^q$ is a homotopy equivalence. Consequently, the characteristic map for all smooth Haefliger structures on manifolds of sufficiently high dimension is given functorially by the Chern-Weil map for Haefliger-singular foliations.

### 6 Discussion

Our work opens up a number of directions for future research, which we briefly discuss now.
Noncommutative geometry: The study of singular foliations has gained substantial traction in recent years, following the seminal construction of the holonomy groupoid of a Stefan-Sussmann singular foliation by I. Androulidakis and G. Skandalis [2] (as we show in Appendix A, all foliations we consider fall into this class). As is well-known, the Gel’fand-Fuks classes of a regular foliation play an important role in noncommutative geometry, where they define cyclic cocycles that pair with $K$-theory elements of the convolution algebra associated to the holonomy groupoid to yield numerical invariants [15]. We anticipate that our theory will facilitate the construction of analogous cocycles to pair with groupoid algebras of Haefliger-singular foliations, allowing deeper insight into the structure of the noncommutative leaf spaces of such objects.

Residue formulæ: Recall Bott’s celebrated residue formula [7], which expresses characteristic numbers of a Riemannian manifold in terms of quantities localised to the zeros of a Killing field. This may have an analogy for singular foliations. Similar residue formulæ for Chern forms associated to the normal bundle of (not necessarily Haefliger-) singular foliations have already appeared in the literature and aided the study of characteristic classes [6, 27]. Motivated by this, we ask whether there exists a Haefliger-singular foliation whose singular set is an embedded submanifold, which admits adapted geometries that are preserved by leafwise vector fields vanishing along the singular set. By similar arguments to those in [7], the Godbillon-Vey invariant of such a foliation (thought of as a current) could be localised to the singular set in a residue formula, which may shed new light on the topological dynamics of singular foliations.

A Haefliger structures and Stefan-Sussmann singular foliations

Here we fill in a gap in the literature by showing that all smooth Haefliger structures on manifolds induce Stefan-Sussmann foliations. We begin by recalling the definition of a Stefan-Sussmann foliation. Start with a distribution on a smooth manifold $M$, here meaning some fiberwise linear subset $\Delta \subseteq TM$, so a choice of linear subspace $\Delta_x \subseteq T_x M$ (of possibly varying dimension) for each point $x \in M$. Now, take the smooth functions $C^\infty(M)$ as a sheaf on $M$, and the vector fields $\mathfrak{X}(M)$ as a module over this sheaf. We consider sub-modules $D$ of $\mathfrak{X}(M)$, subject to the condition that each point $M$ be contained in some open set $U$ so that $D(U)$ is not empty (the module covers $M$; the zero vector field is valid). We will want $D$ to be involutive, closed under the local application of Lie brackets, but $D$ can always be replaced with its closure under Lie brackets if necessary. There is a map from such tangent subsheaves to distributions; the subsheaf $D$ determines $\Delta$ which for each $x \in M$ is spanned by the germinal vector fields in the stalk of $D$ at $x$.

Any distribution that is so spanned by a tangent subsheaf is called a smooth distribution. An integral submanifold of a smooth distribution is an imbedded submanifold $\iota: \Sigma \rightarrow M$ (not necessarily homeomorphic onto its image) that through each image point has tangent plane equal to the distribution, $d\iota(T_x \Sigma) = \Delta_{\iota(x)}$. A smooth distribution is integrable if it has an integral submanifold through each point; in this case, the maximal integral submanifolds are unique through each point, and determine a partition of $M$ into imbedded submanifolds. Stefan [15] and Sussmann [16] independently and near simultaneously discovered the conditions under
which a smooth distribution is integrable, the statement as follows.

**Theorem A.1** ([45], [46]). *On a smooth manifold \( M \), a smooth distribution \( \Delta \) (spanned by \( D \)) is integrable if and only if \( \Delta \) is invariant under the local flows of (local) vector fields in \( D \). \( \square \)

Notice that the requirement that sections of \( \Delta \) be closed under Lie bracket is implicit in the condition that \( \Delta \) be invariant under the local flows of vector fields valued in \( \Delta \).

Now, let \( h: \tilde{U} \to \Gamma_q \) be a Haefliger cocycle over an open cover \( U = \{U_\alpha\}_{\alpha \in A} \) of a manifold \( M \) (Definition 3.3). We will denote by \( f_\alpha := s \circ h_{\alpha \alpha} : U_\alpha \to \mathbb{R}^q \) the corresponding Haefliger charts. There is a naturally associated differential ideal \( \mathcal{I} \) on \( M \), which is generated locally, in each Haefliger chart \( f_\alpha : U_\alpha \to \mathbb{R}^q \), by \( \omega^i_\alpha := f^*_\alpha dx^i \) (where the \( x^i \) are standard coordinates on \( \mathbb{R}^q \)). This ideal is clearly *formally Frobenius*, in the sense that it is generated algebraically by the 1-forms \( \omega^i_\alpha \); equivalently, for each \( i = 1, \ldots, q \),

\[
d\omega^i_\alpha \equiv 0 \mod \omega^j_\alpha, \quad j = 1, \ldots, q.
\]

(In fact, \( d\omega^i_\alpha \equiv 0 \) identically, but it is the weaker displayed condition that is basis independent, and crucial.)

There is associated to this ideal a sheaf of vector fields; for \( U \) open in \( M \),

\[
\mathcal{D}(U) = \{ X \in \mathfrak{X}(U) : \omega(X) = 0 \text{ all } \omega \in \mathcal{I}(U) \}.
\]

That this sheaf is involutive follows from the fact that \( \mathcal{I} \) is formally Frobenius, viz.

\[
\omega([X,Y]) = d\omega(X,Y) - X\omega(Y) + Y\omega(X) = 0
\]

for any \( X,Y \in \mathcal{D}(U) \) and any \( \omega \in \mathcal{I}(U) \).

The sheaf \( \mathcal{D} \) spans a distribution \( \Delta \subset TM \), and per the Stefan-Sussmann Theorem, \( \Delta \) is seen to be integrable once it is shown to be invariant under flows of elements of \( D \). This is true, as we show now for completeness.

**Proposition A.2.** *The distribution \( \Delta \) associated to a Haefliger cocycle on a manifold \( M \) is integrable.*

**Proof.** It suffices to show for each point \( x \in M \) and vector field \( X \in \mathcal{D} \) defined near \( x \), that \( d\Phi_t(\Delta_x) = \Delta_{\Phi_t(x)} \), with \( \Phi_t = \Phi_t^X \) the local flow of \( X \). But this suffices to be shown for small enough \( t \) that a neighborhood of the curve \( t \mapsto \Phi_t(x) \) is contained in the domain of a single Haefliger chart \( f_\alpha : U_\alpha \to \mathbb{R}^q \).

Now, the subspace \( \Delta_y \) depends only on the stalk of \( \mathcal{D} \) at \( y \), which in turn depends only on the stalk (or germ) of \( \mathcal{I} \) at \( y \). The latter is invariant under local flows along \( D \), so the former is too. To be explicit, we have first \( (\Phi_t^X \mathcal{I})_x = \mathcal{I}_x \), as follows from the observation that \( f_\alpha \circ \Phi_t = f_\alpha \) and the implication that

\[
(\Phi_t^* \omega^i_\alpha)_x = \Phi_t^* f_\alpha^* dx^i = (f_\alpha \Phi_t)^* dx^i = f_\alpha^* dx^i = (\omega^i_\alpha)_x \Rightarrow (\Phi_t^X \mathcal{I})_x \supseteq \mathcal{I}_x,
\]

plus the opposite inclusion deduced from an application of \( \Phi_{-t}^* \). Then, given \( v \in \Delta_{\Phi_t(x)} \) and \( V \in \mathcal{D} \) extending \( v \), let \( W = d\Phi_{-t}V \) be the vector field pullback of \( V \). For any \( \omega \in \mathcal{I} \) near \( x \),
let \( \omega' = \Phi^* \omega \in \mathcal{I} \); we find, for \( y \) near \( x \), that

\[
\omega_y(W) = (\Phi_t^\ast \omega')(y)(W) = \omega_{\Phi_t(y)}(V) = 0.
\]

So, \( W \) is in \( \mathcal{D} \), its evaluation at \( x \) is in \( \Delta_x \), and we see that \( \Delta_{\Phi_t(x)} \subseteq d\Phi_t(\Delta_x) \). The reverse inclusion follows analogously, as required.

\[
\Box
\]

B  The diffeological generalisation of Haefliger’s classifying theorem

In this appendix we detail our claim that our diffeological classifying space theorem (Theorem 3.7) generalises Haefliger’s topological classifying theorem, which we now recall for the reader’s convenience.

**Theorem B.1** (Haefliger’s classifying theorem). [25, Theorem 7] Let \( \Gamma \) be a topological groupoid, and regard \( B\Gamma \) with the strong topology (the weakest topology making the canonical partition of unity and cocycle maps on \( B\Gamma \) continuous).

1. The canonical \( \Gamma \)-structure \( \gamma \) on \( B\Gamma \) is numerable.

2. For any numerable \( \Gamma \)-structure \( h \) on a topological space \( X \), there is a continuous map \( \eta : X \to B\Gamma \) such that \( h = \eta^* \gamma \).

3. If \( \eta_0, \eta_1 : X \to B\Gamma \) are continuous maps, then \( \eta_0^* \gamma \) and \( \eta_1^* \gamma \) are numerably homotopic if and only if \( \eta_0 \) and \( \eta_1 \) are homotopic.

To see that Theorem B.1 follows from Theorem 3.7, equip the topological groupoid \( \Gamma \) with the continuous diffeology. Then the canonical partition of unity on the diffeological space \( B\Gamma \) is smooth, hence continuous with respect to the D-topology, hence continuous with respect to the strong topology, which is contained in the D-topology. That the second and third items of Haefliger’s classifying theorem follow from the corresponding items of Theorem 3.7 can be seen immediately by the application of the following lemma, whose proof is an elementary consequence of the definitions.

**Lemma B.2.** Let \( \Gamma \) be a topological groupoid, equipped with the continuous diffeology. Then the corresponding Mostow diffeology on \( B\Gamma \) coincides with the continuous diffeology on \( B\Gamma \) induced by the strong topology. Consequently, if \( X \) is a topological space with the continuous diffeology, a map \( \eta : X \to B\Gamma \) is smooth if and only if it is strongly continuous.

Suppose now that \( h \) is a numerable \( \Gamma \)-structure on a topological space \( X \). Equipping \( X \) with the continuous diffeology, \( h \) is smoothly numerable, hence by Theorem 3.7 is associated to a smooth (hence strongly continuous) map \( \eta : X \to B\Gamma \) such that \( h = \eta^* \gamma \). Conversely, by Lemma B.2, any strongly continuous map \( \eta : X \to B\Gamma \) is automatically smooth when \( X \) is equipped with the continuous diffeology and \( B\Gamma \) with the Mostow diffeology, and Theorem 3.7 then applies to yield a corresponding continuous \( \Gamma \)-structure \( \eta^* \gamma \) on \( X \).
Finally suppose that \( \eta_0, \eta_1 : X \to B\Gamma \) are strongly continuous maps. By Lemma \[3.2\] they are then smooth for the continuous diffeology on \( X \) and the Mostow diffeology on \( B\Gamma \), and Theorem \[3.7\] then applies to show that \( \eta_0^* \gamma \) and \( \eta_1^* \gamma \) are numerably homotopic if and only if \( \eta_0 \) and \( \eta_1 \) are homotopic.

### C Sections of Haefliger bundles mod \( O(q) \)

Here we describe an explicit construction, inspired by Dupont \[17\], of a smooth section of the bundle \( Fr_{\infty}(\gamma)/O(q) \to B\Gamma_q \), from which the existence of analogous sections for all smoothly numerable Haefliger structures follows. Note that the existence of continuous sections follows from the fact that \( B\Gamma_q \) (with the strong topology) has the homotopy type of a CW-complex together with contractibility of the fibre, but the existence of smooth sections in the diffeological setting requires additional argumentation.

To this end, we show in this appendix how to construct a section \( \sigma \) of \( Fr_{\infty}(\gamma)/O(q) \) from any section \( \sigma_0 \) of \( Fr_{\infty}(\mathbb{R}^q) \to \mathbb{R}^q \). In fact, we will identify \( Fr_{\infty}(\gamma) \) with the classifying space \( B\Gamma_{q,\infty} \) of a smooth groupoid \( \Gamma_{q,\infty} \), and then \( \sigma \) will given functorially as \( B\sigma_0 \). The construction will furthermore show that any two smooth sections of a Haefliger bundle mod \( O(q) \) are smoothly homotopic, so that their induced characteristic maps (Definition \[3.10\]) coincide.

**Proposition C.1.** For \( k \in \mathbb{N} \cup \{\infty\} \), define \( \Gamma_{q,k} \) to be the action groupoid \( \Gamma_q \ltimes Fr_k(\mathbb{R}^q) \). Then \( B\Gamma_{q,k} \to B\Gamma_q \) is a principal \( G_q^k \)-bundle over \( B\Gamma_q \) that is canonically isomorphic to \( Fr_k(\gamma) \to B\Gamma_q \).

**Proof.** The groupoid \( \Gamma_{q,k} = \Gamma_q \ltimes Fr_k(\mathbb{R}^q) = \Gamma_q \times_{s,n_k} Fr_k(\mathbb{R}^q) \) is equipped with the subspace diffeology of the product \( \Gamma_q \times Fr_k(\mathbb{R}^q) \), meaning that a parameterisation \( P : U \to \Gamma_q \ltimes Fr_k(\mathbb{R}^q) \) is a plot if and only if each of its component maps \( U \to \Gamma_q \) and \( U \to Fr_k(\mathbb{R}^q) \) are plots.

A smooth action of a diffeological group \( G \) on a diffeological space \( X \) is by definition principal if and only if the action map \( a : X \times G \ni (x, g) \mapsto (x, x \cdot g) \in X \times X \) is a diffeological induction \[31\], Section 8.11], meaning that \( a \) is injective (the action is free) and each parameterisation \( P : U \to \Gamma_q \ltimes Fr_k(\mathbb{R}^q) \) is a plot if and only if \( a \circ P \) is. Since \( Fr_k(\mathbb{R}^q) \to \mathbb{R}^q \) is a principal \( G_q^k \)-bundle, the action map \( a : Fr_k(\mathbb{R}^q) \times G_q^k \to Fr_k(\mathbb{R}^q) \times Fr_k(\mathbb{R}^q) \) is an induction. It induces an action map \( a_\Gamma : \Gamma_{q,k} \times G_q^k \to \Gamma_{q,k} \times \Gamma_{q,k} \) by the rule

\[
a_\Gamma(\gamma, \phi, g) = ((\gamma, \phi), (\gamma, a(\phi, g))),
\]

which is inductive because \( a \) is and by definition of the diffeology on \( \Gamma_{q,k} \). The map \( a_\Gamma \) induces in turn an action map \( B\alpha_\Gamma : B\Gamma_{q,k} \times G_q^k \to B\Gamma_{q,k} \times B\Gamma_{q,k} \), given by the formula

\[
B\alpha_\Gamma\left(\tilde{t}; \tilde{a}; \overline{(\gamma, \phi)}\right), g := \left(\tilde{t}; \tilde{a}; \overline{(\gamma, \phi)}, \left[\tilde{t}; \tilde{a}; a_\Gamma(\gamma, \phi, g)\right]\right).
\]

Here the composable tuple

\[
\overline{(\gamma, \phi)} = ((\gamma_1, \phi_1), \ldots, (\gamma_n, \phi_n)) \in \Gamma_{q,k}^{(n)}
\]
is mapped to the composable tuple
\[ a_\Gamma(\gamma, \phi, g) = ((\gamma_1, a(\phi_1, g)), \ldots, (\gamma_n, a(\phi_n, g))). \]

The map \( a_\Gamma \) preserves each open \( U_\alpha \) of the canonical cover of \( B\Gamma_{q,k} \), and is such that for any \( \alpha, \beta \in \mathbb{N} \), the diagram
\[
\begin{array}{ccc}
(U_\alpha \cap U_\beta) \times G_q^k & \xrightarrow{\text{Bar}} & (U_\alpha \cap U_\beta) \\
\downarrow_{\gamma_{a,\beta} \times \text{id}} & & \downarrow_{\gamma_{a,\beta} \times \gamma_{a,\beta}^k} \\
\Gamma_{q,k} \times G_q^k & \xrightarrow{a_\Gamma} & \Gamma_{q,k} \times \Gamma_{q,k}
\end{array}
\]
commutes. Indeed, this follows because the right action of \( G_q^k \) on \( \text{Fr}_k(\mathbb{R}^q) \) commutes with the left action of \( \Gamma_q \). That \( \text{Bar}_\Gamma \) is an induction then follows from the inductivity of \( a_\Gamma \) by a diagram chase.

Finally, we come to identifying \( B\Gamma_{q,k} \) with \( \text{Fr}_k(\gamma) \). For this, observe that for any \( \alpha \in \mathbb{N} \) we have a canonical, \( G_q^k \)-equivariant identification of \( U_\alpha \subset B\Gamma_{q,k} \) with \( (s \circ \gamma_\alpha)^* \text{Fr}_k(\mathbb{R}^q) \), defined by
\[
(\bar{\tilde{\gamma}}; \tilde{\alpha}; (\gamma, \phi)) \mapsto (\bar{\tilde{\gamma}}; \tilde{\alpha}; \bar{\gamma}); \phi_j, \quad (17)
\]
where \( \alpha = \alpha_j \in \tilde{\alpha} \). Furthermore, if \( (\bar{\tilde{\gamma}}; \tilde{\alpha}; (\gamma, \phi)) \in U_\alpha \cap U_\beta \), where \( \beta = \alpha_k = \tilde{\alpha} \), then one has \( \phi_j = \gamma_{\alpha_\beta}(\bar{\tilde{\gamma}}; \tilde{\alpha}; \bar{\gamma}) \cdot \phi_k \). It follows that the local identifications of Equation (17) patch together to a global identification of \( B\Gamma_{q,k} \) with \( \text{Fr}_k(\gamma) \).

Recall the natural identification of \( G_q^1 \) with \( \text{GL}(q, \mathbb{R}) \). Recall too the inclusions
\[
\text{O}(q) \hookrightarrow \text{GL}(q, \mathbb{R}) \hookrightarrow G_q^\infty,
\]
defined by taking the infinite jet at 0 of diffeomorphisms that fix the metric and linear structures of \( \mathbb{R}^q \) respectively. The construction of a section of \( \text{Fr}_\infty(\gamma)/\text{O}(q) \to B\Gamma_q \) requires the following lemma, which defines a canonical ‘exponential’ path in \( G_q^\infty/\text{O}(q) \) to each element from the identity equivalence class. For this we make use of the fact that, as a projective limit of manifolds, \( G_q^\infty \) admits a natural tangent structure [1, Chapter 1].

**Lemma C.2.** There exists a \( G_q^1 \)-equivariant diffeomorphism \( e: T_{\text{O}(q)}(G_q^\infty/\text{O}(q)) \to G_q^\infty/\text{O}(q) \).

**Proof.** The natural projection \( \pi: G_q^\infty \to G_q^1 \) of groups is split, so, letting \( N = \ker(\pi) \), there is an identification of \( G_q^\infty \) as the semidirect product \( G_q^1 \ltimes N \). Explicitly, the identification is given by
\[
G_q^\infty \ni g \mapsto (\pi(g), g\pi(g)^{-1}) \in G_q^1 \ltimes N.
\]
The right action of \( \text{O}(q) \) only disturbs the first factor, so induces a diffeomorphism \( G_q^\infty/\text{O}(q) \cong G_q^1/\text{O}(q) \times N \). This diffeomorphism is equivariant for the left action of \( G_q^1 \), in that
\[
A[g] \mapsto (A[\pi(g)], Ag\pi(g)^{-1}A^{-1})
\]
for $A \in G^1_q$ and $[g] \in G^\infty_q / O(q)$. This gives also the identification

$$T_{O(q)}(G^\infty_q / O(q)) \cong (g^1_q / \mathfrak{so}(q)) \oplus n.$$ 

There is a canonical bi-$O(q)$-invariant, left-$G^1_q$-invariant Riemannian structure on $G^1_q$, which descends to a Riemannian structure on the homogeneous space $G^1_q / O(q)$. The Riemannian exponential map for this metric induces a diffeomorphism $\exp : T_{O(q)}(G^1_q / O(q)) \to G^1_q / O(q)$, which is equivariant for the left multiplication actions of $G^1_q$ \cite{ChapVI, Thm 1.1}. From \cite{34, Prop 13.4}, the exponential map $\exp_N : n \to N$ is a global diffeomorphism, equivariant for the adjoint actions of $G^1_q$ on domain and codomain. Combining these two facts, we have the diffeomorphism

$$e = \exp_R \times \exp_N : (g^1_q / \mathfrak{so}(q)) \oplus n \longrightarrow (G^1_q / O(q)) \times N,$$

which is equivariant in each factor, hence equivariant.

The associated bundle construction works internal to the diffeological category \cite{Dif 8.16}. In particular, given a principal $G^\infty_q$-bundle $Y \to X$, the quotient $Y / O(q)$ is the associated $G^\infty_q / O(q)$-bundle to the action of $G^\infty_q$ on $G^\infty_q / O(q)$. Define its \textit{vertical tangent bundle} $V(Y / O(q))$ as the associated bundle $Y \times T_{O(q)}(G^\infty_q / O(q))$. We have a commutative diagram,

\[
\begin{array}{ccc}
V(Y / O(q)) & \longrightarrow & Y / O(q) \\
\downarrow & & \downarrow \\
Y / O(q) & \xrightarrow{\sigma} & X
\end{array}
\]

where the lower three arrows are bundle projections, and the top arrow is given, using the associated bundle construction on both sides, by the rule

$$[y, [g, v]] = [y \cdot g, [\text{id}, g^{-1} \cdot v]] \longmapsto [y \cdot g, [e(g^{-1} \cdot v)]]$$

for $y \in Y$ and $[g, v] \in T(G^\infty_q / O(q)) \cong G^\infty_q / O(q) \times T_{O(q)}(G^\infty_q / O(q))$. The top arrow defines fibrewise diffeomorphisms, in that it maps the fibre over any $y \in Y / O(q)$ diffeomorphically to the fibre over $p(y)$. The next lemma follows immediately.

\textbf{Lemma C.3.} For any section $\sigma : X \to Y / O(q)$, there is a canonical diffeomorphism of fibre bundles

$$e_\sigma : \sigma^* V(Y / O(q)) \to Y / O(q).$$

Using the equivalence between $\text{Fr}_\infty(\gamma)$ and $B\Gamma_q$, we construct a section of $\text{Fr}_\infty(\gamma) / O(q) \to B\Gamma_q$. This is done semi-simplicially, using Lemma \textbf{C.3} in the construction of sections $\sigma_k : \Delta_k \times \mathbb{H}^k \times \Gamma^k_q \to \Delta_k \times \mathbb{H}^k \times (\Gamma_q / O(q))$ that satisfy

$$\left(id \times \partial_j\right) \circ \sigma_k \circ (d_j \times \text{id}) = (d_j \times \text{id}) \circ \sigma_{k-1} \circ (\text{id} \times \partial_j) \quad j = 0, \ldots, k,$$

(18)

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guaranteeing the sections glue to a global section. We take inspiration from Dupont [17].

Fix any section $\sigma$ of

$$\Gamma_{q,\infty}^{(0)} \cong \text{Fr}_\infty(\mathbb{R}^q) \longrightarrow \mathbb{R}^q \cong \Gamma_q^{(0)},$$

and let $\sigma_0 : \Delta_0 \times \mathbb{N}^{(0)} \times \Gamma_{q,\infty}^{(0)} \to \Delta_0 \times \mathbb{N}^{(0)} \times (\Gamma_{q,\infty}^{(0)}/O(q))$ be the induced map

$$\sigma_0 : (\ast; \alpha; \vec{x}) \longmapsto (\ast; \alpha; [\sigma(\vec{x})]), \quad \vec{x} \in \mathbb{R}^q.$$

We now construct $\sigma_k$ for $k > 0$. The diffeomorphism $e : \sigma^*V(\text{Fr}_\infty(\mathbb{R}^q)/O(q)) \to \text{Fr}_\infty(\mathbb{R}^q)/O(q)$ of Lemma [C.3] allows the definition of a fibrewise contraction: for $t \in [0, 1]$ and $\vec{x} \in \mathbb{R}^q$, define $g_{t,\vec{x}} : \text{Fr}_\infty(\mathbb{R}^q)/O(q)_{\vec{x}} \to \text{Fr}_\infty(\mathbb{R}^q)/O(q)_{\vec{x}}$ by the formula

$$g_{t,\vec{x}}(b) := e_a(t e_{\sigma}^{-1}(b)), \quad b \in \text{Fr}_\infty(\mathbb{R}^q)/O(q)_{\vec{x}}.$$ 

Now letting $(t_0, \ldots, t_k)$ denote the barycentric coordinates on the standard simplex $\Delta_k$, write $s_i := t_i + \cdots + t_k$ for $i = 1, \ldots, k$, and define $\sigma_k : \Delta_k \times \mathbb{N}^{(k)} \times \Gamma_q^{(k)} \to \Delta_k \times \mathbb{N}^{(k)} \times (\Gamma_q^{(k)}/O(q))$ by the formula

$$\sigma_k(\vec{t}; \vec{\alpha}; \vec{\gamma}) := (\vec{t}; \vec{\alpha}; \vec{\gamma}_k(\vec{t}; \vec{\gamma}), \ldots, \vec{\alpha}_k(\vec{t}; \vec{\gamma})),$$

where

$$\vec{\gamma}_k(\vec{t}; \vec{\gamma}) := \vec{\gamma}_k \cdot g_{\vec{s}_1,\vec{r}(\vec{\gamma})} \left( g_{\vec{2},\vec{r}(\vec{\gamma})} \left( \cdots g_{\vec{k},\vec{r}(\vec{\gamma})} \left( \gamma_k \cdot \sigma(s(\vec{\gamma})) \right) \cdots \right) \right). \quad (19)$$

As in [17, p. 241], $\sigma_k$ may be assumed to be smooth, and a routine calculation shows that they satisfy the identities of Equation [18]. As a consequence, the $\sigma_k$ glue to a smooth section $B\sigma$ of $\text{Fr}_\infty(\gamma)/O(q) \cong B\Gamma_{q,\infty}/O(q) \to B\Gamma_q$.

We can now easily prove Theorem 3.9 According to the above construction, smooth sections of $\text{Fr}_\infty(\gamma)/O(q) \to B\Gamma_q$ exist. Now let $h$ be a smoothly numerable Haefliger structure on a diffeological space $X$, associated to a smooth map $\eta : X \to B\Gamma_q$. From the isomorphism $\text{Fr}_\infty(h)/O(q) \cong \eta^* \text{Fr}_\infty(\gamma)/O(q)$ (Proposition [J.3]), any smooth section of $\text{Fr}_\infty(\gamma)/O(q) \to B\Gamma_q$ induces a corresponding smooth section of $\text{Fr}_\infty(h)/O(q) \to X$. Finally, any two smooth sections are smoothly homotopic via an exponentiated-linear homotopy using Lemma [C.3].

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