Susceptibility and vertex corrections for a square Fermi surface

D. Djajaputra and J. Ruvalds
Department of Physics, University of Virginia, Charlottesville, VA 22903
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We investigate the response of an electron system which exhibits ideal nesting features. Using the standard Matsubara formalism we derive analytic expressions for the imaginary and real parts of the bare particle-hole susceptibility. The imaginary part has sharp peaks whose maxima at the nesting momenta approximately scale with \( \omega/T \). The peak lineshapes resemble neutron scattering data on chromium and some copper oxide superconductors. The real part of the bare susceptibility at the nesting vectors diverges logarithmically at low temperatures. Analytic formulas for the first vertex correction to the susceptibility are derived for a Hubbard interaction, and its momentum and temperature variations are calculated numerically. This term detracts substantially from the ordinary RPA terms for intermediate values of the Coulomb repulsion. Exact cancellation of a certain class of diagrams at half filling is shown to result from particle-hole symmetry. We discuss the consequences of these results for spin fluctuation theories of high temperature superconductors and spin density wave instabilities.

I. MOTIVATION

Nested Fermi surfaces with nearly parallel orbit segments are traditionally associated with electronic instabilities. A peak in the susceptibility at a nesting momentum can create a charge density wave for strong electron-phonon coupling, or alternately induce a spin density wave (SDW) when the Coulomb repulsion dominates. These instabilities often appear in systems with reduced dimensionality for electron dynamics like the quasi one-dimensional organic metals and layered structure materials.

The present work on a square Fermi surface is motivated by the discoveries of nesting phenomena in high temperature superconductors. In these materials, the anomalous normal state quasiparticle damping that is linear in frequency and temperature can be explained by nesting, providing that electron-electron scattering is the primary damping source. The microscopic nested Fermi liquid (NFL) theory derives this unconventional damping from electron scattering across nested regions of Fermi surface that approximately satisfy the nesting condition \( \varepsilon(k + Q) + \varepsilon(k) = 2\mu \), where \( Q \) is the nesting vector. This implies scaling of the spin susceptibility at the nesting vector in frequency divided by temperature \( (\omega/T) \) which has been observed by neutron scattering experiments. By contrast, the weak Fermi liquid damping that is the hallmark of ordinary metals owes its quadratic frequency and temperature variation to a susceptibility that is essentially independent of temperature. Within a self-consistent scheme, the NFL theory has provided a physical explanation of the optical conductivity, electronic Raman spectrum, and many other normal state features of cuprate superconductors. Fermi surface nesting has been discovered by photoemission in \( \text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8 \) and some other cuprate compounds.

Our present study is relevant to the spin fluctuation mechanism of \( d \)-wave superconductivity which was originally examined thirty years ago by Berk and Schrieffer. The basic Fermi sphere yields extremely small transition temperatures \( T_c \) for \( d \)-waves, and thus this concept became dormant until evidence for an unconventional mechanism in heavy fermion metals revived interest. The possibility of raising \( T_c \) by a susceptibility enhancement treated in the random phase approximation (RPA) was suggested by Scalapino et al. and subsequently used by many groups in 2D tight-binding model calculations for cuprates. Nesting enhances \( d \)-wave electron pairing in leading order exchange of antiferromagnetic spin fluctuations. Schrieffer, however, has stressed the need to examine vertex corrections and other terms beyond the simple RPA. If the RPA series yields a large susceptibility enhancement, it is reasonable to expect that self energy and vertex contributions become correspondingly important.

Within the nesting approximation, Viroztek and Ruvalds have derived analytic expressions for the self energy and vertex corrections to the susceptibility at the nesting vector. These higher order contributions preserve the scaling of the susceptibility and they become comparable to the RPA terms for intermediate values of the Coulomb repulsion. Thus the RPA method by itself can be misleading, especially when the system is close to an SDW instability.

In this paper we study these issues by using a 2D linearized dispersion model which gives a square Fermi surface at any filling. The linearization allows the imaginary and real parts of the noninteracting susceptibility to be obtained analytically and the perfect nesting allows us to examine the validity of the NFL theory.
The linear dispersion model was first introduced by Mattis [13] as a simplified version of the 2D tight-binding model. A similar model with square Fermi surface has also been studied by Luther [14] and Hlubina [15]. Mattis and Hlubina have claimed that the model is exactly solvable using bosonization and that the system is a 2D Luttinger liquid, with spin-charge separation, rather than a Fermi liquid. Indeed, bosonization is one of the powerful methods that can be used to solve the 1D version of this model—which is a variant of the Tomonaga-Luttinger model. In higher dimensions, however, bosonization (at least in the simple version used by Mattis and Hlubina) is expected to give an incomplete picture of the system due to the presence of the particle-hole continuum. Haldane [16] has pointed out that the distinctive feature of 1D fermion systems is the nonexistence of low-energy particle-hole pairs with small momenta, whereas in higher dimensions this continuum always exists. This absence of excitation decay channels allows the response of a 1D system to be completely described in terms of bosonic collective excitations. In the language of susceptibility, the imaginary part of the spin and charge susceptibilities in 1D will consist only of delta function peaks corresponding to these bosonic modes. In higher dimensions, we get a qualitatively different picture. In addition to peaks from the collective excitations, there is also a regular contribution to the susceptibility, corresponding to excitations of particle-hole pairs which may also serve as a decay channel of the collective excitations if their dispersions lie within the particle-hole continuum. This feature appears clearly in our analytic results for the susceptibility, and may shed light to the current controversy concerning the existence of Luttinger liquids in dimensions higher than one.

We define the square model and our convention for units in Sec. II. In Sec. III we proceed to evaluate the imaginary and real parts of the bare susceptibility and display the key features emanating from this square model in the noninteracting limit. Effects of electron-electron interaction are studied in Sec. IV where we numerically compute the leading vertex correction to the susceptibility and compare it with the RPA term of the same order. In this section we also prove a particle-hole theorem that is relevant to a large class of diagrams near half filling. We discuss the results of our investigation in Sec. V.

II. SQUARE FERMI SURFACE MODEL

We consider a modified Hubbard model on a square lattice defined by the Hamiltonian

\[ H = H_0 + H_U = \sum_{\mathbf{k},\sigma} (\varepsilon(\mathbf{k}) - \mu) \hat{c}^+_{\mathbf{k},\sigma} \hat{c}_{\mathbf{k},\sigma} + U \sum_i n_{i,\uparrow} n_{i,\downarrow}, \]

with a noninteracting energy dispersion given by

\[ \varepsilon(\mathbf{k}) = v_F (|k_x| + |k_y|), \]

for momenta in the first Brillouin zone \(-\pi/a \leq k_x, k_y \leq \pi/a\). In the coordinate space the noninteracting Hamiltonian has the form

\[ H_0 = -\sum_{i,\sigma} \sum_{m=\infty}^{\infty} t(m) (c_{i,\sigma}^\dagger c_{i+m,\sigma} + c_{i,\sigma}^\dagger c_{i+m,\sigma}^\dagger), \]

with \(x_m \equiv (2m-1, 0)\), \(y_m \equiv (0,2m-1)\), and hopping amplitudes

\[ t(m) = \frac{2v_F}{\pi a} \frac{1}{(2m-1)^2}. \]

Note that the electrons are only allowed to hop from one sublattice to another, but not between sites in the same sublattice.

In the rest of this paper we will use the following convention for units: \(v_F = \pi/a = k_B = 1\), where \(k_B\) is the Boltzmann constant. The unit of energy is therefore \(v_F(\pi/a) = W/2\), where \(W\) is the noninteracting bandwidth. If the bandwidth were 2 eV, temperature \(T = 0.03\) would correspond to 360 K. The symbol \(\mu\) will be reserved for the chemical potential. We also define a useful variable \(\gamma \equiv 1/(2T) = \beta/2\).

The Fermi surface for this model with \(\mu = 0.9\) is the square shown in Fig. [1]. The arrows specify the nesting vectors for this value of the chemical potential that will be used throughout this paper. Assuming the velocity to be independent of momentum avoids the logarithmic divergence in the density of states that characterize other square Fermi surface models (such as the 2D tight binding band at half filling). This allows us to study the characteristic features due to nesting separately from those caused by the Van Hove singularities. For the present model, the density of states (per spin) is
\[ N(\varepsilon) = \varepsilon \Theta(\varepsilon)\Theta(1 - \varepsilon) + (2 - \varepsilon) \Theta(\varepsilon - 1)\Theta(2 - \varepsilon). \]  

We use the standard Matsubara finite temperature formalism to calculate the bare and the first vertex correction to the spin susceptibility of this model. The bare susceptibility is given by

\[ \chi_0(q, i\omega) = -\frac{T}{N} \sum_{k, \nu} G_0(k, i\nu)G_0(k + q, i\omega + i\nu), \]

where \( G_0(k, i\omega) = [i\omega - \varepsilon(k)]^{-1} \) is the bare Green's function. Doing the frequency sum and performing analytic continuation to real frequency, we get the standard result for the bare particle-hole susceptibility

\[ \chi_0(q, \omega) = \frac{1}{4} \int dq \frac{f[\varepsilon(k + q)] - f[\varepsilon(k)]}{\omega - \varepsilon(k + q) + \varepsilon(k) + id}, \]

where \( f(x) = [\exp \beta(x - E) + 1]^{-1} \) is the Fermi function. At \( \omega = 0 \) and \( q \to 0 \) this reduces to the temperature independent Pauli susceptibility \( \chi(0, 0) = N(\mu) \).

III. SUSCEPTIBILITY FOR NONINTERACTING ELECTRONS

A. Imaginary Part of Susceptibility

The susceptibility comprises real and imaginary parts \( \chi_0(q, \omega) = \chi_0'(q, \omega) + i\chi_0''(q, \omega) \). The expression for the imaginary part is

\[ \chi_0''(q, \omega) = \frac{\pi}{4} \int_{-1}^{1} dx \int_{-1}^{1} dy \left\{ \frac{1}{e^{\beta(x,y) - \mu} + 1} - \frac{1}{e^{\beta(x+a,y+b) - \mu} + 1} \right\} \delta(\omega - \varepsilon(x + a, y + b) + \varepsilon(x, y)), \]

where \( a = \max(|q_x|, |q_y|) \) and \( b = \min(|q_x|, |q_y|) \) with \( 0 \leq b < a \leq 1 \) for \( q \in 1BZ \). The integral can be carried out directly by dividing the region \(-1 \leq x, y \leq 1\) into 16 regions separated by the lines \( x = \{-1, -a, 0, 1 - a, 1\} \) and \( y = \{-1, -b, 0, 1 - b, 1\} \). The result is conveniently expressed in terms of the following functions:

\[ S_1(x, \omega) = \frac{\pi}{16} \left\{ f\left(\frac{x - \omega}{2}\right) - f\left(\frac{x + \omega}{2}\right) \right\} = -S_1(x, -\omega), \]

\[ S_2(x) = \frac{\pi^2}{24} - \frac{\ln 2}{2} \left(\frac{x - \mu}{T}\right) + \frac{1}{8} \left(\frac{x - \mu}{T}\right)^2 + \frac{1}{2} \text{Li}_2\left[ -\exp\left(\frac{\mu - x}{T}\right) \right], \]

\[ S_3(x, \omega) = \frac{\pi \omega}{16} + \left(\frac{\pi T}{8}\right) \ln \left[ \frac{\cosh \frac{x + \omega - 2\mu}{4T}}{\cosh \frac{x - \omega - 2\mu}{4T}} \right] = -S_3(x, -\omega), \]

\[ K_1(a, b, \omega) = (a + b - |\omega|) \Theta(a + b - |\omega|) \Theta(|\omega| - a + b) + 2b \Theta(a - b - |\omega|), \]

\[ K_2(a, b, \omega) = \Theta(a + b - \omega) \Theta(\omega - a + b) + \Theta(a + b + \omega) \Theta(-\omega - a + b), \]

\[ K_3(a, b, \omega) = \Theta(a + b - \omega) \Theta(\omega + a - b) + \Theta(a + b + \omega) \Theta(-\omega + a - b), \]

\[ D_1(a, b) = \left(\frac{\pi T^2}{2}\right) \left\{ S_2(1 - a) - S_2(1 + a) + S_2(1 - b) - S_2(1 + b) + S_2(2) - S_2(0) + S_2(a + b) - S_2(2 - a - b) \right\}, \]

\[ D_2(a, b) = \left(\frac{\pi T^2}{2}\right) \left\{ S_2(1 + a - b) - S_2(1 - a + b) + S_2(2 - a) - S_2(2 - b) + S_2(b) - S_2(a) \right\}. \]
In the definition of $S_2(x)$, we have used the dilogarithm function which is defined by

$$
\text{Li}_2(x) = \int_0^x \frac{\ln(1-t)}{t} dt.
$$

(17)

Using these functions, the imaginary part of the susceptibility can be written as

$$
\chi'''_{0}(q, \omega) = \chi'''_{0, \text{reg}}(q, \omega) + \chi'''_{0, \text{sing}}(q, \omega),
$$

(18)

with a regular part

$$
\chi'''_{0, \text{reg}}(q, \omega) = \left\{ S_1(4 - a - b, \omega) + S_2(2 + a - b, \omega) + S_1(2 - a + b, \omega) + S_1(a + b, \omega) \right\} K_1(a, b, \omega)
$$

$$
+ \left\{ S_3(4 - a - b, \omega) - S_3(2 + a - b, \omega) + S_3(2 - a + b, \omega) - S_3(a + b, \omega) \right\} K_2(a, b, \omega)
$$

$$
+ \left\{ S_3(4 - a - b, \omega) + S_3(2 + a - b, \omega) - S_3(2 - a + b, \omega) - S_3(a + b, \omega) \right\} K_3(a, b, \omega),
$$

(19)

and a singular contribution

$$
\chi'''_{0, \text{sing}}(q, \omega) = D_1(a, b) \left\{ \delta(\omega - a - b) - \delta(\omega + a + b) \right\} + D_2(a, b) \left\{ \delta(\omega + a - b) - \delta(\omega - a + b) \right\}.
$$

(20)

For small $(\omega/T)$ we can expand $S_1(x, \omega)$ and $S_3(x, \omega)$

$$
S_1(x, \omega) = \left( \frac{\pi \omega}{64T} \right) \text{sech}^2 \left( \frac{x - 2\mu}{4T} \right) + O((\omega/T)^3),
$$

(21)

$$
S_3(x, \omega) = \left( \frac{\pi \omega}{16} \right) \left[ 1 + \tanh \left( \frac{x - 2\mu}{4T} \right) + O((\omega/T)^2) \right].
$$

(22)

These expansions can be used to obtain the leading term in $(\omega/T)$ of the regular part at the nesting vector $Q = (\mu, \mu)$

$$
\chi'''_{0, \text{reg}}(Q, \omega) = \Theta(2\mu - |\omega|) \left( \frac{\mu \pi \omega}{32T} \right) \left\{ 1 + 2 \text{sech}^2 \left( \frac{2 - 2\mu}{4T} \right) + \text{sech}^2 \left( \frac{4 - 4\mu}{4T} \right) + \frac{4T}{\mu} \tanh \left( \frac{4 - 4\mu}{4T} \right) \right\} + O\left( \frac{|\omega| \omega}{\mu T} \right).
$$

(23)

For a half-filled system $(\mu = 1)$, this reduces to

$$
\chi'''_{0, \text{reg}}(Q, \omega) = \Theta(2 - |\omega|) \left( \frac{\pi \omega}{8T} \right) + O\left( \frac{|\omega| \omega}{\mu T} \right),
$$

(24)

which is just the leading term of the exact expression that can be obtained using the nesting condition

$$
\chi'''_{0, \text{reg}}(Q, \omega) = (\pi/2) N(1 - \omega/2) \tanh(\omega/4T).
$$

(25)

The momentum variation of $\chi'''_{0, \text{reg}}(Q, \omega)$ is sharply peaked at the nesting vectors as shown in Fig. 3. Neutron scattering experiments have observed this type of peak structure in chromium and also in high temperature superconductors. The strong temperature variation of the peak lineshape is caused by nesting since the quasiparticle damping does not enter in the noninteracting susceptibility. A 3D plot shows the full momentum dependence of $\chi'''_{0, \text{reg}}(Q, \omega)$ in Fig. 5.

The imaginary part of the susceptibility at the nesting vector in Eqs. (23)–(25) approximately scales with frequency divided by temperature. The weak energy dependence of the density of states, which is nowhere singular in this model, introduces only a slow modulation which can be neglected for small $\omega$, leaving only the temperature $T$ to set the effective energy scale. Scaling of the susceptibility was first obtained by means of the nesting approximation and it determines the anomalous quasiparticle damping that distinguishes the cuprates from ordinary metals. Fig. 4 shows the frequency variation of $\chi'''_{0, \text{reg}}(Q, \omega)$ at several temperatures.

The singular part of the susceptibility comes from forward scattering processes which are strongly enhanced in this model by the synergetic combination of the linear dispersion and the flatness of the Fermi surface. These terms are key ingredients for collective charge and spin modes and they are relevant to the issue of charge and spin separation. In Fig. 5 we show the frequency dependence of $\chi_0'(q, \omega)$ for small $q$. The delta function peak carries a spectral weight which is two orders of magnitude larger than the integrated weight of the continuum. Nonlinear energy dispersion or finite curvature of the Fermi surface, however, will remove this singularity and replace it with a very sharp, but nonetheless finite, peak. The 2D nearest-neighbor tight-binding model, for example, gives a finite susceptibility for all $q$ except for the half-filled case when there is a Van Hove singularity in the density of states.
B. Real Part of Susceptibility

The real part of the susceptibility determines the SDW instability and is a vital component in the exchange processes that may create a d-wave superconducting state for a system with highly anisotropic Fermi surface. Taking the real part of Eq. (7) we obtain the expression for the real part of susceptibility

$$\chi'_0(q, \omega) = \frac{1}{8} \mathcal{P} \int_{-1}^{1} dx \int_{-1}^{1} dy \frac{\tanh[\gamma (x, y) - \mu]}{\omega - \varepsilon(x, y)}.$$

Both integrals are principal value integrals as indicated by the symbol $\mathcal{P}$. The integral can be done by dividing the region of integration into 16 regions as was previously done for the imaginary part. The result can be expressed in terms of functions $R_1$ and $R_2$ which are defined in the Appendix. We first define several auxiliary functions:

$$L_1(a, b) = \ln[\cosh \gamma(2 - a - \mu)] - \ln[\cosh \gamma(2 - \mu)] - \ln[\cosh \gamma(2 - a - b - \mu)] + \ln[\cosh \gamma(2 - b - \mu)]$$
$$+ \ln[\cosh \gamma(1 - \mu)] - \ln[\cosh \gamma(1 - b - \mu)] + \ln[\cosh \gamma(1 + a - b - \mu)] - \ln[\cosh \gamma(1 + a - \mu)]$$
$$+ \ln[\cosh \gamma(a - \mu)] - \ln[\cosh \gamma(a + b - \mu)] + \ln[\cosh \gamma(b - \mu)] - \ln[\cosh \gamma(-\mu)],$$

$$L_2(a, b, \omega) = \left( \frac{b}{16} \right) \left\{ R_1(1; 1 - a + b; 2 - a + b - \omega) + R_1(1 + a - b; 1; 2 + a - b - \omega) \right\}$$
$$+ \left( \frac{b}{16} \right) \left\{ R_1(a; b; a + b - \omega) + R_1(2 - b; 2 - a; 4 - a - b - \omega) \right\},$$

$$L_3(a, b, \omega) = \frac{(a + b - \omega)}{32} \left\{ R_1(2 - a; 2 - a - b; 4 - a - b - \omega) + R_1(2; 2 - b; 4 - a - b + \omega) \right\}$$
$$+ \frac{(a + b - \omega)}{32} \left\{ R_1(1 - a + b; 1 - a; 2 - a + b - \omega) + R_1(1 + b; 1; 2 - a + b + \omega) \right\}$$
$$+ \frac{(a + b - \omega)}{32} \left\{ R_1(1 + a; 1 + a - b; 2 + a - b + \omega) + R_1(1; 1 - b; 2 + a - b - \omega) \right\}$$
$$+ \frac{(a + b - \omega)}{32} \left\{ R_1(0; a + b - \omega) + R_1(a + b; a; a + b + \omega) \right\},$$

$$L_4(a, b, \omega) = R_2(2 - a; 2 - a - b; 4 - a - b - \omega) + R_2(2; 2 - b; 4 - a - b + \omega)$$
$$+ R_2(1 - a + b; 1 - a; 2 - a + b - \omega) + R_2(1 + b; 1; 2 - a + b + \omega)$$
$$- R_2(1 + a; 1 + a - b; 2 + a - b + \omega) - R_2(1; 1 - b; 2 + a - b - \omega)$$
$$- R_2(0; a + b - \omega) - R_2(a + b; a; a + b + \omega).$$

With these functions, the real part of the susceptibility simplifies to

$$\chi'_0(q, \omega) = (T/4) L_1(a, b) + \left( \frac{D_1(a, b)}{\pi} \right) \left\{ \frac{1}{\omega + a + b} - \frac{1}{\omega - a - b} \right\} + \left( \frac{D_2(a, b)}{\pi} \right) \left\{ \frac{1}{\omega - a + b} - \frac{1}{\omega + a - b} \right\}$$
$$+ \left\{ L_2(a, b, \omega) + L_2(a, b, -\omega) + L_3(a, b, \omega) + L_3(a, b, -\omega) + (T/4) (L_4(a, b, \omega) + L_4(b, a, \omega)) \right\}.$$
IV. COULOMB INTERACTIONS

In this section we will use the standard diagrammatic analysis to discuss the effects of the interaction Hamiltonian $H_U$ in Eq. (1). The Hubbard onsite Coulomb repulsion $U$ only couples electrons with opposite spin on the same lattice site. This, in particular, means that all exchange diagrams vanish since there is no interaction between electrons with equal spin.

A. Random Phase Approximation

The RPA is one of the most useful approximation methods in many-body physics and has been used effectively to describe plasma oscillations and density wave instabilities. In this approximation, the longitudinal spin susceptibility for the interacting Hamiltonian is given by the simple formula

$$
\chi^s_{\text{rpa}}(\mathbf{q}, i\omega_0) = \frac{2\chi_0(\mathbf{q}, i\omega_0)}{1 - U\chi_0(\mathbf{q}, i\omega_0)},
$$

(34)

The diagrams corresponding to the first three terms in this series are shown in Fig. (a),(b),(c) where the bare susceptibility “bubble” is (a). Note that for each diagram there is another diagram in which all the spins are flipped. In the paramagnetic state, which will always be assumed in the following calculations, this spin degeneracy gives rise to the factor of two in Eq. (34).

Despite its popularity, the RPA should be used with caution since occasionally its application can lead to serious mistakes whose remedy requires a study of vertex corrections. For example, one may incorrectly infer that a single vertex correction to the bare particle-hole susceptibility for the square model.

For our discussion, it is useful to define the spin-dependent particle-hole susceptibilities

$$
\chi^{\sigma\sigma'}(\mathbf{q}, i\omega_0) = \frac{1}{N} \int_0^\beta d\tau e^{i\omega_0\tau} \langle T_\tau[n_\sigma(\mathbf{q}, \tau)n_{\sigma'}(-\mathbf{q}, 0)] \rangle,
$$

(35)

where

$$
n_\sigma(\mathbf{q}, 0) = \sum_k c_{\mathbf{q} + \mathbf{k}, \sigma}^\dagger c_{\mathbf{k}, \sigma}.
$$

(36)

From this susceptibility we can get the charge and spin susceptibilities

$$
\chi^c(\mathbf{q}, i\omega_0) = \chi^{\uparrow\uparrow}(\mathbf{q}, i\omega_0) + \chi^{\downarrow\downarrow}(\mathbf{q}, i\omega_0) + 2\chi^{\uparrow\downarrow}(\mathbf{q}, i\omega_0),
$$

(37)

$$
\chi^s(\mathbf{q}, i\omega_0) = \chi^{\uparrow\uparrow}(\mathbf{q}, i\omega_0) + \chi^{\downarrow\downarrow}(\mathbf{q}, i\omega_0) - 2\chi^{\uparrow\downarrow}(\mathbf{q}, i\omega_0).
$$

(38)

At zeroth order in $U$, we have $\chi^{\uparrow\uparrow}(\mathbf{q}, i\omega_0) = \chi^{\downarrow\downarrow}(\mathbf{q}, i\omega_0) = \chi_0(\mathbf{q}, i\omega_0)$ and $\chi^{\uparrow\downarrow}(\mathbf{q}, i\omega_0) = 0$. It is useful to define separate RPA series for these susceptibilities

$$
\chi^{\uparrow\uparrow}_{\text{rpa}}(\mathbf{q}, i\omega_0) = \frac{\chi_0(\mathbf{q}, i\omega_0)}{1 - U\chi_0(\mathbf{q}, i\omega_0)} = \chi^{\uparrow\uparrow}_{\text{rpa}}(\mathbf{q}, i\omega_0),
$$

(39)

$$
\chi^{\downarrow\downarrow}_{\text{rpa}}(\mathbf{q}, i\omega_0) = \frac{U\chi_0(\mathbf{q}, i\omega_0)}{1 - U\chi_0(\mathbf{q}, i\omega_0)^2} = \chi^{\downarrow\downarrow}_{\text{rpa}}(\mathbf{q}, i\omega_0).
$$

(40)

Within RPA, the charge and longitudinal spin susceptibilities are therefore

$$
\chi^{c}_{\text{rpa}}(\mathbf{q}, i\omega_0) = 2(\chi^{\uparrow\uparrow}_{\text{rpa}}(\mathbf{q}, i\omega_0) + \chi^{\downarrow\downarrow}_{\text{rpa}}(\mathbf{q}, i\omega_0)),
$$

(41)

$$
\chi^{s}_{\text{rpa}}(\mathbf{q}, i\omega_0) = 2(\chi^{\uparrow\uparrow}_{\text{rpa}}(\mathbf{q}, i\omega_0) - \chi^{\downarrow\downarrow}_{\text{rpa}}(\mathbf{q}, i\omega_0)).
$$

(42)

The RPA would give an exact susceptibility if in the formula we replace the bare susceptibility $\chi_0(\mathbf{q}, i\omega_0)$ with the exact irreducible susceptibility which requires all possible self energy and vertex corrections. Clearly this is very difficult. For the tight-binding model, the spin and charge susceptibilities have been calculated to second order in $U$. The calculation, however, was done on a finite lattice and the susceptibilities were calculated only at $\mathbf{q} = (0, 0)$ and $\mathbf{q} = (\pi/a, \pi/a)$. In the next subsection we will discuss the momentum and temperature dependence of the leading order vertex correction to the bare particle-hole susceptibility for the square model.
B. Vertex Correction to $\chi^{↑↑}(q, i\eta_0)$

The first vertex correction to $\chi^{↑↑}(p, i\eta_0)$ is shown in Fig. 4(d) and is given by the expression

$$\chi_{corr}(p, i\eta_0) = -\frac{T}{N} \sum_k \sum_{ik_0} \Gamma(k, p, ik_0, i\eta_0) G_0(k + p, ik_0 + i\eta_0) G_0(k, ik_0),$$

where

$$\Gamma(k, p, ik_0, i\eta_0) = \frac{U^2 T}{N} \sum_q \sum_{i\eta_0} \chi_0(q, i\eta_0) G_0(k + q, ik_0 + i\eta_0) G_0(k + p + q, ik_0 + i\eta_0 + i\eta_0).$$

Note that this is the only vertex correction to $\chi^{↑↑}(q, i\eta_0)$ in the second order. To calculate this correction we define

$$\chi_{corr}(p, i\eta_0) = -\frac{U^2}{N} \sum_k F(k, p, i\eta_0),$$

with

$$F(k, p, i\eta_0) = \frac{1}{N} \sum_q G^{13} - G^{23} - G^{14} + G^{24},$$

$$G^{\alpha\beta} = T^2 \sum_{ik_0} \frac{1}{i\eta_0 - \xi_\beta} \sum_{i\eta_0} \frac{\chi_0(q, i\eta_0)}{i\eta_0 + ik_0 - \xi_\alpha},$$

where $\alpha \in \{1, 2\}; \beta \in \{3, 4\}; \xi_1 = \xi_{k+q}; \xi_2 = (\xi_{k+p+q} - i\eta_0); \xi_3 = \xi_k; \xi_4 = (\xi_{k+p} - i\eta_0)$. Note that $\eta_0$ and $\eta_0$ are boson frequencies, $k_0$ is a fermion frequency, and $\chi_0(q, i\eta_0) = \chi_0(0, q, \omega \rightarrow i\eta_0)$. The two frequency summations in $G^{\alpha\beta}$ can be performed using the analytic expressions for $\chi_0(q, i\eta_0)$ that we have obtained previously. First note that for a simple pole $1/(i\eta_0 - z)$ we have

$$T^2 \sum_{ik_0} \frac{1}{i\eta_0 - \xi_\beta} \sum_{i\eta_0} \frac{1}{i\eta_0 + ik_0 - \xi_\alpha} \cdot \frac{1}{i\eta_0 - z} =$$

$$\left\{ \frac{\coth(\gamma z) - \tanh(\gamma \xi_\alpha)}{\xi_\alpha - z - \xi_\beta} \right\}.$$  

Now observe that in the expressions for $\chi_0(q, i\eta_0)$ each term has exactly one simple pole of the form $1/(i\eta_0 - z)$ where $z_i$ varies from term to term. ($L_1(a, b)$ and $L_2(a, b, i\eta_0)$ clearly are not in this form, but one can check easily that their sum can be written in this form.) Except for the poles multiplying $D_1(a, b)$ and $D_2(a, b)$, all the other poles reside in an integral. We assume that the integration and the frequency summation operations commute so that we can perform the frequency summation first. With all this, $G^{\alpha\beta}$ can be obtained from $\chi_0(q, i\eta_0)$ by a simple substitution rule

$$\frac{1}{i\eta_0 - z_i} \rightarrow \left\{ \frac{\coth(\gamma z_i) - \tanh(\gamma \xi_\alpha)}{\xi_\alpha - z_i - \xi_\beta} \right\}.$$  

Performing this substitution, $G^{\alpha\beta}$ is now expressed as a sum of one dimensional integrals. These integrals can be calculated analytically using the piecewise polynomial approximation to the functions $\tanh(\omega)$, $\ln(\cosh(\omega))$, and $\coth(\omega)$ as presented in the Appendix. Note that the number of terms that need to be calculated in this method increases very rapidly compared to the number of terms in the bare susceptibility. The analytic expressions, however, allow us to perform the principal value integrations exactly within this approximation. With this analytic expression for $G^{\alpha\beta}$, the computation of $\chi_{corr}(p, i\eta_0)$ is reduced to a numerical computation of a four dimensional integral.

The integral over $q$ was carried out numerically to give the function $F(k, p, 0)$ defined in Eq. (44). We have performed these computation using SP2 parallel processors at the Maui High Performance Computing Center and the
University of Virginia. Each of the plots of \( F(\mathbf{k}, \mathbf{p}, 0) \) shown in Fig. 2 typically requires about 4 days of CPU time on 10 processors. The vertex correction \( \chi_{\text{corr}}(\mathbf{p}, 0) \) is then obtained by integrating \( F(\mathbf{k}, \mathbf{p}, 0) \) over \( \mathbf{k} \).

In Fig. 2 we show the result for \( \chi_{\text{corr}}(\mathbf{p}, 0) \) at \( T = 0.03 \) at several points in the first Brillouin zone. The influence of this vertex correction to the static susceptibility is demonstrated in Fig. 3 by the dashed curve, in comparison to the bare susceptibility (dotted curve) and the second order RPA term (dot-dashed curve). Combining these corrections yields the solid curve for the total susceptibility, which exceeds the bare curve by roughly a factor of two when \( U = 0.5 \).

The temperature variation of the \( \chi_{\text{corr}}(\mathbf{p}, 0) \) at the nesting vector \( \mathbf{Q} = (\mu, \mu) \) is displayed in Fig. 4. The bare susceptibility (solid curve) and the second order RPA (dot-dashed curve) diverge at low temperatures as \( \ln(T) \) and \( \ln^3(T) \) respectively. The standard RPA spin susceptibility Eq. (34) yields a Néel temperature for an SDW instability at \( T_N = 60 \text{ K} \) for \( W = 2 \text{ eV} \) and \( U = 0.5 \text{ eV} \). The vertex correction \( \chi_{\text{corr}}(\mathbf{p}, 0) \) is expected to diverge also as \( \ln^3(T) \) and we have fitted the computed points (circles) using a third order polynomial in \( \ln(T) \).

C. Vertex Corrections to \( \chi^{\uparrow\downarrow}(\mathbf{q}, i\nu_0) \)

At second order in \( U \), there are two vertex correction diagrams to \( \chi^{\uparrow\downarrow}(\mathbf{q}, i\nu_0) \) which are shown in Fig. 11. The expressions for these diagrams are

\[
\chi_A(q) = U^2 \sum_{k_1} \sum_{k_2} \sum_p G_0(k_1) G_0(k_1 + q) G_0(p) G_0(k_2 - k_1 + p) G_0(k_2) G_0(k_2 + q), \tag{50}
\]

\[
\chi_B(q) = U^2 \sum_{k_1} \sum_{k_2} \sum_p G_0(k_1) G_0(k_1 + q) G_0(p) G_0(k_2 + k_1 - p) G_0(k_2) G_0(k_2 - q) \tag{51}
\]

where we have used \( q \equiv (\mathbf{q}, i\nu_0) \) and

\[
\sum_q \equiv T \sum_{i\nu_0} \sum_{\mathbf{q}} . \tag{52}
\]

These diagrams are usually expressed in terms of the bare particle-hole susceptibility Eq. (3) and the particle-particle susceptibility

\[
\phi_0(\mathbf{q}, i\nu) = \frac{T}{N} \sum_{\mathbf{k}, i\nu} G_0(\mathbf{k}, i\nu) G_0(\mathbf{q} - \mathbf{k}, i\nu - i\nu). \tag{53}
\]

We remark that at half filling these two susceptibilities are related by

\[
\phi_0(\mathbf{Q} - \mathbf{q}, i\nu) = \chi_0(\mathbf{q}, i\nu), \tag{54}
\]

where \( \mathbf{Q} \) is the nesting vector at half filling: \( \mathbf{Q} = (1, 1) \). For the following discussion we will use the forms given in Eq. (50)-(53). The fact that the Hubbard interaction vertex does not depend on momentum directly implies

\[
\chi_A(\mathbf{q}, i\nu_0) = \chi_{\text{corr}}(\mathbf{q}, i\nu_0), \tag{55}
\]

therefore our previous results for \( \chi_{\text{corr}}(\mathbf{q}, i\nu_0) \) are still useful for this section. The other diagram \( \chi_B(\mathbf{q}, i\nu_0) \) differs from \( \chi_A(\mathbf{q}, i\nu_0) \) only in that the particle lines in one of its two loops are replaced by hole propagators. In general these two diagrams must be calculated separately. We show, however, that at half filling, where we have exact particle-hole symmetry, these two diagrams exactly cancel each other. To show this, we write

\[
\chi_A(q) + \chi_B(q) = U^2 \sum_{k_1} \sum_p G_0(k_1) G_0(k_1 + q) G_0(p) \Delta(k_1, p, q), \tag{56}
\]

\[
\Delta(k_1, p, q) = \sum_{k_2} G_0(k_2 - k_1 + p) G_0(k_2) G_0(k_2 + q) + \sum_{k_2} G_0(k_2 + k_1 - p) G_0(k_2) G_0(k_2 - q). \tag{57}
\]

The symmetry between the two terms can be seen more clearly if in the second sum we write \( k_2 = -k_3 \)
\[
\Delta(k_1, p, q) = \sum_{k_2} G_0((k_2 - k_1 + p)) \ G_0((k_2 + q)) \\
+ \sum_{k_3} G_0((-k_3 - k_1 + p)) \ G_0(-k_3) \ G_0(-(k_3 + q)).
\]  

(58)

The momenta in the second sum are just the negative of the momenta that appear in the first sum. Now consider a propagator \(G(-k) = (-ik_0 - \xi(-\mathbf{k}))^{-1} = (-ik_0 - \xi(-\mathbf{k}))^{-1}\). If we perform the following transformation

\[
\begin{align*}
  ik_0 &\to ip_0; \quad k \to p + Q; \quad \xi(k) \to -\xi(p);
\end{align*}
\]

we get \(G(-k) \to (-ip_0 + \xi(p))^{-1} = -G(p)\). This means that if we do the same transformation in the second sum in Eq. (58) by writing \(ik_{30} = ik_{40}\) and \(\mathbf{k}_3 = \mathbf{k}_4 + Q\), each propagator becomes the negative of the corresponding propagator in the first sum. Since there are three propagators in each sum, this directly implies that

\[
\Delta(k_1, p, q) = 0.
\]  

(60)

We thus conclude that \(\chi_A(q, iq_0) + \chi_B(q, iq_0) = 0\) for a half-filled system. This cancellation is valid for any external momentum and frequency. For non half-filled systems, we expect the sum to vanish in powers of \((\tilde{\mu}/W)\) where \(\tilde{\mu}\) is the chemical potential measured from the center of the band, and \(W\) is the bandwidth. The absolute scale is determined by \(\chi_A(q, iq_0)\) which, as we have shown, is exactly equal to \(\chi_{\text{corr}}(q, iq_0)\) which has been analyzed in the previous section.

**D. Particle-hole Symmetry**

The cancellation of diagrams presented in the previous section clearly can be generalized to higher order diagrams. This leads to a simple theorem: for a Hubbard interaction at half filling, a diagram which has a loop consisting of \(n > 1\) particle lines is equal to \((-1)^n\) times a similar diagram where the direction of all particle lines in the loop is reversed. There are three things to note: (a) The theorem does not hold for the simple Hartree loop (\(n = 1\)) but applies to any other particle loop. (b) This cancellation depends only on the particle-hole symmetry which is exactly satisfied at half filling. It is therefore valid for any symmetric dispersion in any dimension. (c) In any diagram, particle lines which are not continuously connected to the external lines must reside in a loop. Therefore the theorem has relevance for a large class of diagrams.

As a simple application of this theorem, consider the self-energy diagrams in Fig. 12. The second order diagram in Fig. 12(a) is the first nontrivial self-energy diagram in a Hubbard model. The expression for this diagram is

\[
\Sigma(k) = U^2 \sum_q G_0(k - q) \chi_0(q).
\]  

(61)

For a nested Fermi surface, Virosztek and Ruvalds have shown that this diagram yields a quasiparticle damping which is linear in frequency and temperature. The RPA method, which is usually used to selectively include higher order diagrams, replace the bare susceptibility \(\chi_0(q)\) with \(\chi_{\text{pp}}^{(2)}(q)\). The self-energy in this approximation is given by

\[
\Sigma_{\text{rpa}}(k) = \sum_{n=1}^{\infty} \Sigma_{\text{rpa}}^{(2n)}(k),
\]

(62)

\[
\Sigma_{\text{rpa}}^{(2n)}(k) = U^{2n} \sum_q G_0(k - q)\chi_{\text{pp}}^{2n-1}(q).
\]  

(63)

Note that there is no odd order term in this series. The fourth order RPA self-energy, \(\Sigma_{\text{rpa}}^{(4)}(k)\), is shown in Fig. 12(b).

For a Hubbard interaction, which does not depend on momentum, this diagram is exactly equal to another self-energy diagram due to the particle-hole ladder with four rungs, \(\Sigma_{\text{pp}}^{(4)}(k)\), which is shown in Fig. 12(c). In general, we have

\[
\Sigma_{\text{rpa}}^{(2n)}(k) = \Sigma_{\text{ph}}^{(2n)}(k).
\]  

(64)

At half filling, we can use the previous theorem to relate the self-energy diagrams due to the particle-hole ladder with the corresponding diagrams from the particle-particle ladder. The exact relation is

\[
\Sigma_{\text{ph}}^{(m)}(k) = (-1)^m \Sigma_{\text{pp}}^{(m)}(k).
\]  

(65)
Combining these results, we conclude that at half filling

$$\Sigma^{(2n)}_{rpa}(k) = \Sigma^{(2n)}_{ph}(k) = \Sigma^{(2n)}_{pp}(k),$$  \hfill (66)

$$\Sigma^{(2n+1)}_{ph}(k) + \Sigma^{(2n+1)}_{pp}(k) = 0.$$  \hfill (67)

Using these relations, together with \(\Sigma^{(2n+1)}_{rpa} = 0\), we can write the sum of the contributions to the self-energy from the RPA, particle-hole ladder, and particle-particle ladder at half filling to be

$$\Sigma_{\text{sum}}(k) = U^2 \sum_q G_0(k - q) \chi_{\text{sum}}(q),$$  \hfill (68)

where

$$\chi_{\text{sum}}(q) = \frac{3\chi_0(q)}{1 - (U\chi_0(q))^2} - 2\chi_0(q).$$  \hfill (69)

We have substracted 2\(\chi_0(q)\) since there is only one diagram for the three series at the second order.

V. DISCUSSION

The present results for a square Fermi surface in general support the nested Fermi liquid theory of the anomalous damping that distinguishes high temperature superconductors from ordinary metals. Analytic forms for the imaginary part of the susceptibility reveal a sharp peak structure in momentum space which will assure the dominance of electron collisions across nested regions. Scaling of the susceptibility as a function of \(\omega/T\) is modified slightly by the energy dependence of the density of states, but otherwise fits the requirements to produce a linear frequency and temperature variation of the quasiparticle damping.

The real part of the susceptibility for the square model demonstrates a broader peak shape in momentum space and its maxima at the nesting vectors display a slow divergence in \(\ln(T)\) that is important for the spin density wave and the competing \(d\)-wave superconducting phase instabilities. The analytic results presented in this paper may provide a basis for estimating the transition temperatures for Hubbard models.

The cumbersome task of evaluating the multidimensional integrals for the first vertex correction is slightly simplified by the analytic formulas of the bare susceptibility and some approximations that we develop for the required hyperbolic functions. The numerical evaluation of the resulting expressions, however, still requires considerable computing power and time. The calculated vertex correction counteracts the susceptibility enhancement that comes from the simple RPA and therefore should not be neglected in computations of \(d\)-wave superconducting transition temperatures.

Particle-hole symmetry in a half-filled energy band leads to a theorem for the cancellation of a general class of diagrams for a Hubbard interaction. This theorem eliminates the need to evaluate diagrams containing loops with an odd number of propagators at half filling.

In contrast to the standard BCS theory in simple metals, where Migdal’s theorem allows the neglect of certain vertex corrections for the electron-phonon coupling, spin fluctuation theories normally cannot rely on a disparity between the energy scales for the interaction medium and the bandwidth. Hence the reliability of many calculations for the \(d\)-wave superconducting transition temperatures are difficult to estimate because vertex corrections for tight-binding models are particularly difficult to compute. Our proof for cancellation of many types of higher order diagrams applies also to the tight-binding models at half filling and it is reasonable to expect that these diagrams will be less important even in the realistic cases such as the copper oxide superconductors. Nevertheless it would be interesting to derive these corrections in terms of the chemical potential placement away from half filling.

A quantitative constraint on our weak-coupling analysis is the strength of the Coulomb repulsion \(U\) which should remain much less than the bandwidth. For \(U/W \ll 1\), the combined RPA and first vertex correction yield a net enhancement of the susceptibility. Higher order vertex correction terms clearly should be included for larger \(U\). Self energy corrections have been examined using the nesting approximation in the nested Fermi liquid theory. The self energy corrections for the square model will be explored in future work.

Finally, the controversial issue of charge and spin separation in two dimensional systems with nesting may benefit from the complete analytic results for the bare susceptibility which includes the regular part in addition to the singular terms emanating from the flat sides of the Fermi surface. Both terms should be included in future computations of the electron self energy, with particular emphasis on particle-hole decay channels and their impact on the spectral
function. Similarly these susceptibility formulas may feature new charge and spin collective modes which may be relevant to a variety of interesting metals.

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APPENDIX

The functions $R_1$ and $R_2$ are defined by (the dependence on $\gamma$ and $\mu$ are implicit)

$$R_i(a; b; c) = M_i(\gamma(a - \mu), \gamma(c/2 - \mu)) - M_i(\gamma(b - \mu), \gamma(c/2 - \mu)), \quad i \in \{1, 2\},$$  \hspace{1cm} (70)

with

$$M_1(x, w) = \mathcal{P} \int_0^x \frac{\tanh(z)}{z - w} \, dz,$$ \hspace{1cm} (71)

$$M_2(x, w) = \mathcal{P} \int_0^x \frac{\ln \cosh(z)}{z - w} \, dz.$$ \hspace{1cm} (72)

These integrals cannot be expressed in terms of known functions. In this paper we have chosen to analyze these integrals by approximating the functions $\tanh(x)$ and $\ln[\cosh(x)]$

$$\tanh(x) \to x \Theta(c_1 - |x|) + \left\{ \left[ 1 + m(|x| - c_2)^3 \right] \Theta(|x| - c_1)\Theta(c_2 - |x|) + \Theta(|x| - c_2) \right\} \text{sign}(x),$$ \hspace{1cm} (73)

$$\ln[\cosh(x)] \to \frac{x^2}{2} \Theta(c_1 - |x|) + \{ |x| + \frac{m}{4} (|x| - c_2)^4 - c_3 \} \Theta(|x| - c_1)\Theta(c_2 - |x|) + \{ |x| - c_3 \} \Theta(|x| - c_2),$$ \hspace{1cm} (74)

with $c_1 = (4/25); m = (625/11907); c_2 = (67/25); c_3 = (1691/2500)$. Using these approximations, the integrals can be analytically calculated. For $M_1(x, w)$ we obtain

$$M_1(x, w) = Q(x, w) \Theta(x) - Q(-x, -w) \Theta(-x),$$ \hspace{1cm} (75)

$$Q(x, w) = Q_1(x, w) \Theta(c_1 - x) + Q_2(x, w) \Theta(x - c_1) \Theta(c_2 - x) + Q_3(x, w) \Theta(c_2 - c_1),$$ \hspace{1cm} (76)

$$Q_0(x, w) = \sum_{n=1}^3 \frac{1}{n} (w - c_2)^{3-n} (x - c_2)^n + (w - c_2)^3 \ln |x - w|,$$ \hspace{1cm} (77)

$$Q_1(x, w) = x + w \ln \left| \frac{x - w}{w} \right|,$$ \hspace{1cm} (78)

$$Q_2(x, w) = Q_1(c_1, w) + m (Q_0(x, w) - Q_0(c_1, w)) + \ln \left| \frac{x - w}{c_1 - w} \right|,$$ \hspace{1cm} (79)

$$Q_3(x, w) = Q_2(c_2, w) + \ln \left| \frac{x - w}{c_2 - w} \right|.$$ \hspace{1cm} (80)

Similarly, for $M_2(x, w)$ we get

$$M_2(x, w) = B(x, w) \Theta(x) + B(-x, -w) \Theta(-x),$$ \hspace{1cm} (81)
\[ B(x, w) = B_1(x, w) \Theta(c_1 - x) + B_2(x, w) \Theta(x - c_1)\Theta(c_2 - x) + B_3(x, w) \Theta(x - c_2), \] (82)

\[ B_0(x, w) = \sum_{n=1}^{4} \frac{1}{n} (w - c_2)^{4-n}(x - c_2)^n + (w - c_2)^4 \ln |x - w|, \] (83)

\[ B_1(x, w) = \frac{x^2}{4} + \frac{wx}{2} + \frac{w^2}{2} \ln \left| \frac{x - w}{w} \right|, \] (84)

\[ B_2(x, w) = B_1(c_1, w) + (m/4) (B_0(x, w) - B_0(c_1, w)) + (x - c_1) + (w - c_3) \ln \left| \frac{x - w}{c_1 - w} \right|, \] (85)

\[ B_3(x, w) = B_2(c_2, w) + (x - c_2) + (w - c_3) \ln \left| \frac{x - w}{c_2 - w} \right|. \] (86)

Finally, in the calculation of the vertex correction, we also need to approximate the function \( \coth(x) \). We do this using the following polynomial

\[ \coth(x) \rightarrow (1/x)\Theta(c_4 - |x|) + \text{sign}(x)(1 + c_5/x^2)\Theta(|x| - c_4), \] (87)

with \( c_4 = 1/2 \) and \( c_5 = 1/4 \).

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FIG. 1. Fermi surface for the square model using a chemical potential \( \mu = 0.9 \). The arrows denote the nesting vectors for this Fermi surface.
FIG. 2. $\chi''_{0,\text{reg}}(q, \omega)$ shows sharp peaks at the nesting vectors $Q_1 = (\mu, \mu)$ and $Q_2 = (2 - \mu, 2 - \mu)$ at low temperatures. The model parameters are chosen to be $W = 2$ eV, $\omega = 1$ meV, and $\mu = 0.9$ eV. The three curves are for $2T/W = 0.01$, 0.02, and 0.03. These nesting peaks contrast with isotropic electronic structure results which yield a susceptibility that has a smooth momentum dependence.

FIG. 3. The momentum dependence of $\chi''_{0,\text{reg}}(k, \omega)$ for $\mu = 0.9$, $\omega = 0.01$, and $T = 0.03$. The peaks at the nesting vectors are strongly temperature dependent.

FIG. 4. Frequency variation of $\chi''_{0,\text{reg}}(Q, \omega)$ is shown at 60 K (solid curve), 120 K (dashed curve), and 240 K (dotted curve) using $W = 2$ eV and $\mu = 0.9$ eV.

FIG. 5. Frequency dependence of $\chi''_{0}(q, \omega)$ for $q = (q, q)$ consists of a regular particle-hole continuum and a singular peak. The peak has the form $A(q)\delta(\omega - 2q)$ with $A(q)$ increases linearly with $q$ as shown in the inset.

FIG. 6. Real part of the susceptibility $\chi'(k, \omega)$ in the static limit ($\omega = 0$) as a function of momentum shows the nesting peaks. We use $T = 0.03$ and $\mu = 0.9$. Note that the vertical scale begins from 0.9, which is the density of states at the chemical potential $N(\mu)$.

FIG. 7. The first three terms of the RPA series (a–c) and the second order vertex correction diagram (d) for $\chi^{\uparrow\downarrow}$. Note that diagram (b) belongs to $\chi^{\uparrow\downarrow}$.

FIG. 8. Clockwise from top left: the function $F(k, p, 0)$ plotted as a function of $k$ for $p = (0.1, 0), (0.5, 0), (0.9, 0.9), (0.9, 0)$.

FIG. 9. The static susceptibility as a function of momentum showing the bare susceptibility $\chi(q, 0)$ (dotted curve), second order RPA contribution $U^2\chi^3(q, 0)$ (dashed curve), vertex correction $\chi_{\text{core}}(q, 0)$, and the total of these three terms (solid curve). $U = 0.5$ (W/2).

FIG. 10. The static susceptibility at a nesting vector $\chi(Q, 0)$ diverges as $\ln(T)$ at low temperatures as shown by the solid curve. The second order RPA contribution $U^2\chi^3(Q, 0)$, which diverges as $\ln^3(T)$, is shown by the dot-dashed curve. The circles represent the results of our numerical computation for $\chi_{\text{core}}(Q, 0)$. This vertex correction is expected to diverge as $\ln^3(T)$ and we have drawn a fit (dotted curve) for the data using a polynomial $3.766 + 3.483x + 0.9530x^2 + 0.08844x^3$ where $x = \ln(2T/W)$. We have used $\mu = 0.9$ (W/2) and $U = 0.5$ (W/2).

FIG. 11. The first two vertex correction diagrams for $\chi^{\uparrow\downarrow}(q, iq_0)$. The diagrams only differ in the orientation of lines in the second loop. These two diagrams exactly cancel each other at half filling.

FIG. 12. Diagrams (a) and (b) are the first two self energy diagrams in the RPA; diagram (c) is the third term in the self energy series due to the particle-hole ladder; diagram (d) is the third term in the particle-particle ladder. For a Hubbard onsite interaction diagrams (b) and (c) are equal and at half filling diagram (d) is also equal to diagram (c).
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