Closed Strings in the Skyrme Gauge Model

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Abstract. In the framework of the skyrme gauge SU(2) model, closed chiral strings (vortices) are considered, whose closure radius $a$ is considered large in comparison with the characteristic size of the vortex determined by the parameters of the model. In this approximation, it is possible to find the chiral and gauge fields inside the vortex, estimate its radius and energy as functions of the topological charge $Q$.

1. Introduction

The $SU(2)$ skyrme gauge model \cite{1} considers a chiral field $U$ with values in the group $SU(2)$ interacting with a gauge field $A_{\mu} = iA_{\mu a}^a \tau_a/2$, where $\tau_a$ are Pauli matrices. In the low-energy approximation, the chiral field describes nucleons, and the gauge field $\rho$-mesons \cite{2}. Previously, spherically symmetric solitons with a unit topological charge $Q$, interpreted as a baryon number \cite{3,4} and axially symmetric solitons \cite{5} were studied in such a model. In this case, a special ansatz for the chiral field \cite{6} was used, which made it possible to significantly advance in the description of the solitons of the model. In this paper, the results obtained in \cite{5} will be used to further simplify the model within the large closure radius approximation $a$.

2. Basic equations of the model

The Lagrangian of the skyrme gauge model \cite{2} has the form:

$$
\mathcal{L} = -\frac{1}{4\lambda^2} S_{\mu\nu} L_{\mu\nu}^2 + \frac{\varepsilon^2}{16} S_p [L_{\mu}, L_{\nu}]^2 + \frac{1}{2e^2} S_p F_{\mu\nu}^2 + \frac{m^2}{2e^2} \left( A_{\mu}^a \right) - \frac{m^2}{2\lambda^2} S_p (I - U) \tag{1}
$$

where we have used the covariant chiral currents

$$
L_{\mu} = U^+ D_{\mu} U, \quad D_{\mu} = \partial_{\mu} U - A_{\mu} U
$$

and the field strength of Yang-Mills

$$
F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - [A_{\mu}, A_{\nu}]
$$

and the vector $A_{\mu}$ takes values in the Lie algebra of the group $SU(2)$. 
The parameter $\lambda$ in (1) is the characteristic length of the model, $\epsilon$ and $\epsilon$ are dimensionless constants $a$, $m_\pi$ and $m_p$ respectively, are inverse Compton lengths of the pion and $\rho$-meson.

The works [5,6] used a special ansatz for the chiral field $U$, the structure of which is dictated by the desire to obtain a closed charged string based on the Hopf map $S^3 \rightarrow S^2$:

$$U = \cos \psi e^{ir_3 \varphi} + i \tau_1 \sin \psi e^{ir_3 x}$$

(2)

where $\psi$, $\varphi$ and $\chi$ are the corresponding chiral angles. In the axially symmetric case in cylindrical coordinates $\rho$, $\alpha$, $z$ we have:

$$x = -k\alpha, \quad k \in \mathbb{Z}$$

(3)

and the angles $\psi$ and $\varphi$ depend of $\rho$ and $z$. In this case, the vector field has only the azimuthal component

$$A_{\alpha} = i \tau_3 S + i \tau_1 w e^{-ir_3 \kappa}$$

(4)

where $s$ and $w$ are some functions from $\rho$ and $z$.

Substituting (2), (3), and (4) into (1) yields the following expression for the energy $E$ of the system:

$$E = 2\pi \int_0^\infty \rho d\rho \int_0^\infty dz \left( \frac{1}{2\lambda^2} (T^2 + R^2) + \epsilon^2 (\cos^2 \psi [\nabla \psi \nabla \varphi]^2 + R^2 T^2) + \frac{1}{2\epsilon^2} \left[ (\nabla w)^2 + (\nabla S)^2 + \left( m_\rho^2 + \frac{1}{\rho^2} + \frac{1}{\rho} \partial_\rho \right) (w^2 + s^2) \right] + \frac{m_{\pi}^2}{\lambda^2} (1 - \cos \psi \cos \varphi) \right)$$

(5)

where indicated $T^2 = (\nabla \psi)^2 + \cos^2 \psi (\nabla \varphi)^2$, $R = \left( 2s - \frac{k}{\rho} \right) \sin \psi - 2w \cos \psi \sin \varphi$.

The topological charge $Q$ is calculated as the degree of mapping $S^3 \rightarrow S^3$:

$$Q = \frac{k}{2\pi} \int d\sin^2 \psi \wedge d\varphi$$

(6)

To consider closed strings (vortices), we proceed to the toroidal coordinates $r$, $\theta$ by putting

$$\rho = a + r \cos \theta \quad z = r \sin \theta$$

While $\theta \in [-\pi, \pi]$, and inside the vortex $r \ll a$ and $\psi \equiv \pi/2$. Thus, inside the vortex one can put $\cos \psi \ll \sin \psi$, i.e.

$$R^2 \approx \left( 2s - \frac{k}{\rho} \right)^2 \sin^2 \psi$$

As a result, only even terms $w$ remain in the expression for energy (5), which allows us to put $w = 0$.

As a result, we notice that for $a \rightarrow \infty$, the main term in the energy density containing $s$ is reduced to the expression:

$$m_{\rho}^2 \frac{2s^2}{2\epsilon^2} + \frac{1}{2\lambda^2} \left( 2s - \frac{k}{\rho} \right)^2 \sin^2 \psi$$

minimizing which by $s$, we find the gauge field inside the vortex:

$$s = \frac{2k}{a\lambda^2} \sin^2 \psi \left( \frac{m_{\pi}^2}{\epsilon^2} + \frac{4}{\lambda^2} \sin^2 \psi \right)^{-1}$$

(7)
Therefore, the main term that determines the energy $E$ has the form:

$$E \approx 2\pi \int_0^a dr \int_{-\pi}^{\pi} d\theta \left( \frac{1}{2\lambda^2} \left[ (\nabla \psi)^2 + \cos^2 \psi (\nabla \psi)^2 \right] + \epsilon^2 \cos^2 \psi [\nabla \psi \nabla \varphi]^2 + \frac{2\pi^2 a^3}{\lambda^2 m^2} \right)$$  \hspace{1cm} (8)

It can be seen that the functional (8) is invariant with respect to the transformation group:

$$\varphi \to \varphi + \delta_1 \quad \theta \to \theta + \delta_2$$  \hspace{1cm} (9)

which corresponds to the invariant field

$$\varphi = n\theta \quad \psi = \psi(r)$$  \hspace{1cm} (10)

where $n$ is an integer that will be assumed to be large enough ($n \gg K$) that our approximation of the larger radius makes sense. Substituting (10) into (6) and assuming boundary conditions for $\psi$:

$$\psi(0) = \frac{\pi}{2} \quad \psi(a) = 0$$  \hspace{1cm} (11)

find topological charge closed string

$$Q = nk$$

Substituting (10) into (8), by variation in the $\psi$ it is easy to obtain the equation for the chiral angle $\psi(r)$, which is convenient to solve it by setting $r = a \exp(-\tau)$:

$$\frac{d^2 \psi}{d\tau^2} + n^2 \sin \psi \cos \psi = 0$$  \hspace{1cm} (12)

Equation (12) admits the integral

$$\left( \frac{d\psi}{d\tau} \right)^2 + n^2 \sin^2 \psi = C$$

While from the boundary conditions (11), we derive the value of the constant of integration $C = n^2$, which should be more simple equation

$$\frac{d\psi}{d\tau} = n \cos \psi$$

with the obvious solution

$$\psi = \arcsin[\text{th}(n\tau)]$$  \hspace{1cm} (13)

With (13) from (7) and (8) it is easy to evaluate the energy of the vortex $E$ as a function of the radius of the circuit $a$

$$E(a) \approx 4\pi^2 a \left[ \frac{n}{\lambda^2} + \frac{m^2}{2\lambda^2 a^2} + \frac{2\epsilon^2}{3a^2 n^3} \right]$$  \hspace{1cm} (14)

Minimizing energy (14) by $a$ and considering that $n \gg 1$, find

$$a^2 \approx \frac{2\epsilon \lambda}{3m^2 n^{3/2}}$$  \hspace{1cm} (15)

So the energy (14) with (15) is equal to

$$E \approx \frac{16}{3} \pi^2 \left( \frac{2\epsilon m^2}{3\lambda^3} \right)^{1/2} n^{9/4}$$  \hspace{1cm} (16)
3. Conclusion

Based on the addition of a large radius $a$, which is equivalent to a large topological charge ($n \gg k$), it is possible to find explicit expressions for the chiral and gauge fields inside a closed string (vortex) and use them to calculate the closure radius and vortex energy as a function of the topological number $n$, which determines the number of twists of the chiral angle $\varphi$ around the vortex axis. Interestingly, the energy of the vortex, as seen from (16), is at $n \gg 1$ proportional to $n^{9/4}$ and independent of the parameters of the vector field.

4. References

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