Wall-crossing for vortex partition function and handsaw quiver variety

Ryo Ohkawa \(^1,\ast\), Yutaka Yoshida\(^2\)

Abstract

We investigate vortex partition functions defined from integrals over the handsaw quiver varieties of type \(A_1\) via wall-crossing phenomena. We consider vortex partition functions defined by two types of cohomology classes, and get functional equations for each of them. We also give explicit formula for these partition functions. This gives proofs to formula suggested by physicists. In particular, we obtain geometric interpretation of formulas for multiple hypergeometric functions including rational limit of the Kajihara transformation formula.

Keywords: Wall-crossing formula, Vortex partition function, Handsaw quiver variety

2020 MSC: 14D21, 33C80, 81T30

1. Introduction

We investigate vortex partition functions defined from integrals over the handsaw quiver varieties of type \(A_1\) via wall-crossing phenomena. We get functional equations for two types of partition functions defined from different cohomology classes. We also give explicit formula for these partition functions. This gives proofs to formula suggested by physicists Gomis-Floch \(^4\) and Honda-Okuda \(^3\) (see Section \(2.3\)). Furthermore, we obtain geometric interpretations of formula for multiple hypergeometric functions. They are considered as rational limits of the Kajihara transformation \(^13\) and formula obtained in Langer-Schlosser-Warnaar \(^15\) and Hallnäs-Langmann-Noumi-Rosengren \(^1\), \(^8\) from various contexts (see Section \(2.4\)).

\(^\ast\)Corresponding author

\(^1\)Osaka Metropolitan University, 3-3-138 Sumiyoshi-ku, Osaka, 558-8585, Japan: ohkawa.ryo@omu.ac.jp, Research Institute for Mathematical Sciences, Kyoto University, Oiwake-cho, Sakyo-ku, Kyoto, 606-8502, Japan: ohkawa@kurims.kyoto-u.ac.jp

\(^2\)Department of Current Legal Studies, Faculty of Law, Meiji gakuin university, 1518 Kamikurata-cho, Totsuka-Ku, Yokohama 244-8539, Japan: yutakayy@law.meijigakuin.ac.jp
These partition functions are analogue of the Nekrasov partition function. The Nekrasov partition function \[18\] is given by integrals over the quiver varieties of Jordan type. These varieties are non-compact, and hence integrations are defined by counting torus fixed points. Furthermore descriptions of fixed points depend on stability conditions for quiver representations. From this observation, Nakajima-Yoshioka \[21\] derived the blow-up formula using wall-crossing formula developed by Mochizuki \[16\].

Nekrasov’s conjecture states that these partition functions give deformations of the Seiberg-Witten prepotentials for four-dimensional (4d) \(\mathcal{N} = 2\) supersymmetric gauge theories. This conjecture is proven in Braverman-Etingof \[2\], Nekrasov-Okounkov \[19\] and Nakajima-Yoshioka \[20\] independently. In \[20\], they study relationships with similar partition functions defined for blow-up \(\mathbb{P}^2\) of \(\mathbb{P}^2\) along the origin, and get the blow-up formula. These are bilinear relations of \(Z_{\mathbb{P}^2}(\varepsilon_1, \varepsilon_2 - \varepsilon_1, a, q)\) and \(Z_{\mathbb{P}^2}(\varepsilon_1 - \varepsilon_1, \varepsilon_2, a, q)\), which correspond to \(T^2\)-fixed points on \((-1)\) curve of \(\mathbb{P}^2\). Furthermore these arguments are extended in \[21\] to various cohomology classes other than 1 using the theory of perverse coherent sheaves. This method also gives functional equations of Nekrasov functions \[22, 23\].

In physics literature, vortex analogues of instanton partition functions called vortex partition functions are known. The vortex partition functions are given by the partition functions of the moduli data of vortex solutions in supersymmetric gauge theories, where Higgs branch vacua agrees with the D-brane realization of the vortex moduli space. When the gauge group is \(U(r)\), vortex moduli space in the brane construction \[10\] coincides with handsaw quiver varieties of type \(A_1\), where \(r\) corresponds to the rank of a framing. When supersymmetric localization is applied, one can show that vortex partition functions in two (resp. three) dimensions have combinatorial descriptions similar to Nekrasov formula of instanton partition functions in four (resp. five) dimensions. It was shown in \[11\] that, when the Fayet-Iliopoulos (FI) parameter for the moduli space changes from positive to negative, vortex partition functions in certain classes of three-dimensional (3d) \(\mathcal{N} = 2\) gauge theories change discontinuously, i.e., wall-crossing phenomena of Witten indices in \[6\] occur. The authors of \[11\], see also \[12\], derived wall-crossing formulas of generating functions of vortex partition functions in 3d \(\mathcal{N} = 2\) gauge theories.

In two-dimensional (2d) \(\mathcal{N} = (2, 2)\) gauge theories, relations similar to wall-crossing formula in \[11\] were originally found in the study of Seiberg-like dualities \[4, 9\], see also earlier works \[1, 3\]. These relate the vortex partition functions of two different gauge theories. However the precise relation between the formulas in \[4, 9\] and wall-crossing phenomena has not been studied even in
the physics literature. In this paper, we study algebraic and geometric aspects of vortex partition functions of 2d $\mathcal{N} = (2, 2)$ gauge theories and prove the wall-crossing formula of the 2d $\mathcal{N} = (2, 2)$ gauge theories, which establishes the precise relation between wall-crossing phenomena and the formulas in [4, 9]. The wall-crossing formula we will derive is regarded as a rational degeneration of the wall-crossing formula in [11] and identical to a finite type version of the wall-crossing formulas in [22, 23].

Organization of the paper is the following. In Section 2, we introduce the handsaw quiver variety of type $A_1$, and state our main results Theorem 2.2 and Theorem 2.4. In Section 3, we give a proof of Theorem 2.2 using combinatorial description of framed moduli on $\mathbb{P}^2$ due to Nakajima-Yoshioka [20]. In Section 4, we briefly recall Mochizuki method to prove Theorem 2.4. In Section 5, we give a proof of Theorem 2.4.

2. Main results

For a fixed integer $r > 0$, we consider partitions $J = (J_0, J_1)$ of $[r] = \{1, \ldots, r\}$ such that $[r] = J_0 \cup J_1$, and put $r_0 = |J_0|$ and $r_1 = |J_1|$. In our notation, we often use a vector $\vec{r} = (r_0, r_1)$ instead of $J = (J_0, J_1)$, and consider the case where $J_0 = \{1, 2, \ldots, r_0\}$ and $J_1 = \{r_0+1, r_0+2, \ldots, r\}$. For full generality, we can simply change variables.

2.1. Handsaw quiver variety of type $A_1$

For a vector space $W = \mathbb{C} w_1 \oplus \cdots \oplus \mathbb{C} w_r$, we put

$$W_0 = \bigoplus_{\alpha \in J_0} \mathbb{C} w_\alpha, \quad W_1 = \bigoplus_{\alpha \in J_1} \mathbb{C} w_\alpha. \quad (1)$$

For a vector space $V = \mathbb{C}^n$, we consider an affine space

$$\mathbb{M}(W, V) = \mathbb{M} = \operatorname{End}_\mathbb{C}(V) \times \operatorname{Hom}_\mathbb{C}(W_0, V) \times \operatorname{Hom}_\mathbb{C}(V, W_1).$$

**Definition 2.1.** For $\zeta \in \mathbb{R}$, a datum $A = (B, z, w) \in \mathbb{M}$ is said to be $\zeta$-stable if any sub-space $P$ of $V$ with $B(P) \subset P$ satisfies the following two conditions:

1. If $P \subset \ker w$ and $P \neq 0$, we have $\zeta \dim P < 0$.
2. If $\text{im } z \subset P$ and $P \neq V$, we have $\zeta \dim V/P > 0$. 

3
We put $M_+(\vec{r}, n) = [\mathcal{M}^+ / \text{GL}(V)]$, $M_-(\vec{r}, n) = [\mathcal{M}^- / \text{GL}(V)]$, where

$$\mathcal{M}^\pm = \{(B, z, w) \in \mathcal{M} \mid \zeta\text{-stable}\}$$

for a parameter $\zeta$ satisfying $\pm \zeta > 0$. We have an isomorphism from $M_-(\vec{r}, n)$ to Laumon space of type $A_1$ by [17]. We also consider tautological bundles $V = \mu^{-1}(0)^\pm \times V / \text{GL}(V)$, $W_0 = M_+(\vec{r}, n) \otimes W_0$, and $W_1 = M_+(\vec{r}, n) \otimes W_1$ on $M_+(\vec{r}, n)$.

For an algebraic torus $T = \mathbb{C}^*$, we put $T = T^{2r+2}$. We write by $(q, e, e^m, e^\theta)$ an element in $T = T^1 \times T^r \times T^r \times T^1$, where $e = (e_1, \ldots, e_r)$ and $e^m = (e_{m1}, \ldots, e_{mr})$. We write by a monomial $m$ of these variables the corresponding character of $T$, and by $C_m$ the weight space, that is, one-dimensional $T$-representation with the eigenvalue $m$. If we rewrite

$$\mathcal{M}(W, V) = \mathcal{M} = \text{End}_C(V) \otimes C_q \times \text{Hom}_C(W_0, V) \times \text{Hom}_C(V, W_1) \otimes C_q,$$

then we have a natural $T$-action on $M_\pm(\vec{r}, n) = \mathcal{M}^\pm(W, V) / \text{GL}(V)$ by regarding $e = (e_1, \ldots, e_r)$ as an diagonal elements in $\text{GL}(W)$ via [11]. Here this action does not depend on $e^m, e^\theta$, but in the following we will consider vector bundles on $M_\pm(\vec{r}, n)$ tensored with these weights.

2.2. Partition functions

Putting

$$\psi_{\text{fund}} = \text{Eu} \left( \bigoplus_{f=1}^r V \otimes C_{q^{-1}e^{mf}} \right), \quad \psi_{\text{adj}} = \text{Eu} (TM_\pm(\vec{r}, n) \otimes C_{e^\theta}),$$

we consider two types of partition functions

$$Z^J_{\pm \text{fund}}(\varepsilon, a, m, p) = \sum_{n=0}^{\infty} p^n \int_{M_\pm(\vec{r}, n)} \psi_{\text{fund}},$$

$$Z^J_{\pm \text{adj}}(\varepsilon, a, \theta, p) = \sum_{n=0}^{\infty} p^n \int_{M_\pm(\vec{r}, n)} \psi_{\text{adj}}$$

where $a = (a_1, \ldots, a_r), m = (m_1, \ldots, m_r)$.

One of main results of this paper is the following explicit presentation of these partition functions.
Theorem 2.2. (a) The partition function $Z_{\text{fund}}^J(\varepsilon, a, m, p)$ is equal to
\[
\sum_{k \in \mathbb{Z}^J_0} p^{|k|} \prod_{\beta \in J_1} (-1)^{|k|} \prod_{f=1}^J \left( (a_\beta + m_f) / \varepsilon \right) k_\beta \prod_{\alpha \in J_0} (a_\alpha - a_\beta) / \varepsilon - k_\beta, \tag{3}
\]
and the partition function $Z_{-\text{fund}}^J(\varepsilon, a, m, p)$ is equal to
\[
\sum_{k \in \mathbb{Z}^J_0} p^{|k|} \prod_{\beta \in J_1} (-1)^{|k|} \prod_{f=1}^J \left( (a_\beta + m_f) / \varepsilon - k_\beta \right) k_\beta \prod_{\alpha \in J_0} (a_\alpha - a_\beta) / \varepsilon - k_\beta, \tag{4}
\]
where $(x)_k = x(x + 1) \cdots (x + (k - 1))$ is the Pochhammer symbol.

(b) The partition function $Z_{\text{adj}}^J(\varepsilon, a, m, p)$ is equal to
\[
\sum_{k \in \mathbb{Z}^J_0} p^{|k|} \prod_{\beta < \alpha \in J_1} \prod_{\alpha \in J_0} (a_\alpha - a_\beta) / \varepsilon - k_\beta) k_\alpha \prod_{\beta \in J_1} (a_\alpha - a_\beta - \theta) / \varepsilon - k_\beta) k_\beta, \tag{5}
\]
and the partition function $Z_{-\text{adj}}^J(\varepsilon, a, m, p)$ is equal to
\[
\sum_{k \in \mathbb{Z}^J_0} p^{|k|} \prod_{\beta < \alpha \in J_1} \prod_{\alpha \in J_0} (a_\alpha - a_\beta - \theta) / \varepsilon - k_\beta) k_\alpha \prod_{\beta \in J_1} (a_\alpha - a_\beta) / \varepsilon - k_\beta) k_\beta. \tag{6}
\]

These explicit formula are proved in Section 3 using combinatorial descriptions of fixed points sets $M_{\pm}(\mathbf{r}, n)^T$.

Remark 2.3. Using the elementary identity
\[
\prod_{1 \leq \alpha < \beta \leq m} \frac{x_\alpha - x_\beta + k_\alpha - k_\beta}{x_\alpha - x_\beta} \prod_{\alpha, \beta = 1}^m \frac{1}{(x_\alpha - x_\beta + 1) k_\alpha} = \prod_{\alpha, \beta = 1}^m \frac{(-1)^{|k|}}{(x_\alpha - x_\beta - k_\beta) k_\alpha}
\]
as in the argument preceding [8, Theorem 2.1], the above explicit formula (3) and (4) are rewritten as follows.

We put $u = (-1)^{|r_1 - 1|} p$. Then $Z_{+\text{fund}}^J(\varepsilon, a, m, u)$ is equal to
\[
\sum_{k \in \mathbb{Z}^J_0} u^{|k|} \frac{\Delta J_1(a/\varepsilon + k)}{\Delta J_1(a/\varepsilon)} \prod_{\alpha \in J_1} \prod_{\beta \in J_1} \frac{(a_\alpha + m_\beta)}{a_\alpha - a_\beta + 1} k_\alpha \prod_{\beta \in J_1} \frac{a_\beta}{a_\beta - a_\alpha + 1} k_\beta.
\]
and $Z^J_{+\text{fund}}(\varepsilon, a, m, u)$ is equal to

$$
\sum_{k \in \mathbb{Z}_{\geq 0}} u^{|k|} \frac{\Delta_{J_0}(-a/\varepsilon + k)}{\Delta_{J_0}(-a/\varepsilon)} \prod_{\beta \in J_0 \atop \alpha \in J_0} \left( -\frac{a_{\beta}}{\varepsilon} - \frac{m_{\alpha}}{\varepsilon} + 1 \right)_{k_{\beta}} \prod_{\beta \in J_1 \atop \alpha \in J_0} \left( \frac{a_{\beta}}{\varepsilon} - \frac{a_{\alpha}}{\varepsilon} + 1 \right)_{k_{\beta}},
$$

where $\Delta_K(x) = \prod_{\alpha < \beta} (x_\alpha - x_\beta)$ for any subset $K \subset [r]$.

Another main result is the following wall-crossing formula.

**Theorem 2.4.** (a) For $A = \sum_{\alpha=1}^r a_{\alpha}/\varepsilon + m_{\alpha}/\varepsilon$, we have

$$
Z^J_{+\text{fund}}(\varepsilon, a, m, p) = (1 + (-1)^r p)^{-A+r_0} Z^J_{-\text{fund}}(\varepsilon, a, m, p).
$$

(b) **Conjecture 5.5** implies the following formula

$$
Z^J_{+\text{adj}}(\varepsilon, a, \theta, p) = (1 - p)^{(r_0-r_1)(\theta/\varepsilon+1)} Z^J_{-\text{adj}}(\varepsilon, a, \theta, p).
$$

Here **Conjecture 5.5** is a combinatorial identity. We prove this theorem in Section 5 using Mochizuki method. In particular, we have

$$
Z^{(0,[r])}_{+\text{fund}}(\varepsilon, a, m, p) = (1 + (-1)^r p)^{-A}, \quad Z^{(r,[\theta])}_{-\text{fund}}(\varepsilon, a, m, p) = (1 + p)^{A-r},
$$

$$
Z^{(r,[\theta])}_{+\text{adj}}(\varepsilon, a, \theta, p) = Z^{(0,[r])}_{-\text{adj}}(\varepsilon, a, \theta, p) = (1 - p)^{-r(\theta/\varepsilon+1)}.
$$

### 2.3. Comparison with physical computations

From (5) and (6) we can see that

$$
Z^J_{+\text{adj}}(\varepsilon, a, \theta, p) = Z^J_{-\text{adj}}(\varepsilon, -a, \theta, p).
$$

Then our main result (7) gives a proof of the formula proposed by Gomis-Floch [4, (B.1.27)] after substituting $r = N_F, r_0 = N$, and

$$
p = (-1)^{N_{f}-N-1}x, \quad m_f - 1/\varepsilon = \hat{m}_f, \quad a_f/\varepsilon = \hat{m}_f.
$$
where the left (resp. right) hand side of (9) denotes the symbols in this paper (resp. [4]). Our main result (8) also gives a proof of the formula proposed by Honda-Okuda [9, (6.3)] after substituting $r = N_F, r_0 = N, and$

$$p = (-1)^N \cdot e^{-t}, \quad \theta / \varepsilon = -m_{ad}, \quad a_f / \varepsilon = m_f.$$\n
Here we also mention on the result in [11], where Hwang, Yi and the second named author studied wall-crossing formulas of 3d $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ gauge theories in the physics literature. In the rational limit: $\sinh(x/2) = \frac{x}{2} + \cdots$, vortex partition functions (4.7) (resp. (4.10)) of a 3d $\mathcal{N} = 2$ gauge theory in [11] agree with (4) (resp. (3)) under the following identification of the parameters:

$$r_0 = N_c, \quad r = N_f = N_a, \quad k = (n_1, n_2, \cdots, n_{N_c}), \quad a_\alpha = m_\alpha, \quad \varepsilon = -2\gamma, \quad m_f = -\tilde{m}_\alpha + 2\mu - 2\gamma, \quad 0 = \kappa,$$

and also vortex partition functions (4.51) (resp. (4.54)) of a 3d $\mathcal{N} = 2^*$ gauge theory in [11] agree with (6) (resp. (5)) under the following identification of the parameters:

$$r_0 = N_c, \quad k = (n_1, n_2, \cdots, n_{N_c}), \quad a_\alpha = m_\alpha, \quad \varepsilon = -2\gamma, \quad \theta = -2\mu.$$

Then the wall-crossing formulas (4.45) and (4.59) in [11] can be regarded as the trigonometric version of the wall-crossing formulas studied in this article.

### 2.4. Kajihara transformation

Using Remark 2.3 and (a) in Theorem 2.4 together, we have

$$\sum_{k \in \mathbb{Z}_{\geq 0}} u^{|k|} \frac{\Delta_{J_1}(\alpha / \varepsilon + k)}{\Delta_{J_1}(\alpha / \varepsilon)} \prod_{\alpha \in J_1} \frac{(m_\beta / \varepsilon + a_\alpha / \varepsilon)_{k_\alpha}}{(1 + a_\alpha / \varepsilon - a_\beta / \varepsilon)_{k_\alpha}} \prod_{\beta \in J_0} \frac{(m_\alpha / \varepsilon + a_\beta / \varepsilon)_{k_\alpha}}{(1 + a_\alpha / \varepsilon - a_\beta / \varepsilon)_{k_\beta}}$$

$$= (1 - u)^{-A + r_0}$$

$$\cdot \sum_{k \in \mathbb{Z}_{\geq 0}} u^{|k|} \frac{\Delta_{J_0}(-\alpha / \varepsilon + k)}{\Delta_{J_0}(-\alpha / \varepsilon)} \prod_{\alpha \in J_0} \frac{(-m_\alpha / \varepsilon + 1 - a_\alpha / \varepsilon)_{k_\alpha}}{(1 - a_\alpha / \varepsilon + a_\beta / \varepsilon)_{k_\alpha}} \prod_{\beta \in J_0} \frac{(-m_\beta / \varepsilon + 1 - a_\beta / \varepsilon)_{k_\beta}}{(1 - a_\alpha / \varepsilon - a_\beta / \varepsilon)_{k_\beta}},$$

where $u = (-1)^{r_1-1} p$, and $A = \sum_{\alpha = 1}^{r} a_\alpha / \varepsilon + m_\alpha / \varepsilon$. 

7
This formula (12) is equivalent to the rational limit \[13, (4.8)\] of the Kajihara transformation after substituting
\[
\begin{align*}
  x_\alpha &= a_\alpha / \varepsilon + 1 & \alpha \in J_1 \\
  y_\beta &= -a_\beta / \varepsilon & \beta \in J_0 \\
  t_\alpha &= -a_\alpha / \varepsilon - m_\alpha / \varepsilon & \alpha \in J_1 \\
  s_\beta &= a_\beta / \varepsilon + m_\beta / \varepsilon - 1 & \beta \in J_0.
\end{align*}
\]

It is a natural question whether we can obtain similar geometric interpretation for the Kajihara transformation \[13, Theorem 1.1\], or the Kajihara-Noumi transformation \[14, Theorem 2.2\] in elliptic case. See also \[14, Remark 2.4\].

When the radius of the circle \(S^1\) goes to zero, vortex partition functions in 3d \(\mathcal{N} = 2\) theories on \(S^1 \times \mathbb{R}^2\) reduce to vortex partition functions in 2d \(\mathcal{N} = (2, 2)\) theories on \(\mathbb{R}^2\). In this limit, the wall-crossing formula of 3d vortex partition functions \[11\] reduces to that of 2d vortex partition functions \[4\] which is identical to (7) and (12). At this moment, we have not identified the wall-crossing formula of 3d vortex partition functions in \[11\] with the original Kajihara transformation \[13, Theorem 1.1\]. It would be interesting to clarify the relation between the wall-crossing formula of 3d vortex partition functions and Kajihara transformation.

On the other hand, using the explicit formulas (5) and (6), Theorem 2.4 (b) is essentially a rational version of a transformation formula for multiple elliptic hypergeometric series proposed in Langer-Schlosser-Warnaar \[15, Cor. 4.3\] in the context of Kawanaka’s conjecture, and Hallnäs-Langmann-Noumi-Rosengren \[8, (6.7)\] in relation to deformed elliptic Ruijsenaars models. In fact, from the special case of \[8, (6.7)\], we can derive the following formula
\[
\sum_{k \in \mathbb{Z}_{\geq 0}^r} p^{\lvert k \rvert} \prod_{\alpha, \beta = 1}^{r_1} \frac{(q^{\alpha_\beta+1} x_\alpha / t x_\beta; q)_{k_\alpha}}{(q^{-\alpha_\beta} x_\alpha / x_\beta; q)_{k_\alpha}} \prod_{\beta = 1}^{r_0} \prod_{\alpha = 1}^{r_0} \frac{(x_\beta y_\alpha; q)_{k_\beta}}{(q x_\beta y_\alpha / t; q)_{k_\beta}}
\]
\[
= \prod_{s = 1}^{r_1 - r_0} \frac{(q^s p / t^{s-1}; q)_\infty}{(q^s p / t^s; q)_\infty}
\cdot \sum_{k \in \mathbb{Z}_{\geq 0}^r} \frac{(q^{r_1 - r_0} p)}{t^{r_1 - r_0}} \prod_{\alpha, \beta = 1}^{r_0} \frac{(q^{-k_\beta+1} y_\alpha / y_\beta; q)_{k_\alpha}}{(q^{-k_\beta} y_\alpha / y_\beta; q)_{k_\alpha}} \prod_{\beta = 1}^{r_0} \prod_{\alpha = 1}^{r_1} \frac{(y_\beta x_\alpha; q)_{k_\beta}}{(y_\beta x_\alpha / t; q)_{k_\beta}}
\]
due to M. Noumi (private communication). Substituting \(t = q^{\theta / \varepsilon + 1}\), and
\[
\begin{align*}
  x_\alpha &= q^{\alpha_\alpha / \varepsilon} & \alpha \in J_0 \\
  y_\alpha &= q^{-\alpha_\alpha / \varepsilon + \theta / \varepsilon + 1} & \alpha \in J_1
\end{align*}
\]
and taking limit $q \to 1$ in (13), we get (8).

3. Proof of explicit formula

Here we give proofs of explicit formulas for $Z^J_{\pm \text{fund}}(\varepsilon, \mathbf{a}, m, p)$ and $Z^J_{\pm \text{adj}}(\varepsilon, \mathbf{a}, \theta, p)$ using combinatorial description due to [20] and [17].

3.1. Combinatorial description

As in [17, Section 2 (ii)], we can embed handsaw quiver varieties into certain framed moduli of sheaves on $\mathbb{P}^2$ whose fixed points sets are described by Young diagrams [20, Proposition 2.9]. As a result, we also get combinatorial description of handsaw quiver varieties [17, Section 4]. Here we summarize it for the handsaw quiver variety of type $A_1$, that is, $M_{-}(\vec{r}, n)$, and its dual space $M_{+}(\vec{r}, n)$.

The fixed points set $M_{-}(\vec{r}, n)^T$ can be identified with $\{ \mathbf{k} \in (\mathbb{Z}_{\geq 0})^{J_0} | ||\mathbf{k}|| = n \}$, where we regard $(\mathbb{Z}_{\geq 0})^{J_0}$ as a subset of $(\mathbb{Z}_{\geq 0})^{r}$ and put $||\mathbf{k}|| = \sum_{\alpha=1}^{r} k_{\alpha}$. The $T$-fixed point in $M_{-}(\vec{r}, n)$ corresponding to $\mathbf{k}$ is described by

$$ V = \bigoplus_{\alpha \in J_0} \bigoplus_{i=1}^{k_{\alpha}} \mathbb{C}e_{\alpha}q^{-i+1}, \quad W_0 = \bigoplus_{\alpha \in J_0} \mathbb{C}e_{\alpha}, \quad W_1 = \bigoplus_{\alpha \in J_1} \mathbb{C}e_{\alpha}, \quad B(C_{e_{\alpha}q^{-i+1}}) = \begin{cases} C_{e_{\alpha}q^{-i}} & \text{if } 1 \leq i < k_{\alpha} \\ 0 & \text{if } i = k_{\alpha} \end{cases}, \quad z(C_{e_{\alpha}}) = C_{e_{\alpha}}, \quad \text{and } w = 0. $$

Hence the tangent space $T_{\mathbf{k}}M_{-}(\vec{r}, n)$ at the point corresponding to $\mathbf{k}$ is

$$ \text{Hom}_{\mathbb{C}}(V, V) \otimes \mathbb{C}_q + \text{Hom}_{\mathbb{C}}(W_0, V) + \text{Hom}_{\mathbb{C}}(V, W_1) \otimes \mathbb{C}_q - \text{Hom}_{\mathbb{C}}(V, V) $$

$$ = \sum_{\alpha, \beta \in J_0} e_{\alpha}e_{\beta}^{-1} \left( \sum_{i=1}^{k_{\alpha}} \sum_{j=1}^{k_{\beta}} (q^{j-i+1} - q^{j-i}) + \sum_{i=1}^{k_{\alpha}} q^{-i+1} \right) + \sum_{\alpha \in J_1} \sum_{\beta \in J_0} e_{\alpha}e_{\beta}^{-1} \sum_{i=1}^{k_{\beta}} q^i. $$

Here we calculate in $K_T(\text{pt}) = \mathbb{Z}[q, e, e^m, e^\theta]$.

Similarly, we can identify $M_{+}(\vec{r}, n)^T$ with $\{ \mathbf{k} \in (\mathbb{Z}_{\geq 0})^{J_1} | ||\mathbf{k}|| = n \}$. The $T$-fixed point in $M_{+}(\vec{r}, n)$ corresponding to $\mathbf{k}$ is described by

$$ V = \bigoplus_{\alpha \in J_1} \bigoplus_{i=1}^{k_{\alpha}} \mathbb{C}e_{\alpha}q^i, \quad W_0 = \bigoplus_{\alpha \in J_0} \mathbb{C}e_{\alpha}, \quad W_1 = \bigoplus_{\alpha \in J_1} \mathbb{C}e_{\alpha}, $$

$$ B(C_{e_{\alpha}q^i}) = \begin{cases} C_{e_{\alpha}q^i} & \text{if } 1 \leq i < k_{\alpha} \\ 0 & \text{if } i = k_{\alpha} \end{cases}, \quad z(C_{e_{\alpha}}) = C_{e_{\alpha}}, \quad \text{and } w = 0. $$
3.2. Proof of Theorem 2.2

Proof. For the first term in $\sum_{i=1}^{k_{\beta}} q^{i} + \sum_{i=1}^{k_{\alpha}} q^{i+1}$, we have

$$B(\mathbb{C}_{q^{i-1}}) = \begin{cases} \mathbb{C}_{q^{i-1}} & \text{if } i > 1 \\ 0 & \text{if } i = 1 \end{cases}, \quad w(\mathbb{C}_{q^{i-1}}) = \begin{cases} 0 & \text{if } i > 1 \\ \mathbb{C}_{q^{i-1}} & \text{if } i = 1 \end{cases},$$

and $z = 0$. Hence the tangent space $T_{k}M_{+}(\vec{r}, n)$ at the point corresponding to $k$ is

$$\text{Hom}_{\mathbb{C}}(V, V) \otimes \mathbb{C}_{q} + \text{Hom}_{\mathbb{C}}(W_{0}, V) + \text{Hom}_{\mathbb{C}}(V, W_{1}) \otimes \mathbb{C}_{q} - \text{Hom}_{\mathbb{C}}(V, V)$$

when $\ell = 0$. When $\ell = 1$, we divide the range of the summation into small pieces according to $\ell = j - i = -m$. When $k_{\beta} \geq k_{\alpha}$, we have

$$\sum_{i=1}^{k_{\alpha}} q^{i} + \sum_{i=1}^{k_{\beta}} q^{i+1} = q^{k_{\beta}} + \sum_{\ell=k_{\beta}-k_{\alpha}+1}^{k_{\beta}-\ell+1} \left( \sum_{i=1}^{k_{\beta}-\ell} q^{i} - \sum_{i=1}^{k_{\beta}-\ell} q^{i} \right) + \sum_{\ell=1}^{k_{\beta}-k_{\alpha}} \left( \sum_{i=1}^{k_{\alpha}} q^{i} \right) + \sum_{m=0}^{k_{\alpha}-1} \left( \sum_{i=m+2}^{k_{\alpha}} q^{i-m} - \sum_{i=m+1}^{k_{\alpha}} q^{i-m} \right) + \sum_{i=1}^{k_{\beta}} q^{i-1} = \sum_{\ell=k_{\beta}-k_{\alpha}+1}^{k_{\beta}} q^{i}.$$
When $k_\beta < k_\alpha$, the similar arguments hold. Hence in both cases, we have
\[
e_\alpha e_\beta^{-1} \sum_{i=1}^{k_\alpha} \sum_{j=1}^{k_\beta} (q^j - i + 1 - q^j - i) + \sum_{i=1}^{k_\alpha} e_\alpha e_\beta^{-1} (q^j - i + 1) = e_\alpha e_\beta^{-1} \sum_{\ell=k_\beta-k_\alpha+1}^{k_\beta} q^\ell.
\]

For $T_k M_+(\vec{r}, n)$, the assertion follows from \([18]\). □

Our integral is defined by
\[
\int_{M_\pm(\vec{r}, n)^T} \psi = \sum_{k \in M_\pm(\vec{r}, n)^T} \frac{\psi|_k}{\text{Eu}(T_k M_\pm(\vec{r}, n))},
\]
in the similar way as in \([20, \text{Section 4}]\) (see also Section 4.6 for geometric background), where $M_\pm(\vec{r}, n)^T$ are identified as subsets of $\mathbb{Z}_{\geq 0}$ as in the preceding argument. Here the Euler class is defined by a group homomorphism $K(\text{pt}) = \mathbb{Z}[q, e, \mu] \to \mathbb{Q}(\varepsilon, a, m)^\times$ via
\[
\text{Eu}(\pm q^j e_r^1 \cdots e_r^r \mu_1^{k_1} \cdots \mu_r^{k_r}) = \left( i\varepsilon + \sum_{\alpha=1}^r (j_\alpha a_\alpha + k_\alpha m_\alpha) \right)^{\pm 1}.
\]

Since combinatorial descriptions of tautological bundles $\mathcal{V} = \mu^{-1}(0)^{\pm \zeta} \times V/\text{GL}(V)$ over each point $k \in M_\pm(\vec{r}, n)$ are given by \([14]\) and \([16]\), this gives a proof of Theorem 2.2.

4. Mochizuki method

Here we briefly recall methods developed in \([16]\), and \([21]\) for quiver setting. These are similar to \([22], [23]\), and we will also summarize in \([25]\) in more general setting.

4.1. $\ell$-stability

For any subset $\mathcal{I}$ of $[n] = \{1, \ldots, n\}$ and $\mathbb{C}^n = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n$, we put $\mathbb{C}^3 = \bigoplus_{i \in \mathcal{I}} \mathbb{C}v_i$. In this section, we fix $\mathcal{I} \subset [n]$, and often put $V = \mathbb{C}^3$. We enhance data $\mathcal{A} = (B, z, w) \in \mathbb{M}(W, \mathbb{C}^3)$ with flags $F_\bullet = (F_k)$ of $\mathbb{C}^3$ such that $F_0 = 0$, $F_n = \mathbb{C}^3$, and $\dim F_k/F_{k-1} \leq 1$ for any $k$. Hence $F_\bullet$ may have repetitions. For $\ell = 0, 1, \ldots, n$, we introduce $\ell$-stability condition for such pairs $(\mathcal{A}, F_\bullet)$.

**Definition 4.1.** A pair $(\mathcal{A}, F_\bullet)$ of $\mathcal{A} \in \mathbb{M}(W, \mathbb{C}^3)$ and a flag $F_\bullet$ of $V = \mathbb{C}^3$ is said to be $\ell$-stable if any sub-space $P$ of $V = \mathbb{C}^3$ with $B(P) \subset P$ satisfies the following two conditions:
(1) If \( P \subset \ker w \) and \( P \neq 0 \), we have \( P \cap F^\ell = 0 \).

(2) If \( \text{im} \ z \subset P \) and \( P \neq V \), we have \( F^\ell \not\subset P \).

To construct moduli spaces \( \tilde{M}^\ell(\vec{r}, \mathcal{I}) \) of \( \ell \)-stable pairs, we put \( \mathcal{I}^\leq k = \{ i \in \mathcal{I} \mid i \leq k \} \) and

\[
\tilde{M} = \tilde{M}(W, \mathbb{C}^3) = \mathbb{M}(W, \mathbb{C}^3) \times \prod_{k=0}^{n} \text{Hom}_\mathbb{C}(\mathbb{C}^{\mathcal{I}^\leq k}, \mathbb{C}^{\mathcal{I}^\leq k+1}),
\]

and \( G_3 = \text{GL}(V) \times \prod_{k=1}^{n} \text{GL}(\mathbb{C}^{\mathcal{I}^\leq k}) \). For an element \( f = (f_k)_{k=0}^{n} \in \prod_{k=0}^{n} \text{Hom}_\mathbb{C}(\mathbb{C}^{\mathcal{I}^\leq k}, \mathbb{C}^{\mathcal{I}^\leq k+1}) \), we put \( F_k = f_n \cdots f_k(\mathbb{C}^{\mathcal{I}^\leq k}) \subset \mathbb{C}^3 \) and \( F_* = (F_k)_{k=0}^{n} \) when all \( f_k \) are injective. We write by \( \tilde{M}^\ell(W, \mathbb{C}^3) \) a subset consisting of \((A, f) \in \tilde{M}(W, \mathbb{C}^3)\) such that all \( f_k \) are injective and \((A, F_*) = (\text{im} f_n \cdots f_k)_{k=0}^{n}\) is \( \ell \)-stable.

**Definition 4.2.** For \( \zeta \in \mathbb{Q} \) and \( \eta = (\eta_k)_{k=1}^{n} \in (\mathbb{Q}_{>0})^n \), an element \((B, z, w, f) \in \tilde{M}\) is said to be \((\zeta, \eta)\)-semistable if all \( f_k \) are injective and any sub-space \( P \) of \( V = \mathbb{C}^3 \) with \( B(P) \subset P \) satisfies the following two conditions:

(1) If \( P \subset \ker w \) and \( P \neq 0 \), we have

\[
\frac{\zeta \dim P + \sum_{k=1}^{n} \eta_k \dim P \cap F_k}{\dim P} \leq \frac{\sum_{k=1}^{n} \eta_k \dim F_k}{n+1}.
\]

(2) If \( \text{im} \ z \subset P \) and \( P \neq V \), we have

\[
\frac{\zeta \dim V/P + \sum_{k=1}^{n} \eta_k \dim (F_k/P \cap F_k)}{\dim V/P} \geq \frac{\sum_{k=1}^{n} \eta_k \dim F_k}{n+1}.
\]

The condition of injectivity for all \( f_k \) also follows from (1) if we remove the assumption \( P \neq 0 \) and suitably treat the infinity. We put \( \tilde{M}^{(\zeta, \eta)}(W, \mathbb{C}^3) = \{(B, z, w, f) : (\zeta, \eta)\)-semistable\}. For \( \ell = 1, 2, \ldots, n \), we consider the following condition

\[
\zeta + \sum_{k=\ell+1}^{n} \eta_k \cdot \dim \mathbb{C}^{\mathcal{I}^\leq k} < \frac{\sum_{k=1}^{n} \eta_k \cdot \dim \mathbb{C}^{\mathcal{I}^\leq k}}{n+1} < \zeta + \frac{\eta_\ell}{n+1}.
\]

**Remark 4.3.** We assume (21) for a fixed \( \ell \). Then we have

\[
\tilde{M}^{(\zeta, \eta)}(W, \mathbb{C}^3) = \tilde{M}^\ell(\vec{r}, \mathbb{C}^3).
\]
Proof. When $P \subset \ker w$, we have

$$\zeta \dim P + \sum_{k=1}^{n} \eta_k \dim P \cap \mathbb{C}^{\leq k} \begin{cases} > \zeta + \frac{\eta}{n+1} & \text{if } P \cap \mathbb{C}^{\leq \ell} \neq 0 \\ < \zeta + \sum_{k=\ell+1}^{n} \eta_k \cdot \dim \mathbb{C}^{\leq k} & \text{if } P \cap \mathbb{C}^{\leq \ell} = 0. \end{cases}$$

Hence in this case, the inequality in Definition 4.2 (1) is equivalent to $S_0 \cap \tilde{V}^\ell = 0$ by (21).

When $P \supset \text{im} z$, we have

$$\zeta \dim V/P + \sum_{k=1}^{n} \eta_k \dim \mathbb{C}^{\leq k}/P \cap \mathbb{C}^{\leq k} \begin{cases} > \zeta + \frac{\eta}{n+1} & \text{if } P \cap V^{\leq \ell} \neq V^{\leq \ell} \\ < \zeta + \sum_{k=\ell+1}^{n} \eta_k \cdot \dim \mathbb{C}^{\leq k} & \text{if } P \cap V^{\leq \ell} = V^{\leq \ell}. \end{cases}$$

Hence in this case, the inequality in Definition 4.2 (2) is equivalent to $P \cap V^{\leq \ell} \neq V^{\leq \ell}$ by (21).

In both cases, we get strict inequalities if $(B, z, w, f)$ is $(\zeta, \eta)$ semi-stable. Hence we get the desired isomorphism.

Thus we get moduli spaces $\widetilde{M}^\ell(\vec{r}, \mathcal{I})$ of $\ell$-stable pairs as $\widetilde{M}^{(\zeta, \eta)}(W, \mathbb{C}^3)/G_3$. Isomorphism classes of $\widetilde{M}^\ell(\vec{r}, \mathcal{I})$ do not depend on $\mathcal{I}$, but $\dim \mathbb{C}^3 = |\mathcal{I}|$. Hence we also use $\widetilde{M}^\ell(\vec{r}, |\mathcal{I}|)$ to denote this moduli space when we emphasize the number $|\mathcal{I}|$ of elements in $\mathcal{I}$, but not $\mathcal{I}$ itself.

4.2. Enhanced master space

We consider the following two conditions on $\eta$:

$$\sum_{k=1}^{n} \eta_k l_k \neq 0 \quad \text{for any } (l_1, \ldots, l_n) \in \mathbb{Z}^n \setminus \{0\} \text{ with } |I_k| \leq 2n^2 \quad (22)$$

$$\eta_m > \dim V \sum_{k=m+1}^{n} \eta_k \dim F^k \quad \text{for } m = 1, 2, \ldots, n. \quad (23)$$

The condition (22) are called 2-stability condition in [16].

We take $\zeta_- < 0 < \zeta_+ \in \mathbb{Q}$ and $\eta \in (\mathbb{Q}_{>0})^n$ such that $(\zeta_+, \eta)$ satisfies (21) for a fixed $\ell = 1, \ldots, n$,
and \( \eta \) satisfies (22) and (23). If necessary we multiply enough divisible positive integer so that we can assume \( \zeta \pm \) and \( \eta \) are all integers. Then we consider ample \( G \)-linearizations

\[
L_{\pm} = \mathcal{O}_{\tilde{M}} \otimes \bigotimes_{k=1}^{n} \mathbb{C}^{(\det_{\mathcal{I} \leq k}) \theta_{k}^{\pm}}
\]
on \( \tilde{M} = \tilde{M}(W, \mathcal{I}) \), where for any character \( \chi: G \rightarrow \mathbb{C}^* \) we write by \( \mathbb{C}^{\chi} \) the weight space. Here for any \( \mathcal{I}' \subset [n] \), we write by \( \det_{\mathcal{I}'}: GL(\mathbb{C}^{\mathcal{I}'}) \rightarrow \mathbb{C}^* \) the determinant, and \( \theta_{n}^{\pm} = \zeta^{\pm} + \eta_{n} - \sum_{k=1}^{n-1} \eta_{k} \dim \mathbb{C}^{\mathcal{I} \leq k} \) and \( \theta_{k}^{\pm} = \eta_{k}(n+1) \) for \( k < n \). See [24] for this choice of \( \theta^{\pm} = (\theta_{k}^{\pm})_{k=1}^{n} \).

We put \( \tilde{M} = \tilde{M}(W, \mathcal{I}) = \text{ProjSym}(L_{-} \oplus L_{+}) \) and consider the semi-stable locus \( \tilde{M}^{ss} \) with respect to \( \mathcal{O}_{\tilde{M}}(1) \). We define an enhanced master space by \( M = \tilde{M}^{ss}/G \). We consider \( \mathbb{C}^{*}_{\hbar} \)-action on \( M \) defined by

\[
(A, F_{\bullet}, [x_{-}, x_{+}]) \mapsto (A, F_{\bullet}, [e^{h}x_{-}, x_{+}]).
\]

(24)

Remark 4.4. We have

\[
M^{\mathcal{I}_{\sharp}} = M_{+} \sqcup M_{-} \sqcup \bigcup_{\mathcal{I} \in D^{\mathcal{I}}(\mathcal{I})} M_{\mathcal{I}_{\delta}},
\]

where \( D^{\mathcal{I}}(\mathcal{I}) = \{ \mathcal{I}^{\sharp} \subset \mathcal{I} \mid \mathcal{I}^{\sharp} \neq \emptyset, \min(\mathcal{I}^{\sharp}) \leq \ell \} \).

We see that \( M_{-} = \{ x_{+} = 0 \} \) is isomorphic to the full flag bundle \( Fl(V, \mathcal{I}) \) of the tautological bundle \( V \) on \( M_{-}(\vec{r}, |\mathcal{I}|) \) with \( \{ k \in [n] \mid F^{k}/F^{k-1} = \mathcal{I} \} = \mathcal{I} \), and \( M_{+} = \{ x_{-} = 0 \} \) is isomorphic to \( \tilde{M}^{\mathcal{I}}(\vec{r}, \mathcal{I}) \).

We call \( \mathcal{I}^{\sharp} \in D^{\mathcal{I}}(\mathcal{I}) \) decomposition data, and we identify them with a pair \( (\mathcal{I}^{\sharp}, \mathcal{I}^{\flat}) \), where \( \mathcal{I}^{\flat} = \mathcal{I} \setminus \mathcal{I}^{\sharp} \). We take a decomposition \( \mathbb{C}^{3} = \mathbb{C}^{3^{\sharp}} \oplus \mathbb{C}^{3^{\flat}} \), and describe \( M_{\mathcal{I}^{\sharp}} \) as follows.

4.3. Modified \( \mathbb{C}^{*}_{\hbar} \)-action

We put \( D = \zeta^{+} - \zeta^{-} \in \mathbb{Z} \). For each \( \mathcal{I}^{\sharp} \in D^{\mathcal{I}}(\mathcal{I}) \), we put \( d^{\sharp} = |\mathcal{I}^{\sharp}| \). To describe \( M_{\mathcal{I}^{\sharp}} \), we consider a modified action \( \mathbb{C}^{*}_{\hbar} \times M \rightarrow M \) induced by

\[
(A, F_{\bullet}, [x_{-}, x_{+}]) \mapsto \left( e^{\frac{d^{\sharp}}{\hbar} \cdot \text{id}_{\mathbb{C}^{3^{\sharp}}} \oplus \text{id}_{\mathbb{C}^{3^{\flat}}}} \right) (A, F_{\bullet}, [e^{h}x_{-}, x_{+}]).
\]

(25)
This action is equal to the original $\mathbb{C}_h^*$-action \((24)\), since the difference is absorbed in $G$-action. Then \((A_z \oplus A_y, f_2 \oplus f_y, [1, \rho])\) is fixed by this $\mathbb{C}_h^*$-action for \((A_z, f_z) \in \widetilde{M}(0, \mathbb{C}^3), (A_y, f_y) \in \widetilde{M}(W, \mathbb{C}^3)\), and $\rho \neq 0$. Hence if this element is stable, then it represents a $\mathbb{C}_h^*$-fixed point in $\mathcal{M}$. This observation together with analysis for stability condition in \([22]\) implies that $\mathcal{M}_{3\tilde{z}}$ is isomorphic to the quotient stack of

$$\widetilde{M}^\sharp(0, \mathbb{C}^3) \times (\mathbb{C}_{t^{-1}(\det_{\tilde{g}})} D)^\times \times \widetilde{M}^{\min(3)} - 1(W, \mathbb{C}^3) \times (\mathbb{C}_{t(\det_{\tilde{g}})} D)^\times$$

by a group $G_{3\tilde{z}} \times G_\infty \times \mathbb{C}_t^*$, where $\widetilde{M}^\sharp(0, \mathbb{C}^3) = \{(B, F_*) \in \widetilde{M}(0, \mathbb{C}^3) \mid \mathbb{C}[B] F^1 = \mathbb{C}^3\}$, and the superscript “$\times$” denotes the complement of zero.

If we replace the group $\mathbb{C}_t^*$ with $\mathbb{C}_{t/\tilde{d}}^*$, then we have an étale cover $\Phi_{3\tilde{z}} : \mathcal{M}_{3\tilde{z}}' \to \mathcal{M}_{3\tilde{z}}$ of degree $1/\tilde{d} D$. Then by a group automorphism of $\text{GL}(V_{3\tilde{z}}) \times \mathbb{C}_{t/\tilde{d}}^*$ sending $(g, t^{1/\tilde{d}} D)$ to $(t^{-1/\tilde{d}} D g, t^{1/\tilde{d}} D)$, we get an isomorphism $\mathcal{M}_{3\tilde{z}}' \cong \mathcal{M}_z \times \mathcal{M}_b$, where

$$\mathcal{M}_z = \left[ \widetilde{M}^\sharp(0, \mathbb{C}^3) \times (\mathbb{C}_{(\det_{\tilde{g}})} D)^\times / G_{3\tilde{z}} \right],$$

$$\mathcal{M}_b = \left[ \widetilde{M}^{\min(3)} - 1(W, \mathbb{C}^3) \times (\mathbb{C}_{t(\det_{\tilde{g}})} D)^\times / G_{3\tilde{z}} \times \mathbb{C}_{t/\tilde{d}}^* \right].$$

4.4. Moduli stack of destabilizing objects $\mathcal{M}_z$

To study $\mathcal{M}_z$, we give explicit description of $M_-(1, 0, \tilde{d}^\alpha) = \mathbb{M}^-(W_2, V_2)/\text{GL}(V_2)$, where $W_2$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with $W_{20} = \mathbb{C}$ and $W_{21} = 0$. In fact, we do not need the next proposition, but we give a proof for the sake of explanation.

**Proposition 4.5.** We have an isomorphism $M_-(1, 0, \tilde{d}^\alpha) \cong A_{\tilde{d}^\alpha} = \text{Spec} \mathbb{C}[x_1, \ldots, x_{\tilde{d}^\alpha}]$ such that the tautological bundle $\mathcal{V}_z$ corresponds to $\mathbb{C}[x_1, \ldots, x_{\tilde{d}^\alpha}]$-module $\mathbb{C}[x_1, \ldots, x_{\tilde{d}^\alpha}, y]/(y^{\tilde{d}^\alpha} + x_1 y^{\tilde{d}^\alpha - 1} + \ldots + x_{\tilde{d}^\alpha})$ via this isomorphism. Furthermore $q \in T$ acts by $q^{-1} y$ and $q^i x_i$ for $i = 1, \ldots, \tilde{d}^\alpha$.

**Proof.** We regard $\mathbb{C}[x_1, \ldots, x_{\tilde{d}^\alpha}]$-module $\mathbb{C}[x_1, \ldots, x_{\tilde{d}^\alpha}, y]/(y^{\tilde{d}^\alpha} + x_1 y^{\tilde{d}^\alpha - 1} + \ldots + x_{\tilde{d}^\alpha})$ as family of data with $(B, z, w) = (y, 1, 0)$. Then we have a morphism $A_{\tilde{d}^\alpha} = \text{Spec} \mathbb{C}[x_1, \ldots, x_{\tilde{d}^\alpha}] \to M_-(1, 0, \tilde{d}^\alpha)$, and we can check that this induces bijection between the sets of closed points. Hence this is an isomorphism.

We consider the corresponding torus action on $A_{\tilde{d}^\alpha}$ as follows. For $\mathbf{x} = (x_1, \ldots, x_{\tilde{d}^\alpha}) \in A_{\tilde{d}^\alpha}$, we
put \( f_x = y^d + x_1 y^{d-1} + \cdots + x_d y \). We have a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}[y]/\langle f_x(y) \rangle & \xrightarrow{ty} & \mathbb{C}[y]/\langle f_x(y) \rangle \\
y = ty & & y = ty \\
\mathbb{C}[y]/\langle f_x(t^{-1}y) \rangle & \xrightarrow{y} & \mathbb{C}[y]/\langle f_x(t^{-1}y) \rangle
\end{array}
\]

If we put \( q, x = (q x_1, q^2 x_2, \ldots, q^{d-1} x_{d-1}, q^d x_d) \), we have \( f_x(q^{-1}y) = q^{-d} f_q x(y) \). Hence we get the assertion for \( T \)-action on \( \mathbb{A}^d \).

Similarly we can easily see that \( GL(W_{\ell}) \) trivially acts on \( M_{\ell}((1, 0), d^\ell) \) by the following commutative diagram for \( e \in GL(W_{\ell}) \):

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{e^{-1}} & \mathbb{C}[y]/\langle f_x(y) \rangle \\
& & y \\
\mathbb{C} & \xrightarrow{e^{-1}} & \mathbb{C}[y]/\langle ef_x(y) \rangle
\end{array}
\]

In the following, we write by \( \mathcal{V}_\ell \) the tautological bundle over \( M_{\ell}((1, 0), d^\ell) \), and by \( \mathcal{V}_\ell \) one on \( \widetilde{M}_{\ell}^{\min(3\ell)}(\ell^\ell) \). We consider the full flag bundle \( Fl(\mathcal{V}_\ell/\mathcal{O}_{M_{\ell}((1, 0), d^\ell)}, \ell^\ell) \) of the tautological bundle \( \mathcal{V}_\ell \) on \( M_{\ell}((1, 0), d^\ell) \), where \( \ell^\ell = \ell^\ell \backslash \{\min(3\ell)\} \). We also write by the same letter \( \mathcal{V}_\ell \) the pull-back to \( Fl(\mathcal{V}_\ell/\mathcal{O}_{M_{\ell}((1, 0), d^\ell)}, \ell^\ell) \) and consider the determinant line bundle \( \det \mathcal{V}_\ell \) over \( Fl(\mathcal{V}_\ell/\mathcal{O}_{M_{\ell}((1, 0), d^\ell)}, \ell^\ell) \).

Then for \( \mathcal{M}_\ell \) defined in [26], we have the following proposition.

**Remark 4.6.** We have an isomorphism

\[
\mathcal{M}_\ell \cong \left[ (\det \mathcal{V}_\ell)^{\otimes D} \otimes \mathbb{C}_{u^D} \right]/\mathbb{C}_u^*.
\]

In particular, we have an étale morphism \( \mathcal{M}_\ell \to Fl(\mathcal{V}_\ell/\mathcal{O}_{M_{\ell}((1, 0), d^\ell)}, \ell^\ell) \) of degree \( 1/(d^2 D) \).

**Proof.** We use [26] and consider a map

\[
\tilde{M}^\ell(0, V_{2\ell}) \to M^- (W_{\ell}, V_{\ell}), \quad (B_{\ell}, f_{\ell}) = (f_{\ell k}) \mapsto (B_{\ell}, f_{\ell k n-1} \cdots f_{\ell \min(3\ell)}, 0).
\]

Then a group automorphism \( GL(V_{2\ell}) \times GL(V_{\ell}^{\min(3\ell)}) \) mapping \((g, u)\) to \((u^{-1} g, u)\) gives morphism \( \mathcal{M}_\ell \to M^-((1, 0), d) \), which induce the desired isomorphism.
We consider the universal flags $\mathcal{F}_n$ on $Fl(\mathcal{V}_i / \mathcal{O}_{M_-(1,0,d)}, \mathfrak{J})$ and $\mathcal{F}_n$ on $\tilde{M}^{\min(\mathfrak{J})-1}(\mathfrak{r}, \mathfrak{J})$. On $Fl(\mathcal{V}_i / \mathcal{O}_{M_-(1,0,d)}, \mathfrak{J})$, we define a full flag $\mathcal{F}_n$ of $\mathcal{V}_i$ by the pull-back of $\mathcal{F}_n$ to $\mathcal{V}_i$ for $i \neq \min(\mathfrak{J})$ and $\mathcal{F}_n = \mathcal{F}_n^{\min(\mathfrak{J})}$ of $\mathcal{V}_i \times \tilde{M}^{\min(\mathfrak{J})-1}(\mathfrak{r}, \mathfrak{J})$ by the same letter $\mathcal{F}_n$, $\mathcal{F}_n$.

By [23] and [27] and Proposition [4.6] we have a natural étale morphism $\Psi_{\mathfrak{J}} : \mathcal{M}^{\mathfrak{J}} \to Fl(\mathcal{V}_i / \mathcal{O}_{M_-(1,0,d)}, \mathfrak{J})$ of degree $1/(d^2D)^2$.

4.5. Decomposition of $\mathcal{M}^{\mathfrak{J}}$

Summarizing, we have the following theorem.

Theorem 4.7. We have

$$\mathcal{M}^{\mathfrak{J}} = \mathcal{M}_+ \sqcup \mathcal{M}_- \sqcup \bigsqcup_{\mathfrak{J} \in D(\mathcal{J})} \mathcal{M}_{\mathfrak{J}}$$

such that the following hold.

(i) We have $\mathcal{M}_+ \cong \tilde{M}(\mathfrak{r}, \mathfrak{J})$ and $\mathcal{M}_- \cong \tilde{M}(\mathfrak{r}, \mathfrak{J})$, that is, the full flag bundle $Fl(\mathcal{V}, \mathfrak{J})$ of the tautological bundle $\mathcal{V}$ over $M_-(\mathfrak{r}, \mathfrak{J})$.

(ii) For each $\mathfrak{J} \in D$, we have finite étale morphisms $\Phi_{\mathfrak{J}} : \mathcal{M}_{\mathfrak{J}} \to \mathcal{M}_{\mathfrak{J}}$ of degree $1/(d^2D)$, and $\Psi_{\mathfrak{J}} : \mathcal{M}_{\mathfrak{J}} \to Fl(\mathcal{V}_i / \mathcal{O}_{M_-(1,0,d)}, \mathfrak{J}) \times \tilde{M}^{\min(\mathfrak{J})-1}(\mathfrak{r}, \mathfrak{J})$ of degree $1/(d^2D)^2$, where $d^2 = |\mathfrak{J}|$ and $D = \zeta_+ - \zeta_- \in \mathbb{Z}$.

(iii) As $\mathbb{C}_{h/d^2D}^\ast$-equivariant vector bundles on $\mathcal{M}_{\mathfrak{J}}$, we have

$$\Phi_{\mathfrak{J}}^\ast \mathcal{V}_i \cong \Psi_{\mathfrak{J}}^\ast \mathcal{V}_2 \otimes e_{h/d^2D} \otimes (L_{\mathfrak{J}})^\vee, \quad \Phi_{\mathfrak{J}}^\ast \mathcal{V}_{3\infty} \cong \Psi_{\mathfrak{J}}^\ast \mathcal{V}_0$$

for a line bundle $L_{\mathfrak{J}}$ on $\mathcal{M}_{\mathfrak{J}}$ such that $(L_{\mathfrak{J}})^\otimes D \cong \Psi_{\mathfrak{J}}^\ast (\det \mathcal{V}_2 \otimes \det \mathcal{V}_0)^\otimes D$.

We write by $N_+, N_-$, and $N_{\mathfrak{J}}$ normal bundles of $\mathcal{M}_+, \mathcal{M}_-$ and $\mathcal{M}_{\mathfrak{J}}$ in $\mathcal{M}$ respectively. For the following computations, we introduce some notation and describe these normal bundles here. For a vector bundle $\mathcal{E}$ and a finite set $\mathfrak{J} \subset \mathbb{Z}$ with $\text{rk} \mathcal{E}$ elements, we write by $Fl(\mathcal{E}, \mathfrak{J})$ the full flag bundle $F_i = (F_i)_{i \in \mathbb{Z}}$ of $\mathcal{E}$ such that

$$\{i \in \mathbb{Z} \mid F_i / F_{i-1} \neq 0\} = \mathfrak{J}.$$  

These are all isomorphic to $Fl(\mathcal{E}, [\text{rk} \mathcal{E}])$, but we use products of these flag bundles and combinatorial description in Section [5.4].
For two flags $F_\bullet, F'_\bullet$ of sheaves on the same Deligne-Mumford stack, we put
\[
\Theta(F_\bullet, F'_\bullet) = \sum_{i > j} \text{Hom}(F_j/F_{j-1}, F'_i/F'_{i-1}),
\]
\[
\tilde{\gamma}(F_\bullet, F'_\bullet) = \Theta(F_\bullet, F'_\bullet) + \Theta(F'_\bullet, F_\bullet).
\]
When $F_\bullet = F'_\bullet$, we put $\Theta(F_\bullet) = \Theta(F_\bullet, F_\bullet)$.

**Lemma 4.8.** (1) We have $N_\pm = L_\pm \otimes e^\hbar$.

(2) We have $\Phi^\ast I^\bullet N^\bullet \sim \Psi^\ast I^\bullet N^\bullet$ where

\[
N^\bullet = \text{Hom}(V^\bullet, V^\bullet \otimes C_q) + \text{Hom}(W^0, V^\bullet) + \text{Hom}(V^\bullet, W^1 \otimes C_q) - \text{Hom}(V^\bullet, V^\bullet) - \text{Hom}(V^\bullet, W^1 \otimes C_q)
\]

for vector bundles $V, V^\bullet$, and $Z_2$-graded vector bundle $W = W^0 \oplus W^1$.

**4.6. Localization**

We put $M_0 = \text{Spec } (\mathcal{M}(W, V), \mathcal{O}_M)^{GL(V)}$. By the main result \[4.8, (1)] and Theorem \[4.7\] we have the following diagram

\[
\begin{array}{ccc}
\lim_{\leftarrow m} A^\ast_{\mathbb{C}_h^\pm}(M \times_T E_m) \otimes_{\mathbb{C}[h]} \mathbb{C}[h^{\pm 1}] & \xrightarrow{\sim} & \lim_{\leftarrow m} A^\ast(M^{\mathbb{C}_h^\pm} \times_T E_m) \otimes_{\mathbb{C}} \mathbb{C}[h^{\pm 1}] \\
\Pi_\ast(\cdot) \otimes [M]^\otimes & \downarrow & \Pi_\ast(\cdot) \otimes [(M_\pm)^\otimes] + [M_\pm]^\otimes + \sum_{\mathbb{Z}_2} [M_{\mathbb{Z}_2}]^\otimes \\
\lim_{\leftarrow m} A_\ast(M_0 \times_T E_m) \otimes_{\mathbb{C}} \mathbb{C}[h, h^{-1}] & \xrightarrow{\sim} & \lim_{\leftarrow m} A_\ast(M_0 \times_T E_m) \otimes_{\mathbb{C}} \mathbb{C}[h, h^{-1}]
\end{array}
\]

where the upper horizontal arrow is given by

\[
\frac{\iota^\ast_+}{\text{Eu}(\mathcal{N}(M_+))} + \frac{\iota^\ast_-}{\text{Eu}(\mathcal{N}(M_-))} + \sum_{\mathbb{Z}_2} \frac{\iota^\ast_{\mathbb{Z}_2}}{\text{Eu}(\mathcal{N}(M_{\mathbb{Z}_2}))}.
\]

Here $h$ corresponds to the first Chern class in $A^{\mathbb{C}_h^\ast}(pt)$ of the weight $e^\hbar \in \mathbb{C}_h^\ast$, and $\iota_\pm$ and $\iota_{\mathbb{Z}_2}$ are embeddings of $M_\pm$ and $M_{\mathbb{Z}_2}$ into $M$. 

18
Then $\text{Eu}(N_\pm), \text{Eu}(N_\mp)$ are invertible in $A^*_{\mathbb{T} \times \mathbb{C}_h^*}(\mathcal{M})[\hbar, \hbar^{-1}])$, and we have the following localization formula

$$
\int_{\mathcal{M}} \phi = \int_{\mathcal{M}_+} \frac{\phi|_{\mathcal{M}_+}}{\text{Eu}(N_+)} + \int_{\mathcal{M}_-} \frac{\phi|_{\mathcal{M}_-}}{\text{Eu}(N_-)} + \sum_{\mathcal{I} \in D'(\mathcal{I})} \int_{\mathcal{M}_{\mathcal{I}}} \frac{\phi|_{\mathcal{M}_{\mathcal{I}}}}{\text{Eu}(N_{\mathcal{I}})}.
$$

(29)

4.7. Cohomology classes

We take two cohomology classes

$$
\text{Eu} \left( \bigoplus_{f=1}^r V \otimes \frac{e^m_f}{q} \right), \quad \text{Eu}^\theta (\Lambda(V)) \in A^*_{\mathbb{T} \times \mathbb{C}_h^*}(\mathcal{M}),
$$

(30)

where $\Lambda(V) = \text{End}(V) \otimes \mathbb{C}_q + \mathcal{H}om(W_0, V) + \mathcal{H}om(V, W_1) \otimes \mathbb{C}_q - \text{End}(V)$. For $\alpha = [E] - [F] \in K(\mathcal{M})$ with vector bundles $E, F$ on $\mathcal{M}$, we define $\text{Eu}^\theta(\alpha) = c_{tk}E(E \otimes e^\theta)/c_{tk}F(F \otimes e^\theta)$.

In the following we consider one of classes in (30) and write it by $\psi = \psi(V)$. Furthermore, we put

$$
\tilde{\psi} = \tilde{\psi}(V) = \frac{\psi \cdot \text{Eu}^\theta (\Theta(F^\bullet))}{|\mathcal{I}|!} \in A^*_{\mathbb{C}_h}(\mathcal{M} \times T E_m)
$$

and substitute $\phi = \tilde{\psi}$ in (29). The left hand side in (29) is a polynomial in $\hbar$ while the right hand side has a power series part in $\hbar^{-1}$. Hence if the symbol $\text{Res}_{\hbar=\infty}$ denotes the operation taking the coefficient in $\hbar^{-1}$, we have

$$
\int_{\tilde{M}^\phi(\vec{r}, \mathcal{I})} \tilde{\psi} - \int_{\tilde{M}_-(\vec{r}, |\mathcal{I}|)} \psi = - \text{Res}_{\hbar=\infty} \sum_{\mathcal{I} \in D'(\mathcal{I})} \int_{\mathcal{M}_{\mathcal{I}}} \frac{\tilde{\psi}|_{\mathcal{M}_{\mathcal{I}}}}{\text{Eu}(N_{\mathcal{I}})}
$$

By Lemma 4.8 (2), the last summand is equal to

$$
\frac{|\mathcal{I}|!}{|\mathcal{I}|!} \int_{\tilde{M}_{\text{min}(\mathcal{I})} - (\vec{r}, |\mathcal{I}|)} \frac{1}{|\mathcal{I}|!} \int_{F_k(V_{\mathcal{I}}/\mathcal{O}_{M_-(1,0,d^2)}, \mathcal{I})} \frac{\psi(V_{\mathcal{I}} \otimes V_{\mathcal{I}} \otimes e^{\hbar}) \cdot \text{Eu}^\theta (\Theta(F^\bullet_{\mathcal{I}} \otimes e^\hbar \otimes F^\bullet_{\mathcal{I}}))}{\mathcal{N}(V_{\mathcal{I}} \otimes e^\hbar, V_{\mathcal{I}}) \cdot \text{Eu} (\widetilde{\mathcal{F}}^\bullet_{\mathcal{I}} \otimes e^\hbar, F^\bullet_{\mathcal{I}})}
$$

(31)
where $\mathcal{F} = \mathcal{F} \setminus \{\min \mathcal{F}\}$. We have
\[
\Theta(\mathcal{F} \otimes e^h \otimes \mathcal{F}) = \Theta(\mathcal{F}^\circ) + \Theta(\mathcal{F}^\circ) + \mathcal{F}(\mathcal{F}^\circ \otimes e^h \otimes \mathcal{F}),
\]
and
\[
\Theta(\mathcal{F}^\circ) = \Theta(\mathcal{F}^\circ) + \mathcal{V}/\mathcal{O}_{M_-(1,0,d^\circ)}.
\]
(32)

We note that $\Theta(\mathcal{F}^\circ)$ is the relative tangent bundle of $Fl(V/\mathcal{O}_{M_-(1,0,d^\circ)} \to M_-(1,0,d^\circ)$.

In this expression, we deleted some line bundles and a parameter $d^\circ D$, since we have \(\text{Res} f(h) = \text{Res} f(d^\circ Dh + a)\) (cf. [21, Section 8.2]), and integrals are taken over $1/d^\circ D$-degree étale covering

5. Wall-crossing formula

In the following, we use wall-crossing formula deduced from analysis in the previous section to get functional equations (7). These are the similar calculations to [21, Section 6], hence we omit detail explanation.

For $\mathbf{d} = (d_1, \ldots, d_j) \in \mathbb{Z}_{>0}$, we put $|\mathbf{d}| = d_1 + \cdots + d_j$. Let $\text{Dec}^n_j$ be the set of collections $\mathcal{J} = (\mathcal{J}_1, \ldots, \mathcal{J}_j)$ of non-empty subsets of $[n]$ such that

- $\mathcal{J}_i \cap \mathcal{J}_k = \emptyset$ for $i \neq k$,
- $\min(\mathcal{J}_1) > \cdots > \min(\mathcal{J}_j)$.

We note that $\text{Dec}^n_j = \mathcal{D}^n_j$. We identify $\mathcal{J} = (\mathcal{J}_1, \ldots, \mathcal{J}_j)$ with $(\mathcal{J}_1, \ldots, \mathcal{J}_j, \mathcal{J}_\infty)$ where $\mathcal{J}_\infty = [n] \setminus \bigcup_{i=1}^j \mathcal{J}_i$. We consider maps $\sigma : \text{Dec}^n_{j+1} \to \text{Dec}^n_j$,

$$(\mathcal{J}' = (\mathcal{J}'_1, \ldots, \mathcal{J}'_{j+1}) \mapsto \sigma(\mathcal{J}') = (\mathcal{J}'_1, \ldots, \mathcal{J}'_j)$$

and $\rho : \text{Dec}^n_j \to S^n_j = \left\{ \mathbf{d} = (d_1, \ldots, d_j) \in \mathbb{Z}_{>0}^j \mid |\mathbf{d}| \leq n \right\}$, $\mathcal{J} \mapsto \mathbf{d}_\mathcal{J} = (|\mathcal{J}_1|, \ldots, |\mathcal{J}_j|)$ for $j = 1, \ldots, n$.

5.1. Iterated cohomology classes

For $j > 0$ and $\mathbf{d} = (d_1, \ldots, d_j) \in \mathbb{Z}_{>0}^j$, we consider a product

$$M_\mathbf{d} = M_-(\mathbf{r}, n - |\mathbf{d}|) \times \prod_{i=1}^j M_-(1,0,d_i).$$
We write by $\mathcal{V}^{(i)}$ the tautological bundle on the component $M_{-((1,0),d_i)}$ of $M_d$.

For $\mathfrak{I} = (\mathfrak{I}_1, \ldots, \mathfrak{I}_j) \in \rho^{-1}(d)$, we also put

$$\tilde{M}_2 = \tilde{M}(\bar{r}, \mathfrak{I}_\infty) \times \text{Fl}(\mathcal{V}^{(1)}/\mathcal{O}_{M_{-((1,0),d_1)}}, \mathfrak{I}_1) \times \cdots \times \text{Fl}(\mathcal{V}^{(j)}/\mathcal{O}_{M_{-((1,0),d_j)}}, \mathfrak{I}_j)$$

where $\tilde{\mathfrak{I}_i} = \mathfrak{I}_i \setminus \{\text{min}(\mathfrak{I}_i)\}$, and write by $\mathcal{F}^{(i)} = \text{the pull-back to } \tilde{M}_2$ of the universal flag on each component $\text{Fl}(\mathcal{V}^{(i)}/\mathcal{O}_{M_{-((1,0),d_i)}}, \tilde{\mathfrak{I}_i})$. We consider tori $\mathbb{C}_{h_i}^*$ for $i = 1, 2, \ldots, n$, and take a coordinate $e^{h_i} \in \mathbb{C}_{h_i}^*$. By abuse of notation $e^{h_i}$ also denotes a trivial bundle with $e^{h_i}$-weight. We write by $\mathcal{F}_\infty$ the pull-back of the flag $\mathcal{F}$ on $\tilde{M}(\bar{r}, \mathfrak{I}_\infty)$, and put $\mathcal{F}^{(i)} = \mathcal{F}_\infty \oplus \bigoplus_{k > i} \mathcal{F}^{(k)} \otimes e^{h_k}$.

We do not need the obstruction theory to define fundamental cycle $[\tilde{M}_2]$ since we only consider smooth Deligne-Mumford stacks in this paper. For $\alpha \in A^\bullet_{T \times \mathbb{C}^*_a \times \prod_{i=1}^l \mathbb{C}_{h_i}^*} (\tilde{M}_2)$, we write by $\int_{[\tilde{M}_2]} \alpha \in A^\bullet_{T \times \mathbb{C}^*_a \times \prod_{i=1}^l \mathbb{C}_{h_i}^*} (\tilde{M}(\bar{r}, n - |d_3|))$ the Poincare dual of the push-forward of $\alpha \cap [\tilde{M}_2]$ by the projection $\tilde{M}_2 \to \tilde{M}(\bar{r}, \mathfrak{I}_\infty)$.

We write by the same letters $\mathcal{V}, \mathcal{W}_0, \mathcal{W}_1$ the pull-backs of ones on $\tilde{M}(\bar{r}, \mathfrak{I}_\infty)$ to the product $\tilde{M}_2$. For $\mathfrak{I} = (\mathfrak{I}_1, \ldots, \mathfrak{I}_j) \in \text{Dec}_j^n$, we write by $\tilde{\psi}_\mathfrak{I}(\mathcal{V})$ the following cohomology class

$$\int_{[\tilde{M}_2]} \frac{\psi \left( \mathcal{V} \oplus \bigoplus_{i=1}^j \mathcal{V}^{(i)} \otimes e^{h_i} \right) \text{Eu}^\theta(\Theta(\mathcal{F}_\infty^{(i)}))}{\text{Eu} \left( \bigoplus_{i=1}^j \mathfrak{N}(\mathcal{V}^{(i)} \otimes e^{h_i}, \mathcal{V} \oplus \bigoplus_{k=i+1}^j \mathcal{V}^{(k)} \otimes e^{h_k}) \right)} \frac{1}{|\mathfrak{I}_\infty|!}$$

$$= \int_{[\tilde{M}_2]} \frac{\psi \left( \mathcal{V} \oplus \bigoplus_{i=1}^j \mathcal{V}^{(i)} \otimes e^{h_i} \right) \text{Eu}^\theta(\Theta(\mathcal{F}_\infty^{(i)})) / |\mathfrak{I}_\infty|!}{\text{Eu} \left( \bigoplus_{i=1}^j \mathfrak{N}(\mathcal{V}^{(i)} \otimes e^{h_i}, \mathcal{V} \oplus \bigoplus_{k=i+1}^j \mathcal{V}^{(k)} \otimes e^{h_k}) \right)} \cdot \prod_{i=1}^j \frac{\text{Eu}^\theta(\Theta(\mathcal{F}_\infty^{(i)})) \text{Eu}^\theta(\tilde{\mathcal{F}}^{(i)} \otimes e^{h_i}, \mathcal{F}_\infty^{(i)})}{\text{Eu}(\tilde{\mathcal{F}}^{(i)} \otimes e^{h_i}, \mathcal{F}_\infty^{(i)})}$$

in $A^\bullet_{T \times \mathbb{C}^*_a \times \prod_{i=1}^l \mathbb{C}_{h_i}^*} (M)$. Here $\mathfrak{N}(\mathcal{V}_2, \mathcal{V}_3)$ is defined by (28). By modified $\mathbb{C}_{h_i}^*$-action (25), we need to multiply $\mathcal{V}^{(i)}$ with $e^{h_i}$ in (33).

By the projection formula, we have a polynomial $f_3(x) \in \mathbb{Q}(\epsilon, a, m, \theta)[x]$ independent of $M$ such that $f_3(\mathcal{V}) = \tilde{\psi}_\mathfrak{I}(\mathcal{V})$ since we have finite $T$-fixed points sets of $\text{Fl}(\mathcal{V}^{(i)}/\mathcal{O}_{M_{-((1,0),d_i)}}, \tilde{\mathfrak{I}_i})$. 

21
5.2. Recursions

For $\mathfrak{J} = (\mathfrak{J}_1, \ldots, \mathfrak{J}_j) \in \mathbb{Z}_+^j$, we put $\mathfrak{J} = \mathfrak{J}_\infty$, $\ell = \min(\mathfrak{J}_j) - 1$, and take an equivariant cohomology classes $\varphi = \tilde{\psi}_\mathfrak{J}(\mathcal{V})$ on $\mathcal{M}$. For the convenience, we also put $\tilde{\psi}(\cdot) = \psi$ for $j = 0$. Then (31) is equal to

\[
\|\mathfrak{J}\|! \int_{\tilde{M}^{\min(\mathfrak{J}^\prime) - 1}(\vec{r}, \mathfrak{J}^\prime)} \tilde{\psi}(\mathfrak{J}, \mathfrak{J}^\prime)(\mathcal{V}),
\]

where $(\mathfrak{J}, \mathfrak{J}^\prime) \in \text{Dec}^n_{j+1}$. Using this argument repeatedly, we deduce recursion formula.

Lemma 5.1. For $l \geq 1$, we have

\[
\int_{M_+(\vec{r}, n)} \psi(\mathcal{V}) - \int_{M_-(\vec{r}, n)} \psi(\mathcal{V}) = \sum_{j=1}^{l-1} \text{Res}_{h_j=\infty} \cdots \text{Res}_{h_{j-1}=\infty} \sum_{\mathfrak{J} \in \text{Dec}^n_j} \frac{|\mathfrak{J}_\infty|!}{n!} \int_{\tilde{M}^{0}(\vec{r}, \mathfrak{J}_\infty)} \tilde{\psi}_\mathfrak{J}(\mathcal{V})
\]

\[
+ \text{Res}_{h_1=\infty} \cdots \text{Res}_{h_l=\infty} \sum_{\mathfrak{J} \in \text{Dec}^n_l} \frac{|\mathfrak{J}_\infty|!}{n!} \int_{\tilde{M}^{\min(\mathfrak{J}_1) - 1}(\vec{r}, \mathfrak{J}_\infty)} \tilde{\psi}_\mathfrak{J}(\mathcal{V}).
\]

Proof. We prove by induction on $j$. For $l = 1$, (35) is nothing but (34) for $j_0 = 0$ and $\ell = n$. For $l \geq 1$, we assume the formulas (35). Then again by (34), the last summand for each $\mathfrak{J} \in \text{Dec}^n_l$ is equal to

\[
\frac{|\mathfrak{J}_\infty|!}{n!} \left( \int_{\tilde{M}^{0}(\vec{r}, \mathfrak{J}_\infty)} \tilde{\psi}_\mathfrak{J}(\mathcal{V}) \right)
\]

\[
+ \text{Res}_{h_{l+1}=\infty} \sum_{\mathfrak{J} \in \text{Dec}^n_{l+1}} \frac{|\mathfrak{J}_\infty|!}{|\mathfrak{J}_\infty|!} \int_{\tilde{M}^{\min(\mathfrak{J}_1) - 1}(\vec{r}, \mathfrak{J}_\infty)} \tilde{\psi}_{\mathfrak{J}'}(\mathcal{V}),
\]

where $\mathfrak{J}' = (\mathfrak{J}, \mathfrak{J}_{l+1})$ and $\mathfrak{J}_\infty = [n] \setminus \bigcup_{i=1}^{l+1} \mathfrak{J}_i$. Hence we have (35) for general $l \geq 1$.

For $l > n$, the set $\text{Dec}^n_l$ is empty. Thus we get the following theorem.
Theorem 5.2. We have

\[ \int_{M^+(\vec{r},n)} \psi(\mathcal{V}) - \int_{M^-(\vec{r},n)} \psi(\mathcal{V}) \]

\[ = \sum_{j=1}^{n} \sum_{\mathcal{I} \in \text{Dec}_j^n} \frac{\mathcal{I}_\infty}{n!} \ \text{Res} \cdots \text{Res} \int_{\tilde{M}^0(\vec{r},\mathcal{I}_\infty)} \tilde{\psi}_3(\mathcal{V}). \]  

(36)

5.3. Euler classes of tautological bundles

For \( \psi = \text{Eu}(\bigoplus_{j=1}^{r} \mathcal{V} \otimes e^{m_j}/q) \), we substitute \( \theta = 0 \) in (31) and (33). So for \( d = (d_1, \ldots, d_j) \in (\mathbb{Z}_{>0})^j \), we put

\[ \psi_d(\mathcal{V}) = \int_{\prod_{i=1}^j M^-(1,0,d_i)} \psi \left( \mathcal{V} \oplus \bigoplus_{i=1}^{j} \mathcal{V}^{(i)} \otimes e^{h_i} \right) \text{Eu} \left( \bigoplus_{i=1}^{j} \mathcal{V}^{(i)}/\mathcal{O}_{M^d} \right), \]

where \( \int_{\prod_{i=1}^j M^-(1,0,d_k)} \) denotes the push-forward by the projection

\[ M^-(\vec{r},n - |d_3|) \times \prod_{i=1}^j M^-(1,0,d_i) \rightarrow M^-(\vec{r},n - |d_3|). \]

Since \( \tilde{M}^0(\vec{r},\mathcal{I}_\infty) \) is a full flag bundle of \( \mathcal{V} \) over \( M^-(\vec{r},n - |d_3|) \), we have

\[ \int_{M^+(\vec{r},n)} \psi(\mathcal{V}) - \int_{M^-(\vec{r},n)} \psi(\mathcal{V}) \]

\[ = \sum_{j=1}^{n} \sum_{\mathcal{I} \in \text{Dec}_j^n} \frac{\mathcal{I}_\infty}{n!} \prod_{k=1}^{j} (|\mathcal{I}_k| - 1)! \ \text{Res} \cdots \text{Res} \int_{M^-(\vec{r},n - |d_3|)} \psi_d(\mathcal{V}). \]  

(37)

Each summand in the right hand side of (37) depends only on \( d = d_3 \).

Lemma 5.3. For \( d \in S^n_j = \left\{ d = (d_1, \ldots, d_j) \in \mathbb{Z}_{>0}^j \mid |d| \leq n \right\} \) and \( \mathcal{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_j) \in \rho^{-1}(d) \), we have

\[ |\rho^{-1}(d)| = \frac{1}{\prod_{i=1}^j \sum_{1 \leq k \leq i} d_k |\mathcal{I}_\infty|! \prod_{k=1}^{j} (|\mathcal{I}_k| - 1)!}. \]  

23
Proof. This follows from [21, Lemma 6.8] since \(|J_k| = d_k|.

By this lemma and (37), we get
\[
\int_{M_+ (\vec{r}, n)} \psi (V) - \int_{M_- (\vec{r}, n)} \psi (V)
= \sum_{j=1}^{n} \sum_{d \in \mathbb{Z}_{\geq 0}^{|J_k|}} \frac{1}{d_i \varepsilon} \operatorname{Res}_{h_i = \infty} \cdots \operatorname{Res}_{h_j = \infty} \int_{M_- (\vec{r}, n - |d|)} \psi_d (V) \cdot \prod_{i=1}^{j} (-1)^{d_i - 1} \prod_{l=0}^{r} \prod_{\alpha \in J_o} (h_i - (l + 1) \varepsilon + m_f) \prod_{\alpha \in J_1} (-h_i + (l + 1) \varepsilon + a_\alpha)
\]
\[
= \psi (V) \cdot \prod_{i=1}^{j} (-1)^{r_i d_i + d_i - 1} \varepsilon \cdot \left( - \sum_{\alpha=1}^{r} (a_\alpha + m_\alpha + r_0 \varepsilon) \right) \tag{39}
\]
since we have \(\operatorname{Res}_{h_i = \infty} \cdots \operatorname{Res}_{h_j = \infty} \frac{\psi (V) \cdot \prod_{i=1}^{j} (-1)^{d_i - 1} \prod_{l=0}^{r} \prod_{\alpha \in J_o} (h_i - (l + 1) \varepsilon + m_f) \prod_{\alpha \in J_1} (-h_i + (l + 1) \varepsilon + a_\alpha)}{\alpha = 1} = 0\), where we put \(V^{> i} = V \oplus \bigoplus_{k=1}^{r} V^{(k)} \otimes e^{h_k}. \) Here (39) is equal to \(\psi (V) \cdot (-1)^{r_i |d| + |d| j} (-A + r_0)^j\) where \(A = \sum_{\alpha=1}^{r} (a_\alpha + m_\alpha)\). If we put \(\alpha_n = \int_{M_+ (\vec{r}, n)} \psi \) and \(\beta_n = \int_{M_- (\vec{r}, n)} \psi, \) then by (38) we have
\[
\beta_n = \alpha_n + \sum_{j=1}^{n} \sum_{d \in \mathbb{Z}_{\geq 0}^{|J_k|}} \frac{(-1)^{r_i |d| + |d| j} (-A + r_0)^j}{\prod_{i=1}^{j} \prod_{1 \leq k \leq i} d_k} \alpha_{n-|d|}
\]
\[
= \sum_{l=0}^{n} \frac{(-1)^{r_i l} (-A + r_0) (-A + r_0 - 1) \cdots (-A + r_0 - l + 1)}{l!} \alpha_{n-l},
\]
Since \(Z_{\text{fund}}^f (\varepsilon, a, m, p) = \sum_{n=0}^{\infty} \alpha_n p^n \) and \(Z_{\text{fund}}^f (\varepsilon, a, m, p) = \sum_{n=0}^{\infty} \beta_n p^n, \) this gives (7).
5.4. Euler classes of tangent bundles

To show (3), let us take a class

\[ \Lambda(V) = \mathcal{E}nd(V) \otimes \mathbb{C}_q + \mathcal{H}om(W_0, V) + \mathcal{H}om(V, W_1) \otimes \mathbb{C}_q - \mathcal{E}nd(V) \in K_T(M), \]

and put \( \psi(V) = \text{Eu}^\theta(\Lambda(V)) \) in \( A_{\mathbb{C}_q \times \mathbb{C}_h}(M) \). We also put \( u = (r_1 - r_0)(\theta/\varepsilon + 1). \) In (31), we have

\[ \tilde{\psi}(V_S \oplus V_\psi \otimes e^h) = \tilde{\psi}(V_S \psi \otimes e^h) \text{Eu}^\theta(S(V_S, V_\psi) \otimes \mathbb{C}_q - S(V_S, V_\psi)), \]

\[ \mathcal{M}(V_S \otimes e^h, V_\psi) = S(V_S, V_\psi) \otimes \mathbb{C}_q - S(V_S, V_\psi) + \mathcal{H}om(W_0, V_\psi) + \mathcal{H}om(V_\psi, W_1) \otimes \mathbb{C}_q, \]

where \( S(V_S, V_\psi) = \mathcal{H}om(V_S, V_\psi) + \mathcal{H}om(V_\psi, V_S). \)

To compute the push-forward \( \int_{Fl(V_S/\mathcal{O}_{M_{-((1,0),p)}}, \mathcal{F}_\theta)} \) in (31), we divide the computations into four cases:

**Lemma 5.4.** We have the following.

1. \( \text{Res}_{h=\infty} \text{Eu}^\theta(S(V_S, V_\psi) \otimes \mathbb{C}_q - S(V_S, V_\psi))/ \text{Eu}(S(V_S, V_\psi) \otimes \mathbb{C}_q - S(V_S, V_\psi)) = 0. \)

2. \( \text{Res}_{h=\infty} \text{Eu}^\theta(\tilde{\mathcal{M}}(V_S \otimes e^h, V_\psi))/ \text{Eu}((\tilde{\mathcal{M}}(V_S \otimes e^h, V_\psi))) = -s(\mathcal{J}, \mathcal{K})\theta, \) where \( \mathcal{J} = [n] \setminus \mathcal{J} \) for \( \mathcal{J} \subset [n], \) and

\[ s(\mathcal{J}, \mathcal{K}) = \left| \{(i, j) \in \mathcal{J} \times \mathcal{K} \mid i < j\} \right| - \left| \{(i, j) \in \mathcal{J} \times \mathcal{K} \mid i > j\} \right|. \]

3. \( \text{Res}_{h=\infty} \frac{\text{Eu}^\theta(\mathcal{H}om(W_0, V_\psi) + \mathcal{H}om(V_\psi, W_1) \otimes \mathbb{C}_q)}{\text{Eu}(\mathcal{H}om(W_0, V_\psi) + \mathcal{H}om(V_\psi, W_1) \otimes \mathbb{C}_q)} = (r_0 - r_1)d^\mathcal{F}_\theta. \)

4. \( \int_{Fl(V_S/\mathcal{O}_{M_{-((1,0),p)}}, \mathcal{F}_\theta)} \text{Eu}^\theta(\mathcal{E}nd(V_S) \otimes \mathbb{C}_q - \mathcal{E}nd(V_\psi) + \Theta(\mathcal{F}_\theta)) = \frac{(\theta/\varepsilon + 1)d^\mathcal{F}_\theta}{d\mathcal{F}_\theta}. \)

**Proof.** We only give a proof for (4) since the other statements are direct computations. To prove (4) we remark that from (32) we have

\[ \mathcal{E}nd(V_S) \otimes \mathbb{C}_q - \mathcal{E}nd(V_\psi) + \Theta_\mathcal{F} = TM_{-((1,0), d^\mathcal{F})} - \mathcal{O}_{M_{-((1,0), d^\mathcal{F})}} + \Theta(\mathcal{F}_\theta). \]

Then the assertion follows from \( \int_{M_{-((1,0), d^\mathcal{F})}} \text{Eu}^\theta(TM_{-((1,0), d^\mathcal{F})}) = \frac{(\theta/\varepsilon + 1)d^\mathcal{F}}{d^\mathcal{F}}, \) and

\[ \int_{Fl(\mathbb{C}^3, \mathcal{F}_\theta)} \text{Eu}^\theta(TM((1,0), d^\mathcal{F})) = (d^\mathcal{F} - 1)!. \]

Here the last integral does not depend on \( \theta \) since \( Fl(\mathbb{C}^3, \mathcal{F}_\theta) \) is compact. \( \square \)
Computing (31) by this proposition, we have
\[
\int_{\tilde{M}^+(\vec{r},\mathcal{J})} \tilde{\psi} - \int_{\tilde{M}^-(\vec{r},|\mathcal{J}|)} \text{Eu}^\theta(T\tilde{M}^-(\vec{r}, |\mathcal{J}|))
\]
\[
= \sum_{\mathcal{J} \in \mathcal{D}^+(\mathcal{J})} \frac{(n - d^2)! (s(\mathcal{J}^z, |\mathcal{J}|) - \bar{r}d^2)(\theta/\varepsilon + 1)dt}{n!d^2} \int_{\tilde{M}^{|\mathcal{J}^z|-1}(\vec{r},\mathcal{J}^z)} \tilde{\psi},
\]
where \( \bar{r} = r_0 - r_1 \) and we put \( d^2 = |\mathcal{J}| \) as before.

For \( \mathcal{J} = (\mathcal{J}_1, \ldots, \mathcal{J}_j) \in \text{Dec}_n^j \), we use a symbol \( \mathcal{J}_{>i} = \mathcal{J}_\infty \cup \bigcup_{k=i}^j \mathcal{J}_k \). By substituting similar computations into (36) in Theorem 5.2, we have
\[
\int_{M^+(\vec{r},n)} \text{Eu}^\theta(T\tilde{M}^+(\vec{r}, n)) - \int_{M^-(\vec{r},n)} \text{Eu}^\theta(T\tilde{M}^-(\vec{r}, n))
\]
\[
= \sum_{j=1}^n \sum_{\mathcal{J} \in \text{Dec}_n^j} \frac{|\mathcal{J}_\infty|!}{n!} \prod_{i=1}^j \frac{(s(\mathcal{J}_i, \mathcal{J}_{>i}) - \bar{r}|\mathcal{J}_i|)(\theta/\varepsilon + 1)dt}{|\mathcal{J}_i|} \int_{M^-(\vec{r},n-|\mathcal{J}_j|)} \psi.
\]

5.5. Vortex partition function

In the following, we only consider the case where \( J_0 = \{1, \ldots, r_0\} \) and \( J_1 = \{r_0 + 1, \ldots, r\} \). Hence we use \( Z^\vec{r}_{\pm \text{adj}} \) instead of \( Z^J_{\pm \text{adj}} \).

We put \( \alpha_n = \int_{M^-(\vec{r},n)} \tilde{\psi}, \beta_n = \int_{M^+(\vec{r},n)} \tilde{\psi} \), and rewrite (41) to analyze vortex partition functions
\[
Z^\vec{r}_{\text{fund}}(\varepsilon, a, m, p) = \sum_{n=0}^\infty \alpha_n p^n \quad \text{and} \quad Z^\vec{r}_{\text{fund}}(\varepsilon, a, m, p) = \sum_{n=0}^\infty \beta_n p^n.
\]

We put \( a_j(\bar{r}) = \prod_{i=1}^j \frac{(s(\mathcal{J}_{>i}, |\mathcal{J}_i|)(\theta/\varepsilon + 1)dt}{|\mathcal{J}_i|} \) for \( \mathcal{J} = (\mathcal{J}_1, \ldots, \mathcal{J}_j) \). We also write by \( \text{Dec}(i, n) \) the set of elements \( \mathcal{J} = (\mathcal{J}_1, \ldots, \mathcal{J}_j) \) in \( \text{Dec}_n^j \) for \( j = 1, \ldots, n \) such that \( |\mathcal{J}_\infty| = i \). For \( i = 0 \), we put
\[
A_n(\bar{r}) = \sum_{\mathcal{J} \in \text{Dec}(0, n)} a_j(\bar{r}).
\]

By computer experiments, we can check the following conjecture.

**Conjecture 5.5.** For any \( i = 0, 1, \ldots, n \), we have \( \sum_{\mathcal{J} \in \text{Dec}(i, n)} a_j(\bar{r}) = \binom{n}{i} A_{n-i}(\bar{r}) \).
In the following, we assume that this conjecture holds. Substituting these into (41), we get

$$\beta_n - \alpha_n = \sum_{i=0}^{n} \frac{i!}{n!} \sum_{\mathcal{I} \in \text{Dec}(i,n)} \alpha_{\mathcal{I}}(\bar{r}) \alpha_i = \sum_{i=0}^{n} \frac{A_{n-i}(\bar{r})}{(n-i)!} \alpha_i.$$  

(42)

This is equivalent to saying that

$$Z^r_{r_0}(\varepsilon, a, \theta, p) = \left( \sum_{n=0}^{\infty} \frac{A_n(\bar{r})}{n!} p^n \right) Z^r_{\bar{r}}(\varepsilon, a, \theta, p).$$  

(43)

Since we have $Z^{(r,0)}_{-}(\varepsilon, a, \theta, p) = Z^{(0,r)}_{-}(\varepsilon, a, \theta, p) = 1$, from (43) we get

$$\sum_{n=0}^{\infty} \frac{A_{n}(\bar{r})}{n!} p^n = \begin{cases} Z^{(r,0)}_{-}(\varepsilon, a, \theta, p)^{-1} & \text{if } \bar{r} = r_0 - r_1 \geq 0, \\ Z^{(0,r)}_{+}(\varepsilon, a, \theta, p) & \text{if } \bar{r} = r_0 - r_1 \leq 0. \end{cases}$$

In particular, we get

$$\sum_{n=0}^{\infty} \frac{A_{n}(\bar{r})}{n!} p^n = (1 - p)^{\pm(\theta/\varepsilon+1)},$$

and

$$Z^{(r_0,\bar{r})}_{-}(\varepsilon, a, \theta, p) = (1 - p)^{\pm(\theta/\varepsilon+1)} Z^{(r_0,\bar{r})}_{+}(\varepsilon, a, \theta, p).$$  

(44)

We take limit $a_r/\varepsilon \to \infty$ in (6) and (5):

$$\lim_{a_r/\varepsilon \to \infty} Z^{(r_0,\bar{r})}_{-}(\varepsilon, a, \theta, p) = Z^{(r_0,\bar{r})}_{-}(\varepsilon, a_1, \ldots, a_{r-1}, \theta, p),$$

$$\lim_{a_r/\varepsilon \to \infty} Z^{(r_0,\bar{r})}_{+}(\varepsilon, a, \theta, p) = (1 - p)^{\theta/\varepsilon-1} Z^{(r_0,\bar{r})}_{+}(\varepsilon, a_1, \ldots, a_{r-1}, \theta, p).$$

Applying this to (44), we have

$$Z^{(r_0,\bar{r})}_{-}(\varepsilon, a, \theta, p) = Z^{(r_0,\bar{r})}_{+}(\varepsilon, a, \theta, p),$$

and

$$Z^{(r_0,\bar{r})}_{-}(\varepsilon, a_1, \ldots, a_{r-1}, \theta, p) = (1 - p)^{2(\theta/\varepsilon+1)} Z^{(r_0,\bar{r})}_{+}(\varepsilon, a_1, \ldots, a_{r-1}, \theta, p).$$

Repeating this procedure, we have

$$Z^{(r_0,\bar{r})}_{-}(\varepsilon, a, \theta, p) = (1 - p)^{-i(\theta/\varepsilon+1)} Z^{(r_0,\bar{r})}_{+}(\varepsilon, a, \theta, p)$$

for any positive integer $i \leq r_0$. In particular, we have $Z^{(r_0,\bar{r})}_{-}(\varepsilon, a, \theta, p) = (1 - p)^{-r_0(\theta/\varepsilon+1)}$. Similarly, we have

$$Z^{(0,\bar{r})}_{+}(\varepsilon, a, \theta, p) = \sum_{n=0}^{\infty} \frac{A_{n}(-\bar{r})}{n!} p^n = (1 - p)^{-\bar{r}(\theta/\varepsilon+1)}.$$ 

This gives the following proposition.
Remark 5.6. For any integer $\bar{r}$, we have
\[
\sum_{n=0}^{\infty} \frac{A_n(\bar{r})}{n!} p^n = (1 - p)^{\bar{r}(\theta/\varepsilon+1)}.
\]

Thus we get Theorem 2.4 (b).

Acknowledgements
The authors thanks Hidetoshi Awata, Ayumu Hoshino, Hiroaki Kanno, Hitoshi Konno, Takuro Mochizuki, Hiraku Nakajima, Masatoshi Noumi, Takuya Okuda, Yusuke Ohkubo, Yoshihisa Saito and Junichi Shiraishi for discussion. He is grateful for Masatoshi Noumi for directing his attention to relationships between our results and transformation formulas for multiple hypergeometric functions including the Kajihara transformations. RO is partially supported by Grant-in-Aid for Scientific Research 21K03180 and 17H06127, JSPS. He has had the generous support and encouragement of Masa-Hiko Saito. YY is supported by Grant-in-Aid for Scientific Research 21K03382 and 21H05190, JSPS. This work was partly supported by Osaka Central Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849, and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The authors have no conflicts of interest directly relevant to the content of this article. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] F. Benini, S. Cremonesi, \textit{Partition functions of }$\mathcal{N}=(2,2)$\textit{ gauge theories on }$S^2$\textit{ and vortices}, Comm. Math. Phys. 334 (2015), no. 3, 1483–1527.

[2] A. Braverman and P. Etingof, \textit{Instanton counting via affine Lie algebras II: from Whittaker vectors to the Seiberg-Witten prepotential}, In: Bernstein J., Hinich V., Melnikov A., (eds), Studies in Lie theory, Progr. Math. 243, 61–78, Birkhauser, Boston (2006)

[3] Giulio Bonelli, Antonio Sciarappa, Alessandro Tanzini, Petr Vasko, \textit{Vortex partition functions, wall crossing and equivariant Gromov-Witten invariants}, Comm. Math. Phys. 333 (2014), no. 3, 717-760.

[4] J. Gomis and B. Le Floch, \textit{M2-brane surface operators and gauge theory dualities in Toda}, JHEP 04 (2016), 183
[5] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. 135 (1999), 487–518.

[6] K. Hori, H. Kim and P. Yi, *Witten Index and Wall Crossing*, JHEP 01 (2015), 124.

[7] M. Hallnäs, E. Langmann, M. Noumi and H. Rosengren, *From Kajihara’s transformation formula to deformed Macdonald-Ruijsenaars and Noumi-Sano operators*, Selecta Math. (N.S.) 28 (2022), no. 2, Paper no. 24, 36 pp.

[8] M. Hallnäs, E. Langmann, M. Noumi and H. Rosengren, *Higher order deformed elliptic Ruijsenaars operators*, Comm. Math. Phys. 392 (2022), no. 2, 659–689.

[9] D Honda, T Okuda, *Exact results for boundaries and domain walls in 2d supersymmetric theories*, JHEP 09 (2015), 140.

[10] A. Hanany and D. Tong, *Vortices, instantons and branes*, JHEP 07 (2003), 037.

[11] C Hwang, P. Yi and Y Yoshida, *Fundamental Vortices, Wall-Crossing, and Particle-Vortex Duality*, JHEP 05 (2017), 099.

[12] C. Hwang and J. Park, *Factorization of the 3d superconformal index with an adjoint matter*, JHEP 11 (2015), 028.

[13] Y. Kajihara, *Euler transformation formula for multiple basic hypergeometric series of type A and some applications*, Adv. in Math. 187 (2004), 53–97.

[14] Y. Kajihara, M. Noumi, *Multiple elliptic hypergeometric series. An approach from the Cauchy determinant*, Indag. Mathem., N. S., 14 (3, 4), 395–421.

[15] R. Langer, M.J. Schlosser and S.O. Warnaar, *Theta functions, elliptic hypergeometric series, and Kawanaka’s Macdonald polynomial conjecture*, SIGMA 5 (2009), Paper 055.

[16] T. Mochizuki, *Donaldson Type Invariants for Algebraic Surfaces: Transition of Moduli Stacks*, Lecture Notes in Math. 1972, Springer, Berlin, 2009.

[17] H. Nakajima, *Handsaw quiver varieties and W-algebras*, Moscow Math. J. 12 (2012), no.3, 633–666.
[18] N. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. 7 (2003), no. 5, 831–864.

[19] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, In: Etingof P., Retakh V. S., Singer, I.M., (eds), The unity of mathematics, Progr. Math. 244, 525–596, Birkhauser Boston, Boston, MA, (2006)

[20] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. I. 4-dimensional pure gauge theory*, Invent. Math. 162 (2005), no. 2, 313–355.

[21] H. Nakajima and K. Yoshioka, *Perverse coherent sheaves on blowup. III. Blow-up formula from wall-crossing*, Kyoto J. Math. 51(2011), no. 2, 263–335.

[22] R. Ohkawa, *Wall-crossing between stable and co-stable ADHM data*, Lett. Math. Phys. 108 (2018), no. 6 1485–1523.

[23] R. Ohkawa, *Functional Equations of Nekrasov Functions Proposed by Ito, Maruyoshi, and Okuda*, Moscow Math. J. 20 (2020), no. 3, 531–573.

[24] R. Ohkawa, *Residue formula for flag manifold of type A from wall-crossing*, in preparation.

[25] R. Ohkawa, *Wall-crossing formula for framed quiver moduli*, in preparation.