CHARACTERIZATION OF PRIME SUBMODULES OF A FREE MODULE OF FINITE RANK OVER A VALUATION DOMAIN

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Abstract. Let $F = R^{(n)}$ be a free $R$-module of finite rank $n \geq 2$. In this paper, we characterize the prime submodules of $F$ with at most $n$ generators when $R$ is a Prufer domain. We also introduce the notion of prime matrix and we show that when $R$ is a valuation domain, every finitely generated prime submodule of $F$ with at least $n$ generators is the row space of a prime matrix.

0. Introduction

Prime submodules of a module over a commutative ring have been studied in [1, 7, 8, 9, 10] and prime submodules of a finitely generated free module over a PID have been studied in [5]. The authors in [5], have described prime submodules of a free module of finite rank $n$ ($n \geq 2$) and with at most $n$ generators over a UFD. They have characterized the prime submodules of a free module of finite rank over a PID. In [9] we have extended some results obtained in [4] to a Dedekind domain. In this paper we extend these results to a Prufer domain. Moreover, we define the notion of prime matrix and show that when $R$ is a valuation domain, every finitely generated prime submodule of a free $R$-module of finite rank $n$ ($n \geq 2$), with at least $n$ generators is the row space of a prime matrix.

Throughout this paper all rings are assumed to be commutative with identity and $F$ denotes a free $R$-module of finite rank $n$ ($n \geq 2$). We use the notation $R^{(n)}$ for $\underbrace{R \oplus \cdots \oplus R}_{n}$-times. Let $M$ be a unitary $R$-module. A proper submodule $N$ of $M$ is called $P$-prime if $rm \in N$ for some $r \in R$ and $m \in M$ implies $m \in N$ or $r \in P = (N : M)$, where $(N : M) = \{r \in R \mid rM \subseteq N\}$.

Let $R$ be a commutative domain and $K$ be the quotient field of $R$. Then $R$ is a valuation domain if for every $x \in K$, either $x \in R$ or $x^{-1} \in R$. Equivalently, the set of all ideals of $R$ is totally ordered by inclusion. Let $R$ be a commutative

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domain and \( I \) be an ideal of \( R \). Let \( I^{-1} = (R :_K I) = \{ r \in K \mid rI \subseteq R \} \). Then \( I \) is invertible if \( II^{-1} = R \). An integral domain \( R \) is a Prüfer domain if each non-zero finitely generated ideal of \( R \) is invertible. It can be shown that an integral domain \( R \) is a Prüfer domain if and only if \( R_P \) is a valuation domain for every maximal ideal \( P \) of \( R \) (see [4]).

1. Prime submodules of \( F = R^{(n)} \)

Let \( X_i = (x_{i1}, \ldots, x_{in}) \in F = R^{(n)} \) for some \( x_{ij} \in R, 1 \leq i \leq m, 1 \leq j \leq n \).

We put

\[
B_{m \times n} = [X_1 \ldots X_m] = \begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1n} \\
  x_{21} & x_{22} & \cdots & x_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix} \in M_{m \times n}(R).
\]

Thus the \( j \)th row of the matrix \([X_1 \ldots X_m]\) consists of the components of \( X_j \) in \( F \). We use \( N = \langle B \rangle \) to denote a non-zero submodule of \( F \) generated by the rows of \( B \). Also \( B(j_1, \ldots, j_k) \in M_{m \times k}(R) \) will denote a submatrix of \( B \) consisting of the columns \( j_1, \ldots, j_k \in \{1, \ldots, n\} \) of \( B \).

**Lemma 1.1.** Let \( R \) be a domain. Let \( B \in M_{n \times n}(R) \), \( \det B \neq 0 \) and \( B' = (b'_{ij}) \) be the adjoint matrix of \( B \). Then \( (x_{11}, \ldots, x_{1n}) \in \langle B \rangle \) for some \( x_{ij} \in R (1 \leq i \leq n) \) if and only if \( \sum_{k=1}^{n} x_{ij} b'_{kj} \in \langle \det B \rangle \) for every \( j \), \( 1 \leq j \leq n \).

**Proof.**

\[
(x_{11}, \ldots, x_{1n}) \in \langle B \rangle \iff (x_{11}, \ldots, x_{1n}) = (r_1, \ldots, r_n)B; \exists r_i \in R \\
\iff (x_{11}, \ldots, x_{1n})B' = (r_1, \ldots, r_n)(\det B)I_n \\
\iff \sum_{i=1}^{n} x_{ij} b'_{ij} = (\det B)r_j; \forall j \in \{1, \ldots, n\} \\
\iff \sum_{i=1}^{n} x_{ij} b'_{ij} \in \langle \det B \rangle; \forall j \in \{1, \ldots, n\}.
\]

**Proposition 1.2.** Let \( R \) be an integral domain and \( F = R^{(n)} \) (\( n \geq 2 \)). Let \( B = [X_1 \ldots X_m] \) for some \( X_i \in F (1 \leq i \leq m, m < n) \) and \( \text{rank } B = m \). If the ideal \( J \) of \( F \) generated by determinants of all \( n \times m \) submatrices of \( B \) is \( R \), then \( N = \langle B \rangle \) is a prime submodule of \( F \).

**Proof.** Assume that \( J = R \). It follows that

\[
1 = \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} r_{i_1 \ldots i_m} \det B(i_1, \ldots, i_m)
\]

for some \( r_{i_1 \ldots i_m} \in R \) and \( 1 \leq i_j \leq n, 1 \leq j \leq m \). Put

\[
M = \{ X \in F \mid \det \beta(i_1, \ldots, i_{m+1}) = 0 \text{ for every } i_1, \ldots, i_{m+1} \in \{1, \ldots, n\} \}
\]
where $\beta = [XX_1 \cdots X_m]$. Since $X_i \in M$ ($1 \leq i \leq m$), then $N \subseteq M$. Now suppose that $X \in M$. Then by [9, Lemma 1.5], we have $(\det B(i_1, \ldots, i_m))X \in N$ for every $i_1, \ldots, i_m \in \{1, \ldots, n\}$. So

$$X = \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} (r_{i_1 \cdots i_m} \det B(i_1, \ldots, i_m))X \in N.$$ 

Thus $N = M$ and $N$ is a prime submodule of $F$ [9, Corollary 1.9].

**Proposition 1.3.** Suppose $R$ is a domain and $F = R^{(n)}$ $(n \geq 2)$. Let $B \in M_{n \times n}(R)$ and rank $B = n$. If there exist a maximal ideal $P$ of $R$ and a positive integer $\alpha$ such that $(\det B) = P^\alpha$ and the ideal $J'$ of $R$ generated by entries of $B'$ is $P^{\alpha-1}$, where $B'$ is the adjoint matrix of $B$, then $N = (B)$ is a prime submodule of $F$.

**Proof.** Suppose there exist a maximal ideal $P$ of $R$ and a positive integer $\alpha$ such that $(\det B) = P^\alpha$ and $J' = P^{\alpha-1}$. Let $B' = (b'_{ij})$ and $r x_1, \ldots, x_n \in N$ for some $r, x_i \in R$. Thus by Lemma 1.1, $r \sum_{i=1}^{n} x_i b'_{ij} \in (\det B), 1 \leq j \leq n$. If $\sum_{i=1}^{n} x_i b'_{ij} \in (\det B)$ for every $1 \leq j \leq n$, then by Lemma 1.1, $(x_1, \ldots, x_n) \subseteq N$. Let $\sum_{i=1}^{n} x_i b'_{ij} \not\in (\det B)$ for some $1 \leq j \leq n$. Since $(\det B)$ is $P$-primary, $r \in P$. But $b'_{ij} \in P^{\alpha-1}$, $1 \leq i, j \leq n$. So $rb'_{ij} \in (\det B)$, $1 \leq i, j \leq n$. It follows that $(0, \ldots, 0, r, 0, \ldots, 0) \in N$, with $r$ as the $i$th component $(1 \leq i \leq n)$. Thus $rF \subseteq N$ and so $N$ is a prime submodule of $F$. \hfill \Box

2. Characterization of finitely generated prime submodules of $F = R^{(n)}$ over a valuation domain $R$

In this section we characterize the finitely generated prime submodules of $F = R^{(n)}$ $(n \geq 2)$, when $R$ is a valuation domain.

**Theorem 2.1.** Let $R$ be a valuation domain and $F = R^{(n)}$ $(n \geq 2)$. Let $B = [X_1 \cdots X_m] \in M_{m \times n}(R)$ for some $X_i \in F$ ($1 \leq i \leq m, m < n$) and rank $B = m$. Then $N = (B)$ is a prime submodule of $F$ if and only if the determinant of one of the $m \times m$ submatrices of $B$ is a unit.

**Proof.** Let $N$ be a prime submodule of $F$ and $J$ be the ideal of $R$ generated by determinants of all $m \times m$ submatrices of $B$. Since $R$ is a valuation domain, there exists a $m \times m$ submatrix $A = B(j_1, \ldots, j_m)$ of $B$ for some $j_1 < j_2 < \cdots < j_m$ of $\{1, \ldots, n\}$ such that $J = (\det A)$.

By [5, Lemma 2.2], $\det A \neq 0$. Let $A' = (a'_{ij})$ be the adjoint matrix of $A$. For the moment, fix $1 \leq i \leq m$. Consider the element $(x_1, \ldots, x_m) = (a'_{i1}, \ldots, a'_{im})B \in N$. Since $A'A = (\det A)I_m$, then $x_{j_1} = \det A$ and $x_{j_k} = 0$ ($1 \leq k \leq m, k \neq i$). Also, if $C_j = B(j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_m)$, then $x_j = \pm \det C_j$ and so $x_j \in (\det A)$ for all $j \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_m\}$. Hence $(\frac{x_1}{\det A}, \ldots, \frac{x_n}{\det A}) \in F$. Note that $\det A (\frac{x_1}{\det A}, \ldots, \frac{x_n}{\det A}) \in N$. Since $N$ is prime, $(\det A)F \subseteq N$ or $(\frac{x_1}{\det A}, \ldots, \frac{x_n}{\det A}) \in N$. If $(\det A)F \subseteq N$, then for
and so by [3, Corollary 3.1.8], \((R, \det A, 0, \ldots, 0) \in N\) with \(\det A\) as the \(j_0\)th component. Hence there are \(r_j \in R\) \((1 \leq j \leq m)\) such that \((0, \ldots, 0, \det A, 0, \ldots, 0) = (r_1, \ldots, r_m)A\).

It follows that \(r_j \det A = 0(1 \leq j \leq m)\) and hence \(r_j = 0(1 \leq j \leq m)\). Thus \(\det A = 0\), which is a contradiction. So \((\frac{a_{1,1}}{\det A}, \ldots, \frac{a_{m,m}}{\det A}) \in N\), i.e., there are \(s_j \in R\) \((1 \leq j \leq m)\) such that \((\frac{a_{1,1}}{\det A}, \ldots, \frac{a_{m,m}}{\det A}) = (s_1, \ldots, s_m)B\). We conclude that \((a_{1,1}', \ldots, a_{m,m}')B = (x_1, \ldots, x_n) = (\det A)(s_1, \ldots, s_m)B\), so that \((a_{1,1}', \ldots, a_{m,m}')A = (\det A)(s_1, \ldots, s_m)A\). Thus \(a_{1,1}' \det A = s_j(\det A)^2\) and hence \(a_{1,1}' = s_j A\) \((1 \leq j \leq m)\). Thus \(\det A' = (\det A)^{n-1}\) if \(n \neq 0\). But \(\det A' = (\det A)^{n-1}\). It follows that \(\det A\) is a unit. Conversely, let the determinant of one of the \(m \times m\) submatrices of \(B\) be a unit. Then the ideal \(J \subseteq R\) generated by determinants of all \(m \times m\) submatrices of \(B\) is \(R\). So, by Proposition 1.2, \(N\) is prime.

\[\square\]

**Proposition 2.2.** Let \(R\) be a Prüfer domain and \(F = R^{(n)}\) \((n \geq 2)\). Let \(l \geq n\) be a positive integer and \(\Psi \subseteq F\) be a finite subset of \(F\) with \(|\Psi| = l\). If \(N = \langle \Psi \rangle\) is a prime submodule of \(F\), then \(P = (N : F)\) is a finitely generated ideal of \(R\).

**Proof.** For \(N = P^{(n)}\), the assertion is clear. Now suppose that \(N \neq P^{(n)}\).

Then by [9, Theorem 1.6], there exist a positive integer \(k < n\) and a matrix \(B = [X_1, \ldots, X_k] \in M_{k \times n}(R), X_i \in \Psi, 1 \leq i \leq k\) such that determinant of one of its \(k \times k\) submatrices is not in \(P\).

Without loss of generality, we can assume that \(d = \det B(1, \ldots, k) \notin P\). Put

\[N = \{X \in F | \det b(i_1, \ldots, i_{k+1}) \in P \text{ for every } i_1, \ldots, i_{k+1} \in \{1, \ldots, n\}\},\]

where \(b = [XX_1 \ldots X_k]\). By [9, Lemma 1.5], \(dX_i = \sum_{j=i}^{k} r_{i,j} X_j + Y_j\) for some \(r_{i,j} \in R\) \((k + 1 \leq i \leq l, 1 \leq j \leq k)\) and \(Y_j = (0, \ldots, 0, y_{j+k+1}, \ldots, y_{n}) \in P^{(n)}\).

Let \(M\) be the submodule of \(F\) generated by the set \(\{X_i, Y_j; 1 \leq i \leq k, k + 1 \leq j \leq l\}\). Then \(dN \subseteq M\). Now fix \(p \in P\). Then \(d(0, \ldots, p) = \sum_{i=1}^{k} r_{i,p} X_i + \sum_{j=k+1}^{l} l_j Y_j\) for some \(r_{i,j} \in R\) \((1 \leq i \leq k, k + 1 \leq j \leq l)\). Thus \((r_1, \ldots, r_k)B(1, \ldots, k) = (0, \ldots, 0)\). It follows that \(r_i \det B(1, \ldots, k) = 0\) and hence \(r_i = 0\) \((1 \leq i \leq k)\). Let \(I\) be the ideal of \(R\) generated by the set \(\{y_{kn}; k + 1 \leq i \leq l\}\). Then \(dP \subseteq I\). Since \(R\) is a valuation domain, \(PR_P = PR = (\langle y_{kn} \rangle)_{P}\) for some \(k + 1 \leq t \leq l\). So \(s_i y_{kn} \in \langle y_{kn} \rangle\) for some \(s_i \in R - P\), \(k + 1 \leq i \leq l\). Let \(s = \prod_{i=k+1}^{l} s_i\), then \(sdP \subseteq sI \subseteq \langle y_{kn} \rangle\). Thus \(sdP \subseteq \langle y_{kn} \rangle\) (\(R : \langle y_{kn} \rangle\)). If \(P\) is not finitely generated, it is not an invertible ideal, and so by [3, Corollary 3.1.8], \((R : \langle K \rangle P) = (P : \langle K \rangle P)\). Hence \(sdP \subseteq \langle P \rangle\). It follows that \(sdP \subseteq P^2\).

Now, since \(R\) is a Prüfer domain, by [4, Theorem 4.23.3], \(P = P[P + (sd)]\). It follows that \(P = P^2\) and hence \(PR_P = P^2 R_P\), which is a contradiction. Thus \(P\) is finitely generated and by [4, Proposition 4.23.3], it is maximal.

\[\square\]

**Corollary 2.3.** Suppose \(R\) is a valuation domain and \(F = R^{(n)}\) \((n \geq 2)\). Let \(l \geq n\) be a positive integer and \(\Psi \subseteq F\) be a finite subset of \(F\) with \(|\Psi| = l\). If
\[ N = \langle \Psi \rangle \text{ is a prime submodule of } F \text{ then } P = \langle N : F \rangle \text{ is a finitely generated ideal of } R \text{ and } N = \langle B \rangle \text{ for some matrix } B \in M_{n \times n}(R). \]

**Proof.** By Proposition 2.2, \( P \) is a finitely generated ideal of \( R \). Since \( R \) is a valuation domain, \( P = \langle p \rangle \) for some \( p \in R \). If \( N = P^{(n)} \), then \( N = \langle B \rangle \), where \( B = pI_n \). Now let \( P^{(n)} \subseteq N \). Then by the proof of Proposition 2.2, there exist a positive integer \( k < n \) and \( X_i \in \Psi(1 \leq i \leq k) \), \( Y_t = (0, \ldots, 0, y_{kt+1}, \ldots, y_{kn}) \in P^{(n)}(k + 1 \leq t \leq l) \), \( k + 1 \leq t \leq l \), such that \( N = \langle \{X_i, Y_t \mid 1 \leq i \leq k, k + 1 \leq t \leq l \} \rangle \). Let \( X_i = (0, \ldots, p, \ldots, 0) \) with \( p \) as \( i \)th component, \( k + 1 \leq i \leq n \). We show that the submodule \( M_1 \) of \( F \) generated by \( \{Y_t \mid k + 1 \leq t \leq l \} \) is equal to the submodule \( M_2 \) of \( F \) generated by \( \{X_i \mid k + 1 \leq i \leq n \} \). Since \( Y_t \in M_2 \), \( k + 1 \leq t \leq n \), hence \( M_1 \subseteq M_2 \). Now since \( X_i \in N \), \( k + 1 \leq i \leq n \), we have \( X_i = \sum_{j=1}^{k} r_{ij} X_j + \sum_{t=k+1}^{l} l_{it} Y_t \) for some \( r_{ij}, l_{it} \in R \), \( 1 \leq j \leq k, k + 1 \leq t \leq l, k + 1 \leq i \leq n \). By an argument similar to that in the proof of Proposition 2.2, \( r_{ij} = 0, 1 \leq j \leq k, k + 1 \leq i \leq n \). So \( X_i \in M_1 \), \( k + 1 \leq i \leq n \) and \( M_2 \subseteq M_1 \). Now let \( B = [X_1 \ldots X_n] \), then \( N = \langle B \rangle \). \( \square \)

**Theorem 2.4.** Suppose \( R \) is a valuation domain with maximal ideal \( m \) and \( F = R^{(n)} \) (\( n \geq 2 \)). Let \( l \geq n \) be a positive integer and \( \Psi \subseteq F \) a finite subset of \( F \) with \( | \Psi | = l \). Let \( N = \langle \Psi \rangle \). Then \( N \) is a prime submodule of \( F \) if and only if there exist a matrix \( B \in M_{n \times n}(R) \) and a positive integer \( \alpha \leq n \) such that \( N = \langle B \rangle \), \( m^\alpha = (\det B) \) and the ideal \( J' \) of \( R \) generated by entries of \( B' \) is \( m^{\alpha-1} \), where \( B' \) is the adjoint matrix of \( B \).

**Proof.** Let \( N = \langle \Psi \rangle \) be a prime submodule of \( F \). By Corollary 2.3, \( N = \langle B \rangle \) for some matrix \( B \in M_{n \times n}(R) \) and \( (N : F) \) is a finitely generated ideal of \( R \). By [4, Theorem 4.23.3], \( m = (N : F) \) is principal. Assume that \( m = \langle p \rangle \) for some \( p \in R \). By [9, Lemma 1.1], \( (\det B) \subseteq \langle m \rangle \). If \( (\det B) \subseteq \langle m^k \rangle \) for every positive integer \( k \geq 1 \), then \( (\det B) \subseteq \bigcap_{k=1}^{\infty} \langle m^k \rangle \). So by [4, Theorem 3.17.1] and [9, Corollary 1.3], \( m = \bigcap_{k=1}^{\infty} \langle m^k \rangle \). Hence \( m^2 = m \), which is a contradiction. Thus there exist a positive integer \( \alpha \) and a unit \( u \in R \) such that \( \det B = u \alpha \). So \( (\det B) = \langle m^\alpha \rangle \). Now since \( p \in (N : F) \), by Lemma 1.1, \( pB_{ij} \in \langle p \rangle \) and hence \( b_{ij}' \in \langle p^{\alpha-1} \rangle \) for every \( 1 \leq i, j \leq n \). Thus \( \det B' = (\det B)^{\alpha-1} \in \langle p^{n(n-1)} \rangle \). Therefore \( (up^\alpha)^{n-1} = sp^{n(n-1)} \) for some \( s \in R \). Since \( p \) is not a unit, \( n(\alpha - 1) \leq \alpha(n - 1) \) and \( \alpha \leq n \). Let \( J' \) be the ideal of \( R \) generated by the entries of \( B' \). Then \( J' = \langle b_{ij}' \rangle \) for some \( 1 \leq i, j \leq n \). Since \( \langle p^{\alpha} \rangle \subseteq \langle b_{ij}' \rangle \subseteq \langle p^{\alpha-1} \rangle \), \( J' \) is \( m \)-primary and since \( m \neq m^2 \), then \( J' = m^t \) for \( t = \alpha \) or \( \alpha - 1 \) [4, Theorem 3.17.3]. If \( J' = m^\alpha \), then \( \det B' = (\det B)^{n-1} \in \langle p^{n^2} \rangle \). Hence \( p \) is a unit, which is a contradiction. So \( J' = m^{\alpha-1} \). \( \square \)

In the following we assume that \( (R, m) \) is a valuation domain with principal maximal ideal \( m \). We introduce the notion of prime matrix and show that every finitely generated prime submodule of \( R^{(n)} \) (\( n \geq 2 \)), with at least \( n \) generators is the row space of a prime matrix. Note that, \( R \) is not necessarily a PID.
Example. Take $Z \oplus Z = Z^{(2)}$ with lexicographic order. Let $K$ be a field and define the valuation $v : K[x, y] \to Z^{(2)}$ with $v(x) = (1, 0) \leq v(y) = (0, 1)$ and take the value of a polynomial as the minimal value among those of its monomials. Then by [4, Proposition 3.18.1], $v : K(x, y) \to Z^{(2)}$ with $v'(\frac{f}{g}) = v(f) - v(g)$; $f, g \in K[x, y]$ is a valuation on $K(x, y)$. In this case, the maximal ideal consists of all the elements whose valuations are strictly greater than $(0, 0)$. But the valuation of any such element is at least $(0, 1)$ and therefore any element of value $(0, 1)$ gives a generator of the maximal ideal. Also, since the value group is $Z^{(2)}$, the valuation ring is not a DVR.

Definition. Suppose $R$ is a valuation domain with principal maximal ideal $m$ and $m = (p)$ for some $p \in R$. Let $J = \{j_1, \ldots, j_\alpha\}$ be a subset of $\{1, \ldots, n\}$. A matrix $B = (b_{ij}) \in M_{n \times n}(R)$ is said to be a $p$-prime matrix if it satisfies the following conditions:

i) $B$ is upper triangular.

ii) For all $i, 1 \leq i \leq n$, $a_{ii} = p$, if $i \in J$ and $a_{ii} = 1$, if $i \notin J$.

iii) For all $i, j \in \{1, \ldots, n\}$, $a_{ij} = 0$ except possibly when $i \notin J$ and $j \in J$.

Sometimes we call $J$ the set of integers associated with $B$ and denote it by $J_B$. By (i) and (ii), it is clear that $\det(B) = p^\alpha$.

Lemma 2.5. Suppose $R$ is a valuation domain with principal maximal ideal $m = (p)$ and $r_i \in R$, $1 \leq i \leq n$. Let $J = \{j_1, \ldots, j_\alpha\}$ be a subset of $\{1, \ldots, n\}$ and $J_k = \{0, 1, \ldots, j_k\} - J$, $1 \leq k \leq \alpha$. Then $(r_1, \ldots, r_n) \in (B)$ for some $p$-prime matrix $B \in M_{n \times n}(R)$ with $J_B = J$ if and only if for every $k, 1 \leq k \leq \alpha$ the equation $\sum_{j \in J_k} r_j x_j \equiv r_{j_k} (\text{mod } p)$ has a solution.

Proof. Let $B = (b_{ij})$ be a $p$-prime matrix with $J_B = \{j_1, \ldots, j_\alpha\}$ and let $B' = (b'_{ij})$. For all $1 \leq i, j \leq n$, it is easy to see that $b'_{ii} = p^{\alpha - 1}$ if $i \in J_B$, $b'_{ii} = p^\alpha$ if $i \notin J_B$ and $b'_{ij} = -p^{\alpha - 1} b_{ij}$ if $i \neq j$. Hence by Lemma 1.1,

$$(r_1, \ldots, r_n) \in (B) \iff p^\alpha \sum_{j=1}^n r_j b'_{\ell j}, \ 1 \leq \ell \leq n$$

$$\iff p^\alpha \sum_{j=0}^{\ell - 1} r_j (-p^{\alpha - 1} b_{j \ell}) + p^{\alpha - 1} r_\ell \text{ for every } \ell \in J_B$$

$$\iff p \sum_{j \in J_k} -r_j b_{j j_k} + r_{j_k}, \ 1 \leq k \leq \alpha$$

$$\iff \sum_{j \in J_k} r_j b_{j j_k} \equiv r_{j_k} (\text{mod } p) \text{ for every } k, 1 \leq k \leq \alpha. \quad \square$$

Lemma 2.6. Suppose $R$ is a valuation domain with principal maximal ideal $m = (p)$ and $s$ and $n$ are positive integers such that $s < n$. Also, suppose that $A \in M_{n \times s}(R)$, $Y \in M_{n \times 1}(R)$ and $X = (x_1, \ldots, x_s) \in R^s$. Let $C \in M_{n \times (s+1)}(R)$ be the augmented matrix $[A : Y]$. If $p$ does not divide the determinant of at least one $s \times s$ submatrix of $A$, then the system of equations
AX ≡ Y (mod p) has a solution if and only if p divides the determinants of all \((s+1) \times (s+1)\) submatrices of C.

**Proof.** Suppose \(AX \equiv Y \pmod{p}\) has a solution and \(C_0\) is an \((s+1) \times (s+1)\) submatrix of C. If \(Y_0\) is the last column of \(C_0\) and \(A_0\) consists of all columns of \(C_0\) except for \(Y_0\), then \(A_0X \equiv Y_0 \pmod{p}\). So that \(C_0^tA_0X \equiv C_0^tY_0 \pmod{p}\).

The last equation of this system is \(0 \equiv \det(C_0) \pmod{p}\). Hence \(p \mid \det(C_0)\).

Conversely, let \(X_1, \ldots, X_s \in M_{n \times 1}(R)\) be the columns of A. Then \(A' = [X_1 \ldots X_s] \in M_{s \times n}(R)\) and \(C' = [X_1 \ldots X_s1] \in M_{s+1 \times n}(R)\). Now let \(p \nmid \det(A')\) and \(p \not\mid \det(A')\). Then by \(\langle 9,\text{ Lemma 1.5(ii)} \rangle\), \(\det(A'(i_1, \ldots, i_s))Y^t \in \langle p \rangle F + \langle A' \rangle\). Since \(\det(A'(i_1, \ldots, i_s))\) is unit, \(Y' \in \langle p \rangle F + \langle A' \rangle\) and so the system of equations \(AX \equiv Y \pmod{p}\) has a solution. \(\square\)

**Theorem 2.7.** Suppose \(R\) is a valuation domain with principal maximal ideal \(m = \langle p \rangle\). Let \(s, n\) and \(\alpha\) be positive integers such that \(s \leq n\) and \(1 \leq \alpha \leq n\) and \(A \in M_{s \times n}(R)\). Then \(\langle A \rangle \subseteq \langle B \rangle\) for some \(p\)-prime matrix \(B \in M_{n \times n}(R)\) with \(\det(B) = p^\alpha\) if and only if \(p\) divides the determinants of all \((n-\alpha+1) \times (n-\alpha+1)\) submatrices of A.

**Proof.** Let \(\langle A \rangle \subseteq \langle B \rangle\) for some \(p\)-prime matrix \(B\) with \(\det(B) = p^\alpha\). So there exists \(C \in M_{s \times n}(R)\) such that \(A = CB\). Let \(A_0\) be an \((n-\alpha+1) \times (n-\alpha+1)\) submatrix of A. Thus there exists an \((n-\alpha+1) \times n\) submatrix \(C_0\) of \(C\) and an \(n \times (n-\alpha+1)\) submatrix \(B_0\) of \(B\) such that \(A_0 = C_0B_0\). Suppose that \(B_1\) is an \((n-\alpha+1) \times (n-\alpha+1)\) submatrix consisting of rows \(i_1, \ldots, i_{n-\alpha+1}\) of \(B_0\). Since \(J_B\) has \(\alpha\) elements, \(i_k \in J_B\) for some \(k, 1 \leq k \leq n - \alpha + 1\).

It follows that the entries of the row \(i_k\) of \(B_0\) are 0 or \(p\). Thus \(p \mid \det(B_1)\). By the Binet–Cauchy formula \(\langle 6,\text{ Theorem 1} \rangle\), \(\det(A)\) may be expressed as a linear combination of the determinants of all \((n-\alpha+1) \times (n-\alpha+1)\) submatrices of \(B_0\), hence \(p \mid \det(A_0)\). Conversely, assume that \(p\) divides the determinants of all \((n-\alpha+1) \times (n-\alpha+1)\) submatrices of \(A\). By adding some zero rows to \(A\) if necessary, we may suppose that \(A \in M_{n \times n}(R)\). We use induction on \(\alpha\).

By assumption for \(\alpha = 1\), \(p \mid \det(A)\). Let \(k\) be the smallest integer such that \(p\) divides the determinants of all \(k \times k\) submatrices of \(A_k\), where \(A_k \in M_{n \times k}(R)\) consists of the first columns of \(A\). If \(A = (a_{ij})\) then by Lemma 2.6, the system of equations

\[
\left\{ \sum_{j=0}^{k-1} a_{ij}x_j \equiv a_{ik} \pmod{p} \mid 1 \leq i \leq n \right\}
\]

has a solution. Therefore by Lemma 2.5, there exists a prime matrix \(B\) with \(J_B = \{k\}\) such that \(\langle A \rangle \subseteq \langle B \rangle\). Now suppose that the assertion is true for some \(\alpha, 1 \leq \alpha \leq n-1\). Assume that \(p\) divides the determinants of all \((n-\alpha) \times (n-\alpha)\) submatrices of \(A = (a_{ij})\). Hence \(p\) divides the determinants of all \((n-\alpha+1) \times (n-\alpha+1)\) submatrices of \(A\). Therefore by the induction hypothesis, there exists a prime matrix \(B\) with \(\det(B) = p^\alpha\) such that \(\langle A \rangle \subseteq \langle B \rangle\). Let \(J_B = \{j_1, \ldots, j_\alpha\}\) and \(J_k = \{0, 1, \ldots, j_k\} - J_B, 1 \leq k \leq \alpha\). Fix \(k\) for the
moment. By Lemma 2.5, the system of equations
\[
\begin{aligned}
\sum_{j \in J_k} a_{ij} x_j & \equiv a_{ijk} \pmod{p} \ | \ 1 \leq i \leq n
\end{aligned}
\]
has a solution, say \( x_j = r_j \) for some \( r_j \in R, j \in J_k \). Thus we have
\[
(1) \quad \sum_{j \in J_k} a_{ij} r_j \equiv a_{ijk} \pmod{p} \forall i, \ 1 \leq i \leq n.
\]
Let \( A_0 \) be the \( n \times (n - \alpha) \) submatrix obtained by deleting columns \( j_1, \ldots, j_\alpha \) from \( A \). Let \( \ell \) be the smallest integer such that \( p \) divides the determinants of all \( \ell \times \ell \) submatrices of \( A_\ell \in M_{n \times \ell}(R) \) consisting of the first \( \ell \) columns of \( A_0 \). Assume that \( j_0 \) is the integer such that column \( \ell \) of \( A_0 \) is column \( j_0 \) of \( A \). Clearly \( j_0 \not\in J_B \). Let \( J_0 = \{0, \ldots, j_0 - 1\} - J_B \). By Lemma 2.6, the system of equations
\[
\begin{aligned}
\sum_{j \in J_0} a_{ij} x_j & \equiv a_{ijo} \pmod{p} \ | \ 1 \leq i \leq n
\end{aligned}
\]
has a solution, say \( x_j = s_j \) for some \( s_j \in R, j \in J_0 \). Therefore we have
\[
(2) \quad \sum_{j \in J_0} a_{ij} s_j \equiv a_{ijo} \pmod{p} \forall i, \ 1 \leq i \leq n.
\]
Put \( J' = \{j_1, \ldots, j_\alpha, j_0\} \) and let \( J'_k = \{0, 1, \ldots, j_k\} - J' \). If \( j_k > j_0 \), then combining (1) and (2) yields
\[
a_{ijk} \equiv \sum_{j \in J'_k} a_{ij} r_j + \left( \sum_{j \in J_0} a_{ij} s_j \right) r_{j_0} \pmod{p},
\]
for every \( i, 1 \leq i \leq n \). Hence the system of equations
\[
\begin{aligned}
\sum_{j \in J'_k} a_{ij} x_j & \equiv a_{ijk} \pmod{p} \ | \ 1 \leq i \leq n
\end{aligned}
\]
has a solution. On the other hand, if \( j_k \leq j_0 \), then obviously the above system has a solution by (1). Since \( k \) is arbitrary, by Lemma 2.5, there exists a prime matrix \( B_0 \) with \( \det(B_0) = p^{\alpha + 1} \) such that \( \langle A \rangle \subseteq \langle B_0 \rangle \) and \( j_{B_0} = J' \). Thus the assertion is true for \( \alpha + 1 \) and hence by induction for every \( \alpha, 1 \leq \alpha \leq n \). \( \square \)

**Proposition 2.8.** Suppose \( R \) is a valuation domain with principal maximal ideal \( m = \langle p \rangle \) and \( n \) a positive integer. Let \( A \in M_{n \times n}(R) \) and \( 1 \leq \alpha \leq n \), be the greatest integer such that \( p^{\alpha} \mid \det(A) \) and \( p^{\alpha - 1} \) divides all entries of \( A' \). Then \( p \) divides the determinants of all \( (n - \alpha + 1) \times (n - \alpha + 1) \) submatrices of \( A \).

**Proof.** By [2, Lemma 4.4], there exist a diagonal matrix \( C = (c_{ii}) \) and invertible matrices \( D, E \in M_{n \times n}(R) \) such that \( AE = DC \), so that \( E'A' = C'D' \). By hypothesis \( p^{\alpha - 1} \) divides all entries of \( A' \) and hence those of \( C'D' \). Let \( C' =
\((c_j')\). If \(p^2 \mid c_{jj}\) for some \(j, 1 \leq j \leq n\), then \(p^{a-1} \mid c_j'\). Hence \(p\) divides all entries of row \(j\) of \(D'\). Thus \(p \mid \det(D')\), which contradicts the fact that \(D\) is invertible. Since \(p^a \mid \det(C)\), \(p\) divides at least \(a\) entries of the diagonal of \(C\), therefore we conclude that \(p\) divides all the entries of at least one column of every \((n-a+1) \times (n-a+1)\) submatrix of \(DC\). Thus \(p\) divides the determinants of all \((n-a+1) \times (n-a+1)\) submatrix of \(DC\) and by the Binet-Cauchy formula it is easy to see that \(p\) divides the determinants of all \((n-a+1) \times (n-a+1)\) submatrices of \(A = (DC)E^{-1}\).

Theorem 2.9. Suppose \(R\) is a valuation domain with maximal ideal \(m\) and \(F = R^{(n)} (n \geq 2)\). Let \(N\) be a finitely generated submodule of \(F\) with at least \(n\) generators. Then \(N\) is a prime submodule of \(F\) if and only if \(m\) is a principal ideal of \(R\) and \(N\) is the row space of a prime matrix.

Proof. Let \(N\) be a prime submodule of \(F\). Then, by Corollary 2.3 and Theorem 2.4, \((N : F) = m\) is a principal ideal of \(R\) and there exist a matrix \(A \in M_{n \times n}(R)\) and a positive integer \(a \leq n\) such that \(N = \langle A \rangle\), \((\det A) = m^a\) and the ideal \(J'\) of \(R\) generated by entries of \(A'\) is \(m^{a-1}\). Let \(m = \langle p \rangle\) for some \(p \in R\). By Proposition 2.8 and Theorem 2.7, \(N \subseteq \langle B \rangle\) for some prime matrix \(B\) with \(\det(B) = p^a\). Thus \(A = CB\) for some \(C \in M_{n \times n}(R)\) and therefore \(ap^a = \det(A) = \det(C) \det(B) = \det(C)p^a\). Thus \(\det(C) = u\) and so \(C\) is invertible. Hence \(C^{-1}B = A\). It follows that \(\langle B \rangle \subseteq N = \langle A \rangle\). Therefore \(N = \langle B \rangle\). Conversely, by Theorem 2.4, the row space of every prime matrix is a prime submodule.

3. Prime submodules of \(F = R^{(n)}\) with at most \(n\)-generators over a Prüfer domain \(R\)

In this section we characterize the prime submodules of \(F = R^{(n)} (n \geq 2)\) with at most \(n\)-generators over a Prüfer domain.

Theorem 3.1. Suppose \(R\) is a Prüfer domain and \(F = R^{(n)} (n \geq 2)\). Let \(B = [X_1 \cdots X_m]\) for some \(X_i \in F\) \((1 \leq i \leq m, m < n)\) and rank \(B = m\). Then \(N = \langle B \rangle\) is a prime submodule of \(F\) if and only if the ideal \(J\) generated by the determinants of all \(m \times m\) submatrices of \(B\) is \(R\).

Proof. Let \(N\) be a prime submodule of \(F\). Then by [9, Proposition 1.2], \((N : F) = \langle 0 \rangle\). Suppose that \(J \neq R\) and \(P\) is a prime ideal of \(R\) with \(J \subset P\). Then by [5, Lemma 2.2], \(P \neq 0\) and \(NP\) is a prime submodule of \(FP\) with \((NP : FP) = \langle 0 \rangle\). Since \(R\) is a Prüfer domain, \(RP\) is a valuation domain [4, Theorem 4.22.1]. Therefore by Theorem 2.1, \(RP = JP\). It follows that \(1 = \frac{1}{s} \det(B)_{j_1 \cdots j_m}\) for some \(1 \leq j_1 < \cdots < j_m < n\), \(r \in R\) and \(s \in R\setminus P\). So \(s = r \det(B)_{j_1 \cdots j_m} \in J \subset P\), which is a contradiction. Therefore \(J = R\). The converse follows from Proposition 1.2.

Theorem 3.2. Suppose \(R\) is a Prüfer domain and \(F = R^{(n)}\). Let \(B \in M_{n \times n}(R)\) and rank \(B = n\). Then \(N = \langle B \rangle\) is prime in \(F\) if and only if
there exist a maximal ideal $P$ of $R$ and a positive integer $\alpha \leq n$ such that $(\det B) = P^\alpha$ and the ideal $J'$ of $R$ generated by entries of $B'$ is $P^{\alpha - 1}$, where $B'$ is the adjoint matrix of $B$.

Proof. Let $N$ be a prime submodule of $F$. By Proposition 2.2, $P = (N : F) = \sqrt{(\det B)}$ is a finitely generated ideal of $R$ and so by [4, Theorem 4.23.3], is maximal. Since $R$ is a Prüfer domain, $R_P$ is a valuation domain. Since $N_P$ is a $P_P$-prime submodule of $F_P$, by Theorem 2.4, $(\det B)_P = P^\alpha_P$ and $J'_P = P^{\alpha - 1}_P$ for some positive integer $\alpha \leq n$.

Let $\phi : R \to R_P$ be the natural homomorphism. Since $(\det B)$ is $P$-primary, $\varphi^{-1}((\det B)_P) = (\det B)$. So $(\det B) = P^\alpha$. Now let $r \in \varphi^{-1}(J'_P)$. Then $r \in J'_P$ and hence $sr \in J'$ for some $s \in R - P$. Since $P$ is a maximal ideal of $R$, $1 = sx + y^\alpha$ for some $x \in R$ and $y \in P$. So $r = sxr + y^\alpha r \in J'$. Therefore $\varphi^{-1}(J'_P) = J'$. Thus $J' = P^{\alpha - 1}$.

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