Spaces of generalized splines over T-meshes

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Abstract

We consider spaces locally spanned both by polynomial and by non-polynomial functions, and we use them to define a new case of spline space on T-meshes, that is, generalized splines over T-meshes. For such spaces, we provide, under certain conditions, a dimension formula and a basis based on the notion of minimal determining set. We explicitly examine some relevant cases, which enjoy a noteworthy behaviour with respect to differentiation and integration; finally, we also study the approximation power of the just constructed spline spaces.

Keywords: T-mesh, generalized splines, dimension formula, basis functions, approximation power.

1 Introduction

The concept of T-splines, first introduced in \cite{14} and \cite{15}, was a significant advancement in CAD and CAGD techniques, also with important applications to differential problems, in particular in the framework of isogeometric analysis (see, e.g., \cite{4} and \cite{1}). The spline spaces over T-meshes are a closely related notion, first introduced by Deng et al. in \cite{6} and further studied by the same authors and several others (see, e.g., \cite{9}, \cite{10}, \cite{12} and \cite{13}). The basic idea is to consider spaces of spline functions which are polynomials of a certain degree in each of the cells of the T-mesh, which, unlike the classical tensor-product meshes, allows T-junctions, that is, vertices where only three edges meet.

In this paper we introduce a new class of non-polynomial spline spaces defined over T-mesh, which we will refer to as \textit{Generalized spline spaces over T-meshes}: such spaces are locally spanned by polynomials and some non-polynomial functions satisfying certain conditions allowing to construct a Bernstein-like local basis (see \cite{3}). These kind of non-polynomial functions have been recently used also to construct non-polynomial T-splines (see \cite{2}). The introduction of such spaces is justified by at least two reasons. First of all, the presence of non-polynomial functions allows to exactly reproduce certain shapes which can only be approximated by polynomial splines or NURBS (for example relevant curves like helices, cycloids, catenaries, or other transcendental curves). Moreover, as we
will also point out in Section 4, using certain non-polynomial functions also allows an easier computation of derivatives and integrals of certain surfaces with respect to using polynomial splines or NURBS (see also [11], [7]).

After having made some assumptions on the regularity of the spline functions, we will provide a dimension formula for the generalized spline spaces over a T-mesh and we propose the construction of a basis by using Bernstein-Bézier techniques similar to the ones used for the polynomial case in [13]. Moreover, we also show and discuss some noteworthy cases of generalized spline spaces over a T-mesh which have a good behaviour w.r.t. differentiation and integration, which is a useful feature for the use of such spaces in the isogeometric analysis framework. Finally, we will give some results about the approximation power of the just constructed spline spaces by defining and presenting the properties of a quasi-interpolant.

The paper is organized as follows. Section 2 includes several preliminary arguments about the non-polynomial spaces we will use to define the new spline spaces, including some important properties about the derivatives of the basis functions and the basic concepts about T-meshes. Section 3 presents the new generalized spline spaces over T-meshes, and includes a detailed proof of the dimension formula and of the construction of the basis. Section 4 deals with the issue of suitable choices of the spaces in order to get a good behaviour of the spaces themselves w.r.t. differentiation and integration. Finally, Section 5 is devoted to the study of the approximation power of the constructed generalized spline space.

2 Preliminaries

The spaces we will consider are of the type

\[ P_{u,v}^n([a,b]) := \text{span}\{1, s, \ldots, s^{n-2}, u(s), v(s)\}, \quad s \in [a,b], \quad 2 \leq n \in \mathbb{N}, \]  

where \( u, v \in C^{n+1}([a,b]) \); for \( n = 1 \) we set

\[ P_{u,v}^1([a,b]) := \text{span}\{u(s), v(s)\}, \quad s \in [a,b]. \]

We assume that \( \dim(P_{u,v}^n([a,b])) = n + 1 \); moreover, in order to prove some of the properties we are about to present, we will sometimes require the following additional conditions on \( P_{u,v}^n([a,b]) \):

\[ \forall \psi \in P_{u,v}^n([a,b]), \text{ if } \psi^{(n-1)}(s_1) = \psi^{(n-1)}(s_2) = 0, \quad s_1, s_2 \in [a,b], \quad s_1 \neq s_2 \]
\[ \text{then } \psi^{(n-1)}(s) = 0, \quad s \in [a,b]; \]  

\[ \forall \psi \in P_{u,v}^n([a,b]), \text{ if } \psi^{(n-1)}(s_1) = \psi^{(n)}(s_1) = 0, \quad s_1 \in (a,b), \]
\[ \text{then } \psi^{(n-1)}(s) = 0, \quad s \in [a,b]. \]  

In the following, we will explicitly mention when such conditions are needed.
2.1 Bernstein basis and its properties

In this subsection we construct a basis for the space \( P_{u,v}^n([a,b]) \); since it shares its main features with the classical Bernstein polynomials (see Definition 2.1), we will refer to it as the Bernstein basis of \( P_{u,v}^n([a,b]) \). The procedure to obtain the basis and the fundamental properties are already known and can be found in [3], so in this subsection we will just recall the main results of [3] omitting the proofs; we will instead prove Property 2.5, which is an original result and will be crucial in order to obtain some results later in the paper. In this Section we assume that the condition (2) holds.

Definition 2.1. Given the space \( P_{u,v}^n([a,b]) \), \( n \geq 2 \), a basis of functions \( \{B_{i,n}\}_{i=0}^n \) is called Bernstein basis if

\[
\sum_{i=0}^n B_{i,n}(s) = 1, \quad s \in [a,b],
\]

\[
B_{i,n}(s) > 0, \quad s \in (a,b), \quad i = 0, ..., n.
\]

The Bernstein basis can be constructed by using the following integral recurrence relation (see also [3]). By (2), there exist unique elements \( U_{0,1,n} \) and \( U_{1,1,n} \) belonging to \( \text{span}(u^{(n-1)}, v^{(n-1)}) \) satisfying

\[
U_{0,1,n}(a) = 1, \quad U_{0,1,n}(b) = 0, \\
U_{1,1,n}(a) = 0, \quad U_{1,1,n}(b) = 1,
\]

(4)

and

\[
U_{0,1,n}(s), U_{1,1,n}(s) > 0, \quad s \in (a,b).
\]

(5)

Moreover, we define, for \( k = 2, ..., n \) and \( n \geq 2 \)

\[
U_{0,k,n}(s) = 1 - V_{0,k-1,n}(s), \\
U_{i,k,n}(s) = V_{i-1,k-1,n}(s) - V_{i,k-1,n}(s), \quad 1 \leq i \leq k-1, \\
U_{k,k,n}(s) = V_{k-1,k-1,n}(s),
\]

(6)

where

\[
V_{i,k,n}(s) = \int_a^s U_{i,k,n}/d_{i,k,n}dt,
\]

(7)

and

\[
d_{i,k,n}(s) = \int_a^b U_{i,k,n}dt,
\]

for \( i = 0, ..., k, \quad k = 1, ..., n - 1 \). Note that (4) and (5) hold also in the particular case \( n = 1 \), and then \( U_{0,1,1} \) and \( U_{1,1,1} \) are a positive basis for \( P_{u,v}^1([a,b]) \). The following results can be proved about the just defined functions (see [3]).

Theorem 2.2. For \( k = 2, ..., n \) and \( n \geq 2 \), the set of functions \( \{U_{0,k,n}, ..., U_{k,k,n}\} \) is a basis for the space

\[
\text{span}(1, s, ..., s^{k-2}, u^{(n-k)}(s), v^{(n-k)}(s)).
\]

Moreover, it is a Bernstein basis, that is, satisfies the conditions \( \sum_{i=0}^k U_{i,k,n}(s) = 1 \) and \( U_{i,k,n}(s) > 0 \) for \( s \in (a,b), i = 0, ..., k \).
Corollary 2.3. The set of functions \( \{U_{0,n,n}, \ldots, U_{n,n,n}\} \) is a Bernstein basis for the space \( P^n_{u,v}([a,b]) \), \( n \geq 2 \), that is, \( U_{i,n,n} = B_{i,n} \), where \( \{B_{i,n}\}_{i=0}^n \) is the basis of Definition 2.1. For \( n = 1 \), the set \( \{U_{0,1,1}, U_{1,1,1}\} \) is a positive basis of \( P^1_{u,v}([a,b]) \).

Since in the case \( n = 1 \) we cannot, in general, guarantee the construction of a Bernstein basis, in the following we will assume \( n \geq 2 \).

Property 2.4. For \( i = 0, \ldots, k, k = 2, \ldots, n \) and \( n \geq 2 \), we have
\[
U^{(j)}_{i,k,n}(a) = 0, \quad j = 0, \ldots, i - 1, \\
U^{(j)}_{i,k,n}(b) = 0, \quad j = 0, \ldots, k - i - 1.
\]

In particular, if we consider \( k = n \), we have
\[
B^{(j)}_{i,n}(a) = 0, \quad j = 0, \ldots, i - 1, \\
B^{(j)}_{i,n}(b) = 0, \quad j = 0, \ldots, n - i - 1.
\]

As far as we know, the following property was not available in literature, so we will prove it, since it is essential to get some of the results in the following sections.

Property 2.5. For \( k = 2, \ldots, n \) and \( n \geq 2 \), we have
\[
U^{(i)}_{i,k,n}(a) \neq 0, \quad i = 0, \ldots, k - 1, \\
U^{(k-i)}_{i,k,n}(b) \neq 0, \quad i = 1, \ldots, k.
\]

In particular, if we consider \( k = n \), we have
\[
B^{(i)}_{i,n}(a) \neq 0, \quad i = 0, \ldots, n - 1, \quad (8) \\
B^{(n-i)}_{i,n}(b) \neq 0, \quad i = 1, \ldots, n. \quad (9)
\]

Proof. First, let us prove (8) for induction. For \( k = 2 \), (8) holds, since from (4), (6) and (7) we get
\[
U_{0,2,n}(a) = 1 - V_{0,1,n}(a) = 1 - \int_a^a U_{0,1,n}(t)/d_{0,1,n}dt = 1 - 0 = 1,
\]
\[
U^{(1)}_{1,2,n}(a) = D[V_{0,1,n}(s) - V_{1,1,n}(s)]_{s=a} = \frac{U_{0,1,n}(a)}{d_{0,1,n}} - \frac{U_{1,1,n}(a)}{d_{1,1,n}} = \frac{1}{d_{0,1,n}} - 0 \neq 0.
\]

Now, if (8) holds for \( k \), it must be true for \( k + 1 \) as well, since we have
\[
U_{0,k+1,n}(a) = 1 - V_{0,k,n}(a) = 1 - \int_a^a U_{0,k,n}(t)/d_{0,k,n}dt = 1 - 0 = 1,
\]
\[
U^{(i)}_{i,k+1,n} = \frac{U^{(i-1)}_{i-1,k,n}(a)}{d_{i-1,k,n}} - \frac{U^{(i-1)}_{i,k,n}(a)}{d_{i,k,n}} = \frac{U^{(i-1)}_{i-1,k,n}(a)}{d_{i-1,k,n}} \neq 0,
\]

where we used (6), (7), Property 2.4 and the induction hypothesis. Analogously we can prove (9). □
2.2 T-meshes

We will now recall the definition of T-mesh and of some related objects, using the notations of \cite{13}. Note that the concept of T-mesh we will consider here may slightly differ from other ones in the literature, such as the more general used in \cite{5}, which allows the presence not only of T-junctions, but of L-junctions and I-junctions as well.

**Definition 2.6.** A T-mesh is a collection of axis-aligned rectangles $\Delta = \{R_i\}_{i=1}^{N}$ such that the domain $\Omega \equiv \bigcup_i R_i$ is connected and any pair of rectangles (which we will call cells) $R_i, R_j \in \Delta$ intersect each other only at points on their edges.

![Figure 1: An example of T-mesh where $\Omega = [-1, 6] \times [-1, 5]$](image)

Note that this definition does not imply that the domain $\Omega$ is rectangular and allows the presence of holes in it. Tensor-product meshes are a particular case of T-meshes. If a vertex $v$ of a cell belonging to $\Delta$ lies in the interior of an edge of another cell, then we call it a T-junction.

**Definition 2.7.** Given a T-mesh $\Delta$, a line segment $e = \langle w_1, w_2 \rangle$ connecting the vertices $w_1$ and $w_2$ is called edge segment if there are no vertices lying in its interior. Instead, if all the vertices lying in its interior are T-junctions and if it cannot be extended to a longer segment with the same property, then we call it a composite edge.

In the following, we will consider T-meshes which are regular and have no cycles, in the sense of the following definitions (see \cite{13} for more details).
Definition 2.8. A T-mesh $\Delta$ is regular if for each of its vertices $w$ the set of all rectangles containing $w$ has a connected interior.

Definition 2.9. Let $w_1, ..., w_n$ be a collection of T-junctions in a T-mesh $\Delta$ such that $w_i$ lies in the interior of a composite edge having one of its endpoints at $w_{i+1}$ (we assume $w_{n+1} = w_1$). Then $w_1, ..., w_n$ are said to form a cycle.

3 Spaces of generalized splines on T-meshes

3.1 Basics

Let $\Delta$ be a regular T-mesh without cycles, and let $0 \leq r_1 < n_1 - 1, 0 \leq r_2 < n_2 - 1$, where $r_1, r_2, n_1, n_2$ are integers and $n_1, n_2 \geq 2$. Later on, we will also use the notation
\( r = (r_1, r_2) \) and \( n = (n_1, n_2) \). We define the space of generalized splines over the T-mesh \( \Delta \) of bi-order \( n \) and smoothness \( r \) as

\[
GS_{n,r}^n(\Delta) = \{ p(s, t) \in C^r(\Omega) : p|_R \in \mathcal{P}_{u,v}^n(R) \ \forall R \in \Delta \}, \tag{10}
\]

where \( \Omega = \bigcup_{R \in \Delta} R \), \( C^r(\Omega) \) denotes the space of functions \( p \) such that their derivatives \( D_i^r D_j^r p \) are continuous for all \( 0 \leq i \leq r_1 \) and \( 0 \leq j \leq r_2 \), and the space \( \mathcal{P}_{u,v}^n \) is defined as

\[
\mathcal{P}_{u,v}^n(R) = \text{span}(f(s)g(t) : f \in \mathcal{P}_{u_1,v_1}^{n_1-1}([a_R, b_R]), g \in \mathcal{P}_{u_2,v_2}^{n_2-1}([c_R, d_R])), \tag{11}
\]

with \( R = [a_R, b_R] \times [c_R, d_R] \) and \( u_1, v_1 \in C^{n_1}([a, b]), u_2, v_2 \in C^{n_2}([c, d]) \) such that \( \dim(\mathcal{P}_{u_1,v_1}^{n_1-1}([a_R, b_R])) = n_1 \) and \( \dim(\mathcal{P}_{u_2,v_2}^{n_2-1}([c_R, d_R])) = n_2 \), and satisfying both (2) and (3). In other words, \( GS_{n,r}^n(\Delta) \) is a space of spline functions which, restricted to each cell \( R \), are products of functions belonging to spaces of type (1).

We introduce now on each cell \( R \) a Bernstein-Bézier representation for the elements of \( GS_{n,r}^n(\Delta) \) based on the Bernstein basis of \( \mathcal{P}_{u_1,v_1}^{n_1-1}([a_R, b_R]) \) and \( \mathcal{P}_{u_2,v_2}^{n_2-1}([c_R, d_R]) \) constructed in Theorem 2.3; therefore, we need to assume that (2) is satisfied both by \( \mathcal{P}_{u_1,v_1}^{n_1-1}([a_R, b_R]) \) and \( \mathcal{P}_{u_2,v_2}^{n_2-1}([c_R, d_R]) \). Let us denote by \( \{ B_{i,n_1-1}^R \}_{i=0}^{n_1-1} \) and \( \{ B_{i,n_2-1}^R \}_{i=0}^{n_2-1} \) the Bernstein basis of, respectively, \( \mathcal{P}_{u_1,v_1}^{n_1-1}([a_R, b_R]) \) and \( \mathcal{P}_{u_2,v_2}^{n_2-1}([c_R, d_R]) \), to stress the dependence of the basis on the coordinates \( a_R, b_R, c_R, d_R \) of the vertices of the cell \( R \). For any \( p \in GS_{n,r}^n(\Delta) \), we can then give on the cell \( R \) the following representation:

\[
p|_R(s, t) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} c_{ij}^R B_{i,n_1-1}^R(s) B_{j,n_2-1}^R(t), \tag{12}
\]

where \( c_{ij}^R \in \mathbb{R} \) are suitable coefficients. Let us define the set of domain points associated to \( R \):

\[
D_{n,R} = \{ \xi_{ij}^R : i=0, j=0 \} = \left( \frac{(n_1 - 1 - i) a_R + i b_R}{n_1 - 1}, \frac{(n_2 - 1 - j) c_R + j d_R}{n_2 - 1} \right) , \quad i = 0, ..., n_1 - 1, j = 0, ..., n_2 - 1.
\]

We can then define the set of domain points for a given T-mesh \( \Delta \) as

\[
D_{n,\Delta} = \bigcup_{R \in \Delta} D_{n,R},
\]

where we assume that multiple appearances of the same point are allowed. If we set

\[
B_{\xi}^R(s, t) = B_{i,n_1-1}^R(s) B_{j,n_2-1}^R(t), \quad \text{where } \xi_{ij}^R = \xi,
\]

then, for each \( R \in \Delta \), we can re-write (12) in the more compact form

\[
p|_R(s, t) = \sum_{\xi \in D_{n,R}} c_{\xi}^R B_{\xi}^R(s, t),
\]

which we call Bernstein-Bézier form; we refer to the \( c_{\xi}^R \) as the B-coefficients. It is then clear that any element of the space \( GS_{n,r}^n(\Delta) \) is completely determined by a set of B-coefficients \( \{ c_{\xi} \}_{\xi \in D_{n,\Delta}} \). Of course, not every choice of the B-coefficients corresponds to an element in the spline space, since smoothness conditions must be satisfied.
3.2 Smoothness conditions

In order to study the consequences of the smoothness conditions required for $GS_{u,v}^n(\Delta)$ on the determination of the B-coefficients of an element of the space, first we need to recall some more concepts about domain points.

Let $R= [a_R, b_R] \times [c_R, d_R] \in \Delta$, $w = (a_R, c_R)$, and $\mu = (\mu_1, \mu_2)$ with $\mu_1 \leq n_1 - 1$ and $\mu_2 \leq n_2 - 1$. We call the set $D^R_{\mu}(w) = \{ \xi_{ij} \}_{i=0,j=0}^{\mu_1,\mu_2}$ the disk of size $\mu$ around $w$. The disks around the other vertices of $R$ can be defined analogously. Moreover, we say that the points $\xi_{ij}^R$ with $0 \leq i \leq \nu$ lie within a distance $\nu$ from the edge $e = \{ a_R \} \times [c_R, d_R]$ and we use the notation $d(\xi_{ij}^R, w) \leq \nu$. Analogous notations hold for the other edges of $R$.

Moreover, we can define the set of domain points

$$D_{\mu}(w) = \bigcup_{R \in \Delta_w} D^R_{\mu},$$

where $\Delta_w \subset \Delta$ contains only the cells having $w$ as one of their vertices and multiple appearances of a point are allowed in the union. Given a composite edge $e$, an edge $e^\prime$ lying on $e$ and a vertex $w$ of $e^\prime$, if $d(w, e^\prime) \leq \nu$, then we write that $d(w, e) \leq \nu$ as well.

The following lemma is a key step to be able to understand the influence of the smoothness conditions and to get a dimension formula for the space.

**Lemma 3.1.** Let $p \in GS_{u,v}^n(\Delta)$ and let $w$ be a vertex of $\Delta$. Let us consider two cells $R$ and $\tilde{R}$ with vertices (in counter-clockwise order) $w, w_2, w_3, w_4$ and $w, w_5, w_6, w_7$, respectively. If the coefficients $c_{\xi}, \xi \in D^R_{\nu}(w)$ are given, then the coefficients $c_{\eta}, \eta \in D^\tilde{R}_{\nu}(w)$ are uniquely determined by the smoothness conditions at $w$.

**Proof.** We prove the lemma for the case where $w$ is the upper-right corner of $R = [a_R, b_R] \times [c_R, d_R]$ and the lower-left corner of $\tilde{R} = [a_{\tilde{R}}, b_{\tilde{R}}] \times [c_{\tilde{R}}, d_{\tilde{R}}]$, that is, $w = (b_R, d_R) = (a_{\tilde{R}}, c_{\tilde{R}})$. First, let us consider the partial derivatives of $p|_{\tilde{R}}$:

$$D^h_s D^k_t p|_{\tilde{R}}(a_{\tilde{R}}, c_{\tilde{R}}) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} c_{ij}^R D^h_s D^k_t B_{i,n_1-1}(a_{\tilde{R}}) D^R_{j,n_2-1}(c_{\tilde{R}}), \quad 0 \leq h \leq r_1, 0 \leq k \leq r_2.$$

Since by Corollary 2.3, $B_{i,n_1-1}(s) = U_{i,n_1-1,n_1-1}(s)$ and $B_{j,n_2-1}(t) = U_{j,n_2-1,n_2-1}(t)$, using Property 2.4 gives that

$$D^h_s B_{i,n_1-1}(a_{\tilde{R}}) = 0, \quad h < i \leq n_1 - 1,$n

$$D^k_t B_{j,n_2-1}(c_{\tilde{R}}) = 0, \quad k < j \leq n_2 - 1.$$

Therefore,

$$D^h_s D^k_t p|_{\tilde{R}}(a_{\tilde{R}}, c_{\tilde{R}}) = \sum_{i=0}^{h} \sum_{j=0}^{k} c_{ij}^R D^h_s D^R_{i,n_1-1}(a_{\tilde{R}}) D^k_t B_{j,n_2-1}(c_{\tilde{R}}).$$

Now let us compute the partial derivatives of $p|_{R}$:

$$D^h_s D^k_t p|_{R}(b_R, d_R) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} c_{ij}^R D^h_s D^R_{i,n_1-1}(b_R) D^k_t B_{j,n_2-1}(d_R), \quad 0 \leq h \leq r_1, 0 \leq k \leq r_2,$$
Since by Corollary \( \text{Corollary 2.3} \) \( B^R_{i,n_1-1}(s) = U_{i,n_1-1,n_1-1}(s) \) and \( B^R_{j,n_2-1}(t) = U_{j,n_2-1,n_2-1}(t) \), using Property \( \text{Property 2.4} \) gives that
\[
D^h_i B^R_{i,n_1-1}(b_R) = 0, \quad 0 \leq i < n_1 - 1 - h, \\
D^k_i B^R_{j,n_2-1}(d_R) = 0, \quad 0 \leq j < n_2 - 1 - k.
\]

Therefore,
\[
D^h_i D^k_j p|_R(b_R, d_R) = \sum_{i=n_1-1-h}^{n_2-1} \sum_{j=n_2-1-k}^{n_2-1} c^h_j D^h_i B^R_{i,n_1-1}(b_R) D^k_j B^R_{j,n_2-1}(d_R).
\]

Requiring the \( C^r \) smoothness at \( v \) is then equivalent to the linear system composed of the equations
\[
\sum_{i=0}^{h} \sum_{j=0}^{k} c^{R}_{ij} D^h_i B^R_{i,n_1-1}(a_R) D^k_j B^R_{j,n_2-1}(c_R) = 0
\]
for \( h = 0, ..., r_1, \ k = 0, ..., r_2 \).

Note that in this case we have \( \{c_{ij}\} \in D^h_{(v)} = \{c^R_{ij}\}_{i=n_1-1-h,j=n_2-1-k} \), that is, the \( (r_1 + 1) \times (r_2 + 1) \) B-coefficients associated to \( R \) given by hypothesis are exactly the ones on the right-hand of equations (13). Analogously, \( \{c_{ij}\} \in D^h_{(v)} = \{c^R_{ij}\}_{i=0,j=0} \), which means that the \( (r_1 + 1) \times (r_2 + 1) \) B-coefficients associated to \( \tilde{R} \) are the unknowns of the system (13). It is easy to observe that, if we organize the equations according to the order of the derivatives, the matrix of the system is lower triangular. Moreover, the entries on the diagonal of the matrix, that is,
\[
D^h_i B^R_{h,n_1-1}(a_R) D^k_j B^R_{k,n_2-1}(c_R), \quad h = 0, ..., r_1, \ k = 0, ..., r_2,
\]
are not zero because of Property \( \text{Property 2.5} \). \( \square \)

After having studied the influence of smoothness around a vertex, we now study the situation around edges. Given a composite edge \( e \), we will use the following notation:
\[
r_e = \begin{cases} r_1, & \text{if } e \text{ is vertical,} \\ r_2, & \text{if } e \text{ is horizontal,} \end{cases}
\]
\[
D_e = \begin{cases} D^s_i, & \text{if } e \text{ is vertical,} \\ D^t_i, & \text{if } e \text{ is horizontal,} \end{cases}
\]
\[
n_e = \begin{cases} n_2, & \text{if } e \text{ is vertical,} \\ n_1, & \text{if } e \text{ is horizontal,} \end{cases}
\]
\[
u_e = \begin{cases} u_2, & \text{if } e \text{ is vertical,} \\ u_1, & \text{if } e \text{ is horizontal,} \end{cases}
\]
\[
v_e = \begin{cases} v_2, & \text{if } e \text{ is vertical,} \\ v_1, & \text{if } e \text{ is horizontal.} \end{cases}
\]
Moreover, to get the following results we will assume that $u_1,v_1$ are such that for each horizontal edge segment $f = [a_f,b_f] \times \{c_f\}$
\[
\dim \mathcal{P}_{u_1,v_1}^{n_1-1}([a_f,b_f]) = n_1,
\]
and that $u_2,v_2$ are such that for each vertical edge segment $f = \{c_f\} \times [c_f,d_f]$
\[
\dim \mathcal{P}_{u_2,v_2}^{n_2-1}([c_f,d_f]) = n_2.
\]

**Lemma 3.2.** Let $e$ be a composite edge of $\Delta$ with endpoints $w_1$ and $w_5$, and let $p \in GS_{u,v}^{n\nu}(\Delta)$. For any $0 \leq \nu \leq r_e$, $D^\nu_e p|_e$ is a univariate function belonging to the space $\mathcal{P}_{u_e,v_e}^{n_e-1}([w_1,e,w_5,e])$, where $w_1,e,w_5,e \in \mathbb{R}$ are the abscissas/ordinates of $w_1,w_5$.

**Proof.** The Lemma can be trivially proved with the same arguments used in [13] for the polynomial case, thanks to the assumptions (14) and (15).

Let us now consider a cell $R_e$ with vertices $w_1,w_2,w_3,w_4$ and another cell $\tilde{R}_e$ with vertices $w_5,w_6,w_7,w_8$. Moreover we assume that $w_2$ and $w_8$ lie on $e$ as well (the other cases are analogous). Let us use the notation
\[
\mathcal{M}_e = \\begin{cases} 
\{x_{ij}\}^r_{i,j=0} & \text{if } e \text{ is vertical,} \\
\{x_{ij}\}^r_{i=0,j=r_2+1} & \text{if } e \text{ is horizontal.}
\end{cases}
\]

In other words, the set $\mathcal{M}_e$ includes all the domain points $\xi$ lying outside the disks $D^{r_e}_r(w_1)$ and $D^{r_e}_r(w_2)$ and satisfying $d(\xi,e) \leq r_e$.

**Lemma 3.3.** Let $e$ be an edge of the T-mesh $\Delta$ with endpoints $w_1$ and $w_5$, and let us assume that $\mathcal{P}_{u_e,v_e}^{n_e-1}([w_1,e,w_5,e])$ satisfies (2). If the B-coefficients of a spline $p \in GS_{u,v}^{n\nu}(\Delta)$ corresponding to the domain points belonging to the set
\[
\mathcal{M}_e = D^{r_e}_r(w_1) \cup D^{r_e}_r(w_5) \cup \mathcal{M}_e
\]
are given, then the coefficient of $p$ associated to domain points $\xi$ such that $d(\xi,e) \leq r_e$ are uniquely determined.

**Proof.** We will suppose that $e$ is horizontal (the case where $e$ is vertical is analogous). We consider a cell $R$ with an edge lying on $e$: let us assume, for instance that $R$ has vertices $z_1,z_2,z_3,z_4$ and that $z_3$ and $z_4$ lie on $e$, like in Figure 5. We will show that the B-coefficients corresponding to the domain points $\xi$ belonging to $D_{n,R}$ and such that $d(\xi,e) \leq r_e$ are uniquely determined.

Let $p \in GS_{u,v}^{n\nu}(\Delta)$. First of all, since the B-coefficients corresponding to the domain points in $\mathcal{M}_e$ are given, we can compute the derivatives
\[
\{D^i_j D^i_j p(w_1)\}^r_{i,j=0}, \quad \{D^i_j D^i_j p(w_5)\}^r_{i,j=0}.
\]

In fact, by Property (2,3) the computation of these derivatives involves just the B-coefficients contained in $\mathcal{M}_e$. Now, let us consider the smallest rectangle $\tilde{R}$ containing $R$ and $e$ (the
rectangle with vertices $\hat{w}_1, \hat{w}_5, w_3, w_1$ in Figure 5). Note that $\hat{R}$ does not necessarily belong to the T-mesh $\Delta$; however, since we assumed that $P_{u_e,v_e}^{-1}([w_1,e,w_5,e])$ satisfies (2), we can temporarily assume it does and consider the corresponding B-coefficients. In order to obtain the B-coefficients $c_{ij}^{\hat{R}}$, $i = 0, \ldots, n_1 - 1, j = 0, \ldots, r_2$, associated to $\hat{R}$, that is, the ones corresponding to domain points $\xi$ in $\hat{R}$ s.t. $d(\xi,e) \leq r_e$, it’s sufficient to solve the linear systems

$$
D_s^iD_l^j p|_{\hat{R}}(w_1) = D_s^iD_l^j p(w_1), \quad i = 0, \ldots, n_1 - r_1 - 2, j = 0, \ldots, r_2,
$$

$$
D_s^iD_l^j p|_{\hat{R}}(w_5) = D_s^iD_l^j p(w_5), \quad i = 0, \ldots, r_1, j = 0, \ldots, r_2, \tag{17}
$$

where on the right side of the equality we have the already computed derivatives. Note that, by Properties 2.4 and 2.5, the matrices associated to these systems are both triangular with non-zero elements on the diagonal. From Lemma 3.2 we know that $p|_e \in P_{u_e,v_e}^{-1}([w_1,e,w_5,e])$; in particular, this allows us to state that

$$
D_s^iD_l^j p|_{\hat{R}}(w) = D_s^iD_l^j p(w), \quad i = 0, \ldots, n_1 - r_1 - 2, j = 0, \ldots, r_2,
$$

$$
D_s^iD_l^j p|_{\hat{R}}(\hat{w}) = D_s^iD_l^j p(\hat{w}), \quad i = 0, \ldots, r_1, j = 0, \ldots, r_2, \tag{18}
$$

for any two points $w$ and $\hat{w}$ lying on $e$ between $z_4$ and $z_3$. Let us choose $w = z_4$ and $\hat{w} = z_3$: we observe that the derivatives on the right side of equations (18) can be computed because we have enough B-coefficients of $p|_{\hat{R}}$ (previously obtained solving (17)). Then, we get from (18) two linear systems where the unknowns are the B-coefficients $c_{ij}^{R}$, $i = 0, \ldots, n_1 - 1, j = 0, \ldots, r_2$, associated to the cell $R$, that is, the ones corresponding to domain points $\xi$ in $R$ s.t. $d(\xi,e) \leq r_e$. These systems can be solved, since the associated matrices are triangular and with non-zero elements on the diagonals (again because of Properties 2.4 and 2.5). Since the same procedure can be repeated for any cell with one edge lying on $e$, the Lemma is proved.

Figure 5: the cells considered in the proof of Lemma 3.3.
3.3 Basis and dimension formula

We will prove the construction of the basis and a dimension formula for $GS_{n,v}^r(\Delta)$ in the case $n_1 - 1 \geq 2r_1 + 1$ and $n_2 - 1 \geq 2r_2 + 1$. One of the main tools which we will need to prove the dimension formula is the concept of minimal determining set.

**Definition 3.4.** Let $\mathcal{M} \subset D_{n,\Delta}$. $\mathcal{M}$ is a determining set for $GS_{n,v}^r(\Delta)$ if for any spline function $p$ belonging to $GS_{n,v}^r(\Delta)$

$$c_\xi = 0, \quad \forall \; \xi \in \mathcal{M} \; \text{implies} \; p \equiv 0,$$

(19)

where for any $\xi \in \mathcal{M}$, $c_\xi$ is the corresponding B-coefficient of $p$. If there is no smaller set satisfying this property, $\mathcal{M}$ is called a minimal determining set.

Using simple linear algebra tools, it is easy to verify that, for any determining set $\mathcal{M}$, $\dim(GS_{n,v}^r(\Delta)) \leq \#\mathcal{M}$ and, for any minimal determining set $\mathcal{M}$, $\dim(GS_{n,v}^r(\Delta)) = \#\mathcal{M}$.

Let us denote by $J_{NT}$ and $C$ the sets of vertices which are not T-junctions and the set of composite edges of $\Delta$, respectively. For any $w$ in $J_{NT}$, let $R_w$ be a cell with an edge $e_w$ with one endpoint at $w$, and such that any other edge with an endpoint at $w$ has length at most equal to $e_w$. Moreover, let

$$\mathcal{M}_w = D_r^{R_w}(w), \quad \text{for any } w \in J_{NT},$$

$$\mathcal{M}_R = \left\{ \xi \in \bigcup_{i=r_1+1,r_2+1} \xi_i n_{i-r_1-2,n_2-r_2-2} \right\}, \quad \text{for any } R \in \Delta,$$

$$\mathcal{M} = \bigcup_{w \in J_{NT}} \mathcal{M}_w \cup \bigcup_{e \in C} \mathcal{M}_e \cup \bigcup_{R \in \Delta} \mathcal{M}_R,$$

where $\mathcal{M}_e$, $e \in C$, is defined by (16).

**Lemma 3.5.** The subset of domain points $\mathcal{M} \subset D_{n,\Delta}$ is a determining set for $GS_{n,v}^r(\Delta)$.

**Proof.** In order to prove the lemma is sufficient to show that (19) holds. Let $p \in GS_{n,v}^r(\Delta)$ with $c_\xi = 0$ for any $\xi \in \mathcal{M}$. First of all, by hypothesis, for any $w \in J_{NT}$ $c_\xi = 0$ for all $\xi \in \mathcal{M}_w = D_r^{R_w}(w)$, which implies, by Lemma 3.4, that $c_\xi = 0$ for all $\xi \in D_r(w)$. As a consequence, we can state that for any composite edge $e$ having both the endpoints in $J_{NT}$ we have $c_\xi = 0$ for all $\xi \in \mathcal{M}_e$, since by hypothesis $c_\xi = 0$ for all $\xi \in \mathcal{M}_e$. Therefore, by using Lemma 3.3, we also obtain that $c_\xi = 0$ for all $\xi$ such that $d(\xi,e) \leq r_e$. We will refer to the already considered vertices and edges as already determined. To determine the remaining B-coefficients we now use an iterative procedure consisting of two steps:

1. for each T-junction $w$ on an already determined composite edge, we can use Lemma 3.1 to show that $c_\xi = 0$ for all $\xi \in D_r(w)$;

2. for each composite edge $e$ with already determined endpoints $w_1$ and $w_2$, since $c_\xi = 0$ for all $\xi \in \mathcal{M}_e$ by hypothesis and $c_\xi = 0$ for all $\xi \in D_r(w_1) \cup D_r(w_2)$, because these two vertices are already determined, we can use Lemma 3.3 to show that $c_\xi = 0$ for all $\xi$ such that $d(\xi,e) \leq r_e$. 

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Since we consider only T-meshes without cycles, this two steps can be repeated until all the vertices and all the edges are already determined. Then, at this point, we can state that all the B-coefficients corresponding to domain points within a distance $r_e$ from any edge $e$ are determined and are zero. The remaining coefficients are zeros as well, since they correspond to domain points whose distance from any edge $e$ is greater than $r_e$, which are exactly the domain points contained in $\bigcup_{e \in \Delta} R_e$.

We are now able to provide the construction of a basis space and a dimension formula for the generalized spline $GS_{u,v}^{n,r}(\Delta)$. Since the following results can be proved in the same way as the analogous ones for the polynomial case in [13], here we will omit the proofs.

**Lemma 3.6.** For each $\xi \in \mathcal{M}$, there is a unique $\psi_\xi \in GS_{u,v}^{n,r}(\Delta)$ such that

$$\gamma_\eta \psi_\xi = \delta_{\xi,\eta}, \quad \eta \in \mathcal{M},$$

where $\delta_{\xi,\eta}$ is the Kronecker delta and, for any $\eta \in \mathcal{D}_{n,\Delta}$, $\gamma_\eta : GS_{u,v}^{n,r}(\Delta) \rightarrow \mathbb{R}$ is the functional defined by

$$\gamma_\eta p = c_\eta, \quad \text{with } c_\eta \text{ B-coefficient of } p \text{ associated to } \xi, \quad p \in GS_{u,v}^{n,r}(\Delta). \quad (20)$$

**Theorem 3.7.** The set $\mathcal{M}$ is a minimal determining set for $GS_{u,v}^{n,r}(\Delta)$, the set $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a basis for $GS_{u,v}^{n,r}(\Delta)$, and

$$\dim(GS_{u,v}^{n,r}(\Delta)) = (r_1 + 1)(r_2 + 1)J_{NT} + (r_2 + 1)(n_1 - 2r_1 - 2)E_{hor}$$

$$+ (r_1 + 1)(n_2 - 2r_2 - 2)E_{ver} + (n_1 - 2r_1 - 2)(n_2 - 2r_2 - 2)N,$$

where

- $J_{NT}$ = number of vertices of $\Delta$ which are not T-junctions,
- $E_{hor}$ = number of horizontal composite edges of $\Delta$,
- $E_{ver}$ = number of vertical composite edges of $\Delta$,
- $N$ = number of cells of $\Delta$.

**Corollary 3.8.** Let $n = (n,n)$ and $r = (r,r)$ with $n - 1 \geq 2r + 1$; then

$$\dim(GS_{u,v}^{n,r}(\Delta)) = (r + 1)^2J_{NT} + (r + 1)(n - 2r - 2)(E_{hor} + E_{ver}) + (n - 2r - 2)^2N.$$

4 Noteworthy cases of generalized spline spaces

Given a T-mesh $\Delta$, it is clear, from [10] and [11], that the corresponding generalized spline space $GS_{u,v}^{n,r}(\Delta)$ depends on the choice of the functions $u_1, v_1$ and $u_2, v_2$. Some noteworthy choices are, for example, the trigonometric functions, that is, $u_1(s) = \cos(s)$, $v_1(s) = \sin(s)$, $u_2(t) = \cos(t)$, $v_2(t) = \sin(t)$, or the hyperbolic functions, that is, $u_1(s) = \cosh(s)$, $v_1(s) = \sinh(s)$, $u_2(t) = \cosh(t)$, $v_2(t) = \sinh(t)$, because of their properties.
to exactly reproduce certain shapes (conic sections, helices, cycloids, catenaries; see also [11]). With particular reference to the context of isogeometric analysis (see, e.g., [4] and [1]), it is of a certain interest to choose $u_1$, $v_1$ and $u_2$, $v_2$ such that the space of generalized splines over the T-mesh has a nice behaviour with respect to the fundamental derivation and integration operators, that is, such that the derivatives and integrals belong to spaces of the same type. Such a feature can be hardly formalized exactly, but in our case we can obtain some favourable cases by examining which spaces $GS_{u,v}^{n,r}((\Delta))$ satisfy the following conditions:

$$
\psi(s,t) \in GS_{u,v}^{n,r}((\Delta)) \implies D_\alpha \psi(s,t) \in GS_{u,v}^{n,r}((\Delta)) \quad (21)
$$

$$
\psi(s,t) \in GS_{u,v}^{n,r}((\Delta)) \implies \int \psi(s,t)ds \in GS_{u,v}^{n,r}((\Delta)) \quad (22)
$$

$$
\psi(s,t) \in GS_{u,v}^{n,r}((\Delta)) \implies D_\beta \psi(s,t) \in GS_{u,v}^{n,r}((\Delta)) \quad (23)
$$

$$
\psi(s,t) \in GS_{u,v}^{n,r}((\Delta)) \implies \int \psi(s,t)dt \in GS_{u,v}^{n,r}((\Delta)), \quad (24)
$$

where $\hat{n}_s = (n_1 - 1, n_2)$, $\hat{n}_r = (n_1 + 1, n_2)$, $\hat{n}_t = (n_1, n_2 - 1)$, $\hat{n}_r = (n_1, n_2 + 1)$, and $\hat{r}_s = (r_1 - 1, r_2)$, $\hat{r}_t = (r_1 + 1, r_2)$, $\hat{r}_t = (r_1, r_2 - 1)$, $\hat{r}_t = (r_1, r_2 + 1)$. Although we are dealing with spline spaces in two dimensions defined over a T-mesh, we can study this property in the univariate case over a single interval, because, in every cell, the function is a product between two functions depending just by one variable. Then, from now on we will consider the space $P_{u,v}^n([a,b])$, $n \geq 1$, requiring that

$$
\psi(s) \in P_{u,v}^n([a,b]) \implies \psi'(s) \in P_{u,v}^{n-1}([a,b]) \quad (25)
$$

$$
\psi(s) \in P_{u,v}^n([a,b]) \implies \int \psi(s)ds \in P_{u,v}^{n+1}([a,b]) \quad (26)
$$

First of all, we note that it’s not restrictive assuming that the derivatives of $u$ and $v$ are linear combinations of the two functions themselves. In fact, if we require that the derivative of $u$ belongs to the space $P_{u,v}^{n-1}([a,b])$, with $n \geq 2$, we can write it as

$$
u'(s) = \alpha_1 \cdot u(s) + \alpha_2 \cdot v(s) + \alpha_3 \cdot P_{n-3}(s), \quad \text{where } P_{n-3}(s) \text{ is a polynomial of degree at most } n-3, \text{ obtained by differentiating the polynomial parts, of degree at most } n-2, \text{ of } u \text{ and } v.
$$

However, we can assume without loss of generality that both $u$ and $v$ are free from polynomial parts of degree at most $n-2$, because, in the case they are not, the corresponding space $P_{u,v}^n([a,b])$ would not change. Therefore, we can suppose $P_{n-3}(s) \equiv 0$ and, as a consequence, $u'(s) = \alpha_1 \cdot u(s) + \alpha_2 \cdot v(s)$. Similar remarks hold for $v$ and for both the functions when considering integration. When differentiating a function $\psi(s) \in P_{u,v}^n([a,b])$, the derivative of its polynomial part of course belongs to $P_{u,v}^{n-1}([a,b])$. Therefore, condition (25) reduces to

$$
\begin{cases}
u'(s) = \alpha_1 \cdot u(s) + \alpha_2 \cdot v(s) \\
v'(s) = \beta_1 \cdot u(s) + \beta_2 \cdot v(s)
\end{cases}
$$

Following the classical theory of vector-valued ordinary differential equations, we can consider $A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}$ and compute its nonzero eigenvalues $\lambda_1$ and $\lambda_2$, the associated
eigenvectors $W_1 = \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix}$, $W_2 = \begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix}$, and examine the possible cases.

- **A has distinct real eigenvalues.** In this case, the solutions are given by:
  \[
  \begin{align*}
  u(s) &= c_1 w_{11} e^{\lambda_1 s} + c_2 w_{12} e^{\lambda_2 s}, \\
  v(s) &= c_1 w_{21} e^{\lambda_1 s} + c_2 w_{22} e^{\lambda_2 s},
  \end{align*}
  \]
  where $c_1, c_2 \in \mathbb{R}$. Having assumed the linear independence of $u$ and $v$, necessarily we have $c_1 \neq 0, c_2 \neq 0$; without loss of generality for the space spanned by $u$ and $v$, we can set:
  \[
  u(s) = e^{\lambda_1 s}, \quad v(s) = e^{\lambda_2 s}
  \]
  A notable example arises if we set $\lambda_1 = c, \lambda_2 = -c$: we obtain the same space generated by $\cosh(cs), \sinh(cs)$. The corresponding Bernstein-like basis with $c = 1$, constructed according to Corollary 2.3, is shown in Figures 6-7 for the cases $n = 2, 4$.

- **A has coincident eigenvalues, and it is diagonalizable.** The previous scheme for the solutions can be used, but, due to the fact that $\lambda_1 = \lambda_2$, and $u$ and $v$ cannot be linearly independent. So this case is not of our interest.

- **A has coincident eigenvalues, and it is not diagonalizable.** It corresponds, for our purposes, to $u(s) = e^{\lambda s}, v(s) = se^{\lambda s}$.

- **A has distinct complex conjugate eigenvalues.** If $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$, then:
  \[
  \begin{align*}
  u(s) &= c_1 w_{11} e^{(\alpha + i\beta)s} + c_2 w_{12} e^{(\alpha - i\beta)s}, \\
  v(s) &= c_1 w_{21} e^{(\alpha + i\beta)s} + c_2 w_{22} e^{(\alpha - i\beta)s}
  \end{align*}
  \]
  Assuming the linear independence of $u$ and $v$, for our goals it is equivalent considering
  \[
  u(s) = e^{\alpha s} \cos(\beta s), \quad v(s) = e^{\alpha s} \sin(\beta s)
  \]
For $\alpha = 0$, we obtain the trigonometric case $u_0(s) = \cos(\beta s), v_0(s) = \sin(\beta s)$. The corresponding Bernstein-like basis with $\beta = 1$, constructed according to Corollary 2.3, is shown in Figures 8-9 for the cases $n = 2, 4$.

Figure 8: The Bernstein basis of $P^2_{u,v}([0,\pi/2])$ with $u(s) = \cos(s), v(s) = \sin(s)$.

Figure 9: The Bernstein basis of $P^4_{u,v}([0,\pi/2])$ with $u(s) = \cos(s), v(s) = \sin(s)$.

It can be easily verified that each of the just obtained couples of $u$ and $v$ corresponds to spaces $P^n_{u,v}([a,b])$ satisfying (26).

Now we show that, if (25) and (26) hold, (2) is verified for sure in some cases and conditionally in other ones, while (3) is always verified. We will do this by studying these two conditions for each of the just obtained cases of $u,v$ satisfying (25) and (26). Note that, due to (25), for any element $\psi$ of the space, the $(n-1)$-th derivative $\psi^{(n-1)}$ can be written in the form $a_{n-1}u^{(n-1)} + b_{n-1}v^{(n-1)}$, considering the fact that a possibly polynomial part vanishes if differentiated $(n-1)$ times.

- $u(s) = e^{\lambda_1 s}, v(s) = e^{\lambda_2 s}$ with $\lambda_1 \neq \lambda_2$: linear combinations $a_{n-1}u^{(n-1)} + b_{n-1}v^{(n-1)}$ are always nonzero if $a_{n-1}u^{(n-1)}$ and $b_{n-1}v^{(n-1)}$ have the same sign, while they are null just once on $\mathbb{R}$ if they have opposite signs. So in this case (2) holds, regardless of the choice of parameters or interval.

For (3), by differentiating again once, we obtain the $n$-th derivative $\psi^{(n)}$ as $\lambda_1 \cdot a_{n-1}e^{\lambda_1 s} + \lambda_2 \cdot b_{n-1}e^{\lambda_2 s}$. Now, let us suppose the existence of an $s_1$ such that $\psi^{(n-1)}(s_1) = \psi^{(n)}(s_1) = 0$, i.e.:

$$\begin{cases} 
  e^{\lambda_1 s_1}a_{n-1} + e^{\lambda_2 s_1}b_{n-1} = 0 \\
  \lambda_1 e^{\lambda_1 s_1}a_{n-1} + \lambda_2 e^{\lambda_2 s_1}b_{n-1} = 0
\end{cases}$$

The determinant of the matrix associated to the system is:

$$\begin{vmatrix} 
  e^{\lambda_1 s_1} & e^{\lambda_2 s_1} \\
  \lambda_1 e^{\lambda_1 s_1} & \lambda_2 e^{\lambda_2 s_1}
\end{vmatrix} = (\lambda_2 - \lambda_1)e^{(\lambda_1 + \lambda_2)s_1} \neq 0$$

since $\lambda_1 \neq \lambda_2$.

So the system has only the solution $a_{n-1} = b_{n-1} = 0$, that is, $\psi^{(n-1)}(s) \equiv 0$.  

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• \( u(s) = e^{\lambda s} \), \( v(s) = se^{\lambda s} \): concerning (2), we note that any linear combination of \( u^{(n-1)} \) and \( v^{(n-1)} \) is of the form \((a_{n-1} + b_{n-1})e^{\lambda s}\). If we assume \( a_{n-1} \neq 0 \), such combination is zero if and only if \( s = -b_{n-1}/a_{n-1} \); if \( a_{n-1} = 0 \), it is never zero (if \( b_{n-1} \neq 0 \)) or always zero (if \( b_{n-1} = 0 \), which corresponds to considering the null function). Condition (2) is then verified. Condition (3) corresponds, because of the same considerations made above and of the fact that \( e^{\lambda s} \) never vanishes, to the system

\[
\begin{align*}
s_1a_{n-1} + b_{n-1} &= 0 \\
(\lambda s_1 + 1)a_{n-1} + \lambda b_{n-1} &= 0
\end{align*}
\]

The determinant of the associated matrix is

\[
\begin{vmatrix}
s_1 & 1 \\
\lambda s_1 + 1 & \lambda
\end{vmatrix} = -1 \neq 0
\]

and again the only solution is the trivial one corresponding to \( \psi^{(n-1)}(s) \equiv 0 \).

• \( u(s) = e^{\alpha s} \cos(\beta s) \), \( v(s) = e^{\alpha s} \sin(\beta s) \): the condition for which (2) holds was widely treated in the case \( \alpha = 0 \) (see, for example, [11]), and it corresponds to requiring that \( \beta (b-a) < \pi \). The same constraint on \( \beta \) guarantees that (2) also for any other value of \( \alpha \). In fact, a linear combination of \( u^{(n-1)} \) and \( v^{(n-1)} \)

\[
a_{n-1}e^{\alpha s} \cos(\beta s) + b_{n-1}e^{\alpha s} \sin(\beta s) = e^{\alpha s}(a_{n-1} \cos(\beta s) + b_{n-1} \sin(\beta s))
\]

vanishes if and only if the second factor does, since \( e^{\alpha s} \) is obviously nonzero. Finally, taking into account the positiveness of the exponential functions, condition (3) corresponds to the linear system

\[
\begin{align*}
a_{n-1} \cos(\beta s_1) + b_{n-1} \sin(\beta s_1) &= 0 \\
a_{n-1}(\alpha \cos(\beta s_1) - \beta \sin(\beta s_1)) + b_{n-1}(\alpha \sin(\beta s_1) + \beta \cos(\beta s_1)) &= 0
\end{align*}
\]

with

\[
\begin{vmatrix}
\cos(\beta s_1) & \sin(\beta s_1) \\
\alpha \cos(\beta s_1) - \beta \sin(\beta s_1) & \alpha \sin(\beta s_1) + \beta \cos(\beta s_1)
\end{vmatrix} = \beta \neq 0.
\]

As a consequence, (3) is verified in this case, too.

Finally, we mention another possible choice of \( u \) and \( v \) (see, e.g., [3]) not satisfying the conditions (21) - (24), but anyway having a good behaviour with respect to differentiation and integration:

\[ u(s) = s^{m_0}, \quad v(s) = (1 - s)^{m_1} \]

where \( m_0 \) and \( m_1 \) are sufficiently large to guarantee the linear independence of the set \( \{1, s, ..., s^{n-2}, s^{m_0}, (1 - s)^{m_1}\} \).

Also note that all the choices of \( u, v \) considered in this section are suitable to construct spaces of generalized splines over a T-mesh (10) satisfying the conditions (14)-(15).
5 Approximation power

This section is devoted to the study of approximation properties of the generalized spline spaces over T-meshes. First, we need to obtain some results about the approximation properties of the tensor-product spaces on each cell, that is, the spaces of type $\mathcal{P}_{u,v}^n([a,b] \times [c,d])$ defined in (11).

Given a function $f \in \mathcal{C}^n(\Omega)$ and $(s_0, t_0) \in [a,b] \times [c,d]$, we define the interpolant $Q_L(f; s_0, t_0)(s, t)$ as the function satisfying the two following conditions:

1. it belongs to $\mathcal{P}_{u,v}^n([a,b] \times [c,d])$,
2. its polynomial expansion of coordinate bi-degree $(n_1 - 1, n_2 - 1)$ coincides with the polynomial expansion of $f$ of the same bi-degree, that is, it is a Hermite interpolant of coordinate bi-degree $(n_1 - 1, n_2 - 1)$.

Since $Q_L(f; s_0, t_0)$ is a Hermite interpolant, the Taylor expansion of the difference $f - Q_L(f; s_0, t_0)$ does not contain any term of degree smaller than or equal to $k$, where $k = \min\{n_1-1, n_2-1\}$, and then $\|f - Q_L(f; s_0, t_0)\| = \mathcal{O}(\|h\|^{k+1})$, where $h = \text{diam}([a,b] \times [c,d])$.

In order to show that $Q_L(f; s_0, t_0)(s, t)$ exists and is unique for any $f \in \mathcal{C}^n(\Omega)$ and $(s_0, t_0) \in [a,b] \times [c,d]$, let us write the explicit expressions of a generic element belonging to $\mathcal{P}_{u,v}^n([a,b] \times [c,d])$

\[
\begin{align*}
&\sum_{i=0}^{n_1-3} a_{ij}(s - s_0)^i(t - t_0)^j + \sum_{i=0}^{n_1-3} b_i(s - s_0)^i u_2(t) + \sum_{i=0}^{n_1-3} c_i(s - s_0)^i v_2(t) \\
&+ \sum_{j=0}^{n_2-3} d_{ij} u_1(s)(t - t_0)^j + \sum_{j=0}^{n_2-3} e_j v_1(s)(t - t_0)^j \\
&+ \nu_1 u_1(s) u_2(t) + \nu_2 v_1(s) v_2(t) + \nu_3 u_1(s) u_2(t) + \nu_4 v_1(s) v_2(t)
\end{align*}
\]

and of its Taylor expansion of coordinate bi-degree $(n_1 - 1, n_2 - 1)$

\[
\begin{align*}
&\sum_{i=0}^{n_1-3} \sum_{j=0}^{n_2-3} a_{ij}(s - s_0)^i(t - t_0)^j + \sum_{i=0}^{n_1-3} \sum_{j=0}^{n_2-3} b_i(s - s_0)^i u_2(t) + \sum_{i=0}^{n_1-3} \sum_{j=0}^{n_2-3} c_i(s - s_0)^i v_2(t) \\
&+ \sum_{i=0}^{n_1-3} \sum_{j=0}^{n_2-3} d_{ij} u_1(s)(t - t_0)^j + \sum_{i=0}^{n_1-3} \sum_{j=0}^{n_2-3} e_j v_1(s)(t - t_0)^j \\
&+ \nu_1 \sum_{i=0}^{n_1-3} \sum_{j=0}^{n_2-3} D_i^j u_1(s)(s - s_0)^i(t - t_0)^j \\
&+ \nu_2 \sum_{i=0}^{n_1-3} \sum_{j=0}^{n_2-3} D_i^j v_1(s)(s - s_0)^i(t - t_0)^j \\
&+ \nu_3 \sum_{i=0}^{n_1-3} \sum_{j=0}^{n_2-3} D_i^j u_1(s)(s - s_0)^i(t - t_0)^j \\
&+ \nu_4 \sum_{i=0}^{n_1-3} \sum_{j=0}^{n_2-3} D_i^j v_1(s)(s - s_0)^i(t - t_0)^j.
\end{align*}
\]

Then, the condition requiring that $Q_L(f; s_0, t_0)$ is a Hermite interpolant of coordinate bi-degree $(n_1 - 1, n_2 - 1)$ corresponds to the following equations:

\[
\begin{align*}
&\quad a_{ij} + b_i D^j u_2(t_0) + c_i D^j v_2(t_0) = D^j u_1(s_0) + e_j D^j v_1(s_0) + \nu_1 D^j u_1(s_0) D^j u_2(t_0) \\
&\quad + \nu_2 D^j u_1(s_0) D^j v_2(t_0) + \nu_3 D^j v_1(s_0) D^j u_2(t_0) + \nu_4 D^j v_1(s_0) D^j v_2(t_0) = D^j f(s_0, t_0),
\end{align*}
\]
for $0 \leq i \leq n_1 - 3, 0 \leq j \leq n_2 - 3,$

$$b_i D^j u_2(t_0) + c_i D^j v_2(t_0) + \nu_1 D^i u_1(s_0) D^j u_2(t_0)$$

$$+ \nu_2 D^i u_1(s_0) D^j v_2(t_0) + \nu_3 D^i v_1(s_0) D^j u_2(t_0) + \nu_4 D^i v_1(s_0) D^j v_2(t_0) = D^i D^j f(s_0, t_0),$$

for $0 \leq i \leq n_1 - 3, j = n_2 - 2, n_2 - 1,$

$$d_j D^i u_1(s_0) + e_j D^i v_1(s_0) + \nu_1 D^i u_1(s_0) D^j u_2(t_0)$$

$$+ \nu_2 D^i u_1(s_0) D^j v_2(t_0) + \nu_3 D^i v_1(s_0) D^j u_2(t_0) + \nu_4 D^i v_1(s_0) D^j v_2(t_0) = D^i D^j f(s_0, t_0),$$

for $i = n_1 - 2, n_1 - 1, 0 \leq j \leq n_2 - 3,$ and

$$\nu_1 D^i u_1(s_0) D^j u_2(t_0) + \nu_2 D^i u_1(s_0) D^j v_2(t_0)$$

$$+ \nu_3 D^i v_1(s_0) D^j u_2(t_0) + \nu_4 D^i v_1(s_0) D^j v_2(t_0) = D^i D^j f(s_0, t_0),$$

for $i = n_1 - 2, n_1 - 1, j = n_2 - 2, n_2 - 1.$ By using a suitable reordering of the unknowns $a_{ij}, b_i, c_i, d_j, e_j, \nu_k$, we obtain a system whose matrix is

$$A = \begin{bmatrix}
I & \ast & \ast & \ast \\
0 & A_1 & 0 & \ast \\
0 & 0 & A_2 & \ast \\
0 & 0 & 0 & A_3
\end{bmatrix}$$

where $I$ is the identity matrix of size $(n_1 - 2)(n_2 - 2) \times (n_1 - 2)(n_2 - 2)$, and

$$A_1 = \begin{bmatrix}
D^{n_2-2} u_2(t_0) & 0 & \ldots & 0 & D^{n_2-2} v_2(t_0) & 0 & \ldots & 0 \\
D^{n_2-1} u_2(t_0) & 0 & \ldots & 0 & D^{n_2-1} v_2(t_0) & 0 & \ldots & 0 \\
0 & D^{n_2-2} u_2(t_0) & \ldots & 0 & 0 & D^{n_2-2} v_2(t_0) & \ldots & 0 \\
0 & D^{n_2-1} u_2(t_0) & \ldots & 0 & 0 & D^{n_2-1} v_2(t_0) & \ldots & 0 \\
0 & 0 & \ddots & 0 & 0 & \ddots & 0 & \vdots \\
0 & 0 & \ldots & D^{n_2-2} u_2(t_0) & 0 & 0 & \ldots & D^{n_2-2} v_2(t_0) \\
0 & 0 & \ldots & D^{n_2-1} u_2(t_0) & 0 & 0 & \ldots & D^{n_2-1} v_2(t_0)
\end{bmatrix}$$

$$A_2 = \begin{bmatrix}
D^{n_1-2} u_1(s_0) & 0 & \ldots & 0 & D^{n_1-2} v_1(s_0) & 0 & \ldots & 0 \\
D^{n_1-1} u_1(s_0) & 0 & \ldots & 0 & D^{n_1-1} v_1(s_0) & 0 & \ldots & 0 \\
0 & D^{n_1-2} u_1(s_0) & \ldots & 0 & 0 & D^{n_1-2} v_1(s_0) & \ldots & 0 \\
0 & D^{n_1-1} u_1(s_0) & \ldots & 0 & 0 & D^{n_1-1} v_1(s_0) & \ldots & 0 \\
0 & 0 & \ddots & 0 & 0 & \ddots & 0 & \vdots \\
0 & 0 & \ldots & D^{n_1-2} u_1(s_0) & 0 & 0 & \ldots & D^{n_1-2} v_1(s_0) \\
0 & 0 & \ldots & D^{n_1-1} u_1(s_0) & 0 & 0 & \ldots & D^{n_1-1} v_1(s_0)
\end{bmatrix}$$

$$A_3 = \begin{bmatrix}
D^{n_1-2} u_1(s_0) & D^{n_2-2} u_2(t_0) & D^{n_1-2} u_1(s_0) & D^{n_2-2} v_2(t_0) & D^{n_1-2} v_1(s_0) & D^{n_2-2} u_2(t_0) & D^{n_1-2} v_1(s_0) & D^{n_2-2} v_2(t_0) \\
D^{n_1-1} u_1(s_0) & D^{n_2-2} u_2(t_0) & D^{n_1-1} u_1(s_0) & D^{n_2-2} v_2(t_0) & D^{n_1-1} v_1(s_0) & D^{n_2-2} u_2(t_0) & D^{n_1-1} v_1(s_0) & D^{n_2-2} v_2(t_0) \\
D^{n_1-2} u_1(s_0) & D^{n_2-1} u_2(t_0) & D^{n_1-2} u_1(s_0) & D^{n_2-1} v_2(t_0) & D^{n_1-2} v_1(s_0) & D^{n_2-1} u_2(t_0) & D^{n_1-2} v_1(s_0) & D^{n_2-1} v_2(t_0) \\
D^{n_1-1} u_1(s_0) & D^{n_2-1} u_2(t_0) & D^{n_1-1} u_1(s_0) & D^{n_2-1} v_2(t_0) & D^{n_1-1} v_1(s_0) & D^{n_2-1} u_2(t_0) & D^{n_1-1} v_1(s_0) & D^{n_2-1} v_2(t_0)
\end{bmatrix}$$
The matrix $A_1$ has size $2(n_1 - 2) \times 2(n_1 - 2)$, $A_2$ has size $2(n_2 - 2) \times 2(n_2 - 2)$, $A_3$ has size 4, $*$ stand for blocks of suitable size, and 0 stand for null matrices of suitable sizes. The existence and uniqueness of the interpolation operator $QL$ is then equivalent to the non-singularity of this matrix. Since $A$ is an upper triangular block matrix, its non-singularity can be proved by studying $A_1$, $A_2$, $A_3$ ($I$ is obviously non-singular). The matrices $A_1$ and $A_2$ are not singular, due to their structure, and the fact that (2) and (3) hold. In fact, the determinants of $A_1$ and $A_2$ are

\[
\det(A_1) = (-1)^{f(n_1 - 2)}[D^{n_2 - 2}u_2(t_0)D^{n_1 - 2}v_2(t_0) - D^{n_1 - 2}u_2(t_0)D^{n_2 - 2}v_2(t_0)]^{n_1 - 2}
\]

\[
\det(A_2) = (-1)^{f(n_2 - 2)}[D^{n_1 - 2}u_1(t_0)D^{n_2 - 2}v_1(t_0) - D^{n_1 - 1}u_1(t_0)D^{n_2 - 2}v_1(t_0)]^{n_2 - 2}
\]

where $f(m) = \left\lfloor \frac{m}{2} \right\rfloor$. The basis of these powers cannot vanish because of those facts holding.

In order to investigate the non-singularity of $A_3$, we set:

\[
D_1 := \begin{bmatrix} D^{n_1 - 2}u_1 & D^{n_1 - 2}v_1 \\ D^{n_1 - 1}u_1 & D^{n_1 - 1}v_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

and:

\[
D_2 := \begin{bmatrix} D^{n_2 - 2}u_2 & D^{n_2 - 2}v_2 \\ D^{n_2 - 1}u_2 & D^{n_2 - 1}v_2 \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}
\]

So we can write:

\[
A_3 = \begin{bmatrix} ac & af & be & bf \\ ce & cf & de & df \\ ag & ah & bg & bh \\ cg & ch & dg & dh \end{bmatrix}
\]

whose determinant is $-(ad - bc)^2(ch - fg)^2 = -[\det(D_1)]^2[\det(D_2)]^2$, which is nonzero if and only if both $\det(D_1) \neq 0$ and $\det(D_2) \neq 0$.

If we assume that (3) holds for both univariate spans, it can be shown (see [3]) that $\det(D_1) \neq 0$ and $\det(D_2) \neq 0$, and so $\det(A_3) \neq 0$.

Given a function $f \in C^n(\Omega)$, we now define the following quasi-interpolant belonging to the generalized spline space $GS_{u,v}^{n,r}(\Delta)$:

\[
Qf = \sum_{\xi \in \mathcal{M}} \gamma_\xi(Q_L(f; s_\xi, t_\xi))\psi_\xi
\]

where:

- $\mathcal{M}$ is the minimal determining set constructed in (34);
- $\psi_\xi$ are the elements of the basis of the spline space on the T-mesh $\Delta$ associated to $\mathcal{M}$;
- $\gamma_\xi$ are the linear functionals defined in (20) that associate to a spline $p \in GS_{u,v}^{n,r}(\Delta)$ the correspondent B-coefficients, needed to express $p$ as a linear combination of the basis $\psi_\xi$:

\[
p = \sum_{\xi \in \mathcal{M}} \gamma_\xi p \psi_\xi, \quad \forall p \in GS_{u,v}^{n,r}(\Delta)
\]
Lemma 5.2. Given a rectangle $K$, the result is then achieved by setting $n$ depending only on $\xi$.

Note that $Q$ is a linear operator, being the functionals $\gamma_\xi$ linear, and it is a projection onto $GS_{u,\xi}^n(\Delta)$, that is, $Qf = f$ for every $f \in GS_{u,\xi}^n(\Delta)$. In the following, we will use the generalization to our non-polynomial setting of Lemmas 3.1 and 3.2 in [13].

**Lemma 5.1.** Let $p \in GS_{u,\xi}^n(\Delta)$. Let $R \in \Delta$, and let $p|_R = \sum_{\eta \in \mathcal{D}_{n,R}} c_\eta B_\eta^R(s,t)$. We denote by $c$ the vector containing all the coefficients $c_\eta^R$, $\eta \in \mathcal{D}_{n,R}$. Then, there exists a constant $K_1$, depending only on $n_1$ and $n_2$, such that:

$$\frac{||c||_\infty}{K_1} \leq ||p||_R \leq ||c||_\infty,$$

where $||c||_\infty$ stands for the max-norm of $c$ and $|| \cdot ||_R$ for the sup-norm of a function restricted to $R$.

**Proof.** This is a straightforward generalization of the polynomial case: the upper bound follows from the fact that the basis functions are nonnegative and sum to one, while the lower bound follows from the following argument: the matrix $M := [B_\eta^R(\xi)]_{\xi,\eta \in \mathcal{D}_{n,R}}$ is a non-singular matrix, since the functionals $\{\lambda_\xi\} \subset \mathcal{D}_{n,R}$, defined by $\lambda_\xi(f) = f(\xi)$, $f \in C^0(R)$, are a dual basis of $\{B_\eta^R\}_{\eta \in \mathcal{D}_{n,R}}$. Then $Mc = r$, where $r$ is the vector $\{p(\xi)\}_{\xi \in \mathcal{D}_{n,R}}$. As a consequence, we have

$$||c||_\infty \leq ||M^{-1}r||_\infty \leq ||M^{-1}||_\infty||r||_\infty \leq ||M^{-1}||_\infty||p||_R = K_1||p||_R.$$

The results is then achieved by setting $K_1 = ||M^{-1}||_\infty$. □

**Lemma 5.2.** Given a rectangle $R$, let $A_R$ be its area. Then there exists a constant $K_2$, depending only on $n_1$ and $n_2$, such that:

$$\frac{A_R^{1/q}}{K_2} ||c||_q \leq ||p||_{q,R} \leq A_R^{1/q} ||c||_q,$$

where $||c||_q$ stands for the $q$-norm of the vector $c$ and $|| \cdot ||_{q,R}$ for the $q$-norm of a function restricted to $R$.

**Proof.** It is sufficient to use equivalence of norms on finite dimensional spaces, considering that both a classical polynomial space and the more general space in which we work have finite dimension. This allows us to establish stability in the $q$-norm for any $1 \leq q < \infty$, by generalizing Theorem 2.7 in [8]. □

To prove the approximation property of the quasi-interpolant, we will need the following result about the minimal determining set and the B-coefficients.

**Definition 5.3.** Let $e$ be a composite edge of $\Delta$, and let $e_1, ..., e_m$ be a maximal sequence of composite edges such that for each $i = 1, ..., m$, one endpoint of $e_i$ lies in the interior of $e_{i+1}$, where we assume $e_{m+1} = e$. We call $e_1, ..., e_m$ a chain ending at $e$. We call $m$ the length of the chain.

**Theorem 5.4.** For every composite edge $e$ consisting of $m$ edge segments $e_1, ..., e_m$ with $m \geq 1$, let $\alpha_e := \max\{|e|/|e_1|, |e|/|e_m|\}$, and let $\beta_e$ be the length of the longest chain ending on $e$. For each rectangle $R$ in $\Delta$, let $\kappa_R$ be the ratio of the length of its longest edge to the length to
its shortest edge, and by recalling that $E$ is the set of all composite edges of $\Delta$, we let $\alpha_\Delta := \max_{e \in E} \alpha_e, \beta_\Delta := \max_{e \in E} \beta_e, \kappa_\Delta := \max_{e \in E} \kappa_e$. Then, for any $p \in G_{u,v}^{n,r}(\Delta)$, its associated B-coefficients satisfy:

$$|c_\eta| \leq K_3 \max_{\xi \in M} |c_\xi|, \quad \eta \in D_{n,\Delta}$$

where $K_3$ is a constant depending only on $n, \alpha_\Delta, \beta_\Delta, \kappa_\Delta$.

**Proof.** The proof essentially is a generalization of Theorem 6.1 in [13], since analogous relations between the directional derivatives of a spline belonging to $G_{u,v}^{n,r}(\Delta)$ can be found. □

Let $\xi \in M$ and $F \in C^n(\Omega)$. By applying Lemma 5.2 with $c = \gamma(\xi(L(F; s_\xi, t_\xi)), p = Q_L(F; s_\xi, t_\xi))$, we obtain:

$$|\gamma(\xi(L(F; s_\xi, t_\xi)))| \leq \frac{K_2}{A^{1/q}_{R_\xi}} ||Q_L(F; s_\xi, t_\xi)||_{q,R_\xi}.$$

An analogous use of Lemma 5.1 leads to:

$$|\gamma(\xi(Q_L(F; s_\xi, t_\xi)))| \leq K_1 ||Q_L(F; s_\xi, t_\xi)||_{\max,R_\xi}.$$

If we denote by $T^{(n_1-1,n_2-1)}Q_L(F; s_\xi, t_\xi)$ the Taylor expansion of $Q_L(F; s_\xi, t_\xi)$ at $(s_\xi, t_\xi)$ of bi-degree $(n_1 - 1, n_2 - 1)$, for $1 \leq q < \infty$, we get:

$$|\gamma(\xi(Q_L(F; s_\xi, t_\xi)))| \leq \frac{K_2}{A^{1/q}_{R_\xi}} ||Q_L(F; s_\xi, t_\xi)||_{q,R_\xi} \leq \frac{K_2}{A^{1/q}_{R_\xi}} ||Q_L(F; s_\xi, t_\xi)||_{\max,R_\xi} \leq K_2 \max_{(x,y) \in R_\xi} |T^{(n_1-1,n_2-1)}Q_L(F; s_\xi, t_\xi)(x,y)| + O((\max\text{diam}(R_\xi)^{k+1}),$$

that is,

$$|\gamma(\xi(Q_L(F; s_\xi, t_\xi)))| \leq K_\xi(F) + O((\max\text{diam}(R_\xi)^{k+1})$$

where $K_\xi(F) = K_2 \max_{(x,y) \in R_\xi} |T^{(n_1-1,n_2-1)}Q_L(F; s_\xi, t_\xi)|$. An analogous bound can be trivially obtained for $q = \infty$.

For $\eta \in D_{n,R}$, by using Theorem 5.3 we have

$$|\gamma(\eta)| \leq K_3 \max_{\xi \in M} |\gamma(\xi)| \leq K_3 \max_{\xi \in M} K_\xi(F) + O((\max\text{diam}(R_\xi)^{k+1})$$

That allows us to obtain a limitation for $QF$:

$$||QF||_{q,R} \leq A^{1/q}_{R} ||QF||_{\max,R} = A^{1/q}_{R} \sum_{\eta \in D_{n,R}} \gamma_\eta B^R_\eta ||QF||_{\max,R} \leq A^{1/q}_{R} K_3 \max_{\xi \in M} K_\xi(F) + O((\max\text{diam}(R_\xi)^{k+1}).$$

(27)

Now, we can finally get a result showing the approximation power of the quasi-interpolant $Q$. Given a cell $R_\xi \in \Delta$, we have

$$||f - Qf||_{q,R_\xi} \leq ||f - Q_L(f; s_\xi, t_\xi)||_{q,R_\xi} + ||Q_L(f; s_\xi, t_\xi) - Qf||_{q,R_\xi} = ||f - Q_L(f; s_\xi, t_\xi)||_{q,R_\xi} + ||Q(f - Q_L(f; s_\xi, t_\xi))||_{q,R_\xi} \leq O((\max\text{diam}(R_\xi)^{k+1}) + A^{1/q}_{R} K_3 \max_{\xi \in M} K_\xi(F) + O((\max\text{diam}(R_\xi)^{k+1})$$

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where we used the fact that $Q$ is linear and it is a projection on $GS_{u,v}^n(\Delta)$, and we applied inequality (27) to $F = f - Q_L(f; s_\xi, t_\xi)$. Since $F = f - Q_L(f; s_\xi, t_\xi), K_\xi(F) = 0, \forall \xi \in \mathcal{M}$, because $F^{(0,0)} = F^{(0,1)} = F^{(1,0)} = \ldots = F^{(n_1-1,n_2-1)} = 0$, and so also $\max_{\xi \in \mathcal{M}} K_\xi = 0$. Therefore:

$$||f - Qf||_{q,R_\xi} \leq O((\max_{\xi \in \mathcal{M}} \text{diam}(R_\xi))^{k+1})$$

If we define the mesh size of $\Delta$ as $H = \max_{R \in \Delta} \text{diam}(R)$, then we get:

$$||f - Qf||_{q,R_\xi} \leq O(H^{k+1})$$

### 6 Conclusions

In this paper we have generalized the concept of spline space over T-meshes to the case of the non-polynomial tensor-product spaces of type (11). We have shown that, in spite of the different functions locally considered, the overall behaviour of the new spline spaces is analogous to the classical polynomial case: we can get a basis and a dimension formula associated to a minimal determining set, and the approximation order given by using a quasi-interpolant based on the local Hermite interpolants is essentially the same as in the polynomial case. We’ve also provided a discussion about some noteworthy choices of the non-polynomial functions to be chosen in order to get a good behaviour with respect to differentiation and integration.

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