Abstract

There are two well known Stochastic Approximation techniques that are known to have optimal rate of convergence (measured in terms of asymptotic variance): the Ruppert-Polyak averaging technique, and stochastic Newton-Raphson (SNR) (a matrix gain algorithm that resembles the deterministic Newton-Raphson method). The “Zap” algorithms introduced by the authors are a version of SNR designed to behave more closely like their deterministic cousin. It is found that estimates from the Zap Q-learning algorithm converge remarkably quickly, but the per-iteration complexity can be high.

This paper introduces an entirely new class of stochastic approximation algorithms based on matrix momentum. For a special choice of the matrix momentum and gain sequences, it is found in simulations that the parameter estimates obtained from the algorithm couple with those obtained from the more complex stochastic Newton-Raphson algorithm. Conditions under which coupling is guaranteed are established for a class of linear recursions. Optimal finite-\(n\) error bounds are also obtained.

The main objective of this work is to create more efficient algorithms for applications to reinforcement learning. Numerical results illustrate the value of these techniques in this setting.

1 Introduction

The general goal of this paper is the efficient computation of the root of a vector valued function: obtain the solution \(\theta^* \in \mathbb{R}^d\) to the \(d\)-dimensional equation \(\bar{f}(\theta^*) = 0\). It is assumed that the function \(\bar{f} : \mathbb{R}^d \to \mathbb{R}^d\) is expressed as an expectation: \(f(\theta) = E[f(\theta, \mathcal{X})]\), where \(f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d\) and \(\mathcal{X}\) is an \(\mathbb{R}^m\)-valued random variable with distribution denoted \(\omega\). The stochastic approximation (SA) literature contains a large collection of tools to construct algorithms and obtain bounds on their convergence rate. In this paper we show how algorithms with optimal rate of convergence can be constructed based on a synthesis of techniques from classical SA theory combined with variants of momentum algorithms pioneered by Polyak [25, 26].

The algorithms and analysis in this paper admit application to both optimization and reinforcement learning. In such applications, it is commonplace to assume that there is an aperiodic and positive Harris recurrent Markov chain \(\mathcal{X}\) whose steady-state distribution is \(\omega\). Let \(f_n(\theta) = f(\theta, \mathcal{X}_n)\) for
\( n \geq 0 \) and \( \theta \in \mathbb{R}^d \). Under suitable bounds on the function \( f \), the following limits hold:

\[
\bar{f}(\theta) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_k(\theta) = \lim_{n \to \infty} \mathbb{E}[f_n(\theta)], \quad \theta \in \mathbb{R}^d, \quad \text{ (first limit in the a.s. sense [21]).} \tag{1}
\]

Three general classes of algorithms are investigated. Each is defined with respect to a non-negative scalar gain sequence \( \{\alpha_n\} \), and two include \( d \times d \) matrix sequences \( \{G_n, M_n\} \). For each algorithm, the difference sequence is denoted \( \Delta \theta_n := \theta_n - \theta_{n-1}, n \geq 0 \), with given initial condition \( \theta_0 = \theta_{-1} \).

1. **Stochastic approximation with matrix gain**

\[
\Delta \theta_{n+1} = \alpha_{n+1} G_{n+1} f_{n+1}(\theta_n). \tag{2}
\]

2. **Matrix Heavy-Ball Stochastic approximation (PolSA)**

\[
\Delta \theta_{n+1} = M_{n+1} \Delta \theta_n + \alpha_{n+1} G_{n+1} f_{n+1}(\theta_n) \tag{3}
\]

3. **Nesterov Stochastic approximation (NeSA)**

For a fixed scalar \( \zeta > 0 \),

\[
\Delta \theta_{n+1} = \Delta \theta_n + \zeta (f_{n+1}(\theta_n) - f_{n+1}(\theta_{n-1})) + \zeta \alpha_{n+1} f_{n+1}(\theta_n) \tag{4}
\]

If \( G_n \equiv I \), then (2) is the classical algorithm of Robbins and Monro [6]. In Stochastic Newton Raphson (SNR) and the more recent Zap SNR [11, 12, 29], the matrix sequence is chosen to be an approximation of \(-[\partial f(\theta_n)]^{-1}\). Stability of the algorithm has been demonstrated in application to Q-learning [11, 12]; a non-trivial result, given that Q-learning is cast as root finding and not a minimization problem. The PolSA algorithm coincides with the heavy-ball method when \( \{M_n\} \) is a sequence of scalars [20, 25, 26]. Justification for the special form (4) in NeSA is provided in the next section.

As in many previous papers in the context of high-dimensional optimization [20] and SA [6, 17, 19], parameter error analysis is restricted to a linear setting:

\[
f_n(\theta) = A_n \theta - b_n = A(\theta - \theta^*) + \Delta_n \tag{5}
\]

in which \( (A_n, b_n) \) is a stochastic process with common mean \( (A, b) \), and

\[
\Delta_n := \tilde{A}_n (\theta - \theta^*) + \Delta_n^* \quad \text{with} \quad \Delta_n^* := f_n(\theta^*), \quad n \geq 1, \tag{6}
\]

where the tilde always denotes deviation: \( \tilde{A}_n = A_n - A \).

**Goals**

The main goal is to design algorithms with 1. **fast convergence** to zero of the error sequence \( \theta_n = \theta_n - \theta^* \), and 2. **low computational complexity**.

Rates of convergence are well understood for the SA recursion. It is known that the Central Limit Theorem and Law of the Iterated Logarithm hold under general conditions, and the **asymptotic covariance** appearing in these results can be expressed as the limit

\[
\Sigma^\theta = \lim_{n \to \infty} \Sigma_n^\theta := \lim_{n \to \infty} n \mathbb{E}[\tilde{\theta}_n \tilde{\theta}_n^T]. \tag{7}
\]

The LIL is most interesting in terms of bounds:

\[
|\tilde{\theta}_n(i)| \leq \left( \sigma(i) + o(1) \right) \sqrt{\frac{2 \log \log(n)}{n}} \tag{8}
\]

where \( o(1) \to 0 \) as \( n \to \infty \), and \( \sigma^2(i) = \Sigma^\theta(i, i) \).

The a.s. bound (8) may not be as satisfying as a Hoeffding or PAC-style finite-\( n \) bound, but presently there are no such bounds for Markovian models with useful constants (see e.g. [15]). Applications to reinforcement learning are typically cast in a Markov setting.

A **necessary** condition for quick convergence is that the LIL holds with small \( \sigma(i) \). Again, for the SA recursion, optimization of this parameter is well-understood. Denote by \( \Sigma^\theta \) the asymptotic
covariance for \( (2) \) with \( G_n := G \). When this is finite, it admits a representation in terms of the asymptotic covariance of the noise:

\[
\Sigma^\Delta = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \left( \sum_{k=1}^{n} \Delta_k \right) \left( \sum_{k=1}^{n} \Delta_k^* \right)^T \right]
\]  

(9)

In particular, the choice \( G^* = -A^{-1} \) is a special case of SNR, for which asymptotic covariance admits the explicit form

\[
\Sigma^* := A^{-1} \Sigma^\Delta A^{-1T}.
\]  

(10)

This is optimal: the difference \( \Sigma^G - \Sigma^* \) is positive semi-definite for any \( G \) [5, 6, 19].

What about computational complexity? In realistic applications of SNR the gain sequence will be of the form \( G_n = -\hat{A}_n^{-1} \), where \( \{\hat{A}_n\} \) are approximations of \( A \). In a nonlinear model, \( \hat{A}_n \) is an approximation of \( \partial f(\theta_n) \), obtained using the two time-scale algorithm of [11, 12]. The resulting complexity is a barrier to application in high dimension. Steps towards resolving this obstacle are presented in this paper:

(i) The parameters in the PolSA algorithm can be designed so that the error sequence enjoys all the attractive properties of SNR, but without the need for any matrix inversion.

(ii) NeSA is often simpler than PolSA in applications to RL. A formula for the asymptotic covariance of a variant of NeSA is obtained in this paper. While not equal to \( \Sigma^* \), the reduced complexity makes it a valuable option.

These conclusions are established in Propositions 2.2, 3.1 and 3.2 for linear recursions, and illustrated in numerical examples for new Q-learning algorithms introduced in Section 4.

The techniques introduced in this paper are not directly applicable to batch gradient descent – an explanation is given in Section 2.2.

Portions of this article are taken from the recent survey [10].

**Literature survey** The reader is referred to [11, 12] for a survey on SNR and the more recent Zap SNR algorithms.

The present paper is built on a vast literature on optimization [23–26] and stochastic approximation [6,17,19,27–29]. The work of Polyak is central to both thrusts: the introduction of momentum, and techniques to minimize variance in SA algorithms.

Many papers with similar titles focus on high dimensional optimization in which randomness is introduced to ease computational burden; an example is batch gradient descent — see [36] for an extensive survey. Again, as explained in Section 2.2, we do not know if the concepts introduced in this paper can be applied to batch gradient descent.

Most closely related to the present work is the literature on ERM (empirical risk minimization) in which the sample path limit in (1) is replaced by a finite average, \( \hat{f}_n(\theta) = \frac{1}{n} \sum_{k=0}^{n-1} f_k(\theta) \) [1,9,16]. Under general conditions it can be shown that the sequence of ERM optimizers \( \{\hat{\theta}_k^*\} \) is convergent, and has optimal asymptotic covariance (a survey and further discussion is presented in [16]). The papers [4,13,14,16,22] establish the optimal convergence rate of \( O(1/\sqrt{n}) \) for various algorithms.

The recent paper [16] is most closely related to the present work, considering the shared goal of optimization of the asymptotic covariance, along with rapidly vanishing transients through algorithm design. The paper restricts to nonlinear optimization rather than the root finding problems considered here, which rules out application to many reinforcement learning algorithms. The metric for performance is slightly different, focusing on the rate of convergence of the expected loss, for which they obtain bounds for each iteration \( n \) of the algorithm. Optimal asymptotic covariance is not established, but rather they obtain tight bounds on the regret.

The algorithms presented here do achieve the optimal asymptotic covariance, are not restricted to ERM, and we believe that in many applications they will be simpler to implement. This is especially true for the NeSA algorithm applied to Q-learning.
2 Momentum methods and applications

Consider first the deterministic root-finding problem. This will bring insight into the relationship between the three algorithms (2–4) discussed in the introduction. Since the Markovian disturbance is absent, the notation \( f : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is used in place of \( f \). The goal remains the same: find the vector \( \theta^* \in \mathbb{R}^d \) such that \( f(\theta^*) = 0 \).

Deterministic variants of (2–4) commonly considered in the literature are, respectively,

- Successive approximation: \( \Delta \theta_{n+1} = \alpha f(\theta_n) \) (11)
- Polyak’s heavy ball: \( \Delta \theta_{n+1} = \mu \Delta \theta_n + \alpha f(\theta_n) \) (12)
- Nesterov’s acceleration: \( \Delta \theta_{n+1} = \mu \Delta \theta_n + \zeta[f(\theta_n) - f(\theta_{n-1})] + \alpha f(\theta_n) \) (13)

where \( \alpha, \mu, \zeta \) are positive constants. Nesterov’s algorithm was designed for extremal seeking, which is the special case \( f = -\nabla J \) for a real-valued function \( J \). The recursion (13) is the natural extension to the root-finding problem considered here.

The questions asked in this paper are posed in a stochastic setting, but analogous questions are:

(i) Why restrict to a scalar momentum term \( \mu \)? Can a \( d \times d \) “momentum matrix” \( M \) be designed to improve performance?

(ii) Can online algorithms be designed to approximate the optimal momentum matrix? If so, we require tools to investigate the performance of a matrix sequence \( \{M_n\} \):

\[
\Delta \theta_{n+1} = M_{n+1} \Delta \theta_n + \alpha f(\theta_n)
\]

(14)

Potential answers are obtained by establishing relationships between these deterministic recursions.

Consider the successive approximation algorithm (11) under the assumption of global convergence: \( \theta_n \rightarrow \theta^* \) as \( n \rightarrow \infty \). Assume moreover that \( f \in C^1 \). We obtain by the mean-value theorem,

\[
\Delta \theta_{n+1} - \Delta \theta_n = \alpha \partial f(\bar{\theta}_n) \Delta \theta_n = \alpha^2 \partial f(\bar{\theta}_n) f(\theta_{n-1})
\]

where \( \bar{\theta}_n \in \mathbb{R}^d \) lies on the line connecting the vectors \( \theta_{n+1} \) and \( \theta_n \). It follows that \( \| \Delta \theta_{n+1} - \Delta \theta_n \| = O(\min\{\alpha^2, \alpha \| \Delta \theta_n \| \}) \). This suggests a heuristic: swap \( \Delta \theta_{n+1} \) and \( \Delta \theta_n \) in a given convergent algorithm to obtain a new algorithm that is simpler, but with desirable properties. This is similar in spirit to the coupling concept introduced in [2]. Applying this heuristic to (14) results in

\[
\Delta \theta_{n+1} = M_{n+1} \Delta \theta_n + \alpha f(\theta_n) \approx M_{n+1} \Delta \theta_{n+1} + \alpha f(\theta_n)
\]

Assuming that an inverse exists, this becomes

\[
\Delta \theta_{n+1} \approx \alpha [I - M_{n+1}]^{-1} f(\theta_n)
\]

We thus arrive at a possible answer to the question of optimal momentum: For the matrix sequence \( M_{n+1} = I + \alpha \partial f(\theta_n) \), the algorithm (14) can be expressed

\[
\Delta \theta_{n+1} = [I + \alpha \partial f(\theta_n)] \Delta \theta_n + \alpha f(\theta_n)
\]

(15)

The foregoing approximations suggest that this is an approximation of Newton-Raphson:

\[
\Delta \theta_{n+1} \approx -[\partial f(\theta_n)]^{-1} f(\theta_n)
\]

A Taylor series argument shows that the recursion (15) is approximated by

\[
\Delta \theta_{n+1} = \Delta \theta_n + \alpha [f(\theta_n) - f(\theta_{n-1})] + \alpha f(\theta_n)
\]

(16)

This is the special case of Nesterov’s algorithm (13) with \( \mu = 1 \) and \( \zeta = \alpha \).

In this paper, the focus is root-finding in the stochastic setting described in the introduction. Strong justification for the stochastic analog of (15) is provided through a coupling bound between the respective algorithms; see Prop. 2.2. It is found that similar transformations lead to new algorithms for reinforcement learning and other applications.
2.1 Optimal matrix momentum for PolSA

Returning to the stochastic setting, the following special case of PolSA is the analog of (15) considered in this paper

\[ \Delta \theta_{n+1} = [I + \zeta \hat{A}_{n+1}] \Delta \theta_n + \alpha_{n+1} \zeta f_{n+1}(\theta_n) \quad (17) \]

where \( \zeta > 0 \) is a scalar constant, and \( \{ \hat{A}_n \} \) are estimates of \( A = E[A_n] \). The SNR algorithm is (2) in which \( G_n = \hat{A}_n^{-1} \) (the Moore–Penrose pseudo inverse when necessary).

Certain simplifying assumptions are imposed to ease analysis. The linear model (5) is adopted, which is subject to the following stability assumptions: For any eigenvalue \( \lambda \) of \( A \),

\[ \text{Real}(\lambda) < 0 \quad \text{and} \quad |1 + \zeta \lambda| < 1 \quad (18) \]

It is assumed without loss of generality that \( \zeta = 1 \). The algorithms are also simplified by replacing the estimate \( \hat{A}_n \) with its true value \( A \). The simplified SNR and PolSA algorithms are defined as follows:

\[ \Delta \theta^*_n = -\alpha f_n(\theta^*_n) \quad (19) \]
\[ \Delta \theta_{n+1} = [I + A] \Delta \theta_n + \alpha_{n+1} f_{n+1}(\theta_n) \quad (20) \]

Theorem 2.1 shows that the SNR algorithm is in some sense optimal under general conditions. Its proof is contained in Section A, and is based on [8, 18, 21].

**Theorem 2.1.** Suppose that \( \{ A_n, b_n \} \) is a bounded sequence, and each entry is a function of a \( V \)-uniformly ergodic Markov chain [21]. Then,

(i) The asymptotic covariance of SNR is given by (10), where \( \Sigma^\Delta \) is the asymptotic covariance of the zero-mean sequence \( \{ \Delta_n = f_n(\theta^*) \} \).

(ii) The Central Limit Theorem and Law of the Iterated logarithm hold for \( \{ \tilde{\theta}_n \} \) with respect to this covariance sequence. In particular, (8) holds with \( \sigma^*(i) = \sqrt{\Sigma^*(i,i)} \).

(iii) Consider any other algorithm of the form (2) for which the sequences \( \{ \theta_n, G_n \} \) are convergent, and the limit (7) holds. Then the matrix inequality holds: \( \Sigma^\theta \geq \Sigma^* \). \( \square \)

A drawback with SNR is complexity introduced by the matrix inverse. The algorithm (2.1) is simpler and enjoys the same attractive properties. This is established through coupling under slightly stronger assumptions:

**Proposition 2.2.** Suppose that the assumptions of Theorem 2.1 hold, and that \( \{ \hat{A}_n, \hat{b}_n \} \) are bounded martingale difference sequences. Let \( \{ \theta^*_n \} \) denote the iterates using SNR (19) and \( \{ \theta_n \} \) the iterates obtained using (20), with identical initial conditions.

Then, there is a square-integrable random variable \( \tilde{B} \) such that

\[ ||\theta_n - \theta^*_n|| \leq \tilde{B} n^{-1}, \quad n \geq 1. \quad (21) \]

Consequently, the conclusions of Theorem 2.1 (i) and (ii) also hold for the PolSA algorithm. \( \square \)
Other than the two techniques introduced by Polyak, SNR and the Polyak-Ruppert averaging technique, PoLSA is the only other SA algorithm that achieves optimal asymptotic variance.

The proof of Prop. 2.2, contained in Section B, is based on a justification of the heuristic used to construct the deterministic recursion (15).

An illustration is provided in Fig. 1 for the linear model \( f_n(\theta) = A\theta + \Delta_n \) in which \(-A\) is symmetric and positive definite, with \( \lambda_{\text{max}}(-A) = 1 \), and \( \{\Delta_n\} \) is i.i.d. and Gaussian. Shown are the trajectories of \( \{\theta_n(1) : n \leq 10^5\} \) (a short run for this example, since \( \Sigma^* (1, 1) \) is over one million).

### 2.2 Applications

**Reinforcement learning** Section 4 describes application to Q-learning, and includes numerical examples. Section D contains a full account of TD-learning. In particular, the LSTD algorithm can be regarded as an instance of SNR: (2) with \( G_n = -\tilde{A}_n^{-1} \) an estimate of \( G^* = -A^{-1} \).

**Optimization** A common application of stochastic approximation is convex optimization. In this setting, \( f(\theta) = \nabla E[J_n(\theta)] \) for a sequence of smooth functions \( \{J_n\} \), and then \( f_n = -\nabla J_n \). The theory developed in this paper is applicable to this general class of optimization problems, except in degenerate cases. For comparison, consider the quadratic optimization problem in which \( f_n(\theta) = A\theta - b + \Delta_n \), with \(-A > 0\). The stability condition (18) holds provided \( \zeta < 1/\lambda_{\text{max}}(-A) \) a condition familiar in the convex optimization literature.

**Batch gradient descent** Consider the following special case in which a deterministic function \( J : \mathbb{R}^d \rightarrow \mathbb{R} \) is given. The goal is again to compute a stationary point \( \nabla f(\theta^*) = 0 \). A sequence of sparse matrices \( \{I_n\} \) is constructed, and then for each \( n \)

\[
 f_n := -I_n \nabla J
\]

It is assumed that \( E[I_n] = I \), so that \( \{f_n\} \) are unbiased samples of the gradient; see [20] for a recent survey. Randomness vanishes as \( \theta \rightarrow \theta^* \):\[
 \|f_n(\theta^*)\|^2 = (\nabla J(\theta^*))^T I_n^2 \nabla J(\theta^*) = 0
\]

The covariance matrix \( \Sigma^A \) is zero, and hence also the asymptotic covariance (10). There is no known CLT or LIL in this case, and hence the results of this paper do not apply to this special setting.

### 3 Variance analysis of the NeSA algorithm

The NeSA algorithm has a finite asymptotic covariance that can be expressed as the solution to a Lyapunov equation. We again restrict to the linear model, so that the recursion (4) (with \( \zeta = 1 \)) becomes

\[
\Delta \theta_{n+1} = [I + A_{n+1}] \Delta \theta_n + \alpha_{n+1}[A_{n+1} \theta_n - b_{n+1}]
\]

Stability of the recursion requires a strengthening of (18). Define the linear operator \( \mathcal{L} : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d} \) as follows: For any matrix \( Q \in \mathbb{R}^{d \times d} \),

\[
 \mathcal{L}(Q) := E[(I + A_n)Q(I + A_n)^T]
\]

The following assumptions are imposed throughout:

(N1) \( \{\tilde{A}_n, \tilde{b}_n\} \) are bounded martingale difference sequences. Moreover, for any matrix \( Q \),

\[
 E[(I + A_n)Q(I + A_n)^T | \mathcal{F}_{n-1}] = \mathcal{L}(Q)
\]

where \( \mathcal{F}_n \) is the natural filtration: \( \mathcal{F}_n = \sigma\{A_k, b_k : k \leq n\} \).

(N2) The bounds in (18) hold, and the linear operator \( \mathcal{L} \) has spectral radius bounded by unity.

Define the 2d-dimensional vector processes

\[
\Phi_n := \left( \frac{\sqrt{n} \theta_n}{n \Delta \theta_n} \right) \quad \text{and} \quad \Delta \Phi_n := \left( \frac{\sqrt{n} \Delta \theta_n}{n \Delta \theta_n} \right)
\]

A 2d \times 2d covariance matrix sequence is the focus of this section:

\[
\Sigma_n := E[\Phi_n \Phi_n^T] = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}
\]

6
Proposition 3.1. Suppose that (N1) and (N2) hold. Then, the covariance sequence is convergent:

$$\lim_{n \to \infty} \Sigma_n = \begin{bmatrix} \Sigma_{11}^\infty & 0 \\ 0 & \Sigma_{22}^\infty \end{bmatrix}$$ (26)

in which the second limit is the solution to the Lyapunov equation

$$\Sigma_{22}^\infty = L(\Sigma_{22}^\infty) + \Sigma$$ (27)

(an explicit solution is given in eqn. (53)), and

$$\Sigma_{11}^\infty = -\Sigma_{22}^\infty - A^{-1}\Sigma_{22}^\infty - \Sigma_{22}^\infty A^{-1}$$ (28)

The following result is a corollary to Prop. 3.1, with an independent proof provided in Section C.1.

Proposition 3.2. Under (N1)–(N3) the conclusions of Prop. 3.1 hold for the PolSA recursion (20).

In this case the solution to the Lyapunov equation is the optimal covariance:

$$\Sigma_{11}^\infty = \Sigma^* := A^{-1}\Sigma\Delta A^{-1}$$ (29)

and $$\Sigma_{22}^\infty \geq 0$$ is the unique solution to the Lyapunov equation

$$\Sigma_{22}^\infty = (I + A)\Sigma_{22}^\infty (I + A)^T + \Sigma$$ (30)

The main step in the proof of Prop. 3.1 involves a finer look at the off-diagonal blocks of the covariance matrix. The proofs of the following Lemmas are postponed to Section C.

Lemma 3.3. The following approximations hold for $$\Sigma_{22}^n$$ and the scaled covariance $$\psi_n := \sqrt{n}\Sigma_{21}^n$$:

$$\Sigma_{22}^{n+1} = L(\Sigma_{22}^n) + \Sigma + o(1), \quad \psi_n = -\Sigma_{11}^n - A^{-1}\Sigma_{22}^\infty + o(1), \quad n \geq 1.$$ (31)

The second iteration is used together with the following result to obtain (28).

Lemma 3.4. The following approximation holds: $$\Sigma_{11}^{n+1} =$$

$$\Sigma_{11}^n + \alpha_{n+1} \left( \Sigma_{11}^n + A\Sigma_{11}^n + \Sigma_{11}^n A^T + \psi_n (I + A)^T + (I + A)\psi_n + L(\Sigma_{22}^\infty) + \Sigma + o(1) \right)$$ (32)

Proof of Prop. 3.1. The first approximation in (31) combined with Assumption (N2) implies that the sequence $$\{\Sigma_{22}^n\}$$ is convergent, and the limit is the solution to (27) (details are provided in Section C.2).

Substituting the approximation (31) for $$\psi_n$$ into (32) gives

$$\Sigma_{11}^{n+1} = \Sigma_{11}^n + \alpha_{n+1} \left( -\Sigma_{11}^n - \Sigma_{22}^\infty - A^{-1}\Sigma_{22}^\infty - \Sigma_{22}^\infty A^{-1} + o(1) \right)$$

This can be regarded as a Euler approximation to the ODE:

$$\frac{dx_t}{dt} = -x_t - \Sigma_{22}^\infty - A^{-1}\Sigma_{22}^\infty - \Sigma_{22}^\infty A^{-1}$$

The convergence techniques of stochastic approximation theory can be applied to establish that the limits of $$\{\Sigma_{11}^n\}$$ and $$\{x_t\}$$ coincide with the stationary point, which is (28). □
4 Application to Q-learning

Consider the discounted-cost control problem with state-action process denoted \( \{(X_n, U_n) : n \geq 0\} \) evolving on the finite set \( X \times U \), cost function \( c: X \to \mathbb{R} \), and discount factor \( \beta < 1 \). The Q-function solves the Bellman equation: 
\[
Q^\ast(x, u) = c(x, u) + \beta \mathbb{E}[\min_{u'} Q^\ast(X_{n+1}, u')].
\]
For a \( d \)-dimensional basis of functions \( \{\psi_k\} \) and a given parameter \( \theta \in \mathbb{R}^d \), the corresponding estimate is defined by 
\[
Q^n(x, u) = \sum_{k} \theta_k \psi_k(x, u),
\]
Watkins’ Q-learning algorithm is designed to compute the exact Q-function that solves the corresponding Bellman errors [34,35]. In this setting the basis is taken to be the set of indicator variables: 
\[
\psi_k(x, u) = \mathbb{I} \{x = x^k, u = u^k\}, \ 1 \leq k \leq d, \text{ with } d = |X \times U|.
\]
The basic algorithm of Watkins can be written 
\[
\Delta \theta_{n+1} = \alpha_{n+1} \hat{D}_{n+1} [A_{n+1} \theta_n - b_{n+1}]
\]
in which the matrix gain is diagonal:
\[
\hat{D}_n(i, i) = \left[ \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{I} \{(X_k, U_k) = (x^i, u^i)\} \right]^{-1},
\]
\( A_{n+1} = \psi(X_n, U_n) \{\beta \psi(X_{n+1}, \phi_n(X_{n+1})) - \psi(X_n, U_n)\}^T \) and \( b_{n+1} = c(X_n, U_n) \psi(X_n, U_n) \), where \( \phi_n(x) = \arg \min_u Q^n(x, u) \) (see [31] for more details). 

Among the other algorithms compared are 

- **SNR**: 
  \[
  \Delta \theta_{n+1} = -\alpha_{n+1} \hat{A}_{n+1}^{-1} [A_{n+1} \theta_n - b_{n+1}]
  \]
- **PolSA**: 
  \[
  \Delta \theta_{n+1} = [I + \hat{A}_{n+1} \hat{A}_{n+1}^{-1}] \Delta \theta_n + \alpha_{n+1} [A_{n+1} \theta_n - b_{n+1}]
  \]
- **PolSA-D**: 
  \[
  \Delta \theta_{n+1} = [I + \hat{D}_{n+1} \hat{A}_{n+1} \hat{A}_{n+1}^{-1}] \Delta \theta_n + \alpha_{n+1} \hat{D}_{n+1} [A_{n+1} \theta_n - b_{n+1}]
  \]
- **NeSA**: 
  \[
  \Delta \theta_{n+1} = [I + \hat{A}_{n+1}] \Delta \theta_n + \alpha_{n+1} [A_{n+1} \theta_n - b_{n+1}]
  \]

![Figure 3: Bellman error for the 6 node example](image)

The SNR algorithm considered coincides with the Zap Q-learning algorithm of [11,12]. Histograms for a particular 6-state MDP model with \( d = 18 \) were considered in this prior work. Fig. 2 shows that the histograms for \( \{\sqrt{n} \hat{\theta}_n\} \) using PolSA-D and SNR nearly coincide with \( n = 10^6 \) (performance for PolSA is similar). The histogram for NeSA shows a much higher variance, but this algorithm requires by far the least computation per iteration.

Fig. 3 shows the maximal Bellman error as a function of iterations for several algorithms.

Experiments were performed for larger examples. Results from two such experiments are shown in Fig. 4. The MDP model is again a stochastic shortest path problem. The model construction was based on the creation of a graph with \( N \) nodes, in which the probability of an edge between a pair of nodes is i.i.d. with probability \( p \). Additional edges \((i, i+1)\) are added, for each \( i < N \), to ensure the resulting graph is strongly connected.

The controls are as in the 6-state example: with probability 0.8 the particle moves in the desired direction, and with remaining probability any of the neighboring nodes is chosen uniformly. Two
exploration rules were considered: the “online” version described above (asynchronous version), and the offline “clock sampling” approach in which state-action pairs \((x_i, u_i)\) are chosen sequentially (synchronous version). At stage \(n\), if \((x, u)\) is the current pair, a random variable \(X'_{n+1}\) is chosen according to the distribution \(P_n(x, \cdot)\), and the \((x, u)\) entry of the Q-function is updated according to the particular algorithm using the triple \((x, u, X'_{n+1})\). A significant change to Watkins’ iteration (33) in the synchronous setting is that \(\tilde{D}_n\) is replaced by \(d^{-1} I\). This combined with deterministic sampling results in significant reduction in variance.

The synchronous speedy Q-learning recursion of [3] appears similar to the NeSA algorithm with clock sampling, following the heuristic swapping of arguments used to motivate the algorithms introduced in this paper. We do not know if the swapping is justified for their algorithm.

Two graphs were used in the survey experiments, one resulting in an MDP with \(d = 19\) state-action pairs and another resulting in a larger MDP with \(d = 117\) state-action pairs. The plots in Fig. 4 show Bellman error as a function of iteration \(n\) for the two cases. Comparison of the performance of algorithms in a deterministic exploration setting versus the online setting is also shown.

We have tested examples with over 500 variables and obtain similar performance for the clock-sampling version of the algorithm. Performance for the PolSA-D algorithm degrades due to estimation error for \(\tilde{D}_n\) in (34). Projection may be needed to improve reliability in high dimensions.

5 Conclusions

It is exciting to see how the intuitive transformation from SNR to PolSA and NeSA can be justified theoretically and in simulations. While the covariance of NeSA is not optimal, it is the simplest of the three algorithms and does well in the experiments we have conducted.

An important next step is to create adaptive techniques to ensure fast coupling or other ways to ensure fast forgetting of the initial condition. It is possible that techniques in [16] may be adapted. We do not have a natural notion of expected loss in the general root finding problem, but it will be of interest to pursue analysis in the special case of nonlinear optimization.

References

[1] Z. Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. ArXiv e-prints, Mar. 2016.
[2] Z. Allen-Zhu and L. Orecchia. Linear Coupling: An Ultimate Unification of Gradient and Mirror Descent. ArXiv e-prints, July 2014.
[3] M. G. Azar, R. Munos, M. Ghavamzadeh, and H. Kappen. Speedy Q-learning. In Advances in Neural Information Processing Systems, 2011.
[4] F. Bach and E. Moulines. Non-strongly-convex smooth stochastic approximation with convergence rate \(o(1/n)\). In Advances in Neural Information Processing Systems 26, pages 773–781. Curran Associates, Inc., 2013.
[5] A. Benveniste, M. Métivier, and P. Priouret. *Adaptive algorithms and stochastic approximations*, volume 22 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1990. Translated from the French by Stephen S. Wilson.

[6] V. S. Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Hindustan Book Agency and Cambridge University Press (jointly), Delhi, India and Cambridge, UK, 2008.

[7] J. A. Boyan. Technical update: Least-squares temporal difference learning. *Mach. Learn.*, 49(2-3):233–246, 2002.

[8] X. Chen. On the limit laws of the second order for additive functionals of Harris recurrent Markov chains. *Probability Theory and Related Fields*, 116(1):89–123, Jan 2000.

[9] A. Defazio, F. Bach, and S. Lacoste-Julien. Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. In *Advances in neural information processing systems*, pages 1646–1654, 2014.

[10] A. M. Devraj, A. Bušić, and S. Meyn. Zap Q Learning – a user’s guide. In *Proceedings of the fifth Indian Control Conference*, January 2019.

[11] A. M. Devraj and S. Meyn. Zap Q-Learning. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, *Advances in Neural Information Processing Systems 30*, pages 2235–2244. Curran Associates, Inc., 2017.

[12] A. M. Devraj and S. P. Meyn. Fastest convergence for Q-learning. *ArXiv e-prints*, July 2017.

[13] J. Duchi. Introductory lectures on stochastic optimization. *Stanford Lecture Series*, 2016.

[14] S. Gadat, F. Panloup, and S. Saadane. Stochastic heavy ball. *Electron. J. Statist.*, 12(1):461–529, 2018.

[15] P. W. Glynn and D. Ormoneit. Hoeffding’s inequality for uniformly ergodic Markov chains. *Statistics and Probability Letters*, 56:143–146, 2002.

[16] P. Jain, S. M. Kakade, R. Kidambi, P. Netrapalli, and A. Sidford. Accelerating Stochastic Gradient Descent. *ArXiv e-prints (and to appear, COLT 2018)*, Apr. 2017.

[17] V. R. Konda and J. N. Tsitsiklis. Convergence rate of linear two-time-scale stochastic approximation. *Ann. Appl. Probab.*, 14(2):796–819, 2004.

[18] V. Koval and R. Schwabe. A law of the iterated logarithm for stochastic approximation procedures in d-dimensional euclidean space. *Stochastic Processes and their Applications*, 105(2):299–313, 2003.

[19] H. J. Kushner and G. G. Yin. *Stochastic approximation algorithms and applications*, volume 35 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1997.

[20] N. Loizou and P. Richtárik. Momentum and Stochastic Momentum for Stochastic Gradient, Newton, Proximal Point and Subspace Descent Methods. *ArXiv e-prints*, Dec. 2017.

[21] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Cambridge University Press, Cambridge, second edition, 2009. Published in the Cambridge Mathematical Library. 1993 edition online.

[22] E. Moulines and F. R. Bach. Non-asymptotic analysis of stochastic approximation algorithms for machine learning. In *Advances in Neural Information Processing Systems 24*, pages 451–459. Curran Associates, Inc., 2011.

[23] Y. Nesterov. A method of solving a convex programming problem with convergence rate $O(1/k^2)$. In *Soviet Mathematics Doklady*, 1983.

[24] Y. Nesterov. Efficiency of coordinate descent methods on huge-scale optimization problems. *SIAM Journal on Optimization*, 22(2):341–362, 2012.

[25] B. T. Polyak. Some methods of speeding up the convergence of iteration methods. *USSR Computational Mathematics and Mathematical Physics*, 4(5):1–17, 1964.

[26] B. T. Polyak. *Introduction to Optimization*. Optimization Software Inc, New York, 1987.

[27] B. T. Polyak. A new method of stochastic approximation type. *Automatika i telemekhanika (in Russian), translated in Automat. Remote Control*, 51 (1991), pages 98–107, 1990.

[28] B. T. Polyak and A. B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM J. Control Optim.*, 30(4):838–855, 1992.

[29] D. Ruppert. A Newton-Raphson version of the multivariate Robbins-Monro procedure. *The Annals of Statistics*, 13(1):236–245, 1985.

[30] R. S. Sutton. Learning to predict by the methods of temporal differences. *Mach. Learn.*, 3(1):9–44, 1988.

[31] C. Szepesvári. Algorithms for reinforcement learning. *Synthesis lectures on artificial intelligence and machine learning*, 4(1):1–103, 2010.

[32] C. Szepesvári. *Algorithms for Reinforcement Learning*. Synthesis Lectures on Artificial Intelligence and Machine Learning. Morgan & Claypool Publishers, 2010.
[33] J. N. Tsitsiklis and B. Van Roy. An analysis of temporal-difference learning with function approximation. *IEEE Trans. Automat. Control*, 42(5):674–690, 1997.

[34] C. J. C. H. Watkins. *Learning from Delayed Rewards*. PhD thesis, King’s College, Cambridge, Cambridge, UK, 1989.

[35] C. J. C. H. Watkins and P. Dayan. *Q-learning*. *Machine Learning*, 8(3-4):279–292, 1992.

[36] A. C. Wilson, B. Recht, and M. I. Jordan. A Lyapunov Analysis of Momentum Methods in Optimization. *ArXiv e-prints*, 2016.
Appendix

A Limit theory for Markov chains and SA: Proof of Theorem 2.1

The error recursion for the linear SNR algorithm (19) can be expressed
\[
\tilde{\theta}_n^{*+1} = \tilde{\theta}_n^{*} + \alpha_n^{+1}[-\tilde{\theta}_n^{*} + A^{-1}\tilde{A}_n\tilde{\theta}_n^{*} + A^{-1}\Delta_n^{*}]
\]
for which \(\tilde{\theta}_n^{*} = \theta_n^{*} - \tilde{\theta}_n^{*}\). This linear recursion is convergent under the assumptions of the theorem.

The sequence \(\{\Delta_n^{*}\}\) satisfies the CLT with covariance \(\Sigma^{\Delta}\) [21], and also the LIL [8]. Each of these results is established by first constructing a martingale:

\[
M_n = H_n - H_0 + \sum_{k=1}^{n} \Delta_k^{*}
\]

where \(\{H_n = h(X_n)\}\) are functions of the underlying Markov chain \(\{X_n\}\), where \(h\) solves a certain Poisson’s equation. We have \(h \leq \text{const.}(1 + \log V)\) under the assumption that \(\Delta_n^{*}\) is a bounded function of \(X_n\) and the Markov chain is \(V\)-uniformly ergodic (Foster’s criterion holds using the function \(V_0 = 1 + \log V\)). Since \(V(X_n)\) has bounded mean, the assumptions of [18] hold to establish the LIL for the SA recursion, and the CLT can be found in [19] (see Theorem 2.1 of Chapter 10 and the following discussion in Section 10.2.2).

B Coupling

The proof of Prop. 2.2 is based on a transformation of SNR so that it resembles PolSA with a vanishing disturbance sequence. This is essentially a reversal of the manipulations applied to derive (13) from an approximation of (14) at the start of Section 2.

For simplicity we take \(\zeta = 1\) (this is without loss of generality by re-defining the matrix \(A\)).

It is simplest to first prove the result when \(\{A_n\}\) is deterministic: \(\tilde{A}_n \equiv 0\).

B.1 Deterministic matrix sequence

Lemma B.1. The SNR and PolSA recursions with deterministic \(\{A_n\}\) can be expressed, respectively

\[
\Delta\theta_{n+1}^{*} = [I + A]\Delta\theta_{n}^{*} + \alpha_n^{+1}[A\tilde{\theta}_n^{*} + \Delta_n^{*+1}] + \xi_{n+1}
\]

\[
\Delta\theta_{n+1} = [I + A]\Delta\theta_{n} + \alpha_n^{+1}[A\tilde{\theta}_n + \Delta_n^{*+1}],
\]

where \(\Delta_n = b_n - b\) and

\[
\xi_{n+1} = [I + A]\{\Delta\theta_{n+1}^{*} - \Delta\theta_n^{*}\}, \quad n \geq 0.
\]

Proof. The recursion (36) is the definition of PolSA in this special case.

The recursion (35) is obtained by expressing SNR as follows:

\[
0 = A\Delta\theta_{n+1}^{*} + \alpha_n^{+1}[A\tilde{\theta}_n^{*} + \Delta_n^{*+1}]
\]

\[
= A\Delta\theta_{n}^{*} + \alpha_n^{+1}[A\tilde{\theta}_n^{*} + \Delta_n^{*+1}] - \{\Delta\theta_{n+1}^{*} - \Delta\theta_n^{*}\} + \xi_{n+1}
\]

Moving \(\Delta\theta_{n+1}^{*}\) to the left-hand side completes the proof. \(\square\)

Denote \(\tilde{\theta}_n = \theta_n - \theta_n^{*}\) and \(\xi_n = n\tilde{\theta}_n\). The proof of Prop. 2.2 requires that we establish uniform bounds on these sequences.

Lemma B.2. The error sequence evolves according to the recursion

\[
\Delta\tilde{\theta}_{n+1} = [I + A]\Delta\tilde{\theta}_n + \alpha_n A\tilde{\theta}_n - \xi_{n+1}
\]

in which the sequence (37) satisfies the following:
(i) \( \{n \xi_n : n \geq 1\} \) is a bounded sequence

(ii) The partial sums are also bounded:

\[
S_n^\xi := \sum_{k=1}^{n} k \xi_k, \quad n \geq 1.
\]

**Proof.** Recall that \( \Delta \theta_{n+1}^* = -(n + 1)^{-1}(\theta_n^* + A^{-1} \Delta_n + 1) \). Consequently,

\[
(n + 1) (\Delta \theta_{n+1}^* - \Delta \theta_n^*) = (n + 1) \Delta \theta_{n+1}^* - n \Delta \theta_n^* - \Delta \theta_n^* = - (\theta_n^* + A^{-1} \Delta_n + 1) + (\theta_{n-1}^* + A^{-1} \Delta_n) - \Delta \theta_n^* = -2 \Delta \theta_n^* - A^{-1}(\Delta_n + 1 - \Delta_n)
\]

The right hand side is a bounded and telescoping sequence; the bounds on \( \{n \xi_n, S_n^\xi\} \) easily follow. \( \square \)

The proof of the following lemma is routine.

**Lemma B.3.** The normalized error sequence evolves according to the recursion

\[
\Delta \xi_{n+1} = \frac{1}{n(n + 1)} ([I + A] \xi_{n-1} + [I + A] \Delta \xi_n + 2 \theta_n - \bar{\theta}_{n-1}) - (n + 1) \xi_{n+1}
\]

**Proof of Prop. 2.2 – deterministic case.** On summing each side of the identity in Lemma B.3 we obtain, for any \( n > m \geq 2 \),

\[
\xi_{n+1} - \xi_m = [I + A] \sum_{k=m}^{n} \frac{1}{k(k+1)} \xi_{k-1}
\]

\[
+ [I + A] \left( \xi_n - \xi_{m-1} + 2 \theta_n - \bar{\theta}_{m-1} \right) - \sum_{k=m}^{n} (k+1) \xi_{n+1}
\]

or

\[
\xi_{n+1} = [I + A] \left( (1 + 2n^{-1}) \xi_n + \sum_{k=m}^{n} \frac{1}{k(k+1)} \xi_{k-1} \right) + b_{n+1}^\xi
\]

where the final term is bounded in \( n \) for any fixed \( m \):

\[
b_{n+1}^\xi = \xi_m - (1 - 2(m - 1)^{-1})[I + A] \xi_{m-1} - \sum_{k=m}^{n} (k+1) \xi_{n+1}
\]

To prove that the sequence is bounded, let \( M > 0 \) and \( \delta \in (0, 1) \) satisfy the matrix inequality

\[
[I + A] M [I + A]^T \leq \delta M
\]

Let \( \|x\|_\infty := \sup_n \sqrt{x_n^\top M x_n} \).

Let \( \mathcal{H} \) denote the Banach space of sequences with finite norm

\[
\mathcal{H}(\mathbf{x}) = 0 \quad \text{for} \quad n \leq m, \quad \text{and for} \quad n > m
\]

\[
\mathcal{H}(\mathbf{x})_{n+1} = [I + A] \left( (1 + 2n^{-1}) \xi_n + \sum_{k=m}^{n} \frac{1}{k(k+1)} \xi_{k} \right) + b_{n+1}^\xi
\]

For sufficiently large \( m \), this is a contraction in \( \ell_\infty \): for some \( \varphi < 1 \) and any pair of bounded sequences,

\[
\| \mathcal{H}(\mathbf{x}) - \mathcal{H}(\mathbf{x}') \|_\infty := \sup_n \| \mathcal{H}(\mathbf{x})_n - \mathcal{H}(\mathbf{x}')_n \| \leq \varphi \| \mathbf{x} - \mathbf{x}' \|_\infty.
\]

It follows from the Banach fixed point theorem that the \( \{\xi_n\} = \mathcal{H}(\{\xi_n\}) \) is a bounded sequence. \( \square \)

13
B.2 Proof of Prop. 2.2 for the general linear algorithm

The major difference in the case of random \( \{A_n\} \) is the identity (38) holds with a modified error sequence:

\[
\Delta \tilde{\theta}_{n+1} = [I + A] \Delta \tilde{\theta}_n + \alpha_{n+1} A \tilde{\theta}_n - \xi_{n+1} + \alpha_{n+1} A_{n+1} \tilde{\theta}_n
\]

Lemma B.3 must be modified accordingly, and from this we obtain

\[
\xi_{n+1} = [I + A] (1 + 2n^{-1}) \xi_n + \sum_{k=m}^{n} \frac{1}{k(k+1)} \xi_{k-1} + b_{n+1}^i + M_{m,n}
\]

where for fixed \( m \), the sequence \( \{M_{m,n} : n \geq m\} \) is a martingale:

\[
\sum_{k=m}^{n} \frac{1}{k(k+1)} A_{k+1} \xi_k
\]

The proof then proceeds as in the previous deterministic setting.

C Variance analysis of the NeSA algorithm

Throughout this section it is assumed that the assumptions of Section 3.1 hold. We will initially impose an additional assumption:

(N3) The covariance sequence \( \{\Sigma_n\} \) defined in (25) is bounded.

This simplifies the proof of convergence. Following this proof, we explain how the same arguments leading to convergence can be elaborated to establish boundedness.

Even with under this assumption, the convergence proof appears complex. We first provide a proof for the simpler PolSA algorithm (20).

C.1 Variance analysis of PolSA

The recursion (20) is expressed in state space form as follows:

\[
\begin{bmatrix}
\hat{\theta}_{n+1} \\
\Delta \theta_{n+1}
\end{bmatrix} = \begin{bmatrix} I & (I + A) \\ 0 & (I + A) \end{bmatrix} \begin{bmatrix} \hat{\theta}_{n} \\
\Delta \theta_{n}
\end{bmatrix} + \alpha_{n+1} \begin{bmatrix} A & 0 \\ A & A \end{bmatrix} \begin{bmatrix} \hat{\theta}_{n} \\
\Delta \theta_{n}
\end{bmatrix} + \begin{bmatrix} \Delta_{n+1} \\
\Delta_{n+1}
\end{bmatrix}
\]

(39)

Using the Taylor series approximation:

\[
\sqrt{n+1} = \sqrt{n} + \frac{n}{2n} O\left(\frac{1}{n^{3/2}}\right),
\]

it follows that the process (24) evolves as

\[
\Phi_{n+1} = M \Phi_n + \alpha_{n+1} (B \Phi_n + \Delta_{n+1}^\Phi + \varepsilon_n^\Phi),
\]

(41)

involving the \( 2d \times 2d \) matrices \( M \) and \( B \), and the \( 2d \times 1 \) column vector \( \varepsilon_n^\Phi \):

\[
M := \begin{bmatrix} I & \frac{(I + A)}{\sqrt{n}} \\ \frac{A}{\sqrt{n}} & I + A \end{bmatrix}, \quad B := \begin{bmatrix} \frac{1}{2} I + A & 0 \\ 0 & I + A \end{bmatrix}, \quad \varepsilon_n^\Phi = \frac{1}{\sqrt{n}} \begin{bmatrix} O(\|\hat{\theta}_n\| + \|\theta_n\|) \end{bmatrix}
\]

(42)

Once again, the main step in the proof of Prop. 3.2 to obtain sharp results for the off-diagonal blocks of the covariance matrix. The proofs of the following lemmas are contained in Section C.2.

Lemma C.1. Under the conditions of Prop. 3.2, the following approximations hold for \( \Sigma_{n+1}^{22} \) and the scaled covariance \( \psi_n := \sqrt{n} \Sigma_{n+1}^{22} \):

\[
\Sigma_{n+1}^{22} = (I + A) \Sigma_n^{22} (I + A)^T + \Sigma^\Delta + O(\alpha_{n+1})
\]

\[
\psi_n = -\Sigma^{11} - A^{-1} \Sigma_{n+1}^{22} + o(1), \quad n \geq 1.
\]

\[
\Sigma_{n+1}^{11} = \Sigma^{11} + \alpha_{n+1} \left( \Sigma^{11} + A \Sigma^{11} + A^T \psi_n (I + A)^T + (I + A) \psi_n + \Sigma^\Delta + o(1) \right)
\]

(43)
Proof of Prop. 3.2. From Assumption A.1, which implies that the eigenvalues of matrix \((I + A)\) lie within the open unit disc, it follows that the first recursion in (43) of Lemma C.1 can be approximated by a geometrically stable discrete-time Lyapunov recursion with a time-invariant, bounded input \(\Sigma^\Delta\). The limit (30) directly follows.

Substituting the approximation for \(\psi_n\) in (43) into the right hand side of the \(\Sigma_{n+1}^{11}\) recursion in (43) gives

\[
\Sigma_{n+1}^{11} = \Sigma_{n}^{11} + \alpha_{n+1} \left( -\Sigma_{n}^{11} + A^{-1} \Sigma^\Delta (A^{-1})^T + o(1) \right)
\]

As before, this can be regarded as a Euler approximation to the ODE:

\[
\frac{d}{dt} x_t = -x_t + A^{-1} \Sigma^\Delta (A^{-1})^T
\]

The limits of \(\{\Sigma_n^{11}\}\) and \(\{x_t\}\) coincide with the stationary point \(-x^* + A^{-1} \Sigma^\Delta (A^{-1})^T = 0\).

C.2 Recursion approximations

The proof of Lemmas C.1, 3.3 and 3.4 are provided here. Recall that these results are first established under the assumption that the covariance sequence is bounded.

All of these results require the following simple consequence of the (N1):

Recall the definition of \(\Sigma_n^\Delta\) and \(\Sigma^\Delta\) in (6). We begin with the following consequence of the definitions:

Lemma C.2. Under Assumption (N1), the covariance \(\Sigma_n^\Delta\) satisfies:

\[
\Sigma_n^\Delta = \Sigma^\Delta + O(n^{-1} \text{trace } \Sigma_{n}^{11})
\]

Bounds for PolSA  From equations (41) and (25), the \(2d \times 2d\) matrix sequence \(\{\Sigma_n\}\) satisfies:

\[
\Sigma_{n+1} = M \Sigma_n M^T + \alpha_{n+1} (B \Sigma_n M^T + M \Sigma_n B^T + \Sigma_n^\Delta_{n+1}),
\]

where \(M\) and \(B\) are defined in (42), and \(\Sigma_n^\Delta = E[\Delta_n^\Phi (\Delta_n^\Phi)^T] \) is also a \(2d \times 2d\) matrix:

\[
\Sigma_n^\Delta := \begin{bmatrix} \Sigma_n^\Delta & \sqrt{n} \Sigma_n^\Delta \\ \sqrt{n} \Sigma_n^\Delta & n \Sigma_n^\Delta \end{bmatrix}
\]

with \(\Sigma_n^\Delta\) defined in (6).
Based on (25), it is simpler to view (46) as four parallel interdependent matrix recursions:

\[
\Sigma_{n+1}^{11} = \Sigma_n^{11} + \frac{\Sigma_n^{12}(I + A)^T}{\sqrt{n}} + \frac{(I + A)\Sigma_n^{21}}{\sqrt{n}} + \alpha_{n+1} \left( \left( \frac{1}{2}I + A \right)\Sigma_n^{11} + \frac{1}{2}I + A \right)^T + \frac{(I + A)\Sigma_n^{22}(I + A)^T + \Sigma^\Delta + \varepsilon_n^{11}}{\sqrt{n}}
\]

\[
\Sigma_{n+1}^{12} = \Sigma_n^{12}(I + A)^T + \frac{1}{n}(I + A)^T + \frac{1}{\sqrt{n}} \left( \Sigma_n^{11}A^T + (I + A)\Sigma_n^{22}(I + A)^T + \Sigma^\Delta \right) + \alpha_{n+1} \left( \left( \frac{1}{2}I + A \right)\Sigma_n^{11}A^T + \frac{3}{2}I + A \right)^T + \frac{(I + A)\Sigma_n^{22}(I + A)^T + \varepsilon_n^{12}}{\sqrt{n}}
\]

\[
\Sigma_{n+1}^{21} = \left( \Sigma_n^{12} \right)^T
\]

\[
\Sigma_{n+1}^{22} = (I + A)^T \Sigma_n^{22}(I + A)^T + \Sigma^\Delta + \frac{1}{n} A \Sigma_n^{11} A^T + \frac{1}{\sqrt{n}} \left( A \Sigma_n^{12}(I + A)^T + (I + A)\Sigma_n^{21}A^T \right) + \alpha_{n+1} \left( \frac{A \Sigma_n^{12}(I + A)^T}{\sqrt{n}} + \frac{(I + A)\Sigma_n^{21}A^T}{\sqrt{n}} + 2(I + A)\Sigma_n^{22}(I + A)^T + \varepsilon_n^{22} \right)
\]

in which the error terms satisfy the following:

\[
\varepsilon_n^{11} = O(n^{-1} \text{trace} \Sigma_n^{11}), \quad \varepsilon_n^{12} = O(n^{-1/2} \text{trace} \Sigma_n^{11}), \quad \varepsilon_n^{22} = O(\text{trace} \Sigma_n^{11})
\]

\[
(49)
\]

Proof of Lemma C.1. Under the boundedness assumption (N3), the first approximation in (43) follows from the recursion for \( \Sigma_{n}^{22} \) in (48). The stability condition (18) implies that this sequence is convergent, and the limit solves the Lyapunov equation

\[
\Sigma^\infty = (I + A)\Sigma^\infty (I + A)^T + \Sigma^\Delta
\]

A finer analysis shows that

\[
\Sigma_n = \Sigma^\infty + O(n^{-1} \text{trace} \Sigma_n^{11})
\]

We next prove that the second recursion in (43) holds. Multiplying both sides of the recursion for \( \Sigma_{n}^{21} \) in (48) with \( \sqrt{n+1} \), and using the Taylor series approximation (40), we obtain:

\[
\psi_{n+1} = A \Sigma_n^{11} + (I + A)\psi_n + (I + A)\Sigma_n^{22}(I + A)^T + \Sigma^\Delta + O(\alpha_{n+1}(1 + \|\psi_n\|))
\]

where once again we have used the assumption that \{\Sigma_n\} is a bounded sequence.

Substituting the approximation for \( \Sigma_n^{22} \) into the recursion (50):

\[
\psi_{n+1} = (I + A)\psi_n + (I + A)\Sigma^\infty (I + A)^T + \Sigma^\Delta + O(\|\Sigma_n^{12}\|n^{-1/2})
\]

where \( \varepsilon_n^\psi = o(\text{trace} \Sigma_n^{11} + \|\Sigma_n^{12}\|) \). Using the Bellman-Gronwall Lemma, it can be shown that the limit supremum of \( \{\|\psi_n\|\} \) coincides with the sequence obtained with the final term removed:

\[
\psi_{n+1}^\psi = (I + A)\psi_n^\psi + u(n)
\]

\[
u(n) = A \Sigma_n^{11} + \Sigma^\infty + \varepsilon_n^\psi
\]

where

Expressed as a convolution,

\[
\psi_{n+k} = (I + A)^k \psi_n + \sum_{j=0}^{k-1} (I + A)^j u(n + k - 1 - j)
\]

16
The next step is to replace \( u(n + k - 1 - j) \) with \( u(n + k) \), and bound the error \( \{ \varepsilon(n, k) \} \):

\[
\psi_{n+k}^\circ = (I + A)^k \psi_n^\circ + \sum_{j=0}^{k-1} (I + A)^j u(n + k) + \varepsilon(n, k),
\]

where:

\[
\varepsilon(n, k) = \sum_{j=0}^{k-1} (I + A)^j \left( u(n + k - 1 - j) - u(n + k) \right).
\]

From the recursion for \( \Sigma_n^{11} \) in (48), it follows that for some constant \( c_0 < \infty \):

\[
\| \Sigma_n^{11} - \Sigma_n^{11} \| \leq c_0 \ln \left( \frac{n + j + 1}{n} \right)
\]

It follows that for \( c < \infty \), the input sequence \( \{ u(n) \} \) satisfies:

\[
\| u(n + j) - u(n) \| \leq c \left( \frac{j + 1}{n} \right), \quad k \geq 0, \quad n \geq 1.
\]

Using this in the expression for \( \varepsilon(n, k) \), we obtain,

\[
\| \varepsilon(n, k) \| \leq \sum_{j=0}^{k-1} \| (I + A)^j (u(n + k - 1 - j) - u(n + k)) \|
\]

\[
\leq c \sum_{j=0}^{k-1} \left\| (I + A)^j \left( \frac{j + 2}{n + k - 1 - j} \right) \right\|
\]

\[
\leq c \sum_{j=0}^{\infty} \| (j + 2)(I + A)^j \|
\]

\[
\leq \frac{c}{n} \sum_{j=0}^{\infty} \| (j + 2)(I + A)^j \|
\]

This together with the eigenvalue bound for \( (I + A) \) in (18) gives

\[
\lim_{n \to \infty} \sup_k \| \varepsilon(n, k) \| = 0.
\]

Using this in (52), we have:

\[
\psi_{n+k}^\circ = (I + A)^k \psi_n^\circ + \sum_{j=0}^{k-1} (I + A)^j u(n + k)
\]

\[
= \sum_{j=0}^{\infty} (I + A)^j u(n + k) + o(1) + O(\rho^k),
\]

where \( 0 < \rho < 1 \). That is,

\[
\psi_n^\circ = -A^{-1} u(n) + o(1)
\]

Substituting for \( u(n) \) from (51) gives the desired result:

\[
\psi_n^\circ = -\Sigma_n^{11} - A^{-1} \Sigma_{\infty}^{22} + o(1)
\]

The final approximation for the \( \{ \Sigma_n^{11} \} \) recursion in (43) follows by substituting \( \psi_n = \sqrt{n} \Sigma_n^{21} \) and \( \psi_n^T = \sqrt{n} \Sigma_n^{12} \) in the recursion for \( \Sigma_n^{11} \) in (48) and then using (43).

**Boundedness of the covariance** Lengthier arguments show that without the boundedness assumption we obtain a modified version of (44):

\[
\Sigma_{n+1}^{11} = \Sigma_n^{11} + \alpha_{n+1} \left( -\Sigma_n^{11} + A^{-1} \Sigma_{\infty}^{11} \right) + o(1 + \text{trace} (\Sigma_n^{11}))
\]

This is enough to obtain the desired limit for \( \{ \Sigma_n^{11} \} \). Boundedness of this sequence implies boundedness of the remaining covariance components.
The matrix $H$ taking expectations results in the interdependent matrix recursions:

Similar to (39), based on (22) and (5), the pair of sequences

The solution to (27) is thus

Using the Taylor series (40) we again obtain a state space recursion for the normalized error sequences:

Bounds for NeSA

It is very complex to keep track of every error term – to simplify exposition we impose (N3), and write $o(1)$ in place of $o(1 + \text{trace} (\Sigma_i^o))$ for any $i, j$.

Assumption (N2) ensures that the following representation is well defined:

The matrix $H = [I - L]^{-1}(Q)$ solves the Lyapunov equation $H = L(H) + Q$ for any matrix $Q$.

The solution to (27) is thus

Similar to (39), based on (22) and (5), the pair of sequences $\{\hat{\theta}_n\}$ and $\{\Delta \theta_n\}$ of the NeSA algorithm satisfy the following recursion:

Using the Taylor series (40) we again obtain a state space recursion for the normalized error sequences:

To begin the covariance analysis, consider first the outer-product without expectation:

Then, based on (55), the $2d \times 2d$ matrix sequence $\{\hat{\Sigma}_n\}$ satisfies the following recursion:

Taking expectations results in the interdependent matrix recursions:

Taking expectations results in the interdependent matrix recursions:

Taking expectations results in the interdependent matrix recursions:
where $\Sigma_n^W := \mathbb{E}[(I + A_{n+1})\Phi_n^1\Delta_{n+1}]$, and $\ell_n^{22} = \mathbb{E}[\mathcal{L}(\hat{\Sigma}_n^{22})]$ with $\mathcal{L}$ is defined in (23). Linearity implies that expectation and operation can be interchanged:

$$\ell_n^{22} = \mathbb{E}[(I + A_{n+1})Q\hat{\Delta}_{n+1}^T] = \mathbb{E}[\mathcal{L}(\hat{\Sigma}_n^{22})]$$

Two more linear operators are required in the following: For any $Q \in \mathbb{R}^{d \times d}$ define

$$\hat{\mathcal{L}}(Q) := \mathbb{E}[(I + A_{n+1})Q\hat{\Delta}_{n+1}^T]. \quad (60)$$

The second operator maps vectors to matrices: for any $v \in \mathbb{R}^d$,

$$\mathcal{M}(v) = \mathbb{E}[(I + A_{n+1})(\hat{\Delta}_{n+1} + \hat{\Delta}_{n+1}\theta^*)^T] \quad (61)$$

The following result will be used to show that $\{\Sigma_n^W\}$ converges to 0:

**Lemma C.3.** Under (N1)–(N3) we have $\lim_{n \to \infty} \Sigma_n^W = 0$.

**Proof.** Using the definition of $\Delta_n$ in (6) gives

$$\Sigma_n^W = \mathbb{E}[(I + A_{n+1})\Phi_n^1\hat{\Delta}_{n+1}^T] = \mathbb{E}[(I + A_{n+1})\Phi_n^1(\hat{\Delta}_{n+1} + \hat{\Delta}_{n+1}\theta^*)^T]$$

where the last equality follows from (N1). Linearity of $\hat{\mathcal{L}}$ and $\mathcal{M}$ then implies

$$\Sigma_n^W = \frac{1}{\sqrt{n}}\hat{\mathcal{L}}(\mathcal{E}[\hat{\Sigma}_n^{21}]) + \mathcal{M}(\Phi_n^1)$$

$$= \frac{1}{\sqrt{n}}\hat{\mathcal{L}}(\Sigma_n^{21}) + \mathcal{M}(\Phi_n^1)$$

$$= \frac{1}{n}\hat{\mathcal{L}}(\psi_n) + \mathcal{M}(\Phi_n^1)$$

where the last equality used the definition $\psi_n = \sqrt{n}\Sigma_n^{21}$. The first term vanishes under (N3) since $\psi_n/n = O(n^{-1/2})$. It remains to show that the second term converges to 0. Based on (55) it is straightforward to establish the following limit under (N1)–(N2):

$$\lim_{n \to \infty} \mathbb{E}[\Phi_n^1] = 0$$

Using the definition (61), and taking expectations completes the proof that $\mathbb{E}[\mathcal{M}(\Phi_n^1)] = o(1)$. \qed

**Proof of Lemma 3.3.** We first prove the first recursion in (31). Based on the assumption that each $\Sigma_1, \Sigma_2$ and $\psi_n = \sqrt{n}\Sigma_2$ in (59) are bounded, the recursion for $\Sigma_n^{22}$ can be written as

$$\Sigma_n^{22} = \mathcal{L}(\Sigma_n^{22}) + \Sigma_n^W + \Sigma_n^{W^T} + O(\alpha_{n+1})$$

where the $O(\alpha_{n+1})$ term includes all terms in the recursion that are multiplied with $\alpha_{n+1}$. Lemma C.3 then implies the first recursion in (31),

$$\Sigma_n^{22} = \mathcal{L}(\Sigma_n^{22}) + \Sigma_n^W + O(1/\sqrt{n})$$
This is regarded as a Lyapunov recursion with time varying forcing term $\Sigma^\Delta + O(1/\sqrt{n})$. Under (N2) convergence follows, giving (27).

We next prove that the second recursion in (31) holds. Multiplying both sides of the recursion for $\Sigma_n^{21}$ in (59) by $\sqrt{n}+1$, and using the Taylor series approximation (40), we obtain:

$$\psi_{n+1} = (I + A)\psi_n + A\Sigma_n^{11} + L(\Sigma_n^{22}) + \Sigma_n^\Delta + \Sigma_n^W + \Sigma_n^{WT} + O(\alpha_{n+1}),$$

where once again, the $O(\alpha_{n+1})$ terms are due to the boundedness assumption on $\Sigma_n^{11}, \Sigma_n^{22}$ and $\psi_n$. From Lemma C.3, we have $\Sigma_n^W \to 0$, and furthermore, using $\Sigma_n^{22} = \Sigma_n^{22} + o(1)$,

$$\psi_{n+1} = (I + A)\psi_n + A\Sigma_n^{11} + L(\Sigma_n^{22}) + \Sigma_n^\Delta + o(1)$$

Once we have the above form, the rest of the proof follows steps exactly same as the proof of Lemma C.1: By viewing the recursion as state evolution of a discrete-time stable linear system with bounded input sequence $\{A\Sigma_n^{11} + \Sigma_n^{22}\}$ and vanishing additive noise, we can show that $\{\psi_n\}$ satisfies the second recursion in (31).

$\square$

**Proof of Lemma 3.4.** The Lemma follows directly by substituting $\psi_n = \sqrt{n}\Sigma_n^{21}$ and $\psi_n^T = \sqrt{n}\Sigma_n^{21}$ in the recursion for $\Sigma_n^{11}$ in (59) and then using (31).

### D NeSA and PolSA TD-learning algorithms

In this section of the Appendix, we briefly give details on how to apply the algorithms introduced in this paper to solve value-function estimation problems in Reinforcement Learning. For simplicity, we consider the TD($\lambda$)-learning algorithm with $\lambda = 0$. Extension to $\lambda \in (0, 1]$ is straightforward.

Let $\{P^n\}$ denote the transition semigroup for the Markov chain $X$: For each $n \geq 0$, $x \in X$, and $A \in B(X)$ (where $B(\cdot)$ denotes the Borel set),

$$P^n(x, A) := P_x \{X_n \in A\} := \Pr\{X_n \in A \mid X_0 = x\}.$$

The standard operator-theoretic notation is used for conditional expectation: for any measurable function $f: X \to \mathbb{R}$,

$$P^n f(x) = E_x[f(X_n)] := E[f(X_n) \mid X_0 = x].$$

In a finite state space setting, $P^n$ is the $n$-step transition probability matrix of the Markov chain, and the conditional expectation appears as matrix-vector multiplication:

$$P^n f(x) = \sum_{x' \in X} P^n(x, x') f(x'), \quad x \in X.$$

Let $c: X \to \mathbb{R}_+$ denote a cost function, and $\beta \in (0, 1)$ a discount factor. The discounted-cost value function is defined as

$$h(x) = \sum_{n=0}^\infty \beta^n P^n c(x), \quad x \in X.$$

It is known that the value function is the unique solution to the Bellman equation

$$c(x) + \beta Ph(x) = h(x) \quad (62)$$

Consider the case of a $d$-dimensional linear parameterization: A function $\psi: X \to \mathbb{R}^d$ is chosen, which is viewed as a collection of $d$ basis functions. Given a parameter vector $\theta \in \mathbb{R}^d$, the corresponding approximation to the value function is defined as:

$$h^\theta(x) = \sum_i \theta_i \psi_i(x) = \theta^T \psi(x)$$

The goal of TD-learning is to approximate the solution to (62) by $h^\theta(x)$ [30, 33]. In particular, the TD(0) algorithm intends to solve the Galerkin relaxation of the problem [12, 32]: Find $\theta^*$ such that

$$0 = E\left[\left(-h^\theta(X_n) + c(X_n) + \beta h^\theta(X_{n+1})\right)\psi_i(X_n)\right], \quad 1 \leq i \leq d,$$

(63)
where the expectation is with respect to the steady state distribution of the Markov chain. This model is of the form considered in Theorem 2.1, with Markov chain defined by $X_n = (X_n, X_{n-1})$, and $\overline{f}(\theta) = A\theta - b$, with $A := E[A_n]$, $b := E[b_n]$ (expectations in steady-state), and

$$A_n := \psi(X_{n-1})(\beta\psi(X_n) - \psi(X_{n-1}))^T, \quad b_n := -\psi(X_n)c(X_n) \quad (64)$$

The TD(0) algorithm is stochastic approximation in the form (2), with $G_n \equiv I$.

**TD(0) algorithm:** For initialization $\theta_0 \in \mathbb{R}^d$, the sequence of estimates are defined recursively:

$$\begin{align*}
\theta_{n+1} &= \theta_n + \alpha_{n+1}\psi(X_n) d_{n+1} \\
\alpha_{n+1} &= c(X_n) + \left[\beta\psi(X_{n+1}) - \psi(X_n)\right]^T \theta_n
\end{align*} \quad (65)$$

The sequence $\{d_n\}$ also appears in the algorithms described next. The parameter recursion can be expressed in the more suggestive form

$$\theta_{n+1} = \theta_n + \alpha_{n+1}\left[A_{n+1}\theta_n - b_n\right]$$

The LSTD algorithm of [7] is a stochastic approximation algorithm of the form (2), with $G_n$ equal to a Monte-Carlo estimate of $-A^{-1}$ [12].

**LSTD(0) algorithm:** For initialization $\theta_0 \in \mathbb{R}^d$, the sequence of estimates are defined recursively:

$$\begin{align*}
\theta_{n+1} &= \theta_n - \alpha_{n+1}\overline{A}_{n+1}\theta_n d_{n+1} \\
\alpha_{n+1} &= c(X_n) + \left[\beta\psi(X_{n+1}) - \psi(X_n)\right]^T \theta_n \\
\overline{A}_{n+1} &= \overline{A}_n + \gamma_{n+1}[A_{n+1} - \overline{A}_n]
\end{align*} \quad (66)$$

The non-negative gain sequence $\{\gamma_n\}$ is an ingredient in the “Zap” algorithms of [11, 12], where it is assumed to satisfy standard assumptions, but is relatively large:

$$\sum_{n=1}^{\infty} \gamma_n = \infty, \quad \sum_{n=1}^{\infty} \gamma_n^2 < \infty, \quad \lim_{n \to \infty} \frac{\gamma_n}{\alpha_n} = \infty$$

There is a single gain sequence in the LSTD(0) algorithm, $\alpha_n \equiv \gamma_n$, and in this case the matrix recursion is equivalent to the simple average:

$$\overline{A}_n = \frac{1}{n} \sum_{i=1}^{n} A_i = \frac{1}{n} \sum_{i=1}^{n} \psi(X_{i-1})\left[\beta\psi(X_i) - \psi(X_{i-1})\right]^T$$

The PolSA and NeSA algorithms for TD(0)-learning are given as follows. In each case, the initialization $\theta_0 \in \mathbb{R}^d$, and gain $\zeta$ satisfying (18), are pre-specified.

**PolSA TD(0) algorithm:**

$$\begin{align*}
\theta_{n+1} &= \theta_n + (I + \zeta\overline{A}_{n+1})\Delta\theta_n + \alpha_{n+1}\zeta\psi(X_n) d_{n+1} \\
\alpha_{n+1} &= c(X_n) + \left[\beta\psi(X_{n+1}) - \psi(X_n)\right]^T \theta_n
\end{align*} \quad (67)$$

**NeSA TD(0) algorithm:**

$$\begin{align*}
\theta_{n+1} &= \theta_n + (I + \zeta\overline{A}_{n+1})\Delta\theta_n + \alpha_{n+1}\zeta\psi(X_n) d_{n+1} \\
\alpha_{n+1} &= c(X_n) + \left[\beta\psi(X_{n+1}) - \psi(X_n)\right]^T \theta_n
\end{align*} \quad (68)$$

The matrix momentum term, highlighted in red, is the only difference between the two algorithms.