PERFECT FORMS AND THE MODULI SPACE OF ABELIAN VARIETIES

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Toroidal compactifications of the moduli space $A_g$, or the stack $A_g$, of principally polarized abelian $g$-folds have been constructed over $\mathbb{C}$ in [AMRT] and over any base in [FC]. Roughly speaking, each such compactification corresponds to choosing a way of decomposing the cone of real positive quadratic forms in $g$ variables. The choice made here is the perfect cone decomposition, also called the first Voronoi decomposition, which leads to the compactification $A_{gF}$. This carries divisor classes $M$, the line bundle of weight 1 modular forms, and $D$, the reduced boundary $A_{gF} \setminus A_g$. (Sometimes, but not in this paper, $M$ is denoted instead by $\omega$.) It is easy to see that $D$ is geometrically irreducible, and it follows ([M] or [F2]) that the classes $M$ and $D$ generate $\text{NS}(A_{gF}) \otimes \mathbb{Q}$. Here is the first main result of this paper, where a divisor class $E$ on a projective variety $X$ is nef if $E.C \geq 0$ for all curves $C$ on $X$:

**Theorem 0.1** $aM - D$ is nef if and only if $a \geq 12$ and ample if and only if $a > 12$.

This extends the the picture when $g = 1$, where there is a unique cusp form of weight 12 and level 1, the discriminant, and it has no zeroes away from the cusp. A better generalization would be the statement that $12M - D$ eventually has no base points (that is, the complete linear system $|m(12M - D)|$ has no base points for $m >> 0$), but I can’t prove that except when $g \leq 11$ and the base is $\mathbb{C}$.

One consequence is that the cone of curves $\overline{\text{NE}}(A_{gF})$ is the closed convex cone generated by curves $C_1, C_2$, where $C_1$ is the closure of the set of points $B \times E$, where $B$ is a fixed principally polarized abelian ($g - 1$)-fold and $E$ is a variable elliptic curve, and $C_2$ is any exceptional curve of the contraction $A_{gF} \to A_{gSat}$. Note that the properties of $\Delta$ just described appear in this context as the formula $(12M - D).C_1 = 0$. Another consequence is that for any value of $a > 12$, the graded ring $\bigoplus_{n \geq 0} H^0(A_{gF}Z, O(n(aM - D)))$ of Siegel modular forms of weight $an$, vanishing along the principal cusp to order at least $n$ (that is, of slope $a$) and with Fourier coefficients in $\mathbb{Z}$ is finitely generated over $\mathbb{Z}$.

The second main result derives from 0.1. It concerns the nature of $A_g$, now over $\mathbb{C}$, in the context of arbitrary quasi-projective varieties.

If $X$ is a quasi-projective complex algebraic variety of general type, then its canonical model (in the sense of Mori and Reid, not of Shimura) $X_{can}$ is a normal complex projective variety, birationally equivalent to $X$, with canonical singularities and ample canonical class. It is known that $X_{can}$ exists if the dimension of $X$ is at most 3, since this is when flips are known to exist and terminate.
However, even when the canonical model is known to exist it is not always easy to find or describe explicitly, even for such concrete examples as Hilbert modular surfaces [vdG]. The existence of $X_{can}$ is equivalent to the canonical ring $igoplus H^0(\tilde{X}, \mathcal{O}(nK_{\tilde{X}}))$ of $X$ being of finite type for some, or any, smooth projective model $\tilde{X}$ of $X$. In any case, this ring is determined uniquely by the function field $\mathbb{C}(X)$ of $X$. There is also a local version of this notion: if $X$ is a normal complex variety, then its relative canonical model is a proper birational morphism $f : Y \to X$ such that $Y$ has canonical singularities and $K_Y$ is ample relative to $f$. Again, this is uniquely determined by $X$ if it exists.

Freitag [F1], Tai [T] and Mumford [M] have shown that, over $\mathbb{C}$, the coarse moduli space $A_g$ of principally polarized abelian $g$-folds, is of general type when $g \geq 7$.

**Theorem 0.2**

1. $A^F_g$ is the relative canonical model of the Satake-Baily-Borel compactification $A^\text{Sat}_g$ when $g \geq 5$.
2. When $g \geq 12$, the canonical model of $A_g$ exists and equals $A^F_g$.
3. $A_{11, \text{can}}$ exists and arises as the contraction of a certain extremal ray in the cone of curves $NE(A^F_{11})$.

Deriving 0.2 from 0.1 requires the local statement that $A^F_g$ has canonical singularities over $\mathbb{C}$ when $g \geq 5$. For $A_g$ this is due to Tai [T]; there are hints given there that this is true for $A^F_g$, but it seemed to be worth making it explicit. From the formula $K_{A^F_g} = (g+1)M - D$, the ampleness of $K_{A^F_g}$ for $g \geq 12$ is immediate.

Namikawa [N] has already raised the question of the geometric meaning of the first Voronoi compactification (which is also known as the perfect cone compactification). Alexeev [A] has shown that, over any base, the second Voronoi compactification $A^\text{Second}_g$ has an interpretation as a moduli space of generalized principally polarized abelian varieties; more precisely, he defines and solves a moduli problem for generalized ppav’s, shows that the moduli space is proper and then shows that the second Voronoi compactification is an irreducible component of this space. By contrast, there is no reason to believe that $A^F_g$ is a moduli space; it is not even clear that there is an equidimensional family of projective schemes over it that extends the universal family of abelian varieties. Another thing is that the natural birational equivalence between $A^F_g$ and $A^\text{Second}_g$ is regular in neither direction, because $[ER]$ when $g \geq 6$ neither of the two Voronoi decompositions is a refinement of the other.

Hulek and Sankaran [HS02] raise the issue of describing the nef cone of $A^\text{Second}_g$. This seems much harder. In [HS04] they have already described the nef cones of $A^F_g$ and $A^\text{Second}_g$ for $g \leq 4$ and, which is most relevant to this paper, the nef cone of Mumford’s partial compactification $A^\text{part}_g$, the open subvariety lying over the open subvariety $A_g \bigsqcup A_{g-1}$ of $A^\text{Sat}_g$. That is, they deal with complete curves in $A^\text{part}_g$. (This inverse image is the same for all toroidal compactifications.)
I am grateful to Tom Fisher for his help in elucidating one of Tai’s calculations and to Klaus Hulek for his valuable remarks.

1 Curves on the first Voronoi compactification.

Fix a copy $X_g$ of $\mathbb{Z}^g$. Denote by $X_g^\vee$ its dual and $B(X_g)$ the lattice of symmetric bilinear $\mathbb{Z}$-valued forms on $X_g$. Let $\overline{C}(X_g)$ denote the cone in $B(X_g) \otimes \mathbb{R}$ of positive semi-definite forms with rational radical. According to [FC], for every ring $R$, any basic $GL(X_g)$-admissible decomposition of $\overline{C}(X_g)$ defines a smooth Deligne–Mumford stack $\tilde{A}_{g,R}$ over $\text{Spec} R$ that is a toroidal compactification of $A_{g,R}$, such that $\tilde{A}_{g,R} = A_g \otimes_\mathbb{Z} R$. It also defines a toroidal compactification $\tilde{A}_{g,R}$ of the coarse moduli space $A_{g,R}$ over $\text{Spec} R$. (When $R = \mathbb{Z}$ it is omitted.) The Satake compactification $A_{Sat,g,R}$ of $A_{g,R}$ is constructed in [FC] as a blowing down of $\tilde{A}_{g,R}$. However, as Alexeev remarks [A], it follows from this and the general principles of torus embeddings that any $GL(X_g)$-admissible decomposition of $\overline{C}(X_g)$ defines a toroidal compactification over $\text{Spec} R$. (Alternatively, note that the only use made in [FC] of the assumption that the decomposition be basic is to ensure that the sheaf of 1-forms with logarithmic poles along the boundary is locally free; they use this to analyze the Kodaira-Spencer maps of the semi-abelian schemes that they construct. Since this sheaf is locally free for any toroidal scheme over $\text{Spec} R$, the arguments of [FC] carry over.) Moreover, when $n$ is an integer such that $R$ contains $1/n$ and a primitive $n$th root of unity, there are corresponding compactifications of the moduli objects $A_{g,n}$ and $A_{g,n}$ for ppav’s with full level $n$ structure.

From the viewpoint of constructing a toroidal resolution of $A_{g,R}$, the choice of resolution over the 0-dimensional cusp determines what happens over all cusps, in that a choice of admissible decomposition of $\overline{C}(X_g)$ determines an admissible decomposition of $\overline{C}(X_r)$ for every quotient $X_g \twoheadrightarrow X_r$. There is a unique cusp (a copy of $A_{g,1}$) in $A_{g,R}$ of maximal dimension and over this cusp there is a unique exceptional divisor $D$ in $\tilde{A}$, which is generically the universal Kummer variety of dimension $g - 1$. This divisor corresponds to the primitive rank 1 forms in $\overline{C}(X_g)$, which are all equivalent under $GL(X_g)$.

**Theorem 1.1** (1) Taking the cones over the faces of the convex hull of $(B(X_g) - \{0\}) \cap \overline{C}(X_g)$ provides a $GL(X_g)$-admissible decomposition of $\overline{C}(X_g)$.

(2) The convex hull of $(B(X_g) - \{0\}) \cap \overline{C}(X_g)$ equals the convex hull of the set of primitive rank 1 forms in $\overline{C}(X_g)$.

**Proof:** For (1) see pp. 144-150 of [AMRT]. In their notation, this decomposition is provided by the perfect co-core. Part (2) is the main result of [BC]; in fact, they prove that every positive semi-definite form of rank at least two is in the interior of the convex hull of the set of primitive rank one forms. □
Following Namikawa [N] we shall refer to this decomposition of \( \overline{C}(X_g) \) as the first Voronoi decomposition or the perfect cone decomposition and the resulting compactification of \( A_g \), or of any finite level cover \( A_{g,n} \) of it or of the stack \( A_g \), as the first Voronoi compactification or the perfect cone compactification, denoted by \( A_g^F \) or \( A_{g,n}^F \). The reason is that \( B(X_g)^\vee \) is naturally isomorphic to the lattice \( Q(X_g) = \text{Sym}^2(X_g) \) of quadratic forms on \( X_g^\vee \), so that the maximal faces of this decomposition are level sets of some element \( q \) of \( Q(X_g) \) that is unique up to a scalar. By definition, and the Barnes-Cohn result, these forms \( q \) are exactly the perfect ones.

By the construction given in [AMRT] or [FC], over a 0-dimensional cusp, \( A_g^F \) is formally isomorphic to the quotient stack \([\overline{E}/GL(X_g)]\), where \( \overline{E} \) is the locally finite torus embedding defined by taking \( \mathbb{X}_*(E) = B(X_g) \) and then choosing the first Voronoi decomposition of \( \overline{C}(X_g) \). Over an arbitrary cusp, \( A_g^F \) is formally isomorphic to \([\overline{\Xi}/GL(X_{g-h})]\), where \( X_{g-h} \) is a rank \( g-h \) quotient lattice of \( X_g \), \( E_1 \hookrightarrow \overline{E} \) is the torus embedding described by replacing \( X_g \) by \( X_{g-h} \) in the description of \( E \hookrightarrow \overline{E} \) and \( \overline{\Xi} \) is the contracted fibre product \( \overline{E}_1 \times_{E_1} \overline{\Xi} \), where \( \overline{\Xi} \) is a certain \( E_1 \)-torsor over the \((g-h)\)-fold fibre product of the universal principally polarized abelian scheme \( U_h \to A_h \). We shall recall further details of this construction later on.

**Theorem 1.2** Assume that \( g \geq 2 \).

1. The contraction \( \pi : A_g^F \to A_g^{Sat} \) has a unique exceptional divisor \( D \). For every prime \( p \), \( D \otimes \mathbb{F}_p \) is absolutely irreducible.

2. \(-D\) is ample relative to \( \pi \).

**PROOF:** The exceptional divisors correspond to the \( GL(X_g) \)-orbits of the vectors that form the 1-skeleton of the decomposition. The vectors in the 1-skeleton are the primitive rank 1 forms, which are \( GL(X_g) \)-equivalent.

The projectivity criterion of [FC], V.5, shows that the inverse image of \(-D\) on \( A_{g,n}^1 \) is ample relative to the contraction \( A_{g,n}^1 \to A_{g,n}^{Sat} \) for any \( n \geq 3 \). The result follows for \( A_g^F \to A_g^{Sat} \) by taking quotients.

The passage from a Deligne–Mumford stack to its geometric quotient [KM] does not always commute with base change. The next lemma spells this out.

**Lemma 1.3** For any toroidal compactification \( \tilde{A}_g \) as above and any field \( k \), there is a natural morphism \( \tilde{A}_{g,k} \to \tilde{A}_Z \otimes k \) that is birational and finite.

**PROOF:** This is an immediate consequence of the fact that for any \( g \) and over any algebraically closed field there are ppav’s whose automorphism groupscheme is precisely \( \mathbb{Z}/2\mathbb{Z} \) (for example, take the Jacobian of a curve with trivial automorphism groupscheme). So the natural morphism \( \tilde{A}_{g,k} \to \tilde{A}_Z \otimes k \), which certainly exists, is an isomorphism over the locus where the automorphism groupscheme is \( \mathbb{Z}/2\mathbb{Z} \). ■
Theorem 1.4 For every field \( k \) the variety \( A^F_g \otimes k \) is \( \mathbb{Q} \)-factorial and the \( \mathbb{Q} \)-vector spaces \( \text{Pic}(A^F_g \otimes k) \otimes \mathbb{Q} \) and \( \text{Pic}(A^F_g \otimes k) \otimes \mathbb{Q} \) are 2-dimensional and generated by the classes \( M \) and \( D \).

PROOF: It follows from Corollary 1.6 of [M] that the natural homomorphism \( H^2(A^F_{g,C}, \mathbb{Q}) \to \text{Pic}(A^F_{g,C}) \otimes \mathbb{Q} \) is an isomorphism, and the result then holds for any field of characteristic zero.

In general, choose a prime \( l \) different from \( \text{char} \ k \). We can assume \( k \) to be algebraically closed. Then the specialization map

\[
H^2(A^F_g \otimes \overline{\mathbb{Q}}, \mathbb{Q}_l(1)) \to H^2(A^F_g \otimes k, \mathbb{Q}_l(1))
\]

is surjective. By 1.3, the natural homomorphism

\[
H^2(A^F_g \otimes k, \mathbb{Q}_l(1)) \to H^2(A^F_{g,k}, \mathbb{Q}_l(1))
\]

is an isomorphism. Since \( \rho(A^F_{g,k}) \geq 2 \), from the existence of a contractible divisor, we are done.

From now on we shall be careless in distinguishing between \( A^F_g \otimes k \) and its normalization \( A^F_{g,k} \). The justification for this is 1.3.

Now suppose that the base is a field \( k \). Pick any curve \( C_2 \subset D \) that is contracted to a point in \( A^F_{g,Sat} \). Fix a principally polarized abelian \((g - 1)\)-fold \( B \) and let \( C_1 \) be the closure in \( A^F_g \) of the curve \( \{ B \times E \} \), where \( E \) is a varying elliptic curve and \( B \times E \) is given the product principal polarization. The rest of this section is devoted to showing that these curves generate the cone \( \overline{NE}(A^F_g) \).

Lemma 1.5 \( M.C_1 = 1/12, M.C_2 = 0, D.C_1 = 1 \) and \( D.C_2 < 0 \).

PROOF: The formulae \( M.C_1 = 1/12 \) and \( D.C_1 = 1 \) follow from the existence of the discriminant in the case \( g = 1 \). The other formulae are obvious.

Lemma 1.6 Suppose that \( f : X \to Y \) is a non-constant morphism of normal projective varieties over a field and that \( C \) is a reducible curve in \( X \) that is contracted to finitely many points by \( f \). Then for every component \( C' \) of \( C \) the ray \( \mathbb{R}_+[C'] \) lies in the boundary \( \partial \overline{NE}(X) \) of \( \overline{NE}(X) \).

PROOF: Choose an ample divisor class \( H \) on \( Y \). Then \( \mathbb{R}_+[C'] \) lies in \( \overline{NE}(X) \cap (f^*H)^\perp \), which is contained in \( \partial \overline{NE}(X) \).

Lemma 1.7 \( C_2 \) lies in \( \partial \overline{NE}(A^F_{g,k}) \).

PROOF: This follows from 1.6 and the choice of \( C_2 \) as a curve that is contracted by the morphism \( A^F_{g,k} \to A^F_{g,Sat} \).
2 $12M - D$ is nef

In this section the base is a field $k$; there is no loss of generality if this is taken to be algebraically closed.

To show that a $\mathbb{Q}$-divisor class $E$ on a projective variety is nef, it is enough to show that $E$ is nef when restricted to the base locus of $|nE|$ for some $n$. Here is a sketch of how we do this for $12M - D$ on $A_g^F$.

(1) A consideration of Kummer varieties shows that the base locus of the linear system $|m(12M - D)|$ lies over the copy of $A_g^{Sat}$ in $A_g^{Sat}$.

(2) The irreducible components $Z$ of the part of $D$ lying over cusps of higher codimension are closures of proper bundles $Z^0 \to U^r_{g-r}$, where $U^r_{g-r}$ is the $r$-fold fibre product of the universal abelian scheme over $A_{g-r}$. The fibre of each $Z^0 \to U^r_{g-r}$ is the boundary of some torus embedding, and the action of this torus extends to $Z$. Then, via the Borel fixed point theorem, the torus action pushes the Chow point $[C]$ of a negative curve into the Chow point $[C_0]$ of a curve in some boundary stratum that corresponds to some cone $\sigma$ that meets the interior $C(X_r)$ of $\mathcal{C}(X_r)$, in such a way that $[C] - [C_0]$ is effective and is supported on curves $\Gamma$ that are contracted to points in $A_g^{Sat}$. Since $M.\Gamma = 0$ and $D.\Gamma < 0$, by 1.22, we can replace $C$ by $C_0$. Moreover, from the nature of $\sigma$, this boundary stratum is a closure of a copy of $U^r_{g-r}$.

(3) The normalized closure $\tilde{U}$ of this copy $U$ of $U^r_{g-r}$ is, after deleting a closed subset of codimension 2 (here the argument depends on the fact that we’re dealing with the perfect compactification), a semi-abelian scheme $U^r_{g-r, part} \to A^r_{g-r, part}$. So when we pull $12M - D$ back to $\tilde{U}$, it will follow that any negative curve in $\tilde{U}$ will lie in $\tilde{U} \setminus U$ provided that $m(12M - D)$ when restricted to the open $U$ has no base points.

(4) By considering various projections $U^r_{g-r, part} \to U^r_{g-r}$ that are determined by the geometry of the cone $\sigma$ (here again we rely on having the perfect compactification) we reduce the problem of proving that the restriction of $m(12M - D)$ to $U$ has no base points to what was proved in (1) and conclude by induction.

Write $D_g$ or $\mathcal{D}_g$ for the (irreducible) boundary divisor in $A_g^F$ or $A_g^{Sat}$.

Theorem 2.1 Suppose that the base is an algebraically closed field $k$. For some $m > 0$ the linear system $|m(12M - D_g)|$ has no base points in $A_g^{part}$.

Proof: First, assume that char $k$ is not 2. We shall work at level 2.

The inverse image $D_g^{part}$ in $A_g^{part}$ of $A_{g-1}$ is the locus of (torus) rank 1 degenerations. (In fact, $A_g^{part}$ and $A_g^{Sat}$ are independent of the choice of toroidal compactification.) The universal abelian scheme over $A_g$ extends to an equivariant relative compactification $\pi : U_g \to A_g^{part}$ of a semi-abelian scheme. Each fibre $\pi^{-1}(x)$ over a geometric point $x$ of $\mathcal{D}^0$ is constructed from a $\mathbb{P}^1$-bundle over a principally polarized abelian $(g - 1)$-fold by identifying two disjoint sections and is an equivariant compactification of the semi-abelian variety that is the smooth locus.
Consider the full level 2 version $\mathcal{A}^F_{g,2}$ of the stack $\mathcal{A}^F_g$ and the corresponding open substack $\mathcal{A}^\text{part}_{g,2}$. Again the universal abelian scheme over $\mathcal{A}^\text{part}_{g,2}$ extends to an equidimensional projective family $\pi_2 : \mathcal{U}_{g,2} \to \mathcal{A}^\text{part}_{g,2}$. This time, a fibre $F$ over a geometric point of $\mathcal{D}^\text{part}$ is the sum of two components $F_1$ and $F_2$. Each $F_i$ is a copy of $X \times \mathbb{P}^1$, where $X$ is a principally polarized abelian $(g - 1)$-fold, $F_1 \times \{0\}$ is identified with $F_2 \times \{\infty\}$ by a translation determined by a chosen point $P_0 \in X$ and similarly for $F_1 \times \{\infty\}$ and $F_2 \times \{0\}$.

There is a symmetric line bundle $\mathcal{L}$ on $\mathcal{U}_{g,2}$ that defines twice the principal polarization on the smooth fibres. On a singular fibre $F$ it is of degree 1 on each copy of $\mathbb{P}^1$ that appears and defines twice the principal polarization on $X$.

At this point we need a lemma.

**Lemma 2.2** For any $x \in \mathcal{A}^\text{part}_{g,2}$ the specialization map

$$\phi_x : \pi_2^* \mathcal{L} \to H^0(\mathcal{U}_x, \mathcal{L}_x)$$

is surjective and $\mathcal{L}_x$ is generated by its sections.

**PROOF:** The surjectivity of $\phi_x$ is a consequence of the vanishing of $H^i(\mathcal{U}_x, \mathcal{L}_x)$ for $i > 0$ and the standard base change results.

For smooth fibres the absence of base points is well known. For any fibre the space $H^0(\mathcal{U}_x, \mathcal{L}_x)$ is a representation of the Heisenberg group $\text{Heis}_{g,2}$ (the central extension of $\mathbb{Z}/2\mathbb{Z}^g \times \mu_2^g$ by $\mu_2$ determined by the Weil pairing) and for a singular fibre $F = \mathcal{U}_x$ in which $X$ appears as a component of the singular locus, the restriction map $H^0(F, \mathcal{L}_x) \to H^0(X, \mathcal{L}_x)|_X$ is equivariant for the subgroup $\text{Heis}_{g-1,2}$ of $\text{Heis}_{g,2}$. Since $H^0(X, \mathcal{L}_x)|_X$ is an irreducible representation of $\text{Heis}_{g-1,2}$, restriction is surjective. (If it were zero, then it would be zero on the other component of the singular locus, which is impossible since $\mathcal{L}_x$ has degree 1 on each $\mathbb{P}^1$.) In particular, there are no base points in the singular locus of $F$.

Suppose that $P$ is a base point in the smooth locus. Consider the copy $\Gamma$ of $\mathbb{P}^1$ in $F$ that contains $P$. Then there is a copy of $\mu_2$ in $\text{Heis}_{g,2}$ that preserves $\Gamma$ and acts on the restriction of $H^0(F, \mathcal{L}_x)$ to $\Gamma$; it follows that $\Gamma$ lies in the base locus, and the lemma is proved. □

Since $\text{Heis}_{g,2}$ rigidifies the projective space $\mathbb{P}(H^0(\mathcal{L}_x)) = \mathbb{P}^N$ (which is independent of the geometric point $x$), where $N = 2^g - 1$. Therefore there is a morphism $Km : \mathcal{A}^\text{part}_{g,2} \to \text{Chow}(\mathbb{P}^N)$ that sends each point $x$ to the cycle that is the image, counted with appropriate multiplicity, under the $2\theta$ linear system, of the scheme $f_2^{-1}(x)$. If $x$ corresponds to an irreducible principally polarized abelian variety, then $Km(x)$ is the associated Kummer variety, counted with multiplicity 1 ([LB], Th. 8.1, p. 99; the assumption there that the case field is $\mathbb{C}$ is unnecessary for their argument), so that $Km$ is radicial (that is, generically an étale homeomorphism) onto its image. However, $Km$ is constant along the inverse image in $\mathcal{A}^\text{part}_{g,2}$ of the curve $C_1$; this is the statement that the Kummer variety of an elliptic curve is $\mathbb{P}^1$, so that for $x = [E \times B] \in C_1$ the image of
Km\((E \times B)\) is just \(\mathbb{P}^1 \times \text{Km}(B)\) embedded in \(\mathbb{P}^N\) via the Segre embedding of \(\mathbb{P}^1 \times \mathbb{P}^{2g-1-1}\). Since \(Km\) factors through the geometric quotient \(A_{g,2}^{\text{part}}\) of \(\mathcal{A}_{g,2}^{\text{part}}\), we have a non-constant morphism defined on \(A_{g,2}^{\text{part}}\) that collapses the inverse image of \(C_1\). To get the morphism that we want on \(A_g^{\text{part}}\), take quotients by the finite group \(Sp_{2g}(\mathbb{F}_2)\): acts on \(\mathbb{P}^N\), so on \(\text{Chow}(\mathbb{P}^N)\), and gives rise to a commutative diagram

\[
A_{g,2}^{\text{part}} \xrightarrow{H} \text{Chow}(\mathbb{P}^N) \\
\downarrow \quad \downarrow \\
A_g^{\text{part}} \xrightarrow{h} \text{Chow}(\mathbb{P}^N)/Sp_{2g}(\mathbb{F}_2).
\]

Observe that the morphism \(h\) factors through a morphism \(H\) defined on \(A_g^{\text{part}}\) that contracts \(C_1\).

Regard \(H\) as a rational map on \(A_g^{\text{part}}\). Since \(\rho(A_g^{\text{part}}) = 2\) and \(A_g^{\text{part}}\) is regular, \(H\) is defined by some linear system \([aM - bD]\). Since \(H\) contracts \(C_1\) and \(M.C_1 = \frac{1}{12}\), \(D.C_1 = 1\), by 1.5, \(am - bD\) is proportional to \(12M - D\).

Now suppose that \(\text{char } k = 2\). The proof follows the same lines, except that we consider a full level 3 structure, the linear system \(|3\Theta|\) and the action of the level 3 Heisenberg group on (the dual of) this projective space. To carry this out requires the introduction of a theta level structure, since otherwise there is no line bundle \(\mathcal{O}(\Theta)\).

According to [FC], p. 132 et seq., there is, over any ring \(R\) that contains \(\mathcal{O}\) and in which 3 is invertible, a stack \(\mathcal{N}_{g,3}^{\text{part}}\), finite and flat over \(A_{g,3}\), with a universal abelian scheme \(\mathcal{U}^0 \to \mathcal{N}_{g,3}^{\text{part}}\) with a full level 3 structure and a symmetric ample line bundle \(\mathcal{O}(\Theta)\) that induces a principal polarization of \(\mathcal{U}_g^0\). Now suppose that \(R\) is a field. Then, in the notation of loc. cit. (except that we write \(L\) instead of \(X\)), the perfect cone decomposition, for a choice of \(\rho \in L\), of the cone \(\overline{C}(X_g)\), where now \(Q_\rho(L)\) is the lattice instead of \(B(X_g)\), so that the perfect cone decomposition is the convex hull of the rank 1 forms in \(Q_\rho(L)\) instead of \(B(X_g)\), determines a normal toroidal compactification \(\mathcal{N}_{g,3}^{\text{norm}}\) of any component \(\mathcal{N}_{g,3}^{\text{part}}\) of the normalization of \(\mathcal{N}_{g,3}\). Moreover, the abelian scheme \(\mathcal{U}^0 \to \mathcal{N}_{g,3}\) extends to a semi-abelian scheme over \(\mathcal{N}_{g,3}^{\text{part}}\) that possesses an equivariant compactification \(\mathcal{U} \to \mathcal{N}_{g,3}^{\text{part}}\) on which there is a line bundle \(L\) that is a relative \(\mathcal{O}(\Theta)\).

A slight modification of the proof of 2.2 shows that the line bundle \(\mathcal{O}(\Theta)\) is relatively very ample over \(\mathcal{N}_{g,3}^{\text{part}}\). When \(X\) is, as a \(g\)-dimensional principally polarized abelian variety, of the form \(X = E \times B\) with \(E\) an elliptic curve, the intersection of the quadrics containing the \(|3\Theta|\)-image of \(X\) is \(\mathbb{P}^2 \times B'\), where \(B'\) is the intersection of the quadrics that contain the \(|3\Theta|\)-image of \(B\). In particular, the \(|3\Theta|\)-image of \(X\) lies in the image \(\Sigma\) of the Segre embedding of \(\mathbb{P}^2 \times \mathbb{P}^{3g-1-1}\) in \(\mathbb{P}^{3g-1}\). On the other hand, if \(X\) is irreducible and \(g \geq 3\), then its \(|3\Theta|\)-image cannot lie in \(\Sigma\), for then \(X\) would possess a non-constant morphism to \(\mathbb{P}^2\). (The case where \(g = 2\) is left to the reader.) Finally, instead of \(\text{Chow}(\mathbb{P}^N)\) we take the Grassmannian of \(M\)-dimensional subspaces of the vector space \(V\).
of quadrics in $\mathbb{P}^{3g-1}$, where $M = \dim H^0(\mathbb{P}^{3g-1}, \mathcal{I}_{X/\mathbb{P}^{3g-1}}(2))$. (Note that $M$ is independent of the principally polarized abelian $g$-fold $X$, since the $|3\Theta|$-image of $X$ is projectively normal and $\dim H^0(X, \mathcal{O}(6\Theta)) = 6g$, so is independent of $X$.) This space $V$ is rigidified by the finite Heisenberg group determined by a full level 3 structure, and now we argue as in the first case. □

**Corollary 2.3** Any curve $C$ in $A_g^F$ with $(12M - D_g)C < 0$ lies in $A_g^F \setminus A_g^{part}$ and has 1-dimensional image in $A_g^{Sat}$.

**PROOF:** The first part follows from 2.1 and the second from the fact that $-D_g$ is ample relative to $A_g^F \to A_g^{Sat}$. □

So we must consider what happens higher up in the boundary. For this, recall further details of the toroidal construction.

We have a lattice $X_g = \mathbb{Z}^g$ with quotient lattices $X_g \to X_{g-1} \to \cdots \to X_0 = 0$. Put $\overline{C}(X_r) = C_r$. Then $C_r$ is a subcone of $C_{r+1}$. Fix an admissible decomposition $\{\sigma\}$ of $\overline{C}(X_g)$; this determines an admissible decomposition of $\overline{C}(Y)$ for every $X_g \to Y$. Over $A_{g-r} \subset A_g^{Sat}$, the picture is this ([FC], p. 105): let $U_{g-r} \to A_{g-r}$ be universal, $U_{g-r}^r = \text{Hom}(X_r, A_r)$, a copy of the $r$-fold fibre product of $U_{g-r} \to A_{g-r}$ and take a certain torsor $\Xi \to U_{g-r}^r$ under the $r(r+1)/2$-dimensional torus $E = E_r$ with $\mathbb{X}_r(E_r) = B(X_r)$; the characterization of this torsor in terms of the $\mathbb{G}_m$-bundles over $U_{g-r}$ associated to given characters of $E_r$ is given in the last paragraph of *loc.cit.* and will be needed later in 2.11. Take the locally finite torus embedding $E_r \to \overline{E}_r$ associated to the admissible decomposition, with boundary $\partial E_r = \overline{E}_r \setminus E_r$. Notice that the irreducible components of $\partial E_r$ correspond to the minimal cones in the decomposition of $C_{r+1}$ that meet the relative interior $C(X_r)$. Then the inverse image of the locally closed subvariety $A_{g-r} \subset A_g^{Sat}$ in the toroidal compactification $A_{g,\{\sigma\}}$ is the quotient by $GL(X_r)$ of the associated bundle $\Xi \times_{E_r} \partial E_r \to U_{g-r}^r$. Note that this quotient by $GL(X_r)$ does exist in the category of schemes locally of finite type, and the analogous statement holds at the level of stacks. From this, and consideration of the embedding $\overline{C}(X_r) \hookrightarrow \overline{C}(X_{r+1})$, the following lemma, except maybe for 2.4 1, whose significance is that the $Z$ that appears there is proper, is clear.

**Lemma 2.4** (1) At level $n \geq 3$, the torus $E_r$ acts on each component $Z$ of the inverse image of $A_{g-r,n}^{Sat}$ in $A_{g,n,\{\sigma\}}$ such that the inverse image of $A_{g-r,n}$ is finitely stratified with each stratum being a bundle over $U_{g-r,n}^r$ whose fibre is a quotient of $E_r$.

(2) The strata correspond to equivalence classes, under the principal congruence subgroup of level $n$ in $GL(X_r)$, of the cones in $\{\sigma\}$ that lie in $\overline{C}(X_r)$ and meet $C(X_r)$.

(3) The strata corresponding to maximal cones in $\overline{C}(X_r)$ are copies of $U_{g-r,n}^r$.

**PROOF:** For (1), take the morphism $Z \to A_{g-r,n}^{Sat}$; we have seen that $E_r$ acts on the inverse image $Z^0$ of $A_{g-r,n}$. It is also clear, from considering the embedding
\( C(X_r) \hookrightarrow \overline{C}(X_{r+s}) \), that \( E_r \) acts on the formal completion of \( Z \) along its fibre over each copy of \( A_{g-r-s, n} \) that is a cusp in \( A_{g-r-s, n}^{\text{Sat}} \) compatibly with its action on \( Z^0 \). Since \( Z \) is normal, it is enough to show that if \( Z \) is a normal \( k \)-variety \( U \) is open in \( Z \), \( F = Z \setminus U \) and \( E \) is a \( k \)-torus that acts compatibly on \( U \) and on the \( F \)-adic completion \( \widehat{Z} \) of \( Z \), then \( E \) acts on \( Z \).

By assumption, there is a commutative diagram
\[
\begin{array}{ccc}
E \times \widehat{Z} & \rightarrow & \widehat{Z} \\
\downarrow & & \downarrow \\
E \times Z & \rightarrow & Z
\end{array}
\]
where the products are over \( k \) and the horizontal arrows are the actions; the lower one is a rational map whose base locus lies in \( E \times F \). This shows that the base locus of the rational map \( E \times Z \rightarrow Z \) disappears after making the cover \( E \times \widehat{Z} \rightarrow E \times Z \) of normal schemes; since this is cover is faithfully flat over \( E \times F \), the base locus is empty.

The rest of the lemma has already been proved in the discussion just preceding it.

Corollary 2.5 If there is a curve \( C \) in \( A_g^F \) with \( (12M - D_g).C < 0 \), then for some \( r \geq 2 \) there is such a curve lying in the closure \( \overline{U}_{g-r}^r \), of the copy of \( U_{g-r}^r \) that corresponds, under 2.4 3, to some cone \( \sigma \) in the perfect cone decomposition of \( \overline{C}(X_g) \) that lies in \( \overline{C}(X_r) \) and is maximal there.

Proof: Work at level \( n \), with \( n \geq 3 \) and prime to \( \text{char} \ k \). There is a natural cover \( \pi: A_{g,n}^F \rightarrow A_g^F \) with \( \pi^*D_g = nD_{g,n} \), where \( D_{g,n} \) is the reduced sum of the boundary divisors in \( A_{g,n}^F \). So having such a curve is equivalent, at level \( n \), to having a curve \( C \) in \( A_{g,n}^F \) with \( (12 \frac{M}{n} - D_{g,n}).C < 0 \).

We know that \( C \) is in \( A_{g,n}^F \setminus A_{g,n}^{\text{part}} \). So there is a unique value of \( r \) such that the image of \( C \) in \( A_{g,n}^{\text{Sat}} \) lies in \( A_{g-r,n}^{\text{Sat}} \setminus A_{g-r-1,n}^{\text{Sat}} \). Then the result follows from 2.4 by applying the Borel fixed point theorem to the action of \( E_r \) on the Chow scheme of curves in the inverse image of \( A_{g-r,n}^{\text{Sat}} \) in \( A_{g,n}^F \), as was explained at the start of this section.

From now on, assume that there is a curve on which \( 12M - D_g \) is negative. Take a cone \( \sigma \subset B(X_r) \otimes \mathbb{R} \) and corresponding closure \( \overline{U}_{g-r}^r \) that contains such
a negative curve, as provided by 2.5. Suppose that \( g = q_\sigma \in Q(X_r) \) is the perfect form (unique up to a scalar) that defines \( \sigma \). Let \( \tilde{U}_{g-r}^r \) be the normalization of \( \mathcal{U}^r_{g-r} \). Then, in order to get further control on these negative curves, it is enough to show that the complete linear system given by some large multiple of the pullback of the divisor class \( 12M - D_g \) to \( \tilde{U}_{g-r}^r \) has no base points in the open subvariety \( U_{g-r}^r \).

**Proposition 2.6** Suppose that \( \tilde{U}_{g-r}^r \) is the normalization of \( \mathcal{U}^r_{g-r} \). Then the complement \( \tilde{U}_{g-r}^r \setminus U_{g-r}^r \) is an irreducible divisor.

**Proof:** This is a consequence, via the usual rules governing the construction of torus embeddings, of the statement that the cones \( \tau \) in \( C(X_{r+1}) \) such that
1. \( \sigma \) is a proper face of \( \tau \)
2. \( \tau \) is minimal with respect to this property
form a single orbit under the subgroup of \( GL(X_{r+1}) \) that preserves the quotient homomorphism \( X_{r+1} \to X_r \) and acts trivially on \( X_r \).

To prove this, suppose that \( f = f(x_1, \ldots, x_r) \) is a perfect form in \( r \) variables and minimum value \( a \). Then the set \( \min(f_1) \) of minimal vectors of \( f_1 := f + ax_{r+1}^2 \) is given by
\[
\min(f_1) = \min(f) \cup \{ \pm(0, \ldots, 0, 1) \},
\]
so that, by the defining property of the perfect cone decomposition, \( f_1 \) defines one of the cones \( \tau \) in question.

To show that this construction accounts for all the \( \tau \), suppose that \( \tau' \) is any one of the cones in question. There is a form \( F = F(x_1, \ldots, x_r, x_{r+1}) \) the squares of whose minimal vectors span \( \tau' \), again by the defining property of the perfect cone decomposition. Since \( \sigma \) is spanned by a subset \( S \) of these squares, the minimality of \( \tau' \) ensures that it is spanned by \( S \) and just one more square. So we have accounted for all the \( \tau \).

**Corollary 2.7** There is an open substack \( \tilde{U}_{g-r, part}^r \) of \( \tilde{U}_{g-r}^r \) whose complement has codimension 2 and that contains \( \mathcal{U}^r_{g-r} \) and is a semi-abelian scheme over \( A^r_{g-r, part} \) extending \( \mathcal{U}^r_{g-r} \to A^r_{g-r} \).

**Proof:** This follows directly from the restatement of 2.6 in terms of stacks.

Here is a summary of some of these objects and morphisms:
\[
U_{g-r, part}^r \hookrightarrow \tilde{U}_{g-r, part}^r \hookrightarrow \tilde{U}_{g-r}^r \to \mathcal{U}^r_{g-r} \hookrightarrow D_g \hookrightarrow A^F_g,
\]
where the embeddings are open except for \( D_g \hookrightarrow A^F_g \), which is closed. Let \( A_{g,r} : U_{g-r, part}^r \to A^F_g \) denote the composite. When \( r = 1 \), \( \mathcal{U}^r_{g-r} = D_g^F \). In addition, there is a 0-section \( N_{g,r}^0 \subset U_{g-r, part}^r \). Let \( N_{g,r} \) denote the normalized closure of \( N_{g,r}^0 \) in \( \tilde{U}_{g-r}^r \) and \( N_{g,r, part}^r = N_{g,r} \cap \tilde{U}_{g-r, part}^r \).
Lemma 2.8 \( N_{g,r} \) is naturally isomorphic to \( A_{g-r}^F \).

PROOF: First, the closure of \( A_{g-r} \times A_r \) in \( A_g^F \) is just \( A_{g-r}^F \times A_r^F \); this is the statement that if \( q_1 \) is a positive quadratic form in \( g-r \) variables, \( q_2 \) a positive quadratic form in \( r \) further variables and \( q_1, q_2 \) have equal minimum values, then \( \min(q_1 + q_2) = \min(q_1) \cup \min(q_2) \).

Then inside \( A_{g-r}^F \), \( N_{g,r} \) is the limit of the locus \( A_{g-r}^F \times \{[B]\} \), where \([B] \in A_r \) and \([B] \) tends to a point in \( D_r \) in a way specified by the choice of \( \sigma \) as a maximal cone in the perfect cone decomposition of \( C_r \).

The next thing is to understand the restriction of the class \( [D_g] \) to its subvariety \( N_{g,r} \) in two different ways.

Lemma 2.9 The restriction of \( D_g \) to \( N_{g,r} \) is linearly equivalent to \( D_{g-r} \) via the identification \( N_{g,r} = A_{g-r}^F \) of 2.8.

PROOF: This follows at once from the limiting description of \( N_{g,r} \) given in the proof of 2.8.

For another view of this restriction, look again at the embedding \( \overline{U}_{g-r} \hookrightarrow D_g \) determined by the cone \( \sigma \). Along this subvariety, \( D_g \) is the union of branches \( \delta_1, \ldots, \delta_p \) of which \( \overline{U}_{g-r} \) is the intersection. At the stack level, these branches correspond, once a basis of \( X \) has been chosen, so that \( X \) is identified with \( X^\vee \), to the squares \( x_1^2, \ldots, x_p^2 \) of the minimal vectors of the perfect form \( q_\sigma \) in \( r \) variables that determines \( \sigma \), or, equivalently, to the rank 1 quadratic forms that span \( \sigma \). The number \( p \) is half the kissing number of \( q_\sigma \).

Denote by \( L_{g-r} \) the symmetric divisor class on \( U_{g-r} \rightarrow A_{g-r} \) that defines \( (-2) \) times the principal polarization on each fibre. Let \( \gamma : U_{g-r}^{r,part} \rightarrow A_{g-r}^{r,part} \) denote the structural morphism.

Lemma 2.10 Two divisor classes on \( U_{g-r}^{r,part} \) or \( U_{g-r}^{r,part} \) are equivalent, modulo torsion, if they are equivalent on the zero-section and on the generic fibre.

PROOF: Since the complement of \( U_{g-r}^{r,part} \) both in \( U_{g-r}^{r,part} \) and in the normalized closure \( \overline{U}_{g-r} \) is an irreducible divisor, it is enough to prove the result for classes in \( H^2 \) on \( \overline{U}_{g-r} \) over \( \mathbb{C} \).

In this case it is enough, by the Hochschild–Serre spectral sequence, to show that \( H^1(Sp_{2g}, M) \) is torsion, where \( M = \mathbb{Z}^{2g} \) is the standard representation. As explained on p. 135 of [BMS], this follows from the linear reductivity of the algebraic group \( Sp_{2g} \) over \( \mathbb{Q} \).

Proposition 2.11 (1) The boundary divisor \( D_{g-r+1} \) in \( A_{g-r+1}^F \) cuts out \( L_{g-r} \) via the identification of \( U_{g-r} \) with an open substack of \( D_{g-r+1} \).

(2) There are positive rational numbers \( c, b_i \) and projections \( \pi_i : U_{g-r}^{r,part} \rightarrow U_{g-r}^{r,part}, \) for \( i = 1, \ldots, p \), with composites \( \rho_i = \alpha_{g-r+1,1} \circ \pi_i : U_{g-r}^{r,part} \rightarrow A_{g-r+1} \) such that \( \sum b_i \rho_i^*(12M - D_{g-r+1}) + c\gamma^*(12M - D_{g-r}^{part}) \) is numerically equivalent to \( \alpha_{g,r}^*(12M - D_g) \).
PROOF: First, take the affine torus embedding $E \hookrightarrow \mathbb{E}$ that corresponds to $\sigma$. In particular, $B(X_r) = X_r(E)$.

Since the 1-skeleton of $\sigma$ is generated by vertices that lie in the same affine hyperplane, there is a morphism $f : (E \hookrightarrow \mathbb{E}) \to (\mathbb{G}_m \hookrightarrow \mathbb{A}^1)$ of torus embeddings such that the divisor $f^{-1}(0)$ is an integral multiple of the reduced boundary divisor $\partial \mathbb{E} = \mathbb{E} \setminus E$. Fix a basis of $X$ and then identify $X$ with $X^\vee$. The lattice homomorphism $h : \mathbb{Z}^p = \sum \mathbb{Z} e_i \to B(X_r)$, where $\tilde{h}(e_i) = x_i^2$ and the $x_i$ are the minimal vectors of $q$, gives a morphism $h : (\mathbb{G}_m^p \hookrightarrow \mathbb{A}^p) \to (E \hookrightarrow \mathbb{E})$ of torus embeddings. Composing this with $f$ shows that such that $h^{-1}(\partial \mathbb{E})$ is the sum of the co-ordinate hyperplanes in $\mathbb{A}^p$.

Now look at various associated bundles over $U^r_{g-r}$ constructed over the $E$-torsor $\Xi \to U^r_{g-r}$ described on p. 105 of [FC]. Also, consider $U^r_{g-r}$ as an open subscheme of $U^r_{g-r}$, so inside $\mathcal{A}^F$. This copy $U^r_{g-r,\text{origin}}$ of $U^r_{g-r}$ appears inside these associated bundles as the origin. Moreover, the associated $\mathbb{A}^p$-bundle $V \to U^r_{g-r}$ that we get is the direct sum of certain line bundles $H_1, \ldots, H_p$. There is a morphism $h : V \to \Xi$ of bundles over $U^r_{g-r}$, and the restriction of the $\mathbb{Q}$-divisor class $h^{-1}(\partial \Xi)$ to $U^r_{g-r,\text{origin}}$ is a rational multiple of $\sum c_1(H_i)$.

Note that, by the nature of the toroidal compactification $A^F$, the $\mathbb{Q}$-divisor classes $\Xi$ and $D_g$ have the same restriction to $U^r_{g-r,\text{origin}}$. So the restriction of $D_g$ to $U^r_{g-r,\text{origin}}$ is the same as the restriction of $\sum c_1(H_i)$, say $s \sum c_1(H_i)$.

We must describe these bundles $H_i$. There is an identification of $X_r \otimes (\mathcal{A}^t)$ with $(X_r \otimes \mathcal{A})^t$, so that if $\lambda : \mathcal{A} \to \mathcal{A}^t$ is the given principal polarization, then $1_{X_r} \otimes \lambda : X_r \otimes \mathcal{A} \to X_r \otimes (\mathcal{A}^t)$ is a principal polarization. It also identifies $X_r$ with $X_r^\vee$ and embeds $B(X_r)$ into its dual $Q(X_r) = \text{Sym}^2(X_r)$. Moreover, the elements $h(e_i)$ of $B(X_r)$ are then characters of $E$ and the $\mathbb{G}_m$-bundle $\Xi \times E, e_i \mathbb{G}_m \to U^r_{g-r}$ is $H_i \setminus \{0\}$. Since $\tilde{h}(e_i)$ is, as an element of $Q(X_r^\vee)$, the square of a primitive vector in $X_r$, it follows from the description given on p. 105 of [FC] that $H_i$ is the pullback of the inverse Poincaré bundle $P^{-1}$ under a composite morphism

$$U^r_{g-r} \xrightarrow{f_i} U^r_{g-r} \xrightarrow{\Delta} U^r_{g-r} \times_{A^r_{g-r}} U^r_{g-r},$$

where $\Delta$ is the diagonal embedding and the rank 1 projection $f_i$ depends upon $e_i$. Since $\Delta^*P$ defines twice the principal polarization, (1) follows by taking $r = 1$.

For (2), we have now constructed $f_i : U^r_{g-r} \to U^r_{g-r}$ such that the class $H_i$ cuts out $f_i^* L_{g-r}$ on the generic fibre of $\gamma : U^r_{g-r} \to A^r_{g-r}$. By [FC], p. 9, $f_i$ extends uniquely to $\pi_i : U^r_{g-r,\text{part}} \to U^r_{g-r,\text{part}}$.

Let $b, c \in \mathbb{Q}_{>0}$ and $n \in \mathbb{N}_{>0}$, to be determined later. Put $\pi_i = [n] \circ \pi_i$, where $[n]$ is multiplication by $n$ on $U^r_{g-r}$. We also let $\pi_i$ denote the induced morphism of geometric quotients and $\rho_i = \alpha_{g-r+1,1} \circ \pi_i$.

Put $F = b \sum r_i^*(12M - D^F_{g-r+1}) + c_r^*(12M - D_{g-r,\text{part}})$ and $G = \alpha^*_{g,r}(12M - D_g)$. It is enough, by 2.10, to show that these classes are equal when restricted to the 0-section $N^\text{part}_{g,r}$ of $\gamma$ and to the generic fibre $\Phi$ of $\gamma$. 

On the zero-section $N_{g,r}^\text{part}$, these restrictions are independent of $n$. The restriction of $F$ is $(pb+c)(12M-D_{g-r}^\text{part})$, since $N_{g,r}^\text{part}$ maps isomorphically under $\pi_i$ to its image in $U_{g-r}^\text{part}$, the 0-section of $U_{g-r}^\text{part} \to A_{g-r}^\text{part}$. On the other hand, $G|_{N_{g,r}^\text{part}} = 12M - D_{g-r}^\text{part}$, so the restrictions to $N_{g,r}^\text{part}$ are equal provided that $c+pb=1$.

On $\Phi$, the class $M$ is trivial, so the restrictions are $F|_\Phi = -bn^2 \sum H_i$, since $[n]^* H_i = n^2 H_i$, and $G|_\Phi = -s \sum H_i$.

So we need to find $n, b, c$ such that $c+pb=1$ and $bn^2 = s$. Clearly this is possible.

The next lemma removes the distinction between numerical and rational equivalence in our context.

**Lemma 2.12** $\tilde{U}_{g-r}$ is regular. That is, its Albanese variety is trivial.

**PROOF:** It is enough to prove this at some level $n \geq 3$ that is prime to the characteristic.

Put $U_{g-r,n}^\text{part} = U$ and $\tilde{U}_{g-r,n}^\text{part} = \tilde{U}$. There is an open immersion $j : U \to \tilde{U}$ and $U$ is semi-abelian over $A_{g-r,n}^\text{part}$. Then there is a commutative diagram

$$
\begin{array}{ccc}
U & \longrightarrow & A_{g-r,n}^\text{part} \\
\downarrow & & \downarrow \\
U^* & \longrightarrow & A_{g-r,n}^\text{Second}
\end{array}
$$

where the vertical arrows are open immersions, $f$ is proper and is an equivariant compactification of a semi-abelian scheme over $A_{g-r,n}^\text{Second}$. Over any point $P$ in $A_{g-r,n}^\text{Second}$ that lies over a 0-dimensional cusp in $A_{g-r,n}^\text{Sat}$, the fibre $f^{-1}(P)$ is stratified into (torsors under) algebraic tori, and so is collapsed under the natural morphism $\alpha : \text{Alb}(U^*) \to \text{Alb}(A_{g-r,n}^\text{Second})$. It follows that $\alpha$ is an isomorphism.

To prove the vanishing of $\text{Alb}(A_{g-r,n}^\text{Second})$ it is enough, by the usual comparison theorems, to work over $\mathbb{C}$. Suppose first that $g-r \geq 2$. Then $[Kn] A_{g-r,n}^\text{Second}$ is simply connected, so regular. If $g-r = 1$, then take $n=3$ or $5$ and use the fact that $X(3)$ and $X(5)$ are rational.

Finally, the restriction homomorphism $\text{Pic} \tilde{U} \to \text{Pic} U$ is an isomorphism, since $\tilde{U} \setminus U$ has codimension at least 2, while $\text{Pic} U^* \to \text{Pic} U$ is surjective.

Now we can prove the main result of this section. This is just a matter of assembling the pieces.
Theorem 2.13  $12M - D_g$ is nef on $A_g^F$.

PROOF: If there is a curve $C$ in $A_g^F$ with $(12M - D_g).C < 0$, then for some $r \geq 2$ and some maximal cone $\sigma$ in $\overline{C}(X_r)$, $C$ lies in the subvariety $\overline{U}^r_{g-r}$ corresponding to $\sigma$. But 2.1 (replacing $g$ by $g-r$), 2.11 and 2.7 together show that the restriction of $12M - D_g$ to $\overline{U}^r_{g-r}$ gives (after taking a large multiple) a linear system whose base locus lies over $A_{g-r-1}^{\text{Sat}}$. □

Corollary 2.14 The cone $\text{NE}(A_g^F)$ of curves on $A_g^F$ is closed and is the rational polyhedral cone generated by the curves $C_1$ and $C_2$.

PROOF: Since $\rho(A_g^F) = 2$, by 1.4, it’s enough to show that both curves lie in the boundary of $\overline{\text{NE}}(A_g^F)$.

For $C_2$ this is 1.7. For $C_1$ it follows from 2.13 and the formula $(12M - D_g).C_1 = 0$.

Remark: As examples, consider the cases where $g = 2$ and $n = 2$ or 3.

(i): $A_{g,2}^F$ is the blow-up of the Segre cubic threefold $S$ in its 10 nodes. The contraction $A_g^F = A_{g,2}^F/Sp_4(F_2) \to S/Sp_4(F_2)$ is the contraction of the ray generated by $C_1$.

(ii): $A_{g,3}^F$ is the blow-up of the Burkhardt quartic $B$ in its 45 nodes and $A_g^F = A_{g,3}^F \to B/Sp_4(F_3)$ is again the contraction of the ray generated by $C_1$.

Corollary 2.15 (1) The divisor class $aM - bD$ is ample on $A_g^F$ if and only if $12a > b > 0$.

(2) $aM - bD$ is nef if and only if $12a \geq b \geq 0$.

(3) The canonical class of $A_g^F$ is ample if and only if $g \geq 12$.

PROOF: Immediate from the result that $12M - D$ is nef and Kleinman’s criterion for ampleness: a Cartier divisor class on the projective variety $X$ is ample if and only if it is strictly positive on $\partial \overline{\text{NE}}(X)$ ([K], Th. 2, p. 326).

When $g \leq 3$ 2.15 1 is due to Hulek and Sankaran ([HS02], Theorem II.2.4).

Definition 2.16 The slope of a Siegel cusp form is its weight divided by its order of vanishing. (This is the reciprocal of what Weissauer [W] calls the order of vanishing of a modular form.)

Corollary 2.17 Fix a slope $a \in \mathbb{Q}$. Then the ring of Siegel modular forms of degree $g$ and slope $a$ and with Fourier coefficients in $\mathbb{Z}$ is a finitely generated $\mathbb{Z}$-algebra provided that $a > 12$. The same is true for $a = 12$ provided that $g \leq 11$ and we consider Fourier coefficients in $\mathbb{C}$.

PROOF: For $a > 12$ this follows from 2.15. Over $\mathbb{C}$ it follows from 2.15 and the base point free theorem for complex projective varieties with canonical singularities [CKM]. □

Weissauer has shown ([W], p. 220) that for every $a > 12$ and point $x \in A_g$, there is a slope $a$ cusp form that does not vanish at $x$. This can be extended to include $a = 12$ (but not $a < 12$, as he remarked).
Corollary 2.18 For every $a \geq 12$ and every point $x \in \mathcal{A}^{\text{part}}$ there is a slope $a$ cusp form that does not vanish (as a slope $a$ cusp form) at $x$.

PROOF: When $a > 12$ this follows from the ampleness on $A^F_g$ of the bundle of slope $a$ cusp forms. When $a = 12$ we do not know, for $g \geq 12$, whether this bundle is eventually base-point-free on $A^F_g$; however, we do know, by the proof of 2.1, that it has no base points on $A^{\text{part}}_g$.

Remark: One natural problem is whether the total co-ordinate ring of $A^F_g$, that is, the bigraded ring $\bigoplus_{a,b \geq 0} H^0(A^F_g, \mathcal{O}(aM - bD))$ is of finite type, over $\mathbb{Z}$ or any other base ring. Another is to compute the various intersection numbers $M^aD^b$ for $a + b = \frac{g(g+1)}{2}$.

3 Canonical models

Over a field of characteristic zero the results of the previous section lead to statements about the field of Siegel modular functions. Over other fields the basic questions about resolving singularities, even quotients of toroidal singularities by finite groups, are still too hard. So in this section the base is $\text{Spec} \mathbb{C}$. Identify $A_g$ with $\mathcal{H}_g/\Gamma$, where $\mathcal{H}_g$ is the Siegel upper half-space of degree $g$ and $\Gamma = \text{Sp}_{2g}(\mathbb{Z})$. There is a birational morphism $\pi: \widetilde{A} \to A$. (Since $\Gamma$ is not neat, $\widetilde{A}$ will have non-trivial quotient singularities even if the cone decompositions chosen for the construction of $\widetilde{A}$ are basic. For quotients by neat arithmetic groups, these decompositions are basic if and only if the toroidal compactification is smooth.)

Let us say that a Deligne-Mumford stack $X$ has canonical or terminal singularities if there is an étale surjective cover $X \to \mathcal{X}$ from a scheme $X$ such that $X$ has canonical or terminal singularities.

Lemma 3.1 The stack $A^F_g$ has terminal singularities.

PROOF: Suppose that $T \hookrightarrow X$ is an affine torus embedding. This corresponds to a rational polyhedral convex cone $\sigma$ in $X_*(T) \otimes \mathbb{R}$. Then $X$ has $\mathbb{Q}$-Gorenstein singularities if and only if $\sigma$ is the cone over a rational polyhedral convex polytope $\tau$ whose vertices are in $X_*(T)$ and that lies in an affine hyperplane ($z = 1$) of $X_*(T) \otimes \mathbb{R}$. Moreover, $X$ has canonical singularities if and only if in addition there are no points in $X_*(T) \cap \sigma$ satisfying $z < 1$ and has terminal singularities if and only if the only points of $X_*(T) \cap \sigma$ that satisfy $z \leq 1$ are the vertices of $\tau$.

To prove the result, we again use 1.1 2: if $f$ is a quadratic form on $X_g$, then, when $f$ is regarded as a linear function on $\overline{C}(X_g)$, its minima are all rank one elements of $B(X_g)$. The statement now follows from the description, which we have already recalled in Section 1, of toroidal resolutions.

To deal with the passage from the stack to its geometric quotient requires analysis of the isotropy group actions. Tai shows [T] that the quotient map
$A_g \to A_g$ is unramified in codimension 1 when $g \geq 2$ and that $A_g$ has canonical singularities if $g \geq 5$; his argument shows that in fact $A_g$ has terminal singularities if $g \geq 6$. (Note that his proof of Lemma 4.3 of loc. cit. needs slight amendment. He states, without giving details, that

$$\sum \{ t_i + t_j \} \geq \frac{r(r+1)^2}{4m},$$

where $\{ x \}$ is the fractional part of $x$ and $r = \frac{1}{2}\phi(m)$; this leaves special cases, such as $m = 30$, requiring individual treatment beyond those that he gives. Attempts to fill in these details led to the estimate

$$\sum \{ t_i + t_j \} \geq \frac{r^3 + \frac{3}{2}r^2 + \frac{1}{2}r}{4m};$$

this leaves the cases $m = 8, 10, 12, 18, 30$ requiring individual treatment, since then the last expression is less than 1. Rather than check this by hand, Tom Fisher wrote a Magma routine to do it. For $m = 12$ he found $\sum \{ t_i + t_j \} = 1$ and $\sum \{ t_i + t_j \} > 1$ for the other values of $m$.)

Tai’s extension of this argument to deal with singularities in the boundary hints that $A_F^g$ has canonical singularities if $g \geq 5$, and terminal singularities if $g \geq 6$. Rather than explain this, we derive it from the facts that $A_g$ and $A_F^g$ have canonical or terminal singularities and the following proposition, which was proved by Snurnikov [S] when $V$ is a point.

**Proposition 3.2** Suppose that $T \hookrightarrow X$ is a torus embedding on which the finite group $G$ acts as a group of algebraic torus automorphisms and that the $G$-action preserves a certain neighbourhood $U$ of the boundary divisor $X \setminus T$. Put $U \cap T = U_0$. Assume also that $V$ is a smooth variety on which $G$ acts freely in codimension 1 on $U_0 \times V$ and that $U$ and $(U_0 \times V)/G$ have canonical or terminal singularities. Then $G$ acts freely in codimension 1 on $Z := U \times V$ and $Z/G$ has canonical or terminal singularities.

**PROOF:** Choose a $T \times G$-equivariant resolution $\tilde{X} \to X$. Let $\tilde{U}$ denote the inverse image of $U$. We shall show that $G$ acts freely in codimension 1 on $\tilde{Z} := \tilde{U} \times V$. If not, then there is a non-trivial subgroup $H$ of $G$ and an irreducible divisor $D$ on $\tilde{Z}$ such that $H$ acts trivially on $D$. Let $P \in D$ be general. Then there is an $H$-equivariant isomorphism $T_{\tilde{Z}}(P) \to T_{\tilde{U}}(P) \oplus T_V(P)$. So either $D = \tilde{U} \times D_2$ or $D = D_1 \times V$. In the first case $D \cap (U_0 \times V)$ is non-empty, which is impossible, and so $D = D_1 \times V$, where $D_1$ is a boundary divisor. Then $D_1$ corresponds to a 1-PS $\lambda : \mathbb{G}_m \to T$ and the torus embedding $T \hookrightarrow \tilde{X}$ gives a torus embedding $T_1 := T/\lambda(\mathbb{G}_m) \hookrightarrow D_1$ on which $H$ acts trivially. But this contradicts the fact that $H$ acts freely in codimension 1 on $U_0 \times V$. So $G$ acts freely in codimension 1 on both $Z$ and $\tilde{Z}$. 
Consider the commutative diagram

\[ \begin{array}{ccc}
\tilde{Z} & \longrightarrow & \tilde{Z}/G \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Z/G.
\end{array} \]

For some integer \( r \geq 1 \) there is a \( G \)-invariant generator \( \sigma \) of \( O(rK_{\tilde{Z}/G}) \), which is then a generator of \( O(rK_{\tilde{Z}}) \), and is either regular or zero along every exceptional divisor of \( \tilde{Z} \to Z \); since \( G \) acts freely in codimension 1 on \( \tilde{Z} \), it is enough to show that \( \tilde{Z}/G \) has canonical or terminal singularities. Then we can assume that \( \tilde{X} = X = \mathbb{A}^n \) and that \( T \) is the complement of the co-ordinate hyperplanes, so that \( G \) acts on \( X \) by permuting the co-ordinates. Let \( L = \{(t, \ldots, t)\} \) be the diagonal copy of \( \mathbb{A}^F \) in \( \mathbb{A}^n \). There is a \( G \)-equivariant decomposition \( \mathbb{A}^n = L \times M \), with \( M = \mathbb{A}^{n-1} \), so that \( (X \times V)/G \) is isomorphic to \( ((M \times V)/G) \times L \). Since \( (U_0 \times V)/G \) has canonical or terminal singularities, we are done.

**Corollary 3.3** \( A_g^F \) has canonical singularities if \( g \geq 5 \) and terminal singularities if \( g \geq 6 \).

**PROOF:** According to the local description given immediately after 1.1, over an arbitrary cusp, \( A_g^F \) is a quotient stack \( [\{(X_1 \times T_1 V)/GL(L_1)\}] \). Now the corollary is a consequence of 3.2, 3.1 and Tai’s result, recalled above, that \( A_g \) has canonical or terminal singularities if \( g \geq 5 \) or \( g \geq 6 \).

**Corollary 3.4**

1. \( A_g^F \to A_g^{\text{Sat}} \) is the relative canonical model of \( A_g^{\text{Sat}} \) if \( g \geq 5 \).
2. \( A_g^F \) is the canonical model of \( A_g \) if \( g \geq 12 \).
3. The canonical model of \( A_{11} \) exists and is the result of contracting the extremal ray \( \mathbb{R}_+[C_1] \).

**PROOF:** (1) and (2) follow from 2.15 and 3.3. For (3), we also need the base point free theorem. Note that since \( K_{A_{11}} \cdot C_2 > 0 \), the result of the contraction, which is guaranteed to have terminal singularities, has ample canonical class.

In particular, we recover the result, weaker than that of Freitag, Mumford and Tai, that \( A_g \) is of general type if \( g \geq 11 \). If \( 5 \leq g \leq 10 \), then \( C_1 \) spans an extremal ray on which \( K \) is negative. This ray is then contractible, from general results on complex varieties. It would be interesting to understand how to carry forward the minimal model program for these varieties, especially when \( g = 6 \), since this is the only case where the Kodaira dimension of \( A_g \) is still unknown.

### 4 Higher level

In this section we work over a field \( k \) that contains \( \zeta_n \), and \( n \) is an integer with \( n \geq 2 \) that is not divisible by \( \text{char} \, k \).
Theorem 4.1 (1) \(aM - D^{(n)}\) is nef on the first Voronoi compactification \(A^F_{g,n}\) of \(A_{g,n}\), where \(D^{(n)}\) is the reduced boundary divisor, if and only if \(a \geq 12/n\) and ample if and only if \(a > 12/n\).

(2) The canonical class of \(A^F_{g,n}\) is ample if \(g + 1 > 12/n\).

(3) Assume that \(\text{char } k = 0\) and that either \(g \geq 3\) or \(n \geq 3\). Then \(A^F_{g,n}\) has canonical singularities and is the relative canonical model of \(A^{Sat}_{g,n}\). Moreover, \(A_{g,n}\) is of general type if \(g + 1 \geq 12/n\) and \(A^F_{g,n}\) is its canonical model if \(g + 1 > 12/n\).

**Proof:** Consider the projection \(\pi : A^F_{g,n} \to A^F_g\). Then \(\pi^*D = nD^{(n)}\), so that \(K_{A^F_{g,n}} = \pi^*((g + 1)M - \frac{1}{n}D)\), and now (1) and (2) follow from the description of the ample cone on \(A^F_g\). The proof of (3) is also immediate. \(\Box\)

**Remark:** In fact, \(A_{g,n}\) is known to be of general type in this range, and more besides; see [HS02], Theorem II.2.1, p. 106.

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