On the thermodynamical limit of self-gravitating systems

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ABSTRACT

It is shown that the diluted thermodynamical limit of a self-gravitating system proposed by de Vega and Sánchez suffers from the same problems as the usual thermodynamical limit and leads to divergent thermodynamical functions. This question is also discussed from the point of view of mean field theory.

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Recently, de Vega and Sánchez have pointed out that a kind of thermodynamical limit of a self-gravitating system can be defined if one considers what they call the diluted limit: send the number of particles, $N$, and the volume, $V$, to infinity, keeping constant the ratio $N/V^{1/3}$ instead of the density, $N/V$ [1]. This is a rather surprising –and very interesting– suggestion, since it is well known that the usual thermodynamical limit of self-gravitating systems leads to singular thermodynamical functions (due to the gravitational instability) [2, 3]. However, we will show in this paper that the diluted regime leads to divergent thermodynamical functions in a way similar to the usual thermodynamical limit. Before demonstrating this statement, we will remember the differences, in what concerns the thermodynamical limit, between a thermodynamical stable system [4] and a self-gravitating system, and present the arguments that support the idea that the diluted limit gives well defined thermodynamical functions for self-gravitating systems. Afterwards, we shall prove that this is not possible, and we will point out the loophole in the arguments that lead to the wrong conclusion. We will end the paper with a brief discussion on mean field theory.

Consider a system of $N$ classical particles, endowed with a hard core, the interactions of which are described by a potential energy which is a sum of two body contributions:

$$\Phi_N(r_1, \ldots, r_N) = \sum_{i<j} \phi(r_i - r_j). \quad (1)$$

Using the notation $\phi_{ij} \equiv \phi(r_i - r_j)$, the microcanonical and canonical partition functions, $Z_{MC}$ and $Z_C$, respectively, can be written as

$$Z_{MC} = \frac{1}{N! \Gamma(3N/2)} \int_{V^N} \prod_{i=1}^{N} d^3 r_i \left[ E - \sum_{i<j} \phi_{ij} \right]^{3N/2-1}, \quad (2)$$

$$Z_C = \frac{1}{N!} \beta^{-3N/2} \int_{V^N} \prod_{i=1}^{N} d^3 r_i \exp(-\beta \sum_{i<j} \phi_{ij}), \quad (3)$$

where $E$ is the energy, $V$ the volume, $\beta$ the inverse temperature, $[x]_+ = x$ if $x > 0$ and $[x]_+ = 0$ if $x \leq 0$, and we have ignored some irrelevant factors involving powers of the particle mass. Introducing the family of functions

$$G_{V}^{(\alpha)}(u) = \int_{V^N} \prod_{i=1}^{N} d^3 r_i \delta(u - \frac{1}{N^\alpha} \sum_{i<j} \phi_{ij}), \quad (4)$$
the partition functions can be obviously written as
\[ Z_{\text{MC}} = \frac{1}{N! \Gamma(3N/2)} \int du \left[ E - N^\alpha u \right]^{3N/2-1}, \tag{5} \]
\[ Z_{\text{C}} = \frac{1}{N!} \beta^{-3N/2} \int du G^{(\alpha)}(u) \exp[-\beta N^\alpha u]. \tag{6} \]

Assuming that, as \( N \to \infty \), keeping the density \( \rho = N/V \) fixed, the following asymptotic behavior holds
\[ G^{(1)}_V(u) \approx e^{Ng(u,\rho)}, \tag{7} \]
we can get the partition functions with the aid of the saddle point method. For the microcanonical ensemble (MC) we have the equation
\[ \frac{\partial g(u,\rho)}{\partial u} = \frac{3}{2} \frac{1}{E/N - u}. \tag{8} \]
The solution is, obviously, a function of \( \epsilon = E/N \) and \( \rho = N/V \), \( u = \bar{u}_m(\epsilon, \rho) \), and the entropy scales with \( N \). The corresponding saddle point equation for the canonical ensemble (CE) is
\[ \frac{\partial g(u,\rho)}{\partial u} = \beta, \tag{9} \]
the solution of which is a function of \( \beta \) and \( \rho \), \( u = \bar{u}_c(\beta, \rho) \), and the corresponding thermodynamical potential (usually called the Helmholtz free energy) is extensive. It is not difficult to realize that the saddle point equations imply the equivalence between the MC and the CE.

Let us formally see that the scaling (7) holds if the two body potential is short ranged. We will use the notation
\[ \langle f(r_1, \ldots, r_N) \rangle = \frac{1}{V^N} \int_{V^N} \prod_i d^3r_i f(r_1, \ldots, r_N). \tag{10} \]
Using the Fourier representation of the Dirac delta we can write
\[ G^{(\alpha)}_V(u) = V^N \int_{-\infty}^{\infty} d\omega \frac{\omega}{2\pi} e^{i\omega u} \tilde{G}^{(\alpha)}_V(\omega), \tag{11} \]
where, with our definition (10),
\[ \tilde{G}^{(\alpha)}_V(\omega) \equiv \langle \exp(-i\frac{\omega}{N^\alpha} \sum_{i<j} \phi_{ij}) \rangle = \frac{1}{V^N} \int_{V^N} \prod_{i=1}^{N} d^3r_i \exp(-i\frac{\omega}{N^\alpha} \sum_{i<j} \phi_{ij}). \tag{12} \]
We can now apply the cumulant expansion to the above expression

\[ \langle \exp(-i\omega \sum_{i<j} \phi_{ij}) \rangle = \exp \left( \sum_{n=0}^{\infty} \frac{(-i\omega)^n}{n!N^{\alpha n}} \sum_{i_1<j_1} \cdots \sum_{i_n<j_n} \langle \phi_{i_1j_1} \cdots \phi_{i_nj_n} \rangle_c \right), \]

(13)

where \( \langle \phi_{i_1j_1} \cdots \phi_{i_nj_n} \rangle_c \) is the connected correlation function. It is well known that only those sequences of coordinates \( \{i_1j_1, \ldots, i_nj_n\} \) for which the integral of Eq. (14) below cannot be split into the product of two (or more) integrals contribute to the connected correlation function. We will call such sequences connected sequences. For large \( N \) and \( n \ll N \), the number of connected sequences asymptotically is \( c_n N^{n+1} \), where \( c_n \) is a number independent of \( N \). Let us define \( \xi_l = r_{i_l} - r_{j_l} \), for \( l = 1, \ldots, n \). The number of connected sequences for which the \( \xi_l \)'s are linearly dependent scales with a lower power of \( N \), and, therefore, they do not contribute as \( N \to \infty \). Only connected sequences with the \( \xi_l \)'s linearly independent contribute to the leading term in \( N \) of the \( n \)-th order of the cumulant expansion. For such sequences, if the potential decays sufficiently fast at long distances, the integral of \( \phi_{i_1j_1} \cdots \phi_{i_nj_n} \) over the \( \xi_l \)'s gives a number, \( \chi_{i_1j_1 \cdots i_nj_n} \), that is independent of the volume if \( V \) is large. The integral over the remaining variables gives a power of the volume, \( V^{N-n} \). Therefore, we have:

\[ \frac{1}{V^N} \int_{V^N} \prod_{i=1}^{N} d^3r_i \phi_{i_1j_1} \cdots \phi_{i_nj_n} \approx \frac{\chi_{i_1j_1 \cdots i_nj_n}}{V^n}. \]

(14)

The connected correlation function is a sum of products of integrals of the above type and has the same behavior. Collecting all the results, we have

\[ \sum_{i_1<j_1} \cdots \sum_{i_n<j_n} \langle \phi_{i_1j_1} \cdots \phi_{i_nj_n} \rangle_c \approx N^{n+1} \frac{c_n \chi_n}{V^n}, \]

(15)

where \( \chi_n \) is the average of \( \chi_{i_1j_1 \cdots i_nj_n} \) over the connected sequences. This behavior is broken down for \( n \sim N \), but the contribution of these terms to the cumulant expansion is negligible. Then, the \( n \)-th term is

\[ (-i)^n \frac{c_n \chi_n}{n!} \frac{\omega^n}{N^{\alpha n-1}} \left( \frac{N}{V} \right)^n. \]

(16)

Performing the change of variables \( \omega \to N\omega \) in the integral of Eq. (11), we get for \( \alpha = 1 \)

\[ G_V^{(1)}(u) \approx \frac{V^N}{N} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{N[i\omega u + \Omega(i\omega)\rho]}, \]

(17)
where
\[ \Omega(x) = \sum_{n=0}^{\infty} (-1)^n \frac{c_n x^n}{n!}. \] (18)

The integral of Eq. (17) can be evaluated by the saddle point method, and gives
\[ G_{V}^{(1)}(u) \approx \frac{V^N}{N} \exp\left[Ng(u/\rho)\right], \] (19)
where
\[ g(u/\rho) = \tilde{\omega}(u/\rho)u/\rho + \Omega[\tilde{\omega}(u/\rho)], \] (20)
and \( \tilde{\omega}(u/\rho) \) is the solution of
\[ \Omega'(\tilde{\omega}) = -u/\rho. \] (21)

Therefore, if the potential is short ranged, \( G^{(1)}(u) \) verifies the scaling of Eq. (7) and the thermodynamical functions are well behaved in the thermodynamical limit.

What about a self-gravitating system? Since the potential decays as the inverse distance, we have that, in the same conditions as before
\[ \frac{1}{V^N} \int_{V^N} \prod_{k=1}^{N} d^3 r_k \phi_{i_1 j_1} \cdots \phi_{i_n j_n} \approx \frac{\chi_{i_1 j_1 \cdots i_n j_n}}{R^n}, \] (22)
where \( R \) is the linear size of the system \( (V = R^3) \). The same arguments of the above paragraphs lead to the following expression for the \( n \)-th term of the cumulant expansion:
\[ (-i)^n \frac{\chi_n}{n!} \frac{\omega^n}{N^{n-1}} V^{2n/3} \left( \frac{N}{V} \right)^n. \] (23)

The above expression shows that, as expected, the asymptotic behavior (7) does not hold. However, we see that for \( \alpha = 5/3 \) the \( n \)-th order term of the cumulant expansion is
\[ (-i)^n \frac{\chi_n}{n!} \frac{\omega^n}{N^{n-1}} \left( \frac{N}{V} \right)^{n/3}. \] (24)

Hence, changing the variable \( \omega \rightarrow N\omega \) in the integral of Eq. (11) and using the saddle point method, leads to
\[ G_{V}^{(5/3)}(u) \approx \frac{V^N}{N} \exp\left[Ng(u/\rho^{1/3})\right]. \] (25)
for a self-gravitating system.

Now, the saddle point equation that gives the MC partition function is

$$\frac{\partial g(u/\rho^{1/3})}{\partial u} = \frac{3}{2} \frac{1}{E/N^{5/3} - u}. \tag{26}$$

The solution is a function of $E/N^{5/3}$ and $\rho$, $u = \rho^{1/3} \bar{u}_m[E/(\rho^{1/3} N^{5/3})]$, and therefore the entropy is not extensive$^1$. The MC temperature, $T_{MC} = (\partial \ln Z_{MC}/\partial E)^{-1}$, is

$$T_{MC} = \frac{2E}{3N} - \frac{2}{3} N^{2/3} \rho^{1/3} \bar{u}_m[E/(N^{5/3} \rho^{1/3})]. \tag{27}$$

We see that the MC temperature diverges as the number of particles increases. This is a manifestation of the so called gravothermal catastrophe [5].

Let us analyze the canonical ensemble. If the temperature is fixed, the contribution of $G_V^{(5/3)}(u)$ is negligible compared with the contribution of the Boltzmann weight, $\exp[-\beta N^{5/3} u]$, and the saddle point solution is given by $u = \bar{u}_c = u_0$, where $u_0$ is the minimum possible value of $u$, that is, the minimum potential energy. Hence, the system is completely collapsed. The only way to avoid the complete collapse of the system as the number of particles increases is to increase the temperature by a factor $N^{2/3}$, in agreement with the MC analysis.

The thermodynamical potentials for self-gravitating systems are non extensive and the system will be inhomogeneous. The thermodynamical limit is singular and gives ill behaved thermodynamical functions.

Up to here, everything is known. There is, however, another interesting possibility: as de Vega and Sánchez pointed out [1], it seems that the function $G_V^{(1)}(u)$ will have the asymptotic behavior as $\exp[N g(u, \sigma)]$ if we keep $\sigma = N/R$ constant (instead of $\rho = N/V$) as $N \to \infty$. They called this the diluted limit. Indeed, plugging $V = R^3$ in Eq. (23), the $n$-th term of the cumulant expansion of $G_V^{(1)}(\omega)$ reads

$$(-i)^n \frac{\chi_n}{n!} \frac{\omega^n}{N^{n-1}} \left( \frac{N}{R} \right)^n. \tag{28}$$

Then, $G_V^{(1)}(u) \approx \exp[N g(u, \sigma)]$, and the thermodynamical potentials will scale with $N$. In particular, the canonical partition function will scale as

$$Z_C \approx \frac{V^N}{N} e^{-N f(\beta, \sigma)}. \tag{29}$$

$^1$It is not an homogeneous function of $N$, $E$, and $V$. 
The system will not be extensive, however, and will develop inhomogeneities, but the thermodynamical functions will be well behaved in the thermodynamical (diluted) limit [1].

The above proof of the existence of the diluted thermodynamical limit is formal, since it relies on a series expansion, the convergence of which has not been demonstrated. Indeed, we are going to show that the diluted limit cannot give well behaved thermodynamical functions. Let us consider the canonical partition function on a volume $V = R^3$ in the diluted regime, so that $N \sim R$, and let us take a portion of such volume of linear size $R_0 < R$, such that $N \sim R_0^3$. Then, we obviously have

$$Z_C \geq \frac{1}{N!} \beta^{-3N/2} \int_{V_0^N} \prod_{i=1}^N d^3r_i \exp[-\beta \sum_{i<j} \phi_{ij}] \geq V_0^N N! \exp[-\beta \sum_{i<j} \langle \phi_{ij} \rangle_{V_0}],$$

(30)

where

$$\langle \phi_{ij} \rangle_{V_0} = \frac{1}{V_0^N} \int_{V_0^N} \prod_{k=1}^N d^3r_k \phi_{ij},$$

(31)

and the last inequality follows from a well known property of the exponential function: $\langle \exp(y) \rangle \geq \exp(\langle y \rangle)$. Since $\langle \phi_{ij} \rangle_{V_0} = -\kappa/R_0$, where $\kappa > 0$ is a geometrical number independent of $R_0$ if $R_0$ is large, we have

$$Z_C \geq \frac{V_0^N}{N!} \exp[\beta N(N-1)\kappa/R_0].$$

(32)

Recalling that $R_0 \sim L_0 N^{1/3}$, where $L_0$ is a fixed number with dimensions of length, we have

$$Z_C \geq \frac{N^N}{N!} \exp[N^{5/3} \beta \kappa / L_0].$$

(33)

Therefore, the canonical partition function cannot behave as $\exp[-N f(\beta, \sigma)]$ in the diluted thermodynamical limit. Rather, it behaves as in the usual thermodynamical limit. The reason is clear: the partition function is dominated by collapsed configurations even in the diluted regime. The gain in entropy provided by the dilution, which is of the order $N \ln V$, cannot compete with the energy gain due to collapse, which of the order $N^{5/3}$.

If the canonical partition function does not scale as $\exp[-N f(\beta, \sigma)]$, it is impossible that $C^{(1)}_V(u) \approx \exp[N g(u, \sigma)]$. This is in conflict with the result of the cumulant expansion. The inequality (32) that originates this conflict is rigorous, so that the fallacy must be found in the cumulant expansion.
The solution of the paradox is that the cumulant expansion for $G_V^{(1)}(u)$ is dominated by terms of the order of $N$. It should converge for any finite $N$, but the radius of convergence shrinks to zero as $N \to \infty$. Indeed, $\tilde{G}_V^{(1)}(N\omega)$ is the analytical continuation of the $Z_C(\beta)$ to the imaginary axis. If the radius of convergence of the cumulant expansion were finite in the diluted thermodynamical limit, it would imply [cf. Eq. (17)]

$$\tilde{G}_V^{(1)}(N\omega) \approx e^{N\Omega(i\omega)} .$$

(34)

The asymptotic behavior of the above equation is not compatible with Eq. (33). The cumulant expansion for $\Omega(x)$ given by Eq. (18) must have a vanishing convergence radius in the diluted thermodynamical limit of a self-gravitating system.

There is a clear explanation of the failure of the cumulant expansion. For imaginary $\omega$, $\tilde{G}_V^{(1)}(N\omega)$ is a canonical partition function at temperature $1/\text{Im}(\omega)$. The cumulant expansion relies on connected correlation functions computed with a flat measure, $V^{-N} \prod_i d^3r_i$, instead of the Boltzmann weight, $Z_C^{-1} \prod_i d^3r_i \exp[-\text{Im}(\omega) \sum \phi_{ij}]$. The cumulant expansion, then, will be valid when the two measures are similar, i.e., in a gas phase. In a self-gravitating system, Eq. (33) indicates that the canonical partition function is completely dominated by the potential energy, and therefore that collapse takes place, at temperatures smaller or of the order of $N^{2/3}$. Hence, for imaginary $\omega$, the cumulant expansion will be only valid for $\text{Im}(\omega) < N^{-2/3}$. Thus, the convergence radius of the cumulant expansion shrinks to zero as $N^{-2/3}$ in the diluted thermodynamical limit.

The collapse in the thermodynamical limit can be avoided by rescaling properly the radius of the hard core, $a$. Indeed, it is known that in mean field theory the collapse phase covers the whole phase diagram in the limit in which the "filling" parameter, $Na^3/R^3$, tends to zero [6]. The "filling" parameter remains constant in the usual thermodynamical limit. To keep it constant in the diluted limit, the hard core must be scaled as $a \sim N^{2/3}$. Then, previous argument does not apply, since such big particles do not fit in a volume of linear size $R_0 \sim N^{1/3}$. The minimum size of a region able to enclose the $N$ particles must have a linear size scaling as $N$. Hence, the diluted thermodynamical limit can exist. This is, however, a rather trivial and ad hoc way of avoiding collapse: particles are forced to remain far away one from another, and it is physically difficult to justify the scaling of the hard core. Moreover, this way of preventing collapse in the thermodynamical limit cannot apply to other types of short distance regularizations, such

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as softened potentials [7]. Our proof of the non-existence of the diluted thermodynamical limit is also valid for softened potentials.

It is generally believed that mean field theory is exact for self-gravitating systems. Since in mean field theory any thermodynamical function depends on the thermodynamical variables only through the dimensionless combinations \( \Lambda = -ER/(Gm^2N^2) \) (MC) or \( \eta = \beta Gm^2N/R \) (CE), the thermodynamical limit must be taken keeping either \( \Lambda \) or \( \eta \) finite. Then, it could be argued that the dependence of \( \Lambda \) or \( \eta \) on \( N \) can be absorbed in any of the quantities entering \( \Lambda \) or \( \eta \). For instance, one could take \( R \propto N \) and \( E \propto N \), as in the diluted limit. We will show in the following that this is not correct.

Consider a softened gravitational potential, for instance \( -\bar{\phi}(r) = \frac{GM^2\bar{r}}{R} \), with \( \bar{\phi}(r) = \frac{R}{\sqrt{r^2 + s^2}} \), where \( s \) is the softening scale. Dividing the finite region that encloses the system in \( W \) cells of linear size \( w^3 = R^3/W \), and denoting by \( n_i \) the occupation number of the \( i \)th cell, the partition functions (2) and (3) can be approximated by [2]:

\[
Z_{MC} = \frac{1}{\Gamma(3N/2)} \sum_{n_1=0}^{N} \cdots \sum_{n_W=0}^{N} \frac{\delta(\sum_l n_l - N)}{\prod_l n_l!} \left\{ -\Lambda + \sum_{i,j} n_i n_j \bar{\phi}_{ij} \right\}^{3N/2}, \tag{35}
\]

\[
Z_C = \frac{1}{\Gamma(3N/2)} \sum_{n_1=0}^{N} \cdots \sum_{n_W=0}^{N} \frac{\delta(\sum_l n_l - N)}{\prod_l n_l!} \exp\left\{ \eta \sum_{i,j} n_i n_j \bar{\phi}_{ij} \right\}, \tag{36}
\]

where \( \bar{\phi}_{ij} = \bar{\phi}(r_i - r_j) \) and \( r_i \) is the position of the center of the \( i \)th cell (we have again ignored constant factors). The above equalities hold rigorously in the limit \( W \to \infty \). For large \( N \), the factorials can be approximated by their asymptotic form. Defining the density by \( \rho_i = n_i/(Nw^3) \), so that \( \rho_i \in [0, 1/w^3] \) becomes a continuum variable as \( N \to \infty \), we have, ignoring constant factors

\[
Z_{MC} = \int \prod_{k=1}^{W} d\rho_k \delta(\sum_l w^3 \rho_l - 1) \times \\
\exp\left\{ N[-w^3 \sum_i (\rho_i \ln \rho_i - \rho_i) + 3\frac{1}{2} \ln[-\Lambda + \sum_{i,j} w^6 \rho_i \rho_j \bar{\phi}_{ij}]] \right\}, \tag{37}
\]

\[
Z_C = \int \prod_{k=1}^{W} d\rho_k \delta(\sum_l w^3 \rho_l - 1) \times \\
\exp\left\{ N[-w^3 \sum_i (\rho_i \ln \rho_i - \rho_i) + \eta \sum_{i,j} w^6 \rho_i \rho_j \bar{\phi}_{ij}] \right\}. \tag{38}
\]
Mean field theory is obtained by taking $N \to \infty$ before $W \to \infty$. In such case, the integrals in the above equations are saturated by the maximum of the integrands, provided that the number of cells, $W$, and $\Lambda$ or $\eta$, respectively, are kept constant. Obviously, the maximum of the integrand is given by the maximum of the entropy per particle,

\[
S = - \int d^3r [\rho(r) \ln \rho(r) - \rho(r)] + \frac{3}{2} \ln [-\Lambda + \int d^3r d^3r' \rho(r) \rho(r') \tilde{\phi}(r-r')],
\tag{39}
\]

in the MC, and with the minimum of the free energy per particle,

\[
F = - \int d^3r [\rho(r) \ln \rho(r) - \rho(r)] + \eta \int d^3r d^3r' \rho(r) \rho(r') \tilde{\phi}(r-r'),
\tag{40}
\]

in the CE, with the constraint $\int d^3r \rho(r) = 1$, if the grid is fine enough. The thermodynamical functions depend therefore on two dimensionless parameters: $s/R$ and $\Lambda$ or $\eta$. However, to derive mean field theory rigorously, one has to take the limit $N \to \infty$ in (37) and (38), keeping the number of cells constant. Otherwise, if $W$ grows with $N$, the integrals of Eqs. (37) and (38) cannot be evaluated by the saddle point. This implies that $R$ must be kept constant. Therefore, the factors $N$ entering $\Lambda$ and $\eta$ cannot be absorbed in $R$. They can be absorbed in $G$ or in $m^2$. Hence, mean field theory does not support the diluted limit either.

In conclusion, the thermodynamical limit of a self-gravitating system does not exist, either in the usual form or in the diluted regime of Ref. [1]. The meaning of non-existence of the thermodynamical limit is that the thermodynamical potentials do not scale properly with $N$ and thus thermodynamical functions, such as temperature, diverge. Nevertheless, it is possible to take the usual thermodynamical limit and, consequently, to use safely the usual thermodynamical tools by first regularizing the long distance behavior of the gravitational potential, introducing a very large screening length. The system is then thermodynamically stable and the thermodynamical limit does exist. Afterwards, one can study the limit in which the screening length tends to infinity [8].

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