STRICHARTZ ESTIMATES FOR 2D-SCALING INVARIANT ELECTROMAGNETIC WAVES

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Abstract. We prove Strichartz estimates for the 2D-wave equation with a scaling-critical electromagnetic potential. This problem is doubly critical, because of the scaling invariance of the model and the singularities of the potentials, which are not locally integrable. In particular, the diamagnetic phenomenon allows to consider negative electric potential which can be singular in the same fashion as the inverse-square potential.

Key Words: Decay estimates, Strichartz estimates, Aharonov-Bohm magnetic field, scaling-critical electromagnetic potential, wave equation

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1. Introduction

Let us consider the following initial-value problem for the wave equation
\[
\begin{aligned}
\partial_{tt} u + L_{A,a} u &= 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\
u(0, x) &= f(x), \quad \partial_t u(0, x) = g(x).
\end{aligned}
\]
Here, the operator $L_{A,a}$ is defined by
\[
L_{A,a} = \left( i \nabla + \frac{A(\hat{x})}{|x|} \right)^2 + \frac{a(\hat{x})}{|x|^2},
\]
where $\hat{x} = \frac{x}{|x|} \in S^1$, $a \in W^{1,\infty}(S^1, \mathbb{R})$ and $A \in W^{1,\infty}(S^1; \mathbb{R}^2)$ satisfies the transversality condition
\[
A(\hat{x}) \cdot \hat{x} = 0, \quad \text{for all } x \in \mathbb{R}^2.
\]

Our two main examples are the following:
- the Aharonov-Bohm potential
  \[
a \equiv 0, \quad A(\hat{x}) = \alpha \left( -\frac{x_2}{|x|}, \frac{x_1}{|x|} \right), \quad \alpha \in \mathbb{R},
\]
  introduced in [1], in the context of Schrödinger dynamics, to show that scattering effects can even occur in regions in which the electromagnetic field is absent (see also [27]);
- the inverse-square potential
  \[
A \equiv 0, \quad a(\hat{x}) \equiv a > 0.
\]

Throughout this paper, we will always assume that
\[
\|a_\cdot\|_{L^\infty(S^1)} < \min_{k \in \mathbb{Z}} \{(|k - \Phi_A|)^2, \Phi_A \notin \mathbb{Z}, \}
\]
where $a_- := \max\{0, -a\}$ is the negative part of $a$, and $\Phi_A$ is the total flux along the sphere

$$\Phi_A = \frac{1}{2\pi} \int_0^{2\pi} A(\theta) \, d\theta,$$

where $A(\theta)$ is defined by (2.4) below. Indeed, thanks to the Hardy inequality

$$\min_{k \in \mathbb{Z}} \{ |k - \Phi_A| \}^2 \int_{\mathbb{R}^2} \frac{|f|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^2} |\nabla_A f|^2 \, dx, \quad \nabla_A := i\nabla + A,$$

(see [22], and [16, cf. (27)]), thanks to assumption (1.6) the Hamiltonian $\mathcal{L}_{A,a}$ can be defined as a self-adjoint operator on $L^2$, via Friedrichs’ Extension Theorem (see e.g. [20, Thm. VI.2.1] and [26, X.3]), on the natural form domain, which in 2D turns out to be equivalent to

$$H^1_{A,0} := \left\{ f \in L^2(\mathbb{R}^2; \mathbb{C}) : \int_{\mathbb{R}^2} |\nabla_A f|^2 \, dx < +\infty \right\}$$

(see [16] cf. Lemma 23 - (ii) for details). Therefore, the Spectral Theorem allows us to consider the dispersive propagators $e^{it\mathcal{L}_{A,a}}, \cos(t\sqrt{\mathcal{L}_{A,a}}), \frac{\sin(t\sqrt{\mathcal{L}_{A,a}})}{\sqrt{\mathcal{L}_{A,a}}}$, as one-parameter groups of operators on $L^2$. In particular, the unique solution to (1.1) can be represented by

$$u(t, \cdot) = \cos(t\sqrt{\mathcal{L}_{A,a}}) f(\cdot) + \frac{\sin(t\sqrt{\mathcal{L}_{A,a}})}{\sqrt{\mathcal{L}_{A,a}}} g(\cdot).$$

One of the main mathematical features of the wave equation (1.1) is the scaling invariance, namely

$$u_\lambda(t, x) := \lambda^2 u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right) \quad \Rightarrow \quad (\partial_t + \mathcal{L}_{A,a}) u_\lambda(t, x) = (\partial_t u + \mathcal{L}_{A,a} u)(\frac{t}{\lambda}, \frac{x}{\lambda}).$$

This makes the dispersive evolution in (1.1) critical with respect to a large class of phenomena, as e.g. time-decay, Strichartz and smoothing estimates. The validity of such properties has been object of deep investigation in the last decades, due to their relevance in the description of linear and nonlinear dynamics. We now briefly sketch the state of the art about these problems.

**Purely electric case $A \equiv 0$.** The first available results are due to Burq, Planchon, Stalker, and Tahvildar-Zadeh in [23], in which they proved the validity of Strichartz estimates for the Schrödinger and wave equations, in space dimension $n \geq 2$. Assumption (1.6) is replaced by the natural one which involves the usual Hardy inequality. For the inverse-square potential $a(\hat{x}) \equiv a \in \mathbb{R}$ it reads, in dimension $n \geq 3$, as $a > -(n-2)^2/4$, while $a \geq 0$ is needed in dimension $n = 2$, due to the lack of Hardy inequality. More recently, Mizutani treated in [24] the analog problem for Schrödinger for the critical inverse-square $a = -(n-2)^2/4$, in dimension $n \geq 3$. Later, Fanelli, Felli, Fontelos, and Primo proved in [12,13] investigated the validity of the time-decay estimate for the Schrödinger evolution, and proved that it holds, in some specific cases, including the inverse square potential. A quite interesting remark in [12,13] is that the usual time-decay for the Schrödinger equation does not hold in the range $-\frac{(n-2)^2}{4} < a < 0$, while Strichartz estimates are true.
Electromagnetic case. If a magnetic field is present, the picture is much more unclear. After a sequel of papers (see [7–11, 29] and the references therein) in which time-decay or Strichartz estimates are studied, with subcritical magnetic potentials, in [13], the author noticed that the space dimension \( n = 2 \) is very peculiar for this kind of problems. Indeed, from one side the critical potential \( A/|x| \) is not in \( L^2_{\mathrm{loc}} \), which means that the domain \( H^1_{A,0} \) is strictly contained in \( H^1 \); from the other side, since the associated spherical problem is 1-dimensional, several explicit expansions are available, leading to quite complete results. For examples, the time-decay estimate

\[
\|e^{it\mathcal{L}_{A,a}}\|_{L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)} \lesssim |t|^{-1}
\]

is proved in [13], provided \((1.3)\) holds, and \( a(\hat{x}) > 0 \). This implies Strichartz estimates for \( e^{it\mathcal{L}_{A,a}} \), by the usual Keel-Tao argument [21]. We also mention that the behavior \( A \sim |x| \) is known to be critical for the validity of Strichartz estimates, as proved e.g. in [17] in the case of the Schrödinger equation.

A crucial role in [12, 13] is played by the pseudoconformal invariance of the Schrödinger equation, which together with a suitable transformation permits to pass to a Hamiltonian with an explicit, purely discrete spectrum, obtaining a nice representation formula for the solution in the physical space. This argument is still not available for the wave equation. Very recently, in [31], the authors were able to prove the validity of the time-decay estimate for the 2D-wave equation with an Aharonov-Bohm field, but this argument does not seem to be extendable to more generale critical electromagnetic fields.

In view of the above comments, the aim of this paper is to prove Strichartz estimates for equation \((1.1)\). In order to do this, let us introduce some preliminary notations. In the following, the Sobolev spaces will be denoted by

\[
\begin{align*}
\dot{H}^{s}_{A,a}(\mathbb{R}^2) &:= L^2_{\text{loc}}(\mathbb{R}^2), \\
\dot{H}^{s}(\mathbb{R}^2) &:= \dot{H}^{s}_{0,0}(\mathbb{R}^2), \\
H^{s}_{A,a}(\mathbb{R}^2) &:= L^2(\mathbb{R}^2) \cap \dot{H}^{s}_{A,a}(\mathbb{R}^2), \\
H^{s}(\mathbb{R}^2) &:= H^{s}_{0,0}(\mathbb{R}^2). 
\end{align*}
\]

We say \((q, r)\) is a (2D)-wave-admissible pair, if

\[
(q, r) \in \Lambda_s^W := \{(q, r) \in [2, \infty] \times [2, \infty), \quad \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}, \quad s = 2\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q}\}, \quad s \in \mathbb{R}. 
\]

We remark that \( 0 \leq s < 1 \), otherwise the set \( \Lambda_s^W \) is empty (see Figure [1]). It is well known by [21] that there exists a constant \( C > 0 \) such that the solution to the free wave equation \( u(t, \cdot) := \cos(t\sqrt{-\Delta})f(\cdot) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g(\cdot) \) satisfies the Strichartz estimate

\[
\|u\|_{L^q_t(\mathbb{R}; L^r(\mathbb{R}^2))} \leq C(\|f\|_{\dot{H}^{s}_0(\mathbb{R}^2)} + \|g\|_{\dot{H}^{s-1}_0(\mathbb{R}^2)}),
\]

for any wave-admissible pair \((q, r)\).
We are now ready to state our main result.

**Theorem 1.1.** Let \( a \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R}) \), \( A \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2) \), assume (1.3), (1.6), and let \( u \) be as in (1.9). Then there exists a constant \( C \) such that

\[
\|u\|_{L^q_t(L^r(R^2))} \leq C \left( \|f\|_{\dot{H}^s_A(R^2)} + \|g\|_{\dot{H}^{-1,s}_A(R^2)} \right),
\]

for any \( s \in \mathbb{R} \) and \((q,r) \in \Lambda^W_s\).

**Remark 1.1.** Theorem 1.1 is a completely new result for the wave equation with a critical magnetic field. We stress that a time-decay estimate as (1.10) is not available, in this setting. We find particularly interesting the inequality (1.13), in the case of a negative inverse-square electric potential \( a(\hat{x}) = a \), with

\[
-\min_{k \in \mathbb{Z}} (|k - \Phi_A|^2)^2 < a < 0,
\]

for which the role played by the magnetic potential is crucial. In analogy with the case of the Schrödinger equation, time-decay should not hold in the range (1.14), but this is an open question.

**Remark 1.2.** Related to the above remark, another interesting open question is concerned with the critical inverse square potential

\[ a(\hat{x}) = \min_{k \in \mathbb{Z}} (|k - \Phi_A|^2)^2. \]

In this case, in analogy with [24], one may ask about the validity of (1.13), and in particular of the endpoint estimate.

**Remark 1.3.** Notice that the magnetic Sobolev norms \( \dot{H}^s_{A,0} \) at the right-hand side of (1.13) cannot be replaced by the usual Sobolev norms \( \dot{H}^s \), since \( \dot{H}^s(\mathbb{R}^2) \setminus \dot{H}^s_{A,0}(\mathbb{R}^2) \neq \emptyset \), for critical magnetic potentials, hence the evolution cannot be well defined on \( \dot{H}^s \). On the other hand, in dimension \( n \geq 3 \), the spaces \( \dot{H}^s_{A,0} \) and \( \dot{H}^s \) are equivalent (see [16, cf. Lemma 2.3 - (i)] for details), therefore one can wonder whether Strichartz estimates like

\[
\|u\|_{L^q_t(L^r(R^2))} \leq C \left( \|f\|_{\dot{H}^s(\mathbb{R}^2)} + \|g\|_{\dot{H}^{-1,s}(\mathbb{R}^2)} \right)
\]

hold, in dimension \( n \geq 3 \), for \( n \)-wave admissible pairs

\[
(q,r) \in \Lambda^W_s := \left\{ 2 \leq q, r \leq \infty, r \neq \infty, \frac{2}{q} + \frac{n-1}{r} \leq \frac{n-1}{2}, s = n \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{1}{2} \right\}, \quad s \in \mathbb{R}. \]

The validity of (1.14) for \( A \neq 0 \) is a completely open problem, at our knowledge.
The proof of Theorem 1.1 is inspired to the perturbation arguments in [2, 3]. Nevertheless, to treat $L_{A,a}$ as a perturbation of $-\Delta$, as in that case, involving local smoothing estimates, would require an estimate like
$$\| x^{-\frac{1}{2}} e^{it\sqrt{L_{A,a}}} f \|_{L^2_x(\mathbb{R}^2)} \leq \| f \|_{L^2(\mathbb{R}^2)},$$
to handle the first-order term coming from the magnetic potential. Unfortunately, this estimate is known to be false, even in the free case, by the standard Agmon-Hörmander Theory. To overcome this difficulty, we treat $L_{A,a}$ as an electric perturbation of the purely magnetic operator $L_{A,0}$. For $L_{A,0}$, thanks to the transversality condition (1.3) and the geometric features of the dimension $n = 2$, we can apply the distorted Fourier transform argument, which leads to the desired results.

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## 2. Harmonic Analysis on the Operator $L_{A,a}$

In this section, we study the harmonic analytical features of the operator $L_{A,a}$, relying on the distorted Fourier transform introduced in [31] and the spectral properties proved in [12].

First of all, by using polar coordinated, we can write
$$L_{A,a} = -\frac{\partial^2}{r^2} - \frac{1}{r} \partial_r + \frac{L_{A,a}}{r^2}, \quad (2.1)$$
where
$$L_{A,a} = (i\nabla_{S^1} + A(\hat{x}))^2 + a(\hat{x})$$
$$= -\Delta_{S^1} + \left( (|A(\hat{x})|^2 + a(\hat{x}) + i \text{div}_{S^1} A(\hat{x})) \right) + 2iA(\hat{x}) \cdot \nabla_{S^1}. \quad (2.2)$$

Let $\hat{x} = (\cos \theta, \sin \theta)$: then
$$\partial_\theta = -\hat{x}_2 \partial_{\hat{x}_1} + \hat{x}_1 \partial_{\hat{x}_2}, \quad \partial_\theta^2 = \Delta_{S^1}. \quad (2.3)$$
We define $A(\theta) : [0, 2\pi] \rightarrow \mathbb{R}$ as follows
$$A(\theta) = A(\cos \theta, \sin \theta) \cdot (-\sin \theta, \cos \theta). \quad (2.4)$$
Hence by (1.3) we can write
$$A(\cos \theta, \sin \theta) = A(\theta)(-\sin \theta, \cos \theta), \quad \theta \in [0, 2\pi], \quad (2.5)$$
therefore we obtain
$$L_{A,a} = -\Delta_{S^1} + \left( (|A(\theta)|^2 + a(\theta) + i \text{div}_{S^1} A(\hat{x})) \right) + 2iA(\hat{x}) \cdot \nabla_{S^1}$$
$$= -\partial_\theta^2 + \left( |A(\theta)|^2 + a(\theta) + iA'(\theta) \right) + 2iA(\theta) \partial_\theta$$
$$=: L_{A,a}. \quad (2.6)$$
2.1. The Fourier analysis associated with $L_{A,0}$. We now focus our attention on the purely magnetic operator $L_{A,0}$. We start with a result about the distorted plane wave associated with $L_{A,0}$.

**Proposition 2.1.** Let $x = r \dot{x} = r(\cos \theta, \sin \theta) \in \mathbb{R}^2, \xi = \lambda \dot{\xi} = \lambda(\cos \omega, \sin \omega)$, where $\theta, \omega \in [0, 2\pi]$. Then, there holds

$$\varphi(x, \xi) = \varphi(r, \theta; \lambda, \omega) = e^{-ir\xi} e^{i \int_0^\lambda A(\theta')d\theta'},$$

such that

$$L_{A,0}(\varphi(x, \xi)) = |\xi|^2 \varphi(x, \xi),$$

which implies that $\varphi(x, \xi)$ is the distorted plane wave of the operator $L_{A,0}$.

**Proof.** Since $x \cdot \xi = r\lambda \cos(\theta - \omega)$, we get

$$\varphi(r, \lambda, \theta, \omega) = e^{-ir\lambda \cos(\theta - \omega)} e^{i \int_0^\lambda A(\theta')d\theta'}.$$

Then,

$$\partial_r \varphi = -i\lambda \cos(\theta - \omega) \varphi$$
$$\partial_\theta^2 \varphi = -\lambda^2 \cos^2(\theta - \omega) \varphi$$
$$\partial_\theta \varphi = [ir\lambda \sin(\theta - \omega) + iA(\theta)] \varphi$$
$$\partial_\theta^2 \varphi = \{[ir\lambda \cos(\theta - \omega) + iA'(\theta)] + [ir\lambda \sin(\theta - \omega) + iA(\theta)]^2\} \varphi$$

$$= \{[ir\lambda \cos(\theta - \omega) + iA'(\theta)] - 2r^2 \lambda^2 \sin^2(\theta - \omega) - 2r\lambda \sin(\theta - \omega) - |A(\theta)|^2\} \varphi$$

and hence

$$-\partial_\theta^2 + 2iA(\theta) \partial_\theta + \left(\frac{\lambda^2}{r^2} \cos(\theta - \omega) - \frac{i\lambda}{r} \cos(\theta - \omega)\right) \varphi = \left[\lambda^2 \sin^2(\theta - \omega) - \frac{i\lambda}{r} \cos(\theta - \omega)\right] \varphi.$$

From (2.4) and (2.6), this implies

$$L_{A,0} \varphi = |\xi|^2 \varphi. \tag{2.9}$$

Thus, we conclude the proof of Proposition 2.1. $\square$

We can now define the distorted Fourier transform associated with the operator $L_{A,0}$.

**Definition 2.1** (Distorted Fourier transform). For the function $f, g \in L^2 \cap L^1$, we define the distorted Fourier transform of $f$ as follows

$$\mathcal{F}(f) = \hat{f}(\xi) = \hat{f}(\lambda \cos \omega, \lambda \sin \omega)$$

$$= \int_0^\infty \int_0^{2\pi} e^{-ir\lambda \cos(\theta - \omega)} e^{i \int_0^\lambda A(\theta')d\theta'} f(r, \theta) rdrd\theta, \tag{2.10}$$

and the inverse distorted Fourier transform of $g$ is defined by

$$\mathcal{F}^{-1}(g) = \hat{g}(x) = \hat{g}(r \cos \theta, r \sin \theta)$$

$$= \int_0^\infty \int_0^{2\pi} e^{ir\lambda \cos(\theta - \omega)} e^{-i \int_0^\lambda A(\theta')d\theta'} g(\lambda, \omega) \lambda d\lambda d\omega. \tag{2.11}$$
Lemma 2.1 (Properties of the distorted Fourier transform). The distorted Fourier transform satisfies the following properties:

1. \( \mathcal{F}^{-1} \mathcal{F} = \text{Id} \) on \( L^2 \cap L^1 \);
2. the Plancherel identity holds on \( L^2 \cap L^1 \): \( \langle f, f \rangle = \langle \hat{f}, \hat{f} \rangle \).

Proof. To prove (1), let \( y = (\rho \cos \phi, \rho \sin \phi) \). We have

\[
\mathcal{F}^{-1}(\hat{f})(x) = \int_0^\infty \int_0^{2\pi} e^{ir\lambda \cos(\theta-\omega)} e^{-i \int_0^\theta A(\theta') d\theta'} \hat{f}(\lambda, \omega) \lambda \, d\lambda \, d\omega
\]

Using the distorted Fourier transform, we get the following explicit formula for the transform that satisfies the following properties:

\[
\mathcal{F}^{-1}(\hat{f})(x) = \int_0^\infty \int_0^{2\pi} e^{ir\lambda \cos(\theta-\omega)} e^{-i \int_0^\theta A(\theta') d\theta'} \hat{f}(\lambda, \omega) \lambda \, d\lambda \, d\omega
\]

The proof of (2) is completely analogous. \( \square \)

Using the distorted Fourier transform, we get the following explicit formula for the functional calculus.

Lemma 2.2. Let \( F \) be the Borel measure function, and \( x = r(\cos \theta, \sin \theta) \) and \( y = \rho(\cos \phi, \sin \phi) \). Then, the kernels of the operator \( F(\sqrt{\mathcal{L}}_{A,0}) \) satisfy

\[
F(\sqrt{\mathcal{L}}_{A,0})(x, y) = \int_0^\infty \int_0^{2\pi} e^{ir\lambda \cos(\theta-\omega)} e^{-i \int_0^\theta A(\theta') d\theta'} \lambda \, d\lambda \, d\omega
\]

where

\[
F(\sqrt{-\Delta})(x, y) = \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} F(|\xi|) d\xi.
\]

2.2. Magnetic Besov and Sobolev spaces. Let \( \varphi \in C_c^\infty(\mathbb{R} \setminus \{0\}) \), with \( 0 \leq \varphi \leq 1 \), \( \text{supp} \varphi \subset [1/2, 1] \), and

\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \lambda) = 1.
\]
and from the magnetic potential \( f \) (2.21), we write

\[
\varphi_j(\sqrt{L_{\mathbf{A},0}}) f(x) = \int_{\mathbb{R}^2} e^{-i \int_0^y A(\theta') d\theta'} \int_{\mathbb{R}^2} e^{-i(x-y) \cdot \xi} \varphi(2^{-j} |\xi|) \, d\xi \, f(y) \, dy, \tag{2.15}
\]

where \( x = r(\cos \theta, \sin \theta) \) and \( y = \rho(\cos \phi, \sin \phi) \).

**Definition 2.2** (Magnetic Besov and Sobolev spaces associated with \( L_{\mathbf{A},0} \)). For \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), the norm of \( \| \cdot \|_{B^s_{p,r}(\mathbb{R}^2)} \) is given by

\[
\|f\|_{B^s_{p,r}(\mathbb{R}^2)} = \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \| \varphi_j(\sqrt{L_{\mathbf{A},0}}) f\|_{L^p(\mathbb{R}^2)}^{1/r} \right)^{1/r}. \tag{2.16}
\]

In particular, for \( p = r = 2 \), we have

\[
\|f\|_{\dot{H}^s_{p,r}(\mathbb{R}^2)} := \|L_{\mathbf{A},0}^s f\|_{L^2(\mathbb{R}^2)} = \left( \sum_{j \in \mathbb{Z}} 2^{2js} \| \varphi_j(\sqrt{L_{\mathbf{A},0}}) f\|_{L^2(\mathbb{R}^2)} \right)^{1/2} = \|f\|_{B^s_{2,2}(\mathbb{R}^2)}. \tag{2.17}
\]

**Remark 2.1.** By Lemma 2.2, we easily see that we have the following equivalence

\[
\|f(r, \theta)\|_{\dot{B}^s_{p,r}(\mathbb{R}^2)} \sim \|e^{-i \int_0^y A(\theta') d\theta'} f(r, \theta)\|_{\dot{B}^s_{p,r}(\mathbb{R}^2)}. \tag{2.18}
\]

2.3. **Spectral properties of \( L_{\mathbf{A},0} \).** In this subsection, we consider the perturbation from the magnetic potential \( \mathbf{A} \) and electrical potential \( a \).

First of all, recall (2.6). The operator \( L_{\mathbf{A},a} \) on \( L^2(\mathbb{S}^1) \) has a compact inverse, hence by classical Spectral Theory, its spectrum is purely discrete, and it is made by a countable family of real eigenvalues with finite multiplicity. We denote them with \( \{ \mu_k(\mathbf{A}, a) \}_{k=1}^\infty \), enumerated such that

\[
\mu_1(\mathbf{A}, a) \leq \mu_2(\mathbf{A}, a) \leq \cdots \tag{2.19}
\]

and we repeat each eigenvalue as many times as its multiplicity, and \( \lim_{k \to \infty} \mu_k(\mathbf{A}, a) = +\infty \) (see [16, Lemma A.5] for further details).

**Remark 2.2.** We remark that

\[
\mu_1(\mathbf{A}, 0) = \min_{k \in \mathbb{Z}} \{ |k| - \Phi_\mathbf{A} |^2 \} \tag{2.20}
\]

(see [16, (28)]). Notice that assumption (1.6) implies that \( \mu_1(\mathbf{A}, a) > 0 \).

For each \( k \in \mathbb{N}, k \geq 1 \), let \( \psi_k(\hat{x}) \in L^2(\mathbb{S}^1) \) be the normalized eigenfunction of the operator \( L_{\mathbf{A},a} \) corresponding to the \( k \)-th eigenvalue \( \mu_k(\mathbf{A}, a) \), i.e. satisfying that

\[
\begin{cases}
L_{\mathbf{A},a} \psi_k(\hat{x}) = \mu_k(\mathbf{A}, a) \psi_k(\hat{x}) & \text{on } \mathbb{S}^1; \\
\int_{\mathbb{S}^1} |\psi_k(\hat{x})|^2 d\hat{x} = 1.
\end{cases} \tag{2.21}
\]

For \( f \in L^2(\mathbb{R}^2) \), based on the \( \{ \psi_k(\theta) \}_{k=0}^\infty \) where \( \psi_0(\theta) = 1/\sqrt{2\pi} \) and \( \psi_k(\theta) \) given in (2.21), we write \( f \) into the form of separating variables

\[
f(x) = \sum_{k=0}^\infty c_k(r) \psi_k(\theta) \tag{2.22}
\]
where
\[ c_k(r) = \int_0^{2\pi} f(r, \theta) \psi_k(\theta) d\theta, \]
then
\[ \|f(r, \theta)\|_{L^2([0, 2\pi])}^2 = \sum_{k=0}^{\infty} |c_k(r)|^2. \tag{2.23} \]
From the fact that
\[ \mathcal{L}_{A,a} = -\partial_r^2 - \frac{1}{r} \partial_r + \frac{L_{A,a}}{r^2}, \tag{2.24} \]
then, on each space \( \mathcal{H}^k = \text{span}\{\psi_k\}\), the action of the operator is given by
\[ \mathcal{L}_{A,a} = -\partial_r^2 - \frac{1}{r} \partial_r + \frac{\mu_k}{r^2}. \]
Let \( \nu = \nu_k = \sqrt{\mu_k} \), for \( f \in L^2(\mathbb{R}^2) \), we define the Hankel transform of order \( \nu \)
\[ (H_{\nu} f)(\rho, \theta) = \int_0^\infty J_{\nu}(r\rho) f(r, \theta) r dr, \tag{2.25} \]
where the Bessel function of order \( \nu \) is given by
\[ J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_{-1}^{1} e^{irs} (1 - s^2)^{(2\nu - 1)/2} ds, \quad \nu > -1/2, r > 0. \tag{2.26} \]

**Lemma 2.3.** Let \( J_\nu(r) \) be the Bessel function defined in (2.26) and \( R \gg 1 \), then there exists a constant \( C \) independent of \( \nu \) and \( R \) such that
\[ |J_\nu(r)| \leq \frac{Cr^\nu}{2^n \Gamma(\nu + \frac{1}{2})} \left( 1 + \frac{1}{\nu + 1/2} \right), \tag{2.27} \]
and
\[ \int_R^{2R} |J_\nu(r)|^2 dr \leq C. \tag{2.28} \]

**Proof.** The first one is obtained by a direct computation. The inequality (2.28) is a direct consequence of the asymptotically behavior of Bessel function; see [23, Lemma 2.2]. \( \square \)

The Hankel transform satisfies the following properties (see [22,28]):

**Lemma 2.4.** Let \( \mathcal{H}_\nu \) be defined as above and \( A_\nu := -\partial_r^2 - \frac{1}{r} \partial_r + \frac{\nu^2}{r^2} \). Then
(i) \( \mathcal{H}_\nu = \mathcal{H}_{-\nu}^{-1} \),
(ii) \( \mathcal{H}_\nu \) is self-adjoint, i.e. \( \mathcal{H}_\nu = \mathcal{H}_\nu^* \),
(iii) \( \mathcal{H}_\nu \) is an \( L^2 \) isometry, i.e. \( \|\mathcal{H}_\nu \phi\|_{L^2} = \|\phi\|_{L^2} \),
(iv) \( \mathcal{H}_\nu(A_\nu \phi)(\xi) = |\xi|^2 (\mathcal{H}_\nu \phi)(\xi) \), for \( \phi \in L^2 \).

Briefly recalling the functional calculus for well-behaved functions \( F \) (see [30]),
\[ F(\mathcal{L}_{A,a}) f(r, \theta) = \sum_{k=0}^{\infty} \psi_k(\theta) \int_0^\infty F(\rho^2) J_{\nu_k}(r\rho) b_k(\rho) \rho d\rho \tag{2.29} \]
where \( b_k(\rho) = (\mathcal{H}_{\nu_k} a_k)(\rho) \) and \( f(r, \theta) = \sum_{k=0}^{\infty} a_k(r) \psi_k(\theta) \).
2.4. Sobolev spaces.

Lemma 2.5. Let \( a \in W^{1,\infty}(S^1, \mathbb{R}) \), \( A \in W^{1,\infty}(S^1, \mathbb{R}^2) \), and assume (1.6). Then

\[
\|f\|_{\dot{H}^{-s}_{A,a}(\mathbb{R}^2)} \simeq \|f\|_{\dot{H}^{-s}_{A,a}(\mathbb{R}^2)},
\]
for all \( s \in [-1, 1] \).

Proof. For \( s = 1 \), the proof is an immediate consequence of assumption (1.8). Indeed, we have

\[
\|f\|_{\dot{H}^{1}_{A,a}(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} (|\nabla A f|^2 + \frac{a(\hat{x})}{|x|^2} |f|^2) dx
\]
then

\[
\int_{\mathbb{R}^2} (|\nabla A f|^2 - \frac{a_-(\hat{x})}{|x|^2} |f|^2) dx \leq \|f\|_{\dot{H}^{1}_{A,a}(\mathbb{R}^2)}^2 \leq \int_{\mathbb{R}^2} (|\nabla A f|^2 + \frac{a_+(\hat{x})}{|x|^2} |f|^2) dx
\]
where \( a_- := \max\{0, -a\} \) and \( a_+ := \max\{0, a\} \). From (1.6) and (1.8), there exist a small constant \( c \) and a large constant \( C \) such that

\[
\int_{\mathbb{R}^2} (|\nabla A f|^2 - \frac{a_-(\hat{x})}{|x|^2} |f|^2) dx \geq (1 - \frac{\|a_-\|_{L^\infty(S^1)}}{\min_{k \in \mathbb{Z}} \{|k - \Phi A|^2\}}) \int_{\mathbb{R}^2} |\nabla A f|^2 dx
\]
\[
\geq c \int_{\mathbb{R}^2} |\nabla A f|^2 dx
\]
and

\[
\int_{\mathbb{R}^2} (|\nabla A f|^2 + \frac{a_+(\hat{x})}{|x|^2} |f|^2) dx \leq (1 + \frac{\|a_+\|_{L^\infty(S^1)}}{\min_{k \in \mathbb{Z}} \{|k - \Phi A|^2\}}) \int_{\mathbb{R}^2} |\nabla A f|^2 dx
\]
\[
\leq C \int_{\mathbb{R}^2} |\nabla A f|^2 dx.
\]

Then, by duality and interpolation, one obtains the full range \( s \in [-1, 1] \). \( \square \)

Finally, we derive the following Sobolev embedding.

Lemma 2.6. Let \( a \in W^{1,\infty}(S^1, \mathbb{R}) \), \( A \in W^{1,\infty}(S^1, \mathbb{R}^2) \), and assume (1.6). Then,

\[
\|f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{\dot{H}^{1-s}_{A,a}(\mathbb{R}^2)}
\]
for any \( 2 \leq p < +\infty \).

Proof. In the purely magnetic case \( a \equiv 0 \), this immediately follows by the usual Sobolev embedding \( H^{1-s}_{\mathbb{R}^2} \hookrightarrow L^p(\mathbb{R}^2) \), and the diamagnetic inequality

\[
|\nabla_A f(x)| \leq |\nabla f(x)|.
\]
Then the proof follows by Lemma 2.5. \( \square \)
3. The proof of Strichartz estimates

We devote this section to the proof of Theorem 1.1. First notice that, by the representation formula (1.9) and the equivalence in Lemma 2.5, it is sufficient to prove the following estimate

\[
\|e^{it\sqrt{\mathcal{L}_A}} f\|_{L^q_t L^r_x(\mathbb{R}^2)} \lesssim \|f\|_{\dot{H}^s_{A,0}(\mathbb{R}^2)}. \tag{3.1}
\]

We will first prove (3.1) in the purely magnetic case \(a = 0\), and then in the general case, as a consequence of a local smoothing estimate.

3.1. **Strichartz estimates for purely magnetic waves.** Let us start with the purely magnetic case \(a = 0\). Our first step is to prove the following claim

\[
\|e^{it\sqrt{\mathcal{L}_A}} f\|_{L^q_t L^r_x(\mathbb{R}^2)} \lesssim \|f\|_{\dot{H}^s_{A,0}(\mathbb{R}^2)}, \tag{3.2}
\]

for \(s \in \mathbb{R}\), any wave-admissible pair \((q, r) \in \Lambda^W_A\) as in (1.12), \(f \in \dot{H}^s_{A,0}(\mathbb{R}^2)\), and for some \(C > 0\) independent on \(f\). To this aim, we follow a similar argument as in [31].

**Lemma 3.1.** Let \(\varphi \in C^\infty_0(\mathbb{R} \setminus \{0\})\), with \(0 \leq \varphi \leq 1\), and supp \(\varphi \subset [1/2, 2]\), as in (2.14). Then for all \(j \in \mathbb{Z}\), there exists a constant \(C\) independent of \(x\) and \(y\) such that

\[
\left| \int_{\mathbb{R}^2} e^{-i(x-y)\xi} e^{it|\xi|} \varphi(2^{-j}|\xi|) d\xi \right| \leq C \|2^{\frac{3j}{2}} (2^{-j} + |t|)^{-\frac{3}{2}}. \tag{3.3}
\]

The proof is obtained by Stationary Phase, and can be found in [31, Lemma 2.3].

**Proposition 3.1.** Let \(U(t) = e^{it\sqrt{\mathcal{L}_A}} \) and \(f = \varphi_j(\sqrt{\mathcal{L}_A}) f\) as in (2.15) for \(j \in \mathbb{Z}\), then

\[
\|U(t)f\|_{L^q_t L^r_x(\mathbb{R}^2)} \lesssim 2^j \|f\|_{L^2(\mathbb{R}^2)}, \tag{3.4}
\]

where \(s \in \mathbb{R}\) and \((q, r) \in \Lambda^W_A\) defined in (1.12).

**Proof.** The proof follows the argument of [31, Proposition 3.1] with minor modification. For the sake of completeness, we provide the details. Let \(\tilde{\varphi} \in C^\infty_0(\mathbb{R} \setminus \{0\})\), with \(0 \leq \tilde{\varphi} \leq 1\) such that \(\tilde{\varphi}\varphi = \varphi\). We can write

\[
U(t)f = U(t)\varphi_j(\sqrt{\mathcal{L}_A}) \tilde{\varphi}_j(\sqrt{\mathcal{L}_A}) f
= \int_{\mathbb{R}^2} e^{-i \int_0^t A(\theta)d\theta} \int_{\mathbb{R}^2} e^{-i((x-y)\xi) e^{it|\xi|}} \varphi(2^{-j}|\xi|) d\xi \tilde{\varphi}_j(\sqrt{\mathcal{L}_A}) f(y) dy.
\]

Define the operator \(U_j(t) : L^2 \to L^2\)

\[
U_j(t) = e^{-i \int_0^t A(\theta)d\theta} \int_{\mathbb{R}^2} e^{-i((x-y)\xi) e^{it|\xi|}} \varphi(2^{-j}|\xi|) d\xi,
\]

and notice that

\[
U_j(t)U_j^*(s) = e^{-i \int_0^s A(\theta)d\theta} \int_{\mathbb{R}^2} e^{-i((x-y)\xi) e^{i(t-s)|\xi|}} \varphi(2^{-j}|\xi|) d\xi.
\]

From (3.3) and the unitary property of \(U_j\), we see there exists a constant \(C\) such that

\[
\|U_j(t)\|_{L^2 \to L^2} \leq C, \quad t \in \mathbb{R},
\]

\[
\|U_j(t)U_j(s)^*\|_{L^1 \to L^\infty} \leq C 2^{\frac{3j}{2}} (2^{-j} + |t-s|)^{-\frac{3}{2}}. \tag{3.5}
\]
Now we prove (3.4). We first consider the estimates on the board line, that is, \((q,r)\) satisfies \(\frac{2}{q} + \frac{1}{r} = \frac{1}{2}\). This will be done by following the method of Keel-Tao [21]. Indeed, the Keel-Tao’s argument [21, Sections 3-7] shows (3.4) since we can replace \(|t-s| + 2^{-j}\) by \(|t-s|^{-1/2}\) to satisfy the condition [21 (2)] with \(\sigma = 1/2\).

Next we only consider \(\frac{2}{q} + \frac{1}{r} < \frac{1}{2}\). By the \(TT^*\) argument, it suffices to show

\[\left| \int \int (U_j(s)^* f(s), U_j(t)^* g(t)) ds dt \right| \lesssim 2^{2sj} \| f \|_{L^q_t L^r_x} \| g \|_{L^q_t L^r_x}.\]

Using the bilinear interpolation of (3.5), we have

\[\langle U_j(s)^* f(s), U_j(t)^* g(t) \rangle \leq C 2^{\frac{s}{2}} (1 - \frac{q}{2})^j (2^{-j} + |t-s|)^{-\frac{1}{2}(1 - \frac{q}{2})} \| f \|_{L^r_t} \| g \|_{L^r_t}.
\]

Therefore, we see by Hölder’s and Young’s inequalities for \(\frac{2}{q} + \frac{1}{r} < \frac{1}{2}\)

\[\left| \int \int (U_j(s)^* f(s), U_j(t)^* g(t)) ds dt \right| \lesssim 2^{\frac{s}{2}} (1 - \frac{q}{2})^j \left( \int \int (2^{-j} + |t-s|)^{-\frac{1}{2}(1 - \frac{q}{2})} \| f \|_{L^r_t} \| g \|_{L^r_t} ds dt \right) \lesssim 2^{\frac{s}{2}} (1 - \frac{q}{2})^j \| f \|_{L^q_t L^r_x} \| g \|_{L^q_t L^r_x}.
\]

Note \(s = 2(\frac{1}{2} - \frac{1}{q}) - \frac{1}{q}\), this proves (3.4).

\[\square\]

**Proposition 3.2** (Littlewood-Paley square function inequality). Let \(\mathcal{L}_{A,0}\) be the Schrödinger operator as in (1.2). Then for \(1 < p < \infty\), there exist constants \(c_p\) and \(C_p\) depending on \(p\) such that

\[c_p \| f \|_{L^p(\mathbb{R}^2)} \leq \left( \sum_{j \in \mathbb{Z}} \| \varphi_j(\sqrt{\mathcal{L}_{A,0}}) f \|_2^2 \right)^{\frac{1}{2}} \leq C_p \| f \|_{L^p(\mathbb{R}^2)} \quad (3.6)\]

where the Littlewood-Paley operator \(\varphi_j(\sqrt{\mathcal{L}_{A,0}})\) is defined in (2.15).

**Proof.** From (2.12), we see the relationship between the two kernels

\[\varphi_j(\sqrt{\mathcal{L}_{A,0}})(x,y) = e^{-it \int_0^\theta A(\theta')d\theta'} \varphi_j(\sqrt{-\Delta})(x,y), \quad (3.7)\]

where \(x = (r \cos \theta, r \sin \theta), \ y = (r \cos \phi, r \sin \phi)\). Then we see

\[|\varphi_j(\sqrt{\mathcal{L}_{A,0}}) f| = |\varphi_j(\sqrt{-\Delta}) g|, \quad g(r, \theta) = e^{it \int_0^\theta A(\theta')d\theta'} f(r, \theta). \quad (3.8)\]

Using the Littlewood-Paley square function estimates associated with \(-\Delta\) and the fact \(\| g \|_{L^p} = \| f \|_{L^p}\), we obtain (3.6).

\[\square\]

We are now ready to prove the claim (3.2), in the case \(a \equiv 0\). Indeed, \(q, r \geq 2\), and by (3.6) and Minkowski’s inequality we get

\[\| e^{it \sqrt{\mathcal{L}_{A,0}} f} \|_{L^q(\mathbb{R}; L^r_x(\mathbb{R}^2))} \lesssim \left( \sum_{j \in \mathbb{Z}} \| e^{it \sqrt{\mathcal{L}_{A,0}}} \varphi_j(\sqrt{\mathcal{L}_{A,0}}) f \|_{L^q(\mathbb{R}; L^r_x(\mathbb{R}^2))}^2 \right)^{\frac{1}{2}}. \quad (3.9)\]
Proposition 3.3. Let \( L \) be the spherical operator in (2.2), with first eigenvalue \( \mu_1(A, a) \) as in (2.19), and denote by \( \nu_0 := \sqrt{\mu_1(A, a)} \). Then there exists a constant \( C > 0 \) such that, for any \( f \in \dot{H}^{\beta - \frac{1}{2}} \), 
\[
\| \nu^{-\beta} e^{it \sqrt{L}} f \|_{L^2_t(L^r_x)} \leq C \| f \|_{\dot{H}^{\beta - \frac{1}{2}}},
\]
for any \( \beta \in \left( \frac{1}{2}, 1 + \nu_0 \right) \).

Remark 3.1. The first endpoint \( \beta = \frac{1}{2} \) in (3.11) is known to be false, even in the free case \( A \equiv a \equiv 0 \), by the usual Agmon-Hörmander Theory (see e.g. [19] and the references therein). As for the second endpoint \( \beta = 1 + \nu_0 \), this equals 1, in the free case. In the perturbed case, thanks to assumption (1.6), we have \( \mu_1(A, a) > 0 \), hence \( \nu_0 > 0 \) is well defined and we get an improvement in the range of validity of the estimate. This fact has been already observed in several papers, for different evolution models (see e.g. [4, 5, 14, 15, 18, 25]). In addition, a further improvement occurs for higher frequencies. Indeed, if \( f(x) \) belongs to \( \bigoplus_{\nu > k} \dot{H}^\nu \cap \dot{H}^{\beta - \frac{1}{2}}(\mathbb{R}^2) \) where \( k > \nu_0 \), then one can relax the upper restriction on \( \beta \) to \( \beta < 1 + k \).

Proof. Suppose that 
\[
f(x) = \sum_{k=0}^{\infty} a_k(r) \psi_k(\theta), \quad b_k(\rho) = (\mathcal{H}_\nu a_k)(\rho).
\]
We want to estimate 
\[
e^{it \sqrt{L} A} f = \sum_{k=0}^{\infty} \psi_k(\theta) \int_0^{\infty} J_{\nu_k}(r \rho) e^{it \rho} b_k(\rho) d\rho, \quad \nu_k = \sqrt{\mu_k}.
\]
By the Plancherel theorem with respect to time \( t \), it suffices to estimate the term 
\[
\int_{\mathbb{R}^2} \int_0^{\infty} | \sum_{k=0}^{\infty} \psi_k(\theta) J_{\nu_k}(r \rho) b_k(\rho) \rho |^2 d\rho |x|^{-2\beta} dx
\]
\[
= \sum_{k=0}^{\infty} \int_0^{\infty} \int_0^{\infty} | J_{\nu_k}(r \rho) b_k(\rho) \rho |^2 d\rho r^{1-2\beta} dr.
\]
Let \( \chi \) be a smoothing function supported in \([1, 2]\), we make dyadic decompositions to obtain
\[
\sum_{k=0}^\infty \sum_{M \in 2^k} \int_0^\infty \int_0^\infty |J_{\nu_k}(r\rho)b_k(\rho)\rho|^2 d\rho r^{-1-2\beta} dr
\]
\[
\lesssim \sum_{k=0}^\infty \sum_{M \in 2^k} \sum_{R \in 2^z} M^{1+2\beta} R^{1-2\beta} \int_R^{2R} \int_0^\infty |J_{\nu_k}(r\rho)b_k(M\rho)\chi(\rho)|^2 d\rho dr.
\]
Let
\[
Q_k(R, M) = \int_R^{2R} \int_0^\infty |J_{\nu_k}(r\rho)b_k(M\rho)\chi(\rho)|^2 d\rho dr.
\]
We now claim that the following holds inequality:
\[
Q_k(R, M) \lesssim \begin{cases} R^{2\nu_k+1} M^{-2} ||b_k(\rho)\chi(\frac{\rho}{M})\rho^{1/2}||^2_{L^2}, & R \lesssim 1; \\ M^{-2} ||b_k(\rho)\chi(\frac{\rho}{M})\rho^{1/2}||^2_{L^2}, & R \gg 1. \end{cases}
\]

**Proof of (3.16).** We consider two different cases.

- **Case 1:** \( R \lesssim 1 \). Since \( \rho \sim 1 \), thus \( r\rho \lesssim 1 \). By (2.27), we obtain
\[
Q_k(R, M) \lesssim \int_R^{2R} \int_0^\infty \frac{(r\rho)^\nu}{2^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} b_k(M\rho)\chi(\rho)^2 d\rho dr
\]
\[
\lesssim R^{2\nu_k+1} M^{-2} ||b_k(\rho)\chi(\frac{\rho}{M})\rho^{1/2}||^2_{L^2}.
\]

- **Case 2:** \( R \gg 1 \). Since \( \rho \sim 1 \), thus \( r\rho \gg 1 \). We estimate by (2.28) in Lemma 2.3
\[
Q_k(R, M) \lesssim \int_0^\infty |b_k(M\rho)\chi(\rho)|^2 dr \int_R^{2R} |J_{\nu}(r\rho)|^2 dr \rho
\]
\[
\lesssim \int_0^\infty |b_k(M\rho)\chi(\rho)|^2 d\rho M^{-2} ||b_k(\rho)\chi(\frac{\rho}{M})\rho^{1/2}||^2_{L^2}.
\]
This concludes the proof of (3.16). \( \square \)

With (3.16) in hand, we can now estimate
\[
\sum_{k=0}^\infty \sum_{M \in 2^k} \sum_{R \in 2^z} M^{1+2\beta} R^{1-2\beta} \int_R^{2R} \int_0^\infty |J_{\nu_k}(r\rho)b_k(M\rho)\chi(\rho)|^2 d\rho dr
\]
\[
\lesssim \sum_{k=0}^\infty \sum_{M \in 2^k} \sum_{R \in 2^z} M^{1+2\beta} R^{1-2\beta} Q_k(R, M)
\]
\[
\lesssim \sum_{k=0}^\infty \sum_{M \in 2^k} \left( \sum_{R \in 2^z, R \leq 1} M^{1+2\beta} R^{1-2\beta} R^{2\nu_k+1} M^{-2} + \sum_{R \in 2^z, R \gg 1} M^{1+2\beta} R^{1-2\beta} M^{-2} \right) ||b_k(\rho)\chi(\frac{\rho}{M})\rho^{1/2}||^2_{L^2}
\]
\[
\lesssim \sum_{k=0}^\infty \sum_{M \in 2^k} \left( \sum_{R \in 2^z, R \leq 1} M^{2\beta-1} R^{2(1+\nu_k-\beta)} + \sum_{R \in 2^z, R \gg 1} M^{2\beta-1} R^{1-2\beta} \right) ||b_k(\rho)\chi(\frac{\rho}{M})\rho^{1/2}||^2_{L^2}.
\]
Under the assumption: $\frac{1}{2} < \beta < 1 + \nu_0$, we sum in $R$ to get
\[
\sum_{k=0}^{\infty} \sum_{M \in 2^\mathbb{Z}} M^{2^3-1} \left\| b_k(\rho) \chi \left( \left( \frac{\rho}{M^2} \right) \right) \rho^{\frac{1}{2}} \right\|^2_{L^2} = \left\| f \right\|^2_{L^2(R^2)}.
\]
(3.17)

Indeed, it follows from (2.29) that
\[
\mathcal{L}_{A,a}^2 f (r, \theta) = \sum_{k=0}^{\infty} \int_{0}^{\rho} \rho^s \psi_k(\tau) b_k(\rho) \rho d\rho = \sum_{k=0}^{\infty} \psi_k(\theta) \mathcal{H}_{\nu(k)} \left( \rho^s b_k(\rho) \right)(r).
\]

And so we obtain
\[
\left\| f \right\|^2_{L^2(R^2)} = \left\| \mathcal{L}_{A,a}^2 f \right\|^2_{L^2(R^2)} = \sum_{k=0}^{\infty} \int_{0}^{\rho} \rho^s \psi_k(\omega) \psi_k(\omega) \rho d\rho = \sum_{k=0}^{\infty} \int_{0}^{\rho} \rho^s \psi_k(\omega) \psi_k(\omega) \rho d\rho.
\]

Using Lemma 2.4 (iii), we get
\[
\left\| \mathcal{L}_{A,a}^2 f \right\|^2_{L^2(R^2)} = \sum_{k=0}^{\infty} \int_{0}^{\rho} \rho^s b_k(\rho) \psi_k(\omega) \rho d\rho = \sum_{k=0}^{\infty} \int_{0}^{\rho} \rho^s b_k(\rho) \rho d\rho.
\]

Applying the unit decomposition, one has
\[
\left\| \mathcal{L}_{A,a}^2 f \right\|^2_{L^2(R^2)} = \sum_{k=0}^{\infty} \int_{0}^{\rho} \sum_{M \in 2^\mathbb{Z}} \chi \left( \left( \frac{\rho}{M^2} \right) \right) \rho^s b_k(\rho) \rho d\rho
\]
\[
\approx \sum_{k=0}^{\infty} \sum_{M \in 2^\mathbb{Z}} \int_{0}^{\rho} \chi \left( \left( \frac{\rho}{M^2} \right) \right) \rho^s b_k(\rho) \rho d\rho
\]
\[
\approx \sum_{k=0}^{\infty} \sum_{M \in 2^\mathbb{Z}} M^s \left\| \chi \left( \left( \frac{\rho}{M^2} \right) \right) \rho^s b_k(\rho) \rho \right\|^2_{L^2}.
\]

This implies (3.17), hence we proved (3.11), and the proof of (3.11) is complete. □

3.3. Conclusion of the proof of Theorem 1.1. Let $u$ be the solution of (1.1), given by (1.3). The case $q = +\infty$ in Theorem 1.1 immediately follows by Spectral Theory and the Sobolev embedding in Lemma 2.6. Indeed, one has
\[
\left\| u(t, z) \right\|_{L^\infty(R; L^r(R^2))} \lesssim \left\| \mathcal{L}_{A,a}^2 u(t, x) \right\|_{L^\infty(R; L^2(R^2))}
\]
\[
\lesssim \left\| f \right\|_{L^2(R^2)} + \left\| g \right\|_{L^2(R^2)}
\]

where $s = 1 - \frac{2}{r}$ and $2 \leq r < +\infty$. 
Now, let \( v \) be the purely magnetic wave

\[
v(t, \cdot) := \cos(t \sqrt{\mathcal{L}_{A,0}}) f(\cdot) + \sin(t \sqrt{\mathcal{L}_{A,0}}) g(\cdot).
\]

By the Duhamel formula, we can hence write

\[
u(t, \cdot) = v(t, \cdot) - \int_0^t \sin(t - \tau) \sqrt{\mathcal{L}_{A,0}} \left( \frac{\alpha(x)}{|x|^2} u(\tau, \cdot) \right) d\tau.
\]  \quad (3.18)

By (3.2), it follows that

\[
\|v(t, x)\|_{L^q(R; L^r(R^2))} \leq C \left( \|f\|_{H^s_{A,0}} + \|g\|_{H^{s-1}_{A,0}} \right),
\]

for \( s \in \mathbb{R} \), any wave-admissible pair \((q, r) \in \Lambda^W_s\) as in (1.12), and for some \( C > 0 \) independent on \( f, g \). Therefore we get

\[
\|u(t, x)\|_{L^q(R; L^r(R^2))} \leq C \left( \|f\|_{H^s_{A,0}} + \|g\|_{H^{s-1}_{A,0}} \right) + \left\| \int_0^t \sin(t - \tau) \sqrt{\mathcal{L}_{A,0}} \left( \frac{\alpha(x)}{|x|^2} u(\tau, x) \right) d\tau \right\|_{L^q(R; L^r(R^2))}
\]  \quad (3.19)

Now our main task is to estimate the \( TT^* \)-term

\[
\left\| \int_0^t \sin(t - \tau) \sqrt{\mathcal{L}_{A,0}} \left( \frac{\alpha(x)}{|x|^2} u(\tau, x) \right) d\tau \right\|_{L^q(R; L^r(R^2))}.
\]  \quad (3.20)

Notice that if the set \( \Lambda^W_s \) is not empty, we must have \( 0 \leq s < 1 \). And when \( s = 0 \), we must have \((q, r) = (+\infty, 2)\). Hence we only need to study the range \( 0 < s < 1 \). We will treat separately the following two cases:

(i) \( 0 < s < \min \{1, \frac{2}{3} + \nu_0\} \),

(ii) \( \frac{1}{2} + \nu_0 \leq s < 1 \) with \( \nu_0 < \frac{1}{2} \).

**Case 1:** \( 0 < s < \min \{1, \frac{2}{3} + \nu_0\} \). Define the operator

\[
T : L^2(R^2) \to L^2(R; L^2(R^2)), \quad Tf = r^{-\beta} e^{i\theta} \sqrt{\mathcal{L}_{A,0}} \frac{1}{2} (1 - \beta) f.
\]

Thus from the proof of the local smoothing estimate, it follows that \( T \) is a bounded operator. By duality, we obtain that for its adjoint \( T^* \)

\[
T^* : L^2(R; L^2(R^2)) \to L^2, \quad T^* F = \int_{\tau \in \mathbb{R}} \mathcal{L}_{A,0}^{1/2} e^{-i\theta} \sqrt{\mathcal{L}_{A,0}} r^{-\beta} F(\tau) d\tau
\]

is also bounded. Define the operator

\[
B : L^2(R; L^2(X)) \to L^q(R; L^r(R^2)), \quad BF = \int_{\tau \in \mathbb{R}} e^{i\theta(t-\tau)} \sqrt{\mathcal{L}_{A,0}} r^{-\beta} F(\tau) d\tau.
\]
Hence by the Strichartz estimate (3.2) with $s = \frac{3}{2} - \beta$, one has

$$
\|BF\|_{L^{q}(\mathbb{R};L^{r}(\mathbb{R}^{2}))} = \left\| e^{it\sqrt{\mathcal{L}_{A,0}}} \int_{\mathbb{R}} e^{-it\sqrt{\mathcal{L}_{A,0}}} r^{-\beta} F(\tau) d\tau \right\|_{L^{q}(\mathbb{R};L^{r}(\mathbb{R}^{2}))}
\leq \left\| \int_{\mathbb{R}} e^{-it\sqrt{\mathcal{L}_{A,0}}} r^{-\beta} F(\tau) d\tau \right\|_{L^{q}(\mathbb{R};L^{r}(\mathbb{R}^{2}))} = \|T^{A} F\|_{L^{2} \lesssim \|F\|_{L^{2}(\mathbb{R};L^{2}(\mathbb{R}^{2}))}}.
$$

(3.21)

Now we estimate (3.20). Note that

$$
\sin(t-\tau)\sqrt{\mathcal{L}_{A,0}} = \frac{1}{2i} (e^{i(t-\tau)}\sqrt{\mathcal{L}_{A,0}} - e^{-i(t-\tau)}\sqrt{\mathcal{L}_{A,0}}),
$$

thus by (3.21), we have a minor modification of (3.20)

$$
\left\| \int_{\mathbb{R}} \frac{\sin(t-\tau)\sqrt{\mathcal{L}_{A,0}}}{\sqrt{\mathcal{L}_{A,0}}} (\frac{a(\xi)}{\xi^{2}} u(\tau, x)) d\tau \right\|_{L^{q}(\mathbb{R};L^{r}(\mathbb{R}^{2}))}
\leq \|B(r^{\beta} \frac{a(\xi)}{\xi^{2}} u(\tau, x))\|_{L^{q}(\mathbb{R};L^{r}(\mathbb{R}^{2}))}
\leq \|r^{\beta-2} u(\tau, x)\|_{L^{2}(\mathbb{R};L^{2}(\mathbb{R}^{2}))}
\leq \|f\|_{H^{\frac{3}{2}-\beta}_{A,0}(\mathbb{R}^{2})} + \|g\|_{H^{\frac{3}{2}-\beta}_{A,0}(\mathbb{R}^{2})}
$$

where we use the local smoothing estimate in Proposition 3.3 again in the last inequality and we need $1 - \nu_{0} < \beta < 3/2$ such that $1/2 < 2 - \beta < 1 + \nu_{0}$. Therefore the above statement holds for all $\max\{1/2, 1 - \nu_{0}\} < \beta < 3/2$. By the Christ-Kiselev lemma [6], thus we have showed that for $q > 4$ and $(q, r) \in \Lambda_{s}^{W}$ with $s = \frac{3}{2} - \beta$

$$
3.20 \lesssim \|f\|_{H^{\frac{3}{2}}_{A,s}(\mathbb{R}^{2})} + \|g\|_{H^{\frac{3}{2}-1}_{A,s}(\mathbb{R}^{2})}.
$$

(3.22)

Therefore we have proved all $(q, r) \in \Lambda_{s}^{W}$ when $s$ satisfies $0 < s < \min\{1, \frac{1}{2} + \nu_{0}\}$.

**Case 2:** $\frac{1}{2} + \nu_{0} \leq s < 1$ with $\nu_{0} < \frac{1}{2}$. To this end, we split the initial data into two parts: one is projected to $\mathcal{H}^{k}$ with $k \leq 1 + \nu_{0}$ and the other is the remaining terms. Without loss of generality, we assume $g = 0$ and divide $f = f_{1} + f_{h}$ where $f_{h} = f - f_{1}$ and

$$
f_{1}(x) = \sum_{k=0}^{1} a_{k}(r) \psi_{k}(\theta).
$$

(3.23)

For the part involving $f_{h}$, we can repeat the argument of **Case 1**. In this case, as remarked in Remark 3.1 we can use Proposition 3.3 with $1/2 < 2 - \beta < 2 + \nu_{0}$. Thus we obtain the Strichartz estimate on $e^{it\sqrt{\mathcal{L}_{A,0}}} f_{h}$ for $\Lambda_{s}^{W}$ with $s \in [\frac{1}{2} + \nu_{0}, 1)$.  

**STRICHARTZ ESTIMATES** 17
Next we consider the Strichartz estimate on \( e^{it\sqrt{L_{\Lambda^a}}} f_i \). We follow the argument of [28] which treated a radial function. Recall from (2.29)

\[
e^{it\sqrt{L_{\Lambda^a}}} u_{0,1}(x) = \sum_{k=0}^{\infty} \psi_k(\theta) \int_0^\infty J_{\nu(k)}(r \rho) e^{it \rho \mathcal{H}_{\nu(k)}(a_k) \rho} \, d\rho,
\]

\[
= \sum_{k=0}^{\infty} \psi_k(\theta) \mathcal{H}_{\nu(k)}[e^{it \rho \mathcal{H}_{\nu(k)}(a_k)}](r).
\]

Since \( \psi_k(\theta) \in L^r(S^1) \), we get

\[
\|e^{it\sqrt{L_{\Lambda^a}}} u_{0,1}\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^2))} \leq C \sum_{k=0}^{\infty} \|\mathcal{H}_{\nu(k)}[e^{it \rho \mathcal{H}_{\nu(k)}(a_k)}](r)\|_{L^q(\mathbb{R}; L^r_{rdr})}.
\]

Recall \( \mathcal{H}_0 \mathcal{H}_0 = Id \), then it suffices to estimate

\[
\sum_{k=0}^{\infty} \|\mathcal{H}_{\nu(k)}(\mathcal{H}_0 \mathcal{H}_0)[e^{it \rho \mathcal{H}_{\nu(k)}(a_k)}](r)\|_{L^q(\mathbb{R}; L^r_{rdr})}.
\]

For our purpose, we recall [28, Theorem 3.1] which claimed that the operator \( K_{0, \nu}^0 := \mathcal{H}_\mu \mathcal{H}_\nu \) is continuous on \( L^p_{r=1,dr} ([0, \infty)) \) if

\[
\max \left\{ \frac{1}{n} \left( \frac{n^2}{2} - \mu \right), 0 \right\} < \frac{1}{n} < \min \left\{ \frac{1}{2} \left( \frac{n^2}{2} + \nu + 2 \right), 1 \right\}.
\]

Notice \( n = 2 \), we obtain that \( K_{0, \nu}^0 \) and \( K_{\nu, 0}^0 \) are bounded in \( L^p_{rdr} ([0, \infty)) \) provided \( p > 2 \) and \( \nu > 0 \). On the other hand, \( \mathcal{H}_0[e^{it \rho \mathcal{H}_0}] \) is a classical half-wave propagator in the radial case which has Strichartz estimate with \( (q, r) \in \Lambda^W_q \). In sum, for \( (q, r) \in \Lambda^W_q \), we have

\[
\|e^{it\sqrt{L_{\Lambda^a}}} f_i\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^2))} \leq C \sum_{k=0}^{\infty} \|\mathcal{H}_{\nu(k)}(\mathcal{H}_0 \mathcal{H}_0)[e^{it \rho \mathcal{H}_{\nu(k)}(a_k)}](r)\|_{L^q(\mathbb{R}; L^r_{rdr})}
\]

\[
\leq C \sum_{k=0}^{\infty} \|\mathcal{H}_0 \mathcal{H}_{\nu(k)}(a_k)(r)\|_{L^q_{r^2 \rho^2}} \leq C \left( \sum_{k=0}^{\infty} \|a_k(r)\|_{L^q_{r^2 \rho^2}}^2 \right)^{1/2} \|f_i\|_{L^q_{r^2 \rho^2}}.
\]

In the second inequality, we use [28, Theorem 3.8]. Therefore, we conclude the proof of Theorem 1.1.

REFERENCES

[1] Y. Aharonov, and D. Bohm, Significance of electromagnetic potentials in quantum theory, Phys. Rev. Lett. 115 (1959), 485–491
[2] N. Burq, F. Planchon, J. Stalker, and A. S. Tahvildar-Zadeh, Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential, J. Funct. Anal. 203 (2003), 519-549
[3] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay, Indiana Univ. Math. J., 53 (2004), 1665–1680.
[4] F. Cacciafesta, and L. Fanelli, Dispersive estimates for the Dirac equation in an Aharonov-Bohm field, Journal of Differential Equations 263 (2017), 4382–4399.
F. Cacciafesta, and L. Fanelli, Weak dispersive estimates for fractional Aharonov-Bohm-Schrödinger groups, *Dynamics of PDE*, **10** (2013), 379–392.

M. Christ, and A. Kiselev, Maximal functions associated to filtrations, *J. Funct. Anal.*, **179** (2001), 409–425.

S. Cuccagna, and Schirmer, On the wave equation with a magnetic potential, *Comm. Pure Appl. Math.*, **54** (2001), 135–152.

P. D’Ancona, and L. Fanelli, Decay estimates for the wave and Dirac equations with a magnetic potential, *Comm. Pure Appl. Math.*, **60** (2007), 357–392.

P. D’Ancona, L. Fanelli, L. Vega, and N. Visciglia, Endpoint Strichartz estimates for the magnetic Schrödinger equation, *J. Funct. Anal.*, **258** (2010), 3227–3240.

M.B. Erdogan, M. Goldberg and W. Schlag, Strichartz and Smoothing Estimates for Schrödinger Operators with Almost Critical Magnetic Potentials in Three and Higher Dimensions, *Forum Math.*, **21** (2009), 687–722.

M.B. Erdogan, M. Goldberg and W. Schlag, Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in $\mathbb{R}^3$, *J. European Math. Soc.*, **10** (2008), 507–531.

L. Fanelli, V. Felli, M. A. Fontelos, and A. Primo, Time decay of scaling invariant electromagnetic Schrödinger equations on the plane, *Comm. Math. Phys.*, **337** (2015), 1515–1533.

L. Fanelli, V. Felli, M. A. Fontelos, and A. Primo, Time decay of scaling critical electromagnetic Schrödinger flows, *Comm. Math. Phys.*, **324** (2013), 1033–1067.

L. Fanelli, V. Felli, M. A. Fontelos, and A. Primo, Frequency-dependent time decay of Schrödinger flows, *J. Spectral Theory*, **8** (2018), 509–521.

L. Fanelli, G. Grillo, and H. Kovarik, Improved time-decay for a class of scaling-critical Schrödinger flows, *J. Func. Anal.*, **269** (2015), 3336–3346.

V. Felli, A. Ferrero, S. Terracini, Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential. *J. Eur. Math. Soc.* **13** (2011), 119-174.

M. Goldberg and W. Schlag, Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in $\mathbb{R}^3$, *J. European Math. Soc.*, **10** (2008), 507–531.

L. Fanelli and A. García, Counterexamples to Strichartz estimates for the magnetic Schrödinger equation, *Comm. Cont. Math.*, **13** (2011), 213–234.

G. Grillo, and H. Kovarik, Weighted dispersive estimates for two-dimensional Schrödinger operators with Aharonov-Bohm magnetic field, *J. Differential Equations* **256** (2014), 3889–3911.

A. Ionescu, and W. Schlag, Agmon-Kato-Kuroda theorems for a large class of perturbations *Duke Math. J.*, **131** (2006), 397–440.

T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.

M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math.*, **120** (1998), 955–980.

A. Laptev, and T. Weidl, Hardy inequalities for magnetic Dirichlet forms, *Mathematical results in quantum mechanics (Prague, 1998)*, 299–305; *Oper. Theory Adv. Appl.*, **108**, Birkhäuser, Basel, 1999.

C. Miao, J. Zhang and J. Zheng, A note on the cone restriction conjecture, *Proc. Amer. Math. Soc.*, **140** (2012), 2091–2102.

H. Mizutani, Remarks on endpoint Strichartz estimates for Schrödinger equations with the critical inverse-square potential, *J. Diff. Eq.*, **263** (2017), 3832–3853.

H. Mizutani, J. Zhang, and J. Zheng, Uniform resolvent estimates for Schrödinger operator with an inverse-square potential, *J. Func. Anal.*, **278**(2020), 108350.

M. Reed, and B. Simon., *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press, New York-London, 1975.

M. Peshkin, and A. Tonomura. *The Aharonov-Bohm Effect*. Lect. Notes Phys. **340** (1989).

F. Planchon, J. Stalker and A. S. Tahvildar-Zadeh, $L^p$ estimates for the wave equation with the inverse-square potential, *Discrete Contin. Dynam. Systems* **9** (2003), 427–442.
[29] W. Schlag. *Dispersive estimates for Schrödinger operators: a survey*. Mathematical aspects of nonlinear dispersive equations, 255–285, Ann. of Math. Stud. 163, Princeton Univ. Press, Princeton, NJ, 2007.

[30] M. Taylor, *Partial Differential Equations, Vol II*, Springer, 1996.

[31] J. Zhang, and J. Zheng, Decay and Strichartz estimates for dispersive equations in an Aharonov-Bohm field, arXiv:2003.03086.

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