SEVERAL EXPLICIT FORMULAE OF SUMS AND HYPER-SUMS OF POWERS OF INTEGERS

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Abstract. In this paper, we present several explicit formulas of the sums and hyper-sums of the powers of the first \((n+1)\)-terms of a general arithmetic sequence in terms of Stirling numbers and generalized Bernoulli polynomials.

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1. Introduction

The problem of finding formulas for sums of powers of integers has attracted the attention of many mathematicians and has been developed in several different directions. For a recent treatment and references, see [1, 2, 7, 10, 11, 12]. This paper is concerned both with sums \(S_{p,(a,d)}(n)\) and hyper-sums \(S_{p,(a,d)}^{(r)}(n)\) of the \(p\)-the powers of the first \((n+1)\)-terms of a general arithmetic sequence. Let

\[
S_{p,(a,d)}(n) = a^p + (a + d)^p + \cdots + (a + nd)^p
\]

be the power sum of arithmetic progression with \(n, p\) are non-negative integers and \(a\) and \(d\) are complex numbers with \(d \neq 0\).

For the most studied case \(a = 0\) and \(d = 1\)

\[
S_{p,(0,1)}(n) = \begin{cases} 
n + 1 & (p = 0) 
1^p + 2^p + 3^p + \cdots + n^p & (p > 0)
\end{cases}
\]

there have been a considerable number of results.

The basic properties for the \(S_{p,(a,d)}(n)\) can be obtained from the following generating function [8]

\[
\sum_{p \geq 0} S_{p,(a,d)}(n) \frac{z^p}{p!} = \sum_{k=0}^{n} e^{(a + kd)z},
\]

and we can easily verify that [13]

\[
S_{p,(a,d)}(n) = \frac{d^p}{p+1} \left( B_{p+1} \left( n + \frac{a}{d} + 1 \right) - B_{p+1} \left( \frac{a}{d} \right) \right),
\]

where \(B_n(x)\) denotes the classical Bernoulli polynomials, which are defined by the following generating function

\[
\frac{ze^{xz}}{e^z - 1} = \sum_{n \geq 0} B_n(x) \frac{z^n}{n!}.
\]
Recall that the weighted Stirling numbers $S^i_n(x)$ of the second kind are defined by (see \cite{5, 6})

\begin{align}
S^i_n(x) &= \frac{1}{i!} \Delta^i x^n \\
&= \frac{1}{i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} (x+j)^n,
\end{align}

where $\Delta$ denotes the forward difference operator. The exponential generating function of $S^i_n(x)$ is given by

\begin{equation}
\sum_{n=i}^{\infty} \frac{S^i_n(x) z^n}{n!} = \frac{1}{i!} e^{xz} (e^z - 1)^i
\end{equation}

and $S^i_n(x)$ satisfy the following recurrence relation:

\[ S^i_{n+1}(x) = S^{i-1}_n(x) + (x+i)S^i_n(x) \quad (1 \leq i \leq n). \]

In particular, we have for nonnegative integer $r$

\[ S^i_n(0) = \left\{ \begin{array}{c} n \\ i \end{array} \right\} \quad \text{and} \quad S^i_n(r) = \left\{ \begin{array}{c} n+r \\ i+r \end{array} \right\}_r, \]

where $\left\{ \begin{array}{c} n \\ i \end{array} \right\}_r$ denotes the $r$-Stirling numbers of the second kind \cite{4}. These numbers counts the number of partitions of a set of $n$ objects into exactly $k$ nonempty, disjoint subsets, such that the first $r$ elements are in distinct subsets.

For any positive integer $m$. The $r$-Whitney numbers of the second kind $W_{m,r}(n,i)$ are the coefficients in the expansion

\[ (mx+r)^n = \sum_{i=0}^{n} m^i W_{m,r}(n,i)x(x+1)\cdots(x+i-1), \]

and given by their generating function

\[ \sum_{n\geq i} W_{m,r}(n,i) \frac{z^n}{n!} = \frac{1}{m!} e^{rz} (e^{mz} - 1)^i. \]

Clearly, we have

\[ W_{1,0}(n,i) = \left\{ \begin{array}{c} n \\ i \end{array} \right\}, \quad W_{1,r}(n,i) = \left\{ \begin{array}{c} n+r \\ i+r \end{array} \right\}_r \]

and

\[ W_{m,r}(n,i) = m^{n-i} S^i_n \left( \frac{r}{m} \right). \]

For more details of these numbers see \cite{13}.
2. The sums of powers of integers $S_{p,(a,d)}(n)$

An explicit formula for $S_{p,(a,d)}(n)$ is given by the following:

**Theorem 1.** For all integers $n, p \geq 0$ and $a, d$ are complex numbers with $d \neq 0$, we have

$$S_{p,(a,d)}(n) = d^p \sum_{k=0}^{p} k! \left( \frac{n + 1}{k + 1} \right) S_{p}^{k} \left( \frac{a}{d} \right).$$

**Proof.** It follows from (1.5) that

$$\sum_{p \geq 0} \left( d^p \sum_{k=0}^{p} k! \left( \frac{n + 1}{k + 1} \right) S_{p}^{k} \left( \frac{a}{d} \right) \right) \frac{z^p}{p!} = \sum_{k \geq 0} k! \left( \frac{n + 1}{k + 1} \right) \sum_{p \geq 0} S_{p}^{k} \left( \frac{a}{d} \right) \frac{(dz)^p}{p!}$$

$$= e^{az} \sum_{k \geq 0} \left( \frac{n + 1}{k + 1} \right) (e^{dz} - 1)^k$$

$$= e^{az} \frac{e^{(n+1)dz} - 1}{e^{dz} - 1}$$

$$= \sum_{k=0}^{n} e^{(a+kd)z}$$

and the proof is complete. $\Box$

The following Corollary immediately follows from Theorem 1.

**Corollary 1.** If we assume that $d$ divides $a$, then we have for $p > 0$

$$S_{p,(a,d)}(n) = d^p \sum_{k=0}^{p} k! \left( \frac{n + 1}{k + 1} \right) \left\{ \frac{p + \frac{a}{d}}{k + \frac{a}{d}} \right\}. $$

The next Corollary contains an explicit formula for $S_{p,(a,d)}(n)$ expressed in terms of the $r$-Whitney numbers of the second kind $W_{m,r}(n,k)$.

**Corollary 2.** If we assume that $a$ and $d$ are coprime integers, then we have for $p \geq 0$

$$S_{p,(a,d)}(n) = \sum_{k=0}^{p} k! d^k \left( \frac{n + 1}{k + 1} \right) W_{a,d}(p,k).$$

An explicit formula for $S_{p,(a,d)}(n)$ involving Bernoulli polynomials is given by the following Theorem.

**Theorem 2.**

$$S_{p,(a,d)}(n) = \frac{d^p}{p + 1} \sum_{s=0}^{p} \binom{p + 1}{s} (n + 1)^{p+1-s} B_{s} \left( \frac{a}{d} \right),$$

**Proof.** It follows from (14) that

$$B_n(x) = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} S_{n}^{k}(x).$$

(2.1)
Thus (1.2) becomes
\[
S_{p,(a,d)}(n) = \frac{d^p}{p + 1} \sum_{k=0}^{p+1} (-1)^k \frac{k!}{k+1} \left( S_{p+1}^k \left( n + \frac{a}{d} + 1 \right) - S_{p+1}^k \left( n + \frac{a}{d} + 1 \right) \right).
\]

Now, from (1.4), we get
\[
S_{p,(a,d)}(n) = \frac{d^p}{p + 1} \sum_{k=0}^{p+1} (-1)^k \frac{k!}{k+1} \left( \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left[ \sum_{s=0}^{p} \binom{p + 1}{s} \left( \frac{a + j}{d + j} \right)^s \right] \right)
\]
\[
= \sum_{s=0}^{p} \binom{p + 1}{s} (n + 1)^{p+1-s} \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left( \frac{a + j}{d + j} \right)^s
\]
\[
= \frac{d^p}{p + 1} \sum_{k=0}^{p+1} (-1)^k \frac{k!}{k+1} \left( \sum_{s=0}^{p} \binom{p + 1}{s} (n + 1)^{p+1-s} S_{p+1}^k \left( \frac{a}{d} \right) \right)
\]
\[
= \frac{d^p}{p + 1} \sum_{s=0}^{p} \binom{p + 1}{s} (n + 1)^{p+1-s} \sum_{k=0}^{p+1} (-1)^k \frac{k!}{k+1} S_{p+1}^k \left( \frac{a}{d} \right).
\]

Using again (2.1), we get the desired result.

\[\square\]

3. The hyper-sums of powers of integers \( S_{p,(a,d)}^{(r)}(n) \)

The hyper-sums of powers of integers \( S_{p,(a,d)}^{(r)}(n) \) \((p \geq 0)\) (or the \(r\)-fold summation of \(p\)th powers) are defined recursively as
\[
S_{p,(a,d)}^{(0)}(n) = \sum_{i=0}^{n} (a + id)^p
\]
\[
S_{p,(a,d)}^{(r)}(n) = \sum_{j=0}^{n} S_{p,(a,d)}^{(r-1)}(j).
\]

In this section, we generalize the results obtained recently by the same authors in [9]. An explicit formula for \( S_{p,(a,d)}^{(r)}(n) \) is given in the following Theorem.

**Theorem 3.** The hyper-sums of powers of integers \( S_{p,(a,d)}^{(r)}(n) \) is given by
\[
S_{p,(a,d)}^{(r)}(n) = \sum_{i=0}^{n} \binom{n + r - i}{r} (a + id)^p.
\]

**Proof.** These facts are easily verified by induction on \(r\) with
\[
\sum_{j=1}^{n} \binom{j - i + r - 1}{r - 1} = \binom{n + r - i}{r}.
\]

\[\square\]

We will now derive a few further consequences of Theorem 3.
Corollary 3. The exponential generating function of the hyper-sums of powers of integers $S_{p(a,d)}^{(r)}(n)$ is given by

\[
\sum_{p \geq 0} S_{p(a,d)}^{(r)}(n) \frac{z^p}{p!} = \sum_{k=0}^{n} \binom{n + r - k}{r} e^{(a+kd)z}. \tag{3.1}
\]

Proof. We have

\[
\sum_{p \geq 0} S_{p(a,d)}^{(r)}(n) \frac{z^p}{p!} = \sum_{p \geq 0} \left( \sum_{k=0}^{n} \binom{n + r - k}{r} (a+kd)^p \right) \frac{z^p}{p!}
\]

\[
= \sum_{k=0}^{n} \binom{n + r - k}{r} \sum_{p \geq 0} \frac{((a+kd)z)^p}{p!}
\]

\[
= \sum_{k=0}^{n} \binom{n + r - k}{r} e^{(a+kd)z}.
\]

\[\square\]

Theorem 4. The exponential generating function of the hyper-sums of powers of integers $S_{p(a,d)}^{(r)}(n)$ is

\[
\sum_{p \geq 0} S_{p(a,d)}^{(r)}(n) \frac{z^p}{p!} = \left( \frac{n + r + 1}{r + 1} \right) e^{az} {}_2F_1 \left( \frac{1}{r+2}, 1-e^{dz}; 1-e^{dz} \right), \tag{3.2}
\]

where $\, _2F_1 \left( \frac{a}{c} ;z \right)$ denotes the Gaussian hypergeometric function defined by

\[
\sum_{n \geq 0} \frac{(a)^n (b)_n}{(c)^n n!} z^n,
\]

and $(x)^\pi$ denotes the Pochhammer symbol defined by

\[
(x)^\pi = 1 \quad \text{and} \quad (x)^\pi = x(x+1) \cdots (x+n-1).
\]

Proof. From (3.1), we have

\[
\sum_{p \geq 0} S_{p(a,d)}^{(r)}(n) \frac{z^p}{p!} = e^{az} \sum_{k=0}^{n} \binom{k + r}{r} e^{d(n-k)z}
\]

\[
= \frac{(n + r + 1)! e^{az}}{n! r!} \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)! (k+r)!}{(n + r + 1)!} e^{d(n-k)z}
\]

\[
= \left( \frac{n + r + 1}{r + 1} \right) (r + 1) e^{az} \sum_{k=0}^{n} \binom{n}{k} e^{d(n-k)z} \int_{0}^{1} (1-x)^{r+k} x^{n-k-1} dx
\]

\[
= \left( \frac{n + r + 1}{r + 1} \right) (r + 1) e^{az} \int_{0}^{1} (1-x)^{r} \left( \sum_{k=0}^{n} \binom{n}{k} (xe^{dz})^{n-k} (1-x)^{k} \right) dx
\]
\[
= \left(\frac{n + r + 1}{r + 1}\right) e^{az} (r + 1) \int_{0}^{1} \left(1 - x\right)^{r} \left(1 - x + xe^{dz}\right)^{n} dx.
\]

It follows from the theory of hypergeometric functions that the Gaussian hypergeometric function \( \mathbf{2F1} \left( \frac{1, -n}{r + 2}; 1 - e^{dz} \right) \) has an integral representation given by
\[
\mathbf{2F1} \left( \frac{1, -n}{r + 2}; 1 - e^{dz} \right) = (r + 1) \int_{0}^{1} \left(1 - x\right)^{r} \left(1 - x + xe^{dz}\right)^{n} dx.
\]

which implies \( (3.2) \).

**Theorem 5.** The ordinary generating function of the hyper-sums of powers of integers \( S_{p(a,d)}^{(r)}(n) \) is given by
\[
(3.3) \quad \sum_{r \geq 0} S_{p(a,d)}^{(r)}(n) z^{r} = \frac{1}{(1 - z)^{n+1}} \sum_{i=0}^{n} \frac{(1 - z)^{i} (a + i d)^{p}}{z^{r}}
\]

**Proof.** Since
\[
\sum_{r \geq 0} \binom{n + r - i}{r} z^{r} = (1 - z)^{i - n - 1},
\]
which implies \( (3.3) \).

**Theorem 6.** The double generating function of the hyper-sums of powers of integers \( S_{p(a,d)}^{(r)}(n) \) is given by
\[
\sum_{r \geq 0} \sum_{p \geq 0} S_{p(a,d)}^{(r)}(n) \frac{z^{p}}{p!} t^{r} = \frac{e^{az} - (1 - t)^{n+1} e^{(a+(n+1)d)z}}{(1 - t)^{n+1} (1 - (1 - t) e^{dz})}.
\]

**Proof.** From \( (3.1) \) and \( (3.3) \), we obtain
\[
\sum_{r \geq 0} \sum_{p \geq 0} S_{p(a,d)}^{(r)}(n) \frac{z^{p}}{p!} t^{r} = \sum_{s=0}^{n} \sum_{r \geq 0} \binom{n + r - s}{r} t^{r} e^{(a+s d)z}
\]
\[
= \frac{e^{az}}{(1 - t)^{n+1}} \sum_{s=0}^{n} ((1 - t) e^{dz})^{s}
\]
\[
= \frac{e^{az}}{(1 - t)^{n+1}} \left[ \frac{1 - (1 - t)^{n+1} e^{(n+1)d}z}{1 - (1 - t) e^{dz}} \right]
\]
\[
= \frac{e^{az} - (1 - t)^{n+1} e^{(a+(n+1)d)z}}{(1 - t)^{n+1} (1 - (1 - t) e^{dz})}.
\]

Now, according to the well-known formula, for \( n \in \mathbb{N} \) and \( m \in \mathbb{N}^{*} \)
\[
\mathbf{2F1} \left( \frac{-n, 1}{m}; z \right) = \frac{n! (z - 1)^{m-2}}{(m)^{n}} \sum_{k=0}^{m-2} \frac{(n + 1)^{k}}{k!} \left(\frac{z}{z - 1}\right)^{k} - (1 - z)^{n+1}.
\]
we can rewrite the exponential generating function of the hyper-sums of powers of integers $S_{p}^{(r)}(n)$ as

\begin{equation}
\sum_{p \geq 0} S_{p,(a,d)}^{(r)} \frac{z^{p}}{p!} = \frac{e^{(a+d(r+(n+1)))z}}{(e^{dz} - 1)^{r+1}} - \sum_{k=0}^{r} \binom{n+k}{k} \frac{e^{(a+(r-k)d)z}}{(e^{dz} - 1)^{r-k+1}}.
\end{equation}

The next result gives an explicit formula for $S_{p,(a,d)}^{(r)}(n)$ involving the generalized Bernoulli polynomials. Recall that the generalized Bernoulli polynomials $B_{n}^{(a)}(x)$ of degree $n$ in $x$ are defined by the exponential generating function

\begin{equation}
\left( \frac{z}{e^{z} - 1} \right)^{\alpha} e^{xz} = \sum_{n \geq 0} B_{n}^{(a)}(x) \frac{z^{n}}{n!}
\end{equation}

for arbitrary parameter $\alpha$. In particular, $B_{n}^{(1)}(x) := B_{n}(x)$ denotes the classical Bernoulli polynomials with $B_{1}(0) = -\frac{1}{2}$. For a recent treatment see \[3\] \[15\].

**Theorem 7.** For all $n, p, r \geq 0$, we have

\begin{equation}
S_{p,(a,d)}^{(r)}(n) = \frac{p!d^{p}}{(p+r+1)!} B_{p+r+1}^{(r+1)} \left( \frac{a}{d} + (r+(n+1)) \right)
- \frac{p!d^{p}}{(p+r+1)!} \sum_{k=0}^{r} \binom{n+k}{k} \frac{1}{(p+r+1-k)!} B_{p+r+1-k}^{(r-k+1)} \left( \frac{a}{d} + (r-k) \right)
\end{equation}

**Proof.** By (3.4) and (3.5) we have

\begin{align*}
\sum_{p \geq 0} S_{p,(a,d)}^{(r)}(n) \frac{z^{p}}{p!} &= -\sum_{k=0}^{r} \binom{n+k}{k} \sum_{p \geq 0} d^{p-r-k+1} B_{p}^{(r-k+1)} \left( \frac{a}{d} + (r-k) \right) \frac{z^{p-r-k+1}}{p!} \\
&\quad + \sum_{p \geq 0} d^{p-r-1} B_{p}^{(r+1)} \left( \frac{a}{d} + (r+(n+1)) \right) \frac{z^{p-r-1}}{p!}
\end{align*}

After some rearrangement, we find

\begin{align*}
\sum_{p \geq 0} S_{p,(a,d)}^{(r)}(n) \frac{z^{p}}{p!} &= \sum_{p \geq 0} \frac{z^{p}}{p!} \left( \frac{p!d^{p}}{(p+r+1)!} B_{p+r+1}^{(r+1)} \left( \frac{a}{d} + (r+(n+1)) \right) \\
&\quad - \frac{p!d^{p}}{(p+r+1)!} \sum_{k=0}^{r} \binom{n+k}{k} \frac{1}{(p+r+1-k)!} B_{p+r+1-k}^{(r-k+1)} \left( \frac{a}{d} + (r-k) \right) \right)
\end{align*}

Equating the coefficient of $\frac{z^{p}}{p!}$, we get the result. \[\square\]

When $r = 0$, Theorem 7 reduces to (1.2).

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