Dissipative solitons stabilized by a quantum Zeno-like effect

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The solitary wave1 (or the soliton2) is a ubiquitous nonlinear phenomenon, which has been widely observed in many systems, particularly, in optical fibers3,4 and Bose-Einstein condensates (BEC)5,6,7. It was recognized that the stability of a soliton is due to a subtle balance between competing features like dispersion and nonlinearity in a medium. The practical applications of the soliton raise the requirement for optimal soliton management8,9,10. However, due to the presence of dissipation, the soliton inevitably fades away. Here we show that the dissipative effect can be completely suppressed by an analogue of the quantum Zeno effect11 (in short we will call it quantum Zeno-like effect) and propose a novel way to stabilize the soliton. The proposal opens a new perspective to realize an ideal optical soliton transmission, a seemingly impossible dream in communication technologies. It is also of fundamental interest to the quantum Zeno effect itself and the matter-wave dynamics.

In quantum mechanics, the quantum Zeno effect shows that an unstable particle can be completely stabilized by frequent measurements11, which has been confirmed by many experiments12,13,14 and its possible applications have been extensively explored15,16. Solitons are localized, finite energy states in a medium. They behave like elementary particles, can pass through each other and preserve their shapes and speeds after collisions. However, the presence of dissipation effect makes the soliton behave like an unstable particle, so it will inevitably decay and eventually disappears in a medium. The insight gained from the quantum Zeno effect in quantum mechanics provides a possible mechanism to stabilize such a soliton if it is frequently “measured”. Through a rigorous mathematical analysis, the soliton is, indeed, shown to become stable if the external potential applied and/or the Feshbach resonance along with the dispersion is periodically modulated with high frequency. This is due to the fact that the “inevitable” dissipation effect is completely suppressed by such a frequent “measurement”, even in the presence of the dissipation in reality.

We study the dynamics of the soliton governed by a one-dimensional (1D) dimensionless nonautonomous nonlinear Schrödinger (NLS) equation

\[
i \frac{\partial u(x, t)}{\partial t} + \varepsilon f(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} + \delta g(x, t)|u(x, t)|^2 u(x, t) - \frac{1}{2} V(t)x^2 u(x, t) = i \frac{\gamma(t)}{2} u(x, t).
\]

Here \(u(x, t)\) can represent either the envelope of pulses in nonlinear optics with different notations or the macroscopic wave function of BEC with strong transverse confinement (i.e., effective 1D model) at the mean-field level. Therefore, the soliton satisfying equation (1) can be the temporal (if \(V(t) = 0\)) or the spatial optical soliton or the matter-wave soliton. For convenience, below we use the notations for the matter-wave solitons. \(f(x, t)\) and \(g(x, t)\) are dimensionless and denote the time- and space-dependent dispersion and nonlinearity managements, respectively. Here the space- and time-coordinates \(x\) and \(t\) are measured in units of \(a_r\) and \(\omega_r^{-1}\), where \(a_r\) and \(\omega_r\) being the transverse harmonic oscillator length and the transverse confining frequency of BEC, respectively. \(\varepsilon\) and \(\delta\) are constants in the standard NLS equation (see Method). \(V(t)\) represents the external harmonic potential applied, whose strength and sign can be tuned experimentally. The linear dissipation (\(\gamma < 0\)) or gain (\(\gamma > 0\)) rate is also time-dependent. Generally, these coefficients should be real. Due to the introduction of these control parameters, the soliton obeying equation (1) can be called as nonautonomous17 or dissipative18 soliton or similar or in nonlinear optics19,20,21.

The integrability study of the nonautonomous NLS equation has a long history, see, for example, ref. (9). A complete integrability condition of equation (1) neglecting dissipation and/or gain (i.e., \(\gamma(t) \equiv 0\)) was clearly shown in ref. (17) and further studied recently by the Painlevé analysis22,23,24. However, it is nontrivial to explore the integrability of equation (1) in the presence of the dissipation and/or gain effect. Below we show a generalized “integrability” condition of equation (1). It can be obtained by the Painlevé analysis22. For details of the Painlevé analysis one can refer to ref. (23). It reads
Here we would like to point out that the dispersion and nonlinearity managements are not allowed to be space-dependent, even if their space-dependence assumptions are initially made in equation (1). This is a necessary condition for equation (1) to pass through the Painlevé test. Hereafter the coordinate \( x \) in \( f(x, t) \) and \( g(x, t) \) is omitted. In addition, we also mention that equation (2) is essentially not a complete integrability condition in a strict physical sense, namely, the infinite conservation laws are not assumed. It is the condition under which the Painlevé analysis can pass through. In this sense, we will call it the Painlevé integrability condition, which sheds a new light on the “integrability” behavior of equation (1).

The seemingly complicated Painlevé integrability condition obtained for the first time shows a subtle balance between four parameters \( f(t), g(t), V(t) \) and \( \gamma(t) \) affecting the dynamics of the nonautonomous soliton, and thus has a profound implication to the control of the soliton dynamics. Any three parameters out of them can be set independently, while the remaining one can be tuned according to equation (2). In this case, the soliton satisfied equation (1) can be managed as a whole in the sense of the Painlevé integrability. This feature provides many possibilities to control the soliton dynamics in different fields such as nonlinear optics and BEC, deserving further exploration.

Here we focus on the dynamical stability of the soliton, which is of fundamental importance to the study of the matter-wave dynamics and in particular to the propagation of the optical solitons in transmission lines. In this case, \( f(t), g(t) \) and/or \( V(t) \) are set independently and \( \gamma(t) \) is effectively determined by these three independent parameters. At first glance, this assumption seems to be logically incorrect since the dissipation and/or gain mechanism always physically exists in reality. However, when some control parameters are modulated, the actual \( \gamma(t) \) acting on solitons might be different from that without such modulations. Here it is quite instructive to think of a comparison with the quantum Zeno effect. A physically unstable particle always decays with a natural decay rate named as \( R_0 \). However, it is well known that that frequent measurements, for example, with a frequency \( \omega \), on the unstable particle can slow down its decay. This fact tells us that the actual dissipation rate named as \( R_\omega \) of the unstable particle with frequent measurements must be different from \( R_0 \). In this case, \( R_0 \) is only a nominal, while \( R_\omega \) is the actual dissipation rate of the unstable particle. The soliton case is similar. The only difference is that the soliton is not a real particle, though it behaves like an elementary particle. For convenience, we take the dissipation and/or gain as \( \gamma_\omega(t) \) when dispersion, nonlinearity or the external potential is modulated periodically with frequency \( \omega \).

Focusing on the dynamical stability of the soliton, we rewrite equation (2) as

\[
\frac{d}{dt} \gamma_\omega(t) - \gamma_\omega^2(t) - \alpha(t) \gamma_\omega(t) + \beta(t) = 0, \tag{3}
\]

where \( \alpha(t) = \frac{1}{\gamma(t)} \frac{df(t)}{dt} - \frac{2}{g(t)} \frac{dg(t)}{dt} \) and \( \beta(t) \) represents the remaining terms in (2) independent of \( \gamma_\omega(t) \). This is a standard Riccati equation\(^\text{25}\). Mathematically, the Riccati equation can not be solved analytically in a general case. Here we resort to the Runge-Kutta method to solve numerically this equation. We first consider the effect of the external potential \( V(t) \) by fixing \( f(t) \) and \( g(t) \) to \( f(t) = g(t) = \pm 1 \) and let \( \varepsilon = 1/2 \) as usual. Thus, \( \alpha(t) = 0 \) and \( \beta(t) = \mp V(t) \).

Figure 1a shows \( \gamma_\omega(t) \) as a function of time. At low frequencies, \( \gamma_\omega(t) \) oscillates and turns slowly up with time. Finally, at certain moment \( \gamma_\omega(t) \) diverges. With increasing frequency, the diverging point of \( \gamma_\omega(t) \) is postponed rapidly and finally, at the time interval we studied (up to 100 here and not fully shown for clarity) no divergence is observed at least for \( \omega > 128 \). Meanwhile, the magnitude of oscillation decreases quickly. This clearly indicates that the matter-wave soliton can be stabilized by applying a modulated external potential with high frequency. This can also be seen from the behavior of \( \Gamma_\omega(t) \) with increasing frequencies, as shown in Figure 1b. When \( \omega = 2048 \), no visible deviation from zero is observed for the whole time domain studied. The explicit time evolution of the nonautonomous bright soliton is shown in Figure 1e-f with frequencies of \( \omega = 2, 8, \) and 32. With increasing frequencies, the nonautonomous bright soliton approaches gradually to the canonical bright soliton shown in Figure 1d. The overlap integral of the evolving bright soliton with its initial state is shown in Figure 1c for different frequencies. At high frequencies the soliton evolves slowly and the deviation from its initial state keeps invisible. Analytically, the nonautonomous soliton can be effectively reduced to the canonical soliton form since \( \Gamma_\omega \to 0 \) at the limit of high frequency. As a result, \( a(x, t) = 0, X(x, t) = x, T(t) = t, \) and \( c(t) = 0 \) neglecting the constants \( C_1, C_2, \) and \( C_3 \) (see Method). In this case, the Painlevé integrability condition recovers consistently to the complete integrability condition. As a result, the nonautonomous soliton becomes stable and completely integrable. This is a quantum Zeno-like effect.
FIG. 1: The effective dissipation and/or gain rates and the time evolution of the nonautonomous solitons in the periodically modulated external potentials. a. The effective dissipation and/or gain rate $\gamma_\omega(t)$. b. The effective accumulated dissipation and/or gain rate, which is defined by $\Gamma_\omega(t) = \int_0^t \gamma_\omega(t') dt'$. c. The normalized overlap integral defined by $W(t) = \frac{\int u^*(x,0) u(x,t) dx}{\int |u(x,0)|^2 dx}$. Here $u(x, t)$ obeys equation (1) and $u(x, 0)$ is its initial state, which is equivalent to the canonical bright soliton. d. The canonical bright soliton $Q(X,T) = \text{sech}(X) \exp(iT/2)$. e - f. The explicit evolution of the nonautonomous bright soliton in periodically modulated harmonic external potentials with increasing frequencies from $\omega = 2$ to 32. The time interval for the soliton evolution is from 0 to 10 (4 in the case of $\omega = 2$). Here the frequency and time are measured in units of $\omega_r$ and $\omega_r^{-1}$, respectively. The strength of the harmonic external potential is modulated by $V(t) = \cos(\omega t)$ with $V = 1$. Other parameters used are $f(t) = g(t) = 1$, $\varepsilon = 1/2$ and for all cases the initial value of $\gamma_\omega(t)$ is set to be zero. The result of $f(t) = g(t) = -1$ is similar.

FIG. 2: The effective dissipation or gain rates and the time evolution of the nonautonomous solitons with periodically modulated nonlinearity. a - g fully correspond to those in Figure 1. The time interval for the soliton evolution in e - g is plotted from 0 to 10. The nonlinearity modulation is given by $g(t) = \exp(\delta a \cos(\omega t))$, where $\delta a = 1/\omega^2$ is taken. Other physical variables used are $V(t) = 0$ and $f(t) = 1$.

in the context of nonlinear dynamics found here for the first time. It builds up a clear relationship between the nonautonomous and the canonical solitons. This surprising result unambiguously shows the particle nature of the nonautonomous solitons and provides a dynamical way to stabilize them.

Experimentally, it is easy to modulate the external potential by changing periodically the direction and magnitude of the magnetic field controlling the magnetic traps for BEC. Actually, an easier way to stabilize the matter-wave solitons is to use the Feshbach resonance management. To show this, let $f(t) = 1$ and $g(t) = \exp(\delta a \cos(\omega t))$, where $\delta a$ and $\omega$ are the amplitude and the frequency of the Feshbach resonance modulation.
simplicity, we also take $V(t) = 0$. In this case, one has

$$\alpha(t) = -2\delta_a \omega \sin(\omega t),$$
$$\beta(t) = -\delta_a \omega^2 \left( \cos(\omega t) - \delta a \sin^2(\omega t) \right).$$

Limiting to weak disturbances, we take $\delta a = 1/\omega^2$. The result is shown in Figure 2a & b. Similarly, the matter-wave soliton can be stabilized by a high frequency modulation Feshbach resonance. As an example, Figure 2e-f shows the nonautonomous bright solitons evolving with time. Likewise, with increasing frequencies, the nonautonomous soliton approaches gradually to the canonical soliton evolution, as shown in Figure 2d. For $\omega = 2048$, the nonautonomous soliton has no visible deviation from its initial state, as shown in Figure 2c. This is another example that the nonautonomous soliton can be stabilized by a quantum Zeno-like effect. Theoretically, it is also possible to modulate periodically the dispersion to stabilize the nonautonomous soliton, which is quite readily realized in the context of optical soliton transmission.

The above discussions are entirely based on the Painlevé integrability condition (2) and are not limited to the bright soliton solution. In fact, any allowed solution of the standard NLS equation, including the bright and dark solitons, the multi-soliton solutions, and even the periodic plane wave solutions, can be stabilized by such a scheme. This provides a novel approach to study in detail the matter-wave properties and related soliton dynamics.

The same scheme can also be applied to the optical soliton. The present result has an important implication that with the help of the quantum Zeno-like effect a dissipationless, ideal optical soliton transmission might be possible in reality. Once it is realized, the impact to modern information technology is indeed far-reaching.

The above discussion is based on an intuitive physics consideration and the Painlevé integrability condition obtained by a rigorous mathematical derivation. One may question its realizability, even be suspicious of it as an artifact. In fact, some previous results have an implication that our result is physically reasonable. For example, the earlier numerical simulations of the nonautonomous NLS equation in BEC$^{27,28}$ have clearly indicated that the 2D bright soliton can be stabilized for some time by tuning periodically the nonlinearity, even if the instability of multidimensional solitons is eventually inevitable$^{29}$. A recent experiment in optics$^{30}$ also clearly demonstrated that the propagation of femtosecond pulses was stabilized in the layered Kerr media consisting of glass and air (the nonlinearity management). Loosely speaking, the fact that the dispersion management in nonlinear optics can effectively improve the optical soliton transmission quality is also compatible to the present result. It is expectable that an appropriately designed dispersion management can further improve the optical soliton transmission quality, and ideally yield a dissipationless transmission.

**Method**

Once the Painlevé integrability condition equation (2) is satisfied, an exact analytical solution of the nonautonomous NLS equation (1) can be obtained from the corresponding canonical solution of the standard NLS equation

$$\frac{\partial}{\partial T} Q(X, T) + \varepsilon \frac{\partial^2}{\partial X^2} Q(X, T) + \delta |Q(X, T)|^2 Q(X, T) = 0. \tag{4}$$

Here $\varepsilon$ and $\delta$ are constants. When $\delta > 0 (<0)$, the standard NLS equation (4) has bright (dark) soliton solutions. This is materialized by a general transformation$^{24}$

$$u(x, t) = Q(X(x, t), T(t)) e^{i a(x, t) + c(t)}, \tag{5}$$

where $X(x, t), T(t), a(x, t)$ and $c(t)$ are real functions to be determined by the requirement that $u(x, t)$ and $Q(X, T)$ satisfy equations (1) and (4), respectively. Inserting equation (5) into equation (1) and comparing with equation (4), we obtain a set of ordinary differential equations for these transformation parameters, whose solutions read

$$a(x, t) = \frac{1}{4\varepsilon f(t)} \left( \frac{d}{dt} \ln \left( \frac{f(t)}{g(t)} \right) - \gamma(t) \right) x^2 + C_1 \frac{g(t)}{f(t)} e^{\Gamma(t)} x - C_2 \varepsilon \int dt' \frac{g(t')^2}{f(t')} e^{2\Gamma(t')} + C_3, \tag{6}$$

$$X(x, t) = \frac{g(t)}{f(t)} e^{\Gamma(t)} x - C_1 \int dt' \frac{g(t')^2}{f(t')} e^{2\Gamma(t')} + C_2, \tag{7}$$

$$T(t) = \int dt' \frac{g(t')^2}{f(t')} e^{2\Gamma(t')} + C_3, \tag{8}$$

$$c(t) = \frac{1}{2} \ln \frac{g(t)}{f(t)} + \Gamma(t), \tag{9}$$

where $\Gamma(t) = \int_0^t \gamma(t') dt'$ is accumulated dissipation and/or gain effect. $C_1, C_2,$ and $C_3$ are arbitrary constants and are set to be zero for simplicity. Through this transformation, all exact solutions, including the canonical solitons, of the standard NLS equation can be recast into the corresponding solutions of the nonautonomous NLS equation (1). This result yields a mapping between the canonical and the nonautonomous solitons in a straightforward way.

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