ABSTRACT

We propose a formulation of an absorbing boundary for a quantum particle. The formulation is based on a Feynman-type integral over trajectories that are confined by the absorbing boundary. Trajectories that reach the absorbing wall are instantaneously terminated and their probability is discounted from the population of the surviving trajectories. This gives rise to a unidirectional absorption current at the boundary. We calculate the survival probability as a function of time. Several modes of absorption are derived from our formalism: total absorption, absorption that depends on energy levels, and absorption of non-interacting particles. Several applications are given: the slit experiment with an absorbing screen and with absorbing lateral walls, and one dimensional particle between two absorbing walls. The survival probability of a particle between absorbing walls exhibits decay with beats.
1. Introduction

There are many different descriptions of absorption in electromagnetic theory, nuclear physics, solid state physics, and so on. These descriptions are mostly based on scattering theory. In scattering theory absorption occurs when a jet of particles hits an obstacle and is partially absorbed. The reduction in the population of particles is described in scattering theory by a complex potential. The introduction of a complex potential in scattering theory is supported by and large by phenomenological considerations [1].

The introduction of a complex potential in scattering theory, that is, of phase shift, is analogous to the introduction of a complex dielectric coefficient in Maxwell’s equations. There exist several models for the calculation of the complex part of the dielectric constant in Maxwell’s theory. The basis for these models is the possibility to write equations of motion of a damped charged particle in an electromagnetic field and to identify the complex part of the dielectric constant as the damping coefficient [2, 3]. Another approach to decay is based on the decoherent histories approach [4].

The purpose of this paper is to propose a Feynman-type integral to describe total or partial absorption of particles in a surface bounding a domain. The need for such a description stems from the reflecting property of potentials of all kinds, real and complex, finite and infinite. This property eliminates potentials as means of absorbing all particles that reach a given absorbing surface such as a photographic plate. Additional postulates are needed to incorporate absorbing surfaces into quantum mechanics. The method of complex potentials, as used in Maxwell’s equations, does not carry over to quantum mechanics due to the fact that the wave function of a particle does not interact with the medium the way an electromagnetic wave does. In particular, in classical quantum theory, unlike in electromagnetic theory, the wave function does not transfer energy to the medium.

In our approach to absorption, we postulate an absorbing boundary that absorbs instantaneously all Feynman trajectories at the instant they reach the boundary. The probability of these trajectories is discounted from the wave function of the surviving trajectories, much like the procedure adopted in the derivation of Fermi’s Golden Rule. We find that the wave function of the surviving Feynman trajectories satisfies Schrödinger’s equation with reflecting boundary conditions on the absorbing boundary. We define a discounted wave function whose total probability within the absorbing boundaries is the survival probability of the un-absorbed trajectories. We find the decay law of the survival probability for different modes of absorption: total absorption, absorption that depends on energy levels, and absorption of non-interacting particles. Total absorption is obtained if the wave function is discounted by the probability of the absorbed trajectories. In this mode of absorption the survival probability decays in time at a rate proportional to the total energy of the system with beats at Bohr frequencies. Absorption that depends on energy levels apportions different decay rates to modes corresponding to different energies in a such a away that the total discounted probability is the same as in the former case. Absorption of non-interacting particles produces a discounted wave function which is the
product of the discounted wave functions of each particle separately.

In Section 2, we formulate the postulates of absorption in a surface and explain the discounting procedure. In Section 3, we calculate the Feynman integral over the class of trajectories that are bounded within a given boundary and show that it satisfies Schrödinger’s equation with total reflection on the boundary. In Section 4, we calculate the probability of Feynman trajectories that propagate into the absorbing boundary and obtain the survival probability of the un-absorbed trajectories. In Section 5, we describe other modes of absorption. In Section 6, we calculate the absorption current at the absorbing boundary and consider the slit experiment with an absorbing screen. Section 7 contains the following examples: a particle between absorbing walls, a Gaussian wave packet incident on an absorbing wall, and the slit experiment with absorbing lateral boundaries. Experiments are proposed to examine the various decay modes. Finally, a discussion and summary are offered in Section 8.

2. The postulates of absorption in a surface

The two simplest types of instantaneous absorption at a wall are the absorption of all Feynman trajectories when they reach the wall for the first time, or absorption of trajectories that propagate across the wall. It was shown in [5] that Feynman integrals over the set of trajectories that terminate at a given wall produce wave functions that vanish on and beyond the wall. The resulting wave functions are continuous, though have discontinuous derivatives on the boundaries of their supports. It was shown in [6] that for this type of initial wave functions the probability that propagates across the boundary of the support in a short time $\Delta t$ is proportional to $\Delta t^{3/2}$ so that a continuous discounting of the wave function by the probability of the trajectories that crossed into the absorbing domain beyond the wall leads to total reflection, much like in the Zeno effect [7]. It is shown below that the probability density that propagates into an absorbing wall in a short time $\Delta t$ is proportional to $\Delta t$, so that discounting the probability of these trajectories leads to a decay law, consequently, we adopt the concept of instantaneous absorption at a wall for wave functions that vanish on and beyond the wall.

More specifically, in our Feynman-type integral trajectories that propagate into the surface for the first time are considered to be absorbed instantaneously or reflected with a given probability which is a property of the interaction between the particle and the absorbing surface. The absorbed trajectories are therefore terminated at that surface. The population of the surviving trajectories is discounted by the probability of the absorbed trajectories at each time step. The instantaneous absorption rate is assumed proportional to the number (probability) of particles at the absorbing surface at a given moment of time. The proportionality constant is a characteristic length, $\lambda$, determined by the absorbing material, e.g., photographic plate or fluorescent screen, and by the absorbed particle, e.g., electron or neutron. The length $\lambda$ is assumed to depend weakly on the absorbed particle’s energy so, in this paper, we approximate it by a constant. Under this assumption $\lambda$ can be determined experimentally by measuring the absorption rate of a single energy level and
than be used to calculate the absorption rate at any other energy level, in some energy range, or the simultaneous absorption of several levels in this range. The constant $\lambda$ can be assumed to contain a multiplicative factor that represents the probability of absorption of a trajectory when it hits the wall.

In the context of the decay of an energy level into a continuum, usually described by the Wigner-Weiskopf theory and Fermi’s Golden Rule [8], the characteristic length $\lambda$ in our theory, can be related to the size of the support of the perturbing Hamiltonian. In the context of $\alpha$-decay in nuclear physics, the length $\lambda$ can be related to the width of the potential barrier [9].

A detailed theory of $\lambda$ can be expected to involve the atomic detail of the absorbing wall and the details of the interaction between the particle and the atoms of the absorber. In contrast, our theory involves the trajectories of the particles before they hit the absorbing wall so that $\lambda$ is an external input into our theory. This situation is analogous to the theory of frequency dependent refraction coefficient in electromagnetic theory where the frequency dependence is determined from a detailed analysis of the interaction between the electromagnetic wave and the atoms of the medium. Thus the frequency dependent refraction coefficient is an external input into the Helmholtz equation. In this theory, the knowledge of the response to a single frequency does not determine in a simple way the response to a wave containing several frequencies. Only in the case of a frequency independent dielectric constant its measurement at one frequency allows to predict the response to a wave containing several frequencies. In different theories of absorption the parameter $\lambda$ can depend on any number of physical properties of the system, such as the energy of the system or some average energy per mode, and so on.

In general, if the trajectories are partitioned into two subsets, the part of the wave function obtained from the Feynman integral over one subset cannot be used to calculate the probability of this subset, due to interference between the wave functions of the two subsets. However, in the physical situation under consideration, such a calculation may be justified as follows. Our procedure, in effect, assumes a partition of all the possible trajectories at any given time interval $[t, t+\Delta t]$ into two classes. One is a class of restricted trajectories that have not reached the absorbing surface by time $t+\Delta t$ and remain in the domain, and the other is a class of trajectories that arrived at the surface for the first time in the interval $[t, t+\Delta t]$. We assume that the part of the wave function obtained from the Feynman integral over trajectories that reached the surface in this time interval no longer interferes with the part of the wave function obtained from the Feynman integral over the class of restricted trajectories in a significant way. That is, the interference is terminated at this point so that the general population of trajectories can be discounted by the probability of the terminated trajectories. This assumption makes it possible to calculate separately the probability of the absorbed trajectories in the time interval $[t, t+\Delta t]$.

The assumptions discussed above can be summarized in the form of the following postulates:
I. The absorption of Feynman trajectories represents absorption of actual trajectories of particles.

II. The population of Feynman trajectories can be discounted by the probability of the absorbed trajectories.

3. The Feynman integral over bounded trajectories

If a quantum particle is constrained in space to a finite (or semi finite) domain, the Feynman integral has to be confined to Feynman trajectories that stay forever in this domain. This implies the following modification in the definition of the Feynman integral. The function space is now the class

\[ \sigma_{a,b} = \{ x(\cdot) \in C[0,t] \mid a \leq x(\tau) \leq b, 0 \leq \tau \leq t \} \]

and the definition of the Feynman integral over the class \( \sigma_{a,b} \) is

\[
K_{a,b}(x,t) = \int_{\sigma_{a,b}} \exp \left\{ \frac{i}{\hbar} S[x(\cdot), t] \right\} \mathcal{D}x(\cdot)
\]

\[
\equiv \lim_{N \to \infty} \alpha_N \int_a^b \ldots \int_a^b \exp \left\{ \frac{i}{\hbar} S(x_0, \ldots, x_N, t) \right\} \prod_{j=1}^{N-1} dx_j.
\]

(3.1)

Next, following the method of [10], we show that \( K_{a,b}(x,t) \) satisfies the Schrödinger equation and determine the boundary conditions at the endpoints of the interval \([a,b]\). We begin with a derivation of a recursion relation that defines \( K(x,t) \). We set

\[
K_N(x_N, t) \equiv \alpha_N \int_a^b \ldots \int_a^b \exp \left\{ \frac{i}{\hbar} S(x_0, \ldots, x_N, t) \right\} \prod_{j=1}^{N-1} dx_j,
\]

(3.2)

then by the definition (3.1), \( K(x,t) = \lim_{N \to \infty} K_N(x,t) \). The definition (3.2) implies the recursion relation

\[
K_N(x,t) = \left\{ \frac{m}{2\pi i \hbar \Delta t} \right\}^{1/2} \int_a^b \exp \left\{ \frac{i}{\hbar} \left[ \frac{m(x-x_{N-1})^2}{2\Delta t} - V(x) \Delta t \right] \right\} K_{N-1}(x_{N-1}, t_{N-1}) dx_{N-1}.
\]

(3.3)

The following derivation is formal. A strict derivation can be constructed along the lines of [10]. We expand the function \( K_{N-1}(x_{N-1}, t_{N-1}) \) in (3.3) in Taylor’s series about \( x \) to obtain
\[ K_N(x, t) = \left( \frac{m}{2\pi i\hbar \Delta t} \right)^{1/2} e^{-iV(x)\Delta t/\hbar} \int_{a}^{b} \exp \left\{ \frac{im}{2\hbar \Delta t} (x - x_{N-1})^2 \right\} \]

\[ \left[ K_{N-1}(x, t_{N-1}) - (x - x_{N-1}) \frac{\partial K_{N-1}(x, t_{N-1})}{\partial x} + \frac{1}{2} (x - x_{N-1})^2 \frac{\partial^2 K_{N-1}(x, t_{N-1})}{\partial x^2} + O \left( (x - x_{N-1})^3 \right) \right] dx_{N-1}. \]

The integrals in eq. (3.4) are evaluated separately for \( x \) inside the interval \([a, b]\) and on its boundaries. When \( x \) is inside the interval, the integrals become the Fresnel integrals over the entire line in the limit \( \Delta t \to 0 \). This recovers the Schrödinger equation inside the interval. When \( x = a, b \), the first integral in eq. (3.4) becomes in the limit \( \Delta t \to 0 \) the Fresnel integral over half the real line while the other integrals vanish. We obtain

\[ K(a, t) = \frac{1}{2} K(a, t), \quad (3.5) \]

hence \( K(a, t) = 0 \) and similarly, \( K(b, t) = 0 \). More specifically, consider the normalized \( n \)-th moment of the Gaussian integral (\( n = 0, 1, 2 \))

\[ \alpha^n m_n(\alpha, a, b, y) = \left( \frac{\alpha}{i\pi} \right)^{1/2} \int_{a}^{b} (x - y)^n \exp \left\{ -i\alpha(x - y)^2 \right\} dx, \]

where

\[ \frac{m}{2\hbar \Delta t} = \alpha. \]

We change variable to

\[ \sqrt{\alpha}(x - y) = u \]

and get

\[ m_n(\alpha, a, b, y) = \left( \frac{1}{i\pi} \right)^{1/2} \int_{\sqrt{\alpha}(a-y)}^{\sqrt{\alpha}(b-y)} u^n \exp \left\{ -iu^2 \right\} du. \]

Note that the limits of integration become

\[ \sqrt{\alpha}(a - y) \to -\infty, \quad \sqrt{\alpha}(b - y) \to \infty \quad \text{as} \quad \alpha \to \infty \]

so that in the limit \( \Delta t \to 0 \), that is, as \( \alpha \to \infty \) and for \( a < y < b \), we obtain

\[ m_n(a, b, y) = \lim_{\alpha \to \infty} m_n(\alpha, a, b, y) = \int_{\sqrt{\alpha}(a-y)}^{\sqrt{\alpha}(b-y)} u^n \exp \left\{ -iu^2 \right\} du \]

\[ = \int_{-\infty}^{\infty} u^n \exp \left\{ -iu^2 \right\} du. \quad (3.7) \]
Note that $m_1 = 0$ while $m_0 = m_2 = 1$. Now, transferring $K_{N-1}$ to the left hand side of equation (3.4), dividing by $\Delta t$, taking the limit, and applying eq.(3.7), we recover the Schrödinger equation.

At the boundary point $y = b$, the substitution (3.6) transforms the upper limit of integration to 0. In the limit $\Delta t \to 0$, we obtain

$$m_n(a,b,b) = \lim_{\alpha \to \infty} m_n(\alpha,a,b,b) = \lim_{\alpha \to \infty} \int_0^\alpha u^n \exp\{-iu^2\} \, du = \int_{-\infty}^0 u^n \exp\{-iu^2\} \, du. \tag{3.8}$$

That is, the Fresnel integrals are extended only over half the line. Thus, we obtain that the coefficient of $K_{N-1}$ in eq.(3.4) is

$$m_0(a,b,b) = \frac{1}{2},$$

while that of the first derivative of $K_{N-1}$ is

$$\alpha^{-1}m_1(\alpha,a,b,b) \to 0$$

and that of the second derivative of $K_{N-1}$ is

$$\alpha^{-2}m_2(\alpha,a,b,b) \to 0.$$ 

Thus, setting $x = b$ in eq.(3.4) and taking the limit $\Delta t \to 0$, we obtain eq.(3.5), which in turn implies the boundary condition eq.(3.10).

When $a < x < b$, the derivation given in [10] leads to the Schrödinger equation. Thus

$$i\hbar \frac{\partial K(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 K(x,t)}{\partial x^2} + V(x)K(x,t) \quad \text{for } a < x < b \tag{3.9}$$

$$K(a,t) = K(b,t) = 0 \quad \text{for } t > 0 \tag{3.10}$$

$$K(x,0) = \delta(x - x_I) \quad \text{for } a < x < b. \tag{3.11}$$

In the case $a = -\infty$, $b = \infty$ every $x$ is an internal point so that the Schrödinger equation is satisfied for all $x$ and thus the Feynman integral (3.1) is equivalent to the Schrödinger equation (3.9) and the initial condition (3.11) on the entire real line.

This result shows that bounded trajectories imply in effect an infinite potential barrier on the boundary. In contrast, the Wiener integral over bounded trajectories leads to the diffusion equation with zero boundary condition and to the decay of the population. This means that the boundary is absorbing. This difference between the Wiener and the Feynman integrals over the same class of functions is illustrative of the different roles that trajectories play in quantum and classical theories.
The same result has been derived in [11] and [12]. In [11] trajectories are extended into the entire line as periodic functions. The analysis of this section provides a simpler derivation of the result. The derivation in [12] is done by converting the Wiener integral into the Feynman integral by analytic continuation of $t$ into the imaginary axis.

4. Feynman integrals with absorbing boundaries

Assume now that a trajectory that reaches the boundary $x = a$ or $x = b$ for the first time is instantaneously absorbed. This means that the wave function outside the interval $(a, b)$ vanishes identically and the population inside the interval $(a, b)$ is reduced at a rate determined by the current at the boundary, as described below. The vanishing wave function outside the interval $(a, b)$ expresses the assumption that once outside the interval the particle no longer participates in the quantum evolution of the particles inside the interval. This may occur, for example, in the scattering of particles on a target (e.g., a nucleus). Particles absorbed in the nucleus are discounted from the scattered population. Another example is that of a particle that enters a bath, such as a photographic plate, and leaves an irreversible trace. Consequently, its quantum interaction with the particles inside the interval becomes negligible. Also in this case the population inside the interval is discounted at the rate particles are absorbed.

The absorption process at a given time $t$ is described as the limiting process as $\Delta t \to 0$ of the propagation of a trajectory that survived in the interval $[a, b]$ till time $t$ to a boundary point in the time interval $[t, t + \Delta t]$. In order to incorporate this behavior into the Feynman formulation, we adopt the procedure that leads to the Feynman-Kac formula for the probability density function for a diffusion process with a killing (absorption) measure (see, e.g., [13]). We consider separately the trajectories that reach the boundary in the time interval $[0, \Delta t]$, then those that survived till $\Delta t$, but reach the boundary in the time interval $[\Delta t, 2\Delta t]$, and so on. In each time step, the total population of trajectories has to be discounted by the probability of the absorbed trajectories. This leads to a modified expression for the discretized Feynman integral.

First, we calculate the discretized Feynman integral in the time interval $[0, \Delta t]$, 

$$
\psi_1(x, \Delta t) = \left\{ \frac{m}{2\pi i \hbar \Delta t} \right\}^{1/2} \int_a^b \psi_0(x_0) \exp \left\{ \frac{i}{\hbar} S(x_0, x, \Delta t) \right\} dx_0.
$$

Therefore, the probability density of finding a trajectory at $x = a$ in the time interval $[0, \Delta t]$ is $|\psi(a, \Delta t)|^2$, and there is an analogous expression for the probability density of finding a trajectory at $x = b$ in the time interval $[0, \Delta t]$.

For simplicity, we assume that $b = 0$ and consider the interval $[-a, 0]$, where $a > 0$. In any time interval $[t, t + \Delta t]$, the probability density propagated from this interval into the absorbing boundary at $x = 0$ is calculated next. We begin with an initial wave function $\psi(x, t)$ that is a polynomial 

$$
Q(x, t) = \sum_{j=1}^{N} q_j(t) x^j
$$
in the interval \([-a, 0]\), such that \(Q(-a, t) = Q(0, t) = 0\) and \(\psi(x, t) = 0\) otherwise. The free propagation from the interval \([-a, 0]\) is given by

\[
\psi(y, t + \Delta t) = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int_{-a}^{0} Q(x, t) \exp\left\{ \frac{i m(x - y)^2}{2\hbar \Delta t} \right\} dx.
\]

The boundary condition at the left end of the support of \(\psi(x, t)\) is written explicitly as

\[
\sum_{j=1}^{N} q_j(t) (-a)^j = 0.
\]

Setting

\[
\alpha = \frac{\hbar \Delta t}{m},
\]

the probability mass propagated freely into the absorbing boundary point \(x = 0\) in time \(\Delta t\) is given by

\[
|\psi(0, t + \Delta t)|^2 = \frac{1}{2\pi \alpha} \left| \int_{-a}^{0} Q(x, t) e^{ix^2/2\alpha} dx \right|^2.
\]

We change variable by setting \(x = \sqrt{\alpha} \xi\) to get

\[
|\psi(0, t + \Delta t)|^2 = \frac{1}{2\pi} \left| \int_{-a/\sqrt{\alpha}}^{0} Q\left(\sqrt{\alpha} \xi\right) e^{i\xi^2/2} d\xi \right|^2. \tag{4.1}
\]

First, we evaluate the inner integral,

\[
I_N = \sum_{j=1}^{N} q_j \sqrt{\alpha^j} \int_{-a/\sqrt{\alpha}}^{0} \xi^j e^{i\xi^2/2} d\xi.
\]

All limits of the type

\[
\lim_{\alpha \to 0^+} \alpha^{-(j+1)/2} \int_{-a/\sqrt{\alpha}}^{0} x^j e^{i x^2/2\alpha} dx,
\]

are understood in the sense

\[
\lim_{\alpha \to 0^+} \alpha^{-(j+1)/2} \int_{-a/\sqrt{\alpha}}^{0} x^j e^{i x^2/2\alpha} dx = \lim_{\epsilon \to 0^+} \lim_{\alpha \to 0^+} \alpha^{-(j+1)/2} \int_{-a/\sqrt{\alpha}}^{0} x^j e^{-(\epsilon+\epsilon^2)x^2/2\alpha} dx. \tag{4.2}
\]

The first term in the sum \(I_N\) gives

\[
-i q_1(t) \sqrt{\alpha} \int_{-a/\sqrt{\alpha}}^{0} d\xi e^{i\xi^2/2} = -i q_1(t) \sqrt{\alpha} \left(1 - e^{ia^2/2\alpha}\right).
\]

The second term gives

\[
-i q_2(t) a \sqrt{\alpha} e^{a^2/2\alpha} + i q_2(t) \alpha \int_{-a/\sqrt{\alpha}}^{0} e^{i\xi^2/2} d\xi.
\]
Setting
\[ S_j = \alpha^{j/2} \int_{-a/\sqrt{\alpha}}^{0} \xi^j e^{i\xi^2/2} d\xi, \]
integration by parts gives the recursion relation
\[ S_j = i\sqrt{\alpha}(-a)^{j-1} e^{ia^2/2\alpha} + i(j-1)\alpha S_{j-2}. \]

Proceeding by induction, we find that for \( j > 2 \)
\[ S_j = O \left( \sqrt{\alpha} e^{ia^2/2\alpha} \right). \]

Now, using the definition (4.2), we find that
\[
\lim_{\Delta t \to 0} \frac{1}{\alpha} |\psi(0, t + \Delta t)|^2 = \frac{1}{2\pi} |q_1(t)|^2. \tag{4.3}
\]

Keeping in mind that
\[ q_1(t) = \frac{\partial \psi(0, t)}{\partial x}, \]
we can write
\[
|\psi(0, t + \Delta t)|^2 = \frac{\hbar \Delta t}{2\pi m} \left| \frac{\partial \psi(0, t)}{\partial x} \right|^2 + o \left( \frac{\hbar \Delta t}{2\pi m} \right) \quad \text{for} \ \alpha \ll 1. \tag{4.4}
\]

Returning to the propagation from an interval \([a, b]\) to its absorbing boundaries, the\nFeynman trajectories in the time interval \([0, \Delta t]\) consist of those that have not reached\nthe absorbing boundaries and of those that have. We calculate the Feynman integral\separately on each one of these two classes of trajectories. The integral over the first class\is the one calculated in Section 3 above. That over the latter class is the integral calculated\in the above paragraphs in this section. According to our assumptions, trajectories that\propagate into the absorbing boundary never return into the interval \([a, b]\) so that the\Feynman integral over these trajectories is supported outside the interval. On the other\hand, the Feynman integral over the bounded trajectories in the interval is supported\inside the interval. Thus the two integrals are orthogonal and give rise to no interference.\nThe Feynman integral over the bounded trajectories represents the wave function\conditioned on not exiting the interval in \([0, \Delta t]\). According to eq. (4.4) and to our\assumptions, the probability defined by the Feynman integral over the trajectories that\propagated into the absorbing boundary in this time interval is given by
\[
P_1(\Delta t) = \lambda(a) |\psi_1(a, \Delta t)|^2 + \lambda(b) |\psi_1(b, \Delta t)|^2 + o(\Delta t)
= \frac{\hbar \Delta t}{2\pi m} \left\{ \lambda(a) \left| \frac{\partial \psi_1(a, 0)}{\partial x} \right|^2 + \lambda(b) \left| \frac{\partial \psi_1(b, 0)}{\partial x} \right|^2 \right\} + o(\Delta t),
\]
where $\lambda(a)$ and $\lambda(b)$ are the characteristic lengths at $x = a$ and $x = b$, respectively. The survival probability in this time interval is

$$S_1(\Delta t) = 1 - P_1(\Delta t). \quad (4.5)$$

Next, we calculate the discretized Feynman integral in the time interval $[\Delta t, 2\Delta t]$ and again break it into the same classes as above. The discretized integral over the class of bounded trajectories in the interval $[a, b]$ is

$$\psi_2(x, 2\Delta t) = \left\{ \frac{m}{2\pi i\hbar \Delta t} \right\}^{1/2} \int_a^b \psi_1(x_1, \Delta t) \exp \left\{ \frac{i}{\hbar} S(x_1, x, \Delta t) \right\} dx_1.$$

The probability calculated from the Feynman integral over the absorbed trajectory in the time interval $[\Delta t, 2\Delta t]$ is

$$P_2(2\Delta t) = \lambda(a) |\psi_2(a, 2\Delta t)|^2 + \lambda(b) |\psi_2(b, 2\Delta t)|^2 + o(\Delta t).$$

This is the conditional probability of trajectories that propagate into the absorbing boundaries in the time interval $[\Delta t, 2\Delta t]$, given that they did not reach the boundaries in the previous time interval. Thus the discretized survival probability in the time interval $[0, 2\Delta t]$ is

$$S_2(2\Delta t) = (1 - P_1(\Delta t))(1 - P_2(2\Delta t)).$$

Proceeding this way, we find that the discretized Feynman integral over the class of bounded trajectories in the time interval $[0, N\Delta t]$ is

$$\psi_N(x, N\Delta t) = \left\{ \frac{m}{2\pi i\hbar \Delta t} \right\}^{1/2} \int_a^b \psi_{N-1}(x_{N-1}, (N - 1)\Delta t) \exp \left\{ \frac{i}{\hbar} S(x_{N-1}, x, \Delta t) \right\} dx_{N-1}. \quad (4.6)$$

As in eq.(3.3), we find that $\psi_N(x, N\Delta t) \to \psi(x, t)$ as $N \to \infty$, where $\psi(x, t)$ is the solution of Schrödinger’s equation in $(a, b)$ with the boundary conditions $\psi(a, t) = \psi(b, t) = 0$.

The probability that propagates into the absorbing walls in the time interval $[(j - 1)\Delta t, j\Delta t]$ is given by

$$P_j(j\Delta t) = \lambda(a) |\psi_j(a, j\Delta t)|^2 + \lambda(b) |\psi_j(b, j\Delta t)|^2 + o(\Delta t). \quad (4.7)$$

It follows that the survival probability of trajectories inside the interval is

$$S(t) = \lim_{N \to \infty} \prod_{j=1}^N (1 - P_j(j\Delta t)). \quad (4.8)$$

According to eqs.(4.4) and (4.7), $P_j(j\Delta t)$ is given by

$$P_j(j\Delta t) = \frac{\hbar \Delta t}{2\pi m} \left[ \lambda(a) \left| \frac{\partial}{\partial x} \psi_{j-1}(a, (j - 1)t) \right|^2 + \lambda(b) \left| \frac{\partial}{\partial x} \psi_{j-1}(b, (j - 1)t) \right|^2 + o(1) \right],$$
so that eq. (4.8) gives the survival probability

\[
S(t) = \exp \left\{ -\frac{\hbar}{\pi m} \int_0^t \left[ \lambda(a) \left| \frac{\partial}{\partial x} \psi(a, t') \right|^2 + \lambda(b) \left| \frac{\partial}{\partial x} \psi(b, t') \right|^2 \right] dt' \right\}. \tag{4.9}
\]

The wave function of the trajectories that have not been absorbed by time \( t \) is the wave function of a particle, conditioned on not reaching the absorbing boundary by time \( t \). The conditioning renormalizes the wave function inside the domain at all times and thus it remains \( \psi(x, t) \). The survival probability is \( 1 - P(t) \). If a particle is known not to have been absorbed by time \( t_1 \), its survival probability till time \( t_2 > t_1 \), denoted \( S(t_2, t_1) \), is given by

\[
S(t_2, t_1) = \exp \left\{ -\frac{\hbar}{\pi m} \int_{t_1}^{t_2} \left[ \lambda(a) \left| \frac{\partial}{\partial x} \psi(a, t) \right|^2 + \lambda(b) \left| \frac{\partial}{\partial x} \psi(b, t) \right|^2 \right] dt \right\}. \tag{4.10}
\]

The survival probability \( S(t) \) and the wave function \( \psi(x, t) \) can be combined into a discounted wave function

\[
\Psi(x, t) = \sqrt{S(t)} \psi(x, t). \tag{4.11}
\]

The discounted wave function should be used for all purposes if an absorbing surface is present in the system. It is normalized to \( S(t) \) and decays in time. The wave function \( \psi(x, t) \) is recovered from \( \Psi(x, t) \) by conditioning on survival of the particle by time \( t \).

5. Other modes of absorption

There are different phenomenological descriptions of absorption and decay in the literature. These may include different absorption rates at different energies, absorption of two non-interacting particles at a wall, and so on. It is not a priori obvious which mode of absorption applies in a given physical situation without a detailed analysis of the absorption mechanism. We show below that some of the phenomenological descriptions of absorption can be derived from our formalism.

5.1 Absorption in energy windows

Some of the descriptions are based on the premise that each mode is absorbed (or decays) at a rate proportional to its energy. This type of absorption can be obtained from the model of absorption of Feynman trajectories at an absorbing wall. Since the trajectories that reach the wall in time \( \Delta t \) cannot be separated into those that belong to one mode or another, the overall discounted probability has to be a function of the rate and probability of propagation of trajectories into the wall and the overall energy of the system. The way the discounting is spread among the different components of the surviving wave function can be chosen according to any number of criteria.
Therefore, we adopt the following absorption principles: the overall probability discounted at each time step is a function of the rate of propagation of probability into the wall and the average energy of the system; the probability discounted from each mode is, in addition, a function of the energy of that mode. According to these principles, the factor \( \lambda \) is allowed to be a function of the propagation rate, that is,

\[
\lambda = \lambda \left( \frac{\hbar}{\pi m} \left| \frac{\partial}{\partial x} \psi (0, t) \right| \right)^2.
\]

In addition, it can depend on the average energy, \( \langle \mathcal{E} \rangle \), defined as follows. Consider a single absorbing wall placed at the origin and assume that the wave function in the presence of an infinite potential wall at the origin is

\[
\psi (x, t) = \sum_{k=1}^{\infty} a_k \psi_k (x) \exp \left\{ \frac{-i}{\hbar} E_k t \right\},
\]

where the eigenfunctions \( \psi_k (x) \), corresponding to energies \( E_k \), form a complete set. The sum is replaced with an integral if the spectrum is continuous. The normalization condition is

\[
\sum_{k=1}^{\infty} |a_k|^2 = 1.
\]

The energy of a mode is defined as

\[
E_k = |a_k|^2 E_k
\]

and we define the average energy per mode as

\[
\langle \mathcal{E} \rangle = \sum_{k=1}^{\infty} |a_k|^2 \mathcal{E}_k = \sum_{k=1}^{\infty} |a_k|^4 E_k.
\]

Thus the factor \( \lambda \) has the form

\[
\lambda = \lambda \left( \frac{\hbar}{\pi m} \left| \frac{\partial}{\partial x} \psi (0, t) \right|^2, \langle \mathcal{E} \rangle \right).
\]

The overall probability propagating into the wall in the time interval \([t, t + \Delta t]\) is given by

\[
P(t) = \frac{\lambda \hbar}{\pi m} \left| \frac{\partial}{\partial x} \psi (0, t) \right|^2 \Delta t + O \left( \Delta t^{3/2} \right).
\]

Now, we spread this discounted probability among the different modes. The mode \( k \) is discounted by

\[
a_k \psi_k (x) \exp \left\{ -\frac{i}{\hbar} E_k t \right\} \left( 1 - \frac{\lambda_k}{2} \frac{\hbar}{\pi m} \left| \frac{\partial}{\partial x} \psi (0, t) \right|^2 \right) \Delta t + O \left( \Delta t^{3/2} \right).
\]
where $\lambda_k$ is defined below. The value of $\lambda_k$ has to be chosen so that the overall discounted probability is as given in eq. (5.2). The total surviving population is

$$
\int_{-\infty}^{0} \sum_{k=1}^{\infty} a_k \psi_k(x) \exp \left\{ -\frac{i}{\hbar} E_k t \right\} \left( 1 - \frac{\lambda_k}{\pi m} \left| \frac{\partial}{\partial x} \psi(0, t) \right|^2 \Delta t + O\left( \Delta t^{3/2} \right) \right)^2 dx
$$

$$
= \sum_{k=1}^{\infty} |a_k|^2 \left( 1 - \frac{\lambda_k}{\pi m} \left| \frac{\partial}{\partial x} \psi(0, t) \right|^2 \Delta t + O\left( \Delta t^{3/2} \right) \right),
$$

so that the overall discounted probability is

$$
\sum_{k=1}^{\infty} |a_k|^2 \lambda_k \frac{\hbar}{\pi m} \left| \frac{\partial}{\partial x} \psi(0, t) \right|^2 \Delta t + O\left( \Delta t^{3/2} \right) = \frac{\lambda \hbar}{\pi m} \left| \frac{\partial}{\partial x} \psi(0, t) \right|^2 \Delta t + O\left( \Delta t^{3/2} \right).
$$

Simplification leads to

$$
\sum_{k=1}^{\infty} |a_k|^2 \lambda_k = \lambda.  \tag{5.3}
$$

We set

$$
\lambda_k = \lambda \alpha_k
$$

so that eq. (5.3) becomes

$$
\sum_{k=1}^{\infty} |a_k|^2 \alpha_k = 1.
$$

To obtain absorption rates proportional to energies, we have to choose

$$
\alpha_k = \frac{\mathcal{E}_k}{\langle \mathcal{E} \rangle}.
$$

It follows that

$$
\lambda_k = \lambda \frac{\mathcal{E}_k}{\langle \mathcal{E} \rangle} \tag{5.4}
$$

and the absorption law for each mode reduces to

$$
\exp \left\{ - \int_{0}^{t} \lambda \frac{\mathcal{E}_k}{\langle \mathcal{E} \rangle} \frac{\hbar}{\pi m} \left| \frac{\partial}{\partial x} \psi(0, t') \right|^2 dt' \right\}.
$$

Now, we choose $\lambda$ so that

$$
\lambda \frac{\hbar}{\pi m} \left| \frac{\partial}{\partial x} \psi(0, t) \right|^2 = \frac{\langle \mathcal{E} \rangle}{\hbar},
$$

we obtain the discounted wave function of the surviving trajectories in the form

$$
\Psi(x, t) = \sum_{k=1}^{\infty} a_k \psi_k(x) \exp \left\{ -\frac{i}{\hbar} E_k t \right\} \exp \left\{ -\frac{\mathcal{E}_k}{2\hbar} t \right\}.  \tag{5.5}
$$
The survival probability of mode $k$ is

$$S_k(t) = |a_k|^2 \exp \left\{ -\frac{\mathcal{E}_k}{\hbar} t \right\}$$

and the total survival probability is the sum

$$S(t) = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = \sum_{k=1}^{\infty} |a_k|^2 \exp \left\{ -\frac{\mathcal{E}_k}{\hbar} t \right\}. \quad (5.6)$$

This mode of decay is similar to that in diffusion, where each mode decays separately at a different decay rate.

Now, we consider an interface between two media, one of which absorbs particles with energies in a given band $E_L < E < E_U$ and reflects all particles with energies outside the band. This situation is described by an infinite potential at the interface for particles with energies outside the band and an absorbing wall for particles with energies inside the band.

According to the formalism developed above, all particles see effectively an infinite wall at the interface, but only those with energies in the band are absorbed. To describe the absorption process, we separate the initial wave function $\psi_0(x)$ into its projection, $\psi_{\text{band}}(x)$, on the subspace spanned by eigenfunctions corresponding to the infinite wall whose eigenvalues (energies) are in the band, and to its projection, $\psi_{\text{c}}(x)$, on the orthogonal complement of this subspace. The wave function, $\psi(x,t)$, is also decomposed into analogous projections,

$$\psi(x,t) = \psi_{\text{band}}(x,t) + \psi_{\text{c}}(x,t),$$

where $\psi_{\text{band}}(x,t)$ decays according to the above absorption formalism, whereas $\psi_{\text{c}}(x,t)$ evolves according to the Schrödinger equation with zero boundary conditions on the interface. The $\lambda_k$ for this case are chosen by eq.\,[5.4] for energies in the band and 0 outside the band. The mean energy per mode is defined only for the modes that are absorbed, that is,

$$\langle \mathcal{E} \rangle = \sum_{k \in \text{band}} |a_k|^4 \mathcal{E}_k.$$

5.2 Absorption of two non-interacting particles at a wall

The Hamiltonian for two non-interacting particles has the structure

$$H(x_1, p_1; x_2, p_2) = H_1(x_1, p_1) + H_2(x_2, p_2)$$

so that the wave function has the form

$$\psi(x_1, x_2, t) = \psi_1(x_1, t)\psi_2(x_2, t).$$

If an absorbing wall for both particles is placed at $x = 0$ and initially both particles are to the left of the wall, the configuration space is the third quadrant in the $(x_1, x_2)$-plane.
The absorbing boundary, denoted $\partial D$, consists of the negative $x_1$ and $x_2$ axes. Each particle can have a different $\lambda$ so that a two dimensional analog of the calculation of the absorption rate gives the instantaneous rate

$$\frac{\lambda_1 \hbar}{\pi m_1} \int_{-\infty}^{0} \left| \frac{\partial \psi_1(0,t)}{\partial x_1} \right|^2 dx_2 + \frac{\lambda_2 \hbar}{\pi m_2} \int_{-\infty}^{0} \left| \frac{\partial \psi_2(0,t)}{\partial x_2} \right|^2 dx_1 = \lambda_1 \hbar \pi m_1 \left| \frac{\partial \psi_1(0,t)}{\partial x_1} \right|^2 + \lambda_2 \hbar \pi m_2 \left| \frac{\partial \psi_2(0,t)}{\partial x_2} \right|^2.$$  

(5.7)

Thus the discounted wave function is given by

$$\Psi(x_1, x_2, t) = \Psi_1(x_1, t)\Psi_2(x_2, t),$$

where

$$\Psi_j(x_j, t) = \psi_j(x_j, t) \exp \left\{ \frac{\lambda_j \hbar}{\pi m_j} \left| \frac{\partial \psi_j(0,t)}{\partial x_j} \right|^2 \right\}, \quad (j = 1, 2).$$

Thus, the population decays together at a rate that is the sum of the two rates. The decay law is different than that of a single particle in a superposition of two states in that there are no beats in the decay law of two independent particles.

It is apparent from the above examples that different modes of decay can be obtained from the model of an absorbing wall. It is not a-priory clear which mode of decay is appropriate for a given physical system. The choice of the decay mode depends on the particular physical system.

6. The absorption current and the slit experiment

An absorbing boundary engenders an absorption probability current into the boundary $\Gamma$. The simplest definition of the absorption current at a single absorbing point, $x = 0$, in one dimension, is (see eq.(4.3))

$$\mathcal{J}(0, t) = \frac{d}{dt} [1 - S(t)] = \lambda \hbar \left| \frac{\partial \psi(0,t)}{\partial x} \right|^2 \exp \left\{ -\lambda \hbar \int_{0}^{t} \left| \frac{\partial \psi(0,t')}{\partial x} \right|^2 dt' \right\},$$  

(6.1)

In higher dimensions the discounted wave function in a domain $\mathcal{D}$ in the presence of an absorbing boundary $\Gamma$ is given by

$$\Psi(\mathbf{x}, t) = \psi(\mathbf{x}, t) \exp \left\{ -\frac{\lambda \hbar}{2\pi m} \int_{0}^{t} \int_{\Gamma} \left| \frac{\partial \psi(\mathbf{x}', t')}{\partial \mathbf{n}} \right|^2 dS_{\mathbf{x}'} dt' \right\},$$  

(6.2)
where $\psi(x, t)$ is the solution of Schrödinger’s equation in $\mathcal{D}$ with zero boundary condition on $\Gamma$ and $n$ is the unit outer normal to the boundary. Adopting the interpretation of the squared modulus of the wave function as the probability density of finding a particle at the point $x$ at time $t$ (whatever that means), the discounted wave function can be used to calculate the joint probability density of surviving by time $t$ and finding the particle at $x$ at the same time. The squared modulus of the wave function, conditioned on surviving by time $t$, is found by dividing the joint probability density $|\Psi(x, t)|^2$ by the probability of the condition, $S(t)$. In the multi-dimensional case at hand

$$S(t) = \exp\left\{ -\frac{\lambda \hbar}{\pi m} \int_0^t \oint_{\Gamma} \left| \frac{\partial \psi(x', t')}{\partial n} \right|^2 dS_{x'} dt' \right\}.$$  \hfill (6.3)

It follows from eqs. (6.2) and (6.3) that the conditioned wave function is $\psi(x, t)$.

Generalizing the definition (6.1) to higher dimensions, the total current at the absorbing boundary is defined as

$$J(\Gamma) = \frac{d}{dt}[1 - S(t)]$$

$$= \frac{\lambda \hbar}{\pi m} \oint_{\Gamma} \left| \frac{\partial \psi(x, t)}{\partial n} \right|^2 dS_x \exp\left\{ -\frac{\lambda \hbar}{\pi m} \int_0^t \oint_{\Gamma} \left| \frac{\partial \psi(x', t')}{\partial n} \right|^2 dS_{x'} dt' \right\}.$$  \hfill (6.4)

The normal component of the multi-dimensional probability current density at any point $x$ on $\Gamma$ is given by

$$J(x, t) \cdot n(x)|_{\Gamma} = \frac{\lambda \hbar}{\pi m} \left| \frac{\partial \psi(x, t)}{\partial n} \right|_{\Gamma} \exp\left\{ -\frac{\lambda \hbar}{\pi m} \int_0^t \oint_{\Gamma} \left| \frac{\partial \psi(x', t')}{\partial n} \right|^2 dS_{x'} dt' \right\}.$$  \hfill (6.5)

Next, we consider the slit experiment with an absorbing screen (e.g., photographic plate or a fluorescent screen). In this case, unlike the usual assumption in quantum mechanics [14], the pattern that appears on the absorbing screen cannot be the squared modulus of the wave function in the entire space (with or without a finite potential), as the above analysis implies. It cannot be the modulus of the wave function with an absorbing boundary because the wave function vanishes on an absorbing boundary. According to the absorption principles, the pattern on the screen represents the probability density that propagates into the screen. This probability density is the unidirectional absorption current on the screen. Thus, the instantaneous pattern at time $t$ is $J(x, t) \cdot n(x)|_{\Gamma}$ and the cumulative pattern, after the arrival of many particles, is

$$J(x) \cdot n(x)|_{\Gamma} = \int_0^\infty J(x, t) \cdot n(x)|_{\Gamma} dt.$$  \hfill (6.6)

The instantaneous pattern $J(x, t) \cdot n(x)|_{\Gamma}$ is obtained as the histogram of points on the screen collected at time $t$ after releasing the particle in the slit. The cumulative current density $J(x) \cdot n(x)|_{\Gamma}$ is the histogram of points on the screen collected at all times.
We consider the following experimental setup for the slit experiment with an absorbing screen. A planar screen is placed in the plane $x = 0$ and another screen it placed in the plane $x = x_0$ and it is slit along a line parallel to the $z$–axis. Due to the invariance of the geometry of the problem in $z$ the mathematical description of the slit is, following [14], an initial truncated Gaussian wave packet in the $(x, y)$–plane, concentrated around the initial point, $(x_0, 0)$. To describe the interference pattern on the screen, we assume it is an absorbing line on the $y$–axis and apply the formalism developed above. The wave function, as given by our formalism, evolves from the initial packet according to eq.(6.2), as

$$\Psi(x, y, t) = \psi(x, y, t) \exp \left\{ -\frac{\lambda \hbar}{2\pi m} \int_0^t \int_{-\infty}^{\infty} \left| \frac{\partial \psi(0, y', t')}{\partial x} \right|^2 dy' dt' \right\},$$

where $\psi(x, y, t)$ is the solution of the Schrödinger equation in the half plane $x > 0$ with $\psi(0, y, t) = 0$ and $\psi(x, y, 0)$ is the given initial packet.

The probability current density on the screen is given by

$$J(0, y, t) = \frac{\lambda \hbar}{\pi m} \left| \frac{\partial \psi(0, y, t)}{\partial x} \right|^2 \exp \left\{ -\frac{\lambda \hbar}{\pi m} \int_0^t \int_{-\infty}^{\infty} \left| \frac{\partial \psi(0, y', t')}{\partial x} \right|^2 dy' dt' \right\} \quad (6.7)$$

This is the probability density of a collapse of the wave function occurring at the point $y$ on the screen at time $t$. The total current

$$J(y) = \frac{\lambda \hbar}{\pi m} \int_0^\infty \left| \frac{\partial \psi(0, y, t)}{\partial x} \right|^2 \exp \left\{ -\frac{\lambda \hbar}{\pi m} \int_0^t \int_{-\infty}^{\infty} \left| \frac{\partial \psi(0, y', t')}{\partial x} \right|^2 dy' dt' \right\} dt \quad (6.8)$$

is the probability density that the collapse of the wave function occurs at the point $y$ on the screen (ever). If the initial distribution of velocities in the $x$ direction is concentrated about $v_0$, the density $J(y)$ is approximately the same as $J(0, y, \bar{t})$, where [14]

$$\bar{t} = \frac{x_0}{v_0}.$$
where \( \psi^1(x, t) \) is the function \( K(x, t) \) in eqs. (3.9, 3.10, 3.11) given above and by eq. (7.4) given below.

Note that according to our theory \( \psi^1(0, t) = 0 \), because the absorbing screen is placed at \( x = 0 \). The component \( \psi^2(y, t) \) of the wave function is given by the freely propagating Gaussian slit [14]

\[
\psi^2(y, t) = \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2_y} e^{-im(y-z)^2/2\hbar t} \sqrt{2\pi i\sigma_y} \frac{1}{\sqrt{2\pi i\hbar t/m}} dz.
\]

Evaluation of the integral gives

\[
|\psi^2(y, t)|^2 = \frac{1}{2\pi\sigma_y} \left( \frac{\hbar^2 t^2}{\sigma^2_y m^2} + \sigma^2_y \right)^{-1/2} \exp \left\{ -\frac{y^2}{\frac{\hbar^2 t^2}{\sigma^2_y m^2} + \sigma^2_y} \right\}
\]

If \( \sigma_x \ll |x_0| \), the upper limit of integration in eq. (4.9) can be replaced by \( \infty \) with a transcendentally small error.

According to eq. (6.7), the instantaneous absorption rate at time \( t \) at a point \((0, y)\) on the screen is given by

\[
\mathcal{J}(0, y, t) = \left| \frac{\partial K(0, t)}{\partial x} \right|^2 \left| \psi^2(y, t) \right|^2,
\]

where \( |\partial K(0, t)/\partial x|^2 \) is given in eq. (7.5) below. Thus the pattern on the screen at time \( t \) is given by \( \mathcal{J}(0, y, t) \) in eq. (6.9).

To compare eq. (6.9) with that given in [14], we reproduce the derivation of [14] with an initial two-dimensional Gaussian wave packet. The result gives the wave function as

\[
\psi_F(x, y, t) = \psi^1(x, t)\psi^2(y, t)
\]

and probability density at the screen at time \( t \) as

\[
|\psi_F(0, y, t)|^2 = \frac{1}{2\pi\sigma_x} \left( \frac{\hbar^2 t^2}{\sigma^2_x m^2} + \sigma^2_x \right)^{-1/2} \exp \left\{ -\frac{x_0^2}{\frac{\hbar^2 t^2}{\sigma^2_x m^2} + \sigma^2_x} \right\} \left| \psi^2(y, t) \right|^2.
\]

Comparing equation (6.10) with eq. (6.9), we see that the pattern predicted by Feynman’s theory differs from that in the present theory by a time dependent factor only. The instantaneous intensity of the diffraction pattern in the absence of an absorbing screen, given in [14], is defined as \( |\psi_F(0, y, t)|^2 \). Thus, the introduction of an absorbing screen, according to these interpretations, gives the relative brightness as

\[
\lambda \left| \frac{\partial \psi(0, y, t)}{\partial x} \right|^2 \left| \psi_F(0, y, t) \right|^2,
\]

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which is a function of time only. The decay in time of the quotient reflects the fact that the absorbing screen depresses the entire wave function in time. Thus, the part of the packet that arrives later is already attenuated by the preceding absorption, relative to the unattenuated wave function in the absence of absorption.

7. Examples

In the following examples possible experiments are discussed, that can be used to verify the predictions of the above theory in the different modes of absorption. The examples include a particle between absorbing walls, a free particle incident on an absorbing wall, and the slit experiment bounded by absorbing walls. The latter indicates that the pattern on the screen changes when absorbing walls are present.

7.1 A particle between two absorbing walls

First, we consider a particle with symmetric absorbing walls at \( x = 0, a \) and zero potential. We assume that \( \lambda_+ = \lambda_0 \). The wave function is given by

\[
\psi(x, t) = \sum_{n=1}^{\infty} A_n \exp\left\{ -\frac{i\hbar n^2 \pi^2}{2ma^2} t \right\} \sin \frac{n\pi}{a} x.
\]

It was shown in [5] that for a particle with a single energy level the wave function decays at an exponential rate proportional to the energy. However, if there are more than just one level, the exponent contains beats. For example, for a two level system with real coefficients, we obtain the survival probability

\[
S(t) = \exp \left\{ -\frac{\lambda_0 \hbar}{m \pi} \left[ \frac{\pi^2}{a^2} (A_k^2 k^2 + A_n^2 n^2) t - \frac{4m(-1)^{k+n} kn A_k A_n}{\hbar (n^2 - k^2)} \sin \frac{\hbar (n^2 - k^2) \pi^2}{2ma^2} t \right] \right\}.
\]

The strongest beats occur for \( k = 2, n = 1 \) with frequency \( \omega_{1,2} = 3\hbar \pi^2 / 2ma^2 \). Setting \( A_1 = A_2 = \sqrt{1/2a} \) and introducing the dimensionless time \( \tau = \omega_{1,2} t \), we find that the survival probability is

\[
S(t) = \exp \left\{ -\frac{\lambda_0}{3\pi} (5\tau - 2 \sin \tau) \right\}.
\]

The function

\[
\log S(t) = -\frac{\lambda_0}{3\pi} (5\tau - 2 \sin \tau)
\]

is qualitatively similar to that given in [15].

If absorption in windows of energy is assumed, the discounted wave function (5.5) is

\[
\Psi(x, t) = \sum_{n=1}^{\infty} A_n \exp \left\{ -\frac{i\hbar n^2 \pi^2}{2ma^2} t \right\} \exp \left\{ -\frac{|A_n|^2 \hbar n^2 \pi^2}{2ma^2} t \right\} \sin \frac{n\pi}{a} x.
\]
Writing

\[ A_n = \frac{a_n}{\sqrt{a}} \]

such that \( \sum |a_n|^2 = 1 \), we obtain that the survival probability (5.6) is

\[ S_E(t) = \sum_{n=1}^{\infty} |a_n|^2 \exp \left\{ -\frac{|a_n|^2 \hbar n^2 \pi^2}{ma^2} t \right\}. \]  

(7.3)

In the particular case that there is only one energy level in the box, both results (7.1) and (7.3) should be the same. In this case, we obtain

\[ \lambda = \pi a. \]

**7.2 A wave packet incident on an absorbing wall**

Next, we consider a Gaussian-like wave packet of free particles traveling toward an absorbing wall at \( x = 0 \) with positive mean velocity \( k_0 \). The wave function is given by

\[ \Psi(x, t) = \left( \frac{2a^2}{\pi} \right)^{1/4} \frac{A^{-1/2} \psi}{(a^4 + 4\hbar^2 t^2)^{1/4}} \times \left[ \exp \left\{ -\frac{(x + x_0 - \frac{\hbar k_0}{m} t)^2}{a^2 + \frac{2\hbar t}{m}} + ik_0 (x + x_0) \right\} \right] - \left[ \exp \left\{ -\frac{(x - x_0 + \frac{\hbar k_0}{m} t)^2}{a^2 + \frac{2\hbar t}{m}} - ik_0 (x - x_0) \right\} \right], \]

where

\[ A = \left[ 1 - \exp \left( -\frac{2x_0^2}{a^2} \right) \exp \left( -\frac{1}{2} a^2 k_0^2 \right) \right] \]

and \( \varphi \) is a phase (see \[8, p.64\]). It follows that

\[ \frac{\lambda \hbar}{m \pi} \left| \frac{\partial}{\partial x} \psi(0, t) \right|^2 = \]  

\[ \frac{\sqrt{2} \lambda \hbar m^2 (4x_0^2 + a^4 k_0^2)}{(\sqrt{\pi})^3 A (a^4 m^2 + 4t^2 \hbar^2)} \exp \left( -\frac{2a^2 x_0^2 m^2 + \hbar^2 k_0^2 t^2 - 2k_0 mx_0 \hbar}{a^4 m^2 + 4t^2 \hbar^2} \right) \],

hence

\[ \int_{0}^{\infty} \left| \frac{\partial}{\partial x} \psi(0, t) \right|^2 dt < \infty. \]  

(7.6)

Thus

\[ R = S(\infty) = \lim_{t \to \infty} [1 - P(t)] > 0, \]
that is, the wave packet is only partially absorbed. This means that the “reflected” wave consists of trajectories that turned around before propagating past the absorbing wall. The absorption occurs when the trajectory propagates into the medium inside the wall. The discount of the wave function occurs when the packet is at the wall, as can be seen from eqs. (7.5) and (7.6). Thus $R$ plays the role of a reflection coefficient. This is neither the usual reflection coefficient for a finite potential barrier nor that for an infinite barrier.

The reflection coefficient is a function of the packet group velocity, $k_0$, the width of the packet, $a$, its initial distance from the absorbing screen $x_0$, the parameter $\lambda$, and its mass $m$. This dependence is experimentally measurable.

If absorption in energy windows is assumed, the discounted wave function, as given in eq.(5.3), is calculated as follows. The modes are the functions

$$\psi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx$$

and the energies $E_k$ are

$$E_k = \frac{\hbar^2 k^2}{2m}.$$

The discounted wave function is given by eq.(5.5) with summation replaced by integration with respect to $k$ over the entire real line. The coefficient $a_k$ is the sine transform of the initial wave function (7.4) with $t = 0$,

$$a_k = -i(2\pi)^{1/4} \sqrt{\frac{a}{A}} \times$$

$$\exp \left\{ -\frac{x_0^2}{a^2} \right\} \left( \exp \left\{ \frac{(x_0 - ika^2 + ik_0 a^2)^2}{a^2} \right\} - \exp \left\{ \frac{(x_0 - -ika^2 + ik_0 a^2)^2}{a^2} \right\} \right)$$

and the decay rate of the $k$-th mode, as given in eqs.(5.1), (5.5), is

$$\frac{\mathcal{E}_k}{\hbar} = \frac{|a_k|^2 \hbar^2 k^2}{2m\hbar}$$

$$= \frac{a\hbar k^2 \sqrt{2\pi}}{4mA} \exp \left\{ -\frac{k^2 a^2}{2} \right\} \exp \left\{ -\frac{k^2 a^2}{2} \right\} \left[ \cosh (kk_0 a^2) - \cos (2x_0 k) \right].$$

According to Section 5.1, the survival probability is the same as in the previous case.

### 7.3 The slit experiment with lateral absorbing boundaries

Here, we propose a simple device for performing a measurement of times of arrival of particles at an absorbing boundary in one dimension, as well as that of the unidirectional current at the absorbing boundaries. In addition, the proposed device demonstrates the effect of additional absorbing boundaries on the slit experiment. The proposed measurement can discriminate between the various modes of absorption described above.
Consider the setup of the slit experiment enclosed between two parallel absorbing walls, symmetric with respect to the slit and perpendicular to the planes of the screen and the slit. For example, the walls can be made of photographic plates. Particles are given a constant initial velocity, \( v_x \), in the \( x \) direction (perpendicular to the planes of the screen and slit), within the constraints of uncertainty. The time a particle leaves the slit is also measurable within the constraints of uncertainty.

The initial packet is Gaussian in the \( x \) direction and is uniform inside the slit (in the \( y \) direction). This means that the initial velocities in the \( y \) direction have the density

\[
|\Psi(k)|^2 = \left| \frac{\sin \frac{\pi k}{2}}{\frac{\pi k}{2}} \right|^2 . \tag{7.7}
\]

The plane of the slit is \( x = x_0 \), the plane of the screen is \( x = 0 \), the slit is the interval \(-\pi/2 < y < \pi/2\). The absorbing planes are \( y = \pm y_0 \) with \( y_0 > \pi/2 \). In this setup the motion of the particles in the \( x \) direction is independent of that in the \( y \) direction. The latter is the object of the proposed experiment.

Particles that hit the planes \( y = \pm y_0 \) leave traces at points \( x_1, x_2, \ldots, x_N \). These distances are proportional (within the constraints of uncertainty) to the times of arrival at the absorbing walls of particles that start out in the interval \(-\pi/2 < y < \pi/2\) with initial velocities distributed as in eq.(7.7). The histogram obtained from these points, on an axis normalized with the velocity \( v_x \), is that of the times of arrival of one dimensional particles moving on the \( y \) axis.

The wave function for this configuration is given by

\[
\psi(x, y, t) = \psi^1(x, t)\psi^2(y, t) ,
\]

where \( |\partial\psi^1(0, t)/\partial x|^2 \) is the same as given in eq.(7.6) and

\[
\psi^2(y, t) = \sum_{n=1}^{\infty} \frac{2}{n\pi^{3/2}} \cos \frac{n\pi^2}{4y_0} \sin \frac{n\pi}{y_0} y \exp \left\{ -\frac{i n^2 \pi^2}{\hbar y_0^2} t \right\} .
\]

According to our theory, the pattern on each lateral wall is given by

\[
\mathcal{J}(x, \pm y_0) = \int_0^\infty \mathcal{J}(x, \pm y_0, t) \, dt = \int_0^\infty \left| \frac{\partial\psi^1(x, t)}{\partial x} \right|^2 |\psi^2(\pm y_0, t)|^2 \, dt . \tag{7.8}
\]

Next, we consider the pattern observed on the wall \( x = 0, -\pi/2 < y < \pi/2 \). The histogram of the traces of the particles on the screen at \( x = 0 \) is given by

\[
\mathcal{J}(y) = \int_0^\infty \left| \frac{\partial\psi^1(0, t)}{\partial x} \right|^2 \sum_{n=1}^{\infty} \frac{2}{n\pi^{3/2}} \cos \frac{n\pi^2}{4y_0} \sin \frac{n\pi}{y_0} y \exp \left\{ -\frac{i n^2 \pi^2}{\hbar y_0^2} t \right\}^2 \, dt . \tag{7.9}
\]
If the velocities in the $x$ direction are concentrated around $v_x$, the histogram will be approximately

$$J(y) \approx J(y, \bar{t}) = \left| \partial \psi_1^1(0, \bar{t}) / \partial x \right|^2 \psi_2^2(y, \bar{t})$$

This is not the same as the expression obtained in eq.(6.8). The difference is due to the effect of the lateral absorbing boundaries. Thus, according to our theory, absorbing boundaries cannot be ignored, as usually done in quantum mechanics.

If absorption in energy windows is assumed, the function $\psi_1^1(x, t)$ is the same as calculated in Section 7.1 above,

$$\psi_2^2(y, t) = \sum_{n=1}^{\infty} \frac{2}{n\pi^{3/2}} \cos \frac{n\pi^2}{4y_0} \sin \frac{n\pi}{y_0} y \exp \left\{ -\frac{i}{\hbar} \frac{n^2\pi^2}{2y_0^2} t \right\}.$$

The patterns on the absorbing walls are given by eqs.(7.8) and (7.9).

8. Discussion and summary

We observe that according to eq.(4.5) there is conservation of probability: the probability of the absorbed Feynman trajectories and that of the surviving trajectories sum to 1. This is the result of our postulate that the Feynman integrals over the two classes of trajectories have disjoint supports, that is, the absorbed trajectories never return to the interval $[a, b]$. This conservation of probability persists for all times. This result is different from that obtained in decoherent state theory [4].

The survival probability $S(t)$ is the probability obtained by repeating the experiment of observing the particle between the absorbing boundaries at time $t$ and constructing a histogram of the number of times the particle is observed. This probability is not a quantum mechanical quantity in the sense that it is not the integral of the squared modulus of a probability amplitude defined by Schrödinger’s equation.

The wave function of the particle at time $t$, given that it has not been absorbed by that time, is the wave function $\psi(x, t)$ defined by Schrödinger’s equation inside the given domain with reflecting boundary conditions. That is, if the absorbing boundary represents a detector, the wave function of the particle is $\psi(x, t)$ as long as the particle has not been detected.

The concept of absorption of particles, energy, momentum, and so on, is an aspect of time irreversible processes in quantum mechanics. Examples of irreversible processes in quantum mechanics are the notion of collapse of the wave function that is caused by measurement (this will discuss at [10]), the decay of a state of a particle, a particle that enters into a bulk and loses its momentum due to interactions with the particles of the bulk, and so on. This paper is an attempt to construct a formalism for the description of such phenomena.
The process of absorption of quantum particles can be illustrated by a packet of identical non-interacting particles that hit an absorbing surface, for example, a photographic plate. At the moment a particle hits the plate it is absorbed in the sense that its wave function no longer evolves according to the Hamiltonian of the particles that have not been absorbed so far. Thus the absorbing surface separates the particles of the packet into two sets, those that have reached the surface by a given time and those that have not. The interference between the Feynman integral over the trajectories of the absorbed particles and that over the trajectories of the surviving ones vanishes. The probability of the absorbed particles is discounted from the total probability of the packet. This is also the case for any other detector that absorbs particles. The surviving trajectories, those that have not reached the absorbing boundaries so far, give rise to a reflected wave, as if the absorbing boundary were an infinite potential wall. This fact becomes apparent not from a solution of a wave equation, but rather from the calculation of the Feynman integral over a class of restricted trajectories.

Our derivation does not start with a Hamiltonian, but rather with an action of trajectories in a restricted class.

Quantum mechanics without absorption is recovered from our formalism when the absorbing boundaries are moved to infinity or when the absorption constant $\lambda$ vanishes.

The examples demonstrate the expected phenomenon that particles that reach the absorbing boundary are partially reflected and partially absorbed. In either case the decay pattern of the wave function seems to be new.

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