A quaternary diophantine inequality by prime numbers of a special type

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Abstract

Let $1 < c < 832/825$. For large real numbers $N > 0$ and a small constant $\vartheta > 0$, the inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c - N| < \vartheta$$

has a solution in prime numbers $p_1, p_2, p_3, p_4$ such that, for each $i \in \{1, 2, 3, 4\}$, $p_i + 2$ has at most 32 prime factors.

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1 Introduction and statements of the result.

In 1952 I. I. Piatetski-Shapiro [10] investigated the inequality

$$|p_1^c + p_2^c + \cdots + p_r^c - N| < \varepsilon$$

where $c > 1$ is not an integer, $\varepsilon$ is a fixed small positive number, and $p_1, \ldots, p_r$ are primes. He proved the existence of an $H(c)$, depending only on $c$, such that for all sufficiently large real $N$, (1) has a solution for $H(c) \leq r$. He established that

$$\limsup_{c \to \infty} \frac{H(c)}{c \log c} \leq 4$$

and also that $H(c) \leq 5$ if $1 < c < 3/2$.

In 1992 Tolev [13] proved that (1) has a solution for $r = 3$ and $1 < c < 15/14$. The interval $1 < c < 15/14$ was subsequently improved by several authors [3], [7], [8], [1].

In 2003 Zhai and Cao [15] proved that (1) has a solution for $r = 4$ and $1 < c < 81/68$. Their result was improved to $1 < c < 97/81$ by Mu [9].
In 2016 Dimitrov [4] showed that (1) has a solution for $r = 3$, $0 < c < 4/21$ and primes $p_1, p_2, p_3$ such that, for each $i \in \{1, 2, 3\}$, $p_i + 2$ has at most 10 prime factors.

Recently Tolev [14] proved that (1) has a solution for $r = 3$, $1 < c < 15/14$ and primes $p_1, p_2, p_3$ such that, for each $i \in \{1, 2, 3\}$, $p_i + 2$ has at most \[\left\lfloor \frac{369}{180 - 168c} \right\rfloor\] prime factors.

Let $P_l$ is a number with at most $l$ prime factors. Motivated by [14], we shall prove the following theorem.

**Theorem 1.** Let $A$ be an arbitrary large and fixed, and let $1 < c < 832/825$. There exists a number $N_0(c) > 0$ such that for each real number $N > N_0(c)$ the inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c - N| < \frac{1}{(\log N)^A}$$

has a solution in prime numbers $p_1, p_2, p_3$ such that

$$p_1 + 2 = P'_3, \quad p_2 + 2 = P''_3, \quad p_3 + 2 = P'''_3, \quad p_4 + 2 = P'^{***}_3.$$

By choosing the parameters in a different way we may obtain other similar results, for example $1 < c < 51/50$, $p_i + 2 = P_r$, $i = 1, 2, 3, 4$, where $r$ is large. Obviously the enlargement of the range for $c$ leads to increase of the number of the prime factors of $p_i + 2$.

### 2 Notations and some lemmas.

As usual $\varphi(n)$ and $\mu(n)$ denote respectively, Euler’s function and Möbius’ function. We denote by $\tau(n)$ the number of the positive divisors of $n$. Let $(m_1, m_2)$ be the greatest common divisor. Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n \,(k)$. As usual $[y]$ denotes the integer part of $y$, $e(y) = e^{2\pi iy}$. Let $c$ be a fixed real number such that
1 < c < 832/825 and N be a sufficiently large number.

\[ X = (N/3)^{1/c}; \]  \hspace{1cm} (2)

\[ \tau = X^{57/275-c}; \]  \hspace{1cm} (3)

\[ \vartheta = \frac{1}{(\log X)^{4+1}}, \quad A > 20 \text{ is arbitrary large}; \]  \hspace{1cm} (4)

\[ K = \frac{\log^2 X}{\vartheta}; \]  \hspace{1cm} (5)

\[ D = X^{1/11-\varepsilon_0}, \quad \varepsilon_0 = 0.001; \]  \hspace{1cm} (6)

\[ \eta = \frac{\varepsilon_0}{9}; \]  \hspace{1cm} (7)

\[ z = X^\beta, \quad 0 < \beta < 1/33; \]  \hspace{1cm} (8)

\[ P(z) = \prod_{2<p\leq z} p, \quad p \text{-prime number}; \]  \hspace{1cm} (9)

\[ I(\alpha) = \int_{X/2} e(\alpha t^c)dt. \]  \hspace{1cm} (10)

The value of \( \beta \) will be specified latter.

Let \( \lambda^\pm(d) \) be the lower and upper bounds Rosser’s weights of level \( D \), hence

\[ |\lambda^\pm(d)| \leq 1, \quad \lambda^\pm(d) = 0 \quad \text{if} \ d \geq D \quad \text{or} \ \mu(d) = 0. \]  \hspace{1cm} (11)

For further properties of Rosser’s weights we refer to [5], [6].

**Lemma 1.** Let \( \vartheta \in \mathbb{R} \) and \( k \in \mathbb{N} \). There exists a function \( \theta(y) \) which is \( k \) times continuously differentiable and such that

\[ \theta(y) = 1 \quad \text{for} \quad |y| \leq 3\vartheta/4; \]

\[ 0 \leq \theta(y) < 1 \quad \text{for} \quad 3\vartheta/4 < |y| < \vartheta; \]

\[ \theta(y) = 0 \quad \text{for} \quad |y| \geq \vartheta. \]

and its Fourier transform

\[ \Theta(x) = \int_{-\infty}^{\infty} \theta(y)e(-xy)dy \]

satisfies the inequality

\[ |\Theta(x)| \leq \min \left( \frac{7\vartheta}{4}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left( \frac{k}{2\pi|x|\vartheta/8} \right)^k \right). \]
Proof. See [11].

Lemma 2. Let \( n \in \mathbb{N} \). Then

\[
\tau(n) \ll n^\varepsilon,
\]

where \( \varepsilon \) is an arbitrary small positive number.

Lemma 3. Let \( X \in \mathbb{R}, X \geq 2 \). We have

\[
\sum_{n \leq X} \frac{1}{\varphi(n)} \ll \log X.
\]

Lemma 4. Assume that \( F(x), G(x) \) are real functions defined in \([a, b]\), \( |G(x)| \leq H \) for \( a \leq x \leq b \) and \( G(x)/F'(x) \) is a monotonous function. Set

\[
I = \int_a^b G(x)e(F(x))dx.
\]

If \( F'(x) \geq h > 0 \) for all \( x \in [a, b] \) or if \( F'(x) \leq -h < 0 \) for all \( x \in [a, b] \) then

\[
|I| \ll H/h.
\]

If \( F''(x) \geq h > 0 \) for all \( x \in [a, b] \) or if \( F''(x) \leq -h < 0 \) for all \( x \in [a, b] \) then

\[
|I| \ll H/\sqrt{h}.
\]

Proof. See ([12], p. 71).

3 Outline of the proof.

Consider the sum

\[
\Gamma = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \log p_1 \log p_2 \log p_3 \log p_4.
\]

Any non-trivial estimate from below of \( \Gamma \) implies the solvability of \( |p_1^\varepsilon + p_2^\varepsilon + p_3^\varepsilon + p_4^\varepsilon - N| < \vartheta \) in primes such that \( p_1 + 2 = P_h, \ h = [\beta^{-1}] \).

We have

\[
\Gamma \geq \tilde{\Gamma} = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \theta(p_1^\varepsilon + p_2^\varepsilon + p_3^\varepsilon + p_4^\varepsilon - N) \log p_1 \log p_2 \log p_3 \log p_4.
\]
On the other hand

\[ \tilde{\Gamma} = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \log p_1 \log p_2 \log p_3 \log p_4, \]  

where

\[ \Lambda_i = \sum_{d \mid (p_i + 2, P(z))} \mu(d) \text{, } i = 1, 2, 3, 4. \]

We denote

\[ \Lambda_i^\pm = \sum_{d \mid (p_i + 2, P(z))} \lambda^\pm(d) \text{, } i = 1, 2, 3, 4. \]

From the linear sieve we know that \( \Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+ \) (see [2], Lemma 10). Then we have a simple inequality

\[ \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \geq \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^- \Lambda_4^- + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- \Lambda_4^+ + \Lambda_1^- \Lambda_2^- \Lambda_3^+ \Lambda_4^- - 3 \Lambda_1^+ \Lambda_2^+ \Lambda_3^- \Lambda_4^- \]  

(see [2], Lemma 13).

Using (14) and (16) we obtain

\[ \tilde{\Gamma} \geq \Gamma_0 = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} (p_1^c + p_2^c + p_3^c + p_4^c - N) \times \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^- \Lambda_4^- + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- \Lambda_4^+ + \Lambda_1^- \Lambda_2^- \Lambda_3^+ \Lambda_4^- - 3 \Lambda_1^+ \Lambda_2^+ \Lambda_3^- \Lambda_4^- \times \log p_1 \log p_2 \log p_3 \log p_4. \]

Let

\[ \Gamma_0 = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 - 3 \Gamma_5, \]  

where for example

\[ \Gamma_1 = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \log p_1 \log p_2 \log p_3 \log p_4 \]  

(19)

and so on.

It is easy to see that \( \Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 \). We shall consider the sum \( \Gamma_1 \). The sum \( \Gamma_5 \) can be considered in the same way.

From (15) and (19) we have

\[ \Gamma_1 = \sum_{d_i \mid P(z)} \lambda_i^-(d_1) \lambda_i^+(d_2) \lambda_i^+(d_3) \lambda_i^+(d_4) \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \times \log p_1 \log p_2 \log p_3 \log p_4. \]
Using the inverse Fourier transform for the function $\theta(x)$ we get

$$
\Gamma_1 = \sum_{d_i \mid P(z)} \lambda^- (d_1) \lambda^+ (d_2) \lambda^+ (d_3) \lambda^+ (d_4) \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X \atop p_i + 2 = 0 (d_i), i = 1, 2, 3, 4} \log p_1 \log p_2 \log p_3 \log p_4
$$

$$
\times \int_{-\infty}^{ \infty} \Theta(t) e((p_1^c + p_2^c + p_3^c + p_4^c - N)t) dt
$$

$$
= \int_{-\infty}^{ \infty} \Theta(t) e(-Nt) L_1(t, X) L_2^3(t, X) dt ,
$$

where

$$
L_1(t, X) = \sum_{d \mid P(z)} \lambda^-(d) \sum_{X/2 < p \leq X \atop p + 2 \equiv 0 (d)} e(p^c t) \log p ,
$$

$$
L_2(t, X) = \sum_{d \mid P(z)} \lambda^+(d) \sum_{X/2 < p \leq X \atop p + 2 \equiv 0 (d)} e(p^c t) \log p .
$$

We divide $\Gamma_1$ into three parts

$$
\Gamma_1 = \Gamma_1^{(1)} + \Gamma_1^{(2)} + \Gamma_1^{(3)} .
$$

where

$$
\Gamma_1^{(1)} = \int_{|t| < \tau} \Theta(t) e(-Nt) L_1(t, X) L_2^3(t, X) dt ,
$$

$$
\Gamma_1^{(2)} = \int_{\tau \leq |t| \leq K} \Theta(t) e(-Nt) L_1(t, X) L_2^3(t, X) dt ,
$$

$$
\Gamma_1^{(3)} = \int_{|t| > K} \Theta(t) e(-Nt) L_1(t, X) L_2^3(t, X) dt .
$$

We shall estimate $\Gamma_1^{(3)}$, $\Gamma_1^{(1)}$, $\Gamma_1^{(2)}$ respectively in the sections 4, 5, 6. In section 7 we shall complete the proof of the Theorem.

4 Upper bound for $\Gamma_1^{(3)}$.

Arguing as in [14] we obtain

Lemma 5. For the sum $\Gamma_1^{(3)}$, defined by (27), we have

$$
\Gamma_1^{(3)} \ll 1 .
$$
5 Asymptotic formula for $\Gamma^{(1)}_1$.

The first lemma we need in this section gives us asymptotic formula for the sums $L_j(\alpha, X)$ denoted by (20) and (21).

**Lemma 6.** Let $D$ is defined by (17), and $\lambda(d)$ be complex numbers defined for $d \leq D$ such that
\[
|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \quad \text{if} \quad 2|d| \quad \text{or} \quad \mu(d) = 0.
\]
If
\[
L(\alpha, X) = \sum_{d \leq D} \lambda(d) \sum_{\substack{X/2 < p \leq X \\ p+2 \equiv 0 (d)}} e(p^{\alpha}) \log p
\]
then for $|\alpha| < \tau$ we have
\[
L(\alpha, X) = I(\alpha) \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} + O\left(\frac{X}{(\log X)^A}\right),
\]
where $A > 0$ is an arbitrary large constant.

**Proof.** See ([14], Lemma 10).

The next lemma is the following

**Lemma 7.** Using the definitions (11), (20) and (21) we have
\[
(i) \quad \int_{-\tau}^{\tau} |L_j(\alpha, X)|^2 d\alpha \ll X^{2-\epsilon} \log^6 X, \quad j = 1, 2
\]
\[
(ii) \quad \int_{-\tau}^{\tau} |I(\alpha)|^2 d\alpha \ll X^{2-\epsilon} \log X.
\]

**Proof.** See ([14], Lemma 11).

Let
\[
L_j = L_j(t, X), \quad j = 1, 2
\]
\[
\mathcal{M}_1 = \mathcal{M}_1(t, X) = I(t) \sum_{d \leq D} \frac{\lambda^-(d)}{\varphi(d)},
\]
\[
\mathcal{M}_2 = \mathcal{M}_2(t, X) = I(t) \sum_{d \leq D} \frac{\lambda^+(d)}{\varphi(d)}.
\]
where \( L_j(t, X) \) are denoted by (20) and (21). We use the identity

\[
L_1 L_2^3 = M_1 M_2^2 + (L_1 - M_1)M_3^2 + L_1(L_2 - M_2)M_2^2 + L_1L_2(L_2 - M_2).
\] (31)

Replace

\[
J_1 = \int_{|t|<\tau} \Theta(t)e(-Nt)M_1(t, X)M_2(t, X)dt.
\] (32)

Then from Lemma 1, Lemma 6, (20), (21), (23), (29) – (32) we obtain

\[
\Gamma^{(1)}_1 - J_1 = \int_{|t|<\tau} \Theta(t)e(\eta t)(L_1(t, X) - M_1(t, X))M_2^2(t, X)dt
\]

\[
+ \int_{|t|<\tau} \Theta(t)e(\eta t)L_1(t, X)(L_2(t, X) - M_2(t, X))M_2(t, X)dt
\]

\[
+ \int_{|t|<\tau} \Theta(t)e(\eta t)L_1(t, X)L_2(t, X)(L_2(t, X) - M_2(t, X))M_2(t, X)dt
\]

\[
+ \int_{|t|<\tau} \Theta(t)e(\eta t)L_1(t, X)L_2^2(t, X)(L_2(t, X) - M_2(t, X))dt
\]

\[
\ll \vartheta \frac{X}{(\log X)^A} \left( \int_{|t|<\tau} |M_2^2(t, X)|dt + \int_{|t|<\tau} |L_1(t, X)M_2^2(t, X)|dt
\]

\[
+ \int_{|t|<\tau} |L_1(t, X)L_2(t, X)M_2(t, X)|dt + \int_{|t|<\tau} |L_1(t, X)L_2^2(t, X)|dt \right). \] (33)

On the other hand (11), (30) and Lemma 3 give us

\[
|M_2(t, X)| \ll |I(t)| \log X. \] (34)

Using (33) and (34) we find

\[
\Gamma^{(1)}_1 - J_1 \ll \vartheta \frac{X}{(\log X)^{A-3}} \left( \int_{|t|<\tau} |I(t)|^3dt + \int_{|t|<\tau} |L_1(t, X)||I(t)|^2dt
\]

\[
+ \int_{|t|<\tau} |L_1(t, X)L_2(t, X)I(t)|dt + \int_{|t|<\tau} |L_1(t, X)L_2^2(t, X)|dt \right). \] (35)
Bearing in mind the definitions (10), (20) and (21) we get the trivial estimates

\[ |I(t)| \ll X ; \quad |L_j(t, X)| \ll X \log^2 X, \quad j = 1, 2. \]  

(36)

Now from (35), (36) and Lemma 7 we obtain

\[ \Gamma_1^{(1)} - J_1 \ll \frac{X^2}{(\log X)^{A - \varepsilon}} \left( \int_{|t| < \tau} |I(t)|^2 dt + \int_{|t| < \tau} |L_1(t, X)|^2 dt \right) \ll \frac{X^{4 - \varepsilon}}{(\log X)^{A - 11}}. \]  

(37)

Let us consider \( J_1 \). According to Lemma 4 we have

\[ |I(\alpha)| \ll \frac{X^{1 - \varepsilon}}{|\alpha|}. \]  

(38)

Therefore by Lemma 11, Lemma 3, (29), (30), (32) and (38) we find

\[ J_1 = \sum_{d_i \mid P(z)} \sum_{i=1,2,3,4} \frac{\lambda^{-1}(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)\varphi(d_4)} \int_{|t| < \tau} \Theta(t) e(-Nt) \]

\[ \times \left( \int \frac{X}{X/2} \int \frac{X}{X/2} \int \frac{X}{X/2} \int \frac{X}{X/2} e(t(y_1^c)dy_1 e(t(y_2^c)dy_2 e(t(y_3^c)dy_3 e(t(y_4^c)dy_4) \right) dt \]

\[ = \sum_{d_i \mid P(z)} \sum_{i=1,2,3,4} \frac{\lambda^{-1}(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)\varphi(d_4)} \left[ \int_{-\infty}^{\infty} \Theta(t) e(-Nt) \right] \]

\[ \times \left( \int \frac{X}{X/2} \int \frac{X}{X/2} \int \frac{X}{X/2} \int \frac{X}{X/2} e(t(y_1^c + y_2^c + y_3^c + y_4^c))dy_1 dy_2 dy_3 dy_4 \right) dt \]

\[ + \mathcal{O}\left( \frac{\vartheta X^{4 - 4\varepsilon}}{\tau^{\varepsilon}} \int_{-\infty}^{\infty} \frac{dt}{t^3} \right) \]

\[ = \sum_{d_i \mid P(z)} \sum_{i=1,2,3,4} \frac{\lambda^{-1}(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)\varphi(d_4)} \left( \int \frac{X}{X/2} \int \frac{X}{X/2} \int \frac{X}{X/2} \int \frac{X}{X/2} \Theta(t) e(t(y_1^c + y_2^c + y_3^c + y_4^c - N)) dtdy_1 dy_2 dy_3 dy_4 + \mathcal{O}\left( \vartheta X^{4 - 4\varepsilon} \tau^{-3} \right) \right) \]
\[
\prod_{i=1,2,3,4} \frac{\lambda^-(d_i) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4)}{\varphi(d_1) \varphi(d_2) \varphi(d_3) \varphi(d_4)} \times \left( \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \theta(y^c_1 + y^c_2 + y^c_3 + y^c_4 - N) dy_1 dy_2 dy_3 dy_4 + \mathcal{O}(\vartheta X^{4-c} \tau^{-3}) \right)
\]

\[= \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \theta(y^c_1 + y^c_2 + y^c_3 + y^c_4 - N) dy_1 dy_2 dy_3 dy_4 \times \sum_{d_i \mid \mathcal{P}(z)} \frac{\lambda^-(d_i) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4)}{\varphi(d_1) \varphi(d_2) \varphi(d_3) \varphi(d_4)} \mathcal{O}(\vartheta X^{4-c} \tau^{-3} \log^4 X).
\]

The last formula, (3) and (37) imply

\[
\Gamma^{(1)}_1 = B(X) \sum_{d_i \mid \mathcal{P}(z)} \frac{\lambda^-(d)}{\varphi(d)} \left( \sum_{d_i \mid \mathcal{P}(z)} \frac{\lambda^+(d)}{\varphi(d)} \right)^3 + \mathcal{O}\left( \vartheta \frac{X^{4-c}}{(\log X)^{A-11}} \right),
\]

where

\[
B(X) = \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \theta(y^c_1 + y^c_2 + y^c_3 + y^c_4 - N) dy_1 dy_2 dy_3 dy_4.
\]

According to ([15], Lemma 8) we have

\[
B(X) \gg \vartheta X^{4-c}.
\]

Let

\[
G^\pm = \sum_{d_i \mid \mathcal{P}(z)} \frac{\lambda^\pm(d)}{\varphi(d)}.
\]

Thus from (39) and (42) it follows

\[
\Gamma^{(1)}_1 = B(X)G^- (G^+)^3 + \mathcal{O}\left( \vartheta \frac{X^{4-c}}{(\log X)^{A-11}} \right).
\]

6 Upper bound for \( \Gamma^{(2)}_1 \).

The treatment of the intermediate region depends on the following four lemmas.
Lemma 8. For the sums denoted by (20) and (21) we have
\[ \int_0^1 |L_j(t, X)|^2 dt \ll X \log^5 X, \quad j = 1, 2. \]

Proof. See [14], Lemma 11.

Lemma 9. Let \( 1 < c < 1603/1033 \). Then
\[ \sum_{X/2 < n_1, n_2, n_3, n_4 \leq X} \min \left( 1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \ll X^{4-c} \log^5 X. \]

Proof. See [15], Theorem 1.

Lemma 10. For the sums denoted by (20) and (21) we have
\[ \int_0^1 |L_j(t, X)|^4 dt \ll X^{4-c+\eta}, \quad j = 1, 2, \]
where \( \eta \) is defined by (7).

Proof. We only prove for \( j = 1 \). The case for \( j = 2 \) is analogous.

From (11), (20), Lemma 2 and Lemma 9 it follows
\[
\begin{align*}
\int_0^1 |L_1(t, X)|^4 dt &= \sum_{d_i \mid P(z)} \lambda^-(d_1) \cdots \lambda^-(d_4) \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \log p_1 \cdots \log p_4 \int_0^1 e((p_1^c + p_2^c - p_3^c - p_4^c)t)dt \\
&\ll \sum_{d_i \leq D} \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \log p_1 \cdots \log p_4 \min \left( 1, \frac{1}{|p_1^c + p_2^c - p_3^c - p_4^c|} \right) \\
&\ll (\log X)^4 \sum_{d_i \leq D} \sum_{X/2 < n_1, n_2, n_3, n_4 \leq X} \min \left( 1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \\
&= (\log X)^4 \sum_{X/2 < n_1, n_2, n_3, n_4 \leq X} \min \left( 1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \tau(n_1 + 2) \cdots \tau(n_4 + 2) \\
&\ll (\log X)^4 \sum_{X/2 < n_1, n_2, n_3, n_4 \leq X} \min \left( 1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \tau(n_1 + 2) \cdots \tau(n_4 + 2)
\end{align*}
\]
\[ \ll X^{\eta/2} \sum_{X/2 < n_1, n_2, n_3, n_4 \leq X} \min \left( 1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \ll X^{4-c+\eta}. \]

**Lemma 11.** Assume that \( \tau \leq |\alpha| \leq K \). Let \( \beta(d) \) be complex number defined for \( d \leq D \), and let
\[
\beta(d) \ll 1. \quad (44)
\]
Then for the sum
\[
L(\alpha, X) = \sum_{d \leq D} \beta(d) \sum_{X/2 < p \leq X \atop p \equiv 0 \pmod{(d)}} e(p^c \alpha) \log p \quad (45)
\]
we have
\[
L(\alpha, X) \ll X^{\eta} \left( X^{1/3+c/2}DK^{1/2} + X^{3/4+c/6}D^{2/3}K^{1/6} + X^{1-c/6}D^{1/3}\tau^{-1/6} \right),
\]
where \( \eta \) is defined by (7).

**Proof.** See ([14], Lemma 15).

We next treat \( \Gamma_1^{(2)} \), defined by (24). We have
\[
\Gamma_1^{(2)} \ll \max_{\tau \leq t \leq K} |L_1(t, X)| \int_{\tau}^{K} |\Theta(t)||L_2(t, X)|^3 dt. \quad (46)
\]
Using Cauchy’s inequality we obtain
\[
\int_{\tau}^{K} |\Theta(t)||L_2(t, X)|^3 dt \ll \left( \int_{\tau}^{K} |\Theta(t)||L_2(t, X)|^2 dt \right)^{1/2} \left( \int_{\tau}^{K} |\Theta(t)||L_2(t, X)|^4 dt \right)^{1/2}. \quad (47)
\]
On the one hand from (4), (5), Lemma 1 and Lemma 8 it follows
\[
\int_{\tau}^{K} |\Theta(t)||L_2(t, X)|^2 dt \ll \theta \int_{\tau}^{K} |L_2(t, X)|^2 dt + \int_{\tau}^{K} |L_2(t, X)|^2 dt \int_{t}^{K} 1 dt
\]
\[
\ll \theta \sum_{0 \leq n \leq 1/\theta} \int_{\tau}^{K} |L_2(t, X)|^2 dt + \sum_{1/\theta \leq n \leq K} \frac{1}{n} \int_{\tau}^{K} |L_2(t, X)|^2 dt
\]
\[
\ll X \log^6 X. \quad (48)
\]
On the other hand (4), (5), Lemma 1 and Lemma 10 give us
\[
\int_{\tau} \kappa |\Theta(t)||L_2(t, X)|^4 dt \ll \vartheta \int_{\tau} |L_2(t, X)|^4 dt + \int_{1/\vartheta}^{\kappa} |L_2(t, X)|^4 dt\]
\[
\ll \vartheta \sum_{0 \leq n \leq 1/\vartheta} \int |L_2(t, X)|^4 dt + \sum_{1/\vartheta - 1 \leq n \leq \kappa} \frac{1}{n} \int |L_2(t, X)|^4 dt
\]
\[
\ll X^{4-c+\eta} \log X ,
\] (49)
where \eta is defined by (7).

Therefore by (3) – (7), (46) – (49) and by Lemma 11 we obtain
\[
\Gamma(2) \ll \vartheta X^{4-c} \log^5 X .
\] (50)

Summarizing (22), (26), (43) and (50) we find
\[
\Gamma_1 = B(X)G - (G+)^3 + O\left( \vartheta \frac{X^{4-c}}{\log^5 X} \right) ,
\] (51)

7 Proof of the Theorem.

Since \Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 and \Gamma_5 is estimated in the same way then from (12), (13), (17), (18) and (51) we obtain
\[
\Gamma \geq B(X)W + O\left( \vartheta \frac{X^{4-c}}{\log^5 X} \right) ,
\] (52)
where
\[
W = 4 (G^+)^3 \left( G^- - \frac{3}{4} G^+ \right) .
\] (53)

We put
\[
\mathcal{F}(z) = \prod_{2 < p \leq z} \left( 1 - \frac{1}{p - 1} \right) , \quad s = \frac{\log D}{\log z} .
\] (54)

Let \( f(s) \) and \( F(s) \) are the lower and the upper functions of the linear sieve. Using (42) and (2, Lemma 10) we obtain
\[
\mathcal{F}(z) \left( f(s) + O\left( (\log X)^{-1/3} \right) \right) \leq G^- \leq \mathcal{F}(z) \leq G^+ \leq \mathcal{F}(z) \left( F(s) + O\left( (\log X)^{-1/3} \right) \right) .
\] (55)
To estimate \( W \) from below we shall use the inequalities (see (55)):

\[
G^- - \frac{2}{3}G^+ \geq F(z) \left( f(s) - \frac{3}{4}F(s) + \mathcal{O} \left( \log X \right)^{-1/3} \right)
\]

(56)

\[
G^+ \geq F(z).
\]

Then from (53) and (56) it follows

\[
W \geq 4F^4(z) \left( f(s) - \frac{3}{4}F(s) + \mathcal{O} \left( \log X \right)^{-1/3} \right).
\]

(57)

Hence, using (52) and (57) we get

\[
\Gamma \geq 4BF^4(z) \left( f(s) - \frac{2}{3}F(s) + \mathcal{O} \left( \log X \right)^{-1/3} \right) + \mathcal{O} \left( \frac{X^{4-c}}{\log^6 X} \right).
\]

(58)

For \( 2 \leq s \leq 3 \) we have

\[
f(s) = \frac{2e^\gamma \log(s-1)}{s}, \quad F(s) = \frac{2e^\gamma}{s}
\]

(\( \gamma \) denotes Euler’s constant). We choose

\[s = 2.95.\]

Then by (58), (60) and (54) we find

\[
\beta = 0.030477.
\]

It is not difficult to compute that for sufficiently large \( X \) we have

\[
f(s) - \frac{2}{3}F(s) > 10^{-5}.
\]

(59)

It remains to notice that

\[
F(z) \approx \frac{1}{\log X}.
\]

(60)

Therefore, using (8), (11), (58) – (60) we obtain

\[
\Gamma \gg \beta \frac{X^{4-c}}{\log^4 X}.
\]

(61)

From (11) and (61) it follows that \( \Gamma \to \infty \) as \( X \to \infty \).

Bearing in mind (4), (12) and (61) we conclude that for some constant \( c_0 > 0 \) there are at least \( c_0X^{4-c}\log^{-A-9}X \) triples of primes \( p_1, p_2, p_3 \) satisfying \( X/2 < p_1, p_2, p_3, p_4 \leq X, \ |p_1^2 + p_2^2 + p_3^2 + p_4^2 - N| < \beta \) and such that for every prime factor \( p \) of \( p_j + 2, \ j = 1, 2, 3, 4 \) we have \( p \geq X^{0.030477} \).

The proof of the Theorem is complete.
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