Research Article

Yong Xu* and Xianhua Li

On $CSQ$-normal subgroups of finite groups

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Abstract: We introduce a new subgroup embedding property of finite groups called $CSQ$-normality of subgroups. Using this subgroup property, we determine the structure of finite groups with some $CSQ$-normal subgroups of Sylow subgroups. As an application of our results, some recent results are generalized.

Keywords: $CSQ$-normal subgroup, Nilpotent subgroup, Supersolvable subgroup

MSC: 20D10, 20D15

1 Introduction

All groups in this paper are finite. Let $\pi(G)$ stand for the set of all prime divisors of the order of a group $G$. The other notations and terminologies in this paper are standard (see [1]).

Let $H \leq G$ and $g \in G$, then $H \leq \langle H, H^g \rangle \leq \langle H, g \rangle$. It is clear that $H = \langle H, H^g \rangle$ for all $g \in G$ if and only if $H \normal G$. In [2], $H$ is called abnormal in $G$ if $\langle H, H^g \rangle = \langle H, g \rangle$ for all $g \in G$. In [3], the famous Wielandt Theorem shows that $H \trianglelefteq \langle H, H^g \rangle$ for all $g \in G$ if and only if $H \trianglelefteq G$. In [4], $H$ is called pronormal in $G$ if $H$ is conjugate to $H^g$ in $\langle H, H^g \rangle$ for all $g \in G$. These show that the normalities of a subgroup $H$ in $G$ may be determined by the normalities of a subgroup $H$ in $\langle H, H^g \rangle$. This leads us to investigate the properties of $G$ from the relationship between the subgroup $H$ of $G$ and the union of $\langle H, H^g \rangle$ for all $g \in G$. On the other hand, Kegel in [5] introduced the concept of $S$-quasinormal subgroups. A subgroup $H$ of a group $G$ is said to be $s$-permutable, $S$-quasinormal, or $\pi$-quasinormal in $G$ if $PH = HP$ for all Sylow subgroups $P$ of $G$. In this paper, we introduce a new generalized normality of subgroups, $CSQ$-normality, and obtain a criterion for nilpotency and supersolvability of a group by using the $CSQ$-normality of subgroups. Now we recall the following definitions. Let $G$ be a finite group. For every $n \mid |G|$, if $G$ has a subgroup of order $n$, then $G$ is called a $CLT$-group. Furthermore, $G$ is called a $QCLT$-group if the image of $G$ under every homomorphism is a $CLT$-group. As an application of our results, some recent results are generalized. For example, Humphreys [6] proved that a $QCLT$-group of odd order is supersolvable, and we will prove that a $QCLT$-group of even order is also supersolvable if the maximal subgroups of its Sylow 2-subgroup are all $CSQ$-normal subgroups.

Definition 1.1. Let $H$ be a subgroup of a group $G$. We say that $H$ is $CSQ$-normal in $G$ if $H$ is $S$-quasinormal in $\langle H, H^g \rangle$ for all $g \in G$.

By [5, Lemma 3], we know that all $S$-quasinormal subgroups are $CSQ$-normal subgroups. The following example shows that a $CSQ$-normal subgroup is not necessarily a $S$-quasinormal subgroup.

Example 1.2. Let $G = A_4$, $H = \langle (12)(34) \rangle$. Obviously, $H$ is not $S$-quasinormal in $G$ but $CSQ$-normal in $G$.

*Corresponding Author: Yong Xu: School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, Henan 471003, China and School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China, E-mail: xuy2011@163.com

Xianhua Li: School of Mathematical Science, Soochow University, Suzhou, Jiangsu 215006, China
2 Basic definitions and preliminary results

The lemma presented below is crucial in the sequel. The proof is a routine check, and we omit its details.

Lemma 2.1. Let $H$ be a CSQ-normal subgroup of a group $G$ and $N \trianglelefteq G$. Then
(a) If $H \leq K$, then $H$ is CSQ-normal in $K$.
(b) $HN/N$ is CSQ-normal in $G/N$.

Lemma 2.2. Suppose that every proper subgroup of a group $G$ is nilpotent but $G$ itself is not nilpotent. Then
(a) By the hypothesis, $G$ has cyclic subgroups.
(b) If $G$ is a group with $G = \langle a \rangle$, then $a^{2n} = e$ for all $n$.
(c) Let $G$ be a group with subgroups $H$ and $K$. Then $G/HK$ is normal in $G$.
(d) If $G$ is a group with $G = \langle a \rangle$, then $a^{2n} = e$ for all $n$.
(e) Let $G$ be a group with subgroups $H$ and $K$. Then $G/HK$ is normal in $G$.
(f) If $G$ is a group with subgroups $H$ and $K$, then $G/HK$ is normal in $G$.

Proof. By [7, Theorem 1.1], the result is true.

As in [8], a minimal non-supersolvable group is a group whose proper subgroups and quotients are supersolvable.

Lemma 2.3. Suppose that a group $G$ is minimal non-supersolvable. Then $G$ is isomorphic to a group of the form $G_t$ for $1 \leq t \leq 6$, where the groups $G_t$ are defined in the following way.
(I) $G_1$ is a minimal nonabelian group and $|G_1| = pq^t$, where $p \nmid q - 1$, $\beta \geq 2$.
(II) $G_2 = \langle a, x \rangle$ and $|G_2| = p^{\alpha + \beta}q$, where $a = p^\alpha xq^\beta + r - 1$, where $a \geq 2$, $\alpha = 2$, $a^{\alpha + \beta} = a^r = c_1^2 = \cdots = c_{\beta}^r = 1$.
(III) $G_3 = \langle a, b, c \rangle$ and $|G_3| = t$, where $\alpha = 1$, $\beta = 1$, $\gamma = 2$, $\delta = 1$.
(IV) $G_4 = \langle a, b \rangle$ and $|G_4| = p^{\alpha + \beta}q$, where $\alpha = 2$, $\beta = 1$.
(V) $G_5 = \langle a, b \rangle$ and $|G_5| = t$, where $\alpha = 2$, $\beta = 1$.
(VI) $G_6 = \langle a, b, c \rangle$ and $|G_6| = t$, where $\alpha = 2$, $\beta = 1$.

Proof. See [8, Corollary 2.2].

Lemma 2.4. Let $H$ be a CSQ-normal subgroup of $G$. Then
(a) $H^x$ is also a CSQ-normal subgroup of $G$ for any $x \in G$.
(b) $H$ is subnormal in $G$.

Proof. (a) By the hypothesis, $H$ is S-quasinormal in $(H, H^x)$ for all $g \in G$. Then for any $x \in G$, we have that $H^x$ is S-quasinormal in $(H^x, (H^x)^x) = (H^x, (H^x)^x)$ for all $g \in G$. Then one checks easily that $\tau : G \rightarrow G$, defined by

$$\tau(g) = g^x$$

is a bijective map. Since $g^x$ runs over $G$ as $g$ does for fixed $x$, we get that $H^x$ is S-quasinormal in $(H^x, (H^x)^x)$ for all $g \in G$. Thus $H^x$ is a CSQ-normal subgroup of $G$. 
(b) By the hypothesis, $H$ is $S$-quasinormal in $(H, H^g)$ for all $g \in G$. By [5, Theorem 1], we know that $H$ is subnormal in $(H, H^g)$ for all $g \in G$, so $H$ is subnormal in $G$ by Wielandt’s theorem.

\section{3 Main results}

Let $Z$ be a complete set of Sylow subgroups of a group $G$, that is, for each prime $p$ dividing the order of $G$, $Z$ contains exactly one Sylow $p$-subgroup of $G$. Let $Z \cap E = \{P \cap E \mid P \in Z\}$.

\textbf{Theorem 3.1.} Let $G$ be a group and $Z$ be a complete set of Sylow subgroups of $G$. Suppose that $E \trianglelefteq G$ such that $G/E$ is nilpotent and $G$ is $G_1$-free. If every cyclic subgroup of a Sylow subgroup of $E$ contained in $Z \cap E$ is a CSQ-normal subgroup of $G$, then $G$ is nilpotent.

\textbf{Proof.} Assume that the result is false, and let $G$ be a counterexample with least $([G] + |E|)$.

Let $H < G$. Of course, $H$ is $G_1$-free. Obviously, $H/H \cap E \cong HE/E$ is nilpotent. Suppose that $K = H \cap E$ and $K_p$ is a Sylow $p$-subgroup of $K$, so $Z = \{K_p \mid p \in \pi(H \cap E)\}$ is a complete set of Sylow subgroups of $H \cap E$. Assume that $T$ is a cyclic subgroup of $K_p$. Since $K \leq E$, there exists $x \in E$ such that $K_p^x \leq P \cap E$, where $P \in Z$. By the hypothesis and Lemma 2.4 (a), we get that $T$ is CSQ-normal in $G$. Then $T$ is CSQ-normal in $H$ by Lemma 2.1 (a). Hence all cyclic subgroups of $K_p$ contained in $Z$ are CSQ-normal in $H$, and thus $H$ and its normal subgroup $K$ satisfy the hypothesis. By the minimal choice of $|G| + |E|$, $H$ is nilpotent. By Lemma 2.2, we may assume that $G = P^*Q$, where $Q$ is a normal Sylow $q$-subgroup of $G$ and $P^*$ is a cyclic Sylow $p$-subgroup of $G$.

Suppose that $N \trianglelefteq G$. We shall prove that $(G/N, EN/N)$ satisfies the hypothesis. Clearly, $(G/N)/(EN/N) \cong G/EN$ is nilpotent and $G/N$ is $G_1$-free. Let $H/N$ be a cyclic subgroup of a Sylow subgroup of $EN/N \cap ZN/N$. Then we may assume $H = \langle xN \rangle$ and $\langle x \rangle$ is a cyclic subgroup of a Sylow subgroup in $E \cap Z$. By the hypothesis, $\langle x \rangle$ is CSQ-normal in $G$ and by Lemma 2.1 (b), $H/N$ is CSQ-normal in $G/N$. Then $(G/\Phi(G), E/\Phi(G))$ satisfies the hypothesis of the theorem. The minimality of $|G| + |E|$ implies that $G/\Phi(G)$ is nilpotent and so is $G$, a contradiction. Thus $\Phi(G) = 1$ and so $G \cong G_1$, again a contradiction. This shows that there exists no counterexample, so the result is true. \hfill \Box

\textbf{Remark 3.2.} We cannot replace the condition “cyclic subgroup of Sylow subgroup” by “minimal subgroup of a Sylow subgroup” in Theorem 3.1. For example, let $G = E = (Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2) \rtimes Z_9$. Obviously, the pair $(G, E)$ satisfy the hypothesis. Nevertheless, it is not nilpotent.

\textbf{Remark 3.3.} The condition of “$G$ is $G_1$-free” cannot be removed. For example, let $G = S_3$ and choose $E = A_3$. Then the pair $(S_3, A_3)$ satisfy the hypothesis of Theorem 3.1. Nevertheless, $S_3$ is not nilpotent.

\textbf{Corollary 3.4.} Let $G$ be a group and $Z$ be a complete set of Sylow subgroups of $G$. If every cyclic subgroup of a Sylow subgroup of $G$ contained in $Z$ is a CSQ-normal subgroup of $G$, then $G$ is nilpotent.

\textbf{Proof.} By the proof of Theorem 3.1, we just need to check that $G \cong G_1$. By the hypothesis, we have that a $p$-Sylow subgroup $G_p$ is a CSQ-normal subgroup of $G$. Then $G_p \trianglelefteq G$ by Lemma 2.4 (b), thus $G_p \trianglelefteq G$, so $G$ is nilpotent. The proof is completed. \hfill \Box

To prove Theorem 3.6, we need the following Lemma 3.5.

\textbf{Lemma 3.5.} Let $G$ be a group and $Z$ be a complete set of Sylow subgroups of $G$. Suppose that $P$ is a Sylow $p$-subgroup of $G$ contained in $Z$, where $p$ is a prime divisor of $|G|$ with $\left(|G|, p - 1\right) = 1$. If every maximal subgroup of $P$ is CSQ-normal in $G$, then $G/O_p(G)$ is $p$-nilpotent and hence $G$ is solvable.

\textbf{Proof.} Assume that the result is false and let $G$ be a counterexample of smallest order.
First of all, we show that $O_p(G) = 1$. Assume that $O_p(G) = P$. Then $G/O_p(G)$ is a $p'$-group and of course it is $p$-nilpotent, a contradiction. Assume that $1 < O_p(G) < P$. Obviously, $O_p(G)Z/O_p(G)$ is a complete set of Sylow subgroups of $G/O_p(G)$ and $G/O_p(G)$ satisfies the hypothesis by Lemma 2.1 (b). The minimal choice implies that $G/O_p(G) \cong (G/O_p(G))/(O_p(G)/O_p(G))$ is $p$-nilpotent, a contradiction. Thus we have $O_p(G) = 1$.

Let $P_1$ be a maximal subgroup of $P$. By the hypothesis, $P_1$ is CSQ-normal subgroup of $G$. Then $P_1$ is subnormal in $G$ by Lemma 2.4, and thus $P_1 \leq O_p(G) = 1$. Hence $P$ is a cyclic subgroup of order $p$. Since $N_G(P)/C_G(P) \cong \text{Aut}(P)$, we get that the order of $N_G(P)/C_G(P)$ must divide $\langle |G|, p - 1 \rangle = 1$. Then $N_G(P) = C_G(P)$. Thus $G$ is $p$-nilpotent by [1, Burnside’s theorem], a contradiction. We conclude that there is no counterexample and Lemma 3.5 is proved.

**Theorem 3.6.** Let $G$ be a group and $Z$ be a complete set of Sylow subgroups of $G$. Suppose that $G$ is $G_1$-free with $t \in \{1, 2, 6\}$ and every maximal subgroup of any non-cyclic Sylow subgroup of $G$ contained in $Z$ is CSQ-normal in $G$. Then $G$ is supersolvable.

**Proof.** Assume that the theorem is false and let $G$ be a counterexample of smallest order. We proceed in a number of steps.

If every Sylow subgroup of $G$ contained in $Z$ is cyclic, then every Sylow subgroup of $G$ is cyclic, thus $G$ is supersolvable. Next we assume that there is a non-cyclic Sylow $p$-subgroup contained in $Z$.

Step 1. $G$ is solvable.

Let $p = \min \pi(G)$ and $P$ be a Sylow $p$-subgroup of $G$ contained in $Z$. If $P$ is cyclic, then $G$ is $p$-nilpotent, so $G$ is solvable. If $P$ is not cyclic, then $G/O_p(G)$ is $p$-nilpotent by Lemma 3.5, thus $G$ is solvable. Hence we have Step 1.

Step 2. $G$ has a unique minimal normal subgroup $N$ and $\Phi(G) = 1$.

Let $N$ be a minimal normal subgroup of $G$. Then $ZN/N$ be a complete set of Sylow subgroups of $G/N$. Let $PN/N \in \text{Syl}_p(G/N)$, where $P \in Z$ and $PN/N$ is non-cyclic. (Of course, $P$ is non-cyclic.) Assume that $T/N$ be a maximal subgroup of $PN/N$. Then $T = T \cap PN = (T \cap P)N$. Suppose that $T \cap P = P_1$. Then $P_1 \cap N = T \cap P \cap N = P \cap N$. Hence $|P : P_1| = |PN/N : P_1N/N| = |P/N : T/N| = p$.

By the hypothesis, $P_1$ is CSQ-normal in $G$, so $P_1N/N = T/N$ is CSQ-normal in $G/N$ by Lemma 2.1 (b). Thus $G/N$ satisfies the hypothesis. By the choice of $G$, we obtain that $G/N$ is supersolvable. Similarly, if $N_1$ is another minimal normal subgroup of $G$. Then $G/N_1$ is also supersolvable. Now it follows that $G \cong G/N \cap N_1$ is supersolvable, a contradiction. Hence, $N$ is the unique minimal normal subgroup of $G$. If $N \leq \Phi(G)$, then the supersolvability of $G/N$ implies the supersolvability of $G$. Hence, $\Phi(G) = 1$. Therefore, we have Step 2.

Step 3. $N = O_p(G) = P$, $C_G(N) = N$ and $|G| = p^{1_2^\alpha_1}r_2^\alpha_2 \cdots r_s^\alpha_s$, the Sylow $r_i$-subgroup of $G$ is cyclic, where $1 \leq i \leq s$, $\alpha_i \geq 1$.

By Step 1 and Step 2, we know that $N$ is an elementary abelian $p$-subgroup and $N = F(G) = O_p(G) \leq P$, so $C_G(N) = N$. Assume that $N < P$. Given a maximal subgroup $P_1$ of $P$, by the hypothesis, $P_1$ is a CSQ-normal subgroup of $G$, then $P_1$ is subnormal in $G$ by Lemma 2.4, so $P_1 \leq O_p(G) = N < P$. If $N = P_1 < G$, we get that $P$ has a unique maximal subgroup, so $P$ is cyclic and hence so is $N$. By Step 2, we obtain that $G/N$ is supersolvable, hence so is $G$, a contradiction. Therefore, we have $N = P$. Suppose that $R_i$ is a non-cyclic Sylow $r_i$-subgroup of $G$ contained in $Z$ for some natural number $i$, $1 \leq i \leq s$, and $|R_i| = r_i^{\alpha_i}$. Then $\alpha_i \geq 2$, so we can choose $1 \neq R_{i1}$ to be a maximal subgroup of $R_i \in \text{Syl}_{r_i}(G)$. By the hypothesis, $R_{i1}$ is CSQ-normal in $G$, so $R_{i1}$ is subnormal in $G$ by Lemma 2.4, so $1 \neq R_{i1} \leq O_p(G)$. By the uniqueness of $N$, this is impossible. Hence $R_i$ is cyclic, and thus all Sylow subgroups of $G$ are cyclic except $B = P$. Hence we have the assertion in Step 3.

Step 4. Let $E$ be a maximal subgroup of $G$. We show that $|G : E| = |P| = p^{\beta_1}r_1^\beta_1$, where $\beta_1 \leq \alpha_i$. Then $E$ satisfies the hypothesis, so $E$ is supersolvable.
Since $G$ is solvable, $|G : E| = p^j$ or $r_i^{\beta_i}$, where $j \leq n$, $\beta_i \leq \alpha_i$. Suppose that $|G : E| = p^j$. By Step 2 and Step 3, it is easy to show $G = NE$ and $N \cap E = 1$, so $E = R_1 R_2 \cdots R_s$ and $j = n$, where $R_i \in Syl_{r_i}(G)$ ($1 \leq i \leq s$). It is clear that $E$ satisfies the hypothesis by Lemma 2.1 (a), so $E$ is supersolvable.

Step 5. Final contradiction.

By Step 2 and Step 4, we know that $G$ is minimal nonsupersolvable. On the other hand, by Step 4 and the hypothesis, $G$ is not isomorphic to any group $G_i$ in Lemma 2.3. We conclude that there is no minimal counterexample and Theorem 3.6 is proved.

If we remove “non-cyclic” in the hypothesis of Theorem 3.6, we can get the following Theorem.

**Theorem 3.7.** Let $G$ be a group and $Z$ be a complete set of Sylow subgroups of $G$. Suppose that $G$ is $G_1$-free and $G_6'$-free, where $G_6' \subseteq G_6$ and $|G_6'| = p^a q^b r^c$, that is, the case $\alpha = 1$. If every maximal subgroup of every $S$ylow subgroup of $G$ contained in $Z$ is a $CSQ$-normal subgroup of $G$, then $G$ is supersolvable.

**Proof.** By the proof of Theorem 3.6, we only need to check $G \cong G_2$ and $G \cong G_5$, where $|G_5| = p^a q^b r^c$ and $p^a q^b r^c | r - 1$, $p | q - 1$, $\alpha \geq 2$. Assume that $G \cong G_2$. Using the same description as in Lemma 2.3, let $V_1 = \langle a^p \rangle$. Then it is a maximal subgroup of $P$. By the hypothesis $V_1$ is a $CSQ$-normal subgroup of $G$, so $V_1$ is $S$-quasinormal in $\langle V_1, V_1^{c_1} \rangle$ for all $g \in G$. Choosing $g = c_{i^1}$. Then $((a^p)^{-1} c_{i^1} = c_{i^1}^{-1} (a^p)^{-1} c_{i^1} \in \langle V_1, V_1^{c_1} \rangle$, so

$$c_{i^1}^{-1} (a^p)^{-1} c_{i^1} a^p = c_{i^1}^{-1} a^p c_{i^1} = c_{i^1}^{-1} (a^p)^{a^p} = c_{i^1}^{-1} a_{i^1 + 1} = \cdots = c_{i^1}^{-1} c_{i^1} \in \langle V_1, V_1^{c_1} \rangle$$

where the exponent of $t$ (mod $r$) is $p^{\alpha - 1}$. Thus $r$ divides

$$t^{p^{\alpha - 1}} - 1 = (t - 1)(t^{p^{\alpha - 1}} - 1 + t^{p^{\alpha - 2}} + \cdots + 1).$$

If $r | t - 1$, then $c_{i^1}$ commutes with $V_1$, of course, $c_{i^1}$ normalizes $V_1$. If $r \nmid t - 1$, then $(t - 1, r) = 1$, we get

$$c_{i^1} = c_{i^1}^{m(t - 1) + nr} = (c_{i^1}^{-1})^m \in \langle V_1, V_1^{c_1} \rangle.$$

It follows that $\langle V_1, V_1^{c_1} \rangle = \langle a^p, c_{i^1} \rangle$. Since $V_1$ is $S$-quasinormal in $\langle V_1, V_1^{c_1} \rangle$, we have that $V_1 R_i = \langle a^p \rangle R_i$ is a subgroup of $G$, where $R_i \in Syl_{r_i}((a^p, c_{i^1}))$. By [5, Theorem 1], $V_1$ is normal in $V_1 R_i$, hence $V_1 \lhd V_1 R_i$. Therefore, $R_i$ normalizes $V_1$, and, of course, $c_{i^1}$ normalizes $V_1$. Since $i$ was arbitrary, we conclude that $V_1$ is normalized by $P$ and $R$, where $P \in Syl_p(G)$, $R \in Syl_r(G)$. If $\alpha \geq 2$, then $1 \neq V_1 \lhd G$, which is impossible. If $\alpha = 1$, then $G \cong G_1$, a contradiction. Hence $G$ is not isomorphic to $G_1$. As in a similar argument above, we also get that $G$ is not isomorphic to $G_6$, where $|G_6| = p^a q^b r^c$ and $p^a q^b | r - 1$, $p | q - 1$, $\alpha \geq 2$. The proof is completed.

**Corollary 3.8.** [9, Theorem 2] Let $G$ be a group with the property that maximal subgroups of Sylow subgroups are $\pi$-quasinormal in $G$ for $\pi = \pi(G)$. Then $G$ is supersolvable.

**Proof.** By the proof of Theorem 3.6 and Theorem 3.7, we only need to check that $G \cong G_1$ and $G \cong G_6$, where $|G_6| = p^a q^b r^c$, and $p^a q^b r^c | r - 1$, $p | q - 1$. Assume that $G \cong G_1$. By Lemma 2.3, we have $G_1 = PQ$, where $|P| = p$ and $|Q| = q^b \beta \geq 2$. By Step 2 and Step 3 of Theorem 3.6, $Q$ is a minimal normal subgroup of $G_1$. Choosing $Q_1$ to be a maximal subgroup of $Q$, by the hypothesis, we obtain that $Q_1$ is $\pi$-quasinormal in $G_1$. Then $O^\pi(G) \leq N_G(Q_1)$, so $P$ normalizes $Q_1$, and thus $1 \neq Q_1 \lhd G$, contrary to the minimality of $Q$. Hence $G \not\cong G_1$. Using a similar argument as above, we also get that $G$ is not isomorphic to $G_6$. The proof is completed.

**Corollary 3.9.** [9, Theorem 1]. Let $G$ be a group with the property that maximal subgroups of Sylow subgroups are normal in $G$. Then $G$ is supersolvable.

**Theorem 3.10.** Let $G$ be a $QCLT$-group. If every maximal subgroup of a Sylow 2-subgroup of $G$ is $CSQ$-normal in $G$, then $G$ is supersolvable.

**Proof.** Assume that the Theorem is false and let $G$ be a counterexample of smallest order.
Assume first that $G$ has odd order. Since $G$ is a $QCLT$-group, by [6], we have that $G$ is supersolvable. Now we assume that $2 \mid |G|$. By Lemma 3.5, we have that $G$ is solvable. For any $1 \neq N \trianglelefteq G$, if $2 \nmid |G/N|$, then $G/N$ is a $QCLT$-group of odd order and hence $G/N$ is supersolvable. Suppose that $2 \mid |G/N|$. Without loss of generality, we assume that every maximal subgroup of a Sylow 2-subgroup of $G/N$ is of the form $P_1N/N$, where $P_1$ is a maximal subgroup of a Sylow 2-subgroup of $G$. Then $P_1$ is $CSQ$-normal in $G$ by hypothesis, so $P_1N/N$ is $CSQ$-normal in $G/N$ by Lemma 2.1 (b). Hence the quotient group $G/N$ satisfies the hypothesis. By the choice of $G$, we have that $G$ is a solvable outer-supersolvable group. Then, by [7, Theorem 7.1], $G = ML$, where $M$ is a maximal subgroup of $G$, $M \cap L = 1$, $L$ is an elementary abelian $p$-group and is also the unique minimal normal subgroup of $G$ with order $p^\alpha$, $\alpha > 1$, the Sylow $p$-subgroup of $M$ is an abelian $p$-group and $\Phi(G) = 1$.

If $|G_2| \leq 4$, where $G_2 \in Syl_2(G)$, then $G_2$ is a cyclic subgroup or an elementary abelian 2-subgroup. It follows that $G$ is $S_4$-free, then $G$ is supersolvable by [10, Theorem 4], a contradiction. Hence we may choose $1 \neq P_1$ to be a maximal subgroup of $G_2$. By hypothesis, $P_1$ is a $CSQ$-normal subgroup of $G$. Then $P_1$ is subnormal in $G$ by Lemma 2.4, thus $1 \neq P_1 \subseteq O_2(G)$, hence $L \leq O_2(G)$, so we get $p = 2$. By [7, §6.1, Main lemma], we also get $O_2(G) = F(G) = L$.

Let $M_2$ be a Sylow 2-subgroup of $M$. Then $G_2 = M_2L$ is a Sylow 2-subgroup of $G$. Assume that $P_1$ is a maximal subgroup of $M_2N$ containing $M_2$. Then $M_2 < P_1$ since $|L| = 2^\alpha$, where $\alpha > 1$. Then $P_1$ is $CSQ$-normal in $G$ by the hypothesis, so $P_1$ is subnormal in $G$ by Lemma 2.4. Thus $P_1 \leq O_2(G) = L$, hence $G_2 = M_2L = P_1L = L$ is an elementary abelian Sylow 2-subgroup of $G$. It follows that $G$ is $S_4$-free, so $G$ is supersolvable by [10, Theorem 4], a contradiction. Hence the minimal counterexample does not exist. Therefore $G$ is supersolvable.

**Theorem 3.11.** Let $G$ be a $QCLT$-group. If every 2-maximal subgroup of a Sylow 2-subgroup of $G$ is $CSQ$-normal in $G$. Then $G$ is supersolvable.

**Proof.** The proof is similar to Theorem 3.10 and omitted here. 

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