A primal-dual interior-point method capable of rapidly detecting infeasibility for nonlinear programs

Yu-Hong Dai,† Xin-Wei Liu,‡ and Jie Sun§

Abstract. With the help of a logarithmic barrier augmented Lagrangian function, we can obtain closed-form solutions of slack variables of logarithmic-barrier problems of nonlinear programs. As a result, a two-parameter primal-dual nonlinear system is proposed, which corresponds to the Karush-Kuhn-Tucker point and the infeasible stationary point of nonlinear programs, respectively, as one of two parameters vanishes. Based on this distinctive system, we present a primal-dual interior-point method capable of rapidly detecting infeasibility of nonlinear programs. The method generates interior-point iterates without truncation of the step. It is proved that our method converges to a Karush-Kuhn-Tucker point of the original problem as the barrier parameter tends to zero. Otherwise, the scaling parameter tends to zero, and the method converges to either an infeasible stationary point or a singular stationary point of the original problem. Moreover, our method has the capability to rapidly detect the infeasibility of the problem. Under suitable conditions, not only the method can be superlinearly or quadratically convergent to the Karush-Kuhn-Tucker point as the original problem is feasible, but also it can be superlinearly or quadratically convergent to the infeasible stationary point when a problem is infeasible. Preliminary numerical results show that the method is efficient in solving some simple but hard problems and some standard test problems from the CUTE collection, where the superlinear convergence is demonstrated when we solve two infeasible problems and one well-posed feasible counterexample presented in the literature.

Key words: nonlinear programming, constrained optimization, infeasibility, interior-point method, global and local convergence

AMS subject classifications. 90C26, 90C30, 90C51
1. Introduction

Developing effective methods for nonlinear programs has always been an active area in optimization research. There are many interesting works in this area in recent years, which focus on various aspects of nonlinear programs. It is well known that, without assuming any constraint qualification, a local solution of nonlinear programs can be either a Karush-Kuhn-Tucker (KKT) point or a Fritz-John (FJ) point. The method is said to have strong global convergence if it can find either a KKT point or an FJ point, or even an infeasible point with first-order stationarity (i.e., an infeasible stationary point) for minimizing some kind of measure of constraint violations.

There are already many methods for nonlinear programs in the literature which are proved to have strong global convergence (see, for example, [1, 5, 6, 13, 24, 25, 26, 38]). Some of them are also shown to be of locally superlinear/quadratic convergence to the KKT point. However, it has been an open problem whether these methods are capable of rapidly converging to an infeasible stationary point before Byrd, Curtis and Nocedal [8] creatively presented a set of conditions to guarantee the superlinear convergence of their SQP algorithm to an infeasible stationary point. More recently, Burke, Curtis and Wang [5] considered the general program with equality and inequality constraints, and proved that their SQP method has strong global convergence and locally can have rapid convergence to the KKT point, and have superlinear/quadratic convergence to an infeasible stationary point.

The aim of this paper is to present a primal-dual interior-point method capable of converging to an infeasible stationary point when a nonlinear constrained optimization problem is infeasible. It should also be of strong global convergence and can be of locally rapid convergence to the KKT point when the problem is feasible. For simplicity, we consider the nonlinear program with general inequality constraints

\[
\begin{align*}
\text{minimize} & \quad \min f(x) \\
\text{subject to} & \quad c_i(x) \leq 0, \ i \in \mathcal{I},
\end{align*}
\]

where \( x \in \mathbb{R}^n, \mathcal{I} = \{1, 2, \ldots, m\} \) is an index set, \( f \) and \( c_i (i \in \mathcal{I}) \) are twice continuously differentiable real-valued functions defined on \( \mathbb{R}^n \). Our method can easily be extended to the nonlinear program with general equality and inequality constraints (see Section 6 for implementation). By introducing slack variables to the inequality constraints, problem (1.1)–(1.2) is reformulated as the program with equality and nonnegative constraints as follows:

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad c_i(x) + y_i = 0, \ i \in \mathcal{I}, \\
& \quad y_i \geq 0, \ i \in \mathcal{I},
\end{align*}
\]

where \( y_i (i \in \mathcal{I}) \) are slack variables.

The interior-point approach has been shown to be robust and efficient in solving linear and nonlinear programs (for example, see [2, 8, 9, 10, 13, 14, 22] and [24, 26, 27, 30, 31, 32, 33, 35]. Among all interior-point methods, the primal-dual interior-point methods have
drawn considerable attention. It is noted that, other than some feasible interior-point methods which requires all iterates to be (strictly) feasible for constraints, most of efficient interior-point methods for nonlinear programs are presented with combining a distinctive penalty strategy. These methods can roughly and mainly be summarized into three kinds by the order of using the penalty technique. The first kind of methods firstly reformulate the original program to a problem with only equality constraints by interior barrier technique and then prompt the global convergence of these methods by different penalty functions, such as [9, 10, 13, 15, 21, 24, 26, 27, 30]. The second kind of methods first use the penalty strategy to obtain a new formulation of the original program with only inequality constraints and then use the interior-point methods to solve the formulation, such as [22]. The third kind of methods use both penalty strategy and interior-point technique to transform the original problem with inequality constraints into a new formulation with only equality constraints (see [14]). The co-existence of penalty and barrier parameters brings new challenge to this kind of methods. As a return to the challenge, the last kind of methods can be expected to have some exclusive global and/or local convergence properties such as the rapid detection of infeasibility.

Although every interior-point method has its novelty, they share some common points, for example, the iterates are usually the approximate solutions of some parametric primal-dual nonlinear system which converges to the KKT conditions of the original problem as the barrier parameter tends to zero, and should be interior points for nonnegative constraints. The interior-point condition can result in the truncation of the step, which may cause the failure of global convergence to the KKT point even for a well-posed problem (see [34] for a counterexample) and make the local convergence analysis of the primal-dual interior-point methods much complicated and sophisticated (e.g., [2, 11, 18, 20, 21, 36, 37]). By introducing the null-space technique, some interior-point methods such as [13, 24, 26] have been proved not to suffer the failure of global convergence. They have strong global convergence and can converge to an infeasible stationary point when the problem is infeasible, but they cannot detect the infeasibility rapidly.

Similar to the first kind of interior-point methods mentioned above, we consider the logarithmic-barrier problem

\[
\begin{align*}
\min \ f(x) &- \beta \sum_{i \in \mathcal{I}} \ln y_i \\
\text{s.t.} \ &c_i(x) + y_i = 0, \ i \in \mathcal{I},
\end{align*}
\]  

(1.6) (1.7)

where \( \beta > 0 \) is the barrier parameter, \( y_i > 0 \) \( (i \in \mathcal{I}) \) are slack variables. With the help of a logarithmic barrier augmented Lagrangian function, we can obtain closed-form solutions of slack variables of logarithmic-barrier problems of nonlinear programs. As a result, a two-parameter primal-dual nonlinear system is proposed, which corresponds to the KKT point and the infeasible stationary point of nonlinear programs, respectively, as one of two parameters vanishes. Based on this distinctive system, we present a primal-dual interior-point method capable of rapidly detecting infeasibility of nonlinear programs. Our method generates interior-point iterates without truncation of the step and can detect the infeasibility of the problem rapidly. Rapid detection of infeasibility is also one of important features of newly developed
penalty-interior-point algorithm (see [14]) and SQP methods (see [5, 8]), and is a very useful property in practice.

Our method has similarity to the existing interior-point methods for nonlinear programs. Similar to [9, 10, 24, 26], we consider the problem with slack variables (1.6)–(1.7) and use similar null-space technique and the technique for updating slack variables. But unlike those methods, our method is based on a distinctive primal-dual system and uses a different merit function dependent on both primal and dual variables, which is similar to [18, 21]. We note that [18, 21] also use augmented Lagrangian functions in developing their interior-point methods, but they are not based on the problem (1.6)–(1.7) and have a different flavor with our method. A recent work on interior-point methods is [14] which solves a two-parameter subproblem (or correspondingly a two-parameter primal-dual nonlinear system), but his system can only be proved to be asymptotically approximate the KKT conditions of the original problem as the barrier parameter tends to zero. Curtis [14] and Nocedal, Öztoprak and Waltz [27] have shown by numerical experiments that their interior-point methods have the ability to detect the infeasibility, but no theoretical proof is provided to show that those methods can detect infeasibility at quadratic or superlinear rate.

Without assuming any constraint qualification or requiring any feasibility of constraints, it is proved that our method globally converges to a KKT point of the original problem as the barrier parameter tends to zero. Otherwise, the scaling parameter tends to zero, and the method globally converges to either an infeasible stationary point or a singular stationary point of the original problem. Under suitable local conditions, we prove that the method can be not only superlinearly or quadratically convergent to the KKT point as the original problem is feasible, but also superlinearly or quadratically convergent to the infeasible stationary point when a problem is infeasible. Preliminary numerical results show that the method is efficient in solving some simple but hard problems and some standard test problems from the CUTE collection. The superlinear convergence have also been observed when we solve the infeasible problems given by [8] and the well-posed feasible problem presented as a counterexample to show the failure of global convergence of some interior-point methods by [33].

This paper is organized as follows. In Section 2, we first give a closed-form solution on slack variables of the KKT system of the logarithmic barrier problem (1.6)–(1.7). A corresponding two-parameter primal-dual nonlinear system is followed. Then we describe our algorithm for the original problem in Section 3. The strong global convergence results on the algorithm are proved in Section 4. In Section 5, under suitable assumptions, we show that the algorithm can be of locally quadratic or superlinear convergence to the KKT point or the infeasible stationary point of the original problem. The algorithm is implemented in Section 6, and preliminary numerical results for some simple but hard problems from literature and some standard test problems from the CUTE collection [4] are reported. We conclude our paper in Section 7.

Throughout the article, a letter with subscript \( k \) (or \( l \)) is related to the \( k \)th (or \( l \)th) iteration, the subscript \( i \) indicates the \( i \)th component of a vector or the \( i \)th column of a matrix, and the subscript \( k \) (or \( l \)) is the \( k \)th component of a vector or the \( l \)th column of a matrix at the \( k \)th (or \( l \)th) iteration. All vectors are column vectors, and \( z = (x, u) \) means \( z = [x^T, u^T]^T \).
The expression $\theta_k = O(\tau_k)$ means that there exists a constant $M$ independent of $k$ such that $|\theta_k| \leq M|\tau_k|$ for all $k$ large enough, and $\theta_k = o(\tau_k)$ indicates that $|\theta_k| \leq \epsilon_k|\tau_k|$ for all $k$ large enough with $\lim_{k \to 0} \epsilon_k = 0$. If it is not specified, $I$ is an identity matrix whose order may be showed in the subscript or be clear in the context, $\| \cdot \|$ is the Euclidean norm, $|S|$ is the cardinality of set $S$. For simplicity, we also use simplified notations for functions, such as $f_k = f(x_k)$, $\nabla f_k = \nabla f(x_k)$, $c_{ki} = c_i(x_k)$, $\nabla c_{ki} = \nabla c_i(x_k)$ and so on.

2. A two-parameter primal-dual system for nonlinear programs

With the help of a logarithmic barrier augmented Lagrangian function, we can derive closed-form solutions on slack variables of the logarithmic barrier problem (1.6)–(1.7). A primal-dual nonlinear system with barrier and scaling parameters is then followed. Its solution corresponds to the KKT point and the infeasible stationary point of program (1.1)–(1.2), respectively, as one of two parameters vanishes. Based on this system, we present our primal-dual interior-point algorithm for nonlinear programs (1.1)–(1.2).

We consider the augmented Lagrangian function for the logarithmic barrier problem (1.6)–(1.7)

$$P(\beta, \rho)(x, y, u) = \rho \left[ f(x) - \beta \sum_{i \in I} \ln y_i + u^T (c(x) + y) \right] + \frac{1}{2} \|c(x) + y\|^2,$$

(2.1)

where $\beta > 0$ is the barrier parameter, $\rho > 0$ is a scaling parameter, $c(x) = (c_i(x), i \in I) \in \mathbb{R}^m$, $y = (y_i, i \in I) \in \mathbb{R}^m$, $u$ is a vector in $\mathbb{R}^m$. The stationary conditions on $P(\beta, \rho)(x, y, u)$ suggest the following equations:

$$\begin{cases}
\rho \nabla f(x) + \sum_{i \in I} [\rho u_i + c_i(x) + y_i] \nabla c_i(x) = 0, \\
-\rho \beta y_i^{-1} + pu_i + c_i(x) + y_i = 0, & i \in I, \\
\rho (c(x) + y) = 0.
\end{cases}
$$

(2.2)

By multiplying $y_i$ on both sides of the second equation, one has the equation

$$y_i^2 + (c_i(x) + pu_i)y_i - \rho \beta = 0, \quad i \in I.
$$

(2.3)

Thus, we have closed-form solutions on slack variables

$$y_i = \frac{1}{2} \left[ \sqrt{(c_i(x) + pu_i)^2 + 4 \rho \beta} - (c_i(x) + pu_i) \right], \quad i \in I,
$$

(2.4)

where the negative root is not taken since $y_i > 0$. Therefore,

$$c_i(x) + y_i = \frac{1}{2} \left[ \sqrt{(c_i(x) + pu_i)^2 + 4 \rho \beta} + (c_i(x) + pu_i) \right], \quad i \in I.
$$

(2.5)

If we set $\lambda_i = pu_i + c_i(x) + y_i$ for $i \in I$, one has

$$\lambda_i = \frac{1}{2} \left[ \sqrt{(c_i(x) + pu_i)^2 + 4 \rho \beta} + (c_i(x) + pu_i) \right], \quad i \in I.
$$

(2.5)
Using (2.4) and (2.5), equations in (2.2) can be reformulated as the following system of equations on unknowns \((x,u)\):

\[
\begin{align*}
\rho \nabla f(x) + \sum_{i \in I} \frac{1}{2} \left[ (c_i(x) + \rho u_i)^2 + 4\rho \beta + (c_i(x) + \rho u_i) \right] \nabla c_i(x) &= 0, \\
\frac{1}{2}\rho \left[ (c_i(x) + \rho u_i)^2 + 4\rho \beta + (c_i(x) - \rho u_i) \right] &= 0, \quad i \in I,
\end{align*}
\] (2.6)

where \(\beta > 0\) and \(\rho > 0\) are two parameters.

It is noted that, if \(\beta = 0\) and \(\rho > 0\), equations in (2.6) are reduced to the equations

\[
\rho \nabla f(x) + \sum_{i \in I} \frac{1}{2} \left[ |c_i(x) + \rho u_i| + (c_i(x) + \rho u_i) \right] \nabla c_i(x) = 0, \quad (2.7)
\]

\[
\frac{1}{2} \left[ |c_i(x) + \rho u_i| + (c_i(x) - \rho u_i) \right] = 0, \quad i \in I. \quad (2.8)
\]

Define index sets \(A(x) = \{i \in I | c_i(x) + \rho u_i \geq 0\}\) and \(N(x) = \{i \in I | c_i(x) + \rho u_i < 0\}\). Then, by (2.8), for any solution \((x,u)\) of the system (2.6) (if there exists), one has \(c_i(x) = 0\) for \(i \in A(x)\) and \(u_i = 0, \; i \in N(x)\). Thus, \(c_i(x) < 0\) for \(i \in N(x)\), and (2.7) implies

\[
\nabla f(x) + \sum_{i \in A(x)} u_i \nabla c_i(x) = 0. \quad (2.9)
\]

Consequently, \((x,u)\) is a KKT pair of the original problem (1.1)–(1.2).

If \(\rho = 0\), the first equation in (2.6) is reduced to the equation

\[
\sum_{i \in I} \frac{1}{2} |c_i(x)| + c_i(x) \nabla c_i(x) = 0, \quad (2.10)
\]

which shows that, if \((x,u)\) satisfies the system (2.6), and \(x\) is infeasible to the problem (1.1)–(1.2), then \(x\) is a stationary point for minimizing \(\frac{1}{2} \| \max(0, c(x)) \|^2\), i.e., a stationary point for minimizing the \(\ell_2\) measure of residuals of the constraints, which is also called as an infeasible stationary point of problem (1.1)–(1.2) (see Definition 4.1).

The preceding argument shows that the proposed system (2.6) can not only reduce to the KKT conditions of the original problem as parameter \(\beta\) vanishes, but also can reduce to the stationary condition of an infeasible stationary point of the original problem as parameter \(\rho\) is zero. This feature is distinguished from all primal-dual systems used by the existing interior-point methods. It turns out that is a favorable and important characterization, since we want to develop an interior-point method which can not only converge to a KKT point of the original problem as the problem is feasible, but also can converge to an infeasible stationary point of the original problem as it is infeasible.

In next section, we will develop our primal-dual interior-point method for nonlinear programs based on the two-parameter system (2.6). For convenience of statement, we denote, for \(i \in I\),

\[
y_i(x, u; \beta, \rho) = \frac{1}{2} \left[ \sqrt{\left( c_i(x) + \rho u_i \right)^2 + 4\rho \beta - (c_i(x) + \rho u_i)} \right], \quad (2.11)
\]

\[
\lambda_i(x, u; \beta, \rho) = \frac{1}{2} \left[ \sqrt{\left( c_i(x) + \rho u_i \right)^2 + 4\rho \beta + (c_i(x) + \rho u_i)} \right]. \quad (2.12)
\]
That is, $\lambda_i$ and $y_i$ ($i \in I$) are functions on $(x, u)$, and are dependent on parameters $\beta$ and $\rho$. If it is not confused in the context, we may use $\lambda_i = \lambda_i(x, u; \beta, \rho)$ and $y_i = y_i(x, u; \beta, \rho)$ for simplicity. Thus, $\lambda_i y_i = \rho \beta$ for $i \in I$. Using (2.11) and (2.12), the two-parameter system (2.6) can be written as the concise form

$$
\begin{cases}
\rho \nabla f(x) + \sum_{i \in I} \lambda_i(x, u; \beta, \rho) \nabla c_i(x) = 0, \\
c_i(x) + y_i(x, u; \beta, \rho) = 0, \quad i \in I.
\end{cases}
$$

(2.13)

We need the following preliminary results for our method and its global and local analysis.

**Lemma 2.1** For $i \in I$, let $y_i$ and $\lambda_i$ be defined by (2.11) and (2.12), respectively.

1. If $c_i(x)$ is differentiable, then $y_i$ and $\lambda_i$ are differentiable on $(x, u)$, and

$$
\nabla_x y_i = -\frac{y_i y_i}{y_i + \lambda_i} \nabla c_i(x), \quad \nabla_x \lambda_i = \frac{\lambda_i}{y_i + \lambda_i} \nabla c_i(x),
$$

(2.14)

By (2.12), $\lambda_i = \rho u_i + c_i(x) + y_i$. Thus,

$$
\nabla_x \lambda_i = \nabla c_i(x) + \nabla_x y_i.
$$

(2.17)

Substituting (2.17) into (2.16),

$$
\nabla_x y_i = -\frac{y_i}{y_i + \lambda_i} \nabla c_i(x).
$$

Again by (2.17), $\nabla_x \lambda_i = \frac{\lambda_i}{y_i + \lambda_i} \nabla c_i(x)$.

Similar to (2.16) and (2.17), one has

$$
y_i \frac{\partial \lambda_i}{\partial u_i} + \lambda_i \frac{\partial y_i}{\partial u_i} = 0, \quad \frac{\partial \lambda_i}{\partial u_i} = \rho + \frac{\partial y_i}{\partial u_i}.
$$

Thus,

$$
\frac{\partial y_i}{\partial u_i} = -\rho \frac{y_i}{y_i + \lambda_i}, \quad \frac{\partial \lambda_i}{\partial u_i} = \rho \frac{\lambda_i}{y_i + \lambda_i}.
$$

For $i \neq i'$, $\frac{\partial y_i}{\partial u_{i'}} = \frac{\partial \lambda_i}{\partial u_{i'}} = 0$ since $y_i$ and $\lambda_i$ do not depend on $u_{i'}$. 
(2) The result follows immediately since \( \frac{\partial y_i}{\partial u_i} < 0 \) and \( \frac{\partial \lambda_i}{\partial u_i} > 0 \).

(3) It is obvious from (2.11) that \( y_i \) is smaller as \( \beta \) is smaller. If \( c_i(x) + y_i > 0 \), then \( u_i y_i = \frac{1}{\beta} (\lambda_i y_i - y_i (c_i + y_i)) < \beta \) which implies

\[
\frac{\partial y_i}{\partial \rho} = \frac{\beta - u_i y_i}{y_i + \lambda_i} > 0.
\]

Hence, \( y_i \) is a nondecreasing function on \( \rho \).

3. Our algorithm

Our algorithm consists of the inner algorithm and the outer algorithm, where the inner algorithm tries to find an approximate solution of the system (2.6) for given parameters \( \beta \) and \( \rho \), while the outer algorithm updates the parameters by the information derived from the inner algorithm.

3.1. A well-behaved quadratic programming subproblem.

A quadratic programming subproblem is presented for deriving our search direction in this subsection. The subproblem is well-behaved since it is always feasible. Suppose that \((x_k, u_k)\) is the current iterate. For given \( \beta > 0 \) and \( \rho > 0 \), let

\[
B_k = H_k + \sum_{i \in I} \frac{\lambda_{ki}}{y_{ki} + \lambda_{ki}} \nabla c_{ki} \nabla c_{ki}^T,
\]

where \( H_k \) is the Hessian of the Fritz-John function \( L_\rho(x, \lambda) = \rho f(x) + \lambda^T c(x) \) at \((x_k, \lambda_k)\). In order to avoid the computation of second-order derivatives, we may take \( H_k \) to be an approximation to the Hessian in our algorithm. Using (2.14)–(2.15), the Newton’s equations for (2.13) have the form

\[
\begin{align*}
B_k d_x + \sum_{i \in I} \rho \frac{\lambda_{ki}}{y_{ki} + \lambda_{ki}} d_{ui} \nabla c_{ki} &= -\rho \nabla f_k + \sum_{i \in I} \lambda_{ki} \nabla c_{ki}, \\
\rho \frac{\lambda_{ki}}{y_{ki} + \lambda_{ki}} \nabla c_{ki}^T d_x - \rho^2 \frac{y_{ki}}{y_{ki} + \lambda_{ki}} d_{ui} &= -\rho (c_{ki} + y_{ki}), \quad i \in I.
\end{align*}
\]

For simplicity of statement, let

\[
R_k = \begin{pmatrix}
\frac{\lambda_{k1}}{y_{k1} + \lambda_{k1}} \nabla c_{k1} & \ldots & \frac{\lambda_{km}}{y_{km} + \lambda_{km}} \nabla c_{km} \\
-\rho \frac{y_{k1}}{y_{k1} + \lambda_{k1}} & \ldots & 0 \\
\ldots & \ldots & \ldots \\
0 & \ldots & -\rho \frac{y_{km}}{y_{km} + \lambda_{km}}
\end{pmatrix},
\]

and \( r_k = c_k + y_k \). The following result shows that one can obtain the solution of the system (3.2) by solving the feasible quadratic programming (QP) subproblem (3.3)–(3.4).

**Lemma 3.1** (1) For given \( \beta > 0 \) and \( \rho > 0 \), the solution to the QP problem

\[
\begin{align*}
\min q_k(d) := (\nabla_x L_\rho(x_k, \lambda_k))^T d_x + \frac{1}{2} d^T Q_k d \\
st. \quad R_k^T d = -(c_k + y_k)
\end{align*}
\]
satisfies the system (3.2), where \( d = (d_x, d_u) \in \mathbb{R}^{n+m}, \) \( \nabla_x L_k(x_k, \lambda_k) = \rho \nabla f_k + \sum_{i \in I} \lambda_{ki} \nabla c_{ki}, \)

\[
Q_k = \begin{pmatrix}
H_k + \sum_{i \in I} \frac{\rho \beta}{(y_{ki} + \lambda_{ki})} \nabla c_{ki}^T \nabla c_{ki} & \frac{\rho \beta}{(y_{ki} + \lambda_{ki})} \nabla c_{ki}^T \nabla c_{k1} & \cdots & \frac{\rho \beta}{(y_{km} + \lambda_{km})} \nabla c_{ki}^T \nabla c_{km} \\
\frac{\rho \beta}{(y_{ki} + \lambda_{ki})} \nabla c_{k1}^T & \frac{\rho \beta}{(y_{ki} + \lambda_{ki})} \nabla c_{k1}^T & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\rho \beta}{(y_{km} + \lambda_{km})} \nabla c_{km}^T & 0 & \cdots & \frac{\rho \beta}{(y_{km} + \lambda_{km})} \nabla c_{km}^T 
\end{pmatrix}.
\]

(2) If

\[
d_k^T H_k d_x + \sum_{i \in I} \frac{\lambda_{ki}}{y_{ki} + \lambda_{ki}} \|\nabla c_{ki}^T d_x\|^2 > 0, \quad \forall d_x \in \mathbb{R}^n,
\]

the above QP has a unique solution, which implies that the system (3.2) is consistent.

Proof. (1) In addition to (3.4), the KKT conditions of the above QP contain the equations:

\[
\nabla_x L_k(x_k, \lambda_k) + (H_k + \sum_{i \in I} \frac{\rho \beta}{(y_{ki} + \lambda_{ki})^2} \nabla c_{ki} \nabla c_{ki}^T) d_x + \sum_{i \in I} \frac{\rho \beta}{(y_{ki} + \lambda_{ki})^2} d_{ui} \nabla c_{ki} + \sum_{i \in I} \frac{\lambda_{ki}}{y_{ki} + \lambda_{ki}} \nabla c_{ki} = 0,
\]

\[
\frac{\rho \beta}{(y_{ki} + \lambda_{ki})^2} \nabla c_{ki}^T d_x + \frac{\rho \beta}{(y_{ki} + \lambda_{ki})^2} d_{ui} - \frac{y_{ki}}{y_{ki} + \lambda_{ki}} \dot{\lambda}_{ki} = 0, \quad i \in I,
\]

where \( \dot{\lambda}_{ki} (i \in I) \) are the associated multipliers with (3.4). One can first have \( \dot{\lambda}_k \) from (3.7), and then substitute it into (3.6) to derive the first equation of the system (3.2).

(2) If (3.5) holds, then \( \nabla^2 q_k \) is positive definite in the null space of \( R_k^T \) since

\[
d_k^T Q_k d = d_k^T (H_k + \sum_{i \in I} \frac{\lambda_{ki}}{y_{ki} + \lambda_{ki}} \nabla c_{ki} \nabla c_{ki}^T) d_x + \sum_{i \in I} \frac{y_{ki}}{y_{ki} + \lambda_{ki}} \rho \beta d_{ui}^2 > 0,
\]

for all \( d \in \mathbb{R}^{n+m} \) such that \( R_k^T d = 0 \). It follows from Lemma 16.1 of [25] that QP (3.3)–(3.4) has a unique solution. By (1), the unique solution also solves the system (3.2).

The null-space technology in nonlinear optimization was initially presented by Byrd [2] for trust-region methods. It has been proved to be very efficient in trust-region and line-search SQP and interior-point methods (for example, see [10, 24, 26]). In order to obtain the strong global convergence properties, we introduce this technique to the subproblem. Firstly, \( d_k^e \in \mathbb{R}^{n+m} \) is computed to satisfy some prescribed mild conditions presented in Assumption 4.3 and \( d_k^e = 0 \) as \( r_k = 0 \) (we refer the readers to [24, 26] for more details). Then we solve the following null-space quadratic programming subproblem

\[
\min \quad \hat{q}_k(d) := \rho \nabla f_k^T d_x + \sum_{i \in I} \frac{\rho \beta}{y_{ki} + \lambda_{ki}} (\nabla c_{ki}^T d_x + \rho d_{ui}) + \frac{1}{2} d_k^T H_k d_x \\
+ \frac{1}{2} \sum_{i \in I} \frac{\rho \beta}{(y_{ki} + \lambda_{ki})^2} (\nabla c_{ki}^T d_x + \rho d_{ui})^2
\]

s.t. \( R_k^T d = R_k^T d_k^e \),

\[
(3.8)
\]

\[
(3.9)
\]
where the right-hand-side term $-(c_k + y_k)$ of (3.4) is replaced by $R_k^T d_k$ and the scalar $(\lambda_k)^T R_k^T d_k$ of the objective in (3.3) is removed.

3.2. The merit function.

In order to prompt global convergence of the algorithm, we introduce the merit function

$$\Phi_\xi(x, u; \beta, \rho) = \xi \rho f(x) - \xi \rho \beta \sum_{i \in I} \ln y_i + ||c(x) + y||,$$

where $\xi > 0$ is a penalty parameter which is updated in accordance with the directional derivative of $\Phi_\xi(x, u; \beta, \rho)$ along the search direction. The update of the scaling parameter $\rho$ in the outer algorithm depends on the value of $\xi$. Although it has a similar form to those used in some existing primal-dual interior-point methods such as [13, 24, 26], it is essentially different in that $y$ is a function on $x$ and $u$.

The following result is helpful for us to select an appropriate penalty parameter $\xi$ so that the search direction is a descent direction of the merit function.

**Lemma 3.2** For given $\beta > 0$ and $\rho > 0$, let $z_k = (x_k, u_k)$, and let $d_k = (d_{xk}, d_{uk})$ be the solution of subproblem (3.8)–(3.9), $\Phi_\xi'(z_k; d_k)$ be the directional derivative of $\Phi_\xi(z_k; \beta, \rho)$ at $z_k$ along the direction $d_k$.

1. If $r_k = 0$, then $\Phi_\xi'(z_k; d_k) \leq -\frac{1}{\xi} d_k^T Q_k d_k$.

**Proof.** (1) Let

$$\Theta(x, u) = ||c(x) + y||.$$  \hspace{1cm} (3.10)

Then, by the proof of Proposition 3.1 of [24], $\Theta'(z_k; d_k) \leq ||r_k + R_k^T d_k|| - ||r_k||$. Therefore,

$$\Phi_\xi'(z_k; d_k) \leq \xi \rho f_k^T d_{xk} - \xi \rho \beta \sum_{i \in I} y_{ki}^{-1} ((\nabla x y_i)^T d_{xk} + (\nabla u y_i)^T d_{uk}) + ||r_k + R_k^T d_k|| - ||r_k||$$

$$\quad = \xi (\rho f_k^T d_{xk} + \sum_{i \in I} y_{ki} \frac{\rho \beta}{\lambda_{ki}} (\nabla c_{ki}^T d_{xk} + \rho d_{uki})) + ||r_k + R_k^T d_k|| - ||r_k||,$$

where the equality follows from Lemma 2.11(1).

(2) If $r_k = 0$, then, by (1), $\Phi_\xi'(z_k; d_k) \leq \xi (\rho f_k^T d_{xk} + \sum_{i \in I} y_{ki} \frac{\rho \beta}{\lambda_{ki}} (\nabla c_{ki}^T d_{xk} + \rho d_{uki}))$. The result follows immediately since $d = 0$ is a feasible solution to the QP (3.8)–(3.9). \hfill \square

Certain additional update techniques are used in primal-dual interior-point methods for nonlinear programs with strong global convergence (for example, see [9, 24, 26, 31]). A technique, which was introduced first in Byrd, Gilbert and Nocedal [9] and was examined to be efficient later, is to update $y_{k+1} = y_k + \alpha_k d_{yk}$ to $y_{k+1} = \max\{y_k + \alpha_k d_{yk}, -c(x_{k+1})\}$, so that $c(x_{k+1}) + y_{k+1} \geq 0$ at the $(k + 1)$th iteration. However, this technique cannot be applied to our method straightforward here since $y_k$ depends on both $x_k$ and $u_k$. The following result shows that $c(x_{k+1}) + y_{k+1} \geq 0$ can still hold provided $u_{k+1}$ is appropriately updated, thus the strong global convergence is attained.
Lemma 3.3 For given $\beta > 0$ and $\rho > 0$, if $c_i(x_{k+1}) \geq 0$, or $c_i(x_{k+1}) < 0$ but $u_{k+1,i} \leq -\frac{\beta}{c_i(x_{k+1})}$ for any $i \in I$, then $c_i(x_{k+1}) + y_{k+1,i} \geq 0$, where $y_{k+1,i} = y_i(x_{k+1}, u_{k+1}; \beta, \rho)$ is given by (2.11).

Proof. If $c_i(x_{k+1}) \geq 0$, then $c_i(x_{k+1}) + y_{k+1,i} > 0$ since $y_{k+1,i} > 0$. In the remainder, we consider the case $c_i(x_{k+1}) < 0$.

If $c_i(x_{k+1}) - \rho u_{k+1,i} \geq 0$, by (2.14), one has $c_i(x_{k+1}) + y_{k+1,i} \geq 0$. In this case,

$$u_{k+1,i} < 0 < -\frac{\beta}{c_i(x_{k+1})}.$$

If $c_i(x_{k+1}) - \rho u_{k+1,i} < 0$, by (2.14), $c_i(x_{k+1}) + y_{k+1,i} \geq 0$ if and only if

$$\sqrt{(c_i(x_{k+1}) + \rho u_{k+1,i})^2 + 4\rho \beta} \geq -(c_i(x_{k+1}) - \rho u_{k+1,i}),$$

which is equivalent to $c_i(x_{k+1})u_{k+1,i} \geq -\beta$.

Due to $c_i(x_{k+1})u_{k+1,i} \geq -\beta$, the result follows immediately. $\square$

3.3. The framework of our algorithm.

We denote by $\mathcal{F}$ the class of continuous functions $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim_{t \to 0} \theta(t) = 0$, and

$$\phi(\beta, \rho)(x, u) = \left( \rho \nabla f(x) + \sum_{i \in I} \lambda_i(x, u; \beta, \rho) \nabla c_i(x) \right).$$

Now we are ready to describe our algorithmic framework for problem (1.1)–(1.2). The details on implementation of the algorithm will be provided in Section 6.

Algorithm 3.4 (The algorithm for problem (1.1)–(1.2))

Step 1. Given $z_0 = (x_0, u_0) \in \mathbb{R}^{n+m}$, $\beta_0 > 0$, $\rho_0 > 0$, $\delta \in (0, 1)$, $\sigma \in (0, \frac{1}{2})$, $\epsilon > 0$, and functions $\theta_1, \theta_2 \in \mathcal{F}$. Set $k := 0$.

Step 2. While $\beta_i > \epsilon$ and $\rho_i > \epsilon$, start the following inner algorithm.

Step 2.0 Given $H_i \in \mathbb{R}^{n \times n}$, $\xi_0 = 1$, let $z_0 = (x_1, u_1)$. Evaluate $y_0$ and $\lambda_0$ by (2.11) and (2.12) with $\beta = \beta_i$ and $\rho = \rho_i$. Let $k := 0$.

Step 2.1 Obtain $d_{\xi_k}$, and solve the QP subproblem (3.10), (3.11) to derive $(d_{x_k}, d_{u_k})$.

Step 2.2 Choose $\xi_{k+1}$ with either $\xi_{k+1} = \xi_k$ or $\xi_{k+1} \leq 0.5\xi_k$ such that

$$\pi_{\xi_{k+1}}(z_k; d_k) + (1 - \delta)(\|r_k\| - \|r_k + R_k d_k\|) \leq -0.5\xi_{k+1} d_k^T Q_k d_k, \quad (3.11)$$

where

$$\pi_{\xi}(z_k; d_k) = (\rho \nabla f_{x_k}^T d_{x_k} + \sum_{i \in I} \frac{\rho \delta}{y_{k+1,i}} (\nabla c_i^T_{x_k} d_{x_k} + \rho d_{u_k,i})) + \|r_k + R_k d_k\| - \|r_k\|.$$

Step 2.3 Choose the step-size $\alpha_k \in (0, 1]$ to be the maximal in $\{1, \delta, \delta^2, \ldots\}$ such that

$$\Phi_{\xi_{k+1}}(x_k + \alpha_k d_{x_k}, u_k + \alpha_k d_{u_k}; \beta_i, \rho_i) - \Phi_{\xi_{k+1}}(x_k, u_k; \beta_i, \rho_i) \leq \sigma \alpha_k \pi_{\xi_{k+1}}(z_k; d_k). \quad (3.12)$$

Step 2.4 Set $x_{k+1} = x_k + \alpha_k d_{x_k}$ and $u_{k+1} = u_k + \alpha_k d_{u_k}$.

Step 2.5 Set

$$u_{k+1,i} = \begin{cases} 
\hat{u}_{k+1,i}, & \text{if } c_i(x_{k+1}) \geq 0; \\
\min\{\hat{u}_{k+1,i}, -\frac{\beta}{c_i(x_{k+1})}\}, & \text{otherwise}
\end{cases} \quad (3.13)$$

for every $i \in I$. Set $z_{k+1} = (x_{k+1}, u_{k+1})$. 

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Step 3 If \( \| \phi(\beta_t, \rho_l)(x_{k+1}, u_{k+1}) \|_\infty \leq \rho_l \theta_1(\beta_t) \), then update \( \beta_t \) to \( \beta_{t+1} \leq 0.1 \beta_t \), \( \rho_{t+1} = \rho_t \); else if \( \xi_{k+1} \leq 0.1 \min(\rho_l^{0.5}, 1) \), then update \( \rho_t \) to \( \rho_{t+1} \leq \xi_{k+1} \rho_t \), \( \beta_{t+1} = \beta_t \). In these two cases, the inner algorithm is stopped. Let \( z_{t+1} = z_{k+1} \), \( l := l + 1 \) and go to Step 2. Otherwise, evaluate \( y_{k+1} = y(x_{k+1}, u_{k+1}; \beta_t, \rho_t) \) and \( \lambda_{k+1} = \lambda(x_{k+1}, u_{k+1}; \beta_t, \rho_t) \), update \( H_k \) to \( H_{k+1} \), let \( k := k + 1 \) and go to Step 2.1.

Due to Step 2.2 of Algorithm 3.4, \( \Phi'_{\xi_{k+1}}(z_k; d_k) < 0 \). Thus, there is always a sufficiently small number \( \alpha_k > 0 \) such that (3.12) holds (for example, see Lemma 2.7 of [3]). That is, the inner algorithm of Algorithm 3.4 is well-defined.

Let \( \hat{y}_{k+1,i} = y_t(x_{k+1}, \hat{u}_{k+1}; \beta_t, \rho_t) \) for \( i \in I \). It follows from (3.13) and the proof of Lemma 3.3 that \( c_i(x_{k+1}) + \hat{y}_{k+1,i} \geq 0 \) if and only if \( u_{k+1,i} = \hat{u}_{k+1,i} \) (thus \( y_{k+1,i} = \hat{y}_{k+1,i} \)). If \( c_i(x_{k+1}) + \hat{y}_{k+1,i} < 0 \), then \( u_{k+1,i} = -\frac{\beta_i}{c_i(x_{k+1})} \) and \( c_i(x_{k+1}) + y_{k+1,i} = 0 \) (in this case \( y_{k+1,i} > \hat{y}_{k+1,i} \)). Therefore,

\[
\| c(x_{k+1}) + y_{k+1} \| \leq \| c(x_{k+1}) + \hat{y}_{k+1} \|. \tag{3.14}
\]

and \( \| c(x_{k+1}) + y_{k+1} \| \leq \| c(x_{k+1}) + \hat{y}_{k+1} \| \). Since the logarithmic function is monotonically nondecreasing, and, for any \( i \in I \), \( y_{k+1,i} \geq \hat{y}_{k+1,i} \), one has \( \ln y_{k+1,i} \geq \ln \hat{y}_{k+1,i} \) for every \( i \in I \). Note that the line search procedure guarantees \( \Phi_{\xi_{k+1}}(x_{k+1}, \hat{u}_{k+1}; \beta_t, \rho_t) \leq \Phi_{\xi_{k+1}}(x_k, u_k; \beta_t, \rho_t) \).

Hence, for every \( k \geq 0 \),

\[
\Phi_{\xi_{k+1}}(x_{k+1}, u_{k+1}; \beta_t, \rho_t) \leq \Phi_{\xi_{k+1}}(x_k, u_k; \beta_t, \rho_t). \tag{3.15}
\]

The well-definedness of the whole algorithm is based on the global convergence results of Algorithm 3.4. It will be proved, in the next section, that either the inner algorithm converges to a solution satisfying the system (2.13), in this situation the terminating condition \( \| \phi(\beta_t, \rho_l)(z_{k+1}) \|_\infty \leq \rho_l \theta_1(\beta_t) \) will hold in a finite number of iterations, or \( \xi_{k+1} \rightarrow 0 \) and the terminating condition \( \xi_{k+1} \leq 0.1 \min(\rho_l^{0.5}, 1) \) for the inner algorithm will be satisfied. Since the inner algorithm will always be terminated in a finite number of iterations, by Step 3 of Algorithm 3.4, either \( \beta_t \) or \( \rho_l \) will be reduced at least to a fixed fraction.

### 4. Global convergence

We present our global convergence results on Algorithm 3.4 in this section. Firstly, we consider the global convergence of the inner algorithm. For given \( \beta_t > 0 \) and \( \rho_l > 0 \), suppose that the inner algorithm does not terminate in a finite number of iterations. We prove that, if \( \{\xi_k\} \) is bounded away from zero, then every limit point of sequence \( \{(x_k, u_k)\} \) is a solution of the system (2.13); otherwise, \( \xi_k \rightarrow 0 \) as \( k \rightarrow \infty \). It shows that the supposition will never happen. After that, the global convergence results of the whole algorithm are presented. The results show that the whole algorithm converges to a KKT point of the original problem provided \( \beta_t \rightarrow 0 \) but \( \rho_l \not\rightarrow 0 \), otherwise \( \rho_l \rightarrow 0 \) and there is one of the limit points of the sequence \( \{x_l\} \) which is an infeasible stationary point or a singular stationary point of problem (1.1)–(1.2).

We need the following definitions.
Definition 4.1 \( x^* \in \mathbb{R}^n \) is called an infeasible stationary point of problem (1.1)–(1.2) if \( x^* \) is an infeasible point and
\[
\sum_{i \in I} a^*_i \nabla c_i(x^*) = 0,
\]
(4.1)
where \( a^*_i = \max\{c_i(x^*), 0\}, \ i \in I. \)

Definition 4.2 \( x^* \in \mathbb{R}^n \) is called a singular stationary point of problem (1.1)–(1.2) if there is a nonzero vector \( b^* \in \mathbb{R}^m \) such that
\[
\sum_{i \in I} b^*_i \nabla c_i(x^*) = 0,
\]
(4.2)
\[
b^*_i \geq 0, \ c_i(x^*) \leq 0, \ b^*_i c_i(x^*) = 0, \ i \in I.
\]
(4.3)

While Definition 4.1 shows that \( x^* \) is a stationary point for minimizing the constraint violations
\[
\frac{1}{2} \sum_{i \in I} |\max\{c_i(x), 0\}|^2,
\]
(4.4)
Definition 4.2 implies that \( x^* \) is a Fritz-John point of problem (1.1)–(1.2) at which the Mangasarian-Fromovitz constraint qualification (MFCQ) does not hold.

It should be noticed that various definitions have been given for infeasible and singular stationary points, see [5, 6, 8, 13, 24, 25, 38]. These stationary points may either belong to a set of minimizers of the problem minimizing the measure of constraint violations like problem (4.4) or be the optimal solutions of some degenerate nonlinear programs, see Section 6.1 for the details. For example, [8] considered the infeasible stationary point to be a first-order optimal solution \( x^* \) of the problem
\[
\min \sum_{i \in \{i \in I | c_i(x^*) > 0\}} c_i(x)
\]
s.t. \( c_i(x) = 0, \ i \in \{i \in I | c_i(x^*) = 0\}, \)
whereas [24] identifies some singular stationary points at which the linear independence constraint qualification (LICQ) does not hold.

4.1. Global convergence of the inner algorithm.

We consider the global convergence of the inner algorithm. Suppose that, for parameters \( \beta_l > 0 \) and \( \rho_l > 0 \), the inner algorithm of Algorithm 3.4 does not terminate in a finite number of iterations and \( \{(x_k, u_k)\} \) is an infinite sequence generated by the algorithm. For the sake of global convergence analysis, we need the following blanket assumptions.

Assumption 4.3
(1) The functions \( f \) and \( c_i(i \in I) \) are twice continuously differentiable on \( \mathbb{R}^n \);
(2) The iterative sequence \( \{x_k\} \) is in an open bounded set;
(3) The sequence \( \{H_k\} \) is bounded, and for all \( k \geq 0 \) and \( d \in \mathbb{R}^n \), \( d^T H_k d \geq \rho_l \gamma \|d\|^2 \), where \( \gamma > 0 \) is a constant;

(4) For all \( k \geq 0 \), \( d_k^\infty \) satisfies the conditions:

\[
\begin{align*}
(i) \quad \|d_k^\infty\| &\leq \eta_1 \|R_k r_k\|, \\
(ii) \quad \|r_k\| - \|r_k + R_k^T d_k^\infty\| &\geq \eta_2 \|R_k r_k\|^2 / \|r_k\|, \text{ where } \eta_1 > 0 \text{ and } \eta_2 > 0 \text{ are two constants.}
\end{align*}
\]

The conditions in Assumption 4.3 (1)–(3) are the same as those commonly used in global convergence analysis of iterative methods for nonlinear optimization (for example, see [5, 9, 13, 22, 24, 26]). Assumption 4.3 (4) is for the strong global convergence of the algorithm, which is very mild and can be satisfied easily (see Section 2.2 of [24]).

The following results depend only on the merit function and can be proved in the same way as Lemma 5 of [9] and Lemma 4.2 of [26].

**Lemma 4.4** Suppose that Assumption 4.3 holds. Then \( \{y_k\} \) is bounded, \( \{\lambda_k\} \) is componentwise bounded away from zero and \( \{u_k\} \) is lower bounded. Furthermore, if the penalty parameter \( \xi_k \) remains constant for all sufficiently large \( k \), then \( \{y_k\} \) is componentwise bounded away from zero, \( \{\lambda_k\} \) and \( \{u_k\} \) are bounded.

**Proof.** The results on \( \{y_k\} \) can be derived by [9, 26]. Due to \( \lambda_k y_k = \rho_l \beta_l \), the results on \( \{\lambda_k\} \) follow immediately.

For given \( \beta_l > 0 \) and \( \rho_l > 0 \), if \( \{y_k\} \) is bounded, then, by (2.11), \( u_ki > -\infty \) for all \( k \geq 0 \) and \( i \in I \). Otherwise, if \( u_ki \to -\infty \) for some \( i \), then \( y_ki \to \infty \), which is a contradiction. If \( \{y_k\} \) is componentwise bounded away from zero, then, by (2.11), \( u_ki < \infty \) for all \( k \geq 0 \) and \( i \in I \). Thus, the results on \( \{u_k\} \) are proved.

The update rule on \( \xi_k \) is adaptive. It implies that the sequence \( \{\xi_k\} \) is monotonically nonincreasing, which either remains a positive constant after a finite number of iterations or tends to zero as \( k \) tends to infinity. The next two results show that, if \( \xi_k \) is bounded away from zero, all step-sizes can be selected to be bounded away from zero.

**Lemma 4.5** Suppose that Assumption 4.3 holds. Let \( d_k = (d_{xk}, d_{uk}) \in \mathbb{R}^{n+m} \) be the solution of quadratic programming subproblem (3.8)–(3.9), and let \( g_k \in \mathbb{R}^m \) be the associated Lagrangian multiplier. If \( \xi_k \) remains a positive constant after a finite number of iterations, then \( \{\|d_k\|\} \) and \( \{|R_k g_k|\} \) are bounded.

**Proof.** Since \( \nabla f_k \) and \( d_k^\infty \) are bounded, \( H_k \) is bounded and uniformly positive definite, \( \|d_{xk}\| \) and \( \left| \sum_{i \in I} \frac{\nabla c_i^T d_{xk} + \rho d_{uk}}{y_{ki} + \lambda_{ki}} \right| \) are bounded due to \( \hat{q}(d_k) \leq \hat{q}(d_k^\infty) \).

If \( \xi_k \) is bounded away from zero, in view of Lemma 4.4 both \( y_{ki} \) and \( \lambda_{ki} \) are bounded above and bounded away from zero. Thus, \( \|d_k\| \) is bounded since \( 1/(y_{ki} + \lambda_{ki}) \) for every \( i \in I \) is bounded away from zero.

In view of

\[
\rho_l \nabla f_k + H_k d_{xk} + \sum_{i \in I} \frac{\rho_l y_{ki} \beta_l}{y_{ki} + \lambda_{ki}} (1 + \frac{\nabla c_i^T d_{xk} + \rho d_{uk}}{y_{ki} + \lambda_{ki}}) \nabla c_i + \sum_{i \in I} \frac{\lambda_{ki}}{y_{ki} + \lambda_{ki}} g_{ki} \nabla c_{ki} = 0, \quad (4.5)
\]

\[
\frac{\rho_l^2 y_{ki} \beta_l}{y_{ki} + \lambda_{ki}} (1 + \frac{\nabla c_i^T d_{xk} + \rho d_{uk}}{y_{ki} + \lambda_{ki}}) - \rho_l \frac{y_{ki}}{y_{ki} + \lambda_{ki}} g_{ki} = 0, \quad i \in I, \quad (4.6)
\]
and \( \frac{\rho_i \beta_i}{y_{ki} + \lambda_{ki}} \leq \sqrt{\frac{\rho_i \beta_i}{2}} \) for \( i \in \mathcal{I} \), and note that \( \|d_k\| \) is bounded, one can deduce that \( \|R_k g_k\| \) is bounded.

**Lemma 4.6** Suppose that Assumption 4.3 holds. Let \( \{\alpha_k\} \) be the sequence of step-sizes derived from (3.12) of Algorithm 3.4. If \( \xi_k \) remains a positive constant after a finite number of iterations, and

\[
\|R_k r_k\| \geq \hat{\eta} \|r_k\| \tag{4.7}
\]

for some constant \( \hat{\eta} > 0 \) and for all \( k \geq 0 \), then \( \{\alpha_k\} \) is bounded away from zero.

**Proof.** Due to Lemmas 4.4 and 4.5 for every \( i \in \mathcal{I} \), one has

\[
\begin{align*}
- \ln y_i(z_k + \alpha d_{x_k}; \beta_i, \rho_i) + \ln y_{ki} & - \frac{1}{y_{ki} + \lambda_{ki}} (\nabla e_{ki}^T d_{x_k} + \rho_i d_{uki}) = o(\alpha), \\
\Theta(x_k + \alpha d_{x_k}, u_k + \alpha d_{uk}) & = \|r_k + \alpha R_k^T d_k\| + o(\alpha)
\end{align*}
\]

for all \( \alpha > 0 \) sufficiently small, where \( \Theta(x, u) \) is defined by (5.11). Therefore,

\[
\Phi_{\xi_{k+1}}(x_k + \alpha d_{x_k}, u_k + \alpha d_{uk}; \beta_i, \rho_i) - \Phi_{\xi_{k+1}}(x_k, u_k; \beta_i, \rho_i) = \alpha \pi_{\xi_{k+1}}(z_k; d_k) + o(\alpha) \tag{4.8}
\]

for all \( \alpha \in [0, \tilde{\alpha}] \), where \( \tilde{\alpha} > 0 \) is a sufficiently small scalar. Note that, due to (4.7),

\[
(1 - \sigma) \alpha \pi_{\xi_{k+1}}(z_k; d_k) \leq (1 - \sigma)(1 - \delta)(\|r_k + R_k^T d_k\| - \|r_k\|) \leq -\alpha \eta_3 \|r_k\|, \tag{4.9}
\]

where \( \eta_3 = \eta_2 \hat{\eta}^2 (1 - \sigma)(1 - \delta) \). It follows from (4.8) and (4.9) that there exists a scalar \( \tilde{\alpha} \in (0, \tilde{\alpha}] \) such that

\[
\Phi_{\xi_{k+1}}(x_k + \alpha d_{x_k}, u_k + \alpha d_{uk}; \beta_i, \rho_i) - \Phi_{\xi_{k+1}}(x_k, u_k; \beta_i, \rho_i) \leq \sigma \alpha \pi_{\xi_{k+1}}(z_k; d_k)
\]

for all \( \alpha \in (0, \tilde{\alpha}] \) and for all \( k \geq 0 \). Thus, by Step 2.3 of Algorithm 3.4, \( \alpha_k \geq \tilde{\alpha} \) for all \( k \geq 0 \). \( \square \)

We prove that, if condition (4.7) holds, the penalty parameter \( \xi_k \) in the merit function will remain a positive constant after a finite number of iterations.

**Lemma 4.7** Suppose that Assumption 4.3 holds. If (4.7) holds for some scalar \( \hat{\eta} > 0 \) and for all \( k \geq 0 \), there is a constant \( \hat{\xi} > 0 \) such that \( \xi_k = \xi \) for all sufficiently large \( k \).

**Proof.** We achieve the result by proving that (3.11) holds with \( \xi_k = \hat{\xi} \) as \( \xi \) is small enough.

Note that \( y_{ki} + \lambda_{ki} \),

\[
\frac{1}{y_{ki}} = \frac{y_{ki}}{y_{ki} + \lambda_{ki}} \leq \frac{1}{\rho_i \beta_i} y_{ki}.
\]

Hence, due to \( \hat{q}_k(d_k) \leq \hat{q}_k(d_k^*) \), Assumption 4.3 (4) (ii) and Lemma 4.3 one has

\[
\begin{align*}
\pi_{\xi_{k+1}}(z_k; d_k) & + (1 - \delta)(\|r_k\| - \|r_k + R_k^T d_k\|) + \frac{1}{2} \xi_{k+1} d_k^T Q_k d_k \\
& \leq \xi_{k+1}(z_k; d_k) + \delta(\|r_k + R_k^T d_k\| - \|r_k\|) \\
& \leq \xi_{k+1}(z_k; d_k) + \delta(\|r_k + R_k^T d_k\| - \|r_k\|) \\
& \leq \gamma_1 \xi_{k+1} \|d_k^*\| - \delta \eta_2 \hat{\eta}^2 \|r_k\|.
\end{align*}
\]
where $\gamma_1 > 0$ is a scalar. Finally, it follows from Assumption 4.3 (4) (i) that (3.11) holds with $\xi_{k+1} = \hat{\xi}$ as $\hat{\xi} \leq \delta \eta_2 \hat{\eta}^2 / (\gamma_1 \eta_1)$.

Now we prove that sequence $\{(x_k, u_k)\}$ generated by the inner algorithm of Algorithm 3.4 will converge to a solution of the system (2.13) provided (4.7) holds.

**Lemma 4.8** Let $\{(x_k, u_k)\}$ be the infinite sequence generated by the inner algorithm of Algorithm 3.4. Suppose that Assumption 4.3 holds, and assume that (4.7) holds for some scalar $\hat{\eta} > 0$ and for all $k \geq 0$. Then any limit point of $\{(x_k, u_k)\}$ is a solution of the system (2.13).

**Proof.** Firstly, we prove that $\lim_{k \to \infty} \|r_k\| = 0$ and $\lim_{k \to \infty} \|d_k\| = 0$. (4.10)

Without loss of generality, we suppose that $\xi_k = \hat{\xi}$ for all $k \geq 0$. Then, by (3.15), $\{\Phi_{\hat{\xi}}(z_k; \beta_l, \rho_l)\}$ is a monotonically nonincreasing sequence. Note that it is also a bounded sequence. Thus,

$$\lim_{k \to \infty} \pi_{\hat{\xi}}(z_k; d_k) = 0 \quad (4.11)$$

since $\alpha_k$ is bounded away from zero. Using the last inequality of (4.9), one has

$$\lim_{k \to \infty} \|r_k\| = 0,$$

which implies $\lim_{k \to \infty} \|d_k\| = 0$. In view of (3.11), $\lim_{k \to \infty} \|d_k\| = 0$.

It follows from Lemma 4.4 that $\{y_k\}$ and $\{\lambda_k\}$ are bounded above and componentwise bounded away from zero, $\{u_k\}$ is bounded. Without loss of generality, let $z^* = (x^*, u^*)$ be any limit point of $\{z_k\}$ and suppose that $\lim_{k \to \infty} x_k = x^*$ and $\lim_{k \to \infty} u_k = u^*$. Since $\lim_{k \to \infty} \|r_k\| = 0$ and note $c_k + y_k = \lambda_k - \rho_l u_k$, one has

$$\lim_{k \to \infty} \lambda_k = \rho_l u^* > 0, \quad \lim_{k \to \infty} y_k = -c^* > 0.$$ 

In view of $\lim_{k \to \infty} \|d_k\| = 0$, by taking the limit on $k \to \infty$ in both sides of (4.5) and (4.6), there holds $\lim_{k \to \infty} g_{ki} y_{ki} = \rho_l \beta_l$ for $i \in I$ and $\lim_{k \to \infty} (\rho_l \nabla f_k + \sum_{i \in I} g_{ki} \nabla c_{ki}) = 0$. Thus,

$$\lim_{k \to \infty} (g_k - \lambda_k) = 0 \quad \text{and} \quad \lim_{k \to \infty} (\rho_l \nabla f_k + \sum_{i \in I} \lambda_{ki} \nabla c_{ki}) = 0.$$ 

That is, $\phi_{(\beta_l, \rho_l)}(x^*, u^*) = 0$. □

Now we are ready to present our global convergence results on the inner algorithm of Algorithm 3.4. It indicates that, for any given $\beta_l > 0$ and $\rho_l > 0$, the inner algorithm of Algorithm 3.4 will be terminated in a finite number of iterations.

**Theorem 4.9** Given $\beta_l > 0$ and $\rho_l > 0$ are two scalars. Let $\{(x_k, u_k)\}$ be the infinite sequence generated by the inner algorithm of Algorithm 3.4. Suppose that Assumption 4.3 holds. Then
one of the following statements is true:

(1) \( \| R_k r_k \| \geq \eta \| r_k \| \) for some scalar \( \eta > 0 \) and for all \( k \geq 0 \), \( \xi_k \) remains a positive constant for all sufficiently large \( k \), and any limit point of \( \{ (x_k, u_k) \} \) is a solution of the system \( (2.13) \);

(2) \( \xi_k \to 0 \) as \( k \to \infty \), and there exists some infinite index subset \( K \) such that

\[
\lim_{k \in K, k \to \infty} \frac{\| R_k r_k \|}{\| r_k \|} = 0,
\]

where \( r_k \geq 0 \) for all \( k \geq 0 \).

Proof. The result (1) follows from the preceding Lemma 4.7 and Lemma 4.8. The result (2) is straightforward and can be taken as a corollary of Lemma 4.7 where \( r_k \geq 0 \) since \( (3.14) \).

4.2. Global Convergence results of the whole algorithm.

Now we consider the global convergence of the whole algorithm. Without loss of generality, we let \( \epsilon = 0 \) and let \( \{ (x_l, u_l) \} \) be an infinite sequence generated by the outer algorithm of Algorithm 3.4. It is shown that, either we have \( \beta_l \to 0 \) and \( \rho_l \geq \hat{\rho} \) for some positive scalar \( \hat{\rho} \) and for all \( l \), and every limit point \( (x^*, u^*) \) of sequence \( \{ (x_l, u_l) \} \) is a KKT pair of the original problem \( (1.1) - (1.2) \), or we have \( \rho_l \to 0 \) and there exists a limit point \( x^* \) of the sequence \( \{ x_l \} \) which is either an infeasible stationary point or a singular stationary point of the problem \( (1.1) - (1.2) \).

**Theorem 4.10** Suppose that Assumption 4.3 holds for every given parameter \( \beta_l > 0 \) and \( \rho_l > 0 \). Let \( \epsilon = 0 \), and let \( \{ (x_l, u_l) \} \) be an infinite sequence generated by the outer algorithm of Algorithm 3.4. Then one of the following two cases will happen.

(1) \( \rho_l \geq \hat{\rho} \) for some positive scalar \( \hat{\rho} \) and for all \( l \), \( \beta_l \to 0 \) as \( l \to \infty \), every limit point \( (x^*, u^*) \) of sequence \( \{ (x_l, u_l) \} \) is a KKT pair of the original problem \( (1.1) - (1.2) \),

(2) \( \rho_l \to 0 \) as \( l \to \infty \), and there exists a limit point \( x^* \) of the sequence \( \{ x_l \} \) which is either an infeasible stationary point or a singular stationary point of the problem \( (1.1) - (1.2) \).

Proof. Since, for every given parameters \( \beta_l > 0 \) and \( \rho_l > 0 \), the inner algorithm of Algorithm 3.4 is terminated in a finite number of iterations, we have either the case with \( \| \phi(\beta_l, \rho_l)(z_{l+1}) \|_\infty \leq \rho_l \theta_1(\beta_l) \) for all sufficiently large \( l \) or the case that there exists an infinite subsequence \( \{ \rho_{l_k} \} \) of sequence \( \{ \rho_l \} \) such that \( \rho_{l_k} \leq 0.1 \rho_{l_{k-1}}^{1.5} \) for all \( k \).

If \( l_0 \) is a positive integer such that \( \| \phi(\beta_l, \rho_l)(z_{l+1}) \|_\infty \leq \rho_l \theta_1(\beta_l) \) for all \( l \geq l_0 \), then, by Step 3 of Algorithm 3.4, \( \rho_l \geq \rho_{l_0} \) for all \( l \) and \( \beta_l \to 0 \) as \( l \to \infty \). Thus,

\[
\lim_{l \to \infty} \| \phi(\beta_l, \rho_l)(x_{l+1}, u_{l+1}) \|_\infty = 0.
\]

In view of the argument on the system \( (2.6) \) in section 2, the above equation implies that every limit point \( (x^*, u^*) \) of sequence \( \{ (x_l, u_l) \} \) is a KKT pair of the original problem \( (1.1) - (1.2) \).

In the following, we consider the latter case. If \( \rho_{l_k} \leq 0.1 \rho_{l_{k-1}}^{1.5} \) for all \( k \), then \( \rho_{l_k} \leq 0.1 \rho_{l_{k-1}}^{1.5} \) for all \( k \) since \( \{ \rho_l \} \) is a nonincreasing sequence. Thus, \( \rho_l \to 0 \) as \( l \to \infty \). It follows from the result (2) of Theorem 4.9 that

\[
\lim_{k \to \infty} \| R_k r_k \| / \| r_k \| = 0,
\]
which is equivalent to

\[
\lim_{k \to \infty} \sum_{i \in \mathcal{I}} \frac{\lambda_{k,i}}{y_{k,i} + \lambda_{k,i}} \left( c_{l,i} + y_{k,i} \right) \nabla c_{l,i} = 0, \tag{4.12}
\]

\[
\lim_{k \to \infty} \frac{y_{k,i}}{y_{k,i} + \lambda_{k,i}} \left( c_{l,i} + y_{k,i} \right) \left\| r_{k} \right\| = 0, \quad i \in \mathcal{I}. \tag{4.13}
\]

Since \( \{x_l\} \) and \( \{y_l\} \) are bounded sequences, there exist convergent subsequences, for which, without loss of generality, we suppose

\[
\lim_{k \to \infty} x_{l_k} = x^* \quad \text{and} \quad \lim_{k \to \infty} y_{l_k} = y^*.
\]

If \( \lim_{k \to \infty} \left\| r_{l_k} \right\| = 0 \), then \( x^* \) is a feasible point of the original problem (1.1)–(1.2). Without loss of generality, we suppose

\[
\lim_{k \to \infty} \frac{\lambda_{l_k,i}}{y_{l_k,i} + \lambda_{l_k,i}} = \nu^*_i, \quad i \in \mathcal{I}, \quad \lim_{k \to \infty} \frac{c_{l_k,i} + y_{l_k,i}}{\left\| r_{l_k} \right\|} = b^*.
\]

Then \( b^* \neq 0 \). Since \( c_l + y_l \geq 0 \) for all \( l \geq 0 \), one has \( b^*_l \geq 0 \). By (4.13), \( (1 - \nu^*_l)b^*_l = 0 \), \( i \in \mathcal{I} \). Thus, for \( i \in \mathcal{I} \), \( b^*_l = \nu^*_l b^*_i \), i.e., \( b^*_l = 0 \) as \( \nu^*_l = 0 \), \( \nu^*_l = 1 \) as \( b^*_l \neq 0 \). Note that \( \nu^*_l = 1 \) implies \( y^*_l = 0 \) and \( c^*_l = 0 \). Hence, \( b^*_l c^*_l = 0, \forall i \in \mathcal{I} \). Finally, by (4.12), (4.2) holds. That is, \( x^* \) is a singular stationary of the problem (1.1)–(1.2).

Due to (4.13),

\[
y^*_l (c^*_l + y^*_l) = 0, \quad i \in \mathcal{I}. \tag{4.14}
\]

Since \( c_l + y_l \geq 0 \) for all \( l \geq 0 \), it follows from (4.14) that, for \( i \in \mathcal{I} \), \( c^*_l + y^*_l = \max\{c^*_l, 0\} \). If \( \lim_{k \to \infty} \left\| r_{l_k} \right\| \neq 0 \), then \( x^* \) is an infeasible point of the original problem (1.1)–(1.2). By (4.12) and (4.13), one has \( \sum_{i \in \mathcal{I}} \max\{c^*_l, 0\} \nabla c^*_l = 0 \). Therefore, one has (4.1) immediately.

The preceding theorem shows that, for any given \( \epsilon > 0 \), Algorithm 3.4 will be terminated at either the case \( \beta_l \leq \epsilon \) or the case \( \rho_l \leq \epsilon \).

5. Local convergence

In this section, we prove that, under suitable conditions, the step at \( x_l \) in Algorithm 3.4 can be a superlinearly or quadratically convergent step, no matter whether the sequence \( \{x_l\} \) converges to a KKT point or an infeasible stationary point of the original problem. Thus, the whole algorithm is capable of rapidly converging to a KKT point when the problem is feasible, and, in particular, rapidly converging to an infeasible stationary point when a problem is infeasible.

Let \( \rho_l \to \rho^* \) and \( \beta_l \to \beta^* \) as \( l \to \infty \), \( \nu_l \in \mathbb{R}^m \) be a vector with components \( \nu_{li} = \lambda_{li}/(y_{li} + \lambda_{li}), \quad i \in \mathcal{I} \). For simplicity, we suppose that \( \|\rho_l u_l\| \leq M \) for some constant \( M > 0 \) and for all \( l \geq 0 \). This supposition is reasonable from the global convergence analysis in previous
Lemma 5.3 is nonsingular, where $B^* = H^* + \sum_{i \in I} \nu_i^* \nabla c_i(x^*), \nabla c_i(x^*)^T, [\nu_i^* \nabla c_i(x^*), i \in W^*]$, $\nu_i^* \nabla c_i(x^*)$, $i \in W^*$ is a matrix with $\nu_i^* \nabla c_i(x^*)$ ($i \in W^*$) as its column vectors, $\text{diag}(1 - \nu_i^*, i \in W^*)$ is a diagonal matrix with $(1 - \nu_i^*)$ ($i \in W^*$) as its diagonal entries, $I_{|\mathcal{I} \setminus W^*|}$ is an identity matrix of order $|\mathcal{I} \setminus W^*|$.  

5.1. Rapid convergence to a KKT point. We focus on how the barrier parameter $\beta_l$ is updated at $(x_l, u_l)$ results in that $(d_{xl}, d_{ul})$ is a superlinearly or quadratically convergent step, so that our algorithm is capable of rapidly converging to the KKT point. In addition to Assumption 5.1, we also need the following general conditions.

Assumption 5.2

(1) $\rho^* > 0$ and $\beta^* = 0$;
(2) $u_l \to u^*$ as $l \to \infty$. Thus, $z_l \to z^*$ as $l \to \infty$.

The following index sets are used throughout this subsection: $\mathcal{P}^* = \{i \in \mathcal{I}| c_i(x^*) + \rho^* u_i^* > 0\}$, $\mathcal{Z}^* = \{i \in \mathcal{I}| c_i(x^*) + \rho^* u_i^* = 0\}$, $\mathcal{N}^* = \{i \in \mathcal{I}| c_i(x^*) + \rho^* u_i^* < 0\}$. Assumption 5.2 shows that $(x^*, u^*)$ is a KKT pair, and $c_i(x^*) = 0$ for $i \in \mathcal{P}^* \cup \mathcal{Z}^*$, $c_i(x^*) < 0$ for $i \in \mathcal{N}^*$. It follows from (2.11) and (2.12) that $y_i^* = 0$ and $\lambda_i^* > 0$ for $i \in \mathcal{P}^*$, and $y_i^* > 0$ and $\lambda_i^* = 0$ for $i \in \mathcal{N}^*$. They imply that $\nu_i^* = 1$ for $i \in \mathcal{P}^*$, $\nu_i^* = 0$ for $i \in \mathcal{N}^*$. Hence, $\mathcal{W}^* \subseteq \mathcal{I}^*$ and $\mathcal{E}^* = \mathcal{E}$.

Similar to Lemma 16.1 of [28], one can prove the following result. We omit the proof for brevity.

Lemma 5.3 Suppose that Assumptions 5.1 and 5.2 hold. Then the matrix

$$
\Omega^* = \begin{pmatrix}
B^* & [\nu_i^* \nabla c_i(x^*), i \in W^*] & 0 \\
[\nu_i^* \nabla c_i(x^*), i \in W^*]^T & \text{diag}(1 - \nu_i^*, i \in W^*) & 0 \\
0 & 0 & -I_{|\mathcal{I} \setminus W^*|}
\end{pmatrix}
$$

is nonsingular, where $B^* = H^* + \sum_{i \in \mathcal{I}} \nu_i^* \nabla c_i(x^*) \nabla c_i(x^*)^T$, $[\nu_i^* \nabla c_i(x^*), i \in W^*]$ is a matrix with $\nu_i^* \nabla c_i(x^*)$ ($i \in W^*$) as its column vectors, $\text{diag}(1 - \nu_i^*, i \in W^*)$ is a diagonal matrix with $(1 - \nu_i^*)$ ($i \in W^*$) as its diagonal entries, $I_{|\mathcal{I} \setminus W^*|}$ is an identity matrix of order $|\mathcal{I} \setminus W^*|$.  

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For simplicity of notations, we suppose that $\rho_l = \rho^*$ for all $l \geq 0$ in this subsection. Let $J^* = \lim_{t \to \infty} J_t$, where $J_t = \nabla \phi_{(\beta_t, \rho_t)}(z_t)$. Then $J^* = D^* \Omega^* D^*$, where $D^*$ is a diagonal matrix with $n$ 1s and $m$ $p^*$s. Due to Lemma 5.3, $J^*$ is nonsingular. It follows from the Implicit Function Theorem (p.128 of [29]) that there exists a $\hat{\beta} > 0$ such that the equation $\phi_{(\beta_l, \rho_l)}(z) = 0$ has a unique solution $z^*(\beta_l)$ for all $\beta_l \leq \hat{\beta}$, and there holds

$$\|z^*(\beta_l) - z^*\| \leq M \beta_l < \epsilon,$$  \hspace{1cm} \text{(5.1)}

where $\epsilon > 0$ is small enough and

$$M = \max_{\|z - z^*\| < \epsilon} \left\| \left[ \nabla \phi_{(\beta_l, \rho_l)}(z) \right]^{-1} \frac{\partial \phi_{(\beta_l, \rho_l)}(z)}{\partial \beta} \right\|$$

is a constant independent of $\beta_l$.

The following two lemmas can be attained in a way similar to Lemmas 2.1 and 2.3 in [11]. We will not give their proofs for brevity.

**Lemma 5.4** Suppose that Assumptions 5.1 and 5.2 hold. Then there are sufficiently small scalars $\epsilon > 0$ and $\hat{\beta} > 0$, and positive constants $M_0$ and $L_0$, such that, for all $\beta_l \leq \hat{\beta}$ and $z \in \{z \| z - z^*(\beta_l) \| < \epsilon \}$, $\nabla \phi_{(\beta_l, \rho_l)}(z)$ is invertible,

$$\left\| \left[ \nabla \phi_{(\beta_l, \rho_l)}(z) \right]^{-1} \right\| \leq M_0,$$  \hspace{1cm} \text{(5.2)}

and

$$\left\| \nabla \phi_{(\beta_l, \rho_l)}(z) - \nabla \phi_{(\beta_l, \rho_l)}(z^*(\beta_l)) \right\| \leq L_0 \| z - z^*(\beta_l) \|.$$  \hspace{1cm} \text{(5.3)}

Moreover, for $z \in \{z \| z - z^*(\beta_l) \| < \epsilon \}$ and $\beta_l \leq \hat{\beta}$, one has

$$\left\| \nabla \phi_{(\beta_l, \rho_l)}(z)^T (z - z^*(\beta_l)) - \phi_{(\beta_l, \rho_l)}(z) \right\| \leq L_0 \| z - z^*(\beta_l) \|^2.$$  \hspace{1cm} \text{(5.4)}

**Lemma 5.5** Suppose that Assumptions 5.1 and 5.2 hold. Then there are sufficiently small scalars $\epsilon > 0$ and $\hat{\beta} > 0$, such that for $z \in \{z \| z - z^*(\beta_l) \| < \epsilon \}$ and $\beta_l \leq \hat{\beta}$,

$$\| z - z^*(\beta_l) \| \leq 2M_0 \| \phi_{(\beta_l, \rho_l)}(z) \|, \hspace{1cm} \| \phi_{(\beta_l, \rho_l)}(z) \| \leq 2M_1 \| z - z^*(\beta_l) \|,$$  \hspace{1cm} \text{(5.5)}

where $M_1 = \sup_{\| z - z^*(\beta_l) \| < \epsilon} \| \nabla \phi_{(\beta_l, \rho_l)}(z) \|$.

Using Lemmas 5.4 and 5.5 we can prove the following results.

**Theorem 5.6** Suppose that Assumptions 5.1 and 5.2 hold, and $\beta_l = O(\| z_l - z^* \|^2)$. If $d_l^*$ is computed such that $\| r_l + R_l^T d_l^* \| = O(\| r_l \|^2)$, then

$$\| z_l + d_l - z^* \| = O(\| z_l - z^* \|^2).$$  \hspace{1cm} \text{(5.6)}

That is, $d_l$ is a quadratically convergent step. If, instead, $\beta_l = o(\| z_l - z^* \|), \| r_l + R_l^T d_l^* \| = o(\| r_l \|)$, then $\| z_l + d_l - z^* \| = o(\| z_l - z^* \|)$. That is, the step is superlinear.
Proof. In order to prove the result, we show
\[
\limsup_{k \to \infty} \frac{\|z_l + d_l - z^*\|}{\|z_l - z^*\|^2} \leq \gamma,
\]
where \(\gamma > 0\) is a constant.

Let \(\phi_l = \phi_{(\beta_l, \rho_l)}(z_l)\), \(J_l = \nabla \phi_{(\beta_l, \rho_l)}(z_l)\), \(\hat{d}_l\) be the vector which is different from \(\phi_l\) in that the last \(m\) components are replaced by \(-\rho_l R_l^T d_l\), \(\hat{d}_l\) be the unique solution of the equation \(J_l \hat{d} = -\phi_l\). Then, due to \(\|r_l\| \leq \|\phi_l\|\), by (5.2) and (5.5),
\[
\|d_l - \hat{d}_l\| = \|J_l^{-1} \begin{pmatrix} 0 \\ r_l + R_l^T d_l^c \end{pmatrix}\| = O(\|z_l - z^*(\beta_l)\|^2).
\]

Furthermore, by (5.2) and (5.4),
\[
\|z_l + \hat{d}_l - z^*(\beta_l)\| \leq \|J_l^{-1}\| \|J_l(z_l - z^*(\beta_l)) - \phi_l\| \leq M_0 L_0 \|z_l - z^*(\beta_l)\|^2.
\]

Using (5.1), (5.8) and (5.9), one has
\[
\|z_l + d_l - z^*\| \leq \|z_l + \hat{d}_l - z^*(\beta_l)\| + \|d_l - \hat{d}_l\| + \|z^*(\beta_l) - z^*\| \leq M_0 L_0 \|z_l - z^*(\beta_l)\|^2 + O(\|z_l - z^*(\beta_l)\|^2) + M \beta_l.
\]

If \(\beta_l = O(\|z_l - z^*\|^2)\), that is, \(\beta_l \leq M_2 \|z_l - z^*\|^2\) for some constant \(M_2 > 0\), then
\[
\|z_l - z^*(\beta_l)\| \leq \|z_l - z^*\| + \|z^* - z^*(\beta_l)\| \leq (1 + M M_2 \|z_l - z^*\|) \|z_l - z^*\|.
\]

Thus, (5.7) follows immediately from (5.10).

The result for the case \(\beta_l = o(\|z_l - z^*\|)\) can be proved similarly. \qed

5.2. Rapid convergence to an infeasible stationary point. In this subsection, we consider the rate of convergence to an infeasible stationary point. We prove that \(d_{xl}\) can be a superlinearly or quadratically convergent step provided the penalty parameter \(\rho_l\) is appropriately updated at \(z_l\). The barrier parameter \(\beta_l \in (0, \beta_0]\) can be any finite number.

Other than the general assumptions in Assumption 5.1, we also need some additional conditions for local analysis in this subsection.

Assumption 5.7
(1) \(\rho^* = 0\), \(x^*\) is an infeasible stationary point;
(2) \(\rho_l u_l \to 0\) as \(l \to \infty\).

The above assumption does not prevent \(\|u_l\|\) from tending to \(\infty\). Since the inner algorithm is terminated finitely for every \(l\), one can update \(\rho_l\) appropriately such that Assumption 5.7 (2) holds. With this assumption, \(\|\lambda_l\|\) is bounded.
For simplicity, we set \( \hat{u}_i = \rho u_i \). Let \( \mathcal{P}^* = \{ i \in \mathcal{I} | c_i(x^*) > 0 \} \), and \( \mathcal{N}^* = \{ i \in \mathcal{I} | c_i(x^*) < 0 \} \). In virtue of (2.11) and (2.12), \( \lambda_i^2 > 0 \) and \( y_i^* = 0 \) for \( i \in \mathcal{P}^* \), \( \lambda_i^2 = 0 \) and \( y_i^* > 0 \) for \( i \in \mathcal{N}^* \). They imply \( \nu_i^* = 1 \) for \( i \in \mathcal{P}^* \), and \( \nu_i^* = 0 \) for \( i \in \mathcal{N}^* \). Thus, \( \mathcal{P}^* \subseteq \mathcal{W}^* \subseteq \mathcal{P}^* \cup \mathcal{I}^* \).

Let us consider the system

\[
F_{(\beta, \rho)}(x, \hat{u}) = 0,
\]

where \( F_{(\beta, \rho)}(x, \hat{u}) = \left( \rho \nabla f + \sum_{i \in \mathcal{I}^*} \lambda_i \nabla c_i + \sum_{i \in \mathcal{P}^*} (c_i + \hat{u}_i) \nabla c_i + \hat{y}_i, \quad i \in \mathcal{I}^* \right), \quad \mathcal{I}^* = \mathcal{I} \cap \mathcal{W}^*, \quad \lambda_i = \frac{1}{2} (\sqrt{(c_i + \hat{u}_i)^2 + 4\beta} + c_i + \hat{u}_i), \quad \hat{y}_i = \frac{1}{2} (\sqrt{(c_i + \hat{u}_i)^2 + 4\beta} - c_i - \hat{u}_i) \). Obviously, when \( \rho = 0 \) and \( x^* \) is an infeasible stationary point of problem (1.1)–(1.2), \( (x^*, 0) \) is a solution of (5.11).

Although our algorithm is totally different from those in [5, 8], we can similarly establish the following local convergence results.

**Lemma 5.8** Suppose that Assumptions 5.1 and 5.7 hold. Let \( \hat{u}_{\mathcal{I}^*} = 0 \). Then there exists a constant \( \hat{\rho} > 0 \) such that, for \( \rho \leq \hat{\rho} \), the system (5.11) has a unique solution \( (x^*(\rho), \hat{u}^*(\rho)) \) with \( \hat{u}^*_i(\rho) = 0 \) for \( i \in \mathcal{I} \setminus \mathcal{I}^* \), and

\[
\left\| \begin{pmatrix} x^*(\rho) - x^* \\ \hat{u}^*(\rho) \end{pmatrix} \right\| \leq M\rho
\]

for some positive constant \( M \) independent of \( \rho \).

**Proof.** Let \( \hat{F}_{(\beta, \rho)}(x, \hat{u}_{\mathcal{I}^*}) = F_{(\beta, \rho)}(x, \hat{u}) \) with \( \hat{u}_{\mathcal{I} \setminus \mathcal{I}^*} = 0 \). Note that \( \hat{F}_{(\beta^*, 0)}(x^*, 0) = 0 \) and \( \hat{F}_{(\beta, \rho)}(x, \hat{u}_{\mathcal{I}^*}) \) is continuously differentiable on \((x, \hat{u}_{\mathcal{I}^*})\). Furthermore,

\[
\nabla \hat{F}_{(\beta, \rho)}(x, \hat{u}_{\mathcal{I}^*}) = \begin{pmatrix} G(x, \hat{u}_{\mathcal{I}^*}) \\ [\nu_i^* \nabla c_i(x^*), i \in \mathcal{I}^*] \end{pmatrix}^T - \text{diag}(1 - \nu_i^*, i \in \mathcal{I}^*),
\]

where

\[
G(x, \hat{u}_{\mathcal{I}^*}) = \rho \nabla^2 f + \sum_{i \in \mathcal{I}^*} \lambda_i \nabla^2 c_i + \sum_{i \in \mathcal{P}^*} c_i \nabla^2 c_i + \sum_{i \in \mathcal{I}^*} \hat{u}_i \nabla c_i \nabla c_i^T + \sum_{i \in \mathcal{P}^*} \nabla c_i \nabla c_i^T.
\]

Let \( J_F^* = \lim_{\rho \to 0} \nabla \hat{F}_{(\beta, \rho)}(x, \hat{u}_{\mathcal{I}^*}) \). By items (3) and (4) of Assumption 5.1 and Assumption 5.7, \( J_F^* \) is nonsingular. Thus, the result follows immediately by applying the Implicit Function Theorem (p.128 of [29]).

Corresponding to the mapping \( F_{(\beta, \rho)}(x, \hat{u}) \), we set

\[
\hat{\phi}_{(\beta, \rho)}(x, u) = \begin{pmatrix} \rho \nabla f + \sum_{i \in \mathcal{I}^*} \lambda_i \nabla c_i \\ c_i + \hat{u}_i \end{pmatrix}, \quad i \in \mathcal{I}^*.
\]

**Lemma 5.9** Suppose that Assumptions 5.1 and 5.7 hold. Then, for all sufficiently large \( l \),

\[
\| \hat{\phi}_{(\beta^l, \rho^l)}(x_l, u_l) - F_{(\beta, \rho)}(x_l, \hat{u}_l) \| \leq M\rho_l
\]

for some positive constant \( M \) independent of \( \rho_l \).
Proof. For sufficiently large \( l \), one has \( c_{i l} < 0 \), \( i \in \mathcal{N}^* \) and \( c_{i l} > 0 \), \( i \in \mathcal{P}^* \). Thus, for \( i \in \mathcal{N}^* \),

\[
\lambda_{i l} = \frac{1}{2} \left( \sqrt{(c_{i l} + \rho_l u_{i l})^2 + 4\rho_l \beta_i} - c_{i l} - \rho_l u_{i l} \right)
\]

\[
\leq \frac{\sqrt{M}}{m + \rho_l}, \tag{5.15}
\]

and for \( i \in \mathcal{P}^* \),

\[
(\lambda_{i l} - c_{i l} - \hat{u}_{i l}) = \frac{1}{2} \left( \sqrt{(c_{i l} + \rho_l u_{i l})^2 + 4\rho_l \beta_i} - c_{i l} - \rho_l u_{i l} \right)
\]

\[
\leq \frac{\sqrt{M}}{m + \rho_l}, \tag{5.16}
\]

for some positive constant \( M \) independent of \( \rho_l \). Therefore, for sufficiently large \( l \), there holds

\[
\| \hat{\phi}_{(\beta_l, \rho_l)}(x_l, u_l) - F_{(\beta_l, \rho_l)}(x_l, \hat{u}_l) \| \leq \| \sum_{i \in \mathcal{P}^*} (\lambda_{i l} - c_{i l} - \hat{u}_{i l}) \nabla c_{i l} + \sum_{i \in \mathcal{N}^*} \lambda_{i l} \nabla c_{i l} \| \leq M \rho_l \tag{5.17}
\]

provided \( \| \nabla c_{i l} \| \leq \sqrt{M} \) for \( i \in \mathcal{P}^* \cup \mathcal{N}^* \). Then the result follows immediately from items (1) and (3) of Assumption 5.1.

For simplicity, we denote \( \hat{z}^*(\rho) = (x^*(\rho), \hat{u}_{x*}^*(\rho)), \hat{z} = (x, \hat{u}_{x*}) \), \( \hat{w}^* = (x^*, 0_{\mathbb{Z}_*}) \). The following two lemmas can be obtained in a way similar to Lemmas 5.4 and 5.5 and hence their proofs are neglected here for brevity.

**Lemma 5.10** Suppose that Assumptions 5.1 and 5.7 hold. Let \( \hat{F}_l(\hat{z}) = \hat{F}_{(\beta_l, \rho_l)}(\hat{z}) \). Then there are sufficiently small scalars \( \epsilon > 0 \) and \( \hat{\rho} > 0 \), and positive constants \( M_0 \) and \( L_0 \), such that, for all \( \rho_l \leq \hat{\rho} \) and \( \hat{z} \in \{ \hat{z} \| \hat{z} - \hat{z}^*(\rho_l) \| < \epsilon \} \), \( \nabla \hat{F}_l(\hat{z}) \) is invertible,

\[
\| [\nabla \hat{F}_l(\hat{z})]^{-1} \| \leq M_0, \tag{5.18}
\]

and

\[
\| \nabla \hat{F}_l(\hat{z})^T (\hat{z} - \hat{z}^*(\rho_l)) - \hat{F}_l(\hat{z}) \| \leq L_0 \| \hat{z} - \hat{z}^*(\rho_l) \|^2. \tag{5.19}
\]

**Lemma 5.11** Suppose that Assumptions 5.1 and 5.7 hold. Then there are sufficiently small scalars \( \epsilon > 0 \) and \( \hat{\rho} > 0 \), such that, for all \( \rho_l \leq \hat{\rho} \) and \( \hat{z} \in \{ \hat{z} \| \hat{z} - \hat{z}^*(\rho_l) \| < \epsilon \} \),

\[
\| \hat{F}_l(\hat{z}) \| \leq 2M_1 \| \hat{z} - \hat{z}^*(\rho_l) \|,
\]

where \( M_1 = \sup_{\| \hat{z} - \hat{z}^*(\rho_l) \| < \epsilon} \| \nabla \hat{F}_l(\hat{z}) \| \).
Denote \((r_i)\hat{\beta} = (c_i + y_i, i \in \hat{I}^*)\), \((R_i^T)\hat{\beta} = (R_i^T, i \in \hat{I}^*)\), \(d_i = (d_{x_i}, \rho_i(d_{ul})\hat{\beta}^*)\), where \((d_{x_i}, d_{ul})\) is the solution of QP \((5.8)-(5.9)\). Let \(d_i\) be the unique solution of the equation \(\nabla F_i(\hat{z}_i)^T d = -F_i(\hat{z}_i)\). Now we are ready to provide the following local convergence result when the whole algorithm converges to an infeasible stationary point.

**Theorem 5.12** Suppose that Assumptions \([5.7]\) and \([5.7]\) hold. If \(\rho_i = O(\|x_i - x^*\|)^2\), and \(d_i^*\) is computed such that \(\|(r_i)\hat{\beta} + (R_i^T)\hat{\beta}, d_i^*\| = O((\|(r_i)\hat{\beta}\|)^2)\), then

\[
\|x_i + d_{x_i} - x^*\| = O(\|x_i - x^*\|^2). \tag{5.20}
\]

If, instead, \(\rho_i = o(\|x_i - x^*\|)\), and \(\|(r_i)\hat{\beta} + (R_i^T)\hat{\beta}, d_i^*\| = o((\|(r_i)\hat{\beta}\|, x^*))\), then the convergence is superlinear.

**Proof.** Assume that \(\rho_i = O(\|x_i - x^*\|^2)\). In order to prove the result, we first show that

\[
\limsup_{i \to \infty} \|\hat{z}_i + \hat{d}_i - \hat{z}^*\|/\|\hat{z}_i - \hat{z}^*\|^2 \leq \gamma, \tag{5.21}
\]

where \(\gamma > 0\) is a constant.

Due to \(\|\hat{z}_i + \hat{d}_i - \hat{z}^*(\rho_i)\| \leq \|\nabla \hat{F}_i^T\|^{-1}\|\nabla \hat{F}_i^T(\hat{z}_i - \hat{z}^*(\rho_i)) - \hat{F}_i\|\), by \((5.18)\) and \((5.19)\), one has

\[
\|\hat{z}_i + \hat{d}_i - \hat{z}^*(\rho_i)\| = O((\|\hat{z}_i - \hat{z}^*(\rho_i)\|)^2). \tag{5.22}
\]

Note that

\[
\|\hat{d}_i - \hat{d}_i\| \leq \|\nabla \hat{F}_i^T\|^{-1}(\hat{z}_i - \hat{F}_i)\| + M_0\|(r_i)\hat{\beta}, (R_i^T)\hat{\beta}, d_i^*\|
\]

(by \((5.18)\), Lemmas \([5.9]\) and \([5.11]\) and

\[
\|\hat{z}_i + \hat{d}_i - \hat{z}^*(\rho_i)\| \leq \|\hat{z}_i + \hat{d}_i - \hat{z}^*(\rho_i)\| + \|\hat{d}_i - \hat{d}_i\| + \|\hat{z}^*(\rho_i) - \hat{z}^*\|,
\]

it follows from \((5.12)\) and \((5.22)\) that

\[
\|\hat{z}_i + \hat{d}_i - \hat{z}^*\| = O((\|\hat{z}_i - \hat{z}^*\|)^2). \tag{5.23}
\]

Therefore, \((5.21)\) is obtained.

Since \(\|\hat{z}_i + \hat{d}_i - \hat{z}^*\|^2 = \|x_i + d_{x_i} - x^*\|^2 + \rho_i^2(\|(u_i)\hat{\beta}^* + (d_{ul})\hat{\beta}^*\|^2)\) and \(\|\hat{z}_i - \hat{z}^*\|^2 = \|x_i - x^*\|^2 + \rho_i^2(\|(u_i)\hat{\beta}^*\|^2)\), if \(\rho_i = O((x_i - x^*\|)^2)\), then

\[
\|x_i + d_{x_i} - x^*\| = O((x_i - x^*\|)^2). \tag{5.24}
\]

One can similarly prove the result for the case of \(\rho_i = o((x_i - x^*\|)\). \qed

6. Numerical experiments

We implemented our algorithm in MATLAB (version R2008a). The numerical tests were conducted on a Lenovo laptop with the LINUX operating system (Fedora 11). Two kinds of test
problems from the literature were solved. Firstly, we solved some simple but hard problems, which may be infeasible, or feasible but LICQ and MFCQ failing to hold at the solution, or a well-posed one but some class of interior-point methods was demonstrated not to be globally convergent. Secondly, we applied our algorithm to solve some standard test problems from the CUTE collection [3].

The initial parameters were chosen as follows: \( \beta_0 = 0.1, \delta = 0.5, \sigma = 10^{-4}, \) and \( \epsilon = 10^{-8}. \) The initial penalty selection was \( \rho_0 = \min\{100, \max(1, \|\max(0, c(x_0))\|/\|f(x_0)\|)\}, \) which depended on the initial point. Simply, we took \( H_0 = \rho I \) (where \( I \in \mathbb{R}^{n \times n} \) is the identity matrix), \( H_k \) was updated similarly by the well-known Powell’s damped BFGS update formula (for example, see [3, 28]). For CUTE test problems, the initial iterate was provided by the test problem itself. We used the standard initial point \( x_0 \) for the simple test problems, and set \( u_0 = 0 \) for the problem with inequality constraints. Algorithm 3.4 was also extended to solve problems with both inequality and equality constraints, where we only needed to change the items associated with constraints including the constraints of subproblems and the merit function.

The vector \( d_k^0 \) was derived by Algorithm 6.1 of [24]. For solving the QP subproblem (3.8)–(3.9), we first computed the null-space matrix \( W \) of \( R_k^T \) by the MATLAB null-space routine, then computed the solution of the QP by formulating the reduced Hessian explicitly and using the MATLAB routine of bi-conjugate gradients method with preconditioner generated by the sparse incomplete Cholesky-Infinity factorization, which was presented by Zhang [39] for avoiding numerically zero pivots in the sparse incomplete Cholesky factorization.

In the inner algorithm, \( \xi_{k+1} \) is further updated such that \( \|\xi_{k+1}g_k\|_\infty \leq 0.1, \) where \( g_k \) is the multiplier of the QP (see Lemma 4.5). We require \( \xi_{k+1}^T (\max(\max(\rho_1 u_k, 0)))^{1.1} \leq 1 \) so that \( \xi_{k+1}^T \rho_1 u_k \rightarrow 0 \) as \( \xi_{k+1} \rightarrow 0. \) Functions \( \theta_1 \) and \( \theta_2 \) are defined as \( \theta_1(\beta) = 10 \beta \) and \( \theta_2(\rho) = \rho, \) respectively. In order to obtain rapid convergence, we update \( \beta_l \) to \( \beta_{l+1} = \min(0.1 \beta_l, \|\phi(\beta_l, \rho_l)(z_{k+1})\|_{\infty}^{-5}) \) when we need to reduce \( \beta_l. \) If \( \rho_l \) needs to be updated, \( \rho_l \) is reduced to

\[
\rho_{l+1} = \min\{\xi_{k+1}^T \rho_l, \|\psi(\beta_l, \xi_{k+1} \rho_l)(z_{k+1})\|_\infty^2, (\|\lambda(z_{k+1}; \beta_l, \rho_l)\|_\infty/\rho_l)^{-2}\}
\]

provided \( \|r_k\| - \|r_k + R_k^T d_k\| < 0.01 \|r_k\|, \) otherwise \( \rho_{l+1} = \xi_{k+1} \rho_l, \) where

\[
\psi(\beta_l, \xi_{k+1} \rho_l)(z_{k+1}) = \xi_{k+1} \rho_l \nabla f_{k+1} + \sum_{i \in I} \lambda_i(z_{k+1}; \beta_l, \xi_{k+1} \rho_l) \nabla c_{k+1,i}.
\]

We use \( \|\psi(\beta_l, \xi_{k+1} \rho_l)(z_{k+1})\|_\infty \) to measure the convergence to the infeasible stationary point, which is the same as [3]. It is easy to note that \( \|\psi(\beta_l, \xi_{k+1} \rho_l)(z_{k+1})\|_\infty \rightarrow 0 \) as \( \rho_l \rightarrow 0 \) due to Theorem 4.10. The whole algorithm was terminated as either \( \beta_l < \epsilon \) or \( \rho_l < \epsilon, \) or the total number of iterations (that is, the number of solving QP (3.8)–(3.9)) was greater than 1000.

6.1. Some simple but hard problems. In this subsection, we applied our algorithm to solve four simple but hard examples taken from the literature. The results for these examples are reported respectively in Tables 1, 6 where Tables 1, 3 and 5 show the numerical results derived from the outer algorithm of Algorithm 3.4 for four test problems, in which the numbers in column \( l \) are the order numbers of outer iterations, \( f_l = f(x_l), v_l = \max\{0, c(x_l)\}, \|\phi_l\|_\infty = \)
\[ \| \phi(\beta_l, \rho_l)(z_{k+1}) \|_\infty, \quad \| \psi_l \|_\infty = \| \psi(\beta_l, \xi_{k+1} \rho_l)(z_{k+1}) \|_\infty, \]

\[ k \] is the number of inner iterations needed for changing parameters. It is noted that Algorithm 3.4 needs more inner iterations before termination for (TP3) and (TP4). In order to have an insight into the local convergence of our algorithm, we also report some counterparts on some inner iterations generated by the inner algorithm of Algorithm 3.4 in Tables 4 and 6, in which the last \( n \) columns are the components of iterates.

The first two examples are infeasible problems presented by Byrd, Curtis and Nocedal [8]. The following is their isolated problem:

\[
\begin{align*}
\text{min} \quad & x_1 + x_2 \\
\text{subject to} \quad & x_1^2 - x_2 + 1 \leq 0, \\
& x_1^2 + x_2 + 1 \leq 0, \\
& -x_1 + x_2^2 + 1 \leq 0, \\
& x_1 + x_2^2 + 1 \leq 0.
\end{align*}
\]

The standard initial point is \( x_0 = (3, 2) \), its solution \( x^* = (0, 0) \) is a strict minimizer of the infeasibility measure [4.4]. The algorithm presented in [8] found this point. Our algorithm terminates at an approximate point to it. Table 1 shows that, when \( \rho_3 = 3.1595e-06 \) is reduced to \( \rho_4 = 1.0000e-09 \), rapid convergence emerged since \( \| \psi_3 \|_\infty \) is reduced superlinearly.

The second example is the inactive problem in [8]:

\[
\begin{align*}
\text{min} \quad & x_1 \\
\text{subject to} \quad & \frac{1}{2}(x_1 + x_2^2 + 1) \leq 0, \\
& -x_1 + x_2^2 \leq 0, \\
& x_1 - x_2^2 \leq 0.
\end{align*}
\]

The given initial point is \( x_0 = (-20, 10) \). The point \( x^* = (0, 0) \) derived by [8] was an infeasible stationary point with \( \| \max(0, c^*) \| = 0.5 \). Algorithm 3.4 terminates at a point approximating an infeasible stationary point \( x^* = (-0.2000, 0.0000) \). Similar to that for (TP1), Table 2 indicates that, when \( \rho_6 = 2.6880e-06 \) is reduced to \( \rho_7 = 1.0000e-09 \), rapid convergence emerged since \( \| \psi_6 \|_\infty \) is reduced superlinearly.
The third example is a counterexample presented by Wächter and Biegler [33] and further discussed by Byrd, Marazzi and Nocedal [12]:

\[
\begin{align*}
\min_x & \quad x_1 \\
\text{subject to} & \quad x_1^2 - x_2 - 1 = 0, \\
& \quad x_1 - x_3 - 2 = 0, \\
& \quad x_2 \geq 0, \quad x_3 \geq 0.
\end{align*}
\]

The initial point is \( x_0 = (-4, 1, 1) \). This problem has a unique global minimizer \((2, 3, 0)\), at which gradients of the active constraints are linearly independent, and MFCQ holds. However, [33] showed that many line search interior-point methods could not find the minimizer, even failed to find a feasible solution. Our algorithm terminates at the approximate solution \( x^* = (2.0000, 3.0000, 0.0000) \) in 16 iterations (including all numbers of inner iterations). Table 3 illustrates that we still have rapid convergence for outer iterations since, when \( \beta_6 = 0.1 \) is reduced in turn to \( \beta_7 = 6.5080 e^{-07} \) and \( \beta_8 = 1.0000 e^{-09} \), the KKT measure \( \|\phi_6\|_\infty = 0.0562 \) is reduced superlinearly in turn to \( \|\phi_7\|_\infty = 7.5098 e^{-05} \) and \( \|\phi_8\|_\infty = 3.6687 e^{-10} \) after 3 and 5 inner iterations.

In order to identify that the rapid convergence for outer iterations is not an accumulation of linear convergence of inner iterations, we report the numerical results on the inner iterations when \( \beta_7 = 6.5080 e^{-07} \) and \( \beta_8 = 9.8465 e^{-05} \) in Table 4. It is noted that, since \( \beta_6 = 0.1 \) is replaced by \( \beta_7 = 6.5080 e^{-07} \), the initial KKT measure \( \|\phi_1\|_\infty = 0.0056 \) in Table 3 becomes much larger than \( \|\phi_7\|_\infty = 7.5098 e^{-05} \) in Table 3 but we can still observe the superlinear convergence in Table 4.

The last example in this subsection is a standard test problem taken from [23] Problem 13:

\[
\begin{align*}
\min \quad (x_1 - 2)^2 + x_2^2 \\
\text{subject to} & \quad (1 - x_1)^3 - x_2 \geq 0, \\
& \quad x_1 \geq 0, \quad x_2 \geq 0.
\end{align*}
\]
Table 3: Output for test problem (TP3)

| $l$ | $f_l$ | $v_l$ | $\|\phi_l\|_\infty$ | $\|\psi_l\|_\infty$ | $\beta_l$ | $\rho_l$ | $k$ |
|-----|-------|-------|----------------------|----------------------|----------|--------|-----|
| 0   | -4    | 15.6525 | 54.7837              | 0.7296               | 0.1000   | 3.9131 | -   |
| 1   | -2.3137 | 8.0327 | 20.3266              | 0.3014               | 0.1000   | 1.2792 | 1   |
| 2   | -1.6196 | 4.6562 | 5.3301              | 0.1682               | 0.1000   | 0.6396 | 1   |
| 3   | -1.3615 | 5.1399 | 2.1352              | 0.0650               | 0.1000   | 0.3198 | 1   |
| 4   | -1.2150 | 5.7561 | 0.8127              | 0.1901               | 0.1000   | 0.1599 | 1   |
| 5   | -1.2243 | 5.5496 | 0.4558              | 0.1474               | 0.1000   | 0.0217 | 2   |
| 6   | -1.2355 | 5.5172 | 0.0562              | 0.0099               | 0.1000   | 9.8465e-05 | 1 |
| 7   | 1.3024  | 1.1379 | 7.5098e-05         | 3.9305e-05           | 6.5080e-07  | 9.8465e-05 | 3 |
| 8   | 2.0000  | 6.0330e-06 | 3.6687e-10 | 2.3390e-11 | 1.0000e-09  | 9.8465e-05 | 5 |
| 9   | 2.0000  | 6.0156e-06 | 3.6515e-10 | 3.8200e-15 | - | - | - |

Table 4: The inner iterations corresponding to $l = 8$ for test problem (TP3)

| $k$ | $f_k$  | $v_k$  | $\|\phi_k\|_\infty$ | $\|\psi_k\|_\infty$ | $x_{k1}$ | $x_{k2}$ | $x_{k3}$ |
|-----|--------|--------|----------------------|----------------------|----------|----------|----------|
| 1   | 1.2025 | 0.7976 | 0.0056               | 0.0056               | 1.2025   | 0.4359   | -0.7975  |
| 2   | 1.9999 | 0.6359 | 0.0040               | 0.0040               | 1.9999   | 2.3636   | -0.0002  |
| 3   | 2.0000 | 4.3817e-06 | 7.1859e-04 | 2.0000   | 3.0000   | -0.0000  |
| 4   | 2.0000 | 4.0547e-06 | 1.3612e-07 | 2.0000   | 3.0000   | -0.0000  |
| 5   | 2.0000 | 4.0791e-06 | 3.6687e-10 | 2.3390e-11 | 2.0000 | 3.0000   | -0.0000  |

Table 5: Output for test problem (TP4)

| $l$ | $f_l$ | $v_l$ | $\|\phi_l\|_\infty$ | $\|\psi_l\|_\infty$ | $\beta_l$ | $\rho_l$ | $k$ |
|-----|-------|-------|----------------------|----------------------|----------|--------|-----|
| 0   | 20    | 2.8284 | 9.9557              | 9.9557               | 0.1000   | 1      | -   |
| 1   | 0.2305 | 0.4167 | 0.8900              | 0.7008               | 0.0100   | 1      | 4   |
| 2   | 0.1652 | 0.1687 | 0.1631              | 0.0771               | 0.0100   | 0.3268 | 4   |
| 3   | 0.1690 | 0.1630 | 0.0503              | 0.0022               | 0.0100   | 4.7328e-06 | 1 |
| 4   | 0.8561 | 2.9531e-04 | 3.1379e-06 | 3.1379e-06 | 0.0100   | 1.0000e-09 | 14 |
| 5   | 0.9028 | 1.2372e-04 | 9.3463e-08 | 9.3463e-08 | -   | -   | -   |

Table 6: The last 4 inner iterations corresponding to $l = 4$ for test problem (TP4)

| $k$ | $f_k$ | $v_k$ | $\|\phi_k\|_\infty$ | $\|\psi_k\|_\infty$ | $x_{k1}$ | $x_{k2}$ |
|-----|-------|-------|----------------------|----------------------|----------|----------|
| 11  | 0.8500 | 5.7136e-04 | 5.6703e-04 | 5.6703e-04 | 1.0780 | 0.0001  |
| 12  | 0.8548 | 3.0434e-04 | 1.2222e-05 | 1.2222e-05 | 1.0754 | -0.0002 |
| 13  | 0.8556 | 2.9845e-04 | 6.2125e-06 | 6.2125e-06 | 1.0750 | -0.0002 |
| 14  | 0.8561 | 2.9531e-04 | 3.1379e-06 | 3.1379e-06 | 1.0747 | -0.0002 |
The standard initial point \( x_0 = (-2, -2) \) is an infeasible point. This problem was not solved in \([30, ?]\), but the algorithms in \([10, 31]\) got its approximate solution. Its optimal solution \( x^* = (1, 0) \) is not a KKT point but is a singular stationary point, at which the gradients of active constraints are linearly dependent. Numerical results show that Algorithm 3.4 terminates at an approximate point to the solution, but it does not suggest rapid convergence for either inner or outer iterations. In fact, we still do not have any theoretical result on rapid convergence to a singular stationary point of nonlinear programs in the literature.

In summary, the preceding numerical results not only demonstrate our global convergence results on Algorithm 3.4 for infeasible, well-posed and degenerate nonlinear programs, but also demonstrate our locally rapid convergence results on Algorithm 3.4 with convergence to the KKT point of a feasible nonlinear program and to an infeasible stationary point of a nonlinear program which is infeasible.

6.2. The test problems from CUTE collection. A set of 55 small- and medium-size test problems \((n \leq 100 \text{ and } m \leq 200)\) with general inequality constraints from the CUTE collection \([4]\) were solved. These problems were selected since they had actual numbers of problem variables and general inequality constraints (i.e., not only bound constraints). Besides general inequality constraints, some test problems may also have equality constraints.

The numerical results are reported in Tables 7 and 8, where the columns “n” and “m” are the numbers of variables and constraints of test problems, respectively. While the columns of \( f \) and \( v \) show, respectively, the values of objective functions and the infinite norms of constraint violations at the terminating points, “iter” represents the total number of iterations needed for obtaining those values, and \( \|\phi\|_\infty \) is the infinite norm of the residual of the system \((2.13)\) at the terminating point. The last two columns “\( N_f \)” and “\( N_g \)” of Tables 7 and 8 are, respectively, the numbers of evaluations of functions and gradients needed by the algorithm.

The preliminary results show that our algorithm can be efficient for most of test problems, where there are only 3 problems for which the total number of iterations are out of the restrictions, and for only 3 problems, the algorithm does not find an approximate feasible solution. There are 8 problems, for which the residual of system \((2.13)\) is greater than \(10^{-6}\) (where there are 5 problems greater than \(10^{-4}\)). Since our MATLAB implementation uses MATLAB routines simply, it has much work to do including improving the computations of subproblems using some more advanced techniques. It is believed that further improvements can be achieved with those techniques. We agree that, in order to further examine the efficiency of our method, it is important to make some comparison with IPOPT and some other well performed algorithm such as that in \([15]\) in the future.

7. Conclusion

Upon great success in solving large-scale linear programming problems, the interior-point approach has effectively been extended to solving general convex programming (such as semidefinite and cone programming) and nonconvex programming problems. The research on interior-point
Table 7: Results for CUTE problems using approximate Hessians, part 1.

| Problem     | n  | m  | $f$  | $v$ | iter | $\|\phi\|_{\infty}$ | $N_f$ | $N_g$ |
|-------------|----|----|------|-----|------|----------------------|-------|-------|
| CB2         | 3  | 3  | 1.9522 | 0   | 10   | 8.8741e-08           | 11    | 11    |
| CB3         | 3  | 3  | 2.0000 | 0   | 10   | 1.7477e-08           | 11    | 11    |
| CHACONN1    | 3  | 3  | 1.9522 | 0   | 12   | 4.8926e-07           | 13    | 13    |
| CHACONN2    | 3  | 3  | 2.0000 | 0   | 662  | 1.1460e-11           | 8383  | 663   |
| CONGIMZ     | 3  | 5  | 28.0562 | 0   | 29   | 3.3364e-09           | 72    | 30    |
| DEMYMALO    | 3  | 3  | -3.0000 | 0   | 17   | 9.3215e-09           | 25    | 18    |
| DIFIGR      | 7  | 4  | 680.6301 | 0   | 46   | 3.4528e-07           | 88    | 47    |
| EXPFITA     | 5  | 22 | 0.0014 | 0   | 35   | 1.0288e-10           | 64    | 36    |
| EXPFITB     | 5  | 102| 0.3704 | 0   | 46   | 2.8463e-07           | 68    | 47    |
| GIGOMEZ1    | 3  | 3  | -3.0000 | 0   | 13   | 1.6223e-09           | 16    | 14    |
| GIGOMEZ2    | 3  | 3  | 1.9522 | 0   | 11   | 2.0767e-09           | 12    | 12    |
| GIGOMEZ3    | 3  | 3  | 2.0000 | 0   | 14   | 1.0296e-09           | 15    | 15    |
| GOFFIN      | 51 | 50 | 3.9795e-05 | 0   | 248  | 0.0031               | 485   | 249   |
| HAIFAS      | 13 | 9  | -0.4500 | 0   | 25   | 1.3212e-07           | 35    | 26    |
| HALDMADS    | 6  | 42 | 0.4163 | 0   | 15   | 1.1128e-10           | 34    | 16    |
| HS10        | 2  | 1  | -1.0000 | 0   | 15   | 1.5957e-09           | 18    | 16    |
| HS11        | 2  | 1  | -8.4985 | 0   | 8    | 1.7862e-07           | 9     | 9     |
| HS12        | 2  | 1  | -30.0000 | 0   | 19   | 5.1527e-10           | 28    | 20    |
| HS14        | 2  | 2  | 1.3935 | 2.2204e-16 | 8 | 6.2753e-09 | 9 | 9 |
| HS22        | 2  | 2  | 1.0000 | 0   | 13   | 7.1423e-09           | 27    | 14    |
| HS29        | 3  | 1  | -22.6266 | 0   | 50   | 0.0013               | 414   | 51    |
| HS43        | 4  | 3  | -4.4000 | 0   | 28   | 2.6610e-08           | 56    | 29    |
| HS88        | 2  | 1  | 1.3625 | 1.4230e-07 | 21 | 4.5241e-08 | 40 | 22 |
| HS89        | 3  | 1  | 1.3627 | 0   | 27   | 1.1073e-16           | 33    | 28    |
| HS90        | 4  | 1  | 1.3627 | 0   | 23   | 1.4424e-16           | 40    | 24    |
| HS91        | 5  | 1  | 1.3627 | 0   | 27   | 1.0819e-17           | 40    | 28    |
| HS92        | 6  | 1  | 1.3627 | 0   | 26   | 1.9847e-16           | 38    | 27    |
| HS100       | 7  | 4  | 680.6301 | 0   | 46   | 3.4592e-07           | 88    | 47    |
| HS100MOD    | 7  | 4  | 678.6796 | 0   | 43   | 2.0270e-08           | 69    | 44    |
| HS113       | 10 | 8  | 24.3062 | 0   | 54   | 2.6954e-05           | 150   | 55    |
| KIWCRESC    | 3  | 2  | 2.0000e-09 | 0   | 16   | 3.7972e-07           | 20    | 17    |
| MADSSEN     | 3  | 6  | 0.6164 | 0   | 16   | 5.4372e-05           | 19    | 17    |
| MAKELA1     | 3  | 2  | -1.4019 | 0   | 11   | 5.5732e-10           | 35    | 12    |
| MAKELA2     | 3  | 3  | 7.2000 | 0   | 14   | 2.2852e-07           | 15    | 15    |
| MAKELA3     | 21 | 20 | 2.0000e-08 | 0   | 293  | 4.0113e-11           | 549   | 294   |
| MAKELA4     | 21 | 40 | 11.9543 | 0   | 27   | 8.3854e-11           | 42    | 28    |
| MIFFLN1     | 3  | 2  | -1.0000 | 0   | 10   | 4.7956e-09           | 13    | 11    |
| MIFFLN2     | 3  | 2  | -0.9993 | 0   | 47   | 7.1572e-11           | 227   | 48    |
| MINMAXBD    | 5  | 20 | 138.0075 | 0   | 1001 | 0.0168               | 5863  | 1002  |
| MINMAXRB    | 3  | 4  | 6.7208e-08 | 2.3231e-08 | 72 | 0.3366 | 108 | 73   |
| PENTAGON    | 6  | 15 | 7.8592e-04 | 0   | 14   | 4.0903e-12           | 27    | 15    |
| POLAK1      | 3  | 2  | 2.7183 | 0   | 1001 | 9.1480e-07           | 11219 | 1002  |
| POLAK3      | 12 | 10 | 5.9410 | 0   | 96   | 9.9531e-10           | 291   | 97    |
Table 8: Results for CUTE problems using approximate Hessians, part 2.

| Problem | n  | m  | f          | v  | iter | $\|\phi\|_\infty$ | $N_f$ | $N_g$ |
|---------|----|----|------------|----|------|----------------|------|------|
| POLAK5  | 3  | 2  | 50.0139    | 0  | 1001 | 2.0053e-04   | 5973 | 1002 |
| POLAK6  | 5  | 4  | -43.8595   | 0  | 128  | 9.9592e-10   | 335  | 129  |
| ROSENMMX | 5  | 4  | -44.0000   | 0  | 31   | 2.5387e-06   | 79   | 32   |
| S268    | 5  | 5  | 2.3837e-06 | 0  | 19   | 4.1462e-12   | 68   | 20   |
| SPIRAL  | 3  | 2  | 1.9984e-09 | 0  | 146  | 4.3318e-07   | 319  | 147  |
| TFI1    | 3  | 101| 21.6437    | 0  | 34   | 7.7655e-09   | 60   | 35   |
| TFI2    | 3  | 101| 0.6490     | 0  | 209  | 7.4972e-10   | 292  | 210  |
| TFI3    | 3  | 101| 4.3012     | 0  | 147  | 2.2122e-14   | 554  | 148  |
| VANDERM1| 100| 199| 0          | 0.7532| 5 | 1.3314e-11 | 8   | 6   |
| VANDERM2| 100| 199| 0          | 0.7532| 5 | 1.3314e-11 | 8   | 6   |
| VANDERM3| 100| 199| 0          | 0.7770| 7 | 2.1786e-11 | 10  | 8   |
| WOMFLET | 3  | 3  | 0.0031     | 0  | 295  | 9.9111e-10   | 2331 | 296  |

methods for nonlinear programs has been one of focuses of optimization area in recent years. Based on a distinctive two-parameter primal-dual nonlinear system, which corresponds to the KKT point and the infeasible stationary point of nonlinear programs, respectively, as one of two parameters vanishes, we have presented a new interior-point method for nonlinear programs in this paper. Our method always produces interior-point iterates without truncation of the step. Not only the method can be globally convergent and locally quick convergent to KKT points when the problem is feasible, but also it can globally converge to an infeasible stationary point and rapidly detect the infeasibility of the solved problem when the problem is infeasible. A possible future topic of the subsequent research is to consider similar methods in solving linear programming or semidefinite programming problems.

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