A strong contradiction in the multi-layer Hele-Shaw model.

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Abstract. The Saffman-Taylor instability occurs when a less viscous fluid is displacing a more viscous one in a rectangular Hele-Shaw cell. A surface tension on the interface between the two fluids is improving the stability. The multi-layer Hele-Shaw model, consisting of \( N \) intermediate fluids with constant viscosities, was studied in some previous papers and very low growth constants were obtained for large \( N \). We prove that this model leads us to a significant instability, even if \( N \) is very large. The maximum value of growth constants can not decrease under a certain value, not depending on the surface tensions on the interfaces. This contradiction with the Saffman-Taylor result makes us have some doubts concerning the correctness of multi-layer model.

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1. Introduction

We consider a Stokes flow in a Hele-Shaw cell: the narrow space between two parallel plates, first studied in [18]. The velocities averages are verifying the Darcy’s law for the flow in a porous medium with the permeability \( (b^2/12) \), where \( b \) is the distance between the plates - see [3], [20].

Saffman and Taylor [23] proven the well-known result: the interface between two immiscible fluids is unstable when the displacing fluid is less viscous. Moreover, the growth rate is unbounded with respect to the wave numbers if the surface tension on the interface is missing. A surface tension on the interface is limiting the range of unstable disturbances - see the formula (11) in [23]. The fingering phenomenon (which appears for unstable disturbances) and the selection problem in Hele-Shaw displacements are studied in a large number of papers - see [19], [24], [32] and references therein. In [29] was studied the case when the surface tension is very small. Some singular effects due to the zero-surface-tension problem are studied in [30].

The optimization of displacements in porous media were studied in [1], [2], [4], [12], [27]. An intermediate fluid with variable viscosity in a middle layer between the displacing fluids can minimize the Saffman-Taylor instability if surface tensions exist - see experimental and numerical results in [15], [16], [17], [25], [26], [31]. In [9], [10], [11] are given theoretical results concerning the linear stability of such three-layer Hele-Shaw flow. Some exact formulas of the growth constants were given, for variable and constant intermediate viscosities.

The Hele-Shaw displacement with \( N \) intermediate layers (the multi-layer Hele-Shaw model) was studied in [5], [6], [7], [8]. Only upper bounds of the growth rates were obtained. In the case of intermediate viscosities with positive jumps in the flow direction, in [5] was proved that the
corresponding growth rates tend to zero when the number of the intermediate layers is very large and the surface tensions satisfy some conditions.

In this paper we study the multi-layer Hele-Shaw model with constant $N$ intermediate viscosities. A new upper estimate of the growth rates is obtained, for a bounded range of the wavenumbers and large $N$. We show that the maximum value of the growth constants is not depending on the surface tensions on the interfaces and can not be less than the difference between the viscosities of the initial displacing fluids. Therefore the stability can not be improved, even if the surface tensions on the interfaces are very large. This contradiction with the Saffman-Taylor result makes us have some doubts concerning the correctness of multi-layer model.

An important new element is given by the terms of Dirac type, appearing in the estimates of the growth rates. These terms are related with the derivatives of the viscosity across the interfaces.

The paper is laid out as follows. In section 2 we describe the three-layer Hele-Shaw model with variable intermediate viscosity, first studied in [16]. In section 3 we get the formula of the growth rates corresponding to a fourth-layer Hele-Shaw flow with constant intermediate viscosities. This result is used in section 4, for a model with $N$ intermediate layers with constant viscosities. We conclude in section 5.

2. The three-layer Hele-Shaw model with variable intermediate viscosity.

The three-layer Hele-Shaw flow with variable intermediate viscosity was first described in [16], [17]. The cell is parallel with the $xOy$ plane. An intermediate region between the initial immiscible fluids is considered, where a given amount of polymer-solute exists. The adsorption, dispersion and diffusion of the solute in the equivalent porous medium are neglected. The intermediate viscosity can be considered as a powers series with respect to the concentration $C$ of the polymer-solute - see [13], [15]. For a dilute solute, the viscosity is a linear expression with respect to $C$, then is invertible. We consider $\mu = 12\nu/b^2$, where $\nu$ is the viscosity on the intermediate region. The continuity equation for the solute gives us the “continuity” equation $\mu_t + u\mu_x + v\mu_y = 0$, where $(u, v)$ are the velocities and the indices $t, x, y$ denote the partial derivatives with respect to time and spatial variables.

During the displacement process, the initial sharp interfaces change over time and the finger phenomenon appears. We consider small enough time intervals, to avoid large deformations of the initial interfaces.

Mungan [22] used an intermediate polymer-solute with an exponentially-decreasing viscosity (from the front interface) and obtained an almost stable flow. The displacements with variable viscosity in Hele-Shaw cells and porous media are studied in [21], [28]. On the page 3 of [14] is considered a linear viscosity profile in a porous medium.

In this paper, the displacing and displaced fluids are denoted with the lower indices $w, o$. 

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Suppose the intermediate region is the interval \( x \in (Ut - Q, Ut) \), moving with the constant velocity \( U \) far upstream. We have three incompressible fluids with viscosities \( \mu_W \) (displacing fluid), \( \mu \) (intermediate layer) and \( \mu_O \) (displaced fluid). In a large number of papers, the flow equations, quite similar with Darcy’s law for flow in a porous medium, are written in simpler form

\[
\begin{align*}
p_x &= -\mu_d u; \quad p_y = -\mu_d v; \quad p_z = 0; \quad u_x + v_y = 0; \\
\mu_d &= \mu_W, \quad x < Ut - Q; \quad \mu_d = \mu, \quad x \in (Ut - Q, Ut); \quad \mu_d = \mu_O, \quad x > Ut.
\end{align*}
\]

(1)

The viscosities \( \nu_W, \nu, \nu_O \) are given by

\[
\begin{align*}
\mu_W &= 12 \nu_W / b^2, \\
\mu &= 12 \nu / b^2, \\
\mu_O &= 12 \nu_O / b^2,
\end{align*}
\]

(3)

and the permeability of the equivalent porous medium is \( b^2/12 \). The velocities appearing in (2) are the average of the real (effective) fluid velocities - see [3], [18], [20].

The basic velocity and interfaces are \( u = U, \; v = 0; \; x = Ut - Q, \; x = Ut \).

On the interfaces we consider the Laplace’s law: the pressure jump is given by the surface tension multiplied with the curvature of the interface. The component \( u \) of the velocity is continuous and the interface is a material one.

The basic interfaces are straight lines, then the basic pressure \( P \) is continuous (but his gradient is not) and

\[
\begin{align*}
P_x &= -\mu_d U, \quad P_y = 0.
\end{align*}
\]

(4)

We use the “continuity” equation for \( \mu \) (see the end of the first paragraph of this section), then the basic (unknown) \( \mu \) in the middle layer verifies the equation

\[
\begin{align*}
\mu_t + U \mu_x &= 0.
\end{align*}
\]

(5)

We introduce the moving reference frame \( \bar{x} = x - Ut, \; \tau = t \). The equation (5) leads to \( \mu_\tau = 0 \), then \( \mu = \mu(\bar{x}) \). The middle region in the moving reference frame is the segment \( -Q < \bar{x} < 0 \).

However, we still use the notation \( x, \; t \) instead of \( \bar{x}, \tau \).

The perturbations \( u', v', p', \mu' \) of the basic velocity, pressure and viscosity are governed by the system (see [16])

\[
\begin{align*}
p'_x &= -\mu u' - \mu' U, \\
p'_y &= -\mu u', \\
u'_x + v'_y &= 0, \\
\mu'_t + u' \mu_x &= 0.
\end{align*}
\]

(6)\( (7)\)\( (8)\)

A Fourier decomposition for the perturbation \( u' \) is used:

\[
u'(x, y, t) = f(x)[\cos(ky) + \sin(ky)]e^{\sigma t}, \quad k \geq 0,
\]

(9)
where \( f(x), \sigma, k \) are the amplitude, the growth constant and the wave numbers.

The velocity along the axis \( Ox \) is continuous, then the amplitude \( f(x) \) is continuous. From (6) - (9) we get the Fourier decompositions for the perturbations \( v', p', \mu' \):

\[
v' = (1/k)f_x[-\sin(ky) + \cos(ky)]e^{\sigma t},
\]

\[
p' = (\mu/k^2)f_x[-\cos(ky) - \sin(ky)]e^{\sigma t},
\]

\[
\mu' = (-1/\sigma)\mu_x f[\cos(ky) + \sin(ky)]e^{\sigma t}.
\]  

(10)

The cross derivation of the relations (6), (7) leads us to

\[
\mu u_y' + \mu_y'U = \mu_x v' + \mu v_x' .
\]  

(11)

From (9), (10), (11) we get the equation which governs the amplitude \( f \):

\[
-(\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} U k^2 f \mu_x , \quad \forall x \notin \{-Q,0\}.
\]  

(12)

Outside the intermediate region we have constant viscosities, then (12) becomes

\[-f_{xx} + k^2 f = 0, \quad x \notin (-Q,0)\]

and in the far field we have

\[
f(x) = \begin{cases} 
  f(-Q)e^{k(x+Q)}, & \forall x \leq -Q; \\
  f(0)e^{-kx}, & \forall x \geq 0.
\end{cases}
\]  

(13)

Suppose that a viscosity jump exists at a point \( a \). The perturbed interface near \( a \) is denoted by \( \eta(a,y,t) \). In the first approximation we have \( \eta_t = u \), therefore

\[
\eta(a,y,t) = (1/\sigma)f(a)[\cos(ky) + \sin(ky)]e^{\sigma t}.
\]  

(14)

The right and left limit values of the pressure in the point \( a \) are denoted by \( p^+(a), \ p^-(a) \). We use \( P \) in the point \( a \), the Taylor first order expansion of \( P \) near \( a \) and \( p'(a) \) given by (10). From (4) it follows \( P_x^+(a) = -\mu^+(a)U \) and \( P_x^-(a) = -\mu^-(a)U \), then we get

\[
p^+(a) = P^+(a) + P_x^+(a)\eta + p^+(a) = \\
\]

\[
P^+(a) - \mu^+(a)\left(\frac{U f(a)}{\sigma} + \frac{f_x^+(a)}{k^2}\right)[\cos(ky) + \sin(ky)]e^{\sigma t},
\]  

(15)

\[
p^-(a) = P^-(a) - \mu^-(a)\left(\frac{U f(a)}{\sigma} + \frac{f_x^-(a)}{k^2}\right)[\cos(ky) + \sin(ky)]e^{\sigma t},
\]  

(16)

The Laplace’s law is \( p^+(a) - p^-(a) = T(a)\eta_{yy} \), where \( T(a) \) is the surface tension and \( \eta_{yy} \) is the approximate value of the curvature of the perturbed interface. As \( P^-(a) = P^+(a) \) (see the line before (14)), from the jump relation equations (15) - (16) we get

\[
-\mu^+(a)\left[\frac{U f(a)}{\sigma} + \frac{f_x^+(a)}{k^2}\right] + \mu^-(a)\left[\frac{U f(a)}{\sigma} + \frac{f_x^-(a)}{k^2}\right] = -\frac{T(a)}{\sigma} f(a)k^2.
\]  

(17)
The growth constant for three-layer case is obtained as follows. We multiply with $f$ in the amplitude equation (12), we integrate on $(-Q, 0)$ and obtain

$$-\int_{-Q}^{0} (\mu f_x f)_x + \int_{-Q}^{0} \mu f_x^2 + k^2 \int_{-Q}^{0} \mu f^2 = \frac{k^2 U}{\sigma} \int_{-Q}^{0} \mu_x f^2.$$ 

We have not jumps of $\mu$ inside the intermediate region, then we get

$$\mu^+(-Q)f^+_x(-Q)f(-Q) - \mu^-(0)f^-_x(0)f(0) + \int_{-Q}^{0} \mu f_x^2 + k^2 \int_{-Q}^{0} \mu f^2 = \int_{-Q}^{0} \mu_x f^2. \quad (18)$$

From the relations (13) we have

$$f^-_x(-Q) = kf_1 := kf(-Q), \quad f^+_x(0) = -kf_0 := kf(0). \quad (19)$$

Recall $\mu^-(Q) = \mu_W$, $\mu^+(0) = \mu_O$, then from (17), (18), (19) it follows

$$\sigma = \frac{S_0 f_0^2 + S_1 f_1^2 + k^2 U \int_{-Q}^{0} \mu_x f^2}{\mu_O k f_0^2 + \mu_W k f_1^2 + I},$$

$$S_0 = k^2 U [\mu]_0 = k^4 T_0, \quad S_1 = k^2 U [\mu]_1 = k^4 T_1, \quad I = \int_{-Q}^{0} [\mu f_x^2 + k^2 \mu f^2], \quad (20)$$

where $T_0, T_1$ are the surface tensions in $x = 0, x = -Q$ and

$$[\mu]_0 = (\mu^+ - \mu^-)_0 = \mu_O - \mu^+(-Q),$$

$$[\mu]_1 = (\mu^+ - \mu^-)_1 = \mu^+(0) - \mu_W. \quad (21)$$

Remark 1. From (17) we can recover the Saffman - Taylor formula

$$\sigma^{ST} = \frac{kU(\mu_O - \mu_W) - T(a)k^3}{\mu_O + \mu_W}. \quad (22)$$

Indeed, we have

$$\mu^+(a) = \mu_O; \quad \mu^-(a) = \mu_W;$$

$$f(x) = f(a)e^{k(x-a)}, \quad x \leq a \Rightarrow f^-_x(a) = kf(a);$$

$$f(x) = f(a)e^{-k(x-a)}, \quad x \geq a \Rightarrow f^+_x(a) = -kf(a). \quad \square$$

Remark 2. It is possible to inject polymer-solutes with constant concentrations $c_1, c_2, \ldots, c_N$ during some the time intervals $t_1, t_2, \ldots, t_N$. We obtain a steady flow of $N$ thin layers of immiscible fluids with constant viscosities $\nu_i, \quad i = 1, 2, \ldots, N$. This is the multi-layer model studied in [5], [7], [8], [6]. \square
3. The fourth-layer Hele-Shaw model with constant intermediate viscosities.

Consider two intermediate layers \((-Q, -Q/2)\) and \((-Q/2, 0)\) with constant \(\mu\):

\[
\mu(x) = \mu_W, \quad x < -Q, \quad \mu(x) = \mu_O, \quad x > 0,
\]

\[
\mu(x) = \mu_2, \quad x \in (-Q, -Q/2), \quad \mu(x) = \mu_1, \quad x \in (-Q/2, 0).
\]

The basic interfaces are \(x_0 = 0, x_1 = -Q/2, x_2 = -Q\). This time, the amplitude equation is

\[
-(\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} U k^2 f \mu_x, \quad \forall x \notin \{-Q, -Q/2, 0\}. \tag{23}
\]

Inside the intermediary region, \(\mu\) is a Heaviside function. The derivative \(\mu_x\) on the interface \(x = -Q/2\) is a Dirac distribution, then

\[
\int_{-Q}^{0} \mu_x f^2 = f^2(x_1)(\mu_2 - \mu_1). \tag{24}
\]

The term \([24]\) is not appearing in \([7]\). We multiply with \(f\) the above relation and integrate on \((-Q, 0)\), then it follows

\[
-\int_{-Q}^{-Q/2} (\mu f_x f)_x - \int_{-Q/2}^{0} (\mu f_x f)_x + \int_{-Q}^{0} \mu f^2_x + k^2 \int_{-Q}^{0} \mu f^2 = \frac{k^2 U}{\sigma} f^2(x_1)(\mu_2 - \mu_1).
\]

We use the notations \((FG)(x) := F(x)G(x), \quad f_i := f(x_i)\) and get

\[
(\mu^+ f_x^+ f)(-Q) - (\mu^- f_x^- f)(-Q/2) + (\mu^+ f_x^+ f)(-Q/2) - (\mu^- f_x^- f)(0) +
\]

\[
+ I_1 + I_2 = \frac{k^2 U}{\sigma} f^2(x_1)(\mu_2 - \mu_1),
\]

\[
I_i = \mu_i \int_{x_i}^{x_{i+1}} (f_x^2 + k^2 f^2)dx, \quad i = 1, 2. \tag{25}
\]

Recall

\[
\mu^-(x_2) = \mu_W, \quad \mu^+(x_0) = \mu_O, \quad (f_1)^-(Q) = kf_2, \quad (f_1)^+(0) = -kf_0.
\]

The jump relations \([17]\) in the points \(a = x_2, x_1, x_0\) are

\[
-(\mu^+ f_x^+)(x_2) + \mu_W kf_2 + f_2 \frac{U k^2}{\sigma} [\mu_W - \mu_2] = -\frac{T_2}{\sigma} f_2 k^4,
\]

\[
-(\mu^+ f_x^+)(x_1) + (\mu^- f_x^-)(x_1) = f_1 \frac{U k^2}{\sigma} [\mu_1 - \mu_2] - \frac{T_1}{\sigma} f_1 k^4,
\]

\[
\mu_0 k f_0 + (\mu^- f_x^-)(x_0) + f_0 \frac{U k^2}{\sigma} [\mu_1 - \mu_O] = -\frac{T_0}{\sigma} f_0 k^4. \tag{26}
\]

In \([25]\) - \([26]\) we use the viscosities \(\nu_W, \nu_O\) given in \([3]\), then \(\nu_i = b^2 \mu_i/12\) and it follows

\[
\sigma = \frac{S_0 f_0^2 + S_1 f_1^2 + S_2 f_2^2 + k^2 U f_1^2 (\nu_2 - \nu_1)}{\nu_O k f_0^2 + I_1 + I_2 + \nu_W k f_1^2},
\]

\(0 < \sigma < \frac{U k^2}{\sigma}\),
\[ S_i = Uk^2[v]_i - k^2 T_i b^2 / 12, \quad i = 0, 1, 2, \]

\[ [v]_2 = \nu_2 - \nu_W, \quad [v]_1 = \nu_1 - \nu_2, \quad [v]_0 = \nu_0 - \nu_1, \]

\[ I_1 = \nu_1 \int_{x_1}^{x_0} \left( f_x^2 + k^2 f^2 \right) dx, \quad I_2 = \nu_2 \int_{x_2}^{x_1} \left( f_x^2 + k^2 f^2 \right) dx. \] (27)

The following dimensionless quantities denoted by \( \prime \) are introduced:

\[ x' = x/Q, \quad y' = y/Q, \quad f' = f/U, \quad \epsilon = b/Q \approx 10^{-3}, \]

\[ u' = u/U, \quad v' = v/U, \quad T' = T/(\mu W U) \]

\[ \nu' = \nu/\mu W, \quad k' = kQ, \quad \sigma' = \sigma/(U/Q). \] (28)

In the rest of this paper we will omit the \( \prime \). The dimensionless intermediate region is the interval \((-1, 0)\). The relation (27) gives us the dimensionless growth rate, denoted by \( \sigma_2 \) (recall \( \mu W = 1 \)):

\[ \sigma_2 = \frac{\sum_{i=0}^{i=2} \left( k^2 [v]_i - \epsilon^2 k^4 T_i / 12 \right) f_i^2 + k^2 f_i^2 [v]_1}{k \nu O f_0^2 + I_1 + I_2 + k \cdot 1 \cdot f_2^2} \]

\[ I_1 = \nu_1 \int_{x_1}^{x_0} \left( f_x^2 + k^2 f^2 \right) dx, \quad I_2 = \nu_2 \int_{x_2}^{x_1} \left( f_x^2 + k^2 f^2 \right) dx, \]

\[ x_2 = -1, x_1 = -1/2, x_0 = 0. \] (29)

The factor \( \epsilon^2 \) in front of the surface tensions \( T_i \) is very important for the stability analysis. In [3] is given a similar formula, but with dimensional quantities, then without the parameter \( \epsilon \).

The dimensionless Saffman-Taylor growth rate and its maximal value are obtained from (22) and (28):

\[ \sigma_D = \frac{k(\nu_O - 1) - k^3 T e^2 / 12}{\nu_O + 1} \leq \sigma_{DM} = \frac{4(\nu_O - 1)^{3/2}}{3(\nu_O + 1) \epsilon \sqrt{T}}. \] (30)

4. The N-layer Hele-Shaw model with constant intermediate viscosities.

We divide the intermediate region in \( N \) small layers with equal length \((1/N)\). The interfaces are \( x_0 = x_1 = \ldots = -i/N, \quad i = 1, \ldots, N \). In the layer \((x_i, x_{i-1})\) we consider the constant viscosity \( \nu(x) = \nu_i \) such that \( \nu_0 = \nu_O, \quad \nu_{N+1} = 1 \) (recall \( \nu_W = 1 \)) and

\[ \nu_i = \nu_0 - i(\nu_0 - 1)/(N + 1), \quad (\nu^+ - \nu^-)_i = (\nu_0 - 1)/(N + 1). \] (31)

The amplitude equations are

\[ -(\nu f_x)_x + k^2 \nu f = \frac{1}{\sigma} U k^2 f \nu_x, \quad \forall x \notin \{-j/N\}, \quad j = 0, 1, \ldots, N. \] (32)

The growth constants, denoted by \( \sigma_N \), are obtained just like formula (29) in section 3:

\[ \sigma_N = \frac{\sum_{i=0}^{i=N} \left( k^2 (\nu^+ - \nu^-)_i - k^4 T_i \epsilon^2 / 12 \right) f_i^2 + \sum_{i=1}^{i=N-1} k^2 (\nu^+ - \nu^-)_i f_i f_i^2}{k \nu O f_0^2 + \sum_{i=1}^{i=N} I_i + k f_N^2}. \] (33)
\[ I_i = \int_{x_i}^{x_{(i-1)}} \nu_i(f_x^2 + k^2 f^2), \quad f_i = f(x_i). \]

Our growth constants are real. An instability result is obtained if only one growth constant is positive. It is much more difficult to prove a stability result: all growth constants must be negative. In this paper we consider some particular eigenfunctions \( f \) and analyse the corresponding growth rates given by (33). We prove that even if the number of intermediate layers is very large, the maximum value of the growth constants is not so small (in a bounded range of \( k \)) and is not depending on the surface tensions.

4.1 An upper bund of \( \sigma_N \). From (33) we see that \( \sigma_N \) is negative beyond a finite value of \( k \), then the “dangerous” wave numbers \( k \) are bounded.

**Lemma 1.** If \( k \in [0, 1] \) and \( N = 1/(c - a) \) is large enough s.t. \( k^2(c-a)^2 \approx 0 \), then we have

\[ f(x) = e^{kx} \quad \forall x \in (a, c) \Rightarrow J(a, c) := \int_a^c (f_x^2 + k^2 f^2) \approx (k^2/N)[f^2(a) + f^2(c)]. \] (34)

**Proof.** As \( f(x) = e^{kx} \) we get \( f_x^2 + k^2 f^2 = (f_x f)_x \) and

\[ J(a, c) = (f_x f)(c) - (f_x f)(a) = k[f^2(c) - f^2(a)]. \] (35)

We use the trapezoidal rule for \( F \in C^2(a, c) \):

\[ \int_a^c F(x) dx = \frac{c-a}{2} [F(a) + F(c)] - R, \quad R = \frac{(c-a)^3}{12} F''(\chi), \quad \chi \in (a, c). \]

Consider \( F(x) = f^2(x) = e^{2kx} \), then \( F''(x) = 4k^2 e^{2kx} \) and for bounded \( k \) and small enough \( (c-a) \) we get an arbitrary small \( R \). As \( I = 2k^2 \int_a^c f^2 \), we use (35) and we have to prove

\[ k[f^2(c) - f^2(a)] \approx k^2(c-a)[f^2(a) + f^2(c)]. \] (36)

For this, we neglect \( k^2(c-a)^2 \) and use the first order Taylor expansion of \( f(x) = e^{kx} \):

\[ f(c) \approx f(a) + k f(a)(c-a), \quad f^2(c) \approx f^2(a) + 2k f^2(a)(c-a), \]

\[ f^2(c) - f^2(a) \approx 2k f^2(a)(c-a), \quad f^2(c) + f^2(a) \approx 2 f^2(a) + 2k f^2(a)(c-a). \] (37)

The approximation (36) is equivalent with

\[ k \cdot 2k(c-a) \approx k^2(c-a)[2 + 2k(c-a)] \]

which holds because \( k^3(c-a)^2 \leq k^2(c-a)^2 \approx 0. \]

We use Lemma 1 for computing the integrals \( I_i \), with \( (a, c) = (x_i, x_{i-1}) \), \( i = 1, 2, ..., N \).

We consider \( k \in [0, 1] \) and \( N = 10^4 \), then \( k/N \leq 10^{-4} \) and the approximation (36) holds. From (33) we get

\[ \sigma_N \approx \frac{G_0 f_0^2 + \sum_{i=1}^{N-1} G_i f_i^2 + G_N f_N^2}{kvF f_N^2 + kvO f_0^2 + \sum_{i=1}^{i=N} (k^2/N) \nu_i(f_{i-1}^2 + f_i^2)}, \]
\[ G_i = 2k^2(\nu^+ - \nu^-)_i - k^4T_i\epsilon^2/12, \quad i = 1, 2, \ldots, N - 1, \]
\[ G_j = k^2(\nu^+ - \nu^-)_j - k^4T_j\epsilon^2/12, \quad j = 0, N. \]  

(38)

An important new element appearing in this formula is based on the Dirac distributions corresponding to the derivative \( \nu_x \) on the interfaces. As a consequence, the “middle” terms \( G_i, 1 \leq i \leq N - 1 \), are larger, compared with \( G_0, G_N \).

We recall the well-known inequality

\[ B_i, x_i > 0 \Rightarrow \min \{ \frac{A_i}{B_i} \} \leq \frac{\sum_{i=0}^{i=M} A_ix_i}{\sum_{i=0}^{i=M} B_ix_i} \leq \max \{ \frac{A_i}{B_i} \}, \]  

(39)

From (38) and (39) it follows

\[ \sigma_N \leq \frac{2k^2(\nu^+ - \nu^-)_{N-1} - k^4T_{\text{min}}\epsilon^2/12}{k^2(\nu_N + \nu_{N-1})/N}. \]

As a consequence, in the range \( k \in [0, 1] \), the upper bound of \( \sigma_N \) is a polynomial of order 2, and not of order 3 as in [5] and [23]. We have

\[ \nu_N + \nu_{N-1} = (2N - 1 + 3\nu_O)/(N + 1), \]

then from the last estimate we get

\[ \sigma_N < 2(\nu_O - 1)\frac{N}{2N - 1 + 3\nu_O} - k^2T_{\text{min}}\epsilon^2/12N(N + 1)/\nu_O). \]  

(40)

\[ \sigma_M \leq \sigma_{NM} = 2(\nu_O - 1)\frac{N}{2N - 1 + 3\nu_O}. \]  

(41)

The estimate (40) holds for \( k \in [0, 1] \) and large enough values of \( N \), but his maximum value (41) is not depending on the surface tension \( T_{\text{min}} \). Here is a strong contradiction with the Saffman-Taylor result: it is very natural to have an improvement of stability when the surface tensions are large enough. From this point of view, the present multi-layer model is wrong. Moreover, the maximum value (41) can not be arbitrary small for large \( N \).

For \( k \in (0, 1), N = 10^4, T = 1/\epsilon^2 \), the relations (30) and (41) give us

\[ \sigma_{NM} \approx (\nu_O - 1) = 99; \quad \sigma_{DM} = \frac{4(\nu_O - 1)^{3/2}}{3(\nu_O + 1)} \approx 13.3. \]  

(42)

Then the Saffman-Taylor formula gives us a more stable displacement for a large enough surface tension. From this point of view, the multi-layer Hee-Shaw model with constant intermediate viscosities is useless. Future research is needed to see the cause of the strong contradiction with the Saffman-Taylor result.
5. Conclusions

The interface between two Newtonian immiscible fluids in a rectangular Hele-Shaw cell is unstable when the displacing fluid is less viscous. A surface tension on the interface can improve the stability - see the formula (22).

An intermediate fluid with a variable viscosity between the displacing fluids can minimize the Saffman-Taylor instability when the surface tensions are different from zero - see the papers [15], [16], [17], [25], [26], [31].

A continuous function can be approximated by a step function. For this reason, the multi-layer Hele-Shaw model, consisting of $N$ intermediate fluids with constant viscosities was studied in [5], [6], [7], [8]. Upper bounds of the growth rates were obtained in these papers. If all surface tensions verify some conditions, an arbitrary small (positive) upper bound of the growth rates can be obtained, if $N$ is large enough.

In this paper we study the multi-layer Hele-Shaw displacements in rectangular cells. The three-layer case is considered in section 2. We get a formula of the growth rates in the fourth-layer case with constant intermediate viscosity - see section 3. We use this result for $N$ intermediate constant-viscosity layers and get a new upper bound for the growth constants in section 4, by using the dimensionless quantities (28). We prove that even if the number of intermediate layers is very large, the maximum value of the growth constants is not so small - see (40).

The most important result of our paper is the upper bound (41) of the growth constant (which holds only for bounded $k$ and large $N$). This result is based on three new elements: the terms due to the Dirac distributions $\mu_x$ on interfaces, the dimensionless quantities and the new estimate of the growth rates given in section 4.

The maximum value of (41) is not depending on the surface tensions and can not be arbitrary small for a large enough number of intermediate layers. Then we have a significant instability, even if the surface tensions on the interfaces are very large. This contradiction with the Saffman-Taylor formula raises questions about the validity of the multi-layer model.

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