Dynamics of test particles in thin-shell wormhole spacetimes

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Abstract
Geodesic motion in traversable Schwarzschild and Kerr thin-shell wormholes constructed by the cut-and-paste method introduced by Visser (1989 \textit{Nucl. Phys. B} \textbf{328} 203; 1995 \textit{Wormholes: from Einstein to Hawking} (Woodbury, MN: American Institute of Physics)) is studied. The orbits are calculated exactly in terms of elliptic functions and visualized with the help of embedding diagrams.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The idea to travel in the twinkling of an eye from one region of the Universe to another, no matter how far, or even to travel in time has been fascinating people since many years. Wormholes are believed to make such dreams possible. The first solution providing such a sought for connection between distant locations was discovered by Einstein and Rosen [1], the Einstein–Rosen bridge. Wheeler gave it the name wormhole [2]. But unfortunately this first wormhole was not traversable [3].

For Sagan’s novel ‘Contact’ Morris and Thorne devised a wormhole which could in principle be traversed by humans [4]. Visser [5, 6] provided further examples of traversable wormholes. In particular, he constructed such wormholes by surgically manipulating the Schwarzschild spacetime so that no event horizon was present. In both cases, the price to pay is the need for an exotic form of matter that must be present at the throat of such wormholes. Moreover, such wormholes should be stable if they are traversable. Visser [6] stressed that certain equations of state would lead to stable wormholes. He shows that a traveler can prevent from interacting with the exotic matter and feels no tidal forces during a journey through a thin-shell wormhole. A detailed study of the stability of these traversable wormholes against radial perturbations can be found in [7] where the conditions on the wormhole mass, throat
radius and equation of state parameter defined by a particular model of the exotic matter are derived. Linearized radial stability of charged thin-shell wormholes is studied for example in [21] and generalization to the presence of a cosmological constant is considered in [23]. Stability of the general thin-shell wormholes with spherical symmetry and its dependence on a suitable exotic material on the wormhole throat are investigated in [22, 24, 25].

The material comprising the throat of the wormhole was addressed in many scientific works and is still an active field of research. For example, a phantom scalar field was discussed in [8–12], ‘tachyon matter’ as a source term in the field equations with a positive cosmological constant was studied in [13], or the wormhole geometry with a Chaplygin gas in the exotic equation of state was explored in [14–16]. An interesting discussion of a perfect fluid and an anisotropic fluid as candidates for the exotic matter was given in [17]. Since the metric is continuous on a throat but not its first derivatives, the energy conditions on the matter supporting the wormhole can be derived by considering the Riemann curvature on both sides of the throat. Thus, from the conditions on the surface energy density, surface pressures and angular momentum density follow that a perfect fluid can only support a spherically symmetric thin-shell wormhole [5, 6], while an anisotropic fluid can maintain the geometry of a rotating thin-shell wormhole [17]. But the matter in both cases has negative surface energy density violating the weak energy conditions. Estimates on the amount of exotic matter necessary for the traversability were given in [18]. A combined model comprising ordinary and quintessential matter can support a traversable wormhole in Einstein–Maxwell gravity as shown in [19]. Wormholes in the framework of the Brans–Dicke gravity were constructed in [20]. Further references can be found in the book of Visser [26] and in the overview by Lobo [27]. The appearance of wormholes in string theory was investigated in [28–33].

Stable spherical wormholes which do not need any form of exotic matter for their existence were recently obtained numerically in dilatonic Einstein–Gauss–Bonnet theory in four spacetime dimensions [34, 35]. Besides their stability the authors studied their domain of existence, they investigated their geodesics, determining the possible types of trajectories, and performed a study of the acceleration and tidal forces that a traveler crossing such a wormhole would feel. Further numerically obtained configurations of wormholes were explored in [36–38], where wormholes were filled by a perfect fluid (ordinary matter) and a phantom scalar field. This model was applied to describe stars as well as neutron stars with a nontrivial topology. Traversable wormholes without violation of energy conditions in the geometries of charged shells are constructed analytically in [39]. However, these wormhole spacetimes have closed time-like curves.

A comprehensive investigation of the geodesics in the Morris–Thorne wormhole spacetime was carried out in [40], where for visualization of the trajectories embedding diagrams were constructed. Moreover, gravitational lensing and illumination calculations were addressed. Further properties of the propagation of particles and fields in static spherically symmetric wormhole spacetimes were studied in [41]. Some aspects of the electromagnetic fields and charged particle motion around slowly rotating magnetized wormholes are discussed in [42].

Rotating wormholes are a natural generalization of the initially studied, static spherically symmetric wormholes, that are interesting from a physical point of view. For instance, one can construct a thin-shell rotating wormhole [17] by the same cut-and-paste method [6] used for the thin-shell static wormhole. On the other hand, generalizations of the Morris–Thorne wormhole [4] to rotating wormholes were presented by Teo [43] and Kuhfittig [44]. It was shown that these wormholes are traversable (tidal forces are not larger than on Earth) and due to the rotation and a possible ergoregion (for not too wide a throat) an extraction of energy by the Penrose process is possible. Slowly rotating wormholes with a phantom scalar field
were studied in [45, 46]. The electromagnetic field generation around rotating wormholes surrounded by charged particles was studied in [47]. In [48] a class of rotating and magnetized wormholes with Einstein–Maxwell and phantom fields was explored.

The conversion of a wormhole into a time machine, the problems appearing thereby and accompanying physical effects were discussed in [49–51]. The associated problems of causality violations were addressed in [52, 53].

In this paper we study the motion of massive and massless particles in thin-shell Schwarzschild and Kerr-wormhole spacetimes, constructed by the cut-and-paste method [6, 17]. Thus the paper contains two main sections. Section 2 explores the dynamics of null and time-like geodesics for static spherically symmetric thin-shell wormholes and section 3 the geodesics for rotating thin-shell wormholes. We also study the influence of the ergoregion, existing for sufficiently thin throats, on the properties of the orbits. The trajectories are given by exact analytical solutions of the geodesic equations in terms of the Jacobi and Weierstrass elliptic functions.

2. Schwarzschild wormhole

2.1. The geodesic equation

We start with the Schwarzschild metric in the form (see e.g. [54, 57, 58])

$$\text{d}s^2 = -\left(1 - \frac{1}{r}\right)\text{d}t^2 + \left(1 - \frac{1}{r}\right)^{-1}\text{d}r^2 + r^2(\sin^2 \theta \text{d}\varphi^2 + \text{d}\theta^2),$$

(1)

where the radial coordinate $r$ is normalized to $2M$ and is dimensionless.

For free particles moving along geodesics in the equatorial plane the Lagrangian $L$ takes the form [54, 59]

$$2L = -\left(1 - \frac{1}{r}\right)i^2 + \left(1 - \frac{1}{r}\right)^{-1}i^2 + r^2\dot{\varphi}^2 \quad \text{with} \quad 2L = -\delta,$$

(2)

where $\dot{x}^\alpha$ denotes the differentiation with respect to the affine parameter $\lambda$ and $\delta = 0$ for null and $\delta = 1$ for time-like geodesics.

With the conserved energy $E = -\frac{\partial L}{\partial i} = (1 - \frac{1}{r})\dot{i}$ and conserved angular momentum $L = \frac{\partial L}{\partial \varphi} = r^2\dot{\varphi}$, a free test particle moves along a geodesic defined by the differential equation

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{L^2} \left(E^2 - \left(\delta + \frac{L^2}{r^2}\right)(1 - \frac{1}{r})\right),$$

(3)

where the quantities $E$ and $L$ are dimensionless.

To solve the differential equation (3) we refer to the theory of elliptic functions [60] and make few substitutions in order to reduce the problem to the standard form. A first substitution $r = u^{-1}$ reduces (3) to the form

$$\left(\frac{du}{d\varphi}\right)^2 = u^3 - u^2 + \frac{\delta}{L^2} u + \frac{E^2 - \delta}{L^2}.$$

(4)

The differential (4) is elliptic of the first kind and can be integrated by Jacobi elliptic functions. To reduce it to the standard form we make two transformations. The first one $u = 4v + \frac{1}{4}$ transforms (4) into the Weierstrass form

$$\left(\frac{dv}{d\varphi}\right)^2 = 4\prod_{i=1}^3(v - v_i) = 4v^3 - g_2v - g_3,$$

(5)
where \( v_i, i = 1, 2, 3 \), are the roots of the cubic polynomial in (5) and satisfy the condition
\[
\sum_{i=1}^{3} v_i = 0, \quad g_3 = 4v_1v_2v_3 \text{ and } g_2 = 4v_3^2 - 4v_1v_2.
\]

The second transformation \( v = (v_2 - v_1)s^2 + v_1 \) turns (5) into
\[
\frac{1}{v_2 - v_1} \left( \frac{ds}{d\phi} \right)^2 = (1 - s^2)(1 - m^2s^2),
\]
where \( m^2 = (v_3 - v_1)(v_2 - v_1)^{-1} \), and \( m \) is the modulus of the Jacobian elliptic functions [60].

With Jacobi’s elliptic function \( sn(x) \) we find from (6) \( s \) as a function of \( \psi \):
\[
s = sn((\psi - \psi')\sqrt{v_2 - v_1}),
\]
where
\[
\psi' = \varphi_0 + \frac{1}{\sqrt{v_2 - v_1}} \int_{\varphi_0}^{\psi} \frac{ds}{\sqrt{(1 - s^2)(1 - m^2s^2)}}
\]
is a constant which can be expressed in terms of the periods of \( sn(x) \), and \( \varphi_0 \) is the initial value of \( \psi \).

We then get the function \( r(\psi) \) [54, 60]:
\[
r = \left( \frac{4(v_2 - v_1)}{sn^2((\psi - \psi')\sqrt{v_2 - v_1})} + 4v_1 + \frac{1}{3} \right)^{-1}.
\]

2.2. Construction of the wormhole and embedding

To obtain the Schwarzschild thin-shell wormhole we follow the ‘surgery’ method of Visser [5, 6]. This method was also applied by Kashgarin and Sushkov [17] for the construction of the Kerr thin-shell wormhole (see section 3.2). The procedure is as follows: we cut the Schwarzschild spacetime outside the event horizon \( h = 1 \) at some value of the radial coordinate \( b_0 \), where \( b_0 \) is now the throat parameter and \( b_0 > h \). Gluing two copies of the asymptotically flat region \( r \geq b_0 \), which are then connected by a wormhole with the throat hypersurface \( \Sigma \), we obtain a geodesically complete manifold. The coordinate \( |l| = |r - b_0| \) such that \( l \in (-\infty, \infty) \) then covers the whole spacetime (the upper and the lower universe). The throat of the wormhole is located at \( l = 0 \).

To visualize the topology of the wormhole we consider a two-dimensional hypersurface \( (t = \text{const, } \theta = \pi/2) \) [40]. Its inner geometry yields
\[
d\Sigma_0 = \left( 1 - \frac{1}{r} \right)^{-1} dr^2 + r^2 d\psi^2.
\]
This two-dimensional hypersurface can be embedded into the Euclidean space given by
\[
d\Sigma_E = dr^2 + r^2 d\psi^2 + dz^2 = \left( 1 + \left( \frac{dz}{dr} \right)^2 \right) dr^2 + r^2 d\psi^2
\]
in the cylindrical coordinates \( (r, \psi, z) \).

Comparing the coefficients of \( dr^2 \) in (10) and (11) we find that the shape of the embedding diagram in the Euclidean space is given by
\[
z(r) = 2\sqrt{r - 1} - 2\sqrt{b_0} - 1.
\]

It is visualized in figure 2 in coordinates \( (x, y, z) \), where \( x = r \cos(\psi) \) and \( y = r \sin(\psi) \). For the embedding diagram we choose the initial value of \( r \) to be \( b_0 \) and \( \psi \in [0, 2\pi] \). (In the
next section we discuss the orbits in the thin-shell Schwarzschild-wormhole spacetime and show an example for the embedding diagram of an orbit.)

2.3. Geodesics

Let us introduce an effective potential in equation (3)

\[
\left( \frac{dr}{d\varphi} \right)^2 = \frac{r^4}{L^2} (E^2 - V_{\text{eff}}),
\]

i.e. [54]

\[
V_{\text{eff}} = \left( \delta + \frac{L^2}{r^2} \right) \left( 1 - \frac{1}{r} \right),
\]

where intersections of the function \(E^2\) and \(V_{\text{eff}}\) specify the turning points of the motion.

Figure 1 shows examples of the effective potential (14) for massive test particles with \(\delta = 1\) for different values of \(b_0\). Here the coordinate \(l\) was used. The regions which do not satisfy the condition \(E^2 \geq V_{\text{eff}}\) are forbidden and colored gray.

The following types of orbits exist in the Schwarzschild-wormhole spacetime for massive test particles.

**Type A**: two-world escape (TWE) orbit which connect two parts of the Universe. These orbits would be ending at the singularity in usual Schwarzschild spacetime.

**Type B**: two-world bound (TWB) orbits when a test particle moves on a bound orbit (BO) stretching from one into another part of the Universe. This orbit connects the upper and lower parts of the universe similar to the orbit of type A.

**Type C**: this type contains two possible orbits: TWB orbit and escape orbit (EO). EOs exist in both parts of the Universe. For growing values of the throat parameter \(b_0\) this type reduces to \(C_0\) containing only EOs. See figure 1(b).

**Type D**: this type includes TWB orbit and BO. BOs exist in both parts of the Universe. Here again \(D\) reduces to type \(D_0\) for growing values of the throat parameter \(b_0\). See figure 1(d).

For large \(b_0\)-values only orbits of type \(B\) and \(A\) survive (figure 1(c)). For very large values—only type \(A\) is possible.

Figure 1(d) shows a set of potential plots for large \(L^2 = 9\), when no planetary BOs are possible (also in the Schwarzschild black hole spacetime), and different values of the throat parameter \(b_0\).

Geodesics of massive test particles described by the solution (9) are shown in figure 2. In this figure, the TWB orbit (a) and the BO (b) correspond to type D. The TWB orbit (c) and the EO (d) are of type C, and the TWE orbits (e) and (f) are of type A. The orbits are embedded into the three-dimensional space with the coordinates \((x, y, z)\), where \(x = r \cos(\varphi)\), \(y = r \sin(\varphi)\), and \(r\) and \(z\) are given by (9) and (12) correspondingly.

**Geodesics for \(\delta = 0\)**. Similar to the Schwarzschild black hole spacetime no planetary BOs exist in the Schwarzschild thin-shell wormhole spacetime. The orbits of type \(A\) (TWE), \(B\) (TWB) and \(C\) (TWB and EO) are possible. We show some examples in figure 3.
3. Kerr wormhole

3.1. The geodesic equations

We start with the Kerr metric in Boyer–Lindquist coordinates \[54\]

\[
dr^2 = - \left(1 - \frac{r}{\rho^2}\right) dt^2 + \frac{r^2}{\rho^2} dr^2 + \rho^2 d\vartheta^2 - \frac{2ar}{\rho^2} \sin^2 \vartheta d\varphi\ dr

+ \left(r^2 + \alpha^2 + \frac{a^2 r}{\rho^2} \sin^2 \vartheta\right) \sin^2 \vartheta \ d\varphi^2,
\]

(15)
Figure 2. Orbits of a test particle with $\delta = 1$ for different values of angular momentum and energy in the thin-shell Schwarzschild-wormhole spacetime. The TWB and TWE orbits extend from the upper to the lower part of the Universe connected by a wormhole with throat parameter $b_0 = 1.01$.

Figure 3. Orbits in the thin-shell Schwarzschild-wormhole spacetime for a massless test particle ($\delta = 0$). The TWB and TWE orbits are located in the upper and lower parts of the Universe which are connected by a wormhole with throat parameter $b_0 = 1.01$. 
where $\Delta = r^2 + a^2 - r$ and $\varphi^2 = r^2 + a^2 \cos^2 \vartheta$. The radial coordinate $r$ and the rotation parameter $a$ are normalized to the mass parameter $2M$ and are dimensionless.

The Lagrangian for a free test particle following a geodesic in the equatorial plane $\vartheta = \frac{\pi}{2}$ yields

$$2L = -\left(1 - \frac{1}{r}\right) i^2 + \frac{r^2}{\Delta} i^2 - \frac{2a}{r} i \dot{\phi} + \left(r^2 + a^2 + \frac{a^2}{r}\right) \dot{\phi}^2$$ with $2L = -\delta$, \hspace{1cm} (16)

where $\delta = 0$ for null geodesics and $\delta = 1$ for time-like geodesics, and $\dot{\chi}$ is the derivative with respect to the affine parameter $\lambda$.

The conserved and dimensionless energy $E = -\frac{\delta L}{\delta \varphi}$ and angular momentum $L = \frac{\delta L}{\delta \varphi}$ of a free test particle are

$$E = \left(1 - \frac{1}{r}\right) i + \frac{a}{r} \dot{\psi},$$ \hspace{1cm} (17)

$$L = -\frac{a}{r} + \left(r^2 + a^2 + \frac{a^2}{r}\right) \dot{\psi}.$$ \hspace{1cm} (18)

Solving the equations (17) and (18) with respect to $i$ and $\psi$ and substituting them into the Lagrangian (16) we obtain the differential equations describing the motion of a free test particle in the equatorial plane [54, 55]

$$\left(\frac{dr}{\dot{\gamma}}\right)^2 = E^2 r^4 + r^2 (a^2 E^2 - L^2) - \delta \Delta r^2 + r (aE - L)^2,$$ \hspace{1cm} (19)

$$\frac{d\varphi}{d\gamma} = \frac{r}{\Delta} (Lr + (aE - L)).$$ \hspace{1cm} (20)

Here $\gamma$ is a new affine parameter related to $\lambda$ by $d\gamma = r^{-2} d\lambda$.

The ergosphere, defined by the condition $g_{rr} = 0$, is located at $r = 1$ for the equatorial plane. The condition $g_{rr} \to \infty$ corresponding to the vanishing of $\Delta$ defines the two horizons of the Kerr spacetime: $h_{1,2} = (1 \pm \sqrt{1 - 4a^2})/2$.

In the following we will solve the differential equations (17) and (18) using the theory of elliptic functions [60].

**Radial equation.** By the transformation $r = a_1 (4y - a_2/3)^{-1}$, the equation (19) reduces to the Weierstrass form [60]

$$\left(\frac{dy}{\dot{\gamma}}\right)^2 = 4y^3 - v_2 y - v_3 = P_3(y),$$ \hspace{1cm} (21)

where $v_2 = (a_2^2/3 - a_1 a_3)/4$ and $v_3 = a_1 a_2 a_3/48 - a_0 a_3^2/16 - a_2^3/216$. The coefficients $a_i$ read:

$$a_3 = (aE - L)^2,$$

$$a_2 = a^2 (E^2 - \delta) - L^2,$$

$$a_1 = \delta,$$

$$a_0 = E^2 - \delta.$$ \hspace{1cm} (22)

The solution of the differential equation (21) is the Weierstrass $\varphi$-function $y = \varphi(y - y')$, where $y' = y_0 + \int_{0}^{\infty} \left(4y^3 - v_2 y - v_3\right)^{-1/2} dy$ is a constant defined by the initial conditions of the geodesic. With this result, the final solution for the coordinate $r$ yields [54, 55, 60]:

$$r = \frac{3a_3}{12 \varphi(y - y') - a_2}.$$ \hspace{1cm} (23)
Detailed representation of the potential (a).

Figure 4. The effective potential (equation (32)) for $\delta = 1$ and $L = 2.1, a = 0.2$ in the Kerr-wormhole spacetime $b_0 > h_2$. Filled grey regions denote forbidden energy values. See section 3.3 for details.

Azimuthal equation. The transformation $r = a_3(4y - a_2/3)^{-1}$ reduces the equation (20) to a sum of two differentials of the third kind:

$$d\phi = \sum_{i=1}^{2} \frac{K_i}{y - y_i} \frac{dy}{\sqrt{P_3(y)}}, \quad (24)$$

where the constants $K_1$ and $K_2$ are defined as

$$K_i = \frac{(-1)^{i-1}}{\sqrt{1 - 4a^2}} \frac{a_3}{4} \left( L + \frac{(aE - L)(2a^2)^{-1}}{(1 + (-1)^{i-1}\sqrt{1 - 4a^2})^{-1}} \right), \quad (25)$$

and $i = 1, 2$.

To proceed, we introduce $w = \gamma - \gamma'$ and replace $y$ in (24) by $y = \phi(w)$. This yields

$$d\phi = \sum_{i=1}^{2} \frac{K_i}{\phi(w) - \phi_{y_i}} dw, \quad (26)$$

where $\phi_{y_i}, i = 1, 2$, is defined by the equation $\phi_{y_i} = \phi(w_{y_i}) = y_i$. 

\[9\]
The integration of (26) in terms of elliptic \( \zeta \) and \( \sigma \) functions reads \[60–63\]
\[
\varphi = \varphi_0 + \sum_{i=1}^{2} \frac{K_i}{\gamma_i(w_i)} \left( 2\zeta(w_i)(w - w_0) + \ln \frac{\sigma(w - w_i)}{\sigma(w + w_i)} - \ln \frac{\sigma(w_0 - w_i)}{\sigma(w_0 + w_i)} \right),
\]
where \( w_0 = w(\gamma_0) \).

3.2. Construction of the wormhole and embedding

The throat of the Kerr-wormhole lies behind the horizons: \( b_0 > h_2 \). Following [17] we assume that the thin-shell Kerr-wormhole with the throat parameter \( b_0 \) does not depend on the proper time and, thus, is constant. Similar to the section 2.2 we use the surgery method to construct the thin-shell Kerr-wormhole. Namely, we cut the Kerr spacetime at some \( b_0 > h_2 \) and glue two copies of the manifold \( M_{1,2} = (t, r, \vartheta, \varphi | r \geq b_0) \) onto each other. Identifying the boundaries of these copies we get a new geodesically complete manifold with two regions connected by a wormhole with the throat characterized by the parameter \( b_0 \). The new coordinate \(|l| = |r - b_0|\) running from \(-\infty\) to \(\infty\) and vanishing at the throat is a practical coordinate to portray the effective potential (32) (see figures 4 and 5 in the section 3.3).

To understand the Kerr-wormhole topology for the equatorial plane motion described by the functions (23) and (27), consider the two-dimensional hypersurface \((t = \text{const}, \vartheta = \pi/2)\) described by the geometry:
\[
d s_0 = \frac{r^2}{\Delta} \, dr^2 + \left( r^2 + a^2 + \frac{a^2}{r} \right) d\varphi^2 \\
= \frac{r^2}{\Delta} \, dr^2 + R^2 \, d\varphi^2,
\]
where

\( \Delta = r^2 - a^2 \cos^2 \vartheta \).
where $R^2 = r^2 + a^2 + z^2$ and $\Delta = r^2 + a^2 - r$. This two-dimensional hypersurface can be embedded into the Euclidean space given by

$$d\mathbf{s}_E = dR^2 + R^2 \, d\varphi^2 + dz^2 = \left( \frac{dR}{dr} \right)^2 + \left( \frac{dz}{dr} \right)^2 \, dr^2 + R^2 \, d\varphi^2$$

(29)

in the cylindrical coordinates $(r, \varphi, z)$.

Comparing the coefficients of $dr^2$ in (28) and (29) we find that the shape of the embedding diagram in the Euclidean space is given by the integral

$$z(r) = \int_{b_0}^{r} \sqrt{\frac{r''^2}{\Delta} - \frac{(2r' a^3 - a^2)^2}{4r^3(r'^2 + a^2(r' + 1))}} \, dr'$$

(30)

and is visualized in figure 6 in coordinates $(x, y, z)$, where $x = r \cos(\varphi)$ and $y = r \sin(\varphi)$ (to the discussion of the orbits we will come in the next section). The integral in (30) is calculated numerically. For the embedding diagram we choose the initial value of $r$ to be $b_0$ and $\varphi \in [0, 2\pi]$. 

Figure 6. Orbits for a test particle with $\delta = 1$ for different values of angular momentum and energy in the thin-shell Kerr-wormhole spacetime. The TWB and TWE orbits stretch across the two regions of a Universe connected by a wormhole with throat parameter $b_0 = 0.99$ in ((a)-(c)) and $b_0 = 0.96$ in ((d)-(e)). Kerr-parameter $a = 0.2$. 

(a) $E = 1.11722, L = 2.1$. TWB. (b) $E = 1.11722, L = 2.1$. EO. (c) $E = 1.11723, L = 2.1$. TWE. (d) $E = 0.975, L = 2.1$. BO. (e) $E = 0.975, L = 2.1$. TWB.
Figure 7. Orbits for a test particle with \( \delta = 1 \) for different values of angular momentum and energy in the thin-shell Kerr-wormhole spacetime. Kerr-parameter \( a = 0.4 \). The orbit (a) is the \( x-y \) projection of the orbit (b). The black circle is the throat \( b_0 = 0.81 \). A test particle with the negative value of the angular momentum is dragged in the direction of the Kerr-wormhole rotation. This happens in the vicinity of the ergoregion which surrounds the throat of the wormhole. The third orbit (c) is a BO for the same \( E \)- and \( L \)-values. The described dragging can be also seen in the orbit shown in (e) and its \( x-y \) projection (d).

3.3. Geodesics

We introduce the effective potential via equation (19)

\[
\left( \frac{dr}{d\gamma} \right)^2 = r(r^3 + a^2(r + 1))(E - V_{\text{eff}}^1)(E - V_{\text{eff}}^{-1}).
\]  

Thus, the effective potential in the Kerr spacetime is given by [54–56]

\[
V_{\text{eff}} \equiv V_{\text{eff}}^{\pm} = \frac{aL \pm \sqrt{r^3 + a^2(r + 1)(r^3 + a^2(r + 1))}}{r^3 + a^2(r + 1)}.
\]  

The intersections of the function \( E \) and \( V_{\text{eff}} \) specify the turning points of the motion.

The graphical representation of the effective potential (32) is shown in figure 4, where the coordinate \(|l| = |r - b_0|\) is used. The regions which do not satisfy the condition \( E \geq V_{\text{eff}} \) and \( E \leq V_{\text{eff}} \) are forbidden and colored gray.

The orbit types in the thin-shell Kerr-wormhole are very similar to those in the thin-shell Schwarzschild wormhole. Namely,
Figure 8. Orbits for a massless test particle ($\delta = 0$) in the thin-shell Kerr-wormhole spacetime. Kerr-parameter $a = 0.49$. Plot (a) shows a TWB orbit and plot (b) two EO in the upper and lower parts of the Universe. Plot (c) shows a test particle moving in the upper and lower parts of the Universe on a TWE orbit. Here $b_0 = 0.61$.

Type A: TWE orbit which connects two regions of the Universe. In the Kerr black hole spacetime this orbit would continue into the negative radial coordinate region.

Type B: TWB orbits. A test particle moves on a BO which is partly located in the lower (respectively, upper) Universe, so that, a particle flies through the throat.

Type C: here TWB orbit and EO are possible. EOs exist in both parts of the Universe. For growing values of the throat parameter $b_0$ this type reduces to $C_0$ containing only EOs. See figure 4(c).

Type D: TWB and BOs are possible. BOs exist in both parts of the Universe. For growing values of the throat parameter $b_0$ the type D reduces to $D_0$ containing only BOs (see figure 4(c)).

For a further growing throat parameter $b_0$, TWB (type B) and TWE (type A) exist (see figure 4(d)). For larger $b_0$ only the TWE (type A) orbits remain. In figure 5 we make $L$ (in (a)) and $a$ (in (b)) change. In the first case, similar to the Kerr black hole spacetime, for large angular momenta of a test particle no planetary BOs exist. In the second case the form of the effective potential transforms when $a$ tends to the critical value $a_{\text{crit}} = 0.5$ when the horizons in the Kerr black hole spacetime merge.

We illustrate these orbit types in figures 6 and 7. Figure 7, plotted for opposite signs of the rotation parameter of the wormhole and the angular momentum of a test particle, also shows the influence of the ergosphere. It can be recognized in the directional change taking place shortly
before the black circle is approached, which symbolizes the wormhole throat in the pictures (a) for the TWB and (d) for the TWE orbit. The corresponding orbits are shown in (b) and (e). The trajectories are embedded into the three-dimensional space with the coordinates \((x, y, z)\), where \(x = r \cos(\varphi)\), \(y = r \sin(\varphi)\), and \(r, \varphi\) and \(z\) are given by (23), (27) and (30) correspondingly.

**Geodesics for \(\delta = 0\).** Like in the Kerr black hole spacetime no planetary BOs exist in the Kerr thin-shell wormhole spacetime, and only the orbits of type A TWE, B (TWB) and C (TWB and EO) can be found. Figure 8 visualizes some orbits for corotating massless test particles and figure 9 for counterrotating ones. In the last case a particle is dragged into the direction of wormhole rotation in the vicinity of ergosphere which is best seen in the projection (figure 9(a)) for the embedded TWB orbit (figure 9(b)) and the projection (figure 9(d)) for the embedded TWE orbit (figure 9(e)).

**4. Conclusion**

We have studied the motion of massive and massless test particles in the Schwarzschild and Kerr thin-shell wormhole spacetimes. These traversable wormhole spacetimes were constructed by
the cut-and-paste method and represent geodesically complete manifolds. The solution of the geodesic equations is given analytically in terms of elliptic Jacobi and Weierstrass functions. We have shown that in the traversable Schwarzschild and Kerr-wormhole spacetimes bound and escape orbits connecting the upper and lower parts of the Universe exist. Planetary bound orbits exist only for not too wide a throat of the wormhole in both cases. If the throat is sufficiently narrow (i.e. the throat parameter is chosen to be smaller than the ergosphere), a test particle is dragged into the direction of the wormhole rotation in the vicinity of the ergosphere. Here extraction of energy due to the Penrose process is possible.

Free particles moving on geodesics in the thin-shell wormhole spacetimes considered here encounter no exotic matter needed to maintain the wormhole and feel no tidal forces so that they will simply be transferred into the other universe. This possibility for travelers to avoid regions of exotic material in their traversal of the wormhole was discussed by Visser [5, 6] and Teo [43].

We will discuss the general geodesics—without restriction to the equatorial plane—for the traversable Kerr thin-shell wormhole spacetime in [64] where the solution of the geodesic equations is also given by elliptic \( \wp, \zeta \), and \( \sigma \)-functions.

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