Exact solutions admitting isometry groups $G_r \supseteq$ Abelian $G_3$

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Abstract
Metrics admitting a minimal three dimensional Abelian isometry group, $G_3$ are classified according to their Petrov types and metrics, giving all type $O$ and $D$ metrics explicitly, without imposing a source condition. The corresponding maximal Lie algebras for these metrics are obtained and identified as well. The type $O$ metrics admit a maximal $G_r \supseteq G_3$ with $r = 4, 6, 7$ and $10$, whereas the classes of metrics of type $D$ admit $G_r \supseteq G_3$ with $r = 3, 4, 5$ and $6$ as the maximal isometry groups. Type $O$ metrics with a perfect fluid source are then found explicitly and are shown to admit a maximal $G_r$ with $r = 4, 7$ and $10$. Type $D$ perfect fluid metrics are found explicitly which admit either a maximal $G_3$ or $G_4$. This classification also proves that the only non-null Einstein-Maxwell field admitting a maximal $G_4 \supseteq G_3$ is the type $D$ metric $(6.7)$ which is of Segre type $[(1, 1) (1 1)]$ and is isometric to the McVittie solution.

1. Introduction

The successes of the Schwarzschild solution in giving a clear understanding of the problems like perihelion shift of Mercury and gravitational deflection of light, besides giving a theoretical understanding for the existence of black holes and some of the global properties of a spacetime manifold appeared as a great motivation for finding explicit exact solutions of the Einstein’s field equations (EFEs). This further followed to have the more general Reissner-Nordstrom metric, Friedmann solutions for cosmology (Robertson-Walker metrics), Kerr metric for the existence and interpretation of circular black holes and the plane wave solutions for the gravitational radiation. This motivated many to make attempts to find exact solutions using different techniques which may relate either to the geometry of the spacetime manifold ($g_{ab}$) or to its matter contents ($T_{ab}$) or to both the geometry and the matter. These techniques have been given a comprehensive treatment in the Exact Solutions Book [1]. The metric of a spacetime manifold if admits a group of motions, can be reduced to a diagonal/block diagonal or a triangular form, which may in turn make the system of EFEs, simpler to solve [1–10]. The groups of motion have also proved instrumental in the construction of more general cosmologies than the Friedmann’s [1]. Petrov classification of the Weyl tensor [11–16] and Segre and Plebański classifications of the Ricci tensor [15, 17–19] have also proved effectively useful in finding exact solutions besides understanding the properties of the algebraically special (other than Petrov type $I$) metrics. Cartan theory has been successfully used to address the local equivalence problem and to establish the invariance of the gravitational fields besides providing information of the Lie algebraic structure of a spacetime manifold [20–23]. A general review of the achievements made in the field and their detailed applications in astrophysics and cosmology are available in the references [1, 24–26].

If a metric is known explicitly, one can use computer algebra programs like Maple and Mathematica to explore most of its geometrical and physical properties. Moreover, one can also try to find an asymptotic link of a metric to one of the Friedmann cosmological models in order to explore its role (if exists) from cosmological or astrophysical point of view. Finding explicit exact solutions of EFEs, is not at all trivial. This amounts to solving a system of ten partial differential equations of degree eight to find twenty unknown functions $g_{ab}(10)$ and $T_{ab}(10)$, of four variables, if no restrictions are made on the metric or on the stress-energy tensor components. There are at least two approaches: One, a pure mathematician’s approach of finding exact solutions as an art, without bothering about their physical implications and the other, considers understanding the exact solutions as more
important [25]. According to the remarks of Ehlers and Kundt, understanding exact solutions means to understand them with respect to their local geometry, symmetries, singularities, sources, extensions, completeness, topology and stability [27, 28]. This may in turn give new insights in the understanding of our universe. For example the Petrov classification have been used in the Kundt solutions which played an important role in the development of gravitational radiation theory [1, 3, 28]. Petrov classification and the Newman-Penrose formalism based on null tetrad basis paved the way to the discovery of the Taub-NUT, Kerr, and Robinson-Trautman solutions [1, 28]. This is the motivation for obtaining Petrov classification of gravitational fields using null tetrad basis for their metrics.

Here the focus would be to classify the gravitational fields in terms of their Petrov types and metrics by considering a general class of metrics which admit a certain minimal isometry group. By doing so, the required minimal isometry group at the first stage, reduces the most general metric to a nice form (a diagonal/block diagonal or a triangular form) having arbitrary coefficients which are functions of less than four variables. Petrov classification then puts further new differential constraints on the resulting metric coefficients whose solution (although may be challenging in many cases) would provide metrics in explicit form with their Petrov types. Later this classification may be used to find exact solutions by putting a desired source condition. As a start we choose the gravitational fields which admit an Abelian group $G_3$ as the minimal group of motions. The exact solutions with this property with a given source as a perfect fluid have been studied by many authors from time to time using different techniques. A brief review (whose source is mainly reference [1]) of these exact solutions is given before proceeding to do classification of gravitational fields in the present perspective.

The first empty solution admitting a $G_4 = so(1, 2) \times R \supset G_3$ was achieved by Levi-Civita [29] and later by Kasner [30] whereas the vacuum solutions admitting a maximal three dimensional group were found by Taub [31]. The exact solutions admitting a maximal $G_4$ with a perfect fluid source has a long history which goes back to a metric found by Taub [32] (metric 15.76 given in reference [1]) in an implicit form with an equation of state $\omega = \omega(p)$, where $\omega$ represents the energy density and $p$, the pressure. Barnes [1, 33] considered the problem of finding all degenerate (Petrov types O and D) static perfect fluid solutions and proved that the only positive density conformally flat static perfect fluid metrics are locally isometric to the interior Schwarzschild solution (which admits a maximal $G_4$) or one of the Einstein (admitting maximal $G_7$) or deSitter universes (admitting maximal $G_{10}$), besides providing a list of static perfect fluid solutions of type D in table 1 of his paper [33]. This list of solutions has also been provided on page 286 in [1]. According to this list, there appear metrics admitting $G_r (r = 2, 3, 4$ and 6) as the maximal isometry groups. The only metric admitting an Abelian $G_3$, is the static cylindrically symmetric perfect fluid solution (iii of table 1, which appears here as the metric $(5.2.26)$). There appear three metrics (ii, iic and iii in table 1 of [33]) admitting a $G_4$ as the minimal isometry group. Metric iii admits a maximal $G_6$ (with spherical or hyperbolic symmetry) whereas metrics id and iv admit a maximal $G_2$. All metrics except metric ii appearing in [33] are explicit with explicitly given equations of state, whereas the plane symmetric static metric iii admitting a minimal $G_4$ (with a slightly modified coordinate system), given as

$$ds^2 = \exp(2U(x))dt^2 - V^2(x)dx^2 - x^2(dy^2 + dz^2), \quad (1.1)$$

depends on unknown functions $U(x)$ and $V(x)$ which are subject to three coupled ordinary differential equations $(29(a)-(29(c))$ according to [33]. These equations are obtained essentially from the Einstein field equations for the metric $(1.1)$ and are simplified as

$$-V^{-2}(x^{-1}U' - (xV)G^{-1}V' + x^{-2}) = 0.5(p - \omega), \quad (1.2)$$

$$-V^{-2}(U'' + U'^2 - V^{-1}V'U' - 2(xV)^{-1}V') = 0.5(p - \omega), \quad (1.3)$$

$$-V^{-2}(U'' + U'^2 - V^{-1}V'U' + 2x^{-1}U') = 0.5(3p + \omega) \quad (1.4)$$

with $\omega$ and $p$ appearing as arbitrary functions of the independent variable $x$ whereas $V$ and $\omega$ are shown to relate as

$$V^{-2} = x^{-1}\int_0^x \omega(s)s^2ds + 2mx^{-1}, \quad (1.5)$$

with $m$ a constant of integration. Other perfect fluid solutions which admit a maximal $G_6 \supset G_3$ include: Tabensky and Taub’s Type D metric (given here by equation $(5.3.54)$), with the equation of state $p = \omega [34]$; Type I metrics given by: Taub with $\omega = \text{constant} [32]$ and Teixeira et al with $3p = \omega [35]$. Hojman and Santamarina [36] and later Collins probed further the implicit condition $(15.76b)$ given in [1], of the Taub’s metric [32] and were able to find an explicit Type D solution which admits $G_6 \supset G_3$ as the maximal group, with the equation of state $p = (\gamma - 1)\omega [37]$. Evans [38], later found a maximal $G_6$ type I solution with the equation of state $3p = \omega$. Soon after, a Type I exact solution admitting a $G_4$ as the maximal group, for an equation of state $p = (\gamma - 1)\omega$ was found in an implicit form by Bronnikov [39], which was later found independently in an implicit form by Kramer [40]. Kramer also found in the same paper a type D metric (given here by equation $(5.4.18)$) which admits a maximal $G_3$ with an equation of state $\omega = -\alpha^2e^{-\beta x} = p [40]$. Maximal $G_3$ type I solutions of Philbin [41], Haggag and Desokey [42], and Haggag [43] were found by using the implicit pair
of equations of Kramer established in [40] whereas other such solutions found by Narain [44] and Davidson [45] were obtained by making some adhoc assumptions [1].

For a non-null Einstein-Maxwell field, an exact solution with maximal $G_4 \supset G_3$ was found by Kar [46] and later by McVittie [47], which is expressed by metric (15.27) in [1]. This metric corresponds to an electrostatic field uniform in the direction of the field and nearly constant but with a slight exponential change of strength as we go along the field [47].

The plan is to focus on finding all degenerate metrics explicitly (without putting a source condition a priori) which admit an isometry group $G_\omega$ which is the Abelian isometry group of dimension 3. All static cylindrically symmetric metrics are known to admit a $G_3$ as the minimal isometry group and are known to be of Petrov type $D$, or $O$ [1, 11, 27, 48]. The plane symmetric static metrics are known to admit a minimal $G_3$ and are of Petrov type $D$ or $O$ [1].

The approach would be to choose the most general metric admitting a $G_4$ as the Abelian group of motions and by using the algorithm of d’Inverno and Russell-Clark [1, 11], classify the metric according to its Petrov types, metrics and the maximal groups of motion. This approach establishes a set of differential constraints on the metric coefficients for each type $O$ and $D$ metrics (section 2). The solution of these constraints provides an explicit list of all type $O$ and $D$ metrics (sections 3 and 4). These include: the type $O$ metrics (5.10)–(5.30) admitting $G_4$ with $r = 4, 6, 7$ and 10 as the maximal isometry groups; the type $D$ metrics (4.5), (4.6), (4.8) admitting a maximal $G_5$, (4.10) representing the static plane symmetric metric which is isometric to (1.1) and may admit a maximal $G_4 \supset G_3$ with $r = 4, 5$ and 6 [49–54]. The union of the two sets of type $D$ metrics (4.10) and (4.12)–(4.14) cover the type $D$ metrics (4.5), (4.6) and (4.8).

In section 5, we use the classification obtained in sections 2–4 and identify the perfect fluid solutions in the case of specific metrics of type $O$, whereas in the case of classes of metrics, depending on arbitrary functions, the field equations are solved in closed analytic forms to find the perfect fluid solutions of type $D$. It appears that the type $O$ metrics with a perfect fluid source admit a maximal $G_4$ with $r = 4, 7$ and 10, representing interior Schwarzschild, Einstein and Minkowski or deSitter metrics respectively, which are completely in line with the findings made in [33]. Type $D$ metrics with a perfect fluid source admit either a maximal $G_3$ or $G_4$. There appear two independent type $D$ perfect fluid solutions which admit a maximal $G_4$; metric (5.2.26) and a new class of metrics (5.4.16) (up to an arbitrary constant $a$. This class contains Kramer’s type $D$ metric (5.4.18) for $a > 0$ [40] as a special case). Metric (5.4.16) was missed in the investigations made in [33] and satisfies the equation of state $p = \omega$ with $\omega > 0$ for a < 0. Exact solutions with this type of equation of state describe radiation, relativistic degenerate Fermi gas and probably very dense baryon matter according to the investigations made by Zeldovich and Novikov [55]. For a maximal $G_4$, the requirement for a perfect fluid with an equation of state $\omega = \epsilon p$ reduces to a third order non-linear ODE (5.3.9). A complete solution of this ODE in closed analytic form is achieved, which gives three independent classes of perfect fluid spacetime metrics: (5.3.25) which may be considered a new class up to two arbitrary constants $k_1$ and $k_2$ when $\epsilon > 1$. This class reduces to the empty solution of Levi Civita and Kasner (5.3.27) [29, 30] for $k_1 = 0$. For $k_2 = 0$, to the perfect fluid solution (5.3.29) with the equation of state $\omega = \epsilon p$. Evidently the spacetime metric (5.3.29) serves as a source for the vacuum metric (5.3.27). Metric (5.3.29) has been shown to reduce to the perfect fluid metric found by Taub in an explicit form, later found by Hojman and Santamarina, and Collins explicitly [32, 36, 37] for $k_2 = 0$; (5.3.53) appears to be a new class up to an arbitrary constant $k$ when $\epsilon = 1$ and reduces to the Tabenski and Taub metric with $p = \omega$ ($\omega < 0$) [34] for $k < 0$. The other metric from this class gives $p = \omega$ with ($\omega > 0$) for $k > 0$. Zeldovich and Novikov interpretation is also applicable on this metric while Tabenski and Taub [34] added that if in addition the motion of the gas is irrotational, then such a source has the same stress-energy tensor as that of the massless scalar field. Wainwright et. al. noted that the mass-less scalar field solutions can in turn be interpreted as vacuum solutions in the Brans-Dicke theory [24] and (5.3.59) (which appears as a new class up to two arbitrary functions satisfying a constraint (5.3.58). This class is capable of generating new exact solutions other than the metrics: (5.3.66), (5.3.67) and (5.3.70) obtained here. All these metrics arise as explicit solutions of the class of metrics (4.10) and hence of (1.1)). Metrics (5.3.66) and (5.4.6) appear isometric and represent perfect fluid spacetime metrics with $\omega > 0$ where $p + \omega = 0$. The Ricci tensor in this case is of $-\text{term type thus reducing}$ the perfect fluid spacetime metrics to Einstein spaces [1]. Metrics (5.3.67) and (5.3.70), each appear to represent two infinite tubes of perfect fluids. In section 6, type $D$ exact solutions with non-null Einstein-Maxwell fields are also probed and it is proved that the only metric with this property is the metric (6.7), which appears to be equivalent to the McVittie solution [47, 56].

2. Petrov types of the metrics admitting $G_\omega \supset$ Abelian $G_3$

Static metrics in the isotropic coordinates, admitting a minimal $G_3$ generated by three commuting Killing vectors (KVs), $X_1 = -\partial_t$, $X_2 = a^{-2}\partial_\theta$, $X_3 = \partial_\varphi$, can be written as [1]
If $\theta$ is not considered periodic, $a d \theta = d y$ and the metric becomes plane symmetric static. In that case, the metric admits an additional Killing vector $X_4 = y \partial_y - z \partial_z$ which forces the metric (2.1) to have $\lambda = \mu$ and the minimal isometry group is $G_4 \supset G_3$, generated by $X_4$.

In the null tetrad basis, given by

$$k^a = \frac{1}{\sqrt{2}} \left( e^{-v/2} \frac{\partial}{\partial t} - \frac{\partial}{\partial \rho} \right),$$

$$l^a = \frac{1}{\sqrt{2}} \left( -e^{-v/2} \frac{\partial}{\partial t} + \frac{\partial}{\partial \rho} \right),$$

$$m^a = \frac{1}{\sqrt{2}} \left( \frac{e^{-\lambda/2}}{a} \frac{\partial}{\partial \theta} + ie^{-v/2} \frac{\partial}{\partial z} \right),$$

$$\bar{m}^a = \frac{1}{\sqrt{2}} \left( \frac{e^{-\lambda/2}}{a} \frac{\partial}{\partial \theta} - ie^{-v/2} \frac{\partial}{\partial z} \right),$$

the non zero components of the Weyl tensor for the metric (2.1) reduce only to:

$$16\Psi_0 = [2\lambda'' - 2\mu'' - \lambda'^2 - \mu'^2 - \nu' \lambda' + \nu' \mu'] = 16\Psi_0,$$

$$48\Psi_2 = -[4\nu'' - 2\lambda'' - 2\mu'' + 2\nu'^2 - \lambda'^2 - \mu'^2 - \nu' \lambda' + 2\lambda' \mu'].$$

The invariants $I, J, K, L$ and $N$ defined in [1, 11], can be expressed in terms of these components as:

$$I = \Psi_0^2 + 3\Psi_2^2,$$

$$J = \Psi_0(\Psi_0^2 - \Psi_2^2),$$

$$K = 0,$$

$$L = \Psi_0\Psi_2,$$

$$N = -\Psi_0(\Psi_0^2 + 3\Psi_2^2)(\Psi_0 - 3\Psi_2),$$

which give

$$I^3 - 27J^2 = \Psi_0^2(\Psi_0 + 3\Psi_2^2)(\Psi_0 - 3\Psi_2).$$

Now taking the view that all the invariants depend only on $\Psi_0$ and $\Psi_2$, There could be following four possibilities:

(a) $\Psi_0 = 0 = \Psi_2$; (b) $\Psi_0 = 0$, $\Psi_2 \neq 0$; (c) $\Psi_0 \neq 0$, $\Psi_2 = 0$; and (d) $\Psi_0 \neq 0 \neq \Psi_2$.

Possibilities (a) and (b) readily give the invariant

$$I^3 - 27J^2 = 0,$$

thus giving all algebraically special gravitational fields. The possibility (a), gives all $\Psi_i \equiv 0$, thus giving all Petrov type O metrics; the possibility (b), gives

$$I = 0 = J \quad \text{and} \quad K = 0 = N,$$

thus giving only, Petrov type D metrics; the possibility (c), gives

$$I^3 - 27J^2 = 0,$$

thus giving only, type I metrics; in the case of possibility (d),

$$I^3 - 27J^2 = 0 \Leftrightarrow \Psi_0 + 3\Psi_2 = 0 \neq \Psi_0 - 3\Psi_2,$$

thus, if

$$\Psi_0 + 3\Psi_2 \neq 0 \neq \Psi_0 - 3\Psi_2,$$

one gets type I metrics, otherwise, the type is D.

3. Petrov type O metrics

writing

$$2\nu'' - 2\lambda'' + \nu'^2 - \lambda'^2 - \nu' \lambda' + \lambda' \mu' \equiv Q(\rho),$$

$$2\nu'' - 2\mu'' + \nu'^2 - \mu'^2 - \nu' \lambda' + \lambda' \mu' \equiv R(\rho),$$
gives
\[ \Psi_0 = \frac{1}{16} (R - Q) \quad \text{and} \quad \Psi_2 = -\frac{1}{48} (R + Q). \tag{3.3} \]

Now for a Petrov type O solution, the necessary conditions are, \( \Psi_0 = 0 = \Psi_2 \). These instantly give, \( Q = 0 = R \). Thus to achieve all type O metrics, it is sufficient to find a complete solution of the following system of three, coupled second order, quasi linear ordinary differential equations (ODEs):

\[
2\lambda'' - 2\mu'' + \lambda'\lambda' - \mu'\mu' + \nu'\mu' + \nu'\lambda' = 0 = 4 \left( \frac{\lambda'}{2} - \frac{\mu'}{2} \right) e^{x_{\lambda'\mu'}}
\tag{3.4}
\]

\[
2\nu'' - 2\mu'' + \nu'\nu' - \mu'\mu' + \nu'\lambda' + \lambda'\lambda' = 0 = 4 \left( \frac{\nu'}{2} - \frac{\mu'}{2} \right) e^{x_{\nu'\mu'}}
\tag{3.5}
\]

\[
2\nu'' - 2\lambda'' + \nu'\nu' - \lambda'\lambda' - \nu'\mu' + \lambda'\lambda' = 0 = 4 \left( \frac{\nu'}{2} - \frac{\lambda'}{2} \right) e^{x_{\nu'\lambda'}}
\tag{3.6}
\]

Equations (3.4)–(3.6) readily give

\[
\left( \frac{\lambda'}{2} - \frac{\mu'}{2} \right) e^{x_{\lambda'\mu'}} = k_0,
\tag{3.7}
\]

\[
\left( \frac{\nu'}{2} - \frac{\mu'}{2} \right) e^{x_{\nu'\mu'}} = k_2,
\tag{3.8}
\]

\[
\left( \frac{\nu'}{2} - \frac{\lambda'}{2} \right) e^{x_{\nu'\lambda'}} = k_3.
\tag{3.9}
\]

First we assume that \( \exists \) a coordinate neighborhood in which none of \( \nu', \lambda' \) and \( \mu' \) is zero. Now if \( k_i = 0, \forall i \), then without loss of generality, one can choose \( \nu = \lambda = \mu = \sigma (\rho) \), with \( \sigma' \neq 0 \). This gives a class of metrics

\[
ds^2 = e^{2\sigma} [dt^2 - a^2d\theta^2 - dz^2] - dp^2,
\tag{3.10}
\]

of the type O. Metric (3.10) admits a six dimensional maximal isometry group whose Lie algebra is generated by:

\[
X_1 = - \partial_t, \quad X_2 = a^{-2} \partial_\theta, \quad X_3 = \partial_z, \quad X_4 = - a^{-2} z \partial_\theta + \theta \partial_z,
\]

\[
X_5 = z \partial_t + t \partial_z, \quad X_6 = \theta \partial_t + a^{-2} t \partial_\theta.
\tag{3.11}
\]

If any one of the \( k_i \) is \( 0 \), this gives any of the following three classes of Petrov type O metrics:

\[
\mu = \lambda \quad \text{and} \quad e^{\bar{\sigma}} = e^{\bar{\sigma}} \left[ k_2 \int e^{-2z} dp + k_4 \right], \quad (k_2 = k_3)
\tag{3.12}
\]

\[
\nu = \lambda \quad \text{and} \quad e^{\bar{\sigma}} = e^{\bar{\sigma}} \left[ -k_1 \int e^{-2z} dp + k_4 \right], \quad (k_2 = k_3)
\tag{3.13}
\]

\[
\nu = \mu \quad \text{and} \quad e^{\bar{\sigma}} = e^{\bar{\sigma}} \left[ k_2 \int e^{-2z} dp + k_4 \right], \quad (k_2 = -k_3)
\tag{3.14}
\]

Metrics (3.12) reduces to

\[
ds^2 = dt^2 - dp^2 - e^{4\sigma}(dy^2 + dz^2),
\tag{3.15}
\]

when \( k_4 = 0 \) and \( \nu, \lambda \) and \( \mu \) are linear functions of \( \rho \). Other two metrics corresponding to (3.13)–(3.14) are isometric to the metric (3.15) and can be directly obtained from this metric by using the complex transformations: \( z \rightarrow it, \ t \rightarrow iz \) and \( y \rightarrow it, \ t \rightarrow iy \) respectively. Metrics (3.12)–(3.14) admit a \( G_4 \) as the maximal isometry group whereas the metric (3.15) and its isometric metrics admit \( G_7 = so(1, 3) \otimes R \) as the maximal isometry group with generators of the Lie algebra given as

\[
X_1 = - \partial_t, \quad X_2 = a^{-2} \partial_\theta, \quad X_3 = \partial_z, \quad X_4 = \partial_\theta - \frac{A\theta}{2} \partial_\phi - \frac{A\phi}{2} \partial_\theta,
\tag{3.16}
\]

\[
X_5 = \theta \partial_\theta + a^{-2} \left( A^{-1} e^{-A\rho} - a^{-2} \partial_\phi + \frac{A\phi}{4} \right) \partial_\theta - \frac{A\theta}{2} \partial_\theta,
\]

\[
X_6 = z \partial_t - \frac{A\theta}{2} \partial_\theta + \left( A^{-1} e^{-A\rho} - \frac{A\phi}{4} + a^{-2} \frac{A\phi}{4} \right) \partial_\theta, \quad X_7 = - a^{-2} z \partial_\theta + \theta \partial_z.
\tag{3.18}
\]

If any two of the \( k_i \) are zero, then the third one is necessarily zero. This again reduces to the case of the metric (3.10). Now the remaining possibility is to have \( k_1 = 0, \forall i \). This leads equations (3.7)–(3.9) to give

\[
k_1 e^{\nu} + k_2 e^{\mu} + k_3 e^{\lambda} = 0.
\tag{3.19}
\]

Let \( k_3 \neq 0 \). Then equation (3.19) gives

\[
\frac{\lambda'}{2} = - \frac{k_1 \nu'}{k_3} e^{\nu - \lambda} - \frac{k_2 \mu'}{k_3} e^{\mu - \lambda}.
\tag{3.20}
\]
Using the value of \( 2l \) from equation (3.20) to rewrite equation (3.7) as

\[
-k_k k_3 = \left( k_1 e^{-\frac{2l}{z}} + k_2 e^{-\frac{2l_1}{z}} + k_3 e^{-\frac{2l_2}{z}} \right). \tag{3.21}
\]

Now using equations (3.7)–(3.9) gives

\[
-k_k k_3 e^{-\mu} = \left[ \frac{\nu'}{2} - \frac{\nu''}{z} + \frac{\mu'}{2} - \frac{\mu''}{z} \right]. \tag{3.22}
\]

On the other hand equations (3.7) and (3.9) give

\[
k_k k_3 e^{-\mu} = \left( \frac{\nu'}{2} - \frac{\nu''}{z} \right). \tag{3.23}
\]

Comparing equations (3.22)–(3.23), gives: \( \nu'' = 0 \). The possibility, \( \nu' = 0 \) gives \( k_3 = 0 \), a contradiction whereas the possibility \( \nu' = 0 \) and the other such possibilities are discussed as follows.

Choosing one of the \( \nu', \lambda' \) and \( \mu' \), as zero, gives the following specific metrics:

\[
\begin{align*}
\nu &= 0, \quad \lambda = 2 \ln \left[ \sinh \left( \frac{\rho}{\rho_0} \right) \right], \\
\mu &= 2 \ln \left[ \cosh \left( \frac{\rho}{\rho_0} \right) \right].
\end{align*} \tag{3.24}
\]

\[
\begin{align*}
\nu &= 0, \quad \lambda = 2 \ln \left[ \sin \left( \frac{\rho}{\rho_0} \right) \right], \\
\mu &= 2 \ln \left[ \cos \left( \frac{\rho}{\rho_0} \right) \right].
\end{align*} \tag{3.25}
\]

\[
\begin{align*}
\nu &= 2 \ln \left[ \sin \left( \frac{\rho}{\rho_0} \right) \right], \\
\lambda &= 0, \quad \mu = 2 \ln \left[ \cosh \left( \frac{\rho}{\rho_0} \right) \right].
\end{align*} \tag{3.26}
\]

\[
\begin{align*}
\nu &= 2 \ln \left[ \sin \left( \frac{\rho}{\rho_0} \right) \right], \\
\lambda &= 0, \quad \mu = 2 \ln \left[ \cos \left( \frac{\rho}{\rho_0} \right) \right].
\end{align*} \tag{3.27}
\]

\[
\begin{align*}
\nu &= 2 \ln \left[ \sin \left( \frac{\rho}{\rho_0} \right) \right], \\
\lambda &= 2 \ln \left[ \cosh \left( \frac{\rho}{\rho_0} \right) \right], \quad \mu = 0,
\end{align*} \tag{3.28}
\]

\[
\begin{align*}
\nu &= 2 \ln \left[ \sin \left( \frac{\rho}{\rho_0} \right) \right], \\
\lambda &= 2 \ln \left[ \cos \left( \frac{\rho}{\rho_0} \right) \right], \quad \mu = 0.
\end{align*} \tag{3.29}
\]

Metrics (3.24)–(3.29) admit seven isometries identified as \( \text{so}(1, 3) \otimes R \).

The cases of any two of the \( \nu', \lambda' \) and \( \mu' \) being zero is trivial to give three possibilities, each with \( \nu, \lambda, \) and \( \mu \) given by:

\[
\nu = 2 \ln \left[ \frac{\rho}{\rho_0} \right], \quad \lambda = 0 = \mu; \quad \nu = 2 \ln \left[ \frac{\rho}{\rho_0} \right], \quad \lambda = 0 = \mu; \quad \text{and} \quad \nu = 2 \ln \left[ \frac{\rho}{\rho_0} \right], \quad \nu = 0 = \lambda. \tag{3.30}
\]

All of these three metrics turn out to be flat. The last possibility, in which all the derivatives of \( \nu, \lambda \) and \( \mu \) are zero just reduces to a flat metric too and admits \( \text{so}(1, 4) \) as the maximal isometry group. This completes the classification of the type O metrics.

4. Petrov type D metrics

For Petrov type D solutions, the possibility (b) (refer to section 2), gives the necessary conditions as:

\[
\Psi_0 = 0 = \Psi_2;
\]

whereas, the possibility (d), gives the necessary conditions:

\[
\begin{align*}
\text{either (i) } & \Psi_0 + 3\Psi_2 = 0 = Q, \quad \Psi_0 - 3\Psi_2 = 0 = R, \\
& \text{or (ii) } \Psi_0 - 3\Psi_2 = 0 = R, \quad \Psi_0 + 3\Psi_2 = 0 = Q.
\end{align*}
\]

For the possibility (b) the invariants:

\[
I = 3(\Psi_2)^2 \text{ and } J = -(\Psi_0)^2;
\]

whereas for the possibility (d):

\[
I = 12(\Psi_2)^2 \text{ and } J = 8(\Psi_0)^3,
\]

therefore the metrics arising from these two possibilities are non-isometric.

For the possibility (b) the invariants, emerging from the possibility (b), it is to be noted that \( \Psi_0 = 0 \), gives equation (3.7) to provide type D solutions relating, three arbitrary functions, whereas, \( \Psi_2 = 0 \), by virtue of equation (2.7), puts constraint on these functions. Equation (3.7), can simplify the expression (2.7), for \( \Psi_2 \), in two different ways, given by

\[
\Psi_2 = -\frac{1}{12} \left[ \nu'' - \lambda'' + \frac{\nu'}{2} (\nu' - \lambda') + k_1 (\nu' - \lambda') e^{\nu - \mu} \right]. \tag{4.1}
\]
\[ \Psi_2 = -\frac{1}{12}\left[ \nu'' - \mu'' + \frac{\nu'}{2}(\nu' - \mu') - k_0(\nu' - \mu')e^\nu e^{-\mu} \right]. \] (4.2)

Equations (4.1)–(4.2), readily give that \( \Psi_2 = 0 \), if \( \nu = \lambda \) or \( \nu = \mu \). Thus, in this case, two of the necessary conditions appear as, \( \nu = \lambda \) and \( \nu = \mu \). Equations (4.1)–(4.2), now give
\[ \Psi_2 = - \frac{(\nu' - \lambda')(\nu' - \mu')}{12\nu'' - \lambda' - \mu'} \ln [(\nu' - \lambda')|\nu' - \mu'|e^{\nu'}]'. \] (4.3)

It is to be noted that \( \nu - \lambda \approx 0 \) and \( \nu - \mu \approx 0 \), give \( 2\nu - \lambda - \mu \approx 0 \). Moreover, if \( \nu' = 0 \), then \( \lambda'\mu' \approx 0 \). Thus,
\[ \Psi_2 = 0 \Leftrightarrow \ln [(\nu' - \lambda')(\nu' - \mu')e^{\nu'}]' \approx 0 \]
\[ \Leftrightarrow (\nu' - \lambda')(\nu' - \mu')e^{\nu'} \approx 0. \] (4.4)

Thus the possibility (b) gives, a class of solutions of type \( D \), given by the metric
\[ ds^2 = e^{\nu(\rho)}dt^2 - d\rho^2 - e^{\mu(\rho)} \left[ k_4 + k_1 \int e^{\frac{\nu}{2}\rho} d\rho \right]^2 d\theta^2 + dz^2, \] (4.5)

where \( \nu, \lambda, \) and \( \mu \) are subject to the constraint given by the relation (4.4).

On the same lines, the cases (i) and (ii) are dealt with to give, respectively,
\[ ds^2 = e^{\lambda(\rho)} \left[ k_4 + k_2 \int e^{\frac{-\nu}{2}\rho} d\rho \right]^2 dt^2 - a^2d\theta^2 - d\rho^2 - e^{\mu(\rho)}dz^2, \] (4.6)

subject to the constraint
\[ [(\mu' - \lambda')(\mu' - \nu')e^{\nu'}] = 0, \] (4.7)

and
\[ ds^2 = e^{\nu(\rho)} \left[ k_4 + k_3 \int e^{\frac{-\lambda}{2}\rho} d\rho \right]^2 dt^2 - a^2d\theta^2 - d\rho^2 - a^2e^{\lambda(\rho)}d\theta^2, \] (4.8)

subject to the constraint
\[ [(\lambda' - \nu')(\lambda' - \mu')e^{\lambda}] = 0. \] (4.9)

Metrics (4.5), (4.6) and (4.8) admit a \( G_3 \) as the maximal isometry group. The eigen values corresponding to the metric (4.5) are \( (\Psi_2, \Psi_3, -2\Psi_2) \) whereas for the metrics (4.6) and (4.8) are \( (-2\Psi_2, 4\Psi_3, -2\Psi_2) \). Six of the Newman-Penrose spin coefficients: \( \pi, \nu, \alpha, \tau, \kappa, \) and \( \beta \) for both of these types of classes of metrics are zero.

One can readily find by putting \( k_1 = 0 \) \((i = 1, 2, 3)\) and \( ad\theta = dy \) in (4.5), (4.6) and (4.8) respectively and by using the complex transformations used in section 3, three classes of type \( D \) metrics isometric to the plane symmetric static metric
\[ ds^2 = e^{\nu(\rho)}dt^2 - d\rho^2 - e^{\lambda(\rho)}(dy^2 + dz^2) \] s.th. \([\nu' - \lambda']e^{\nu} = 0. \] (4.10)

Metric (4.10) admits maximal: \( G_3 \) if \( \lambda(\rho) = 0 \) and \( \nu(\rho) = 2 \ln\cosh(\alpha p + b) \) or \( 2 \ln\cos(\alpha p + b) \) or \( \alpha p; \) \( G_3 \) if \( \lambda(\rho) = b p \) and \( \nu(\rho) = 0 \) or if \( \nu(\rho) = \alpha p (a = b) \) otherwise it admits a maximal \( G_2 \). If one uses the transformation \( e^{\tau} = x, \) metrics (1.1) and (4.10) become isometric.

In the case when \( k_1 = 0 \) \((i = 1, 2, 3)\), one can choose \( k_1 = 1 \) and \( k_4 = 0 \), without loss of generality, and then setting
\[ \int e^{\frac{-\nu}{2}\rho} d\rho = \beta(\rho) \] and \( \int e^{\frac{-\lambda}{2}\rho} d\rho = \beta(\rho), \] (4.11)

\( \mu(\rho) = \alpha(\rho), \lambda(\rho) = \alpha(\rho) \) and \( \nu(\rho) = \alpha(\rho) \) in the metrics (4.5), (4.6) and (4.8) respectively, one can rewrite them in another form, given by
\[ ds^2 = e^{\alpha(\rho)}[\beta^2e^{\rho}dt^2 - \beta^2d\theta^2 - \beta^2dz^2] - d\rho^2, \] (4.12)
\[ ds^2 = e^{\alpha(\rho)}[\beta^2dt^2 - a^2d\theta^2 - \beta^2e^{\rho}dz^2] - d\rho^2, \] (4.13)
\[ ds^2 = e^{\alpha(\rho)}[dt^2 - \beta^2e^{\rho}d\theta^2 - \beta^2dz^2] - d\rho^2, \] (4.14)

such that
\[ \left( \beta^2e^{\alpha(\rho)} \right) \left( \ln \beta^2e^{\alpha(\rho)} \right)^' = 0. \] (4.15)

This completes the classification of the type \( D \) metrics. The residue of the metrics or classes of metrics obtained in sections 3–4, then give, Petrov type I metrics.
5. Perfect fluid solutions admitting Abelian $G_3$ or a $G_4 \supset G_3$

The stress energy tensor for a perfect fluid is given by
\[ T_{ab} = (\omega + p)u_a u_b - p g_{ab}, \]  \hspace{1cm} (5.1)
where $u_a$ is the timelike four velocity of the fluid corresponding to a space-time metric $g_{ab}$. For the the metric (2.1), the energy density reduces to
\[ \omega = T^{00} = \frac{\lambda''}{2} + \frac{\mu''}{2} + \frac{\lambda'^2}{4} + \frac{\mu'^2}{4} + \frac{\lambda\mu'}{4}, \]  \hspace{1cm} (5.2)
whereas for an isotropic pressure $p$, it is required to have $-T^1_1 = -T^2_2 = -T^3_3 = p$. The expressions for $T^1_1$, $T^2_2$ and $T^3_3$ are simplified to
\[ T^1_1 = \frac{1}{4} (\nu'' \lambda' + \lambda' \mu' + \mu' \nu'), \] \hspace{1cm} (5.3)
\[ T^2_2 = \frac{\mu''}{2} + \frac{\mu'^2}{4} + \frac{\mu' \nu'}{4}, \] \hspace{1cm} (5.4)
\[ T^3_3 = \frac{\lambda''}{2} + \frac{\lambda'^2}{4} + \frac{\lambda \nu'}{4}. \] \hspace{1cm} (5.5)

5.1. Perfect fluid solutions of Type O

The metric (3.10) with $\sigma(\rho) = A \rho$, gives $p = -(3/4)A^2 = -\omega$ and appears as a conformally flat static perfect fluid spacetime of constant curvature, admitting a maximal $so(2, 3)$. For the metric (3.12), after absorbing $k_2$ in the definition of time and writing $(k_4/k_3) = \beta$, the above procedure provides a perfect fluid spacetime metric
\[ ds^2 = \left(\beta e^{\frac{x}{2}} - \frac{2}{\alpha}\right) dt^2 - d\rho^2 - e^{\alpha \rho} (dy^2 + dz^2), \quad (\alpha \neq 0). \] \hspace{1cm} (5.1.1)
One can readily observe that the transformation: $\frac{2}{\alpha} e^{-\frac{x}{2}} = \beta + x$, reduces the metric to the form
\[ ds^2 = (\beta + x)^{-2} [x^2 dt^2 - dx^2 - dy^2 - dz^2], \] \hspace{1cm} (5.1.2)
which gives: $\omega = 3$, $p = (2\beta - x)/x$, therefore the equation of state is $\omega = 3x \rho/(2\beta - x)$. Here $R = 6(\beta - x)/x$, which highlights an essential singularity at $x = 0$. A coordinate singularity at $x = -\beta$ is apparent from here. Thus this line source could be interpreted as an interior solution for the exterior vacuum solution (3.27). The metric admits $G_4$ as the maximal isometry group and is plane symmetric static locally isometric to the interior Schwarzschild solution. The requirement of a perfect fluid solution reduces the other two metrics (3.13) and (3.14) to the Minkowski spacetime metrics. Metric (3.15) turns out to be a perfect fluid spacetime with $\omega > 0$ and admits a maximal $G_7 = so(1, 3) \oplus R$. The other two metrics obtained from (3.15) by complex transformations turn out to have anisotropic stresses with positive energy density and admit a maximal $G_7$. Metrics (3.24)–(3.25) also turn out to be isometric to the metric (3.15) whereas (3.26)–(3.29) are tachyonic fluids with Segre type [(1, 11)1].

5.2. Type D metrics (4.5), (4.6), (4.8)

For the type D metric (4.5), one gets
\[ \frac{\omega}{3} = \frac{\mu''}{2} + \frac{\mu'^2}{4} = -p, \] \hspace{1cm} (5.2.1)
subject to the constraints given by: $\nu' = 0$ and
\[ \frac{\mu''}{2} - \frac{k_0}{2} \mu' e^{\frac{3p}{2}} = 0, \] \hspace{1cm} (5.2.2)
\[ \frac{1}{2} (\lambda' - \mu') = k_0 e^{\frac{3p}{2}}. \] \hspace{1cm} (5.2.3)
$\nu' = 0$ gives $\nu = $ a constant, which can then be absorbed in the definition of time. The other two constraints are the coupled non-linear ordinary differential equations, whose solution gives
\[ e^\gamma = \sqrt{k_0} \sinh(\alpha \rho + \beta), \quad e^\sigma = \frac{k_0}{\alpha} \cosh(\alpha \rho + \beta), \] \hspace{1cm} (5.2.4)
which is the metric (3.24). Thus the requirement of a perfect fluid solution forces the type D metric (4.5) to reduce to the type O metric (3.25). The equation (5.2.1), then instantly gives $\frac{p}{\omega} = -0.3$. 

8
For the metric (4.6)
\[ e^\tau = k_4 e^{\lambda^2} + k_2 e^{\lambda^2} \int e^{\tau - \lambda} d\rho. \]  
(5.2.5)

This gives
\[ \frac{1}{2} \nu' = \frac{1}{2} \lambda' + k_2 e^{\frac{\lambda^2}{2}}. \]  
(5.2.6)

and
\[ \frac{1}{2} \nu'' = \frac{1}{2} \lambda'' + k_2 e^{\frac{\lambda^2}{2}} \left( \frac{1}{2} \mu' - \lambda' \right) - k_2^2 e^{\frac{\lambda^2}{2}}. \]  
(5.2.7)

Using equations (5.2.5)–(5.2.6) in equations (5.3)–(5.5) and comparing the resulting expressions of \( T_1 \) with \( T_2 \) and of \( T_2 \) with \( T_3 \) give respectively:
\[ \frac{1}{4} (2 \lambda'' + 2 \mu' + \mu^2 - \lambda' \mu') - \frac{k_2}{2} (\lambda' - \mu') e^{\frac{\lambda^2}{2}} = 0, \]  
(5.2.8)
\[ \frac{1}{4} (2 \lambda'' - 2 \mu' + 2 \lambda^2 - \mu^2 - \lambda' \mu') + \frac{k_2}{2} (\lambda' - \mu') e^{\frac{\lambda^2}{2}} = 0. \]  
(5.2.9)

Equations (5.2.8)–(5.2.9) get reduce to
\[ 2 \lambda'' + \lambda'^2 - \lambda' \mu' = 0, \]  
(5.2.10)

which solves to give
\[ e^{\frac{\mu}{\alpha}} = \frac{1}{\alpha} \left( e^{\frac{\lambda}{\alpha}} \right)', \]  
(5.2.11)

where \( \alpha \) is a constant of integration and can be absorbed in the definition of \( z \). This equation transforms equation (5.2.5) to give
\[ e^\tau = k_4 e^{\lambda^2} - k_2. \]  
(5.2.12)

This in return simplifies the metric (4.6) to the form
\[ ds^2 = \left( k_4 e^{\lambda^2} - k_2 \right) dt^2 - d\rho^2 - e^{\lambda^2} d\theta^2 - \left[ \left( e^{\lambda^2} \right)' \right]^2 dz^2. \]  
(5.2.13)

For this metric
\[ p = - \frac{3}{4} \lambda^2 + \lambda - \frac{k_2}{k_4 e^{\lambda^2} - k_2} \left( \lambda^2 + \lambda^2 \right), \]  
(5.2.14)
\[ \omega = \frac{3}{4} \lambda^2 + \lambda'' - \frac{k_2^2}{k_4 e^{\lambda^2} - k_2}, \]  
(5.2.15)

subject to the constraint
\[ \lambda'' = - \frac{3}{2} \lambda'' - \frac{k_2^2}{k_4 e^{\lambda^2} - k_2}, \]  
(5.2.16)

whose solution gives
\[ \lambda'' = \frac{\beta}{e^\lambda (k_4 e^{\lambda^2} - k_2)}, \lambda'' = 0, \]  
(5.2.17)

which is further solved to have
\[ \frac{\lambda^2}{4} = \frac{2}{k_4^2} e^{-\lambda^2} + \frac{\beta}{2k_4} e^{-\lambda} + \frac{\beta k_4^2}{2k_4^2} \ln \left( k_4 - k_2 e^{-\lambda^2} \right)^2 + C, \]  
(5.2.18)

where \( \beta \) and \( C \) are constants of integration. Using the transformation:
\[ k_2 e^{-\lambda^2} = k_4 + x(p), \]  
(5.2.19)

one gets
\[ \frac{\lambda'}{2} = - \frac{1}{k_4 + x} \frac{dx}{d\rho} \]  
(5.2.20)

and
\[ \frac{\lambda^2}{4} = C + \frac{\beta}{2k_4^2} \left[ k_4^2 \ln x^2 + 2k_4 (k_4 + x) + (k_4 + x)^2 \right] / F(x). \]  
(5.2.21)
Equations (5.2.20)–(5.2.21), now give
\[ dp^2 = \frac{dx^2}{(k_4 + x)^2 F(x)}, \] (5.2.22)
and the metric (5.2.13), is finally transformed to
\[ ds^2 = \frac{1}{(k_4 + x)^2} \left[ x^2 dt^2 - \frac{dx^2}{F(x)} - a^2 d\theta^2 - F(x) dz^2 \right], \] (5.2.23)
with \( \omega \) and \( p \) for this metric are given as
\[ p = \frac{F(x)(2k_4 - x)}{x} + \frac{\beta}{2k_3^2} (x + k_4)^3 (k_4 - x), \] (5.2.24)
\[ \omega = 3F(x) - \frac{\beta}{2k_3^2} (x + k_4)^3 (3x + k_4). \] (5.2.25)
Now the solution for the metric (4.8) can be readily obtained by using the transformation \( \lambda \lref \mu \) in the solution of the metric (4.6), which instantly reduces to the metric:
\[ ds^2 = \frac{1}{(k_4 + x)^2} \left[ x^2 dt^2 - \frac{dx^2}{F(x)} - a^2 F(x) dt^2 - dz^2 \right], \] (5.2.26)
given as class (iii) in table 1 of reference [33] and admits a \( G_3 \) as the maximal group of motions. The possibilities, \( k_4, \beta = 0 \) \( (k_4 = n \text{ and } \beta/2k_3^2 = \epsilon, \text{ in the Barnes notation}) \) reduce the above metrics (5.2.23) and (5.2.26)) to the conformally flat metric (5.1.2).

5.3. Type D Metric (4.10)
Metric (4.10) which is isometric to the metric (1.1), if admits a \( G, \text{ with } r > 4, \) cannot give a perfect fluid solution. We concentrate when it admits a maximal \( G_4 \supset G_3 \) Equations (5.2)–(5.5) then reduce to
\[ \omega = T_0^0 = \lambda'' + \frac{3\lambda'^2}{4}, \] (5.3.1)
\[ -T_1^1 = -\frac{\nu'\lambda'}{2} - \frac{\lambda'^2}{4}, \] (5.3.2)
and
\[ -T_2^2 = -\frac{\nu''}{2} - \frac{\lambda''}{2} - \frac{\nu'^2}{4} - \frac{\lambda'^2}{4} - \frac{\nu'\lambda'}{4} = -T_3^3. \] (5.3.3)
Thus to have an isotropic pressure, equations (5.3.2)–(5.3.3) give the constraint
\[ \frac{\nu''}{2} + \frac{\lambda''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} = 0, \] (5.3.4)
which matches to the condition for static equilibrium \( (T^a_{\mu a} = 0) \), which turns out to be the only constraint on the metric to be a perfect fluid. Thus in this case one has more liberty to try to have perfect fluid spacetimes with \( \omega, p \) and \( T \) all positive, which is attempted as follows.

For a barotropic equation of state we first assume \( \omega = \epsilon p \) and equations (5.3.1)–(5.3.2) therefore give
\[ \lambda'' + \frac{\epsilon + 3}{4} \nu'' + \frac{\epsilon}{2} \nu'\lambda' = 0, \] (5.3.5)
where for physical considerations, \( \epsilon \geq 1 \). Equation (5.3.5) is subject to the constraint given by equation (5.3.4). In order to solve the coupled non-linear system of equations (5.3.4)–(5.3.5) to give \( \nu(\rho) \) and \( \lambda(\rho) \), we proceed as follows: Transforming the equation (5.3.5) to the form
\[ e^{\alpha \tau} \left( \frac{e^{\alpha \tau}}{e^{\alpha \tau}} \right)'' + e^{\alpha \tau} \left( \frac{e^{\alpha \tau}}{e^{\alpha \tau}} \right)' = 0, \] (5.3.6)
gives
\[ \left( e^{\alpha \tau} \left( \frac{e^{\alpha \tau}}{e^{\alpha \tau}} \right)' \right)' = 0, \] (5.3.7)
and finally, one gets a relation between \( \nu \) and \( \lambda \) as
\[ e^{\alpha \tau} = \left( \frac{e^{\alpha \tau}}{e^{\alpha \tau}} \right)'. \] (5.3.8)
Equation (5.3.5) then reduces to a third order non-linear ODE in $\lambda$ given as

$$\lambda'' = \frac{\varepsilon + 1}{\varepsilon} \lambda' + \frac{\varepsilon^2 + \varepsilon + 6}{4\varepsilon} \lambda'' + \frac{3\varepsilon + (\varepsilon + 3)}{16\varepsilon} (\lambda')^3 = 0. \quad (5.3.9)$$

To be able to find solution of equation (5.3.9). Writing

$$a = \frac{\varepsilon + 1}{\varepsilon}, \quad b = \frac{\varepsilon^2 + \varepsilon + 6}{4\varepsilon}, \quad c = \frac{3\varepsilon + (\varepsilon + 3)}{16\varepsilon}$$

and adding $f(\varepsilon)\lambda'\lambda''$ to both sides of equation (5.3.9) to get

$$\lambda'' + f(\varepsilon)\lambda'\lambda'' = a\frac{\lambda''}{\lambda'} + (f(\varepsilon) + b)\lambda'' + c(\lambda')^3. \quad (5.3.10)$$

Equation (5.3.10) can be transformed to

$$\left(\lambda'' + f(\varepsilon)\lambda'\lambda''\right)' = \frac{a\lambda''}{\lambda'}\left(\lambda'' + f(\varepsilon)\lambda'\lambda''\right) + \frac{(2-a)f(\varepsilon) + 2b}{2}\lambda'\lambda'' + c(\lambda')^3,$$ \hspace{1cm} (5.3.11)

which can then be expressed as

$$\left(\lambda'' + f(\varepsilon)\lambda'\lambda''\right)' = \frac{a\lambda''}{\lambda'}\left(\lambda'' + f(\varepsilon)\lambda'\lambda''\right) + \frac{(2-a)f(\varepsilon) + 2b}{2}\lambda'\left(\lambda''\left(\lambda'' + f(\varepsilon)\lambda'\lambda''\right)\right)^{-1}(\lambda')^2.$$ \hspace{1cm} (5.3.12)

Choosing

$$c\left(\frac{(2-a)f(\varepsilon) + 2b}{2}\right)^{-1} = \frac{f(\varepsilon)}{2}$$

gives

$$f(\varepsilon) = \frac{3}{2} \quad \text{or} \quad \frac{(\varepsilon + 1)(\varepsilon + 3)}{2(\varepsilon - 1)}, \quad (\varepsilon > 1).$$

Thus we need to explore the solutions of the equation (5.3.12) in the cases:

(A) $f(\varepsilon) = \frac{3}{2}$, (B) $f(\varepsilon) = -\frac{(\varepsilon + 1)(\varepsilon + 3)}{2(\varepsilon - 1)}$, (C) $\varepsilon = 1$.

Equation (5.3.12) is now expressed as

$$\left(\lambda'' + f(\varepsilon)\lambda'\lambda''\right)' = \frac{a\lambda''}{\lambda'}\left(\lambda'' + f(\varepsilon)\lambda'\lambda''\right) + \frac{2c}{f}\lambda'\left(\lambda''\left(\lambda'' + f(\varepsilon)\lambda'\lambda''\right)\right)^{-1}(\lambda')^2,$$ \hspace{1cm} (5.3.13)

which gives

$$\lambda'' + \frac{f}{2}\lambda' = k\lambda'' \left(\frac{\varepsilon + 3}{2(\varepsilon - 1)}\right), \quad (k \text{ is an arbitrary constant}). \quad (5.3.14)$$

Multiplying throughout by $\frac{2}{f}\lambda'^2$, equation (5.3.14) can be written as (after absorbing some additional integration constants in $k$)

$$\left(c_{\lambda'}^{\varepsilon}\right)'' = k\lambda'' \left(\frac{\varepsilon + 3}{2(\varepsilon - 1)}\right) \left(c_{\lambda'}^{\varepsilon}\right)' = k\left(\left(c_{\lambda'}^{\varepsilon}\right)\right)^{a}, \quad (5.3.15)$$

where the expression on right hand side of equation (5.3.15) can be further simplified to give

$$k\left(c_{\lambda'}^{\varepsilon}\right)'' = k\left(\left(c_{\lambda'}^{\varepsilon}\right)^{a}\right)^{a} = k\left(c_{\lambda'}^{\varepsilon}\right)^{a(4\varepsilon - f)^{-2}} \left(c_{\lambda'}^{\varepsilon}\right)' \left(c_{\lambda'}^{\varepsilon}\right)^{-1}. \quad (5.3.16)$$

Thus using equations (5.3.15) and (5.3.16), we have

$$\left(c_{\lambda'}^{\varepsilon}\right)'' = k\left(c_{\lambda'}^{\varepsilon}\right)^{a(4\varepsilon - f)^{-2}} \left(c_{\lambda'}^{\varepsilon}\right)', \quad (5.3.17)$$

or

$$\left(c_{\lambda'}^{\varepsilon}\right)' \left(c_{\lambda'}^{\varepsilon}\right)^{1-a} = k\left(c_{\lambda'}^{\varepsilon}\right)^{a(4\varepsilon - f)^{-2}} \left(c_{\lambda'}^{\varepsilon}\right)' \left(c_{\lambda'}^{\varepsilon}\right)^{-1}. \quad (5.3.18)$$
which solves to give
\[
\left( e^{\epsilon \lambda} \right)' = \left[ k_1 \left( e^{\epsilon \lambda} \right)^{2\left(4c - df^2 + 2f / \epsilon^2 \right) / 2} + k_2 \right]^{1/2}. \tag{5.3.19}
\]

We write this equation as
\[
\lambda' = e^{-\frac{f}{\epsilon}} \left[ k_1 \left( e^{\epsilon \lambda} \right)^{2\left(4c - df^2 + 2f / \epsilon^2 \right) / 2} + k_2 \right]^{1/2}, \tag{5.3.20}
\]
which can be further constructed in the form
\[
\lambda' e^{\epsilon \lambda / 2} = e^{-\frac{f}{\epsilon}} \left[ k_1 \left( e^{\epsilon \lambda} \right)^{2\left(4c - df^2 + 2f / \epsilon^2 \right) / 2} + k_2 \right]^{1/2} = \left( e^{\epsilon \lambda} \right)'. \tag{5.3.21}
\]

Equations (5.3.8) and (5.3.21) are used now to write
\[
\left( e^{\epsilon \lambda} \right)' = e^{-\frac{f}{\epsilon}} \left[ k_1 e^{\left(4c - df^2 + 2f / \epsilon^2 \right) \lambda} + k_2 \right]^{1/2} = e^{-\frac{f}{\epsilon}}. \tag{5.3.22}
\]
This finally gives
\[
\epsilon' = \left[ k_1 e^{\left(4c + 6f - 3f / \epsilon + df^2 / \epsilon \right) \lambda} + k_2 \right]^{1/2}. \tag{5.3.23}
\]

Now we discuss the cases (A)–(C).

5.3.1. Case (A) $f = \frac{3}{2}$

In this case, equation (5.3.23), after using the values
\[
a = \frac{\epsilon + 1}{\epsilon}, \quad c = \frac{3c + 1}{16 \epsilon}
\]
reduces to
\[
\epsilon' = (k_1 e^{rac{\lambda (c + 3)}{\epsilon}} + k_2 e^{rac{\lambda (c - 1)}{\epsilon}})^{\frac{1}{2}}, \tag{5.3.24}
\]
giving the metric
\[
ds^2 = (k_1 e^{rac{\lambda (c + 3)}{\epsilon}} + k_2 e^{rac{\lambda (c - 1)}{\epsilon}})^{-\frac{2}{\epsilon}} dt^2 - d\rho^2 = e^\lambda \left( dy^2 + dz^2 \right), \tag{5.3.25}
\]
where by using equation (5.3.19), $e^\lambda$ is given by
\[
\int \frac{d\left( e^{\epsilon \lambda} \right)}{k_1 \left( e^{\epsilon \lambda} \right)^{2\left(4c - df^2 + 2f / \epsilon^2 \right) / 2} + k_2} = \alpha \rho + \beta. \tag{5.3.26}
\]

Thus metric (5.3.25) gives a class of perfect fluid solutions up to two arbitrary constants $k_1$ and $k_2$ with the equation of state $\omega = \epsilon p$, where $\omega, p > 0$.

To be more specific: if we choose $k_1 = 0$, equation (5.3.25) instantly gives $e^\lambda = (\alpha \rho + \beta)^2$ and the metric (5.3.25) reduces to
\[
ds^2 = (\alpha \rho + \beta)^{-\frac{2}{\epsilon}} dt^2 - d\rho^2 = (\alpha \rho + \beta)^2 (a^2 dt^2 + dz^2), \tag{5.3.27}
\]
which is the well known vacuum metric of Levi-Civita [29], later found by Kasner [30]; For $k_2 = 0$, equation (5.3.26) reduces to
\[
\int \frac{d\left( e^{\epsilon \lambda} \right)}{k_1 \left( e^{\epsilon \lambda} \right)^{2\left(4c - df^2 + 2f / \epsilon^2 \right) / 2} + k_2} = \alpha \rho + \beta, \tag{5.3.28}
\]
which solves to give
\[
e^\lambda = (\alpha \rho + \beta)^{-4\epsilon / \left(\epsilon + 1\right)},
\]
thus finally giving the metric
\[
ds^2 = (\alpha \rho + \beta)^{\frac{2}{\epsilon}} dt^2 - d\rho^2 = (\alpha \rho + \beta)^{\frac{4\epsilon}{\left(\epsilon + 1\right)}} (dy^2 + dz^2), \quad \epsilon > 1. \tag{5.3.29}
\]

Metric (5.3.29) is a perfect fluid solution with energy density ($\omega > 0$) and pressure ($p > 0$) given by
\[
\omega = \frac{4\epsilon (\epsilon - 1)(\epsilon + 7)}{(\epsilon + 1)^2(\epsilon + 3)^2(\alpha \rho + \beta)^2} = p, \quad \epsilon \in (1, \infty). \tag{5.3.30}
\]
The regularity of the solution can be observed at $p = 0$, whereas the Ricci scalar
\[ R = \frac{4a^2e - 1(e+1)(e+2)}{e+1} \frac{4a^2e - 21}{e+1} \]

The metric (5.3.29) serves as a perfect fluid source for the vacuum metric (5.3.27). If one puts $e = (\gamma - 1)^{-1}$ in the metric (5.3.29), the equation of state $\omega = e p$ is readily transformed to $p = (\gamma - 1)\omega$, whereas the solution (5.3.29) gets the form
\[ ds^2 = (\alpha e + \beta)\frac{4a^2e - 10}{e+1} dt^2 - d\rho^2 - (\alpha e + \beta)\frac{4a^2e - 10}{e+1} [dy^2 + dz^2]. \]

One may note that $\varepsilon > 1$ instantly gives $\gamma < 2$, on the other hand as $\varepsilon \to \infty$, $\gamma \to 1$. Thus in this case $\gamma$ gets necessarily the range $1 < \gamma < 2$. The metrics (5.3.29) and hence (5.3.31) are of Petrov type D and admit a $G_4$ as the maximal group of isometries.

Using $e^2 = x$ and $e = (\gamma - 1)^{-1}, 1 < \gamma < 2$, metric (5.3.25) reduces to the Taub’s metric [32] (metric 15.76, page 243 of reference [1]) in the explicit form as
\[ ds^2 = x^{-1} \left( ax^{\frac{2\gamma - 4}{\gamma - 2}} + b \right) \frac{1}{2} dt^2 - x \left( ax^{\frac{2\gamma - 4}{\gamma - 2}} + b \right)^{\frac{1}{2}} dx^2 - x^2(dy^2 + dz^2), \]

which satisfies the equation of state: $p = (\gamma - 1)\omega$, $1 < \gamma < 2$. Comparing the metrics (5.3.32) and (15.76a, reference [1]) yields:
\[ F(x) = \left( ax^{\frac{2\gamma - 4}{\gamma - 2}} + b \right)^{\frac{1}{2}}, \omega(x) = \frac{a(\gamma - 6)x^{\frac{2\gamma - 4}{\gamma - 2}}}{(\gamma - 2)x^{\frac{2\gamma - 4}{\gamma - 2}} - 1} = \frac{bx^4}{F} = 2x^\omega, \]

and the constraints given by (15.76b, reference [1]) are readily satisfied giving
\[ \frac{2xp}{\omega + p} = 4 - \frac{6\gamma}{\gamma - 2} + \frac{b(\gamma - 6)}{(\gamma - 2)}\left( ax^{\frac{2\gamma - 4}{\gamma - 2}} + b \right)^{-1} = 1 - \frac{bx^4}{F} = 2x^\omega, \]

and
\[ F'(x) = -\frac{a(\gamma - 6)}{(\gamma - 2)(\gamma - 2)} x^{\frac{2\gamma - 4}{\gamma - 2}} \left( ax^{\frac{2\gamma - 4}{\gamma - 2}} + b \right)^{\frac{1}{2}} = -x^2\omega. \]

One may like to probe the conditions given by Hojan and Santamaria and later by Collins [36, 37], where the metric (5.3.32) verifies the condition
\[ G(x) = \frac{b(2 - \gamma)}{a(\gamma - 6)} x^{\frac{2\gamma - 4}{\gamma - 2}} + \frac{(2 - \gamma)}{(\gamma - 6)} x^{\gamma}. \]

If we choose $e^2 = f$ and $e = (\gamma - 1)^{-1}, 1 < \gamma < 2$, metric (5.3.25) reduces to the Collins metric [37]
\[ ds^2 = f^{-1} \left( af^{\frac{2\gamma - 4}{\gamma - 2}} + b \right) \frac{1}{2} dt^2 - f \left( af^{\frac{2\gamma - 4}{\gamma - 2}} + b \right)^{\frac{1}{2}} dx^2 - f^2(dy^2 + dz^2), \]

with minor adjustments of the constants.

If one chooses
\[ U(x) = \frac{1}{2} \ln x + \ln(x^{\frac{2\gamma - 4}{\gamma - 2}} + b), \quad V(x) = \frac{1}{2x} \left( ax^{\frac{2\gamma - 4}{\gamma - 2}} + b \right)^{\frac{1}{2}}, \]

these appear explicit solutions of the metric (1.1) ((iii), table 1, [33]).

5.3.2 Case (B) $f = -\frac{e^{\gamma + 1} + e^{\gamma + 3}}{2(\gamma - 1)}$

In this case equation (5.3.23) reduces to equation (5.3.24) with $k_1 \leftrightarrow k_2$ and $e^\lambda$ (given by equation (5.3.19)) simplifies to give
\[ (e^{\frac{\gamma + 1 + 3\lambda}{e^{\gamma + 3}}})' = \left[ k_1 e^{-\gamma + \lambda} + k_2 \right]^{\frac{1}{2}}, \]

which can be written as
\[ \int \frac{d(e^{\frac{\gamma + 1 + 3\lambda}{e^{\gamma + 3}}} = (\alpha e + \beta). \]

For $k_1 = 0$,
\[ e^\lambda = (\alpha e + \beta) e^{\frac{\gamma - 3}{e^{\gamma + 3}}} \]

and one gets again the metric (5.3.31). For $k_2 = 0$,
\[ (e^{\frac{\gamma + 1 + 3\lambda}{e^{\gamma + 3}}})' = k_2 e^{\frac{\gamma + 7}{e^{\gamma + 3}}}, \]

(5.3.41)
which can be written as
\[
(e^{-\frac{1}{3}(\sqrt{1+4\lambda}^2)})' = k\left(e^{\frac{1}{3}(\sqrt{1+4\lambda}^2)}\right)\frac{\sqrt{1+4\lambda}}{3},
\] (5.3.42)
which gives
\[
\left(e^{-\frac{1}{3}(\sqrt{1+4\lambda}^2)}\right)\frac{\sqrt{1+4\lambda}}{3} = \alpha \rho + \beta.
\] (5.3.43)
This simplifies to give
\[
\left(e^{-\frac{1}{3}(\sqrt{1+4\lambda}^2)}\right)\frac{\sqrt{1+4\lambda}}{3} = \alpha \rho + \beta
\] (5.3.44)
which reduces to \(e^\lambda = (\alpha \rho + \beta)^2\) and \(e'^\omega = (\alpha \rho + \beta)^{-2}\), and gives again the Levi-Civita metric (5.3.27).

5.3.3. Case (C) \(\varepsilon = 1\)
Choosing \(\varepsilon = 1\), equation (5.3.9) reduces to
\[
\lambda'^\omega = \frac{2x^\lambda}{\lambda} + \frac{4}{1+4\lambda} \lambda'^\nu + \frac{3}{2}(\lambda')^3.
\] (5.3.45)
which can be written as
\[
\lambda'^\omega + \frac{3}{2} x' x'^\lambda = \frac{2x^\lambda}{\lambda} + \frac{7}{2} \lambda'^\nu + \frac{3}{2}(\lambda')^3.
\] (5.3.46)
and can be reduced to the form
\[
\left(\lambda'^\omega + \frac{3}{2} x'^\lambda\right)' = \frac{2x^\lambda}{\lambda} \left(\lambda'^\nu + \frac{3}{4} x'^\lambda\right) + 2\lambda'\left(\lambda'^\nu + \frac{3}{4} x'^\lambda\right).
\] (5.3.47)
If \(\lambda'^\nu + \frac{3}{4} x'^\lambda \neq 0\) (otherwise the metric reduces to the Levi-Civita metric), the above equation can be written as
\[
\frac{\lambda'^\nu + \frac{3}{4} x'^\lambda}{\lambda'^\nu + \frac{3}{4} x'^\lambda} = \frac{2x^\lambda}{\lambda} + 2\lambda'
\] (5.3.48)
which gives
\[
(e^{\lambda'}\nu) = k e^{\lambda'}((e^{\lambda'})')^2 = k(e^{\lambda'}\nu)^2 (e^{\lambda'})'.
\] (5.3.49)
Equation (5.3.49) is solved to obtain
\[
(e^{\lambda'})' = e^\nu (e^{\lambda'})^4,
\] (5.3.50)
which gives the transformation
\[
dx = e^{-k} (\tau^4) (e^{\lambda'})^4 d(e^{\lambda'}).\] (5.3.51)
Equation (5.3.8) now gets simplified to
\[
e' = \left(e^{\lambda'}\right)^{-1} e^{-2k} (e^{\lambda'})^4
\] (5.3.52)
Now using the transformation \(e^{\lambda'} = x\), metric (4.10) gets the form
\[
ds^2 = \frac{e^{-2x't} x^2}{x} dt^2 - x e^{-2x't} dx^2 - x^2 (dy^2 + dz^2),
\] (5.3.53)
with the equation of state \(p = 8k e^{2x't} = \omega\). Metric (5.3.53) gives a class of perfect fluid solutions up to an arbitrary constant \(k\). Clearly for \(k > 0\), \(p\) and \(\omega\) are positive. On the other hand if one chooses \(k = -\frac{1}{2}\) and \(x^2 = u\), metric (5.3.53) reduces to the Tabensky and Taub metric [34]
\[
ds^2 = \frac{e^{x^2} (dt^2 - dx^2)}{\sqrt{x}} - x (dy^2 + dz^2),
\] (5.3.54)
with the equation of state
\[
p = -\sqrt{x} e^{-x^2} = \omega
\] (5.3.55)
giving \(\omega\) and \(p\) both negative. Metric (5.3.53) admits \(G_4\) as the maximal isometry group. Metric (5.3.53) with \(k > 0\) gives \(p = \omega\) with \(\omega > 0\). Such metrics have been interpreted by Zeldovich and Novikov [54], describing radiation, relativistic degenerate Fermi gas and very dense baryon matter. Motion of the gas in this case cannot be rotational due to presence of exponential terms in the metric. Thus this source has the same stress-energy tensor as that of the massless scalar field.
The isotropy condition given by equation (5.3.4) can also be expressed as

\[ e^2 [e^{-2}(e^2)]' = -e^{-2} \frac{\lambda''}{2}. \]  
(5.3.56)

Using the transformation \( e^{-\frac{1}{2}} \frac{d}{d\rho} = e^{-\frac{1}{2}} \frac{d}{d\rho} (\mathbf{e}^2) \) and \( \frac{d}{d\rho}(\mathbf{e}^2) \) gives

\[ e^{-\frac{1}{2}} \frac{d}{d\rho} (\mathbf{e}^2) = e^{-\frac{1}{2}} \left[ e^{-\frac{1}{2}} \frac{d}{d\rho} \left( e^{-\frac{1}{2}} \frac{d}{d\rho} (\mathbf{e}^2) \right) \right] = e^{-\frac{1}{2}} \frac{d^2(\mathbf{e}^2)}{dx^2}. \]  
(5.3.57)

Thus equation (5.3.56) is transformed to

\[ e^{-\frac{1}{2}} \frac{d^2(\mathbf{e}^2)}{dx^2} = -e^{-\frac{1}{2}} \frac{d^2(\mathbf{e}^2)}{dx^2}, \]

(5.3.58)

which is a type of a perfect fluid metric whereas the non-zero curvature invariants for the metric are:

\[ \text{up to two arbitrary functions related by the constraint given by equation (5.3.58).} \]

The energy density and the pressure for this class of metrics reduce to

\[ \omega = e^{\mathbf{f}} \left[ \frac{\mathbf{d}(\mathbf{g})}{\mathbf{dx}} + \frac{\mathbf{f}}{4} \left( \frac{\mathbf{d}(\mathbf{g})}{\mathbf{dx}} \right)^2 \right], \]

(5.3.60)

and

\[ p = -e^{\mathbf{f}} \left[ \frac{\mathbf{d}(\mathbf{f})}{\mathbf{dx}} \right] \left( \frac{\mathbf{d}(\mathbf{g})}{\mathbf{dx}} + \frac{\mathbf{f}}{4} \left( \frac{\mathbf{d}(\mathbf{g})}{\mathbf{dx}} \right)^2 \right]. \]

(5.3.61)

The class of metrics (5.3.59) serves for generating perfect fluid sources for the vacuum metric (5.3.27). There may be at least two independent ways, one may try to find perfect fluid solutions from this class of metrics; either by fixing one of the \( f(x) \) and \( g(x) \) arbitrarily in equation (5.3.58) to find the other; or by fixing \( \omega \) in equation (5.3.60) to find \( g(x) \) and then using equation (5.3.58) to find \( f(x) \). For example, if we choose to find a solution for the equation of state \( p = -\omega (\omega > 0) \), equations (5.3.60)–(5.3.61) instantly give a condition

\[ \left( pe^{-\mathbf{f}} + \frac{\mathbf{f}}{4} \right) \left( pe^{-\mathbf{f}} + g'' + \frac{5g'''}{4} \right) = 0. \]

(5.3.62)

If \( pe^{-\mathbf{f}} + \frac{\mathbf{f}}{4} = 0 \), equation (5.3.61) gives \( f'g' = 0 \) and \( f' = 0, g' = 0 \) gives a conformally flat metric whereas \( f' \neq 0, g' = 0 \) gives a flat metric. If \( pe^{-\mathbf{f}} + g'' + \frac{5g'''}{4} = 0 \), equation (5.3.60) gives \( \omega + p = 0 \) for which equations (5.3.60)–(5.3.61) simplify to give

\[ g'' + g'''' - \frac{1}{2} f'g' = 0. \]

(5.3.63)

which solves to have

\[ e^\mathbf{f} = k(e^\mathbf{f})'. \]

(5.3.64)

Now equation (5.3.58) solves to give

\[ (e^\mathbf{f})' = (e^\mathbf{f})^{-\frac{1}{2}} \sqrt{a(e^\mathbf{f})^3 + b} \quad \text{and} \quad e^\mathbf{f} = e^{-\frac{1}{2}} \left[ a(e^\mathbf{f})^3 + b \right]. \]

(5.3.65)

This readily transforms the metric (5.3.59) to the form

\[ ds^2 = \frac{(ax^3 + b)}{x} dt^2 = x^2(dx^2 - x^2(dy^2 + dz^2)), \]

(5.3.66)

which is a type D metric satisfying: \( p = -\omega = -3a \). The invariants:

\[ I = \frac{3b^2}{4x^6}, \quad J = -\frac{b^3}{8x^9} \]

whereas the non-zero curvature invariants for the metric are:

\[ W_1 = \frac{3b^2}{2x^6} \quad \text{and} \quad W_2 = -\frac{3b^3}{4x^9}. \]

The metric (5.3.66) gives another explicit solution of the metric (1.1), the class ii of table 1 in [33]. This is the case of a perfect fluid where \( p + \omega = 0 \). The Ricci tensor in this case is of \( \Lambda - \) term type thus reducing the perfect fluid spacetime metric to an Einstein space [1].

One may readily find other two solutions of type D if one fixes each side of equation (5.3.58) as an arbitrary constant. The two possibilities of the arbitrary constant (>0 or <0) respectively give rise to two metrics:
\[ ds^2 = \cos^2(\alpha x + \beta) dt^2 - \frac{ds^2}{\cosh^2(\alpha x + \beta)} \]

\[ \omega = \alpha^2 [2 \cosh^2(\alpha x + \beta) + 3 \sinh^2(\alpha x + \beta)], \]

\[ p = \alpha^2 [\tan(\alpha x + \beta) \sinh(2(\alpha x + \beta)) - \sinh(\alpha x + \beta)], \]

where \( \omega, \ p > 0 \) on the domain

\[ x \in \left( -\infty, \frac{\beta - 3.5}{\alpha} \right) \cup \left( \frac{\beta - 3.5}{\alpha}, \infty \right); \]

\[ ds^2 = \cosh^2(\alpha x + \beta) dt^2 - \frac{dx^2}{\cosh^2(\alpha x + \beta)} - \cos^2(\alpha x + \beta) [dy^2 + dz^2], \]

\[ \omega = \alpha^2 [3 - 5 \cos^2(\alpha x + \beta)], \]

\[ p = \alpha^2 [\sinh(\alpha x + \beta) \sin(2(\alpha x + \beta)) - \cosh(\alpha x + \beta) \sin^2(\alpha x + \beta)], \]

where \( \omega > 0 \) on \( x \in \left( \frac{\beta - 2.5}{\alpha}, \frac{\beta - 4}{\alpha} \right) \) or \( x \in \left( \frac{\beta + 4}{\alpha}, \frac{\beta + 4.25}{\alpha} \right) \) only, giving two infinite tubes of perfect fluids when \( x \in \left( \frac{\beta - 4.25}{\alpha}, \frac{\beta - 4}{\alpha} \right) \) or \( x \in \left( \frac{\beta + 4}{\alpha}, \frac{\beta + 4.25}{\alpha} \right) \). Metrics (5.3.67)-(5.3.70) are plane symmetric static and admit \( G_4 \) as the maximal isometry group. It may be noted that the metric (5.3.67) is regular everywhere on the interval of validity and energy density is positive and regular everywhere, however pressure becomes infinite at a value which has been excluded from the domain. Metric (5.3.70) appears to have infinitely many singularities and so are the energy density and the pressure. By removing these singular points from both of these metrics, one may visualize these as tubes like perfect fluid sources surrounded by vacuum.

### 5.4. Type D metrics (4.12)-(4.14)

Finally the forms (4.12)-(4.14) of the type D metrics are explored according to their Segre types. We use the comparison of the Segre types with the non-zero Newman-Penrose components, \( \Phi_{AB} \) given in [54]: Segre type:

\[
\begin{align*}
1, 1 & 11 \quad 0; 11 00 22 02 00 \\
& \Leftrightarrow F^1 F^1 = F^0 F^0 = F^0 \\
1, 1 & 11 1 \quad 0; 11 00 22 02 \\
& \Leftrightarrow F^1 F^1 = F^0 F^0 = F^0 \\
1, 1 & 11 1 2 \quad 00 11 22 \\
& \Leftrightarrow F = F = F \\
1, 1 & 11 1 2 \quad 11 \\
& \Leftrightarrow F = F = F \\
\end{align*}
\]

For the metric (4.12) the surviving \( \Phi_{AB} \) are:

\[
\Phi_{00} = \frac{1}{8\beta'} [\alpha'^2 \beta' - 2\beta' \alpha'' + 2\alpha' \beta''] = \Phi_{22}, \quad (5.4.1)
\]

\[
\Phi_{02} = -\frac{1}{2\beta'} [\beta'' + \beta' \alpha'], \quad (5.4.2)
\]

\[
\Phi_{11} = \frac{1}{16\beta'^3} [4\beta' \beta'' + 8\beta' \alpha' \beta' + 3\beta' \beta'' + 3\beta' \alpha'^2 - 2\alpha' \beta'^2]. \quad (5.4.3)
\]

Thus for the type \( [(1, 1) 11] \), equation (5.4.1) gives either: \( \alpha \), a constant, which can be taken as zero without loss of generality and the resulting metric in this case reduces to

\[ ds^2 = \beta'^2 dt^2 - \beta^2 d\theta^2 - dz^2 - d\rho^2, \quad (5.4.4) \]

which gives an-isotropic stresses and an attempt to have isotropy reduces it to a flat metric; or \( \alpha \) is arbitrary with

\[ \beta'^2 (\alpha')^{-2} e^{\alpha} = \frac{b^2}{4}, \quad \beta = \frac{b}{2} e^{-z} + a. \]

This gives the metric

\[ ds^2 = b^2 (e^z)^2 dt^2 - (b + ae^z)^2 d\theta^2 - e^{\alpha} dz^2 - d\rho^2. \quad (5.4.5) \]

It is straightforward to check that the transformation

\[ d\rho = \frac{d\alpha}{\sqrt{E - Fe^{-2\alpha}}}, \quad (5.4.5a) \]

with \( \alpha(\rho) = x \) reduces the metric (5.4.5) to

\[ ds^2 = (Ee^z + Fe^{-z}) dt^2 - e^z \left( \frac{dx^2}{Ee^z + Fe^{-z}} + dy^2 + dz^2 \right). \quad (5.4.6) \]

which admits a maximal isometry group \( G_{6W} \), \( \omega = \frac{3}{4} E = -p \), and the invariants:

\[ I = \frac{3}{512} F^3 e^{-3z}, \quad J = \frac{F^3}{512} e^{-\frac{3}{2}z}. \]
whereas the non-zero curvature invariants:

$$W_1 = \frac{3}{32}F^2e^{-3x}, \quad W_2 = -\frac{3}{256}F^3e^{-\frac{3}{2}x},$$

show that the metrics (5.4.6) and (5.3.66) are isometric. Otherwise there are anisotropic stresses.

For the type \([1, (11)]\), to have an isotropic pressure the metric reduces to the type \(\mathbf{O}\) metric (3.26) whereas for the other types, the metric either gives anisotropic stresses or otherwise reduces to a flat metric.

For the metric (4.13), the surviving \(\Phi_{AB}\) are

$$\begin{align*}
\Phi_{00} &= \frac{1}{8\beta^3}[-\beta\beta'\alpha'^2 - 3\beta\beta'\alpha'' - 3\beta'\alpha'^2 - 2\beta'\alpha'' + 3\alpha'\beta'^2 + 2\beta'\beta''] = \Phi_{22}, \\
\Phi_{02} &= \frac{1}{8\beta^3}[2\beta\beta'\alpha'^2 + \beta\beta'\alpha'' + 5\beta'\alpha'^2 + 2\beta'\beta' + 2\alpha'^2 + 2\beta'\beta''], \\
\Phi_{11} &= \frac{1}{16\beta^3}[-\beta\beta'\alpha'^2 + 2\beta\beta'\alpha'' - 2\beta'\alpha' + 4\alpha'\beta'^2 + 4\beta'\beta''].
\end{align*}$$

The energy density and the pressures for this metric reduce to

$$\begin{align*}
\omega &= \frac{1}{4\beta^3}[7\beta'\alpha'^2 + 6\beta\alpha'' + 10\alpha'\beta'' + 4\beta''], \\
p_1 &= -\frac{1}{4\beta^3}[6\alpha'\beta'^2 + 4\beta'\beta'' + 5\beta'\alpha'^2 + 4\alpha'\beta''], \\
p_2 &= -\frac{1}{4\beta^3}[8\alpha'\beta'^2 + 7\beta\beta'\alpha'^2 + 6\beta'\alpha'' + 10\alpha'\beta'' + 8\beta'\beta'' + 4\beta'^2], \\
p_3 &= -\frac{1}{4\beta^3}[4\beta'' + 6\alpha'\beta''] + 3\beta'\alpha'^2 + 4\beta'\beta''.
\end{align*}$$

The isotropy requirement gives

$$\begin{align*}
\alpha'\beta'^2 + 2\beta'\beta'' + 3\beta'\alpha'^2 + 3\beta'\alpha'' + 2\beta'\beta'' &= 0, \\
\alpha'\beta'^2 + 2\beta'\beta'' + 2\beta'\alpha'^2 + 5\beta'\alpha'' + 2\beta'\beta'' &= 0, \\
\beta'\alpha'^2 + 2\beta'\beta'' - 2\beta'\beta'' &= 0.
\end{align*}$$

For the type \([1, (11)]\), one gets anisotropic stresses or otherwise, the equations (5.4.13)–(5.4.15) together with the equations \(\Phi_{00} = 0 = \Phi_{22}\) force to give a flat metric.

For the type \([1, (11)]\), the requirement \(\Phi_{02} = 0\) is identical to the equation (5.4.14) and equation (5.4.13) together whereas equation (5.4.15) also reduces to equation (5.4.14). Thus in order to obtain an isotropic solution one needs to solve the coupled system of two ODEs (5.4.14) and (5.4.15). From equation (5.4.15), \(\beta' \neq 0\) due to metric constraint and we have two possibilities:

(A) \(\alpha' = 0\); (B) \(\alpha' \neq 0\).

The possibility (A) gives

$$ds^2 = \beta^2 dt^2 - dp^2 - C^2 d\theta^2 - (2a \ln \beta + b) dz^2,$$

where

$$\beta' = \sqrt{b + 2a \ln \beta}$$

giving the transformation

$$dp = (b + 2a \ln \beta)^{-\frac{1}{2}} d\beta,$$

which reduces the metric to the form

$$ds^2 = \beta^2 dt^2 - \frac{d\beta^2}{b + 2a \ln \beta} - C^2 d\theta^2 - (2a \ln \beta + b) dz^2,$$

with \(\omega = -\frac{\gamma}{4x} = p\) (\(a\) is arbitrary). Metric (5.4.16) gives a class of metrics up to parameter \(a\). For \(a < 0\), \(\omega\) and \(p\) are positive for the metric (5.4.16). Metric (5.4.16) admits \(G_3\) as the maximal isometry group; the non-zero Newman- Penrose spin coefficients for the metric are:

$$\begin{align*}
\gamma &= \frac{\sqrt{2}(b + 2a \ln \beta)}{4x}, \\
-\rho &= -\mu = \sigma = \lambda = \frac{a\sqrt{2}}{4x\sqrt{b + 2a \ln \beta}}.
\end{align*}$$
The invariants:
\[ I = \frac{a^2}{3\beta^4}, \quad J = \frac{a^3}{27\beta^6} \quad \text{and} \quad R = -\frac{2a}{\beta^2}. \]

The transformation
\[ a\beta^2 = e^{\alpha^2} r^2 (a > 0) \quad \text{with} \quad b = 0 \]
and redefinition of \( \theta \) and \( z \), reduces metric (5.4.16) to the form
\[ ds^2 = e^{\alpha^2 r^2} (dr^2 - dr^2) - r^2 d\theta^2 - dz^2, \quad (5.4.18) \]
with
\[ \omega = -\alpha^2 e^{-\alpha^2 r^2} = p, \]
showing that \( \omega \) and \( p \) are negative for the metric (5.4.18). The type D metric (5.4.18) earlier found by Kramer [40] can also readily be obtained directly from the metric (4.8), if one chooses to put:
\[ e^{- r} d \rho d r = e^{\alpha^2 r^2} dr, \quad k_4 = 0, \quad a = 1 \]
and absorbs a constant in the definition of \( t \).

The possibility (B) gives
\[ \beta(x) = b - ae^{-\frac{\omega}{a}} \]
and the metric (4.13) becomes
\[ ds^2 = (be^x - a)^2 dt^2 - dx^2 - e^{\alpha^2} d\theta^2 - \frac{a^2}{4} e^{\alpha^2} dz^2, \quad (5.4.19) \]
with the constraint
\[ \left( \left( b - ae^{-\frac{\omega}{a}} \right) e^{\alpha^2 \alpha''} \right)' = 0. \]

This solves to give
\[ \frac{\alpha'^2}{4} = \frac{kb}{a^2} e^{-\frac{\omega}{a}} + \frac{k}{2a} e^{-\alpha} + \frac{kb^2}{2a^3} \ln \left( b - ae^{-\frac{\omega}{a}} \right)^2 + C. \]

The transformation
\[ \frac{\alpha'}{2} = \frac{1}{k_4 + x \, d\rho} \]
then reduces (5.4.19) to the metric (5.2.26).

For the metric (5.4.14), the surviving \( \Phi_{AB} \) are
\[ \Phi_{00} = -\frac{1}{8\beta} \left[ \beta \alpha'^2 + 3\beta' \alpha'' + 3\alpha \beta' \alpha'' + 2\beta \alpha'' + \alpha \beta' + 2\beta' \alpha'' \right] = \Phi_{22}, \]
\[ \Phi_{02} = -\frac{1}{8\beta} \left[ 2\beta \alpha'^2 + \beta \alpha'' + 3\alpha \beta' \alpha'' + 2\beta \alpha'' - 3\alpha \beta' + 2\beta' \alpha'' \right] = \Phi_{20}, \]
\[ \Phi_{11} = -\frac{1}{16\beta} \left[ \beta \alpha'^2 - 2\beta \alpha'' + 2\alpha \beta' \alpha'' + 4\alpha \beta' + 4\beta' \alpha'' \right]. \]

For a perfect fluid solution, we use the Segre type [1, (111)]. Using \( \Phi_{02} = 0 \) and \( \Phi_{00} = 2\Phi_{11} = \Phi_{22} \), one gets
\[ 2\beta \alpha'^2 + \beta \alpha'' + 3\alpha \beta' \alpha'' + 2\beta \alpha'' - 3\alpha \beta' + 2\beta' \alpha'' = 0, \]
\[ 5\beta \alpha'' + \beta \alpha' + 2\beta' \alpha'' + 3\alpha \beta' - 2\beta' \alpha'' = 0. \]

From these equations, one gets
\[ \beta \alpha'^2 - 2\beta \alpha'' + 2\alpha \beta' = 0. \]

There are two possibilities: either \( \alpha' = 0 \), which gives \( \beta \alpha'' = \beta' \beta'' = 0 \) from equations (5.4.25)–(5.4.26). This reduces metric (4.14) to the type O metric (3.25); otherwise, equation (5.4.27) gives \( \beta(x) = b - ae^{-\frac{\omega}{a}} \) and the metric reduces to
\[ ds^2 = e^{\alpha(x)} dt^2 - dx^2 - a^2 \left( e^{\alpha(x)} \right)^2 d\theta^2 - \left( e^{\alpha(x)} - a \right)^2 dz^2, \quad (5.4.28) \]
where \( \alpha(x) \) satisfies the constraint (obtained using equation (5.4.25) or equation (5.4.26))
\[ \alpha'^2(x) + \frac{3}{2} \alpha'(x) \alpha''(x) = \frac{a(\alpha'(x)^2 + \alpha'(x)\alpha''(x))}{2(\beta \alpha'' - a)}. \]
The energy density and the pressure for this metric reduce to
\[
\omega = \frac{1}{4(b - ae^{-\frac{a}{b}})}[ae^{-\frac{a}{b}(2\alpha'' + \alpha'^2)} + b(4\alpha'' + 3\alpha'^2)],
\]
\[
p = \frac{1}{4(b - ae^{-\frac{a}{b}})}[ae^{-\frac{a}{b}(2\alpha'' + \alpha'^2)} - b(4\alpha'' + 3\alpha'^2)].
\]

Now equation (5.4.29) can be transformed to
\[
(b^{-\frac{a}{b}} - a)\omega + a \left( \frac{b^{-\frac{a}{b}} - a}{b^{-\frac{a}{b}} - a} \right)^{-\frac{\alpha''}{\alpha'}} - \left( \frac{b^{-\frac{a}{b}} - a}{b^{-\frac{a}{b}} - a} \right)^{\alpha'} = 0.
\]

Using the transformation
\[
\ln(b^{-\frac{a}{b}} - a) = y(x),
\]
equation (5.4.32) reduces to
\[
(y''e^{2y}(e^y + a))' = 0,
\]
whereas the metric (5.4.28) gets reduce to
\[
ds^2 = \left( \frac{e^{y(x)} + a}{b} \right)^2 dt^2 - dx^2 - \frac{a^2}{b^2}[(e^{y(x)})']^2 d\theta^2 - e^{2y(x)}dz^2.
\]

Equation (5.4.33) then solves to give a transformation
\[
\frac{dy}{dx} = \sqrt{\frac{2e^{-y}}{a^2} - e^{-2y} - \frac{1}{a^2}\ln(1 + ae^{-y})^2 + C}
\]
which reduces the metric (5.4.34) to the metric (5.2.26).

### 6. Metrics representing non-null Einstein-Maxwell fields

Metrics admitting maximal \( G_3 \) or \( G_4 \) do not admit a null Killing vector, thus can not admit a null Einstein-Maxwell (EM) field. The Segre type \((1, 1; 1, 1) \Leftrightarrow \Phi_{11}\) gives metrics which represent non-null EM fields. Rainich conditions [57] for such metrics give \( T = 0 = R \), whereas the Segre types provide the conditions \( \Phi_{00} = 0 = \Phi_{22} \) and \( \Phi_{02} = 0 \). The union of the two sets of type \( \mathbf{D} \) metrics (4.10) and (4.12)–(4.14) cover the type \( \mathbf{D} \) metrics (4.5), (4.6) and (4.8). In order to explore the metrics representing non-null EM fields, we explore the metrics: (4.10) and then (4.12)–(4.14).

For the metric (4.10), \( \Phi_{03} \equiv 0 \) whereas from \( \Phi_{00} = 0 = \Phi_{22} \) we have
\[
\frac{\lambda''}{2} + \frac{\lambda'^2}{4} - \nu'\lambda' = 0,
\]
which solves to give
\[
e^{-\frac{\nu(x)}{2}} = \left( e^{\frac{\lambda(x)}{2}} \right)^\lambda.
\]

Thus the metric (4.10) reduces to
\[
ds^2 = \left( e^{\frac{\lambda(x)}{2}} \right)^2 dt^2 - dx^2 - e^{\lambda(x)}(dy^2 + dz^2).
\]

Now the Rainich condition, \( R = 0 \) for this metric simplifies to
\[
2\lambda'' + 7\lambda'\lambda'' + 3\lambda'^2 = 0.
\]

Equation (6.4) can be transformed to
\[
(\lambda'' + \lambda'^2)' + \frac{3}{2}\lambda'(\lambda'' + \lambda'^2) = 0,
\]
which then solves to give
\[
(e^\lambda)' = e^{-\frac{\lambda}{2}}\sqrt{a\lambda^2 + b},
\]
which transforms the metric (6.3) to
\[
ds^2 = \frac{ax + b}{x^2} dt^2 - \frac{x^2}{ax + b} dx^2 - x^2(dy^2 + dz^2).
\]
The metric gives the non-zero stress-energy tensor components as:

\[ T_{0}^{0} = - \frac{b}{x^4} = T_{1}^{1}; \quad T_{2}^{2} = \frac{b}{x^4} = T_{3}^{3}. \tag{6.8} \]

The surviving NP components for this metric are

\[ \Psi_{2} = \frac{ax + 2b}{2x^4}; \quad \rho = -\frac{\sqrt{ax + b}}{\sqrt{2}x^2} = \mu; \quad \gamma = -\frac{ax + 2b}{4\sqrt{2}x^2\sqrt{ax + b}} = \varepsilon. \tag{6.9} \]

The electric and magnetic field intensities for the metric are given as:

\[ E = \begin{bmatrix} \sqrt{-b}/x^2, 0, 0, 0 \end{bmatrix}; \quad H = \begin{bmatrix} \sqrt{-b}, 0, 0, 0 \end{bmatrix} \]

respectively, which shows that \( b < 0 \). The metric satisfies the conditions: \( \kappa\kappa + \sigma\sigma = 0; \rho \neq 0 \), is of Petrov type \( D \) and admits \( G_{4} \supset G_{3} \) as the maximal isometry group. The invariants \( I \) and \( J \) for the metric (6.7) reduce to:

\[ I = 3\left(\frac{ax + 2b}{2x^4}\right)^2 \quad \text{and} \quad J = -\left(\frac{ax + 2b}{2x^4}\right)^3. \tag{6.10} \]

The curvature invariants for the metric (6.7) appear to be non-zero except the invariants \( R_{2} \) and \( M_{4} \) like the McVittie metric [47], showing the equivalence of the two metrics.

For the metrics (4.12)–(4.14), one can show that there does not exist a non-trivial solution. Here we show it for the metric (4.13) and the same argument is applicable for the metrics (4.12) and (4.14) as well.

For the metric (4.13), the conditions \( \Phi_{00} = 0 = \Phi_{22}, \Phi_{02} = 0 \) are obtained from equations (5.4.7)–(5.4.8) whereas the condition \( R = 0 \) gives

\[ (12\beta\beta'' + 10\beta'^{2})\alpha' + 11\beta\beta'\alpha'' + 8\beta\beta'\alpha'' + 4\beta\beta'' + 8\beta'\beta'' = 0. \tag{6.11} \]

The conditions

\[ \Phi_{00} = 0 = \Phi_{22}, \Phi_{02} = 0 \]

And equation (6.11) together form a coupled system of three non-linear ODEs to give \( \alpha(x) \) and \( \beta(x) \). We view these equations as a linear algebraic system of three equations in the unknowns \( \alpha', \alpha'' \) and \( \beta'' \). Noting \( \beta \neq 0 \neq \beta' \) due to the metric constraints, \( \beta'' = 0 \) appears as the necessary condition for the system to be homogeneous. The determinant of the coefficients matrix:

\[ 40\beta'^{2}(2\beta\beta'' - \beta') \]

of the homogeneous system then reduces to \(-40\beta'^{4}\), which cannot vanish in order to have a non-trivial solution. Thus in this case one gets \( \alpha' = 0 \) and the metric (4.13) then reduces to a flat metric. If we choose to have \( \beta'' 
eq 0 \) and assume \( 2\beta\beta'' - \beta' > 0 \), the system (5.4.7)–(5.4.8), (6.11) then gives

\[ \alpha' = -\frac{\beta\beta''}{2\beta\beta'' - \beta'}, \quad \alpha'' = \frac{2}{5}\left(2\beta'\beta'^{2} + 5\beta'\beta'' - 8\beta'\beta'' + 4\beta\beta'^{2} + 4\beta\beta''\beta''\right), \tag{6.12} \]

\[ \alpha'' = -\frac{1}{5}\left(3\beta'^{2} + 5\beta'\beta'' - 12\beta'\beta'^{2} + 6\beta'\beta''\right). \tag{6.13} \]

We need now to check the the compatibilities of \( \alpha', \alpha'' \) and \( \beta'' \). The compatibility of \( \alpha', \alpha'' \) give

\[ \alpha'' = \frac{4}{5}\left(\frac{2\beta\beta'' - \beta''}{\beta'}\right) \quad \text{or} \quad \beta' = \frac{2\beta'\beta'^{2} - \beta'^{2}}{\beta'} \tag{6.14} \]

Case (a) reduces (6.12)–(6.13) to

\[ \alpha' = -\frac{4\beta'}{5\beta}, \quad \alpha'' = \frac{16\beta'^{2}}{25\beta'^{2}}, \quad \alpha'' = -\frac{2}{25}\beta'^{2} < 0. \tag{6.15} \]

Now the compatibility of \( \alpha' \) and \( \alpha'' \) instantly gives \( \beta' = 0 \) but then this contradicts to the consistency of the metric. Case (b) gives

\[ \alpha' = -2\frac{\beta'}{\beta}, \quad \alpha'' = 4\frac{\beta'^{2}}{\beta'^{2}}, \quad \alpha'' = -2\frac{\beta'^{2}}{\beta'^{2}}. \tag{6.16} \]

and the compatibility of \( \alpha' \) and \( \alpha'' \) gives \( \beta'^{2} - 2\beta'^{2} = 0 \). This solves to give

\[ \beta(x) = \frac{1}{ax + b}, \quad \alpha(x) = 2\ln(ax + b), \tag{6.17} \]

and the metric (4.13) in this case reduces to the Minkowski metric. This proves that the metric (6.7) is the only spacetime metric of type \( D \) which represents a non-null Einstein-Maxwell field and admits a maximal \( G_{4} \supset G_{3} \).
The uniqueness of the metric (6.7) and the fact that it has features similar to the McVittie metric proves that the metric (6.7) and the McVittie metrics are equivalent. Thus the only spacetime metric which represents a non-null Einstein-Maxwell field and admits a maximal \( G_1 \supset G_3 \) is the McVittie metric. This feature of the McVittie solution has already been discussed in [56], where the basis of the argument was different.

7. Conclusion

The classification of the Lorentzian manifolds admitting minimal Abelian isometry group of dimension three according to their Petrov types and metrics is achieved and is used to find the perfect fluid solutions of these manifolds, besides proving that the type D metric (6.7), admitting a maximal \( G_1 \supset G_3 \) is unique to represent the non-null Einstein-Maxwell field and the metric (6.7) is equivalent to the McVittie metric [47, 56].

As a result of this study there appear metrics (3.10)–(3.30) which are conformally flat and admit maximal \( G_1 \supset G_3 \), where \( r = 4, 6, 7 \) and 10. Metrics (4.5), (4.6) and (4.8) are the only classes of metrics which are of type D and admit a maximal \( G_5 \). Metric (4.10) (which is isometric to the class (1.1)) appears as a special case of the metrics (4.5)–(4.8) and admits maximal: \( G_0 \), if \( \lambda (\rho) = 0 \) and \( \nu (\rho) = 2 \cosh (ap + b) \) or \( 2 \cos (ap + b) \) or \( ap; G_5 \) if \( \lambda (\rho) = bp \) and \( \nu (\rho) = 0 \) or if \( \nu (\rho) = ap (a \neq b) \) otherwise it admits a maximal \( G_4 \).

Conformally flat perfect fluid spacetimes are then explored to have: metric (3.10) with \( \sigma (\rho) = A \rho \) admitting a maximal \( G_0 \) so \((2, 3) \supset G_0 \), isometric to the deSitter metric; the metric (3.12) shown to reduce to metric (5.4.16) was missed in the investigations made in [33] and satisfies the equation of state \( p = \omega \) with \( \omega > 0 \) for \( a < 0 \). Exact solutions with this type of equation of state describe radiation, relativistic degenerate Fermi gas and probably very dense baryon matter according to the investigations made by Zeldovich and Novikov [54]. For a maximal \( G_0 \), the requirement for a perfect fluid with an equation of state: \( \omega = \varepsilon p \) reduces to a third order non-linear ODE (5.3.9). A complete solution of this ODE in closed analytic form is achieved, which gives three independent classes of perfect fluid spacetime metrics: (5.3.25) which may be considered a new class up to two arbitrary constants \( k_1 \) and \( k_2 \) when \( \varepsilon > 1 \). This class reduces to the empty solution of Levi Civita and Kasner [5.3.27] [29, 30] for \( k_1 = 0 \). For \( k_2 = 0 \), the class reduces to the perfect fluid solution (5.3.29) with the equation of state \( \omega = \varepsilon p \). Evidently the spacetime metric (5.3.29) serves as a source for the vacuum metric (5.3.27). Metric (5.3.29) was then shown to reduce to the perfect fluid metric found implicitly by Taub, later found by Hojman and Santamaria, and Collins explicitly [32, 36, 37] for \( k_2 = 0 \); (5.3.53) to a new class up to an arbitrary constant \( k \) when \( \varepsilon = 1 \) and reduces to the Tabenski and Taub metric with \( P = \omega (\omega < 0) \) [34] for \( k < 0 \). The other metric from this class gives \( p = \omega \) with \( (\omega > 0) \) for \( k > 0 \). Zeldovich and Novikov interpretation is also applicable on this metric while Tabenski and Taub [34] added that if in addition the motion of the gas is irrotational, then such a source has the same stress-energy tensor as that of the massless scalar field. Wainwright et al. noted that the mass-less scalar field solutions can in turn be interpreted as vacuum solutions in the Brans-Dicke theory [24.1] and (5.3.59) (which appears as a new class up to two arbitrary functions satisfying a constraint (5.3.58). This class is capable of generating new exact solutions other than the metrics: (5.3.66), (5.3.67) and (5.3.70) obtained here. All these metrics arise as explicit solutions of the class of metrics (4.10) and hence of (1.1)). Metrics (5.3.66) and (5.4.6) appear isometric and represent perfect fluid spacetime metrics with \( \omega > 0 \) where \( p + \omega = 0 \). The Ricci tensor in this case is of \( \Lambda \)–term type thus reducing the perfect fluid spacetime metrics to Einstein spaces [1]. Metrics (5.3.67) and (5.3.70), each appear to represent two infinite tubes of perfect fluids. In section 6, type D exact solutions with non-null Einstein-Maxwell fields are also probed and it is proved that the only metric with this property is the metric (6.7), which appears to be equivalent to the McVittie solution [47, 56].

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