Abstract

We give an alternative proof and improve upon a result of S.M. Kozlov [7]. It deals with the asymptotic of the integrated density of states of the acoustic operator $H_\omega = -\nabla \rho_\omega \nabla$, at the bottom of the spectrum.

2000 Mathematics Subject Classification : 81Q10, 35P05, 37A30, 47F05.

Keywords and phrases : spectral theory, random operators, integrated density of states, Lifshitz tails, homogenization.

1 Introduction

Let $H_\omega$, be the self adjoint operator on $L^2(\mathbb{R}^d)$ formally defined by:

$$H_\omega = H(\rho_\omega) = -\nabla \cdot \rho_\omega \cdot \nabla,$$

(1.1)

where $\rho_\omega$ is a positive and bounded function.

$H_\omega$ is called the acoustic operator, see [1] for the physical interpretations.

Let us start by defining the main object of our study: the integrated density of states. For this, we consider $\Lambda$ a cube of $\mathbb{R}^d$. We note by $H_{\omega,\Lambda}$ the restriction of $H_\omega$ to $\Lambda$ with self-adjoint boundary conditions. As $H_\omega$ is elliptic, the resolvent of $H_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $H_{\omega,\Lambda}$

---

1Researches partially supported by CMCU N 02/F1511 and N 04/S1404 projects.
is discrete and is made of isolated eigenvalues of finite multiplicity \[13\]. We define 
\[
N_\Lambda(E) = \frac{1}{\text{vol}(\Lambda)} \cdot \# \{\text{eigenvalues of } A_\omega, \Lambda \leq E \}. \tag{1.2}
\]
Here \(\text{vol}(\Lambda)\) is the volume of \(\Lambda\) in the Lebesgue sense and \(#E\) is the cardinal of \(E\).

It is shown that the limit of \(N_\Lambda(E)\) when \(\Lambda\) tends to \(\mathbb{R}^d\) exists almost surely and is independent of the boundary conditions. It is called the \textbf{integrated density of states} of \(A_\omega\) (IDS as acronym). See \[12\].

The question we are interested in here regards the behavior of \(N\) at the bottom of the spectrum of \(H_\omega\). In previous works \[8, 9, 10, 11\], the author gives the behavior of \(N\) at the internal band edges of the spectrum of \((1.1)\). It was a Lifshitz behavior (\(N\) decreases exponentially fast). In the present situation, for the bottom of the spectrum, it is known that it can’t decrease more than polynomially fast, \[7\]. Here we compare the behavior of \(N\) to the behavior of the IDS of some periodic operator with exponentially precision.

\textbf{Acknowledgements.} The author would like to thank professor Frédéric Klopp for interesting discussion concerning this work and professor Mabrouk Ben Ammar for many helpful.

1.1 The model

Consider the random Schrödinger operator
\[
H_\omega = -\nabla \frac{1}{\rho_\omega} \nabla. \tag{1.3}
\]
Where \(\rho_\omega\) is a bounded, \(\mathbb{Z}^d\)-ergodic random field such that there exists some constant \(\rho_* > 1\), satisfying
\[
\rho_* \leq \rho_\omega \leq \rho_* \tag{1.4}
\]
We assume that \(\rho_\omega\) is of Anderson type i.e. it has the form
\[
\rho_\omega(x) = \rho^+(x) + \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma \rho^0(x - \gamma) \tag{1.5}
\]
where
• $\rho^+$ is a $\mathbb{Z}^d$-periodic measurable function,

• $\rho^0$ is a compactly supported measurable function,

• $(\omega_{\gamma})_{\gamma \in \mathbb{Z}^d}$ are non trivial, i.i.d. random variables.

The choice of our model ensures that $A_\omega$ is a measurable family of self-adjoint operators and ergodic [4, 12]. Indeed, if $\tau_\gamma$ refers to the translation by $\gamma$, then $(\tau_\gamma)_{\gamma \in \mathbb{Z}^d}$ is a group of unitary operators on $L^2(\mathbb{R}^d)$ and for $\gamma \in \mathbb{Z}^d$ we have

$$\tau_\gamma A_\omega \tau_{-\gamma} = A_{\tau_\gamma \omega}.$$ 

According to [4, 12], we know that there exists $\Sigma, \Sigma_{pp}, \Sigma_{ac}$ and $\Sigma_{sc}$ closed and non-random sets of $\mathbb{R}$ such that $\Sigma$ is the spectrum of $A_\omega$ with probability one and such that if $\sigma_{pp}$ (respectively $\sigma_{ac}$ and $\sigma_{sc}$) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of $A_\omega$, then $\Sigma_{pp} = \sigma_{pp}, \Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability one. $H_\omega$ is defined as the Friedrichs extension of the following positive quadratic form

$$H_\omega[\psi] = \langle \rho_\omega \nabla \psi, \nabla \psi \rangle, \psi \in H^1_{loc}(\mathbb{R}^d).$$

1.2 The result

We shall prove

Theorem 1.1. There exists $\alpha$, $\tau > 0$ and $C > 1$ such that when $E \to 0^+$ we have,

$$\overline{\mathcal{N}}(E - E^\alpha) - Ce^{-E^{-\tau}} \leq \mathcal{N}(E) \leq \overline{\mathcal{N}}(E + E^\alpha) + Ce^{-E^{-\tau}} \quad (1.6)$$

where $\overline{\mathcal{N}}$ is the integrated density of states of the following periodic operator

$$\mathcal{H} = -\nabla \overline{\rho} \nabla; \quad (1.7)$$

and $\overline{\rho} = \mathbb{E}(\rho_\omega)$.

Remark 1.2. 1) The improvement over Kozlov’s result essentially consists in the estimate of the remainder term and the exponential precision .

$$\mathcal{N}(E) \to \overline{\mathcal{N}}(E + o(E))$$ exponentially as $E \to 0^+$. 

3
2) We don't believe this estimate to be optimal: namely, we expect the exponent \( \alpha \) to be larger than the one we found in the present study.

2 Proof of Theorem 1.1

2.1 The periodic approximation

Pick \( n \in \mathbb{N} \setminus \{0\} \) and define the following periodic Schrödinger operator

\[
H^n_\omega = -\nabla \rho^n_\omega \nabla.
\]

Here,

\[
\rho^n_\omega = \rho^{+,n} + \rho^{0,n} = \rho^+(x) + \sum_{\gamma \in \Lambda_n \cap \mathbb{Z}^d} \omega \sum_{\beta \in (2n+1)\mathbb{Z}^d} \rho^0(x - \gamma + \beta).
\]

Where \( \Lambda_k \) is the cube

\[
\Lambda_n = \{ x \in \mathbb{R}^d; \forall 1 \leq j \leq d, -\frac{2k+1}{2} < x_j \leq \frac{2k+1}{2} \}.
\]

For \( \omega \) fixed and \( n \in \mathbb{N}^* \), \( H^n_\omega \) is a \((2n+1)\mathbb{Z}^d\)-periodic self-adjoint Schrödinger operator.

Let \( \omega = E(\omega_0) \) and \( \overline{\rho^{0,n}} = \sum_{\gamma \in \Lambda_n \cap \mathbb{Z}^d} \sum_{\beta \in (2n+1)\mathbb{Z}^d} \rho^0(x - \gamma + \beta) \).

2.2 Some Floquet Theory

Now we review some standard facts from the Floquet theory for periodic operators. Basic references of this material are in [13].

Let the torus \( \mathbb{T}^{2n+1} = \mathbb{R}^d / 2\pi(2n+1)\mathbb{Z}^d \). We define \( \mathcal{H}_n \) by

\[
\mathcal{H}_n = \{ u(x, \theta) \in L^2_{\text{loc}}(\mathbb{R}^d) \otimes \mathcal{L}(\mathbb{T}^{2n+1}_d); \forall (x, \theta, \gamma) \in \mathbb{R}^d \times \mathbb{T}^{2n+1}_d \times (2n+1)\mathbb{Z}^d; u(x + \gamma, \theta) = e^{i\gamma \theta} u(x, \theta) \}.
\]

There exists \( U \) a unitary isometry from \( L^2(\mathbb{R}^d) \) to \( \mathcal{H}_n \) such that \( H^n_\omega \) admits the following Floquet decomposition [13]

\[
U H^n_\omega U^* = \int_{\mathbb{T}^{2n+1}_d} H^n_\omega(\theta) d\theta.
\]
Here $H_\omega^n(\theta)$ is the self adjoint operator on $\mathcal{H}_{n,\theta}$ defined as the operator $H_\omega^n$ acting on $\mathcal{H}_{n,\theta}$ with

$$\mathcal{H}_{n,\theta} = \{ u \in L^2_{\text{loc}}(\mathbb{R}^d); \forall \gamma \in (2n+1)\mathbb{Z}^d, u(x+\gamma) = e^{i\gamma\theta} u(x) \},$$

and

$$\mathcal{H}^1_{n,\theta} = \{ u \in \mathcal{H}_{n,\theta}; \partial_x u \in \mathcal{H}_{n,\theta}; |\alpha| = 1 \}.$$

As $H_\omega^n$ is elliptic, we know that, $H_\omega^n(\theta)$ has a compact resolvent; hence its spectrum is discrete $[13]$. We denote its eigenvalues, called Floquet eigenvalues of $H_\omega^n(\theta)$, by

$$E_0(n,\omega,\theta) \leq E_1(n,\omega,\theta) \leq \cdots \leq E_k(n,\omega,\theta) \leq \cdots.$$

The corresponding eigenfunctions are denoted by $(w(x,\cdot)_k)_{k \in \mathbb{N}}$. The functions $(\theta \to E_k(n,\omega,\theta))_{k \in \mathbb{N}}$ are Lipschitz-continuous, and we have

$$E_k(n,\omega,\theta) \to +\infty \text{ as } k \to +\infty \text{ uniformly in } \theta.$$

The spectrum $\sigma(A_\omega^n)$ of $A_\omega^n$ has a band structure, (i.e $\sigma(A_\omega^n) = \bigcup_{k \in \mathbb{N}} E_k(n,\omega,T^n)$).

Let $\mathcal{N}_\omega^n$ be the integrated density of states of $H_\omega^n$; it satisfies

$$\mathcal{N}_\omega^n(E) = \sum_{k \in \mathbb{N}} \frac{1}{(2\pi)^d} \int_{\mathcal{E}_k(n,\omega,\theta) \leq E} d\theta = \frac{1}{(2\pi)^d} \int_{T^n_{2n+1}} \mathcal{V}(H_\omega^n(\theta),E)d\theta.$$

(2.8)

Here $\mathcal{V}(B,E)$ is the number of eigenvalues of $B$ less or equal to $E$. Let $d\mathcal{N}_\omega^n$ be the derivative of $\mathcal{N}_\omega^n$ in the distribution sense. As $\mathcal{N}_\omega^n$ is increasing, $d\mathcal{N}_\omega^n$ is a positive measure; it is the density of states of $H_\omega^n$. We denote by $d\mathcal{N}$ the density of states of $H_\omega$. For all $\varphi \in C^\infty_0(\mathbb{R}), d\mathcal{N}_\omega^n$ verifies $[6],

$$\langle \varphi, d\mathcal{N}_\omega^n \rangle = \frac{1}{(2\pi)^d} \int_{\mathcal{E}_k(n,\omega,\theta) \leq E} \text{tr}_{H_\theta} \left( \varphi(H_\omega^n(\theta)) \right) d\theta,$$

$$= \frac{1}{\text{vol}(C_k)} \text{tr} \left( \chi_{C_k} \varphi(H_\omega^n) \chi_{C_k} \right),$$

(2.9)

where for $\Lambda \subset \mathbb{R}^d$, $\chi_\Lambda$ will design the characteristic function of $\Lambda$ and tr($A$) is the trace of $A$ (we index by $H_\theta$ if the trace is taken in $H_\theta$).
Lemma 2.1 ([6]). For any $\varphi \in C_c^\infty(\mathbb{R})$ and for almost all $\omega \in \Omega$ we have

$$\lim_{n \to \infty} \mathbb{E}((\langle \varphi, dN_n^\omega \rangle)) = \langle \varphi, dN \rangle.$$ 

Moreover, we have that the IDS of $H_\omega$ is exponentially well-approximated by the expectation of the IDS of the periodic operators $H_n^\omega$ when $n$ is polynomial in $\varepsilon^{-1}$. More precisely we have

Theorem 2.2 ([6]). Pick $\eta > 0$ and $I \subset \mathbb{R}$, a compact interval. There exists $\varepsilon_0 > 0$ and $\rho > 0$ such that, for $E \in I$, $\varepsilon \in (0, \varepsilon_0)$ and $n \geq \varepsilon^{-\rho}$, one has

$$0 \leq N(E + \varepsilon) - N(E) \leq \mathbb{E}(N_n^\omega(E + 2\varepsilon)) - \mathbb{E}(N_n^\omega(E - 2\varepsilon)) + e^{-\varepsilon - \eta}. \quad (2.10)$$

Remark 2.3. This lemma is proven in [6] for the Schrödinger case. It is still true for our case. The proof is based on the Helffer-Sjöstrand formula and the resolvent equation with the exponential decay of the resolvent kernels (the Combes-Thomas argument).

Now we study the periodic approximations. For a vector space $E$, we note by $\dim(E)$ the dimension of $E$. We have,

$$\mathcal{V}(H_n^\omega(\theta), E) = \sup \dim \{ \mathcal{E} \subset \mathcal{H}_n^\omega, \text{ such that, } \forall u \in \mathcal{E}; \langle H_n^\omega(\theta)u, u \rangle \leq E \|u\|^2 \}$$

$$= \sup \dim \{ \mathcal{E} \subset \mathcal{H}_n^\omega, \text{ such that, } \forall u \in \mathcal{E};$$

$$\langle \left( H_n^\omega(\theta) - \overline{H}^\alpha(\theta) \right)u, u \rangle + \langle \overline{H}^\alpha(\theta)u, u \rangle \leq E \|u\|^2 \}. \quad (2.11)$$

Let, $\alpha > 0$ and

$$\mathcal{E}_1^\alpha(\theta) = \{ u \in \mathcal{H}_n^1; \left| \left( H_n^\omega(\theta) - \overline{H}^\alpha(\theta) \right)u, u \right| \leq E^\alpha \|u\|^2 \},$$

and

$$\mathcal{E}_2^\alpha(\theta) = \{ u \in \mathcal{H}_n^1; \left| \left( H_n^\omega(\theta) - \overline{H}^\alpha(\theta) \right)u, u \right| \geq E^\alpha \|u\|^2 \}.$$
Then we have

\[
\mathcal{V}(H^n_\omega(\theta), E) \leq \sup \dim \{ \mathcal{E} \subset \mathcal{E}_1^n(\theta), \text{ such that, } \forall u \in \mathcal{E}; \langle H^n_\omega(\theta)u, u \rangle \leq E\|u\|^2 \} \\
+ \sup \dim \{ \mathcal{E} \subset \mathcal{E}_2^n(\theta), \text{ such that, } \forall u \in \mathcal{E}; \langle H^n_\omega(\theta)u, u \rangle \leq E\|u\|^2 \} \\
\leq \sup \dim \{ \mathcal{E} \subset \mathcal{E}_1^n(\theta), \text{ such that, } \forall u \in \mathcal{E}; \langle \overline{T}^n(\theta)u, u \rangle \leq (E + E^\alpha)\|u\|^2 \} \\
+ \sup \dim \{ \mathcal{E} \subset \mathcal{E}_2^n(\theta), \text{ such that, } \forall u \in \mathcal{E}; \langle H^n_\omega(\theta)u, u \rangle \leq E\|u\|^2 \}.
\]

So, we get

\[
\mathcal{V}(H^n_\omega(\theta), E) \leq \mathcal{V}(\overline{T}^n(\theta), (E + E^\alpha)) \\
+ \sup \dim \{ \mathcal{E} \subset \mathcal{E}_2^n(\theta), \text{ such that, } \forall u \in \mathcal{E}; \langle H^n_\omega(\theta)u, u \rangle \leq E\|u\|^2 \}. \tag{2.11}
\]

Now integrating both sides of (2.11) over \( \mathbb{T}_{2n+1}^* \) and taking into account (2.8), we get that

\[
\mathcal{N}_n^\omega(E) \leq \overline{\mathcal{N}}^n((E + E^\alpha)) + \\
\frac{1}{(2\pi)^d} \int_{\mathbb{T}_{2n+1}^*} \dim \{ \mathcal{E} \subset \mathcal{E}_2^n(\theta), \text{ such that, } \forall u \in \mathcal{E}; \langle H^n_\omega(\theta)u, u \rangle \leq E\|u\|^2 \} d\theta.
\]

\( \tag{2.12} \)

Where \( \overline{\mathcal{N}}^n \) is the IDS of \( \overline{T}^n \).

Notice that \( \dim \{ \mathcal{E} \subset \mathcal{E}_2^n(\theta), \text{ such that, } \forall u \in \mathcal{E}; \langle H^n_\omega(\theta)u, u \rangle \leq E\|u\|^2 \} \) is bounded by the number of eigenvalues of \( -\Delta^n(\theta) \) less than \( E\rho^* \) which is it self bounded by \( Cn^d \) (\( C \) depends only on \( E \)). As the volume of \( \mathbb{T}_{2n+1}^* \) is \( (2\pi(2n+1))^{-d} \) we get that for some \( C > 0 \) we have

\[
E(\mathcal{N}_n^\omega(E)) \leq \overline{\mathcal{N}}^n((E + E^\alpha)) + C\mathcal{P}(\Omega_{n,E,\alpha}). \tag{2.13}
\]

With

\[
\Omega_{n,E,\alpha} = \left\{ \omega; \exists u \in \mathcal{E}_2^n(\theta); \|u\|_{L^2(\mathbb{R}^d)} = 1; \langle \nabla u, u \nabla \rangle \leq E\rho^*\|u\|^2 \right\}.
\]

Now let us consider

\[
\mathcal{V}(\overline{T}^n(\theta), (E - E^\alpha)) = \\
\sup \dim \{ \mathcal{E} \subset \mathcal{H}_{n,\theta}, \text{ such that, } \forall u \in \mathcal{E}; \langle \overline{T}^n(\theta)u, u \rangle \leq (E - E^\alpha)\|u\|^2 \}.
\]

7
\[ \sup \dim \{ E \subset H_{n,\theta}^1, \text{such that}, \forall u \in E; \] 
\[ \langle (\overline{H}_n^\omega(\theta) - H_n^\omega(\theta))u, u \rangle + \langle H_n^\omega(\theta)u, u \rangle \leq (E - E^\alpha)\|u\|^2 \} \]

\[ \leq \sup \dim \{ E \subset E_1^\alpha(\theta), \text{such that}, \forall u \in E; \langle H_n^\omega(\theta)u, u \rangle \leq E\|u\|^2 \} + \sup \dim \{ E \subset E_2^\alpha(\theta), \text{such that}, \forall u \in E; \langle \overline{H}_n^\omega(\theta)u, u \rangle \leq (E - E^\alpha)\|u\|^2 \}. \]

Using the same computation carried out from (2.11) to (2.13), we get that

\[ N((E - E^\alpha)) - C_P(\Omega_{n,E,\alpha}) \leq N(E). \]  

(2.14)

Now we have to estimate the following probability, \( P(\Omega_{n,E,\alpha}) \). It is the purpose of the following Lemma. It is a large deviation argument.

**Lemma 2.4.** There exists \( \tau > 0 \) such that for \( E \) sufficiently small and \( n \) large, we have

\[ P(\Omega_{n,E,\alpha}) \leq e^{-E^{-\tau}}. \]

Now the proof of Theorem 1.1 is just to take into account Theorem 2.2 and Lemma 2.4.

**The proof of Lemma 2.4**

We prove this Lemma using techniques of [5]. We have \( \Omega_{n,E,\alpha} \subset \Omega'_{n,E,\alpha} \).

With

\[ \Omega'_{n,E,\alpha} = \left\{ \omega; \exists u \in H^1(\mathbb{R}); \|u\|_{L^2(\mathbb{R}^d)} = 1; \|\nabla u\|^2 \leq E\rho; \right. \]

\[ \left. \text{and } \left| \langle (H_n^\omega(\theta) - \overline{H}_n^\omega(\theta))u, u \rangle \right| \geq E^\alpha\|u\|^2 \right\}. \]

Let us estimate the probability of the latest events. Notice that we asked that

\[ \left| \langle (H_n^\omega(\theta) - \overline{H}_n^\omega(\theta))u, u \rangle \right| = \sum_{i=1}^d \left| \langle \rho_n^\omega(\theta) - \rho_i(\theta) \rangle \partial_x u, \partial_x u \rangle \right| \geq E^\alpha\|u\|^2. \]

(2.15)

Let \( u \in H^1(\mathbb{R}^d) \), then \( u \) can be written using the Floquet decomposition as:

\[ u = \sum_{k \in \mathbb{N}} \int_{T_n^d} \chi_k(\theta)w_k(x, \theta)d\theta. \]
Where \((w(\cdot, \theta)k)_{k \in \mathbb{N}}\) are the Floquet eigenfunctions of \(-\Delta^\theta_n\) associated to \((E_k(\theta))_{k \in \mathbb{N}}\).

By this, for \(u\) such that \(\langle -\Delta u, u \rangle \leq E\rho\) we have:

\[
\left( \sum_{k \geq 0} \int_{T_{2n+1}} |E_k(\theta)|^2 |\chi_k(\theta)|^2 d\theta \right) \leq CE^2.
\]  
(2.16)

0 is the bottom of the spectrum of \(-\Delta\). It is a simple non-degenerate Floquet eigenvalue \([13]\). Hence there exists \(C > 0\) such that

- For \(k \neq 0\), \(\forall \theta \in T_{2n+1}^*\)
  \[|E_k(\theta)| \geq 1/C,\]  
(2.17)

- and \(\exists Z = \{\theta_j \in T_{2n+1}^*; 1 \leq j \leq n_0\}\) such that \(E_0(\theta_j) = 0\).

\[|E_0(\theta)| \geq 1/C \inf_{1 \leq j \leq n_0} |\theta - \theta_j|^2.\]  
(2.18)

Let \((2l + 1) = \lfloor E^{-1/2+2\rho'} \rfloor \cdot \lfloor E^{-\rho'} \rfloor\) and \((2k + 1) = \lfloor E^{-\eta} \rfloor\), where \(\alpha < \rho' < \frac{d}{(d+1)}\) and \(\eta > 0\) such that \((2l + 1) \cdot (2k + 1) = 2n + 1\). Here \(\lfloor \cdot \rfloor\) denotes the largest odd integer smaller than \(\cdot\).

From (2.16), (2.17) and (2.18) we get that

\[
\sum_{k \geq 1} \int_{T_{2n+1}} |\chi_k(\theta)|^2 d\theta + \sum_{j=1}^{n_0} \int_{|\theta - \theta_j| > \frac{1}{4}} |\chi_0(\theta)|^2 d\theta \leq CE^2l^2 \leq CE^{2\rho'}.
\]  
(2.19)

Hence we write

\[u = \sum_{j=1}^{n_0} u_j + u^e, \text{ where } u_j = \int_{|\theta - \theta_j| \leq \frac{1}{4}} \chi_0(\theta)w_0(\cdot, \theta_j)d\theta; \quad \|u^e\| \leq CE^{2\rho'},\]  
(2.20)

and we have

\[
\sum_{j=1}^{n_0} \|u_j\|^2 = \|u\|^2 - CE^{\rho'} = 1 - CE^{2\rho'}.
\]

(2.21)

Now using (2.20) in (2.15), we get that for \(E\) small we have

\[
\sum_{1 \leq j, j' \leq n_0} \left| \langle \left( \rho_0^n - \bar{\rho} \right) \nabla u_j, \nabla u_{j'} \rangle \right| \geq E^{\alpha}/4.
\]  
(2.21)
So, for some $1 \leq j, j' \leq n_0$, one has
\[
|\langle \left( p^a_n - \rho^a \right) \nabla u_j, \nabla u_{j'} \rangle | \geq E^\alpha / (2n_0)^2.
\] (2.22)

Now we state a Lemma based on the Uncertainty principle and proved in [5].

**Lemma 2.5.** Fix $1 \leq j \leq n_0$. For $1 \leq l' \leq l$, there exists $\tilde{u}_j \in L^2(\mathbb{R}^d)$ such that

1) $\tilde{u}_j$ is constant on each cube
\[\Lambda_{\gamma, l'} = \{ x = (x_1, \cdots, x_d); \forall 1 \leq i \leq d - l' - \frac{1}{2} \leq x_i - (2l' + 1)\gamma_i < l' + \frac{1}{2} \}\]
where $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{Z}^d$.

2) $\exists C > 0$ depending only on $w_0(\cdot, \theta)$ such that
\[
\| u_j - \tilde{u}_j \cdot w_0(\cdot, \theta_j) \|_{L^2(\mathbb{R}^d)} \leq C l'/l,
\] (2.23)
where $w_0(\cdot, \theta)$ is the periodic component of $w_0(\cdot, \theta)$ i.e. $w_0(\cdot, \theta) = e^{i\theta \cdot x} w_0(\cdot, \theta)$.

Let
\[
\psi_j(x) = \tilde{u}_j(x) \overline{w}_0(x, \theta_j) = \overline{w}_0(x, \theta_j) \sum_{\beta \in \mathbb{Z}^d} (2l' + 1)^{-d/2} a_j(\beta) 1_{(2l' + 1)\beta + \Lambda_{\theta_j}}.
\]
$\tilde{u}_j(x) \overline{w}_0(x, \theta_j) \in L^2(\mathbb{R}^d)$ using the periodicity of $\overline{w}_0(x, \theta_j)$ we get,
\[
\| \psi_j \|_{L^2(\mathbb{R}^d)} = \| \tilde{u}_j(x) \overline{w}_0(x, \theta_j) \|_{L^2(\mathbb{R}^d)} = \sum_{\beta \in \mathbb{Z}^d} |a_j(\beta)|^2 \int_{\Lambda_{\theta_j}} |\overline{w}_0(\cdot, \theta_j)|^2 dx.
\] (2.24)

Then using (2.23) and the fact that
\[
\int_{\Lambda_{\theta_j}} |w_0(x, \theta_j)|^2 dx = \int_{\Lambda_{\theta_j}} |\overline{w}_0(x, \theta_j)|^2 dx;
\]
we get that there exists $C > 0$ such that
\[
\sum_{\beta \in \mathbb{Z}^d} |a_j(\beta)|^2 \leq C \| u_j \|_{L^2(\mathbb{R}^d)} < +\infty.
\] (2.25)
We set $2l' + 1 = [E^{-1/2 + 2\rho'}]_0$ and $2k' + 1 = [E^{-\rho'}]_0 [E^{-\eta}]_0$, for $\alpha < \rho' < \frac{d}{4(d+1)}$ and $\eta > 0$ so that $(2n + 1) = (2l' + 1) \cdot (2k' + 1)$. So taking into account (2.22), (2.23) and the choice of $l$ and $l'$, we get

$$
\sum_{i=1}^{d} \left| \langle (\rho_0^0 - \rho_0^n) \partial x_i \psi_j, \partial x_i \psi_{j'} \rangle \right| \geq E^\alpha / (2n_0)^2 - CE^{\rho'} \geq E^\alpha / (4n_0)^2. \quad (2.26)
$$

We set

$$
\sum_{i=1}^{d} \left| \langle (\rho_0^0 - \rho_0^n) \partial x_i \psi_j, \partial x_i \psi_{j'} \rangle \right| = \sum_{1 \leq i \leq d} |A_{ij, j'}^i|.
$$

With

$$
A_{ij, j'}^i = \langle (\rho_0^0 - \rho_0^n) \partial x_i \psi_j, \partial x_i \psi_{j'} \rangle.
$$

We have

$$
A_{ij, j'}^i = \sum_{\beta \in \mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}_{2n_0+1}^d} (2l' + 1)^{-d} a_j(\beta) \cdot \overline{\alpha_i(\beta)} \cdot \int_{(2l' + 1)\beta + \Lambda_0, \theta} (\rho_0^0 - \rho_0^n)(x - \gamma) \partial x_i \overline{\omega_0}(x, \theta_j) \cdot \partial x_i \overline{\omega_0}(x, \theta_{j'}) dx 
$$

$$
= \sum_{\beta \in \mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}_{2n_0+1}^d} a_j(\beta) \cdot \overline{\alpha_j(\beta)} \cdot \int_{(2l' + 1)^d \Lambda_0, \theta} (\rho_0^0 - \rho_0^n)(x - \gamma + (2l' + 1)\beta) \partial x_i \overline{\omega_0}(x, \theta_j) \cdot \partial x_i \overline{\omega_0}(x, \theta_{j'}) dx.
$$

(2.27)

As $\rho_0^n$ is $(2n + 1)\mathbb{Z}^d$-periodic and $(2l' + 1)(2k' + 1) = (2n + 1)$ we get that

$$
A_{ij, j'}^i = \sum_{\beta \in \mathbb{Z}_{2k' + 1}^d} \sum_{\beta' \in \mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}_{2n_0+1}^d} a_j(\beta + (2k' + 1)\beta') \cdot \overline{\alpha_j(\beta + (2k' + 1)\beta')} \cdot \int_{(2l' + 1)^d \Lambda_0, \theta} (\rho_0^0 - \rho_0^n)(x - \gamma + (2l' + 1)\beta) \partial x_i \overline{\omega_0}(x, \theta_j) \cdot \partial x_i \overline{\omega_0}(x, \theta_{j'}) dx.
$$

(2.28)
Using the expression of $\rho_\omega$ we get that.

$$A_{j,j'} = \sum_{\beta \in \mathbb{Z}_{2k'+1}} \sum_{\gamma \in \mathbb{Z}_{2n+1}} (\omega_\gamma - \overline{\omega}) a_j(\beta + (2k' + 1)\beta') a_{j'}(\beta + (2k' + 1)\beta').$$

$$\frac{1}{(2l' + 1)^d} \int_{\Lambda_{0,l'}} \rho^0(x - \gamma + (2l' + 1)\beta) \partial_x \overline{w_0}(x, \theta_j) \cdot \partial_x \overline{w'(x, \theta_{j'})} dx. \quad (2.29)$$

We set

$$B_{j,j'}^i(\beta) = \sum_{\beta' \in \mathbb{Z}^d} a_j(\beta + (2k' + 1)\beta') a_{j'}(\beta + (2k' + 1)\beta'). \quad (2.30)$$

Then we get that

$$A_{j,j'} = (2l' + 1)^{-d} \sum_{\gamma \in \mathbb{Z}^d_{2n+1}} (\omega_\gamma - \overline{\omega}) \left( \sum_{\beta \in \mathbb{Z}_{2k'+1}} B_{j,j'}^i(\beta) \cdot \int_{\Lambda_{0,l'}} \rho^0(x - \gamma + (2l' + 1)\beta) \overline{w_0}(x, \theta_j) \cdot \overline{w(x, \theta_{j'})} dx \right)$$

$$= (2l' + 1)^{-d} \sum_{\gamma \in \mathbb{Z}^d_{2l' + 1}} \left[ \sum_{\gamma' \in \mathbb{Z}^d_{2l' + 1}} (\omega_{\gamma + (2l' + 1)\gamma'} - \overline{\omega}) \left( \sum_{\beta \in \mathbb{Z}_{2k'+1}} B_{j,j'}^i(\beta) \cdot \int_{\Lambda_{0,l'}} \rho^0(x - \gamma + (2l' + 1)(\beta - \gamma')) \partial_x \overline{w_0}(x, \theta_j) \cdot \partial_x \overline{w(x, \theta_{j'})} dx \right) \right]. \quad (2.31)$$

We set

$$C_{j,j'}^i(\gamma, \gamma') = \sum_{\beta \in \mathbb{Z}^d_{2k'+1}} B_{j,j'}^i(\beta) \int_{\Lambda_{0,l'}} \rho^0(x - \gamma + (2l' + 1)(\beta - \gamma')) \partial_x \overline{w_0}(x, \theta_j) \cdot \partial_x \overline{w(x, \theta_{j'})} dx$$

and

$$Y_{j,j'}^i(\gamma) = \sum_{\gamma' \in \mathbb{Z}^d_{2l' + 1}} (\omega_{\gamma + (2l' + 1)\gamma'} - \overline{\omega}) C_{j,j'}^i(\gamma, \gamma').$$

Then

$$A_{j,j'} = \frac{1}{(2l' + 1)^d} \sum_{\gamma \in \mathbb{Z}^d_{2l' + 1}} Y_{j,j'}^i(\gamma). \quad (2.32)$$
Notice that \((Y^i_{j,j'}(\gamma))_{\gamma \in \mathbb{Z}^{2l'+1}_d}\) are bounded random and independent variables with \(\mathbb{E}(Y^i_{j,j'}(\gamma)) = 0\). Indeed, using the fact that \(\rho^0\) is compactly supported and \([2]\) we get that
\[
|Y_{j,j'}(\gamma)| \leq ||u^0||^2.
\]
So, to estimate the probability of \(\Omega(n, E, \alpha)\) it suffices to estimate the probability that
\[
E^\alpha/(4n_0)^2 \leq \frac{1}{(2l'+1)^d} \sum_{\gamma \in \mathbb{Z}^{2l'+1}_d} Y^i_{j,j'}(\gamma).
\]
This probability is given by the large deviation principle which gives that \([2]\)
\[
\mathbb{P}
\left(\frac{E^\alpha}{(4n_0)^2} \leq \frac{1}{(2l'+1)^d} \sum_{\gamma \in \mathbb{Z}^{2l'+1}_d} Y^i_{j,j'}(\gamma)\right) \leq e^{-c(l')d E^{2\alpha}} \leq e^{-c E^{d/2+2d\rho'+2\alpha}}.
\]
Here we have used the expression of \(l'\). Using the fact that for our choice of \(\rho'\) we have \(-d/2 + 2d\rho' + 2\alpha < 0\), so for some \(\tau > 0\) and \(E\) sufficiently small, one has
\[
\mathbb{P}
\left(\frac{E}{(2l'+1)^d} \sum_{\gamma \in \mathbb{Z}^{2l'+1}_d} Y^i_{j,j'}(\gamma)\right) \leq e^{-E^{-\alpha}}.
\]
As the probability of \(\Omega(n, E, \alpha)\) is bounded by the sum over \(1 \leq i \leq d\) and \(1 \leq j,j' \leq n_0\) of the probability estimate previously, we get the result of the Lemma \([2,3]\) \(\square\)

References

[1] A. Figotin and A. Klein : *Localization of Classical Waves I : Acoustic Waves*. Commu. Math. Phys. 180, (1996) p 439-482.

[2] A. Dembo and O. Zeitouni. *Large deviation and applications*. Jones and Bartlett Publishers, Boston, 1992.

[3] J-M. Deuschel and D. Stroock. *Large deviation*. volum 137 of Pure and applied Mathematics. Academic Press, 1989.
[4] W. Kirsch: *Random Schrödinger operators* A Course Lecture Notes In Phy 345 Springer-Verlag, Berlin (1989) p 264-370.

[5] F. Klopp. *Weak disorder localization and Lifshitz tails: continuous Hamiltonians*. Ann. I.H.P. (3):711-737, 2002.

[6] F. Klopp. *Internal Lifshits Tails For long range single site potentials*. Jour. Math. Phys. 43 (2002) N° 6 p 2948-2958.

[7] N. Kozlov. *Averaging Random Structure*. Soviet Math. Dokjl. Vol. 19 (1978) N° 4 p 950-954.

[8] H. Najar: *Asymptotique de la densité d'états intégrée des opérateurs acoustiques aléatoires*. C. R. Acad. Sci. Paris, 333 I, p 191-194 (2001).

[9] H. Najar: *Lifshitz tails for random acoustic operators*. Jour. Math. Phys. 44 N° 4 (2003) p1842-1867.

[10] H. Najar: *Asymptotic behavior of the integrated density of states of acoustic operator with long range random perturbations*. Jour. Stat. Phys. 115 N° 4 (2003) p 977-996.

[11] H. Najar: *2-Dimensional localization of acoustic waves in random perturbation of periodic media*. accepted for publication in Jour. Math. Ana. App. (2004).

[12] L. Pastur and A. Figotin. *Spectra of Random and Almost-Periodic Operators*. Springer Verlag, Berlin, 1992.

[13] M. Reed and B. Simon. *Methods of Modern Mathematical vol IV:Analysis of Operators*. Academic Press, New York, 1978.