Generating the Mapping Class Group of a Punctured Surface by Involutions

Naoyuki MONDEN
Osaka University

(Communicated by Y. Matsumoto)

Abstract. Let $\Sigma_{g,b}$ denote a closed oriented surface of genus $g$ with $b$ punctures and let $\text{Mod}(\Sigma_{g,b})$ denote its mapping class group. Kassabov showed that $\text{Mod}(\Sigma_{g,b})$ is generated by 4 involutions if $g > 7$ or $g = 7$ and $b$ is even, 5 involutions if $g > 5$ or $g = 5$ and $b$ is even, and 6 involutions if $g > 3$ or $g = 3$ and $b$ is even. We proved that $\text{Mod}(\Sigma_{g,b})$ is generated by 4 involutions if $g = 7$ and $b$ is odd, and 5 involutions if $g = 5$ and $b$ is odd.

1. Introduction

Let $\Sigma_{g,b}$ be an closed oriented surface of genus $g \geq 1$ with arbitrarily chosen $b$ points (which we call punctures). Let $\text{Mod}(\Sigma_{g,b})$ be the mapping class group of $\Sigma_{g,b}$, which is the group of homotopy classes of orientation-preserving homeomorphisms preserving the set of punctures. By $\text{Mod}^0(\Sigma_{g,b})$ we will denote the subgroup of $\text{Mod}(\Sigma_{g,b})$ which fixes the punctures pointwise.

The question of generating mapping class groups by involutions was first investigated by McCarthy and Papadopoulos (see [MP]). In [MP], they proved that $\text{Mod}(\Sigma_{g,0})$ is generated by infinitely many conjugates of a single involution for $g \geq 3$. Luo described the finite set of involutions which generate $\text{Mod}(\Sigma_{g,b})$ for $g \geq 3$ (see [Lu]). He also proved that $\text{Mod}(\Sigma_{g,b})$ is generated by torsion elements in all cases except $g = 2$ and $b = 5k + 4$, but this group is not generated by involutions if $g \leq 2$. Brendle and Farb proved that $\text{Mod}(\Sigma_{g,b})$ can be generated by 6 involutions for $g \geq 3$, $b = 0$ and $g \geq 4$, $b \leq 1$ (see [BF]). In [Ka], Kassabov showed that $\text{Mod}(\Sigma_{g,b})$ is generated by 4 involutions if $g > 7$ or $g = 7$ and $b$ is even, 5 involutions if $g > 5$ or $g = 5$ and $b$ is even, and 6 involutions if $g > 3$ or $g = 3$ and $b$ is even.

We show that when $b$ is odd and $g \geq 7$, $\text{Mod}(\Sigma_{g,b})$ is generated by 4 involutions by improving two involutions which are constructed in Section 2 of [Ka]. Furthermore, by using the argument in Section 3.4 of [Ka] we show that when $b$ is odd, $\text{Mod}(\Sigma_{5,b})$ is generated by 5 involutions. We prove these results by the arguments similar to [Ka]. When we combine Kassabov’s theorem with these results, we get the following results:

**Main Theorem.** For all $b \geq 0$, $\text{Mod}(\Sigma_{g,b})$ is generated by:
(a) 4 involutions if $g \geq 7$,
(b) 5 involutions if $g \geq 5$.

2. Preliminaries

Let $c$ be a simple closed curve on $\Sigma_{g,b}$. We will denote by $T_c$ the (right handed) Dehn twists about the curve $c$.

We record the following lemmas.

**Lemma 1.** For all $h \in \text{Mod}(\Sigma_{g,b})$,
\[ T_{h(c)} = hT_ch^{-1}. \]

**Lemma 2.** Let $c$ and $d$ be two simple closed curves on $\Sigma_{g,b}$. If $c$ is disjoint from $d$, then
\[ T_cT_d = T_DT_c \]

It is clear that we have the exact sequence:
\[ 1 \rightarrow \text{Mod}^0(\Sigma_{g,b}) \rightarrow \text{Mod}(\Sigma_{g,b}) \xrightarrow{\pi} \text{Sym}_b \rightarrow 1. \]

Therefore, we see the following lemma;

**Lemma 3.** Let $H$ denote a subgroup of $\text{Mod}(\Sigma_{g,b})$, which contains $\text{Mod}^0(\Sigma_{g,b})$. If $\pi(H) = \text{Sym}_b$, then $H$ is equal to $\text{Mod}(\Sigma_{g,b})$.

3. Proof of Main Theorem

Hereafter, we assume that $g \geq 5$, and that the number of punctures $b = 2l + 1$ is odd.

We will construct two involutions $\rho_1, \rho_2$ by modifying the involutions $\rho_1, \rho_2$ which are constructed in Section 2 of [Ka]. We note that we change the action of $\rho_1, \rho_2$ on punctures and swap the top parts of Figure 1 of [Ka].

Let us embed our surface $\Sigma_{g,b}$ in the Euclidean space in two different ways as shown on Figure 1. (In these pictures we will assume that genus $g = 2k + 1$. In the case of even genus we only have to swap the top parts of the pictures.) In Figure 1 we have also marked the punctures as $x_1, \ldots, x_b$ and we have the curves $\alpha_i, \beta_i, \gamma_i$ and $\delta$. The curves $\alpha_i, \beta_i, \gamma_i$ are non separating curves and $\delta$ is a separating curve.

Let $\rho_1$ and $\rho_2$ denote the involutions which are rotation by $\pi$ about the axises indicated in Figure 1. Then we get the following lemma;

**Lemma 4.** The subgroup of $\text{Mod}(\Sigma_{g,b})$ be generated by $\rho_1, \rho_2$ and 3 Dehn twists $T_\alpha, T_\beta$ and $T_\gamma$ around one of the curve in each family contains the subgroup $\text{Mod}^0(\Sigma_{g,b})$. 
MAPPING CLASS GROUP OF A PUNCTURED SURFACE

We postpone the proof of lemma 4 until Section 4.

Let $\pi$ be the homomorphism and $H$ be the subgroup of $\text{Mod}(\Sigma_{g,b})$ mentioned in Lemma 3. Showing the surjectivity of $\pi$ from $H$ to $\text{Sym}_b$ is equivalent to showing that the $\text{Sym}_b$ can be generated by involutions;

$$r_1 = (1, b-1)(2, b-2) \cdots (l, l+1)(b)$$
$$r_2 = (2, b-1)(3, b-2) \cdots (l, l+2)(1)(l+1)(b)$$
$$r_3 = (1, b)(2, b-1)(3, b-2) \cdots (l, l+2)(l+1)$$

corresponding to 3 involutions in $H$ by $\pi$. The group generated by $r_i$ contains the long cycle $r_3 r_1 = (1, 2, \ldots, b)$ and transposition $r_3 r_2 = (1, b)$. These two elements generate the whole symmetric group, therefore the involutions $r_i$ ($i = 1, 2, 3$) generate $\text{Sym}_b$. We note that the images of $\rho_1$ and $\rho_2$ to $\text{Sym}_b$ are $r_1$ and $r_2$.

3.1. Generating Dehn twists by 4 involutions. By using the arguments similar to Section 3.4 of [Ka] we generate Dehn twists by 4 involutions.

We assume that $g \geq 5$.

Let $S_{0,4}$ be a surface of genus 0 with 4 boundary components. Denote by $a_1, a_2, a_3$ and $a_4$ the four boundary curves of the surface $S_{0,4}$ and let the interior curves $y_1, y_2$ and $y_3$ be as shown in Figure 2.
The lantern relation is the following relation:

\[ T_{y_1} T_{y_2} T_{y_3} = T_{a_1} T_{a_2} T_{a_3} T_{a_4}. \]  

(1)

Notice that the curves \( a_i \) do not intersect any other curve and that the Dehn twists \( T_{a_i} \) commute with every twist in this relation. Thus we have

\[ T_{a_4} = (T_{y_1} T_{a_1}^{-1})(T_{y_2} T_{a_2}^{-1})(T_{y_3} T_{a_3}^{-1}). \]  

(2)

Let \( R \) denote the product \( \rho_2 \rho_1 \). By Figure 1 we can see that \( R \) acts as follows:

\[
\begin{align*}
R \alpha_i &= \alpha_{i+1}, & (1 \leq i < g) \\
R \beta_i &= \beta_{i+1}, & (1 \leq i < g) \\
R \gamma_i &= \gamma_{i+1}, & (1 \leq i < g-1).
\end{align*}
\]

(3)

Let \( S \) be a lantern whose boundary components are \( a_1, a_2, a_3, a_4 \), and \( R^{-2} \) a lantern whose boundary components are \( R^{-2} a_1, R^{-2} a_2, R^{-2} a_3, R^{-2} a_4 \). We identify \( a_1 \) with \( R^{-2} a_2 \). Then we obtain a surface \( S_2 \) homeomorphic to a sphere with 6 boundary components.

By Figure 3 we see that there exists an involution \( \bar{J} \) of \( S_2 \) which takes \( S \) to \( R^{-2} S \). In [Ka] \( R^2 \) is used instead of \( R^{-2} \), since \( g \) is even in [Ka].

Let us embed the surface \( S_2 \) in \( \Sigma_{g,b} \) as shown on Figure 4. We note \( a_1 = \alpha_{k+1}, a_2 = \alpha_{k+3}, a_3 = \gamma_{k+2}, a_4 = \gamma_{k+1}, R^{-2} a_1 = \alpha_{k-1}, R^{-2} a_2 = \alpha_{k+1}, R^{-2} a_3 = \gamma_k, R^{-2} a_4 = \gamma_{k-1} \) and \( \gamma_1 = \alpha_{k+2} \). Figure 4 shows the existence of the involution \( \bar{J} \) on the complement of \( S_2 \) which is a surface of genus \( g - 5 \) with 6 boundary components. Gluing together \( \bar{J} \) and \( \tilde{J} \) gives us the involution \( J \) of \( \Sigma_{g,b} \). By Figure 3 \( J \) acts as follows

\[
\begin{align*}
J(a_1) &= R^{-2} a_2, & J(a_3) &= R^{-2} a_1, & J(y_1) &= R^{-2} y_2, & J(y_3) &= R^{-2} y_1.
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
R^2 J(a_1) &= a_2, & R^2 J(y_1) &= y_2 \\
JR^{-2}(a_1) &= a_3, & JR^{-2}(y_1) &= y_3.
\end{align*}
\]

(4)
Let $\rho_3$ denote $T_{a_1}^{-1}T_{a_1}^{-1}$. In [Ka] $T_{a_1}^{-1}T_{a_1}^{-1}$ is used instead of $T_{a_1}^{-1}T_{a_1}^{-1}$. By Lemma 1, the relation (4) and $\rho_2(a_1) = \rho_2(\alpha_{k+1}) = \alpha_{k+2} = \gamma_1$, we have

$$T_{a_1}^{-1}T_{a_1}^{-1} = \rho_2 T_{a_1}^{-1}T_{a_1}^{-1} = \rho_2 \rho_3$$

$T_{a_1}^{-1}T_{a_1}^{-1} = R^2 J \rho_2 \rho_3 J R^{-2}$,

$T_{a_1}^{-1}T_{a_1}^{-1} = J R^{-2} \rho_2 \rho_3 R^2 J$.

By the relation (2) and (5) we have

$$T_{a_1}^{-1}T_{a_1}^{-1} \in G_1.$$ We since $J(T_{a_1}^{-1}) = J(\alpha_{k+1}) = \gamma_{k+2}$ and $R(T_{a_1}^{-1}) = J R T_{a_1}^{-1} R^{-1} J^{-1} \in G_1$. Moreover, since $R^2(\alpha_{k+1}) = \alpha_{k+1}$ and $I(\alpha_{k+1}) = \beta_{k+1}$, we have $T_{a_1}^{-1}T_{a_1}^{-1} \in G_1$. By the construction of $J$, the image of $J$ to $\text{Sym}_b$ is $r_3$. We note that the images of $\rho_1$ and $\rho_2$ to $\text{Sym}_b$ are $r_1$ and $r_2$. Therefore, there is the surjection from $G_1$ to $\text{Sym}_b$. By Lemma 3 and 4 we see that $G_1$ is equal to $\text{Mod}(\Sigma_{g,b})$. □

### 3.2. In the case of genus 5.

We assume that $g \geq 5$ and $b = 2l + 1$.

We proof that $\text{Mod}(\Sigma_{g,b})$ is generated by 5 involutions.

The five involutions are $\rho_1$, $\rho_2$, $\rho_3$, $J$ and another involution $I$ which was constructed in Section 3.2 of [Ka]. We note that since we assume that $g$ is odd, $I$ maps $\alpha_{k+1}$ to $\beta_{k+2}$.

**Theorem 5.** If $g \geq 5$ and $b = 2l + 1$, the group $G_1$ generated by $\rho_1$, $\rho_2$, $\rho_3$, $J$ and $I$ is the whole mapping class group $\text{Mod}(\Sigma_{g,b})$.

**Proof.** By the relation (6) we have $T_{a_1}^{-1}T_{a_1}^{-1} \in G_1$. Since $J(\alpha_{k-1}) = \gamma_{k+2}$ and $R(\gamma_{k+1}) = \gamma_{k+2}$, we see that $T_{a_1}^{-1}T_{a_1}^{-1} = J R T_{a_1}^{-1} R^{-1} J^{-1} \in G_1$. Moreover, since $R^2(\alpha_{k-1}) = \alpha_{k+1}$ and $I(\alpha_{k+1}) = \beta_{k+2}$, we have $T_{a_1}^{-1}T_{a_1}^{-1} \in G_1$. By the construction of $J$, the image of $J$ to $\text{Sym}_b$ is $r_3$. We note that the images of $\rho_1$ and $\rho_2$ to $\text{Sym}_b$ are $r_1$ and $r_2$. Therefore, there is the surjection from $G_1$ to $\text{Sym}_b$. By Lemma 3 and 4 we see that $G_1$ is equal to $\text{Mod}(\Sigma_{g,b})$. □

### 3.3. In the case of genus 7.

We assume that $g \geq 7$ and $b = 2l + 1$.

We will construct the involution $J'$ which acts on the punctures as the involution $r_3$ by the method similar to Section 3.4 of [Ka]. We note that the action of $J'$ on punctures is different.
Figure 4. The involution $J$ on $\Sigma_{g,b}$.

from that of $J$ which is constructed in Section 3.4 of [Ka].

The $S_2$ and two pairs of pants have common boundary components $R^{-1}a_1$ and $a_3$ and their union is a surface $S_3$ homeomorphic to a sphere with 8 boundary components. Figure 5 shows the existence of the involution $\tilde{J}'$ on $S_3$ which extends the involution $\bar{J}$ on $S_2$.

Let us embed $S_3$ in the $\Sigma_{g,b}$ as shown on Figure 5. We note that the embedding of $S_2$ is
similar to that of Section 3.1. From Figure 5 we can find the involution $\tilde{J}'$ of the complement of $S_3$. Let $J'$ be the involution obtained by gluing together $\tilde{J}'$ and $\bar{J}'$. Moreover, from Figure 5 we find that $J'$ acts on the punctures as the involution $r_3$.

**Theorem 6.** If $g \geq 7$ and $b = 2l + 1$, the group $G_2$ generated by $\rho_1, \rho_2, \rho_3$ and $J'$ is
the whole mapping class group $\text{Mod}(\Sigma_{g,b})$.

PROOF. The proof is the argument similar to Section 3.4 of [Ka]. We omit the proof.

\[ \square \]

4. The subgroup generated by 2 involutions and 3 Dehn twists, which contains $\text{Mod}^0(\Sigma_{g,b})$

In this section we prove Lemma 4.

Let the subgroup $G$ of $\text{Mod}(\Sigma_{g,b})$ be generated by $\rho_1$, $\rho_2$ and 3 Dehn twists $T_\alpha$, $T_\beta$ and $T_\gamma$ around one of the curve in each family. We will show that $G$ contains $\text{Mod}^0(\Sigma_{g,b})$. Let $\delta'$, $\eta'$, $\delta''$, $\eta''$, $\delta_j$, $\eta_j$ ($j = 1, \ldots, l - 1, l + 1, \ldots, b - 2$) be the curves illustrated in Figure 6. In [Ge] it is shown that $\text{Mod}^0(\Sigma_{g,b})$ is generated by Dehn twists about the curves $\alpha_i$-es, $\beta_i$-es, $\gamma_i$-es, $\delta'$, $\delta''$ and $\delta_j$-es, for $j = 1, \ldots, l - 1, l + 1, \ldots, b - 2$.

We recall that $R = \rho_2\rho_1$. By Lemma 1 and the relation (3) we see that $T_{\alpha_i}, T_{\beta_i}, T_{\gamma_i} \in G$ for all $i$.

From the action of $\rho_1$ and $\rho_2$ we can find that $R^{-1}(\delta_j) = \eta_{j-1}$ ($l + 2 \leq j \leq b - 1$) and $R^{-1}(\delta_{l+1}) = \eta'$.

\[ \text{FIGURE 6. The curves } \delta_i \text{-es, } \eta_i \text{-es}. \]
LEMMA 7. \(T_{b_j}, T_{\delta'}, T_{\delta''} \in G \ (j = 1, \ldots, l-1, l+1, \ldots, b-2)\).

PROOF. We will prove \(T_{b_j} \in G \ (j = l+1, \ldots, b-1)\) by induction on \(j\) and \(T_{\delta'} \in G\).

The base case, \(j = b-1\), is clear because \(G\) contains \(T_{\delta_{b-1}} = T_{\alpha_1}\). Suppose that \(G\) contains the twist \(T_{\delta_j}\). By \(R^{-1}(\delta_j) = \eta_{j-1}\) we have \(T_{\eta_{j-1}} = R^{-1}T_{\delta_j}R \in G\).

Let \(U \in G\) denote the product
\[
U = T_{\rho_1^{-1}}T_{\gamma_1^{-1}}T_{\rho_2^{-1}} \cdots T_{\gamma_{g-1}^{-1}}T_{\alpha_g^{-1}}T_{\rho_1}T_{\gamma_1}T_{\rho_2} \cdots T_{\gamma_{g-1}}T_{\alpha_{g-1}}T_{\rho_g}.
\]
We find that
\[
\begin{align*}
U(\eta') &= \delta' \\
U(\eta'') &= \delta'' \\
U(\eta_j) &= \delta_j \quad (j = 1, \ldots, l-1, l+1, \ldots, b-2).
\end{align*}
\]
Therefore, we see that \(T_{\delta_{j-1}} = UT_{\eta_{j-1}}U^{-1} \in G \ (j = l+2, \ldots, b-1)\). Moreover, since \(R^{-1}(\delta_{l+1}) = \eta'\) and \(U(\eta') = \delta'\), we have that \(T_{\delta'} \in G\).

By Figure 6, we find that \(\rho_1(\delta'') = \eta', \rho_1(\delta_j) = \eta_{b-1-j} \ (1 \leq j \leq l-1)\). Therefore, we see that \(T_{\delta_j} = \rho_1^{-1}T_{\eta_{b-1-j}} \rho_1, T_{\delta'} = \rho_1^{-1}T_{\eta'} \rho_1 \in G\). We finished proving Lemma 7. \(\square\)

COROLLARY 8. The group \(G\) contains the subgroup \(\text{Mod}^0(\Sigma_{g,b})\).

Therefore, we can prove Lemma 4.

ACKNOWLEDGMENT. I would like to thank Professor Hisaaki Endo for careful readings and for many helpful suggestions and comments. And I would like to thank Hitomi Fukushima, Yeonhee Jang and Kouki Masumoto for many advices.

References

[BF] T. E. Brendle and B. Farb, Every mapping class group is generated by 3 torsion elements and by 6 involutions, J. Algebra 278 (2004), 187–198.

[Ge] S. Gervais, A finite presentation of the mapping class group of a punctured surface, Topology 40 (2001), No. 4, 703–725.

[Ka] M. Kassabov, Generating Mapping Class Groups by Involutions, arXiv:math.GT/0311455 v1 25 Nov 2003.

[Lu] N. Lu, On the mapping class groups of the closed orientable surfaces, Topology Proc. 13 (1988), 293–324.

[MP] J. McCarthy and A. Papadopoulos, Involutions in surface mapping class groups, Enseign. Math. 33 (1987), 275–290.
Present Address:
DEPARTMENT OF MATHEMATICS,
GRADUATE SCHOOL OF SCIENCE,
OSAKA UNIVERSITY, TOYONAKA, OSAKA, 560–0043 JAPAN.
e-mail: n-monden@cr.math.sci.osaka-u.ac.jp