The Topological Charges of the $a_n^{(1)}$ Affine Toda Solitons.

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Abstract

The topological charges of the $a_n^{(1)}$ affine Toda solitons are considered. A general formula is presented for the number of charges associated with each soliton, as well as an expression for the charges themselves. For each soliton the charges are found to lie in the corresponding fundamental representation, though in general these representations are not filled. Each soliton’s topological charges are invariant under cyclic permutations of the simple roots plus the extended root or equivalently, under the action of the Coxeter element (with a particular ordering). Multisolitons are considered and are found to have topological charges filling the remainder of the fundamental representations as well as the entire weight lattice. The article concludes with a discussion of some of the other affine Toda theories.
1 Introduction.

Affine Toda field theory is one of the three Toda field theories each distinguished by the underlying Lie algebra to which it is associated. All of the theories are integrable, though only two of the three are conformal. Conformal Toda field theory, is a conformal theory associated with a finite dimensional Lie algebra. The conformal invariance can be broken in such a way as to preserve its integrability. The remaining theory is affine Toda (AT) field theory which is associated with loop algebras. Finally, extra fields can be added to the AT model to regain conformal invariance. The resulting theory is known as Conformal Affine Toda (CAT) field theory. This theory is associated with the Kac-Moody algebras.

Solitons were first constructed by Hollowood \cite{hollowood} using Hirota’s method for the $a_n^{(1)}$ series of AT models, with imaginary coupling. In \cite{hollowood}, not only were the soliton energy and momenta found to be real despite the complex nature of the solutions, but the ratios of their masses were calculated and shown to be equal to the ratios of the fundamental Toda particles of the real coupling theory – that is those excitations obtained by expanding the potential term of the theory’s Lagrangian density about its minimum. The topological charges of the $a^{th}$ soliton were claimed to lie in the $a^{th}$ fundamental representation though not, in general, to fill it. Aspects of quantisation of the solitons were considered and taken up in later papers \cite{2, 3}.

As well as extending the methods of \cite{hollowood} to the remaining theories \cite{7}, other authors have approached the subject from differing directions. Firstly, Hirota’s method has been applied to the CAT model from which the AT solitons can be derived \cite{1, 2, 4}. Secondly, Bäcklund transformations have been constructed for the $a_n^{(1)}$ AT model \cite{5}. Finally, following the methods of Leznov and Saveliev \cite{20} a general solution to all of the Toda models associated with a simple Lie algebra has been presented \cite{9}, and investigated \cite{11, 12, 13}.

Except in \cite{1, 13}, and more recently \cite{12} which proves a conjecture appearing in \cite{10} relating to the charges, there has been little information gathered on the topological charges of the solitons. The purpose of this article, therefore, is to obtain a better understanding of the topological charges of the solitons in the simplest AT theory, that of $a_n^{(1)}$. 

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An upper bound for the number of charges of the \( a^{th} \) soliton is found to be

\[
\tilde{h}_a = \frac{h}{\gcd(a,h)}
\]

where \( h \) is the Coxeter number. Indeed, \( \tilde{h}_a \) turns out to be equal to the number of topological charges of the \( a^{th} \) soliton. The origin of this formula lies in the dependence of the analytic expression for the soliton on the \( a^{th} \) power of an \( h^{th} \) root of unity. From studying the soliton solutions, the relationship between the topological charges is deduced to be the map

\[
\tau : \alpha_j \rightarrow \alpha_{(j-1) \mod h} \quad (0 \leq j \leq h-1), \tag{1.0a}
\]

which is also an automorphism of the extended Dynkin diagram, \( \Delta(a_n^{(1)}) \). Therefore, for each soliton, once one topological charge is calculated the rest immediately follow by application of (1.0a). The map \( \tau \) has the same effect as the action of the Coxeter element \( [24] \) with the following ordering:

\[
\omega_{tc} = r_n r_{n-1} r_{n-2} \ldots r_3 r_2 r_1.
\]

As a result, the topological charges lie in the same representation, which is shown to be the \( a^{th} \) fundamental representation. The expression for the topological charges themselves is also derived, and found to be given by

\[
th_a^{(k)} = \sum_{j=1}^{n} \frac{a(h-j) \mod h}{h} \alpha_j - \sum_{l=1}^{k-1} \sum_{j=1}^{n} \delta_{a(h-j) \mod h, h-l \gcd(a,h)} \alpha_j \quad \tag{1.0b}
\]

where \( k = 1, \ldots, \tilde{h}_a \). This allows calculation of charges to be carried out much more easily than through the use of \( \tau \).

When there are widely separated solitons it is intuitive to expect the total topological charge to be the sum of the topological charges of the individual solitons. This statement is proved to be true. Using this result, a double soliton composed of solitons whose topological charges fill up the first and \( n \)-th fundamental representations can be constructed, which has charges filling up the adjoint representation, and in particular has \( \{ \pm \alpha_1, \pm \alpha_2, \ldots, \pm \alpha_n \} \) as topological charges. Further combinations of solitons can therefore be constructed which fill up the fundamental representations and the entire weight lattice itself.

The article closes with a short discussion of some of the other AT theories.
The Lagrangian density of affine Toda field theory can be written in the form

\[ \mathcal{L} = \frac{1}{2} (\partial \mu \phi) \cdot (\partial ^\mu \phi) - \frac{m^2}{\beta^2} n \sum_{j=0}^{n} n_j (e^{i \beta \alpha_j \cdot \phi} - 1). \]

The field \( \phi(x,t) \) is an \( n \)-dimensional vector, \( n \) being the rank of the finite Lie algebra \( g \). The \( \alpha_j \)'s, for \( j = 1, ..., n \), are the simple roots of \( g \); \( \alpha_0 \) is chosen such that the inner products among the elements of the set \( \{ \alpha_0, \alpha_j \} \) are described by one of the extended Dynkin diagrams. It is expressible in terms of the other roots by the equation

\[ \alpha_0 = - \sum_{j=1}^{n} n_j \alpha_j \quad (2.0a) \]

where the \( n_j \)'s are positive integers. Both \( \beta \) and \( m \) are constants, \( \beta \) being the coupling constant.

The inclusion of \( \alpha_0 \) distinguishes affine Toda field theory from Toda field theory. Toda field theory is conformal and integrable, its integrability implying the existence of a Lax pair, infinitely many conserved quantities and exact solubility [14, 15, 16] (for further references see [17]). The extended root is chosen in such a way as to preserve the integrability of Toda field theory (though not the conformal property), with the enlarged set of roots \( \{ \alpha_0, \alpha_j \} \) forming an admissible root system [14].

Also, the conformal invariance can be recovered via the introduction of two additional fields \( \eta \) and \( \nu \). The resulting theory is once again both conformally invariant and integrable. This is the CAT model [18, 19].

When the coupling constant \( \beta \) is replaced by \( i \beta \), the potential term,

\[ V(\phi) \sim \sum_{j=0}^{n} n_j (e^{i \beta \alpha_j \cdot \phi} - 1) \]

which in the real coupling case, upon considering real fields, is zero only for \( \phi = 0 \), now has zeros for \( \phi \in \frac{2\pi i}{\beta} \Lambda^*_W \), (\( \Lambda^*_W \) being the co-weight lattice). The appearance of many minima of the potential is an indication that soliton solutions, interpolating from one minimum at \( x = -\infty \) to another at \( x = +\infty \), may exist. The change in the field between \( x = \pm \infty \) is therefore proportional to an element of the co-weight lattice.
With complex coupling, the equations of motion

$$\partial^2 \phi - \frac{im^2}{\beta} \sum_{j=0}^{n} n_j \alpha_j e^{i\beta \alpha_j \phi} = 0 \quad (2.0b)$$

can, under the following substitution for the field \( \phi(x,t) \),

$$\phi = -\frac{1}{i\beta} \sum_{i=0}^{n} \frac{2}{\alpha_i \cdot \alpha_i} \ln \tau_i \quad (2.0c)$$

be reduced to the following form:

$$\sum_{j=0}^{n} \alpha_j Q_j = 0$$

where

$$Q_j = \left( \frac{\eta_j}{\tau_j^2} (D_t^2 - D_x^2) \tau_j \cdot \tau_j - 2m^2 n_j \left( \prod_{k=0}^{n} \tau_k^{-\eta_k \alpha_k \cdot \alpha_j} - 1 \right) \right).$$

The operators \( D_x \) and \( D_t \) are introduced to ease calculation. They are Hirota derivatives, defined by

$$D_x^m D_t^n f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x,t)g(x',t') \mid_{x=x',t=t'}.$$

It is assumed \[4\] that \( Q_j = 0 \ \forall j \), although this is not the most general decoupling. (The existence of \( n + 1 \) \( \tau \)-functions (compared to the \( n \)-component field \( \phi \)) can be traced back to the \( \nu \) field in the CAT model \[4\]). The equations of motion can now be reduced to the form,

$$\eta_j(D_t^2 - D_x^2)\tau_j \cdot \tau_j - 2m^2 n_j \left( \prod_{k=0}^{n} \tau_k^{-\eta_k \alpha_k \cdot \alpha_j} - 1 \right) \tau_j^2 = 0. \quad (2.0d)$$

In the spirit of Hirota’s method for finding soliton solutions \[21\], it is assumed that

$$\tau_j = 1 + \delta_{j}^{(1)} e^{\Phi} \epsilon + \delta_{j}^{(2)} e^{2\Phi} \epsilon^2 + \ldots + \delta_{j}^{(p_j)} e^{p_j \Phi} \epsilon^{p_j}$$

\(2.0e\)

where \( \Phi = \sigma(x - vt) + \xi \) and \( \delta_{j}^{(k)} (1 \leq k \leq p_j) \), \( \sigma \), \( v \) and \( \xi \) are arbitrary complex constants.

The constant \( p_j \) is a positive integer and \( \epsilon \) a dummy parameter. The method employed is to solve (2.3) at successive orders in \( \epsilon \).

The parameters \( \sigma \), \( v \) and \( m \) are related by

$$\sigma^2(1 - v^2) = m^2 \lambda.$$ 

where \( \lambda \) is an eigenvalue of \( NC \). The matrices \( N \) and \( C \) are defined as

- \( N = \text{diag}(n_0, n_1, \ldots, n_n) \),
- \( (C)_{ij} = \alpha_i \cdot \alpha_j \).
This result is used in showing that the ratios of the soliton masses are equal to the ratios of the masses of the fundamental Toda particles, as the non-zero eigenvalues of NC were shown in [22, 23] (for the A, D and E theories) to be the squared masses of the fundamental particles. The requirement that \( \tau_j \) be bounded as \( x \to \pm \infty \), requires \( n_0 \eta_j p_j = n_j \eta_0 p_0 \).

Finally, it is unnecessary to consider the solution corresponding to \( \lambda = 0 \), as it is always \( \phi = 0 \).

### 2.1 The \( a_n^{(1)} \) solitons

The Dynkin diagram for \( a_n^{(1)} \) is shown in Figure 1. The matrix \( NC \) has eigenvalues given by

\[
\lambda_a = 4 \sin^2 \left( \frac{\pi a}{h} \right), \quad \text{where } h = n + 1 \text{ and } a = 1, \ldots, h - 1.
\]

![Figure 1: Affine Dynkin diagram for \( a_n^{(1)} \).](image)

With \( \eta_j = 1 \ \forall j \), the tau equations of motion are

\[
(D_t^2 - D_x^2) \tau_j \cdot \tau_j = 2m^2 (\tau_{j-1} \tau_{j+1} - \tau_j^2)
\]

i.e. those of [1]. Using the expansion (2.4) with \( p_0 = 1 \) for the single soliton solutions, it is found that

\[
\tau_j = 1 + \omega^j e^\Phi
\]

where \( \omega \) is an \( h^{th} \) root of unity. There are \( n \) non-trivial solutions [1] (equal to the number of fundamental particles) with \( \omega_a = \exp(2\pi i a/h) \) where \( 1 \leq a \leq n \). These \( n \) solutions to \( a_n^{(1)} \) can be written in the form

\[
\phi(a) = -\frac{1}{i\beta} \sum_{k=1}^{r} \alpha_j \ln \left( \frac{1 + w^j e^\Phi}{1 + e^\Phi} \right).
\]
The general $N$-soliton solution can be built up from the single soliton solutions, having
\[
\tau_j(x,t) = \sum_{\mu_1=0}^1 \cdots \sum_{\mu_N} \exp \left( \sum_{p=1}^N \mu_p \omega_p^j \Phi_p + \sum_{1 \leq p < q \leq N} \mu_p \mu_q \ln A^{(pq)} \right)
\]  
where
\[
A^{(pq)} = \frac{(\sigma_p - \sigma_q)^2 - (\sigma_p v_p - \sigma_q v_q)^2 - 4m^2 \sin^2 \frac{\pi}{h} (a_p - a_q)}{(\sigma_p + \sigma_q)^2 - (\sigma_p v_p + \sigma_q v_q)^2 - 4m^2 \sin^2 \frac{\pi}{h} (a_p + a_q)}
\]
is the ‘interaction constant’. As well as non-static multisolitons there exist also static configurations composed of different species (i.e. $a_p \neq a_q$) of single solitons. The number of such configurations is given by
\[
\sum_{k=1}^n \binom{n}{k} = 2^n - 1.
\]
For example, static double solitons take the form
\[
\tau_j(x,t) = 1 + \omega_1^j e^{\Phi_1} + \omega_2^j e^{\Phi_2} + A^{(12)} \omega_{a_1}^j \omega_{a_2}^j e^{\Phi_1 + \Phi_2}.
\]
With both solitons having the same velocity, $\sigma_1 \sqrt{\lambda_2} = \sigma_2 \sqrt{\lambda_1}$, and so
\[
\Phi_2 = \sqrt{\frac{\lambda_2}{\lambda_1}} (\Phi_1 - \xi_1) + \xi_2.
\]
When the coefficient of $\Phi_1$ in (2.1b) is a positive integer then these solutions arise directly from the Hirota method, that is when
\[
\sin^2 \left( \frac{a_1 \pi}{h} \right) = k^2 \sin^2 \left( \frac{a_2 \pi}{h} \right).
\]
If $a_2 = h - a_1$, then
\[
A^{(12)} = \cos^2 \left( \frac{a_1 \pi}{h} \right) = 1 - \frac{1}{4} \lambda_{a_1},
\]
and after the shift $\Phi_1 \to \Phi_1 + \xi_1$,
\[
\tau_j(x,t) = 1 + y_1 \omega_1^j e^{\Phi_1} + y_2 \omega_{a_1}^j e^{\Phi_1} + y_1 y_2 \left( 1 - \frac{1}{4} \lambda_{a_1} \right) e^{2\Phi_1}
\]
where $y_1 = e^{\xi_1}$, and $y_2 = e^{\xi_2}$. Notice that when $y_1 = 0$ (or $y_2 = 0$), that is when the first (or second) soliton is sent off to infinity, the above solution reduces to a that of a single soliton. These ‘mass degenerate’ solutions which appear for $h = 2p$ and $h = 6p$ [4] (another two such configurations appear in [3] for $a_1^{(1)}$, but are similarly derived by the method above) were discussed in [3, 5] when they where viewed as different to those constructed in [4]. In fact, as shown above, they are special cases of more general static configurations lying within the $N$-soliton solution (2.1a).
3 Topological Charge.

The topological charge of the solitons is defined by

\[ t = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \phi = \frac{\beta}{2\pi} (\lim_{x \to \infty} - \lim_{x \to -\infty}) \phi(x, t), \]

which, using the ansatz (2.0c), can be written in the following form:

\[ t = -\frac{1}{2\pi i} \sum_{j=0}^{n} \frac{2}{\alpha_j \cdot \alpha_j} (\lim_{x \to \infty} - \lim_{x \to -\infty}) \ln \tau_j(x, t), \]

(3.1)

3.1 The \( a_n^{(1)} \) solitons.

In this subsection, the topological charges of the single solitons will be calculated. The method used is to find a relationship between the charges, which is then used to deduce all the charges from just one – the highest charge (so called, as all charges are subsequently shown to be obtainable from the highest charge by subtracting a sum of simple roots). An explicit expression for the topological charges associated to each soliton is constructed, before the relationship between the charges is shown to be equivalent to the action of the Coxeter element (with a particular ordering), so implying that the topological charges lie in the same representation. This is shown for the \( a^{th} \) soliton to be the \( a^{th} \) fundamental representation. Lastly, the topological charges of the multisolitons is considered.

3.1.1 Single soliton topological charge.

It will prove convenient in the subsequent discussion to write equation (3.1) in the form,

\[ t = -\frac{1}{2\pi i} \sum_{j=1}^{n} (\lim_{x \to \infty} - \lim_{x \to -\infty}) (\ln |f_j(x, t)| + i \arg f_j(x, t)) \alpha_j. \]

(3.1a)

where

\[ |f_j(x, t)| = \frac{\tau_j(x, t)}{\tau_0(x, t)} \to 1 \quad \text{as} \quad x \to \pm \infty. \]

The topological charge is therefore given by

\[ t = -\frac{1}{2\pi} \sum_{j=1}^{n} \lim_{x \to \infty} - \lim_{x \to -\infty} \arg f_j(x, t) \alpha_j. \]
In order to calculate the topological charges it is necessary to understand the behaviour of the complex functions \( f_j(x, t) \). At \( t = 0 \) (assuming throughout that \( \sigma > 0 \)), with \( \xi = \xi_1 + i\xi_2 \), it is convenient to write

\[
e^{\sigma x + \xi_1 + i\xi_2} = ye^{i\xi_2},
\]

where \( y \to 0 \) as \( x \to -\infty \), \( y \to \infty \) as \( x \to \infty \), and \( \xi_2 \) is chosen such that \(-\pi < \xi_2 \leq \pi\). It is also convenient to write

\[
\omega_a^j = e^{i\mu} \quad \text{where} \quad \mu = \frac{2\pi aj}{h} \mod (2\pi).
\]

Before proceeding, it is worthwhile to make clear the idea behind the calculation which follows. The function

\[
f_j(x, t) = \frac{1 + ye^{i(\mu + \xi_2)}}{1 + ye^{i\xi_2}}
\]

has zeros whenever \( \mu + \xi_2 = \pi \) and \( y = 1 \), and is undefined when \( \xi_2 = \pi \) and \( y = 1 \). In either case, \( \phi(x, t) \) is undefined. The range of \( \xi \) can then be divided into sectors, the boundary of each sector being the values of \( \xi_2 \) for which \( f_j(x, t) \) is either zero or undefined. Now, the topological charge is a measure of the change in the argument of \( f_j(x, t) \) as \( x \) goes from \(-\infty \) to \(+\infty \), and so the topological charge can change only when the curve traced out by \( f_j(x, t) \) in the complex plane, is either (i) undefined, or (ii) passes through the origin. The implication therefore is that the topological charge of the soliton is constant on each of the sectors in the range of \( \xi_2 \) mentioned above. Indeed, it will be shown that the topological charge takes on a unique value in each sector. An expression for topological charge in one particular sector, that of the highest charge, is calculated and from it (in the following subsection) an expression for the remaining charges is deduced.

The number of sectors \( \tilde{h}_a \), which the range of \( \xi_2 \) is divided into is rather straightforward to calculate. It is equal to the number of different values that

\[
\frac{2\pi ia j}{h} \mod 2\pi i
\]

can take, or in other words the smallest value of \( q \) for which

\[
\frac{2\pi iqa}{h} = 2\pi ik \quad \text{where} \quad q, k \in \mathbb{N}.
\]
Rewriting this as

\[ q \tilde{a} = k \tilde{h}_a \quad \text{where} \quad \tilde{a} = \frac{a}{\gcd(a, h)} \quad \text{and} \quad \tilde{h}_a = \frac{h}{\gcd(a, h)} \]

are coprime, then \( q = \tilde{h}_a \) and \( k = \tilde{a} \). So the maximum number of values that the topological charge can take for the \( a^{th} \) soliton is

\[ \tilde{h}_a = \frac{h}{\gcd(a, h)} \].

The range of \( \xi_2 \) can therefore be split into \( \tilde{h}_a \) regions

\[ I_p = \left( -\pi + \frac{2\pi p}{\tilde{h}_a}, -\pi + \frac{2\pi (p + 1)}{\tilde{h}_a} \right) \]

with \( 0 \leq p \leq \tilde{h}_a - 1 \). Consider now the transformation

\[ \xi_2 \rightarrow \left( \xi_2 + \frac{2\pi a}{h} \right) \mod (2\pi) \in [-\pi, \pi). \quad (3.1.1b) \]

If \( \xi_2 \) originally resides in the region \( I_0 \) then repeated application of (3.1.1b) will send \( \xi_2 \) to each of the other regions in turn, before returning to \( I_0 \) on the \( \tilde{h}_a^{th} \) application. As will now be shown, the above transformation is equivalent to a cyclic permutation of the simple roots \( \{\alpha_j\} \) plus the extended roots \( \alpha_0 \). The \( a^{th} \) soliton solution takes the form

\[ \phi(a)(x, t) = -\frac{1}{i\beta} \sum_{j=0}^{n} \alpha_j \ln(1 + \omega^j_ye^{i\xi_2}) \]

which, under the above transformation, becomes

\[ \phi(a)(x, t) = -\frac{1}{i\beta} \sum_{j=0}^{n} \alpha_j \ln(1 + \omega^j_ye^{i\xi_2}) = -\frac{1}{i\beta} \sum_{j=0}^{n} \alpha_{j-1} \ln(1 + \omega^j_ye^{i\xi_2}), \]

the labelling of the roots being mod \( h \). Therefore, to calculate the full set of topological charges of a single soliton, all that is required is to calculate the topological charge for one value of \( \xi_2 \) and then cyclically permute the labelling on the \( \alpha_j \) (\( 0 \leq j \leq h - 1 \)) to generate the others.

Consider now the function \( f_j(x, t) \). Splitting it up into its real and imaginary parts,

\[ f_j(x, t) = \frac{1 + y [\cos(\mu + \xi_2) + \cos \xi_2]}{|1 + ye^{i\xi_2}|^2} + i \frac{y [\sin(\mu + \xi_2) - \sin \xi_2]}{|1 + ye^{i\xi_2}|^2}. \]
The imaginary part is zero for \( y = 0 \) (i.e. at \( x = -\infty \)) and at one other point given by
\[
y = -\frac{\sin(\mu + \xi_2) - \sin(\xi_2)}{\sin \mu} \quad \text{(provided } y > 0), \tag{3.1.1b}
\]
where \( \mu \neq 0, \pi \). If \( \xi_2 = -\pi + \epsilon \), where \( \epsilon > 0 \) is an infinitesimal. Then,
\[
\text{Im}(f_j(x,t)) = 0 \quad \text{for} \quad y = 1 - \frac{(1 - \cos \mu)}{\sin \mu} \epsilon + O(\epsilon^2),
\]
and
\[
\text{Re}(f_j(x,t))|1 + ye^{i\xi_2}|^2 = \frac{2\epsilon}{\sin \mu}(1 - \cos \mu) + O(\epsilon^2).
\]
Therefore the complex function \( f_j(x,t) \) crosses the real axis positively for \( 0 < \mu < \pi \) and negatively for \( \pi < \mu < 2\pi \). Also, for small positive \( y \),
\[
\text{Im}(f_j(x,t))|1 + ye^{i\xi_2}|^2 = -y \sin \mu + \text{(higher order terms)}
\]
i.e. the function starts off with negative imaginary part for \( 0 < \mu < \pi \) and positive imaginary part for \( \pi < \mu < 2\pi \). Finally, if \( \mu = 0 \) then \( f_j = 1 \) and contributes zero to the topological charge, whereas if \( \mu = \pi \), the change in the function’s argument is \( +\pi \). Taking all of this information together the topological charge in this sector is deduced to be
\[
t^{(1)}_a = -\frac{1}{2\pi i} \sum_{j=0}^{n} \left( \frac{2\pi i a j}{h} \mod 2\pi i \right) \alpha_j
\]
\[
= \sum_{j=0}^{n} \frac{a(h - j) \mod h}{h} \alpha_j.
\]
This topological charge will be called the ‘highest charge’ since the difference between it and all subsequent topological charges, is proportional to a sum of positive roots. The remaining charges are therefore generated under \( \tau : \alpha_j \rightarrow \alpha_{(j-1) \mod h} \). The order of \( \tau \) acting on the highest charge is the smallest value of \( q \) such that
\[
a(h - (j + q)) \mod h = a(h - j) \mod h.
\]
This is given by \( q = \tilde{h}_a \), confirming that \( \tilde{h}_a \) is in fact equal to the number of charges for the \( a^{th} \) soliton.
3.1.2 An explicit formula for the charges.

Consider the highest charge, which is written for convenience in the following form:

$$\lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \cdots + \lambda_n \alpha_n$$

Each $$\lambda_j$$ is equal to one of $$0, 1/\tilde{h}_a, 2/\tilde{h}_a, \ldots, (\tilde{h}_a - 1)/\tilde{h}_a$$. The other $$\tilde{h}_a - 1$$ charges are obtained by cyclically permuting the labelling of the simple roots so that

$$\lambda_0 = 1/\tilde{h}_a, \lambda_0 = 2/\tilde{h}_a, \ldots, \lambda_0 = (\tilde{h}_a - 1)/\tilde{h}_a.$$  

Consider now, the permutation that results in $$\lambda_0 = k/\tilde{h}_a$$ where $$1 \leq k \leq \tilde{h}_a - 1$$. This is in effect equivalent to adding, modulo $$h$$,

$$\frac{k}{\tilde{h}_a} (\alpha_0 + \alpha_1 + \cdots + \alpha_n)$$

to the highest charge. Therefore,

$$\lambda_j \rightarrow \begin{cases} 
\lambda_j + k/\tilde{h}_a, & \text{if } \lambda_j + k/\tilde{h}_a < 1; \\
\lambda_j + k/\tilde{h}_a - 1, & \text{if } \lambda_j + k/\tilde{h}_a \geq 1.
\end{cases} \quad (3.1.2a)$$

Using (2.0a) to set $$\lambda_0$$ equal to zero, the overall effect of the permutation is the subtraction of 1 from $$\lambda_j$$ where $$\lambda_j + k/\tilde{h}_a \geq 1$$. The expression for the topological charges is therefore deduced to be

$$t_a^{(k)} = \sum_{j=1}^{n} \frac{a(h - j) \mod h}{h} \alpha_j - \sum_{l=1}^{k-1} \sum_{j=1}^{n} \delta_{a(h-j) \mod h, h-l \gcd(a,h)} \alpha_j$$

where $$k = 1, \ldots, \tilde{h}$$. The example of the $$A_5$$ theory is given in Appendix A.1.

3.1.3 The highest charge and its fundamental representation.

In this section it will be shown that the topological charge of the $$a^{th}$$ soliton lies in the $$a^{th}$$ fundamental representation. This will be used in the next section when the remaining solitons will be shown to lie in the same representation as the highest charge and so imply that all the topological charges lie in the same fundamental representation.
It will be convenient to write $a = h - b$, $b = \tilde{b} \gcd(b, h)$, and $h = \tilde{h} \gcd(b, h)$.

Due to the symmetry of the $a_n^{(1)}$ theory under $\alpha_i \rightarrow \alpha_{h-i}$ for $1 \leq i \leq n$, it is necessary only to consider

$$b \leq \begin{cases} \frac{1}{2}(h-1), & \text{if } h \text{ is odd;} \\ \frac{1}{2}h, & \text{if } h \text{ is even.} \end{cases}$$

The highest charge is then given by

$$t_{\alpha}^{(1)}(a) = \sum_{j=1}^{n} \frac{bj \mod h}{h} \alpha_j.$$ 

The inner products of $t^{(1)}$ with each of the simple roots will be considered and shown to be transformable via Weyl reflections to the highest weight of the $a$th fundamental representation. Consider firstly the case of $\gcd(b, h) = 1$, i.e. $b$ and $h$ coprime. Defining,

$$\Omega(k) = \left\lfloor \frac{hk}{b} \right\rfloor$$

where $[...]$ denotes the integer part, then

$$t_{\alpha}^{(1)} \cdot \alpha_j = \begin{cases} 1 & \text{for } j = \Omega(k), \text{ where } k = 1, \ldots, b-1, \\ -1 & \text{for } j = \Omega(k) + 1, \text{ where } k = 1, \ldots, b-1, \\ 1 & \text{for } j = h-1, \\ 0 & \text{otherwise.} \end{cases}$$

Also, $\Omega(k) + 1 < \Omega(k+1)$ for $k = 1, \ldots, b-2$, and $\Omega(b-1) < h-1$. Therefore, in general, $t_{\alpha}$ has inner products with the simple roots of the form

$$t_{\alpha} \cdot \{\alpha_j\} = (0, 0, \ldots, 0, 1, -1, 0, \ldots, 0, 1, -1, 0, \ldots, 0, 0, 1),$$

the notation indicating that the $j^{th}$ component of the row vector is given by $t_{\alpha} \cdot \alpha_j$. There are two things that can be immediately shown to be true. Notice that if a weight, $w$ has inner products with the simple roots given by

$$w \cdot \{\alpha_j\} = (\ldots, 0, 1, -1, 0, 0, \ldots)$$

then under a Weyl reflection in the root which has inner product $-1$ with $w$, $w \rightarrow w'$, where

$$w' \cdot \{\alpha_j\} = (\ldots, 0, 1, -1, 0, 0, \ldots).$$
Applying this to the case of $t_a$, then a series of Weyl reflections will result in $t_a \rightarrow t_a'$ where

$$t_a' \cdot \{\alpha_j\} = (0, \ldots, 0, 1, -1, 1, -1, \ldots, 1, -1, 1).$$

If a weight $w$ has inner product with the simple roots now given by

$$w \cdot \{\alpha_j\} = (\ldots, 1, -1, 1, \ldots)$$

then again performing a Weyl reflection in the simple root which has inner product $-1$ with $w$, $w \rightarrow w'$ where

$$w' \cdot \{\alpha_j\} = (\ldots, 0, 1, 0, \ldots)$$

This last procedure combined with the previous one, can be applied to $t_a'$ to finally give $t_a''$ which is expressed via

$$t_a'' \cdot \{\alpha_j\} = (0, \ldots, 0, 1, 0, \ldots, 0)$$

with the $1$ appearing in the $d^{th}$ position, $d$ being given by,

$$d = n - [b - 1] = h - b = a.$$

Therefore, $t_a$ lies in the same representation as $t_a''$, i.e. the $a^{th}$ fundamental representation. The generalisation to the case of $\gcd(b, h) \neq 1$ is straightforward. If $\tilde{t}^{(1)}$ is the highest charge in the theory with Coxeter number $\tilde{h}$ of the $\tilde{a} = \tilde{h} - \tilde{b}$ soliton, then the highest charge of the current soliton is given by

$$t^{(1)} = (\tilde{t}^{(1)}, 0, \tilde{t}^{(1)}, 0, \ldots, \tilde{t}^{(1)}).$$

Then by the results of the above discussion the inner products of $t^{(1)}$ with the simple roots is also of the form (3.1.2b), and again by the above lies in the $a^{th}$ fundamental representation.

3.1.4 The Topological charges, the Coxeter element and the fundamental representations.

In the last two subsections, the topological charges of the $a^{th}$ soliton were calculated, with the highest charge shown to lie in the $a^{th}$ fundamental representation. In this subsection
it will be shown that the cyclical permutation of the roots used to connect the topological charges is in fact equivalent to the application of the Coxeter element

\[ \omega_{tc} = r_n r_{n-1} r_{n-2} \ldots r_3 r_2 r_1. \]  

(3.1.4b)

where \( r_i \) is a Weyl reflection in the \( i^{th} \) simple root \( \alpha_i \). It is important to note that the ordering of the Weyl reflections is not arbitrary – other orderings do not necessarily connect the charges. This can be shown for the case of \( A_5 \) discussed in the previous subsection (see Appendix A.1 for an illustration of this).

Firstly, consider the effect of \( \omega_{tc} \) on the set of simple roots \( \{ \alpha_j \} \). It can be shown

\[
\begin{align*}
\alpha_0 & \rightarrow \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} + 2\alpha_n, \\
\alpha_1 & \rightarrow -\alpha_1 - \alpha_2 - \alpha_3 - \ldots - \alpha_n, \\
\text{and} \quad \alpha_i & \rightarrow \alpha_{i-1} \quad \text{for} \quad 2 \leq i \leq n,
\end{align*}
\]

and so an arbitrary linear combination of the simple roots plus the extended root

\[ u = \lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \ldots + \lambda_n \alpha_n \]

is transformed thus:

\[
\begin{align*}
u & \rightarrow \lambda_0 \alpha_0 + (\lambda_0 - \lambda_1) \alpha_1 + \ldots + (\lambda_0 - \lambda_1 + \lambda_n) \alpha_{n-1} + (2\lambda_0 - \lambda_1) \alpha_n \\
&= \lambda_1 \alpha_0 + \lambda_2 \alpha_1 + \lambda_3 \alpha_2 + \ldots + \lambda_n \alpha_{n-1} + \lambda_0 \alpha_n
\end{align*}
\]

by equation (2.0a). Using the notation

\[ \lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \ldots + \lambda_n \alpha_n = (\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n), \]

then

\[ \omega_{tc}(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n) = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \lambda_0), \]

i.e. the action of the Coxeter element cyclically permutes the \( \lambda_j \)'s. This invariance of the set of topological charge under the action of the Coxeter element means that the topological charges lie in the same representation as the highest charge i.e. the \( a^{th} \) fundamental representation.
3.1.5 The other $a_n^{(1)}$ automorphisms

In subsection 3.1.1 it was shown that the set of topological charges corresponding to each soliton was invariant under the automorphisms of the extended Dynkin diagram which cyclically permute the elements of the extended root system. There are other automorphisms of the extended diagram for $a_n^{(1)}$. In this subsection, the effect of these mappings (up to a sign), on a soliton’s topological charges will be discussed.

Sending $\xi_2 \rightarrow -\xi_2$ in the soliton solution is equivalent to evaluating the topological charge in region $I_{\tilde{h}_a-1-p}$ rather than $I_p$ where $0 \leq p \leq \tilde{h}_a - 1$. The form of the $a^{th}$ soliton solution is

$$\phi(a)(x, t) = -\frac{1}{i\beta} \sum_{j=1}^{n} \alpha_j \ln \left( \frac{1 + \omega_j^a y e^{-i\xi_2}}{1 + ye^{-i\xi_2}} \right) = -\frac{1}{i\beta} \sum_{j=1}^{n} \alpha_j \ln \left( \frac{\omega_j^a y^h e^{-i\xi_2}}{1 + ye^{-i\xi_2}} \right).$$

The last expression can be recast into the form

$$\phi(a)(x, t) = -\frac{1}{i\beta} \sum_{j=1}^{n} \alpha_{h-j} \ln \left( \frac{\omega_{h-j}^a y^{-1} e^{i\xi_2}}{1 + y^{-1} e^{i\xi_2}} \right).$$

The topological charge in region $I_{\tilde{h}_a-1-p}$ is therefore obtained from the topological charge in region $I_p$ ($1 \leq p \leq \tilde{h}_a - 1$) via the mapping

$$\sigma_0 : \alpha_j \rightarrow -\alpha_{h-j}.$$

Combining this with the map of cyclic permutations of the extended root system, $\tau$, it is found that the set of topological charges of each soliton is invariant under

$$\sigma_k : \alpha_j \rightarrow -\alpha_{(k-j) \mod h} \quad (0 \leq j, k \leq h - 1).$$

It is compelling to associate these mappings with the automorphisms of the extended Dynkin diagrams which reflect the diagram in a line splitting it in two as shown in figure 2, below.

\[\begin{align*}
\sigma_0 & : \alpha_j \rightarrow -\alpha_{h-j}. \\
\sigma_k & : \alpha_j \rightarrow -\alpha_{(k-j) \mod h} \quad (0 \leq j, k \leq h - 1).
\end{align*}\]

\hspace{1cm}$a_n^{(1)}$ when $n$ is even.

\hspace{1cm}$a_n^{(1)}$ when $n$ is odd.

Figure 2: Reflection symmetry of the $a_n^{(1)}$ Dynkin diagram.
3.1.6 Multisoliton solutions.

In this section, a multisoliton configuration composed of $N$ widely separated solitons is considered. In this large separation approximation the topological charge of the configuration as a whole is found to be the sum of the topological charges of the individual solitons. This will be done via an inductive argument. In [1] the tau functions of the multisolitons were found to be

$$
\tau_j(x, t) = \sum_{\mu_1 = 0}^{1} \cdots \sum_{\mu_N} \exp \left( \sum_{p=1}^{N} \mu_p \omega^j_p \phi_p + \sum_{1 \leq p < q \leq N} \mu_p \mu_q \ln A(pq) \right)
$$

where

$$
A(pq) = -\frac{(\sigma_p - \sigma_q)^2 - (\sigma_p v_p - \sigma_q v_q)^2 - 4m^2 \sin^2 \frac{\pi}{n+1} (a_p - a_q)}{(\sigma_p + \sigma_q)^2 - (\sigma_p v_p + \sigma_q v_q)^2 - 4m^2 \sin^2 \frac{\pi}{n+1} (a_p + a_q)}
$$

is the ‘interaction constant’. Relabelling the solitons, if necessary, then

$$
\sigma_1 v_1 < \sigma_2 v_2 < \ldots < \sigma_{N-1} v_{N-1} < \sigma_N v_N. \quad (3.1.5a)
$$

It will be convenient to write $e^{\sigma_i(x-v_i t+\xi^{(i)})} = ye^{-\mu_i(t)}e^{i\xi_2^{(i)}}$, where $\mu_i(t) = \sigma_i v_i t - \xi^{(i)}_1$. If $t = T$ is fixed for sufficiently large $T$, then write $\mu_i(T) = \mu_i$ so that

$$
\mu_1 \ll \mu_2 \ll \ldots \ll \mu_{N-1} \ll \mu_N. \quad (3.1.5b)
$$

It is worthwhile to find the range of $y$ for which the soliton field $\phi(x, t)$ has its most rapid variation (and so where the soliton is located). This is done via the parameter $k \gg 1$, and the imposition that

$$
\frac{1}{k} < |\omega^j_{a_i} y^{\sigma_i} e^{-\mu_i} e^{i\xi_2^{(i)}} | < k,
$$

since below the lower limit $\tau_j/\tau_0 \sim 1$, and above the upper limit $\tau_j/\tau_0 \sim \omega^j_{a_i}$. The corresponding limits in the range of $y$ are:

$$
y \sim \left( \frac{1}{k} e^{\mu} \right)^{1/\sigma_i} \quad \text{for} \quad |\omega^j_{a_i} y^{\sigma_i} e^{-\mu_i} e^{i\xi_2^{(i)}} | \sim \frac{1}{k},
$$

and

$$
y \sim (k e^{\mu})^{1/\sigma_i} \quad \text{for} \quad |\omega^j_{a_i} y^{\sigma_i} e^{-\mu_i} e^{i\xi_2^{(i)}} | \sim k.
$$

The point in time considered $T$, can be chosen large enough so that each of the above regions are far apart, that is

$$
\left( \frac{1}{k} e^{\mu_1} \right)^{1/\sigma_1} \ll (k e^{\mu_1})^{1/\sigma_1} \ll \left( \frac{1}{k} e^{\mu_2} \right)^{1/\sigma_2} \ll (k e^{\mu_2})^{1/\sigma_2} \ll \ldots
$$

$$
\ldots \left( \frac{1}{k} e^{\mu_{N-1}} \right)^{1/\sigma_{N-1}} \ll (k e^{\mu_{N-1}})^{1/\sigma_{N-1}} \ll \left( \frac{1}{k} e^{\mu_N} \right)^{1/\sigma_N} \ll (k e^{\mu_N})^{1/\sigma_N}.
$$
The scene is now set for a straightforward calculation of the multisoliton topological charge. Consider the two soliton solution

\[
\frac{\tau_j}{\tau_0} = \frac{1 + \omega_1^j y^\sigma_1 e^{-\mu_1 e^{i\xi_1}} + \omega_2^j y^\sigma_2 e^{-\mu_2 e^{i\xi_2}} (1 + A_{12} \omega_{a1}^j y^\sigma_1 e^{-\mu_1 e^{i\xi_1}})}{1 + y^\sigma_1 e^{-\mu_1 e^{i\xi_1}} + y^\sigma_2 e^{-\mu_2 e^{i\xi_2}} (1 + A_{12} y^\sigma_1 e^{-\mu_1 e^{i\xi_1}})}.
\]

Here the first soliton is located in the range \(\left(\frac{1}{k e^{\mu_1}}\right)^{1/\sigma_1} \leq y \leq (k e^{\mu_1})^{1/\sigma_1}\), and the second in the range \(\left(\frac{1}{k e^{\mu_1}}\right)^{1/\sigma_1} \leq y \leq (k e^{\mu_1})^{1/\sigma_1}\). Outside these regions \(\tau_j/\tau_0\) is effectively constant and equal to (in order of increasing \(y\)), \(1, \omega_1^j, \omega_2^j, \omega_{a1}^j, \omega_{a2}^j\), respectively. In the range \(\left(\frac{1}{k e^{\mu_1}}\right)^{1/\sigma_1} \leq y \leq (k e^{\mu_1})^{1/\sigma_1}\),

\[
\frac{\tau_j}{\tau_0} \sim \frac{1 + \omega_1^j y^\sigma_1 e^{-\mu_1 e^{i\xi_1}}}{1 + y^\sigma_1 e^{-\mu_1 e^{i\xi_1}}}
\]

(i.e. it is effectively the \(j^{th}\) component of the first soliton) which contributes to the topological charge by \(t_1\). Finally, for \(\left(\frac{1}{k e^{\mu_2}}\right)^{1/\sigma_2} \leq y \leq (k e^{\mu_2})^{1/\sigma_2}\),

\[
\frac{\tau_j}{\tau_0} \sim \omega_{a1}^j \frac{1 + A_{12} \omega_{a2}^j y^\sigma_2 e^{-\mu_2 e^{i\xi_2}}}{1 + A_{12} y^\sigma_2 e^{-\mu_2 e^{i\xi_2}}}
\]

which contributes \(t_2\) to the topological charge. Therefore the topological charge of the double soliton is given by

\[t = t_1 + t_2.\]

Suppose now that the \((N - 1)\)-soliton solution has topological charge given by

\[t_{(N-1)} = t_1 + t_2 + \ldots + t_{(N-1)}.\]

If \(\tau_j^{(p)}\) is the \(j^{th}\) tau function of the \(p\)-soliton solution, then for \(0 < y \leq (k e^{\mu_{N-1}})^{1/\sigma_{N-1}}\),

\[
\frac{\tau_j^{(N)}}{\tau_0^{(N)}} \sim \frac{\tau_j^{(N-1)}}{\tau_0^{(N-1)}},
\]

contributing \(t_{(N-1)}\) to the topological charge.

In the regions \((k e^{\mu_{N-1}})^{1/\sigma_{N-1}} < y < (k e^{\mu_N})^{1/\sigma_N}\), and for \(y > (k e^{\mu_N})^{1/\sigma_N}\), the functions \(\tau_j^{(N)}/\tau_0^{(N)}\) are effectively constant and equal to \(\omega_1^j \omega_2^j \ldots \omega_{a1}^j \omega_{a2}^j \ldots \omega_{a_{N-1}}^j \omega_{a_N}^j\) respectively. For \(\left(\frac{1}{k e^{\mu_N}}\right)^{1/\sigma_N} \leq y \leq (k e^{\mu_N})^{1/\sigma_N}\),

\[
\frac{\tau_j^{(N)}}{\tau_0^{(N)}} \sim \omega_{a1}^j \omega_{a2}^j \ldots \omega_{a_{N-1}}^j \frac{1 + A_{a1a2} \ldots A_{a_{N-1}a_{N-1}} \omega_{a_N}^j y^\sigma_N e^{-\mu_N e^{i\xi_N}}}{1 + A_{a1a2} \ldots A_{a_{N-1}a_{N-1}} y^\sigma_N e^{-\mu_N e^{i\xi_N}}}.
\]
which contributes $t_N$ to the topological charge. Therefore the topological charge of the $N$-soliton solution is

$$t = t_1 + t_2 + \ldots + t_{N-1} + t_N.$$ 

This result still holds if the strict inequalities of (3.1.5a) are relaxed to allow for solitons to have $\sigma_i v_i$ equal, provided $\xi_1^{(i)}$ is large enough so that (3.1.5b) holds.

### 3.1.7 Multisolitons and representation theory.

Having established in the previous subsection that the topological charge of a multisoliton is the sum of the topological charges of its constituent solitons, the representations in which the topological charges of these solitons lie will be discussed.

Denote the set of topological charges of the $N$-soliton solution, which is composed of the $\{a_1, a_2, \ldots, a_N\}$-solitons, and the $a^{th}$ fundamental representation by $T_{(a_1, \ldots, a_N)}$ and $\mathcal{R}_a$ respectively. As the topological charges of two widely separated solitons are equal to the pairwise sums of the topological charges of the individual solitons, then for the special case of a double soliton composed of the single solitons associated to the first and $n$-th fundamental representations, the resulting topological charges are the weights of the tensor product representation $\mathcal{R}_1 \otimes \mathcal{R}_n$. However, this tensor product of representations contains the adjoint representation, and so contains $\{\pm \alpha_1, \pm \alpha_2, \ldots, \pm \alpha_n\}$. As a result, further multisoliton configurations can be constructed that employ these solitons having charges equal to the simple roots, and so fill up all the fundamental representations as well as the entire weight lattice.

### 4 Discussion and conclusions.

Having discussed the $a_n^{(1)}$ theory in detail it is worthwhile to consider what can be deduced about the other theories. Firstly consider $c_n^{(1)}$ whose solitons are multisolitons of the $a_{2m-1}^{(1)}$ theory. As a result it may be expected that the results of §3.1.5 can be used to calculated the topological charges. This is not true, as the results on the multisolitons depend on the
individual solitons being widely spaced – the $c_n^{(1)}$ single solitons are $a_{2n-1}^{(1)}$ double solitons of zero separation. Indeed it can be shown that the number of topological charges of the $a^\text{th}$ soliton ($1 \leq a \leq n - 1$) in the $c_n^{(1)}$ theory is given by

$$2(\tilde{n} - \tilde{a} + 1)(1 - \frac{1}{2}\delta_{0,\tilde{a} \mod 2})$$

where

$$\tilde{n} = \frac{n}{\gcd(a, n)} \quad \text{and} \quad \tilde{a} = \frac{a}{\gcd(a, n)}$$

and two topological charges for the $n^\text{th}$ soliton. Therefore, in general, the number of topological charges in the $c_n^{(1)}$ theory is less than that calculated naively from the $a_n^{(1)}$ theory. The set of topological charges corresponding to each soliton of the $c_n^{(1)}$ theory is also invariant under mappings related to the automorphisms of the theory. However, there are not enough mappings to allow all of the charges to be deduced from just one.

For the $d_4^{(1)}$ and $d_5^{(1)}$ theories, the topological charges of the single solitons have been calculated and can be shown to lie in the corresponding fundamental representation (for example see [1] for $d_4^{(1)}$) though, as in $a_n^{(1)}$, not fill it. It is possible, as in the $a_n^{(1)}$ case to deduce relationships between the charges from the soliton solution, and so allow deduction of all the charges from just one. There also exist products of reflections whose orbits are made up of the charges, though the results are not as neat as those for $a_n^{(1)}$.

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Appendix

A.1 The topological charges of $a_5^{(1)}$ single solitons.

Consider the case of the $a_5^{(1)}$ theory whose Dynkin diagram is shown in figure 2, below.

![Dynkin diagram for $a_5^{(1)}$]

The number above each spot on the diagram is the number of topological charges corresponding to that soliton.

First soliton (a=1).

\[
\begin{align*}
t^{(1)} &= \frac{5}{6} \alpha_1 + \frac{2}{3} \alpha_2 + \frac{1}{2} \alpha_3 + \frac{1}{3} \alpha_4 + \frac{1}{6} \alpha_5, \\
t^{(2)} &= -\frac{1}{6} \alpha_1 + \frac{2}{3} \alpha_2 + \frac{1}{2} \alpha_3 + \frac{1}{3} \alpha_4 + \frac{1}{6} \alpha_5, \\
t^{(3)} &= -\frac{1}{6} \alpha_1 - \frac{1}{3} \alpha_2 + \frac{1}{2} \alpha_3 + \frac{1}{3} \alpha_4 + \frac{1}{6} \alpha_5, \\
t^{(4)} &= -\frac{1}{6} \alpha_1 - \frac{2}{6} \alpha_2 - \frac{3}{6} \alpha_3 + \frac{2}{6} \alpha_4 + \frac{1}{6} \alpha_5, \\
t^{(5)} &= -\frac{1}{6} \alpha_1 - \frac{2}{6} \alpha_2 - \frac{3}{6} \alpha_3 - \frac{4}{6} \alpha_4 + \frac{1}{6} \alpha_5, \\
t^{(6)} &= -\frac{1}{6} \alpha_1 - \frac{2}{6} \alpha_2 - \frac{3}{6} \alpha_3 - \frac{4}{6} \alpha_4 - \frac{5}{6} \alpha_5.
\end{align*}
\]

Second soliton (a=2).

\[
\begin{align*}
t^{(1)} &= \frac{2}{3} \alpha_1 + \frac{1}{3} \alpha_2 + \frac{2}{3} \alpha_4 + \frac{1}{3} \alpha_5, \\
t^{(2)} &= -\frac{1}{3} \alpha_1 + \frac{1}{3} \alpha_2 - \frac{1}{3} \alpha_4 + \frac{1}{3} \alpha_5, \\
t^{(3)} &= -\frac{1}{3} \alpha_1 - \frac{2}{3} \alpha_2 - \frac{1}{3} \alpha_4 - \frac{2}{3} \alpha_5. \\
t^{(4)} &= -\frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_3 + \frac{1}{2} \alpha_5, \\
t^{(5)} &= -\frac{1}{2} \alpha_1 - \frac{1}{2} \alpha_3 - \frac{1}{2} \alpha_5.
\end{align*}
\]

Third soliton (a=3).

The topological charges of the fourth and fifth solitons are obtained from those of the second and first solitons respectively, via $\alpha_j \rightarrow \alpha_{h-j}$ ($1 \leq j \leq n$).

A.2 The action of the Coxeter element.

It is shown in section 3.1.4 that the action of the Coxeter element with the ordering of (3.1.4a) generates all of the topological charges from just one. Consider now the action of the Coxeter element

\[ w = r_4 r_2 r_5 r_3 r_1 \]

on the Weyl orbit of the second fundamental weight $\lambda_2$ of the $a_5^{(1)}$ theory (this ordering is the familiar ‘black-white’ ordering of [24] i.e. the sets of simple roots $\{\alpha_1, \alpha_3, \alpha_5\}$ and $\{\alpha_2, \alpha_4\}$ corresponding to $\{r_1, r_3, r_5\}$ and $\{r_2, r_4\}$ are composed of elements which are orthogonal to each other). The Weyl orbit is partitioned into three Coxeter orbits, say $C_+$, $C_-$, and $C_0$. This is visualised in Figure 3 where
• each spot corresponds to a weight in the Weyl orbit of \( \lambda(2) \),

• if two spots are joined by a line, they are Weyl reflections of each other in the simple root \( \alpha_j \) where \( j \) is the number on the line,

• spots with the same labelling (either +, −, or 0) lie in the same Coxeter orbit.

Therefore, whereas \( t^{(1)} \) and \( t^{(3)} \) lie in the same Coxeter orbit, \( t^{(2)} \) lies in a different one.

![Diagram of Weyl orbit partition]

Figure 3 : Partition of Weyl orbit of \( \lambda(2) \).

A.3 The interaction parameter.

It will be shown that the interaction parameter is always positive. The interaction parameter is given by

\[
A = \frac{-(\sigma_1 - \sigma_2)^2 - (\sigma_1 v_1 - \sigma_2 v_2)^2 - 4m^2 \sin^2 \left( \frac{\pi}{n} (a_p - a_q) \right)}{(\sigma_1 - \sigma_2)^2 - (\sigma_1 v_1 - \sigma_2 v_2)^2 - 4m^2 \sin^2 \left( \frac{\pi}{n} (a_1 - a_2) \right)}
\]

Using the parameterisation \( v_{1,2} = \tanh \theta_{1,2} \) (so that for \(-1 < v_{1,2} < 1, -\infty < \theta_{1,2} < \infty\) where \( \theta_{1,2} \) is the rapidity of the first and second soliton respectively, and writing \( \theta = \theta_1 - \theta_2 \) the above can be rewritten

\[
A = \frac{\sin \left( \frac{\theta}{2} + \frac{\pi (a_p - a_q)}{2n} \right) \sin \left( \frac{\theta}{2} - \frac{\pi (a_p - a_q)}{2n} \right)}{\sin \left( \frac{\theta}{2} + \frac{\pi (a_p + a_q)}{2n} \right) \sin \left( \frac{\theta}{2} - \frac{\pi (a_p + a_q)}{2n} \right)}
\]

\[
= \frac{\cos \left( \frac{\pi}{n} (a_p - a_q) \right) - \cos \left( \frac{\theta}{2} \right)}{\cos \left( \frac{\pi}{n} (a_p + a_q) \right) - \cos \left( \frac{\theta}{2} \right)}
\]

\[
= \frac{\cos \left( \frac{\pi}{n} (a_p - a_q) \right) - \cosh \theta}{\cos \left( \frac{\pi}{n} (a_p + a_q) \right) - \cosh \theta}
\]

As \( \theta \) is real, \( \cosh \theta \geq 1 \), and so the interaction parameter is always positive.
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