Hierarchical Identity-Based (Lossy) Trapdoor Functions

Alex Escala¹, Javier Herranz¹, Benoît Libert², and Carla Ràfols³

¹ Universitat Politècnica de Catalunya, Dept. Matemàtica Aplicada IV (Spain)
e-mail: {alex.escala,jherranz}@ma4.upc.edu
² Université catholique de Louvain, ICTEAM Institute – Crypto Group (Belgium)
e-mail: benoit.libert@uclouvain.be
³ Ruhr-Universität Bochum, Horst Görtz Institut für IT-Sicherheit (Germany)
e-mail: Carla.Rafols@rub.de

Abstract. Lossy trapdoor functions, introduced by Peikert and Waters (STOC’08), have received a lot of attention in the last years, because of their wide range of applications in theoretical cryptography. The notion has been recently extended to the identity-based setting by Bellare et al. (Eurocrypt’12). We provide one more step in this direction, by considering the notion of hierarchical identity-based (lossy) trapdoor functions (HIB-TDFs). Hierarchical identity-based cryptography has proved very useful both for practical applications and to establish theoretical relations with other cryptographic primitives. The notion of security for IB-TDFs put forward by Bellare et al. easily extends to the hierarchical scenario, but an (H)IB-TDF secure in this sense is not known to generically imply other related primitives with security against adaptive-id adversaries, not even IND-ID-CPA secure encryption. Our first contribution is to define a new security property for (H)IB-TDFs. We show that functions satisfying this property imply secure cryptographic primitives in the adaptive identity-based setting: these include encryption schemes with semantic security under chosen-plaintext attacks, deterministic encryption schemes, and (non-adaptive) hedged encryption schemes that maintain some security when messages are encrypted using randomness of poor quality. We emphasize that some of these primitives were unrealized in the (H)IB setting previous to this work.

As a second contribution, we describe the first pairing-based HIB-TDF realization. This is also the first example of hierarchical trapdoor function based on traditional number theoretic assumptions: so far, constructions were only known under lattice assumptions. Our HIB-TDF construction is based on techniques that differ from those of Bellare et al. in that it uses a hierarchical predicate encryption scheme as a key ingredient. The resulting HIB-TDF is proved to satisfy the new security definition, against either selective or, for hierarchies of constant depth, adaptive adversaries. In either case, we only need the underlying predicate encryption system to be selectively secure.

Keywords. Lossy trapdoor functions, hierarchical identity-based encryption, partial lossiness.

1 Introduction

1.1 (Identity-Based) Lossy Trapdoor Functions

Lossy trapdoor functions, as introduced by Peikert and Waters in [32], have been proved very powerful in theoretical cryptography and received a lot of attention in the recent years (see, e.g., [21, 26, 29, 13, 27, 37]). Roughly speaking, a lossy trapdoor function is a family of functions that can be instantiated in two different modes. In the injective mode, the function is injective and can be inverted using the corresponding trapdoor. In lossy mode, the function is (highly) non-injective since its image size is much smaller than the size of the domain. The key point is that lossy instantiations of the function must be indistinguishable from injective instantiations.

In their seminal paper [32], Peikert and Waters showed that lossy trapdoor functions provide black-box constructions of chosen-ciphertext secure (IND-CCA) public-key encryption schemes as well as universal one-way and collision-resistant hash functions. Later on, other applications of lossy trapdoor functions were discovered: they gave rise to deterministic encryption schemes [5]
in the standard model [10], public-key encryption hedged schemes maintaining some security in
the absence of reliable encryption coins [6] and even public-key encryption with selective-opening
security [8] (i.e., which offer certain security guarantees in case of sender corruption).

Recently, Bellare, Kiltz, Peikert and Waters [9] introduced the notion of identity-based (lossy)
trapdoor function (IB-TDF), which is the analogue of lossy trapdoor functions in the setting of
identity-based cryptography [35]. In the identity-based scenario, users’ public keys are directly
derived from their identities, whereas secret keys are delivered by a trusted master entity. In this
way, the need for digital certificates, which usually bind public keys to users in traditional public-
key cryptography, is drastically reduced. Throughout the last decade, several generalizations of
identity-based cryptography were put forth, including hierarchical identity-based cryptography [22],
attribute-based cryptography [33, 23] or predicate-based cryptography [11, 28]. In the setting of hi-
erarchical identity-based cryptography, identities are organized in a hierarchical way, so that a user
who holds the secret key of an identity id can generate, use and distribute valid secret keys for any
identity that is a descendant of id in the hierarchy. Hierarchical identity-based encryption (HIBE)
is of great interest due to both practical and theoretical reasons. On the practical side, many or-
ganizations and systems that may need (identity-based) cryptographic solutions are organized in
a hierarchical way. On the theoretical side, generic constructions [15, 16] are known to transform a
weakly secure HIBE scheme (i.e., IND-CPA security against selective adversaries) into (public-key)
encryption schemes with strong security properties, like chosen-ciphertext security [16] or forward-
security [4, 15], where private keys are updated in such a way that past encryptions remain safe
after a private key exposure.

Bellare et al. [9] proposed instantiations of identity-based lossy trapdoor functions based on
bilinear maps and on lattices (as noted in [9], almost all IBE schemes belong to these families). The
former makes clever use of an anonymous IBE system (where the ciphertext hides the receiver’s
identity) with pseudorandom ciphertexts whereas the latter relies on lossiness properties of learning-
with-error-based cryptosystems. Moreover, they show that their definition of partial-lossiness for
identity-based trapdoor functions leads to the same cryptographic results as lossy trapdoor func-
tions, but in the selective identity-based setting only, where the attacker must choose the target
identity upfront in the attack game. Namely, in the case of selective adversaries, IB-TDFs satisfying
their definition imply identity-based encryption with semantic security, identity-based deterministic
encryption and identity-based hedged encryption. In [9], it was left as an open problem to prove
that the same results hold in the case of adaptive adversaries.

1.2 Our Contributions

This paper extends to the hierarchical setting the notion of identity-based (lossy) trapdoor function.

New Definition of Partial Lossiness. From a theoretical standpoint, we first define a new
security property for hierarchical identity-based trapdoor functions (HIB-TDFs). We show that a
HIB-TDF which satisfies this property can be used to obtain the same kind of results that are
derived from standard lossy trapdoor functions [32]. Namely, they lead to standard encryption
schemes, to deterministic encryption schemes, and to non-adaptive hedged encryption schemes,
which are secure in the hierarchical identity-based setting, in front of adaptive-id adversaries. Since
HIB-TDFs contain IB-TDFs as a particular case, our results for adaptive adversaries solve an open
problem in [9]. Interestingly, the pairing-based IB-TDF of Bellare et al. [9] can be proved to also
satisfy the new security property. As a consequence, it provides adaptively secure deterministic
and hedged IBE schemes. Recently, Xie et al. [38] designed an adaptively secure D-IBE system using
lattices. The construction of [9] is thus the first adaptively secure pairing-based realization.

**Construction of a Pairing-Based Hierarchical Trapdoor Function.** On the constructive side, we focus on pairing-based systems where, as already mentioned in [9], greater challenges are faced. Indeed, in the hierarchical scenario, anonymity—which was used as an important ingredient by Bellare et al. [9]—has been significantly harder to obtain in the world of bilinear maps than with lattices [17, 1, 2]: for example, the first pairing-based anonymous HIBE system [12] appeared four years after the first collusion-resistant HIBE [22]. Moreover, very few anonymous IBE systems seem amenable for constructing IB-TDFs, as noted in [9] where a new scheme was specially designed for this purpose.

Using bilinear maps, we thus construct a HIB-TDF and prove that it satisfies our new definition of partial lossiness under mild hardness assumptions in groups of prime order. As an intermediate step, we design a hierarchical predicate encryption (HPE) system [34, 30] with suitable anonymity properties, which may be of independent interest. Perhaps surprisingly, although this scheme is proved secure only against weak selective adversaries (who select their target attribute set before seeing the public parameters), we are able to turn it into a HIB-TDF providing security (namely, our new version of partial lossiness) against adaptive adversaries for hierarchies of constant depth. To the best of our knowledge, our HIB-TDF gives rise to the first hierarchy of trapdoor functions which does not rely on lattices: realizing such a hierarchy using traditional number theoretic techniques was identified as an open problem by Cash et al. [17].

Beyond its hierarchical nature, our construction brings out an alternative design principle for (H)IB-TDFs. The idea is to rely on hierarchical predicate encryption (HPE) to deal with hierarchies. Namely, public parameters consist of a matrix of HPE encryptions and, when the function has to be evaluated, the latter matrix is turned into a matrix of (anonymous) HIBE ciphertexts. The homomorphic properties of the underlying HIBE then make it possible to evaluate the function while guaranteeing a sufficient amount of lossiness in lossy mode.

While the pairing-based construction of Bellare et al. [9] builds on an adaptively secure anonymous IBE, our (hierarchical) IB-TDF is obtained from a selectively weakly attribute-hiding HPE system. This result is somewhat incomparable with [9]: on one hand, we start from a more powerful primitive—because predicate encryption implies anonymous IBE—but, on the other hand, we need a weaker security level to begin with. Both (H)IB-TDF constructions rely on specific algebraic properties in the underlying IBE/HPE and neither is generic. It would be interesting to see if a more general approach exists for building such functions.

### 1.3 Discussion on the Implications

Combining our HIB-TDF with the theoretical implications of our new security property, we obtain:

1. a semantically secure HIBE encryption scheme under adaptive adversaries,
2. the first secure deterministic HIBE scheme, and
3. the first HIBE scheme that (non-adaptively) hedges against bad randomness, as advocated by Bellare et al. [6].

In the case of adaptive adversaries, these results only hold for hierarchies of constant depth (said otherwise, we do not provide full security). However, using our definition of partial lossiness or that of Bellare et al. [9], this appears very difficult to avoid. The reason is that both definitions seem inherently bound to the partitioning paradigm. Namely, they assume the existence of alternative public parameters, called lossy parameters, where the identity space is partitioned into subsets of

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4 See [38] for a recent and independent construction, in the (non-hierarchical) IBE case.
injective and lossy identities. The definition of [9] intuitively captures that a fraction $\delta$ of identities are lossy in the case of lossy parameters. In the hierarchical setting, the analogy with HIBE schemes suggests that all ancestors of a lossy identity be lossy themselves. Hence, unless one can make sure that certain lossy identities only have lossy descendants, the fraction $\delta$ seems doomed to exponentially decline with the depth of the hierarchy.

Finally, due to the results of Canetti, Halevi and Katz [15], our construction also implies the first forward-secure deterministic and hedged public-key encryption schemes (note that selective security suffices to give forward-secure public-key cryptosystems). Although our scheme is not practical due to very large ciphertexts and key sizes, it provides the first feasibility results in these directions.

2 Hierarchical Identity-Based (Lossy) Trapdoor Functions, and Applications

In this section we extend to the hierarchical scenario the definitions for identity-based (lossy) trapdoor functions given in [9].

**Syntax.** A hierarchical identity-based trapdoor function (HIB-TDF) is a tuple of efficient algorithms $HF = \langle HF.Setup, HF.MKg, HF.Kg, HF.Del, HF.Eval, HF.Inv \rangle$. The setup algorithm $HF.Setup$ takes as input a security parameter $\lambda \in \mathbb{N}$ and outputs a set of global public parameters $pms$, which specifies an input space $InpSp$, an identity space $IdSp$ and the necessary mathematical objects and hash functions. The master key generation algorithm $HF.MKg$ takes as input $pms$ and outputs a master public key $mpk$ and a master secret key $msk$. The key generation algorithm $HF.Kg$ takes as input $pms$, $msk$ and a hierarchical identity $(id_1, \ldots, id_\ell) \in IdSp$, for some $\ell \geq 1$ and outputs a secret key $SK_{(id_1, \ldots, id_\ell)}$. The delegation algorithm $HF.Del$ takes as input $pms$, $msk$, a hierarchical identity $(id_1, \ldots, id_\ell)$, a secret key $SK_{(id_1, \ldots, id_\ell)}$ for it, and an additional identity $id_{\ell+1}$; the output is a secret key $SK_{(id_1, \ldots, id_{\ell}, id_{\ell+1})}$ for the hierarchical identity $(id_1, \ldots, id_{\ell}, id_{\ell+1})$. The evaluation algorithm $HF.Eval$ takes as input $pms$, $msk$, an identity $id = (id_1, \ldots, id_\ell)$ and a value $X \in InpSp$; the result of the evaluation is denoted as $C$. Finally, the inversion algorithm $HF.Inv$ takes as input $pms$, $msk$, a hierarchical identity $id = (id_1, \ldots, id_\ell)$, a secret key $SK_{id}$ for it and an evaluation $C$, and outputs a value $X \in InpSp$.

A HIB-TDF satisfies the property of correctness if

$$HF.Inv(pms, mpk, id, SK_{id}, HF.Eval(pms, mpk, id = (id_1, \ldots, id_\ell), X)) = X,$$

for any $X \in InpSp$, any $pms, (mpk, msk)$ generated by $HF.Setup$ and $HF.MKg$, any hierarchical identity $(id_1, \ldots, id_\ell) \in IdSp$ and any secret key $SK_{(id_1, \ldots, id_\ell)}$ generated either by running $HF.Kg(pms, msk, (id_1, \ldots, id_\ell))$ or by applying the delegation algorithm $HF.Del$ to secret keys of shorter hierarchical identities.

Before formalizing the new definition of partial lossiness for a HIB-TDF, let us recall the notion of lossiness: if $f$ is a function with domain $\text{Dom}(f)$ and image $\text{Im}(f) = \{ f(x) : x \in \text{Dom}(f) \}$, we say that $f$ is $\omega$-lossy if $\lambda(f) \geq \omega$, where $\lambda(f) = \log \frac{|\text{Dom}(f)|}{|\text{Im}(f)|}$.

To properly define lossiness in the IB setting, it will also be useful to consider extended HIB-TDFs, which differ from standard HIB-TDFs in that, in the latter, the algorithm $HF.Setup$ specifies in $pms$ an auxiliary input space $AuxSp$, and $HF.MKg$ takes as additional auxiliary input $aux \in AuxSp$. Also, given some HIB-TDF $HF = \langle HF.Setup, HF.MKg, HF.Kg, HF.Del, HF.Eval, HF.Inv \rangle$, a sibling for $HF$ is an extended HIB-TDF $LHF = \langle HF.Setup, LH.FMKg, LH.Kg, HF.Del, HF.Eval, HF.Inv \rangle$ whose delegation, evaluation and inversion algorithms are those of $HF$, and where an auxiliary space
AuxSp is contained in \( \text{pms} \leftarrow \text{HF.Setup}(\varrho) \), so that \( \text{IdSp} \subset \text{AuxSp} \).

Looking ahead, we will define, as in [9], two different experiments: one corresponding to the standard setup and one corresponding to the lossy setup, in one of them the experiment will interact with a standard HIB-TDF, in the other one with a sibling in which some identities lead to lossy evaluation functions. The notion of extended HIB-TDF will serve to construct both of these functions as an extended HIB-TDF but with different auxiliary inputs \( g^{(0)}, g^{(1)} \).

2.1 A New Security Definition for HIB-TDFs

The basic security property of a trapdoor function is one-wayness, which means that the function is hard to invert without the suitable secret key. In the identity-based setting, one-wayness is required to hold even when the adversary has oracle access to the secret keys for some identities. Partial lossiness for identity-based trapdoor functions was introduced in [9], where it was proved to imply one-wayness. Roughly speaking, partial lossiness requires that the weighted difference of the probability that any adversary outputs 1 in the lossy or in the real setup is negligible. For the selective case, the weights can simply be set to 1 and it can be proved that an IB-TDF satisfying their notion of partial lossiness in the selective scenario can be used to build: (1) identity-based encryption (IBE) schemes with selective IND-CPA security, (2) selectively secure deterministic IBE schemes, (3) selectively secure hedged IBE schemes. However, these results are not known to be true in the adaptive setting, in particular, the definition is not even known to yield an IND-ID-CPA secure encryption scheme.

To address this question, we propose an alternative definition for the partial lossiness of (hierarchical) identity-based trapdoor functions — in particular, they also result in a new definition when the hierarchy depth is equal to 1, the case considered by Bellare et al. [9]. We will show that a HIB-TDF satisfying this new definition gives a secure construction of the same primitives we mentioned for the selective case.

As in [9], we define two different experiments, a lossy experiment and a real experiment. For any adversary \( \mathcal{A} \) against a HIB-TDF, and any \( \zeta \in (0, 1) \), the \( \text{REAL}_{\text{HF},LHF,\mathcal{P},\omega,\zeta} \) experiment and the \( \text{LOSSY}_{\text{HF},LHF,\mathcal{P},\omega,\zeta} \) experiment are parameterized by the security parameter \( \varrho \) (which we will usually omit in the notation) and values \( \zeta(\varrho), \omega(\varrho) \). The value \( \zeta \) will be important to show that our construction implies other cryptographic primitives. Intuitively, \( \zeta \) will be the advantage of an adversary against a cryptographic scheme the security of which is reduced to the security of the HIB-TDF. The experiment also takes as input the specification of some efficient algorithm \( \mathcal{P} \) which takes as input \( \zeta, \text{pms}, \text{mpk}_1, \text{msk}_1, IS, \text{id}^* \), and outputs a bit \( d_2 \). This procedure \( \mathcal{P} \) can be, in general, any probabilistic and polynomial-time algorithm. In the security analysis of the selectively secure version of our HIB-TDF, \( \mathcal{P} \) will be the trivial algorithm that always outputs \( d_2 = 1 \). In other cases, \( \mathcal{P} \) could be a more complicated algorithm; for instance, in order to prove the security of the adaptive-id version of our HIB-TDF, we will take as pre-output stage \( \mathcal{P} \) Waters’ artificial abort step. Actually, procedure \( \mathcal{P} \) is a way to relax the security requirements in Denition 1, in order to allow the possibility of building cryptographic schemes from secure HIB-TDFs, in a black-box way.

Finally, the value \( \omega \) is related to the lossiness of the trapdoor function. To simplify notation, we will simply write \( \text{REAL} \) instead of \( \text{REAL}_{\text{HF},LHF,\mathcal{P},\omega,\zeta} \) and \( \text{LOSSY} \) instead of \( \text{LOSSY}_{\text{HF},LHF,\mathcal{P},\omega,\zeta} \).

For compactness, we present the two experiments as a single experiment depending of a bit \( \beta \): the challenger \( \mathcal{C} \), who interacts with the adversary \( \mathcal{A} \), runs either \( \text{REAL} \) if \( \beta = 0 \) or \( \text{LOSSY} \) if \( \beta = 1 \). Also, some instructions of both experiments depend on whether selective or adaptive security is being considered. We say that a hierarchical identity \( \text{id} = (\text{id}_1, \ldots, \text{id}_\ell) \) is a prefix of another one \( \text{id}^* = (\text{id}_1^*, \ldots, \text{id}_\ell^*) \) if \( \ell \leq \ell^* \) and \( \text{id}_i = \text{id}_i^* \) for every \( i = 1, \ldots, \ell \). We denote it by \( \text{id} \leq \text{id}^* \).
0. \( C \) chooses global parameters \( \text{pms} \) by running \( \text{HF.Setup} \). The parameters \( \text{pms} \) are given to \( A \), who replies by choosing a hierarchical identity \( \text{id}^0 = (\text{id}_1^1, \ldots, \text{id}_{\ell}^1) \), for some \( \ell \leq d \).

1. \( C \) runs \((\text{mpk}_0, \text{msk}_0) \leftarrow \text{HF.MKg}(\text{pms})\) and \((\text{mpk}_1, \text{msk}_1) \leftarrow \text{LHF.MKg}(\text{pms}, \text{aux} = \text{id}^1)\). The adversary \( A \) receives \( \text{mpk}_\beta \) and lists \( \text{IS} \leftarrow \emptyset \), \( \text{QS} \leftarrow \emptyset \) are initialized.

2. \( A \) can make adaptive queries for hierarchical identities \( \text{id} = (\text{id}_1, \ldots, \text{id}_\ell) \) and identities \( \text{id}_{\ell+1} \).

   - **Create-key**: \( A \) provides \( \text{id} \) and \( C \) creates a private key \( SK_{\text{id}} \). If \( \beta = 0 \), \( SK_{\text{id}} \) is created by running \( \text{HF.Kg}(\text{pms}, \text{msk}_0, \text{id}) \). If \( \beta = 1 \), it is created by running \( \text{LHF.Kg}(\text{pms}, \text{msk}_1, \text{id}) \). The list \( QS \) is updated as \( QS = QS \cup \{ \text{id} \} \).

   - **Create-delegated-key**: \( A \) provides \( \text{id} = (\text{id}_1, \ldots, \text{id}_\ell) \) and \( \text{id}_{\ell+1} \) such that \( \text{id} \in QS \). The challenger \( C \) then computes \( SK_{\text{id}'} \) for \( \text{id}' = (\text{id}_1, \ldots, \text{id}_{\ell+1}) \) by running the delegation algorithm \( \text{HF.Del}(\text{pms}, \text{mpk}_\beta, SK_{\text{id}}, \text{id}_{\ell+1}) \). The list \( QS \) is updated as \( QS = QS \cup \{ \text{id}' \} \).

   - **Reveal-key**: \( A \) provides \( \text{id} \) with the restriction that if \( A \) is selective, then \( \text{id} \not\in \text{id}^1 \). \( C \) returns \( \bot \) if \( \text{id} \not\in QS \). Otherwise, \( SK_{\text{id}} \) is returned to \( A \) and the list \( IS \) is updated as \( IS = IS \cup \{ \text{id} \} \).

3. The adversary \( A \) outputs a hierarchical identity \( \text{id}^* = (\text{id}_1^*, \ldots, \text{id}_\ell^*) \) and a bit \( d_A \in \{0, 1\} \). If \( A \) is selective, then \( \text{id}^* = \text{id}^1 \). In the adaptive case, no element of \( IS \) can be a prefix of \( \text{id}^* \). Let \( d_1 \) be the bit \( d_1 := (\forall \text{id} \in IS, (\lambda (\text{HF.Eval}(\text{pms}, \text{mpk}_1, \text{id}, \cdot)) = 0) \land (\lambda (\text{HF.Eval}(\text{pms}, \text{mpk}_1, \text{id}^*, \cdot)) \geq \omega) \).

4. \( C \) sets \( d_2 \) to be the output of the pre-output stage \( P \) with input \( \zeta, \text{pms}, \text{mpk}_1, \text{msk}_1, IS, \text{id}^* \).

5. The final output of the experiment consists of \( \{ d_A, d_{\text{exp}} \} \), where \( d_{\text{exp}} = d_1 \land d_2 \in \{0, 1\} \).

For notational convenience, from now on, let us define \( d_{\text{exp}}^A = d_A \land d_{\text{exp}} \).

Note that, in the lossy experiment, some identities may lead to lossy functions, which can be detected by \( A \) if it queries the secret key for such an identity. This causes an asymmetry when comparing the real and the lossy experiments. For this reason, Bellare et al. defined the advantage of a distinguisher among the lossy and real experiments as the weighted difference of the probability of outputting 1 in the real case minus the same probability in the lossy case. Our solution is different: we force the experiment to adopt the same behavior in the real and lossy settings: if a query would force the LOSSY experiment to set the bits \( d_1 \) or \( d_2 \) to 0 (and, as a consequence, setting \( d_{\text{exp}} = 0 \)), then it also forces the REAL experiment to set \( d_1 \) or \( d_2 \) to 0, respectively. Therefore, the difference of probabilities in our definition (see condition (i) below) is not weighted.

**Definition 1.** A \( \text{HIB-TDF} \) is \((\omega, \delta)\)-partially lossy if it admits a sibling and an efficient pre-output stage \( P \) such that for all PPT adversaries \( A \) and for all non-negligible \( \zeta \), there exist two non-negligible values \( \epsilon_1, \epsilon_2 \) with \( \delta = \epsilon_1 \epsilon_2 \) such that the following three conditions hold:

(i) the following advantage function is negligible in the security parameter \( q \):

\[
\text{Adv}_{\text{HF,LHF},P,\omega,\zeta}^{\text{lossy}}(A) = \left| \Pr[d_{\text{exp}}^A = 1 \mid \text{REAL}] - \Pr[d_{\text{exp}}^A = 1 \mid \text{LOSSY}] \right| \]

(ii) \( \Pr[d_{\text{exp}}^A = 1 \mid \text{REAL}] \geq \epsilon_1 \).

(iii) if \( A \) is such that \( \Pr[d_A = 1 \mid \text{REAL}] - \frac{1}{2} > \zeta \), then

\[
\Pr[d_A = 1 \mid \text{REAL} \land d_{\text{exp}}^A = 1] - \frac{1}{2} > \epsilon_2 \cdot \zeta,
\]

where \( \delta \) may be a function of \( q \) the maximal number of secret key queries of \( A \).
As said above, condition (i) is a simple modification of the definition of partial lossiness of Bellare et al. [9]. We add condition (ii) to rule out some cases in which the definition would be trivial to satisfy, like the case where the procedure \( \mathcal{P} \) aborts with overwhelming probability or the case where the sibling admits only lossy identities: in any of these scenarios, we would have \( d_{\text{abort}} = 1 \) with negligible probability, which would render the scheme useless with the sole condition (i). Then, we add a third condition (iii) which allows reducing \( \text{HIB-TDFs} \) to other primitives. Roughly speaking, this condition guarantees that the probability of aborting is somewhat independent of the behavior of any computationally bounded adversary. Interestingly, it is possible to prove (by proceeding exactly as in the proof of our \( \text{HIB-TDF} \)) that the pairing-based IB-TDF described by Bellare et al. [9] satisfies our new partial-lossiness definition.

2.2 Implications of Lossy (H)IB-TDFs: the Example of (H)IBE

Using the same argument as in [9], it is quite easy to prove that a \( \text{HIB-TDF} \) which enjoys the new version of the partial lossiness property is already one-way, in both the selective and adaptive settings. In this section we prove that a \( \text{HIB-TDF} \) which satisfies our new security definition can be used to build other primitives in the hierarchical identity-based security, with security against adaptive adversaries. We detail the example of hierarchical identity-based encryption (HIBE) with IND-CPA security\(^5\). The construction is the direct adaptation of the Peikert-Waters construction [32] in the public-key setting.

Let \( HF \) be a \( \text{HIB-TDF} \) with message space \( \{0,1\}^n \) and lossiness \( \omega \), and \( H \) a family of pairwise independent hash functions from \( \{0,1\}^n \) to \( \{0,1\}^l \) where \( l \leq \omega - 2\lg(1/\epsilon_{\text{LHL}}) \) for some negligible \( \epsilon_{\text{LHL}} \). The HIBE scheme has message space \( \{0,1\}^l \). Its setup, key generation and key delegation algorithms are basically the same ones as those for \( HF \), the rest are as follows:

| MKGen(pms) | Enc(pms, mpk, m, id) | Dec(pms, mpk, SK_{id}, C, id) |
|------------|---------------------|-------------------------------|
| (mpk', msk) \leftarrow HF.MKg(1^k) | \( x \leftarrow \{0,1\}^n \) | \( x = HF.inv(pms, mpk, SK_{id}, c_1, id) \) |
| \( h \leftarrow H \) | \( c_1 = HF.eval(pms, mpk, id, x) \) | \( m = c_2 \oplus h(x) \) |
| mpk = (mpk', h) | \( c_2 = h(x) \oplus m \) | Return \( C = (c_1, c_2) \) |
| Return mpk | Return m | Return C |

We prove the following theorem.

**Theorem 1.** If \( HF \) is \((\omega, \delta)\)-partially lossy for some non-negligible value of \( \delta \), then the HIBE scheme \( \Pi \) described is IND-ID-CPA secure. In particular, for every IND-ID-CPA adversary \( B \) against \( \Pi \) there exists a PPT adversary \( A \) against \( HF \) such that

\[
\text{Adv}_{HF,LHF,\mathcal{P},\omega,\delta}(A) \geq \frac{2}{3} \cdot \delta \cdot \text{Adv}_{\text{ind-CPA}}(B) - \nu(\varrho)
\]

for some negligible function \( \nu; \) both adversaries \( A \) and \( B \) run in comparable times.

**Proof.** Let us assume that an adversary \( B \) has advantage at least \( \zeta \) in breaking the IND-ID-CPA security of the HIBE scheme \( \Pi \), for some non-negligible \( \zeta \). We build an adversary \( A \) that breaks the condition (i) of Definition 1 assuming that conditions (ii) and (iii) are satisfied. Our adversary \( A \), who interacts with a challenger that runs either the experiment REAL or the experiment LOSSY, proceeds to simulate the challenger in the IND-ID-CPA game with \( B \) as follows.

\(^5\) The cases of deterministic HIBE and hedged HIBE are discussed in Appendices C and D.
\( \mathcal{A} \) forwards an identity \( \text{id}^† \) to its challenger, which is some random identity in the adaptive case or corresponds to the challenge identity chosen by \( \mathcal{B} \) in the selective case. When the challenger runs the setup and gives the output to \( \mathcal{A} \), \( \mathcal{A} \) forwards this information to \( \mathcal{B} \) together with a hash function \( h \leftarrow \mathcal{H} \). When \( \mathcal{B} \) asks for a secret key for a hierarchical identity \( \text{id} \), \( \mathcal{A} \) forwards the query to the experiment and forwards the reply to \( \mathcal{B} \). At some point, \( \mathcal{B} \) outputs \( (m_0, m_1, \text{id}^*) \), with \( \text{id}^* = \text{id}^α \) in the selective case. Adversary \( \mathcal{A} \) then forwards \( \text{id}^* \) to its challenger, chooses \( \gamma \leftarrow \{0, 1\} \) at random and encrypts \( m_\gamma \) under the identity \( \text{id}^* \). After some more secret key queries, \( \mathcal{B} \) outputs a guess \( \gamma' \) and \( \mathcal{A} \) outputs \( d_A = 1 \) if \( \gamma = \gamma' \) and \( d_A = 0 \) otherwise.

In the \( \text{REAL} \) setting, we will have
\[
\Pr[\gamma' = \gamma | \text{REAL}] - \frac{1}{2} = \Pr[d_A = 1 | \text{REAL}] - \frac{1}{2} \geq \zeta,
\]
since \( \mathcal{A} \) perfectly simulated the IND-ID-CPA game with \( \mathcal{B} \). This inequality can be combined with conditions (ii) and (iii) of the definition of \((\omega, \delta)\)-partial lossiness (which we assume to be satisfied by \( \text{HF} \)), and we obtain
\[
\Pr[d_A = 1 | \text{REAL}] \cdot \left( \Pr[d_A = 1 | \text{REAL} \land d_A = 1] - \frac{1}{2} \right) > \epsilon_1 \epsilon_2 \zeta. \tag{3}
\]

On the other hand, as proved in [9], in the \( \text{LOSSY} \) setting when \( \text{id}^* \) is lossy, the advantage of \( \mathcal{B} \) in guessing \( \gamma \) is negligible. Indeed, since we are using a pairwise independent hash function, the Leftover Hash Lemma [24] (more precisely, its variant proved in [18]) implies that the distribution of guessing so that there exists \( \tilde{\nu} \in \text{negl}(\lambda) \) such that
\[
\text{Adv}_{\text{HF}, \text{LHF}, \rho, \omega, \zeta}(\mathcal{A}) \geq \frac{2}{3} \cdot \left| \Pr[d_A = 1 | \text{REAL}] \cdot \left( \Pr[d_A = 1 | \text{REAL} \land d_A = 1] - \frac{1}{2} \right) \right| - \tilde{\nu}.
\]

\( \text{Adv}_{\text{HF}, \text{LHF}, \rho, \omega, \zeta}(\mathcal{A}) \) is statistically close to the uniform distribution. We thus have \( \Pr[d_A = 1 | \text{LOSSY} \land d_A = 1] \leq \frac{1}{2} + \epsilon_{\text{LHL}} \), for some negligible function \( \epsilon_{\text{LHL}} \). Since \( d_{\text{exp}} = d_{a\text{abort}} \land d_A \), we find
\[
\Pr[d_{\text{exp}} = 1 | \text{LOSSY}] = \Pr[d_A = 1 | \text{LOSSY} \land d_{a\text{abort}} = 1] \Pr[d_{a\text{abort}} = 1 | \text{LOSSY}]
\leq \left( \frac{1}{2} + \epsilon_{\text{LHL}} \right) \cdot \Pr[d_{a\text{abort}} = 1 | \text{LOSSY}]
\leq \frac{1}{2} \cdot \left( \Pr[d_{a\text{abort}} = 1 | \text{REAL}] + \text{Adv}_{\text{HF}, \text{LHF}, \rho, \omega, \zeta}(\mathcal{A}) \right) + \nu, \tag{4}
\]
for some negligible function \( \nu \in \text{negl}(\lambda) \). The last equality follows from the fact that we can assume that \( \Pr[d_{a\text{abort}} = 1 | \text{LOSSY}] - \Pr[d_{a\text{abort}} = 1 | \text{REAL}] \leq \text{Adv}_{\text{HF}, \text{LHF}, \rho, \omega, \zeta}(\mathcal{A}) \); otherwise, we can easily build a distinguisher\(^6\) against condition (i) of the partial lossiness definition. If we plug (4) into the definition of \( \text{Adv}_{\text{HF}, \text{LHF}, \rho, \omega, \zeta}(\mathcal{A}) \), we obtain
\[
\text{Adv}_{\text{HF}, \text{LHF}, \rho, \omega, \zeta}(\mathcal{A}) = \left| \Pr[d_{\text{exp}} = 1 | \text{REAL}] - \Pr[d_{\text{exp}} = 1 | \text{LOSSY}] \right|
\geq \left| \Pr[d_{a\text{abort}} = 1 | \text{REAL}] \cdot \left( \Pr[d_A = 1 | \text{REAL} \land d_A = 1] - \frac{1}{2} \right) \right|
\geq \frac{1}{2} \cdot \text{Adv}_{\text{HF}, \text{LHF}, \rho, \omega, \zeta}(\mathcal{A}) - \nu, \tag{5}
\]
so that there exists \( \tilde{\nu} \in \text{negl}(\lambda) \) such that
\[
\text{Adv}_{\text{HF}, \text{LHF}, \rho, \omega, \zeta}(\mathcal{A}) \geq \frac{2}{3} \cdot \left| \Pr[d_{a\text{abort}} = 1 | \text{REAL}] \cdot \left( \Pr[d_A = 1 | \text{REAL} \land d_A = 1] - \frac{1}{2} \right) \right| - \tilde{\nu}.
\]

\(^6\) This distinguisher \( \mathcal{A}_1 \) is obtained from \( \mathcal{A} \) by ignoring \( d_A \in \{0, 1\} \) and replacing it by a 1, so that \( d_{a\text{abort}} = d_{\text{exp}} \).
This means that \( \text{Adv}^{\text{lossy}}_{HF, LHF, P, \omega, \zeta}(A) \) is non-negligible since \( \delta = \epsilon_1 \epsilon_2 \) and the right-hand-side member of the above expression is at least \( (2/3) \cdot \delta \cdot \zeta - \nu \). In other words, any adversary guessing \( \gamma' = \gamma \) with non-negligible advantage \( \zeta \) in REAL necessarily contradicts condition (i).

\[ \square \]

3 Interlude: Hierarchical Predicate Encryption

In this section we propose a new hierarchical predicate encryption scheme with the attribute-hiding property, which will be used as an ingredient to build, in the next section, a hierarchical identity-based (lossy) trapdoor function. The syntax and security model for hierarchical predicate encryption schemes are recalled in Appendix A.1.

3.1 Some Complexity Assumptions

We consider groups \((G, \hat{G}, G_T)\) of prime order \(p\) for which an asymmetric bilinear map \(e : G \times \hat{G} \rightarrow G_T\) is efficiently computable. We will assume that the DDH assumption holds in both \(G\) and \(\hat{G}\), which implies that no isomorphism is efficiently computable between \(G\) and \(\hat{G}\). The assumptions that we need are sometimes somewhat stronger than DDH. However, they have \(constant\ size\) (i.e., we do not rely on \(q\)-type assumptions) and were previously used in [20].

**The Bilinear Diffie Hellman Assumption (BDH):** in bilinear groups \((G, \hat{G}, G_T)\) of prime order \(p\), the distribution \(D_1 = \{(g, g^a, g^c, \hat{g}, \hat{g}^a, \hat{g}^b, e(g, \hat{g})^{abc}) | a, b, c \in \mathbb{Z}_p^*\}\) is computationally indistinguishable from \(D_2 = \{(g, g^a, g^c, \hat{g}, \hat{g}^a, \hat{g}^b, e(g, \hat{g})^{c} | a, b, c, z \in \mathbb{Z}_p^*\}\).

**The \(\mathcal{P}\)-BDH Assumption:** in asymmetric bilinear groups \((G, \hat{G}, G_T)\) of prime order \(p\), the distribution \(D_1 = \{(g, g^a, g^b, \hat{g}, g^a, g^b, g^{abc}) | a, b, c \in \mathbb{Z}_p^*\}\) is computationally indistinguishable from \(D_2 = \{(g, g^a, g^b, g^{ab}, g^b, g^{ab}, g^{abc}) | a, b, c \in \mathbb{Z}_p^*\}\).

**The DDH2 Assumption:** in asymmetric bilinear groups \((G, \hat{G}, G_T)\) of prime order \(p\), the distribution \(D_1 = \{(g, \hat{g}, g^a, g^b, \hat{g}^{ab}) | a, b, \hat{g} \in \mathbb{Z}_p^*\}\) is computationally indistinguishable from the distribution \(D_2 = \{(g, \hat{g}, g^a, g^b, \hat{g}^{ab}) | a, b, \hat{g} \in \mathbb{Z}_p^*\}\).

3.2 A Selectively Secure Weakly Attribute-Hiding Hierarchical Predicate Encryption Scheme

The construction is inspired by the Shi-Waters delegatable predicate encryption scheme [34] (at a high-level, it also bears similarities with the lattice-based scheme of [3]). However, we have to turn it into a predicate encryption scheme for inner product relations (like the one of Okamoto and Takashima [30]) instead of a hidden vector encryption [11]. Another difficulty to solve is that we cannot use composite order groups as in [34] because, in our HIB-TDF of Section 4, one of the subgroups would eventually leak information on the input in lossy mode (this is actually what happened with our initial attempt). For this reason, we chose to work with prime-order groups and used asymmetric pairing configurations to anonymize ciphertexts. As a benefit, we obtain a better efficiency than by using the techniques of [11] by reducing the number of pairing evaluations.

**Setup\((g, d, \mu)\):** given a security parameter \(g \in \mathbb{N}\), the \(constant\) desired number of levels in the hierarchy \(d \in \mathbb{N}\) and the desired length \(\mu\) of the attribute vectors at each level, choose asymmetric bilinear groups \((G, \hat{G}, G_T)\) of order \(p\), where \(p > 2^\ell\). Choose \(g \in \mathbb{G}\), \(\hat{g} \in \hat{G}\). Then, pick \(\alpha, \alpha_v, \alpha_w \in \mathbb{Z}_p^*\) and set \(v = g^{\alpha_v}, \hat{v} = g^{\alpha_v}, w = g^{\alpha_w}\) and \(\hat{w} = g^{\alpha_w}\). For \(i_1 = 1, \ldots, d\) and
For all $i_1 = 1, \ldots, \mu$, pick $\alpha_{i_1,i_2} \overset{R}{\leftarrow} \mathbb{Z}_p^*$ and compute $h_{i_1,i_2} = g^{\alpha_{i_1,i_2}} \in \mathbb{G}$ and $\hat{h}_{i_1,i_2} = \hat{g}^{\alpha_{i_1,i_2}} \in \mathbb{G}$. The master public key is defined to be

$$\text{mpk} := (v, w, e(g, \hat{v})^a, \{h_{i_1,i_2}\}_{i_1 \in \{1,\ldots,d\}, i_2 \in \{0,\ldots,\mu\}})$$

while the master secret key is $\text{msk} := (\hat{g}, \hat{v}, \hat{w}, \{\hat{h}_{i_1,i_2}\}_{i_1 \in \{1,\ldots,d\}, i_2 \in \{0,\ldots,\mu\}})$.

**Keygen**($\text{msk}, (\hat{X}_1, \ldots, \hat{X}_\ell)$): to generate a private key for vectors $(\hat{X}_1, \ldots, \hat{X}_\ell)$ with $\ell \leq d$, parse $\text{msk}$ as $(\hat{g}, \hat{v}, \hat{w}, \{\hat{h}_{i_1,i_2}\} _{i_1 \in \{1,\ldots,d\}, i_2 \in \{0,\ldots,\mu\}})$. For $i_1 = 1, \ldots, \mu$, parse $\hat{X}_{i_1}$ as $(x_{i_1,1}, \ldots, x_{i_1,\mu}) \in \mathbb{Z}_p^\mu$.

Let $r_{w} \overset{R}{\leftarrow} \mathbb{Z}_p^*$ and $r_{1,\ldots,\ell} \overset{R}{\leftarrow} \mathbb{Z}_p^*$, for $i_1 \in \{1,\ldots,\ell\}$. Then, compute the decryption component $SK_D = (D, D_w, \{D_{i_1}\}_{i_1 = 1}^\ell)$ of the key as

$$D = \hat{g}^a \cdot \prod_{i_1=1}^\ell \prod_{i_2=1}^\mu \hat{h}_{i_1,i_2}^{\ell x_{i_2+1,\ell}} \cdot \hat{w}^{r_w}, \quad D_w = \hat{w}^{r_w}, \quad D_{i_1} = \hat{v}^{r_{1,i_1}}.$$ 

To define the elements of its delegation component $SK_{DL}$

$$\{(K_{j,k}, L_j, L_{j,k,i_1}, D_{w,j,k})\}_{j \in \{\ell+1,\ldots,d\}, k \in \{1,\ldots,\mu\}, i_1 \in \{1,\ldots,\ell\}}$$

pick $s_j \overset{R}{\leftarrow} \mathbb{Z}_p^*$, $s_{j,k,i_1} \overset{R}{\leftarrow} \mathbb{Z}_p^*$, $s_{w,j,k} \overset{R}{\leftarrow} \mathbb{Z}_p^*$ for $i_1 \in \{1,\ldots,\ell\}$, $i_2 \in \{1,\ldots,\mu\}$ and set:

$$K_{j,k} = \prod_{i_1=1}^\ell \prod_{i_2=1}^\mu \hat{h}_{i_1,i_2}^{x_{i_2+1,\ell}} \cdot \hat{h}_{j,k}^{s_{j,k,i_1}} \cdot \hat{w}^{s_{w,j,k}}, \quad L_j = \hat{v}^{s_j}, \quad L_{j,k,i_1} = \hat{v}^{s_{j,k,i_1}}, \quad L_{w,j,k} = \hat{v}^{s_{w,j,k}}.$$ 

Output the private key $SK(\hat{X}_1, \ldots, \hat{X}_\ell) = (SK_D, SK_{DL})$.

**Delegate**($\text{mpk}, (\hat{X}_1, \ldots, \hat{X}_\ell), SK(\hat{X}_1, \ldots, \hat{X}_\ell)_{\ell+1}$): parse the key $SK(\hat{X}_1, \ldots, \hat{X}_\ell)$ as $(SK_D, SK_{DL})$. Given, $\hat{X}_{\ell+1} = (x_{\ell+1,1}, \ldots, x_{\ell+1,\mu}) \in \mathbb{Z}_p^\mu$, do the following.

1. Randomize $SK_{DL}$ by raising all its component to some $z \overset{R}{\leftarrow} \mathbb{Z}_p^*$. Call this new key $\hat{SK}_{DL}$ and write its elements with a hat (e.g., $\hat{K}_{j,k} = K_{j,k}^z$).

2. Compute a partial decryption key

$$K_{\ell+1} = \prod_{k=1}^{\mu} \hat{K}_{\ell+1,k} = \prod_{i_1=1}^{\ell} \prod_{i_2=1}^{\mu} \hat{h}_{i_1,i_2}^{\ell x_{i_2+1,\ell}} \cdot \left( \prod_{k=1}^{\mu} \hat{h}_{\ell+1,k}^{x_{\ell+1,k}} \right)^{s_{\ell+1,i_1}} \cdot \hat{w}^{s_{w,\ell+1}}, \quad L_{\ell+1,i_1} = \hat{L}_{\ell+1,i_1} = \hat{L}_{\ell+1,i_1}^z,$$

where we define the exponents $s_{\ell+1,i_1} = z \cdot \sum_{k=1}^{\mu} s_{\ell+1,k,i_1} \cdot x_{\ell+1,k}$ for $i_1 \in \{1,\ldots,\ell\}$, and $s_{w,\ell+1} = z \cdot \sum_{k=1}^{\mu} s_{w,\ell+1,k} \cdot x_{\ell+1,k}$.

3. For all $j \in \{\ell + 2, \ldots, d\}$, $k \in \{1,\ldots,\mu\}$, compute re-randomized versions of the partial decryption key by raising the partial decryption key to a random power $r_{j,k} \overset{R}{\leftarrow} \mathbb{Z}_p^*$.

$$K_{\ell+1}^{(j,k)} = K_{\ell+1}^{r_{j,k}}, \quad L_{\ell+1,i_1}^{(j,k)} = L_{\ell+1,i_1}^{r_{j,k}}, \quad \{L_{\ell+1,i_1}^{(j,k)} = L_{\ell+1,i_1}^{r_{j,k}}\}_{i_1 = 1}^\ell.$$

These values will be used to compute the delegation component of the new key at step 5.
4. Compute a decryption component $SK'_D = (D', D'_w, \{D'_{i_1}\}_{i_1=1}^{\ell+1})$ for the delegated key by setting $D' = D \cdot K_{\ell+1}$, $D'_w = D_w \cdot L\omega, \ell+1$. Then, define $D'_{\ell+1} = L_{\ell+1, \ell+1}$ and, for each $i_1 \in \{1, \ldots, \ell\}$, set $D'_i = D_{i_1} \cdot L_{\ell+1,i_1}$.

5. Compute a delegation component for the delegated key. For each $j \in \{\ell + 2, \ldots, d\}$, set $L'_j = \hat{L}_j$. Then, for $k = 1$ to $\mu$ and $i_1 = 1$ to $\ell + 1$, set

$$K'_{j,k} = \hat{K}_{j,k} \cdot K_{\ell+1}, \quad L'_{w,j,k} = \hat{L}_{w,j,k} \cdot L_{\omega, \ell+1}, \quad L'_{j,k,i_1} = \hat{L}_{j,k,i_1} \cdot L_{\ell+1,i_1}^{(j,k)},$$

where $\hat{L}_{j,k,\ell+1} = 1$ for all $j, k$. The new delegation component $SK'_D(L)$ is

$$\{\{K'_{j,k}, L'_{j,k,i_1}, L'_{w,j,k}\}_{j \in \{\ell + 2, \ldots, d\}}, k \in \{1, \ldots, \mu\}, i_1 \in \{1, \ldots, \ell\}\}$$

Return the delegated private key $SK'_{D} = (SK'_D, SK'_D(L))$.

**Encrypt**($mpk, (\bar{Y}_1, \ldots, \bar{Y}_\kappa), M$): given $mpk$, a plaintext $M \in \mathbb{G}_T$ as well as a hierarchy of vectors $Y_1 = (y_{1,1}, \ldots, y_{1,\mu}), \ldots, Y_\kappa = (y_{\kappa,1}, \ldots, y_{\kappa,\mu})$, choose $s \in \mathbb{Z}_p^*$ and compute

$$C_0 = M \cdot e(g, \bar{v})^{\alpha s}, \quad C_v = v^s, \quad C_w = w^s, \quad \{C_{i_1,i_2} = (h_{i_1,0}^{y_{1,i_2}} \cdot h_{i_1,i_2})^s\}_{i_1 \in \{1, \ldots, \kappa\}, i_2 \in \{1, \ldots, \mu\}}.$$  

The ciphertext is $C = (C_0, C_v, C_w, \{C_{i_1,i_2}\}_{i_1 \in \{1, \ldots, \kappa\}, i_2 \in \{1, \ldots, \mu\}})$.

**Decrypt**($mpk, (\bar{X}_1, \ldots, \bar{X}_\ell), SK_{(X_1, \ldots, X_\ell)}), C$): parse the private key $SK_{(X_1, \ldots, X_\ell)}$ as $(SK_D, SK_{DL})$, where $SK_D = (D, D_w, \{D_{i_1}\}_{i_1=1}^{\ell+1})$ and the ciphertext $C$ as $(C_0, C_v, C_w, \{C_{i_1,i_2}\}_{i_1 \in \{1, \ldots, \kappa\}, i_2 \in \{1, \ldots, \mu\}})$.

1. For each $i_1 \in \{1, \ldots, \ell\}$, compute $C_{i_1} = \prod_{i_2=1}^{\mu} C_{i_1,i_2}^{x_{1,i_2} y_{1,i_2}} = (h_{i_1,0}^{\bar{X}_{i_1}} \cdot \prod_{i_1=1}^{\mu} h_{i_1,i_2})^s$.
2. Return $M$ if $M = C_0 \cdot e(C_v, D)^{-1} \cdot e(C_w, D_w) \cdot \prod_{i_1=1}^{\ell} e(C_{i_1}, D_{i_1})$ is in the appropriate subspace$^7$ of $\mathbb{G}_T$. Otherwise, return $\bot$.

We show that our scheme is correct in Appendix A.2. Our HIB-TDF uses the predicate-only variant of the above scheme, obtained by discarding the ciphertext component $C_0$ (which contains the payload) and the factor $\bar{g}^a$ from the private key component $D$.

We remark that the number of pairing evaluations only depends on the depth $\ell$ of the predicate $(\bar{X}_1, \ldots, \bar{X}_\ell)$ encoded in the private key and not on the dimension $\mu$ of vectors at each level. Except a recent fully secure construction [31], all previous schemes required $O(\ell \cdot \mu)$ pairing evaluations to decrypt. In our HIB-TDF of Section 4, this will make it possible to invert a function admitting $n$-bit inputs by computing $O(\ell \cdot n)$ pairings, instead of $O(\ell \cdot n \cdot \mu)$.

The new HPE scheme is selectively weakly attribute-hiding under the BDH, $\mathcal{P}$-BDH$_1$ and DDH$_2$ assumptions, as established by Theorem 2. The security of its predicate-only variant (which is the one used as a key ingredient in the design of our HIB-TDF) relies only on the latter two assumptions.

**Theorem 2.** The HPE scheme is selectively weakly attribute-hiding (in the sense of Definition 2 in Appendix A.1) if the BDH, $\mathcal{P}$-BDH$_1$ and DDH$_2$ assumptions hold in $(\mathbb{G}, \mathbb{G}_T)$. (The proof is given in Appendix A.3).

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$^7$ As in [11,28], the plaintext space is restricted to have a size much smaller than $|\mathbb{G}_T|$ to make sure that the decryption algorithm returns $\bot$ if an unauthorized key is used to decrypt.
4 A Hierarchical Identity-Based (Lossy) Trapdoor Function

From the HPE scheme of Section 3.2, our lossy function is obtained by including a $n \times n$ matrix of HPE ciphertexts in the master public parameters. As in the DDH-based function of [32], each row of the matrix is associated with an encryption exponent, which is re-used throughout the entire row. Each column corresponds to a different set of public parameters in the HPE system.

The HIB-TDF that we construct is actually an extended HIB-TDF, and so the master key generation protocol takes an auxiliary input. Depending on the value of this auxiliary input, we obtain the trapdoor (injective) function or a partially lossy function, used in the security proofs. Actually, all HPE ciphertexts in the above-mentioned matrix correspond to different hierarchical vectors $(y_1, \ldots, y_d) \in \mathbb{Z}^{d \times n}_p$, depending on the auxiliary input. The selective weak attribute-hiding property of the HPE scheme guarantees that the two setups are computationally indistinguishable.

In order to evaluate a function for some hierarchical identity $id = (id_1, \ldots, id_d)$, the first step of the evaluation algorithm computes a transformation on HPE ciphertexts so as to obtain a matrix of HIBE ciphertexts. During this transformation, a set of inner products $\{\langle y_{i_1}, id_{i_1} \rangle \}_{i_1=1}^d$ is calculated in the exponent in the diagonal entries of the matrix. The transformation provides a $n \times n$ matrix (8) of anonymous HIBE ciphertexts that are always well-formed in non-diagonal entries. As for diagonal entries, they contain “perturbed” HIBE ciphertexts: at each level, one ciphertext component contains a perturbation factor of the form $\alpha_i \cdot y_{i_1}$. In this matrix of HIBE ciphertexts, random encryption exponents are again re-used in all positions at each row.

The function evaluation is then carried out as in [32], by computing a matrix-vector product in the exponent. The function is thus inverted by testing the well-formedness of each HIBE ciphertext using the private key.

4.1 Description

**HF.Setup**$(\varrho, d, n, \mu)$: given a security parameter $\varrho \in \mathbb{N}$, the (constant) desired number of levels in the hierarchy $d \in \mathbb{N}$ and integers $\mu, n \in \text{poly}(\varrho)$ specifying the length of identities and that of function inputs, respectively, choose asymmetric bilinear groups $((\mathbb{G}, \mathbb{G}_T), \varrho, (\mathbb{G}_T))$ of prime order $p > 2^\varrho$. Define $\text{InpSp} = \{0, 1\}^n$, $\Sigma_{id} = \{(1, x) : x \in \mathbb{Z}_p^{\mu-1}\}$, $\text{IdSp} = \mathbb{Z}_p^{\leq d}$ and $\text{AuxSp} = \mathbb{Z}_p^{d \times n}$. The public parameters are $\text{pms} = (p, (\mathbb{G}, \mathbb{G}_T), d, n, \mu, \text{InpSp}, \text{IdSp}, \text{AuxSp})$.

Since HF is an extended HIB-TDF, the master key generation algorithm of our HIB-TDF receives an auxiliary input $y \in \text{AuxSp}$. Here, it is seen as a concatenation of $d$ row vectors $y_1, \ldots, y_d \in \mathbb{Z}_p^\mu$.

**HF.MKg**(pms, y): parse the auxiliary input as $y = [y_1, \ldots, y_d] \in \mathbb{Z}_p^{d \times \mu}$, and proceed as follows.

1. Choose $\alpha_v \leftarrow \mathbb{G}_p^\mu$, $\alpha_w \leftarrow \mathbb{G}_p^n$, and $\alpha_h \leftarrow \mathbb{G}_T^{d \times (\mu + 1) \times n}$. Define $v = g^{\alpha_v}$, $w = g^{\alpha_w} \in \mathbb{G}^\mu$ and $\hat{w} = g^{\alpha_w} \in \mathbb{G}^n$. Likewise, set up vectors $h = g^{\alpha_h} \in \mathbb{G}_T^{d \times (\mu + 1) \times n}$ and $\hat{h} = g^{\alpha_h} \in \mathbb{G}_T^{d \times (\mu + 1) \times n}$. Define $PP_{\text{core}} := \left\{v, \{\langle w | l \rangle\}_{l=1}^n, \{\langle h | l_1, i_2, l \rangle\}_{i_1 \in \{1, \ldots, d\}, i_2 \in \{0, \ldots, \mu\}, l \in \{1, \ldots, n\}}\right\}$.

2. For $i_1 = 1$ to $d$, parse $y_{i_1}$ as $[y_{i_1}[1], \ldots, y_{i_1}[\mu]] \in \mathbb{Z}_p^{\mu}$. For $l_2 = 1$ to $n$, do the following.
a. Choose $s[l_2] \in \mathbb{Z}_p^*$ and compute $J[l_2] = v^{s[l_2]}$ as well as
\[
C_{w[l_2,l_1]} = w[l_1]s[l_2], \quad C_{[i_1,i_2,l_2,l_1]} = (h[i_1,0,l_1]y_{i_2}^{-}\Delta(l_2,l_1) \cdot h[i_1,i_2,l_1])^{s[l_2]}
\]
for each $i_1 \in \{1, \ldots, d\}$, $i_2 \in \{1, \ldots, \mu\}$, $l_1 \in \{1, \ldots, n\}$.

b. Define a $n \times n$ matrix $\{CT[l_2,l_1]\}_{l_2,l_1 \in \{1,\ldots,n\}}$ of HPE ciphertexts
\[
CT[l_2,l_1] = (J[l_2], C_{w[l_2,l_1]}, \{C_{[i_1,i_2,l_2,l_1]}\}_{i_1 \in \{1,\ldots,d\}, i_2 \in \{1,\ldots,\mu\}})
\]
The master public key consists of $\text{mpk} := (PP_{\text{core}}, \{CT[l_2,l_1]\}_{l_2,l_1 \in \{1,\ldots,n\}})$ while the master secret key is $\text{msk} := (\hat{v}, \hat{w}, \hat{h})$. For each $l_1 \in \{1, \ldots, n\}$, it will be convenient to view $(PP_{\text{core}}, \text{msk})$ as a vector of HPE master key pairs $(\text{mpk}[l_1], \text{msk}[l_1])$, with
\[
\text{mpk}[l_1] = (v, w[l_1], \{h[i_1,i_2,l_1]\}_{i_1 \in \{1,\ldots,d\}, i_2 \in \{0,\ldots,\mu\}})
\]
\[
\text{msk}[l_1] = (\hat{v}, \hat{w}[l_1], \{\hat{h}[i_1,i_2,l_1]\}_{i_1 \in \{1,\ldots,d\}, i_2 \in \{0,\ldots,\mu\}})
\]

**HF.Kg**(pms, msk, (id$_{1_1}, \ldots, id$_{\ell}))\,: to generate a key for an identity $(id$_{1_1}, \ldots, id$_{\ell}) \in \text{IdSp}$, parse msk as $(\hat{v}, \hat{w}, \hat{h})$ and id$_{1_1}$ as id$_{1_1}[1] \ldots id$_{1_1}[\mu]$ for $i_1 = 1$ to $\ell$. Choose $r_w, r_1, \ldots, r_\ell \in (\mathbb{Z}_p^*)^n$. For each $l_1 \in \{1, \ldots, n\}$, compute the decryption component $\text{SK}_D = (D, D_w, \{D_{[j,k,l_1]}\}_{j,k,l_1 = 1}^{\ell})$ of the key as
\[
D[l_1] = \prod_{i_1=1}^{\ell} \left( \prod_{i_2=1}^{\mu} \hat{h}[i_1,i_2,l_1]^{id_{1_1}[i_2]} \right) r_{1}[l_1] \cdot \hat{w}[l_1]^{r_w[l_1]}, \quad D_w[l_1] = \hat{v}^{r_w[l_1]}, \quad D_{[j,k,l_1]}[l_1] = \hat{v}^{r_{1_1}[l_1]}
\]
and the delegation component $\text{SK}_D = (\{K[j,k,l_1]\}_{j,k,l_1}, \{L[j,k,l_1]\}_{j,k,l_1}, \{L_w[j,k,l_1]\}_{j,k,l_1})$, with $j \in \{\ell + 1, \ldots, d\}$, $k \in \{1, \ldots, \mu\}$ and $i_1 \in \{1, \ldots, \ell\}$ as
\[
K[j,k,l_1] = \prod_{i_1=1}^{\ell} \left( \prod_{i_2=1}^{\mu} \hat{h}[i_1,i_2,l_1]^{id_{1_1}[i_2]} \right)^{s[j,k,i_1,l_1]} \cdot \hat{w}[l_1]^{s_{w,j,k,l_1}},
\]
\[
L[j,k,l_1] = \hat{v}^{s[j,k,l_1]}, \quad L_{[j,k,i_1,l_1]} = \hat{v}^{s[j,k,i_1,l_1]} \quad \text{and} \quad L_w[j,k,l_1] = \hat{v}^{s_{w,j,k,l_1}}
\]
Output $\text{SK}_{(id$_{1_1}, \ldots, id$_{\ell})} = (\text{SK}_D, \text{SK}_D)$. **HF.Del**(pms, mpk, (id$_{1_1}, \ldots, id$_{\ell}), $\text{SK}_{(id$_{1_1}, \ldots, id$_{\ell})}$, id$_{\ell+1}$)\,: parse $\text{SK}_{(id$_{1_1}, \ldots, id$_{\ell})}$ as a HF private key of the form $(\text{SK}_D, \text{SK}_D)$, and id$_{\ell+1}$ as a string $id$_{\ell+1}[1] \ldots id$_{\ell+1}[\mu] \in \Sigma_D$.

1. For $l_1 = 1$ to $n$, define $\ell$-th level HPE keys $\text{SK}_{(id$_{1_1}, \ldots, id$_{\ell})}[l_1] = (\text{SK}_D[l_1], \text{SK}_D[l_1])$ where
\[
\text{SK}_D[l_1] = (D[l_1], D_w[l_1], \{D_{[j,k,l_1]}\}_{j,k,l_1 = 1}^{\ell})
\]
\[
\text{SK}_D[l_1] = (\{K[j,k,l_1]\}_{j,k,l_1}, \{L[j,k,l_1]\}_j, \{L_w[j,k,l_1]\}_{j,k,l_1})
\]
2. For $l_1 = 1$ to $n$, run Delegate$\text{mpk}[l_1], (id$_{1_1}, \ldots, id$_{\ell}), \text{SDK}(id$_{1_1}, \ldots, id$_{\ell})[l_1], id_{\ell+1}$) (as specified in Section 3.2) to get $\text{SK}_{(id$_{1_1}, \ldots, id$_{\ell}, id_{\ell+1})}[l_1] = (\text{SK}_D[l_1], \text{SK}_D[l_1])$.
Finally return $\{\text{SK}_{(id$_{1_1}, \ldots, id$_{\ell}, id_{\ell+1})}[l_1]\}_{l_1 = 1}^{n}$.

**HF.Eval**(pms, mpk, (id$_{1_1}, \ldots, id$_{\ell}), X)\,: Given a $n$-bit input $X = x_1 \ldots x_n \in \{0,1\}^n$, for $i_1 = 1$ to $\ell$, parse $id$_{1_1} as $id$_{1_1}[1] \ldots $id$_{1_1}[\mu]$. For $l_1 = 1$ to $n$, do the following.
1. For each \( l_2 \in \{1, \ldots, n\} \), compute modified HPE ciphertexts by defining
\[
C_{id[i_1, l_2, l_1]} = \prod_{i_2=1}^{\mu} C_{[i_1, i_2, l_2, l_1]^{[id[i_1][i_2]]}} = \left(h[i_1, 0, l_1] \cdot v[i_1, id[i_1]] \cdot \prod_{i_2=1}^{\mu} h[i_1, i_2, l_1]^{[id[i_1][i_2]]}ight)^{s[l_2]}
\]
for each \( i_1 \in \{1, \ldots, \ell\} \), \( l_1, l_2 \in \{1, \ldots, n\} \). The modified ciphertexts are
\[
C_{id[i_1, l_2, l_1]} = (J[l_2], \{C_{id[i_1, l_2, l_1]\}_{i_1=1}^{\ell}) \in G^{\ell+1}.
\]

The resulting \( \{C_{id[i_2, l_2, l_1]\}_{l_2, i_1 \in \{1, \ldots, n\}} \) thus form a \( n \times n \) matrix of anonymous HIBE ciphertexts for the identity \( id = (id_1, \ldots, id_\ell) \).

2. Compute \( C_{id, w} = \prod_{l_2=1}^{n} J[l_2]^{x_{l_2}} = v^{(s, X)} \), \( C_{id, w}[l_1] = \prod_{l_2=1}^{n} C_{w[l_2, l_1]}^{x_{l_2}} = w[l_1]^{(s, X)} \) and
\[
C_{id[i_1, l_1]} = \prod_{l_2=1}^{n} C_{id[i_1, l_2, l_1]}^{x_{l_2}} = h[i_1, 0, l_1]^{w[l_1]^{x_{l_1}} \cdot |D[i_1]|} \cdot \left(\prod_{l_2=1}^{\mu} h[i_1, i_2, l_1]^{[id[i_1][i_2]]}\right)^{(s, X)}
\]

Then, output \( C = (C_{id, w}, \{C_{id, w}[l_1]\}_{i_1=1}^{\ell}, \{C_{id[i_1, l_1]\}_{i_1 \in \{1, \ldots, \ell\}, l_1 \in \{1, \ldots, n\}} \in G^{n+1+n \times \ell}.
\]

HF.Inv(pms, mpk, (id_1, \ldots, id_\ell), SK_{id, \ldots, id_\ell}, C); parse the decryption component \( SK_D \) of the private key as a tuple of the form \( (D, D_w, D_w, \{D_i\}_{i=1}^{\ell}) \) and the output \( C \) as per (10). Then, for \( l_1 = 1 \) to \( n \), set \( x_{l_1} = 0 \) if
\[
e(C_{id, w}, D[l_1]) \cdot e(C_{id, w}[l_1], D_w[l_1])^{-1} \cdot \prod_{i_1=1}^{\ell} e(C_{id[i_1, l_1], D_i[l_1])^{-1} = 1_{G^\mu}.
\]

Otherwise, set \( x_{l_1} = 1 \). Eventually, return \( X = x_1 \ldots x_n \in \{0, 1\}^n \).

From (9), we notice that, with overwhelming probability, if there exists some \( i_1 \in \{1, \ldots, d\} \) such that \( \langle y_{i_1}, id[i_1]\rangle \neq 0 \), relation (11) is satisfied if and only if \( x_{i_1} = 0 \). Indeed, in this case, the output (10) is distributed as a vector of \( n \) Boneh-Boyen HIBE ciphertexts (in their anonymous variant considered in [20]). These ciphertexts correspond to the same encryption exponent \( \langle s, X\rangle \) and are generated under \( n \) distinct master public keys sharing the same component \( v \in G \).

When the function is implemented in injective mode, the auxiliary input consists of a vector \( y_{0}^{(0)} = [(1, 0, \ldots, 0)] \) such that \( y_{0}^{(1)} = [(1, 0, \ldots, 0)] \) in RL_0, and auxiliary input \( y_{1}^{(1)} = [y_{1}^{(1)} \ldots y_{d}^{(1)}] \) in RL_n, where \( y_{1}^{(1)} \) is produced by an auxiliary input generator Aux(id) taking as input a special hierarchical identity \( id = (id_1, \ldots, id_\ell) \). The master public key mpk is given to the \( A \). \( A \) can request secret keys
for identities $id$, which will be answered using $HF.Kg$ and $HF.Del$ and will be added to $IS$, initialized to $IS = \{\emptyset\}$. Also, $A$ will output a guess $d_A$. Both in $RL_0$ and $RL_n$, the experiment will halt and output $d' = 0$ if: a) for any $id = (id_1, \ldots, id_\ell) \in IS$ we have that $\langle y^{(1)}_1, \ldots, y^{(1)}_{d_A} \rangle = 0$ for each $i_1 \in \{1, \ldots, \ell\}$, or b) if $\langle y^{(1)}_1, \ldots, y^{(1)}_{d_A} \rangle \neq 0$ for some $i_1 \in \{1, \ldots, \ell\}$. If the experiment has not aborted, it will output the bit $d' = d_A$. The result in the following lemma will be used to prove that $HF$ enjoys partial lossiness.

**Lemma 1.** Under the $P-BDH_1$ and $DDH_2$ assumptions, the experiments $RL_0$ and $RL_n$ return 1 with nearly identical probabilities. Namely, there exist PPT algorithms $B_1$ and $B_2$ such that

$$\left|Pr[RL_0 \Rightarrow 1] - Pr[RL_n \Rightarrow 1]\right| \leq n \cdot \left((d \cdot \mu + 1) \cdot Adv^{P-BDH_1}(B_1) + q \cdot Adv^{P-DDH_2}(B_2)\right),$$

where $q$ is the number of “Reveal-key” queries made by $A$. (The proof is in Appendix B.1).

**Adaptive-id Security.** The details on the security analysis of $HF$ with respect to selective adversaries are given in Appendix B.2. For adaptive security we consider the construction of $HF$ with a restricted identity space, namely taking $\Sigma_{ID} = \{(1, x) : x \in \{0, 1\}^{\mu - 1}\}$. Define a sibling $LHF$ with $AuxSp = \Xi_{P-d}$, where the auxiliary input $y^{(1)} = [y^{(1)}_1, \ldots, y^{(1)}_d]$, for any $i_1$ from 1 to $d$, is defined as:

$$y^{(1)}_{i_1} = \{0, \ldots, 2q - 1\}, \quad \xi_{i_1} \leftarrow \{0, \ldots, \mu + 1\}, \quad y^{(1)}_{i_1}(1) = y^{(1)}_{i_1} - 2\xi_{i_1}q, \quad \{y^{(1)}_{i_1}[i] \leftarrow \{0, \ldots, 2q - 1\}\}^{\mu-2}.$$

The pre-output stage $P$ associated to the scheme $HF$ in the adaptive case will be the artificial abort used in [36]: if $IS$ is the set of queried identities (only taking into account “Reveal key” queries) and $id^*$ is the challenge identity, let $E(IS, id^*)$ be the event that the REAL or the LOSSY experiments set $d_1$ to be 1, and let $\eta(IS, id^*) = Pr[E(IS, id^*)]$. We prove in Appendix B.3:

**Lemma 2.** $\eta_{low} = 1 / (2 \cdot (2q^{\mu})^d) \leq \eta(IS, id^*)$.

$P$ computes an approximation $\eta'(IS, id^*)$ of $\eta(IS, id^*)$, by using $O(\zeta^2 \ln(\zeta^{-1}) \eta_{low}^{-1} \ln(\eta_{low}^{-1}))$ samples, for a non-negligible $\zeta$, which is part of the input to $P$. If $\eta'(IS, id^*) \leq \eta_{low}$, then $d_2$ is always set to 1. If $\eta'(IS, id^*) > \eta_{low}$, then $d_2 = 0$ with probability $1 - \eta_{low} / \eta'(IS, id^*)$ and $d_2 = 1$ with probability $\eta_{low} / \eta'(IS, id^*)$. Adaptive security of $HF$ is established in the following theorem.

**Theorem 3.** Let $n > \log p$ and let $\omega = n - \log p$. Let $HF$ be the HIB-TDF with parameters $n, d, \mu, \Sigma_{ID} = \{(1, x) : x \in \{0, 1\}^{\mu - 1}\}$, $\Sigma_{ID} = \Sigma_{ID}^{\leq d}$. Let $LHF$ be the sibling with auxiliary input as above, let the pre-output stage associated to $HF$ be the one specified above. In front of adaptive-id adversaries that make a maximal number of $q \leq p/(2\mu)$ queries, $HF$ is $(\omega, \delta)$-partially lossy, for $\delta = 13 / (32 \cdot (2q^{\mu})^d)$.

Specifically regarding condition (i), for any such adaptive-id adversary $A$ there exist algorithms $B_1$ and $B_2$ such that $Adv_{HF,LHF,\omega,\zeta}(A) \leq 2n \cdot ((d \cdot \mu + 1) \cdot Adv^{P-BDH_1}(B_1) + q \cdot Adv^{P-DDH_2}(B_2))$, where the running time of $B_1$ and $B_2$ is that of $A$ plus a $O(\zeta^2 \ln(\zeta^{-1}) \eta_{low}^{-1} \ln(\eta_{low}^{-1}))$ overhead, with $\eta_{low} = 1 / (2 \cdot (2q^{\mu})^d)$. (The proof is given in Appendix B.4).

The fact that $\eta_{low}^{-1}$ depends exponentially on $d$ is what makes our adaptive-id security result valid only for hierarchies of constant depth $d$. Whether this restriction can be avoided at all is an interesting open question.
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A. Definitions for Hierarchical Predicate Encryption

A.1 Definitions for Hierarchical Predicate Encryption

A tuple of integers $\vec{\nu} = (\nu_1, \ldots, \nu_d)$ such that $\nu_0 = 0 \leq \nu_1 < \nu_2 < \cdots < \nu_d = \mu_1$ is called a format hierarchy of depth $d$. Such a hierarchy is associated with attribute spaces $\{\Sigma_i\}_{i=0}^d$. In our setting, we set $\Sigma_i = \mathbb{Z}_{p_i}^{\nu_i-\nu_{i-1}} \setminus \{0\}$ for $i = 1$ to $d$, for some $p_i \in \mathbb{N}$, and also define the universe of hierarchical attributes as $\Sigma := \cup_{i=1}^d (\Sigma_1 \times \cdots \times \Sigma_i)$. For vectors $\vec{X}_i \in \Sigma_i$, we will consider the (inner-product) hierarchical predicate $f_{\vec{X}_1, \ldots, \vec{X}_\ell} (\vec{Y}_1, \ldots, \vec{Y}_\nu) = 1$ iff $\ell \leq \kappa$ and $\vec{X}_i \cdot \vec{Y}_i = 0$ for all $i \in \{1, \ldots, \ell\}$. The space of hierarchical predicates is defined to be

$\mathcal{F} = \{f_{\vec{X}_1, \ldots, \vec{X}_\ell} | \{\vec{X}_i \in \Sigma_i \}_{i \in \{1, \ldots, \ell\}}\}$

and the integer $\kappa$ (resp. $\ell$) is called the depth (resp. the level) of $(\vec{Y}_1, \ldots, \vec{Y}_\nu)$ (resp. $(\vec{X}_1, \ldots, \vec{X}_\ell)$). Most of the time we will assume that $\nu_i/\nu_{i-1} = \mu$ for any $i$, and also define the hierarchy $\vec{\nu} = (\nu_1, \ldots, \nu_d)$. A hierarchical predicate encryption (HPE) scheme for a predicate family $\mathcal{F}$ consists of these algorithms.

Setup($\varrho, \vec{\nu}$): takes as input a security parameter $\varrho \in \mathbb{N}$ and a format hierarchy $\vec{\nu} = (\nu_1, \ldots, \nu_d)$. It outputs a master secret key $\text{msk}$ and a master public key $\text{mpk}$ that includes the description of a hierarchical attribute space $\Sigma$.

Keygen($\text{msk}, (\vec{X}_1, \ldots, \vec{X}_\ell)$): takes as input predicate vectors $(\vec{X}_1, \ldots, \vec{X}_\ell) \in \Sigma_1 \times \cdots \times \Sigma_\ell$ and the master secret key $\text{msk}$. It outputs a private key $SK_{\vec{X}_1, \ldots, \vec{X}_\ell}$.

Encrypt($\text{mpk}, (\vec{Y}_1, \ldots, \vec{Y}_\nu), M$): takes as input attribute vectors $(\vec{Y}_1, \ldots, \vec{Y}_\nu) \in \Sigma_1 \times \cdots \times \Sigma_k$, the master public key $\text{mpk}$ and a message $M$. It outputs a ciphertext $C$.

Decrypt($\text{mpk}, (\vec{X}_1, \ldots, \vec{X}_\ell), SK_{\vec{X}_1, \ldots, \vec{X}_\ell}, C$): takes in a private key $SK_{\vec{X}_1, \ldots, \vec{X}_\ell}$ for the vectors $(\vec{X}_1, \ldots, \vec{X}_\ell)$, the master public key $\text{mpk}$ and a ciphertext $C$. It outputs a plaintext $M$ or $\perp$. 

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Delegate \((\text{mpk}, (\vec{X}_1,\ldots,\vec{X}_\ell), \text{SK}(\vec{X}_1,\ldots,\vec{X}_\ell), \vec{X}_{\ell+1})\): takes in a \(\ell\)-th level private key \(\text{SK}(\vec{X}_1,\ldots,\vec{X}_\ell)\), the corresponding vectors \((\vec{X}_1,\ldots,\vec{X}_\ell)\) and a vector \(\vec{X}_{\ell+1}\). It outputs a \((\ell + 1)\)-th level private key \(\text{SK}(\vec{X}_1,\ldots,\vec{X}_\ell,\vec{X}_{\ell+1})\).

Correctness mandates that, for any message \(M\) and hierarchical vectors \((\vec{X}_1,\ldots,\vec{X}_\ell), (\vec{Y}_1,\ldots,\vec{Y}_\kappa)\), we have \(\text{Decrypt}(\text{mpk}, (\vec{X}_1,\ldots,\vec{X}_\ell), \text{SK}(\vec{X}_1,\ldots,\vec{X}_\ell), \text{Encrypt}(\text{mpk}, (\vec{Y}_1,\ldots,\vec{Y}_\kappa), M)) = M\) whenever the condition \(f_{(\vec{X}_1,\ldots,\vec{X}_\ell)}(\vec{Y}_1,\ldots,\vec{Y}_\kappa) = 1\) is satisfied.

Following [30], we write \(f' \leq f\) to express that the predicate vector of \(f\) is a prefix of that for \(f'\), meaning that \(f'\) is at least as constraining as \(f\).

**Definition 2.** A Hierarchical Predicate Encryption scheme is selectively weakly attribute-hiding if no PPT adversary has non-negligible advantage in the following game:

1. The adversary \(A\) chooses a format hierarchy \(\vec{\nu} = (n_1,d;\nu_1,\ldots,\nu_d)\) and vectors \((\vec{Y}^0_1,\ldots,\vec{Y}^0_d), (\vec{Y}^1_1,\ldots,\vec{Y}^1_d)\), for some \(d^* \leq d\). The challenger generates a master key pair \((\text{msk}, \text{mpk}) \leftarrow \text{Setup}(\rho, \vec{\nu})\) and \(\text{mpk}\) is given to \(A\).

2. \(A\) is allowed to make a number of adaptive queries.

   - **Create-key:** \(A\) provides a predicate \(f \in \mathcal{F}\) and the challenger creates a private key \(\text{SK}_f\) for \(f\) without revealing it to \(A\).

   - **Create-delegated-key:** \(A\) chooses a private key that was previously created for some predicate \(f\) and also specifies another predicate \(f' \leq f\) that \(f\) is a prefix of. The challenger then computes a delegated key \(\text{SK}_{f'}\) for \(f'\) without revealing it to \(A\).

   - **Reveal-key:** \(A\) asks the challenger to give out a previously created key.

For each Reveal-key query \((\vec{X}_1,\ldots,\vec{X}_\ell)\), it is required that

\[
f_{(\vec{X}_1,\ldots,\vec{X}_\ell)}(\vec{Y}^0_1,\ldots,\vec{Y}^0_d) = f_{(\vec{X}_1,\ldots,\vec{X}_\ell)}(\vec{Y}^1_1,\ldots,\vec{Y}^1_d) = 0.
\]

3. \(A\) outputs messages \(M_0, M_1\). Then, the challenger chooses \(\beta \in \{0,1\}\) and computes a challenge ciphertext \(C^* = \text{Encrypt}(\text{mpk}, (\vec{Y}^\beta_1,\ldots,\vec{Y}^\beta_d), M_\beta)\), which is sent to \(A\).

4. \(A\) makes further private key queries for hierarchical vectors \((\vec{X}_1,\ldots,\vec{X}_\ell)\) under the same restriction as above.

5. \(A\) outputs a bit \(\beta' \in \{0,1\}\) and wins if \(\beta' = \beta\).

\(A\)’s advantage is quantified as the distance \(\text{Adv}(A) = |\Pr[\beta' = \beta] - 1/2|\).

**A.2 Correctness of the HIPE scheme**

**Lemma 3.** The HIPE scheme of Section 3.2 is correct. Namely, for any message \(M\) and any vectors \(\vec{X}_1,\ldots,\vec{X}_\ell, \vec{Y}_1,\ldots,\vec{Y}_\kappa\), we have

\[
\text{Decrypt}(\text{mpk}, (\vec{X}_1,\ldots,\vec{X}_\ell), \text{SK}(\vec{X}_1,\ldots,\vec{X}_\ell), \text{Encrypt}(\text{mpk}, (\vec{Y}_1,\ldots,\vec{Y}_\kappa), M)) = M
\]

whenever \(f_{(\vec{X}_1,\ldots,\vec{X}_\ell)}(\vec{Y}_1,\ldots,\vec{Y}_\kappa) = 1\). This fact does not depend on whether the key \(\text{SK}(\vec{X}_1,\ldots,\vec{X}_\ell)\) was created using Delegate or Keygen.
Proof. Let $SK(\vec{X}_1, \ldots, \vec{X}_\ell) = (SK_D, SK_{DL})$ be obtained by running $Keygen(msk, (\vec{X}_1, \ldots, \vec{X}_\ell))$, where each attribute vector is $\vec{X}_i = (x_{i1}, \ldots, x_{i,\mu}) \in \mathbb{Z}_p^\mu$, for each $i \in \{1, \ldots, \ell\}$. Let us write the decryption component of the key as $SK_D = (D, D_w, \{D_{i_1}\}_{i_1=1}^{\ell})$.

Let $C = (C_0, C_v, C_w, \{C_{i_1,i_2}\}_{i_1\in\{1,\ldots,\kappa\}, i_2\in\{1,\ldots,\mu\}})$ be the output of the encryption algorithm $Encrypt(mpk, (\vec{Y}_1, \ldots, \vec{Y}_\kappa), M)$, for vectors $\vec{Y}_1 = (y_{1,1}, \ldots, y_{1,\mu}), \ldots, \vec{Y}_\kappa = (y_{\kappa,1}, \ldots, y_{\kappa,\mu})$.

Since $f(\vec{X}_1, \ldots, \vec{X}_\ell)(\vec{Y}_1, \ldots, \vec{Y}_\kappa) = 1$, and by the definition of hierarchical inner-product predicates, we know that $\ell \leq \kappa$ and $\vec{X}_i \cdot \vec{Y}_i = 0$ for all $i \in \{1, \ldots, \ell\}$. Therefore, when the decryption protocol $Decrypt(mpk, (\vec{X}_1, \ldots, \vec{X}_\ell), SK(\vec{X}_1, \ldots, \vec{X}_\ell), C)$ computes $C_{i_1}$, for each $i_1 \in \{1, \ldots, \ell\}$, the obtained value equals $C_{i_1} = (\prod_{i_2=1}^{\mu} h_{\vec{x}_{i_1,i_2}})^{s_{i_1}}$. The decryption protocol computes then the pairing of this element $C_{i_1}$ with $D_{i_1} = \hat{v}^{r_{i_1}}$. Multiplying all these pairings, for indices $i_1 \in \{1, \ldots, \ell\}$, one obtains $e(v^s, \prod_{i_1=1}^{\ell} (\prod_{i_2=1}^{\mu} h_{\vec{x}_{i_1,i_2}})^{r_{i_1}})$. This value is canceled out with one of the factors of $e(C_v, D)$, when computing the final decryption operation $C_0 \cdot e(C_v, D)^{-1} \cdot e(C_w, D_w) \cdot \prod_{i_1=1}^{\ell} e(C_{i_1}, D_{i_1})$. The other two factors of $e(C_v, D)$ are $e(v^s, \hat{w}^{s\alpha})$ and $e(v^s, \hat{w}^{s\beta})$, which cancel out the factor $e(g, \hat{v})^{\alpha s}$, contained in $C_0 = M \cdot e(g, \hat{v})^{\alpha s}$, and the factor $e(C_w, D_w) = e(w^s, \hat{w}^{s\alpha})$, respectively. Therefore, the final computation of the decryption protocol results in the plaintext $M$ contained in $C_0$.

In this way, we have proved that the encryption and decryption protocols work correctly when the original secret keys (resulting from Keygen) are used. The fact that the decryption protocol works fine also with delegated secret keys (resulting from Delegate) is a consequence of Lemma 4, inside the proof of Theorem 2. If a delegated secret key could lead to an incorrect decryption, then this fact could be used to distinguish original secret keys from delegated ones, which would contradict the statement of Lemma 4.

A.3 Proof of Theorem 2

The proof considers a sequence of games starting with the real game and ending with a game where the adversary has no advantage and wins with probability exactly 1/2.

For each $i$, we denote by $S_i$ the event that the adversary wins in Game$_i$. In the whole sequence of games, we call $d^*$ the depth of the challenge hierarchical vectors $(\vec{Y}_0^1, \ldots, \vec{Y}_0^{d^*})$ and $(\vec{Y}_1^1, \ldots, \vec{Y}_1^{d^*})$ and $C^* = (C_0^*, C_v^*, C_w^*, \{C_{i_1,i_2}^*\}_{i_1\in\{1,\ldots,d^*\}, i_2\in\{1,\ldots,\mu\}})$ denotes the challenge ciphertext.

Game$_0$: is the real attack game at the end of which the challenger outputs 1 in the event, called $S_0$, that the adversary $A$ manages to output $\beta' \in \{0, 1\}$ such that $\beta' = \beta$, where $\beta \in \{0, 1\}$ is the challenger’s hidden bit in the challenge phase. If $\beta' \neq \beta$, the challenger outputs 0.

Game$_1$: is identical to Game$_0$ with the difference that the challenger always answers private key queries by returning fresh private keys (i.e., keys produced by Keygen) instead of deriving those keys using the delegation algorithm.

Game$_2$: is like Game$_1$ but the challenge ciphertext $C^*$ is now an encryption under $(\vec{Y}_1^0, \ldots, \vec{Y}_1^{d^*})$ of a random plaintext $M \overset{\$}{\leftarrow} \mathcal{G}_T$, which is chosen independently of $M_0$ and $M_1$.

Game$_3$: is identical to Game$_2$ with the difference that, in the challenge ciphertext, $C_{i_1,i_2}^*$ is replaced by a random group element chosen uniformly and independently in $\mathcal{G}$.

Game$_{4,i,j}$ ($1 \leq i \leq d^*, 1 \leq j \leq \mu$): is identical to Game 3 with the difference that, in the challenge ciphertext, $C_{i_1,i_2}^*$ are replaced by random elements of $\mathcal{G}$ if $i_1 < i$ or $(i_1 = i) \land (i_2 \leq j)$. Other group elements (i.e., for which $i > i_1$ or $(i = i_1) \land (i_2 > j)$) are still computed as in a normal challenge ciphertext.
In Game$_{4,d^\star,\mu}$, it is easy to see that the adversary $A$ cannot guess $\beta \in \{0,1\}$ with higher probability than $\Pr[S_{4,d^\star,\mu}] = 1/2$ since the challenge ciphertext $C^*$ is completely independent of $\beta \in \{0,1\}$.

**Lemma 4.** Game$_0$ and Game$_1$ are computationally indistinguishable if the DDH$_2$ assumption holds in $(G, \hat{G})$.

**Proof.** The lemma will be proved by a hybrid argument. We define Game$_{0,i}$ for all $0 \leq i \leq q$. Game$_{0,i}$ differs from Game$_0$ in the fact that, when the adversary issues the first $i$ delegation queries, instead of generating the delegated keys faithfully using the Delegate algorithm, the challenger calls the Keygen algorithm to generate these delegated keys. For all the remaining queries, the challenger computes keys and responds faithfully as in Game$_0$. Under the above definition, Game$_{0,0}$ is the same as Game$_0$ and Game$_{0,q}$ is the same as Game$_1$. We will prove that Game$_{0,\kappa}$ is indistinguishable from Game$_{0,\kappa+1}$ for all $0 \leq \kappa \leq q - 1$. To this end, we will proceed similarly to [34] and rely on a generalized version (called GDDH hereafter) of the DDH problem in $\hat{G}$.

Given a group generator $G\hat{G}$, define the following distribution $P(\varrho)$:

$$(p, (\hat{G}, \hat{G}_T, e)) \xlongleftarrow{} R \hat{G}(\varrho, 1),$$
$$g \xlongleftarrow{} \hat{G}, \quad \hat{g} \xlongleftarrow{} \hat{G},$$
$$\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_\ell \xlongleftarrow{} \hat{G},$$
$$\tau \xlongleftarrow{} \mathbb{Z}_p,$$
$$X \leftarrow ((p, (\hat{G}, \hat{G}_T, e), g, \hat{g}, \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_\ell),$$
$$Q \leftarrow (\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_\ell),$$
$$\text{Output } (X, Q).$$

For an algorithm $A$, define $A$’s advantage in solving the above problem:

$$\ell-\text{GDDH Adv}_{G\hat{G},A}(\varrho) := |Pr[A(X, Q) = 1] - Pr[A(X, R)]|$$

where $(X, Q) \leftarrow P(\varrho)$ and $R \leftarrow \hat{G}^\ell$. It is immediate$^8$ that GDDH is not easier than DDH$_2$ and that the latter advantage function is negligible if the DDH$_2$ assumption holds in $(G, \hat{G})$.

To prove that Game$_{0,\kappa}$ is indistinguishable from Game$_{0,\kappa+1}$ we will use another hybrid argument. We define Game$'_{0,\kappa}$, which differs from Game$_{0,\kappa}$ in that, for the $(\kappa + 1)$-th delegation query, $SK_{DL}$ is the delegation component of a fresh key, instead of a delegation component obtained by raising every element in $SK_{DL}$ to the same random power $z \in \mathbb{Z}_p$. We show that a PPT adversary cannot distinguish between the two games.

We also define Game$''_{0,\kappa}$, which differs from Game$'_{0,\kappa}$ in that, for the $(\kappa + 1)$-th delegation query, instead of re-randomizing the components of the partial decryption key with the same exponent $\tau_{j,k}$, $\{L_{\ell+1,i}^{(j,k)}\}_{i=1}^{\ell+1}, \{L_{\ell+1,i}^{(j,k)}\}_{i=1}^{\ell+1}$ are randomized with different independently chosen exponents, while $K'^{j,k}_{j,k}$ is chosen in such a way that the resulting key is still valid. We also prove that no PPT adversary can notice the difference.

$^8$ The straightforward reduction computes a GDDH instance from a DDH$_2$ instance $(g, \hat{g}, \hat{g}^a, \hat{g}^b, \hat{g}^z)$ by setting $\hat{h}_i = \hat{g}^{\alpha_i} \cdot (\hat{g}^b)^{\beta_i}$ and $Q_i = (\hat{g}^z)^{\alpha_i} \cdot \hat{g}^{\beta_i}$ for $i = 1$ to $\ell$ with $\alpha_1, \ldots, \alpha_\ell \xlongleftarrow{} \mathbb{Z}_p, \beta_1, \ldots, \beta_\ell \xlongleftarrow{} \mathbb{Z}_p$. If $\hat{g} = \hat{g}^ab$, we have $Q = (\hat{h}_1, \ldots, \hat{h}_\ell)$ whereas, if $\eta \in \mathbb{R} \hat{G}$, $Q$ is a random vector of $\hat{G}^{\ell}$. 

20
We will argue that Game\textsuperscript{0,κ}″ = Game\textsuperscript{0,κ+1}. Indeed, in the first step, we change $\overline{SK}_{DL}$ so that, in step 2 of the Delegate algorithm, we obtain a randomized decryption key (except for the $g^a$ term). When multiplied by $SK_D$, it gives a randomized decryption key for $(\hat{X}_1, \ldots, \hat{X}_{L+1})$. On the other hand, in step 2 of the hybrid proof we change the partial decryption keys so that they also are randomized keys except the $g^a$ term.

**Claim.** Game\textsuperscript{0,κ} is computationally indistinguishable from Game\textsuperscript{0,κ}″.

**Proof.** Let $q_0$ denote the maximum number of secret key queries (taking into account both the “Create-key” and “Create-delegated-key” queries) made by the adversary. We build a simulator $B$ that uses $A$ to break the following $(q_0d\mu(d+1))$-GDDH assumption.

$$(p, \mathbb{G}, \hat{G}, \mathbb{G}_T, c) \xleftarrow{\$} \mathbb{G}(q, 1),$$

$$g \xleftarrow{\$} \mathbb{G}, \quad \hat{g} \xleftarrow{\$} \hat{G}$$

$${\hat{v}_{i,1,i_2,k}} \leftarrow \hat{G}_{i \in \{1, \ldots, q_0\}, i_1 \in \{1, \ldots, d\}, i_2 \in \{0, \ldots, \mu\}, k \in \{1, \ldots, d+1\}}$$

$$\tau \xleftarrow{\$} \mathbb{Z}_p$$

$$(p, \mathbb{G}, \hat{G}, \mathbb{G}_T, c), \quad g, \hat{g}, \quad {\hat{v}_{i,1,i_2,k}} \leftarrow \hat{G}_{i \in \{1, \ldots, q_0\}, i_1 \in \{1, \ldots, d\}, i_2 \in \{0, \ldots, \mu\}, k \in \{1, \ldots, d+1\}}$$

$$Q \leftarrow (\{\hat{v}_{i,1,i_2,k}\}_{i \in \{1, \ldots, q_0\}, i_1 \in \{1, \ldots, d\}, i_2 \in \{0, \ldots, \mu\}, k \in \{1, \ldots, d+1\}})$$

Then, the challenger randomly decides to give $(X, Q' = Q)$ or $(X, Q' = R)$, where $R$ is a random vector of elements in $\hat{G}$ of the size of $Q$. The simulator will use $A$ as a subroutine to break the above problem.

**Init and Setup.** At the beginning of the security game, the adversary commits to two hierarchical vectors $(\hat{Y}_1^0, \ldots, \hat{Y}_d^0)$ and $(\hat{Y}_1^1, \ldots, \hat{Y}_d^1)$, where $d^* \leq d$. The simulator chooses the public and the secret key as usual according to the Setup algorithm. Let $c = \log_w(w)$ and $a_{i_1,i_2} = \log_w(h_{i_1,i_2})$. Note that these values are known to the simulator since they are easily computable from msk.

**Secret key queries.** We distinguish three cases:

- When a “create-key” query or one of the first $\kappa$ delegated secret key queries is made, the simulator computes and saves a private key, which is given to $A$ when a “reveal-key” query is made. To compute this secret key, the simulator uses the elements from the GDDH instance, in such a way that the exponents are distributed at random. In particular, if it is the $i$-th query, the simulator defines the components of the decryption component of the key as:

$$D = g^a \prod_{i_1=1}^{\ell} D_{i_1}^{(\sum_{i_2=1}^{\mu} a_{i_1,i_2} \cdot x_{i_1,i_2})} \cdot D_{w,1} \cdot D_{w,0,1,d+1} \cdot D_{w,0,1,i_1},$$

$$D_{i_1} = \hat{v}_{i,0,1,i_1} \quad i_1 \in \{1, \ldots, \ell\}.$$  

For the delegation component of the key, for all $j \in \{\ell+1 \ldots d\}$, all $k \in \{1, \ldots, \mu\}$, the simulator lets:

$$L_{j} = \hat{v}_{i,j,1,\ell+1}, \quad L_{j,k,i_1} = \hat{v}_{i,j,k,i_1} \quad i_1 \in \{1, \ldots, \ell\}, \quad \quad L_{w,j,k} = \hat{v}_{i,j,k,d+1}.$$ 

As the simulator knows the discrete logarithms $c = \log_w(w)$ and $a_{i_1,i_2} = \log_w(h_{i_1,i_2})$, for each $j \in \{\ell+1 \ldots d\}$ and all $k \in \{1, \ldots, \mu\}$, it can compute the remaining components of the key.
as follows:

\[ K_{j,k} = \prod_{i_1=1}^{\ell} L_{j,k,i_1}^{a_{i_1,i_2} x_{i_1,i_2}} \cdot L_{j,k}^{a_{j,k}} \cdot L_{w,j,k}^c. \]  

\( (12) \)

- When the adversary makes the \((\kappa + 1)\)-th delegation query, it specifies a parent key and asks to fix the level of the hierarchy to some vector \(X_{\ell+1} \). In particular, assume that the parent key was created in the \(i\)-th query. When performing Step 1 of the Delegate algorithm, for all \(j \in \{\ell + 2, \ldots , d\}, k \in \{1, \ldots , \mu\} \), the simulator sets

\[
\begin{align*}
\hat{L}_j &= Q'_{i,j,1,\ell+1}, \\
\hat{L}_{j,k,i_1} &= Q'_{i,j,k,i_1} & i_1 & \in \{1, \ldots , \ell\}, \\
\hat{L}_{w,j,k} &= Q'_{i,j,k,d+1}
\end{align*}
\]

and computes \(K_{j,k}\) for each \(j \in \{\ell + 2, \ldots , d\}, k \in \{1, \ldots , \mu\}\) exactly in the same way as in expression \((12)\).

- For all the remaining queries, the simulator responds faithfully as in the real game.

Clearly, if \(Q' = Q\) in the GDDH instance, then the above simulation is identical to \(\text{Game}_{0,\kappa}\). Otherwise, it is identical to \(\text{Game}''_{0,\kappa}\) since, in the the \((\kappa + 1)\)-th delegation query, \(Q = R\) implicitly defines a set of fresh random values for \(a_{j,i_1,i_2}, x_{i_1,i_2}, a_{j,k}\) for the appropriate values of \(j, k, i_1\).

**Challenge** The simulator generates the challenge ciphertext as normal.

**Guess** If the adversary has a difference of \(\epsilon\) in its advantage in \(\text{Game}_{0,\kappa}\) and \(\text{Game}''_{0,\kappa}\), the simulator has a comparable advantage in solving the GDDH instance.

**Claim.** \(\text{Game}''_{0,\kappa}\) is computationally indistinguishable from \(\text{Game}''_{0,\kappa}\).

**Proof.** To prove this claim, we will appeal to a nested hybrid argument. Let \(\text{Game}'_{0,\kappa,0,0} = \text{Game}'_{0,\kappa}\), and for \(1 \leq \eta \leq (d-\ell-1), 1 \leq \nu \leq \mu\), define \(\text{Game}'_{0,\kappa,\eta,\nu}\) as the game that differs from \(\text{Game}_{0,\kappa}\) in the following: in the step of the delegation algorithm where the components \(\{L_{\ell+1,i_1}^{(j,k)}\}_{i_1 \in \{1,\ldots,\ell+1\}}, L_{j,k}^{(j,k)}\) are created by re-randomizing the partial decryption key with some exponent \(\tau_{j,k}\), we re-randomize instead each of these components with a different exponent chosen uniformly and independently at random whenever \((j,k) \leq (\eta + \ell + 1, \nu)\) (in lexicographic order). Observe that, by definition, \(\text{Game}'_{0,\kappa,d-\ell-1,\mu} = \text{Game}''_{0,\kappa}\). We will show that an adversary cannot distinguish between one game and the next. That is, if we define \(\text{Game}'_{0,\kappa,\eta,\mu+1} = \text{Game}'_{0,\kappa,\eta+1,0}\) to simplify the notation, what we will show is that no polynomial time adversary can distinguish between \(\text{Game}'_{0,\kappa,\eta,\nu}\) and \(\text{Game}'_{0,\kappa,\eta,\nu+1}\).

The simulator tries to solve the following \((\mu(d+1))\)-GDDH instance.

\[
\begin{align*}
(p, G, \hat{G}, \hat{G}_T, e) &\xleftarrow{R} \text{GG}(q, 1), \\
g &\xleftarrow{R} G, \ \hat{g} &\xleftarrow{R} \hat{G} \\
\{\hat{v}_{i,k} \xleftarrow{\$} \hat{G}\}_{i \in \{1,\ldots,\mu\}, k \in \{1,\ldots,d+1\}} \\
\tau &\xleftarrow{R} Z_p \\
X &\xleftarrow{\$} (n, G, \hat{G}, \hat{G}_T, e), g, \hat{g}, \ \{\hat{v}_{i,k}\}_{i \in \{1,\ldots,\mu\}, k \in \{1,\ldots,d+1\}} \\
Q &\xleftarrow{\$} (\{\hat{v}_{i,k}\}_{i \in \{1,\ldots,\mu\}, k \in \{1,\ldots,d+1\}})
\end{align*}
\]

\[22\]
The simulator tries to distinguish between \((X, Q = Q)\) and \((X, Q' = R)\) where \(R\) is a random vector from \(\hat{G}\). The simulator uses as a subroutine an adversary \(\mathcal{A}\) who can distinguish between \(\text{Game}'_{0,\kappa,\eta,\nu}\) and \(\text{Game}'_{0,\kappa,\eta,\nu+1}\).

The simulator runs the Setup algorithm as usual in such a way that it knows the discrete logarithms \(c = \log_\hat{v}(\hat{w}) = \log_\hat{v}(\hat{w})\) and \(a_{i_1i_2} = \log_\hat{v}(\hat{h}_{i_1i_2}) = \log_\hat{v}(\hat{h}_{i_1i_2})\) for any \(1 \leq i_1 \leq d, 0 \leq i_2 \leq \mu\).

To answer secret key queries, the simulator proceeds as follows:

- For the first \(\kappa\)-th delegation queries and all of the “create-key” queries, the simulator computes the keys freshly at random.
- At the \((\kappa + 1)\)-th delegation query, the adversary specifies a parent key and requests to fix the \((\ell + 1)\)-th level of the hierarchy to \(\hat{X}_{\ell+1}\). To answer this query, the simulator first generates some components of \(SK_{DL}\) simply by choosing at random the values \(\hat{L}_j, \hat{L}_{w,j,k}\) for all \(j \in \{\ell + 2, \ldots, d\}, k \in \{1, \ldots, \mu\}, i_1 \in \{1, \ldots, \ell\}\). To compute the remaining components and the decryption key component, the simulator sets

\[
L_{\ell+1,i_1} = \prod_{k=1}^{\mu} \hat{L}_x^{x+1,k,i_1} = \prod_{k=1}^{\mu} (\hat{v}_{k,i_1})^{x+1,k} \quad i_1 \in \{1, \ldots, \ell\}
\]

\[
L_{\ell+1,\ell+1} = \hat{L}_{\ell+1} = \hat{v}_{1,\ell+1}
\]

\[
L_{w,\ell+1} = \prod_{k=1}^{\mu} \hat{L}_{w,\ell+1,k} = \prod_{k=1}^{\mu} (\hat{v}_{k,d+1})^{x+1,k}.
\]

Since the simulator knows the discrete logarithms \(c = \log_\hat{v}(\hat{w})\) and \(a_{i_1i_2} = \log_\hat{v}(\hat{h}_{i_1i_2})\), the remaining components of \(SK_{DL}\) and those of the partial decryption key can be generated efficiently in the same way as in the proof of indistinguishability of \(\text{Game}_{0,\kappa}\) and \(\text{Game}'_{0,\kappa}\). In particular, the simulator can compute

\[
K_{\ell+1} = \prod_{i_1=1}^{\ell} \prod_{i_2=1}^{\mu} L_{\ell+1,i_1}^{a_{i_1i_2}} \cdot L_{\ell+1,\ell+1}^{\sum_{k=1}^{\mu} a_{\ell+1,k}} \cdot L_{w,\ell+1}^{c}.
\]

To create \(K_{\ell+1}^{(j,k)}\), \(L_{\ell+1,i_1}^{(j,k)}\), \(L_{w,\ell+1}^{(j,k)}\), if \((j, k) \leq (\eta + \ell + 1, \nu)\) (in lexicographic order), the values are chosen as fresh random delegation keys. For the \((\eta + \ell + 1, \nu+1)\) partial decryption key, the simulator lets

\[
L_{\ell+1,i_1}^{(\eta+\ell+1,\nu+1)} = \prod_{k=1}^{\mu} (Q_{j,k,i_1}^{c})^{x+1,k} \quad i_1 \in \{1, \ldots, \ell\}
\]

\[
L_{\ell+1,\ell+1}^{(\eta+\ell+1,\nu+1)} = Q_{\ell+1}^{1}\n\]

\[
L_{w,\ell+1}^{(\eta+\ell+1,\nu+1)} = \prod_{k=1}^{\mu} (Q_{j,k,d+1}^{c})^{x+1,k}.
\]

Again, as the simulator knows the discrete logarithm of \(\hat{w}, \hat{h}_{i_1i_2}\) w.r.t. the base \(\hat{v}\), the remaining terms – including \(K_{\ell+1}^{(\eta+\ell+1,\nu+1)}\) – can be generated efficiently. For the remaining partial decryption keys, the simulator generates them faithfully using the Delegate algorithm.
• The remaining delegated key queries are generated faithfully.

Clearly, if \( Q' = Q \) in the GDDH instance, then the above simulation is identically distributed as Game'_{0,\kappa,\eta,\nu}, otherwise it is identically distributed as Game'_{0,\kappa,\eta,\nu+1}.

The simulator generates the challenge ciphertext as normal and sends it to the adversary. If the adversary has \( \epsilon \) difference in its advantage in Game'_{0,\kappa,\eta,\nu} and Game'_{0,\kappa,\eta,\nu+1}, it is not hard to see that the simulator has a comparable advantage in solving the GDDH instance. \( \square \)

**Lemma 5.** Game\(_1\) and Game\(_2\) are computationally indistinguishable if the BDH assumption holds in \((G, \hat{G}, G_T)\).

**Proof.** Assume that there’s an adversary \( \mathcal{A} \) that can distinguish between Game\(_1\) and Game\(_2\). We build an adversary \( \mathcal{B} \) that uses \( \mathcal{A} \) as a subroutine to break the BDH assumption. The simulator \( \mathcal{B} \) receives as input

\[
(g, g^a, g^c, \hat{g}, \hat{g}^a, \hat{g}^b, Q'),
\]

where \( Q' \) is either \( e(g, \hat{g})^{abc} \) or an element chosen uniformly at random in \( G_T \).

The adversary \( \mathcal{A} \) commits to two vectors \((\hat{Y}^0_1, \ldots, \hat{Y}^0_d)\) and \((\hat{Y}^1_1, \ldots, \hat{Y}^1_d)\). The challenger \( \mathcal{B} \) picks \( \beta \in \{0, 1\} \). If \( d^* < d \), the simulator picks at random \( d - d^* \) vectors \( Y^\beta_{d^*+1}, \ldots, Y^\beta_d \in \mathbb{Z}_p^\mu \). The simulator runs the Setup algorithm as usual except that it implicitly sets \( \alpha \) to be \( ab \) by defining \( e(g, \hat{v})^\alpha = e(g^a, \hat{g}^b)^{\alpha_w} \), and then, for \( i_1 = 1, \ldots, d \) and \( i_2 = 1, \ldots, \mu \), it defines \( h_{i_1,i_2} = (g^a)^{z_{i_1}y_{i_1,i_2}}g^{i_1,i_2} \) for \( i_1 \in \{1, \ldots, d\} \), \( i_2 \in \{1, \ldots, \mu\} \) and \( h_{i_1,0} = (g^a)^{-z_{i_1}} \), for some random exponents \( z_{i_1}, t_{i_1,i_2} \). The values of \( h_{i_1,i_2} \) for \( i_1 \in \{1, \ldots, d\} \), \( i_2 \in \{0, \ldots, \mu\} \) are defined similarly from the value \( \hat{g}^a \) of the BDH instance. Observe that \( \beta \) remains hidden from \( \mathcal{A} \) and that the parameters are correctly distributed.

For the secret key queries, in the last lemma we have just proven that delegated keys are indistinguishable from freshly generated ones. Therefore, \( \mathcal{B} \) will generate the secret keys using algorithm Keygen when a reveal query is made. Note that \( \alpha \) is defined as \( ab \), which is not known to \( \mathcal{B} \). However, \( \mathcal{B} \) needs to simulate secret keys for \((\hat{X}_1, \ldots, \hat{X}_\ell)\) as long as \( f(\hat{X}_1, \ldots, \hat{X}_\ell)(\hat{Y}^\beta_1, \ldots, \hat{Y}^\beta_{d^*}) = 0 \). As \( \alpha \) does not appear in the delegating component of the key, the delegation component of the secret keys \( SK_{DL} \) can be created using the parameters as usual. Therefore, from now on we focus on how to create the decryption component of the secret key. Denote by \( \ell' \) the index of the smallest element of the vector \((\hat{X}_1, \ldots, \hat{X}_\ell)\) for which \( \hat{X}_{\ell'} \cdot \hat{Y}^\beta_{d^*} \neq 0 \). This value \( \ell' \) always exists if \( \ell \leq d^* \) by hypothesis and exists with overwhelming probability if \( \ell > d^* \), since the vectors \( Y^\beta_{d^*+1}, \ldots, Y^\beta_d \in \mathbb{Z}_p^\mu \) are completely hidden from \( \mathcal{A} \)'s view.

The simulator creates the decryption component of the secret key as follows:

\[
D = \hat{g}^{ab} \prod_{i_1=1}^{\ell} \left( \prod_{i_2=1}^{\mu} h_{i_1,i_2}^{x_{i_1,i_2}} \right)^{r_{i_1}} \hat{w}^{r_w}
\]

where the term \( \hat{g}^{ab} \left( \prod_{i_2=1}^{\mu} h_{i_1,i_2}^{x_{i_1,i_2}} \right)^{r_{i_1}} \) is computed as

\[
\left( \prod_{i_2=1}^{\mu} h_{i_1,i_2}^{x_{i_1,i_2}} \right)^{r_{i_1}} = (g^a)^{-(\sum_{i_2=1}^{\mu} x_{i_1,i_2}t_{i_1,i_2})}/(z_{i_1}X_{i_1}Y^\beta_{i_1}),
\]

24
while the other terms in the product are just computed as usual. It is not hard to see that, if 
we define \( r' = r - b/(z \cdot X') \), then the computation is correct. All the other terms in the
decryption component can be computed efficiently, since the simulator knows all the parameters
needed and it also knows \( \hat{g}^h \).

At the challenge step, the adversary \( A \) gives \( B \) two messages, \( M_0 \) and \( M_1 \). \( B \) then computes
\[
C_0 = M_0 \cdot (Q')^\alpha, \quad C_v = (g^c)^\alpha, \quad C_w = (g^c)^\omega,
\]
\[
\{C_{i_1, j_2} = (g^c)^{\gamma_{i_1, j_2}}\}_{i_1 \in \{1, \ldots, d^*\}, \ j_2 \in \{1, \ldots, \mu\}}
\]
It is not hard to check that the challenge ciphertext is correctly distributed.

Finally, when the adversary \( A \) outputs a guess \( \beta' \), if \( \beta = \beta' \), then \( B \) guesses that \( Q' = e(g, \hat{g})^{abc} \)
and if \( \beta \neq \beta' \) guesses that \( Q' = R \). If \( A \) has advantage in distinguishing between the two cases,
then \( B \) also has \( \epsilon \) advantage in solving the assumption G3DH instance, except in the case that \( A \)
managed to output some secret key query for vector \((\vec{X}_1, \ldots, \vec{X}_\ell)\) for which \( \vec{X}_i \cdot \vec{Y}_i^\beta = 0 \) for all
\( i = d^* + 1, \ldots, d \) was zero for all \( i \in \{1, \ldots, d\} \), which occurs only with negligible probability.

**Lemma 6.** Game2 and Game3 are computationally indistinguishable if the \( \mathcal{P}-\text{BDH}_1 \) assumption holds in \((\mathbb{G}, \hat{\mathbb{G}})\).

**Proof.** Assuming that the adversary \( A \) outputs \( \beta' = \beta \) with noticeably different probabilities in
Game2 and Game3, we build an distinguisher \( B \) for the \( \mathcal{P}-\text{BDH}_1 \) assumption.

Namely, algorithm \( B \) receives as input a tuple
\[
(g, \ g^b, g^{ab}, g^c, \ \hat{g}, \ \hat{g}^a, \ \hat{g}^b, \ \hat{g}^c),
\]
where \( a, b, c \in \mathbb{Z}_p \). Its goal is to decide if \( z = abc \) or \( z \in \mathbb{P} \mathbb{Z}_p \).

To this end, \( B \) interacts with \( A \) as follows. It first receives the vectors \((\vec{Y}_0^0, \ldots, \vec{Y}_0^\mu), \ (\vec{Y}_1^0, \ldots, \vec{Y}_1^\mu)\),
at some depth \( d^* \leq d \), that \( A \) wishes to be challenged upon. We may assume w.l.o.g. that \( B \) chooses
its challenge bit \( \beta \in \{0, 1\} \) at the beginning of the game. Also, if \( d^* < d \), for each \( i \in \{d^*+1, \ldots, d\} \),
\( B \) defines the vector \( \vec{Y}_i^{\beta} \) as a random vector of \( \mathbb{Z}_p^\mu \).

It defines the master public key by setting \( e(g, \hat{v})^\alpha = e(g, \hat{g})^{\alpha' \cdot \gamma_0} \) and
\[
v = g^{\gamma_0}, \quad h_{i_1,0} = (g^b)^{\gamma_{i_1,0}} \quad i_1 \in \{1, \ldots, d\},
\]
\[
h_{i_1, i_2} = (g^b)^{-\gamma_{i_1, 0} \cdot y_{i_1, i_2}^b} \cdot g^{\gamma_{i_1, i_2}} \quad i_1 \in \{1, \ldots, d\}, \ i_2 \in \{1, \ldots, \mu\}
\]
\[
w = g^y \cdot (g^{ab})^x,
\]
where \( \alpha, \gamma_0, y, z \in \mathbb{Z}_p \) and \( \gamma_{j_1, j_2} \in \mathbb{Z}_p^\mu \), for each \( i_1 \in \{1, \ldots, d\}, \ i_2 \in \{0, \ldots, \mu\} \). For each \( i_1 \), we also
define the vector \( \vec{\gamma}_{i_1} = (\gamma_{i_1,1}, \ldots, \gamma_{i_1, \mu}) \in \mathbb{Z}_p^\mu \). We observe that \( B \) does not know \( \hat{v} \) (which depends
on the unavailable term \( \hat{g}^{ab} \)) but can compute \( \{h_{i_1, i_2}\}_{i_1 \in \{1, \ldots, d\}, \ i_2 \in \{1, \ldots, \mu\}} \).

When the adversary \( A \) requests a key for a hierarchical vector \( \vec{X} = (\vec{X}_1, \ldots, \vec{X}_\ell) \), \( B \) parses \( \vec{X}_{i_1} \)
as \( (x_{i_1,1}, \ldots, x_{i_1, \mu}) \in \mathbb{Z}_p^\mu \) for each \( i_1 \in \{1, \ldots, \ell\} \). Then, \( B \) responds as follows.

- If \( \ell \leq d^* \), let \( i \in \{1, \ldots, \ell\} \) be the smallest index such that \( \vec{X}_{i} \cdot \vec{Y}_{i}^{\beta} \neq 0 \) (by hypothesis, this index
must exist). By choosing \( r_w \in \mathbb{Z}_p \) and \( r_i \in \mathbb{Z}_p \), \( B \) implicitly defines the exponent
\[
\hat{r}_i = r_i + \frac{a \cdot r_w \cdot x}{\gamma_{i_0,0} \cdot \vec{X}_i \cdot \vec{Y}_{i}^{\beta}}
\]
and can compute
\[
(\prod_{i=1}^{\mu} \hat{h}_{x_{i,1},i}^{r_{1i}})^{r_{1i}} \cdot \hat{w}^{r_{w}} = \left( \prod_{i=1}^{\mu} \hat{h}_{x_{i,1},i}^{r_{1i}} \right)^{r_{1i}} \cdot \left( (\hat{g}^{ab})^{r_{w,x}} \cdot \hat{g}^{r_{w,y}} \right)
\]
(14)
without knowing \(\hat{g}^{ab}\). Similarly, it can compute
\[
D_i = \hat{v}^{r_{1i}} = \hat{g}^{r_{v}} \cdot \left( (\hat{g}^{a})^{r_{w,x}} \cdot (\hat{g}^{b})^{r_{w,y}} \right)
\]
as well as \(D_w = \hat{v}^{r_{w}}\).

We now turn to indices \(i_1 \in \{1, \ldots, \ell\} \setminus \{i\}\), for which \(B\) can trivially compute \(\left( \prod_{i=1}^{\mu} \hat{h}_{x_{i,1},i}^{r_{1i}} \right)^{r_{1i}}\) and \(D_{i_1} = \hat{v}^{r_{1i}} \cdot S_{v,11}\) since it knows \(\hat{h}_{i_1,i}^{r_{1i}}\) and \(v\). This suffices for computing the whole decryption component \(SK_D\) of the private key.

As for the delegation component \(SK_{DL}\), \(B\) can compute \(\{K_{j,k}\}_{j \in \{\ell+1, \ldots, d\}, k \in \{1, \ldots, \mu\}}\), \(\{L_j\}_{j=\ell+1}\) and \(\{L_{w,j,k}\}_{j,k}\) by applying exactly the same procedure as for \(D, D_1\) and \(D_w\) (and taking advantage of the fact that \(\hat{X}_i \cdot \hat{Y}_{i,1}^{\beta} \neq 0\) for at least one \(i \in \{1, \ldots, \ell\}\)). Remaining pieces of \(SK_{DL}\) are then trivially computable since \(B\) has \(\hat{h}_{i_1,i}^{r_{1i}}\) and \(v\) at disposal.

- If \(\ell > d^*\), it can be the case that \(\hat{X}_i \cdot \hat{Y}_{i,1}^{\beta} = 0\) for \(i = 1\) to \(\ell\). However, with overwhelming probability, there must exist \(i \in \{d^* + 1, \ldots, \ell\}\) such that \(\hat{X}_i \cdot \hat{Y}_{i,1}^{\beta} \neq 0\) since the vectors \(\hat{Y}_{d^*+1, \ldots, d}\) have been chosen at random and, due to the generation of the public key as per (13), they are completely independent of \(A\)'s view. It comes that \(B\) can generate a private key in the same way as in the case \(\ell < d^*\) (see equation (14)).

When \(B\) has to construct the challenge ciphertext, \(B\) sets \(C_0 = M \cdot e(\hat{g}^c, \hat{g}^{r_{v}})^{a_i}\), where \(M \in \mathbb{G}_T\), and
\[
C_v = (\hat{g}^{c})^{r_{v}}, \quad C_w = (\hat{g}^{c})^{r_{w,x}} \cdot (\hat{g}^{b})^{r_{w,y}},
\]
\[
C_{i_1,i_2} = (\hat{g}^{c})^{r_{1i_1,i_2}} \quad i_1 \in \{1, \ldots, d^*\}, \quad i_2 \in \{1, \ldots, \mu\}
\]
We observe that, if \(z = abc\), \(\{C_0, C_v, C_w, \{C_{i_1,i_2}\}_{i_1 \in \{1, \ldots, d^*\}, i_2 \in \{1, \ldots, \mu\}}\}\) corresponds to a valid ciphertext with the encryption exponent \(s = c\). In this situation, \(B\) is playing Game2 with \(A\).

In contrast, if \(z \in R \mathbb{Z}_p\), we have \(z \neq c\) with overwhelming probability. In this case, we have \(g^z = g^{abc+\theta}\), for some \(\theta \neq 0\), and we can thus write \(C_w = w^c \cdot g^{\theta \cdot x}\). This means that \(C_w\) looks uniformly random and independent from \(A\)'s view. Indeed, until the challenge phase, \(A\) has no information about \(\theta \in \mathbb{Z}_p\) (recall that public parameters do not depend on \(c\)) and the value \(x \in \mathbb{Z}_p\) is also independent of \(A\)'s view. We conclude that, if \(g^z\) is such that \(z \in R \mathbb{Z}_p\), we are in Game3.

**Lemma 7.** For each \(\delta_1 \in \{1, \ldots, d\}\) and each \(\delta_2 \in \{2, \ldots, \mu\}\), \(\text{Game}_{4,\delta_1,\delta_2-1}\) and \(\text{Game}_{4,\delta_1,\delta_2}\) are computationally indistinguishable if the \(P \cdot \text{BDH}_1\) assumption holds in \((\mathbb{G}, \hat{\mathbb{G}})\).

**Proof.** Towards a contradiction, we assume there exists \(\delta_1, \delta_2\) such that the adversary \(A\) outputs \(\beta' = \beta\) with significantly different probabilities in \(\text{Game}_{4,\delta_1,\delta_2-1}\) and \(\text{Game}_{4,\delta_1,\delta_2}\). We show that \(A\)
implies a distinguisher \( B \) against \( \mathcal{P}\text{-BDH}_1 \).

Our distinguisher \( B \) receives as input \((g, g^b, g^{ab}, g^a, \hat{g}, \hat{g}^b, \hat{g}^a, \hat{g}^b, \hat{g}^a)\), with \( a, b, c \in \mathbb{F}_p \). It aims to decide if \( z = abc \) or \( z \in R \mathbb{Z}_p \).

To do this, \( B \) runs the adversary \( A \) as follows. It first receives the challenge vectors \((\vec{Y}_0, \ldots, \vec{Y}_d)\), \((\vec{Y}_0^1, \ldots, \vec{Y}_d^1)\), at some depth \( d^* \leq d \), that are chosen by \( A \). We assume that \( B \) chooses its challenge bit \( \beta \in \{0, 1\} \) at the outset of the game. Also, if \( d^* < d \), for each \( i \in \{d^* + 1, \ldots, d\} \), \( B \) defines the vector \( \vec{Y}_i^\beta \) as a random vector of \( \mathbb{Z}_p^n \).

It defines the master public key by setting \( e(g, \hat{g})^\alpha = e(g, \hat{g})^{\alpha_{\gamma \gamma \gamma}} \) and

\[
v = g^{\gamma \gamma}, \\
h_{1,0} = g^{\gamma_{1,0}}, \\
h_{1,i_2} = g^{\gamma_{1,i_2}} \cdot g^{\gamma_{1,i_2}} \\
h_{\delta_1,\delta_2} = g^{\gamma_{\delta_1,\delta_2}} \cdot (g^{ab})^{\gamma_{\delta_1,\delta_2}}
\]

for \( i_1 \in \{1, \ldots, d\} \) or \( i_2 \in \{1, \ldots, \mu\} \) \( \backslash \{\delta_2\} \)

where \( \alpha, \gamma \gamma \gamma, y \in \mathbb{F}_p, x \in \mathbb{Z}_p \) and \( \gamma_{1,i_2} \in \mathbb{Z}_p \) for each \( i_1 \in \{1, \ldots, d\} \), \( i_2 \in \{0, \ldots, \mu\} \). For each \( i_1 \), we also define the vector \( \vec{\gamma}_i = (\gamma_{i,1}, \ldots, \gamma_{i,\mu}) \in \mathbb{Z}_p^\mu \). Note that, in the implicitly master secret key \( \text{msk} \), the distinguisher \( B \) knows all the components but \( h_{\delta_1,\delta_2} \), which depends on the unknown term \( \hat{g}^{ab} \).

When the adversary \( A \) requests a key for a hierarchical vector \( \vec{X} = (\vec{X}_1, \ldots, \vec{X}_\ell) \), \( B \) parses \( \vec{X}_{\delta_1} \) as \((\vec{x}_{i,1}, \ldots, \vec{x}_{i,\mu}) \in \mathbb{Z}_p^\mu \) for each \( i_1 \in \{1, \ldots, \ell\} \). Then, \( B \) responds as follows.

- If \( \ell \geq \delta_1 \), \( B \) chooses \( r'_{\ell} \in \mathbb{F}_p \) and \( r_{\delta_1} \in \mathbb{F}_p \), \( B \) implicitly defines the exponent

\[
r_w = r'_w - \frac{a \cdot r_{\delta_1} \cdot \gamma_{\delta_1,\delta_2}}{x}
\]

and can compute the product

\[
\left( h_{\delta_1,\delta_2}^{y_{\delta_1,\delta_2}} \cdot h_{\delta_1,\delta_2} \right)^{r_{\delta_1}} \cdot \hat{w}^r_w = (\hat{g}^{ab})^{\gamma_{\delta_1,\delta_2}} \cdot \hat{w}^r_w \cdot (\hat{g}^y)^{\gamma_{\delta_1,\delta_2}} \cdot (\hat{g}^{ab})^{\gamma_{\delta_1,\delta_2}}/x
\]

which is the only factor of \( D \) that it cannot trivially compute without knowing \( \hat{g}^{ab} \). Similarly, it can compute \( D_w = r_w = \hat{g}^{\alpha} \cdot r_{\delta_1} \cdot \gamma_{\delta_1,\delta_2}/x \).

To generate the delegation component \( SK_{DL} \) of the key, the reduction \( B \) is able to compute \( \{K_{j,k}, L_{w,j,k}\}_{j \in \{1, \ldots, d\}, k \in \{1, \ldots, \mu\}} \) by repeating \( (d - \ell) \cdot \mu \) times the same procedure as for computing \( D \) and \( D_w \).

- If \( \ell < \delta_1 \), \( B \) can directly compute \( (D, D_w, \{D_i\})_{i=1}^{\ell} \) since it knows \( \{h_{i_1,i_2}\}_{i_1 \in \{1, \ldots, \ell\}, i_2 \in \{1, \ldots, \mu\}} \). \( \hat{w}^{\alpha} \) and \( \hat{w}^{\beta} \). The difficulty is to compute the delegation components \( \{K_{\delta_1,k}\}_{k=1}^{\mu} \) without knowing \( \hat{g}^{ab} \). In fact, among these components, the only factor of \( K_{\delta_1,\delta_2} \) that \( B \) cannot trivially compute is \( h_{\delta_1,\delta_2} \).

However, similarly to (16)-(17), it can choose \( s_{\delta_1} = s'_{\delta_1,\delta_2} \in \mathbb{F}_p \), define \( s_{w,\delta_1,\delta_2} = s'_{w,\delta_1,\delta_2} - \frac{\alpha \cdot r_{\delta_1} \cdot \gamma_{\delta_1,\delta_2}}{x} \), and compute the product

\[
h_{\delta_1,\delta_2}^{s_{\delta_1}} \cdot \hat{w}^{s_{w,\delta_1,\delta_2}} = (\hat{g}^{\gamma_{\delta_1,\delta_2}} \cdot (\hat{g}^{ab})^{s_{\delta_1}} \cdot (\hat{g}^y)^{\gamma_{\delta_1,\delta_2}} \cdot (\hat{g}^{ab})^{\gamma_{\delta_1,\delta_2}}/x \cdot \hat{w}^{s'_{w,\delta_1,\delta_2}}
\]

(18)
In the same way, \( \mathcal{B} \) computes
\[
L_{w;\delta_1,\delta_2} = \hat{v}_{w;\delta_1,\delta_2} = \hat{g}_{\gamma_{v,\delta_1,\delta_2}} \cdot (\hat{g}_a)^{-\gamma_{v,\delta_1,\delta_2}/x}.
\]

Note that, for each \( k \in \{1, \ldots, \mu\} \setminus \{\delta_2\} \), \( \mathcal{B} \) has to generate \( K_{\delta_1,k} \) by computing \( \hat{s}_{\delta_1,k} \) using the same random exponent \( s_{\delta_1} \) as in (18). This is always possible since \( \mathcal{B} \) knows that exponent.

When it comes to construct the challenge ciphertext, algorithm \( \mathcal{B} \) first sets \( C_0 = M \cdot (g^r \cdot \hat{g}^{\gamma_v})^\alpha \), where \( M \overset{\$}{\leftarrow} \mathbb{G}_T \). It also chooses \( \Gamma_w \overset{\$}{\leftarrow} \mathbb{G} \) and computes
\[
C_v = (g^c)^{\gamma_v}, \quad C_w = \Gamma_w,
\]
as well as
\[
C_{i_1,i_2} = (g^c)^{\gamma_{i_1,i_2}} \quad \text{if} \quad (i_1 > \delta_1) \lor ((i_1 = \delta_1) \land (i_2 > \delta_2))
\]
\[
C_{\delta_1,\delta_2} = (g^x)^{\gamma_{\delta_1,\delta_2}}.
\]

If \( i_1 < \delta_1 \) or \( i_1 = \delta_1 \) and \( i_2 < \delta_2 \), then \( C_{i_1,i_2} \) is chosen uniformly in \( \mathbb{G} \).

We observe that, in the situation where \( z = abc \), \( (C_0, C_v, C_w, \{C_{i_1,i_2}\}_{i_1 \in \{1, \ldots, d\}, i_2 \in \{1, \ldots, \mu\}}) \) is distributed in the same way as in Game\( _4\delta_1,\delta_2 \). We also observe that, in the situation where \( z \in R \mathbb{Z}_p \), we have \( z \neq c \) with overwhelming probability. In this case, \( C_{\delta_1,\delta_2} \) looks random to the adversary and \( \mathcal{B} \) is thus playing Game\( _4\delta_1,\delta_2 \).

\[\square\]

**B Deferred Proofs for the Security of HF**

**B.1 Proof of Lemma 1**

We consider a sequence of \( n+1 \) hybrid experiments \( RL_0, \ldots, RL_n \). For each \( k \in \{0, \ldots, n\} \), \( RL_k \) is defined to be an experiment where public parameters are generated as follows. First, the simulator \( \mathcal{B} \) chooses \( u \overset{\$}{\leftarrow} \mathbb{G} \), \( w \overset{\$}{\leftarrow} \mathbb{Z}_p^n \), \( \mathbf{C} \overset{\$}{\leftarrow} \mathbb{G}_T^d \), \( \mathbf{h} \overset{\$}{\leftarrow} \mathbb{G}_T^d \times (\mu+1) \times n \), \( \mathbf{g} \overset{\$}{\leftarrow} \mathbb{G}_T^d \times (\mu+1) \times n \), and computes \( \mathbf{PP}_{\text{core}} \) in the same way as in the real scheme.

In the second step of the setup procedure, the simulator \( \mathcal{B} \) chooses a vector \( \mathbf{s} \overset{\$}{\leftarrow} (\mathbb{Z}_p^n) \). For \( l_1, l_2 \in \{1, \ldots, n\} \), it first computes
\[
J[l_2] = u^{s[l_2]}, \quad C_{w[l_2,l_1]} = w[l_1]^{s[l_2]}.
\]

Then, for each pair \( (l_1, l_2) \) such that \( l_1 \neq l_2 \), \( \mathcal{B} \) sets
\[
C[l_1,l_2,l_1] = h[l_1,l_2,l_1]^{s[l_2]}.
\]

Finally, for each \( l \in \{1, \ldots, n\} \), \( i_1 \in \{1, \ldots, d\} \) and \( i_2 \in \{1, \ldots, \mu\} \), \( \mathcal{B} \) defines
\[
\mathbf{C}[l_1,l_2,l] = (h[l_1,l_2,l]^{\gamma_{l_1}^{(0)}[1]} \cdot h[l_1,l_2,l])^{s[l]} \quad \text{if} \quad l \leq k,
\]
\[
\mathbf{C}[l_1,l_2,l] = (h[l_1,l_2,l]^{\gamma_{l_1}^{(1)}[1]} \cdot h[l_1,l_2,l])^{s[l]} \quad \text{if} \quad l > k.
\]

Lemma 8 below demonstrates that, for each \( k \in \{1, \ldots, n\} \), experiment \( RL_k \) is computationally indistinguishable from experiment \( RL_{k-1} \).

If we assume that the statement of Lemma 1 is false, there must exist \( k \in \{1, \ldots, n\} \) such that the adversary can distinguish \( RL_k \) from \( RL_{k-1} \) and we obtain a contradiction. \[\square\]
Lemma 8. If the HPE scheme described in Section 3.2 is selectively weakly attribute-hiding, then Game $RL_k$ is indistinguishable from Game $RL_{k-1}$ for each $k \in \{1, \ldots, n\}$. Namely, for each $k$, there exist algorithms $B_1$ and $B_2$ such that

$$|\Pr[RL_k \Rightarrow 1] - \Pr[RL_{k-1} \Rightarrow 1]| \leq (d \cdot \mu + 1) \cdot \text{Adv}^{P} \cdot \text{BDH}_1(B_1) + q \cdot \text{Adv}^{P} \cdot \text{BDH}_2(B_2),$$

where $q$ is the number of “Reveal-key” queries made by $A$.

Proof. For the sake of contradiction, let us assume that there exist two auxiliary hierarchical vector $y^{(0)} = [y^{(0)}_1 \ldots y^{(0)}_d]$, $y^{(1)} = [y^{(1)}_1 \ldots y^{(1)}_d]$ and an index $k \in \{1, \ldots, n\}$ such that the adversary $A$ has noticeably different behaviors in experiments $RL_k$ and $RL_{k-1}$. Using $A$, we construct a selective weakly attribute-hiding adversary $B$ against the HPE scheme described in Section 3.2 (in its predicate-only variant).

Our adversary $B$ first declares $y^{(0)}, y^{(1)} \in \mathbb{Z}_p^{d \times \mu}$ as the vectors that it wishes to be challenged upon. Then, the HPE challenger provides $B$ with public parameters

$$\text{mpk}_{\text{HPE}} = (v, w, \{h_{i_1,i_2}\}_{i_1 \in \{1,\ldots,d\}, i_2 \in \{0,\ldots,\mu\}}).$$

Then, $B$ chooses a vector $\zeta \overset{R}{\leftarrow} \mathbb{Z}_p^n$ and a matrix $\gamma \in \mathbb{Z}_p^{d \times (\mu+1) \times n}$, which it uses to compute

$$\begin{align*}
w[l_1] &= v^{\zeta[l_1]} \quad \text{for} \quad l_1 \in \{1,\ldots,n\} \setminus \{k\} \\
h[i_1,i_2,l_1] &= v^{\gamma[i_1,i_2,l_1]} \quad \text{for} \quad i_1 \in \{1,\ldots,d\}, \ i_2 \in \{0,\ldots,\mu\}, \ l_1 \in \{1,\ldots,n\} \setminus \{k\}.
\end{align*}$$

It also sets $w[k] = w$ as well as

$$h[i_1,i_2,k] = h_{i_1,i_2} \quad \text{for} \quad i_1 \in \{1,\ldots,d\}, \ i_2 \in \{0,\ldots,\mu\}.$$ 

Then, $B$ defines core public parameters

$$\text{pp}_{\text{core}} = (v, \{w[l_1]\}, \{h[i_1,i_2,l_1]\}_{i_1 \in \{1,\ldots,d\}, i_2 \in \{0,\ldots,\mu\}, \ l_1 \in \{1,\ldots,n\}});$$

that correspond to the master secret key $\text{msk} = (v, w, h)$, which is not completely known to $B$ (specifically, $v, w[k]$ and $h[..,k]$ are not available). Then, $B$ notifies its HPE challenger that it wishes to directly enter the challenge phase without making any pre-challenge query. The challenger replies with the challenge ciphertext

$$C^* = (C_v, C_w, \{C_{i_1,i_2}\}_{i_1 \in \{1,\ldots,d\}, i_2 \in \{0,\ldots,\mu\}}),$$

where

$$\begin{align*}
C_v &= v^s, \\
C_w &= w^s, \\
\{C_{i_1,i_2}\} &= \{h_{i_1,i_2}^{(\beta)} \cdot C_{i_1,i_2}\}^{(\beta)} \quad \text{for} \quad i_1 \in \{1,\ldots,d\}, \ i_2 \in \{0,\ldots,\mu\}, \ \beta \in \{0,1\}.
\end{align*}$$

for a random element $s \overset{R}{\leftarrow} \mathbb{Z}_p^n$ and a random bit $\beta \in \{0,1\}$. Here we are using, for $\beta \in \{0,1\}$, the notation $y^{(\beta)} = [y^{(\beta)}_1 \ldots y^{(\beta)}_d] \in \mathbb{Z}_p^{d \times \mu}$, where each $y^{(\beta)}_i = (y^{(\beta)}_{i,1}, \ldots, y^{(\beta)}_{i,d}) \in \mathbb{Z}_p^{d}$, for $i_1 \in \{1,\ldots,d\}$. At this point, $B$ constructs the matrix $\{CT[i_1,i_2]\}_{i_1,i_2 \in \{1,\ldots,n\}}$ of HPE ciphertexts by setting

$$\begin{align*}
J[k] &= C_v \\
C_{w}[k,k] &= C_w \\
C[i_1,i_2,k] &= C_{i_1,i_2} \quad \text{for} \quad i_1 \in \{1,\ldots,d\}, \ i_2 \in \{1,\ldots,\mu\}.
\end{align*}$$

29
and, for each \( l_1 \in \{1, \ldots, n \}\setminus\{k\} \),
\[
C_w[k, l_1] = C_\zeta[l_1]
\]
\[
C[i_1, i_2, k, l_1] = C_\gamma[i_1, i_2, k, l_1] \quad \text{for } i_1 \in \{1, \ldots, d\}, \ i_2 \in \{1, \ldots, \mu\}.
\]

Note that this implicitly sets \( s[k] = s \), where \( s \) is the encryption exponent chosen by the HPE challenger to compute \( C^* \). Then, for each \( l_2 \in \{1, \ldots, n\}\setminus\{k\} \), \( \mathcal{B} \) chooses a random exponent \( s[l_2] \leftarrow \mathbb{Z}_p^* \) and computes
\[
\begin{align*}
J[l_2] & = s[l_2] \\
C_w[l_2, l_1] & = w[l_1]^{s[l_2]} \quad \text{for } l_1 \in \{1, \ldots, n\}\setminus\{l_2\} \\
C[i_1, i_2, l_2, l_1] & = h[i_1, i_2, l_1]^{s[l_2]} \quad \text{for } i_1 \in \{1, \ldots, d\}, \ i_2 \in \{1, \ldots, \mu\}, \ l_1 \in \{1, \ldots, n\}\setminus\{l_2\}
\end{align*}
\]

As for entries of the form \( \{C[i_1, i_2, l, l]\}_{i_1, i_2, l \neq k} \), \( \mathcal{B} \) computes them as
\[
\begin{align*}
C[i_1, i_2, l] & = (h[i_1, 0, l]^{\gamma(i_1)}[i_2] \cdot h[i_1, i_2, l])^{s[l]} \quad \text{if } l < k \\
C[i_1, i_2, l] & = (h[i_1, 0, l]^{\gamma(i_2)}[i_1] \cdot h[i_1, i_2, l])^{s[l]} \quad \text{if } l > k.
\end{align*}
\]

using the exponents \( s[l] \in \mathbb{Z}_p^* \) that were chosen in (19). Finally, our adversary \( \mathcal{B} \) defines the \( n \times n \) matrix \( \{CT[l_2, l_1]\}_{l_2, l_1 \in \{1, \ldots, n\}} \) of HPE ciphertexts
\[
CT[l_2, l_1] = (J[l_2], C_w[l_2, l_1], \{C[i_1, i_2, l_2, l_1]\}_{i_1 \in \{1, \ldots, d\}, i_2 \in \{1, \ldots, \mu\}}).
\]

Finally, \( \mathcal{B} \) defines \( mpk := (PP_{core}, \{CT[l_2, l_1]\}_{l_2, l_1 \in \{1, \ldots, n\}}) \) and sends it to the adversary \( \mathcal{A} \).

When it comes to answer \( \mathcal{A} \)’s private key queries for hierarchical identities \( (id_1, \ldots, id_\ell) \), \( \mathcal{B} \) first encodes each level’s identity \( id_i \in \{0, 1\}^\mu \) as a \( \mu \)-vector \( \tilde{X}_{id_i} = (id_i[1], \ldots, id_i[\mu]) \) for each \( i \in \{1, \ldots, \ell\} \). Although \( \mathcal{B} \) does not entirely know \( msk \), the decryption components \( \{D[l_1], D_w[l_1], D_{\gamma(l_1)}\} \) of the private key are always directly computable when \( l_1 \neq k \): namely, \( \mathcal{B} \) chooses \( D_w[l_1] \leftarrow \mathcal{G} \) and \( D_{\gamma(l_1)} \leftarrow \mathcal{G} \), for \( i_1 = 1 \) to \( \ell \), and computes
\[
D[l_1] = \prod_{i_1=1}^\ell D_{\gamma(l_1)}[l_1]^{\sum_{i_2=1}^\mu \gamma(i_1, i_2, l_1)id_i[i_2]} \cdot D_w[l_1]^{\zeta(l_1)}.
\]

It is easy to see that \( \{D[l_1], D_w, \{D_{\gamma(l_1)}\}_{l_1=1}^\ell\} \) forms a decryption component of the form (7). Moreover, the delegation components can be obtained exactly in the same way.

As for the remaining coordinate \( l_1 = k \), the simulator \( \mathcal{B} \) aborts if, for any \( \tilde{\gamma} \in \{0, 1\} \), the obtained hierarchical vector \( \langle \tilde{X}_1, \ldots, \tilde{X}_\ell \rangle \) is one for which \( \langle y^{(\tilde{\gamma})}_i, X_{\gamma_i} \rangle = 0 \) for each \( i \in \{1, \ldots, \ell\} \) (which translates into
\[
f(\tilde{X}_1, \ldots, \tilde{X}_\ell)(y^{(\tilde{\gamma})}_1, \ldots, y^{(\tilde{\gamma})}_d) = 1
\]
in the predicate encryption language). Otherwise, \( \mathcal{B} \) can obtain the missing private key components by invoking its HPE challenger to obtain a complete private key \( SK_{\langle id_1, \ldots, id_\ell \rangle} = (SK_D, SK_{DL}) \).

It is easy to check that, if the HPE challenger’s bit is \( \beta = 0 \), \( mpk \) is distributed as in Game \( RL_k \). In contrast, if \( \beta = 1 \), \( mpk \) has the same distribution as in Game \( RL_{k-1} \).  \qed
B.2 Selective Security of HF

We consider our HF with $\mu = 2$, $\Sigma_{ID} = \{(1,x) : x \in \mathbb{Z}_p^*\}$ and $\text{IdSp} = (\Sigma_{ID})^{(\leq d)}$. Let $(id^{*},\ldots,id^*) \in \text{IdSp}$ be the identity chosen by the adversary. Define the sibling $LHF$ with $\text{AuxSp} = \mathbb{Z}_p^{2d}$ and auxiliary input $y^{(1)} = [y^{(1)}_1\ldots y^{(1)}_d] \in \mathbb{Z}_p^{2d}$ where $y^{(1)}_i = (-id^*_i,1)$ for all $i_1 \in \{1,\ldots,\ell\}$ and $y^{(1)}_i = (1,0)$ for $i_1 \in \{\ell^*+1,\ldots,d\}$. Selective security is established in the following theorem. In this selective setting, we will omit $\zeta$ in the notation, because the trivial pre-output stage that always outputs $d_2 = 1$ will be enough in this case and, as a consequence, the experiments do not depend on $\zeta$.

**Theorem 4.** Let $n > \log p$ and let $\omega = n - \log p$. Let $HF$ be the HIB-TDF with parameters $n,d,\mu = 2$, $\Sigma_{ID} = \{(1,x) : x \in \mathbb{Z}_p^*\}$ and $\text{IdSp} = (\Sigma_{ID})^{(\leq d)}$. Let $LHF$ be the sibling associated with it as above. Then $HF$ is $(\omega,1)$-partially lossy against selective-id adversaries.

Specifically regarding condition (i), for any selective-id adversary $A$ there exist algorithms $B_1$ and $B_2$ such that

$$\text{Adv}_{\text{HF,LHF,}\omega}(A) \leq n \cdot ((2d+1) \cdot \text{Adv}_{P,\text{BDH}}(B_1) + q \cdot \text{Adv}_{P,\text{DDH}}(B_2))$$

The running time of $B_1$ and $B_2$ are comparable to the running time of $A$.

**Proof.** Let $RL_0$ and $RL_n$ be the games specified above. We claim that both

$$\Pr[d^A_{\text{exp}} = 1 \mid \text{REAL}_{\text{HF,LHF,}\omega}^A] = \Pr[RL_0 \Rightarrow 1]$$

and

$$\Pr[d^A_{\text{exp}} = 1 \mid \text{LOSSY}_{\text{HF,LHF,}\omega}^A] - \Pr[RL_n \Rightarrow 1] \in \text{negl}(\omega),$$

which implies condition (i) of partial lossiness, when combined with Lemma 1.

To prove that, we just need to justify that, when using $y^{(0)}$ as the auxiliary input, the function $HF.Eval$ is injective for all identities $id$. On the other hand, when using $y^{(1)}$, the function $HF.Eval$ will be lossy for identities $id = (id_1,\ldots, id_\ell)$ such that $\langle y^{(1)}_i, id_i \rangle = 0$ for all $i_1 \in \{1,\ldots,\ell\}$ and injective for all other identities.

To see that, note that all the terms of the output of $HF.Eval$ are determined by $\langle s, X \rangle$ except the term $CT_{id}[i_1, i_1]$, which is also determined by $s[i_1] \cdot x$, $y^{(1)}_i$. Therefore, when $\langle y^{(1)}_i, id_i \rangle = 0$ for all $i_1 \in \{1,\ldots,\ell\}$, the function will be determined only by $\langle s, X \rangle$, which can take only $p$ values, so $HF.Eval$ will be lossy with lossiness $\lambda$ ($HF.Eval(pms, mpk, id_1,\ldots, id_\ell) \rangle = n - \log p = \omega$. On the other hand, if there is at least one index $i_1$ for which $\langle y^{(1)}_i, id_i \rangle \neq 0$, then $HF.Eval$ will be injective with overwhelming probability and $HF.Inv$ will be its inverse, also with overwhelming probability. Finally, observe that when the auxiliary input is $y^{(0)}$, then for all identities $(id_1,\ldots, id_\ell)$ and for all $i_1 \in \{1,\ldots,\ell\}$, $\langle y^{(0)}_i, id_i \rangle \equiv 1 \pmod{1}$ by construction.

With this, we conclude that $y^{(0)}$ makes $HF.Eval$ injective for all identities and $y^{(1)}$ makes $HF.Eval$ lossy only for identities such that $\langle y^{(1)}_i, id_i \rangle = 0$ for all $i_1 \in \{1,\ldots,\ell\}$, and in this case the lossiness is $n - \log p$ (where when we claim that $HF.Eval$ is injective, it holds with overwhelming probability). This concludes the proof of condition (i).

Regarding conditions (ii) and (iii), when the auxiliary input is $y^{(1)}$, then $HF.Eval$ is injective for all valid queried id in “Reveal key” queries and $HF.Eval$ is lossy for id* = $(id_1^*,\ldots, id^*_\ell)$. Indeed, we are setting $y^{(1)}_i = (-id^*_i,1)$ for all $i_1 \in \{1,\ldots,\ell\}$ and $y^{(1)}_i = (1,0)$ for $i_1 \in \{\ell^*+1,\ldots,d\}$. This implies that if id = $(id_1,\ldots, id_\ell)$ is an identity, then $\langle y^{(1)}_i, id_i \rangle = 0$ for all $i_1 \in \{1,\ldots,\ell\}$ if and only if id is a prefix of id*. As the adversary is not allowed to make “Reveal key” queries for
prefixes of \( id^* \), then the condition is satisfied. This implies that \( d_1 = 1 \) with overwhelming probability. Also, the pre-output stage \( \mathcal{P} \) in this case simply outputs \( d_2 = 1 \), always. This implies that \( d_{\text{abort}}^- \) is 1 with overwhelming probability and, as a consequence, \( \Pr[d_{\text{abort}}^- = 1 \mid \text{REAL}_A^-_{HF, \omega, \zeta}] \) is negligibly close to 1 and \( (\Pr[d_A = 1 \mid \text{REAL}_A^+_{HF, \omega, \zeta} \wedge d_{\text{abort}}^- = 1] - \frac{1}{2}) \) is negligibly close to \( \Pr[d_A = 1 \mid \text{REAL}_A^+_{HF, \omega, \zeta}] - \frac{1}{2} \). These are precisely the requirements for \( HF \) to fulfill conditions (ii) and (iii), with \( \delta = 1 \).

### B.3 Adaptive Security: Proof of Lemma 2

Fix the view of the adversary \( A \), which implies fixing the queried identities \( id^{(1)}, \ldots, id^{(q)}, id^* \). Although we are assuming that the adversary \( A \) makes the maximum number of queries, with a smaller number of queries we would have the same bounds. We abbreviate \( \eta = \eta(IS, id^*), y = y^{(1)} \) and also call \( \ell^* \) the depth of the challenge identity \( id^* \). For an integer \( t \), define the event

\[
E_t : \bigwedge_{i=1}^{\ell^*} \left( \bigvee_{i_1=1}^{\ell^{(i)}} (\langle id_{i_1}^{(i)} \rangle, y_{i_1} \rangle \neq 0 \mod t) \right) \wedge \bigwedge_{i_1=1}^{\ell^*} (\langle id_{i_1}^*, y_{i_1} \rangle = 0 \mod t)
\]

We denote \( Y = \{y_1', \ldots, y_d', y_1, \ldots, y_d, \xi_1, \ldots, \xi_d \} \). For all \( i_1 \in \{1, \ldots, \ell^*\} \), we have

\[
\langle id_{i_1}, y_{i_1} \rangle = y_{i_1}' + \sum_{k=2}^{\mu} y_{i_1}[k]id_{i_1}[k] - 2\xi_{i_1}q
\]

for some \( 0 \leq \xi_{i_1} \leq \mu - 1 \). In particular, observe that

\[
0 \leq y_{i_1}' + \sum_{k=2}^{\mu} y_{i_1}[k]id_{i_1}[k] < 2q\mu < p.
\]

Let us define the value \( \xi_{i_1}^* := \lfloor (y_{i_1}' + \sum_{k=2}^{\mu} y_{i_1}[k]id_{i_1}^*[k])/2q \rfloor \). If we have the two conditions

\[
\xi_{i_1} = \xi_{i_1}^*, \quad \langle id_{i_1}^*, y_{i_1} \rangle = 0 \mod p,
\]

then clearly \( \langle id_{i_1}^*, y_{i_1} \rangle = 0 \mod p \). Also, if \( \langle id_{i_1}, y_{i_1} \rangle \neq 0 \mod 2q \), then we also have \( \langle id_{i_1}, y_{i_1} \rangle \neq 0 \mod p \). Using these observations, we have

\[
\eta \geq \Pr[\xi_{i_1} = \xi_{i_1}^*, \forall i_1 \in \{1, \ldots, \ell^*\}] \Pr[Y_p \mid \xi_{i_1} = \xi_{i_1}^*, \forall i_1 \in \{1, \ldots, \ell^*\}]
\]

\[
= \frac{1}{\mu^{\ell^*}} \Pr[Y_p \mid \xi_{i_1} = \xi_{i_1}^*, \forall i_1 \in \{1, \ldots, \ell^*\}]
\]

\[
\geq \frac{1}{\mu^{\ell^*}} \Pr[Y_{2q} \mid \xi_{i_1} = \xi_{i_1}^*, \forall i_1 \in \{1, \ldots, \ell^*\}]
\]

\[
= \frac{1}{\mu^{\ell^*}} \Pr[Y_{2q} \mid \xi_{i_1} = \xi_{i_1}^*, \forall i_1 \in \{1, \ldots, \ell^*\}]
\]

where \( Y' \) contains \( \{y_1', \ldots, y_d'\} \) and \( y_{i_1}' = (y_{i_1}', y_{i_1}[2], \ldots, y_{i_1}[\mu]) \) for all \( i \in \{1, \ldots, d\} \). Note that the second inequality above holds because of the condition \( \xi_{i_1} = \xi_{i_1}^* \). If we now define
\( \eta_{2q} = \Pr_{Y'}[E_{2q} | \xi_{i_1} = \xi_{i_1}^* \ \forall i_1 \in \{1, \ldots, \ell^*\}] \), we just showed that \( \eta \geq \frac{1}{\mu_{\ell^*}} \cdot \eta_{2q} \).

Now, we observe some facts about \( \langle id_{i_1}, y_{i_1} \rangle \). First, observe that \( \langle id_{i_1}, y_{i_1} \rangle \) and \( \langle id_{i_1}', y_{i_1}' \rangle \) are independent for \( i_1 \neq i_1' \). This is because of the way \( Y \) is chosen.

Also, note that for any \( id_{i_1}, a \in \mathbb{Z} \), \( \Pr_Y[\langle id_{i_1}, y_{i_1} \rangle = a \mod 2q] = 1/2q \). This is because for any choice of \( y_{i_1}[2], \ldots, y_{i_1}[\mu] \), there is only one value of \( y_{i_1}' \) for which the equality holds.

Consider \( id = (id_1, \ldots, id_\ell) \neq id' = (id'_1, \ldots, id'_\ell) \) and \( id \) not being a prefix of \( id' \) and \( a, b \in \mathbb{Z} \). First, if \( \ell > \ell' \), for each \( i_1 \in \{\ell' + 1, \ldots, \ell\} \) such that \( id_{i_1} \neq id_{i_1}' \), we have

\[
\Pr_Y[\langle id_{i_1}, y_{i_1} \rangle = a \mod 2q \mid \bigwedge_{k=1}^{\ell'} \langle id_k', y_k \rangle = b \mod 2q] = \Pr_Y[\langle id_{i_1}, y_{i_1} \rangle = a \mod 2q] = 1/2q.
\]

This happens because \( \bigwedge_{k=1}^{\ell'} \langle id_k', y_k \rangle = b \mod 2q \) does not impose any condition on \( \langle id_{i_1}, y_{i_1} \rangle \) and we can apply the same arguments as previously.

On the other hand, for all \( i_1 \leq \ell' \), \( \Pr_Y[\langle id_{i_1}, y_{i_1} \rangle = a \mod 2q \mid \bigwedge_{k=1}^{\ell'} \langle id_k', y_k \rangle = b \mod 2q] \) is either 0, if \( id_{i_1} = id_{i_1}' \) or 1/2q, if \( id_{i_1} \neq id_{i_1}' \). The second fact is because, if \( id_{i_1} \neq id_{i_1}' \), there exists an index \( j \) for which \( id_{i_1}[j] = 1 \) and \( id_{i_1}'[j] = 0 \) or the other way around. We see that, if we fix all \( y_{i_1}[i] \) for \( i \neq j \) so that \( \langle id_{i_1}', y_{i_1} \rangle = b \mod 2q \), then there is only one value for \( y_{i_1}[j] \) so that \( \langle id_{i_1}, y_{i_1} \rangle = a \mod 2q \).

With all these observations, we calculate the following bound on \( \eta_{2q} \):

\[
\eta_{2q} = \Pr_Y \left[ \bigwedge_{i_1=1}^{\ell} \left( \bigvee_{i_1=1}^{q} \langle id_{i_1}, y_{i_1} \rangle \neq 0 \mod 2q \right) \right] \bigwedge_{i_1=1}^{\ell^*} \langle id_{i_1}^*, y_{i_1} \rangle = 0 \mod 2q \right] \).
\]

\[
= 1/(2q)^{\ell^*} \Pr_Y \left[ \bigwedge_{i_1=1}^{\ell} \left( \bigvee_{i_1=1}^{q} \langle id_{i_1}, y_{i_1} \rangle \neq 0 \mod 2q \right) \right] \bigwedge_{i_1=1}^{\ell^*} \langle id_{i_1}^*, y_{i_1} \rangle = 0 \mod 2q \right] \]
\]

\[
= 1/(2q)^{\ell^*} \left( 1 - \Pr_Y \left[ \bigwedge_{i_1=1}^{\ell} \left( \bigvee_{i_1=1}^{q} \langle id_{i_1}, y_{i_1} \rangle = 0 \mod 2q \right) \right] \right) \bigwedge_{i_1=1}^{\ell^*} \langle id_{i_1}^*, y_{i_1} \rangle = 0 \mod 2q \right] \]
\]

\[
\geq 1/(2q)^{\ell^*} \left( 1 - \sum_{i=1}^{\ell^*} \Pr_Y \left[ \bigwedge_{i_1=1}^{\ell} \langle id_{i_1}, y_{i_1} \rangle = 0 \mod 2q \right] \right) \bigwedge_{i_1=1}^{\ell^*} \langle id_{i_1}^*, y_{i_1} \rangle = 0 \mod 2q \right] \].

We now focus on how to bound

\[
\Pr_Y \left[ \bigwedge_{i_1=1}^{\ell} \langle id_{i_1}, y_{i_1} \rangle = 0 \mod 2q \right] \bigwedge_{i_1=1}^{\ell^*} \langle id_{i_1}, y_{i_1} \rangle = 0 \mod 2q \right] \].
First, observe that $\langle id_{i_1}, y_{i_1} \rangle = 0 \mod 2q$ is independent to $\langle id_{i_1}, y_{i_1}' \rangle = 0 \mod 2q$ if $i_1 \neq i_1'$. As a consequence, we only need to compute $\Pr_{Y'}\left[\langle id_{i_1}, y_{i_1} \rangle = 0 \mod 2q \right]$. To this end, we consider two cases: that $\ell^{(i)} > \ell^*$ or $\ell^{(i)} \leq \ell^*$. In the first case, for each $i_1 \in \{\ell^* + 1, \ldots, \ell^{(i)}\}$ such that

$$\Pr_{Y'}\left[\langle id_{i_1}, y_{i_1} \rangle = 0 \mod 2q \right] = \frac{1}{2q}.$$ 

For all indices $i_1 \in \ell^*$, the same probability is either 1 or $1/(2q)$.

If $\ell^{(i)} \leq \ell^*$, for each $i_1 \in \{1, \ldots, \ell^{(i)}\}$, we have

$$\Pr_{Y'}\left[\langle id_{i_1}, y_{i_1} \rangle = 0 \mod 2q \right] = \frac{1}{2q}$$

which is 1 if $id_{i_1}^{(i)} = id_{i_1}^*$ and $1/(2q)$ otherwise due to the fact stated above.

Define $\chi_1^{(i)} = \max(\ell^{(i)} - \ell^*, 0)$ and $\chi_2^{(i)} = \#\{1 \leq i_1 \leq \min(\ell^*, \ell^{(i)})|id_{i_1}^{(i)} \neq id_{i_1}^*\}$. Note that, by the restrictions imposed on $id^{(i)}$, we have $\chi_1^{(i)} + \chi_2^{(i)} \geq 1$ for all $i \in \{1, \ldots, q\}$. Putting it all together, we find

$$\Pr_{Y'}\left[\bigwedge_{i_1=1}^{\ell^{(i)}}\left(\langle id_{i_1}^{(i)}, y_{i_1} \rangle = 0 \mod 2q \right) \bigwedge_{i_1=1}^{\ell^*}\left(\langle id_{i_1}^*, y_{i_1} \rangle = 0 \mod 2q \right)\right] = \frac{1}{(2q)^{\chi_1^{(i)}+\chi_2^{(i)}}} \leq \frac{1}{(2q)^n}.$$

We can conclude that $\eta_{2q} \geq 1/(2^{\ell^*+1}q^{\ell^*})$. Combining this bound with the bound on $\eta$, we get the statement of the Lemma.

\[\square\]

### B.4 Adaptive Security: Proof of Theorem 3

Regarding condition (i) of partial lossiness, let $RL_0$ and $RL_n$ be the games specified in Section 4.2. Let $\overline{RL}_0$ and $\overline{RL}_n$ be the games which are the same as $RL_0$ and $RL_n$ except that they include the artificial abort stated described above. First, we claim that

$$\Pr[\overline{RL}_0 \Rightarrow 1] - \Pr[\overline{RL}_n \Rightarrow 1] \leq n \cdot ((d \cdot \mu + 1) \cdot \text{Adv}^{P\cdot\text{BDH}}(B_1) + q \cdot \text{Adv}^{P\cdot\text{DDH}}(B_2))$$

The proof for this statement is almost identical to the proof for Lemma 1. We now claim that both $\Pr[d_{exp}^A = 1 | \text{REAL}] = \Pr[\overline{RL}_0 \Rightarrow 1]$ and

$$|\Pr[d_{exp}^A = 1 | \text{LOSSY}] - \Pr[\overline{RL}_n \Rightarrow 1]| \in \text{negl}(\rho).$$
These statements imply condition (i) of partial lossiness.

The proof of the last two statements is identical to the proof for Theorem 4: when using the auxiliary input \( y^{(1)} \), \( \text{HF.Eval} \) will always be injective. On the other hand, when using \( y^{(2)} \), \( \text{HF.Eval} \) will be lossy for identities \((id_1, \ldots, id_{\ell})\) such that \( \langle y^{(1)}_{i_1}, id_{i_1} \rangle = 0 \) for all \( i_1 \in \{1, \ldots, \ell\} \), with lossiness \( \lambda \left( \text{HF.Eval}(\text{pms}, mpk, (id_1, \ldots, id_{\ell})), \right) \geq n - \log p = \omega \), and \( \text{HF.Eval} \) will be injective for the other identities.

Seeing that condition (ii) is fulfilled is straightforward: from Lemma 2, we know that \( \eta_{\text{low}} \leq \Pr[d_1 = 1 \mid \text{REAL}] \) and, by construction, \( \eta_{\text{low}} \leq \Pr[d_2 = 1 \mid d_1 = 1 \land \text{REAL}] \). This gives us the lower bound \( \eta_{\text{low}}^2 \) on \( \Pr[d_{\text{abort}} = 1 \mid \text{REAL}] \), which is non-negligible. However, the value of \( \epsilon_1 \) will not be this lower bound. Instead, we will show that another condition (which we give at the end of this proof) is satisfied and guarantees the existence of \( \epsilon_1 \) and \( \epsilon_2 \). The proof for this is actually very similar to the proof in [36]. Here, we briefly outline the details. First, due to Chernoff bounds we have the following:

\[
\Pr[d_{\text{abort}} = 1 \mid d_A = 1 \land \text{REAL}] \geq \eta_{\text{low}} \left( 1 - \frac{1}{4}\zeta \right)
\]

and

\[
\Pr[d_{\text{abort}} = 1 \mid d_A = 0 \land \text{REAL}] \leq \eta_{\text{low}} \left( 1 + \frac{3}{8}\zeta \right)
\]

We refer the reader to [36] for details on how to compute these bounds.

Let us now assume the existence of a PPT adversary \( A \) such that

\[
\Pr[d_A = 1 \mid \text{REAL}] - \frac{1}{2} > \zeta
\]

for some non-negligible \( \zeta \). This implies that we can write \( \Pr[d_A = 1 \mid \text{REAL}] > 1/2 + \zeta \) and \( \Pr[d_A = 0 \mid \text{REAL}] < 1/2 - \zeta \). Combining these inequalities with the two inequalities above (from Chernoff bounds) we obtain

\[
\Pr[d_A = 1 \mid \text{REAL}] \cdot \Pr[d_{\text{abort}} = 1 \mid d_A = 1 \land \text{REAL}] \\
- \Pr[d_A = 0 \mid \text{REAL}] \cdot \Pr[d_{\text{abort}} = 1 \mid d_A = 0 \land \text{REAL}] > 2\delta \zeta
\]

with \( \delta = 13\eta_{\text{low}} / 16 = 13 / (32 \cdot (2q \mu)^4) \). Now, let us observe that

\[
\Pr[d_A = 1 \mid \text{REAL}] \cdot \Pr[d_{\text{abort}} = 1 \mid d_A = 1 \land \text{REAL}] = \Pr[d_{\text{abort}} = 1 \mid d_A = 1 \land \text{REAL}]
\]

and

\[
\Pr[d_A = 0 \mid \text{REAL}] \cdot \Pr[d_{\text{abort}} = 1 \mid d_A = 0 \land \text{REAL}] = \\
\Pr[d_{\text{abort}} = 1 \land d_A = 0 \mid \text{REAL}]
\]

\[
= \Pr[d_{\text{abort}} = 1 \mid \text{REAL}] \cdot \Pr[d_A = 0 \mid d_{\text{abort}} = 1 \land \text{REAL}]
\]

\[
= \Pr[d_{\text{abort}} = 1 \mid \text{REAL}] \cdot (1 - \Pr[d_A = 1 \mid d_{\text{abort}} = 1 \land \text{REAL}])
\]

\[
= \Pr[d_{\text{abort}} = 1 \mid \text{REAL}] - \Pr[d_{\text{abort}} = 1 \land d_A = 1 \mid \text{REAL}]
\]

Combining these two equalities with inequality (22), we obtain

\[
2\delta \zeta < 2 \Pr[d_{\text{abort}} = 1 \land d_A = 1 \mid \text{REAL}] - \Pr[d_{\text{abort}} = 1 \mid \text{REAL}].
\]

35
Dividing this inequality by 2 and using the fact that
\[ \Pr[d_{\text{abort}} = 1 \land d_A = 1 \mid \text{REAL}] = \Pr[d_{\text{abort}} = 1 \mid \text{REAL}] \cdot \Pr[d_A = 1 \mid d_{\text{abort}} = 1 \land \text{REAL}], \]
we obtain the relation
\[ \delta \zeta < \Pr[d_{\text{abort}} = 1 \mid \text{REAL}] \cdot \left( \Pr[d_A = 1 \mid d_{\text{abort}} = 1 \land \text{REAL}] - \frac{1}{2} \right) \]

Finally, this shows that there exist \( \epsilon_1 \) and \( \epsilon_2 \) such that (ii) and (iii) are satisfied and that their product is \( \delta \).

\[ \square \]

C Adaptive-id Secure Deterministic (H)IBE

A hierarchical identity-based deterministic encryption scheme (HIB-DE) is a tuple of efficient algorithms \( \text{HIB-DE} = (\text{HIB-DE.Setup}, \text{HIB-DE.MKg}, \text{HIB-DE.Kg}, \text{HIB-DE.Del}, \text{HIB-DE.Enc}, \text{HIB-DE.Dec}) \). The setup algorithm \( \text{HIB-DE.Setup} \) takes as input a security parameter \( q \in \mathbb{N} \), the (constant) number of levels in the hierarchy \( d \in \mathbb{N} \), the length of the identities \( \mu \in \text{poly}(q) \) and the length of the plaintexts \( s \in \text{poly}(q) \), and outputs a set of global public parameters \( \text{pms} \), which specifies an identity space \( \text{IdSp} \) and the necessary mathematical objects and hash functions. The master key generation algorithm \( \text{HIB-DE.MKg} \) takes as input \( \text{pms} \) and outputs a master public key \( \text{mpk} \) and a master secret key \( \text{msk} \). The key generation algorithm \( \text{HIB-DE.Kg} \) takes as input \( \text{pms} \), \( \text{msk} \) and a hierarchical identity \( (id_1, \ldots, id_{\ell}) \in \text{IdSp} \), for some \( \ell \geq 1 \) and outputs a secret key \( \text{SK}_{(id_1,\ldots,id_{\ell})} \). The delegation algorithm \( \text{HIB-DE.Del} \) takes as input \( \text{pms} \), \( \text{msk} \), a hierarchical identity \( (id_1, \ldots, id_{\ell}) \), a secret key \( \text{SK}_{(id_1,\ldots,id_{\ell})} \) for it, and an additional identity \( id_{\ell+1} \); the output is a secret key \( \text{SK}_{(id_1,\ldots,id_{\ell},id_{\ell+1})} \) for the hierarchical identity \( (id_1, \ldots, id_{\ell}, id_{\ell+1}) \) iff \( (id_1, \ldots, id_{\ell}, id_{\ell+1}) \in \text{IdSp} \). The evaluation algorithm \( \text{HIB-DE.Enc} \) takes as input \( \text{pms} \), \( \text{msk} \), a hierarchical identity \( id = (id_1, \ldots, id_{\ell}) \) and a value \( m \in \{0,1\}^s \); the result of the evaluation is denoted as \( C \). Finally, the inversion algorithm \( \text{HIB-DE.Dec} \) takes as input \( \text{pms} \), \( \text{msk} \), a hierarchical identity \( id = (id_1, \ldots, id_{\ell}) \), a secret key \( \text{SK}_{id} \) for it and an evaluation \( C \), and outputs a value \( \tilde{m} \in \{0,1\}^s \).

A HIB-DE satisfies the property of correctness if
\[ \text{HIB-DE.Dec}(\text{pms}, \text{mpk}, \text{id}, \text{SK}_{id}, \text{HIB-DE.Enc}(\text{pms}, \text{mpk}, \text{id} = (id_1, \ldots, id_{\ell}), m)) = m, \]
for any \( m \in \{0,1\}^s \), any \( \text{pms}, (\text{mpk}, \text{msk}) \) generated by \( \text{HIB-DE.Setup} \) and \( \text{HIB-DE.MKg} \), any hierarchical identity \( (id_1, \ldots, id_{\ell}) \in \text{IdSp} \) and any secret key \( \text{SK}_{(id_1,\ldots,id_{\ell})} \) generated either by running \( \text{HIB-DE.Kg}(\text{pms}, \text{msk}, (id_1, \ldots, id_{\ell})) \) or by applying the delegation algorithm \( \text{HIB-DE.Del} \) to secret keys of shorter hierarchical identities.

Bellare et al. [7] gave several definitions for deterministic encryption and proved them equivalent. In the case of block sources\(^9\) [10], Boldyreva et al. [10] proved that single-challenge security (called PRIV1 security in [5, 10]), where the adversary is given a vector of challenge ciphertexts) is equivalent to multi-challenge security (referred to as PRIV security [5], where the adversary is given a vector of challenge ciphertexts) in the sense of indistinguishability-based definitions. The simplified indistinguishability-based notion, called PRIV1-IND hereafter, is somewhat handier to work with and we thus use this one.

We define PRIV1-IND-ID security, the natural analogue of PRIV1-IND security in the IBE scenario. The security notion is defined via the following experiment between a challenger and an adversary. Some instructions depend on whether we are in the selective or adaptive security case.

\(^9\) Informally, a block source is a distribution of message vectors where each component has high min-entropy conditionally on previous ones.
Definition 3. We define Guess$^A_{\text{HIB-DE}}(M)$, for a random variable $M$ as follows:

0. The challenger $C$ chooses global parameters $\text{pms}$ by running $\text{HIB-DE.Setup}$. The parameters $\text{pms}$ are given to $A$, who replies by choosing a hierarchical identity $\text{id}^1 = (\text{id}_{1}^1,\ldots,\text{id}_{\ell}^1)$, for some $\ell^1 \leq d$.
1. The challenger runs $(\text{mpk},\text{msk}) \leftarrow \text{HIB-DE.MKg(\text{pms})}$ and sends $\text{mpk}$ to $A$. Also, two lists $\text{QS} \leftarrow \emptyset$, $\text{IS} \leftarrow \emptyset$ are initialized.
2. $A$ is allowed to make a number of adaptive queries for hierarchical identities $\text{id} = (\text{id}_1,\ldots,\text{id}_\ell)$ (where hierarchical identities are encoded as hierarchical vectors $(X_1^1,\ldots,X_\ell^1)$).
   - **Create-key**: $A$ provides $\text{id}$ and the challenger $C$ creates a private key $\text{SK}_{\text{id}}$ by running $\text{HIB-DE.Kg(\text{pms}, \text{msk}, \text{id})}$. The list $\text{QS}$ is updated as $\text{QS} = \text{QS} \cup \{\text{id}\}$.
   - **Create-delegated-key**: $A$ provides $\text{id} = (\text{id}_1,\ldots,\text{id}_\ell)$ and $\text{id}_{\ell+1}$ such that $\text{id} \in \text{QS}$. The challenger $C$ then computes $\text{SK}_{\text{id}'}$ for $\text{id}' = (\text{id}_1,\ldots,\text{id}_{\ell+1})$ by running the delegation algorithm $\text{HIB-DE.Del(\text{pms}, \text{mpk}_j, \text{SK}_{\text{id}}, \text{id}_{\ell+1})}$. The list $\text{QS}$ is updated as $\text{QS} = \text{QS} \cup \{\text{id}'\}$.
   - **Reveal-key**: $A$ provides $\text{id}$ with the restriction that if $A$ is selective, then $\text{id} \not\in \text{id}^1$. $C$ returns $\perp$ if $\text{id} \not\in \text{QS}$. Otherwise, $\text{SK}_{\text{id}}$ is returned to $A$ and the list $\text{IS}$ is updated as $\text{IS} = \text{IS} \cup \{\text{id}\}$.
3. The adversary $A$ outputs a hierarchical identity $\text{id}^*$ = $(\text{id}_1^*,\ldots,\text{id}_j^*)$. In the selective setting, we impose $\ell^* = \ell$ and $\text{id}_j^* = \text{id}_j^1$ for each $j \in \{1,\ldots,\ell^*\}$. In the adaptive case, no element of $\text{IS}$ is a prefix of $\text{id}^*$.
4. The challenger encrypts a message $m$ sampled from the given message distribution $M$. The resulting ciphertext is sent to $A$.
5. $A$ outputs a bit $b' \in \{0,1\}$, which is the output of the experiment.

Recall that a random variable $X$ over $\{0,1\}^*$ is called a $(t,s)$-source if $H_\infty(X) \geq t$, where $H_\infty(X)$ is the min-entropy of $X$; $H_\infty(X) = -\log(\max_x P_X(x))$. We now give the identity-based version of IND-PRIV1 security of [10].

**Definition 4.** An $s$-bit encryption scheme $\text{HIB-DE}$ is PRIV1-IND-ID secure for $(t,s)$-sources if for any $(t,s)$-sources $M_0$ and $M_1$ and all polynomial time adversaries $A$, the PRIV1-IND-ID advantage

$$\text{Adv}^{\text{priv1-ind-id}}_{\text{HIB-DE}}(A, M_0, M_1) = \Pr[\text{Guess}^A_{\text{HIB-DE}}(M_0) = 1] - \Pr[\text{Guess}^A_{\text{HIB-DE}}(M_1) = 1]$$

of $A$ against $\text{HIB-DE}$ is negligible.

**C.1 Universal HF implies deterministic encryption**

As it can be seen from the definition, a hierarchical identity-based encryption scheme is very close to an HIB-TDF, even syntactically. Indeed, as shown in [10], in the public key setting a natural way to construct a deterministic DE encryption scheme is by defining the algorithms of DE as their natural counterparts in some lossy LTDF. Boldyreva et al. [10] show that if the lossy function is also universal, this construction is PRIV1-IND secure. For functions not satisfying this property, the result follows also directly by an extension of the Crooked Leftover Hash Lemma of Dodis and Smith [19] given by Boldyreva et al. for this purpose. Similarly, in the identity-based setting, one can construct a hierarchical deterministic identity-based encryption scheme HIB-DE from HF, a lossy HIB-TDF, by defining the algorithms of HIB-DE from those of HF in the natural way (in this case, we say that HIB-DE is defined by HF). We will show that with the additional requirement of
universality, such a construction is secure in some natural extension of PRIV1-IND security. To do without this extra condition, it is possible to use the same techniques of Boldyreva et al.

More specifically, universality requires that if LHF is the lossy sibling of a \((\omega, \delta)\)-lossy HIB-TDF function, then HF.Setup, LHF.MKg and HF.Eval(pms, mpk, id*, ·) induce a universal hash function when we have a lossy function for id*, i.e., \(\lambda(HF.Eval(pms, mpk_1, (id^*_1, \ldots, id^*_\ell)), \cdot) \geq \omega\). It is not difficult to see that both [9] and our construction induce a universal hash in the lossy mode.

**Theorem 5.** If HF is a universal \((\omega, \delta)\)-lossy HIB-TDF function with input \(\{0, 1\}^n\), then the hierarchical deterministic encryption function HIB-DE defined by it is secure for poly-time sampleable \((t, s)\)-sources such that \(t \geq n - \omega + 2\log(1/\epsilon)\) for some negligible \(\epsilon\). In particular, for any two \((t, s)\)-sources \(M_0\) and \(M_1\) and for every PRIV1-IND-ID adversary \(B\) against HIB-DE there exists a PPT adversary \(A\) against HF such that

\[
\text{Adv}_{\text{HF,LHF,\omega,\epsilon}}(A) \geq \frac{1}{3} \cdot \delta \cdot \text{Adv}_{\text{HIB-DE}}^{\text{priv1-ind-id}}(B, M_0, M_1) - \nu(\epsilon).
\]

**Proof.** We use similar ideas as those of [10]. Namely, our proof goes as follows: assume that there is an adversary \(B\) that breaks PRIV1-IND-ID security of the deterministic encryption scheme. Namely, there exist two \((t, s)\)-sources \(M_0, M_1\) and a non-negligible \(\zeta\) such that

\[
\text{Adv}_{\text{HIB-DE}}^{\text{priv1-ind-id}}(B, M_0, M_1) = \Pr[\text{Guess}_{\text{HIB-DE}}^B(M_0) = 1] - \Pr[\text{Guess}_{\text{HIB-DE}}^B(M_1) = 1] \geq \zeta.
\]

Then, we build an adversary \(A\) that breaks the security of the underlying HF, that is, \(A\) will interact with a challenger according to the lossy or the real experiment and will tell the difference between both scenarios with non-negligible probability.

\(A\) forwards an identity id\(^{\dagger}\) to its challenger, which is some random identity in the adaptive case. When the challenger runs the setup and gives the output to \(A\), \(A\) forwards this information to \(B\). When \(B\) asks for a secret key for a hierarchical identity id, \(A\) forwards the query to the experiment and forwards the reply to \(B\). At some point, \(B\) outputs id*. Adversary \(A\) then forwards id* to its challenger, chooses \(\gamma \leftarrow \{0, 1\}\) at random and encrypts \(m_\gamma \leftarrow M_\gamma\) under the identity id*. Note that this corresponds to an execution of Guess\(_{\text{HIB-DE}}^B(M_\gamma)\). After some more secret key queries, \(B\) outputs a bit \(\gamma' \in \{0, 1\}\) and \(A\) outputs \(d_A = 1\) if \(\gamma = \gamma'\) and \(d_A = 0\) otherwise.

The security analysis is very similar to the one in Section 2.2. A simple argument shows that

\[
\Pr[\gamma' = \gamma | \text{REAL}] - \frac{1}{2} = \Pr[d_A = 1 | \text{REAL}] - \frac{1}{2} \geq \zeta/2,
\]

since \(A\) perfectly simulated the experiment Guess\(_{\text{HIB-DE}}^B(M_\gamma)\) with \(B\). On the other hand, in the LOSSY setting when id* is lossy, the advantage of \(B\) in guessing \(\gamma\) is negligible because the universality property means that the output distribution of the encryption algorithm is independent of the input distribution, and this holds regardless of whether \(d_{\text{abort}}^A = 1\), therefore:

\[
\Pr[d_A = 1 | \text{LOSSY} \land d_{\text{abort}}^A = 1] \leq \frac{1}{2} + \nu,
\]

for some negligible \(\nu\). At this point the analysis follows as in the analysis of the IND-ID-CPA case (proof of Theorem 1).
Related Work

In a recent, independent work [38], Xie, Xue and Zhang gave a lattice-based deterministic IBE construction in the auxiliary input setting [14]. While their scheme does provide adaptive security in the auxiliary input setting, it was described as a single-level IBE. Our results are incomparable to theirs: on one hand, we do not consider auxiliary inputs; on the other hand, we are concerned with pairing-based schemes in a hierarchical setting and focus on a more powerful primitive.

D On Hedged (H)IBE

Bellare et al. [6] studied the problem of designing hedged public key encryption schemes, which remain secure even if the randomness used for encryption is relatively bad. They consider two notions of hedged security: non-adaptive and adaptive. A non-adaptive attacker can make only a single challenge encryption query, whereas an adaptive attacker can make up to $q$ challenge adaptive encryption queries, all of them for the same public key.

Bellare et al. give in [6] different constructions of public key encryption scheme enjoying non-adaptive hedged security. For instance, their construction RtD combines an IND-CPA secure encryption scheme and a deterministic PRIV1-IND secure encryption scheme to achieve non-adaptive hedged security. Then they prove a general result stating that a public key encryption scheme which enjoys both anonymity and non-adaptive hedged security, already enjoys adaptive hedged security. They show how to provide the necessary anonymity property by using universal lossy trapdoor functions.

The notion of non-adaptive hedged security, with only one challenge encryption query, can be easily extended to the identity-based setting, as well as the non-adaptively secure constructions in [6]. In particular, since our new notion of partial lossiness for (H)IB-TDFs implies both IND-CPA secure IBE and PRIV1-IND secure (H)IB deterministic encryption, we have that the existence of (H)IB-TDFs enjoying partial lossiness implies (through the identity-based version of RtD) non-adaptive hedged (H)IBE secure against adversaries who choose the challenge identity in an adaptive way.

Regarding the notion of adaptive hedged security, the extension to the identity-based setting is more challenging, because the adversary could choose different identities for each of the $q$ challenge adaptive encryption queries. The partial lossiness definitions in [9] and in this paper do not seem to imply any positive result in that case, because these definitions consider only one challenge (lossy) identity. Maybe more general definitions for the partial lossiness notion of (H)IB-TDFs are needed to overcome this obstacle, in the line of all-but-$N$ trapdoor functions [25] and all-but-many trapdoor functions [27]. We leave the problem of designing (H)IBE schemes with adaptive hedged security as an interesting open problem.