We analyze the set $A_Q^N$ of mixed unitary channels represented in the Weyl basis and accessible by a Lindblad semigroup acting on an $N$-level quantum system. General necessary and sufficient conditions for a mixed Weyl quantum channel of an arbitrary dimension to be accessible by a semigroup are established. The set $A_Q^N$ is shown to be log-convex and star-shaped with respect to the completely depolarizing channel. A decoherence supermap acting in the space of Lindblad operators transforms them into the space of Kolmogorov generators of classical semigroups. We show that for mixed Weyl channels the hyper-decoherence commutes with the dynamics, so that decohering a quantum accessible channel we obtain a bistochastic matrix form the set $A_C^N$ of classical maps accessible by a semigroup. Focusing on 3-level systems we investigate the geometry of the sets of quantum accessible maps, its classical counterpart and the support of their spectra. We demonstrate that the set $A_Q^3$ is not included in the set $U_Q^3$ of quantum unistochastic channels, although an analogous relation holds for $N = 2$. The set of transition matrices obtained by hyper-decoherence of unistochastic channels of order $N \geq 3$ is shown to be larger than the set of unistochastic matrices of this order, and yields a motivation to introduce the larger sets of $k$-unistochastic matrices.

Keywords: Quantum operations, Lindblad dynamics, Quantum semigroups, Pauli channels, Unistochastic matrices
I. INTRODUCTION

Open quantum systems have attracted a lot of interest from different perspectives [1, 2]. Two major methods to describe the dynamics of a quantum system interacting with an environment are frequently used. The first one takes place in continuous time, as one deals with differential equations of motion governing the time evolution of the system. The most general form of such a non-unitary quantum dynamics was introduced by Gorini, Kossakowski, Sudarshan, and independently by Lindblad [3, 4]. For a historical account on these fundamental results, see the recent review [5].

In the second, stroboscopic approach, time changes in a discrete way and the evolution of a quantum system is described by quantum operations: linear completely positive maps which preserve the trace of the density matrix. Let \( \Phi : M_N \to M_N \) be a quantum operation acting on an \( N \)-dimensional system described in the complex Hilbert space \( \mathcal{H}_N \). The action of this map on an \( N \)-level density matrix \( \rho \) in terms of Kraus operators, \( K_j \), reads, \( \Phi(\rho) = \sum K_j \rho K_j^\dagger \). In an alternative approach, one can represent a channel by the corresponding superoperator \( \Phi \) acting on an extended Hilbert space, \( \mathcal{H}_N \otimes \mathcal{H}_N \). The superoperator, denoted by the same symbol \( \Phi \) as the map it represents, can be determined by its entries in a product basis, \( \Phi_{m\mu} = \langle m\mu | \Phi | n\nu \rangle = \sum_j (K_{j})_{mn}(\overline{K}_{j})_{\mu\nu} \). It is known that a given quantum channel may have many corresponding sets of Kraus operators, while the superoperator associated with a channel is unique.

Although the superoperator \( \Phi \) is not necessarily Hermitian, reshufling its entries results in a positive semidefinite dynamical matrix [6], also known as the Choi matrix [7] of the channel, \( D_\Phi = \Phi^R \), where \( X_{n\nu}^R = X_{mn} - \text{see} [8] \). It is customary to refer to the \( N^2 \) eigenvalues of \( \Phi \) as the channel eigenvalues. Due to Hermiticity and trace preserving conditions, the eigenvalues of a channel are either real or appear in complex conjugate pairs. There exists at least one leading eigenvalue equal to unity, which corresponds to the invariant state [9]. It is interesting to study relations between both approaches [10] and investigate which discrete channel \( \Phi \) can be accessed by a continuous dynamics in the Lindblad form. This general question, sometimes called embedding problem or Markovianity problem, is known to be hard for a large system size [11]. A necessary condition is the divisibility property [12–14], so that the map \( \Phi \) can be represented as a composition of two other quantum maps, \( \Phi = \Psi_2 \Psi_1 \), which are completely positive and trace preserving such that one of them is not unitary. There exist non-divisible quantum maps, which correspond to non-Markovian dynamics [15, 16].

In the simplest case of \( N = 2 \) and one-qubit unital maps, which are unitary equivalent to Pauli channels, \( \Phi_p = \sum p_i \sigma_i \otimes \overline{\sigma}_i \), necessary and sufficient conditions for accessibility by a dynamical semigroup were established [14, 17, 18]. If all eigenvalues \( \lambda_i \) of \( \Phi_p \) are real and positive, the map \( \Phi \) is shown to be accessible by a semigroup if and only if the relation for subleading eigenvalues, \( \lambda_i \geq \lambda_j \lambda_k \), holds for any choice of three different indices \( i, j, \) and \( k \) from the set \( \{1, 2, 3\} \), as the leading eigenvalue reads \( \lambda_0 = 1 \). These relations are equivalent to the following inequalities for the weights in the convex combination of Pauli matrices, \( p_0 p_i \geq p_j p_k \), where the weight \( p_0 \) standing by \( \sigma_0 = I_2 \) is assumed to be the largest. Furthermore, it was shown that the set \( \mathcal{A}_N^C \) of quantum channels accessible by a semigroup, forms a fourth part of the set \( \mathcal{U}_2^Q \) of quantum unistochastic channels [18], determined by unitary matrices of size \( N^2 = 4 \) which couple the principal system with the environment initially in the maximally mixed state [8, 19]. Further geometric properties of the set of one qubit Pauli maps were recently investigated in [20].

The aim of the present work is to generalize earlier results on the structure of the set \( \mathcal{A}_N^Q \) of one-qubit accessible channels [17, 18] for maps acting on \( N \)-level systems. In this work we concentrate on a class of mixed unitary channels (also called random external fields [21]), defined as convex combinations of rotations by Weyl matrices, which can be considered as unitary generalizations of the Pauli matrices. The set \( \mathcal{W}_N \) of Weyl mixed unitary channels of size \( N \) was recently investigated to study the degree of non-Markovianity [16]. Here we find necessary and sufficient conditions for a quantum Weyl channel acting on an \( N \) dimensional system to be accessible by a Lindblad semigroup. Furthermore, we investigate the set \( \mathcal{A}_N^Q \) of Weyl channels accessible by a semigroup and demonstrate that it is log-convex and star-shaped. This property is also inherited by the corresponding set \( \mathcal{A}_N^C \) of circulant bistochastic matrices accessible by classical semigroups. As the relative volumes of different classes of quantum maps were investigated [20, 22, 23], we analyze the ratio of the volume of the set \( \mathcal{A}_N^Q \) of accessible maps to the total volume of the set \( \mathcal{W}_N \) of Weyl channels.

This paper is organized as follows. In Section II we fix the notation, introduce the basis of unitary Weyl matrices and define channels accessible by a Lindblad semigroup. Furthermore, we recall the definitions of stochastic, bistochastic and unistochastic quantum channels, and discuss the corresponding classical transition matrices. In Section III, we establish a condition for the Weyl channels to be accessible by a semigroup. Geometric properties of the set \( \mathcal{A}_N^Q \) of accessible quantum channels are analyzed in Section IV.

In Section V we investigate properties of classical transition matrices generated from the Weyl channels by hyper-decoherence, and demonstrate that these bistochastic matrices are circulant. As decoherence acts in the space of quantum states [24], an analogous process acting in the larger space of quantum maps and described by a supermap is sometimes called hyper-decoherence. We show that the semigroup dynamics commutes with hyper-decoherence...
for mixed Weyl channels, so that any accessible quantum map subjected to decoherence leads to a bistochastic matrix accessible by a classical semigroup. In mathematical literature such Markov stochastic matrices are also called embeddable \cite{25, 26} and this property, related to classical analogue of divisibility of a channel \cite{27}, was also studied in quantum context \cite{28}.

Explicit conditions for a circulant bistochastic matrix, obtained by hyper-decoherence of mixed Weyl channels, to be accessible by classical semigroup are derived and geometry of the set $\mathcal{A}_N^C$ of these matrices is analyzed. In Section VI, we focus on 3-level systems, called qudits, to emphasize prominent difference with respect to the simpler case of maps acting on 2-level systems. Possible relations between the sets of accessible channels, unistochastic channels, and quantum channels, such that the corresponding classical transition matrix is unistochastic are analyzed in Section VII.

In particular, we demonstrate that for $N = 3$ there exists a quantum map accessible by a semigroup, which is not unistochastic. The work is concluded in Section VIII, in which we summarize results achieved and present a list of open questions. Some properties of unistochastic matrices of order $N = 3$ and unistochastic channels acting in this dimension are discussed in Appendices.

II. SETTING THE SCENE

A. Unitary Weyl matrices

In this Section we review necessary results on discrete quantum maps and continuous dynamics of the Lindblad form. We will be mainly concerned with mixed unitary channels, defined by a convex combination of unitary transformations,

$$\Phi_p(\rho) = \sum_{n=1}^{M} p_n V_n \rho V_n^\dagger.$$  \hspace{1cm} (1)

Here $\{V_n\}_{n=1}^{M}$ denotes an arbitrary collection of $M$ unitary matrices of order $N$, while $\bar{p} = (p_1, \ldots, p_M)^T$ represents a probability vector. By definition, any mixed unitary channel is unital, i.e. $\Phi(1) = 1$. In this work we focus our attention on the set $\mathcal{W}_N$ of mixed unitary channels in the form of Eq. (1), where matrices $V_n$ form the basis of Weyl unitary matrices,

$$U_{kl} = X^k Z^l, \quad k,l = 0, \ldots, N-1.$$ \hspace{1cm} (2)

Here $X$ denotes the shift operator, $X|i\rangle = |i+1\rangle$, and $Z$ is a diagonal unitary matrix, $Z = \text{diag}\{1, \omega, \omega^2, \ldots, \omega^{N-1}\}$ with $\omega = e^{i\frac{2\pi}{N}}$. The notation $i \oplus j$, refers to addition $i + j$ modulo $N$. For convenience, we often use a single index $\mu = 0, \ldots, N^2 - 1$, instead of two, $\mu = Nk + l$.

This set of unitary matrices, applied in context of quantum theory by Weyl \cite{29}, and later popularized by Schwinger \cite{30}, was invented by Sylvester \cite{31} already in 1867. These matrices form $N$-dimensional unitary representation of the elements of the Weyl-Heisenberg group and they are sometimes called clock–and–shift matrices, which refers to the operators $Z$ and $X$, respectively.

With the exception of the identity matrix, $U_{00} = \mathbb{I}_N$, all remaining Weyl matrices are traceless. Full set of $N^2$ matrices $U_{\mu}$ forms an orthogonal basis composed of unitary matrices of size $N$, which can be considered as a unitary generalization of Pauli matrices of size $N = 2$. It is known that any one-qubit bistochastic channel $\Phi_B$ is unitarily equivalent to a Pauli channel, $\Phi_B(\rho) = U_2(\Phi_{p}(U_1^\dagger \rho U_1^\dagger))U_2^\dagger$, which is not the case for Weyl channels acting in higher dimensions. Furthermore, for one-qubit systems all unital channels are mixed unitary, which is not true in higher dimensions \cite{32}.

It is easy to see that the Weyl unitary matrices of an arbitrary order $N$ enjoy the following property:

a) $U_{kl} U_{kl'} = \omega_k^{l l'} U_{k \oplus k' l \oplus l'}$, which will be crucial in the further analysis. In turn it implies,

b) Commutativity up to a phase, $U_{kl} U_{kl'} = \omega_k^{l l'} U_{kl'} U_{kl}$,

c) Reflection symmetry, $U_{k,l} U_{k,-l} U_{-l} \cdot -l \oplus N = \omega^{-l k} \mathbb{I}_N$.

B. Mixed unitary channels accessible by a Lindblad semigroup

Through celebrated GKLS theory \cite{3, 4}, the time evolution of a quantum system experiencing Markovian dynamics can be described by a Lindblad generator $\mathcal{L}$ and represented as $\rho(t) = e^{\mathcal{L}t}[\rho(0)] = \Lambda_t[\rho(0)]$. The action of the generator $\mathcal{L}$ on a quantum state $\rho$ of size $N$ can be written in terms of at most $N^2 - 1$ jump operators $L_j$,

$$\mathcal{L}(\rho) = \sum_{j=1}^{N^2-1} \left( L_j \rho L_j^\dagger - \frac{1}{2} L_j^\dagger L_j \rho - \frac{1}{2} \rho L_j^\dagger L_j \right).$$  \hspace{1cm} (3)
The corresponding superoperator $\mathcal{L}$ can be represented as a matrix acting in the extended space $\mathcal{H}_N \otimes \mathcal{H}_N$,

$$\mathcal{L} = \sum_{j=1}^{N^2-1} L_j \otimes \bar{L}_j - \frac{1}{2} \sum_j L_j^T L_j \otimes \mathbb{I}_N - \frac{1}{2} \mathbb{I}_N \otimes \sum_j L_j^T \bar{L}_j. \quad (4)$$

With these preliminaries in mind, let $\Phi_\rho$ be a mixed unitary quantum channel written in terms of the Weyl unitary matrices:

$$\Phi_\rho = \sum_{k,l=0}^{N-1} p_{kl} U_{kl} \otimes \bar{U}_{kl} = \sum_{\mu=0}^{N^2-1} p_\mu \rho_\mu \otimes \bar{\rho}_\mu. \quad (5)$$

Probabilities in this convex combination form an $(N^2 - 1)$-dimensional simplex $\bar{\rho} \in \Delta_{N^2-1} \subset \mathbb{R}^{N^2-1}$. In this paper we find out, for which $\rho$ the map is semigroup accessible, $\Phi_\rho \in \mathcal{A}_N^Q$, so that there exists a Lindblad generator $\mathcal{L}$ and $t > 0$ such that $e^{t\mathcal{L}} = \Phi_\rho$.

A quantum channel $\Phi$ belongs to a dynamical semigroup if and only if there is a Lindblad generator $\mathcal{L}$ and $t > 0$ such that $\Phi = e^{t\mathcal{L}}$, where $\mathcal{L}$ fulfills the following three conditions [14]:

i) Hermiticity preserving, $\mathcal{L}[Y^\dagger] = \mathcal{L}[Y]^\dagger$ for any operator $Y$. This in turn results in the following condition for entries of the Lindblad generator, $\mathcal{L}_{nm} = \bar{\mathcal{L}}_{mn}$.

ii) Trace preserving, $\text{Tr}(\mathcal{L}[Y]) = 0$ for any $Y$, which implies, $\forall n, \nu: \sum_m \mathcal{L}_{nm} = 0$.

iii) Conditionally completely positive, $(\mathbb{I}_{N^2} - |\psi_+\rangle\langle\psi_+|)(\mathcal{L} \otimes \mathbb{I}_N)[|\psi_+\rangle\langle\psi_+|](\mathbb{I}_{N^2} - |\psi_+\rangle\langle\psi_+|)|\psi_+\rangle \geq 0$, where $|\psi_+\rangle = \frac{1}{\sqrt{N}} \sum |ii\rangle$ is the maximally entangled state acting on the composite space $\mathcal{H}_A \otimes \mathcal{H}_B$. This condition is equivalent to $(\mathbb{I}_{N^2} - |\psi_+\rangle\langle\psi_+|)\mathcal{L}^R(\mathbb{I}_{N^2} - |\psi_+\rangle\langle\psi_+|)|\psi_+\rangle \geq 0$ in which $\mathcal{L}^R$ is the reshuffled form of $\mathcal{L}$.

However, as logarithm of a matrix is usually not unique [33, 34], it is not straightforward to verify the last condition for an arbitrary map $\Phi$.

### C. Bistochastic quantum maps and classical bistochastic matrices

Any completely positive map $\Phi$ acting on a quantum system of size $N$ can be represented by the corresponding dynamical matrix, determined by the Choi–Jamiołkowski isomorphism, $D_\Phi = N(\Phi \otimes \mathbb{I}_N)|\psi_+\rangle\langle\psi_+|$. As discussed in the previous Section, the dynamical matrix is related to the superoperator $\Phi$ by a particular reordering of the entries of the matrix, called reshuffling, $D_\Phi = \Phi^R$.

A coarse-graining channel sends any density matrix $\rho$ into its diagonal, $D(\rho) = \hat{\rho} := \text{diag}(\rho)$, which can be interpreted as a probability vector. Hence such a decoherence process [24], equivalent to stripping away all off-diagonal elements of the density matrix, projects the set of quantum states $\Pi_N$ into the classical probability simplex $\Delta_N$.

An analogous process applied to any Choi matrix $D_\Phi$ produces a diagonal matrix $\hat{D}$. Its diagonal forms a vector $\hat{t}$ of length $N^2$ which can be reshaped into a classical transition matrix $T(\Phi) = (T(D))$. More formally, writing $\hat{t} = \text{diag}(D_\Phi)$, we have $T_{ij} = \hat{t}_k$, where $k = N(i - 1) + j$, with $i, j = 1, \ldots, N$.

The process of stripping of the off-diagonal elements of the Choi matrix can thus be called decoherence process or also hyper-decoherence, to emphasize that it acts on a larger space of generalized states or quantum maps [35]. More precisely, we describe it in the formalism of super-maps, which send the set of all quantum operations into itself, $\Phi' = \Gamma(\Phi)$ – see [35–37]. The hyper-decoherence $D_\Phi$ acting on a map $\Phi$ corresponds to a coarse graining supermap and technically it can be described by the procedure of reshuffling, $D = \Phi^R$, decohering and then reshaping the diagonal matrix $D$, so one can put the above recipe in a nutshell as: $T(\Phi) = D_\Phi(\Phi)$. Alternatively, one may write equations for the discrete evolution of populations, $T_{ij} = \text{Tr}[|\langle i | \langle j | \Phi(\psi_+\rangle\langle\psi_+|)|\psi_+\rangle \langle\psi_+|]|\psi_+\rangle \langle\psi_+|]$. In this way we represent the process of hyper-decoherence acting in the space of quantum maps by a projection from $N^2$ dimensions to $N$. Let $\Pi_N = \sum_{i=1}^N |i, i\rangle\langle i, i|$ denote a projection operator of dimension $N$. Then the classical transition matrix $T(\Phi)$ and its entries can be written by

$$T(\Phi) = \Pi_N \Phi \Pi_N, \quad \text{and} \quad T_{ij} = \Phi_{i,i}. \quad (6)$$

where the four index notation introduced in the previous section is used. Note that no sum is performed over the repeated indices.

If a Choi matrix $D$ is positive and satisfies the partial trace condition, $\text{Tr}_A D = \mathbb{I}$, it represents a stochastic map $\Phi$. Then the corresponding transition matrix $T(\Phi)$ is stochastic, so that $T_{ij} \geq 0$ and $\sum_i T_{ij} = 1$. $\mathcal{S}_N^Q$ and $\mathcal{S}_N^C$ represent the
sets of $N$-dimensional quantum and classical stochastic processes, respectively. The condition of unitality, $\Phi(I) = I$, analogous to the trace preserving property, is equivalent to the dual partial trace condition, $\text{Tr}_B D = I$. Any completely positive, trace preserving and unital map $\Psi_B$ is called bistochastic and $B_N^Q$ will denote the set of quantum bistochastic maps acting on $N$ dimensional system. Decohering any bistochastic map one obtains a bistochastic transition matrix $B = (\Psi_B)$, such that $B_{ij} \geq 0$ and $\sum_i B_{ij} = \sum_j B_{ij} = 1$ – see [9]. Due to the theorem of Birkhoff any bistochastic matrix can be represented as a convex combination of permutation matrices. The set of bistochastic matrices, denoted by $B_N^C$, has $(N-1)^2$ dimensions and is called the Birkhoff polytope.

Consider any unitary matrix $V$ of order $N$. Unitarity condition, $VV^\dagger = I$, implies that the matrix given by the Hadamard (element-wise) product, $B = V \otimes V$, is bistochastic, as its entries read $B_{ij} = |V_{ij}|^2$. Thus one may ask, whether any bistochastic matrix $B$ can be represented in such a way. The answer is negative, so one introduces the set $U_N^Q$ of unistochastic matrices, for which there exist a unitary $V$ such that $B = V \otimes V$.

It is easy to check that for $N = 2$ any bistochastic matrix is unistochastic, so both sets coincide, $U_2^C = B_2^C$. However, already for $N = 3$ there exist bistochastic matrices which are not unistochastic, so that $U_3^C \subset B_3^C$. As a simple example let us mention the bistochastic matrix considered by Schur, $B_S = \frac{1}{3}(X_3 + X_3^2) \in U_3^C$, where $X_3$ is the three-element cycle permutation matrix. In general, to verify whether a given bistochastic matrix $B \in B_3^C$ is unistochastic one can compute the quantity $Q$, related to the Jarlskog invariant [38, 39], and check if it is positive – see Appendix A. For larger dimensions no explicit criteria necessary and sufficient for unistochasticity are known [40], but in the case $N = 4$ one can rely on an efficient numerical procedure based on the algorithm of Haagerup [41].

As unitary $V$ of size $N$ determines a bistochastic matrix $B = V \otimes V \in B_N^C$, a unitary matrix $U$ of size $N^2$ determines a certain quantum bistochastic channel. Physically, such an operation corresponds to a coupling of the principal system with an $d$-dimensional ancillary subsystem $E$, initially in the maximally mixed state, by a non-local unitary matrix $U$, followed by the partial trace over the environment,

$$\Psi_U = \text{Tr}_E\left[U\left(\rho \otimes \frac{I_d}{d}\right)U^\dagger\right], \quad (7)$$

By construction such maps are trace preserving and unital for any dimension $d$ of the environment. The case $d = N$ is distinguished, as in this case the rank of the corresponding Choi matrix, $r \leq d^2$ can be full, so in analogy to the unistochastic matrices such quantum maps were called unistochastic channels [8]. The set of unistochastic maps, denoted by $U_N^Q$, forms a proper subset of the set $B_N^Q$ of bistochastic maps. In the one-qubit case, the set of bistochastic maps forms the tetrahedron of Pauli channels $\Phi_{bij}$, while $U_2^Q$ forms its non-convex subset, bounded by ruled surfaces [19, 42] and called Steinhaus tetrahedron [43] – see Fig. 1 and ref. [18]. The set of one-qubit accessible maps was analyzed recently in [44, 45] from a slightly different perspective.

The set of unistochastic channels was found important by studying the maps accessible by a semigroup, as it was shown [18] that $\mathcal{A}_2^Q \subset U_2^Q$. In this work we show that an analogous relation does not hold for $N = 3$ and analyze classical analogues of these sets. For any unitary dynamics, $\Psi_T^Q = V \otimes V$, the corresponding classical transition matrix is unistochastic, $T = V \otimes \bar{V}$. A stronger property states [46] that a classical transition matrix $T$ can be coherified to a unitary dynamics if and only if it is unistochastic, so the set $\mathcal{I}_N$ of isometries - unitary quantum rotations - decoheres to the set of classically unistochastic matrices.

Therefore, although the word unistochastic in both cases refers to the underlying unitary matrix, the sets of quantum unistochastic operations $\mathcal{U}_N^Q$ and classical unistochastic matrices $U_N^C$ are not directly related by decoherence. To explain this fact consider an arbitrary unistochastic map $\Psi_U$, determined in (7) by a unitary matrix $U$ of order $N^2$. It is convenient to represent it in the four-index notation, $U_{\alpha \beta} = \langle \alpha, \beta | U | \gamma, \delta \rangle$. It is known [8] that the corresponding Choi matrix reads, $D\Psi_U = (U^R)^\dagger U^R/N$. Taking its diagonal and reshaping it we arrive at the classical transition matrix $T(\Psi_U)$ with entries

$$T_{ij} = \frac{1}{N} \sum_{b,c=1}^N |U_{ib}|^2 |j_c|^2, \quad i, j = 1, \ldots, N. \quad (8)$$

Note that each entry of $T$ can be written as an average of the entries of a block of size $N$ of a unistochastic matrix $U \otimes \bar{U}$ of order $N^2$. The above observation provides a clear motivation to introduce a family of generalized unistochastic matrices. Define an auxiliary $kN \times N$ rectangular matrix $L_{N,k}$, being a suitable extension of identity,

$$L_{N,k} = I_N \otimes |\tilde{\phi}_k\rangle, \quad (9)$$

where the $k$-dimensional state of the uniform superposition reads, $|\tilde{\phi}_k\rangle = \frac{1}{\sqrt{k}} \sum_{j=1}^k |j\rangle$. For instance, for $N = 2$ and $k = 1$ one has $L_{2,1} = I_2$ while for $k = 2$ such a coarse graining matrix reads $L_{2,2}^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.
FIG. 1. Any one-qubit mixed unitary channel is unitarily equivalent to a convex mixture of Pauli rotations, represented by the tetrahedron $\mathcal{B}_2^Q$ of Pauli channels. The set of $N = 2$ unistochastic channels, denoted by $\mathcal{U}_2^Q$, forms a non-convex set with ruled surfaces, called Steinhaus tetrahedron. Its quarter containing the corner representing identity, forms the set $\mathcal{A}_2^Q$ of Pauli channels accessible by a quantum semigroup. This set is star–shaped with respect to the maximally depolarizing channel $\Phi_*$. The set of classical bistochastic matrices $\mathcal{B}_2^C$, represented by the (red) interval joining classical identity $I_c$ and classical permutation matrix $P_c$, includes the set $\mathcal{A}_2^C$ of matrices accessible by a classical semigroup. This set can be obtained by hyper-decoherence (denoted by dashed arrows), which squeezes the 3D set $\mathcal{A}_2^Q$ into the interval $\mathcal{A}_2^C$.

**Definition.** A bistochastic matrix $B$ or order $N$ will be called $k$–unistochastic if there exists an unitary matrix $U_{kN}$ of size $kN$ such that $B$ is given by the following matrix coarse graining [47] of the larger unistochastic matrix of order $kN$,

$$B = L_{N,k}^T (U_{kN} \otimes \bar{U}_{kN}) L_{N,k}.$$  

(10)

Since $L_{N,1} = I_N$ this definition implies that any standard unistochastic matrix is 1-unistochastic, while expression (8) for the classical transition matrix is equivalent to $T = L_{N,N}^T (U_{N^2} \otimes \bar{U}_{N^2}) L_{N,N}$. Thus a classical transition matrix $T$ obtained by decoherence of any unistochastic channel is $N$–unistochastic. By construction the set $\mathcal{U}_{N,N}^C$ of $N$–unistochastic matrices of order $N$ includes the original set $\mathcal{U}_N^C$ of unistochastic matrices. For $N = 2$ every bistochastic matrix is unistochastic, but already for $N = 3$ there exits a unistochastic operation $\Psi_U$ which decoheres to a non–unistochastic classical transition matrix $T$, which is 3–unistochastic – see Appendix B.

A list of sets of quantum operations discussed further in this work and the corresponding sets of classical transition matrices is presented in Table I. Note that the inclusion relations for quantum maps, visible in the table, correspond to analogous relations in the classical case. Natural decoherence relations, like $\mathcal{D}_h(S_N^Q) = S_N^C$ and $\mathcal{D}_h(B_N^Q) = B_N^C$, connect the sets of unistochastic channels and $N$-unistochastic matrices, $\mathcal{D}_h(\mathcal{U}_N^Q) = \mathcal{U}_N^C$. The latter one contains the set $\mathcal{U}_N^C$ of unistochastic matrices arising from isometric unitary quantum maps by hyper-decoherence, $\mathcal{D}_h(I_N) = \mathcal{U}_N^C$, which can be modeled by Schur superchannels [48]. The set of mixed unitary Weyl channels decoheres to the set of circulant bistochastic matrices, $\mathcal{D}_h(W_N) = C_N$, while quantum accessible Weyl channels decohere to classical accessible bistochastic matrices, $\mathcal{D}_h(\mathcal{A}_N^Q) = \mathcal{A}_N^C$. 


TABLE I. Set \( S^Q_N \) of quantum stochastic maps acting on \( N \) dimensional states contains subsets of bistochastic maps \( B^Q_N \), unistochastic maps \( U^Q_N \), isometric unitary rotations \( T^Q_N \), mixed unitary Weyl channels \( W_N \) defined in (1) and its subset \( A^Q_N \) of maps accessible by a semigroup. Their classical analogues forming subsets of the set \( S^C_N \) of stochastic transition matrices of order \( N \) read: bistochastic matrices \( B^C_N \), N-unistochastic matrices \( U^C_{N,N} \), unistochastic matrices \( U^C_N \), circulant bistochastic matrices \( C_N \) and circulant transition matrices \( A^C_N \) accessible by a classical semigroup. Vertical arrows represent process of hyper-decoherence, e.g. \( D_h(S^Q_N) = S^C_N \) and \( D_h(\Phi_p) = T_q \) with \( q \) being the marginal (35) of the vector \( p \), the rectangular coarse graining matrix \( L_{N,N^2} \) of order \( N \times N^2 \) is defined in (9), while \( U \) and \( V \) represent unitary matrices of size \( N^2 \) and \( N \), respectively.

| Quantum operation          | \( S^Q_N \) | \( B^Q_N \) | \( U^Q_N \) | \( T^Q_N \) | \( W_N \) | \( A^Q_N \) |
|----------------------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Stochastic map, \( \Phi^R \) \( = D \geq 0 \) | \( \text{Tr}_A D = I \) | \( \text{Tr}_B D = I \) | \( D = N^{-1}(U^R)^\dagger U^R \) | \( D = (V \otimes \bar{V})^R \) | \( \Phi_p \) | \( \Phi = \exp(t\mathcal{L}) \) |
| Hyper-decoherence \( D_{h}(\cdot) \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) |
| Classical transition       | \( S^C_N \) | \( B^C_N \) | \( U^C_{N,N} \) | \( U^C_N \) | \( C_N \) | \( A^C_N \) |
| Stochastic matrix, \( T_{ij} \geq 0 \) | \( \sum_i T_{ij} = 1 \) | \( \sum_j T_{ij} = 1 \) | \( T = L^T(U \otimes \bar{U})L \) | \( T = V \otimes \bar{V} \) | \( T_q \) | \( T = \exp(t\mathcal{K}) \) |

Any unitary rotation forms a unistochastic map, so that \( U^Q_N \supset T^Q_N \). A classical analogue of unitary rotations is played by permutations matrices and their set sits inside the set of unistochastic matrices, \( P^C_N \subset U^C_N \subset B^C_N \). In the case \( N = 2 \) the following relations hold: \( A^C_2 \subset U^C_2 \) and \( U^C_2 = B^C_2 \), which are not true for higher \( N \). The set \( U^Q_N \) of unistochastic channels and the set \( W_N \) of the Weyl channels are incomparable. So are the sets of Weyl channel and accessible channels, so in this work we will restrict our attention to the Weyl channels which are accessible by a semigroup.

III. GENERAL CASE: ACCESSIBILITY BY A SEMIGROUP FOR AN ARBITRARY DIMENSION \( N \)

A. Channel eigenvalues and probabilities

In order to answer the question, whether a given map is accessible by a semigroup we need to analyze further properties of channels in the form (5). The following lemma presents a key feature of mixed unitary channels written in the Weyl basis.

**Lemma 1.** Weyl unitary operators satisfy \([U_{ij} \otimes U_{kl}, U_{kl} \otimes \bar{U}_{ij}] = U_{ij}U_{kl}U_{kl} \otimes \bar{U}_{ij}U_{kl} - U_{ij}U_{kl} \otimes \bar{U}_{ij}U_{kl} = 0 \) for any choice of the indices \( i,j,k,l = 0,\ldots,N-1 \). The equality holds as Weyl unitaries are commutative up to a phase, which is not relevant here as the expressions above contain \( U \) and its complex conjugate, \( \bar{U} \).

Accordingly, all combinations (not necessarily convex) of tensor products of Weyl unitaries and their complex conjugates are compatible, so are all mixed unitary channels \( \Phi_p \), i.e. \( \forall \tilde{p}, \tilde{p}' \ [\Phi_{\tilde{p}}, \Phi_{\tilde{p}'}] = 0 \). For the sake of brevity, from now on we may drop subscript \( \tilde{p} \) and denote these channels by \( \Phi \). The common eigenbasis of the channels defined in Eq. (5) is given by the maximally entangled states proportional to the vectorized form of Weyl unitaries. For any matrix \( A = \sum A_{ij}|i\rangle\langle j| \) let us use the shorthand notation \( |A\rangle = \sum A_{ij}|i\rangle|j\rangle \) to represent its vectorized form. It enjoys the following properties: \( (A \otimes C^T)|B\rangle = |ABC\rangle \) and \( \langle [B|A] \rangle = \text{Tr}(B^\dagger A) \), which defines the Hilbert-Schmidt inner product in the space of matrices. A matrix element of the superoperator \( \Phi_\beta \) in the non-normalized basis \( \{|U_{ij}\rangle\} \) reads,

\[
\langle [U_{ij}|\Phi_\beta|U_{mn}\rangle = \sum_{kl}p_{kl}\langle [U_{ij}|U_{kl} \otimes \bar{U}_{kl}|U_{mn}\rangle = \sum_{kl}p_{kl}\text{Tr}(U_{ij}^\dagger U_{kl}U_{mn}U_{kl}^\dagger) = \sum_{kl}p_{kl}\omega_m^{l-kn}\text{Tr}(U_{ij}^\dagger U_{mn}) = N\sum_{kl}p_{kl}\omega_N^{m-l-kn}\delta_{im}\delta_{jn}.
\]

(11)

The above equalities holds due to orthogonality of Weyl unitaries and their commutativity up to a phase mentioned above. For any mixed unitary channel (1) represented by a superoperator in the form \( \Psi_\beta = \sum p_n V_n \otimes \bar{V}_n \), the dynamical matrix \( D \), obtained by reshuffling its entries,

\[
D_\Psi = \Psi_\beta^R = \sum_{n}p_n(V_n \otimes \bar{V}_n)^R = \sum_{n}p_n|V_n\rangle\langle V_n|,
\]

(12)

coincides with the map by the Choi-Jamiołkowski isomorphism [9]. Note that the last form yields the spectral decomposition of the Choi matrix \( D_\Psi \) if and only if the set \( \{V_n, V_n \in U(N)\}_{n=1}^N \) forms an orthogonal basis. Observing
that $\{\langle U_{kl}|U_{ij}\rangle\} = N\delta_{ik}\delta_{jl}$ and $\Phi^R_p = \sum_{kl} p_{kl}|U_{kl}\rangle\langle U_{kl}|$ we infer that all the channels $\Phi_p$ are compatible and also that they commute with the set of the corresponding dynamical maps. This fact implies the following proposition clarifying the relation between the eigenvalues $\lambda_{kl}$ of the superoperator $\Phi_p$ and the probabilities, $p_{kl}$, which are proportional to the eigenvalues of the corresponding dynamical matrix $D = \Phi^R_p$.

**Proposition 2.** There exists a complex Hadamard matrix $H$ (unitary up to rescaling with all unimodular entries), which transforms probabilities $p_{kl}$ into eigenvalues $\lambda_{nn}$ of the superoperator $\Phi_p$ defined in Eq. (5), namely $\lambda = HH^\dagger = N^{2}\varepsilon$, which means that $H$ is a complex Hadamard matrix [49], self inverse up to the constant $N^2$. Hermiticity of $H$ is obvious from definition, and for unitarity we have

$$H_{mn} = \omega^m_n - k, l, m, n = 0, \ldots, N - 1.$$ (13)

To demonstrate the above statement one needs to show that $H^2 = HH^\dagger = N^2 I$, which means that $H$ is a complex Hadamard matrix [49], self inverse up to the constant $N^2$. Hermiticity of $H$ is obvious from definition, and for unitarity we have

$$\sum_{k,l=0}^{N-1} H_{mn} H_{kl} p_{kl} = \sum_{k,l=0}^{N-1} \omega^m_n \omega^k_l p_{kl} = \sum_l \omega^l_m p_{kl} \sum_k \omega^k_n p_{kl} = N^2 \delta_{ml} \delta_{nl}.$$ (13)

**Remark 3.** For any quantum channel of the form $\Phi_p = \sum_{kl} p_{kl} U_{kl} \otimes \overline{U}_{kl} = \frac{1}{N} \sum_{kl} \lambda_{kl}|U_{kl}\rangle\langle U_{kl}|$, the following facts can be verified with the help of the matrix $H$,

1. For any superoperator $\Phi_p$ of this form its leading eigenvalue (corresponding to invariant state $\frac{1}{\sqrt{N}}|1\rangle$) is equal to unity, $\lambda_{00} = 1$. 
2. For completely depolarizing channel $\Phi_\ast$, corresponding to the flat probability vector, $p_{kl} = \frac{1}{N^2}$ for $k, l = 0, \ldots, N - 1$, all subleading eigenvalues vanish, $\lambda_{kl} = 0$ for $k \neq 0$ and $l \neq 0$.
3. $\lambda_{k,l} = \lambda_{-k,-l} = \lambda_{N-k,N-l}$, so apart from $\lambda_{00}$, for an even $N$ there exist three other eigenvalues, $\lambda_{0,0}, \lambda_{0,\frac{N}{2}},$ and $\lambda_{\frac{N}{2},\frac{N}{2}}$ which are always real.
4. As the entries of $H$ are unimodular and the eigenvalues of the superoperator $\Phi_p$ are their convex combinations, they belong to the unit disk, $|\lambda_{k,l}| \leq 1$ for any $k,l$.

**B. Lindblad generators**

Knowing all we need about the spectral form of quantum channels and their dynamical maps, consider now a special family of dynamical semigroups $\Lambda_{t_\mu} = e^{t_\mu \mathcal{L}_\mu}$, defined by the non-negative interaction time $t_\mu$, and the Lindblad generator $\mathcal{L}_\mu$ specified by a single jump operator $L_i = U_i \delta_{i\mu}$. Due to Eq. (4), the generator $\mathcal{L}_\mu$ is

$$\mathcal{L}_\mu = U_\mu \otimes \overline{U}_\mu - \mathbb{I}_{N^2}.$$ (14)

Note that for the case $\mu = 0$ Lindblad generator is trivial, $\mathcal{L}_0 = 0$. By the definition of $\mathcal{L}_\mu$ and Lemma 1, the Lindblad generators do commute, $[\mathcal{L}_\mu, \mathcal{L}_\nu] = 0$. The commutativity of Lindblad generators implies commutativity on their respective dynamical semigroups, $\Lambda_{t_\mu} \Lambda_{t_\nu} = \Lambda_{t_\nu} \Lambda_{t_\mu}$. So it is convenient to write:

$$\Lambda_{t_0} \Lambda_{t_1} \ldots \Lambda_{t_{N-1}} = e^{t_0 \mathcal{L}_0} e^{t_1 \mathcal{L}_1} \ldots e^{t_{N-1} \mathcal{L}_{N-1}} = e^{\sum_{\mu} t_\mu \mathcal{L}_\mu} = e^{t \mathcal{L}} = \Lambda_t.$$ (15)

Here $t = \sum_{\mu} t_\mu$ and

$$\mathcal{L} = \sum_{\mu} p_\mu' \mathcal{L}_\mu, \text{ where: } p_\mu' = \frac{t_\mu}{t}.$$ (16)

Regarding that $\mathcal{L}_0 = 0$, $t_0$ has no physical meaning and one can set it to zero, $t_0 = 0$, which means the first component of the $N^2$ dimensional probability vector $\bar{p}$ is zero, $p_0' = 0$. Above representation of the semigroup $\Lambda_t$ implies the following statement.

**Corollary 4.** For any dimension $N$ the set $\mathcal{A}_N^Q$ of quantum channels accessible by a dynamical semigroup is log convex: any generator $\mathcal{L}$ in the exponent can be represented by a convex combination of $N^2 - 1$ generators $\mathcal{L}_\mu$. 
Moreover, it is known that any concatenation of quantum channels is a quantum channel itself. However, for the family of semigroups in Eq. (15), the stronger fact is that quantum channels gained by an arbitrary concatenation of dynamical semigroups, $e^{t_\mu H_\mu}$, are mixed unitary channels in the sense of Eq. (5), i.e. $\Lambda_t = \Phi$, and the order of dynamical semigroups is not important due to commutativity. To prove the claim, by applying Eq. (14), we obtain $\mathcal{L} = \sum_\mu p_\mu U_\mu \otimes \overline{U}_\mu - I_{N^2}$, so the spectral form of the Lindblad generator is:

$$\mathcal{L} = \frac{1}{N} \sum_{\nu=0}^{N^2-1} (\lambda_\nu - 1)|U_\nu\rangle\langle U_\nu|,$$

where $\lambda_\nu$ are the eigenvalues of the operator $\sum_\mu p_\mu U_\mu \otimes \overline{U}_\mu$ obtained by acting with the matrix $H$ defined in Eq. (13) on the probability vector $p_\mu$ of the rescaled interaction times. In conclusion, the dynamical semigroup associated with $\mathcal{L}$ has the form:

$$\Lambda_t = \frac{1}{N} \sum_{\nu=0}^{N^2-1} e^{t(\lambda_\nu - 1)}|U_\nu\rangle\langle U_\nu|.$$

Now we will show that the semigroup (18) can be written in the form (5) as a convex combination of Weyl unitary matrices. The probabilities $p_\mu$ in the latter equation are obtained by applying $H$ to the eigenvalues of $\Lambda_0$, and they are proportional to the eigenvalues of $\Lambda_0^t$, the reshuffled form of $\Lambda_t$. This implies that every entry of the probability vector is non-negative, as required. Accordingly, quantum channels accessible by a dynamical semigroup $\Lambda_t$ form the subset $\mathcal{A}^N_{t\nu-1}$ within the simplex $\Delta_{N^2-1}$ of mixed unitary channels in the Weyl basis.

Note that if $\lambda_\nu$ is real, then $e^{t(\lambda_\nu - 1)}$, the corresponding eigenvalue of $\Lambda_t$, is positive. Also, if $\lambda_\nu$ and $\overline{\lambda_\nu}$ form a complex conjugate pair, the corresponding eigenvalues of $\Lambda_t$ are again a complex conjugate pair. In this case, at $t_+ = \frac{2n\pi}{\text{Im}(\lambda_\nu)}$ and $t_- = \frac{(2n-1)\pi}{\text{Im}(\lambda_\nu)}$ for any natural number $n \in \mathbb{N}$, the equality $e^{t_+(\lambda_\nu - 1)} = e^{t_-\lambda_\nu - 1}$ holds. So $\Lambda_t$ becomes a map with a positive degenerate or a negative degenerate eigenvalue, respectively, at $t_+$ and $t_-$. Finally, since $|\lambda_\nu| \leq 1$ (see Remark 3), the moduli of the eigenvalues of $\Lambda_t$ decay exponentially with respect to time $t$ or remain unchanged and equal to unity.

**Theorem 5. Consider a mixed unitary channel represented in the Weyl basis, written in the vectorized form $|U_\mu\rangle$,

$$\Phi = \frac{1}{N} \sum_{\mu=0}^{N^2-1} \lambda_\mu |U_\mu\rangle\langle U_\mu|.$$  

In general the eigenvalues $\lambda_\mu$ of the superoperator are complex, and a suitable choice of $\log \lambda_\mu$ is discussed below. The necessary and sufficient condition for $\Phi$ to be accessible by a dynamical semigroup (15) is that the sum on the right hand side of Eq. (19) is real and positive,

$$t_\mu = \frac{1}{N^2} \sum_{\nu=1}^{N^2-1} (\Pi H \Pi)_{\mu\nu} \log \lambda_\nu \geq 0,$$

for $\mu = 1, 2, \ldots, N^2 - 1$. Here $H$ denotes the complex Hadamard matrix defined in (13), while $\Pi = I_{N^2} - |00\rangle\langle 00|$ is a projector operator onto $(N^2 - 1)$-dimensional space and $|00\rangle\langle 00|$ written as a product basis projects $\hat{\lambda}$ to its leading element $\lambda_0 = 1$.

Before proceeding to the proof note that if there exists a single real negative eigenvalue of $\Phi$, its logarithm is complex, so the above condition cannot be satisfied, hence the map is not accessible. On the other hand, if negative eigenvalue is degenerated (an even number of times) then the imaginary parts of $\log \lambda_\nu$ can cancel and the sum (19) can be real and positive. The fact that a one-qubit map with a negative degenerated eigenvalue of the superoperator can be accessible by a semigroup was already reported in [17].

**Proof.** To prove the “if” part, we will assume $\Phi$ is accessible by a dynamical semigroup (15), $\Phi = \Lambda_t$ for some interaction time $t \geq 0$, and apply Eq. (18):

$$\log \lambda_\mu = (\lambda_\nu - 1)t = (\sum_\nu H_{\mu\nu} p_\nu - 1)t = \sum_\nu (H_{\mu\nu} - 1) t_\nu = \sum_\nu H'_{\mu\nu} t_\nu.$$

where $H' = H - W$ with $W_{\mu\nu} = 1$. Recalling the definition of matrix $H$, see Eq. (13), it is clear that the entries of its first row and column are equal to 1, so the corresponding entries of $H'$ vanish. Consequently, $H'$ has the same structure of a projector onto a $(N^2 - 1)$-dimensional subspace, in other words, $H' = \Pi H \Pi$. This means that $H'$ is
not invertible, however, one can define its inverse on the \((N^2 - 1)\)-dimensional subspace \(\Pi H \Pi\). We proceed to show that this inverse exists and is equal to \(\Pi H \Pi\), up to the constant \(N^2\),

\[
(\Pi H \Pi)(\Pi H \Pi) = \sum_{m,n,k,l} \omega_n^{m-n} \left( \omega_n^{m'} \delta_{m'n'} \delta_{mn}(k'l') - 1 \right) \delta_{k,m'} \delta_{l,n'} |mn \rangle \langle k'l' | \\
- \sum_{m,n,k,l} \left( \omega_n^{m-n} - 1 \right) \left( \delta_{m'0} \delta_{n'0} |mn \rangle \langle k'l' | + \delta_{mn'} \delta_{n'0} |00 \rangle \langle k'l' | \right) \\
+ \sum_{m',n',k',l'} \left( \omega_n^{m'-n'} - 1 \right) \delta_{m'0} \delta_{n'0} |00 \rangle \langle k'l' | = N^2 \left( \sum_{mn} |mn \rangle \langle mn | - |00 \rangle \langle 00 | \right). 
\]

By multiplying both sides of Eq. (20) by \(\Pi H \Pi\), one gets the equality in Eq. (19). Because \(t_\mu \geq 0\) for \(\mu = 1, \ldots, N^2 - 1\), we completely recover Eq (19). Note that Eq. (20) implies \(\log \lambda_\mu = \log \bar{\lambda}_\mu\).

To prove “then” part, we should show that if inequalities (19) between eigenvalues hold, the channel is accessible by dynamical semigroups (15). This fact is actually obvious as we have explicitly introduced the interaction time and this completes the proof.

Note that the above result allows us not only to check, whether a given channel is accessible, but if the answer is positive, one can also construct the desired semigroup. For any probability vector \(\vec{p}\), determining the mixed unitary channel (1), we find the complex vector, \(\lambda = H \vec{p}\), check if conditions (19) are satisfied, and if it is the case, we use the non-negative vector \(\vec{t}\) of interaction times to write the semigroup (15).

Moreover, as it should be, Eq. (19) is equivalent of existence of a proper generator for the quantum channel \(\Phi\) that satisfies properties mentioned in the section II B. For accessible channels from \(A_{NC}^2\), the (non-real) logarithm of \(\Phi\) providing such a generator is equal to:

\[
\log \Phi = \frac{1}{N} \sum_\mu \log \lambda_\mu |U_\mu \rangle \langle U_\mu |. 
\]

The above equation fulfils trace preserving condition and by considering \(\log \bar{\lambda}_\mu = \overline{\log \lambda_\mu}\), Hermiticity preserving property also holds. Now, we need to demonstrate that \(\log \Phi\) is conditionally completely positive. Note that \((\log \Phi)^R = \sum_\mu C_\mu |U_\mu \rangle \langle U_\mu |\)

where \(C_\mu = \frac{1}{N^2} \sum_\nu H_{\mu \nu} \log \lambda_\nu\), and that the maximally entangled state \(|\psi_\mu\rangle\) is equal to \(\frac{1}{\sqrt{N}} \langle U_\mu |\)

So conditionally completely positivity for \(\log \Phi\) defined by Eq. (22) means that all of eigenvalues of its reshuffled form are positive besides the one associated with \(|U_0 \rangle\), i.e. \(C_\mu \geq 0\) for \(\mu = 1, \ldots, N^2 - 1\). Observing that \(\log \lambda_0 = 0\), we get \(\sum_\nu (\Pi H \Pi)_{\mu \nu} \log \lambda_\nu \geq 0\) which is satisfied as we assumed \(\Phi \in A_{NC}^2\).

Thus far, we have shown which generator for channels in \(A_{NC}^2\) is suitable, however, for logarithm of \(\lambda_\nu\) one has \(\log \lambda_\nu = \log r_\nu + i \theta_\nu + i2\pi M_\nu\) with \(M_\nu \in \mathbb{Z}\) and \(\theta_\nu \in [-\pi, \pi]\). Note that both \(\pi\) and \(-\pi\) are included in the interval, so for a negative degenerate eigenvalue one can assign \(\pi\) to one of them and \(-\pi\) to another one. Recalling the condition imposed on the logarithm of eigenvalues, one can verify (with two indices) \(M_{-k \oplus N, -l \oplus N} = -M_{k,l}\). This implies:

\[
t_{m,n} = \frac{1}{N^2} \sum_{k,l} H_{mn}^{kl} \left( \log r_{kl} + i \theta_{kl} + i2\pi M_{k,l} \right) \geq 0, 
\]

for \(m, n \in \{0, \ldots, N - 1\}\) which are not simultaneously zero, and so for \(m' = -m \oplus N\) and \(n' = -n \oplus N\):

\[
t_{-m \oplus N, -n \oplus N} = \frac{1}{N^2} \sum_{k,l} H_{mn}^{kl} \left( \log r_{kl} - i \theta_{kl} - i2\pi M_{k,l} \right) \geq 0. 
\]

These two relations result in the following bounds:

\[
- \sum_{k,l} H_{mn}^{kl} \left( \log r_{kl} + i \theta_{kl} \right) \leq i2\pi \sum_{k,l} H_{mn}^{kl} M_{k,l} \leq \sum_{k,l} H_{mn}^{kl} \left( \log r_{kl} - i \theta_{kl} \right). 
\]

In the next Remark we summarize further properties of the accessible channels fulfilling relation (19) and properties of \(M_{k,l}\).

**Remark 6.** Regarding this fact that \(\lambda_\nu\)'s themselves are eigenvalues of a quantum Weyl channel—see Eq. (17), so they have the properties mentioned in Remark 3, one can achieve the following results for accessible maps, \(\Phi = \Lambda_{\nu}^r\):
1. As it is mentioned earlier in this section for a real \( \lambda'_N \) dynamical semigroup \( \Lambda_t \) has a positive eigenvalue. Due to Remark 3 we know for an even \( N \) there are three real subleading eigenvalues \( \lambda'_N \). Consequently, for an even \( N \) the essentially real subleading eigenvalues of any accessible map \( \Phi \in A^N_N \) are necessarily positive, i.e. \( 0 \leq \lambda_{mn} \leq 1 \) for \( \{(m,n)\} = \{(0, \frac{N}{2}), (\frac{N}{2}, 0), (\frac{N}{2}, \frac{N}{2})\} \).

2. Adding Eq. (23) to Eq. (24), we get the following necessary condition for eigenvalues of an accessible map:

\[
\sum_{k,l} H_{kl} \log r_{kl} \geq 0, \tag{26}
\]

where \( r_{kl} \) is the modulus of \( \lambda_{kl} \), i.e. \( \lambda_{kl} = r_{kl}e^{i\theta_{kl}} \).

3. If \( \forall m,n : \sum_{k,l} H_{kl} \log(r_{kl} + i\theta_{kl}) \geq 0 \), we do not need to check any \( M_{k,l} \) to see if the channel is accessible. However, if there exist \( M_{k,l} \) for which inequality (23) holds for any \( m,n \), then the associated quantum channel is accessible not by a unique Lindblad generator.

4. If \( \sum_{k,l} H_{kl} \log(r_{kl} + i\theta_{kl}) \leq 0 \) for both \( (m,n) = (m',n') \) and \( (m,n) = (-m' \oplus N,-n' \oplus N) \) the associated channel is not accessible.

5. Applying Eq. (20) for an even \( N \), we can show \( M_{k,l} = 0 \) for \( \{(k,l)\} = \{(0, \frac{N}{2}), (\frac{N}{2}, 0), (\frac{N}{2}, \frac{N}{2})\} \).

Eq. (20) shows the eigenvalues of a quantum channel belonging to a dynamical semigroup as an explicit function of time. By applying matrix \( H \), we can transform eigenvalues into probabilities and gain the following relation representing the probabilities as an explicit function of time.

\[
p_\mu = \frac{1}{N^2} \sum_\nu H_{\mu \nu} e^{\Sigma_\nu H_{\nu \tau} t_{\tau \eta}}, \tag{27}
\]

Rewriting the previous equation using two indices \( kl \) rather than \( \mu \), one has \( p_{kl} = \frac{1}{N^2} \sum_{mn} H_{mn} e^{\sum_{\tau\nu} H_{\nu \tau} t_{\tau \eta}}. \) Computing the partial derivative with respect to \( t_{ij} \), we obtain the following useful relations

\[
\frac{\partial p_{kl}}{\partial t_{ij}} = \frac{1}{N^2} \sum_{mn} H_{kl} e^{\sum_{\nu} H_{\nu \tau} t_{\tau \eta}} = \frac{1}{N^2} \sum_{mn} H_{kl} \left( H_{ij} - 1 \right) e^{\sum_{\nu} H_{\nu \tau} t_{\tau \eta}} = \frac{1}{N^2} \sum_{mn} H_{kl} e^{\sum_{\nu} H_{\nu \tau} t_{\tau \eta}} - \frac{1}{N^2} \sum_{mn} H_{kl} e^{\sum_{\nu} H_{\nu \tau} t_{\tau \eta}} - p_{kl}.
\]

In the next step we get

\[
\frac{p_{kl} + \partial p_{kl}}{\partial t_{ij}} = \frac{1}{N^2} \sum_{mn} \omega_n^{(km-mn)} \omega_n^{(mj-ni)} e^{\sum_{\nu} H_{\nu \tau} t_{\tau \eta}} = \frac{1}{N^2} \sum_{mn} \omega_n^{(kn-ml)} \omega_n^{(mi-nj)} e^{\sum_{\nu} H_{\nu \tau} t_{\tau \eta}} = \sum_{mn} H_{-i(k,m-nj)} e^{\sum_{\nu} H_{\nu \tau} t_{\tau \eta}},
\]

which implies the relation

\[
p_{kl} + \frac{\partial p_{kl}}{\partial t_{ij}} = p_{-i(k,m-nj)} \tag{28}
\]

This equality shows that it is enough to know only one component of the probability vector, say \( p_{00} \), to compute the rest of the probability vector, \( p_{00} + \frac{\partial p_{00}}{\partial t_{ij}} = p_{-i(k,m-nj)} \). By a proper substitution, this can be applied to eigenvalues.

To demonstrate this approach in action consider first the case \( N = 2 \), in which the superoperator is expressed by the Pauli matrices, \( \Phi = \sum_\mu P_\mu \sigma_\mu \otimes \sigma_\mu \). In this case, the Hadamard matrix \( H \) of order \( N^2 \) defined in (5) reads,

\[
H = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 \\
\end{pmatrix}, \tag{29}
\]
By acting with $H$ on the probability vector $\hat{p} = (p_0, p_1, p_2, p_3)^T$, one obtains eigenvalues as a function of the probabilities [18]. Furthermore, by applying Eq. (19) in qubit case, one obtains relation

$$\begin{pmatrix}
\frac{1}{4} 
\frac{1}{4} 
\frac{1}{4}
\end{pmatrix}
\begin{pmatrix}
1 & -1 & -1
1 & 0 & -1
1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
\log \lambda_1 
\log \lambda_2 
\log \lambda_3
\end{pmatrix}
= \frac{1}{4}
\begin{pmatrix}
\log \frac{\lambda_1}{\lambda_2^2} 
\log \frac{\lambda_2}{\lambda_3^2} 
\log \frac{\lambda_3}{\lambda_1^2}
\end{pmatrix},$$

(30)
in which only the ‘core’ of size $N^2 - 1$ [49] of the complex Hadamard matrix $H$ appears. Non-negativity of all components of the vector $\hat{t} = (t_1, t_2, t_3)$ implies the desired set of three inequalities

$$\lambda_i \geq \lambda_j \lambda_k, \quad \text{for} \quad i \neq j \neq k.$$  

(31)

These relations form the necessary and sufficient condition for a quantum channel to be a seed for Pauli semigroups [17, 18]. By applying Eq. (27), one can easily find the time evolution of the probability vector,

$$p_0 = \frac{1}{4} \left(1 + e^{-2(t_2 + t_3)} + e^{-2(t_1 + t_3)} + e^{-2(t_1 + t_2)}\right),$$

$$p_1 = \frac{1}{4} \left(1 + e^{-2(t_2 + t_3)} - e^{-2(t_1 + t_3)} - e^{-2(t_1 + t_2)}\right),$$

$$p_2 = \frac{1}{4} \left(1 - e^{-2(t_2 + t_3)} - e^{-2(t_1 + t_3)} + e^{-2(t_1 + t_2)}\right),$$

$$p_3 = \frac{1}{4} \left(1 - e^{-2(t_2 + t_3)} - e^{-2(t_1 + t_3)} + e^{-2(t_1 + t_2)}\right),$$

(32)

where the total interaction time reads, $t = \sum_{\mu=1}^3 t_\mu$.

**IV. GEOMETRY OF THE SET $A_N^Q$ OF MAPS ACCESSIBLE BY A SEMIGROUP**

The set $A_N^Q$ of mixed unitary Weyl quantum channels acting on $N$-dimensional states and accessible by a dynamical semigroup is not convex [14]. In Corollary 4 we have shown that for any dimension $N$ the set $A_N^Q \subset \mathcal{W}_N$ forms a log-convex subset of the probability simplex $\Delta_{N^2-1}$ representing all Weyl mixed unitary channels. Another important property of this set is established below.

**A. The set $A_N^Q$ of accessible maps is star-shaped**

We show that the log-convex set $A_N^Q$ of quantum Weyl channels accessible by a dynamical semigroup is star-shaped.

**Proposition 7.** The set $A_N^Q \in \Delta_{N^2-1}$ of mixed unitary Weyl channels accessible by a semigroup has the star-shape property with respect to completely depolarizing channel $\Phi_*$.

**Proof.** The aim is to show for a quantum channel $\Phi \in A_N^Q$ any convex combination with $\Phi_*$ belongs to $A_N^Q$, $\Phi' = m \Phi + (1 - m) \Phi_* \in A_N^Q$, where $0 \leq m \leq 1$. For any $t_\mu \geq 0$ associated with $\Phi \in A_N^Q$ through Eq. (15), one can define the non-negative parameter $t'_\mu = t_\mu - N^{-2} \log m$ for which:

$$\Phi = e^\sum_{\mu} t'_\mu E_\mu = e^{\sum_{\mu} t_\mu E_\mu} e^{-\frac{\log(m)\mu}{N^2}} \sum_{\mu} \mu E_\mu + \Phi_1 e^{-\frac{\log(m)\mu}{N^2}} \sum \left(U_\mu \otimes \I_{N^2-1} \right)$$

$$= \Phi_1 e^{\log(m)1_{N^2-2} - \log(m)\Phi_*} = m \Phi_1 e^{-\log(m)\Phi_*} = m \Phi_1 \left( -1 + \frac{1}{m} \Phi_* + 1_{N^2} \right) = m \Phi_* + (1 - m) \Phi_*.$$  

Here we use the fact that all subleading eigenvalues of $\Phi_*$ are zero, and that the composition of any unital channel with the completely depolarizing channel $\Phi_*$ is equal to $\Phi_*$.

Note that the boundaries of the set $A_N^Q$ consist of channels accessible by a semigroup with at least one of the non-trivial ($\mu \neq 0$) parameters $t_\mu$ in Eq. (15) vanishing. This means that all quantum channels of the form $\exp\left(\sum_{\mu=1}^{N^2-2} L_\mu t_\mu\right)$ belong to $\partial A_N^Q$, for any choice of $N^2 - 2$ out of $N^2 - 1$ non-trivial Lindblad generators and those channels using $N^2 - 1$ Lindblad generators, $\Phi_\beta = \exp\left(\sum_{\mu=1}^{N^2-1} L_\mu t_\mu\right)$, lie inside $A_N^Q$. To see that, note that if $\Phi_\beta$ belongs
to $A_N^Q$, it is connected to $I_N$ through a trajectory indicated by non-negative interaction time and each point of this trajectory corresponds to a channel belonging to $A_N^Q \in \Delta_{N^2-1}$. This in turn leads us to the fact that for any $t \geq 0$ corresponding to a point on the trajectory connecting $I_N$ to $\Phi_{p}$, the dynamical map is positive, also for small time $t$. Expanding the dynamical map around $t = 0$, positivity condition of the corresponding Choi matrix, $D_\Phi = \Phi^R$, implies:

$$
\left( I_{N^2} + \epsilon \frac{d\Phi}{dt} |_{t=0} \right)^R \geq 0,
$$

(33)

which holds for any $\epsilon > 0$ small enough for any $\Phi \in A_N^Q$. Applying Eq. (15), it is easy to see that the above equation is equal to $(I_{N^2} + \epsilon L)^R = (I_{N^2} + \epsilon (\sum \mu p^*_\mu U_\mu \otimes \overline{U_\mu} - I_{N^2}))^R$, which must remain positive for small $\epsilon$. Due to Proposition 2, its eigenvalues belong to the set $\{ N(1 - \epsilon (1 - p^*_0)), \epsilon N p^*_1, \ldots, \epsilon N p^*_{N^2-1} \}$ and each of which should be non-negative. Noting that $\epsilon$ is small, the first eigenvalue is always positive. However, if $p^*_\mu = 0$ for any $\mu = 1, 2, \ldots, N^2 - 1$, the corresponding eigenvalue is zero which means that Eq. (33) is saturated, so that this subspace belongs to the boundary of $A_N^Q$. Due to Eq. (16) this fact implies that the boundary of $A_N^Q$ is achieved if we set to zero the interaction time $t_\mu$ associated with any selected generator $L_\mu$. The above statement is valid for any $N$. In particular, it implies Proposition (2) shown in [18] for the case of a single qubit.

V. HYPER–DECOHERENCE OF QUANTUM WEYL CHANNELS

A. Corresponding circulant bistochastic transition matrices

As discussed in Sec. II, one can assign a classical stochastic transition matrix $T$ to any quantum channel by decohering and reshaping the dynamical matrix, $D_\Phi = \Phi^R$, so that $T$ forms a minor of order $N$ of the superoperator $\Phi$ of order $N^2$. If $\{ K_n \}$ represents the set of Kraus operators related to a channel, it is straightforward to see the associated stochastic map is gained by $T = \sum_n K_n \otimes \overline{K_n}$. Thus for the Weyl channels defined by Eq. (5) the associated bistochastic transition $T$ is given by

$$
T(\Phi_{p}) = \sum_{k,l=0}^{N-1} p_{kl} U_{kl} \otimes \overline{U_{kl}} = \sum_{k,l=0}^{N-1} p_{kl} X^k Z^l = \sum_{k,l=0}^{N-1} p_{kl} X^{k},
$$

(34)

To prove above equation we used the fact that for two arbitrary matrices $A$ and $C$ and two diagonal matrices $B$ and $D$, the equality $AB \otimes CD = (A \otimes C)(B \otimes D)$ holds. Since the operators $Z^l$ do not influence the terms contributing to the transition matrix $T$, it is convenient to introduce an $N$-point probability vector $\tilde{q}$, given by the marginal of the vector $p$ of length $N^2$,

$$
q_k = \sum_{l} p_{kl}.
$$

(35)

Note that $\tilde{q}$ can also be considered as a reduction of $\tilde{p}$, analogous to partial trace. Thus the transition matrix $T_{\tilde{q}} = D_h(\Phi_{\bar{p}})$ corresponding the Weyl channel (1) has the following form

$$
T_{\tilde{q}}(\Phi) = \sum_{k=0}^{N-1} q_k X^k \in C_N.
$$

(36)

The set $C_N$ of classical transitions defined by Eq. (36) consists of circulant bistochastic matrices of order $N$, i.e., bistochastic matrices whose rows (columns) are cyclic permutations of its first row (column). Hence the set $C_N$ forms an $(N-1)$-dimensional simplex $\Delta_{N-1} = \{ I_N, X, X^2, \ldots, X^{N-1} \}$ embedded inside the Birkhoff polytope $\mathbb{B}_N^C$ of dimension $(N-1)^2$ which contains all permutation matrices of order $N$. Note that the shift operator $X$, also written $X_N$, is actually the $N$-element cycle permutation, thus hyper-decoherence of the Weyl channels gives the set of circulant bistochastic matrices, $D_h(W_N) = C_N$.

Lemma 8. Arbitrary powers of cycle permutation do commute, $[X^k, X^{k'}] = 0$. This implies that all (not necessarily convex) combinations of $X^k$ are compatible, so the transitions matrices in Eq.(36) satisfy relation $T(\Phi_{\tilde{p}})T(\Phi_{\tilde{p}}) = T(\Phi_{\tilde{p}})$. 

B. Spectral properties of classical and quantum maps

The Fourier matrix of order $N$, with entries $F_{mn} = \exp(-i2\pi mn/N)$, diagonalizes any combination of powers $X_N^{kl}$ of the cyclic permutation matrix $X_N$. The common eigenbasis is given by $|x_j\rangle = \frac{1}{\sqrt{N}} |j\rangle$. Also, eigenvalues and coefficients of expansion can be transformed to each other applying $F$ and $F^\dagger$. For instance, for a transition matrix $T$ in Eq. (36), its eigenvalues $\xi_i$ and the probabilities $q_k$ satisfy relations, $\xi = F^\dagger \tilde{q}$ and $\tilde{q} = \frac{1}{N} F^\dagger \xi$. To see this let us rewrite $T$ in the basis $\{|x_j\rangle\}$ and use the notation, $\omega_N = \exp(2\pi i/N)$,

$$
\langle x_m|T|x_n\rangle = \frac{1}{N} \sum_k q_k \sum_i \sum_j \omega_N^{-im} \omega_N^{jm} |i|x\oplus k\rangle \langle x|j\rangle \\
= \frac{1}{N} \sum_k q_k \omega^{-mk} \sum_x \omega^{x(n-m)} = \sum_k \omega^{-mk} q_k \delta_{mn}.
$$

(37)

As any circulant bistochastic transition matrix $T \in \mathcal{C}$ contains non-negative real entries and can be diagonalized by the Fourier matrix $F_N$, the following spectral properties hold:

i) Any circulant bistochastic matrix $T$ has the leading (Frobenius–Perron) eigenvalue $\xi_0 = 1$.

ii) For the matrix $T_N$, located at the center of the Birkhoff polytope $B_N$ of bistochastic matrices and corresponding to the flat probability vector, $q_k = \frac{1}{N}$ for $k = 0, \ldots, N - 1$, all subleading eigenvalues vanish, $\xi_i = 0$ for $i \neq 0$.

iii) For a cyclic permutation matrix $X_N$ its spectrum belongs to the unit circle and forms the $N$-sided regular polygon.

iv) Eigenvalues of an arbitrary $T$ are either real or appear in complex conjugate pairs.

v) $\xi_{i \oplus N} = \xi_i$, so for an even $N$ apart from the leading eigenvalue $\xi_0$, there exists another real eigenvalue $\xi_{\omega}$.

vi) As the modulus of the entries of $F$ is unity and eigenvalues of $T$ are just a convex combination of these elements, so the spectrum belongs to the unit disk, $|\xi_i| \leq 1$ for any $i$. Furthermore, as the entries of $F_N$ are given by powers of the $N$-th root of unity, $\omega_N = e^{i2\pi/N}$, then all eigenvalues $\xi_i$ belong to the regular $N$-sided polygon containing $\xi_0 = 1$.

The above properties of the circulant bistochastic matrices – classical maps – can be extended for certain quantum maps.

Proposition 9. Consider a Weyl channel $\Phi_\rho$ from the simplex $\Delta(\Phi_1, \Phi_X, \ldots, \Phi_{X^{N-1}})$. The set of eigenvalues of the superoperator $\Phi_\rho$ is then equal (up to degeneracy) to the spectrum of the corresponding circulant bistochastic matrix $T(\Phi_\rho)$. Moreover, the spectrum of $\Phi_\rho$ is $N$-fold degenerate.

Proof. Let $p_{kl}$ denote the probability vector of size $N^2$ defining the Weyl channel $\Phi_\rho$. Due to Proposition 2 the eigenvalues of the superoperator $\Phi_\rho$ read

$$
\lambda_{mn} = \sum_{kl} \omega_N^{ml-nk} p_{kl}.
$$

(38)

Since hyper-decoherence transforms this channel into the classical map, $T(\Phi_\rho) = \sum X^k q_k$, with $q_k = \sum_l p_{kl}$ the spectrum of the circulant bistochastic matrix $T(\Phi_\rho)$ can be expressed by the Fourier matrix $F$ of size $N$,

$$
\xi_n = \sum_k F_{nk} q_k.
$$

Thus Proposition 9 is equivalent to showing that $\forall m$, $\lambda_{mn} = \xi_n$. We start with expression (38) for eigenvalues $\lambda_{mn}$ and notice that since $\Phi_\rho$ belongs to the the simplex $\Delta(\Phi_1, \Phi_X, \ldots, \Phi_{X^{N-1}})$ and $p_{kl} \neq 0$ only for $l = 0$, then

$$
\lambda_{mn} = \sum_{kl} \omega_N^{ml-nk} p_{kl} = \sum_k \omega_N^{-nk} p_{k0} \\
= \sum_k \omega_N^{-nk} q_k = \sum_k F_{nk} q_k = \xi_n.
$$

In the second row we used that $q_k = \sum_l p_{kl} = p_{k0}$ which is due to the fact that $\Phi_\rho$ belongs to the simplex $\Delta$. This shows that both set of eigenvalues are equal.
Proposition 10. Consider an arbitrary Weyl channel $\Phi_{p}$. There exists a change of basis such that $\Phi_{p}$ is block diagonal, each block of order $N$ forms a circulant matrix. Thus the spectrum of the superoperator $\Phi_{p}$ is contained in the regular $N$-polygon, with its rightmost vertex being 1. In addition, one of these blocks forms matrix $T(\Phi_{p})$, so the spectrum of $\Phi_{p}$ contains the spectrum of the corresponding transition matrix $T$.

Proof. Consider the basis given by $\{j \oplus k, j\}$ for $k, j \in \{0, \ldots, N - 1\}$, corresponding to a permutation of the original basis. The next step is to compute an entry of $\Phi_{p}$ in the new basis:

$$\langle j \oplus k, j|\Phi_{p}|j' \oplus k', j'\rangle = \langle j \oplus k, j|\left(\sum_{mn} p_{mn}U_{mn} \otimes \mathcal{U}_{mn}\right)|j' \oplus k', j'\rangle = \sum_{mn} p_{mn}\langle j \oplus k|U_{mn}|j' \oplus k'\rangle \langle j|\mathcal{U}_{mn}|j'\rangle = \sum_{n} p_{j' \oplus j, n}n^{\omega_{N}^{kn}}\delta_{k, k'}. \tag{39}$$

Thus the following decomposition holds

$$\Phi_{p} = \sum_{k=0}^{N-1} \sum_{j, j'=0}^{N-1} \left(\sum_{l} p_{j' \oplus j, l}Z_{N}^{kl}|j \oplus k, j\rangle\langle j' \oplus k, j'\rangle\right). \tag{40}$$

Fixing $k$, the remaining sum on $j$ and $j'$ corresponds to a block $\Phi_{p}^{(k)}$ of size $N$, given by

$$\Phi_{p}^{(k)} = \sum_{k'}^{N-1} \left(\sum_{l=0}^{N-1} p_{kl}Z_{N}^{kl}|X^{k'} = \sum_{k'} X^{k'}(\sum_{ml} (Z^{k'})_{ml}p_{kl}) = \sum_{k'} X^{k'}q_{k'}^{m}, \tag{40}$$

where we introduced a shorthand notation, $q_{m}^{k} = \sum_{k'}(Z^{k'})_{k'm}p_{ml}$. Therefore, we can write

$$\Phi_{p} = \bigoplus_{k} \Phi_{p}^{(k)}. \tag{40}$$

Given that each block $\Phi_{p}^{(k)}$ is circulant and can be diagonalized by the $N$ dimensional Fourier matrix, the spectrum of $\Phi_{p}$ is contained in the regular $N$-polygon. Finally, observing that $q_{m}^{0} = q_{m}$, we realize that the first block of the superoperator is equal to its classical analogue, $\Phi_{p}^{(0)} = T(\Phi_{p})$. \qed

C. Kolmogorov generators and accessible bistochastic matrices

In analogy with the quantum case, it is interesting to find which transition matrices out of those described in (36) are accessible by a classical dynamical semigroup. From a mathematical perspective such stochastic matrices are also called embeddable and their properties were analyzed in [25, 26]. In our physics–oriented approach it will be convenient to make use of the notion of strictly incoherent operations (SIO), defined as these stochastic maps, which are not able to generate and to use quantum coherence [50, 51]. Therefore an arbitrary quantum operation is strictly incoherent if and only if its Kraus operators are of the form, $K_{n} = \sum_{i} c_{ni} P_{n}(i)\langle i|$, where $P_{n}(i)$ is a permutation matrix [52] and $c_{ni}$ are arbitrary complex coefficients satisfying the normalization condition.

In the analyzed case of mixed Weyl channels the corresponding Kraus operators read, $K_{kl} = \sqrt{p_{kl}} \sum_{m} \omega_{N}^{ml}|m \oplus k\rangle\langle m|$, which proves that these channels are strictly incoherent. This property is equivalent to the following decoupling property,

$$\Phi_{p}(\mathcal{D}(\rho)) = \mathcal{D}(\Phi_{p}(\rho)). \tag{41}$$

It states that the decoherence map $\mathcal{D}$, acting in the space of density matrices – see Sec. II – is not coupled with the dynamics generated by a Weyl channel $\Phi_{p}$. Thus coherence of input states of Weyl channels plays no role in the evolution of probability vector of their diagonal entries. Applying a mixed unitary Weyl channel (5) on an input state $\rho$ we see that the evolution of its diagonal entries is governed by the classical transition matrix $T$ given in (36) and related to the Weyl channel through Eq. (6). If $d_{Y}$ denotes the vector of diagonal elements of a matrix $Y$, written $d_{Y} = \text{diag}(Y)$, then the following equality holds,

$$Td_{\rho} = d_{\Phi(\rho)}. \tag{42}$$
This relation, not working for an arbitrary quantum channel, shows that in the case of a Weyl channel, the corresponding transition matrix $T(\Phi)$ is responsible for the evolution of the diagonal elements of $\rho$. This fact can be restated in the following way.

**Proposition 11.** The evolution of diagonal elements of input states for the channels defined by Eq. (5) that also are accessible by a dynamical semigroup, i.e. those Weyl channels describing a Markovian evolution, is governed by Markovian Pauli master equation:

$$\frac{d}{dt}d_\rho = Kd_\rho, \quad (43)$$

where $K$ is the Kolmogorov operator.

So $d_\rho(t)$ in the above equation is determined as $d_\rho(t) = e^{Kt}d_\rho(0)$. For future references let us explicitly mention necessary properties of a Kolmogorov operator represented by a real matrix of order $N$ – see [53],

a) $\forall i \neq j$ $K_{ij} \geq 0$,

b) $\forall j$ $\sum_i K_{ij} = 0$, or equivalently:

b') $\forall j$ $K_{jj} = -\sum_{i \neq j} K_{ij}$.

As discussed in Section II, the process of decoherence $D$ brings a quantum state into a classical probability vector, while the hyper-decoherence $D_h$ sends a quantum channel $\Phi$ into a classical transition matrix $T$. In a similar manner a Lindblad operator $L$ governing the quantum dynamics is transformed by process of hyper-decoherence into a Kolmogorov operator $K$ generating a classical semigroup. Consider first a Lindblad generator determining a quantum semigroup (16) related to accessible Weyl channels.

**Lemma 12.** Decohering and then reshaping the diagonal elements of $L^n$, such that $L$ is defined in Eq. (16), gives a proper Kolmogorov operator corresponding to the Lindblad generator.

**Proof.** It is straightforward to see the aforementioned method results in the following operator,

$$K = \sum_{k,l} q'_k X^k - \mathbb{I}_N = \sum_k q'_k K_k \quad (44)$$

where $q'_k = \sum_i p'_{ki}$ and $K_k = X^k - \mathbb{I}_N$ is the Kolmogorov operator associated to $L_{kl}$ and is independent of $l$. Note that $K_0 = 0$ is the trivial case. It is easy to see $K$ and $K_k$ satisfy properties (a), (b) and (c) of a valid Kolmogorov operator which completes the proof.

In general, the process of hyper-decoherence acting in the space of Lindblad operators produces a valid Kolmogorov generator [53, 54], with entries $K_{ij} = \text{Tr}([i\rangle\langle i| L(|j\rangle\langle j|))]$. Thus any Lindblad operator produces, by the $N$-dimensional projection operator $\Pi_N$ used in Eq. (6), a corresponding classical Kolmogorov generator,

$$K(L) = D_h(L) = \Pi_N L \Pi_N, \quad \text{and} \quad K_{ij} = L_{jj}. \quad (45)$$

However, the analyzed case of Weyl channels and the corresponding Lindblad operators is rather special, as all the Lindblad generators are commutative - see Eq. (15). This property is also inherited in the classical setup, as due to Lemma 8 all the generators $K_k$ are commutative, so are all $K$ and $K'$ in the form of Eq. (44). The commutativity of Kolmogorov operators $K_k$ imposes commutativity on their respective dynamical semigroups, $T_{tk} T_{tk'} = T_{tk'} T_{tk}$ where $T_{tk} = e^{tK_k}$.

Moreover, classical transition matrices gained by any concatenation of classical dynamical semigroups, $e^{tK_k}$, belong to the set $\mathcal{C}_N$ of circulant bistochastic matrices (36), i.e.

$$T_t = T_{t_0} T_{t_1} \ldots T_{t_{N-1}} = e^{t_0 K_0} e^{t_1 K_1} \ldots e^{t_{N-1} K_{N-1}} = e^{\sum_{k=0}^{N-1} t_k K_k} = e^{t \sum K_k} = e^{tK} = T_q, \quad (46)$$

where the total interaction time $t = \sum t_k$ and $q'_k = t_k/t$. Observe that the order of classical dynamical semigroups is not important because of their commutativity. To see that, one can apply the Fourier matrix $F$ to diagonalize the Kolmogorov operator and to find eigenvalues, so $T_t$ has the following form:

$$T_t = \sum_j e^{t(\xi_j - 1)} |x_j\rangle \langle x_j|, \quad (47)$$
where $\xi_j$ are the eigenvalues of $\sum q_k X^k$ obtained by applying $F$ on the probabilities $q_k$. Therefore, the above classical dynamical semigroup can be rewritten in the form (36) in which probabilities $q_k$ are given by,

$$ q_k = \frac{1}{N} \sum_j F^\dagger_{kj} e^{\xi_j} = \frac{1}{N} \sum_j \psi_j \exp \left( \sum_m \left( \omega^{-jm} - 1 \right) t_m \right). \quad (48) $$

In Theorem 15 we will prove that this equality provides a valid probability vector with non-negative entries. Observe that according to Eq. (47), for any real $\xi_j$, the corresponding eigenvalue of $T_t$ is positive. For the complex conjugate pair $\xi_j$ and $\xi_{-j}$, the corresponding eigenvalues of $T_t$ are complex conjugates as well. Moreover, at $t_+ = \frac{2\pi}{|\text{Im}(\xi)|}$ and $t_+ = \frac{2\pi}{|\text{Im}(\xi)|}$ for any $n \in \mathbb{N}$, $T_t$ will possess positive degenerate and negative degenerate eigenvalues respectively. Finally, as $|\xi_j| \leq 1$, the moduli of the eigenvalues of $T_t$ are either unity or decay exponentially with respect to $t$.

We denote by $\mathcal{A}_N \subset \mathcal{C}_N$ the set of circulant bistochastic transitions satisfying Eq. (46) and forming the set of transition matrices accessible by a classical dynamical semigroup with the Kolmogorov operator in the form of Eq. (44). The aforementioned equation also implies the following result, analogous to the quantum counterpart formulated in Corollary 4.

**Corollary 13.** The set $\mathcal{A}_N$ of classical transition matrices acting on $N$-point probability vectors and accessible by a classical dynamical semigroup generated by $\mathcal{K}$ in Eq. (44) is log convex.

**Theorem 14.** Assume now a circulant bistochastic transition in the form of

$$ T = \sum_n \xi_n |x_n)(x_n|. \quad (49) $$

Again, we choose logarithm such that $\text{log } \xi_n = \text{log } \bar{\xi}_n = \text{log } \bar{\xi}_{-n}\otimes \mathbb{N}$. The necessary and sufficient condition for $T$ to be accessible by a classical dynamical semigroup (46), $T \in \mathcal{A}_N$, is that the sum on the right hand side of Eq. (50) is real and positive,

$$ t_m = \frac{1}{N} \sum_n (QF^\dagger Q)_{mn} \log \xi_n \geq 0, \quad (50) $$

for $m = 1, \ldots, N-1$. Here $F$ denotes the Fourier matrix, $Q = \mathbb{I}_N - |0\rangle \langle 0|$ is the projector operator on $(N-1)$-dimensional subspace and $|0\rangle \langle 0|$ projects $\bar{\xi}$ to $\bar{\xi}_0 = 1$. Note that there is no condition on $t_0$ as it is the coefficient of $K_0 = 0$.

**Proof.** Let us first show that if $T$ is an accessible classical map, then Eq. (50) holds. So we start by the dynamical semigroup in Eq. (47) that implies the following relation for the eigenvalues of an accessible map:

$$ \log \xi_n = t(\xi_n - 1) = t \left( \sum_{m=0}^{N-1} \omega_n^{-mn} q_m - 1 \right) = \sum_{m=0}^{N-1} \left( \omega_n^{-mn} - 1 \right) t_m = \sum_{m=0}^{N-1} F^\dagger t_m, \quad (51) $$

where $F' = F - V$ with $V_{mn} = 1$. As the entries of the first row and column of $F'$ are all zero, $F'$ is not invertible and $F' = QF^\dagger Q$. It is again possible to define the inverse on the $(N-1)$-dimensional subspace $Q$. In the following we show this inverse exists and equals to $QF^\dagger Q$, up to the constant $N$:

$$ (QF^\dagger Q)(QF'Q) = \sum_{m,n=1}^{N-1} \sum_{i,j=0}^{N-1} \omega_{ij}^m (\omega_n^{-kl} - 1) m(i) m(j)n(k) l $$

$$ = \sum_{m,n=1}^{N-1} \sum_{i,j=0}^{N-1} \omega_{mn}^i (\omega_n^{-nl} - 1) m(l) $$

$$ = N \sum_{m=1}^{N-1} \delta_{ml} m(l) = N \sum_{m=1}^{N-1} m = N(I_N - |0\rangle \langle 0|). \quad (52) $$

So by acting $(QF^\dagger Q)$ on the both sides of Eq. (51) the condition in Eq. (50) is achieved for interaction time $t_m$ which should be non-negative. Moreover, note that Eq. (51) implies $\text{log } \xi_n = \text{log } \bar{\xi}_n$. On the other hand, if the inequality (50) holds between eigenvalues of a classical map, as the interaction time is explicitly presented, it is clear that the map belongs to $\mathcal{A}_N$. \qed
As in the quantum version, condition (50) for \( T \) implies existence of a proper logarithm of \( T \) satisfying properties of a Kolmogorov operator mentioned right after Prop. 11. In analogy to the quantum case one can simply write \( \log T \) as follows:

\[
\log T = \sum_{n=0}^{N-1} \log \xi_n(x_n)(x_n) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} \omega_n^{mn} \log \xi_n[m \otimes j](j).
\] (53)

For this generator \( \forall j : \sum_i (\log T)_{ij} = 0 \) is straightforwardly verified. In addition, \( \forall i \neq j : (\log T)_{ij} \geq 0 \) means

\[
\forall m \neq 0 : \frac{1}{N} \sum_{n=0}^{N-1} \omega_n^{mn} \log \xi_n \geq 0.
\] (54)

Interestingly, the interaction times \( t_m \) entering Eq. (50) can be obtained by summation over the second index of \( t_{mn} \) introduced in Eq. (19) when it is rewritten by two indices \( m, n \) rather than \( \mu \). This observation suggests a closer relation between quantum accessible channels and classical accessible transition matrices. The following theorem shows that \( A_N^C \) is the image of \( A_N^Q \) in classical space, i.e. \( D_h(A_N^Q) = A_N^C \), where hyper-decoherence \( D_h \) is defined by the projection (6).

**Theorem 15.** Let \( \Phi_\beta \) be a Weyl channel accessible by a Lindblad generator in Eq. (16), i.e. \( \Phi_\beta = e^{\lambda \xi} \in A_N^Q \). Then the corresponding circulant bistochastic matrix \( T(\Phi) \) belongs to the set of transition matrices accessible by a classical semigroup whose generator \( K \) is defined by Eq. (44) based on \( L \), i.e. \( T = e^{tK} \in A_N^C \) and \( K \) is related to \( L \) through Lemma 12. Hence for a generator \( L \) related to a Weyl channel the hyper-decoherence commutes with the time evolution,

\[
D_h(e^{tL}) = e^{tD_h(L)} = e^{tK}.
\] (55)

**Proof.** We start the proof by finding classical transition of \( \Phi = e^{t\xi} \). Assuming \( t_{mn} = tp'_{mn} \), and rewriting Eq. (27) using two indices \( k, l \) one has:

\[
\Phi_\beta = \sum_{k,l} p_{k,l} U_{kl} \otimes U_{kl},
\] (56)

where

\[
p_{k,l} = \frac{1}{N^2} \sum_{ij} \omega_N^{kj-il} \exp \left( \sum_{mn} (\omega_N^{-jm} - 1) t_{mn} \right).
\] (57)

Hence \( T \) can be gained by Eq. (36), in which the probabilities \( q_k \) read,

\[
q_k = \sum_l p_{k,l} = \frac{1}{N^2} \sum_{ij} \omega_N^{kj} \sum_l (\omega_N^{-il}) \exp \left( \sum_{mn} (\omega_N^{-jm} - 1) t_{mn} \right)
\]

\[
= \frac{1}{N} \sum_j \omega_N^{kj} \exp \left( \sum_m (\omega_N^{-jm} - 1) t_m \right),
\] (58)

where \( t_m = \sum_n t_{mn} = t \sum_n p'_{mn} = tq'_m \), while \( q'_m \) is defined by Eq. (44). Equalities (48) and (58) complete the proof. Furthermore, Eq. (48) explicitly represents a valid probability vector \( q \) with non-negative components, as it was claimed before.

Thus the classical image of a quantum accessible Weyl channel is a circulant bistochastic matrix accessible by a classical semigroup. An immediate consequence of this fact and Prop. 7 is the following Corollary.

**Corollary 16.** The set \( A_N^C \) of circulant bistochastic transition matrices, accessible by a semigroup generated by Kolmogorov operators (44), is star-shaped with respect to the flat matrix, \( T_* = \frac{1}{N} \sum_k X_k \), in analogy to its quantum counterpart \( A_N^Q \).

Already we have seen the set of classical accessible transitions are log-convex and star-shaped. The following Proposition shows how one can get the boundaries of the set \( A_N^C \).

**Proposition 17.** The boundaries of the set, \( \partial A_N^C \), are accessible when at least one of the probabilities \( q_k \) related to non-trivial Kolmogorov operators \( (k \neq 0) \) (44) is zero. In other words, if one takes at most \( N - 2 \) out of \( N - 1 \) non-trivial Kolmogorov operators, the resultant transitions, \( e^{tK} \), form the boundaries \( \partial A_N^C \).
Proof. To prove the claim, note that if $T$ belongs to $A_{N}^{C}$ it is connected to $I_{N}$ through a trajectory that at each $t \geq 0$ belongs to the set, so does for small time $t = \epsilon$. Expanding $e^{tK}$ around $t = 0$, one has:

$$e^{tK} = \left( I_{N} + \epsilon \frac{dT}{dt} |_{t=0} \right) = \left( I_{N} + \epsilon K \right) = (1 - \epsilon (1 - q_{0}')) I_{N} + \epsilon \sum_{k=1}^{N-1} q_{k}' X^{k},$$

that should describe a proper bistochastic transition for small $\epsilon > 0$. This means all of its entries should be non-negative and sum of them over each row and column equals to unity. The latter is satisfied automatically. The first condition means $1 - \epsilon (1 - q_{0}') \geq 0$ and $\epsilon q_{k}' \geq 0$ for $k \in \{1, \ldots, N - 1\}$. Regarding that $\epsilon$ is small $1 - \epsilon (1 - q_{0}')$ is always positive, however, if $q_{k}'$ is taken to be zero for $k \in \{1, \ldots, N - 1\}$, the second inequality is satisfied with equality. So in that case we get the boundaries of the set. □

D. Spectra of accessible quantum Weyl channels and circular bistochastic matrices

In this subsection we analyze the boundary of certain cross-sections of the set $A_{N}^{Q}$ of accessible Weyl channels and characterize the support of their spectra in the complex plane. As Proposition 9 relates the eigenvalues of a superoperator $\Phi_{p}$, corresponding to a Weyl channel, with eigenvalues of its classical action, described by a bistochastic matrix, our results concern also support of spectra of transition matrices accessible by a classical semigroup. Circulant bistochastic matrices of order $N$ are spanned by the cycle permutation matrix $X$ and its powers, $X, X^{2}, ..., X^{N}$, where $X^{N} = I_{N}$.

We consider the spectrum of accessible Weyl channels $A_{N}^{Q}$, i.e., for which there exists a trajectory, $z(t)$, such that $z(t = 0) = \Phi_{0}$ and belongs to the set of Weyl channels at all times $t$. This is equivalent to the statement that the trajectory in the space of transition matrices corresponding to $z(t)$ by hyper-decoherence belongs to the simplex $\Delta$ for all times $t$. In order to obtain an analytical description, it suffices to study the behavior of the curve at times $t$ close to 0. Using this observation, we get the boundary of spectrum $A_{N}^{Q}$ by a parametric expression,

$$e^{\pi t \tan(\pi/N)} \text{ for } t \in [0, \pi].$$ (59)

Note that in the region described above, equal to the support of spectra of accessible classical transition matrices from $A_{N}^{C}$, there is a line segment of negative values. More precisely, the intersection of the spectra of accessible classical and quantum maps with the real line is $A_{N} \cap \mathbb{R} = \left[ -e^{-\pi \tan(\pi/N)}, 1 \right]$. Thus, the set $A_{N}^{C}$ of bistochastic matrices accessible by a classical semigroup contains also matrices with degenerated eigenvalues which are real and negative. The minimum real eigenvalue for a matrix $T$ belonging to $A_{N}^{C}$, or for a superoperator $\Phi$ from $A_{N}^{Q}$, reads

$$x_{\min}(N) = -e^{-\pi \tan(\pi/N)} = -1 + \frac{\pi^{2}}{N} + O \left( \frac{1}{N^{2}} \right).$$ (60)

For $N = 3$ this number is close to zero, $x_{\min}(3) = -e^{-\sqrt{3}\pi} \approx -0.00433$, but for large dimension one has, $x_{\min}(N) \to -1$, and the support of the spectrum of accessible maps covers completely the unit disk. In order to calculate the area of this set, we consider real and imaginary part of the parametric expression, $x(t)$ and $y(t)$, respectively

$$x(t) = \cos(t) e^{-\pi \tan(\pi/N)},$$
$$y(t) = \sin(t) e^{-\pi \tan(\pi/N)}.$$ (61)

Next, we calculate the area enclosed by this curve using standard analytical methods

$$V_{N} = 2 \int_{0}^{\pi} x(t) y'(t) dt = \frac{1}{2} \left( 1 - e^{-2\pi \tan(\pi/N)} \right) \cot \left( \frac{\pi}{N} \right) = \pi - \frac{\pi^{3}}{N} + O \left( \frac{1}{N^{2}} \right),$$ (62)

which for $N = 3$ gives $V_{3} = 1.5e^{-2\sqrt{3}\pi} \approx 0.28867$. Observe that for any $z \in \mathbb{C} \setminus \mathbb{R}$, with $|z| \leq 1$, the curve $z^{t}$ tends to 0 as $t \to \infty$. The trajectory is a logarithmic spiral ($Spira mirabilis$), and therefore, it crosses the negative real axis infinitely many times.

It should be mentioned here that in the mathematical literature, stochastic matrices accessible by a classical semigroup are called embeddable and an expression equivalent to Eq. (59) appeared in the literature [25, 26]. The characteristic heart-like set plotted in Fig. 2a was analyzed recently [28] in the context of Markovian evolution of quantum coherence for $N = 3$ systems. Note that the logarithmic spiral (59) plotted in the case $N = 3$ in Fig. 2a, is
FIG. 2. Shaded polygons, spanned by roots of unity, represent support of the spectra of superoperators $\Phi_p$ representing all Weyl channels acting in dimension $N = 3, 4, 5, 10$ for panels a)-d), respectively. Support of the spectra of accessible Weyl channels from $\mathcal{A}_N^Q$ is denoted by a dark set bounded by two logarithmic spirals (59), which cross at $x_{\text{min}} = -e^{-\pi \tan(\pi/N)}$, so that the origin, $z = 0$, belongs to its interior. The same shapes correspond to spectra of all circulant bistochastic matrices and matrices accessible by a classical semigroup from $\mathcal{A}_N^C$.

relevant for our study in several ways: it characterizes a) the boundary of the set of quantum accessible Weyl channels $\mathcal{A}_3^Q$ at the triangular face $\Delta(\Phi_2, \Phi_X, \Phi_{X^2})$ – see Fig. 3a; b) the boundary of the set $\mathcal{A}_3^C$ of classical accessible matrices in the triangle $\Delta(X, X^2, I_3)$ of circular bistochastic matrices – see Fig. 5c; the boundary of the support of the spectrum of members of c) quantum accessible maps from $\mathcal{A}_3^Q$; d) classical accessible transition matrices from $\mathcal{A}_3^C$ – see Fig. 6c. The two latter statements concerning the boundary of the spectra in the complex plane are also valid for higher dimensions $N$ – see Fig. 2b-d.

VI. LINDBLAD DYNAMICS FOR QUTRITS

A. Explicit criterion for accessibility by a Lindblad semigroup

Let us now apply previous results for the special case $N = 3$. Although the explicit form of Weyl unitary matrices is not relevant here, we wish to explain the notation for future reference. For the sake of brevity, we are going to use one index $\mu \in \{0, \ldots, 8\}$ instead of two $k, l \in \{0, 1, 2\}$, related by $\mu = 3k + l$. This means, up to a phase, $U_1$ is equal to $U_1$, $U_3$ is $U_6$, $U_4$ equals to $U_8$, and $U_5$ is $U_7$ which in turn implies following relations on the eigenvalues: $\lambda_2 = \lambda_1$, $\lambda_3 = \lambda_6$, $\lambda_4 = \lambda_8$, and $\lambda_5 = \lambda_7$.

In the case $N = 3$ the Hermitian matrix $H$ of size $N^2 = 9$, defined in Eq. (13) can be represented in the element-wise
These functions are analogous to Eq. (32) valid for logarithm notation useful for complex Hadamard matrices [49].

\[
\log H = \frac{2\pi i}{3} \left( \begin{array}{cccc} 
\bullet & \bullet & 2 & 2 \\
\bullet & 2 & 1 & 1 \\
2 & 2 & 1 & 1 \\
1 & 1 & 2 & 2 \\
\bullet & 1 & 2 & 1 \\
\bullet & 1 & 2 & 2 \\
1 & 2 & 1 & 1 \\
2 & 1 & 1 & 1 \\
\end{array} \right),
\] (63)

where \( \bullet \) represents zero, so that \( H_{11} = \exp(0) = 1 \), while \( H_{24} = \exp(i4\pi/3) = \omega_3^2 \). Applying \( H \) on the probability vector \( p_\mu \) defining the mixed unitary channel (5) yields eigenvalues \( \lambda_\mu \) of the superoperator \( \Phi_\rho \). To describe a legitimate quantum channel accessible by a semigroup, \( \Phi_\rho \in \mathcal{A}_3^Q \), the interaction times need to be non-negative, \( t_\mu \geq 0 \), which due to Eq. (19) yields constraints for the complex eigenvalues \( \lambda_\mu \).

\[
\log \left( \frac{|\lambda_1|^2}{|\lambda_3||\lambda_4||\lambda_5|} \right) \pm \sqrt{3}(\theta_3 + \theta_4 + \theta_5 + M_3 + M_4 + M_5) \geq 0,
\]

\[
\log \left( \frac{|\lambda_2|^2}{|\lambda_1||\lambda_4||\lambda_5|} \right) \pm \sqrt{3}(\theta_1 + \theta_4 - \theta_5 + M_1 + M_4 - M_5) \geq 0,
\]

\[
\log \left( \frac{|\lambda_4|^2}{|\lambda_1||\lambda_3||\lambda_5|} \right) \pm \sqrt{3}(\theta_1 - \theta_3 + \theta_5 + M_1 - M_3 + M_5) \geq 0,
\]

\[
\log \left( \frac{|\lambda_5|^2}{|\lambda_1||\lambda_3||\lambda_4|} \right) \pm \sqrt{3}(\theta_1 + \theta_3 - \theta_4 + M_1 + M_3 - M_4) \geq 0.
\] (64)

Here \( \theta_i \) denotes the phase of \( \lambda_i \) and \( M_i \in \mathbb{Z} \) is an integer. Since the arguments of the logarithm have to be positive the second line implies \( |\lambda_3| \geq \sqrt{|\lambda_1||\lambda_4||\lambda_5|} \). Substituting this to an analogous equation from the first line, \( |\lambda_1|^2 \geq \sqrt{|\lambda_3||\lambda_4||\lambda_5|} \), we obtain an inequality, \( |\lambda_1|^{3/2} \geq |\lambda_4|^{3/2} |\lambda_5|^{3/2} \). Taking both sides to the power \( 2/3 \) and repeating these steps with the other equations, we arrive at the following necessary condition on the eigenvalues:

\[
|\lambda_\alpha| \geq |\lambda_\beta||\lambda_\gamma|,
\] (65)

for any choice of indices but the leading one, \( \alpha \neq \beta \neq \gamma \neq 0 \). These conditions are analogous to relations (31) defining the set \( \mathcal{A}_3^Q \) of qubit channels accessible by a semigroup [17, 18], except that for \( N = 3 \) these conditions are necessary but not sufficient.

### B. Product probability vectors

Applying Eq. (20), we obtain eigenvalues of a dynamical semigroup as an explicit function of time:

\[
\lambda_0 = 1,
\]

\[
\lambda_{1,2} = e^{-\frac{2}{3}(t_3 + t_4 + t_5 + t_6 + t_7 + t_8)} e^{\frac{2i}{3}(t_3 - t_4 - t_5 + t_6 + t_7 + t_8)},
\]

\[
\lambda_{3,6} = e^{-\frac{2}{3}(t_1 + t_2 + t_4 + t_5 + t_6 + t_7 + t_8)} e^{\frac{2i}{3}(t_1 - t_2 + t_4 - t_5 + t_6 + t_7)},
\]

\[
\lambda_{4,8} = e^{-\frac{2}{3}(t_1 + t_2 + t_3 + t_5 + t_6 + t_7)} e^{\frac{2i}{3}(t_1 - t_2 - t_3 + t_5 + t_6 + t_7)},
\]

\[
\lambda_{5,7} = e^{-\frac{2}{3}(t_1 + t_2 + t_3 + t_4 + t_6 + t_8)} e^{\frac{2i}{3}(t_1 - t_2 - t_3 - t_4 + t_6 + t_7)}
\]

For channels \( \Phi_\rho \) accessible by a semigroup, the probabilities can be expressed by Eq. (27) as explicit functions of time. These functions are analogous to Eq. (32) valid for \( N = 2 \), but contain many more terms. For instance, in the case
$N = 3$ the time dependence of $p_0$ has the form
\[
p_0 = \frac{1}{9} \left( 1 + 2e^{-\frac{3i}{2}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7)} \cos \left( \frac{\sqrt{3}}{2}(t_1 - t_2 - t_3 - t_4 + t_5 - t_6 - t_7) \right) 
+ 2e^{-\frac{3i}{2}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8)} \cos \left( \frac{\sqrt{3}}{2}(t_1 - t_2 + t_3 - t_4 + t_5 + t_6 + t_7) \right) 
+ 2e^{-\frac{3i}{2}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8)} \cos \left( \frac{\sqrt{3}}{2}(t_1 - t_2 + t_3 - t_4 + t_5 + t_6 - t_7) \right) \right).
\]
\[
p_0 p_i = p_j p_k,
\]
for any different choice of $i, j, k$ from the set $\{1, 2, 3\}$. In the case $N = 3$ we are dealing with 8-dimensional simplex and product probability vectors form at most a 4-dimensional subset of the 7-dimensional boundary of $A_3$. Therefore, there are other channels at the boundary represented by a probability vector $p$ without the product structure.

**Proposition 18.** Consider any four out of eight Lindblad generators $L_\mu$ defined by Eq. (14) whose corresponding unitaries $U_\mu$ are Hermitian conjugates up to a phase. Let us represent them by $L_\alpha, L_\beta, L_\gamma, L_\delta$ for which, up to a phase, $U_\alpha$ equals to $U_\beta^\dagger$ and $U_\alpha$ equals to $U_\gamma^\dagger$. For this set the associated dynamical semigroup defined by Eq. (15) belongs to the 4-dimensional boundaries of $A_3^Q$ and the corresponding probability vector possesses the product structure.

**Proof.** The corresponding 8-dimensional vector $t_\mu$ of interaction times has four components equal to zero and four non-zero components, so the channel belongs to the boundary of $A_3^Q$. To see the product structure of the probability vector $p_\mu$ one can express probabilities as explicit functions of time through Eq. (27). For the probabilities appearing in $A_3^Q$, one has $p_\mu = \hat{w}(a, b) \times \hat{w}(c, d)$ in which $\hat{w}$ is a permutation of the probability vector $\hat{p}$, while $\hat{w}(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y))$ is a local probability vector of size three given by
\[
f_1(x, y) = \frac{1}{3} \left( 1 + 2e^{-\frac{3i}{2}(t_x + t_y)} \cos \left( \frac{\sqrt{3}}{2}(t_x - t_y) \right) \right),
\]
\[
f_2(x, y) = \frac{1}{3} \left( 1 - e^{-\frac{3i}{2}(t_x + t_y)} \cos \left( \frac{\sqrt{3}}{2}(t_x - t_y) \right) \right) + \sqrt{3}e^{-\frac{3i}{2}(t_x + t_y)} \sin \left( \frac{\sqrt{3}}{2}(t_x - t_y) \right),
\]
\[
f_3(x, y) = \frac{1}{3} \left( 1 - e^{-\frac{3i}{2}(t_x + t_y)} \cos \left( \frac{\sqrt{3}}{2}(t_x - t_y) \right) \right) - \sqrt{3}e^{-\frac{3i}{2}(t_x + t_y)} \sin \left( \frac{\sqrt{3}}{2}(t_x - t_y) \right).
\]
\]

A particular case of Proposition 18 corresponds to the channels with real eigenvalues $\lambda_\mu$. Note that this case is not generic: sampling uniformly vectors $\vec{r}, \vec{s} \in \Delta_3$ generates channels with complex eigenvalues with probability one. This can be seen by decomposing the eigenvalues of $\Phi \mu$ into its real and imaginary parts. Equating the imaginary part of each eigenvalue to 0 impose a large list of restrictions on $\vec{r}$ and $\vec{s}$, leaving at most one free parameter of the original four $(r_1, r_2, s_1, s_2)$. Thus, the region in the space of mixed unitary channels, for which all the eigenvalues $\lambda_\mu$ are positive has zero measure.

An interesting implication of Proposition 18 is that inside the set $A_3^Q$ of accessible maps there are no channels with rank 2. In other words, the edges connecting identity map $U_0$ with the Weyl unitary rotations $U_\mu$ contain no accessible maps. This fact is visualized in Fig. 3 which presents several cross-sections of the 8D simplex $\Delta_8$ of mixed unitary Weyl channels and its subset $A_3^Q$ of accessible maps.

Another difference concerning the structure of the set of accessible maps for $N = 2$ and $N = 3$ is that in the former case $p_0$ is the largest component for the probability vector of $\Phi \in A_3^Q$ for any $t \geq 0$ [18]. This is not the case for $N = 3$, even if we choose only one interaction time $t_\mu$ to be nonzero – see Fig. 3a.

**C. Accessible maps for qubits and qutrits**

To describe properties of the set $A_N^Q$ of maps accessible by a semigroup we shall compare the volumes of these sets for $N = 2$ and $N = 3$ relative to the volumes of the simplex $\Delta_{N+1}$ of Weyl mixed unitary channels. In the qubit case the set of mixed unitaries corresponds to the tetrahedron spanned by the identity operation, $\Phi_0 = I \otimes I$, and unitary
rotations associated to three Pauli matrices, \( \Phi_1 = \sigma_x \otimes I, \Phi_2 = \sigma_y \otimes I, \) and \( \Phi_3 = \sigma_z \otimes I. \) As the length of each edge of the regular tetrahedron can be set to unity, its volume reads, \( V_{\Delta_3} = \sqrt{2}/12. \)

To calculate the volume of its subset \( A_3^Q \) we use Eq. (32) and work with three-dimensional Cartesian coordinates. Each point inside the tetrahedron \( \Delta_3 \) can specified by its coordinates \( X, Y, Z. \) For channels in \( \Delta_3 \) which also belong to \( A_3^Q, \) the probabilities \( p_0, p_1, p_2, \) and \( p_3 \) are defined by Eq. (32) suggesting \( X, Y, \) and \( Z \) as an explicit function of \( t_1, t_2, \) and \( t_3. \) Let \( |J| = \exp(-A(t_1 + t_2 + t_3)) \sqrt{2} \) denotes the Jacobian determinant associated to such a change of variables. The volume of the set of one-qubit accessible maps reads then

\[
V_{A_2} = \iiint_{A_2} dX dY dZ = \iiint |J| dt_1 dt_2 dt_3 = \frac{1}{\sqrt{2}} \int_0^\infty e^{-4t_1} dt_1 \int_0^\infty e^{-4t_2} dt_2 \int_0^\infty e^{-4t_3} dt_3 = \frac{1}{4^3 \sqrt{2}}.
\]  

Hence the relative volume of the set of maps accessible by a semigroup is \( V_{A_2}/V_{\Delta_3} = 3/32 = 0.09375, \) as it forms a quarter of the volume of the set \( U_2^Q \) of one-qubit unistochastic channels investigated in [20, 42, 55]. On the other hand, for the qutrit channels the analogous ratio gained numerically is much smaller, \( V_{A_2}/V_{\Delta_3} = 9.02 \times 10^{-4} \pm 1.8 \times 10^{-5}. \) This estimation was obtained by sampling 10^6 different random probability vectors \( \vec{p} \in \Delta_3 \) and verifying whether the corresponding mixed unitary channel satisfies all conditions (19) for accessibility. As for qubit channels this ratio is larger by two orders of magnitude, one can expect a significant decrease of the relative volume of accessible channels with their dimension \( N. \)

To analyze the structure of the set \( A_3^Q \) of accessible maps we shall analyze cross-sections of the simplex \( \Delta_8 \) determined by fixing the weight \( p_0 \) of the identity component of a mixed unitary channel (1). Consider an 8-point random vector \( \vec{\varphi} \in \Delta_7 \) from the 7 dimensional simplex and define a 9 dimensional vector, \( \vec{p} = [p_0, (1-p_0)\vec{\varphi}]. \) The corresponding mixed unitary channel \( \Phi_\vec{p} \) is thus given by Eq. (5).

Generating auxiliary vectors \( \vec{\varphi} \) according to the flat measure in the simplex \( \Delta_7 \) we obtain an ensemble of random vectors \( \vec{p} \) and thus random unitary channels with a fixed size of the identity component \( p_0 \) and study the fraction of
probability vectors corresponding to accessible channels. Numerical results presented in Fig. 4 show that for $N = 3$ the relative volume of the analyzed set $A_3^Q$, grows monotonically to 1 as a function of the weight $p_0$ of the probability vector defining the channel. For comparison we plotted analogous data obtained for the case $N = 2$ for which the relative volume grows faster with the weight $p_0$. However, in this case the curve has an inflection point which does not occur for $N = 3$. The inset shows behavior for small values of $p_0$, as the relative volume is positive already for $p_0 < 1/9$. This observation is consistent with Fig. 3a, showing that in the case $N = 3$ there exist accessible maps $\Phi_p$, for which the component $p_0$ is not the largest one.

![Fig. 4](image)

**FIG. 4.** Relative volume $V_{\text{acc}}$ of the set $A_3^Q$ of accessible channels at a cross-section of the probability simplex $\Delta_8$ of mixed unitary channels specified by the fixed value of the identity component $p_0$. Dotted blue line represents numerical results for $N = 3$ estimated by the fraction of accessible maps. Each point was calculated using samples of size $10^5$ channels, so the error bars are smaller than the size of the dot. Inset zooms in the part of the plot, for which the relative volume is positive. Continuous orange curve represents analytical results obtained for $N = 2$.

### D. Bistochastic matrices accessible by $N = 3$ classical semigroup

As discussed in Section V, any mixed Weyl channel decoheres to a classical circulant bistochastic matrix. We shall analyze the issue here for $N = 3$ in some more detail. Related question of describing the subset of the Birkhoff polytope containing these bistochastic matrices of order three, so that their square root is bistochastic, was studied in [56].

Consider a random Weyl channel $\Phi_\rho$ distributed uniformly from the triangle $\Delta(\Phi_0, \Phi_X, \Phi_X^2)$ shown in Fig. 3a – a 2D face of the $8$D simplex $\Delta_8$ of all Weyl channels. Circulant bistochastic transition matrices, obtained by hyper-decoherence, $T = D_h(\Phi_\rho)$ cover the triangle $\Delta(1, \omega_3, \omega_3^2)$ uniformly – see Fig. 5a. This is a consequence of the fact that in this subspace the supermap $D_h$ is linear and transforms a regular triangle into itself,

$$D_h\left(\sum_{j=0}^2 q_j \Phi_{X^j}\right) = \sum_{j=0}^2 q_j X_j = T_q.$$ 

On the other hand, if one takes an average over the entire set $E_3^Q$ of Weyl channels and samples the probability vector $\tilde{\rho}$ uniformly in the simplex $\Delta_3$, the distribution of classical matrices exhibits maximum at the flat matrix $T_*$ with all entries equal to $1/3$ – see Fig. 5b. If the sampling is restricted to the set of accessible quantum channels only, its image under decoherence is supported inside the characteristic ‘heart–like’ shape presented in panel Fig. 5c.

In Eq. (37) we used the fact that a circulant matrix $T$ can be diagonalized by the Fourier matrix $F$. If the transition matrix is obtained by the convex combination of permutation matrices, $T = \sum_{i=0}^2 q_i X_i^j$, then its spectrum consists of $1, z, \bar{z}$, where $z = \sum_{i=0}^2 q_i \omega_3^j$. Therefore, if the distribution of the probability vector $q$ is uniform in the triangle of bistochastic matrices, so is the level density of subleading eigenvalues in the regular triangle $\Delta(1, \omega_3, \omega_3^2)$ inscribed in the unit disk — compare Fig. 6a.

To analyze the shape of the set $A_3^C$ of circulant bistochastic matrices accessible by a classical semigroup we may use Proposition 17. Since the hyper-decoherence transformation, $T = D_h(\Phi_\rho)$, acts linearly (69) in the triangle of quantum Weyl channels $\Delta(\Phi_0, \Phi_X, \Phi_X^2)$, the shape of the set $A_3^C$ of classically accessible matrices is identical with cross-section of the set $A_3^Q$ of quantum accessible maps presented in Fig. 3d. This set is uniformly covered when the
FIG. 5. Distribution of random circulant bistochastic matrices of size $N = 3$ inside the regular triangle $\Delta(3, X_3, X_3^2)$ obtained by hyper-decoherence of: a) random Weyl channels distributed uniformly from the 2D triangle $\Delta(\Phi_3, \Phi_{X_3}, \Phi_{X_3^2})$; b) random channels distributed uniformly from the entire 8D simplex of Weyl channels; c) random channels distributed uniformly from the set $A_Q^3$ of quantum accessible channels. Light blue line represents the boundary of the set $A_C^3$ of matrices accessible by classical semigroups – observe similarity with Fig. 2a.

same region, corresponding to quantum channels, is uniformly sampled. The structure of this star-shaped set, bounded by fragments of two symmetric logarithmic spirals, is discussed in Section V D. This set appeared in the literature in context of analyzing Markovian evolution of quantum coherence and $N = 3$ embeddable stochastic matrices [28].

FIG. 6. Probability distribution for superimposed spectra of $10^4$ quantum maps acting on 3-level systems: a) Weyl channels uniformly distributed in the triangle $\Delta(\Phi_3, \Phi_{X_3}, \Phi_{X_3^2})$, b) random Weyl channels $\Phi_3$ choosing $\hat{p}$ from the flat distribution on $\Delta_3$, c) random Weyl channels distributed uniformly from the set $A_Q^3$ of accessible quantum channels and spectra of the corresponding random circulant bistochastic matrices of size $N = 3$: d) flat measure on the simplex $\Delta_3$ of circulant bistochastic matrices, e) obtained by hyper-decoherence $T(\Phi_3)$ of random Weyl channels $\Phi_3$ from $\Delta_3$, f) accessible circulant bistochastic matrices $A_C^3$, obtained by hyper-decoherence $T(\Phi_3)$ of accessible quantum channels from $A_Q^3$ – the boundaries of the support are shown in Fig. 2a. Due to Proposition 9 panels d), e) and f) coincide with panels a), b), and c) respectively, so they are not plotted.

VII. SET $U_Q^3$ OF UNISTOCHASTIC CHANNELS ACTING ON $N = 3$ SYSTEMS

As for $N = 2$ the set $A_Q^2$ of quantum accessible channels forms a quarter of the set $U_Q^2$ of unistochastic channels, so the relation $A_Q^2 \subset U_Q^2$ holds [18]. Hence in this Section we compare the corresponding sets for $N = 3$. A similar analysis will be performed for the analogous classical sets inside the Birkhoff polytope $B_C^3$ of bistochastic matrices of this size.

A. Unistochastic channels and channels accessible by a semigroup

We start demonstrating that for one-qutrit system the set of quantum accessible channels is not included in the set of unistochastic channels, written $A_Q^3 \not\subset U_Q^3$. 


**Proposition 19.** For 3-level systems, there exists a channel accessible by a Lindblad semigroup which is not unistochastic.

**Proof.** To show a counterexample we analyze Weyl channels being a convex combination of $I$, $Z$, and $Z^2$,

$$
\Phi_p = p_1 I \otimes I + p_2 Z \otimes Z + (1 - p_1 - p_2)Z^2 \otimes Z^2.
$$

(70)

We wish to find a vector $p = (p_1, p_2, p_3)$ such that there is no unitary dilation matrix $U \in U(9)$ for which Eq. (7) produces the analyzed channel $\Phi_p$. Given that $Z$ is diagonal, the dilation matrix $U$ is block diagonal, with three unitary blocks $W_1, W_2, W_3$, each of size 3,

$$
U = W_1 \otimes W_2 \otimes W_3 = \text{diag}(1, 0, 0) \otimes W_1 + \text{diag}(0, 1, 0) \otimes W_2 + \text{diag}(0, 0, 1) \otimes W_3.
$$

By unistochasticity of the channel $\Phi_p$ the corresponding Choi matrix $D = \Phi_p^R$ has to satisfy $D = \frac{1}{N}(U^{R})^\dagger U^{R}$. This leads to the following conditions for $W_1, W_2$ and $W_3$:

$$
\text{Tr} W_1^\dagger W_1 = s, \quad \text{Tr} W_2^\dagger W_1 = s^*, \quad \text{Tr} W_3^\dagger W_2 = s,
$$

(71)

where $s = \frac{1}{2}(3p_1 - 1) + \frac{i\sqrt{3}}{2}(p_1 + 2p_2 - 1)$.

Consider a probability vector with three non zero entries given by $p_c = (p_1, p_2, 1 - p_1 - p_2) = (35, 4, 1)/40$. Using $p_c$ in Eq. (70), we can verify that $\Phi_{p_c}$ satisfies the three properties mentioned at the end of Subsection II B, which implies that the channel is accessible, $\Phi_{p_c} \in \mathcal{A}_2^Q$. Making use of the parametrization of the group of unitary matrices of size 3 one can show that there are no unitary matrices $W_1, W_2, W_3 \in U(3)$ which satisfy conditions (71) for the vector $p_c$ selected above: if any two of these equalities are satisfied, the third one is not. Hence for this $p$ there is no unitary dilation matrix $U \in U(9)$ entering expression (7), so the channel $\Phi_p$ is not unistochastic.

Although for 2-level systems, the following inclusion relations holds $\mathcal{A}_2^Q \subset \mathcal{U}_2^Q$ [18], we have shown that the analogous property is not valid for $N = 3$.

**B. Unistochastic and N-unistochastic matrices**

Due to the process of hyper-decoherence a quantum stochastic map undergoes transition to a classical stochastic matrix. We shall analyze relations between subsets of the set $\mathcal{S}_N$ of stochastic matrices introduced in Section II C.

**Proposition 20.** For any $N$-level system, the set of unistochastic matrices $\mathcal{U}_N^S$ is contained in the set $\mathcal{D}_N(\mathcal{U}_N^Q)$ of transition matrices obtained by the hyper-decoherence [9] of a unistochastic channel.

**Proof.** It is enough to take a dilation matrix of the product form, $U = V \otimes I_N$. Then due to Eq. (8) one obtains $T = V \otimes V$, where $\otimes$ denotes the Hadamard product, so the classical transition matrix is unistochastic. Thus the set of classical transition matrices obtained by hyper-decoherence of unistochastic channels $\mathcal{U}_N^Q$ containing $N$-unistochastic matrices, includes the set of unistochastic matrices, $\mathcal{D}_N(\mathcal{U}_N^Q) = \mathcal{U}_{N,N} \supset \mathcal{U}_N^S$ — compare table I.

**C. Quantum unistochastic maps and classical unistochastic matrices**

To explore relations between unistochastic maps and matrices, we will restrict our attention to the face of the set $\mathcal{W}_N$ of mixed Weyl channels determined by three points: $I$, $U_X$, and $U_{X^2}$. In this equilateral triangle we need to distinguish two regions:

i) the region bounded by the 3-hypocycloid [39] denoted by $\mathbb{H}_3$. For $p_1, p_2 \in \Delta_2$, the set $\mathbb{H}_3$ includes points satisfying the inequality,

$$
4p_1^2(1 - p_1 - p_2)p_2 - (p_1 - p_1^2 - (1 - p_1 - p_2)p_2)^2 \geq 0.
$$

The 3-hypocycloid determines the boundary of the set of unistochastic matrices at the cross-section of the $N = 3$ Birkhoff polytope [39], as the above expression implies positivity of the quantity (A2), which determines the Jarlskog invariant of the corresponding bistochastic matrix—see Appendix A.

ii) the region resembling the Star of David, formed by two equilateral triangles, $\Delta_A$ and $\Delta_B$, each one described by three vertices on the plane:

$$
\Delta_A: (1/3, 0), (1/3, 1/\sqrt{3}), (5/6, 1/2\sqrt{3}), \quad \Delta_B: (2/3, 1/\sqrt{3}), (2/3, 0), (1/6, 1/2\sqrt{3}).
$$
These subsets of the triangle $\Delta(\mathbb{I}, \Phi_X, \Phi_{X^2})$ are represented in Fig. 7. The 3–hypocycloid plays a key role [39] in characterizing the set of unistochastic matrices of order three. We will show now that it is also relevant to describe the set of unistochastic channels and its cross-sections.

**Proposition 21.** Mixed unitary Weyl channel $\Phi_\beta$ belonging the the 3–hypocycloid, $\beta \in \mathbb{H}_3 \subset \Delta(\Phi_\beta, \Phi_X, \Phi_{X^2})$, is unistochastic, $\Phi_\beta \in \mathcal{U}_3^{Q}$, and the corresponding transition matrix is unistochastic, $T(\Phi_\beta) \in \mathcal{U}_3^{Q}$.

**Proof.** Consider a matrix $U$ of order $N^2 = 9$,

$$U = \sqrt{p_1} e^{i\varphi_3} \mathbb{I} \otimes \mathbb{I} + \sqrt{p_2} e^{i\varphi_3} X \otimes X + \sqrt{p_3} e^{i\varphi_3} X^2 \otimes X^2,$$

where $\{\varphi_i\}_{i=1}^3$ are real phases to be determined. Substituting $U$ in Eq. (7), we observe that it generates a superoperator $\Phi_\beta = p_1 \mathbb{I} \otimes \mathbb{I} + p_2 X \otimes X + (1-p_1-p_2) X^2 \otimes X^2$. Thus we have constructed a channel of the desired form, provided matrix $U$ is unitary. To analyze under what conditions it is the case we will compute the scalar product between each pair of rows. Let us introduce auxiliary real parameters, $L_1 = \sqrt{1-p_1-p_2} \sqrt{p_1}$, $L_2 = \sqrt{1-p_1-p_2} \sqrt{p_2}$, and $L_3 = \sqrt{p_1} \sqrt{p_2}$. Then the resulting expression takes the form,

$$\exp(i\varphi_1) L_1 + \exp(i\varphi_2) L_2 + \exp(i\varphi_3) L_3 = 0,$$

where $\{\varphi_i\}_{i=1}^3$ correspond to linear combinations of phases $\{\varphi_i\}$. Left-hand side of Eq. (73), can be pictured as three joined line segments on the complex plane, of length $L_1$, $L_2$, and $L_3$, respectively. Inequality holds if these three sides form a triangle. Thus Eq. (73) is equivalent to the triangle inequality:

$$|L_1 - L_2| \leq L_3 \leq L_1 + L_2,$$

This inequality is satisfied for any point $p$ in the interior of the hypocycloid, $\beta \in \mathbb{H}_3$. In such a case the columns of $U$ form an orthonormal set, hence $U$ is unitary [39]. This implies that if $\beta \in \mathbb{H}_3$ then due to Eq. (7) unitary $U$ leads to a unistochastic $\Phi_\beta$, while $T(\Phi_\beta)$ is a unistochastic matrix.

Further analysis shows that the cross-section of the set $\mathcal{U}_3^{Q}$ of unistochastic channels by the plane containing the triangle $\Delta(\Phi_\beta, \Phi_X, \Phi_{X^2})$ is larger than the hypocycloid $\beta \in \mathbb{H}_3$. Numerical results allow us to formulate the following conjecture.
Conjecture VII.1. In the face of the simplex $W_3$ of the Weyl channels formed by $\Delta_Q = \Delta(\Phi_1, \Phi_X, \Phi_X^2)$, the set $U_3^Q$ of unistochastic channels corresponds to the region formed by the union $H_3 \cup \Delta_A \cup \Delta_B$.

The triangle $\Delta_Q$ of quantum Weyl channels shown in Fig. 7a corresponds to classical triangle of bistochastic matrices $\Delta_C = \Delta(\mathbb{I}, X, X^2)$, which forms a cross-section of the Birkhoff polytope [39]. Its subset $A_C^Q$ of matrices accessible by a classical semigroup has the same structure as the set $A_Q^Q$ of accessible quantum maps shown in Fig. 5c. The region union $H_3 \cup \Delta_A \cup \Delta_B$ corresponds to the set $U_{4,3}^C$ of 3-unistochastic matrices while $H_3$ denotes its subset $U_{3}^C$ of unistochastic matrices – see Fig. 7b.

To conclude note that not every unitary matrix $U \in U(9)$ produces by Eq. (7) a unistochastic channel of order three, which decoheres to a unistochastic transition matrix. An exemplary channel $\Phi \in U_3^Q$ such that $T(\Phi) \notin U_3^C$ is described in Appendix VIII.

VIII. CONCLUDING REMARKS

In this work we investigated properties of mixed unitary Weyl channels and circulant bistochastic matrices which describe corresponding classical dynamics. Extending one-qubit results presented in [17, 18] we characterized the set $A_Q^Q$ of Weyl channels accessible by a quantum semigroup. We showed that this set is log-convex and star-shaped with respect to the completely depolarizing channel.

The standard process of decoherence, occurring due to inevitable interaction of the system analyzed with an environment, transforms any quantum state $\rho$ into the classical probability vector, $p = D(\rho) = \text{diag}(\rho)$, and destroys all quantum effects. In a similar way one analyzes analogous transitions between a quantum operation $\Phi$ and a classical transition matrix, $T = D_h(\Phi)$. Such an effect, sometimes called hyper-decoherence, can also act on a Lindblad operator $L$, which generates a quantum semigroup. The image gives a legitimate Kolmogorov operator, $K = D_h(L)$, related to a classical semigroup [53].

We showed that any Weyl channel $\Phi_p$ subjected to hyper-decoherence yields a circulant transition bistochastic matrix $T_{\bar{q}}$, where the $N$-point probability vector $\bar{q}$ arises as the marginal of the $N^2$-point initial probability vector $\bar{p}$. Furthermore, for mixed Weyl channels the dynamics commutes with decoherence, so that $D_h(\rho(L)) = e^{LD_h(L)}$. Hence the set $A_N^Q$ of transition matrices accessible by a classical semigroup, arises by hyper-decoherence of its quantum counterpart, $A_N^Q = D_h(A_N^Q)$. This implies that this classical set inherits key properties of the its quantum relative $A_Q^Q$ and is also log-convex and star-shaped.

For concreteness, we set the dimension $N = 3$ and in this case investigated geometry of the above sets. In particular, we studied various cross-sections of $8D$ set $A_Q^Q$ of accessible quantum channels and described in details the corresponding set $A_C^Q$ of embeddable bistochastic matrices [28], which are accessible by a classical semigroup. Motivated by the relation between maps accessible by a quantum semigroup and unistochastic channels established in the single qubit case, $A_Q^Q \subset U_2^Q$ [18], we studied the unistochastic operations. A constructive example of a qutrit channel is presented, which is accessible by a quantum semigroup but it cannot be obtained by the partial trace over the environment of the same size, initially prepared in the maximally mixed state. Such a counterexample allows us to conclude that for higher dimensions such a relation between unistochastic and accessible channels breaks down, as the analogous inclusion relation does not hold already for $N = 3$.

A notion of higher order, $k$–unistochastic matrices of size $N$ has been introduced. We have shown that unistochastic channels acting on $N$-level systems decohere to $N$–unistochastic matrices, $D_h(U_Q^N) = U_N^C, U_N^C$. It contains the set of unistochastic matrices, which arise by hyper-decoherence of isometric unitary transformations, $U_N^C = D_h(T_Q^N)$. Let us conclude the paper presenting a list of some open problems.

1. Results of this work are obtained for mixed unitary channels written in the Weyl basis (2). Analyze which of them can be generalized for a) mixed unitary channels written in other orthogonal basis, b) any bistochastic maps, or c) any stochastic maps.

2. For one-qubit case, channels belonging to Pauli semigroups are the only ones connected to the identity map through a trajectory formed by mixed unitary channels. Is an analogous property true for higher dimensions?

3. One-qubit Pauli maps belonging to $A_2^Q$ (accessible by a semigroup) are of rank 1, 2 or 4. For qutrits, we identified only accessible channels of rank 1, 3, and 9, so existence of $N = 3$ accessible channels with rank 2, 4, ..., 8 remains open.

4. Check if the rank of the operation does not change (or does not decrease) during the time evolution induced by a quantum semigroup $L$ specified in Eq. (15).
5. Find necessary or sufficient criteria for a $N=3$ bistochastic map $\Phi$ to be unistochastic, so there exists a unitary matrix $U \in U(9)$ leading to the representation (7).

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Appendix A: Jarlskog invariant for $N=3$ bistochastic matrices

Consider the Birkhoff polytope $B^C_3$ of bistochastic matrix of order three. Any matrix from this four-dimensional set can be parameterized by four non-negative numbers $b_1, b_2, b_3, b_4$,

$$B = \begin{pmatrix} b_1 & b_2 & 1-b_1-b_2 \\ b_3 & b_4 & 1-b_3-b_4 \\ 1-b_1-b_3 & 1-b_2-b_4 & b_1 + b_2 + b_3 + b_4 - 1 \end{pmatrix}. \quad (A1)$$

Note that these parameters do satisfy some constraints, as any entry of $B$ has to be positive and not larger than one. A bistochastic matrix $B$ is called unistochastic if there exists a unitary matrix $V$ such that $B$ can be represented by the Hadamard (entrywise) product, $B = V \odot V$, so its entries read $B_{ij} = |V_{ij}|^2$.

For any matrix $B \in B^C_3$ one introduces the quantity

$$Q(B) = 4b_1b_2b_3b_4 - (b_1 + b_2 + b_3 + b_4 - 1 - b_1b_4 - b_2b_3)^2, \quad (A2)$$

which serves as an effective tool to detect unistochasticity of bistochastic matrices of size $N=3$. If $Q(B) \geq 0$, the matrix $B$ is unistochastic [38], as the triangle inequality (74) is satisfied and there exists a corresponding unitary matrix $V$ such that $B = V \odot V$. Conversely, if $Q(B)$ is negative, the unitarity triangle cannot be constructed and $B$ is not unistochastic. This is a consequence of the fact that the triangle conditions necessary to find two orthogonal vectors forming first two columns of $V$ are not satisfied [39].

Quantity (A2) can be written as $Q = 4J^2$, where $J(B) = J(B(V))$, is called the Jarlskog invariant, originally invented for the corresponding unitary $V$ [38]. The name of this quantity is related to the following invariance property: For two unitary matrices $V$ and $V'$, equivalent with respect to permutations and multiplication from left and right by two diagonal unitary matrices, the value of $J$ computed for the associated bistochastic matrices $B$ and $B'$ is constant.

Interestingly, the maximal value of $Q$, equal to $1/27$, is attained by the flat bistochastic matrix, $B_{ij} = 1/3$, which forms the center of the Birkhoff polytope $B^C_3$. The minimal value reads, $Q_{\min} = -1/16$, and corresponds to the combination of two permutation matrices $B_S = (X_3 + X_3^T)/2$, introduced in Section II C, which is most distant from the set $U^C_3$ of unistochastic matrices [39].

Appendix B: Not every transition matrix corresponding to unistochastic channel is unistochastic

Consider the unitary matrix $U \in U(9)$ defined by

$$U = I_2 \oplus H_2 \oplus \text{adig}(1,-1,1) \oplus I_2,$$

where $H_2$ is the Hadamard matrix representing the Hadamard gate, and $\text{adig}(1,-1,1)$ is the matrix with non-zero entries in the antidiagonal. Such a two-qutrit unitary matrix $U$ defines by Eq. (7) a single-qutrit unistochastic channel $\Psi_U$. The corresponding classical transition matrix generated by hyper-decoherence, $T = D_h(\Psi_U)$, reads

$$T = \frac{1}{6} \begin{pmatrix} 5 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$
Expression (A2) implies that its squared Jarlskog invariant is negative, \( Q(T) = -\frac{1}{32} \). Therefore, the matrix \( U \in U(9) \) determines a single-qutrit unistochastic channel \( \Psi_U \), such that the corresponding classical transition matrix \( T(U) \) of size \( N = 3 \) is not unistochastic. This example shows that the set \( U_3^Q \) of unistochastic channels (yellow region in Fig. 7a) is not included inside the blue set of isometric unitary channels \( I_3^Q \) which decohere to unistochastic transition matrices \([46]\).

Appendix C: Unistochastic channels at the corners of the star of David.

In this appendix we present six unitary matrices of order \( N^2 = 9 \) which produce unistochastic channels represented in Fig. 7a by points at six corners of the Star of David for the 2-face of \( \Delta_k \). Additionally, we will show the relation between the unistochastic channels corresponding to faces spanned by the triads \((\Phi_1, \Phi_Z, \Phi_Z^z)\) and \((\Phi_1, \Phi_X, \Phi_X^z)\).

First, let us introduce a diagonal unitary matrix \( U_c \in U(9) \),

\[
U_c = \text{diag}(1, 1, 1, 1, 1, 1, \omega, 1, \omega^2),
\]

where \( \omega = e^{2\pi i/3} \). Using definition Eq. (7), we obtain the Choi matrix for the corresponding unistochastic channel,

\[
D_{U_c} = \frac{1}{3} \left( U_c^R \right)^\dagger U_c^R = \begin{pmatrix}
1 & 0 & 0 & 0 & (2 + \omega^2)/3 & 0 & 0 & 0 & (2 + \omega)/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2 + \omega)/3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (2 + \omega^2)/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2 + \omega^2)/3 & 0 & 0 & 0 & (2 + \omega)/3 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

which can be represented as \( D_{U_c} = \left( \frac{2}{3} \Phi_1 + \frac{1}{3} \Phi_Z \right)^R \). This implies that the unistochastic channel \( \Phi_{U_Z} = \frac{2}{3} \Phi_1 + \frac{1}{3} \Phi_Z \), so that it is represented by a point at one of the six corners of the star of David inscribed in the triangle \( \Delta(\Phi_1, \Phi_Z, \Phi_Z^z) \).

In a similar way one can construct unistochastic channels corresponding to the remaining corners of the star.

Let us now consider the other face of the simplex of Weyl channels determined by \((\Phi_1, \Phi_X, \Phi_X^z)\). The channel \( \frac{2}{3} \Phi_1 + \frac{1}{3} \Phi_X \) at a corner of the star of David inscribed into this triangle and plotted in Fig. 7a, corresponds to the unitary matrix \( \overline{U_cF} \). We use here the tensor product of two Fourier matrices, \( F = F_3 \otimes F_3 \), while \( (\overline{\phantom{F}}) \) denotes the complex conjugation.

Combining the tensor product of matrix \( Z \) of order three with Fourier matrices and matrix \( U_c \) introduced above, we obtain the remaining channels at six corners of the star of David at this face, listed in Tab. II.

| Unitary matrix \( U \in U(9) \) | Unistochastic channel \( \Phi_U = [(U^R)^\dagger U^R]^R \) |
|---------------------------------|-----------------------------------------------|
| \( \mathbb{F}U_c \mathbb{F} \)   | \( \frac{2}{3} \Phi_1 + \frac{1}{3} \Phi_X \) |
| \( \mathbb{F}\overline{U_c} \mathbb{F} \) | \( \frac{2}{3} \Phi_1 + \frac{1}{3} \Phi_X^z \) |
| \( \mathbb{F}(Z \otimes Z)U_c \mathbb{F} \) | \( \frac{2}{3} \Phi_1 + \frac{1}{3} \Phi_X \) |
| \( \mathbb{F}(Z \otimes Z)\overline{U_c} \mathbb{F} \) | \( \frac{2}{3} \Phi_1 + \frac{1}{3} \Phi_X^z \) |
| \( \mathbb{F}(Z^2 \otimes Z)U_c \mathbb{F} \) | \( \frac{2}{3} \Phi_X + \frac{1}{3} \Phi_Z \) |
| \( \mathbb{F}(Z^2 \otimes Z)\overline{U_c} \mathbb{F} \) | \( \frac{2}{3} \Phi_X^z + \frac{1}{3} \Phi_Z \) |

TABLE II. Unitary matrix \( U \) of size 9 (left) determining by (7) the \( N = 3 \) unistochastic channel \( \Phi_U \) (right) corresponding to a corner of the Star of David in Fig. 7a. Matrix \( U_c \) is defined in Eq. (C1), while \( \mathbb{F} = F_3 \otimes F_3 \), where \( F_3 \) denotes the Fourier matrix of order three.

As demonstrated in Table II all the corners of the star correspond to unistochastic channels. Numerical results support the conjecture that all points inside the star also represent unistochastic channels. Not being able to prove this statement so far, we can show that the problem of finding the boundary of the sets of unistochastic channels in the triangles \( \Delta(\Phi_1, \Phi_Z, \Phi_Z^z) \) and \( \Delta(\Phi_1, \Phi_X, \Phi_X^z) \) are equivalent.

**Proposition 22.** Consider a unistochastic channel \( \Phi_p = p_1 \Phi_1 + p_2 \Phi_Z + p_3 \Phi_Z^z \), associated with an unitary matrix \( U \) of order nine. Then the local transformation of \( U' = (F_3 \otimes F_3)U(F_3 \otimes F_3)^\dagger \), corresponds to the channel \( \Phi'_p = p_1 \Phi_1 + p_2 \Phi_X + p_3 \Phi_X^z \) belonging to the triangle \( \Delta(\Phi_1, \Phi_X, \Phi_X^z) \).
Proof. We consider a unistochastic channel \( \Phi_p = p_1 \Phi_1 + p_2 \Phi_Z + p_3 \Phi_Z^2 \), with associated unitary matrix \( V \). Our aim is to show that there exists a local transformation that transforms it into \( \Phi'_p \). In other words, we show that a unistochastic channel in the cross-section \((I, Z, Z^2)\) can be transformed into a unistochastic channel in the cross-section \((I, X, X^2)\) by means of a local transformation.

Firstly, we find the expression for the channel entries in four-index notation:

\[
\langle e_m f_p | \Phi | e_n f_q \rangle = \delta_{mn} \delta_{pq} (p_1 + p_2 \omega^{n-q} + p_3 \omega^{2(n-q)}) = \Phi_{mn pq} 
\]

Then the corresponding reshuffling is

\[
(\Phi R_{mn pq}) = \delta_{mm} \delta_{nn} (p_1 + p_2 \omega^{n-m} + p_3 \omega^{2(n-m)}).
\]

On the other hand, Choi matrix entries corresponding to a unitary matrix \( V \) are

\[
(\Phi_V)_{st} = \frac{1}{3} \sum_{\lambda \tau} V_{sl} \lambda V_{\tau \lambda}.
\]

Thus the relation between \( V \) entries and \( \tilde{p} = (p_1, p_2, p_3) \) is

\[
\frac{1}{3} \sum_{l \tau} V_{ml} \lambda V_{n \lambda} = \delta_{mm} \delta_{nn} (p_1 + p_2 \omega^{n-m} + p_3 \omega^{2(n-m)}).
\]

Now we proceed to find the entries of the matrix \((F_3 \otimes F_3) U (F_3 \otimes F_3)\),

\[
\langle e_m f_p | (F_3 \otimes F_3) U (F_3 \otimes F_3) | e_n f_q \rangle = \sum_{l \tau} F_{ml} \lambda U_{l \lambda} V_{\tau \lambda} = \sum_{l \tau} F_{ml} \lambda U_{l \lambda} V_{\tau \lambda} = \frac{1}{3^2} \sum_{n \lambda} \delta_{mm} \delta_{nn} (p_1 + p_2 \omega^{n-m} + p_3 \omega^{2(n-m)}) = V_{mn \nu}.
\]

What follows now is to substitute Eq. (C3)—\( V = (F_3 \otimes F_3) U (F_3 \otimes F_3)\)—in the Choi matrix definition of a unistochastic channel, Eq. (7),

\[
D_{st} = (\Phi_V)_{st} = \frac{1}{3} \sum_{l \tau} \sum_{\lambda} V_{sl} \lambda \sum_{\lambda} V_{\tau \lambda} = \frac{1}{3} \frac{1}{3^4} \sum_{l \tau} \sum_{d_3 d_4} \sum_{d_3 d_4} \omega^d_3 \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 U_{d_3 d_4}
\]

\[
\quad \times \left( \sum_{c_1 c_2 \leq c_3 c_4} \delta^c_{c_1 c_2} \delta^c_{c_3 c_4} \right) \]

\[
= \frac{1}{3^3} \sum_{l \tau} \sum_{d_3 d_4} \sum_{d_3 d_4} \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 U_{d_3 d_4} \sum_{c_1 c_2 \leq c_3 c_4} \delta^c_{c_1 c_2} \delta^c_{c_3 c_4}.
\]

Making use of the following identity, \( \sum_{\lambda} \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 U_{d_3 d_4} \), the above formulae can be reduced,

\[
D_{st} = \frac{1}{3^5} \sum_{l \tau} \sum_{d_3 d_4} \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 U_{d_3 d_4}
\]

\[
\times \left( \sum_{c_1 c_2 \leq c_3 c_4} \delta^c_{c_1 c_2} \delta^c_{c_3 c_4} \right) \]

\[
= \frac{1}{3^3} \sum_{l \tau} \sum_{d_3 d_4} \sum_{d_3 d_4} \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 U_{d_3 d_4} \sum_{d_3 d_4} \sum_{c_1 c_2 \leq c_3 c_4} \delta^c_{c_1 c_2} \delta^c_{c_3 c_4}.
\]

Now using Eq. (C2) we can rewrite the entries of the Choi matrix \( D \) corresponding to the map \( \Phi_V \)

\[
D_{st} = \frac{1}{3^2} \sum_{d_3 d_4} \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 \omega^{d_3}_3 \omega^{d_4}_3 \sum_{d_3 d_4} \sum_{c_1 c_2 \leq c_3 c_4} \delta^c_{c_1 c_2} \delta^c_{c_3 c_4} p_1 + p_2 \omega^{c_1 - c_2} + p_3 \omega^{2(c_1 - c_2)}
\]

\[
= \frac{1}{3^2} \sum_{x y} \omega^x_3 \omega^y_3 \delta^x_{x+y} (p_1 + p_2 \omega^{x-y} + p_3 \omega^{2(x-y)}).
\]

The last equation corresponds—after reshuffling—to the desired channel \( \Phi_V = \Phi'_p = p_1 \Phi_1 + p_2 \Phi_X + p_3 \Phi_X^2 \).
In general, for any pair of triangles, $\Delta(\Phi, \Phi_{\mu}, \Phi_{\nu})$ and $\Delta(\Phi, \Phi_{\nu}, \Phi_{\delta})$ with $\nu \neq \mu$, there exists a similar relationship determined by unitary matrices belonging to the Clifford group [57]. The proof of this is similar to the one given above.

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