Riesz transforms associated with Schrödinger operators acting on weighted Hardy spaces

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Abstract
Let \( L = -\Delta + V \) be a Schrödinger operator acting on \( L^2(\mathbb{R}^n), n \geq 1 \), where \( V \not\equiv 0 \) is a nonnegative locally integrable function on \( \mathbb{R}^n \). In this article, we will introduce weighted Hardy spaces \( H^p_w(\mathbb{R}^n) \) associated with \( L \) by means of the area integral function and study their atomic decomposition theory. We also show that the Riesz transform \( \nabla L^{-1/2} \) associated with \( L \) is bounded from our new space \( H^p_w(\mathbb{R}^n) \) to the classical weighted Hardy space \( H^p_w(\mathbb{R}^n) \) when \( n + 1 < p < 1 \) and \( w \in A_1 \cap RH_{(2/p)'} \).

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1 Introduction
Let \( n \geq 1 \) and \( V \) be a nonnegative locally integrable function defined on \( \mathbb{R}^n \), not identically zero. We define the form \( \mathcal{Q} \) by

\[
\mathcal{Q}(u, v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^n} V uv \, dx
\]

with domain \( \mathcal{D}(\mathcal{Q}) = \mathcal{V} \times \mathcal{V} \) where

\[
\mathcal{V} = \{ u \in L^2(\mathbb{R}^n) : \frac{\partial u}{\partial x_k} \in L^2(\mathbb{R}^n) \text{ for } k = 1, \ldots, n \text{ and } \sqrt{V} u \in L^2(\mathbb{R}^n) \}.
\]

It is well known that this symmetric form is closed. Note also that it was shown by Simon [17] that this form coincides with the minimal closure of the form given by the same expression but defined on \( C_0^\infty(\mathbb{R}^n) \) (the space of
functions with compact supports). In other words, \( C_0^\infty(\mathbb{R}^n) \) is a core of the form \( Q \).

Let us denote by \( L \) the self-adjoint operator associated with \( Q \). The domain of \( L \) is given by

\[
\mathcal{D}(L) = \{ u \in \mathcal{D}(Q) : \exists v \in L^2 \text{ such that } Q(u, \varphi) = \int_{\mathbb{R}^n} v \varphi \, dx, \forall \varphi \in \mathcal{D}(Q) \}.
\]

Formally, we write \( L = -\Delta + V \) as a Schrödinger operator with potential \( V \). Let \( \{e^{-tL}\}_{t>0} \) be the semigroup of linear operators generated by \( -L \) and \( p_t(x,y) \) be their kernels. Since \( V \) is nonnegative, the Feynman-Kac formula implies that

\[
0 \leq p_t(x,y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}
\]

for all \( t > 0 \) and \( x,y \in \mathbb{R}^n \).

The operator \( \nabla L^{-1/2} \) is called the Riesz transform associated with \( L \), which is defined by

\[
\nabla L^{-1/2}(f)(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{-tL}(f)(x) \frac{dt}{\sqrt{t}}.
\]

This operator is bounded on \( L^2(\mathbb{R}^n) \) (see [11]). Moreover, it was proved in [1,3] that by using the molecular decomposition of functions in the Hardy space \( H^1_L(\mathbb{R}^n) \), the operator \( \nabla L^{-1/2} \) is bounded from \( H^1_L(\mathbb{R}^n) \) into \( L^1(\mathbb{R}^n) \), and hence, by interpolation, is bounded on \( L^p(\mathbb{R}^n) \) for all \( 1 < p \leq 2 \). Now assume that \( V \in RH_q \) (Reverse Hölder class). In [15], Shen showed that \( \nabla L^{-1/2} \) is a Calderón-Zygmund operator if \( q \geq n \). When \( \frac{n}{2} \leq q < n \), \( \nabla L^{-1/2} \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p \leq p_0 \), where \( 1/p_0 = 1/q - 1/n \), and the above range of \( p \) is optimal. For more information about the Hardy spaces \( H^p_L(\mathbb{R}^n) \) associated with Schrödinger operators for \( 0 < p \leq 1 \), we refer the readers to [4,5,6].

In [18], Song and Yan introduced the weighted Hardy spaces \( H^1_L(w) \) associated to \( L \) in terms of the area integral function and established their atomic decomposition theory. In the meantime, they also showed that the Riesz transform \( \nabla L^{-1/2} \) is bounded on \( L^p(w) \) for \( 1 < p < 2 \), and bounded from \( H^1_L(w) \) to the classical weighted Hardy space \( H^1(w) \).

As a continuation of [18], the main purpose of this paper is to define the weighted Hardy spaces \( H^p_L(w) \) associated to \( L \) for \( 0 < p < 1 \) and study their atomic characterizations. We also obtain that \( \nabla L^{-1/2} \) is bounded from \( H^p_L(w) \) to the classical weighted Hardy space \( H^p(w) \) for \( \frac{n}{n+1} < p < 1 \). Our main result is stated as follows.
Theorem 1.1. Suppose that \( L = -\Delta + V \). Let \( \frac{n}{n+1} < p < 1 \) and \( w \in A_1 \cap RH(2/p) \). Then the operator \( \nabla L^{-1/2} \) is bounded from \( H^p_L(w) \) to the classical weighted Hardy space \( H^p(w) \).

It is worth pointing out that when \( L = -\Delta \) is the Laplace operator on \( \mathbb{R}^n \), then the space \( H^p_L(w) \) coincides with the classical weighted Hardy space \( H^p(w) \). Therefore, in this particular case, we derive that the classical Riesz transform \( \nabla (-\Delta)^{-1/2} \) is bounded on \( H^p(w) \) for \( \frac{n}{n+1} < p < 1 \), which was already obtained by Lee and Lin in [12].

2 Notations and preliminaries

First, let us recall some standard definitions and notations. The classical \( A_p \) weight theory was first introduced by Muckenhoupt in the study of weighted \( L^p \) boundedness of Hardy-Littlewood maximal functions in [13]. A weight \( w \) is a locally integrable function on \( \mathbb{R}^n \) which takes values in \((0, \infty)\) almost everywhere, \( B = B(x_0, r) \) denotes the ball with the center \( x_0 \) and radius \( r \).

We say that \( w \in A_p, 1 < p < \infty \), if

\[
\left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n,
\]

where \( C \) is a positive constant which is independent of \( B \).

For the case \( p = 1, w \in A_1 \), if

\[
\frac{1}{|B|} \int_B w(x) \, dx \leq C \cdot \text{ess inf}_{x \in B} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n.
\]

A weight function \( w \) is said to belong to the reverse Hölder class \( RH_r \) if there exist two constants \( r > 1 \) and \( C > 0 \) such that the following reverse Hölder inequality holds

\[
\left( \frac{1}{|B|} \int_B w(x)^r \, dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) \, dx \right) \quad \text{for every ball } B \subseteq \mathbb{R}^n.
\]

If \( w \in A_p \) with \( 1 < p < \infty \), then we have \( w \in A_r \) for all \( r > p \), and \( w \in A_q \) for some \( 1 < q < p \). It follows from Hölder’s inequality that \( w \in RH_r \) implies \( w \in RH_s \) for all \( 1 < s < r \). Moreover, if \( w \in RH_r, r > 1 \), then we have \( w \in RH_{r+\varepsilon} \) for some \( \varepsilon > 0 \).

Given a ball \( B \) and \( \lambda > 0 \), \( \lambda B \) denotes the ball with the same center as \( B \) whose radius is \( \lambda \) times that of \( B \). For a given weight function \( w \), we
denote the Lebesgue measure of $B$ by $|B|$ and the weighted measure of $B$ by $w(B)$, where $w(B) = \int_B w(x) \, dx$.

We give the following results that we will use in the sequel.

**Lemma 2.1 ([8]).** Let $w \in A_p$, $p \geq 1$. Then, for any ball $B$, there exists an absolute constant $C$ such that

$$w(2B) \leq C w(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \leq C \cdot \lambda^p w(B),$$

where $C$ does not depend on $B$ nor on $\lambda$.

**Lemma 2.2 ([8,9]).** Let $w \in A_p \cap RH_r$, $p \geq 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1 \left( \frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{(r-1)/r}$$

for any measurable subset $E$ of a ball $B$.

Given a Muckenhoupt’s weight function $w$ on $\mathbb{R}^n$, for $0 < p < \infty$, we denote by $L^p(w)$ the space of all functions satisfying

$$\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

Throughout this article, we will use $C$ to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{A}{B} \leq C$. Moreover, we denote the conjugate exponent of $q > 1$ by $q' = q/(q - 1)$.

### 3 Weighted Hardy spaces $H^p_L(w)$ for $0 < p < 1$ and their atomic decompositions

Let $L = -\Delta + V$. For any $t > 0$, we define $P_t = e^{-tL}$ and

$$Q_{t,k} = (-t)^k \frac{q^k P_s}{ds^k} \bigg|_{s=t} = (tL)^k e^{-tL}, \quad k = 1, 2, \ldots.$$

We denote simply by $Q_t$ when $k = 1$. First note that Gaussian upper bounds carry over from heat kernels to their time derivatives.
Lemma 3.1 ([2,14]). For every \( k = 1, 2, \ldots \), there exist two positive constants \( C_k \) and \( c_k \) such that the kernel \( p_{t,k}(x,y) \) of the operator \( Q_{t,k} \) satisfies

\[
|p_{t,k}(x,y)| \leq \frac{C_k}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4ct}}
\]

for all \( t > 0 \) and almost all \( x, y \in \mathbb{R}^n \).

Set

\[
H^2(\mathbb{R}^n) = \overline{R(L)} = \{Lu \in L^2(\mathbb{R}^n) : u \in L^2(\mathbb{R}^n)\},
\]

where \( \overline{R(L)} \) stands for the range of \( L \). We also set

\[
\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1} : |x-y| < t\}.
\]

For a given function \( f \in L^2(\mathbb{R}^n) \), we consider the area integral function associated to Schrödinger operator \( L \)

\[
S_L(f)(x) = \left( \iint_{\Gamma(x)} |Q_L^2(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n.
\]

Given a weight function \( w \) on \( \mathbb{R}^n \), for \( 0 < p < 1 \), we shall define the weighted Hardy spaces \( H^p_L(w) \) as the completion of \( H^2(\mathbb{R}^n) \) in the norm given by the \( L^p(w) \)-norm of area integral function; that is

\[
\|f\|_{H^p_L(w)} = \|S_L(f)\|_{L^p(w)}.
\]

Let \( M \in \mathbb{N} \) and \( 0 < p < 1 \). As in [18], we say that a function \( a(x) \in L^2(\mathbb{R}^n) \) is called a \((p,M)\)-atom with respect to \( w \)(or a \( w-(p,M)\)-atom) if there exist a ball \( B = B(x_0, r) \) and a function \( b \in D(L^M) \) such that

(a) \( a = L^M b \);

(b) \( \text{supp} \, L^k b \subseteq B, \quad k = 0, 1, \ldots, M \);

(c) \( \| (r^2L)^k b \|_{L^2(B)} \leq r^{2M}|B|^{1/2}w(B)^{-1/p}, \quad k = 0, 1, \ldots, M \).

Let \( M \in \mathbb{N} \) and \( \frac{\alpha}{\beta+1} < p < 1 \). For any \( w-(p,M) \)-atom \( a \) associated to a ball \( B = B(x_0, r) \), \( \|a\|_{L^2(B)} \leq |B|^{1/2}w(B)^{-1/p} \), we will show that \( a \in H^p_L(w) \) and its \( H^p_L(w) \)-norm is uniformly bounded; precisely

**Theorem 3.2.** Let \( M \in \mathbb{N} \), \( \frac{\alpha}{\beta+1} < p < 1 \) and \( w \in A_1 \cap RH(2/p)' \). Then there exists a constant \( C > 0 \) independent of \( a \) such that

\[
\|S_L(a)\|_{L^p(w)} \leq C.
\]
Proof. We write

$$\|S_L(a)\|_{L^p(w)}^p = \int_{2B} |S_L(a)(x)|^p w(x) \, dx + \int_{(2B)^c} |S_L(a)(x)|^p w(x) \, dx$$

$$= I_1 + I_2.$$ 

Set $q = 2/p$. Note that $w \in RH_{q'}$, then it follows from Hölder’s inequality, Lemma 2.1 and the $L^2$ boundedness of $S_L$ (see (3.2) below) that

$$I_1 \leq \left( \int_{2B} |S_L(a)(x)|^2 \, dx \right)^{p/2} \left( \int_{2B} |w(x)|^{q'} \, dx \right)^{1/q'}$$

$$\leq C \|a\|_{L^2(B)}^p \frac{w(2B)}{|2B|^{1/q}}$$

$$\leq C.$$ 

We turn to deal with $I_2$. By using Hölder’s inequality and the fact that $w \in RH_{q'}$, we can get

$$I_2 = \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |S_L(a)(x)|^p w(x) \, dx$$

$$\leq C \sum_{k=1}^{\infty} \left( \int_{2^{k+1}B \setminus 2^k B} |S_L(a)(x)|^2 \, dx \right)^{p/2} \cdot \frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/q'}}.$$ 

For any $x \in 2^{k+1}B \setminus 2^k B$, $k = 1, 2, \ldots$, we write

$$|S_L(a)(x)|^2$$

$$= \int_0^r \int_{|y-z|<t} |t^2 Le^{-t^2} a(y)| \frac{2 \, dy \, dt}{t^{n+1}} + \int_r^{\infty} \int_{|y-z|<t} |t^2 Le^{-t^2} a(y)| \frac{2 \, dy \, dt}{t^{n+1}}$$

$$= \text{I} + \text{II}.$$ 

For the term I, note that $0 < t < r$. By a simple calculation, we obtain that for any $(y, t) \in \Gamma(x)$, $x \in 2^{k+1}B \setminus 2^k B$, $z \in B$, then $|y - z| \geq 2^{k-1}r$. Hence, by using Hölder’s inequality and Lemma 3.1, we deduce

$$|t^2 Le^{-t^2} a(y)| \leq C \cdot \frac{t}{(2^{k-1}r)^{n+1}} \int_B |a(z)| \, dz$$

$$\leq C \cdot \frac{t}{(2^k r)^{n+1}} \|a\|_{L^2(\mathbb{R}^n)} |B|^{1/2}$$

$$\leq C \cdot w(B)^{-1/p} \frac{t}{2^{k(n+1)} \cdot r}.$$
Consequently
\[
I \leq C \left( \frac{1}{2^{k(n+1)}w(B)^{1/p}} \right)^2 \cdot \frac{1}{r^2} \int_0^r t \, dt 
\leq C \left( \frac{1}{2^{k(n+1)}w(B)^{1/p}} \right)^2.
\]

We now estimate the other term II. In this case, a direct computation shows that for any \((y, t) \in \Gamma(x), x \in 2^{k+1}B \setminus 2^k B \) and \(z \in B\), we have \(t + |y - z| \geq 2^{k-1}r\). Since there exists a function \(b \in D(L^M)\) such that \(a = L^M b\), then by Hölder’s inequality and Lemma 3.1 again, we get
\[
|t^2 L e^{-t^2 L} a(y)| = |(t^2 L)^{M+1} e^{-t^2 L} b(y)| \cdot \frac{1}{t^{2M}} 
\leq C \cdot \frac{1}{(2^{k-1}r)^{n+1}} \int_B |b(z)| \, dz \cdot \frac{1}{t^{2M-1}} 
\leq C \cdot \frac{1}{(2^{k}r)^{n+1}} \|b\|_{L^2(\mathbb{R}^n)} |B|^{1/2} \cdot \frac{1}{t^{2M-1}} 
\leq C \cdot \frac{r^{2M-1}}{2^{k(n+1)}w(B)^{1/p}} \cdot \frac{1}{t^{2M-1}}.
\]
Therefore
\[
II \leq C \left( \frac{1}{2^{k(n+1)}w(B)^{1/p}} \right)^2 \cdot r^{4M-2} \int_r^\infty \frac{dt}{t^{4M-1}} 
\leq C \left( \frac{1}{2^{k(n+1)}w(B)^{1/p}} \right)^2.
\]

Combining the above estimates for I and II, we thus obtain
\[
|S_L(a)(x)| \leq C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}}, \quad \text{when } x \in 2^{k+1}B \setminus 2^k B.
\]

Then it follows immediately from Lemma 2.1 that
\[
I_2 \leq C \sum_{k=1}^{\infty} \frac{1}{2^{k(p(n+1)-kn)}} \cdot w(2^{k+1}B) 
\leq C \sum_{k=1}^{\infty} \frac{1}{2^{k(p(n+1)-kn)}} 
\leq C,
\]
where in the last inequality we have used the fact that \(p > n/(n+1)\).

Summarizing the estimates for \(I_1\) and \(I_2\) derived above, we complete the proof of Theorem 3.2. \(\square\)
For every bounded Borel function $F : [0, \infty) \to \mathbb{C}$, we define the operator $F(L) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by the following formula

$$F(L) = \int_0^\infty F(\lambda) \, dE_L(\lambda),$$

where $E_L(\lambda)$ is the spectral decomposition of $L$. Therefore, the operator $\cos(t\sqrt{L})$ is well-defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from [16] that there exists a constant $c_0$ such that the Schwartz kernel $K_{\cos(t\sqrt{L})}(x, y)$ of $\cos(t\sqrt{L})$ has support contained in $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq c_0 t\}$. By the functional calculus for $L$ and Fourier inversion formula, whenever $F$ is an even bounded Borel function with $\hat{F} \in L^1(\mathbb{R})$, we can write

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^\infty \hat{F}(t) \cos(t\sqrt{L}) \, dt.$$

**Lemma 3.3** ([10]). Let $\varphi \in C_0^\infty(\mathbb{R})$ be even and $\text{supp} \varphi \subseteq [-c_0^{-1}, c_0^{-1}]$. Let $\Phi$ denote the Fourier transform of $\varphi$. Then for each $j = 0, 1, \ldots$, and for all $t > 0$, the Schwartz kernel $K_{(t^2L)^j\Phi(t\sqrt{L})}(x, y)$ of $(t^2L)^j\Phi(t\sqrt{L})$ satisfies

$$\text{supp} K_{(t^2L)^j\Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}.$$

For a given $s > 0$, we set

$$\mathcal{F}(s) = \left\{ \psi : \mathbb{C} \to \mathbb{C} \text{ measurable}, |\psi(z)| \leq C \frac{|z|^s}{1 + |z|^{2s}} \right\}.$$ 

Then for any nonzero function $\psi \in \mathcal{F}(s)$, we have the following estimate (see [18])

$$\left( \int_0^\infty \|\psi(t\sqrt{L})f\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right)^{1/2} \leq C \|f\|_{L^2(\mathbb{R}^n)}.$$  \hspace{1cm} (3.1)

In particular, we have

$$\|S_L(f)\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}.$$  \hspace{1cm} (3.2)

We are going to establish the atomic decomposition for the weighted Hardy spaces $H^p_L(w)(0 < p < 1)$.

**Theorem 3.4.** Let $M \in \mathbb{N}$, $0 < p < 1$ and $w \in A_1$. If $f \in H^p_L(w)$, then there exist a family of $w$-(p, M)-atoms $\{a_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq C \|f\|_{H^p_L(w)}^p$ such that $f$ can be represented in the form

$$f(x) = \sum_j \lambda_j a_j(x),$$

and the sum converges both in the sense of $L^2(\mathbb{R}^n)$-norm and $H^p_L(w)$-norm.
Proof. First assume that \( f \in H^p_w(w) \cap H^2(\mathbb{R}^n) \). We follow the same constructions as in [18]. Let \( \varphi \) and \( \Phi \) be as in Lemma 3.3. We set \( \Psi(x) = x^{2M}\Phi(x), x \in \mathbb{R} \). By the \( L^2 \)-functional calculus of \( L \), for every \( f \in H^2(\mathbb{R}^n) \), we can establish the following version of the Calderón reproducing formula

\[
f(x) = c_\psi \int_0^\infty \Psi(t\sqrt{L})t^2L^{-t^2L}(f)(x)\frac{dt}{t},
\]

where the above equality holds in the sense of \( L^2(\mathbb{R}^n) \)-norm. For any \( k \in \mathbb{Z} \), set

\[
\Omega_k = \{ x \in \mathbb{R}^n : S_{L,10\sqrt{n}}(f)(x) > 2^k \}.
\]

Let \( D \) denote the set formed by all dyadic cubes in \( \mathbb{R}^n \) and let

\[
D_k = \left\{ Q \in D : w(Q \cap \Omega_k) > \frac{w(Q)}{2}, w(Q \cap \Omega_{k+1}) \leq \frac{w(Q)}{2} \right\}.
\]

Obviously, for any \( Q \in D \), there exists a unique \( k \in \mathbb{Z} \) such that \( Q \in D_k \). We also denote the maximal dyadic cubes in \( D_k \) by \( Q^l_k \). Note that the maximal dyadic cubes \( Q^l_k \) are pairwise disjoint, then it is easy to check that

\[
\sum_l w(Q^l_k) \leq C \cdot w(\Omega_k).
\]

Set

\[
\tilde{Q} = \left\{ (y,t) \in \mathbb{R}^{n+1} : y \in Q, \frac{l(Q)}{2} < t \leq l(Q) \right\},
\]

where \( l(Q) \) denotes the side length of \( Q \). If we set \( \tilde{Q}_k = \bigcup_{Q \in Q_k} \tilde{Q} \), then we have \( \mathbb{R}^{n+1} = \bigcup_{k,l} \tilde{Q}_k \). Hence, by the formula (3.3), we can write

\[
f(x) = \sum_k \sum_l c_\psi \int_{Q^l_k} \Psi(t\sqrt{L})(x,y)t^2L^{-t^2L}f(y)\frac{dydt}{t} = \sum_k \sum_l \lambda_k a^l_k(x),
\]

where \( a^l_k = L^M b^l_k \),

\[
b^l_k(x) = c_\psi \lambda_k^{-1} \int_{Q^l_k} t^{2M}\Phi(t\sqrt{L})(x,y)t^2L^{-t^2L}f(y)\frac{dydt}{t}
\]

and

\[
\lambda_k = w(Q^l_k)^{1/p-1/2} \left( \int_{Q^l_k} |t^2L^{-t^2L}f(y)|^2 \frac{w(Q^l_k)}{|Q^l_k|} \frac{dydt}{t} \right)^{1/2}.
\]
By using Lemma 3.3, the authors in [18] showed that for every \( j = 0, 1, \ldots, M \), \( \text{supp}(L^{j}b_{k}^{j}) \subseteq 3Q_{k}^{j} \). In [18], they also obtained the following estimate

\[
\sum_{j} \int_{Q_{k}^{j}} |t^{2}Le^{-t^{2}L}f(y)|^{2} \frac{w(Q_{k}^{j})}{|Q_{k}^{j}|} dydt \leq C \cdot 2^{2k}w(\Omega_{k}). \tag{3.5}
\]

Since

\[
\|((LQ_{k}^{j})^{2}L)^{j}b_{k}(x)\|_{L^{2}(3Q_{k}^{j})} = \sup_{\|h\|_{L^{2}(3Q_{k}^{j})} \leq 1} \left| \int_{\mathbb{R}^{n}} ((LQ_{k}^{j})^{2}L)^{j}b_{k}(x)h(x) \, dx \right|
\]

Let \( \Psi_{j}(x) = x^{2j}\Phi(x) \), \( j = 0, 1, \ldots, M \). Then we can easily verify that \( \Psi_{j} \in \mathcal{F}(2j) \). Observe that when \((y,t) \in \hat{Q}_{k}^{j}\), we have \( t \sim l(Q_{k}^{j}) \). Then it follows from Hölder’s inequality and the estimate (3.1) that

\[
\left| \int_{\mathbb{R}^{n}} ((LQ_{k}^{j})^{2}L)^{j}b_{k}(x)h(x) \, dx \right|
\leq \frac{C \cdot l(Q_{k}^{j})^{2M}}{\lambda_{kl}} \left( \int_{Q_{k}^{j}} |t^{2}Le^{-t^{2}L}f(y)|^{2} \frac{dydt}{t} \right)^{1/2}
\times \left( \int_{Q_{k}^{j}} |(t^{2}L)^{j}\Phi(t\sqrt{L})(h\chi_{3Q_{k}^{j}})(y)|^{2} \frac{dydt}{t} \right)^{1/2}
\leq C \cdot l(Q_{k}^{j})^{2M}|Q_{k}^{j}|^{1/2}w(Q_{k}^{j})^{-1/p} \left( \int_{0}^{\infty} \|\Psi_{j}(t\sqrt{L})(h\chi_{3Q_{k}^{j}})\|_{L^{2}(\mathbb{R}^{n})}^{2} \frac{dt}{t} \right)^{1/2}
\leq C \cdot l(Q_{k}^{j})^{2M}|Q_{k}^{j}|^{1/2}w(Q_{k}^{j})^{-1/p}.
\]

Hence

\[
\|((LQ_{k}^{j})^{2}L)^{j}b_{k}\|_{L^{2}(3Q_{k}^{j})} \leq C \cdot l(Q_{k}^{j})^{2M}|Q_{k}^{j}|^{1/2}w(Q_{k}^{j})^{-1/p}.
\]

From the above discussions, we have proved that these functions \( a_{k}^{j} \) are all \( w_{-}(p,M) \)-atoms up to a normalization by a multiplicative constant. Finally, by using Hölder’s inequality, the estimates (3.4) and (3.5), we obtain

\[
\sum_{k} \sum_{l} |\lambda_{kl}|^{p} = \sum_{k} \sum_{l} \left( w(Q_{k}^{j}) \right)^{1-p/2} \left( \int_{Q_{k}^{j}} |t^{2}Le^{-t^{2}L}f(y)|^{2} \frac{w(Q_{k}^{j})}{|Q_{k}^{j}|} dydt \right)^{p/2}
\leq C \sum_{k} \left( \sum_{l} w(Q_{k}^{j}) \right)^{1-p/2} \left( \sum_{l} \int_{Q_{k}^{j}} |t^{2}Le^{-t^{2}L}f(y)|^{2} \frac{w(Q_{k}^{j})}{|Q_{k}^{j}|} dydt \right)^{p/2}
\leq C \sum_{k} \left( w(\Omega_{k}) \right)^{1-p/2} \left( 2^{2k}w(\Omega_{k}) \right)^{p/2}
\leq C \|S_{L}(f)\|_{L^{p}(w)}^{p}.
\]
Therefore, we have established the atomic decomposition for all functions in the space $H^p_L(w) \cap H^2(\mathbb{R}^n)$. By a standard density argument, we can show that the same conclusion holds for $H^p_L(w)$. Following along the same arguments as in [18], we can also prove that the sum $f = \sum_j \lambda_j a_j$ converges both in the sense of $L^2(\mathbb{R}^n)$-norm and $H^p_L(w)$-norm, the details are omitted here. This completes the proof of Theorem 3.4.

4 Proof of Theorem 1.1

We shall need the following Davies-Gaffney estimate which can be found in [10,18].

**Lemma 4.1.** For any two closed sets $E$ and $F$ of $\mathbb{R}^n$, there exist two positive constants $C$ and $c$ such that

$$
\|t \nabla e^{-t^2 L} f\|_{L^2(F)} \leq C \cdot e^{-\frac{\text{dist}(E,F)^2}{ct^d}} \|f\|_{L^2(E)}
$$

for every $f \in L^2(\mathbb{R}^n)$ with support contained in $E$.

**Theorem 4.2.** Let $\frac{n}{n+1} < p < 1$ and $w \in A_1 \cap RH(\frac{2}{p})'$. Then the operator $\nabla L^{-1/2}$ is bounded from $H^p_L(w)$ to $L^p(w)$.

**Proof.** By Theorem 3.4 we just proved, it is enough for us to show that for any $w$-$(p,M)$-atom $a$, $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, there exists a constant $C > 0$ independent of $a$ such that $\|\nabla L^{-1/2}(a)\|_{L^p(w)} \leq C$. Let $a$ be a $w$-$(p,M)$-atom with $\text{supp} a \subseteq B = B(x_0, r)$, $\|a\|_{L^2(B)} \leq |B|^{1/2} w(B)^{-1/p}$. We write

$$
\|\nabla L^{-1/2}(a)\|_{L^p(w)}^p = \int_{2B} |\nabla L^{-1/2}(a)(x)|^p w(x) \, dx + \int_{(2B)^c} |\nabla L^{-1/2}(a)(x)|^p w(x) \, dx
$$

$$
= J_1 + J_2.
$$

Set $q = 2/p$. Applying Hölder’s inequality, the $L^2$ boundedness of $\nabla L^{-1/2}$, Lemma 2.1 and $w \in RH_{q'}$, we thus have

$$
J_1 \leq \left( \int_{2B} |\nabla L^{-1/2}(a)(x)|^2 \, dx \right)^{p/2} \left( \int_{2B} w(x)^{q'} \, dx \right)^{1/q'}
$$

$$
\leq C \|a\|_{L^2(B)}^p \cdot \frac{w(2B)}{|2B|^{1/q}}
$$

$$
\leq C.
$$
On the other hand, it follows from Hölder’s inequality and $w \in RH_q$ that

$$J_2 = \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^k B} |\nabla L^{-1/2}(a)(x)|^p w(x) \, dx$$

$$\leq C \sum_{k=1}^\infty \left( \int_{2^{k+1}B \setminus 2^k B} |\nabla L^{-1/2}(a)(x)|^2 \, dx \right)^{p/2} \cdot \frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/q}}. \tag{4.1}$$

By a change of variable $s = t^2$, we can rewrite (1.2) as

$$\nabla L^{-1/2}(a)(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty s^{-1/2} e^{-s^2 L(a)(x)} \frac{ds}{s}. \tag{4.2}$$

For any $k = 1, 2, \ldots$, it follows immediately from Minkowski’s integral inequality that

$$\left( \int_{2^{k+1}B \setminus 2^k B} |\nabla L^{-1/2}(a)(x)|^2 \, dx \right)^{1/2}$$

$$\leq C \int_r^r \|s^{-2} e^{-s^2 L} a\|_{L^2(2^{k+1}B \setminus 2^k B)} \frac{ds}{s} + C \int_r^\infty \|s^{-2} e^{-s^2 L} a\|_{L^2(2^{k+1}B \setminus 2^k B)} \frac{ds}{s}$$

$$= \text{III+IV.}$$

Observe that $M > \frac{q}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$. Then we are able to choose a positive number $N$ such that $\frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right) < N < M$. By using Lemma 4.1, we can get

$$\text{III} \leq C \int_0^r e^{-\frac{(2^k)^2}{s^2}} \|a\|_{L^2(B)} \frac{ds}{s}$$

$$\leq C \int_0^r \frac{s^{2N}}{(2^k r)^{2N}} \frac{ds}{s} \cdot \|a\|_{L^2(B)}$$

$$\leq C \cdot 2^{-2kN} |B|^{1/2} w(B)^{-1/p}. \tag{4.3}$$

We now turn to estimate the term IV. Since $a = L^M b$ and $\|b\|_{L^2(B)} \leq r^{2M} |B|^{1/2} w(B)^{-1/p}$. Using Lemma 3.1 and Lemma 4.1, we deduce

$$\text{IV} = C \int_r^\infty \|s^{-2} e^{-s^2 L} (L^M b)\|_{L^2(2^{k+1}B \setminus 2^k B)} \frac{ds}{s}$$

$$= C \int_r^\infty \|s^{-2} e^{-s^2 L} (s^2 L)^M e^{-s^2 L} b\|_{L^2(2^{k+1}B \setminus 2^k B)} \frac{ds}{s^{2M+1}}$$

$$\leq C \int_r^\infty e^{-\frac{(2^k)^2}{s^2}} \|(s^2 L)^M e^{-s^2 L} b\|_{L^2(B)} \frac{ds}{s^{2M+1}}$$

$$\leq C \int_r^\infty \frac{s^{2N}}{(2^k r)^{2N}} \frac{ds}{s^{2M+1}} \cdot \|b\|_{L^2(B)}$$

$$\leq C \cdot 2^{-2kN} |B|^{1/2} w(B)^{-1/p}. \tag{4.4}$$
Combining the above inequality (4.4) with (4.3), we thus obtain
\[
\left( \int_{2^{k+1}B \setminus 2^k B} |\nabla L^{-1/2}(a)(x)|^2 \, dx \right)^{1/2} \leq C \cdot 2^{-2kN} |B|^{1/2} w(B)^{-1/p}. \tag{4.5}
\]

Substituting the above inequality (4.5) into (4.1) and using Lemma 2.1, then we have
\[
J_2 \leq C \sum_{k=1}^{\infty} \left( 2^{-2kN} |B|^{1/2} w(B)^{-1/p} \right)^p \cdot \frac{w(2^{k+1}B)}{|2^{k+1}B|^{p/2}}
\]
\[
\leq C \sum_{k=1}^{\infty} \frac{1}{2^{k(2pN+\frac{np}{2}-n)}}
\]
\[
\leq C,
\]
where the last series is convergent since \( N > \frac{p}{2} \left( \frac{1}{p} - \frac{1}{2} \right) \). Summarizing the estimates for \( J_1 \) and \( J_2 \), we get the desired result. \( \square \)

The real-variable theory of classical weighted Hardy spaces have been extensively studied by many authors. In 1979, Garcia-Cuerva studied the atomic decomposition and the dual spaces of \( H^p(w) \) for \( 0 < p \leq 1 \). In 2002, Lee and Lin gave the molecular characterization of \( H^p(w) \) for \( 0 < p \leq 1 \), they also obtained the \( H^p(w) \) \( (\frac{n}{n+1} < p < 1) \) boundedness of the Hilbert transform and the \( H^p(w) \) \( (\frac{n}{n+1} < p \leq 1) \) boundedness of the Riesz transforms. For the results mentioned above, we refer the readers to [7,12,19] for further details.

Let \( \frac{n}{n+1} < p < 1 \) and \( w \in A_1 \). A real-valued function \( a(x) \) is called a \( w \)-\( (p, 2, 0) \)-atom if the following conditions are satisfied (see [7,19]):

(a) \( \text{supp} a \subseteq B \);
(b) \( \|a\|_{L^2(B)} \leq |B|^{1/2} w(B)^{-1/p} \);
(c) \( \int_{\mathbb{R}^n} a(x) \, dx = 0 \).

**Theorem 4.3.** Let \( \frac{n}{n+1} < p < 1 \) and \( w \in A_1 \). For each \( f \in H^p(w) \), there exist a family of \( w \)-\( (p, 2, 0) \)-atoms \( \{a_j\} \) and a sequence of real numbers \( \{\lambda_j\} \) with \( \sum_j |\lambda_j|^p \leq C \|f\|_{H^p(w)}^p \) such that \( f = \sum_j \lambda_j a_j \) in the sense of \( H^p(w) \) norm.

Next, as in [18], we shall also define the new weighted molecules for \( H^p(w) \). Let \( \frac{n}{n+1} < p < 1 \), \( w \in A_1 \) and \( \varepsilon > 0 \). A function \( m(x) \in L^2(\mathbb{R}^n) \) is called a \( w \)-\( (p, 2, 0, \varepsilon) \)-molecule associated to a ball \( B \) if the following conditions are satisfied:

(A) \( \int_{\mathbb{R}^n} m(x) \, dx = 0 \);
(B) \( \|m\|_{L^2(2B)} \leq |B|^{1/2} w(B)^{-1/p} \).
Let \( \frac{m}{n+1} < p < 1 \) and \( w \in A_1 \).

(i) If \( f \in H^p(w) \), then there exist a family of \( w-(p,2,0,\varepsilon) \)-molecules \( \{ m_j \} \) and a sequence of real numbers \( \{ \lambda_j \} \) with \( \sum_j |\lambda_j|^p \leq C \| f \|_{H^p(w)}^p \) such that \( f = \sum_j \lambda_j a_j \) in the sense of \( H^p(w) \) norm.

(ii) Suppose that \( w \in RH_{(2/p)^\prime} \) and \( \varepsilon > n/2 \), then every \( w-(p,2,0,\varepsilon) \)-molecule \( m \) is in \( H^p(w) \). Moreover, there exists a constant \( C > 0 \) independent of \( m \) such that \( \| m \|_{H^p(w)} \leq C. \)

**Proof.** (i) is a straightforward consequence of Theorem 4.3.

(ii) We follow the idea of [18]. Denote \( m_0(x) = m(x) \chi_{2B}(x) \), \( m_k(x) = m(x) \chi_{2^{k+1}B \setminus 2^{k}B}(x) \), \( k = 1, 2, \ldots \). Then we can decompose \( m(x) \) as

\[
m(x) = \sum_{k=0}^{\infty} m_k(x) = \sum_{k=0}^{\infty} (m_k(x) - N_k(x)) + \sum_{k=0}^{\infty} N_k(x),
\]

where \( N_0(x) = \frac{1}{|2B|} \int_{\mathbb{R}^n} m_0(y) dy \chi_{2B}(x) \) and \( N_k(x) = \frac{1}{|2^{k+1}B \setminus 2^{k}B|} \int_{\mathbb{R}^n} m_k(y) dy \chi_{2^{k+1}B \setminus 2^{k}B}(x), k = 1, 2, \ldots \). Following along the same lines as in [18], we can also show that each \( (m_k - N_k) \) is a multiple of \( w-(p,2,0) \)-atom with a sequence of coefficients in \( l^p \). We set \( \eta_k = \int_{\mathbb{R}^n} m_k(y) dy, k = 0, 1, \ldots \). In [18], Song and Yan established the following identity

\[
\sum_{k=0}^{\infty} N_k(x) = \sum_{k=0}^{\infty} p_k \cdot \psi_k(x),
\]

where \( p_k = \sum_{j=k+1}^{\infty} \eta_j \) and \( \psi_k(x) = \frac{N_{k+1}(x)}{\eta_{k+1}} - \frac{N_k(x)}{\eta_k} \). Then we have

\[
|p_k| \leq \sum_{j=k+1}^{\infty} \int_{2^{j+1}B \setminus 2^jB} |m(y)| dy \leq \sum_{j=k+1}^{\infty} \|m\|_{L^2(2^{j+1}B \setminus 2^jB)} |2^{j+1}B|^{1/2} \leq C \sum_{j=k+1}^{\infty} 2^{-\varepsilon j} \cdot |2^j B| w(2^j B)^{-1/p}.
\]
When \( j \geq k + 1 \), then \( 2^k B \subseteq 2^j B \). Since \( w \in RH_{(2/p)'} \), then by Lemma 2.2, we can get

\[
\frac{w(2^k B)}{w(2^j B)} \leq C \left( \frac{|2^k B|}{|2^j B|} \right)^{p/2}.
\]

Hence

\[
|p_k| \leq C \cdot \frac{|2^k B|}{w(2^k B)^{1/p}} \sum_{j=k+1}^{\infty} 2^{-j\varepsilon} \left( \frac{|2^j B|}{|2^k B|} \right)^{1/2}
\]

\[
\leq C \cdot \frac{|2^k B|}{w(2^k B)^{1/p}} \left( \sum_{j=k+1}^{\infty} 2^{-j(\varepsilon - n/2)} \right) \cdot 2^{-kn/2}
\]

\[
\leq C \cdot 2^{-k\varepsilon} \frac{|2^k B|}{w(2^k B)^{1/p}},
\]

where the last inequality holds since \( \varepsilon > n/2 \). As in [18], we can easily check that \( 2^{-k\varepsilon} p_k \psi(x) \) are all \( w-(p, 2, 0) \)-atoms associated to \( 2^{k+1}B \). Therefore the sum \( \sum_{k=0}^{\infty} N_k \) can be write as an infinite linear combination of \( w-(p, 2, 0) \)-atoms with a sequence of coefficients in \( l^p \). Summarizing the above discussions, we complete the proof of Theorem 4.4.

We are now in a position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Theorem 3.4 and Theorem 4.4, it suffices to show that for every \( w-(p, M) \)-atom \( a \) with \( \text{supp} a \subseteq B \), then \( \nabla L^{-1/2}a \) is a \( w-(p, 2, 0, \varepsilon) \)-molecule, where \( M > n(\frac{1}{p} - \frac{1}{2}) \) and \( \varepsilon > n/2 \). It is easy to see that \( \int_{\mathbb{R}^n} \nabla L^{-1/2}a(x) \, dx = 0 \). It remains to verify the estimates (B) and (C).

Hölder’s inequality and the definition of \( w-(p, M) \)-atom imply

\[
\| \nabla L^{-1/2}(a) \|_{L^2(2B \setminus 2^k B)} \leq C \| a \|_{L^2(B)} \leq C \cdot |B|^{1/2} w(B)^{-1/p}.
\]

For \( k = 1, 2, \ldots \), it follows from the previous estimates (4.3) and (4.4) that

\[
\| \nabla L^{-1/2}(a) \|_{L^2(2^{k+1}B \setminus 2^k B)} \leq C \cdot 2^{-2kN} |B|^{1/2} w(B)^{-1/p},
\]

where \( N > 0 \) is chosen such that \( n(\frac{1}{p} - \frac{1}{2}) < N < M \). By using Lemma 2.2, we get

\[
\frac{w(B)}{w(2^k B)} \geq C \cdot \frac{|B|}{|2^k B|}.
\]

Hence

\[
\| \nabla L^{-1/2}(a) \|_{L^2(2^{k+1}B \setminus 2^k B)} \leq C \cdot 2^{-k(2N-n/p+n/2)} |2^k B|^{1/2} w(2^k B)^{-1/p}.
\]

Therefore, we have proved \( \nabla L^{-1/2}a \) is a \( w-(p, 2, 0, 2N-n/p+n/2) \)-molecule. This concludes the proof of Theorem 1.1. \( \square \)
References

[1] P. Auscher, X. T. Duong, A. McIntosh, Boundedness of Banach space valued singular integral operators and Hardy spaces, preprint, 2004.

[2] E. B. Davies, Heat Kernels and Spectral Theory, Cambridge Univ. Press, 1989.

[3] X. T. Duong, E. M. Ouhabaz and L. X. Yan, Endpoint estimates for Riesz transforms of magnetic Schrödinger operators, Ark. Mat, 44(2006), 261–275.

[4] J. Dziubański and J. Zienkiewicz, Hardy space $H^1$ associated to Schrödinger operator with potential satisfying reverse Hölder inequality, Rev. Mat. Iberoamericana, 15(1999), 279–296.

[5] J. Dziubański and J. Zienkiewicz, $H^p$ spaces for Schrödinger operators, Fourier Analysis and Related Topics, Vol 56, 2002, 45–53.

[6] J. Dziubański and J. Zienkiewicz, $H^p$ spaces associated with Schrödinger operators with potentials from reverse Hölder classes, Colloq. Math, 98(2003), 5–38.

[7] J. Garcia-Cuerva, Weighted $H^p$ spaces, Dissertations Math, 162(1979), 1–63.

[8] J. Garcia-Cuerva and J. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland, Amsterdam, 1985.

[9] R. F. Gundy and R. L. Wheeden, Weighted integral inequalities for nontangential maximal function, Lusin area integral, and Walsh-Paley series, Studia Math, 49(1974), 107–124.

[10] S. Hofmann, G. Z. Lu, D. Mitrea, M. Mitrea, L. X. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, preprint, 2007.

[11] T. Kato, Perturbation Theory for Linear Operators, Second Edition, Springer-Verlag, Berlin, 1984.

[12] M. Y. Lee and C. C. Lin, The molecular characterization of weighted Hardy spaces, J. Funct. Anal, 188(2002), 442–460.

[13] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc, 165(1972), 207–226.
[14] E. M. Ouhabaz, Analysis of Heat Equations on Domains, London Math. Soc. Monographs, Vol 31, Princeton Univ. Press, Princeton, NJ, 2005.

[15] Z. Shen, $L^p$ estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier(Grenoble), 45(1995), 513–546.

[16] A. Sikora, Riesz transform, Gaussian bounds and the method of wave equation, Math. Z, 247(2004), 643–662.

[17] B. Simon, Maximal and minimal Schrödinger forms, J. Operator Theory, 1(1979), 37–47.

[18] L. Song and L. X. Yan, Riesz transforms associated to Schrödinger operators on weighted Hardy spaces, J. Funct. Anal, 259(2010), 1466–1490.

[19] J. O. Stömberg and A. Torchinsky, Weighted Hardy spaces, Lecture Notes in Math, Vol 1381, Springer-Verlag, 1989.