Compact U(1) on a spatial lattice: duality, photons, and confinement

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Abstract. I present a Hamiltonian approach to compact U(1) gauge theory on a regular spatial lattice in (2 + 1) and (3 + 1) dimensions. The diagonalization of the Hamiltonian in a "charge network" basis (eigenstates of the electric field) leads in a natural way to an electromagnetic duality transformation where the dual variables can be associated with the dual lattice. I present the solutions, in particular the vacuum state, in the weak-coupling limit and compare with the (shadow) states obtained recently by Varadarajan in a UV-regularized version of loop-quantized (non-compact) continuum U(1) gauge theory. Finally, I comment on the existence of an instanton condensate which destroys this simple picture in the (2 + 1)-dimensional compact case and leads to confinement.

1. Introduction

Why would anybody interested in quantum gravity bother about abelian lattice gauge theory? Well, first of all, the formulation in Ashtekar's variables converts (quantum) gravity into a SU(2) gauge theory plus diffeomorphism invariance plus a Hamiltonian constraint. The combination of a gauge theory with diffeomorphism invariance leads very naturally to a formulation in terms of loop variables (see, e.g., Rovelli's contribution to this conference) or "spin networks". From there, any intent to formulate a renormalization group in the Wilsonian sense (elimination of "high-energy" or "short-wavelength" degrees of freedom) leads almost inevitably to the introduction of a regular or irregular lattice. In fact, the only known non-perturbative renormalization schemes that respect gauge symmetries use the lattice as a regulator. The simplest example of a lattice gauge theory is an abelian gauge theory on a cubic lattice. There is another important motivation for investigating these theories: the confinement mechanism of non-abelian gauge theories, at least in the abelian gauge, apparently has its origin, somewhat paradoxically, in the abelian subsector(s). The condensation of instantons in (2 + 1) dimensions and of magnetic monopoles in (3+1) dimensions lead to confinement via a dual superconductor mechanism. This can conveniently be studied in a formulation of the theory on a regular lattice [1].

I will begin by defining the theory and establishing my conventions. Every vertex on the lattice is parametrized (in (3 + 1) dimensions) by three integer coordinates \((r, s, t)\), hence the physical position is obtained by multiplying with the lattice spacing \(a\). A link is described through its base point (a vertex) and its direction, given by one the unit vectors \(\hat{i}, \hat{j}, \hat{k}\). Finally, a plaquette is specified by its "lower left" vertex and, in (3 + 1) dimensions, by the direction of its normal.
In the Hamiltonian formulation of the theory it is convenient to use the radiation gauge $A^0 = 0$. The natural configuration space variables on a lattice are the holonomies along the lattice links,

$$U_l = e^{ie \int_l dx A(x)} \equiv e^{i \theta_l} \in U(1),$$

which defines the dimensionless link variables $\theta_l$. The values of $\theta_l$ (that describe inequivalent configurations $U_l$) are restricted to $-\pi < \theta_l \leq \pi$, which is why this $U(1)$ gauge theory is called “compact”.

The remaining gauge freedom,

$$U_{v,\hat{i}} \rightarrow g_{v,\hat{i}} U_{v,\hat{i}} g_{v,\hat{i}}^{-1} \tag{2}$$

(analogously for links in directions $\hat{j}$ and $\hat{k}$) for arbitrary values of $g_v \in U(1)$ at every vertex, is fixed by demanding invariance of the state functionals under the transformations (2), in differential form

$$\left( \nabla \cdot \frac{\partial}{\partial \theta_l} \right) \psi(\theta_l) \equiv \left( \frac{\partial}{\partial \theta_{v,\hat{i}}} - \frac{\partial}{\partial \theta_{v-\hat{i},\hat{i}}} + \frac{\partial}{\partial \theta_{v,\hat{j}}} - \frac{\partial}{\partial \theta_{v-\hat{j},\hat{i}}} \right) \psi(\theta_l) = 0 \tag{3}$$

at all vertices $v$ (in (2+1) dimensions, with the obvious analogue in (3+1) dimensions). Equation (3) also introduces the definition of the (dimensionless) lattice divergence illustrated in Fig. 1.

In canonical quantization (in the radiation gauge), the electric field is represented by ($i$ times) the derivative with respect to the potential, hence it is proportional to the derivative with respect to $\theta_l$, and Eq. (3) is simply Gauss’ law for a theory without electrical charges.

Now consider the product of holonomies around one plaquette,

$$U_{v,\hat{k}} = U_{v,\hat{j}}^{-1} U_{v,\hat{j}+1} U_{v+1,\hat{j}} U_{v,\hat{i}} = e^{i \phi_{v,\hat{k}}}, \quad \phi_{v,\hat{k}} = \nabla \times \theta_l \equiv \theta_{v,\hat{i}} + \theta_{v+i,\hat{j}} - \theta_{v+\hat{j},\hat{i}} - \theta_{v,\hat{j}}, \tag{4}$$

thus defining a dimensionless plaquette angle $\phi_p$ corresponding to ($e$ times) the magnetic flux through the plaquette. The definition of the lattice curl in Eq. (4) is also shown in Fig. 1.

The dynamics of the theory is determined by the Kogut-Susskind Hamiltonian [2]

$$H = \frac{1}{a} \left[ -\frac{g^2}{2} \sum_l \frac{\partial^2}{\partial \theta_l^2} + \frac{1}{2g^2} \sum_p (2 - U_p - U_p^{-1}) \right] \tag{5},$$

where I have introduced the dimensionless coupling constant $g^2 = a^{3-d} e^2$. 

**Figure 1.** Representation of lattice divergence (left) and curl (right), Eqs. (3) and (4).
Figure 2. Left: the curl for plaquette variables in (3 + 1) dimensions. Represented is the \( i \)-component of the curl \( \nabla \times m_p \) given in Eq. (9). Note the orientations of the plaquettes relative to the link \((v, i)\). Right: the corresponding quantities on the dual lattice, where the plaquette variable dual to the \( i \)-component of the curl \( \nabla \times m_p \) is given by the curl of the link variables dual to the \( m_p \) as defined in Eq. (4).

2. Duality

In order to try and diagonalize the Hamiltonian, one possibility is to look for the eigenstates of the “kinetic term” in Eq. (5). The expansion of the state functionals \( \psi(\theta_l) \) in the basis of eigenstates is simply a Fourier decomposition,

\[
\psi(\theta_l) = \sum_{\{n_l\}} \hat{\psi}(n_l) e^{i \sum_l n_l \theta_l},
\]

where the outer sum is over all \( n_l \in \mathbb{Z} \) for all links \( l \). The representation of the operators in the new basis is

\[
\frac{1}{i} \frac{\partial}{\partial \theta_l} \psi(\theta_l) \rightarrow n_l \hat{\psi}(n_l),
\]

\[
e^{i \phi} \hat{\psi}(\theta_l) \rightarrow \hat{\psi}(n_v, i - n_v - \hat{i}, n_v + \hat{j} - n_v - \hat{j}, n_v + \hat{j} + 1; n_{v'}) .
\]

In particular, \( n_l \) corresponds to the electric field in direction \( l \). The result of the application of \( e^{i \phi} \) is evident from Eqs. (4) and (6) (the last argument \( n_{v'} \) of \( \hat{\psi} \) represents all the link variables that are left unchanged).

Gauss’ law (3) now becomes

\[
\nabla \cdot n_l \equiv n_{v,i} - n_{v,-i,i} + n_{v,j} - n_{v,-j,j} = 0 ,
\]

to be understood in the sense that \( \hat{\psi}(n_l) = 0 \) unless (8) is fulfilled at every vertex. This constraint makes it possible to express \( n_l \) in terms of an integer plaquette variable \( m_p \) via

\[
n_l = \nabla \times m_p , \quad n_{v,i} = m_{v,k} - m_{v,j,k} - m_{v,j} + m_{v,-k,j}
\]

and analogously for the other components in directions \( j, k \) (in (3 + 1) dimensions), as illustrated in Fig. 2. In (2 + 1) dimensions, only the component \( m_{v,k} \) contributes. Depending on the global topology of the lattice, there may be configurations of the \( n_l \) that comply with Gauss’ law but cannot be represented as curls (such as Polyakov loops). We will not be concerned with global topological effects in this contribution.

Given the interpretation of \( n_l \) as an electric field, \( m_p \) plays, according to Eq. (9), the role of a dual potential in the sense of electromagnetic duality. In (3 + 1) dimensions, this fits in nicely
Figure 3. \textit{Left:} the divergence for plaquette variables $m_p$ associated with a given cube $v$ in $(3+1)$ dimensions. \textit{Right:} the corresponding quantities on the dual lattice, where the divergence at the vertex $v$ is given in terms of the link variables dual to the $m_p$ via the definition (3).

with the representation of Eq. (9) on the dual lattice. Here, the plaquettes of the original lattice become the links of the dual lattice. Conversely, links of the original lattice go into plaquettes of the dual lattice. Analogously, vertices are dual to cubes. Under this prescription, the definition for the curl of plaquette variables shown in Fig. 2 is equivalent to the definition used before for link variables when applied to the dual lattice.

In $(3+1)$ dimensions, Eq. (9) does not fix $m_p$ entirely in terms of $n_l$. The residual “dual gauge freedom” in fixing $m_p$ can be eliminated in different ways, the most natural one in the present context being the “dual Coulomb gauge” which demands

$$\nabla \cdot m_p = 0$$

for every cube. The natural definition of the divergence of plaquette variables is illustrated in Fig. 3, and again corresponds to the definition used before for link variables when considered on the dual lattice.

With the plaquette variables being defined, we can rewrite $\hat{\psi}(n_l)$ as $\hat{\psi}(m_p)$ (for all the configurations $n_l$ with $\hat{\psi}(n_l) \neq 0$). Then

$$\hat{\psi}(n_{v,i} - 1, n_{v+i,j} - 1, n_{v+j,i} + 1, n_{v,j} + 1; n_l) = \hat{\psi}(m_{v,k} - 1; m_p'),$$

the other variables $n_l$ and $m_p'$ being unchanged. Hence the “potential term” in Eq. (5) becomes particularly simple when expressed in terms of the plaquette variables.

The different variables associated with the links and plaquettes of the original or the dual lattice are related as shown in the following “duality scheme”:

The variables on the left-hand-side of this scheme are elements of $U(1)$ (or rather their exponents), while the variables on the right-hand-side are elements of the “dual group” $\mathbb{Z}$. The link and plaquette variables on both sides are conjugate to each other with respect to Fourier transformation (F.T.). This was established for the pair $(\theta_l, n_l)$ in Eq. (6) and extends to $(\phi_p, m_p)$ due to the identity

$$\sum_l n_l \theta_l = \sum_p m_p \phi_p.$$
3. PHOTONS

I will now consider the weak-coupling limit \( g^2 \ll 1 \) where the theory turns out to be analytically solvable. In this limit, the “potential term” in the Hamiltonian is enhanced by a factor \( 1/g^2 \), hence for the low-energy eigenstates the expression multiplying this factor must be small. Consequently, the variation of \( \psi \) with \( m_p \) must be very slow, and it is plausible that we can replace \( m_p \) by a continuous variable (cf. Eqs. (4), (7), and (11)),

\[
\frac{1}{2g^2} \sum_p \left[ 2\hat{\psi}(m_p; m_{p'}) - \hat{\psi}(m_p - 1; m_{p'}) - \hat{\psi}(m_p + 1; m_{p'}) \right] \rightarrow -\frac{1}{2g^2} \sum_p \frac{\partial^2}{\partial m_p^2} \hat{\psi}(m_p; m_{p'}) . \quad (13)
\]

The Hamiltonian then reduces to the one of a set of coupled harmonic oscillators where the coupling is effected by the “kinetic term” (see Eqs. (7) and (9)). As usual, diagonalization is achieved in momentum space, defined on the lattice by the Fourier transformation of the variables,

\[
\hat{m}_{q,k} = \frac{1}{N^{d/2}} \sum_v e^{-\frac{2\pi i q \cdot v}{N}} m_{v,k} , \quad (14)
\]

and analogously for the components in directions \( \hat{i}, \hat{j} \). The “wave vector” \( q \) is integer-valued (with three components in \( (3 + 1) \) dimensions), and \( N \) denotes the number of vertices in one spatial direction.

It is now an easy task to calculate the (approximate) eigenstates and eigenvalues of the Hamiltonian. I will give here the explicit result for the vacuum state, reexpressed in terms of the lattice by the Fourier transformation of the variables,

\[
\hat{\psi}_0(n_l) = \prod_v \delta(n_v, \hat{i} - n_v - n_{v-j}, 0) 
\times \exp \left[ -\frac{g^2}{2} \sum_{v,v'} G(v - v') (n_{v,j} n_{v',\hat{i}} + n_{v,j} n_{v',\hat{j}}) \right] , \quad (15)
\]

where I have used the notation \( \delta(i,j) \) for the Kronecker symbol \( \delta_{ij} \), and

\[
G(v - v') = \frac{1}{N^2} \sum_q \frac{1}{2\omega_q} e^{2\pi i q \cdot (v-v')/N} , \quad \omega_q = \sqrt{\sin^2 \left( \frac{\pi q_x}{N} \right) + \sin^2 \left( \frac{\pi q_y}{N} \right)} . \quad (16)
\]

The product of Kronecker symbols in Eq. (15) is a consequence of Gauss’ law. The analogous calculation in \( (3 + 1) \) dimensions leads to a similar expression.

It is interesting to compare this result with the vacuum state in the loop quantization of the continuum (non-compact) theory, which has recently been determined by Varadarajan [3] (see also Ref. [4]). More precisely, I will compare the lattice result with the corresponding “shadow state” which is the loop quantization vacuum restricted to the charge network basis corresponding to a fixed graph, the lattice in our case. Varadarajan’s result reads

\[
\tilde{\psi}_0(n_l) = \exp \left[ -\frac{g^2}{2} \sum_{l,l'} \left( \int \frac{d^2 q}{(2\pi)^2} \frac{F_{l^*}^r(q) \cdot F_l^r(q)}{|q|} n_l n_{l'} \right) \right] , \quad (17)
\]

in \( (2 + 1) \) dimensions and with the present conventions (inserting appropriate powers of \( g \)). The “smeared form factor” \( F_l^r \) is defined as

\[
F_l^r(q) = e^{-q^2 r^2/2} F_l(q) \quad (18)
\]
in terms of the continuum Fourier transform of the form factor
\[ F_l(x) = \int l(s) \delta(x - l(s)) \] (19)
for the link \( l \) considered as a curve \( l(s) \) in position space (here \( x \) denotes a vector in (two-dimensional) position space). The parameter \( 1/r \) plays the role of an UV cutoff [3, 4]. The product of Kronecker symbols that implements Gauss’ law is already implicitly contained in \( \tilde{\psi}_0(n_l) \) as a restriction on the proper charge network basis in the loop quantization approach.

It is straightforward to evaluate the smeared form factors in the exponent of Eq. (17) for our two-dimensional lattice. The result is
\[ \int \frac{d^2 q}{(2\pi)^2} \frac{F_{\nu,i}^*(q) \cdot F_{\nu',j}^*(q)}{|q|} = \int \frac{d^2 q}{(2\pi)^2} e^{-q^2 r^2} \frac{4}{q^2} \sin^2 \left( \frac{qa}{2} \right) e^{iaq \cdot (v-v')} \] (20)
with an analogous expression for \( F_{\nu,i}^*(q) \cdot F_{\nu',j}^*(q) \), while \( F_{\nu,i}^*(q) \cdot F_{\nu',j}^*(q) = 0 \). In the double limit
\[ q^2 \ll \frac{1}{a^2}, \frac{1}{r^2}, \] (21)
hence in particular for wavelengths \( 2\pi/q \) much larger than the lattice spacing \( a \), the integral (20) tends to \( G(v-v') \), and \( \tilde{\psi}_0(n_l) \) and \( \tilde{\psi}_0(n_l) \) coincide. This is an important result for any intent to formulate a renormalization scheme in the loop quantized theory, and it also sheds some light on the form of Varadarajan’s states.

Finally, I would like to point out that precisely in \((2+1)\) dimensions, the result obtained for the vacuum state on a spatial lattice is not entirely correct because the presence of an instanton condensate destroys the simple picture of a perturbative vacuum state [1]. In fact, the study of compact \( U(1) \) gauge theory on a spatial lattice in a classical paper by Drell, Weinstein et al. [5] has shown this theory to be confining in \((2+1)\) dimensions even in the weak-coupling limit. However, the approximation used in Ref. [5] is quite rough and, in particular, does not make any use of the duality transformation developed in this contribution. It is expected that the description of confinement and other properties of the theory becomes simpler in the dual variables where a confining theory is supposed to correspond to an abelian Higgs model.

In \((3+1)\) dimensions, a phase transition associated with the formation of a monopole condensate separates a confining phase at strong coupling from the massless photon phase at weak coupling. It is hoped that the present work can contribute to a quantitative description of these phenomena.

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