Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds I

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1 Introduction

This is the first of a series of papers devoted to the study of how Gromov-Witten invariants of 3-folds transform under surgery. Special attention will be paid to Calabi-Yau 3-folds. In [R1], [RT1], [RT2], Ruan-Tian established the mathematical foundations of the theory of quantum cohomology or Gromov-Witten invariants for the semipositive symplectic manifolds. Recently, semipositivity condition has been removed by the work of many authors: [LT2], [B], [FO], [LT3], [R5], [S]. The focus now is on calculation and applications. As quantum cohomology was developed, many examples were calculated by direct computation. In particular, the localization technique has had many successes [Gi], [LLY]. In this article, we take a new direction by computing the change of Gromov-Witten invariants under some well-known surgeries in algebraic geometry. Then, our formula will calculate the Gromov-Witten invariants of any 3-fold obtained by performing these surgeries from known examples. Another outcome of this paper is the appearance of the relation between the naturality problem of quantum cohomology and birational geometry. This newly discovered relation is both surprising and intriguing. We hope to investigate this more in the future.

1 partially supported by a NSFC grant and a Qiu Shi grant
2 partially supported by a NSF grant and a Sloan fellowship
To motivate our choice of surgeries, let’s review the classification of Calabi-Yau (CY) 3-folds. A first step is to classify CY 3-folds in the same birational class. Any two CY 3-folds are birationally equivalent to each other iff they can be connected by a sequence of flops $[\text{Ka}], [K]$. A flop is a kind of surgery: two CY 3-folds $M$ and $M_f$ are said to be related by a flop if there is a a singular CY 3-fold $M_s$ obtained from $M$ by a “small contraction” (contracting finitely many rational curves) with $M_f$ obtained from $M_s$ by blowing up these singularities differently. Flopping can be defined for any 3-fold once the corresponding small contraction exists. A conjecture by Morrison [Mo1] is that flopping among CY 3-folds induces an isomorphism on quantum cohomology. We will give an affirmative answer to Morrison’s conjecture for any 3-fold. It is well-known that flopping does not induce an isomorphism on ordinary cohomology. Birational geometry is a central topic in algebraic geometry. It is often difficult to construct birational invariants. A proof of Morrison’s conjecture would provide the first truly quantum birational invariant. It seems to us that it will be an important problem to study quantum cohomology under other type of birational transformations, for example flips. This requires an extension of quantum cohomology to orbifolds. We shall leave this to future research.

For nonbirationally equivalent CY 3-folds, an influential conjecture by Miles Reid (Reid’s fantasy) states that any two Calabi-Yau 3-folds are connected to each other by a sequence of so-called contract-deform or deform-resolve surgery. Contract-deform means that we contract certain curves or divisors to obtain a singular Calabi-Yau 3-fold and deform the result to a smooth CY 3-fold. Deform-resolve is the reverse. Reid’s original conjecture was stated in the category of non-Kähler manifolds. To stay in the Kähler category, the contractions must be extremal in the sense of Mori. The smoothing theory for the case of ordinary double points was treated in [3], [11]. In the general case, we refer to [Grd] for references. An extremal contraction-smoothing or its opposite surgery is called an extremal transition or transition. We call a transition small if the corresponding contraction is small. Furthermore, a small transition can be performed over any 3-fold. A modified version of Reid’s conjecture is that any two smooth CY 3-folds are connected to each other by a sequence of flops or extremal transitions. Large classes of CY 3-folds are indeed connected to each other in this way.

The classification of CY 3-folds involves the study of surgeries. It would be desirable to study the effect of these surgeries on mirror symmetry. This paper takes a step in this direction. Recall that the mirror symmetry conjecture asserts that every CY 3-fold $X$ has a mirror partner $Y$ such that $h_{1,1}(X) = h_{1,2}(Y), h_{1,2}(X) = h_{1,1}(Y)$. Furthermore, the quantum cohomology of $X$ is the same as the Yukawa coupling of $Y$. It was known that the most general form of the mirror symmetry conjecture is false due to the existence of rigid CY 3-folds with $h_{1,2} = 0$. The conjectured mirror partner of $X$ will have $h_{1,1} = 0$. Hence, it can not be Kähler. We believe that the most difficult problem in mirror symmetry is to find the precise category of CY 3-folds where mirror symmetry holds. An interesting speculation [Mo2] here was that, except for the obvious counterexamples, each extremal transition has a mirror surgery which preserves mirror symmetry. It can be summarized as follows: Mirror surgery conjecture: Every extremal transition $L$ has a mirror surgery $L_m$ with the following property: If we have a mirror pair $(X, Y)$ and perform an extremal transition $L$ on $X$ and obtain $\tilde{X}$, then one of the following is true: (1) $\tilde{X}$ has no large complex structure limit. In this case, $X$ has no mirror. (2) The mirror of $\tilde{X}$ can be obtained by performing $L_m$ on $Y$. Indeed, all the known examples of rigid CY 3-folds can be obtained by extremal transitions. This gives a nice explanation of the failure of mirror symmetry for rigid CY 3-folds. This conjecture grew from a discussion of the second author with P. Aspinwall. A closely related conjecture was also proposed by D. Morrison [Mo2]. Once we understand how relevant invariants change under extremal transitions, we can extend mirror symmetry to a large classes of CY 3-folds, and hopefully find the precise category where mirror symmetry holds. One can view the mirror surgery conjecture
as a combination of the classification problem and mirror symmetry. Any results on the mirror surgery conjecture would yield a deeper understanding to both the classification problem and mirror symmetry. If we want to prove Morrison’s conjecture or the mirror surgery conjecture, it is clear that we have to calculate the change of GW-invariants under flops and transitions. In this paper, we study flops and small transitions over any 3-fold. In subsequent papers, we will focus on CY 3-folds to deal with other type of transitions where there is a good classification theory.

The invariants we consider here are primitive GW-invariants \( \Psi^M_{(A, g, m)}(\mathcal{M}_{g, m}; \{\alpha_i\}) \) for the stable range \( 2g + m \geq 3 \). It was conjectured [RT2] that any GW-invariants can be reduced to primitive ones. We shall drop \( \mathcal{M}_{g, m} \) to simplify the notation. If \( m = 0 \), We will drop \( m \) as well. For primitive invariants, it is also convenient to drop the stable range condition. The construction is standard and we leave it to the readers. Then, one can eliminate the divisor class \( \alpha \in H^2(M, \mathbb{R}) \) by the relation

\[
(1.1) \quad \Psi^M_{(A, g, m+1)}(\alpha, \alpha_1, \cdots, \alpha_m) = \alpha(A) \Psi^M_{(A, g, k)}(\alpha_1, \cdots, \alpha_m),
\]

for \( A \neq 0 \).

Choose a basis \( A_1, \cdots, A_m \) of \( H_2(M, \mathbb{Q}) \). For \( A = \sum_i a_i A_i \), we define the formal product \( q^A = (q_{A_i})^{a_1} \cdots (q_{A_m})^{a_m} \). We can define a quantum 3-point function

\[
(1.2) \quad \Psi^M_w(\alpha_1, \alpha_2, \alpha_3) = \sum_j \frac{1}{j!} \sum_A \Psi^M_{(A, 0, j+3)}(\alpha_1, \alpha_2, \alpha_3, w, \cdots, w) q^A,
\]

where \( w \) appears \( m \) times. Here, we view \( \Psi^M \) as a power series in the formal variables \( p_i = q^{A_i} \). Clearly, an isomorphism on \( H_2 \) will induce a change of variables \( p_i \). One can define the quantum product from the quantum 3-point function. They contain the same information. For our purpose, it is convenient to work directly with the quantum 3-point function.

**Definition 1.1:** Suppose that

\[
\varphi : H_2(X, \mathbb{Z}) \to H_2(Y, \mathbb{Z}), \quad H^{even}(Y, \mathbb{R}) \to H^{even}(X, \mathbb{R})
\]

are vector space homomorphisms such that the maps on \( H_2, H^2 \) are dual to each other. We say \( \varphi \) is natural with respect to (big) quantum cohomology if \( \varphi^* \Psi^N_Y = \Psi^N_X (\varphi^* \Psi^N_Y = \Psi^N_{(A, g, m)} \) up to a change of formal variable \( q^A \to q^{\varphi(A)} \). If \( \varphi \) is also an isomorphism, we say \( \varphi \) induces an isomorphism on (big) quantum cohomology or they have the same (big) quantum cohomology.

Here, two power series \( F, G \) are treated as the same if \( F = H + F', G = H + G' \) such that \( G' \) is an analytic continuation of \( F' \). For example, we can expand \( \frac{1}{1-t} = \sum_{i=0}^\infty t^i \) at \( t = 0 \) or \( \frac{1}{1-t} = -\frac{1}{t(1-t)} \) at \( t = \infty \). Hence, we will treat \( \sum_{i=0}^\infty t^i, \sum_{i=0}^\infty t^{-i} \) as the same power series.

When \( X, Y \) are 3-folds, such a \( \varphi \) is completely determined by maps on \( H_2 \). For example, the dual map of \( \varphi : H_2(X, \mathbb{Z}) \to H_2(Y, \mathbb{Z}) \) gives a map \( H^2(Y, \mathbb{R}) \to H^2(X, \mathbb{R}) \). A map \( H^*(Y, \mathbb{R}) \to H^*(X, \mathbb{R}) \) is Poincaré dual to a map \( H_2(Y, \mathbb{R}) \to H_2(X, \mathbb{R}) \). In the case of flops, the natural map \( H_2(X, \mathbb{Z}) \to H_2(Y, \mathbb{Z}) \) is an isomorphism. Therefore, we can take the map \( H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z}) \) as its inverse. The maps on \( H^0, H^6 \) are obvious.

Suppose that \( M_f \) is obtained after a flop on \( M \). There is a natural isomorphism (see section 2)

\[
(1.3) \quad \varphi : H_2(M, \mathbb{Z}) \to H_2(M_f, \mathbb{Z}).
\]
The manifolds $M$ and $M_f$ have the same set of exceptional curves. Suppose that $\Gamma$ is an exceptional curve and $\Gamma_f$ is the corresponding exceptional curve on $M_f$. Then,

$$(1.4) \quad \varphi([\Gamma]) = -[\Gamma_f].$$

Our first theorem is that

**Theorem A** If $A \neq n[\Gamma]$ for any exceptional curve $\Gamma$, then

$$(1.5) \quad \Psi^M_{(A,g,m)}(\{\varphi^*\alpha_i\}) = \Psi^M_{(\varphi(A),g,m)}(\{\alpha_i\}).$$

Moreover,

$$(1.6) \quad \Psi^M_{[n[\Gamma],g]} = \Psi^M_{[n[\Gamma_f],g]}.$$

When $M$ is a Calabi-Yau 3-fold, Theorem A takes a particularly simple form.

**Corollary A.1:** Suppose that $M$ is a Calabi-Yau 3-fold. If $A \neq n[\Gamma]$ for any exceptional curve $\Gamma$, then

$$\Psi^M_{(A,g)} = \Psi^M_{(\varphi(A),g)}.$$ 

Moreover,

$$\Psi^M_{[n[\Gamma],g]} = \Psi^M_{[n[\Gamma_f],g]}.$$

Using our formula in genus zero case and Morrison’s argument [Mo1], we have the following corollary.

**Corollary A.2:** $\varphi$ induces an isomorphism on quantum cohomology.

Recall that a minimal model is a projective variety with terminal singularities and nef canonical bundle. It is known that in higher dimension there are many different minimal models in the same birational class. However, in dimension three, they are related by flops. Then the above corollary yields

**Corollary A.3:** Any two smooth minimal models in dimension three have the same quantum cohomology.

The second author conjectures that two smooth minimal models over any dimension have isomorphic quantum cohomology [R6]. When $M$ is a CY-3-fold, the above corollary implies Morrison’s conjecture. In the case of a CY 3-fold, Corollary A.2 admits another interpretation. Instead of considering $M, M_f$ as different manifolds, we can use $\phi$ to map the cohomology of different birational models into a single space called the movable cone. Then, the Corollary A.2 can be restated as follows: the quantum 3-point function extends analytically over the movable cone. In fact, this was how Morrison stated his conjecture.

Let $M_e$ be the 3-fold obtained after a small transition. The small transition induces a surjective map

$$(1.8) \quad \varphi_e : H_2(M, \mathbb{Z}) \to H_2(M_e, \mathbb{Z}).$$
We choose a right inverse, which is a map $H_2(M_e, \mathbb{R}) \to H_2(M, \mathbb{R})$. Using the general gluing theory we establish, we obtain the following formula:

**Theorem B** Let $M_e$ be obtained by a small transition. For $B \neq 0$,

$$
\Psi_{(B, g, m)}^{M_e}(\{\alpha_i\}) = \sum_{\varphi_e(A) = B} \Psi_{(A, g, m)}^M(\{\varphi^*(\alpha_i)\}).
$$

Again, Theorem B takes a simple form for Calabi-Yau 3-fold.

**Corollary B.1** Let $M_e$ be obtained by a small transition performed on a Calabi-Yau 3-fold $M$. Then, for $B \neq 0$

$$
\Psi_{(B, g)}^{M_e} = \sum_{\varphi_e(A) = B} \Psi_{(A, g)}^M.
$$

Then, we can use the genus zero case of Theorem B and generalize Tian’s argument \cite{T} to compare quantum cohomology.

**Corollary B.2:** $\varphi$ is natural with respect to big quantum cohomology.

We remark that, in the case of Calabi-Yau 3-folds, the genus zero cases of both Theorems A and B have been studied in physics. However, it is rather surprising to us that it exhibits such a naturality relation. From known computation, this is the first case with this property. Higher genus GW-invariants are not enumerative invariants in general. The nature of these invariants is quite mysterious even today. We were surprised that the same formula is also true for higher genus GW-invariants. Several years ago, there was a mysterious Kodaira-Spencer quantum field theory in physics which dealt with the higher genus case \cite{BCOV}. We hope that our calculation will shed some light on the structure of higher genus invariants.

One of the main ideas of this paper is a symplectic interpretation of the above algebraic geometric surgeries. Then, we can use symplectic techniques, which are more flexible. We know of two instances where surgeries were used to calculate GW-invariants. In an approach quite different from ours, G. Tian studied the degeneration of rational curves under symplectic degeneration \cite{T} which is an analogue of the degeneration in algebraic geometry. The method McDuff used in \cite{M2}, \cite{Lo}, is similar to that of this paper. However, she studied the holomorphic curves completely inside the symplectic manifold with contact boundary, which is much easier to deal with. The main difficulty in this paper is to handle holomorphic curves which intersect the boundary.

When we search for the formula for general surgery, Theorems A and B are rather misleading. The natural invariants appearing in general surgery are log GW-invariants instead of absolute GW-invariants. To describe log GW-invariants, let’s first review the definition of symplectic cutting \cite{E}.

Suppose that $H : M^0 \to \mathbb{R}$ is a proper Hamiltonian function such that the Hamiltonian vector field $X_H$ generates a circle action, where $M^0 \subset M$ is an open domain. By adding a constant, we can assume that 0 is a regular value. Then $H^{-1}(0)$ is a smooth submanifold preserved by circle action. The quotient $Z = H^{-1}(0)/S^1$ is the famous symplectic reduction. Namely, it has an induced symplectic structure. We can cut $M$ along $H^{-1}(0)$. Suppose that we obtain two disjoint components $M^\pm$ which have the boundary $H^{-1}(0)$. We can collapse the $S^1$-action on $H^{-1}(0)$ to obtain $\overline{M^\pm}$ containing a real codimension two submanifold $Z$. Moreover, there is a map

$$
\pi : M \to \overline{M^+} \cup_Z \overline{M^-},
$$

**(1.10)**
where $\overline{M}^+ \cup_Z \overline{M}^-$ is the union of $\overline{M}^\pm$ along $Z$. It induces a homomorphism
\begin{equation}
\pi^*: H^*(\overline{M}^+ \cup_Z \overline{M}^-, \mathbb{Q}) \to H^*(M, \mathbb{Q}).
\end{equation}

It was shown by Lerman [L] that the restriction of the symplectic structure $\omega$ on $M^\pm$ extends to symplectic structures $\omega^\pm$ over $\overline{M}^\pm$ such that $\omega^+|_Z = \omega^-|_Z$ is the induced symplectic structure from symplectic reduction. By the Mayer-Vietoris sequence, a pair of cohomology classes $(\alpha^+, \alpha^-) \in (H^*(\overline{M}^+, Z), H^*(\overline{M}^-, Z))$ with $\alpha^+|_Z = \alpha^-|_Z$ defines a cohomology class of $\overline{M}^+ \cup_Z \overline{M}^-$, denoted by $\alpha^+ \cup_Z \alpha^-$. It is clear from Lerman’s construction that $\pi^*(\omega^+ \cup_Z \omega^-) = \omega$.

Suppose that $B$ is a real codimension two symplectic submanifold of $M$. By a result of Guillemin and Sternberg [GS], the symplectic structure of a tubular neighborhood of $B$ is modeled on a neighborhood of $Z$ in either $\overline{M}^+$ or $\overline{M}^-$. We can define a log GW-invariant $\Psi^{(M, Z)}$ by counting the number of log stable holomorphic maps intersecting $Z$ at finitely many points with prescribed tangency. Let $T_m = (t_1, \ldots, t_m)$ be a set of positive integers such that $\sum_i t_i = Z^*(A)$, where $Z^*$ is the Poincaré dual of $Z$. We order them such that $t_1 = \cdots = t_l = 0$ and $t_i > 0$ for $i > l$. Consider the moduli space $\mathcal{M}_A(g, T_m)$ of genus $g$ pseudo-holomorphic maps $f$ such that $f$ has marked points $(x_1, \ldots, x_m)$ with the property that $f$ is tangent to $Z$ at $x_i$ with order $t_i$. Here, $t_i = 0$ means that there is no intersection. Then, we compactify $\mathcal{M}_A(g, T_m)$ by $\overline{\mathcal{M}}_A(g, T_m)$, the space of log stable maps. We have evaluation map
\begin{equation}
e_i : \overline{\mathcal{M}}_A(g, T_m) \to M
\end{equation}
for $i \leq l$ and
\begin{equation}
e_j : \overline{\mathcal{M}}_A(g, T_m) \to Z
\end{equation}
for $j > l$. Roughly, the log GW-invariants are defined as
\begin{equation}
\Psi^{(M, Z)}_{(A, g, T_m)}(\alpha_1, \ldots, \alpha_l; \beta_{l+1}, \ldots, \beta_m) = \int_{\overline{\mathcal{M}}_A(g, T_m)} \prod_i e^*_i \alpha_i \wedge \prod_j e^*_j \beta_j.
\end{equation}

To carry out the virtual integration, we need to construct a virtual neighborhood (see Section 5 of $\overline{\mathcal{M}}_A(g, T_m)$). Then we take the ordinary integrand (1.13) over the virtual neighborhood.

To justify our notation, we remark that when $T_m = (0, \ldots, 0, 1, \ldots, 1)$, $\Psi^{(M, Z)}$ is different from an ordinary or absolute GW-invariant in general. So $\Psi^{(M, Z)}$ is not a generalized GW-invariant.

**Theorem C (Theorem 5.3):**

(i). $\Psi^{(M, Z)}_{(A, g, T_m)}(\alpha_1, \ldots, \alpha_l; \beta_{l+1}, \ldots, \beta_m)$ is well-defined, multilinear and skew symmetric with respect to $\alpha_i$ and $\beta_j$ respectively.

(ii). $\Psi^{(M, Z)}_{(A, g, T_m)}(\alpha_1, \ldots, \alpha_l; \beta_{l+1}, \ldots, \beta_m)$ is independent of the choice of forms $\alpha_i, \beta_j$ representing the cohomology classes $[\beta_j], [\alpha_i]$, and the choice of virtual neighborhoods.

(iii). $\Psi^{(M, Z)}_{(A, g, T_m)}(\alpha_1, \ldots, \alpha_l; \beta_{l+1}, \ldots, \beta_m)$ is independent of the choice of almost complex structures on $Z$ and on the normal bundle $v_Z$, and hence is an invariant of $(M, Z)$.

**Theorem D (Theorem 5.6, 5.7):** Suppose that $\alpha_i^+|_Z = \alpha_i^-|_Z$ and hence $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{M}^+ \cup_Z \overline{M}^-, \mathbb{R})$. Let $\alpha_i = \pi^*(\alpha_i^+ \cup_Z \alpha_i^-)$ (1.11). There is a gluing formula to relate $\Psi^M(\{\alpha_i\})$ to the log invariants $\Psi^{(M^+, Z)}(\{\alpha_i^+\})$ and $\Psi^{(M^-, Z)}(\{\alpha_i^-\})$. 
Although it is not needed in this paper, it is also important to consider descendant GW-invariant in general quantum cohomology theory. Our gluing theory extends to this more general setting with the same proof. Let’s give a brief description on the formulation. Recall that the cotangent space of each marked point $x_i$ generates an orbifold line bundle $L_i$ over the moduli space of stable map $\overline{M}_A(g,k)$. The descendant GW-invariant is defined as

\begin{equation}
\Psi^M_{(A,g,k)}(\mathcal{O}_{h_1}(\alpha_1), \ldots, \mathcal{O}_{h_k}(\alpha_k)) = \int_{\overline{M}_A(g,k)} \prod_i C_l^i(L_i)e^*_i \alpha_i.
\end{equation}

One can define descendant log-GW-invariants $\Psi^M,Z_{(A,g,T_k)}(\mathcal{O}_{h_1}(\alpha_1), \ldots, \mathcal{O}_{h_i}(\alpha_I); \mathcal{O}_{h_{i+1}}(\beta_{i+1}), \ldots, \mathcal{O}_{h_m}(\beta_m))$ in the same fashion and Theorem C extends to this more general descendant log-GW-invariants. In gluing theory, we only need $\Psi^X,Z_{(A,g,T_k)}(\mathcal{O}_{h_1}(\alpha_1), \ldots, \mathcal{O}_{h_i}(\alpha_I); \beta_{i+1}, \ldots, \beta_m)$. Theorem D can be easily generalized (by the identical proof) to express descendant invariant $\Psi^M_{(A,g,k)}(\{\mathcal{O}_{h_i}(\alpha_i)\})$ in terms of descendant log-invariant $\Psi^{(M^\pm,Z)}(\{\mathcal{O}_{h_i}(\alpha_i)\})$ by the same formula as the case of ordinary GW-invariants.

We also remark that different $\alpha_i^\pm$ may give the same $\alpha_i$. Then we obtain different gluing formulas for the same invariant. For example, if $\alpha_i$ is Poincaré dual to a point, its support could be in $\overline{M}^+$ or $\overline{M}^-$. This is very important in the applications.

The exact statement of the general formula is rather complicated. We refer the reader to section 5 for the details.

In a subsequent paper [LQR], we will apply our general gluing formula to study the change of GW-invariants for other types of transition. The other types of transitions are closely related to ordinary blow-up. It is known that the blow-up formula of quantum cohomology is complicated. Hence, one can not expect to have such a simple formula as in the case of small transition. However, our general gluing formula gives a complete answer to the case of the other type of transition for Calabi-Yau 3-folds. We refer to our next paper [LQR] for the detail.

Let’s briefly describe the idea of the proof. The first step is to reinterpret the flop and extremal transition as a combination of contact surgery and symplectic cutting (section 2). Then, we stretch symplectic manifolds along a hypersurface admitting a local $S^1$-Hamiltonian action. Then we study the behavior of pseudoholomorphic curves under the stretching. In the case of a contact manifold, every contact manifold with a fixed contact form possesses a unique vector field called the Reeb vector field. Hofer observed that the boundary of a finite energy pseudoholomorphic curve will converge to a periodic orbit of the Reeb vector field. Furthermore, Hofer and his collaborators established the analysis of the moduli space of pseudoholomorphic curves whose ends converge to periodic orbits. We will generalize it to the case of symplectic cutting. We will do so by casting it into the language of stable maps (section 3). Furthermore, we establish the gluing theorem which is the reverse of the stretching construction (section 4). The last piece of information we need is a vanishing theorem for certain relative GW-invariants. This is done by a simple index calculation.

We should point out that the log GW-invariants also appeared in the work of Caporaso-Harris [CH]. They are closely related to the blow-up formula for Seiberg-Witten invariants.

This paper has been revised many times due to referee’s suggestions. In many ways, current version is much better written than original version. We would like to thank referee for the insistence which made this possible. But our underline approach has always been the same. Namely, we explored the local hamiltonian circle action near the symplectic submanifold and established a gluing theorem in this setting. In the original version of our paper, in addition to the results of this paper, we also proved a compactness theorem for contact manifolds whose periodic orbits are of Bott type. The contact case is irrelevant to our paper. It was included in our original version.
only because it can be proved by the same technique. Latter, Hofer informed us that it has been dealt with already in [HWZ1]. Then, the contact case was taken out in the revised version. There are many papers in physics that discuss both flop and extremal transitions. We refer to [Mo2] for the relevant references. But they discuss only genus zero invariants. The paper by P. Wilson [Wi3] calculates GW-invariants of extremal rays. His paper is complementary to ours. During the preparation of this paper, we received an article [BCKS] which is related to our results. We are working on the setting of the local Hamiltonian $S^1$-action, which is not contact in general. In the setting of contact manifold, a general gluing formula over a contact boundary requires a definition of contact Floer homology, which is developed by Eliashberg and Hofer. We are also informed that Ionel and Parker are developing a gluing theory independently using a different approach. Apparently, they only used one component of the moduli space of log-stable map to define their relative invariants instead of using full moduli space as we did. Then, their relative invariants is different from ours, which results in different gluing formula.

This paper is organized as follows. We will review the constructions of various surgery operations in symplectic geometry and interpret both the flop and extremal transitions as symplectic surgeries. The gluing theory will be established in sections 3-5. Theorems A and B will be proved in section 6.

The main results of this paper were announced in a conference in Kyoto in December, 1996. The second author would like to thank S. Mori for the invitation. The second author also wish to thank H. Clemens, Y. Eliashberg, M. Gross, J. Kollár, K. Kawamata, S. Katz, E. Lerman, S. Mori, K. Oguiso, M. Reid, Z. Qin and P. Wilson for valuable discussions. In the original version of our paper, Log GW-invariants was called relative GW-invariants. Both author would like to thank Qi Zhang to point out to us that our invariant belongs to log category in algebraic geometry. Both authors would like to thank Yihong Gao for valuable discussions and the referee for many suggestions to improve the presentation of this paper. Thanks also to J. Robbin and A. Greenspoon for editorial assistance.

2 Symplectic surgery, flops and extremal transition

Symplectic surgeries have been extensively studied by a number of authors. Many such surgeries already appeared in [Gr]. One of the oldest ones is the gluing along a contact boundary. We do not know who was the first to propose contact surgeries; however, the second author benefited from a number of stimulating discussions with Y. Eliashberg on contact surgeries over the years. During the last ten years, symplectic surgeries have been successfully used to study symplectic topology, for example, symplectic blow-up and blow-down by McDuff [MS1] and symplectic norm sum by Gompf [Go2] and McCarthy and Wolfson [MW]. Very recently, Lerman [L] introduced an operation called “symplectic cutting” which plays an important role in this article. For the reader’s convienience, we will give a quick review of each of these operations. The second half of this section is devoted to studying algebro-geometric surgeries of flop and small transition in terms of symplectic surgery.

Definition 2.1 A contact structure $\xi$ on an odd dimensional manifold $N^{2n+1}$ is a codimension one distribution defined globally by a one-form $\lambda$ ($\text{Ker}\lambda = \xi$) such that $\lambda \wedge (d\lambda)^n$ is a volume form. We call $\lambda$ a contact form.

Any contact manifold has a canonical orientation defined by the volume form $\lambda \wedge (d\lambda)^n$. If $\lambda$ is a contact form, so is $f\lambda$ for a positive function $f$. Any contact form defines the Reeb vector field $X_\lambda$ by the equation

$$i_{X_\lambda} \lambda = 1, i_{X_\lambda} d\lambda = 0.$$
In general, the dynamics of $X_\lambda$ depends on $\lambda$. There is a version of Moser’s theorem for contact manifolds as follows. Let $\lambda_t$ be a family of contact forms. There is a family of diffeomorphisms
\[ \varphi_t : N \rightarrow N \]
such that $\varphi_0 = Id$ and $\varphi_t^* \lambda_t = f_t \lambda$, where $f_t$ is a family of functions such that $f_0 = 1$.

**Example 2.2:** The sphere $S^{2n-1} \subset \mathbb{C}^n$ with the standard symplectic structure $\omega = \sum dx_i \wedge dy_i$ is a contact hypersurface. The contact form is the restriction of $\lambda = \sum (x_i dy_j - y_j dx_i)$. The closed orbit of the Reeb vector field is generated by complex multiplication by $e^{i\theta}$. Furthermore, if $Q$ is a symplectic submanifold intersecting $S^{2n-1}$ transversely, $Q \cap S^{2n-1}$ is a contact hypersurface of $Q$.

**Definition 2.3:** $(M, \omega)$ is a compact symplectic manifold with boundary such that $\partial M = N$. $M$ has a contact boundary $N$ if in a tubular neighborhood $N \times [0, \epsilon)$ of $N$, $\omega = d(f \lambda)$, where $f$ is a function. Suppose that $X$ is an outward transverse vector field on $N$. We say that $N$ is a convex contact boundary if $\omega(X_N, X) > 0$. Otherwise, we call $N$ a concave contact boundary. In other words, the induced orientation of a convex contact boundary coincides with the canonical orientation. But the induced orientation of a concave contact boundary is the opposite of the canonical orientation.

Let $(M^+, \omega^+), (M^-, \omega^-)$ be symplectic manifolds with contact boundary $N^+, N^-$. Furthermore, suppose that $N^+$ is a convex boundary of $M^+$ and $N^-$ is a concave boundary of $M^-$. For any contact diffeomorphisms $\varphi : N^+ \rightarrow N^-$, we can glue them together to form a closed symplectic manifold $M_{gl} = M^+ \cup_\varphi M^-$. One can find the details of the gluing construction from $[E]$ (Proposition 3.1).

Recall that GW-invariants only depend on the symplectic deformation class.

**Definition 2.4:** Two symplectic manifolds $(M, \omega), (M', \omega')$ with contact boundaries $(N, \lambda), (N', \lambda')$ are deformation equivalent if there is a diffeomorphism $\varphi : M \rightarrow M'$, a family of symplectic structures $\omega_t$ and a family of contact structures $\lambda_t$ such that $(M, \omega_t)$ is a symplectic manifold with contact boundary $(N, \lambda_t)$ and
\[
\omega_0 = \omega, \quad \omega_1 = \varphi^* \omega', \quad \lambda_0 = \lambda, \quad \lambda_1 = \varphi^* \lambda'.
\]

The proof of the following Lemma is trivial. We omit it.

**Lemma 2.5:** If we deform the symplectic structures of $M^\pm$ according to Definition 2.3, the glued-up manifolds $M_{gl}$ are deformation equivalent.

Another closely related surgery is symplectic cutting described in the introduction, where the Hamiltonian vector field plays the role of the Reeb vector field. The converse gluing process has been studied previously by McCarthy and Wolfson $[MW]$ in connection with symplectic norm sum. By (1.10), we have a map
\[ \pi : M \rightarrow M^+ \cup_\mathbb{Z} M^- . \]
It induces a map
\[
\pi_* : H_i(M, \mathbb{Z}) \rightarrow H_i(M^+ \cup_\mathbb{Z} M^-, \mathbb{Z}).
\]
A class in ker $\pi_*$ is called a vanishing cycle. The virtual neighborhood construction requires that the symplectic forms $\omega, \omega^+, \omega^-$ on $M, M^+, M^-$ have integral periods. We can deform $\omega^+, \omega^-$ slightly (still denoted by $\omega^+, \omega^-$) such that $\omega^+|_\mathbb{Z} = \omega^-|_\mathbb{Z}$ and $\omega^\pm$ has rational periods. Then, we use the
Gompf-McCarthy-Wolfson gluing construction (an inverse operation of the symplectic cutting) to obtain $\omega$. Then, $\omega$ has rational periods since $\omega = \pi^*(\omega^+ \cup \omega^-)$. To get integral periods, we just multiple $\omega$ by a large integer.

**Example 2.6:** We consider the Hamiltonian action of $S^1$ on $(\mathbb{C}^n, \omega_0)$ corresponding to multiplication by $e^{it}$ with the Hamiltonian function $H(z) = |z|^2 - \epsilon$. Then we have a Hamiltonian action of $S^1$ on $((\mathbb{C}^n \times \mathbb{C}, -i(dw \wedge d\bar{w} + \Sigma_{j=1}^n dz_j \wedge d\bar{z}_j))$ given by

\begin{equation}
(2.4) \quad e^{it}(z, w) = (e^{it}z, e^{it}w)
\end{equation}

with moment map

\begin{equation}
(2.5) \quad \mu(z, w) = H(z) + |w|^2.
\end{equation}

We have

\begin{equation}
(2.6) \quad \mu^{-1}(0) = \{(z, w)|0 < |w|^2 \leq \epsilon, H(z) = -|w|^2\} \cup \{(z, 0)|H(z) = 0\}
\end{equation}

\begin{equation}
(2.7) \quad \overline{\mathcal{M}}^+ = \mu^{-1}(0)/S^1.
\end{equation}

where $M^+ = \{z||z| \leq \epsilon\}$ and $\overline{\mathcal{M}}^+$ is the corresponding symplectic cut. Choose the canonical complex structure $i$ in $\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}$. The following fact is well-known (see [MS1]):

*For $\mathbb{C}^n$ the symplectic cut $\overline{\mathcal{M}}^+$ is $\mathbb{P}^n$ with symplectic form

\begin{equation}
(2.8) \quad -i\epsilon d\left(\frac{zd\bar{z} - \bar{z}dz}{1 + ||z||^2}\right).
\end{equation}

**Example 2.7:** We consider $\mathcal{O}(-1) + \mathcal{O}(-1)$ over $\mathbb{P}^1$. One can construct a symplectic form as follows. Choose a hermitian metric on the vector bundle $\mathcal{O}(-1) + \mathcal{O}(-1)$. The $||z||^2$ for $z \in \mathcal{O}(-1) + \mathcal{O}(-1)$ is a smooth function. $i\partial\bar{\partial}||z||^2$ is a 2-form nondegenerate on the fiber. Suppose that $\omega_0$ is a symplectic form on $\mathbb{P}^1$.

\begin{equation}
(2.9) \quad \omega = \pi^*\omega_0 + \epsilon i\partial\bar{\partial}||z||^2
\end{equation}

is a symplectic form on the total space in a neighborhood of the zero section, where $\pi : \mathcal{O}(-1) + \mathcal{O}(-1) \to \mathbb{P}^1$ is the projection and $\epsilon$ is a small constant. The Hamiltonian function is

\begin{equation}
(2.10) \quad H(x, z_1, z_2) = |z_1|^2 + |z_2|^2 - \epsilon.
\end{equation}

It is a routine computation that the $S^1$-action is given by

\begin{equation}
(2.11) \quad e^{it}(x, z_1, z_2) = \left(x, e^{it}z_1, e^{it}z_2\right).
\end{equation}

We perform the symplectic cutting along the hypersurface $\tilde{\mathcal{M}} = H^{-1}(0)$. By example 1, we conclude that

*For $\mathcal{O}(-1) + \mathcal{O}(-1)$ over $\mathbb{P}^1$ the symplectic cut $\overline{\mathcal{M}}^+$ is $P(\mathcal{O}(-1) + \mathcal{O}(-1) + \mathcal{O})$. }
Example 2.8: Consider the algebraic variety $M$ defined by a homogeneous polynomial $F$ in $\mathbb{C}^n$. Let $S^{2n-1}(\epsilon)$ be the sphere of radius $\epsilon$. Let $\overline{M} = S^{2n-1}(\epsilon) \cap \{F = 0\}$. Consider the Hamiltonian action of $S^1$ on $(\mathbb{C}^n, \omega_0)$ corresponding to multiplication by $e^{it}$ with the Hamiltonian function

$$H(z) = |z|^2 - \epsilon.$$  

Since $F$ is a homogeneous polynomial, $S^1$ acts on $S^{2n-1}(\epsilon) \cap \{F = 0\}$. By example 1 we have

For $M = \{F = 0\}$ the symplectic cut $\overline{M}^+$ is symplectic deformation to the variety defined by $F = 0$ in $\mathbb{P}^n$.

We have finished our digression about symplectic surgery. Next we study flop and extremal transition from a symplectic point of view. Recall that two projective manifolds $M$, $M'$ are birational equivalent iff some Zariski open sets are isomorphic. If $M$, $M'$ are smooth CY 3-folds, $M$, $M'$ are related by a sequence of flops [Ka], [K]. Namely, there is a sequence of smooth CY 3-folds $M_1, \ldots, M_k$ such that $M_1$ is obtained by a flop from $M$, $M_{i+1}$ is obtained by a flop from $M_i$, and $M'$ is obtained by a flop from $M_k$. Furthermore, we can choose $M_i$ and the corresponding singular CY 3-folds $(M_i)_s$ to be projective. Moreover, we can assume that the contractions are primitive. In this paper, we only deal with the small transition, which starts from a small contraction. Suppose that $M, M_f$ is a pair of 3-folds connected by a flop and $M_b$ is the corresponding singular projective 3-fold. The singularities of $M_b$ are rational double points. Wilson observed that we can choose a local complex deformation around singularities to split a rational double point into a collection of ordinary double points (ODP). With its simultaneous resolution, it gives local complex deformations of $M, M_f$ to deform an exceptional curve to a collection of rational curves with $O(-1) + O(-1)$ normal bundle. We can patch them with complex structures outside to define almost complex structures of $M, M_f$ which are tamed with symplectic structures. To relate $M, M_f$ as almost complex manifolds, we can blow up the exceptional rational curves (they are all $O(-1) + O(-1)$ curves) to obtain the same almost complex manifold $M_b$. To summarize the construction,

Proposition 2.9: As almost complex manifolds, $M, M_f$ described above have the same blow-up $M_b$.

This is the model that we will use to compare invariants. Namely, we will compare GW-invariants of $M, M_f$ with the log GW-invariants of $M_b$. Clearly, $M, M_f$ have the same set of exceptional curves. It is easy to see that the exceptional divisors of $M_b$ is a disjoint union of $\mathbb{P}^1 \times \mathbb{P}^1$’s.

Next, we study the topology of flops. As we mentioned in the introduction, there is a homomorphism

$$\varphi : H_2(M, \mathbb{Z}) \to H_2(M_f, \mathbb{Z})$$

such that $\varphi$ flips the sign of fundamental class of exceptional rational curves. The original proof used algebraic geometry. Here, we give a topological proof, which also fits with our gluing construction.

For any $A \in H_2(M, \mathbb{Z})$, it is represented by a 2-dimensional pseudo-submanifold $\Sigma$ of $M$. Since the exceptional set is a union of finitely many curves, we can perturb $\Sigma$ so that $\Sigma$ does not intersect any exceptional curve. Any two different perturbations are pseudo-submanifold cobordant. Moreover, the cobordism can also be pushed off the exceptional curves. Hence, two different perturbations represent the same homology class. Let $\{\Gamma_k\}$ be the set of exceptional curves. We have shown that the inclusion

$$i : M - \bigcup \Gamma_k \to M$$
induces an isomorphism on $H_2$. The same thing is true for $M_f$. Moreover, $M - \cup \Gamma_k$ is the same as $M_f - \cup (\Gamma_f)_k$ where $\{(\Gamma_f)_k\}$ is the corresponding set of exceptional rational curves of $M_f$. Hence, $H_2(M,\mathbb{Z})$ is isomorphic to $H_2(M_f,\mathbb{Z})$ and $\varphi$ is the isomorphism. Moreover, the same argument defines an injective homomorphism

$$\phi_b : H_2(M,\mathbb{Z}) \to H_2(M_b,\mathbb{Z})$$

whose image consists of elements with zero intersection with exceptional divisors. Let $\Gamma \subset M$, $\Gamma_f \subset M_f$ be a pair of exceptional curves obtained by blowing down the same exceptional divisor of $M_b$ over different rulings. We claim

$$\varphi([\Gamma]) = -[\Gamma_f].$$

Note that the normal bundle of an exceptional divisor of $M_b$ is $N = O(-1) \otimes O(1)$ over $\mathbb{P}^1 \times \mathbb{P}^1$, where $O(-1)$ means the $O(-1)$ over the $i$-th factor of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ be the antipodal map which reverses the orientation. Then the restriction of $N$ over $E = \{(x, \tau(x))\}$ is trivial. One can push $E$ off $\mathbb{P}^1 \times \mathbb{P}^1$. We obtain the push-off of the exceptional curve of $M$ by projecting the perturbation of $E$ to $M$. By the construction, $\Gamma, -\Gamma_f$ have the same push-off, where $-\Gamma_f$ means the opposite orientation. Hence

$$\phi([\Gamma]) = -[\Gamma_f].$$

It was already observed in \[\text{[1]}\] that the blow-up along a symplectic submanifold can be viewed as the symplectic cutting. For the reader’s convenience, let’s construct the Hamiltonian $S^1$-action explicitly. By the symplectic neighborhood theorem, the symplectic structure of a neighborhood of $\Gamma$ is uniquely determined by the symplectic structure of $\Gamma$ and the almost complex structure of the symplectic normal bundle. Moreover, a symplectic structure of $\Gamma$ is uniquely determined by the symplectic structure of $\Gamma$ and the almost complex structure of $\Gamma$.

Proposition 2.10: When we perform a symplectic cutting along the boundary of a tubular neighborhood of exceptional curves, we obtain $M_b$ and a collection of $P(O(-1) + O(-1) + O)$’s. Here, $Z$ is a collection of $\mathbb{P}^1 \times \mathbb{P}^1$’s.

The following lemma concerns the vanishing two-cycles.

Lemma 2.11: In the case of blow-up over a complex codimension two submanifold, there is no vanishing 2-cycle.

Proof: For any 2-cycle $\Sigma \subset M$, using PL-transversality, we can assume that $\Sigma \subset \overline{M}$. Hence, $\Sigma$ defines a homology class $\Sigma_b \in H_2(\overline{M} \setminus \mathbb{Z})$. However, $\overline{M} = M_b$ and there is a map $R : M_b \to M$. Moreover, $R_*(\Sigma_b) = \Sigma_b$. Hence, $\Sigma \neq 0$ implies $\Sigma_b \neq 0$.

Next, we discuss the small transition. Again, Wilson’s argument reduces to the consideration of ODP. Let $U_b \subset M_b$ be a neighborhood of the singularity and $U_b \subset M_b$ be its resolution. We first standardize the symplectic form over $0 \in U_b \subset M_b$. Since $M_b$ is projective, we assume that $M_b$ is embedded into a projective space. In the standard affine coordinates, the Fubini-Study form can be written as

$$\omega_0 = \frac{i}{4\pi} \left( \sum_i (x_i d\bar{x}_i - \bar{x}_i dx_i) \right).$$
However, we must choose an analytic change of coordinates to obtain the standard form given by a quadratic equation. Suppose that the change of coordinate is

\[ x_i = f_i(z_1, \cdots, z_4), \]

where \( f_i(0) = 0 \) and \( (\frac{\partial f_i}{\partial z_j}(0)) \) is nondegenerate. We use \( L(f_i) \) to denote the linear term of \( f_i \) and \( L(\omega_0) \) to denote the two-form obtained from \( \omega_0 \) by the linear change of coordinates \( (\frac{\partial f_i}{\partial z_j}(0)) \). Under such a change of coordinates,

\[ \omega = L(\omega_0) + d(\tilde{\alpha}), \]

where \( \tilde{\alpha} = O(|z|^2)(dz_i + d\bar{z}_i) \). Let \( \beta_r \) be a cut-off function which equals 1 when \( |z| > 2r \) and equals zero when \( |z| < r \). Let

\[ \omega_r = \omega - d(\beta_r \tilde{\alpha}). \]

\( \omega_r \) is closed by construction and equal to \( L(\omega_0) \) when \( |z| > 2r \) and equal to \( \omega \) when \( |z| < r \). Moreover,

\[ d(\beta_r \tilde{\alpha}) = (d\beta_r)\tilde{\alpha} + \beta_r d\tilde{\alpha} = O(r)(dz_i \wedge d\bar{z}_j). \]

Therefore, \( \omega_r \) is nondegenerate for small \( r \). Furthermore, by Moser’s theorem, it is the same symplectic structure as \( \omega \) on \( P^N \). Hence, we can assume that \( \omega = \omega_r \). We use the inverse of \( (\frac{\partial f_i}{\partial z_j}(0)) \) to change \( \omega \) to the standard form \( \omega_0 \) and the equation to some homogeneous polynomial \( F_2 \) of degree 2.

Let \( S_r \) be the sphere of radius \( r \). Here we choose \( r \) small enough such that \( \omega = \omega_0 \). Then, \( S_r \) is a contact hypersurface of \( P^N \). Let \( N = S_r \cap \{F_2 = 0\} \). \( N \) is a contact hypersurface of \( M_s \).

To obtain \( M_t \), we have to deform the defining equation of \( M_s \) to smooth the singularity. This step can be described by a contact surgery. Let \( M^-_s = D_r \cap \{F_2 = 0\} \) where \( D_r \) is the ball of radius \( r \) and \( M^+_s = M_s - M^-_s \). The local smoothing is simply

\[ \{F_2 = t\}. \]

Let

\[ M^- _{t_0} = D^r \cap \{F_2 = t_0\} \]

for small \( t_0 \neq 0 \). The global smoothing is well-understood by [F], [T] which requires some linear conditions on the homology classes of the rational curves contracted. Here, we assume that the global smoothing exists and that smoothing is \( M_t \). We use the previous construction to decompose

\[ M_t = M^+_t \cup_{id} M^-_t, \]

where \( M^-_t = D_r \cap M_t \) and \( M^+_t \) is the complement. However, \( M^+_t \) is symplectic deformation equivalent to \( M^+_s \). \( M^-_t \) is deformation equivalent to \( M^-_{t_0} \). Hence, \( M_t \) is symplectic deformation equivalent to

\[ M_e = M^+_s \cup M^-_{t_0}. \]

This is the symplectic model which we will use to calculate GW-invariants. Clearly, if we choose \( r' \) slightly larger than \( r \), the manifold \( M_e \) also admits an \( S^1 \)-Hamiltonian function near \( M_e \cap S^{r'} \).
Then we can perform the symplectic cutting to obtain two symplectic manifolds. We observed that one of them is precisely $M_b$. The other one is symplectic deformation equivalent to

$$\{F_2 - t_0 x_5^2 = 0\} \subset \mathbb{P}^4.$$ 

We summarize the previous construction as

**Proposition 2.12:** We can perform a symplectic cutting on $M_e$ to obtain $M_b$ and a collection of quadric 3-folds.

By the previous argument, we can push any 2-dimensional homology class of $M$ off the exceptional loci, and hence off the singular points of $M_s$. Therefore, we have a map

$$(2.24) \quad \varphi : H_2(M, \mathbb{Z}) \to H_2(M_e, \mathbb{Z}).$$

The topological description of small transition along ODP was known to H. Clemens a long time ago. Note that the boundary of a tubular neighborhood of $O(-1)+O(-1)$ is $S^2 \times S^3$. The transition is an interchange of the handle $S^2 \times D^4$ with $D^3 \times S^3$. The singular 3-fold $M_s$ corresponds to the cone over $S^2 \times S^3$. In particular, Clemens showed that $M_{t_0}^-$ is homeomorphic to $S^3 \times D^3$. It doesn’t carry any nontrivial 2-dimensional homology class in the interior. Therefore, $\varphi (2.24)$ is surjective.

Previous argument also show

**Lemma 2.13:** There is no vanishing 2-cycle for small transition.

Unfortunately, the authors do not know any direct symplectic surgery from $M$ to $M_e$ without going through $M_b$.

Recall that a holomorphic small transition exists iff $\sum_i a_i [\Gamma_i] = 0$ for $a_i \neq 0$, where $\{\Gamma_i\}$ is the set of exceptional curves $[F]$, $[T_1]$. However, we have obtained a stronger result in the symplectic category.

**Theorem 2.14:** Suppose that $V_s$ is projective with rational double points. We always have a symplectic manifold $V_e$ obtained by gluing with local smoothing of singularities. In the holomorphic case, $V_e$ is symplectic deformation equivalent to a holomorphic small transition.

**Remark 2.15:** Finally, we make a remark about general extremal transitions. Many interesting examples are constructed by nonprimitive extremal transitions. A natural question is if we can decompose an extremal transition as a sequence of primitive transitions. Our theorems actually give a criterion to this question. After we perform a primitive extremal transition, we need to study the change of the Kähler cone. Suppose that $E$ gives a different extremal ray. Then it is easy to observe that $E$ will remain an extremal ray if its GW-invariants are not zero. Our theorem shows that the GW invariant can be calculated using the GW-invariant of the Calabi-Yau 3-fold before the transition. Moreover, GW-invariants of a crepant resolution depend only on a neighborhood of the exceptional loci and can be calculated independently from Calabi-Yau 3-folds.

**Remark 2.16:** We would like to make another remark about the smoothing of singular Calabi-Yau 3-folds. One can study the smoothing theory by studying the local smoothings of singularities and the global extension of a local smoothing. The singular CY 3-folds obtained by contracting a smooth CY 3-fold can only have so called canonical singularities.

There are many interesting phenomena in the smoothing theory of CY 3-folds with canonical singularities. For example, M. Gross showed that the same singular CY 3-fold could have two different smoothings [Gro1]. They give a pair of examples where two diffeomorphic CY 3-folds carry different quantum cohomology structures [VI]. By our arguments, it is clear that if a singular
Calabi-Yau 3-fold has a local smoothing for its singularity, it has a global symplectic smoothing. This is not true in the algebro-geometric category by Friedman’s results. This indicates an exciting possibility that there is perhaps a theory of symplectic Calabi-Yau 3-folds, which is broader than the algebro-geometric theory. Such a theory would undoubtedly be important for the classification of CY 3-folds itself.

3 Compactness Theorems

Let \((M, \omega)\) be a compact symplectic manifold of dimension \(2n+2\), \(H : M \to \mathbb{R}\) a local Hamiltonian function such that there is a small interval \(I = (-\ell, \ell)\) of regular values. Denote \(\bar{M} = H^{-1}(0)\). Suppose that the Hamiltonian vector field \(X_H\) generates a circle action on \(H^{-1}(I)\). There is a circle bundle \(\pi : \bar{M} \to Z = \bar{M}/S^1\) and a natural symplectic form \(\tau_0\) on \(Z\). We may choose a connection 1-form \(\lambda\) on \(\bar{M}\) such that \(\lambda(X_H) = 1\) and \(d\lambda\) represents the first Chern class for the circle bundle (see [MS1]). Denote \(\xi = ker(\lambda)\). Then \(\xi\) is an \(S^1\)-invariant distribution and \((\xi, \pi^*\tau_0) \to \bar{M}\) is a \(2n\)-dimensional symplectic vector bundle. We identify \(H^{-1}(I)\) with \(I \times \bar{M}\). By a uniqueness theorem on symplectic forms (see [MS1], [MW]) we may assume that the symplectic form on \(\bar{M}\) is expressed by

\[
\omega = \pi^*(\tau_0 + yd\lambda) - \lambda \wedge dy.
\]

We assume that the hypersurface \(\bar{M} = H^{-1}(0)\) devides \(M\) into two parts \(M^+\) and \(M^-\), which can be written as

\[
M^+_0 \cup \{(\ell, 0) \times \bar{M}\},
\]

\[
M^-_0 \cup \{(0, \ell) \times \bar{M}\},
\]

where \(M^+_0\) and \(M^-_0\) are compact manifolds with boundary. We mainly discuss \(M^+\); the discussion for \(M^-\) is identical. Let \(\phi^+: [0, \infty) \to [-\ell, 0)\) be a function satisfying

\[
(\phi^+)' > 0, \quad \phi^+(0) = -\ell, \quad \phi^+(a) \to 0 \quad as \quad a \to \infty.
\]

Through \(\phi^+\) we consider \(M^+\) to be \(M^+ = M^+_0 \cup \{[0, \infty) \times \bar{M}\}\) with symplectic form \(\omega|_{M^+_0} = \omega\), and over the cylinder \(\mathbb{R} \times \bar{M}\)

\[
\omega_{\phi^+} = \pi^*(\tau_0 + \phi^+d\lambda) - (\phi^+)'\lambda \wedge da.
\]

Moreover, if we choose the origin of \(\mathbb{R}\) tending to \(\infty\), we obtain \(\mathbb{R} \times \bar{M}\) in the limit. Later we will fix the value of \(\phi^+\) on \([0, 1]\). Choose \(\ell_0 < \ell\) and denote

\[
\Phi^+ = \{\phi : [1, \infty) \to [\ell_0, 0)|\phi^+ > 0\}.
\]

Let \(\ell_1 < \ell_2\) be two real numbers satisfying \(-\ell_0 \leq \ell_1 < \ell_2 \leq 0\). Let \(\Phi_{\ell_1, \ell_2}\) be the set of all smooth functions \(\phi : \mathbb{R} \to [\ell_1, \ell_2]\) satisfying

\[
\phi^+ > 0, \quad \phi(a) \to \ell_2 \quad as \quad a \to \infty, \quad \phi(a) \to \ell_1 \quad as \quad a \to -\infty.
\]

To simplify notations we use \(\Phi\) to denote both \(\Phi^+\) and \(\Phi_{\ell_1, \ell_2}\), in case this does not cause confusion.

We choose a compatible almost complex structure \(J\) for \(Z\) such that

\[
g_{\bar{J}(x)}(h, k) = \tau_0(x)(h, \bar{J}(x)k),
\]
for all \( h, k \in TZ \), defines a Riemannian metric. \( \tilde{J} \) and \( \tilde{g} \) are lifted in a natural way to \( \xi \). We define an almost complex structure \( J \) on \( R \times \tilde{M} \) as follows:

\[
J \frac{\partial}{\partial a} = X_H, \quad JX_H = -\frac{\partial}{\partial a},
\]

(3.4)

\[
J\xi = \xi, \quad J|_\xi = \tilde{J}.
\]

(3.5)

Since \( g_{\tilde{J}} \) is positive, and \( d\lambda \) is a 2-form on \( Z \) (the curvature form), by choosing \( \ell \) small enough we may assume that \( \tilde{J} \) is tamed by \( \tau_0 + yd\lambda \) for \( |y| < \ell \), and there is a constant \( C > 0 \) such that

\[
\tau_0(v, \tilde{J}v) \leq C \left( \tau_0(v, \tilde{J}v) + yd\lambda(v, \tilde{J}v) \right)
\]

(3.6)

for all \( v \in TZ, |y| \leq \ell \). Then \( J \) is \( \omega_\phi \)-tamed for any \( \phi \in \Phi \) over the tube.

We denote by \( N \) one of \( M^+, M^- \) and \( R \times \tilde{M} \). We may choose an almost complex structure \( J \) on \( N \) such that

i) \( J \) is tamed by \( \omega_\phi \) in the usual sense,

ii) Over the tube \( R \times \tilde{M} \), (3.4) and (3.5) hold.

Then for any \( \phi \in \Phi \)

\[
\langle v, w \rangle_{\omega_\phi} = \frac{1}{2} (\omega_\phi(v, Jw) + \omega_\phi(w, Jv)) \quad \forall \ v, w \in TN
\]

(3.7)

defines a Riemannian metric on \( N \). Note that \( \langle , \rangle_{\omega_\phi} \) is not complete. We choose another metric \( \langle , \rangle \) on \( N \) such that

\[
\langle , \rangle = \langle , \rangle_{\omega_\phi} \quad \text{on} \quad M_0^+
\]

and over the tubes

\[
\langle (a, v), (b, w) \rangle = ab + \lambda(v)\lambda(w) + g_{\tilde{J}}(\Pi v, \Pi w),
\]

(3.8)

where we denote by \( \Pi : T\tilde{M} \to \xi \) the projection along \( X_H \). It is easy to see that \( \langle , \rangle \) is a complete metric on \( N \).

Let \( (\Sigma, i) \) be a compact Riemann surface and \( P \subset \Sigma \) be a finite collection of points. Denote \( \hat{\Sigma} = \Sigma \setminus P \). Let \( u : \hat{\Sigma} \to N \) be a \( J \)-holomorphic map, i.e., \( u \) satisfies

\[
du \circ i = J \circ du.
\]

(3.9)

Following [HWZ1] we impose an energy condition on \( u \). For any \( J \)-holomorphic map \( u : \hat{\Sigma} \to N \) and any \( \phi \in \Phi \) the energy \( E_\phi(u) \) is defined by

\[
E_\phi(u) = \int_{\hat{\Sigma}} u^* \omega_\phi.
\]

(3.10)

A \( J \)-holomorphic map \( u : \hat{\Sigma} \to N \) is called a finite energy \( J \)-holomorphic map if

\[
\sup_{\phi \in \Phi} \left\{ \int_{\hat{\Sigma}} u^* \omega_\phi \right\} < \infty.
\]

(3.11)
We shall see later that the condition is natural in view of our surgery. For a $J$-holomorphic map $u : \hat{\Sigma} \to \mathbb{R} \times \tilde{M}$ we write $u = (a, \bar{u})$ and define

$$(3.12) \quad \tilde{E}(u) = \int_{\Sigma} \bar{u}^* (\pi^* \tau_0).$$

Let $z = e^{s+2\pi it}$. One computes over the cylindrical part

$$(3.13) \quad u^* \omega_\phi = ((\tau_0 + \phi d\lambda)((\pi \bar{u})_s, (\pi \bar{u})_t)) + \phi'(a_s^2 + a_t^2)ds \wedge dt,$$

which is a nonnegative integrand.

In this section we shall prove a compactness theorem for $J$-holomorphic curves in $M^\pm$, and a convergence theorem for $J$-holomorphic curves when $M$ is stretched to infinity along $\tilde{M}$. There are various existing compactness theorems over compact manifolds. (see [RT1],[FO],[MS],[Ye]). In our situation the manifold is not compact, so we have to analyse the behaviour of sequences of holomorphic curves at infinity. In the case of contact manifolds, a similar analysis has been done by Hofer and his collaborators [HWZ1], [HWZ2], [HWZ3]. However, the case under consideration is not contact in general and their analysis does not directly apply. Here, we carry out the analysis in the case of a Hamiltonian $S^1$-action where we follows closely Hofer’s approach. We remark that the existence of an $S^1$-action simplifies many arguments so that we have a simpler proof than in the case of contact manifolds.

### 3.1 Convergence to periodic orbits

We fix an integer $r > 2$, let $W^2_r(S^1, \tilde{M})$ be the space of $W^2_r$- maps from $S^1$ into $\tilde{M}$, which is a Hilbert manifold. For any $\gamma \in W^2_r(S^1, \tilde{M})$ the tangent space $T_\gamma W^2_r(S^1, \tilde{M})$ is the space of vector fields $\eta \in W^2_r(\gamma^* T \tilde{M})$ along $\gamma$. The Riemannian metric $(, )$ on $W^2_r(S^1, \tilde{M})$ is defined by

$$(3.14) \quad (\eta_1, \eta_2) := \int_{S^1} \langle \eta_1, \eta_2 \rangle dt.$$

Let $x(t) \in Z$ be an orbit of the circle action. For any integer $k$ let $x_k(t) := x(kt)$, which is called a $k$-periodic orbit. We can consider $x_k(t)$ as a element in $W^2_r(S^1, \tilde{M})$. Now we define an action functional in a neighbourhood of $x_k(t)$ in $W^2_r(S^1, \tilde{M})$. Choose an $\epsilon$-ball $O_{x_k, \epsilon}$ of $0$ in the Hilbert space $T_{x_k} W^2_r(S^1, \tilde{M})$ such that the exponential map

$$\exp_{x_k} : O_{x_k, \epsilon} \to W^2_r(S^1, \tilde{M})$$

identifies $O_{x_k, \epsilon}$ with a neighbourhood of $x_k(t)$ in $W^2_r(S^1, \tilde{M})$. Since $x_k : S^1 \to \tilde{M}$ is an immersion and the points in $O_{x_k, \epsilon}$ are near $x_k$ with respect to the $W^2$-norm, we can assume that for any $\gamma \in O_{x_k, \epsilon}$, $\gamma : S^1 \to \tilde{M}$ is an immersion. For any $W^2_r$-loop $\gamma$ in this neighbourhood let $\eta \in O_{x_k, \epsilon}$ be the corresponding vector field along $x_k(t)$. Denote by $W : S^1 \times [0,1] \to \tilde{M}$ the annulus defined by $\{\exp_{x_k(t)} s\eta | 0 \leq s \leq 1, t \in S^1\}$. We define an action functional by

$$(3.15) \quad \mathcal{A}(\gamma) = -\int_{S^1 \times [0,1]} W^* \pi^* \tau_0.$$

Suppose $\gamma_p$ is a smooth curve in $W^2_r(S^1, \tilde{M})$, with $\gamma_0 = \gamma$. Then $\eta = (d\gamma_p/dp)_{p=0}$ is a vector field along $\gamma$. One can easily derive the first variational formula:

$$(3.16) \quad d\mathcal{A}(\gamma) \eta = \left. \frac{d\mathcal{A}(\gamma_p)}{dp} \right|_{p=0} = -\int_{S^1} \pi^* \tau_0 (\gamma, \eta) dt = \int_{S^1} \langle \Pi \gamma, \bar{J} \Pi \eta \rangle dt.$$
It follows that \( \gamma \) is a critical point if and only if \( \dot{\gamma} \) is parallel to \( X_H \) everywhere. So every \( k \)-orbit is a critical point of \( A \).

Denote by \( S_k \) the set of \( k \)-periodic orbits. It is standard to prove that \( S_k \) is compact. Obviously, \( A(\gamma) \) is constant on the set \( S_k \). We now calculate the second variation for \( A \). Consider a 2-parameter variation \((t, w, v)\) of \( \gamma \), i.e., a map \( F : Q \to \bar{M} \) such that \( F(t, 0, 0) = \gamma(t) \), where 
\[
Q = S^1 \times (-\epsilon, \epsilon) \times (-\delta, \delta),
\]
and \( F \) is smooth with respect to \((w, v)\). Let \( \tau, \eta, \zeta \) be vector fields corresponding to the first, second and third variable of \( F \), respectively. Suppose that \( \gamma \) is a critical point of \( A \); then \( \tau(t, 0, 0) \) is parallel to \( X_H \). We have
\[
\frac{\partial A}{\partial w} = \int_{S^1} \langle \Pi \tau, \bar{J} \Pi \eta \rangle dt
\]
where \( \nabla \) denotes the covariant derivative with respect to the Levi-Civita connection of \((\ , \ )\) on \( \bar{M} \). Denote \( \zeta^\perp := \Pi \zeta \), the normal component of \( \zeta \), and \( T_{\gamma}^\perp := \{ \zeta | \langle \zeta, \dot{\gamma} \rangle = 0 \} \). Note that at \((t, 0, 0)\)
\[
\nabla \zeta (\Pi \tau) = \nabla_\tau \zeta = \nabla (\langle \tau, X_H \rangle) X_H
\]
\[
= \nabla_\tau \zeta - \langle \tau, X_H \rangle \nabla \zeta X_H \mod X_H
\]
\[
= \nabla_\tau \zeta^\perp - \langle \tau, X_H \rangle \nabla \zeta^\perp X_H \mod X_H.
\]
By our choice of the Riemannian metric and the almost complex structure
\[
\frac{D \zeta}{dt} \in T_{\gamma}^\perp \forall \zeta \in T_{\gamma}^\perp.
\]
Define the linear transformation \( S : T_{\gamma}^\perp \to T_{\gamma}^\perp \) by \( \zeta \to \bar{J} \nabla \zeta X_H \). We get the following variational formula:
\[
\frac{\partial^2 A}{\partial v \partial w} \bigg|_{(0,0)} = -\int_{S^1} \langle (\bar{J} \Pi \frac{D}{dt} + \langle \dot{\gamma}, X_H \rangle) S \rangle \zeta^\perp, \eta^\perp \rangle dt
\]
where \( D/dt \) denotes the covariant derivative along \( \gamma \). It is easy to show that the second variation formula is a symmetric bilinear form on the space of smooth vector fields along \( \gamma \), denoted by \( I(\zeta, \eta) \). Let \( P = -\bar{J} \Pi \frac{D}{dt} - \langle \dot{\gamma}, X_H \rangle S \). We have
\[
I(\zeta, \eta) = (P \zeta, \eta).
\]
Since the inclusion \( W^2_1(S^1) \to L^2(S^1) \) is compact, and \( P \) is a compact perturbation of the selfadjoint operator \( J_0 \frac{d}{dt} \), the spectrum \( \sigma(P) \) consists of isolated eigenvalues, which accumulate at \( \pm \infty \).

**Remark 3.1** Let \( \varphi : S^1 \to S^1 \) be a diffeomorphism. Then \( x_k \circ \varphi \) is also a critical point of \( A \), and the second variational formula (3.18) remains valid.

Now we consider a smooth map \( F : S^1 \times (-\epsilon, \epsilon) \to \bar{M} \) such that \( F(t, 0) = \gamma(t) \) and such that at every point \((t, w)\), \( \Pi \epsilon = 0 \), i.e., for every \( w \), \( F(\cdot, w) \) is a critical point of \( A \). It follows from (3.17) that \( \zeta \) satisfies the following equation:
\[
\nabla_\tau \zeta^\perp - \langle \tau, X_H \rangle \nabla \zeta^\perp X_H = 0.
\]
Put $T_{x_k}^\perp = \{ \eta \in T_{x_k}W^r_2(S^1, \tilde{M}) | \eta \perp \dot{x}_k \}$. $T_{x_k}^\perp$ is a Hilbert space. Let $C_r(A)$ be the set of critical points of $A$, and denote $K =: C_r(A) \cap T_{x_k}^\perp$.

**Proposition 3.2**

1) $K$ is a smooth submanifold in $T_{x_k}^\perp$.

2) At any $\gamma \in K$, the restriction of the index $I(\cdot, \cdot)$ to the normal direction of $K$ in $T_{x_k}^\perp$ is nondegenerate.

**Proof:** It is easy to show that $K$ is a smooth submanifold of dimension $2n$ in $T_{x_k}^\perp$ (for the definition of submanifold in a Hilbert space see [Kli]). We prove 2). Let $\gamma \in K$. It follows from (3.20) that any tangent vector $\zeta$ of $K$ at $\gamma$ satisfies

$$
(3.21) \quad \dot{J} \frac{D\zeta}{dt} + \langle \dot{\gamma}, X_H \rangle S\zeta = 0.
$$

On the other hand, any solution of the equation (3.21) is determined by its value $\zeta^\perp(0)$ at $t = 0$. Since $\zeta^\perp(0) \perp \dot{\gamma}$ we have $\dim \ker(P) \leq 2n$. Hence the tangent space of $K$ at $\gamma$ coincides with $\ker P$.

The conclusion follows. \(\square\)

**Remark 3.3** The set of critical points satisfying 1) and 2) in Proposition 3.2 is usually called of Bott-type.

**Proposition 3.4** Let $x_k(t) \in S_k$. There exists a neighborhood $O$ of $\{x_k(t + d), 0 \leq d \leq 1\}$, in $W^r_2(S^1, \tilde{M})$ and a constant $C > 0$ such that the inequality

$$
(3.22) \quad \| \nabla A \|_{L^2(S^1)} \geq C|A|^\frac{1}{2}
$$

holds in $O$.

**Proof:** We first prove that (3.22) holds in a neighborhood of 0 in $T_{x_k}^\perp$. Using the Morse lemma with parameters we can find a neighborhood $O_{x_{\tilde{t}}} \subset O_t$ of 0 such that the following holds: there are diffeomorphisms $\varphi_x : N_x \to N_x$, which depend continuously on $x \in K \cap O_{\tilde{t}}$, such that under the diffeomorphisms the function $A|_{N_x}$ has the form

$$
A(y) = (P'(x)y, y), \quad \forall y \in N_x
$$

where $P'$ denotes the restriction of $P(x)$ to $N_x$. It suffices to prove the inequality for the quadratic function $A(y) = (P'y, y)$. By the nondegeneracy of $I(\cdot, \cdot)$ we can find a constant $C_2 > 0$ such that $|\lambda_i| \geq C_2$ for all $x \in K \cap O_{\tilde{t}}$ and all eigenvalues $\lambda_i$. Then

$$
\| \nabla A \|_{L^2(S^1)} \geq \| \nabla_y A \|_{L^2(S^1)} = 2(P'(x)y, P'(x)y)^\frac{1}{2}
$$

$$
\geq \sqrt{C_2}(P'(x)y, y)^\frac{1}{2} = \sqrt{C_2}|A|^\frac{1}{2},
$$

where $\nabla_y A$ denotes the gradient of $A|_{N_x}$ as a functional on $N_x$. Now let $\eta \in W^r_2(x_k^*T\tilde{M})$ be a vector field along $x_k$ such that $|\eta||_r$ is very small. For any $t \in S^1$ we can find a unique $t' \in S^1$ and a vector $\eta' \in T_{x_k(t')}^\perp \tilde{M}$ such that

$$
\exp_{x_k(t)} \eta = \exp_{x_k(t')} \eta'.
$$

This induces a map $\varphi : S^1 \to S^1$, $t \mapsto t'$. When $|\eta||_r$ is very small $\varphi$ is a diffeomorphism with $\varphi' \approx 1$. Since $A$ is invariant under the diffeomorphism group $S^1 \to S^1$ we can find a neighborhood $O$ of $x_k(t + d)$, $0 \leq d \leq 1$, in $W^r_2(S^1, \tilde{M})$ such that (3.22) holds. The proposition follows. \(\square\)
We are interested in the behaviors of the finite energy \( J \)-holomorphic maps near a puncture. There are two different types of punctures: the removable singularities and the non-removable singularities. If \( u \) is bounded near a puncture, then this puncture is a removable singularity. In the following, we assume that all punctures in \( P \) are non-removable. Then \( u \) is unbounded near the punctures. By using the same method as in [II], one can prove the following two lemmas:

**Lemma 3.5** Let \( u = (a, \tilde{u}) : C \to \mathbf{R} \times \tilde{M} \) be a \( J \)-holomorphic map with finite energy. If \( \int_C \tilde{u}^*(\pi^* \tau_0) = 0 \), then \( u \) is a constant.

**Lemma 3.6** Let \( u = (a, \tilde{u}) : C \to \mathbf{R} \times \tilde{M} \) be a nonconstant \( J \)-holomorphic map with finite energy. Put \( z = e^{\delta + 2\pi i t} \). Then for any sequence \( s_i \to \infty \), there is a subsequence, still denoted by \( s_i \), such that

\[
\lim_{i \to \infty} \tilde{u}(s_i, t) = x(kt)
\]

in \( C^\infty(S^1) \) for some \( k \)-periodic orbit \( x_k \).

**Theorem 3.7** Let \( u = (a, \tilde{u}) : C \to \mathbf{R} \times \tilde{M} \) be a \( J \)-holomorphic map with finite energy. Put \( z = e^{\delta + 2\pi i t} \). Then

\[
\lim_{s \to \infty} \tilde{u}(s, t) = x(kt)
\]

in \( C^\infty(S^1) \) for some \( k \)-periodic orbit \( x_k \). Moreover, the convergence is to be understood as exponential decay uniformly in \( t \).

**Proof:** Denote

\[
\tilde{E}(s) = \int_s^\infty \int_{S^1} \tilde{u}^*(\pi^* \tau_0).
\]

Then

\[
\tilde{E}(s) = \int_s^\infty \int_{S^1} |\Pi \tilde{u}_t|^2 ds dt,
\]

\[
(3.23) \quad \frac{d\tilde{E}(s)}{d s} = - \int_{S^1} |\Pi \tilde{u}_t|^2 dt.
\]

We choose a sequence \( s_i \) such that

\[
\lim_{i \to \infty} \tilde{u}(s_i, t) = x(kt) =: x_k(t)
\]

for some \( k \)-periodic orbit \( x \). For every loop \( \tilde{u}(s, \cdot) \in O_{x_k, \epsilon} \) we draw an annulus \( W(s) \) as before and define the functional \( \mathcal{A}(s) := \mathcal{A}(\tilde{u}(s, \cdot)) \). Since the annulus \( W(s) \) varies continuous as \( s \) varies we have \( \tilde{E}(s) - \tilde{E}(s_i) = |\mathcal{A}(s) - \mathcal{A}(s_i)| \), where \( s < s_i \). Let \( i \to \infty \); then

\[
(3.24) \quad \tilde{E}(s) = |\mathcal{A}(s)|.
\]

By Proposition 3.4, there is a \( W^2 \)-neighbourhood \( O \) of \( \{ x_k(t+d), 0 \leq d \leq 1 \} \) in which the inequality (3.22) holds. For \( \tilde{u}(s, \cdot) \in O \) we have

\[
(3.25) \quad \frac{d\tilde{E}(s)}{d s} = - \int_{S^1} |\Pi \tilde{u}_t|^2 dt = - \| \nabla \mathcal{A}(s) \|^2_{L^2(S^1)} \leq - C^2 |\mathcal{A}(s)| \leq - C^2 \tilde{E}(s)
\]
\( \frac{d\tilde{E}(s)}{ds} \leq -C\|\Pi\tilde{u}\|_{L^2(S^1)} \tilde{E}(s)^{1/2}. \)

Suppose that \( \tilde{u}(s, \cdot) \in O \) for \( s_0 \leq s \leq s_1 \), and \( \tilde{E}(s) \neq 0 \) \( \forall s \in (s_0, s_1) \). It follows from (3.25) and (3.26) that

\[ \tilde{E}(s_1) \leq \tilde{E}(s_0)e^{-C^2(s_1-s_0)} \]

(3.27)

\[ \int_{s_0}^{s_1} \|\Pi\tilde{u}\|_{L^2(S^1)} ds \leq \frac{1}{C} \left( \tilde{E}(s_0)^{1/2} - \tilde{E}(s_1)^{1/2} \right) \]

(3.28)

Denote by \( \tilde{d} \) the distance function defined by the metric \( g_\tilde{\Pi} \) on \( Z \). We have

\[ \int_{S^1} \tilde{d}(\tilde{u}(s, t), \tilde{u}(s_i, t)) dt \leq \int_{S^1} \|\Pi\tilde{u}\|_{L^2} ds \]

\[ \leq \frac{1}{C} \left( \tilde{E}(s)^{1/2} - \tilde{E}(s_i)^{1/2} \right) \leq \frac{1}{C} \left( \tilde{E}(s) - \tilde{E}(s_i) \right)^{1/2}. \]

(3.29)

We show that for any \( C^\infty \)-neighbourhood \( U \) of \( \{x_k(t+d), 0 \leq d \leq 1\} \) there is a \( N > 0 \) such that if \( s > N \) then \( \tilde{u}(s, \cdot) \in U \). If not, we could find a neighbourhood \( U \subseteq O \) and a subsequence of \( s_i \) (still denoted by \( s_i \)) and a sequence \( s_i' \) such that

\[ \tilde{u}(s, \cdot) \in O, \text{ for } s_i \leq s \leq s_i', \]

(3.30)

\[ \tilde{u}(s_i', \cdot) \notin U. \]

(3.31)

By Lemma (3.6) and by choosing a subsequence we may assume that

\[ \tilde{u}(s_i', t) \to x'(k't) \text{ in } C^\infty(S^1, \tilde{M}) \]

for some \( k' \)-periodic solution \( x'(t) \in O \). We may assume that \( O \) is so small that there is no \( k' \)-periodic solution in \( O \) with \( k' \neq k \). Then, we have \( k' = k, x' \in S_k \). We assume that \( \tilde{E}(s) \neq 0 \). From (3.29) we have

\[ \int_{S^1} \tilde{d}(\tilde{u}(s_i', t), \tilde{u}(s_i, t)) dt \leq \frac{1}{C} \left( \tilde{E}(s_i') \right)^{1/2}, \]

where \( \tilde{d} \) denotes the distance function defined by the metric \( g_\tilde{\Pi} \) on \( Z \). Taking the limit \( i \to \infty \), we get

\[ \int_{S^1} \tilde{d}(x'(kt), x(kt)) dt = 0. \]

It follows that

\[ x'(kt) = x(kt + \theta_0) \]

for some constant \( \theta_0 \). This contradicts (3.31). If there is some \( s_0 \) such that \( \tilde{E}(s_0) = 0 \), then \( \|\Pi\tilde{u}\|^2 = \|\Pi\tilde{u}\|^2 = 0 \) \( \forall s \geq s_0 \). We still have a contradiction.

We may choose a local coordinate system \( y = (y_1, ..., y_{2n}) \) on \( Z \) and a local trivialization of \( \tilde{M} \to Z \) around \( x \) such that \( x_k = \{0 \leq \theta \leq 1, y = 0\} \), and

\[ \lambda = d\theta + \sum b_i(y) dy_i, \]

(3.32)
where \( b_i(0) = 0 \). Obviously, \( \xi(\theta, 0) \) is spanned by \( \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_{2n}} \). For \( y \) small enough we may choose a frame \( e_1, ..., e_{2n} \) for \( \xi(\theta, y) \) as follows: in terms of the coordinates \( (\theta, y_1, ..., y_{2n}) \)

\[ e_i = (a_i(\theta, y), 0, ..., 0), \quad i = 1, ..., 2n. \]

We write \( u(s, t) = (a(s, t), \theta(s, t), y(s, t)) \).

Denote by \( L \) the matrix of the almost complex structure \( \tilde{J} \) on \( \xi \) with respect to the frame \( e_1, ..., e_{2n} \), and set \( \tilde{J}(s, t) = L(u(s, t)) \). We can write the equation (3.9) as follows:

\[
\begin{align*}
(3.33) & \quad a_s = \lambda(u_t) = \theta_t + \sum b_i(y) y_t \\
(3.34) & \quad a_t = -\lambda(u_s) = -\theta_s + \sum b_i(y) y_s \\
(3.35) & \quad y_s + \tilde{J}(s, t) y_t = 0.
\end{align*}
\]

By the discussion above, we have \( y(s, t) \to 0 \) in \( C^\infty(S^1) \) as \( s \to \infty \). Hence we have \( |y(s, t)| < 1, |y_s(s, t)| < 1, |y_t(s, t)| < 1 \) for \( s \) large enough. Then

\[
\|y(s, t)\|_{L^2(S^1)}^2 \leq \int_{S^1} |y(s, t)| dt \leq \int_{S^1} \int_s^\infty |y_s(s, t)| ds dt \\
\leq C_1 \int_s^\infty \|\Pi \tilde{u}_t\|_{L^2(S^1)} ds
\]

for some constant \( C_1 > 0 \). Together with (3.29) we have

\[
(3.36) \quad \|y(s, t)\|_{L^2(S^1)} \leq \frac{1}{C} \tilde{E}(s)^{1/4},
\]

\[
(3.37) \quad \|\sum b_i(y) y_s\|_{L^2(S^1)} \leq \frac{1}{C} \tilde{E}(s)^{1/4},
\]

\[
(3.38) \quad \|\sum b_i(y) y_t\|_{L^2(S^1)} \leq \frac{1}{C} \tilde{E}(s)^{1/4}.
\]

for some constant \( C > 0 \) independent of \( u(s, t) \). By using the same argument as in [HWZ1] one can prove that there are constants \( \delta > 0, \ell_0 \) and \( \theta_0 \) such that for all \( r = (r_1, r_2) \in Z^2 \)

\[
(3.39) \quad |\partial^r [a(s, t) - ks - \ell_0]| \leq C_r e^{-\delta|r|}
\]

\[
(3.40) \quad |\partial^r [\theta(s, t) - kt - \theta_0]| \leq C_r e^{-\delta|r|}
\]

\[
(3.41) \quad |\partial^r w(s, t)| \leq C_r e^{-\delta|r|}
\]

where \( C_r \) are constants. \( \square \)

We remark that since \( S_k \) is compact, we may choose \( \delta, C_r \) to be independent of \( x_k \in S_k \)

Let \( u : \Sigma - \{p\} \to N \) be a \( J \)-holomorphic map with finite energy, and \( p \) be a nonremovable singularity. If, in terms of local coordinates \( (s, t) \) around \( p \), \( \lim_{s \to \infty} \tilde{u}(s, t) = x(kt) \), we say simply that \( u(s, t) \) converges to the \( k \)-periodic orbit \( x_k \). We call \( p \) a positive (resp. negative ) end, if \( a(z) \to \infty \) (resp. \( -\infty \)) as \( z \to p \).
3.2 Some technical lemmas

Recall that if we collapse the $S^1$-action on $\tilde{M} = H^{-1}(0)$ we obtain symplectic cuts $\tilde{M}^+$ and $\tilde{M}^-$. The reduced space $Z$ is a codimension 2 symplectic submanifold of both $\tilde{M}^+$ and $\tilde{M}^-$ (see [L]). There is another way to look at this. The length of every orbit of the $S^1$ action on $\tilde{M}$ with respect to the metric $\langle \cdot, \cdot \rangle_{\omega_\phi}$ is $\phi'$, which converges to zero as $a \to \pm \infty$. Hence we can view $\tilde{M}^\pm$ as the completions of $M^\pm$. We will mainly discuss $M^+$, the discussion for $M^-$ is identical. Since $\tilde{M}^+$ is a closed symplectic manifold, and we fix the values of $\phi$ on $[0, 1]$, the following lemma is well-known.

Lemma 3.8 There exist constants $\varepsilon_0 > 0$ and $C > 0$ independent of $\phi$ such that the following holds, for each $\varepsilon < \varepsilon_0$ and each metric ball $D_p(\varepsilon)$ centered at $p \in M^+_0$ and of radius $\varepsilon$. Let $u' : \Sigma' \to M^+$ be a $J$-holomorphic map. Suppose $u'(\Sigma') \subseteq D_p(\varepsilon), u'(\partial\Sigma') \subset \partial D_p(\varepsilon), \partial\Sigma' \neq \emptyset$ and $p \in u'(\Sigma')$. Then

\begin{equation}
\int_{\Sigma'} u'^* \omega_\phi > C\varepsilon^2.
\end{equation}

Lemma 3.9 There is a constant $h_0 > 0$ such that for every finite energy $J$-holomorphic map $u = (a, \bar{u}) : \tilde{\Sigma} \to \mathbb{R} \times \tilde{M}$ with $\tilde{E}(u) \neq 0$ we have $\tilde{E}(u) \geq h_0$.

Proof: Consider the $\bar{J}$-holomorphic map

$$\bar{v} = \pi \circ \bar{u} : \Sigma \to Z.$$  

$\bar{v}$ extends to a $\bar{J}$-holomorphic curve from $\Sigma$ to $Z$. Then the assertion follows from a standard result for compact symplectic manifolds. \hfill \Box

Let $(\Sigma; y_1, ..., y_t, p_1, ..., p_\nu) \in M_{g, l+\nu}$, and $u : \tilde{\Sigma} \to M^+$ be a J-holomorphic map. Suppose that $u(z)$ converges to a $k_i$-periodic orbit $x_{k_i}$ as $z$ tends to $p_i$. By using the removable singularities theorem we get a $J$-holomorphic map $\bar{u}$ from $\Sigma$ into $\tilde{M}^+$. Let $A = [\bar{u}(\Sigma)]$. It is obvious that

\begin{equation}
E_{\phi}(u) = \omega_\phi(A)
\end{equation}

which is independent of $\phi$. We fix a homology class $A \in H_2(\tilde{M}^+, \mathbb{Z})$ and a fixed set \{ $k_1, ..., k_\nu$ \}. We denote by $\mathcal{M}_A(M^+, g, l, k)$ the moduli space of all $J$-holomorphic curves representing the homology class $A$ and converging to $k$-periodic orbits. There is a natural map

$$P : \mathcal{M}_A(M^\pm, g, l, k) \to \prod_{i=1}^\nu S_{k_i}$$

given by

\begin{equation}
P(u, \Sigma; y, \{p_1, ..., p_\nu\}) = \{x_{k_1}, ..., x_{k_\nu}\}.
\end{equation}

We will identify $S_k$ with $Z$. For $J$-holomorphic maps in $\mathcal{M}_A(M^+, g, l, k)$, there is a uniform bound independent of $\phi$ on the energy $E_{\phi}(u)$. Similarly, for a map $u$ from $\Sigma$ into $\mathbb{R} \times \tilde{M}$, we define $\mathcal{M}_A(\mathbb{R} \times \tilde{M}, g, l, k^+, k^-)$ and $P^\pm$ in the obvious way, where $\pm$ denote positive or negative ends. For any $J$-holomorphic curves in $\mathcal{M}_A(\mathbb{R} \times \tilde{M}, g, l, k^+, k^-)$, there is a uniform bound on $\tilde{E}$.

Following McDuff and Salamon [MS] we introduce the notion of singular points for a sequence $u_i$ and the notion of mass of singular points. Suppose that $(\Sigma_i; y_i, p_i)$ are stable curves and converge to $(\Sigma; y, p)$ in $\mathcal{M}_{g, l+\nu}$. A point $q \in \Sigma - \{\text{double points}\}$ is called regular for $u_i$ if there exist $q_i \in \Sigma_i$, $q_i \to q$, and $\epsilon > 0$ such that the sequence $|du_i|_{h_i}$ is uniformly bounded on $D_{q_i}(\epsilon, h_i)$, where
\[|du_i|_{h_i}\] denotes the norm with respect to the metric \(\langle,\rangle\) on \(N\) and the metric \(h_i\) on \(\Sigma_i\). A point \(q \in \Sigma - \{\text{double points}\}\) is called singular for \(u_i\) if it is not regular. A singular point \(q\) for \(u_i\) is called rigid if it is singular for every subsequence of \(u_i\). It is called tame if it is isolated and the limit
\[
m_\epsilon(q) = \lim_{i \to \infty} E_{\phi}(u_i; D_{q_i}(\epsilon, h_i))
\]
exists for every sufficiently small \(\epsilon > 0\). The mass of the singular point \(q\) is defined to be
\[
m(q) = \lim_{\epsilon \to 0} m_\epsilon(q).
\]

**Lemma 3.10** There is a constant \(h > 0\) such that every rigid singular point \(y\) for \(u_i\) has mass
\[
m(y) \geq h.
\]

**Proof:** By using the standard rescaling argument (see [MS]) we may construct for every rigid singular point a nonconstant \(J\)-holomorphic map \(v : \mathbb{C} \to M^+\) (or \(\mathbb{R} \times \tilde{M}\)). In case \(u(\mathbb{C}) \cap M^+_2 \neq \emptyset\) we use Lemma 3.9. Now we assume that \(u(\mathbb{C})\) lies in the cylindrical part. By Lemma 3.5 \(E(v) \neq 0\). We have \(E(v) \geq h_0\) by lemma 3.9. Then Lemma 3.10 follows by (3.6) and (3.13). \(\square\)

Denote by \(P \subset \Sigma\) the set of singular points for \(u_i\), the double points and the puncture points. By Lemma 3.10 and (3.43), \(P\) is a finite set. By definition, \(|du_i|_{h_i}\) is uniformly bounded on every compact subset of \(\Sigma - P\). We assume that
\[
u_i(\Sigma_i) \cap M^+_l = \emptyset
\]
for some positive number \(d\) independent of \(i\), where \(M^+_d = M^+_0 \cup \{[0, d] \times \tilde{M}\}\). If there is no \(d\) satisfying (48), we will make a translation along \(\mathbb{R}\); then the discussion is similar. It follows from (3.48) and the uniformly boundedness of \(|du_i|_{h_i}\) on compact sets that \(u_i\) maps every compact subset of \(\Sigma - P\) uniformly into a bounded subset of \(M^+\) (recall the definition (3.8) of the metric \(\langle,\rangle\)). By passing to a further subsequence we may assume that \(u_i\) converges uniformly with all derivatives on every compact subset of \(\Sigma - P\) to a \(J\)-holomorphic map \(u : \Sigma - P \to M^+\). Obviously, \(u\) is a finite energy \(J\)-holomorphic map.

We need to study the behaviour of the sequence \(u_i\) near each singular point for \(u\), and near each double point. We consider singular points for \(u_i\). The same method with only minor changes applies to double points. Let \(q \in \Sigma\) be a rigid singular point for \(u_i\). Then \(q\) is a puncture point for \(u\). If \(q\) is a removable singularity of \(u_i\), i.e., there are \(\epsilon > 0, d > 0\) such that \(u(D_q(\epsilon)) \subset M^+_d\), we consider the compact manifold \(\overline{M}^+\), and construct bubbles as usual for a compact symplectic manifold (see [RT1], [PW], [MS]). In the following, we assume that \(q\) is a nonremovable singularity. We may identify each neighborhood of \(q_i\) in \(\Sigma_i\) with a neighborhood of 0 in \(\mathbb{C}\), say \(D(1)\). The sequence \(u_i\) is considered to be a sequence of \(J\)-holomorphic maps from \(D(1)\) into \(M^+\), and \(q_i \in D(1), q_i \to 0\). Without loss of generality, we may assume that 0 is the unique singular point and is tame (see [MS]). For \(|z|\) sufficiently small, \(u(z)\) lies in the cylindrical part. It is convenient to use cylindrical coordinates \(z = e^{s+2\pi it}\). We write
\[
u_i(s, t) = (a_i(s, t), \bar{u}_i(s, t)) = (a_i(s, t), \theta_i(s, t), x_i(s, t))
\]
\[u(s, t) = (a(s, t), \bar{u}(s, t)) = (a(s, t), \theta(s, t), x(s, t)).
\]
Let \(S^2\) denote the standard sphere in \(\mathbb{R}^3\) with two distinguished points \(0 = (0, 0, 0)\) and its antipodal north pole \((0, 0, 1)\). For any \(\epsilon > 0\), through a conformal transformation of \(S^2\) we may assume that
$u_i$ are defined in the disk $D(\epsilon) = \{z ||z| < \epsilon\}$ such that the center of mass of the measure $|du_i|^2$ is on the $z$-axis. By Theorem 3.7. we have

$$\lim_{s \to -\infty} \tilde{u}(s,t) = x(kt)$$

in $C^\infty(S^1)$, where $x(\ ,\ )$ is a $k$-periodic orbit on $\tilde{M}$. Choosing $\epsilon$ small enough we have $|\partial_\theta \delta - k| < 1/10$ for all $(s,t) \in (-\infty, \log \epsilon) \times S^1$. By choosing $i$ large enough we may assume that $|\partial_\theta (\log \epsilon, t) - k| < 1/5$, $|\partial_s (\log \epsilon, t) - k| < 1/5$ for all $i$ and all $t \in S^1$. For every $i$ there exists $\delta_i > 0$ such that

$$E_\phi(u_i; D(\delta_i)) = m_0 - \frac{1}{2}h_0.$$ 

By definition of the mass $m_0$, the sequence $\delta_i$ converges to 0. We discuss two cases:

**Case 1:** There is an $L > 0$ such that for all $i > L$

$$(3.49) \quad |\partial_{\delta} \tilde{u}(s,t) - k| \leq 1/2, \quad |\partial_s \tilde{u}(s,t) - k| \leq 1/2$$

hold for all $\log \delta_i \leq s \leq \log \epsilon$, $t \in S^1$; we set $\delta_i = \delta_i'$. 

**Case 2:** Otherwise, we could find a subsequence, still denoted by $u_i$, and $\delta_i > \delta_i'$ such that for all $i$ (3.49) holds for all $\log \delta_i \leq s \leq \log \epsilon$, $t \in S^1$, and there is $t_0 \in S^1$ such that

$$(3.50) \quad |\partial_{\delta} \tilde{u}(\delta_i, t_0) - k| = 1/2, \quad \text{or} \quad |\partial_s \tilde{u}(\delta_i, t_0) - k| = 1/2.$$ 

Since $u_i$ converges uniformly with all derivatives to $u$ on any compact set of $D(\epsilon) - \{0\}$, $\delta_i$ must converge to 0. Put $z = \delta_i w = \delta_i e^{r+2\pi it}$, and define the $J$-holomorphic curve $v_i(r,t)$ by

$$v_i(r,t) = (b_i(r,t), \bar{v}_i(r,t)) = (a_i(\log \delta_i + r, t) - a_i(\log \delta_i, t_0), \bar{u}_i(\log \delta_i + r, t)).$$

where $t_0 \in S^1$ is a fixed point.

**Lemma 3.11** Suppose that $\theta$ is a nonremovable singular point of $u$. Define the $J$-holomorphic map $v_i$ as above. Then there exists a subsequence (still denoted by $v_i$) such that

1. The set of singular points $\{w_1, \cdots, w_N\}$ for $v_i$ is finite and tame, and is contained in the disk $D(1) = \{w ||w| \leq 1\}$;

2. The subsequence $v_i$ converges with all derivatives uniformly on every compact subset of $C\setminus\{w_1, \cdots, w_N\}$ to a nonconstant $(J,i)$-holomorphic map $v : C\setminus\{w_1, \cdots, w_N\} \to \mathbb{R} \times \tilde{M};$

3. $$\lim_{s \to -\infty} \tilde{u}(s,t) = \lim_{r \to \infty} \tilde{v}(r,t);$$

4. $$\bar{E}(u) + \sum_{i=1}^{N} m(w_i) = m_0.$$ 

**Proof:** (1) By definition of $m_0$ and $\delta_i$ we have

$$\lim_{i \to \infty} \sup_{R \to \infty} \bar{E}(v_i; D(R) - D(r)) \leq \frac{h_0}{2}$$

for any $R > r > 1$. We claim that $|dv_i|$ is uniformly bounded on $C - D(r)$. Otherwise, by a standard rescaling argument (see [MS]) we could construct a nonconstant $J$-holomorphic curve
\( v : \mathbb{C} \to \mathbb{R} \times \tilde{M} \) with \( \bar{E}(v; \mathbb{C}) \leq \frac{h_0}{2} \). It follows from Lemma 3.9 that \( \bar{E}(v; \mathbb{C}) = 0 \). Then \( v \) must be constant by Lemma (3.5), a contradiction.

(2) The proof is standard (see [MS]).

(3) Consider the \( \tilde{J} \)-holomorphic map
\[
f_{i} = \pi \tilde{u}_{i} : \Sigma \to Z.
\]
Write \( A(r, R) = D(R) - D(r) \). Since \( E(f_{i}; A(R\delta_{i}, \epsilon)) \leq \frac{1}{2} h_0 \), by a standard result about the energy of a pseudo-holomorphic curve on an arbitrarily long cylinder (see [MS]), there exists a \( T_0 > 0 \) such that for \( T > T_0 \)

\[
E(f_{i}; A(R\delta_{i}e^{T}, ee^{-T})) \leq \frac{C}{T} E(f_{i}; A(R\delta_{i}, \epsilon))
\]

and

\[
\int_{S^1} d(f_{i}(ee^{-T+it}), f_{i}(R\delta_{i}e^{T+it}))dt \leq \frac{C}{\sqrt{T}} \sqrt{E(f_{i}; A(R\delta_{i}, \epsilon))}.
\]

We choose \( T > C \). It follows from (3.52) that

\[
\tilde{E}(u_{i}; A(R\delta_{i}, \epsilon)) \leq \frac{1}{1 - \frac{1}{T}} \left( \tilde{E}(u_{i}; A(\epsilon e^{-T}, \epsilon)) + \tilde{E}(v_{i}; A(R, Re^{T})) \right).
\]

Since \( u_{i} \to u \) and \( v_{i} \to v \) uniformly on compact sets, the above inequality implies that

\[
\lim_{\epsilon \to 0, R \to \infty} \lim_{i \to \infty} \tilde{E}(u_{i}; A(\epsilon, R\delta_{i})) = 0.
\]

It is easy to see that \( v \) is a finite energy \( J \)-holomorphic curve. Suppose that \( v \) converges to a \( k' \)-periodic orbit \( x' \). From (3.53) and (3.55), we obtain

\[
\int_{S^1} \tilde{d}(x(kt), x'(kt))dt = 0.
\]

From (3.49), it follows that \( |k - k'| \leq 1/2 \), and therefore \( k = k' \).

(4) For \( \epsilon > 0 \) sufficiently small and \( R \) large enough,

\[
E_{\phi}(u_{i}, D(\epsilon)) = E_{\phi}(u_{i}, D(\epsilon) - D(R\delta_{i})) + E_{\phi}(v_{i}, D(R)).
\]

Obviously,

\[
\lim_{R \to \infty} \lim_{i \to \infty} E_{\phi}(v_{i}, D(R)) = \bar{E}(v) + \sum_{i=1}^{N} m(w_{i}).
\]

By (3.49),

\[
\lim_{\epsilon \to 0} \lim_{i \to \infty} \min \{ a_{i}(s, t) \} \to \infty.
\]

Together with (3.55), we have

\[
\lim_{\epsilon \to 0, R \to \infty} \lim_{i \to \infty} E_{\phi}(u_{i}, D(\epsilon) - D(R\delta_{i})) = 0.
\]
Then (4) follows. □

It is possible for \( v \) to have \( \tilde{E}(v) = 0 \). As in the case of compact manifolds (see [PW]) we call \( v \) a
ghost bubble.

**Lemma 3.12** If \( \tilde{E}(v) = 0 \) then \( N \geq 2 \).

**Proof:** For Case 1, we use the same argument as in [PW]. We now prove our lemma for Case 2.
If \( 0 \) is the only singularity, then \( v \) is a \( J \)-holomorphic map from \( \mathbb{R} \times S^1 \) to \( \mathbb{R} \times \tilde{M} \) with \( \tilde{E}(v) = 0 \).
It must be \( (kr + d, kt + \theta_0) \) by Lemma 3.5. But from (3.50), it follows that
\[
|\frac{\partial \theta}{\partial t}(0, t_0) - k| = 1/2, \quad \text{or} \quad |\frac{\partial b}{\partial r}(0, t_0) - k| = 1/2.
\]
We get a contradiction. Hence there is a singular point \( z \neq 0 \). The lemma follows from symmetry. □

**Remark 3.13** We need also study sequences of \( J \)-holomorphic curves in \( \mathbb{R} \times \tilde{M} \). The method is
the same as we did for \( M^+ \). In particular, if \( q \) is a removable singular point of \( u \), we collapse
the \( S^1 \) action on \( \tilde{M} \) at \( \infty \) and \( -\infty \) to obtain a compact manifold and construct a bubble as usual
procedure for compact manifold (see [RT1],[MS],[PW]). If \( q \) is a nonremovable singular point of \( u \),
we construct bubble as above.

For any \( r > 0 \), we construct \( M_r \) as follows: The parts \( |a^\pm| \geq 3r \) are cut out from \( M^\pm \), and the
remainders are glued along a collar of length \( 2r \) by the formula
\[(3.56) \quad a^- = a^+ - 4r \]
Let \( r \to \infty \). We may consider the limit \( M_{\infty} \) of \( M_r \) to be a compact manifold obtained by gluing
\( M^+ \) and \( M^- \) along \( \tilde{M} \) at \( +\infty \) and \( -\infty \). Let \( u_r : \Sigma_r \to M_r \) be a \( J \)-holomorphic curve. Suppose
that \( \Sigma_r \to \Sigma = \Sigma^+ \cup \Sigma^- \). Both \( \Sigma^+ \) and \( \Sigma^- \) may contain several
connected nodal Riemann surfaces, and \( \Sigma^+ \) and \( \Sigma^- \) may have several intersect points. For simplicity, we assume that both \( \Sigma^+ \) and \( \Sigma^- \) have one component and \( \Sigma^+ \) and \( \Sigma^- \) intersect at one point \( p \). For the general case the treatment
is identical. We may consider \( \Sigma_r \) to be obtained from \( \Sigma \) by resolving the singularity using the
parameter \( \delta_r, \delta_r \to 0 \), i.e., \( \Sigma_r = \Sigma_1 \# \sqrt{\Sigma_2} \) with gluing formulas \( t = t', s' + \log \delta_r = s \). The same
method as we used before, with only minor changes, applies.

### 3.3 Compactness theorems

We introduce a terminology. Let \( \Sigma_1 \) and \( \Sigma_2 \) join at \( p \), and \( (u_1, u_2) : \Sigma_1 \cup \Sigma_2 \to \mathbb{R} \times \tilde{M} \) a map.
Choose holomorphic cylindrical coordinates \( z_1 = (s_1, t_1) \) on \( \Sigma_1 \) and \( z_2 = (s_2, t_2) \) on \( \Sigma_2 \) near \( p \)
respectively. Suppose that
\[
\lim_{s_1 \to -\infty} \tilde{u}_1(s_1, t_1) = x_1(k_1 t_1) \quad \text{and} \quad \lim_{s_2 \to +\infty} \tilde{u}_2(s_2, t_2) = x_2(k_2 t_2).
\]
We say \( u_1 \) and \( u_2 \) converge to a same periodic orbit, if \( k_1 = k_2 \), and \( P(x_1) = P(x_2) \), where \( P \)
denotes the projection to \( Z \).

**Definition 3.14** Let \((\tilde{\Sigma}; \mathbf{y}, \mathbf{p})\) be a Riemann surface of genus \( g \) with \( l \) marked points \( \mathbf{y} \) and \( \nu \) ends \( \mathbf{p} \).
A log stable holomorphic map with \( \{k_1, \ldots, k_\nu\} \) -ends from \((\tilde{\Sigma}; \mathbf{y}, \mathbf{p})\) into \( M^+ \) is an equivalence class
of continuous maps \( u \) from \( \tilde{\Sigma} \) into \( (M^+)' \), modulo the automorphism group \( \text{stab}_u \), the translations,
and \( S^1 \)-action on \( \mathbb{R} \times \tilde{M} \), where \( \tilde{\Sigma}' \) is obtained by joining chains of \( \mathbb{P}^1 \)’s at some double points of
\[ \Sigma \text{ to separate the two components, and then attaching some trees of } \mathbb{P}^1 \text{s}; (M^+)′ \text{ is obtained by attaching some } \mathbb{R} \times \widetilde{M} \text{ to } M^+. \text{ We call components of } \overset{\circ}{\Sigma} \text{ principal components and others bubble components. Furthermore,} \\

(1) \text{ If we attach a tree of } \mathbb{P}^1 \text{ at a marked point } y_i \text{ or a puncture point } p_i, \text{ then } y_i \text{ or } p_i \text{ will be replaced by a point different from the intersection points on a component of the tree. Otherwise, the marked points or puncture points do not change;} \\

(2) \overset{\circ}{\Sigma}' \text{ is a connected curve with normal crossings;} \\

(3) \text{ Let } m_j \text{ be the number of points on } \Sigma_j \text{ which are nodal points or marked points or puncture points. If } u|_{\Sigma_j} \text{ is constant or } \pi u|_{\Sigma_j} \text{ is constant (for maps into } \mathbb{R} \times \tilde{M}), \text{ then } m_j + 2g_j \geq 3; \\

(4) \text{ The restriction of } u \text{ to each component is } J\text{-holomorphic.} \\

(5) \text{ }\text{ }u \text{ converges exponentially to some periodic orbits } (x_{k_1}, ..., x_{k_\nu}) \text{ as the variable tends to the puncture } (p_1, ..., p_\nu); \text{ more precisely, } u \text{ satisfies (3.39)-(3.41);} \\

(6) \text{ Let } q \text{ be a nodal point of } \Sigma'. \text{ Suppose } q \text{ is the intersection point of } \Sigma_i \text{ and } \Sigma_j. \text{ If } q \text{ is a removable singular point of } u, \text{ then } u \text{ is continuous at } q; \text{ If } q \text{ is a nonremovable singular point of } u, \text{ then } u|_{\Sigma_i} \text{ and } u|_{\Sigma_j} \text{ converge exponentially to the same periodic orbit on } \tilde{M} \text{ as the variables tend to the nodal point } q. \]

If we drop the condition (4), we simply call \( u \) a log stable map. Put \( m = l + \nu \) and let \( T_m \) be as in introduction. Let \( \overline{M}_A(M^+, g, T_m) \) be the space of equivalence classes of log stable holomorphic maps with ends, and \( \overline{B}_A(M^+, g, T_m) \) be the space of stable maps with ends. \( \overline{M}_A(M^+, g, T_m) \) has an obvious stratification indexed by the combinatorial type of the domain with the following data:

(1) the topological type of the domain as an abstract Riemann surface with marked points and puncture points;

(2) sets of periodic orbits corresponding to each puncture point;

(3) a decomposition of the integral 2-dimensional class \( A = \sum A_i \).

Suppose that \( D_{g,T_m}^{J,A} \) is the set of indices.

**Lemma 3.15** \( D_{g,T_m}^{J,A} \) is a finite set.

**Proof:** By Lemma 3.10, 3.12 and a standard argument, one can show that there are finitely many combinatorial types of the domain as an abstract nodal surface and integral 2-dimensional classes \( A_i \). The class \( A \) determines the bound on \( T_m \). Since for a \( J \)-holomorphic curve in \( \mathcal{M}_{A_i}(\mathbb{R} \times \widetilde{M}, g_i, m_i, k^+, k^-) \) the difference

\[ \sum_j k^+_j - \sum_j k^-_j \]

is determined by the class \( A_i \), we conclude that the set of periods is finite. \( \Box \)

By using the above lemmas one immediately obtains

**Theorem 3.16** Let \( \Gamma = (u_i, \Sigma_i; y_i, p_i) \in \mathcal{M}_A(M^+, g, T_m) \) be a sequence. Then there is a subsequence which weakly converges to a log stable \( J \)-holomorphic curve in \( \overline{M}_A(M^+, g, T_m) \). Here,
by weak convergence, we mean the Gromov-Uhlenbeck convergence with possible translation and \(S^1\)-action on \(\mathbb{R} \times \mathcal{M}\).

**Corollary 3.17** \(\overline{\mathcal{M}}_A(M^+, g, T_m)\) is compact.

To define stable \(J\)-holomorphic maps in \(M_\infty\) we need to extend log stable maps into \(M^\pm\) to include non-connected components. Suppose that \(\Sigma^\pm\) has \(l^\pm\) connected components \(\Sigma_i^\pm, i = 1, ..., l^\pm\) of genus \(g_i^\pm\) with \(m_i^\pm\) marked points and \(\nu_i^\pm\) ends. We have

\[
\sum \nu_i^+ = \sum \nu_i^- = \nu, \quad \sum m_i^+ = m^+, \quad \sum m_i^- = m^-, \quad m^+ + m^- = m.
\]

Put

\[
\overline{\mathcal{M}}_A^\pm(M^\pm, g^\pm, T_m^\pm) = \bigoplus_{i=1}^{l^\pm} \overline{\mathcal{M}}_{A_i^\pm}^\pm(M^\pm, g_i^\pm, T_{m_i^\pm}^\pm),
\]

where \(A^\pm = \{A_1^\pm, ..., A_{l^\pm}^\pm\}, g^\pm = \{g_1^\pm, ..., g_{l^\pm}^\pm\}, m^\pm = \{m_1^\pm, ..., m_{l^\pm}^\pm\}\), \(k = \{k_1, ..., k_{l^\pm}\}\). The maps \(e_j^\pm\) are extended to \(\overline{\mathcal{M}}_A^\pm(M^\pm, g^\pm, T_m^\pm)\) in a natural way.

**Definition 3.18** A stable \(J\)-holomorphic map of genus \(g\) and class \(A\) into \(M_\infty\) is a triple \((\Gamma^-, \Gamma^+, \rho)\), where \(\Gamma^\pm \in \overline{\mathcal{M}}_A^\pm(M^\pm, g^\pm, T_m^\pm)\) and \(\rho : \{p_1^\pm, ..., p_{\nu^\pm}\} \to \{p_1^-, ..., p_{\nu^-}\}\) is a one-to-one map satisfying

1. If we identify \(p_i^+\) and \(\rho(p_i^+)^\pm\) then \(\Sigma^+ \cup \Sigma^-\) forms a connected closed Riemann surface of genus \(g\);
2. \(\tilde{u}^+(z)\) and \(\tilde{u}^-(w)\) converge to the same periodic orbit when \(z \to p_i^+\) and \(w \to \rho(p_i^+)\) respectively;
3. \((\Gamma^-, \Gamma^+, \rho)\) represents the homology class \(A\).

**Remark 3.19** If \(\{p_1^+, ..., p_{\nu^\pm}\}\) in the above definition is the empty set then \((\Sigma, u)\) is a log stable \(J\)-holomorphic map into \(M^+\) or \(M^-\).

Denote by \(\overline{\mathcal{M}}_A(M_\infty, g, m)\) the moduli space of stable \(J\)-holomorphic maps in \(M_\infty\). Using Lemmas in subsection 3.2 we immediately obtain the following convergence theorem:

**Theorem 3.20** \(\overline{\mathcal{M}}_A(M_\infty, g, m)\) is compact.

**Theorem 3.21** Let \(\Gamma_r \in \overline{\mathcal{M}}_A(M_r, g, m)\) be a sequence. Then there is a subsequence which weakly converges to a stable \(J\)-holomorphic map in \(\overline{\mathcal{M}}_A(M_\infty, g, m)\).

## 4 Log invariants

In this section we define the log invariants for a pair \((V, B)\), where \(V\) is a compact symplectic manifold, and \(B\) a codimension two symplectic submanifold of \(V\). Since the symplectic structure of a tubular neighborhood of \(B\) is modeled on a neighborhood of \(Z\) in \(\overline{\mathcal{M}}^+\). So we consider \((\overline{M}^+, Z)\). To define the log GW-invariants we use the virtual neighborhood technique developed in [R5]. We remark that other constructions [FO], [LT3], [S] can be applied to our case as well.
4.1 Stabilization equations

We will describe $\mathcal{B}_A(M^+, g, T_m)$, $\mathcal{F}_A(M^+, g, T_m)$, then we construct a finite dimensional $V$-bundle $\mathcal{E}$ over $\mathcal{B}_A(M^+, g, T_m)$, following Siebert’s construction, and define a stabilization equation $\mathcal{S}_L = 0$. Note that we need only to consider a neighborhood $\mathcal{U}$ of $\overline{M}_A(M^+, g, T_m)$ in the configuration space $\mathcal{B}_A(M^+, g, T_m)$ of $C^\infty$-log stable (holomorphic or not) maps. We first consider the case that $\Sigma$ is a smooth component. Denote by $N$ the one of $M^+$ and $\mathbb{R} \times \mathbb{M}$. We consider $\mathcal{B}_A(N, g, T_m)$. Let $b = (u, (\Sigma, j); p) \in \mathcal{B}_A(N, g, T_m)$. Here $j$ is a complex structure (including marked points), which is standard near each puncture point. We introduce the holomorphic cylindrical coordinates on $\Sigma$ near each puncture points $p_i$. Then we may consider $(\Sigma, j)$ as a Riemann surface with some cylindrical ends. Note that we will consider a nodal point of two Riemann surfaces as a puncture point of both Riemann surface and use the cylindrical model. So there are two types of puncture points: orbit type puncture points and non-orbit puncture points, which must be nodal points. By using the removable singularities theorem we may consider $u$ as a $J$-holomorphic map from $(\Sigma, j)$ into $\overline{M}^+$ or $\mathbb{R}$, where we denote by $\mathbb{R}$ the space obtained from $\mathbb{R} \times \mathbb{M}$ by collapsing the $S^1$-action on the $\pm \infty$ ends. Then for any non-orbit puncture point $p$, $u(p)$ lies in $\overline{M}^+ - Z$ (or $\mathbb{R} - Z$), while for any orbit type puncture point $p$, $u(p)$ lies in $Z$. For each puncture point $p_i$, we choose the holomorphic cylindrical coordinates $(s, t)$ on $\Sigma$ near each $p_i$. Over the tube the linearized operator

$$D_u = D\tilde{\partial}(u) : C^\infty(\Sigma; u^*TN) \to \Omega^{0,1}(u^*TN)$$

takes the following form

$$D_u = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S = \tilde{\partial} + S. \tag{4.1}$$

For each orbit type puncture point we choose a local frame field $e_1 = \frac{\partial}{\partial s}, e_2 = X_H$ and $e_3, \ldots, e_{2n+2} \in \xi$ near the periodic orbit $x_i(k, t)$. Denote by $\nabla$ the Levi-Civita connection with respect to the metric $(\cdot, \cdot)$. Then

$$S_{k\ell} = \left\langle e_k, \nabla_s e_\ell + J \nabla_t e_\ell + (\nabla e_\ell J) \frac{\partial u}{\partial t} \right\rangle. \tag{4.2}$$

Recall that for a $J$-holomorphic map $u$, $D_u$ is independent of the choice of connection. Note that $S_{1\ell} = S_{2\ell} = S_{12} = 0$. Since $\frac{\partial u}{\partial s} \to 0$, $\frac{\partial u}{\partial t} \to X_H$ exponentially and uniformly in $t$ as $s \to \pm \infty$, we have

$$|S| \leq Ce^{-\delta s} \tag{4.3}$$

for some constant $C > 0$ for $s$ big enough. Therefore, the operator $H_s = J_0 \frac{\partial}{\partial t} + S$ converges to $H_\infty = J_0 \frac{\partial}{\partial t}$. Obviously, the operator $D_u$ is not a Fredholm operator because over each orbit puncture end the operator $H_\infty = J_0 \frac{\partial}{\partial t}$ has zero eigenvalue. The kernel $H_\infty$ consists of constant vectors. This is also true for each non-orbit puncture end. To recover a Fredholm theory we use weighted function spaces. We choose a weight $\alpha$ for each end. Fix a positive function $W$ on $\Sigma$ which has order equal to $e^{\alpha|\cdot|}$ on each end, where $\alpha$ is a small constant such that $0 < \alpha < \delta$ and over each end $H_\infty - \alpha = J_0 \frac{\partial}{\partial t} - \alpha$ is invertible. We will write the weight function simply as $e^{\alpha|\cdot|}$. For any section $h \in C^\infty(\Sigma; u^*TN)$ and section $\eta \in \Omega^{0,1}(u^*TN)$ we define the norms

$$\|h\|_{1, p, \alpha} = \left( \int_\Sigma (|h|^p + |\nabla h|^p) d\mu \right)^{1/p} + \left( \int_\Sigma e^{2\alpha|\cdot|}(|h|^2 + |\nabla h|^2) d\mu \right)^{1/2} \tag{4.4}$$

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(4.5) \[ \| \eta \|_{p, \alpha} = \left( \int_{\Sigma} |\eta|^p d\mu \right)^{1/p} + \left( \int_{\Sigma} e^{2\alpha s^2} |\eta|^2 d\mu \right)^{1/2} \]

for \( p \geq 2 \), where all norms and covariant derivatives are taken with respect to the metric \( \langle \ , \ \rangle \) on \( u^* T N \) defined in (3.8), (3.9), and the metric on \( \Sigma \). Denote

\[ \mathcal{C}(\Sigma; u^* T N) = \{ h \in C^\infty(\Sigma; u^* T N); \| h \|_{1, p, \alpha} < \infty \}, \]

\[ \mathcal{C}(u^* T N \otimes \wedge^{0,1}) = \{ \eta \in \Omega^{0,1}(u^* T N); \| \eta \|_{p, \alpha} < \infty \}. \]

Denote by \( W^{1,p,\alpha}(\Sigma; u^* T N) \) and \( L^{p,\alpha}(u^* T N \otimes \wedge^{0,1}) \) the completions of \( \mathcal{C}(\Sigma; u^* T N) \) and \( \mathcal{C}(u^* T N \otimes \wedge^{0,1}) \) with respect to the norms (4.4) and (4.5) respectively.

For each puncture point \( p_i \), \( i = 1, \ldots, \nu \), let \( h_{i0} \in \ker H_{i0} \). Put \( H_{\infty} = (H_{1\infty}, \ldots, H_{\nu\infty}) \), \( h_0 = (h_{10}, \ldots, h_{\nu0}) \). We choose a normal coordinate system near each non-orbit type puncture, and choose a Darboux coordinate system near each orbit type puncture. \( h_0 \) may be considered as a vector field in the coordinate neighborhood. We fix a cutoff function \( \varrho \):

\[ \varrho(s) = \begin{cases} 1, & \text{if } |s| \geq d, \\ 0, & \text{if } |s| \leq \frac{d}{2} \end{cases} \]

where \( d \) is a large positive number. Put

\[ \hat{h}_0 = \varrho h_0. \]

Then for \( d \) big enough \( \hat{h}_0 \) is a section in \( C^\infty(\Sigma; u^* T N) \) supported in the tube \( \{(s,t)||s| \geq \frac{d}{2}, t \in S^1 \} \). Denote

\[ \mathcal{W}^{1,p,\alpha} = \{ h + \hat{h}_0 | h \in W^{1,p,\alpha}, h_0 \in \ker L_{\infty} \}. \]

The operator \( D_u : \mathcal{W}^{1,p,\alpha} \to L^{p,\alpha} \) is a Fredholm operator so long as \( \alpha \) does not lie in the spectrum of the operator \( H_{i\infty} \) for all \( i = 1, \ldots, \nu \).

**Remark 4.1** The index \( \text{ind}(D_u, \alpha) \) does not change if \( \alpha \) is varied in such a way that \( \alpha \) avoids the spectrum of \( L_{\infty} \). Conversely, the index will change if \( \alpha \) is moved across an eigenvalue. We will choose \( \alpha \) slightly larger than zero such that at each end it does not across the first positive eigenvalue.

The proof of the following lemma is almost the same as in [R3], we omit it.

**Lemma 4.2** \( B_{A_i}(N, g_i, T_{m_i}) \) is a Hausdorff Frechet V-manifold for any \( 2g_i + m_i \geq 3 \) or \( g_i = 0, m_i < 3, A_i \neq 0 \).

**Remark 4.3** Let \( b = (u, \Sigma, j; p) \in \mathcal{M}_{A_i}(N, g_i, T_{m_i}) \). We describe the neighborhoods of \( b \).

1. \( N = M^+ \). When \( 2g_i + m_i \geq 3 \), \( \Sigma \) is stable and \( \mathcal{M}_{g_i,m_i} \) is a V-manifold. Hence, the automorphism group \( \text{Aut}_{\Sigma} \) of \( \Sigma \) is finite. Denote by \( O_j \) a neighborhood of complex structures on \( (\Sigma, j) \). Note that we change the complex structure in a compact set \( K_{\text{deform}} \) of \( \Sigma \) away from the puncture points. A neighborhood \( \mathcal{U}_b \) of \( b \) can be described as

\[ O_j \times \{ \exp_u(h + \hat{h}_0); h \in \mathcal{C}(\Sigma; u^* TM^\pm), h_0 \in \ker H_{\infty}, \| h \|_{1, p, \alpha} + |h_0| < \epsilon \} / \text{stab}_u. \]

For the case \( g_i = 0, m_i \leq 2, A_i \neq 0 \), \( \Sigma \) is no longer stable and the automorphism group \( \text{Aut}_{\Sigma} \) is infinite. One must construct a slice \( W_u \) of the action \( \text{Aut}_{\Sigma} \) such that \( W_u / \text{stab}_u \) is a neighborhood of \( (\Sigma, u) \). We can write again \( \mathcal{U}_b = O_j \times W_u / \text{stab}_u \) with \( O_j = \text{point} \) (see [R3]).
2: \( N = \mathbb{R} \times \tilde{M} \). We must mod the group \( \mathbb{C}^* \) generated by the \( S^1 \)-action and the translation along \( \mathbb{R} \). We fix a point \( y_0 \in \Sigma \) different from the marked points and puncture points. Fix a local coordinate system \( a, \theta, w \) on \( \mathbb{R} \times \tilde{M} \) such that \( u(y_0) = (0, 0, 0) \). We use \( C(\Sigma; u^*TN) = \{ h \in C(\Sigma; u^*TN) | h(y_0) = (0, 0, +) \} \) instead of \( C(\Sigma; u^*TN) \), then the construction of \( U_b \) is the same as for \( M^+ \).

We can define a bundle

\[
\mathcal{F}_{A_1}(N, g_i, T_{m_i}) \rightarrow \mathcal{B}_{A_1}(N, g_i, T_{m_i})
\]

whose fiber at \( b = (u, \Sigma, j; p) \) is an infinite dimensional vector space \( \mathcal{C}(u^*TN \otimes \wedge^0, 1) \).

For any \( D \in D_{g,T_m}^A \), let \( \mathcal{B}_D(M^+, g, T_m) \subset \mathcal{B}_A(M^+, g, T_m) \) be the set of \( C^\infty \) stable maps whose domain and the corresponding fundamental class of each component and the sets of periodic orbits corresponding to each puncture point have type \( D \). Then, \( \mathcal{B}_D(M^+, g, T_m) \) is a strata of \( \mathcal{B}_A(M^+, g, T_m) \).

To define the bundle \( \mathcal{F}_D(M^+, g, T_m) \rightarrow \mathcal{B}_D(M^+, g, T_m) \) and describe a neighborhood \( U_b \) for any \( b \in \mathcal{B}_D(M^+, g, T_m) \), we consider two simple cases, the discussion for general case is the same.

**Case 1.** Assume that \( D \) has two components \( (\Sigma_1, j_1) \) and \( (\Sigma_2, j_2) \) joining at \( p \), and any \( b \in \mathcal{B}_D(M^+, g, T_m) \) has the form: \( b = (u_1, u_2; \Sigma_1 \wedge \Sigma_2, j_1, j_2) \), where \( u_i : \Sigma_i \rightarrow M^+ \) are \( C^\infty \) stable maps with \( u_1(p) = u_2(p) \). A neighborhood \( U_b \) of \( b \) can be described as

\[
O_{j_1} \times O_{j_2} \times \left\{ \left( \exp_{u_1}(h_1 + \hat{h}_{10}), \exp_{u_2}(h_2 + \hat{h}_{20}) \right) \mid h_i \in C(\Sigma; u_i^*TM^+) \right\},
\]

\[
h_{10} = h_{20} \in T_{u_1(p)}M^+, \|h_i\|_{1,p,\alpha} + |h_0| < \epsilon \} / \text{stab},
\]

The fiber of \( \mathcal{F}_D(M^+, g, T_m) \rightarrow \mathcal{B}_D(M^+, g, T_m) \) at \( b \) is \( \mathcal{C}(u_1^*TM^+ \otimes \wedge^0, 1) \times \mathcal{C}(u_2^*TM^+ \otimes \wedge^0, 1) \).

**Case 2.** Assume that \( D \) has two components \( (\Sigma_1, j_1) \) and \( (\Sigma_2, j_2) \) joining at \( p \), and any \( b \in \mathcal{B}_D(M^+, g, T_m) \) has the form: \( b = (u_1, u_2; \Sigma_1 \wedge \Sigma_2, j_1, j_2) \), where \( u_1 : \Sigma_1 \rightarrow M^+ \) and \( u_2 : \Sigma_2 \rightarrow \mathbb{R} \times \tilde{M} \) are \( C^\infty \) stable maps such that \( u_1 \) and \( u_2 \) converge to a same \( k \)-periodic orbit when \( z_1 \) and \( z_2 \) converge to \( p \). A neighborhood \( U_b \) of \( b \) can be described as

\[
O_{j_1} \times O_{j_2} \times \left\{ \left( \exp_{u_1}(h_1 + \hat{h}_{10}), \exp_{u_2}(h_2 + \hat{h}_{20}) \right) | h_1 \in C(\Sigma; u_1^*TM^+), h_2 \in C(\Sigma; u_2^*R(\mathbb{R} \times \tilde{M})) \right\},
\]

\[
h_{10} = \hat{h}_{20} \in T \Sigma, \|h_i\|_{1,p,\alpha} + |h_{0i}| < \epsilon \} / \text{stab},
\]

where we write \( h_0 = (\ast, \ast, \hat{h}_0) \) with \( \hat{h}_0 \in T \Sigma \). The fiber of \( \mathcal{F}_D(M^+, g, T_m) \rightarrow \mathcal{B}_D(M^+, g, T_m) \) at \( b \) is \( \mathcal{C}(u_1^*TM^+ \otimes \wedge^0, 1) \times \mathcal{C}(u_2^*(\mathbb{R} \times \tilde{M}) \otimes \wedge^0, 1) \).

The following lemma is obviously holds.

**Lemma 4.4:** \( \mathcal{B}_D(M^+, g, T_m) \) is a Hausdorff Frechet V-manifold. The bundle \( \mathcal{F}_D(M^+, g, T_m) \rightarrow \mathcal{B}_D(M^+, g, T_m) \) is smooth.

Now we construct a finite dimensional V-bundle \( \mathcal{E}_D \) over \( \mathcal{B}_D(M^+, g, T_m) \). The construction imitates Siebert’s construction. First of all, we can slightly deform \( \omega \) such that \( [\omega] \) is a rational class. By taking multiple, we can assume \( [\omega] \) is an integral class on \( \overline{M}^+ \). Therefore, it is the Chern class of a complex line bundle \( L \) over \( \overline{M}^+ \). We choose a unitary connection \( \nabla \) on \( L \). There is a line bundle associated with the domain of stable maps called dualized tangent sheaf \( \lambda \). The restriction of \( \lambda \) on \( (\Sigma, j) \) is \( \lambda_{(\sigma,j)} \), the sheaf of meromorphic 1-form with at worst simple pole at the special points (marked points, puncture points and intersection points) and for each intersect point \( p \), say \( \Sigma_1 \) and \( \Sigma_2 \) intersects at \( p \),

\[
\text{Res}_p(\lambda_{(\Sigma_1,j_1)}) = \text{Res}_p(\lambda_{(\Sigma_2,j_2)}).
\]

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Finally, we define Lemma 4.5: $b \in B(D^{+}, g, T_{m})$ is a smooth Frechet $V$-bundle.

We use gluing arguments to patch different strata together (see below for the details). Note that $\mathcal{F}$ (resp. $\mathcal{E}$) is not a bundle in general because there is no local trivialization. However, for the gluing argument, it behaves as an infinite dimensional (resp. finite dimensional) vector bundle. By abusing the notation, we will call $\mathcal{F}$ and $\mathcal{E}$ bundles.

Next, we define a map $\eta : \mathcal{E} \to \mathcal{F}$, and a stabilization $S_{e}$ of $\bar{\partial}$ over a neighborhood $U$ of $M^{+}, g, T_{m}$:

$$S_{e} : \mathcal{E} \to \mathcal{F}$$

$$S_{e} = \bar{\partial}_{J} + \eta$$

such that

1. The restriction of $\eta$ to every strata is a smooth bundle map;
2. For any $b = (u, (\Sigma, j); p) \in U$, $D S_{e} = D \bar{\partial}_{J} + \eta$ is surjective.

Here we consider $\bar{\partial}_{J}(u)$ as a map from $\mathcal{E}_{b}$ into $\mathcal{F}_{b}$ in a natural way. For any $b = (u, (\Sigma, j); p) \in M_{A}(M^{+}, g, T_{m})$, $\text{coker} D_{u}$ is a representation space of $\text{stab}_{u}$. Hence it can be decomposed as sum of irreducible representations. There is a result in algebra saying that the irreducible factors of group ring contain all the irreducible representations of finite group. Hence, it is enough to find a copy of group ring in $E_{b}$. This is done by algebraic geometry. Recall that $\mathcal{E}_{b} = H^{0}(\Sigma, u^{*}L \otimes \lambda_{(\Sigma, j)})$. By taking power of $u^{*}L \otimes \lambda_{(\Sigma, j)}$ if necessary, we can assume that $u^{*}L \otimes \lambda_{(\Sigma, j)}$ induces an embedding of $\Sigma$ into $CP^{N}$ for some $N$. Furthermore, since $u^{*}L \otimes \lambda_{(\Sigma, j)}$ is invariant under $\text{stab}_{u}$, $\text{stab}_{u}$ also acts effectively naturally on $CP^{N}$. Pick any point $x_{0} \in \text{im}(\Sigma) \subset CP^{N}$ such that $\sigma_{j}(x_{0})$ are mutually
different for any $\sigma_i \in \text{stab}_u$. Then, we can find a homogeneous polynomial $F$ of some degree, say $\ell$ such that $F(x_0) \neq 0, F(\sigma_i(x_0)) = 0$ for $\sigma_i \neq 1$. Notes that $\sigma_i^* F$ generates a group ring. Next, we note that $F \in H^0(O(\ell))$. By pull back over $\Sigma$, $F$ induces a section of $H^0(\Sigma, (u^* L \otimes \lambda_{(\Sigma, j)})^\ell)$. Therefore, if we replace $u^* L \otimes \lambda_{(\Sigma, j)}$ by $(u^* L \otimes \lambda_{(\Sigma, j)})^\ell$ and redefine $\mathcal{E}_b = H^0(\Sigma, (u^* L \otimes \lambda_{(\Sigma, j)})^\ell)$, $\mathcal{E}_b$ contains a copy of group ring. Now, it is clear how to construct a map $\eta$. For $b \in \mathcal{M}_A(M^+, g, T_m)$, we decompose $\text{Coker} D_u$ as irreducible representations. Pick a factor and project group ring to this irreducible factor. If there is more than one factors, we take direct sum of $\mathcal{E}$ to get a surjective map $\eta_b$ to $\text{Coker} D_u$.

Now we extend the map $\eta_b$ to a neighborhood of $b$. If $\Sigma$ is smooth, we may choose a neighborhood $O(b)$ of $b$ and take local travilizations for both $\mathcal{E}$ and $\mathcal{F}$ over $O(b)$, and extend the map $\eta_b$ to $O(b)$ in a natural way. By choosing $O(b)$ so small we may assume that $\eta$ is surjective on $O(b)$. Then we use cut-off function to extend $\eta$ over whole $\mathcal{U}$. In the following we assume that $\Sigma$ is a nodel surface. We will consider two types of stratas \textbf{Case 1} and \textbf{Case 2} as above. We use the holomorphic cylindrical coordinates $(s_i, t_i)$ on $\Sigma_i$ near $p$, and write

\begin{align*}
\Sigma_1 - \{p\} &= \Sigma_{10} \bigcup \{[0, \infty) \times S^1\}, \\
\Sigma_2 - \{p\} &= \Sigma_{20} \bigcup \{(-\infty, 0] \times S^1\}.
\end{align*}

We discuss the two cases separately.

\textbf{Case 1.} For any $(r, \tau)$ we construct a surface $\Sigma_{(r)} = \Sigma_1 \#_{(r)} \Sigma_2$ as follows, where and later we use $(r)$ to denote gluing parameters. We cut off the part of $\Sigma_i$ with cylindrical coordinate $|s_i| > 3r$ and glue the remainders along the collars of length $2r$ of the cylinders with the gluing formulas:

\begin{align*}
s_1 &= s_2 + 4r, \\
t_1 &= t_2 + \tau.
\end{align*}

We glue the map $(u^+, u^-)$ to get a map $u_{(r)}$ from $\Sigma_{(r)}$ to $M^+$ as follows. Set

\begin{align*}
u_{(r)} &= \begin{cases}
u_1 & \text{on } \Sigma_{10} \bigcup \{(s_1, t_1) | 0 \leq s_1 \leq r, t_1 \in S^1\} \\
u_1(p) &= u_2(p) & \text{on } \{(s_1, t_1) | \frac{3r}{2} \leq s_1 \leq \frac{5r}{2}, t_1 \in S^1\} \\
u_2 & \text{on } \Sigma_{20} \bigcup \{(s_2, t_2) | 0 \geq s_2 \geq -r, t_2 \in S^1\}
\end{cases}
\end{align*}

To define the map $u_{(r)}$ in the remaining part we fix a smooth cutoff function $\beta : \mathbb{R} \rightarrow [0, 1]$ such that

\begin{align*}
\beta(s) &= \begin{cases}
1 & \text{if } s \geq 1 \\
0 & \text{if } s \leq 0
\end{cases}
\end{align*}

and $|\beta'(s)| \leq 2$. We assume that $r$ is large enough such that $u_i$ maps the tube $\{(s_i, t_i) | |s_i| \geq r, t_i \in S^1\}$ into a normal coordinate domain of $u_i(p)$. We define

\begin{align*}
u_{(r)} &= u_1(p) + \left(\beta(3 - \frac{2s_1}{r})(u_1(s_1, t_1) - u_1(p)) + \beta(\frac{2s_1}{r} - 5)(u_2(s_1, t_1) - u_1(p))\right).
\end{align*}

\textbf{Case 2.} We choose a Darboux coordinate system $a_i, \theta_i, w_i$ on the cylinder near the periodic orbit. Suppose that

\begin{align*}
a_i(s_i, t_i) - ks_i - \ell_1 &\rightarrow 0 \quad \theta_i(s_i, t_i) - kt_i - \theta_{10} \rightarrow 0
\end{align*}
In view of the condition \( u_2(y_0) = (0, 0, *) \) for some point \( y_0 \), we may assume that \( \ell_2 = 0, \theta_2 = 0 \). For any \((r, \theta_0)\) we glue \( M^+ \) and \( \mathbb{R} \times \tilde{M} \) with parameter \((r, \theta_0)\) to get again \( M^+ \) with gluing formula:

\[
(4.12) \quad a_1 = a_2 + 4kr + \ell
\]

\[
(4.13) \quad \theta_1 = \theta_2 + \theta_0 \mod 1.
\]

Now we construct a surface \( \Sigma_{(r)} = \Sigma_1 \# (r) \Sigma_2 \) with gluing formulas:

\[
(4.14) \quad s_1 = s_2 + 4r
\]

\[
(4.15) \quad t_1 = t_2 + \frac{\theta_0 - \theta_{10} + n}{k}
\]

for some \( n \in \mathbb{Z} \). To get a pregluing map \( u_{(r)} \) from \( \Sigma_{(r)} \) we set

\[
u_{(r)} = \begin{cases} 
  u_1 & \text{on } \Sigma_{10} \cup \{ (s_1, t_1) | 0 \leq s_1 \leq r, t_1 \in S^1 \} \\
  (ks_1, x(kt_1)) & \text{on } \{ (s_1, t_1) | \frac{3r}{2} \leq s_1 \leq \frac{5r}{2}, t_1 \in S^1 \} \\
  u_2 & \text{on } \Sigma_{20} \cup \{ (s_2, t_2) | 0 \geq s_2 \geq -r, t_2 \in S^1 \}.
\end{cases}
\]

We assume that \( r \) is large enough such that \( u_i \) maps the tube \{ \((s_i, t_i)|s_i| \geq \frac{r}{k}, t_i \in S^1 \} \) into a domain with Darboux coordinates \((a_i, \theta_i, w_i)\). We write in in terms the coordinate system \((a_1, \theta_1, w_1)\)

\[
u_{(r)} = \left( a_{(r)}, \overline{u}_{(r)} \right) = \left( a_{(r)}, \theta_{(r)}, w_{(r)} \right)
\]

and define

\[
a_{(r)} = ks_1 + \left( \beta(3 - \frac{2s_1}{r})(a_1(s_1, t_1) - ks_1 - \ell) + \beta(\frac{2s_1}{r} - 5)(a_2(s_1, t_1) - ks_1 - \ell) \right)
\]

\[
\overline{u}_{(r)} = x(kt_1) + \left( \beta(3 - \frac{2s_1}{r})(\overline{u}_1(s_1, t_1) - x(kt_1)) + \beta(\frac{2s_1}{r} - 5)(\overline{u}_2(s_1, t_1) - x(kt_1)) \right).
\]

It is easy to check that in both cases \( u_{(r)} \) is a smooth function. By using the exponential decay of \( u_i \) one can easily prove that \( u_{(r)} \) are a family of approximate \( J \)-holomorphic map, precisely the following lemma holds.

**Lemma 4.6**

\[
(4.16) \quad \| \partial_{\Sigma_{(r)}} u_{(r)} \|_{p, \alpha, r} \leq C e^{-(\beta - \alpha)r}.
\]

The constants \( C \) in the above estimates are independent of \( r \).

Let \( b_{(r)} = (u_{(r)}, \Sigma_{(r)}, \xi, j_1, j_2) \). For any section \( \xi \in C^\infty(\Sigma_{(r)}; u^* L \otimes \lambda(\Sigma_{(r)}, j_1, j_2)) \), we denote

\[
(4.17) \quad \beta \xi = (\xi_1, \xi_2) = (\xi_1(2\alpha r + 1 - \alpha s_1), \xi_2(2\alpha r + 1 + \alpha s_2)).
\]

We may consider \( \beta \xi \) as a section of the bundle \( u^* L \otimes \lambda(\Sigma, j) \), where \( u = (u_1, u_2) \), \( \Sigma = \Sigma_1 \wedge \Sigma_2 \), \( j = (j_1, j_2) \). Denote by \( \pi \) the projection from \( W^{1, p, \delta}(\Sigma, u^* L \otimes \lambda(\Sigma, j)) \) into \( H^0(\Sigma, u^* L \otimes \lambda(\Sigma, j)) \). We have a map

\[
i_r : E_{b_{(r)}} \to E_{\delta}
\]

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Lemma 4.7: \( i_r \) is a isomorphism for \( r \) big enough.

By multiplying a cut-off function we may assume that

\[
\eta_b = 0 \quad \text{for} |s_i| \geq R
\]

and \( D_u + \eta_b \) remains surjective for \( R \) big enough. Consider a neighborhood \( U_\epsilon(b_{(r)}) = O_{j_1, \epsilon} \times O_{j_2, \epsilon} \times \{ \exp_{u_{(r)}} h \| h \|_{1,p} < \epsilon \} \), where \( O_{j_i, \epsilon} \) is an \( \epsilon \)-neighborhood of \( j_i \) with respect to a metric in the space of complex structures. Let \( b' = (u'_{(r)}, \Sigma_{(r)}, j'_1, j'_2) \in U_\epsilon(b_{(r)}) \). To simplify notations, without loss of generality, we assume that \( j'_i = j_i \). The parallelog transport to \( u_{(r)} \) along geodesics with respect to the unitary connection on \( L \) induces a map \( T \) from \( u^*_{(r)} L \otimes \lambda(\Sigma_{(r)}, j_1, j_2) \) into \( u^*_{(r)} L \otimes \lambda(\Sigma_{(r)}, j_1, j_2) \). For any \( \xi \in \mathcal{E}_{b'} \) we define

\[
\eta_{b'}(\xi) = \eta_b(I_r(T\xi)).
\]

We define in the neighborhood \( U_r(b_{(r)}) \)

\[
S_c = \bar{\partial}_f + \eta.
\]

Then

\[
DS_c = D\bar{\partial}_f + \eta.
\]

For any \( \eta \in C^\infty(\Sigma_{(r)}; u_{(r)}^* TM^+ \otimes \Lambda^{0,1}) \), let \( \eta_h \) be its restriction to the part \( \Sigma_i \cup \{(s_i, t_i) \mid |s_i| < 3r \} \), extended by zero to yield a section over \( \Sigma_i \). Define

\[
\|\eta\|_{p,\alpha,r} = \|\eta_1\|_{\Sigma_1, p, \alpha} + \|\eta_2\|_{\Sigma_2, p, \alpha}.
\]

Denote the resulting completed spaces by \( L^p_{r,1} \). For any section \((\xi, h) \in \mathcal{E}_{b_{(r)}} \times C^\infty(\Sigma_{(r)}; u_{(r)}^* TM^+) \), denote

\[
h_0 = \int_{S^1} h(2r, t) dt
\]

(4.22) \( h_1 = h/\beta(2\alpha r + 1 - \alpha s_1) \)

(4.23) \( h_2 = h/\beta(2\alpha r + 1 + \alpha s_2). \)

We define

\[
\|(\xi, h)\|_{1,p,\alpha,r} = \|\xi\|_{1,p,\alpha} + \|h_1\|_{\Sigma_1,1,p,\alpha} + \|h_2\|_{\Sigma_2,1,p,\alpha} + |h_0|
\]

where \( |\cdot|_{1,p,\alpha} \) is the norm on \( \mathcal{E}_{b_{(r)}} \). Denote the resulting completed spaces by \( \mathcal{W}_{r,1}^{1,p,\alpha} \).

Lemma 4.8 \( DS_{c,b_{(r)}} \) is surjective for \( r \) large enough and \( \epsilon \) small enough. Moreover, there are a right inverses \( Q_{b_{(r)}} \) such that

\[
DS_{c,b_{(r)}} Q_{b_{(r)}} = Id
\]

(4.26) \( \|Q_{b_{(r)}} - Q_{b_r}\| \leq \frac{C}{r} \)
for some constant $C > 0$ independent of $(r)$ and $\epsilon$. Here and later the inequality (4.27) means for any $\eta$, 

\begin{equation}
\left\| \left( \frac{\partial}{\partial r} Q_{b(r)} \right) \eta \right\|_{s_i \leq r} \leq \frac{Ce}{r^2}
\end{equation}

\textbf{Proof:} The proof is similar to the proof in \cite{MS}. We first construct an approximate right inverse $Q'_{b(r)}$ such that the following estimates hold:

\begin{equation}
\left\| Q'_{b(r)} \right\| \leq C
\end{equation}

\begin{equation}
\left\| D\mathcal{S}_{e,b(r)} Q'_{b(r)} - Id \right\| \leq \frac{1}{2}.
\end{equation}

Then the operator $D\mathcal{S}_{e,b(r)} Q'_{b(r)}$ is invertible and a right inverse $Q_{b(r)}$ of $D\mathcal{S}_{e,b(r)}$ is given by

\begin{equation}
Q_{b(r)} = Q'_{b(r)} (D\mathcal{S}_{e,b(r)} Q'_{b(r)})^{-1}.
\end{equation}

Given $\eta \in L^p_0$, we have a pair $(\eta_1, \eta_2)$. Let $(\xi, h) = Q_b(\eta_1, \eta_2)$. We may write $h$ as $(h_1 + \hat{h}_0, h_2 + \hat{h}_0)$, and define

\begin{equation}
h(r) = \hat{h}_0 + h_1 (3/2 - \frac{s_1}{2r}) + h_2 \left( 1 - \beta(3/2 - \frac{s_1}{2r}) \right).
\end{equation}

Then we define

\begin{equation}
Q_{b(r)}' \eta = (\hat{i}_r^{-1}(\xi), h(r)).
\end{equation}

We must prove that

\begin{equation}
\left\| \eta_{b(r)} (\hat{i}_r^{-1}(\xi)) + D_{u(r)} h(r) - \eta \right\|_{p,0,r} \leq \frac{1}{2} \left\| \eta \right\|_{p,0,r}.
\end{equation}

By definition $\eta_{b(r)} (\hat{i}_r^{-1}(\xi)) = \eta_b(\xi)$. Since $\eta_b(\xi) + D_{\xi} h = \eta$ the term on the left hand side vanishes for $|s_i| \leq r$. It suffices to estimate the left hand side in the annulus $r \leq |s_i| \leq 3r$. Denote $\beta_1 = \beta(3/2 - \frac{s_1}{2r})$, $\beta_2 = (1 - \beta(3/2 - \frac{s_1}{2r}))$. Note that in this annulus

\begin{equation}
\beta_1 + \beta_2 = 1, \quad \eta_b(\xi) = 0, \quad D_{u(r)} (h_1 + \hat{h}_0) = \eta.
\end{equation}

Since near $u_1(p) = u_2(p)$ (or near the periodic orbit $x(kt)$), $D = \tilde{D} + S$, we have

\begin{equation}
D\mathcal{S}_{e,b(r)} Q'_{b(r)} \eta - \eta = \eta_{b(r)} (i_r^{-1}(\xi)) + D_{u(r)} h(r) - \eta
\end{equation}

\begin{equation}
= (\tilde{\partial} \beta_1) h_1 + \beta_1 (S_{u(r)} - S_{u_1}) h_1 + (\tilde{\partial} \beta_2) h_2 + \beta_2 (S_{u(r)} - S_{u_2}) h_2 - \beta_1 S_{u_1} \hat{h}_0 - \beta_2 S_{u_2} \hat{h}_0 + S_{u(r)} \hat{h}_0.
\end{equation}

By the exponential decay of $S$ we get

\begin{equation}
\left\| \eta_{b(r)} (i_r^{-1}(\xi)) + D_{u(r)} h(r) - \eta \right\|_{p,0,r} \leq \frac{C_1}{r} (\left\| h_1 \right\|_{p,0} + \left\| h_2 \right\|_{p,0} + \left\| h_0 \right\|_{p,0}) \leq \frac{C_2}{r} \left\| \eta \right\|_{p,0,r}.
\end{equation}
In the last inequality we used that \( \|Q_b\| \leq C \) and (4.20). Then (4.30) follows by choosing \( r \) big enough. The proof of (4.27) is easy, we omit it. Here we prove (4.28). From (4.35) and the exponential decay of \( S \) we get

\[
(4.47) \quad \left\| \frac{\partial}{\partial r} \left( DS_{e,b(r)}Q_{b(r)} \right) \right\| \leq \frac{C}{r^2}.
\]

From

\[
DS_{e,b(r)}Q_{b(r)} \left( DS_{e,b(r)}Q_{b(r)}^t \right)^{-1} = Id
\]

and (4.37) we conclude that

\[
(4.38) \quad \left\| \frac{\partial}{\partial r} \left( (DS_{e,b(r)}Q_{b(r)}^t)^{-1} \right) \right\| \leq \frac{C}{r^2}.
\]

Since for any \( \eta \in L_{p,\alpha}^r \) the restriction of \( Q_{b(r)}^t \) to the parts \( |s_i| \leq r \) is \( \eta \), from (4.31) and (4.38) we get

\[
\left\| \frac{\partial}{\partial r} Q_{b(r)} \right\|_{|s_i| \leq r} \leq \frac{C}{r^2} \quad \square
\]

Thus for every point \( b \in \overline{\mathcal{M}}_A(M^+, g, T_m) \) we may choose a neighborhood \( O(b) \) such that \( DS_e \) is surjective on \( O(b) \). Then we use cut-off function to extend \( \eta \) over whole \( U \). By the compactness of \( \overline{\mathcal{M}}_A(M^+, g, T_m) \), we may choose finitely many such neighborhoods \( O(b_j), j = 1, 2, \ldots, N \) such that \( \overline{\mathcal{M}}_A(M^+, g, T_m) \subset \bigcup_{j=1}^N O(b_j) \). Then we take \( E^N \) instead of \( E \) and put

\[
\eta = \sum \eta_{b_j}.
\]

Our stabilization operator is the bundle map

\[
(4.39) \quad S_e = \bar{\partial}f + \eta : E \to F.
\]

It is easy to see that the restriction of \( \eta \) to every strata is a smooth bundle map, and for any \( b = (u, \Sigma, j, p) \in U \) \( DS_e = D\bar{\partial}f + \eta \) is surjective.

We continue to discuss the gluing for the **Case1** and the **Case2**. We define a map \( I_r : \ker DS_{e,b(r)} \to \ker DS_{e,b} \) as in (4.17). For any \( h \in \ker DS_{e,b(r)} \) we denote by \( h_i \) the restriction of \( h \) to the part \( |s_i| \leq 2r + \frac{1}{\alpha} \). We get a pair \((h_1, h_2)\). We put

\[
\beta h = (h_1\beta(2\alpha r + 1 - \alpha s_1), h_2\beta(2\alpha r + 1 + \alpha s_2))
\]

and define

\[
(4.40) \quad I_r(\xi, h) = (i_r(\xi), \beta h) - Q_b DS_e(i_r(\xi), \beta h)
\]

where \( Q_b \) is a right inverse of \( DS_{e,b} \). We also construct a map \( I_r' : \ker DS_{e,b} \to \ker DS_{e,b(r)} \). Let \( (\xi, h) \in \ker DS_{e,b} \). We write \( h = (h_1 + \hat{h}_0, h_2 + \hat{h}_0) \), and define

\[
(4.41) \quad h_{(r)} = \hat{h}_0 + h_1\beta(3/2 - \frac{s_1}{2r}) + h_2 \left( 1 - \beta(3/2 - \frac{s_2 + 4r}{2r}) \right).
\]

We define

\[
(4.42) \quad I_r'(\xi, h) = (i_{r}^{-1}(\xi), h_{(r)}) - Q_{b(r)} DS_{e,b(r)}(i_{r}^{-1}(\xi), h_{(r)}).
\]

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Lemma 4.9: Both \( I_r \) and \( I'_r \) are isomorphisms for \( r \) big enough. Moreover

\[
\frac{|\partial I'_r|}{|\partial I_r|} \leq \frac{C}{r^2}.
\]

Proof: The proof is basically a similar gluing argument as in an unpublished book of Donaldson. The proof is devides into 2 steps.

Step 1. We show that \( I_r \) is injective for \( r \) big enough. We have from (4.40)

\[
\|I_r(\xi, h) - (i_r(\xi), \beta h)\|_{1,p,\alpha} \leq C_1\|\eta b(i_r(\xi)) + D_u(\beta h)\|_{p,\alpha}
\]

\[
= C_1\|\eta b(i_r(\xi)) + (\bar{\partial}\beta)h + \beta(D_u h + D_u(r) h + \eta b(r)(\xi) - D_u(r) h - \eta b(r)(\xi))\|_{p,\alpha}
\]

\[
= C_1\|((\bar{\partial}\beta)h + \beta((S_u - S_u(r))h))\|_{p,\alpha},
\]

where we used the fact that \((\xi, h) \in \ker D_S e,b, \eta b(i_r(\xi)) = \beta \eta b(r)(\xi)\) for \( r > R \). Note that

\[
S_u = S_u(r) \quad \text{if} \quad s_1 \leq r, \text{ or } s_2 \geq -r
\]

\[
\beta(2\alpha r + 1 - \alpha s_1) = 1 \quad \text{if} \quad s_1 \leq 2r
\]

\[
\beta(2\alpha r + 1 + \alpha s_2) = 1 \quad \text{if} \quad s_2 \geq -2r.
\]

By exponential decay of \( S \) we have

\[
\|(S_u - S_u(r))\beta h\|_{p,\alpha} \leq C e^{-\delta r}\|\beta h\|_{1,p,\alpha}
\]

for some constant \( C > 0 \). Since \((\bar{\partial}\beta(2\alpha r + 1 - \alpha s_1))h\) supports in \( 2r \leq s_1 \leq 2r + \frac{1}{\alpha} \), and over this part

\[
|\bar{\partial}\beta(2\alpha r + 1 - \alpha s_1)| \leq |\alpha|
\]

\[
\beta(2\alpha r + 1 + \alpha s_2) = 1, \quad e^{2\alpha|s_1|} \leq e^4 e^{2\alpha|s_2|},
\]

we obtain

\[
\|((\bar{\partial}\beta(2\alpha r + 1 - \alpha s_1))h\|_{p,\alpha} \leq |\alpha|e^4\|h_2\|_{p,\alpha} \leq |\alpha|e^4\|\beta h\|_{p,\alpha}.
\]

Similar inequality for \((\bar{\partial}\beta(2\alpha r + 1 + \alpha s_2))h\) also holds. So we have

\[
\|((\bar{\partial}\beta)h\|_{p,\alpha} \leq 2|\alpha|e^4\|\beta h\|_{1,p,\alpha}.
\]

Hence

\[
\|I_r(\xi, h) - (i_r(\xi), \beta h)\|_{1,p,\alpha} \leq C_3(|\alpha| + e^{-\delta r})\|\beta h\|_{1,p,\alpha} \leq 1/2\|\beta h\|_{1,p,\alpha}
\]

for some constant \( C_3 > 0 \), here we choose \( r \) big enough and \( |\alpha| \) so small that \( C_3(|\alpha| + e^{-\delta r}) < 1/2 \).

Now suppose that \( I_r(\xi, h) = 0 \), then (4.44) gives us \( |i_r(\xi)|_{1,p,\alpha} = 0, \|\beta h\|_{1,p,\alpha} = 0 \). It follows that

\( \xi = 0, \quad h = 0. \)

Step 2. By a similar calculation as the proof of Lemma 4.8 we obtain

\[
\|I'_r(\xi, h) - (i_r^{-1}(\xi), h(r))\|_{p,\alpha,r} \leq C\|h\|_{p,\alpha} + |h_0|).
\]

In particular, it holds for \( p = 2 \). It remains to show that \( \|h(r)\|_{2,\alpha,r} \) is close to \( \|h\|_{2,\alpha} \). Denote \( \pi \) the projection into the second component, that is, \( \pi(\xi, h) = h \). Then \( \pi(\ker D_S e,b) \) is a finite dimensional
space. Let \( f_i, i = 1, \ldots, d \) be an orthonormal basis. Then \( F = \sum f_i^2 e^{2\alpha |s_i|} \) is an integrable function on \( \Sigma \). For any \( \epsilon > 0 \), we may choose \( R_0 \) so that

\[
\int_{|s_i| \geq R_0} F \leq \epsilon.
\]

Then the restriction of \( h \) to \( |s_i| \geq R_0 \) satisfies

\[
\|h\|_{|s_i| \geq R_0} \leq \epsilon \|h\|_{2,\alpha},
\]

therefore

\[
\|h(r)\|_{2,\alpha,r} \geq \|h\|_{|s_i| \leq R_0} \|2,\alpha + |h_0| \geq (1 - \epsilon)\|h\|_{2,\alpha} + |h_0|,
\]

for \( r > R_0 \). Suppose that \( I'(\xi, h) = 0 \). Then (4.45) and (4.46) give us \( h = 0 \), and so \( \xi = 0 \).

The step 1 and step 2 together show that both \( I_f \) and \( I'_f \) are isomorphisms for \( r \) big enough. Now we prove (4.43). By definition \( \eta_{(r)}(i_r^{-1}(\xi)) \) is independent of \( r \), by a direct calculation we get

\[
(4.47) \quad \left\| \frac{\partial}{\partial r} \left( DS_{e,b}(i_r^{-1}(\xi), h(r)) \right) \right\| \leq \frac{C}{r^2}.
\]

Since \( h(r)\|_{|s_i| \leq r} = h \), using (4.47) we get (4.43). \( \square \)

### 4.2 Virtual neighborhood and log invariants

Let \( U_{S_e} = (S_e)^{-1}(0) \). Following [R5], we can assume that \( U_{S_e} \) is compatible with the stratification of \( \mathcal{E} \) by defining \( \eta_i \) inductively over the strata of \( \overline{B}_A \). Namely, we define \( \eta_i \) on lower strata first. Then, we extend it over \( \mathcal{E} \) such that \( \eta_i \) is supported in a neighborhood of lower strata. Then, we define \( \eta_{i+1} \) supported away from lower strata. Once \( \eta_i \) is defined in such a fashion.

**Proposition 4.10** \( U_{S_e} \) has the property

1. **Each strata of** \( U_{S_e} \) **is a smooth V-manifold.**
2. **If** \( B_{D'} \subset \overline{B}_D \) **is a lower stratum,**

\[
U_{S_e} \cap \mathcal{E}|_{B_{D'}} \subset U_{S_e} \cap \mathcal{E}|_{\overline{B}_D}
\]

**is a submanifold of codimension at least 2.**

**Proof:**

1. We prove only for the top strata, for lower strata the proof is the same. Let \( b = (u, (\Sigma, j)) \in \mathcal{M}_A(M^+, g, T_m) \). Consider a map

\[
F : \mathcal{E}_b \times W^{1,p,\alpha}(\Sigma; u^*TM^+) \to L^{p,\alpha}(u^*TM^+ \otimes \wedge^{0,1})
\]

\[
F(\xi, h) = P_{exp_ah}(\partial_t exp_u h + \eta(T\xi))
\]

where \( P_{exp_ah} \) denotes the parallel transport from \( exp_u h \) to \( u \) along the geodesics \( t \to exp_u(\theta t) \) and \( T \) is a trivialization of \( \mathcal{E} \) near \( b \). Denote by \( Q_b \) a right inverse of \( DS_{e,b} \). By the Implicit Function Theorem there is a smooth map

\[
f : ker DS_{e,b} \to L^{p,\alpha}(u^*TM^+ \otimes \wedge^{0,1})
\]

such that the zero set of \( F \) is locally the form \( (\zeta, Q_b f(\zeta)) \), where \( \zeta \in ker DS_{e,b} \). So \( \zeta \) gives a local coordinate system of the top strata of \( U_{S_e} \). If we choose another \( b' \) we may obtain another local
coordinate system. We may show the coordinate transformation is smooth.

2. Because of the existence of 2-dimensional gluing parameters for both Case 1 and Case 2 in above. For general case the situation is the same.

There are two maps, the inclusion map

\[ I : U_{S_b} \to \mathcal{E} \]

and the projection

\[ \pi : U_{S_b} \to \mathcal{U}. \]

\( I \) can be viewed as a section of the bundle \( E = \pi^* \mathcal{E}, \) and \( I^{-1}(0) = \mathcal{M}_A(M^+, g, T_m). \)

**Lemma 4.11** \( I \) is a proper map.

The proof is the same as the proof of the compactness theorem for \( \mathcal{M}_A(M^+, g, T_m). \)

Using the virtual neighborhood we can define the log GW-invariants. Recall that we have two natural maps

\[ e_i : \mathcal{B}_A(M^+, g, T_m) \to M^+ \]

for \( i \leq l \) defined by evaluating at marked points and

\[ e_j : \mathcal{B}_A(M^+, g, T_m) \to Z \]

for \( j > l \) defined by projecting to its periodic orbit. To define the log GW-invariants, choose an \( r \)-form \( \Theta \) on \( \mathcal{E} \) supported in a neighborhood of the zero section, where \( r \) is the dimension of the fiber, such that

\[ \int_{E_x} i^* \Theta = 1 \]

for any \( x \in U_{S_b} \), where \( i \) is the inclusion map \( E_x \to E \). We call \( \Theta \) a Thom form. The log GW-invariant can be defined as

\[ \Psi_{(A, g, T_m)}^{(V,B)}(\alpha_1, \ldots, \alpha_l; \beta_{l+1}, \ldots, \beta_m) = \int_{U_{S_b}} \prod_i e_i^* \alpha_i \wedge \prod_j e_j^* \beta_j \wedge I^* \Theta. \]

for \( \alpha_i \in H^*(M^+, \mathbb{R}) \) and \( \beta_j \in H^*(Z, \mathbb{R}) \) represented by differential form. Clearly, \( \Psi = 0 \) if \( \sum \deg(\alpha_i) + \sum \deg(\beta_j) \neq \text{ind} \). We must prove the convergence of the integral (4.50) near each lower strata. We prove this for Case 2, for other case the proof is the same. We have proved in Lemma (4.8) that there is a uniform right inverse \( Q_{(r)} \). Consider a map

\[ F_{(r)} : \mathcal{E}_{\theta_{(r)}} \times W^{1,p,\alpha,r}(\Sigma_{(r)}; u_{(r)}^* TM^+) \to L^{p,\alpha,r}(u_{(r)}^* TM^+ \otimes \wedge^{0,1}) \]

\[ F_{(r)}(\xi, h) = P_{exp_{u_{(r)}}} h \left( \bar{\partial}_j exp_{u_{(r)}} h + \eta(T \xi) \right) \]

By the Implicit Function Theorem there is a smooth map

\[ f_{(r)} : \ker DS_{e,b_{(r)}} \to L^{p,\alpha,r}(u_{(r)}^* TM^+ \otimes \wedge^{0,1}) \]

such that the zero set of \( F_{(r)} \) is locally the form \( (\zeta, Q_{b_{(r)}} f_{(r)}(\zeta)), \) i.e

\[ F_{(r)}(\zeta + Q_{b_{(r)}} f_{(r)}(\zeta)) = 0 \]
where \( \zeta \in \ker DS_{e,b(r)} \). Since there is an isomorphism \( I'_r : \ker DS_{e,b} \to \ker DS_{e,b(r)} \), we have

**Lemma 4.12** There is a neighborhood \( O \) of \((0,0)\) in \( \ker DS_{e,b} \) and \( R_0 > 0 \) such that

\[
f_{(r,\theta)} \circ I'_{(r,\theta)} : O \times Z_k \to US_{e,b(\theta)}
\]

for \( r > R_0 \) is a family of orientation preserving local diffeomorphisms. Moreover, \((R_0,\infty) \times Z_k \times (O/(\text{stab}_{b_1} \times \text{stab}_{b_2})) / \cong \) is a local chart around \( b \), where \( \cong \) is an equivalence relation at \( \infty \):

\[
(a_1,b_1,n_1) \cong (a_2,b_2,n_2) \iff (a_1,b_1) = (a_2,b_2).
\]

**Lemma 4.13** Restricting to the part \(|s_i| \leq r\) we have

\[
(4.52) \quad \left\| \frac{\partial}{\partial r} (f_{(r)} \circ I'_r(\xi)) \right\| \leq \frac{C}{r^2}.
\]

**Proof**: Restricting to the part \(|s_i| \leq r\), \( F_{(r)} \) is independent of \( r \). From (4.51) we get

\[
DF_{(r)} \left( \frac{\partial}{\partial r} I'_r(\xi) + \frac{\partial}{\partial r} (Q_{b(r)} f_{(r)}(I'_r(\xi)) + Q_{b(r)} \frac{\partial}{\partial r} (f_{(r)}(I'_r(\xi))) \right) = 0.
\]

Using (4.43), (4.28) we get (4.52). \( \square \)

We prove the convergence of the integral (4.50). Since the marked points \( \{y_i\} \) and the end points (puncture points) \( \{p_j\} \) vary in a compact set \( K \) out of the gluing part, the inequality (4.52) holds in \( K \). By the standard elliptic estimate we get

\[
(4.53) \quad \left\| \frac{\partial}{\partial r} (f_{(r)} \circ I'_r(\xi)) \right\| \leq \frac{C}{r^2}.
\]

By using a similar argument it is easy to get

\[
(4.54) \quad \left| \frac{\partial}{\partial v} (f_{(r)} \circ I'_r(\xi)) \right| \leq C \quad \left| \frac{\partial}{\partial v} I'_r(\xi) \right| \leq C \quad \text{for other parameter } v.
\]

Let \( \alpha \in H^*(M^+,\mathbb{R}) \) and \( \beta \in H^*(Z,\mathbb{R}) \) represented by differential form. We may write

\[
\prod_i e^*_i \alpha_i \wedge \prod_j e^*_j \beta_j \wedge I^* \Theta = ydr \wedge dt \wedge d\xi \wedge dj,
\]

where \( d\xi \) and \( dj \) denote the volume forms of \( \ker DS_{e,b} \) and the space of complex structures respectively and \( y \) is a function. Then (4.53) implies that \( |y| \leq \frac{C}{r^2} \). Then the convergence of the integral (4.50) follows.

By the same argument as in [6], one can easily show that

**Theorem 4.14**

(i). \( \Psi^{(M,Z)}_{(A,g,T_m)}(\alpha_1,...,\alpha_l;\beta_{l+1},...,\beta_m) \) is well-defined, multi-linear and skew symmetric.

(ii). \( \Psi^{(M,Z)}_{(A,g,T_m)}(\alpha_1,...,\alpha_l;\beta_{l+1},...,\beta_m) \) is independent of the choice of forms \( \alpha_i, \beta_j \) representing the cohomology classes \( [\beta_j], [\alpha_i] \), and the choice of virtual neighborhoods.

(iii). \( \Psi^{(M,Z)}_{(A,g,T_m)}(\alpha_1,...,\alpha_l;\beta_{l+1},...,\beta_m) \) is independent of the choice of \( \bar{J} \) and \( J_v \).
5 A gluing formula

We first prove an addition formula for operator \( D_u \). Let \( u = (u^+, u^-) : (\Sigma^+, \Sigma^-) \to (M^+, M^-) \) be \( J \)-holomorphic curves such that \( u^+ \) and \( u^- \) have \( \nu \) ends and they converge to the same periodic orbits at each end. Note that according to our convention \( \Sigma^\pm \) may not be connected (see the end of Section 4). In this case \( \text{Ind}(D_{u^\pm}, \alpha) \) denotes the sum of the indices of its components. Suppose that \( \Sigma = \Sigma^+ \cup \Sigma^- \) has genus \( g \) and \( [\mu(\Sigma)] = A \). Denote by \( j \) (resp. \( j^\pm \)) the complex structure on \( \Sigma \) (resp. \( \Sigma^\pm \)), and by \( \text{Ind}(D_{u, j}) \) (resp. \( \text{Ind}(D_{u^\pm, j^\pm}, \alpha) \)) the corresponding index. From the proof of Lemma 4.9 we get the following index addition formula of Bott-type:

\[
\text{Ind}(D_{u^+, j^+}, \alpha) + \text{Ind}(D_{u^-, j^-}, \alpha) - 2(n + 1)\nu = 2C_1(A) + (n + 1)(2 - 2g) + 6g - 6,
\]

where we used the fact that at every end \( \dim\ker L_\infty = 2(n + 1) \). Considering the variation of the complex structures on the Riemann surfaces we have from (3.63) that

**Theorem 5.1**

\[
\text{Ind}(D_{u^+, \alpha}) + \text{Ind}(D_{u^-, \alpha}) = 2n\nu + 2C_1(A) + (n + 1)(2 - 2g) + 6g - 6.
\]

**Proof:** Let us consider a simple case, the general case is identical. Suppose that \( \Sigma^+ \) is a connected smooth Riemann surface of genus \( g^+ \), and \( \Sigma^- \) consists of two components: \( \Sigma^-_1 \) of genus \( g^-_1 \) and \( \Sigma^-_2 \) of genus \( g^-_2 \). Suppose that \( \Sigma^+ \) intersects \( \Sigma^-_1 \) at \( \nu_1 \) points, and intersects \( \Sigma^-_2 \) at \( \nu_2 \) points. \( \Sigma^+ \) and \( \Sigma^- \) form a Riemann surface \( \Sigma \) of genus

\[
g = g^+ + g^-_1 + g^-_2 + \nu_1 + \nu_2 - 2.
\]

We add \( 6g - 6 \) to both sides of (3.63). Note that

\[
\text{Ind}(D_{u^+, \alpha}) = \text{Ind}(D_{u^+, j^+}, \alpha) + 6g^+ - 6 + 2(\nu_1 + \nu_2),
\]

\[
\text{Ind}(D_{u^-, \alpha}) = \text{Ind}(D_{u^-, j^-}, \alpha) + 6g^-_1 - 6 + 2\nu_1 + \text{Ind}(D_{u^-, j^-}, \alpha) + 6g^-_2 - 6 + 2\nu_2.
\]

Then (5.2) follows from the above equalities. □

**Remark 5.2** Let \( u \) be a \( J \)-holomorphic map from \( (\hat{\Sigma}; y_1, \ldots, y_m, p_1, \ldots, p_\nu) \) into \( M^\pm \) such that each end converges to a periodic orbit. By using the removable singularities theorem we get a \( J \)-holomorphic map \( \bar{u} \) from \( \Sigma \) into \( \overline{M^\pm} \). Therefore, we have a natural identification of finite energy pseudo-holomorphic maps into \( M^\pm \) and closed pseudo-holomorphic maps into the closed symplectic manifolds \( \overline{M^\pm} \). Moreover, the operator \( D_u \) is identified with the operator \( D_{\bar{u}} \) in a natural way. Under this identification, the condition that \( u \) converges to a \( k \)-multiple periodic orbit at a marked point \( p \) is naturally interpreted as \( \bar{u} \) being tangent to \( Z \) at \( p \) with order \( k \). Since \( \ker L_\infty \) consists of constant vectors, we can identify the vector fields in \( W^{1,p,\alpha}_\pm \) along \( u \) with the vector fields in \( W^{1,p,\alpha}_\pm \) along \( \bar{u} \), the space \( L^{p,\alpha}_\pm \) along \( u \) is also identified with the space \( L^{p,\alpha}_\pm \) along \( \bar{u} \). In the case of closed manifolds the definitions of \( L^{p,\alpha}_\pm \) and \( W^{1,p,\alpha}_\pm \) are the same as that of \([\text{Liu}]\).

Thus we have

**Proposition 5.3**

\[
\text{Ind}(D_{u, \alpha}) = \text{Ind}(D_{\bar{u}}).
\]

We next prove a general gluing formula relating GW-invariants of a closed symplectic manifold in terms of log GW-invariants of its symplectic cut.
First, we make a remark about an error in the draft concerning the homology class of the general gluing formula. The gluing theorem in the previous section shows that one can glue two pseudo-holomorphic curves \((f_+, f_-)\) in \(M^+, M^-\) with the same end point to a pseudo-holomorphic curve \(f\) in \(M\). Suppose that the homology classes of \(f_+, f_-\) are \(A^+, A^-\). Then, we carelessly wrote \(A = A^+ + A^-\). R. Fintushel and E. Ionel pointed to us that in general the homology class of \(f\) depends on the pseudo-holomorphic curve representatives \(f_+, f_-\) instead of the homology classes \(A^+, A^-\). One can also understand it as follows. Recall that there is a map 
\[
\pi : M \to \overline{M}^+ \cup \overline{M}^-.
\]
\(\pi\) induces a homomorphism 
\[
\pi_* : H_2(M, \mathbb{Z}) \to H_2(\overline{M}^+ \cup \overline{M}^-, \mathbb{Z}).
\]
Using the Mayer-Vietoris sequence for \((\overline{M}^+, \overline{M}^-, \overline{M}^+ \cup \overline{M}^-), (f_+, f_-)\) defines a homology class 
\[
[f^+ + f^-] \in H_2(\overline{M}^+ \cup \overline{M}^-, \mathbb{Z}).
\]
The existence of the glued map \(f\) implies \([f^+ + f^-] = \pi_*([f])\). If \((f'_+, f'_-)\) is another representative and glued to \(f'\), 
\[
\pi_*([f']) = [f'_+ + f'_-] = [f_+ + f_-] = \pi_*([f]).
\]
When \(\ker \pi_* \neq 0\), \([f], [f']\) could be different from a vanishing 2-cycle in \(\ker \pi_*\). For the application of our gluing formula to the main theorems, there is no vanishing cycle (Lemmas 2.11, 2.13, section 2). Hence, this problem does not arise.

However, the original statement of our general gluing formula is incorrect. Instead, our argument yields the following modified statement. Let \([A] = A + \ker \pi_*\). Then, we define 
\[
\Psi_{([A], \ldots)} = \sum_{B \in [A]} \Psi_{([B], \ldots)}.
\]
Note that for \(B, B' \in [A]\), \(\omega(B) = \omega(B')\). By the compactness theorem, there are only finitely many such \(B\) to be represented by stable \(J\)-holomorphic maps. Hence the summation on the right hand side is finite. By abuse of notation, we use \([A] = A^+ + A^-\) to represent the set of homology classes of glued maps. Then we replace \(\Psi_{([A], \ldots)}\) by \(\Psi_{([A], \ldots)}\) in all the statements in this section and the original proof is still valid.

The proof of our gluing formula is similar to the proof of the composition law of GW-invariants and has two steps. The first step is to define an invariant for \(M_\infty\) and prove that it is the same as the invariant of \(M_r\). Then, we write the invariant of \(M_\infty\) in terms of log invariants of \(M^\pm\).

We first construct a virtual neighborhood for \(M_\infty\). Suppose that \(\Sigma^\pm\) has \(l^\pm\) connected components \(\Sigma^\pm_i, i = 1, \ldots, l^\pm\) of genus \(g^\pm_i\) with \(m^\pm_i\) marked points and \(n^\pm_i\) ends. The moduli space \(\overline{M}_{[A]}(M_\infty, g, m)\) consists of the components indexed by the following data:

1. The combinatorial type of \((\Sigma^\pm, u^\pm)\): \(\{A^\pm_i, g^\pm_i, m^\pm_i, (k^\pm_1, \ldots, k^\pm_i)\}, i = 1, \ldots, l^\pm\);
2. A map \(\rho : \{p^+_1, \ldots, p^+_\nu\} \to \{p^-_1, \ldots, p^-_\nu\}\), where \((p^+_1, \ldots, p^+_\nu)\) denote the puncture points of \(\Sigma^\pm\).

Suppose that \(C^J_{\phi, m}\) is the set of indices. Let \(C \in C^J_{\phi, m}\). Denote by \(\mathcal{M}_C\) the set of stable maps corresponding to \(C\). The following lemma is obvious.

**Lemma 5.4** \(C^J_{\phi, m}\) is a finite set.

We use the same method as in the above subsection to construct a virtual neighborhood \(U_C\) for each component \(C\) and get a virtual neighborhood \((U, E, I)\) for \(\overline{M}_{[A]}(M_\infty, g, m)\) starting inductively
from the lowest stratum. But \( U \) is usually not a smooth manifold. To see this, we observe that the configuration space \( \overline{\mathcal{B}}_{[A]}(M_\infty, g, m) \) can be identified as

\[
\cup_C \overline{\mathcal{B}}_C(M_\infty, g, m).
\]

\( \overline{\mathcal{B}}_C(M_\infty, g, m), \overline{\mathcal{B}}_{C'}(M_\infty, g, m) \) may intersect each other at lower strata. Hence, \( U = \cup_C U_C \), where \( U_C \) is a virtual neighborhood of \( \mathcal{M}_C \). Note that \( \mathcal{M}_C \) and \( \mathcal{M}_{C'} \) may intersect each other, where the intersection corresponds to a stable map with some component in \( \mathbb{R} \times \tilde{M} \). Then, \( U_C \) may intersect each other. By our construction, \( U \) has the same stratification structure as that of \( \overline{\mathcal{M}}_{[A]}(M_\infty, g, m) \). Hence, for \( C \neq C' \), \( U_C \cap U_{C'} \) is a stratum of \( U_C, U_{C'} \) codimension at least 2.

The integration theory can be extended to such a space in an obvious fashion. Namely, we take the sum of integrals over each \( U_C \). Choose a Thom-form \( \Theta \) of \( E \). Then, we can define GW-invariants \( \Psi_{(M_\infty, [A], g, m)} \) using the same integral formula. We can also define GW-invariants \( \Psi_C \) for each component \( C \). By the same argument as in Section 4 we can show the convergence of the integrals. It is easy to see that

\[
(5.4) \quad \Psi_{(M_\infty, [A], g, m)} = \sum_{C \in \mathcal{C}_{[A]}(M_\infty, g, m)} \Psi_{C}.
\]

**Remark 5.5** It is easy to see that

(i) For \( C = \{A^+, g^+, m^+\} \), we have

\[
(5.5) \quad \Psi_C(\alpha^+) = \Psi_{(\mathcal{M}^+, \tilde{Z})}(A^+, g^+, m^+)(\alpha^+);
\]

(ii) For \( C = \{A^-, g^-, m^-\} \), we have

\[
(5.6) \quad \Psi_C(\alpha^-) = \Psi_{(\mathcal{M}^-, \tilde{Z})}(A^-, g^-, m^-)(\alpha^-).
\]

**Theorem 5.6** For any \( r, 0 < r < \infty \), we have

\[
(5.7) \quad \Psi_{(M_\infty, [A], g, m)} = \Psi_{(M_r, [A], g, m)}.
\]

**Proof:** For each \( r \) we have \( \overline{\mathcal{M}}_A(M_r, g, T_m) \), denoted by \( \overline{\mathcal{M}}_r \). We may construct a virtual neighborhood \((U_r, E_r, I_r)\). Put \( \mathcal{M}_{(r)} = \bigcup_r \mathcal{M}_r \times \{r\} \). Repeating the previous argument, we may construct a virtual neighborhood \((U_{(r)}, E_{(r)}, S_{(r)})\) such that \((U_1, E_1, I_1)\) is a virtual neighborhood at \( r = 1 \) and \((U_\infty, E_\infty, I_\infty)\) is a virtual neighborhood at \( r = \infty \). In general \( U_\infty \) is not a smooth boundary of \( U_{(r)} \). In order to apply Stokes’ Theorem, we need a clear description of neighborhoods of each lower stratum of \( U_\infty \) in \( U_{(r)} \). This is basically a gluing argument. We consider a simple case; the argument for the general case is similar. Suppose that \( U_C \cap U_{C'} = \mathcal{M}^{(+,0,-)} \), where

\[
\mathcal{M}^{(+,0,-)} = \{(a, b, d) \in \mathcal{M}^+_S(A^+, 1) \times \mathcal{M}_S^-(A^-, 1) | P^+(a) = P^-(b), P^+(b) = P^-(d)\}
\]

Suppose that \((\Gamma^+, \Gamma_0, \Gamma^-) \in \mathcal{M}^{(+,0,-)}\), where

\[
\Gamma^\pm = ((u^\pm, \Sigma^\pm, p^\pm), \eta^\pm),
\]

\[
\Gamma_0 = ((u_0, \Sigma_0, q^+, q^-), \eta_0).
\]
Let \( O_{(+,0,-)} \) be a neighborhood of \((\Gamma^+, \Gamma_0, \Gamma^-)\) in \( \mathcal{M}^{(+,0,-)} \). For any \((\theta, r_1, r_2)\) we glue \( M^+, \mathbb{R} \times \tilde{M} \) and \( M^- \) to get \( M_{(\theta, r_1, r_2)} \) with the following gluing formulas

\[
a^+ = a_0 + 4r_1, \quad a^- = a_0 + 4r_2
\]

\[
\theta^+ = \theta^- = \theta_0 + \theta \mod 1.
\]

Given a complex structure \( \xi = (\xi^+, \xi_0, \xi^-) \), we construct \( \Sigma(\xi, \nu, \mu) = \Sigma^{+ \# r_1} \Sigma^{0 \# r_2} \Sigma^- \) and a pre-gluing map \( u_{(\theta, r_1, r_2)} : \Sigma(\xi, \nu, \mu) \to M_{(\theta, r_1, r_2)} \) in a similar way as in Section 4 (Recall that we perturb \( u^\pm \) and \( u_0 \) only in the gluing domain). Then we use a similar method as in Section 4 to prove that a neighborhood of \((\Gamma^+, \Gamma_0, \Gamma^-)\) in \( U_{(\ell)} \) can be described as \( O_{(+,0,-)} \times N \), where \( N : S^1 \times (R_1, \infty) \times (R_2, \infty) \to \mathbb{R}^4 \) is a local hypersurface given by

\[
y = (e^{-r_1} \cos \theta, e^{-r_1} \sin \theta, e^{-r_2} \cos \theta, e^{-r_2} \sin \theta).
\]

Both

\[
O_{(+,0,-)} \times N(S^1 \times \{\infty\} \times (R_2, \infty))
\]

and

\[
O_{(+,0,-)} \times N(S^1 \times (R_1, \infty) \times \{\infty\})
\]

are its boundary. We show that the Stokes theorem still applies to such a space. Without loss of generality we consider a domain \( D, (0, 0, 0, 0) \in D \subset N \). The boundary \( \partial D \) consists of three parts:
1. \( \gamma_1 \): the boundary of \( D \) in the hypersurface \( N \); 1. \( \gamma_1 \): the boundary of \( D \) in the hypersurface \( N \); we assume it to be smooth;
2. \( \gamma_2 = D \cap N(S^1 \times \{\infty\} \times (R_2, \infty)) \);
3. \( \gamma_3 = D \cap N(S^1 \times (R_1, \infty) \times \{\infty\}) \).

We draw a small ellipsoid \( B(\epsilon_1, \epsilon_2) : \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \) around \((0, 0, 0, 0)\) in \( \mathbb{R}^4 \), then \( D - B(\epsilon_1, \epsilon_2) \) is a polyhedron (union of simplices) in Euclidean space, for which the Stokes Theorem applies. We use Stokes’ Theorem and then let \( \epsilon_i \to 0 \). To simplify notations we don’t consider \( O_{(+,0,-)} \). For any differential form \( \alpha \) we have

\[
\int_D d\alpha = \int_{\partial D} \alpha - \int_{N_\epsilon} \alpha
\]

where \( N_\epsilon \) denotes \( N \cap \partial B(\epsilon_1, \epsilon_2) \). If we can show that the integral is convergent as \( \epsilon_i \to 0 \), we can obtain

\[
\int_D d\alpha = \int_{\partial D} \alpha.
\]

We first let \( \epsilon_1 \to 0 \), then \( \epsilon_2 \to 0 \). By the estimate (4.53), (4.54) and the same argument as in the subsection 4.2 the convergence follows. This argument obviously works for \( O_{(+,0,-)} \times N \). Then we use Stokes’ Theorem. Theorem 5.6 is proved. \( \square \)

We derive a gluing formula for the component \( C = \{A^+, g^+, m^+, k; A^-, g^-, m^-, k\} \). For any component \( C \) we can use repeatedly this formula. Choose a homology basis \( \{\beta_0\} \) of \( H^*(Z, \mathbb{R}) \). Let \( (\delta_{ab}) \) be its intersection matrix.

**Theorem 5.7** Let \( \alpha^+_i \) be differential forms with \( \deg \alpha^+_i = \deg \alpha^-_i \) even. Suppose that \( \alpha^+_i|_Z = \alpha^-_i|_Z \) and hence \( \alpha^+_i \cup \alpha^-_i \in H^*(M^+ \cup Z \tilde{M}^-, \mathbb{R}) \). Let \( \alpha_i = \pi^*(\alpha^+_i \cup \alpha^-_i) \) (1.11). For \( C = \{A^+, g^+, m^+, k; A^-, g^-, m^-, k\} \), we have the gluing formula

\[
\psi_C(\alpha_1, ..., \alpha_{m^+ + m^-}) = k \sum_{\beta_0} \delta_{ab} \psi_{(A^+, g^+, m^+, k)}(\alpha^+_1, ..., \alpha^+_m, \beta_0) \psi_{(A^-, g^-, m^-, k)}(\alpha^-_m, ..., \alpha^-_{m^+ + m^-}, \beta_0).
\]

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where we use \( \Psi_{(A^\pm,g^\pm,m^\pm,k)}^{(M^\pm,Z)} \) to denote \( \Psi_{(A^\pm,g^\pm,T_{m^\pm})}^{(M^\pm,Z)} \).

**Proof:** We denote \( U_C = U|_C, U(A^\pm,k) = (S^\pm_e)^{-1}(0)/stb_{u^\pm}, \) and

\[
U(A^+,A^-) = \{(a,b) \in U(A^+,k) \times U(A^-,k) : P^+(a) = P^-(b) \}.
\]

There is a natural map of degree \( k \)

\[
Q : U_C \rightarrow U(A^+,A^-)
\]

defined by

\[
Q(a,b,n) = (a,b),
\]

and a map

\[
P : U(A^+,k) \times U(A^-,k) \rightarrow Z \times Z
\]

defined by

\[
P(a,b) = (P^+(a), P^-(b)).
\]

Note that

\[
\mathcal{M}_{S^e}(A^+,A^-) = P^{-1}(\Delta),
\]

where \( \Delta \subset Z \times Z \) is the diagonal. The Poincaré dual \( \Delta^* \) of \( \Delta \) is

\[
\Delta^* = \sum \delta_{ab} \beta_a \wedge \beta_b.
\]

Choose a Thom form \( \Theta = \Theta^+ \wedge \Theta^- \), where \( \Theta^\pm \) are Thom forms in \( F^\pm \) supported in a neighborhood of the zero section. By perturbing \( \alpha_i^\pm \) we may assume that \( \alpha_i \) are smooth forms. Then

\[
\Psi_C(\alpha_1, \ldots, \alpha_{m^+ + m^-}) = \int_{U_C} \prod_{i=1}^{m^+} \alpha_i \wedge \prod_{j=m^+}^{m^+ + m^-} \alpha_j \wedge I^*\Theta
\]

\[
= k \int_{U(A^+,A^-)} \prod_{i=1}^{m^+} \alpha_i \wedge \prod_{j=m^+}^{m^+ + m^-} \alpha_j \wedge I^*\Theta
\]

\[
= k \int_{U(A^+,k) \times U(A^-,k)} \sum \delta_{ab} \prod_{i=1}^{m^+} \alpha_i^+ \wedge I^*\Theta^+ \wedge \beta_a \wedge \prod_{j=m^+}^{m^+ + m^-} \alpha_j^- \wedge I^*\Theta^- \wedge \beta_b
\]

\[
= k \sum \delta_{ab} \Psi_{(A^+,g^+,m^+,k)}^{(M^+,Z)}(\alpha_1^+, \ldots, \alpha_{m^+}^+, \beta_a) \Psi_{(A^-,g^-,m^-,k)}^{(M^-,Z)}(\alpha_{m^+}^-, \ldots, \alpha_{m^++m^-}^-, \beta_b). \quad \Box
\]

For general \( C \) with \((k_1, \ldots, k_l)\)-periodic orbits we may easily obtain

**Theorem 5.8**

\[
(5.9) \quad \Psi_C(\alpha) = k \sum_{I,J} \Psi_{(A^+,g^+,m^+,k)}^{(M^+,Z)}(\alpha_I^+, \beta_I) \delta_I^J \Psi_{(A^-,g^-,m^-,k)}^{(M^-,Z)}(\alpha^-_J, \beta_J),
\]

where we associate \( \beta_i \delta_I^J \beta_j \) to every periodic orbit as in Theorem 5.7, and put \( |k| = k_1 \ldots k_l, \delta_I^J = \delta_i^1 \delta_j^1 \ldots \delta_i^l \delta_j^l \), and denote by \( \Psi_{(A^\pm,g^\pm,m^\pm,k)}^{(M^\pm,Z)}(\alpha^\pm, \beta_J) \) the product of log invariants corresponding to each component.
For example, for $C = \{ A^+, g^+, m^+, k_1, k_2; A_1^-, g_1^-, m_1^-, k_1, A_2^-, g_2^-, m_2^-, k_2 \}$, our formula (5.9) reads:

$$\Psi_C(\alpha) = k_1k_2 \sum_{i_1, i_2, j_1, j_2} \psi^{(M^+, Z)}_{(A^+, g^+, m^+, k_1, k_2)}(\alpha^+, \beta_{i_1}, \beta_{i_2}) \delta_{i_1, j_1} \delta_{i_2, j_2} \psi^{(M^-, Z)}_{(A^-, g^-, m^-, k_1, k_2)}(\alpha^-, \beta_{j_1}) \psi^{(M^-, Z)}_{(A^-, g^-, m^-, k_2)}(\alpha^-, \beta_{j_2}).$$

**Remark 5.9:** The definition of log invariants and gluing formula extend to the case that $Z$ is a disjoint union of smooth codimension two symplectic submanifolds in an obvious fashion. Furthermore, if symplectic cutting only obtains one symplectic manifolds, gluing formula also extends to this case in an obvious fashion.

### 6 Proofs of the Main Theorems

**Proof of Theorem A**

Let $M$ be a 3-fold and $M_f$ be obtained by a flop. By Proposition 2.9, after a perturbation of complex structures to almost complex structures, $M, M_f$ have the same blow-up $M_b$. Without the loss of generality, we can assume that $M_b$ is the blow up along a $O(-1) + O(-1)$ rational curve $\Gamma \subset M$ or $\Gamma_f \subset M_f$. By proposition 2.10, we can relate $M, M_b$ by a symplectic cutting such that $M^+ = M_b$, and $M^- = M$. The same procedure applies to $M_f$. We have

$$\overline{M^+} = \overline{M_f^+} = P(\mathcal{O}(-1) + \mathcal{O}(-1) + \mathcal{O}),$$

$$\overline{M^-} = \overline{M_f^-} = M_b.$$

Now, we compare GW-invariants of $M, M_f$ with log GW-invariants of $M_b$. By Lemma 2.11, there is no vanishing 2-cycle in our cases. Now we use the gluing formula (5.8) to prove our assertion. First, we assume that $\deg(\alpha_i) \geq 4$ and is Poincaré dual to a point or a 2-dimensional homology class $\Sigma$. Then, we can choose the pseudo-submanifold representative of $\Sigma$ such that $\Sigma$ is in $M^-$. Hence, $\alpha_i$ is supported in $M^-$. For simplicity we consider a special component:

$$C = \{ A^+, g^+, m^+, k; A^-, g^-, m^-, k \}.$$

The general case can be treated in the same way.

The following argument depends only on an index calculation. The index has an additive property. If a relative stable map has more than one component, we can always construct a pre-gluing map and use the index of the pre-gluing map. Hence, we can assume that the stable map under consideration has only two components $(u^+, u^-)$, where $u^\pm$ is a $J$-map in $M^\pm$.

Since $M$ is a 3-fold, using the addition formula (5.2) for the index we have

$$\text{Ind}(D_{u^-}, \alpha) + \text{Ind}(D_{u^+}, \alpha) - 4 = \text{Ind}(D_u).$$

By (5.3),

$$\text{Ind}(D_{u^+}, \alpha) = \text{Ind}(D_{\bar{u}^+}).$$
Note that $\overline{M}^+ = P(\mathcal{O}(-1) + \mathcal{O}(-1) + \mathcal{O})$. Next we claim that

\[(6.5) \quad \text{Ind}(D_{u^+}) \geq 6.\]

Note that $\bar{u}^+$ can be identified as a holomorphic curve $h$ in $P(\mathcal{O}(-1) + \mathcal{O}(-1) + \mathcal{O})$ over $\mathbb{P}^1$ by remark 3.24. Then,

\[(6.6) \quad \text{Ind}(D_h) = 2(C_1([h]) - k + 1),\]

where $C_1(M^+)$ is represented precisely by $3\mathcal{Z}_\infty$, where $\mathcal{Z}_\infty$ is the infinity divisor of $P(\mathcal{O}(-1) + \mathcal{O}(-1) + \mathcal{O})$. A simple index calculation shows that $C_1([h]) = 3k$. Hence, if $k > 0$

\[\text{Ind}(D_{u^+}) \geq 6.\]

From (6.3)-(6.5) we have

\[(6.7) \quad \text{Ind}(D_{u^+}, \alpha) \leq -2 + \text{Ind}(D_u).\]

Hence $\text{Ind}(D_{u^+}, \alpha) < \text{Ind}(D_u)$. However, the representatives of $\alpha_i$ (hence $\phi^*(\alpha_i)$) is supported in $M^+$. It is clear that $\alpha_i^+ = 0$ and the only nontrivial terms in the gluing formula are

\[k \sum \delta_{ab} \Psi_{(A^+, g^+, m^+, k)}(\beta_a) \Psi_{(A^-, g^-, m^-, k)}(\alpha_i^-), \beta_b).\]

However, $\sum_i \deg(\alpha_i) = \text{Ind}(D_u) > \text{Ind}(D_{u^+})$. Then for any $\beta_b$,

\[\Psi_{(A^-, g^-, m^-, k)}(\alpha_i^-), \beta_b) = 0.\]

If $u^\pm$ has more than one end, say $\nu$ ends, $\text{Ind}(D_{u^+})$ increases faster than $4\nu$. It will force $\text{Ind}(D_{u^+})$ to become even more negative. It follows that $\Psi_C = 0$ except for $C = \{A^+, g, m\}$ or $C = \{A^-, g, m\}$ i.e., $C$ stays completely on the one side.

Suppose that $A \neq n[\Gamma]$. We claim that $C = \{A^-, g, m\}$. If $C = \{A^+, g, m\}$, $A^+$ is represented by a stable map whose image is completely inside the total space of $\mathcal{O}(-1) + \mathcal{O}(-1)$. Then, it must be homologous to $n[\Gamma]$ for some $n$. Hence, $\pi_k(A) = \pi_k(n[\Gamma])$, where $\pi_k$ is defined in (2.3). However, there is no vanishing 2-cycle. This contradicts the assumption that $A \neq n[\Gamma]$. Hence,

\[(6.8) \quad \Psi^M_{(A, g, m)}(\{\phi^*(\alpha_i)\}) = \Psi^M_{(A^-, g, m)}(\{\alpha_i^-\}).\]

It is easy to observe that $A^- = \phi_*(A)$, where $\phi : H_2(M, Z) \to H_2(M_b, Z)$ is defined in (2.14). The same argument shows that

\[(6.9) \quad \Psi^M_{(\phi(A), g, m)}(\{\alpha_i^+\}) = \Psi^M_{(\phi_*(A), g, m)}(\{\alpha_i^-\}).\]

Hence,

\[(6.10) \quad \Psi^M_{(A, g, m)}(\{\phi^*(\alpha_i)\}) = \Psi^M_{(\phi(A), g, m)}(\{\alpha_i\}).\]

When $\deg(\alpha_i) = 2$, we can use (1.1) to reduce it to the previous case.

When $A = n[\Gamma]$, we do not need the gluing formula. By the assumption, $\Gamma$ generates an extremal ray. If $f$ is a stable map representing $A$, $\text{im}(f) = \Gamma$. $\Gamma, \Gamma_f$ have isomorphic neighborhoods as complex manifolds. It follows from the definition that

\[\Psi^M_{(n[\Gamma], g)} = \Psi^M_{(n[\Gamma_f], g)}.\]
Remark 8.1: Although it is not needed in our proof, a similar gluing argument can show that

\[(6.11) \quad \Psi_{(\phi_b(A),9)}^{(M_b,Z)}(\{\alpha_i^+\}) = \Psi_{(\phi_b(A),9)}^{M_b}(\{\alpha_i^-\})\]

Only Corollary A.2 needs a proof. The others are immediate consequences of Theorem A and Corollary A.2.

Proof of Corollary A.2

For simplicity, we assume that all the exceptional curves are in the same homology class \([\Gamma]\). Let \(n_\Gamma = n_{\Gamma_f}\) be the number of such curves, \([\Gamma]\) and \([\Gamma_f]\) be the homology classes of the exceptional curves respectively in \(M\) and \(M_f\). Then for any \(0 < k \in \mathbb{Z}\), by the formula for multiple cover maps,

\[\Phi_{(k[\Gamma],0)}^M = \frac{n_{\Gamma_f}}{n_\Gamma} \quad \text{(see [V]).}\]

Then the total 3-point function can be written in the form

\[(6.12) \quad \Psi^M(\beta_1,\beta_2,\beta_3) = \beta_1 \wedge \beta_2 \wedge \beta_3 + \sum_{A \neq k[\Gamma]} \Psi_{(A,0,3)}^M(\beta_1,\beta_2,\beta_3)q^A \]

As for the last term, it is zero except that \(deg(\beta_i) = 2\) for the dimension reasons. If \(deg(\beta_i) = 2\),

\[(6.13) \quad \sum_{n[\Gamma],n \neq 0} \Psi_{(A,0,3)}^M(\beta_1,\beta_2,\beta_3)q^{n[\Gamma]} = \frac{q^{[\Gamma]} \beta_1([\Gamma])\beta_2([\Gamma])\beta_3([\Gamma])n_{\Gamma}}{1 - q^{[\Gamma]}\beta_1([\Gamma])\beta_2([\Gamma])\beta_3([\Gamma])n_{\Gamma}}.\]

As for the first term, it is also zero except that \((deg(\beta_1), deg(\beta_2), deg(\beta_3)) = (2,2,2), (4,2,0), (6,0,0)\). There is a similar expression for the 3-point function of \(M_f\). We see that, by Theorem A, only the first and last terms are different under identification \(q^A \rightarrow q^{\phi(A)}\). Suppose that \(\beta_i = \varphi^*\alpha_i\). Let’s first consider the case that \((deg(\beta_1), deg(\beta_2), deg(\beta_3)) = (4,2,0), (6,0,0)\). In these cases, the last terms are zero. We claim that the first terms are the same as well. This is obvious for the case of \((6,0,0)\). For the case \((4,2,0)\), we can assume that \(\alpha_3 = 1\). By definition, the map \(H^4(M_f, \mathbb{R}) \rightarrow H^4(M, \mathbb{R})\) is Poincaré dual to the inverse of the map \(H_2(M, \mathbb{Z}) \rightarrow H_2(M_f, \mathbb{Z})\). By our construction, \(\alpha_1, \alpha_2\) are Poincaré dual to the same 2-manifold \(\Sigma\) disjoint from the exceptional locus. Hence,

\[\varphi^*\alpha_2 \wedge \varphi^*\alpha_1 = \varphi^*\alpha_2(\Sigma) = \alpha_2(\Sigma) = \alpha_2 \wedge \alpha_1.\]

For the case \((2,2,2)\), both the first term and last term are non-zero. Let \(p : M_b \rightarrow M_f; p_f : M_b \rightarrow M_f\) be the projection. We observe that \(p_f^*\alpha_i = -p^*\beta_i = -\alpha_i(\Gamma_f)Z\), where \(Z\) is the exceptional divisor. A routine calculation shows that

\[(6.14) \quad \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \beta_1 \wedge \beta_2 \wedge \beta_3 = \alpha_1(\Gamma_f)\alpha_2(\Gamma_f)\alpha_3(\Gamma_f).\]

For \(q^{[\Gamma_f]} \neq 1\),

\[(6.15) \quad \frac{q^{[\Gamma_f]}}{1 - q^{[\Gamma_f]}} + \frac{q^{-[\Gamma_f]}}{1 - q^{-[\Gamma_f]}} = -1.\]

Recall that

\[(6.16) \quad \beta_1(\Gamma)\beta_2(\Gamma)\beta_3(\Gamma) = -\alpha_1(\Gamma_f)\alpha_2(\Gamma_f)\alpha_3(\Gamma_f)\]

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and

\[ \phi([\Gamma]) = -[\Gamma_f]. \]

We have

\[ \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \frac{q^{[\Gamma_f]}}{1 - q^{[\Gamma_f]}} \alpha_1(\Gamma_f)\alpha_2(\Gamma_f)\alpha_3(\Gamma_f)n_{\Gamma_f} = \beta_1 \wedge \beta_2 \wedge \beta_3 + \frac{q^{\phi([\Gamma])}}{1 - q^{\phi([\Gamma])}} \beta_1(\Gamma)\beta_2(\Gamma)\beta_3(\Gamma)n_{\Gamma_f}. \]

The assertion follows. \(\square\)

**Proof of Theorem B**

The proof is similar to that of Theorem A. By Proposition 2.12, we can perform a symplectic cutting such that one part \(\overline{M_e}^-\) is \(M_b\). The other part \(\overline{M_e}^+\) is a collection of quadric 3-folds. Without the loss of generality, we can assume that \(\overline{M_e}^+\) has only one component. Again, we first argue the case that \(\text{deg}(\alpha_i) \geq 4\), where we choose the support of \(\alpha_i\) inside \(M^-\). By a simple index calculation, if \(h\) is a holomorphic curve of \(Y\) tangent to the infinity divisor with order \(k\), its index is \(2(3k - k + 1) \geq 6\) if \(k > 0\). By the same argument as in Theorem A we conclude that we need only consider those \(J\)-holomorphic curves which don’t go through \(\tilde{M}\). In this case, there is no holomorphic curves in \(\overline{M_e}^+\) disjoint from \(Z\). This shows that for \(B \neq 0\)

\[ \Psi_{B_{g,m}}^{M_e}(\{\alpha_i\}) = \sum_{A^-} \Psi_{A^-_{g,m}}^{M_b,Z}(\{\alpha_i^\pm\}), \]

where summation is taken over \(A^-\) homologous to \(\pi_*(A)\). A moment of thought tells us that these are the set of \(\phi_b(A)\) such that \(\phi_c(A) = B\). Then, the Theorem B follows from (6.19) and (6.8). Then, we use formula (1.1) to reduce the case \(\text{deg}(\alpha_i) = 2\) to the previous case. \(\square\)

Only Corollary B.2 needs a proof. The others are immediate consequences of Theorem B and Corollary B.2.

**Proof of Corollary B.2**

The proof is a generalization of Tian’s argument [T]. The surjective map

\[ \varphi : H_2(M, \mathbb{R}) \to H_2(M_e, \mathbb{R}) \]

induces an injective map

\[ \varphi^* : H^2(M_e, \mathbb{R}) \to H^2(M, \mathbb{R}). \]

By definition, the map on \(H^4\) is Poincaré dual to a right inverse of (6.20). We claim that the ordinary cup product remains the same after transition. Assume that \(\beta_i = \varphi^*\alpha_i\). Again, we need to consider the case that \((\text{deg}(\beta_1), \text{deg}(\beta_2), \text{deg}(\beta_3)) = (2, 2, 2), (4, 2, 0), (6, 0, 0)\). The case \((6, 0, 0)\) is obvious. The proof of the case \((4, 2, 0)\) is similar to that of Corollary of Theorem A. \(\beta_1\) is Poincaré dual to \(A_1 \in H_2(M, \mathbb{R})\) such that \(\alpha_1\) is Poincaré dual to \(\varphi_*(A_1)\). Hence,

\[ \varphi^*\alpha_2 \wedge \varphi^*\alpha_1 = \varphi^*\alpha_2(A_1) = \alpha_2(\varphi_*(A_1)) = \alpha_2 \wedge \alpha_1. \]

For the case \((2, 2, 2)\), clearly \(\varphi^*(\beta_i)(\Gamma) = 0\). Without loss of generality, assume that there is a 4-manifold representing \(\varphi^*(\beta_i)\) which is disjoint from \(\Gamma\). Hence, it can be viewed as a submanifold of \(M_e\). Clearly, the same 4-manifold represents \(\beta_i\). Hence,

\[ \beta_1 \wedge \beta_2 \wedge \beta_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3. \]
Therefore,

\[(6.24) \quad \Psi^M_{\varphi^*w}(\varphi^*(\beta_1), \varphi^*(\beta_2), \varphi^*(\beta_3)) = \varphi^*(\beta_1) \wedge \varphi^*(\beta_2) \wedge \varphi^*(\beta_3) \]

\[+ \sum_{A \neq k[\Gamma]} \sum_{m} \frac{1}{m!} \Psi^M_{(A,0,m+3)}(\varphi^*(\beta_1), \varphi^*(\beta_2), \varphi^*(\beta_3), \varphi^*w, \cdots, \varphi^*w)q^A \]

\[+ \sum_{k[\Gamma], k \neq 0} \sum_{m} \frac{1}{m!} \Psi^M_{(A,0,m+3)}(\varphi^*(\beta_1), \varphi^*(\beta_2), \varphi^*(\beta_3), w, \cdots, w)q^{k[\Gamma]}.\]

By the previous argument, the first term is the same as \(\alpha_1 \wedge \alpha_2 \wedge \alpha_3\). The last term is always zero.

Now we change the formal variable by \(q^A \rightarrow q^{e^*(A)}\) and apply Theorem B. Then we prove

\[(6.25) \quad \Psi^M_{\varphi^*w}(\varphi^*(\alpha_1), \varphi^*(\alpha_2), \varphi^*(\alpha_3)) = \Psi^M_{w}(\alpha_1, \alpha_2, \alpha_3).\]

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