Population kinetics of many-electron atoms in ionizing plasmas studied using a continuous collisional radiative model

Akira Nishio, Julian C. Berengut, Masahiro Hasuo, and Keisuke Fujii

1 Department of Mechanical Engineering and Science, Graduate School of Engineering, Kyoto University, Kyoto 615-8540, Japan
2 School of Physics, University of New South Wales, New South Wales 2052, Australia

(Dated: June 15, 2020)

Collisional–radiative (CR) models based on ab initio atomic structure calculation have been utilized over 20 years to analyze many-electron atomic and ionic spectra. Although the population distribution of the excited states in plasmas and their emission spectra are computed using the CR models, systematic and analytical understanding of the population kinetics are still lacking. In this work, we present a continuous CR model (CCRM), in which we approximate the dense energy structure of complex many-electron atoms by a continuum. Using this model, we predict asymptotic population distributions of many-electron atoms in plasmas and their electron-density and temperature dependence. In particular, the CCRM shows that the population distribution of highly excited states of many-electron atoms in plasmas resembles a Boltzmann distribution but with an effective excitation temperature. We also show the existence of three typical electron-density regions and two electron-temperature regions where the parameter dependence of the excitation temperature is different. Analytical representations of the effective excitation temperature and the boundaries of these phases are also presented.

I. INTRODUCTION

The spectra of many-electron atomic ions can be seen in various optically thin plasmas. In the stellar atmosphere, neutral and singly charged iron (Fe) are the dominant components in the absorption spectra in terms of the number of lines [1]. Many of Fe absorption lines have been identified to study the stellar atmosphere [2-4]. Highly charged tin and actinoide ions play an important role in realizing ultraviolet light sources [5-7], in which quasi-continuum emission in laser-produced plasmas is used. Since the radiative power should be concentrated into a particular energy region for the commercial light source realization, many works have been carried out to understand the population dynamics in plasmas [5-7]. In fusion tokamak plasmas, highly charged tungsten ions convert electron kinetic energy to strong radiation and therefore need to be controlled [8,9]. The thermalization process of the nuclei in kilonovae, which has recently been probed from the emission of neutral transition metals, is yet to be understood [10,11].

The collisional–radiative (CR) model is the key tool to study the population kinetics of many-electron atoms in plasmas and their emission and absorption spectra. This model solves the steady state equation of the excited state population of ions by taking into account the rates of elementary processes in plasmas. In order to perform accurate predictions, accurate atomic data are required, i.e., energy levels and transition rates of many elementary processes, including electron-impact excitations and spontaneous transitions. Therefore, many works have been dedicated to develop and improve ab initio calculation of these atomic data [12-14]. Although this first-principles approach has been successful in many cases [7,9,13,16], accurate calculations are still difficult and computationally demanding, particularly for many-valence-electron atoms and ions. This difficulty comes from their strong wavefunction mixing, which requires an unacceptably large Hilbert basis space to represent their wavefunctions. Due to the complexity and difficulty of the first-principle computation of the atomic structure, it is difficult to understand and validate the CR model result.

A probabilistic model may provide a complementary approach to the first-principle calculation. For a system with sufficiently strong mixing of basis states, i.e., systems exhibiting many-body quantum chaos, it is known that some of the properties of their atomic structure can be represented using a statistical theory [17]. Although its applications to plasma diagnostics are very limited, we have recently shown that the intensity statistics of many-electron atoms can be understood from this structure, and can be used to measure electron temperature in plasmas [18]. Since the use of probabilistic nature of many-electron atoms requires only a small amount of atomic data, this approach is not only robust against possible numerical errors, but also gives us a systematic insight of the population kinetics in plasmas.

In this work, we develop a continuous CR model (CCRM), in which we approximate the dense energy levels of many-electron atoms by a continuum based on the statistical theory. For our simplified theory, only two atomic parameters are used to represent the spectrum: the energy scale of the level density growth; and another energy scale that describes the decay of transition strengths. Based on this model, we will show that the population distribution of highly excited states of many-electron atoms is similar to Boltzmann’s distribution but with an effective excitation temperature $T_{\text{ex}}$. The dependence of this excitation temperature on electron density ($n_e$) and temperature ($T_e$) are then studied using the CCRM, revealing the existence of three typical $n_e$ regions.
and two $T_e$ regions. In particular, it is shown that in low $T_e$ regions, the excitation temperature becomes almost $T_e$ even in low $n_e$ plasmas. This property indicates much wider applicability of the Boltzmann’s method, which is a well-known method to estimate $T_e$ values from the emission spectra in high $n_e$ plasmas. It also indicates the wider applicability of the new temperature diagnostics based on the line intensity statistics, in which the Boltzmann population distribution is assumed. The population kinetics is also compared with that of hydrogen (H)-like ions, which has been extensively studied as shown in Fig. 1 (a) (see also Appendix A for details).

In section II, we briefly describe the principle of the CR model and show some simulation results obtained using an ab initio calculation code for several many-electron atomic ions. In section III, we present our CCRM to study the population kinetics of many-electron atoms and compare it with the ab initio simulation result. In section IV, we discuss its parameter dependence.

II. CRM FOR MANY-ELECTRON ATOMS

A. Principle of the CR Model

The CR model is a balance equation of the population of excited states in plasmas. In optically thin plasmas, the dominant excitation/de-excitation processes are radiative decay and electron-impact excitation and de-excitation. The temporal evolution of the population in an excited state $p$ can be written as

$$\frac{dn_p}{dt} = A_p^{in} - A_p^{out} + C_p^{in} - C_p^{out} + F_p^{in} - F_p^{out},$$

where $A_p^{in}$ and $A_p^{out}$ are the population influx from upper levels to state $p$ and the outflux from state $p$ to lower levels by spontaneous decay, respectively. Similarly, $C_p^{in}$ and $C_p^{out}$ are the influx and the outflux by electron-impact excitation, respectively, and $F_p^{in}$ and $F_p^{out}$ are the influx and the outflux by electron-impact de-excitation, respectively. Except for extreme cases, the time scale of the excited state population is very short compared with that of the bulk plasma parameters (e.g., $n_e$ and $T_e$). Therefore, the steady state of the population of excited states can be assumed, $dn_p/dt = 0$.

Many other elementary processes can be included in Eq. (1), such as ionization, recombination, photoionization, and photoexcitation. For highly charged ions, dielectronic recombination and auto-ionization may be important. However, for simplicity, we mainly focus on electron-impact excitation, de-excitation, and spontaneous decay in this work.

The excited state population is divided into two components, i.e., ionizing and recombining plasma components, depending on whether the population contributions of the ground state or the next ionized stage are dominant. However, in this work, we neglect the ionization and recombination processes, and therefore only consider the ionizing plasma component.

Each term in Eq. (1) is explicitly written as follows:

$$A_p^{in} = \sum_{q>p} A_{pq}^{in} n_q, \quad A_p^{out} = \sum_{q<p} A_{qp}^{out} n_p$$

$$C_p^{in} = \sum_{q<p} C_{pq}^{in} n_q n_e, \quad C_p^{out} = \sum_{q>p} C_{qp}^{out} n_p n_e$$

$$F_p^{in} = \sum_{q<p} F_{pq}^{in} n_e n_q, \quad F_p^{out} = \sum_{q>p} F_{qp}^{out} n_e n_p$$

where $A_{pq}^{in}$ is the spontaneous transition rate from $q$ to $p$ states, $C_{pq}^{in}$ is the electron-impact excitation rate coefficient from $q$ state to $p$ state, and $F_{pq}^{in}$ is the deexcitation rate coefficient. Here, $\sum_{q<p}$ and $\sum_{q>p}$ indicate the summation over states $q$ with higher and lower excited energies than the energy of $p$, respectively. Then, Equation (1) becomes a linear equation of $n_p$, which can be solved if we know all the rates.

The radiative transition rate $A_{pq}$ is related to the line strength $S_{pq}$ between $p$ and $q$ states. The transition rate by electric dipole transitions, which are almost always dominant, can be written as

$$A_{pq} = \frac{1}{g_q} \frac{\omega_{pq}^3}{\pi} S_{pq}$$

with

$$\gamma = \frac{4}{3} \frac{e^4}{a_0^2 \alpha^2} \frac{1}{E_H^2} \frac{1}{e^2 a_0^2}$$

where $\omega_{pq} = E_q - E_p$ is the energy difference between states $p$ and $q$, $g_q$ is the statistical weight of state $q$, $E_H \approx 27.2$ eV is the Hartree energy, $\alpha$ is the fine structure constant, $e$ is the elementary charge, $a_0$ is the Bohr radius, and $c$ is the light speed.

The electron-impact excitation rate coefficient is more complicated. Various methods of its approximation are available, such as Born method, close-coupling method, and distorted-wave approximation. One of the simplest approximations is as follows:

$$C_{pq} \approx \frac{1}{g_q} \frac{\beta}{\sqrt{E_T}} S_{pq} \exp \left[ -\frac{\omega_{pq}}{kT_e} \right],$$

where $k$ is the Boltzmann constant. Several authors have proposed this simple form. Griem and Fujimoto et al. used

$$\beta = \frac{2^{5/2}}{3} \frac{\pi^{3/2} e^3}{\alpha a_0^2} \sqrt{E_H} \frac{1}{\epsilon^2 a_0^2},$$

Mewe has proposed a slightly different form

$$\beta = \frac{2^{7/2}}{3\sqrt{3}} \frac{\pi^{3/2} e^3}{\alpha a_0^2} \sqrt{E_H} \frac{1}{\epsilon^2 a_0^2} \xi,$$
Since the electron-impact de-excitation is the inverse process of the excitation, the following relation can be deduced from the detailed balance principle,

$$F_{p \leftarrow q} = \frac{g_p}{g_q} C_{q \rightarrow p} \exp \left[ \frac{\omega_{pq}}{kT_e} \right], \quad (8)$$

where $F_{p \leftarrow q}$ is the electron-impact de-excitation rate coefficient.

### B. Flexible Atomic Code

There are several integrated packages of the atomic structure calculation and CR model for studying many-electron atom spectra [12,13]. One common package is Flexible Atomic Code (FAC) [12]. In the FAC, the electron wavefunction of a many-electron atom is approximated by a linear combination of single-body product wavefunctions. Their mixing coefficients are calculated based on the configuration interaction method, and the line strengths are computed from these mixing coefficients. The electron-impact excitation cross sections are computed via distorted wave approximation, which is believed to be more accurate than Eq. (5) [12]. The electron-impact ionization cross sections and autoionization rate coefficients are also computed using the FAC. Using these cross sections and rates, Eq. (1) is solved, and the population distribution is simulated with given pairs of $n_e$ and $T_e$.

As examples, we present in this work simulation results for neutral iron (FeI), manganese-like iron (FeII), and nickel-like krypton (KrIX), which have different numbers of electrons. The details of the configurations included in each calculation are presented in Appendix B. We start by presenting the level density. The vertical bars in Fig. 2 (a), (b), and (c) show the energy levels calculated with FAC. For comparison, the energy levels compiled in Atomic Spectra Database by National Institute of Standards and Technology (NIST ASD [24]) are also shown in Fig. 2 by vertical bars.

The population distributions normalized by the statistical weight with several $T_e$ and $n_e$ combinations are shown in Fig. 3. In high-electron density plasmas, the distributions become closer to the Boltzmann distribution.

$$n_p/g_p \propto \exp \left( -\frac{E_p}{kT_e} \right), \quad (9)$$

where $E_p$ is the excited energy of state $p$. In low-electron density plasmas, the distribution deviates from Eq. (9), but it still decreases exponentially against the excited energy particularly in highly excited states. In lower density plasma, the slope becomes steeper and the scatter becomes bigger.
FIG. 2. Level densities of (a) FeI, (b) FeII (Mn-like Fe), and (c) KrIX (Ni-like krypton). Actual energy levels compiled in NIST ASD and computed using the FAC are shown by vertical bars in each figure. The gray step lines are histograms for the computed energy levels with three different bin sizes. The black solid lines are fit by the constant-temperature model (Eq. (10)). The vertical dashed lines indicate the first ionization energy.

FIG. 3. Population distribution \( n/g \) of (a) FeI, (b) FeII, and (c) KrIX computed using the FAC. The upper panels show the computation with \( T_e \) values smaller than \( \epsilon_0 \), and the lower panels show those with larger values. Three different colored dots in each panel show the results with different \( n_e \) values. In higher density, the population distribution stays on one line, Eq. (9). At lower density, the population is scattered around an exponentially decaying line (a straight line in the semilogarithmic plot) with a steeper slope. The gray solid lines in each panel show the result of our simplified model with Eqs. (16)–(23) with the same \( T_e \) and \( n_e \) values. Note that the scales of these lines are adjusted to fit those of the FAC result, and therefore, only the slope is important.

III. CONTINUOUS CR MODEL FOR MANY-ELECTRON ATOMS

In this work, we present an analytic form of Eq. (1), using the statistical theory of the atomic structure of many-electron atoms. In particular, we assume the following two properties:

1. exponential increase in the level density over the excited energy, and

2. independently and identically distributed line strength.

These probabilistic assumptions, as well as the huge number of energy levels and transitions, allow us to approximate the summations in Eq. (2) as integrals.

In subsection IIIA we present the details of the probabilistic assumptions. In subsection IIIB we construct the CCRM by approximating the summations in Eq. (2) by integrals. In subsection IIIC we focus on highly excited states, which allows us to further simplify the CCRM.
A. Atomic Structure Approximation

1. Level density of many-electron atoms

It is known that in fermionic many-body systems, the level density $\rho(E)$, the number of excited states per unit energy, has a nearly exponential dependence on the excited energy. Step lines in Fig. 2(a), (b), and (c) show the level densities of FeI, FeII, and KrIX, respectively, which are computed from the simulated energy levels using the FAC. In order to present an overall dependence without the finite bin-size effect, we show three histograms with different bin sizes for each atom. It is clear that the level density of these many-electron atoms increases nearly exponentially.

Two most common models of the level densities of another fermionic many-body system, heavy nuclei, are back-shifted fermi gas model and constant-temperature model \[26, 27\]. Applications of these models to the level densities of many-electron atoms have been also reported by several authors \[17, 28\]. In the constant-temperature model, the level density is expressed as follows:

$$\rho(E) = \rho_0 \exp \left( \frac{E}{\epsilon_0} \right),$$ \hspace{1cm} (10)

where $\epsilon_0$ is an energy scale parameter indicating the inverse of the level density increase rate. This value in principle can be estimated based on the number of valence electrons and shell-separation energy \[28\].

The solid lines in Fig. 2 show the fit using the constant-temperature model, where $\rho_0$ and $\epsilon_0$ are adjusted so that Eq. (10) matches the computed histogram. The histograms shown in the figure are in good agreement with Eq. (10), particularly in the highly excited energy regions.

Figure 4 shows the ionization-energy dependence of the scale parameter $\epsilon_0$ for several atomic ions. In the figure, the computed values of $\epsilon_0$ for neutral and singly charged ions of transition metals and the isoelectronic sequence of Fe-like and Ni-like ions are shown. The first ionization energy data $\chi$ are taken from NIST ASD \[25\]. $\epsilon_0/\chi$ is almost constant over the wide variety of atoms and charges.

$$\frac{\epsilon_0}{\chi} \approx 0.2$$ \hspace{1cm} (11)

may be a good empirical approximation for transition metals. Atomic parameters for the atoms used in this work are presented in Table I. Note that the uncertainty of $\epsilon_0$ is $\approx 20\%$, which originates from the variation of the FAC computation result over the change in hyperparameters, e.g. the number of basis states and central potential.

2. Line strength distribution

The line strength distribution of fermionic many-body systems has been approximated as independent and iden-

| Ion      | $\epsilon_0$ (eV) | $\sigma$ (eV) | $\chi$ (eV) |
|----------|-------------------|---------------|-------------|
| FeI      | 1.6               | 0.62          | 7.902       |
| FeII     | 3.9               | 2.3           | 16.2        |
| KrIX     | 36                | 23            | 233.0       |

FIG. 4. The scale factors of the level density ($\epsilon_0$) and line strength ($\sigma$) for several atomic ions. The values are normalized by their first ionization energy $\chi$. Data points around $\chi \approx 8$ eV are for neutral transition metals from Sc to Ni, and those around $\chi \approx 15$ eV are for singly charged ions of transition metals. The other five points are the isoelectronic sequence of Fe and Ni.
the oscillator strength,

\[ N = \sum_j f_{i\rightarrow j} \]

\[ \approx \int_0^E f(E, E - \omega)\rho(E - \omega)d\omega \]

\[ + \int_0^\infty f(E, E + \omega)\rho(E + \omega)d\omega, \quad (13) \]

where \( N \) is the number of electrons in the atom, and \( f(E, E + \omega) \) is the oscillator strength from the state at \( E \) to that at \( E + \omega \). With \( f \propto S \omega \propto \omega_1 \), the second integral in the right hand side of Eq. (13) diverges. Although there are no well-established model distributions for line strengths of many-electron atoms, we assume an exponentially decreasing function,

\[ S(\omega) = S_0 \exp\left(-\frac{|\omega|}{\sigma}\right) \quad (14) \]

where \( \sigma \) is a scale parameter. With \( \sigma < \epsilon_0 \), Eq. (13) does not diverge. Eq. (14) can be understood from the statistical theory because the line strength between orbitals near \( E \) and orbitals near \( E + \omega \) is distributed over the ergodically mixed states, resulting in an exponential decay.

Figure 5 (a) shows the density distribution of \( S \) values computed using the FAC for FeI, FeII, and KrIX, as a function of the energy difference \( \omega \) and its actual value. The strength function values are computed by averaging these points in certain energy bins, and are shown by solid lines. This function decreases exponentially against the transition energy. Our modeled strength functions Eq. (14) are shown by dotted lines in the figure, \( \sigma \) values of which are estimated from the computed \( S \) values. In Fig. 5 (b), the strength functions computed from the different energy range of the initial states are shown. Although a slight initial-energy dependence can be seen, this approximation agrees well with the FAC computation.

In Fig. 4, we also plot \( \sigma \) values for several transition metals and their isoelectronic sequence. Although it shows more scatter than that of \( \epsilon_0 \),

\[ \sigma/\chi \approx 0.1, \quad (15) \]

may be a reasonable estimate.

**B. Continuous Balance Equation**

The spontaneous transition rate can be directly computed from Eq. (3) with given \( S \) distributions. Let us approximate \( A_p^{\text{in}} \) by using Eq. (10) and Eq. (3).

\[ A_p^{\text{in}} = \sum_{q>p} A_{p-q}n_q \]

\[ \approx \int_0^\infty \gamma \omega^3 S(\omega)\rho(E_p + \omega)n(E_p + \omega)d\omega \]

\[ = \gamma \rho_0 S_0 e^{E_p/\epsilon_0} \int_0^\infty \omega^3 e^{-\omega/\delta} n(E_p + \omega)d\omega \quad (16) \]

with

\[ \delta = \frac{1}{\sigma - \frac{1}{\epsilon_0}}. \quad (17) \]

\( n(E_p) \) is the population of the state \( p \). Here, we approximate \( g \approx \bar{g} \) and \( S \approx \bar{S}(\omega) \), where \( \bar{g} \) is the mean value of the statistical weight. Note that this approximation comes from the central limit theorem and is valid if the distribution of \( S \) is independent. Therefore, the shape of the distribution is not restricted to Eq. (12), but is rather arbitrary.

Similarly, the total radiative outflux \( A_p^{\text{out}} \) can be approximated as

\[ A_p^{\text{out}} = \sum_{q<p} A_{q-p}n_p \approx \int_0^{E_p} \gamma \omega^3 \bar{S}(\omega)\rho(E_p - \omega)n(E_p)d\omega \]

\[ = \gamma \rho_0 S_0 e^{E_p/\epsilon_0} \left[ 6\mu^4 - (6\mu^4 + 6\mu^3 E_p + 3\mu^2 E_p^2 + \mu E_p^3) e^{-E_p/\mu} \right] n(E), \quad (18) \]

with

\[ \mu = \frac{1}{\sigma} + \frac{1}{\epsilon_0}. \quad (19) \]

Using Eq. (11) and Eq. (15) we may further approximate \( \delta \approx \chi/5 \) and \( \mu \approx \chi/15 \).

Although the electron impact excitation rate has a much more complex dependence on \( S \), we adopt the simpler approximation Eq. (7) with \( \xi = 0.6 \) [24]. With this dependence, we can also approximate \( C_{\text{in}}, C_{\text{out}}, F_{\text{in}}, \) and \( F_{\text{out}} \) as
FIG. 5. Strength functions for the line strengths of FeI, FeII, and KrIX. (a) The density distribution of the line strength values as a function of the transition energy $\omega$ and their actual values. Note that the density scale is different in $\omega < 0$ and $\omega > 0$. The solid lines indicate their averaged values, i.e., the strength function $S(\omega)$, computed from these data. The dotted slope lines indicate Eq. (14), the energy scale $\sigma$ of which is computed from the weighted expectation. (b) Initial energy dependence of $S(\omega)$. 4–6, 6–8, and 8–10 eV for FeI; 6–10, 10–14, and 14–18 eV for FeII; and 140–200 and 200–240 eV for KrIX are shown.

follows:

$$C_{p}^{\text{in}} = \sum_{q<p} C_{p\rightarrow q}(T_{e})n_{e}n_{q} \approx \int_{0}^{E_{p}} C(T_{e}, \omega)n_{e}n(E_{p} - \omega)\rho(E_{p} - \omega)d\omega$$

$$= \beta \frac{n_{e}}{\sqrt{kT_{e}}} \rho_{0} S_{0} E_{p}/\epsilon_{0} \int_{0}^{E_{p}} \exp \left[ - \left( \frac{1}{kT_{e}} + \frac{1}{\mu} \right) \omega \right] n(E_{p} - \omega)d\omega$$

(20)

$$C_{p}^{\text{out}} = \sum_{q>p} C_{q\rightarrow p}(T_{e})n_{e}n_{p} \approx \int_{0}^{E_{p}} C(T_{e}, \omega)n_{e}n(E_{p} + \omega)\rho(E_{p} + \omega)d\omega$$

$$= \beta \frac{n_{e}}{\sqrt{kT_{e}}} \rho_{0} S_{0} E_{p}/\epsilon_{0} \frac{1}{kT_{e} + \frac{1}{\mu}} n(E_{p})$$

(21)

$$F_{p}^{\text{in}} = \sum_{q>p} F_{p\rightarrow q}(T_{e})n_{e}n_{q} \approx \int_{0}^{E_{p}} F(T_{e}, \omega)n_{e}n(E_{p} + \omega)\rho(E_{p} + \omega)d\omega$$

$$= \beta \frac{n_{e}}{\sqrt{kT_{e}}} \rho_{0} S_{0} E_{p}/\epsilon_{0} \int_{0}^{E_{p}} e^{-\omega/\delta} n(E_{p} + \omega)d\omega$$

(22)

$$F_{p}^{\text{out}} = \sum_{q<p} F_{q\rightarrow p}(T_{e})n_{e}n_{p} \approx \int_{0}^{E_{p}} F(T_{e}, \omega)n_{e}n(E_{p})\rho(E_{p} - \omega)d\omega$$

$$= \beta \frac{n_{e}}{\sqrt{kT_{e}}} \rho_{0} S_{0} E_{p}/\epsilon_{0} \mu \left( 1 - e^{-E_{p}/\mu} \right) n(E_{p})$$

(23)

Here, we again use the averaged value of $S_{pq} \approx S(\omega_{pq}) = S_{0} \exp(-|\omega_{pq}|/\sigma)$. $C$ and $F$ are the rate coefficients defined in Eq. (5) and Eq. (8), respectively, with $S$ substituted by $S$.

We solve Eq. (1) by substituting Eqs. (16–23). Solid lines in Fig. 3 show the numerical solutions, whose vertical scales ($n_{0}$) are chosen to fit the FAC results. Although our CCRM assumes a smooth population distribution and therefore does not reproduce the vertical scatter (relative scatter up to $10^{2}$ depending on $n_{e}$ and $T_{e}$), its slope agrees well with the FAC results, particularly in highly excited states. Recall that in our model, we only use two parameters to represent the atomic structure, $\epsilon_{0}$ and $\sigma$. In light of this huge simplification, the consistency between our CCRM and the FAC computation is surprising. This suggests that the probabilistic approximation is reasonable for many-electron atoms.
C. Further simplification for highly excited states

In highly excited states with \( E \gg \epsilon_0 \), the following approximation for the finite-range integration in Eq. (18), Eq. (20), and Eq. (23) may be valid,

\[
\int_0^E \cdot d\omega \approx \int_0^\infty \cdot d\omega,
\]

(24)
as \( S(\omega) \) decreases quickly for large \( \omega \). Then, all Eqs. (16)–(23) become translation-invariant, i.e., \( n(E) \propto n(E+\Delta) \) holds for any energy \( E \). Therefore, \( n(E) \) should have the following form

\[
n(E) = n_0 \exp \left( -\frac{E}{kT_{ex}} \right),
\]

(25)
with an effective excitation temperature \( T_{ex} \) and an arbitrary scale \( n_0 \).

By substituting Eq. (25) into Eqs. (16)–(23) and assuming \( E \gg \epsilon_0 \), we get

\[
A_{in} \approx \gamma_0 \rho_0 S_0 n_0 e^{-E/\tau} \left( \frac{1}{kT_{ex} + \frac{1}{\delta}} \right)^4
\]

(26)

\[
A_{out} \approx \gamma_0 \rho_0 S_0 n_0 e^{-E/\tau} 6 \mu^4
\]

(27)

\[
C_{in} \approx \beta n_e \sqrt{kT_e \rho_0 S_0 n_0} e^{-E/\tau} \left( 1 + \frac{\epsilon}{\mu} - \frac{kT_e}{kT_{ex}} \right)
\]

(28)

\[
C_{out} \approx \beta n_e \sqrt{kT_e \rho_0 S_0 n_0} e^{-E/\tau} \left( 1 + \frac{\epsilon}{\mu} \right)
\]

(29)

\[
F_{in} \approx \beta n_e \sqrt{kT_e \rho_0 S_0 n_0} e^{-E/\tau} \frac{kT_e}{kT_{ex}}
\]

(30)

\[
F_{out} \approx \beta n_e \sqrt{kT_e \rho_0 S_0 n_0} e^{-E/\tau} \frac{\mu}{kT_e}
\]

(31)

with

\[
\tau = \frac{1}{kT_{ex}} - \frac{1}{\epsilon_0}
\]

(32)

By substituting Eqs. (26)–(31) into Eq. (1), we have the following equation for \( T_{ex} \),

\[
6 \gamma \left\{ \frac{1}{\left( \frac{1}{kT_{ex} + \frac{1}{\delta}} \right)} - \mu^4 \right\} + \beta n_e \sqrt{kT_e} \left\{ \frac{1}{1 + \frac{\epsilon}{\mu} - \frac{kT_e}{kT_{ex}}} - \frac{1}{1 + \frac{\epsilon}{\mu} + \frac{kT_e}{kT_{ex}}} - \frac{\mu}{kT_e} \right\} = 0.
\]

(33)

Although Eq. (33) is not analytically solvable, its numerical solution can be found with given \( T_e, n_e, \epsilon_0 \), and \( \sigma \).

In Fig. 6, we present the excitation temperatures for the FAC results and the CCRM. In order to estimate \( T_{ex} \) from the result obtained using FAC, we choose a certain excited energy range and fit the result in this region by Eq. (25) (see caption of Fig. 6 for the details). In the high-density limit, \( T_{ex} \) approaches \( T_e \). On the other hand, in the low-density limit, \( T_{ex} \) approaches a different value. There is a density region where the transition between these two phases takes place. These tendencies, and the actual values of \( T_{ex} \), are consistent between FAC and our CCRM.

IV. DISCUSSIONS

In Fig. 3, we see that the excited state population distribution changes depending on \( n_e \) and \( T_e \) values of the plasma. In this section, we aim to understand this population kinetics using our CCRM.

The discussion in this section is largely inspired by the proceeding works for H-like ions by Fujimoto [19]. They summarized the population kinetics of H-like ions and its \( n_e \) and \( T_e \) dependence. Figure 1(b) shows the diagram of population kinetics of H-like ions. There are two typical density regions, corona phase and saturation phase, in which the population kinetics is systematically different. In the corona phase, excited state atoms are dominantly generated by electron-impact excitation from the ground state, while being dominantly depopulated by radiative decay. On the other hand, in higher density plasmas, the dominant population path is the excitation from the next lower level and the dominant depopulation path is the excitation to the next higher level. The density boundary is given by

\[
n_e \approx 0.7 \cdot 3 \cdot 2^4 \frac{\gamma}{\beta} \left( \frac{z^2 E_H}{2} \right)^3 (kT_e)^{1/2} \nu^{-8.5},
\]

(34)

where \( \nu \) represents the principal quantum number of the level and \( z^2 E_H \) corresponds to the ionization energy of an H-like ion with nuclear charge \( z \). More details can be found in Appendix A.

We will discuss the population kinetics of many-electron atoms. As we will see below, their population kinetics is also systematically different depending on the \( n_e \) and \( T_e \) values in plasmas. Although the population distribution of many-electron atoms is found to be Boltzmann-like Eq. (25), which is in contrast with the power-law distribution of H-like ions (see Eq. (A4) and Eq. (A6) in Appendix), the density boundaries Eq. (36) and Eq. (40) show a similar form to that for H-like ions Eq. (34).

A. Highly Excited States

1. Low-temperature plasmas

In Fig. 7(a), we show the flux composition for highly excited states of FeI with \( T_e = 0.3 \) eV. The upper part of the figure shows the influx to a certain level, and the bottom part is the outflux from this level. The total
influx and outflux are normalized to unity, so that their compositions can be clearly seen.

On the low-\(n_e\) side, the dominant influx is collisional excitation from lower levels, and the dominant outflux is a spontaneous decay to lower levels. On the high-\(n_e\) side, the dominant influx does not change, collisional excitation, and the dominant outflux becomes collisional de-excitation to lower levels.

\textit{a. Low-density region: corona phase} — The low-\(n_e\) region is similar to the corona phase, which is defined for H-like ions. In the original corona phase, the dominant influx is collisional excitation from the ground state, whereas in our case, the excitation from all lower levels is considered.

\textit{b. High-density region: saturation phase} — In the high-\(n_e\) region, the effect of the spontaneous decay is negligible, and \(C^\text{in} = F^\text{out}\) and \(F^\text{in} = C^\text{out}\) are satisfied based on the principle of detailed balance. We call this phase the saturation phase, similar to that in H-like ions. The population influx and outflux with lower levels are dominant compared with those with higher levels. This can be understood by considering the asymptotic value of \(F^\text{in}\) and \(F^\text{out}\). If \(T_e \approx T_{ex} \ll \delta\),

\[
\frac{F^\text{in}}{F^\text{out}} \approx \frac{\mu}{kT_e}
\]

Therefore, if \(kT_e \ll \mu\), the population balance with lower levels is dominant, and if \(\mu \ll kT_e\), the population balance with higher levels becomes dominant (compare with Fig. 7 (b)).

\textit{c. Boundary densities} — Here, we define two boundary densities. The lower boundary density (\(n_{e}^{\text{cor}}\)) is at the end of the corona phase, and another one (\(n_{e}^{\text{sat}}\)) is at the start of the saturation phase.

\(n_{e}^{\text{cor}}\) may be defined as where \(A^\text{out} = F^\text{out}\) is satisfied. Thus, the boundary density is evaluated as

\[
n_{e}^{\text{cor}} \approx \frac{6\gamma}{\beta} \mu^3 (kT_e)^{1/2}.
\]  

These density boundaries are shown by the vertical bars in Fig. 7 (a) and Fig. 6. This well reproduces the \(T_{ex}\) behavior predicted by the \textit{ab initio} calculation.

If we substitute Eq. (11) and Eq. (15) into Eq. (36), this becomes

\[
n_{e}^{\text{cor}} \approx 2 \times 10^{-3} \frac{\gamma}{\beta} \chi^3 (kT_e)^{1/2}.
\]  

This form can be directly compared with the boundary density of H-like ions Eq. (34). Both boundary densities scale as \(\chi^3 T_e^{1/2}\). However, the boundary for many-electron atoms does not depend on the excited energy (except for that discussed in subsection IV B), in contrast with the \(\nu^{-1.5}\)-dependence in Eq. (34).

FIG. 6. \(n_e\) dependence of \(T_{ex}\) for (a) FeI, (b) FeII, and (c) KrIX. Markers are generated from the FAC results, and lines are computed from Eq. (35), at \(E \approx 0.7\chi\). The two horizontal dotted lines in each panel show \(T_{ex} = T_e\) and \(T_{ex} = \epsilon_0/2k\). The four vertical dotted lines indicate the boundary densities, Eq. (36), Eq. (38), Eq. (40), and Eq. (41), respectively.

FIG. 7. Flux composition for FeI at (a) \(T_e = 0.3\) eV and (b) \(T_e = 10\) eV. Here \(A\) is due to spontaneous emission processes, \(C\) is collisional excitation, and \(F\) is collisional de-excitation.
We also define \( n_{e}^{\text{sat}} \) as the boundary density where \( T_{\text{ex}} = 0.9T_{e} \). By substituting this into Eq. (33) we obtain
\[
 n_{e}^{\text{sat}} \approx \frac{60\gamma}{\beta} \mu^{2}(kT_{e})^{3/2}.
\] (38)

This boundary density scales as \( T_{e}^{3/2} \) in contrast with Eq. (39). However, as can be seen in Fig. 6 and Fig. 9 these two boundaries have similar values in the low-temperature region. This is because in this region, \( T_{e} < \epsilon_{0}/2 \approx \mu \) should be satisfied, as we will see later.

2. High-temperature plasmas

In Fig. 7 (b), we show the flux composition for highly excited states of FeI with \( T_{e} = 10 \text{ eV} \). On the low-\( n_{e} \) side, the dominant influx is a spontaneous decay from higher levels, and the dominant outflux is a spontaneous decay to lower levels. On the high-\( n_{e} \) side, the dominant influx is collisional de-excitation from higher levels, and the dominant outflux becomes collisional excitation to higher levels.

a. Low-density region: cascade phase — In the low-density region, the population balance is established at \( A_{\text{in}} \approx A^{\text{out}} \). From Eq. (26) and Eq. (27), \( T_{\text{ex}} \) can be reduced as follows:
\[
 T_{\text{ex}} = \epsilon_{0}/2k.
\] (39)

The value of \( \epsilon_{0}/2k \) is plotted in Fig. 6 by horizontal dotted lines. This excitation temperature agrees well with the FAC result.

b. High-density limit: saturation phase — In this density region, the population balance is established at \( F_{\text{in}} \approx F^{\text{out}} \). Because the effect of spontaneous decay is negligible in this region, \( T_{\text{ex}} = T_{e} \) is established, as in low-temperature plasmas.

c. Boundary densities — We define two boundary densities for high-temperature plasmas, as in the low-temperature plasmas. The lower boundary density \( (n_{e}^{\text{cas}}) \) is at the end of the cascading phase, and another one \( (n_{e}^{\text{sat}}) \) is at the start of the saturation phase.

As the dominant outflux changes from \( A^{\text{out}} \) to \( F^{\text{out}} \) when the corona phase changes to the saturation phase, \( n_{e}^{\text{cas}} \) may be defined, where \( A^{\text{in}} = F^{\text{in}} \) is satisfied. From the equality and Eq. (27) and Eq. (51), the boundary density is reduced as
\[
 n_{e}^{\text{cas}} \approx \frac{6\gamma}{\beta} \mu^{3} \beta^{1/2} (kT_{e})^{1/2}.
\] (40)

This density boundary is shown by the vertical bars in Fig. 7 (a) and Fig. 6. This boundary density also scales as \( \chi^{1}T_{e}^{1/2} \), similar to Eq. (34) and Eq. (36).

We define another boundary density \( n_{e}^{\text{sat}} \), where \( T_{\text{ex}} = 0.9T_{e} \) is satisfied. By substituting this into Eq. (33),
\[
 n_{e}^{\text{sat}} \approx \frac{60\gamma}{\beta} \delta^{2}(kT_{e})^{3/2}.
\] (41)

These density boundaries are shown by the vertical bars in Fig. 7 (a) and Fig. 6. They well reproduce the \( T_{\text{ex}} \) behavior predicted by the \textit{ab initio} calculation.

3. Temperature boundary

In Fig. 8, we show \( T_{\text{ex}}/T_{e} \) for the low-density region computed from the FAC results, as a function of \( kT_{e}/\epsilon_{0} \). The \( n_{e} \) values and atomic elements are shown in the figure. All results are on the same curve. In the figure, we also show the lines \( T_{\text{ex}} = T_{e} \) and \( kT_{\text{ex}} = \epsilon_{0}/2 \). It can be seen that these are good estimates of \( T_{\text{ex}} \) in the low- and high-\( T_{e} \) regions, respectively. Therefore, the temperature boundary is,
\[
 T_{e}^{b} = \frac{\epsilon_{0}}{2k}.
\] (42)

A diagram illustrating these phases and their boundaries is shown in Fig. 9.
B. Low Excited States

In the discussion above, we have focused on highly excited states. In this section, we discuss the energy dependence of the population kinetics. In Fig. 10 (a-1) and (b-1), we show the population distribution of FeX computed using the FAC (markers) and our continuous model (lines) with $T_e = 0.3$ eV and 10 eV, respectively, and $n_e = 10^{16}$ m$^{-3}$. Both results of FAC and the CCRM deviate from a Boltzmann-like distribution Eq. (25) in the low-energy region ($E < 5$ eV for the FAC and $E < 2.5$ eV for CCRM). This deviation is clearer in the high-temperature case.

The observed inconsistency between FAC and our model is also more prominent in the low-energy region. This indicates that in this energy range the assumptions that we use to derive our model, i.e., sufficiently large level density and wavefunction mixing, are not valid. For example, as there are only even-parity levels in FeX at $E < 2$ eV, other even levels at higher excited energy cannot decay to these levels by electric-dipole transitions. This situation is far different from what we consider in Section III. Although the applicability is limited, for completeness, here we discuss the population kinetics in low-energy levels based on our continuous CR model.

In Fig. 10 (a-2) and (b-2), we show the energy dependence of $kT_{ex}(E) = -n(E)\frac{d\ln(E)}{dE}$. In low-temperature plasmas, $T_{ex}$ is almost constant for all excited energies, whereas the drop in $T_{ex}$ in the low-energy region is significant in high-temperature plasmas. The flux decomposition is shown in Fig. 10 (a-3) and (b-3) as a function of the excited energy.

In high-temperature plasmas, the dominant influx and outflux are both spontaneous decay except for $E < 0.5$ eV. However, if $E \lesssim \delta$, the approximation of Eq. (24) is not valid, and thus Eq. (27) cannot be approximated as follows, from the Taylor expansion of $e^{-E/\mu}$ up to the fourth order:

$$A^{out} \approx \gamma_{j0} S_j \sigma_0^{E/\tau} E^4/4.$$  

From $A^{in} = A^{out}$, the excitation temperature may be approximated by

$$kT_{ex} \approx \frac{1}{2^{1/4} E^{-3/2}}.$$  

where $T_{ex}$ has $E$ dependence.

As in highly excited states, $kT_{ex} \approx \epsilon_0/2$, the boundary energy may be defined at $\frac{1}{2^{1/4} E^{-3/2}} = \epsilon_0/2$. This gives the boundary energy,

$$E^b = 24^{1/4} \mu$$

This boundary energy is shown in Fig. 10 (b-1) and (b-2) by the vertical bars.

In low-temperature plasmas, the violation of the infinite-range-integration approximation Eq. (24) becomes significant also for the influx, i.e., electron-impact excitation. The decrease in the influx due to the boundary effect compensates for the decrease in the outflux, and therefore, the change in $T_{ex}$ is smaller than that in high-temperature plasmas.

V. SUMMARY

In this work, we studied the population kinetics of many-electron atoms in plasmas. From the statistical theory of the many-electron-atom structure, we constructed a continuous CR model that has only two atom-specific energy scales as parameters, $\epsilon_0$ and $\sigma$. From this model, the population distribution in highly excited states was found to be Boltzmann-like, but with excitation temperature sometimes smaller than the electron temperature. We also clarified that there are different phases depending on values of $n_e$ and $T_e$ and derived analytical representations of the boundaries.

Some of our findings can be directly used for plasma diagnostics. For example, the Boltzmann method has been frequently used to estimate $T_{ex}$, based on the slope of the population distribution and the assumption of the saturation phase, and therefore, the applicability of this method has been limited only to high-density plasmas. However, as shown in Fig. 6, if $T_{ex} < \epsilon_0/2k$, then $T_{ex} \approx T_e$ can be inferred. This clearly shows much wider applicability of the Boltzmann method for low-temperature plasmas. This property also enables us to use a new temperature diagnostics using line intensity statistics, which has been proposed in Ref. [18]. By contrast, if $T_{ex} \gtrsim \epsilon_0/2k$, then the inference of $T_e$ may be difficult without knowing $n_e$.

For highly charged ions in low-density and high-temperature plasmas, such as heavy ions in tokamak core plasmas or in electron-beam ion traps, the population is mostly concentrated in the low excited states, to which our model is not applicable. However, our finding Eq. (39) may still be useful to estimate the cascade contribution from very highly excited states, which is difficult to consider from first-principles owing to the enormous requirement of the computation resources.

In this work, we only compared our model with another simulation model, FAC. Comparison with experimental observation is desirable; however, because of the difficulty in the level identification and accurate computation of the transition rates for highly excited states, it is not available at the current stage. We leave it for future studies.

In principle, our CCRM could be further developed to include additional atomic structure data, such as more sophisticated line-strength functions, based on individual orbitals within the statistical theory of many-body quantum chaos [34]. While this would not add significant computational overhead, the simplicity of our current formulation, Eq. (14), allows for analytical exploration of the effective excitation temperature through phase space.
FIG. 10. Excited energy dependence of the population kinetics for FeI with \( n_e = 10^{16} \, \text{m}^{-3} \) and (a) \( T_e = 0.3 \, \text{eV} \) and (b) \( T_e = 10 \, \text{eV} \). (a-1) and (b-1) show the population distributions computed using the FAC (markers) and our continuous model. (a-2) and (b-2) show the local excitation temperature computed from the derivative of the model result. (a-3) and (b-3) show the flux compositions as functions of the excited temperature \( E \). In low-temperature plasmas, the constant \( T_{ex} \) is reasonable, whereas in high-temperature plasmas, \( T_{ex} \) significantly drops in low excited states because of the contribution of electron-impact excitation.

ACKNOWLEDGMENTS

This work was partly supported by JSPS KAKENHI Grant Number 19K14680, the grant of Joint Research by the National Institutes of Natural Sciences (NINS program No. 01111905), and partly by the Max-Planck Society for the Advancement of Science. JCB is supported by the Alexander von Humboldt Foundation. We thank José Crespo Lopáz-Urrutia for useful discussions.

Appendix A: Population Kinetics of H-like Ions in Plasmas

The study of CR models was started from the simplest atoms, H- and He-like ions [19, 23, 39–41]. It was expanded to analyze more complex ions and molecules as detailed calculation of the atomic and molecular structure became available.

For H and other simple ions such as He or alkali metals, the population kinetics has been understood systematically by Griem and Fujimoto et al. [19, 38]. As shown in Fig. 1 (a), they clarified the existence of typical two phases for ionizing plasmas. The analytical representation of the boundary has been also studied. Here, we briefly summarize their works.

The line strength of an H-like atom is known to have the following asymptotic dependence [42, 43],

\[
S_{pq} = \frac{2^6}{\sqrt{3\pi}} p^{-3} q^{-3} (p^{-2} - q^{-2})^{-4} z^{-2} e^2 a_0^2 g_{bb} \quad (A1)
\]

where \( z \) is the nuclear charge of the H-like ion and \( g_{bb} \) is the bound–bound gaunt factor, which is close to 1. Although in Eq. (2), \( p \) and \( q \) should represent any states, and they are not necessarily quantum numbers, only in this subsection, we assume \( p \) and \( q \) to correspond to the principal quantum numbers for H-like ions.

In the low-density limit of ionizing plasmas, the population is more concentrated on the ground state, because of the strong population flow to lower levels by spontaneous transitions. Therefore, collisional excitation from the ground state is the dominant population process for excited states. In this case, \( C_{in} \) and \( A_{out} \) can be written as follows:

\[
C_{in} \approx C_{p-1} n_e n_1
\]

\[
\approx \frac{1}{g_1} \frac{\beta n_e}{\sqrt{kT_e}} \exp \left( -\frac{\omega_{p1}}{kT_e} \right) S_{1p}
\]

\[
\approx \frac{1}{2} \frac{\beta n_e}{\sqrt{kT_e}} p^{-3} z^{-2} \exp \left( -\frac{\omega_{p1}}{kT_e} \right) \frac{2^6}{\sqrt{3\pi}} e^2 a_0^2 \quad (A2)
\]
and
\[ A_{\text{out}} \approx \sum_{q<p} A_{q\rightarrow p} n_p \]
\[ \approx \sum_{q<p} \frac{1}{g_p} \omega_p^3 S_{pq} n_p \approx 0.7 \cdot \frac{3}{2} \frac{3}{2} E_{\text{H}}^3 \gamma^3 \zeta^{-4} \frac{2^6}{\sqrt{3\pi}} e^2 a_0^2 n_p \]  
(A3)

Here, we substituted Eq. (A1) into Eq. (3) and Eq. (5) and assumed \( p \gg 1 \) and log \( p \approx 0 \) \(^{(19)}\). From the flux balance \( C_{\text{in}} \approx A_{\text{out}} \), the population at state \( p \) can be written as follows:
\[ n_p/g_p \propto p^{-0.5} \]  
(A4)

In high-density plasmas, the dominant population process becomes electron-impact excitation from the next lower level, and the dominant depopulation process is electron-impact excitation to the next higher level,
\[ C_{p+1\rightarrow p} n_p \approx C_{p\rightarrow p-1} n_p n_{p-1} \approx (\text{const.}) \]  
(A5)

This leads \( n_p \propto (C_{p+1\rightarrow p})^{-1} \), and thus \(^{(19)}\)
\[ n_p/g_p \propto p^{-6} \]  
(A6)

In this phase, the population distribution does not depend on \( n_e \). We call this the saturation phase.

The transition boundary between the corona and saturation phases may be defined when the dominant outflux changes from spontaneous transition to electron-impact excitation,
\[ \sum_{q<p} A_{q\rightarrow p} \approx C_{p+1\rightarrow p} n_e, \]  
(A7)

which leads to \(^{(19, 38, 44)}\) Eq. (34). A schematic diagram of the population kinetics is shown in Fig. 1(a).

**Appendix B: FAC Computation**

In this section, we show the detailed setup for the atomic structure calculation using the FAC.

The FAC utilizes the configuration interaction method, where a many-body wavefunction is approximated by a linear combination of single-body wavefunctions. In principle, more basis wavefunctions give better accuracy. Table II shows the configurations used to simulate the atomic structure of FeI, FeII, and KrIX. The total numbers of the basis functions for FeI, FeII, and KrIX are 5427, 6997, and 3489, respectively.

The configuration interaction method should converge to the true value if we include infinite number of basis wavefunctions with any arbitrary central potential. However, in reality, we may need to tune the potential. We tune them, so that some of the computed low-lying levels and the ionization potential match to those data compiled in NIST ASD.

*Table II. List of configurations to simulate the atomic structure using the FAC. The notation of \( n^m \) indicates the use of all possible orbital combinations of \( m \) electrons in the shell with the principal quantum number \( n \).*

| \( n^m \) | 3d\(^{10} \) | 3d\(^{8} \) | 3d\(^{6} \) |
|---|---|---|---|
| 3d\(^{8} \) 4\(^{2} \) | 3d\(^{8} \)4\(^{2} \) | 3d\(^{8} \)4\(^{2} \) | 3d\(^{8} \)4\(^{2} \) |
| 3d\(^{10} \)| 3d\(^{8} \) | 3d\(^{6} \) | 3d\(^{4} \) |
| 3d\(^{8} \)5\(^{1} \) | 3d\(^{6} \)5\(^{1} \) | 3d\(^{4} \)5\(^{1} \) | 3d\(^{4} \)5\(^{1} \) |
| 3d\(^{4} \)5\(^{1} \) | 3d\(^{4} \)5\(^{1} \) | 3d\(^{4} \)5\(^{1} \) | 3d\(^{4} \)5\(^{1} \) |

---

[1] R. Tousey, Journal of the Optical Society of America B 5, 2330 (1988).
[2] G. Nave, S. Johansson, R. C. M. Learner, A. P. Thorne, and J. W. Brault, A new multiplet table for Fe I, Tech. Rep. (1994).
[3] F. Castelli and R. L. Kurucz, Astronomy and Astrophysics 520, A57 (2010).
[4] R. C. Peterson, R. L. Kurucz, and T. R. Ayres, The Astrophysical Journal Supplement Series 229, 23 (2017).
[5] G. O’sullivan, B. Li, R. D’arcy, P. Dunne, P. Hayden, D. Kilbane, T. Mccormack, H. Ohashi, F. O’reilly, P. Sheridan, E. Sokell, C. Suzuki, and T. Higashiguchi, Journal of Physics B: Atomic, Molecular and Optical Physics J. Phys. B: At. Mol. Opt. Phys 48, 144025 (2015).
[6] C. Suzuki, F. Koike, I. Murakami, N. Tamura, and S. Sudo, Journal of Physics B: Atomic, Molecular and Optical Physics 45, 135002 (2012).
[7] F. Torretti, J. Sheil, R. Schupp, M. M. Basko, M. Bayraktar, R. A. Meijer, S. Witte, W. Ubachs, R. Hoelestra, O. O. Versolato, A. J. Neukirch, and J. Colgan, Nature Communications 11, 2334 (2020).
[8] T. Pütterich, R. Neu, R. Dux, A. D. Whiteford, and M. G. O’Mullane, Plasma Physics and Controlled Fusion 50, 085016 (2008).
[9] I. Murakami, H. Sakaue, C. Suzuki, D. Kato, M. Goto, N. Tamura, S. Sudo, and S. Morita, Nuclear Fusion 55, 093016 (2015).
[10] E. Pian, P. D’Avanzo, S. Benetti, M. Branchesi, E. Brocato, S. Campana, E. Cappellaro, S. Covino, V. D’Elia, J. P. U. Fynbo, F. Getman, G. Ghirlanda, G. Ghisellini, A. Grado, G. Greco, J. Hjorth, C. Kouveliotou, A. Levan, L. Limatola, D. Malesani, P. A. Mazzali, A. Melandri, P. Møller, L. Nicastro, E. Palazzi, S. Piras.
