The effective electromagnetic interaction in a dense fermionic medium in $QED_{2+1}$.

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Abstract

The effective Lagrangian of arbitrary varying in space electromagnetic field in a dense medium is derived. It has been used for investigation of interaction between charged fermions in the medium. It is shown the possibility for the formation of metastable electron bound states in the medium when external magnetic field is applied.

1 Introduction

The method of effective Lagrangians has been well developed for the cases of homogeneous [1] and/or smoothly varying [2] external fields. However, in some problems, for example, such as photon splitting in the electron-positron plasma [3] it is necessary to consider gauge fields rapidly varying in space or time. For such a type of conditions the calculation of the effective Lagrangian ($EL$) is not trivial. In the paper [4] some important properties and the dependence on chemical potential $\mu$ of the polarization tensor and the three-photonic vertex in a dense medium were discovered. The results obtained allow to construct $EL$ for arbitrary inhomogeneous static electromagnetic fields. The goal of the present paper is to obtain $EL$ of electromagnetic field in a dense fermionic medium by making use the discovered properties of polarization tensors and investigate the corresponding interaction of charged fermions in a dense environment. Although proposed procedure has a general character and is expected to be applicable for various gauge theories, here we shall consider two-dimensional quantum electrodynamics which is widely used for investigation of high temperature superconductivity [5] and quantum Hall effect [6].

2 The one-loop effective Lagrangian of electromagnetic field in a dense fermionic medium

The effective action of electromagnetic field in the one-loop approximation can be expressed in terms of infinite series containing polarization tensors [7]

$$S^{(n)}(A) = \frac{(-1)^n e^n}{n} \int \hat{A}(x_1)G(x_2 - x_1)\hat{A}(x_2)...\hat{A}(x_n)G(x_1 - x_n)d^3x_1d^3x_2...d^3x_n =$$

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\[
\frac{(-1)^n e^n}{n(2\pi)^m} \int A(k_1)G(p_1)\ldots A(k_n)G(p_n)e^{ik_1x_1}e^{ip_1(x_2-x_1)} \ldots \times \\
e^{ik_nx_n}e^{ip_n(x_1-x_n)}d^3x_1\ldots d^3x_n d^3k_1\ldots d^3k_n d^3p_1\ldots d^3p_n = \\
\frac{(-1)^n}{n(2\pi)^{3(n-1)}} \int A_{\mu_1}(k_1)\ldots A_{\mu_n}(k_n)\Pi^{\mu_1\ldots \mu_n}(k_1\ldots k_n)\delta(\sum_{i=1}^n k_i)d^3k_1\ldots d^3k_n = \\
\frac{(-1)^n}{n(2\pi)^{3n}} \int A_{\mu_1}(k_1)\ldots A_{\mu_n}(k_n)\Pi^{\mu_1\ldots \mu_n}(k_1\ldots k_n)e^{ix} \sum_{i=1}^n k_i d^3k_1\ldots d^3k_n d^3x,
\]

where \(A_{\mu}\) is a potential and \(G\) is Green’s function of electromagnetic field, \(\Pi^{\mu_1\ldots \mu_n}(k_1\ldots k_n)\) are polarization tensors with \(n\) external photonic lines carrying momenta \(k_i\). For \(\Pi^{\mu_1\ldots \mu_n}\) being arbitrary functions of momenta the integration over \(k_i\) in eq.(1) is impossible. However, as it was shown in [4] for static case \(k_0 = 0, k \neq 0\) in the limit of \(\mu \gg m\) the tensors tend to constants proportional to certain degrees of \(\mu\). This important property allows to integrate over all momenta in eq.(1) and express \(EL\) as follows

\[
\mathcal{L}'(A) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} A_{\mu_1}(x)\ldots A_{\mu_n}(x)\Pi^{\mu_1\ldots \mu_n}.
\]

Moreover, the \(\mu\)-dependence of tensors occurred to be a decreasing function of the number of external photonic lines. So, only a few terms with non-negative degrees of \(\mu\) contribute to eq.(2).

Now let us consider in the limit \(\mu \gg m\) the realization of eq.(2) for the two-dimensional quantum electrodynamics. In this case we should take into account in eq.(2) the first three terms containing tensors \(\Pi_{\mu}(k_1), \Pi_{\mu\nu}(k_1,k_2), \Pi_{\mu\nu\lambda}(k_1,k_2,k_3)\). They have been calculated in [4] for static limit for interval of momenta \(|k| \subset [0; 2\sqrt{\mu^2 - m^2}]\):

\[
\Pi_0 = \frac{e}{2\pi}(\mu^2 - m^2)\theta(\mu^2 - m^2),
\]

\[
\Delta \Pi_{00} = -\frac{e^2}{2\pi}\theta(\mu^2 - m^2)\mu, \Delta \Pi_{ij} = -\frac{e^2}{2\pi}\theta(\mu^2 - m^2)(\mu - m) \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)
\]

\[
\Pi_{000} = \frac{e^3}{3\pi}\theta(\mu^2 - m^2), \Pi_{i0j} = -\left(\delta_{ij} - \frac{k_i k_j k^2 + k'_i k'_j k^2 - k_i k'_j (kk')}{k^2 k'^2}\right) \frac{\Pi_{000}}{4},
\]

where \(\Delta \Pi_{ij}\) is the statistical part of the polarization tensor which completely determines its properties in the limit considered, \(\theta(x)\) is a step function. In eq.(5) the momentum conservation, \(k_1 + k_2 + k_3 = 0\), has been taken into account.

As it can be seen tensors have manifestly transversal structures that have to guarantee the gauge invariance of \(EL\).

Let us consider the electroneutrality condition of the medium and clarify the question: whether or not the contribution to \(\mathcal{L}'\) of a heavy positively charged subsystem considered as a limiting form of fermions with mass \(M > m\) cancels the contribution of light negatively charged system? The answer follows immediately from observation that all multiphotonic vertices with a number of external lines \(n > 1\) tend to zero in the limit \(M \to \infty\). The only diagrams
giving non-zero contributions are tadpoles and those contributions can be identified with usual interaction of static potential with charge. Hence we can have neutral system with non-zero three photonic interaction.

Substituting expressions (3)-(5) into eq.(2) one obtains $EL$

$$L'(x) = B_1 A_0(x) - C_1 A_0^2(x) - C_2 A_1(x)(\delta^{ij} \Delta - \partial^i \partial^j) \tilde{A}_j(x)$$

$$+ D_1 A_0^3(x) + D_2 A_0(x) \left((\delta^{ij} \Delta - \partial^i \partial^j) \tilde{A}_j(x)\right)^2,$$

where notations are introduced: $B = \Pi_0$, $C_1 = \frac{\varepsilon^2}{4\pi} \theta(\mu^2 - m^2) \mu$, $C_2 = \frac{\varepsilon^2}{4\pi} \theta(\mu^2 - m^2) (\mu - m)$, $D_1 = \frac{\varepsilon^3}{3\pi} \theta(\mu^2 - m^2)$, $D_2 = -\frac{\varepsilon^3}{12\pi} \theta(\mu^2 - m^2)$ and $\tilde{A}_j(x) = \frac{1}{(2\pi)^7} \int \frac{A_j(k)}{k^2} e^{iBx}dk$.

Constructed $EL$ leads to non-linear field equations and can be used for the arbitrary dependence of $A_\mu(x)$ on spatial coordinates.

3 Generation of magnetic field in a dense medium

Now, as an application of eq.(3), we are going to consider the modification of the magnetic field generated by static electric charge \[8\] in a dense medium.

The selfconsistent investigation of this problem requires consideration of the effective Lagrangian including classical part $L_0 = -\frac{1}{4\gamma} F^{\mu\nu} F_{\mu\nu}$, ($\gamma$ is dimensional constant caused by 2-dimensional nature of theory) and the part generated by integration over fermionic fields. The latter part contains two kinds of terms appeared in one-loop approximation. First of them is the vacuum contribution which includes well-known Chern-Simons term \[9, 10\], $L_{CS} = \frac{m_{CS}}{4\pi} \epsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha$ ($m_{CS}$-Chern-Simons mass). Second one is the contribution of medium $L'$, calculated herein (3).

Electric $\phi \equiv A_0$ and vector $A$ potentials generated by the point charge possess axial symmetry ($A_\rho = 0, A_\varphi \neq 0, H(\rho) = \frac{A_\varphi}{\rho} + \frac{\partial A_\varphi}{\partial \rho}$) and thus in cylindrical coordinates we have the following equation of motion

$$\Delta \phi + m_{CS} H(\rho) - 2\gamma C_1 \phi + 3\gamma D_1 A_0^2 + \gamma D_2 \left(\frac{1}{\rho^2} \int_0^\rho A_\varphi(\rho') \rho' d\rho'\right)^2 = 0,$$  

$$\frac{\partial H}{\partial \rho} + m_{CS} \frac{\partial \phi}{\partial \rho} - \gamma C_2 \left(A_\varphi + \frac{1}{\rho^2} \int_0^\rho A_\varphi(\rho') \rho' d\rho'\right) - 2\gamma D_2 \phi \left(\frac{1}{\rho^2} \int_0^\rho A_\varphi(\rho') \rho' d\rho'\right) = 0.$$  

If we consider the case of large value of the chemical potential and induced Chern-Simons mass ($\mu \gg m_{CS}$), eqs.(8) can be simplified as follows:

$$\Delta \phi - 2\gamma C_1 \phi = 0,$$

$$\frac{\partial H}{\partial \rho} + m_{CS} \frac{\partial \phi}{\partial \rho} - \gamma C_2 \left(A_\varphi + \frac{1}{\rho^2} \int_0^\rho A_\varphi(\rho') \rho' d\rho'\right) = 0.$$  

Actually, this simplification implies that screening of electric potential in a dense medium is completely determined by chemical potential, and the Chern-Simons term is important only for generation of magnetic field.

From eq.(9) we obtain

$$\phi = \gamma eK_0(\lambda \rho),$$

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where $K_0$ is MacDonald’s function, $\lambda = e^{\sqrt{\mu \gamma / 2\pi}}$.

After substitution eq.(11) in eq.(10) we have

$$\frac{\partial^2 A_\phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \rho} - \gamma C_2 \left( A_\phi + \frac{1}{\rho^2} \int_0^\rho A_\phi(\rho') \rho' d\rho' \right) = m_{CS}\gamma e\lambda K_1(\lambda \rho).$$  \hspace{1cm} (12)

By introducing of the new function $h(\rho) = \rho A_\phi(\rho)$ and differentiating eq.(12) can be transformed to the linear differential equation of third order:

$$\rho h'''(\rho) - \gamma C_2 \rho h'(\rho) - 2\gamma C_2 h(\rho) = m_{CS}\gamma e\lambda \rho \left( K_1(\lambda \rho) + \lambda \rho K_0(\lambda \rho) \right).$$  \hspace{1cm} (13)

For eq.(13) it is possible to derive the asymptotic solution for $\rho \gg 2\sqrt{\gamma C_2} \sim \frac{2}{e} \sqrt{\frac{\pi}{\gamma \mu}}$. Obviously, this condition can easily be satisfied as long as we deal with a high density medium.

Finally, we obtain the following expression for the magnetic field $H(\rho)$:

$$H(\rho) \simeq \pi e m_{CS}\gamma e^{-\sqrt{\pi\rho}} \left[ K_0(\lambda \rho) L_1(\lambda \rho) \left( \pi + 1 \right) + K_1(\lambda \rho) L_0(\lambda \rho) \left( \pi + \frac{3}{\lambda \rho} \right) - \frac{6}{\pi} K_1(\lambda \rho) \right]$$  \hspace{1cm} (14)

where $L_n$ is modified Struve’s function. The leading term in eq.(14) gives the estimation for $H(\lambda \rho)$ at large distance

$$H(\rho) \sim \pi e m_{CS}\gamma e^{-\sqrt{\pi\rho}} \frac{e^{-\frac{\sqrt{\pi\rho}}{\sqrt{2}}}}{\lambda \rho}$$  \hspace{1cm} (15)

As it is seen, in a dense medium the function describing generated magnetic field is different from that of at $\mu = 0$ [8] and has a higher rate of decreasing.

4 Static interaction between the electric charges in the presence of the external magnetic field

At $\mu \neq 0$ the Furry theorem is violated [11] and three-photonic interaction mixing $\phi$ and spatial components of potential $A$ is not zero (See eq.(5)). It should modify the effective interaction between static charges, especially perceptibly when external fields is applied.

For the sake of simplicity let us consider the external homogeneous magnetic field by potential $A_\phi = \rho H/2$. As it was shown above (See eqs.(14), (15)), generated magnetic field decreases very rapidly from the source and can be neglected starting from the distance $\rho > 1/\lambda \sim \frac{2}{e} \sqrt{\frac{\pi}{\gamma \mu}}$. In this case we can deal with eq. (7) only and rewrite it as follows

$$\Delta \phi + m_{CS}H - 2\gamma C_1 \phi + \gamma D_2 \left( \frac{1}{\rho} \int_0^\rho A_\phi(\rho') \rho' d\rho' \right)^2 = 0$$  \hspace{1cm} (16)

For this equation one can easily derive the approximate solution, describing electric potential produced by the point charge:

$$\phi(z) \simeq \left( 1 - \frac{\pi m_{CS}}{\gamma e^2} \left( \frac{H}{\gamma e \mu} \right) z^2 + \frac{\pi}{72} \left( \frac{H}{\gamma e \mu} \right)^2 z^4 \right) \gamma e K_0(z),$$  \hspace{1cm} (17)

where $z = \rho/\lambda \simeq \frac{1}{e} \sqrt{\frac{\pi \mu}{\gamma \mu}} \rho$. If the Chern-Simons mass is induced one and $\frac{H}{\gamma e \mu} > 1$, the three-photonic interaction term dominates in eq.(17). As it is seen (Fig.1), potential (17) has a local minimum, which provides an attraction between electrons at distances dependent on $H$ and $\mu$. 

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This picture shows a potential ability of the formation of the metastable bound electronic states which would be interesting for high temperature superconductivity.

5 Conclusions

The \textit{QED} in medium in two spatial dimensions has intensively been investigated in recent time (See for example \cite{12}). It is expected that in this way such phenomena as spontaneous magnetization and quantum Hall effect can be adequately described. In papers \cite{12} the uniform magnetic field was considered and corresponding effective Lagrangians have been calculated. The present paper is an endeavour of new approach to the investigation of more general field configurations which can be considered in connection with mentioned phenomena.

The main result of the present paper is the construction of the gauge invariant \textit{EL} for static electromagnetic field with an arbitrary dependence on coordinates. This remarkable possibility can be realized due to important properties of the polarization tensors \cite{4}: in a dense medium the degree of \( \mu \) in the asymptotics of the tensors decreases with an increasing of a number external photonic lines. It provides a rapid convergence of the series of the one-loop diagrams determining \textit{EL} owing to the presence of the small parameter \( \sim \frac{1}{\mu} \). Moreover, just this property leads to the fact that a few first diagrams in eq.(1) adequately describe an effective non-linear interaction of electromagnetic field. Obviously, such a dependence has a general character and is not connected with the number of the spatial dimensions but provided by structure of the fermionic propagator in medium. Similar procedure can be realized in other gauge theories: \textit{QED}_{3+1}, \textit{QCD}. The only necessary condition for this is the presence of dense environment.

The most interesting application of proposed \textit{EL} considered here is the modification of the electrostatic potential in the presence of external magnetic field. For some range of \( H \) and \( \mu \) it is possible to form metastable electronic bound states in planar structures.

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Figure caption:
(Fig.1.) The electric potential of the point charge in the presence of the external magnetic field.
Fig. 1

\[ \phi(\rho) \]

\[ H = 7e^\gamma \mu \]
\[ H = 5e^\gamma \mu \]
\[ H = 3e^\gamma \mu \]
\[ H = e^\gamma \mu \]