On an Algorithm for Obtaining All Binary Matrices of Special Class Related to V. E.
Tarakanovs Formula

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Abstract

An algorithm for obtaining all \( n \times n \) binary matrices having exactly 2 units in every row and every column is described in the paper. After analysing the work of the algorithm a formula for calculating the number of these matrices has been obtained. This formula is known and has been obtained using other methods, which by their nature are purely analytical and not constructive. Thus a new, constructive proof of this known formula has been obtained.

Keywords: Binary matrix, Algorithm, Constructive proof, Tarakanovs formula

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1 Introduction

Everywhere in the current work \([n]\) will denote the set \([n] = \{1, 2, \ldots, n\}\). If \(M\) is a finite set, as usual \(|M|\) will denote the cardinality of \(M\), and with \(\mathcal{P}(M)\) we will be denoting the set of all permutations of the elements from \(M\). With \(S_n\) we will be denoting the symmetric group, \(S_n \cong \mathcal{P}([n])\). If \(\rho \in S_n\), then \(\rho(i)\) is the image of \(i \in [n]\) by the mapping \(\rho\).

Binary is called a matrix whose elements belong to the set \([0, 1]\).

\(\Lambda_k^n\)-matrices we will call all \(n \times n\) binary matrices in which in every row and every column there exist exactly \(k\) identities and \(n - k\) zeros. In the current work we will examine the following

Problem 1 Obtain all \(\Lambda_2^n\)-matrices

Let us denote with \(\lambda_n\) the number of all \(\Lambda_2^n\)-matrices. In [3], the next formula is shown:

\[
\lambda_n = \sum_{2x_2 + 3x_3 + \cdots + nx_n = n} \frac{(n!)^2}{\prod_{k=2}^{n} x_k!(2k)^{x_k}}
\] (1)

Let us notice that formula (1) does not give a solution to problem 1, it only calculates the number of the objects which are being sought. We will give a solution to problem 1 with the help of algorithm 1 described below.

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Furthermore, formula (1) has been obtained by V. E. Tarakanov when examining an equivalence relation in the set of all ordered pairs of derangements \((\rho, \sigma) \in S_n \times S_n\). Let us remind that \(\rho, \sigma \in S_n\) is called derangement, if \(\rho(i) \neq \sigma(i)\) for every \(i = 1, 2, \ldots, n\). In the current work we will suggest a new constructive proof of formula (1) by suggesting an algorithm for obtaining all \(\Lambda_n^2\)-matrices.

2 The main results

A partition of the set \(M\) we will be calling every family of sets \(\{M_i\}_{i \in I}\), such that \(M_i \subseteq M\) for every \(i \in I\), \(\bigcup_{i \in I} M_i = M\), \(M_i \cap M_j = \emptyset\) when \(i \neq j\). The subsets \(M_i\) are called blocks of partition. We say that a partition is of type \(1^{x_1}2^{x_2} \ldots n^{x_n}\), if there exist exactly \(x_k\) blocks with cardinality \(k\), \((k = 1, 2, \ldots, n)\).

As it is well known (1) there exists an one-to-one correspondence between the set of all types of partitions of \(M\), \(|M| = n\) and the set of all solutions to the equation \(1 \cdot x_1 + 2 \cdot x_2 + \cdots + n \cdot x_n = n\).

Algorithm 1 Obtaining all \(\Lambda_n^2\)-matrices.

1. For every solution to the equation \(2x_2 + 3x_3 + \cdots + nx_n = n\) Do begin
2. For every partition \(C\) of \([n]\) of type \(1^{02^2}3^{03^2} \ldots n^{x_n}\) Do begin
3. For every partition \(A\) of \([n]\) of type \(1^{02^2}3^{03^2} \ldots n^{x_n}\) Do begin
4. For every \(k = 2, 3, \ldots, n\), for which \(x_k \neq 0\) Do begin
5. Form the sets
   \[C := \{C^{(1)}_1, C^{(2)}_2, \ldots, C^{(k)}_{x_k}\}\]
   and
   \[A := \{A^{(1)}_1, A^{(2)}_2, \ldots, A^{(k)}_{x_k}\}\]
   where \(C^{(k)}_i \in C, |C^{(k)}_i| = k, A^{(k)}_i \in A, |A^{(k)}_i| = k, i = 1, 2, \ldots, k\);
6. For every \(\rho \in S_{x_k}\) Do begin
7. For every \(i = 1, 2, \ldots, x_k\) Do begin
8. \(l := 0; L^{(k)}_i := \emptyset\)
/* \(l\) - integer; \(L^{(k)}_i\) - set of integers */
9. \(c_1 := \min\{c \mid c \in C^{(k)}_i\}\);
10. For every permutation \(c_2c_3 \ldots c_k \in \mathcal{P}C^{(k)}_i \setminus \{c_1\}\) Do begin
11. For every permutation \(a_1a_2 \ldots a_k \in \mathcal{P}A^{(k)}_i \setminus \{\rho(i)\}\) Do begin
12. If \(a_1 > a_2\) Then return to 11;
13. \(l := l + 1; L^{(k)}_i := L^{(k)}_i \cup \{l\}\);
14. Obtain the set of three-dimensional vectors:
   \[N^{(k)}_i[l] := \left\{ \begin{pmatrix} c_1 \\ a_1 \\ a_2 \\ a_3 \\ c_2 \\ a_2 \\ a_3 \\ \vdots \\ c_k \\ a_k \end{pmatrix} \right\} ;\]
15. End 11
16. End 10
17. End 7
18. Form the set \(Q_{x_k} := L^{(k)}_1 \times L^{(k)}_2 \times \cdots \times L^{(k)}_{x_k}\);
19. For every \(x_k\)-tuple \(\overline{q} = (l_1, l_2, \ldots, l_{x_k}) \in Q_{x_k}\) Do begin
20. Form the set of three-dimensional vectors:

\[ M[\rho; \overline{q}] := \bigcup_{i=1}^{x} N_i^{(k)}[l_i] \]

21. End 19
22. End 6
23. End 4
24. For every \((n - 1)\)-tuple \((\rho_2, \rho_3, \ldots, \rho_n) \in S_{x_2} \times S_{x_3} \times \cdots \times S_{x_n}\) Do begin

/* If \(x_k = 0\), then by definition \(S_{x_k} = \emptyset\) and \(|S_{x_k}| = 0! = 1\) */
25. For every \((n - 1)\)-tuple \((\overline{q_2}, \overline{q_3}, \ldots, \overline{q_n}) \in Q_{x_2} \times Q_{x_3} \times \cdots \times Q_{x_n}\) Do begin

/* If \(x_k = 0\), then by definition \(Q_{x_k} = \emptyset\) \(|Q_{x_k}| = 0! = 1\) */
26. Form the set of three-dimensional vectors:

\[ R = \bigcup_{k=2}^{n} M[\rho_k; \overline{q_k}] ; \]

/* If \(x_k = 0\), then by definition \(M[\rho_k; q_k] = \emptyset\) \(|M[\rho_k; q_k]| = 0! = 1\) */
27. From \(R\) obtain only \(\Lambda_n^2\)-matrix \(\alpha\), such that if \(\begin{pmatrix} c & u \\ u & v \end{pmatrix} \in R\), then in \(\alpha\) in c-th column the only two units stand in rows with numbers \(u\) and \(v\);
28. End 25
29. End 24
30. End 3
31. End 2
32. End 1

**Theorem 1** With the help of algorithm \(\overline{7}\) all \(\Lambda_n^2\)-matrices are obtained, moreover during every execution of statement 27 a different \(\Lambda_n^2\)-matrix is obtained.

Proof. Let \(w' = \begin{pmatrix} c' & u' \\ u' & v' \end{pmatrix}\) and \(w'' = \begin{pmatrix} c'' & u'' \\ u'' & v'' \end{pmatrix}\) be two vectors from set \(R\), which is obtained during execution of statement 26 from algorithm \(\overline{1}\). If there exist \(i, k, l, t\), such that \(w', w'' \in N_i^{(k)}[l]\) (i.e. \(w'\) and \(w''\) are obtained during single execution of statement 14), then according to statements 10 and 14 \(c' \neq c''\). According to statements 20 and 26 it is not possible to exist \(i, k_1, k_2, l, t\), such that \(w' \in N_i^{(k_1)}[l]\), \(w'' \in N_i^{(k_2)}[l]\) and \(w'\) and \(w''\) to be from one and the same set \(R\), which is obtained during execution of statement 26. If \(w' \in N_i^{(k_1)}[l]\), \(w'' \in N_j^{(k_2)}[l]\) when \(i \neq j\), then \(c'\) and \(c''\) are from different blocks of the partition \(C\), i.e. again \(c' \neq c''\). Furthermore, obviously for every \(c \in [n]\) there exists a block \(C_i^{(k)} \in C\), such that \(c \in C_i^{(k)}\) and according to statements 10 and 14 there exist a natural number \(l\) and \(u, v \in [n]\), such that \(\begin{pmatrix} c & u \\ u & v \end{pmatrix} \in N_i^{(k)}[l]\). Therefore the first elements of every vector in the set \(R\) during single execution of statement 26 correctly indicate a number of a column in a matrix.
Let $a \in [n]$. Since statements 11, 12, 14 and 26 are inner for loops with numbers 2 and 3, then according to these statements there exist unique three-dimensional vectors $w'$ and $w''$, $w' \neq w''$ from the set $R$ during single execution of statement 26, such that $a$ be at second position in $w'$ and at third position in $w''$. Therefore in the matrices which are obtained in statement 27 there are exactly two units in every row and according to statement 27 exactly two units in every column.

We assume that there exist two sets $R_1$ and $R_2$, obtained during different executions of statement 26, which are equal to each other. Then $R_1 = R_2 = N_1^{(2)}[l_1] \cup \cdots \cup N_2^{(2)}[l_2] \cup \cdots \cup N_2^{(n)}[l_n]$, which is impossible, bearing in mind the courses of action of statements 14, 20 and 26, and with what objects they work during each of their executions. Therefore during every execution of statement 27 a different $\Lambda_2^n$-matrix is obtained.

Let us denote with $T$ the set of the matrices obtained during the work of statement 27. Since in $T$ there are not any duplicate elements, then $T \subseteq \Lambda_2^n$.

Let $\alpha$ be an arbitrary $\Lambda_2^n$-matrix. We will show that $\alpha \in T$. Let in the beginning $\alpha$ have not any marked rows and columns. In $\alpha$ we build a walk, starting from the upper "1" of the leftmost unmarked column of the matrix, we continue to the next below "1" in the column, after that we reach the second "1" from the same row, then the second "1" from the same column and so on until we return to the initial cell. Moving on this walk we mark the rows and the columns we pass, and when passing through $c$-th column we obtain the vector $\begin{pmatrix} c \\ u \\ v \end{pmatrix}$ (we have first passed through the "1" in $u$-th row, then through the "1" in $v$-th row). We continue the same process with the unmarked rows and columns until we mark the whole matrix. Let $c_1$ be the minimum number among the numbers standing at first position of the vectors corresponding to a particular walk. Since the beginning of the loop starts from the upper "1" of the leftmost column of the walk, then for the vector $\begin{pmatrix} c_1 \\ u \\ v \end{pmatrix}$ the inequality $u < v$ is satisfied. Therefore for every walk we obtain a set of vectors, which are obtained in statement 14 as well. Uniting the sets of all vectors which are obtained from walks in $\alpha$ with the same length, we obtain a set, which is obtained in statement 20 as well. The union of all such sets corresponding to a particular matrix, is a set, which is obtained in statement 26 as well. Therefore $|\Lambda_2^n| \leq |T|$, whence the assertion of the theorem follows.

With the help of Algorithm 1 and Theorem 1 we will also prove the next theorem, proven in [3] in a different way.

**Theorem 2** [3] The number $\lambda_n$ of all $\Lambda_2^n$-matrices is given by the formula:

$$
\lambda_n = \sum_{2x_2 + 3x_3 + \cdots + nx_n = n} \frac{(n!)^2}{\prod_{k=2}^{n} x_k! (2k)^{x_k}}
$$

**Proof.** Let us denote with $\mu_j$ the number of the iterations of loop with number $j$ in Algorithm 1. Since Algorithm 1 has been maid so that every time
during execution of statements 14, 20, 26 and therefore also statement 27 (it follows from Theorem 1) different objects are obtained, then

$$\lambda_n = \mu_1 \mu_2 \mu_3 \mu_{24} \mu_{25}, \quad (2)$$

Is easy to see that

$$\mu_1 = \sum_{2x_2 + 3x_3 + \cdots + nx_n = n} 1.$$  

According to a well-known formula [1, (3.15.ii)] the next equation is true:

$$\mu_2 = \mu_3 = n! \prod_{k=2}^n x_k! (k!)^{x_k}$$

For $$\mu_{24}$$ and $$\mu_{25}$$ we obtain respectively:

$$\mu_{24} = \prod_{k=2}^n |S_{x_k}| = \prod_{k=2}^n x_k!$$

$$\mu_{25} = \prod_{k=2}^n |Q_{x_k}| = \prod_{k=2}^n \left( \prod_{j=1}^{x_k} L_{j}^{(k)} \right) = \prod_{k=2}^n \left( \prod_{j=1}^{x_k} \mu_{10} \mu_{11} \right) = \prod_{k=2}^n \left( \prod_{j=1}^{x_k} \frac{(k-1)!}{k!} \cdot \frac{k!}{2} \right) = \prod_{k=2}^n \frac{(k!)^{2x_k}}{(2k)^{x_k}}$$

Substituting in (2) we obtain the assertion of the theorem as well. \(\Box\)

3 Conclusion

Comparing the proof of theorem to formula we are convinced of the high efficiency of algorithm 1. This is practically a proof of the fact that algorithm 1 works exactly as much as necessary.

References

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