Aspects of a planar nonbirefringent and CPT-even electrodynamics stemming from the Standard Model Extension

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We have studied a (1+2)-dimensional Lorentz-violating model which is obtained from the dimensional reduction of the nonbirefringent sector of the CPT-even electrodynamics of the standard model extension (SME). The planar theory contains a gauge sector and a scalar sector which are linearly coupled by means of a Lorentz-invariance violating (LIV) vector, $S^\mu$, while the kinetic terms of both sectors are affected by the components of a Lorentz-violating symmetric tensor, $k^\mu\nu$. The energy-momentum tensor reveals that both sectors present energy stability for sufficiently small values of the Lorentz-violating parameters. The full dispersion relation equations are exactly determined and analyzed for some special configurations of the LIV backgrounds, showing that the planar model is entirely nonbirefringent at any order in the LIV parameters. At first order, the gauge and scalar sectors are described by the same dispersion relations. Finally, the equations of motion have been solved in the stationary regime and at first order in the LIV parameters. It is observed that the Lorentz-violating parameters do not alter the asymptotical behavior of the electric and magnetic fields but induce an angular dependence which is not present in Maxwell’s planar theory.

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I. INTRODUCTION

Since the establishment of the special theory of relativity as a true of nature, Lorentz symmetry has been taken as a key ingredient of theoretical physics. A motivation for studies involving the violation of Lorentz symmetry is the demonstration that string theories may support spontaneous violation of this symmetry [1], with important and interesting connections with the physics in the Planck energy scale. The Standard Model Extension (SME) [2] has arisen as theoretical framework for addressing Lorentz violation (LV) in a broader context than the usual Standard Model, in an attempt of scrutinizing remanent effects of this violation in several low energy systems. In this way, the SME incorporates Lorentz-violating coefficients to all sectors of the standard model and to general relativity, representing a suitable tool for investigating Lorentz violation in several distinct respects.

The violation of Lorentz symmetry in the gauge sector of the SME is governed by a CPT-odd and a CPT-even tensor, yielding some unconventional phenomena such as vacuum birefringence and Cherenkov radiation. The LV coefficients are usually classified in accordance with the parity and birefringence. The CPT-odd term is represented by the Carroll-Field-Jackiw (CFJ) background [3], which is also parity-odd and birefringent. This electrodynamics has been much investigated, encompassing aspects as diverse as: consistency aspects and modifications induced in QED [4–6], supersymmetry [7], vacuum Cherenkov radiation emission [8], finite-temperature contributions and Planck distribution [9, 10], electromagnetic propagation in waveguides [11], Casimir effect [12].

The CPT-even gauge sector of the SME is represented by the CPT-even tensor, $(k_F)_{\alpha\beta\mu\nu}$, composed of 19 independent coefficients, with nine nonbirefringent and ten birefringent ones. This sector has been studied since 2002 [13, 14, 15, 16], being represented by the following Lagrangian:

$$\mathcal{L}_{(1+3)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} (k_F)_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} - J_{\mu} A^\mu, \tag{1}$$

where the indices with hat, $\hat{\mu}, \hat{\nu}$, run from 0 to 3, $A^\mu$ is the four-potential and $F_{\mu\nu}$ is the usual electromagnetic field tensor. The tensor $(k_F)_{\mu\nu\lambda\kappa}$ stands for the Lorentz-violating coupling and possesses the symmetries of the Riemann tensor, $(k_F)_{\mu\nu\lambda\kappa} = -(k_F)_{\nu\mu\lambda\kappa}$, $(k_F)_{\mu\lambda\kappa} = -(k_F)_{\lambda\mu\kappa}$, $(k_F)_{\mu\nu\lambda\kappa} = -(k_F)_{\lambda\nu\mu\kappa}$, $(k_F)_{\mu\lambda\kappa} = -(k_F)_{\lambda\mu\kappa}$, $(k_F)_{\mu\lambda\kappa} + (k_F)_{\lambda\mu\kappa} + (k_F)_{\mu\nu\lambda\kappa} = 0$, and a double null trace, $(k_F)_{\mu\nu} \hat{\rho} \hat{\sigma} = 0$. A very useful parametrization for addressing this electrodynamics is the one...
Planar theories have been investigated since the beginning of 80’s \cite{23}, and have gained much attention due its connection with the Chern-Simons theories \cite{24}, planar superconductivity, anyons and quantum Hall effect \cite{25}, and This CPT-even sector has also been recently investigated in connection with consistency aspects in Refs. \cite{21, 22}.

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The nonbirefringent components are severely constrained by astrophysical tests involving high-quality cosmological spectropolarimetry data, which have yielded stringent upper bounds at the level of 1 part in 10^{32} \cite{13, 14} and 1 part in 10^{37} \cite{15}. The nonbirefringent components are embraced by the matrices $\widetilde{\kappa}_e$ (six elements) and $\widetilde{\kappa}_{\phi+}$ (three elements), and can be constrained by means of laboratory tests \cite{17} and the absence of emission of Cherenkov radiation by UHECR (ultrahigh energy cosmic rays) \cite{18, 19}. These coefficients also undergo restriction at the order of 1 part in 10^{37} considering their sub-leading birefringent role \cite{20}. This CPT-even sector has also been recently investigated in connection with consistency aspects in Refs. \cite{21, 22}.

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A CPT-even field theory in (1+2)-dimensions model with Lorentz violation was recently attained by means of the dimensional reduction of the CPT-even gauge sector of the Standard Model Extension \cite{28}. The resulting planar electrodynamics is composed of a gauge and scalar sectors, both endowed with Lorentz violation, whose planar Lagrangian is

$$L_{1+2} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - C_{\mu\lambda} \partial^\mu \phi \partial^\lambda \phi + T_{\mu\lambda\kappa} \partial^\mu \phi F^{\lambda\kappa},$$

where $Z_{\mu\nu\lambda\kappa}, C_{\mu\lambda}, T_{\mu\lambda\kappa}$ are LIV tensors which have together 19 components and present the following symmetries:

$$Z_{\mu\nu\lambda\kappa} = Z_{\lambda\kappa\mu\nu}, \quad Z_{\mu\nu\lambda\kappa} = -Z_{\nu\mu\lambda\kappa}, \quad Z_{\mu\nu\lambda\kappa} = -Z_{\mu\nu\kappa\lambda},$$

$$Z_{\mu\nu\lambda\kappa} + Z_{\mu\lambda\kappa\nu} + Z_{\nu\mu\kappa\lambda} = 0,$$

$$T_{\mu\lambda\kappa} + T_{\lambda\kappa\mu} + T_{\kappa\mu\lambda} = 0,$$

$$C_{\mu\lambda} = C_{\lambda\mu}, \quad T_{\mu\lambda\kappa} = -T_{\mu\kappa\lambda}.$$

Some aspects of this model, involving wave equations and dispersion relations, were addressed in Ref. \cite{28}, having shown that the pure abelian gauge or electromagnetic sector presents nonbirefringence at any order. The birefringence in this model is associated with the elements of the coupling tensor, $T_{\mu\lambda\kappa}$.

In the present work, we accomplish the dimensional reduction of the nonbirefringent gauge sector of the SME, represented by 9 components which can be incorporated in a symmetric and traceless tensor $\kappa_{\bar{\nu}\bar{\rho}}$, defined as the contraction \cite{24}: $\kappa_{\bar{\nu}\bar{\rho}} = (k_F)_{\bar{\alpha}\bar{\nu}\bar{\rho}}$. The nonbirefringent components of the tensor $(k_F)_{\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\kappa}}$ are parametrized as

$$(k_F)_{\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\kappa}} = \frac{1}{2} \left( g_{\bar{\mu}\bar{\lambda}} \kappa_{\bar{\nu}\bar{\rho}} - g_{\bar{\mu}\bar{\rho}} \kappa_{\bar{\nu}\bar{\lambda}} - g_{\bar{\nu}\bar{\lambda}} \kappa_{\bar{\mu}\bar{\rho}} + g_{\bar{\nu}\bar{\rho}} \kappa_{\bar{\mu}\bar{\lambda}} \right),$$

which implies

$$(k_F)_{\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\kappa}} F^{\bar{\mu}\bar{\nu}} F^{\bar{\lambda}\bar{\kappa}} = 2 \kappa_{\bar{\nu}\bar{\rho}} F_{\bar{\lambda}} \bar{\nu} F^{\bar{\lambda}\bar{\rho}},$$

so that the Lagrangian \cite{11} takes on the form

$$L_{1+3} = -\frac{1}{4} F_{\bar{\mu}\bar{\nu}} F^{\bar{\mu}\bar{\nu}} - \frac{1}{2} \kappa_{\bar{\nu}\bar{\rho}} F_{\bar{\lambda}} \bar{\nu} F^{\bar{\lambda}\bar{\rho}} - J_{\bar{\mu}} A^{\bar{\mu}}.$$
Some properties of this nonbirefringent electrodynamics were investigated in Ref. [22], in which the corresponding Feynman gauge propagator was evaluated and some of its consistency properties (causality and unitarity) were analyzed.

In the present work, we perform the dimensional reduction of Lagrangian (11), which produces a nonbirefringent planar theory composed of 9 LIV parameters instead of the 19 ones attained in Ref. [28]. In this simpler framework, Lorentz violation is controlled only by a rank-2, which modifies the kinetic part of the scalar and gauge sectors, and a rank-1 tensor, which couples both sectors. The density of energy was evaluated, revealing that the model presents positive-definite energy for small values of the Lorentz-violating parameters. We work out the complete dispersion relations of this planar model from the vacuum-vacuum amplitude, showing that all theory is nonbirefringent. Such planar model provides a more direct way to analyze consistency aspects associated with the Feynman propagator and the effects of the LIV parameters on some planar systems of interest.

This work is organized as follows. In Sec. II, we accomplish the dimensional reduction of Lagrangian (11), obtaining a planar scalar electrodynamics in which the Lorentz violation is controlled by the symmetric tensor, \( \kappa \), the counterpart of the original tensor \( \kappa_{\mu\nu} \) defined in (1+2) dimensions, and a three-vector denoted as \( S_\nu \). The energy-momentum tensor is computed and the density of energy is analyzed. The Sec. III is devoted to the analysis of the dispersion relation in two situations: considering the complete model and regarding the gauge and scalar sector as decoupled. In Sec. IV, we write the corresponding equations of motion and wave equations for the model. The wave equations for the gauge and scalar sectors are solved in the stationary regime at first-order in the LIV parameters. In Sec. V, we present our Conclusions.

II. THE DIMENSIONAL REDUCTION PROCEDURE

In this section, we perform the dimensional reduction of the model represented by Lagrangian (11). There are some distinct procedures for accomplishing the dimensional reduction of a theory. In the present case, we adopt the one that freezes the third spacial component of the position four-vector and any other four-vector. This is done requiring that the physical fields \( \{ \chi \} \) do not depend anymore on this component, that is, \( \partial_3 \chi = 0 \). Besides this, we split out the fourth component of the four-vectors. This procedure is employed in Ref. [28]. The electromagnetic four-potential is written as

\[
A^\nu \rightarrow (A^\nu; \phi),
\]

where \( A^{(3)} = \phi \) is now a scalar field and the Greek indices (without hat) run from 0 to 2, \( \mu = 0, 1, 2 \). Carrying out this prescription for the terms of Lagrangian (11), one then obtains:

\[
\begin{align*}
F_{\mu\nu} F^{\mu\nu} &= F_{\mu\nu} F^{\mu\nu} - 2 \partial_\mu \phi \partial^\mu \phi, \\
\kappa_{\nu\rho} F^{\nu\rho} F^{\lambda\phi} &= \kappa_{\nu\rho} F^{\nu\rho} F^{\lambda\phi} - 2 S_\nu F^{\nu\lambda} \partial_\lambda \phi + \eta \partial_\lambda \phi \partial^\lambda \phi - \kappa_{\nu\rho} \partial^\rho \phi \partial^\rho \phi,
\end{align*}
\]

where we have defined \( F^{\nu\lambda} = \partial^\nu \phi, \; F_{\mu\lambda} = -\partial_\mu \phi \). Also, we have renamed the set of LIV parameters, they now are represented by a second rank tensor \( \kappa_{\nu\rho} \) which is the (1+2)-dimensional counterpart of the tensor \( \kappa_{\rho\sigma} \), a vector \( S_\nu \) and a scalar quantity \( \eta \) which are defined as

\[
S_\nu = \kappa_{\nu3}, \; \eta = \kappa_{33},
\]

respectively. Thus, after the dimensional reduction procedure we attain the following Lagrangian density:

\[
\mathcal{L}_{1+2} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \kappa_{\nu\rho} F^{\nu\rho} F^{\lambda\phi} + \frac{1}{2} [1 - \eta] \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \kappa_{\nu\rho} \partial^\rho \phi \partial^\rho \phi + S_\nu F^{\nu\lambda} \partial_\lambda \phi - A_\mu J^\mu - J \phi,
\]

it is composed of a gauge sector (\( \mathcal{L}_{EM} \)), a scalar sector (\( \mathcal{L}_{scalar} \)), and a coupling sector (\( \mathcal{L}_{coupling} \)) ruled by the Lorentz-violating vector \( S_\nu \) that contains three LIV parameters. The Lorentz-violating symmetric tensor \( \kappa_{\nu\rho} \) presents
six independent coefficients, which modify both the electromagnetic and scalar sectors, altering the dynamics of the Maxwell field and yielding a noncanonical kinetic term for the scalar field. The LIV noncanonical kinetic term present in the scalar sector has been recently investigated in scenarios involving topological defects in (1+1) dimensions and acoustic black holes with Lorentz-violation in (1+2) dimensions. A similar term is also found in the Lagrangian of Ref. [28]. The present work provides a possible origin for this kind of term.

Our planar model [16] has ten dimensionless Lorentz-violating parameters contained in the tensors $\kappa_{\nu\rho}$, $S_\nu$ and in the scalar $\eta$. The traceless condition of the original tensor, $\kappa^\mu_\mu = 0$, gives one constraint between the $\kappa_{\nu\rho}$—components

$$\kappa_{00} - \kappa_{ii} = \eta, \quad (17)$$

so, the model possesses nine independent Lorentz-violating parameters, the same number of the original four-dimensional theory. It demonstrates the consistency in the dimensional reduction procedure.

We define the components of the electric field as $E^i = F_{0i}$, the magnetic field by $B = -\frac{1}{2} \epsilon_{ij} F_{ij}$ and $\epsilon_{012} = \epsilon_{12} = 1$, then the Lagrangian [16] can be written in terms of fields of the electric and magnetic field in the form:

$$\mathcal{L}_{1+2} = \mathcal{L}_{EM} + \mathcal{L}_{scalar} + \mathcal{L}_{coupling}, \quad \tag{18}$$

where

$$\mathcal{L}_{EM} = \frac{1}{2} \left( 1 + \kappa_{00} \right) E^2 - \frac{1}{2} (1 - \kappa_{i}) B^2 - \frac{1}{2} \kappa_{ij} E^i E^j + \kappa_{0i} \epsilon_{ij} F_{ij}, \quad \tag{19}$$

$$\mathcal{L}_{scalar} = \frac{1}{2} (1 - \eta) \left[ (\partial_t \phi)^2 - (\partial_i \phi)^2 \right] + \frac{1}{2} \kappa_{00} (\partial_0 \phi)^2 - \kappa_{0i} \partial_0 \phi \partial_i \phi + \frac{1}{2} \kappa_{ij} \partial_i \phi \partial_j \phi, \quad \tag{20}$$

$$\mathcal{L}_{coupling} = - S^0 E^i \partial_j \phi - S^i E^0 \partial_i \phi + \epsilon_{ij} S^j \partial_j \phi B. \quad \tag{21}$$

The above decomposition allows to determine the parity-properties of the LIV couplings. In (1+2)-dimension, the parity operator acts doing $r \rightarrow (-x, y)$, it changes the fields as $A_0 \rightarrow A_0$, $A \rightarrow (-A_x, A_y)$, $E \rightarrow (-E_x, E_y)$, $B \rightarrow -B$. For more details, see Ref. [23]. Here, we consider that the field $\phi$ behaves as a scalar, $\phi \rightarrow \phi$. Since the Lagrangian density is parity-even, we can conclude that the planar model possesses nine independent coefficients, six are parity-even $(\kappa_{00}, \kappa_{02}, \kappa_{11}, \kappa_{22}, S_0, S_2)$, and three are parity-odd $(\kappa_{01}, \kappa_{12}, S_1)$. The fact that the components of the vector $S^\mu$ transform distinctly is a consequence of the way as the vectors $E, B$ and the field $\phi$ behave under parity.

An issue that deserves some attention is the energy stability, once it is known that the Lorentz violation yields energy instability in some models, for example, the Carroll-Field-Jackiw electrodynamics [3]. A preliminary analysis concerning this point can be performed by means of the energy-momentum tensor for the full planar theory,

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \partial^\nu A_\rho + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}, \quad \tag{22}$$

which is carried out as

$$\Theta^{\mu\nu} = - F^{\mu\rho} F^\rho_\nu - \kappa^{\rho\beta} F^\mu_\beta F^\rho_\nu + \kappa^{\mu\beta} F^\rho_\beta F^\nu_\rho + S^\mu F^{\rho\nu} \partial^\rho \phi + S_\rho F^{\rho\nu} \partial^\rho \phi + S_\beta F^{\rho\nu} \partial^\rho \phi + (1 - \eta) \partial^\mu \phi \partial^\nu \phi + \kappa^{\mu\beta} \partial^\beta \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}. \quad \tag{23}$$

We now specialize our evaluation for the density of energy

$$\Theta^{00} = \frac{1}{2} M_{jk} E_j E_k + \frac{1}{2} (1 - \kappa_{jj}) B^2 + B \epsilon_{jk} S_j \partial_k \phi - S_j E_j \partial_0 \phi + \frac{1}{2} (1 + \kappa_{jj}) (\partial_0 \phi)^2 + \frac{1}{2} N_{jk} \partial_j \phi \partial_k \phi, \quad \tag{24}$$

where we have defined the symmetric matrices

$$M_{jk} = (1 + \kappa_{00}) \delta_{jk} - \kappa_{jk}, \quad N_{jk} = (1 - \kappa_{00} + \kappa_{ii}) \delta_{jk} - \kappa_{jk}, \quad \tag{25}$$

and used $\eta = \kappa_{00} - \kappa_{jj}$. We see that the energy density for the electromagnetic and scalar fields, when regarded as isolated, are

$$\Theta_{EM}^{00} = \frac{1}{2} M_{ij} E_j E_k + \frac{1}{2} (1 - \kappa_{jj}) B^2, \quad \tag{26}$$

$$\Theta_{scalar}^{00} = \frac{1}{2} (1 + \kappa_{jj}) (\partial_0 \phi)^2 + \frac{1}{2} N_{jk} \partial_j \phi \partial_k \phi. \quad \tag{27}$$
Both the gauge and scalar energy densities will be positive-definite if \(|\kappa_{jj}| < 1\) and the matrices \(M_{ij}\) and \(N_{ij}\) are positive-definite. As the LV parameters are usually much smaller than the unit, we conclude that the scalar and gauge sectors, as regarded separately, are stable. However, the energy positivity of the full model seems to be spoiled by the mixing terms, \(S_j E_j \partial_0 \phi\) and \(\eta \epsilon_{jk} S_j \partial_k \phi\). In order to have more clarity, we write Eq. (24) in the following way

\[
\Theta^{00} = \frac{1}{2} \left[ E_j - (M^{-1})_{ja} S_a \partial_0 \phi \right] M_{jk} \left[ E_k - (M^{-1})_{ka} S_a \partial_0 \phi \right] + \frac{1}{2} (1 - \kappa_{ii}) \left[ B + \epsilon_{jk} S_j \partial_k \phi \right] \\
+ \frac{1}{2} \left[ 1 + \kappa_{jj} - (M^{-1})_{ij} S_i S_j \right] \left( \partial_0 \phi \right)^2 + \frac{1}{2} \left[ N_{jk} - \frac{(S_a)^2 \delta_{jk} - S_j S_k}{1 - \kappa_{ii}} \right] \partial_j \phi \partial_k \phi.
\]  

\(28\)

It shows that the energy density is positive-definite whenever the LV parameters are sufficiently small.

### III. DISPERSION RELATIONS

In this section, we compute the dispersion relations of the model described by the Lagrangian density \(\text{[16]}\). Our approach follows an alternative way by evaluating the vacuum-vacuum amplitude (VVA) of the model. After the Hamiltonian analysis, the well-defined vacuum-vacuum amplitude (VVA) for the model, in the generalized Lorentz gauge, can be written as

\[
Z = \text{det} \left( \xi^{-1/2} \right) \int \text{DA}_\mu \text{D} \phi \exp \left\{ i \int dx \frac{1}{2} A_\mu D^{\mu \nu} A_\nu - \frac{1}{2} \phi \Box \phi + \phi \mathbb{J} A_\mu \right\},
\]

\(29\)

where \(\xi\) is the gauge-fixing parameter and we have defined the following operators:

\[
D^{\mu \nu} = (\Box + \kappa^\rho\sigma \partial_\rho \partial_\sigma) g^{\mu \nu} + (\xi^{-1} - 1) \partial^\mu \partial^\nu + \kappa^{\mu \nu} \Box - \kappa^{\mu \rho} \partial_\rho \partial^\nu - \kappa^{\nu \rho} \partial_\rho \partial^\mu;
\]

\(30\)

\[
\Box = (1 - \eta) \Box + \kappa^{\mu \nu} \partial_\mu \partial_\nu, \quad \mathbb{J} = S^\mu \Box - S^\nu \partial_\nu \partial^\mu.
\]

\(31\)

With the purpose of understanding the dispersion relations of the full model, we first analyze the dispersion relations of the gauge and scalar sectors when considered uncoupled.

#### A. Uncoupled dispersion relations

For \(S^\mu = 0\) the vacuum-vacuum amplitude \(\text{[20]}\) is factored as \(Z = Z_{A_\mu} Z_\phi\), where \(Z_{A_\mu}\) and \(Z_\phi\) are the vacuum-vacuum amplitudes for the pure gauge and pure scalar fields, respectively.

1. Dispersion relation for the pure gauge field

The vacuum-vacuum amplitude for the pure gauge field is

\[
Z_{A_\mu} = \text{det} \left( \xi^{-1/2} \right) \int \text{DA}_\mu \exp \left\{ i \int dx \frac{1}{2} A_\mu D^{\mu \nu} A_\nu \right\} = \text{det} \left( \xi^{-1/2} \right) (\text{det} D^{\mu \nu})^{-1/2},
\]

\(32\)

with the operator \(D^{\mu \nu}\) defined by \(\text{[30]}\). By computing the functional determinant,

\[
\text{det} D^{\mu \nu} = \text{det} \left( \xi^{-1} \Box \right) \text{det} (\Box),
\]

\(33\)

the VVA for the pure gauge field is

\[
Z_{A_\mu} = \text{det} (\Box)^{-1/2},
\]

\(34\)
where the operator $\Box$ in momentum space reads as

$$\tilde{\Box} = \alpha p_0^2 + \beta p_0 + \gamma,$$  

(35)

with the coefficients defined as

$$\alpha = (1 + \kappa_{\mu\nu})(1 + \kappa_{00} - \text{tr} K) + \text{det} K, \quad K = [\kappa_{ij}],$$  

(36)

$$\beta = -2\kappa_{0i} Q_{ij} p_j, \quad Q_{ij} = [(1 + \kappa_{00})\delta_{ij} - \kappa_{ij}],$$  

(37)

$$\gamma = (1 - \text{tr} K)[\kappa_{ij} p_i p_j - (1 + \kappa_{00})p^2] - (\epsilon_{ij} p_i \kappa_{0j})^2.$$  

(38)

The dispersion relations for the pure gauge field are obtained from the condition $\tilde{\Box} = 0$, which yields

$$p_0 = \frac{\kappa_{0i} Q_{ij} p_j}{\alpha} \pm \frac{\sqrt{(\kappa_{0i} Q_{ij} p_j)^2 - \alpha(1 - \text{tr} K)[\kappa_{ij} p_i p_j - (1 + \kappa_{00})p^2] + \alpha(\epsilon_{ij} p_i \kappa_{0j})^2}}{\alpha}.$$  

(39)

It is easy to show that this relation implies nonbirefringence at any order in the LIV parameters, once it yields the same phase velocity for the left and right modes traveling at the same sense. For similar situations, see Ref. [28]. At first order, it is given by

$$p_0 = \kappa_{0i} p_i \pm |p| \left(1 - \frac{1}{2} \kappa_{00} - \frac{\kappa_{ij} p_i p_j}{2p^2}\right).$$  

(40)

The gauge dispersion relation (39) can specialized for some particular cases. For $\kappa_{ij} = 0, \kappa_{0j} = 0$, the Lorentz-violating coefficients are represented by the parity-even element $\kappa_{00}$ and the Eq. (39) yields the relation

$$p_0 = \pm \frac{|p|}{(1 + \kappa_{00})^{1/2}},$$  

(41)

which is the isotropic parity-even dispersion relation. Adopting $\kappa_{00} = 0, \kappa_{0j} = 0$, we achieve the anisotropic dispersion relation,

$$p_0 = \pm N_0 |p| \sqrt{1 - \kappa_{ij} p_i p_j / p^2},$$  

(42)

where $N_0 = \sqrt{(1 - \text{tr} K)/(1 - \text{tr} K + \text{det} K)}$. This relation involves parity-even and parity-odd coefficients.

For $\kappa_{ij} = 0, \kappa_{00} = 0$, we attain other anisotropic dispersion relation,

$$p_0 = \kappa_{0i} p_i \pm |p| \sqrt{1 + (\kappa_{0i})^2}.$$  

(43)

The energy-momentum tensor of the pure gauge field shows that the electromagnetic sector represents a stable theory. The relations (41, 42, 43), however, could anticipate a noncausal electrodynamics for some values of the LIV coefficients. The spoil of causality may be inferred from the evaluation of the group velocity ($u_g = dp_0/d|p|$) associated with each dispersion relation.

2. Dispersion relation of the pure scalar sector

In the same way, the vacuum-vacuum amplitude for the uncoupled scalar field is

$$Z_\phi = \int D\phi \exp \left\{ -\frac{i}{2} \int dx \phi \Box \phi \right\} = (\text{det} \Box)^{-1/2},$$  

(44)

with the operator $\Box$ defined in Eq. (31). In the momentum space it is read as

$$\tilde{\Box} = (1 - \eta) p^2 + \kappa^{\rho\sigma} p_\rho p_\sigma.$$  

(45)
The dispersion relation of the scalar field are computed by the condition \( \Box = 0 \), taking into account the relation (17), which provides the following equation for \( p_0 \):

\[
(1 + \text{tr} K) p_0^2 - 2 (\kappa_0 p_i) p_0 - (1 - \kappa_{00} + \text{tr} K) p^2 + \kappa_{ij} p_i p_j = 0,
\]

whose roots are

\[
p_0^{(\pm)} = \lambda \left[ \kappa_0 p_i \pm \sqrt{(\kappa_0 p_i)^2 + (1 + \text{tr} K) \left[ (1 - \kappa_{00} + \text{tr} K) p^2 - \kappa_{ij} p_i p_j \right]} \right],
\]

where \( \lambda = [1 + \text{tr} K]^{-1} \). This is a nonbirefringent relation at any order in LIV parameters. At first order such relation is given by

\[
p_0 = \kappa_0 p_i \pm |p| \left( 1 - \frac{1}{2} \kappa_{00} - \frac{\kappa_{ij} p_i p_j}{2 p^2} \right),
\]

which is exactly the first-order gauge dispersion relation given in Eq. (40). Although the exact dispersion relations of the scalar and gauge sectors, (39) and (47), are clearly different, at first order in the LIV parameters both sectors are governed by the same dispersion relation. A direct analysis of the relation (47) indicates that the scalar sector can support noncausal modes, similarly as it occurs in the gauge sector.

### B. Full dispersion relations

In order to examine the complete dispersion relations, we evaluate the vacuum-vacuum amplitude by considering the presence of the coupling vector, \( S^\mu \). We first integrate the \( \phi - \)field, obtaining

\[
Z = \det (\xi^{-1/2} \Box) \det (\Box)^{-1/2} \int D A_\mu \exp \left\{ i \int dx A_\mu \Box^{\mu\nu} A_\nu \right\},
\]

where the operator \( \Box^{\mu\nu} \) is defined as

\[
\Box^{\mu\nu} = D^{\mu\nu} + \bar{g}^{\mu\nu}. \tag{50}
\]

By integrating the gauge field, we achieve

\[
Z = \det (\xi^{-1/2} \Box) \det (\Box)^{-1/2} \det (\Box^{\mu\nu})^{-1/2}, \tag{51}
\]

which can be rewritten as

\[
Z = \det (\xi^{-1/2} \Box) \det (\Box) \det (\Box^{\mu\nu} + \bar{g}^{\mu\nu})^{-1/2}. \tag{52}
\]

We now compute the functional determinant of the term \( (\Box D^{\mu\nu} + \bar{g}^{\mu\nu}) \),

\[
\det (\Box D^{\mu\nu} + \bar{g}^{\mu\nu}) = \det \left( \xi^{-1/2} \Box \right)^2 \det (\Box)^2 \det (\otimes), \tag{53}
\]

which replaced in Eq. (52) leads to the simpler result

\[
Z = \det (\otimes)^{-1/2}. \tag{54}
\]

In the momentum space the operator \( \otimes \) is represented by \( \otimes (p) \) and the dispersion relations for the full model are obtained from the equation \( \otimes (p) = 0 \). In our case, we have the exact equation for the dispersion relations,

\[
\otimes (p) = a_4 (p_0)^4 + a_3 (p_0)^3 + a_2 (p_0)^2 + a_1 p_0 + a_0 = 0, \tag{55}
\]
with \( a_k \) \((k = 0, 1, 2, 3, 4)\) being functions of the LIV parameters having the following structure

\[
a_4 = 1 + a_4^{(1)} + a_4^{(2)} + a_4^{(3)}, \quad (56)
\]

\[
a_3 = a_3^{(1)} + a_3^{(2)} + a_3^{(3)}, \quad (57)
\]

\[
a_2 = -2p^2 + a_2^{(1)} + a_2^{(2)} + a_2^{(3)}, \quad (58)
\]

\[
a_1 = a_1^{(1)} + a_1^{(2)} + a_1^{(3)}, \quad (59)
\]

\[
a_0 = p^2 + a_0^{(1)} + a_0^{(2)} + a_0^{(3)}, \quad (60)
\]

where \( a_k^{(n)} \) \((n = 1, 2, 3)\) represents the contribution to \( n \)th order in the LIV parameters to the coefficient \( a_k \), whose explicit expressions are given in the appendix A. Below we present some configurations of the LIV parameters which allow to factorize and solve exactly the full dispersion relation equation given in Eq. (55).

We first analyze the pure contribution of the coupling vector \( S^\mu \) to the dispersion relations of the scalar and gauge fields. For this purpose, we set \( \kappa^{\mu\nu} = 0 \) in the full vacuum-vacuum amplitude \([54]\), obtaining

\[
Z = \det (\nabla)^{-1/2} \det \left[ (1 + S^2) \nabla - (S \cdot \partial)^2 \right]^{-1/2}. \quad (61)
\]

It describes two bosonic degrees of freedom; a first one is a gauge field governed by the usual dispersion relation,

\[
p_0 = \pm |p|, \quad (62)
\]

while the second one describes a massless scalar field

\[
(p_0)_\pm = -S_0 (S \cdot p) \pm \frac{\sqrt{p^2 (1 + S^2) (1 - S^2) + (S \cdot p)^2 (1 - S^2) + (S_0)^2 (S \cdot p)^2}}{1 - S^2}, \quad (63)
\]

which also is compatible with absence of birefringence. At leading-order the above dispersion relation reads as

\[
(p_0)_\pm = -S_0 (S \cdot p) \pm |p| \left( 1 + \frac{1}{2} (S_0)^2 + \frac{1}{2} \frac{(S \cdot p)^2}{p^2} \right), \quad (64)
\]

showing that the contributions of the vector \( S_\mu \) to the dispersion relations only begin at second order.

The second case corresponds to the general isotropic dispersion relation, provided by fixing \( \kappa_{ij} = 0 \), \( \kappa_{0i} = 0 \) and \( S_i = 0 \). The partition function \([54]\) factorizes as

\[
Z = \det \left[ (1 + \kappa_{00}) \nabla + \kappa_{00} \nabla^2 \right]^{-1/2} \det \left[ (1 + \kappa_{00}) \nabla - \left\{ (S_0)^2 - (k_{00})^2 - \kappa_{00} \right\} \nabla^2 \right]^{-1/2}, \quad (65)
\]

describing two bosonic degree of freedom supporting the following dispersion relations:

\[
p_0 = \pm |p|/\sqrt{1 + \kappa_{00}}, \quad (66)
\]

\[
p_0 = \pm |p| \sqrt{\frac{1 - (\kappa_{00})^2 + (S_0)^2}{1 + \kappa_{00}}}. \quad (67)
\]

The relation \((66)\) describes the gauge field, while the relation \((67)\) is associated to the massless scalar field. This association comes from Eqs. \((39)\) and \((47)\), when properly written for the pure isotropic coefficient, \( \kappa_{00} \).

A third case is obtained by considering \( \kappa_{0i} \) and \( S_0 \) as non-null, which provides the following vacuum-vacuum amplitude

\[
Z = \det \left[ \nabla - 2\kappa_{0i} \partial_i \partial_0 \right]^{-1/2} \det \left[ \nabla - 2\kappa_{0i} \partial_i \partial_0 - \left\{ (S_0)^2 + (k_{0i})^2 \right\} \nabla^2 \right]^{-1/2}. \quad (68)
\]
The specialization of the exact relations (39) and (47) for the coefficients $\kappa_0$ and the gauge field dispersion relation is the scalar and the gauge field dispersion relation, governed by the same dispersion relations.

Here, it is important to highlight that at first-order in the LIV backgrounds the dispersion relations (77,78) are free from the influence of the vector $S$ and the gauge field $A$, confirming our previous computations. We thus verify that the all the dispersion relations of this planar model are free from the influence of the vector $S^\mu$ at first-order in the LIV parameters.

Both dispersion relations can expressed at second order in the LIV coefficients, yielding

$$p_0 = \pm |p| \left(1 + \frac{A(2) - 2B^{(4)}}{8p^2}\right),$$

$$p_0 = \pm |p| \left(1 + \frac{A(2) + 2B^{(4)}}{8p^2}\right),$$

where

$$A(2) = \mathbf{p}^2 (\kappa_0)^2 + 2 (\mathbf{S} \cdot \mathbf{p})^2,$$

$$B^{(4)} = \mathbf{p}^4 (\kappa_0)^4 + 4 \mathbf{p}^2 (\kappa_0)^2 \mathbf{S}^2 - 6 \mathbf{p}^2 (\kappa_0)^2 (\mathbf{S} \cdot \mathbf{p})^2 + (\mathbf{S} \cdot \mathbf{p})^4.$$

Here, it is important to highlight that at first-order in the LIV backgrounds the dispersion relations (77,78) are the same one, confirming the results of the previous subsections: at first order the scalar and the gauge sectors are governed by the same dispersion relations.

For arbitrary configurations of the LIV backgrounds, it is convenient to compute the roots of the dispersion relations in a perturbative way. At first order in the LIV parameters, we obtain

$$p_0^{(s,g)} = \kappa_0 p_i \pm |p| \left(1 - \frac{1}{2}\kappa_0 - \frac{1}{2} \kappa_{ij} p_j p_i \right),$$

for the dispersion relations of the gauge and massless scalar fields. This is the same expression of Eqs. (79,78), confirming our previous computations. We thus verify that the all the dispersion relations of this planar model are free from the influence of the vector $S^\mu$ at first-order in the LIV parameters.
IV. EQUATIONS OF MOTION AND STATIONARY SOLUTIONS

The classical behavior of this theory is governed by the equations of motion stemming from the Euler-Lagrange equations, that is

\[ \partial_\alpha F^{\alpha \beta} + \kappa^{\beta \rho} \partial_\rho F^{\alpha \beta} - \kappa^{\alpha \rho} \partial_\rho F^{\beta \beta} + S^3 \square \phi - S^0 \partial_\alpha \partial^2 \phi = J^3, \]  

(82)

\[ [1 - \eta] \square \phi + \kappa^{\alpha \rho} \partial_\alpha \phi + S_\rho \partial_\alpha F^{\mu \alpha} = -J. \]  

(83)

In terms of the gauge potential and by using the Lorentz gauge, \( \partial \cdot A = 0 \), these equations are written as

\[ [\square g^{\beta \rho} + \square \kappa^{\beta \rho} + g^{\beta \rho} \kappa^{\alpha \sigma} \partial_\sigma \partial_\rho - \kappa^{\alpha \sigma} \partial_\rho \partial^2 \] \( \phi = J^3, \)  

(84)

\[ [(1 - \eta) \square + \kappa^{\alpha \rho} \partial_\alpha \phi] + S_\rho \partial_\alpha A^\nu = -J. \]  

(85)

The modified Maxwell equations stems from Eq.(82) lead to the altered forms for the Gauss’s and Ampere’s laws,

\[ (1 + \kappa_{00}) \partial_t E_i + \epsilon^{ij} \kappa_0 \partial_0 B - \kappa_{ij} \partial_j E^3 - S_0 \nabla^2 \phi - S^i \partial_i \partial_0 \phi = \rho, \]  

(86)

\[ \left( \epsilon^{ij} - \kappa_{il} \epsilon^{ij} - \kappa_{jl} \epsilon^{il} \right) \partial_j B + \kappa_{00} \epsilon^{ij} \partial_0 B - \partial_0 E^i + \kappa_{0l} \partial_0 E^j - \kappa_{il} \partial_j E^j \]  

(87)

\[ + \kappa_{j0} \partial_j E^3 - S^i \nabla^2 \phi + S^j \partial_0^2 \phi - S^j S_0 \partial_0 \partial_0 \phi = J^i, \]  

while the scalar sector evolves in accordance with

\[ [1 - \eta + \kappa_{00}] \partial_t^2 \phi - [1 - \eta] \nabla^2 \phi + \kappa^3 \partial_0 \partial_j \phi + 2 \kappa^{0j} \partial_0 \partial_j \phi - S_0 \partial_0 E^i + S_l \partial_0 E^j - \epsilon_{ij} S_0 \partial_0 B = -J. \]  

(88)

In order to solve this electrodynamics, Eqs.(83) should be considered jointly with the Faraday’s law,

\[ \partial_t A + \nabla \times E = 0. \]  

(89)

which comes from the tensor form of Bianchi identity, \( \partial_\mu F^{\mu \nu} = 0 \). Here, \( F^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} F_{\alpha \beta} \) is the the dual of the electromagnetic field tensor in \((1 + 2)\)-dimensions.

At first order in LIV parameters, the solutions of the equations of motion (84) and (85) are

\[ A_\mu = \frac{1}{\square} \left( g_{\mu \rho} - \kappa_{\mu \rho} - g_{\mu \rho} \kappa^{\alpha \beta} \partial_\alpha \partial_\beta + \kappa_{\rho \alpha} \partial^\alpha \partial_\mu \right) J^\rho + \frac{1}{\square} \left( S_\mu - S^\rho \partial_\rho \partial_\mu \right) J, \]  

(90)

\[ \phi = -\frac{1}{\square} \left[ 1 + \eta - \kappa^{\alpha \beta} \partial_\alpha \partial_\beta \right] J + \frac{1}{\square} S_\rho J^\rho. \]  

(91)

The pure Green’s functions for the gauge and the scalar fields read

\[ G_{\mu \rho} (x - x') = \frac{1}{\square} \left[ g_{\mu \rho} - \kappa_{\mu \rho} - g_{\mu \rho} \kappa^{\alpha \beta} \partial_\alpha \partial_\beta + \kappa_{\rho \alpha} \partial^\alpha \partial_\mu \right] \delta (x - x'), \]  

(92)

\[ G_\mu (x - x') = \frac{1}{\square} \left( S_\mu - S^\rho \partial_\rho \partial_\mu \right) \delta (x - x'), \]  

(93)

\[ G (x - x') = -\frac{1}{\square} \left[ 1 + \eta - \kappa^{\alpha \beta} \partial_\alpha \partial_\beta \right] \delta (x - x'), \]  

(94)

respectively, where \( x = (x_0, r) \). The above equations show the both sources \( J^\mu \) and \( J \) can be generate electromagnetic phenomena.
A. Static solutions for the pure gauge field

The stationary solution for the gauge field in \((99)\) can be expressed as

\[
A_\mu (r) = \int dr' G_{\mu \rho} (r - r') J^\rho (r') + \int dr' G_\mu (r - r') J (r'),
\]

where \(G_{\mu \rho} (r - r')\) is the stationary Green’s function whose components obtained from \((92)\) are

\[
G_{00} (R) = -\frac{1}{2\pi} \left( 1 - \kappa_{00} + \frac{1}{2} \kappa_{aa} \right) \ln R - \frac{1}{4\pi} \kappa_{ab} R_a R_b \frac{R}{R^2},
\]

\[
G_{0i} (R) = \frac{1}{2\pi} \kappa_{0i} \ln R, \quad G_{i0} (R) = \frac{1}{4\pi} \kappa_{0i} \ln R - \frac{1}{4\pi} \kappa_{0a} R_a R_i \frac{R}{R^2},
\]

\[
G_{ij} (R) = \frac{1}{2\pi} \left[ \delta_{ij} \left( 1 + \frac{1}{2} \kappa_{aa} \right) + \frac{1}{2} \kappa_{ij} \right] \ln R + \frac{1}{4\pi} \delta_{ij} \kappa_{ab} R_a R_b \frac{R}{R^2} + \frac{1}{4\pi} \kappa_{ja} R_a R_i \frac{R}{R^2},
\]

and \(G_\mu (r - r')\) is the Green’s functions describing the contribution of the scalar source \(J\) to the electromagnetic field given by

\[
G_0 (R) = -\frac{1}{2\pi} S_0 \ln R, \quad G_i (R) = -\frac{1}{4\pi} S_i \ln R + \frac{1}{4\pi} S_a R_a R_i \frac{R}{R^2},
\]

where we have denoted \(R = r - r'\).

The non-diagonal Green’s function components reveal that charges yield electric and magnetic fields, as well as currents do. We now compute the electric and magnetic fields for some special configurations of charge and current densities. In accordance with Eq. \((95)\), the scalar and vector potentials are

\[
A_0 (r) = -\frac{1}{2\pi} \left( 1 - \kappa_{00} + \frac{1}{2} \kappa_{aa} \right) \int dr' \rho (r') \ln |r - r'| - \frac{1}{4\pi} \kappa_{ab} \int dr' \frac{(r-r')_a (r-r')_b}{(r-r')^2} \rho (r')
\]

\[+ \frac{1}{2\pi} \kappa_{0a} \int dr' J^a (r') \ln |r - r'| - \frac{1}{2\pi} S_0 \int dr' J (r') \ln |r - r'| \]

\[
\text{and}
\]

\[
A_j = \frac{1}{4\pi} \kappa_{0j} \int dr' \rho (r') \ln |r - r'| - \frac{1}{4\pi} \kappa_{0a} \int dr' \frac{(r-r')_a (r-r')_j}{|r-r'|^2} \rho (r')
\]

\[+ \frac{1}{2\pi} \left[ \delta_{jb} \left( 1 + \frac{1}{2} \kappa_{aa} \right) + \frac{1}{2} \kappa_{jb} \right] \int dr' J^b (r') \ln |r - r'| \]

\[+ \frac{1}{4\pi} \delta_{jc} \kappa_{ab} \int dr' \frac{(r-r')_a (r-r')_b J^c (r')}{|r-r'|^2} - \frac{1}{4\pi} \kappa_{ab} \int dr' \frac{(r-r')_a (r-r')_b J^b (r')}{|r-r'|^2}
\]

\[- \frac{1}{4\pi} S_j \int dr' J (r') \ln |r - r'| + \frac{1}{4\pi} S_a \int dr' \frac{(r-r')_a (r-r')_j}{|r-r'|^2} J (r'),
\]

respectively.

For a pointlike static charge distribution, \(\rho (r') = q \delta (r') \quad [J_i (r') = 0 = J (r')]\), the scalar potential and the potential vector are

\[
A_0 (r) = -\frac{q}{2\pi} \left[ \left( 1 - \kappa_{00} + \frac{1}{2} \kappa_{aa} \right) \ln r + \frac{1}{2} \kappa_{ab} \frac{r_a r_b}{r^2} \right],
\]

\[
A_j (r) = \frac{q}{4\pi} \left( \kappa_{0j} \ln r - \kappa_{0a} \frac{r_a r_j}{r^2} \right),
\]
respectively. The solution differs from the usual scalar potential generated by a pointlike charge in (1+2) dimensions mainly by the term \(\kappa^{ab} r_a r_b / r^2\), which yields an anisotropic behavior for it. The electric field produced by the pointlike charge is,

\[
E_i (\mathbf{r}) = -\frac{q}{2\pi} \left[ \left( 1 - \kappa_{00} + \frac{1}{2} \kappa_{aa} \right) \frac{r_i}{y^2} + \kappa_{ib} \frac{r_b}{r^2} - \kappa_{ab} \frac{r_a r_b}{r^4} r_i \right],
\]

which in addition to its radial behavior \(r^{-1}\) presents anisotropies, due to the two last terms \(\kappa_{ib} r_b / r^2\) and \(\kappa_{ab} r_a r_b / r^4\), produced by the LIV backgrounds but these Lorentz-violating corrections do not modify the global asymptotic behavior of the electric field in (1+2) dimensions: it remains decaying as \(1/r\).

From the potential vector \((101)\) we compute the associated magnetic field produced by a pointlike charge,

\[
B (\mathbf{r}) = \frac{q}{2\pi} \kappa_{0i} r_i. \quad \text{(103)}
\]

Here, we observe that the LIV parameter \(\kappa_{0i}\) engenders an anisotropic magnetic field whose asymptotic behavior goes as \(r^{-1}\). It can be used to impose an upper-bound for the \(\kappa_{0i}\) coefficients by using the experimental data concerning the two-dimensional physics.

For a pointlike charge with velocity \(\mathbf{u}\), \(J^i (\mathbf{r'}) = q\delta (\mathbf{r'}) u^i\), \([\rho (\mathbf{r'}) = 0 = J (\mathbf{r'})]\), the scalar potential is

\[
A_0 (\mathbf{r}) = -\frac{q}{2\pi} \kappa_{0i} u_a \ln r, \quad \text{(104)}
\]

while the vector potential is

\[
A_j (\mathbf{r}) = -\frac{q}{2\pi} \left[ \left( 1 + \frac{1}{2} \kappa_{aa} \right) u_j + \frac{1}{2} \kappa_{ja} u_a \right] \ln r - \frac{q}{4\pi} \kappa_{ab} u_j \frac{r_a r_b}{r^2} + \frac{q}{4\pi} \kappa_{ab} u_b \frac{r_a r_j}{r^2}. \quad \text{(105)}
\]

The respective electric and magnetic field are

\[
E_i (\mathbf{r}) = -\frac{q}{2\pi} \kappa_{0i} u_a \frac{r_i}{r^2}, \quad \text{(106)}
\]

\[
B (\mathbf{r}) = \frac{q}{2\pi} \left[ \left( 1 + \frac{1}{2} \kappa_{aa} \right) \epsilon_{ij} \frac{r_i u_j}{r^2} - \epsilon_{ij} \kappa_{ab} \frac{r_a r_b r_i u_j}{r^4} + \epsilon_{ij} \kappa_{ja} \frac{3r_i u_a - r_a u_i}{r^2} \right]. \quad \text{(107)}
\]

In this model a pointlike scalar source, \(J (\mathbf{r'}) = q\delta (\mathbf{r'})\), \([\rho (\mathbf{r'}) = 0 = J_i (\mathbf{r'})]\), also generates electromagnetic phenomena whose scalar and vector potentials are given by

\[
A_0 (\mathbf{r}) = -\frac{q}{2\pi} S_0 \ln r, \quad A_j (\mathbf{r}) = -\frac{q}{4\pi} S_j \ln r + \frac{q}{4\pi} S_a r_a r_j, \quad \text{(108)}
\]

leading to the following electric and magnetic field solutions:

\[
E_i (\mathbf{r}) = -\frac{q}{2\pi} S_0 \frac{r_i}{r^2}, \quad B (\mathbf{r}) = \frac{q}{2\pi} \epsilon_{ij} \frac{S_j r_i}{r^2}. \quad \text{(109)}
\]

**B. Static solutions for the pure scalar field**

From \(\text{[111]}\), the stationary solution for the scalar field in can be expressed as

\[
\phi (\mathbf{r}) = \int d\mathbf{r'} G (\mathbf{r} - \mathbf{r'}) J (\mathbf{r'}) - \frac{1}{2\pi} S_\mu \int d\mathbf{r'} J^\mu (\mathbf{r'}) \ln |\mathbf{r} - \mathbf{r'}| \quad \text{(110)}
\]

where \(G (\mathbf{r} - \mathbf{r'})\) is the stationary scalar Green’s function obtained from Eq. \(\text{[114]}\), we attain

\[
G (\mathbf{R}) = \frac{1}{2\pi} \left( 1 + \eta + \frac{1}{2} \kappa_{aa} \right) \ln R + \frac{1}{4\pi} \kappa_{ab} \frac{R_i R_j}{R^2}. \quad \text{(111)}
\]
The scalar field generated by a pointlike scalar source, \( J(r') = q \delta(r') \), is

\[
\phi(r) = \frac{q}{2\pi} \left[ \left(1 + \eta + \frac{1}{2} \kappa_{aa} \right) \ln r + \frac{1}{2} \kappa_{ij} r_i r_j \right].
\] (112)

We thus confirm that scalar field presents a very similar behavior to the one of the scalar potential, given by Eq. (100).

Similarly, the scalar field produced by a pointlike charge scalar source, \( \rho(r') = q \delta(r') \), and a pointlike charge with constant velocity \( u \), \( J^i(r') = q \delta(r') u^i \), are

\[
\phi(r) = -\frac{q}{2\pi} S_0 \ln r, \quad \phi(r) = \frac{q}{2\pi} S_i u_i \ln r,
\] (113)

respectively, showing similar radial behavior.

V. CONCLUSIONS

In this work, we have performed the dimensional reduction of the nonbirefringent CPT-even electrodynamics of the standard model extension. Such procedure generates a planar Lorentz-violating electrodynamics composed of a gauge field and a scalar field linearly coupled by a LIV 3-vector \( S^\mu \). Both fields have kinetic terms modified by the Lorentz violating symmetric tensor, \( \kappa^{\nu\rho} \). This planar model possesses nine independent LV components including six parity-even and three parity-odd, being more simpler than the one of Ref. [28], in which the Lorentz-violation is governed by 19 parameters (see Lagrangian (1)).

The evaluation of the energy-momentum tensor has shown that the density of energy of the full theory can be positive definite whenever the LV parameters are sufficiently small. This indicates that the full theory is endowed with energy stability. The same conclusion is valid for both the pure gauge and the pure scalar sectors. A complete study on the dispersion relations was performed. Initially, we have evaluated the dispersion relations of the gauge and scalar sector (regarded as uncoupled) from the vacuum-vacuum amplitude, revealing that, at first order, these two fields are described by the same dispersion relations. After, we have carried out the full dispersion relations, which were exactly computed for some special combinations of the LIV parameters. The coupling vector \( S^\mu \) contributes only at second order for the dispersion relations. All the expressions confirm that the planar model is nonbirefringent at any order, whereas the original (1+3)-dimensional model is nonbirefringent only at leading order. From these relations we also conclude the gauge and scalar sector are stable, but endowed with causality illness. A more careful analysis about the physical consistency of this model (stability, causality, unitarity) is under progress.

We have established the wave equations for the gauge and scalar field and we have achieved their stationary solutions, via the Green’s function technique, at first-order in the LIV coefficients. The Lorentz-violating terms induce an anisotropic character to these stationary solutions which now exhibit an explicit angular dependence. However, the LIV coefficients do not modify the long distance profile of the solutions, keeping the \( r^{-1} \) asymptotic behavior of the pure Maxwell planar electrodynamics (a fact compatible with dimensionless nature of the LIV coefficients). The scalar and vector potential generated by a pointlike scalar charge were carried out as well, showing that it generates electromagnetic fields. An analogous calculation was accomplished for the scalar sector, demonstrating that it obeys stationary solutions similar to the ones of the scalar potential \( A_0 \).

This kind of theoretical framework can find applications in usual planar systems, such as vortex and Hall systems. At moment, we are particularily interested in analyzing effects of LIV coefficients in stable vortex configurations, having already verified that the gauge sector represented by Lagrangian \( \mathcal{L}_{EM} \), when properly coupled to the Higgs sector endowed with a fourth-order self-interacting potential, supports BPS (Bogomol’nyi, Prasad, Sommerfeld) solutions. Advances will be reported elsewhere.

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Appendix A: The $a^{(n)}$ coefficients

\[ a_4^{(1)} = 2\kappa_{00}, \quad (A1) \]
\[ a_4^{(2)} = (\kappa_{00})^2 + \kappa_{00} \text{tr} (\kappa_{ij}) - S^2 - [\text{tr} (\kappa_{ij})]^2 + \det (\kappa_{ij}), \quad \mathbb{K} = [\kappa_{ij}], \quad (A2) \]
\[ a_4^{(3)} = -\kappa_{00} S^2 - \kappa_{00} \text{tr} \kappa_{ij} - \kappa_{ij} S_i S_j + \left( (\kappa_{00})^2 + (\det \mathbb{K}) + S^2 \right) \text{tr} (\kappa_{ij}). \quad (A3) \]
\[ a_4^{(1)} = -4 (\kappa_{0i} p_i), \quad (A4) \]
\[ a_3^{(2)} = 2 S_0 (\mathbf{S} \cdot \mathbf{p}) + 2 (\kappa_{0i} \kappa_{ij} p_j) - 6 \kappa_{00} (\kappa_{0i} p_i), \quad (A5) \]
\[ a_3^{(3)} = -2 (\kappa_{00})^2 (\kappa_{0i} p_i) + 2 (\mathbf{S} \cdot \mathbf{p}) \left[ \kappa_{00} S_0 - S_0 (\text{tr} \mathbb{K}) + (S_j \kappa_{0j}) \right] + 2 S_0 (\kappa_{ij} S_i p_j) + 2 (\kappa_{ia} \kappa_{0a} (\kappa_{ib} p_b) \quad (A6) \]
\[ a_2^{(1)} = 2 (\kappa_{ij} p_i p_j) - 2 \kappa_{00} \mathbf{p}^2, \quad (A7) \]
\[ a_2^{(2)} = (\kappa_{00})^2 \mathbf{p}^2 + 2 \kappa_{00} (\epsilon_{ia} \kappa_{ab} \epsilon_{bj} p_i p_j) - (S_0)^2 \mathbf{p}^2 + (S_i)^2 \mathbf{p}^2 + (\epsilon_{ij} S_i p_j)^2 \]
\[ - 4 (\kappa_{0i} p_i)^2 - (\epsilon_{ij} \kappa_{0j} p_j) (\text{tr} \mathbb{K}), \quad (A8) \]
\[ a_2^{(3)} = (\kappa_{00})^3 \mathbf{p}^2 - (\kappa_{00})^2 [2 \mathbf{p}^2 (\text{tr} \mathbb{K}) - \kappa_{ij} p_i p_j] + \kappa_{00} \left[ 2 (\epsilon_{ij} S_i p_j)^2 - \kappa_{ia} \kappa_{aj} p_i p_j + 4 (\kappa_{0i} p_i)^2 + 2 (\text{tr} \mathbb{K})^2 \mathbf{p}^2 \right] \]
\[ - \kappa_{00} (S_0)^2 \mathbf{p}^2 - (S_0)^2 (\epsilon_{ia} \kappa_{ab} \epsilon_{bj} p_i p_j) - 2 S_0 \left[ (\kappa_{0i} p_i) (S_j p_j) + \mathbf{p}^2 (\kappa_{0i} S_i) \right] + \mathbf{p}^2 (\epsilon_{ia} \kappa_{ab} \epsilon_{bj} S_i S_j) \]
\[ - (S_k)^2 (\kappa_{ij} p_i p_j) - 2 (\kappa_{0a} p_a) (\kappa_{0i} \kappa_{ij} p_j) - (\kappa_{0k})^2 (\kappa_{ij} p_i p_j) - \mathbf{p}^2 (\kappa_{0i} \kappa_{ij} \kappa_{0j} - (\text{tr} \mathbb{K})^2) p_i p_j \]
\[ - 2 \mathbf{p}^2 (\text{tr} \mathbb{K}) \det (\mathbb{K}). \quad (A9) \]
\[ a_1^{(1)} = 4 \mathbf{p}^2 (\kappa_{0i} p_i) \quad (A10) \]
\[ a_1^{(2)} = 2 \mathbf{p}^2 \left[ \kappa_{00} (\kappa_{0i} p_i) - (\kappa_{0i} \kappa_{ij} p_j) - S_0 (S_i p_j) \right] - 4 (\kappa_{0a} p_a) (\kappa_{ij} p_j), \quad (A11) \]
\[ a_1^{(3)} = -2 \mathbf{p}^2 (\kappa_{00})^2 (\kappa_{0i} p_i) - 2 \kappa_{00} (\epsilon_{ia} \kappa_{aj} p_i p_j) (\epsilon_{bc} \kappa_{0b} p_c) + 2 \mathbf{p}^2 (S_0)^2 (\kappa_{0i} p_i) + 2 S_0 (S_i p_i) (\kappa_{ij} p_j) \]
\[ - 2 \mathbf{p}^2 S_0 (\epsilon_{ia} \kappa_{ab} \epsilon_{bj} S_i S_j) + 2 (\kappa_{ij} p_i p_j) (\kappa_{ij} \kappa_{0j} p_j) - 2 \mathbf{p}^2 (S_i \kappa_{0i} (S_i p_i) + 2 (S_i p_i)^2 (\kappa_{0i} p_i) \]
\[ + 2 (\text{tr} \mathbb{K}) (\epsilon_{ia} \kappa_{aj} p_j) (\epsilon_{bc} \kappa_{0b} p_c) + 2 (\kappa_{0i} p_i) (\epsilon_{bc} \kappa_{0b} p_c)^2. \quad (A12) \]
\[ a_0^{(1)} = -2 \mathbf{p}^2 (\kappa_{ij} p_i p_j), \quad (A13) \]
\[ a_0^{(2)} = - (\kappa_{00})^2 \mathbf{p}^4 + \mathbf{p}^4 \kappa_{00} (\text{tr} \mathbb{K}) + (S_0)^2 \mathbf{p}^4 - \mathbf{p}^4 (\text{tr} \mathbb{K})^2 - \mathbf{p}^2 (\epsilon_{ij} S_i p_j)^2 \]
\[ + \mathbf{p}^2 (\epsilon_{ij} \kappa_{0j} p_j)^2 + (\kappa_{ij} p_i p_j)^2 + \mathbf{p}^2 (\text{tr} \mathbb{K}) (\kappa_{ij} p_i p_j), \quad (A14) \]
\[ a_0^{(3)} = \mathbf{p}^4 (\kappa_{00})^2 (\text{tr} \mathbb{K}) - \mathbf{p}^2 \kappa_{00} \left[ \mathbf{p}^2 (\text{tr} \mathbb{K})^2 + (\epsilon_{ij} S_i p_j)^2 + (\epsilon_{ij} \kappa_{0j} p_j)^2 \right] - \mathbf{p}^4 (S_0)^2 (\text{tr} \mathbb{K}) + (\kappa_{ij} p_i p_j) (\epsilon_{ij} S_i p_j)^2 \]
\[ + 2 S_0 (\epsilon_{ij} S_i p_j) (\epsilon_{ij} \kappa_{0j} p_j) - (\kappa_{ij} p_i p_j - \mathbf{p}^2 \text{tr} \mathbb{K}) (\epsilon_{ij} \kappa_{0i} p_j)^2 - (\kappa_{ij} p_i p_j - \mathbf{p}^2 \text{tr} \mathbb{K}) (\epsilon_{ij} S_i p_j)^2. \]
