Transverse Weitzenböck formulas and curvature dimension inequalities on Riemannian foliations with totally geodesic leaves

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Abstract

We prove a family of Weitzenböck formulas on a Riemannian foliation with totally geodesic leaves. These Weitzenböck formulas are naturally parametrized by the canonical variation of the metric. As a consequence, under natural geometric conditions, the horizontal Laplacian satisfies a generalized curvature dimension inequality. Among other things, this curvature dimension inequality implies Li-Yau estimates for positive solutions of the horizontal heat equation sharp eigenvalue estimates and a sub-Riemannian Bonnet-Myers compactness theorem whose assumptions only rely on the intrinsic geometry of the horizontal distribution.

Keywords: Sub-Riemannian geometry, Weitzenböck formula, Bochner method, Riemannian foliation

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1 Introduction

In the recent few years, there has been a great deal of interest in developing geometric analysis methods in sub-Riemannian geometry that parallel the methods which are available in Riemannian geometry (see for instance [1, 2] and the references therein). One of these fundamental methods is the Bochner’s method which connects the geometry and topology of the ambient manifold to the analysis of a second order differential operator, the Laplace-Beltrami operator. In all generality, there is no canonical second-order operator on a sub-Riemannian manifold. However, in many interesting situations, like in the Hopf fibrations, the sub-Riemannian geometry of interest is associated to the horizontal distribution of a Riemannian foliation. In that particular case, there is a canonical sub-Riemannian Laplacian: the horizontal Laplacian of the foliation. With such an operator in hand, it is then natural to wonder if a suitable analogue of the Bochner’s would make the bridge between the analysis of this horizontal Laplacian and the sub-Riemannian geometry and the topology of the ambient manifold.

In the present paper, we prove that on a Riemannian foliation with totally geodesic leaves and a bracket generating horizontal distribution many fundamental tools are available to study the horizontal Laplacian and the associated sub-Riemannian geometry. These tools include Li-Yau type gradient estimates for positive solutions of the heat equation and associated Harnack inequalities, volume comparison theorems for the metric balls, sharp Sobolev inequalities, sharp first eigenvalue estimates and corresponding rigidity results. Our work here builds on several previous works, notably [7, 8, 9], where it is proved that, if a certain curvature dimension inequality and some further assumptions are satisfied, then the above tools are automatically available. So, our main contribution here is to prove that this inequality and assumptions are satisfied in any Riemannian foliation with totally geodesic leaves if the following natural geometric conditions are fulfilled: completeness of the metric, uniform bracket generating condition, global lower bound on the horizontal Ricci curvature and global upper bound on the torsion of the Bott’s connection. Proving the curvature dimension estimate under these assumptions is not easy and our analysis relies on new taylor-made Bochner-Weitzenböck type inequalities for a suitable one-parameter family of sub-Laplacians on one-forms.

It is now time to give more details on our contribution. In the work [9] the authors proved that on a sub-Riemannian manifold with transverse symmetries, assuming natural geometric conditions, the sub-Laplacian satisfies a generalized curvature dimension inequality. Among other things, this curvature dimension estimate implies Li-Yau inequalities for positive solutions of the heat equation [7, 9], Gaussian lower and upper bounds for the subelliptic heat kernel [7, 9], log-Sobolev and isoperimetric inequalities [6, 11], volume and distance comparison estimates [8] and a Bonnet-Myers type theorem [9]. Recently, it has been pointed out by Elworthy [15] that sub-Riemannian manifolds with transverse symmetries can be seen as Riemannian manifolds with bundle like metrics which are foliated by totally geodesic leaves. The goal of the present work is two-fold:
We actually prove that on any Riemannian foliation with bundle like metric and totally geodesic leaves, under natural geometric conditions, the horizontal Laplacian satisfies the curvature dimension estimate introduced in \cite{9}. As a consequence, all the results proved in \cite{6, 7, 8, 9, 10, 11, 17} apply in this much more general case.

We simplify the original approach of \cite{9} by working out new Weitzenböck type identities for the horizontal Laplacian which we think are interesting in themselves. These Weitzenböck identities easily imply not only the curvature dimension estimate but also the stochastic completeness of the heat semigroup, which is a crucial ingredient to run the machinery developed in \cite{9}.

The paper is organized as follows. In Section 2, we give the basic definitions and conventions that will be used throughout the text. In Section 3, we introduce a canonical one parameter family of horizontal Laplacians on one-forms and prove Weitzenböck-Bochner’s type inequalities for this family of horizontal Laplacians. In Section 4, we prove the generalized curvature dimension inequality. We point out that, unlike many previous works on Riemannian foliations (see \cite{19, 20} and the references therein), our results in Section 4 actually concern the sub-Riemannian geometry associated to the horizontal distribution and are not restricted to the transversal geometry of the foliation.

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2 Preliminaries

Let $\mathbb{M}$ be a smooth, connected manifold with dimension $n + m$. We assume that $\mathbb{M}$ is equipped with a Riemannian foliation $\mathcal{F}$ with bundle like metric $g$ and totally geodesic $m$-dimensional leaves (see the classical monograph by Tondeur \cite{19} for the basic properties of such foliations).

The sub-bundle $\mathcal{V}$ formed by vectors tangent to the leaves is referred to as the set of vertical directions. The sub-bundle $\mathcal{H}$ which is normal to $\mathcal{V}$ is referred to as the set of horizontal directions. The metric $g$ can be split as

$$g = g_{\mathcal{H}} \oplus g_{\mathcal{V}},$$

and for later use, we introduce the one-parameter family of Riemannian metrics:

$$g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon}g_{\mathcal{V}}, \quad \varepsilon > 0,$$

which is going to play a pervasive role in the sequel. It is called the canonical variation of $g$, see Chapter 9 G in the monograph by Besse \cite{14}.

There is a canonical connection on $\mathbb{M}$, the Bott connection, which is given as follows:
\[ \nabla_X Y = \begin{cases} 
\pi_H(\nabla^R_X Y), & X, Y \in \Gamma^\infty(\mathcal{H}) \\
\pi_H([X, Y]), & X \in \Gamma^\infty(\mathcal{V}), Y \in \Gamma^\infty(\mathcal{H}) \\
\pi_V([X, Y]), & X \in \Gamma^\infty(\mathcal{H}), Y \in \Gamma^\infty(\mathcal{V}) \\
\pi_V(\nabla^R_X Y), & X, Y \in \Gamma^\infty(\mathcal{V}) 
\end{cases} \]

where \(\nabla^R\) is the Levi-Civita connection and \(\pi_H\) (resp. \(\pi_V\)) the projection on \(\mathcal{H}\) (resp. \(\mathcal{V}\)). It is easy to check that for every \(\varepsilon > 0\), this connection satisfies \(\nabla g_\varepsilon = 0\).

We define the horizontal gradient \(\nabla_H f\) of a function \(f\) as the projection of the Riemannian gradient of \(f\) on the horizontal bundle. Similarly, we define the vertical gradient \(\nabla_V f\) of a function \(f\) as the projection of the Riemannian gradient of \(f\) on the vertical bundle. The horizontal Laplacian is the generator of the symmetric Dirichlet form

\[ E(\mathcal{H})(f, g) = \int_M \langle \nabla_H f, \nabla_H g \rangle_H d\mu. \]

It is a diffusion operator \(L\) on \(\mathcal{M}\) which is symmetric on \(C_0^\infty(\mathcal{M})\) with respect to the volume measure \(\mu\).

We now introduce some tensors that will play an important role in the sequel.

For \(Z \in \Gamma^\infty(TM)\), there is a unique skew-symmetric endomorphism \(J_Z : \mathcal{H}_x \to \mathcal{H}_x\) such that for all horizontal vector fields \(X\) and \(Y\),

\[ g_H(J_Z(X), Y) = g_V(Z, T(X, Y)). \tag{2.1} \]

where \(T\) is the torsion tensor of \(\nabla\). We then extend \(J_Z\) to be 0 on \(\mathcal{V}_x\). If \(Z_1, \cdots, Z_m\) is a local vertical frame, the operator \(\sum_{i=1}^m J_{Z_i}J_{Z_i}\) does not depend on the choice of the frame and shall concisely be denoted by \(J^2\). For instance, if \(\mathcal{M}\) is a \(K\)-contact manifold equipped with the Reeb foliation, then \(J\) is an almost complex structure, \(J^2 = -\text{Id}_H\).

The horizontal divergence of the torsion \(T\) is the \((1,1)\) tensor which is defined in a local horizontal frame \(X_1, \cdots, X_n\) by

\[ \delta_H T(X) = \sum_{j=1}^n (\nabla_{X_j} T)(X_j, X). \]

The \(g\)-adjoint of \(\delta_H\) will be denoted \(\delta_H^* T\).

By declaring a one-form to be horizontal (resp. vertical) if it vanishes on the vertical bundle \(\mathcal{V}\) (resp. on the horizontal bundle \(\mathcal{H}\)), the splitting of the tangent space

\[ T_x \mathcal{M} = \mathcal{H}(x) \oplus \mathcal{V}(x) \]

gives a splitting of the cotangent space.
The metric $g_\varepsilon$ induces then a metric on the cotangent bundle which we still denote $g_\varepsilon$. By using similar notations and conventions as before we have for every $\eta$ in $T^*_xM$,

$$\|\eta\|_\varepsilon^2 = \|\eta\|^2_H + \varepsilon \|\eta\|_V^2.$$ 

By using the duality given by the metric $g$, $(1,1)$ tensors can also be seen as linear maps on the cotangent bundle $T^*\mathbb{M}$. More precisely, if $A$ is a $(1,1)$ tensor, we will still denote by $A$ the fiberwise linear map on the cotangent bundle which is defined as the $g$-adjoint of the dual map of $A$. The same convention will be made for any $(r,s)$ tensor.

We define then the horizontal Ricci curvature $\mathfrak{Ric}_H$ as the fiberwise symmetric linear map on one-forms such that for every smooth functions $f, g$,

$$\langle \mathfrak{Ric}_H(df), dg \rangle = \text{Ricci}(\nabla_H f, \nabla_H g),$$

where $\text{Ricci}$ is the Ricci curvature of the connection $\nabla$.

If $V$ is a horizontal vector field and $\varepsilon > 0$, we consider the fiberwise linear map from the space of one-forms into itself which is given for $\eta \in \Gamma^\infty(T^*\mathbb{M})$ and $Y \in \Gamma^\infty(TM)$ by

$$\mathfrak{T}_V^\varepsilon \eta(Y) = \begin{cases} \frac{1}{\varepsilon} \eta(J_Y V), & Y \in \Gamma^\infty(V) \\ -\eta(T(V,Y)), & Y \in \Gamma^\infty(H) \end{cases}$$

We observe that $\mathfrak{T}_V^\varepsilon$ is skew-symmetric for the metric $g_\varepsilon$ so that $\nabla - \mathfrak{T}_\varepsilon$ is a $g_\varepsilon$-metric connection.

If $Z_1, \cdots, Z_m$ is a local vertical frame of the leaves, we denote

$$\mathfrak{J}(\eta) = \sum_{\ell=1}^m J_{Z_\ell}(\iota_{Z_\ell} d\eta_V),$$

where $\eta_V$ is the the projection of $\eta$ to the vertical cotangent bundle and $\iota$ the usual interior product. Of course, $\mathfrak{J}$ does not depend on the choice of the frame.

If $\eta$ is a one-form, we define the horizontal gradient in a local adapted frame of $\eta$ as the $(0,2)$ tensor

$$\nabla_H \eta = \sum_{i=1}^n \nabla_X \eta \otimes \theta_i.$$ 

We denote by $\nabla^\#_H \eta$ the symmetrization of $\nabla_H \eta$.

Similarly, we will use the notation

$$\mathfrak{T}_H^\varepsilon \eta = \sum_{i=1}^n \mathfrak{T}_X^\varepsilon \eta \otimes \theta_i.$$
Finally, we will still denote by $L$ the covariant extension on one-forms of the horizontal Laplacian. In a local horizontal frame, we have thus

$$L = \sum_{i=1}^{d} \nabla_{X_i} \nabla_{X_i} \nabla_{\nabla_{X_i} X_i}.$$

3 Bochner-Weitzenböck formulas for the horizontal Laplacian

For $\varepsilon > 0$, we consider the following operator which is defined on one-forms by

$$\Box_{\varepsilon} = -(\nabla_{\mathcal{H}} - \mathcal{T}_{\mathcal{H}}^\varepsilon)^* (\nabla_{\mathcal{H}} - \mathcal{T}_{\mathcal{H}}^\varepsilon) - \frac{1}{\varepsilon} J^2 + \frac{1}{\varepsilon} \delta_{\mathcal{H}} T - \mathcal{Ric}_{\mathcal{H}},$$

where the adjoint is understood with respect to the metric $g_\varepsilon$. It is easily seen that, in a local horizontal frame,

$$-(\nabla_{\mathcal{H}} - \mathcal{T}_{\mathcal{H}}^\varepsilon)^* (\nabla_{\mathcal{H}} - \mathcal{T}_{\mathcal{H}}^\varepsilon) = \sum_{i=1}^{n} (\nabla_{X_i} - \mathcal{T}_{X_i}^\varepsilon)^2 - (\nabla_{\nabla_{X_i} X_i} - \mathcal{T}_{\nabla_{X_i} X_i})^2.$$

The following theorem is the main result of the section

**Theorem 3.1** For every $f \in C^\infty(\mathbb{M})$, we have

$$dLf = \Box_{\varepsilon} df,$$

and for every $\eta \in \Gamma^\infty(T^*\mathbb{M})$,

$$\frac{1}{2} L\|\eta\|^2_{\varepsilon} - \langle \Box_{\varepsilon} \eta, \eta \rangle_{\varepsilon}$$

$$= \|\nabla_{\mathcal{H}} \eta - \mathcal{T}_{\mathcal{H}}^\varepsilon \eta\|^2_{\varepsilon} + \langle \mathcal{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}} - \langle \delta_{\mathcal{H}} T(\eta), \eta \rangle_{\mathcal{V}} + \frac{1}{\varepsilon} \langle J^2(\eta), \eta \rangle_{\mathcal{H}}$$

$$\geq \frac{1}{n} \left( \text{Tr}_{\mathcal{H}}(\nabla_{\mathcal{H}}^\# \eta) \right)^2 - \frac{1}{4} \text{Tr}_{\mathcal{H}}(J^2_{\eta}) + \langle \mathcal{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}} - \langle \delta_{\mathcal{H}} T(\eta), \eta \rangle_{\mathcal{V}} + \frac{1}{\varepsilon} \langle J^2(\eta), \eta \rangle_{\mathcal{H}}$$

The remainder of the section is devoted to the proof of this result.

We proceed in several steps and divide the proof into several lemmas. Since the statement is local, we can assume that the Riemannian foliation comes from a Riemannian submersion with totally geodesic fibers. We fix $x \in \mathbb{M}$ throughout the proof.

Let $X_1, \cdots, X_n$ be a local orthonormal horizontal frame around $x$ consisting of basic vector fields for the submersion. We can assume that, at $x$, $\nabla_{X_i} X_j = 0$. Let now $Z_1, \cdots, Z_m$ be
a local orthonormal vertical frame around $x$. Since $X_i$ is basic, the vector field $[X_i, Z_m]$ is tangent to the leaves. We write the structure constants in that local frame:

$$[X_i, X_j] = \sum_{k=1}^{n} \omega_{ij}^k X_k + \sum_{k=1}^{m} \gamma_{ij}^k Z_k$$

$$[X_i, Z_k] = \sum_{j=1}^{m} \beta_{ik}^j Z_j,$$

and observe that at the center $x$ of the frame, we have $\omega_{ij}^k = 0$. Moreover, since $X_i$ is basic and the submersion has totally geodesic fibers, the flow generated by $X_i$ induces an isometry between the leaves (see Besse [14], Chapter 9), as a consequence we have the skew-symmetry,

$$\beta_{ik}^j = -\beta_{ij}^k.$$

It is easy to see that we can also assume that, at the center $x$, $\beta_{ij}^k = 0$ (see for instance [16], Corollary 2.22).

The dual coframe of $\{X_1, \ldots, X_n, Z_1, \ldots, Z_m\}$ will be denoted $\{\theta_1, \ldots, \theta_n, \nu_1, \ldots, \nu_m\}$ and a generic one-form $\eta$ will be written, $\eta = \sum_{i=1}^{n} f_i \theta_i + \sum_{\ell=1}^{m} g_{\ell} \nu_{\ell}$.

**Lemma 3.2** At $x$,

- $\text{Ricci}(X_i, X_k) = \sum_{j=1}^{n} \left( \frac{1}{2} X_j (\omega_{ik}^j - \omega_{ij}^k - \omega_{jk}^i) - X_i \omega_{jk}^i \right)$. 
- $\text{Ricci}(Z_\ell, X_k) = -\sum_{j=1}^{n} Z_\ell \omega_{jk}^j = 0$. 
- $\mathfrak{J}(\eta) = \sum_{i,j=1}^{n} \sum_{\ell=1}^{m} \gamma_{ij}^\ell (X_i g_\ell) \theta_j$. 
- $\delta_H T(\eta) = -\sum_{i,j=1}^{n} \sum_{\ell=1}^{m} (X_i \gamma_{ij}^\ell) f_j \nu_\ell$. 
- $\delta_H^* T(\eta) = \sum_{i,j=1}^{n} \sum_{k=1}^{m} (X_j \gamma_{ij}^k) g_k \theta_i$. 
- $\mathfrak{T}_{X_i}^\epsilon \eta = \sum_{j=1}^{n} \sum_{\ell=1}^{m} \left( \gamma_{ij}^\ell \epsilon g_\ell \theta_j - \frac{1}{\epsilon} \gamma_{ij}^\ell f_j \nu_\ell \right)$

**Proof.** The computations are routine. We just point out that the vanishing of $\text{Ricci}(Z_\ell, X_k)$ comes from the fact that since $X_k$ and $\nabla_{X_i} X_k$ basic and $[X_i, Z_\ell]$ is tangent to the leaves, we have at $x$, $\nabla_{Z_\ell} X_k = \nabla_{Z_\ell} \nabla_{X_i} X_k = \nabla_{[X_i, Z_\ell]} X_k = 0$. 

**Lemma 3.3** Let $\Box_\infty$ be the operator defined on one-forms by

$$\Box_\infty = L + 2\mathfrak{J} - 2\text{Ric}_H + \delta_H^* T,$$

then for any $f \in C^\infty(M)$,

$$dL f = \Box_\infty df.$$
Lemma 3.4

For 

\[ \text{by,} \]

Let us consider the map

\[ T \]

This completes the proof.

\[ \text{Lemma 3.2 that at } x \text{ of the frame, we have } \nabla_{X_j} \theta_i = \nabla_{X_j} \nu \theta = 0, \text{ and} \]

\[ L \theta_i = \sum_{j,k=1}^n (-X_j \Gamma^i_{jk}) \theta_k, \quad L \nu \theta = \sum_{j=1}^m \sum_{k=1}^m (-X_j \beta^i_{jk}) \nu_k. \quad (3.4) \]

where the $\Gamma^k_{ij}$’s are the Cristoffel symbols of the Bott’s connection. We also easily compute

\[ [Z_{\ell}, L]f = \sum_{j=1}^n \left(- \sum_{i=1}^n X_{i \beta^j_{ik}}\right) Z_j f \]

and

\[ [X_i, L]f = \sum_{j=1}^n \sum_{l=1}^m \gamma^j_{ij} X_j Z_{\ell} f + \sum_{\ell=1}^n \left( \sum_{j=1}^n X_j \gamma^j_{ij} \right) Z_\ell f + \sum_{j,k=1}^m \left( X_k \omega^j_{ik} - X_i \omega^k_{jk} \right) X_j f. \quad (3.5) \]

If we plug this in (3.3), then the second line of (3.3) turns out to be 0 and we deduce from Lemma 3.2 that at $x$, we have

\[ dLf - Ldf = 2\Theta(df) - 2Ric_H(df) + \delta T(df). \]

This completes the proof. \qed

Let us consider the map $T: \Gamma^\infty(\wedge^2 T^* M) \to \Gamma^\infty(T^* M)$ which is given in the local frame by,

\[ \mathcal{T}(\theta_i \wedge \theta_j) = -\gamma^j_{ij} \nu_\ell, \quad \mathcal{T}(\theta_i \wedge \nu_k) = \mathcal{T}(\nu_k \wedge \nu_\ell) = 0. \]

Lemma 3.4 For $\varepsilon > 0$

\[ \Box_{\varepsilon} = \Box_{\infty} - \frac{2}{\varepsilon} \mathcal{T} \circ d. \]

Proof. A direct computation shows that, at the center $x$, for any $\eta = \sum_{j=1}^n f_j \theta_j + \sum_{k=1}^m g_k \nu_k$,

\[ - (\nabla_{\mathcal{H}} - \mathcal{T}_\varepsilon) (\nabla_{\mathcal{H}} - \mathcal{T}_\varepsilon) \eta \]

\[ = \sum_{i,j=1}^n \left( X_i f_j \theta_j - \sum_{\ell=1}^m X_i (\gamma^\ell_{ij} g_\ell) \right) \theta_j + \sum_{\ell=1}^m \left( \sum_{i=1}^n X_i^2 g_\ell + \frac{1}{\varepsilon} \sum_{i,j=1}^n X_i (\gamma^j_{ij} f_j) + \sum_{i=1}^n \sum_{k=1}^m X_i (\beta^j_{ik} g_k) \right) \nu_\ell \]

\[ + \frac{1}{\varepsilon} \sum_{i,j=1}^n \sum_{k=1}^m \left( X_i f_j - \sum_{\ell=1}^m \gamma^\ell_{ij} g_\ell \right) \gamma^k_{ij} \nu_k - \sum_{i,h=1}^n \sum_{\ell=1}^m \left( X_i g_\ell + \frac{1}{\varepsilon} \sum_{j=1}^n \gamma^j_{ij} f_j \right) \gamma^\ell_{ih} \theta_h \]
Therefore, we have
\[
\left(-\nabla_L - \mathcal{T}_L^\varepsilon \right)^* \left(\nabla_L - \mathcal{T}_L^\varepsilon \right) - \frac{1}{\varepsilon} J^2 \right) \eta = \left( L + 2J - \frac{2}{\varepsilon} T \circ d + \delta H^T - \frac{1}{\varepsilon} \delta_T \right) \eta.
\]
By using the definition of \( \square_{\infty} \) we immediately obtain the conclusion.

**Lemma 3.5** For any \( \eta \in \Gamma^\infty(T^*M) \),
\[
\frac{1}{2} L \| \eta \|^2 - \langle \square_{\varepsilon} \eta, \eta \rangle_{\varepsilon} = \| \nabla L \eta - \mathcal{T}_L^\varepsilon \eta \|^2 + \langle \text{Ric}_H(\eta), \eta \rangle_H - \langle \delta_H^T(\eta), \eta \rangle_{\varepsilon} + \frac{1}{\varepsilon} \langle J^2(\eta), \eta \rangle_H.
\]

**Proof.** First note that
\[
\square_{\varepsilon} = \square_{\infty} - \frac{2}{\varepsilon} T \circ d = L + 2J - \text{Ric}_H + \delta_H^T - \frac{2}{\varepsilon} T \circ d,
\]
therefore it is equivalent to show that
\[
\frac{1}{2} L \| \eta \|^2 - \left\langle \left( L + 2J - \frac{2}{\varepsilon} T \circ d \right) \eta, \eta \right\rangle = \| \nabla L \eta - \mathcal{T}_L^\varepsilon \eta \|^2 + \frac{1}{\varepsilon} \langle J^2(\eta), \eta \rangle_H.
\]
We now compute that, at the center \( x \),
\[
\frac{1}{2} L \| \eta \|^2 - \left\langle \left( L + 2J - \frac{2}{\varepsilon} T \circ d \right) \eta, \eta \right\rangle = \| \nabla L \eta \|^2_H + \varepsilon \| \nabla \eta \|^2_V + 2 \sum_{i,j=1}^n \sum_{k,\ell=1}^m \gamma_{ij}^\varepsilon (X_i g_k f_j - \frac{1}{2} \gamma_{ij}^\varepsilon g_k) g_k.
\]
and
\[
\| \nabla L \eta - \mathcal{T}_L^\varepsilon \eta \|^2 = \sum_{i,j=1}^n \left( X_i f_j - \sum_{\ell=1}^m \gamma_{ij}^\varepsilon g_k \right)^2 + \varepsilon \sum_{i,j=1}^n \sum_{k,\ell=1}^m \left( X_i g_k + \frac{1}{\varepsilon} \sum_{j=1}^m \gamma_{ij}^\varepsilon f_j \right)^2
\]
\[
= \| \nabla L \eta \|^2_H + \varepsilon \| \nabla \eta \|^2_V - 2 \sum_{i,j=1}^n \sum_{k,\ell=1}^m (X_i f_j) \gamma_{ij}^\varepsilon g_k + \sum_{i,j=1}^n \sum_{k,\ell=1}^m \gamma_{ij}^\varepsilon \gamma_{ij}^\varepsilon g_k
\]
\[
+ 2\varepsilon \sum_{i=1}^n \sum_{\ell=1}^m (X_i f_j) \left( \frac{1}{\varepsilon} \sum_{j=1}^m \gamma_{ij}^\varepsilon f_j \right)^2 + \varepsilon \sum_{\ell=1}^m \sum_{i=1}^n \left( \frac{1}{\varepsilon} \sum_{j=1}^m \gamma_{ij}^\varepsilon f_j \right)^2
\]
The claim easily follows.

**Proposition 3.6** For every \( \eta \in \Gamma^\infty(T^*M) \),
\[
\frac{1}{2} L \| \eta \|^2 - \langle \square_{\varepsilon} \eta, \eta \rangle_{\varepsilon} \\
\geq \frac{1}{n} \left( \text{Tr}_H \nabla H^2 \right) - \frac{1}{4} \text{Tr}_H (J_H^2) + \langle \text{Ric}_H(\eta), \eta \rangle_H - \langle \delta_H^T(\eta), \eta \rangle_{\varepsilon} + \frac{1}{\varepsilon} \langle J^2(\eta), \eta \rangle_H
\]
Proof. Due to Lemma [3.5] it amounts to prove that

\[ \| \nabla H \eta - \nabla^\varepsilon H \eta \|_2^2 \geq \frac{1}{n} (\text{Tr}_H \nabla^\# H \eta)^2 - \frac{1}{4} \text{Tr}_H (J^2_\eta). \]

For any \( \eta \in \Gamma^\infty(T^*M) \), we have at \( x \)

\[ \nabla^\#_H(\eta) = \frac{1}{2} \sum_{i,j=1}^n ((X_i f_j) \theta_j + (X_j f_i) \theta_i) + \sum_{\ell=1}^m (X_i g_\ell) \nu_\ell. \]

Easy computations show that

\[ \| \nabla H \eta - \nabla^\varepsilon H \eta \|_2^2 = \| \nabla^\#_H \eta \|_2^2 + \frac{1}{4} \sum_{i,j=1}^n \left( \sum_{\ell=1}^m \gamma^\ell_{ij} g_\ell \right)^2, \]

therefore, from Cauchy-Schwarz inequality, we have

\[ \| \nabla H \eta - \nabla^\varepsilon H \eta \|_2^2 = \| \nabla^\#_H \eta \|_2^2 + \frac{1}{4} \sum_{i,j=1}^n \left( \sum_{\ell=1}^m \gamma^\ell_{ij} g_\ell \right)^2 \geq \frac{1}{n} (\text{Tr}_H \nabla^\# H \eta)^2 - \frac{1}{4} \text{Tr}_H (J^2_\eta). \]

4 Stochastic completeness

Throughout the section we consider, as above, a smooth connected manifold \( M \) which is equipped with a Riemannian foliation with bundle like metric \( g \) and totally geodesic leaves. We moreover assume that the metric \( g \) is complete and that the horizontal distribution \( H \) of the foliation is bracket-generating and Yang-Mills (see Besse [14], Definition 9.35).

The hypothesis that \( H \) is bracket generating implies that the horizontal Laplacian \( L \) is subelliptic and it is easily seen that with our notations the Yang-Mills condition is equivalent to the fact that

\[ \delta_H T = 0. \]

The operator

\[ \Box^\varepsilon = -\nabla_H - \nabla^\varepsilon_H (\nabla_H - \nabla^\varepsilon_H) - \frac{1}{\varepsilon} J^2 - \mathcal{Ric}_H. \]

that we introduced in the previous section is then symmetric for the metric \( g_\varepsilon \).

In this section, we also assume that for every horizontal one-form \( \eta \in \Gamma^\infty(H^*) \),

\[ \langle \mathcal{Ric}_H(\eta), \eta \rangle_H \geq -K \| \eta \|_H^2, \quad -\langle J^2 \eta, \eta \rangle_H \leq \kappa \| \eta \|_H^2, \]

with \( K, \kappa \geq 0 \).
The completeness of the metric $g$ implies that the horizontal Laplacian $L$ is essentially self-adjoint on the space of smooth and compactly supported functions and that the operator $\Box_\varepsilon$ is essentially self-adjoint on the space of smooth and compactly supported one-forms (see the argument in Lemma 4.3 of [5]).

Since $\Box_\varepsilon$ is essentially self-adjoint, it admits a unique self-adjoint extension which generates thanks to the spectral theorem a semigroup $Q^\varepsilon_t = e^{\Box_\varepsilon t}$. We will denote by $P_t = e^{tL}$ the semigroup generated by $L$. We have the following commutation property:

**Lemma 4.1** If $f \in C^\infty_0(M)$, then for every $t \geq 0$,

$$dP_t f = Q^\varepsilon_t df.$$

*Proof.* Let $\eta_t = Q^\varepsilon_t df$. By essential self-adjointness, it is the unique solution in $L^2$ of the heat equation

$$\frac{\partial \eta}{\partial t} = \Box_\varepsilon \eta,$$

with initial condition $\eta_0 = df$. From the fact that

$$dL = \Box_\varepsilon d,$$

we see that $\alpha_t = dP_t f$ solves the heat equation

$$\frac{\partial \alpha}{\partial t} = \Box_\varepsilon \alpha$$

with the same initial condition $\alpha_0 = df$. In order to conclude, we thus just need to prove that for every $t \geq 0$, $dP_t f$ is in $L^2$.

Let us denote by $L^V$ the vertical (leaf) Laplacian. The Laplace-Beltrami operator of $M$ is therefore $\Delta = L + L^V$. Since the leaves are totally geodesic, $\Delta$ commutes with $L$ on $C^2$ functions (see [13]). Moreover from the spectral theorem, $Le^{t\Delta}$ maps $C^\infty_0(M)$ into $L^2(M)$. We deduce by essential self-adjointness that $Le^{t\Delta} = e^{tL}e^{t\Delta}$. Similarly we obtain $e^{sL}e^{t\Delta} = e^{(s+t)L}$ which implies $\Delta e^{sL} = e^{sL}\Delta$. As a consequence we have that for every $t \geq 0$, $dP_t f$ is in $L^2$. \hfill \Box

**Theorem 4.2** For every $\varepsilon > 0$, $t \geq 0$, $x \in M$ and $f \in C^\infty_0(M)$,

$$\|dP_t f(x)\|_\varepsilon \leq e^{(K + \frac{\varepsilon}{2})t}P_t\|df\|_\varepsilon(x).$$

As a consequence, the heat semigroup is conservative that is for every $t \geq 0$, $P_t 1 = 1$.

*Proof.* The idea is to use the Feynman-Kac stochastic representation of $Q^\varepsilon_t$. We denote by $(X_t)_{t \geq 0}$ the symmetric diffusion process generated by $\frac{1}{2}L$ and denote by $e$ its lifetime. Consider the process $\tau^\varepsilon_t : T^*_x M \to T^*_x M$ which is the solution of the following covariant Stratonovitch stochastic differential equation:

$$d[\tau^\varepsilon_t \alpha(X_t)] = \tau^\varepsilon_t \left( \nabla_{odX_t} - \Sigma^\varepsilon_{odX_t} - \frac{1}{2} \left( \frac{1}{\varepsilon} J^2 + \mathfrak{Ric}_H \right) dt \right) \alpha(X_t), \quad \tau^\varepsilon_0 = \text{Id},$$

(4.8)
where $\alpha$ is any smooth one-form. By using Gronwall’s lemma, we have for every $t \geq 0$,
\[
\|\tau_t^\varepsilon \alpha(X_t)\| \leq e^{\frac{1}{2}(K + \kappa) t} \|\alpha(X_0)\|.
\]
By the Feynman-Kac formula, we have for every smooth and compactly supported one-form
\[
Q_{t/2} \eta(x) = E_x (\tau_t \eta(X_t) 1_{t < \varepsilon}).
\]
Since $dP_t = Q_t d$, it follows easily that
\[
\|dP_t f(x)\| \leq e^{(K + \kappa)} t \|f\| \varepsilon(x).
\]
It is well-known that this type of gradient bound implies the stochastic completeness of $P_t$ (see [3]). \qed

5 Curvature-dimension inequality, Li-Yau estimates and Bonnet-Myers type theorem

Let $M$ be a smooth, connected manifold with dimension $n + m$. We assume that $M$ is equipped with a Riemannian foliation $\mathcal{F}$ with bundle like metric $g$ and totally geodesic $m$-dimensional leaves for which the horizontal distribution is Yang-Mills. We also assume that $M$ is complete and that globally on $M$, for every $\eta_1 \in \Gamma^\infty(\mathcal{H}^\ast)$ and $\eta_2 \in \Gamma^\infty(\mathcal{V}^\ast)$,
\[
\langle \text{Ric}_\mathcal{H}(\eta_1), \eta_1 \rangle_{\mathcal{H}} \geq \rho_1 \|\eta_1\|^2_{\mathcal{H}}, \quad -\langle \mathcal{J}^2 \eta_1, \eta_1 \rangle_{\mathcal{H}} \leq \kappa \|\eta_1\|_{\mathcal{H}}^2, \quad -\frac{1}{4} \text{Tr}_\mathcal{H}(\mathcal{J}_{\eta_2}^2) \geq \rho_2 \|\eta_2\|_{\mathcal{V}}^2,
\]
for some $\rho_1 \in \mathbb{R}$, $\kappa, \rho_2 > 0$. The second assumption can be thought as a uniform bracket generating condition of the horizontal distribution $\mathcal{H}$ and from Hörmander’s theorem, it implies that the horizontal Laplacian $L$ is a subelliptic diffusion operator. We insist that for the following results below to be true, the positivity of $\rho_2$ is required.

We introduce the following operators defined for $f, g \in C^\infty(M)$,
\[
\Gamma(f, g) = \frac{1}{2}(L(fg) - gLf - fLg) = \langle \nabla_\mathcal{H} f, \nabla_\mathcal{H} g \rangle_{\mathcal{H}}
\]
\[
\Gamma^\mathcal{V}(f, g) = \langle \nabla_\mathcal{V} f, \nabla_\mathcal{V} g \rangle_{\mathcal{V}}
\]
and their iterations which are defined by
\[
\Gamma_2(f, g) = \frac{1}{2}(L(\Gamma(f, g)) - \Gamma(g, Lf) - \Gamma(f, Lg))
\]
\[
\Gamma_2^\mathcal{V}(f, g) = \frac{1}{2}(L(\Gamma^\mathcal{V}(f, g)) - \Gamma^\mathcal{V}(g, Lf) - \Gamma^\mathcal{V}(f, Lg))
\]
As a consequence of Theorem 3.1, we obtain the curvature dimension inequality introduced in [9].

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**Theorem 5.1** For every \( f, g \in C^\infty(\mathbb{M}) \), and \( \varepsilon > 0 \),
\[
\Gamma_2(f, f) + \varepsilon \Gamma^V_2(f, f) \geq \frac{1}{n} (L f)^2 + \left( \rho_1 - \frac{\kappa}{\varepsilon} \right) \Gamma(f, f) + \rho_2 \Gamma^V(f, f),
\]
and
\[
\Gamma(f, \Gamma^V(f)) = \Gamma^V(f, \Gamma(f)).
\]

**Proof.** From Theorem 3.1, we have for every \( \eta \in \Gamma^\infty(T^*\mathbb{M}) \),
\[
\frac{1}{2} L \|\eta\|_2^2 - \langle \Box \varepsilon \eta, \eta \rangle \geq \frac{1}{n} \left( \text{Tr}_H \nabla^2 \eta \right)^2 - \frac{1}{4} \text{Tr}_H (J^2_\eta) + \langle \text{Ric}_H(\eta), \eta \rangle_H + \frac{1}{\varepsilon} (J^2(\eta), \eta)_H.
\]
Using this inequality with \( \eta = df \) and taking into account the assumptions
\[
\langle \text{Ric}_H(\eta_1), \eta_1 \rangle_H \geq \rho_1 \|\eta_1\|_H^2, \quad -\langle J^2(\eta_1), \eta_1 \rangle_H \leq \kappa \|\eta_1\|_H^2, \quad -\frac{1}{4} \text{Tr}_H (J^2_\eta) \geq \rho_2 \|\eta_2\|_V^2,
\]
immediately yields the expected result. The intertwining \( \Gamma(f, \Gamma^V(f)) = \Gamma^V(f, \Gamma(f)) \) is proved in Appendix A in [15] and easy to check in a local frame.

Combining Theorems 4.2 and 5.1, we see then that all the results proved in the works [6, 7, 8, 9, 10, 11, 12] apply. We obtain therefore, among many other things, the following results which are completely new in the context of Riemannian foliations:

1) **Li-Yau type inequalities**: [9] For any bounded \( f \in C^\infty(\mathbb{M}) \), such that \( f, \sqrt{\Gamma(f)} \in L^2_\mu(\mathbb{M}), f \geq 0, f \neq 0 \), the following inequality holds for \( t > 0 \):
\[
\Gamma(\ln P_t f) + \frac{2 \rho_2}{3} t \Gamma^V(\ln P_t f)
\leq \left( 1 + \frac{3 \kappa}{2 \rho_2} - \frac{2 \rho_1}{3} \right) \frac{L P_t f}{P_t f} + \frac{n \rho_1^2}{6} t - \frac{n \rho_1}{2} \left( 1 + \frac{3 \kappa}{2 \rho_2} \right) + \frac{n \left( 1 + \frac{3 \kappa}{2 \rho_2} \right)^2}{2t}.
\]

2) **Gaussian lower and upper bounds for the horizontal heat kernel**: [7] If \( \rho_1 \geq 0 \), then for any \( 0 < \varepsilon < 1 \) there exists a constant \( C(\varepsilon) = C(n, \kappa, \rho_2, \varepsilon) > 0 \), which tends to \( \infty \) as \( \varepsilon \to 0^+ \), such that for every \( x, y \in \mathbb{M} \) and \( t > 0 \) one has
\[
\frac{C(\varepsilon)^{-1}}{\mu(B(x, \sqrt{t}))} \exp \left( -\frac{D d(x, y)^2}{n(4 - \varepsilon)t} \right) \leq p(x, y, t) \leq \frac{C(\varepsilon)}{\mu(B(x, \sqrt{t}))} \exp \left( -\frac{d(x, y)^2}{(4 + \varepsilon)t} \right).
\]
Here \( D = \left( 1 + \frac{3 \kappa}{2 \rho_2} \right) n \) and \( d(x, y) \) is the sub-Riemannian distance between \( x \) and \( y \).

3) **Bonnet-Myers theorem**: [9] Suppose that \( \rho_1 > 0 \). Then, the manifold \( \mathbb{M} \) is compact and the sub-Riemannian diameter of \( \mathbb{M} \) satisfies the bound
\[
\text{diam } \mathbb{M} \leq 2\sqrt{3\pi} \sqrt{\frac{\kappa + \rho_2}{\rho_1 \rho_2} \left( 1 + \frac{3 \kappa}{2 \rho_2} \right)^2} n.
\]
We mention that in the Bonnet-Myers theorem, the bound

\[ \text{diam } M \leq 2\sqrt{3\pi} \left( \frac{\kappa + \rho_2}{\rho_1 \rho_2} \left( 1 + \frac{3\kappa}{2\rho_2} \right) \right)^n. \]

is not sharp, as can be checked in some examples like the Hopf fibrations. This is because the method we use in [9] is an adaption of the energy-entropy inequality methods developed by Bakry in [4]. Even in the Riemannian case, Bakry methods are known to lead to non sharp constants.

Finally, at last, we observe that the methods of [12] can be adapted to the present framework and that the following result can be proved:

**Proposition 5.2** Assume \( \rho_1 > 0 \). Then the first eigenvalue \( \lambda_1 \) of the horizontal Laplacian \(-L\) satisfies

\[ \lambda_1 \geq \frac{\rho_1}{1 - \frac{1}{\pi} + \frac{3\kappa}{4\rho_2}}. \]

As a consequence of the Obata theorem proved in [12], this bound is sharp.

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