Weak decay of magnetized pions

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The leptonic decay of charged pions is investigated in the presence of background magnetic fields. It is found that the background field opens up a new decay channel that proceeds via the vector part of the electroweak current, and is characterized by a new fundamental decay constant. We calculate the magnetic field-dependence of both the usual and the new decay constant non-perturbatively on the lattice. We employ both Wilson and staggered quarks and extrapolate the results to the continuum limit. With this non-perturbative input we calculate the tree-level electroweak amplitude for the full decay rate in strong magnetic fields. We find that the muonic decay of the charged pion is enhanced drastically by the magnetic field. We comment on possible astrophysical implications.

I. INTRODUCTION

Strong (electro)magnetic fields bear a significant impact on the physics of various systems ranging from off-central heavy-ion collisions through the evolution of the early universe to magnetized neutron stars (magnetars). In particular, many novel phenomena emerge from the competition between electromagnetism and color interactions if the magnetic field $B$ becomes similar in magnitude to the strong interaction scale: $eB \sim \Lambda^2_{\text{QCD}}$. If the time-scale of the fluctuations in $B$ is larger than other relevant scales of the problem, it is reasonable to treat the magnetic field classically as a background field. For reviews on this subject, see for example Refs. [1, 2].

Such a background magnetic field is known, for instance, to affect the phase diagram of quantum chromodynamics (QCD) [3–5]. For cold astrophysical environments, the low-temperature (hadronic) phase of QCD is particularly relevant. In this regime, a prime role is played by the lightest hadrons, i.e. pions and kaons. Specifically, their masses appear in the nuclear equation of state within compact stellar objects and, thus, influence their mass-radius relations. For stability and equilibrium analyses, the respective decay rates are equally important. Dominant cooling mechanisms for magnetars [6] involve (inverse) $\beta$-decay, photo-meson interactions and pion decay [7]. Pions radiate energy via inverse Compton scattering until they decay, imprinting the spectrum of the subsequently produced neutrinos [8]. Strong electromagnetic fields are also created in violent astrophysical processes such as neutron star mergers and supernova events, where weak nuclear reactions and decays govern cooling mechanisms and affect the neutrino spectrum [9].

The $B$-dependence of pion masses has been investigated in various settings, ranging from chiral perturbation theory [10, 11] through numerical lattice QCD simulations [3, 12–14] to model approaches [15–20]. Less is known about the decay rates for nonzero magnetic fields. The decay constant for the neutral pion has been studied in chiral perturbation theory [10, 11, 21] and in model settings [15–19, 22]. The decay constant of the charged pion has only been discussed so far in chiral perturbation theory [11].

In this letter we investigate the magnetic field-dependence of the decay rate of charged pions at zero temperature. We demonstrate that the previous studies in this direction are incomplete: in the presence of the magnetic field both neutral and charged pions have two independent decay constants, of which only one has been investigated up to now. We determine both decay constants for charged pions non-perturbatively on the lattice, employing two different fermionic discretizations. Using this QCD input, we proceed to calculate the weak decay rate using leading-order electroweak perturbation theory. For this calculation we employ the lowest Landau level (LLL) approximation for the outgoing charged lepton state, which is a viable simplification for strong background magnetic fields.

In Sec. II we provide the definition of the usual pion decay constant and describe how a second decay constant emerges if a background magnetic field is present. This is followed by Sec. III, where we calculate the full decay rate perturbatively. In Sec. IV we describe our Wilson and staggered lattice setups, followed by the presentation of the results for both decay constants in Sec. V. Preliminary results for Wilson fermions on a reduced set of lattice spacings were presented in Ref. [23].

II. PION DECAY CONSTANTS

The pion decay constant is related to the hadronic matrix elements $H_\mu$ of the weak interaction current between the vacuum and a pion state with momentum $p_\mu$. For $B = 0$, parity dictates that the matrix element $\langle 0 | \bar{d} \gamma_\mu u | \pi^- \rangle$ vanishes, since the only Lorentz-structure available is $p_\mu$:

$$H_\mu \equiv \langle 0 | \bar{d} (x) \gamma_\mu (1 - \gamma^5) u (x) | \pi^- (p) \rangle = -i e^{ipx} f_\pi p_\mu .$$

(1)
The coefficient $f_2$ is the pion decay constant, which coincides for negatively and positively charged pions due to charge conjugation symmetry. Throughout this letter we use the normalization where $f_2 \approx 131$ MeV for a physical pion in the vacuum.

In the presence of a background electromagnetic field $F_{\mu \nu}$ the relation (1) takes a more general form. Exploiting Lorentz-covariance, using the tensor $F_{\mu \nu}$ and the vector $p_\mu$, additional vector and axial vector combinations can be formed:

\[
\langle 0 | \bar{d} (x) \gamma_{\mu} \gamma_5 u (x) | \pi^- (p) \rangle = i e^{i p x} \left[ f_\pi p_\mu + f_\pi^\mu e F_{\mu \nu} p^\nu \right],
\]

\[
\langle 0 | \bar{d} (x) \gamma_{\mu} u (x) | \pi^- (p) \rangle = i e^{i p x} \left[ 2 f_\pi^\mu \varepsilon_{\mu \nu \rho \sigma} e F^{\nu \rho} p^\sigma \right],
\]

where $e$ denotes the elementary charge and we follow the convention $\epsilon^{0123} = +1$. Charge conjugation implies that the decay rate is the same for positively and negatively charged pions and is also independent of the direction of the magnetic field. This is ensured by the ratios $f_\pi / f_\pi^\mu$ and $f_\pi^\mu / f_\pi^\nu$ being real, as we will see below in Sec. III.

In our conventions, all three decay constants are real and positive. We remark that the new Lorentz structures also differ from $f_{\rho \pi} = 0$.

We work at the tree level of electroweak perturbation theory and employ the effective, four-fermion interaction with Fermi’s constant $G$ as coupling. Due to the current-current structure of the effective electroweak Lagrangian [25], the decay amplitude factorizes into leptonic and hadronic parts,

\[
\mathcal{M} \equiv \frac{G}{\sqrt{2}} \cos \theta_c \, L^\mu H_\mu,
\]

where the Cabibbo angle $\theta_c$ entered due to the mixing between the down and strange quark mass eigenstates. The relevant hadronic components $H_\mu$ are shown in Eq. (3). The leptonic component reads, in terms of the bispinor solutions $u_\ell$ and $v_\nu$:

\[
L^\mu \equiv \bar{u}_\ell (k) \gamma^\mu \left( 1 - \gamma^5 \right) v_\nu (q).
\]

The decay rate $\Gamma$ involves the modulus square of the amplitude, integrated over the phase space and summed over the intrinsic quantum numbers of the outgoing asymptotic states. To find the latter for the charged lepton, we need the bispinor solutions of the Dirac equation for $B > 0$. These are the so-called Landau levels — orbits localized in the spatial plane perpendicular to $\mathbf{B}$ with quantized radii. The Landau levels come with a multiplicity proportional to the flux $\Phi = |eB| \cdot L^2$ of the magnetic field. In order to regulate this multiplicity, we need to assume that the outgoing states are defined in a finite spatial volume $V = L^3$. For the decay rate such volume factors will cancel.

For strong fields the dominant contribution stems from the lowest Landau level. The sum over the multiplicity of the LLL states is calculated in App. A, giving

\[
\sum_{\text{LLL}} u_\ell (k) \bar{u}_\ell (k) = \frac{\Phi}{2 \pi} \cdot (\vec{k} \parallel + m_\ell) \cdot \frac{1 - \sigma^{12}}{2},
\]

where $\vec{k} \parallel = k^0 \gamma^0 - k^3 \gamma^3$ and $\sigma^{12} = i \gamma^1 \gamma^2$ is the relativistic spin operator. Eq. (7) reflects the fact that the LLL solutions have their spin anti-aligned with the magnetic field (since the lepton has negative charge) and are characterized only by the momentum along the z direction (i.e. along $\mathbf{B}$). Due to angular momentum conservation the antineutrino spin is also aligned with the magnetic field. Moreover, the right-handedness of the antineutrino also sets the direction of its momentum to be parallel to the magnetic field.
Having determined $|\mathcal{M}|^2$, we finally need to integrate over the phase space for the outgoing particles. The kinematics and further details of the calculation are discussed in App. B. The resulting decay rate reads
\[ \Gamma(B) = |eB| \frac{G^2}{4\pi} \cos^2 \theta_c |f_\pi + i f'_\pi eB|^2 \frac{m_\pi^2}{M_\pi}. \] (8)

As anticipated above, the decay rate only depends on the magnitude of $B$, due to the absence of an interference term in $|f_\pi + i f'_\pi eB|^2 = f_\pi^2 + |f'_\pi eB|^2$. Dividing by the $B = 0$ result [25], the dependence on $G$ and $\theta_c$ cancels:
\[ \frac{\Gamma(B)}{\Gamma(0)} = \left( f_\pi^2 + |f'_\pi eB|^2 \right) \frac{1 - \frac{m_\pi^2}{M_\pi^2}(0)^2}{\frac{2|eB|}{M_\pi(0)M_\pi(B)}}. \] (9)

We stress that this result was obtained using the LLL approximation, which is in general valid for strong fields [2, 26]. For the leading-order perturbative decay rate, higher Landau-levels turn out to give zero contribution for $eB > M_\pi^2(0) - m_\pi^2$, see App. B.

IV. LATTICE SETUP

Eq. (9) contains three non-perturbative parameters, which describe the response of the pion to the background field: $M_\pi(B)$, $f_\pi(B)$ and $f'_\pi(B)$. We calculate these via two independent sets of lattice QCD simulations. First, we work with quenched Wilson quarks. The zero-temperature ensembles generated and analyzed in Ref. [14] are supplemented by a fourth, finer lattice ensemble, so that the lattice spacing spans 0.047 fm $\leq a \leq 0.124$ fm. The $B = 0$ pion mass is set to $M_\pi(0) \approx 415$ MeV. To remove $B$-dependent $O(a^2)$ effects on quark masses, the bare mass parameters are tuned to fall on the magnetic field-dependent line of constant physics determined in Ref. [14].

In the second set of simulations we work with $N_f = 2 + 1$ flavors of dynamical staggered fermions, using the ensembles of Refs. [3, 27]. The employed lattice spacings lie in the range 0.1 fm $\leq a \leq 0.22$ fm, and the quark masses are set to their physical values [28] such that $M_\pi(0) \approx 135$ MeV. For both formulations we perform a continuum extrapolation based on the available four lattice spacings. This enables us to quantify the systematicatics related to the differences between the two approaches: heavier-than-physical versus physical pion mass and quenched versus dynamical quarks. We remark that simulations with dynamical Wilson quarks in the presence of a background magnetic field would require computational resources that are by orders of magnitude larger than those used for the current study.

The general measurement strategy involves the analysis of the matrix elements $H_0$ and $H_3$ of Eq. (3). These are encoded in the zero momentum-projected Euclidean correlators
\[ C_{\mathcal{O}P}(t) = \left\langle \sum_{\mathbf{x}} O(\mathbf{x}, t) P^t(\mathbf{y}, 0) \right\rangle, \] (10)

with $\mathcal{O} \in \{ P, A, V \}$ and
\[ P = \bar{u}\gamma^5 d, \quad A = \bar{u}\gamma_0\gamma^5 d, \quad V = \bar{u}\gamma_3 d. \] (11)

In the large-$t$ limit, the dominant contribution to the spectral representation of all three correlators comes from a pion state. We fit the three correlators using
\[ C_{\mathcal{O}P}(t) = c_{\mathcal{O}P} \left[ e^{-M_\pi t} \pm e^{-M_\pi(N_t-t)} \right], \] (12)

where the positive sign is taken for $C_{AP}$ and the negative for $C_{VP}$ and $C_{VP}$ due to the time reversal properties of the correlators. The decay constants are extracted via
\[ f_\pi = Z_A \cdot \frac{\sqrt{2} c_{AP}}{\sqrt{M_\pi c_{PP}}}, \quad f'_\pi eB = Z_V \cdot \frac{\sqrt{2} c_{VP}}{\sqrt{M_\pi c_{PP}}}, \] (13)

where $Z_A$ and $Z_V$ are the multiplicative renormalization constants of the axial vector and vector currents.

For Wilson quarks we employ smeared pseudoscalar sources (for more details, see Ref. [14]) and fit all three correlators simultaneously. For the staggered analysis we work with point sources and fit the $C_{AP}$ and $C_{PP}$ correlators to find $M_\pi$ and $f_\pi$. In a second step, volume sources are employed for $C_{VP}/C_{AP}$ to enhance the signal in $f'_\pi/f_\pi$. The staggered discretization of $A$ and $V$ requires operators nonlocal in Euclidean time and has been worked out in Ref. [29]. For staggered quarks and the currents we use the renormalization constants are trivial, $Z_A = Z_V = 1$. For Wilson quarks this is not the case, nevertheless, these ultraviolet quantities are expected to be independent of the magnetic field. We employ the $B = 0$ non-perturbative results of Ref. [30] (see also Ref. [31]) and fit these in combination with the asymptotic perturbative two-loop results of Ref. [32] (see also Ref. [33]) to a Padé parametrization.

V. RESULTS

The three correlators of Eq. (10) are plotted in Fig. 1 for both fermion formulations for a high magnetic field and meson masses approximately equal in lattice units (for the staggered case a kaon correlator is shown). A clear signal is visible for the $C_{VP}$ correlator, which would vanish on average at $B = 0$.

The mass and the decay constants are extracted using the fits described in Eqs. (12) and (13). For the complete magnetic field range, the pion mass is found to be described within 5% by the formula
\[ M_\pi/M_\pi(0) = \sqrt{1 + |eB|/M_\pi^2(0)}, \] (14)

which assumes pions to be point-like free scalars. This has been observed many times in the literature, both using dynamical staggered [3], quenched Wilson [12, 14] and quenched overlap quarks [13].

The normalized combinations $f_\pi/f_\pi(0)$ and $f'_\pi eB/f_\pi(0)$ are shown for four lattice spacings in
For the pion mass we use the analytic results, using a rescaling to the physical point as explained in Ref. 1. The continuum extrapolation is carried out by including lattice artefacts of ${\cal O}(a)$ (for Wilson) and ${\cal O}(a^2)$ (for staggered) in the coefficients. Specifically, the parameterizations of the individual decay constants read

$$f_\pi/f_\pi(0) = [1 + c_1 |eB|] \cdot M_\pi(0)/M_\pi,$$

$$f_\pi'/f_\pi(0) = [d_0 + d_1 |eB| + d_2 |eB|^2] \cdot M_\pi(0)/M_\pi,$$ (15)

and $M_\pi/M_\pi(0)$ is taken from Eq. (14). The quality of the staggered data for $f_\pi/f_\pi(0)$ only allows for a fit with $d_1 = d_2 = 0$ and increased systematic errors as $B$ grows.

Motivated by the dependence of $M_\pi/M_\pi(0)$ on the scaling variable $eB_M^2(0)$, we compare the continuum extrapolated Wilson results (obtained for $M_\pi(0) = 415$ MeV) to the staggered data (obtained for physical pion masses), after rescaling the magnetic field for the latter. In particular, we take the staggered results for $f_\pi/f_\pi(0)$ at the magnetic field $eB \cdot (415/135)^2$. The resulting curve is also included in the lower panel of Fig. 2, revealing nice agreement between the two approaches. In particular, the slope at the origin is found to be $-16.9(3)$ GeV$^{-2}$ for staggered and $-1.7(2)$ GeV$^{-2}$ for Wilson — the ratio of which is consistent with the squared pion mass ratio. A similar rescaling of $f_\pi'/f_\pi(0)$ also shows qualitative agreement, although the errors of the staggered data for this decay constant increase quickly as $B$ grows. For low magnetic fields we find consistent results: $f_\pi'/f_\pi(0) = 0.8(2)$ GeV$^{-2}$ for staggered and $1.2(3)$ GeV$^{-2}$ for Wilson, respectively.

To determine the decay rate (9) we employ the continuum extrapolated staggered results. On the basis of the above comparisons, we also consider the Wilson results, using a rescaling to the physical point as explained above. For the pion mass we use the analytic dependence (14), including a 5% systematic error. The so-obtained curves for the muonic decay rate are shown in Fig. 3 for magnetic fields $eB \leq 0.45$ GeV$^2$, where both staggered and (rescaled) Wilson results are available. The decay rate is enhanced drastically by the magnetic field: for $eB \approx 0.3$ GeV$^2$ we observe an almost fifty-fold increase with respect to $B = 0$. We note that while the ordinary decay mechanism dominates in our study, the contribution of the new vector decay constant $f_\pi'$ grows to about 10% of the total decay rate at the largest magnetic field of Fig. 3.

We remark that Eq. (9), supplemented by our staggered lattice results, suggests the decay rate $K^- \to \mu^- \bar{\nu}_\mu$ to be enhanced only by a factor of about two at the same field strength. This is mainly due to the larger mass of the kaon. Finally, the pion decay rate into electrons undergoes an enhancement by a factor of about ten. In fact,
carried out to eliminate lattice discretization errors. For quarks. For both cases, continuum extrapolations were quenched Wilson simulations with heavier-than-physical masses, and also compared to the results of lattice simulations employing dynamical staggered quarks with physical masses, and correspondingly, a drastic reduction of the mean lifetime \( \tau_\pi = 1/\Gamma \). A typical \( B > 0 \) lifetime is

\[
\tau_\pi = 5 \cdot 10^{-10} \, \text{s} \quad \text{for} \quad B \approx 0.3 \, \text{GeV}^2/c \approx 5 \cdot 10^{15} \, \text{T}.
\]

Since lifetimes of magnetic fields in off-central heavy-ion collisions are by \( 14 - 15 \) orders of magnitude smaller [1], it is clear that this effect will not result in any observable predictions for heavy-ion phenomenology. However, the \( B \)-dependence of weak decays is expected to be essential in astrophysical environments. (Notice that the upper limit for magnetic field strengths in the core of magnetized neutron stars is thought to be around \( B = 10^{14-16} \, \text{T} \) [36, 37].) Indeed, for \( B = 0 \) the pion mean lifetime and the time-scale for cooling via inverse Compton scattering are roughly comparable [8]. Thus, a reduction in \( \tau_\pi \) will inevitably decrease radiation energy loss of pions and result in a harder neutrino spectrum.

Similarly to the pion decay rate, the magnetic field will have an impact on (inverse) \( \beta \)-decay rates and nucleon electroweak transition form-factors. Indeed, an enhancement by the magnetic field is expected for processes involving nucleons as well [38–40], see, e.g., the review [41]. The tools developed in the present letter will also be useful to study these effects that are relevant for cold and magnetized environments.

**VI. CONCLUSIONS**

In this letter we computed the rate for the leptonic decay of charged pions in the presence of strong background magnetic fields. Our main result is Eq. (9), for which we employed electroweak perturbation theory and the lowest-Landau-level approximation for the outgoing charged lepton \( \ell \), valid for strong fields. Including higher-order terms in the electroweak calculation (see, e.g., Refs. [34, 35]), as well as going beyond the lowest Landau level is possible, allowing to systematically improve this result. In this case also the constant \( f_\pi' \), that we have not determined here, may enter.

We demonstrated that — besides the ordinary pion decay constant \( f_\pi \) — the decay rate depends on an additional fundamental parameter \( f_\pi' \). The latter decay constant characterizes a new channel that becomes available for nonzero magnetic fields. We calculated both decay constants, together with the pion mass, using lattice simulations employing dynamical staggered quarks with physical masses, and also compared to the results of quenched Wilson simulations with heavier-than-physical quarks. For both cases, continuum extrapolations were carried out to eliminate lattice discretization errors. For low magnetic fields, we obtain for the new decay constant

\[
f_\pi' = 0.10(2) \, \text{GeV}^{-1} + \mathcal{O}(B) \, .
\]  

Our final result for the full decay rate is visualized in Fig. 3, revealing a dramatic enhancement of the rate or, correspondingly, a drastic reduction of the mean lifetime \( \tau_\pi = 1/\Gamma \). A typical \( B > 0 \) lifetime is

\[
\tau_\pi = 5 \cdot 10^{-10} \, \text{s} \quad \text{for} \quad B \approx 0.3 \, \text{GeV}^2/c \approx 5 \cdot 10^{15} \, \text{T}.
\]

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**Appendix A: Dirac bispinors for \( B > 0 \)**

We work with the metric \( g^{\mu\nu} = \text{diag}(1,-1,-1,-1) \), and use Minkowskian Dirac matrices with the conventions \( \sigma^{12} = i\gamma^1\gamma^2 \) and \( \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \).

To write down the bispinor states for the outgoing charged lepton, we need to solve the Dirac equation for \( B > 0 \). The solutions are the so-called Landau levels, indexed by the Landau index \( n \). These have the energies

\[
E_{n,k_3,s_3} = \pm \sqrt{(2n + 1 + 2s_3)eB + k_3^2 + m_\ell^2},
\]  

where \( k_3 \) is the momentum along the magnetic field — which is chosen to lie in the positive \( z \) direction — and \( s_3 \) is the spin projection on the \( z \) axis. Note that the lowest energy is \( n = k_3 = 0 \) with \( s_3 = -1/2 \), reflecting the fact that the lepton spin (due to its negative charge \(-e\)) tends to align itself anti-parallel to \( B \). Higher-lying levels have

\[
\text{FIG. 3. The muonic decay rate in units of its } B = 0 \text{ value using the continuum extrapolated staggered results with physical quark masses (green). For comparison, the continuum extrapolated Wilson data at higher-than-physical quark masses are also included after a rescaling of the magnetic field by the squared pion mass (yellow). The LLL approximation we employed for the decay rate is valid for } eB > M_\pi^2(0) - m_\mu^2, \text{ marked by the dashed vertical line.}
\]
energies of at least $\sqrt{2eB}$ and are in general not expected to be relevant at low temperatures and densities, as long as $eB \gg m_\ell^2$. Here we employ the lowest Landau level (LLL) approximation ($n = 0$, $s_3 = -1/2$), which simplifies the discussion considerably. The calculation can be generalized to take into account all levels.

We start from the LLL modes calculated in Ref. [42] in the gauge $A_1 = -Bx_2$ and in the infinite volume (see also Ref. [2]). Here we consider the volume to be large but finite to regulate the number of modes. In a volume $V = L^3$ this degeneracy factor is $\Phi/(2\pi)$ with $\Phi = |eB|L^2$ being the flux of the magnetic field through the area of the system. The modes in coordinate space read,\footnote{Compared to Ref. [42] we insert a factor $\sqrt{k_0 + m_\ell} \cdot V$ for a more convenient normalization.}

$$u_\ell(k) = \sqrt{k_0 + m_\ell} \cdot \left( \frac{eB L^2}{\pi} \right)^{1/4} \cdot \left( 0, 1, 0, -\frac{k_3}{k_0 + m_\ell} \right)^T \cdot \exp \left[ i k_0 x_0 - i k_1 x_1 - i k_3 x_3 - eB \left( \frac{x_2}{2} + \frac{k_1}{cB} \right) \right],$$

where the energy is $k_0 = E_0 k_3 - 1/2 = \sqrt{k_1^2 + k_3^2}$. Note that while $k_3$ is a momentum, $k_1$ is a quantum number indexing the degeneracy of the Landau-modes and has no momentum interpretation. Also note that the LLL can only have negative spin so that

$$\sigma^{12} u_\ell(k) = -u_\ell(k).$$

The normalization of the state (A2) is

$$\int d^3x \, u_\ell^\dagger(p) \, u_\ell(k) = (2\pi)^2 \delta_{k_3}^2 \delta_{k_1}^2 \frac{2 k_0 L}{\pi}.$$ 

Thus, the states with different $k_1$ quantum numbers are orthogonal to each other. Regulating the $\delta$-functions in a finite volume,\footnote{The Fourier-transform of the $\delta$-function in a finite linear size $L$ is $(2\pi) \delta(p) = \int dx \, e^{ipx}$ so that $\delta(0) = L/(2\pi)$.} we have

$$|u_\ell(k)|^2 = \int d^3x \, u_\ell^\dagger(k) \, u_\ell(k) = 2 k_0 V,$$

which coincides with the usual normalization of bispinors (cf. Ref. [43]).

Having specified the normalization of the charged lepton state, we can calculate the sum over the intrinsic quantum numbers. For our LLL states this does not involve a spin-sum, since only $s_3 = -1/2$ is allowed. However, we have to sum over the unspecified quantum number $k_1$. In a finite volume, this index takes $\Phi/(2\pi)$ different values and can be approximated as $L/(2\pi)$ times the integral over $k_1$,

$$\sum_{\text{LLL}} \equiv \sum_{k_1} = L \int \frac{dk_1}{2\pi}.$$ 

After some algebra we obtain (see also Ref. [42]) the result that we quoted in Eq. (7) in the body of the text.

### Appendix B: Decay rate for $B > 0$

As explained in Sec. III, the antineutrino has fixed spin (in the positive $z$ direction) as well as fixed chirality. The corresponding bispinor solution $v_\nu(q)$ necessarily has vanishing perpendicular momenta $q_1 = q_2 = 0$, satisfies the on-shell relation $q_0 = |q_3|$ and fulfills

$$v_\nu(q)\bar{v}_\nu(q) = P_\sigma \, \delta(q), \quad P_\sigma = \frac{1 - \sigma^{12}}{2},$$

keeping in mind that the physical spin for antiparticles is the opposite of the eigenvalue of $P_\sigma$. Inserting (B1) and the spin sum (7) into the modulus square of the summed amplitude (5), we obtain

$$\mathcal{A} = \sum_{\text{LLL}} \mathcal{M} \mathcal{M}^* = \frac{G^2}{4\pi} \cos^2 \theta_c \, H_\rho H_\rho^* \, T^{\mu\rho},$$

where

$$T^{\mu\rho} \equiv \text{tr} \left[ P_\sigma q_{\rho} \gamma^\rho(1 - \gamma^5)(k_3 + m) P_\sigma \gamma^\mu(1 - \gamma^5) \right].$$

The trace is simplified using standard identities, revealing that $T^{\mu\rho}$ is only nonzero if the indices $\mu$ and $\rho$ assume the values 0 or 3. Thus, only $H_0$ and $H_3$ contribute to the decay rate, as anticipated in Sec. II above. Making use of Eq. (3) we arrive at

$$\mathcal{A} = \frac{G^2}{\pi} \cos^2 \theta_c \, M_\pi^2 \, (k^0 + k^3)(q^0 + q^3) \, |f_x + i f'_x eB|^2.$$ 

We proceed by specifying the kinematics in the frame where the pion vanishing momentum along the magnetic field (and is in the LLL). Together with the conservation rules, the on-shell conditions for the outgoing leptons fully specify the magnitude of $q^0 = -k^3$ in terms of the masses $M_\pi$ and $m_\ell$, the decay only being possible for $M_\pi > m_\ell$. Due to the right-handedness of the antineutrino and the fixed spins of the leptons, $k^3$ must in fact be negative, also visible from the $q^0 + q^3 = |q^3| + q^3 = |k^3| - k^3$ factor of Eq. (B4). Thus we have

$$k^0 = \frac{M_\pi^2 + m_\ell^2}{2M_\pi}, \quad k^3 = \frac{m_\ell^2 - M_\pi^2}{2M_\pi}.$$

Inserting this into Eq. (B4), we obtain

$$\mathcal{A} = \frac{G^2}{\pi} \cos^2 \theta_c \, |f_x + i f'_x eB|^2 (M_\pi^2 - m_\ell^2) m_\ell^2.$$ 

Finally, we need the phase space integration factor to obtain the full decay rate $\Gamma$. This is the integral of the differential probability $dP$ over a time interval $T$ [43],

$$\Gamma = \frac{1}{T} \int dP,$$

where, working with $k_3 = -k^3 > 0$,

$$dP = A \cdot \frac{1}{2M_\pi V} \frac{1}{2q_0 V} \frac{1}{2k_0 V} \frac{V}{(2\pi)^3} d^3q \frac{L}{2\pi} \, dk_3 \cdot TV(2\pi)^4 \delta(M_\pi - q_0 - k_0) \delta(q_3 + k_3) \delta(q_1) \delta(q_2).$$

(B8)
Here the normalization of the one-particle states (for the charged lepton, see Eq. (A5)) was used, and the \( \delta \)-functions ensure that the perpendicular momenta of the antineutrino vanish.\(^3\) Note that the phase space for the charged lepton is one-dimensional due to its LLL nature. Performing the integral over \( q \) and expressing the energies through \( k_3 \) we obtain

\[
\Gamma = \frac{A}{4M_\pi L^2} \int_0^\infty \frac{dk_3}{k_3 \sqrt{k_3^2 + m_\pi^2}} \delta \left( M_\pi - k_3 - \sqrt{k_3^2 + m_\pi^2} \right). \tag{B9}
\]

After the variable substitution \( y = k_3 + \sqrt{k_3^2 + m_\pi^2} \), we insert \( A \) from Eq. (B6), to finally arrive at Eq. (8) of the main text.\(^4\) As anticipated, all volume factors have canceled. We note that while the inclusion of higher Landau levels is considerably more involved, their contribution is constrained via energy conservation. Indeed, for the charged lepton to be created with quantum numbers \( n \) and \( s_3 \), it is required that \( M_\pi^2 > E_{n,0,s_3}^2 = (2n + 1 + 2s_3) \frac{eB}{2} + m_\pi^2 \), see Eq. (A1). Thus, for magnetic fields \( eB > M_\pi^2(0) - m_\pi^2 \), only the LLL \((n = 0, s_3 = -1/2)\) can contribute. For higher orders in electroweak perturbation theory soft photons become relevant and this simple picture changes.

The decay rate may be compared to the \( B = 0 \) result \([25]\),

\[
\Gamma(0) = \frac{G^2}{8\pi} \cos^2 \theta_c f_\pi^2(0) \left[ M_\pi^2(0) - m_\pi^2 \right] \frac{m_\pi^2}{M_\pi^2(0)} , \tag{B10}
\]

where we explicitly indicated that \( M_\pi \) and \( f_\pi \) are to be understood at \( B = 0 \). The ratio of Eqs. (8) and (B10) gives Eq. (9) in the body of the text. For charged kaon decay the calculation is completely analogous, only the substitutions \( M_\pi \to M_K \), \( f_\pi \to f_K \), \( f_\pi' \to f'_K \) and \( \cos \theta_c \to \sin \theta_c \) have to be made.

\[\footnotesize{\begin{align*}
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\end{align*}\]
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