RELATIVISTIC CORRECTIONS TO THE SUNYAEV-ZELDOVICH EFFECT FOR CLUSTERS OF GALAXIES

NAOKIITOHI
Department of Physics, Sophia University, 7-1 Kioi-cho, Chiyoda-ku, Tokyo, 102, Japan; nito@hoffman.cc.sophia.ac.jp

YASUHARUKOHYAMA
Fuji Research Institute Corporation, 2-3 Kanda-Nishiki-cho, Chiyoda-ku, Tokyo, 101, Japan; kohyama@crab.fuji-ric.co.jp

AND

SATOSHI NOZAWA
Josai Junior College for Women, 1-1 Keyakidai, Sakado-shi, Saitama, 350-02, Japan; snozawa@venus.josai.ac.jp

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ABSTRACT

We study the generalized Kompaneets equation (the kinetic equation for the photon distribution function taking into account Compton scattering by electrons) using a relativistically covariant formalism. Using the generalized Kompaneets equation, we derive an analytic expression for the Sunyaev-Zeldovich effect, which takes into account up to \(O(\theta^2)\) terms, where \(\theta_c = k_B T_e/mc^2\) is the relativistic expansion parameter and \(T_e\) is the electron temperature. We carefully study the applicable region of the obtained analytic expression by comparing it with the result of the direct numerical integration. We conclude that the present analytic expression can be reliably applied to the calculation of the Sunyaev-Zeldovich effect for \(k_B T_e \leq 15\) keV, which is the highest electron temperature in the clusters of galaxies presently known. Therefore, the present analytic expression can be applied to all known clusters of galaxies.

Subject headings: cosmic microwave background — cosmology: theory — distance scale — galaxies: clusters: general — radiation mechanisms: nonthermal — relativity

1. INTRODUCTION

Compton scattering of the cosmic microwave background (CMB) radiation by hot intracluster gas—the Sunyaev-Zeldovich effect (Zeldovich & Sunyaev 1969; Sunyaev & Zeldovich 1972, 1980)—provides a useful method to measure the Hubble constant \(H_0\) (Gunn 1978; Silk & White 1978; Birkinshaw 1979; Cavaliere, Danese, & De Zotti 1979; Birkinshaw, Hughes, & Arnaud 1991; Birkinshaw & Hughes 1994; Myers et al. 1995; Herbig et al. 1995; Jones 1995; Markevitch et al. 1996; Holzapfel et al. 1997). The Sunyaev-Zeldovich formula has been derived from a kinetic equation for the photon distribution function taking into account the Compton scattering by electrons: the Kompaneets equation (Kompaneets 1957; Weymann 1965). The Kompaneets equation has been derived with a nonrelativistic approximation for the electron. However, the electrons in clusters of galaxies are extremely hot, \(k_B T_e = 5–15\) keV (Arnaud et al. 1994; Markevitch et al. 1994; Markevitch et al. 1996; Holzapfel et al. 1997).

Recently, attempts have been made to include the relativistic corrections in the Sunyaev-Zeldovich effect (Rephaeli 1995; Rephaeli & Yankovitch 1997). However, it appears that the calculations have not been carried out in a manifestly covariant form. For example, equation (4) in Rephaeli (1995), which comes from Chandrasekhar (1950), is a nonrelativistic formula. Since the extension of the Kompaneets equation to the relativistic regime is extremely important in view of many recent measurements of the Hubble constant \(H_0\) with the use of the Sunyaev-Zeldovich effect, we will solve the kinetic equation for the photon distribution function in a manifestly covariant form, taking into account the Compton scattering by electrons.

Very recently a generalized Kompaneets equation has been derived by two groups (Stebbins 1997; Challinor & Lasenby 1997). By using the generalized Kompaneets equation, analytic expressions for the Sunyaev-Zeldovich effect have been derived as a power series of \(\theta_c = k_B T_e/mc^2\), where \(T_e\) and \(m\) are the electron temperature and the electron mass, respectively. It has been shown that the results obtained by a power series expansion agree with the previous numerical calculations by Rephaeli (1995) and Rephaeli & Yankovitch (1997). Analytic expressions are compact and extremely useful to study the Sunyaev-Zeldovich effect. On the other hand, it has been pointed out by Challinor & Lasenby (1997) that the convergence of the power series expansion in \(\theta_c\) is slow. They suspect that the expansion is an asymptotic expansion. Therefore, it is extremely important to study the extent of the \(\theta_c\) region where the analytic expressions can be applied. This is the main subject in the present paper. Following the approach of Challinor & Lasenby (1997), we will derive analytic expressions for the intensity change of the photon spectrum. In the derivation we will take into account the relativistic terms up to the fifth order in \(\theta_c\). In order to examine the accuracy of the analytic expressions derived from the power series expansion, we will directly integrate the Boltzmann equation numerically. Comparing these results, we will carefully study the valid region for the electron temperature \(T_e\) and for the photon angular frequency \(\omega\) where the present analytic expressions can be reliably applied.

The present paper is organized as follows: extension of the Kompaneets equation to the relativistic regime will be treated, and an analytic expression for the intensity change of the photon spectrum will be derived, in § 2. In § 3 we will study the accuracy of the analytic expressions by comparing them with the numerical results. Concluding remarks will be given in § 4.

2. GENERALIZED KOMPANEETS EQUATION

In this section we will extend the Kompaneets equation to the relativistic regime. We will formulate the kinetic equation for the photon distribution function using a relativistically covariant formalism (Berestetskii, Lifshitz, & Pitaevskii 1982; Buchler
& Yueh 1976). As a reference system, we choose the system that is fixed to the center of mass of the cluster of galaxies. This choice of reference system allows us to carry out all the calculations in the most straightforward way. We will use the invariant amplitude for the Compton scattering as given by Berestetskii et al. (1982) and by Buchler & Yueh (1976).

The time evolution of the photon distribution function $n(\omega)$ is written as

$$\frac{\partial n(\omega)}{\partial t} = -2 \int \frac{d^3p}{(2\pi)^3} d^3p' d^3k' W[n(\omega)[1 + n(\omega')]f(E) - n(\omega')[1 + n(\omega)]f(E')] ,$$

$$W = \frac{(e^2/4\pi^2)X\delta^4(p + k - p' - k')}{2\omega_0'E'} ,$$

$$X = -\left(\frac{\kappa}{\kappa'} + \frac{\kappa'}{\kappa}\right) + 4m^2\left(\frac{1}{\kappa} + \frac{1}{\kappa'}\right) - 4m^2\left(\frac{1}{\kappa} + \frac{1}{\kappa'}\right) ,$$

$$\kappa = 2(\mathbf{p} \cdot \mathbf{k}) = -2\omega E\left(1 - \frac{p}{E}\cos \alpha\right) ,$$

$$\kappa' = 2(\mathbf{p} \cdot \mathbf{k'}) = 2\omega E\left(1 - \frac{p}{E}\cos \alpha'\right).$$

In the above, $W$ is the transition probability corresponding to the Compton scattering. The four-momenta of the initial electron and photon are $p = (E, \mathbf{p})$ and $k = (\omega, \mathbf{k})$, respectively. The four-momenta of the final electron and photon are $p' = (E', \mathbf{p}')$ and $k' = (\omega', \mathbf{k}')$, respectively. The angles $\alpha$ and $\alpha'$ are the angles between $p$ and $k$ and between $p$ and $k'$, respectively. Throughout this paper, we use the natural unit $\hbar = c = 1$ unit, unless otherwise stated explicitly.

By ignoring the degeneracy effects, we have the relativistic Maxwellian distribution for electrons with temperature $T_e$ as follows:

$$f(E) = \left(e^{(E - m - (\mu - m))/k_BT_e} + 1\right) \approx e^{-(K - (\mu - m))/k_BT_e} ,$$

where $K \equiv (E - m)$ is the kinetic energy of the initial electron and $(\mu - m)$ is the nonrelativistic chemical potential of the electron. We now introduce the quantities

$$\chi \equiv \frac{\omega}{k_BT_e} ,$$

$$\Delta \chi \equiv \frac{\omega' - \omega}{k_BT_e} .$$

Substituting equations (2.6)–(2.8) into equation (2.1), we obtain

$$\frac{\partial n(\omega)}{\partial t} = -2 \int \frac{d^3p}{(2\pi)^3} d^3p' d^3k' Wf(E)[1 + n(\omega')]n(\omega) - [1 + n(\omega)]n(\omega')e^{\chi}\cdot$$

Equation (2.9) is our basic equation. One can numerically integrate this equation directly. We will perform this integration in § 3.

Following Challinor & Lasenby (1997), we expand equation (2.9) in powers of $\Delta \chi$ by assuming $\Delta \chi \ll 1$. We obtain the Fokker-Planck expansion

$$\frac{\partial n(\omega)}{\partial t} = 2\left[\frac{\partial n}{\partial \chi} + n(1 + n)\right]I_1 + 2\left[\frac{\partial^2 n}{\partial \chi^2} + 2(1 + n)\frac{\partial n}{\partial \chi} + n(1 + n)\right]I_2 + 2\left[\frac{\partial^3 n}{\partial \chi^3} + 3(1 + n)\frac{\partial^2 n}{\partial \chi^2} + 3(1 + n)\frac{\partial n}{\partial \chi} + n(1 + n)\right]I_3 + 2\left[\frac{\partial^4 n}{\partial \chi^4} + 4(1 + n)\frac{\partial^3 n}{\partial \chi^3} + 6(1 + n)\frac{\partial^2 n}{\partial \chi^2} + 4(1 + n)\frac{\partial n}{\partial \chi} + n(1 + n)\right]I_4 + \cdots ,$$

where

$$I_k \equiv \frac{1}{k!} \int \frac{d^3p}{(2\pi)^3} d^3p' d^3k' Wf(E)(\Delta \chi)^k .$$

Analytic integration of equation (2.11) is not possible except by doing power series expansions of the integrand. Technically speaking, there are several choices for the expansion parameter of the integrand of equation (2.11). They are, for example, $p/m$, $K/m \equiv (E/m - 1)$, and $v \equiv p/E$. It is important to note that obtained analytic expressions of $I_k$ after the integration do not depend on the choice of the expansion parameter. It is also extremely important to note that the expansions in terms of these variables are asymptotic expansions in $I_k$. Therefore, not only is the convergence very slow, but the accuracy of the analytic expressions also has to be carefully examined for the parameter region considered. This is one of our main subjects in the present paper.
Challinor & Lasenby (1997) carried out a calculation up to $O(\theta_3^2)$ terms. We will carry out a calculation up to $O(\theta_4^2)$ terms in the present paper. The calculation of $I_k$ has been performed with a symbolic manipulation computer algebra package MATHEMATICA. We obtain

\[ I_1 = \frac{1}{2} \sigma_T N_e \theta_c x \left[ 4 - x + \theta_c \left( 10 - \frac{47}{2} x + \frac{21}{5} x^2 \right) + \theta_c^2 \left( \frac{15}{2} - \frac{1023}{8} x + \frac{567}{5} x^2 - \frac{147}{10} x^3 \right) \right. \]
\[ + \theta_c^2 \left( -\frac{15}{32} - \frac{2505}{5} x + \frac{9891}{10} x^2 - \frac{9551}{20} x^3 + \frac{1616}{35} x^4 \right) \]
\[ + \theta_c^2 \left( 135 - \frac{30375}{128} x + \frac{177849}{40} x^2 - \frac{472349}{80} x^3 + \frac{63456}{35} x^4 - \frac{940}{7} x^5 \right) \left. \right] , \quad (2.12) \]

\[ I_2 = \frac{1}{2} \sigma_T N_e \theta_c x^2 \left[ 1 + \theta_c \left( \frac{47}{5} x - \frac{7}{10} x^2 \right) + \theta_c^2 \left( \frac{1023}{8} x - \frac{1302}{5} x + \frac{161}{2} x^2 - \frac{22}{5} x^3 \right) \right. \]
\[ + \theta_c^2 \left( \frac{2505}{8} - \frac{10647}{5} x + \frac{38057}{20} x^2 - \frac{2829}{7} x^3 + \frac{682}{35} x^4 \right) \]
\[ + \theta_c^2 \left( \frac{30375}{128} - \frac{187173}{20} x + \frac{1701803}{80} x^2 - \frac{44769}{4} x^3 + \frac{61512}{35} x^4 - \frac{510}{7} x^5 \right) \left. \right] , \quad (2.13) \]

\[ I_3 = \frac{1}{2} \sigma_T N_e \theta_c x^3 \left[ \theta_c \left( \frac{42}{5} x - \frac{7}{5} x^2 \right) + \theta_c^2 \left( \frac{868}{5} x - \frac{658}{5} x + \frac{88}{5} x^2 - \frac{11}{30} x^3 \right) \right. \]
\[ + \theta_c^2 \left( \frac{7098}{5} - \frac{14253}{x} + \frac{8084}{7} x^2 - \frac{3503}{28} x^3 + \frac{64}{21} x^4 \right) \]
\[ + \theta_c^2 \left( \frac{62391}{10} - \frac{614727}{20} x + \frac{28193}{16} x^2 - \frac{123083}{16} x^3 + \frac{14404}{21} x^4 - \frac{344}{21} x^5 \right) \left. \right] , \quad (2.14) \]

\[ I_4 = \frac{1}{2} \sigma_T N_e \theta_c x^4 \left[ \theta_c \left( \frac{7}{10} \theta_c + \theta_c^2 \left( \frac{329}{5} - \frac{22}{10} x \right) \right) \right. \]
\[ + \theta_c^2 \left( \frac{14253}{10} - \frac{9297}{7} x + \frac{7781}{28} x^2 - \frac{320}{21} x^3 + \frac{16}{105} x^4 \right) \]
\[ + \theta_c^2 \left( \frac{614727}{10} - \frac{124389}{4} x + \frac{239393}{16} x^2 - \frac{7010}{3} x^3 + \frac{12676}{105} x^4 - \frac{11}{7} x^5 \right) \left. \right] , \quad (2.15) \]

\[ I_5 = \frac{1}{2} \sigma_T N_e \theta_c x^5 \left[ \theta_c^2 \left( \frac{44}{5} - \frac{11}{10} x \right) + \theta_c^2 \left( \frac{18594}{35} - \frac{36177}{140} x + \frac{192}{7} x^2 - \frac{64}{105} x^3 \right) \right. \]
\[ + \theta_c^2 \left( \frac{124389}{10} - \frac{1067109}{80} x + \frac{3696 x^2}{15} - \frac{5032}{15} x^3 + \frac{66}{7} x^4 - \frac{11 x^5}{210} \right) \left. \right] , \quad (2.16) \]

\[ I_6 = \frac{1}{2} \sigma_T N_e \theta_c x^6 \left[ \frac{1}{30} \theta_c^2 + \theta_c^2 \left( \frac{12059}{140} - \frac{64}{3} x + \frac{32}{35} x^2 \right) \right. \]
\[ + \theta_c^2 \left( \frac{355703}{80} - \frac{8284}{3} x + \frac{6688}{15} x^2 - \frac{22 x^3 + 11}{42} x^4 \right) \left. \right] , \quad (2.17) \]

\[ I_7 = \frac{1}{2} \sigma_T N_e \theta_c x^7 \left[ \theta_c^2 \left( \frac{128}{21} - \frac{64}{105} x \right) + \theta_c^2 \left( \frac{16568}{21} - \frac{30064}{105} x + \frac{176}{7} x^2 - \frac{11}{21} x^3 \right) \right. \left. \right] , \quad (2.18) \]

\[ I_8 = \frac{1}{2} \sigma_T N_e \theta_c x^8 \left[ \frac{16}{105} \theta_c^2 + \theta_c^2 \left( \frac{7516}{105} - \frac{99}{7} x + \frac{11}{21} x^2 \right) \right. \left. \right] , \quad (2.19) \]

\[ I_9 = \frac{1}{2} \sigma_T N_e \theta_c x^9 \left[ \theta_c^2 \left( \frac{22}{7} - \frac{11}{42} x \right) \right. \left. \right] , \quad (2.20) \]

\[ I_{10} = \frac{1}{2} \sigma_T N_e \theta_c x^{10} \left[ \theta_c^2 \left( \frac{11}{210} \theta_c^2 \right) \right. \left. \right] , \quad (2.21) \]
where \( \sigma_T \) is the Thomson scattering cross section and \( N_e \) is the electron number density. The expansion parameter \( \theta_e \) is defined by

\[
\theta_e = \frac{k_B T_e}{m c^2}.
\] (2.22)

In deriving equations (2.12)–(2.21), we have ignored \( O(\theta_e^5) \) contributions. Using equations (2.12)–(2.21), one can show that the photon number is conserved order by order in terms of the expansion parameter \( \theta_e \).

We now apply the present result of the generalized Kompaneets equation to the Sunyaev-Zeldovich effect for clusters of galaxies. We assume the initial photon distribution of the CMB radiation to be Planckian with temperature \( T_0 \):

\[
n_0(X) = \frac{1}{e^X - 1},
\] (2.23)

where

\[
X = \frac{\omega}{k_B T_0}.
\] (2.24)

Substituting equation (2.23) and equations (2.12)–(2.21) into equation (2.10), and assuming \( T_0 / T_e \ll 1 \), one obtains the following expression for the fractional distortion of the photon spectrum:

\[
\frac{\Delta n(X)}{n_0(X)} = \frac{y \theta_e X e^X}{e^X - 1} \left( Y_0 + \theta_e Y_1 + \theta_e^2 Y_2 + \theta_e^3 Y_3 + \theta_e^4 Y_4 \right),
\] (2.25)

\[
Y_0 = -4 + \frac{X}{2},
\] (2.26)

\[
Y_1 = -10 + \frac{47}{2} X - \frac{42}{5} X^2 + \frac{7}{10} X^3 + \mathcal{S}^2 \left( -\frac{21}{5} + \frac{7}{5} X \right),
\] (2.27)

\[
Y_2 = -\frac{15}{8} + \frac{1023}{8} X - \frac{868}{5} X^2 + \frac{329}{5} X^3 - \frac{44}{5} X^4 + \frac{11}{30} X^5
\]

\[
+ \mathcal{S}^2 \left( -\frac{434}{5} + \frac{658}{5} X - \frac{242}{30} X^2 + \frac{143}{30} X^3 \right) + \mathcal{S}^4 \left( -\frac{44}{5} + \frac{187}{60} X \right),
\] (2.28)

\[
Y_3 = \frac{15}{2} + \frac{2505}{8} X - \frac{7098}{5} X^2 + \frac{14253}{10} X^3 - \frac{18594}{35} X^4 + \frac{12059}{140} X^5 - \frac{128}{21} X^6 + \frac{16}{105} X^7
\]

\[
+ \mathcal{S}^2 \left( -\frac{7098}{10} + \frac{14253}{5} X - \frac{102267}{35} X^2 + \frac{156767}{140} X^3 - \frac{1216}{7} X^4 + \frac{64}{7} X^5 \right)
\]

\[
+ \mathcal{S}^4 \left( -\frac{18594}{35} + \frac{205003}{280} X - \frac{1920}{7} X^2 + \frac{1024}{35} X^3 \right) + \mathcal{S}^6 \left( -\frac{544}{21} + \frac{992}{105} X \right),
\] (2.29)

\[
Y_4 = -\frac{135}{32} + \frac{30375}{128} X - \frac{62391}{10} X^2 + \frac{614727}{40} X^3 - \frac{124389}{10} X^4
\]

\[
+ \frac{355703}{80} X^5 - \frac{16568}{21} X^6 + \frac{7516}{105} X^7 - \frac{22}{7} X^8 + \frac{11}{210} X^9
\]

\[
+ \mathcal{S}^2 \left( -\frac{62391}{20} + \frac{614727}{20} X - \frac{1368279}{20} X^2 + \frac{4624139}{80} X^3 - \frac{157396}{7} X^4
\]

\[
+ \frac{30064}{7} X^5 - \frac{2717}{7} X^6 + \frac{2761}{210} X^7 \right)
\]

\[
+ \mathcal{S}^4 \left( -\frac{124389}{10} + \frac{6046951}{160} X - \frac{248520}{7} X^2 + \frac{481024}{35} X^3 - \frac{15972}{7} X^4 + \frac{18689}{140} X^5 \right)
\]

\[
+ \mathcal{S}^6 \left( -\frac{70414}{21} + \frac{465992}{105} X - \frac{11792}{7} X^2 + \frac{19778}{105} X^3 \right) + \mathcal{S}^8 \left( -\frac{682}{7} + \frac{7601}{210} X \right),
\] (2.30)
where

\[ y \equiv \sigma_T \int d/N_e , \quad (2.31) \]

\[ \bar{X} \equiv X \coth \left( \frac{X}{2} \right) , \quad (2.32) \]

\[ \bar{S} \equiv \frac{X}{\sinh \left( X/2 \right)} . \quad (2.33) \]

Note that the analytic forms of \( Y_0, Y_1, \) and \( Y_2 \) in equations (2.26)-(2.28) agree with the results obtained by Challinor & Lasenby (1997). Finally, we define the distortion of the spectral intensity as follows:

\[ \Delta I = \frac{X^3}{e^x - 1} \frac{\Delta n(X)}{n_0(X)} . \quad (2.34) \]

3. ANALYSIS OF THE CONVERGENCE OF THE POWER SERIES

We now carefully study the convergence of the analytic expressions of equations (2.25) and (2.34). In order to do the task, first of all, we integrate equation (2.9) directly by numerical integration. We confirm that the total photon number is conserved with excellent accuracy \(( < 10^{-9} )\) in the numerical integration. We are now ready to compare the present numerical results with those obtained by the analytic expressions of equations (2.25) and (2.34) for various \( X-T_e \) regions and investigate the accuracy of the analytic expressions.

3.1. Rayleigh-Jeans Region

In the Rayleigh-Jeans limit where \( X \to 0 \), equation (2.25) is further simplified:

\[ \frac{\Delta n(X)}{n_0(X)} \to -2\gamma \theta \left( 1 - \frac{17}{10} \theta_e + \frac{123}{40} \theta_e^2 - \frac{1989}{280} \theta_e^3 + \frac{14403}{640} \theta_e^4 \right) . \quad (3.1) \]

As is seen explicitly from equation (3.1), the convergence of the power expansion is very fast in the \( X \to 0 \) limit. Furthermore, we show in Figure 1 the \( T_e \)-dependence of the spectral intensity distortion equation (2.34) for \( X = 1 \). As is expected, the convergence is extremely fast for \( k_B T_e \leq 50 \text{ keV} \). Relativistic corrections higher than \( O(\theta_e^2) \) terms are almost negligible in this region. So far the Sunyaev-Zeldovich effects have been measured in the Rayleigh-Jeans region. Therefore one can reliably apply the analytic expressions of equations (2.25), (2.34), and (3.1) to the analysis of the observed data.

In passing we remark that the form of equation (3.1) is meaningful only in an idealized situation. In order that the higher order terms have a physical meaning, it is necessary that the electron distribution is rigorously given by the relativistic Maxwellian distribution (eq. [2.6]) with a precisely determined temperature \( T_e \). In real observation, the electron temperature \( T_e \) has a significant amount of observational error. This thereby restricts the precision of equation (3.1).

![Figure 1](image_url)

**Fig. 1.**—Spectral intensity distortion \( \Delta I/y \) as a function of \( k_B T_e \) for \( X = 1 \). The solid curve shows the result of the numerical integration, the dotted curve shows the contribution of the first two terms in eq. (2.25), the long-short-dashed curve shows the contribution of the first three terms, the dash-dotted curve shows the contribution from the first four terms, and the dashed curve shows the contribution from all the terms in eq. (2.25).
3.2. $X \approx 4$ Region

As one can see from equation (2.26), the leading order contribution $Y_0$ vanishes at $\bar{X} = 4 \approx X$. Therefore, higher order corrections become more important in this region. In Figure 2 we have plotted the $T_e$-dependence of the fractional spectral distortion at $X = 4$. It is seen that the dash-dotted line is closest to the exact result. It should be emphasized here that the dash-dotted line is the contribution that includes only the first four terms in equation (2.25). The dashed curve, which includes all the terms in equation (2.25), shows a poorer agreement with the exact result. This means that the power series expansion in $\theta_e$ in this region is not convergent, but asymptotic for large $T_e$. We conclude that an analytic expression that includes up to $O(\theta_e^3)$ terms is reliable for $k_BT_e \leq 15$ keV in the $X \approx 4$ region. We recommend that an analytic expression that includes up to $\theta_e^3 Y_3$ terms (dash-dotted curve) be used for the analysis of the observational data for 15 keV $< k_BT_e < 30$ keV, $X \approx 4$.

3.3. Wien Region

We now study the Wien region, where $X > 4$. As is mentioned earlier, the Sunyaev-Zeldovich effects have so far been studied observationally in the $X < 1$ region. However, the effects will be observed in the Wien region in the future. For an illustrative purpose, we show the $T_e$-dependence of the spectral intensity distortion of equation (2.34) at $X = 8$ in Figure 3. The convergence is very slow. All curves are diverging quickly from the solid curve (exact result) for $k_BT_e > 30$ keV. We conclude that the analytic expression including up to $O(\theta_e^2)$ terms is reliable for $k_BT_e \leq 15$ keV. In Figures 4-5 we show $\Delta I/y$ for $k_BT_e = 15$ keV and $k_BT_e = 20$ keV, respectively. We confirm the good accuracy of the analytic expression for $k_BT_e = 15$ keV.
3.4. Crossover Frequency

Finally, we study the crossover frequency $X_0$, where the spectral intensity distortion vanishes. It is known that the accurate determination of the $X_0$ values is extremely important for the study of the Sunyaev-Zeldovich effects (Rephaeli 1995). In Figure 6, we have plotted the $T_e$-dependence of $X_0$ for $k_B T_e \leq 50$ keV calculated by the analytic expressions and also by the numerical integration (solid curve). The numerical result is well approximated as a linear function of $\theta_e$ for $k_B T_e < 20$ keV. It starts to deviate from the linear form for $k_B T_e > 20$ keV. We have fitted the numerical result as follows:

$$X_0 \approx 3.830(1 + 1.1674\theta_e - 0.8533\theta_e^2) .$$

The errors of this fitting function are less than $1 \times 10^{-3}$ for $0 \leq k_B T_e \leq 50$ keV.

Rephaeli (1995) discusses the consequence of the relativistic shift of the crossover frequency on the value of the peculiar velocity of the cluster measured with the use of the kinematic Sunyaev-Zeldovich effect. For a cluster with $k_B T_e = 13.8$ keV, one obtains an error of $20$ km s$^{-1}$ in the deduced value of the peculiar velocity for the SUZIE experiment—corresponding to an accuracy of $1 \times 10^{-3}$ in equation (3.2)—by making an interpolation of Rephaeli’s estimation. Therefore, one concludes that the current level of the observational accuracy has not reached the accuracy provided by the theoretical equation (3.2).

Another implication of the accuracy $1 \times 10^{-3}$ of equation (3.2) will be the following: Let us consider a cluster with the plasma temperature $k_B T_e = 10$ keV, making a proper motion with the velocity $v = 1000$ km s$^{-1}$. Then the effect of this proper motion on the Sunyaev-Zeldovich effect will be of order $(v/c)^2/(k_B T_e/\rho c^2) = 5 \times 10^{-4}$. Thus the kinematic effect of the proper motion of the cluster will not exceed the accuracy of equation (3.2).
Finally, we estimate an error $\delta$ of the power series approximation in the $X$-$T_e$ parameter space. We define

$$\delta \equiv 1 - \frac{\Delta I}{\Delta I_{\text{num}}}$$

where $\Delta I$ is given by equation (2.34) with equations (2.25)–(2.33) and $\Delta I_{\text{num}}$ is the result of the direct numerical integration of equation (2.1). In Figure 7, we have plotted the results for $\delta = 1\%$, $5\%$, and $10\%$. It should be remarked that the sharp peaks in Figure 7 correspond to the points of $X$ where the analytic curve and the numerical curve are crossing. This can be easily seen in Figure 5, where the dashed curve (analytic curve) and the solid curve (numerical curve) are crossing at these points of $X$. It is obvious from Figure 7 that the power series approximation is extremely accurate up to 50 keV for the Rayleigh-Jeans region ($X < 1.5$). The approximation has a larger error in the $X \approx 4$ region, where the leading order contribution vanishes. Finally, the power series approximation is good up to 15 keV for the Wien region.

4. CONCLUDING REMARKS

We have calculated the relativistic corrections to the Sunyaev-Zeldovich effect including terms up to $O(h^5)$. The results of the obtained analytic expressions have been compared with the result of the direct numerical integration. The extent of the applicability of these analytic expressions has been carefully examined. It has been shown that the present analytic expressions
have an excellent accuracy in the Rayleigh-Jeans region where $X \leq 1$. There the convergence is extremely fast, and the results are reliable at least for $k_B T_e \leq 50$ keV. On the other hand, the applicability of the analytic expressions becomes limited for cases, both of the $X \approx 4$ region and of the Wien region, where $X > 4$. In these regions, the presently obtained analytic expression, which takes into account up to $O(\theta_x^2)$ terms, is reliably applicable for $k_B T_e \leq 15$ keV. This is the highest electron temperature of the presently known clusters of galaxies. Therefore, the present analytic expressions can be reliably applied to the known clusters of galaxies.

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