CLASSIFICATION OF NONNEGATIVE SOLUTIONS TO AN EQUATION INVOLVING THE LAPLACIAN OF ARBITRARY ORDER

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Abstract. We classify all nonnegative nontrivial classical solutions to the equation

\[-\Delta u = c_1 \left( \frac{1}{|x|^{n-\beta}} * f(u) \right) g(u) + c_2 h(u) \quad \text{in} \ \mathbb{R}^n,
\]

where \(0 < \alpha, \beta < n\), \(c_1, c_2 \geq 0\), \(c_1 + c_2 > 0\) and \(f, g, h \in C([0, +\infty), [0, +\infty))\) are increasing functions such that \(f(t)/t^{\alpha+n+\beta}, g(t)/t^{\alpha+n+\beta}, h(t)/t^{n+\alpha}\) are non-increasing in \((0, +\infty)\). We also derive a Liouville type theorem for the equation in the case \(\alpha \geq n\). The main tool we use is the method of moving spheres in integral forms.

1. Introduction. The objective of the present paper is to classify all nonnegative nontrivial classical solutions to the equation

\[-\Delta u = c_1 \left( \frac{1}{|x|^{n-\beta}} * f(u) \right) g(u) + c_2 h(u) \quad \text{in} \ \mathbb{R}^n, \tag{1}
\]

where \(n \geq 2, \alpha, \beta > 0, c_1, c_2 \geq 0, c_1 + c_2 > 0\) and \(f, g, h \in C([0, +\infty), [0, +\infty))\) satisfy some mild conditions. Such an equation is said to be in critical order if \(\alpha = n\), in sub-critical order if \(0 < \alpha < n\) and in super-critical order if \(\alpha > n\). In particular, our results can be applied to the model

\[-\Delta u = c_1 \left( \frac{1}{|x|^{n-\beta}} * u^{q_0} \right) u^{q_1} + c_2 u^{q_2} \quad \text{in} \ \mathbb{R}^n, \tag{2}
\]

where \(q_0, q_1, q_2 > 0\).

When \(0 < \alpha < 2\), the operator \(-\Delta^{\alpha}\) is the well-known fractional Laplacian which is defined as

\[-\Delta^{\alpha} u(x) = C_{n,\alpha} \lim_{\varepsilon \to 0^+} \int_{|y-x|\geq \varepsilon} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy.
\]
for functions $u \in C^1_{\text{loc}}(\mathbb{R}^n) \cap L_\alpha$ with arbitrarily small $\varepsilon > 0$. Here $[\alpha]$ denotes the integer part of $\alpha$, $\{\alpha\} := \alpha - [\alpha]$, the normalization constant $C_{n,\alpha} = \left(\int_{\mathbb{R}^n} 1_{|\xi|^{n+\alpha}}d\xi\right)^{-1}$ and the space

$$L_\alpha = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}}dx < \infty \right\}.$$

More generally, for $\alpha = 0$, we adopt the convention that $(-\Delta)^0 u = u$ and for any positive real number $\alpha$, we define

$$(-\Delta)^{\frac{\alpha}{2}} u = (-\Delta)^{\lfloor \frac{\alpha}{2} \rfloor} (-\Delta)^{\frac{\alpha}{2} - \lfloor \frac{\alpha}{2} \rfloor} u.$$

Such an operator is well-defined for

$$u \in \begin{cases} C^n(\mathbb{R}^n) & \text{if } \frac{\alpha}{2} \text{ is an integer}, \\ C^1_{\text{loc}}(\mathbb{R}^n) \cap L_\alpha & \text{otherwise,} \end{cases}$$

where $\varepsilon > 0$ is arbitrarily small. Throughout this paper, we study nonnegative solutions $u$ of (1) and (2) in that classical sense.

First, we discuss the case $c_1 = 0$. In this case, equation (2) has the form

$$(-\Delta)^{\frac{\alpha}{2}} u = u^p \quad \text{in } \mathbb{R}^n,$$

which was studied by several authors in the past. For $\alpha = 2$ and $1 < p < \frac{n+2}{n-2}$ (i.e. $\infty$ if $n = 2$), Liouville type theorem for positive solutions of (3) was established by Gidas-Spruck in their seminal paper [19]. This nonexistence theorem can be complemented with a classification result in the critical case $p = \frac{n+2}{n-2}$, see Caffarelli-Gidas-Spruck [2]. Later, the proof was remarkably simplified by Chen-Li [6] using the Kelvin transform and the method of moving planes. Analogous results for the case $0 < \alpha < 2$ and $1 < p \leq \frac{n+2}{n-2}$ was established recently by Chen-Li-Li [8] using a direct method of moving planes. Later, this method was used by Zhang-Wang in [43] to derive a classification result for equation (1) with $0 < \alpha < 2$, $c_1 = 0$ and $h$ is an increasing positive function such that $h(t)/t^{\frac{n+2}{n-2}}$ is nonincreasing in $(0, +\infty)$. Notice that Zhang-Wang’s result is the fractional counterpart of classical results for $\alpha = 2$ in [1, 16].

As to the case $\frac{\alpha}{2} = m \in (1, \frac{n}{2}) \cap \mathbb{N}$, equation (3) is usually regarded as a higher order or polyharmonic equation. Wei-Xu [41] proved a Liouville type theorem for nonnegative $C^{2m}(\mathbb{R}^n)$ solutions of (3) when $1 < p < \frac{n+2m}{n-2m}$ and classified all such solutions in the case $p = \frac{n+2m}{n-2m}$. Before that, Lin [32] established a similar result for the special case $m = 2$. We also notice that the higher fractional order case $\frac{\alpha}{2} \in (1, \frac{n}{2}) \setminus \mathbb{N}$ was tackled recently by Cao-Dai-Qin in [4]. All these results indicate that every positive solution of (3) in the critical case $p = \frac{n+2}{n-2}$ must assume the form

$$u(x) = C \left( \frac{\mu}{1 + \mu^2|x - \varpi|^2} \right)^{\frac{n-\alpha}{2}}$$

for some $C, \mu > 0$ and $\varpi \in \mathbb{R}^n$.

Now we turn our attention to the case $c_1 \neq 0$. In this case, equation (2) is analogous to the stationary Choquard-Pekar equation

$$-\Delta u + Vu = \left( \frac{1}{|x|^2} + u^2 \right) u \quad \text{in } \mathbb{R}^3.$$

In some references, it is also known as the static Hartree or Schrödinger-Newton equation. Equation (5) was used by Pekar [39] to describe a polaron at rest in
quantum mechanics. Later, Lieb [31] used the equation to model an electron trapped in its own hole. Equation (5) was also proposed in [35] to characterize the particle moving in its own gravitational field. We stress that equation (2) is also related to the Thomas–Fermi–Von Weizacker model. In the last decades, more and more researchers have studied the existence and qualitative properties of solutions to elliptic problems of type (5) by variational methods under various assumptions of the potential $V$. Without any intention to cite all papers on the topic, we refer the reader to recent works [17, 18, 34, 36–38] and the references therein.

Recently, Dai and Liu [13] exploited the direct method of moving spheres, which was introduced by Chen-Li-Zhang in [10], to classify all nonnegative solutions of (2) in two different cases:

- $0 < \alpha \leq 2$, $\beta = n - 2\alpha$, $q_0 = 2$, $0 < q_1 \leq 1$, $0 < q_2 \leq \frac{n + \alpha}{n - \alpha}$.
- $0 < \alpha \leq 2$, $\beta = \alpha$, $q_0 = \frac{n + \alpha}{n - \alpha}$, $0 < q_1 \leq \frac{2\alpha}{n - \alpha}$, $0 < q_2 \leq \frac{n + \alpha}{n - \alpha}$.

They showed that every positive solution must assume the form (4) in the critical case and such a solution does not exist in the sub-critical cases. Another approach to equation (2) in the case $0 < \alpha < 2$ is the direct method of moving planes. Indeed, some classification results for equation (2) with $0 < \alpha < 2$ and $c_2 = 0$ was established in [11, 24] using the latter method. However, these two methods only work for the case $0 < \alpha \leq 2$. There is another approach - the integral equation method, which was initiated by Chen-Li-Ou in [9]. One advantage of the integral equation method is that it works indiscriminately for all real values $\alpha$ between 0 and $n$. For instance, equation (1) is closely related to the integral equation

$$u(x) = R_{n,\alpha} \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha}} \left[ c_1 \left( \frac{1}{|x|^{\alpha-n}} * f(u) \right)(y) + c_2 h(u(y)) \right] dy,$$  \hspace{1cm} (6)

where $R_{n,\alpha} = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}$ is the Riesz potential's constant and $\Gamma$ denotes the gamma function. When $0 < \alpha \leq 2$ and $f, g, h$ satisfy some conditions, one can show that every nonnegative classical solution of (1) also satisfies (6) and vice versa. Then the method of moving planes in integral forms [9] can be used to study the qualitative properties of solutions. Using the integral equation method, equation (2) with $\alpha = 2$ and $c_2 = 0$ was recently investigated in [27]. Some classification results for nonnegative $H^\alpha(\mathbb{R}^n)$ solutions of (2) with $0 < \alpha < n$ and $c_2 = 0$ were also established in [12, 22, 25, 26, 28, 33, 42] using the integral equation method.

To show that (1) is equivalent to (6) in the case $\alpha > 2$ for classical solutions, one may follow the ideas in [9, Theorem 4.3] by deriving the super polyharmonic property of solutions $u$ to (1), namely, $(-\Delta)^{2^{-t}}u \geq 0$ in $\mathbb{R}^n$ for every $t = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$, where $\lceil \cdot \rceil$ denotes the ceiling function, i.e., $\lceil t \rceil$ is the smallest integer which is not smaller than $t$. In recent papers, Dai-Liu-Qin derived such the super polyharmonic property for equation (2) with either $c_1 = 0$ (see [4]) or $c_2 = 0$ (see [14]). Then they classified all nonnegative solutions to the same equation.

Motivated by the above works, in this paper, we will prove the super polyharmonic property for equation (1). To establish such the property in the case that $\frac{\alpha}{2}$ is an integer greater than 1, we need to impose the following assumption on $f$ and $h$:

(FH) $f(t) \geq \varphi(t)$ and $h(t) \geq ct^q$ for all $t \geq 0$, where $c > 0$, $q > 1$ and $\varphi$ is a nonnegative convex function such that $\lim_{t \to +\infty} \varphi(t) > 0$.

Then we establish the equivalence between (1) and (6) in the class of nonnegative classical solutions. This enable us to use the method of moving spheres in integral
forms to classify nonnegative nontrivial solutions to equation (1) under some mild assumptions of \(\alpha, \beta, f, g, h\). Our first result reads as follows.

**Theorem 1.1.** Assume \(0 < \alpha, \beta < n\), \(c_1, c_2 \geq 0, c_1 + c_2 > 0\) and \(f, g, h \in C([0, +\infty), [0, +\infty))\) such that the following conditions hold

(i) \(f, g, h\) are increasing in \([0, +\infty)\),
(ii) \(f(t)/t^{n-\alpha}, g(t)/t^{n-\beta}, h(t)/t^{n-\beta}\) are nonincreasing in \((0, +\infty)\),
(iii) \(f, h\) satisfy (FH) if \(\frac{\alpha}{2}\) is an integer greater than 1.

Suppose that \(u\) is a nonnegative nontrivial solution of equation (1). Then \(u\) attains its maximum in \(\mathbb{R}^n\) and for some \(d_0, d_1, d_2 > 0\),

\[
f(t) = d_0 t^{\frac{n-\beta}{2}}, \quad g(t) = d_1 t^{\frac{n-\alpha}{2}} \quad \text{for all } t \in [0, \max u] \quad \text{if } c_1 > 0,
\]

\[
h(t) = d_2 t^{\frac{n-\alpha}{2}} \quad \text{for all } t \in [0, \max u] \quad \text{if } c_2 > 0.
\]

Moreover, \(u\) must have the form

\[
u(x) \equiv c_0 \left( \frac{\mu}{1 + \mu^2 |x - \bar{x}|^2} \right)^{\frac{n-\alpha}{2}} \quad \text{for some } \mu > 0 \text{ and } \bar{x} \in \mathbb{R}^n,
\]

where \(c_0 > 0\) satisfies

\[
R_{n,\alpha} \left[ \frac{2(\alpha + \beta)}{c_0} c_1 d_0 d_1 I \left( \frac{n-\beta}{2} \right) + c_0^{-2\alpha} c_2 d_2 \right] I \left( \frac{n-\alpha}{2} \right) = 1
\]

with \(I(s) = \frac{\pi^{\frac{n}{2}} \Gamma \left( \frac{n-s}{2} \right)}{\Gamma(n-s)}\) for \(0 < s < \frac{\alpha}{2}\).

**Remark 1.** Notice that the assumption (FH) is only required in the case that \(\frac{\alpha}{2}\) is an integer greater than 1. When \(c_1 = 0, c_2 = 1\) and \(0 < \alpha < 2\), Theorem 1.1 recovers [43, Theorem 1.1]. We remind that the technique used in [43] is the direct method of moving planes, which does not work in the case \(\alpha > 2\).

As a consequence of Theorem 1.1, we have the following classification result for equation (2).

**Corollary 1.** Let \(0 < \alpha, \beta < n, c_1, c_2 \geq 0\) be such that \(c_1 + c_2 > 0\). If \(\frac{\alpha}{2}\) is an integer greater than 1, we assume \(1 \leq q_0 \leq \frac{n+\beta}{n-\alpha}, 0 < q_1 \leq \frac{\alpha+\beta}{n-\alpha}\) and \(1 < q_2 \leq \frac{n+\alpha}{n-\alpha}\). Otherwise, we assume \(0 < q_0 \leq \frac{n+\beta}{n-\alpha}, 0 < q_1 \leq \frac{\alpha+\beta}{n-\alpha}\) and \(0 < q_2 \leq \frac{n+\alpha}{n-\alpha}\). Let \(u\) be a nonnegative solution of equation (2).

(i) If \(c_1 \left( \frac{n+\beta}{n-\alpha} - q_0 \right) + c_1 \left( \frac{\alpha+\beta}{n-\alpha} - q_1 \right) + c_2 \left( \frac{n+\alpha}{n-\alpha} - q_2 \right) > 0\), then \(u \equiv 0\).

(ii) If \(c_1 \left( \frac{n+\beta}{n-\alpha} - q_0 \right) + c_1 \left( \frac{\alpha+\beta}{n-\alpha} - q_1 \right) + c_2 \left( \frac{n+\alpha}{n-\alpha} - q_2 \right) = 0\), then \(u \equiv 0\) or \(u\) has the form as in Theorem 1.1 with \(d_0 = d_1 = d_2 = 1\).

**Remark 2.** Corollary 1 unifies and improves several results in the previous works.

- When \(\beta = n - 2\alpha, \alpha \in (0, 2]\) and \(q_0 = 2\), Corollary 1 becomes [13, Theorem 1.1].
- When \(\beta = \alpha \in (0, 2]\) and \(q_0 = \frac{n+\alpha}{n-\alpha}\), Corollary 1 reduces to [13, Theorem 1.3].
- When \(c_1 = 1, c_2 = 0\), Corollary 1 recovers [14, Corollary 1.5], which in turn extends and improves previous results in [3,11–13,15,24–27].

In our next result, we prove a Liouville type theorem for equation (1) when \(\alpha \geq n\).
Theorem 1.2. Assume $\alpha \geq n \geq 3$, $\beta > 0$, $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$ and $f, g, h \in C([0, +\infty), [0, +\infty))$ such that the following conditions hold

(i) $f \not\equiv 0$ and $g(t), h(t) > 0$ for $t > 0$,
(ii) $f, h$ satisfy (FH) if $\frac{\alpha}{2}$ is an integer.

Suppose that $u$ is a nonnegative solution of equation (1). Then $u \equiv 0$.

As an immediate consequence, we can state the following Liouville type result for equation (2).

Corollary 2. Let $\alpha \geq n \geq 3$, $\beta > 0$, $c_1, c_2 \geq 0$ be such that $c_1 + c_2 > 0$. If $\frac{\alpha}{2}$ is an integer, we assume $q_0 \geq 1$, $q_1 \geq 0$ and $q_2 > 1$. Otherwise, we assume $q_0, q_1, q_2 \geq 0$.

Suppose that $u$ is a nonnegative solution of equation (1), then $u \equiv 0$.

The rest of this paper is organized as follows. In Section 2, we derive the super polyharmonic property for nonnegative solutions of (1). In Section 3, we use the super polyharmonic property to establish the equivalence between PDE (1) and integral equation (6). In Section 4, we exploit the method of moving spheres in integral forms to prove Theorem 1.1, i.e., to classify all nonnegative solutions of (1) in the case $0 < \alpha < n$. The last section is devoted to the proof of Theorem 1.2, which is concerned with the nonexistence of nonnegative nontrivial solutions for equation (1) in the case $\alpha \geq n$.

Throughout the paper, we use $C$ to denote various positive constants whose values may vary from line to line. We denote by $B_R(x^0)$ the ball of radius $R > 0$ with center $x^0 \in \mathbb{R}^n$. For brevity, we also write $B_R = B_R(0)$.

2. Super polyharmonic property. To establish the equivalence of (1) and (6) when $\alpha > 2$, the following super polyharmonic property is the key ingredient.

Theorem 2.1. Assume $\alpha > 2$, $\beta > 0$, $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$ and $f, g, h \in C([0, +\infty), [0, +\infty))$. If $\frac{\alpha}{2}$ is an integer, we further assume that $f, h$ satisfy (FH).

Suppose that $u$ is a nonnegative solution of equation (1). Then, for every $i = 1, 2, \ldots, \left\lceil \frac{\alpha}{2} \right\rceil - 1$, we have

\[ (-\Delta)^{\frac{\alpha}{2} - i} u \geq 0 \quad \text{in } \mathbb{R}^n. \]

Proof. If $\frac{\alpha}{2}$ is not an integer, then the super polyharmonic property follows from [4, Theorem 1.1]. Moreover, if $\frac{\alpha}{2}$ is an integer and $c_2 > 0$, then the super polyharmonic property was established in [7, 41]. Therefore, it suffices to consider the case that $\frac{\alpha}{2} = m$, where $m$ is an integer, $c_1 > 0$ and $c_2 = 0$.

Setting $u_i = (-\Delta)^i u$ for $i = 1, 2, \ldots, m - 1$. We first show that $u_{m-1} \geq 0$ by contradiction arguments. Indeed, if this is not the case, then there exists $x^0 \in \mathbb{R}^n$ such that $u_{m-1}(x^0) < 0$. Let

\[ \overline{w}(r) = \frac{1}{|\partial B_r(x^0)|} \int_{\partial B_r(x^0)} w(x) \, d\sigma \]

be the spherical average of a function $w$ with respect to the center $x^0$. From the well-known property $\overline{\Delta w} = \Delta \overline{w}$, we have, for any $r \geq 0$,

\[
\begin{cases}
-\Delta \overline{\pi}(r) = \overline{\pi}_1(r), \\
-\Delta \overline{\pi}_1(r) = \overline{\pi}_2(r), \\
\quad \cdots \\
-\Delta \overline{\pi}_{m-1}(r) = c_1 \left( \frac{1}{|x|^{n-\alpha}} \ast f(u) \right) g(u)(r) \geq 0.
\end{cases}
\]
From the last equation in (7), one has
\[-r^{1-n}(r^{n-1}u_{m-1}'(r))' \geq 0 \quad \text{for any } r \geq 0.
\]
Integrating this twice yields
\[\pi_{m-1}(r) \leq \pi_{m-1}(0) = u_{m-1}(x^0) := -c_0 < 0 \quad \text{for any } r \geq 0.\]

Then from the \((m - 1)\)-th equation in (7), we derive
\[-r^{1-n}(r^{n-1}u_{m-2}'(r))' \leq -c_0 \quad \text{for any } r \geq 0.
\]
Again, by integrating both sides of this inequality twice, we arrive at
\[\pi_{m-2}(r) \geq \pi_{m-2}(0) + c_1 r^2 \quad \text{for any } r \geq 0,
\]
where \(c_1 = \frac{c_0}{2m} > 0\).

Continuing this way, we finally obtain that
\[(-1)^m \pi(r) \geq a_{m-1}r^{2(m-1)} + a_{m-2}r^{2(m-2)} + \cdots + a_0 \quad \text{for any } r \geq 0, \tag{8}\]
where \(a_{m-1} > 0\).

If \(m\) is odd, then (8) implies \(\pi(r) < 0\) for sufficiently large \(r\). This contradicts the nonnegativity of \(u\). Hence, \(m\) is even.

In view of (FH), we have \(\varphi(t) \geq c\) for all \(t \geq t_0\), where \(c, t_0 > 0\). From (8), there exists \(R_0 > 1\) sufficiently large such that \(\pi(r) \geq t_0\) for all \(r \geq R_0\). Hence \(\varphi(\pi(r)) \geq c\) for all \(r \geq R_0\).

As \(\varphi\) is convex, by Jensen’s inequality, we have
\[+\infty > \left(\frac{1}{|x|^{n-\beta}} * f(u)(x^0)\right)(x^0) = \int_{\mathbb{R}^n} \frac{f(u(y))}{|x^0 - y|^{n-\beta}} dy \geq \int_{\mathbb{R}^n} \frac{\varphi(u(y))}{|x^0 - y|^{n-\beta}} dy = \int_0^{+\infty} \int_{\partial B_r(x^0)} \frac{\varphi(u(y))}{|x^0 - y|^{n-\beta}} d\sigma dr \]
\[= C \int_0^{+\infty} \varphi(u)(r)r^{\beta-1} dr \geq C \int_0^{+\infty} \varphi(\pi)(r)r^{\beta-1} dr \geq C \int_{R_0}^{+\infty} r^{\beta-1} dr = +\infty,
\]
which is a contradiction. Therefore, we must have \(u_{m-1} \geq 0\). Then we can prove that \(u_{m-2} \geq 0\) though entirely similar procedure. Continuing this way, we obtain that \(u_i \geq 0\) for every \(i = 1, 2, \ldots, m - 1\). \(\square\)

3. Equivalence between the PDE and the integral equation. We recall the following maximum principle for \(\gamma\)-superharmonic functions and Liouville theorem for \(\gamma\)-harmonic functions, where \(0 < \gamma < 2\).

**Lemma 3.1** (Maximum principle [8, 40]). Let \(0 < \gamma < 2\) and \(\Omega\) be a bounded domain in \(\mathbb{R}^n\). Assume that \(w \in L_\gamma \cap C_{\text{loc}}^{1,1}(\Omega)\) and is lower semi-continuous on \(\overline{\Omega}\). If
\[
\begin{align*}
(-\Delta)^{\frac{\gamma}{2}} w(x) & \geq 0, \quad x \in \Omega, \\
w(x) & \geq 0, \quad x \in \mathbb{R}^n \setminus \Omega,
\end{align*}
\]
then \( w(x) \geq 0 \) for all \( x \in \mathbb{R}^n \). Moreover, if \( w = 0 \) at some point in \( \Omega \), then \( w(x) = 0 \) almost everywhere in \( \mathbb{R}^n \). These conclusions hold for unbounded region \( \Omega \) if we further assume that \( \liminf_{|x| \to \infty} w(x) \geq 0 \).

**Lemma 3.2** (Liouville theorem [44]). Assume \( 0 < \gamma < 2 \) and \( w \) is a classical solution of

\[
\begin{align*}
(-\Delta)^\frac{\gamma}{2} w(x) &= 0, & x &\in \mathbb{R}^n, \\
w(x) &\geq 0, & x &\in \mathbb{R}^n.
\end{align*}
\]

Then, \( u \equiv C \) for some constant \( C \geq 0 \).

Using some ideas in [9], we prove a general result concerning with the equivalence between a PDE and the corresponding integral equation under the super polyharmonic property of solutions.

**Lemma 3.3.** Assume \( 0 < \alpha < n \) and \( f \in C(\mathbb{R}^n) \) is a nonnegative function. Suppose that \( u \) is a nonnegative solution of the equation

\[
(-\Delta)^{\frac{\alpha}{2}} u = f \quad \text{in } \mathbb{R}^n
\]

such that \( (-\Delta)^{\frac{\alpha}{2}-i} u \geq 0 \) in \( \mathbb{R}^n \) for every \( i = 1, 2, \ldots, \lceil \frac{\alpha}{2} \rceil - 1 \). Then \( u \) is also a nonnegative solution of the integral equation

\[
u(x) = R_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy + C \quad \text{for all } x \in \mathbb{R}^n,
\]

and vice versa, where \( C \) denotes a nonnegative constant.

**Proof.** Suppose that \( u \) is a nonnegative solution of (9).

We denote \( m = \lceil \frac{\alpha}{2} \rceil - 1 \) and \( u_i = (-\Delta)^{\frac{\alpha}{2}-i} u \geq 0 \) for every \( i = 0, 1, 2, \ldots, m \).

If \( n = 2 \), then \( m = 0 \) and we can go directly to Case 2 below with \( \gamma = \alpha \). Hence, in deriving formula (11) below, we may assume \( n \geq 3 \).

Notice that \( u_1 \) is a nonnegative solution of the equation \(-\Delta u_1 = u_0 = f \) in \( \mathbb{R}^n \). For arbitrary \( R > 0 \), let

\[
u_{1,R}(x) = \int_{B_R} G_R^2(x,y) u_0(y) dy,
\]

where \( G_R^2 \) is the Green’s function for \(-\Delta \) on \( B_R \), which is given by

\[
G_R^2(x,y) = R_{n,2} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|R^{-1}x - |x||^{-n-2}} \right) \quad \text{if } x, y \in B_R
\]

and \( G_R^2(x,y) = 0 \) if \( x \) or \( y \in \mathbb{R}^n \setminus B_R \) (see [20, Chapter 2]). Then \( u_{1,R} \in C^2(\mathbb{R}^n) \) and satisfies

\[
\begin{align*}
-\Delta u_{1,R} &= u_0 & \text{in } B_R, \\
u_{1,R} &= 0 & \text{in } \mathbb{R}^n \setminus B_R.
\end{align*}
\]

Hence

\[
\begin{align*}
-\Delta (u_1 - u_{1,R}) &= 0 & \text{in } B_R, \\
u_1 - u_{1,R} &\geq 0 & \text{in } \mathbb{R}^n \setminus B_R.
\end{align*}
\]

By the maximum principle, we deduce that

\[
u_1 \geq u_{1,R}
\]

in \( \mathbb{R}^n \).
for any $R > 0$. For each fixed $x \in \mathbb{R}^n$, letting $R \to \infty$, we have

$$u_1(x) \geq u_{1,\infty}(x) := R_{n,2} \int_{\mathbb{R}^n} \frac{u_0(y)}{|x - y|^{n-2}} dy.$$ 

Notice that $u_{1,\infty} \in C^2(\mathbb{R}^n)$ and satisfies $-\Delta u_{1,\infty} = u_0$ in $\mathbb{R}^n$. Hence

$$\begin{cases} -\Delta(u_1 - u_{1,\infty}) = 0 & \text{in } \mathbb{R}^n, \\ u_1 - u_{1,\infty} \geq 0 & \text{in } \mathbb{R}^n. \end{cases}$$

From the Liouville theorem for nonnegative harmonic functions, we can deduce that $u_1 - u_{1,\infty} \equiv C_1 \geq 0$. That is,

$$u_1(x) = R_{n,2} \int_{\mathbb{R}^n} \frac{u_0(y)}{|x - y|^{n-2}} dy + C_1.$$ 

Similarly, using the fact that $u_i$ is a nonnegative solution of the equation $-\Delta u_i = u_{i-1}$ in $\mathbb{R}^n$ for $i = 1, 2, \ldots, m$, we deduce that

$$u_i(x) = R_{n,2} \int_{\mathbb{R}^n} \frac{u_{i-1}(y)}{|x - y|^{n-2}} dy + C_i \quad \text{for } i = 1, 2, \ldots, m, \quad (11)$$

where $C_i \geq 0$.

Setting $\gamma = \alpha - 2m$, then $\gamma \in (0, 2]$. We consider two cases.

**Case 1.** $\gamma = 2$.

In this case, $u$ is a nonnegative solution of the equation $-\Delta u = u_m$ in $\mathbb{R}^n$. Hence we can use the above reasoning to obtain

$$u(x) = R_{n,2} \int_{\mathbb{R}^n} \frac{u_m(y)}{|x - y|^{n-2}} dy + C,$$

where $C \geq 0$.

**Case 2.** $\gamma \in (0, 2)$.

In this case, $u$ is a nonnegative solution of the fractional equation

$$(-\Delta)^\gamma u = u_m \quad \text{in } \mathbb{R}^n.$$ 

For arbitrary $R > 0$, let

$$u_R(x) = \int_{B_R} G_R^\gamma(x, y) u_m(y) dy,$$

where $G_R^\gamma$ is the Green’s function for $(-\Delta)^\gamma$ on $B_R$, which is given by

$$G_R^\gamma(x, y) = \begin{cases} \frac{R_{n,\gamma}}{|x-y|^{n-\gamma}} \int_0^{b\frac{R_{n,\gamma}}{s}} \frac{b^{\frac{n-\gamma}{2}} - 1}{(1 + 1/b)^{\frac{n-\gamma}{2}}} db, & \text{if } x, y \in B_R, \\ 0, & \text{if } x \text{ or } y \in \mathbb{R}^n \setminus B_R, \end{cases}$$

where $s_R = \frac{|x-y|^2}{R^2}$ and $t_R = \left(1 - \frac{|x|^2}{R^2}\right) \left(1 - \frac{|y|^2}{R^2}\right)$ (see [23]). Then $u_R$ satisfies

$$\begin{cases} (-\Delta)^\gamma u_R = u_m & \text{in } B_R, \\ u_R = 0 & \text{in } \mathbb{R}^n \setminus B_R. \end{cases}$$

Hence

$$\begin{cases} (-\Delta)^\gamma (u - u_R) = 0 & \text{in } B_R, \\ u - u_R \geq 0 & \text{in } \mathbb{R}^n \setminus B_R. \end{cases}$$
By the maximum principle for $\gamma$-superharmonic functions (Lemma 3.1), we deduce that

$$u \geq u_R \quad \text{in } \mathbb{R}^n$$

for any $R > 0$. For each fixed $x \in \mathbb{R}^n$, letting $R \to \infty$, we have

$$u(x) \geq u_\infty(x) := R_{n,\gamma} \int_{\mathbb{R}^n} \frac{u_m(y)}{|x-y|^{n-\gamma}} dy,$$

Notice that $u_\infty$ satisfies $(-\Delta)^{\frac{\gamma}{2}} u_\infty = u_m$ in $\mathbb{R}^n$. Hence

$$\begin{cases} (-\Delta)^{\frac{\gamma}{2}} (u - u_\infty) = 0 \quad \text{in } \mathbb{R}^n, \\ u - u_\infty \geq 0 \quad \text{in } \mathbb{R}^n. \end{cases}$$

From the Liouville theorem for $\gamma$-harmonic functions (Lemma 3.2), we can deduce that $u - u_\infty \equiv C \geq 0$. That is,

$$u(x) = R_{n,\gamma} \int_{\mathbb{R}^n} \frac{u_m(y)}{|x-y|^{n-\gamma}} dy + C. \quad (12)$$

Hence, in both cases, we have formulae (11) (if $m > 0$) and (12). Moreover, we must have

$$C_i = 0 \quad \text{for } i = 1, 2, \ldots, m. \quad (13)$$

Indeed, if $C_i > 0$ for some $i \in \{1, 2, \ldots, m-1\}$, then

$$u_{i+1}(x) = R_{n,2} \int_{\mathbb{R}^n} \frac{u_{i+1}(y)}{|x-y|^{n-\gamma}} dy + C_{i+1} \geq R_{n,2}C_i \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} dy + C_{i+1} = +\infty,$$

which is absurd. Similarly, if $C_m > 0$, then

$$u(x) = R_{n,\gamma} \int_{\mathbb{R}^n} \frac{u_m(y)}{|x-y|^{n-\gamma}} dy + C \geq R_{n,\gamma}C_m \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\gamma}} dy + C = +\infty,$$

which is also a contradiction.

From (11), (12) and (13), we deduce

$$u(x) = \int_{\mathbb{R}^n} \frac{R_{n,\gamma}}{|x-y|^{m+1}|x-y|^{n-\gamma}} \int_{\mathbb{R}^n} \frac{R_{n,2}}{|y^{m+1}-y^{m}|^{\alpha_1+\alpha_2}} \ldots \int_{\mathbb{R}^n} \frac{R_{n,2}}{|y^{2+1}-y^{2}|^{\alpha_2}} dy \int_{\mathbb{R}^n} f(y_1) \cdots dy m^{m+1} dy + C,$$

where in the last equality, we consecutively used the well-known Selberg formula

$$\int_{\mathbb{R}^n} \frac{R_{n,\alpha_1}}{|x-y|^{\alpha_1}} \frac{R_{n,\alpha_2}}{|y-z|^{\alpha_2}} dy = \frac{R_{n,\alpha_1+\alpha_2}}{|x-z|^{\alpha_1+\alpha_2}}$$

for any $\alpha_1, \alpha_2 \in (0, n)$ such that $\alpha_1 + \alpha_2 \in (0, n)$, see [21]. That means that $u$ satisfies (10).

Conversely, assume that $u$ is a nonnegative solution of (10). Then we have

$$(-\Delta)^{\frac{\gamma}{2}} u(x) = \int_{\mathbb{R}^n} (-\Delta)^{\frac{\gamma}{2}} \left( \frac{R_{n,\alpha}}{|x-y|^{n-\alpha}} \right) f(y) dy = \int_{\mathbb{R}^n} \delta(x,y) f(y) dy = f(x)$$

for all $x \in \mathbb{R}^n$, where $\delta$ is the Dirac distribution. That is, $u$ is a nonnegative solution of (9).

The main goal of this section is the following.
**Theorem 3.4.** Assume $0 < \alpha < n$, $\beta > 0$, $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$ and $f, g, h \in C([0, +\infty), [0, +\infty])$ such that $\inf_{s \in [0, +\infty)} f > 0$ and $\inf_{s \in [0, +\infty)} h > 0$ for all $s > 0$. If $\frac{\alpha}{2}$ is an integer greater than 1, we further assume that $f, h$ satisfy (FH). Suppose that $u$ is a nonnegative solution of equation (1), then $u$ is also a nonnegative solution of integral equation (6), and vice versa.

**Proof.** Suppose that $u$ is a nonnegative solution of equation (1). By Theorem 2.1, we have $(-\Delta)^{\frac{\alpha}{2}} u \geq 0$ in $\mathbb{R}^n$ for every $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$. Therefore, we can apply Lemma 3.3 with $f = c_1 \left(\frac{1}{|x|^{n-\alpha}} * f(u)\right) g(u) + c_2 h(u)$ to deduce

$$u(x) = R_{n, \alpha} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} \left[ c_1 \left(\frac{1}{|x|^{n-\beta}} * f(u)\right) (y) g(u(y)) + c_2 h(u(y)) \right] dy + C,$$

for some $C \geq 0$. This implies $u(x) \geq C$.

If $C > 0$ and $c_1 > 0$, then

$$\left(\frac{1}{|x|^{n-\beta}} * f(u)\right)(x) = \int_{\mathbb{R}^n} \frac{f(u(y))}{|x-y|^{n-\beta}} dy \geq \inf_{|C, +\infty)} f \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\beta}} dy = +\infty,$$

which is absurd.

Otherwise, if $C > 0$ and $c_2 > 0$, then

$$u(x) \geq R_{n, \alpha} c_2 \left(\inf_{|C, +\infty)} h\right) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} dy + C = +\infty,$$

which is another contradiction.

Hence $C = 0$ and $u$ is a solution of (6). The reverse is also true due to Lemma 3.3. \hfill \Box

4. **Classification of nonnegative solutions for sub-critical order equations.** Throughout this section, let $u$ be a nonnegative nontrivial solution of equation (1) with $0 < \alpha, \beta < n$. By Theorem 3.4, $u$ satisfies (6) and hence $u$ is positive. Moreover, by setting $v = \frac{1}{|x|^{n-\beta}} * f(u)$, we observe that $(u, v)$ is a positive continuous solution of the integral system

$$
\begin{aligned}
u(x) &= R_{n, \alpha} \int_{\mathbb{R}^n} \frac{c_1 g(u(y)) v(y) + c_2 h(u(y))}{|x-y|^{n-\alpha}} dy, \\
v(x) &= \int_{\mathbb{R}^n} \frac{f(u(y))}{|x-y|^{n-\beta}} dy.
\end{aligned}
$$

We define $F(t) := f(t)/t^{\frac{n+\beta}{n-\alpha}}$, $G(t) := g(t)/t^{\frac{n+\beta}{n-\alpha}}$ and $H(t) := h(t)/t^{\frac{n+\alpha}{n-\alpha}}$. Then $F, G, H$ are nonincreasing in $(0, +\infty)$ and system (14) can be rewritten as

$$
\begin{aligned}
u(x) &= R_{n, \alpha} \int_{\mathbb{R}^n} \frac{c_1 G(u(y)) u(y)^{\frac{n+\beta}{n-\alpha}} v(y) + c_2 H(u(y)) u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}} dy, \\
v(x) &= \int_{\mathbb{R}^n} \frac{F(u(y)) u(y)^{\frac{n+\beta}{n-\alpha}}}{|x-y|^{n-\beta}} dy.
\end{aligned}
$$

We will exploit the method of moving spheres in integral forms to prove Theorem 1.1. To this end, for any $x^0 \in \mathbb{R}^n$ and $\lambda > 0$, we denote by

$$x^{x^0, \lambda} = \frac{\lambda^2 (x-x^0)}{|x-x^0|^2} + x^0.$$
the inversion of \( x \in \mathbb{R}^n \setminus \{x^0\} \) about the sphere \( \partial B_\lambda(x^0) \). We then define the Kelvin transform of \( u \) and \( v \) with respect to \( \partial B_\lambda(x^0) \) by

\[
u_{x^0,\lambda}(x) = \left( \frac{\lambda}{|x - x^0|} \right)^{n-\alpha} u(x^{0,\lambda}) \quad \text{and} \quad v_{x^0,\lambda}(x) = \left( \frac{\lambda}{|x - x^0|} \right)^{n-\beta} v(x^{0,\lambda}).
\]

We also define

\[
U_{x^0,\lambda}(x) = u_{x^0,\lambda}(x) - u(x) \quad \text{and} \quad V_{x^0,\lambda}(x) = v_{x^0,\lambda}(x) - v(x).
\]

**Proposition 1.** For any \( x^0 \in \mathbb{R}^n \), the set

\[
\Gamma_{x^0} = \{ \lambda > 0 \mid U_{x^0,\mu} \geq 0 \text{ in } B_\mu(x^0) \setminus \{x^0\} \text{ for all } \mu \in (0, \lambda]\}
\]

is not empty. Moreover, if \( \lambda_{x^0} := \sup \Gamma_{x^0} < \infty \), then \( U_{x^0,\lambda_{x^0}} = 0 \) in \( B_{\lambda_{x^0}}(x^0) \setminus \{x^0\} \).

Since equation (1) is invariant by translations, it suffices to prove Proposition 1 for \( x^0 = 0 \). For brevity, we will drop the script \( x^0 \) in notations when \( x^0 = 0 \). That is, we will write

\[
x^\lambda = \frac{\lambda^2 x}{|x|^2}, \quad u_\lambda(x) = \left( \frac{\lambda}{|x|} \right)^{n-\alpha} u(x^\lambda), \quad v_\lambda(x) = \left( \frac{\lambda}{|x|} \right)^{n-\beta} v(x^\lambda),
\]

\[
U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x).
\]

The Kelvin transform of \( u \) and \( v \) with respect to \( \partial B_\lambda \) satisfy

\[
\begin{aligned}
u_\lambda(x) &= R_{n,\alpha} \int_{\mathbb{R}^n} \frac{c_1 G (u(y^\lambda)) u_\lambda(y)^{\frac{n-\alpha}{n}} v_\lambda(y) + c_2 H (u(y^\lambda)) u_\lambda(y)^{\frac{n+\alpha}{n}}}{|x - y|^n} dy, \\
v_\lambda(x) &= \int_{\mathbb{R}^n} \frac{F (u(y^\lambda)) u_\lambda(y)^{\frac{n-\beta}{n}} d\lambda}{|x - y|^n} dy,
\end{aligned}
\]

for all \( x \in \mathbb{R}^n \setminus \{0\} \).

Indeed, for any \( x \in \mathbb{R}^n \setminus \{0\} \), using the first equation in (15), we have

\[
u_\lambda(x) = R_{n,\alpha} \left( \frac{\lambda}{|x|} \right)^{n-\alpha} \int_{\mathbb{R}^n} \frac{c_1 G (u(y)) u(y)^{\frac{n+\alpha}{n}} v(y) + c_2 H (u(y)) u(y)^{\frac{n+\alpha}{n}}}{|x^\lambda - y|^n} dy
\]

\[
= R_{n,\alpha} \left( \frac{\lambda}{|x|} \right)^{n-\alpha} \int_{\mathbb{R}^n} \frac{c_1 G (u(y^\lambda)) u_\lambda(y)^{\frac{n+\alpha}{n}} v_\lambda(y) + c_2 H (u(y^\lambda)) u_\lambda(y)^{\frac{n+\alpha}{n}}}{|x^\lambda - y^\lambda|^n} d\lambda
\]

\[
= R_{n,\alpha} \int_{\mathbb{R}^n} \frac{c_1 G (u(y^\lambda)) u_\lambda(y)^{\frac{n+\alpha}{n}} v_\lambda(y) + c_2 H (u(y^\lambda)) u_\lambda(y)^{\frac{n+\alpha}{n}}}{|x - y|^n} dy,
\]

where we have used the following identities in the last line

\[
d\lambda^\lambda = \left( \frac{\lambda}{|y|} \right)^{2n} dy \quad \text{and} \quad |x^\lambda - y^\lambda| = \frac{\lambda^2 |x - y|}{|x| |y|} \quad \text{for all } x, y \in \mathbb{R}^n \setminus \{0\}.
\]

Similarly, from the second equation in (15), we derive

\[
v_\lambda(x) = \left( \frac{\lambda}{|x|} \right)^{n-\beta} \int_{\mathbb{R}^n} \frac{F (u(y^\lambda)) u(y^\lambda)^{\frac{n+\beta}{n}}}{|x^\lambda - y^\lambda|^n} d\lambda
\]

\[
= \left( \frac{\lambda}{|x|} \right)^{n-\beta} \int_{\mathbb{R}^n} \frac{F (u(y^\lambda)) u_\lambda(y)^{\frac{n+\beta}{n}}}{|x - y|^n} d\lambda.
\]
This proves (16). Next, we denote for each \( \lambda > 0 \),
\[
B^\lambda = \{ x \in B_\lambda \setminus \{0\} \mid U_\lambda(x) < 0 \}, \quad B^\lambda_\lambda = \{ x \in B_\lambda \setminus \{0\} \mid V_\lambda(x) < 0 \}.
\]
We prove a key inequality which will be used in the method of moving spheres.

**Lemma 4.1.** Let \( \overline{\lambda} > 0 \). For any \( 0 < \lambda < \overline{\lambda} \), we have
\[
\| U_\lambda \|_{L^{\frac{2n+n}{n-\alpha}}(B^\lambda_\lambda)} \leq C \left( c_1 \left\| u \right\|_{L^{\frac{2(n+n+1)}{2n+n}}(B^\lambda_\lambda)} + c_1 \left\| u \right\|^{\frac{2n+n}{n-\alpha}}_{L^{\frac{2n+n}{n-\alpha}}(B^\lambda_\lambda)} + c_2 \left\| u \right\|_{L^{\frac{2n+n}{n-\alpha}}(B^\lambda_\lambda)} \right) \| U_\lambda \|_{L^{\frac{2n+n}{n-\alpha}}(B^\lambda_\lambda)},
\]
where \( C > 0 \) depends on \( \overline{\lambda} \) but is independent of \( \lambda \).

**Proof.** Let \( x \in B_\lambda \setminus \{0\} \). From the first equation in (15), we have
\[
u(x) = R_{n,\alpha} \int_{B_\lambda} c_1 G(u(y)) u(y) \frac{\alpha+\beta}{n-\alpha} v(y) + c_2 H(u(y)) u(y) \frac{n+\alpha}{n-\alpha} dy
+ R_{n,\alpha} \int_{B_\lambda} c_1 G(u(y)) u(\lambda) \frac{\alpha+\beta}{n-\alpha} v(y) + c_2 H(u(y)) u(\lambda) \frac{n+\alpha}{n-\alpha} dy.
\]
From the first equation in (16), we obtain
\[
u_\lambda(x) = R_{n,\alpha} \int_{B_\lambda} c_1 G(u(y^\lambda)) u(\lambda) \frac{\alpha+\beta}{n-\alpha} v_\lambda(y) + c_2 H(u(y^\lambda)) u(\lambda) \frac{n+\alpha}{n-\alpha} dy
+ R_{n,\alpha} \int_{B_\lambda} c_1 G(u(y)) u(y^\lambda) \frac{\alpha+\beta}{n-\alpha} v(y) + c_2 H(u(y)) u(y^\lambda) \frac{n+\alpha}{n-\alpha} dy.
\]
Combining the above two formulae, we derive
\[
u_\lambda(x) = R_{n,\alpha} \int_{B_\lambda} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x-y^\lambda|^{n-\alpha}} \right) [c_1 r_1(y) + c_2 r_2(y)] dy, \quad (17)
\]
where
\[
r_1(y) = G(u(y^\lambda)) u(\lambda) \frac{\alpha+\beta}{n-\alpha} v_\lambda(y) - G(u(y)) u(\lambda) \frac{n+\alpha}{n-\alpha} v(y),
\]
\[
r_2(y) = H(u(y^\lambda)) u(\lambda) \frac{n+\alpha}{n-\alpha} - H(u(y)) u(\lambda) \frac{n+\alpha}{n-\alpha}.
\]
Notice that
\[
\frac{|y|}{\lambda} \left| x - \frac{y}{|y|} \right|^2 - |x - y|^2 = \frac{(|x|^2 - \lambda^2)(|y|^2 - \lambda^2)}{\lambda^2} > 0 \quad \text{for} \ x, y \in B_\lambda \setminus \{0\}. \quad (18)
\]
For any \( 0 < a \leq b \) and \( q \geq 0 \), we have the following inequality
\[
a^q - b^q \geq \max\{q, 1\} b^{q-1} (a - b). \quad (19)
\]
Indeed, (19) can be proved by using the mean value theorem for \( q > 1 \) and direct computation for \( 0 \leq q \leq 1 \). We also observe that for \( y \in B_\lambda \setminus \{0\} \), we have
\[
u_\lambda(y) > u(y^\lambda) \quad \text{and} \quad v_\lambda(y) > v(y^\lambda). \quad (20)
\]
Using (19) and (20), we estimate \( r_1(y) \) for each \( y \in B_\lambda \setminus \{0\} \) by considering four possible cases:
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If \( u_\lambda(y) \geq u(y) \) and \( v_\lambda(y) \geq v(y) \), then

\[
\begin{align*}
  r_1(y) & \geq G(u_\lambda(y)) \left[ u_\lambda(y) \frac{\alpha + \beta}{n - \alpha} - u(y) \frac{\alpha + \beta}{n - \alpha} \right] v(y) \\
  & = g(u_\lambda(y)) v_\lambda(y) - g(u(y)) v(y) \\
  & \geq 0.
\end{align*}
\]

If \( u_\lambda(y) < u(y) \) and \( v_\lambda(y) \geq v(y) \), then

\[
\begin{align*}
  r_1(y) & \geq G(u_\lambda(y)) \left[ u_\lambda(y) \frac{\alpha + \beta}{n - \alpha} - u(y) \frac{\alpha + \beta}{n - \alpha} \right] v(y) \\
  & \geq G(u(y)) \left[ u_\lambda(y) \frac{\alpha + \beta}{n - \alpha} - u(y) \frac{\alpha + \beta}{n - \alpha} \right] v(y) \\
  & \geq \max \left\{ \frac{\alpha + \beta}{n - \alpha}, 1 \right\} G(u(y)) u(y) \frac{2n + \beta - n}{n - \alpha} v(y) U_\lambda(y) \\
  & \geq C_u(y) \frac{2n + \beta - n}{n - \alpha} v(y) U_\lambda(y).
\end{align*}
\]

If \( u_\lambda(y) \geq u(y) \) and \( v_\lambda(y) < v(y) \), then

\[
\begin{align*}
  r_1(y) & = \left[ G(u(y)) \right] u_\lambda(y) \frac{\alpha + \beta}{n - \alpha} - G(u(y)) u(y) \frac{\alpha + \beta}{n - \alpha} \right] v(y) + G(u(y)) u(y) \frac{\alpha + \beta}{n - \alpha} V_\lambda(y) \\
  & \geq \left[ G(u_\lambda(y)) u_\lambda(y) \frac{\alpha + \beta}{n - \alpha} - G(u(y)) u(y) \frac{\alpha + \beta}{n - \alpha} \right] v_\lambda(y) + G(u(y)) u(y) \frac{\alpha + \beta}{n - \alpha} V_\lambda(y) \\
  & = [g(u_\lambda(y)) - g(u(y))] v_\lambda(y) + G(u(y)) u(y) \frac{\alpha + \beta}{n - \alpha} V_\lambda(y) \\
  & \geq C_u(y) \frac{2n + \beta - n}{n - \alpha} v(y) U_\lambda(y) + C_v(y) \frac{\alpha + \beta}{n - \alpha} V_\lambda(y).
\end{align*}
\]

Therefore, for any \( y \in \mathcal{B}_\lambda \setminus \{0\} \), we have

\[
  r_1(y) \geq C_u(y) \frac{2n + \beta - n}{n - \alpha} v(y) U_\lambda(y)^{-1} + C_v(y) \frac{\alpha + \beta}{n - \alpha} V_\lambda(y)^{-1}.
\]

To estimate \( r_2 \) for \( y \in \mathcal{B}_\lambda \setminus \{0\} \), we consider two cases:

- If \( u_\lambda(y) \geq u(y) \), then

  \[
  r_2(y) \geq H(u_\lambda(y)) u_\lambda(y) \frac{n + \alpha}{n - \alpha} - H(u(y)) u(y) \frac{n + \alpha}{n - \alpha} \\
  = h(u_\lambda(y)) - h(u(y)) \\
  \geq 0.
  \]

- If \( u_\lambda(y) < u(y) \), then

  \[
  r_2(y) \geq H(u(y)) \left[ u_\lambda(y) \frac{n + \alpha}{n - \alpha} - u(y) \frac{n + \alpha}{n - \alpha} \right] \\
  \geq \frac{n + \alpha}{n - \alpha} H(u(y)) \frac{n + \alpha}{n - \alpha} U_\lambda(y) \\
  \geq C_u(y) \frac{n + \alpha}{n - \alpha} U_\lambda(y).
  \]
That is, for any \( y \in B_\lambda \setminus \{0\} \),
\[
 r_2(y) \geq C_\lambda u(y)^{\frac{2\alpha}{n}} U_\lambda(y).
\]  
(24)

From (17), (18), (22) and (24), we deduce
\[
 C_\lambda U_\lambda(x) \geq c_1 \int_{B_\lambda} \frac{u(y)^{\frac{a+b}{n}} V_\lambda(y)^{-}}{|x-y|^{n-\alpha}} dy + c_1 \int_{B_\lambda} \frac{u(y)^{\frac{2\alpha+b-n}{n}} v(y) U_\lambda(y)^{-}}{|x-y|^{n-\alpha}} dy
+ c_2 \int_{B_\lambda} \frac{u(y)^{\frac{2\alpha}{n}} U_\lambda(y)^{-}}{|x-y|^{n-\alpha}} dy.
\]

Now we apply the Hardy-Littlewood-Sobolev and Hölder’s inequalities to get
\[
 C_\lambda \|U_\lambda\|_{L^{\frac{2n}{n-\alpha}}(B_\lambda^n)}
\leq c_1 \left\| u^{\frac{a+b}{n}} V_\lambda \right\|_{L^{\frac{2n}{n-\alpha}}(B_\lambda^n \cap B_\lambda^n)} + c_1 \left\| u^{\frac{2\alpha+b-n}{n}} v U_\lambda \right\|_{L^{\frac{2n}{n-\alpha}}(B_\lambda^n)}
+ c_2 \left\| u^{\frac{2\alpha}{n}} U_\lambda \right\|_{L^{\frac{2n}{n-\alpha}}(B_\lambda^n)}
\leq c_1 \left\| u \right\|_{L^{\frac{a+b}{n-\alpha}}(B_\lambda^n)} \left\| V_\lambda \right\|_{L^{\frac{2n}{n-\alpha}}(B_\lambda^n)} + c_1 \left\| u \right\|_{L^{\frac{2\alpha+b-n}{n}}(B_\lambda^n)} \left\| U_\lambda \right\|_{L^{\frac{2n}{n-\alpha}}(B_\lambda^n)}.
\]  
(25)

In a similar way, from the second equation in (15), we have
\[
v(x) = \int_{B_\lambda} \frac{F(u(y)) u(y)^{\frac{a+b}{n}}}{|x-y|^{n-\beta}} dy + \int_{B_\lambda} \frac{F(u(y)) u_\lambda(y)^{\frac{a+b}{n}}}{|x-y|^{n-\beta}} dy.
\]

From the second equation in (16), we obtain
\[
v_\lambda(x) = \int_{B_\lambda} \frac{F(u(y)) u_\lambda(y)^{\frac{a+b}{n}}}{|x-y|^{n-\beta}} dy + \int_{B_\lambda} \frac{F(u(y)) u(y)^{\frac{a+b}{n}}}{|x-y|^{n-\beta}} dy.
\]

Combining the above two formulae, we derive
\[
 V_\lambda(x) = \int_{B_\lambda} \left( \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x-y|^{n-\beta}} \right) r_3(y) dy,
\]  
(26)

where
\[
r_3(y) = F(u(y)) u_\lambda(y)^{\frac{a+b}{n}} - F(u(y)) u(y)^{\frac{a+b}{n}}.
\]

Similar to \( r_2 \), we consider two cases:

- If \( u_\lambda(y) \geq u(y) \), then
\[
r_3(y) \geq F(u_\lambda(y)) u_\lambda(y)^{\frac{a+b}{n}} - F(u(y)) u(y)^{\frac{a+b}{n}}
= f(u_\lambda(y)) - f(u(y))
\geq 0.
\]  
(27)
• If $u_\lambda(y) < u(y)$, then
  \[
  r_3(y) \geq F(u(y)) \left[ u_\lambda(y) \frac{\alpha+\beta}{\alpha} - u(y) \frac{\alpha+\beta}{\alpha} \right] \\
  \geq \frac{n+\beta}{n-\alpha} F(u(y)) u(y) \frac{\alpha+\beta}{\alpha} U_\lambda(y) \\
  \geq C \bar{\lambda} u(y) \frac{\alpha+\beta}{\alpha} U_\lambda(y).
  \]

  Hence for all $y \in B_\lambda \setminus \{0\}$,
  \[
  r_3(y) \geq C \bar{\lambda} u(y) \frac{\alpha+\beta}{\alpha} U_\lambda(y)^-.
  \]

  From (26) and (28), we have
  \[
  \mathcal{C} \bar{\lambda} \nabla(x) \geq \int_{B_\lambda} \frac{u(y) \frac{\alpha+\beta}{\alpha} U_\lambda(y)^-}{|x-y|^{n-\beta}} \, dy.
  \]

  Applying the Hardy-Littlewood-Sobolev and Hölder’s inequalities, we obtain
  \[
  \mathcal{C} \bar{\lambda} \|V_\lambda\|_{L^\frac{2n-\alpha}{\alpha} (B_\lambda)} \leq \left\| \frac{\alpha+\beta}{\alpha} U_\lambda \right\|_{L^\frac{2n-\alpha}{\alpha} (B_\lambda)} \left\| u \right\|_{L^{\frac{2n}{\alpha}} (B_\lambda)} \left\| U_\lambda \right\|_{L^\frac{2n}{\alpha} (B_\lambda)}.
  \]

  The conclusion follows from (25) and (29).

  \[\Box\]

  \textit{Proof of Proposition 1.} As mentioned before, we only need to prove the proposition for $x^0 = 0$.

  \textbf{Step 1.} (Start dilating the sphere $\partial B_\lambda$ from near $\lambda = 0$)

  In this step, we prove $\Gamma_0 \neq \emptyset$, i.e., for $\lambda > 0$ sufficiently small,
  \[
  U_\lambda \geq 0 \quad \text{in } B_\lambda \setminus \{0\}.
  \]

  Indeed, since $u$ and $v$ are continuous and positive, there exists $\varepsilon_0 \in (0, 1)$ small enough, such that
  \[
  c_1 \left\| u \right\|_{L^{\frac{2(n+\beta)}{\beta}} (B_\lambda^\varepsilon)} + c_1 \left\| u \frac{2\alpha+\beta-n}{\alpha} v \right\|_{L^{\frac{2n}{\alpha}} (B_\lambda^\varepsilon)} + c_2 \left\| u \right\|_{L^{\frac{2n}{\alpha}} (B_\lambda^\varepsilon)} \leq \frac{1}{2C}
  \]

  for all $0 < \lambda < \varepsilon_0$, where the constant $C$ is the same as in Lemma 4.1 with $\bar{\lambda} = 1$.

  Hence, Lemma 4.1 indicates $\|U\|_{L^{\frac{2n}{\alpha}} (B_\lambda^\varepsilon)} = 0$, which means $|B_\lambda^\varepsilon| = 0$. Therefore, (30) holds for all $\lambda < \varepsilon_0$. This completes Step 1.

  \textbf{Step 2.} (Dilate the sphere $\partial B_\lambda$ outward to the critical scale $\lambda_0$)

  Step 1 provides us a starting point to dilate the sphere $\partial B_\lambda$ from near $\lambda = 0$.

  Now we dilate the sphere $\partial B_\lambda$ outward as long as (30) holds. Let
  \[
  \lambda_0 = \sup \{\lambda > 0 \mid U_{\mu} \geq 0 \text{ in } B_{\mu} \setminus \{0\} \text{ for all } \mu \in (0, \lambda]\}.
  \]

  In this step, we show that
  \[
  \text{if } \lambda_0 < \infty, \text{ then } U_{\lambda_0} = 0 \text{ in } B_{\lambda_0} \setminus \{0\}.
  \]

  By contradiction, in what follows, we assume $\lambda_0 < \infty$ and $U_{\lambda_0} \neq 0$ in $B_{\lambda_0} \setminus \{0\}$.

  Since $U_\lambda$ is continuous with respect to $\lambda$, we already have $U_{\lambda_0} \geq 0$ in $B_{\lambda_0} \setminus \{0\}$.

  From (26) and (27), we have
  \[
  V_{\lambda_0}(x) \geq \int_{B_{\lambda_0}} \left( \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|y|^{\lambda_0}} \right) [f(u_{\lambda_0}(y)) - f(u(y))] \, dy > 0
  \]
for all $x \in B_{\lambda_0} \setminus \{0\}$ due to the strict monotonicity of $f$. Then (17), (21) and (23) imply

$$U_{\lambda_0}(x) \geq R_{n,\alpha} \int_{B_{\lambda_0}} \left( \frac{1}{|x - y|^{\alpha - \alpha}} - \frac{1}{|\lambda y|^{\alpha - \alpha}} \right) \times \{c_1 [g(u_{\lambda_0}(y))v_{\lambda_0}(y) - g(u(y))v(y)] + c_2 [h(u_{\lambda_0}(y)) - h(u(y))]\} \, dy > 0$$

(32)

for all $x \in B_{\lambda_0} \setminus \{0\}$.

Next, we claim that there exist $C > 0$ and $\eta > 0$ such that

$$U_{\lambda_0} \geq C \quad \text{in } B_{\eta} \setminus \{0\}. \quad (33)$$

Indeed, from (32) and Fatou’s lemma, we have

$$\liminf_{x \to 0} U_{\lambda_0}(x) \geq C \int_{B_{\lambda_0}} \left( \frac{1}{|y|^{\alpha - \alpha}} - \frac{1}{\lambda_0|y|^{\alpha - \alpha}} \right) \times \{c_1 [g(u_{\lambda_0}(y))v_{\lambda_0}(y) - g(u(y))v(y)] + c_2 [h(u_{\lambda_0}(y)) - h(u(y))]\} \, dy > 0.$$ 

Hence for $x \in B_{\eta} \setminus \{0\}$, where $\eta$ is sufficiently small, we have $U_{\lambda_0}(x) \geq C$. This proves (33).

Now we fix $0 < r_0 < \frac{\lambda_0}{\lambda}$ small enough, such that

$$c_1 \|u\|_{L^{\alpha} B_{\lambda_0} \setminus B_{\lambda_0} \setminus \{0\}}^{2(\alpha + \beta)} + c_1 \|v\|_{L^{\alpha} B_{\lambda_0} \setminus B_{\lambda_0} \setminus \{0\}}^{2(\alpha + \beta - \alpha)} \leq \frac{1}{2C},$$

(34)

where the constant $C$ is the same as in Lemma 4.1 with $\lambda = 2\lambda_0$.

It follows from (32), (33) and the continuity of $U_{\lambda_0}$ that, there exists a constant $C > 0$ such that

$$U_{\lambda_0} \geq C \quad \text{in } B_{\lambda_0} \setminus \{0\}.$$ 

Since $u$ is uniformly continuous on an arbitrary compact set, there exists $\rho_0 \in (0, r_0)$ such that, for any $\lambda \in (\lambda_0, \lambda_0 + \rho_0),

$$U_{\lambda} \geq \frac{C}{2} > 0 \quad \text{in } B_{\lambda_0} \setminus \{0\}.$$ 

Therefore, for any $\lambda \in (\lambda_0, \lambda_0 + \rho_0),

$$B^u_{\lambda} \subseteq B_{\lambda_0 + r_0} \setminus B_{\lambda_0}.$$ 

Hence, Lemma 4.1 and (34) yield $\|U_{\lambda}\|_{L^{2(\alpha + \beta)}(B^u_{\lambda})} = 0$, which means $|B^u_{\lambda}| = 0$. Thus, for any $\lambda \in (\lambda_0, \lambda_0 + \rho_0),

$$U_{\lambda} \geq 0 \quad \text{in } B_{\lambda} \setminus \{0\}.$$ 

However, this contradicts the definition of $\lambda_0$. Therefore, (31) is proved.

This completes the proof of Proposition 1. \hfill \Box

**Proposition 2.** One of the following two assertions holds:

(i) $\lambda_{x^0} < \infty$ for all $x^0 \in \mathbb{R}^n$,

(ii) $\lambda_{x^0} = \infty$ for all $x^0 \in \mathbb{R}^n$. 

Proof of Proposition 2. Assume that there exist \( x^0, y^0 \in \mathbb{R}^n \) such that \( \lambda_{x^0} = \infty \) and \( \lambda_{y^0} < \infty \).

Since \( \lambda_{x^0} = \infty \), we have, for any \( \lambda > 0 \),

\[
U_{x^0, \lambda} \geq 0 \quad \text{for all } x \in B_\lambda(x^0) \setminus \{x^0\}.
\]

This implies that, for any \( \lambda > 0 \),

\[
u(x) \geq u_{x^0, \lambda}(x) \quad \text{for all } x \in \mathbb{R}^n \setminus B_\lambda(x^0).
\]

That is,

\[
|x - x^0|^{-\alpha} \nu(x) \geq \lambda^{n-\alpha} u(x^{x^0, \lambda}) \quad \text{for all } x \in \mathbb{R}^n \setminus B_\lambda(x^0).
\]

Hence

\[
\liminf_{|x| \to \infty} |x|^{-\alpha} \nu(x) \geq \lambda^{n-\alpha} u(x^0).
\]

Due to the arbitrariness of \( \lambda > 0 \), we must have

\[
\lim_{|x| \to \infty} |x|^{-\alpha} \nu(x) = \infty.
\]

On the other hand, since \( \lambda_{y^0} < \infty \), we may use Proposition 1 to get

\[
u_{y^0, \lambda_{y^0}}(x) = \nu(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \{y^0\}.
\]

This indicates that

\[
\lim_{|x| \to \infty} |x|^{-\alpha} \nu(x) = \lambda_{y^0}^{-\alpha} u(y^0) < \infty.
\]

This contradiction concludes the proof of Proposition 2. \( \Box \)

To classify all nonnegative solutions, we need the following calculus lemmas.

Lemma 4.2 (see [29, 30]). Let \( n \geq 1 \), \( \nu \in \mathbb{R} \) and \( w \in C^1(\mathbb{R}^n) \). For every \( x^0 \in \mathbb{R}^n \) and \( \lambda > 0 \), we define

\[
w_{x^0, \lambda}(x) = \left( \frac{\lambda}{|x - x^0|} \right)^\nu w(x^{x^0, \lambda})
\]

for all \( x \in \mathbb{R}^n \setminus \{x^0\} \). Then we have

(i) If for every \( x^0 \in \mathbb{R}^n \), there exists \( \lambda_{x^0} < \infty \) such that

\[
w_{x^0, \lambda_{x^0}}(x) = \nu(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \{x^0\},
\]

then for some \( C \in \mathbb{R} \), \( \mu > 0 \) and \( \bar{x} \in \mathbb{R}^n \),

\[
w(x) = C \left( \frac{\mu}{1 + \mu^2 |x - \bar{x}|^2} \right) \bar{x}^\nu.
\]

(ii) If for every \( x^0 \in \mathbb{R}^n \),

\[
w_{x^0, \lambda}(x) \geq \nu(x) \quad \text{for all } x \in B_\lambda(x^0) \setminus \{x^0\} \text{ and all } \lambda \in \mathbb{R},
\]

then \( w \equiv C \) for some constant \( C \in \mathbb{R} \).

To classify nonlinearities \( f, g, h \), we need the following elementary lemma.

Lemma 4.3. Let \( K \in C((0, +\infty), \mathbb{R}) \) be a nonincreasing function and \( w \in C(\mathbb{R}^n) \) be a positive function such that \( \lim_{|x| \to \infty} w(x) = 0 \). Assume that, for every \( x^0 \in \mathbb{R}^n \),

there exists \( 0 < \lambda_{x^0} < \infty \) such that \( K \left( w(x^{x^0, \lambda_{x^0}}) \right) = K \left( w(x) \right) \) for all \( x \in \mathbb{R}^n \setminus \{x^0\} \).

Then \( K \) is constant in \((0, \max_{\mathbb{R}^n} w] \).
Proof of Lemma 4.3. Taking \( x^0 \) such that \( w(x^0) = \max_{x \in \mathbb{R}^n} w \). By assumptions,
\[
K \left( w(x^0, \lambda_{x^0}) \right) = K (w(x)) \quad \text{for all } x \in \mathbb{R}^n \setminus \{ x^0 \}.
\]
Letting \( |x| \to \infty \) and using the continuity and monotonicity of \( K \), we derive
\[
K (w(0)) = \lim_{|x| \to \infty} K (w(x)) \geq K (\varepsilon)
\]
for every \( \varepsilon > 0 \). Since \( K \) is nonincreasing, this implies that \( K \) is constant in \( (0, \max_{x \in \mathbb{R}^n} w] \).

We are ready to prove the main result in this section, namely, Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2, there are two possibilities.

Case 1: For every \( x^0 \in \mathbb{R}^n \), the critical scale \( \lambda_{x^0} = \infty \).
By Lemma 4.2 and the positivity of \( u \), we have \( u \equiv C \) for some constant \( C > 0 \). This is absurd since positive constant functions do not satisfy (6).

Case 2: For every \( x^0 \in \mathbb{R}^n \), the critical scale \( \lambda_{x^0} < \infty \).
By Lemma 4.2, \( u \) must assume the form
\[
u(x) = c_0 \left( \frac{\mu}{1 + \mu^2 |x - \pi|^2} \right)^{\frac{\alpha}{\pi - \alpha}},
\]
for some \( c_0, \mu > 0 \) and \( \pi \in \mathbb{R}^n \).

We consider \( x^0 = 0 \). By Proposition 1, we have \( U_{\lambda_0} = 0 \) in \( \mathbb{R}^n \setminus \{ 0 \} \). Then using (26) and (27), we derive \( V_{\lambda_0} \geq 0 \) in \( \mathbb{R}^n \setminus \{ 0 \} \).

Now formula (17) yields
\[
0 = \int_{B_{\lambda_0}} \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|w| \lambda_0^{\alpha} x - \lambda_0 y |n - \alpha|} \right) \times \left\{ c_1 \left[ G (u(y_{\lambda_0})) v_{\lambda_0} (y) - G (u(y)) v(y) \right] u(y)^{\frac{n + \beta}{\pi - \alpha}} 
+ c_2 \left[ H (u(y_{\lambda_0})) - H (u(y)) \right] u(y)^{\frac{n + \beta}{\pi - \alpha}} \right\} dy 
\geq 0.
\]

Consequently, we must have
\[
G (u(y_{\lambda_0})) = G (u(y)) \quad \text{and} \quad v_{\lambda_0} (y) = v(y) \quad \text{for all } y \in B_{\lambda_0} \setminus \{ 0 \} \quad \text{if } c_1 > 0,
H (u(y_{\lambda_0})) = H (u(y)) \quad \text{for all } y \in B_{\lambda_0} \setminus \{ 0 \} \quad \text{if } c_2 > 0.
\]

Moreover, if \( v_{\lambda_0} (y) = v(y) \) for \( y \in B_{\lambda_0} \setminus \{ 0 \} \), we may use (26) to derive
\[
0 = \int_{B_{\lambda_0}} \left( \frac{1}{|x - y|^{n - \beta}} - \frac{1}{|w| \lambda_0^{\beta} x - \lambda_0 y |n - \beta|} \right) \left[ F (u(y_{\lambda_0})) - F (u(y)) \right] u(y)^{\frac{n + \beta}{\pi - \alpha}} dy \geq 0.
\]
This implies \( F (u(y_{\lambda_0})) = F (u(y)) \) for all \( y \in B_{\lambda_0} \setminus \{ 0 \} \).

Obviously, similar arguments also hold for the case \( x^0 \neq 0 \). Therefore, for any \( x^0 \in \mathbb{R}^n \), we have the following assertions
- If \( c_1 > 0 \), then \( G \left( u(x^0, \lambda_{x^0}) \right) = G (u(y)) \) and \( F \left( u(y, \lambda_{x^0}) \right) = F (u(y)) \) for all \( y \in \mathbb{R}^n \setminus \{ x^0 \} \).
- If \( c_2 > 0 \), then \( H \left( u(x^0, \lambda_{x^0}) \right) = H (u(y)) \) for all \( y \in \mathbb{R}^n \setminus \{ x^0 \} \).
By Lemma 4.3, we conclude that

• If $c_1 > 0$, then $F(t) = d_0$ and $G(t) = d_1$ for all $t \in [0, \max_{\mathbb{R}^n} u]$, where $d_0, d_1 > 0$.

• If $c_2 > 0$, then $H(t) = d_2$ for all $t \in [0, \max_{\mathbb{R}^n} u]$, where $d_2 > 0$.

That means that $f, g, h$ have the forms stated in Theorem 1.1. Then $u$ satisfies the integral equation

$$u(x) = R_{n, \alpha} \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha}} \times \left[ c_1 d_0 d_1 \left( \frac{1}{|x|^{n-\beta}} * u^{\frac{n+\alpha}{n}} \right) (y) u(y)^{\frac{\alpha+\beta}{n}} + c_2 d_2 u(y)^{\frac{n+\alpha}{n}} \right] dy \quad \text{(36)}$$

Next, we calculate the constant $c_0$ in (35). Notice that, if $u$ is a solution of (1), then $\mu^{\frac{n-\alpha}{\alpha}} u(\mu(x - \overline{x}))$ is also a solution for all $\mu > 0$ and $\overline{x} \in \mathbb{R}^n$. Therefore, it suffices to determine $c_0 > 0$ such that

$$Q(x) = c_0 \left( \frac{1}{1 + |x|^2} \right)^{\frac{n-\alpha}{2}}$$

is a solution of (1).

The following identity holds for any $0 < s < \frac{n}{2}$ (see (37) in [12])

$$\int_{\mathbb{R}^n} \frac{1}{|x - y|^{2s}} \left( \frac{1}{1 + |y|^2} \right)^{n-\beta} \ dy = I(s) \left( \frac{1}{1 + |x|^2} \right)^{s} \quad \text{(37)}$$

Using (37), we have

$$\left( \frac{1}{|x|^{n-\beta}} * Q^{\frac{n+\beta}{n}} \right)(x) = \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\beta}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{n+\beta}{n}} \ dy$$

$$= c_0^{\frac{n+\beta}{n}} I \left( \frac{n - \beta}{2} \right) \left( \frac{1}{1 + |x|^2} \right)^{\frac{n+\beta}{n}}.$$

Hence

$$R_{n, \alpha} \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha}} \left[ c_1 d_0 d_1 \left( \frac{1}{|x|^{n-\beta}} * Q^{\frac{n+\beta}{n}} \right)(y) Q(y)^{\frac{n+\beta}{n}} + c_2 d_2 Q(y)^{\frac{n+\alpha}{n}} \right] dy$$

$$= R_{n, \alpha} \left[ c_1 d_0 d_1 c_0^{\frac{n+\alpha+2\beta}{n}} I \left( \frac{n - \beta}{2} \right) + c_2 d_2 c_0^{\frac{n+\alpha}{n}} \right] \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{n+\alpha}{2}} \ dy$$

$$= R_{n, \alpha} \left[ c_1 d_0 d_1 c_0^{\frac{n+\alpha+2\beta}{n}} I \left( \frac{n - \beta}{2} \right) + c_2 d_2 c_0^{\frac{n+\alpha}{n}} \right] I \left( \frac{n - \alpha}{2} \right) \left( \frac{1}{1 + |x|^2} \right)^{\frac{n-\alpha}{2}}.$$

In order $Q$ to be a solution of (36), we must have

$$R_{n, \alpha} \left[ c_1 d_0 d_1 c_0^{\frac{n+\alpha+2\beta}{n}} I \left( \frac{n - \beta}{2} \right) + c_2 d_2 c_0^{\frac{n+\alpha}{n}} \right] I \left( \frac{n - \alpha}{2} \right) = c_0.$$

That is,

$$R_{n, \alpha} \left[ c_0^{\frac{2(n+\beta)}{n-\alpha}} c_1 d_0 d_1 I \left( \frac{n - \beta}{2} \right) + c_0^{\frac{2\alpha}{n-\alpha}} c_2 d_2 \right] I \left( \frac{n - \alpha}{2} \right) = 1.$$

This completes the proof of Theorem 1.1. \qed
5. Liouville type theorem for critical and super-critical order equations. Motivated by the ideas in [5], we prove a general Liouville type theorem for nonnegative solutions having the super polyharmonic property to critical and super-critical order equations.

Lemma 5.1. Assume $\alpha \geq n \geq 3$ and $f \in C(\mathbb{R}^n)$ is a nonnegative function such that $f \neq 0$. Then the equation

$$(-\Delta)^{\frac{\alpha}{2}} u = f \quad \text{in } \mathbb{R}^n$$

has no nonnegative solution $u$ such that $(-\Delta)^{\frac{\alpha}{2} - i} u \geq 0$ in $\mathbb{R}^n$ for every $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$.

Proof. Assume by contradiction that such a solution exists.

Let $m = \lceil \frac{n}{2} \rceil - 1$. We denote $u_i = (-\Delta)^{\frac{\alpha}{2} - i} u$ for $i = 0, 1, 2, \ldots, m$. We also denote $u_{m+1} = (-\Delta)^{\frac{\alpha}{2} - m - \frac{\alpha}{2}} u$, where $\gamma = \min\{\alpha - 2m, 2\}$. Notice that $\frac{n}{2} - m - \frac{\alpha}{2} \leq 0 \leq \frac{n}{2} - m - \frac{\alpha}{2}$.

By assumptions, $u_i \geq 0$ for every $i = 0, 1, 2, \ldots, m + 1$.

Since $0 < m < \frac{n}{2}$ and $(-\Delta)^m u_m = f$ in $\mathbb{R}^n$, by Lemma 3.3, we have

$$u_m(x) = R_n,2m \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2m}} dy + C_0 \quad \text{for all } x \in \mathbb{R}^n,$$

where $C_0 \geq 0$. Again, by Lemma 3.3, $0 < \gamma < n$ and $(-\Delta)^{\frac{\alpha}{2}} u_{m+1} = u_m$, we deduce

$$u_{m+1}(x) = R_n,\gamma \int_{\mathbb{R}^n} \frac{u_m(y)}{|x-y|^{n-\gamma}} dy + C_1 \quad \text{for all } x \in \mathbb{R}^n,$$

where $C_1 \geq 0$. Then we derive from (38) and (39) that

$$u_{m+1}(x) \geq R_n,\gamma R_n,2m \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\gamma}} \int_{\mathbb{R}^n} \frac{f(z)}{|y-z|^{n-2m}} dz dy.\quad (40)$$

Choosing $R_0 > 0$ such that $\int_{B_{R_0}} f(z) dz > 0$. Notice that, if $|y| \geq 2R_0$ and $|z| < R_0$, then

$$|y-z| \leq |y| + |z| < |y| + R_0 \leq \frac{3}{2} |y|.$$  

Consequently, for any $|y| \geq 2R_0$, we have

$$\int_{\mathbb{R}^n} \frac{f(z)}{|y-z|^{n-2m}} dz \geq \left( \frac{2}{3|y|} \right)^{n-2m} \int_{B_{R_0}} f(z) dz = \frac{C}{|y|^{n-2m}}.$$

Substituting this inequality into (40) and using the fact $\frac{n}{2} - m - \frac{\alpha}{2} \leq 0$, we deduce

$$u_{m+1}(0) \geq C \int_{|y| \geq 2R_0} \frac{1}{|y|^{2n-2m-\gamma}} dy = +\infty,$$

which is a contradiction. This proves the lemma.

Proof of Theorem 1.2. Let $u$ be a nonnegative solution of equation (1). Assume $u \neq 0$. By Theorem 2.1, we have

$$(-\Delta)^{\frac{\alpha}{2} - \lceil \frac{\alpha}{2} \rceil + 1} u \geq 0 \quad \text{in } \mathbb{R}^n.$$  

Notice that $0 < \frac{n}{2} - \lceil \frac{n}{2} \rceil + 1 \leq 1$. By the strong maximum principle (see Lemma 3.1 if $0 < \frac{n}{2} - \lceil \frac{n}{2} \rceil + 1 < 1$), we have $u > 0$. Hence $c_1 \left( \frac{1}{|z|^{\alpha-2}} \ast f(u) \right) g(u) + c_2 h(u) > 0$ in $\mathbb{R}^n$.

By combining Theorem 2.1 and Lemma 5.1, we derive a contradiction. Therefore, $u \equiv 0$. This completes the proof.
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