NUTATIONAL TWO-DIMENSIONAL STRUCTURES IN MAGNETS

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New types of magnetic structures in the Heisenberg model are found. Analytical methods are used to describe spiral structures, spiral vortex structures, and their interaction. Methods for obtaining these structures in real systems, including nanomagnets, are discussed.

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INTRODUCTION

There has been great interest in vortices and other nonlinear structures in low-dimension magnets over the last two decades. Many investigators have noted the important role of such structures in magnetic topological phase transitions (see, for example, \cite{1} and \cite{2}). Now that a large class of new quasi-one- and quasi-two-dimensional (2D) ferromagnets (layered magnets, intercalated compounds, graphite, cuprate planes in HTSCs \cite{3}), where the magnetic interaction within crystallographic planes is much stronger than the interaction between planes have been discovered and synthesized, the theoretical description of nonlinear structures and magnets is not only of academic interest.

Aside from different types of dynamical solitons, many stationary and autowave structures of condensed media, whose order parameter is a complex scalar field $\Psi$ just as in the Heisenberg model ($\Psi = e^{i\Phi}\cot(\theta)$ in the Heisenberg model) and which are invariant with respect to global changes at the phase $\Phi$, have now been studied. The most popular and well studied equations of this form are the Gross-Pitaevskii equation in the theory of superfluidity \cite{4}

\begin{equation}
\frac{i}{\partial t} \Psi = \Delta \Psi + (1 + \lambda) \Psi \pm |\Psi|^2
\end{equation}

and the complex Landau-Ginsburg equation \cite{5}

\begin{equation}
\frac{i}{\partial t} \Psi = \Delta \Psi + (1 + ib)\Psi \pm (1 + ic)\Psi|\Psi|^2,
\end{equation}

with real parameters $b$ and $c$ characterizing the linear and nonlinear dispersion. For $\Psi = S_x + iS_y$ Eq. \cite{6} also describes on the basis of the Landau-Lifshitz equation the lowamplitude dynamics of the spin system of a uniaxial ferromagnet with a distinguished axis $z$. A nontrivial structure described by Eq. \cite{6} is the nonstationary vortex (for example, the Pitaevskii vortex or magnetic vortex). In a polar coordinates system the function $\Psi$ has the form $\Psi = r S_x + i S_y$ with a distinguished axis $z$. A nontrivial structure described by Eq. \cite{6} is the nonstationary vortex (for example, the Pitaevskii vortex or magnetic vortex).

Other nontrivial spatial structures observed in autooscillatory active media are $N$-turn spiral waves. They correspond to solutions of Eq. \cite{6} in the form $\rho = \rho(r)$, $\Phi = N\varphi + \omega t + f(r)$. The phase of the spiral at large distances from the center of the spiral is proportional to the distance $f(r \to \infty) = kr$, the frequency $\omega = c + (b - c)^2 k^2$ is determined by the characteristics of the auto-oscillatory system itself, $\rho(0) \to const$ as $r \to \infty$, and the lines of constant phase are Archimedean spirals. The wave amplitude decreases near the center and vanishes as $r \to 0$. Numerical methods are used to determine the form of the functions $\rho(r)$ and $f(r)$ \cite{7,8,9,10,11,12,13,14}. Spiral structures are the richest class of spatial structures in active media (spiral galaxies, mollusks, spiral waves in the Belousov-Zhabotinskii reactions).

Such media are characterized by a continuous inflow of energy from a source to each physically small element and dissipation of this energy, and various stationary or time-dependent spatial structures, which lie at the foundation of self-organization, are observed to form in them under certain conditions.

It has been established experimentally \cite{12,13,14,15} that new structures — target-type guiding centers, spiral and spiral domains, observed by means of the magneto-optic Kerr effect — are formed in thin magnetic films with strong perpendicular easy-axis anisotropy under the action of harmonic or pulsed magnetic fields. Static stability and strong nonlinearity are characteristic experimentally observed features of magnetic structures. They do not vanish after the magnetic field is switched off — the lifetimes of the targets and spiral domains are several orders of magnitude greater.
than the period of the magnetic field. This makes it possible to study magnetic structures of the target and spiral-domain type as magnetic defects which are excited by energy pumping and relax to the thermodynamic equilibrium state over quite long periods of time.

A detailed investigation of spiral vortex structures for the $XY$ and Heisenberg models (since they are universal) is of interest for investigating structures in nanomagnets, where spiral structures have been found, as well as for possible applications in the physics of liquid crystals and the quantum Hall effect and for studying a number of biological systems where self-organizing spiral structures have been found [13–16].

Very simple types of two-dimensional spiral vortex structures have been obtained in analytic form for the Heisenberg model of a ferromagnet [17, 18]. The most general type of spiral structures which are formed by the main (exchange) approximation are found in next section. A wide class of new, exact solutions of the corresponding equations is discussed, the structure and interaction of spiral vortices are investigated, and the possibility of obtaining such structures experimentally is discussed.

**TWO-DIMENSIONAL SPIRAL VORTEICES**

We shall examine a model of an isotropic Heisenberg ferromagnet with spin $S$ described by a Hamiltonian of the form

$$H = - \sum_{p,n} J_{pn} S_p S_n, \tag{3}$$

where $S_p$ is the spin operator at the site $p$ of a two- or three-dimensional lattice and $a$ is the distance between the nearest neighbors with constant exchange interaction between them $J_{pn} = J \delta_{n,p+a} (J > 0)$. The nonlinear differential equations describing the dynamics of the model can be derived by examining the diagonal matrix element of the equation of motion of the $p$-spin operator $S_p^+ = S_p^z + i S_p^y$:

$$-i \hbar \frac{dS_p^+}{dt} = [H, S_p^+] \tag{4}$$

in the representation of spin coherent states $|\Omega\rangle = \prod_p |\theta_p, \Phi_p\rangle$, where $0 \leq \theta_p \leq \pi$ and $0 \leq \Phi_p < 2\pi$ parameterize the spin states on a sphere with unit radius [20]. For a Hamiltonian with bilinear interactions this results in a system of equations for the classical variables $\theta_p$ and $\Phi_p$ parameterizing the spin vector $S_p = S(sin\theta_p cos\Phi_p, sin\theta_p sin\Phi_p, cos\theta_p)$,

$$sin\theta_p \frac{d\Phi_p}{dt} = -\frac{S}{\hbar} \sum_n J_{np} sin\theta_p cos\theta_p cos(\Phi_p - \Phi_n) + sin\theta_p \frac{S}{\hbar} \sum_n J_{np} cos\theta_n; \tag{5}$$

$$\frac{d\theta_p}{dt} = \frac{S}{\hbar} \sum_n J_{np} sin\theta_n sin(\Phi_n - \Phi_p), \tag{6}$$

here the index $n$ enumerates the nearest neighbors of the spin under consideration. In the continuum limit in the two-dimensional case we introduce the fields $\theta(x, y)$ and $\Phi(x, y)$, which are defined in the $(x, y)$ plane. The equations for the static solutions $\frac{d\theta_p}{dt} = \frac{d\Phi_p}{dt} = 0$ can be obtained by passing to the continual approximation in the equations for spins on a discrete lattice

$$\begin{cases} \Delta \theta = sin(\theta)cos(\theta)(\nabla^2 \Phi)^2, \\ \nabla (sin(\theta)^2 \nabla \Phi) = 0. \end{cases} \tag{7}$$

It is shown in [17, 18] that spiral structures exist when the contour lines of the fields $\Phi(x, y)$ and $\theta(x, y)$ are orthogonal ($\nabla \Phi \cdot \nabla \theta = 0$) or parallel ($\nabla \Phi \propto \nabla \theta$) to one another. Here we shall examine the case where the derivative fields $\Phi(x, y)$ are linear combinations of the derivatives of the field $\theta$ with coefficients that depend on this field

$$\begin{cases} \frac{\partial \Phi}{\partial x} = -F_1(\theta) \frac{\partial \theta}{\partial x} + F_2(\theta) \frac{\partial \theta}{\partial y}, \\ \frac{\partial \Phi}{\partial y} = F_1(\theta) \frac{\partial \theta}{\partial x} + F_2(\theta) \frac{\partial \theta}{\partial y}, \end{cases} \tag{8}$$

where, because the exchange interactions are invariant under the rotation group, we assume the functions $F_1$ and $F_2$ to depend only on the field $\theta(x, y)$. The compatibility condition for the system (5) and (7) yields a closed system of
three nonlinear equations for the fields $\theta(x, y)$, $F_1$, and $F_2$. The solution of two of them determines the fields $F_1$ and $F_2$ in explicit form:

$$ F_2 = c_1 \frac{F_1}{\sin(\theta)^2}, \quad F_1 = \frac{2\sin(\theta)}{\sqrt{-4c_1^2 - c_2 \sin(\theta)^2 - \sin(2\theta)^2}} $$

with arbitrary constants $c_1$ and $c_2$, and the last equation has the form

$$ \Delta \theta = -(\nabla \theta)^2 \frac{F_1'}{F_1}. $$

Redefining the constants $c_1$ and $c_2$ and introducing the auxiliary field $a(x, y)$, this equation reduces to the Laplace equation

$$ \Delta a(x, y) = 0, $$

where the field $a(x, y)$ determines the field $\theta(x, y)$ as follows:

$$ \cos[\theta(x, y)] = c \cdot \text{sn}[a(x, y), k], \quad (0 < k < 1). $$

Here $\text{sn}[a(x, y), k]$ is the Jacobi elliptic function (elliptic sine) with modulus $k$.

The solution of the system (8) becomes

$$ \Phi(x, y) = -\sqrt{(1 - c^2)(c^2 - k^2)} \int_0^a(x, y) \frac{dX}{(1 - c^2 \text{sn}(X, k)^2)} + \Psi(x, y). $$

where $a(x, y) + i\Psi(x, y)$ is an analytic function of the complex variable $z = x + iy$. As a result, the relations (11), (12), (13) give a new class of exact solutions of the equations (7) which is determined by the analytic function $a(x, y) + i\Psi(x, y)$ and the two parameters $k, c$ whose ranges are

$$ 0 \leq k \leq 1, \quad k \leq c \leq 1. $$

The limits of the intervals of these parameters correspond to known classes of solutions. In the limit $k \to 0$ we obtain a family of solutions which depend only on the harmonic function $a(x, y)$

$$ \cos[\theta(x, y)] = c \cdot \text{sn}[a(x, y), k], \quad (0 < k < 1) $$

and were investigated in [18]. For $c = 1$ the substitution (8) reduces to the Cauchy-Riemann equation for the analytic function $\Omega = \Phi(x, y) + i\Psi(x, y)$ and the corresponding class of solutions found in [17]. Finally, for $c = 1$, $k \to 1$, and

$$ \Omega = \sum_{j=1}^n (Q_j)\ln(x + iy - c_j), \quad (Q_j \in \mathbb{Z}) $$

we represent the solutions as follows:

$$ \cot \left( \frac{\theta}{2} \right) e^{i\Phi} = \prod_{j=1}^n \left( \frac{x + iy - c_j}{A_j} \right)^{Q_j} $$

These solutions describe the structure and interaction of instantonsmagnetic vortices [20].

Here we shall discuss the choice of $\frac{1}{c}a(x, y) + i\Psi(x, y)$ in the form of a potential of vortex sources in hydrodynamics:

$$ a(x, y) = \sum_{i=1}^n \alpha_i \cdot \ln \left( \sqrt{(x - x_0i)^2 + (y - y_0i)^2} \right) + q_i \cdot \arctan \left( \frac{y - y_0i}{x - x_0i} \right), $$

where $x_0i$ and $y_0i$ are the positions of the sources and $\alpha_i$ are their strengths.
\[ \Psi(x, y) = \sum_{i=1}^{n} \left( -\frac{k}{c} \right) q_i \cdot \ln \left( \frac{\sqrt{(x-x_0i)^2 + (y-y_0i)^2}}{\sqrt{x-x_0i}^2 + (y-y_0i)^2} \right) + \left( \frac{k}{c} \right) \alpha_i \cdot \arctan \left( \frac{y-y_0i}{x-x_0i} \right), \] (15)

with singularities at the points \((x_0i, y_0i)\) — the centers of magnetic defects. It follows from the fact that the magnetization is single-valued, the form of the solution \([13]\), and the symmetry of the elliptic functions \(sn(u, k) = sn(u+4K, k)\) and \(sn(u, k) = sn(2K - u, k)\) (where \(K = K(k)\) is the complete elliptic integral of the first kind) that the changes in the fields \(a, \Phi\) with one revolution about a closed contour around the point \((x_0i, y_0i)\) must satisfy the conditions

\[ \delta a = 4KN_i, \quad \delta \Phi = 2\pi Q_i, \quad (N_i, Q_i \in Z). \] (16)

Hence follows immediately the macroscopic quantization of the parameter \(q_i\):

\[ q_i = \frac{2KN_i}{\pi}. \] (17)

Taking account of the change of the field \(\Phi(x, y)\) with such a revolution, equal to

\[ \delta \Phi(x, y) = -\sqrt{\frac{(1-c^2)(c^2-k^2)}{c}} \int_0^{\delta a(x,y)} \frac{dX}{(1-c^2 sn(X, k)^2)} + \delta \Psi(x, y), \] (18)

and the relation \([16]\), we obtain a relation between the parameters \(\alpha_i\) and \(Q_i\):

\[ \alpha_i = \frac{c\pi Q_i + 2\sqrt{1-c^2}\sqrt{c^2-k^2}N_i P(-c^2, k)}{k\pi}, \] (19)

where

\[ P(-c^2, k) = \int_0^K \frac{dX}{(1-c^2 sn(X, k)^2)} \] (20)

is a complete elliptic integral of the third kind.

Finally, the relations \([2], [13], [11], [13], [17], [18], [19]\) and \([20]\) describe new types of magnetic structures in ferromagnets in the exchange approximation. For \(n = 1\) we obtain in a polar coordinate system

\[ \cos(\theta) = c \cdot sn(\alpha \cdot \ln(r) + \frac{2KN}{\pi} \varphi, k), \]

\[ \Phi = \frac{k\alpha}{c} \varphi - \frac{2KN}{c\pi} \ln(r) + \frac{\sqrt{1-c^2}\sqrt{c^2-k^2}}{c} \int_0^{\alpha \cdot \ln(r) + \frac{2KN}{\pi} \varphi} \frac{dX}{(-1+c^2 sn(X, k)^2)}, \] (21)

where

\[ \alpha = \frac{kQ}{k} + \frac{2N\sqrt{1-c^2}\sqrt{c^2-k^2}}{k\pi} \int_0^K \frac{dX}{(1-c^2 sn(X, k)^2)} \]

The parameter \(c \leq 1\) controls the amplitude with which the spins leave the \(xy\) plane. The structures \([2]\) can be called nutational, since the angle \(\theta\) lies in the range \(\theta_{\text{max}} \leq \theta < \pi - \theta_{\text{max}}\) with the maximum value \(\theta_{\text{max}} = \arccos(c)\). They include several types of structures. The case \(N = 0\) corresponds to a vortex magnetic target (Fig.1). The distribution of the azimuthal angle \(\Phi\) is characteristic for vortex structures with radial dependence, and the component \(S_z\) has the form of infinite concentric (with respect to the variable \(r\)) ring domains forming an independent striped domain structure with respect to the variable \(\ln r\). For \(Q = 0\) the azimuthal angle of magnetization \(\Phi\) depends on \(r, \varphi\) and does not have a vortical dependence. The component \(S_z\) is a spiral structure, since it is constant on the curves in the \((x, y)\) plane which are logarithmic spirals (Fig.3)

\[ r = C_{\exp} \left( -\frac{2K \varphi N}{\pi \alpha} \right). \] (22)

An isolated magnetic defect with discrete parameters \(N \neq 0\) and \(Q \neq 0\) is a spiral vortex with a vortical distribution of the field \(\Phi\) and a spiral structure for \(S_z\). For \(N = 1\) the \(S_z\) distribution consists of two domains have opposite directions of magnetization and separated by two logarithmic spirals (Fig.3). The width of spiral solitons (domain
walls) depends on \( k \) and increases away from the center of the vortex. Since \( K = K(k) \) is a monotonically increasing function of the parameter \( k \), this parameter determines the degree of twisting of the spiral (Fig. 1). The chirality of the spiral (direction of twist) is determined by the sign of the quantity \( N/\alpha \). The parameter \( N \) determines the number of arms of the logarithmic spiral. A plot of the field \( S_z \) and the configuration of the domains for a two-turn (two-arm) spiral are presented in Fig. 2. We shall show that the exchange interaction forms logarithmic spirals \( \ln(r) \propto (\varphi - \varphi_0) \).

Indeed, the equations (21) are invariant under scale transformations \( r = r e^{\beta} \) and rotations \( \varphi = \varphi + \gamma \) with the parameters \( \gamma \) and \( \beta \), respectively. Consequently, the curve of constant values of \( \theta(x, y) \) can be invariant with respect to a single-parameter group of transformations \( r = r e^{\beta}, \varphi = \varphi + \rho \beta \) (spiral rotation group (21)) with the parameter \( \beta \). In our case \( \rho = -\frac{\pi \alpha}{2KN} \).

Direct calculations show that the energy density \( \frac{\nabla S \cdot \nabla S}{2} \) of the structure (21) is proportional to \( (\nabla a)^2 \):

\[
\frac{\nabla S \cdot \nabla S}{2} = \frac{1}{2} (1 + k^2 - 2k^2 sn(a, k)^2)(\nabla a)^2.
\] (23)
To calculate the energy $H$ of a spiral vortex we employ the expansion [22]

$$sn(u,k)^2 = \frac{1}{k^2 K^2} \left( K^2 - KE - 2\pi^2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos \left( \frac{2n\pi u}{2K} \right) \right).$$

Here $E = E(k)$ is a complete elliptic integral of the second kind and $q = \exp(-\frac{K^2}{K})$ ($K^2 = K\sqrt{1 - k^2}$). Then the energy of a spiral vortex, just as that of other nonlocal structures of a similar type (vortices and hydrodynamics, dislocations in the crystal lattice), depends logarithmically on the size $L$ of the system and the radius $d$ of the vortex core (of the order of the lattice constant):

$$H = \frac{(2E + (-1 + k^2)K)(\pi^2 \alpha^2 + 4N^2 K^2)}{\pi K} \ln \left( \frac{L}{d} \right)$$

(24)

where $N \neq 0$. 

FIG. 3: Structure of the core of a one-turn spiral ($Q = 1, N = 1, c = 0.5, k = 0.15$).

FIG. 4: Structure of the core of a one-turn spiral ($Q = 1, N = 1, c = 0.5$); $k = 0.15$ (a), $k = 0.1$ (b).
Since a spiral is characterized by two integers \((N, Q)\), spiral dipoles have structurally more diverse than vortices. We shall examine as an example some types of vortex dipoles. In contrast to many-instanton solutions, the energy of multispiral configurations with \(k \neq 1\) depends on the distances between the centers of the spiral vortices, which results in their interaction.

As an example, we shall now examine certain types of spiral dipoles consisting of vortex spirals with the numbers \((N_1, Q_1)\) and \((N_2, Q_2)\). At large distances such a dipole transforms into a definite spiral configuration with the numbers \((N_1 + N_2, Q_1 + Q_2)\). A dipole consisting of two spiral structures with the numbers \((1, 1)\) forms at large distances a two-turn spiral (Fig. 5), a dipole with \((1, 1)\) and \((1, -1)\) forms a \((2, 0)\) structure of a vortex-free spiral (Fig. 6), and a dipole with \((1, 1)\) and \((-1, 1)\) forms a magnetic target structure (Fig. 8).

The interaction of two vortices with the parameters \((N, Q)\) and \((-N, -Q)\) is attractive. The corresponding solution is localized and is displayed in Fig. 5. The energy of such a dipole does not depend on the size \(L\) of the system and at large distances the energy density is inversely proportional to \(r^4\). Since the activation energy is low, such spiral dipoles can be generated by thermal fluctuations and contribute to the thermodynamic properties of a system.

We shall discuss briefly the possibility of observing experimentally the magnetic structures found in the present work. The rapid advancement of the technology for growing thin films has made it possible to produce artificially ordered ASM alloys (artificially structured materials). As a result of the influence of symmetry and low-dimension effects, new phases can arise in such materials during growth of thin films. Then a uniform state, conventionally considered to be the ground state for a two-dimensional Heisenberg ferromagnet, is simply impossible to obtain in practice if the magnetic structure possesses a nonzero momentum or angular momentum.

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FIG. 6: Spiral dipole consisting of two one-turn spirals ($N = 1, Q = 0$ and $N = 1, Q = 0$), respectively, with $c = 0.5$, $k = 0.15$. Regions are shown on different scales: structure with large distances (a) and structure of the cores (b).

FIG. 7: Spiral dipole consisting of two one-turn spirals ($N = 1, Q = 0$ and $N = 1, Q = -2$), respectively, with $c = 0.5$, $k = 0.15$. 
FIG. 8: Spiral dipole consisting of two one-turn spirals \((N = 1, Q = 0\) and \(N = -1, Q = 2\)), respectively, with \(c = 0.4, k = 0.25\).

FIG. 9: Spiral dipole consisting of one-turn spirals \((N = 1, Q = 0\) and \(N = -1, Q = 0\)), respectively, with \(c = 0.5, k = 0.15\).