Noether Symmetry Approach in Energy-Momentum Squared Gravity

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Abstract

In this paper, we investigate the newly developed $f(R, T^2)$ theory ($R$ is the Ricci scalar and $T^2 = T_{\alpha\beta}T^{\alpha\beta}$, $T_{\alpha\beta}$ demonstrates the energy-momentum tensor) to explore some viable cosmological models. For this purpose, we use the Noether symmetry approach in the context of flat Friedmann-Robertson-Walker (FRW) universe. We solve the Noether equations of this modified theory for two types of models and obtain the symmetry generators as well as corresponding conserved quantities. We also evaluate exact solutions and investigate their physical behavior via different cosmological parameters. For the prospective models, the graphical behavior of these parameters indicate consistency with recent observations representing accelerated expansion of the universe. In the first case, we take a special model of this theory and obtain new class of exact solutions with the help of conserved quantities. Secondly, we consider minimal and non-minimal coupling models of $f(R, T^2)$ gravity. We conclude that conserved quantities are very useful to derive the exact solutions that are used to study the cosmic accelerated expansion.

Keywords: $f(R, T^2)$ gravity; Noether symmetries; Conserved quantities; Exact solutions.

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1 Introduction

The accelerated expansion of the universe has been the most unexpected and surprising result for the scientific community for the last two decades. Since gravity being an attractive force will lead the universe and all the matter present inside it to contract, hence the expansion of the universe would gradually slow down. However, this is against the observational evidences and hence we need to search for some new physics which is consistent with our observations. The well-known approach is modifying the geometry of space-time, i.e., general theory of relativity (GR) at large distances, specifically beyond our solar system to produce accelerating cosmological solutions [1]. Modified theories can be formulated by adding the functions of curvature invariants in the geometric part of the Einstein-Hilbert action. The natural modification is obtained by replacing an arbitrary function of the Ricci scalar \( R \) in the Einstein-Hilbert action, so called \( f(R) \) theory of gravity. There have been a crucial literature [2] available to understand the viable characteristics of this gravity.

The \( f(R) \) theory of gravity has further been generalized by introducing some couplings between the geometrical quantities and the matter sector. The non-minimally coupling between the curvature invariant and matter lagrangian density (\( \mathcal{L}_m \)) has been established in [3] dubbed as \( f(R, \mathcal{L}_m) \) theory of gravity. These curvature-matter couplings explain various cosmic eras as well as the rotation curves of galaxies. Such interactions also include non-conserved energy-momentum tensor indicating the existence of an additional force. These theories play a significant role to understand the expanding behavior of the universe and dark matter/energy interactions [4]. One such modifications gave rise to \( f(R, T) \) theories (\( T \) represents the trace of energy-momentum tensor) [5]. A more generic theory in which matter is nonminimally coupled to geometry was proposed [6], referred to as \( f(R, T, R_{\alpha\beta}T^{\alpha\beta}) \) gravity (\( R_{\alpha\beta} \) is the Ricci tensor and \( T_{\alpha\beta} \) is the energy-momentum tensor).

Sharif and Ikram [7] formulated such a coupling in \( f(\mathcal{G}) \) gravity known as \( f(\mathcal{G}, T) \) theory, here \( \mathcal{G} \) defines the Gauss-Bonnet invariant. Moraes and Santos [8] established \( f(R, T^\phi) \) theory, where \( T^\phi \) demonstrates the trace of the energy-momentum of the scalar field.

This generalization procedure for the \( f(R, \mathcal{L}_m) \) theory can also modify the corresponding Lagrangian by including some analytic function of \( T_{\alpha\beta}T^{\alpha\beta} \). This choice of the corresponding Lagrangian will lead to \( f(R, T_{\alpha\beta}T^{\alpha\beta}) \) theory of gravity, also called energy-momentum squared gravity. Katirci and
Kavuk [9] proposed such a theory for the first time in 2014, which allows the existence of a term proportional to $T_\alpha{}^\beta T^{\alpha\beta}$ in the action functional. Different researchers have carried out further studies on this theory. There has been a recent literature [10] that indicates various cosmological applications of this modified theory.

Roshan and Shojai [11] found that this theory has a bounce at early times and avoids the existence of singularity. Further, they argued that the “repulsive” nature of the cosmological constant plays a significant role at early times for resolving the singularity only after matter-dominated era. Board and Barrow [12] investigated the range of exact solutions for isotropic spacetime, presence of singularities, cosmic accelerated expansion as well as evolution with a particular model of this theory. Morares and Sahoo [13] studied non-exotic matter wormholes while Akarsu et al. [14] explored possible constraints from neutron stars in this framework. Bahamonde et al. [15] studied different cosmological models to investigate the ambiguous cosmic characteristics. Akarsu et al. [16] investigated the minimal and non-minimal curvature-matter coupling models of $f(R, T^2)$ theory and observed that these models describe the current cosmic accelerated expansion. Narim and Roshan [17] studied physical viability and stability of compact stars in this framework. Bahamonde et al. [18] studied dynamical system analysis of this theory and found that this theory can explain the current evolution of the universe and the emergence of the accelerated expansion as a geometrical consequence. This literature clearly motivates that $f(R, T^2)$ gravity requires more focus and there are many open issues that can be studied. This would add and improve our current knowledge about different modified theories of gravity.

Symmetry is a well-known significant aspect of cosmology as well as theoretical physics. In this regard, Noether symmetry technique helps to find exact solutions of the defined Lagrangian. It is an interesting approach that suggests a correlation between conserved quantities as well as symmetry generators of a dynamical system [19]. Such symmetries enable us to find analytical solutions of nonlinear partial differential equations (PDEs) by reducing them to a linear one. The main motivation comes from various conservation laws (energy, momentum, angular momentum, etc.) which are outcomes of some kind of symmetry being present in a system. The conservation laws are the key factors in the study of various physical processes and Noether theorem implies that every differentiable symmetry of the action leads to the law of conservation. This theorem is significant because it provides a
correlation between conserved quantities and symmetries of a physical system \[20\]. Capozziello and Ritis \[21\] investigated the Noether symmetries and also found the exact cosmological solutions in non-minimally coupled gravitational theory. Capozziello et al. \[22\] examined Noether symmetry approach in the phantom quintessence universe. Sharif and his collaborators \[23\] analyzed the current cosmic expansion and evolution by using this approach.

Capozziello et al. \[24\] used Noether symmetry approach to find static and non-static spherical solutions in \(f(R)\) theory. Roshan and Shojai \[25\] studied Palatini \(f(R)\) cosmology using Noether symmetry approach for the matter-dominated universe. Hussain et al. \[26\] used this technique to analyze the Noether gauge symmetry in the background of \(f(R)\) theory. Shamir et al. \[27\] applied this symmetry approach to analyze the stability criteria of \(f(R)\) gravity models for spherically symmetric as well as FRW universe. Kucukakca et al. \[28\] applied the Noether symmetry technique to obtain analytic solutions of the Bianchi type-I spacetime. Shamir and Ahmad \[29\] discussed some cosmological models with isotropic as well as anisotropic matter distribution though this technique in \(f(G,T)\) theory. Bahamonde et al. \[30\] used this approach to obtain various exact solutions of teleparallel gravity with boundary term.

In this paper, we study the existence of Noether symmetry of flat FRW universe in \(f(R,T^2)\) theory of gravity. We determine possible symmetries as well as corresponding conserved quantities and evaluate exact solutions for two \(f(R,T^2)\) models to analyze cosmic evolution through cosmological parameters. The paper is planned as follows. In section 2, we study some basic facts of this theory. Section 3 gives a brief description about symmetry minimized Lagrangian and Noether equations. Section 4 provides cosmological solutions based on the conserved quantities. A brief summary and discussion of the results is given in the last section.

## 2 Basics of \(f(R,T^2)\) Gravity

In this section, we formulate the field equations for \(f(R,T^2)\) theory in the presence of perfect fluid. The action for this gravity can be expressed as \[18\]

\[
S = \frac{1}{2\kappa^2} \int f(R,T^2) \sqrt{-g} d^4x + \int \mathcal{L}_m \sqrt{-g} d^4x, \tag{1}
\]
where $\kappa^2$, $g$ and $L_m$ represent the coupling constant, determinant of the metric tensor and the Lagrangian density of matter, respectively. We consider coupling constant as a unity for the sake of simplicity. The action indicates that this theory has extra degrees of freedom. Therefore, the possibility of exact solutions is enhanced as compared to GR. Due to matter dominated era, it is expected that some useful consequences would be obtained to study the issues of dark energy and current cosmic expansion in this gravity. The variation of the action with respect to the metric tensor yields the following field equations

$$
R_{\alpha\beta}f_R + g_{\alpha\beta}\Box f_R - \nabla_\alpha \nabla_\beta f_R - \frac{1}{2}g_{\alpha\beta}f = T_{\alpha\beta} - \Theta_{\alpha\beta}fT^2,
$$

(2)

where $\Box = \nabla_\alpha \nabla^\alpha$, $f \equiv f(R, T^2)$, $fT^2 = \frac{\partial f}{\partial T}$, $f_R = \frac{\partial f}{\partial R}$, and

$$
\Theta_{\alpha\beta} = -2L_m \left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) - 4\frac{\partial^2 L_m}{\partial g^{\alpha\beta} \partial g^{\mu\nu}}T^{\mu\nu} - TT_{\alpha\beta} + 2T^\mu_{\alpha}T_{\beta\mu}.
$$

(3)

For $f(R, T^2) = f(R)$, the field equations of this gravity reduces to $f(R)$ theory and GR is recovered when $f(R, T^2) = R$ \cite{32,33}.

We consider the matter configuration as a perfect fluid

$$
T^m_{\alpha\beta} = (\rho + p)U_\alpha U_\beta + pg_{\alpha\beta},
$$

(4)

where $U_\alpha$, $p$ and $\rho$ depict the four velocity, energy density and pressure, respectively. We assume the matter Lagrangian as $L_m = p$ so that

$$
\Theta_{\alpha\beta} = -U_\alpha U_\beta \left(4\rho p + \rho^2 + 3p^2\right).
$$

Rearranging Eq.(2), we obtain

$$
G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{1}{f_R} \left(T^c_{\alpha\beta} + T^m_{\alpha\beta}\right),
$$

(5)

where $T^c_{\alpha\beta}$ are the correction terms of $f(R, T^2)$ theory given as follows

$$
T^c_{\alpha\beta} = \frac{1}{2}g_{\alpha\beta}(f - Rf_R) - g_{\alpha\beta}\Box f_R + \nabla_\alpha \nabla_\beta f_R - \Theta_{\alpha\beta}fT^2.
$$

(6)

Equation (5) indicates that the stress-energy tensor of gravitational fluid $(T^c_{\alpha\beta})$ gives matter contents of the spacetime. Consequently, this technique
includes all the components of matter that might be significant to uncover the cosmic mysteries. By contracting Eq.\[1\], we have
\[Rf_R - 2f + 3\Box f_R = T - \Theta f_{T^2}.\] (7)

The flat FRW universe model is given by
\[ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right),\] (8)

where \(a(t)\) defines the cosmic scale factor. The corresponding dynamical quantities \(R\) and \(T^2\) are
\[R = 6\frac{\ddot{a}}{a^2} + 6\frac{\dot{a}^2}{a^2}, \quad T^2 = 3\rho^2 + \rho^2.\] (9)

The respective field equations turn out to be
\[-3\frac{\ddot{a}}{a} f_R + \frac{1}{2} f + 3\frac{\dot{a}}{a} \dot{f}_R = \rho - (3\rho^2 + \rho^2 + 4\rho \dot{\rho}) f_{T^2},\] (10)
\[
\left( \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \right) f_R - \frac{1}{2} f - \dot{f}_R - 2\frac{\dot{a}}{a} \dot{f}_R = p,\] (11)

where dot defines the rate of change with respect to time.

The field Eqs.\[10\] and \[11\] are highly non-linear as well as complicated due to the presence of multivariate function and its derivatives. In order to solve these equations, we consider the Noether symmetry approach and determine exact solutions of \(f(R, T^2)\) field equations. Since the conservation law does not hold in this theory but we obtain conserved quantities in the background of Noether symmetry approach. These are helpful to obtain physically viable analytic or numeric solutions as well as to analyze the mysterious universe. We analyze some feasible models of cosmology through Noether symmetry approach.

### 3 Symmetry Reduced Lagrangian and Noether Equations

Noether symmetry provides a fascinating procedure to develop new cosmological models and related geometries in modified gravitational theories. Here, we formulate the point-like Lagrangian for FRW universe in the background
of $f(R, T^2)$ theory. We determine the corresponding equations by using Noether symmetry technique. This method provides a unique nature of the vector field within the tangent space associated with it. Hence, the vector field behaves as a symmetry generator and gives conserved quantities which are then useful to examine exact solutions of the modified field equations.

The canonical form of the action (1) gives

$$S = \int L \left( a, \dot{a}, R, \dot{R}, T^2, \dot{T}^2 \right) dt. \quad (12)$$

Using Lagrange multiplier approach, we have

$$S = \int \sqrt{-g} \left\{ f - (R - \bar{R})\nu_1 - (T^2 - \bar{T}^2)\nu_2 + p(a) \right\} dt, \quad (13)$$

where $\bar{R} = 6 \left( \frac{\ddot{a}^2 + a\dot{a}^2}{a^2} \right)$, $\bar{T}^2 = 3p^2 + \rho^2$ and $\sqrt{-g} = a^3$. We see that if $R = \bar{R}$ and $T^2 = \bar{T}^2 = 0$, then the above action reduces to the action (1) for FRW universe. Varying Lagrange multipliers $\nu_1$ and $\nu_2$ with respect to $R$ and $T^2$, we obtain

$$\nu_1 = f_R, \quad \nu_2 = f_{T^2}. \quad (14)$$

The corresponding action (13) yields

$$S = \int a^3 \left\{ f - \left( R - 6\frac{\dot{a}^2}{a^2} - 6\frac{\dot{a}\ddot{a}}{a^2} \right) f_R - \left( T^2 - 3p^2 - \rho^2 \right) f_{T^2} + p \right\} dt. \quad (15)$$

Eliminating the boundary terms with the help of integration by parts, we have

$$L(a, \dot{a}, R, \dot{R}, T^2, \dot{T}^2) = a^3 \left( f - Rf_R - T^2f_{T^2} + (3p^2 + \rho^2) f_{T^2} + p \right) - 6a\dot{a}^2f_R - 6a^2\ddot{a}\left( \dot{R}f_{RR} + \dot{T}^2f_{RT^2} \right). \quad (16)$$

The Euler-Lagrange equations is given by

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = 0, \quad i = 1, 2, 3, \ldots, n, \quad (17)$$

where $q^i$ represent the generalized coordinates of $n$-dimensional configuration space. By using Lagrangian (16), Eqs.(17) turn out to be

$$\frac{1}{2} \left( f - Rf_R - T^2f_{T^2} + (3p^2 + \rho^2) f_{T^2} + p \right) + 2\frac{\dot{a}}{a} f_R$$
\[ + \frac{a}{6} \left\{ (6p_{p,a} + 2\rho_{p,a}) f_{T^2} + p,a \right\} + \left( \frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} \right) f_R + \ddot{f}_R = 0, \quad (18) \]

\[ \left( R - 6\frac{\dot{a}^2}{a^2} - 6\frac{\ddot{a}}{a} \right) f_{RR} + (T^2 - 3p^2 - \rho^2) f_{RT^2} = 0, \quad (19) \]

\[ \left( R - 6\frac{\dot{a}^2}{a^2} - 6\frac{\ddot{a}}{a} \right) f_{R T^2} + (T^2 - 3p^2 - \rho^2) f_{T^2 T^2} = 0. \quad (20) \]

The Hamiltonian of the Lagrangian is expressed as

\[ H = \dot{q}^i \left( \frac{\partial L}{\partial \dot{q}^i} \right) - L. \quad (21) \]

Using Eq.(16), it turns out to be

\[ H = -a^3 \left( f - R f_R - T^2 f_{T^2} + (3p^2 + \rho^2) f_{T^2} + p \right) - 6a^2 \dot{a} \left( \dot{R} f_{RR} + \dot{T}^2 f_{RT^2} \right) - 6a \dddot{a} f_R. \quad (22) \]

The generators of Lagrangian (16) are considered as

\[ K = \tau \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial q^i}, \quad (23) \]

where \( \tau \equiv \tau(t, a, R, T^2) \) and \( \xi^i \equiv \xi^i(t, a, R, T^2) \) for \( i = 1, 2, 3, 4 \) are unknown coefficients of the vector field \( K \). The Lagrangian must fulfill the condition of invariance for unique vector field \( K \) over the tangent space to assure the existence of Noether symmetries. In this regard, the vector field acts as a symmetry generator that constructs the conserved quantities. The invariance condition can be expressed as

\[ K^{[1]} L + (D\tau) L = D\psi, \quad (24) \]

where \( \psi \) represents the boundary term, \( K^{[1]} \) is the first order prolongation and \( D \) demonstrates the total derivative. Further, it can be expressed as

\[ K^{[1]} = K + \dot{\xi}^i \frac{\partial}{\partial \dot{q}^i}, \quad D = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial \dot{q}^i}, \quad (25) \]

here \( \dot{\xi}^i = D\xi^i - \dot{q}^i D\psi \).
The first integral of motion corresponds to Noether symmetry generator $K$ determined as

$$ I = -\tau H + \xi^i \frac{\partial L}{\partial \dot{q}^i} - \psi. \quad (26) $$

This is the most significant part of Noether symmetries which is also known as a conserved quantity. It is interesting to mention here that the first integral plays a remarkable role to obtain physically viable solutions. By considering Eq. (24) and comparing the coefficients, we obtain a set of PDEs as follows

$$ \tau_a = 0, \quad \tau_R = 0, \quad \tau_{T^2} = 0, \quad \xi_{1,R} f_{RR} = 0, \quad (27) $$
$$ 6a^2 \xi_1 f_{RR} + \psi_R = 0, \quad 6a^2 \xi_1 f_{RT^2} + \psi_{T^2} = 0, \quad (28) $$
$$ 3\xi_1 f_{T^2} + a(\xi^2 f_{RT^2} + \xi^3 f_{T^2T^2}) + a \tau_{,i} f_{T^2} = 0, \quad (29) $$
$$ 12a \xi_1 f_{R} + 6a^2 (\xi_1 f_{RR} + \xi_3 f_{RT^2}) + \psi_a = 0, \quad (30) $$
$$ \xi_1 f_{R} + a (\xi^2 f_{RR} + \xi^3 f_{RT^2}) + 2a \xi_{,a} f_{R} - a \tau_{,i} f_{R} $$
$$ + a^2 (\xi_{2,R} f_{RR} + \xi_{2,R} f_{RT^2}) = 0, \quad (31) $$
$$ 2\xi_1 f_{RR} + a (\xi^2 f_{RR} + \xi^3 f_{RT^2}) + 2\xi_{1,R} f_{R} - a \tau_{,i} f_{RR} $$
$$ + a (\xi_{1,R} f_{RR} + \xi_{2,R} f_{RR} + \xi_{3,R} f_{RT^2}) = 0, \quad (32) $$
$$ 2\xi_1 f_{RT^2} + a (\xi^2 f_{RT^2} + \xi^3 f_{RT^2T^2}) + 2\xi_{1,T^2} f_{R} - a \tau_{,i} f_{RT^2} $$
$$ + a (\xi_{1,T^2} f_{RT^2} + \xi_{2,T^2} f_{RR} + \xi_{3,T^2} f_{RT^2}) = 0, \quad (33) $$

$$ 3a^2 \xi_1 \left\{ f - R f_R - T^2 f_{T^2} + (3p^2 + \rho^2) f_{T^2} + p \right\} $$
$$ + a^3 \xi^2 \left\{ (6pp_a + 2\rho p_a) f_{T^2} + p_a \right\} + a^3 \xi^3 \left\{ - R f_{RR} - T^2 f_{RT^2} + (3p^2 + \rho^2) f_{T^2T^2} + (3p^2 + \rho^2) f_{T^2T^2} \right\} $$
$$ + a^3 \tau_{,i} \left\{ f - R f_R - T^2 f_{T^2} + (3p^2 + \rho^2) f_{T^2} + p \right\} - \psi_{,i} = 0. \quad (34) $$

In the next section, we solve the above system of equations for various cases.

### 4 Conserved Quantities

In this section, we manipulate the system of PDEs (27) to (34) to obtain Noether symmetries $K = \tau \partial_{\tau} + \xi^i \partial_{\dot{q}^i}$. Equation (27) provides a trivial symmetry $I = \partial_t$ for any $f(R, T^2)$ model. However, it is complicated to derive a non-trivial solution without taking any particular $f(R, T^2)$ model. In the following, we take different models to reduce complexity of the system.
4.1 \( f(R) \) Gravity

This case helps us to re-examine the usual \( f(R) \) theory. The last expression in Eq. (27) implies that either \( \xi^1_R = 0 \) or \( f_{RR} = 0 \). If we consider \( f_{RR} = 0 \) and \( \xi^1_R \neq 0 \), then Eq. (32) yields \( f_R = 0 \). Hence, Eq. (34) with \( \tau = \tau(t) \) and \( \psi = \psi(t) \) gives

\[
3a^2\xi^1(f + p) + a^3\tau_t(f + p) + a^3\xi^1\rho - \psi_t = 0.
\] (35)

Differentiating this with respect to \( R \), we have \( \xi^1_{RR} = 0 \), that yields contradiction to the fact that \( \xi^1_R \neq 0 \). So, our supposition is wrong and hence \( f_{RR} \neq 0 \) for \( f(R) \) theory. For the sake of convenience, we consider \( f(R, T^2) = f_0 R^2 \) [31] which has already been studied in the literature [27], [32]-[34]. By solving Eqs. (27) to (34), we obtain

\[
\xi^1 = \frac{12c_2a^2f_0 - 2c_1c_3t - 9c_1c_5f_0}{18c_1a_0}, \quad \xi^2 = \left( \frac{2c_3t}{9a^2f_0} - \frac{2c_2}{c_1} + \frac{c_5}{a^2} \right) R, \quad \xi^3 = \rho = 0.
\] (36)

The Noether symmetry generators can be expressed as

\[
K_1 = -\frac{t}{9a_0} \frac{\partial}{\partial a} + \frac{2tR}{9a^2f_0} \frac{\partial}{\partial R}, \quad K_2 = -\frac{1}{2a} \frac{\partial}{\partial a} + \frac{R}{a^2} \frac{\partial}{\partial R}, \quad K_3 = \frac{\partial}{\partial t}. \quad (38)
\]

Using Eq. (26), the conserved quantities can be found as

\[
I_1 = 9a^2\hat{a}^2\sqrt{R}f_0 + \frac{9a^2}{2} \hat{R}f_0 - \frac{a^3}{2}R^{3/2}f_0, \quad I_2 = \frac{a\hat{R}t}{2\sqrt{R}} + \hat{a}\sqrt{R}t - a\sqrt{R}, \quad I_3 = \frac{9a\hat{R}t}{8\sqrt{R}} + \frac{9}{2} \hat{a}\sqrt{R}f_0. \quad (40)
\]

These conserved quantities are the key aspects to determine the cosmological solutions.

Now, we provide an important solution corresponding to the last conserved quantity \( I_3 \) that can be written as

\[
\hat{a} - \frac{2}{9\sqrt{R}f_0} \left( I_3 - \frac{9a\hat{R}}{4\sqrt{R}}f_0 \right) = 0. \quad (41)
\]
We can obtain a numerical solution by assuming $I_3 = 2f_0$ with some suitable initial conditions. The exact solution of Eq. (41) is of the following form

$$a = R^{-1/2} \left( a_0 + \frac{2I_3 t}{9f_0} \right). \tag{42}$$

Using the value of $R$ from Eq. (9), the above equation provides the exact solution for the cosmic scale factor

$$a(t) = \sqrt{c_0 + c_1 t + c_3 t^2 + c_3 t^3 + c_4 t^4}, \tag{43}$$

where $c_i$ are the combinations of initial conditions that help to discuss cosmic evolution. If $c_4 \neq 0$, then it gives a power-law inflation while the radiation-dominated era is achieved for the linear term in $c_1$ [31].

### 4.2 $f(R, T^2)$ Gravity

Here, we use curvature-matter coupling model to examine the Noether symmetry technique in $f(R, T^2)$ theory. We consider a specific type of a generic function both minimal as well as non-minimal coupling between curvature and matter. We analyze the cosmic evolution for the dust fluid.

#### 4.2.1 Minimal Coupling Models

We take two minimal coupling models to find exact solutions. The first minimal coupling model is given by [17]

$$f(R, T^2) = \alpha R^n + \beta (T^2)^m, \quad n, m \neq 0, 1, \quad \tag{44}$$

where $\alpha, \beta, m$ and $n$ are constants. We consider $\alpha = 1 = \beta$ for the sake of convenience. Solving the system (27)-(33), we have

$$\xi^1 = \frac{ac_2 + \frac{c_1}{a}}{a}$$
$$\xi^2 = \frac{R^{2-n}c_3}{a} - \frac{R(3a^2 c_2 + c_1)}{(n-1)a^2}, \quad \tag{45}$$
$$\xi^3 = -T^2 \frac{3a^2 c_2 + 3c_1}{(m-1)a^2}, \quad \tau = \psi = 0. \quad \tag{46}$$

Using Eqs. (45)-(46) in (34), we obtain

$$\rho = \left\{ R c_3 (T^2)^{1-m} \tan^{-1} \left( \frac{c_2 a}{\sqrt{c_1 c_2}} \right) n (n-1) + R^n (T^2)^{1-m} \right\}$$
\times n\sqrt{c_1c_2} \left(2\ln(a) - \ln(c_1 + a^2c_2)\right) - 3T^2\ln(a)
\times \sqrt{c_1c_2} \left(R^n(T^2)^{-m} + 1\right) \right)^{\frac{1}{2}}.

(47)

The symmetry generators take the following form

\begin{align*}
K_1 &= \frac{1}{a} \frac{\partial}{\partial a} - \frac{R}{(n-1)a^2} \frac{\partial}{\partial R} - 3T^2 \frac{\partial}{\partial T^2}, \\
K_2 &= a \frac{\partial}{\partial a} - \frac{3R}{(n-1)} \frac{\partial}{\partial R} - \frac{3T^2}{(m-1)} \frac{\partial}{\partial T^2}, \quad K_3 = \frac{R^{2-n}}{a} \frac{\partial}{\partial R}.
\end{align*}

(48)

(49)

The corresponding first integrals become

\begin{align*}
I_1 &= -6n \left(a\dot{R}R^{n-2}(n-1) + \dot{a}R^{n-1}\right), \quad I_2 = -6a\dot{a}(n-1), \\
I_3 &= -6a^2n \left(a\dot{R}R^{n-2}(n-2) - \dot{a}R^n\right).
\end{align*}

(50)

(51)

In order to establish cosmological analysis of the constructed model experiencing minimal coupling with matter, we evaluate second conserved quantity of Eq.(50) as

\[ a = \left(a_0 - \frac{I_2t}{3n(n-1)}\right)^{\frac{1}{2}}. \]

(52)

For the sake of simplicity, we consider \( n = 2 \). Substituting the value of second conserved quantity in the above equation, we obtain exact solution of the scale factor as

\[ a = \sqrt{c_4(c_4c_5 - t)} \]

(53)

To investigate this solution, we discuss the behavior of some significant cosmological parameters, i.e., Hubble, deceleration and equation of state (EoS) parameters which play a crucial role in the study of current accelerated expansion of the universe. The Hubble parameter \((H)\) measures the rate of expansion whereas the value of deceleration parameter \((q)\) determines accelerated \((q < 0)\), decelerated \((q > 0)\), or constant expansion \((q = 0)\) of the universe. For the isotropic universe model, the Hubble and deceleration parameters are defined as

\[ H = \frac{\dot{a}}{a}, \quad q = -\frac{\ddot{H}}{H^2} - 1. \]
For the explicit form of \( f(R, T^2) \) model and scale factor, the corresponding Hubble and deceleration parameters turn out to be \( H = -\frac{1}{2(c_1 c_2-t)} \) and \( q = 1 \). The graphical behavior of the scale factor and Hubble parameter is shown in Figure 1. The left plot indicates that the universe experiences accelerated expansion as the scale factor grows continuously while the right plot identifies decreasing rate of expansion. The positivity of deceleration parameter ensures the decelerating universe. Furthermore, the first integral provides a solution of the form

\[
a = R^{1-n} \left( a_0 - \frac{I_1 t}{6n} \right). \tag{54}
\]

The scale factor is physically viable due to its increasing behavior, i.e., it describes cosmic accelerated expansion as shown in Figure 2. When \( n = 3/2 \), this differential equation gives the identical solution in \( f(R) \) theory and GR is recovered for \( n = 1 \).

The second minimal coupling model is taken as

\[ f(R, T^2) = R + \eta(T^2)^n, \quad n \neq 0, \]
where $\eta$ is a constant $^{[26]}$. The simultaneous solutions of Eqs. $(27)-(33)$ yield
\[
\xi^1 = \frac{c_1}{\sqrt{a}}, \quad \xi^3 = -\frac{6T^2c_1}{(n-1)a^3}, \quad \xi^2 = \tau = \psi = 0.
\] (55)
Substituting these values in Eq. $(34)$, we have
\[
\rho = \sqrt{\frac{3T^2 \ln(a)}{n}}.
\] (56)
The corresponding generators of the Noether symmetry and conserved quantities become
\[
K_1 = \frac{1}{\sqrt{a}} \frac{\partial}{\partial a} - \frac{6T^2}{(n-1)a^3} \frac{\partial}{\partial T^2},
\] (57)
\[
I_1 = -12a\sqrt{a},
\] (58)
respectively. Using Eq. $(58)$, we formulate exact solution of the scale factor as
\[
a = \frac{\{(c^3e^{c_2^2} + t)(e^{c_2^2})^\frac{3}{2}\}}{(e^{c_2^2})^2}.
\] (59)
For this cosmological solution, Hubble and deceleration parameters become
\[
H = \frac{2}{a(c^3e^{c_2^2} + t)} \quad \text{and} \quad q = \frac{1}{2},
\] respectively. The EoS parameter $(\omega = \frac{\rho_{\text{eff}}}{p_{\text{eff}}})$
characterizes the universe into different eras and also distinguishes DE era into distinct phases like \( \omega = -1 \) describes cosmological constant, while \(-1 < \omega \leq 1/3\) and \( \omega < -1 \) correspond to quintessence and phantom phases, respectively. In Figure 3, the left plot indicates that the increasing behavior of the scale factor describes accelerated expansion whereas the right plot represents that Hubble parameter measures decreasing rate of cosmic expansion. The positivity of deceleration parameter defines the decelerating universe. Figure 4 shows that the universe possesses an elegant exit from matter dominated era to phantom phase which leads to quintessence phase with the passage of time.

4.2.2 Non-Minimal Coupling Model

This case explores the dynamical behavior with non-minimal coupling model given as

\[
f(R, T^2) = f_0 R^n (T^2)^m, \quad n, m \neq 0, 1,
\]

where \( f_0 \) is a real constant [17]. Solving Eqs. (27)-(33), we have

\[
\xi^1 = ac_1, \quad \xi^2 = \frac{1}{a} \left\{ -\frac{3Rac_1}{(n + m - 1)} + R^{2-n} (T^2)^m c_2 \right\}, \quad \tau = 0 = \psi, (60)
\]
\[ \xi^3 = \frac{T^2}{(n+m-1)(m-1)a^2R} \left\{ ac_2 (n+m-1) n R^{2-n} (T^2)^{-m} \right. \]
\[ + \left. 3a^2 (m-1) R c_1 \right\}. \quad (61) \]

Putting these values in Eq. (34), we obtain
\[ \rho = \left\{ -\frac{1}{m(m-1)ac_1} \left\{ 3T^2 ac_1 (m-1) \ln(a) \right. \right. \]
\[ + \left. \left. (T^2)^{1-m} R^{1-n} c_2 (1-n-m) \right\} \right\}^{\frac{1}{2}}. \quad (62) \]

The corresponding generators of Noether symmetry can be written as
\[ K_1 = a \frac{\partial}{\partial a} - \frac{3R}{(n+m-1) \partial R} - \frac{3T^2}{(n+m-1) \partial T^2}, \quad (63) \]
\[ K_2 = \frac{R^{2-n}}{(T^2)^{m} \partial R} + \frac{nR^{1-n}(T^2)^{1-m}}{(1-m) \partial T^2}, \quad (64) \]
and the conserved quantities turn out to be
\[ I_1 = \frac{1}{(n+m-1)} \left\{ 18a^2 \dot{a} R^{m-1} (T^2)^m (n^2 - n + m - 1) f_0 \right\} \]
Figure 5: The evolutionary behavior of the scale factor for $f_0 R^m(T^2)^m$.

$$+ 6a^3 \dot{R} R^{n-2}(T^2)^m n(n-1)f_0 - 6a^3 R^{n-1}(T^2)^m \dot{T}^2 f_0$$

$$- 12a^2 \dot{a} n R^{n-1}(T^2)^m f_0,$$

$$I_2 = \frac{6 \dot{m} n^2}{(m-1)} f_0 - 6a \dot{a} n (n-1) f_0.$$  \hspace{1cm} (65)

Equation (66) can be rearranged as

$$a \left\{ \frac{6mn^2 f_0}{(m-1)} - 6an(n-1)f_0 \right\} - I_2 = 0.$$  \hspace{1cm} (67)

Using the initial condition $a(0) = 0$ with $I_2 = 6f_0$, a numerical solution is obtained. Figure 5 shows that the scale factor describes the cosmic evolution for appropriate values of $m$ and $n$. Equation (67) gives analytic solution of the form

$$a = \left\{ a_0 + \frac{I_2 t(m-1)}{3n(n+m-1)f_0} \right\}^{\frac{1}{2}}.$$  \hspace{1cm} (68)

When $m = 0$, this equation provides the same solution as given for the first minimal coupling model and reduces to GR if $m, n = 0$. However, some interesting solutions can be established by taking suitable values of $n$ and $m.$
5 Concluding Remarks

Modified gravitational theories are considered as the most fascinating and promising approach to investigate the current cosmic expansion due to the additional higher-order curvature terms. In this paper, we have discussed Noether symmetries of $f(R, T^2)$ theory for flat FRW universe model. Such symmetries not only manage solutions of the dynamical system but also their presence can provide some viable conditions so that cosmological models can be selected according to current observations [35]. In particular, the characteristics of mysterious energy associated with Noether symmetries can be identified [36]-[39]. The Lagrangian multipliers are used to minimize the dynamical system that ultimately help to evaluate analytical solutions. We have formulated the Lagrangian of $f(R, T^2)$ gravity and evaluated the conserved quantities to investigate the exact solutions of modified equations of motion. The analytic solutions of Noether equations have been studied for minimal and non-minimal coupling models of this theory by assuming dust fluid just for the sake of simplicity. We summarize the results obtained as follows.

• Firstly, we have discussed exact solutions of Noether equations for $f(R)$ model. The $f(R)$ theory is recovered for $f_{RR} \neq 0$. We have considered $f(R, T^2) = f_0 R^{3/2}$ with $f_0 \neq 0$ and derived Noether symmetries that are consistent with those already present in the literature [31, 35]. We have applied the conserved quantities to analyze numerical as well as exact solutions for cosmic evolution. We have then formulated a numeric solution after applying some suitable initial condition with appropriate values of the parameters. The scale factor indicates that the universe is expanding with an accelerating phase (Figure 1). The analytical approach provides an exact solution for $f(R)$ gravity model [32, 37].

• There has been a significant literature [40]-[42] that indicates various cosmological applications corresponding to this cosmological model. Newtonian gravity is the weak-field limit of general relativity and Modified Newtonian Dynamics (MOND) is the weak-field limit of a particular extended theory of gravity. It has been found that Noether symmetry approach yields a conserved quantity coherent with the relativistic MONDian extension. The MOND regime can be fully recovered as the
weak-field limit of a particular theory of gravity formulated in the metric approach. This is possible when Milgrom’s acceleration constant is taken as a fundamental quantity which couples to the theory in a very consistent manner. The power-law $f(R)$ gravity model demonstrates the existence of a new fundamental gravitational radius. This radius plays an analog role for weak gravitational field at galactic scales and using the new radius, $f(R)$ gravity provides a theoretical foundation for rotation curve of galaxies as well as empirical baryonic Tully-Fisher relation. In particular, for $f(R) = R^{3/2}$, the MOND acceleration regime is recovered.

- In cosmology, perfect fluid can represent the effective behavior of Hubble flow ranging from inflation to dark energy epochs. Therefore, compatibility of perfect fluid solutions with modified or extended theories of gravity is a crucial issue to be investigated. The $n$-dimensional generalized Robertson-Walker spacetime with divergence-free conformal curvature tensor exhibits a perfect fluid stress-energy tensor for any $f(R)$ gravity model. Furthermore, a conformally flat generalized Robertson-Walker spacetime is still a perfect fluid in both $f(R)$ and quadratic gravity.

- Secondly, we have studied minimal and non-minimal curvature-matter coupling models of this theory. We have taken two minimal and one non-minimal models. For the first minimal model, $f(R, T^2) = \alpha R^n + \beta (T^2)^m$, $n, m \neq 0, 1$, we have obtained three conserved quantities out of which two give a new framework of analytic solutions. In this case, we have found cosmological solution of the scale factor whose physical interpretation is established through cosmological parameters like Hubble, deceleration and EoS parameters. The graphical analysis of scale factor and rate of expansion is found to be increasing. The deceleration parameter remains negative. The EoS parameter characterize phantom phase which leads to quintessence phase with the passage of time. For the second minimal model $f(R, T^2) = R + \eta (T^2)^n$ with $n \neq 0$, we have different solutions using conserved quantities for different values of $n$. For the non-minimal model $f(R, T^2) = f_0 R^n (T^2)^m$, we have found two generators. It is clear that the scale factor is rapidly increasing which indicates the cosmic accelerated expansion for all cases (Figures 1-5).

- It is worthwhile to mention here that the results of this theory are
compatible with each other. In the first minimal coupling model, the solution of scale factor \( a = R^{-1/2} \left( a_0 - \frac{I_{1t}}{9} \right) \) for \( n = 3/2 \) which is similar to that obtained in \( f(R) \) theory (42). Similarly, GR is recovered when \( n = 1 \). For non-minimal coupling model, we have the solution of scale factor \( a = \left( a_0 + \frac{I_{2t}(m-1)}{3m(n+m-1)f_0} \right)^{\frac{1}{2}} \) which reduces to \( a = \left( a_0 - \frac{I_{2t}}{3n(n-1)} \right)^{\frac{1}{2}} \) for \( m = 0 \). It is identical to the one discussed in (52). Substituting \( m, n = 0 \) in Eq. (68) the scale factor of GR is recovered.

We would like to mention here that our results reduce to some other models of \( f(R, T^2) \) gravity for different values of parameters.

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