Eigenmodes of a rotating spherical Stokes flow with a radial stratification and radial buoyancy

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Abstract. The eigenmodes of a rotating spherical Stokes flow with a radial stratification are computed. The radial buoyancy is taken into account to fit to a geophysical model. We study the stability of the eigenmodes in the parameters space. Indeed, the flow depends on the Froude and the Schmidt numbers which describe respectively the buoyancy and the mass diffusivity.

1. Introduction

The Couette flows allow to study the transition to the turbulence which occurs in shear flows. The geometrical shape may be planar, circular or spherical. For geophysical problems, the Taylor-Couette model allows to study the mixing and the flow in the ocean (see [1]). Indeed, the equatorial area can be approximated to a rotating cylinder. In a previous study [2], the authors have developed an effective numerical method to address a parametric study of the primary instability of a radially stratified Taylor-Couette. Depending on the Froude number and the Schmidt number, i.e. the gravity strength and the massic diffusivity, three unstable modes were identified [2]. Nevertheless, to fit to a geophysical model, the spherical shape is the most natural choice. So, in the present work, we consider a spherical flow which improves the geometric representation into the geophysical model. Although the spherical Couette flow has been investigated (for instance in [3, 4, 5, 6, 7]), the buoyancy in the radial direction has never been taken into account because the laboratory gravity imposes an axial stratification.

The present work considers the flow induced in a spherical shell by fixing the outer sphere and rotating the inner one. The gap size $d$ between the inner and the outer spheres is defined using the radius ratio $\tilde{\eta} = R_1/R_2$, where $R_1$ and $R_2$ are respectively the inner and the outer radii (see fig. 1). In our study, the fluid undergoes a radial buoyancy and is supposed to be radially stratified.

In order to reduce the number of parameters describing the flow configuration, the governing equations are nondimensionalized. To do so in a standard way, the following characteristics of the spherical flow are chosen as references:

$$\rho_{ref} = \rho_{min}, \quad L_{ref} = d, \quad V_{ref} = 2d\sqrt{-A\Omega_1}$$

(1)

$\rho_{min}$ is the minimal density taken initially at the boundary of the outer sphere. The reference velocity $V_{ref}$ is obtained from the laminar solution of the circular Taylor-Couette flow in order to keep the same Taylor number definition as in the circular Taylor-Couette flow. Indeed, for a
small gap length, the equatorial region of a spherical Couette flow may be modelized as a circular flow. This modelization was supported for geophysical studies by several authors [8, 9, 1, ]. So, we have

$$A = \frac{(\Omega_2 R_2^2 - \Omega_1 R_1^2)/(R_2^2 - R_1^2)}{\Omega_1}$$

$$\Omega_1$$ and $$\Omega_2$$ stand for the angular velocity of the inner and the outer sphere respectively.

The Boussinesq approximation consists in neglecting the variation of the inertial term in the left hand side (LHS) of the equation (2). Then, the remaining effect of the density variation is taken into account by the buoyancy term in the right hand side (RHS) of the equation (2), namely $$-\rho \frac{c_r}{Fr}$$. This approach makes the set of equations (2–4) linear with respect to the relative density. The set of equations is thus written as:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \left(\frac{1}{\sqrt{T}}\Delta u - \frac{\rho}{Fr} c_r \mathbf{e}_r \right)$$

(2)

$$\nabla \cdot \mathbf{u} = 0$$

(3)

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \frac{1}{S_c \sqrt{T}} \Delta \rho$$

(4)

where $$\mathbf{u}$$, $$p$$ and $$\rho$$ are respectively the flow velocity, the pressure and the density. The three parameters of the previous equations are $$S_c = \frac{\kappa}{\nu}$$ the Schmidt number, $$T = \left(\frac{dV_{ref}}{\eta} \right)^2$$ the Taylor number and $$F_r = \frac{\nu^2}{\eta \kappa}$$ the Froude number. $$\nu$$ is the kinematic viscosity, constant in accordance with the Boussinesq approximation, and $$\kappa$$ is the diffusive coefficient of $$\rho$$. Along the radial direction, the kinematic boundary conditions are:

$$\mathbf{u} \left( r = \frac{\hat{\eta}}{1 - \hat{\eta}} \right) = \frac{\hat{\eta}}{2} \sqrt{\frac{1 + \hat{\eta}}{\hat{\eta}^2 - \hat{\bar{\mu}} \mid (1 - \hat{\eta})}} \sin(\theta) e_\phi$$

(5)

$$\mathbf{u} \left( r = \frac{1}{1 - \hat{\eta}} \right) = \frac{\hat{\bar{\mu}}}{2} \sqrt{\frac{1 + \hat{\eta}}{\hat{\bar{\mu}}^2 - \hat{\eta} \mid (1 - \hat{\eta})}} \sin(\theta) e_\phi$$

(6)

in the spherical coordinates where $$\hat{\bar{\mu}} = \Omega_2 / \Omega_1$$ is the ratio between the angular velocity of the outer and inner cylinders respectively. To keep a stratification, the following conditions are imposed to the density:

$$\rho \left( r = \frac{\hat{\eta}}{1 - \hat{\eta}} \right) = \rho_{\text{max}}, \quad \rho \left( r = \frac{1}{1 - \hat{\eta}} \right) = 1$$

(7)

In the following, we set $$\rho_{\text{max}} = 1.015$. 

Figure 1. Spherical flow configuration. The inner sphere rotates and the outer one is steady. The fluid domain states in the gap $$d$$ between the inner and the outer spheres.
2. Stokes flow

The basic flow of a spherical Couette system is generally considered to be the axisymmetric 0-vortex flow as observed by several authors (see [5, 10, 6]). The 0-vortex flow has two contra-rotating vortices which correspond to a circulation between the pole and the equator and the azimuthal velocity is close to the Stokes one [6]. The Taylor-Görtler vortices appear after a pinching phenomenon of the flow [3, 4]. Then, the 0-vortex flow becomes the 1-vortex flow which is the last axisymmetric flow before the transition to non-axisymmetric flows [7]. In the following, we consider the Stokes flow as a reference to compute the eigenmodes to explain the 0-vortex flow, the pinching phenomenon and the 1-vortex flow onset. So, our model differs only in terms of geometry from the cylindrical Couette system.

The Stokes flow supposes that only the azimuthal velocity \( W \) is non-zero and that the stratification is radial. Then, the set of equations (2), (3) and (4) provides the following axisymmetric solution:

\[
W(r, \theta) = \left( \frac{K_1}{r^2} + K_2 r \right) \sin(\theta), \quad \rho_0(r) = \left( \frac{C_1}{r} + C_2 \right)
\] (8)

The boundary conditions (5–7) provide the constants:

\[
K_1 = \frac{\hat{\eta}^3(\hat{\mu} - 1)\sqrt{1 - \hat{\eta}^2}}{2(1 - \hat{\eta})(\hat{\eta}^3 - 1)\sqrt{\hat{\eta}^2 - \hat{\mu}}}, \quad K_2 = \frac{(\hat{\eta}^3 - \hat{\mu})\sqrt{1 - \hat{\eta}^2}}{2(\hat{\eta}^3 - 1)\sqrt{\hat{\eta}^2 - \hat{\mu}}},
\]

\[
C_1 = \frac{\hat{\eta}(\rho_{\text{max}} - 1)}{(1 - \hat{\eta})^2}, \quad C_2 = \frac{\hat{\eta}\rho_{\text{max}} - 1}{\hat{\eta} - 1}
\] (9)

3. Azimuthal decomposition and U(1)-representation

In order to withdraw the singularity which appears on the axis of the sphere in the Navier-Stokes equations (2) written in the spherical coordinates system, we use the U(1)-representation defined as in [11]:

\[
v_z = u \cos(\theta) - v \sin(\theta), \quad v_{\pm} = u \sin(\theta) + v \cos(\theta) \pm iw
\] (11)

where \((u, v, w)\) are the spherical components of the velocity perturbations \(\vec{v}\). More, to have a bidimensional problem, the velocity and the density are decomposed in the azimuthal direction for obvious geometrical reasons:

\[
\vec{v} = \sum_{m \in \mathbb{Z}} \vec{v}_m(r, \theta)e^{im\phi}
\] (12)

So, the linear operators of the Navier-Stokes equations (2) have to be written for a given azimuthal mode \(m\) in the U(1)-representation. The array representation of the gradient is:

\[
G_m = \begin{bmatrix} \cos(\theta) \frac{\partial}{\partial r} & - \sin(\theta) \frac{\partial}{\partial \theta} & \sin(\theta) \frac{\partial}{\partial r} + \cos(\theta) \frac{\partial}{\partial \theta} \pm \frac{m}{r \sin(\theta)} \end{bmatrix}'
\] (13)

where each element refers respectively to the components \(v_z\) and \(v_{\pm}\). Similarly, the divergence is written as:

\[
D_m = \begin{bmatrix} \cos(\theta) \frac{\partial}{\partial r} & - \sin(\theta) \frac{\partial}{\partial \theta} & \frac{1}{2} \left( \sin(\theta) \frac{\partial}{\partial r} + \cos(\theta) \frac{\partial}{\partial \theta} \pm \frac{m}{r \sin(\theta)} \right) \end{bmatrix}
\] (14)

Finally, the laplacian operator is given by the following expression:

\[
\Delta_m = [\Delta_{z,m}, \Delta_{\pm,m}]
\] (15)
where each element stands for:

\[
\Delta_{z,m} = \frac{\partial^2}{\partial r^2} + \frac{2\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2} + \cot(\theta)\frac{\partial}{\partial r} - \left(\frac{m}{r\sin(\theta)}\right)^2
\]

(16)

\[
\Delta_{\pm,m} = \frac{\partial^2}{\partial r^2} + \frac{2\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2} + \cot(\theta)\frac{\partial}{\partial r} - \left(\frac{m \pm 1}{r\sin(\theta)}\right)^2
\]

(17)

By introducing the sum of the Stokes solution (8) and the perturbations (11) in the set of equations (2), (3) and (4) and after a linearization and the azimuthal decomposition (12), the following system is obtained for each azimuthal mode \(m\):

\[
\frac{\partial U_m}{\partial t} = (I - G_m \times (D_m \times G_m)^{-1} \times D_m) \times L_m U_m
\]

(18)

where we define \(U_m = [v_z, v_+, v_-, \rho]'_m\). \(I\) stands for the identity operator and \(L_m\) is the operator which includes the Laplacian and the linearized terms. The linearized operator is written in the array representation as following:

\[
L_m = \begin{bmatrix}
\Delta_{z,m} / \sqrt{T} - \hat{W}_m & 0 & 0 & -\frac{\cos(\theta)}{r_T} \\
L_{++} & \Delta_{z,m} / \sqrt{T} + L_{++} - \hat{W}_{m+2} & L_{++} & -\frac{\sin(\theta)}{r_T} \\
-L_{++} & -L_{++} & \Delta_{z,m} / \sqrt{T} - L_{++} - \hat{W}_{m-2} & -\frac{\sin(\theta)}{r_T} \\
-C_1 \frac{\cos(\theta)}{r^2} & -C_1 \frac{\sin(\theta)}{2r^2} & -C_1 \frac{\sin(\theta)}{2r^2} & \Delta_{z,m} / \sqrt{T} - \hat{W}_m
\end{bmatrix}
\]

(19)

where

\[
\hat{W}_m = \frac{imW}{r\sin(\theta)}
\]

(20)

\[
L_{++} = \frac{3iK_1 \sin(\theta) \cos(\theta)}{r^3}
\]

(21)

\[
L_{+-} = \frac{3iK_1 \sin^2(\theta)}{2r^3}
\]

(22)

The pressure, which appears in the set of equations (2–3), is now given by:

\[
P_m = (D_m \times G_m)^{-1} \times D_m \times L_m U_m
\]

(23)

To compute the operators defined in the equations (13, 14 and 19), we use the centered finite difference scheme for the spatial derivations taken at the second order. We consider rigid boundary conditions for the perturbations (Dirichlet condition). The following consists in calculating the eigenfields of the right hand term of the equation (18) and to extract the most unstable one which corresponds to the eigenvalue with the greatest real part. The calculation algorithm uses the LAPACK’s libraries [12] to provide the eigenvalues and the corresponding eigenfields. The computational cost of this operation is equivalent to only one time step of a Navier-Stokes solver [2].

4. Threshold and most unstable mode

The 0-vortex corresponds to the onset of the eigenmode shown by the figure (2-a and b) at a Taylor number which tends to zero. Indeed, the angular velocity centrifuges the fluid from the pole as it would happen between two rotating disks. The stratification does not change the mode (fig. 2-b). Nevertheless, at higher Taylor numbers, the most unstable eigenmode changes and
becomes as shown in figure (2-c and d). The vortices appear in the equatorial region whereas the polar vortices vanish. We find a threshold where the equatorial vortices mode becomes the most unstable mode. This explains the pinching phenomenon and next the 1-vortex onset.

In a homogeneous fluid, the threshold of the Taylor mode instability increases with the gap size (fig. 3-a). Indeed, the instability onset is due to the shear flow rate which grows when the gap size decreases, i.e. when the radius ratio \( \hat{\eta} \) tends to 1. For the considered cases (0.5 \( \leq \hat{\eta} \leq 0.9 \)), we notice that the non-axisymmetric modes \( (m > 0) \) are stable at the threshold of the axisymmetric Taylor mode, i.e. that the maximum of the real part of the eigenvalues is negative when \( m \neq 0 \). The 1-vortex threshold is found to be at \( Ta = 6021 \) by Wang and Li [6] for \( \hat{\eta} = 0.8475 \). Our threshold value is lower but it is a minimum value because the energy absorbed by the 0-vortex flow is not taken into account in our computation. Although the 0-vortex flow does not modify significantly the Stokes azimuthal velocity [6], the 0-vortex flow may stabilize the system in comparison of our basic Stokes flow.

When we compare the threshold for several Froude numbers, it shows that, as expected [2], the buoyancy stabilizes the flow (fig. 3-b). More, the buoyancy destroys the steady state and the angular frequency of the oscillating mode grows when the buoyancy strength grows (i.e. towards small Froude numbers \( Fr < 1 \), fig. 3-c). Indeed, the density and the velocity fields have
not the same shape (fig. 2-c and d) and the competition between the radial velocity and the
density variation makes oscillate the flow. This phenomenon is similar to the one observed in
a circular Taylor-Couette with a radial stratification [2] even if our simple model provides quite
low value for the threshold in comparison.

Although the Stokes flow becomes unstable for non-zero Taylor number, the linear analysis
of stability of this flow is sufficient to predict the onset of the Taylor-Görtler vortices around the
equatorial line. Even if the value of the threshold is too low for the onset of the Taylor-Görtler
vortices, the linear analysis of the stability of the spherical Stokes flow allows to capture the
main effects of the buoyancy in the first stage of the flow transition. So, this result shows that
the cylindrical model is relevant to describe the mixing around the equatorial line in supercritical
regimes (i.e. not in the 0-vortex regime) even if the radial buoyancy is taken into account for \( \eta \)
close to 1.

References

[1] Ermanyuk E and Flór J B 2005 Taylor-Couette flow in a two-layer stratified fluid: instabilities and mixing
Dyn. Atmos. Oceans 40 57–69
[2] Jenny M and Nsom B 2007 Primary instability of a Taylor-Couette flow with a radial stratification and radial
buoyancy Phys. Fluids 19
[3] Marcus P and Tuckerman L 1987 Simulation of flow between two concentric rotating spheres. Part 1. Steady
states. J. Fluid Mech. 185 1–30
[4] Marcus P and Tuckerman L 1987 Simulation of flow between two concentric rotating spheres. Part 2.
Transitions. J. Fluid Mech. 185 31–66
[5] Mamun C and Tuckerman L 1995 Asymmetry and hopf bifurcation in spherical Couette flow Phys. Fluids 7
80–91
[6] Wang H and Li K 2004 Numerical simulation of spherical couette flow Computers & Mathematics with
Applications 48 109–116
[7] Hollerbach R, Junk M and Egbers C 2006 Non-axisymmetric instabilities in basic state spherical Couette
flow Fluid Dynamics Research 38 257–273
[8] Boubnov B, Gledzer E, Hopfinger E and Orlandi P 1996 Layer formation and transitions in stratified circular
Couette flow Dyn. Atmos. Oceans 23 139–153
[9] Hua B, Gentil S and P Orlandi 1997 First transition in circular couette flow with axial stratification Phys.
Fluids A 9 365
[10] Sha W, Nakabayashi K and Ueda H 1998 An accurate second-order approximation factorization method for
time-dependent incompressible Navier-Stokes equations in spherical polar coordinates J. Comput. Phys.
142 47–66
[11] Orszag S and Patera A 1983 Secondary instability of wall-bounded shear flows J. Fluid Mech. 128 347–385
[12] Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J, Croz J D, Green-
baum A, Hammarling S, McKenney A and Sorensen D 1999 LAPACK User’s Guide 3rd ed
(http://www.netlib.org/lapack/lug/lapack_lug.html)