Computations of HOMFLY homology

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Abstract

We give an algorithm to compute the reduced HOMFLY homology for knots. We determine the homologies for 695 prime knots, including all prime knots with up to 10 crossings, and all prime knots with 11 crossings and braid length up to 13.

Contents

1 Introduction 1
2 Preliminaries 4
  2.1 Rasmussen’s construction 4
  2.2 MFW bounds and dualities 6
3 Algorithm 7
  3.1 Free variables of the edge ring 8
  3.2 Reinterpretation as cube complexes 9
  3.3 Slicing by $q$-degrees 11
  3.4 Exclusion of variables 13
  3.5 Overall procedure 14
4 Computations 15
5 Future Prospects 16
  5.1 On spectral sequences 16
  5.2 On homologies for general knot diagrams 17

1 Introduction

The HOMFLY polynomial [Fre+85; Jon87; PT87] $P(L)(q, a)$ is an invariant of oriented links in $S^3$, defined by the skein relation

$$a^{-1}P(\overline{\gamma}) - aP(\gamma) = (q^{-1} - q)P(\gamma),$$

where $\gamma$ is a knot diagram and $\overline{\gamma}$ is its mirror image.
and the normalization $P(\text{trivial knot}) = 1$. The invariant specializes to the Alexander polynomial $\Delta(L)(q) = P(L)(q,1)$ and to the $sl(N)$ polynomials $P_N(L)(q) = P(L)(q,q^N)$ for $N > 0$. In particular the $sl(2)$ polynomial is the Jones polynomial.

The (reduced) HOMFLY homology $\mathcal{H}^{\ast,\ast,\ast}(L)$ is a categorification of the HOMFLY polynomial, introduced by Khovanov and Rozansky [KR08]. It is a triply-graded homological link invariant whose graded Euler characteristic is the HOMFLY polynomial $P(L)$. That is to say, if we denote the Poincaré series of $\mathcal{H}(L)$ by

$$P(L)(q,a,t) = \sum_{i,j,k} q^i a^j t^{(k-j)/2} \dim \mathcal{H}^{i,j,k}(L),$$

then we have $P(L)(q,a,-1) = P(L)(q,a)$.

The HOMFLY homology is determined for some specific classes of links. For instance, Rasmussen showed that the homology of two-bridge knots are determined by the HOMFLY polynomial and the signature [Ras07]. Mellit and Hogancamp gave recursion formulas to compute the homology of torus links [HM19].

As for direct computations, although it is technically possible, naive implementation will easily lead to combinatorial explosion. There is a program written by Webster [Web]1, based on the construction using the Rouquier complexes. Rasmussen [Ras15] used this program together with the theoretical machinery he developed, and determined the homologies for all prime knots with up to 9 crossings. Further computations seem to be out of reach.

In this paper, we give a more effective algorithm to compute the reduced HOMFLY homology of knots, based on Rasmussen’s construction [Ras15]. By some reinterpretations of the chain complex, we show that the computation is achieved by a sequence of homology computations over $\mathbb{Q}$. Furthermore, several reduction methods are applicable, which are based on known facts on the HOMFLY homology.

With the implementation of the algorithm [NS] and the braid representations of prime knots provided by KnotInfo Database [LM] as the input data, we computed the homology for all prime knots with up to 10 crossings, and for all prime knots with 11 crossings and braid length up to 13 (the bolded parts in Table 1).

\begin{table}
| $n \setminus l$ | 8 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----------------|---|----|----|----|----|----|----|----|
| $\leq 8$        | 35|     |    |    |    |    |    |    |
| 9               | 43| 5  | 1  |    |    |    |    |    |
| 10              | 126| 31 | 2  | 6  |    |    |    |    |
| 11              | 237| 135| 74 | 81 | 14 | 9  | 2  |    |
\end{table}

Table 1: Targets of computation ($n$: crossing number, $l$: braid length)

1At the time of writing, the program given at the referred URL doesn’t seem to work.
Table 2: $H(5_{1})$ and $H(m(10_{132}))$

| $k \setminus j$ | 4   | 6   | $k \setminus j$ | 2   | 4   | 6   |
|------------------|-----|-----|------------------|-----|-----|-----|
| 4                | $q^{-4}$ |     | 2                | $q^{-2}$ | $q^{-4}$ |
| 0                | $1$ | $q^{-2}$ | 0                | $q^{-2}$ | $q^{-4}$ |
| $-4$             | $q^{4}$ | $q^{2}$ | $-2$             | $q^{2}$ | $1$   |
|                  |     |       | $-4$             | $q^{2}$ | $2$   | $q^{-2}$ |
|                  |     |       | $-8$             | $q^{4}$ | $q^{2}$ |

Observations on the results give direct proofs of some known facts. Table 2 shows the reduced HOMFLY homology of $5_{1}$ and $m(10_{132})^{2}$, where $m(\cdot)$ indicates mirroring of knots. It is known that $P(5_{1}) = P(m(10_{132}))$ but we obviously see that $H(5_{1}) \neq H(m(10_{132}))$. This implies that the reduced HOMFLY homology is strictly stronger than the HOMFLY polynomial (as verified in several contexts, for example, see [Kaw09]). This pair appears in [Bar02] as an example that the two knots has the same Jones polynomial but distinct Khovanov homology. $11n_{79}$ and $m(11n_{138})$ gives another such pair.

Another observation is that the Conway knot $11n_{34}$ and the Kinoshita–Terasaka knot $11n_{42}$ have identical reduced HOMFLY homology. This is proved in [MV08] using spectral sequence arguments. The result is interesting in that the two knots have distinct knot Floer homology [OS04], while it is conjectured that there is a spectral sequence from the reduced HOMFLY homology to the knot Floer homology [DGR06]. More observations are given in Section 4. The whole computation results can be found at [NS].

Our original motivation for developing an effective computer program was to investigate how the homology $\overline{H}(D)$ defined in [Ras15] varies among diagrams $D$ of the same link $L$. Although $\overline{H}(D)$ is defined for an arbitrary link diagrams, its invariance is only proved for braidlike Reidemeister moves, and hence the reduced HOMFLY homology $\overline{H}(L)$ is defined as $\overline{H}(D)$ for a braid closure diagram $D$ of $L$. In fact, it is known that $\overline{H}(D)$ is generally not invariant under the RI Ib move (Figure 1) [Abe17; Nak20].

Compared with the fact that the HOMFLY polynomial $P(L)$ can be computed from arbitrary link diagrams, a natural question arises:

**Question 1.1.** Is it possible to refine the construction of the HOMFLY homology so that it can be computed directly from arbitrary link diagrams?

Currently we haven’t obtained an answer to this question. We expect that observations on computational results will help us find a clue. More considerations on general link diagrams are given in the final section.

This paper is organized as follows. In Section 2, we review the definition of the reduced HOMFLY homology $\overline{H}(L)$ given by Rasmussen in [Ras15]. We also recall several important properties of $\overline{H}(L)$ which are essential in proving the finiteness of the algorithm. In Section 3 we describe the proposed algorithm,
and in Section 4 we discuss the computational results. Finally in Section 5 we discuss future prospects, regarding the conjecture of [DGR06] and the homology $\overline{H}(D)$ for general link diagrams.

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2 Preliminaries

2.1 Rasmussen’s construction

We review the definition of the reduced HOMFLY homology, following Rasmussen [Ras15]. Let $D$ be an oriented link diagram, and $G(D)$ the underlying projection regarded as an oriented 4-valent graph. For simplicity, we assume that $D$ is non-trivial (i.e. has at least one crossing) and that $D$ is connected (i.e. $G(D)$ is connected).

First we define the reduced edge ring $R(D)$ of $D$. Associate an indeterminate $x_e$ to each edge $e$ of $G(D)$, and let $R'(D)$ be the multivariate polynomial ring $\mathbb{Q}[x_e]$ generated by them. Endow a $\mathbb{Z}$-grading on $R'(D)$, called the $q$-grading, by declaring that every generator $x_e$ has degree 2. Define $\theta = \sum_e x_e \in R'(D)$, and for each crossing $p$ of $D$,

$$\rho_p = x_a + x_b - x_c - x_d \in R'(D),$$

where $a, b$ are outgoing edges at $p$ and $c, d$ are incoming edges at $p$ (see Figure 2). Let $I(D)$ be the ideal of $R'(D)$ generated by $\theta$ and $\rho_p$ for all crossings $p$ in $D$. $R(D)$ is defined by the quotient ring $R'(D)/I(D)$. Since $I(D)$ is a homogeneous ideal, the $q$-grading on $R'(D)$ is inherited to $R(D)$.

Remark 2.1. Since we only deal with the reduced theory, we simply write $R(D)$ instead of $R_r(D)$ as written in [Ras15].

\(^3\)https://academist-cf.com/fanclubs/121
Next we define a double chain complex $C_p(D)$ over $R = R(D)$. We start with some conventions and terminologies. The complex $C(D)$ has two homological gradings, called the horizontal grading and the vertical grading. It also has an internal grading, again called the $q$-grading, which is compatible with the grading on $R$. We denote by $C^{\ast,j,k}(D)$ the $R$-module in $C(D)$ on horizontal degree $j$ and vertical degree $k$, and by $C^{i,j,k}(D)$ the homogeneous $\mathbb{Q}$-subspace of $C(D)$ with $q$-degree $i$ in $C^{\ast,j,k}(D)$.

There are two differentials $d_H, d_V$ on $C(D)$, which are called the horizontal differential and the vertical differential respectively. The differentials $d_H, d_V$ are homogeneous with triple degrees $(2,2,0)$ and $(0,0,2)$, respectively. The complex $C^{\ast,j,k}(D)$ will be supported on even $j,k$.

The construction of $C(D)$ follows. For a crossing $p$ in $D$, we define a double complex $C_p(D)$ by the diagrams in Figure 2, depending on the sign of $p$. In the figure, the horizontal arrows indicate $d_H$ and the vertical arrows indicate $d_V$. Each arrow is labeled by a polynomial $f \in R$, which indicates that the map is given by the multiplication by $f$. The subscripts $a,b,c,d$ indicates the edges around $p$, as shown on the right of the figure. The notation $R\{i,j,k\}$ indicates a copy of $R$ on homological bidegree $(j,k)$ with $q$-grading shifted so that $1 \in R$ has $q$-degree $i$.

We immediately see that $d_H$ and $d_V$ commute in $C_p(D)$ and that the triple degrees of $d_H$ and $d_V$ are indeed $(2,2,0)$ and $(0,0,2)$, respectively. Define $C(D)$ as the tensor product (over $R$) of $C_p(D)$ taken among all crossings $p$ of $D$. 

Figure 2: The definition of $C_p(D)$
Now define the (unshifted) homology

\[ \hat{H}(D) = H(H(C(D), d_H), (d_V)_*). \]

This reads: first take the homology of \( C(D) \) by the horizontal differential \( d_H \), and then take homology of \( H(C(D), d_H) \) by the induced differential \( (d_V)_* \).

**Definition 2.2.** Let \( w \) be the writhe of \( D \), and let \( s \) be the number of Seifert circles of \( D \). The homology \( \overline{H}(D) \) of \( D \) is given by

\[ \overline{H}(D) = \hat{H}(D) \{ -w + s - 1, w + s - 1, w - s + 1 \}, \tag{2.1} \]

where here the notation \( \{ i, j, k \} \) indicates a shift of triple gradings.

**Theorem 1** ([KR08; Ras15]). Let \( L \) be a link. The isomorphism class of \( \overline{H}(D) \) as a triply graded vector space is independent of the choice of \( D \), provided that \( D \) is a closed braid diagram of \( L \).

**Definition 2.3.** For a link \( L \), we denote by \( \overline{H}(L) \) the homology group \( \overline{H}(D) \) for a closed braid diagram \( D \) of \( L \), and call it the reduced HOMFLY homology of \( L \).

**Remark 2.4.** Unless \( D \) is a closed braid diagram of \( L \), the homology \( \overline{H}(D) \) generally does not give the HOMFLY homology \( \overline{H}(L) \). Further considerations are given in Section 5.

**Definition 2.5.** The Poincaré series of \( \overline{H}(L) \) is denoted by

\[ \mathcal{P}(L) = \sum_{i,j,k} q^i a^j t^{(k-j)/2} \dim \overline{H}^{i,j,k}(L). \]

The following theorem states that the graded Euler characteristic of the HOMFLY homology \( \overline{H}(L) \) gives the HOMFLY polynomial \( P(L) \).

**Theorem 2** ([KR08; Ras15]). For any oriented link \( L \), we have

\[ P(L)(q, a) = \mathcal{P}(L)(q, a, -1). \]

### 2.2 MFW bounds and dualities

We recall several important results on the reduced HOMFLY homology of knots, which are essential in proving the finiteness of our algorithm.

The following is a categorified version of the Morton-Franks-Williams Inequality [Mor86; FW87].

**Proposition 2.6** ([Kho07; Wu08]). For a link \( L \) with braid closure diagram \( D \), the HOMFLY homology \( \overline{H}^{j,*}(L) \) is supported in

\[ j \in [w - s + 1, w + s - 1], \]

where \( w, s \) denote the writhe and the number of Seifert circles of \( D \) respectively.
The following is a categorified version of the symmetry of the HOMFLY polynomial $P_K(q, a) = P_K(-q^{-1}, a)$, which had been conjectured since the foundation of HOMFLY homology theory.

**Proposition 2.7 ([DGR06; OR20; GHM21]).** For a knot $K$, $\overline{P}^{i,j,k}(K) \cong \overline{P}^{-i,j,k+2i}(K)$.

From this, we obtain a categorified version of the Morton bound [Mor86].

**Proposition 2.8.** For a knot $K$ with braid closure diagram $D$, $\overline{H}^{i,*,*}(K)$ is supported in $i \in [-n + s - 1, n - s + 1]$, where $n, s$ denote the number of crossings and the number of Seifert circles of $D$ respectively.

**Proof.** Let $n^-$ be the number of negative crossings of $D$. From Figure 2, it is obvious that the lowest $q$-degree with non-trivial chain group in $C(D)$ is $-2n^-$. The corresponding shifted degree in $\overline{H}(D)$ gives the lower bound. The upper bound follows from the symmetry.

Finally, the following is the duality on mirrors of knots.

**Proposition 2.9 ([Gor+19]).** For a knot $K$ and its mirror $m(K)$, $\overline{H}^{i,j,k}(m(K)) \cong \overline{H}^{-i,-j,-k}(K)$.

**Proof.** One can use the Künneth universal coefficient spectral sequence to deduce this isomorphism from the duality given in Corollary 1.12 of [Gor+19]. The spectral sequence converges at the $E_2$ page, since any variable in the reduced base polynomial ring acts as zero on the homology $\overline{H}(K)$ of a knot $K$ (which is an analogous fact of [Ras15, Lemma 5.16]).

### 3 Algorithm

The basic idea of the algorithm is similar to the one given in [SS20]: if $F$ is a field, $R$ is a multivariate polynomial ring over $F$, and $C$ is a freely and finitely generated chain complex over $R$ whose differential is homogeneous with respect to the polynomial grading, we may regard $C$ as a chain complex over $F$ and decompose it into (infinitely many) finitely generated subcomplexes. Having proved that these subcomplexes are acyclic except for finitely many ones, it follows that $H(C)$ can be fully computed algorithmically. For the reduced HOMFLY homology, the key to proving the finiteness is the symmetry (Proposition 2.7).

Throughout this section, let $D$ be a non-trivial, connected, oriented link diagram with $n$ crossings.
3.1 Free variables of the edge ring

Lemma 3.1 ([Ras15, Lemma 2.4]). The reduced edge ring $R(D) = R'(D)/I(D)$ is isomorphic to a polynomial ring with $n$ variables.

Proof. $R'(D)$ is generated by $2n$ indeterminates. The ideal $I(D)$ is generated by $1 + n$ linear polynomials, and they have a unique linear relation $\sum_p \rho_p = 0$ since we assumed that $D$ is connected.

Next we show that a generating set of $R(D)$ can be given by considering a spanning tree of the Seifert graph of $D$. For each crossing $p$ of $D$, we define $X_p, X'_p \in R(D)$ by

$$X_p = x_b - x_c = -(x_a - x_d),$$
$$X'_p = x_a - x_c = -(x_b - x_d),$$

where $a, b$ are outgoing edges at $p$ and $c, d$ are incoming edges at $p$, and $a, c$ are placed to the left of $p$ (see Figures 2 and 3).

Lemma 3.2 ([Nak20]). Let $T$ be a spanning tree of the Seifert graph of $D$, and $S$ the set of crossings in $D$ corresponding to the edges of $T$. Then, $X_T = \{X_p\}_{p \in S} \cup \{X'_p\}_{p \notin S}$ is an algebraically independent generating set of $R(D)$.

Proof. Working over $\mathbb{Q}$ allows us to have the equation

$$\mathbb{Q}[x_1, \ldots, x_m] = \mathbb{Q}[x_1 - x_2, x_2 - x_3, \ldots, x_{m-1} - x_m, x_1 + \cdots + x_m].$$

Thus it follows that $R(D)$ is generated by elements of the form $x_e - x_f$, where $e, f$ are edges of $G(D)$. Since $|X_T| = n$, it is enough to show that each $x_e - x_f$
can be written as a linear sum of elements in $X_T$. Since $D$ is connected, by resolving all crossings in $D$ which are not in $S$, we obtain a diagram $D'$ of the trivial knot (see Figure 4 for an easy case). There is a unique oriented path $\gamma$ in $G(D')$ from $f$ to $e$, which may be also regarded as a path in $G(D)$. Trace $\gamma$ from $x_f$ to $x_e$, and each time $\gamma$ passes a crossing $p$ in $D$, take a term $\pm X_p$ or $\pm X'_p$ according to how $\gamma$ passes $p$ (see Figure 3). It is obvious that these terms belong to $X_T$ and that the sum gives $x_e - x_f$.

### 3.2 Reinterpretation as cube complexes

Next we reinterpret the chain complex $C(D)$ given in Section 2 as a “double cube complex”. Precise statement follows.

The $n$-cube is a poset $\{0, 1\}^n$ with the product order of $0 < 1$, considered as a category. An object $v \in \{0, 1\}^n$ is called a vertex, and the Manhattan norm of $v$ is denoted by $|v| = v_1 + \cdots + v_n$. A morphism $v \to w$ such that $|v| + 1 = |w|$ is denoted $v \to_1 w$ and called an edge. Define vertices $\overline{0} = (0, \ldots, 0)$ and $\overline{1} = (1, \ldots, 1)$.

Let $\mathcal{A}$ be an additive category.

**Definition 3.3.** For $n \geq 0$, an $n$-cube in $\mathcal{A}$ is a functor $C: \{0, 1\}^n \to \mathcal{A}$.

Given an $n$-cube $C$ in $\mathcal{A}$, one obtains a cube complex $(C^*, d^*)$ in $\mathcal{A}$ with

$$C^* = \bigoplus_{\|v\| = i} C(v)$$

and differentials

$$d^i = \sum_{\|v\| = i} s(e)C(e),$$

by choosing $s(e) \in \{-1, 1\}$ for each edge $e$ of the cube so that for any square consisting of four edges $e_1, \ldots, e_4$, we have $s(e_1) \cdots s(e_4) = -1$. 

Figure 5: A cube complex with factors $(a_1, \ldots, a_n)$. 

![Figure 5: A cube complex with factors (a1, ..., an).](image-url)
Remark 3.4. The isomorphism class of the above cube complex is independent of the choice of $s$.

Now we put $C(D)$ in this framework. First we will ignore the original homological gradings, and lately relate them with the newly introduced gradings. We recall that $C(D)$ is a tensor product of (vertical) two-terms complexes of (horizontal) complexes. Hence $C(D)$ can be regarded as a cube complex of complexes. We write

$$C(D) = \bigoplus_{v \in \{0,1\}^n} C^*_v(D),$$

where $C^*_v(D)$ is the complex at a vertex $v \in \{0,1\}^n$, called the horizontal complex of $D$ at $v$. Since $C^*_v(D)$ is also a tensor product of two-terms complex over $R(D)$, it can be regarded as a cube complex again. We write

$$C^*_v(D) = \bigoplus_{h \in \{0,1\}^n} R_{v,h}(D),$$

where $R_{v,h}(D)$ is the module at a vertex $h \in \{0,1\}^n$, and it is indeed a shifted copy of $R(D)$. The horizontal differential

$$d_H : C^*_v(D) \to C^*_{v+1}(D)$$

has the following form. For each standard unit vector $e_i$ of $\mathbb{R}^n$, there exists an element $a_i$ of $R(D)$ such that for any edge $v \to w$ in the direction of $e_i$ in the $n$-cube, the corresponding map $R_{v,h}(D) \to R_{w,h}(D)$ summed in $d_H$ is the multiplication map by $a_i$ as specified by the horizontal arrows of Figure 2 according to $v$ (with some appropriate sign assignment). We call $a_1, \ldots, a_n$ the factors of $C^*_v(D)$ (see Figure 5).

By taking the homology with respect to $d_H$, we get a cube complex of homologies

$$\bigoplus_{v \in \{0,1\}^n} H^*(C_v(D), d_H)$$

with differential $(d_V)_*$. Then it is obvious that $\hat{H}(D)$ is given by the homology of this complex

$$\hat{H}(D) = H(\bigoplus_v (H(C_v(D), d_H), (d_V)_*).$$

The homological bigrading $(\alpha, \beta)$ of the double cube complex $C(D)$ correspond to the original bigrading $(j, k)$ as

$$(\alpha, \beta) = \frac{1}{2} (j - j_0, k - k_0)$$

where $(j_0, k_0)$ is the (original) homological bidegree of the module $R_{\bar{0}, \bar{0}}(D)$ placed at $(v, h) = (0, 0)$. For later use, let $i_0$ denote the $q$-grading shift of $R_{\bar{0}, \bar{0}}(D)$. Then from Figure 2 we have

$$(i_0, j_0, k_0) = (2n^+, -2n, -2n^+)$$

where $n_+$ and $n_-$ are the number of positive and negative crossings of $D$ respectively.
3.3 Slicing by $q$-degrees

As described in Lemma 3.2, the ring $R(D)$ can be represented by a multivariate polynomial ring $\mathbb{Q}[X_i]$ with $n$ variables, so we have a homogeneous decomposition of $R(D)$ as a $\mathbb{Q}$-vector space

$$R(D) \cong \mathbb{Q}(1) \oplus \mathbb{Q}(X_i)_{1 \leq i \leq n} \oplus \mathbb{Q}(X_iX_j)_{1 \leq i \leq j \leq n} \cdots.$$  

Let $i_{v,h} \in \mathbb{Z}$ be the $q$-grading shift of $R_{v,h}(D) = R(D)\{i_{v,h}\}$. Decompose $R_{v,h}(D)$ as

$$R_{v,h}(D) = \bigoplus_{l=-\infty}^{\infty} V_{l,v,h}(D)$$

so that each $V_{l,v,h}(D)$ is homogeneous of $q$-degree $2(l + |h|) + i_{v,h}$. It is generated by monomials of degree (in the usual sense)

$$e(l, v, h) = l + |h| + \frac{i_{0} - i_{v,h}}{2}. \quad (3.2)$$

Now define

$$C_l(D) = \bigoplus_{v,h} V_{l,v,h}(D)$$

then it is obvious that both $d_H, d_V$ are closed in $C_l(D)$. Regarding $C(D)$ as a chain complex over $\mathbb{Q}$, we get a decomposition

$$C(D) = \bigoplus_{l=-\infty}^{\infty} C_l(D).$$

We call each subcomplex $C_l(D)$ the level-$l$ slice of $C(D)$. Note that the decomposition is defined so that $V_{l,0,0}(D)$ is generated by monomials of degree $e = l$.

**Lemma 3.5.** $C_l(D) = 0$ for $l < -2n$.

**Proof.** The claim is clear since $e$ is non-decreasing with respect to $v,h$ and $e(l,1,1) = l + 2n$. \qed

From the previous arguments, it is obvious that each $C_l(D)$ may also be regarded as a double cube complex. Define the level-$l$ slice $C_{l,v}(D)$ of the horizontal complex $C_v(D)$ at $v \in \{0,1\}^n$ by

$$C_{l,v}(D) = \bigoplus_{h \in \{0,1\}^n} V_{l,v,h}(D).$$

Then we get a decomposition

$$\hat{H}(D) = \bigoplus_{l=-2n}^{\infty} \hat{H}_l(D),$$

11
where
\[ \tilde{H}_l(D) = H(\oplus_v (H(C^*_{l,v}(D), d_H), (d_V)_{\ast})). \]

We claim that each \( \tilde{H}_l(D) \) is algorithmically computable. To take the homology twice, we find a basis on each chain group \( C^i = C^i_{l,v}(D) \) that gives a decomposition
\[ C^i = H^i \oplus B^i \oplus (d_H^{i+1})^{-1}(B^{i+1}) \]
so that we get representative cycles of the generators of \( H^i = H^i(C^*_{l,v}(D)) \) and that the secondary homology computation can be done on the chain level. This is achieved by standard methods such as the Gaussian elimination.

**Proposition 3.6.** For a link diagram \( D \), the homology \( \tilde{H}(D) \) has a decomposition
\[ \tilde{H}(D) = \bigoplus_{l=-2n}^{\infty} \tilde{H}_l(D) \]
where each summand \( \tilde{H}_l(D) \) can be computed algorithmically. Moreover for each triple degree \( (i,j,k) \) we have
\[ \tilde{H}^{i,j,k}(D) = H^\beta(\oplus_v (H^\alpha(C^{*}_{v,l}(D), d_H), (d_V)_{\ast})) \] (3.3)
where
\[ (l,\alpha,\beta) = \frac{1}{2}((i-i_0) - (j-j_0), j-j_0, k-k_0) \]
and \( (i_0,j_0,k_0) \) are given as in (3.1). Similar statement holds for \( \overline{P}(D) \). \( \square \)

Now we restrict to the case where \( D \) is a knot diagram, and in particular where it is a braid closure. In such case, it is well-known that \( \overline{P}(D) \) is finite dimensional [Ras15]. In order to perform the actual calculation, we must specify an actual upper bound for the level \( l \).

**Proposition 3.7.** If \( K \) is a knot with braid closure diagram \( D \), then
\[ \overline{P}(K) \cong \bigoplus_{l=-2n}^{-n} \overline{P}_l(D). \]
Moreover, \( \overline{P}(K) \) can be obtained from the computations of \( \overline{H}^{i,j,k}(D) \) within the following ranges:
\[ i \in [-n+s-1,0], \] (1)
\[ j \in [w-s+1,w+s-1], \] (2)
\[ k, k+2i \in [-n+s-1,n-s+1]. \] (3)
Proof. From Propositions 2.6 and 2.8 and equation (2.1), it follows that $\overline{H}(D)$ is supported in
\[
2l = (i - i_0) - (j - j_0) \\
\leq \{(n - s + 1) - (-w + s - 1) - 2n^+\} \\
\quad - \{(w - s + 1) - (w + s - 1) - (-2n)\} \\
= -2n.
\]
For the latter statement: (1) the lower bound is given by the Morton bound (Proposition 2.8), and from Proposition 2.7 it suffices to compute $H_{i^*}^{\alpha^*}(D)$ within $i \leq 0$. (2) This is exactly Proposition 2.6. (3) The lower bound of $k$ is obvious from the definition of $C(D)$, and from Proposition 2.9 we also have the upper bound. From Proposition 2.7, $k + 2i$ must also lie in this range. $\blacksquare$

3.4 Exclusion of variables

As apparent from (3.2), the number of generators increases combinatorially as $l$ increases. In order to reduce the computational cost, we use the process called “exclusion of variables” as described in [Ras15, Lemma 3.8].

Again we assume $D$ is a link diagram. Put $R = R(D)$. Take any $v \in \{0,1\}^n$ and consider the horizontal cube complex $C = C_v(D)$. Take any factor $f \in R$ of $C$. Note that $f$ is either linear or quadratic.

After choosing an appropriate sign assignment, $C$ may be regarded as the mapping cone of the endomorphism
\[
\begin{align*}
C' \xrightarrow{d'} C'
\end{align*}
\]
of an $n - 1$ dimensional cube complex $C'$. Let $d,d'$ denote the differentials of $C,C'$ respectively. The complex $C$ is isomorphic to $C' \oplus C'$ as an $R$-module, and we can write
\[
d(x_0, x_1) = (-d'x_0, fx_0 + d'x_1).
\]

Now $f$ is monic with respect to some variable $X_k$. Define
\[
R_0 = \mathbb{Q}[X_1, \ldots, \hat{X}_k, \ldots, X_n].
\]
Let $\pi: R \to R_1 = R/(f)$ be the quotient map, $\iota: R_1 \to R$ be the map that sends any residue class $[g] \in R_1$ to the remainder of $g$ by $f$ with respect to the variable $X_k$. We note that $\pi$ and $\iota$ are homomorphisms of modules over $R$ and $R_0$ respectively. Let $C'' = C' \otimes_R R_1$, and let $d''$ be the differential of $C''$. Two chain maps $\phi: C \to C''$ and $\psi: C'' \to C$ over $R_0$ are given by
\[
\begin{align*}
\phi(x_0, x_1) &= \pi(x_1), \\
\psi(y) &= \left(\frac{\iota(d''y) - d'\iota(y)}{f}, \iota(y)\right),
\end{align*}
\]
and it is not hard to verify that $\psi \phi \simeq \text{id}_C$ and $\phi \psi = \text{id}_{C''}$. The complex $C''$ admits the obvious triple grading so that both $\phi, \psi$ are degree preserving. Thus the two maps restrict as chain homotopy equivalences between the two sliced complexes $C_l$ and $C''_l$. The effect is that $H(C_l) \cong H(C''_l)$ can be computed with fewer generators, which is drastic when $l$ becomes large.

Furthermore, this process of reduction can be repeated for other directions as long as the target variable is algebraically independent in the base ring. When $f$ is linear, we have $R_1 \cong R_0$ and all other variables remain independent. When $f$ is quadratic, then $R_1 \cong R_0 \otimes \mathbb{Q}\{1, X_k\}$ as $\mathbb{Q}$-vector spaces, and the variables that does not appear in the coefficients of $f$ remains independent.

In particular, when all factors $f$ of $C$ are linear, then the exclusion can be continued until the differential becomes trivial. In general, to exclude the variables as much as possible, it is preferable to perform exclusions on linear polynomials first, and then on the remaining quadratics.

For the computation of $\overline{H}(D)$, we can replace each horizontal complex $C_{l,v}$ with a reduced one, compute those homologies, and replace the differential $(d_V)_*$ on the vertical complex with $(\phi_{v'} \circ d_V \circ \psi_v)_*$ for each edge $v \to v'$.

### 3.5 Overall procedure

We summarize the overall procedure for computing (the dimensions of) the reduced HOMFLY homology $h_{i,j,k} = \{\dim \overline{H}^{i,j,k}_l(k)\}_{i,j,k}$ of a knot $K$.

**Algorithm 3.8.** Given a braid closure diagram $D$ of a knot $K$,

1. Compute the Seifert graph $G(D)$ and find a spanning tree $T$ of $G(D)$.

2. Using $T$, identify the edge ring $R(D)$ as a multivariate polynomial ring, and represent both $d_H$ and $d_V$ as $n$-tuples of polynomials in $R(D)$ as in the proof of Lemma 3.2.

3. For each $(l, \alpha, \beta)$ in $[-2n, -n] \times [0, n] \times [0, n]$:
   
   (a) Convert $(l, \alpha, \beta)$ to $(i, j, k)$. Skip to the next iteration if $(i, j, k)$ is not in the range of Proposition 3.7.

   (b) Setup the generators of the level-$l$ horizontal complex $C_l(D)$.

   (c) Perform exclusion of variables on $C_l(D)$ to get a reduced complex together with the chain homotopy equivalences.

   (d) Compute $h_{i,j,k} = \dim \overline{H}^{i,j,k}_l(D)$ from (3.3), with the horizontal complexes replaced with the reduced ones.

   (e) Assign $h_{-i,j,k+2i} = h_{i,j,k}$ if $i < 0$.

4. Return $h$.

Reusable data should be cached within the iteration. Further improvement can be achieved by concerning mirrors of knots. To explain this, assume that $D$ has only one crossing $p$. Recall that the vertex module $V_{i,v,h}(D)$ at $(v, h) \in \{0, 1\}^2$
is spanned by the monomials of degree given by (3.2). From Figure 2, we see that $e(l,1,0) = e(l,0,1) = e(l,0,0) + 1$ if $p$ is positive, and $e(l,1,0) = e(l,0,1) = e(l,0,0)$ if $p$ is negative (in both cases $e(l,1,1) = e(l,0,0) + 1$). This means that $C_l(D)$ has a smaller generating set when $p$ is negative. In general if $w(D) > 0$, then $C_l(m(D))$ has a smaller generating set than $C_l(D)$. This difference becomes intense as $n$ increases. Thus the improved version is:

**Algorithm 3.9.** Given a braid closure representative $D$ of a knot $K$,

1. If $w(D) \leq 0$, compute and return $h(D)$ by Algorithm 3.8.
2. Otherwise, compute $h(m(D))$ by Algorithm 3.8 and return its dual.

### 4 Computations

The implementation of the algorithm is made public at [NS]. As is apparent from the algorithm, the computational cost primarily depends on length $l$ of the input braid representation. Our program is capable of computing the reduced HOMFLY homology for knots with braid length $l$ up to 13 on an ordinary personal computer. Using the braid representatives given in the KnotInfo Database, we have computed the homology for all prime knots with crossing number $n \leq 10$, and for all prime knots with $n = 11$ and braid length $l \leq 13$ (see Table 1 in Section 1). The whole results can be found at [NS]. Here we pick some notable ones.

**Proposition 4.1.** $\{5_1, m(10_{132})\}$ and $\{11n_{79}, m(11n_{138})\}$ are pairs such that the two knots have identical HOMFLY polynomial but distinct reduced HOMFLY homology.

**Proposition 4.2.** $9_{42}, 10_{125}, 11n_{24}, 11n_{82}$ are knots $K$ such that $K$ and $m(K)$ have identical HOMFLY polynomial but distinct reduced HOMFLY homology.

**Proposition 4.3** ([MV08]). The Conway knot and the Kinoshita–Terasaka knot ($11n_{34}$ and $11n_{42}$) have identical reduced HOMFLY homology.

Recall that a knot $K$ is called KR-thin if its reduced HOMFLY homology $\overline{\mathcal{P}}^{i,j,k}(K)$ is supported on $i + j + k = \sigma(K)$, where $\sigma$ denotes the signature. $K$ is called KR-thick if it is not KR-thin.

**Proposition 4.4.** The following 19 are the only KR-thick knots with up to 10 crossings:

\[8_{19}, 9_{42}, 9_{43}, 9_{47}, 10_{124}, 10_{125}, 10_{128}, 10_{132}, 10_{134}, 10_{136}, 10_{138}, 10_{139}, 10_{142}, 10_{145}, 10_{152}, 10_{153}, 10_{154}, 10_{160}, 10_{161}.\]
Proposition 4.5. The following 73 are the only KR-thick knots with 11 crossings and braid length up to 13:

\[11a_{263}, 11n_{2}, 11n_{6}, 11n_{9}, 11n_{12}, 11n_{13}, 11n_{16}, 11n_{19}, 11n_{20}, 11n_{23},
11n_{24}, 11n_{27}, 11n_{30}, 11n_{31}, 11n_{34}, 11n_{36}, 11n_{38}, 11n_{39}, 11n_{41}, 11n_{42},
11n_{44}, 11n_{45}, 11n_{47}, 11n_{49}, 11n_{57}, 11n_{60}, 11n_{61}, 11n_{64}, 11n_{69}, 11n_{70},
11n_{73}, 11n_{74}, 11n_{76}, 11n_{77}, 11n_{78}, 11n_{79}, 11n_{80}, 11n_{81}, 11n_{82}, 11n_{88},
11n_{90}, 11n_{92}, 11n_{93}, 11n_{95}, 11n_{96}, 11n_{102}, 11n_{104}, 11n_{107}, 11n_{111}, 11n_{116},
11n_{118}, 11n_{120}, 11n_{126}, 11n_{133}, 11n_{135}, 11n_{138}, 11n_{143}, 11n_{145}, 11n_{147}, 11n_{148},
11n_{149}, 11n_{150}, 11n_{151}, 11n_{152}, 11n_{153}, 11n_{155}, 11n_{158}, 11n_{166}, 11n_{169}, 11n_{173},
11n_{177}, 11n_{182}, 11n_{183}.\]

Proposition 4.4 extends [Ras15, Proposition 7.10] which lists KR-thick knots for \( n \leq 9 \). Also note that an alternating knot \( 11a_{263} \) is KR-thick, as pointed out in [Ras07]. This is in contrast to the fact that alternating knots are necessarily homologically thin for Khovanov homology [Lee05] and for knot Floer homology [OS03].

5 Future Prospects

5.1 On spectral sequences

The celebrated conjecture given by Dunfield, Gukov, and Rasmussen [DGR06] claims that the HOMFLY homology unifies categorifications of the skein link polynomials. More precisely, they conjectured that for each knot \( K \), there are differentials \( \{d_N\}_{N \in \mathbb{Z}} \) on the HOMFLY homology \( \overline{H}(K) \) and an involution \( \Phi \) on \( \overline{H}(K) \) such that

- the homology of \( \overline{H}(K) \) with respect to the differential \( d_N \) gives the (reduced) \( sl(N) \) homology \( \overline{H}_N(K) \) for \( N > 0 \),
- the homology of \( \overline{H}(K) \) with respect to \( d_0 \) gives the knot Floer homology of \( K \),
- the involution \( \Phi \) gives isomorphisms \( \Phi: \overline{H}^{i,j,k}(L) \cong \overline{H}^{-i,j,k+2i}(L) \),
- \( d_N d_M = -d_M d_N \) and \( \Phi d_N = d_{-N} \Phi \) for all \( N, M \in \mathbb{Z} \).

Rasmussen [Ras15] found a spectral sequence from the HOMFLY homology to the \( sl(N) \) homology for each \( N > 0 \) which partially supports the first statement. The \( m \)-th differential \( d_m(N) \) in the spectral sequence has triple degree \((2mN, -2m, 2-2m)\). If the sequence converges after the first differential \( d_1(N) \), then the first statement is verified.

For the knots we have computed, for \( N > 2 \) the differentials \( d_m(N) \) are necessarily trivial by degree reasons. In other words, the HOMFLY homology \( \overline{H}(K) \) directly gives the \( sl(N) \) homology \( \overline{H}_N(K) \) for \( N > 2 \).
For \( N = 2 \), the first differential \( d_1(2) \) is non-trivial for many KR-thick knots listed in Propositions 4.4 and 4.5. For some of them, we have verified by hand that \( d_m(2) = 0 \) for \( m > 1 \) by observing their reduced Khovanov homology and using the anti-commutativity of \( d_1(1) \) and \( d_1(2) \).

A future objective for the authors is to refine the algorithm so that we can compute Rasmussen’s spectral sequences and check the Dunfield–Gukov–Rasmussen conjecture for more knots. It will also allow us to compute the integer valued invariant \( S(K) \) [DGR06], which comes from the spectral sequence for \( N = 1 \) in a similar way with Rasmussen’s \( s \)-invariant [Ras10].

### 5.2 On homologies for general knot diagrams

As noted in Section 2, \( \overline{H}(D) \) does not generally give the reduced HOMFLY homology unless \( D \) is a braid closure diagram. Still our program can partially compute \( \overline{H}(D) \) for an arbitrary knot diagram \( D \). For example, consider the following diagram of 8\( _{15} \) represented by the planar diagram code [LM; BMa]

\[
D = X_{1726}X_{3,15,4,14}X_{5968}X_{9,13,10,12}X_{11,1,12,16}X_{13,5,14,4}X_{15,11,16,10}.
\]

Table 3 shows the homologies \( \overline{H}(8_{15}) \) and \( \overline{H}(D) \) (possibly partial). It is obviously not isomorphic to \( H(8_{15}) \), nor is it symmetric as in Proposition 2.7.

| \( k \) \( j \) | 4 | 6 | 8 | 10 |
|---|---|---|---|---|
| 4 | \( q^4 \) | \( q^6 \) | \( q^8 \) | \( q^{10} \) |
| 2 | \( 2q^{-2} \) | \( 2q^{-4} \) | \( 2q^{-6} \) | \( 2q^{-8} \) |
| 0 | \( 3q^{-2} \) | \( 3q^{-4} \) | \( 3q^{-6} \) | \( 3q^{-8} \) |
| -2 | \( q^2 2q^{-4} \) | \( q^2 2q^{-6} \) | \( q^2 2q^{-8} \) | \( q^2 2q^{-10} \) |
| -4 | \( 2q^4 \) | \( 2q^6 \) | \( 2q^8 \) | \( 2q^{10} \) |
| -6 | \( 3q^2 1 \) | \( 3q^4 2 \) | \( 3q^6 3 \) | \( 3q^8 4 \) |

Table 3: \( \overline{H}(8_{15}) \) and \( \overline{H}(D) \) for a diagram \( D \) of \( 8_{15} \)

At the time of writing, we do not know how \( \overline{H}(D) \) varies under the RIIb moves (Question 1.1). Nevertheless, we can possibly extend the usage of \( \overline{H}(D) \). For \( \overline{H}(D) \) to give the correct HOMFLY homology of the corresponding knot \( K \), it is not necessary that \( D \) be a closed braid diagram of \( K \). Indeed, if a spectral sequence from \( \overline{H}(D) \) to \( \overline{H}_N(D) \) can be constructed for each \( N > 0 \) in Rasmussen’s manner [Ras15], then for sufficiently large \( N > 0 \) we have \( \overline{H}(D) \approx \overline{H}_N(D) \approx \overline{H}_N(K) \approx \overline{H}(K) \). This holds when a variant of the MOY calculus used in [Ras15; Wu08] runs until the end for \( D \). We say \( D \) is good if \( \overline{H}(D) \approx \overline{H}(K) \).

Finding a good diagram for a knot \( K \) can be beneficial for homology computation. If there is a good diagram of a knot \( K \) which has less crossings than

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\( ^4 \)Since the symmetry of the homology can not be assumed, it is possible that there are more non-trivial summands in level \( l > -n \).

17
its braid length, then the computational cost for $\overline{H}(K)$ can be reduced. For instance, the following diagram of $11n_{39}$

$$D = \prod X_{4251}X_{8493}X_{11,17,12,16}X_{12,5,13,8}X_{13,7,14}X_{17,22,18,1}X_{18,10,19}X_{21,10,22,11}X_{15,21,16,20}X_{19,15,20,14}X_{2837}$$

is good and has 11 crossings, while the braid length of $11n_{39}$ is 13.

**Question 5.1.** Can we give geometrical or combinatorial characterization of good diagrams? Is there a systematic (or algorithmic) way to produce good diagrams for any knot $K$?

**Question 5.2.** Are the spectral sequences for good diagrams also knot invariants?

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