INFINITE IMAGE PARTITION REGULAR MATRICES - SOLUTION IN C-SETS

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ABSTRACT. A finite or infinite matrix $A$ is image partition regular provided that whenever $\mathbb{N}$ is finitely colored, there must be some $\vec{x}$ with entries from $\mathbb{N}$ such that all entries of $A\vec{x}$ are in the same color class. Comparing to the finite case, infinite image partition regular matrices seem more harder to analyze. The concept of centrally image partition regular matrices were introduced to extend the results of finite image partition regular matrices to infinite one. In this paper, we shall introduce the notion of C-image partition regular matrices, an interesting subclass of centrally image partition regular matrices. Also we shall see that many of known centrally image partition regular matrices are C-image partition regular.

1. INTRODUCTION

The classical theorems of Ramsey Theory can be spontaneously stated as statements about image partition regular matrices which is why the study of Image partition regular matrices seeks remarkable attention. For example, Schurs Theorem [Schur [1917]] and the length 4 version of van der Waerdens Theorem [Van der Waerden [1927]] assures us that the matrices
\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
1 & 2 \\
1 & 3
\end{pmatrix}
\]
are image partition regular.

Let us recall the following well known definition of the image partition regularity.

Definition 1.1. Let $A$ be a $m \times n$ matrix with entries from $\mathbb{Q}$ for some $m, n \in \mathbb{N}$. We call the matrix $A$ to be an image partition regular matrix over $\mathbb{N}$ if and only if there exists $i \in \{1, 2, \cdots, p\}$ and $\vec{x} \in \mathbb{N}^n$ such that $A\vec{x} \in E_i^m$, whenever we are able to write $\mathbb{N} = \bigcup_{i=1}^p E_i$ for some $p \in \mathbb{N}$.

Several characterizations of finite image partition regular matrices involve the notion of “first entries matrix”, a concept based on Deubers $(m, p, c)$ sets.

We recall the following definition from Hindman and Strauss [2000].

Definition 1.2. Let $A$ be a $p \times q$ matrix with rational entries. Then $A$ is a first entries matrix if and only if no row of $A$ is $0$ and there exist strictly positive rational numbers $t_1, t_2, \cdots, t_q$ such that $a_{i,j} = t_j$, where $i \in \{1, 2, \cdots, p\}$ and $j = \min\{l \in \{1, 2, \cdots, q\} : a_{i,l} \neq 0\}$.

We call $t_j$ to be the first entry of $A$ if there exists $i \in \{1, 2, \cdots, p\}$ such that $j = \min\{l \in \{1, 2, \cdots, q\} : a_{i,l} \neq 0\}$.

Some of the known characterizations of finite image partition regular matrices involve the notion of central sets while the famous concept was introduced in [Furstenberg [1981]] and defined in terms of the view point of topological dynamics. These sets enjoy very strong combinatorial properties. (See [Furstenberg [1981], Proposition 8.21] or [Hindman and Strauss [1998]],...
Chapter 14). They have a nice characterization in terms of the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. We shall present this characterization below, after introducing the necessary background information.

Let $(S, +)$ be an infinite discrete semigroup. The points of $\beta S$ are taken to be the ultrafilters on $S$ with the understanding that the principal ultrafilters are being identified with the points of $S$. It is a folklore that for a given set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{A : A \subseteq S\}$ turns out to be a basis for the open sets (as well as a basis for the closed sets) of $\beta S$.

One can naturally extend the operation $+$ of $S$ to the whole of $\beta S$ by making $\beta S$ a compact right topological semigroup with its topological center containing $S$. This says that for each $p \in \beta S$ the function $\mu_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\nu_x : \beta S \to \beta S$ is continuous, where $\mu_p(q) = q + p$ and $\nu_x(q) = x + q$. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p + q$ if and only if $\{x \in S : -x + A \subseteq q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$.

If a non-empty subset $I$ of a semigroup $(T, +)$ satisfies $T + I \subseteq I$, we call $I$ to be a left ideal of $T$ and a right ideal if $I + T \subseteq I$. A two sided ideal (or simply an ideal) is both a left and a right ideal. A minimal left ideal is a left ideal that does not contain any proper left ideal. Now it is very obvious to analogously define a minimal right ideal and the smallest ideal.

Any compact Hausdorff right topological semigroup $(T, +)$ contains idempotents and therefore has a smallest two sided ideal

$$K(T) = \bigcup \{\mathcal{L} : \mathcal{L} \text{ is a minimal left ideal of } T\} = \bigcup \{\mathcal{R} : \mathcal{R} \text{ is a minimal right ideal of } T\}.$$ 

Given a minimal left ideal $\mathcal{L}$ and a minimal right ideal $\mathcal{R}$, it turns out that $\mathcal{L} \cap \mathcal{R}$ is a group and therefore contains an idempotent. If $p$ and $q$ are idempotents in $T$, we write $p \leq q$ if and only if $p + q = q + p = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal $K(T)$ of $T$.

A beautiful inauguration to the algebra of $\beta S$ is given in Hindman and Strauss [1998].

**Definition 1.3.** Let $(S, +)$ be an infinite discrete semigroup. A subset in $S$ is said to be Central if and only if it is contained in some minimal idempotent of $(\beta S, +)$.

Now we state the most general version of Central Sets Theorem from De et al. [2008]. We state it here only for the commutative subgroups.

**Theorem 1.4.** Let $(S, +)$ be a commutative semigroup and denote the set of all sequences in $S$ by $\tau$. Let $C \subseteq S$ be central, then there exists functions $\alpha : \mathcal{P}_f(\tau) \to S$ and $H : \mathcal{P}_f(\tau) \to \mathcal{P}_f(\mathbb{N})$ such that

1. if $F, G \in \mathcal{P}_f(\tau)$ and $F \subseteq G$ then $\max H(F) < \min H(G)$, and
2. whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\tau)$, $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m$ and for each $i \in \{1, 2, \cdots, m\}$, $f_i \in G_i$, one has

$$\sum_{i=1}^{m} (\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)) \in C.$$

Recently a lot of attention has been paid to those sets which satisfy the conclusion of the latest Central Sets Theorem. Like Central sets they also contain the image of finite image partition regular matrices.

**Definition 1.5.** Let $(S, +)$ be a commutative semigroup. Also let $\tau = \mathbb{N}_S$ be the set of all sequences in $S$ and $C \subseteq S$. If there exists functions $\alpha : \mathcal{P}_f(\tau) \to S$ and $H : \mathcal{P}_f(\tau) \to \mathcal{P}_f(\mathbb{N})$ such that the conditions 1 and 2 of Theorem 1.4 is satisfied, then we call $C$ to be a $C$ set.
Therefore we can readily observe that Central sets are in particular C-sets. We now present some notations from De et al. [2008].

**Definition 1.6.** Let \((S, +)\) be a commutative semigroup, and let \(\tau\) be as in the Definition 1.5.

1. \(A \subseteq S\) is said to be a J-set if for every \(F \in \mathcal{P}_f(\tau)\) there exists \(a \in S\) and \(H \in \mathcal{P}_f(\mathbb{N})\) such that for all \(f \in F\),
   \[a + \sum_{t \in H} f(t) \in A.\]

2. \(J(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a J-set})\} \text{.}

**Theorem 1.7.** Let \((S, +)\) be a discrete commutative semigroup and \(A\) be a subset of \(S\). Then \(A\) is a J-set if and only if \( \text{cl}A \cap J(S) \neq \emptyset \).

**Proof.** The theorem follows from Theorem 3.11 of Hindman and Strauss [1998] while noting that the collection of J sets form a partition regular family. \(\square\)

The following is a consequence of Theorem 3.8 of De et al. [2008].

**Theorem 1.8.** Let \((S, +)\) be a commutative semigroup and \(A \subseteq S\). Then \(A\) is a C-set if and only if \(\text{cl}A \cap J(S)\) contains at least an idempotent \(p\).

We also state the following theorem which is Theorem 3.5 in De et al. [2008].

**Theorem 1.9.** Let \((S, +)\) be a discrete commutative semigroup, then \(J(S)\) is a closed two sided ideal of \(\beta S\) and \(\text{cl}K(\beta S) \subseteq J(S)\).

It is known that finite image partition regular matrices are compatible with respect to central sets. A similar result holds true for C-sets.

**Theorem 1.10.** Let \(A\) be a \(p \times q\) matrix with rational entries for some natural numbers \(p\) and \(q\). The following statements are equivalent:

1. \(A\) is an image partition regular matrix.
2. For any C-set \(C\) of \(\mathbb{N}\), \(A\overrightarrow{x} \in C^p\) for some \(\overrightarrow{x} \in \mathbb{N}^q\).
3. For any C-set \(C\) of \(\mathbb{N}\), \(\{\overrightarrow{x} \in \mathbb{N}^q : A\overrightarrow{x} \in C^p\}\) is also a central subset \(\mathbb{N}^q\).

**Proof.** The proof is similar as that of Theorem 1.2 of Hindman et al. [2003]. \(\square\)

The notion of image partition regular matrices extends naturally to infinite \(\omega \times \omega\) matrices provided each row of the matrix contains only finitely many non-zero entries. (Here \(\omega\), the first infinite cardinal, is also the set of nonnegative integers.)

It is an immediate consequence of Theorem 1.5(b) of Hindman and Strauss [2000] that whenever \(A\) and \(B\) are finite image partition regular matrices, so is \(\begin{pmatrix} A & 0 \\ O & B \end{pmatrix}\), where \(O\) represents a matrix of the appropriate size with all zero entries. However, the analogous result does not hold true for infinite image partition regular matrices because of Theorem 3.14 of Deuber et al. [1995].

Motivated by this distinction and by the condition of Theorem 1.5(l) of Hindman and Strauss [2000], Hindman, Leader and Strauss came up with the notion of “Centrally image partition regular matrices” and “Strongly Centrally image partition regular matrices” in Definition 2.7 and in Definition 2.10 respectively in Hindman et al. [2003].

In Section 2, we shall introduce the notion of C-image partition regularity and see that the behaviour of this infinite image partition regularity almost same like Centrally image partition regularity. In Section 3 we shall give some classes of C-image partition regular matrices which also had occurred in the case of Centrally image partition regularity.
2. C-IMAGE PARTITION REGULARITY

Centrally image partition regular matrices were introduced in order to extend the results of finite image partition regular matrices to infinite one. In this section we shall introduce the notion of C-image partition regularity and see that parallel results for this type of image partition regularity also holds true.

**Definition 2.1.** Let $A$ be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$.

1. The matrix $A$ is C-image partition regular if and only if for every C-set $C$ of $\mathbb{N}$, one has $A\vec{x} \in C^\omega$ for some $\vec{x} \in \mathbb{N}^\omega$.
2. The matrix $A$ is strongly C-image partition regular if and only if for every C-set $C$ of $\mathbb{N}$, one has $\vec{y} = A\vec{x} \in C^\omega$ for some $\vec{x} \in \mathbb{N}^\omega$ and entries of $A\vec{x}$ corresponding to distinct rows of $A$ are distinct i.e., for all $i$, $j$ if row $i$ and row $j$ of $A$ are unequal then $y_i \neq y_j$.

Like Centrally image partition regular matrices, there is a simple necessary condition for a matrix to be strongly C-image partition regular which is as follows:

**Theorem 2.2.** Let $A$ be a strongly C-image partition regular matrix having no repeated rows. Then,

$$\{i : \text{for all } j \geq k, a_{i,j} = 0\}$$

is finite for all $k \in \mathbb{N}$.

**Proof.** Suppose that $\{i : \text{for all } j \geq k, a_{i,j} = 0\}$ is finite. Then by discarding the other rows we may presume that $A$ is an $\omega \times k$ matrix. Let $D = \{\vec{x} \in \mathbb{N}^k : \text{all entries of } A\vec{x} \text{ are distinct}\}$. Enumerate $D$ as $\langle \vec{x}^{(n)} \rangle_{n=1}^\infty$. Inductively choose distinct $y_n$ and $z_n$ in $A\vec{x}^{(n)}$ with $\{y_n, z_n\} \bigcap \{y_t : t \in \{1, 2, \cdots, n-1\}\} \neq \phi$ if $n > 1$. Let $C = \{y_n : n \in \mathbb{N}\}$. Then there exists no $\vec{x} \in D$ with $A\vec{x} \in C^\omega$ and no $\vec{x} \in D$ with $A\vec{x} \in (\mathbb{N}\setminus C)^\omega$. \qed

**Theorem 2.3.** Let $p$ be an idempotent in $R$ where $R$ is a right ideal of $(\beta \mathbb{N}, +)$. Then for each $C \in p$, there are $2^c$ minimal idempotents in $R \cap C$.

**Proof.** Let $C \in p$ and $C^* := \{x \in C : -x + C \in p\}$. Then notice that by Lemma 4.14 of Hindman and Strauss [1998], $-x + C^* \in p$, for each $x \in C^*$. For each $m \in \mathbb{N}$,

$$S_m = 2^m \mathbb{N} \bigcap C^* \bigcap \{k \in C^* : k \in C^* \bigcap \{1, 2, \cdots, m\}\}.$$

Let $V = \bigcap_{m=1}^\infty S_m$. For every $m \in \mathbb{N}$, $2^m \mathbb{N} \in p$ by Lemma 6.6 of Hindman and Strauss [1998] and so $m \in p$. Thus $p \in V$. We show that $V$ is a subsemigroup of $\beta \mathbb{N}$, using Theorem 4.20 of Hindman and Strauss [1998]. So, let $m \in \mathbb{N}$ and let $n \in S_m$. It is sufficient to show that $n + S_{m+n} \subseteq S_m$. Let $r \in S_{m+n}$. Obviously $n + r \in 2^n \mathbb{N}$. We have $n + r \in C^*$ because $n \in C^* \bigcap \{1, 2, \cdots, m + n\}$. Let $k \in C^* \bigcap \{1, 2, \cdots, m\}$. Then $n \in -k + C^*$. So $k + n \in C^* \bigcap \{1, 2, \cdots, m + n\}$ and thus $r \in -(k + n) + C^*$ so that $n + r \in -k + C^*$ as required.

Since $p \in V$ we have by Theorem 6.32 of Hindman and Strauss [1998] that $V$ contains a copy of $\mathbb{H} = \bigcap_{n=1}^\infty \mathbb{N}^{2^n}$. By Theorem 6.9 of Hindman and Strauss [1998], $(\beta \mathbb{N}, +)$ has $2^c$ minimal left ideals. Thus there is a subset $W$ of $\beta \mathbb{N}$ containing idempotents such that $|W| = 2^c$. The subset $W$ will also have the property that whenever $u$ and $v$ are distinct members of $W$, $u + v = u$ and $v + u = v$. Following Lemma 6.6 of Hindman and Strauss [1998], $W \subseteq H$ and $V$ contains a copy of $\mathbb{H}$. Therefore we have a set $E \subseteq V$ of idempotents such that $|E| = 2^c$ and $u + v \neq u$ and $v + u \neq v$ for all distinct members $u$ and $v$ of $E$.

By Theorem 6.20 of Hindman and Strauss [1998], $(\beta \mathbb{N} + u) \bigcap (\beta \mathbb{N} + v) = \phi$ whenever $u$ and $v$ are distinct members of $E$. So we can further say $(V + u)(V + v) = \phi$. For each $u \in E$ pick an idempotent $\alpha_u \in (p + V) \bigcap (V + u)$ with the property that $\alpha_u$ minimal in $V$. 

By Corollary 2.6 and Theorem 2.7 of Hindman and Strauss [1998], \( p + V \) contains a minimal right ideal \( R_1 \) of \( V \) and \( V + u \) contains a minimal left ideal \( L_1 \) of \( V \). Then, \( R_1 \cap L_1 \) is a group. Let \( \alpha_u \) be the identity element of this group.) Then \( \{\alpha_u : u \in E\} \) is a set of 2\( ^{\omega} \) minimal idempotents of \( V \) in \( p + V \leq R \).

**Corollary 2.4.** Let \( C \) be a C-set in \( \mathbb{N} \). Then there exists a sequence \( \langle C_n \rangle_{n=1}^{\infty} \) of pairwise disjoint C-sets in \( \mathbb{N} \) such that \( \bigcup_{n=1}^{\infty} C_n \subseteq C \).

**Proof.** By Theorem 2.3, the set of idempotents in \( \overline{C} \cap J(\mathbb{N}) \) is infinite. Therefore \( \overline{C} \cap J(\mathbb{N}) \) contains an infinite strongly discrete subset. (Alternatively, there are two idempotents in \( \overline{C} \cap J(\mathbb{N}) \) so that \( C \) can be divided into C-sets \( C_1 \) and \( D_1 \). Then \( D_1 \) can again be divided into two C-sets, \( C_2 \) and \( D_2 \) and so on.)

**Corollary 2.5.** For each \( n \in \mathbb{N} \), let \( A_n \) be a strongly C-image partition regular matrix. Then the matrix

\[
M = \begin{pmatrix}
A_1 & 0 & 0 & \cdots \\
0 & A_2 & 0 & \cdots \\
0 & 0 & A_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

is also strongly C-image partition regular.

**Proof.** Let \( C \) be a C-set. By Corollary 2.4, choose a sequence \( \langle C_n \rangle_{n=1}^{\infty} \) of pairwise disjoint C-sets in \( \mathbb{N} \) such that \( \bigcup_{n=1}^{\infty} C_n \subseteq C \). For each \( n \in \mathbb{N} \) choose \( \overrightarrow{y} \in \mathbb{N}^\omega \) such that \( \overrightarrow{y} = A_n \overrightarrow{x} \in C_n^\omega \) and if row \( i \) and \( j \) of \( A_n \) fails to be equal, then \( \overrightarrow{y}_i \neq \overrightarrow{y}_j \). Let

\[
\overrightarrow{z} = \begin{pmatrix}
\overrightarrow{y}^{(1)} \\
\overrightarrow{y}^{(2)} \\
\vdots
\end{pmatrix}.
\]

Then all entries of \( M \overrightarrow{z} \) are in \( C \) and entries from distinct rows of \( M \overrightarrow{z} \) are unequal.

Surely, Corollary 2.5 remains valid if “strongly C-image partition regular” is replaced by “C-image partition regular”. The same proof applies and one does not need to introduce the pairwise disjoint C-sets, which were required to guarantee that the entries of \( M \overrightarrow{z} \) from distinct rows were distinct.

Notice that trivially, if \( A \) is an \( \omega \times \omega \) matrix with entries from \( \mathbb{Q} \) and there is some positive rational number \( m \) such that each row of \( A \) sums to \( m \), then \( A \) is centrally image partition regular. (Given a central set \( C \), simply pick \( d \in \mathbb{N} \) such that \( d_m \in C \), which one can do because for each \( n \in \mathbb{N} \), \( N_n \) is a member of every idempotent by Lemma 6.6 of Hindman and Strauss [1998]. Then let \( x_i = d \) for each \( i \in \omega \).

**Theorem 2.6.** Let \( k \in \mathbb{N} \) and \( m \in \mathbb{Q} \) such that \( m > 0 \). Let \( A \) be an \( \omega \times \omega \) with rational entries such that

1. the sum of each row of \( A \) is \( m \), and
2. for each \( l \in \omega \), \( \{\langle a_{i,o}, a_{i,1}, \cdots, a_{i,l} \rangle : i \in \omega \} \) is finite.
Let \( \vec{r}^{(1)}, \vec{r}^{(2)}, \ldots, \vec{r}^{(k)} \in \mathbb{Q}^\omega \setminus \{0\} \) such that each \( \vec{r}^{(i)} \) has only finitely many non-zero entries. Then there exist \( b_1, b_2, \ldots, b_k \in \mathbb{Q} \setminus \{0\} \) such that

\[
\begin{pmatrix}
  b_1 \vec{r}^{(1)} \\
b_2 \vec{r}^{(2)} \\
  \vdots \\
b_k \vec{r}^{(k)} \\
  \vec{A}
\end{pmatrix}
\]

is \( C \)-image partition regular.

**Proof.** Pick \( l \in \mathbb{N} \) such that \( r_i^{(j)} = 0 \), for every \( j \in \{1, 2, \ldots, k\} \) and every \( i \geq l \). Let \( s^{(j)} = \langle r_0^{(j)}, r_1^{(j)}, \ldots, r_l^{(j)} \rangle \), for \( j \in \{1, 2, \ldots, k\} \). Enumerate

\[
\{ \langle a_{i_0}, a_{i_1}, \ldots, a_{i_l} \rangle : i \in \omega \}
\]

as \( \vec{w}^{(0)}, \vec{w}^{(1)}, \ldots, \vec{w}^{(u)} \). Let \( d_i = m - \sum_{j=0}^{l-1} w_j^{(i)} \), for \( i \in \{1, 2, \ldots, u\} \). Let \( E \) be the \((u+1) \times (l+1)\) matrix with entries

\[
e_{i,j} = \begin{cases} 
  w_j^{(i)}, & \text{if } j \in \{0, 1, \ldots, l-1\} \\
  d_i, & \text{if } j = l.
\end{cases}
\]

Then \( E \) is image partition regular because \( E \) has constant row sums. By applying Theorem 1.2(d) of Hindman et al. [2003] \( u+1 \) times, pick \( b_1, b_2, \ldots, b_k \in \mathbb{Q} \setminus \{0\} \) such that the matrix

\[
\begin{pmatrix}
  b_1 s^{(1)} \\
b_2 s^{(2)} \\
  \vdots \\
b_k s^{(k)} \\
  \vec{E}
\end{pmatrix}
\]

is image partition regular. Hence by Theorem 1.2(b) of Hindman et al. [2003] the above matrix is image partition regular.

Let \( C \) be a \( C \)-set and pick \( \langle z_0, z_1, \ldots, z_l \rangle \in \mathbb{N}^{l+1} \) with \( H \vec{z} \in C^{u+1} \). For \( n \in \{0, 1, \ldots, l-1\} \), let \( x_n = z_n \). For \( n \in \{l, l+1, l+2, \ldots\} \), let \( x_n = z_l \). Consequently,

\[
\begin{pmatrix}
  b_1 \vec{r}^{(1)} \\
b_2 \vec{r}^{(2)} \\
  \vdots \\
b_k \vec{r}^{(k)} \\
  \vec{A}
\end{pmatrix}
\]

\( \vec{x} \in C^{\omega} \).

\[ \square \]

3. Some classes of \( C \)-image partition regular matrices

We know that an extension of “first entries matrix” to infinite matrices does not essentially produce image partition regular matrices. Therefore we introduce a sparse version of the notion of first entries matrix which is studied quite elaborately in this section.
Firstly we recall the following definition which is definition 3.1 in Hindman and Strauss [2000].

**Definition 3.1.** Let $A$ be an $\omega \times \omega$ matrix with rational entries. Then $A$ is said to be a *segmented image partition regular matrix* if and only if

1. $A$ contains no row as $\top$;
2. the set $\{ j \in \omega : a_{i,j} \neq 0 \}$ is finite, for each $i \in \omega$; and
3. there is an increasing sequence $\langle a_{n} \rangle_{n=0}^{\infty}$ with elements from $\omega$ satisfying $a_{0} = 0$ and for each $n \in \omega$,

\[
\{ \langle a_{i,0}, a_{i,1}, \ldots, a_{i,n+1} \rangle : i \in \omega \} \setminus \{ \top \}
\]

is either empty or forms the set of rows of a finite image partition regular matrix.

We shall say that $A$ is a *segmented first entries matrix* if each of these finite image partition regular matrices is a first entries matrix. Moreover $A$ is said to be a *monic segmented first entries matrix* if in addition the first non-zero entry of each $\langle a_{i,0}, a_{i,1}, \ldots, a_{i,n+1} \rangle$, if any, is 1.

The most celebrated example of segmented first entries matrices are the finite sums matrix which generates the $(M, P, C)$-systems of Hindman and Lefmann [1993].

**Theorem 3.2.** Any segmented image partition regular matrix is strongly $C$-image partition regular.

**Proof.** Let $\overrightarrow{c_{0}}, \overrightarrow{c_{1}}, \overrightarrow{c_{2}}, \ldots$ denote the columns of a segmented image partition regular matrix $A$ and choose $\langle a_{n} \rangle_{n=0}^{\infty}$ according to the definition of a segmented image partition regular matrix. Suppose $A_{n}$ is the matrix containing columns $\overrightarrow{c_{0}}, \overrightarrow{c_{1}}, \ldots, \overrightarrow{c_{n-1}}$ for each $n \in \omega$. Then the set of non-zero rows of $A_{n}$ is finite and if it is non-empty then it is the set of rows of a finite image partition regular matrix. Let $B_{n} = (A_{0} A_{1} \cdots A_{n})$.

Take a $C$-set $C$ of $\mathbb{N}$. By Theorem 1.8 choose an idempotent $p \in J(\mathbb{N})$ with $C \subseteq p$. Let $C^{*} = n \in C : n + C \subseteq p$. Then $C^{*} \in p$ and $n + C^{*} \in p$ for every $n \in C^{*}$ by [9, Lemma 4.14].

By Theorem 1.10, we can choose $\overrightarrow{x}^{(0)} \in \mathbb{N}^{\omega}$ with the property that, if $\overrightarrow{y} = A_{0} \overrightarrow{x}^{(0)}$, then $y_{i} \in C^{*}$ for every $i \in \omega$ with the $i$-th row of $A_{0}$ is non-zero, and entries of $\overrightarrow{y}$ which correspond to unequal rows of $A_{0}$ are distinct.

We now make the inductive assumption that, for some $m \in \omega$, we have chosen $\overrightarrow{x}^{(0)}, \overrightarrow{x}^{(1)}, \ldots, \overrightarrow{x}^{(m)}$ such that $\overrightarrow{x}^{(i)} \in \mathbb{N}^{\omega}$ for every $i \in \{0, 1, 2, \ldots, m\}$, and, if $\overrightarrow{y} = B_{m} \overrightarrow{x}^{(0)}, \overrightarrow{x}^{(1)}, \ldots, \overrightarrow{x}^{(m)}$ , then $y_{j} \in C^{*}$ for every $j \in \omega$ for which the $j$-th row of $B_{m}$ is non-zero and “$t$” denotes the matrix transpose. We further suppose that entries of $\overrightarrow{y}$ which correspond to unequal rows of $B_{m}$ are distinct.

Let $D = \{ j \in \omega : \text{row } j \text{ of } B_{m+1} \text{ is not } \top \}$. It follows that for each $j \in \omega$, $y_{j} + C^{*} \subseteq p$. (Either $y_{j} = 0$ or $y_{j} \in C^{*}$.) Let $l = \max\{ y_{j} : i \in \omega \} + 1$ and note that $\mathbb{N} \subseteq p$ by Hindman and Strauss [1998], Lemma 6.6. Thus by Theorem 1.10, we can choose $\overrightarrow{x}^{(m+1)} \in \mathbb{N}^{\omega}$ such that, if $\overrightarrow{z} = A_{m+1} \overrightarrow{x}^{(m+1)}$, then $z_{j} \in \mathbb{N} \cap \bigcap_{i \in D} (y_{j} + C^{*})$ for every $j \in D$, and $z_{j} \neq z_{k}$ whenever rows $j$ and $k$ of $A_{m+1}$ are distinct and not equal to $\top$. Since each $z_{j} \in \mathbb{N}$, we also get that $y_{j} + z_{j} \neq y_{k} + z_{k}$ whenever $j, k \in D$ and distinct rows $j$ and $k$ of $B_{m+1}$.

Thus we can choose an infinite sequence $\langle \overrightarrow{x}^{(i)} \rangle_{i \in \omega}$ with the property that for every $i \in \omega$, $\overrightarrow{x}^{(i)} \in \mathbb{N}^{\omega}$, and, if $\overrightarrow{y} = B_{i} \overrightarrow{x}^{(0)}, \overrightarrow{x}^{(1)}, \ldots, \overrightarrow{x}^{(i)}$ , then $y_{j} \in C^{*}$ for every $j \in \omega$ for which
the \( j \)-th row of \( B_i \) is non-zero. Moreover, entries of \( \vec{y} \) corresponding to distinct rows of \( B_i \) are distinct.

Let \( \vec{y} = A\vec{x} \) where \( \vec{x} = (x^{(0)}, x^{(1)}, x^{(3)}, \ldots)^t \). We note that, for every \( j \in \omega \) and for \( i > m \), there exists \( m \in \omega \) such that \( y_j \) is the \( j \)-th entry of \( B_i(x^{(0)}, x^{(1)}, x^{(3)}, \ldots, x^{(i)})^t \). Thus all the entries of \( \vec{y} \) are in \( C^n \) and entries corresponding to distinct rows are distinct. \( \square \)

Now we recall the following definition which is Definition 4.1 in Hindman and Strauss [2000].

**Definition 3.3.** An \( \omega \times \omega \) matrix \( A \) is said to be a restricted triangular matrix if and only if all entries of \( A \) are from \( \mathbb{Z} \) and there exist \( d \in \mathbb{N} \) and an increasing function \( \gamma : \omega \rightarrow \omega \) such that for all \( i \in \omega \),

1. \( a_{i,j(i)} \in \{1, 2, \ldots, d\} \),
2. \( a_{i,l} = 0 \), whenever \( l > j(i) \), and
3. for all \( k > i \) and all \( t \in \{1, 2, \ldots, d\} \), \( t | a_{k,j(i)} \).

**Theorem 3.4.** A restricted triangular matrix \( A \) is strongly C-image partition regular. In particular, if \( p \in \bigcap_{n \in \mathbb{N}} \mathcal{C}_{\mathbb{Z}}(n\mathbb{N}) \) and \( P \in p \), then there exists \( \vec{x} \in \mathbb{N}^\omega \) such that the entries of \( A\vec{x} \) are distinct elements of \( P \).

**Proof.** The proof is essentially done in the proof of Theorem 4.2 in Hindman and Strauss [2000]. \( \square \)

**Corollary 3.5.** Let \( A \) be an \( \omega \times \omega \) matrix with entries from \( \mathbb{Z} \). Suppose exists an increasing function \( \gamma : \omega \rightarrow \omega \) such that for all \( i \in \omega \) the following properties are satisfied:

1. \( a_{i,j(i)} = 1 \) and
2. \( a_{i,l} = 0 \) for all \( l > j(i) \).

Then \( A \) is strongly C-image partition regular.

**Proof.** The corollary is immediate because \( A \) is a restricted triangular matrix with \( d = 1 \). \( \square \)

**Corollary 3.6.** Let \( A \) be an \( \omega \times \omega \) matrix with entries from \( \mathbb{Z} \). In addition suppose \( A \) contains only finitely many non-zero entries in each row. Suppose there exist \( d \in \mathbb{N} \) and a function \( \gamma : \omega \rightarrow \omega \) such that for all \( i \in \omega \), the following are satisfied:

1. \( a_{i,j(i)} \in \{1, 2, \ldots, d\} \) and
2. \( a_{k,j(i)} = 0 \) for all \( k \neq i \).

Then \( A \) is strongly C-image partition regular.

**Proof.** The proof of this corollary follows by noting a possible rearrangement of the columns using condition (2). \( \square \)

**Theorem 3.7.** Let \( A \) be a restricted triangular matrix with finitely many non-zero entries. Let \( \vec{r} \in \mathbb{Z}^\omega \setminus \{0\} \). Then \( \left( \frac{b\vec{r}}{A} \right) \) is a strongly C-image partition regular matrix for some \( b \in \mathbb{Q} \setminus \{0\} \).

**Proof.** Pick \( d \in \mathbb{N} \) and \( j : \omega \rightarrow \omega \) according as Definition 1.3. Take \( l \geq j(0) \) such that \( r_i = 0 \) for all \( i > l \). Also pick \( \gamma \in \omega \) such that \( j(\gamma) \leq l < j(\gamma + 1) \).

Call \( B \) to be the upper left \((\gamma + 1) \times (l + 1)\) corner of \( A \). By Theorem 1.4, \( A \) is C-image partition regular and therefore \( B \) is image partition regular. Applying Theorem 3.2, \( l + 2 \) times,
pick \(b_0, b_1, \ldots, b_l, b\) in \(\mathbb{Q}\) such that

\[
D = \begin{pmatrix}
br_0 & br_1 & br_2 & \cdots & br_l \\
b_0 & 0 & o & \cdots & 0 \\
0 & b_1 & o & \cdots & 0 \\
0 & 0 & b_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & b_l \\
B
\end{pmatrix}
\]

is image partition regular. We claim that \(\begin{pmatrix} br \cr A \end{pmatrix}\) is C-image partition regular. For that let \(C\) be a C-set and let \(c\) be a common multiple of the numerators of \(b_0, b_1, \ldots, b_l\). Then \(C \cap \mathbb{N}cd!\) is again a C-set. By Theorem 3.2, pick \(x_0, x_1, \ldots, x_l\) such that all entries of \(D\) are in \(C \cap \mathbb{N}cd!\) and are distinct. For \(t \in \{0, 1, \ldots, l\}\), one has in particular that \(b_t x_t \in \mathbb{N}cd!\) and consequently \(x_t \in \mathbb{N}d!\). For \(t > l\), choose \(x_t = d!\) exactly as in the proof of Theorem 1.4. One concludes immediately that all entries of \(\begin{pmatrix} br \cr A \end{pmatrix}\) are in \(C\) and are unequal. \(\square\)

**Theorem 3.8.** Let \(A\) be a C-image partition regular matrix and let \(\langle b_n \rangle_{n=0}^\infty\) be a sequence of positive integers. Let

\[
B = \begin{pmatrix}
b_0 & 0 & o & \cdots \\
0 & b_1 & o & \cdots \\
0 & 0 & b_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots \\
A
\end{pmatrix},
\]

Then the matrix \(\begin{pmatrix} O & B \\
A & O \\
A & B
\end{pmatrix}\) is C-image partition regular.

**Proof.** Let \(C\) be a C-set of \(\mathbb{N}\). Take a minimal idempotent \(p\) in \(\beta \mathbb{N}\) such that \(C \in p\). Let \(D := \{x \in C : x + C \in p\}\). Then by [Hindman and Strauss 1998], Lemma 4.14 \(D \in p\) and therefore \(D\) is C-set. So we can get \(\vec{x} \in N^\omega\) such that \(A \vec{x} \in D^\omega\).

Define \(c_n = \sum_{t=0}^\infty a_{n,t} \cdot x_t\), for any given \(n \in \omega\). Then \(C \cap (c_n + C) \in p\), so pick \(z_n \in C \cap (c_n + C) \cap N b_n\) and let \(y_n = \frac{z_n}{b_n}\). Thus we get

\[
\begin{pmatrix} O & B \\
A & O \\
A & B
\end{pmatrix} \begin{pmatrix} \vec{x} \\
\vec{y} \end{pmatrix} \in C^{\omega+\omega+\omega}.
\]

\(\square\)

Let us quickly recall the following definition which is Definition 4.8 in Hindman and Strauss [2000].

**Definition 3.9.** Let \(C\) be a \(\gamma \times \delta\) matrix with finitely many non-zero entries in each row, for some \(\gamma, \delta \in \omega \cup \{\omega\}\). For each \(t < \delta\), let \(B_t\) be a \(u_t \times v_t\) (finite) matrix. Let \(R = \{(i, j) : i < \gamma\) and \(j \in \times_{t<\delta}\{0, 1, \ldots, u_t1\}\). Given \(t < \delta\) and \(k \in \{0, 1, \ldots, u_t1\}\), denote the \(k\)-th row of \(B_t\) by the notation \(b_k\). Then \(D\) is said to be an insertion matrix of \(\langle B_t \rangle_{t<\delta}\) into \(C\) if and only
if the rows of $D$ are all rows of the form
\[ c_{i,0} \cdot b_{j(0)} \sim c_{i,1} \cdot b_{j(1)} \sim \cdots \]
where $(i, j) \in R$.

For example we can consider that one which is given in Hindman and Strauss [2000]. Suppose $C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $B_0 = \begin{pmatrix} 1 & 1 \\ 5 & 7 \end{pmatrix}$, and $B_1 = \begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix}$, then the following matrix
\[
D = \begin{pmatrix}
1 & 1 & 0 & 0 \\
5 & 7 & 0 & 0 \\
2 & 2 & 0 & 1 \\
2 & 2 & 3 & 3 \\
10 & 14 & 0 & 1 \\
10 & 14 & 3 & 3
\end{pmatrix}
\]
is an insertion matrix of $\langle B_t \rangle_{t<2}$ into $C$.

**Theorem 3.10.** Let $C$ be a segmented first entries matrix. Also let $B_t$ be a $u_t \times v_t$ (finite) image partition regular matrix, for each $t < \omega$. Then any insertion matrix of $\langle B \rangle_{t<\omega}$ into $C$ is $C$-image partition regular.

**Proof.** Let us take $A$ to be an insertion matrix of $\langle B \rangle_{t<\omega}$ into $C$. For each $t \in \omega$, pick by Theorem 1.5(g) of Hindman and Strauss [2000], some $m_t \in \mathbb{N}$ and a $u_t \times m_t$ first entries matrix $D_t$ with the property that for all $\vec{y} \in \mathbb{N}^{m_t}$ there exists $\vec{x} \in \mathbb{N}^{u_t}$ such that $B_t \vec{x} = D_t \vec{y}$. Let $E$ be another insertion matrix of $\langle D_t \rangle_{t<\omega}$ into $C$ where the rows occur in the corresponding position to those of $A$. That is, if $i < \omega$ and $j \in \times_{t<\omega} \{0, 1, \cdots, u_t \}$ and
\[
\begin{align*}
&c_{i,0} \sim c_{i,1} \sim \cdots \\
&d_{j(0)} \sim d_{j(1)} \sim \cdots
\end{align*}
\]
is row $k$ of $A$, then
\[
\begin{align*}
&c_{i,0} \sim c_{i,1} \sim \cdots \\
&d_{j(0)} \sim d_{j(1)} \sim \cdots
\end{align*}
\]
is row $k$ of $E$.

Let $H$ be a C-set of $\mathbb{N}$. By Lemma 4.9 of Hindman and Strauss [2000], $E$ is a segmented first entries matrix. Therefore, pick $\vec{y} \in \mathbb{N}^\omega$ such that all entries of $E\vec{y}$ are contained in $H$. Let $\delta_0 = \gamma_0 = 0$ and for $n \in \mathbb{N}$ take $\delta_n := \sum_{i=0}^{n} u_i$ and $\gamma_n := \sum_{i=0}^{n} m_i$. For each $n \in \omega$, pick
\[
\begin{pmatrix}
x_{\delta_n} \\
x_{\delta_n+1} \\
\vdots \\
x_{\delta_n+n-1}
\end{pmatrix} \in \mathbb{N}^n \text{ such that } B_t \begin{pmatrix}
x_{\delta_n} \\
x_{\delta_n+1} \\
\vdots \\
x_{\delta_n+n-1}
\end{pmatrix} = D_t \begin{pmatrix}
y_{\gamma_n} \\
y_{\gamma_n+1} \\
\vdots \\
y_{\gamma_n+n-1}
\end{pmatrix}.
\]
Then it is clear that $A \vec{x} = E \vec{y}$. □

At the end of the paper we raise the following question.

**Question:** Is it true that every $C$-image partition regular matrices are Centrally image partition regular?
REFERENCES

D. De, N. Hindman, and D. Strauss. A new and stronger central sets theorem. *Fund. Math.*, 199 (2):155–175, 2008.

W. A. Deuber, N. Hindman, I. Leader, and H. Lefmann. Infinite partition regular matrices. *Combinatorica*, 15(3):333–355, 1995. ISSN 0209-9683. URL https://doi.org/10.1007/BF01299740.

H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory*. Princeton University Press, Princeton, N.J., 1981. ISBN 0-691-08269-3. M. B. Porter Lectures.

N. Hindman and H. Lefmann. Partition regularity of $(M, P, C)$-systems. *J. Combin. Theory Ser. A*, 64(1):1–9, 1993. ISSN 0097-3165. URL https://doi.org/10.1016/0097-3165(93)90084-L.

N. Hindman and D. Strauss. *Algebra in the Stone-Chu compactification: theory and applications*, volume 27. Walter de Gruyter, 1998.

N. Hindman and D. Strauss. Infinite partition regular matrices. II. Extending the finite results. In *Proceedings of the 15th Summer Conference on General Topology and its Applications/1st Turkish International Conference on Topology and its Applications (Oxford, OH/Istanbul, 2000)*, volume 25, pages 217–255 (2002), 2000.

N. Hindman, I. Leader, and D. Strauss. Infinite partition regular matrices: solutions in central sets. *Trans. Amer. Math. Soc.*, 355(3):1213–1235, 2003. ISSN 0002-9947. URL https://doi.org/10.1090/S0002-9947-02-03191-4.

I. Schur. Über kongruenz $x\equiv \ldots (mod. p.)$. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 25:114–116, 1917.

B. L. Van der Waerden. Beweis einer baudetschen vermutung. *Nieuw Arch. Wiskunde*, 15:212–216, 1927.

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