Strict s-numbers of non-compact Sobolev embeddings into continuous functions

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Abstract. For limiting non-compact Sobolev embeddings into continuous functions we study behavior of Approximation, Gelfand, Kolmogorov, Bernstein and Isomorphism s-numbers. In the one dimensional case the exact values of the above-mentioned strict s-numbers were obtained and in the higher dimensions sharp estimates for asymptotic behavior of strict s-numbers were established. As all known results for s-numbers of Sobolev type embeddings are studied mainly under the compactness assumption then our work is an extension of existing results and reveal an interesting behavior of s-numbers in the limiting case when some of them (Approximation, Gelfand and Kolmogorov) have positive lower bound and others (Bernstein and Isomorphism) are decreasing to zero. From our results also follows that such limiting non-compact Sobolev embeddings are finitely strictly singular maps.

1. Introduction and main results

This paper is devoted to the study of strict s-numbers of certain type of Sobolev embedding. As far as we know, all known results deal only with embeddings under the assumption of compactness. Since some s-numbers may be regarded as measurements of compactness, such assumption seems to be reasonable. However not every s-number has so strong connection with compactness. For instance, Bernstein numbers represent a method of quantification the finite strictly singularity, the property weaker than compactness.

Let us take a look at known results. Denote by Q a cube in \( \mathbb{R}^d \), let \( 1 \leq p < d \) and let \( p \leq q < \frac{dp}{d-p} \) or \( p = d \) and \( p \leq q < \infty \). Then

\[
b_n(V_0^1L^p(Q) \hookrightarrow L^q(Q)) \asymp n^{-1/d}
\]

and notice that the decay of \( b_n \) is the same, regardless how close to the limiting (i.e. non-compact) case we are [see Ngu15]. Here \( \asymp \) means that left and right hand sides are bounded by each other up to multiplicative constants independent of \( n \). By \( V_0^1X(Q) \), we mean the space of functions \( u \) defined on \( Q \) such that its continuation

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by zero outside $Q$ is weakly differentiable and $|\nabla u| \in X(\Omega)$. The norm is defined as $\|u\|_{V^1_0(Q)} = \|\nabla u\|_{X(Q)}$ and the relation “$\hookrightarrow$” represents continuous inclusion.

This motivates us to ask the question what is happening on this borderline. In our result, we show that in the case when $q$ tends to infinity, the answer is in consent with the forehead discussion. In the situation when $q = \infty$, the proper domain space is not $V^1_0 L^d(Q)$, but the Sobolev-Lorentz space $V^1_0 L^{d,1}(Q)$, the largest rearrangement invariant Banach function space, which is continuously embedded into $L^{\infty}(Q)$. Moreover, every function in $V^1_0 L^{d,1}(Q)$ has continuous representative, hence

$$V^1_0 L^{d,1}(Q) \hookrightarrow C(Q). \quad (1.1)$$

The embedding (1.1) is thus an example of sharp non-compact embedding which is the subject of our first main result.

**Theorem 1.1.** Let $Q$ be a cube in $\mathbb{R}^d$, $d \geq 2$. Then for every $n \in \mathbb{N}$

$$s_n(V^1_0 L^{d,1}(Q) \hookrightarrow C(Q)) \asymp n^{-\frac{1}{d}}, \quad (1.2)$$

where $s_n$ stands for $n$-th Bernstein or Isomorphism number and

$$s_n(V^1_0 L^{d,1}(Q) \hookrightarrow C(Q)) \asymp 1 \quad (1.3)$$

for $s_n$ representing Approximation, Gelfand or Kolmogorov numbers.

The definitions of the above-mentioned numbers are recalled in Section 2.2. The deep part of this result is contained in (1.2). In particular, the decay of Bernstein numbers to zero implies that (1.1) is finitely strictly singular. The relation (1.3) is, on the other hand, rather clear and we include it just for the sake of completeness. As a consequence of our approach, we also get (1.2) where $L^{d,1}$ is replaced by any smaller Banach function space.

**Corollary 1.2.** Let $X(Q)$ be any Banach function space over the cube $Q$ in $\mathbb{R}^d$, $d \geq 2$, satisfying $X(Q) \hookrightarrow L^{d,1}(\Omega)$. Then for every $n \in \mathbb{N}$

$$s_n(V^1_0 X(Q) \hookrightarrow C(Q)) \asymp n^{-\frac{1}{d}},$$

in which $s_n$ stands for $n$-th Bernstein or Isomorphism number.

In (1.2) of Theorem 1.1, we do not focus on obtaining estimates on constants involved in “$\asymp$”, although they might be tracked from the proofs by careful reader. The error is strongly dependent on the used method and it is very unlikely to get sharp estimates.

Considering the one-dimensional case, the situation is completely different. Using caref ul estimates, we obtain the exact values of various $s$-numbers in the non-compact embedding $V^1_0 L^1(I) \hookrightarrow C(I)$. The result reads as follows.

**Theorem 1.3.** Let $I$ be a bounded interval. Then for every $n \in \mathbb{N}$

$$s_n(V^1_0 L^1(I) \hookrightarrow C(I)) = \frac{1}{2n}, \quad (1.4)$$

where $s_n$ stands for $n$-th Bernstein or Isomorphism number,

$$s_n(V^1_0 L^1(I) \hookrightarrow C(I)) = \frac{1}{2}, \quad (1.5)$$
if $s_n$ is Approximation or Gelfand number and for every $n \geq 2$

$$d_n(V_{0,1}^{1,1}(I) \hookrightarrow C(I)) = \frac{1}{4},$$

(1.6)
in which $d_n$ denotes the $n$-th Kolmogorov number.

The asymptotic behaviour of involved $s$-numbers are in the correspondence with the above-mentioned vision. Furthermore, we would like to notice the difference between Gelfand and Kolmogorov numbers. This in particular implies that the space $V_{0,1}^{1,1}(I)$, consisting of absolute continuous functions with zero boundary condition, does not have the lifting property [cf. Pin85, p. 36].

Let us now briefly comment the method of proof. The crucial ingredient of the proof of (1.4) is the following proposition based on the deep result [CFP+09, Theorem 1].

**Proposition 1.4.** Let $E$ be a $n$-dimensional subspace of $C(I)$ where $I$ is any bounded interval. Then to every $\varepsilon > 0$ there exist a function $g \in E$, $\|g\|_{\infty} \leq 1 + \varepsilon$, and $n$-tuple of points $t_1 < t_2 < \cdots < t_n$ in $I$ such that

$$g(t_k) = (-1)^k \quad \text{for } 1 \leq k \leq n.$$
The paper is structured as follows. In Section 2 we recall the definitions we use and we collect all necessary later-needed material. Section 3 is devoted to some preliminary results including the proof of Proposition 1.4. Section 4 deals with the one-dimensional case contained in Theorem 1.3 and Section 5 treats the higher-dimensional situation collected in Theorem 1.1.

2. BACKGROUND MATERIAL

2.1. Banach spaces. For Banach spaces $X$ and $Y$, we denote by $B(X, Y)$ the set of all bounded linear operators acting between $X$ and $Y$. For any $T \in B(X, Y)$, we use just $\|T\|$ for its operator norm, since the domain and target spaces are always clear from the context. By $B_X$ and $S_X$, we mean the closed unit ball of $X$ and unit sphere of $X$, respectively.

Let $Z$ be a closed subspace of the Banach space $X$. The quotient space $X/Z$ is the collection of the sets $[x] = x + Z = \{x + z; z \in Z\}$ equipped with the norm $\|[x]\|_{X/Z} = \inf\{\|x - z\|_X; z \in Z\}$.

We sometimes adopt the notation $\|x\|_{X/Z}$ when no confusion is likely to happen. Recall the notion of canonical map $Q_Z : X \to X/Z$, given by $Q_Z(x) = [x]$.

Let $Z$ be a closed subspace of a Banach space $X$. Then $Z^*$ is isometric to $X^*/Z^\perp$, (2.1) where $Z^\perp = \{\phi \in X^*; \phi(z) = 0\text{ for all } z \in Z\}$ [see FH+11, Proposition 2.6].

2.2. The $s$-numbers. Let $X$ and $Y$ be Banach spaces. To a given operator $T \in B(X, Y)$, one can assign a scalar sequence $s_n(T), n \in \mathbb{N}$, satisfying for every $n \in \mathbb{N}$ the following conditions

(S1) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$, (monotonicity),
(S2) $s_n(T + S) \leq s_n(T) + \|S\|$ for every $S \in B(X, Y)$,
(S3) $s_n(B \circ T \circ A) \leq \|B\|s_n(T)\|A\|$ for every $A \in B(X_1, X)$ and $B \in B(Y, Y_1)$, (ideal property),
(S4) $s_n(\text{Id}: \ell^2_n \to \ell^2_n) = 1$, (norming property),
(S5) $s_n(T) = 0$ whenever rank $T < n$, (rank property).

The number $s_n(T)$ is then called the $n$-th $s$-number of the operator $T$. When (S4) is replaced by a stronger condition

(S6) $s_n(\text{Id}: E \to E) = 1$ for every Banach space $E$, dim $E = n$,

we say that $s_n(T)$ is the $n$-th strict $s$-number of $T$. We also adopt the notation $s_n(T : X \to Y)$, or $s_n(X \hookrightarrow Y)$ when $T$ is identity, to emphasize the spaces involved.

Note that the definition of $s$-numbers is not unified in literature. The original definition of $s$-numbers, which was introduced by Pietsch in [Pie74], uses the condition (S6) which was later modified to accommodate wider class of $s$-numbers (like Weyl, Chang and Hilbert numbers). Some authors also use more strict version of (S2). For a detailed account of $s$-numbers, one is referred for instance to [Pie07], [CS90] or [LE11].
We shall recall some particular strict s-numbers. Let $T \in B(X, Y)$ and $n \in \mathbb{N}$. The $n$-th Approximation, Gelfand, Kolmogorov, Isomorphism and Bernstein numbers of $T$ are defined by

$$a_n(T) = \inf_{F \in B(X, Y)} \{ \| T - F \| : \text{rank } F < n \},$$

$$c_n(T) = \inf_{M \subseteq X} \{ \sup_{x \in B_M} \| Tx \|_Y : \text{codim } M < n \},$$

$$d_n(T) = \inf_{N \subseteq Y} \{ \sup_{x \in B_X} \| Tx \|_{Y/N} : \text{dim } N < n \},$$

$$i_n(T) = \sup \{ \| A \|^{-1} \| B \|^{-1} : A \in B(Y, E), B \in B(E, X), A \circ T \circ B \text{ is the identity map on } E \},$$

where the supremum is taken over all Banach spaces $E$ with $\text{dim } E \geq n$ and $A \in B(Y, E), B \in B(E, X)$ such that $A \circ T \circ B$ is the identity map on $E$, and

$$b_n(T) = \sup_{M \subseteq X} \{ \inf_{x \in \mathcal{S}_M} \| Tx \|_Y : \text{dim } M \geq n \},$$

respectively. It is known that Approximation numbers are the largest s-numbers while the Isomorphism numbers are the smallest among all strict s-numbers. Let us mention that for any $T \in B(X, Y)$ and any $n \in \mathbb{N}$ we have

$$i_n(T) \leq b_n(T) \leq \max\{c_n(T), d_n(T)\} \leq a_n(T).$$

Moreover, if $X$ has lifting property, then $a_n(T) = d_n(T)$ for every $n \in \mathbb{N}$ [see Pie07, LE11].

2.3. Function spaces. Given an open set $\Omega \subseteq \mathbb{R}^d$, we denote by $C(\Omega)$ the space of all continuous functions on $\Omega$ equipped with the supremum norm.

We will, at some stage, use the setting of so-called Banach function spaces. This wide class of function spaces provides handy tools commonly used in study of Sobolev embeddings. Roughly speaking, a Banach function space $X(\Omega)$ over an measurable set $\Omega \subseteq \mathbb{R}^d$ is a collection of all real-valued measurable functions, for which the functional $\| \cdot \|_{X(\Omega)}$ is finite, where $\| \cdot \|_{X(\Omega)}$ is a Banach function norm satisfying given set of axioms. We also sometimes write $X$ instead of $X(\Omega)$ for brevity. We kindly refer the reader who is not familiar with this notion to the monograph [BS88, Chapter 1 and 2].

Basic examples of Banach function spaces are Lorentz spaces $L^{p,q}(\Omega)$, where the norm is given by

$$\| f \|_{p,q} = \left( p \int_0^\infty \left[ \mu_f(s) \right]^q s^{-1} \frac{1}{s} \right)^{\frac{1}{q}}.$$

Here, $\mu_f$ is the distribution function of $f$, defined as

$$\mu_f(t) = \left| \{ x \in \Omega; |f(x)| > t \} \right| \quad \text{for } 0 \leq t < \infty.$$
We consider only the case $1 \leq q \leq p$. When $p = q$, then $L^{p,p} = L^p$, the customary Lebesgue space. Recall that if $q \leq p$ then
\[
\sum_k \|f \chi_{E_k}\|_{p,q}^p \leq \|f\|_{p,q}^p
\] (2.2)
for any $f \in L^{p,q}(\Omega)$ and any pairwise disjoint measurable subsets $E_k$ of $\Omega$.

Let $X(\Omega)$ be a Banach function space over $\Omega \subseteq \mathbb{R}^d$. By Sobolev space $V_0^1 X(\Omega)$ we mean the collection of all functions $u$ such that their continuation by zero outside $\Omega$ is weakly differentiable in $\mathbb{R}^d$ and satisfy $|\nabla u| \in X(\Omega)$. On $V_0^1 X(\Omega)$ we use the norm
\[
\|u\|_{V_0^1 X(\Omega)} = \|\nabla u\|_{X(\Omega)}.
\]
When $X(\Omega) = L^p(\Omega)$, we use the abbreviate notation $V_0^{1,p}(\Omega)$. Recall that if $|\Omega| < \infty$ then, thanks to Poincaré inequality, $\|u\|_{V_0^1 X(\Omega)}$ is equivalent to the $\|u\|_{X(\Omega)} + \|\nabla u\|_{X(\Omega)}$, the usual norm in Sobolev spaces denoted by $W_0^{1,1}(\Omega)$.

### 2.4. John domains and Sobolev embeddings
We say that a bounded domain $\Omega \subseteq \mathbb{R}^d$ is a John domain if there is a constant $C_J \geq 1$ and a distinguished point $x_0 \in \Omega$ (called central point) so that each point $x \in \Omega$ can be joined to $x_0$ by a curve (called John curve) $\gamma: [0,1] \rightarrow \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and
\[
\text{dist}(\gamma(t), \partial \Omega) \geq C_J^{-1}|x - \gamma(t)|
\] (2.3)
for every $t \in [0,1]$.

The John domains provides a wide class of domains for which the classical Poincaré inequality hold. We restrict our attention only to the limiting case
\[
\|u - u_\Omega\|_{\infty(\Omega)} \leq C\|\nabla u\|_{d,1(\Omega)},
\]
where we want to control the size of the constant $C$ appearing here. Here, $u_\Omega$ represents the integral average of $u$ over $\Omega$. It could be shown that $C$ depends on $C_J$, the John constant of the domain $\Omega$, and its dimension $d$. However, this result seems not to be available in the literature hence we sketch the proof here.

Let us come out of the potential estimate obtained independently in [Mar88] and [Res80] (see also [Boj88]) which reads as follows. Suppose that $\Omega \subseteq \mathbb{R}^d$ is a John domain. Then
\[
|u(x) - u_\Omega| \leq C(C_J, d) \int_{\Omega} \frac{|\nabla u(z)|}{|x - z|^{d-1}} \, dz
\] (2.4)
for every locally Lipschitz function $u$ in $\Omega$ and all $x \in \Omega$. Assuming that $\nabla u \in L^{n,1}(\Omega)$, we can follow on the right hand side of (2.4) by
\[
\int_{\Omega} \frac{|\nabla u(z)|}{|x - z|^{d-1}} \, dz = \int_0^\infty \int_{\{|\nabla u| > s\}} |x - z|^{1-d} \, dz \, ds 
\leq \int_0^\infty \left[\mu_{|\nabla u|}(s)\right]^{\frac{1}{n}} \, ds = \frac{1}{d} \|\nabla u\|_{d,1(\Omega)}
\]
One can thus show that every weakly differentiable $u$ such that $\nabla u \in L^{d,1}(\Omega)$ has continuous representative which satisfies

$$\sup_{x,y \in \Omega} |u(x) - u(y)| \leq C(C_J, d)\|\nabla u\|_{d,1(\Omega)}. \quad (2.5)$$

For a different approach, where $C$ depends on the constants in corresponding isoperimetric inequalities, we refer to [CP98].

3. Preliminaries

Throughout this section, assume that $I$ is any bounded interval. In one-dimensional case, we can reduce the $s$-numbers of Sobolev embeddings to the $s$-numbers of an integral operator. This is the objective of the following proposition.

**Proposition 3.1.** Let $I$ be a bounded interval. Then for every $n \in \mathbb{N}$

$$s_n(V_0^{1,1}(I) \hookrightarrow C(I)) = s_n(V : L_0^1(I) \hookrightarrow C(I)),$$

where

$$V f(t) = \int_{I \cap (-\infty, t)} f(s) \, ds \quad \text{for } t \in I$$

and $L_0^1(I)$ is a subspace of $L^1(I)$ given by

$$L_0^1(I) = \{ f \in L^1(I); \int_I f = 0 \}.$$

**Proof.** To prove “$\leq$”, consider the composition

$$L_0^1 \xrightarrow{V} V_0^{1,1} \hookrightarrow C.$$

By the ideal property (S3) of $s$-numbers,

$$s_n(V : L_0^1 \hookrightarrow C) \leq \|V\| s_n(V_0^{1,1} \hookrightarrow C)$$

and $\|V\| = 1$ since $\|V f\|_{V_0^{1,1}} = \|f\|_1$ for every $f \in L_0^1$.

To prove the reverse inequality, we use the chain

$$V_0^{1,1} \xrightarrow{V^{-1}} L_0^1 \xrightarrow{V} C.$$

Note that $V$ is one-to-one mapping from $L_0^1$ onto $V_0^{1,1}$, $V^{-1}$ is well-defined and thus

$$s_n(V_0^{1,1} \hookrightarrow C) \leq s_n(V : L_0^1 \hookrightarrow C) \|V^{-1}\|.$$

Next $\|V^{-1}u\|_{L_0^1} = \|u'\|_{L_0^1} = \|u\|_{V_0^{1,1}}$, whence $\|V^{-1}\| = 1$. \qed

**Remark 3.2.** As a consequence of Proposition 3.1, we have the exact value of the norm of the embedding $V_0^{1,1}(I) \hookrightarrow C(I)$, namely

$$\sup_{f \in V_0^{1,1}(I)} \frac{\|f\|_{\infty}}{\|f'\|_1} = \|V\| = \frac{1}{2}.$$
Proof of Proposition 1.4. The unit ball $B_E$ is compact and, by the Arzelà-Ascoli theorem, equi-continuous. Thus, to a given $\varrho \in (0, 1)$ there is a $\delta > 0$ such that

$$\sup_{f \in B_E} |f(u) - f(v)| < \varrho$$  \hspace{1cm} (3.1)

for any $u, v \in [0, 1]$ satisfying $|v - u| < \delta$. To this $\delta$, choose points $s_k$ in $I$, such that $s_0 < s_1 < \cdots < s_{N+1}$ and $s_{k+1} - s_k < \delta$, ($0 \leq k \leq N$). We also require $N \geq n$.

Now, define a linear mapping $\Phi: E \to \ell_\infty^N$ by

$$(\Phi f)_k = f(s_k), \quad (1 \leq k \leq N), \quad \text{for } f \in E.$$  

We show that

$$(1 - \varrho)\|f\|_\infty \leq \|\Phi f\|_\infty \leq \|f\|_\infty \quad \text{for } f \in E.$$  \hspace{1cm} (3.2)

The latter inequality is obvious. As for the former, suppose that $f \in E$ is given and attains its norm at some point $s \in I$. To such $s$, there is a unique $k$ such that $s_k \leq s < s_{k+1}$, whence

$$\|f\|_\infty - |f(s_k)| = |f(s)| - |f(s_k)| \leq |f(s) - f(s_k)| \leq \varrho \|f\|_\infty,$$

thanks to (3.1). The inequality (3.2) therefore follows by taking the supremum over $1 \leq k \leq N$. By (3.2), $\Phi$ maps $E$ onto $\Phi(E)$ isomorphically, whence $\Phi(E)$ forms a $n$-dimensional subspace in $\ell_\infty^N$. By the Zigzag Theorem [CFP+09, Theorem 1], there is an element $g \in E$ and indices $1 \leq k_1 < k_2 < \cdots < k_n \leq N$ such that $\|\Phi g\|_\infty = 1$ and

$$g(s_{k_j}) = (-1)^j \quad \text{for } 1 \leq j \leq n.$$  

Also, by (3.2),

$$\|g\|_\infty \leq \frac{1}{1 - \varrho} \leq 1 + \varepsilon$$

for $\varrho$ taken sufficiently small. The proposition now follows by taking $t_j = s_{k_j}$ for $1 \leq j \leq n$. \hfill \Box

Lemma 3.3. Let $T: X(I) \to C(I)$ be any bounded linear operator acting on Banach function space $X(I)$ over an interval $I$ taking its values in continuous functions. Suppose that $f_k$, ($k \in \mathbb{N}$), is a collection in $X$ having the following properties.

(i) It holds $\|f_k\|_X = 1$ for each $k \in \mathbb{N}$,

(ii) There is a point $\xi \in I$, two sequences $s_k \to \xi$ and $t_k \to \xi$ and two points $a < b$ in $I$ such that $Tf_k(s_k) \to a$ and $Tf_k(t_k) \to b$.

Then we have the following lower bound for the Kolmogorov numbers for every $n \geq 2$.

$$d_n(T: X \to C) \geq \frac{b - a}{2}.$$  \hspace{1cm} (3.3)

Proof. Let $n \geq 2$ and any $\varepsilon > 0$ be given. By the definition of the $n$-th Kolmogorov number, there is a subspace $N$ with $\dim N < n$ such that

$$d_n(T) + \varepsilon \geq \sup_{f \in B_X} \|Tf\|_{C/N}.$$
Now, by the definition of the quotient norm, to every \( f_k \) one can attach a function \( g_k \in N \) in a way that
\[
\|T f_k - g_k\|_\infty \leq \|T f_k\|_{C/N} + \varepsilon.
\]
Observe that the set of all the functions \( g_k \) is bounded in \( N \). Indeed,
\[
\|g_k\|_\infty \leq \|T f_k - g_k\|_\infty + \|T f_k\|_\infty \leq \|T f_k\|_{L^\infty/N} + \varepsilon + \|f_k\|_{X} \leq d_n(T) + 2\varepsilon + \|T\|
\]
for every \( k \in \mathbb{N} \). Thus, since \( N \) is finite-dimensional, there is a convergent subsequence of \( \{g_k\} \) which we denote \( \{g_k\} \) again. Hence \( g_k \) converges to, say, \( g \in N \), i.e., there is an index \( k_0 \) such that \( \|g_k - g\|_\infty < \varepsilon \) for every \( k \geq k_0 \). The limiting function \( g \) then satisfies
\[
\|T f_k - g\|_\infty \leq \|T f_k - g_k\|_\infty + \|g_k - g\|_\infty \leq \|T f_k\|_{L^\infty/N} + 2\varepsilon
\]
for \( k \geq k_0 \) and thus
\[
\sup_{k \geq k_0} \|T f_k - g\|_\infty \leq d_n(V) + 3\varepsilon. \tag{3.4}
\]
Next, we estimate the left hand side of (3.4) by taking the value attained at the points \( s_k \), i.e.,
\[
\sup_{k \geq k_0} \|T f_k - g\|_\infty \geq \sup_{k \geq k_0} |T f_k(s_k) - g(s_k)| \geq |a - g(\xi)|, \tag{3.5}
\]
or at the points \( t_k \), i.e.,
\[
\sup_{k \geq k_0} \|T f_k - g\|_\infty \geq \sup_{k \geq k_0} |T f_k(t_k) - g(t_k)| \geq |b - g(\xi)|, \tag{3.6}
\]
where we used the continuity of \( g \). Combining (3.4), (3.5) and (3.6), we get
\[
d_n(V) + 3\varepsilon \geq \max\{|a - g(t)|, |b - g(t)|\} \geq \frac{b-a}{2}.
\]
Therefore we obtain (3.3) by the arbitrariness of \( \varepsilon \). \( \square \)

4. Embeddings on the interval

Throughout this section, all function spaces will be defined over the closed unit interval unless explicitly stated.

Lemma 4.1. Let \( n \in \mathbb{N} \). Then
\[
i_n(V : L^1_0 \to C) \geq \frac{1}{2n}.
\]

Proof. Let \( n \in \mathbb{N} \) be fixed. Consider the chain
\[
\ell^\infty_n \xrightarrow{B} L^1_0 \xrightarrow{V} C \xrightarrow{A} \ell^\infty_n
\]
where we define \( A \) as
\[
(Af)_k = f\left(\frac{2k - 1}{2n}\right), \quad (1 \leq k \leq n), \quad \text{for } f \in C.
\]
and \( B \) by
\[
B(x) = 2n \sum_{k=1}^{n} x_k \left(\chi_{I_{2k+1}} - \chi_{I_{2k}}\right) \quad \text{for } x \in \ell^1_n,
\]
where the intervals \( I_1, I_2, \ldots, I_{2n} \) are the non-overlapping intervals of the same length \( \frac{1}{2n} \), i.e.,

\[
I_k = \left[ \frac{k - 1}{2n}, \frac{k}{2n} \right] \quad \text{for } 1 \leq k \leq 2n.
\]

Both \( A \) and \( B \) are well-defined and the composition \( A \circ V \circ B \) is an identity mapping on \( n \)-dimensional space \( \ell_\infty^n \). The operator norms are \( \|A\| = 1 \) and \( \|B\| = 2n \) thus, by the definition of the \( n \)-th Isomorphism number,

\[
i_n(V) \geq \|A\|^{-1}\|B\|^{-1} \geq \frac{1}{2n}.
\]

\[\square\]

**Lemma 4.2.** Let \( n \in \mathbb{N} \). Then

\[
b_n(V : L^1_{\infty} \to C) \leq \frac{1}{2n}.
\]

**Proof.** By the definition of the \( n \)-th Bernstein number, it is enough to show that for any \( \varrho > 0 \) and any \( n \)-dimensional subspace \( E \) of \( L^1_{\infty} \) satisfying

\[
\|Vf\|_\infty \geq \varrho\|f\|_1 \quad \text{for every } f \in E
\]

we have

\[
\varrho \leq \frac{1}{2n}.
\]

To prove this, fix some arbitrary \( \varepsilon \in (0, 1) \). Since \( V \) is linear and injective, \( V(E) \) is a \( n \)-dimensional subspace of \( C \). By Proposition 1.4 there is an element \( h \in L^1_{\infty} \) such that \( \|Vh\|_\infty \leq 1 + \varepsilon \) and

\[
Vh(t_k) = (-1)^k \quad \text{for } 1 \leq k \leq n.
\]

for some points \( 0 \leq t_1 < t_2 < \cdots < t_n \leq 1 \). Let us estimate the norm \( \|h\|_1 \). We have

\[
\int_0^1 |h(s)| \, ds = \int_0^{t_1} |h(s)| \, ds + \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} |h(s)| \, ds + \int_{t_n}^1 |h(s)| \, ds
\]

\[
\geq |Vh(t_1)| + \sum_{k=1}^{n-1} |Vh(t_{k+1}) - Vh(t_k)| + |Vh(t_n)|
\]

\[
= 1 + 2(n - 1) + 1 = 2n.
\]

Note, that for the third term, we used that \( \int_{t_n}^1 h = \int_0^{t_n} h \) thanks to the boundary condition in \( L^1_{\infty} \). Next, by (4.1),

\[
1 + \varepsilon \geq \|Vh\|_\infty \geq \varrho \int_0^1 |h(s)| \, ds,
\]

whence

\[
\varrho \leq \frac{1 + \varepsilon}{2n}
\]

and the result follows by the arbitrariness of \( \varepsilon \). \[\square\]
Lemma 4.3. Let \( n \in \mathbb{N} \). Then
\[
c_n(V : L^1_0 \to C) \geq \frac{1}{2}.
\]

**Proof.** We will use an alternative expression for Gelfand numbers, proved in [CS90, Proposition 2.3.2], which proposes that \( c_n(V) \) is the infimum of all \( \varrho > 0 \) such that
\[
\|Vf\|_\infty \leq \sup_{1 \leq k \leq m} |\varphi_k(f)| + \varrho \|f\|_1 \quad \text{for all } f \in L^1_0,
\]
where \( m < n \) and \( \varphi_k, (1 \leq k \leq m) \), are any continuous linear functionals on \( L^1_0 \).

First, we explicitly describe all possible \( \varphi \), i.e., we find the dual space of \( L^1_0 \). Since \( L^1_0 \) is closed subspace of \( L^1 \), by (2.1), its dual is isometric to the quotient
\[
(L^1_0)^* \simeq L^\infty/(L^1_0)^\perp.
\]
Here, we have
\[
(L^1_0)^\perp \simeq \{ g \in L^\infty; \int_0^1 fg = 0 \text{ for all } f \in L^1_0 \}
\]
\[
\simeq \{ g \text{ is a constant function} \} \simeq \mathbb{R},
\]
so \( (L^1_0)^* \) is isometric to \( L^\infty/\mathbb{R} \) via the mapping \( \sigma: L^\infty/\mathbb{R} \to (L^1_0)^* \) given by
\[
\sigma[g](f) = \int_0^1 f(t)g(t) \, dt \quad \text{for } [g] \in L^\infty/\mathbb{R} \text{ and } f \in L^1_0.
\]
Note that \( \sigma \) is well-defined, i.e., the definition does not depend on the choice of the representative \( g \), since, if \( [g_1] = [g_2] \), then there is a constant \( c \) such that \( g_1 = g_2 + c \) a.e. and hence
\[
\sigma[g_1](f) = \int_0^1 f(t)g_1(t) \, dt = c \int_0^1 f(t) \, dt + \int_0^1 f(t)g_2(t) \, dt = \sigma[g_2](f)
\]
for every \( f \in L^1_0 \) and therefore \( \sigma[g_1] = \sigma[g_2] \). In conclusion, (4.2) rewrites as
\[
\|Vf\|_\infty \leq \sup_{1 \leq k \leq m} \left| \int_0^1 f(t)g_k(t) \, dt \right| + \varrho \|f\|_1,
\]
where \( m < n \) and \( g_k \) are bounded functions.

Next, fix \( \varrho > 0 \), \( m < n \) and any \( g_k \in L^\infty \), \((1 \leq k \leq m)\), such that (4.3) holds for every \( f \in L^1_0 \). We have to show that \( \varrho \geq \frac{1}{2} \). Let \( \varepsilon > 0 \) and denote \( M = \max_{1 \leq k \leq m} \|g_k\|_\infty \). All the values of the functions \( |g_k| \) thus essentially range in the interval \([0, M]\). Let \( I_1, I_2, \ldots, I_r \) be sets of diameter less than \( \varepsilon \) and covering \([0, M]\). Denote by \( M_{k,l} \) the preimage of \( I_l \) under \( |g_k| \), i.e.,
\[
M_{k,l} = |g_k|^{-1}[I_l], \quad (1 \leq k \leq n, 1 \leq l \leq d).
\]
Clearly
\[
\bigcup_{\pi \in \Pi_d} \bigcap_{k=1}^n M_{k,\pi(k)} = \bigcap_{k=1}^n \bigcup_{l=1}^r M_{k,l} = [0,1] \quad \text{a.e.,}
\]
(4.4)
where \( \Pi_r \) denotes the set of all the permutations of the elements 1, 2, \ldots, r. Therefore, by (4.4), there is a \( \pi \in \Pi_r \), such that

\[
\Omega = \bigcap_{k=1}^{n} M_{k, \pi(k)}
\]

has positive measure. Now, fix any \( i_l \in I_l \), \( (1 \leq l \leq r) \). We have

\[
|g_k - i_{\pi(k)}| \leq \text{diam } I_{\pi(k)} < \varepsilon \quad \text{on } \Omega, \quad (1 \leq k \leq n).
\]

One can also choose a point \( x \) in \([0, 1]\) such that both \( \Omega_1 = (0, x) \cap \Omega \) and \( \Omega_2 = (x, 1) \cap \Omega \) have positive measure. Now, there is a function \( f \) having support in \( \Omega \), positive on \( \Omega_1 \), negative on \( \Omega_2 \) and satisfying

\[
\int_{\Omega_1} f(t) \, dt = -\int_{\Omega_2} f(t) \, dt = \frac{1}{2}.
\]

Observe that also \( \|f\|_1 = 1 \) and

\[
Vf(x) = \int_0^x f(t) \, dt = \frac{1}{2}.
\]

Since, by (4.6), \( \int_0^1 f = 0 \) and, thanks to (4.5), \( g_k \) is almost constant on \( \Omega \), we claim that \( \int_0^1 fg_k \) is small. Indeed, for any \( 1 \leq k \leq n \),

\[
\int_0^1 f(t)g_k(t) \, dt = \int_{\Omega_1} f(t)g_k(t) \, dt + \int_{\Omega_2} f(t)g_k(t) \, dt \\
\leq (i_{\pi(k)} + \varepsilon) \int_{\Omega_1} f(t) \, dt + (i_{\pi(k)} - \varepsilon) \int_{\Omega_2} f(t) \, dt \\
\leq \frac{1}{2} (i_{\pi(k)} + \varepsilon) - \frac{1}{2} (i_{\pi(k)} - \varepsilon) \leq \varepsilon.
\]

Therefore, by (4.3) together with (4.7) and (4.8),

\[
\frac{1}{2} = Vf(x) \leq \|Vf\|_\infty \leq \sup_{1 \leq k \leq m} \left| \int_0^1 f(t)g_k(t) \, dt \right| + q \|f\|_1 \leq \varepsilon + q
\]

and \( q \geq \frac{1}{2} \) by the arbitrary choice of \( \varepsilon \).

Lemma 4.4. Let \( n \geq 2 \). Then

\[
d_n(V : L_0^1 \to C) = \frac{1}{4}.
\]

Proof. Let us first establish the inequality “\( \leq \)”. By the monotonicity of \( s \)-numbers, it is enough to show that \( d_2(V) \leq \frac{1}{4} \). Let \( N \) be the one-dimensional subspace of \( L^\infty \) consisting of constant functions. Then, by the definition of the second Kolmogorov number,

\[
d_2(V) \leq \sup_{f \in B_{L_0^1}} \inf_{c \in \mathbb{R}} \|Vf - c\|_\infty.
\]
Now, to a given $\varepsilon > 0$, there is $f \in B_{L^1_0}$ such that

$$d_2(V) - \varepsilon \leq \inf_{c \in \mathbb{R}} \|V f - c\|_\infty.$$

We claim that

$$|V f(u) - V f(v)| \leq \frac{1}{2}$$

for any $u, v \in [0, 1]$. To observe that, let $E$ be any measurable subset of $[0, 1]$ and $E^c$ its complement. Then

$$\int_E f = \int_E f(t) \, dt - \frac{1}{2} \int_{E \cup E^c} f(t) \, dt$$

$$= \frac{1}{2} \int_E f(t) \, dt - \frac{1}{2} \int_{E^c} f(t) \, dt$$

$$\leq \frac{1}{2} \int_0^1 |f(t)| \, dt \leq \frac{1}{2}$$

and the claim follows once we set $E$ as the interval with endpoints $u$ and $v$. Next, on setting

$$c = \frac{1}{2} (\sup_{[0,1]} V f - \min V f),$$

we obtain that

$$\|V f - c\|_\infty \leq \frac{1}{4}.$$

Since $\varepsilon$ was arbitrary, we have $d_n(V) \leq \frac{1}{4}$.

For the opposite inequality, consider the functions

$$f_k = 2^k \left( \chi_{(2^{-k-1},2^{-k})} - \chi_{(1-2^{-k},1-2^{-k-1})} \right) \quad \text{for } k \in \mathbb{N}.$$

These are linearly independent elements of $L^1_0$ with $\|f_k\|_1 = 1$ and satisfying $V f_k(0) = 0$ and $V f_k(2^{-k}) = \frac{1}{2}$. The assertion thus follows by Lemma 3.3.

**Proof of Theorem 1.3.** By Proposition 3.1, the $s$-numbers of Sobolev embedding $V_{0,1}^1(I) \hookrightarrow C(I)$ coincide with $s$-numbers of the operator $V: L^1_0(I) \to C(I)$. We can assume that $I$ is the unit interval because in the non-compact setting the scaling does not affect the norm and the values of $s$-numbers.

The upper bound for the Bernstein numbers $b_n(V) \leq \frac{1}{2n}$ follows from Lemma 4.2 and the lower bound for the Isomorphism numbers reads as $i_n(V) \geq \frac{1}{2n}$, thanks to Lemma 4.1. Since the Isomorphism numbers are the smallest among strict $s$-numbers, the equalities in (1.4) follow.

Next, since the Approximation numbers are the largest $s$-numbers, we have $\frac{1}{2} = \|V\| \geq a_n(V) \geq c_n(V) \geq \frac{1}{2}$, where the last inequality is due to Lemma 4.3. This proves (1.5). Finally, the relation (1.6) is subject of Lemma 4.4.
5. Higher-dimensional Sobolev embeddings

In this section we denote by $Q = (0,1)^d$ the $d$-dimensional cube where $d \geq 2$. All the function spaces are considered over $Q$, unless explicitly stated.

Lemma 5.1. Let $X$ be any Banach function space satisfying $X \subseteq L^{d,1}$. Then there exists a positive constant $c$, such that for every $n \in \mathbb{N}$
\[ i_n(V_0^1 X \hookrightarrow C) \geq cn^{-\frac{1}{d}}. \]

Proof. Let $n = m^d$ for some $m \in \mathbb{N}$. Denote $B_1, B_2, \ldots, B_n$ disjoint balls in $Q$ having radii $r = \frac{1}{2m}$ and. Let us label the centre point of $B_k$ by $x_k$, ($1 \leq k \leq n$). On setting
\[ u_k(x) = (r - |x - x_k|) \chi_{B_k} \text{ for } x \in Q, \]
we have that $\|u_k\|_{\infty} = 1$ and $|\nabla u_k| = \chi_{B_k}$ a.e. for every $1 \leq k \leq n$. Consider now the chain
\[ \ell_n^\infty \xrightarrow{B} V_0^1 X \hookrightarrow C \xrightarrow{A} \ell_n^\infty, \]
where $A$ is defined as
\[ (Au)_k = u(x_k), \quad (1 \leq k \leq n), \quad \text{for } u \in C \]
and $B$ is given by
\[ B(y) = \frac{1}{r} \sum_{k=1}^{n} y_k u_k \text{ for } y \in \ell_n^\infty. \]
Both $A$ and $B$ are well-defined and the composition $A \circ \text{Id} \circ B$ forms an identity on $\ell_n^\infty$. As for the norms, clearly $\|A\| = 1$ and
\[ \|B(y)\|_{V_0^1 X} = \frac{1}{r} \left\| \sum_{k=1}^{n} y_k \nabla u_k \right\|_X \leq \frac{1}{r} \|y\| \left\| \sum_{k=1}^{n} \nabla u_k \right\|_X \leq \frac{\|\chi_Q\|_X}{r} \|y\|, \]
therefore, by the definition of the $n$-th Isomorphism number,
\[ i_n(V_0^1 X \hookrightarrow C) \geq \frac{1}{\|A\| \|B\|} \geq \frac{r}{\|\chi_Q\|_X} = \frac{1}{2\|\chi_Q\|_X} n^{-\frac{1}{d}}. \]
The result for arbitrary $n$ then follows by the monotonicity (S1) of the Isomorphism numbers. \hfill \Box

Lemma 5.2. There is a positive constant $c = c(d)$ such that for every $n \in \mathbb{N}$ we have the following estimate of the Bernstein numbers
\[ b_n(V_0^1 L^{d,1} \rightarrow C) \leq cn^{-\frac{1}{d}}. \]

Before we prove this lemma, we will make a preliminary discussion. Let $d$ and $k$ be given. We shall work with the dyadic cubes in $Q$. For $j = 0, 1, \ldots, k$ we denote by $Q_j$ the family of all the dyadic cubes in $Q$ with side length $2^{-j}$, i.e.,
\[ Q_j = \{x \in Q; 2^j x - z \in Q}; z \in \mathbb{Z}^d\}. \quad (5.1) \]
We shall also refer to $Q_j$ as the $j$-th generation of dyadic cubes in $Q$. 

We would like to define an arrangement of the cubes in $Q_k$ in a way that we could transform the cube $Q$ into the straight strip. We moreover require that every sub-sequence of consecutive cubes forms a set which is “plump” enough. More precisely, denote by $Q: \{1, 2, \ldots, 2^{dk}\} \to Q_k$ a numbering of cubes in $Q_k$ with the property that consecutive cubes $Q_j$ and $Q_{j+1}$ neighbourhood by a face and define

$$\Omega_{ij} = \bigcup_{l=i}^{j} Q_l \quad \text{for } 1 \leq i \leq j \leq 2^{dk}. \quad (5.2)$$

We would like to have the inequality

$$\sup_{x,y \in \Omega_{ij}} |u(x) - u(y)| \leq C\|\nabla u\|_{d,1(\Omega_{ij})} \quad (5.3)$$

valid for every continuous $u$ having $|\nabla u| \in L^{d,1}(\Omega)$ and for every $1 \leq i \leq j \leq 2^{dk}$ with the absolute constant $C$ independent of $i$, $j$ and $k$.

By (2.5) the constant $C$ in (5.3) might be controlled by the John constant $C_J$ of the John domain (2.3) and by the dimension $d$. Therefore it is enough to have $C_J(\Omega_{ij})$ uniformly bounded in $i$, $j$ and $k$. However not every arrangement $Q$ may possess this property. For instance, one has to avoid the numbering “by lines” since any such thin joist would have the constant $C_J$ proportional to $2^k$.

As an example of proper numbering we introduce the arrangement inspired by the $k$-th approximation of Hilbert space-filling curve which we denote by $H^d_k: [1, 2^{dk}] \to Q$. Such mappings are well-studied usually in the field of computer science where they are used for instance in an implementation of a multiattribute access methods, data clustering [MJFS01] or as a heuristics for Traveling Salesman Problem [PB89].

Unfortunately, it is not easy to give the exact definition of $H^d_k$ and we will not do it here. For those readers who are interested in this topic, we refer to [But69] or [AN00] for analytical definition or [MJFS01] for recurrent geometric view. We will use only one basic fact of those curves which can be formulated as follows.

Once the curve $H^d_k$ enters any cube $\tilde{Q}$ from the family $Q_l$, it does not leave $\tilde{Q}$ until it visits all the subcubes of $\tilde{Q}$ contained in the refined family $Q_{l+1}$ (and hence $Q_m$ for any $m > l$). For illustration, see Figure 2.

Let us summarize the above-mentioned thoughts into the following lemma.

**Lemma 5.3.** Let $Q = (0, 1)^d$, $d \geq 2$, and let $Q_k$ be the family of dyadic cubes (5.1) for some $k \in \mathbb{N}$. Suppose that the cubes in $Q_k$ are labeled along the path $H^d_k$, i.e., set $Q: \{1, 2, \ldots, 2^{dk}\} \to Q_k$ as $Q_l = \tilde{Q}$ where $\tilde{Q} \in Q_k$ is such that $H^d_k(l) \in \tilde{Q}$. Then all the sets $\Omega_{ij}$ defined by (5.2) are John domains with comparable constants depending on $d$ and independent of $k$, $i$ and $j$, i.e., there is $C = C(d)$ such that

$$C_J(\Omega_{ij}) \leq C \quad \text{for every } 1 \leq i \leq j \leq 2^{dk}. \quad \text{Proof.}$$

Let $d$, $k$, $i$ and $j$ be fixed. We may imagine $\Omega_{ij}$ as a chain of “small cubes” from the youngest family $Q_k$. As the chain twists, it also fills the larger dyadic cubes from older families $Q_m$, ($m < k$).
Denote by $Q_{k_1}$ the oldest family for which there exists any dyadic cube contained in $\Omega_{ij}$ and label any of such cube as $Q_{k_1}^{k_1}$. The curve $\mathcal{H}_k^d$ passes through $Q_{k_1}^{k_1}$ and exits by two different faces. Let us follow $\mathcal{H}_k^d$ on one of its exists (the other one would be analogous) and watch for the largest filled dyadic cubes. We may still meet the dyadic cubes from $Q_{k_1}^{k_1}$. Observe that $m_1 \leq 2(2^d - 2)$. Continuing our journey along $\mathcal{H}_d$, the next largest filled dyadic cube must belong to younger family $Q_{k_2}$, $(k_2 > k_1)$. Denote by $Q_{k_2}^{k_2}$ the cubes in $Q_{k_2}$ the curve $\mathcal{H}_d$ fills consecutively. Their total amount satisfies $m_2 \leq 2^{d-1}$. Let us continue in this manner inductively. We obtain the sequence of families $Q_{k_1}^{k_1}, \ldots, Q_{k_r}^{k_r}$ and the sequences of dyadic cubes $\{Q_{k_1}^{k_1}, \ldots, Q_{m_1}^{k_1}\} \subseteq Q_{k_1}^{k_1}, \ldots, \{Q_{k_r}^{k_r}, \ldots Q_{m_r}^{k_r}\} \subseteq Q_{k_r}^{k_r}$. Except for the $k_1$-th family, all such sequences are no longer than $2^d - 1$, i.e., $m_l \leq 2^d - 1$, $(2 \leq l \leq r)$. See Figure 3 for illustration.

![Figure 3. Labeling of dyadic cubes in $\Omega_{ij}$](image)

Next, we construct a John curve for a given $x \in \Omega_{ij}$. We may assume that $x \in Q_{m_r}^{k_r}$ since otherwise we can stop the process of the labeling cubes as we reach the first dyadic cube containing $x$. Denote by $x_1^{k_1}, \ldots, x_{m_1}^{k_1}$, the central points of the cubes $Q_{k_1}^{k_1}, \ldots, Q_{m_1}^{k_1}$, $(1 \leq l \leq r)$, respectively. The image of the curve $\gamma : [0, 1] \rightarrow \Omega_{ij}$ is now defined as a polyline as follows. We choose $x$, the starting point, and we connect it to the center $x_{m_1}^{k_1}$, then we join all the points $x_\xi^{k_1}$, $(1 \leq \xi \leq m_1)$, in the reverse order. From $x_1^{k_1}$ we go to the point $x_{m_1}^{k_1}$, through the center of the face of $Q_1^{k_1}$. Denote this point by $y^{k_1}$ and similarly all the others by $y^{k_2}, \ldots, y^{k_{r-1}}$. We then follow this manner until we finally reach the point $x_1^{k_1}$, the central point of the John domain $\Omega_{ij}$. See Figure 3 again.

Let now $\gamma$ be parametrized arbitrarily and choose any $t \in [0, 1]$. We distinguish two cases. If $\gamma(t)$ lays between $x$ and $x_{m_r}^{k_r}$ then (2.3) holds with $C_J = C_J(Q)$ trivially. In the remaining cases, assume that $\gamma(t) \in Q_l^{k_l}$ for some $1 \leq l \leq r$ and some $1 \leq m \leq m_l$. 
We have, by the construction,

\[ |x - \gamma(t)| \leq \sum_{\lambda=l+1}^{r} \sum_{\xi=1}^{m_{\lambda}-1} |x_{\xi}^{k_{\lambda}} - x_{\xi+1}^{k_{\lambda}}| + \sum_{\xi=m}^{m_{l}-1} |x_{\xi}^{k_{l}} - x_{\xi+1}^{k_{l}}| + \sum_{\lambda=l+1}^{r} |x_{1}^{k_{\lambda}} - y^{k_{\lambda}}| + \sum_{\lambda=l+1}^{r} |y^{k_{\lambda}} - x_{m_{\lambda-1}}^{k_{\lambda-1}}| + |x_{m}^{k_{l}} - \gamma(t)| \]

\[ \leq \sum_{\lambda=l+1}^{r} (2^{d} - 1)2^{-k_{\lambda}} + (2^{d} - 1)2^{-k_{l}} \]

\[ + \sum_{\lambda=l+1}^{r} 2^{-k_{\lambda}-1} + \sum_{\lambda=l+1}^{r} \sqrt{d}2^{-k_{\lambda}-1} + \sqrt{d}2^{-k_{l}-1} \]

\[ \leq c(d)2^{-k_{l}} \]

and

\[ \text{dist}\left(\gamma(t), \partial\Omega_{ij}\right) \geq 2^{-k_{l}-2}. \]

Therefore, by the definition of the John domain, \( C_{J}(\Omega_{ij}) \leq 4c. \)

\[ \square \]

**Proof of Lemma 5.2.** It is enough to show that for every \( n \)-dimensional subspace \( E \) of \( V_{0}^{1}L^{d,1}(Q) \) satisfying

\[ \|u\|_{\infty} \geq \varrho\|\nabla u\|_{d,1} \quad \text{for} \quad u \in E, \]

there is the inequality

\[ \varrho \leq cn^{-\frac{1}{d}}, \]

where \( c = c(d) \) is the absolute constant.

We proceed similarly as in the proof of Lemma 4.2. We may assume that all functions in \( E \) are continuous and, since \( E \) is of finite dimension, they are uniformly continuous. Formally written, to any given \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that

\[ |u(x) - u(y)| \leq \varepsilon\|u\|_{\infty} \quad \text{whenever} \quad |x - y| < \delta \quad \text{and} \quad u \in E. \]

Let \( Q_{k} \) be the family of dyadic cubes as in (5.1) where \( k \in \mathbb{N} \) is chosen sufficiently large to ensure that the cubes in \( Q_{k} \) have diameter less than \( \delta \). Denote \( N = 2^{dk} \), the total number of the cubes in \( Q_{k} \). We may also assume that \( N > n \). We label the cubes along the \( k \)-th approximation of Hilbert curve as is in Lemma 5.3 by the labeling map \( Q: \{1, 2, \ldots, N\} \rightarrow Q_{k} \). The center point of a cube \( Q_{l} \) will be denoted by \( x_{l} \).

Let us define the mapping \( \Phi: E \rightarrow \ell_{N}^{\infty} \) by

\[ (\Phi u)_{l} = u(x_{l}) \quad \text{for} \quad 1 \leq l \leq N. \]

Function \( \Phi \) is well defined, since \( E \) consists of continuous functions. Clearly \( \|\Phi u\|_{\infty} \leq \|u\|_{\infty} \) and also \( (1 - \varepsilon)\|u\|_{\infty} \leq \|\Phi u\|_{\infty} \) by the same procedure as in (3.2) of Proposition 1.4. Thus \( \Phi \) is an isomorphism from \( E \) onto a \( n \)-dimensional subspace of \( \ell_{N}^{\infty} \). On applying Zigzag theorem [CFP+09, Theorem 1], there is an element \( v \in E \) with \( \|\Phi v\|_{\infty} = 1 \) and a subsequence of \( \{1, 2, \ldots, N\} \) of length \( n \) satisfying

\[ v(x_{i_{j}}) = (-1)^{j} \quad \text{for} \quad 1 \leq j \leq n. \]
Denote $\Omega_j = \Omega_{l_j, l_{j+1}}$ using the definition of the latter symbol given in (5.2). We have

\[
2(n - 1) = \sum_{j=1}^{n-1} |v(x_l_j) - v(x_{l_{j+1}})|
\leq \sum_{j=1}^{n-1} C_J(\Omega_j) \| \nabla u \chi_{\Omega_j} \|_{d,1}
\leq \left( \sum_{j=1}^{n-1} C_J(\Omega_j)^{d'} \right)^{\frac{1}{d'}} \left( \sum_{j=1}^{n-1} \| \nabla v \chi_{\Omega_j} \|_{d,1}^d \right)^{\frac{1}{d'}}
\leq C(n - 1)^{\frac{1}{d'}} \| \nabla v \|_{d,1},
\]

where, in the first term, we used Lemma 5.3 and, in the second term, we used the summation property of the Lorentz norm (2.2). Note that the sets $\Omega_j$ are not disjoint, however $\sum_j \chi_{\Omega_j} \leq 2$ on $Q$ which only affects the constant $C$.

Now, combining (5.6) and (5.4), we get

\[
2(n - 1)^{\frac{1}{d'}} \leq C \| \nabla v \|_{d,1} \leq C \frac{\| v \|_{\infty}}{\ell(1 - \varepsilon)}
\]

and the inequality (5.5) follows by letting $\varepsilon \to 0^+$.

**Proof of Theorem 1.1.** The proof of (1.2) follows from Lemma 5.1 and Lemma 5.2. The relation (1.3) is a consequence of the non-compactness of the embedding in question.

**Proof of Corollary 1.2.** Let $s_n$ represent Isomorphism or Bernstein numbers. We have

\[
 cn^{-1/d} \leq i_n(V_0^1 X \hookrightarrow C) \leq s_n(V_0^1 X \hookrightarrow C) \leq s_n(V_0^1 L^{d,1} \hookrightarrow C) \leq cn^{-1/d},
\]

where the first inequality follows by Lemma 5.1, the second by the ideal property (S3) of $s$-numbers and the last one is due to Theorem 1.1.

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