Localization of Matters on Pure Geometrical Thick Branes

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ABSTRACT: In the literatures, several types of thick smooth brane configurations in a pure geometric Weyl integrable 5-dimensional space time have been presented. The Weyl geometry is a non-Riemannian modification of 5-dimensional Kaluza–Klein (KK) theory. All these thick brane solutions preserve 4-dimensional Poincaré invariance, and some of them break $\mathbb{Z}_2$–symmetry along the extra dimension. In this paper, we study localization of various matter fields on these pure geometrical thick branes, which also localize the graviton. We present the shape of the potential of the corresponding Schrödinger problem and obtain the KK modes. Just as the case of scalar, the massless and several massive modes of spin 1 vectors are found to be normalizable on the brane. The result is different from the famous case of RS model in $AdS_5$ space, in which vector fields are not localized neither on a brane with positive tension nor on a brane with negative tension. We also show that, for the case of massive fermions, stronger Yukawa coupling and weaker gravity can trap more massive states. Furthermore, an interesting result is that, even without the Yukawa coupling, there is still at least one bounded massive state in the case of weak gravity (i.e., small $k$). But for the massless left or right fermion localization, there must be some kind of Yukawa coupling. These features show that these thick branes arising from a pure geometric Weyl integrable manifold are different from others.

KEYWORDS: Large Extra Dimensions, Field Theories in Higher Dimensions.

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1. Introduction

Recently, there has been increasing interest and considerable activity in the study of higher-dimensional space-times with large extra dimensions [1, 2]. Suggestions that extra dimensions may not be compact [2, 3, 4, 5, 6, 7] or large [1, 8] can provide new insights for the solution of some relevant problems of high–energy physics such as the mass hierarchy problem, dark matter, non–locality and the cosmological constant [4, 7, 9]. In the framework of brane scenarios, an important ingredient is that gravity is free to propagate in all dimensions, whereas all the matter fields are confined to a 3–brane with no contradiction with present time gravitational experiments [1, 4, 6, 10, 11].

In the brane world scenario, an important question is how to realize the brane world idea, in which a key ingredient is localization of various bulk fields on a brane by a natural mechanism. It is well known that the massless scalar field [12] and the graviton [2] are localized on branes of different types, and that the spin 1 Abelian vector fields can not be localized on the Randall-Sundrum(RS) model in five dimensions but can be localized in some higher-dimensional cases [13]. For fermions, they do not have normalizable zero modes in five and six dimensions [12, 13, 14, 15, 16, 17, 18]. Meanwhile, for the brane with inclusion of scalar backgrounds [19] and minimal gauged supergravity [20] in higher dimensions, localized chiral fermions can be obtained under some conditions.

Recently, thick brane scenarios based on gravity coupled to scalars have been constructed [21, 22, 23, 24]. An interesting feature of these models is that one can obtain branes naturally without introducing them by hand in the action of the theory [21]. Furthermore, these scalar fields do not play the role of bulk fields but provide the “material” from which the thick branes are made of. By considering a non-Riemannian modification of 5-dimensional Kaluza–Klein (KK) theory (in a pure geometric Weyl integrable 5-dimensional space time), the generalized models based on gravity coupled to scalars have
been studied in Refs. [25, 26, 27]. In this scenario, spacetime structures with pure geometric thick smooth branes separated in the extra dimension arise. The authors obtained a single bound state which represents a stable 4D graviton and proved that the spectrum of massive modes of KK excitations is not discrete or quantized at all, but continuous without mass gap due to the asymptotic behavior of the quantum mechanics potential [26, 27]. This gives an very important conclusion: the claim that Weylian structures mimic classically quantum behavior does not constitute a generic feature of these geometric manifolds [26].

The aim of the present article is to investigate localization of various matters on the pure geometrical thick branes obtained in Refs. [25, 26, 27]. The paper is organized as follows: In section 2, we first give a review of the thick branes arising from a pure geometric Weyl integrable 5-dimensional space-time, which is a non-Riemannian modification of 5-dimensional KK theory. Then, in section 3, we study localization of various matters on the pure geometrical thick branes in 5 dimensions. Finally, a brief conclusion and discussion are presented.

2. Review of thick brane worlds arising from pure geometry

Let us start with a non–Riemannian generalization of KK theory, i.e., a pure geometrical Weyl action in five dimensions

\[
S_W^5 = \int_{M_5^W} \frac{d^5x \sqrt{\hat{g}}}{16\pi G_5} e^{\frac{\omega}{2}} \left[ \hat{R} + 3 \hat{\xi}(\nabla \omega)^2 + 6 \hat{U}(\omega) \right],
\]

where \( M_5^W \) is a 5-dimensional Weyl-integrable manifold specified by the pair \((\hat{g}_{AB}, \omega)\), \( \hat{g}_{AB} \) is a 5-dimensional metric and \( \omega \) is a Weyl scalar function. In such manifolds the Weylian Ricci tensor is given by \( \hat{R}_{MN} = \hat{\Gamma}_{A}^{A}{}_{M,N} - \hat{\Gamma}_{A}^{A}{}_{M,N} + \hat{\Gamma}_{P}^{M}{}_{Q} \hat{\Gamma}_{Q}^{P}{}_{N} - \hat{\Gamma}_{P}^{M}{}_{Q} \hat{\Gamma}_{N}^{P}{}_{Q} \), with \( \hat{\Gamma}_{A}^{A} = \hat{\Gamma}_{A}^{A} - \hat{\xi}(\omega, \delta_B^A + \omega, B \delta_A^C - g_{AB} \omega^C) \) the affine connections on \( M_5^W \) and \( \Gamma_{C}^{C} = \{C_{MN}\} \) the Christoffel symbols. The parameter \( \hat{\xi} \) is a coupling constant, and \( \hat{U}(\omega) \) is a self-interaction potential for \( \omega \), which, in general, breaks the invariance of the action (2.1) under Weyl rescaling,

\[
\hat{g}_{AB} \to \Omega^{-2} \hat{g}_{AB}, \quad \omega \to \omega + \ln \Omega^2, \quad \hat{\xi} \to \hat{\xi}/(1 + \partial_{\omega} \ln \Omega^2)^2,
\]

where \( \Omega^2 \) is a smooth function on \( M_5^W \). \( \hat{U}(\omega) = \lambda e^{\omega} \), where \( \lambda \) is a constant parameter, is the only functional form which preserves the scale invariance of the Weyl action (2.1). When the Weyl invariance is broken, the scalar field transforms from a geometrical object into an observable degree of freedom which generates the smooth thick brane configurations, namely, \( \omega \) is not a bulk field playing the role of the modulus for the extra dimension. The Weyl action is of pure geometrical nature since the scalar field \( \omega \) enters in the definition of the affine connections of the Weyl manifold.

The ansatz for the line-element which results in a 4-dimensional Poincaré invariance of the Weyl action (2.1) is given by

\[
d\hat{s}_5^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,
\]

where \( A(y) \) is a smooth function. In section 2, we first give a review of the thick branes arising from a pure geometric Weyl integrable 5-dimensional space-time, which is a non-Riemannian modification of 5-dimensional KK theory. Then, in section 3, we study localization of various matters on the pure geometrical thick branes in 5 dimensions. Finally, a brief conclusion and discussion are presented.
where $e^{2A(y)}$ is the warp factor, and $y$ stands for the extra coordinate.

In search of a solution to the setup defined by (2.1) and (2.3), we shall use the conformal technique. Via a conformal transformation, $g_{AB} = e^{2\phi} \tilde{g}_{AB}$, we go from the Weyl frame to the Riemann one, $M^W_5 \to M^R_5$. The action (2.1) is mapped into the following Riemannian form

$$S^R_5 = \int_{M^R_5} d^5x \sqrt{-g} \left[ R + 3\xi (\nabla \omega)^2 + 6U(\omega) \right],$$

(2.4)

where $\xi = \hat{\xi} - \frac{1}{2}$, $U(\omega) = e^{-\omega} \hat{U}(\omega)$. Thus, in this frame, we have a theory which describes 5-dimensional gravity coupled to a scalar field with a self-interaction potential. After this transformation, the line element reads

$$ds^2_5 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{\omega(y)} dy^2,$$

(2.5)

where $2\sigma = 2A + \omega$. If we introduce a new pair of variables $X \equiv \omega'$ and $Y \equiv 2A'$, then the field equations that are derivable from (2.4) with the ansatz (2.5) reduce to the following pair of coupled equations

$$X' + 2YX - \frac{3}{2} X^2 = \frac{1}{\xi} \frac{dU}{d\omega} e^{-\omega},$$

(2.6)

$$Y' - \frac{3}{2} XY + 2Y^2 = \left( \frac{1}{\xi} \frac{dU}{d\omega} + 4U \right) e^{-\omega}.$$

(2.7)

As pointed out in [25], this system of equations can be easily solved if one uses the restriction $X = kY$, where $k$ is an arbitrary constant parameter which is not allowed to adopt the value $k = 1$ because the system would be incompatible. This condition leads to a Riemannian potential of the form $U = \lambda e^{\frac{4k}{1-k} \omega}$. It turns out that this constrain leads to the following simple brane configurations:

**Configuration 1: $Z_2$-symmetric thick brane**

In this case, $-\infty < y < +\infty$ (we recall that, due to orbifold symmetry of the solution, only one half of the extra dimension, say $0 \leq y < +\infty$, is physically relevant). The expressions for the warp factor and the scalar field read [25]

$$e^{2A(y)} = \left[ \cosh(ay) \right]^b, \quad \omega = kb \ln \cosh(ay).$$

(2.8)

where

$$a = \sqrt{\frac{4 - 3k}{1 - k}} 2\lambda, \quad b = \frac{2}{4 - 3k},$$

(2.9)

and

$$\lambda > 0, \quad k > 4/3.$$  

(2.10)

Hence, $b$ is negative and the warp factor is concentrated near of the origin $y = 0$. The energy density of the scalar matter is [26]

$$\mu(y) = -\frac{3a^2b}{4\pi G_5} (e^{ay} - e^{-ay})^{b-2} \left[ 1 + \frac{b}{4} (e^{ay} - e^{-ay})^2 \right].$$

(2.11)
This function has two negative minima and a positive maximum at \( y = 0 \) between them at some \( y \neq 0 \), and finally it vanishes asymptotically (see Fig. 1). The fact can be compared with Randall-Sundrum thin brane case, where one of the branes has a positive brane tension meanwhile the second brane has a negative one \[2\].

![Figure 1: The shape of the energy density function with \( k = 5/3 \) and \( \lambda = 0.01 \). A thick brane with positive energy density is centered at the origin \( y = 0 \).](image)

**Configuration 2: non \( Z_2 \)-symmetric thick brane**

The non \( Z_2 \)-symmetric thick brane solution was found by Barbosa-Cendejas and Herrera-Aguilar \[26\]

\[
e^{2A(y)} = k_3 \left(e^{ay} + k_1 e^{-ay}\right)^b, \quad \omega = \ln \left[k_2 \left(e^{ay} + k_1 e^{-ay}\right)^{kb}\right], \tag{2.12}
\]

where \( k_2 \) and \( k_3 \) are arbitrary constants, and

\[
\lambda > 0, \quad k > 4/3, \quad k_1 > 0. \tag{2.13}
\]

The \( Z_2 \)-symmetric solution \[2.8\] is the particular case of this solution with \( k_1 = 1, \ k_2 = 2^{-kb} \) and \( k_3 = 2^{-b} \). The parameter \( k_1 \) represents the \( Z_2 \)-asymmetry of the solution through a shift along the extra coordinate. This has a quite important physical implication, i.e., the space time is not restricted to be an orbifold geometry, it allows for a more general type of manifolds.

The 5-dimensional curvature scalar in the Riemann frame and in the Weyl frame are \[26\]

\[
R_5 = \frac{-64\lambda k_1 (1 + k)}{1 - k} (e^{ay} + k_1 e^{-ay})^{-(kb+2)} \left[1 + \frac{b(5 + 3k)}{16k_1} (e^{ay} - k_1 e^{-ay})^2\right] \tag{2.14}
\]

and

\[
\hat{R}_5 = \frac{-16a^2 bk_1}{(e^{ay} + k_1 e^{-ay})^2} \left[1 + \frac{5b}{16k_1} (e^{ay} - k_1 e^{-ay})^2\right], \tag{2.15}
\]

respectively. It is worth to note that the latter is always bounded but the former is not bounded. Hence we have a 5-dimensional manifold which is singular in the Riemann frame but is regular in the Weyl one.
Configuration 3: another non $Z_2$–symmetric thick brane

In above three configurations, the parameter $\xi$ has been chosen as $\xi = (1 - k)/(4k)$ with $k$ arbitrary but $k \neq 4/3$. In Ref. [27], the condition $k = 4/3$ is imposed and the corresponding solution is read

$$e^{2A} = \left[ \frac{\sqrt{-8\lambda p}}{c_1} \cosh (c_1(y - c_2)) \right]^{\frac{3}{4p}}, \quad \omega = \frac{2}{p} \ln \left[ \frac{\sqrt{-8\lambda p}}{c_1} \cosh (c_1(y - c_2)) \right],$$  \hspace{1cm} (2.16)

where $p = 1 + 16\xi$, $c_1$ and $c_2$ are arbitrary integration constants, and

$$\lambda > 0, \quad p < 0, \quad c_1 > 0.$$  \hspace{1cm} (2.17)

From the solution, one can get the energy density of the scalar matter, which behavior is similar to that of (2.11). So, it represents a thick brane with positive energy density centered at $y = c_2$.

These solutions would be utilized to analyze localization of various matter fields on pure geometrical thick branes in the next section.

3. Localization of various matters

Now, we ask the question of whether various bulk fields with spin ranging from 0 to 1 can be localized on thick branes by means of only the gravitational interaction. Of course, we have implicitly assumed that various bulk fields considered below make little contribution to the bulk energy so that the solutions given in previous section remain valid even in the presence of bulk fields. In this section, we shall investigate the localization problem in the Riemann frame and the line element is given by (2.3).

3.1 Spin 0 scalar field

In this subsection we study localization of a real scalar field on pure geometrical thick branes in the backgrounds (2.8)-(2.16). Let us consider the action of a massless real scalar coupled to gravity

$$S_0 = \frac{1}{2} \int d^5x \sqrt{-g} g^{MN} \partial_M \Phi \partial_N \Phi,$$  \hspace{1cm} (3.1)

from which the equation of motion can be derived

$$\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} \partial_N \Phi) = 0.$$  \hspace{1cm} (3.2)

For simplicity, we define $P(y) = e^{2\sigma(y)}$ and $Q(y) = e^{\omega(y)}$. Then the background metric is determined by

$$ds_5^2 = P(y)\eta_{\mu\nu} dx^\mu dx^\nu + Q(y)dy^2,$$  \hspace{1cm} (3.3)

and the equation of motion (3.2) becomes

$$PQ^{-\frac{1}{2}} \eta^{\mu\nu} \partial_\mu \partial_\nu \Phi + \partial_y (P^2 Q^{-\frac{1}{2}} \partial_y \Phi) = 0.$$  \hspace{1cm} (3.4)
Let us first focus on massive scalars. With the decomposition

$$\Phi(x, y) = \sum_n \phi_n(x) \chi_n(y),$$

and demanding $\phi_n(x)$ satisfies the 4-dimensional massive Klein–Gordon equation $(\eta^{\mu
u} \partial_\mu \partial_\nu - m^2)\phi_n(x) = 0$, we obtain the equation for $\chi_n(y)$

$$\partial_y(P^2 Q^{-\frac{1}{2}} \partial_y \chi_n(y)) + m^2 P Q^{-\frac{1}{2}} \chi_n(y) = 0.$$  

The 5-dimensional action (3.1) reduces to the standard 4-dimensional action for the massive scalars, when integrated over the extra dimension under the conditions that Eq. (3.6) is satisfied and the orthonormality condition

$$\int_{-\infty}^{\infty} P Q^{1/2} \chi_n \chi_m dy = \delta_{mn}$$

for $\chi_n$, or

$$\int_{-\infty}^{\infty} \tilde{\chi}_n \tilde{\chi}_m dy = \delta_{mn}$$

for $\tilde{\chi}_n = P^{1/2} Q^{1/4} \chi_n$, is obeyed. In order to obtain the Schrödinger-like equation, we define $\tilde{\chi}_n = P Q^{1/4} \chi_n$ and get

$$\left[ \partial_y^2 - \left( \frac{m^2}{P} + \frac{P'}{2P} - \frac{3Q'}{16Q^2} + \frac{P''}{P} + \frac{Q''}{4Q} \right) \right] \tilde{\chi}_n = 0.$$  

Here we only consider the first brane configuration (2.8), the cases of the other types are similar. For the first case, the effective potential is reduced to

$$V_{eff} = \frac{1}{4}a^2 b(5k + 4) - m^2 \cosh^{-b(k+1)}(ay) + \frac{1}{16} a^2 b(5kb + 4b - 4) \tanh^2(ay).$$

The value of the potential at $y = 0$ is $V_0 = -\frac{4+5k}{k-1} \lambda - m^2 < 0$. The shape of the above potential is shown in Fig. (2) for different values of $k$ and $\lambda$. All the plots show a minimum at the position of the brane, which indicates that the massive scalar could be trapped on the brane. It can be seen that for smaller $k$ and larger $\lambda$ there has a deeper and higher barrier which may trap more massive states.

Now, we expand the effective potential around $y = 0$ and retain terms up to order $y^2$ for the purpose of obtaining the massive KK modes solutions. The differential equation is reduced to

$$\{ \partial_y^2 - [A_1 - m^2 + A_2 y^2] \} \tilde{\chi}_n = 0,$$

where

$$A_1 = -\frac{5k + 4}{k-1} \lambda, \quad A_2 = \frac{(5k + 4)(11k - 4)}{(k-1)^2} \lambda^2 - \frac{2k + 1}{k-1} m^2 \lambda.$$ 

(3.12)
Figure 2: The shape of the effective potential $V_{\text{eff}}$ for massive scalars. The parameters are set to $k = 5/3, \lambda = 1, m = 1$ (red thin line), $k = 2, \lambda = 1, m = 1$ (green thick line) and $k = 2, \lambda = 2, m = 1$ (blue thick line).

This is the harmonic oscillator approximation in the neighborhood of the brane and the solution is

$$\tilde{\chi}_n \propto e^{-\frac{1}{2}\sqrt{A_2 y^2}} H_n \left( A_2^{1/4} y \right),$$

(3.13)

here $n = \frac{m^2 - A_1^2}{2\sqrt{A_2}} - \frac{1}{2}$ is a nonnegative integer and $H_n(y)$ are the Hermite polynomials. The possible values of $m^2$ are given by

$$m_n^2 = \frac{\lambda}{k - 1} \left[ (2n + 1)\sqrt{4n(n+1)(k+1)^2 + 22k(3k+2) - 7} - [4n(n+1) + 6](k+1) + 1 \right]$$

(3.14)

with $n = 0, 1, 2, \cdots$.

Now the important thing we must to know is that whether these states are normalized and bound or not. Note that the wave functions we needed are $\tilde{\alpha}_n$ but not $\hat{\alpha}_n$. According to the relation of $\tilde{\chi}_n = P^{-1/2}\tilde{\alpha}_n$ and Eq. (3.13), the solution of $\tilde{\chi}_n$ is

$$\tilde{\chi}_n \propto \cosh^{-\frac{1}{2}b(k+1)}(ay) e^{-\frac{1}{2}\sqrt{A_2 y^2}} H_n \left( A_2^{1/4} y \right),$$

(3.15)

which shows that these massive states can be normalized. On the other hand, it is necessary to have a potential barrier higher than zero in order to have trapped states. The potential barrier is decreasing with the increasing of $m^2$. So there is an upper limit on the mass of the scalars which are bounded by the effective potential. So there only finite number of bound massive states for given $\lambda$ and $k$. The similar but more complex problem also emerges in the case of massive fermions which will be discussed later. We will solve the problem in section (3.3) for fermions.

For the case of zero mode, the solution is $\tilde{\chi} = P^{1/2}Q^{1/4}\chi = C_0 \cosh^{-\frac{2(k+1)}{3(k-1)}}(ay)$ with $C_0$ the normalized constant. It is easy to check that for $k > 4/3$ the condition (3.8) is satisfied, so the solution $\tilde{\chi}$ is normalized. But in order to decide whether localization takes
place, we still need to compare the profile of the zero mode with the profile of the low lying KK modes. The comparison of the profiles is plotted in Fig (3). It is shown that the zero mode is localized on the brane.

![Figure 3: The comparison of the zero mode $\tilde{\chi}$ (red thick line) of the scalar with the low lying KK modes $\tilde{\chi}_n (n = 0, 1, 2)$. We have set $k = 2, \lambda = 1.$](image)

### 3.2 Spin 1 vector field

Let us turn to spin 1 vector field. Here we consider the action of $U(1)$ vector field

$$S_1 = -\frac{1}{4} \int d^5 x \sqrt{-g} g^{MN} F_{MR} F_{NS},$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$ as usual. From this action the equation motion is given by

$$\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} g^{RS} F_{NS}) = 0.$$  \hspace{1cm} (3.17)

From the background geometry (3.3), this equation is reduced to

$$\eta^{\mu\nu} \partial_\mu F_{\nu y} = 0,$$

$$\partial^\mu F_{\mu\nu} + Q^{-\frac{1}{2}} \partial_\mu \left( PQ^{-\frac{1}{2}} F_{\nu y} \right) = 0.$$ \hspace{1cm} (3.19)

We assume that the $A_\mu$ are $Z_2$-even and that $A_4$ is $Z_2$-odd with respect to the extra dimension $y$, which results in that $A_4$ has no zero mode in the effective 4D theory. Furthermore, in order to consistent with the gauge invariant equation $\oint dy A_4 = 0$, we use gauge freedom to choose $A_4 = 0$. Under these assumption, the action (3.16) is reduced to

$$S_1 = -\frac{1}{4} \int d^5 x \left( \sqrt{\eta} \eta^{\mu\lambda} \eta^{\rho\sigma} F_{\mu\rho} F_{\lambda\sigma} - 2 \eta^{\mu\nu} A_\mu \partial_\nu (P Q^{-\frac{1}{2}} \partial_\gamma A_\nu) \right).$$ \hspace{1cm} (3.20)

By decomposing the vector field as

$$A_\mu (x, y) = \sum_n a_\mu^{(n)} (x) \rho^{(n)} (y),$$  \hspace{1cm} (3.21)
and importing the orthonormality condition
\[ \int_{-\infty}^{+\infty} dy \sqrt{Q} \rho^{(m)} \rho^{(n)} = \delta^{mn}, \] (3.22)
or equivalently
\[ \int_{-\infty}^{+\infty} dy \tilde{\rho}^{(m)} \tilde{\rho}^{(n)} = \delta^{mn} \] (3.23)
for \( \tilde{\rho} \) with \( \tilde{\rho} = Q^{1/4} \rho \), the action (3.24) is read
\[ S_1 = \int d^4x \sum_n \left( -\frac{1}{4} \eta^{\mu\nu} f^{(n)}_{\mu\nu} - \frac{1}{2} m_n^2 \eta^{\mu\nu} a^{(n)}_\mu a^{(n)}_\nu \right). \] (3.24)
where \( f^{(n)}_{\mu\nu} = \partial_\mu a^{(n)}_\nu - \partial_\nu a^{(n)}_\mu \) is the 4-dimensional field strength tensor, and we have required that the \( \rho^{(n)}(y) \) satisfy the differential equation
\[ \partial_y \left( \frac{P}{\sqrt{Q}} \partial_y \rho^{(n)} \right) = -\sqrt{Q} m_n^2 \rho^{(n)}. \] (3.25)

For massive vectors, by defining \( \hat{\rho}_n = P^{1/2} Q^{-1/4} \rho_n \), Eq. (3.26) changes into
\[ \left[ \partial_y^2 - \left( \frac{Q}{P} m_n^2 - \frac{P^2}{4P^2} - \frac{P'Q'}{4PQ} + \frac{5Q'^2}{16Q^2} + \frac{P''}{2P} - \frac{Q''}{4Q} \right) \right] \hat{\rho}_n = 0. \] (3.26)
For the first brane configuration, the effective potential is reduced to
\[ V_{eff} = \frac{1}{4} a^2 b(k + 2) - m^2 \cosh^{-b}(ay) + \frac{1}{16} a^2 b(k + 2)b(3k - 4) \tanh^2(ay). \] (3.27)
The potential is very similar to the one given in Eq. (3.27). Hence, we encounter the same analyise and similar result. Here, we only give the expression of \( m_n^2 \)
\[ m_n^2 = \frac{\lambda}{k - 1} \left[ (2n + 1) \sqrt{4n^2 + 4n + 7k^2 + 10k - 7} - \left( 4n^2 + 4n + k + 3 \right) \right]. \] (n = 0, 2, 4, \cdots) (3.28)
For large \( n \), the upper limit of \( m_n^2 \) is
\[ m_n^2 \rightarrow \frac{(2 + k)(7k - 6)\lambda}{2(k - 1)}. \] (3.29)

It is worth noting that, for zero mode, the solution is \( \tilde{\rho} = Q^{1/4} \rho \propto \cosh^{\frac{k}{2(3k - 4)}}(ay) \) and the orthonormality condition (3.23) is satisfied. The comparison of the profile of the zero mode with the lowest KK mode is plotted in Fig (4). It is turned out that the result is same as the scalar case, i.e. the zero mode of the spin 1 vector field can be localized on the thick brane without additional condition on the parameter \( k \). It was shown in the RS model in \( AdS_5 \) space that spin 1 vector field is not localized neither on a brane with positive tension nor on a brane with negative tension so the Dvali-Shifman mechanism [28] must be considered for the vector field localization [12]. Here, we do not need to introduce additional mechanism for the vector field localization in the case at hand. The reason is that the factor \( Q \) in current set-up does not trivially equal to 1.
Figure 4: The comparison of the zero mode $\bar{\chi}$ (red thick line) of the spin 1 vector field with the lowest KK mode $\bar{\chi}_0$. We have set $k = 2, \lambda = 1$.

3.3 Spin 1/2 fermionic field

Localization of fermions in general spacetimes has been studied for example in [19]. In Ref. [29], it is found that fermions can escape into the bulk by tunneling, and the rate depends on the parameters of the scalar field potential. In Ref. [30], Melfo et al. studied the localization of fermions on various different scalar thick branes. They showed that only one massless chiral mode is localized in double walls and branes interpolating between different $AdS_5$ spacetimes whenever the wall thickness is keep finite, while chiral fermionic modes cannot be localized in $dS_4$ walls embedded in a $M_5$ spacetime. In this subsection we study localization of a spin 1/2 fermionic field on the pure geometrical thick branes.

Let us consider the Dirac action of a massless spin 1/2 fermion coupled to gravity and scalar

$$S_{1/2} = \int d^5x \sqrt{-g} \left( \bar{\Psi} i \Gamma^M D_M \Psi - \eta \bar{\Psi} F(\omega) \Psi \right),$$

from which the equation of motion is given by

$$\left[ i \Gamma^M (\partial_M + \omega_M) - \eta F(\omega) \right] \Psi = 0,$$

where $\omega_M = \frac{1}{4} \omega^{\bar{M}\bar{N}}_M \Gamma_{\bar{M}} \Gamma_{\bar{N}}$ is the spin connection with $\bar{M}, \bar{N}, \cdots$ denoting the local Lorentz indices, $\Gamma^M$ and $\Gamma^M$ are the curved gamma matrices and the flat gamma ones, respectively, and have the relations $\Gamma^M = e^M_{\bar{M}} \Gamma^\bar{N} = (P^{-1/2} \gamma^\mu, -iQ^{-1/2} \gamma^5)$ with $e^M_{\bar{M}}$ being the vielbein. The spin connection $\omega^{\bar{M}\bar{N}}_M$ in the covariant derivative $D_M \Psi = (\partial_M + \frac{1}{4} \omega^{\bar{M}\bar{N}}_M \Gamma_{\bar{M}} \Gamma_{\bar{N}}) \Psi$ is defined as

$$\omega^{\bar{M}\bar{N}}_M = \frac{1}{2} e^{\bar{M}\bar{N}} (\partial_M e_{\bar{N}} - \partial_{\bar{N}} e_{\bar{M}}) - \frac{1}{2} e^{\bar{N}\bar{M}} (\partial_M e_{\bar{M}} - \partial_{\bar{M}} e_{\bar{N}}) - \frac{1}{2} e^{\bar{P}\bar{M}} e^{\bar{Q}\bar{N}} (\partial_P e_{\bar{Q}R} - \partial_Q e_{\bar{P}R}) e_{\bar{M}}^R.$$

(3.30)

(3.31)

(3.32)
The non-vanishing components of $\omega_M$ are

$$\omega_{\mu} = \frac{P'}{4\sqrt{PQ}} \gamma_{\mu} \gamma_5. \quad (3.33)$$

And the Dirac equation (3.31) then becomes

$$0 = \left[ i\Gamma^M (\partial_M + \omega_M) - \eta F(\omega) \right] \Psi$$
$$= \left\{ \frac{1}{\sqrt{P}} i\gamma^\mu \partial_\mu + \frac{1}{\sqrt{Q}} \gamma_5 \left( \partial_y + \frac{P'}{P} \right) - \eta F(\omega) \right\} \Psi \quad (3.34)$$

where $i\gamma^\mu \partial_\mu$ is the Dirac operator on the brane. We are now ready to study the above Dirac equation for 5-dimensional fluctuations, and write it in terms of 4-dimensional effective fields. From the equation of motion (3.34), we will search for the solutions of the chiral decomposition

$$\Psi(x, y) = \psi_L(x) \alpha_L(y) + \psi_R(x) \alpha_R(y), \quad (3.35)$$

where $\psi_L(x)$ and $\psi_R(x)$ are the left-handed and right-handed components of a 4-dimensional Dirac field. Let us assume that $\psi_L(x)$ and $\psi_R(x)$ satisfy the 4-dimensional massive Dirac equations

$$i\gamma^\mu \partial_\mu \psi_L(x) = m \psi_R(x),$$
$$i\gamma^\mu \partial_\mu \psi_R(x) = m \psi_L(x).$$

Then $\alpha_L(y)$ and $\alpha_R(y)$ satisfy the following eigenvalue equations

$$\sqrt{\frac{P}{Q}} \left( \partial_y + \frac{P'}{P} + \eta F(\omega) \sqrt{Q} \right) \alpha_L(y) = m \alpha_R(y), \quad (3.36a)$$
$$\sqrt{\frac{P}{Q}} \left( \partial_y + \frac{P'}{P} - \eta F(\omega) \sqrt{Q} \right) \alpha_R(y) = -m \alpha_L(y). \quad (3.36b)$$

In order to obtain the standard four dimensional action for the massive chiral fermions, we need the following orthonormality conditions

$$\int_{-\infty}^{\infty} \sqrt{P^3Q} \alpha_{Lm} \alpha_{Ln} dy = \int_{-\infty}^{\infty} \sqrt{P^3Q} \alpha_{Rm} \alpha_{Rn} dy = \delta_{mn}, \quad (3.37a)$$
$$\int_{-\infty}^{\infty} \sqrt{P^3Q} \alpha_{Lm} \alpha_{Rn} dy = 0. \quad (3.37b)$$

for $\alpha_{Ln}$ and $\alpha_{Rn}$, or

$$\int_{-\infty}^{\infty} \tilde{\alpha}_{Lm} \tilde{\alpha}_{Ln} dy = \int_{-\infty}^{\infty} \tilde{\alpha}_{Rm} \tilde{\alpha}_{Rn} dy = \delta_{mn}, \quad (3.38a)$$
$$\int_{-\infty}^{\infty} \tilde{\alpha}_{Lm} \tilde{\alpha}_{Rn} dy = 0. \quad (3.38b)$$

for $\tilde{\alpha}_{Ln} = P^{3/4} Q^{1/4} \alpha_{Ln}$ and $\tilde{\alpha}_{Rn} = P^{3/4} Q^{1/4} \alpha_{Rn}$. 
By defining \( \tilde{\alpha}_L = \sqrt{P/Q} \tilde{\alpha}_L \), we get the Schrödinger-like equation for the left chiral fermions

\[
(\partial_y^2 - V_{\text{eff}}) \tilde{\alpha}_L = 0 \quad (3.39)
\]

with the effective potential

\[
V_{\text{eff}} = -m^2 \frac{P}{Q} + \eta^2 F^2(\omega)Q - \eta F'(\omega(y)) \sqrt{Q} - \frac{1}{2} \eta F(\omega) \sqrt{Q} \frac{P'}{P}
\]

\[
- \frac{3P'^2}{16P^2} + \frac{5Q'^2}{16P^2} - \frac{P'Q'}{8PQ} + \frac{P''}{4P} - \frac{Q''}{4Q}. \quad (3.40)
\]

For localization of massive fermions around the brane, the effective potential \( V_{\text{eff}} \) should have a minimum at the brane. Furthermore, we also demand a symmetry for \( V_{\text{eff}} \) about the position of the brane. This requires \( F(\omega(y)) \) to be an odd function of \( y \). So we set \( F(\omega(y)) = -\omega'(y) \). Here we only discuss the first configuration of brane (2.3) (for others configurations, the corresponding discuss is similar). Now the potential is reduced to

\[
V_{\text{eff}} = \frac{1}{4} a^2 b - m^2 \text{sech}^2(ay) + a^2 b k \eta \cosh \frac{1}{2} b k (ay) + a^2 b^2 k^2 \eta^2 \cosh b k (ay) \tanh^2 (ay)
\]

\[
+ \frac{1}{16} a^2 b (b - 4) \tanh^2 (ay) + \frac{1}{2} a^2 b^2 \eta k (k + 1) \cosh \frac{1}{2} b k (ay) \tanh^2 (ay). \quad (3.41)
\]

The shape of the above effective potential is shown in Fig. (5) for different values of \( \eta \). All the plots show a minimum at the position of the brane, which indicates that the massive fermions could be trapped on the brane. Furthermore, it is worth noting that the value of the potential at \( y = 0 \) is given by

\[
V_{\text{eff}}(y = 0) = \frac{1}{4} a^2 b - m^2 + a^2 b k \eta. \quad (3.42)
\]

\[\text{Figure 5: The shape of the effective potential } V_{\text{eff}} \text{ for left chiral fermions. The parameters are set to } a = 2, b = -1, \eta = 1 \text{ (red thin line) and } a = 2, b = -1, \eta = 2 \text{ (blue thick line).} \]

It is worth mentioning that even without the Yukawa coupling \( (\eta = 0) \), the potential is negative at the location of the brane because of \( b < 0 \). In this case we may obtain the
solution of bound states. However, in the case of Yukawa coupling, we can see that with increasing $\eta$, the depth of the well at the location of the brane becomes larger, resulting in that more trapped states are possible for stronger Yukawa coupling. Additionally, the depth also increases for small $k$ and large $\lambda$. Some work related to bound states in such 'volcano' potentials has been studied in Refs. [16, 31].

In order to get the massive KK modes, we expand the effective potential around $y = 0$ and retain terms up to order $y^2$. This is the harmonic oscillator approximation in the neighborhood of the brane. The differential equation for left chiral fermions is reduced to

$$\left\{ \partial_y^2 - \left[ B_1 - m^2 + \left( B_2 + \frac{1}{2} a^2 b m^2 \right) y^2 \right]\right\} \tilde{\alpha}_L = 0,$$

where

$$B_1 = \left( \frac{1}{4} + k \eta \right) a^2 b, \tag{3.44a}$$

$$B_2 = \frac{1}{16} a^4 b \left[ -4(1 + 4k\eta) + b(1 + 8k\eta + 4k^2\eta(3 + 4\eta)) \right]. \tag{3.44b}$$

Considering that the full five dimensional function $\Psi(x, y)$ should have a definite parity because of the overall symmetry of the problem under $y \to -y$, the left chiral wavefunctions will be even [15]. The possible values of $m^2$ are given by

$$m^2_n = B_1 + \left( n + \frac{1}{2} \right)^2 a^2 b + \left( n + \frac{1}{2} \right) \sqrt{2a^2 b B_1 + 4 B_2 + a^4 b^2 \left( n + \frac{1}{2} \right)}. \quad (n = 0, 2, 4, \ldots) \tag{3.45}$$

Substituting the expressions of $B_1$ and $B_2$ in Eq. (3.44) with $a$ and $b$ given by Eq. (2.9) into the above equation, $m^2_n$ are explicitly expressed as

$$m^2_n = \frac{2\lambda}{k-1} \left( (2n+1) \sqrt{n^2 + n - 1 + \frac{3}{2} k + (9 + 4\eta)k^2\eta - 4k\eta - (2n^2 + 2n + 1 + 2k\eta)} \right). \tag{3.46}$$

The relations of $m^2_n$ and $n, k, \eta$ are shown in Fig. (6). For large $n$, $m^2$ tends to a positive constant $(-7 + 6k - 24k\eta + 36k^2\eta + 16k^2\eta^2)\lambda/(k-1)$. The discrete mass spectrum is decided by the three parameters $k, \lambda$ and $\eta$. For large $k, \lambda$ or $\eta$, the values of $m^2$ will increase. For the case of $\eta = 0$, we get

$$m^2_n = \frac{2\lambda}{k-1} \left( (2n+1) \sqrt{n^2 + n - 1 + \frac{3}{2} k - (2n^2 + 2n + 1)} \right), \tag{3.47}$$

and from Fig. (6) the higher states approach continuum very soon and the limit is $(6k - 7)\lambda/(k - 1)$.

Now let us check whether there really exist normalized and bound massive states. Note that the wave functions we needed are $\tilde{\alpha}_L$, but not $\hat{\alpha}_L$. The solution of $\tilde{\alpha}_L$ is found to be

$$\tilde{\alpha}_L \propto \cosh^{-b}(ay) e^{-\frac{4B_3 y^2}{2}} H_n(\sqrt{B_3} y) \tag{3.48}$$
with $B_3 = \sqrt{B_2 + \frac{1}{7}a^2bn^2}$. The orthonormality conditions (3.38) are obviously satisfied by these massive states. The corresponding normalized wave functions for lower even $n$ are plot in Fig. (7). On the other hand, it is necessary to have a potential barrier higher than zero in order to have trapped states. It can be seen from the expression of the effective potential (3.41) that the potential barrier is decreasing with the increasing of $m^2$. So there is an upper limit on the mass of the fermions which are bounded by the effective potential. Far away from the location of the brane, the effective potential reduces to

$$V_{\text{eff}} \sim -m^2 \text{sech}^b(ay) + \frac{1}{32}a^2b^2 \left(1 + 8k\eta + 8k^2\eta(1 + \eta)\right). \quad (3.49)$$

Matching the two approximate effective potentials given in Eqs. (3.43) and (3.49), one obtain the conjoined point to be at $y = y_m \sim \frac{1}{2a}$ in the limit $\lambda \ll 1, k > 2, \eta \gg 1$. Then from the condition $V_{\text{eff}}(m_{\text{max}}, y_m) \geq 0$, we get the maximum value of mass of the bound state

$$m_{\text{max}} = \frac{1}{2} \sqrt{\frac{8k^2\eta(1+\eta) + 8k\eta + 1}{3k^2 - 7k + 4}} \text{Sech} \frac{1}{a} \left(\frac{1}{2}\right). \quad (3.50)$$

The corresponding maximum number of discrete massive bound states is

$$N_{\text{max}} = \text{Int} \left(\sqrt{\frac{\sqrt{2}(m_{\text{max}}^2 - B_1)}{4\sqrt{a^2bn_{\text{max}}^2 + 2B_2} - \frac{1}{4}}}\right). \quad (3.51)$$

where we have introduced an extra $1/2$ factor to account for the even states only. The relation of $N_{\text{max}}$ and $k, \eta$ is shown in Fig. (8). From the figure one can see that stronger coupling (large $\eta$) and weaker gravity (small $k$) can trap more massive states. For large $k$ and small $\eta$, there may have not bound states.

![Figure 6](image_url)

**Figure 6:** The relations of $m^2$ and $n, k, \eta$. The parameters are set to $k = 2, \lambda = 1$ and $\eta = 0, 1/2, 1$ for red, green and blue lines respectively in the left figure, and $k = 4, \lambda = 1$ and $\eta = 0, 1/2, 1$ for red, green and blue lines respectively in the right figure. Mass spectrum approach continuum very quickly for small $k$ and $\eta$.

Furthermore, it is interesting to investigate problem that whether there exist bounded massive states for the case without coupling ($\eta = 0$). This is a complex problem and we only
Figure 7: The normalized wave functions corresponding to the massive left chiral fermions with $\eta = 0, n = 0$ and $\eta = 1/2, n = 0, 2, 4$ for left panel and right panel, respectively. The parameters are set to be $\lambda = 1$ for both panels, and $k = 2$ in right panel, and $k = 2, 1.5$ and 1.35 for red, green and blue lines respectively in left panel. For the case of $\eta = 0$, namely, the case without coupling, we only plot the first level state ($n = 0$) because there is only one state at the most for the case.

Figure 8: The relation of $N_{\text{max}}$ and $k, \eta$. We have set $\lambda = 0.01$.

study the case of ground state ($n = 0$). From Eq. (3.47), by defining $h = 3k - 4$ ($h > 0$), the expression of $m_0$ is decided by

$$m_0^2 = \frac{3\lambda}{h + 1} \left( \sqrt{2h + 4} - 2 \right). \quad (3.52)$$

By defining $z = ay$ and substituting the above expression of $m_0^2$ into Eq. (3.41) yields

$$V_{\text{eff}} = \frac{3\lambda}{h + 1} \left[ \left( 1 + \frac{1}{2h} \right) \tanh^2 z - 1 - \left( \sqrt{2h + 4} - 2 \right) \cosh^2 z \right]. \quad (3.53)$$

In order to obtain bound massive states, the maximum of the effective potential $V_{\text{eff}}$ should be positive. Considering $\lambda > 0$ and $h > 0$, it can be seen that the sign of the maximum of $V_{\text{eff}}$ is only decided by the parameter $h$ (or $k$). The profile of $V_{\text{eff}}$ is plotted by a
new function $V_{eql}$ with $V_{eff} = \frac{3\lambda}{n+1}V_{eql}$ in Fig. (8). From the figures, one can see that for small $h$, there exist positive maximum of $V_{eql}$, and the maximum increase with the decreasing of $h$. We find that the maximum are positive for $0 < h < 0.05398$. Hence, for $4/3 < k < 1.35133$, there exists a bound massive ground ($n = 0$) state.

**Figure 9:** The profile of $V_{eql}$.

Until now, there is still another question. Since the effective potential becomes negative at infinity, the massive modes trapped on the brane are in metastable states and can tunnel from the brane to the bulk. The problem had been investigated in many literatures such as ([16, 31, 32]). In this paper we don’t discuss the issue.

At last, let us discuss massless fermions for a Yukawa coupling of general form $\eta \bar{\Psi}F(\omega(y))\Psi$. In the case, $\alpha_L(y)$ and $\alpha_R(y)$ satisfy the following decoupled equations respectively

\[
\begin{align*}
\left\{ \partial_y + \frac{P'}{P} + \eta F(\omega(y))\sqrt{Q} \right\} \alpha_L(y) &= 0, \\
\left\{ \partial_y + \frac{P'}{P} - \eta F(\omega(y))\sqrt{Q} \right\} \alpha_R(y) &= 0.
\end{align*}
\]

The solution is turned out to be

\[
\begin{align*}
\alpha_L(y) &\propto \frac{1}{P} e^{-\eta \int F(\omega(y))\sqrt{Q} dy}, \\
\alpha_R(y) &\propto \frac{1}{P} e^{\eta \int F(\omega(y))\sqrt{Q} dy}.
\end{align*}
\]

Substituting the zero mode (3.55) into the Dirac action (3.30) and considering odd $F(\omega(y))$, we obtain

\[
S_{1/2}^{(0)} = \int d^5 x \sqrt{-g} \bar{\Psi} i\Gamma^M D_M \Psi = I_L^\frac{1}{2} \int d^4 x \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + I_R^\frac{1}{2} \int d^4 x \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R,
\]

(3.56)
where

\[ I_{L\frac{1}{2}} \propto \int dy \sqrt{\frac{Q}{P}} e^{-2\eta \int dy F(\omega)\sqrt{Q}}, \quad (3.57) \]

\[ I_{R\frac{1}{2}} \propto \int dy \sqrt{\frac{Q}{P}} e^{2\eta \int dy F(\omega)\sqrt{Q}}, \quad (3.58) \]

In order to get bounded left or right chiral fermions in this framework, the integral (3.57) or (3.58) should be finite. For the case without Yukawa coupling, these integrals are infinite, so both left and right chiral fermions are not bounded. But for coupled cases, one of them may be bounded as long as the integral (3.57) or (3.58) finite. For example, if we choose the coupling form \( F = -\partial \omega e^{-\frac{1}{2} \omega} \), we will get

\[ I_{L\frac{1}{2}} \propto \int dy \cosh^{\frac{1}{2}(4k\eta-1)}(ay), \quad (3.59) \]

\[ I_{R\frac{1}{2}} \propto \int dy \cosh^{\frac{1}{2}(-4k\eta-1)}(ay). \quad (3.60) \]

The comparison of the profile of the zero mode of left chiral fermion \( \tilde{\alpha}_L = P^{-\frac{3}{2}}Q^{\frac{1}{2}}\alpha_L \propto \cosh^{b(k\eta-\frac{1}{2})}(ay) \) with the lower lying left chiral KK modes \( \tilde{\alpha}_{Ln} \) is plotted in Fig (10). Hence providing that \( \eta > \frac{1}{4k} \) yields the bounded left chiral fermion. Indeed, the right-handed zero mode could have been obtained by considering the coupling form of \( F = \partial \omega e^{-\frac{1}{2} \omega} \).

![Figure 10: The comparison of the zero mode \( \tilde{\alpha}_L \) (red thick line) of the left chiral fermion with the lower KK modes \( \tilde{\alpha}_{Ln} \) (n = 0, 2, 4). We have set \( k = 2, \lambda = 1, \eta = 2. \)](image)

4. Discussions

In this paper, we have investigated the possibility of localizing various matter fields on pure geometrical thick branes, which also localize the graviton, from the viewpoint of field theory. We first give a brief review of several types of thick smooth brane configurations in a pure geometric Weyl integrable 5-dimensional space time. Some of these thick branes break \( Z_2 \)-symmetry along the extra dimension. Then, we check localization of various
matter fields on these pure geometrical thick branes from the viewpoint of field theory. Localization of massive and massless spin 0 scalars, spin 1 vectors and spin 1/2 fermions are studied in the first kind of background geometry. For those matter fields, the massless modes and several massive modes are found to be normalizable on the brane. It is worth noting that the result for spin 1 vectors is different from the case of RS model in $AdS_5$ space, in which vector fields are not localized neither on a brane with positive tension nor on a brane with negative tension so the Dvali-Shifman mechanism should be considered for the vector field localization. It is shown that, for the case of massive fermions, stronger Yukawa coupling and weaker gravity can trap more massive states. Furthermore, it is also shown that, even without the Yukawa coupling, there is still at least one bounded massive state for small $k$. But for the massless left or right fermion localization, there must be some kind of Yukawa coupling. These situations can be compared with the case of the domain wall in the RS framework [12] where for localization of spin 1/2 field additional localization method by Jackiw and Rebbi [33] was introduced.

Localizing the fermionic degrees of freedom on branes or defects requires us to introduce other interactions but gravity. Recently, Parameswaran et al studied fluctuations about axisymmetric warped brane solutions in 6-dimensional minimal gauged supergravity and proved that, not only gravity, but Standard Model fields could “feel” the extent of large extra dimensions, and still be described by an effective 4-Dimensional theory [20]. Moreover, there are some other backgrounds could be considered besides gauge field [34] and supergravity [33], for example, vortex background [36, 37]. The topological vortex coupled to fermions may result in chiral fermion zero modes [38]. More recently, Volkas et al had extensively analyzed localization mechanisms on a domain wall. In particular, in Ref. [39], they proposed a well-defined model for localizing the SM, or something close to it, on a domain wall brane. Their paper made use of preparatory work done in Refs. [40, 41].

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References

[1] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, The hierarchy problem and new dimensions at a millimeter, Phys. Lett. B 429 (1998) 263, arxiv:hep-ph/9803315; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, New dimensions at a millimeter to a Fermi and superstrings at a TeV, Phys. Lett. B 436 (1998) 257, arxiv:hep-ph/9804398.
[2] L. Randall and R. Sundrum, *A Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev Lett. **83** (1999) 3370, arxiv:hep-ph/9905221; *An alternative to compactification*, Phys. Rev. Lett. **83** (1999) 4690, arxiv:hep-th/9906064; J. Lykken and L. Randall, *The Shape of Gravity*, JHEP **0006** (2000) 014, arxiv:hep-th/9908076.

[3] V.A. Rubakov and M.E. Shaposhnikov, *Do we live inside a domain wall?*, Phys. Lett. **B 125** (1983) 136.

[4] V.A. Rubakov and M.E. Shaposhnikov, *Extra space-time dimensions: towards a solution to the cosmological constant problem*, Phys. Lett. **B 125** (1983) 139.

[5] K. Akama, *Proceedings of the symposium on gauge theory and gravitation*, Nara, Japan, eds. K. Kikkawa, N. Nakanishi and H. Nariai (Springer-Verlag, 1983).

[6] M. Visser, *An exotic class of Kaluza-Klein models*, Phys. Lett. **B 159** (1985) 22, arxiv:hep-th/9910093.

[7] S. Randjbar-Daemi and C. Wetterich, *Kaluza-Klein solutions with noncompact internal spaces*, Phys. Lett. **B 166** (1986) 65.

[8] I. Antoniadis, *A possible new dimension at a few Tev*, Phys. Lett. **B 246** (1990) 377.

[9] A. Kehagias, *A conical tear drop as a vacuum-energy drain for the solution of the cosmological constant problem*, Phys. Lett. **B 600** (2004) 133, arxiv:hep-th/0406025.

[10] E.J. Squires, *Dimensional reduction caused by a cosmological constant*, Phys. Lett. **B 167** (1986) 286.

[11] M. Gogberashvili, *Four dimensionality in noncompact Kaluza-Klein model*, Mod. Phys. Lett. **A 14** (1999) 2025; *Gravitational trapping for extended extra dimension*, Int. J. Mod. Phys. **D 11** (2002) 1635.

[12] B. Bajc and G. Gabadadze, *Localization of matter and cosmological constant on a brane in anti de Sitter space*, Phys. Lett. **B 474** (2000) 282, arxiv:hep-th/9912232.

[13] I. Oda, *Localization of matters on a string-like defect*, Phys. Lett. **B 496** (2000) 113, arxiv:hep-th/0006203.

[14] Y. Grossman and N. Neubert, *Neutrino masses and mixings in non-factorizable geometry*, Phys. Lett. **B 474** (2000) 361, arxiv:hep-ph/9912408; S. Ichinose, *Fermions in Kaluza-Klein and Randall-Sundrum theories*, Phys. Rev. **D 66** (2002) 104015, arxiv:hep-th/0206187; R. Koley and S. Kar, *A novel braneworld model with a bulk scalar field*, Phys. Lett. **B 623** (2005) 244, arxiv:hep-th/0507277; [Erratum *ibid.* **631** (2005) 199].

[15] C. Ringeval, P. Peter and J.P. Uzan, *Localization of massive fermions on the brane*, Phys. Rev. **D 65** (2002) 044416, arxiv:hep-th/0109194.

[16] R. Koley and S. Kar, *Scalar kinks and fermion localisation in warped spacetimes*, Class. Quantum Grav. **22** (2005) 753, arxiv:hep-th/0407158.

[17] T. Gherghetta and M. Shaposhnikov, *Localizing gravity on a string-like defect in six dimensions*, Phys. Rev. Lett. **85** (2000) 240, arxiv:hep-th/0004014.

[18] I.P. Neupane, *Consistency of higher derivative gravity in the brane background*, JHEP **0009** (2000) 040, arxiv:hep-th/0008190; I.P. Neupane, *Completely localized gravity with higher curvature terms*, Class. Quant. Grav. **19** (2002) 5507, arXiv:hep-th/0106100.
[19] S. Randjbar-Daemi and M. Shaposhnikov, *Fermion zero-modes on brane-worlds*, Phys. Lett. B 492 (2000) 361, arxiv:hep-th/0008079.

[20] S.L. Parameswaran, S. Randjbar-Daemi and A. Salvio, *Gauge Fields, Fermions and Mass Gaps in 6D Brane Worlds*, Nucl. Phys. B 767 (2007) 54, arxiv:hep-th/0608074.

[21] O. DeWolfe, D.Z. Freedman, S.S. Gubser and A. Karch, *Modeling the fifth dimension with scalars and gravity*, Phys. Rev. D 62 (2000) 046008, arxiv:hep-th/9909134.

[22] M. Greem, *Four-dimensional gravity on a thick domain wall*, Phys. Lett. B 478 (2000) 434; Thick domain walls and singular spaces, Phys. Rev. D 62 (2000) 044017.

[23] C. Csaki, J. Erlich, T. Hollowood and Y. Shirman, *Universal Aspects of gravity localized on thick branes*, Nucl. Phys. B 581 (2000) 309; R. Emparan, R. Gregory and C. Santos, *Black holes on thick branes*, Phys. Rev. D 63 (2001) 104022; S. Kobayashi, K. Koyama and J. Soda, *Thick brane worlds and their stability*, Phys. Rev. D 65 (2002) 064014; A. Campos, *Critical phenomena of thick brane in warped space-time*, Phys. Rev. Lett. 88 (2002) 141602; A. Wang, *Thick de Sitter Branes, dynamic black holes and localization of gravity*, Phys. Rev. D 66 (2002) 024024; R. Guerrero, A. Melfo and N. Pantoja, *Selfgravitating domain walls and the thin wall limit*, Phys. Rev. D 65 (2002) 125010; A. Melfo, N. Pantoja and A. Skirzewski, *Thick domain wall space-time with and without reflection symmetry*, Phys. Rev. D 67 (2003) 105003; K.A. Bronnikov and B.E. Meierovich, *A general thick brane supported by a scalar field*, Grav. Cosmol. 9 (2003) 313; O. Castillo–Felisola, A. Melfo, N. Pantoja and A. Ramirez, *Localizing gravity on exotic thick three-branes*, Phys. Rev. D 70 (2004) 104029; M. Minamitsuji, W. Naylor and M. Sasaki, *Quantum fluctuations on a thick de Sitter brane*, arxiv:hep-th/0508093; T.R. Slattery and R.R. Volkas, *Cosmology and fermion confinement in a scalar-field generated domain wall brane in five dimensions*, JHEP 0704 (2007) 062, arxiv:hep-ph/0609003.

[24] V. Dzhunushaliev, V. Folomeev, D. Singleton and S. Aguilar-Rudametkin, *Thick branes from scalar fields*, arxiv:hep-th/0703043; V. Dzhunushaliev, V. Folomeev, K. Myrzakulov and R. Myrzakulov *Thick brane in 7D and 8D spacetimes*, arxiv:0705.4014.

[25] O. Arias, R. Cardenas and I. Quiros, *Thick Brane Worlds Arising From Pure Geometry*, Nucl. Phys. B 643 (2002) 187, arxiv:hep-th/0202130.

[26] N. Barbosa-Cendejas and A. Herrera-Aguilar, *4D gravity localized in non $Z_2$–symmetric thick branes* JHEP 0510 (2005) 101, arxiv:hep-th/0511050.

[27] N. Barbosa-Cendejas and A. Herrera-Aguilar, *Localization of 4D gravity on pure geometrical thick branes*, Phys. Rev. D 73 (2006) 084022, arxiv:hep-th/0603184.

[28] G. Dvali and M. Shifman, *Domain walls in strongly coupled theories*, Phys. Lett. B 396 (1997) 64, arxiv:hep-th/9612128.

[29] S.L. Dubovsky, V.A. Rubakov and P.G. Tinyakov, *Brane world: disappearing massive matter*, Phys. Rev. D 62 (2000) 105011, arxiv:hep-th/0006046;

[30] A. Melfo, N. Pantoja and J.D. Tempo, *Fermion localization on thick branes*, Phys. Rev. D 73 (2006) 044033, arxiv:hep-th/0601161.

[31] E Caliceti, V. Grechii and M. Maioli, *Double wells: Nevanlinna analyticity, distributional Borel sum and asymptotics*, Commun. Math. Phys. 176 (1996) 1; J. Zambastil, J. Cizek and L. Scala, *WKB Approach to Calculating the Lifetime of Quasistationary States*, Phys. Rev. Letts. 84 (2000) 5683; F. J. Gomez and L. Sesma, Phys. Letts. A 301 (2002) 184; M. M.
Nieto, *A Simple Volcano Potential with an Analytic, Zero-Energy, Ground State*, Phys. Letts. **B 486** (2000) 414.

[32] H. Davoudiasl, J.L. Hewett and T.G. Rizzo, *Bulk Gauge Fields in the Randall-Sundrum Model*, Phys. Lett. **B 473** (2000) 43.

[33] R. Jackiw and C. Rebbi, *Solitons with fermion number 1/2*, Phys. Rev. **D 13** (1976) 3398.

[34] Y.X. Liu, L. Zhao, Y.S. Duan, *Localization of Fermions on a String-like Defect*, JHEP **0704** (2007) 097, arxiv:hep-th/0701010.

[35] G. de Pol, H. Singh and M. Tonin, *Action with manifest duality for maximally supersymmetric six-dimensional supergravity*, Int. J. Mod. Phys. **A 15** (2000) 4447, arxiv:hep-th/0003106.

[36] Y.X. Liu, L. Zhao, X.H. Zhang and Y.S. Duan, *Fermions in Self-dual Vortex Background on a String-like Defect*, Nucl. Phys. **B 785** (2007) 234, arxiv:0704.2812[hep-th].

[37] Y.Q. Wang, T.Y. Si, Y.X. Liu and Y.S. Duan, *Fermionic zero modes in self-dual vortex background*, Mod. Phys. Lett. **A 20** (2005) 3045, arxiv:hep-th/0508111; Y.S. Duan, Y.X. Liu and Y.Q. Wang, *Fermionic Zero Modes in Gauge and Gravity Backgrounds on $T^2$*, Mod. Phys. Lett. **A 21** (2006) 2019, arxiv:hep-th/0602157; Y.X. Liu, Y.Q. Wang and Y.S. Duan, *Fermionic zero modes in self-dual vortex background on a torus*, accepted by Commun. Theor. Phys. (2007).

[38] R. Jackiw and P. Rossi, *Zero modes of the vortex-fermion system*, Nucl. Phys. **B 190** (1981) 681.

[39] R. Davies, D.P. George, and R.R. Volkas, *Standard model on a domain wall brane*, arxiv:0705.1584[hep-ph].

[40] D.P. George, and R.R. Volkas, *Kink modes and effective four dimensional fermion and Higgs brane models*, Phys. Rev. **D 75** (2007) 105007, arxiv:hep-ph/0612270.

[41] R. Davies and D.P. George, *Fermions, scalars and Randall-Sundrum gravity on domain-wall branes*, to appear in Phys. Rev. D, arxiv:0705.1391[hep-ph].