LINEAR MAPS PRESERVING THE DIMENSION OF FIXED POINTS OF OPERATORS

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Abstract. Let $B(\mathcal{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 3$. In this paper, we characterize the forms of surjective linear maps on $B(\mathcal{X})$ which preserve the dimension of the vector space containing of all fixed points of operators, whenever $\mathcal{X}$ is a finite dimensional Banach space. Moreover, we characterize the forms of linear maps on $B(\mathcal{X})$ which preserve the vector space containing of all fixed points of operators.

1. Introduction And Statement of the Results

The study of maps on operator algebras preserving certain properties is a topic which attracts much attention of many authors (see [1]–[13] and the references cited there.) Some of these problems are concerned with preserving a certain property of usual products or other products of operators (see [1]–[8] and [12]–[14]).

Let $B(X)$ denotes the algebra of all bounded linear operators on a complex Banach space $X$ with $\dim X \geq 3$. Recall that $x \in X$ is a fixed point of an operator $T \in B(X)$, whenever we have $Tx = x$. Denote by $F(T)$, the set of all fixed points of $T$. It is clear that for an linear operator $T$, $F(T)$ is a vector space. Denote by $\dim F(T)$, the dimension of $F(T)$. In [12], authors characterized the forms of surjective maps on $B(\mathcal{X})$ such that preserve the dimension of the vector space containing of all fixed points of products of operators. In this paper, we characterize the forms of surjective linear maps on $B(\mathcal{X})$ which preserve the dimension of the vector space containing of all fixed points of operators.

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points of operators, whenever $\mathcal{X}$ is a finite dimensional Banach space. Moreover, we characterize the forms of linear maps on $\mathcal{B}(\mathcal{X})$ which preserve the vector space containing of all fixed points of operators. The statements of our main results are the follows.

**Theorem 1.1.** Let $\mathcal{X}$ be a complex Banach space with $\dim \mathcal{X} \geq 3$. Suppose $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ is a surjective linear map which satisfies the following condition:

$$F(A) = F(\phi(A)) \quad (A \in \mathcal{B}(\mathcal{X})).$$

Then $\phi(A) = A$ for all $A \in \mathcal{B}(\mathcal{X})$.

**Theorem 1.2.** Let $n \geq 3$. Suppose $\phi : \mathcal{M}_n \longrightarrow \mathcal{M}_n$ is a surjective linear map which satisfies the following condition:

$$(2) \quad \dim F(A) = \dim F(\phi(A)) \quad (A \in \mathcal{M}_n).$$

Then there exists an invertible matrix $S \in \mathcal{M}_n$ such that $\phi(A) = SAS^{-1}$ or $\phi(A) = -SAS^{-1}$ for all $A \in \mathcal{M}_n$.

We recall some notations. $X^*$ denotes the dual space of $X$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank one operator in $\mathcal{B}(\mathcal{X})$ can be written in this way. The rank one operator $x \otimes f$ is idempotent if and only if $f(x) = 1$.

Let $x \otimes f$ be a rank-one operator. It is easy to check that $x \otimes f$ is an idempotent if and only if $F(x \otimes f) = \langle x \rangle$ (the linear subspace spanned by $x$). If $x \otimes f$ isn’t idempotent, then $F(x \otimes f) = \{0\}$.

Given $P, Q \in \mathcal{P}$, we say that $P$ and $Q$ are orthogonal if and only if $PQ = QP = 0$.

2. **Linear maps preserving the fixed points of operators**

Assume that $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ is a surjective linear map which satisfies the following condition:

$$(1) \quad F(A) = F(\phi(A)) \quad (A \in \mathcal{B}(\mathcal{X})).$$
First we prove some elementary results which are useful in the proof of Theorem 1.1.

**Lemma 2.1.** Let $x \in \mathcal{X}$ and $A \in \mathcal{B}(\mathcal{X})$. If $x$ and $Ax$ are linear independent vectors, then there exists a rank one idempotent $P$ such that $x \in F(A + P)$.

*Proof.* Since $x$ and $Ax$ are linear independent vectors, there exists a linear functional $f$ such that $f(x) = 1$ and $f(Ax) = 0$. Set $P = (x - Ax) \otimes f$. We have

$$(A + P)x = (A + (x - Ax) \otimes f)x = Ax + (x - Ax)f(x) = x$$

which completes the proof. \hfill \Box

**Lemma 2.2.** $\phi(P) = P$ for every rank one idempotent $P$.

*Proof.* Since $\mathcal{X} = F(I) = F(\phi(I))$, we obtain $\phi(I) = I$. This implies that $\ker(A) = \ker(\phi(A))$, for every $A \in \mathcal{B}(\mathcal{X})$, because

$$x \in \ker(A) \iff Ax + x = x \iff x \in F(A + I) = F(\phi(A) + I)$$

$$\iff x = \phi(A)x + x \iff x \in \ker(\phi(A)).$$

Let $P = x \otimes f$, for some $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$ such that $f(x) = 1$. We have

$$\ker f(x) = \ker x \otimes f = \ker(\phi(x) \otimes f)$$

which implies that $\phi(x \otimes f)$ is a rank one operator, because its null space is a hyperspace of $\mathcal{X}$. Thus there exists a $y \in \mathcal{X}$ such that $\phi(P) = y \otimes f$, because its null space is equal to the null space of $f$. By assumption we have $F(x \otimes f) = F(y \otimes f)$ which implies that $x$ and $y$ are linear dependent and also $f(y) = 1$. Hence $x = y$ and this completes the proof. \hfill \Box

**Lemma 2.3.** For any rank one idempotent $P$, there exists an $\eta(P) \in \mathbb{C}$ such that $\phi(A) + P = \eta(P)(A + P)$, for every $A \in \mathcal{B}(\mathcal{X})$.

*Proof.* Let $P$ be a rank one idempotent. Suppose there exists an $x \in \mathcal{X}$ such that $(\phi(A) + P)x$ and $(A + P)x$ are linear independent. If $x$ and $(A + P)x$ are linear independent, by Lemma 2.1 there exists a rank
one idempotent $Q$ such that $x \in F(A + P + Q)$ which by Lemma 2.2 implies that $x \in F(\phi(A) + P + Q)$. Thus $\phi(A)x = Ax$, which is a contradiction, because we assume that $(\phi(A) + P)x$ and $(A + P)x$ are linear independent. Therefore, $\phi(A) + P$ and $A + P$ are locally linear dependent for any rank one idempotent $P$. By [9, Theorem 2.4], there is an $\eta(P) \in \mathbb{C}$ such that $\phi(A) + P = \eta(P)(A + P)$. □

**Proof of Theorem 1.1.** Let $A \in \mathcal{B}(\mathcal{X})$ and $P$ be a rank one idempotent. It is enough to prove that $\eta(P)$ in Step 2 is equal to 1. If $A + P$ is a scalar operator, then by Lemma 2.3, $\eta(P) = 1$, because $A + P = \phi(A + P) = \phi(A) + P$. If $A + P$ is a non-scalar operator, then there exists an $x \in \mathcal{X}$ such that $x$ and $(A + P)x$ are linear independent. By Lemma 2.1 we can find a rank one idempotent $Q_1$ such that $x \in F(A + P + Q_1)$ which implies that $x \in F(\phi(A) + P + Q_1) = F(\eta(P)(A + P) + Q_1)$

and so

$$(A + P)x = \eta(P)(A + P)x.$$ 

Since $(A + P)x$ is nonzero, $\eta(P) = 1$ and this completes the proof.

3. **Linear maps preserving the dimension of fixed points of operators**

Let $n \geq 3$. Assume that $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ is a surjective linear map which satisfies the following condition:

$$(2) \quad \dim F(A) = \dim F(\phi(A)) \quad (A \in \mathcal{M}_n).$$

First we prove some elementary results which are useful in the proof of Theorem 1.12.

**Lemma 3.1.** $\phi(A) = I$ if and only if $A = I$.

*Proof.* Since $n = \dim F(I) = \dim F(\phi(A)) = \dim F(A)$, we obtain $F(A) = \mathcal{X}$ and so $A = I$. □

**Lemma 3.2.** $\phi$ preserves the rank one idempotents in both directions.
Proof. It is easy to check that
\[
(*) \quad \ker(A) = F(A + I),
\]
for every \( A \in \mathcal{B} (\mathcal{X}) \). Let \( x \in \mathcal{X} \) and \( f \in \mathcal{X}^* \). Lemma 3.1 together with \((*)\) and (2) implies that
\[
n - 1 = \dim \ker(x \otimes f) = \dim F(x \otimes f + I) = \dim F(\phi(x \otimes f) + I) = \dim \ker(\phi(x \otimes f)).
\]
Thus \( \phi(x \otimes f) \) is a rank one operator. If \( f(x) = 1 \), then we have
\[
1 = \dim F(x \otimes f) = \dim F(\phi(x \otimes f))
\]
which implies that \( \phi(x \otimes f) \) is idempotent. In a similar way can show that \( \phi \) preserves the rank one idempotents in other direction and this completes the proof. \(\square\)

**Lemma 3.3.** \( \phi \) preserves the orthogonality of rank one idempotents in both directions.

*Proof.* It is easy to check that
\[
(**) \quad \text{rank}(A) \geq \dim F(A)
\]
for every \( A \in \mathcal{M}_n \). Let \( P \) and \( Q \) be two orthogonal idempotents. This together with (2) implies that
\[
2 = \dim F(P + Q) = \dim F(\phi(P) + \phi(Q)).
\]
So by \((**)\) we obtain \( \text{rank}(\phi(P) + \phi(Q)) \geq 2 \). On the other hand, since \( \phi(P) \) and \( \phi(Q) \) are rank one idempotents, \( \text{rank}(\phi(P) + \phi(Q)) \leq 2 \). Thus we obtain \( \text{rank}(\phi(P) + \phi(Q)) = 2 \) and this yields the orthogonality of \( \phi(P) \) and \( \phi(Q) \). In a similar way can show that \( \phi \) preserves the orthogonality of rank one idempotents in other direction and this completes the proof. \(\square\)
Proof of Theorem 1.2. By Lemmas 3.2 and 3.3, it is clear that $\phi$ preserves idempotent operators in both directions. So assertion follows from [11, Theorem 2.2].

In the end of this paper, we pose the following problem.

Problem. In this paper, we discuss the linear maps on matrix algebra preserving the dimension of fixed points of elements. One natural problem is how one should characterize linear maps on $B(\mathcal{X})$ preserving the dimension of fixed points of elements, where $\mathcal{X}$ is infinite dimensional Banach space.

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