SIGNATURES, SUMS OF HERMITIAN SQUARES AND
POSITIVE CONES ON ALGEBRAS WITH INVOLUTION

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(Communicated by Robert L. Griess)

ABSTRACT. We provide a coherent picture of our efforts thus far in extending
real algebra and its links to the theory of quadratic forms over ordered fields in
the noncommutative direction, using hermitian forms and “ordered” algebras
with involution.

1. Introduction

Hilbert, in his famous Paris talk in 1900 [15], posed the following as his 17th prob-
lem: ob nicht jede definite Form als Quotient von Summen von Formenquadraten
dargestellt werden kann. Translated: is every nonnegative n-ary polynomial over a
field \( F \) a sum of squares of rational functions over \( F \)? Partial answers were known
to Hilbert, as well as the fact that a positive semidefinite polynomial need not be
a sum of squares of polynomials.

The full, affirmative answer was obtained by Artin in 1927 and built on joint
work with Schreier, published in the same year. To be precise: let \( F \) be a field
of characteristic different from 2 with space of orderings \( X_F \). The Artin-Schreier
theorem [3] says that \( F \) admits an ordering if and only if \(-1\) is not a sum of squares
in \( F \). Artin’s theorem [2] says that an element is positive at all \( P \in X_F \) if and only
if it is a sum of squares in \( F \). This result is crucial in his solution of Hilbert’s 17th
problem.

Quadratic forms over \( F \) come into the picture via the Witt ring \( W(F) \) (the ring
of Witt equivalence classes of nondegenerate quadratic forms over \( F \)). Orderings
\( P \in X_F \) correspond to signature homomorphisms \( \text{sign}_P : W(F) \to \mathbb{Z} \). This goes
back to Sylvester’s Law of Inertia [33]. In addition to the fundamental ideal of
\( W(F) \), the prime ideals of \( W(F) \) are given by \( \text{sign}_P^{-1}(0) \) and \( \text{sign}_P^{-1}(p\mathbb{Z}) \) (where \( p \)
is an odd prime) with \( P \in X_F \) (Lorenz-Leicht [25]). Given \( q \in W(F) \), the total
signature map \( \text{sign}_*: q : X_F \to \mathbb{Z}, P \mapsto \text{sign}_P q \) is continuous with respect to the
Harrison topology on \( X_F \) and the discrete topology on \( \mathbb{Z} \). Furthermore, the torsion
part of \( W(F) \) consists of those \( q \) with \( \text{sign} \equiv 0 \) (Pfister’s local-global principle [26]).

The above results (well-documented in [21] and [32]), and some of their ex-
tensions to commutative rings, are among the foundations of real algebra, see for
example [12] or Lam’s expository paper [20]. In a series of recent papers [4, 5, 6, 7, 9]
(and also [23]) we extended these results in the noncommutative direction, more precisely to central simple $F$-algebras with involution and hermitian forms over such algebras.

The study of central simple algebras with involution was initiated by Albert in the 1930s [1] and is still a topic of current research as testified by The Book of Involutions [19]; see also [10] and the copious references therein for a list of open problems in this area. A large part of present day research in algebras with involution is driven by the deep connections with linear algebraic groups, first observed by Weil [35]; see also Tignol’s 2 ECM exposition [34]. Some work has been done on algebras with involution over formally real fields, for example [22, 30], but this part of the theory is relatively underdeveloped. This observation, together with the fact that algebras with involution are a natural generalization of quadratic forms, are motivating factors for our research.

This article is an expanded version of the prepublication [8], from the Séminaire de Structures Algébriques Ordonnées, Universities Paris 6 and 7.

2. Signatures

Let $(A, \sigma)$ be an $F$-algebra with involution, by which we mean that $A$ is a finite dimensional simple $F$-algebra with centre a field $K \supseteq F$ and $\sigma$ is an $F$-linear anti-automorphism of $A$ of order 2 (which implies that $[K : F] \leq 2$). Let $W(A, \sigma)$ denote the Witt group of $(A, \sigma)$, i.e., the $W(F)$-module of Witt equivalence classes of nondegenerate hermitian forms $h : M \times M \to A$, where $M$ is a finitely generated right $A$-module (cf. [18, Chap. I] or [32, Chap. 7]). We identify hermitian forms with their Witt class in $W(A, \sigma)$, unless indicated otherwise. Given an ordering $P \in X_F$ we wish to define a signature at $P$, i.e., a morphism of groups

$$W(A, \sigma) \to \mathbb{Z}.$$ 

Following the approach of [11] we do this by extending scalars to a real closure $F_P$ of $F$ at $P$ and realizing that, by Morita equivalence, the Witt group of any $F_P$-algebra with involution is isomorphic to either $\mathbb{Z}$, 0 or $\mathbb{Z}/2\mathbb{Z}$. In the last two cases, the only sensible definition is to take the signature at $P$ to be identically zero. In this case we call $P$ a nil-ordering and we write $\text{Nil}[A, \sigma]$ for the set of all nil-orderings, noting that it only depends on the Brauer class of $A$ and the type of $\sigma$. Furthermore, $\text{Nil}[A, \sigma]$ is clopen in $X_F$, cf. [4, Corollary 6.5].

In the first case, the Witt group $W(A \otimes_F F_P, \sigma \otimes \text{id})$ is isomorphic to one of $W(F_P)$, $W((-1, -1)_{F_P}, -)$, or $W(F_P(\sqrt{-1}), -)$, where $-$ denotes (quaternion) conjugation, each one in turn being isomorphic to $\mathbb{Z}$ via the usual Sylvester signature of quadratic or hermitian forms. The composite map $s_P$, given by

$$W(A, \sigma) \longrightarrow W(A \otimes_F F_P, \sigma \otimes \text{id}) \longrightarrow \mathbb{Z},$$

enables us to define a signature. The map $s_P$ is independent of the choice of $F_P$ [4, Prop. 3.3], but a different choice of Morita equivalence may result in a sign change [4, Prop. 3.4] and, conversely, such a sign change can always be obtained by taking a well-chosen different Morita equivalence.

At first sight, one way to fix a sign would be to demand that $s_P((1)_{\sigma})$ is positive, as is the case for quadratic forms. This is the approach taken in [11], but it may not always work, since it may happen that $s_P((1)_{\sigma})$ is in fact 0, as illustrated in [4, Rem. 3.11 and Ex. 3.12]. Our solution to this dilemma is to show that there exists
a hermitian form $\eta$ over $(A, \sigma)$, called a reference form, such that $s_P(\eta)$ is always nonzero whenever $P \in X_F := X_F \setminus \text{Nil}[A, \sigma]$, cf. [5, Prop. 3.2]. Using this, given $P \in \tilde{X}_F$, we define the signature at $P$ with respect to the reference form $\eta$,

$$\text{sign}^\eta_P : W(A, \sigma) \rightarrow \mathbb{Z},$$

to be the map $s_P$, multiplied by $-1$ in case $s_P(\eta) < 0$, so that $\text{sign}^\eta_P(\eta) > 0$.

The map $\text{sign}^\eta_P$ does not depend on the Morita equivalence used in its computation and so we may use the explicit Morita equivalence presented in [24] in all practical situations.

**Remark 2.1.** In case $(A, \sigma) = (F, \text{id}_F)$, we may take $\eta = \langle 1 \rangle$ and $\text{sign}^\eta_P$ is then the usual Sylvester signature $\text{sign}_P$ of quadratic forms.

**Remark 2.2.** The signature map is defined for all hermitian forms over $(A, \sigma)$, not just the nondegenerate ones as the notation above (which makes use of $W(A, \sigma)$) might suggest. It suffices to replace a form by its nondegenerate part (cf. [7, Prop. A.3]), or alternatively, to replace $W(A, \sigma)$ by $\text{Herm}(A, \sigma)$, the category of hermitian forms over $(A, \sigma)$.

**Remark 2.3.** A reference tuple of hermitian forms of dimension one can be used instead of the reference form, cf. [4, Thm. 6.4] and [5, §3]. In fact, this is the approach used in [4].

We collect some immediate properties of the signature map:

**Proposition 2.4** (Properties of the signature map [5, Thm. 2.6]).

1. Let $h$ be a hyperbolic form over $(A, \sigma)$, then $\text{sign}^\eta_P h = 0$.
2. Let $h_1, h_2 \in W(A, \sigma)$, then $\text{sign}^\eta_P(h_1 \perp h_2) = \text{sign}^\eta_P h_1 + \text{sign}^\eta_P h_2$.
3. Let $h \in W(A, \sigma)$ and $q \in W(F)$, then $\text{sign}^\eta_P(q \cdot h) = \text{sign}_P q \cdot \text{sign}^\eta_P h$.
4. (Going-up) Let $h \in W(A, \sigma)$ and let $L/F$ be an algebraic extension of ordered fields. Then

$$\text{sign}^\eta_{Q \otimes L}(h \otimes L) = \text{sign}^\eta_{Q \cap F} h$$

for all $Q \in \mathcal{X}_L$.

Property (4) is complemented by the following going-down result:

**Theorem 2.5** (Knebusch trace formula [4, Thm 8.1]). Let $L/F$ be a finite extension of ordered fields and assume $P \in X_F$ extends to $L$. Let $h \in W(A \otimes_F L, \sigma \otimes \text{id})$. Then

$$\text{sign}^\eta_P(\text{Tr}^*_A \otimes_F L h) = \sum_{P \subseteq Q \in \mathcal{X}_L} \text{sign}^\eta_{Q \otimes L} h,$$

where $\text{Tr}^*_A \otimes_F L h$ denotes the Scharlau transfer induced by the $A$-linear homomorphism $\text{id}_A \otimes \text{Tr}_L/F : A \otimes_F L \rightarrow A$.

**Theorem 2.6** (Preservation under Morita equivalence [5, Thm 4.2]). Let $(B, \tau)$ be an $F$-algebra with involution, Morita equivalent to $(A, \sigma)$, and assume that $\sigma$ and $\tau$ are of the same type. Let $\zeta : W(A, \sigma) \xrightarrow{\sim} W(B, \tau)$ be the induced isomorphism of Witt groups. Then

$$\text{sign}^\eta_P h = \text{sign}^\zeta_{P} \zeta(h)$$

for all $h \in W(A, \sigma)$ and all $P \in X_F$.

**Theorem 2.7** (Pfister’s local-global principle [23, Thm 4.1]). Let $h \in W(A, \sigma)$. Then $h$ is a torsion form if and only if $\text{sign}^\eta_P h = 0$ for all $P \in X_F$. 
Theorem 2.8 (Continuity of the total signature [4, Thm 7.2]). Let \( h \in W(A, \sigma) \).

The total signature of \( h \), \( \text{sign}^n_{\eta} h : X_F \to \mathbb{Z} \), \( P \mapsto \text{sign}^n_{\eta} h, \) is continuous.

These two theorems motivate the following results, familiar from the quadratic forms case. Let \( C(X_F, \mathbb{Z})_{[A, \sigma]} \) denote the ring of continuous functions from \( X_F \) to \( \mathbb{Z} \), that are zero on \( \text{Nil}[A, \sigma] \).

Theorem 2.9 ([9, Prop. 4.3]). For every \( f \in C(X_F, \mathbb{Z})_{[A, \sigma]} \) there exists \( n \in \mathbb{N} \) such that \( 2^n f \in \text{Im} \text{sign}^n \). In other words, the cokernel of \( \text{sign}^n \) is a 2-primary torsion group.

The stability index of \((A, \sigma)\) is the smallest \( k \in \mathbb{N} \) such that \( 2^k C(X_F, \mathbb{Z})_{[A, \sigma]} \subseteq \text{Im} \text{sign}^n \) if such a \( k \) exists and \( \infty \) otherwise. It is independent of the choice of \( \eta \).

The group coker \( \text{sign}^n \) is up to isomorphism independent of the choice of \( \eta \). We denote it by \( S_\eta(A, \sigma) \) and call it the stability group of \((A, \sigma)\).

Theorem 2.10 ([9, Thm. 4.10]). Let \( W_t(A, \sigma) \) denote the torsion subgroup of \( W(A, \sigma) \). (Recall that it is 2-primary torsion by [31, Thm. 5.1].) The sequence

\[
0 \longrightarrow W_t(A, \sigma) \longrightarrow W(A, \sigma) \longrightarrow \text{sign}^n X_F, \mathbb{Z})_{[A, \sigma]} \longrightarrow S_\eta(A, \sigma) \longrightarrow 0
\]

is exact. The group \( S_\eta(A, \sigma) \) is a 2-primary torsion group.

3. Ideals and Morphisms

Let \( R \) be a commutative ring and let \( M \) be an \( R \)-module. We introduce ideals of \( R \)-modules as follows: An ideal of \( M \) is a pair \((I, N)\) where \( I \) is an ideal of \( R \) and \( N \) is a submodule of \( M \) such that \( I \cdot M \subseteq N \). An ideal \((I, N)\) of \( M \) is prime if \( I \) is a prime ideal of \( R \) (we assume that all prime ideals are proper), \( N \) is a proper submodule of \( M \), and for every \( r \in R \) and \( m \in M \), \( r \cdot m \in N \) implies that \( r \in I \) or \( m \in N \).

These definitions are in part motivated by the following natural example: The pair \((\ker \text{sign}_p, \ker \text{sign}_p)\) is a prime ideal of the \( W(F) \)-module \( W(A, \sigma) \) whenever \( P \in \tilde{X}_F \).

We obtain a classification à la Harrison and Lorenz-Leicht [25]:

Theorem 3.1 ([5, Props. 6.5, 6.7]). Let \((I, N)\) be a prime ideal of the \( W(F) \)-module \( W(A, \sigma) \).

1. If \( 2 \notin I \), then one of the following holds:
   (i) There exists \( P \in \tilde{X}_F \) such that \((I, N) = (\ker \text{sign}_P, \ker \text{sign}_P)\).
   (ii) There exist \( P \in \tilde{X}_F \) and a prime \( p > 2 \) such that \((I, N) = (\ker (\pi_p \circ \text{sign}_P), \ker (\pi \circ \text{sign}_P))\), where \( \pi_p : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \) and \( \pi : \text{Im} \text{sign}_P \to \text{Im} \text{sign}_P/(p \cdot \text{Im} \text{sign}_P) \) are the canonical projections.

2. If \( 2 \in I \), then \( I = I(F), \) the fundamental ideal of \( W(F) \). Furthermore, a pair \((I(F), N)\) is a prime ideal of \( W(A, \sigma) \) if and only if \( N \) is a proper submodule of \( W(A, \sigma) \) with \( I(F) \cdot W(A, \sigma) \subseteq N \).

Remark 3.2. When \( 2 \notin I \), \( N \) is completely determined by \( I \). This is however not the case when \( 2 \in I \), cf. [5, Ex. 6.8].

There is a notion of morphism linked in the usual way to the above notion of ideal, cf. [5, Lemmas 5.6, 5.7 and 5.8]: Let \( R \) and \( S \) be commutative rings, let \( M \) be an \( R \)-module and \( N \) an \( S \)-module. We say that a pair \((\varphi, \psi)\) is an \((R, S)\)-morphism (of modules) from \( M \) to \( N \) if
(1) \( \varphi : R \to S \) is a morphism of rings (and in particular \( \varphi(1) = 1 \));
(2) \( \psi : M \to N \) is a morphism of additive groups;
(3) for every \( r \in R \) and \( m \in M \), \( \psi(r \cdot m) = \varphi(r) \cdot \psi(m) \).

We call an \((R,S)\)-morphism \((\varphi, \psi)\) trivial if \( \psi = 0 \). We denote the set of all \((R,S)\)-morphisms from \( M \) to \( N \) by \( \text{Hom}_{(R,S)}(M,N) \) and its subset of nontrivial \((R,S)\)-morphisms by \( \text{Hom}_{\ast(R,S)}(M,N) \).

Let \((\varphi, \psi_1)\) and \((\varphi, \psi_2)\) be \((R,S)\)-morphisms of modules from \( M \) to \( N_1 \) and \( N_2 \) respectively. We say that \((\varphi, \psi_1)\) and \((\varphi, \psi_2)\) are equivalent if there is an isomorphism of \( \text{Im} \varphi \)-modules \( \vartheta : \text{Im} \psi_1 \to \text{Im} \psi_2 \) such that \( \psi_2 = \vartheta \circ \psi_1 \). We write \( \sim \) for the relation “being equivalent.”

The pair \((\text{sign}_P, \text{sign}_P^0)\) is again a natural example of a \((W(F), \mathbb{Z})\)-morphism from \( W(A, \sigma) \) to \( \mathbb{Z} \) and is trivial if and only if \( P \in \text{Nil}[A, \sigma] \).

The classification of prime ideals of \( W(A, \sigma) \) yields the following description of signatures as morphisms:

**Theorem 3.3 ([5, Prop. 7.4]).** The map that sends \( P \in \tilde{X}_F \) to the pair \((\text{sign}_P, \text{sign}_P^0)\) induces a bijection between \( \tilde{X}_F \) and the equivalence classes with respect to \( \sim \) of \( \text{Hom}_{(W(F), \mathbb{Z})}(W(A, \sigma), \mathbb{Z}) \).

Theorems 3.1 and 3.3 give us the following:

**Corollary 3.4.** There is a bijective correspondence between the prime ideals of \( W(A, \sigma) \), the nonzero signatures of hermitian forms over \( (A, \sigma) \) and the equivalence classes of \( \text{Hom}_{(W(F), \mathbb{Z})}(W(A, \sigma), \mathbb{Z}) \) with respect to \( \sim \).

## 4. Sums of Hermitian squares

In the field case, Pfister’s local-global principle can be used to give a short proof of the fact that sums of squares are exactly the elements that are nonnegative at every ordering. In [7] we showed that the same approach directly yields a similar result for \( F \)-division algebras with involution and, with some extra effort, for all \( F \)-algebras with involution.

Let \( A^\times \) denote the set of invertible elements of \( A \), \( \text{Sym}(A, \sigma) \) the set of \( \sigma \)-symmetric elements of \( A \) and \( \text{Sym}(A, \sigma)^\times := \text{Sym}(A, \sigma) \cap A^\times \). We say that an element \( a \in \text{Sym}(A, \sigma) \) is \( \eta \)-maximal at an ordering \( P \in X_F \) if \( \text{sign}_P^\eta(a)_\sigma \) is maximal among all \( \text{sign}_P^\eta(b)_\sigma \) for \( b \in \text{Sym}(A, \sigma) \). In the field case, this means \( \text{sign}_P^\eta(a) = 1 \), in other words \( a \in P \setminus \{0\} \). For elements \( b_1, \ldots, b_t \in F^\times \) we denote the Harrison set \( \{P \in X_F \mid b_1, \ldots, b_t \in P\} \) by \( H(b_1, \ldots, b_t) \).

**Theorem 4.1 ([7, Thm. 3.6]).** Let \( b_1, \ldots, b_t \in F^\times \) and \( Y = H(b_1, \ldots, b_t) \). Assume that \( a \in \text{Sym}(A, \sigma)^\times \) is \( \eta \)-maximal at all \( P \in Y \). Let \( u \in \text{Sym}(A, \sigma) \). The following statements are equivalent:

(i) \( u \) is \( \eta \)-maximal at all \( P \in Y \).
(ii) \( u \in D_{(A, \sigma)}(k \times \langle b_1, \ldots, b_t \rangle) a_\sigma \) for some \( k \in \mathbb{N} \).

The presence of the element \( a \) as well as the hypothesis on \( \eta \)-maximality correspond in the field case to the fact that \( 1 \) belongs to every ordering. Here \( 1 \) does not play a particular role since it may not have maximal signature at some orderings. We replace it by the element \( a \) and only consider a set of orderings \( Y \) on which \( a \) has maximal signature.
Remark 4.2. As a consequence of our study of positive involutions, given \( Q \in \tilde{X}_F \), there always exist \( a \) and \( Y \) that satisfy the hypothesis of Theorem 4.1 with \( Q \in Y \), cf. Theorem 5.8 below, together with Theorem 2.8 on the continuity of the signature map.

Consider the form \( T_{(A, \sigma, u)}(x, y) := \text{Tr}_A(\sigma(x)uy) \) for \( x, y \in A \) and, following [29, Def. 1.1], let

\[
X_\sigma := \{ P \in X_F \mid \sigma \text{ is positive at } P \} = \{ P \in X_F \mid T_{(A, \sigma, 1)} \text{ is positive semidefinite at } P \}.
\]

Theorem 4.1 is reminiscent of Procesi and Schacher’s theorem [29, Thm. 5.4] and, with the notation just introduced, can also be used to address the question they raised in [29, p. 404], of whether or not the following property is always true:

\((PS)\): for every \( u \in \text{Sym}(A, \sigma) \), the form \( T_{(A, \sigma, u)} \) is positive semidefinite at all \( P \in X_\sigma \) if and only if \( u \in D_{(A, \sigma)}(s \times \langle 1 \rangle_\sigma) \) for some \( s \in \mathbb{N} \).

The general answer to this question is negative as shown in [17], but we can now describe cases where the answer is positive, and also propose a natural reformulation (inspired by signatures of hermitian forms) of the question that has a positive answer.

Proposition 4.3 ([7, Cor. 4.18]). If \( X_\sigma = X_F \), then property \((PS)\) holds.

And, if we introduce the property

\((PS')\): for every \( u \in \text{Sym}(A, \sigma) \), the form \( T_{(A, \sigma, u)} \) is positive semidefinite at all \( P \in \tilde{X}_F \) if and only if \( u \in D_{(A, \sigma)}(s \times \langle 1 \rangle_\sigma) \) for some \( s \in \mathbb{N} \), we obtain

Theorem 4.4 ([7, Thm. 4.19]). Property \((PS')\) holds if and only if \( \tilde{X}_F = X_\sigma \).

5. Positive cones

The results presented thus far suggest that there could be a notion of “ordering” on central simple algebras with involution, whose behaviour would be similar to that of orderings on fields. The purpose of this final section is to present such a notion.

Definition 5.1 ([6, Def. 3.1]). A prepositive cone \( \mathcal{P} \) on \((A, \sigma)\) is a subset \( \mathcal{P} \) of \( \text{Sym}(A, \sigma) \) such that

\begin{align*}
(P1) \quad & \mathcal{P} \neq \emptyset; \\
(P2) \quad & \mathcal{P} + \mathcal{P} \subseteq \mathcal{P}; \\
(P3) \quad & \sigma(a) \cdot \mathcal{P} \cdot a \subseteq \mathcal{P} \text{ for every } a \in A; \\
(P4) \quad & \mathcal{P}_F := \{ u \in F \mid u\mathcal{P} \subseteq \mathcal{P} \} \text{ is an ordering on } F. \\
(P5) \quad & \mathcal{P} \cap -\mathcal{P} = \{0\} \text{ (we say that } \mathcal{P} \text{ is proper).}
\end{align*}

We say that a prepositive cone \( \mathcal{P} \) is over \( P \in X_F \) if \( \mathcal{P}_F = P \).

Remark 5.2. Axiom (P4) is necessary if we want our prepositive cones to consist of either positive semidefinite (PSD) matrices with respect to \( P \), or of negative semidefinite (NSD) matrices with respect to \( P \), in the case of \((M_n(F), t)\), see [6, Rem. 3.13]. If \( \mathcal{P} \) is a prepositive cone, then \( -\mathcal{P} \) is also a prepositive cone. This is due to the fact that prepositive cones are meant to contain elements of maximal
signature, and the sign of the signature can vary with a change of the reference form.

It can be shown that there is a prepositive cone over \( P \in X_F \) on \((A, \sigma)\) if and only if \( P \in \bar{X}_F \), cf. \([6, \text{Prop. } 6.6]\).

**Examples 5.3.**

1. Let \( P \in \bar{X}_F \). We define
   \[
   \mathcal{M}_\eta^P(A, \sigma) := \{a \in \text{Sym}(A, \sigma)^\times \mid a \text{ is } \eta\text{-maximal at } P\} \cup \{0\}. 
   \]
   If \( A \) is a division algebra, \( \mathcal{M}_\eta^P(A, \sigma) \) is a prepositive cone over \( P \) on \((A, \sigma)\).

2. The set of PSD matrices, and the set of NSD matrices with respect to some \( P \in X_F \) are both prepositive cones over \( P \) on \((M_n(F), t)\).

**Remark 5.4.** Other notions of orderings have been introduced for division rings with involution, most notably Baer orderings, *-orderings and their variants and an extensive theory has been developed around them. Craven’s surveys \([13] \) and \([14]\) provide more information on these topics. Without going into the details, the main difference in the definitions is that positive cones were developed to correspond to a pre-existing algebraic notion, namely signatures of hermitian forms (e.g., axiom (P4) reflects the fact that the signature is a morphism of modules, cf. Proposition 2.4(3); see also the sentence after Theorem 5.8) and as a consequence are not required to induce total orderings on the set of symmetric elements.

We obtain the desired results linking prepositive cones and \( W(A, \sigma) \):

**Proposition 5.5** \([6, \text{Prop. } 7.11]\). The following statements are equivalent:

(i) \( (A, \sigma) \) is formally real (i.e., there is at least one prepositive cone on \((A, \sigma)\))

(ii) \( W(A, \sigma) \) is not torsion.

(iii) \( \bar{X}_F \neq \emptyset \).

Prepositive cones are well-behaved under Morita equivalence: If \((A, \sigma)\) and \((B, \tau)\) are Morita equivalent, then there is an inclusion-preserving bijection between their sets of prepositive cones. This bijection can be made explicit in the case of scaling, or when \((A, \sigma) = (D, \vartheta)\) is an \( F\)-division algebra with involution and \((B, \tau) = (M_n(D), \vartheta^t)\), using descriptions of prepositive cones reminiscent of the characterizations of positive semidefinite matrices:

**Proposition 5.6** \([6, \text{§4.1, §4.2}]\). The prepositive cones on \((M_n(D), \vartheta^t)\) are of the form

\[
\text{PSD}_n(\mathcal{P}) := \{M \in \text{Sym}(M_n(D), \vartheta^t) \mid \forall X \in D^n \quad \vartheta(X)^t MX \in \mathcal{P}\},
\]

where \( \mathcal{P} \) is a prepositive cone on \((D, \vartheta)\).

The prepositive cones on \((D, \vartheta)\) are of the form

\[
\text{Tr}_n(\mathcal{P}) := \{\text{tr}(M) \mid M \in \mathcal{P}\},
\]

where \( \mathcal{P} \) is a prepositive cone on \((M_n(D), \vartheta^t)\).

We use prepositive cones to consider the question of the existence of positive involutions:

**Theorem 5.7** \([6, \text{Thm. } 6.8]\). Let \( P \in X_F \). The following statements are equivalent:
on (the NSD matrices over $M$ more detail the maximal prepositive cones, which we simply call positive cones, preordering or Prestel’s pre-semiordering \cite{27, 28}, so it is natural to consider in $\varepsilon$-prepositive cone (namely $\eta$-prepositive cone (in the third statement may not be used in a uniform way. This is included here to point out that while the element $a$ belongs to any prepositive cone on $(A, \sigma)$, contrary to what could be expected from the field case (see \cite[Props. 4.3 and 4.9]{6}). It follows that the PSD matrices over $P$ and the NSD matrices over $P$ are the only positive cones on $(M_n(F), t)$. (See also Proposition 5.6.)

Using this description, it is possible to make the link with the results presented in Section 4, and to obtain results similar to the Artin-Schreier and Artin theorems.

**Theorem 5.9** (\cite[Thm. 7.9]{6}). The following statements are equivalent:

(i) $(A, \sigma)$ is formally real;

(ii) There is $a \in \text{Sym}(A, \sigma)^\times$ and $P \in X_F$ such that $\mathcal{C}_P(a) \cap -\mathcal{C}_P(a) = \{0\}$;

(iii) There is $b \in \text{Sym}(A, \sigma)^\times$ such that $\langle b \rangle_\sigma$ is strongly anisotropic (i.e., the forms $k \times \langle b \rangle_\sigma$ have no nontrivial zeros for all $k \in \mathbb{N}$).

The second statement is a trivial consequence of the first one, but it is still included here to point out that while the element $a$ in it obviously belongs to a prepositive cone (namely $\mathcal{C}_P(a)$), the element $b$ in the third statement may not belong to any prepositive cone on $(A, \sigma)$, contrary to what could be expected from the field case (see \cite[Rem. 7.10]{6}).

**Theorem 5.10** (\cite[Thm. 7.14]{6}). Let $b_1, \ldots, b_t \in F^\times$, let $Y = H(b_1, \ldots, b_t)$ and let $a \in \text{Sym}(A, \sigma)^\times$ be such that, for every $\mathcal{P} \in X_{(A, \sigma)}$ with $\mathcal{P}_F \in Y$, $a \in \mathcal{P} \cup -\mathcal{P}$.

Then

$\bigcap \{ \mathcal{P} \in X_{(A, \sigma)} \mid \mathcal{P}_F \in Y \text{ and } a \in \mathcal{P} \} = \bigcup_{s \in \mathbb{N}} D_{(A, \sigma)}(s \times \langle b_1, \ldots, b_t \rangle \langle a \rangle_\sigma)$.

As a consequence of our study of positive involutions, given $Q \in \tilde{X}_F$, there always exist $a$ and $Y$ that satisfy the hypothesis of Theorem 5.10 with $Q \in Y$, cf. Remark 4.2.

The element $a$ in this theorem plays the same role as the element $a$ in Theorem 4.1, and chooses a prepositive cone from $\{ \mathcal{P}, -\mathcal{P} \}$ in a uniform way. This is
not necessary in the field case, because 1 belongs to every ordering. In the special case where \( a = 1 \) can be used for this purpose, we obtain a result more similar to the usual one:

**Corollary 5.11** ([6, Cor. 7.15]). Assume that for every \( \mathcal{P} \in X_{(A, \sigma)} \), \( 1 \in \mathcal{P} \cup -\mathcal{P} \). Then

\[
\bigcap \{ \mathcal{P} \in X_{(A, \sigma)} \mid 1 \in \mathcal{P} \} = \left\{ \sum_{i=1}^{s} \sigma(x_i)x_i \mid s \in \mathbb{N}, x_i \in A \right\}.
\]

The hypothesis of Corollary 5.11 is exactly \( X_\sigma = \tilde{X}_F \) in the terminology of Section 4. More precisely, as seen therein, this property characterizes the algebras with involution for which there is a positive answer to (PS'), cf. [7, Section 4.2].

A natural question is to ask if signatures of hermitian forms over \((A, \sigma)\) can now also be defined with respect to positive cones on \((A, \sigma)\). As shown in [6, §8.2], this can indeed be done using decompositions of hermitian forms, reminiscent of Sylvester’s decomposition for quadratic forms:

**Theorem 5.12** ([6, Cor. 8.14, Lemma 8.15]). There exists an integer \( t \), depending on \((A, \sigma)\), such that for every \( \mathcal{P} \in X_{(A, \sigma)} \) and for every \( h \in W(A, \sigma) \) there exist \( u_1, \ldots, u_t \in P := \mathcal{P}_F, a_1, \ldots, a_r \in \mathcal{P} \cap A^\times \) and \( b_1, \ldots, b_s \in -\mathcal{P} \cap A^\times \) such that

\[
n_P^2 \times \langle u_1, \ldots, u_t \rangle \otimes h \simeq \langle a_1, \ldots, a_r \rangle_{\sigma} \perp \langle b_1, \ldots, b_s \rangle_{\sigma},
\]

where \( n_P \) is the matrix size of \( A \otimes_F F_P \), and \( r \) and \( s \) are positive integers, uniquely determined by \( \mathcal{P} \) and the rank of \( h \).

This theorem allows us to define

\[
\text{sign}_\mathcal{P} h := \frac{r - s}{n_P t} \in \mathbb{Z},
\]

cf. [6, Def. 8.16], which coincides with the signature defined in Section 2, cf. [6, Prop. 8.17] and also yields total signature maps that are continuous for the topology defined below, cf. [6, Thm. 9.19].

We finish this paper with a presentation of our main results concerning the topology of \( X_{(A, \sigma)} \). We define, for \( a_1, \ldots, a_k \in \text{Sym}(A, \sigma) \),

\[
H_\sigma(a_1, \ldots, a_k) := \{ \mathcal{P} \in X_{(A, \sigma)} \mid a_1, \ldots, a_k \in \mathcal{P} \}.
\]

We denote by \( \mathcal{T}_\sigma \) the topology on \( X_{(A, \sigma)} \) generated by the sets \( H_\sigma(a_1, \ldots, a_k) \), for \( a_1, \ldots, a_k \in \text{Sym}(A, \sigma) \), and by \( \mathcal{T}_\sigma^\times \) the topology on \( X_{(A, \sigma)}^\times \) generated by the sets \( H_\sigma(a_1, \ldots, a_k) \), for \( a_1, \ldots, a_k \in \text{Sym}(A, \sigma)^\times \).

**Proposition 5.13** ([6, Prop. 9.2]). The topologies \( \mathcal{T}_\sigma \) and \( \mathcal{T}_\sigma^\times \) are equal.

Recall that spectral topologies, defined in [16], are precisely the topologies of the spectra of commutative rings, and that a map between spectral spaces (i.e., spaces equipped with spectral topologies) is called spectral if it is continuous and the preimage of a quasicompact open set is quasicompact.

**Theorem 5.14** ([6, Prop. 9.17]). \( \mathcal{T}_\sigma \) is a spectral topology on \( X_{(A, \sigma)} \).

The topology \( \mathcal{T}_\sigma \) is also well-behaved under Morita equivalence:

**Proposition 5.15** ([6, Prop. 9.18]). Let \((A, \sigma)\) and \((B, \tau)\) be two Morita equivalent \( F \)-algebras with involution. The spaces \((X_{(A, \sigma)}, \mathcal{T}_\sigma)\) and \((X_{(B, \tau)}, \mathcal{T}_\tau)\) are homeomorphic via a spectral map.
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