INTEGRATING FACTORS FOR GROUPS OF FORMAL COMPLEX DIFFEOMORPHISMS

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Abstract. We study groups of formal or germs of analytic diffeomorphisms in several complex variables. Such groups are related to the study of the transverse structure and dynamics of Holomorphic foliations, via the notion of holonomy group of a leaf of a foliation. For dimension one, there is a well-established dictionary relating analytic/formal classification of the group, with its algebraic properties (finiteness, commutativity, solvability, ...). Such system of equivalences also characterizes the existence of suitable integrating factors, i.e., invariant vector fields and one-forms associated to the group. In this paper we search the basic lines of such dictionary for the case of several complex variables groups. For abelian, metabelian, solvable or nilpotent groups we investigate the existence of suitable formal vector fields and closed differential forms which exhibit an invariance property under the group action. Our results are applicable in the construction of suitable integrating factors for holomorphic foliations with singularities. We believe they are a starting point in the study of the connection between Liouvillian integration and transverse structures of holomorphic foliations with singularities in the case of arbitrary codimension.

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1. Introduction and main results

The study of groups and germs of complex diffeomorphisms fixing the origin is an important tool in Complex Dynamics and in the theory of Holomorphic Foliations, via the study of holonomy groups (cf. [3]) of its leaves. Indeed, the holonomy groups of (the leaves) of a

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codimension \( n \geq 1 \) holomorphic foliation are (identified with) groups of germs of complex diffeomorphisms fixing the origin of \( \mathbb{C}^n \). In the codimension \( n = 1 \) case these are subgroups of germs of one variable holomorphic maps and there is a well-established dictionary relating topological and dynamical properties of (the leaves of) the foliation to algebraic properties of the group. This is clear in works as \([4], [11], [14] \) and \([20]\).

All these facts are compiled in some works relating the existence of suitable “transverse structures” for the foliation with the transverse dynamics of the foliation \((5, [17], [18])\). In \([2]\) F. Brochero studies groups of germs of complex analytic diffeomorphisms having a fixed point at the origin. For such groups the author gives a nice study mainly focused on the analytical or topological description of the following cases: finite groups, linearizable groups, abelian groups containing a “generic” dicritic diffeomorphism. This important work also motivates some of the concepts and results in our work.

1.1. Preliminaries, notation and definitions. Let us introduce the notation we use throughout this paper. We denote by \( \mathcal{O}_n \) the ring of germs at the origin of holomorphic functions of \( n \) variables and by \( \mathcal{O}_n \) its formal completion\(^1\).

Denote by \( \text{Diff}(\mathbb{C}^n, 0) \) the group of germs of complex diffeomorphisms fixing the origin \( 0 \in \mathbb{C}^n \). We denote by \( z = (z_1, \ldots, z_n) \) a system of complex variables in \( \mathbb{C}^n \). The group of formal diffeomorphisms in \( n \) complex variables fixing the origin is the formal completion \( \text{Diff}(\mathbb{C}^n, 0) \) of \( \text{Diff}(\mathbb{C}^n, 0) \), obtained from the power series of the coordinate functions of elements in \( \text{Diff}(\mathbb{C}^n, 0) \). This way, given a formal diffeomorphism \( \hat{f} \in \text{Diff}(\mathbb{C}^n, 0) \) we write \( \hat{f} = \hat{f}'(0) \cdot z + \sum_{j=2}^{\infty} f_j(z) \) where \( \hat{f}'(0) \in \text{GL}(n, \mathbb{C}) \) and each \( f_j \) is a vector whose coordinates are homogeneous polynomials of degree \( j \) in the variables \( z = (z_1, \ldots, z_n) \). To each formal diffeomorphism \( \hat{f} \in \text{Diff}(\mathbb{C}^n, 0) \) we associate its derivative \( \hat{f}'(0) \in \text{GL}(n, \mathbb{C}) \). We say that the diffeomorphism \( \hat{f} \) is \textit{tangent to the identity} if \( \hat{f}'(0) = \text{Id} \). Denote by \( \text{Diff}_0(\mathbb{C}^n, 0) \) the subgroup of elements tangent to the identity in \( \text{Diff}(\mathbb{C}^n, 0) \). Also put \( \text{Diff}_0(\mathbb{C}^n, 0) = \text{Diff}_0(\mathbb{C}^n, 0) \cap \text{Diff}(\mathbb{C}^n, 0) \). This gives inclusions of \( \text{Diff}(\mathbb{C}^n, 0) \hookrightarrow \text{Diff}(\mathbb{C}^n, 0) \) and \( \text{Diff}_0(\mathbb{C}^n, 0) \hookrightarrow \text{Diff}_0(\mathbb{C}^n, 0) \). A subgroup \( G < \text{Diff}_0(\mathbb{C}^n, 0) \) is called \textit{tangent to the identity}.

The subgroup of formal diffeomorphisms, tangent to the identity with order \( k \), is defined as \( \text{Diff}_k(\mathbb{C}^n, 0) = \{ f \in \text{Diff}(\mathbb{C}^n, 0) \mid \hat{f}(z) = z + f_{k+1}(z) + f_{k+2}(z) + \cdots + f_{k+1} \neq 0 \} \).

Similarly the group of germs of holomorphic diffeomorphisms at the origin \( 0 \in \mathbb{C}^n \), tangent to the identity with order \( k \) is defined as \( \text{Diff}_k(\mathbb{C}^n, 0) = \text{Diff}_k(\mathbb{C}^n, 0) \cap \text{Diff}(\mathbb{C}^n, 0) \).

The classical theory of groups states the following algebraic definitions. Let \( G \) be a group. Given elements \( \alpha, \beta \in G \), the \textit{commutator} of \( \alpha \) and \( \beta \) is defined as \( [\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1} \).

Given subgroups \( H \) and \( L \) of \( G \), we define group of \textit{commutators} \( [H, L] \) as the subgroup of \( G \) generated by the elements of the form \( [\alpha, \beta] \) for \( \alpha \in H \) and \( \beta \in L \). We put

\[
G^{(0)} = G, \quad G^{(n+1)} = [G^{(n)}, G^{(n)}] \quad \forall n \geq 0
\]

the \textit{derived series} of \( G \). We define

\[
C^0G = G, \quad C^{n+1}G = [G, C^nG] \quad \forall n \geq 0
\]

the \textit{descending central series} of \( G \). The group \( G \) is solvable if it has a finite derived series, i.e., if \( G^{(k)} = \{\text{Id}\} \) for some \( k \in \mathbb{N} \). The minimum of such \( k \) is called the \textit{soluble length} \( l(G) \) of \( G \). The group \( G \) is\textit{nilpotent} if it has a finite descending series, i.e., if \( C^0G = \{\text{Id}\} \) for some \( k \in \mathbb{N} \).

\(^1\)We refer to the book of D. Eisenbud (\([2]\)) for a detailed construction of the formal completion of the ring \( \mathcal{O}_n \), as well as for some classical concepts related to the formal objects we deal with in this paper.
1.1.1. Formal vector fields and formal diffeomorphisms. Denote by $\mathcal{X}(C^n,0)$ the $O_n$-module of germs of complex vector fields vanishing at the origin $0 \in C^n$ and by $\hat{\mathcal{X}}(C^n,0)$ its formal counterpart.

**Definition 1.1.** Given a subgroup $G < Diff(C^n,0)$, a formal vector field $\hat{X} \in \hat{\mathcal{X}}(C^n,0)$ is $G$-invariant if we have $\hat{g} \ast \hat{X} = \hat{X}$, $\forall \hat{g} \in G$. We say that $\hat{X}$ is projectively invariant by $G$ if for each $\hat{g} \in G$ there is $c_g \in C$ such that $c_g \hat{X} = \hat{g} \ast \hat{X}$.

The Lie algebra $\hat{\mathcal{X}}_k(C^n,0)$ of formal vector fields of $C^n$ of order $k + 1$ is defined by those vector fields for the form $\hat{X} = \hat{a}_1(z) \frac{\partial}{\partial z_1} + \cdots + \hat{a}_n(z) \frac{\partial}{\partial z_n}; \hat{a}_j \in O_n$, where the minimum order of vanishing of the $\hat{a}_j$ at the origin is $k + 1$. Thus, $\hat{\mathcal{X}}_k(C^n,0)$ is the formal completion of $\mathcal{X}_k(C^n,0)$; the set of germs of complex analytic vector fields which are singular of order $k + 1$ at 0. We denote by $\mathcal{X}_N(C^n,0)$ the subset of $\mathcal{X}(C^n,0)$ of nilpotent vector fields, i.e. vector fields whose first jet has the unique eigenvalue 0. The formal completion of $\hat{\mathcal{X}}_N(C^n,0)$ is denoted by $\hat{\mathcal{X}}_N(C,0)$.

The expression

$$\exp(t\hat{X}) = \left( \sum_{j=0}^{\infty} \frac{t^j \hat{X}^j(z_1), \ldots, \sum_{j=0}^{\infty} \frac{t^j \hat{X}^j(z_n)}}{j!} \right)$$

(1)

defines the exponential of $t\hat{X}$ for $\hat{X} \in \hat{\mathcal{X}}(C^n,0)$ and $t \in C$. Let us remark that $\hat{X}^j(g)$ is the result of applying $j$ times the derivation $\hat{X}$ to the power series $g$. The definition coincides with the classical one if $\hat{X}$ is a germ of convergent vector field.

We denote by $Diff_u(C^n,0)$ the subgroup of unipotent elements of $Diff(C^n,0)$, more precisely $\varphi \in Diff_u(C^n,0)$ if $j^1 \varphi$ is a unipotent linear isomorphism (i.e. $j^1 \varphi - Id$ is nilpotent). Analogously we denote $Diff_u(C^n,0)$ the formal completion of $Diff_u(C^n,0)$.

According to ([12]) the exponential map $exp : \hat{\mathcal{X}}_k(C^n,0) \to Diff_k(C^n,0)$ is a bijection. Also, it induces a bijection from $\hat{\mathcal{X}}_N(C^n,0)$ onto $Diff_u(C^n,0)$.

**Definition 1.2.** Let $\hat{f} \in Diff_k(C^n,0)$. We denote by $\log \hat{f}$ the unique element of $\hat{\mathcal{X}}_k(C,0)$ such that $\hat{f} = \exp(\log \hat{f})$. We say that $\log \hat{f}$ is the infinitesimal generator of $\hat{f}$. Given a map $\hat{h} \in Diff(\mathcal{C},0)$ and $t \in C$ we denote by $\hat{h}[t] = \exp(t\hat{X})$ where $\hat{h} = \exp \hat{X}$.

In general the infinitesimal generator of a (convergent) germ of diffeomorphism is a divergent vector field (see [1]).

1.1.2. Solvable length of a Lie algebra. Let $\mathfrak{g}$ be a complex Lie algebra. Given elements $X, Y \in \mathfrak{g}$ we denote by $[X,Y]$ the Lie bracket of $X, Y \in \mathfrak{g}$. Given Lie subalgebras $\mathfrak{h}$ and $\mathfrak{l}$ of $\mathfrak{g}$. We define $[\mathfrak{h}, \mathfrak{l}]$ the Lie subalgebra of $\mathfrak{g}$ generated by the elements of the form $[X,Y]$ for $X \in \mathfrak{h}$ and $Y \in \mathfrak{l}$. We define

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}] \forall n \geq 0$$

de the derived series of $\mathfrak{g}$. We define

$$C^0\mathfrak{g} = \mathfrak{g}, C^{n+1}\mathfrak{g} = [\mathfrak{g}, C^n\mathfrak{g}] \forall n \geq 0$$

de the descending central series of $\mathfrak{g}$.

**Definition 1.3** (solvable length of a Lie algebra, cf. [13]). Let $\mathfrak{g}$ be a Lie algebra. We define $l(\mathfrak{g})$ the soluble length of $\mathfrak{g}$ as

$$l(\mathfrak{g}) = \min\{k \in \mathbb{N} \cup \{0\} : \mathfrak{g}^{(k)} = \{Id\}\}$$

where $\min \emptyset = \infty$. 

We say that \( \mathfrak{g} \) is **solvable** if \( l(\mathfrak{g}) < \infty \). The Lie algebra \( \mathfrak{g} \) is **nilpotent** if there exists \( j \geq 0 \) such that \( C^j \mathfrak{g} = \{ \{ \} \} \). If \( j \) is the minimum non-negative integer number with such a property we say that \( \mathfrak{g} \) is of **nilpotent class** \( j \).

1.1.3. **Lie algebra of a group of diffeomorphisms** (cf. \[13\]). In \[8\] E. Ghys associates a Lie algebra of formal nilpotent vector fields to any group of unipotent diffeomorphisms (prop. 4.3 in \[8\]). In the same spirit we present a construction, from \[13\], that associates a non-trivial Lie subalgebra of \( \hat{\mathfrak{X}}(\mathbb{C}^n, 0) \) to certain subgroups \( G \) of \( \text{Diff}(\mathbb{C}^n, 0) \), satisfying some connectedness hypothesis. In \[13\] the authors replace \( G \) with a subgroup \( \overline{G}^{(0)} \) of \( \text{Diff}(\mathbb{C}^n, 0) \) containing \( G \) that is, roughly speaking, the algebraic closure of \( G \) (with respect to the Krull topology). Such a group satisfies \( l(\overline{G}^{(0)}) = l(G) \) (i.e., the groups \( G^{(0)} \) and \( G \) have the same solvable length) and it has a non-trivial with analogous algebraic properties. This construction can be performed to every connected group of formal diffeomorphisms.

**Associate Lie algebra.** We introduce the concept of **associate Lie algebra** of a subgroup of \( \text{Diff}(\mathbb{C}^n, 0) \) as presented in \[13\]. First recall that, according to \[13\] Definition 3.8, given a subgroup \( G \) of \( \text{Diff}(\mathbb{C}^n, 0) \), the **lie algebra of** \( G \) is the complex Lie subalgebra \( \log(G) \) of \( \hat{\mathfrak{X}}(\mathbb{C}^n, 0) \) given by

\[
\log(G) = \{ \hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0) : \exp(t\hat{X}) \in G, \forall t \in \mathbb{C} \}.
\]

With this definition the Lie algebra of a discrete group is trivial. However, under some connectedness hypothesis, it is possible to build a group \( \overline{G}^{(0)} \supset G \) with the same algebraic properties of \( G \), having a non-trivial Lie algebra \( \log(\overline{G}^{(0)}) \) and such that, \( l(\overline{G}^{(0)}) = l(G) \) and \( l(\log(\overline{G}^{(0)})) = l(\log G) \). Roughly speaking \( G^{(0)} \) is the algebraic closure of \( G \) (with respect to the Krull topology). In particular this construction associates a non-trivial Lie algebra of formal diffeomorphisms, to every group \( G \) of formal diffeomorphisms with connected linear part \( DG \). In few words, this is done as follows. Let \( \mathfrak{m} \) the maximal ideal of \( \mathbb{C}[x_1, \ldots, x_n] \). A formal diffeomorphism \( \hat{h} \in \text{Diff}(\mathbb{C}^n, 0) \) acts on the space \( \mathfrak{m}/\mathfrak{m}^{k+1} \) of \( k \)-jets in a natural way \( ([g] + \mathfrak{m}^{k+1} \mapsto [g \circ \hat{h}] + \mathfrak{m}^{k+1}) \), defining an element \( \hat{h}_k \) of \( GL(\mathfrak{m}/\mathfrak{m}^{k+1}) \). Let \( G \) be a connected subgroup of \( \text{Diff}(\mathbb{C}^n, 0) \). Fixed \( k \in \mathbb{N} \) we define the group \( C_k = \{ \hat{h}_k : \hat{h} \in G \} \subset GL(\mathfrak{m}/\mathfrak{m}^{k+1}) \) and the matrix group \( G_k \) defined as the smallest algebraic subgroup of \( GL(\mathfrak{m}/\mathfrak{m}^{k+1}) \) containing \( C_k \). We define

\[
\overline{G}^{(0)} = \{ \hat{h} \in \text{Diff}(\mathbb{C}^n, 0) : \hat{h}_k \in G_k \forall k \in \mathbb{N} \}.
\]

Then \( \overline{G}^{(0)} \) is a subgroup of \( \text{Diff}(\mathbb{C}^n, 0) \) containing \( G \) that is closed in the Krull topology. Since a subgroup tangent to the identity is connected we can introduce the following definition:

**Definition 1.4** (Associate Lie algebra, cf. \[13\]). Let \( G \) a subgroup of \( \text{Diff}_{\text{Id}}(\mathbb{C}^n, 0) \). The **associate** Lie algebra of \( G \) is defined as the Lie algebra of \( \overline{G}^{(0)} \).

**Remark 1.5.** It is important to remark that, as observed in \[13\], the Lie algebra of a connected group \( G < \text{Diff}(\mathbb{C}^n, 0) \) shares the usual properties of Lie algebras of Lie groups. This is a consequence of the following proposition where we can consider \( \overline{G}_0^{(0)} \) as the connected component of the identity of \( \overline{G}^{(0)} \).

**Proposition 1.6** (\[13\] Proposition 3.1). Let \( G < \text{Diff}(\mathbb{C}^n, 0) \) be a connected group with associate Lie algebra \( \mathfrak{g} \) as in **Definition 1.4**. Then \( \mathfrak{g} \) is the Lie algebra of \( \overline{G}^{(0)} \). The group

\[2\] a subgroup \( G < \text{Diff}(\mathbb{C}^n, 0) \) is connected if the closure in the Zariski topology of the subgroup of linear parts \( j^1G < \text{GL}(n, \mathbb{C}) \) of \( G \) is connected.
$G_0^{(0)}$ is generated by $\exp(g)$. Moreover if $G$ is unipotent then $g$ is a Lie algebra of formal nilpotent vector fields and $\exp : g \to G_0^{(0)}$ is a bijection.

We denote by $\hat{K}_n$ the field of fractions of the ring of formal power series $\hat{O}_n = \mathbb{C}[[z_1, \ldots, z_n]]$. We have a natural embedding $\hat{K}_n \subset K_n$, where by $K_n$ we mean the field of rational functions in $n$ complex variables, i.e., the fraction field of $\mathbb{C}[z_1, \ldots, z_n]$. Following the above notions it is then natural to define:

**Definition 1.7** ([13] Definition 3.9, dimension of a Lie algebra). For a group tangent to the identity $G < \hat{\text{Diff}}(\mathbb{C}^n, 0)$, by dimension of the associate Lie algebra we mean the dimension of the Lie algebra $\log(G_0)$, viewed as vector space over $\hat{K}_n$.

From now on, by Lie algebra of a group of formal diffeomorphisms tangent to the identity, we shall mean its associate Lie algebra.

### 1.2. The one-dimensional case.

The construction of a dictionary relating algebraic, dynamical and analytic properties of subgroups of diffeomorphisms is a very important tool in Complex Dynamics and Holomorphic foliations. The one-dimensional case has been addressed by several authors. Below we find a compilation of the ir main achievements:

**Theorem 1.8** ([12]). Let $G < \text{Diff}(\mathbb{C}, 0)$ be a subgroup of one-variable formal complex diffeomorphisms.

1. $G$ is abelian if, and only if, $G$ is nilpotent if, and only if, $G$ admits a formal invariant vector field.
2. The following conditions are equivalent:
   a. $G$ is solvable.
   b. $G$ is metabelian, i.e., $G^{(1)}$ is abelian.
   c. All elements tangent to the identity in $G$ have a same order of tangency.
   d. $G$ admits a formal vector field which is projectively invariant by $G$.

Item (1) above is essentially a consequence of two other facts:

(i) For a subgroup $G < \text{Diff}(\mathbb{C}, 0)$ the derivative group $DG = \{ \hat{g}'(0) : \hat{g} \in G \}$ is abelian and therefore the subgroup of commutators $G^{(1)} < G$, which is the set of products of commutators in $G$, is tangent to the identity, i.e., a subgroup of $G_{\text{Id}}$.

(ii) Two elements $\hat{f} = z + a_{k+1}z^{k+1} + \cdots \in G_{\text{Id}}$, and $\hat{g} = z + b_{\ell+1}z^{\ell+1} + \cdots \in G_{\text{Id}}$, $a_{k+1}b_{\ell+1} \neq 0$, tangent to the identity, commute only if we have $k = \ell$.

As we shall see, none of the above facts holds for subgroups of $\text{Diff}(\mathbb{C}^n, 0)$ when $n \geq 2$ (cf. Example 2.3). Therefore it is quite natural to expect that the above mentioned dictionary is much different or much harder to find, in the $n \geq 2$ case. To begin this study is one of the main goals of this work. We also aim on possible applications of our results to the framework of holomorphic foliations (see Section 3.1 and Proposition 3.3).

For some of the reasons mentioned above we divide this work in two parts. The first is mainly, but not only, concerned with the study of subgroups tangent to the identity, i.e., groups with all elements tangent to the identity. The second is about not necessarily groups tangent to the identity, but we require the existence of suitable dicritic ("radial type") elements in the group.

### 1.3. PART I - Lie algebras of groups and vector fields.

As mentioned above, in the first part we focus on the study of the Lie algebra of subgroups $G < \text{Diff}(\mathbb{C}^n, 0)$ under the hypothesis that $G$ is abelian, metabelian or nilpotent.
1.3.1. Existence of an invariant formal vector field. In Section 2 we prove the following (compare with (1) in Theorem 1.8):

Theorem 1.9. Let $G < \text{Diff}(\mathbb{C}^n, 0)$ be an abelian subgroup of formal diffeomorphisms. We have two possibilities:

1. $G$ admits an invariant formal vector field.
2. $G_{\text{Id}} = \{\text{Id}\}$, i.e., the only element of $G$ tangent to the identity is the identity.

Remark 1.10. With respect to the one-dimensional case we observe:

(i) Unlike the one-dimensional case, in general the existence of such an invariant vector field is not enough to assure that the group is abelian (see Example 2.3).

(ii) For $n = 1$, condition (2) above implies that the group is conjugated to its linear part by a formal diffeomorphism. We believe that this also holds for dimension $n \geq 2$.

1.3.2. Abelian subgroups and nilpotent subalgebras. In 2 it is proved (Proposition 4.1) that every nilpotent subalgebra $\mathcal{L}$ of $\mathcal{X}(\mathbb{C}^2, 0)$ is metabelian. As a consequence, if $\mathcal{R}$ is the center of $\mathcal{L}$ then $\mathcal{R} \otimes \hat{K}_2$ is a vector space of dimension 1 or 2 over $\hat{K}_2$. According to [2] Corollary 4.4, if the dimension is one then there is a formal vector field $\hat{X} \in \mathcal{X}(\mathbb{C}^2, 0)$ such that $\exp(\hat{X}) \in G$ and such that for each element $g \in G$ there exists a rational (meromorphic) function $N_g \in K_2$ with $\hat{X}(N_g) = 0$ and $g = \exp(N_g \hat{X})$. In case the dimension is 2 there are (commuting) maps $\hat{f}, \hat{g} \in G$ such that $G \in \{\hat{f}^s \circ \hat{g}^t, s, t \in \mathbb{C}\}$. Using this we obtain the following rephrasing of Brocher’s result:

Theorem 1.11 (cf. [2]). Let $G < \text{Diff}_0(\mathbb{C}^2, 0)$ be an abelian subgroup tangent to the identity. We have the following possibilities:

(i) $G$ leaves invariant an exact rational one-form, say $\hat{\omega} = dT$ for some rational function $T \in K_2$.

(ii) $G$ embeds into the flow of a formal vector field $\hat{X} \in \log(G)$.

(iii) There are two invariant linearly independent commuting formal vector fields $\hat{X}, \hat{Y} \in \log(G)$.

Though this statement is essentially already contained in [2], we give an alternative proof which will indeed allow an extension of this result for higher dimension $n \geq 2$ and we shall prove in Section 7.

Theorem 1.12. Every nilpotent subalgebra $l$ of $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ has length $l(l)$ at most $n$. As a consequence, if $G < \text{Diff}(\mathbb{C}^n, 0)$ is a nilpotent group then the solubility length of $G$ satisfies $l(G) \leq n$.

As a consequence we obtain, the following immediate corollary of (the proof of) Theorem 1.12

Corollary 1.13. Let $G < \text{Diff}_1(\mathbb{C}^n, 0)$ be abelian subgroup tangent to the identity, with Lie algebra $\mathfrak{g}$. Then the possibilities are:

1. There are $\hat{X}_1, \ldots, \hat{X}_n \in \mathfrak{g}$ such that $[\hat{X}_j, \hat{X}_r] = 0, \forall i, j$ and $G < \langle \exp(t_1 \hat{X}_1) \circ \cdots \circ \exp(t_n \hat{X}_n) | t_j \in \mathbb{C} \rangle$.

2. There are $\hat{X}_1, \ldots, \hat{X}_l \in \mathfrak{g}$, for some $l \in \{1, \ldots, n - 1\}$, such that $[\hat{X}_j, \hat{X}_r] = 0, \forall i, j$ and $G < \langle \exp(u_1 \hat{X}_1) \circ \cdots \circ \exp(u_l \hat{X}_l) | u_j \in K_n, \hat{X}_r(u_j) = 0, \forall 1 \leq r, j \leq l \rangle$.

1.3.3. Abelian groups and closed one-forms. Before stating our next main results we observe that in some main applications of the results in Theorem 1.8 (case $n = 1$), the winning strategy is to construct from the information on the holonomy groups of the foliation, some suitable differential forms which allow to “integrate” the foliation (as for instance a foliation
admitting a Liouvillian first integral) (see [5] for instance). More precisely, in dimension $n = 1$ a formal vector field $\hat{X} \in \hat{\mathcal{X}}(\mathbb{C}, 0)$ can be written either as a linear vector field or, in case it has a zero of order $\geq 2$ at the origin, as

$$\hat{X}(z) = \frac{z^{k+1}}{1 + \lambda z^k} \frac{d}{dz}$$

for some $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$. Let us focus on the non-linear case. The duality equation $\hat{\omega} \cdot \hat{X} = 1$ has, in this dimension one case, a single solution $\hat{\omega} = \lambda \frac{dz}{z} + \frac{dz}{z^{k+1}}$.

This expression, is the expression of general closed meromorphic one-form with an isolated pole of order $k + 1$ at the origin $0 \in \mathbb{C}$, residue $\lambda$, in a suitable coordinate system. It is a particular case of the so called Integration Lemma (see for instance [17] Example 1.6 page 174, or Proposition 5.1 in Section 5).

In dimension $n = 1$, given a formal diffeomorphism $\hat{g} \in \hat{\text{Diff}}(\mathbb{C}, 0)$ and $\hat{X}$ and $\hat{\omega}$ satisfying the duality equation as above, we have:

1. $\hat{g}_* \hat{X} = \hat{X} \iff \hat{g}_* \hat{\omega} = \hat{\omega}$
2. $\hat{g}_* \hat{X} = c_\hat{g} \hat{X}$ for some $c_\hat{g} \in \mathbb{C}^*$ $\iff g_\dagger \hat{\omega} = \frac{1}{c_\hat{g}} \hat{\omega}$

Finally, notice that, in dimension-one each formal or meromorphic one-form is closed. This suggests, in view of Theorem 1.18 and all the above, that one may expect to obtain results relating algebraic properties of subgroups of $\text{Diff}(\mathbb{C}^n, 0)$ with the existence of suitable closed one-forms.

We shall see (cf. Theorem 1.15 below) that a subgroup $G < \text{Diff}(\mathbb{C}^2, 0)$ admitting two commuting formal invariant vector fields, exhibits two independent invariant closed formal meromorphic one-forms. A further investigation of this situation involves the following notion:

**Definition 1.14 (Formal separatrices).** A formal curve of 2 complex variables is defined as follows: In the local ring $\hat{O}_2$ we introduce the equivalence relation $\hat{f} \sim \hat{g} \iff \hat{\varphi} = \hat{u} \hat{\psi}$ for some unit $\hat{u} \in \hat{O}_2$, i.e., for some power series $\hat{u}$ with first coefficient $u_0 \neq 0$. By a formal curve we mean an equivalence class of a function $\hat{\varphi} \in \hat{O}_2$ that satisfies $\hat{\varphi}(0) = 0$, that is, a non-invertible formal power series. Such a formal curve is called invariant by a formal complex diffeomorphism $\hat{f} \in \hat{\text{Diff}}(\mathbb{C}^2, 0)$ if $\hat{f}^* \varphi = \varphi \circ \hat{f}$ is equivalent to $\varphi$ in the above sense. Such a formal curve will be called a separatrix of a subgroup $G < \hat{\text{Diff}}(\mathbb{C}^2, 0)$ if it is invariant by each element of this group. The tangent space of a formal curve with representative $\hat{\varphi}$ is defined as the linear subspace of $\mathbb{C}^2$ given by the kernel of $D\hat{\varphi}(0) : \mathbb{C}^2 \to \mathbb{C}$. Two formal curves with representatives $\hat{\varphi}$ and $\hat{\psi}$ are called transverse if:

1. Each tangent space has dimension one.
2. The tangent spaces span $\mathbb{C}^2$.

As a converse of (iii) in Theorem 1.11 we have:

**Theorem 1.15.** Let $G < \hat{\text{Diff}}(\mathbb{C}^2, 0)$ be a subgroup admitting two linearly independent invariant commuting formal vector fields. Then $G$ admits two closed, independent, formal meromorphic, invariant one-forms. The group $G$ is abelian provided that one of the following conditions is satisfied:

1. $G$ is tangent to the identity.
2. $G$ exhibits two formal transverse separatrices.
The notions of formal closed meromorphic one-form and other formal objects are clearly stated in Section 4 (Definition 4.3). As a spolium of the proof of the second part of Theorem 1.5 we obtain normal forms for abelian subgroups admitting two transverse separatrices and having Lie algebra of dimension two (cf. Remark 6.2).

1.3.4. Metabelian groups. All the above is concerned with the abelian case. As for the metabelian non-abelian case we have:

**Theorem 1.16.** Let $G < \text{Diff}(\mathbb{C}^2,0)$ be a metabelian non-abelian subgroup. Assume that the group of commutators $G^{(1)} = [G,G]$ is tangent to the identity, for instance if the derivative group $DG < \text{GL}(2,\mathbb{C})$ is abelian. Denote by $l(G^{(1)})$ the Lie algebra of $G^{(1)} = [G,G]$.

We have the following possibilities:

(i) $l(G^{(1)})$ is one-dimensional. There is a formal vector field $\hat{X}$ with $\exp \hat{X} \in G$, such that, for each $\hat{g} \in G$ there is a rational function $T_{\hat{g}} \in K_2$ that satisfies: $\hat{X}(T_{\hat{g}}) = 0$ and $\hat{y}^*(\hat{X}) = T_{\hat{g}} \cdot \hat{X}$.

(ii) $l(G^{(1)})$ is two-dimensional. There are two $C$-linearly independent formal vector fields $\hat{X}, \hat{Y} \in g = \log G$, such that

- $[\hat{X}, \hat{Y}] = 0$ (i.1)
- $\hat{y}^*(\hat{X}) = s_1 \hat{X} + t_1 \hat{Y}$ and $\hat{y}^*(\hat{Y}) = s_2 \hat{X} + t_2 \hat{Y}$.

Furthermore, in this last case there are two $C$-linearly independent formal vector fields $\hat{y}_j$, $(j = 1,2)$ and $a_j, b_j \in \mathbb{C}^*$ such that

$\hat{y}^*(\hat{y}_j) = a_j \hat{y}_1 + b_j \hat{y}_2$, $\forall \hat{g} \in G$.

Groups as in (ii.2) above are studied in Section 6 (cf. Remark 6.2).

1.4. PART II - Groups containing dicritic diffeomorphisms. The second part of this work is dedicated to the study of subgroups of formal diffeomorphisms under the hypothesis of existence of a suitable dicritic (radial type) element. More precisely, according to [2], a diffeomorphism $\hat{f} \in \text{Diff}_K(\mathbb{C}^n,0)$ is called dicritic if $\hat{f}(z) = z + \hat{f}_{k+1}(z) + \hat{f}_{k+2}(z) + \cdots$, where $\hat{f}_{k+1}(z) = f(z)z$ and $f$ is a homogeneous polynomial of degree $k$. A formal vector field $\hat{X} \in \hat{X}_k(\mathbb{C}^n,0)$, $k \geq 1$ is called dicritic if $\hat{X} = f(z)\partial + (p_{k+1}^{(1)} + \cdots)\partial_{z_{k+1}} + \cdots + (p_{k+2}^{(n)} + \cdots)\partial_{z_{k+2}}$ where $f_{\nu}$ is a homogeneous polynomial of degree $\nu$ and $f$ is homogeneous of degree $k$ and $\partial = z_1\frac{\partial}{\partial z_1} + \cdots + z_n\frac{\partial}{\partial z_n}$ is the radial vector field.

**Definition 1.17** (Regular dicritic vector fields and diffeomorphisms). We also introduce the following useful subclasses of dicritic diffeomorphisms and vector fields. A formal vector field $\hat{X} \in \hat{X}_k(\mathbb{C}^n,0)$ is called regular dicritic if $\hat{X}$ is dicritic and there are $i_0, j_0 \in \{1, \ldots, n\}$, such that $f$ and $z_{j_0}p_{k+2}^{(1)} - z_{i_0}p_{k+2}^{(i_0)}$ are coprime. This implies that $0$ is an isolated singularity of $\hat{X}$. A formal diffeomorphism $\hat{f} \in \text{Diff}_K(\mathbb{C}^n,0)$ is called regular dicritic if its infinitesimal generator is a regular dicritic vector field.

We shall say that a subgroup $G < \text{Diff}(\mathbb{C}^n,0)$ is quasi-abelian if its subgroup $G_{K_1}$ of elements tangent to the identity is abelian.

Our next results are analogous to those in Theorem 1.8 for groups containing a regular dicritic element.

**Theorem 1.18.** For a subgroup $G < \text{Diff}(\mathbb{C}^n,0)$ containing a regular dicritic diffeomorphism, the following conditions are equivalent:

Indeed, this is equivalent to the fact that the singularity is isolated, after a number of blowing-ups at the vector field.
(1) $G$ is quasi-abelian.
(2) $G$ admits a projectively invariant regular dicritic formal vector field.

In particular we obtain:

**Corollary 1.19.** A subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$ tangent to the identity and containing a regular dicritic diffeomorphism is abelian if and only if it admits an invariant formal vector field.

The proof of Theorem 1.18 also shows that

**Proposition 1.20.** Let $G < \text{Diff}(\mathbb{C}^n, 0)$ be a subgroup containing a regular dicritic diffeomorphism $\hat{f} \in G$ and containing its derivative subgroup, $DG < G$. The following conditions are equivalent:

(1) $G$ is abelian
(2) $DG$ is abelian and $\hat{X} = \log \hat{f}$ is invariant by $G$.

Regarding the case of metabelian groups we have:

**Theorem 1.21.** A subgroup of formal diffeomorphisms containing a regular dicritic diffeomorphism and with abelian derivative group is metabelian provided that it admits a projectively invariant formal vector field.

As a partial converse of Theorem 1.21 we have:

**Proposition 1.22.** Let $G < \text{Diff}(\mathbb{C}^n, 0)$ be a metabelian subgroup containing a regular dicritic diffeomorphism with order of tangency $k$. Suppose that $DG$ is abelian and there is a linear diffeomorphism $\hat{h} \in G$, given by $\hat{h}(z) = \lambda \cdot \text{Id}$, where $\lambda \in \mathbb{C}$ is such that $\lambda^k \neq 1$, $\lambda^{k+1} \neq 1$. Then there is a formal vector field $\hat{X} \in \hat{X}_j(\mathbb{C}^n, 0)$, $j \geq 2$, which is projectively invariant by $G$.

As an application we study the case where a group with two generators, one of which is linear, is metabelian (cf. Corollary 9.2).

Next we state an equivalence similar to the dimension one case, but for groups that contain some dicritic diffeomorphism. Theorem 1.23 below is related to Theorem 4.1 and Corollary 4.2 in [2] and to our Example 6.3 of a solvable group of formal diffeomorphisms tangent to the identity which is not metabelian. This example shows the need of our assumption of existence of a dicritic element in the subgroup of commutators in any extension of Theorem 1.8 to higher dimension.

**Theorem 1.23.** For a subgroup $G < \text{Diff}_{\text{Id}}(\mathbb{C}^n, 0)$ of formal diffeomorphisms tangent to the identity and containing a dicritic diffeomorphism tangent to the identity with order $k$, the following statements are equivalent:

(1) The group is abelian.
(2) The group is nilpotent.
(3) Every nontrivial element in the group is tangent to the identity with order $k$.

Our results apply to the study of foliations on complex projective spaces and other ambient manifolds as well. The class of singularities which correspond, via the holonomy of its separatrices, to the class of regular dicritic diffeomorphisms is to be formally introduced and studied in a forthcoming work. Using an adaptation of a classical result due to Hironaka and Matsumura ([10], [9]) we may be able to move from the “formal world” (considered is this paper) to the “analytic/convergent world”, which is the natural ambient to the study of holomorphic foliations with singularities.

A final word should be said about the possible applications of our results. We are interested in the study of Liouvillian integration for holomorphic foliations of codimension $n \geq 1$. 
As suggested by the codimension one cases (see for instance [19]), this passes through the comprehension of algebraic, geometric and formal structures of subgroups of \( \text{Diff}(\mathbb{C}^n, 0) \) in terms we propose in this work.

**PART I - GROUPS OF DIFFEOMORPHISMS TANGENT TO THE IDENTITY**

2. **LIE ALGEBRAS OF GROUPS AND VECTOR FIELDS**

Now we proceed to prove Theorem 1.9. Some steps in the proof of the following well-known lemma will be used later on this paper:

**Lemma 2.1.** If \( \tilde{f} \in \hat{\text{Diff}}(\mathbb{C}^n, 0) \) commutes with the time one flow map of a formal vector field \( \tilde{X} \in \hat{X}_k(\mathbb{C}^n, 0) \), \( k \geq 2 \) then \( \tilde{f} \) commutes with the flow of \( \tilde{X} \) for all time \( t \in \mathbb{C} \).

**Proof.** Let \( \hat{\Phi}_t \) be the (formal) flow of \( \tilde{X} \), which is defined by \( \hat{\Phi}_t := \exp(t\tilde{X}) \in \hat{\text{Diff}}(\mathbb{C}^n, 0) \). Then \( \hat{\Phi}_t \circ \tilde{f} = \tilde{f} \circ \hat{\Phi}_t \). We claim that \( \hat{\Phi}_t \circ \tilde{f} = \tilde{f} \circ \hat{\Phi}_t \) for all \( t \in \mathbb{Z} \). First we prove this by induction, \( \forall t \in \mathbb{N} \). In fact, this is true for \( t = 1 \). Suppose that equality holds for \( n \in \mathbb{N} \). Then

\[
\hat{\Phi}_{n+1} \circ \tilde{f} = \hat{\Phi} \circ \hat{\Phi}_n \circ \tilde{f} = \hat{\Phi} \circ \tilde{f} \circ \hat{\Phi}_n = \tilde{f} \circ \hat{\Phi}_n = \tilde{f} \circ \hat{\Phi}_{n+1}
\]

thus \( \hat{\Phi}_t \circ \tilde{f} = \tilde{f} \circ \hat{\Phi}_t, \forall t \in \mathbb{N} \). On the other hand, if \( \hat{\Phi}_t \circ \tilde{f} = \tilde{f} \circ \hat{\Phi}_t \), then \( \hat{\Phi}_{-t} \circ \tilde{f} = \tilde{f} \circ \hat{\Phi}_{-t} \).

Consequently \( \hat{\Phi}_t \circ \tilde{f} = \tilde{f} \circ \hat{\Phi}_t, \forall t \in \mathbb{Z} \). Now to show that \( \hat{\Phi}_t \circ \tilde{f} = \tilde{f} \circ \hat{\Phi}_t, \forall t \in \mathbb{C} \), is sufficient to prove this equality in the spaces of jets, i.e. in \( \mathcal{J}^k(\mathbb{C}^n, 0) = \mathbb{C}[[z]]/m^{k+1} \) (this has a natural identification with the space of polynomials of degree less than or equal to \( k \)), where \( m = \{ \tilde{f} \in \mathbb{C}[[z]]/\tilde{f}(0) = 0 \} \) is the maximal ideal of \( \mathbb{C}[[z]] \). Indeed, given \( k \in \mathbb{N} \) we have that \( j^k \circ \hat{\Phi}_t \circ \tilde{f} = (f_1, \ldots, f_n) \), where the truncation of formal series

\[
j^k : \mathbb{C}[[z]] \rightarrow \mathcal{J}^k(\mathbb{C}^n, 0),
\]

is defined by \( j^k(\tilde{f}) = \tilde{f} \mod m^{k+1} \), we have that

\[
f_t(z) = \sum_{|N| \leq k} P^k_N(t) z^N
\]

\( e P^k_N(t) \) is a polynomial of degree less than or equal to \( |N| \). Similarly, we have

\[
j^k \circ \tilde{f} \circ \hat{\Phi}_t = (\tilde{f}_1, \ldots, \tilde{f}_n) \text{ where}
\]

\[
\tilde{f}_j(z) = \sum P^j_N(t) z^N
\]

\( e P^j_N(t) \) is a polynomial of degree less than or equal to \( |N| \). now, as \( \hat{\Phi}_t \circ \tilde{f} = \tilde{f} \circ \hat{\Phi}_t, \forall t \in \mathbb{Z} \), for each \( N \in \mathbb{N}^{\mathbb{N}} \) with \( |N| \leq k \), we have that \( P^j_N(t) |_{z = \tilde{P}^k_N(t)} = P^j_N(t) |_{z = \tilde{P}^k_N(t)} \), now as these are polynomial and coincide in \( \mathbb{Z} \), we have that

\[
P^j_N(t) = \tilde{P}^k_N(t), \forall t \in \mathbb{C}.
\]

in consequence \( f_t(z) = \tilde{f}_t(z) \forall X \in \mathbb{C}^{n}, l \in \{1, \ldots, n\} \). therefore \( j^k \circ \hat{\Phi}_t \circ \tilde{f} = j^k \circ \tilde{f} \circ \hat{\Phi}_t, \forall t \in \mathbb{C} \in \mathbb{N} \). So

\[
\hat{\Phi}_t \circ \tilde{f} = \tilde{f} \circ \hat{\Phi}_t, \forall t \in \mathbb{C}.
\]

□

**Proof of Theorem 1.9.** We may assume that \( G_{Id} \) is nontrivial. Thus there is \( \tilde{f} \in G_1 \) which is of the form \( \tilde{f} = \exp(\tilde{X}) \) for some formal vector field \( \tilde{X} \in \hat{X}_j(\mathbb{C}^n, 0), j \geq 2 \). Since \( G \) is abelian, for any \( \tilde{g} \in G \), \( \tilde{g} \circ \tilde{f}(z) = \tilde{f} \circ \tilde{g}(z) \), i.e. \( \tilde{g} \circ \exp(\tilde{X})(z) = \exp(\tilde{X}) \circ \tilde{g}(z) \). Thus, from the previous lemma, we have \( \tilde{g} \circ \exp(t\tilde{X})(z) = \exp(t\tilde{X}) \circ \tilde{g}(z) \), \( \forall t \in \mathbb{C} \) or equivalently

\[
\tilde{g} \circ \exp(t\tilde{X}) \circ \tilde{g}^{-1} = \exp(t\tilde{X}), \forall t \in \mathbb{C} \). Therefore \( \tilde{g}^* \tilde{X} = \tilde{X}, \forall \tilde{g} \in G \). □
Remark 2.2. Regarding case (2) in Theorem 1.9 we observe that, in $DG$ is abelian and algebraic (see definition in [21] or Remark 3.1) and the identity is the only element tangent to the identity, the map $\hat{g} \mapsto D\hat{g}(0)$ gives an (abstract) group isomorphism $G \cong DG$. Now, according to Remark 3.1 either $DG$ is finite (and therefore analytically conjugated to a finite group of diagonal periodic linear maps) or the Zariski closure $\overline{DG}$ contains a linear flow. In this last case, as remarked above, there is a (linear) vector field $\hat{X}$ which is invariant under the action of $DG$.

Example 2.3. Now we give some examples showing that the conditions in our main results, cannot be dropped.

1. The converse of Theorem 1.9 is not always true for dimension $(n \geq 2)$. In fact if $\hat{f}(x, y) = (2x, 4y)$ and $\hat{g}(x, y) = (x, x + y)$ then $G = \langle \hat{f}, \hat{g} \rangle$ is not abelian, however, $G$ is invariant by $\hat{X}$, where $\exp(\hat{X}) = (x, y + x^2)$. As for the tangent to the identity case, let $\hat{f}(x, y) = \exp(x^2 y \frac{\partial}{\partial x})$ and $\hat{g}(x, y) = \exp(x^3 y^2 \frac{\partial}{\partial x})$, then $G = \langle \hat{f}, \hat{g} \rangle$ is not abelian, however, $G$ is invariant by $\hat{X} = -xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$.

2. In dimension $k = 1$, we have that a group $G < \text{Diff}_1(\mathbb{C}, 0)$ of diffeomorphisms tangent to the identity is abelian if and only if there is a formal vector field $\hat{X} \in \mathcal{X}_k(\mathbb{C}, 0)$ $(k \geq 1)$, such that $G < \langle \exp(t\hat{X}) \mid t \in \mathbb{C} \rangle$. For $(n \geq 2)$ if there is a formal vector field $\hat{X} \in \mathcal{X}_k(\mathbb{C}^n, 0)$ $(n \geq 2)$, such that $G < \langle \exp(t\hat{X}) \mid t \in \mathbb{C} \rangle$ then $G$ is abelian, however again the converse is not always true. This is due to the fact that for $n = 1$, if the Lie bracket of two vector field $\hat{X} \in \mathcal{X}_k(\mathbb{C}, 0)$ and $\hat{Y} \in \mathcal{X}_r(\mathbb{C}, 0)$ is zero ($[\hat{X}, \hat{Y}] = 0$) then $r = k$ and there is $c \in \mathbb{C}^*$ such that $\hat{X} = c\hat{Y}$. However this last fact is not always true in dimension $n \geq 2$ as can be seen in the following examples:

a. Let $a \in \mathbb{C}^*$ be constant and take $\hat{X}(x, y) = xy \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}$, $\hat{Y}(x, y) = ax^2 y^2 \frac{\partial}{\partial x} - ax y \frac{\partial}{\partial y}$.

b. $\hat{X}(x, y) = (x^2 + 3xy) \frac{\partial}{\partial x} + (3xy + y^2) \frac{\partial}{\partial y}$, $\hat{Y}(x, y) = (3x^3 - 5x^2 y + xy^2 + y^3) \frac{\partial}{\partial x} + (3x^2 y - 2xy^2 + 3y^3) \frac{\partial}{\partial y}$.

c. For $k \geq 1$, we have: $\hat{X} = (x^{k+1}) \frac{\partial}{\partial x} + (x^k y) \frac{\partial}{\partial y}$, $\hat{Y} = (y^k x) \frac{\partial}{\partial x} + (y^{k+1}) \frac{\partial}{\partial y}$.

d. $\hat{X}(x, y) = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$, $\hat{Y}(x, y) = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$.

3. Holonomy groups and algebraic abelian groups

In the treatment of abelian groups we must pay some attention to the linear case. Given a subgroup $G < \text{Diff}(\mathbb{C}^n, 0)$, the derivative map $D : \text{Diff}(\mathbb{C}^n, 0) \to \text{GL}(\mathbb{C}, n)$, $\hat{f} \mapsto D\hat{f} := \hat{f}'(0)$, induces by restriction to any subgroup $G < \text{Diff}(\mathbb{C}^n, 0)$ a homomorphism $D : G \to \text{GL}(\mathbb{C}, n)$. The kernel of this homomorphism is the subgroup $G_{\text{Id}}$ and the image is the derivative subgroup $DG < \text{GL}(\mathbb{C}, n)$. If $G_{\text{Id}}$ is trivial then we have an injective group homomorphism $G \hookrightarrow \text{GL}(\mathbb{C}, n)$.

Remark 3.1 (Linear algebraic groups). A complex linear algebraic group is a subgroup of the group of invertible $n \times n$ complex matrices (under matrix multiplication) that is defined by complex polynomial equations. Let therefore $G < \text{GL}(n, \mathbb{C})$ be an infinite linear algebraic group. Then its Lie Algebra $\mathcal{L}(G)$ is not trivial and we may choose a (linear) vector field $X \in \mathcal{L}(G)$. The Zariski closure $\overline{\{X_t\}_{t \in \mathbb{C}}}$ of the flow $\{X_t\}_{t \in \mathbb{C}} < \text{GL}(n, \mathbb{C})$ of $X$ in $\mathbb{C}^n$ is a closed abelian subgroup of the closure $G$. Since $\overline{\{X_t\}_{t \in \mathbb{C}}}$ is abelian, there is a closed one-parameter subgroup $H$, it contains a one-dimensional linear algebraic subgroup of $G$.

Let now $G < \text{Diff}(\mathbb{C}^2, 0)$ be an abelian subgroup. The Lie algebra of $G_{\text{Id}} = G \cap \text{Diff}_{\text{Id}}(\mathbb{C}^2, 0)$ has dimension $\leq 2$. If the dimension is zero then $G_{\text{Id}} = \{\text{Id}\}$ and the map $G \to \text{GL}(2, \mathbb{C})$
embeds $G$ into an abelian linear group. If moreover the group $G$ is algebraic, then the derivative group $DG < \text{GL}(n, \mathbb{C})$ is also algebraic and, in the case $G_{1d} = \{\text{Id}\}$, we have from Remark 3.1 that, either $G$ is finite or its image in $\text{GL}(2, \mathbb{C})$ contains a linear flow in its closure.

From Remark 3.1 and Theorem 1.18 we have:

**Corollary 3.2.** Let $G < \text{Diff}(\mathbb{C}^2, 0)$ be an algebraic commutative subgroup. The possibilities are:

1. The Lie algebra of $G_{1d}$ has dimension zero and the possible cases are:
   - $G$ is finite and therefore analytically linearizable.
   - $G$ embeds into a linear flow.

2. The Lie algebra of $G_{1d}$ has dimension one: $G_{1d}$ leaves invariant an exact rational one-form, say $\omega = dT$ for some rational function $T \in K_2$.

3. The Lie algebra of $G_{1d}$ has dimension two: $G_{1d}$ admits two closed, independent, formal meromorphic, invariant one-forms.

Next remark we show how algebraic groups may appear in our framework.

### 3.1. Holonomy groups of holomorphic foliations.

A dimension one holomorphic foliation with singularities $\mathcal{F}$ of a complex manifold $M$ is defined by a pair $(\mathcal{F}', S)$ where $S \subset M$ is a codimension $n$ analytic subset of $M$ and $\mathcal{F}'$ is a dimension one holomorphic foliation of the open manifold $M' = M \setminus S$, in the usual sense. We may choose the subset $S \subset M$ as minimal in the sense that $\mathcal{F}'$ admits no extension as a (nonsingular) foliation to an open subset intersecting $S$ and in this case we say that $S = \text{sing}(\mathcal{F})$ is the singular set of $\mathcal{F}$. This is a discrete subset of $M$. A leaf of $\mathcal{F}$ is by definition a leaf of $\mathcal{F}'$ in $M'$. Given a leaf $L \subset M$ of $\mathcal{F}$, any point $p \in L$ and a small (holomorphic) transverse section $\Sigma \subset M$, with $p \in \Sigma \cap L$, $\Sigma$ transverse to $L$ and $\mathcal{F}$, we may introduce the holonomy group $\text{Hol}(\mathcal{F}, L, \Sigma, p)$ of this leaf of $\mathcal{F}$ as the usual holonomy group of the leaf $L \subset \mathcal{F}'$ of the foliation $\mathcal{F}'$ calculated with respect to the transverse section $\Sigma \subset M'$ at the point $p \in M'$ (see [3] for instance).

In case $M$ is a projective manifold (for instance, the $n + 1$ dimensional complex projective space $M = \mathbb{CP}^{n+1}$), the foliation $\mathcal{F}$ is necessarily algebraic in the sense that it is given by algebraic equations (polynomial vector fields) in any affine subspace of $M$. In this case by an algebraic solution of $\mathcal{F}$ we mean an algebraic curve $\Lambda \subset M$ such that $M \setminus (\Lambda \cap \text{sing}(\mathcal{F}))$ is a union of leaves of $\mathcal{F}$.

**Proposition 3.3.** Let $\mathcal{F}$ be a dimension one holomorphic foliation with singularities of the complex projective space $\mathbb{CP}^{n+1}$. Let $\Lambda \subset \mathbb{CP}^{n+1}$ be an algebraic solution of $\mathcal{F}$. Then the holonomy groups of the leaves $L \subset \Lambda$ are contained in algebraic groups.

**Proof.** Indeed, given a point $p \in \Lambda \setminus \text{sing}(\mathcal{F})$ we may choose a liner hyperplane $E(n) \subset \mathbb{CP}^{n+1}$ such that $\Lambda$ and $E(n)$ meet transversely at $p$. The choice of an affine system of coordinates $\mathbb{CP}^{n+1}$ centered at $p$ gives a polynomial vector field $X_p$ with isolated singularities that defines the foliation $\mathcal{F}|_{\mathbb{CP}^{n+1}}$ in the ordinary sense: the leaves of $\mathcal{F}|_{\mathbb{CP}^{n+1}}$ are the non-singular integral curves of $X_p$ in $\mathbb{CP}^{n+1}$. Then we choose $n$ polynomial one-forms $\omega_1, \ldots, \omega_n$ in the affine space $\mathbb{C}^{n+1}$ with the property that $\omega_j \cdot X = 1$ and such that the exterior product $\omega_1 \land \cdots \land \omega_n \neq 0$. Denote by $\omega^+_j$ the restriction of $\omega_j$ to $E(n) \cap \mathbb{CP}^{n+1} \simeq \mathbb{C}^n$. Then each $\omega^+_j$ is rational. If we denote by $\Sigma$ the germ at $p$ of disc induced by $E(n)$, then for each holonomy diffeomorphism $g \in \text{Hol}(\mathcal{F}, \Sigma, p)$ we have $g^*(\omega_j) \land \omega_1 \land \cdots \land \omega_n \equiv 0$ because the holonomy maps preserve the leaves of the foliation. Therefore, the holonomy group of the leaves contained in $\Lambda$ are contained in algebraic groups. 

---

\[4\text{The group } G \text{ is not necessarily tangent to the identity.}\]
4. Abelian groups tangent to the identity

Now we study the characterization and classification of abelian tangent to the identity groups, which is the subject of Theorem 1.11.

According to [2] (Proposition 4.1) every nilpotent subalgebra \( L \) of \( \hat{X}(\mathbb{C}^2, 0) \) is metabelian. Also following [2], this proposition implies the following characterization of abelian subgroups of \( \text{Diff}_\text{Id}(\mathbb{C}^2, 0) \).

**Proposition 4.1** (cf. [2] Corollary 4.4). If \( G < \text{Diff}_\text{Id}(\mathbb{C}^2, 0) \) is a abelian tangent to the identity (convergent) group, then one of the following items is true:

1. There is a formal vector field \( \hat{X} \) with \( \exp(\hat{X}) \in G \) such that for each \( g \in G \) we have \( g = \exp(T_g \hat{X}) \) where \( T_g \in K_2 \) is a rational function such that \( \hat{X}(T_g) = 0 \);
2. \( G < \langle f^{[s]} \circ g^{[t]}, t, s \in \mathbb{C} \rangle \), where \( f, g \in G \) and \( [f, g] = \text{Id} \).

Notice that Proposition 4.1 above is for convergent (analytic) objects. In this paper we extend an extension of this result, Theorem 1.12 below. This is based also in the following formal version of Proposition 4.1 which is easily obtained by a mimic of its proof:

**Proposition 4.2.** Let \( G < \hat{\text{Diff}}_\text{Id}(\mathbb{C}^2, 0) \) be an abelian subgroup tangent to the identity, we have the two following possibilities:

1. There is a formal vector field \( \hat{X} \), invariant by \( G \), such that for each element \( \hat{f} \in G \) there is a rational function \( T \in K_2 \) depending on \( \hat{f} \), such that \( \hat{X}(N) = 0 \) and \( \hat{f} = \exp(N \hat{X}) \).
2. There are formal commuting vector fields \( \hat{X} \) and \( \hat{Y} \) such that \( \exp(\hat{X}), \exp(\hat{Y}) \in G \) and \( G < \langle \exp(t\hat{X}) \circ \exp(s\hat{Y}) \mid t, s \in \mathbb{C} \rangle \).

**Definition 4.3** (Formal meromorphic one-forms). By a formal meromorphic function of \( n \) complex variables we shall mean a formal quotient \( \hat{R} = \frac{\hat{P}}{\hat{Q}} \) of two formal power series with positive exponents \( \hat{P}, \hat{Q} \in \mathcal{O}_n = \mathbb{C}[[z]] \). In other words, the field of formal meromorphic functions of \( n \) variables \( \hat{K}_n \) will be the fraction field of the domain of integrity \( \mathcal{O}_n \). By a meromorphic formal one-form we mean a formal expression \( \hat{\omega} = \sum_{j=1}^{n} \hat{R}_j \, dz_j \) where each \( \hat{R}_j \) is a formal meromorphic function as defined above. The exterior derivative, wedge product and other concepts are defined for meromorphic formal one-forms in the same way as for analytic one-forms.

**Proof of Theorem 1.11.** Assume that the Lie algebra \( l(G_{\text{Id}}) \) has dimension one. By Proposition 4.2 there is a formal vector field \( \hat{X} \), invariant by \( G \), such that for each \( \hat{f} \in G \) there is a rational function \( T = T_\hat{f} \in K_2 \) with \( \hat{X}(T) = 0 \) and \( \hat{f} = \exp(T \hat{X}) \). Suppose that for some \( \hat{f} \in G \) the function \( T = T_\hat{f} \) is not constant. We consider the one-form \( \hat{\omega} := dT \). This is a non-trivial closed rational one-form and we claim that this is \( G \)-invariant. In fact, take \( \hat{g} \in G \) and write \( G = \exp(S \hat{X}) \) for some rational function \( S \in K_2 \) such that \( \hat{X}(S) = 0 \). Then \( (S \hat{X})(T) = dT(S \hat{X}) = SdT(\hat{X}) = 0 \). Therefore \( T \circ \exp(S \hat{X}) = T \). This gives \( \hat{g}^* (\hat{\omega}) = \hat{g}^* (dT) = d(T \circ \hat{g}) = d(T \circ \exp(S \hat{X})) = dT = \omega \), proving the claim. This corresponds to (i) in Theorem 1.11.

Now we consider the where \( T_\hat{f} \) is constant for each \( \hat{f} \in G \). In this case each element \( \hat{f} \in G \) writes as \( \hat{f} = \exp(c_\hat{f} \hat{X}) \) for some constant \( c_\hat{f} \in \mathbb{C} \). In other words, \( G \) embeds into the flow of \( \hat{X} \) as in (ii) in the statement.
Suppose now that $G$ is as in (2) in Proposition 4.2. Then there are two $\mathbb{C}$-linearly independent, formal, commuting vector fields $\hat{X}_j$, invariant by $G$, such that $\exp(\hat{X}_j) \in G$ and $G < (\exp(t\hat{X}_1) \circ \exp(s\hat{X}_2) \mid t,s \in \mathbb{C})$.

As already mentioned in the Introduction, (iii) in Theorem 1.11 admits a converse, proved as follows:

Proof of Theorem 1.15 - First Part. By hypothesis $G < \hat{\text{Diff}}(\mathbb{C}^n,0)$ admits two linearly independent invariant formal vector fields $\hat{X}_1, \hat{X}_2$. We fix coordinates $(x,y)$ and write $\hat{X}_j = A_j \frac{\partial}{\partial x} + B_j \frac{\partial}{\partial y}$. Since $\hat{X}_j$ is $G$-invariant, we have

$$
\begin{bmatrix}
\frac{\partial y_1}{\partial x} & \frac{\partial y_1}{\partial y} \\
\frac{\partial y_2}{\partial x} & \frac{\partial y_2}{\partial y}
\end{bmatrix}
\begin{bmatrix}
A_1(z) & A_2(z) \\
B_1(z) & B_2(z)
\end{bmatrix}
= 
\begin{bmatrix}
A_1(\hat{g}) & A_2(\hat{g}) \\
B_1(\hat{g}) & B_2(\hat{g})
\end{bmatrix}
$$

Taking transposes, we obtain:

$$
\begin{bmatrix}
A_1(z) & B_1(z) \\
A_2(z) & B_2(z)
\end{bmatrix}
\begin{bmatrix}
\frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} \\
\frac{\partial y_1}{\partial y} & \frac{\partial y_2}{\partial y}
\end{bmatrix}
= 
\begin{bmatrix}
A_1(\hat{g}) & B_1(\hat{g}) \\
A_2(\hat{g}) & B_2(\hat{g})
\end{bmatrix}
$$

thus

$$
\begin{bmatrix}
\frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} \\
\frac{\partial y_1}{\partial y} & \frac{\partial y_2}{\partial y}
\end{bmatrix}
\begin{bmatrix}
A_1(\hat{g}) & B_1(\hat{g}) \\
A_2(\hat{g}) & B_2(\hat{g})
\end{bmatrix}^{-1} = 
\begin{bmatrix}
A_1(z) & B_1(z) \\
A_2(z) & B_2(z)
\end{bmatrix}^{-1}
$$

(2)

Based on this last equation we take:

$$
\begin{bmatrix}
C_1(z) & C_2(z) \\
D_1(z) & D_2(z)
\end{bmatrix}
= 
\begin{bmatrix}
A_1(z) & B_1(z) \\
A_2(z) & B_2(z)
\end{bmatrix}^{-1}
\cdot
\frac{1}{Q(z)}
\begin{bmatrix}
B_2(z) & -B_1(z) \\
-A_2(z) & A_1(z)
\end{bmatrix}
$$

(3)

Where $Q(z) = A_1B_2 - A_2B_1$. Now we define $\hat{\omega}_j := C_j dx + D_j dy$. By the above equations (2) and (3), the one-forms $\hat{\omega}_j$ are invariant for $G$, i.e, for each $\hat{g} \in G$, we have $\hat{g}^*(\hat{\omega}_j) = \hat{\omega}_j$ ($j = 1, 2$). These forms are $\mathbb{C}$-linearly independent (cf. (3)). Let us now show that the $\hat{\omega}_j$ are closed forms. We have to show that $\frac{\partial D_1}{\partial x} - \frac{\partial C_1}{\partial y} = 0$. Since $[\hat{X}_1, \hat{X}_2] = 0$ then

$$
\frac{\partial A_2}{\partial x}A_1 + \frac{\partial A_2}{\partial y}B_1 = \frac{\partial A_1}{\partial x}A_2 + \frac{\partial A_1}{\partial y}B_2
$$

$$
\frac{\partial B_2}{\partial x}A_1 + \frac{\partial B_2}{\partial y}B_1 = \frac{\partial B_1}{\partial x}A_2 + \frac{\partial B_1}{\partial y}B_2
$$
Thus
\[
Q^2 \left( \frac{\partial D_1}{\partial x} - \frac{\partial C_1}{\partial y} \right) = Q^2 \left( - \frac{1}{Q} \frac{\partial A_2}{\partial x} + \frac{A_2}{Q^2} \frac{\partial Q}{\partial x} - \frac{1}{Q} \frac{\partial B_2}{\partial y} + \frac{B_2}{Q^2} \frac{\partial Q}{\partial y} \right)
\]
\[
= A_2 \frac{\partial Q}{\partial x} - Q \frac{\partial A_2}{\partial x} + B_2 \frac{\partial Q}{\partial y} - Q \frac{\partial B_2}{\partial y}
\]
\[
= A_2 B_2 \frac{\partial A_1}{\partial x} + A_2 A_1 \frac{\partial B_2}{\partial x} - A_2 B_1 \frac{\partial A_2}{\partial x} - A_2^2 \frac{\partial B_1}{\partial x} - A_1 B_2 \frac{\partial A_2}{\partial x} + A_2 B_1 \frac{\partial A_2}{\partial y} +
\]
\[
B_2^2 \frac{\partial A_1}{\partial y} + B_2 A_1 \frac{\partial B_2}{\partial y} - B_2 B_1 \frac{\partial A_2}{\partial y} - B_2 A_2 \frac{\partial B_2}{\partial y} - A_1 B_2 \frac{\partial B_2}{\partial y} + A_2 B_1 \frac{\partial B_2}{\partial y}
\]
\[
= B_2 (\frac{\partial A_1}{\partial x} A_2 + \frac{\partial A_1}{\partial y} B_2 - \frac{\partial A_2}{\partial y} A_1 - \frac{\partial A_2}{\partial y} B_1) +
\]
\[
A_2 (\frac{\partial B_2}{\partial x} A_1 + \frac{\partial B_2}{\partial y} B_1 - \frac{\partial B_1}{\partial y} A_2 - \frac{\partial B_1}{\partial y} B_2)
\]
\[
= B_2.0 + A_2.0 = 0.
\]

Therefore \( \hat{\omega}_1 \) is closed. Similarly \( \hat{\omega}_2 \) is a closed one-form. This proves the first part of Theorem 1.15. Assume now that \( G \) is tangent to the identity. Write \( \hat{\omega}_j = A_j dx + B_j dy \) then,
\[
\hat{\omega}_j = \hat{g}^* (\hat{\omega}_j) = A_j (\hat{g}) dg_1 + B_j (\hat{g}) dg_2
\]
\[
= (A_j (\hat{g}) \frac{\partial g_1}{\partial x} + B_j (\hat{g}) \frac{\partial g_2}{\partial x}) dx + (A_j (\hat{g}) \frac{\partial g_1}{\partial y} + B_j (\hat{g}) \frac{\partial g_2}{\partial y}) dy
\]

Thus
\[
\begin{bmatrix}
A_1(\hat{g}) & B_1(\hat{g})
\end{bmatrix}
\begin{bmatrix}
\frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x}
\end{bmatrix}
= \begin{bmatrix}
A_1(z) & B_1(z)
\end{bmatrix}
\begin{bmatrix}
A_2(z) & B_2(z)
\end{bmatrix}
\]

consequently
\[
\begin{bmatrix}
\frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y}
\end{bmatrix}
\begin{bmatrix}
A_1(z) & B_1(z)
\end{bmatrix}^{-1}
= \begin{bmatrix}
A_1(\hat{g}) & B_1(\hat{g})
\end{bmatrix}^{-1}
\begin{bmatrix}
A_2(\hat{g}) & B_2(\hat{g})
\end{bmatrix}
\]

Let us introduce \( \hat{X}_1, \hat{X}_2 \) as follows:
\[
\hat{X}_1 = \frac{1}{Q(z)} (B_2 \frac{\partial}{\partial x} - A_2 \frac{\partial}{\partial y}), \quad \hat{X}_2 = \frac{1}{Q(z)} (-B_1 \frac{\partial}{\partial x} + A_1 \frac{\partial}{\partial y})
\]

where \( Q(z) = A_1(z) B_2(z) - A_2(z) B_1(z) \). Since the \( \hat{\omega}_j \) are closed one-forms we have \( \hat{g}^* (\hat{X}_j) = \hat{X}_j \), for all \( \hat{g} \in G \) and also \( [\hat{X}_1, \hat{X}_2] = 0 \). Also \( \hat{X}_1, \hat{X}_2 \) are \( \mathbb{C} \)-linearly independent in \( \hat{K}_2 \).

Note that \( \{ \hat{X}_1, \hat{X}_2 \} \) is a basis for the vector space \( \mathcal{X}(\mathbb{C}^2, 0) \otimes \hat{K}_2 \) and, since for \( \hat{g} \in G \) we can write \( \hat{g} = \exp (\hat{Y}_h) \) with \( \hat{Y}_h \in \mathcal{X}(\mathbb{C}^2, 0) \), then \( \hat{Y}_h = u_1 \hat{X}_1 + u_2 \hat{X}_2 \), where \( u_j \in \hat{K}_2 \). On the other side \( \hat{g}^* (\hat{X}_j) = \hat{X}_j \) we have that \( [\hat{Y}_h, \hat{X}_1] = 0 \) and \( [\hat{Y}_h, \hat{X}_2] = 0 \), then \( \hat{X}_j(u_k) = 0 \) \( (j, k = 1, 2) \), consequently \( u_j \) are constant in \( C^* \). Now if \( G < \{ \exp (t \hat{Y}_h) \} \ | \ t \in \mathbb{C} \) there is nothing left to prove, thus suppose that there is \( \hat{h} \in G \), \( \hat{h} = \exp (\hat{Y}_h) \) with \( \hat{Y}_h \) and \( \hat{Y}_h \) \( \mathbb{C} \)-linearly independent in \( \hat{K}_2 \) then there are \( v_j \in \mathbb{C} \) such that \( \hat{Y}_h = v_1 \hat{X}_1 + v_2 \hat{X}_2 \) and \( (u_1, u_2) \), \( (v_1, v_2) \) are \( \mathbb{C} \)-linearly independent in \( \mathbb{C}^2 \), therefore \( \hat{X}_j \in \mathcal{X}(\mathbb{C}^2, 0) \) and \( G = \exp (a \hat{X}_1) \circ \exp (b \hat{X}_2) \), with \( a, b \in C^* \), therefore there are formal vector fields \( \hat{X} \) and \( \hat{Y} \) such that \( [\hat{X}, \hat{Y}] = 0 \) and \( G < \{ \exp (t \hat{X}) \circ \exp (s \hat{Y}) \} \ | \ t, s \in \mathbb{C} \). Therefore \( G \) is abelian. This proves the first part of Theorem 1.15. 

The second part will be concluded in the next section (cf. Proposition 5.8).
5. Groups preserving closed one-forms

In this section we finish the proof of Theorem 1.15. Indeed, we proceed studying the classification of groups of formal diffeomorphisms preserving closed meromorphic one-forms in \((\mathbb{C}^2, 0)\). Special attention is given to the "generic" case where the group exhibits two transverse formal separatrices. Before going further into the main subject we recall some classical facts about integration of closed meromorphic one-forms in several complex variables.

**Proposition 5.1** (Integration Lemma, cf. [4], [17] Example 1.6). Let \(\omega\) be a closed meromorphic one-form on \(M\) where \(M\) is a polydisc in \(\mathbb{C}^n\). Then there are irreducible holomorphic functions \(f_1, \ldots, f_r \in \mathcal{O}(M)\), \(n_1, \ldots, n_r \in \mathbb{N}\), complex numbers \(\lambda_1, \ldots, \lambda_r\) and a holomorphic function \(g \in \mathcal{O}(M)\) such that

\[
\omega = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + d\left( \frac{g}{f_1^{n_1} \cdots f_r^{n_r}} \right)
\]

The polar set of \(\omega\) is given in irreducible components by \(\bigcup_{j=1}^{r} \{ f_j = 0 \}\), \(n_j\) is the order of \(\{ f_j = 0 \}\) as a component of the polar set of \(\omega\), \(\lambda_j\) is the residue of \(\omega\) at the component \(\{ f_j = 0 \}\) and the function \(g\) has no common factors with \(f_j\) in \(\mathcal{O}(M)\).

If \(M = \mathbb{C}^n\) and \(\omega\) is a rational closed one-form then we have the same result, where the \(f_j\) are irreducible polynomials and \(g\) is a polynomial without common factors with the \(f_j\). The proof of Theorem 5.1 relies on integration and the fact that the first homology group of the complement of a pure codimension one analytic subset \(\Lambda = \bigcup_{j=1}^{r} \Lambda_j\), where each \(\Lambda_j\) is an irreducible component, of a polydisc \(M\) as above, is generated by small loops around the components \(\Lambda_j\), contained in transverse discs circulating the component. Then a standard argument involving Laurent series implies the result. This cannot be repeated in the formal case, because we cannot rely on integration processes, at first glance. Nevertheless, we still have a formal version of Theorem 5.1 as follows:

**Proposition 5.2** (Formal integration lemma). Let \(\hat{\omega}\) be a closed formal meromorphic one-form in \(n\) complex variables. Denote by \(\hat{f}_j \in \hat{\mathcal{O}}_k\), \(j = 1, \ldots, r\) the formal equations of the set of poles of \(\hat{\omega}\), in independent terms. Then, there are \(\lambda_j \in \mathbb{C}\) and \(n_j \in \mathbb{N}\) and a formal function \(\hat{g} \in \hat{\mathcal{O}}_k\) such that

\[
\hat{\omega} = \sum_{j=1}^{r} \lambda_j \frac{d\hat{f}_j}{\hat{f}_j} + d\left( \frac{\hat{g}}{\hat{f}_1^{n_1} \cdots \hat{f}_r^{n_r}} \right)
\]

The proof is somehow similar to the proof of the local analytic version and it is based on the following:

**Lemma 5.3.** A closed formal meromorphic one-form \(\hat{\omega}\) in \(n\) complex variables, without residues is exact: \(\hat{\omega} = d\hat{f}\) for some meromorphic formal function \(\hat{f} \in \hat{\mathcal{K}}_n\).

This lemma is proved analogously to the following particular case:

**Lemma 5.4.** Let \(\hat{\omega}\) be a closed formal meromorphic one-form in two complex variables and assume that the polar set of \(\hat{\omega}\) consists of two transverse formal curves, and that the residues of \(\hat{\omega}\) are all zero. Then \(\hat{\omega}\) is exact, indeed, in suitable formal coordinates \((x, y)\) we can write

\[
\hat{\omega} = d\left( \frac{\hat{f}}{x^m y^n} \right)
\]

for some \(n, m \in \mathbb{N}\) and some formal function \(\hat{f} \in \hat{\mathcal{O}}_2\).
Proof. Since the polar set of $\dot{\omega}$ consists of two transverse formal curves, we can find formal coordinates $(x,y)$ such that this polar set corresponds to the coordinate axes. We write $\dot{\omega} = (\dot{P}/x^{n+1}y^{m+1})dx + (\dot{Q}/x^{n+1}y^{m+1})dy$ where $\dot{P}, \dot{Q} \in \mathbb{C}[[x,y]]$. We can write $\dot{P} = \sum_{\nu=0}^{\infty} P_{\nu}$ and $\dot{Q} = \sum_{j=0}^{\infty} Q_{j\nu}$ in terms of homogeneous polynomials $P_{\nu}, Q_{j\nu}$ of degree $\nu - n - m$. Then

$$\dot{\omega} = \sum_{\nu=0}^{\infty} (P_{\nu}/x^{n+1}y^{m+1})dx + (Q_{j\nu}/x^{n+1}y^{m+1})dy = \sum_{\nu=-n-m}^{\infty} \omega_{\nu}$$

where $\omega_{\nu} = (P_{\nu}/x^{n+1}y^{m+1})dx + (Q_{j\nu}/x^{n+1}y^{m+1})dy$ is a homogeneous rational one-form of degree $\nu - n - m - 2$. Then $d\dot{\omega} = \sum_{\nu=-n-m-2}^{\infty} d\omega_{\nu}$ where each one-form $d\omega_{\nu}$ is homogeneous of degree $\nu - 1$. Therefore, since $\dot{\omega}$ is closed we have $0 = d\dot{\omega} = \sum_{\nu=-n-m}^{\infty} d\omega_{\nu}$ and then $d\omega_{\nu} = 0, \forall \nu \geq -n - m$. Since $\dot{\omega}$ has no residues, the same holds for $\omega_{\nu}$. Moreover, because each form $\omega_{\nu}$ is of the form $\omega_{\nu} = P_{\nu}/x^{n+1}y^{m+1})dx + (Q_{j\nu}/x^{n+1}y^{m+1})dy$, we conclude from the Integration lemma that $\omega_{\nu} = d(\hat{f}/x^{n}y^{m})$ for some homogeneous polynomial $f_{\nu}$ of degree $\nu$. Thus $\dot{\omega} = d(\sum_{\nu=0}^{\infty} f_{\nu}/x^{n}y^{m}) = d(\hat{f}/x^{n}y^{m})$ where $\hat{f} = \sum_{\nu=0}^{\infty} f_{\nu} \in \hat{O}_{2}$.

As a consequence we obtain the following particular case of Proposition 5.2:

**Proposition 5.5.** Let $\dot{\omega}$ be a closed formal meromorphic one-form in two complex variables and assume that the polar set of $\dot{\omega}$ consists of two transverse formal curves. Then $\dot{\omega}$ writes in suitable formal coordinates $(x,y)$ as

$$\dot{\omega} = \lambda \frac{dx}{x} + \mu \frac{dy}{y} + d(\hat{f}/x^{n}y^{m})$$

for some $\lambda, \mu \in \mathbb{C}$, some $n, m \in \mathbb{N}$ and some formal function $\hat{f} \in \hat{O}_{2}$.

**Proof.** As in the proof of Lemma 5.4 we choose formal coordinates $(x,y)$ such that the polar set of $\dot{\omega}$ corresponds to the coordinate axes. Denote by $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$ the residue of $\dot{\omega}$ at the $x$-axis and $y$-axis respectively. Then $\dot{\theta} = \dot{\omega} - \lambda \frac{dx}{x} + \mu \frac{dy}{y}$ is a closed formal meromorphic one-form with polar set contained in the coordinate axes and zero residues. By Lemma 5.3 we can write $\dot{\theta} = d(\hat{f}/x^{n}y^{m})$ for some formal function $\hat{f} \in \hat{O}_{2}$.

An improvement of the above proposition is the following:

**Lemma 5.6.** Let $\dot{\omega}_{j}, j = 1, 2$ be $\mathbb{C}$-linearly independent closed formal meromorphic one-forms in two variables with polar sets along two transverse formal curves. Then there are formal coordinates $(x,y)$ such that each $\dot{\omega}_{j}$ writes:

$$\dot{\omega}_{j} = a_{j} \frac{dx}{x} + b_{j} \frac{dy}{y} + d(\frac{c_{j}}{x^{n_{j}}y^{m_{j}}})$$

for some constant $a_{j}, b_{j}, c_{j} \in \mathbb{C}$ and some $n_{j}, m_{j} \in \mathbb{N}$.

**Proof.** By Proposition 5.5 we can write $\dot{\omega}_{j} = a_{j} \frac{dx}{x} + b_{j} \frac{dy}{y} + d(\frac{f_{j}}{x^{n_{j}}y^{m_{j}}})$, where $a_{j}, b_{j} \in \mathbb{C}; n_{j}, m_{j} \in \mathbb{N}$ and $f_{j} \in \hat{O}_{k}$. Let us write $n_{1} = n, m_{1} = m$ and $n_{2} = p, m_{2} = q$. We take a model of formal change of coordinates $\hat{\phi} = (xu, yv)$, where we want that $\hat{\phi}^{*}(a_{1} \frac{dx}{x} + b_{1} \frac{dy}{y} +$
Proof.
A diffeomorphism \( \hat{\omega} \) in the normal form of Lemma 5.6.

similar algebraic proprieties, there is no loss of generali
ty in assuming that the forms are as

Now define a formal meromorphic function \( \hat{\omega} \) by

\[
\hat{\omega}(x, y, u, v) = (\hat{\omega}_1(x, y, u, v), \hat{\omega}_2(x, y, u, v))
\]

where

\[
\hat{\omega}_1(x, y, u, v) = (a_1 \ln u + b_1 \ln v)x^n y^m - \hat{f}_1 + \frac{c_1}{u^n v^m} + k_1 x^n y^m
\]

and

\[
\hat{\omega}_2(x, y, u, v) = (a_2 \ln u + b_2 \ln v)x^p y^q - \hat{f}_2 + \frac{c_2}{u^p v^q} + k_2 x^p y^q.
\]

We have \( \hat{\omega}(0, 0, u, v) = (0, 0) \), so that if \( a_1 = \hat{f}_1(0) \) and \( a_2 = \hat{f}_2(0) \) we have \( \frac{1}{u^n v^m}(0, 0) = 1 \)
and \( \frac{1}{u^p v^q}(0, 0) = 1 \) as \( \text{Det}(J2\hat{\omega}(0, (u, v))) = (nq - mp)u^{k+n+1}v^{q+1} \neq 0 \), if \( (m, n) \) and \( (p, q) \) are \( \mathbb{C} \)-linearly independent, from the formal version of the Implicit function theorem
we obtain a unique solution \((u, v)\).

\[ \square \]

Remark 5.7. Let \( G < \text{Diff}_k(\mathbb{C}^2, 0) \) be a subgroup. Given a closed meromorphic 1-form \( \hat{\omega} \) such that \( \hat{\omega} \) is invariant by \( G \), if \( \hat{\omega} \) is conjugated to a 1-form \( \hat{\omega} \) by a diffeomorphism \( \hat{h} \), then the 1-form \( \hat{h} \) is invariant by the group \( \hat{h}^{-1} \circ \hat{g} \circ \hat{h} \). As the groups \( G \) and \( \hat{h}^{-1} \circ \hat{g} \circ \hat{h} \) have similar algebraic proprieties, there is no loss of generality in assuming that the forms are as in the normal form of Lemma 5.6.

The second part of the proof of Theorem 5.5 follows from the following proposition:

Proposition 5.8. Let \( G < \text{Diff}_k(\mathbb{C}^2, 0) \) be a subgroup with two transverse separatrices, if there are \( \mathbb{C} \)-linearly independent closed formal meromorphic formal one-forms \( \hat{\omega}_j \), \( (j = 1, 2) \) which are invariant by \( G \) then \( G \) is abelian. Indeed, \( G \) is formally conjugate to a group of diffeomorphisms generated by any of the following types:

(a) \( \hat{g}(x, y) = (x^{(1 + \frac{k_1}{c_1} x^n y^m)^\frac{m}{n}}, y^{(1 + \frac{k_1}{c_1} x^n y^m)^\frac{m}{n}}) \)

(b) \( \hat{g}(x, y) = (ax^{(1 + \frac{k}{c} x^n y^m)^\frac{m}{n}}, by^{(1 + \frac{k}{c} x^n y^m)^\frac{m}{n}}) \)

(c) \( \hat{g}(x, y) = (b^{\frac{k}{c} x^{(1 + \frac{k}{c} x^n y^m)^\frac{m}{n}}}, by^{(1 + \frac{k}{c} x^n y^m)^\frac{m}{n}}) \)

Proof. A diffeomorphism \( \hat{g} \in G \) writes \( \hat{g}(x, y) = (ux, vy) \) where \( u, v \in \hat{O}_2 \) satisfy \( \hat{u}(0) \neq 0, \hat{v}(0) \neq 0 \). From equation \( 1 \) and \( \hat{g}^*(\hat{\omega}_j) = \hat{\omega}_j \) we obtain:

\[
a_1 \ln u + b_1 \ln v = \frac{c_1}{x^n y^m}(1 - \frac{1}{u^n v^m}) + k_1 \quad \text{and} \quad a_2 \ln u + b_2 \ln v = \frac{c_2}{x^p y^q}(1 - \frac{1}{u^p v^q}) + k_2
\]

If \( a_1 = a_2 = b_1 = b_2 = 0 \) then \( u^n v^m = \frac{1}{1 + \frac{k_1}{c_1} x^n y^m} \) and \( u^p v^q = \frac{1}{1 + \frac{k_2}{c_2} x^p y^q} \), as \( (m, n) \) and \( (p, q) \) must be \( \mathbb{C} \)-linearly independent we have

\[
\hat{g}(x, y) = (x^{(1 + \frac{k_1}{c_1} x^n y^m)^\frac{m}{n}}, y^{(1 + \frac{k_1}{c_1} x^n y^m)^\frac{m}{n}})
\]

with \( D = nq - pm \). The group \( G \) therefore has just linear diffeomorphisms as above and is an abelian group.
Assume now that the left side of equality is holomorphic we have, \( \frac{1}{y^m} = 1 \) and \( \frac{1}{y^n} = 1 \) when \((m, n) \) and \((p, q) \) are \( \mathbb{C} \)-linearly independent, we have that \( u \) and \( v \) are constant, so that \( G \) is linear therefore \( G \) is abelian.

A similar argumentation with the other possible cases gives the forms:

\[
\hat{g}(x, y) = (ax, \frac{a - \frac{n}{m} y}{(1 + k x^n y^m)\frac{m}{n}}), \quad \hat{g}(x, y) = (\frac{b - \frac{m}{n} x}{(1 + k x^n y^m)\frac{m}{n}}, by)
\]

and

\[
\hat{g}(x, y) = (\frac{x}{(1 + k_1 x^n)\frac{m}{n}}, \frac{y}{(1 + k_2 y^m)\frac{m}{n}}).
\]

In particular, on each case, \( G \) is abelian.

\[\square\]

**Remark 5.9** (Holomorphic case). If \( G \) is invariant by two \( \mathbb{C} \)-linearly independent closed formal one-forms (without poles) then \( G = \{\text{Id}\} \).

**Proof.** Let \( \hat{g} \in G \) and write \( \hat{\omega}_j = df_j \) \((j = 1, 2) \) left invariant by \( G \). Take \( \hat{\Phi} = (\hat{f}_1, \hat{f}_2) \), as \( df_1 \) and \( df_2 \) are \( \mathbb{C} \)-linearly independent in this neighborhood of the origin, we have that \( \hat{\Phi} \) is a formal diffeomorphism. Therefore we may assume that \( \hat{\omega}_1 = dx \) and \( \hat{\omega}_2 = dy \), i.e.,

\[
(\hat{\Phi} \circ \hat{g} \circ \hat{\Phi}^{-1})*(dx) = dx \quad \text{and} \quad (\hat{\Phi} \circ \hat{g} \circ \hat{\Phi}^{-1})*(dy) = dy,
\]

because \( (\hat{\Phi} \circ \hat{g} \circ \hat{\Phi}^{-1})*(dx) = (\hat{\Phi}^{-1})^* \circ \hat{g}^* \circ \hat{\Phi}^*(dx) = (\hat{\Phi}^{-1})^* \circ \hat{g}^* \circ \hat{\Phi}^*(dx) = dx \), therefore \( \hat{\Phi} \circ \hat{g} \circ \hat{\Phi}^{-1} = \{\text{Id}\} \) and consequently \( G = \{\text{Id}\} \).

\[\square\]

### 6. Metabelian groups

Now we study metabelian groups in \( \hat{\text{Diff}}(\mathbb{C}^2, 0) \), that is, subgroups \( G < \hat{\text{Diff}}(\mathbb{C}^2, 0) \) such that the subgroup of commutators \( G^{(1)} = [G, G] \) is abelian. Let \( G \) be such a metabelian subgroup. Then, the derivative group \( DG < \text{GL}(\mathbb{C}, 2) \) is also metabelian but not necessarily abelian. For instance, take \( G \) as the linear subgroup of \( 2 \times 2 \) triangular superior matrices. Then \( G \) is not abelian but \( G^{(1)} \) is abelian.

Now if the group \( DG \) is abelian then \( G^{(1)} \) is tangent to the identity, which is a very useful property. For this reason, in our statements related to this case (Theorem \[1.16\]), we may alternatively require that \( DG \) is abelian.

**Lemma 6.1.** Let \( G < \hat{\text{Diff}}(\mathbb{C}^2, 0) \) be a subgroup with \( DG \) abelian. Suppose that, there are two \( \mathbb{C} \)-linearly independent vector fields \( \hat{X} \) and \( \hat{Y} \), projectively invariant by \( G \) and such that \( [\hat{X}, \hat{Y}] = 0 \). Then \( G \) is metabelian.

**Proof.** Since \( \hat{X} \) and \( \hat{Y} \) are projectively invariant by \( G \), for each \( \hat{g} \in G \) there are constants \( a_{\hat{g}}, b_{\hat{g}} \in \mathbb{C} \) such that \( \hat{g} \cdot \hat{X} = a_{\hat{g}} \cdot \hat{X} \) and \( \hat{g} \cdot \hat{Y} = b_{\hat{g}} \cdot \hat{Y} \). Given now a tangent to the identity element \( \hat{h} \in G_{\text{Id}} \) we have \( a_{\hat{h}} = 1 \) and \( b_{\hat{h}} = 1 \), so that \( \hat{h} \cdot \hat{X} = \hat{X} \) and \( \hat{h} \cdot \hat{Y} = \hat{Y} \). This implies that \( G_{\text{Id}} \) is abelian. Since \( DG \) is abelian, we have that \( [G, G] < G_{\text{Id}} \), so that \( [G, G] \) is abelian.

\[\square\]

**Proof of Theorem 1.16.** Let \( G < \hat{\text{Diff}}(\mathbb{C}^2, 0) \) be a metabelian non-abelian subgroup such that the group of commutators \( [G, G] \) is tangent to the identity, \( [G, G] < G_{\text{Id}} \). If \( G_{\text{Id}} \) is trivial then the group is abelian. Therefore, we may assume that the Lie algebra of \( G_{\text{Id}} \) has positive dimension. By Proposition \[1.12\] we have two cases:

**Case 1.** \( [G, G] \leq \langle \exp(N \hat{X}) \mid N \text{ is a rational function}, \hat{X}(N) = 0 \rangle \) and \( \hat{f} = \exp(\hat{X}) \in [G, G] \). Then for all \( \hat{g} \in G \), \( [\hat{g}, \hat{f}] \in [G, G] \) so that, there is a rational function \( N \) such that \( [\hat{g}, \hat{f}] = \exp(\hat{N} \hat{X}) \) then \( \hat{g} \circ \exp(\hat{X}) \circ \hat{g}^{-1} = \exp(\hat{N} \hat{X}) \circ \exp(\hat{X}) = \exp((\hat{N} + 1) \hat{X}) \) therefore \( \hat{g}^*(\hat{X}) = N \hat{X} \)

**Case 2.** \( [G, G] < \langle \exp(s \hat{X}) \circ \exp(t \hat{Y}) \mid s, t \in \mathbb{C} \rangle \). Then \( \hat{f} = \exp(\hat{X}) \). Then for all \( \hat{g} \in G \), \( [\hat{g}, \hat{f}] \in [G, G] \) so that, there are \( \hat{s}_1 \) and \( t_1 \) such that \( [\hat{g}, \hat{f}] = \exp(\hat{s}_1 \hat{X}) \circ \exp(\hat{t}_1 \hat{Y}) \) then...
\( \hat{g} \circ \exp(\hat{X}) \circ \hat{g}^{-1} = \exp(\hat{s}_1 \hat{X} + t_1 \hat{Y}) \circ \exp(\hat{X}) = \exp(s_1 \hat{X} + t_1 \hat{Y}) \). Therefore \( \hat{g}^*(\hat{X}) = s_1 \hat{X} + t_1 \hat{Y} \).

Analogously we have \( \hat{g}^*(\hat{Y}) = s_2 \hat{X} + t_2 \hat{Y} \).

This proves (ii) in Theorem 1.16.

Let us now finish the proof, by constructing in case (ii) the formal closed meromorphic one-forms \( \hat{\omega}_j \), \( j = 1, 2 \). We can write \( \hat{X}_j = A_j \frac{\partial}{\partial x} + B_j \frac{\partial}{\partial y} \) then

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\
\frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y}
\end{bmatrix}
\begin{bmatrix}
A_1(z) & A_2(z) \\
B_1(z) & B_2(z)
\end{bmatrix}
= \begin{bmatrix}
s_1 A_1(\hat{g}) + t_1 A_2(\hat{g}) & s_2 A_1(\hat{g}) + t_2 A_2(\hat{g}) \\
s_1 B_1(\hat{g}) + t_1 B_2(\hat{g}) & s_2 B_1(\hat{g}) + t_2 B_2(\hat{g})
\end{bmatrix}
\]

taking transposes, we have:

\[
\begin{bmatrix}
A_1(z) & B_1(z) \\
A_2(z) & B_2(z)
\end{bmatrix}
\begin{bmatrix}
\frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} \\
\frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y}
\end{bmatrix}
= \begin{bmatrix}
s_1 A_1(\hat{g}) + t_1 A_2(\hat{g}) & s_1 B_1(\hat{g}) + t_1 B_2(\hat{g}) \\
s_2 A_1(\hat{g}) + t_2 A_2(\hat{g}) & s_2 B_1(\hat{g}) + t_2 B_2(\hat{g})
\end{bmatrix}
\]

thus

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} \\
\frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y}
\end{bmatrix}
\begin{bmatrix}
s_1 A_1(\hat{g}) + t_1 A_2(\hat{g}) & s_1 B_1(\hat{g}) + t_1 B_2(\hat{g}) \\
s_2 A_1(\hat{g}) + t_2 A_2(\hat{g}) & s_2 B_1(\hat{g}) + t_2 B_2(\hat{g})
\end{bmatrix}^{-1}
= \begin{bmatrix}
A_1(z) & B_1(z) \\
A_2(z) & B_2(z)
\end{bmatrix}^{-1}
\]

so that we can take:

\[
\begin{bmatrix}
C_1(z) & C_2(z) \\
D_1(z) & D_2(z)
\end{bmatrix}^{-1}
= \begin{bmatrix}
s_1 A_1(z) + t_1 A_2(z) & s_1 B_1(z) + t_1 B_2(z) \\
s_2 A_1(z) + t_2 A_2(z) & s_2 B_1(z) + t_2 B_2(z)
\end{bmatrix}^{-1}
\]

and define \( \hat{\omega}_j = C_j \, dx + D_j \, dy \). Then \( \hat{g}^*(\hat{\omega}_1) = \hat{g}^*(\frac{1}{r(\hat{g})}(s_2 B_1 + t_2 B_2, -(s_1 B_1 + t_1 B_2)) = \frac{1}{Q(\hat{g})}(B_2, -A_2) = s_1 \hat{\omega}_1 + s_2 \hat{\omega}_2 \), where \( Q(x) = C_1(z).D_2(z) - C_2(z).D_1(z) \) and \( r = s_1 t_2 - s_2 t_1 \). Analogously \( \hat{g}^*(\hat{\omega}_2) = t_1 \hat{\omega}_1 + t_2 \hat{\omega}_2 \), clearly \( \hat{\omega}_1 \) and \( \hat{\omega}_2 \) are \( \mathbb{C} \)-linearly independent. It remains to show that the \( \hat{\omega}_j \) are closed forms, i.e. \( \frac{\partial D_j}{\partial x} - \frac{\partial C_j}{\partial y} = 0 \). Since \( [\hat{X}_1, \hat{X}_2] = 0 \) then

\[
\frac{\partial A_2}{\partial x} A_1 + \frac{\partial A_2}{\partial y} B_1 = \frac{\partial A_1}{\partial x} A_2 + \frac{\partial A_1}{\partial y} B_2
\]

\[
\frac{\partial B_2}{\partial x} A_1 + \frac{\partial B_2}{\partial y} B_1 = \frac{\partial B_1}{\partial x} A_2 + \frac{\partial B_1}{\partial y} B_2
\]

Thus outing the value of \( C_j \) and \( D_j \) and using the above equations we can conclude.

Next we study the possible normal forms of groups as in the conclusion of Theorem 1.16

Remark 6.2 (groups leaving invariant a linear system of closed forms). Let \( G < \text{Diff}(\mathbb{C}, 0) \) be a subgroup of formal diffeomorphisms of two variables, that preserves the coordinate axes \( (x = 0) \) and \( (y = 0) \). Suppose that we have

\[ \hat{g}^*(\hat{\omega}_j) = a_j \hat{\omega}_1 + b_j \hat{\omega}_2, \ \forall \hat{g} \in G. \] (5)

where \( a_j, b_j \in \mathbb{C}^* \) and \( \hat{\omega}_j \) is a closed formal meromorphic one-form. A diffeomorphism \( \hat{g} \in G \) writes \( \hat{g}(x, y) = (xu, yv) \) where \( u, v \in \mathcal{O}_2 \). We have the following possibilities for \( \hat{\omega}_1, \hat{\omega}_2 \) in suitable formal coordinates:

(1) (simple poles case) If both forms have simple poles along the coordinate axes we can write \( \hat{\omega}_1 = \alpha_1 \frac{dx}{x} + \beta_1 \frac{dy}{y} \) and \( \hat{\omega}_2 = \alpha_2 \frac{dx}{x} + \beta_2 \frac{dy}{y} \). From equation [5] we get

\[
\alpha_1 \frac{dx}{x} + \beta_1 \frac{dy}{y} + \alpha_1 \frac{du}{u} + \beta_1 \frac{dv}{v} = \hat{g}^*(\hat{\omega}_j) = (a_1 \alpha_1 + b_1 \alpha_2) \frac{dx}{x} + (a_1 \beta_1 + b_1 \beta_2) \frac{dy}{y}
\]
In matrix form we have:

\[
\begin{bmatrix}
\alpha_1 & \beta_1 \\
\alpha_2 & \beta_2
\end{bmatrix}
\begin{bmatrix}
\frac{dx}{x} \\
\frac{dy}{y}
\end{bmatrix}
+ \begin{bmatrix}
\alpha_1 & \beta_1 \\
\alpha_2 & \beta_2
\end{bmatrix}
\begin{bmatrix}
\frac{du}{u} \\
\frac{dv}{v}
\end{bmatrix}
= \begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\begin{bmatrix}
\frac{dx}{x} \\
\frac{dy}{y}
\end{bmatrix}
\]

Comparing the poles we obtain:

\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix} = \text{Id} \quad \text{and} \quad \begin{bmatrix}
\frac{du}{u} \\
\frac{dv}{v}
\end{bmatrix} = 0
\]

Thus \( \hat{g}(x, y) = (xu_0, yv_0) \) with \( u_0 \) and \( v_0 \) constant. In this case the group \( G \) is linear.

(2) **Pure polar case** Assume now that \( \hat{\omega}_j \) has poles of order higher than one and no residues. We can write \( \hat{\omega}_1 = d(\frac{1}{x^m y^n}) \) and \( \hat{\omega}_2 = d(\frac{1}{x^m y^n}) \). Given now a diffeomorphism \( \hat{g}(x, y) = (xu, yv) \) in \( G \) from equation \( \[5\] \) we have:

\[
d\left(\frac{1}{x^m y^n u^p v^q}\right) = a_1 d\left(\frac{1}{x^m y^n}\right) + b_1 d\left(\frac{1}{x^m y^n}\right) \quad \text{and} \quad d\left(\frac{1}{x^m y^n u^p v^q}\right) = a_2 d\left(\frac{1}{x^m y^n}\right) + b_2 d\left(\frac{1}{x^m y^n}\right)
\]

Thus

\[
\frac{1}{x^m y^n u^p v^q} = \frac{a_1}{x^m y^n} + \frac{b_1}{x^m y^n} + k_1 \quad \text{and} \quad \frac{1}{x^m y^n u^p v^q} = \frac{a_2}{x^m y^n} + \frac{b_2}{x^m y^n}
\]

now

\[
\frac{1}{u^p v^q} = a_1 + b_1 x^{-p} y^{-q} + k_1 x^m y^n \quad \text{and} \quad \frac{1}{u^p v^q} = a_2 + b_2 x^{-p} y^{-q} + k_2 x^m y^n
\]

thus

\[
u = \left(\frac{a_2 + b_2 x^{-p} y^{-q} + k_2 x^m y^n}{a_1 + b_1 x^{-p} y^{-q} + k_1 x^m y^n}\right)^{\frac{m}{n}}
\]

and

\[
u = \left(\frac{a_2 + b_2 x^{-p} y^{-q} + k_2 x^m y^n}{a_2 + b_2 x^{-p} y^{-q} + k_2 x^m y^n}\right)^{\frac{m}{n}}
\]

where \( D = qn - pm \), therefore we have:

\[
\hat{g}(x, y) = \left(\frac{x}{a_2 + b_2 x^{-p} y^{-q} + k_2 x^m y^n}, \frac{y}{a_2 + b_2 x^{-p} y^{-q} + k_2 x^m y^n}\right)
\]

Other mixed cases are studied in the same way.

The following example contradicts Corollary 4.2 in [2].

**Example 6.3.** An example of a tangent to the identity group \( G < \hat{\text{Diff}}_H(\mathbb{C}^2, 0) \), which is solvable but not metabelian is \( G = \langle (\hat{h}(x), \hat{a}(x) + \hat{b}(x)y) ; \hat{h} \in H \rangle \), where \( H < \hat{\text{Diff}}_H(\mathbb{C}, 0) \) is any metabelian tangent to the identity subgroup, \( \hat{a}(x) \in \mathbb{C}[[x]] \) has order greater than 2 and \( \hat{b}(x) \in \mathbb{C}[[x]] \) is a unit, \( \hat{b}(0) = 1 \).

7. Nilpotent groups and subalgebras of vector fields

We study nilpotent subgroups of \( \hat{\text{Diff}}_H(\mathbb{C}^n, 0) \), i.e., nilpotent groups of maps tangent to the identity. For \( n = 1 \) it is known that the concepts of solvable and metabelian group are equivalent. Nevertheless, this is not true for dimension \( n \geq 2 \): take \( G < \hat{\text{Diff}}(\mathbb{C}^2, 0) \) as (induced by) the nilpotent linear group of upper triangular matrices. This group has length 2. In general, it is known that for a linear solvable group \( G < \text{GL}(n, \mathbb{C}) \), the solvable length is bounded by the **Newman function** \( \rho(n) \), where \( \rho(n) \leq 2n \), in particular, \( \rho(2) = 4 \) and \( \rho(3) = 5 \) [15]. In this section we prove Theorem 1.12 i.e., that every nilpotent subalgebra \( \mathfrak{l} \) of \( \hat{\mathcal{X}}(\mathbb{C}^n, 0) \) has length \( l(\mathfrak{l}) \) at most \( n \).

Note that \( \hat{\mathcal{X}}(\mathbb{C}^n, 0) \otimes \hat{K}_n \) is a vector space of dimension \( n \) over \( \hat{K}_n \), where \( \hat{K}_n \) is the fraction field of \( \hat{\mathcal{O}}_n \). Denote by \( \mathcal{R} \) the center of \( \hat{\mathfrak{l}} \) and by \( \{ \mathcal{X}_1, \ldots, \mathcal{X}_k \} \) a basis of \( \mathcal{R} \otimes \hat{K}_n \).


Lemma 7.1. Let $I$ a nilpotent subalgebra of $\hat{\mathfrak{g}}(\mathbb{C}^n, 0)$ with dimension $m$ over $\hat{K}_n$. Then there is an ordered basis $\{X_1, \ldots, X_m\}$ for $I$ over $\hat{K}_n$ in the following sense $\mathcal{Z} \in I$:

\[
\begin{align*}
[X, X_j] &= \begin{cases} 
0, & \text{if } l = 1, \ldots, k, \\
\sum_{j=1}^{k} v_j X_j, & \text{if } l = k + 1, \ldots, m.
\end{cases}
\end{align*}
\]

(2) If $\mathcal{Z} = \sum_{j=1}^{l} u_j X_j \in I$ then $X_j(u_r) = 0$ with $(j = 1, \ldots, r)$ and $(r = 1, \ldots, l)$.

Proof. As $I$ a nilpotent subalgebra we have that $k \geq 1$. Take $\mathcal{S} = (\bigcap_{j=1}^{k} \hat{K}_n X_j) \cap I$, we have that $\mathcal{R} \subset \mathcal{S}$ and $\mathcal{S}$ is an abelian subalgebra of $I$. Now for all $\mathcal{Z} \in I$, we have

$\mathcal{Z}, \mathcal{X}_j = \begin{cases} 
0, & \text{if } l = 1, \ldots, k, \\
\sum_{j=1}^{k} v_j X_j, & \text{if } l = k + 1, \ldots, m.
\end{cases}$

because $X_j \in \mathcal{R}, \ j = 1, \ldots, k$ then $[\mathcal{Z}, \sum_{j=1}^{k} w_j X_j] \in \mathcal{S}$, thus $\mathcal{S}$ is an ideal of $I$. Therefore $I/\mathcal{S}$ is a nilpotent Lie algebra, thus $\mathcal{R}_1$ the center of $I_1 = I/\mathcal{S}$ is not trivial, i.e., there are $X_{k+1}, \ldots, X_{p_1} \in I/\mathcal{S}$ such that $\overline{X}_{k+1}, \ldots, \overline{X}_{p_1} \in \mathcal{R}_1$ are the generators the basis of $\mathcal{R}_1 \otimes \hat{K}_n$. Clearly $\{X_1, \ldots, X_{p_1}\}$ are linearly independent in $I \otimes \hat{K}_n$. Now as $\overline{X}_l \in \mathcal{R}_1 (l = k + 1, \ldots, p_1)$, we have $\sum_{j=1}^{k} f_{l,j} X_j = [\mathcal{Z}, \mathcal{X}_l + \sum_{j=1}^{k} w_{l,j} X_j] = [\mathcal{Z}, \mathcal{X}_l] + [\mathcal{Z}, \sum_{j=1}^{k} w_{l,j} X_j] = [\mathcal{Z}, \mathcal{X}_l] + \sum_{j=1}^{k} \mathcal{Z}(w_{l,j}) X_j$, then $[\mathcal{Z}, \mathcal{X}_l] = \sum_{j=1}^{k} (f_{l,j} - \mathcal{Z}(w_{l,j})) X_j = \sum_{j=1}^{k} u_{l,j} X_j, \ (l = k + 1, \ldots, p_1)$. On the other hand, if $\mathcal{Z} = \sum_{j=1}^{p_1} u_j X_j + \sum_{j=1}^{m} u_j Y_j \in I$ for any $Y_j \in I$ completing the basis, we have $X_j(u_r) = 0$ for $j = 1, \ldots, k$ and $r = 1, \ldots, m$, because $X_j \in \mathcal{R}, \ j = 1, \ldots, k$. Now for $l = k + 1, \ldots, p_1$, we have $\sum_{s=1}^{k} u_{s,s} X_s = [\sum_{s=1}^{k} u_{s,s} X_s, \mathcal{X}_s] = [\mathcal{X}_s, \mathcal{X}_s] - \sum_{s=1}^{k} \mathcal{X}_s(u_{s,s}) X_s = - \sum_{s=1}^{k} \mathcal{X}_s(u_{s,s}) X_s$, and the form $\mathcal{Z} = \sum_{j=1}^{k} f_{l,j} X_j = \sum_{j=1}^{k} u_{j,s} X_s - \sum_{j=1}^{k} f_{l,j} X_j = \sum_{j=1}^{k} f_{l,j} X_j = - \sum_{s=k+1}^{l} \mathcal{X}_s(u_{s,s}) X_s = 0$, and as the $X_j$ are linearly independent, we have $X_j(u_r) = 0$ for $r = k + 1, \ldots, p_1$ and $l = 1, \ldots, p_1$, (note that $X_i(u_r)$ for $r = 1, \ldots, k$ can be nonzero ). If $p_1 = m$ we have nothing more to proof, in the other case take $\mathcal{S}_1 = (\bigcap_{j=1}^{p_1} \overline{K}_n \overline{X}_j) \cap I_1$ we have that $\mathcal{R}_1 \subset \mathcal{S}_1$ and $\mathcal{S}_1$ is an abelian subalgebra of $I_1$ and analogously $\mathcal{S}_1$ is an ideal of $I_1$. Therefore $I_1/\mathcal{S}_1$ is a nilpotent Lie algebra, thus $\mathcal{R}_2$ the center of $I_2 = I_1/\mathcal{S}_1$ is not trivial, i.e., there are $\overline{X}_{p_1+1}, \ldots, \overline{X}_{p_2} \in \mathcal{R}_2$ such that $\overline{X}_{p_1+1}, \ldots, \overline{X}_{p_2} \in \mathcal{R}_2$ are the generators the basis of $\mathcal{R}_2 \otimes \hat{K}_n$. Clearly $\{X_1, \ldots, X_{p_2}\}$ are Linearly Independent in $I \otimes \hat{K}_n$, as $\overline{X}_l \in \mathcal{R}_1 (l = p_1 + 1, \ldots, p_2)$, we have $\sum_{j=1}^{p_1} f_{l,j} X_j = \sum_{j=k+1}^{p_1} \mathcal{Z}(X_j) + \sum_{j=k+1}^{p_1} h_{l,r} X_r = \begin{cases} \mathcal{Z}(X_l) + \sum_{j=k+1}^{p_1} w_{l,j} X_j, & \text{if } l = 1, \ldots, k, \\
\sum_{j=k+1}^{p_1} v_{l,j} X_j, & \text{if } l = k + 1, \ldots, m.
\end{cases}$
then \([Z, \mathcal{X}] = \sum_{j=1}^{m} u_{l,j} \mathcal{X}_j, (l = p_1 + 1, \ldots, p_2)\). Similarly we get the second item, repeating this process a finite number of times \((m < n)\) the lemma is proved.

As consequence this lemma we have:

Proof of Theorem 7.12. It is enough to prove that given a nilpotent subalgebra \(\mathfrak{l}\) of \(\hat{\mathcal{X}}(\mathbb{C}^n, 0)\) the \(\mathcal{L}\) has length \(\ell(\mathfrak{l})\) at most \(n\). If the dimension of \(\mathcal{R} \otimes \hat{K}_n\) is \(n\), then for all \(Z \in \mathfrak{l}, Z = \sum_{j=1}^{n} u_j \mathcal{X}_j\) and \(0 = [Z, \mathcal{X}_k] = \sum_{j=1}^{n} [u_j \mathcal{X}_j, \mathcal{X}_k] = \sum_{j=1}^{n} (u_j \mathcal{X}_j \mathcal{X}_k - \mathcal{X}_k (u_j \mathcal{X}_j)) = -\sum_{j=1}^{n} \mathcal{X}_k (u_j) \mathcal{X}_j, \) then \(\mathcal{X}_k (u_j) = 0\) for \((j, k = 1, \ldots, n)\). thus \(u_j\) are constants and therefore \(\mathfrak{l}\) is an abelian Lie algebra. Now if the dimension of \(\mathcal{R} \otimes \hat{K}_n\) is \(m\), where \(m\) is the dimension of \(\mathfrak{l} \otimes \hat{K}_n\) we have nothing to proof. Finally if the dimension of \(\mathcal{R} \otimes \hat{K}_n\) is \(k < m\) by the Lemma 7.1 we have for \(Z_1, Z_2 \in \mathfrak{l}, [Z_1, Z_2] = \left[ \sum_{j=1}^{m} u_j \mathcal{X}_j, \sum_{r=1}^{m} v_r \mathcal{X}_r \right] = \sum_{j=1}^{m} \sum_{r=1}^{m} [u_j \mathcal{X}_j, v_r \mathcal{X}_r] = \sum_{j=1}^{m} \sum_{r=1}^{m} \left( u_j \mathcal{X}_j (v_r) \mathcal{X}_r - v_r \mathcal{X}_r (u_j) \mathcal{X}_j + u_j \mathcal{X}_j [v_r, \mathcal{X}_r] \right) = \sum_{j=1}^{m} \sum_{r=1}^{m} u_j \mathcal{X}_j, \) because \(\mathcal{X}_j (u_m) = 0\) \((j = 1, \ldots, m)\) and \([Z, \mathcal{X}_j] = \sum_{j=1}^{l-1} v_j \mathcal{X}_j\). Thus all \(Z \in \mathfrak{l}^1\) is the form \(Z = \sum_{j=1}^{m-1} w_j \mathcal{X}_j\). Now if \(Z_1, Z_2 \in \mathfrak{l}^1\) we have \([Z_1, Z_2] = \left[ \sum_{j=1}^{m-1} u_j \mathcal{X}_j, \sum_{r=1}^{m-1} v_r \mathcal{X}_r \right] = \sum_{j=1}^{m-1} \sum_{r=1}^{m-1} [u_j \mathcal{X}_j, v_r \mathcal{X}_r] = \sum_{j=1}^{m-1} \sum_{r=1}^{m-1} \left( u_j \mathcal{X}_j (v_r) \mathcal{X}_r - v_r \mathcal{X}_r (u_j) \mathcal{X}_j + u_j \mathcal{X}_j [v_r, \mathcal{X}_r] \right) = \sum_{j=1}^{m-2} w_j \mathcal{X}_j, \) by the Lemma 7.1. Then all \(Z \in \mathfrak{l}^2\) is the form \(Z = \sum_{j=1}^{m-2} w_j \mathcal{X}_j\). Repeating this process at most \(m - 2\) times we can conclude.

Remark 7.2. In [13] a detailed study of the length for solvable subgroups of \(\hat{\mathcal{D}} \mathcal{I} \mathcal{F} \mathcal{F} \mathcal{S} (\mathbb{C}^n, 0)\) is found. The authors correct a statement from [2] and prove the following more general result:

Let \(G < \hat{\mathcal{D}} \mathcal{I} \mathcal{F} \mathcal{F} \mathcal{S} (\mathbb{C}^n, 0)\) be a solvable group. Then the soluble length of \(G\) is at most \(2n - 1 + \rho(n)\) where \(\rho : \mathbb{N} \to \mathbb{N}\) is the Newman function.

Part II - Groups with dicritical maps

8. Dicritic groups with abelian commutators

Unlike the one-dimensional case two commuting tangent to the identity diffeomorphisms may have different orders of tangency to the identity: take \(\hat{f} = \exp(\hat{X})\) and \(\hat{g} = \exp(\hat{Y})\), where the vector fields \(\hat{X}\) and \(\hat{Y}\) are given as in (1) above. This is the main reason why we do not have an equivalence between the concepts of metabelian, quasi-abelian and soluble groups in dimension \(n \geq 2\). From now on we shall take a closer look at this issue. Firstly, in this section, we investigate the characterization of quasi-abelian groups. For this we shall refer to the following concepts, which are two main notions in this paper.

Now we pave the way to Theorem 1.18. For the first part we shall need some lemmas below.

Lemma 8.1. Let \(\hat{X} = f(z) \hat{R}\) and \(\hat{Y} = g(z) \hat{R}\), where \(\hat{R}\) is the radial vector field and \(f\) and \(g\) are homogeneous polynomials of degree \(k\) respectively. Then \([\hat{X}, \hat{Y}] = 0\) if and only if \(k = s\).
Suppose that \( \hat{X} \in \hat{X}_{k-1}(\mathbb{C}^n,0) \) a dicritic vector field. For any vector field \( \hat{Y} \) with order greater than 2, such that \([\hat{X}, \hat{Y}] = 0\), we have that \( \hat{Y} \) is a dicritic vector field with order \( k+1 \).

**Lemma 8.3.** Let \( \hat{X}, \hat{Y} \in \hat{X}_k(\mathbb{C}^n,0), \ k \geq 2 \). Suppose that \( \hat{X} \) is regular dicritic and \( \hat{Y} \) is dicritic. If \([\hat{X}, \hat{Y}] = 0\), then there is \( c \in \mathbb{C} \setminus \{0\} \) such that \( \hat{Y} = c\hat{X} \).

**Proof.** Since \( \hat{X} \) and \( \hat{Y} \) are dicritic, then
\[
\hat{X} = f(z)R + (p^{(1)}_{k+2} + \cdots) \frac{\partial}{\partial z_1} + \cdots + (p^{(n)}_{k+2} + \cdots) \frac{\partial}{\partial z_n},
\]
\[
\hat{Y} = g(z)R + (q^{(1)}_{k+2} + \cdots) \frac{\partial}{\partial z_1} + \cdots + (q^{(n)}_{k+2} + \cdots) \frac{\partial}{\partial z_n},
\]
We have \([\hat{R}, g\hat{R}] = 0\), by Lemma 8.2. Since \([\hat{X}, \hat{Y}] = 0\), then the 2k + 2-jet to Lie bracket is
\[
[g\hat{R}, p^{(1)}_{k+2} \frac{\partial}{\partial z_1} + \cdots + p^{(n)}_{k+2} \frac{\partial}{\partial z_n}] - [f\hat{R}, q^{(1)}_{k+2} \frac{\partial}{\partial z_1} + \cdots + q^{(n)}_{k+2} \frac{\partial}{\partial z_n}] = 0
\]
Now note that
\[
[f\hat{R}, q^{(1)}_{k+2} \frac{\partial}{\partial z_1} + \cdots + q^{(n)}_{k+2} \frac{\partial}{\partial z_n}] = (k+1)f \cdot (q^{(1)}_{k+2} \frac{\partial}{\partial z_1} + \cdots + q^{(n)}_{k+2} \frac{\partial}{\partial z_n}) - (q^{(1)}_{k+2} \frac{\partial}{\partial z_1} + \cdots + q^{(n)}_{k+2} \frac{\partial}{\partial z_n})
\]
Then we have
\[(k+1)(f \cdot q^{(i)}_{k+2} - g \cdot p^{(i)}_{k+2}) = z_i(\nabla f \cdot Q_{k+2} - \nabla g \cdot P_{k+2})
\]
for \( i \in \{1, \ldots, n\} \), thus
\[
f \cdot q^{(i)}_{k+2} - g \cdot p^{(i)}_{k+2} = \frac{z_i}{z_{j_0}} f \cdot q^{(i)}_{k+2} - g \cdot p^{(i)}_{k+2}
\]
or equivalently, \( f \cdot (z_{j_0}q^{(i)}_{k+2} - z_{i_0}q^{(j)}_{k+2}) = g \cdot (z_{j_0}p^{(i)}_{k+2} - z_{i_0}p^{(j)}_{k+2}) \). But by hypothesis \( f \) and \( g \) have the same degree \( g = c \cdot f \) were \( c \in \mathbb{C}^* \). Thus the 2k + 2-jet of Lie bracket is:
\[
[f\hat{R}, (q^{(1)}_{k+2} + c p^{(1)}_{k+2}) \frac{\partial}{\partial z_1} + \cdots + (q^{(n)}_{k+2} + c p^{(n)}_{k+2}) \frac{\partial}{\partial z_n}] = 0
\]
Using the same argument of the previous lemma we have
\[(q_{k+2}^{(1)} - cp_{k+2}^{(1)})z_j = (q_{k+2}^{(j)} - cp_{k+2}^{(j)})z_1\]
so, \(q_{k+2}^{(1)} - cp_{k+2}^{(1)} = 0\), in consequence \(q_{k+2}^{(j)} - cp_{k+2}^{(j)} = 0\), for all \(j = 1, \ldots, n\), or
\[
(q_{k+2}^{(1)} - cp_{k+2}^{(1)}) \frac{\partial}{\partial z_1} + \cdots + (q_{k+2}^{(n)} - cp_{k+2}^{(n)}) \frac{\partial}{\partial z_n} = \frac{(q_{k+2}^{(1)} - cp_{k+2}^{(1)})}{z_1} R
\]
but this latter does not occur, because by the Lemma 8.1 we have \((q_{k+2}^{(1)} - cp_{k+2}^{(1)})\) has degree \(k\), and this is impossible because \(q_{k+2}^{(1)} - cp_{k+2}^{(1)}\) has degree \(k + 2\). Then, we have \(q_{k+2}^{(j)} = cp_{k+2}^{(j)}\), \(\forall j \in \{1, \ldots, n\}\).

Finally suppose that \(Q_{k+j} = cp_{k+j}\) for \(j = 1, \ldots, i\), the \((2k + i + 1)\)-jet of Lie bracket is
\[
[gR, p_{k+i+1}] = 0
\]
by the supposed the following sum is symmetric
\[
\sum_{j=2}^{i} [p_{k+j}^{(1)} \frac{\partial}{\partial z_1} + \cdots + p_{k+j}^{(n)} \frac{\partial}{\partial z_n}, q_{k+i+2-j}^{(1)} \frac{\partial}{\partial z_1} + \cdots + q_{k+i+2-j}^{(n)} \frac{\partial}{\partial z_n}] = 0
\]
Then similarly to the case \(k + 2\) we have that \(Q_{k+j+1} = cp_{k+j+1}\) therefore \(\dot{Y} = c\dot{X}\)

\[\square\]

**Remark 8.4.** We cannot exclude the regularity condition in the previous lemma, since the two vector fields of item 2.(d) in Example 2.3 are dicritic and commute, but they are not regular dicritic and are not \(C\)-linearly dependent.

The following proposition is found in [2].

**Proposition 8.5** ([2] Proposition 4.2). Let \(f \in Diff_{r+1}(\mathbb{C}^n, 0)\) and \(g \in Diff_{s+1}(\mathbb{C}^n, 0)\). Suppose that \(f\) is dicritic and commutes with \(g\). Then \(r = s\) and \(G\) is also dicritic.

We state now the main tool in the proof of Theorem 1.18

**Proposition 8.6.** Let \(G < Diff_{Id}(\mathbb{C}^n, 0)\) be a subgroup of diffeomorphisms tangent to the identity and \(f \in Diff_{Id}(\mathbb{C}^n, 0)\) a regular dicritic diffeomorphism. If \(f\) commutes with \(G\) then \(G < \langle \exp(t\dot{X}) \mid t \in \mathbb{C}\rangle\), where \(\dot{f} = \exp(\dot{X})\). In particular, \(G\) is abelian.

**Proof.** Let \(\hat{g} \in G\) be a diffeomorphism, from Proposition 8.5 \(\hat{g}\) is a dicritic diffeomorphism of same order than \(\dot{f}\), we say \(k + 1\). From the exponential bijection there is \(\dot{Y}\) such that \(\exp(\dot{Y}) = \hat{g}\). Then
\[
\dot{Y} = g(z)\dot{R} + (q_{k+2}^{(1)} + \cdots) \frac{\partial}{\partial z_1} + \cdots + (q_{k+2}^{(n)} + \cdots) \frac{\partial}{\partial z_n}
\]
and thus \(\dot{Y}\) is dicritic. Since \(\dot{f}\) commutes with \(\hat{g}\) it commutes with \(\hat{g}(z) = \exp(\dot{Y})(z)\) and from Lemma 2.4 \(\dot{f}\) commutes with \(\exp(t\dot{Y})(z)\) for all \(t \in \mathbb{C}\). Similarly, for each \(t\) we have that \(\exp(t\dot{Y})(z)\) commutes with \(\exp(s\dot{X})(z)\) and thus \([X, \dot{Y}] = 0\). From Lemma 2.3 there is \(r \in \mathbb{C}\) such that \(\dot{Y} = r\dot{X}\). Consequently \(\hat{g}(z) = \exp(r\dot{X})(z)\) and therefore \(G < \langle \exp(t\dot{X}) \mid t \in \mathbb{C}\rangle\)

The above proposition motivates the following:

**Definition 8.7** (Unimodular diffeomorphism). A formal diffeomorphism tangent to the identity \(\hat{f} \in Diff(\mathbb{C}^n, 0)\) is **unimodular** if its centralizer in \(Diff_{Id}\) is \(\mathbb{C} = \{\hat{f}^t, t \in \mathbb{C}\}\), i.e., generated by \(\hat{f}\).
Proof of Theorem 1.18. Let $G < \hat{\text{Diff}}(\mathbb{C}^n,0)$ be a subgroup containing a regular dicritic diffeomorphism $\hat{f} = \exp(\hat{X}) \in G$. First, suppose that $G$ is quasi-abelian. From Proposition 8.6 we have $G_{\text{id}} < \langle \exp(t\hat{X}) \mid t \in \mathbb{C} \rangle$. Let $\hat{g} \in G$, as $\hat{f} \in G_{\text{id}}$ then $[\hat{f}, \hat{g}] \in G_{\text{id}}$. Thus there is $t_{\hat{g}} \in \mathbb{C}^*$ such that $[\hat{f}, \hat{g}] = \exp(t_{\hat{g}}\hat{X})$, then $\hat{g} \circ \hat{f} \circ \hat{g}^{-1} \circ \hat{f}^{-1} = \exp(t_{\hat{g}}\hat{X})$ so that
\[
\hat{g} \circ \exp(\hat{X}) \circ \hat{g}^{-1} = \exp(t_{\hat{g}}\hat{X}) \circ \exp(\hat{X}) = \exp((t_{\hat{g}} + 1)\hat{X}) = \exp(c_{\hat{g}}\hat{X})
\]
from the same argument used in the proof of Lemma 2.1 we have $\forall s \in \mathbb{C}$, $\hat{g} \circ \exp(s\hat{X}) \circ \hat{g}^{-1} = \exp(s\hat{c}_{\hat{g}}\hat{X})$. Therefore $\hat{g}^*\hat{X} = c_{\hat{g}}\hat{X}$, $\forall \hat{g} \in G$. Conversely, suppose that $\hat{X}$ is projectively invariant with respect to the group $G$. Therefore, for each $\hat{g} \in G$ there exists $c_{\hat{g}}$ such that $\hat{g}^*\hat{X} = c_{\hat{g}}\hat{X}$. We claim that $\forall \hat{g} \in G_{\text{id}}$, $c_{\hat{g}} = 1$. In fact, if $\hat{f}(z) = z + f(z)z + \cdots$ then $\exp(c_{\hat{g}}\hat{X})(z) = z + c_{\hat{g}}f(z)z + \cdots$. Thus if $\hat{g} \in G_{\text{id}}$, $\hat{g} \circ \hat{f}(z) = z + f(z)z + \hat{g}_{k+1}(z) + \cdots$ and $\exp(c_{\hat{g}}\hat{X})(z) \circ \hat{g}(z) = z + c_{\hat{g}}f(z)z + \hat{g}_{k+1}(z) + \cdots$, then $c_{\hat{g}} = 1$. Consequently $\forall \hat{g} \in G_{\text{id}}$, $\hat{g}^*\hat{X} = \hat{X}$, i.e., $G_{\text{id}}$ commutes with $\hat{f}$. From Proposition 8.6 $G_{\text{id}}$ is abelian, i.e., $G$ is quasi-abelian.

In the same way as Theorem 1.18 we have:

Proof of Proposition 1.21. It is immediate to verify that (1) $\Rightarrow$ (2). Let us now prove (2) $\Rightarrow$ (1). Since $DG < G$, for all $\hat{g} \in G$ we have that $\hat{g} = D\hat{g}^{-1}(0) \circ \hat{g} \in G_{\text{id}}$. From (2) we have that $G$ commutes with $\hat{f}$, then $D\hat{g}^{-1}(0) \circ \hat{g} = \exp(c_{\hat{g}}\hat{X})$, therefore $\hat{G} = D\hat{g}(0) \circ \exp(c_{\hat{g}}\hat{X})$, $\forall \hat{g} \in G$. Now let $\hat{g}, \hat{h} \in G$ be diffeomorphisms, as $DG < G$ and from (2), we have that:
\[
\hat{g} \circ \hat{h} = D\hat{g}(0) \circ \exp(c_{\hat{g}}\hat{X}) \circ D\hat{h}(0) \circ \exp(c_{\hat{h}}\hat{X}) = \hat{h} \circ \hat{g}
\]
Therefore $G$ is abelian.

9. Metabelian groups with dicritic elements

Now we shall study metabelian subgroups of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ and prove Theorem 1.21 We strongly rely on the preceding argumentation. The main step is:

Proposition 9.1. Let $G < \hat{\text{Diff}}(\mathbb{C}^n,0)$ be a subgroup with $DG$ abelian, and $\hat{f} = \exp(\hat{X}) \in G$ a regular dicritic diffeomorphism. Then $\hat{X}$ is projectively invariant by $G$ if, and only if, $\hat{f}$ commutes with $[G,G]$.

Proof. Since $DG$ is abelian the subgroup of commutators of $G$ is tangent to the identity i.e., $[G,G] < G_{\text{id}}$. If $\hat{f}$ commutes with $[G,G]$, from Proposition 8.6 we have that $[G,G] < \langle \exp(t\hat{X}) \mid t \in \mathbb{C} \rangle$ and from the proof of Theorem 1.18 we know that $\hat{X}$ is projectively invariant by $G$. Assume now that the vector field $\hat{X}$ is projectively invariant by $G$. Once again, since $[G,G] < G_{\text{id}}$ we know that if $S \in [G,G]$ then $c_S = 1$. Thus $\forall S \in [G,G]$, $S^*\hat{X} = \hat{X}$, therefore $\hat{f} = \exp(\hat{X})$ commutes with $[G,G]$.

Proof of Theorem 1.21. Notice that if $G$ is quasi-abelian and $DG$ is abelian then $G$ is metabelian. Therefore, Theorem 1.21 follows from Propositions 8.6 and 9.1.

Next we prove the announced partial converse of (2) in Theorem 1.21

Proof of Proposition 1.22. By hypothesis $[G,G]$ is an abelian subgroup of diffeomorphisms tangent to the identity and we have that
\[
[\hat{f}, \hat{h}](z) = z + \lambda(\lambda k - 1)f(z)z + \lambda(\lambda k + 1) - 1)P_{k+2}(z) + \cdots
\]
Similarly we have:
\[
\hat{X} = \lambda(\lambda^{k} - 1)f(z)R + (\lambda(\lambda^{k+1} - 1)P_{k+2}^{(1)} + \cdots)\frac{\partial}{\partial z_{1}} + \cdots + (\lambda(\lambda^{k+1} - 1)P_{k+2}^{(n)} + \cdots)\frac{\partial}{\partial z_{n}}
\]
since \( \hat{f} \) is regular dicritic and \( \lambda^{k} \neq 1, \lambda^{k+1} \neq 1 \), the above expression implies that \([\hat{f}, \hat{h}]\) is regular dicritic. According to Proposition 9.1 there is a projectively invariant formal vector field \( \hat{X} = \exp([\hat{f}, \hat{h}]) \).

Now we give an application of our results:

**Corollary 9.2.** Let \( G = \langle \hat{f}, \hat{h} \rangle < \text{Diff}(\mathbb{C}^{n}, 0) \), where \( \hat{f} \) is regular dicritic and \( \hat{h}(z) = \lambda z, \lambda \neq 1, \lambda^{k+1} \neq 1 \). The group \( G \) is metabelian if and only if \([\hat{f}, \hat{h}]\) and \([\hat{f}^{2}, \hat{h}]\) commute with \([\hat{f}, \hat{h}]\).

**Proof.** Let us first introduce some notation. Given two elements \( \hat{\varphi}, \hat{\psi} \in \text{Diff}(\mathbb{C}^{2}, 0) \) we shall write \( \hat{\varphi}^* \hat{\psi} := \hat{\varphi} \circ \hat{\psi} \circ \hat{\varphi}^{-1} \).

If \( G < \text{Diff}(\mathbb{C}^{2}, 0) \) is metabelian, it is immediately seen that \([\hat{f}, \hat{h}^{2}]\) and \([\hat{f}^{2}, \hat{h}]\) commute with \([\hat{f}, \hat{h}]\). We prove the converse, in fact we have that \([\hat{f}, \hat{h}]\) is regular dicritic as noted above. Now, as \([\hat{f}^{2}, \hat{h}^{2}], [\hat{f}^{2}, \hat{h}] \) commutes with \([\hat{f}, \hat{h}]\), we have \([\hat{f}^{2}, \hat{h}] = \exp(c\hat{X}) \) and \([\hat{f}, \hat{h}^{2}] = \exp(r\lambda \hat{X}) \), since \( \hat{f}^{*}[\hat{f}, \hat{h}] \circ [\hat{f}, \hat{h}] = [\hat{f}^{2}, \hat{h}] \) and \([\hat{f}, \hat{h}] \circ \hat{h}^{*}[\hat{f}, \hat{h}] = [\hat{f}, \hat{h}^{2}] \) then

\[
\exp(c\hat{X}) = [\hat{f}^{2}, \hat{h}] = \hat{f}^{*}[\hat{f}, \hat{h}] \circ [\hat{f}, \hat{h}] = \hat{f}^{*} \exp(\hat{X}) \circ \exp(\hat{X})
\]
consequently, \( \hat{f}^{*} \exp(\hat{X}) = \exp(c\hat{X}) \circ \exp(-\hat{X}) = \exp(c\hat{X}) \). Using the same argument in the proof of Lemma 21, \( \hat{f}^{*} \hat{X} = c \hat{X} \). Similarly \( h^{*} \hat{X} = r \hat{X} \). Thus by Theorem 1.21 the group \( G \) is metabelian.

---

**10. Solvable groups with some dicritic element**

The next three lemmas will be used in the proof of Theorem 1.23.

**Lemma 10.1.** Let \( \hat{f} \in \text{Diff}_{r}(\mathbb{C}^{n}, 0) \) and \( \hat{g} \in \text{Diff}_{s}(\mathbb{C}^{n}, 0) \) be formal diffeomorphisms. Then

\[
\hat{f}(\hat{g}(z)) - \hat{g}(\hat{f}(z)) = D\hat{f}_{r+1}(z)\hat{g}_{s+1}(z) - D\hat{g}_{s+1}\hat{f}_{r+1}(z) + O(|z|^{r+s+2})
\]
so \([\hat{f}, \hat{g}] = \text{Id} \) or \([\hat{f}, \hat{g}] \in \text{Diff}_{p}(\mathbb{C}^{n}, 0) \) with \( p \geq r + s \).

**Proof.** Let \( \hat{f}(z) = z + \sum_{k=r}^{r+s} \hat{f}_{k+1}(z) + O(|z|^{r+s+2}) \) and \( \hat{g}(z) = z + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + O(|z|^{r+s+2}) \) then:

\[
\hat{f}(\hat{g}(z)) = z + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + O(|z|^{r+s+2}) + \\
+ \sum_{k=r}^{r+s} \hat{f}_{k+1}(z) + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + O(|z|^{r+s+2}) + O(|z|^{r+s+2})
\]

\[
= z + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + \sum_{k=r}^{r+s} (\hat{f}_{k+1}(z) + D\hat{f}_{k+1}\hat{g}_{s+1}(z) + O(|z|^{k+s+2}))
\]

\[
+ O(|z|^{r+s+2})
\]

\[
= z + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + \sum_{k=r}^{r+s} \hat{f}_{k+1}(z) + D\hat{f}_{r+1}(z)\hat{g}_{s+1}(z) + O(|z|^{r+s+2})
\]

Similarly we have:
Lemma 10.3. Let \( \hat{f} \in \text{Diff}_r(\mathbb{C}^n, 0) \) and \( \hat{g} \in \text{Diff}_s(\mathbb{C}^n, 0) \) be dicritic diffeomorphisms with \( r \neq s \), given by
\[
\hat{f}(z) = z + f(z)z + \cdots \quad \text{and} \quad \hat{g}(z) = z + g(z)z + \cdots
\]
then \([\hat{f}, \hat{g}] \in \text{Diff}_{s+r}(\mathbb{C}^n, 0)\) is dicritic and given by
\[
[\hat{f}, \hat{g}](z) = z + (r-s)g(z)f(z)z + \cdots
\]

Proof. From Lemma 10.1 the term of smaller order of \([\hat{f}, \hat{g}]\) is
\[
D\hat{f}_{r+1}(z)\hat{g}_{s+1}(z) - D\hat{g}_{s+1}\hat{f}_{r+1}(z) = (f(z)I + (z_i \frac{\partial f}{\partial z_j}))\hat{g}_{s+1}(z) - D\hat{g}_{s+1}f(z)z
\]
\[
= (f(z)I + (z_i \frac{\partial f}{\partial z_j}))\hat{g}_{s+1}(z) - (s+1)f(z)\hat{g}_{s+1}
\]
\[
= (-sf(z)I + (z_i \frac{\partial f}{\partial z_j}))\hat{g}_{s+1}(z)
\]
\[
= (-sf(z)I + (z_i \frac{\partial f}{\partial z_j}))g(z)z
\]
Then the i-th component of this is
\[
-sf(z)g(z)z_i + g(z)z_i \nabla f(z)z = -sf(z)g(z)z_i + rf(z)g(z)z_i
\]
thus
\[
D\hat{f}_{r+1}(z)\hat{g}_{s+1}(z) - D\hat{g}_{s+1}\hat{f}_{r+1}(z) = (r-s)f(z)g(z)z
\]
therefore \([\hat{f}, \hat{g}] \in \text{Diff}_{s+r+1}(\mathbb{C}^n, 0)\) and this is dicritic, given by
\[
\hat{h}(z) = z + (r-s)g(z)f(z)z + \cdots
\]

Lemma 10.3. If a subgroup \( G < \text{Diff}(\mathbb{C}^n, 0) \) contains two elements tangent to the identity with different orders then \( G \) is not a solvable group.

Proof. Assume that there are \( \hat{f}_1, \hat{f}_2 \in G \) dicritic diffeomorphisms of different orders, we say \( p_1 + 1 \) and \( p_2 + 1 \) respectively then by above lemma \( \hat{f}_3 = [\hat{f}_1, \hat{f}_2] \) is dicritic of order \( p_3 = p_1 + p_2 + 1 > p_2 + 1 \), similarly, we have that \( \hat{f}_4 = [\hat{f}_3, \hat{f}_2] \) is dicritic of order \( p_4 = p_3 + p_2 > p_3 \) and recurrently \( \hat{f}_n = [\hat{f}_{n-1}, \hat{f}_{n-2}] \) is dicritic of order \( p_n = p_{n-1} + p_{n-2} > p_{n-1} \), thus there is no \( n \in \mathbb{N} \) such that \( G^{(n)} = \{\text{Id}\} \) and this contradicts the fact \( G \) is solvable.

Proof of Theorem 1.22. Let \( G < \text{Diff}(\mathbb{C}^n, 0) \) be a subgroup of diffeomorphisms tangent to the identity containing a dicritic diffeomorphism \( \hat{f} \) with order of tangency \( k \). It is immediate to verify that (1) \( \Rightarrow \) (2). Let us now prove (2) \( \Rightarrow \) (3). Suppose that \( \hat{f}(z) = z + f(z)z + \cdots \). Suppose by contradiction that there is \( \hat{f}^{(1)} \in G \) with order of tangency \( p_k > k \) then obtain
\[
\hat{f}^{(2)} = [\hat{f}^{(1)}, \hat{f}] = z + f^{(2)}_{k_2} + \cdots
\]

We affirm that \( f^{(2)}_{k_2} \neq 0 \) and thus \( f^{(2)} \) has order of tangent \( k_2 = k + k_1 > k_1 + 1 \). In fact, as the j-th coordinate of \( f^{(2)}_{k_2} \), is \( (k_1 - 1)f.q_{k_1}^{(j)} - (\nabla f.Q_{k_1})z_j \), where \( f^{(1)} = z + Q_{k_1} + \cdots \) and \( Q_{k_1} = (q_{k_1}^{(1)}, \ldots, q_{k_1}^{(n)}) \), (in consequence \( k_2 = k + k_1 \), now if \( f^{(2)}_{k_2} = 0 \), then \( (k_1 - 1)f.q_{k_1}^{(j)} = (\nabla f.Q_{k_1})z_j \), for \( j = 1, \ldots, n \). So following the same argument of Lemma 5.2 we have that
\( Q_{k_1} = (g \cdot z_1, \ldots, g \cdot z_n) \), with \( g \) homogeneous polynomial of degree \( p \), thus \( Q_{k_1} \) has degree \( k + 1 \), but this is impossible. Repeating this process we can define:

\[
\hat{f}^{(n)} = [\hat{f}^{(n-1)}, \hat{f}] = z + \hat{f}^{(n)} + \cdots.
\]

Analogously \( \hat{f}^{(n)} \neq 0 \) and thus \( \hat{f}^{(n)} \) has order of tangency \( k_n = k + k_{n-1} > k_{n-1} + 1 > n \), thus there is no \( n \in \mathbb{N} \), such that \( \mathcal{C}^n(G) = \{\text{Id}\} \), what contradicts the fact that \( G \) is nilpotent.

Therefore, we have \( G < \text{Diff}_{k+1}(\mathbb{C}^n,0) \).

Now we prove \( (3) \Rightarrow (1) \). From Lemma 10.1 we have that for \( \hat{h}, \hat{g} \in G, [\hat{h}, \hat{g}] = \{\text{Id}\} \) or \( [\hat{h}, \hat{g}] \in \text{Diff}_\ell(\mathbb{C}^n,0), \ell \geq 2k + 1 \) thus \( [\hat{h}, \hat{g}] = \{\text{Id}\} \) and therefore \( G \) is abelian. \( \square \)

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