LIOUVILLE AND LOGARITHMIC ACTIONS IN LAPLACIAN GROWTH

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Abstract. We discuss and construct an action functional (logarithmic action) for the simply connected Laplacian growth and obtain its variation. This variation admits various interpretations. In particular, we consider a general smooth subordination evolution and give connections with the Virasoro algebra and Neretin polynomials.

1. Introduction

An important quantity in mechanics and the field theory is the action, a local functional over maps \( \phi \) between the space-time \( \Sigma \) and the target space which we assume to be the Euclidean straight line \( \mathbb{R} \). For example, one may consider the action given by the Dirichlet integral

\[
S[\phi] = \int_\Sigma |\nabla \phi|^2 dv,
\]

where the metric structure of \( \Sigma \) and the volume element \( dv \) are to be taken into account as well as certain properties of smoothness of \( \phi \), that give sense to the right-hand side of the above equality. In the classical setup of mechanics this action represents the energy of the system as infinitesimally it is just the scalar product of the field \( \nabla \phi \) and its momentum \( \nabla \phi dv \) for the potential \( \phi \). We note that in general, the action is the integral over the classical Lagrangian. If time does not enter explicitly into the Lagrangian, then the system is closed, and a typical example of such a Lagrangian is the kinetic energy minus the potential energy. One may assume different smooth functions on \( T\Sigma \) as Lagrangians, however physical or geometrical background of the underlying space \( \Sigma \) forces certain restrictions in the choice. Nevertheless, different points of view on the same object can lead to different functionals as Lagrangians.

The classical field theory studies the extremum of the action functional, and its critical value is called the classical action. The critical point \( \phi^* \) satisfies Hamilton’s principle (or the principle of the least action), i.e., \( \delta S[\phi^*] = 0 \), which is the Euler-Lagrange equation for the variational problem defined by the action functional. For the action given by the

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Dirichlet integral the classical action is achieved for the harmonic $\phi^*$ and the principle of the least action leads to the Laplacian equation $\Delta \phi = 0$.

The Liouville action plays a key role in two-dimensional gravity. It is based on a Lagrangian given on the base of the Riemannian geometry of the underlying space. Liouville action describes a highly non-trivial dynamics in quantum field theory and appears in connection with Feynman’s path integral that represents the transition amplitude between two quantum states of a system expressed as a sum over contributions from possible classical histories of that system. Complex transition functions appear naturally in the theory of evolution equations. The original formulation of quantum Liouville theory through path integral has been obtained by Polyakov [29] in 1981 where the domain of integration consisted of all smooth conformal metrics $ds^2$ on an $n$-punctured Riemann sphere ($n \geq 4$ to guarantee the hyperbolicity). A thorough mathematical treatment has been made later by Takhtajan and Zograf (see [36], [33], [43], [44], [45]).

It is not very surprising that several “quantum features” appear in non-linear problems of hydrodynamics, in particular, in the Laplacian Growth problem. In 1898 Hele-Shaw [15] proposed his famous cell that was a device for investigating a flow of viscous fluid in a narrow gap between two parallel plates.

The dimensionless model of a moving viscous incompressible fluid in the Hele-Shaw cell is described by a potential flow with the velocity field $\mathbf{V} = (V_1, V_2)$. The pressure $p$ is the potential for the fluid velocity

$$\mathbf{V} = -\frac{h^2}{12\mu} \nabla p,$$

where $h$ is the cell gap and $\mu$ is the viscosity of the fluid (see, e.g. [27, 34]).

Given an incompressible advancing fluid injected through a point source, the Laplacian growth is formulated as a moving boundary problem for the Laplacian equation for a function $p(z, t)$, supported in a domain $\Omega(t) \subset \mathbb{C}$ as a function of $z$,

$$\Delta p = -2\pi \delta_0(z), \quad z \in \Omega(t),$$

where $t$ is the time variable and $\delta_0$ is the Dirac distribution supported in 0. The dynamic boundary condition is given by putting

$$p \bigg|_{\partial \Omega(t)} = 0,$$

and the kinematic condition for the motion of the boundary $\partial \Omega(t)$ is given by the normal velocity as

$$v_n = -\frac{\partial p}{\partial n}, \quad x \in \partial \Omega(t),$$

where $n$ is the outward unit vector to $\partial \Omega(t)$. 
Through the similarity in the governing equations, Hele-Shaw flows can be used to study models of saturated flows in porous media governed by Darcy’s law. Over the years various particular cases of such a flow have been considered. Different driving mechanisms were employed, such as surface tension or external forces (suction, injection). We mention here a 600-paper bibliography of free and moving boundary problems for Hele-Shaw and Stokes flows since 1898 up to 1998 collected by Gillow and Howison [8].

As it has been shown in [11, 25, 49], the Laplacian growth problem can be embedded into a larger hierarchy of domain variations (Whitham-Toda hierarchy) for which all Richardson’s complex moments [33] are treated as independent variables (generalized time variables), and form an integrable system. Finally, the Laplacian growth has been modeled in the moduli space of Riemann surfaces [21].

In the classical simply connected case of the Laplacian growth without gravity the kinetic energy is given by the Dirichlet integral for the pressure as a potential. However, given the evolution of the phase domain in time as a closed system with a Riemannian metric as the geometrical background, one may construct the action functional based on a different Lagrangian. The idea of the construction of the Liouville action gives us a way to derive an action functional for the Laplacian growth. The aim of this paper is to construct the logarithmic action for the Laplacian growth and to obtain its variation. Then we shall study a general smooth subordination and interpret the variation of the logarithmic action through the infinitesimal version of the action of the Virasoro-Bott group on the space of analytic univalent functions.

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2. Liouville and logarithmic actions

Following the classical approach by Poicaré [28] we consider a compact Riemann surface $S$ of genus $g \geq 2$ that admits uniformization by a Fuchsian group $G$ acting in the unit disk $U$, $S = U/G$. If $z$ is a local analytic coordinate defined on an open set, then the Riemann metric $ds^2$ is represented as $ds^2 = \rho^2 |dz|^2$ for a density $\rho$. A conformal metric corresponds to the invariance under the change of the local coordinate. If $z'$ is another local coordinate defined in an open set and $ds^2 = (\rho')^2 |dz'|^2$ in terms of the local coordinate $z'$, then we have $\rho(z) \equiv \rho'(z'(z)) |dz'/dz|$ in the intersection of these sets.

The Gaussian (sectional) curvature $\kappa$ of this metric is calculated by the formula $\kappa = -\frac{1}{\rho^2} \Delta \log \rho$. It follows from the uniformization theorem that there exists a unique conformal metric of Gaussian curvature $\kappa = -1$ which is called the Poicaré (hyperbolic) metric. In terms of the parameter $z$ this is equivalent to the Liouville equation

$$\varphi_{zz} = \frac{1}{2} e^\varphi,$$
where $\varphi = \log \rho^2$. In the simplest case of the unit disk the Poincaré metric is given as

$$ds^2 = \frac{4|d\zeta|^2}{(1 - |\zeta|^2)^2}.$$  

Considering the universal covering of $S$ by the unit disk and the automorphic (with respect to $G$) projection $h : U \to S$ we deduce that

$$ds^2 = \frac{4|h^{-1}|^2|d\zeta|^2}{(1 - |h^{-1}|^2)^2},$$

where $\zeta$ is taken from the fundamental polygon $U/G$. This metric is complete and the area of $S$ in this metric is $4\pi(g - 1)$ (by the Gauss-Bonnet theorem).

It seems that equation (4) is the Euler-Lagrange equation for the variational problem defined by the functional

$$S[\phi] = \int_{U/G} (|\phi_z|^2 + e^\phi) d\sigma_z,$$

where $d\sigma_z = \frac{|dz \wedge d\bar{z}|}{2}$ which may be chosen as the Liouville action. However, the deal is much more difficult, because the integrant (the first term) does not keep the local 2-form $|\phi_z|^2 dz \wedge d\bar{z}$ invariant under the change of the local parameter from chart to chart on the Riemann surface. Thus, the functional (5) is well-defined only for simply connected domains and the correct Liouville action requires an additional term which has been given by Takhtajan and Teo in [37].

When the underlying Riemann surface have singularities (e.g., punctures, branch points) the action functional is not well defined globally on the surface either. In this case, there are two options to treat the problem. One of them is to change the metric, the other is to make certain regularization. Actually, Poincaré [28] worked in the presence of parabolic singularities. In this case some additional terms appear in (5), see [45]. But certainly one may look for equilibrium between these two options.

Let us turn now to the Laplacian growth. As it was said in Introduction, the problem (1–3) defines the Laplacian growth. More thoroughly we give a strong formulation of this problem. Let $\Omega(t) \subset \mathbb{C}$, $0 \in \Omega(t)$, be a one-parameter family of bounded domains. We call the family $\{\Omega(t)\}$ smooth if $\partial \Omega(t)$ are smooth ($C^\infty$) interfaces for each $t$, and the normal velocity $v_n$ continuously depends on $t$ at any point of $\partial \Omega(t)$. Each $\Omega(t)$ is supposed to be simply connected for any $t \in [0, T)$ fixed. A smooth family $\Omega(t)$, $0 \leq t < T$, as above, is said to be a strong solution for the Laplacian growth if there exists a potential $p(z, t)$, $z \in \Omega(t)$, such that all conditions (1–3) are satisfied. The family of $\Omega(t)$ forms a strong subordination chain of bounded domains: $\Omega(s) \subset \Omega(t)$ for $0 \leq s < t < T$, $0 \in \Omega(0)$. 


It is known that if the initial domain $\Omega(0)$ has an analytic smooth boundary, then the strong solution exists locally in time until the boundary $\partial\Omega(t)$ develops a cusp in a blow-up time or $\Omega(t)$ changes its topology (see, e.g. [16, 32, 46, 48]).

By the Riemann theorem we construct a conformal time-dependent map $z = f(\zeta, t)$ from the unit disk $U$ onto the phase domain $\Omega(t)$, $f(0, t) = 0$, $f'(0, t) > 0$. The function $f(\zeta, 0) = f_0(\zeta)$ parameterizes the initial boundary $\partial\Omega(0) = \{f_0(e^{i\theta}), \theta \in [0, 2\pi]\}$ and the moving boundary is parameterized by $\partial\Omega(t) = \{f(e^{i\theta}, t), \theta \in [0, 2\pi]\}$. We use the notations $\dot{f} = \partial f/\partial t$, $f' = \partial f/\partial \zeta$.

Let us consider the complex potential $W(z, t)$, $\Re W = p$. For each fixed $t$ it is a multivalued analytic function defined in $\Omega(t)$ whose real part solves the Dirichlet problem (1–2). Making use of the Cauchy-Riemann conditions we deduce that $\frac{\partial W}{\partial z} = i \frac{\partial p}{\partial y} - \frac{\partial p}{\partial x}$, $z = x + iy$.

Since Green’s function solves (1,2), we have the representation

$$W(z, t) = -\log z + w_0(z, t),$$

where $w_0(z, t)$ is an analytic regular function in $\Omega(t)$. Because of the conformal invariance of Green’s function we have the superposition

$$(W \circ f)(\zeta, t) = -\log \zeta,$$

and the conformally invariant complex velocity is just $W'(z, t) = -\frac{f^{-1}'}{f^{-1}}(z, t)$, where $\zeta = f^{-1}(z, t)$ is the inverse to our parametric function $f$ and prime means the complex derivative. Rewriting this relation we get

$$(W'(z, t) \, dz)^2 = \frac{d\zeta^2}{\zeta^2}.$$  

The velocity field $(-\nabla p)$ is the conjugation of $(-W')$. In other words the velocity field is directed along the trajectories of the quadratic differential in the left-hand side of (7) for each fixed moment $t$. The equality (7) implies that the boundary $\partial\Omega(t)$ is the orthogonal trajectory of the differential $(W'(z, t) \, dz)^2$ with a double pole at the origin. The dependence on $t$ yields that the trajectory structure of this differential changes in time, and in general, the stream lines are not inherited in time. These lines are geodesic in the conformal metric $|W'(z, t)||dz|$ generated by this differential.

Observe that being thought of as a Riemann surface, the phase domain $\Omega(t) \setminus \{0\}$ is hyperbolic and it admits the Poincaré metric with constant negative curvature

$$ds^2 = \frac{|f^{-1}'|^2}{|f^{-1}'|^2 \log^2 |f^{-1}'| |dz|^2}.$$  

The asymptotics about the origin and close to the hyperbolic boundary implies that the standard expression (5) for the Liouville action can not be used any more. Moreover,
the action integral for the hyperbolic metric in the punctured unit disk has the following asymptotics

\[
\int_{U_\varepsilon} (|\phi_z|^2 + e^\phi) d\sigma_z \sim -2\pi \log \varepsilon_1 - 4\pi \log \log \epsilon_1 - \frac{4\pi}{\log(1 - \varepsilon_2)},
\]

where \( U_\varepsilon = \{ z : \varepsilon_1 < |z| < 1 - \varepsilon_2 \} \). Therefore, the corresponding integral for \( \Omega(t) \setminus \{ 0 \} \) will have a similar asymptotics plus terms containing \( f'(0, t) \) and the boundary distortion (in our case the boundary derivatives) at the unit circle that makes it difficult to operate with.

Let us use the flat logarithmic metric instead generated by (7) which seems to be more natural for the Laplacian growth

\[
ds^2 = \frac{|f^{-1}|'^2}{|f^{-1}|^2} |dz|^2 = |W'|^2 |dz|^2.
\]

The hyperbolic boundary is not singular for this metric whereas the origin is. But it is a parabolic singularity which can be easily regularized.

The density of this metric satisfies the usual Laplacian equation \( \varphi_{zz} = 0 \) in \( \Omega(t) \setminus \{ 0 \} \), where \( \varphi(z) = \log \frac{|f^{-1}|'^2}{|f^{-1}|^2} \). Obviously, the Laplacian equation is the Euler-Lagrange equation for the variational problem defined by the Dirichlet integral

\[
\int_D |\phi_z|^2 d\sigma_z,
\]

locally for any measurable set \( D \subset \Omega(t) \setminus \{ 0 \} \). However, this functional cannot be defined globally in \( \Omega(t) \setminus \{ 0 \} \) because of the parabolic singularity at the origin. To overcome this obstacle we define the classical action in the following way. Let \( \Omega_\varepsilon(t) = \Omega(t) \setminus \{ z : |z| \leq \varepsilon \} \) for a sufficiently small \( \varepsilon, U_\varepsilon = \{ \zeta : \varepsilon < |\zeta| < 1 \} \). The function \( \varphi \) possesses the asymptotics

\[
\varphi \sim \log \frac{1}{|z|^2}, \quad |\varphi_z| \sim \frac{1}{|z|^2} \quad \text{as} \quad z \to 0.
\]

Therefore, the finite limit

\[
(8) \quad \lim_{\varepsilon \to 0} \left\{ \int_{\Omega_\varepsilon(t)} |\varphi_z|^2 d\sigma_z + 2\pi \log \varepsilon \right\} =: S[\varphi]
\]

exists and we call it the logarithmic action for the Laplacian growth.

**Lemma 1.** The Euler-Lagrange equation for the variational problem for the logarithmic action \( S[\varphi] \) is the Laplacian equation \( \Delta \phi = -4\pi \delta_0(z), z \in \Omega(t), \) where \( \delta_0(z) \) is the Dirac
distribution supported at the origin, where \( \phi \) is taken from the class of twice differentiable functions in \( \Omega(t) \setminus \{0\} \) with the asymptotics \( \phi \sim -\log|z|^2 \) as \( z \to 0 \).

**Proof.** Let us consider first the integral

\[
S_\varepsilon[\phi] = \int_{\Omega_\varepsilon(t)} |\varphi_z|^2 \sigma_z = \int_{\Omega_\varepsilon(t)} \chi_{\Omega_{\varepsilon}(t)} |\varphi_z|^2 d\sigma_z,
\]

where \( \chi_{\Omega_{\varepsilon}(t)} \) is the characteristic function of \( \Omega_{\varepsilon}(t) \). Then, due to Green’s theorem,

\[
\lim_{h \to 0} \frac{S_\varepsilon[\phi + hu] - S_\varepsilon[\phi]}{h} = 2 \int_{\Omega_\varepsilon(t)} \chi_{\Omega_{\varepsilon}(t)} \Re \phi_z \overline{u_z} d\sigma_z
\]

in distributional sense for every \( C^\infty(\mathbb{C}) \) test function \( u \) supported in \( \Omega(t) \). On the other hand, we have \( \partial \phi / \partial n \sim -2/\varepsilon \) as \( \varepsilon \to 0 \) and \( u = 0 \) on \( \partial \Omega(t) \). Therefore, the expression \((9)\) tends to

\[
-\frac{1}{2} \int_{\Omega(t)} u \Delta \phi d\sigma_z - 2\pi u(0),
\]

as \( \varepsilon \to 0 \), and the latter must vanish, that is equivalent to the Laplacian equation mentioned in the statement of the lemma. Obviously, the logarithmic term in the definition of \( S[\phi] \) does not contribute into the variation. \( \square \)

Straightforward calculation gives

\[
\varphi_z = -\frac{1}{f'(t)} \left( \frac{f''}{f'} + \frac{1}{\zeta} \right) \circ f^{-1}(z, t).
\]

Hence, the action \( S \) can be expressed in terms of the parametric function as

\[
S[\varphi] \equiv S[f] = \lim_{\varepsilon \to 0} \left\{ \int_{U_\varepsilon} \left| \frac{f''}{f'} + \frac{1}{\zeta} \right| d\sigma_\zeta + 2\pi \log \varepsilon \right\} + 2\pi \log |f'(0, t)|,
\]

or adding the logarithmic term into the integral we obtain

\[
S[f] = \int_U \left( \left| \frac{f''}{f'} + \frac{1}{\zeta} \right|^2 - \frac{1}{|\zeta|^2} \right) d\sigma_\zeta + 2\pi \log |f'(0, t)|.
\]

The functional \((11)\) resembles the universal Liouville action defined by Takhtajan and Teo in [38, 39] for quasicircles which is based on conformal maps from the unit disk and from its exterior.
Observe that the classical kinetic energy $E$ for the harmonic potential $p$ is calculated by the Dirichlet integral

$$\int_{D} |p_z|^2 d\sigma_z = \int_{D} |W'|^2 d\sigma_z,$$

locally for any measurable set $D \subset \Omega(t) \setminus \{0\}$. However, this integral again cannot be defined globally in $\Omega(t) \setminus \{0\}$. Treating $E$ in the same way as $S$ we come to the following finite limit

$$E := \lim_{\varepsilon \to 0} \left\{ \int_{\Omega_{\varepsilon}(t)} |W'|^2 d\sigma_z + 2\pi \log \varepsilon \right\},$$

or in terms of the parametric function $f$

$$E = E[f] = 2\pi \log |f'(0, t)|.$$

The latter expression allows us to think of $E$ as a capacity which exactly corresponds to the physical sense of $E$ as minimal energy.

3. Variation of the logarithmic action

The Laplacian growth problem being rewritten for the parametric time dependent function $f : U \to \Omega(t)$ admits the form of the so-called Polubarinova-Galina equation, which is in principle, a reformulation of the kinematic condition (see, e.g., [16, 46]). Namely, the function $f(\zeta, t)$ satisfies the non-linear first-order partial differential equation

$$\text{Re} \left( \frac{\dot{f}}{f} \frac{\bar{\zeta} f'}{f'} \right) = 1, \quad |\zeta| = 1,$$

with the initial condition $f(\zeta, 0) = f_0(\zeta)$. We denote by $S_f$ the Schwarzian derivative

$$S_f = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2,$$

and by

$$\kappa(\theta, t) = \text{Re} \left( \frac{1 + \frac{e^{i\theta} f''}{f'}}{|f'(e^{i\theta}, t)|} \right)$$

the curvature of the boundary $\partial \Omega(t)$ at the point $f(e^{i\theta}, t)$.

**Theorem 1.** Let $z = f(\zeta, t)$ be the parametric function for the Laplacian growth, $E[f]$ be the kinetic energy, and $S[f]$ be the logarithmic action. Then

$$\frac{d}{dt} E[f] = \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} d\theta,$$
and
\[
\frac{d}{dt}(S[f] + \mathcal{E}[f]) = 2 \int_0^{2\pi} \kappa^2(\theta, t) \, d\theta + \int_0^{2\pi} \frac{2}{|f'(e^{i\theta}, t)|^2} \text{Re} \left( e^{2i\theta} S_f \right) d\theta.
\]

**Proof.** Making use of the Cauchy-Schwarz representation we extend this equation into the unit disk
\[
(13) \quad \dot{f} = \zeta f' p(\zeta, t),
\]
where
\[
(14) \quad p(\zeta, t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \, d\theta.
\]
Immediately, we obtain that
\[
\frac{d}{dt} \mathcal{E}[f] = \frac{d}{dt} 2\pi \log |f'(0, t)| = \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \, d\theta,
\]
and hence,
\[
(15) \quad \frac{d}{dt} S[f] = 2\text{Re} \int_U \left( \frac{f''}{f'} + \frac{1}{\zeta} \right) \left( (1 + \zeta \frac{f''}{f'}) p(\zeta, t) + \zeta p'(\zeta, t) \right)' \, d\sigma_{\zeta} + \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} d\theta.
\]

The integral in the first term of the right-hand side of (15) we rewrite by Green’s theorem as
\[
I = \frac{-1}{2i} \int_{S^1} \left( \frac{f''}{f'} + \frac{1}{\zeta} \right) \left( (1 + \zeta \frac{f''}{f'}) p(\zeta, t) + \zeta p'(\zeta, t) \right) d\bar{\zeta} - \frac{1}{2} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} d\theta,
\]
taking into account a singularity at the origin. Applying the Cauchy-Schwarz formula to the first term in $I$ containing $p$ we arrive at
\[
2 \text{Re} \, I = \int_0^{2\pi} \left| 1 + e^{i\alpha} \frac{f''(e^{i\alpha}, t)}{f'(e^{i\alpha}, t)} \right|^2 \frac{d\alpha}{|f'(e^{i\alpha}, t)|^2}
\]
\[
+ \text{Re} \int_0^{2\pi} \left( 1 + e^{i\alpha} \frac{f''(e^{i\alpha}, t)}{f'(e^{i\alpha}, t)} \right) e^{i\alpha} p'(e^{i\alpha}, t) d\alpha - \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} d\theta,
\]
or

\[ \frac{d}{dt} S[f] = 2 \text{Re } I + \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \, d\theta = \int_0^{2\pi} \left| 1 + e^{i\alpha} \frac{f''}{f'} \right|^2 \frac{d\alpha}{|f'(e^{i\alpha}, t)|^2} \]

\[ + \int_0^{2\pi} \text{Re} \left( 1 + e^{i\alpha} \frac{f''}{f'} \right) \text{Re} e^{i\alpha} p'(e^{i\alpha}, t) \, d\alpha \]

\[ + \int_0^{2\pi} \text{Im} \left( 1 + e^{i\alpha} \frac{f''}{f'} \right) \text{Im} e^{i\alpha} p'(e^{i\alpha}, t) \, d\alpha. \]

These equalities are thought of as limiting values making use of the analyticity of \( f \) on the boundary and \( f' (\zeta, t) \neq 0 \) for all \( \zeta \) in the closure of the unit disk. Let us denote in the latter expression by \( J_2 \) the last integral and by \( J_1 \) the intermediate one. We have,

\[ J_2 = \int_0^{2\pi} \text{Im} \left( 1 + e^{i\alpha} \frac{f''}{f'} \right) \text{Im} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \frac{2e^{i\theta} e^{i\alpha}}{(e^{i\theta} - e^{i\alpha})^2} \, d\theta \, d\alpha. \]

Obviously,

\[ \frac{\partial}{\partial \theta} \left( \frac{e^{i\theta}}{e^{i\theta} - \zeta} \right) = -2 \frac{\zeta i e^{i\theta}}{(e^{i\theta} - \zeta)^2}, \quad \text{and} \quad \frac{\partial}{\partial \theta} \left( \frac{1}{|f'(e^{i\theta}, t)|^2} \right) = \frac{2}{|f'(e^{i\theta}, t)|^2} \text{Im} \frac{e^{i\theta} f''}{f'}. \]

Integrating by parts and applying the Cauchy-Schwarz formula again we obtain

\[ J_2 = -2 \int_0^{2\pi} \left( \text{Im} \left( 1 + e^{i\theta} \frac{f''}{f'} \right) \right)^2 \frac{d\theta}{|f'(e^{i\theta}, t)|^2}. \]

Now we turn to \( J_1 \)

\[ J_1 = \int_0^{2\pi} \text{Re} \left( 1 + e^{i\alpha} \frac{f''}{f'} \right) \text{Re} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \frac{2e^{i\theta} e^{i\alpha}}{(e^{i\theta} - e^{i\alpha})^2} \, d\theta \, d\alpha. \]

Here we change the order of integration and get

\[ J_1 = \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \text{Re} \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left( 1 + e^{i\alpha} \frac{f''}{f'} \right) \frac{2e^{i\theta} e^{i\alpha}}{(e^{i\theta} - e^{i\alpha})^2} \, d\alpha \, d\theta. \]

Integrating by parts we obtain

\[ J_1 = \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \text{Re} \frac{-i}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \alpha} \text{Re} \left( 1 + e^{i\alpha} \frac{f''}{f'} \right) \frac{e^{i\alpha} + e^{i\theta}}{e^{i\alpha} - e^{i\theta}} \, d\alpha \, d\theta. \]
The Cauchy formula gives
\[
J_1 = \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \text{Re} \left( \frac{e^{i\theta} f''}{f'} + \frac{e^{2i\theta} f''}{f'} - \left( \frac{e^{i\theta} f''}{f'} \right)^2 \right) d\theta,
\]
or
\[
J_1 = \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \text{Re} \left( \frac{1}{2} \left( 1 + \frac{e^{i\theta} f''}{f'} \right)^2 + e^{2i\theta} S_f - \frac{1}{2} \right) d\theta,
\]
where \( S_f \) is the Schwarzian derivative.

Summing up all these integrals we come to the conclusion that
\[
\frac{d}{dt} S[f] = \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \text{Re} \left( \frac{3}{2} \left( 1 + \frac{e^{i\theta} f''}{f'} \right)^2 + e^{2i\theta} S_f + \frac{1}{2} \right) d\theta - \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} d\theta.
\]

We observe that
\[
\kappa(\theta, t) = \frac{\text{Re} \left( 1 + \frac{e^{i\theta} f''}{f'} \right)}{|f'(e^{i\theta}, t)|}
\]
is the curvature of the boundary \( \partial \Omega(t) \) at the point \( f(e^{i\theta}, t) \). Integration by parts implies
\[
\int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \left( \text{Im} \left( 1 + \frac{e^{i\theta} f''}{f'} \right) \right)^2 d\theta
\]
\[
= -\frac{1}{2} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \text{Re} \left( \frac{1}{2} \left( 1 + \frac{e^{i\theta} f''}{f'} \right)^2 + e^{2i\theta} S_f - \frac{1}{2} \right) d\theta.
\]

So
\[
\frac{d}{dt} S[f] = \frac{3}{2} \int_0^{2\pi} \kappa^2(\theta, t) d\theta - \frac{3}{2} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \left( \text{Im} \left( 1 + \frac{e^{i\theta} f''}{f'} \right) \right)^2 d\theta
\]
\[
+ \frac{1}{2} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \text{Re} \left( e^{2i\theta} S_f + \frac{1}{2} \right) d\theta - \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} d\theta
\]
\[
= \left( \frac{3}{2} + \frac{3}{8} \right) \int_0^{2\pi} \kappa^2(\theta, t) d\theta - \frac{3}{8} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \left( \text{Im} \left( 1 + \frac{e^{i\theta} f''}{f'} \right) \right)^2 d\theta
\]
\[
+ \frac{1}{2} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \text{Re} \left( \left( 1 + \frac{3}{4} e^{2i\theta} S_f + \left( \frac{1}{2} - \frac{3}{8} \right) \right) \right) d\theta - \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} d\theta.
Repeating this step we get
\[
\frac{d}{dt} S[f] = \frac{3}{2} (1 + \frac{1}{4} + \cdots + \frac{1}{4^n}) \int_0^{2\pi} \kappa^2(\theta, t) \, d\theta - \frac{3}{2} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \left( \Im \left( 1 + \frac{e^{i\theta} f''}{f'} \right) \right)^2 \, d\theta \\
+ \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \Re \left( (1 + \frac{3}{4} + \cdots + \frac{3}{4^n}) e^{2i\theta} S_f + (\frac{1}{2} - \frac{3}{8} - \cdots - \frac{3}{2} \frac{1}{4^n}) \right) \, d\theta \\
- \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \, d\theta,
\]

at \( n \)-th iteration. Taking limit as \( n \to \infty \), we finally obtain
\[
\frac{d}{dt} S[f] = 2 \int_0^{2\pi} \kappa^2(\theta, t) \, d\theta + \int_0^{2\pi} \frac{2}{|f'(e^{i\theta}, t)|^2} \Re (e^{2i\theta} S_f) d\theta - \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \, d\theta,
\]
as claimed in the statement of the theorem.

In the simplest case of the circular evolution \( f(\zeta, t) = \sqrt{2t} \zeta \) we have \( \frac{d}{dt} S[f] = \frac{d}{dt} \mathcal{E}[f] = \frac{\pi}{4} \).

4. Parametric manifold \( \text{Diff} S^1/\text{Rot} S^1 \)

As it has been mentioned in Section 2, starting with a smooth \( (C^\infty) \) initial boundary \( \partial \Omega(0) \) the classical evolution of Laplacian growth is given by domains \( \Omega(t) \) with smooth boundaries \( \partial \Omega(t) \) as long as the classical solution exists. Our aim now is to give an embedding of this evolution into the parametric Kirillov’s space \( \text{Diff} S^1/\text{Rot} S^1 \).

We denote the Lie group of \( C^\infty \) sense preserving diffeomorphisms of the unit circle \( S^1 = \partial U \) by \( \text{Diff} S^1 \). Each element of \( \text{Diff} S^1 \) is represented as \( z = e^{i\phi(\theta)} \) with a monotone increasing, \( C^\infty \) real-valued function \( \phi(\theta) \), such that \( \phi(\theta + 2\pi) = \phi(\theta) + 2\pi \). The Lie algebra for \( \text{Diff} S^1 \) is identified with the Lie algebra \( \text{Vect} S^1 \) of smooth \( (C^\infty) \) tangent vector fields to \( S^1 \) with the Poisson - Lie bracket given by
\[
[\phi_1, \phi_2] = \phi_1 \phi_2' - \phi_2 \phi_1'.
\]

Fixing the trigonometric basis in \( \text{Vect} S^1 \) the commutator relations take the form
\[
[\cos n\theta, \cos m\theta] = \frac{n-m}{2} \sin (n+m)\theta + \frac{n+m}{2} \sin (n-m)\theta, \\
[\sin n\theta, \sin m\theta] = \frac{m-n}{2} \sin (n+m)\theta + \frac{n+m}{2} \sin (n-m)\theta, \\
[\sin n\theta, \cos m\theta] = \frac{m-n}{2} \cos (n+m)\theta - \frac{n+m}{2} \cos (n-m)\theta.
\]

There is no general theory of infinite dimensional Lie groups, example of which is under consideration. The interest to this particular case comes first of all from the string theory
where the Virasoro (vertex) algebra appears as the central extension of \( \text{Vect} \ S^1 \) (see Section 7). The central extension of \( \text{Diff} \ S^1 \) is called the Virasoro-Bott group. Entire necessary background for the construction of the theory of unitary representations of \( \text{Diff} \ S^1 \) is found in the study of Kirillov’s homogeneous Kählerian manifold \( M = \text{Diff} \ S^1 / \text{Rot} \ S^1 \), where \( \text{Rot} \ S^1 \) denotes the group of rotations of \( S^1 \). The group \( \text{Diff} \ S^1 \) acts as a group of translations on the manifold \( M \) with \( \text{Rot} \ S^1 \) as a stabilizer. The Kählerian geometry of \( M \) has been described by Kirillov and Yuriev in [19]. The manifold \( M \) admits several representations, in particular, in the space of smooth probability measures, symplectic realization in the space of quadratic differentials. Let \( A \) stand for the class of all analytic regular univalent functions \( f \) in \( U \) normalized by \( f(0) = 0, \ f'(0) = 1 \). We will use its analytic representation of \( M \) based on the class \( \tilde{A} \) of functions from \( A \) which being extended onto the closure \( \overline{U} \) of \( U \) are supposed to be smooth on \( S^1 \). The class \( \tilde{A} \) is dense in \( A \) in the local uniform topology of \( U \).

Let \( \tilde{\Sigma} \) stand for the class of all univalent regular maps in the exterior \( U^* \) of the unit disk \( U \) normalized by \( g(\zeta) = c_1 \zeta + c_0 + c_1 \zeta^{-1} + \ldots \) which are smooth on \( S^1 \). Then, for each \( f \in \tilde{A} \) there is an adjoint map \( g \in \tilde{\Sigma} \) such that \( \overline{C \setminus f(U)} = g(U^*) \). The superposition \( g^{-1} \circ f \) restricted to \( S^1 \) is in \( M \). Reciprocally, for each element of \( M \) there exist such \( f \) and \( g \).

Observe that a piece-wise smooth closed Jordan curve is a quasicircle if and only if it has no cusps. So any function \( f \) from \( \tilde{A} \) has a quasiconformal extension to \( U^* \). By this realization the manifold \( M \) is naturally embedded into the universal Teichmüler space \( T \). However, defined as a complex Banach manifold the Teichmüller space \( T \) requires additional Hilbert manifold structure to assure the embedding \( M \to T \) to inherit the Kählerian structure of \( M \). This has been done by Takhtajan and Teo in [38]-[40]. The Kählerian structure on \( M \) corresponds to the Kählerian structure on \( T \) given by the analogue of the Weil-Petersson metric.

The Goluzin-Schiffer variational formula lifts the actions from the Lie algebra \( \text{Vect} \ S^1 \) onto \( \tilde{A} \). Let \( f \in \tilde{A} \) and let \( \nu(e^{i\theta}) \) be a \( C^\infty \) real-valued function in \( \theta \in (0, 2\pi] \) from \( \text{Vect} \ S^1 \) making an infinitesimal action as \( \theta \mapsto \theta + \tau \nu(e^{i\theta}) \). Let us consider a variation of \( f \) given by

\[
\delta_\nu f(\zeta) = -\frac{f^2(\zeta)}{2\pi i} \int_{S^1} \left( \frac{w f'(w)}{f(w)} \right)^2 \frac{\nu(w)}{f(w) - f(\zeta)} \frac{dw}{w}.
\]

Kirillov and Yuriev [13], [20] have established that the variations \( \delta_\nu f(\zeta) \) are closed with respect to the commutator and the induced Lie algebra is the same as \( \text{Vect} \ S^1 \). Moreover, Kirillov’s result [17] states that there is the exponential map \( \text{Vect} \ S^1 \to \text{Diff} \ S^1 \) such that the subgroup \( \text{Rot} \ S^1 \) coincides with the stabilizer of the map \( f(\zeta) \equiv \zeta \) from \( \tilde{A} \).
5. **Semigroups of conformal maps**

The basic ideas that we use in this section come from the development of Löwner’s parametric method that emerges at a seminal Löwner’s paper [24]. Löwner was first who proposed to use Lie semigroups of conformal maps to obtain an evolution equation for conformal maps. His ideas have been furthered by many authors among whom we mention Pommerenke [30, 31] as a general reference, and Goryainov’s works [10, 11] especially closed to our consideration, one also may see [35, 47].

We consider the semigroup \( G \) of conformal univalent maps from \( U \) into itself with composition as the semigroup operation. This makes \( G \) a topological semigroup with respect to the topology of local uniform convergence on \( U \). We impose the natural normalization for such conformal maps: \( \Phi(\zeta) = \beta \zeta + b_2 \zeta^2 + \ldots, \zeta \in U, \beta > 0 \). The unit of the semigroup is the identity. Let us construct on \( G \) a one-parameter semi-flow \( \Phi^\tau \), that is, a continuous homomorphism from \( \mathbb{R}^+ \) into \( G \), with the parameter \( \tau \geq 0 \). For any fixed \( \tau \geq 0 \) the element \( \Phi^\tau \) is from \( G \) and is represented by a conformal map \( \Phi(\zeta, \tau) = \beta(\tau) \zeta + b_2(\tau) \zeta^2 + \ldots \) from \( U \) onto the domain \( \Phi(U, \tau) \subset U \). The element \( \Phi^\tau \) satisfies the following properties:

- \( \Phi^0 = id; \)
- \( \Phi^{\tau+s} = \Phi(\Phi(\zeta, \tau), s), \text{ for } \tau, s \geq 0; \)
- \( \Phi(\zeta, \tau) \to \zeta \) locally uniformly in \( U \) as \( \tau \to 0 \).

In particular, \( \beta(0) = 1 \). This semi-flow is generated by a vector field \( v(\zeta) \) if for each \( \zeta \in U \) the function \( w = \Phi(\zeta, \tau), \tau \geq 0 \) is a solution of an autonomous differential equation \( dw/d\tau = v(w) \) with the initial condition \( w|_{\tau=0} = \zeta \). The semi-flow can be extended to a symmetric interval \((-t, t)\) by putting \( \Phi^{-\tau} = \Phi^{-1}(\zeta, \tau) \). Certainly, the latter function is defined on the set \( \Phi(U, \tau) \). Admitting this restriction for negative \( \tau \) we define a one-parameter family \( \Phi^\tau \) for \( \tau \in (-t, t) \).

For a semi-flow \( \Phi^\tau \) on \( G \) there is an infinitesimal generator at \( \tau = 0 \) constructed by the following procedure. Any element \( \Phi^\tau \) is represented by a conformal map \( \Phi(\zeta, \tau) \) that satisfies the Schwarz Lemma for the maps \( U \to U \), and hence,

\[
\operatorname{Re} \frac{\Phi(\zeta, \tau)}{\zeta} \leq \left| \frac{\Phi(\zeta, \tau)}{\zeta} \right| \leq 1, \quad \zeta \in U,
\]

where the equality sign is attained only for \( \Phi^0 = id \simeq \Phi(\zeta, 0) \equiv \zeta \). Therefore, the following limit exists (see, e.g., [10, 11, 35])

\[
\lim_{\tau \to 0} \operatorname{Re} \frac{\Phi(\zeta, \tau) - \zeta}{\tau \zeta} = \operatorname{Re} \frac{\partial \Phi(\zeta, \tau)}{\partial \tau} \bigg|_{\tau=0} \leq 0,
\]

and the representation

\[
\frac{\partial \Phi(\zeta, \tau)}{\partial \tau} \bigg|_{\tau=0} = -\zeta p(\zeta)
\]
holds, where \( p(\zeta) = p_0 + p_1\zeta + \ldots \) is an analytic function in \( U \) with positive real part, and
\[
(17) \quad \frac{\partial \beta(\tau)}{\partial \tau} \bigg|_{\tau=0} = -p_0.
\]
In [12] it was shown that \( \Phi^\tau \) is even \( C^\infty \) with respect to \( \tau \). The function \(-\zeta p(\zeta)\) is an infinitesimal generator for \( \Phi^\tau \) at \( \tau = 0 \), and the following variational formula holds
\[
(18) \quad \Phi(\zeta, \tau) = \zeta - \tau \zeta p(\zeta) + o(\tau), \quad \beta(\tau) = 1 - \tau p_0 + o(\tau).
\]
The convergence is thought of as local uniform. We rewrite (18) as
\[
(19) \quad \Phi(\zeta, \tau) = (1 - \tau p_0) \zeta + \tau \zeta \left(-\zeta p(\zeta) + p_0\right) + o(\tau) = \beta(\tau) \zeta + \tau \zeta (-p(\zeta) + p_0) + o(\tau).
\]
Now let us proceed with the semigroup \( \tilde{\mathcal{G}} \subset \mathcal{G} \) of elements from \( \mathcal{G} \) represented by univalent maps \( \Phi \) smooth on \( S^1 \). By the variation of the identity in \( \tilde{\mathcal{A}} \) given by the formula (16) we get
\[
\frac{\Phi(\zeta, \tau)}{\beta(\tau)} = \zeta - \frac{\zeta^2}{2\pi i} \int_{S^1} \frac{d(w)}{w(w - \zeta)} dw + o(\tau),
\]
for \( \nu(w) \) from \( \text{Vect} S^1 \). Comparing with (19) we come to the conclusion about \( p(\zeta) \):
\[
(20) \quad p(\zeta) = p_0 + \frac{\zeta}{2\pi i} \int_{S^1} \frac{\nu(w)}{w(w - \zeta)} dw.
\]
The constants \( p_0 \) and the function \( \nu(w) \) must be such that \( \text{Re} \, p(z) > 0 \) for all \( z \in U \).

We summarize these observations in the following theorem.

**Theorem 2.** Let \( \Phi^\tau \) be a semi-flow in \( \tilde{\mathcal{G}} \). Then it is generated by the vector field \( \nu(\zeta) = -\zeta p(\zeta) \),
\[
p(\zeta) = p_0 + \frac{\zeta}{2\pi i} \int_{S^1} \frac{\nu(w)}{w(w - \zeta)} dw,
\]
where \( \nu(e^{i\theta}) \in \text{Vect} S^1 \), and the holomorphic function \( p(\zeta) \) has positive real part in \( U \).

This theorem implies that at any point \( \tau \geq 0 \) we have
\[
\frac{\partial \Phi(\zeta, \tau)}{\partial \tau} = -\Phi(\zeta, \tau) p(\Phi(\zeta, \tau)).
\]

6. Evolution families and evolution equations

A subset \( \Phi^{t,s} \) of \( \mathcal{G} \), \( 0 \leq s \leq t \) is called an *evolution family* in \( \mathcal{G} \) if
- \( \Phi^{t,t} = \text{id} \);
- \( \Phi^{t,s} = \Phi^{t,r} \circ \Phi^{r,s} \), for \( 0 \leq s \leq r \leq t \);
- \( \Phi^{t,s} \to \text{id} \) locally uniformly in \( U^* \) as \( t, s \to \tau \).
In particular, if $\Phi^\tau$ is a one-parameter semi-flow, then $\Phi^{t-s}$ is an evolution family. We consider a subordination chain of mappings $f(\zeta, t), \zeta \in U, t \in [0, t_0)$, where the function $f(\zeta, t) = \alpha(t)z + a_2(t)\zeta^2 + \ldots$ is a analytic univalent map $U \to \mathbb{C}$ for each fixed $t$ and $f(U, s) \subset f(U, t)$ for $s < t$. Let us assume that this subordination chain exists for $t$ in an interval $[0, T)$.

Let us pass to the semigroup $\tilde{G}$. So $\Phi^{t, s}$ now has a smooth extension to $S^1$. Moreover, $\Phi^{t, s} \to \text{id}$ locally uniformly in $\mathbb{C}$ as $t, s \to \tau$.

We construct the superposition $f^{-1}(f(\zeta, s), t)$ for $t \in [0, T), s \leq t$. Putting $s = t - \tau$ we denote this mapping by $\Phi(\zeta, t, \tau)$.

Now we suppose the following conditions for $f(\zeta, t)$.

(i) The maps $f(\zeta, t)$ form a subordination chain in $U, t \in [0, T)$.

(ii) The map $f(\zeta, t)$ is holomorphic in $U$, $f(\zeta, t) = \alpha(t)\zeta + a_2(t)\zeta^2 + \ldots$, where $\alpha(t) > 0$ and differentiable with respect to $t$.

(iii) The map $f(\zeta, t)$ admits a smooth continuation onto $S^1$.

The function $\Phi(\zeta, t, \tau)$ is embedded into an evolution family in $\mathcal{G}$. It is differentiable with regard to $\tau$ and $t$ in $[0, T)$, and $\Phi(\zeta, t, 0) = \zeta$. Moreover, $\zeta = \lim_{\tau \to 0} \Phi(\zeta, t, \tau)$ locally uniformly in $U$ and $\Phi(\zeta, t, \tau)$ is embedded now into an evolution family in $\mathcal{G}^w$. The identity map is embedded into a semi-flow $\Phi^\tau \subset \tilde{G}$ (which is smooth) as the initial point with the same velocity vector

\[ \frac{\partial \Phi(\zeta, t, \tau)}{\partial \tau} \bigg|_{\tau = 0} = -\zeta p(\zeta, t), \quad \zeta \in U, \]

that leads to the Löwner-Kufarev equation

\[ \dot{f} = \zeta f' p(\zeta, t), \]

(the semi-flow $\Phi^\tau$ is tangent to the evolution family at the origin). Actually, the differentiable trajectory $f(\zeta, t)$ generates a pencil of tangent smooth semi-flows with starting tangent vectors $-\zeta p(\zeta, t)$ (that may be only measurable with respect to $t$).

Therefore, the conclusion is that the function $f(\zeta, t)$ satisfies the equation $\text{(21)}$ where the function $p(\zeta, t)$ is given by

\[ p(\zeta, t) = p_0(t) + \frac{\zeta}{2\pi i} \int_{S^1} \frac{\nu(w, t)}{w(w - \zeta)} dw, \]

and has positive real part. The existence of $p_0(t)$ comes from the existence of the subordination chain. One may assign the normalization to $f(\zeta, t)$ controlling the change of the conformal radius of the subordination chain by, e.g., $e^t$. In this case, changing variables we obtain $p_0 = 1$.

Summarizing the conclusions about the function $p(\zeta, t)$ we come to the following result.

**Theorem 3.** Let $f(\zeta, t)$ be a subordination chain of maps in $U$ that exists for $t \in [0, T)$ and satisfies the conditions (i–iii). Then, there are a real valued function $p_0(t) > 0$ and a
real valued function \( \nu(\zeta, t) \in \text{Vect } S^1 \), such that \( \text{Re } p(\zeta, t) > 0 \) for \( \zeta \in U \),

\[
p(\zeta, t) = p_0(t) + \frac{\zeta}{2\pi i} \int_{S^1} \frac{\nu(w, t)}{w(w - \zeta)} dw, \quad \zeta \in U,
\]

and \( f(\zeta, t) \) satisfies the L"owner-Kufarev differential equation (21) in \( t \in [0, T) \).

Comparing (13) and (14) with this theorem we come to the following corollary.

**Corollary 1.** Let \( f(\zeta, t) = \alpha(t)\zeta + a_2(t)\zeta^2 + \ldots \) be a subordination chain of maps in \( U \) that parameterizes the classical Laplacian Growth that exists for \( t \in [0, T) \). Then, under the notations of the previous theorem we have

\[
p_0 = \frac{\dot{\alpha}}{\alpha} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} d\theta, \quad \text{and} \quad \nu(e^{i\theta}, t) = \frac{2}{|f'(e^{i\theta}, t)|^2}.
\]

In the view of this corollary we come to an interpretation of the variation of the logarithmic action as

\[
\frac{d}{dt}(S[f] + \mathcal{E}[f]) = 2 \int_0^{2\pi} \nu^2(\theta, t) d\theta + \text{Re} \int_0^{2\pi} e^{2i\theta} \nu(e^{i\theta}, t) S_f d\theta,
\]

where \( \nu \) is a vector from the Lie algebra \( \text{Vect } S^1 \) tangential to \( \text{Diff } S^1 / \text{Rot } S^1 \) at the unity.

7. **Connections with the Virasoro algebra**

In two dimensional conformal field theories [9], the algebra of energy momentum tensor is deformed by a central extension due to the conformal anomaly and becomes the Virasoro algebra. The Virasoro algebra is spanned by elements \( e_k = \zeta^{1+k} \partial, \ k \in \mathbb{Z} \) and \( c \) with \( e_k + e_{-k} \), where \( c \) is a real number, called the central charge, and the Lie brackets are defined by

\[
[e_m, e_n]_{\text{Vir}} = (n - m)e_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{n,-m}, \quad [c, L_k] = 0.
\]

The Virasoro algebra \( (\text{Vir}) \) can be realized as a central extension of \( \text{Vect } S^1 \) by defining

\[
[\phi \partial + ca, \psi \partial + cb]_{\text{Vir}} = (\phi \psi' - \phi' \psi) \partial + \frac{c}{12} \omega(\phi, \psi),
\]

(wheras \( [\phi, \psi] = \phi \psi' - \phi' \psi \)), where the bilinear antisymmetric form \( \omega(\phi, \psi) \) on \( \text{Vect } S^1 \) is given by

\[
\omega(\phi, \psi) = -\frac{1}{4\pi} \int_0^{2\pi} (\phi' + \phi''') \psi d\theta,
\]

and \( a, b \) are numbers. This form defines the Gelfand-Fuks cocycle on \( \text{Vect } S^1 \) and satisfies the Jacobi identity. The factor of 1/12 is merely a matter of convention. The manifold \( M \) being considered as a realization \( \tilde{A} \) admits affine coordinates \( \{c_2, c_3, \ldots\} \), where \( c_k \) is the
$k$-th coefficient of a univalent functions $f \in \tilde{A}$. Due to de Branges’ theorem [5], $M$ is a bounded open subset of $\{|c_k| < k + \varepsilon\}$. 

Taking $\nu = -ie^{ik\theta}$, $k \geq 0$, we obtain the expressions for $L_k = \delta_\nu f$, $f \in \tilde{A} \simeq M$ (see formula (16)), as

$$L_0 = \zeta f'(\zeta) - f(\zeta), \quad L_k = \zeta^{1+k} f'.$$

The computation of $L_k$ for $k < 0$ is more difficult because poles of the integrant. For example,

$$L_{-1} = f' - 1 - 2c_2 f, \quad L_{-2} = \frac{f'}{\zeta} - \frac{1}{f} - 3c_2 + (c_2^2 - 4c_3) f,$$

(see, e.g., [18]). In terms of the coordinates $\{c_2, c_3, \ldots\}$ on $M$

$$L_k = \partial_k + \sum_{n=1}^{\infty} (n+1)c_n \partial_{k+n}, \quad L_0 = \sum_{n=1}^{\infty} nc_n \partial_n,$$

for $k > 0$, where $\partial_k = \partial/\partial c_{k+1}$.

Neretin [26] introduced the sequence of polynomials $P_k$, in the coordinates $\{c_2, c_3, \ldots\}$ on $M$ by the following recurrent relations

$$L_m(P_n) = (n + m)P_{n-m} + \frac{c}{12} m(m^2 - 1) \delta_{n,m}, \quad P_0 = P_1 \equiv 0, \quad P_k(0) = 0,$$

where the central charge $c$ is fixed. This gives, for example, $P_2 = \frac{c}{2}(c_3 - c_2^2)$, $P_3 = 2c(c_4 - 2c_2c_3 + c_3^2)$. In general, the polynomials $P_k$ are homogeneous with respect to rotations of the function $f$. It is worthy to mention that estimates of the absolute value of these polynomials has been a subject of investigations in the theory of univalent functions for a long time, e.g., for $|P_2|$ we have $|c_3 - c_2^2| \leq 1$ (Bieberbach 1916 [4]), for estimates of $|P_3|$ see [13, 23, 41, 42]. For the Neretin polynomials one can construct the generatrix function

$$P(\zeta) = \sum_{k=1}^{\infty} P_k \zeta^k = \frac{c\zeta^2}{12} S_f(\zeta),$$

where $S_f(\zeta)$ is the Schwarzian derivative of $f$. Let $\nu \in \text{Vect } S^1$ and $\nu^g$ be the associated right-invariant tangent vector field defined at $g \in \text{Diff } S^1$. For the basis $\nu_k = -ie^{ik\theta} \partial$, one constructs the corresponding associated right-invariant basis $\nu_k^g$. By $\{\psi_{-k}\}$ we denote the dual basis of 1-forms such that the value of each form on the vector $\nu_k^g$ is given as

$$(\psi_k, \nu_n^g) = \delta_{k+n,0}.$$

Let us construct the 1-form $\Omega$ on $\text{Diff } S^1$ by

$$\Omega = \sum_{k=1}^{\infty} (P_k \circ \pi) \psi_k,$$
where $\pi$ means the natural projection $\text{Diff} \ S^1 \to M$. This form appeared in [2, 3] in the context of the construction of a unitarizing probability measure for the Neretin representation of $M$. It is invariant under the left action of $S^1$. If $f \in \tilde{A}$ represents $g$ and $\nu \in \text{Vect} \ S^1$, then the value of the form $\Omega$ on the vector $\nu$ is

\[
(\Omega, \nu)_f = \int_0^{2\pi} e^{2i\theta} \nu(e^{i\theta}) S_f \, d\theta,
\]

see [2, 3]. So the variation of the logarithmic action given in Theorem 1 becomes

\[
\frac{d}{dt}(S[f] + \mathcal{E}[f]) = 2 \int_0^{2\pi} \kappa^2(\theta, t) \, d\theta + \text{Re} \ (\Omega, \nu)_f.
\]

In this formula, we take into account the first coefficient, the conformal radius of the Laplacian evolution, that does not change the form $\Omega$.

8. SOME OPEN QUESTIONS

(i) One may conjecture that the formula (22) remains true for general smooth subordination evolution.

(ii) The logarithmic action is clearly related to the universal Liouville action suggested by Takhtajan and Teo in [38, 39, 40]. There must be possible to obtain the variation given in Theorem 1 by means of the variation of the universal Liouville action obtained in [39]. This would be yet more interesting because the universal Liouville action is defined for contours without any smoothness hypothesis.

(iii) Another interesting question is whether it is possible to make regularization of the proper Liouville action integral based on the Poincaré metric by the boundary distortion by a univalent function.

(iv) A deeper task is concerned with the Laplacian growth and its embedding into the Whitham-Toda hierarchy. The extended Toda hierarchy (see [6]) admits a non-abelian algebra of infinitesimal symmetries isomorphic to half of the Virasoro algebra [7]. It would be interesting to reveal the connections between the Hamiltonian approach through the Toda hierarchies and the action approach suggested in the present paper.

(v) The multiply connected Laplacian growth is a natural way of generalization of all these results.

REFERENCES

[1] O. Agam, E. Bettelheim, P. Wiegmann, A. Zabrodin, Viscous fingering and a shape of an electronic droplet in the Quantum Hall regime, arXiv: cond-mat/0111333, 2002.

[2] H. Airault, P. Malliavin, Unitarizing probability measures for representations of Virasoro algebra, J. Math. Pures Appl. 80 (2001), no. 6, 627–667.
[3] H. Airault, P. Malliavin, A. Thalmaier, Support of Virasoro unitarizing measures, C. R. Acad. Sci. Paris, Ser. I 335 (2002), 621–626.

[4] L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, S.-B. Preuss. Akad. Wiss. (1916), S. 940–955.

[5] L. de Branges, A proof of the Bieberbach conjecture, Acta Math. 154 (1985), no. 1–2, 137–152.

[6] G. Carlet, B. Dubrovin, Y. Zhang, The extended Toda hierarchy, Mosc. Math. J. 4 (2004), no. 2, 313–332.

[7] B. Dubrovin, Y. Zhang, Virasoro symmetries of the extended Toda hierarchy, Comm. Math. Phys. 250 (2004), no. 1, 161–193.

[8] K. A. Gillow, S. D. Howison, A bibliography of free and moving boundary problems for Hele-Shaw and Stokes flow, Published electronically at URL >http://www.maths.ox.ac.uk/~howison/Hele-Shaw</http://www.maths.ox.ac.uk/~howison/Hele-Shaw>

[9] P. Goddard, D. Olive, Kac-Moody and Virasoro algebras in relation to Quantum Physics, Int. J. Mod. Phys. A 1 (1986), no. 2, 303–414.

[10] V. V. Goryainov, Fractional iterates of functions that are analytic in the unit disk with given fixed points, Mat. Sb. 182 (1991), no. 9, 1281–1299; Engl. Transl. in Math. USSR-Sb. 74 (1993), no. 1, 29–46.

[11] V. V. Goryainov, One-parameter semigroups of analytic functions, Geometric function theory and applications of complex analysis to mechanics: studies in complex analysis and its applications to partial differential equations, 2 (Halle, 1988), Pitman Res. Notes Math. Ser., 257, Longman Sci. Tech., Harlow, 1991, 160–164.

[12] V. V. Goryainov, One-parameter semigroups of analytic functions and a compositional analogue of infinite divisibility, Proceedings of the Institute of Applied Mathematics and Mechanics, Vol. 5, Tr. Inst. Prikl. Mat. Mekh., 5, Nats. Akad. Nauk Ukrainy Inst. Prikl. Mat. Mekh., Donetsk, 2000, 44–57. (in Russian)

[13] L. Gromova, A. Vasil’ev, On the estimate of the fourth-order homogeneous coefficient functional for univalent functions, Ann. Polon. Math. 63 (1996), 7–12.

[14] V. Ya. Gutlijanski˘ı, The method of variations for univalent analytic functions with a quasiconformal extension, Sibirsk. Mat. Zh. 21 (1980), no. 2, 61–78; translation in Siberian Math. J. 21 (1980), no. 2, 190–204.

[15] H. S. Hele-Shaw, The flow of water, Nature 58 (1898), no. 1489, 33–36.

[16] S. D. Howison, Complex variable methods in Hele-Shaw moving boundary problems, European J. Appl. Math. 3 (1992), no. 3, 209–224.

[17] A. A. Kirillov, Kähler structure on the K-orbits of a group of diffeomorphisms of the circle, Funktsional. Anal. i Prilozhen. 21 (1987), no. 2, 42–45.

[18] A. A. Kirillov, Geometric approach to discrete series of unirreps for Vir, J. Math. Pures Appl. 77 (1998), 735–746.

[19] A. A. Kirillov, D. V. Yuriev, Kähler geometry of the infinite-dimensional homogeneous space M = Diff+(S^1)/Rot(S^1), Funktsional. Anal. i Prilozhen. 21 (1987), no. 4, 35–46. (in Russian)

[20] A. A. Kirillov, D. V. Yuriev, Representations of the Virasoro algebra by the orbit method, J. Geom. Phys. 5 (1988), no. 3, 351–363.

[21] I. Krichever, M. Mineev-Weinstein, P. Wiegmann, A. Zabrodin, Laplacian growth and Whitham equations of soliton theory, arXiv: nlin.SI/0311005, 2004.

[22] I. K. Kostov, I. Krichever, M. Mineev-Weinstein, P. B. Wiegmann, A. Zabrodin, The τ-function for analytic curves, Random matrix models and their applications, Math. Sci. Res. Inst. Publ., 40, Cambridge Univ. Press, Cambridge, 2001, 285–299.
[23] P. Lehto, *On fourth-order homogeneous functionals in the class of bounded univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No. 48, (1984), 1–46.
[24] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. **89** (1923), 103–121.
[25] A. Marshakov, P. Wiegmann, A. Zabrodin, *Integrable structure of the Dirichlet boundary problem in two dimensions*, Comm. Math. Phys. **227** (2002), no. 1, 131–153.
[26] Yu. A. Neretin, *Representations of Virasoro and affine Lie algebras*, Encyclopedia of Mathematical Sciences, Vol. 22, Springer-Verlag, 1994, pp. 157–225.
[27] H. Ockendon, J. R. Ockendon, *Viscous Flow*, Cambridge U.P., 1995.
[28] H. Poincaré, *Les fonctions fuchsiennes et l'équation $\Delta u = e^u$*, J. Math. Pure Appl. (5) **4** (1898), 137–230.
[29] A. M. Polyakov, *Quantum geometry of bosonic strings*, Phys. Lett. B **103** (1981), no. 3, 207–210.
[30] Ch. Pommerenke, *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math. **218** (1965), 159–173.
[31] Ch. Pommerenke, *Univalent functions, with a chapter on quadratic differentials by G. Jensen*, Vandenhoeck & Ruprecht, Göttingen, 1975.
[32] M. Reissig, L. Von Woltersdorff, *A simplified proof for a moving boundary problem for Hele-Shaw flows in the plane*, Ark. Mat. **31** (1993), no. 1, 101–116.
[33] S. Richardson, *Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel*, J. Fluid Mech., **56** (1972), no. 4, 609–618.
[34] P. G. Saffman, G. I. Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid*, Proc. Royal Soc. London, Ser. A **245** (1958), no. 281, 312–329.
[35] D. Shoikhet, *Semigroups in geometrical function theory*, Kluwer Academic Publishers, Dordrecht, 2001.
[36] L. A. Takhtajan, *Liouville theory: quantum geometry of Riemann surfaces*, Modern Phys. Lett. A **8** (1993), no. 37, 3529–3535.
[37] L. A. Takhtajan, L.-P. Teo, *Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography*, Comm. Math. Phys. **239** (2003), 183–240.
[38] L. A. Takhtajan, L.-P. Teo, *Weil-Petersson metric on the universal Teichmüller space I. Curvature properties and Chern forms*, arXiv: math. CV/0312172, 2004.
[39] L. A. Takhtajan, L.-P. Teo, *Weil-Petersson metric on the universal Teichmüller space II. Kähler potential and period mapping*, arXiv: math. CV/0406408, 2004.
[40] L. A. Takhtajan, L.-P. Teo, *Weil-Petersson geometry of the universal Teichmüller space*, Progress in Math. **237** (2005), 219–227.
[41] O. Tammi, *Extremum problems for bounded univalent functions*, Lecture Notes in Mathematics, 646. Springer-Verlag, Berlin–New York, 1978.
[42] O. Tammi, *Extremum problems for bounded univalent functions II*, Lecture Notes in Mathematics, 913. Springer-Verlag, Berlin–New York, 1982.
[43] P. G. Zograf, *Liouville action on moduli spaces and uniformization of degenerate Riemann surfaces*, Algebra i Analiz **1** (1989), no. 4, 136–160; translation in Leningrad Math. J. **1** (1990), no. 4, 941–965.
[44] P. G. Zograf, L. A. Takhtajan, *On the Liouville equation, accessory parameters and the geometry of Teichmüller space for Riemann surfaces of genus 0*, Mat. Sb. (N.S.) **132(174)** (1987), no. 2, 147–166; translation in Math. USSR-Sb. **60** (1988), no. 1, 143–161.
[45] P. Zograf, L. Takhtajan, *Hyperbolic 2-spheres with conical singularities, accessory parameters and Kähler metrics on $M_{0,n}$*, Trans. Amer. Math. Soc. **355** (2003), no. 5, 1857–1867.
[46] A. Vasil’ev, *Univalent functions in two-dimensional free boundary problems*, Acta Applic. Math. 79 (2003), no. 3, 249–280.

[47] A. Vasil’ev, *On a parametric method for conformal maps with quasiconformal extensions*, Publ. de l’Institut Math. (Nouvelle Ser.), Belgrad 75(89) (2004), 9–24.

[48] Yu. P. Vinogradov, P. P. Kufarev, *On a problem of filtration*, Akad. Nauk SSSR. Prikl. Mat. Meh. 12 (1948), 181–198. (in Russian)

[49] P. B. Wiegmann, A. Zabrodin, *Conformal maps and integrable hierarchies*, Comm. Math. Phys. 213 (2000), no. 3, 523–538.

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