Heavy Hadrons and QCD Instantons

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Abstract

Heavy hadrons are analyzed in a random and dilute gas of instantons. We derive the instanton-induced interactions between heavy and light quarks at next to leading order in the heavy quark mass and in the planar approximation, and discuss their effects on the hadronic spectrum. The role of these interactions in the formation of exotic hadrons is also discussed.

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I. INTRODUCTION

Hadrons with one or many heavy quarks exhibit a new type of symmetry: invariance under spin-flip of the heavy quark [1]. This invariance can be used to organize the hadronic structure and properties of heavy-light systems. A number of relations follow both in the spectrum and among form factors of heavy hadrons when the mass of the heavy quark is taken to infinity [2].

The interplay between light and heavy degrees of freedom in heavy-light hadrons can be clarified in the heavy-quark limit by combining chiral symmetry with heavy quark symmetry [3]. The basic observation is to note that the hard part in the heavy quark field is kinematical and factorizable. The remaining part is soft and constrained by chiral dynamics. Hence, the soft physics in heavy-light systems can be analyzed in a way similar to the light-light systems. A qualitative understanding of this part can be achieved by using QCD inspired models.

In this paper we discuss the effects of a random gas of instantons and antiinstantons on mesons and baryons containing one or several heavy quarks. We analyze the correlation functions of various hadrons with one or many heavy quarks in inverse powers of the heavy quark mass $m_Q$ using a succession of Foldy-Wouthuysen transformations prior to radiative corrections. In section 2, we give a brief summary of the salient properties of some typical heavy-light systems. In section 3, we show how the heavy meson correlator may be systematically analyzed in inverse powers of the heavy quark mass, in the planar approximation. Recoil and magnetic corrections to both the heavy quark propagator and the heavy-meson correlator are evaluated at next to leading order in the heavy quark mass. These results are quantified in the form of effective interactions between heavy and light constituent quarks. In section 4, we estimate the effects of the induced effective interactions and heavy meson and baryon spectra. The results are in overall agreement with the constituent quark model and heavy solitons. We briefly discuss similar effects in exotic hadronic configurations. Our conclusions are summarized in section 5.
II. GENERALITIES

Throughout, heavy hadrons will be understood as mesons or baryons with at least one heavy quark. Although we will be interested in taking the heavy quark mass to infinity and then relaxing it, we will in fact specifically have in mind for a heavy quark, the bottom quark $b$ with a mass of $4.7 - 5.3$ GeV, the charmed quark $c$ with a mass of $1.3 - 1.7$ GeV, and to some extent the strange quark $s$ with a mass of $100 - 300$ MeV.

Heavy mesons $Q\bar{q}$ may be organized in their ground state into multiplets with $I(J^P) = \frac{1}{2}(0^-, 1^-)$. In the heavy quark limit, the multiplets are invariant representations of the heavy quark symmetry group (essentially left spin rotation). Empirically $(K, K^*) = (493, 892)$ MeV, $(D, D^*) = (1869, 2010)$ MeV and $(B, B^*) = (5278, 5324)$ MeV. $D^*$ decay is mainly through $D\pi$.

The heavy baryons with one heavy quark will be of the type $Qqq$. The conventions being $\Lambda_Q = 0^{1+}$, $\Sigma_Q = 1^{1+}$ and $\Sigma_Q^* = 1^{3+}$. The ones with two heavy quarks will be of the type $QQq$. Few heavy baryons have already been observed. Organizing them in multiplets invariant under heavy quark symmetry, we have $\Lambda_s, (\Sigma_s, \Sigma_s^*) = 1116, (1195, 1385)$ MeV, $\Lambda_c, (\Sigma_c, \Sigma_c^*) = 2284, (2455, 2530\pm5\pm5 \text{[4]})$ MeV, and $\Lambda_b, (\Sigma_b, \Sigma_b^*) = 5641, (\ldots, \ldots)$ MeV. The dots refer to yet to be measured masses. Other measured heavy (charmed) baryons include $(\Xi_c, \Xi_c^*) = (2468, 2642.8\pm2.2 \text{[5]})$ MeV and $\Omega_c = 2704$ MeV. Heavy baryons with more than one heavy quark have not been found yet.

The basic principles at work in a heavy light system are best illustrated using a simple bag model description. If we were to insert a heavy source in a spherical cavity of radius $R$, then the total energy can be organized using the bare heavy quark mass $m_Q$ following $E = m_Q + E_0 m_Q^0 + E_1 m_Q^{-1} + \ldots$. The contribution $E_0$ refers to the energy of the light quarks present in the cavity and is standard [1]. The contribution $E_1$ corresponds to

\footnote{To be confirmed by other experiments.}
where the first term is the recoil of the heavy quark, and the second term is the magnetic interaction between the average magnetic field induced by the light quark at the center of the bag, and the magnetic moment of the heavy quark. While schematic, (1) captures the essence of the $1/m_Q$ corrections in heavy quark physics. Using standard bag model parameters [6], we have for charmed mesons (recoil, spin) $\sim (400, 20)$ MeV, while for bottom mesons (recoil, spin) $\sim (100, 5)$ MeV [7].

### III. HEAVY HADRONS IN AN INSTANTON GAS

In what follows, we will try to understand the origins of $E_0$ and $E_1$ from a microscopic description of the QCD vacuum using a random gas of instantons and antiinstantons.

#### A. Heavy Quark Expansion

Consider the correlation function of a heavy-light meson in the QCD vacuum. In Minkowski space, it reads

$$C_t^\pm(x,x') = \langle T(\bar{\psi}(x)\gamma^\pm q(x')) \rangle$$

with $\Gamma_\pm = (1, \gamma) \times (1 \pm \gamma^0)/2 \times (1, T)$ a non-relativistic source with arbitrary flavor. Here $q(x)$ refers to the light quark, and $\psi(x)$ to the heavy quark with bare mass $m_Q$. For $m_Q$ much larger than the typical scale of the problem $\Lambda_{QCD}$, one may use $\Lambda_{QCD}/m_Q$ expansion to analyze (2). We perform this expansion using a Foldy-Wouthuysen transformation of the heavy quark field [8].

$$\psi(x) \sim e^{-i\gamma_0 m_Q t} e^{-i\sigma_0 |\nabla \cdot \gamma|^2/4 m_Q^2} e^{-i\vec{\gamma} \cdot \hat{\nabla}/2m_Q} Q(x)$$

where $\nabla = \partial - iA$ and with $g$ (the gauge coupling) set to one. The first transformation rescales the momenta, the second eliminates the odd parts, and the third removes the mass
term. The successive transformations in (2) are vector-like, unitary and gauge-covariant. In terms of (3) the QCD part of the action for the heavy field $\psi$ becomes

$$\mathcal{L}_\psi \sim \overline{Q}i\gamma^0\nabla^0Q + \overline{Q}\hat{O}_1Q + \mathcal{O}(m_Q^{-2})$$

(4)

where operator $\hat{O}_1$ is defined as

$$\hat{O}_1 = -\frac{\vec{\nabla}^2}{2m_Q} - \frac{\vec{\sigma} \cdot \vec{B}}{2m_Q}$$

(5)

with $B^i = -ie^{ijk}[\nabla^i, \nabla^j]$. Equation (4) has the expected FW form to order $m_Q^{-1}$. In the first term in (5) we recognize the recoil, in the second we recognize spin effect on the heavy quark in external field. Under (3) the heavy meson source shifts

$$\mathcal{T}_\Gamma \psi \sim \mathcal{T}_\Gamma e^{-i\gamma^0m_Qt} \left(1 - \frac{i\vec{\gamma} \cdot \vec{\nabla}}{2m_Q}\right)Q$$

(6)

As a result, the correlator (2) takes the generic form

$$\mathcal{C}_\Gamma \sim \langle 0|\mathcal{T}_\Gamma e^{-i\gamma^0m_Q(t-t')}Q\bar{Q}\Gamma q|0\rangle + \langle 0|\mathcal{T}_\Gamma e^{-i\gamma^0m_Q(t+t')}\left[\overline{Q\bar{Q}}, \frac{i\vec{\gamma} \cdot \vec{\nabla}}{2m_Q}\right]\Gamma q|0\rangle$$

(7)

Mixing between the particle and the antiparticle content of the correlator (6) drops out in the nonrelativistic limit [9]. Thus, in Euclidean space

$$\mathcal{C}^\pm_\Gamma(x, x') \sim -e^{\mp m_Q(\tau-\tau')}\langle 0|\text{Tr} (\Gamma_{\pm} S_Q(x, x') \Gamma_{\pm} S(x', x))|0\rangle$$

(8)

where $S$ is the propagator of the light quark, and $S_Q$ is the heavy quark propagator,

$$S_Q \sim S_\infty + S_\infty \hat{O}_1 S_\infty + \mathcal{O}(m_Q^{-2})$$

(9)

with $S_\infty = \gamma_4/i \nabla_4$ being the free part. This construction can be carried out to arbitrary orders in $1/m_Q$ [3,9], given that the heavy quark expansion is not upset by renormalization [8].

**B. Heavy Quark Propagator**

The heavy quark propagator (9) may be analyzed in a random instanton gas. In the planar approximation, the infinitely heavy quark propagator satisfies the integral equation [9,10].
\[ S^{-1}_\infty = S^{-1}_* - \sum_{I,J} \langle \left( A_{I,J}^{-1} - S_{\infty} \right)^{-1} \rangle \]  

(10)

where \( A_I = \gamma^4 A^4 \) and \( S_* = i\gamma^4 \partial^4 \). The sum is over all instantons and anti-instantons and the averaging is over the position \( z_I \) and the \( SU(N_c) \) color orientation \( U_I \), with

\[ A_I(x) = U_I A(x - z_I, \rho) U_I^\dagger \]  

(11)

Generically

\[ \sum_{I,J} \rightarrow \frac{N}{2} \left( \frac{1}{V_4} \int d^4z_I \right) \int dU_I + (I \rightarrow T) \sim \frac{N}{2V_4N_c} \int d^4z_I \text{Tr}_C \left( I + T \right) \]  

(12)

where \( \text{Tr}_C \) stands for a trace in color space. For a low instanton density \( n = N/V_4 \sim 1 \text{ fm}^{-4} \), we can iterate (10) in powers of \( n \). The result is

\[ S^{-1}_\infty = S^{-1}_* + i n \left( \Theta_0 + \frac{1}{\rho m_Q} \Theta_1 + \mathcal{O} \left( \frac{1}{\rho^2 m_Q^2} \right) \right) + \mathcal{O}(n^2) \]  

(13)

where the diluteness factor is given by the dimensionless combination \( n\rho^4 \sim 10^{-3} \). Substituting (13) into (10), we obtain

\[ \Theta_0 = \int d^4z_I \text{Tr}_C \left( S^{-1}_* \left( \frac{1}{i\gamma^4 \nabla_{4,I}} - S_* \right) S^{-1}_* + I \rightarrow T \right) \]  

(14)

and

\[ \Theta_1 = \int d^4z_I \text{Tr}_C \left( S^{-1}_* \left( \frac{1}{i\gamma^4 \nabla_{4,I}} - S_* \right) \mathcal{O}_1 \left( \frac{1}{i\gamma^4 \nabla_{4,I}} - S_* \right) S^{-1}_* + I \rightarrow T \right) \]  

(15)

where \( \mathcal{O}_1 = -\vec{\nabla}^2/2 - \vec{\sigma} \cdot \vec{B}/2 \). Both \( \Theta_0 \) and \( \Theta_1 \) are \( \tau \)-dependent. To proceed further, we note that in coordinate space, the heavy quark propagator in the one instanton background reads

\[ <x| \frac{1}{i\nabla_{4,I}} |0> = \delta(\vec{x}) \theta(\tau) \frac{1 + \gamma^4}{2} P e^{i \int_0^\tau ds A_4(x_s - z_I)} \]  

(16)

with \( x_s = (s, \vec{x}) \). Inserting (13) into (14,15) and using the one-instanton configuration

\[ A^a_\mu(x) = +\gamma_{\mu\nu} x_\nu \left( \frac{1}{x^2} - \frac{1}{x^2 + \rho^2} \right) \]  

(17)
yield for large times

\[ < x_{-\infty} | (\Theta_0, \Theta_1) | x_{+\infty} > \sim 8\pi \rho^2 (-4I_0, +I_1) \] (18)

with

\[ (I_0, I_1) = \int_0^\infty x^2 dx \left\{ \cos^2(f_x/2) , \left( [\partial_\mu \cos f_x]^2 - [\partial_\mu \sin f_x]^2 + (\rho A_\mu^a)^2 \cos 2f_x \right) \right\} \] (19)

where \( f_x = \pi |x|/\sqrt{1 + x^2} \). There is no contribution to \( \Theta_1 \) from the spin part \(-\bar{\sigma} \cdot \vec{B}\). From (18) it follows that the instanton induced shift in the heavy quark mass is \( \Delta_0 M_Q \sim 70 \text{ MeV} \) from \( \Theta_0 \) and \( \Delta_1 M_Q \simeq 16 \text{ MeV} \) from \( \Theta_1 \) for a \( c \)-quark with \( m_c = 1350 \text{ MeV} \). The recoil effect \( \Delta_1 M_Q \) is an order of magnitude down compared to the naive bag estimate (1). This can be understood by noting that in the presence of instantons, the energy of a heavy quark can be rewritten schematically as

\[ E = 32\pi \times n\rho^4 \times \left( 1/\rho + \frac{1}{\rho^2} \frac{1}{m_Q^2} + ... \right) \] (20)

where the factors follow from (13) and (18). (20) is the analog of (1). For the instanton parameters used, and a charmed quark, (20) yields

\[ E \sim 32\pi \times 10^{-3} \times \left( 600 + \frac{1}{2} 600 + ... \right) \sim \left( 60 + 30 + ... \right) \text{ MeV} \] (21)

which shows that the zeroth order shift in the mass is about 60 MeV, while the recoil effect is about 30 MeV, as expected.

Similar arguments can be used for the light quark propagator. The result is \( S^{-1} \sim S_0^{-1} + i\sqrt{n}\Sigma(x) \) with an average light quark mass shift \( \Delta M_q \sim 420 \text{ MeV} \) [1,11].

**C. Heavy Quark Correlator**

The correlator (2), written generically as \( \mathcal{C} \sim \langle S \otimes S_\infty \rangle \), receives contributions from both planar and non-planar graphs and is usually hard to analyze in the random gas approximation exactly. In the planar approximation, things simplify. After resummation, the inverse correlator (2) reads
\[ C^{-1} \sim S^{-1} \otimes S_{\infty}^{T^{-1}} - \sum_{I} \langle \langle S - A_{I}^{-1} \rangle^{-1} \otimes (S_{\infty} - A_{I}^{-1})^{T^{-1}} \rangle \]  

(22)

The upper script \( T \) is short for transpose. Here the spin-flavor-color indices are left uncontracted and the space-time indices are omitted. Standard perturbation techniques in the massless sector gives

\[
\frac{1}{A_{I}^{-1} - S} = \sum_{n} S^{-1} \frac{\langle \Phi_{n} \rangle \langle \Phi_{n} \rangle}{\langle \Phi_{n} | S^{-1} - A_{I} | \Phi_{n} \rangle} A_{I}
\]

(23)

where \( |\Phi_{n}\rangle \) is the normalized eigenstate of the Dirac operator in a one-instanton background,

\[
(i\bar{\phi} - A_{I}) \Phi_{n}(x - z_{I}) = \lambda_{n} \Phi_{n}(x - z_{I})
\]

(24)

It follows that the 't Hooft zero mode \( \Phi_{0}(x - z_{I}) \)

\[
\Phi_{0}(x) = \frac{1}{\pi} \frac{\rho}{(x^{2} + \rho^{2})^{3/2}} \sqrt{x^{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \varphi
\]

(25)

is dominant in (23). Thus, to leading order we obtain

\[
C^{-1} \sim S^{-1} \otimes S_{\infty}^{T^{-1}} - n \int d^{4}z_{I} \text{Tr}_{\epsilon} \left[ \left[ L \right]_{I} \otimes \left[ H \right]_{I} + I \rightarrow T \right]
\]

(26)

with

\[
\left[ L \right]_{I} = S_{0}^{-1} \left( \frac{\langle \Phi_{0} \rangle \langle \Phi_{0} \rangle}{i\sqrt{n\Sigma_{0}}} - S_{0} \right) S_{0}^{-1}
\]

(27)

\[
\left[ H \right]_{I} = S_{s}^{-1} \left( \frac{1}{i\gamma^{4} \nabla_{4,I}} - S_{s} \right) S_{s}^{-1} + \frac{1}{m_{Q}} S_{s}^{-1} \left( \frac{1}{i\gamma^{4} \nabla_{4,I}} - S_{s} \right) O_{1} \left( \frac{1}{i\gamma^{4} \nabla_{4,I}} - S_{s} \right) S_{s}^{-1}
\]

where \( \Sigma_{0} = \langle \Phi_{0} | \Sigma | \Phi_{0} \rangle \sim (240 \text{ MeV})^{-1} \).

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D. Effective Interactions

The inverse correlator (26) allows for an immediate translation to effective interactions. In the long wavelength limit, the instanton size is small, and a local interaction between the effective fields \( Q \) and \( q \) can be derived much like the 't Hooft interaction between the light effective fields \( q \) [13,14]. From the Bethe-Salpeter equation associated to (26), we read the vertex
\[ \Gamma_{ba}(x, y, x', y') = -inN_c \int d^4z_I \int dU_I \left( U^a_i \langle x| L_I \rangle_j |x'\rangle U^{1j}_b \otimes U^c_k \langle y| H_I \rangle_l |y'\rangle U^{1l}_d + I \to T \right) \]  

(28)

where the color matrices have been explicitly displayed. This vertex function gives rise to an effective action \( S_I \)

\[ \Gamma_{ab}(x, y, x', y') = \frac{\delta^4 S_I}{\delta q^a(x) \delta q^b(x') \delta Q^c(y) \delta Q^d(y')} \]  

(29)

which is essentially non-local. In the long-wavelength approximation the kernel (28) factorizes into two independent kernels as \( x \to x' \) and \( y \to y' \). The corresponding effective action reads

\[ S_I = -inN_c \int d^4z_I \int dU_I \left[ \int d^4x \ q^\dagger(x) U_I \langle x| L_I \rangle |x\rangle U^\dagger_I q(x) \right. \]

\[ \left. \quad \times \int d^4y \ Q^\dagger(y) U_I \langle y| H_I \rangle |y\rangle U^\dagger_I Q(y) \right] + I \to T \]  

(30)

and yields the effective interaction in Euclidean space (leading order in \( 1/N_c \))

\[ L_{qQ}^E = n \left( -16\pi^3 I_Q \right) \left( \frac{4\pi^2 \rho^2}{\sqrt{n\Sigma_0}} \right) \left( iQ^\dagger x + \gamma^0 2 iQ^\dagger q + \frac{1}{4} iQ^\dagger + \gamma^0 2 \lambda^a Q iQ^\dagger + \gamma^0 \lambda^a Q \right) \]  

(31)

For the detailed construction of the above lagrangian we refer to the Appendix and our previous paper ([9]). The first bracket in (31) arises from the heavy quark part and the second bracket from the light quark part. Wick-rotating to Minkowski space gives

\[ L_{qQ} = -\left( \frac{\Delta M_Q \Delta M_d}{2nN_c} \right) \left( \frac{1 + \gamma^0}{2} Q \bar{q} q + \frac{1}{4} \frac{1 + \gamma^0}{2} \lambda^a Q \bar{q} \lambda^a q \right) \]  

(32)

which is to be compared with the 't Hooft vertex for two light flavors \( q = (u, d) \)

\[ L_{qq} = \left( \frac{\Delta M_d^2}{2nN_c} \right) \left( \text{det} \bar{q}_R q_L + \text{det} \bar{q}_L q_R \right) \]  

(33)

Interaction (32) is dominated by the Coulomb-like second term and has a proper heavy quark spin symmetry. The recoil effect renormalizes the strength of the interaction through \( \Delta M_Q = \Delta_0 M_Q + \Delta_1 M_Q \sim 86\text{MeV} \). The spin part gives rise to a chromomagnetic interaction

\[ L_{qQ}^{\text{spin}} = \frac{\Delta M_d \Delta M_d^{\text{spin}}}{2nN_c} \frac{1}{4} \frac{1 + \gamma^0}{2} \lambda^a \sigma^{\mu \nu} Q \bar{q} \lambda^a \sigma^{\mu \nu} q \]  

(34)
with
\[ \Delta M_Q^{\text{spin}} = \frac{1}{\rho m_Q} n 16\pi \rho^3 \int x^2 \, dx \frac{\sin^2 f_x}{(1 + x^2)^2} \simeq 3 \text{ MeV} \] (35)
for a $c$–quark. As a Coulomb-like term in (32), it has a smooth $N_c^0$ limit for the large $N_c$ and is attractive in the spin zero, color-singlet channel.

Similar arguments may be applied to the heavy mesons $\overline{Q}Q$ as well. To order $1/m_Q$ and in the planar approximation, the effective interaction among the heavy quarks is given by
\[ L_{\overline{Q}Q} = -\left( \frac{\Delta M_Q \Delta M_Q}{2nN_c} \right) \left( \overline{Q} \frac{1 + \gamma^0}{2} Q \overline{Q} \frac{1 + \gamma^0}{2} Q \right) \left( \frac{1}{4} \overline{Q} \frac{1 + \gamma^0}{2} Q \overline{Q} \frac{1 + \gamma^0}{2} Q \right) \] (36)
The recoil effects appear to first order in $1/m_Q$ and renormalize $\Delta M_Q$. The spin effects are of second order in $1/m_Q$, and result in
\[ \Delta L_{\overline{Q}Q}^{\text{spin}} = \left( \frac{\Delta M_Q^{\text{spin}} \Delta M_Q^{\text{spin}}}{2nN_c} \right) \left( \frac{1}{4} \overline{Q} \frac{1 + \gamma^0}{2} \lambda^a \sigma_{\mu\nu} Q \overline{Q} \frac{1 + \gamma^0}{2} \lambda^a \sigma_{\mu\nu} Q \right) \] (37)

For heavy baryons of the type $qqQ$ we have
\[ L_{qqQ} = -\left( \frac{\Delta M_Q \Delta M_Q^2}{2n^2N_c^2} \right) \left( \overline{Q} \frac{1 + \gamma^0}{2} Q \right) (\text{det} \overline{q}_L q_R + \text{det} \overline{q}_R q_L) + \left( \frac{1}{4} \overline{Q} \frac{1 + \gamma^0}{2} \lambda^a Q \right) (\text{det} \overline{q}_L \lambda^a q_R + \text{det} \overline{q}_R \lambda^a q_L) \] (38)
and to second order in $1/m_Q$
\[ L_{qqQ}^{\text{spin}} = -\left( \frac{\Delta M_Q^{\text{spin}} \Delta M_Q^2}{n^2N_c^2} \right) \left( \frac{1}{4} \overline{Q} \frac{1 + \gamma^0}{2} \lambda^a \sigma_{\mu\nu} Q \right) \left( \text{det} \overline{q}_L \lambda^a \sigma_{\mu\nu} q_R + \text{det} \overline{q}_R \lambda^a \sigma_{\mu\nu} q_L \right) \] (39)
The phenomenological implications of these interactions on heavy-light spectra will be discussed next.

**IV. HEAVY HADRON SPECTRA**

The contributions of the various instanton interactions derived above to the heavy hadron spectra, will be discussed using a variational approach. For Mesons, the generic Hamiltonian is
\[ H = \frac{\vec{p}_q^2}{2m_q} + \frac{\vec{p}_Q^2}{2m_Q} + \frac{1}{2} M \omega^2 |\vec{r}_q - \vec{r}_Q|^2 + H^{(2)} \]  

(40)

where \( M \) is the reduced mass of the heavy-light system, with \( m_q = \Delta M_q \sim 420 \) MeV and \( m_Q = m_c + \Delta M_Q \sim (1350 + 86) \) MeV. The harmonic potential provides for a simple mechanism of confinement. The instanton-induced interaction \( H^{(2)} \), derived from Eqs. (32-38) will be treated as a perturbation. The trial wavefunction is

\[ \psi(\chi) = \left( \frac{2\alpha}{\pi} \right)^{3/4} e^{-\alpha \chi^2} \]  

(41)

where \( \chi = \frac{1}{\sqrt{2}}(\vec{r}_q - \vec{r}_Q) \). Minimizing the expectation value of (41) in (41) with respect to \( \alpha \) yields \( \alpha = \frac{1}{2} M \omega \) with the confining energy \( E_{\alpha} = \frac{3}{2} \omega \) as expected. Since the size \( r = \sqrt{\frac{1}{2\alpha}} \) of the ground state is a function of the reduced mass \( M \), we fix our parameters by the size of the heavy-light system \( r_{qQ} = 0.6 \) fm. Then the size of the heavy-heavy system is \( r_{QQ} \approx 0.4 \) fm, and the confining energy is about \( E_{\alpha} \approx 250 \) MeV for both of them.

A. Mesons

For heavy mesons the relevant two-body instanton-induced interactions are for \( Q\bar{q} \)

\[ H_{qQ}^{(2)} = \left( \frac{\Delta M_Q \Delta M_q}{2nN_c} \right) \left( 1 + \frac{1}{4} \lambda_q^a \lambda_Q^a \right) \delta^3(\vec{r}_q - \vec{r}_Q) \]  

(42)

For \( QQ \)

\[ H_{QQ}^{(2)} = \left( \frac{\Delta M_Q \Delta M_Q}{2nN_c} \right) \left( 1 + \frac{1}{4} \lambda_{Q_1}^a \lambda_{Q_2}^a \right) \delta^3(\vec{r}_{Q_1} - \vec{r}_{Q_2}) \]  

(43)

The induced spin-interaction in the \( Q\bar{q} \) configuration is given by

\[ H_{qQ}^{2,\lambda} = -\left( \frac{\Delta M_Q^{spin} \Delta M_q}{2nN_c} \right) \frac{1}{4} \vec{\sigma}_q \cdot \vec{\sigma}_Q \lambda_q^a \lambda_Q^a \delta^3(\vec{r}_q - \vec{r}_Q) \]  

(44)

We recall that \( \Delta M_Q^{spin} \) is down by one power of \( 1/m_Q \). Using (42) and the trial wavefunction (41) we have

\[ \langle H_{qQ}^{(2)} \rangle \sim -C_F \left( \frac{\Delta M_Q \Delta M_q}{2nN_c} \right) |\psi(0)|^2 = -\frac{N_c}{2} \left( \frac{\Delta M_Q \Delta M_q}{2nN_c} \right) \left( \frac{1}{\sqrt{\pi r_{qQ}}} \right)^3 \]  

(45)
and similarly for (43). Thus \( \langle H_{qQ}^{(2)} \rangle \sim -183 \text{ MeV} \) and \( \langle H_{QQ}^{(2)} \rangle \sim -103 \text{ MeV} \). These numbers should be compared respectively to \(-140 \text{ MeV} \) and \(-70 \text{ MeV} \), as quoted in [4] using qualitative arguments. The spin corrections are of order \( \langle H_{qQ}^{2s} \rangle \simeq -42 \text{ MeV} \), a result that is consistent with the constituent Quark Model estimate of \(~-27 \text{ MeV} \) [15]. The spin induced interaction in heavy systems such as charmonium is tiny, \( \langle H_{QQ}^{2s} \rangle \sim -0.7 \text{ MeV} \).

To evaluate the spectrum in heavy-light and heavy-heavy systems, we use the mass formulae

\[
M_{qQ} = \langle H^{(0)} + H^{(1)} + H^{(2)} \rangle
\]  

(46)

\( H^{(0)} \) is the sum of the binding energy \( \mathcal{E}_{\alpha} \) and the current masses \( m_s = 150\text{MeV}, m_c = 1350\text{MeV}, m_b = 4700\text{MeV} \), (we take all light flavors to be massless). \( H^{(1)} \) stands for the induced instanton mass for the light \( \Delta M_q \sim 420\text{MeV} \) and heavy \( \Delta M_Q \sim 86\text{MeV} \) quarks. \( H^{(2)} \) provides for the extra Coulomb binding energy discussed above the \( 1/m_Q \) hyperfine splitting within the multiplets. Our results are summarized in Table 1. Overall, the results are in reasonable agreement with experiment and the constituent quark model [15].

B. Baryons

Heavy baryons may be analyzed in similar fashion using the induced interactions discussed at the end of section 3. First, we note that the analog of the one-gluon exchange in our case, is the induced two-body interaction (32), that is

\[
H_{qq}^{(2)} = \left( \frac{\Delta M_q \Delta M_q}{n N_c} \right) \left( 1 + \frac{3}{32} \lambda_1^a \cdot \lambda_2^a - \frac{9}{32} \sigma_1 \cdot \sigma_2 \lambda_1^a \cdot \lambda_2^a \right) \delta^3(\vec{r}_1 - \vec{r}_2) \]  

(47)

and similarly for baryons with two heavy quarks. This interaction is expected to be overall attractive, thus binding. We note that it scales like \( N_c \) in baryonic configurations. Instantons in heavy baryons, induce also a three-body interaction (38), that is

\[
H_{qqQ}^{(3)} = -\left( \frac{\Delta M_Q \Delta M_q^2}{2n^2 N_c^2} \right) \sum_{i<j} \delta^3(\vec{r}_i - \vec{r}_j) \delta^3(\vec{r}_j - \vec{r}_Q) \]
This interaction scales as $N_c^0$. Although subleading in our previous book-keeping arguments, we will keep it in our $N_c = 3$ arguments.

For baryons, the trial wavefunctions, will be chosen in the form

$$\psi(\chi, \eta) = \left(\frac{2\alpha}{\pi}\right)^{3/2} e^{-\alpha(\chi^2 + \eta^2)}$$

(49)

where $\vec{\chi} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2)$ and $\vec{\eta} = \sqrt{\frac{1}{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_Q)$ are standard Jacobi coordinates. Here we choose $r_{qqQ} = 1$ fm for the size of the heavy-light baryons. The size of the heavier configurations will be set to $r_{QQQ} \approx 0.86$ fm for $QQQ$ and $r_{QQQ} \approx 0.7$ fm for $QQQ$. The confining energy is about $E_\alpha \sim 500$ MeV for all of them.

The addition of one more quark in the baryonic configurations, brings about in the three-body contribution to the energy an additional overall factor of $R_q$ for a light quark, and $R_Q$ for a heavy quark, in comparison to (47). Specifically,

$$R_{q,Q} = -2 \left(\frac{\Delta M_{q,Q}}{2nN_c}\right) \left(\frac{1}{\sqrt{\pi r}}\right)^3$$

The three-body contribution is repulsive, whereas the two-body contribution is attractive. Thus, there is a subtle interplay between two- and three-body interactions in the determination of the overall energy of the heavy-light baryonic systems. For a baryon size $r = 1$ fm, $R_q \sim -0.75$ and $R_Q \sim -0.06$. We note that the results are very sensitive to the size of the hadron. Indeed, for $r = 0.9$ fm, $R_q \sim -1$, and for $r = 0.4$ fm, $R_Q \sim -1$. In other words, two- and three-body contributions, become comparable in strength. The size $r$ is fixed by the choice of the potential (47) and is independent from the character of the induced interaction in our discussion. Here, we chose to work with $r_{qQ} = 0.6$ fm for the heavy-light mesons, and $r_{qqQ} \sim 1$ fm, for the heavy baryons. A smaller size, say $r_{qqQ} \sim 0.5$ fm would not fit the spectrum. Of course, other choices may also be possible, with other choices of the potential in (10).
We present the spectra for heavy baryons in Table 2. Our results are in a reasonable agreement with experiment and other models. Major uncertainty comes from the $1/N_c$ corrections (non-planar graphs) and the approximation for the ground-state wavefunctions. As we have seen, the first order corrections were up to 25% of the leading order. Since we are making two expansions, $\Lambda_{\text{QCD}}/m_Q$ and $1/N_c$, one would also expect substantial contributions coming from the first order corrections in $1/N_c$ for physical $N_c = 3$.

C. Exotics

The rationale of constructing two- and three-body interactions using instanton induced effects, can be extended to multi-quark configurations. For example, for $QQqq$ configurations, the integration over group (cf. (A-9)) leads to the four-body interaction of the form

$$\mathcal{L}_{qqQQ} = -nN_c\left(\frac{\Delta M_q}{nN_c}\right)^2\left(\frac{\Delta M_Q}{2nN_c}\right)^2 \times \left(\begin{array}{c}
\bar{Q}\frac{1 + \gamma^0}{2}Q 
\bar{Q}\frac{1 + \gamma^0}{2}Q 
\end{array}\right) \left(\begin{array}{c}
\text{det} \bar{q}_L q_R + \text{det} \bar{q}_R q_L \\
\text{det} \bar{q}_L q_R + \text{det} \bar{q}_R q_L 
\end{array}\right)
$$

$$+ \frac{1}{4} \left(\begin{array}{c}
\bar{Q}\frac{1 + \gamma^0}{2}Q 
\bar{Q}\frac{1 + \gamma^0}{2}Q 
\end{array}\right) \left(\begin{array}{c}
\text{det} \bar{q}_L \lambda^a q_R + \text{det} \bar{q}_R \lambda^a q_L \\
\text{det} \bar{q}_L \lambda^a q_R + \text{det} \bar{q}_R \lambda^a q_L 
\end{array}\right)
$$

$$+ \frac{1}{4} \left(\begin{array}{c}
\bar{Q}\frac{1 + \gamma^0}{2}Q 
\bar{Q}\frac{1 + \gamma^0}{2}Q 
\end{array}\right) \left(\begin{array}{c}
\text{det} \bar{q}_L \lambda^a q_R + \text{det} \bar{q}_R \lambda^a q_L \\
\text{det} \bar{q}_L \lambda^a q_R + \text{det} \bar{q}_R \lambda^a q_L 
\end{array}\right)$$

(50)

The overall sign is consistent with the naive expectation, that the $n$-body interaction follows from the $(n + 1)$-body interaction by contracting a light quark line, resulting in an overall minus sign (quark condensate).

We recall that for each extra light quark, the penalty factor in the energy is $R_q$. Starting with $r = 1$ fm for three quark states (whether heavy or light), we find that the radius $r$ shrinks to 0.93 fm for one additional light quark (four-quark state), to 0.88 fm for an extra one (five-quark state), and to 0.84 fm for still another one (six-quark state). For $r = 1$ fm, $R_q \sim -0.72$, while for $r = 0.9$ fm, $R_q = -1$. It follows that the three-body interaction (repulsive) will tend to overcome the binding energy provided by the two-body interaction (assumed attractive) in the multiquark configurations of the type ($\bar{q}qqq$), ($\bar{Q}qqq$),
In this respect, we agree with the conclusions of [16] that the H-dibaryon viewed as a six-light-quark state \((qqq qqq)\), will be unbound by the three body-forces induced by instantons.

Adding a heavy quark brings about a penalty factor \(R_Q\) in the energy. This factor is 0.06 for \(r = 1\ fm\), and 1 for \(r = 0.4\ fm\). Using the harmonic potential with two light quarks and four heavy quarks, yield \(r = 0.8\ fm\) and \(R_Q = 0.1\). Thus, for heavy six-quark states the three-body interaction is 10\% of the two-body interaction, hence small. In this respect, if we were to think about the H-dibaryon as a six-heavy-light-quark state \((QQq Qqq)\) will not be unbound by the three-body interaction. Similarly, the four-body-interaction will be expected to be about 1\% of the two-body, about the same order of magnitude as the hyperfine splitting discussed above. Therefore we conclude, that the multi-body effects are only important for multi-quark states near threshold.

We have run specific calculations for exotics containing two and three heavy quarks of the type \((\bar{Q}Q qq)\) and \((\bar{Q}Q Qqq)\). With our choice of parameters, we have found that these configurations were stable against strong decays through \(\bar{Q}Q qq \rightarrow \bar{Q}q + \bar{Q}q\) and \(\bar{Q}Q Qqq \rightarrow \bar{Q}Q + Qqq\) or \(\bar{Q}Q Qqq \rightarrow \bar{Q}q + QQq\). In both cases the binding energy was found to be of the order of 10 MeV, in agreement with other models [17] and [18].

V. CONCLUSIONS

We have presented a general framework for discussing the effects of a dilute and random instanton gas, on the multiquark configurations involving heavy and light flavors. Our instanton-induced interactions obey chiral and heavy quark symmetry to leading order in the bare heavy quark mass. Recoil and spin effects were explicitly worked out and found to be small on single quarks.

We have used the instanton-induced interactions to analyze the spectra of heavy-light

\[2\text{In Ref. [16] the radius } r = 0.5 \text{ fm was used in comparison to } r = 0.84 \text{ fm in our case.}\]
mesons and heavy-light baryons. The results are in overall agreement with the constituent quark model results as well as soliton calculations. The role of the three-body force in multiquark states was also discussed. If the strangeness is viewed as a light degree of freedom, then the H-dibaryon may be unbound by three-body effects. If on the other hand, strangeness is viewed as a heavy degree of freedom, then the faith of the H-dibaryon is controlled by the strength of the two-body forces.

We have analyzed the role of multi-body induced instanton interactions on multi-quark states and found them to be very sensitive to the size of the states in light systems. The size of the system is fixed by long range confining forces, and thus outside the scope of the instanton-based models. In heavy systems, the two-body interaction is dominant whatever the size of the systems considered.

Clearly the present analysis could be extended in several directions. First, the derivation was based on the planar approximation, and that could be lifted as subdominant effects may be considered. Second, the spectrum calculations may be refined, by considering more realistic potentials and trial wavefunctions. A more thorough analysis of the $1/m_Q$ corrections could be carried along the lines we have discussed using Bethe-Salpeter construction. In this respect, it would be interesting to re-investigate directly the present effects on the various correlation functions. Finally, one could use it to calculate magnetic moments and other static characteristics of the hadrons.

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APPENDIX A: U - INTEGRATION

Here we present a graphical shortcut to the derivation of the higher order interactions. Our task is to average over the string of color matrices

\[ \int dU \prod_{k=1}^{n} U_{i_k}^{a_k} U_{b_k}^{\dagger j_k} \equiv \int dU \left[ U_i^{a_i} U_b^{\dagger j} \right]^n \]  

with invariant measure \( \int dU = 1 \). Averaging over color is equivalent to finding all projections onto the singlets of the group, \( i.e., P(3 \otimes \bar{3})^n \rightarrow 1 \). For small \( n = 1, 2, 3 \) the answer can be obtained by a direct calculation using the method developed by Creutz [13] for averaging over links in lattice gauge theory. For higher \( n \) the color integration using this approach becomes very involved and leads to cumbersome expressions involving sums of \( n!^2 \) strings of \( 2n \) Kronecker delta’s. On the other side, since \( 3 \otimes \bar{3} = 1 \oplus 8 \), the problem reduces in practice to finding all projections of the product of \( n \) octets (adjoint representations) onto the singlet, \( i.e., P(\otimes 8)^n \rightarrow 1 \) for \( SU(N) \) with \( N = 3 \). The number of distinct projections \( A_n \) grows with \( n \) like

\[ A_n = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \]  

(A-2)

In order to avoid explicit presentation of the indices, we use the diagrammatic technique, originally proposed by Cvitanovic [20].

Fundamental graph consists of links and vertices based on the following identification (see Fig. 1):

\[ U_i^{a_i} U_b^{\dagger j} \leftrightarrow \frac{1}{N} \delta_{i_j}^{a_b} \delta^{i}_{i_j} + [\lambda^i]^a_b [\lambda^i]^j_i \]  

(A-3)

where \( \lambda^a (a = 1, \ldots, N^2 - 1) \) are the color Gell-Mann matrices, normalized as \( \text{Tr} \lambda^i \lambda^j = 2 \delta^{ij} \).

To each projector and symbol we assign a weight as follows

\[ P_1 = \frac{1}{N}, \quad P_8 = \frac{1}{4(N^2 - 1)} \]  

(A-4)

For \( n = 1 \) one has
\[
\int dUU_i^a U_i^{†j \ b} = P(1 \oplus 8) = P_1 = \frac{1}{N} \delta_i^a \delta_j^b. \tag{A-5}
\]

since averaging over one octet leads to zero. The indices on the r.h.s. follow from the graphical representation. \(n = 2\) makes use of \(P_8\), i.e., projector of two octets onto singlet (Fig. 2)

\[
\int d[U_i^a U_i^{†j \ b}]^2 = P(1 \otimes 1 + 8 \otimes 8) = P_1 \cdot P_1 + P_8(8 \otimes 8) = \left[\frac{1}{N} \delta_i^a \delta_j^b \right]^2 + \frac{1}{4(N^2 - 1)} [\lambda^i_{a_{b_1}}] [\lambda^j_{a_{b_2}}] [\lambda^k_{a_{b_3}}] = P_1 \cdot P_1 + P_8(8 \otimes 8) \tag{A-6}
\]

Contributions to the spin \((ij)\) and color \((ab)\) parts are totally identical, and to simplify the notation, we retain only the color matrices in our formulae and graphs.

For \(n = 3\) the only new ingredient is the projection of the product of three octets onto singlets. Fig. 3 shows how to project \(n = 3\). Explicitly [14],

\[
P(\otimes 8)^3 = P[8 \otimes (1 + 8_a + 8_s + 10 + 10 + 27)] = P_8[8 \otimes 8_a + 8 \otimes 8_s]
\]

\[
= P_8 P_{d} d(\text{color} \cdot \text{spin}) + P_8 P_{f} f(\text{color} \cdot \text{spin})
\]

\[
= \frac{1}{4(N^2 - 1)} \frac{N}{2(N^2 - 4)} d^{ijk}(\lambda_1^a \lambda_2^b \lambda_3^c)_{b_1} \cdot d(\text{spin})
\]

\[
+ \frac{1}{4(N^2 - 1)} \frac{1}{2N} f^{ijk}(\lambda_1^a \lambda_2^b \lambda_3^c)_{b_1} \cdot f(\text{spin}) \tag{A-7}
\]

where \(d(\text{spin}) = d^{abc}(\lambda_1^a \lambda_2^b \lambda_3^c)_{i}\) and we have used values of the symmetric and antisymmetric octet projectors

\[
P_f = \frac{1}{2N}, \quad P_d = \frac{N}{2(N^2 - 4)} \tag{A-8}
\]

Generalization of the above procedure for arbitrary \(n\) is possible due to the classification of all invariant tensors for \(SU(3)\) by Dittner [21]. All of them could be constructed from the combinations of \(f^{abc}\) and \(d^{abc}\), standard structure constants defined by commutator \([\lambda^a, \lambda^b]_- = 2i f^{abc} \lambda^c\) and anticommutator \([\lambda^a, \lambda^b]_+ = 2d^{abc} \lambda^c + \frac{4}{3} \delta^{ab}\), respectively. Assigning a weight for each symbol: for \(f^{abc} \leftrightarrow P_f\), and for \(d^{abc} \leftrightarrow P_d\), we can derive the result for arbitrary \(n\) from the graphical representation.
For \( n = 4 \) one can either have two \( P_8 \) projections or start with \( P_8 \) and then contract using four combinations \( dd, fd, df, \) and \( ff \)

\[
\int dU[U^a_i U^{ij}_b]^4 = \frac{1}{N} \left[ \delta^a_b \delta^i_j \right] \int dU[U^a_i U^{ij}_b]^3 + \text{sym. perm}'s
\]

\[
+ \left( \text{spin} \right)^j_i \cdot (\lambda_1^i \lambda_2^j \lambda_3^k \lambda_4^l)_b \left\{ \frac{1}{4(N^2 - 1)} \right\} \left[ \delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right]
\]

\[
+ \frac{1}{4(N^2 - 1)} P_d P_d \left[ d^{ilm} d^{klm} + d^{kln} d^{plm} + d^{ilm} d^{kln} \right] + d \leftrightarrow f.
\]  \hspace{1cm} (A-9)

Our last example is \( n = 5 \), Fig. 5. Symbolically, it reads

\[
P(\otimes 8)^5 = \left[ P_8 \cdot P_8 P_d + P_8 \cdot P_d P_d P_d + (d \leftrightarrow f) \right] \times (\text{color} \cdot \text{spin}) \]  \hspace{1cm} (A-10)

Any other \( n \) can be considered in a similar fashion, and since the explicit expression becomes lengthy, the use of the diagrams helps greatly in analyzing the properties of the derived formula.

One should keep in mind that not all of possible \((f,d)\) combinations are linearly independent and using appropriate relations between them \([\ref{21}]\), one can reduce their number to the number \((\ref{A-2})\). Our expression have an advantage of being explicitly symmetric. We have tested the procedure in various ways. One way is to take all possible contractions. For example, as a consistency check, one may contract a pair of indices, \( e.g., (b_1 a^2) (i_1 j^2) \) and see if it is reduced to the \( n - 1 \) case

\[
\delta^{b_1}_{a_2} \delta^{i_1}_{j_2} \int dU[U^a_i U^{ij}_b]^n = N \int dU[U^a_i U^{ij}_b]^{n-1}
\]  \hspace{1cm} (A-11)

The same procedure can be used to check formulae for any \( n \). Another useful check is the analysis of the powers of \( N \). The leading behaviour must scale like \( 1/N^n \) for any \( n \) as \( N \to \infty \). The leading contribution in this limit has a form of the determinant build of Kronecker deltas, reproducing, in the instanton model case, ’t Hooft’s result for an arbitrary number of light flavors.
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Figure Captions

**Figure. 1.** Basic relation.

**Figure. 2.** $n = 2$ contraction.

**Figure. 3.** $n = 3$ contraction.

**Figure. 4.** $n = 4$ contraction.

**Figure. 5.** $n = 5$ contraction.
Table Captions

Table I. Mesonic spectrum.

Table II. Baryonic spectrum.
FIGURES

FIG. 1. Basic relation

\[ a \quad b \quad \Rightarrow \quad \frac{1}{N} \quad \begin{bmatrix} a \\ b & d & c \end{bmatrix} + \begin{bmatrix} a \\ b & d \end{bmatrix} \]

FIG. 2. \( n = 2 \) contraction

\[ P_8 \quad = \quad \begin{bmatrix} \end{bmatrix} \]

FIG. 3. \( n = 3 \) contraction

\[ P_8 \quad = \quad \begin{bmatrix} \end{bmatrix} \]

\[ (d, f) \]
FIG. 4. $n = 4$ contraction

FIG. 5. $n = 5$ contraction
### TABLE I. Mesonic spectrum

| $I(J)^P$ | Exp. value | Prediction | QM [15] |
|----------|------------|------------|---------|
| $D^0$    | $\frac{1}{2}(0^-)$ | 1869       | 1881    | 1800 ÷ 1860 |
| $D^*$    | $\frac{1}{2}(1^-)$ | 2010       | 1965    | 1930 ÷ 1990 |
| $D_s^\pm$ | 0(0$^-$) | 1969       | 2031    | 1975    |
| $D_s^\pm$ | 0(1$^-$) | 2110       | 2115    | 2061    |
| $B^0$    | $\frac{1}{2}(0^-)$ | 5278       | 5231    |

### TABLE II. Baryonic spectrum

| I | J | Exp. value | Prediction | QM [15] | Ref. [17] |
|---|---|------------|------------|---------|-----------|
| $\Lambda_c$ | 0 | $\frac{1}{2}$ | (2285) | 2376 | 2200 | 2170 |
| $\Sigma_c$ | 1 | $\frac{1}{2}$ | (2453) | 2502 | 2360 | 2421 |
| $\Xi_c$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 2468 | 2652 | 2420 | 2421 |
| $\Omega_c$ | 0 | $\frac{1}{2}$ | (2704) | 2802 | 2680 | 2645 |
| $\Xi_{cc}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | ? | 3558 | 3550 | 3510 |
| $\Omega_{cc}$ | 0 | $\frac{1}{2}$ | ? | 3708 | 3730 | 3698 |
| $\Omega_{ccc}$ | 0 | $\frac{3}{2}$ | ? | 4808 | 4810 | 4784 |