NONPARAMETRIC ESTIMATION OF LINEAR MULTIPLIER FOR PROCESSES DRIVEN BY MIXED FRACTIONAL BROWNIAN MOTION

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Abstract: We study the problem of nonparametric estimation of linear multiplier function $\theta(t)$ for processes satisfying stochastic differential equations of the type

$$dX_t = \theta(t)X_t dt + \epsilon d\tilde{W}_t^H, X_0 = x_0, 0 \leq t \leq T$$

where $\{\tilde{W}_t^H, t \geq 0\}$ is a mixed fractional Brownian motion with known Hurst index $H$ and study the asymptotic behaviour of the estimator as $\epsilon \to 0$.

Keywords: Nonparametric estimation, Linear multiplier, Mixed Fractional Brownian motion.

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1 Introduction

Statistical inference for fractional diffusion type processes satisfying stochastic differential equations driven by fractional Brownian motion have been studied earlier and a comprehensive survey of various methods is given in Mishura (2008) and Prakasa Rao (2010). There has been a recent interest to study similar problems for stochastic processes driven by a mixed fractional Brownian motion (mfBm). Existence and uniqueness for solutions of stochastic differential equations driven by a mfBm are investigated in Mishura and Shevchenko (2012) and Shevchenko (2014) among others. Maximum likelihood estimation for estimation of drift parameter in a linear stochastic differential equations driven by a mfBm is investigated in Prakasa Rao (2018a). The method of instrumental variable estimation for such parametric models is investigated in Prakasa Rao (2017). Some applications of such models in finance are presented in Prakasa Rao (2015 a,b). For related work on parametric inference for processes driven by mfBm, see Marushkevych (2016), Rudomino-Dusyatska (2003), Song and Lin (2014), Mishra and Prakasa Rao (2017), Prakasa Rao (2009) and Miao (2010) among others. Nonparametric estimation of the trend coefficient in models governed by stochastic
differential equations driven by a mixed fractional Brownian motion is investigated in Prakasa Rao (2018b).

We now discuss the problem of estimating the function \( \theta(t) \), \( 0 \leq t \leq T \) (linear multiplier) based on the observations of a process \( \{X_t, 0 \leq t \leq T\} \) satisfying the stochastic differential equation

\[
dX_t = \theta(t)X_t dt + \epsilon d\tilde{W}_t^H, X_0 = x_0, 0 \leq t \leq T
\]

where \( \{\tilde{W}_t^H, t \geq 0\} \) is mfBM and study the properties of the estimator as \( \epsilon \to 0 \).

2 Mixed fractional Brownian motion

We will now summarize some properties of stochastic processes which are solutions of stochastic differential equations driven by a mixed fractional brownian motion.

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t), P) \) be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the \( P \)-completion of the filtration generated by this process. Let \( \{W_t, t \geq 0\} \) be a standard Wiener process and \( W^H = \{W_t^H, t \geq 0\} \) be an independent normalized fractional Brownian motion with Hurst parameter \( H \in (0, 1) \), that is, a Gaussian process with continuous sample paths such that \( W_0^H = 0 \), \( E(W_t^H) = 0 \) and

\[
E(W_s^H W_t^H) = \frac{1}{2} \left[ s^{2H} + t^{2H} - |s-t|^{2H} \right], t \geq 0, s \geq 0.
\]

Let

\[
\tilde{W}_t^H = W_t + W_t^H, t \geq 0.
\]

The process \( \{\tilde{W}_t^H, t \geq 0\} \) is called the mixed fractional Brownian motion with Hurst index \( H \). We assume here after that Hurst index \( H \) is known. Following the results in Cheridito (2001), it is known that the process \( \tilde{W}^H \) is a semimartingale in its own filtration if and only if either \( H = 1/2 \) or \( H \in (1/4, 1] \).

Let us consider a stochastic process \( Y = \{Y_t, t \geq 0\} \) defined by the stochastic integral equation

\[
Y_t = \int_0^t C(s)ds + \tilde{W}_t^H, t \geq 0
\]

where the process \( C = \{C(t), t \geq 0\} \) is an \((\mathcal{F}_t)\)-adapted process. For convenience, we write the above integral equation in the form of a stochastic differential equation

\[
dY_t = C(t)dt + d\tilde{W}_t^H, t \geq 0
\]
driven by the mixed fractional Brownian motion $\tilde{W}^H$. Following the recent works by Cai et al. (2016) and Chigansky and Kleptsyna (2015), one can construct an integral transformation that transforms the mixed fractional Brownian motion $\tilde{W}^H$ into a martingale $M^H$. Let $g_H(s,t)$ be the solution of the integro-differential equation

\begin{equation}
\frac{d}{ds} g_H(s,t) + H \int_0^t g_H(r,t)|s-r|^{2H-1} \text{sign}(s-r)dr = 1, 0 < s < t.
\end{equation}

Cai et al. (2016) proved that the process

\begin{equation}
M_t^H = \int_0^t g_H(s,t)d\tilde{W}_s^H, t \geq 0
\end{equation}

is a Gaussian martingale with quadratic variation

\begin{equation}
< M^H >_t = \int_0^t g_H(s,t)ds, t \geq 0
\end{equation}

Furthermore the natural filtration of the martingale $M^H$ coincides with that of the mixed fractional Brownian motion $\tilde{W}^H$. Suppose that, for the martingale $M^H$ defined by the equation (2.5), the sample paths of the process $\{C(t), t \geq 0\}$ are smooth enough in the sense that the process

\begin{equation}
Q_H(t) = \frac{d}{d< M^H >_t} \int_0^t g_H(s,t)C(s)ds, t \geq 0
\end{equation}

is well defined. Define the process

\begin{equation}
Z_t = \int_0^t g_H(s,t)dY_s, t \geq 0.
\end{equation}

As a consequence of the results in Cai et al. (2016), it follows that the process $Z$ is a fundamental semimartingale associated with the process $Y$ in the following sense.

**Theorem 2.1**: Let $g_H(s,t)$ be the solution of the equation (2.4). Define the process $Z$ as given in the equation (2.8). Then the following relations hold.

(i) The process $Z$ is a semimartingale with the decomposition

\begin{equation}
Z_t = \int_0^t Q_H(t)d< M^H >_s + M_t^H, t \geq 0
\end{equation}

where $M^H$ is the martingale defined by the equation (2.5).

(ii) The process $Y$ admits the representation

\begin{equation}
Y_t = \int_0^t \hat{g}_H(s,t)dZ_s, t \geq 0
\end{equation}
where

(2.11) \[ \hat{g}_H(s, t) = 1 - \frac{d}{d < M^H>_s} \int_0^t g_H(r, s) dr. \]

(iii) The natural filtrations \((Y_t)\) and \((Z_t)\) of the processes \(Y\) and \(Z\) respectively coincide.

Applying Corollary 2.9 in Cai et al. (2016), it follows that the probability measures \(\mu_Y\) and \(\tilde{\mu}_W^H\) generated by the processes \(Y\) and \(\tilde{W}^H\) on an interval \([0, T]\) are absolutely continuous with respect to each other and the Radon-Nikodym derivative is given by

(2.12) \[ \frac{d\mu_Y}{d\tilde{\mu}_W^H}(Y) = \exp\left[\int_0^T Q_H(s) dZ_s - \frac{1}{2} \int_0^T [Q_H(s)]^2 d < M^H>_s\right] \]

which is also the likelihood function based on the observation \(\{Y_s, 0 \leq s \leq T\}\) Since the filtrations generated by the processes \(Y\) and \(Z\) are the same, the information contained in the families of \(\sigma\)-algebras \((Y_t)\) and \((Z_t)\) is the same and hence the problem of the estimation of the parameters involved based on the observation \(\{Y_s, 0 \leq s \leq T\}\) and \(\{Z_s, 0 \leq s \leq T\}\) are equivalent.

3 Preliminaries

Let \(\tilde{W}^H = \{W_t^H, t \geq o\}\) be a mixed fractional Brownian motion with known Hurst parameter \(H\). Consider the problem of estimating the function \(\theta(t), 0 \leq t \leq T\) (linear multiplier) from the observations \(\{X_t, 0 \leq t \leq T\}\) of process satisfying the stochastic differential equation

(3.1) \[ dX_t = \theta(t)X_t dt + \epsilon d\tilde{W}^H_t, X_0 = x_0, 0 \leq t \leq T \]

and study the properties of the estimator as \(\epsilon \to 0\).

Consider the differential equation in the limiting system of (3.1), that is, for \(\epsilon = 0\), given by

(3.2) \[ dx_t = \theta(t)x_t dt, x_0, 0 \leq t \leq T. \]

Observe that

\[ x_t = x_0 \exp\{\int_0^t \theta(s) ds\}. \]

We assume that the following condition holds:

\((A_1)\): The trend coefficient \(\theta(t)\) over the interval \([0, T]\) is bounded by a constant \(L\).

The condition \((A_1)\) will ensure the existence and uniqueness of the solution of the equation (3.1).
Lemma 3.1: Let the condition \((A_1)\) hold and \(\{X_t, 0 \leq t \leq T\}\) and \(\{x_t, 0 \leq t \leq T\}\) be the solutions of the equations (3.1) and (3.2) respectively. Then, with probability one,

\[
(3.3) \quad |X_t - x_t| < e^{Lt} \epsilon |\tilde{W}_t^H|
\]

and

\[
(3.4) \quad \sup_{0 \leq t \leq T} E(X_t - x_t)^2 \leq e^{2LT} \epsilon^2 (T^{2H} + T).
\]

Proof of (a): Let \(u_t = |X_t - x_t|\). Then by \((A_1)\); we have,

\[
(3.5) \quad u_t \leq \int_0^t |\theta(v)(X_v - x_v)|dv + \epsilon|\tilde{W}_t^H| \leq L \int_0^t u_v dv + \epsilon|\tilde{W}_t^H|.
\]

Applying the Gronwall’s lemma (cf. Lemma 1.12, Kutoyants (1994), p. 26), it follows that

\[
(3.6) \quad u_t \leq \epsilon|\tilde{W}_t^H|e^{Lt}.
\]

Proof of (b): From the equation (3.3), we have

\[
(3.7) \quad E(X_t - x_t)^2 \leq e^{2Lt} \epsilon^2 E(|\tilde{W}_t^H|)^2 = e^{2LT} \epsilon^2 (T^{2H} + T).
\]

Hence

\[
(3.8) \quad \sup_{0 \leq t \leq T} E(X_t - x_t)^2 \leq e^{2LT} \epsilon^2 (T^{2H} + T).
\]

Define

\[
(3.9) \quad Q_{H, \theta}^* (t) = \frac{d}{d < M^H > t} \int_0^t g_H(t, s) \theta(s)x(s)ds = \frac{d}{d < M^H > t} \int_0^t g_H(t, s) \theta(s)x_0 \exp(\int_s^t \theta(u)du)ds
\]

by using the equation (3.2). We assume that

\((A_2)\): the function \(Q_{H, \theta}^* (t)\) is Lipschitz of order \(\gamma\) for any fixed \(\theta(.)\).
Instead of estimation of the function $\theta(t)$, we consider the equivalent problem of estimating the function $Q_{H,\theta}^*(t)$ defined via the equation (3.9). This can be justified by the observation that the process $\{X_t, 0 \leq t \leq T\}$ governed by the stochastic differential equations (3.1) and the corresponding related process $\{Z_t, 0 \leq t \leq T\}$, as defined by (2.8) have the same filtrations by the results in Cai et al. (2016).

We estimate the function $Q_{H,\theta}^*(t)$ by a kernel type estimator defined by

$$
\hat{Q}_{H,\theta}(t) = \frac{1}{h \epsilon} \int_0^T G\left(\frac{s-t}{h \epsilon}\right) dZ_s
$$

by using the equation (2.9) where $G(u)$ is a bounded function with finite support $[A, B]$ satisfying the condition

(A3): $G(u) = 0$ for $u < A, u > B$ and $\int_A^B G(u)du = 1$.

It is obvious that the following conditions are satisfied by the function $G(.)$

(i) $\int_{-\infty}^\infty G^2(u)du < \infty$, and

(ii) $\int_{-\infty}^\infty G(u)|u|^\gamma du < \infty$ for $\gamma > 0$,

Consider a normalizing function $h \epsilon \to 0$ with $\epsilon^2 h \epsilon^{-3/2} \to 0$ as $\epsilon \to 0$.

4 Main Results

Theorem 4.1: Suppose the conditions (A1), (A2) and (A3) are satisfied. Then for any $0 \leq t \leq T$, the estimator $\hat{Q}_{H,\theta}(t)$ is uniformly consistent, that is,

$$
\lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} E(\hat{Q}_{H,\theta}(t) - Q_{H,\theta}^*(t))^2) = 0.
$$

Proof:

From (2.9), we have,

$$
(4. 2)
$$

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\[ E|\hat{Q}_{H,\theta}(t) - Q^*_{H,\theta}(t)|^2 = E\left| \frac{1}{h^2} \int_0^T G\left(\frac{s-t}{h}\right) (Q_{H,\theta}(s) + \epsilon dM^H_s) - Q^*_{H,\theta}(t) \right|^2 \]

\[ = E\left| \frac{1}{h^2} \int_0^T G\left(\frac{s-t}{h}\right) (Q_{H,\theta}(s) - Q^*_{H,\theta}(s)) ds \right| \]

\[ + \frac{1}{h^2} \int_0^T G\left(\frac{s-t}{h}\right) (Q^*_{H,\theta}(s) - Q^*_{H,\theta}(t)) ds \]

\[ + \frac{\epsilon}{h^2} \int_0^T G\left(\frac{s-t}{h}\right) dM^H_s \]

\[ = E[I_1 + I_2 + I_3]^2 \] (denoting the three integrals as \( I_1, I_2 \) and \( I_3 \) respectively)

\[ \leq 3 E(I_1^2) + 3 E(I_2^2) + 3 E(I_3^2). \]

Now

\[ 3 E I_1^2 = 3 E \left| \frac{1}{h^2} \int_0^T G\left(\frac{s-t}{h}\right) (Q_{H,\theta}(t) - Q_{H,\theta}(s)) ds \right|^2 \]

\[ \leq \frac{3}{h^2} \int_0^T G^2\left(\frac{s-t}{h}\right) ds \left[ E \int_0^T (Q_{H,\theta}(s) - Q^*_{H,\theta}(s))^2 ds \right]. \]

Now

\[ E \int_0^T (Q_{H,\theta}(s) - Q^*_{H,\theta}(s))^2 ds \]

\[ = \int_0^T E \left[ \frac{d}{d < M^H >_s} \int_0^s g_H(s,v) \theta(v) (X(v) - x(v)) dv \right]^2 ds \]

\[ = \gamma_H \int_0^T E \left[ \int_0^s \frac{\partial g_H(s,v) \theta(v) (X(v) - x(v)) dv}{\partial s} \right]^2 \beta(s,H) ds \]

\[ \leq \gamma_H \int_0^T \beta(s,H) \left\{ \int_0^s \left( \frac{\partial g_H(s,v)}{\partial s} \right)^2 \theta^2(v) dv \right\} \int_0^s E(X(v) - x(u))^2 dv \} ds \]

where the constant \( \gamma_H \) and the function \( \beta(s,H) \) depend on the quadratic variation of the martingale \( M^H \). Note that

\[ E(X_v - x_v)^2 \leq e^{2Lv \epsilon^2 (v^{2H} + v)} \] (by Lemma 3.1).

Hence, from the equation (4.4) and \( (A_3) \) we get that,

\[ 3 E I_1^2 \leq C \frac{(B-A)}{h^2} \left\{ \int_{-\infty}^{\infty} G^2(u) du \right\} \epsilon^2 h^2 \]
\[
\times \int_0^T \beta(s, H) \left\{ \int_0^s e^{2Lv(v_2 + v)} dv \right\} \left\{ \int_0^s \left( \frac{\partial gH(s, v)}{\partial s} \right)^2 dv \right\} ds \\
\leq \epsilon^2 h^{-1}_c C(T, L, H)
\]

and the last term tends to zero as \( \epsilon \to 0 \).

In addition,

\begin{equation}
I_2^2 = 3 \left\{ \frac{1}{h_c} \int_0^T G \left( \frac{s-t}{h_c} \right) (Q^*(s) - Q^*(t)) ds \right\}^2
\end{equation}

\begin{align*}
= 3 \left\{ \int_{-\infty}^{\infty} G(u)(Q^*(t + h_c u) - Q^*(t)) du \right\}^2 \\
\leq C \left\{ \int_{-\infty}^{\infty} G(u)|h_c u|^\gamma du \right\}^2 \quad \text{(by (A2))}
\leq Ch_{c}^{2\gamma} \left( \int_{-\infty}^{\infty} G(u)|u|^\gamma du \right)^2
\leq Ch_{c}^{2\gamma} \quad \text{by (A3))}
\end{align*}

and the last term tends to zero as \( \epsilon \to 0 \). Furthermore

\begin{equation}
I_3^2 = \frac{3\epsilon^2}{h_c^2} E \left( \int_0^T G \left( \frac{s-t}{h_c} \right) dM_H^s \right)^2
\end{equation}

\begin{align*}
= \frac{3\epsilon^2}{h_c^2} \int_0^T G^2 \left( \frac{s-t}{h_c} \right) d < M^H >_s \\
= \frac{3\epsilon^2}{h_c^2} \int_0^T G^2 \left( \frac{s-t}{h_c} \right) \beta(s, H) ds \\
\leq \frac{3\epsilon^2}{h_c^2} \left\{ \int_0^T G^2 \left( \frac{s-t}{h_c} \right) ds \int_0^T \beta^2(s, H) ds \right\}^{1/2} \\
\leq C \frac{3\epsilon^2}{h_c^2} \left\{ h_c \int_{-\infty}^{\infty} G^2(u) du D(H, T) \right\}^{1/2} \\
\leq C(T, H) \epsilon^2 h^{-3/2}_c.
\end{align*}

for some constants \( C, D(T, H) \) and \( C(T, H) \) depending on \( T \) and \( H \). The result follows from the equations (4.5), (4.6) and (4.7).

**Corollary 4.2:** Under the conditions \((A_1), (A_2)\) and \((A_3), \)
\[
\lim_{\epsilon \to 0} \sup_{|\theta(.)| \leq L, 0 \leq t \leq T} E \left\{ \hat{Q}_{H,\theta}(t) - Q^*_H,\theta(t) \right\}^2 \frac{\epsilon^{\frac{8\gamma}{4\gamma+3}}}{e^{\frac{8\gamma}{4\gamma+3}}} < \infty.
\]

**Proof:** From the inequalities derived in (4.5), (4.6) and (4.7), we get that there exist positive constants \(C_1, C_2\) and \(C_3\) depending on \(T\) and \(H\) such that

\[
(4. \ 8) \quad \sup_{|\theta(.)| \leq L, 0 \leq t \leq T} E \left\{ \hat{Q}_{H,\theta}(t) - Q^*_H,\theta(t) \right\}^2 \leq C_1 \epsilon^2 h^{-1}_\epsilon + C_2 h^{2\gamma}_\epsilon + C_3 \epsilon^2 h^{-\frac{3}{2}}_\epsilon.
\]

Let \(h_\epsilon = \epsilon^\beta, 0 < \beta < \frac{4}{3}\). Then the condition \(h^{2\gamma}_\epsilon = \epsilon^2 h^{-3/2}_\epsilon\) leads to the choice \(\beta = \frac{4}{4\gamma+3}\) and we get an optimum bound in (4.8) and hence

\[
(4. \ 9) \quad \lim_{\epsilon \to 0} \sup_{|\theta(.)| \leq L, 0 \leq t \leq T} E \left[ \hat{Q}_{H,\theta}(t) - Q^*_H,\theta(t) \right]^2 \epsilon^{\frac{8\gamma}{4\gamma+3}} \leq C
\]

for some positive constant \(C\) which implies the result.

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