Is Asymptotically Weyl-Invariant Gravity Viable?

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Abstract

We explore the cosmological viability of a theory of gravity defined by the Lagrangian $f(R) = R^{n(R)}$ in the Palatini formalism, where $n(R)$ is a dimensionless function of the scalar curvature that interpolates between general relativity when $n(R) = 1$ and a locally scale-invariant and renormalizable theory when $n(R) = 2$. The exact form of $n(R)$ is uniquely determined. The low-curvature limit of this theory is found to be the Palatini equivalent of the Starobinsky inflationary model.

We demonstrate that the theory contains no obvious curvature singularities. A phase space analysis yields three fixed points with effective equation of states corresponding to de Sitter, radiation, and matter-dominated phases at low curvatures. Non-standard dynamics are obtained at higher curvatures. The Hubble and deceleration parameters suggest our model is consistent with an early and late period of accelerated expansion, with an intermediate period of decelerated expansion. The eigenvalues of the three fixed points indicate a universe that begins in a de Sitter-like phase before proceeding to radiation and matter-like phases. However, the stability of the matter-like phase appears to prevent the system from accessing the second period of accelerated cosmic expansion. Therefore, despite several positive features, we conclude that the theory presented is unlikely to be viable.

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1 Introduction

The general theory of relativity is currently our best description of gravity. One reason for this is its explanatory power. Assuming little more than a single symmetry principle general relativity can explain a truly astonishing range of experimental phenomena [1].

However, it is at best incomplete. It is often said that general relativity breaks down at high energies or small distances. Yet, it is more accurate to say high energies and small distances. This is an important distinction since it highlights the regime in which we must modify general relativity, namely for large energy densities, or equivalently for large spacetime curvatures. For example, general relativity predicts its own breakdown at curvature singularities, where scalar measures of the spacetime curvature grow without bound. Furthermore, general relativity becomes fundamentally incompatible with quantum field theory at high curvature scales, a failure known as its non-renormalizability [2, 3]. Therefore, theoretical arguments alone are enough to tell us that general relativity must be modified at high curvature scales.

Experimental data also indicates that general relativity must either be augmented or replaced altogether if it is to agree with observation [4]. For example, general relativity by itself is unable to explain the early phase of accelerated cosmic expansion, as evidenced by myriad high-precision measurements [4]; it must be supplemented with unobserved exotic energy sources and scalar fields [5, 6]. Although this top-down approach, as exemplified by the ΛCDM model, is currently our best description of observed...
cosmological dynamics [1], its ad hoc construction has driven attempts to replace general relativity from the bottom-up.

Finding a viable replacement of general relativity is challenging. Such a theory must at the very least be (i) equivalent to general relativity in the low-curvature limit, (ii) renormalizable in the high-curvature limit, (iii) unitary, (iv) stable, (v) contain no curvature singularities, (vi) consistent with observation.

One attempt is that of higher-order gravity, in which the Lagrangian includes terms quadratic in the curvature tensor. Although this approach is perturbatively renormalizable, and hence satisfies criterion (ii), such higher-order theories are not typically unitary or stable, thus failing to satisfy criteria (iii) and (iv). The only higher-order theories that are unitary and stable are so-called $f(R)$ theories, in which the Lagrangian is an arbitrary function of the Ricci scalar only [7].

There are three types of $f(R)$ theory; metric, Palatini and metric-affine variations [7]. Metric $f(R)$ gravity assumes that the affine connection uniquely depends on the metric via the Levi-Civita connection, as in standard general relativity. The Palatini formalism generalises the metric formalism by relaxing the assumption that the connection must depend on the metric. The metric-affine formalism is the most general approach since it even drops the implicit assumption that the matter action is independent of the connection.

Particular metric $f(R)$ models have been shown to conflict with solar system tests [8], give an incorrect Newtonian limit [9], contradict observed cosmological dynamics [10, 11], be unable to satisfy big bang nucleosynthesis constraints [12] and in some cases contain a fatal Ricci scalar instability [13]. Thus, metric $f(R)$ theories do not typically satisfy criteria (iv) or (vi). As for the metric-affine formalism, it is not even a metric theory in the usual sense, meaning diffeomorphism invariance is likely broken [7]. Thus, metric-affine theories do not seem to satisfy criterion (i). However, it has been shown that the Palatini variation is immune to any such Ricci scalar instability [14]. Palatini formulations also appear to pass solar system tests and reproduce the correct Newtonian limit [15]. Remarkably, a Palatini action that is linear in the scalar curvature is identical to regular general relativity [7]. However, this equivalence does not hold for higher-order theories [16, 7]. In particular, a Palatini action that is purely quadratic in the scalar curvature is identical to normal general relativity plus a non-zero cosmological constant [17].

In Ref. [18] we proposed the theory of asymptotically Weyl-invariant gravity (AWIG) within the Palatini formalism. By construction AWIG satisfies criteria (i)-(iv). The present work aims to test whether this theory also satisfies criteria (v) and (vi), and hence to determine if it is a viable replacement for general relativity.

In addition to satisfying criteria (i)-(iv), a major motivation for developing AWIG was finding a theory that possessed the symmetry of local scale invariance. The need for local scale invariance can be seen by recognising that all length measurements are local comparisons. For example, to measure the length of a rod requires bringing it together with some standard unit of length, say a metre stick, at the same point in space and time. In this way the local comparison yields a dimensionless ratio, for example, the rod might be longer than the metre stick by a factor of two. Repeating this comparison at a different spacetime point must yield the same result, even if the metric at this new point were rescaled by an arbitrary factor $\Omega^2(x)$. This is because both the rod and metre stick would be equally rescaled, yielding the same dimensionless ratio. Such a direct comparison cannot be made for two rods with a non-zero space-like or time-like separation [19, 20]. Therefore, it has been argued that the laws of nature must be formulated in such a way as to be invariant under local rescalings of the metric tensor $g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}$, or equivalently under a local change of units. Moreover, since scale-invariant theories of gravity are gauge theories [21, 22], unification with the other three fundamental interactions, which have all been successfully formulated as local gauge theories, becomes tractable.

It is important to establish the standard against which we will judge whether the presented theory is viable. Criterion (v) will be deemed to be satisfied if at least two different curvature invariants can be shown to be divergence-free. To satisfy criterion (vi) we make the maximal demand that the theory reproduces all four observed phases of cosmological evolution in the correct sequence [23], namely an early period of accelerated expansion, followed by radiation and matter-dominated phases, and finally a late
period of accelerated expansion [24].

This paper is organised as follows. In section 2 we define the model of AWIG, including a unique determination of the dimensionless exponent \( n(R) \). Some important features of the model are discussed at the end of section 2. In section 3 we detail the methodology that will be used to test the viability of our model, namely a detailed phase space analysis and examination of curvature invariants. Results are presented in section 4 followed by a concluding discussion in section 5.

2 Model

The class of theories to which our model belongs are defined by the action

\[
S = \frac{1}{\kappa} \int f(R) \sqrt{-g} d^4x,
\]

where \( \kappa \) is the gravitational coupling, \( f(R) \) is an arbitrary function of the Palatini scalar curvature \( R \) and \( g \) is the determinant of the metric tensor. Varying Eq. (1) with respect to the metric and taking the trace gives the field equations [7]

\[
f'(R)R - 2f(R) = \kappa T.
\]

AWIG is defined as the specific case [18]

\[
f(R) = R^{n(R)},
\]

where \( n(R) \) is a dimensionless function of \( R \) that interpolates between general relativity when \( n(R) = 1 \) and a locally scale-invariant and renormalizable theory of gravity when \( n(R) = 2 \). \( n(R) \) is assumed to be a function of \( R \) because corrections to general relativity are expected to become significant at large curvature scales. Moreover, by defining \( n(R) \) in this way the Lagrangian density \( f(R) \) is a pure function of scalar curvature, and hence is guaranteed to be invariant under arbitrary differential coordinate transformations. In 4-dimensional spacetime \( R^{n(R)} \) has a canonical mass dimension \( 2n(R) \). Since \( \sqrt{-g} \) has mass dimension -4, \( \kappa \) must have a mass dimension of \( 2n(R) - 4 \) if Eq. (1) is to be dimensionless. Thus, in the limit \( n(R) \to 2 \) the gravitational coupling becomes dimensionless, as demanded by scale-invariance.

To complete the definition of this model we must now determine the exact functional form of \( n(R) \). We begin by taking the first derivative of \( f(R) \) with respect to \( R \), denoted by \( f'(R) \), finding

\[
f'(R) = R^{n(R)} \left( \frac{n'(R)}{R} + \log(R) n'(R) \right).
\]

Substituting Eqs. (3) and (4) into Eq. (2) and solving for \( n(R) \) yields

\[
n(R) = 2 + \log_R \left( \frac{-\kappa T}{2\kappa^2} + C_1 \right),
\]

where \( C_1 \) is an integration parameter. To determine \( C_1 \) we use the fact that the symmetry of local scale invariance is signalled by the vanishing of the traced energy tensor [25]. Thus, as \( n(R) \to 2 \) we must also have \( T \to 0 \), which sets \( C_1 = 1 \).

Substituting Eq. (2) into Eq. (3) with \( C_1 = 1 \) and solving the resulting partial differential equation gives

\[
n(R) = \log_R (R^2 + C_2),
\]

where \( C_2 \) is another integration parameter. Since in the low-curvature limit \( R \gg R^2 \), we can perform a series expansion of Eq. (6) about a small expansion parameter \( \epsilon \) with \( R^2 = \epsilon R \). Taking \( \epsilon \to 0 \) as \( n(R) \to 1 \) and solving to find \( C_2 \) yields \( C_2 = R \).
Since \( n (\mathcal{R}) \) must be dimensionless we replace \( \mathcal{R} \) with the dimensionless ratio \( \mathcal{R} \to \mathcal{R}_* \equiv \mathcal{R} / M^2 \), where \( M \) is a constant with dimensions of mass. Equation (6) can now be rewritten in the purely dimensionless form

\[
n (\mathcal{R}_*) = \log_{\mathcal{R}_*} (\mathcal{R}_* + \mathcal{R}_*^2).
\]  

(7)

The exponent defined by Eq. (7) is plotted in Fig. 1, showing that \( n (\mathcal{R}_*) \to 2 \) in the high-curvature limit and \( n (\mathcal{R}_*) \to 1 \) in the low-curvature limit. The model presented in this work is therefore defined by the action

\[
S = \frac{1}{\kappa} \int \mathcal{R} \log_{\mathcal{R}_*} (\mathcal{R}_* + \mathcal{R}_*^2) \sqrt{-g} d^4x.
\]  

(8)

![Figure 1: The dimensionless exponent \( n (\mathcal{R}_*) \) as a function of \( \mathcal{R}_* \). The upper (lower) dashed line gives the asymptotic value of \( n (\mathcal{R}_*) \) in the high-curvature (low-curvature) limit.](image)

Using \( \mathcal{R} = M^2 \mathcal{R}_* \) and the logarithmic identity \( b \log_b(a) = a \), the Lagrangian of AWIG can equivalently be written as

\[
f (\mathcal{R}) = M^{2n(\mathcal{R}_*)} (\mathcal{R}_* + \mathcal{R}_*^2) = M^{2n(\mathcal{R}_*)} \left( \frac{\mathcal{R}}{M^2} + \left( \frac{\mathcal{R}}{M^2} \right)^2 \right) = M^{2n(\mathcal{R}_*)-2} \mathcal{R} + M^{2n(\mathcal{R}_*)-4} \mathcal{R}^2,
\]  

(9)

which is consistent with the expected quadratic quantum correction to the Einstein-Hilbert action of general relativity [27, 28]. Since the canonical mass dimension of \( M^{2n(\mathcal{R}_*)} \) is \( 2n (\mathcal{R}_*) \), the dimensionality of AWIG varies with the dimensionless curvature scale \( \mathcal{R}_* \). The high-curvature limit (\( n (\mathcal{R}_*) \to 2 \)) of Eq. (9) then gives \( f (\mathcal{R}) = M^2 \mathcal{R} + \mathcal{R}^2 \), which is Weyl-invariant. However, the low-curvature limit (\( n (\mathcal{R}_*) \to 1 \)) of Eq. (9) yields \( f (\mathcal{R}) = \mathcal{R} + \mathcal{R}^2 / M^2 \), which interestingly is the equivalent of the Starobinsky model \( f (\mathcal{R}) = \mathcal{R} + \mathcal{R}^2 / M^2 \) but in the Palatini formalism. It is therefore apparent that the Weyl symmetry of \( f (\mathcal{R}) \sqrt{-\tilde{g}} \) present at high curvature scales is broken at lower curvatures due to the running dimensionality of \( M^{2n(\mathcal{R}_*)} \), akin to dimensional transmutation in quantum field theory [29].

Finally, a well-defined conformal transformation of the metric tensor \( \tilde{g}_{\mu\nu} = f' (\mathcal{R}) g_{\mu\nu} \) requires that \( f' (\mathcal{R}) > 0 \) for all \( \mathcal{R} \). This condition is only satisfied if \( M \) has a value of one. The term \( M^{2n(\mathcal{R}_*)} \) can therefore only act to set the dimensionality of \( f (\mathcal{R}) \).

1For \( 0 < \mathcal{R}_* < 1 \) the exponent \( n (\mathcal{R}_*) \) takes a negative value. Since the average scalar curvature decreases with cosmological time, negative powers of \( \mathcal{R}_* \) are generally expected to lead to late time cosmological acceleration [29].

2If \( \mathcal{R} \propto 1/r^2 \), where \( r \) is the distance scale at which spacetime is probed, as expected on dimensional grounds and argued in Ref. [30], then \( f' (\mathcal{R}) \equiv \Omega^2 = 1 + \text{const.} / r^2 \), where \( \Omega \) is the conformal factor. This is precisely the factor needed to explain the appearance of dimensional reduction at small distance scales [31, 32, 33, 34].
3 Method

In this section we detail the method used to test the cosmological viability of the model defined by Eq. (8). The methodology presented in this section follows the work of Refs. [23, 35, 18, 7].

Since cosmological observations by the Planck satellite show that our universe is consistent with being spatially flat at late times [1], we begin by assuming a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right),$$

where $a(t)$ is the scale factor of the universe, a function of cosmological time $t$. The evolution of a spatially homogenous and isotropic universe filled with a cosmological fluid composed of pressureless dust and radiation can be described by the modified Friedmann equation [7, 35]

$$\left( H + \frac{f'(\mathcal{R})}{2f'(\mathcal{R})} \right)^2 = \frac{\kappa (\rho_m + 2\rho_r) + f(\mathcal{R})}{6f'(\mathcal{R})}, \quad (11)$$

where $\rho_m$ and $\rho_r$ are the energy density of matter and radiation, respectively. Using Eq. (11), combined with the energy conservation conditions $\dot{\rho}_m + 3H\rho_m = 0$ and $\dot{\rho}_r + 4H\rho_r = 0$, we can express the time derivative of the Palatini scalar curvature as [35]

$$\dot{\mathcal{R}} = -\frac{3H (f'(\mathcal{R}) \mathcal{R} - 2f(\mathcal{R}))}{f''(\mathcal{R}) \mathcal{R} - f'(\mathcal{R})}. \quad (12)$$

Using Eq. (12) we can replace $\dot{\mathcal{R}}$ in Eq. (11) to obtain [35, 7]

$$H = \sqrt{\frac{3f(\mathcal{R}) - f'(\mathcal{R})\mathcal{R}}{6f'(\mathcal{R})\xi}}, \quad (13)$$

where we have used $T = -(\rho_m + \rho_r) = (f'(\mathcal{R}) \mathcal{R} - 2f(\mathcal{R})) / \kappa$ and

$$\xi = \left( \frac{1 - \frac{3}{2} \frac{f''(\mathcal{R})}{f'(\mathcal{R})} \left( f'(\mathcal{R}) \mathcal{R} - 2f(\mathcal{R}) \right)}{f''(\mathcal{R}) \mathcal{R} - f'(\mathcal{R})} \right)^2 . \quad (14)$$

In this work we shall perform a detailed analysis of the phase space of the model defined by Eq. (8). To facilitate this analysis we establish an autonomous system of equations defined by the pair of dimensionless variables [23]

$$y_1 = \frac{f'(\mathcal{R}) \mathcal{R} - f(\mathcal{R})}{6f'(\mathcal{R})\xi H^2}, \quad y_2 = \frac{\kappa \rho_r}{3f'(\mathcal{R})\xi H^2} . \quad (15)$$

The evolution of $y_1$ and $y_2$ as a function of the cosmic scale factor $a$ are established by the differential equations

$$\frac{dy_1}{dN} = y_1 \left( 3 - 3y_1 + y_2 - 3 \frac{(f'(\mathcal{R}) \mathcal{R} - 2f(\mathcal{R})) f''(\mathcal{R}) \mathcal{R}}{(f'(\mathcal{R}) \mathcal{R} - f(\mathcal{R})) (f''(\mathcal{R}) \mathcal{R} - f'(\mathcal{R}))} (1 - y_1) \right) \quad (16)$$

and

$$\frac{dy_2}{dN} = y_2 \left( -1 - 3y_1 + y_2 + 3 \frac{(f'(\mathcal{R}) \mathcal{R} - 2f(\mathcal{R})) f''(\mathcal{R}) \mathcal{R}}{(f'(\mathcal{R}) \mathcal{R} - f(\mathcal{R})) (f''(\mathcal{R}) \mathcal{R} - f'(\mathcal{R}))} y_1 \right), \quad (17)$$

where $N \equiv \ln(a)$. The fixed points of this system correspond to the values $(y_1, y_2)$ that satisfy

$$\frac{dy_1}{dN} = \frac{dy_2}{dN} = 0. \quad (18)$$
By calculating the eigenvalues \((\lambda_1, \lambda_2)\) of the Jacobian matrix at each point \((y_1, y_2)\) the stability of the fixed points can be determined \[23, 36\]. The fixed point is stable when both eigenvalues are real and negative, and unstable when both are real and positive. The fixed point is a saddle point when both eigenvalues are real and of opposite sign. The nature of the fixed point for different eigenvalues \((\lambda_1, \lambda_2)\) is summarized in Tab. 1.

| Eigenvalues | Fixed point |
|-------------|-------------|
| \(\lambda_1 \neq \lambda_2 < 0\) | Stable |
| \(\lambda_1 \neq \lambda_2 > 0\) | Unstable |
| \(\lambda_1 < 0 < \lambda_2\) | Saddle |

Table 1: Fixed point type based on eigenvalue pairs \((\lambda_1, \lambda_2)\).

The values \((y_1, y_2)\) for each corresponding fixed point are then substituted into the effective equation of state \(w_{\text{eff}}\) given by \[23\]

\[
w_{\text{eff}} = -y_1 + \frac{1}{3} y_2 + \frac{\dot{f}'(R)}{3Hf'(R)} + \frac{\dot{\xi}}{3H\xi} - \frac{\dot{f}'(R)R}{18f''(R)\xi H^3}, \tag{19}
\]

where \(\dot{\xi}\) is determined by taking the derivative of Eq. (14) with respect to time and using Eq. (12). \(\dot{f}'(R)\) is given by \[23\]

\[
\dot{f}'(R) = -\frac{3H(f'(R)R - 2f(R))f''(R)}{f''(R)R - f'(R)} = \dot{R}f''(R). \tag{20}
\]

It will also prove useful to define the deceleration parameter \(q\) in terms of the effective equation of state \(w_{\text{eff}}\). Since the deceleration parameter is defined in terms of the Hubble parameter via

\[
q \equiv -\left(\frac{\dot{H}}{H^2} + 1\right), \tag{21}
\]

and since \[23\]

\[
\frac{\dot{H}}{H^2} = -\frac{3}{2} (1 + w_{\text{eff}}), \tag{22}
\]

we then find

\[
q = \frac{1}{2} (1 + 3w_{\text{eff}}). \tag{23}
\]

In order to further evaluate the viability criteria set out in the introduction we must also test whether our theory contains scalar curvature singularities \[18\]. A local rescaling of the metric tensor by a conformal factor \(\Omega^2(x)\) is equivalent to the transformations \[37, 38, 39\]

\[
g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f'(R)g_{\mu\nu}, \quad g^{\mu\nu} \to \tilde{g}^{\mu\nu} = \left(f'(R)\right)^{-1} g^{\mu\nu}. \tag{24}
\]

The Ricci scalar \(R\) defines the simplest possible curvature invariant. Thus, in the Palatini formalism, \(R\) raised to the power of any positive integer \(n\) transforms under (24) via

\[
R^n \to \frac{R^n}{(f'(R))^n}. \tag{25}
\]

\(^3\)As a cross-check of our methodology and computer code we verified that we are able to successfully reproduce the cosmological dynamics found in Ref. \[23\] for the \(\Lambda CDM\) model.
The next simplest curvature invariant involves the Ricci tensor. Since our model is defined in the Palatini variation, the connection $\Gamma^\nu_{\mu\sigma}$ is not assumed to depend on the metric $g_{\mu\nu}$, and so the Ricci tensor

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\rho\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}$$

may remain invariant under the local rescaling transformation of Eq. (24). The Ricci tensor with upper indices, however, is given by

$$R_{\mu\nu} = g^\rho_{\mu\sigma} g^\nu_{\rho\sigma} \text{ and so it does transform under Eq. (24) according to}$$

$$R_{\mu\nu} \rightarrow R_{\mu\nu} \left( f'(R) \right)^{-2}.$$ Therefore, second order curvature invariants involving the Ricci tensor, namely $R_{\mu\nu} R_{\mu\nu}$, to any integer power $n$, will transform under Eq. (24) according to

$$(R_{\mu\nu} R_{\mu\nu})^n \rightarrow \left( R_{\mu\nu} R_{\mu\nu} \right)^n \left( f'(R) \right)^{2n}.$$ 

(27)

It is unclear whether the Kretchmann scalar is in fact a scalar in the Palatini formalism [40], and so we omit this from our analysis.

4 Results

We find that the model defined by the action of Eq. (8), with $|M| = 1$, contains three fixed points. The eigenvalues and stability of these three fixed points, defined by the roots $(y_1, y_2)$ of Eqs. (16) and (17), are displayed in Tab. 2.

| Fixed point | $(y_1, y_2)$ | $(\lambda_1, \lambda_2)$ | Stability |
|-------------|--------------|------------------------|-----------|
| $P_1$       | (1, 0)       | (2, 3)                 | Unstable  |
| $P_2$       | (0, 1)       | (-2, 1)                | Saddle    |
| $P_3$       | (0, 0)       | (-1, -3)               | Stable    |

Table 2: The dimensionless variables $(y_1, y_2)$, eigenvalues $(\lambda_1, \lambda_2)$ and stability of the three fixed points $P_1$, $P_2$ and $P_3$.

Inserting the obtained coordinate pairs $(y_1, y_2)$ into Eq. (19) yields the effective equation of state $w_{eff}$ as a function of the Palatini scalar curvature for each fixed point. The results are displayed in Fig. 2.

![Figure 2: The effective equation of state parameter $w_{eff}$ as a function of the Palatini scalar curvature $R$ for the fixed point $P_1$ (left), $P_2$ (middle) and $P_3$ (right).](image)

The effective equation of state parameter $w_{eff}$ for the three different fixed points in the low and high-curvature limits are summarised in Tab. 3. Thus, at least in the low-curvature limit we can identify $P_1$ as a de Sitter-like phase, $P_2$ as a radiation-like phase, and $P_3$ as matter-like phase.
Table 3: The effective equation of state in the low-curvature limit \( w_{\text{eff}} (R \to 0) \), high-curvature limit \( w_{\text{eff}} (R \to \infty) \) and the phase type for the three fixed points \( P_1 \), \( P_2 \) and \( P_3 \).

| Fixed point | \( w_{\text{eff}} (R \to 0) \) | \( w_{\text{eff}} (R \to \infty) \) | Phase       |
|------------|----------------|----------------|-------------|
| \( P_1 \)  | -1             | 0              | De Sitter   |
| \( P_2 \)  | \( \frac{1}{3} \) | \( \frac{4}{3} \) | Radiation   |
| \( P_3 \)  | 0              | 1              | Matter      |

To further analyse the cosmological evolution of our model we use Eqs. (13) and (23) to illustrate how the Hubble parameter \( H \) and deceleration parameter \( q \) vary as a function of \( R \). The results are shown in Figs. 3 and 4. If \( H > 0 \) and \( q > 0 \) then the universe is expanding but decelerating. If \( H > 0 \) and \( q < 0 \) then the universe is expanding but accelerating \([11]\). Figures 3 and 4 therefore indicate that the de Sitter-like phase undergoes two periods of accelerated expansion, one in the high-curvature regime \( 1 < R < \infty \) and one in the low-curvature regime \( 0 \leq R \leq 0.138 \), mediated by a period of decelerated expansion for \( 0.138 < R < 1 \) (see Fig. 4(left)). Assuming curvature on cosmological scales decreases with cosmological time, this implies an early and late period of accelerating cosmic expansion, with an intermediate period of decelerating expansion. At least in this sense, the dynamics are consistent with cosmological observations.

Yet, a potentially fatal problem remains. The system evolves from the unstable fixed point \( P_1 \) to the saddle point \( P_2 \) and finally to the stable fixed point \( P_3 \). This series of transitions is highlighted by the red trajectory through the phase space of Fig. 5. However, one cannot exit a stable fixed point, such as \( P_3 \), since it is a universal attractor. This means it is not possible for the system to return to \( P_1 \) from \( P_3 \).
and hence it is not possible to enter the second period of accelerated cosmic expansion from a decelerating matter-dominated phase. This is indicated by the absence of a red line connecting (0, 0) to (1, 0) in Fig. 5.

The currently observed phase of cosmic expansion seems to suggest that at some point in the past such a transition must have occurred. This is a serious problem for the model proposed, which we shall return to in section 5.

Figure 5: A slice of constant curvature through the 3-dimensional phase space of AWIG at $R = 0.01$. The red trajectory shows how the system evolves from the unstable fixed point $P_1$ to the saddle point $P_2$, before terminating at the stable fixed point $P_3$.

Because the phase space of AWIG is 3-dimensional, being made of slices of constant curvature, trajectories emanating from the unstable fixed point $P_1$ may also be thought of as flowing perpendicular to the $(y_1, y_2)$ plane when the curvature is rapidly varying. Thus, the system may evolve along a trajectory that passes through the point $P_1$ perpendicular to the $(y_1, y_2)$ plane in the direction of decreasing $R$ until the curvature becomes approximately constant, at which point the system can flow to $P_2$ and then $P_3$.

We now present results for various powers of the Ricci scalar curvature under the local rescaling of Eq. (24). Using Eqs. (9) and (25) we find

$$R^n \rightarrow \frac{R^n}{(f(R))^n} \overset{R \to \infty}{=} \frac{1}{2^n}. \quad (28)$$

The first three powers of the Ricci scalar curvature ($n = 1, 2, 3$) are shown in Fig. 6. As can be seen from Fig. 6 each curvature invariant approaches a constant in the limit $R \to \infty$. Likewise, Eqs. (9) and (27) can be used to show that the curvature invariant $(R_{\mu \nu}R^{\mu \nu})^n$ formed from the Ricci tensor asymptotically approaches $1/2^{2n}$ as $R \to \infty$. Therefore, the model presented contains no curvature singularities in $R$ or $R_{\mu \nu}R^{\mu \nu}$, at any order $n$. 

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Figure 6: The first three powers of the transformed Palatini Ricci scalar curvature \((n = 1, 2, 3)\) as a function of \(\mathcal{R} c\).

5 Discussion

The model presented contains many positive features, such as the apparent absence of curvature singularities and three fixed points with effective equation of states corresponding to de Sitter, radiation and matter-dominated phases at low curvatures. Our model also contains early and late periods of accelerated cosmic expansion, with an intermediate period of decelerated expansion. Note that the early accelerating phase emerges from AWIG without adding a scalar field. This is because AWIG asymptotically approaches the Palatini formulation of pure \(\mathcal{R}^2\) gravity in the high curvature limit, which is equivalent to general relativity plus a non-zero cosmological constant and no massless scalar field [17].

However, the stability of the matter-dominated fixed point prevents the system from reaching the second period of accelerated cosmic expansion. Therefore, going back to the viability criteria laid out in the introduction, we conclude that the presented model satisfies criteria (i)-(v), but not (vi). Given this apparent failure, we are left with two possibilities:

1. The model presented is not a viable theory.

If AWIG is wrong and the universe is not locally scale-invariant then local changes of scale will in principle be physically observable. This is contrary to Weyl’s proposal of a purely infinitesimal geometry in which only local measurements have any meaning [19, 42]. This would also imply that the laws of physics may not be invariant under a local change of units, a conclusion that is hard to fathom [20]. Furthermore, since scale-invariant theories of gravity are also gauge theories, a gravitational theory that is not locally scale-invariant will likely be harder to unify with the other three fundamental forces, which are all locally gauge invariant [21, 22].

If the model is wrong but the universe is locally scale-invariant then we must find an alternative locally scale-invariant action. There are several possible options. One example is the metric-affine variation of \(f(R)\) gravity [4]. This approach is the most general formulation of \(f(R)\) gravity since it generalises the Palatini formalism by dropping the assumption that the matter action is connection independent. The metric-affine variation has also been shown to contain an enriched phenomenology [7]. Despite these positive features, metric-affine gravity is not a metric theory, meaning the energy tensor is not free of divergences and diffeomorphism invariance is likely broken [7], although some of these points are still debated in the literature [7]. Another possibility is that the fundamental action is not of the type \(f(R)\) at all (see Refs. [43, 44] for examples). However, in this case, it is not clear how to avoid the fatal Ostragadsky instability problem [45].

2. The model presented is a viable theory.
If AWIG is correct and the methodology presented is sufficiently accurate to reveal the true dynamics of the theory then we have a clear prediction - there is no late period of accelerated cosmic expansion [46]. In the work of Ref. [46] the authors provide evidence that the accelerated cosmic expansion deduced from supernovae data may be an artefact of us observing from within a region of the universe that is undergoing bulk flow in one particular direction. This issue may be decided definitively by upcoming experimental data from the Euclid Satellite.

However, it is also possible that AWIG is correct but that the methodology is insufficiently accurate to reveal the true dynamics of the theory. In particular, we identify three approximations underlying our methodology that can certainly be improved. First, is the assumption that the universe is spatially homogenous and isotropic on all scales. At best, the universe is only statistically homogenous and isotropic, and only on distance scales greater than approximately 100 Mpc [47]. Second, is the assumption that the matter and energy content of the universe can be accurately modelled by a cosmological fluid composed of dust and radiation with zero pressure. This assumption is known to be wrong. However, the question is how wrong? And would this modification be sufficient to correct the dynamics of our theory? Lastly, our methodology has assumed a purely classical background throughout. Quantum corrections may significantly modify the proposed theory, even in the low-curvature limit. For example, a quantum field theory on a curved background does not generally lead to a vanishing traced energy tensor [25], which would change the derivation of \( n(R) \) presented in section 2.

Going forward, it is logical to first try to definitively rule out AWIG by testing the proposed modifications of the current methodology, before exploring alternative approaches.

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