EXTENSION-LIFTING BIJECTIONS FOR ORIENTED MATROIDS

SPENCER BACKMAN, FRANCISCO SANTOS, CHI HO YUEN

Abstract. Extending the notion of geometric bijections for regular matroids, introduced by the first and third author with Matthew Baker, we describe a family of bijections between bases of an oriented matroid and special orientations. These bijections are specified by a pair of circuit and cocircuit signatures coming respectively from a generic single-element lifting and extension. We then characterize generic single-element liftings and extensions using these bijections.

We also explain the relation of our work with the works of Gioan–Las Vergnas and Ding. Some implications in oriented matroid programming and oriented matroid triangulations are also discussed.

CONTENTS

1. Introduction 1
1.1. Main results and idea of proof 2
1.2. Connections with Other Works 5
1.3. Background and Organization of the Paper 7
2. Preliminaries 7
3. Proof of the Main Results 10
4. Relation with Orientation Activity and Active Bijections 15
5. Relation to Triangulations of Lawrence Polytopes 18
Acknowledgements 21
References 22

1. INTRODUCTION

The number of spanning trees of a graph is an important numerical invariant that often enumerates other objects associated to the graph. There is a long tradition of providing bijective proofs of such equinumerosity, e.g., proving Cayley’s formula by Prüfer sequences. A particularly active direction in recent years is to construct bijections between spanning trees and special orientations or classes of orientations of the graph, a topic that has connections with the chip-firing model.

Date: December 14, 2023.
on graphs and their generalizations \cite{1} \cite{2}: there is a finite abelian group associated to the graph, known as Jacobian (also critical group or sandpile group in the literature), which can be defined using chip-firing and whose size is the number of spanning trees; the group has a canonical simply transitive action on these special orientations/orientation classes, so the bijections allow us to construct a simply transitive action of the Jacobian on the collection of spanning trees via composition.

A family of bijections, called “geometric bijections” for their relation with polyhedral geometry (zonotopal tilings), was proposed by the first and third author together with Matthew Baker in \cite{3}. These bijections are particularly well-behaved with respect to the said group action (see, for example, \cite{36} Section 5.6] and \cite{10}), and they naturally extend to regular matroids, where spanning trees are replaced by bases. Since then, there have been several follow-up works along this direction, such as the work of McDonough which extended the tiling construction to (representable) oriented arithmetic matroids \cite{33}, and the works of Ding, including \cite{9} which extended the bijections to other classes of subgraphs and gave a purely combinatorial proof of bijectivity, and \cite{8} which we elaborate more in Section 1.2 and 5.

In this paper we extract the oriented matroid theoretic essence of these geometric bijections and extend them to all oriented matroids. This is our Theorem A, which we give a preview here; see Section 2 for details and precise definitions.

1.1. Main results and idea of proof. Let $M$ be an oriented matroid with ground set $E$. A (generic) circuit signature $\sigma$ on $M$ is a way to pick, for each circuit $\mathcal{C}$ of the underlying matroid of $M$, one of the two signed circuits of $M$ supported on $\mathcal{C}$. As an example, any (generic) single-element lifting of $M$, i.e., any oriented matroid $\tilde{M}$ such that $M = \tilde{M}/g$ for some element $g$ only lying in spanning circuits of $\tilde{M}$, induces a generic circuit signature. Dually, a generic single-element extension of $M$, i.e., an oriented matroid $M'$ such that $M = M' \setminus f$ for some element $f$ only lying in spanning cocircuits of $M'$, induces a generic cocircuit signature $\sigma^*$ that picks out a signed cocircuit supported on each cocircuit.

An orientation of $M$ is a map $O : E \to \{+, -\}$. One can equivalently speak about a reorientation $-A M$ of $M$ along a subset $A \subset E$ of elements. The two points of view are equivalent by letting $O(e) = -$ if and only if $e \in A$. We say that an orientation $O$ is compatible with a generic circuit signature $\sigma$ if every signed circuit $C$ conformal with $O$ is the signed circuit picked out by $\sigma$ for $C$; we similarly define the compatibility with a generic cocircuit signature $\sigma^*$. When both things happen we say that $O$ is $(\sigma, \sigma^*)$-compatible.

Recall that for a basis $B$ of $M$ and an arbitrary element $e$, there is either a fundamental circuit $C(B, e)$ (if $e \notin B$) or fundamental cocircuit $C^*(B, e)$ (if
Definition 1.1. Let $M$ be an oriented matroid. Let $\sigma$ and $\sigma^*$ be a generic circuit and cocircuit signature of $M$, respectively.

Given a basis $B$, let $O(B)$ be the orientation of $M$ in which we orient each $e \notin B$ according to its orientation in $\sigma(C(B,e))$ and each $e \in B$ according to its orientation in $\sigma^*(C^*(B,e))$. We denote $\beta_{\sigma,\sigma^*}$ the map

$$\beta_{\sigma,\sigma^*} : \{\text{bases of } M\} \rightarrow \{\text{orientations of } M\}$$

$$B \mapsto O(B).$$

Our main result in this paper is:

**Theorem A.** Let $M$ be an oriented matroid, and let $\sigma$ and $\sigma^*$ be the generic circuit and cocircuit signatures induced from a generic single-element lifting and extension, respectively. Then the map $\beta_{\sigma,\sigma^*} : B \mapsto O(B)$ is a bijection between the set of bases of $M$ and the set of ($\sigma,\sigma^*$)-compatible orientations of $M$.

It turns out that the bijections in Theorem A are so closely related to liftings and extensions of oriented matroids that they characterize generic signatures coming from liftings or extensions. More precisely:

**Theorem B.** Let $\sigma$ be a generic circuit signature of an oriented matroid $M$. The following are equivalent:

1. $\sigma$ is the circuit signature induced by some single-element lifting of $M$.
2. $\beta_{\sigma,\sigma^*}$ is a bijection between bases of $M$ and ($\sigma,\sigma^*$)-compatible orientations, for every generic cocircuit signature $\sigma^*$ of $M$ induced by generic single-element extension.
3. $\beta_{\sigma,\sigma^*}$ maps bases of $M$ to $\sigma$-compatible orientations for every generic lexicographic cocircuit signature $\sigma^*$ of $M$.

Let us remark that part (3), saying that “lexicographic signatures suffice”, has a similar flavor as [35, Theorem 3.5].

**Remark 1.2.** As mentioned above, our work was originally motivated by [3], which proves Theorem A for realizable oriented matroids. Let us explain this connection.

Let $A$ be an $r \times n$ real matrix of full row-rank realizing $M$. An extension (respectively, lifting) of $A$ is an $(r + 1) \times n$ matrix $\tilde{A}$ (respectively, an $r \times (n+1)$ matrix $A'$) restricting to $A$ by deletion of its last row (respectively, column). Each such extension/lifting defines a circuit/cocircuit signature of $M$, and the signatures that can be obtained this way are called acyclic in [3]. Theorem 1.4.1 in [3] is exactly Theorem A for the case of acyclic signatures of a realized oriented matroid. The proof there uses (fine) zonotopal tilings of the zonotope of $A$ which, by the Bohne–Dress Theorem, are equivalent to (generic) liftings of $M$ [7, 34].
Figure 1. Left: The affine pseudohyperplane arrangement of $M(K_3)$ (the three curves represent the elements of $M$), together with the extra elements $g$. The regions are labeled by $\sigma$-compatible orientations of $M$. Right: The new curve represents $f$. There are three regions whose optima with respect to $f$ are bounded, and each of these optima is the intersection of curves that form a basis of $M$.

Our proof of Theorem A uses the theory of oriented matroid programs (OMPs) developed by Bland and Lawrence [6, 11]; see details in the next section, and in [5, Chapter 10]. The intuition is as follows.

Let $\tilde{M}$ and $M'$ be a generic lifting and extension of $M$ inducing the signatures $\sigma^*$ and $\sigma$. By the Topological Representation Theorem of Folkman and Lawrence [11], we can picture $\tilde{M}$ (together with its distinguished element $g$) as an affine pseudohyperplane arrangement with $g$ as the hyperplane at infinity. Cells of the arrangement correspond to covectors of $\tilde{M}$, those containing $g$ lying in the affine chart and those not containing $g$ lying at infinity.

In this picture, each region $R$ of the arrangement (or tope of $\tilde{M}$) corresponds to a $\sigma$-compatible orientation of $M$, namely the orientation that makes $R$ the positive region of the arrangement. See the left part of Figure 1, where an affine arrangement (and its corresponding zonotopal tiling) are shown.

The distinguished element $f$ of the extension $M'$ can be included into the picture as an “increasing direction” or “objective function”, with respect to which we consider the optimum of each region; see the right part of Figure 1. In Section 3 (Theorem 3.1), we prove that

the regions of the arrangement which are bounded with respect to $f$ (that is, those whose optima are not lying on $g$) are precisely the $(\sigma, \sigma^*)$-compatible orientations
and (by genericity of $M'$)

bounded vertices of the arrangement (that is, cocircuits of $\tilde{M}$ containing $g$) correspond bijectively to bounded regions, by assigning to each bounded region its optimum vertex.

Finally, by genericity of $\tilde{M}$,

(bounded) vertices of the arrangement correspond bijectively to bases of $M$.

These three bijections together give the bijection stated in Theorem A. It follows by construction that the map from bases to orientations so obtained coincides with the map $\beta_{\sigma,\sigma^*}$.

Let us mention however that, although the pseudo-hyperplane arrangement image is useful for intuition, our proof does not use or need the Topological Representation Theorem. Instead, we directly construct the bijection between $(\sigma, \sigma^*)$-compatible orientations and bounded vertices of the arrangement (that is, cocircuits of $\tilde{M}$ not vanishing at $g$) using the Fundamental Theorem of Oriented Matroid Programming (Theorem 2.5).

1.2. Connections with Other Works. As mentioned at the beginning, there have been several works on constructing bijections between bases and orientations in recent years. Two of them, by Gioan–Las Vergnas and by Ding, respectively, are of particular relevance here. We explain how these works are related to our construction and highlight some technical/conceptual differences. We consider clarifying this picture to be a secondary contribution of this paper.

There is a series of works by Gioan and Las Vergnas on active bijections [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 31], which began with the PhD thesis of Gioan [12]. Given an oriented matroid $M$ and an ordering of its elements, one can define the internal and external activities of a basis and of an orientation. The active bijection maps bases (together with extra specifications) to orientations in an activity preserving manner (hence their name). See Theorem 4.1 for a more precise statement.

A major component in the construction by Gioan and Las Vergnas is to describe the bijection for $(1, 0)$-bases (bases with a single internally active element and no externally active ones). For the proof of this case, the machinery of oriented matroid programming is also applied, in a way quite similar to ours.

In Section 3, we show that the bijection of Theorem A for the original oriented matroid $M$ with a given generic extension $M'$ and lift $\tilde{M}$ coincides with the active bijection of Gioan and Las Vergnas for an oriented matroid $\tilde{M}$ that simultaneously extends/lifts $M'$ and $\tilde{M}$ and in which $g$ and $f$ are the first and second element in the order. The existence of $\tilde{M}'$ is a result of [35] that we also use for the proof of Theorem A (see our Lemma 3.6). This also establishes a connection
between our work with the classical theorem of Greene and Zaslavsky that the number of bounded regions in an affine pseudohyperplane arrangement equals the beta invariant of the corresponding matroid (independent of the choice of pseudohyperplane at infinity) \[26\] Theorem D: the bounded regions of this auxiliary oriented matroid (with \(g\) being at infinity) correspond to the regions of the oriented matroid program considered in the proof of Theorem A, whose optima with respect to \(f\) are bounded.

However, regarding our bijection as a special case of the active bijection would be an oversimplification, since the two bijections have different input data and description, and since many extensions of Theorem A cannot be easily formulated using the language of active bijections; this includes Theorem B, the construction of Jacobian actions for the regular matroid case, and the relation with oriented matroid triangulations (see below). Conversely, our setting cannot capture many features of active bijections either. It is thus appropriate to say that our work gives an alternative way of applying the Main Theorem of OMP to bijective questions.

In [8], Ding unified geometric bijections (as in the original [3]) and the family of bijections by Kálmán–Tóthmérész [28] for regular matroids \(^1\) using the insight that the data needed to define each family of bijections can be interpreted in terms of dissections of the Lawrence polytope \(\Lambda(M)\) of a regular matroid \(M\) (and/or its dual). In particular, he suggested the notion of triangulating signatures that capture the data of arbitrary triangulations of \(\Lambda(M)\), generalizing acyclic signatures (see Remark 1.2) which induce regular triangulations. He then proved in [8, Theorem 1.20] that a pair of triangulation signatures yields a bijection in the same way as [3, Theorem 1.4.1].

We show that Ding’s bijections in the triangulation setting are special cases of Theorem A by showing that his triangulating signatures are precisely signatures induced by generic single-element liftings/extensions, and they induce triangulations of Lawrence polytopes using the abstract formalism of oriented matroid triangulations developed by the second author [35].

Even if the relation between single-element lifts and triangulations of Lawrence polytopes holds for arbitrary oriented matroids, other ingredients in Ding’s proof (or the very formulation) of the results of [8] depend critically on the properties of regular matroids (e.g., realizability), and do not extend, as far as we can see, to an alternative proof of our results for general oriented matroids. Nevertheless, the bijections in [8] are derived from not just a pair of triangulations of the Lawrence polytope, but also when one of them is a dissection, a notion strictly more general than triangulations and with no abstraction in the context of oriented matroids.

\(^1\)The work of Kálmán–Tóthmérész concerns hypergraphs and their associated root polytopes. Ding transferred the construction to Lawrence polytopes of regular matroids.
at this moment; moreover, the formalism (using *atlases*) and arguments in his work are novel and of independent interest. Hence our work does not subsume his either.

We end this section mentioning another result conceptually, but not technically, similar to ours. Given a *strong map* between oriented matroids $M_1 \to M_2$ on the same ground set, Las Vergnas gave a formula to count the number of orientations that are acyclic in $M_1$ and totally cyclic in $M_2$ [29]. Theorem A has a similar flavour in view of Lemma 3.3 although we note that the map $M \to M'$ is not a strong map in general; indeed, while an extension followed by a contraction of the new elements gives rise to a strong map, $\hat{M} \to M'$ can be thought of (by Lemma 3.6) as a single-element extension followed by contracting a *different* element (or equivalently, the map is a contraction followed by an extension).

1.3. **Background and Organization of the Paper.** Theorem A has been the content of an earlier version of this manuscript, accepted as an extended abstract of FPSAC 2019 under the title *Topological Bijections for Oriented Matroids*. The present paper adds substantial new material to it, including Theorem B and an update on the advances along this direction in recent years. We also decided to change the name of our bijections to “extension-lifting bijections” since, as mentioned above, we use the topological representation of oriented matroids as intuition but our results and proofs do not rely on it.

In Section 2 we recall some basic facts from oriented matroid theory, mostly related to extensions, liftings, and oriented matroid programs. Section 3 contains the proof of Theorems A and B and Sections 4 and 5 show how our results connect to the active bijections by Gioan and Las Vergnas, and to the work of Ding on triangulations of Lawrence polytopes.

### 2. Preliminaries

We assume the reader is familiar with the basic definitions in oriented matroid theory, and we refer to [5] for details and notation. We here recall some notions and results that are crucial in our proofs.

Let $M$ be an oriented matroid of rank $r$ on ground set $E$. The set of bases of $M$ will be denoted by $B(M)$, and the set of signed circuits (respectively, signed cocircuits) of $M$ will be denoted by $C(M)$ (respectively, $C^*(M)$). The support of a signed subset $X$ will be denoted by $X$. Whenever we speak of a circuit without the adjective “signed”, we always mean a circuit of the underlying ordinary matroid; same for a cocircuit of $M$. We often write $M \cup f$, $M \setminus f$, etc instead of $M \cup \{f\}$, $M \setminus \{f\}$, etc for simplicity.

**Definition 2.1.** An oriented matroid $M'$ of rank $r$ is a *single-element extension* of $M$ if the ground set of $M'$ is $E \sqcup f$ for some new element $f$ and $M = M' \setminus f$. 

Dually, \( \widetilde{M} \) of rank \( r + 1 \) is a single-element lifting of \( M \) if the ground set of \( \widetilde{M} \) is \( E \sqcup g \) for some new element \( g \) and \( M = \widetilde{M}/g \).

The extension (respectively, lifting) is generic if every circuit of \( M' \) (respectively cocircuit of \( \widetilde{M} \)) containing \( f \) (respectively, \( g \)) is spanning (respectively, has independent complement).

A cocircuit signature is a map \( \sigma^* : \mathcal{C}^*(M) \to \{+, -, 0\} \) satisfying \( \sigma^*(-D) = -\sigma^*(D) \); it is generic if the image lies in \( \{+, -\} \). Let \( M' \) be a single-element extension of \( M \). For every signed cocircuit \( D \) of \( M \) there exists a unique signed cocircuit \( D' \) of \( M' \) such that \( D'|_E = D \). Therefore we can define a cocircuit signature associated to the extension by setting \( \sigma^*(D) := D'(f) \). Clearly, the extension is generic if and only if \( \sigma^* \) is generic. In such case we can understand \( \sigma^* \) as selecting one of the two signed cocircuits of \( M \) supported on each cocircuit of \( M \), namely the one that extends to have \( f \) on its positive side. Thus, we often abuse notation and write \( \sigma^*(D) \) to be such a signed cocircuit for a cocircuit \( D \) of \( M \) and say that such a signed cocircuit is compatible with \( \sigma^* \); this is the terminology being used in some literature such as [3, 9, 8], and should cause no ambiguity as the input is now a (unsigned) cocircuit.

Dually, every generic single-element lifting induces a generic circuit signature \( \sigma \) that sends each circuit \( C \) of \( M \) to the signed circuit \( \sigma(C) \) of \( M \) with that support that extends to have \( g \) on its positive side.

A particularly important class of single-element extensions/liftings are the lexicographic extensions/liftings.

**Definition 2.2.** A set of lexicographic data is an ordered subset \( E' := (e_1, \ldots, e_k) \) of \( E \) together with a sequence of signs \( s_1, \ldots, s_k \in \{+, -\} \). If \( E = E' \) we say the data is full.

The lexicographic cocircuit signature induced by such data is defined as follows: for each signed cocircuit \( D \), \( \sigma^*(D) = 0 \) if \( D \cap E' = \emptyset \), or otherwise \( \sigma^*(D) = s_i \cdot D(e_i) \) where \( i \) is the first index with \( e_i \in D \cap E' \). By definition, this signature is generic if and only if \( E' \) is spanning. In this case \( \sigma^* \) sends each cocircuit \( D \) to the signed cocircuit \( D'(e_i) = s_i \), where \( e_i \) is again the first element in \( D \cap E' \).

Define the lexicographic circuit signature analogously. It is generic if and only if \( E \setminus E' \) is independent.

**Lemma 2.3** (Las Vergnas, see [5, Definition 7.2.3]). Lexicographic cocircuit signatures (respectively, circuit signatures) are extension signatures (respectively, lifting signatures).

As said in the introduction, an orientation of an oriented matroid \( M \) is a map \( \mathcal{O} : E \to \{+, -\} \) and it can be interpreted as a reorientation \( -A M \) of \( M \) along a subset \( A \subset E \) of elements. We say that \( \mathcal{O} \) is conformal with a signed circuit or
signed cocircuit \( C \) if \( \mathcal{O}(e) = C(e) \) for every \( e \in \mathcal{C} \). (This agrees with the usual notion of conformality for arbitrary sign vectors); we also say \( \mathcal{O} \) is conformal with a circuit or cocircuit \( C \) if \( \mathcal{O}|_C \) is a signed circuit or signed cocircuit of \( M \).

**Definition 2.4.** Let \( \sigma \) (respectively, \( \sigma^* \)) be the circuit (respectively, cocircuit) signature induced by some generic single-element lifting \( M \) (respectively, extension \( M' \)). Then an orientation \( \mathcal{O} \) of \( M \) is \( (\sigma, \sigma^*) \)-compatible if \( \sigma \) (respectively, \( \sigma^* \)) is positive at every signed circuit (respectively, cocircuit) conformal with \( \mathcal{O} \). The set of \( (\sigma, \sigma^*) \)-compatible orientations of \( M \) is denoted by \( \lambda(M; \sigma, \sigma^*) \).

For the proof of Theorem [A] we use the machinery of oriented matroid programming. We now recall the Main Theorem of Oriented Matroid Programming, giving some essential definitions along the way.

**Theorem 2.5.** [5, Theorem 10.1.13] Let \( M \) be an oriented matroid with two distinguished elements \( f \neq g \), such that \( f \) is not a coloop and \( g \) is not a loop; the data \( (M, g, f) \) specifies an oriented matroid program.

Suppose the program is (equivalent formulations by [5, Theorem 10.1.9])

1. **Feasible:** there exists a signed cocircuit \( Y \) (a “feasible region”) of \( M \) such that \( Y(e) \geq 0 \) for all \( e \neq f \) and \( Y(g) = + \), and
2. **Bounded:** there exists a signed circuit \( C \) (a “bounded cone”) such that \( C(e) \geq 0 \) for all \( e \neq g \) and \( C(f) = + \).

Then the program has a solution \( Y \), which is a covector of \( M \) that is:

1. **Feasible:** \( Y(e) \geq 0 \) for all \( e \neq f \) and \( Y(g) = + \), and
2. **Optimal:** for every covector \( Z \) such that \( Z(f) = + \), \( Z(g) = 0 \), we have \( (Y \circ Z)(e) < 0 \) for some \( e \neq f, g \).

For Theorem [B] we need the following characterization of signatures of extensions/liftings. This was proved by Las Vergnas [30] and is reproduced also in [5, Theorem 7.1.8] and [35, Lemma 1.3]. (Only the case of extensions is stated in these references, but the case of liftings follows by duality.)

**Lemma 2.6 (Las Vergnas).** Let \( \sigma : \mathcal{C}(M) \to \{+,-,0\} \) and \( \sigma^* : \mathcal{C}^*(M) \to \{+,-,0\} \) be signatures defined on the set of signed circuits and signed cocircuits of an oriented matroid \( M \), respectively. Then:

1. \( \sigma^* \) is the cocircuit signature of a single-element extension if and only if it is so when restricted to every uniform rank two minor on three elements.
2. \( \sigma \) is the circuit signature of a single-element lifting if and only if it is so when restricted to every uniform rank one minor on three elements.

In this statement restricting \( \sigma \) or \( \sigma^* \) means the following. Let \( U = M/B\setminus A \) be a minor of \( M \), where \( A \) and \( B \) are disjoint subsets of the set \( E \) of elements of \( M \). Then, every signed cocircuit \( D \) of \( U \) lifts to a unique signed cocircuit \( \tilde{D} \) of \( M \) whose support is contained in \( E \setminus B \), and every signed circuit \( C \) of \( U \) extends to
a unique signed circuit $C'$ of $M$ whose support is contained in $E \setminus A$. Restricting $\sigma$ and $\sigma^*$ to $U$ means giving $C$ and $D$ the signs that $C'$ and $D$ get in $M$.

### 3. Proof of the Main Results

Throughout this section, let $M$ be an oriented matroid on ground set $E$, and let $M'$ (respectively, $\tilde{M}$) be a generic single-element extension (respectively, lifting) of $M$ on ground set $E \cup f$ (respectively, $E \cup g$).

Theorem A will be deduced from the following theorem, which constructs the inverse of the map $\beta_{\sigma, \sigma^*}$. We use the following notation. For an orientation $O$ of $M$, $O'$ denotes the orientation of $M'$ such that $O'|_E = O$ and $O'(f) = -$; dually, $\tilde{O}_-$ is the orientation of $\tilde{M}$ such that $\tilde{O}_-|_E = O$ and $\tilde{O}_-(g) = -$. Given a sign vector $X$ of $M$, when we consider $M'$ (respectively, $\tilde{M}$), $(X \epsilon) \in \{+, -, 0\}$ is understood as a sign vector that agrees with $X$ over $E$ and is equal to $\epsilon$ over $f$ (respectively, $g$).

**Theorem 3.1.** For every $O \in \chi(M; \sigma, \sigma^*)$, there exists a unique basis $B \in B(M)$ such that $B \cup f$ is a circuit conformal with $O'$ and $(E \setminus B) \cup g$ is a cocircuit conformal with $\tilde{O}_-$.

As explained in the introduction, such a basis corresponds to the optimum (with respect to $f$) of the region corresponding to $O$ in the pseudohyperplane arrangement representing $\tilde{M}$.

We start with a few lemmas.

**Lemma 3.2.** The set of circuits of $M'$ containing $f$ is $\{B \cup f : B \in B(M)\}$. Dually, the set of cocircuits of $\tilde{M}$ containing $g$ is $\{(E \setminus B) \cup g : B \in B(M)\}$.

**Proof.** Let $B \in B(M)$. $B$ is also a basis of $M'$, since $M$ and $M'$ have the same rank and independent sets of $M$ are independent in $M'$. Now, the fundamental circuit $C(B, f)$ must be spanning, by definition of genericity for $f$, so it strictly contains a basis. This implies $C(B, f) = B \cup \{f\}$. In particular, for every basis $B$ of $M$, $B \cup \{f\}$ is a circuit of $M'$.

Conversely, if $C'$ is a circuit of $M'$ containing $f$ then it is spanning. Being a circuit, removing one element (e.g. $f$) from it rank is preserved, so $C' \setminus \{f\}$ is indeed spanning and independent, i.e., a basis.

The dual statement is proven similarly. $\square$

**Lemma 3.3.** An orientation $O$ of $M$ is $\sigma^*$-compatible if and only if $O'_-$ is totally cyclic. Dually, $O$ is $\sigma$-compatible if and only if $\tilde{O}_-$ is acyclic.

**Proof.** Suppose $O'_-$ is conformal with some signed cocircuit $D'$. By Proposition 7.1.4 (ii), $D := D'|_E$ is either (i) a signed cocircuit of $M$, in which $f \in D'$, or (ii) equal to the conformal composition $D_1 \circ D_2$ of signed cocircuits of $M$, in which
σ^*(D_1) = -σ^*(D_2) ≠ 0. For case (i), D is a signed cocircuit conformal with O, but it is not compatible with σ^* as \( D'(f) = O'_-(f) = -1 \); for case (ii), both \( D_1, D_2 \) are conformal with O, but exactly one of them is not compatible with σ^* as \( σ^*(D_1) = -σ^*(D_2) \).

Conversely, if D is a signed cocircuit conformal with O but not compatible with σ^*, then \( (D -) \) is a signed cocircuit of \( M' \) that is conformal with \( O'_- \), hence \( O'_- \) is not totally cyclic. The dual statement can be proven similarly. □

Using the above lemmas, we can give an alternative description of the map \( β_{σ;σ} \), matching the statement of Theorem 3.1.

**Proposition 3.4.** Let \( B \) be a basis of M and let \( O = β_{σ;σ^*}(B) \). Then \( B \cup f \) is a circuit conformal with \( O'_- \) and \( (E \setminus B) \cup g \) is a cocircuit conformal with \( O'_- \).

**Proof.** By Lemma 3.2, \( X := B \cup f \) is a circuit of \( M' \). Denote by \( C \) the signed circuit of \( M' \) whose support is \( X \) and satisfies \( C(f) = -1 \). For every \( e ∈ B \), let \( D_e \) be the fundamental cocircuit of \( e \) with respect to \( B \) in \( M \), oriented according to \( σ^* \).

By the definition of \( σ^* \), the signed subset \( D'_e := (D_e +) \) is a signed cocircuit of \( M' \), and \( X ∩ D'_e = \{ e, f \} \). By the orthogonality of signed circuits and cocircuits as well as the fact that \( D'_e(f) = -C(f) \), we must have \( O(e) = D_e(e) = D'_e(e) = C(e) \), with the first equality from the definition of \( β_{σ;σ^*} \). Therefore \( X \) is oriented as \( C \) in \( O'_- \) and thus a conformal circuit. The second statement is the dual of the first one. □

Now we show that the image of \( β_{σ;σ^*} \) is contained in the set of \( (σ, σ^*) \)-compatible orientations.

**Proposition 3.5.** Let \( O \) be an orientation of M. If there exists a basis \( B ∈ B(M) \) such that \( B \cup f \) is a circuit conformal with \( O'_- \), and \( (E \setminus B) \cup g \) is a cocircuit conformal with \( O'_- \), then \( O ∈ χ(M;σ,σ^*) \).

**Proof.** By Lemma 3.3, it suffices to show that \( O'_- \) is totally cyclic and \( O'_- \) is acyclic. Suppose \( D \) is a signed cocircuit conformal with \( O'_- \). Since \( B \) is also a basis of \( M' \) (see the proof of Lemma 3.2), \( X := D \cap B \) is non-empty, but then \( X \) will be simultaneously in the totally cyclic part and acyclic part of \( O'_- \), contradicting [5, Corollary 3.4.6]. The dual statement can be proven similarly. □

To prove Theorem A via Theorem 3.1 we need one additional construction.

**Lemma 3.6.** [35, Lemma 1.10] Let \( M' \) and \( \tilde{M} \) be oriented matroids with respective ground sets \( E \cup \{ f \} \) and \( E \cup \{ g \} \) and such that \( M := M' \setminus f = \tilde{M}/g \). Then there exists an oriented matroid \( \tilde{M}' \) on ground set \( E \cup \{ f, g \} \) such that \( M' = \tilde{M}'/g \) and \( \tilde{M} = \tilde{M}' \setminus f \).

If \( M' \) and \( \tilde{M} \) are generic as an extension and lift of \( M \), then \( \tilde{M}' \) is a generic extension (resp. lift) of \( \tilde{M} \) (resp. of \( M' \)).
Sketch of proof. Every cocircuit $C$ of $\widetilde{M}$ vanishing on $g$ is a cocircuit of $\widetilde{M}/g = M'/f$ and, hence, it extends to a unique cocircuit (that we still denote $C$) of $M'$. Thus, the following is a well-defined cocircuit signature on $\widetilde{M}$: $\sigma^*(C) = C(g)$ if $C(g) \neq 0$ and $\sigma^*(C) = C(f)$ if $C(g) = 0$.

That this cocircuit signature defines an extension $\widetilde{M}'$ of $\widetilde{M}$ is proven in [35 Lemma 1.10]. Calling $f$ the new element of this extension it is obvious that $\widetilde{M} = \widetilde{M}' \setminus f$ and it is easy to see that $M' = \widetilde{M}'/g$.

If $M'$ is a generic extension of $M$ then $\sigma^*$ is generic: the only way to get a zero in $\sigma^*$ is when $\sigma^*(C) = C(f) = 0$ for a cocircuit of $M' \setminus f$, but this violates genericity of $M'$. Similarly, for $M'$ not to be a generic lift of $M'$ there should be a cocircuit in $\widetilde{M}'$ using $g$ and with dependent complement. The complement cannot contain $f$, because genericity of $f$ implies that all dependent sets containing $f$ are spanning. Hence, the cocircuit (deleting $f$ from it) is also a cocircuit in $\widetilde{M}' \setminus f = \widetilde{M}$. But this violates genericity of $\widetilde{M}$ \hfill \qed

Remark 3.7. The oriented matroid $\widetilde{M}'$ in this lemma is not unique, as can be easily understood looking at Figure 1: one can use instead of $f$ any other pseudo-line intersecting $g$ in the same points as $f$. (The linear programming interpretation is that $f$ does not only play the role of an objective function $c$ but of an affine hyperplane $\{c \cdot x = b\}$. Different choices of $b$ produce different oriented matroids $\widetilde{M}'$, but the same linear program).

Our proof of Theorem 3.1 works regardless of the choice of $\widetilde{M}'$. However, in Section 4 we need the following additional property of the $\widetilde{M}'$ constructed in the proof of the lemma: $f$ and $g$ are (positively) inseparable in it, meaning that there is no signed cocircuit having them in opposite sides.

Proof of Theorem 3.1 “Uniqueness”. Suppose both $B_1$ and $B_2$ are bases satisfying the condition. Let $C_1, C_2$ be the signed circuits of $M'$ obtained from restricting $\mathcal{O}'_-$ to $B_1 \cup f$ and $B_2 \cup f$, respectively; let $D_1, D_2$ be the signed cocircuits of $\tilde{M}$ obtained from restricting $\tilde{O}_-$ to $(E \setminus B_1) \cup g$ and $(E \setminus B_2) \cup g$, respectively. Let $\tilde{M}'$ be the oriented matroid containing both $M'$ and $\tilde{M}$ as guaranteed by Lemma 3.6 and consider the lifting $\tilde{C}_1$ of $C_1$ in $\tilde{M}'$.

Case I: $\tilde{C}_1(g) = +$. Let $D_1', D_2'$ be the extensions of $D_1, D_2$ in $\tilde{M}'$. We must have $D_1'(f) = D_2'(f) = -$ by orthogonality, which in turn forces the lifting $\tilde{C}_2$ of $C_2$ to take value $+$ at $g$. Apply the circuit elimination axiom to $\tilde{C}_1$ and $-\tilde{C}_2$ and eliminate $f$. Denote by $C$ the resulting signed circuit. We have $C \cap D_1' \subset (B_2 \setminus B_1) \cup g$, but $C$ is conformal with $-D_1'$ over $B_2 \setminus B_1$ as $D_1'|_{B_2 \setminus B_1} = \tilde{O}|_{B_2 \setminus B_1} = C_2|_{B_2 \setminus B_1}$, so $C(g) = D_1'(g) = -$ by orthogonality. However, the same orthogonality argument applied to $C$ and $D_2'$ implies that $C(g) = -D_2'(g) = +$, a contradiction.
Case II: $\widetilde{C}_1(g) = -$. The analysis is similar to Case I.

Case III: $\widetilde{C}_1(g) = 0$. This case is also impossible as $\widetilde{C}_1$ cannot be orthogonal to $D'_1$ and $D'_2$.

“Existence”. Let $O \in \chi(M; \sigma, \sigma^*)$. By reorienting $M$ if necessary, we may assume $O \equiv +$. For the sake of matching convention in the literature, we also reorient $f, g$ in $\widetilde{M}'$, so the all positive orientation $O'_+ \subset M'$ is totally cyclic and the all positive orientation $\tilde{O}_+$ is acyclic by Lemma 3.3. Now we consider the oriented matroid program $P := (\widetilde{M}', g, f)$.

This oriented matroid program $P$ is both feasible and bounded from our assumption on $\tilde{O}_+$ and $O'_+$: $\tilde{O}_+$ itself is a (full-dimensional) feasible region of $\widetilde{M}$; any positive circuit of $M'$ whose support is of the form $B \cup f$, with $B \in B(M)$, is a bounded cone. By Theorem 2.5, $P$ has an optimal solution $Y$, which is a covector of $M'$.

By definition, $Y$ is feasible and optimal. Since $Y$ is a covector containing $g$ in $\widetilde{M}'$, we have that $Y \setminus f$ is a covector of $\widetilde{M}$ containing $g$. So $Y \setminus f$ contains a cocircuit (of $\widetilde{M}$), whose support is of the form $(E \setminus B_0) \cup g$ for some $B_0 \in B(M)$ by Lemma 3.2. If the containment is proper, then $Y \setminus f$ contains some cocircuit $Z_0$ of $M$. Since the extension $M'$ is generic, the (ordinary matroidal) extension $\overline{Z_0'}$ of $Z_0$ in $M'$ contains $f$. Write $Z'_0$ as the signed cocircuit of $M'$ (hence $\widetilde{M}'$) supported on $Z'_0$ in which $Z'_0(f) = +$. Now we have a contradiction as $Y \circ Z'_0|_E$ is non-negative. Therefore $Y \setminus f = (E \setminus B_0) \cup g$, and it is a cocircuit of $\widetilde{M}$. We claim that $B_0$ is the basis of $M$ we want.

The second assertion is immediate as $Y|_{E \cup g}$ is non-negative. By Lemma 3.2, $B_0 \cup f$ is a circuit of $M'$. Denote by $C'$ the signed circuit of $M'$ supported on $B_0 \cup f$ such that $C'(f) = +$, it remains to show $C'$ is non-negative. Suppose $C'(e) = -$. Let $Z'_e$ be the fundamental cocircuit of $e$ with respect to $B_0$ in $M$, and let $Z'_e$ be its (ordinary matroidal) extension in $M'$. Since the extension is generic, $f \in Z'_e$. Let $Z'_e$ be the signed cocircuit of $M'$ (hence $\widetilde{M}'$) supported on $Z'_e$ in which $Z'_e(f) = +$. From the choice of $Z'_e, Z'_e \cap C' = \{e, f\}$, so $Z'_e(e) = +$ by orthogonality. In particular, $Y \circ Z'_e|_E$ is non-negative, which is a contradiction. Therefore $B_0 \cup f$ is a positive circuit of $O'_+$ as well. \[\square\]

Proof of Theorem A. By Proposition 3.4 and 3.5, every orientation in the image of $\beta_{\sigma, \sigma^*}$ is $(\sigma, \sigma^*)$-compatible. Injectivity follows from Proposition 3.4 and the uniqueness part of Theorem 3.1. Surjectivity follows from Proposition 3.4 and the existence part of Theorem 3.1. \[\square\]

Remark 3.8. While the map described in Theorem A is very simple and combinatorial, describing an efficient combinatorial inverse appears to be a difficult
task even for special cases such as graphical matroids. In fact, our proof of Theorem A shows that computing the inverse map is essentially equivalent to solving generic oriented matroid programs.

We now prove Theorem B, saying that signatures coming from extensions/liftings are the only ones for which Theorem A works.

**Proof of Theorem B.** (1) implying (2) is Theorem A; (2) implying (3) is the fact that every lexicographic cocircuit signature is induced by the corresponding lexicographic extension.

To prove (3) implies (1), we assume that (1) fails and use Lemma 2.6(ii). The lemma says that there is a rank 1 minor on three elements where \( \sigma \) is not a lifting signature. Such minor is of the form \( M/B | A \) where \( A = \{a_1, a_2, a_3\} \) and \( B = \{b_1, \ldots, b_{r-1}\} \) are disjoint subsets of elements in \( M \) such that \( B_i := B \cup a_i \) is a basis for \( i = 1, 2, 3 \). In particular, \( B \) is independent and, for each \( i \in \{1, 2, 3\} \), there is a unique circuit \( C_i \) contained in \( (B \cup A) \setminus \{a_i\} \) and containing \( \{a_1, a_2, a_3\} \setminus \{a_i\} \).

By reorienting the \( a_i \) if necessary we assume without loss of generality that the three bases \( B_i \) have the same orientation. This implies that for each \( i \in \{1, 2, 3\} \) the two signed circuits supported on \( C_i \) have opposite signs in the two elements \( A \setminus a_i \); let us denote \( C_i \) the one that is positive in \( a_i + 1 \) and negative in \( a_i - 1 \), where \( i \) is regarded modulo three.

With these conventions, saying that \( \sigma \) does not come from a single-element lifting of \( M/B | A \) is the same as saying that \( \sigma \) chooses \( C_i \) as the signed version of \( C_i \) for the three values of \( i \) (or the negation thereof; \( \sigma \) chooses \(-C_i \) for the three values of \( i \), in which case the proof proceeds in a similar manner).

Let \( O \) denote the reorientation provided by \( \beta_{\sigma, \sigma^*} \) for the basis \( B_1 \). We have that \( O(a_2) = - \) (since the fundamental circuit \( C(B_1, a_2) \), oriented according to \( \sigma \), equals \( C_3 \), which is positive at \( a_1 \) and negative at \( a_2 \) and \( O(a_3) = + \) (with the same argument on the (signed) fundamental circuit \( C(B_1, a_3) \) which is \( C_2 \).

It only remains to show that there is a lexicographic cocircuit signature that makes the circuit \( C_1 \) conformal but not compatible with \( O \); that is, with \( O|_{C_1} = -C_1 \). We construct such a \( \sigma^* \) in what follows.

None of what we said above changes under reorientation of elements in \( B \), so we can assume without loss of generality that \( a_2 \) is the only positive element in the signed circuit \( C_1 \). We then define \( \sigma^* \) as the positive lexicographic extension obtained by the elements of \( B \) (in an arbitrary order) followed by \( a_1 \). This choice implies that in \( O \) all elements of \( B \) get a positive orientation, as \( b_i \) is the unique element in \( C^*(B_1, b_i) \cap B_1 \) for every \( i = 1, \ldots, r - 1 \). On the other hand, we said above that \( O|_{C_1} \) is negative at \( a_2 \) and positive at \( a_3 \), which is the opposite of \( C_1 \). This gives \( O|_{C_1} = -C_1 \) as claimed. \( \square \)
4. Relation with Orientation Activity and Active Bijections

In this section we briefly review the work of Gioan and Las Vergnas on active bijections, and show how our bijections are related to theirs. We first recall two notions of activities in (oriented) matroid theory. For this, we fix an ordering of the ground set $E$ of an oriented matroid $M$. Put differently, $M$ is now an ordered oriented matroid.

The first notion is the classical Tutte activities, where an element $e$ is internally (respectively, externally) active for a basis $B$ if $e \in B$ and it is the minimal element in its fundamental cocircuit (respectively, $e \notin B$, fundamental circuit). Denote by $I(B)$ and $E(B)$ the sets of internally and externally active elements of $B$; the number $\iota(B) := |I(B)|, \epsilon(B) := |E(B)|$ of internally/externally active elements is the internal/external activity of $B$.

The second notion is (re)orientation activity of Las Vergnas [32], where an element of $E$ is internally (respectively, externally) active in an orientation $O$ if it is the minimal element in some cocircuit (respectively, circuit) conformal with $O$. Define $I(O), E(O), \iota(O), \epsilon(O)$ for an orientation $O$ analogously.

In the following statement we denote $Y^X$ the set of maps from $X$ to $Y$, and regard $\{+, -, \}^E$ as the set of orientations of $M$.

**Theorem 4.1.** [23] There is an explicit bijection (with explicit inverse)

$$glv : \{(B, \varphi) : B \in \mathcal{B}(M), \varphi \in \{+, -, \}^{I(B) \cup E(B)}\} \longleftrightarrow \{+, -, \}^E$$

satisfying $I(glv(B, \varphi)) = I(B), E(glv(B, \varphi)) = E(B)$, and $glv(B, \varphi)(e) = \varphi(e)$ for all $e \in I(B) \cup E(B)$.

In other words, $glv$ is a bijection that preserves active elements and their specified orientations from the basis side to the orientation side. The bijection in Theorem 4.1 has two components. First, it decomposes the matroid into minors such that the restriction of the basis to each minor is a basis and has exactly one active element: this is the active decomposition of the matroid with respect to the basis [22]. With such a decomposition and up to duality, it suffices to describe the bijection under the assumption that $\iota(B) = 1, \epsilon(B) = 0$. Such bases are known as (1,0)-bases in [20]; their images under the bijections are the (1,0)-orientations that satisfy $\iota(O) = 1, \epsilon(O) = 0$. Restricted to the (1,0) case, Theorem 4.1 reads:

**Corollary 4.2.** There is a bijection between (1,0)-bases and (pairs of opposite) (1,0)-orientations.

The following observation is the key to relate the active bijection to our work.

**Lemma 4.3.** Denote by $g, f$ the first and second element of $M$. Then, every (1,0)-basis contains $g$ and not $f$. 

If $M$ and is a generic extension of $M \setminus f$ and a generic lift of $M/g$ then the converse holds: every basis containing $g$ and not $f$ is a $(1,0)$-basis.

Proof. $g$ is always an active element, so $\epsilon(O) = 0$ implies $g \in B$. Once we know this, $f \in B$ would imply $\iota(O) \geq 2$ because both $f$ and $g$ would be internally active, so we must have $f \not\in B$.

Now, assume genericity and let $B$ be a basis containing $g$ and not $f$. Genericity of $g$ implies that every circuit not containing $g$ is spanning. Hence, every fundamental circuit $C(B,e)$ contains $g$, and no external element is active. Genericity of $f$ implies that every cocircuit not containing $f$ has independent complement. Hence, every fundamental cocircuit $C(B,e)$ contains $f$, and $g$ is the only internally active element. □

So, from now on let us denote by $g,f$ the first and the second element of an oriented matroid $M$ and assume that $M$ is a generic extension of $M \setminus f$ and a generic lift of $M/g$. Observe that if we call $M_0 := M/g \setminus f$ we are in the situation of Section 3 with the oriented matroids $M$ and $M'$ of that section now denoted $M_0$ and $M$. The following statement shows that the sets of bases that appear in Theorem A and Corollary 4.2 biject to one another.

Lemma 4.4. There is a bijection between the bases of $M_0$, cocircuits of $M \setminus f$ that contain $g$, and $(1,0)$-bases of $M$, via $B_0 \leftrightarrow (E_0 \setminus B_0) \cup g \leftrightarrow B_0 \cup g$.

Proof. The correspondence between the first two families is the statement of Lemma 3.2. On the other hand, $(1,0)$-bases of $M$ are the bases containing $g$ and not $f$, which biject (by removing/inserting $g$) to bases of $M_0 := M/g \setminus f$. □

Let us now look at the other side, that of orientations. We say that an acyclic orientation (equivalently, a tope) of $M$ (or $M \setminus f$) is bounded by $g$ if every cocircuit conformal with it contains $g$. The idea behind the “$(1,0)$” case of the active bijection is to interpret the $(1,0)$-orientations of $M$ as topes bounded by $g$, and to relate the $(1,0)$-bases of $M$ with signed cocircuits containing $g$, then to match them with each other by an oriented matroid programming procedure involving (but not limited to) $f$ as the objective function.

In general, the optimal solutions with respect to $f$ are not cocircuits, and further tie-breaking is needed. However, when $g$ and $f$ are generic, the procedure is closely related to our setting, as we now elaborate. Besides genericity, we assume that $f$ and $g$ are positively inseparable in $M$; that is, whenever a tope is bounded by $g$, it is on the same side of $f$ as of $g$. As mentioned in Remark 3.7, the oriented matroid constructed in the proof of Lemma 3.6 has all these properties.

In the following statement we denote $\sigma$ and $\sigma^*$ the generic circuit and cocircuit signatures producing $M \setminus f$ and $M/g$ as a lifting, respectively extension, of $M_0$. 
Lemma 4.5. There is a bijection between the \((1,0)\)-orientations of \(M\) on the negative side of \(g\) and the set \(\chi(M_0; \sigma, \sigma^*)\) of \((\sigma, \sigma^*)\)-compatible orientations of \(M\), via \(\mathcal{O} \leftrightarrow \mathcal{O}/g \setminus f\).

Proof. Suppose \(\mathcal{O}\) is a \((1,0)\)-orientation of \(M\) on the negative side of \(g\), by the inseparable assumption, \(\mathcal{O}(f) = -\) as well. Since \(\mathcal{O}\) has zero externally activity, it is acyclic, the same can be then said for \(\mathcal{O} \setminus f\), so by Lemma 3.3, \(\mathcal{O}/g \setminus f\) is \(\sigma\)-compatible. If a signed cocircuit \(D\) is conformal with \(\mathcal{O}/g \setminus f\) but not \(\sigma^*\), then by the definition of \(\sigma^*\), \((D -)\) is conformal with \(\mathcal{O}/g\) thus with \(\mathcal{O}\), in which its smallest element is an internal active element other than \(g\).

Conversely, starting with a \((\sigma, \sigma^*)\)-compatible orientation \(\mathcal{O}_0\) of \(M_0\), consider the orientation \(\mathcal{O}\) of \(M\) obtained from setting \(\mathcal{O}(f) = \mathcal{O}(g) = -\), and is equal to \(\mathcal{O}_0\) over \(E_0\). If \(\mathcal{O}\) is conformal with a cocircuit \(D\) that does not contain \(g\), then \(\mathcal{O}/g\) is also conformal with \(D\), contradicting Lemma 3.3 so its internal activity is 1. Again by Lemma 3.3, \(\mathcal{O} \setminus f\) is acyclic and has zero external activity. Therefore \(\mathcal{O}\) is of \((1,0)\)-activity and is by construction on the negative side of \(g\). \(\square\)

Summing up, Lemmas 4.4 and 4.5 say that the bijection of Theorem \(\text{A}\) (for the oriented matroid \(M_0\)) “is the same” as the bijection of Corollary 4.2 (for \(M\)), which in turn is the bijection of Theorem 4.1 restricted to the \((1,0)\) case and with the genericity assumptions on the first and second elements of \(M\).

In [17, Section 3] Gioan and Las Vergnas give a simplified proof of the active bijection for the case where \(M\) is uniform, a case in which no tie-breaking is needed. Although our point of view is different, our proof (once the existence of the oriented matroid \(\tilde{M}'\) of Section 3 is established) is in fact quite similar to their uniform case proof. For example, the circuit/cocircuit arguments that we use in the “existence” part of the proof of Theorem 3.1 are close to [17, Lemma 3.2.3]. However, uniformity of \(M\) is much more restrictive than the assumption in our proof, genericity of the first two elements as a lifting and an extension respectively, a situation that we can call “uniformoid”. In particular, our Theorem \(\text{A}\) can be understood as on the one hand extending [17] to this “uniformoid” case and on the other hand showing that any oriented matroid \(M_0\) can be embedded in an uniformoid \(M\) so that the active bijection for \(M\) gives, in \(M_0\), a bijection between bases and \((\sigma, \sigma^*)\)-compatible orientations.

We conclude this section with a discussion of activity classes of an oriented matroid \(M\). An intrinsic definition can be given using the active decomposition of an orientation [23], a counterpart of the aforementioned active decomposition with respect to a basis. However, for the sake of brevity, we define an activity class to be the collection of \(2^{I(B)+E(B)}\) orientations

\[\{ \text{glv}(B, \varphi) : \varphi : I(B) \cup E(B) \to \{+, -\} \}\]
for a basis $B$. In this way, the active bijection can be viewed as a bijection between bases and activity classes of $M$, and thus activity classes play a similar role for active bijections as $(\sigma, \sigma^*)$-compatible orientations play for ours. This illustrates another difference between the two bijections: we are not aware of an analogue of such notion for general $(\sigma, \sigma^*)$ in our setting.

Nonetheless, we describe an instance where our bijections can deduce something about activity. Given a pair $(\sigma, \sigma^*)$ induced by the lifting and extension with respect to the same lexicographic data, a $(\sigma, \sigma^*)$-compatible orientation is called a circuit-cocircuit minimal orientation in $[2]$ and an active fixed and dual-active fixed (re)orientation in $[23]$. We have the following observation relating these compatible orientations and activity classes.

**Proposition 4.6.** Let $\sigma$ and $\sigma^*$ be as above. Then $\chi(M; \sigma, \sigma^*)$ is a system of representatives of the activity classes of $M$.

**Proof.** An orientation $O$ is $(\sigma, \sigma^*)$-compatible if and only if every circuit or cocircuit conformal with $O$ is oriented according to the reference orientation of its minimal element, if and only if every active element of $O$ is oriented according to its reference orientation. By Theorem 4.1, exactly one orientation within an active class has such a property. $\square$

5. Relation to Triangulations of Lawrence Polytopes

**Definition 5.1.** Let $M$ be an oriented matroid on ground set $E$ and rank $r$. The Lawrence polytope or Lawrence lifting $\Lambda(M)$ of $M$ is an oriented matroid on the ground set $E = E \sqcup \overline{E}$, where $\overline{E} = \{ \overline{e} : e \in E \}$, of rank $|E| + r$ and with the following set of signed circuits:

$$C(\Lambda(M)) := \{ (C^+ \cup \overline{C^-}, \overline{C^+} \cup C^-) : (C^+, C^-) \in C(M) \}.$$ 

That is to say, each element $\overline{e} \in \overline{E}$ is “co-antiparallel” (antiparallel in the dual) to the corresponding $e \in E$.

Every sign vector $E \to \{+,-,0\}$ (such as a signed (co)circuit or an orientation) can be interpreted as a subset of $E$ which includes $e$ (respectively, $\overline{e}$) if $e$ is positive (respectively, negative) in the sign vector. Moreover, given a generic circuit signature $\sigma$ and a basis $B$, we can construct a subset $B^\sigma \subset E$ as follows: include both $e$ and $\overline{e}$ if $e \in B$, otherwise include $e$ (respectively, $\overline{e}$) if $e$ is positive (respectively, negative) in $\sigma(C(B,e))$. Observe that such a $B^\sigma$ is a basis of $\Lambda(M)$ and, conversely, every basis can be obtained this way (see Proposition 5.6).

A cocircuit signature $\sigma^*$ is a circuit signature of $M^*$, and we can similarly construct $(E \setminus B)^{\sigma^*} \subset E$ for every basis $B$ of $M$. By duality, $(E \setminus B)^{\sigma^*}$ is a basis of $\Lambda(M^*)$ (not to be confused with $\Lambda(M)^*$).

---

2In $[9]$, such a subset of $E$ is thought as a fourientation of $M$, a notion introduced by the first author and Sam Hopkins in $[3]$. 
Ding introduced the following notion in [8] for regular matroids, which can be steadily extended to all oriented matroids. Let us mention that, as explained in the introduction, [8] goes beyond the setting of triangulations.

**Definition 5.2.** A circuit signature $\sigma$ of $M$ is **triangulating** if for any $B \in \mathcal{B}(M)$, $B^{\sigma}$ does not contain any signed circuit (of $M$) of the form $-\sigma(C)$, interpreted as a subset of $\Lambda(M)$. A triangulating cocircuit signature can be defined analogously.

The orientation $\beta_{\sigma,\sigma^*}(B)$, interpreted as a subset of $\mathcal{E}$, is then equal to $B^\sigma \cap (E \setminus B)^{\sigma^*}$. Ding proved that for a pair of triangulating circuit and cocircuit signatures $(\sigma, \sigma^*)$ of a regular matroid $M$, the map $B \mapsto B^\sigma \cap (E \setminus B)^{\sigma^*}$ is a bijection between $\mathcal{B}(M)$ and $\chi(M; \sigma, \sigma^*)$ [8, Theorem 1.20]. We here show that this result is the special case of Theorem A for regular matroids. The proof in [8] can be extended to show that $\beta_{\sigma,\sigma^*}$ still takes bases to $(\sigma, \sigma^*)$-compatible orientations and is injective for general oriented matroids, but [8] needed to use the circuit-cocircuit reversal system, which is only available for regular matroids, as an intermediate object to deduce surjectivity.

**Proposition 5.3.** A circuit signature is triangulating if and only if it is induced by a generic single-element lifting. Dually, a cocircuit signature is triangulating if and only if it is induced by a generic single-element extension.

**Proof.** Let $\sigma$ be a triangulating circuit signature, we verify (3) of Theorem [8]. Let $\sigma^*$ be a cocircuit signature induced by some lexicographic extension. Suppose for some basis $B$ of $M$, $\beta_{\sigma,\sigma^*}(B)$ is compatible with some $-\sigma(C)$, i.e., $B^\sigma \cap (E \setminus B)^{\sigma^*}$ contains $-\sigma(C)$. Then $B^\sigma$ itself contains $-\sigma(C)$.

Now let $\sigma$ be a circuit signature induced by some generic single-element lifting. Suppose $B^\sigma$ contains some $-\sigma(C)$ for some bases $B$. Consider the lexicographic cocircuit signature $\sigma^*$ with respect to an ordering of $E$ whose elements in $B \cap C$ are the smallest, and the reference orientation of those elements are the same as in $-\sigma(C)$, we have $(E \setminus B)^{\sigma^*}$ contains $-\sigma(C)$, so $\beta_{\sigma,\sigma^*}(B) = B^\sigma \cap (E \setminus B)^{\sigma^*}$ is compatible with $-\sigma(C)$ as well, a contradiction. \qed

When an oriented matroid is realizable, its Lawrence polytope can be realized by the vertex set of an actual polytope in Euclidean space [5, Proposition 9.3.2], which is often known as the Lawrence polytope as well. The name triangulating signature comes from the following result of Ding:

**Theorem 5.4** ([8, Theorem 1.28]). Let $M$ be a regular matroid and $\sigma$ a circuit signature in it. Then, $\{ \text{conv } B^\sigma : B \in \mathcal{B}(M) \}$ is a triangulation of the Lawrence polytope if and only if $\sigma$ is triangulating.

The notion of triangulation can be generalized for any oriented matroid [35], and Ding’s result can also be extended; this is the main content of the rest of this section.
**Definition 5.5.** A **triangulation** of an oriented matroid $M$ is a collection of bases $B_1, \ldots, B_t \in \mathcal{B}(M)$ that satisfies:

1. Pseudo-manifold property: for any $B_i$ and $e \in B_i$, $B_i \setminus e$ is either contained in another $B_j$, or its complement contains a positive cocircuit of $M$.
2. Non-overlapping property: there do not exist $B_i, B_j$ and a signed circuit $C = (C^+, C^-)$ such that $C^+ \subset B_i$ and $C \setminus a \subset B_j$ for some $a \in C^+$.

We have the following description of triangulations of the Lawrence polytope of an oriented matroid.

**Proposition 5.6.** [35] Lemma 4.11 & Proposition 4.12] Let $M$ be an oriented matroid. Then the bases of $\Lambda(M)$ are precisely those of the form $B^A := B \cup \overline{B} \cup A \cup E \setminus (A \cup B)$ for some basis $B$ of $M$ and $A \subset E \setminus B$.

Furthermore, any triangulation of $\Lambda(M)$ contains exactly one basis of the form $B^A$ for each basis $B$ of $M$.[3]

Therefore, as mentioned at the beginning of the section, the $B^\sigma$’s constructed from a circuit signature $\sigma$ are bases of $\Lambda(M)$. Moreover, the definition of triangulating signature (see also the notion of a triangulating atlas as in Definition 1.5(2) of [8]) is a reformulation of the non-overlapping property.

**Lemma 5.7.** Suppose a collection $\{B^A_B : B \in \mathcal{B}(M), A_B \subset E \setminus B\}$ is non-overlapping. Then there exists a triangulating circuit signature $\sigma$ such that $B^\sigma = B^{A_B}$ for every $B$. Conversely, every triangulating circuit signature $\sigma$ gives rise to a non-overlapping collection $\{B^\sigma : B \in \mathcal{B}(M)\}$.

**Proof.** Suppose $B_1, B_2 \in \mathcal{B}(M)$ share the same circuit $C$ as fundamental circuit, but $B_1^{A_B_1}$ contains the signed circuit $(C^+, C^-)$ supported on $C$ while $B_2^{A_B_2}$ contains $(C^-, C^+)$. Then $B_1^{A_B_1}, B_2^{A_B_2}$ overlap on the signed circuit $(C^+ \cup \overline{C^-}, C^+ \cup C^-)$ with the element $a$ being the unique element in $(C^+ \cup \overline{C^-}) \setminus (B_2 \cup \overline{B_2})$, a contradiction. Therefore, there exists a unique way to choose a signed circuits $\sigma(C)$ supported on $C$ such that whenever $B$ has $C$ as a fundamental circuit, $B^{A_B}$ contains $\sigma(C)$, this defines a circuit signature. If $\sigma$ is not triangulating and $B_1^\sigma$ contains $-\sigma(C)$, pick a basis $B_2$ such that $C$ is a fundamental circuit thereof, so $B_2^{A_B}$ contains $\sigma(C)$ from construction. A similar argument as above shows that $B_1^\sigma, B_2^{A_B}$ overlap on the signed circuit of $\Lambda(M)$ coming from $-\sigma(C)$.

Conversely, suppose $\sigma$ is triangulating, but $B_1^\sigma, B_2^{A_B}$ overlap on a signed circuit $(C^+ \cup \overline{C^-}, C^+ \cup C^-)$ of $\Lambda(M)$ that comes from the signed circuit $C = (C^+, C^-)$ of $M$. Without loss of generality, the element $a$ in the definition of non-overlapping property is from $C^+ \subset E$, and we abuse notation to view $a$ as an element of $M$. Since $\{e, \overline{e}\} \subset B_2^{A_B}$ for every $e \in C \setminus a$, we have $C \setminus a \subset B_2$, and $C$ is necessarily the

---

[3] In particular, all the triangulations of $\Lambda(M)$ have the same number of facets, equal to the number of bases of $M$. 

"C" is a fundamental circuit, so such that whenever "B" is necessarily the "C" as fundamental circuit, \(\text{Matroid. Then the bases of } \Lambda(M) \text{ are precisely those of the form } B^A := B \cup \overline{B} \cup A \cup E \setminus (A \cup B) \text{ for some basis } B \text{ of } M \text{ and } A \subset E \setminus B.\) Furthermore, any triangulation of \(\Lambda(M)\) contains exactly one basis of the form \(B^A\) for each basis \(B\) of \(M\).\)

Therefore, as mentioned at the beginning of the section, the \(B^\sigma\)’s constructed from a circuit signature \(\sigma\) are bases of \(\Lambda(M)\). Moreover, the definition of triangulating signature (see also the notion of a triangulating atlas as in Definition 1.5(2) of [8]) is a reformulation of the non-overlapping property.

\[\text{Lemma 5.7.} \text{ Suppose a collection } \{B^A_B : B \in \mathcal{B}(M), A_B \subset E \setminus B\} \text{ is non-overlapping. Then there exists a triangulating circuit signature } \sigma \text{ such that } B^\sigma = B^{A_B} \text{ for every } B. \text{ Conversely, every triangulating circuit signature } \sigma \text{ gives rise to a non-overlapping collection } \{B^\sigma : B \in \mathcal{B}(M)\}.\]

\[\text{Proof.}\] Suppose \(B_1, B_2 \in \mathcal{B}(M)\) share the same circuit \(C\) as fundamental circuit, but \(B_1^{A_B_1}\) contains the signed circuit \((C^+, C^-)\) supported on \(C\) while \(B_2^{A_B_2}\) contains \((C^-, C^+)\). Then \(B_1^{A_B_1}, B_2^{A_B_2}\) overlap on the signed circuit \((C^+ \cup \overline{C^-}, C^+ \cup C^-)\) with the element \(a\) being the unique element in \((C^+ \cup \overline{C^-}) \setminus (B_2 \cup \overline{B_2})\), a contradiction. Therefore, there exists a unique way to choose a signed circuits \(\sigma(C)\) supported on \(C\) such that whenever \(B\) has \(C\) as a fundamental circuit, \(B^{A_B}\) contains \(\sigma(C)\), this defines a circuit signature. If \(\sigma\) is not triangulating and \(B_1^\sigma\) contains \(-\sigma(C)\), pick a basis \(B_2\) such that \(C\) is a fundamental circuit thereof, so \(B_2^{A_B}\) contains \(\sigma(C)\) from construction. A similar argument as above shows that \(B_1^\sigma, B_2^{A_B}\) overlap on the signed circuit of \(\Lambda(M)\) coming from \(-\sigma(C)\).

Conversely, suppose \(\sigma\) is triangulating, but \(B_1^\sigma, B_2^{A_B}\) overlap on a signed circuit \((C^+ \cup \overline{C^-}, C^+ \cup C^-)\) of \(\Lambda(M)\) that comes from the signed circuit \(C = (C^+, C^-)\) of \(M\). Without loss of generality, the element \(a\) in the definition of non-overlapping property is from \(C^+ \subset E\), and we abuse notation to view \(a\) as an element of \(M\). Since \(\{e, \overline{e}\} \subset B_2^{A_B}\) for every \(e \in C \setminus a\), we have \(C \setminus a \subset B_2\), and \(C\) is necessarily the

---

[3] In particular, all the triangulations of \(\Lambda(M)\) have the same number of facets, equal to the number of bases of \(M\).
fundamental circuit of $a$ with respect to $B_2$. By the construction of $B_2^\sigma$, the sign of $a$ in $\sigma(C)$ must be negative as $\overline{a} \in B_2^\sigma$. But then $B_1^\sigma$ contains $C^+ \cup C^- \ni a$, which is the signed circuit $-\sigma(C)$ of $M$, a contradiction. \qed

The following statement is a generalization of [8, Theorem 1.28], and can be thought as a converse of [35, Proposition 4.12] in the sense that “correct number of simplices plus non-overlapping” implies “triangulation”. In [8] this result is proven for regular matroids, where the crucial step is a volume computation to show that the total volume of the simplices corresponding to the $B_\sigma$’s is equal to the volume of the Lawrence polytope (realized as a genuine polytope), hence the non-overlapping property implies that the union of these simplices is the Lawrence polytope itself. Such an argument does not work in the non-realized setting, since there is no natural notion of “volume”.

**Proposition 5.8.** For each basis $B$ consider an $A_B \subset E \setminus B$. If the collection $\{B_A^B : B \in B(M), A_B \subset E \setminus B\}$ satisfies the non-overlapping property, then it is a triangulation of $\Lambda(M)$.

**Proof.** By Lemma 5.7 there exists a triangulating circuit signature $\sigma$ that yields the collection. Such a signature is induced from some generic single-element lifting by Proposition 5.3, so the collection forms a triangulation of $\Lambda(M)$ by [35, Theorem 4.14]. \qed

**Remark 5.9.** It is possible to prove Proposition 5.8 by verifying the pseudo-manifold property directly, which in turns gives a new proof of the generic case of the “single-element liftings yields subdivisions of $\Lambda(M)$” direction in [35, Theorem 4.14(i)].

Conversely, since the collection of bases of a triangulation of $\Lambda(M)$ is non-overlapping, it is induced by a triangulating circuit signature, which comes from a generic single-element lifting. This already gives a new proof of the generic case of the reverse direction of [35, Theorem 4.14(i)]. By the Cayley Trick [27], for realizable oriented matroids it also gives a proof to the generic case of the Bohne–Dress Theorem, stating that all zonotopal tilings of a zonotope $Z(M)$ come from single-element liftings of the oriented matroid $M$ and vice versa.

**Acknowledgements**

Work of S. Backman is supported by a Zuckerman STEM Postdoctoral Scholarship, DFG–Collaborative Research Center, TRR 109 “Discretization in Geometry and Dynamics”, a Simons Collaboration Gift # 854037, and NSF Grant (DMS-2246967). Work of F. Santos is supported by grants PID2019-106188GB-I00, PID2022-137283NB-C21 of MCIN/AEI/10.13039/501100011033 and by project CLaPPo (21.SI03.64658) of Universidad de Cantabria and Banco Santander.
Work of C.H. Yuen was supported by the Trond Mohn Foundation project “Algebraic and Topological Cycles in Complex and Tropical Geometries”; he also acknowledges the support of the Centre for Advanced Study (CAS) in Oslo, Norway, which funded and hosted the Young CAS research project “Real Structures in Discrete, Algebraic, Symplectic, and Tropical Geometries” during the 2021/2022 and 2022/2023 academic years. The authors thank Emeric Gioan for explaining his work with Las Vergnas and their connection with our work, and Changxin Ding for explaining his work on triangulations of Lawrence polytopes and bijections.

REFERENCES

[1] Spencer Backman. Riemann-Roch theory for graph orientations. Adv. Math., 309:655–691, 2017.
[2] Spencer Backman. Partial Graph Orientations and the Tutte Polynomial. Adv. in Appl. Math., 94:103–119, 2018.
[3] Spencer Backman, Matthew Baker, and Chi Ho Yuen. Geometric bijections for regular matroids, zonotopes, and Ehrhart theory. Forum Math. Sigma, 7:e45, 37, 2019.
[4] Spencer Backman and Sam Hopkins. Fourientations and the Tutte polynomial. Res. Math. Sci., 4:Paper No. 18, 57, 2017.
[5] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. Oriented Matroids, volume 46 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, second edition, 1999.
[6] Robert G. Bland. A combinatorial abstraction of linear programming. J. Combinatorial Theory Ser. B, 23(1):33–57, 1977.
[7] J. Bohne. Eine kombinatorische Analyse zonotoper Raumaufteilungen, 1992. Ph.D. thesis, Universität Bielefeld.
[8] Changxin Ding. A framework unifying some bijections for graphs and its connection to Lawrence polytopes, 2023. preprint [arXiv:2306.07376]
[9] Changxin Ding. Geometric bijections between spanning subgraphs and orientations of a graph. Journal of the London Mathematical Society, 108(3):1082–1120, 2023.
[10] Changxin Ding, Alex McDonough, Lilla Tóthmérész, and Chi Ho Yuen. A consistent sandpile torsor algorithm for regular matroids. In preparation.
[11] Jon Folkman and Jim Lawrence. Oriented matroids. J. Combin. Theory Ser. B, 25(2):199–236, 1978.
[12] Emeric Gioan. Correspondance naturelle entre bases et réorientations des matroïdes orientés., 2002. Ph.D. thesis, Université Bordeaux 1.
[13] Emeric Gioan. A survey on the active bijection in graphs, hyperplane arrangements and oriented matroids. In Workshop on the Tutte polynomial, 2015.
[14] Emeric Gioan. On tutte polynomial expansion formulas in perspectives of matroids and oriented matroids. Discrete Mathematics, 345(7):112796, 2022.
[15] Emeric Gioan and Michel Las Vergnas. The active bijection in graphs, hyperplane arrangements, and oriented matroids, 3. linear programming.
[16] Emeric Gioan and Michel Las Vergnas. The active bijection between regions and simplices in supersolvable arrangements of hyperplanes. The Electronic Journal of Combinatorics, pages R30–R30, 2004.
[17] Emeric Gioan and Michel Las Vergnas. Bases, reorientations, and linear programming, in uniform and rank-3 oriented matroids. *Adv. in Appl. Math.*, 32(1-2):212–238, 2004. Special issue on the Tutte polynomial.

[18] Emeric Gioan and Michel Las Vergnas. Activity preserving bijections between spanning trees and orientations in graphs. *Discrete mathematics*, 298(1-3):169–188, 2005.

[19] Emeric Gioan and Michel Las Vergnas. Fully optimal bases and the active bijection in graphs, hyperplane arrangements, and oriented matroids. *Electron. Notes Discret. Math.*, 29:365–371, 2007.

[20] Emeric Gioan and Michel Las Vergnas. The active bijection in graphs, hyperplane arrangements, and oriented matroids. I. The fully optimal basis of a bounded region. *European J. Combin.*, 30(8):1868–1886, 2009.

[21] Emeric Gioan and Michel Las Vergnas. A linear programming construction of fully optimal bases in graphs and hyperplane arrangements. *Electronic Notes in Discrete Mathematics*, 34:307–311, 2009.

[22] Emeric Gioan and Michel Las Vergnas. The Active Bijection 2.a - Decomposition of Activities for Matroid Bases, and Tutte Polynomial of a Matroid in terms of Beta Invariants of Minors, 2018.

[23] Emeric Gioan and Michel Las Vergnas. The Active Bijection 2.b - Decomposition of Activities for Oriented Matroids, and General Definitions of the Active Bijection, 2018.

[24] Emeric Gioan and Michel Las Vergnas. The active bijection for graphs. *Advances in Applied Mathematics*, 104:165–236, 2019.

[25] Emeric Gioan and Michel Las Vergnas. Computing the fully optimal spanning tree of an ordered bipolar directed graph. *arXiv preprint arXiv:1807.06552*, 2018.

[26] Curtis Greene and Thomas Zaslavsky. On the Interpretation of Whitney Numbers through Arrangements of Hyperplanes, Zonotopes, non-Radon Partitions, and Orientations of Graphs. *Trans. Amer. Math. Soc.*, 280(1):97–126, 1983.

[27] Birkett Huber, Jörg Rambau, and Francisco Santos. The Cayley trick, lifting subdivisions and the Bohne-Dress theorem on zonotopal tilings. *J. Eur. Math. Soc. (JEMS)*, 2(2):179–198, 2000.

[28] Tamás Kálmán and Lilla Tóthméresz. Hypergraph polynomials and the Bernardi process. *Algebr. Comb.*, 3(5):1099–1139, 2020.

[29] Michel Las Vergnas. Acyclic and Totally Cyclic Orientations of Combinatorial Geometries. *Discrete Math.*, 20(1):51–61, 1977/78.

[30] Michel Las Vergnas. Extensions ponctuelles d’une géométrie combinatoire orientée. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages pp 265–270. CNRS, Paris, 1978.

[31] Michel Las Vergnas. A correspondence between spanning trees and orientations in graphs. *Graph Theory and Combinatorics*, Academic Press, London, pages 233–238, 1984.

[32] Michel Las Vergnas. The Tutte Polynomial of a Morphism of Matroids. II. Activities of Orientations. In *Progress in Graph Theory (Waterloo, Ont., 1982)*, pages 367–380. Academic Press, Toronto, ON, 1984.

[33] Alex McDonough. A family of matrix-tree multijections. *Algebr. Comb.*, 4(5):795–822, 2021.

[34] Jürgen Richter-Gebert and Günter M. Ziegler. Zonotopal Tilings and the Bohne-Dress Theorem. In *Jerusalem Combinatorics ’93*, volume 178 of *Contemp. Math.*, pages 211–232. Amer. Math. Soc., Providence, RI, 1994.
[35] Francisco Santos. Triangulations of Oriented Matroids. *Mem. Amer. Math. Soc.*, 156(741):viii+80, 2002.

[36] Chi Ho Yuen. *Geometric bijections of graphs and regular matroids*. PhD thesis, Georgia Institute of Technology, 2018.

Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05405, USA
Email address: spencer.backman@uvm.edu

Department of Mathematics, Statistics and Computer Science, University of Cantabria, Spain
Email address: francisco.santos@unican.es

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark
Email address: chy@math.ku.dk