Multi-Objective Trust-Region Filter Method for Nonlinear Constraints using Inexact Gradients

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Abstract

In this article, we build on previous work to present an optimization algorithm for non-linearly constrained multi-objective optimization problems. The algorithm combines a surrogate-assisted derivative-free trust-region approach with the filter method known from single-objective optimization. Instead of the true objective and constraint functions, so-called fully linear models are employed, and we show how to deal with the gradient inexactness in the composite step setting, adapted from single-objective optimization as well. Under standard assumptions, we prove convergence of a subset of iterates to a quasi-stationary point and, if constraint qualifications hold, then the limit point is also a KKT-point of the multi-objective problem.

Keywords Multi-Objective Optimization & Multiobjective Optimization & Nonlinear Optimization & Derivative-Free Optimization & Trust-Region Method & Surrogate Models & Filter Method.

1 Introduction

Multi-objective optimization problems (MOPs) arise naturally in many areas of mathematics, engineering, in the natural sciences or in economics. The goal of multi-objective optimization (MOO) is to find acceptable trade-offs between the competing objectives of a MOP. Generally, there are multiple solutions constituting the so-called Pareto Set in variable space and the Pareto Front in objective space. In this article, we consider the problem

$$\min_{x \in \mathbb{R}^n} \begin{bmatrix} f_1(x) \\ \vdots \\ f_K(x) \end{bmatrix} \quad \text{s.t.} \quad h(x) = [h_1(x), \ldots, h_M(x)]^T = 0, \quad g(x) = [g_1(x), \ldots, g_P(x)]^T \leq 0,$$

(MOP)

where all functions are twice continuously differentiable. A global Pareto-optimal point $x^*$ is feasible and non-dominated, i.e., there is no other feasible $x \neq x^*$ with $f(x) \leq f(x^*)$ and $f_\ell(x) < f_\ell(x^*)$ for some $\ell \in \{1, \ldots, K\}$. Throughout this article we will refer to the feasible set as $\mathcal{X} := \{x \in \mathbb{R}^n : h(x) = 0, g(x) \leq 0\}$.

There is a multitude of methods available to approximate a single solution or the entire Pareto Set/Front (or a superset thereof) and the choice of method is heavily dependent on the structure of the problem at hand and the demands of the person seeking a solution. The references [16, 33, 38, 18] all provide an extensive overview of the topic. Amongst others, there are scalarization approaches [17] and adaptions of single-objective descent methods to the multi-objective case – for different problem classes as defined by the properties of their objectives and constraint functions, see [23, 22, 30, 35, 27, 52, 26] for examples of well-known scalar techniques adjusted for constrained and unconstrained MOPs in the smooth and non-smooth case. For global approximations there are, e.g., evolutionary algorithms (see [6] for an overview and [10] for a prominent example) and structure-exploiting methods [31, 37, 28]. Of course, there is also research in combining global and local techniques [44, 40]. Whilst there are natural applications of MOO for machine learning tasks [45, 1], machine learning techniques can
conversely be used to assist the search for optimal points [39]. In case of expensive objective or constraints, surrogate models can be employed [41, 5, 11].

In our setting we assume (some) objective and constraint functions to be computationally expensive and without exact derivative-information available. This motivates the use of derivative-free optimization methods, which also have been adapted to the multi-objective case. Most prominently, there are direct search algorithms [3] and surrogate-assisted trust-region algorithms [42, 46, 48]. Based on those trust-region algorithms, we have in a previous article [4] presented a trust-region algorithm for problems with a feasible set that is convex and compact (or $\mathcal{X} = \mathbb{R}^n$). The algorithm uses fully linear models (e.g., Lagrange polynomials or radial basis functions (RBFs) as in [51]) to approximate the objective functions and passes the exact constraints to an inner solver. We build upon this work to accommodate general non-linear constraints by also modelling them and solving inexact sub-problems.

To this end, we transfer the techniques from [21] to the multi-objective case with inexact derivatives. Inexact derivative have already been handled in single-objective optimization in a similar manner: For example, the authors of [19] provide strong convergence results for inexact objectives and objective derivatives with similar model accuracy requirements to our case. We will also discuss the single-objective algorithm presented in [14], as it is also based on fully linear models and employs a special kind of criticality check. Our work, however, is more along the lines of [49], where the filter trust-region algorithm and the composite step framework are likewise modified to handle inexact derivatives of both the objective and constraint functions. But in [49] the derivatives (approximated via automatic differentiation) can eventually become exact, in contrast to our setting, which is why – without constraint qualifications – we can only prove convergence to a quasi-stationary point, similar to [15]. The algorithms in [15] and [24] also use fully linear surrogate models satisfying the same error bounds as assumed in this article within single-objective filter algorithms. In [15], convergence to KKT-points is proven under constraint qualifications. We will see, that our result also requires similar constraint qualifications. Such qualifications are not needed in [24], but an additional assumption (eq. (3.10)) on the model accuracy is made to show convergence to critical points. In their case, the additional assumption is justified by domain-specific surrogates.

In next section, we will state the relevant optimality conditions for the multi-objective case. Afterwards, the main building blocks of our algorithm are described in Section 2. The algorithm itself is given in Section 3 and convergence is shown in Sections 4 and 5. Finally, two numerical examples are discussed in Section 6 with a brief discussion in Section 7.

### 1.1 Optimality Conditions

We assume the reader to be familiar with the concept of Pareto-optimality in the context of MOO (else see, for example, [38]). In this subsection we introduce necessary conditions for a point to be locally Pareto-optimal. To state the criteria, we require the following assumption to hold, so that all functions are sufficiently smooth:

**Assumption 1.** The objective functions $f_\ell$, $\ell = 1, \ldots, K$, and the constraint functions $g_\ell$, $1, \ldots, P$, and $h_\ell$, $\ell = 1, \ldots, M$, are twice continuously differentiable in an open domain containing $\mathcal{X}$ and have Lipschitz continuous gradients on $\mathcal{X}$.

There then is a formulation of Fermat’s Theorem for optimization with multiple objectives:

**Theorem 1** (see [2]). Suppose Assumption 1 holds. If $x^* \in \mathcal{X}$ is locally Pareto-optimal for (MOP), then there is no $d \in T_\mathcal{X}(x^*)$ such that for all $\ell \in \{1, \ldots, K\}$ it holds that $\langle \nabla f_\ell(x^*), d \rangle < 0$, where $T_\mathcal{X}(x^*)$ is the tangent cone of $\mathcal{X}$ at $x^*$.

We call a point $x^* \in \mathcal{X}$ satisfying the criterion in Theorem 1 Pareto-critical. Theorem 1 can be used to motivate the following problem to compute a descent direction and check for criticality:

$$\min_{d \in T_\mathcal{X}(x), \|d\|_2 \leq 1} \max_{\ell=1,\ldots,K} \langle \nabla f_\ell(x), d \rangle. \tag{1}$$

For a critical point $x \in \mathcal{X}$ the optimal value is zero, else the minimizer is a multi-descent direction (cf. [2, Th. 1.9.], [22]). The choice of norm in the above problem is not really important, and so we could use a linear norm and assume that at $x \in \mathcal{X}$ certain constraint qualifications hold, ensuring that the tangent cone equals the set of linearized directions

$$L(x) = \{d \in \mathbb{R}^n : d^T \nabla h_\ell(x) = 0, \ell = 1, \ldots, M, d^T \nabla g_\ell(x) \leq 0, \ell \in A(x) \},$$
with \( A(x) := \{ \ell \in \{1, \ldots, P\} : g_{\ell}(x) = 0 \} \), to obtain a linear problem, that also indicates criticality for (MOP). Under constraint qualifications one can also derive KKT conditions which then provide an equivalent definition of Pareto-criticality [32]. In our algorithm, however, we do not use the set \( L \). Instead, at the iterate \( x^{(k)} \) (which does not have to be feasible) we use an approximation of the linearized feasible set,

\[
\mathcal{L}(x^{(k)}) := \left\{ x^{(k)} + d \in \mathbb{R}^n : h(x^{(k)}) + H(x^{(k)}) d = 0, \ g(x^{(k)}) + G(x^{(k)}) d \leq 0 \right\},
\]

where \( H(x^{(k)}) \) and \( G(x^{(k)}) \) denote the full Jacobian matrices of the constraints \( h \) and \( g \), respectively. The set \( \mathcal{L}(x^{(k)}) - x^{(k)} \) is not necessarily a cone, but for feasible \( x^{(k)} \), it is a subset of \( L(x^{(k)}) \), and intuitively it should not matter which of the sets is used near critical points. Indeed, we have the following theorem:

**Theorem 2.** Suppose that \( x \) is feasible and that a suitable constraint qualification holds. If the linear optimization problem

\[
- \min_{d \in \mathcal{L}(x) - x} \max_{\ell=1,\ldots,K} \langle \nabla f_{\ell}(x), d \rangle
\]

has zero as its optimal value, then \( x \) is also a KKT-point of (MOP) (as defined in [37, 32]).

**Proof.** Dropping the argument \( x \) for notational convenience, and denoting by \( F \) the objective Jacobian, the linear problem (2) is equivalent to

\[
\begin{align*}
\max_{\beta^-} & \ [0^T_n, 1] [d] \\
\text{s.t.} & \ d \in \mathbb{R}^n, \ beta^- \in \mathbb{R}, \begin{bmatrix} -I_{n,n} & 0_n \\ I_{n,n} & 0_n \\ F & 1_K \\ H & 0_M \\ G & 0_P \end{bmatrix} \begin{bmatrix} d \\ \beta^- \end{bmatrix} \leq \begin{bmatrix} 1_n \\ 1_n \\ 0_K \\ 0_M \\ 0_P \end{bmatrix}.
\end{align*}
\]

(P)

Consider also the dual problem:

\[
\begin{align*}
\min_{y^1,\ldots,y^5} & \ [1^T_n, 1^T_n, 0^T_K, 0^T_M, -g^T] \begin{bmatrix} y^1 \\ \vdots \\ y^5 \end{bmatrix} \\
\text{s.t.} & \ y^1 \geq 0_n, y^2 \geq 0_n, y^3 \geq 0_K, y^4 \in \mathbb{R}^M, y^5 \geq 0_P \ and \ \\
& \begin{bmatrix} -I_{n,n} & 0_n & F^T & H^T & G^T \\ 0^T_n & 1^T_K & 0^T_M & 0^T_P \end{bmatrix} \begin{bmatrix} y^1 \\ \vdots \\ y^5 \end{bmatrix} = \begin{bmatrix} 0_n \\ 1 \end{bmatrix}.
\end{align*}
\]

(D)

If 0 is the optimal value of (P), then \( \beta^- = 0 \). By strong duality, the dual problem is feasible with optimal value 0, implying \( y^1 = 0 \), \( y^2 = 0 \) and \( -g^T y^5 = 0 \). The KKT equations immediately follow from the remaining constraints and from the complementary slackness property of dual solution pairs. \( \square \)

### 2 Trust-Region Concepts and Surrogates

As mentioned in the previous section, we assume (at least some) of the objectives or constraints to be computationally expensive. To approximate a Pareto-critical point whilst avoiding expensive function evaluations, a trust-region approach is used: The true functions \( f, h \) and \( g \) are modeled by \( \hat{f}^{(k)}, \hat{h}^{(k)} \) and \( \hat{g}^{(k)} \) respectively. These models are constructed to be sufficiently accurate within iteration-dependent trust-regions

\[
B^{(k)} = B(x^{(k)}; \Delta^{(k)}) := \left\{ x \in \mathbb{R}^n : \| x - x^{(k)} \|_{tr,k} \leq \Delta^{(k)} \right\}.
\]

Using the surrogate models, a step \( s^{(k)} \) and a step-size \( \sigma^{(k)} \) are determined – in such a way as to reduce the constraint violation or achieve an objective value reduction at the trial point \( x^{(k)} + \sigma^{(k)} s^{(k)} \), which is tested as a candidate for the next iterate. The step \( \sigma^{(k)} s^{(k)} \) is computed so that the trial point is contained in both the trust-region and the approximate linearized feasible set at \( x^{(k)} \):

\[
\mathcal{L}_h = \mathcal{L}_h(x^{(k)}) = \left\{ x^{(k)} + s \in \mathbb{R}^n : \hat{h}^{(k)}(x^{(k)}) + \hat{H}_h(x^{(k)}) \cdot s = 0, \ \hat{g}^{(k)}(x^{(k)}) + \hat{G}_h(x^{(k)}) \cdot s \leq 0 \right\}.
\]

(3)
where \( \hat{H}_k \) and \( \hat{G}_k \) are now the full model Jacobians. This introduces uncertainty and iterates might no longer be feasible. Hence, we treat the constraints as relaxable, and we might have to evaluate \( f \) (and \( g \) and \( h \)) outside of \( X \), which motivates the next assumption. It states that all functions have to be available in all possible trust-regions. Note, that the algorithm actually can use two trust-region sizes, the preliminary size \( \Delta(k)_0 \) at the beginning of an iteration and \( \Delta(k) \leq \hat{\Delta}(k) \) after the Criticality Routine, which is described in Section 3.1.

**Assumption 2.** There is a constant \( 0 < \Delta_{\text{max}} < \infty \) such that for every \( k \in \mathbb{N}_0 \) the trust region sizes conform to \( 0 < \Delta(k) \leq \hat{\Delta}(k) < \Delta_{\text{max}} \). All functions (the true functions and their models) are defined in these regions, i.e., on the set \( \mathcal{C}(X) = \bigcup_{k \in \mathbb{N}_0} B(\mathbf{x}(k), \hat{\Delta}(k)) \).

### 2.1 Fully Linear Models

In this subsection, we first want to explain what it means for the model functions to be sufficiently accurate. Although the true derivative information is not used in any computation, we will need it for the convergence analysis and hence require the following generalization of Assumption 1:

**Assumption 3.** The objective functions \( f_\ell, \ell = 1, \ldots, K \), the constraint functions \( g_\ell, 1, \ldots, P \), and \( h_\ell, \ell = 1, \ldots, M \), are twice continuously differentiable in an open domain containing \( \mathcal{C}(X) \) and have Lipschitz continuous gradients on \( \mathcal{C}(X) \).

With Assumption 3 we can define models satisfying error bounds w.r.t. to their construction radius:

**Definition 1 (Fully Linear Models).** Let \( \Delta_{\text{max}} > 0 \) be a given constant and let \( f : \mathbb{R}^n \to \mathbb{R} \) be a scalar-valued function satisfying Assumption 3. A set of model functions \( \mathcal{M} = \{ f : \mathbb{R}^n \to \mathbb{R} \} \) is called a fully linear class of models for \( f \) if the following hold:

1. There are positive constants \( \epsilon, \hat{\epsilon} \) and \( L^f \) such that for any \( \Delta \in (0, \Delta_{\text{max}}] \) and for any \( \mathbf{x} \in \mathcal{C}(X) \) there is a model function \( \hat{f} \in \mathcal{M} \), with Lipschitz continuous gradient and corresponding Lipschitz constant bounded by \( L^f \), such that the error between the gradients and the error between the values satisfy
   \[
   \| \nabla f(\xi) - \nabla \hat{f}(\xi) \|_2 \leq \hat{\epsilon} \Delta \quad \text{and} \quad | f(\xi) - \hat{f}(\xi) | \leq \epsilon \Delta^2, \quad \forall \xi \in B(\mathbf{x}; \Delta) \cap \mathcal{C}(X).
   \]

2. For this class \( \mathcal{M} \) there exists a “model-improvement” algorithm that, in a finite, uniformly bounded number of steps can either establish that a given model \( \hat{f} \in \mathcal{M} \) is fully linear on \( B(\mathbf{x}; \Delta) \) or find a model \( \hat{f} \in \mathcal{M} \) that is fully linear on \( B(\mathbf{x}; \Delta) \).

**Definition 2.** Let \( \Delta_{\text{max}} > 0 \) be a given constant and let \( f = [f_1, \ldots, f_K]^T \) be a vector of functions satisfying the requirements of Definition 1 with classes \( \mathcal{M}_\ell \) and constants \( (\mathbf{e}_\ell, \hat{\mathbf{e}}_\ell, L^f_\ell), \ell = 1, \ldots, K \). Then
\[
\mathcal{M} = \{ \hat{f} = [\hat{f}_1, \ldots, \hat{f}_K]^T : \hat{f}_1 \in \mathcal{M}_1, \ldots, \hat{f}_K \in \mathcal{M}_K \}
\]
is a class of fully linear, vector-valued model functions with constants \( \max_\ell \mathbf{e}_\ell > 0, \max_\ell \hat{\mathbf{e}}_\ell > 0 \) and \( \max_\ell L^f_\ell > 0 \). A vector of functions \( \hat{f} \in \mathcal{M} \) is deemed fully linear, if all components \( \hat{f}_\ell \) are fully linear. The improvement algorithms for \( \mathcal{M}_\ell \) are applied component-wise.

A function that is fully linear for \( \Delta(k) > 0 \) is automatically fully linear for any smaller radius \( 0 < \Delta < \Delta(k) \). When the trust-region radius is bounded above, then the constants \( \epsilon > 0 \) and \( \hat{\epsilon} > 0 \) can be chosen large enough such that a fully linear model stays fully linear in enlarged trust-regions:

**Lemma 1 (Lemma 10.25 in [8]).** For \( \mathbf{x} \in \mathcal{C}(X) \) and \( \Delta \in (0, \Delta_{\text{max}}] \) consider a function \( f \) and a fully linear model \( \hat{f} \) of \( f \) with constants \( \epsilon, \hat{\epsilon}, L^f > 0 \). Let \( L^f > 0 \) be a Lipschitz constant of \( \nabla f \). Assume without loss of generality (w.l.o.g.) that
\[
L^f + L^f < \epsilon \quad \text{and} \quad \epsilon/2 < \epsilon.
\]
Then \( \hat{f} \) is fully linear on \( B(\mathbf{x}; \Delta) \) for any \( \Delta \in [\Delta, \Delta_{\text{max}}] \) with the same constants.

**Assumption 4.** For any \( k \in \mathbb{N}_0 \) the models \( \hat{f}^{(k)} \) are fully linear on \( B(\mathbf{x}(k); \Delta(k)) \) as in Definition 2 with constants \( \mathbf{e}_f, \hat{\mathbf{e}}_f \) and \( L^f \) that are chosen large enough such that (4) is fulfilled globally. The same holds for the models \( \hat{g}^{(k)} \) of \( g \) with constants \( \mathbf{e}_g, \hat{\mathbf{e}}_g, L^g \) and the models \( \hat{h}^{(k)} \) of \( h \) with constants \( \mathbf{e}_h, \hat{\mathbf{e}}_h \) and \( L^h \). We also assume that all models are interpolating at \( \mathbf{z}^{(k)} \).
2.2 Composite Step Approach and Sub-Problems

Just like in previous articles on multi-objective trust-region algorithms [46, 4, 48, 42, 43], we use the maximum-scalarization

\[
\Phi(x) = \Phi(f) := \max_{\ell=1,\ldots,K} f_\ell(x) \quad \text{and} \quad \Phi^{(k)}(x) = \Phi(f^{(k)}) := \max_{\ell=1,\ldots,K} f_\ell^{(k)}(x)
\]

to determine objective reduction. The idea then is to find a step \(s^{(k)}\) that approximately solves

\[
\min_{s \in \mathbb{R}^n} \Phi(f^{(k)})(x + s) \quad \text{s.t.} \quad s \in (L_k - x^{(k)}), \quad ||s||_{tr,k} \leq \Delta^{(k)}
\]

with the inexact linearized feasible set defined in (3). Without constraints, inexact line-search can be used. In our case, we use the normal step approach \(s^{(k)}\) is split into a normal component \(n^{(k)}\) towards feasibility and a descent direction \(d^{(k)}\) (see [21, 49] for details). Then, the normal step can be computed with

\[
\min ||n||_2^2 \quad \text{s.t.} \quad n \in (L_k - x^{(k)}).
\] (ITRN\(^{(k)}\))

If a normal step \(n^{(k)}\) has been found – and if \(||n||_{tr,k} \leq \Delta^{(k)}\) – the descent direction \(d^{(k)}\) can be taken as the minimizer of

\[
\omega(x_n^{(k)}, f^{(k)}, L_k, ||\cdot||_k) = \min_{\beta \in \mathbb{R}, d \in \mathbb{R}^n} \beta \quad \text{s.t.} \quad \hat{F}_k(x_n^{(k)}) \cdot d \leq \beta, \quad d \in (L_k - x_n^{(k)}), \quad ||d||_k \leq 1,
\] (ITRT\(^{(k)}\))

where \(L_k\) is the approximate linearized feasible set according to (3) and \(x_n^{(k)} = x^{(k)} + n^{(k)}\).

We actually only compute the descent direction if there is enough movement possible after performing the normal step, i.e., the normal step must not be too large. We call such a step compatible, and it is defined with respect to the preliminary radius \(\Delta^{(k)}\):

**Definition 3.** Let \(c_\Delta \in (0,1], c_\mu > 0 \) and \(\mu \in (0,1)\) be constants. The minimizer \(n^{(k)}\) of (ITRN\(^{(k)}\)) is called compatible if

\[
||n^{(k)}||_{tr,k} \leq c_\Delta \Delta^{(k)} \min \left\{ 1, c_\mu \Delta^{(k)} \right\}.
\]

(5)

Furthermore, we assume a normal step to exist if the true constraint violation is not too large, as measured by the infeasibility function

\[
\theta(x) := \max \left\{ \max_{\ell=1,\ldots,M} |h_\ell(x)|, \max_{\ell=1,\ldots,P} |g_\ell(x)| \right\}.
\]

**Assumption 5** (Existence and Boundedness of Normal Step). If \(\theta_k = \theta(x^{(k)}) \leq \delta_n\), for a constant \(\delta_n > 0\), then \(n^{(k)}\) exists and there is a constant \(c_{ubn} > 0\) such that

\[
||n^{(k)}||_{tr,k} \leq c_{ubn} \theta_k.
\]

(6)

Assumption 5 is a standard assumption and the reasoning behind it can be found in [21]. Finally, the step-size \(\sigma^{(k)}\) is determined in such a way that \(||x^{(k)} + n^{(k)} + \sigma^{(k)}d^{(k)}||_{tr,k} \leq \Delta^{(k)}\) and a sufficient decrease condition (for the objective surrogates) is satisfied, which is described and justified in Section 2.4. The term \(\sigma^{(k)}d^{(k)}\) is also called tangential step and altogether the step \(s^{(k)}\) results in the trial-point \(x^{(k)}_n = x^{(k)} + s^{(k)} = x^{(k)} + \sigma^{(k)}d^{(k)}\).

2.3 Filter Mechanism

As mentioned already, iterates can become infeasible. We employ a so-called Filter to drive them back towards the feasible set. Thus, the trial point \(x^{(k)}_s = x^{(k)} + s^{(k)}\) is tested not only for actual decrease of the original functions but also against previous iterates stored in the filter \(\mathcal{F}\). \(\mathcal{F}\) is a set of tuples \((\theta_j, \Phi_j)\) describing a forbidden area in image space. In fact, the tuples in \(\mathcal{F}\) (w.r.t. \(f\)) are currently non-dominated for the bi-objective optimization problem of minimizing both \(\theta(x)\) and \(\Phi(f)(x)\). The trial point \(x^{(k)}_s\) is only acceptable for \(\mathcal{F}\) if its value tuple is also (sufficiently) non-dominated. An acceptable trial point is further tested and if it sufficiently reduces the true objectives it is kept as the next iterate. The current iterate might be added to the filter if the predicted objective decrease is small compared to the constraint violation, or if the latter is too large. More formally, we use the following definition:
Definition 4 (Multi-Objective Filter). A filter \( \mathcal{F} \) with respect to some function \( \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^K \) is a discrete set of tuples \( \{(\theta_j, \Phi_j)\} \subset \mathbb{R}^2 \), and a point \( \mathbf{x} \) is acceptable for the filter iff
\[
\theta(\mathbf{x}) \leq (1 - \gamma_0)\theta_j \quad \text{or} \quad \Phi(\mathbf{f})(\mathbf{x}) \leq \Phi_j - \gamma_0\theta_j \quad \forall (\theta_j, \Phi_j) \in \mathcal{F}.
\]
When a point \( \mathbf{x} \) is added to the filter the tuple \( (\theta(\mathbf{x}), \Phi(\mathbf{f})(\mathbf{x})) \) is added to the set \( \mathcal{F} \), but all tuples \( (\theta_j, \Phi_j) \) with
\[
\theta_j \geq \theta(\mathbf{x}) \quad \text{and} \quad \Phi_j - \gamma_0\theta_j \geq \Phi(\mathbf{f})(\mathbf{x}) - \gamma_0\theta(\mathbf{x})
\]
are removed from the set.

As can be seen from Definition 4 a filter strengthens non-dominance testing by employing a positive offset \( \gamma_0\theta_j \). Note, that we could also use a stricter, \((K+1)\)-dimensional filter instead of the 2-dimensional filter, similar to [29], by using \( f(\bullet) \) instead of \( \Phi(\mathbf{f})(\bullet) \).

2.4 Sufficient Decrease

In this sub-section we want to explain what is meant by “sufficient decrease”. In short, we have to relate the model predicted objective reduction to the criticality value. The criticality is the optimal value of \((\text{ITRT}^{(k)})\), and it is thus dependent on the norm. As indicated above, we allow for the usage of iteration dependent problem norms \( \|\bullet\|_k \) and trust-region norms \( \|\bullet\|_{tr,k} \). This way, the algorithm can be implemented by using norms that are suited best for the problem geometry or the inner solver(s). We only require that these norms be uniformly equivalent to the Euclidean norm:

Assumption 6. There is a constant \( c \geq 1 \) such that for all \( \mathbf{x} \in \mathcal{C}(\mathcal{X}) \) and all \( k \in \mathbb{N}_0 \) and \( \|\bullet\|_* = \|\bullet\|_{tr,k} \) or \( \|\bullet\|_* = \|\bullet\|_k \) it holds that
\[
\frac{1}{c} \|\mathbf{x}\|_* \leq \|\mathbf{x}\|_2 \leq c \|\mathbf{x}\|_*.
\]

Remark 1. Any two norms that are uniformly equivalent to \( \|\bullet\|_2 \) with constant \( c \) are pairwise equivalent with constants \( 1/c \) and \( c \).

Lemma 2. Suppose Assumptions 2 to 4 and 6 hold. Denote by
\[
\hat{\omega}^{(k)} := \omega(\mathbf{x}^{(k)}; \hat{\mathbf{f}}^{(k)}, \mathcal{L}_k, \|\bullet\|_k),
\]

the optimal value of \( \langle \text{ITRT}^{(k)} \rangle \) and by
\[
\hat{\omega}_2^{(k)} \quad \text{the optimal value if the 2-norm is used,} \quad \hat{\omega}_2^{(k)} = \omega(\mathbf{x}^{(k)}; \hat{\mathbf{f}}^{(k)}, \mathcal{L}_k, \|\bullet\|_2).
\]

There is a constant \( \hat{\omega} \geq 1 \) such that for any \( k \in \mathbb{N}_0 \) for which the normal step exists, it holds that
\[
\frac{1}{\hat{\omega}} \leq \hat{\omega}^{(k)} \leq \hat{\omega}_2^{(k)} \leq \hat{\omega} \cdot \hat{\omega}^{(k)}.
\]

The proof of Lemma 2 can be found in the appendix. The lemma shows that we can relate the iteration dependent inexact critical values \( \omega^{(k)} \) and \( \omega_2^{(k)} \). Furthermore, we show in the appendix that it is then sensible to assume a sufficient decrease condition for the objective surrogate functions as per Assumption 8, as long as an additional (standard) assumption holds:

Assumption 7. The norm of all model Hessians of the objective function surrogates is uniformly bounded above, i.e., there is a positive constant \( H > 0 \) such that for all \( k \in \mathbb{N}_0 \)
\[
\|\mathbf{H} f^{(k)}(\xi)\|_2 \leq H \quad \text{for all } \ell = 1, \ldots, K, \text{ and } \xi \in \mathcal{L}^{(k)}(\mathbf{x}^{(k)}) \cap B^{(k)}.
\]

We know how to find suitable model functions that satisfy Assumption 7, including Lagrange interpolation polynomials or RBFs, so that finally we can state the sufficient decrease assumption:

Assumption 8 (Sufficient Decrease). Suppose Assumptions 2 to 4 hold and that \( \Delta^{(k)} \in (0, \Delta_{\text{max}}) \). Let \( \Phi^{(k)} := \Phi(\hat{\mathbf{f}}^{(k)}) \). If \( \mathbf{n}^{(k)} \) is compatible, and \( \mathbf{d}^{(k)} \) is a minimizer of \( \langle \text{ITRT}^{(k)} \rangle \) at \( \mathbf{x}^{(k)} \) for \( \|\bullet\|_k \) and \( \omega_2^{(k)} \) is the optimal value for \( \langle \text{ITRT}^{(k)} \rangle \) if the 2-norm is used, then there is a step-length \( \sigma(\mathbf{x}) \geq 0 \) such that
\[
\mathbf{x}^{(k)} + \mathbf{n}^{(k)} + \sigma(\mathbf{d}^{(k)}) \in \mathcal{L}_k \cap B^{(k)}
\]
and
\[
\Phi^{(k)}(\mathbf{x}^{(k)}) - \Phi^{(k)}(\mathbf{x}^{(k)} + \sigma(\mathbf{d}^{(k)})) \geq c_{sd} \omega_2^{(k)} \min \left\{ \frac{\omega_2^{(k)}}{\hat{\omega}}, \Delta^{(k)}, 1 \right\},
\]
for constants \( c_{sd} \in (0, 1) \) and \( \hat{\omega} \geq 1. \)
Throughout the rest of this article we use a slightly modified measure for notational convenience:

**Corollary 1** (Modified Criticality Measure). For any \( k \in \mathbb{N}_0 \) and \( \tilde{\omega}^{(k)} \) as in Lemma 2, define the criticality measure \( \chi^{(k)} := \min \{ 1, \tilde{\omega}^{(k)} \} \) and denote by \( \chi^{(k)}_2 \) the corresponding value, if the 2-norm is used instead of \( \| \cdot \|_k \).

Then \( \lim_{k \to \infty} \chi^{(k)}_2 = 0 \) if and only if \( \lim_{k \to \infty} \chi^{(k)} = 0 \), and if Assumption 8 holds, then it also follows (with \( \omega \geq 1 \)) that

\[
\Phi^{(k)}(x_n^{(k)}) - \Phi^{(k)}(x_n^{(k)} + \sigma^{(k)} d^{(k)}) \geq c_{\text{ad}} \chi^{(k)}_2 \min \left\{ \frac{\chi^{(k)}_2}{\omega}, \Delta^{(k)} \right\}.
\]

## 3 Discussion of the Algorithm

The behavior of the algorithm is determined by several additional algorithmic parameters:

| Parameter(s) | Description |
|--------------|-------------|
| \( 0 < \Delta^{(0)} \leq \Delta^{\text{max}} < \infty \) | initial and maximum trust-region radius |
| \( 0 < \gamma_0 \leq \gamma_1 < 1 \leq \gamma_2 \) | shrinking and growing parameters for the trust-regions |
| \( 0 < \nu_1 \leq \nu_0 < 1 \) | acceptance thresholds for trial point test |
| \( 0 < \varepsilon_1 < 1, 0 \leq \varepsilon_2 \leq \delta_n \) | thresholds for criticality test |
| \( \kappa_\theta \in (0, 1), \psi > \frac{1}{1 + \rho} \) | threshold parameters in (10) |
| \( 0 < \beta < \lambda \) and \( \alpha \in (0, 1) \) | Criticality Routine threshold factors and backtracking constant |
| \( c_{\text{ad}} \in (0, 1) \) | sufficient decrease constant in Assumption 8 |
| \( c_{\Delta} \in (0, 1), c_\mu > 0, \mu \in (0, 1) \) | constants defining compatibility in Definition 3 |
| \( \delta_n > 0, c_{\text{shin}} > 0 \) | existence of normal step in Assumption 5 |

Below, the *Criticality Test* is used instead of the original stopping criterion \( \chi(x^{(k)}) = 0 \) because the surrogate models are inexact. If an iterate is nearly feasible and nearly critical for the surrogate problem, then the criticality routine is entered and the trust-region radius is reduced to make the models more precise. At a truly critical point the routine loops infinitely, but if a point is not critical for the true functions, we exit and continue with regular iterations. For further details of the Criticality Test and the Criticality Routine we refer to [8].

### Algorithm

1. **Initialization**: Let \( k \leftarrow 0 \), \( F \leftarrow \emptyset \) and \( n^{(k)} = \text{undef} \). Evaluate \( f(x^{(k)}), g(x^{(k)}), h(x^{(k)}) \) and compute \( \theta_k = \theta(x) \). Build surrogate models \( \tilde{f}^{(k)}, \tilde{g}^{(k)}, \tilde{h}^{(k)} \) that are fully linear in \( B^{(k)} \) with radius \( \bar{\Delta}^{(k)} := \Delta^{(0)} \).

2. **Compatibility Test**: 
   - If \( n^{(k)} \neq \text{undef} \) and \( n^{(k)} \) is compatible w.r.t. \( \bar{\Delta}^{(k)} \); go to step 3.
   - If \( n^{(k)} = \text{undef} \), try to compute \( n^{(k)} \). If \( n^{(k)} \) exists and is compatible, go to step 3.

3. **Restoration**: Add \( x^{(k)} \) to the filter, set \( \Delta^{(k)} = \bar{\Delta}^{(k)} \), \( \chi^{(k)} = \bar{\chi}^{(k)} \), and attempt to find a *restoration step* \( r^{(k)} \) and \( \bar{\Delta}^{(k + 1)} > 0 \) for which (ITRT\(^{(k)}\)) is compatible at \( (x^{(k)} + r^{(k)}, \bar{\Delta}^{(k + 1)}) \) and for which \( x^{(k)} + r^{(k)} \) is acceptable for \( F \). Set \( x^{(k + 1)} = x^{(k)} + r^{(k)} \), keep \( \Delta^{(k + 1)} \) and go to step 8.

4. **Descent Step**: Compute a descent direction \( d^{(k)} \) and \( \bar{\chi}^{(k)} \) with (ITRT\(^{(k)}\)).

5. **Criticality Test**: If \( \theta_k < \varepsilon_2 \) and \( (\bar{\chi}^{(k)} < \varepsilon_\chi \) and \( \bar{\Delta}^{(k)} > n\bar{\chi}^{(k)} \) ), then enter the Criticality Routine to get \( \Delta^{(k)} \) and \( \chi^{(k)} \) and update \( d^{(k)} \) and \( n^{(k)} \). Else, set \( \Delta^{(k)} := \bar{\Delta}^{(k)} \) and \( \chi^{(k)} := \bar{\chi}^{(k)} \).

6. **Acceptance Test**: Compute a step-size \( s^{(k)} > 0 \) such that Assumption 8 is fulfilled. Set \( x_s^{(k)} = x^{(k)} + n^{(k)} + s^{(k)} d^{(k)} \). Compute \( f(x_s^{(k)}), \theta \left( \bar{x}_n^{(k)} \right) \).
   - If \( x_s^{(k)} \) is not acceptable for the augmented filter \( F \cup \{ (\theta_k, \theta \left( \bar{x}_n^{(k)} \right) ) \} \) OR
   - If \( \Phi \left( f^{(k)} \right)(x^{(k)}) - \Phi \left( f^{(k)} \right)(x_s^{(k)}) \geq \kappa_\theta \theta_k^{(0)} \) \[ (10) \]
   AND
   \[
   \rho^{(k)} := \frac{\Phi \left( f \right)(x^{(k)}) - \Phi \left( f \right)(x_s^{(k)})}{\Phi \left( f^{(k)} \right)(x^{(k)}) - \Phi \left( f^{(k)} \right)(x_s^{(k)})} < \nu_1, \]
[11]
When the Criticality Routine starts, we know that the criticality routine is provided with the current models \( \delta \), first condition. From (12) that \( \delta \geq \delta \), it follows that after the routine has stopped finitely the normal step will be compatible for \( \delta \) and for any radius \( \Delta \geq \delta \). Because it always holds that \( \delta \leq \Delta \) and \( \max \{ \delta, B \} \geq \delta \), it follows from (12) that \( n \) is compatible for \( \Delta \).

There are two ways the routine can stop. If the radius is sufficiently small compared to the criticality, i.e., \( \delta \leq \Delta \), or if the next prospective normal step \( n \) is no longer compatible for the smaller radius \( \delta \). The second criterion ensures (inductively), that after the routine has stopped finitely the normal step will be compatible for \( \delta \) and for any radius \( \Delta \geq \delta \). Because it always holds that \( \delta \leq \Delta \) and \( \max \{ \delta, B \} \geq \delta \), it follows from (12) that \( n \) is compatible for \( \Delta \).

One reason for the Criticality Test (and the Criticality Routine) is to have a lower bound on the inexact criticality if the constraint violation is sufficiently small, i.e., \( \theta \leq \varepsilon \theta \leq \delta \) and the routine stops because of the first condition. From \( B \leq \Delta \) as well as \( \delta \leq \Delta \), it follows that after the Criticality Routine has finished due to \( \delta \leq \Delta \), we have \( \chi \leq \chi \). If \( \chi \leq \chi \), then the criticality loop is not entered and \( \chi \geq \varepsilon \chi \). Thus, if the second stopping condition does not apply, it follows from \( \theta \leq \varepsilon \theta \) that \( \chi \geq \min \{ \Delta \chi, \varepsilon \chi \} \).

Finally, note that the models \( \hat{f}, \hat{g}, \hat{h} \) are fully linear when the Criticality Routine has finished after a finite number of iterations – even if the radius is slightly increased above \( \delta \) – because of Assumption 4.
4 Convergence to Quasi-Stationary Points

Note that the problem (ITRT\(^{(k)}\)) defining \(\chi^{(k)}\) is similar to the criticality problem (1), but we compute the descent step starting at \(x^{(k)}_n\) instead of \(x^{(k)}\), the surrogate model gradients are used to determine a model descent step within the approximated linearized feasible set and a variable vector norm \(\|\bullet\|_h\) bounds the problem (which must not necessarily equal the trust-region norm \(\|\bullet\|_{tr,k}\)). For the convergence analysis, the additional uncertainties are dealt with in two steps: The first part (the remainder of this section) is concerned with proving that there is a subsequence \(\{x^{(k)}\}\) of iterates converging to a quasi-stationary point. That is, it holds that

\[
\lim_{\ell \to \infty} \theta(x^{(k)}) = 0 \quad \text{and} \quad \lim_{\ell \to \infty} \omega(x^{(k)}, f^{(k)}, \mathcal{L}_{k,}, \|\bullet\|_{\infty}) = 0.
\]

Afterwards, Section 5 builds on this result to show convergence to actual KKT-points.

4.1 Comparison with Related Algorithms

Most results in this section are “translated” from their single-objective pendants in [21] or [49]. The latter article is also concerned with inexact surrogates, which have error bounds that are slightly different from ours and can become exact eventually. Hence, we cannot simply apply the maximum-scalarization and be done (unfortunately). We also have to take care of the Criticality Routine and other, more subtle differences, such as the iteration-dependent norms. That is why we have decided to cite the results from [21, 49] explicitly and can become exact eventually. Hence, we cannot simply apply the maximum-scalarization and be done otherwise, hints are given on how to adapt them or the proofs are provided wholly for the sake of completeness.

Special mention has to be made of the single-objective algorithm in [14] that is further detailed in the dissertation [13]. This algorithm also employs a Criticality Routine and uses fully linear models. By encoding the surrogate modeling of black-box components as additional constraints to the original problem and using a nonlinear (even non-quadratic) subproblem for the computation of the normal step, the convergence analysis becomes easier due to the resulting inexactness bounds. At the time of writing, the nonlinear normal step computation did not suit our particular needs, but we think it easily possible and very worth-while to also transfer their approach to the multi-objective case.

4.2 Final Definitions and Requirements

To actually investigate limit points of algorithmic iteration sequences we need Assumption 9, which guarantees their existence:

**Assumption 9.** The set \(\mathcal{C}(X)\) is contained in a closed and bounded set.

A detailed discussion of Assumption 9 and alternatives is given in [21]. If the iterates are contained in a bounded domain, the true functions (which are continuous according to Assumption 3) are bounded and so are their Lipschitz-continuous gradients. With Assumption 4 the models are fully linear and have to be bounded at the iterates as well, due to \(\tilde{\Delta}(k) \leq \Delta_{\text{max}} < \infty\):

**Corollary 2.** If Assumptions 3, 4 and 9 and \(\tilde{\Delta}(k) \leq \Delta_{\text{max}} < \infty\) hold, then the norm of all function and model gradients is uniformly bounded above, i.e., there is a positive constant \(c_{\text{obj}} > 0\) such that for \(\varphi \in \{f_1, \ldots, f_K, g_1, \ldots, g_P, h_1, \ldots, h_M, \}\), for all \(k \in \mathbb{N}_0\) and \(\varphi^{(k)} \in \{\tilde{f}^{(k)}, \ldots, \tilde{f}^{(k)}_K, \tilde{g}^{(k)}_1, \ldots, \tilde{g}^{(k)}_P, \tilde{h}^{(k)}_1, \ldots, \tilde{h}^{(k)}_M, \}\) it holds that

\[
\|\nabla\varphi(x^{(k)})\| \leq c_{\text{obj}} \quad \text{and} \quad \|\nabla\varphi^{(k)}(x^{(k)})\|_{\infty} \leq c_{\text{obj}}.
\]

Throughout this section, we consider the following iteration index sets:

**Definition 5.** The set of “successful” iteration indices is \(\mathcal{A} = \{k : x^{(k+1)} = x^{(k)}\}\). The set of restoration indices is defined as \(\mathcal{R} = \{k : n^{(k)} \text{ does not exist or (5) fails, i.e., } n^{(k)} \text{ is incompatible}\}. \) Finally, the set of filter indices (indices of iterations that modify the filter) is \(\mathcal{Z} = \{k : x^{(k)} \text{ is added to the filter }\} \supseteq \mathcal{R}\).

Whenever we require “the Criticality Routine to finish finitely” in subsequent statements, then we want it to finish after a finite number of iterations and explicitly include the case that the routine is not even entered due to the Criticality Test failing in step 4 of the algorithm.
4.3 Convergence Analysis

Similar to [49] we can state accuracy requirements that bound the linearized constraint violation of the steps \( \mathbf{n}(k) \) and \( \mathbf{s}(k) \) by the trust-region radius \( \Delta_{(k)} \):

**Lemma 3** (Accuracy Requirements similar to [49, A. 2.4]). Suppose Assumptions 2 to 4 and 6 hold and that \( k \in \mathbb{N}_0 \) is an iteration index. Then there is a constant \( e_{err} > 0 \) such that, if \( \mathbf{n}(k) \) exists as the solution to (ITRN(k)), it holds that

\[
\max_{\ell} \left\{ h_{\ell}(\mathbf{x}(k)) + \nabla h_{\ell}(\mathbf{x}(k)) \cdot \mathbf{n}(k), \max_{\ell} g_{\ell}(\mathbf{x}(k)) + \nabla g_{\ell}(\mathbf{x}(k)) \cdot \mathbf{n}(k) \right\} \leq e_{err} \bar{\Delta}_{(k)} \left\| \mathbf{n}(k) \right\|_{tr,k}.
\]  

(13)

For any \( k \in \mathbb{N}_0 \) for which \( \mathbf{n}(k) \) exists and satisfies \( \left\| \mathbf{n}(k) \right\|_{tr,k} \leq \bar{\Delta}_{(k)} \) it also holds that

\[
\max_{\ell} \left\{ h_{\ell}(\mathbf{x}(k)) + \nabla h_{\ell}(\mathbf{x}(k)) \cdot \mathbf{n}(k), \max_{\ell} g_{\ell}(\mathbf{x}(k)) + \nabla g_{\ell}(\mathbf{x}(k)) \cdot \mathbf{n}(k) \right\} \leq e_{err} \Delta_{(k)}^2.
\]  

(14)

and if the Criticality Routine finishes finitely and \( \mathbf{s}(k) \) is the step \( \mathbf{n}(k) + \sigma(k) \mathbf{d}(k) \) with \( \mathbf{d}(k) \) computed using (ITRN(k)) and \( \left\| \mathbf{s}(k) \right\|_{tr,k} \leq \bar{\Delta}_{(k)} \), we have

\[
\max_{\ell} \left\{ h_{\ell}(\mathbf{x}(k)) + \nabla h_{\ell}(\mathbf{x}(k)) \cdot \mathbf{s}(k), \max_{\ell} g_{\ell}(\mathbf{x}(k)) + \nabla g_{\ell}(\mathbf{x}(k)) \cdot \mathbf{s}(k) \right\} \leq e_{err} \Delta_{(k)}^2.
\]  

(15)

**Proof.** The constraints of (ITRN(k)) ensure that

\[
\hat{h}(k)(\mathbf{x}(k)) + \hat{H}_k(\mathbf{x}(k)) \cdot \mathbf{n}(k) = 0 \quad \text{and} \quad \hat{g}(k)(\mathbf{x}(k)) + \hat{G}_k(\mathbf{x}(k)) \cdot \mathbf{n}(k) \leq 0.
\]  

(16)

Because the models are interpolating (Assumption 4) we also have \( \hat{h}(k)(\mathbf{x}(k)) = h(\mathbf{x}(k)) \) and \( \hat{g}(k)(\mathbf{x}(k)) = g(\mathbf{x}(k)) \). Hence, additionally using the Cauchy-Schwartz inequality and the error-bounds of the models (for the preliminary radius \( \Delta_{(k)} \)), we obtain

\[
\max_{\ell} \left| h_{\ell} + \nabla h_{\ell} \cdot \mathbf{n}(k) \right| = \left| h_{\ell} + \nabla h_{\ell} \cdot \mathbf{n}(k) - 0 \right| \overset{(16)}{=} \left| \nabla h_{\ell} - \nabla h_{\ell} \right| \cdot \mathbf{n}(k) \leq \left\| \nabla h_{\ell} - \nabla h_{\ell} \right\| _2 \cdot \left\| \mathbf{n}(k) \right\| _2 \leq \hat{e}_h \bar{\Delta}_{(k)} \left\| \mathbf{n}(k) \right\| _{tr,k},
\]

where the argument \( \mathbf{x}(k) \) was dropped to improve readability. Similarly, we find that

\[
\max_{\ell} \left| g_{\ell} + \nabla g_{\ell} \cdot \mathbf{n}(k) \right| \leq \hat{e}_g \bar{\Delta}_{(k)} \left\| \mathbf{n}(k) \right\| _{tr,k}
\]

and (13) follows with \( e_{err} = c \max \{ \hat{e}_h, \hat{e}_g \} \). Moreover, \( \left\| \mathbf{n}(k) \right\| _{tr,k} \leq \bar{\Delta}_{(k)} \) leads to (14).

The second inequality (15) is derived analogously, respecting the fact that the step-size is chosen so that \( \left\| \mathbf{s}(k) \right\| _{tr,k} \leq \Delta_{(k)} \) and that \( \mathbf{x}(k) + \mathbf{s}(k) \) also satisfies the approximated linearized constraints. □

From the preceding accuracy results, a bound on the constraint violation can be derived:

**Lemma 4** ([49, Lemma 4.3],[21, Lemma 3.1]). Assume that the algorithm is applied to (MOP) and that Assumptions 2 to 6 and 9 hold. There is a constant \( c_{\text{obj}} > 0 \) (independent of \( k \)) such that, if the normal step exists, it holds that

\[
\theta_k \leq (e_{err} \bar{\Delta}_{(k)} + c_{\text{obj}}) \left\| \mathbf{n}(k) \right\| _{tr,k}.
\]  

(17)

**Proof.** The proof works very similar to that of Lemma 4.3 in [49]. We only need to take Lemma 3 into account for the different accuracy requirements. □

When the normal step is compatible, i.e., for iterations with \( k \notin \mathcal{R} \), we can further refine the bound on \( \theta_k \):

**Lemma 5** (Lemma 4.4 in [49], Lemma 3.4 in [21]). Suppose that the algorithm is applied to (MOP) and that Assumptions 2 to 6 and 9 hold. Suppose further that \( k \notin \mathcal{R} \) and that (17) holds. Then there is a constant \( c_{\text{obj}} > 0 \) such that

\[
\theta_k \leq c_{\text{obj}} \max \left\{ \Delta_{(k)}^{1+\mu}, \bar{\Delta}_{(k)}^2 \right\} \quad \text{and} \quad \theta \left( \mathbf{x}(k) + \mathbf{s}(k) \right) \leq c_{\text{obj}} \Delta_{(k)}^2.
\]  

(18)
Proof. Since \( k \notin \mathcal{R} \), one obtains from (17) and from (5) that
\[
\theta_k \leq (e_{\text{err}} \Delta(k) + c_{\text{ubj}} \mu) \left\| n^{(k)} \right\|_{\text{tr}, k} \leq (e_{\text{err}} \Delta(k) + c_{\text{ubj}} \mu) c_\Delta \Delta(k) \min \left\{ 1, c_\mu \Delta^{n}(k) \right\}
\]
\[
\leq c_\Delta \left( e_{\text{err}} \Delta^2(k) + c_{\text{ubj}} c_\mu \Delta^{1+\mu}(k) \right) \leq c_\Delta \left( e_{\text{err}} + c_{\text{ubj}} c_\mu \right) \max \left\{ \Delta^{1+\mu}(k), \Delta^2(k) \right\}.
\]
For the second bound let \( c_\ell \) be any constraint component from \( h \) or \( g \). Due to Assumptions 2 and 3 we can construct a Taylor polynomial and use the Mean-Value-Theorem to obtain
\[
c_\ell(x^{(k)} + s^{(k)}) = c_\ell(x^{(k)}) + \nabla c_\ell(x^{(k)})^T s^{(k)} + \frac{1}{2} H_{c_\ell} s^{(k)}, s^{(k)}
\]
(19)
for some \( \xi \) on the line segment from \( x^{(k)} \) to \( x^{(k)} + s^{(k)} \). From (15) we know that \( |c_\ell(x^{(k)}) + \nabla c_\ell(x^{(k)})^T s^{(k)}| \leq e_{\text{err}} \Delta^2(k) \). With Assumptions 3 and 9 the norm off all constraint Hessians in Eq. (19) can be bounded from above globally (say, by \( 2k_H > 0 \)) so that the Cauchy-Schwartz inequality gives
\[
\frac{1}{2} \| H_{c_\ell}(\xi) s^{(k)}, s^{(k)} \| \leq \frac{1}{2} \| s^{(k)} \|_2 \| H_{c_\ell}(\xi) \|_2 \leq \kappa H \Delta^2(k).
\]
Hence, (18) follows with
\[
c_{\text{ub}} = \max \left\{ c^*_\text{ub}, e_{\text{err}} + \kappa H \right\}.
\]
The requirements for the accuracy requirements to hold are also met within the Criticality Routine, allowing for the following corollary:

**Corollary 3.** Let \( k \in \mathbb{N}_0 \) be an iteration index and suppose the same requirements as in Lemma 4 hold. If the Criticality Routine is entered, then for any sub-iteration index \( j \in \mathbb{N}_0 \), it holds that
\[
\theta_k \leq e_{\text{err}} \delta^2_j + c_{\text{ubj}} \left\| n^{(k)}_j \right\|_{2},
\]
(20)
If the Criticality Routine is not entered or if it has completed finitely, then it also holds that
\[
\theta_k \leq e_{\text{err}} \Delta^2_j + c_{\text{ubj}} \left\| n^{(k)} \right\|_{2},
\]
(21)
and if \( k \notin \mathcal{R} \), then
\[
\theta_k \leq c_{\text{ubj}} \max \left\{ \Delta^{1+\mu}(k), \Delta^2(k) \right\}.
\]
(22)

**4.3.1 Convergence in the Criticality Routine**

Our first important observation is that quasi-criticality is approximated if the criticality loop runs infinitely:

**Lemma 6** ([4, Lemma 8]). Assume that Assumptions 2 to 6 and 9 hold. Denote by \( \chi_j \) the inexact criticality value from Corollary 1 in the criticality subroutine. If the criticality routine runs infinitely \((j \to \infty)\) at \( x^{(k)} \), then
\[
\theta \left( x^{(k)} \right) = 0 \text{ and } \lim_{j \to \infty} \delta_j = 0 \text{ and } \lim_{j \to \infty} \chi_j = 0.
\]

Proof. As in the description of the Criticality Routine, let \( j \) be the sub-iteration index and \( \delta^+_j := \alpha^n \Delta(k) \), and \( \delta_j = \delta^+_{j-1} \) after the index \( j \) has been increased. Also, denote by \( \chi_j \) the updated doubly inexact criticality measure from step 2e. There are two logically disjoint stopping criteria for the Criticality Routine. Firstly, the Criticality Routine may stop if \( \mathcal{M} \chi_j \geq \delta_j \) for some \( j \in \mathbb{N}_0 \). Conversely, if the routine loops infinitely, then it must hold that
\[
\chi_j < \frac{\delta_j}{\mathcal{M}_j} = \alpha^n \frac{\Delta(k)}{\mathcal{M}} \quad \text{for all } j \in \mathbb{N}.
\]
Because of \( \alpha \in (0, 1) \) the right side goes to zero and then the second limit in Lemma 6 also follows.
Secondly, the routine stops if no compatible step \( n_j \) for the preliminary radius \( \delta^+_j \) can be found anymore. Vice versa, if the routine runs infinitely, there always is a compatible step with
\[
\| n_j \|_2 \leq c_\Delta \| \delta_j \| \min \left\{ 1, c_\mu \delta^\mu_j \right\}.
\]
This implies \( n_j^{(k)} \to 0 \) for \( j \to \infty \). From (20) it then follows that it must already hold that \( \theta(x^{(k)}) = 0 \). \( \blacksquare \)
4.3.2 Infinitely Many Filter Iterations

Outside of the criticality loop, we can show that feasibility is approached if the number of filter iterations is infinite:

**Lemma 7** ([21, Lemma 3.3]). Suppose that the algorithm is applied to (MOP) and that Assumptions 2, 3 and 9 hold. If $|Z| = \infty$, then $\lim_{k \to \infty} \theta_k = 0$.

**Proof.** The proof is exactly the same as in [21], only $f(\bullet)$ has to be substituted by $\Phi[f](\bullet)$. It also works with a $K + 1$ dimensional filter that uses $f$ instead of $\Phi[f](\bullet)$.

The next result shows that feasibility is approached for any subsequence of iteration indices. The result is a bit stronger than that found in [21] and not necessarily needed for the convergence proofs. A similar theorem can be found in [53].

**Lemma 8.** Suppose that the algorithm is applied to (MOP) and that Assumptions 2, 3 and 9 hold. If $|Z| = \infty$, then $\lim_{k \to \infty} \theta_k = 0$.

**Proof.** For a contradiction, assume that there is a subsequence of indices $\{k_1\}$ for which the constraint violation is bounded below

$$\theta_{k_1} \geq \varepsilon > 0 \quad \forall k.$$  \hspace{1cm} (23)

From Lemma 7 we know that there is some $k_0$ such that

$$\theta_k < \varepsilon \quad \text{for all } k \in Z \text{ with } k \geq k_0.$$  \hspace{1cm} (24)

Because there are infinitely many indices both in $\{k_1\}$ and $Z$, each $k \in \{k_1\}$ (except maybe the first) must lie between a largest filter index $\kappa_1 \in Z$ and a smallest index $\kappa_2 \in Z$, respectively, $\kappa_1(k) \leq k \leq \kappa_2(k)$, and from (23) and (24) we can deduce that there is a $k_1 \geq k_0$ such that

$$k_1 < \kappa_1(k) < k < \kappa_2(k) \quad \text{for all } k \in \{k_1\} \text{ with } k \geq k_1.$$  

Let $k \in \{k_1\}$ be an index with $k \geq k_1$. For all indices $\kappa$ with $\kappa_1(k) < \kappa < \kappa_2(k)$, the iteration is not a filter iteration and hence the function $\Phi$ is “monotonic”:

$$\Phi(x^{(\kappa)}) - \Phi(x^{(\kappa+1)}) \geq 0.$$  

Moreover, there must be some smallest successful index $\kappa_2^{*}(k)$ with $k \leq \kappa_2^{*}(k) < \kappa_2(k)$, reducing the constraint violation from above $\varepsilon$ to below $\varepsilon$ again. $\kappa_2^{*}(k)$ is not a filter index, so (10) succeeds and $\rho(\kappa_2^{*}(k)) \geq \nu_1 > 0$, resulting in

$$\Phi(x^{(\kappa_2^{*}(k))}) - \Phi(x^{(\kappa_2^{*}(k)+1)}) \geq \nu_1 K d \theta_{\kappa_2^{*}(k)} \geq \nu_1 K d \varepsilon > 0.$$  \hspace{1cm} (25)

The function $\Phi$ might be increased eventually in the filter iteration $\kappa_2(k)$ (when accepting $x^{(\kappa_2(k)+1)}$), but due to the monotonicity of the intermediate iterations it holds that $\Phi(x^{(\kappa_2^{*}(k)+1)}) - \Phi(x^{(\kappa_2^{*}(k))}) \geq 0$ and thus

$$\Phi(x^{(\kappa_2^{*}(k))}) - \Phi(x^{(\kappa_2(k))}) \geq \nu_1 K d \varepsilon > 0,$$  \hspace{1cm} (26)

and $x^{(\kappa_2^{*}(k))}$ is added to the filter.

For $k$ from above, let $k_+ \in \{k_1\}$ be the smallest iteration index following $k$ such that it holds for the enclosing filter indices that $\kappa_2(k) \leq \kappa_1(k_+)$. There then must be a largest successful index $\kappa_l^{*}(k_+)$ with $\kappa_1(k_+) \leq \kappa_l^{*}(k_+) < k_+$, such that the constraint violation is increased from below $\varepsilon$ to above $\varepsilon$,

$$\theta_{\kappa_l^{*}(k_+)} < \varepsilon \leq \theta_{\kappa_l^{*}(k_+)+1},$$

and from the filter mechanism we know that then the objective reduction has to be significant compared to all points in the filter with a smaller constraint violation, including that with index $\kappa_2(k)$ and $\theta_{\kappa_2(k)} < \varepsilon$:

$$\Phi(x^{(\kappa_2(k))}) - \Phi(x^{(\kappa_l^{*}(k_+)+1)}) \geq \gamma_d \theta_{\kappa_2(k)} \geq 0.$$  

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Using the same notation as above, (i.e., $\kappa_j(k)$ as in (25)), we deduce from the monotonicity of the intermediate iterations that $\Phi(x^{(\kappa_j(k))}) - \Phi(x^{(\kappa_j(k))}) \geq 0$ and thus, with (26),
\[
\Phi(x^{(\kappa_j(k))}) - \Phi(x^{(\kappa_j(k))}) \geq \nu_k \kappa_\phi \psi > 0.
\] (27)

By repeating the above procedure, we see that it is possible to construct an infinite subsequence $\{k_t\}$ of iteration indices from the $\kappa_j$ values (of indices from $\{k_t\}$), for which $\Phi(x^{(\kappa_j)})$ is strictly monotonically decreasing with a guaranteed (constant) objective reduction (27). This is a contradiction to $\Phi$ being bounded below as per Assumptions 3 and 9.

What follows next is a series of auxiliary lemmata to finally show convergence of the inexact criticality measure when $|Z| = \infty$. First, we have a bound on the surrogate objective change along the normal step:

**Lemma 9** (Similar to a bound in [21, Lemma 3.5]). Suppose the algorithm is applied to (MOP), that Assumptions 2 to 4 and 6 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further that $k \notin R$. Then it holds that
\[
\left| \Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)}) \right| = \left| \Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)}) \right| \leq c_u \left\| n^{(k)} \right\|_2 + \frac{1}{2} H \left\| n^{(k)} \right\|_2^2.
\] (28)

**Proof.** There are maximizing indices $\ell, j \in \{1, \ldots, K\}$ such that
\[
\Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)}) = \tilde{f}_\ell^{(k)}(x^{(k)}) - \tilde{f}_j^{(k)}(x^{(k)}) \leq \Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)}) \leq \tilde{f}_\ell^{(k)}(x^{(k)}) - \tilde{f}_j^{(k)}(x^{(k)}).
\]

Using a 2nd degree Taylor approximation of $\tilde{f}_\ell^{(k)}$ around $x^{(k)}$ at $x^{(k)}$ results in
\[
\Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)}) \leq \tilde{f}_\ell^{(k)}(x^{(k)}) - \tilde{f}_j^{(k)}(x^{(k)})
\]
\[
\leq \left\| \nabla \tilde{f}_\ell^{(k)}(x^{(k)}, n^{(k)}) \right\|_2 \left\| n^{(k)} \right\|_2 + \frac{1}{2} H \tilde{f}_\ell^{(k)}(x^{(k)}) \left\| n^{(k)} \right\|_2^2
\]
\[
\leq c_u \left\| n^{(k)} \right\|_2 + \frac{1}{2} H \left\| n^{(k)} \right\|_2^2,
\]
where the last inequality comes from Assumption 7 and Corollary 2. Analogously, we can show
\[
-\left( \Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)}) \right) \leq \tilde{f}_j^{(k)}(x^{(k)}) - \tilde{f}_j^{(k)}(x^{(k)}) \leq c_u \left\| n^{(k)} \right\|_2 + \frac{1}{2} H \left\| n^{(k)} \right\|_2^2.
\]

The next lemma provides a sufficient decrease bound in case that the doubly inexact criticality measure is bounded below, and the radius is sufficiently small:

**Lemma 10** ([21, Lemma 3.5]). Suppose the algorithm is applied to (MOP), that Assumptions 2 to 4 and 6 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further, that $k \notin R$, that
\[
\chi^{(k)} \geq \frac{1}{w_2(k)} > \frac{1}{w_k}
\] (LB)
for some $\epsilon > 0$ and that
\[
\Delta^{(k)} \leq \delta_m := \min \left\{ \frac{\epsilon}{w}, \left( \frac{2c_u}{H c_\Delta c_\mu} \right)^{\frac{1}{1-p}}, \left( \frac{c_{s, \Delta} \Delta^{(k)}}{4c_u c_\Delta c_\mu} \right)^{\frac{1}{1-p}} \right\}.
\] (29)

Then
\[
\Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)} + s^{(k)}) \geq \frac{1}{2} c_{s, \Delta} \delta^{(k)}.
\] (30)

**Proof.** The proof is similar to that given in [21]. The bounds (LB) and (29) are plugged into the sufficient decrease equation of Assumption 8. The bound (30) follows from Lemma 9, (29) and the fact that $n^{(k)}$ is compatible with some tedious algebra.
Under similar conditions as in Lemma 10 the iteration will be successful:

**Lemma 11** ([21, Lemma 3.6]). Suppose the algorithm is applied to (MOP), that Assumptions 2 to 4 and 6 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further, that \( k \notin R \), that (LB) holds again and that

\[
\Delta(k) \leq \delta_\rho := \min \left\{ \delta_m, \frac{(1 - \nu_0)c_{\text{ef}}}{2e_f} \right\}.
\]

Then \((k)\) is successful, that is, \( \rho(k) \geq \nu_0 \).

**Proof.** We can again follow the proof in [21], because from Assumption 4 we can conclude (cf. [46, Lemma 4.16]) that the model error bound holds also for the scalarization:

\[
\Phi[f](\xi) - \Phi[f(k)](\xi) = \Phi(\xi) - \Phi(k)(\xi) \leq e_f \Delta^2(\xi) \quad \forall \xi \in B(k).
\]

Further, if the radius is sufficiently small, then the test (10) will succeed (prohibiting a \( \theta \)-iteration):

**Lemma 12** ([21, Lemma 3.7]). Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further, that \( k \notin R \), that (LB) holds again, that \( \theta_k \leq \delta_\rho \), and that

\[
\Delta(\xi) \leq \delta_f := \min \left\{ \delta_m, 1, \frac{c_{\text{sef}}}{\frac{1}{\nu_0}c_{\text{ef}} + \frac{1}{\nu_0}c_{\text{ef}}} \right\}.
\]

Then (10) succeeds, i.e., \( \Phi(k)(x(k)) - \Phi(k)(x_k) \geq \kappa_\theta \theta_\rho \).

**Proof.** Lemma 10 provides

\[
\Phi(k)(x(k)) - \Phi(k)(x_k) = \Phi(k)(x(k)) - \Phi(k)(x(k) + s(k)) \geq \frac{1}{2}c_{\text{sef}}\Delta(\xi).
\]

The small constraint violation \( \theta_k \leq \delta_\rho \) and \( k \notin R \) imply (22). Because of \( \Delta(\xi) \leq 1 \) and \( \mu \in (0, 1) \) the bound (22) simplifies to \( \theta_k \leq \Delta^2(\xi) \), and it follows that \( c_{\text{sef}}\Delta(\xi) \geq \theta_\rho \). The final inequality in (33) then leads to

\[
\frac{1}{2}c_{\text{sef}}\Delta(\xi) \geq \kappa_\theta \rho \Delta^2(\xi),
\]

which then implies (10).

If additionally the constraint violation is small enough, then the filter pair at the trial point will not be filter-dominated by the pair at \( x(k) \):

**Lemma 13** ([21, Lemma 3.8]). Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further, that \( k \notin R \), that (LB) holds again, that \( \Delta(\xi) \leq \delta_\rho \) as in (31) and that \( \theta_k \leq \delta_\rho \), and that

\[
\theta_k \leq \delta_\rho := \min \left\{ \frac{1}{c_{\text{ubd}}} \left( \frac{\nu_0}{c_{\text{sef}}} \right)^2, \frac{1}{2c_{\text{ubd}}} \left( \frac{1 + \mu}{\nu_0c_{\text{ef}}} \right) \right\}.
\]

Then \( \Phi(x(k)) - \Phi(x(k) + s(k)) \geq \gamma_\theta \theta_\rho \).

**Proof.** From Lemma 5 we deduce that \( \theta_k \leq \Delta^{1/\mu} \) or \( \theta_k \leq \Delta_\rho^2 \) and thus

\[
\Delta(\xi) \geq \min \left\{ \frac{\theta_k}{c_{\text{ubd}}} \right\} \frac{1}{\nu_0c_{\text{sef}}} \Delta(\xi)
\]

With Lemma 11, it follows that

\[
\Phi(x(k)) - \Phi(x(k) + s(k)) \geq \nu_0 \left( \Phi(k)(x(k)) - \Phi(k)(x(k) + s(k)) \right) \geq \frac{1}{2} \nu_0c_{\text{sef}}\Delta(\xi) \geq \frac{1}{2} \nu_0c_{\text{sef}}\Delta(\xi)
\]

\[
\left( \frac{1}{2} \nu_0c_{\text{sef}} \right) \min \left\{ \frac{\theta_k}{c_{\text{ubd}}} \right\} \frac{1}{\nu_0c_{\text{sef}}} \Delta(\xi) \geq \gamma_\theta \theta_\rho.
\]
In the preceding lemmata, it has always been assumed that the iteration index does not belong to the set of restoration indices $\mathcal{R}$. This is ensured if both the radius and the constraint violation are small enough:

**Lemma 14** ([21, Lemma 3.9]). Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose that (LB) holds again. Suppose further that

$$\Delta(k) \leq \delta \triangleq \min \left\{ \gamma_\theta \theta, \left( \frac{1}{c_{\mu}} \right)^{\frac{1}{\gamma_\theta}}, \left( \frac{(1-\gamma_\theta)\gamma_\theta^3 c_{\Delta} c_{\mu}}{c_{\text{ub}} c_{\text{ub}} \theta} \right)^{\frac{1}{2-\gamma_\theta}} \right\}$$

(36)

and that

$$\theta_k \leq \min \{ \delta, \delta_n \}. \quad (37)$$

If $k > 0$, then $k \notin \mathcal{R}$.

**Proof.** The proof works exactly as in [21], because the Criticality Routine is not entered restoration iterations. □

The previous lemmata allow us to investigate two mutually exclusive cases defined by the number of filter iterations. The first convergence result is obtained for the case that there are infinitely many such iterations.

**Lemma 15.** Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose that (LB) holds again. Suppose further that

$$\Delta(k) \geq \Delta_{\text{min}} > 0 \quad \forall \ell \in \mathbb{N}_0.$$ 

(38)

From Lemma 8 and Assumption 5 it follows that

$$\lim_{\ell \to \infty} \left\| \mathbf{n}^{(k_\ell)} \right\|_{\text{tr}, k} \geq \lim_{\ell \to \infty} \left\| \mathbf{n}^{(k_\ell)} \right\|_2 = 0. \quad (39)$$

Consequently, for sufficiently large $\ell$, the steps will become compatible and satisfy (5) and thus $k_\ell \notin \mathcal{R}$. Hence, Lemma 9 applies and inequality (28) holds again:

$$\left\| \Phi^{(k_\ell)} \left( \mathbf{x}^{(k_\ell)} \right) - \Phi^{(k_\ell)} \left( \mathbf{x}^{(k_\ell)} \right) \right\| \leq c_{\text{ub}} \left\| \mathbf{n}^{(k_\ell)} \right\|_2 + \frac{1}{2} \left\| \mathbf{n}^{(k_\ell)} \right\|_2^2,$$

so that (39) gives

$$\lim_{\ell \to \infty} \left\| \Phi^{(k_\ell)} \left( \mathbf{x}^{(k_\ell)} \right) - \Phi^{(k_\ell)} \left( \mathbf{x}^{(k_\ell)} \right) \right\| = 0. \quad (40)$$

Lemma 8 enables the Criticality Routine. Additionally, from the boundedness of $\Delta(k_\ell)$ as per (38) and from Assumption 5, it also follows that, for large $\ell$, the Criticality Routine will not exit because no compatible normal step exists anymore. This ensures that

$$\chi^{(k_\ell)} \geq \min \left\{ \varepsilon_{\chi}, \frac{\Delta(k_\ell)}{M} \right\} \geq \min \left\{ \varepsilon_{\chi}, \frac{\Delta_{\text{min}}}{M} \right\} =: z > 0 \quad (41)$$

for large $\ell$. We can use this fact in Assumption 8 and obtain

$$\Phi^{(k_\ell)} \left( \mathbf{x}_{\text{n}}^{(k_\ell)} \right) - \Phi^{(k_\ell)} \left( \mathbf{x}_{\text{n}}^{(k_\ell)} \right) \geq c_{\text{ub}} z \min_{\text{const.}} \left\{ \frac{z}{M}, \Delta_{\text{min}} \right\} > 0. \quad (42)$$

If we add zero we obtain the decomposition

$$\Phi^{(k_\ell)} \left( \mathbf{x}^{(k_\ell)} \right) - \Phi^{(k_\ell)} \left( \mathbf{x}^{(k_\ell)} \right) = \left( \Phi^{(k_\ell)} \left( \mathbf{x}^{(k_\ell)} \right) - \Phi^{(k_\ell)} \left( \mathbf{x}_{\text{n}}^{(k_\ell)} \right) \right) + \left( \Phi^{(k_\ell)} \left( \mathbf{x}_{\text{n}}^{(k_\ell)} \right) - \Phi^{(k_\ell)} \left( \mathbf{x}_{\text{s}}^{(k_\ell)} \right) \right). \quad (43)$$
By plugging in (40) for the left set of parentheses, we see that in the limit the difference of values is the same:

\[
\lim_{\ell \to \infty} \Phi^{(k_\ell)}(\mathbf{x}^{(k_\ell)}) - \Phi^{(k_\ell)}(\mathbf{x}^{(k_\ell)}_{\alpha}) = \lim_{\ell \to \infty} \Phi^{(k_\ell)}(\mathbf{x}^{(k_\ell)}_{\alpha}) - \Phi^{(k_\ell)}(\mathbf{x}^{(k_\ell)}_{\beta}).
\]

Because of \(|\{k_\ell\} \cap \mathbb{Z}| = \infty\), there is a subsequence \(\{k_{\ell_j}\} \subseteq \{k_\ell\}\) of filter indices, \(k_{\ell_j} \in \mathbb{Z}\), for which it must hold that \(k_{j} \in \mathcal{R}\) or that \((10)\) fails. We have already shown that \(k_{j} \not\in \mathcal{R}\) for large \(j\). Hence, for \(j\) sufficiently large, it must hold that \(\kappa_{\ell \theta} \geq \Phi^{(k_{j})}(\mathbf{x}^{(k_{j})}) - \Phi^{(k_{j})}(\mathbf{x}^{(k_{j})} + \mathbf{s}^{(k_{j})})\) and Lemma 8 implies that both sides must go to zero. This is a contradiction to (42). Hence, no infinite subsequence \(\{k_{\ell_j}\} \subseteq \{k_\ell\} \cap \mathbb{Z}| = \infty\) and (38) can exist.

**Remark 2.** Without further assumptions on the restoration step it does not seem possible to show \(\lim_{k \to \infty} \Delta(k) = 0\) in the case that \(|\mathbb{Z}| = \infty\). For any element of an index sequence \(\{k_\ell\}\), bounded away from zero and with \(|\{k_\ell\} \cap \mathbb{Z}| < \infty\), there might be a preceding restoration iteration with arbitrarily small radius itself, but possibly increasing it without any restriction, so that no contradiction can be derived. This difficulty has been observed in the literature before, see e.g. [53, Remark 7].

**Lemma 16.** Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose that \(|\mathbb{Z}| = \infty\) and that the doubly inexact criticality is bounded as per (LB) for all \(k \in \mathbb{Z}\). Then the trust-region radius is also bounded below, i.e., there is some \(\Delta_{\min} > 0\) such that \(\Delta(k) \geq \Delta_{\min}\) for all \(k \in \mathbb{Z}\).

**Proof.** If there are infinitely many filter indices \(\mathbb{Z}\), then we know from Lemma 8 that \(\theta_k \to 0\). Suppose \(\{\Delta(k)\}_{k \in \mathbb{Z}}\) is not bounded away from zero. Then there is a subsequence \(\{k_\ell\} \subseteq \mathbb{Z}\) with \(\lim_{\ell \to \infty} \Delta(k_\ell) = 0\). Following the argumentation in [21, Lemma 3.10], we see that for \(\ell\) large enough, Lemma 14 applies and guarantees that \(k_\ell \not\in \mathcal{R}\). At the same time, Lemma 12 applies for large \(\ell\), so that the test (10) always succeeds for \(\mathbf{z}^{(k_\ell)}\). Hence, there is some \(\ell_0 \in \mathbb{N}\) such that for all \(k_\ell\) with \(\ell \geq \ell_0\), it holds that \(k_\ell \not\in \mathbb{Z}\). This contradicts \(\{k_\ell\} \subseteq \mathbb{Z}\).

We are finally able to state the convergence result for the case of infinitely many filter iterations as a cumulative corollary derived from Lemmata 7, 15 and 16:

**Corollary 4 ([21, Lemma 3.10]).** Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further, that \(|\mathbb{Z}| = \infty\). Then there is a subsequence \(\{k_\ell\} \subseteq \mathbb{Z}\) of filter indices with

\[
\lim_{\ell \to \infty} \theta_{k_\ell} = 0, \quad \lim_{\ell \to \infty} \Delta(k_{\ell}) = 0 \quad \text{and} \quad \lim_{\ell \to \infty} \chi(k_{\ell}) = 0.
\]

### 4.3.3 Finitely Many Filter Iterations

We have shown that a quasi-stationary point is approached when there are infinitely many filter indices. We now concentrate on the case that there are only finitely many filter indices (and note that this then implies that there are only finitely many restoration indices as well). From now on, \(k_0\) is the last iteration index for which \(\mathbf{z}^{(k_0-1)}\) is added to the filter.

**Lemma 17 ([21, Lemma 3.11]).** Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further, that \(|\mathbb{Z}| < \infty\). Then

\[
\lim_{k \to k_0} \theta_k = 0.
\]

Furthermore, \(\mathbf{n}^{(k)}\) satisfies (17) for all \(k \geq k_0\) large enough.

**Proof.** First consider the case that there are only finitely many successful indices \(\mathcal{A}\). Then \(\Delta(k) \xrightarrow{k \to \infty} 0\). Because there are no restoration iterations for \(k \geq k_0\), it follows from (5) that \(\mathbf{n}^{(k)} \to 0\). Eq. (21) finally gives \(\theta_k \to 0\).

Now, suppose that \(|\mathcal{A}| = \infty\) and consider any successful iteration with \(k \geq k_0\). Then \(\chi(k)\) is not added to the filter, and it follows from step 5 of the algorithm that \(\rho^{(k)} \geq \nu_1\) and from the definition of \(\rho^{(k)}\) and from (10) that

\[
\Phi(\mathbf{x}^{(k)}) - \Phi(\mathbf{x}^{(k)} + \mathbf{s}^{(k)}) \geq \nu_1(\Phi(\mathbf{x}^{(k)}) - \Phi(\mathbf{x}^{(k)} + \mathbf{s}^{(k)})) \geq \nu_1 \kappa_\rho \theta_k^{(k)} \geq 0.
\]

The sequence \(\{\Phi(\mathbf{x}^{(k)})\}\) \(k \geq k_0\) is bounded below due to Assumptions 3 and 9, and it is monotonically decreasing because (10) always succeeds due to \(k \geq k_0\). Thus, no new iterate is ever chosen with \(\rho^{(k)} < \nu_1\). Hence, for the
right-hand side (RHS) in (45) it follows that
\[
\lim_{k \in \mathcal{A}, k \to \infty} \Phi(x^{(k)}) - \Phi(x^{(k)} + s^{(k)}) = 0.
\] (46)

The limit (44) follows from the RHS in (45) by noticing that \( \theta_j = \theta_k \) for all non-successful indices \( j \geq k \) with \( j^{(j)} < \nu_1 \). The bound (17) holds eventually because \( \theta_k \leq \delta_h \) for large \( k \) and then Lemma 4 applies.

The next auxiliary lemma shows that the trust-region radius is bounded below if the asymptotically feasible iterates do not approach a quasi-stationary point. It is used afterwards to derive a contradiction.

**Lemma 18** ([21, Lemma 3.12]). Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further, that \(|Z| < \infty \) and that the doubly inexact criticality is bounded away from zero as in (LB) for all \( k \geq k_0 \). Then there is a \( \Delta_{\min} > 0 \) such that \( \Delta_{(k)} \geq \Delta_{\min} \) for all \( k \in \mathbb{N}_0 \).

**Proof.** We sketch, how to adapt the proof in [21]: Lemma 17 ensures that for large enough \( k \) we have \( \theta_k \leq \varepsilon_\theta \) and the Criticality Routine affects the criticality value and the radius. For these \( k \) it follows from (12) and (LB) that \( \Delta_{(k)} \geq \min \{ \mathcal{B} \Delta_{(k)}, \Delta_{(k)} \} \geq \min \{ \mathcal{B} \varepsilon, \Delta_{(k)} \} \). Further, if \( k \geq 1 \) and \( k - 1 \notin \mathcal{R} \), then \( \Delta_{(k)} \geq \min \{ \mathcal{B}, \varepsilon, \gamma_0 \Delta_{(k-1)} \} \). Suppose \( k_1 \geq k_0 \) is large enough that \( \theta_k \leq \varepsilon_\theta \) and (37) is fulfilled for all \( k \geq k_1 \), i.e., \( \theta_k \leq \min \{ \varepsilon_\theta, 1 \} \) for all \( k \geq k_1 \). For the purpose of deriving a contradiction, one may now assume that \( j \geq k_1 \) is the first index with
\[
\Delta_{(j)} \leq \gamma_0 \min \left\{ \delta_p, \sqrt{\frac{(1 - \gamma)\theta^F}{c_{ub\theta}}} \Delta_{(k)}, \mathcal{B}_{e} \right\} =: \gamma_0 \delta_x,
\] (47)
where \( \delta_p \) is as defined in Lemma 11 and \( \theta^F := \min_{\ell \in \mathcal{Z}} \theta_\ell \) and proceed as in [21].

**Lemma 19** ([21, Lemma 3.13]). Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further, that \(|Z| < \infty \). Then \( \lim \inf_{k \to \infty} \Delta_{(k)} = 0 \).

**Proof.** If there are only finitely many successful indices, then the result follows immediately from the radius update rules.

Thus, suppose that \( |\mathcal{A}| = \infty \) and for a contradiction assume that there is a constant \( \Delta_{\min} > 0 \) such that
\[
\Delta_{(k)} \geq \Delta_{\min} > 0 \quad \forall k \in \mathbb{N}_0.
\] (48)
As before, we see that from Lemma 17 it follows for large \( k \) that the normal step \( n^{(k)} \) exists and satisfies (6) in Assumption 5. Then the same equations as in Lemma 17 hold, namely (45) and (46). Very much like in the proof of Corollary 4, we can again decompose the model decrease via (43). For the first term, (28) applies again, yielding \( \lim_{k \to \infty} \left( \Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)}) \right) = 0 \). From equations (45), (46) and from the model decrease decomposition it follows that
\[
\lim_{k \to \infty} \left( \Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)}) \right) = 0.
\] (49)
But the sufficient decrease condition (Assumption 8) still holds and (for large \( k \) and assuming (48) holds) the Criticality Routine ensures
\[
\chi^{(k)} \geq \min \left\{ \varepsilon_{\chi}, \frac{\Delta_{(k)}}{M} \right\} \geq \min \left\{ \varepsilon_{\chi}, \frac{\Delta_{\min}}{M} \right\} =: z > 0,
\]
which can be plugged in and gives:
\[
\Phi^{(k)}(x^{(k)}) - \Phi^{(k)}(x^{(k)}) \geq c_{sd} z \min \left\{ \frac{z}{M}, \Delta_{\min} \right\} > 0,
\]
where the RHS is constant, contradicting (49).

**Corollary 5** ([21, Lemma 3.13]). Suppose the algorithm is applied to (MOP), that Assumptions 2 to 9 hold and that the Criticality Routine does not run infinitely. Suppose further, that \(|Z| < \infty \). Then
\[
\lim \inf_{k \to \infty} \chi^{(k)} = 0.
\]
For a subsequence \( \{k_i\} \subseteq \{k\} \) with \( \chi^{(k_i)} \to 0 \) it must also hold that \( \lim_{i \to \infty} \Delta_{(k_i)} = 0 \).
Proof. To derive a contradiction, it is assumed that \( \chi^{(k)} \) is bounded away from zero as per (LB) for all \( k \in \mathbb{N}_0 \). Then the radius must be bounded away from zero due to Lemma 18, in contradiction to \( \lim_{k \to \infty} \Delta(k) = 0 \), as assured by Lemma 19. If \( \{k\} \) is such that \( \chi^{(k)} \to 0 \), but we assume \( \lim_{k \to \infty} \Delta(k) = \Delta_{\min} > 0 \), then the Criticality Routine again ensures (41) for large \( \ell \), which is a contradiction.

**Theorem 3** (Convergence to Quasi-Stationary Point). Suppose the algorithm is applied to (MOP) and that Assumptions 2 to 9 hold. Let \( \{x^{(k)}\} \) be the sequence of iterates produced by the algorithm. Then either the restoration procedure in step 2 terminates unsuccessfully by converging to an infeasible, first-order critical point such that \( \Delta(k) \to \infty \), or there is a subsequence \( \{k_\ell\} \) of indices for which \( \lim_{k \to \infty} x^{(k)} = \bar{x} \) and \( \bar{x} \) is quasi-stationary, i.e.,

\[
\theta(x) = \lim_{k \to \infty} \theta_{k_\ell} = 0 \quad \text{and} \quad \lim_{k \to \infty} \chi^{(k)} = 0,
\]

and for which it also holds that \( \lim_{k \to \infty} \Delta(k) = 0 \).

Proof. If the restoration procedure always terminates successfully, then the convergence to a quasi-stationary point follows from Assumption 9 and Corollaries 4 and 5 and Lemmata 6 and 7.

5. **Convergence to KKT-Points**

In this section, we conclude that a quasi-stationary point is also a KKT-point under suitable constraint qualifications. In the first step, we show that we can easily replace the objective surrogates with their true counterparts.

**Lemma 20.** Suppose Assumptions 2 to 6 hold and suppose that it holds for a subsequence \( \{x^{(k)}\} \) of iterates that

\[
\lim_{k \to \infty} \Delta(k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \omega(k) = \lim_{k \to \infty} \omega(x^{(k)}; \bar{f}, L_k, \|\|_2) = 0.
\]

Then it also holds for the true objective as well as for the true surrogates. In particular, for \( \omega(x^{(k)}; f, L, \|\|_2) \),

\[
\omega(x^{(k)}; f, L_k, \|\|_2) = \omega(x^{(k)}; \bar{f}, L_k, \|\|_2) = 0.
\]

Proof. Similar to [4, Lemma 7], we can show that for \( k \in \mathbb{N}_0 \) it must hold that

\[
\left| \omega(x^{(k)}; f, L_k, \|\|_2) - \omega(x^{(k)}; \bar{f}, L_k, \|\|_2) \right| = \left| \omega_k - \omega_k \right| \leq \epsilon \Delta^{(k)}
\]

for some constant \( \epsilon > 0 \) (independent of \( k \)). The triangle inequality yields

\[
\left| \omega_k \right| \leq \left| \omega_k - \omega_k \right| + \left| \omega_k \right| \leq \epsilon \Delta^{(k)} + \left| \omega_k \right|
\]

and the RHS goes to zero as \( k \to \infty \).

Note that Lemma 20 applies both to a sequence of the main algorithm and to an infinite subsequence of the Criticality Routine (see Lemma 6). In our second step, we now replace the constraint surrogates with the original functions.

**Lemma 21.** Suppose Assumptions 2 to 4 hold. Let \( \{x^{(k)}\} \subseteq \mathbb{R}^n \) be an algorithmic sequence with \( x^{(k)} \to \bar{x} \in X \), \( \omega(x^{(k)}; f, L_k, \|\|_\infty) \to 0 \) and \( \Delta^{(k)} \to 0 \). Further, assume that the Mangasarian-Fromovitz constraint qualification (MFCQ) hold at \( \bar{x} \), i.e., the rows of \( H(\bar{x}) \) are linearly independent, and there is a direction \( d \in \mathbb{R}^n \) such that \( H(\bar{x})d = 0 \) and \( d^T g(\bar{x}) < 0 \) for all \( \ell \) with \( g(\bar{x}) = 0 \). Then \( \bar{x} \) is a KKT-point of (MOP).

Proof. In the following, we use \( \hat{H} \) and \( \hat{G} \) to denote the Jacobians of the surrogates \( \hat{H}^{(k)} \) and \( \hat{G}^{(k)} \), evaluated at \( x^{(k)} \). Likewise, \( F_k \) is the Jacobian of the true objective function at \( x^{(k)} \), while \( \hat{g}_k \leq 0 \) is defined as \( \hat{g}_k(x^{(k)}) + \hat{G}_k \cdot n^{(k)} \).

Assumption 5 gives \( n^{(k)} \to 0 \). Because of this, and the error bounds in Assumption 4, we have that \( x^{(k)} \to \bar{x} \), \( \Delta^{(k)} \to 0 \) as well as \( \hat{H} \to H = D_h(\bar{x}) \), \( \hat{G} \to G = D_h(\bar{x}) \) and \( \hat{g} \to g = g(\bar{x}) \).

By assumption, \( \omega(x^{(k)}; f, L_k, \|\|_\infty) \to 0 \). We thus have a sequence of linear programs,

\[
\omega(x^{(k)}; f, L_k, \|\|_\infty) = \max_{d \in \mathbb{R}^n, b^* \in \mathbb{R}} \left[ \begin{array}{c} 0^n \\ \bar{F}_k \\ \hat{H}_k \\ \hat{G}_k \\ 0_n \end{array} \right] ; \quad \left[ \begin{array}{c} d \\ \beta^* \end{array} \right] \leq \left[ \begin{array}{c} 1_n \\ 0_k \\ 0_M \\ 0_P \end{array} \right] ,
\]

s.t.

\[
\left[ \begin{array}{c} I_{n,n} \\ -I_{n,n} \\ \bar{F}_k \\ \hat{H}_k \\ \hat{G}_k \\ 0_P \end{array} \right] \cdot \left[ \begin{array}{c} d \\ \beta^* \end{array} \right] \leq \left[ \begin{array}{c} 1_n \\ 0_k \\ 0_M \end{array} \right].
\]
the values of which go to zero. The dual problems are

$$\min_{y^1, y^2, y^3 \geq 0, y^4 \in \mathbb{R}^M} \begin{bmatrix} I_n^T & 0_n^T & 0_M^T \end{bmatrix} \begin{bmatrix} 1_n^T & 1_n^T & 0_K^T \end{bmatrix} \begin{bmatrix} y^1 \end{bmatrix}$$

s.t. \( \begin{bmatrix} -I_{n,n} & I_{n,n} & F_k^T & H_k^T & \tilde{G}_k^T \end{bmatrix} \begin{bmatrix} y^1 \end{bmatrix} - \begin{bmatrix} \bar{y}_k^T \end{bmatrix} \begin{bmatrix} 0_n \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \) 

\((D_k)\)

By strong duality, the dual problem for \( k \in \mathbb{N}_0 \) is also always feasible, and its optimal value equals the primal optimal value. Suppose that \( \{y_k^i\}_{i=1}^5 \) is a sequence of dual optimizers. By strong duality, it follows from \( \omega(x_n^k; f, L_k, \|\cdot\|_\infty) \to 0 \) that

$$\lim_{k \to \infty} y_k^1 = 0, \quad \lim_{k \to \infty} y_k^2 = 0, \quad \text{and} \quad \lim_{k \to \infty} -\bar{g}_k^T y_k^5 = 0.$$ 

First, we show that it must also hold for \( i \in \{3, 4, 5\} \) that the sequences \( \{y^i\} \) are bounded (see also [15] for a similar idea). To derive a contradiction, assume that for some \( i \in \{3, 4, 5\} \) the sequence \( \{y_k^i\} \) is not bounded. Define \( \nu_k = \max \{1, \max_{i=3,4,5} \|y_k^i\|_\infty\} \), and we get a bounded sequence of scaled variables

$$\tilde{y}_k^i = \frac{1}{\nu_k} y_k^i \quad i = 1, \ldots, 5.$$ 

Then, \( \nu_k \to \infty \), but we can take a subsequence \( k \in K \) of indices such that for every \( i \in \{3, 4, 5\} \) the sequence \( \{\tilde{y}_k^i\}_{k \in K} \) converges to \( \tilde{y}^i \) and so that for one \( i \in \{3, 4, 5\} \) the limit \( \tilde{y}^i \) is not zero. Consequently, we have

$$\lim_{k \to \infty, k \in K} (y_k^1, \ldots, y_k^5) = (\tilde{y}^1, \ldots, \tilde{y}^5) = (0_n, 0_n, y^3, y^4, y^5), \quad \text{and} \quad \lim_{k \to \infty, k \in K} -\bar{g}_k^T y_k^5 = -\bar{g}^T \tilde{y}^5 = 0.$$ 

(50)

For any dual feasible \( \{y_k^i\}_{i=1}^5 \), the vector \( \{\tilde{y}_k^i\}_{i=1}^5 \) is feasible for the problem

$$\min_{\tilde{y}^1, \tilde{y}^2, \tilde{y}^3, \tilde{y}^4, \tilde{y}^5 \geq 0, \tilde{y}^6 \in \mathbb{R}^M} \begin{bmatrix} I_n^T & 0_n^T & 0_M^T \end{bmatrix} \begin{bmatrix} 1_n^T & 1_n^T & 0_K^T \end{bmatrix} \begin{bmatrix} \tilde{y}^1 \end{bmatrix}$$

s.t. \( \begin{bmatrix} F_k \tilde{y}^T & H_k \tilde{y}^T & \tilde{G}_k \tilde{y}^T \end{bmatrix} \) \( \begin{bmatrix} 0_n \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \).

In the limit, the last constraint becomes \( 1_n^T \tilde{y}^3 = 0 \), the limiting problem is

$$\min_{\tilde{y}^1, \tilde{y}^2, \tilde{y}^3, \tilde{y}^4, \tilde{y}^5 \geq 0, \tilde{y}^6 \in \mathbb{R}^M} \begin{bmatrix} I_n^T & 0_n^T & 0_M^T \end{bmatrix} \begin{bmatrix} 1_n^T & 1_n^T & 0_K^T \end{bmatrix} \begin{bmatrix} \tilde{y}^1 \end{bmatrix}$$

s.t. \( \begin{bmatrix} F_k \tilde{y}^T & H_k \tilde{y}^T & \tilde{G}_k \tilde{y}^T \end{bmatrix} \) \( \begin{bmatrix} 0_n \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \).

\((D^*)\)

The primal thus has constant objective value:

$$\max_{d \in \mathbb{R}^n, \beta^i \in \mathbb{R}} 0 \quad \text{s.t.} \quad \begin{bmatrix} -I_{n,n} & 0_n \end{bmatrix} \begin{bmatrix} d \end{bmatrix} \leq \begin{bmatrix} 1_n \end{bmatrix}, \quad \begin{bmatrix} I_{n,n} & 0_n \end{bmatrix} \begin{bmatrix} \beta \end{bmatrix} \leq \begin{bmatrix} 0_k \end{bmatrix}, \quad \begin{bmatrix} F_k & 1_K \end{bmatrix} \begin{bmatrix} \beta \end{bmatrix} = \begin{bmatrix} 0_m \end{bmatrix}, \quad \begin{bmatrix} H & 0_M \end{bmatrix} \begin{bmatrix} \beta \end{bmatrix} = \begin{bmatrix} 0_M \end{bmatrix}$$

\((P^*)\)

By upper semi-continuity of the feasible set mappings (see [50]), \( \tilde{y} \) is feasible for \((D^*)\), and we see from (50) and \((P^*)\) that it is also optimal. Let \( d \in \mathbb{R}^n \) be a direction with \( \|d\|_\infty \leq 1 \) adhering to the MFCQ. Furthermore, let \( \beta \) be such that \((d, \beta)\) is feasible for \((P^*)\). Then \((d, \beta)\) is optimal for \((P^*)\), and we make the following observations:

- \( \tilde{y}^3 \) must be zero, because of the second constraint in \((D^*)\).
- \( \tilde{y}^3 \) must be zero: Because of the MFCQ it holds that \( F_k \cdot d < 0 \) whenever \( g^T \tilde{y}^3 = 0 \). By complementary slackness, it follows that for these indices \( t \) that the entries in \( \tilde{y}^3 \) must be 0. From (50) it follows for the other indices too, that they must be 0.
- But with \( \tilde{y}^3 \neq \tilde{0} \) it then follows from the first constraint in \((D^*)\) that \( H^T \cdot \tilde{y}^3 = 0 \) in contradiction to the MFCQ.
Hence, for all $i \in \{3, 4, 5\}$ the sequence $\{y^i_k\}$ of (unscaled) Lagrange multipliers must also be bounded! We thus can take a subsequence $\mathcal{K}$ so that $\{y^i_k\}_{k \in \mathcal{K}}$ converges to some $(0, 0, y^3_k, y^4_k, y^5_k)$ and – by upper semicontinuity of the feasible set mapping – the limit point is feasible for the limiting problem of (D$k$), which happens to be (D). The optimal value at $(0, 0, y^3_k, y^4_k, y^5_k)$ is 0 due to strong duality. Because the corresponding primal is (P), it follows from Theorem 2 that $\bar{x}$ is a KKT point of (MOP).

With Theorem 2, it is now easy to derive the main result:

**Theorem 4 (Convergence to KKT-points).** Suppose the same assumptions as in Theorem 3 hold and that $\{x^{(k)}\}$ is a quasi-stationary subsequence with limit point $\bar{x}$. If the MFCQ hold at $\bar{x}$, then $\bar{x}$ is a KKT-point of (MOP).

### 6 Numerical Examples

In this section, we provide numerical examples for which we have applied our algorithm to two test problems. The algorithm is implemented in the Julia language according to the pseudocode in Section 3. We also share our code as a Pluto [47] notebook. Before describing the experiments, we would like to note that the simple implementation neglects many interesting questions and details that could (and should) be explored, such as optimized model construction algorithms. Nonetheless, it serves our primary goal: To demonstrate the general viability of the algorithm.

#### 6.1 Constrained Two-Parabolas Problem

Variations of the two parabolas problem are popular test cases for MOO methods. The objectives are simply two $n$-variate parabolic functions and – in the unconstrained case – the Pareto Set is the line connecting their respective minima. The following constrained version is taken from [27].

$$\min_{x \in \mathbb{R}^2} \left[ \frac{(x_1 - 2)^2 + (x_2 - 1)^2}{(x_1 - 2)^2 + (x_2 + 1)^2} \right] \quad \text{s.t.} \quad g(x) = 1 - x_1^2 - x_2^2 \leq 0. \quad \text{(Ex. 1)}$$

The feasible set of (Ex. 1) is $\mathbb{R}^2$ without the interior of the unit ball. In Fig. 1 the infeasible area is shaded red. The Pareto critical set is the line connecting $[2, -1]$ and $[2, 1]$ and the left boundary of the unit ball:

$$\mathcal{P}_c = \left\{ \begin{bmatrix} 2 \\ s \end{bmatrix} : s \in [-1, 1] \right\} \cup \left\{ \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} : t \in [\pi - \theta, \pi + \theta], \theta = \arctan \left( \frac{1}{2} \right) \right\}.$$ 

We have plotted the critical set with black lines in Fig. 1.

For this problem, we first applied our algorithm three times with the same parameters and beginning at $x_0 = [-2, 0.5]$, but with different model types. We compare RBF models with first and second degree Taylor polynomials of the objective and constraint functions. The RBF models use the cubic kernel and their construction is based on the work in [51]. The derivatives for the Taylor models are approximated using finite differences (to conform to the assumption that gradients are not available exactly). The other parameters are

$$\Delta_{(0)} = 0.5, \Delta_{\text{max}} = 2^5 \cdot \Delta_{(0)}, \gamma_0 = 0.1, \gamma_1 = 0.5, \gamma_2 = 2.0, \nu_1 = 0.01, \nu_0 = 0.9,$$

$$\varepsilon_\chi = 0.1, \varepsilon_\theta = 0.1, \kappa_\theta = 10^{-4}, \psi = 2, B = 1000, M = 3000, \alpha = 0.5, c_\Delta = 0.7, c_\nu = 100, \mu = 0.01.$$ 

These settings largely follow the recommendations in [21]. Additionally, we stop after 100 iterations or if the criticality loop has executed once or if the trial point is accepted, and it holds that $\|x^{(k)} - x^{(k)}_s\| \leq 10^{-5} \|x^{(k)}\|$ or $\|f(x^{(k)}) - f(x^{(k)}_s)\| \leq 10^{-5} \|f(x^{(k)})\|$.

The results are depicted in Fig. 1. As can be seen, all runs converge to the Pareto critical set and avoid the infeasible area. In terms of objective evaluations, the RBF models require significantly less function calls than the Taylor models. (If exact gradients are used the numbers seem to be roughly equal.) In terms of iterations, the second degree Taylor polynomials require the least. Many gradient-based MOO algorithms suffer from bias towards individual minima and, indeed, we also see that the final iterates are close to the minimum [2, 1]. There also is a more notable bias of the RBF models towards that minimum at the beginning of the optimization. This is very likely due to the infinity-norm being used and because of their construction: In the first iteration, only 3 objective evaluations along the coordinate axes around

---

1 https://gist.github.com/manuelbb-upb/69582b2322346485333f01807a2c241c
The same non-linear solver and similar stopping criteria were used to solve a weighted-sum scalarization of (Ex. 2). The second example is problem "W3" taken from [36]. The problem has box constraints and multiple non-linear inequality constraints. Whilst the unconstrained Pareto Front is simply a line in objective space, the constraints make it partially non-convex. We have chosen the problem with 3 variables and 2 objectives to compare our algorithm against a simple weighted-sum scalarization. It reads:

\[
\begin{align*}
\min_{x \in \mathbb{R}^3} & \quad \left[ f_1(x) \right] = \min_{x \in \mathbb{R}^3} \left[ \frac{x_1}{d(x) \cdot \left( 1 - f_1(x) / d(x) \right)} \right] \\
\text{s.t.} & \quad c_1(x) = f_1(x) + f_2(x) - 1.05 - 0.45 \sin(0.75 \pi \cdot l(x))^9 \leq 0, \\
& \quad c_2(x) = -f_1(x) - f_2(x) + 0.85 + 0.3 \sin(0.75 \pi \cdot l(x))^2 \leq 0, \\
& \quad d(x) = 1 + 2(x_2 + (x_1 - 0.5)^2 - 1)^2 + 2(x_3 + (x_2 - 0.5)^2 - 1)^2, \\
& \quad l(x) = \sqrt{2(f_2(x) - f_1(x))}.
\end{align*}
\]

(Ex. 2)

Because the constraints \(c_1\) and \(c_2\) describe the feasible set in terms of the objective values we can show the attainable objective values in objective space in Fig. 2. The critical set is again shown in black.

For the first run (shown on the left), we largely used the same parameters as in the previous section, but relaxed the stopping criteria a bit. For example, the relative stopping tolerances were reduced to \(10^{-6}\) and up to 500 iterations were allowed. Whilst the step problems are modeled using JuMP [12] and solved with any suitable LP or QP solver, like COSMO [25], restoration uses the NLopt solver [34] and its local “COBYLA” algorithm. The same non-linear solver and similar stopping criteria were used to solve a weighted-sum scalarization of (Ex. 2).
with objective $\frac{1}{2}f_1 + \frac{1}{2}f_2$. The blue trajectory in Fig. 2 shows that our method with RBF models can find a critical point on the non-convex part of the Pareto Front while the weighted-sum solution always must belong to the linear part.

Because the initial iterate was not feasible, a restoration step has to be performed right at the beginning. This can cause the final iterate to be far away from the initial site. We see can similarly observe in the left graphic, that there are two additional restorations close to the concave knee of the Pareto Front, which again prevents convergence. For comparison, in a second run, we tried to relax the compatibility parameters. The right plot shows that indeed one restoration can be avoided and the iterates do not deviate that much.

7 Conclusion and Outlook

In this article, we have presented an algorithm for non-linearly constrained MOPs. The method does not necessarily need derivative information of the functions but can use surrogate models instead. Additionally, a Filter ensures convergence towards feasibility. We have proven convergence of an algorithmic sub-sequence to Pareto-criticality and confirm the theoretical results with numerical experiments.

The experiments have shown both promising features of the algorithm compared to naive scalarizations and potential drawbacks.

- Like many gradient-based MOO algorithms, there often seems to be bias towards individual objective minima. Thus, an important task for future research lies in trying to remedy this behavior, maybe by using some gradient scaling.
- In this regard, it would be desirable to be able to use alternative descent direction that allow guidance of the iterates or use momentum to accelerate convergence.
- Moreover, we believe it possible to transfer the model construction optimizations from [51, 4] into our algorithm. This would allow for “model-improvement” steps with models that are not fully linear and could potentially save expensive function evaluations.
- Finally, our algorithm produces only one critical point. After a run has finished, can we leverage the surrogates we already have constructed to warm-start the optimization of a modified MOP to find additional critical points and obtain a covering of (parts) of the critical set?
A more theoretical question is whether or not our convergence results can be strengthened. To this end, using a slanted filter like in [19] might prove beneficial. Furthermore, in Section 4.1 we have already talked about why an approach similar to that in [14] could prove beneficial.

Appendix

Equivalence of Inexact Criticality with Different Norms

To prove Lemma 2 we transfer the corresponding single-objective results from [7] to the multi-objective case. This works relatively straightforward, but to the best of our knowledge the multi-objective results have not yet been published anywhere, so the proofs are given in detail. First, we make use of the following auxiliary result:

**Lemma 22** ([7, Lemma 2]). Suppose Assumptions 2 to 4 and 6 hold and that $\omega$ is defined by

$$\omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k, \vartheta \right) = - \min_{d \in \mathbb{R}^n} \max_{\ell} \left\{ \nabla \tilde{f}^{(k)}(x_n^{(k)}), d \right\} \quad \text{s.t.} \quad d \in L_k, \| d \|_k \leq \vartheta, \tag{51}$$

where $L_k = \left( L_k - x_n^{(k)} \right)$. For any $k \in \mathbb{N}_0$, for which the normal step exits, the following statements hold:

1. The function $\vartheta \mapsto \omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k, \vartheta \right)$ is continuous and non-decreasing for $\vartheta \geq 0$.
2. The function $\vartheta \mapsto \omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k, \vartheta \right)$ is non-increasing for $\vartheta > 0$.

**Proof.** Let $k \in \mathbb{N}_0$ be an iteration index with algorithmic variables $(x_n^{(k)}, L_k, \| \cdot \|_k)$.

1. Same as in [7], the feasible set mapping

$$\vartheta \mapsto D_k(\vartheta) = \left\{ d \in \mathbb{R}^n : \| d \|_k \leq \vartheta, d \in \left( L_k - x_n^{(k)} \right) \right\}$$

is continuous for all $\vartheta \geq 0$. The function $\Psi(\vartheta, d) = \max_{\ell} \left\{ \nabla \tilde{f}^{(k)}(x_n^{(k)}), d \right\}$ is defined on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and for each $\vartheta \geq 0$ it is convex in its second argument. Hence, we can use results from [20], which guarantee the continuity of the optimal value

$$- \min_{d \in D_k(\vartheta)} \Psi(\vartheta, d) = \omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k, \vartheta \right)$$

which is non-decreasing with respect to $\vartheta$ by definition.

2. Consider $0 < \vartheta_1 < \vartheta_2$ and $d_1, d_2 \in \mathbb{R}^n$ such that

$$\omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k, \vartheta_1 \right) = - \max_{\ell} \left\{ \nabla \tilde{f}^{(k)}(x_n^{(k)}), d_1 \right\}, \| d_1 \|_k \leq \vartheta_1, d_1 \in L_k \tag{52}$$

$$\omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k, \vartheta_2 \right) = - \max_{\ell} \left\{ \nabla \tilde{f}^{(k)}(x_n^{(k)}), d_2 \right\}, \| d_2 \|_k \leq \vartheta_2, d_2 \in L_k \tag{53}$$

Because the set $L_k$ is convex, it follows from $\vartheta_1/\vartheta_2 < 1$, $0 \in L_k$ and $d_2 \in L_k$ that also $\vartheta_1/\vartheta_2 d_2 \in L_k$. Moreover, it follows from (52) that

$$\left\| \frac{\vartheta_1}{\vartheta_2} d_2 \right\|_k = \frac{\vartheta_1}{\vartheta_2} \| d_2 \|_k \leq \vartheta_1.$$

Thus, $\vartheta_1/\vartheta_2 d_2$ is feasible for the problem in (53). Consequently,

$$\frac{\omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k, \vartheta_1 \right)}{\vartheta_1} \geq - \frac{1}{\vartheta_1} \max_{\ell} \left\{ \nabla \tilde{f}^{(k)}(x_n^{(k)}), \frac{\vartheta_1}{\vartheta_2} d_2 \right\} = \frac{\omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k, \vartheta_2 \right)}{\vartheta_2}.$$

We can now proof Lemma 2, which states that uniformly equivalent norms imply uniformly equivalent inexact criticality values, i.e.,

$$\frac{1}{\omega} \omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k \right) \leq \omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_2 \right) \leq \omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_1 \right) \leq \frac{1}{\omega} \omega \left( x_n^{(k)}; \hat{f}^{(k)}, L_k, \| \cdot \|_k \right). \tag{54}$$

**Proof.** The proof works similarly to the single-objective case [7, Th. 4]. Let $k \in \mathbb{N}_0$ be such that the normal step exists. We first make the following observations:

1. The ball defined by $\| d \|_2 \leq \frac{1}{\omega}$ is contained in the ball defined by $\| d \|_k \leq 1$, due to Assumption 6.
According to observation 1, it then follows that
\[ \omega_{\text{max}} = \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2, c \right) \quad \text{and} \quad \omega_{\text{min}} = \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2, c^{-1} \right). \]

From the second statement in Lemma 22 it then follows that
\[ \omega_{\text{max}} \leq c^2 \omega_{\text{min}}. \]  
(55)

If \( \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right) = \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right) \) there is nothing to show. Hence, first we assume that
\[ \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right) < \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right), \]
(56)

and we again take the respective minimizers \( \mathbf{d}^{(k)}, \mathbf{d}_2 \in \mathbb{R}^n \) with
\[ \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right) = -\max_{\ell} \|\nabla f\ell (\mathbf{x}_n^{(k)}), \mathbf{d}^{(k)}\|, \|\mathbf{d}^{(k)}\|_k \leq 1, \mathbf{d}^{(k)} \in \mathcal{L}_k \]
(57)
\[ \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right) = -\max_{\ell} \|\nabla f\ell (\mathbf{x}_n^{(k)}), \mathbf{d}_2\|, \|\mathbf{d}_2\|_2 \leq 1, \mathbf{d}_2 \in \mathcal{L}_k \]
(58)

Then
\[ \frac{1}{c} \leq \|\mathbf{d}^{(k)}\|_2 \leq c \quad \text{and} \quad \frac{1}{c} \leq \|\mathbf{d}_2\|_2 \leq c. \]  
(59)

The upper bounds are trivial because of \( c \geq 1 \). Suppose the first lower bound is violated, i.e., \( \|\mathbf{d}^{(k)}\|_2 < 1/c \). According to observation 1, it then follows that \( \|\mathbf{d}^{(k)}\|_2 \leq 1 \). The vector \( \mathbf{d}^{(k)} \in \mathcal{L}_k \) is then also feasible for (58) and the optimality of \( \mathbf{d}_2 \) implies
\[ -\max_{\ell} \|\nabla f\ell (\mathbf{x}_n^{(k)}), \mathbf{d}_2\| \geq -\max_{\ell} \|\nabla f\ell (\mathbf{x}_n^{(k)}), \mathbf{d}^{(k)}\|, \]

in contradiction to (56).

Suppose the second lower bound is violated, i.e., \( \|\mathbf{d}_2\|_2 < 1/c \). According to our observations from above, for \( \mathbf{d}_2 \) we also have \( \|\mathbf{d}_2\|_2 \leq 1 \), and it is therefore feasible for the problem in (57). With (56) we even see that it is strictly optimal, contradicting the optimality of \( \mathbf{d}^{(k)} \) in (57). Thus, the second lower bound in (59) must hold, too.

Equation (59) shows that both vectors \( \mathbf{d}^{(k)} \) and \( \mathbf{d}_2 \) are feasible for the problems defining \( \omega_{\text{max}} \) and \( \omega_{\text{min}} \). We again apply the definition (51) to see that
\[ \omega_{\text{min}} \leq \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right) \leq \omega_{\text{max}} \quad \text{and} \quad \omega_{\text{min}} \leq \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_k \right) \leq \omega_{\text{max}}. \]

Plugging in (56) and (55) results in
\[ \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right) < \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_k \right) \leq c^2 \omega_{\text{min}} \leq c^2 \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right), \]

implying
\[ \frac{1}{c^2} \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_k \right) \leq \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right) \leq c^2 \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_k \right). \]  
(60)

The case
\[ \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_2 \right) < \omega \left( \mathbf{x}_n^{(k)}, \mathbf{f}^{(k)}, \mathcal{L}_k, \|\bullet\|_k \right) \]
is treated analogously and also leads to (60). Thus, (54) holds for any constant \( w \geq c^2 \geq 1 \) and Lemma 2 is valid.
Sufficient Decrease via Backtracking

This section of the appendix is concerned with justifying the sufficient decrease bound (9) in Assumption 8. In the single-objective case, there are many possibilities to achieve the bound, some of which, e.g., exact and inexact line-search, have previously shown to work in the multi-objective as well if the global feasible set is compact and convex [4]. Now at least the approximated linearized feasible sets are convex, which allows us to cite and utilize the following result:

**Lemma 23** (Backtracking Decrease, [4, Lemma 3]). Let $L$ be a convex set and $d$ be a descent direction for $\hat{f}$ at $x \in L$ and let $\sigma \geq 0$ be a step-size such that $x + \sigma |d| \cdot d \in L$ and let $\|\cdot\|$ be any vector norm that fulfills (7). Then, for any fixed constants $a, b \in (0, 1)$ and for $\Phi = \Phi[\hat{f}]$ there is an integer $j \in \mathbb{N}_0$ such that

$$\Phi(x) - \Phi \left( x + \frac{b^j \sigma}{|d|} d \right) \geq a \frac{b^j}{|d|} \omega,$$

where $\omega = -\max_x \langle \nabla \hat{f} (x), d \rangle$. Moreover, there is a constant $c_{sd} \in (0, 1)$ such that if $j$ is the smallest $j \in \mathbb{N}_0$ that satisfies (61), then

$$\Phi(x) - \Phi \left( x + \frac{b^j \sigma}{|d|} d \right) \geq c_{sd} \omega \min \left\{ \frac{\omega}{|d|^2 c^2 H}, \frac{\sigma}{|d|} \right\},$$

where

$$H = \max_{d \in L - x} \frac{1}{t=1, \ldots, K} \left\| H\hat{f}_t (x + d) \right\|_2 s.t. \|d\| \leq b^j \sigma.$$  (62)

For Lemma 23 to be applicable in our setting we require the surrogate function $\hat{f}$ to be twice continuously differentiable and defined on $C(X)$, which is guaranteed by Assumptions 2 to 4. Furthermore, Assumption 7 may be used instead of (62) to bound the model Hessians, which appear due to a Taylor approximation of the components of $\hat{f}$ in the proof of Lemma 23.

We now have to deal with the possibly different norms $\|\cdot\|_{tr,k}$ (defining the trust-region) and $\|\cdot\|_{tr}$ (used in (ITRT(k))) and how to choose the initial backtracking step-length $\sigma$ in Lemma 23. We want to choose $\sigma \geq 0$ as large as possible and so that for a descent direction $d^{(k)}$ it holds that $\|n^{(k)} + \sigma d^{(k)}\|_{tr,k} \leq \Delta^{(k)}$ and $n^{(k)} + \sigma d^{(k)} \in L_k$. Luckily, there is the following bound on the optimal $\sigma$:

**Lemma 24.** For $x^{(k)}$ let $L_k$ be the linearized feasible set and $B^{(k)}$ be a trust-region of radius $\Delta^{(k)}$ w.r.t. $\|\cdot\|_{tr,k}$. Let $d^{(k)}$ be a minimizer of (ITRT(k)) with $\|d^{(k)}\|_{k} \leq 1$. Then there is an initial step-length $\tilde{\sigma} \geq 0$ with $\tilde{x}^{(k)} + \frac{\tilde{\sigma}}{\|d^{(k)}\|_{tr,k}} d^{(k)} \in L_k \cap B^{(k)}$ and

$$\tilde{\sigma} \geq \min \left\{ \Delta^{(k)} - \frac{\|n^{(k)}\|_{tr,k}}{\|d^{(k)}\|_{tr,k}}, \frac{\|d^{(k)}\|_{tr,k}}{\|d^{(k)}\|_{tr,k}} \right\}.  \quad (63)$$

**Proof.** There are, of course, better ways to determine $\tilde{\sigma}$, but if $\|n^{(k)} + d^{(k)}\|_{tr,k} \leq \Delta^{(k)}$ we can always choose $\tilde{\sigma} = \|d^{(k)}\|_{tr,k}$. If $\|n^{(k)} + d^{(k)}\|_{tr,k} > \Delta^{(k)}$ we can equate either side of the triangle inequality

$$\|n^{(k)}\|_{tr,k} + \tilde{\sigma} \geq \|n^{(k)} + \frac{\tilde{\sigma}}{\|d^{(k)}\|_{tr,k}} d^{(k)}\|_{tr,k}$$

with $\Delta^{(k)}$ and solve for $\tilde{\sigma}$. ▶

Finally, we are able to derive the sufficient decrease bound (9) when backtracking is used. Of course, in case that $x^{(k)}$ is critical, the bound (9) is automatically fulfilled. Else, we use Lemma 23 with $\tilde{\sigma}$ satisfying (63). Assumption 6 and $\|d^{(k)}\|_{k} \leq 1$ together imply

$$\|d^{(k)}\|_{tr,k} \leq c^2 \|d^{(k)}\|_{k} \leq c^2.$$

(64)
Lemma 23 and the fact that we assume $\mathbf{n}^{(k)}$ to be compatible lead to
\[
\Phi^{(k)} \left( x_n^{(k)} \right) - \Phi^{(k)} \left( x_n^{(k)} + \sigma^{(k)} \mathbf{d}^{(k)} \right) \\
\geq \tilde{c}_{sd} \tilde{\omega}^{(k)} \min \left\{ \frac{\hat{\omega}^{(k)}}{c^2}, \frac{1}{c^2}, \frac{1 - c \Delta \Delta^{(k)}}{c^2}, \frac{1}{c^2} \right\} \\
\geq \tilde{c}_{sd} \tilde{\omega}^{(k)} \min \left\{ \frac{\hat{\omega}^{(k)}}{c^2}, \frac{1 - c \Delta \Delta^{(k)}}{c^2}, \frac{1}{c^2} \right\} \\
= \frac{1 - c \Delta}{\mathbf{w} \mathbf{c}^2} \tilde{\omega}^{(k)} \min \left\{ \frac{\hat{\omega}^{(k)}}{1 - c \Delta}, \frac{1}{c^2}, \frac{1}{1 - c \Delta} \right\},
\]
and (9) follows with $c_{sd} := \frac{\tilde{c}_{sd} (1 - c \Delta)}{\mathbf{w} \mathbf{c}^2} \in (0, 1)$ and from the fact that $\frac{1}{(1 - c \Delta)} \geq 1$ and with $\mathbf{w} := \mathbf{w} \mathbf{c}^2 \geq 1$, which we may assume w.l.o.g.

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