Trivial Vacua, High Orders in Perturbation Theory and Nontrivial Condensates

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In the limit of an infinite number of colors, an analytic expression for the quark condensate in $QCD_{1+1}$ is derived as a function of the quark mass and the gauge coupling constant. For zero quark mass, a nonvanishing quark condensate is obtained. Nevertheless, it is shown that there is no phase transition as a function of the quark mass. It is furthermore shown that the expansion of $\langle 0 | \bar{\psi} \psi | 0 \rangle$ in the gauge coupling has zero radius of convergence but that the perturbation series is Borel summable with finite radius of convergence. The nonanalytic behavior $\langle 0 | \bar{\psi} \psi | 0 \rangle \sim -N_C \sqrt{G}$ can only be obtained by summing the perturbation series to infinite order. The sum-rule calculation is based on masses and coupling constants calculated from 't Hooft’s solution to $QCD_{1+1}$ which employs LF quantization and is thus based on a trivial vacuum. Nevertheless the chiral condensate remains nonvanishing in the chiral limit which is yet another example that seemingly trivial LF vacua are not in conflict with QCD sum-rule results.

I. INTRODUCTION

What is interesting about the quark condensate in $QCD_{1+1}$? Zhitnitsky [1], using QCD-sum rule techniques, derived an exact result for the condensate in the limit of an infinite number of colors and in the limit $m_q \to 0$

$$\langle 0 | \bar{\psi} \psi | 0 \rangle \bigg|_{m_q=0} = -\frac{N_C}{\sqrt{12}} \frac{g^2 C_F}{\pi},$$

where $C_F = (N_C^2 - 1)/2N_C$ and $G^2 = \frac{g^2 C_F}{\pi}$ is held fixed as $N_C \to \infty$. This result is remarkable in several respects: Firstly, $\langle 0 | \bar{\psi} \psi | 0 \rangle \big|_{m_q=0}$ is nonvanishing, indicating spontaneous breakdown of chiral symmetry. Secondly, the condensate is nonanalytic in the coupling constant $G^2$, thus indicating nonperturbative effects: although it seems natural from dimensional analysis that $\langle 0 | \bar{\psi} \psi | 0 \rangle \propto \sqrt{G^2}$ for small quark masses, it is impossible to obtain such a behavior in perturbation theory where one can only generate terms $\propto G^{2n}$. Thirdly, one may suspect that there is a phase transition in $QCD_{1+1}$.

Since the coupling constant $g$ in $QCD_{1+1}$ carries the dimension of a mass, the theory is super-renormalizable and the scale is set both by the coupling and by the mass. In practice this implies that $\langle 0 | \bar{\psi} \psi | 0 \rangle$ can (up to some dimensionful overall factor) depend on $G^2$ only through the combination $\alpha \equiv G^2/m_q^2$. Therefore, in order to address the abovementioned issues of nonperturbative effects and a possible phase transition, it is necessary to consider nonzero quark masses.

There is another reason why the condensate in $QCD_{1+1}$ is interesting: Zhitnitsky’s result was based on the solutions of ‘t Hooft’s equation [2], which is obtained in the light-front (LF) approach to $QCD_{1+1}$. As is well known, the vacuum (=ground state) in LF quantization is equal to the Fock vacuum and nontrivial condensates seem impossible. However, as has been shown in Ref. [4], QCD sum rules, applied to the spectrum and coupling constants obtained through LF quantization, nevertheless yield nontrivial results for the condensates.

II. $\langle 0 | \bar{\psi} \psi | 0 \rangle$ FROM SUM RULES

For nonzero quark masses, the vacuum expectation value of the scalar density diverges already for free fields

$$\langle 0 | \bar{\psi} \psi | 0 \rangle = \frac{N_C}{2\pi} m_q \ln \frac{\Lambda^2}{m_q^2}. \quad (2.1)$$

However, due to the mild UV behavior in 1 + 1 dimensions ($QCD_{1+1}$ is super-renormalizable) it is sufficient to subtract the free field expectation value (i.e. to “normal order”) to render $\langle 0 | \bar{\psi} \psi | 0 \rangle$ finite. This motivates the definition

$$\langle 0 | \bar{\psi} \psi | 0 \rangle \big|_{ren} \equiv \langle 0 | \bar{\psi} \psi | 0 \rangle - \langle 0 | \bar{\psi} \psi | 0 \rangle \big|_{g=0}. \quad (2.2)$$

The condensate itself can be evaluated using current algebra

$$0 = \lim_{q \to 0} iq^\mu \int d^2 x e^{iqx} \langle 0 | [\bar{\psi} \gamma_\mu \gamma_5 \psi(x) \bar{\psi} \gamma_5 \psi] | 0 \rangle = -\langle 0 | \bar{\psi} \psi | 0 \rangle - 2m_q \int d^2 x \langle 0 | T [\bar{\psi} \gamma_5 \psi(x) \bar{\psi} \gamma_5 \psi] | 0 \rangle. \quad (2.3)$$

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1 Note that Coleman’s theorem [3] prohibits spontaneous breakdown of chiral symmetry for any finite number of colors, since this is a 1 + 1-dimensional model.

2 Fragments of this calculation can be found in Ref. [4].
Upon inserting a complete set of meson states\(^3\) one thus obtains
\[
\langle 0 | \bar{\psi} \psi | 0 \rangle = -m_q \sum_n f_P^2(n) / M_n^2, \tag{2.4}
\]
where
\[
f_P(n) \equiv \langle 0 | \bar{\psi} \gamma_5 \psi | n \rangle = \sqrt{N_C} m_q \int_0^1 dx \frac{1}{x(1-x)} \phi_n(x), \tag{2.5}
\]
and the wavefunctions \(\phi_n\) and invariant masses \(M_n^2\) are obtained from solving ‘t Hooft’s bound state equation for mesons in QCD\(_{1+1}\)
\[
M_n^2 \phi_n(x) = \frac{m_q^2}{x(1-x)} \phi_n(x) + G^2 \int_0^1 dy \frac{\phi_n(x) - \phi_n(y)}{(x-y)^2}. \tag{2.6}
\]
The variable \(x\) corresponds to the light-front momentum fraction carried by the quark in the meson. Note that ‘t Hooft’s equation was been derived using light-front quantization — we will return to this point below.

In the limit of highly excited mesons, the masses and coupling constants scale \([5]\):
\[
M_n^2 \xrightarrow{n \to \infty} n \pi^2 G^2,
\]
and the sum in Eq.(2.4) diverges logarithmically. Of course this only reflects the free field divergence \((2.1)\). In order to regularize Eq.(2.4) in a gauge invariant way we introduce an invariant mass cutoff and obtain
\[
\langle 0 | \bar{\psi} \psi | 0 \rangle \bigg|_{\text{ren}} = -m_q \lim_{\Lambda \to \infty} \left[ \sum_n f_P^2(n) / M_n^2 (1 + M_n^2 / \Lambda^2) \right] \quad "g \to 0" \tag{2.8}
\]
Eq.(2.8) can be used to calculate \(\langle 0 | \bar{\psi} \psi | 0 \rangle \bigg|_{\text{ren}}\) numerically with high precision. However, one must be very careful about the order of limits when trying to evaluate Eq.(2.8) numerically in QCD\(_{1+1}\) or other theories: when one employs DLCQ \([6]\) or a similar regulator to calculate the wavefunctions (and from those the coupling constants \(F_P(n)\)) and spectra then it is crucial to send the DLCQ-cutoff to infinity first — otherwise one gets zero or nonsense for the condensate from the sum rule calculation. Only after sending the DLCQ cutoff to zero one may send the UV-regulator \(\Lambda\) to infinity or the quark mass to zero.

\(^3\) Because we are working at leading order in \(1/N_C\), the sum over one meson states saturates the operator product in Eq.(2.3).

In order to generate an exact result we will use a trick and replace the sum in Eq.(2.8), where both small and large \(n\) contribute, by a sum which is dominated by large \(n\) only. Due to lack of space, only the basic ideas of the derivation will be sketched here — a more detailed discussion of the limit \(\Lambda \to \infty\) can be found in Ref. [4]. For this purpose, let us consider \([6]\)
\[
G(x, \Lambda) \equiv \sqrt{N_C} \sum_n \phi_n(x) \left( \frac{1}{M_n^2} - \frac{1}{M_n^2 + \Lambda^2} \right) f_P(n). \tag{2.9}
\]
Obviously this “Green’s function” can be used to compute the condensate, via
\[
\langle 0 | \bar{\psi} \psi | 0 \rangle \bigg|_{\text{ren}} = -m_q \lim_{\Lambda \to \infty} \int_0^1 dx \left[ G(x, \Lambda) - G(x, \Lambda) \bigg|_{y=0} \right]. \tag{2.10}
\]
From the equation of motion (2.6) one can show that
\[
f_P(n) = \frac{M_n^2}{2m_q} \sqrt{N_C / \pi} \int_0^1 dy \phi_n(y). \tag{2.11}
\]
If one furthermore invokes completeness of ‘t Hooft’s wavefunctions, i.e.
\[
\sum_n \phi_n(x) \phi_n(y) = \delta(x-y) \tag{2.12}
\]
one can simplify the first term in \(G(x, \Lambda)\), yielding
\[
\sqrt{N_C / \pi} \sum_n \phi_n(x) f_P(n) / M_n^2 = \frac{N_C}{\pi} \frac{1}{2m_q} \tag{2.13}
\]
\(\perp\) independent of \(x\) and the coupling constant. This term thus drops out completely when we subtract the free field Green’s function.

The crucial point is that, for \(\Lambda^2 \to \infty\), the remaining term in \(G(x, \Lambda)\) is dominated by the terms with \(n \to \infty\); each individual meson yields a negligible contribution \((M_n^2 + \Lambda^2)^{-1}\) when we send the cutoff to infinity, and a nonzero result arises only from the summation over infinitely many highly excited meson states. We have thus succeeded in converting the low energy sum rule \((2.8)\) into a high energy sum rule and we can now make use of the abovementioned scaling properties of meson masses \(M_n^2\) and coupling constants \(f_P(n)\) as well as of the wavefunction itself
\[
\phi_n(x) \xrightarrow{n \to \infty} \Phi(M_n^2 x). \tag{2.14}
\]
The scaled wavefunction
\[
\Phi(z) \equiv \lim_{n \to \infty} \phi_n(z/M_n^2) \tag{2.15}
\]
satisfies the integral equation \([6]\)
\[ \Phi(z) = \frac{m_q^2}{z} \Phi(z) + G^2 \int_0^\infty dy \frac{\Phi(z) - \Phi(y)}{(x-y)^2}. \]  

(2.16)

In terms of these scaled quantities we thus find

\[ \langle 0 | \bar{\psi} \psi | 0 \rangle_{\text{ren}} = \frac{N_C}{\pi} \frac{m_q^2}{G^2} \int_0^\infty \sum_{n=1,3,5} \frac{\Phi(n \pi^2 G^2 x)}{n \pi^2 G^2 + \Lambda^2} \quad g = 0' \]

(2.17)

Upon performing the substitution \( z = n \pi^2 G^2 x \), replacing \( \sum_{n=1,3,5} \rightarrow (2 \pi^2 G^2)^{-1} \int_0^\infty dz \) and performing the \( x \)-integral one ends up with

\[ \langle 0 | \bar{\psi} \psi | 0 \rangle_{\text{ren}} = \frac{N_C}{\pi} \frac{m_q^2}{G^2} \int_0^\infty dz \frac{1}{z} \Phi(z) \quad g = 0'. \]

(2.18)

Note that \( \frac{1}{\sqrt{G^2}} \int_0^\infty \frac{dz}{z} \Phi(z) = \frac{\pi}{m_q} \) is independent of \( G^2 \) and it does not matter that the argument of the logarithm is dimensionful, since a free theory subtraction is performed. While 't Hooft’s equation cannot be solved exactly, the scaling equation can be solved analytically (here we closely follow Ref. [5]): From Eq. (2.16) it is clear that \( \langle 0 | \bar{\psi} \psi | 0 \rangle_{\text{ren}} \) can be related to the Mellin transform of \( \Phi(z) \), for which a closed form expression has been given in Ref. [3]. The details of the calculation are given in Appendix A. The final result for the renormalized condensate is (Fig. 1)

\[ \langle 0 | \bar{\psi} \psi | 0 \rangle_{\text{ren}} = \frac{m_q N_C}{2 \pi} \left\{ \ln (\pi \alpha) - 1 - \gamma_E \right\} + \left( 1 - \frac{1}{\alpha} \right) [(1 - \alpha) I(\alpha) - \ln 4] \right\}. \]

(2.19)

where \( \alpha = G^2 / m_q^2 \), \( \gamma_E = 0.5772 \ldots \) is Euler’s constant and

\[ I(\alpha) = \int_0^\infty dy \frac{1 - \sinh y \cosh y}{y^2} \frac{\alpha (y \coth y - 1) + 1}. \]

(2.20)

This result is exact for \( N_C \rightarrow \infty \) and all quark masses. In the limit \( \alpha \rightarrow \infty \) one recovers Zhitnitsky’s result Eq. (2.14). Furthermore, one can verify that the exact result coincides with the numerical evaluation of Eq. (2.8). In the limit \( \alpha \rightarrow 0 \) the condensate vanishes, which is not surprising since we have subtracted the free field result. As one can see from Fig. 1, \( \langle 0 | \bar{\psi} \psi | 0 \rangle \) has an infinite derivative as a function of the quark mass for \( m_q = 0 \). It arises from a logarithmic singularity in a strong coupling expansion of Eq. (2.19)

\[ \langle 0 | \bar{\psi} \psi | 0 \rangle_{\text{ren}} = -N_C \left[ \sqrt{\frac{G^2}{12}} \frac{m_q}{2 \pi} \ln \left( \frac{m_q^2}{G^2} \right) + \mathcal{O}(N_C) \right] \]

(2.21)

As will be discussed in the next section, this logarithmic term gives rise to sizable violations of SU(3) flavor symmetry.

FIG. 1. Renormalized quark condensate as a function of the quark mass. Both in units of the effective coupling \( G^2 = g^2 C_F / \pi \). Note the absence of singularities.

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4 This is exact for \( \Lambda \rightarrow \infty \) since the series receives nonvanishing contributions only from the \( n \rightarrow \infty \) region.
This term is interesting for another reason: since the argument of the logarithm is proportional to the $m_q$ (which itself is proportional to $m_π^2$) it looks like a chiral logarithm. However, chiral logarithms usually arise from meson loops which are absent here since we work to leading order in $1/N_C$.

### III. FLAVOR DEPENDENCE OF THE QUARK CONDENSATE

Even though QCD$_{3+1}$ is not QCD$_{3+1}$ it is interesting to test some assumptions that are commonly used (but of course not tested) in the analysis of meson spectra in 3+1 dimensions by making (and testing!) the same assumptions in 1+1 dimensions, where exact results are available.

In QCD$_{3+1}$ one often assumes that the strange quark condensate is about the same as the condensate of light quarks even though the strange quark mass is quite sizable (compared to $Λ_{QCD}$). Based on that (and more) assumption one for example derives the Gell-Mann–Okubo relation

$$0.988 \text{(GeV)}^2 = 4m_0^2 \approx 3m_π^2 + m_π^2 = 0.924 \text{(GeV)}^2,$$

(3.1)

which is surprisingly well satisfied, which is often used as a justification for assuming

$$\langle 0 | \bar{s}s(0) \rangle \approx \langle 0 | \bar{u}u(0) \rangle.$$  

(3.2)

Before one can test the quality of approximations such as Eq. (3.2) in the case of broken flavor symmetry one must fix the mass scale. Fixing the slope of the Regge trajectory in the ’t Hooft model yields $λ$

$$G^2 = (581.6 \text{MeV})^2.$$  

(3.3)

The masses of light and strange quarks are fixed by fitting the masses for $π$ and $K$ mesons, yielding

$$m_q \equiv (m_u + m_d)/2 = 8.6 \text{MeV}$$

$$m_s = 195 \text{MeV} \approx \frac{1}{3} G.$$  

(3.4)

With these quark masses, the Gell-Mann–Okubo mass relation works reasonably well in QCD$_{3+1}$

$$0.984 \text{(GeV)}^2 = 4m_0^2 \approx 3m_n^2 + m_π^2 = 1.1 \text{(GeV)}^2.$$  

(3.5)

However, a glance at Fig$^8$ shows that the strange quark condensate is only about half as large as the light quark condensate for our values for the quark masses, i.e. the SU(3) flavor symmetry for the vacuum condensate of the quarks is violated by 50%. These results improve somewhat when one uses different criteria to set the mass scale. For example, a better fit to the meson spectrum is obtained with $G = 1.16 \text{GeV}$, $m_q = 5.8 \text{MeV}$ and $m_s = 130 \text{MeV}$.

But even for those values there is still a 25% violation of the SU(3) symmetry for the vacuum condensate.

Of course, this is only a toy model and not a serious approximation to QCD$_{3+1}$ and therefore one should be very careful to draw any conclusions from this about the real world. Nevertheless, these results should be taken as a warning and one should be very cautious about assumptions concerning SU(3) symmetry of quark condensates.

### IV. THE GLUON CONDENSATE

In QCD$_{3+1}$ gluons are not a dynamical degree of freedom but appear only in connection with the Coulomb interaction between quarks. Therefore, it should not be surprising that there is a direct connection between the quark condensate and the gluon condensate in QCD$_{3+1}$. In order to find out more, let us first consider the behavior of the theory under scale transformations

$$δψ = \left( \frac{1}{2} + x^μ \partial_μ \right) ψ$$

$$δA_μ = (1 + x^μ \partial_μ) A_μ.$$  

(4.1)

Since the coupling constant carries dimensions of mass, it is not surprising that scale invariance is violated — even in the chiral limit

$$δ \int d^2L = \int d^2 Δ(x),$$  

(4.2)

where

$$Δ(x) = T^μ_μ = m\bar{ψ}ψ - \frac{1}{2} F^μν F^μν.$$  

(4.3)

The Ward identities for scale transformations imply a relation between an arbitrary operator $O(x)$ and its change $δO(x)$ under scale transformations $λ$.

$$\langle 0 | δO(0) \rangle = -i \int d^2x \langle 0 | T[Δ(x)O(0)] | 0 \rangle.$$  

(4.4)

For $O(x) = \bar{ψ}ψ$ this implies

$$i \int d^2x \langle 0 | T \left[ m\bar{ψ}ψ(x) - \frac{1}{2} F^2(x) \right] \bar{ψ}ψ(0) \rangle | 0 \rangle = -\langle 0 | \bar{ψ}ψ(0) \rangle.$$  

(4.5)

When one combines this result with

$$\frac{d}{dm} \langle 0 | F^2 | 0 \rangle = -i \int d^2x \langle 0 | T \left[ \bar{ψ}ψ(x) F^2(0) \right] | 0 \rangle$$  

(4.6)

and

$$\frac{d}{dm} \langle 0 | \bar{ψ}ψ | 0 \rangle = -i \int d^2x \langle 0 | T \left[ \bar{ψ}ψ(x) \bar{ψ}ψ(0) \right] | 0 \rangle$$  

(4.7)

one obtains
which can be easily used to express the gluon condensate in terms of the quark condensate — up to a quark mass independent constant. Notice that the quark mass dependent piece of the gluon condensate is in fact only of $\mathcal{O}(N_C)$, which should, however, not come as a surprise since the gluons appear only through the quarks in QCD. Using the exact result for the quark condensate, it is a straightforward calculation to generate an exact expression for the quark mass dependent part of the gluon condensate in QCD from Eq. (1.8). However, since it was not possible to determine the integration constant, the result will not be displayed here.

Besides the quark and gluon condensate, one can also consider mixed condensates in QCD. Of particular interest are combinations like $\langle \bar{c}c \rangle$ i.e., the asymptotic expansion for $\langle \bar{c}c \rangle$ as the r.h.s. of Eq.(5.1). However, in the strong coupling limit, $f(\alpha) \sim \alpha \ln \alpha$, i.e. looking only at the tail of the asymptotic series yields a strong coupling behavior which is too singular. There is also another way to see that the asymptotic behavior of the series has little do do with the behavior of the exact result in the strong coupling limit: Both in Eq.(2.20) as well as in $f(\alpha)$, the $y \to 0$ region of the integral are crucial for the behavior in the strong coupling limit but it is the $y \to \infty$ part of the integral that is responsible for the behavior of the asymptotic series.

We have started from ‘t Hooft’s equation which is based on light-front quantization. The light-front vacuum is trivial, i.e. identical to the Fock space vacuum. Nevertheless, using current algebra and sum rule techniques, we obtained a nonzero result for the quark condensates. The result we obtained agrees with numerical calculations using equal time quantization (see Refs. [11,14] for $m_q = 0$ and Ref. [13] for the general case). The apparent paradox (nontrivial condensates from trivial vacuum) is clarified by defining the LF field theory with its light-like quantization hypersurface through a limiting procedure in which the quantization hypersurfaces are spacelike [12,14]. The basic result from such studies [13] is that the vacuum in LF front field theory appears to be frozen and correct spectra and structure function (leading twist) results are obtained by solving a suitable LF Hamiltonian which has a trivial vacuum. For some quantities, however, the limiting transition to the LF is not smooth and by working directly on the LF one obtains incorrect results. The quark condensate is such an example. On the other hand, the sum rule calculation based on (leading twist) LF wavefunctions yields the correct results since spectrum and (leading twist) wavefunctions show a smooth LF limit in QCD1+1.

V. DISCUSSION

The exact result which we have obtained (2.20) is an analytic function of $\alpha$ in the complex plane cut along the negative real axis — i.e. there is no phase transition. An asymptotic expansion in powers of $\alpha$ yields

$$\frac{2\pi}{m_q N_C} \langle 0 | \bar{\psi} \psi | 0 \rangle \bigg|_{\text{ren}} = \sum_{\nu=1}^{\infty} c_\nu \alpha^\nu,$$

where the coefficients show factorial growth

$$c_\nu \sim (1)^\nu e^{-2(1-\nu)(\nu - 1)!},$$

i.e., the asymptotic expansion for $\langle 0 | \bar{\psi} \psi | 0 \rangle \bigg|_{\text{ren}}$ is only Borel summable and the Borel series has a finite radius of convergence. Applying the inverse Borel transfer to the Borel summed series one recovers the exact result which reflects the absence of terms like $e^{-\pi \alpha}$. I have compared the first three terms in the asymptotic expansion with the perturbative (Feynman diagrams) expansion and found agreement. Nevertheless Eq.(2.20) is a completely non-perturbative result, because one has to sum up all terms in the perturbation series before one obtains the right scaling behavior (1.1) for small quark mass (large $\alpha$). It is also not sufficient to keep only the asymptotic behavior of the series: it is easy to write down an expression which has the same asymptotic coefficients for large $\nu$ but does not yield the desired $\sqrt{\alpha}$ behavior for $m_q \to 0$:

$$f(\alpha) = -2\alpha e^{-2} \int_0^\infty dy \frac{e^{-y}}{2 + \alpha y}$$

has the same large $\nu$ behavior for the asymptotic series as the r.h.s. of Eq.(5.1). However, in the strong coupling

APPENDIX A: EXACT EXPRESSION FOR THE MELLIN TRANSFORM OF THE SCALING FUNCTION

For $-\beta < \lambda < 1$ one can introduce the Mellin transformed function

$$\Psi(\lambda) = \int_0^\infty dz z^{\lambda - 1} \Phi(z).$$

(A1)

From Eq.(2.18) it is clear that , in order to calculate the condensate, it is sufficient to calculate $\Psi'(0)$, using

$$\langle 0 | \bar{\psi} \psi | 0 \rangle \bigg|_{\text{ren}} = \frac{N_C}{\pi} m_q^2 \left[ \frac{1}{\sqrt{G^2}} \Psi'(0) - "g = 0" \right].$$

(A2)

The Mellin transform $\Psi(\lambda)$ satisfies the difference equation

$$\Psi(\lambda + 1) = (\pi \lambda \cot \pi \lambda - \pi \beta \cot \pi \beta) \Psi(\lambda).$$

(A3)

The analytic continuation of $\Psi(\lambda)$ yields a meromorphic function with poles at $\lambda = 2, 3, 4, \ldots$ as well as
\[ \lambda = -\beta_0, -1 - \beta_1, -2 - \beta_2, \ldots \text{, where } \beta_n \in (0; 1) \text{ is the unique solution of} \]

\[ \pi(n + \beta_n) \cot \pi \beta_n = 1 - \frac{1}{\alpha}, \quad (A4) \]

The solution to this difference equation \((A3)\) is given in Ref. [7]

\[ \Psi(\lambda) = \frac{1}{\sqrt{\alpha}} \psi_0(\lambda) \prod_{n=0}^{\infty} \frac{1 + m_q^2 - G^2}{\tan \pi \lambda} \tan \pi \lambda, \quad (A5) \]

where

\[ \psi_0(\lambda) = \pi^3 \Gamma(\lambda) \exp \left[ -2\pi \int_0^{\lambda-1} du \frac{u + \frac{1}{2} \sin^2 \pi u}{\sin(2\pi u)} \right] \quad (A6) \]

is the solution of the difference equation for \(m_q^2 = G^2\) (i.e. \(\alpha = 1\)). The overall normalization (which is not determined from the difference equation alone since Eq.\((A3)\) is linear in \(\Psi\)). It has been fixed using the known scaling behavior of the pseudoscalar coupling constants \(\lim_{n \to \infty} m_q \int_0^1 dx \frac{\phi_n(x)}{x} = \pi G\), which implies

\[ \Psi(0) = \int_0^{\infty} dz \frac{\Phi(z)}{z} = \pi \sqrt{\alpha}. \quad (A7) \]

From these results it is now straightforward to evaluate the derivative of \(\Psi(\lambda)\) at the origin, yielding

\[ \Psi'(0) = \frac{\pi}{m_q} \left\{ \ln \pi - \gamma_E \right. \]

\[ + (1 - \frac{1}{\alpha}) \left[ \frac{1}{\pi} + \sum_{n=1}^{\infty} \left( \frac{1}{\pi + \beta_n} - \frac{1}{n} \right) \right] \} \right. \quad (A8) \]

This expression is most convenient for \(m_q^2 \ll G^2\) (i.e. \(\alpha \to \infty\)) since there \(\beta_n \to 0\) and the r.h.s. of Eq.\((A8)\) is dominated by \(1/\beta_0\). For the general case it is more convenient to use an integral representation (which can easily be derived from Eq.\((A8)\) using contour integration)

\[ \Psi'(0) = \frac{\pi}{m_q} \left\{ \ln \pi - \gamma_E \right. \]

\[ + (1 - \frac{1}{\alpha}) \left[ - \ln 4 + (1 - \alpha) I(\alpha) \right] \}, \quad (A9) \]

where \(I(\alpha)\) has been defined in Eq.\((2.26)\).

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