Invariant trilinear forms for $\text{SL}_3(\mathbb{R})$

Antonius Deitmar

Abstract: We give a detailed analysis of the orbit structure of the third power of the flag variety attached to $\text{SL}_3(\mathbb{R})$. It turns out that 36 generalized Schubert cells split into 72 orbits plus one continuous family of orbits. On the latter, we construct invariant distributions and thus fill a gap in the literature by giving infinitely many linearly independent triple products of induced representations.

Contents

1 Orbit structure of $X$ 2

2 Principal series 9

Introduction

Invariant trilinear forms on flag varieties give intertwining operators in the category of smooth representations between induced representations and tensor products. Hence the dimension of these spaces represent intertwining numbers or decomposition numbers of tensor products. For applications, it is most interesting to consider cases, where these dimensions are finite, which corresponds to rank one situations, see [AZ12, BC12, BSKZ14, Cla15, CKØP11, CØ11, Cle16, Cle17, DKS15, Dei06, Dei14, KSS16, MØO16, MØ17]. In higher rank cases, it is expected that the space of invariant trilinear forms is infinite-dimensional, however, there is no proof of such an instance in the literature. In the present note we fill this gap.
We consider the case of the group $\text{SL}_3(\mathbb{R})$. In this note we give a complete analysis of the orbit structure of the corresponding triple product of the flag variety

$$X = P\backslash G \times P\backslash G \times P\backslash G.$$  

It turns out, that the 36 generalized Schubert cells contain 72 isolated orbits and one continuous family of orbits. On this family we finally construct infinitely many linearly independent invariant distributions, i.e., invariant triple products.

1 Orbit structure of $X$

Let $P$ be the minimal parabolic subgroup of $G = \text{SL}_3(\mathbb{R})$ consisting of all upper triangular matrices. Then $P$ has Langlands decomposition $P = MAN$, where $A$ is the group of diagonal matrices in $G$ with positive entries, $M$ the group of diagonal matrices in $G$ with entries $\pm 1$, so $M \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Finally, $N$ is the group of all upper triangular matrices with ones on the diagonal.

Let $K = \text{SO}(3)$. Then $K$ is a maximal compact subgroup of $G$. We consider the compact manifold $X = (P\backslash G)^3 = (M\backslash K)^3$.

We write $D = AM$ for the group of diagonal matrices in $G$. Let $N_G(A)$ denote the normalizer of $A$ in $G$. The Weyl group $W = N_G(A)/D$ is isomorphic to the permutation group $\text{Per}(3)$ in three letters. The Bruhat decomposition is the disjoint decomposition of $G$,

$$G = \bigsqcup_{w \in W} PwP = \bigsqcup_{w \in W} PwN.$$  

Every $G$-orbit in $X = (P\backslash G)^3$ contains an element of first coordinate equal to 1, so we get a bijection

$$\begin{align*}
(P\backslash G \times P\backslash G \times P\backslash G) / G &\xrightarrow{\cong} \{1\} \times (P\backslash G)^2 / P.
\end{align*}$$

Using the Bruhat decomposition in the second and third coordinate, this
TRILINEAR FORMS $SL(3)$

gives

\[
X/G \sim \bigcup_{v \in W} \bigcup_{w \in W} \{(1) \times \left( \left| P \backslash PvP \right| \times P \backslash PwP \right)/P, \nonumber
\]

\[
\cong \bigcup_{v \in W} \bigcup_{w \in W} \left( \left(1 \times |v| \times P \backslash PwP/(P \cap v^{-1}Pv) \right) \right) \nonumber
\]

\[
\cong \bigcup_{v \in W} \bigcup_{w \in W} \left( \left(1 \times |v| \times P \backslash PwN/(N \cap v^{-1}Nv) \right) \right). \nonumber
\]

So, if we define the Schubert cells

\[ S_{v,w} = (P \backslash P \cdot 1 \times P \backslash PvP \times P \backslash PwP) G, \]

we get a disjoint $G$-stable decomposition into 36 Schubert cells

\[ X = (P \backslash G)^3 = \bigcup_{v \in W} \bigcup_{w \in W} S_{v,w}. \]

The dimension of a cell is

\[ \dim S_{v,w} = 3 + \dim[v] + \dim[w], \]

where we have written $[v] = P \backslash PvP$. We write the elements of $W$ as $1, s_1, s_2, z_1, z_2, w_0$ where $w_0$ is the long element and the corresponding Weyl chambers are given as in the following picture

![Diagram of Weyl chambers](image)

Then $s_1$ and $s_2$ generate the Weyl group and $z_1 = s_2s_1$, $z_2 = s_1s_2$ as well as $w_0 = s_1s_2s_1 = s_2s_1s_2$. For computations, we choose the following represen-
tatives in $G$,

\[
s_1 = \begin{pmatrix} 1 & 1 \\ & -1 \end{pmatrix} \quad s_2 = \begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix} \\
z_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad z_2 = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix} \\
w_0 = \begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix}
\]

The orbit closure inclusion pattern in $P \setminus G = \bigsqcup_{w \in W} P \setminus PwP$ is given by

![Diagram](image)

where the arrows indicate containment in the closure, so for instance $[1] \subset [s_1]$. Accordingly, this diagram repeats with the $S_{v,w}$'s, which is to say that

\[
\begin{align*}
[v_1] & \subset [v_2] \\
[w_1] & \subset [w_2]
\end{align*}
\]

\[\Leftrightarrow S_{v_1,w_1} \subset S_{v_2,w_2}.
\]

So for instance, $S_{s_1,z_1}$ is contained in the closure of $S_{s_1,v_0}$ and in the closure of $S_{z_1,z_1}$.

For $w \in W$ we write $N_w = N \cap w^{-1}Nw$. For $v, w \in W$ let $R_N^{v,w}$ be a set of representatives in $N$ for the equivalence relation

\[n \sim n' \quad \Leftrightarrow \quad n' = n_dnd^{-1}n_w
\]

for some $d \in D, n_v \in N_v$ and $n_w \in N_w$. We then get a set $R$ of representatives of $X/G$ of the form $R = \{(1, v, wn) : v, w \in W, n \in R_N^{v,w}\}$. We can write this suggestively $S_{v,w}/G \cong 1 \times v \times w(N/\sim)$. The stabilizer of $(1, v, wn)$ equals

\[G_{(1,v,wn)} = P \cap v^{-1}Pv \cap (wn)^{-1}Pwn\].
We clearly have $N_1 = N$. A computation shows that $N_{w_0} = 1$ and

$$N_{s_1} = \begin{pmatrix} 1 & 0 & * \\ 1 & * & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad N_{s_2} = \begin{pmatrix} 1 & * & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$N_{z_1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & * & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad N_{z_2} = \begin{pmatrix} 1 & * & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$  

We now classify all orbits by the Schubert cells. The dimensions of the Schubert cells range from 3 to 9. The orbit dimensions cannot exceed $8 = \dim G$, therefore the open cell $S_{w_0,w_0}$ must contain a continuous family of orbits. We introduce the notation

$$n(x, y, z) = \begin{pmatrix} 1 & x & y \\ 1 & 1 & z \\ 1 & 1 & 1 \end{pmatrix}, \quad d(a, b, c) = \begin{pmatrix} a & b \\ 1 & c \end{pmatrix}.$$  

**Proposition 1.1.** The open cell $S_{w_0,w_0}$ contains a family of orbits of maximal dimension, which is 8, parametrized by $u \in \mathbb{R}^x$ and given by $(1, w_0, w_0 n)$ with $n = n(1, 1, u)$.

There are 7 more orbits: 3 orbits of maximal dimension in $S_{w_0,w_0}$ given by $n(0, 1, 1)$, $n(1, 0, 1)$ and $n(1, 1, 0)$, in each case the stabilizer is trivial. There are 3 orbits of dimension 7, which we list by a representative and the corresponding stabilizer.

| $n$          | stabilizer of $(1, w_0, w_0 n)$ |
|--------------|---------------------------------|
| $n(0, 0, 1)$ | $\{d(a, a, 1/a^2) : a \in \mathbb{R}^x\}$ |
| $n(0, 1, 0)$ | $\{d(1/a^2, a) : a \in \mathbb{R}^x\}$ |
| $n(1, 0, 0)$ | $\{d(1/a^2, a) : a \in \mathbb{R}^x\}$ |

Finally, there is one orbit of dimension 6 given by $n(0, 0, 0)$ with stabilizer $D$.

**Proof.** The given elements form a set of representatives of $N$ modulo $D$-conjugation, so they parametrize the orbits in the cell. The inverse of $n(1, 1, u)$ is $n(-1, u - 1, -u)$ and one notes that for given $d \in D$ in order to have $n(1, 1, u) d n(-1, u - 1, -u) \in D$, one must have $d = 1$. This implies the triviality of the stabilizer. The other cases are treated similarly.  

**Proposition 1.2.** The cell $S_{w_0,z_1}$ comprises 4 orbits which are listed by representa-
tive \((1, w_0, z_1 n)\), dimension of orbit, and stabilizer.

| \(n\)       | \(\text{dimension}\) | \(\text{stabilizer}\)               |
|------------|----------------------|-------------------------------------|
| \((1, 1, 0)\) | 8                    | \(\{1\}\)                           |
| \((1, 0, 0)\) | 7                    | \(\{d(a, 1/a^2, a) : a \in \mathbb{R}^x\}\) |
| \((0, 1, 0)\) | 7                    | \(\{d(1/a^2, a, a) : a \in \mathbb{R}^x\}\) |
| \((0, 0, 0)\) | 6                    | \(D\)                               |

**Proof.** Similar to the last proposition. \(\square\)

**Proposition 1.3.** The cell \(S_{w_0, z_2}\) comprises 4 orbits which are listed by representative \((1, w_0, z_2 n)\), dimension of orbit, and stabilizer.

| \(n\)       | \(\text{dimension}\) | \(\text{stabilizer}\)               |
|------------|----------------------|-------------------------------------|
| \((0, 1, 1)\) | 8                    | \(\{1\}\)                           |
| \((0, 0, 1)\) | 7                    | \(\{d(a, 1/a^2, a) : a \in \mathbb{R}^x\}\) |
| \((0, 1, 0)\) | 7                    | \(\{d(a, a, 1/a^2) : a \in \mathbb{R}^x\}\) |
| \((0, 0, 0)\) | 6                    | \(D\)                               |

**Proposition 1.4.**

- The cell \(S_{w_0, s_1}\) comprises 2 orbits

| \(n\)       | \(\text{dimension}\) | \(\text{stabilizer}\)               |
|------------|----------------------|-------------------------------------|
| \((1, 0, 0)\) | 7                    | \(\{d(a, a, 1/a^2) : a \in \mathbb{R}^x\}\) |
| \((0, 0, 0)\) | 6                    | \(D\)                               |

- The cell \(S_{w_0, s_2}\) comprises 2 orbits

| \(n\)       | \(\text{dimension}\) | \(\text{stabilizer}\)               |
|------------|----------------------|-------------------------------------|
| \((0, 0, 1)\) | 7                    | \(\{d(1/a^2, a, a) : a \in \mathbb{R}^x\}\) |
| \((0, 0, 0)\) | 6                    | \(D\)                               |

- The cell \(S_{w_0, 1}\) equals one orbit given by \((1, w_0, 1)\), the dimension is 6 and the stabilizer is \(D\).
Proposition 1.5.  

- The cell $S_{z_1, z_1}$ comprises 4 orbits

| $n$          | dimension | stabilizer                                                                 |
|--------------|-----------|-----------------------------------------------------------------------------|
| $n(1, 1, 0)$ | 7         | $\begin{cases} \begin{pmatrix} a & 1/a^2 & -1/a^2 \\ 1/a^2 & a & 1/a^2 \\ a & 1/a^2 & a \end{pmatrix} : a \in \mathbb{R} \end{cases}$ |
| $n(1, 0, 0)$ | 7         | $\{ d(a, 1/a^2, a) : a \in \mathbb{R} \}$                                  |
| $n(0, 1, 0)$ | 6         | $\begin{cases} \begin{pmatrix} 1/a^2 & a & \ast \\ a & 1/a^2 & \ast \\ \ast & \ast & 1 \end{pmatrix} : a \in \mathbb{R} \end{cases}$ |
| $n(0, 0, 0)$ | 5         | $D \begin{pmatrix} 1 & \ast \\ 1 & 1 \end{pmatrix}$                         |

- The cell $S_{z_1, z_2}$ comprises 2 orbits

| $n$          | dimension | stabilizer |
|--------------|-----------|------------|
| $n(0, 1, 0)$ | 7         | $\{ d(1/a^2, a, a) : a \in \mathbb{R} \}$                                  |
| $n(0, 0, 0)$ | 5         | $D \begin{pmatrix} 1 & \ast \\ 1 & 1 \end{pmatrix}$                         |

- The cell $S_{z_1, s_1}$ comprises 2 orbits

| $n$          | dimension | stabilizer |
|--------------|-----------|------------|
| $n(1, 0, 0)$ | 6         | $\begin{cases} \begin{pmatrix} a & a & x \\ a & 1/a^2 & \ast \\ \ast & \ast & 1 \end{pmatrix} : a \in \mathbb{R} \end{cases}$ |
| $n(0, 0, 0)$ | 5         | $D \begin{pmatrix} 1 & \ast \\ 1 & 1 \end{pmatrix}$                         |

- The cell $S_{z_1, s_2}$ is one orbit, as is $S_{z_1, 1}$, in both cases we have

| $n$          | dimension | stabilizer |
|--------------|-----------|------------|
| $n(0, 0, 0)$ | 6         | $D$         |

Proposition 1.6.  

- The cell $S_{z_2, z_2}$ comprises 4 orbits

| $n$          | dimension | stabilizer                                                                 |
|--------------|-----------|-----------------------------------------------------------------------------|
| $n(0, 1, 1)$ | 7         | $\begin{cases} \begin{pmatrix} a & a & -1/a^2 \\ 1/a^2 & 1/a^2 & a \end{pmatrix} : a \in \mathbb{R} \end{cases}$ |
| $n(0, 0, 1)$ | 7         | $\{ d(a, 1/a^2, a) : a \in \mathbb{R} \}$                                  |
| $n(0, 1, 0)$ | 6         | $\begin{cases} \begin{pmatrix} a & \ast \\ a & 1/a^2 \\ \ast & z \end{pmatrix} : a \in \mathbb{R} \end{cases}$ |
| $n(0, 0, 0)$ | 5         | $D \begin{pmatrix} 1 & \ast \\ 1 & 1 \end{pmatrix}$                         |
• The cells $S_{22,1}$ and $S_{22,1}$ both are one orbit each. In both cases we have

\[
\begin{array}{ccc}
 n & \text{dimension} & \text{stabilizer} \\
n(0, 0, 0) & 6 & D
\end{array}
\]

• The cell $S_{22,2}$ comprises 2 orbits

\[
\begin{array}{ccc}
 n & \text{dimension} & \text{stabilizer} \\
n(0, 0, 0) & 5 & \left\{ \begin{pmatrix} 1/a^2 & x & y \\ a & \cdot & \cdot \\ a & \cdot & \cdot \end{pmatrix} : a \in \mathbb{R}^\times; x, y \in \mathbb{R} \right\} \\
n(0, 0, 0) & 5 & D \left( \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \\ \cdot & \cdot \end{pmatrix} \right)
\end{array}
\]

**Proposition 1.7.** • The cell $S_{11,11}$ comprises 2 orbits

\[
\begin{array}{ccc}
 n & \text{dimension} & \text{stabilizer} \\
n(1, 0, 0) & 5 & \left\{ \begin{pmatrix} a & y \\ a & z \\ 1/a^2 \end{pmatrix} : a \in \mathbb{R}^\times; x, y \in \mathbb{R} \right\} \\
n(0, 0, 0) & 4 & \left\{ \begin{pmatrix} a & b & y \\ b & z & 1/ab \end{pmatrix} : a, b \in \mathbb{R}^\times; x, y \in \mathbb{R} \right\}
\end{array}
\]

• The cell $S_{11,22}$ is one orbit

\[
\begin{array}{ccc}
 n & \text{dimension} & \text{stabilizer} \\
n(0, 0, 0) & 5 & \left\{ \begin{pmatrix} a & y \\ b & z \\ 1/ab \end{pmatrix} : a, b \in \mathbb{R}^\times; y \in \mathbb{R} \right\}
\end{array}
\]

• The cell $S_{11,1}$ is one orbit

\[
\begin{array}{ccc}
 n & \text{dimension} & \text{stabilizer} \\
n(0, 0, 0) & 4 & \left\{ \begin{pmatrix} a & b & y \\ b & z & 1/ab \end{pmatrix} : a, b \in \mathbb{R}^\times; y, z \in \mathbb{R} \right\}
\end{array}
\]

• The cell $S_{22,2}$ comprises 2 orbits

\[
\begin{array}{ccc}
 n & \text{dimension} & \text{stabilizer} \\
n(0, 0, 1) & 5 & \left\{ \begin{pmatrix} 1/a^2 & x & y \\ a & \cdot & \cdot \\ a & \cdot & \cdot \end{pmatrix} : a \in \mathbb{R}^\times; x, y \in \mathbb{R} \right\} \\
n(0, 0, 0) & 4 & \left\{ \begin{pmatrix} a & x & y \\ b & y & 1/ab \end{pmatrix} : a, b \in \mathbb{R}^\times; x, y \in \mathbb{R} \right\}
\end{array}
\]
The cell $S_{2,1}$ is one orbit

| $n$     | dimension | stabilizer |
|---------|-----------|------------|
| $(0, 0, 0)$ | 4         | $\left\{ \begin{array}{ccc} a & x & y \\ b & 1/ab & \end{array} \right\} : a, b \in \mathbb{R}^\times; x, y \in \mathbb{R}$ |

The cell $S_{1,1}$ is one orbit of dimension 3. The stabilizer is $P$.

**Proposition 1.8.** In $X$, there is one family of orbits parametrized by $u \in \mathbb{R}^\times$ and 72 more orbits. These are distributed over the Schubert cells as in the first of the following tables. The second table gives the dimensions of the orbits in each cell. For instance, $6, 7^2, 8$ stands for one cell of dimension 6, two of dimension 7 and one of dimension 8.

| $n$ | $s_1$ | $s_2$ | $z_1$ | $z_2$ | $w_0$ |
|-----|-------|-------|-------|-------|-------|
| 1   | 1     | 1     | 1     | 1     | 1     |
| $s_1$ | 1     | 2     | 1     | 2     | 1     |
| $s_2$ | 1     | 1     | 2     | 1     | 2     |
| $z_1$ | 1     | 2     | 1     | 4     | 2     |
| $z_2$ | 1     | 1     | 2     | 2     | 4     |
| $w_0$ | 1     | 2     | 2     | 4     | 4     | 7     |

| $n$ | $s_1$ | $s_2$ | $z_1$ | $z_2$ | $w_0$ |
|-----|-------|-------|-------|-------|-------|
| 1   | 1     | 1     | 1     | 1     | 1     |
| $s_1$ | 4     | 5     | 5     | 6     | 6     |
| $s_2$ | 4     | 5     | 4     | 5     | 6     |
| $z_1$ | 5     | 6     | 6     | 5     | 6     |
| $z_2$ | 6     | 6     | 5     | 5     | 6     |
| $w_0$ | 6     | 6     | 6     | 6     | 6     |

Proof. This follows from the previous propositions together with the observation that the orbits structure of $S_{v,w}$ is the same as that of $S_{w,v}$ because of the flip $X \rightarrow X$, $(x, y, z) \mapsto (x, z, y)$. □

## 2 Principal series

Let $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ be the circle group. Fix a character $\lambda \in \hat{A} = \text{Hom}_{\text{cont}}(A, \mathbb{T})$, which we write as $a \mapsto a^\lambda$. Let $\rho$ denote the modular shift, i.e., $\rho$ is the continuous quasicharacter of $A$ given by $(\text{diag}(a, b, c))^\rho = a/b$. Then $P \rightarrow C^\times$, $\text{man} \mapsto a^{\lambda+\rho}$ is a quasi-character of $P$ and induces a $G$-homogeneous vector bundle $E_\lambda$ over the compact manifold $P \backslash G$. Let $V_\lambda$ denote the space of smooth sections of this bundle and let $\pi_\lambda$ denote the induced representation of $G$ on $V_\lambda$. The space $V_\lambda$ can be identified with the space of all smooth functions $f : G \rightarrow \mathbb{C}$ such that $f(\text{man}x) = a^{\lambda+\rho} f(x)$ holds for all $x \in G$, $\text{man} \in P$. In this model, the unitary representation $\pi_\lambda$ of $G$ on $V_\lambda$ is given by $\pi_\lambda(y)f(x) = f(xy)$. 
The space \( V_\lambda = \Gamma^\infty(E_\lambda) \) carries a complete locally convex topology given by the seminorms
\[
\sigma_D(s) = \sup_{x \in \rho \setminus G} \|Ds(x)\|,
\]
where \( D \) is a left-invariant differential operator on \( G \).

For three induction parameters \( \lambda_1, \lambda_2, \lambda_3 \in \hat{A} \) let
\[
\mathcal{T}(\lambda_1, \lambda_2, \lambda_3) = \mathcal{T}(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})
\]
denote the space of all continuous trilinear forms \( T : V_{\lambda_1} \times V_{\lambda_2} \times V_{\lambda_3} \rightarrow \mathbb{C} \) which are \( G \)-invariant, i.e., satisfy
\[
T(\pi_{\lambda_1}(y)f, \pi_{\lambda_2}(y)g, \pi_{\lambda_3}(y)h) = T(f, g, h).
\]

Note that the restriction to \( K \) induces an isomorphism of topological vector spaces \( V_\lambda \cong C^\infty(M \setminus K) \). Therefore, an invariant trilinear form is a distribution on the manifold
\[
X = (P \setminus G)^3 = (M \setminus K)^3.
\]

We use the Iwasawa decomposition \( G = ANK \) and the corresponding map \( a : G \rightarrow A \) uniquely defined by the condition that \( a(g)^{-1}g \in NK \).

**Proposition 2.1.** For every \( u \in \mathbb{R}^k \) we have
\[
\int_{AN} a(w_0 n(u) a)^p a(w_0 n)^q a^{-p} \, da \, dn < \infty.
\]

Here \( n(u)^a = a^{-1} n(u) a \).

**Proof.** We need to make things explicit. first note that for \( a = \text{diag}(e^s, e^t, e^{s-t}) \) we have
\[
a^p = e^{2s+t}.
\]

Next if \( g = ank \in G \), then \( gg^t = ann^t a \). Assume this symmetric matrix is of the form
\[
\begin{pmatrix}
* & * & * \\
* & B & H \\
* & H & C
\end{pmatrix}
\]
then a computation shows
\[
a^p = a(g)^p = \frac{1}{\sqrt{BC - H^2} \sqrt{C}}.
\]
Using that, the Ansatz \( g = w_0 n \) with 
\[
 n = \begin{pmatrix}
 1 & x & y \\
 1 & z & 1 \\
 1 & 1 & 1
\end{pmatrix}
\]
leads to 
\[
 g(w_0 n)^\rho = \left[ (1 + z^2)(1 + x^2 + y^2) - (x + zy)^2 \right] (1 + x^2 + y^2)^{-\frac{1}{2}}.
\]
Call this expression \( f(x, y, z) \). One gets
\[
 f(x, y, z) \ll \frac{1}{\sqrt{1 + z^2}} \frac{1}{1 + x^2 + y^2} \ll \frac{1}{\sqrt{1 + x^2}} \frac{1}{\sqrt{1 + y^2}} \frac{1}{\sqrt{1 + z^2}}.
\]
We also have
\[
 n(u)^{\rho} n = \begin{pmatrix}
 1 & x + e^{-t} & y + ze^{-t} + e^{2s+t} \\
 1 & z + ue^{2s+2t} & 1
\end{pmatrix}
\]
So that the integral in question equals
\[
 I = \int_{R^5} f(x, y, z) f(x + e^{s-t}, y + ze^{s-t} + e^{2s+t}, z + ue^{2s+2t}) \, dx \, dy \, dz \, e^{-2s-t} \, ds \, dt.
\]

**Lemma 2.2.** For any \( 0 < \alpha < 1 \), the function
\[
 \phi(r) = \int_R \frac{1}{\sqrt{1 + x^2}} \frac{1}{\sqrt{1 + (x + r)^2}} \, dx
\]
satisfies
\[
 \phi(r) = O(r^{-\alpha}) \quad \text{as } r \to \infty.
\]

**Proof.** Suppose \( r > 0 \) and set \( y = \frac{x}{r}. \) Then
\[
 \phi(r) = r \int_R \frac{1}{\sqrt{1 + (r^2 y^2)}} \frac{1}{\sqrt{1 + r^2 (y + 1)^2}} \, dy
\]
\[
 = \frac{1}{r} \int_R \frac{1}{\sqrt{\frac{1}{r^2} + y^2}} \frac{1}{\sqrt{\frac{1}{r^2} + (y + 1)^2}} \, dy.
\]
Note that, as \( r \to \infty \), the integrand increases monotonically. Fix \( 0 < \varepsilon < \frac{1}{2} \) and let \( A = A(\varepsilon) \) be the set of all real numbers \( y \) with \( |y| \geq \varepsilon \) and \( |y + 1| \geq \varepsilon \). We decompose the integral into a sum \( \int_A + \int_{|y|<\varepsilon} + \int_{|y+1|<\varepsilon} \). Accordingly, \( \phi \) is a sum \( \phi = \phi_A + \phi_0 + \phi_1 \). As \( r \to \infty \), the integral \( \int_A \) converges by the monotone convergence theorem to some number \( C \), so we get
\[
 \phi_A(r) = O\left(r^{-1}\right), \quad r \to \infty.
\]
For given $0 < \alpha < 1$ we get

$$(r^\alpha \phi_0(r))' = r^{\alpha-1} \left( (\alpha-1)\phi_0(r) + \phi_0(r) + r\phi_0'(r) \right).$$

We have

$$\phi_0(r) = \int_{-\varepsilon r}^{\varepsilon r} \frac{1}{\sqrt{1 + x^2}} \frac{1}{\sqrt{1 + (x + r)^2}} dx,$$

and so

$$\phi_0'(r) = \frac{\varepsilon}{\sqrt{1 + \varepsilon^2 r^2}} \left( \frac{1}{\sqrt{1 + (\varepsilon + 1)^2 r^2}} - \frac{1}{\sqrt{1 + (\varepsilon - 1)^2 r^2}} \right)
- \int_{-\varepsilon r}^{\varepsilon r} \frac{1}{\sqrt{1 + x^2}} \frac{1}{\sqrt{1 + (x + r)^2}} \frac{x + r}{1 + (x + r)^2} dx.
$$

The expression $B(r)$ is $O(r^{-2})$ and $< 0$ for $r$ large enough. As for the second summand, we get

$$\phi_0(r) + r\phi_0'(r) = rB(r) + \int_{-\varepsilon r}^{\varepsilon r} \frac{1}{\sqrt{1 + x^2}} \frac{1}{\sqrt{1 + (x + r)^2}} \frac{1 + x(x + r)}{1 + (x + r)^2} dx.$$

The function $f(x) = \frac{1 + x(x + r)}{1 + (x + r)^2}$ has no local extremum in the interval $(-\varepsilon r, \varepsilon r)$, so can be estimated by the boundary values

$$f(\varepsilon r) = \frac{1 + \varepsilon (\varepsilon + 1) r^2}{1 + (\varepsilon + 1)^2 r^2}, \quad f(-\varepsilon r) = \frac{1 + \varepsilon (\varepsilon - 1) r^2}{1 + (\varepsilon - 1)^2 r^2},$$

which for $r \to \infty$ tend to $\frac{1}{1 \pm \varepsilon r}$. Choosing $\varepsilon > 0$ so small that these lie below $1 - \alpha$, we derive that $(r^\alpha \phi_0(r))' < 0$ for $r$ large enough, which means that $r^\alpha \phi_0$ is bounded. Analogously,

$$\phi_1(r) = \int_{-\varepsilon r}^{\varepsilon r} \frac{1}{\sqrt{1 + x^2}} \frac{1}{\sqrt{1 + (x + r)^2}} dx,$$
so that
\[ \phi'_1(r) = \left( \frac{1}{\sqrt{1 + (\varepsilon - 1)^2 r^2}} - \frac{1}{\sqrt{1 + (\varepsilon + 1)^2 r^2}} \right) \frac{1}{\sqrt{1 + \varepsilon r^2}} \]
\[ - \int_{-r-e}^{r+e} \frac{1}{\sqrt{1 + x^2}} \frac{1}{\sqrt{1 + (x + r)^2}} \frac{x + r}{1 + (x + r)^2} dx, \]
which yields the same result for \( \phi_1 \). The lemma follows. \( \square \)

By the above estimates, and with the notation of the lemma, we get for the inner integrals of \( I \)
\[ \int_{\mathbb{R}^3} f(x, y, z) f(x + e^{s-t}, y + ze^{s-t} + e^{2s+t}, z + ue^{s+2t}) \, dx \, dy \, dz \]
\[ \ll \int_{\mathbb{R}} \phi(e^{s-t}) \phi(ze^{s-t} + e^{2s+t}) \frac{1}{\sqrt{1 + z^2}} \frac{1}{\sqrt{1 + (z + ue^{s+2t})^2}} \, dz \]
\[ \ll \phi(e^{s-t}) \phi(|u| e^{s+2t}) (1 + e^{2s+t})^{-\alpha} \]
\[ \ll (1 + e^{s-t})^{-\alpha} (1 + e^{2s+t})^{-\alpha} (1 + |u| e^{s+2t})^{-\alpha} \]
for every \( 0 < \alpha < 1 \). This implies
\[ I \ll \int_{\mathbb{R}^2} (1 + e^{s-t})^{-\alpha} (1 + e^{2s+t})^{-\alpha} (1 + |u| e^{s+2t})^{-\alpha} e^{-2s-t} \, ds \, dt. \]
Considering the four quadrants \( \pm s > 0, \pm t > 0 \) one sees that this integral is actually finite for \( \alpha > 1/2 \). \( \square \)

**Theorem 2.3.** For any induction parameters \( \lambda_1, \lambda_2, \lambda_3 \in i\mathbb{R} \) the space \( T(\lambda_1, \lambda_2, \lambda_3) \) if invariant distributions is infinite-dimensional.

**Proof.** For every \( u \in \mathbb{R}^x \), the integral over the orbit of \((1, w_0, w_0 n(u))\) defines a distribution, as the integral converges by Proposition 2.1. \( \square \)

**References**

[AZ12] Nalini Anantharaman and Steve Zelditch, *Intertwining the geodesic flow and the Schrödinger group on hyperbolic surfaces*, Math. Ann. 353 (2012), no. 4, 1103–1156.

[BC12] Ralf Beckmann and Jean-Louis Clerc, *Singular invariant trilinear forms and covariant (bi-)differential operators under the conformal group*, J. Funct. Anal. 262 (2012), no. 10, 4341–4376.
[BSKZ14] Salem Ben Said, Khalid Koufany, and Genkai Zhang, Invariant trilinear forms on spherical principal series of real rank one semisimple Lie groups, Internat. J. Math. 25 (2014), no. 3, 1450017, 35.

[Cla15] Pierre Clare, Invariant trilinear forms for spherical degenerate principal series of complex symplectic groups, Internat. J. Math. 26 (2015), no. 13, 1550107, 16. MR3435965

[CKØP11] Jean-Louis Clerc, Toshiyuki Kobayashi, Bent Ørsted, and Michael Pevzner, Generalized Bernstein-Reznikov integrals, Math. Ann. 349 (2011), no. 2, 395–431. MR2753827

[CØ11] Jean-Louis Clerc and Bent Ørsted, Conformally invariant trilinear forms on the sphere, Ann. Inst. Fourier (Grenoble) 61 (2011), no. 5, 1807–1838 (2012) (English, with English and French summaries).

[Cle16] Jean-Louis Clerc, Singular conformally invariant trilinear forms, I: The multiplicity one theorem, Transform. Groups 21 (2016), no. 3, 619–652. MR3531743

[Cle17] ______, Singular conformally invariant trilinear forms, II: The higher multiplicity case, Transform. Groups 22 (2017), no. 3, 651–706. MR3682833

[DKS15] Thomas Danielsen, Bernhard Krötz, and Henrik Schlichtkrull, Decomposition theorems for triple spaces, Geom. Dedicata 174 (2015), 145–154.

[Dei06] Anton Deitmar, Invariant triple products, Int. J. Math. Math. Sci., posted on 2006, Art. ID 48274, 22, DOI 10.1155/IJMMS/2006/48274. MR2251763

[Dei14] ______, Fourier expansion along geodesics on Riemann surfaces, Cent. Eur. J. Math. 12 (2014), no. 4, 559–573.

[KSS16] Bernhard Krötz, Eitan Sayag, and Henrik Schlichtkrull, The harmonic analysis of lattice counting on real spherical spaces, Doc. Math. 21 (2016), 627–660.

[MÖ16] Jan Möllers, Bent Ørsted, and Yoshiki Oshima, Knapp-Stein type intertwining operators for symmetric pairs, Adv. Math. 294 (2016), 256–306. MR3479564

[MO17] Jan Möllers and Bent Ørsted, Estimates for the restriction of automorphic forms on hyperbolic manifolds to compact geodesic cycles, Int. Math. Res. Not. IMRN 11 (2017), 3209–3236. MR3693648

[Rud91] Walter Rudin, Functional analysis, 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

Mathematisches Institut
Auf der Morgenstelle 10
72076 Tübingen
Germany
deitmar@uni-tuebingen.de

September 19, 2018