Free vibration of a rod undergoing finite strain
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Abstract
The finite strain longitudinal free vibration of a rod is studied. Utilizing second Piola-Kirchhoff stress and Green strain tensors, the equation of motion is written in terms of displacement in reference configuration. Three different types of homogenous boundary conditions may be considered for the rod, leading to three nonlinear eigenvalue problems. The series solutions with three terms satisfying the boundary conditions are utilized and the relationships between amplitudes of vibration are obtained by means of the Galerkin method. The backbone curves are drawn and the internal resonance between different modes of vibration is analyzed.

Keywords: Finite strain; Rod; Backbone curve; Internal resonance

1. Introduction
The nonlinear vibration of continuous structures has been the subject of study by numerous investigators. To mention a few articles most pertinent to the present study, Lewandowski [1] used an analytical approach to analyze free vibration of beams with large displacement. In another article Lewandowski [2], analyzed the free vibration of beams, membranes and plates with moderately large deformation. The Galerkin procedure in time and space variables was used to obtain systems of nonlinear equations for the amplitude and frequency of vibration. The backbone curves were drawn and internal resonance between modes of vibration was investigated. The longitudinal vibration of a rod and lateral vibration of a beam subjected to nonlinear response of the surrounding medium, posing weakly nonlinear equations, were studied by Andrianov et al. [3]. In all of these works, structures sustained large deformation but infinitesimal strain. Therefore, nonlinear terms appeared only in strain-displacement relationships and the terms with the highest order of derivative were linear in the ensuing differential equations.

In this work, we consider free vibration of a rod with finite strain. The equation of motion in terms of displacement is strongly nonlinear with quadratic and cubic nonlinear terms. The analysis is carried out for rods with three types of boundary conditions. The application of the Galerkin method results in systems of nonlinear algebraic equations specifying the backbone curves. Moreover, the interaction between modes of vibration i.e., the phenomenon of internal resonance is investigated.

2. Formulation
The deformation gradient tensor is defined as, Malvern [4]

\[ F = u \varepsilon + I \] (1)

where \( u \) is the displacement vector and \( I \) signifies the unit matrix. The equations of motion in the absence of body force, for a medium undergoing finite strain in material coordinates, are

\[ \nabla \{ SF^T \} = \rho_0 \frac{\partial^2 u}{\partial t^2} \] (2)

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where \( \rho_0 \) is the density of the medium in the undeformed configuration and \( S \) denotes the second Piola-Kirchhoff stress tensor. The second Piola-Kirchhoff stress tensor is energetically conjugate to the Green strain tensor. Therefore, Hooke’s law may be used as the constitutive equation of the medium, Malvern [4].

\[
S = cE
\]  
(3)

where \( c \) is a fourth order tensor representing the elastic behavior of material and \( E \) is the Green strain tensor defined as

\[
E = \frac{1}{2} \left[ F^T F - I \right]
\]  
(4)

Let the lateral boundary of the rod be constrained such that only longitudinal deformation \( U(x,t) \) occurs. Consequently, the gradient deformation tensor in Cartesian coordinates becomes a diagonal matrix with components

\[
F_{xx} = 1 + \frac{\partial U}{\partial x}, \quad F_{yy} = 1, \quad F_{zz} = 1
\]  
(5)

Employing (4) the only non-zero component of Green strain tensor is

\[
E_{xx} = \frac{1}{2} \left( 1 + \frac{\partial U}{\partial x} \right) \frac{\partial U}{\partial x}
\]  
(6)

For isotropic materials with modulus of elasticity \( Y \) and Poisson’s ratio \( \nu \), the stress-strain relationships (3) become

\[
S_{xx} = \frac{Y}{1+\nu} \left[ E_{yy} + \frac{\nu}{1-2\nu} E_{zz} \delta_{yy} \right]
\]  
(7)

where \( \delta_{yy} \) is the Kronecker delta. Substituting (6) into (7), we observe that the shear stresses vanish and the normal stress components yield

\[
S_{xx} = \frac{Y}{1+\nu} \left( 1 - \nu \right) \left( 1 + \frac{1}{2} \frac{\partial U}{\partial x} \right) \frac{\partial U}{\partial x}
\]  
(8)

Equation (2) by virtue of Eqs. (5) and (8) becomes

\[
\frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial x} + \frac{2}{3} \frac{\partial^2 U}{\partial x^2} = \frac{2\rho_0 (1+\nu)(1-2\nu)}{Y (1-\nu)} \frac{\partial^2 U}{\partial t^2}
\]  
(9)

In the particular case of infinitesimal deformation of medium, the nonlinear terms in Eq. (9) become insignificant and we arrive at

\[
\frac{\partial^2 U}{\partial x^2} = \frac{\mu (1-\nu)}{\rho_0 (1+\nu)(1-2\nu)} \frac{\partial^2 U}{\partial t^2}
\]  
(10)

To render the equations dimensionless, the following non-dimensional variables are introduced

\[
\Psi = \frac{U}{l}, \quad \zeta = \frac{x}{l}
\]  
(11)

where \( l \) is the undeformed length of the rod. Equations (9) and (10) in view of (11) become

\[
\left( \frac{\partial^2 \Psi}{\partial \zeta^2} + \frac{2}{3} \frac{\partial^2 \Psi}{\partial \zeta^2} + \frac{2}{3} \frac{\partial^3 \Psi}{\partial \zeta^3} \right) = \frac{2}{3} \delta \frac{\partial^2 \Psi}{\partial t^2}
\]  
(12)
\[ \frac{\partial^2 \Psi}{\partial \zeta^2} = \delta \frac{\partial^2 \Psi}{\partial t^2} \]  

(13)

where \( \delta = \rho \pi^2 (1 + \nu)(1 - 2\nu)/\mu (1 - \nu) \).

3. Solution of eigenvalue problem

The solution to nonlinear equation (12) is accomplished by means of Galerkin method in time and space variables by taking series solution with three terms readily satisfying the boundary conditions. It is worth noting that for the analysis of internal resonance a series with three terms representing the first three modes of vibration is sufficient. The ends of a rod are either fixed or free. Consequently, three different cases depending upon the type of boundary conditions may be considered. These are analyzed separately, in what follows.

3.1. Rod with fixed-fixed ends

The boundary conditions in this case are \( \Psi (0, t) = 0 \) and \( \Psi (1, t) = 0 \). The one-harmonic Fourier series expansion of solution of Eq. (12) satisfying the boundary conditions is

\[ \Psi (\zeta, t) = \sum_{n=1} a_n \sin(n \pi \zeta) \cos(\lambda t) \]  

(14)

The application of Galerkin method yields

\[ \frac{2T}{T} \int_0^T R (\zeta, t) \sin(n \pi \zeta) \cos(\lambda t) d\zeta dt = 0 \quad n=1,2,3 \]  

(15)

where \( T = 2\pi/\lambda \) and the residue \( R (\zeta, t) \) is determined by substituting Eq. (14) into (12) leading to

\[ R (\zeta, t) = \pi^2 \left[ (a_1 \pi \cos(n \pi \zeta) + 2a_2 \pi \cos(2n \pi \zeta) + 3a_3 \pi \cos(3n \pi \zeta))^2 \cos(\lambda t) \right]^2 \]

\[ + 2(a_1 \pi \cos(n \pi \zeta) + 2a_2 \pi \cos(2n \pi \zeta) + 3a_3 \pi \cos(3n \pi \zeta))(\cos(\lambda t) + 2/3)[a_1 \sin(n \pi \zeta) \cos(\lambda t) \]

\[ + 4a_2 \sin(2n \pi \zeta) + 9a_3 \sin(3n \pi \zeta)] \cos(\lambda t) \]

\[ - 2/3 \delta \lambda^2 [a_1 \sin(n \pi \zeta) \cos(\lambda t) + 2a_2 \sin(2n \pi \zeta) + a_3 \sin(3n \pi \zeta)] \cos(\lambda t) \]  

(16)

Equation (15) after carrying out the necessary manipulations leads to three algebraic equations for the amplitudes and frequencies of vibration as

\[ \frac{3\pi^4}{4} \left[ \frac{1}{8} a_1^3 + \frac{9}{4} a_2^2 a_1 + \frac{3}{8} a_3^2 a_1 + \frac{3}{2} a_2^2 a_3 \right] + \frac{1}{3} a_1 \left( \pi^2 - \delta \lambda^2 \right) = 0 \]

\[ a_1 \left[ \frac{3\pi^4}{4} \left( \frac{9}{8} a_2^2 a_1 + a_3^2 + 3a_2 a_3 + \frac{3}{2} a_2^2 a_3 \right) + \frac{1}{3} \left( 4\pi^2 - \delta \lambda^2 \right) \right] = 0 \]

\[ \frac{3\pi^4}{4} \left[ \frac{1}{8} a_2^3 + \frac{81}{4} a_2 a_3^2 + \frac{9}{8} a_3^2 a_1 + \frac{3}{2} a_2 a_3^2 \right] + \frac{1}{3} a_1 \left( 9\pi^2 - \delta \lambda^2 \right) = 0 \]  

(17)

The following nontrivial solutions may be obtained for the system of Eqs. (17):

i) \( a_2 = a_3 = 0 \), \( a_1 = \pm \sqrt{\frac{2\sqrt{3}}{3\pi^2}}(1 + \lambda_1^2) \), \( \lambda_1 \geq \frac{\pi}{\sqrt{\delta}} \)  

(18)
ii)  \[ a_1 = a_3 = 0, \quad a_2 = \pm \frac{\sqrt{2}}{3\pi^2} \sqrt{\delta \lambda^2 - 4\pi^2}, \quad \lambda_2 \geq \frac{2\pi}{\sqrt{\delta}} \] 

(19)

iii)  \[ a_1 = a_2 = 0, \quad a_3 = \pm \frac{4}{9\sqrt{6}\pi^2} \sqrt{\delta \lambda^2 - 9\pi^2}, \quad \lambda_3 \geq \frac{3\pi}{\sqrt{\delta}} \] 

(20)

The backbone curve, i.e., the amplitude-frequency relationship for the above cases are depicted in Fig. 1.

Fig. 1: Backbone curves for rods with fixed-fixed and free-free ends

The bifurcation point for each mode is equal to the linear natural frequency of the rod obtained from Eq. (13). The frequency at bifurcation points for second and third modes are integral multiples of the first mode, \( \lambda_2 = 2\lambda_1 \) and \( \lambda_3 = 3\lambda_1 \), thus we expect the phenomenon of internal resonance. First, the possibility of internal resonance between the first and third modes of vibration is investigated.

iv)  \[ a_2 = 0, \quad a_1 \neq 0, \quad a_3 \neq 0 \]

Applying the above conditions to Eqs. (17) and eliminating \( \delta \lambda^2 \), we arrive at the equation specifying the interaction between the first and the third modes of vibration

\[ a_1^3 + 63a_3^3 + 17a_1^2a_3 - 3a_1a_3^2 + \frac{256}{9\pi^2}a_3 = 0 \] 

(21)

The internal resonance occurs when the values of \( a_1 \) and \( a_3 \) satisfying Eq. (21) result in real values for the frequency from the following equation

\[ \hat{\lambda} = \frac{\pi^2}{\sqrt{\delta}} \left[ 1 + \frac{9\pi^2}{16} \left( \frac{1}{2} a_1^2 + 9a_3^2 + \frac{3}{2} a_1a_3 \right) \right]^{1/2} \] 

(22)

The plot of \( a_1 \) versus \( a_3 \) for the internal resonance is shown in Fig. 2, and in Fig. 3, the backbone curves of internal resonance are plotted. It is worth noting that for \( a_1 \neq 0, a_2 \neq 0, a_3 = 0 \), no
real frequency can be found. Thus, the internal resonance between the first and second modes does not exist.

3.2. Rod with free-free ends

The boundary conditions in this case are \( \partial^2 \Psi (0, t) / \partial x = 0 \) and \( \partial^2 \Psi (l, t) / \partial x = 0 \). The solution of Eq. (12) is considered as

\[
\Psi (\zeta, t) = \sum_{n=1}^{3} a_n \cos \left( n \pi \zeta \right) \cos (\lambda t) \tag{23}
\]

Application of Galerkin method leads to the system of algebraic equations

\[
\frac{3\pi^4}{4} \left[ \frac{1}{8} a_1^2 + a_2^2 a_1 + \frac{9}{4} a_3^2 a_1 - \frac{3}{8} a_1^2 a_3 + \frac{3}{2} a_2^2 a_3 \right] + \frac{1}{3} a_1 \left( \pi^2 - \delta \lambda^2 \right) = 0
\]

\[
a_2 \left[ \frac{3\pi^4}{4} \left( 2a_2^2 + a_1^2 + 9a_3^2 + 3a_1 a_3 \right) + \frac{1}{3} \left( 4\pi^2 - \delta \lambda^2 \right) \right] = 0
\]

\[
a_3 \left[ \frac{3\pi^4}{4} \left( -\frac{1}{8} a_1^2 + \frac{81}{8} a_3^2 + \frac{9}{4} a_1^2 a_3 + 9a_2^2 a_3 + \frac{3}{2} a_1 a_3 \right) + \frac{1}{3} a_3 \left( 9\pi^2 - \delta \lambda^2 \right) \right] = 0
\]

The nontrivial solutions of the above equations are identical with those of Eqs. (17). Therefore, Fig. 1 shows the bifurcation points and backbone curves. Internal resonance only between the first and third modes, where \( \lambda_3 = 3\lambda_1 \) exists and the equation for amplitudes of vibration for internal resonance is

\[
a_1^3 - 63a_3^2a_1 + 17a_1^2a_3 - 3a_1a_3^2 - \frac{256}{9\pi^2} a_1 = 0 \tag{25}
\]

The frequency for internal resonance between the first and third modes may be obtained from the following equation

\[
\lambda = \frac{\pi}{\sqrt{\delta}} \left[ 1 + \frac{9\pi^2}{16} \left( \frac{1}{2} a_1^2 + 9a_3^2 - \frac{3}{2} a_1 a_3 \right) \right]^{1/2} \tag{26}
\]
Fig. 4 shows the plot of $a_1$ against $a_3$ for internal resonance. The backbone curves for internal resonance are shown in Fig. 5.

![Fig. 4: Internal resonance for rods with free-free ends](image)

![Fig. 5: Backbone curve for $a_1$ and $a_3$ due to internal resonance for rod with free-free ends](image)

3.3. Rod with fixed-free ends

The solution for a rod fixed at $\zeta = 0$ and free at $\zeta = 1$ is taken as

$$\Psi(\zeta,t) = \sum_{n=1}^{\infty} a_n \sin[(2n-1)\pi \zeta / 2] \cos(\lambda t)$$

(27)

Similar to part (3.1), the system of algebraic equations yield

$$\frac{3\pi^4}{64} \left[ \frac{1}{8} a_1^2 + \frac{9}{4} a_2^2 a_1 + \frac{25}{4} a_3^2 a_1 + \frac{3}{8} a_1^2 a_2 + \frac{15}{4} a_1 a_2 a_3 + \frac{45}{8} a_2^2 a_3 \right] + \frac{1}{3} a_1 (\frac{\pi^2}{4} - \delta \lambda^2) = 0$$

$$\frac{3\pi^4}{64} \left[ \frac{1}{8} a_1^2 - \frac{9}{4} a_1 a_3 + \frac{45}{8} a_2 a_3 + \frac{3}{8} a_1^2 a_2 + \frac{81}{8} a_2 a_1 a_3 + \frac{225}{4} a_2 a_3^2 \right] + \frac{1}{3} a_2 (\frac{9\pi^2}{4} - \delta \lambda^2) = 0$$

$$\frac{3\pi^4}{64} \left[ \frac{45}{8} a_1 a_2 + \frac{15}{8} a_2 a_1 + \frac{25}{4} a_1 a_2 + \frac{225}{8} a_2 a_3 + \frac{625}{4} a_3 \right] + \frac{1}{3} a_3 (\frac{25\pi^2}{4} - \delta \lambda^2) = 0$$

(28)

The nontrivial solutions to the above equations are:

i) $a_2 = a_3 = 0, \quad a_1 = \pm \frac{16\sqrt{2}}{3\pi^2} \sqrt{\delta \lambda^2 - \frac{\pi^2}{4}}, \quad \lambda_1 \geq \frac{\pi}{2\sqrt{\delta}}$

(29)

ii) $a_1 = a_3 = 0, \quad a_2 = \pm \frac{16\sqrt{2}}{27\pi^2} \sqrt{\delta \lambda^2 - \frac{9\pi^2}{4}}, \quad \lambda_2 \geq \frac{3\pi}{2\sqrt{\delta}}$

(30)

iii) $a_1 = a_2 = 0, \quad a_3 = \pm \frac{16\sqrt{2}}{75\pi^2} \sqrt{\delta \lambda^2 - \frac{25\pi^2}{4}}, \quad \lambda_3 \geq \frac{5\pi}{2\sqrt{\delta}}$

(31)

The bifurcation points coincide with the natural frequencies of linear vibration obtained from Eq. (13), and backbone curves are depicted in Fig. 6.

The internal resonance between the first and second modes, $\lambda_2 = 3\lambda_1$, leads to the following case.

iv) $a_3 = 0, \quad a_1 \neq 0, \quad a_2 \neq 0$
From the system of Eqs. (28), after eliminating $\delta \lambda^2$, we arrive at the equation specifying the interaction between the first and the second modes of vibration as

$$3a_1^3 + 189a_2^3 + 51a_1^2a_2 - 9a_1a_2^2 + \frac{1024}{3\pi^2} a_2 = 0$$  \hspace{1cm} (32)

The internal resonance occurs when the values of $a_1$ and $a_2$ satisfying Eq. (32) give real value for the frequency

$$\lambda = \frac{\pi}{2\sqrt{\delta}} \left[ 1 + \frac{9\pi^2}{64} \left( \frac{1}{2}a_1^2 + 9a_2^2 + \frac{3}{2}a_1a_2 \right) \right]^{1/2}$$  \hspace{1cm} (33)

The plot of $a_1$ against $a_2$ is shown in Fig. 7 and the backbone curves are depicted in Fig. 8. It is noteworthy to mention that internal resonance is confined to this case. In Figs. 2, 4 and 7 a saddle point may be observed at the origin. Moreover, since internal resonance occurs between modes of vibration with ratio of frequencies 3 at bifurcation points, Eq. (12) may be considered as an equation with cubic nonlinearity.

Fig. 6: Backbone curves for rods with fixed-free ends

Vertical displacement

Base displacement

Fig. 6: Backbone curves for rods with fixed-free ends
Conclusion:

In this article, the free vibration of a rod is investigated. By solving an eigenvalue problem, the phenomenon of internal resonance is observed between certain modes. This analysis conveys that for the rod with fixed-fixed ends, internal resonance occurs between first and third modes, while no interaction between first and second or second and third modes are observed. This behavior has been reported previously. The same behavior is seen in the rod with free ends. On the other hand, for the rod with a free end and a fixed end, it is seen that there is interaction between first and second modes and no interaction exists between the first and third or second and third modes.

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