MARKOV BASES OF LATTICE IDEALS

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Abstract. Let \( L \subset \mathbb{Z}^n \) be a lattice, \( R = \mathbb{k}[x_1, \ldots, x_n] \) where \( \mathbb{k} \) is a field and \( I_L = \langle x^u - x^v : u - v \in L \rangle \) the corresponding lattice ideal. Most results in the literature on generating sets of lattice ideals concern the case \( L \cap \mathbb{N}^n = \{0\} \). In this paper, using appropriate graphs for each fiber, we characterize minimal generating sets of \( I_L \) of minimal cardinality for all lattices and give invariants for these generating sets. As an application we characterize all binomial complete intersection lattice ideals.

1. Introduction

Let \( R = \mathbb{k}[x_1, \ldots, x_n] \) where \( \mathbb{k} \) is a field, and let \( L \) be a lattice in \( \mathbb{Z}^n \). The lattice ideal \( I_L \) is defined to be the ideal generated by the following binomials:

\[
I_L := \langle x^u - x^v : u - v \in L \rangle.
\]

Let \( \mu(I_L) \) be the least cardinality of any minimal generating set of \( I_L \) consisting of binomials. We call Markov basis of \( I_L \) a minimal system of binomial generators of \( I_L \) of cardinality \( \mu(I_L) \). The study of lattice ideals is a rich subject on its own, see [22, 33] for the general theory and [21] for recent developments. Moreover lattice ideals have applications in diverse areas in mathematics, such as algebraic statistics [8, 26], integer programming [10], hypergeometric differential equations [9], graph theory [25], etc. We note that such ideals were first systematically studied in [11] and that toric ideals are lattice ideals \( I_L \) for which the lattice \( L \) is the kernel of an integer matrix. We note that almost all results in the literature are about lattices \( L \) such that \( L \cap \mathbb{N}^n = \{0\} \), with very few exceptions like in [11, 19, 13, 15, 20].

If \( L \) is such that \( L \cap \mathbb{N}^n = \{0\} \) we say that \( L \) is positively graded. Let \( \mathcal{A} \) be the subsemigroup of \( \mathbb{Z}^n/L \) generated by the elements \( \{a_i = e_i + L : 1 \leq i \leq n\} \), where \( \{e_i : 1 \leq i \leq n\} \) is the canonical basis of \( \mathbb{Z}^n \) and set

\[
\text{deg}_\mathcal{A}(x^v) := v_1 a_1 + \cdots + v_n a_n \in \mathcal{A}
\]

where \( x^v = x_1^{v_1} \cdots x_n^{v_n} \). It follows that

\[
I_L = \langle x^u - x^v : \text{deg}_\mathcal{A}(x^u) = \text{deg}_\mathcal{A}(x^v) \rangle
\]

and that \( I_L \) is \( \mathcal{A} \)-graded. When \( L \) is positively graded, the semigroup \( \mathcal{A} \) is partially ordered:

\[
c \geq d \iff \text{there is } e \in \mathcal{A} \text{ such that } c = d + e.
\]

Then the grading of \( \mathcal{A} \) forces the \( I_L \)-fiber of \( x^v \), i.e. the set \( \{x^v : x^v - x^u \in I_L\} = \{x^v : \text{deg}_\mathcal{A}(x^v) = \text{deg}_\mathcal{A}(x^u)\} \), to be finite. The homogeneous Nakayama Lemma applies and guarantees that all minimal binomial generating sets of \( I_L \) are Markov bases of \( I_L \), since they have the same cardinality. Let \( S \) be a Markov basis of \( I_L \).
and form the multiset of all $I_L$-fibers corresponding to the elements of $S$. This multiset is an invariant of $I_L$ and does not depend on the choice of $S$. Moreover the binomials of $S$ are primitive, see \cite{22,23}, and thus $S$ is a subset of the Graver basis of $I_L$. Since the Graver basis of $I_L$ is a finite set, see \cite{22}, it follows that the Universal Markov basis of $I_L$, (see \cite{17}), i.e. the union of all Markov bases, is finite.

The situation for a general lattice ideal is completely different. Take for example the lattice $L$ generated by $\{(1,1), (5,0)\}$. It can be shown that the following are minimal generating sets of $I_L$ in $k[x,y]$: \{1 - xy, 1 - x^5\}, \{1 - xy, x^3 - y^2\}, \{1 - x^2y^2, 1 - x^3y^3, 1 - x^5\}. It is clear that $I_L$ is not a principal ideal and that $\mu(I_L) = 2$. It is not hard to produce minimal generating sets of $I_L$ of any desired cardinality greater than 2. For example let $p_1, \ldots, p_s$ be $s$ distinct primes and let $a_i = p_1 \cdots p_s / p_i$. The elements $a_1, \ldots, a_s$ are relatively prime and the greatest common divisor of $(1 - z^{a_1}, \ldots, 1 - z^{a_s})$ is $1 - z$, while the greatest common divisor of $\{1 - z^a : j \neq i\}$ is $1 - z^{p_j}$. It follows that $\langle 1 - (xy)^{a_1}, \ldots, 1 - (xy)^{a_s}\rangle = \langle 1 - xy \rangle$ and that $\{1 - x^5, 1 - (xy)^{a_1}, \ldots, 1 - (xy)^{a_s}\}$ is a minimal generating set of $I_L$.

Even if we restrict our attention to Markov bases of $I_L$, see \cite{28}, it is not hard to produce minimal generating sets of $I_L$ of any desired cardinality greater than 2. For example let $p_1, \ldots, p_s$ be $s$ distinct primes and let $a_i = p_1 \cdots p_s / p_i$. The elements $a_1, \ldots, a_s$ are relatively prime and the greatest common divisor of $(1 - z^{a_1}, \ldots, 1 - z^{a_s})$ is $1 - z$, while the greatest common divisor of $\{1 - z^a : j \neq i\}$ is $1 - z^{p_j}$. It follows that $\langle 1 - (xy)^{a_1}, \ldots, 1 - (xy)^{a_s}\rangle = \langle 1 - xy \rangle$ and that $\{1 - x^5, 1 - (xy)^{a_1}, \ldots, 1 - (xy)^{a_s}\}$ is a minimal generating set of $I_L$.

The problem is completely solved, see \cite{4}. In this paper we address this problem for general lattice ideals. We recall that a lattice ideal $I_L$ of height $r$ is a complete intersection if there exist polynomials $P_1, \ldots, P_r$ such that $I_L = \langle P_1, \ldots, P_r \rangle$ and $I_L$ is a binomial complete intersection if there exist binomials $B_1, \ldots, B_r$ such that $I_L = \langle B_1, \ldots, B_r \rangle$. If Nakayama’s lemma applies then complete intersection lattice ideals are automatically binomial complete intersections. The problem is completely
solved when $L$ is positively graded by a series of articles: 14 7 32 18 36 24 30 29 12 31 23. The final conclusion is that $I_L$ is a complete intersection if and only if the matrix $M$ whose rows correspond to a basis of $L$ is “mixed dominating”: every row of $M$ has a positive and negative entry and $M$ contains no square submatrix with this property. When $L$ is not positively graded the situation is less clear. In this paper we characterize all binomial complete intersection lattice ideals.

The structure of this paper is as follows: in Section 2 given the lattice $L \subset \mathbb{Z}^n$ we introduce and examine the properties of $L_{\text{pure}}$, a sublattice of $L$ which will be crucial in our study. Of particular importance is the support of $\sigma$ which we denote by $\sigma_L$. In Section 3 we define an equivalence relation among the $I_L$-fibers. We order the resulting equivalence classes and show that any descending chain of such classes stabilizes. We prove that the multiset of equivalence classes of fibers corresponding to a Markov basis of $I_L$ is an invariant of $I_L$. In Section 4 we characterize all minimal binomial generating sets of pure lattices. Then we describe all Markov bases of $I_L$ for any lattice $L$. We explicitly compute $\mu(I_L)$ in terms of a related graph. We show that if rank $L_{\text{pure}} \geq 1$, there is at most one indispensable binomial. In Section 5 we characterize all lattices $L$ such that $I_L$ is a binomial complete intersection. In section 6 we work out an example in full detail.

2. FIBERS, THE PURE SUBLATTICE AND BASES OF A LATTICE

Let $R = k[x_1, \ldots, x_n]$ where $k$ is a field, $L$ a lattice in $\mathbb{Z}^n$, $I_L = \langle x^u - x^v : u - v \in L \rangle$. We denote by $\mathcal{T}^n$ the set of monomials of $R$ including $1 = x^0$. If $J$ is a monomial ideal of $R$ we denote by $G(J)$ the unique minimal set of monomial generators of $J$. For $r \in \mathbb{N}$ we let $[r] = \{1, \ldots, r\}$. Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$. We write $a \geq b$ if $a_i \geq b_i$ for $i = 1, \ldots, n$. We write $a \geq 0$ if $a \in \mathbb{N}^n$. If either $a \geq 0$ or $-a \geq 0$ we say that $a$ is pure. We say that $a, b$ are incomparable if $a - b$ is not pure. In general we let $\text{supp}(a) = \{i : a_i \neq 0\} \subset [n]$. For any subset $X$ of $\mathbb{Z}^n$ we let

$$\text{supp}(X) := \bigcup_{w \in X} \text{supp}(w).$$

**Definition 2.1.** We say that $F \subset \mathcal{T}^n$ is an $I_L$-fiber if there exists $x^u \in \mathcal{T}^n$ such that $F = \{ x^v \in \mathcal{T}^n : v - u \in L \}$. If $x^u \in F$, and $F$ is an $I_L$-fiber we write $F_u$ or $F_{x^u}$ for $F$. If $B \in I_L$ and $B = x^u - x^v$ we write $F_B$ for $F_u$. When $F$ is an $I_L$-fiber we let $M_F = \langle x^v : x^v \in F \rangle$ be the monomial ideal generated by the elements of $F$.

From the properties of the lattice and the definition of lattice ideals we get the following:

**Proposition 2.2.** If $x^v \in F_u$ then $F_u = F_v$. Moreover $F_u = \{ x^v : x^v - x^u \in I_L \}$. If $x^u - x^v \in I_L$ then $u - v \in L$.

We remark that $F_u$ is a singleton if and only if there is no binomial $0 \neq B \in I_L$ such that $F_B = F_u$. We note that $F \subset M_F$ and $G(M_F) \subset F$. The following proposition follows also from [22] Theorem 8.6.

**Proposition 2.3.** Let $L \subset \mathbb{Z}^n$ be a lattice. The following are equivalent:

1. The lattice $L$ contains a nonzero pure element.
2. All $I_L$-fibers are infinite.
3. There exists an $I_L$-fiber which is infinite.
Proof. (1) $\Rightarrow$ (2) Let $F$ be an $I_L$-fiber and suppose that $0 \neq u \in L \cap \mathbb{N}^n$. It is easy to see that if $v \in \mathbb{N}^n$ and $F$ is the $I_L$-fiber such that $x^v \in F$, then $x^{v+lu} \in F$ for all $l \in \mathbb{N}$. Thus $F$ is infinite. (2) $\Rightarrow$ (3) obvious. (3) $\Rightarrow$ (1) Suppose that an $I_L$-fiber is infinite. Let $x^v \in F$ be such that $x^v \notin G(M_F)$. Note that since $F$ is infinite such a $v$ exists. Since $x^v \in M_F$, there exists a monomial $x^w \in G(M_F)$ such that $x^w | x^v$ and thus $x^v = x^w x^n$ for $0 \neq w \in \mathbb{N}^n$. Since $x^v$, $x^n \in F$, it follows that $w = v - u \in L$, therefore $w \in L \cap \mathbb{N}^n$. □

**Corollary 2.4.** Let $L \subset \mathbb{Z}^n$ be a lattice. The lattice $L$ is positively graded (i.e. $L \cap \mathbb{N}^n = \{0\}$) if and only if $G(M_F) = F$ where $F$ is any $I_L$-fiber.

**Proof.** Suppose that $L \cap \mathbb{N}^n = \{0\}$. Since $M_F = (F)$ to prove that $G(M_F) = F$, it is enough to show that if $x^a \neq x^b \in F$ then $a$, $b$ are incomparable. Suppose otherwise. Then $a - b \in L$ is pure, a contradiction. For the other direction suppose that $G(M_F) = F$. Thus $F$ is finite and the conclusion follows from Proposition 2.3. □

**Notation 2.5.** We let $L^+ = L \cap \mathbb{N}^n$, $\sigma_L = \text{supp}(L^+)$ and $L_{\text{pure}}$ be the subgroup of $L$ generated by $L^+$.

In the course of the proof of Proposition 2.3 we proved the following:

**Proposition 2.6.** Let $L \subset \mathbb{Z}^n$ be a lattice and let $F$ be an $I_L$-fiber. If $G(M_F) = \{x^{a_1}, \ldots, x^{a_s}\}$ then

$$F = \bigcup_{i=1}^{s} \{x^{a_i} x^w : w \in L^+\}.$$

The next proposition considers the support of the elements of $L$ that belong to $L_{\text{pure}}$.

**Proposition 2.7.** There exists an element $w$ in $L^+$ such that $\text{supp}(w) = \sigma_L$. For $u \in L$ we have that $\text{supp}(u) \subset \sigma_L$ if and only if $u \in L_{\text{pure}}$.

**Proof.** The existence of $w$ follows from the observation that if $w_1, w_2 \in L^+$ then $w_1 + w_2 \in L^+$ and $\text{supp}(w_1) \cup \text{supp}(w_2) = \text{supp}(w_1 + w_2)$.

Suppose now that $u \in L$ and $\text{supp}(u) \subset \sigma_L$. Let $w \in L^+$ be such that $\text{supp}(w) = \sigma_L$. It is clear that for $l \in \mathbb{N}$, $l \gg 0$, $u + lw = w' \in \mathbb{N}^n$. Since $u$, $lw \in L$ it follows that $w' \in L$ and thus $w' \in L^+$. Therefore $u = w' - lw \in L_{\text{pure}}$. □

Since $L_{\text{pure}}$ is generated by the elements of $L^+$ it is clear that

$$\text{supp}(L_{\text{pure}}) = \sigma_L.$$

Let $u = (u_i) \in \mathbb{Z}^n$. By $u^{\sigma_L}$ we mean the vector $(u_i)_{i \notin \sigma_L}$. The following is an immediate consequence of Proposition 2.7.

**Corollary 2.8.** Let $L \subset \mathbb{Z}^n$ be a lattice and $u \in L$. Then $u \in L_{\text{pure}}$ if and only if $u^{\sigma_L} = 0$.

**Definition 2.9.** A nonzero vector $u \in L$ is called $L$-primitive if whenever $\lambda u \in L$ where $\lambda \in \mathbb{Q}$ then $\lambda \in \mathbb{Z}$. In other words, $u$ is $L$-primitive if and only if $\mathbb{Q}u \cap L = Zu$.

Equivalently $u$ is $L$-primitive if it is the “smallest” element of $L$ in the direction determined by $u$.

**Proposition 2.10.** Let $0 \neq v \in L$. There is an $L$-primitive vector $u \in L$ such that $v = \lambda u$ for $\lambda \in \mathbb{Z}$.
Proof. If \( v \) is not \( L \)-primitive there is \( v' \in L \) such that \( v' = \frac{1}{m}v \) where \( m \neq 1 \) and \( \gcd(k, m) = 1 \). Thus \( mv' = kv \) and \( m \) divides all coordinates of \( v \). Moreover there are \( t_1, t_2 \in \mathbb{Z} \) such that \( 1 = t_1k + t_2m \). It follows that
\[
\frac{1}{m}v = t_1v' + t_2v =: u = \frac{1}{m}v \in L.
\]
We note that \( v \) is an integer multiple of \( u \). If \( u \) is not \( L \)-primitive we repeat this procedure. Since \( v \in \mathbb{Z}^n \) this procedure has to end in a finite number of steps. \( \square \)

Consider now any basis of \( L \) as a \( \mathbb{Z} \)-module. The next theorem states that the elements of such a basis are necessarily \( L \)-primitive.

**Theorem 2.11.** Let \( L \subset \mathbb{Z}^n \) be a lattice and let \( \mathcal{B} \) be a basis of \( L \) as a \( \mathbb{Z} \)-module. The elements of \( \mathcal{B} \) are \( L \)-primitive.

**Proof.** Since \( L \) is a sublattice of \( \mathbb{Z}^n \), there exists an \( r \in \mathbb{N} \) such that \( L \cong \mathbb{Z}^r \), \( r = \text{rank}(L) \). Let \( \mathcal{B} = \{u_1, \ldots, u_r\} \). Suppose that for some \( i \in [r] \), \( u_i \) is not \( L \)-primitive. By Proposition 2.10 it follows that there is \( v \in L \) and \( m \in \mathbb{N} \), \( m \neq 1 \), such that \( v = \frac{1}{m}u_i \). Since \( \mathcal{B} \) is a basis of \( L \) it follows that \( v = \sum \lambda_j u_j \) where \( \lambda_j \in \mathbb{Z} \) for \( j \in [r] \). Since the \( \mathbb{Q} \)-coordinates of \( v \) are unique it follows that \( \lambda_i = \frac{1}{m} \) and \( \lambda_j = 0 \) for \( j \in [r] \setminus \{i\} \) and thus \( \lambda_i = m = 1 \), a contradiction. \( \square \)

We consider the usual Euclidean inner product in \( \mathbb{Z}^n \): if \( a = (a_i), b = (b_i) \) we let \( a \cdot b = \sum a_i b_i \).

**Theorem 2.12.** Let \( L \) be a lattice and \( u_1 \) an \( L \)-primitive vector. There exists a basis \( \mathcal{B} \) of \( L \) such that \( u_1 \in \mathcal{B} \).

**Proof.** We will do induction on \( r \), the rank of \( L \). If \( r = 1 \) we are done. Assume \( r > 1 \). Let \( w_1 \) be a vector in \( \mathbb{Z}^n \) such that \( w_1 \cdot u_1 = 0 \). Since \( r > 1 \) it is clear that there is \( u_2 \in L \) such that \( w_1 \cdot u_2 \in \mathbb{N} \). Choose \( u_2 \in L \) to be such that \( w_1 \cdot u_2 \) is positive and as small as possible. We will show that for \( u \in L \) there is a \( \lambda \in \mathbb{Z} \) such that \( w_1 \cdot u = \lambda(w_1 \cdot u_2) \). Suppose not. Then
\[
w_1 \cdot u = q(w_1 \cdot u_2) + r, \quad 0 \leq r \leq w_1 \cdot u_2.
\]
It follows that \( w_1 \cdot (u - quu_2) = r \) which contradicts the choice of \( u_2 \). Notice also that \( u_1, u_2 \) are linearly independent. We continue this way and obtain a sequence of linearly independent vectors \( u_1, \ldots, u_r \) and a sequence of vectors \( w_1, \ldots, w_{r-1} \) that satisfy the following properties for \( 1 \leq i \leq r - 1 \):

a) \( w_i \cdot u_j = 0 \) for \( j = 1, \ldots, i \)

b) \( w_i \cdot u_{i+1} > 0 \) and

c) if \( u \in L \) then there is a \( \lambda \in \mathbb{Z} \) such that \( w_i \cdot u = \lambda(w_i \cdot u_{i+1}) \).

It is clear that \( \{u_1, \ldots, u_r\} \) is a \( \mathbb{Q} \)-basis of \( L \). We show that \( \{u_1, \ldots, u_r\} \) is a \( \mathbb{Z} \)-basis of \( L \). Let \( u \in L \), \( u = \sum_{i=1}^r \lambda_i u_i \) where \( \lambda_i \in \mathbb{Q} \). Consider \( w_{r-1} \cdot u \). Then
\[
w_{r-1} \cdot u = \lambda_r(w_{r-1} \cdot u_r).
\]
By \( c) \) above it follows that \( \lambda_r \in \mathbb{Z} \). Next consider \( u' = u - \lambda_r u_r = \sum_{i=1}^{r-1} \lambda_i u_i \).
Since \( w_{r-2} \cdot u' = \lambda_{r-1}(w_{r-2} \cdot u_{r-1}) \) it follows as above that \( \lambda_{r-1} \in \mathbb{Z} \). In this way we get that \( \lambda_r, \ldots, \lambda_2 \in \mathbb{Z} \). Consider now \( v = u - \sum_{i=2}^{r-1} \lambda_i u_i = \lambda_1 u_1 \). Since \( v \in L \) and \( u_1 \) is \( L \)-primitive it follows that \( \lambda_1 \in \mathbb{Z} \). \( \square \)

**Corollary 2.13.** Let \( L \) be a lattice. There exists a basis of \( L_{\text{pure}} \) whose elements are in \( L^+ \) and have support equal to \( \sigma_L \).
Lemma 3.1. Let \( I \) be an ideal of \( \mathbb{Z}^n \). By Theorem 2.12 there exists a basis \( \{ u_1, \ldots, u_r \} \) of \( L_{\text{pure}} \). It is clear that for \( l \geq 0 \), \( u'_i = u_i + lu_1 \in L^+ \) for \( i = 2, \ldots, r \). The set \( \{ u_1, u'_2, \ldots, u'_r \} \) has the desired properties.

Bases of the lattice \( L \) are clearly important for the study of \( I_L \). We note though that it is well known that it is not enough to compute a basis for \( L \) to find a generating set of \( I_L \). Indeed we can associate to each \( u \in L \) the polynomial \( B_u = x^{u^+} - x^{u^-} \) where \( u = u^+ - u^- \) and \( u^+, u^- \in \mathbb{N}^n \). Of course there are many binomials \( x^{w_1} - x^{w_2} \) such that \( u = w_1 - w_2 \) and they all belong to \( I_L \), but \( B_u \) and \( -B_u \) are the only ones with the property that its monomial terms are relatively prime: in some sense \( B_u \) is the lowest term binomial that corresponds to \( u \). It is a basic fact that \( I_L = \langle B_u : u \in L \rangle \). Let \( E \) be a basis of \( L \) and define \( I(E) = \langle E \rangle \). Clearly \( I(E) \subset I_L \) but the equality does not have to hold. For example let \( L \) be the lattice of \( \mathbb{Z}^4 \) with basis \( E = \{ u_1 = (1, -1, -1, 1), u_2 = (1, 2, -2, -1) \} \) and let \( B_1 = xw - yz, B_2 = xz^2 - y^2w \). The lattice \( L \) corresponds to the Macaulay curve:

\[
L = \ker \begin{bmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{bmatrix}
\]

We see that \( u = (2, -3, 1, 0) \in L \) since \( u = u_1 + u_2 \) and that the polynomial \( B = x^2z - y^3 \) of \( \mathbb{k}[x, y, z, w] \) is in \( I_L \). However \( B \) does not belong to \( I(E) = \langle B_1, B_2 \rangle \) since there is no way to create \( x^2z \) from the monomial terms of \( B_1 \) and \( B_2 \). The problem is that the relation

\[
(2, -3, 1, 0) = (1, -1, -1, 1) + (1, 2, 2, -1)
\]

does not translate to a relation among the lowest term binomials. However the above relation on the elements of \( L \) translates to the following relation on elements of \( I(E) \):

\[
wB = xzB_1 + yB_2.
\]

Indeed, as it is shown in [33] Lemma 12.2, for toric ideals the following relation holds:

\[
I_L = I(E): (x_1 \cdots x_n)\infty,
\]

where for \( f \in \mathbb{k}[x_1, \ldots, x_n] \) and \( J \) an ideal of \( \mathbb{k}[x_1, \ldots, x_n] \),

\[
J : f\infty = \{ g \in \mathbb{k}[x_1, \ldots, x_n] : gf^r \in J, r \in \mathbb{N} \}.
\]

In the next two sections we are going to describe minimal generating sets of lattice ideals using properties of the \( I_L \)-fibers and of the bases of \( L_{\text{pure}} \).

3. Fibers and Markov bases of Lattice Ideals

Let \( R = \mathbb{k}[x_1, \ldots, x_n] \) where \( \mathbb{k} \) is a field, \( L \subset \mathbb{Z}^n \) a lattice. For simplicity of notation from now on we write

\[
I := I_L, \quad \sigma := \sigma_L.
\]

If \( G \subset \mathbb{T}^n \) and \( t \in \mathbb{N}^n \) we let

\[
x^tG := \{ x^tu : x^u \in G \}.
\]

Lemma 3.1. Let \( G, F \) be \( I \)-fibers. If there exists \( w_1 \in \mathbb{N}^n \), \( x^u \in G \) such that \( x^{w_1}x^u \in F \) then \( x^{w_1}G \subset F \). Moreover if \( x^{w_2}F \subset G \) for \( w_2 \in \mathbb{N}^n \) then \( w_1 + w_2 \in L^+ \) and \( \text{supp}(w_i) \subset \sigma, i = 1, 2 \).
Proof. Suppose that $x^{w_1}x^u = x^v \in F$ and let $x^{u'} \in G$. Since $u' - u \in L$ it follows that $(w_1 + u') - v \in L$ and thus $x^{w_1+u'} \in F$.

If in addition $x^w F \subset G$ it follows that $x^{w_1+w_2} F \subset F$. Since $x^{w_1+w_2}x^v, x^v \in F$ it follows that $w_1 + w_2 \in L \cap \mathbb{N}^n$. □

**Definition 3.2.** Let $F, G$ be $I$-fibers. We say that $F \equiv_I G$ if there exist $u, v \in \mathbb{N}^n$ such that $x^u F \subset G$ and $x^v G \subset F$.

It is immediate that $F \equiv_I G$ is an equivalence relation among the $I$-fibers. We denote the equivalence class of $F$ by $\overline{F}$. Thus

$$\overline{F} = \{ G : G \text{ is an } I \text{-fiber, } G \equiv_I F \} .$$

We note that $F \equiv_I G$ implies that the cardinality of $F$ is equal to the cardinality of $G$.

**Lemma 3.3.** If $L_{pure} = \{ 0 \}$ and $F$ is an $I$-fiber then $\overline{F} = \{ F \}$.

**Proof.** By Proposition 2.3, $|F| < \infty$. Let $G$ be an $I$-fiber, $G \equiv_I F$. There are $u, v \in \mathbb{N}^n$ such that $x^u F \subset G$ and $x^v G \subset F$. Since $|F| = |x^v F| = |G| = |x^v G|$ it follows that $x^x F = F$ and $x^y = x^z = 1$. □

Next we want to investigate the number of equivalent fibers inside each equivalence class when $L_{pure} \neq \{ 0 \}$ and $\sigma \neq \emptyset$. If $u \in \mathbb{Z}^n$ we let $u_\sigma = (u_i)_{i \in \sigma}$. If $s = |\sigma|$ we can assume that $u_\sigma \subset \mathbb{Z}^s$ and then consider the sublattice $(L_{pure})_\sigma$ of $\mathbb{Z}^s$ generated by the vectors $u_\sigma, u \in L_{pure}$. First we note the following:

**Remark 3.4.** Let $\sigma \neq \emptyset, F$ an $I$-fiber and $u \in \mathbb{N}^n$ such that $\text{supp}(u) \subset \sigma$. If $G$ is an $I$-fiber with the property $x^u F \subset G$ then $G \in F$.

**Proof.** Let $w \in L^+$ be such that $\text{supp}(w) = \sigma$. There exists $l \gg 0$ such that $lw - u \in \mathbb{N}^n$. Since $lw \in L$ it follows that $x^lw F \subset F$. Let $x^v \in G$ such that $x^v = x^w x^p$ for $x^p \in F$. It follows that $x^{lw-u} x^v = x^{lw} x^p \in F$ and thus $x^{lw-u} G \subset F$ by Lemma 3.1. □

**Proposition 3.5.** Let $L_{pure} \neq \{ 0 \}$ and $F$ an $I$-fiber. The cardinality of $\overline{F}$ is equal to $|\mathbb{Z}^s/(L_{pure})_\sigma|$, where $s = |\sigma|$.

**Proof.** For every $G \in \overline{F}$ choose $u_G \in \mathbb{N}^n$ such that $x^{u_G} F \subset G$. Let

$$\phi : \overline{F} \to \mathbb{Z}^s/(L_{pure})_\sigma, \quad \phi(G) = (u_G)_\sigma + (L_{pure})_\sigma .$$

The definition of $\phi$ is independent of the choice of $u_G$. Indeed suppose that $u, v \in \mathbb{N}^n$ are such that $x^u F \subset G$ and $x^v F \subset G$. This implies that $u - v \in L$. By Lemma 3.1 it follows that $u_\sigma = v_\sigma = 0$ and by Proposition 2.7 it follows that $u - v \in L_{pure}$ and $u_\sigma = v_\sigma \in (L_{pure})_\sigma$.

We will show that $\phi$ is a bijection: the only part needing proof is the surjection of $\phi$. Let $u' + (L_{pure})_\sigma$ be an element of $\mathbb{Z}^s + (L_{pure})_\sigma$. First we remark that we can assume without loss of generality that $u' \in \mathbb{N}^s$. Indeed, let $w \in L^+$ be such that $|\text{supp}(u_\sigma)| = s$. It is clear that for $l \gg 0, lw_\sigma + u' \in \mathbb{N}^s$ and thus $u' + (L_{pure})_\sigma = (lw_\sigma + u') + (L_{pure})_\sigma$. Let $u \in \mathbb{N}^n$ be such that $u_\sigma = 0, u_\sigma = u'$ and let $G$ be the $I$-fiber such that $x^u F \subset G$. By Remark 3.4 it follows that $G \in F$, and thus $\phi(G) = u' + (L_{pure})_\sigma$. □
Examples 3.6.
(a) We consider the lattice ideal \( I = \langle 1 - xy \rangle \subset \mathbb{k}[x, y] \) where \( L = \langle (1,1) \rangle \subset \mathbb{Z}^2 \).
There are infinitely many \( I \)-fibers: for any \( c \in \mathbb{Z} \) the set \( F_c = \{ x^i y^j : i - j = c \} \) is an \( I \)-fiber. All \( I \)-fibers are infinite and belong to the same equivalence class: the cardinality of this equivalence class is \( |\mathbb{Z}| \). Indeed \( L_{\text{pure}} = L \), and \( \mathbb{Z}^2/L \cong \mathbb{Z} \).

(b) If we consider the lattice ideal \( I = \langle 1 - xy, 1 - x^5 \rangle \subset \mathbb{k}[x, y] \), where \( L = \langle (1,1), (5,0) \rangle \subset \mathbb{Z}^2 \) then there are exactly five infinite \( I \)-fibers:
\[
F_k = \{ x^i y^j : i - j \equiv k \mod 5 \}, \quad 0 \leq k \leq 4
\]
which are all equivalent. Hence we have only one equivalence class \( \overline{F}_0 = \{ F_0, \ldots, F_4 \} \) which has five equivalent fibers. Indeed \( L_{\text{pure}} = L \) and \( \mathbb{Z}^2/L \cong \mathbb{Z}_5 \).

We define the relation “\( \leq_I \)” among the equivalence classes of \( I \)-fibers.

Definition 3.7. Let \( F, G \) be \( I \)-fibers. We say that \( \overline{F} \leq_I \overline{G} \) if there exists \( u \in \mathbb{N}^n \) such that \( x^u F \subset G \).
It is immediate that “\( \leq_I \)” is well defined and is a partial order among the equivalence classes of \( I \)-fibers. For simplicity of notation we occasionally write \( F \leq_I G \) if \( \overline{F} \leq_I \overline{G} \) and \( F <_I G \) if \( \overline{F} \leq_I \overline{G} \) and \( \overline{F} \neq \overline{G} \). We note that \( F_{\{1\}} \leq_I F \) for any \( I \)-fiber \( F \). We also remark that if \( L_{\text{pure}} = \{ 0 \} \) then \( <_I \) gives the ordering on the fibers of \( I \) induced by the \( \mathbb{Z}^n/L \)-degrees, see [4].

Theorem 3.8. Any descending chain of equivalence classes of \( I \)-fibers is finite.

Proof. Let \( \overline{F}_1 >_I \cdots >_I \overline{F}_k >_I \overline{F}_{k+1} >_I \cdots \) a chain of equivalence classes of fibers with no least element. Choose a representative \( F_i, i \in \mathbb{N} \) for each class. Next consider the corresponding ascending chain of monomial ideals: 
\[
M_{F_1} \subset \cdots \subset M_{F_k} \subset \cdots + M_{F_{k+1}} \subset \cdots
\]
The chain stabilizes at some step, say \( s \), so that 
\[
M_{F_1} + \cdots + M_{F_s} = M_{F_3} + \cdots + M_{F_{s+1}}.
\]
Let \( x^n \in G(M_{F_{s+1}}) \). By the above equality it follows that \( x^n \in M_{F_s} \) for some \( 1 \leq i < s+1 \) and \( x^n = x^a x^b \) where \( x^b \in G(M_{F_s}) \). Since \( x^a x^b \in F_{s+1} \) it follows that \( x^n F_i \subset F_{s+1} \). This leads to a contradiction since \( \overline{F}_{s+1} <_I \overline{F}_i \). \( \square \)

Definition 3.9. A minimal generating set of \( I \) of minimal cardinality is called a Markov basis of \( I \). Let \( F \) be an \( I \)-fiber. We say that \( F \) is a Markov \( I \)-fiber if there exists a Markov basis \( S \) for \( I \) such that \( \overline{F} = \overline{F}_B \) for some \( B \) in \( S \).

If \( L_{\text{pure}} = \{ 0 \} \) then \( I \) is a positively graded lattice ideal. This is the case dealt in [4] and [10]. In order to specify a Markov basis of \( I \), certain subideals of \( I \) were considered, one for each \( I \)-fiber \( F \). We generalize these constructions for arbitrary lattice ideals. Let \( F \) be an \( I \)-fiber. We let
\[
I_{< \overline{F}} = \langle B \in I : B \text{ binomial, } \overline{F}_B <_I \overline{F} \rangle
\]
and
\[
I_{\leq \overline{F}} = \langle B \in I : \overline{F}_B \leq_I \overline{F} \rangle.
\]
We note that $I_{\mathcal{F}} = 0$ if and only if there is no $I$-fiber $G$ such that $G <_1 \mathcal{F}$. It is clear that the definition of these ideals does not depend on the chosen fiber representative. Finally if $S$ is any subset of binomials of $I$ we let

$$S_{\mathcal{F}} = \{ B \in S : F_B \in \mathcal{F} \}.$$  

**Remark 3.10.** We will pay extra attention to the fiber that contains 1, $F_{\{1\}}$. Let $S$ be a set of binomials of $I$. According to the definitions

$$S_{\mathcal{F}_{\{1\}}} = \{ B \in S : F_B \in \mathcal{F}_{\{1\}} \} \text{ and } I_{pure} = I_{\leq \mathcal{F}_{\{1\}}}.$$  

We isolate the following proposition whose proof is within the proof of [14, Lemma A.1].

**Proposition 3.11.** Let $S$ be a minimal generating set of $I$, $F$ an $I$-fiber, $x^{w_1}, x^{w_2} \in F$. There exists a subset $T \subset S_{\mathcal{F}}$ such that

$$x^{w_1} - x^{w_2} = \sum_{i,B} \pm x^{a_i,B} B$$

where $B \in T$ may appear more than once, $a_{i,B} \in \mathbb{N}^n$ and $a_{i,B} \neq a_{j,B}$ for $i \neq j$.

The emphasis of the above statement is that when summing up and factoring out we get an expression

$$x^{w_1} - x^{w_2} = \sum f_i B_i$$

where $B_i \neq B_j$ for $i \neq j$ and all nonzero coefficients of the monomial terms of $f_i$ are $\pm 1$.

Next we describe the ideals $I_{< \mathcal{F}}$ and $I_{\leq \mathcal{F}}$ in terms of the generators of $I$.

**Proposition 3.12.** Let $S$ be a generating system of binomials for $I$. The following hold:

$$I_{< \mathcal{F}} = \langle B : B \in S, \mathcal{F}_B <_1 \mathcal{F} \rangle$$

and

$$I_{\leq \mathcal{F}} = \langle B : B \in S, \mathcal{F}_B \leq \mathcal{F} \rangle.$$  

**Proof.** We will show the statement for $I_{< \mathcal{F}}$, the other one having a similar proof. Let $J = \langle B : B \in S, \mathcal{F}_B <_1 \mathcal{F} \rangle$. We will show that $J = I_{< \mathcal{F}}$. It is clear that $J \subset I_{< \mathcal{F}}$. To show the other containment it is enough to show that if $B = x^u - x^v \in I_{< \mathcal{F}}$ then $B \in J$. Let $B = x^u - x^v \in I_{< \mathcal{F}}$. Since $B \in I$, by Proposition 3.11 it follows that $B = \sum_{i=1}^t \pm x^{a_i,B} B_i$ where $B_i \in S$ are not necessarily distinct while $a_{i,B_i} \neq a_{j,B_j}$ for $i \neq j$. We will do induction on $t$. Without loss of generality we can assume that $B_1 = x^{a_1} - x^{a_2}$ and $x^{a_1} x^{a_2} = x^u$, the other cases being done similarly. First we show the inductive step. Suppose that $t = 1$. Since $x^{a_1} x^{a_2} = x^u$ it follows that $x^{a_1} F_{B_1} \subset F_B$. Thus $F_{B_1} <_1 \mathcal{F}$. Since $F_B <_1 \mathcal{F}$ we see that $F_{B_1} <_1 \mathcal{F}$. Assume now that $t > 1$ and consider $B' = B - x^{a_1} B_1 = x^{a_1} x^{a_2} - x^u$. Since $F_{B'} = F_B$, it follows that $B' \in I_{< \mathcal{F}}$ and we are done by induction. $\square$

**Theorem 3.13.** Let $F$ be an $I$-fiber. $F$ is a Markov $I$-fiber if and only if $I_{< \mathcal{F}} \neq I_{\leq \mathcal{F}}$.

**Proof.** Let $S$ be a Markov basis of $I$. If $I_{< \mathcal{F}} \neq I_{\leq \mathcal{F}}$ then by Proposition 3.12 there exists a $B \in S$ such that $F_B \equiv_1 F$. For the converse assume that there exists $B \in S$ such that $\mathcal{F}_B = \mathcal{F}$. It follows immediately that $I_{< \mathcal{F}} \neq I_{\leq \mathcal{F}}$. $\square$
Corollary 3.14. The set of equivalence classes of Markov fibers of a lattice ideal $I$ is an invariant of $I$.

Theorem 4.14 strengthens the above result. For this a more detailed study of the $I$-fibers is needed, the scopus of section 4.

4. Generating sets of lattice ideals

First we consider the case where the lattice $L \subset \mathbb{Z}^n$ is generated by its pure elements. We call $L$ a pure lattice. We show how to obtain a generating set of $I_L$ of least cardinality $\mu(I_L)$, i.e. a Markov basis of $I_L$. Let $S = \{B_1, \ldots, B_r\}$ be a set of binomials of $I_L$. We say that $S'$ is a rearrangement of $S$ if there is a bijective function $f : S \rightarrow S'$ such that $f(B_i) = \pm B_j$. Compositions of rearrangements is a rearrangement. It is clear that if $S$ is a generating set of $I_L$ then all rearrangements of $S$ are generating sets of $I_L$. The theorem below generalizes Lemma 2.1 of [34].

Theorem 4.1. Let $L = L_{pure}$ be a pure lattice of rank $r$, $S$ a set of $r$ binomials of $I_L$, $\sigma = \sigma_L$. The set $S$ generates $I_L$ if and only if there is a rearrangement \( \{x^{u_1} - x^{v_1}, x^{w_2} - x^{v_2}, \ldots, x^{w_r} - x^{v_r}\} \) of $S$ such that the following three conditions are satisfied:

1. \( \{u_1 - v_1, \ldots, u_r - v_r\} \) is a basis of $L$,
2. for $i \in [r]$, $\text{supp}(u_i) \cup \text{supp}(v_i) \subseteq \sigma$,
3. $x^{v_1} = 1$ and $\text{supp}(v_i) \subseteq \bigcup_{j=1}^{r-1} \text{supp}(u_j)$ for $2 \leq i \leq r$.

Proof. Suppose first that $S = \{B_1, \ldots, B_r\}$ generates $I_L$. We let $B_j = x^{\beta_j} - x^{\gamma_j}$ for $j \in [r]$. Let $u \in L$. According to Proposition 3.1 there is an index set $A$ such that

$$x^{u^+} - x^{u^-} = \sum_{l \in A} \pm x^{a_l} (x^{\beta_l} - x^{\gamma_l}),$$

where $i_t \in [r]$, $a_t \in \mathbb{N}^n$. Expanding the RHS, equating the exponents of the equal monomial terms, subtracting the expressions for $u^+$ and $u^-$ and substituting $a_l$, $l \in A$, one gets that $u \in \mathbb{Z}(\beta_1 - \gamma_1) + \cdots + \mathbb{Z}(\beta_r - \gamma_r)$. This shows that $L = \mathbb{Z}(\beta_1 - \gamma_1) + \cdots + \mathbb{Z}(\beta_r - \gamma_r)$. Since $\text{rank}(L) = r$ it follows that $\{\beta_1 - \gamma_1, \ldots, \beta_r - \gamma_r\}$ is a basis of $L$. Next we remark that $\text{supp}(\beta_i) \subseteq \sigma$ if and only if $\text{supp}(\gamma_i) \subseteq \sigma$. Indeed this is immediate since $\text{supp}(\beta_i) \subseteq \sigma$. Now suppose that for some $i \in [r]$, $\text{supp}(\beta_i) \not\subseteq \sigma$. We claim that $B_i = x^{\beta_i} - x^{\gamma_i}$ is redundant in $S$ as a generator of $I_L$. For this we will show that if $u \in L$ then $x^{u^+} - x^{u^-}$ can be written as a linear combination of the elements of $S \setminus B_i$. Indeed consider again the relation

$$x^{u^+} - x^{u^-} = \sum_l \pm x^{a_l} (B_l),$$

of Proposition 3.1. Substitute the value 0 to any variable $x_j$ where $j \notin \sigma$: the terms in the above relation involving $B_i$ disappear. Thus $S \setminus B_i$ is a generating set of $I_L$. This is of course a contradiction since the height of $I_L$ is $r$, see [33] by the generalized Krull’s Principal Ideal Theorem. To show that there is a rearrangement of $S$ that satisfies the conditions of the theorem we notice that $S$ contains a binomial $B_j$ with 1 as one of its monomial terms. Indeed let $w \in L^+$ with $\text{supp}(w) = \sigma$, see Proposition 2.7. Since $x^{w-1} \in I_L$, $x^{w-1} = \sum \pm x^{a_l} (B_j)$, $l$. It is clear that there exists a value of $l$ such that a monomial term of $B_j$ is equal to 1 (and $a_l = 0$): otherwise $x^{w-1} \in \langle x_1, \ldots, x_n \rangle$, a contradiction. It is immediate that we can rearrange $S$ by a bijective function $f_1$ so that $f_1(B_i) = x^{a_l} - 1$. Next we claim that there is
B = x^β - x^γ \in S$ such that $B \neq B_i$ and $\text{supp}(\beta)$ or $\text{supp}(\gamma) \subset \text{supp}(u_1)$. Indeed, suppose not. Then clearly $\text{supp}(u_1) \neq \sigma$. Consider again the expression

$$x^w - 1 = \sum_{t \in A, i = i_t} x^{at} B_t + \sum_{t \in A, i \neq i_t} \pm x^{at} (B_t) .$$

Substitute the value 1 for all variables whose index is in $\text{supp}(u_1)$ and the value 0 for all other variables. We obtain a contradiction: $-1 = 0$. To avoid the contradiction there must be $B \in S$ so that $\pm B = x^{u_2} - x^{u_2}$ and $\text{supp}(v_2) \subset \text{supp}(u_1)$. We rearrange $f_1(S)$ by $f_2$ which keeps all elements of $f_1(S)$ fixed but $B$: $f_2(B) = x^{u_2} - x^{u_2}$. More generally once $f_s$ has been defined for $s < r$ so that the third condition is satisfied for all $i \leq s$, the same argument produces $f_{s+1}$ with the desired property.

We now prove the converse. Consider a set $S$ of binomials whose rearrangement \{x^{u_1} - 1, x^{u_2} - x^{u_2}, \ldots, x^{u_s} - x^{u_2}\} satisfies the three conditions of the theorem. Let $J = \langle x^{u_1} - 1, \ldots, x^{u_r} - x^{u_r} \rangle$. We will show that $J = I_L$. It is clear that $J \subset I_L$. Since $u_1, \ldots, u_r - v_r$ is a basis of $L$ and $\bigcup_{i=1}^r (\text{supp}(u_i) \cup \text{supp}(v_i)) \subset \sigma$ it is clear that $\bigcup_{i=1}^r (\text{supp}(u_i) \cup \text{supp}(v_i)) = \sigma$. By the third condition it follows that

$$\bigcup_{i=1}^r \text{supp}(u_i) = \sigma .$$

Next we will show that for every $k \in [r]$ there exists $w_k \in L^+$ such that $x^{w_k} - 1 \in J$ and $\text{supp}(w_k) = \bigcup_{j=1}^k \text{supp}(u_j)$. For $k = 1$ we set $w_1 = u_1$. Since $\text{supp}(v_2) \subset u_1$ there exists $\lambda_1 \in \mathbb{N}$, $\lambda_1 \gg 0$ such that $\lambda_1 w_1 > v_2$. We set $w_2 = (\lambda_1 w_1 - v_2) + u_2$; $\text{supp}(w_2) = \text{supp}(u_1) \cup \text{supp}(u_2)$. Moreover

$$x^{w_2} - 1 = x^{\lambda_1 w_1 - v_2}(x^{u_2} - x^{v_2}) + x^{\lambda_1 w_1} - 1 \in J ,$$

as wanted. It is clear that this construction generalizes for all $k \in [r]$. In particular $\text{supp}(w_r) = \sigma$.

We will now show that if $u - v \in L$ then $x^u - x^v \in J$ finishing the proof. Since $J : (x_1 \cdots x_n)^\infty = I_L$ there exists $w \in \mathbb{N}^n$ such that $x^w(x^u - x^v) \in J$. By the second condition it is clear that $w$ can be chosen so that $\text{supp}(w) \subset \sigma$. Since $\text{supp}(w_r) = \sigma$ there exists $\lambda \in \mathbb{N}$, $\lambda \gg 0$ such that $\lambda w_r > w$. Therefore $x^{\lambda w_r}(x^u - x^v) \in J$. It follows that

$$x^u - x^v = (x^{\lambda w_r} - 1)(x^v - x^u) - x^{\lambda w_r}(x^u - x^v) \in J$$

and consequently $I_L = J$. □

We remark that the binomials of a generating set of $I_L$ when $L = L_{pure}$ might have a common monomial factor according to Theorem 4.1. Note also that if $E = \{u_1, \ldots, u_r\}$ is a basis of $L$ such that $u_1 \in L^+$ and $\text{supp}(u_1) = \sigma$ then the set \{1 - x^{u_1}, x^{u_2} - x^{u_2}, \ldots, x^{u_r} - x^{u_r}\} is a Markov basis of $I_L$. The next corollary states that if $L = L_{pure}$, the ideal $I_L$ is always a complete intersection, see also [11]. In section 5 we determine when $I_L$ is a binomial complete intersection ideal for general lattices.

**Corollary 4.2.** Let $L = L_{pure}$ such that $\text{rank}(L) = r$. Then $\mu(I_L) = r$.

**Proof.** A basis $E$ as in the statement of Theorem 4.1 exists by Corollary 2.13. The conclusion follows immediately. □
To characterize the generating sets of lattice ideals we will use the criterion given in [35] and [15]. Let $L \subset \mathbb{Z}^n$ be a lattice, $I = I_L$, $\sigma = \sigma_L$, $S$ a subset of $I$ consisting of vectors of the form $x^u - x^v$ where $u \in L$. Let $F$ be an $I$-fiber. The sequence $(x^{a_1}, x^{a_2}, \ldots, x^{a_k})$ is an $S$-path from $x^u$ to $x^v$ if

- $x^{a_1} = x^u$, $x^{a_k} = x^v$
- for $j = 1, \ldots, k$ each $x^{a_j}$ in the sequence belongs to the fiber $F$ and
- $x^{a_j} - x^{a_{j+1}}$ is equal to $x^{u_j}B_j$ or $-x^{u_j}B_j$ for some $B_j \in S$, $u_j \in \mathbb{N}^n$.

**Theorem 4.3.** ([35] and [15]) The set $S$ of binomials of $I$ is a generating set of $I$ if and only if for every $I$-fiber $F$ there is an $S$-path between any two elements of $F$.

Let $F$ be an $I_L$-fiber and $G(M_F) = \{x^{a_1}, \ldots, x^{a_s}\}$. We define a relation “$\sim$” among the elements of $G(M_F)$ as follows:

$$x^{a_i} \sim x^{a_j} \text{ iff } (a_i + L^+) \bigcap (a_j + L^+) \neq \emptyset.$$  

We note that if $L^+ = \{0\}$ then $x^{a_i} \sim x^{a_j}$ only when $x^{a_i} = x^{a_j}$.

**Lemma 4.4.** “$\sim$” is an equivalence relation among the elements of $G(M_F)$.

**Proof.** It is enough to show transitivity. We can assume that $L_{pure} \neq \{0\}$, the other case being trivial. Suppose that

$$x^{a_i} \sim x^{a_j} \text{ and } x^{a_j} \sim x^{a_k}.$$  

Thus there exist $u_i, u_j, v_j, v_k \in L^+$ such that

$$a_i + u_i = a_j + u_j \text{ and } a_j + v_j = a_k + v_k.$$  

Therefore

$$a_i + (u_i + v_j) = a_k + (u_j + v_k),$$

and we are done. \hfill \Box

**Lemma 4.5.** Let $G(M_F) = \{x^{a_1}, \ldots, x^{a_s}\}$. The following holds for the elements of $G(M_F)$:

$$x^{a_i} \sim x^{a_j} \text{ if and only if } a_i^\sigma = a_j^\sigma.$$  

**Proof.** We can assume that $L^+ = \{0\}$, the other case being trivial. Let $w \in L^+$ such that $\text{supp}(w) = \sigma$. Suppose that $a_i^\sigma = a_j^\sigma$. Since $u = a_i - a_j \in L$ it follows that $w^\sigma = 0$ and thus $\text{supp}(u) \subset \sigma$. Therefore there exists $\lambda \gg 0$ such that $u + \lambda w \in \mathbb{N}^n$. Since $u + \lambda w \in L^+$ and $a_i + \lambda w = a_j + (u + \lambda w)$ it follows that $x^{a_i} \sim x^{a_j}$.

Suppose now that $x^{a_i} \sim x^{a_j}$. There exist $u_i, u_j \in L^+$ such that $a_i + u_i = a_j + u_j$.

Therefore

$$a_i^\sigma = (a_i + u_i)^\sigma = (a_j + u_j)^\sigma = a_j^\sigma.$$  

\hfill \Box

**Lemma 4.6.** Let $G(M_F) = \{x^{a_1}, \ldots, x^{a_s}\}$. The following holds for the elements of $G(M_F)$:

$$x^{a_i} \not\sim x^{a_j} \text{ if and only if } a_i^\sigma, a_j^\sigma \text{ are incomparable.}$$

**Proof.** We will show that $x^{a_i} \not\sim x^{a_j}$ implies that $a_i^\sigma$ and $a_j^\sigma$ are incomparable. Without loss of generality we can assume that $a_i^\sigma - a_j^\sigma > 0$. Let $u = a_i - a_j$. Let $w \in L^+$ such that $\text{supp}(w) = \sigma$. We can find $\lambda \in \mathbb{N}$ large enough so that $(u + \lambda w)_i > 0$ for $i \in \sigma$. Since $\lambda w^\sigma = 0$ and $u^\sigma > 0$ it follows that $u + \lambda w > 0$ and thus $u + \lambda w \in L^+$. Since $a_i + \lambda w = a_j + (u + \lambda w)$ and
is infinite when Lemma 4.7.

Let $G$ and $-x$ of $I$ construct a graph $Γ$. Note that the cardinality of $M$ might be less than the cardinality of $G(M_F)$. In particular

$$G(M_F)^\sigma = \{x^u^\sigma : x^u \in G(M_F)\}.$$

Note that the cardinality of $G(M_F)^\sigma$ might be less than the cardinality of $G(M_F)$. 

**Lemma 4.7.** Let $F, F'$ be two equivalent fibers. Then $G(M_F)^\sigma = G(M_{F'})^\sigma$.

**Proof.** Since $F, F'$ are equivalent $I$-fibers, there exist monomials $x^a, x^b$ such that $x^a F \subset F'$ and $x^b F' \subset F$. Therefore $x^{a+b} F \subset F$ and $x^u \in L^\sigma$. Since $u, v \in \mathbb{N}^n$ it follows that $\text{supp}(u), \text{supp}(v) \subset \sigma$. Let $G(M_F) = \{x^{a_1}, \ldots, x^{a_s}\}$, $G(M_F') = \{x^{b_1}, \ldots, x^{b_r}\}$. To show the desired equality it suffices to show that for any $i \in [s]$ there is $j \in [r]$ so that $a_i^\sigma = b_j^\sigma$, the other inclusion being taken care by symmetry.

Since $x^a = x^b$ is in $F'$ there exists $j \in [r]$ such that $x^{b_j}$ divides $x^a$. Therefore $a_i + u - b_j \in \mathbb{N}^n$. Since $\text{supp}(u) \subset \sigma$ it follows that

$$(a_i + u - b_j)^\sigma = a_i^\sigma - b_j^\sigma \geq 0$$

and $a_i^\sigma \geq b_j^\sigma$. Similarly there exists $k \in [s]$ such that $b_j^\sigma \geq a_k^\sigma$. Therefore $a_i^\sigma \geq b_j^\sigma \geq a_k^\sigma$, and $a_i^\sigma \geq a_k^\sigma$. It follows that $a_i^\sigma = b_j^\sigma$ and thus $a_i^\sigma = b_j^\sigma$. □

We also note the following: let $F$ be an $I$-fiber and let $x^u \in F$. Even though $F$ is infinite when $L_{\text{pure}} \neq \{0\}$, we claim that $x^u$ takes only a finite number of values. The number of values it takes is equal to the cardinality of $G(M_F)^\sigma$.

Indeed, suppose that $G(M_F) = \{x^{a_1}, \ldots, x^{a_s}\}$. Since $x^u \in M_F$ it follows that $x^u$ is divisible by $x^{a_i}$ for some $i \in [s]$. Thus $u - a_i \in \mathbb{N}^n$. Since $u - a_i \in L$ it follows that $u - a_i \in L^\sigma$. Thus $(u - a_i)^\sigma = 0$. It follows that $u^\sigma - a_i^\sigma = 0$ and $x^u^\sigma \in G(M_F)^\sigma$.

Let $F$ be any $I$-fiber. We construct a graph $G_{\mathcal{F}}$ and then we build on $G_{\mathcal{F}}$ to construct a graph $\Gamma_{\mathcal{F}}$ that will be crucial in determining when a set of binomials of $I$ generates $I_{x^u}$.

**Definition 4.8.** Let $F$ be an $I$-fiber, $G(M_F)^\sigma = \{x^{a_1}, \ldots, x^{a_s}\}$ where $x^{a_i} \in G(M_F)$ for $i \in [k]$. We define $G_{\mathcal{F}} = (V(G), E(G))$ to be the graph with $V(G) = [k]$, and

$$E(G) = \{\{i, j\} : \exists x^{u_i}, x^{u_j} \in F \text{ such that } u_i^\sigma = a_i^\sigma, u_j^\sigma = a_j^\sigma, x^{u_i} - x^{u_j} \in I_{x^u}\}.$$

The graph $G_{\mathcal{F}}$ is independent of the fiber representative $F$. This is clear for $V(G)$, by Lemma 4.7. Next we show the independence for $E(G)$. Suppose that $x^{u_i} - x^{u_j} \in I_{x^u}$ where $x^{u_i}, x^{u_j} \in F$ and $u_i^\sigma = a_i^\sigma, u_j^\sigma = a_j^\sigma$. Let $F' \in \mathcal{F}$ and let $x^{u_i}, x^{u_j}$ be such that $x^{u_i} F \subset F'$ and $x^{u_j} F' \subset F$. By Lemma 3.1 $\text{supp}(u_i) \cup \text{supp}(u_j) \subset \sigma$ and thus $u^\sigma = u^\sigma \sigma = 0$. Moreover $x^{u_i} x^{u_j} \in I_{x^u}$, $(u_i + u_j)^\sigma = u^\sigma + u_j^\sigma = u_i^\sigma + u_i^\sigma = a_i^\sigma$, $(u_i + u_j)^\sigma = a_i^\sigma$ and thus $E(G)$ is independent on the choice of the fiber representative $F$.

**Definition 4.9.** We let $\Gamma_{\mathcal{F}}$ to be the complete graph whose vertices are the connected components of $G_{\mathcal{F}}$. Let $B = x^u - x^v \in I$ such that $F_B \notin \mathcal{F}$. We identify $B$ with an edge of $\Gamma_{\mathcal{F}}$ if $x^u^\sigma \neq x^v^\sigma$, $B \in I_{x^u}$ and $B \notin I_{x^v}$. We note that different binomials might correspond to the same edge of $\Gamma_{\mathcal{F}}$. For a subset $S$ of binomials
of $I$ we denote by $\Gamma_{\mathcal{I}}(S)$ the subgraph of $\Gamma_{\mathcal{I}}$ induced by the binomials $B \in S$ such that $F_B \in \mathcal{I}$.

**Lemma 4.10.** Let $L \subset \mathbb{Z}^n$ be a lattice, $I = I_L$, $S$ a binomial subset of $I$ consisting of binomials so that $I_{L_{pure}} = (S_{\mathcal{I}_I})$ and $\Gamma_{\mathcal{I}}(S)$ is a spanning tree of $\Gamma_{\mathcal{I}}$ for every $I$-fiber $F$. Then the set $S$ is a generating set of $I$.

**Proof.** We will show that for any $I$-fiber $F$ and any $x^u, x^v \in F$ there is an $S$-path between $x^u, x^v$. This is clear if $F \in \mathcal{I}_{\{1\}}$. By Theorem 4.3 we can assume that there is an $S$-path between any two elements of $G$ for all $G$ such that $\mathcal{I} < \mathcal{F}$. We note that $u - v \in L$. Suppose that $G(M_F)^{\sigma} = \{x^{a_1}, \ldots, x^{a_k}\}$ where $x^{a_1}, \ldots, x^{a_k} \in G(M_F)$. We examine three cases.

1. If $u^\sigma = v^\sigma$ then $(u - v)^\sigma = 0$ and by Corollary 4.8 it follows that $u - v \in I_{pure}$. Therefore $x^u - x^v \in I_{pure}$ and since $I_{L_{pure}} = (S_{\mathcal{I}_I})$ it follows that $x^u - x^v \in \langle S \rangle$.

2. Suppose that $u^\sigma \neq v^\sigma$ and that the vertices of $G_{\mathcal{F}}$ corresponding to $u^\sigma$ and $v^\sigma$ are in the same connected component of $G_{\mathcal{I}}$. Assume that $u^\sigma = a_1^\sigma$ and $v^\sigma = a_j^\sigma$ and that $i = i_1, \ldots, i_l = j$ is a path in $G_{\mathcal{I}}$. By applying induction on $l$ it is enough to prove the statement when $l = 2$ and $\{i, j\}$ is an edge of $G_{\mathcal{I}}$. It follows from the definition of $G_{\mathcal{I}}$ that there exists a binomial $x^w - x^z \in I_{\mathcal{I}}$ such that $w^\sigma = u^\sigma$, $z^\sigma = v^\sigma$. Moreover $x^w, x^z \in G$ where $\mathcal{I} < \mathcal{F}$.

3. Suppose that $u^\sigma \neq v^\sigma$ and that the vertices of $G_{\mathcal{I}}$ corresponding to $u^\sigma$ and $v^\sigma$ are in disconnected components of $G_{\mathcal{I}}$. Since $S$ determines a spanning tree of $\Gamma_{\mathcal{I}}$ there is a series of edges in $\Gamma_{\mathcal{I}}$ that leads from the component that corresponds to $x^w$ to the component that corresponds to $x^v$. As before it is enough to prove the statement when the components are adjacent. This means that there exists a binomial $B = x^{u'} - x^{v'} \in (I_{\mathcal{I}} \setminus I_{\mathcal{F}}) \cap S$ such that $u'^\sigma, v'^\sigma$ correspond to the same connected component of $G_{\mathcal{I}}$ and similarly for $v'^\sigma, v^\sigma$. The monomials $x^{u'}, x^{v'}$ of $B$, belong to a fiber equivalent to $F$. It follows that there is $b \in \mathbb{N}^n$ such that $x^{u' + b}$ and $x^{v' + b}$ belong to $F$ and thus the sequence $(x^{u' + b}, x^{v' + b})$ is an $S$-path from $x^{u' + b}$ to $x^{v' + b}$. By case (2) above, there is an $S$-path from $x^u$ to $x^{u' + b}$ and an $S$-path from $x^v$ to $x^{v' + b}$. Joining these paths one gets an $S$-path from $x^u$ to $x^v$.

We let $t(\mathcal{F})$ denote the number of vertices of $\Gamma_{\mathcal{I}}$. Thus

$$t(\mathcal{F}) := |V(\Gamma_{\mathcal{I}})|.$$

We note that to construct a spanning tree of $\Gamma_{\mathcal{I}}$ we need exactly $t(\mathcal{F}) - 1$ binomials. To prove the next theorem we will use Theorem 4.3

**Theorem 4.11.** Let $L \subset \mathbb{Z}^n$ be a lattice, $I = I_L$, $S$ a subset of $I$ consisting of binomials. The set $S$ is a Markov basis of $I$ if and only if the following conditions are satisfied:
• $\Gamma F(S)$ is a spanning tree of $\Gamma F$ for every $I$-fiber $F$ and $|S F| = t(F) - 1$,
• $|S F_{(1)}| = \text{rank}(L_{\text{pure}})$ and
• $I_{L_{\text{pure}}} = \langle S F_{(1)} \rangle$.

Proof. Suppose that $S$ is a Markov basis of $I$. We will show that $S$ satisfies the three conditions of the theorem. We note that since $S$ is a generating set of $I$, by Proposition 3.12 and Remark 3.10 it follows that $\langle S F_{(1)} \rangle = I_{L_{\text{pure}}}$. Moreover if $|S F_{(1)}| > \text{rank}(L_{\text{pure}})$, then by Theorem 4.1 one can replace the binomials in $S F_{(1)}$ by a Markov basis of $I_{L_{\text{pure}}}$.

Next we show that for an arbitrary $I$-fiber $F$, $\Gamma F(S)$ is a spanning tree of $\Gamma F$. Indeed, by Theorem 4.3, $S$ induces a spanning graph in $F$. Since $G F$ comes from $F$ by identifying components and similarly for $\Gamma F$ from $G F$, it follows that $\Gamma F(S)$ is a spanning graph of $\Gamma F$. We show that $\Gamma F(S)$ is a tree of $\Gamma F$. If not $\Gamma F(S)$ has a cycle in $\Gamma F$. We omit from $S$ the binomial that induces an edge on this cycle. The resulting set still satisfies the conditions of Lemma 4.10 and is thus a generating set of $I$ of smaller cardinality, a contradiction. Similarly if $|S F| > t(F) - 1$ there is a binomial in $S F$ that does not correspond to an edge of $\Gamma F(S)$ or two binomials that correspond to the same edge. Then one binomial could be omitted from $S$ and the resulting set would still be a generating set of $I$.

Conversely let $S$ be a set that satisfies the three conditions of the theorem. By Lemma 4.10, $S$ is a generating set of $I$. Suppose that there is a Markov basis $S'$ of $I$ such that $|S'| < |S|$. Let $F$ be a Markov fiber such that $|S' F| < |S F|$. We note that $F \notin F_{(1)}$ since $|S F_{(1)}| \geq \text{rank}(L_{\text{pure}}) = |S F_{(1)}|$. Moreover $|S F| = t(F) - 1 = |S' F|$, thus $|S F| = |S' F|$, a contradiction.

□

Remark 4.12. When $L_{\text{pure}} = \{0\}$ then the construction and conditions on $\Gamma F$ coincide with the construction and conditions of $G(M_F) = F$.

We remark that for all but finitely many equivalence classes of fibers $F$, $t(F) = 1$. Indeed by Corollary 3.14 the set consisting of equivalence classes of Markov I-fibers is finite. If an I-fiber $F$ is not a Markov fiber, then by Theorem 3.13 it follows that $I_{\text{pure}} = I_{\text{pure}}$ and hence $G F$ consists of only one connected component. The next result is the main theorem of this section. Its proof is an immediate consequence of Theorem 4.11.

Theorem 4.13. Let $L \subset \mathbb{Z}^n$ be a lattice. Then

$$\mu(I_L) = \text{rank}(L_{\text{pure}}) + \sum_{F \neq F_{(1)}} (t(F) - 1),$$

where the sum runs over all distinct equivalence classes of Markov fibers.

The next corollary follows from Corollary 3.14 and Theorem 4.11. It generalizes the corresponding result for positively graded lattices, see [10].

Corollary 4.14. Let $L \subset \mathbb{Z}^n$ be a lattice, $\mu = \mu(I_L)$, $\{B_1, \ldots, B_\mu\}$ a Markov basis of $I_L$. The multiset

$$\{F B_1, \ldots, F B_\mu\}$$

is an invariant of $I_L$. 

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The following result concerns an arbitrary minimal generating set of $I_L$: the play in the cardinality of such a set only concerns the pure part of $L$. The proof of Corollary 4.15 follows directly from the proof of Theorem 4.11.

**Corollary 4.15.** Let $L$ be a lattice and $S = \{B_1, \ldots, B_t\}$ a minimal generating set of $I_L$. The multiset

$$\{ \overline{F}_{B_i} : F_{B_i} \notin \overline{F}_{\{1\}} \}$$

is an invariant of $I_L$.

Next we examine the binomials that appear in every Markov basis of $I$.

**Definition 4.16.** A binomial is called *indispensable* if it appears in every Markov basis of $I$ up to a constant multiple. A monomial $x^u$ is called *indispensable* if for every Markov basis $S$ of $I$ there is a binomial $B \in S$ so that $x^u$ is a monomial term of $B$.

If $\text{rank}(L_{\text{pure}}) = 0$, the characterization of indispensable binomials is given by [4, Corollary 2.10], see also [4, Theorem 3.4].

**Theorem 4.17.** Let $L$ be a lattice. If $\text{rank}(L_{\text{pure}}) > 1$ then there are no indispensable binomials and only one indispensable monomial, $x^0$. If $\text{rank}(L_{\text{pure}}) = 1$ there exists exactly one indispensable binomial and exactly two indispensable monomials.

**Proof.** Let $L$ be a lattice such that $\text{rank}(L_{\text{pure}}) = r \geq 1$, $S$ a Markov basis of $I$. If $L_{\text{pure}}$ has rank 1 then $L_{\text{pure}} = \langle u \rangle$, where $u \in L^+$ is $L$-primitive and by Theorem 4.1 we have that $S_{\text{pure}} = \{ x^u - 1 \}$.

Suppose that $L_{\text{pure}}$ has rank $r > 1$. Without loss of generality we can assume that $S_{\text{pure}} = \{ x^{u_1} - 1, \ldots, x^{u_r} - x^{v_r} \}$ satisfies directly the three conditions of Theorem 4.1. For every $i \geq 2$ let $u'_i = u_i + u_1$ and $v'_i = v_i + u_1$ and note that $u'_i - v'_i = u_i - v_i$. By Theorem 4.1 it follows that $\{ x^{u_1} - 1, x^{u_i} - x^{v_i}, \ldots, x^{v_r} - x^{v_r} \}$ is also a Markov basis of $I_{\text{pure}}$. Since $\text{rank}(L_{\text{pure}}) > 1$ there are infinitely many $L$-primitive elements of full support and thus infinitely many bases of $L_{\text{pure}}$ as in Corollary 2.13. By applying the above argument to a rearrangement of any of these bases we conclude that there are no indispensable binomials for $I_{\text{pure}}$. The only indispensable monomial of $L_{\text{pure}}$ is $1 = x^0$.

Finally we show that if there is a Markov fiber $F$ such that $\overline{F} \supset 1$ then there are infinitely many distinct choices for the binomials that determine any edge in a spanning tree of $\Gamma_F$. Since $F$ is a Markov fiber, Theorem 4.13 says that $L_{\text{pure}} = I_{\text{pure}}$. Thus there exists $B = x^u - x^v$, such that $F_B \notin \overline{F}$ and $B \notin I_{\text{pure}}$. We note that $u^* \neq v^*$: otherwise $u - v \in L_{\text{pure}}$ and $x^u - x^v \in I_{\text{pure}} \subset I_{\text{pure}}$, a contradiction. Thus $B$ produces an edge in $\Gamma_F$ and can be made part of a Markov basis $S$ of $I$. Let $w_1, w_2 \in L^+$. Then $B' = x^{u+w_1} - x^{u+w_2}$ gives exactly the same edge as $B$ and can replace $B$ in $S$. Therefore, applying Theorem 4.1 we obtain that there are no indispensable binomials of $I$, but there is exactly one indispensable monomial of $I$, that is $1 = x^0$.

We isolate the following result which follows from the proof of Theorem 4.17.

**Theorem 4.18.** Let $L$ be a lattice. If $\text{rank}(L_{\text{pure}}) > 1$ or $\text{rank}(L_{\text{pure}}) = 1$ and $L \neq L_{\text{pure}}$ then the Universal Markov basis of $I_L$ is infinite.

We note that for the lattices $L$ from Theorem 4.18 the Universal Markov basis of $I_L$ is not contained in the Graver basis, since the Graver basis is finite (see [22]).
5. Binomial Complete Intersection Lattice Ideals

In this section we determine binomial complete intersection lattice ideals. This is a problem that engaged mathematicians starting in 1970, see [14]. We recall that \( \sigma = \sigma_L \) is the maximum support of an element of \( L \cap \mathbb{N}^n \) and that \( L_{\text{pure}} \) is the sublattice of \( L \) generated by the elements of \( L \cap \mathbb{N}^n \). By \( L^\sigma \) we mean the sublattice of \( (\mathbb{N}^n)^\sigma \) generated by the vectors \( u^\sigma \) where \( u \in L \). We first show that the lattice ideal of \( L^\sigma \) is positively graded.

**Remark 5.1.** \( L^\sigma \cap (\mathbb{N}^n)^\sigma = \{0\} \).

**Proof.** Let \( w \in L \cap \mathbb{N}^n \) such that \( \text{supp}(w) = \sigma \). If \( 0 \neq u^\sigma \in L^\sigma \cap (\mathbb{N}^n)^\sigma \) then there exists \( k \in \mathbb{N}, k \gg 0 \) such that \( u^\sigma = u + kw \in L \cap \mathbb{N}^n \). Thus \( \sigma \subseteq \text{supp}(u^\sigma) \), a contradiction. \( \Box \)

Next we show that the rank of \( L \) is determined by the ranks of its sublattice \( L_{\text{pure}} \) and the lattice \( L^\sigma \).

**Proposition 5.2.** Let \( L \subset \mathbb{Z}^n \) be a lattice. Then
\[
\text{rank}(L) = \text{rank}(L^\sigma) + \text{rank}(L_{\text{pure}}).
\]

**Proof.** Let \( \mathcal{B}_1 = \{u_1, \ldots, u_k\} \subset L \) be such that \( \mathcal{B}_1' = \{u_1^\sigma, \ldots, u_k^\sigma\} \) is a basis of \( L^\sigma \) and let \( \mathcal{B}_2 = \{v_1, \ldots, v_r\} \) be a basis of \( L_{\text{pure}} \). We will prove our statement by showing that \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \) is a basis of \( L \). In order to prove this we first show that \( \mathcal{B} \) is linearly independent. Indeed, let \( \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_r \) be integers such that
\[
\alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_r v_r = 0.
\]
Then
\[
\alpha_1 u_1^\sigma + \cdots + \alpha_k u_k^\sigma + \beta_1 v_1^\sigma + \cdots + \beta_r v_r^\sigma = 0,
\]
and since \( v_i^\sigma = 0 \) for \( i \in [r] \) it follows that
\[
\alpha_1 u_1^\sigma + \cdots + \alpha_k u_k^\sigma = 0.
\]

Since \( \mathcal{B}_1' \) is a basis of \( L^\sigma \) it follows that \( \alpha_i = 0 \) for \( i \in [k] \). Hence \( \beta_1 v_1 + \cdots + \beta_r v_r = 0 \) and thus \( \beta_j = 0 \) for \( j \in [r] \) since \( \mathcal{B}_2 \) is a basis of \( L_{\text{pure}} \).

It remains to show that \( \mathcal{B} \) is a system of generators for \( L \). Let \( v \in L \) be an arbitrary vector. Then \( v^\sigma = \lambda_1 u_1^\sigma + \cdots + \lambda_k u_k^\sigma \), where \( \lambda_i \in \mathbb{Z} \) for \( i \in [k] \). Consider
\[
u = v - \lambda_1 u_1 - \cdots - \lambda_k u_k.
\]
Since \( u^\sigma = 0 \) it follows that \( \text{supp}(u) \subset \sigma \). By Proposition 2.8 it follows that \( u \in L_{\text{pure}} \). Consequently \( v \in \langle \mathcal{B} \rangle \). \( \Box \)

We consider the lattice ideal \( I_{L^\sigma} \) in \( R^\sigma := \mathbb{k}[x_i : i \notin \sigma] \). We show that in order to compute the \( I_{L^\sigma} \)-fibers, it is enough to consider the generating sets of the corresponding \( I_L \)-fibers.

**Lemma 5.3.** Let \( u \in \mathbb{N}^n \). Then the \( I_{L^\sigma} \)-fiber of \( u^\sigma \) is \( G(M_{F_u})^\sigma \).

**Proof.** Let \( u' = u^\sigma \in (\mathbb{N}^n)^\sigma \) and denote by \( F' \) the \( I_{L^\sigma} \)-fiber of \( u' \). It follows from Remark 5.1 that \( F' \) is finite. We will show that \( F' = F_u^\sigma \) and thus by Proposition 2.4 we obtain \( F' = G(M_{F_u})^\sigma \), since for any vector \( t \in L^+ \) we have \( t^\sigma = 0 \). To prove this, consider first an element \( v \in F_u \). Then \( v - u \in L \) and \( v^\sigma - u^\sigma \in L^\sigma \). Hence \( v^\sigma \in F' \) and we obtain the inclusion \( F' \supseteq F_u^\sigma \). For the converse inclusion, let \( v' \in F' \). Then \( u' - v' \in L^\sigma \) and it follows that there exists a vector \( w \in L \) such
that \( u' - v' = w' \). Therefore \( v' = (u - w)' \). Since \( u - (u - w) = w \in L \) we obtain that \( u - w \in F_u \) and \( v' \in F'_u \), as desired. \( \square \)

Next we show that the cardinality of a Markov basis of \( I_L \) depends on the cardinality of a Markov basis of \( I_{L^\sigma} \).

**Theorem 5.4.** Let \( L \) be a lattice. Then

\[
\mu(I_L) = \mu(I_{L^\sigma}) + \text{rank}(L_{\text{pure}})
\]

**Proof.** If \( F \neq F_{\{1\}} \) then by Lemma 5.3 the graphs \( \Gamma_{F_u} \) and \( \Gamma_{F'_u} \) are equal. The theorem follows from Theorem 4.13 \( \square \)

The following theorem follows directly from Proposition 5.2 and Theorem 5.4 and determines the binomial complete intersection lattice ideals. It is the main theorem of this section.

**Theorem 5.5.** Let \( L \subset \mathbb{Z}^n \) be a lattice. The ideal \( I_L \) is binomial complete intersection if and only if \( I_{L^\sigma} \) is complete intersection.

We can describe the lattices for which \( I_L \) is a binomial complete intersection. Recall that a mixed dominating matrix \( M \) has the property that every row of \( M \) has a positive and negative entry and \( M \) contains no square submatrix with this property.

**Corollary 5.6.** Let \( L \subset \mathbb{Z}^n \) be a lattice. The ideal \( I_L \) is binomial complete intersection if and only if there exists a basis of \( L \) whose vectors give the rows of 

\[
\begin{bmatrix} A & M \\ C & 0 \end{bmatrix}
\]

where \( A \in \mathcal{M}_{(r - r^+) \times |\sigma|}(\mathbb{Z}) \), the matrix \( M \in \mathcal{M}_{(r - r^+) \times (n - |\sigma|)}(\mathbb{Z}) \) is mixed dominating, the matrix \( C \in \mathcal{M}_{r^+ \times |\sigma|}(\mathbb{Z}) \) is a matrix whose rows satisfy the conditions of Theorem 4.11 and \( 0 \) is the zero matrix. For example, any matrix with entries in \( \mathbb{N} \) and independent rows has the desired property for \( C \) above, while for a mixed dominating matrix \( M \) one can use [23, Remark 3.17] or [12, Theorem 2.2]. By working with the appropriate size matrices one can easily obtain a class of lattice ideals that are binomial complete intersections.

**Remark 5.7.** Let \( L \subset \mathbb{Z}^n \) be a lattice, \( r = \text{rank}(L) \), \( r^+ = \text{rank}(L_{\text{pure}}) \), \( \sigma = \sigma_L \). The previous corollary states that \( I_L \) is binomial complete intersection if and only if there is a basis of \( L \) whose vectors give the rows of 

\[
\begin{bmatrix} A & M \\ C & 0 \end{bmatrix}
\]

where \( A \in \mathcal{M}_{(r - r^+) \times |\sigma|}(\mathbb{Z}) \), the matrix \( M \in \mathcal{M}_{(r - r^+) \times (n - |\sigma|)}(\mathbb{Z}) \) is mixed dominating, the matrix \( C \in \mathcal{M}_{r^+ \times |\sigma|}(\mathbb{Z}) \) is a matrix whose rows satisfy the conditions of Theorem 4.11 and \( 0 \) is the zero matrix. For example, any matrix with entries in \( \mathbb{N} \) and independent rows has the desired property for \( C \) above, while for a mixed dominating matrix \( M \) one can use [23, Remark 3.17] or [12, Theorem 2.2]. By working with the appropriate size matrices one can easily obtain a class of lattice ideals that are binomial complete intersections.

**6. Example**

Let \( L \) be the sublattice of \( \mathbb{Z}^5 \) generated by the vectors \( v_1 = (3,0,1,-1,0) \), \( v_2 = (0,1,6,0,-1) \), \( v_3 = (1,1,0,0,0) \) and \( v_4 = (5,0,0,0,0) \). Let \( v_5 = (0,5,0,0,0) \). It is not hard to show that

\[
L^+ = L \cap \mathbb{N}_0^5 = \mathbb{N}_0v_3 + \mathbb{N}_0v_4 + \mathbb{N}_0v_5
\]

while

\[
L_{\text{pure}} = \mathbb{Z}v_3 + \mathbb{Z}v_4.
\]
Thus \( \text{rank}(L_{\text{pure}}) = 2 \), \( \sigma := \sigma_L = \{1, 2\} \) and according to Theorem 4.11
\[
I_{L_{\text{pure}}} = \langle 1 - x_1^5, 1 - x_1x_2 \rangle .
\]
To find a generating set of \( I_L \) in \( R = k[x_1, \ldots, x_5] \) we compute the ideal
\[
\langle x_4 - x_1^3x_3, x_5 - x_2x_4^3, 1 - x_1x_2, 1 - x_1^5 \rangle : (x_1 \cdots x_5)^\infty
\]
using CoCoA. It turns out that \( I_L \) is generated by \( \{B_1, \ldots, B_{16}\} \) where
\[
B_1 = x_3^2 - x_1^2, \quad B_2 = x_1^2 - x_2^2, \quad B_{14} = x_1x_2 - 1, \\
B_3 = x_1^4x_4 - x_3, \quad B_6 = x_2x_3 - x_1x_4, \quad B_{15} = x_2^2x_4 - x_1x_3, \quad B_{16} = x_1^2x_3 - x_2x_4 \\
B_4 = x_2x_3^2 - x_3^2, \quad B_5 = x_1x_2^3 - x_4^3, \quad B_7 = x_1x_4^3 - x_3^3, \quad B_8 = x_3^6 - x_4^4, \\
B_9 = x_3^2x_4 - x_2x_5, \quad B_{10} = x_3x_4^2 - x_2^5, \quad B_{11} = x_3^3x_4^2 - x_2x_5, \quad B_{12} = x_3^3 - x_2x_5, \\
B_{13} = x_3x_4^5 - x_1x_5.
\]
(We wrote the binomials in the order they appear in CoCoA).

Let \( F \) be an \( L \)-fiber. Since \( |Z^2/(L_{\text{pure}})| = 5 \) it follows by Proposition 3.5 that \( F \) consists of 5 equivalent fibers. In particular let \( F_{(1)} \) be the fiber that contains the identity. As in Example 3.6(b) we see that
\[
F_{(1)} = \{F_{x_1}, F_{x_1^2}, F_{x_1^3}, F_{x_1^4}\}
\]
where for each \( 0 \leq k \leq 4 \), the fiber \( F_{x_1^k} \) consists of the monomials \( x_1^k x_2^j \) with \( i - j \equiv k \mod 5 \). Thus if \( x^n \in R \) then
\[
F_{x_1^n} = \{F_{x_1^n}, F_{x_2^n}, F_{x_3^n}, F_{x_4^n}, F_{x_5^n}\}.
\]
Of the generators \( B_i \) of \( I_L \) we notice that
\[
\bullet \quad F_{B_1}, F_{B_2}, F_{B_{14}} \in F_{(1)} \\
\bullet \quad F_{B_3}, F_{B_6}, F_{B_{15}}, F_{B_{16}} \in F_{x_1} \\
\bullet \quad F_{B_4}, F_{B_5} \in F_{x_2} \\
\bullet \quad F_{B_9} \in F_{x_3} \\
\bullet \quad F_{B_{10}} \in F_{x_4} \\
\bullet \quad F_{B_{11}} \in F_{x_5}
\]
It is clear that the above equivalence classes of fibers are pairwise distinct: use Lemma 3.3 and notice that there is no \( w \in N^5 \) with \( \supp(w) \subset \sigma \) such that \( w + (0, 0, 0, k, 0) \in L \). It is easy to see that
\[
F_{(1)} \subset F_{x_1} \subset F_{x_2} \subset F_{x_3} \subset F_{x_4} \subset F_{x_5}.
\]
We also note that \( F_{x_4^6} = F_{x_5} \). For \( i \in \{6\} \) we compute \( I\langle L_{\text{pure}} \rangle \) with the use of CoCoA and apply Theorem 3.13 to conclude that \( F_{x_1}, F_{x_4} \) and \( F_{x_5} \) are Markov fibers. Next we will show how to obtain a Markov basis \( S \) of \( I_L \) using the generating set \( \{B_1, \ldots, B_{16}\} \).

According to Theorem 4.11 \( |S_{F_{(1)}}| = \text{rank}(L_{\text{pure}}) = 2 \) and \( S_{F_{(1)}} \) must generate \( I_{L_{\text{pure}}} = \langle 1 - x_1^5, 1 - x_1x_2 \rangle \). We note that there are infinitely many choices for binomials \( B, B' \) that generate \( I_{L_{\text{pure}}} \); see Theorem 2.12 Theorem 4.11 and Proposition 2.10. We remark that the multiset \( \{F_{B}, F_{B'}\} \) equals \( \{F_{(1)}, F\} \) where \( F \) can be any of the five fibers of \( F_{(1)} \).

Consider now the fiber \( F_{x_4} \). It is an easy exercise that
\[
G(M_{F_{x_4}}) = \{x_2^2x_3, x_1^3x_3, x_4\}.
\]
Thus $G(M_{F_{x_5}})^o = \{x_4, x_3\}$. Since $L_{x_5} = L_{x_4}$ it is immediate that $G_{x_4}$ consists of 2 isolated vertices. Thus $\ell(T_{x_4}) = 2$ and exactly one binomial is needed to construct a spanning tree of $\Gamma_{x_4}$. To obtain a Markov basis $S$ of $L_1$, according to Theorem 4.11, we need to add to $S'_{T_{x_4}}$ a binomial $\pm(x^u - x^v)$ such that $x^u \in x_3 F_{x_4}^{i+1}$ and $x^v \in x_4 F_{x_4}^{i+1}$, where $0 \leq i \leq 4$. For example any of $B_1, B_6, B_{15}, B_{16}$ are of the required type. Let $S' = S'_{T_{x_4}} \cup \{x^u - x^v\}$ be this set.

Next consider the fiber $F_{x_5}$. It can be shown that $G(M_{F_{x_5}})$ is the set $\{x_5, x_4^6, x_1 x_3 x_4, x_3 x_2 x_4, x_1 x_3 x_2 x_4, x_1 x_2 x_3 x_4, x_1 x_2 x_3 x_4, x_2 x_3 x_4, x_1 x_2 x_3 x_4, x_2 x_3 x_4, x_1 x_2 x_3 x_4, x_2 x_3 x_4, x_1 x_2 x_3 x_4, x_2 x_3 x_4\}$ and thus

$$G(M_F)^o = \{x_5, x_3^6, x_3^3 x_4, x_3^4 x_4, x_3^5 x_4, x_3^4 x_4, x_3^5 x_4, x_3^6\}.$$ We claim that the graph $G_{x_5}$ consists of two connected components: the isolated vertex $x_5$ and a component containing all other vertices. We show for example that there is an edge between $x_3^6$ and $x_3^5 x_4$:

$$x_3^6 - x_3^5 x_4 = x_1 x_3^5 (x_3 x_3 x_4 - 1), x_3 x_3 x_4 \in L_{x_4} \text{ and } F_{x_4} < I_{x_5}.$$ We note that $x_3$ is necessarily an isolated vertex, otherwise $G_{x_5}$ would be connected, a contradiction by Theorem 4.11 since $F_{x_5}$ is a Markov fiber. It is easy to see that any of $B_9, . . . , B_{13}$ would produce an edge among the two connected components of $G_{x_5}$. To obtain a Markov basis $S$ of $L_1$, according to Theorem 4.11 we need to add to $S'$ exactly one binomial $\pm(x^w_1 - x^w_2)$ such that $x^w_1 \in x_5 F_{x_1}$ and $x^w_2$ belongs in the union of the sets $x_3^6 F_{x_1}^{i+1}$, $x_3^5 x_4 F_{x_1}^{i+1}$, $x_3^4 x_4 F_{x_1}^{i+1}$, $x_3^3 x_4 F_{x_1}^{i+1}$, $x_3^2 x_4 F_{x_1}^{i+1}$, $x_3 x_4 F_{x_1}^{i+1}$, $x_4 F_{x_1}^{i+1}$, where $0 \leq i \leq 4$. For example any of $B_9, . . . , B_{13}$ are of the required type.

The cardinality of $S$ is 4 and $I_{x_1}$ is a binomial complete intersection. Indeed, this follows immediately from Corollary 5.4 since $L^o$ is generated by $v_1^o, v_2^o$ and the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 6 & 0 & -1 \end{bmatrix}$$

is mixed dominating.

We presented the example in great detail, since the different steps illuminate the various parts of the proof of Theorem 4.11.

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