THE JORDAN ALGEBRAS OF RIEMANN, WEYL
AND CURVATURE COMPATIBLE Tensors

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Abstract. Given the Riemann, or the Weyl, or a generalized curvature tensor $K$, a symmetric tensor $b_{ij}$ is called compatible with the curvature tensor if $b_{i}^{m}K_{jklm} + b_{j}^{m}K_{kilm} + b_{k}^{m}K_{ijlm} = 0$. In addition to establishing some known and some new properties of such tensors, we prove that they form a special Jordan algebra, i.e. the symmetrized product of $K$-compatible tensors is $K$-compatible.

1. Introduction. Let $(M, g)$ be an $n$-dimensional Riemannian or pseudo-Riemannian manifold, and $K_{jklm}$ a generalized curvature tensor (the Riemann tensor, the Weyl tensor, or any tensor with the algebraic properties of the Riemann tensor). In [14] we introduced this concept: a symmetric tensor $b_{ij}$ is $K$-compatible if

\begin{equation}
    b_{i}^{m}K_{jklm} + b_{j}^{m}K_{kilm} + b_{k}^{m}K_{ijlm} = 0.
\end{equation}

We call $(K, b)$ a compatible pair. The motivation was the following theorem [14]: if $b_{ij}$ is $K$-compatible with eigenvectors $X, Y, Z$ and eigenvalues $x, y, z$ with $z \neq x, y$, then

\begin{equation}
    K_{ijlm}X^{i}Y^{j}Z^{m} = 0.
\end{equation}

It extends a result by Derdziński and Shen [6] who proved the same for the Riemann tensor, under the hypothesis that $b_{ij}$ is a Codazzi tensor, $\nabla_{i}b_{jk} = \nabla_{j}b_{ik}$. Despite the increased generality, the replacement of the Codazzi condition with the algebraic condition [1], allowed a much simpler proof of the new theorem.

Equation [1] with Riemann’s tensor originally appeared in a paper by Roter on conformally symmetric spaces [20, Lemma 1]. Riemann and Weyl compatible tensors were studied in [15] [17] [7].

Examples of Riemann compatible tensors are the Codazzi tensors [14], the Ricci tensors of Robertson–Walker space-times or perfect-fluid generalized Robertson–Walker space-times [18], the second fundamental form and

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the Ricci tensor of a hypersurface embedded in a (pseudo)Riemannian manifold \[17\], the Ricci tensors of ‘weakly Z-symmetric’ manifolds (\(\nabla_i Z_{jk} = A_i Z_{jk} + B_j Z_{ik} + D_k Z_{ij}\) with \(Z_{ij} = R_{ij} + \varphi g_{ij}, A_k - B_k\) a closed 1-form) \[16\] that include ‘weakly Ricci-symmetric’ ones (\(\varphi = 0\)) \[24\] and others (see \[3, 2\]), or ‘pseudosymmetric manifolds’ \[9\] (\([\nabla_i, \nabla_j] R_{klmp} = L Q_{k(lm)pj},\) where \(L \neq -1/3\) is a scalar function and \(Q\) is the Tachibana tensor built with the Riemann and Ricci tensors).

A Riemann compatible tensor is also Weyl compatible, but not conversely. The Ricci tensors of Gödel \[10, \text{Th. 2}\] or pseudo-Z symmetric space times \[19\] are Weyl compatible.

In Sections 2 and 3 we review Riemann and Weyl compatible tensors, with some new results and examples, and their relation to known identities due to Lovelock. Then, in Sections 4, 5 and 6, we investigate the algebraic properties of generalized curvature tensors and \(K\)-compatible tensors. The main result is that the latter form a special Jordan algebra, i.e. the set of \(K\)-compatible tensors is closed under the symmetrized product.

2. Riemann compatible tensors. A symmetric tensor is Riemann compatible if

\[ b_i^m R_{jklm} + b_j^m R_{kilm} + b_k^m R_{ijlm} = 0. \]  

This relation may be written as \(b_{(i}^m R_{jk)}^l m = 0\), where \((ijk)\) denotes the sum over cyclic permutations of the indices. Contraction with the metric tensor \(g^{jl}\) gives \(R_{km}^m b_{l}^m - b_k^m R_{mi} = 0\), so that \(b\) commutes with the Ricci tensor. Contraction with \(b_{jl}\) gives \(b_{i}^m R_{jklm} b_{jl}^l + b_k^m R_{ijlm} b_{jl}^l = 0\), and hence \(b\) commutes with the symmetric tensor \(R_{jm} = R_{jklm} b^{kl}\).

**Example 2.1.** Codazzi tensors are Riemann compatible.

**Proof.** In the identity \([\nabla_i, \nabla_j] b_{kl} = -R_{ij}^m b_{km} - R_{ik}^m b_{ml},\) sum over cyclic permutations of \(ijk\). The first Bianchi identity \(R_{(ijk)}^m = 0\) gives

\[ [\nabla_i, \nabla_j] b_{kl} + [\nabla_j, \nabla_k] b_{il} + [\nabla_k, \nabla_i] b_{jl} = -(b_{i}^m R_{jklm} + b_{j}^m R_{kilm} + b_k^m R_{ijlm}). \]

The left-hand side is zero for Codazzi tensors.  

**Example 2.2.** If \(\nabla_j A_k = p_j A_k\), then \(A_i A_j\) is Riemann compatible.

**Proof.** We have \(A_i [\nabla_j, \nabla_k] A_l = A_i (\nabla_j p_k - \nabla_k p_j) A_l = A_l [\nabla_j, \nabla_k] A_i.\) Then \(A_i R_{jklm} A_m = A_l R_{jklm} A_m;\) the sum over cyclic permutations of \(ijk\) gives zero on the right-hand side.

2.1. Codazzi deviation. In \[15\] we introduced the natural concept of Codazzi deviation of a symmetric tensor:

\[ C_{jkl} = \nabla_j b_{kl} - \nabla_k b_{jl}. \]
It satisfies $\mathcal{C}_{jkl} = -\mathcal{C}_{kj l}$, $\mathcal{C}_{jkl} + \mathcal{C}_{klj} + \mathcal{C}_{ljk} = 0$, and
\begin{equation}
\nabla_i \mathcal{C}_{jkl} + \nabla_j \mathcal{C}_{kil} + \nabla_k \mathcal{C}_{ij l} = -(b_{im} R_{jkl}^m + b_{jm} R_{kil}^m + b_{km} R_{ij l}^m).
\end{equation}

Once again we see that a Codazzi tensor is Riemann compatible. By (5) the differential condition $\nabla (i \mathcal{C}_{jk} l) = 0$ is equivalent to the algebraic formula (3). A Veblen-type identity holds:
\begin{equation}
\nabla_i \mathcal{C}_{jkl} + \nabla_j \mathcal{C}_{kil} + \nabla_k \mathcal{C}_{lji} + \nabla_i \mathcal{C}_{ikj} = b_{im} R_{jkl}^m + b_{jm} R_{kil}^m + b_{km} R_{lj i}^m + b_{lm} R_{ikj}^m.
\end{equation}

**Example 2.3.** For a concircular vector field $X$, with $\nabla_i X_j = \rho g_{ij}$, the tensor $X_i X_j$ is Riemann compatible.

**Proof.** One has $\mathcal{C}_{jkl} = (\nabla_j \rho) g_{kl} - (\nabla_k \rho) g_{jl}$ and $\nabla_i \mathcal{C}_{jkl} = (\nabla_i \nabla_j \rho) g_{kl} - (\nabla_i \nabla_k \rho) g_{jl}$. The left-hand side of (5) thus equals zero. \hfill \blacksquare

Note: the existence of a concircular time-like vector field is necessary and sufficient for a space-time to be generalized Robertson–Walker [5].

**Example 2.4** (Lovelock’s identities). 1. The Codazzi deviation of the Ricci tensor is $\mathcal{C}_{jkl} = \nabla_j R_{kl} - \nabla_k R_{jl} = -\nabla^m R_{jklm}$. Property (5) becomes Lovelock’s identity for the Riemann tensor [13, p. 289]:
\begin{equation}
\nabla_i \nabla^m R_{jklm} + \nabla_j \nabla^m R_{kilm} + \nabla_k \nabla^m R_{ij lm} = -R^m (i R_{jkl})_{lm}.
\end{equation}

2. The Codazzi deviation of Schouten’s tensor (1) is $\mathcal{C}_{jkl} = -\frac{1}{n-3} \nabla^m C_{jklm}$. Property (5) reads $\nabla (i \mathcal{C}_{jkl}) = -(n-3) S^m (i R_{jkl})_{lm}$. The term with the metric tensor in $S_{ij}$ does not contribute (due to the Bianchi identity), and one is left with (see [15])
\begin{equation}
\nabla_i \nabla^m C_{jklm} + \nabla_j \nabla^m C_{kilm} + \nabla_k \nabla^m C_{ij lm} = -\frac{n-3}{n-2} R^m (i R_{jkl})_{lm}.
\end{equation}

In particular, for $n > 3$, if $\nabla^m C_{jklm} = 0$ (conformally symmetric spaces, Roter [20]) then the Ricci tensor is Riemann compatible.

**Proposition 2.5.** If $u_i u_j$ is Riemann compatible, and $u^k u_k \neq 0$, then $u_i$ is an eigenvector of the Ricci tensor.

**Proof.** Since $u_i u_j$ is Riemann compatible, it commutes with the Ricci tensor: $R_{ij} u^j u_k = R_{kj} u^j u_i$. Contraction with $u^k$ gives
\[ R_{ij} u^j (u_k u^k) = (R_{kj} u^j u^k) u_i = 0. \]

We extrapolate a simple statement from [7, Proposition 5.1]. A direct proof is possible, by writing (3) for the Ricci tensor in warping coordinates:

(1) Schouten’s tensor is $S_{ij} = \frac{1}{n-2} [R_{ij} - \frac{R}{2(n-1)} g_{ij}]$. It satisfies $\nabla_k S^k_j = \nabla_j S^k_k$, $\nabla^m C_{jklm} = (n-3)(\nabla_k S_{jl} - \nabla_j S_{kl})$. \hfill \(\ \)
Proposition 2.6. In a warped-product spacetime
\[ ds^2 = \pm dt^2 + a(t)^2 g_{\mu\nu} dx^\mu dx^\nu \]
the Ricci tensor is Riemann compatible if and only if the Ricci tensor of the Riemannian submanifold \((M^*, g^*)\) is compatible with the Riemann tensor of the submanifold:
\[ R_{\mu\sigma} R_{\nu\rho\lambda}^* \sigma + R_{\nu\sigma} R_{\rho\mu\lambda}^* \sigma + R_{\rho\sigma} R_{\mu\nu\lambda}^* \sigma = 0. \]

2.2. Geodesic maps. A map \((M, g) \to (M, \bar{g})\) is geodesic if every geodesic line is mapped to a geodesic line. For the identity mapping of \(M\) to be geodesic, it is necessary and sufficient that there exists a 1-form such that the Christoffel symbols are related by
\[ \Gamma^k_{ij} = \Gamma^k_{ij} + \delta^k_i X_j + X_i \delta^k_j \] (Levi-Civita, 1896). The relation between the Riemann tensors then is
\[ R_{jklm} = -\partial_j \Gamma^m_{kl} + \partial_k \Gamma^m_{jl} - \Gamma^d_{kl} \Gamma^m_{jd} + \Gamma^d_{jl} \Gamma^m_{kd} = R_{jklm} - \delta_k^m P_{jl} + \delta_j^m P_{kl}, \]
where \( P_{kl} = \nabla_k X_l - X_k X_l = P_{lk}. \) One has \( \bar{R}_{jl} = R_{jl} + (n - 1)P_{jl}. \)

Geodesic maps preserve the \((3,1)\) projective curvature tensor [23]:
\[ \bar{P}_{jklm} = P_{jklm}, \]
where \( P_{jklm} = R_{jklm} + \frac{1}{n-1} \left( \delta^m_j R_{kl} - \delta^m_k R_{jl} \right). \)

Proposition 2.7 ([15]). For a geodesic map and a symmetric tensor \( b_{ij} = b_{ji} \), the following identity holds:
\[ b_{im} \bar{R}_{jklm} + b_{jm} \bar{R}_{kilm} + b_{km} \bar{R}_{ijlm} = b_{im} R_{jklm} + b_{jm} R_{kilm} + b_{km} R_{ijlm}. \]
Therefore, if \((R, b)\) is a compatible pair, also the pair \((\bar{R}, b)\) is compatible.

3. Weyl compatible tensors. A symmetric tensor is Weyl compatible if
\[ b_{im} C_{jklm} + b_{jm} C_{kilm} + b_{km} C_{ijlm} = 0. \]
The following identity holds for any symmetric tensor [15]:
\[ b_{im} C_{jklm} + b_{jm} C_{kilm} + b_{km} C_{ijlm} = b_{im} R_{jklm} + b_{jm} R_{kilm} + b_{km} R_{ijlm} + \frac{1}{n-1} \left[ g_{kl} (b_{im} R_{jm} - b_{jm} R_{im}) + g_{il} (b_{jm} R_{km} - b_{km} R_{jm}) + g_{jl} (b_{km} R_{im} - b_{im} R_{km}) \right]. \]
A simple consequence is obtained in dimension \( n = 3 \), where the Weyl tensor is zero (see [8], in a less simple manner):

Proposition 3.1. In dimension \( n = 3 \) the Ricci tensor is Riemann compatible.

If \( b_{ij} \) is Riemann compatible, then it commutes with the Ricci tensor. As a result, (11) shows that \( b_{ij} \) is also Weyl compatible. Therefore, Riemann compatibility is a stronger condition than Weyl compatibility. The identity
can be rewritten in terms of the Codazzi deviation:

\[ b_{im}C_{jkl}^m + b_{jm}C_{kil}^m + b_{km}C_{ijl}^m = \nabla_i D_{jkl} + \nabla_j D_{kil} + \nabla_k D_{ijl} \]


\[ -\frac{1}{n-2}\nabla^m (C_{ijm}g_{kl} + C_{jkm}g_{il} + C_{kim}g_{jl}), \]

where \( D_{jkl} = C_{jkl} - \frac{1}{n-2} (C_{jm}m g_{kl} - C_{km}m g_{jl}) \).

Example 3.2. If a vector field is torqued \( \mathbb{H} \), i.e. \( \nabla_i \tau_j = \rho g_{ij} + \alpha_i \tau_j \) with \( \alpha_k \tau^k = 0 \), then \( \tau_i \tau_j \) is Weyl compatible.

Proof. One evaluates \( C_{jkl} = -\rho(\tau_j g_{kl} - \tau_k g_{jl}) \) and \( D_{jkl} = -\frac{1}{n-2} C_{jkl} \). It turns out that the right-hand side of (12) is zero.

Note: the existence of a torqued time-like vector is necessary and sufficient for a space-time to be twisted \( \mathbb{H} \).

Proposition 3.3 (see [11, Remark 4.2]). In a space-time of dimension \( n = 4 \), if \( u_i u_j \) is a Weyl compatible and time-like unit \( (u^k u_k = -1) \) then the Weyl tensor is completely determined by the electric tensor \( E_{kl} = C_{jklm}u^l u^m \):

\[ C_{abcd} = 2(u_a u_d E_{bc} - u_a u_c E_{bd} + u_b u_c E_{ad} - u_b u_d E_{ac}) + g_{ad} E_{bc} - g_{ac} E_{bd} + g_{bc} E_{ad} - g_{bd} E_{ac} \]

Proof. In \( n = 4 \) the following Lovelock identity holds [13, Ex. 4.9, p. 128]:

\[ 0 = g_{ar} C_{bcsr} + g_{br} C_{casr} + g_{cr} C_{asbr} + g_{at} C_{bcrs} + g_{bt} C_{cars} + g_{ct} C_{abrs} + g_{as} C_{bctr} + g_{bs} C_{catr} + g_{cs} C_{abtr} \]

Contraction with \( u^a u^r \) gives

\[ 0 = -C_{bcst} + u_{bt} u^r C_{crst} + u_c u^r C_{rst} + u_t u^r C_{bcrs} + g_{bt} u^a u^r C_{cars} + g_{ct} u^a u^r C_{abrs} + g_{cs} u^a u^r C_{abtr} + g_{ct} E_{cs} - g_{ct} E_{bs} - g_{cs} E_{ct} + g_{cs} E_{bt}. \]

This gives the Weyl tensor in terms of its single and double contractions with \( u^i \). If \( u_i u_j \) is Weyl compatible, the single contraction is \( C_{jklm} u^r = u_k E_{jl} - u_j E_{kl} \), and the result follows. For an extension to \( n > 4 \) see [11].

3.1. Conformal maps. The identity map \( (M, g) \to (M, \hat{g}) \) is conformal if \( \hat{g}_{kl} = e^{2\sigma} g_{kl} \) for some function \( \sigma \). The Christoffel symbols transform according to \( \hat{\Gamma}^m_{ij} = \Gamma^m_{ij} + \delta^m_i X_j + X_i \delta^m_j - g_{ij} X^m \), where \( X_i = \nabla_i \sigma \). A conformal map leaves the (3,1) Weyl tensor unchanged: \( \hat{C}_{jklm} = C_{jklm} \). Therefore, Weyl compatibility is a property invariant under conformal maps.

4. K-compatible tensors. Riemann and Weyl compatibility may be generalized to \( K \)-compatibility, where \( K \) is a generalized curvature tensor (GCT), i.e. a tensor with the algebraic properties of the Riemann tensor.
under permutations of indices [12]:

(14) \[ K_{jklm} = -K_{kjlm} = -K_{jkml}, \]
(15) \[ K_{jklm} + K_{kljm} + K_{ljkm} = 0, \]
(16) \[ K_{jklm} = K_{lmjk}. \]

In analogy with the Riemann tensor, one shows that (14) and (15) imply the symmetry (16), and the identity \( K_{j(klm)} = 0 \). The tensor \( K_{jlm} \) is symmetric.

A symmetric tensor \( b_{ij} \) is \( K \)-compatible if

(17) \[ b_{im} K_{jklm} + b_{jm} K_{kilm} + b_{km} K_{ijlm} = 0, \]
and \((K, b)\) is then called a compatible pair. This can be written as \( b_{m} (i K_{jk})_{lm} = 0 \).

The metric tensor is \( K \)-compatible, by the Bianchi property (15). The tensors \( b_{ij} \) and \( K_{ij} \) commute:

(18) \[ b_{im} K_{mk} - K_{im} b_{mk} = 0 \] (contract (17) with \( g^{jl} \) and use symmetry).

Examples of \( K \)-compatible tensors were obtained by Shaikh et al. (see for example [22, 21]) starting from specific metrics. Bourguignon [1] proved that if \( b_{ij} \) is a Codazzi tensor then \( \dot{R}_{jklm} = R_{jkrs} b_{r}^{s} b_{m}^{s} \) is a GCT. We prove a more general statement:

**Proposition 4.1.** If \( a_{ij} \) and \( b_{ij} \) are \( K \)-compatible, then \( \dot{K}_{jklm} = K_{jkrs} (a^{r} b_{s}^{m} + b^{r} a_{s}^{m}) \) is a GCT.

**Proof.** The properties (14) and (16) are obvious; the Bianchi property (15) completes the proof: \( K_{j(klm)} = a^{r} (i K_{jk})_{rs} b_{s}^{m} + b^{r} (i K_{jk})_{rs} a_{s}^{m} = 0 \) because each term is zero, both \( a \) and \( b \) being \( K \)-compatible. \( \blacksquare \)

**4.1. Properties of \( K \)-compatible tensors.** A linear combination of \( K \)-compatible tensors obviously is \( K \)-compatible. Now we prove:

**Theorem 4.2.** If \( a \) and \( b \) are \( K \)-compatible, then \( \frac{1}{2} (ab + ba) \) is \( K \)-compatible.

**Proof.** Let \( c_{ij} = a_{i}^{k} b_{kj} + b_{i}^{k} a_{kj} \). Then

\[
\begin{align*}
c_{(i K_{jk})rm} &= a_{i}^{s} b_{j}^{m} K_{k} k r m + a_{j}^{s} b_{s}^{m} K_{k} k r m + a_{k}^{s} b_{s}^{m} K_{ij} r m + a \Rightarrow b \\
&= -a_{i}^{s} (b_{j}^{m} K_{k} k s r m + b_{k}^{m} K_{j} i s r m) - a_{j}^{s} (b_{k}^{m} K_{i} k s r m + b_{i}^{m} K_{s} k r m) \\
&= -a_{k}^{s} (b_{i}^{m} K_{j} j s r m + b_{j}^{m} K_{s} i r m) + a \Rightarrow b \\
&= -(a_{i}^{s} b_{j}^{m} - a_{j}^{s} b_{i}^{m}) K_{k} k s r m - (a_{j}^{s} b_{k}^{m} - a_{k}^{s} b_{j}^{m}) K_{i} j s r m \\
&= -(a_{k}^{s} b_{i}^{m} - a_{i}^{s} b_{k}^{m}) K_{j} j s r m + a \Rightarrow b \\
&= -(a_{i}^{s} b_{j}^{m} - a_{j}^{s} b_{i}^{m}) (K_{k} k s r m - K_{k} k m r s) \\
&= (a_{j}^{s} b_{k}^{m} - a_{k}^{s} b_{j}^{m}) (K_{i} j s r m - K_{i} j m r s) \\
&= (a_{k}^{s} b_{i}^{m} - a_{i}^{s} b_{k}^{m}) (K_{j} j s r m - K_{j} j m r s)
\end{align*}
\]
\[\begin{align*}
&= (a_i b_j^m - a_j b_i^m)K_{krsim} + (a_j b_k^m - a_k b_j^m)K_{irsm} \\
&\quad + (a_k b_i^m - a_i b_k^m)K_{jrcsm} \\
&= (a_i b_j^m + b_i a_j^m)K_{krsim} + (a_j b_k^m + b_j a_k^m)K_{irsm} \\
&\quad + (a_k b_i^m + b_k a_i^m)K_{jrcsm} \\
&= \tilde{K}_{krij} + \tilde{K}_{irjk} + \tilde{K}_{jrki} = \tilde{K}_{(kri)j} = 0
\end{align*}\]

because \(\tilde{K}\) is a GCT by Proposition 4.1. □

Therefore, the linear space of \(K\)-compatible tensors is a special Jordan algebra.

In particular, the powers of \(b\) are \(K\)-compatible (powers with exponents \(n, n+1, \ldots\) are linear combinations of lower powers by the Cayley–Hamilton theorem). In particular (by the exchange of indices) the tensor \((b^2)_j^s (b^2)_k^r K_{rslm}\) is a GCT. This enables us to come up with the simple proof of the theorem in [14], so short that we reproduce it here:

**Theorem 4.3 (Extended Derdziński–Shen theorem).** Let \(b_{ij}\) be \(K\)-compatible, and let \(X^i, Y^i, Z^i\) be eigenvectors of \(b_i^m\) with eigenvalues \(x, y, z\). If \(x \neq z\) and \(y \neq z\) then

\[(18)\]

\[K_{ijkl} X^i Y^j Z^k = 0.\]

**Proof.** Consider the identities

\[g^m_{(i} K_{jk)lm} = 0, \quad b^m_{(i} K_{jk)lm} = 0, \quad (b^2)^m_{(i} K_{jk)lm} = 0\]

and contract them with \(X^i Y^j Z^k\). The three algebraic relations are put in matrix form:

\[
\begin{bmatrix}
1 & 1 & 1 \\
x & y & z \\
x^2 & y^2 & z^2
\end{bmatrix}
\begin{bmatrix}
K_{klij} X^i Y^j Z^k \\
K_{kilj} X^i Y^j Z^k \\
K_{ijlk} X^i Y^j Z^k
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

The determinant of the matrix is \((x - y)(x - z)(z - y)\). If the eigenvalues are all different then \(K_{ijkl} X^i Y^j Z^k = 0\) (with contraction of any three indices). If \(x = y \neq z\), the reduced system of equations still implies \(K_{ijkl} X^i Y^j Z^k = 0\). □

**Proposition 4.4.** If \(b\) is \(K\)-compatible and invertible, then \(b^{-1}\) is \(K\)-compatible:

\[(19)\]

\[(b^{-1})^j_s K_{rtl}\]_\[kj] = 0.\]

**Proof.** Multiply (17) by \((b^{-1})^i_r (b^{-1})^j_s\) to obtain the identity \((b^{-1})^j_s K_{jklr} + (b^{-1})^i_r K_{kils} + (b^{-1})^i_r (b^{-1})^j_s b^m_k K_{ijlm} = 0\). Rewrite it as

\[(b^{-1})^j_s K_{rtl}\]_\[kj] - (b^{-1})^j_l K_{srkj} + (b^{-1})^i_r (b^{-1})^j_s b^m_k K_{ijlm} = 0.\]
The last two terms cancel, as shown by
\[
(b^{-1})^j_i K_{srkj} = (b^{-1})^i_r (b^{-1})^j_s b^m_k K_{ijklm} \iff K_{srkb} b^r_a = b^l_i b^m_k K_{aclkm} \iff K_{kbc} = K_{acbk},
\]
which is true as \( \hat{K} \) is a GCT.

We prove a Veblen-type identity:

**Proposition 4.5.** If \( b_{ij} \) is \( K \)-compatible, then

\[
b_i^m K_{jklm} - b_j^m K_{ilkm} + b_k^m K_{ijlm} - b_l^m K_{jkim} = 0.
\]

**Proof.** We have
\[
0 = b_i^m K_{jklm} + b_j^m K_{iklm} + b_k^m K_{ijlm} \\
= b_i^m K_{jklm} - b_j^m (K_{ilk} + K_{lk}) + b_k^m K_{ijlm} \\
= b_i^m K_{jklm} - b_j^m K_{iklm} + b_l^m K_{kijm} + b_k^m K_{jlim} - b_k^m K_{ijlm} \\
= b_i^m K_{jklm} - b_j^m K_{iklm} + b_l^m K_{kijm} - b_k^m K_{ijlm}.
\]

4.2. More on generalized curvature tensors. A linear combination of GCTs is a GCT. Given two compatible pairs \((K, a)\) and \((K, b)\), a new GCT tensor is obtained in Proposition 4.1. In particular, if \( a_{ij} = g_{ij} \) (the metric tensor), the following \( K' \) is a GCT:

\[
K'_jklm = K_{jklr} (b^r_m + b^r_l) = K_{jklm} - K_{jkms} b^s_l.
\]

**Proposition 4.6.** If \( b \) is \( K \)-compatible, then it is \( K' \)-compatible.

**Proof.** The tensor \( K'_{jklm} = K_{jklr} b^r_m - K_{jkms} b^s_l \) is a GCT. Let us evaluate
\[
b_i^m K'_{jklm} = b_i^m K_{jklr} b^r_m - b_i^m K_{jkms} b^s_l = (b^2)^r_i K_{jklr} - K_{jkim}.
\]
Both tensors yield zero if the cyclic sum \((ijk)\) is taken.

**Proposition 4.7.** \((K, b)\) is a compatible pair for every symmetric tensor \( b \) if and only if

\[
K_{ijklm} = \frac{K}{n(n-1)} (g_{ij} g_{jm} - g_{im} g_{jl})
\]
where \( K \) is a scalar.

**Proof.** The symmetry of the tensor is made explicit by writing
\[
b_{ij} = \frac{1}{2} b^{rs} (g_{ir} g_{js} + g_{is} g_{jr}).
\]
The compatibility relation must hold for any \( b^{rs} \), so
\[
0 = g_{ir} K_{jklm} + g_{jr} K_{ikls} + g_{kr} K_{ijls} + g_{is} K_{jklm} + g_{js} K_{ikls} + g_{ks} K_{ijlr}.
\]
Contraction with \( g^t_{ks} \) gives \((n-1)K_{ijlr} = g_{jr} K_{il} - g_{ir} K_{il} \); contraction with \( g^t_{il} \) gives \( K_{jr} = \frac{1}{n} g_{jr} K^t_i \) and (22) follows. The converse, namely, that (22) implies (17), is shown by direct check.
A pseudo-Riemannian manifold of dimension \( n > 2 \) is an Einstein manifold if \( R_{ij} = \frac{1}{n} R g_{ij} \) where \( R \) is the scalar curvature. Since \( \nabla_i R^i_j = \frac{1}{2} \nabla_j R \), the scalar curvature is constant. A manifold is a constant curvature manifold if the Riemann tensor has the form (22). Such manifolds are Einstein manifolds.

**Corollary 4.8.** A manifold is a constant curvature manifold if and only if \( b_i^m R_{jklm} + b_j^m R_{kilm} + b_k^m R_{ijlm} = 0 \) for all symmetric tensors.

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