Perfect 3-Dimensional Lattice Actions for 4-Dimensional Quantum Field Theories at Finite Temperature

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Abstract

We propose a two-step procedure to study the order of phase transitions at finite temperature in electroweak theory and in simplified models thereof.

In a first step a coarse grained free energy is computed by perturbative methods. It is obtained in the form of a 3-dimensional perfect lattice action by a block spin transformation. It has finite temperature dependent coefficients. In this way the UV-problem and the infrared problem is separated in a clean way. In the second step the effective 3-dimensional lattice theory is treated in a nonperturbative way, either by the Feynman-Bogoliubov method (solution of a gap equation), by real space renormalization group methods, or by computer simulations.

In this paper we outline the principles for $\varphi^4$-theory and scalar electrodynamics. The Balaban-Jaffe block spin transformation for the gauge field is used. It is known how to extend this transformation to the nonabelian case, but this will not be discussed here.
1 Introduction

The possibility of restoration of spontaneously broken symmetry in the electroweak theory at high temperature has recently led to a renewed interest in the electroweak phase transition [1].

Since the seminal and pioneering paper by Kirzhnits and Linde [2], considerable effort has been devoted to describe the rather involved processes occurring very close to the critical temperature. The effective potential has shown great usefulness. It gives the free energy as a function of the magnetization. There are perturbative computations combined with $1/N$ expansions for the effective potential [3], and also numerical results based on Monte Carlo simulations and on three dimensional reduced actions [4].

Nevertheless the perturbative calculations have shown to be plagued with problems which are to a large extent due to infrared divergences and are manifested in the appearance of spurious complex terms in the expansion for the effective potential. A great amount of work has been done to obtain meaningful expressions very close to the critical surface. Most of these attempts are related to the resummation of an infinite number of daisy and superdaisy diagrams which have to be taken into account in a consistent perturbative expansion, because they are of the same order of magnitude in the coupling constants [5].

Another very successful approach was proposed by Buchmüller et al. [6], who use an improved perturbation expansion for the effective potential, where the dynamically generated plasma masses are already included in the corresponding expressions from the beginning. These plasma masses are computed up to one loop order by self-consistent gap-equations, and have the effect of damping the infrared divergences. In this context, in [7] the absence of linear terms in the scalar field of the effective potential was explicitly verified for the abelian Higgs model, up to the order $e^4$ and $\lambda^2$, and proved to remain valid at higher orders. Such a linear term contains spurious infrared divergent contributions.

Despite the important and promising contributions to the understanding of the nature of the phase transition that have already been made, there are still ambiguities to be explained and higher order corrections to be included consistently to ensure the survival of the predictions to all orders.

We propose an alternative variational two-step method to study phase transitions at finite temperature in the electroweak theory, and in related models. It consists first in the calculation of a coarse grained free energy by perturbative methods, obtaining a 3-dimensional perfect lattice action as a function of a block spin field. Their coefficients depend explicitly on the temperature. In a second step, the Feynman-Bogoliubov method can be used with this 3-dimensional lattice action to obtain the best quadratic approximation to the perfect action, and therefore the best split into a free part and an interaction. Using this, the constraint effective potential can be computed. To lowest order, the masses are obtained in this way as solutions of a gap equation whose formal solution is the
sum of superdaisy diagrams for the 3-dimensional lattice theory. They depend on temperature. Higher order corrections can be computed in principle. These masses are to be inserted into the expression for the constraint effective potential.

This procedure separates the UV- and IR-problems in a clean way. It allows to study the perfect 3-dimensional lattice action directly by numerical simulations. One could for instance compute the constraint effective potential numerically.

The paper is organized as follows. In section 3 the block spin transformation for the scalar fields at zero temperature is discussed. The evaluation of the perfect scalar action at $T=0$ is explained and the extension to finite temperature is outlined. An explicit computation of the leading terms of the perfect action for the $\lambda \Phi^4$–theory is shown here as well. In section 4 the Feynman-Bogoliubov method is applied to the lattice action of the previous section and a comparison with other approaches is performed here and in appendix [3]. In section 5 we extend the procedure to the Maxwell theory at zero temperature. In particular, the averaging operator for abelian gauge fields, the perfect lattice action, the gauge fixing and the interpolation kernel are discussed. The block spin of Balaban and Jaffe for the abelian gauge field is used [8]. In section 6 the procedure is extended to scalar electrodynamics. Here the main features of the extension of the perfect lattice action to finite temperature are outlined and some quantities which are needed for perturbative calculations are exhibited. A short version of the results of this paper was presented at LATTICE 94 in Bielefeld [9].

2 Scholium : Free propagators at finite temperature

To prepare for the block spin transformation at finite temperature, we recall first the well known relation between free propagators at zero temperature and at finite temperature $T = \beta^{-1}$.

Because of translation invariance, the standard free propagators in Euclidean space time depend on a single variable $x = (r, t)$. The zero temperature propagator has the form

$$v_0(r, t) = \int \frac{d^3p}{(2\pi)^3} \frac{d\omega}{2\pi} \exp(-ip \cdot r - i\omega t) \left[ \omega^2 + p^2 + m^2 \right]^{-1}$$

A finite temperature field theory lives on a Euclidean space time which is periodic in time direction with period $\beta$, i.e. on a tube. Propagators admit a well known random walk representation. The difference between zero and finite temperature stems from the possibility that these random walks may wind around the tube several times. Accordingly, the finite temperature propagator $v_T$ is obtained from $v_0$ by periodizing in time.
\begin{equation}
    v_T(r, t) = \sum_{n \in \mathbb{Z}} v_0(r, t + n\beta) \tag{2}
\end{equation}

One may use Poisson’s resummation formula.

\begin{equation}
    \sum_{n \in \mathbb{Z}} \exp(-i\omega_n(t + n\beta)) = \exp(-i\omega t) \sum_{n \in \mathbb{Z}} 2\pi\beta^{-1}\delta(\omega - \omega_n), \quad \text{with} \quad \omega_n = 2\pi n\beta^{-1} \tag{3}
\end{equation}

As a result, the well known finite temperature propagator is obtained as a sum over Matsubara frequencies \( \omega_n \)

\begin{equation}
    v_T(r, t) = \sum_{n \in \mathbb{Z}} \beta^{-1} \int \frac{d^3p}{(2\pi)^3} \exp(-ip \cdot r - i\omega_n t) \left[ \omega_n^2 + p^2 + m^2 \right]^{-1} \tag{4}
\end{equation}

Later on, we will concentrate on the \( m = 0 \) case.

## 3 Scalar fields only

To introduce our method we start with a theory with scalar fields only. We consider first the \( T = 0 \) case and define the blockspin, the effective action, the interpolation operator and the fluctuation field propagator. Then we deal with the \( T > 0 \) case i.e. with the generalizations of the above mentioned entities. As an illustration we write down the effective action for \( T > 0 \) to leading order in the fluctuation field propagator. Finally we add a note on the cancellation of the \( T \)-dependent UV divergencies.

### 3.1 Block spins for scalar fields at zero temperature

We start from a scalar field theory on the continuum \( \mathbb{R}^4 \) of points \( z \). The continuum is divided into blocks \( x \). Normally one chooses hypercubes, but here we wish to admit different extensions \( L_s \) of the blocks in space direction and \( L_t \) in time direction. We identify the blocks \( x \) with the sites at their centers. In this way a lattice \( \Lambda \) of lattice spacing \( (L_s, L_t) \) in space and time direction is obtained. With a scalar field \( \varphi(z) \) one associates a block spin \( \Phi(x) \). Following Gawedzki and Kupiainien \[16\], we choose them as block averages.

\begin{equation}
    \Phi(x) = C\varphi(x) = \text{av}_{z \in x} \varphi(z) \tag{5}
\end{equation}

\( C \) is called the averaging operator. It has a kernel \( C(x, z) \) which equals the properly normalized characteristic function \( \chi_x(z) \) of the block \( x \). Because of invariance under simultaneous translations of \( z \) and \( x \) by lattice vectors in \( \Lambda \), its
Fourier representation has the form
\[
C(x, z) = \frac{1}{L_s^3 L_t} \chi(x(z)) = \sum_l \int \frac{d^4p}{(2\pi)^4} \exp(-ip(z - x) - ilz) \tilde{C}(l, p)
\] (6)

Summation over \(l = (l_1, l_4)\) runs over \(l_4 \in 2\pi L_t^{-1}Z\) and \(l \in (2\pi L_s^{-1}Z)^3\). Integration over \(p\) is over the first Brillouin zone.

\[
\|p\| \leq \pi L_s^{-1}, \quad |p_4| \leq \pi L_t^{-1}
\] (7)

Explicitly
\[
\tilde{C}(l, p) = \tilde{C}_t(l_4, p_4) \tilde{C}_s(l, p)
\]
\[
\tilde{C}_t(l_4, p_4) = \frac{2}{L_t(p_4 + l_4)} \sin \frac{L_t(p_4 + l_4)}{2}
\]
\[
\tilde{C}_s(l, p) = \prod_{j=1}^3 \frac{2}{L_s(p_j + l_j)} \sin \frac{L_s(p_j + l_j)}{2}
\] (8)

3.2 Definition of the perfect action

Given the continuum action \(L(\varphi)\), the perfect 4-dimensional lattice action \(L_\Lambda(\Phi)\) at zero temperature is defined in Wilson’s way

\[
\exp(-L_\Lambda(\Phi)) = \int D\varphi \delta(C\varphi - \Phi) \exp(-L(\varphi))
\] (9)

In place of the \(\delta\)-function, one may use a Gaussian

\[
\delta_\kappa(C\varphi - \Phi) = \prod_{x \in \Lambda} \left(\frac{2\pi}{\kappa}\right)^{-\frac{d}{2}} \exp\left(-\frac{1}{2\kappa}(C\varphi(x) - \Phi(x))^2\right)
\] (10)

Hasenfratz and Niedermayer \[17\] pointed out that optimal locality properties of the perfect action \(L_\Lambda\) are obtained for a preferred finite value of \(\kappa\).

3.3 Evaluation of the perfect action, interpolation operator and the fluctuation field propagator

To evaluate \(L_\Lambda(\Phi)\) by perturbation expansion one proceeds as follows \[15\]. One splits

\[
L(\varphi) = L_0(\varphi) + V(\varphi)
\] (11)
\[
L_0(\varphi) = \frac{1}{2} \int \partial_\mu \varphi \partial_\mu \varphi = -\frac{1}{2} (\varphi, \triangle \varphi) \tag{12}
\]

It is convenient to include mass terms in the interaction \( V \).

Given the block spin \( \Phi \), one determines a background field \( \psi \) which minimizes \( L_0 \) subject to the constraint \( C \psi = \Phi \). Because \( L_0 \) is quadratic, \( \psi \) is a linear function of \( \Phi \).

\[
\psi(z) = A \Phi(z) = \int_{x \in \Lambda} A(z, x) \Phi(x), \quad \text{with} \quad C A = 1 \tag{13}
\]

and

\[
\int_{x \in \Lambda} = L_t L_s^3 \sum_{n \in N^4}, \quad x = (n L_s, n_4 L_t). \tag{14}
\]

In coordinate space

\[
v^{-1} A = C^\dagger u^{-1}, \quad u = C v C^\dagger, \quad v = (-\Delta)^{-1} \tag{15}
\]

Following Gawedzki and Kupiainen \[16\] one splits the field \( \varphi \) into a low frequency part, which is determined by the block spin \( \Phi \), and a high frequency or fluctuation field \( \zeta \) which has vanishing block average.

\[
\varphi = A \Phi + \zeta, \quad C \zeta = 0 \tag{16}
\]

Because of the extremality property of \( \psi \), the kinetic energy splits

\[
L_0(\varphi) = \frac{1}{2} (A \Phi, -\Delta A \Phi) + \frac{1}{2} (\zeta, -\Delta \zeta) = \frac{1}{2} (\Phi, u^{-1} \Phi) + \frac{1}{2} (\zeta, -\Delta \zeta) \tag{17}
\]

From this one sees that the free massless propagator for the block spin \( \Phi \) equals \( u \). Since \( C A = 1 \), we have \( \delta_\kappa(C \varphi - \Phi) = \delta_\kappa(C \zeta) \).

The measure splits

\[
D \varphi \delta_\kappa(C \varphi - \Phi) \exp(-L_0(\varphi)) = D \zeta \exp(-\frac{1}{2}(\Phi, u^{-1} \Phi)) \delta_\kappa(C \zeta) \exp(-\frac{1}{2}(\zeta, -\Delta \zeta)) \tag{18}
\]

The \( \zeta \)-dependent factor is proportional to a Gaussian measure with covariance (free propagator) \( \Gamma \)

\[
D \zeta \delta_\kappa(C \zeta) \exp(-\frac{1}{2}(\zeta, -\Delta \zeta)) = Z^{-1} d\mu_\Gamma(\zeta), \quad \Gamma = (-\Delta + \kappa C^\dagger C)^{-1} \tag{19}
\]

The limit \( \kappa \to \infty \) can be taken if desired, and results in \[16\]

\[
\Gamma = v - A u A^\dagger = v - A C v C^\dagger A^\dagger, \quad C \Gamma = \Gamma C^\dagger = 0 \tag{20}
\]
The interpolation kernel \( A(z, x) \) of \( A \) has a Fourier expansion like \( C \) in eq. (6) with kernel

\[
\tilde{A}(l, p) = \tilde{v}(k) \tilde{C}^*(l, p) \tilde{u}^{-1}(p)
\]

(21)

\[
\tilde{u}(p) = \sum_l \tilde{v}(k) \left| \tilde{C}(l, p) \right|^2 = \int_{x \in \Lambda} u(x) \exp(-ipx)
\]

(22)

We use the abbreviation \( k = p + l \) here and throughout. We note that

\[
\tilde{u}(p) \sim \frac{1}{p^2} \quad \text{for} \quad p^2 \to 0
\]

(23)

because only the \( l = 0 \) term in (22) is singular at \( p = 0 \).

The fluctuation field propagator \( \Gamma(z, z') \) is invariant under translations by lattice vectors in \( \Lambda \). It admits therefore a Fourier expansion of the form

\[
\Gamma(z, z') = \sum_{l, l'} \int d^4p (2\pi)^4 \exp(i(p + l)z - i(p + l')z') \tilde{\Gamma}(l, p, l')
\]

(24)

Summations and integrations are as explained after (6). All the summations over \( l \) converge well. For \( \kappa = \infty \), eq. (20) yields the explicit expression

\[
\tilde{\Gamma}(l, p, l') = \tilde{v}(k) \delta_{l, l'} - \tilde{A}(l, p)u(p)\tilde{A}^*(l', p)
\]

(25)

with \( k = p + l \), and \( A, u \) from eqs. (21) and (22).

Since \( Z \) is constant, it follows that

\[
L_{eff}(\Phi) = \frac{1}{2}(\Phi, u^{-1}\Phi) + V_{eff}(\Phi) + \text{const.}
\]

(26)

\[
\exp(-V_{eff}(\Phi)) = \int d\mu(\zeta) \exp(-V(\zeta)) \quad \text{with} \quad V(\zeta) = V(A\Phi + \zeta).
\]

(27)

Thus, \( V_{eff} \) is the free energy of a field theory with free propagator \( \Gamma \) and \( \Phi \)-dependent coupling constants. It admits a standard perturbation expansion (cumulant expansion). We will write down the leading terms for the \( T > 0 \) case below. All this is well known.

### 3.4 Scalar fields at finite temperature

At finite temperature, we have periodicity in time with period \( \beta \). The extension \( L_t \) of blocks in time direction must be chosen commensurate with \( \beta \).

A great simplification results if we chose \( L_t = \beta \), so that only one block fits in time direction. The lattice \( \Lambda \) becomes a 3-dimensional lattice.
Our 3-dimensional fields still have the same dimension as the original 4-dimensional ones. This could be remedied by a rescaling by $\beta^{\frac{1}{2}}$. We prefer not to do so because eq. (34) below looks very natural. Since we do not rescale the block spin $\Phi$ in position space the appropriate 3-dimensional integration measure contains a factor $\beta$ and the 3-dimensional $\delta$-functions a corresponding factor $\beta^{-1}$, as follows.

$$\int_{x \in \Lambda} = \beta L_s^3 \sum_{n \in \mathbb{N}^3}, \quad x = (nL_s, 0). \quad (28)$$

In momentum space it is correspondingly.

To adjust to the finite temperature situation, we need to periodize the $T = 0$ quantities in time in the manner described in section 2. Let us begin with the fluctuation field propagator of eq. (20). We write $z = (z, t)$.

$$\Gamma_T(z, t; z', t') = \sum_{n \in \mathbb{Z}} \Gamma(z, t + n\beta; z', t') \quad (29)$$

We insert the Fourier expansion for $\Gamma$. Since $l_4, l_4' \in 2\pi L_i^{-1}Z = 2\pi \beta^{-1}Z$, we have $l_4t = l_4(t + n\beta) \mod 2\pi$. Only a sum $\sum_a \exp(ip_4(t + n\beta))$ needs to be done. This sum was evaluated in section 2. As a result

$$\Gamma_T(z, t; z', t') = \sum_{l, l'} \frac{1}{\beta} \frac{d^3p}{(2\pi)^3} \exp(i(k - ik') \cdot z' + il_4t - il_4't') \tilde{\Gamma}(l, p, 0, l') \quad (30)$$

where $k = p + l$ and $k' = p + l'$.

In other words, periodization has the effect of setting $p_4 = 0$. This reflects the fact that the lattice $\Lambda$ is only 3-dimensional.

Similarly one finds the interpolation and averaging kernels and the block propagator.

$$\tilde{\mathcal{A}}\mathcal{T}(l, p) = \tilde{\mathcal{A}}(l, p, 0), \quad \tilde{\mathcal{C}}\mathcal{T}(l, p) = \tilde{\mathcal{C}}_s(l, p) \delta_{l_4, 0} \quad (31)$$

The block propagator in momentum space is $\tilde{u}(p, 0)$. It is temperature dependent, because the range $l_4 \in 2\pi L_i^{-1}Z$ of the $l_4$-summation depends on $\beta = L_t$. For the same reason, the fluctuation field propagator and the interpolation kernel are $T$-dependent.

It turns out that the temperature dependence of this block spin propagator is a simple factor $\beta^{-1}$. Therefore we agree to extract this factor, writing

$$\tilde{u}_T(p) = \tilde{u}(p, 0) = \beta^{-1} \tilde{u}_{FT}(p) \quad (32)$$
with \( T \)-independent 'finite temperature' propagator \( \tilde{u}_{FT}(p) \). In coordinate space it is expressed in terms of the original bare zero temperature propagator

\[
 u_{FT}(x - y) = \text{av}_{\mathbf{x} \in \mathbf{x}} \text{av}_{\mathbf{y} \in \mathbf{y}} \int_0^\beta dt \, v_T(z - z', t) = \text{av}_{\mathbf{x} \in \mathbf{x}} \text{av}_{\mathbf{y} \in \mathbf{y}} \int_{-\infty}^{\infty} dt \, v_0(z - z', t) \quad (33)
\]

\([\mathbf{x}]\) is the 3-dimensional cube of sidelength \( L_s \) with central point \( \mathbf{x} \).

### 3.5 Leading terms in the perturbation expansion of the perfect action for \( \varphi^4 \)-theory at finite temperature

The perfect lattice action at finite temperature equals

\[
 L_{\text{eff},T}(\Phi) = \frac{\beta}{2} \left( \Phi, u_{FT}^{-1} \Phi \right) + V_{\text{eff},T}(\Phi) \quad (34)
\]

\[
 V_{\text{eff},T}(\Phi) = -\ln \left( \int d\mu_T(\zeta) \exp \left( -V_{\Phi,T}(\zeta) \right) \right) \quad (35)
\]

with

\[
 V_{\Phi,T}(\zeta) = V (\mathcal{A}_T \Phi + \zeta) \quad (36)
\]

The effective interaction (including mass terms) is temperature dependent because the fluctuation field propagator \( \Gamma_T \) and the interpolation kernel \( \mathcal{A}_T \) are both temperature dependent. This temperature dependence is weak and disappears in a zeroth order local approximation.

Let us consider \( \varphi^4 \)-theory with bare mass \( m_0 \).

\[
 L_0(\varphi) = \frac{1}{2} \int (\partial_\mu \varphi \partial^\mu \varphi) \, dz \quad (37)
\]

\[
 V(\varphi) = \int \left( \frac{1}{2} m_0^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right) \, dz + \text{wave function renormalization term} \quad (38)
\]

Inserting the split (16) of the field \( \varphi \), we obtain

\[
 V_{\Phi}(\zeta) = U_{ct}(\Phi) + \sum_{n=1}^4 \int dz \frac{1}{n!} g_n(\Phi, z) \, \zeta(z)^n \quad (39)
\]

with
\[
\begin{align*}
g_1 &= \lambda \frac{1}{3!} \Psi(z)^3 + m_o^2 \Psi(z) \\
g_2 &= \lambda \frac{1}{2!} \Psi(z)^2 + m_o^2 \\
g_3 &= \lambda \Psi(z) \\
g_4 &= \lambda \\
U_{cl}(\Phi) &= \int \left( \frac{1}{2} m_o^2 \Psi(z)^2 + \lambda \frac{3!}{4!} \Psi(z)^4 \right) dz \\
\Psi(z) &= \int_{x \in \Lambda} A_T(z, x) \Phi(x) 
\end{align*}
\]

\(V_{eff,T}\) can be calculated in a loop expansion. When the starting point is a lattice theory, one can use Mayer expansions instead. They are convergent for weak coupling, and the asymptotic expansion of individual terms in powers of the bare coupling constant contains infinite sets of diagrams \([18]\).

We write the perturbative expansion of \(V_{eff,T}\) as

\[
V_{eff,T}(\Phi) = U_{cl}(\Phi) + \sum_{N \geq 1} V_{eff,T}^{(N)}(\Phi) \tag{41}
\]

where \(V_{eff,T}^{(N)}\) scales as \(\gamma^N\) when the fluctuation field propagator \(\Gamma_T \rightarrow \gamma \Gamma_T\).

Let us compute \(V_{eff,T}^{(1)}\). Indicating factors \(\Psi\) by dotted external lines and a fluctuation field propagator by a solid line we have

\[
\begin{align*}
\text{Figure 1 effective potential to first order in the fluctuation propagator}
\end{align*}
\]

\[
\begin{align*}
V_{eff,T}^{(1)} &= \frac{1}{2!} \int dz \left( \frac{1}{2} \lambda \Psi(z)^2 + m_o^2 \right) \Gamma_T(z, z) \\
&\quad - \frac{1}{2!} \int dz_1 dz_2 \left( \frac{1}{3!} \lambda \Psi(z_1)^3 + m_o^2 \Psi(z_1) \right) \Gamma_T(z_1, z_2) \left( \frac{1}{3!} \lambda \Psi(z_2)^3 + m_o^2 \Psi(z_2) \right) \tag{42}
\end{align*}
\]

The fluctuation field propagator \(\Gamma_T(z, z')\), decays exponentially with distance \(|z - z'|\) with decay length one block lattice spacing. That is, it decays with \(|z - z'|\) with decay length \(L_s\). Similarly, \(A_T(z, x)\) decays exponentially in \(|z - x|\) with decay length \(L_s\). As a result, each term in \(V_{eff,T}\) is local in \(\Phi\) modulo exponential tails with decay length of one lattice spacing.
By a process of partial integration (or rather summation) one can exhibit each term in $V_{\text{eff},T}$ as a sum of local terms and small remainders which represent the exponential tails (see appendix A). The terms with coefficients of dimensions up to $(\text{mass})^{-2}$ are

$$V_{\text{eff},T}(\Phi) = \int_x \left\{ \frac{1}{2} m^2 \Phi^2 + \delta_z (\nabla_\mu \Phi)^2 + \frac{\lambda_r}{4!} \Phi^4 + \frac{\lambda_6}{6!} \Phi^6 + \tilde{\gamma} (\nabla_\mu \Phi)^2 \Phi^2 + \cdots \right\}$$  \hspace{1cm} (43)

Since $\tilde{u}_{PT}^{-1}(p) \sim p^2$ as $p \to 0$, we may write the expansion equally well in the form

$$V_{\text{eff},T}(\Phi) = \int_x \left\{ \frac{1}{2} m^2 \Phi^2 + \frac{\lambda_r}{4!} \Phi^4 + \frac{\lambda_6}{6!} \Phi^6 \right\} + \beta \int_x \int_y \Phi(x) \tilde{u}_{PT}^{-1}(x,y) \Phi(y) \left[ 2\delta_z + \gamma \Phi(x)^2 \right] + \cdots$$  \hspace{1cm} (44)

All the coefficients are $T$-dependent and finite.

For computer simulations it is appropriate to consider $V_{\text{eff},T}$ as a function of $\Phi(x)$. For analytical computations it is most convenient to regard it as a function of $\Psi(z)$.

### 3.6 Note on the cancellation of temperature dependent UV-divergent diagrams

The UV-convergence of a quantum field theory concerns its local behaviour and is therefore not temperature dependent. If the proper choice of counter terms makes the theory finite at zero temperature, then also at finite temperature. The cancellation occurs order by order in perturbation theory. This is well known.

Problems can occur when one sums selected classes of diagrams. We do not propose to do so when deriving the perfect action, but stay strictly within the realm of standard perturbation theory. So there can be no problem.

It is nevertheless appropriate to point out that there do exist individual diagrams with temperature dependent UV-divergent pieces. They cancel. We give an example.

Let us split the (fluctuation field) propagator $\Gamma$ into a static part (heavy line) $\Gamma_{\text{stat}}$ and a non static part $\Gamma_{\text{ns}}$ (wavy line). The non static part represents random walks which wind several times around the tube; it is not singular at coinciding arguments : $\Gamma_{\text{ns}}(z,z') < \infty$. But it is not zero and is $T$-dependent. As a result, the diagrams
are both logarithmically UV-divergent, with a temperature dependent coefficient
$\Gamma_{\text{ns}}(z,z')$. They must cancel. $\delta\lambda$ is the logarithmically divergent coupling constant counter term to second order in $\lambda$.

4 The Feynman-Bogoliubov Method

To obtain information on the nature of a phase transition at finite temperature, one wants to compute the effective potential, i.e. the free energy as a function of the magnetization.

$$ M = \int dz \varphi(z) = \int_x \Phi(x) $$

(45)

Alternatively, one may compute the constraint effective potential which gives the probability distribution of $M$.

To do so we should apply nonperturbative methods to solve the lattice theory. One of these methods consists in the solution of gap equations. It is very old and known as the Feynman-Bogoliubov method [12].

The gap equations have "perturbative" solutions which come from summations of superdaisy diagrams. In principle they might also have other solutions. We wish to examine this method in order to see what will be the effect of terms like $(\nabla \mu \Phi)^2 \Phi^2$ etc. in the perfect lattice action.

Let us make it clear that it is not sufficient to find the solutions of these gap equations. In order to justify the perturbative calculation of the perfect action it will be necessary to investigate also the stability properties of the solutions of the gap equations against small perturbations of the lattice action. We will comment on this, but a thorough treatment of this question is beyond the scope of this paper.

Given an action $S(\Phi)$ of some Euclidean field theory, one seeks an optimal quadratic approximation $S_{\text{free}}(\Phi)$ around which to expand

$$ S_{\text{free}}(\Phi) = \frac{1}{2} \int_x \int_y \Phi(x) J(x,y) \Phi(y) $$

(46)

By Peierls inequality [12], the partition functions obey the inequality

$$ \ln(Z) \geq \ln(Z_{\text{free}}) - \langle S - S_{\text{free}} \rangle_{\text{free}} $$

(47)

for any choice of the $S_{\text{free}}$. Herein, $\langle \rangle_{\text{free}}$ is the expectation value in the theory with action $S_{\text{free}}$. The right hand side is the first order approximation to the left hand side in the perturbative expansion around $S_{\text{free}}$. The optimal choice of $S_{\text{free}}$ is that which makes the right hand side of (47) maximal. In other words, it
makes the first order approximation as good as possible. It is asserted that there exists always a unique optimal choice of \( S_{\text{free}} \). It is not asserted that the optimal choice is necessarily a good one. The optimal \( J \) is determined by the extreme value condition

\[
\left\langle \frac{\delta^2 S}{\delta \Phi(x) \delta \Phi(y)} \right\rangle_{\text{free}} = J(x, y) \tag{48}
\]

This is equivalent to the condition that the right hand side of (47) is maximal, viz

\[
\frac{\delta}{\delta J} \left[ \ln(Z_{\text{free}}) - \langle S - S_{\text{free}} \rangle_{\text{free}} \right] = 0 \tag{49}
\]

When applied to the standard \( \Phi^4 \) action, this produces the gap equation whose perturbative solution is the sum of superdaisy diagrams (see below).

Let us consider the gap equation which results from the more complicated lattice action.

\[
S(\Phi) = \frac{\beta}{2}(\Phi, u_{FT}^{-1}\Phi) + V_{\text{eff},T}(\Phi) \tag{50}
\]

with \( V_{\text{eff},T} \) from equation (44).

We obtain

\[
\frac{\delta^2 S(\Phi)}{\delta \Phi(x) \delta \Phi(y)} = \beta u_{FT}^{-1}(x, y) \left[ 1 + 2\delta_z + 3\gamma \Phi(x)^2 \right] \\
+ \beta \delta(x - y) 3\gamma \Phi(x)(u_{FT}^{-1}\Phi)(x) \\
+ \delta(x - y) \left( m^2 + \frac{\lambda_r}{2!} \Phi(x)^2 + \frac{\lambda_6}{4!} \Phi(x)^4 \right) \tag{51}
\]

The expectation value in the theory with action \( S_{\text{free}} = \int \frac{1}{2} \Phi J \Phi \) is

\[
\left\langle \frac{\delta^2 S(\Phi)}{\delta \Phi(x) \delta \Phi(y)} \right\rangle_{\text{free}} = \beta u_{FT}^{-1}(x, y) \left[ 1 + 2\delta_z + 3\gamma J^{-1}(0) \right] \\
+ \beta \delta(x - y) 3\gamma (u_{FT}^{-1}J^{-1})(0) \\
+ \delta(x - y) \left( m^2 + \frac{\lambda_r}{2} J^{-1}(0) + \frac{\lambda_6}{2^22!} J^{-1}(0)^2 \right) \tag{52}
\]

where \( J^{-1}(0) = J^{-1}(x, x) \) is independent of \( x \) by translation invariance, assuming that we seek a translation invariant solution. Other solutions could be of interest.

The gap equation (48) can be solved by the Ansatz
\[ J(x, y) = A u_{FT}^{-1}(x, y) + B \delta(x - y) \quad (53) \]

Inserting the Ansatz results in two transcendental equations for \( A, B \) whose solutions depend on the coefficients \( \beta, \delta_z, \gamma, \lambda_i \).

Basically, the inclusion of the \((\nabla_\mu \Phi)^2 \Phi^2\) -term results in a system of two equations for mass and wave functions renormalization. In standard \( \Phi^4 \)-theory there is only one equation for the mass.

### 4.1 Comparison with other approaches

The standard method of dealing with the finite temperature phase transition of \( \Phi^4 \)-theory is described by Kapusta [1]. It consists in the summation of ring or daisy diagrams which contribute to the self energy part \( \Pi \) of the physical finite temperature propagator.

It would seem natural to improve this by solving the gap equations instead. This would replace the sum over daisy diagrams by a sum over superdaisy diagrams. The temperature dependent mass \( m^2(T) \) would be determined in a self consistent way. But there is a problem with this: There arises a temperature dependent logarithmically divergent contribution to the self energy which cannot be cancelled by a temperature-independent bare mass term. The approach outlined in this paper avoids this problem, because UV-divergences are cancelled before one derives the gap equation. We see here the advantage of making a clean separation between UV- and IR-problems.

Let us explicitly show how the problem arises in a conventional superdaisy approach without a lattice as an UV-cutoff.

We use the standard \( \Phi^4 \)-action in place of (50) and make the usual Ansatz for the solution of the gap equation.

\[ J(x, y) = -\Delta \delta(x - y) + m^2(T) \delta(x - y) \quad (54) \]

We obtain the gap equation (48) as before with \( \delta_x = \gamma = \lambda_0 = 0, u^{-1} = -\Delta \). It is solved if

\[ m^2(T) = m_0^2 + \frac{\lambda}{2} J^{-1}(0) \quad (55) \]

\[ J^{-1}(0) = T \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3} \left[ \omega_n^2 + \mathbf{p}^2 + m^2(T) \right]^{-1} \quad (56) \]

The integral defining \( J^{-1}(0) \) is evaluated in the standard fashion, resulting in
\[ J^{-1}(0) = \Pi_{\text{vac}} + \Pi_{\text{mat}} \]  

\[ \Pi_{\text{vac}} = \int \frac{d^4p}{(2\pi)^4} \left[ p^2 + m^2(T) \right]^{-1} = \frac{1}{16\pi^2} \left[ \Lambda^2 - m^2(T) \ln \frac{\Lambda^2}{m^2(T)} \right] \]  

\[ \Pi_{\text{mat}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega} \left[ \exp(\beta\omega) - 1 \right]^{-1} ; \quad \omega = (p^2 + m^2(T))^{1/2} \]  

The integrals are UV-divergent, therefore a regulator \( \Lambda \) is introduced. It is supposed to be taken to infinity after cancellation against counterterms. We see that without a lattice a temperature dependent counterterm is needed to cancel the logarithmic UV-divergence in (58). This is not acceptable.

Conclusion: The Feynman-Bogoliubov method is designed to deal with infrared problems. It should only be applied after a correct cancellation of UV-divergences has been achieved through the computation of an effective field theory, e.g. on a lattice, or with another low UV-cutoff.

We wish to compare our proposal with another approach. It was proposed to derive first a 3-dimensional continuum theory. When no UV-regularization is introduced, then this 3-dimensional theory is superrenormalizable but not finite. It has linear self energy divergences. These divergences are cancelled by divergent terms in the 3-dimensional action. Again the problem remains that the UV-divergences have not been completely cancelled before nonperturbative methods are applied to deal with infrared aspects. In addition, the 3-dimensional continuum theory is complicated because of nonlocalities with decay length \( \beta \frac{2\pi}{p} \) which appear as soon as one goes beyond the leading order in the perturbative expansion of the 3-dimensional action (see appendix B). The use of a momentum expansion to deal with these nonlocalities would aggravate the UV-problems.

The 'average action' approach of Wetterich [13] is in the same spirit as ours. The difference is that we use a lattice. The lattice has the advantage that one can do computer simulations. Wetterichs work suggests moreover, that one should perform simplifications only on the 1-particle irreducible parts of the perfect action (i.e. those not held together by a single \( \Gamma \)-propagator). It has been known for a long time that the 1-particle reducible second term \( \Psi^3 \Gamma_T \Psi^3 \) in eq.(12) is important inspite of looking irrelevant [16].

5 Perfect lattice action for Maxwell theory

We wish to extend the consideration of the previous sections to scalar electrodynamics. As a preparation for this we consider the free abelian gauge field.

In this section we recall the Balaban-Jaffe block spin transformation for the free abelian gauge field at zero temperature.
5.1 Averaging operators for abelian gauge fields

Given a vector potential \( a(z) = a_\mu(z) dz_\mu \) on the continuum, we define a block spin \( A \) on the block lattice. \( A \) lives on links \( b \) of the block lattice. We use the alternative notation

\[
A[b] = A_\mu(x)
\]

(60)

when \( b \) is the link emanating from \( x \) in \( \mu \)-direction.

We will not distinguish in notation between the averaging operator \( C \) for gauge fields, and for scalars (gauge transformations). Which one is meant will be clear from the context. Thus \( A = Ca \).

Given a link \( b \) from \( x \) in \( \mu \)-direction, and a point \( z \in x \), let \( C_{z,\mu} \) be the straight path of length one block lattice spacing in \( \mu \)-direction starting from \( z \). If \( \mu = 4 \), \( C_{z,\mu} \) connects \( z \) with \( z + L_t e_4 \), and if \( \mu \neq 4 \) it connects \( z \) with \( z + L_s e_\mu \) (\( e_\mu = \) unit vector in \( \mu \)-direction). The blocking procedure is defined by

\[
A[b] = \text{av}_{z \in x} a[C_{z,\mu}]
\]

(61)

\[
a[C_{z,\mu}] = \int_{C_{z,\mu}} dw_\nu a_\nu(w)
\]

(62)

This blocking procedure is covariant under gauge transformations in the following sense. If

\[
a_\mu(z) \rightarrow a_\mu(z) - \partial_\mu \lambda(z) \equiv a_\mu^\lambda(z)
\]

(63)

then

\[
A_\mu(x) \rightarrow A_\mu(x) - \nabla_\mu \Lambda(x)
\]

(64)

\[
\Lambda = C \lambda
\]

(65)

i.e.

\[
\Lambda(x) = \text{av}_{z \in x} \lambda(z)
\]

(66)

We use the notation \( a^\lambda \) for the gauge transform of \( a \), etc. In this notation, the covariance property reads
\[CA^\lambda = (Ca)^C\lambda\]  

We will need the Fourier transform of the \(\mathcal{C}\)-kernel. Let us write

\[A_\mu(x) = \int \mathcal{C}_{\mu\nu}(x,z) a_\nu(z)\]  

(68)

The kernel \(\mathcal{C}_{\mu\nu}\) will have a Fourier expansion just like the scalar kernel. Explicitly

\[\tilde{\mathcal{C}}_{\mu\nu}(l,p) = \delta_{\mu\nu} \exp(-i \frac{k_\nu L_\nu}{2} \sin \frac{k_\nu L_\nu}{2} \tilde{C}(l,p))\]  

(69)

where \(\tilde{C}(l,p)\) is the scalar kernel (6), and no sum over \(\nu\) is implied. \(L_\nu = L_t\) if \(\nu = 4\) and \(L_\nu = L_s\) otherwise; \(k_\nu = p_\nu + l_\nu\) as usual.

### 5.2 Perfect lattice action for the free electromagnetic field

Using the symbol \(\partial\) for the exterior derivative,

\[\partial_a = \frac{1}{2} (\partial_\mu a_\nu - \partial_\nu a_\mu) dz_\mu \wedge dz_\nu,\]

the Maxwell action of the electromagnetic field can be written as

\[S_M(a) = \frac{1}{2} \langle \partial a, \partial a \rangle\]  

(70)

The perfect lattice action associated with this is defined by a formula analogous to eq. (27) for the scalar field. There is one difference, however. In order to give meaning to the functional integral, some amount of gauge fixing is necessary.

We wish to obtain a gauge covariant perfect lattice action. Therefore we wish to retain the freedom of gauge transformations on the lattice, i.e. one gauge degree of freedom per block. Therefore global gauge fixing is not appropriate. Instead, gauge fixing is only used locally within each block.

Consider gauge transformations

\[a_\mu \to a_\mu - \partial_\mu \lambda\]  

with \(C\lambda = 0\)  

(71)

By eq. (67), they leave the block gauge field invariant. These are the gauge degrees of freedom which will be eliminated by fixing the gauge within a block. There remains one gauge degree of freedom per block \(\Lambda(x) = C\lambda(x)\) which is not affected by the fixing. It extends to a global gauge transformation per block,

\[\lambda(z) = \Lambda(x)\]  

for \(z \in x\).

To begin with, a block axial or radial gauge is used. For the purpose of perturbative calculations, a transformation to a block Landau gauge is carried out later.
Suppose for a moment that we start from a theory on a fine lattice $\Lambda_{\text{fine}}$ instead of the continuum. $a$ will then live on links $l$ of $\Lambda_{\text{fine}}$. We describe the block axial gauge for this situation first. The formal continuum limit will be obvious.

For every block $y \in \Lambda$, a maximal tree $Ax(y)$ is selected, for instance the comb within each block shown in the figure below.

![Figure](image.png)

Figure 3: maximal trees within 2-dimensional blocks

$Ax(y)$ consist of links $l \in \Lambda_{\text{fine}}$. In the block axial gauge $a[l] = 0$ for all $l \in Ax(y)$, $y \in \Lambda$.

One defines appropriate $\delta$-functions which are inserted in the functional integral

$$\delta_{Ax(y)}(a) = \prod_{l \in Ax(y)} \delta (a[l]) \quad (72)$$

$$\delta_{Ax}(a) = \prod_{y \in \Lambda} \delta_{Ax(y)}(a) \quad (73)$$

The perfect lattice action $S_{\text{eff}}(A)$ is defined up to an additive constant by

$$\exp(-S_{\text{eff}}(A)) = \int Da \delta_{Ax}(a) \delta (C a - A) \exp(-S_M(a)) \quad (74)$$

Instead of the block axial gauge, the block radial gauge can be used. In the continuum limit it is

$$\sum_{\mu=1}^{4} (z - x)_{\mu} a_{\mu}(z) = 0 \quad \text{for} \quad z \in x \quad . \quad (75)$$

5.2.1 Transformation to the block Landau or $\alpha$-gauge

For perturbative calculations, the block axial gauge is inappropriate because of the bad ultraviolet behaviour of the fluctuation field propagator in this gauge. Therefore the Faddeev-Popov method is used to convert to a block Landau or $\alpha$-gauge.

$S_M$ is invariant under arbitrary gauge transformations while $\delta (C a - A)$ is only invariant under those gauge transformations $a \rightarrow a^\lambda = a - \partial \lambda$ which obey
\[ C \lambda = 0 \quad (76) \]

\( C \) is the averaging of a scalar function over blocks. We wish to eliminate the freedom of performing gauge transformations which obey (76) by gauge fixing.

Recalling that \( \partial \) stands for the exterior derivative of a p-form, \( \partial^\dagger \) is the coderivative. For example \( \partial^\dagger a = \partial_\mu a_\mu \) and \( \partial^\dagger \lambda = 0 \) if \( \lambda \) is a 0-form. In this language the Laplacian is given by \( \partial \partial^\dagger + \partial^\dagger \partial \).

To convert to the block \( \alpha \)-gauge, one multiplies and divides expression (74) by

\[
V(a) = \int D\lambda \, \delta(C\lambda) \exp \left\{-\frac{1}{2\alpha} < \partial^\dagger a^\lambda, \partial^\dagger a^\lambda > \right\}
\]

After interchanging the order of integration and performing a gauge transformation \( a \rightarrow a^{-\lambda} \) in the \( a \)-integral

\[
\exp(-S_{\text{eff}}(A)) = \int Da \frac{1}{V(a)} \delta(Ca-A) \exp(-S_M(a) - \frac{1}{2\alpha} < \partial^\dagger a, \partial^\dagger a >) \int D\lambda \delta(C\lambda) \delta_{Ax}(a^{-\lambda})
\]

The second factor is 1 because \( \lambda \) is fixed up to a global transformation by the condition \( \delta(a^{-\lambda}[l]) \) for every link \( l \in Ax(y) \) in the maximal tree of block \( y \), and the remaining global transformation is absorbed by the \( \delta \)-function \( \delta(C\lambda) \).

It remains to evaluate

\[
V(a) = \int D\lambda \, \delta(C\lambda) \exp \left\{-\frac{1}{2\alpha} < \partial^\dagger a^\lambda, \partial^\dagger a^\lambda > \right\}
\]

\[
= \int D\lambda \delta(C\lambda) \exp \left\{-\frac{1}{2\alpha} < (\Delta^{-1}\partial^\dagger a - \lambda), \Delta^\dagger \Delta(\Delta^{-1}\partial^\dagger a - \lambda) > \right\}
\]

Thinking of the \( \delta \)-function as a limit of a Gaussian, one sees that this is a Gaussian integral. But since the \( \delta \)-function restricts the range of the integration the result is not just a constant. To show this we introduce the projector \( R \) onto those functions \( \lambda \) which satisfy the constraint \( C\lambda = 0 \). \( R \) can be chosen orthogonal with respect to the scalar product \( \langle \cdot, \Delta^\dagger \Delta \cdot \rangle \). The integral is split into two parts, the integration over \((1-R)\lambda\) is restricted by the \( \delta \)-function to \((1-R)\lambda = 0\) whereas the integration over \( R\lambda \) has no further restriction. Since \( R \) is an orthogonal projector there exists no mixed term and the integral is a pure Gaussian.
\[ V(a) = \text{const } \exp \left\{ -\frac{1}{2\alpha} <\partial^\dagger a, \Delta(1 - R)\Delta^{-1}\partial^\dagger a > \right\} \]  

Inserting this into eq. (78) we obtain the final result

\[ \exp(-S_{\text{eff}}(A)) = \int Da \, \delta(Ca - A) \exp(-S_M(a) - \frac{1}{2\alpha} <\partial^\dagger a, \Delta R\Delta^{-1}\partial^\dagger a >) \]  

The projector \( R \) ensures that only the gauge degrees of freedom \( \lambda \) with \( C\lambda = 0 \) – i.e. vanishing block averages of \( \lambda \) – are fixed. There remains the freedom of performing global gauge transformations on each block. This freedom is reflected in the fact that \( \exp(-S_{\text{eff}}(A)) \) is invariant under gauge transformations \( A \rightarrow A^\Lambda = A - \partial \Lambda \), where \( \Lambda \) is a scalar function on the lattice.

### 5.2.2 Explicit formula for \( R \)

We recall from section 3 that the fluctuation field \( \zeta \) associated with a scalar field \( \phi \) satisfies \( C\zeta = 0 \) and can be obtained by applying a projector

\[ \zeta = (1 - A \, C) \, \varphi \]  

\( A \, C \) is an orthogonal projector with respect to the scalar product which is furnished by the kinetic term for \( \varphi \).

Here we look for an orthogonal projector with respect to the scalar product employing the kinetic term for \( \lambda \) which is \( \Delta^2 \) in place of \( -\Delta \). Therefore we may write

\[ R = 1 - A^{(\lambda)} \, C \]  

where \( C \) is the same averaging kernel for scalars as before with its Fourier transform given by eq. (8), while \( A^{(\lambda)} \) is chosen to satisfy

\[ v^{-1}A^{(\lambda)} = C^\dagger u^{-1}; \quad u = CvC^\dagger; \quad v^{-1} = \Delta^2 \]  

in complete analogy to eq. (15).

The Fourier transform is

\[ \tilde{A}^{(\lambda)}(l, p) = \tilde{v}(k)\tilde{C}^*(l, p) \, \tilde{u}^\lambda(p)^{-1} \]  

with \( k = l + p, \quad \tilde{v}(k) = \frac{1}{k^2} \) and

\[ \tilde{u}^\lambda(p) = \sum_l \tilde{v}(k) \left| \tilde{C}(l, p) \right|^2 \]
5.3 Interpolation kernel $\mathcal{A}$ and evaluation of the perfect lattice Maxwell action

The perfect lattice action is determined in essentially the same way as for the scalar field.

Given the block gauge field $\mathbf{A}$, one seeks that gauge field $\hat{\mathbf{a}}$ which minimizes $S_M(a) + \frac{1}{2\alpha} < \partial^i a, \Delta R \Delta^{-1} \partial^i a >$ subject to the constraint $C \hat{a} = \mathbf{A}$. Because the action is quadratic, $\hat{a}$ is a linear function of $\mathbf{A}$:

$$\hat{a} = \mathcal{A} \mathbf{A} \quad \text{i.e.} \quad \hat{a}_\mu(z) = \int_{\text{links } b \in \Lambda} \mathcal{A}_\mu(z, b) \mathbf{A}[b] \quad (88)$$

Balaban and Jaffe show that $\hat{a}$ – and therefore $\mathbf{A}$ – are independent of $\alpha$ and satisfy the block Landau gauge condition

$$\mathcal{R} \partial^i \hat{a} = 0 \quad \text{i.e.} \quad \mathcal{R} \partial^i \mathbf{A} = 0 \quad (89)$$

where $\mathcal{R}$ is a shorthand notation for $\Delta R \Delta^{-1}$. The proof is based on constructing a gauge transformation $\lambda$ with $C \lambda = 0$ which ensures (89). The equation

$$\Delta \lambda = \mathcal{R} \partial^i \hat{a} \quad (90)$$

has a unique solution in the subspace $C \lambda = 0$ because $\mathcal{R}$ is the projector onto those configurations $\Delta \lambda$ which satisfy $C \lambda = 0$.

The kernel $\mathcal{A}_\mu(z, b)$ has exponential decay in the distance of $z$ from the blocks $(x, y) = b$ [8].

The effective action is obtained by shifting the field $a = \mathcal{A} \mathbf{A} + a'$ where $a'$ is called the fluctuation field. The integral over $a'$ produces merely a constant and one obtains

$$\exp(-S_{\text{eff}}(\mathbf{A})) = \exp \left( -\frac{1}{2} < \partial \mathcal{A} \mathbf{A}, \partial \mathcal{A} \mathbf{A} > -\frac{1}{2\alpha} < \partial^i \mathcal{A} \mathbf{A}, \mathcal{R} \partial^i \mathcal{A} \mathbf{A} > \right) \quad (91)$$

$$= \exp \left( -\frac{1}{2} < \partial \mathcal{A} \mathbf{A}, \partial \mathcal{A} \mathbf{A} > \right) \quad (92)$$

independent of $\alpha$ because of eq. (89). Clearly $S_{\text{eff}}(\mathbf{A})$ is a bilinear function of $\mathbf{A}$. One may therefore define an operator $\Delta_1$ acting on block gauge field configurations $\mathbf{A}$, such that

$$S_{\text{eff}}(\mathbf{A}) = \frac{1}{2} < \mathbf{A}, \Delta_1 \mathbf{A} > \quad (93)$$

Explicitly

$$\Delta_1 = \mathcal{A}^\dagger \partial^i \partial \mathcal{A} \quad (94)$$
5.4 Fluctuation field propagator

The probability distribution of the fluctuation field is given up to a normalization factor by \((\kappa = \infty)\)

\[
\delta_\kappa(Ca') \exp \left\{ -\frac{1}{2} < \partial a', \partial a' > -\frac{1}{2\alpha} < \partial^\dagger a', R \partial^\dagger a' > \right\}
\]  

(95)

Thinking again of the \(\delta\)-function as a limit of a Gaussian

\[
\delta_\kappa(Ca') = \left(\frac{2\pi}{\kappa}\right)^{-\frac{3}{2}} \exp \left( -\frac{\kappa}{2} < Ca', Ca' > \right)
\]  

(96)

we see that the probability distribution is given by a Gaussian measure with covariance

\[
\Gamma_\alpha = \lim_{\kappa \to \infty} \Gamma_{\kappa,\alpha}
\]  

(97)

\[
\Gamma_{\kappa,\alpha} = \left( \partial^\dagger \partial + \frac{1}{\alpha} \partial R \partial^\dagger + \kappa C^\dagger C \right)^{-1}
\]  

(98)

\(C\) is again the Balaban-Jaffe averaging kernel for the gauge field \([11]\).

The block Landau gauge is \(\alpha = 0\). There exists a formula which expresses the fluctuation field propagator \(\Gamma\) in the block Landau gauge in terms of the propagator \(\Gamma_{\kappa,\alpha}\) for finite \(\kappa\) (e.g. \(\kappa = 1\)) and arbitrary \(\alpha\) \([8]\).

\[
\Gamma_0 = G - \frac{1}{\alpha} G \partial R \partial^\dagger G - G C^\dagger \left( C G C^\dagger \right)^{-1} C G
\]  

(99)

where \(G = \Gamma_{1,\alpha}\).

The propagators \(G\) and \(\Gamma_0\) decay exponentially in coordinate space with decay length one block lattice spacing. When the original theory lives on a lattice, the proofs of reference \([8]\) apply. One can also convince oneself of the fact by examining the behaviour in momentum space near zero momentum. The explicit formulae given below can be used for that.

The interpolation kernel \(A\) (which is independent of \(\alpha\) as we know) can be expressed in terms of \(G = \Gamma_{1,\alpha}\) as well

\[
A = G \left( CG C^\dagger \right)^{-1}
\]  

(100)

Therefore, the block Landau gauge fluctuation field propagator \(\Gamma_0\) can also be written as
\[ \Gamma_0 = G - \frac{1}{\alpha} G \partial \mathcal{R} \partial^\dagger G - A C G C^\dagger A^\dagger \]  

(101)

This resembles the formula for the scalar case. In place of the propagator for the full theory, which does not exist because we have not fixed the gauge completely, there appears the auxiliary quantity \( G = \Gamma_{1,\alpha} \). The expression for \( \Gamma_0 \) is valid for any choice of \( \alpha \) in this auxiliary propagator.

5.5 Perfect Lattice action for the Maxwell field at Temperature \( T > 0 \)

To go from temperature \( T = 0 \) to finite \( T \) one must periodize the propagators and interpolation kernels in time as in the scalar case.

\[ D = (-\Delta_l)^{-1} \]  

(102)

with Fourier transform \( \tilde{D}(p) \). Then the free lattice photon propagator in the 3-dimensional lattice theory which is obtained from the finite temperature field theory is

\[ \tilde{D}_T(p) = \tilde{D}(p,0) \]  

(103)

Similarly, the interpolation kernel

\[ \tilde{A}_T(l,p) = \tilde{A}(l,p,0) \]  

(104)

and the fluctuation field propagator

\[ \tilde{\Gamma}_{0T}(l,p,l') = \tilde{\Gamma}_0(l,p,0,l'). \]  

(105)

As in the scalar case, the discrete variables \( l, l' \) remain 4-dimensional. Think of them as referring to the original 4-dimensional theory.

6 Scalar electrodynamics

Now we are prepared to deal with scalar electrodynamics. The definitions for the gauge fields can be taken literally from the preceding section. Since the Higgs field is not invariant under a gauge transformation we cannot simply use the results of section 3. So it is the Higgs field we are concerned with in this section.
6.1 Block spin transformation for the Higgs field at zero temperature

Given a Higgs field $\phi(z)$, we wish to define a block Higgs field $\Phi(x)$ on the lattice $\Lambda$ in a gauge covariant way. The blocking procedure for the $\varphi^4$-theory cannot be used as it stands because it is not gauge covariant. In order to maintain gauge covariance, one must use averaging kernels depending on the gauge field $a$.

We denote the averaging operator for the Higgs field by $C^H(a)$

$$\Phi = C^H(a) \phi$$  \hspace{1cm} (106)

i.e.

$$\Phi(x) = \int_{z \in x} dz \ C^H(a|x,z) \phi(z)$$  \hspace{1cm} (107)

The adjoint kernel $C^{H\dagger}(a|z,x) = \overline{C^H}(a|x,z)$

In the $\varphi^4$-theory we used an averaging kernel which was constant on blocks and vanished outside. A constant is the lowest eigenvector of the Laplacian with Neumann boundary conditions on block boundaries.

The natural generalization to the gauge covariant situation is as follows. Let $-\Delta_a = (\partial - a)^\dagger(\partial - a)$ denote the covariant Laplacian and $-\Delta_a^{N,x}$ the covariant Laplacian with Neumann boundary conditions on the block boundary of $x$. We demand that

$$-\Delta_a^{N,x} C^{H\dagger}(a|z,x) = \varepsilon_0(a|x) \ C^{H\dagger}(a|z,x)$$  \hspace{1cm} (108)

and

$$C^{H\dagger}(a|z,x) = 0 \hspace{1cm} \text{for} \hspace{1cm} z \notin x$$  \hspace{1cm} (109)

where $\varepsilon_0(a|x)$ is the lowest eigenvalue of $-\Delta_a^{N,x}$. In addition we impose the normalization condition

$$C^H C^{H\dagger} = 1$$  \hspace{1cm} (110)

This leaves the freedom of multiplying $C^H$ with an $a$-dependent phase factor

$$C^{H\dagger}(a|z,x) \rightarrow C^{H\dagger}(a|z,x) \ \eta(a|x)$$  \hspace{1cm} (111)

\footnote{Here we deviate from the work of Balaban, Jaffe and Imbrie [11]}

24
It follows from the gauge covariance of the eigenvalue problem (108) that under a gauge transformation

\[ C^H(a^\lambda) \phi^\lambda = \left(C^H(a) \phi\right)^\Lambda \] (112)

with some gauge transformation \( \Lambda \) on the lattice which depends on \( \lambda \) and on the choice of conventions to fix \( C^H \) uniquely. The freedom in the choice of an \( a \)-dependent phase factor may be exploited to demand that

\[ \Lambda(x) = C \lambda(x) \] (113)

where \( C \) is the scalar averaging kernel introduced in subsection 3.1. Eq. (112) parallels the gauge covariance property of the blocking procedure for the gauge field.

One can compute the averaging kernel \( C^H(a) \) as a solution of the eigenvalue equation (108) by standard quantum mechanical perturbation theory. A prototype of such a computation is found in reference [14].

6.2 The perfect action of scalar electrodynamics: Definition

The perfect action is defined by

\[ \exp \left(-S_{eff}(\Phi, A)\right) = \int Da \int D\phi \delta_{A_\lambda}(a) \delta(Ca - A) \delta_\kappa \left(C^H(a)\phi - \Phi\right) \exp \left(-S_M(a) - \frac{1}{2}(\phi, -\Delta_\lambda\phi) - V(\phi)\right) \] (114)

where

\[ V(\phi) = \int dz \left[ \frac{1}{2}m_0^2 \phi \phi^* + \frac{\lambda}{4!} (\phi \phi^*)^2 \right] \] (115)

To evaluate it perturbatively, we proceed similarly as in \( \varphi^4 \)-theory and pure abelian gauge theory.

We prefer to use a Gaussian \( \delta_\kappa \left(C^H(a)\phi - \Phi\right) \) in place of a \( \delta \)-function.

6.3 Interpolation kernel for the Higgs field and Higgs fluctuation propagator

The interpolation kernel \( A^H \) for the Higgs field will also depend on the gauge field \( a \) because the averaging kernel does. In order to split the kinetic term for the
Higgs field, we impose the usual demand that \( \Psi = \mathcal{A}^H \Phi \) minimizes \((\Psi, -\Delta_a \Psi)\) subject to the constraint that \( C^H(a) \Psi = \Phi \). This happens if

\[
\Delta_a \mathcal{A}^H(a|z, x) = \int_{y \in \Lambda} C^{H\dagger}(a| z, y) \Delta_{eff}(a| y, x)
\]

for some choice of \( \Delta_{eff} \), and if

\[
C^H(a) \mathcal{A}^H(a) = 1
\]

The last condition implies that

\[
(\Delta_{eff}(a))^{-1} = C^H(a) \Delta_a^{-1} C^{H\dagger}(a)
\]

similarly as before (where we used the notation \( u^{-1} \) in place of \( \Delta_{eff} \)).

If we make a shift of the Higgs field

\[
\phi = \mathcal{A}^H \Phi + \zeta
\]

The fluctuation field propagator for the Higgs field is

\[
\Gamma^H(z, w) = (-\Delta_a + \kappa C^{H\dagger} C^H)^{-1}(z, w)
\]

In the limit \( \kappa \to \infty \) it becomes

\[
\Gamma^H = v^H - \mathcal{A}^H C^H v^H C^{H\dagger} \mathcal{A}^{H\dagger}
\]

where \( v^H = -\Delta_a^{-1} \) is the full gauge covariant free massless propagator for the Higgs field.

The interpolation operator and the gauge covariant fluctuation field propagator can also be computed by standard quantum mechanical perturbation theory.

### 6.4 The perfect action of scalar electrodynamics: Representation as a functional integral in block Landau gauge

We use again the Faddeev-Popov trick to convert from block axial gauge to block Landau or \( \alpha \)-gauge. This is done in exactly the same way as for the pure abelian gauge field theory.

The essential point which makes this possible is that both \( \delta \)-functions

\[
\delta(C a - \mathbf{A}) \delta_{\kappa}(C^H(a) \phi - \Phi)
\]
are invariant under gauge transformations \( a \to a^\lambda, \ \phi \to \phi^\lambda \) which obey the constraint \( C\lambda = 0 \). Proceeding as before, we obtain

\[
\exp \left( -S_{\text{eff}}(\Phi, A) \right) = \int Da \int D\phi \delta (Ca - A) \delta_\kappa (C^H(a)\phi - \Phi) \exp \left( -S_M(a) - S_{gf}(a) - \frac{1}{2} (\phi, -\Delta_a \phi) - V(\phi) \right) \tag{123}
\]

where

\[
S_{gf}(a) = \frac{1}{2\alpha} \left( \partial^\dagger a, R \partial^\dagger a \right) \tag{124}
\]

with the same \( R \) as for the pure abelian gauge field.

Now we shift the fields as

\[
\phi = A^H\Phi + \zeta \\
a = AA + a'
\]

The expression for the perfect action becomes

\[
\exp \left( -S_{\text{eff}}(\Phi, A) \right) = \int Da' \int D\zeta \delta (Ca') \delta_\kappa (C^H(a')\zeta) \exp \left[ -S_{M,\text{eff}}(A) - S_{gf}(AA) - S_M(a') - S_{gf}(a') \\
- \frac{1}{2} (\Phi, -\Delta_{eff}(A) \Phi) - \frac{1}{2} (\zeta, -\Delta_a \zeta) - V \left( A^H(a)\Phi + \zeta \right) \right] \tag{126}
\]

where \( a = AA + a' \).

We separate the terms of zeroth order in \( a' \) and \( \zeta \).

\[
\Delta_{eff}(a) = \Delta_{eff}(AA) + E \left( a', A \right) \equiv \tilde{\Delta}_{eff}(A) + E(a', A) \tag{127}
\]

\[
V(A^H(a)\Phi + \zeta) = U_{cl}(A, \Phi) + W(A, a', \Phi, \zeta) \tag{128}
\]

where

\[
U_{cl}(A, \Phi) = V(A^H(AA)\Phi) \tag{129}
\]

The perfect action becomes
\[ S_{\text{eff}}(\Phi, A) = S_{M,\text{eff}}(A) + S_{gf}(AA) + U_d(A, \Phi) - \frac{1}{2} \left( \Phi, -\Delta_{\text{eff}}(A) \Phi \right) \]
\[ + \tilde{S}_{\text{eff}}(\Phi, A) \] (130)

with

\[ \exp \left( -\tilde{S}_{\text{eff}}(\Phi, A) \right) = \int Da'D\zeta \delta(Ca') \delta_\kappa \left( C^H(AA + a')\zeta \right) \]
\[ \exp \left\{ -S_M(a') - S_{gf}(a') - \frac{1}{2} (\zeta, -\Delta a\zeta) \right\} \]
\[ -\frac{1}{2} (\Phi, E(a', A)\Phi) - W(A, a', \Phi, \zeta) \] (131)

To zeroth order in the fluctuation field propagators \( \tilde{S}_{\text{eff}}(\Phi, A) = 0 \) after subtracting a constant.

### 6.5 Scalar electrodynamics at finite temperature

We wish to adopt the considerations of the preceding section to finite temperature \( T \).

We introduce a block lattice \( \Lambda \) as for the \( \phi^4 \)-theory and the Maxwell theory. The side length of the blocks in time direction is \( \beta \). As a result, the block lattice is 3-dimensional.

The blocking of the gauge field at finite temperature was discussed before.

For the Higgs field, some new features appear, because the averaging and interpolation kernels are gauge field dependent. For this reason, a covariant momentum space description does not exist, and we cannot apply exactly the same periodization procedure as for a purely scalar theory. Nevertheless, the transition to finite temperature by periodization in time is straightforward. The formulae of the preceding section remain valid as they stand when they are properly interpreted.

First of all, we must regard the blocks as coming equipped with periodic boundary conditions in time direction. The Neumann boundary conditions apply only to the boundaries of the block which remain, after periodic boundary conditions in time direction are imposed.

Similarly, the defining equation for the interpolation kernel \( A^H_T \) remains the same as for \( T = 0 \), except that periodic boundary conditions in time direction are imposed, and the previously defined averaging kernel \( C^H_T \) is to be used. This equation together with the normalization condition \( C^H_T A^H_T = 1 \) also defines the propagator

\[ u_{\mu\nu}(x, y) = -\Delta_{\text{eff},\mu\nu}(x, y) \] (132)
for the blocked Higgs field. The fluctuation propagator $\Gamma^H$ is obtained from these quantities as before and will also be temperature dependent. All these quantities depend on the gauge field. This implies a hidden $T$-dependence also for the block Higgs field, because the boundary conditions for the gauge field are $T$-dependent.

The effective action is defined as before, with $C^H$ substituted for $C^H$, etc.. It lives on a 3-dimensional lattice $\Lambda$. Because of the anisotropy, the dependence of the resulting action on $A_0$ and on $A_i, \quad i = 1, 2, 3$ is different. The $A_0$ field behaves like an extra scalar field in the effective 3-dimensional theory.

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A  Partial integration of nonlocal terms in the effective action to obtain a sum of local and irrelevant terms

Consider a term which is quadratic in the field such as

\[ I_2(\Phi) = \int \int_{x_1,x_2} \rho_2(x_2 - x_1)\Phi(x_2)\Phi(x_1) \]  

We are interested in situations where \( \rho_2(x) \) falls off exponentially with decay length one lattice spacing. In this case the sum can be rewritten in the form

\[ I_2(\Phi) = \mu^2 \int \Phi(x_1)^2 + z_{\mu\nu} \int \nabla_\mu \Phi(x_1)\nabla_\nu \Phi(x_1) + \text{irrelevant term} \]  

where the irrelevant term is of the form

\[ \gamma_{\mu\nu\rho\sigma} \int \int_{x_1,x_2} \nabla_\mu \nabla_\nu \Phi(x_1)\rho'_2(x_2 - x_1)\nabla_\rho \nabla_\sigma \Phi(x_2) \]  

\( \rho'_2 \) also decays exponentially with distance \( x_2 - x_1 \). The coefficients are

\[ \mu^2 = \int x_1 \rho_2(x) \]  

\[ z_{\mu\nu} = -\frac{1}{2} \int x_1 \rho_2(x) \]  

Because of the exponential falloff of \( \rho_2 \), its Fourier transform \( \tilde{\rho}_2(p) \) is holomorphic in a strip. Because of the presence of a lattice, it is a periodic and even function of \( p \). Therefore

\[ \tilde{\rho}_2(p) = \mu^2 + z_{\mu\nu} \sin p_\mu \sin p_\nu + \gamma_{\mu\nu\rho\sigma} \sin p_\mu \sin p_\nu \sin p_\rho \sin p_\sigma \tilde{\rho}'_2(p) \]  

where \( \tilde{\rho}'_2(p) \) is also holomorphic, periodic and even.

\[ \mu^2 = \tilde{\rho}_2(0) \]  

\[ z_{\mu\nu} = \frac{1}{2} \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu} \tilde{\rho}_2(p)|_{p=0} \]

Inserting back one gets eq. (134).

B  Dimensional reduction to a 3-dimensional theory in continuous space

The method outlined in this paper can also be used to compute the dimensionally reduced 3-dimensional theory in continuous space by perturbation theory. Its
action is local to zeroth order, but will then start to develop nonlocal terms of the form

\begin{equation}
\int \rho(x_1, \ldots, x_n) \Phi(x_1) \cdots \Phi(x_n)
\end{equation}

(141)

where \( \rho \) decays exponentially with the length of the shortest tree on vertices \( x_1, \ldots, x_n \) with decay length \( \frac{\beta}{2\pi} \).

We outline briefly how this is seen. The method had been used before in the work of M. Nießen [19]. He used it to make d-dimensional quantum statistical systems more palatable to a computer by discretizing time in the d+1-dimensional functional integral representation of the system.

In our case, the 3-dimensional fields are time averages of the 4-dimensional ones.

\[ \Phi(r) = \beta^{-1} \int_0^\beta dt \varphi(r, t) = \int dt' dr' C(r, r', t') \varphi(r', t') \]

\[ C(r, r', t') = \beta^{-1} \delta(r - r') \]

(142)

The block propagator \( u(r - r') \) is therefore given by

\[ u(r) = \beta^{-1} u_{FT}(r) = \beta^{-1} \int_0^\beta dt v_T(r, t) = \beta^{-1} \int_0^\infty dt v_0(r, t) \]

(143)

it depends on \( \beta \) only through the overall factor (note that we did not rescale the fields yet).

The interpolation operator is defined as usual, \( \mathcal{A} = v_T C^\dagger u^{-1} \). Because \( C \) is time independent, \( \mathcal{A} \) comes out trivial.

\[ \mathcal{A}(r', t', r) = \beta^{-1} \delta(r - r') \]

(144)

As a result the fluctuation propagator can also be simplified. It is translation invariant both in \( r \) and \( t \), and

\[ \Gamma_T(r, t) = v_T(r, t) - \beta^{-1} \int_0^\beta dt' v_T(r, t') \]

(145)

The second term can be reexpressed using eq.(143). It is seen that the Fourier-transform of \( \Gamma_T \) is the same as of \( v_T \), except for the absence of the zero Matsubara frequency mode. Thus

\[ \Gamma_T(r, t) = \beta^{-1} \sum_{0 \neq n \in \mathbb{Z}} \int \frac{d^3p}{(2\pi)^3} e^{-ipx - i\omega_n t} (\omega_n^2 + p^2 + m^2)^{-1} \]

\[ \omega_n = 2\pi n \beta^{-1} \]

(146)
The decay in $r$ can be read off the singularities of the integrand on the imaginary $p$-axis ($p = |p|$). The closest singularity is at $p = \pm i 2\pi \beta^{-1}$. Therefore there is exponential decay with decay length $\frac{\beta}{2\pi}$.

$\Gamma_T$ appears as free propagator in the perturbative expansion of the 3-dimensional action. Its decay properties will govern the nonlocalities of the resulting action to all orders of perturbation theory. This is familiar from the work of Gawedzki and Kupiainen.

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