ENUMERATIONS OF VERTEXES AMONG ALL ROOTED ORDERED TREES WITH LEVELS AND DEGREES

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Abstract. In this paper we enumerate and give bijections for the following four sets of vertices among rooted ordered trees of a fixed size: (i) first-children of degree $k$ at level $\ell$, (ii) non-first-children of degree $k$ at level $\ell - 1$, (iii) leaves having $k - 1$ elder siblings at level $\ell$, and (iv) non-leaves of outdegree $k$ at level $\ell - 1$. Our results unite and generalize several previous works in the literature.

1. Introduction

Let $T_n$ be the set of rooted ordered trees with $n$ edges. It is well known that the cardinality of $T_n$ is the $n$-th Catalan number

$$\text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}.$$ 

For example, there are 5 rooted ordered trees with 3 edges, see Figure 1. Clearly the number of vertices among trees in $T_n$ is

$$(n + 1) \text{Cat}_n = \binom{2n}{n}.$$ 

Given a rooted ordered tree, a vertex $v$ is a child of a vertex $u$ and $u$ is the parent of $v$ if $v$ is directly connected to $u$ when moving away from the root. A vertex without children is called a leaf. Note that by definition the root is not a child. Vertices with the same parent are called siblings. Since siblings are linearly ordered, when drawing trees the siblings are put in the left-to-right order. Siblings to the left of $v$ are called an elder siblings of $v$. The leftmost vertex among siblings is called the first-child. In Figure 1, there are 10 first-children as well as 10 leaves, which is precisely a half of all 20 vertices in trees in $T_3$.

In 1999, Shapiro [Sha99] proved the following using generating functions.

Theorem 1. For any positive integer $n$, the following four sets of vertices among trees in $T_n$ are equinumerous:

(i) first-children,
(ii) non-first-children,
(iii) leaves, and
(iv) non-leaves.

Here, the cardinality of each set is

$$\frac{1}{2} \binom{2n}{n},$$
Figure 1. The five trees in $T_3$

which is a half of the number of vertices among trees in $T_n$.

Seo and Shin [SS02] gave an involution proving (iii) and (iv) are equinumerous. The equality with the other two sets is easily seen: (i) and (iv) are equinumerous since each non-leaf has its unique first-child and the union of (i) and (ii) are the same with (iii) and (iv).

1.1. Degree and outdegree. The degree of a vertex is the number of edges incident to it. As every edge has a natural outward direction away from the root, we can have the notion of the outdegree of a vertex $v$, which is the number of edges starting at $v$ and pointing away from the root. Since each vertex has a degree and each non-leaf has a positive outdegree, Theorem 1 can be restated as follows.

**Theorem 1** (Shapiro, 1999). Let $n \geq 1$. Among trees in $T_n$, the number of vertices of positive degree equals twice the number of vertices of positive outdegree.

In 2004, Eu, Liu, and Yeh [ELY04] proved combinatorially the following by constructing a two-to-one correspondence which answered a problem posed by Deutsch and Shapiro [DS01, p. 259].

**Theorem 2** (Eu, Liu, and Yeh, 2004). Let $n \geq 1$. Among trees in $T_n$, the number of vertices of odd degree equals twice the number of vertices of odd positive outdegree.

In light of these two results, it is natural to ask if more can be said. In Corollary 6 we will prove that for any positive integer $k$ the number of vertices of degree $k$ always equals twice the number of that of outdegree $k$ among trees in $T_n$.

1.2. Even and odd level. A vertex $v$ in a rooted tree $T$ is at level $\ell$ if the distance (number of edges) from the root to $v$ is $\ell$. By an involution, the following result was obtained by Chen, Li, and Shapiro [CLS07].

**Theorem 3** (Chen, Li, and Shapiro, 2007). The number of vertices at odd levels equals the number of vertices at even levels among trees in $T_n$.

It is again natural to ask if more can be said. In Corollary 8 of this paper we will prove that for any positive integer $k$ the number of vertices of degree $k$ at odd levels always equals the number of vertices of degree $k$ at even levels among trees in $T_n$. 
1.3. **Main result.** In this paper we generalize the above by regarding both degrees and levels simultaneously. We will give a combinatorial proof for the following main result.

**Theorem 4.** Given \( n \geq 1 \), for any two positive integers \( k \) and \( \ell \), there are one-to-one correspondences between the following four sets of vertices among trees in \( T_n \):

(i) first-children of degree \( k \) at level \( \ell \),
(ii) non-first-children of degree \( k \) at level \( \ell - 1 \),
(iii) leaves having exactly \( (k - 1) \) elder siblings at level \( \ell \), and
(iv) non-leaves of outdegree \( k \) at level \( \ell - 1 \).

Here, the cardinality of each set is

\[
\frac{k + 2\ell - 2}{2n - k} \binom{2n - k}{n + \ell - 1}.
\]  

(1)

The rest of the paper is organized as follows. In Section 2 we derive several corollaries from Theorem 4 including generalizations of Theorem 1 through 3. The proof of Theorem 4 is given in the next two sections: In Section 3 we construct bijections between these four sets while in Section 4 we show combinatorially that the cardinality is given by (1).

2. **Corollaries**

In 2012, Cheon and Shapiro [CS12, Example 2.2] gave a formula for the number of vertices of outdegree \( k \) by generating function arguments. Summing over \( \ell \geq 1 \) in Theorem 4 yields the following, in which the fourth item recovers bijectively the result of Cheon and Shapiro.

**Corollary 5.** Given \( n \geq 1 \) and \( k \geq 1 \), there are one-to-one correspondences between the following four sets of vertices among trees in \( T_n \):

(i) first-children of degree \( k \),
(ii) non-first-children of degree \( k \),
(iii) leaves having exactly \( (k - 1) \) elder siblings, and
(iv) non-leaves of outdegree \( k \).

Here, the cardinality of each set is

\[
\binom{2n - k - 1}{n - 1}.
\]

**Proof.** The result follows by a telescoping summation on \( \ell \geq 1 \) using the formula

\[
\frac{k + 2\ell - 2}{2n - k} \binom{2n - k}{n + \ell - 1} = \binom{2n - k - 1}{n + \ell - 2} - \binom{2n - k - 1}{n + \ell - 1}.
\]

\[\square\]

The result mentioned in Subsection 1.1 can be obtained from (i), (ii), and (iv) of Corollary 5.

**Corollary 6.** Given \( n \geq 1 \) and \( k \geq 1 \), the number of vertices of degree \( k \) equals twice the number of vertices of outdegree \( k \) among trees in \( T_n \).

Note that Corollary 6 is a refinement of Theorem 2. Summing over all \( k \) in Theorem 4 yields the following.
Corollary 7. Given \( n \geq 1 \) and \( \ell \geq 1 \), there are one-to-one correspondences between the following four sets of vertices among trees in \( T_n \):

(i) first-children at level \( \ell \),
(ii) non-first-children at level \( \ell - 1 \),
(iii) leaves at level \( \ell \), and
(iv) non-leaves at level \( \ell - 1 \).

Here, the cardinality of each set is
\[
\frac{\ell}{n} \binom{2n}{n+\ell},
\]

Proof. The result follows by a telescoping summation on \( k \geq 1 \) using the following formula,
\[
\frac{k + 2\ell - 2}{2n - k} \left( \frac{2n-k}{n+\ell-1} \right) = \frac{k + 2\ell - 1}{2n - k + 1} \left( \frac{2n-k+1}{n+\ell} \right) - \frac{k + 2\ell}{2n - k} \left( \frac{2n-k}{n+\ell} \right).
\]

\( \square \)

The result mentioned in Subsection 1.2 can be obtained from (i) and (ii) in Theorem 4.

Corollary 8. Given \( n \geq 1 \), the number of vertices of degree \( k \) at odd levels equals the number of vertices of degree \( k \) at even levels among trees in \( T_n \).

3. Proof of Theorem 4: Bijections

Denote \( TV_n \) by the set of pairs \((T,v)\) satisfying \( T \in T_n \) and \( v \in V(T) \). Given positive integers \( n, k, \) and \( \ell \), we let

(i) \( A \) denote the set of \((T,v)\) such that \( v \) is a first-child of degree \( k \) at level \( \ell \) in \( T \),
(ii) \( B \) denote the set of \((T,v)\) such that \( v \) is a non-first-child of degree \( k \) at level \( \ell - 1 \) in \( T \),
(iii) \( C \) denote the set of \((T,v)\) such that \( v \) is a leaf having exactly \( (k-1) \) elder siblings at level \( \ell \) in \( T \), and
(iv) \( D \) denote the set of \((T,v)\) such that \( v \) is a non-leaf of outdegree \( k \) at level \( \ell - 1 \) in \( T \).

(ii) \( \Leftrightarrow \) (iv). A bijection from \( A \) to \( D \) is constructed as follows: given \((T,v)\) in \( A \), find the parent vertex \( u \) of \( v \). As the Figure 2 consider a subtree \( D_v \) of \( v \) and another subtree \( D_u \) of \( u \) on the right of the edge \((u,v)\). By interchanging \( D_v \) and \( D_u \) we get the new tree \( T' \) with the vertex \( u \) such that
\[
\text{outdeg}(T',u) = \deg(T,v) = k, \quad \text{lev}(T',u) = \text{lev}(T,v) - 1 = \ell - 1,
\]
where \( \text{lev}(T,v) \) (resp. \( \deg(T,v) \), \( \text{outdeg}(T,v) \)) means the level (resp. degree, outdegree) of a vertex \( v \) in a tree \( T \). Thus, \((T',u)\) belongs to \( D \). Since this interchanging action is reversible, it is a one-to-one correspondence.
Figure 2. A bijection from $A$ to $D$

Figure 3. A bijection from $B$ to $D$ if $v$ is not the root of $T$

(iii) $\iff$ (iv). A bijection from $B$ to $D$ is constructed as follow: given $(T, v) \in B$, we perform the following action:

(a) if $v$ is the root of $T$, the pair $(T, v)$ also belongs to $D$ due to

\[ \text{outdeg}(T, v) = \text{deg}(T, v) = k. \]

(b) if $v$ is not the root of $T$, find the parent vertex $u$ of $v$, the first-child $w$ of $u$ and the edge $e = (u, w)$. As the Figure 3 cut and paste the edge $e$ and the subtree $D_u$ consisting of all descendants of $w$ from $u$ to $v$ such that the vertex $w$ is the first-child of $v$. We obtain the tree $T'$ and the vertex $v$ in $T'$ satisfies

\[ \text{outdeg}(T', v) = \text{deg}(T, v) = k; \quad \text{lev}(T', v) = \text{lev}(T, v) = \ell - 1. \]

Thus, $(T', v) \in D$.

Since this action is reversible, it is a one-to-one correspondence.

(iii) $\iff$ (iv). A bijection from $C$ to $D$ is constructed as follows: given $(T, v) \in C$, find the parent vertex $u$ of $v$. As the Figure 4 consider the subtree $D_{uv}$ of $u$ on the right of the edge $(u, v)$. By cutting and pasting the subtrees $D_{uv}$ from $u$ to $v$ we get the new tree $T'$ with the vertex $u$ such that

\[ \text{outdeg}(T', u) = \text{eld}(T, v) + 1 = k, \quad \text{lev}(T', u) = \text{lev}(T, v) - 1 = \ell - 1, \]

where $\text{eld}(T, v)$ means the number of elder siblings of a vertex $v$ in a tree $T$. Thus, $(T', u) \in D$. Since this interchanging action is reversible, it is a one-to-one correspondence.
4. PROOF OF THEOREM \[4\] ENUMERATION

By putting $\ell + 1$ in place of $\ell$ in (iv) of Theorem \[4\] it suffices to show that for any two nonnegative integers $k$ and $\ell$, the number of vertices of outdegree $k$ at level $\ell$ among trees in $T_n$ is

$$\frac{k + 2\ell}{2n - k} \binom{2n - k}{n + \ell}.$$

The following lemma gives a cumulative counting in $k$ and $\ell$.

**Lemma 9** (Main lemma). Given $n \geq 1$, for any two nonnegative integers $k$ and $\ell$, the number of vertices of outdegree at least $k$ and at level at least $\ell$ among trees in $T_n$ is

$$\binom{2n - k}{n + \ell}.$$

**Proof.** Let $V$ be the set of $(T, v) \in T V_n$ such that $v$ is a vertex of outdegree at least $k$ and at level at least $\ell$ in $T$. Let $L$ be the set of lattice paths of length $(2n - k)$ from $(k, k)$ to $(2n, -2\ell)$, consisting of $(n - k - \ell)$ up-steps along the vector $(1, 1)$ and $(n + \ell)$ down-steps along the vector $(1, -1)$. Since

$$\#L = \binom{2n - k}{n - k - \ell, n + \ell} = \binom{2n - k}{n + \ell},$$

it is enough to construct a bijection $\Phi$ between $V$ and $L$.

Before constructing the bijection we first introduce two well-known bijections $\varphi$ and $\psi$ between rooted ordered trees and Dyck paths.

The bijection $\varphi$ corresponds a tree to a Dyck path by recording the steps when the tree is traversed in preorder. Here we record an up-step when we go down an edge and a down-step when going up. An example of the bijection $\varphi$ is shown in the Figure \[5\].

The bijection $\psi$ corresponds a tree to a Dyck path by recording the steps when the tree is traversed in preorder. Here, whenever we meet a vertex of
outdegree $k$, except the last leaf, we record $k$ up-steps and one down-step. An example of the bijection $\psi$ is shown in the Figure 6.

Using two bijections $\phi$ and $\psi$, we will construct another bijection $\Phi$ between $V$ to $L$: given $(T, v) \in V$, let $k' (\geq k)$ be the number of children of $v$ in $T$ and let $\ell' (\geq \ell)$ be the level of $v$ in $T$. We decompose the tree $T$ into $\ell' + 2$ subtrees: the subtree $D_v$ consisting of descendants of $v$, $\ell'$ subtrees $R_1, R_2, \ldots, R_{\ell'}$ on the right hand side of the path from $v$ to the root $r$ of $T$, and the remaining tree $L$ as illustrated in the Figure 7. Clearly, the outdegree of the root of $D_v$ is $k'$. In preorder, the vertex $v$ is the last leaf at level $\ell'$ in the tree $L$.

From $(T, v)$, we define a lattice path $P$ of length $(2n + \ell' + 1)$ from $(0, 0)$ to $(2n + \ell' + 1, -\ell' - 1)$ by

$$P = \psi(D_v) \searrow \varphi(R_1) \searrow \varphi(R_2) \searrow \cdots \searrow \varphi(R_{\ell'}) \searrow \varphi(L),$$

where $\searrow$ means a down-step. By convention, if $\ell' = 0$. We set

$$P = \psi(D_v) \searrow \varphi(\emptyset).$$

Especially, if $\ell' > 0$, the lattice path $P$ always starts with $k'$ consecutive up-steps and ends with one up-step and $\ell'$ consecutive down-steps as the Figure 8.

When we draw the line $y = -\ell - 1$, because the $y$-coordinate of the lowest point of $P$ is $-\ell' - 1$, this line should meet the lattice path $P$. Denote $p$ by the first meeting point of $P$ and $y = -\ell - 1$. Replacing the portion of $P$ from $p$ to $(2n + \ell' + 1, -\ell' - 1)$ by its reflection about $y = -\ell - 1$, we obtain the lattice path $\tilde{P}$ from $(0, 0)$ to $(2n + \ell' + 1, -2\ell + \ell' - 1)$, which also always starts with $k'$ consecutive up-steps and ends with one down-step and
\[ \psi(D_v) \quad \psi(R_1) \quad \psi(R_{\ell+1}) \quad \psi(R_\ell) \]

Figure 8. Outline of a lattice path \( P \) induced from tree decomposition

\[ \varphi(\mathcal{L}) \]

Figure 9. Outline of a lattice path \( \tilde{P} \) and \( \hat{P} \) by reflection about \( y = -\ell - 1 \)

\( \ell' \) consecutive up-steps as the Figure 9 even if \( \ell' = 0 \). Note that, if \( \ell' = 0 \), \( p \) should be the last point \((2n+1, -1)\) of \( P \) and \( \tilde{P} = P \).

By removing the first \( k \) steps and the last \((\ell' + 1)\) steps from \( \tilde{P} \), we obtain the lattice path \( \hat{P} \) from \((k, k)\) to \((2n, -2\ell)\). Thus \( \hat{P} \) belongs to \( \mathcal{L} \) and define \( \Phi(T, v) = \hat{P} \). Since the reflection and the removal are reversible, \( \Phi \) is a bijection.

**Theorem 10.** Given \( n \geq 1 \), for any two nonnegative integers \( k \) and \( \ell \), the number of vertices of outdegree \( k \) at level \( \ell \) among trees in \( T_n \) is

\[
\frac{k + 2\ell}{2n - k} \binom{2n - k}{n + \ell}.
\]

**Proof.** Using a sieve method, we obtain (3) from (2) by

\[
\left( \binom{2n - k}{n + \ell} - \binom{2n - k - 1}{n + \ell} \right) - \left( \frac{2n - k}{n + \ell + 1} \right) + \left( \frac{2n - k - 1}{n + \ell + 1} \right).
\]

Thus, obtaining (1) from (3) by changing the index \( \ell \), we may count the set \( D \) as (iv) of Theorem 4.

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