The space of tight contact structures on $\mathbb{R}^3$ is contractible

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To Emmanuel Giroux on his 60th anniversary

Abstract

One of the results of the paper [5] was the proof that any tight contact structure on $S^3$ is diffeomorphic to the standard one. It was also claimed there without a proof that similar methods could be used to prove a multi-parametric version: the space of tight contact structures on $S^3$, fixed at a point, is contractible. We prove this result in the current paper.

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1 Introduction

A contact structure $\xi$ is called overtwisted, see [4], if there exists an embedded 2-disc $D \subset \mathbb{R}^3$ which is tangent to $\xi$ along $\partial D$. A non-overtwisted contact structure is called tight. It is a fundamental result of D. Bennequin, see [1], that the standard contact structure $\zeta_0 = \{dz - ydx = 0\}$ on $\mathbb{R}^3$ is tight.

The following theorem is the main result of the current paper.

**Theorem 1.1.** The space of standard at infinity tight contact structures on $\mathbb{R}^3$ is contractible.

A contact structure defines an orientation of a contact 3-manifold. Given an oriented contact manifold we call a contact structure positive if the contact orientation is the given one.

**Corollary 1.2.** The space $\text{Tight}_+(S^3)$ of positive tight contact structures on $S^3$ is homotopy equivalent to $\mathbb{R}P^2$. 
Indeed, an evaluation map $\text{Tight}_+(S^3) \to \mathbb{R}P^2$, associating with a contact structure a non-oriented contact plane at a fixed point, is a Serre fibration with a fiber homotopy equivalent to the space of standard at infinity tight contact structures on $\mathbb{R}^3$.

The non-parametric version of Theorem 1.1 i.e. that the space of standard at infinity tight contact structures on $\mathbb{R}^3$ is connected was proven in [5]. Equivalently, that result means that any standard at infinity tight contact structure on $\mathbb{R}^3$ is diffeomorphic to $\zeta_0$ via a compactly supported diffeomorphism of $\mathbb{R}^3$. An approach to the proof of the parametric case using convex surface theory has been suggested and partially implemented by D. Jänichen, see [14]. The proof presented in the current paper does not use the theory of contact convexity.

Denote by $\text{Diff}_0(\mathbb{R}^3)$ the group of compactly supported diffeomorphisms of $\mathbb{R}^3$, and by $\text{Diff}_0(\mathbb{R}^3, \zeta_0)$ the group of compactly supported contactomorphisms of $(\mathbb{R}^3, \zeta_0)$. The non-parametric version Theorem 1.1 from [3] together with Gray’s theorem [11] implies that the evaluation map $f \mapsto f \circ \zeta_0$ is a a Serre fibration $\text{Diff}_0(\mathbb{R}^3) \to \text{Tight}_0(\mathbb{R}^3)$, where $\text{Tight}_0(\mathbb{R}^3)$ is the space of standard at infinity tight contact structures on $\mathbb{R}^3$. The fiber of this fibration is $\text{Diff}_0(\mathbb{R}^3, \zeta_0)$. Hence, Theorem 1.1 equivalently means that the inclusion map $j : \text{Diff}_0(\mathbb{R}^3, \zeta_0) \to \text{Diff}_0(\mathbb{R}^3)$ is a homotopy equivalence, which in view of A. Hatcher’s theorem [15] implies that the group $\text{Diff}_0(\mathbb{R}^3, \zeta_0)$ is contractible.

A recent paper [16] by E. Fernández, J. Martinez-Aguinaga and F. Presas used the main result of the current paper for the study of the topology of the group of contactomorphisms of various other 3-manifolds.

**Scheme of the proof and the plan of the paper**

As in [5], the proof is based on the analysis of characteristic foliations on the 2-sphere induced by a family of tight contact structures on its neighborhood. To make the topology of a characteristic foliation manageable we arrange that its singularities are of Morse or generalized Morse type. This is achieved in Proposition 6.1, which is an analog for our situation of Igusa’s theorem about functions with moderate singularities, see [13] and [7, 9].

The second ingredient in the proof of the main result is a new characterization of characteristic foliations induced on a sphere by a tight contact structure, see Proposition 5.11. It is formulated in terms of existence of a Lyapunov function with special properties. We call Lyapunov functions in this class *simple taming functions*. Let us recall that Giroux’s criterion from [10] for tightness of characteristic foliations
is applicable only to convex (in a contact sense) surfaces, i.e. surfaces admitting a transverse contact vector field. Our characterization is applicable to any generalized Morse foliation on a 2-sphere without the contact convexity assumption. The proof of Proposition 5.11 is based on the analysis of the topology of tight characteristic foliations in Section 3, see there Proposition 3.2. This leads to a Proposition 5.15 which allows us to construct a family of taming simple functions for any family of tight foliations on a 2-sphere.

The third ingredient in the proof is Proposition 4.3 which provides homotopically canonical extension of a family of simple functions to a family of functions on the ball without critical points and with contractible components of its level sets. This proposition is purely topological and has nothing to do with contact geometry.

The final ingredient is Proposition 7.9 which uses the extended to the ball taming functions for construction of homotopically canonical extension of a family of characteristic foliations on the sphere to a family of contact structures on the ball. We apply for this purpose complex geometric techniques of strictly pseudoconvex hypersurfaces, though, probably, it could be achieved by more direct contact geometric methods. We conclude the proof of Theorem 1.1 in Section 8.

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2 Characteristic foliation on a surface in a tight contact manifold

2.1 Singularities of a characteristic foliation

Given a contact structure $\xi = \{ \alpha = 0 \}$ on a neighborhood of a 2-sphere $S$ in a 3-manifold, we use the term characteristic foliation for the singular line field defined by the Pfaffian equation $\{ \alpha|_S = 0 \}$, as well as for the singular foliation $\mathcal{F} := \{ \alpha|_S = 0 \}$ to which it integrates. A characteristic foliation is called Morse if it has no limit cycles and all its singular points are non-degenerate.

More precisely, in a neighborhood of a singular point $p \in S$ we have $d\alpha|_S \neq 0$, and hence $\alpha|_S$ is a Liouville form for the symplectic form $d\alpha|_S$. The corresponding Liouville field $Z$, $\iota(Z)(d\alpha|_S) = \alpha|_S$, integrates to the characteristic foliation $\mathcal{F}$. In any local coordinate system $u = (x, y)$ centered at $p$ the vector field $Z$ is given by a differential equation $\dot{u} = f(u)$, $f(0) = 0$. We are not assuming the coordinate
system canonical (i.e. that $d\alpha = dx \wedge dy$), but require that it defines the symplectic orientation. The linear part $A = d_0f$ has $\text{Tr} A > 0$. The singular point $p$ is called non-degenerate if $A$ is non-degenerate. A non-degenerate point is called elliptic if $\det A > 0$ and hyperbolic otherwise. In the hyperbolic case $A$ has two real eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$. In the elliptic case eigenvalues are either positive real numbers, or conjugate complex numbers with the real part equal to $\frac{\text{Tr} A}{2}$.

A singular point is called an embryo point if $A$ has rank 1 and if the second differential $d^2f : \text{Ker} A \to \text{Coker} A$ does not vanish. We note that the linear map $d^2f : \text{Ker} A \to \text{Coker} A$ is invariantly defined up to a (non-zero) scalar factor.

We call a characteristic foliation generalized Morse if all its singularities are either nondegenerate or embryos. Singularities of a generalized Morse foliation are shown on Fig. 2.1.

It follows from the results of F. Takens, see [18], that in a neighborhood of an embryo the directing Liouville field $Z$ is orbitally equivalent to (i.e. diffeomorphic to a field proportional to) the field $x \frac{\partial}{\partial x} + y^2 f(y) \frac{\partial}{\partial y}$, $f(0) \neq 0$. The above normalization claim also holds in a parametric form.

**Lemma 2.1.** Let $\Lambda$ be a compact parameter space and $Z_\lambda, \lambda \in \Lambda$, a family of $C^\infty$-vector fields on a neighborhood $\Omega_{\mathbb{R}^2,0} \subset \mathbb{R}^2$ of the origin in $\mathbb{R}^2$ with an embryo singularity at the origin. Then for any $k > 0$ there exists a family of germs of $C^k$-diffeomorphisms $h_\lambda : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ such that

$$(h_\lambda)_* Z_\lambda = g_\lambda(x,y) \left( x \frac{\partial}{\partial x} + y^2 f_\lambda(y) \frac{\partial}{\partial y} \right), \quad f_\lambda(0), g_\lambda(0,0) \neq 0.$$  

Fig. 2.1: Elliptic, hyperbolic and embryo points

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1 Moreover, one can choose coordinates in such a way that $f(y)$ is equal either to 1, or $1 + y$, but we will not need this stronger statement.
Note that a 1-parametric deformation $Z_t = Z + t \frac{\partial}{\partial y}$ has no singular points for $t > 0$ and has two singular points, elliptic and hyperbolic for $t < 0$. In fact, one has the following result for any deformation of embryo singularities, see Theorem 5 in [17].

**Proposition 2.2.** Let $Z_t, t \in \mathcal{O}_p \mathbb{R}^n, 0 \subset \mathbb{R}^n$, be a family of vector fields on $\mathcal{O}_p \mathbb{R}^n, 0 \subset \mathbb{R}^2$. Suppose that $Z_0 = x \frac{\partial}{\partial x} + y^2 f(y), f(0) \neq 0$. Then there exist a neighborhood $U \ni 0$ in $\mathbb{R}^2$ such that the family $Z_t|_U$ is orbitally equivalent for sufficiently small $t$ to a family

$$x \frac{\partial}{\partial x} + F(y, t) \frac{\partial}{\partial y}, \quad F(y, 0) = y^2 f(y) \neq 0.$$ 

Proposition 2.2 also holds in a slightly more global parametric form.

**Proposition 2.3.** Let $\Lambda$ be a compact parameter space, $\Lambda_0 \subset \Lambda$ its closed subset, and $Z_\lambda, \lambda \in \Lambda$, be a family of vector fields on $\mathcal{O}_p \mathbb{R}^n, 0 \subset \mathbb{R}^2$ such that for all $\lambda \in \Lambda_0$ we have

$$Z_\lambda = x \frac{\partial}{\partial x} + y^2 f(y, \lambda) \frac{\partial}{\partial y}.$$ 

Then for any $k > 0$ there exist a neighborhood $U \ni 0$ in $\mathbb{R}^2$, a neighborhood $\Omega \supset \Lambda_0$ in $\Lambda$, and a family of germs of $C^k$-diffeomorphisms $h_\lambda : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0), \lambda \in \Omega$, such that

$$(h_\lambda)_* Z_\lambda = g_\lambda(x, y) \left( x \frac{\partial}{\partial x} + F(y, \lambda) \frac{\partial}{\partial y} \right), \lambda \in \Omega,$$

where $F(y, \lambda) = y^2 f(y, \lambda)$ and $g_\lambda(x, y) = 1$ for $\lambda \in \Lambda_0$.

Recall that a contact structure $\xi$ defines an orientation of the 3-dimensional contact manifold. Hence, assuming the surface $S$ and the contact structure $\xi$ oriented (and hence, co-oriented), we can distinguish between positive and negative singular points. At a regular point $p$ of the characteristic foliation choose a vector $\tau_S(p) \in T_pS$ which defines the given co-orientation of $\xi(p)$. Choose a vector $Z(p)$ directing the characteristic foliation in such a way that $(\tau_S(p), Z(p))$ defines the orientation of $S$. Near singular points this orientation is the same as defined by the Liouville field near positive points, and opposite to it near the negative ones. With this convention, positive elliptic points serve as sources and negative as sinks of the characteristic flow.

A positive embryo has 1 incoming separatrix and a half-plane filled with outgoing trajectories. We will refer to the trajectories on the boundary of this half-plane as outgoing separatrices. For a negative embryo there are 2 incoming separatrices and one outgoing.
The contact structure on a neighborhood of a surface is determined by the characteristic foliation up to a contactomorphism, so the tightness can be judged by the characteristic foliation.

For all discussions in Sections 2-3 below only the topological type of the characteristic foliation will be important. In fact, by a $C^1$-small isotopy of the surface in the ambient contact manifold, which is supported in an arbitrary small neighborhood of a singular point, one can arbitrarily change the smooth topology keeping its topological type, see Lemma 5.3 below. See Fig. 2.2 for the case of an elliptic point.

![Diagram showing changing smooth topology of an elliptic point](image)

**Fig. 2.2:** Changing smooth topology of an elliptic point

We will always picture elliptic points as nodes, see Fig. 2.1a) and embryos as half nodes, half saddles, see Fig. 2.1, though the latter picture is not possible up to diffeomorphism.

![Diagram showing topological representation of an embryo](image)

**Fig. 2.3:** Topological representation of an embryo

### 2.2 Manipulating characteristic foliations

In the statements below by an isotopy of a surface we always mean its isotopy in the ambient contact manifold. The next claim is straightforward.
Lemma 2.4. Take a regular point $a$ of $\mathcal{F}$ and a local transverse $T$ to the trajectory $\gamma$ through $a$. Consider an isotopy $\alpha_t : T \to T$, $\alpha_0 = \text{Id}$, supported in $\mathcal{O}_p a$. Then there is a $C^\infty$-small isotopy of the sphere which realizes the holonomy $\alpha_t$ for sufficiently small $t$. See Fig. 2.4.

![Fig. 2.4: Perturbing a characteristic foliation](image1)

Lemma 2.4 implies

Corollary 2.5. Let $h$ be a hyperbolic point, $s_-$ is an incoming separatrix, and $s_+$ outgoing. Then one can $C^\infty$-perturb $\mathcal{F}$ in a neighborhood of $h$ in such a way that $s_+ \cup s_-$ becomes a smooth Legendrian arc bypassing $h$. See Fig. 2.5.

![Fig. 2.5: Bypassing a hyperbolic point](image2)

The statement (i) in Lemma 2.6 below is Giroux-Fuchs elimination lemma, see [10]. Other claims of the lemma are its small variations.

Lemma 2.6. (i) Let $e, h$ be elliptic and hyperbolic points of $\mathcal{F}$ of the same sign. Suppose that $e$ is of the node type. Let $\gamma$ be separatrix of $\mathcal{F}$ connecting $e$ and $h$, $\alpha$ the separatrix of $h$ opposite to $\gamma$, and $\delta$ another trajectory ending at $e$. 
Then $e, h$ can be cancelled in such a way that $\alpha \cup \gamma \cup \delta$ becomes a trajectory of the resulting foliation, see Figure 2.6. The elimination can be realized by a $C^0$-small and supported in $O_p \gamma$ isotopy of the sphere.

![Diagram](image)

**Fig. 2.6: Eliminating an elliptic-hyperbolic pair**

(ii) Let $o$ be an embryo point, $\gamma$ its separatrix, incoming for a positive embryo and outgoing for a negative one, and $\delta$ any non-separatrix trajectory ending at $o$. Then $o$ can be eliminated by a $C^1$-small isotopy of the sphere supported in $O_p \gamma$. The elimination can be done in such a way that $\gamma \cup \delta$ becomes a trajectory of the resulting foliation, see Fig. 2.7, 2.8.

(iii) Let $o$ be an embryo point, $\delta$ any trajectory incoming to $o$ if $o$ is positive and outgoing if $o$ is negative. Then there exists a $C^1$-small supported in $O_p \delta$ isotopy of the sphere which replaces $o$ by an elliptic-hyperbolic pair $(e, h)$ of the same sign, and such that $\delta$ becomes one of the separatrices of $h$, see Fig.

(iv) Let $e$ be an elliptic points and $\gamma, \gamma'$ two adjacent to it trajectories. Then by a $C^1$-small supported in $O_p e$ isotopy one can create an elliptic-hyperbolic pair $e', h'$ of the same sign as $e$ such that $\gamma, \gamma'$ becomes separatrices of $h'$, see Fig. 2.9.
2 Characteristic foliation on a surface in a tight contact manifold

2.3 Invariants $d_\pm$

Let $T$ be either a closed surface in a contact 3-manifold, or a surface which bounds a curve $\Gamma$ transverse to the contact structure $\xi$. In the latter case we assume that the oriented characteristic foliation is outward transverse to $\partial T$ and denote by $e_\pm = e_\pm(T), h_\pm = h_\pm(T)$ the numbers of elliptic and hyperbolic points. Set $d_\pm := e_\pm - h_\pm$.

We have

$$d_+ + d_- = \chi(T); \quad d_+ - d_- = c(T),$$

where $\chi(T)$ is the Euler characteristic and $c(T)$ is the relative Chern (Euler) number, also denoted $\ell(\partial T)$ and called the self-linking number, see [5]. It is an obstruction for extending the vector field tangent to the foliation along the boundary $\partial T$ to a non-vanishing vector field tangent to $\xi$. Thus,

$$d_+ = \frac{1}{2}(\chi + c), \quad d_- = \frac{1}{2}(\chi - c).$$
In particular, for a sphere $S = S^2 \subset \mathbb{R}^3$ we have $d_{\pm} = 1$.

**Lemma 2.7.** (i) $d_{\pm}(S) = 1$ for a tight sphere $S$;

(ii) Suppose $T$ is a genus 0 surface with $k \geq 1$ boundary components. Suppose that $d_{+}(T) = 1$. Then by a $C^0$-small isotopy, fixed near the boundary $\partial T$, one can kill all singular points except 1 positive elliptic and $k - 1$ negative hyperbolic points. In particular, when $T = D$ is a disc one can kill all singular points except 1 positive elliptic.

(iii) Let $A \subset S$ be an annulus in a tight sphere with boundary transverse in the outward sense to the characteristic foliation. Suppose $d_{+}(A) = 1$. Let $D$ be the disc bounded by one of the boundary component $\Gamma$ of $A$ attached to $\Gamma$ from the same side as $A$, see Fig. 2.10. Then $d_{+}(D) = 1$.

**Proof.** (i) The incoming separatrices of positive hyperbolic points begin at positive elliptic points, and the outgoing separatrices of negative hyperbolic points end at
negative elliptic points. Hence, all hyperbolic points can be eliminated using Lemma 2.6. On the other hand, we have $d_+ > 0$ because there should be sources and sinks. But $d_+ + d_- = 2$, and hence, $d_+ = 1$.

(ii) The incoming separatrices of positive hyperbolic points begin at positive elliptic points, and hence can be killed using Lemma 2.6. Suppose that all positive hyperbolic points are killed, and hence only 1 positive elliptic left. Note that $d_-(D) = 2 - k - d_+(D) = 1 - k$, and hence there are at least $k - 1$ negative hyperbolic points. If there is a negative elliptic point $e$, then all its incoming trajectories come either from the positive elliptic point, or from negative hyperbolic points. If there are no incoming separatrices from hyperbolic points then $T$ is the sphere $S$, contradicting to our assumption that $k \geq 1$. Hence, there should be a negative hyperbolic point $h$ whose outgoing separatric ends at $e$. Therefore, we can kill the pair $(e, h)$ using Lemma 2.6.

(iii) The complement $S \setminus A$ is the union of two disjoint discs $D_1 \cup D_2$. We have $d_-(D_1) + d_-(D_2) = 1 - d_-(A) = 2$, but on the hand, each of the discs must have $d_0 > 0$ (as the previous argument shows reversing the orientation). Hence, $d_-(D_1) = d_-(D_2) = 1$ and (ii) implies that $d_+(D_1) = d_+(D_2) = 0$ thus $d_+(D) = d_+(A) - d_+(D_2) = 1$.

2.4 Legendrian polygons

We assume below that $F$ is tight and Morse. Analogous statements hold in the generalized Morse case but we will not need them for our purposes.

A polygon is an embedded domain in $\mathbb{R}^2$ which is a manifold with boundary with corners.

A Legendrian polygon in a sphere $S$ is a continuous map $h : P \to S$ of a polygon $P$ such that

- $h$ is a smooth embedding on the interior of $P$, as well as on the boundary in the complement of vertices;

- each side of $P$ is mapped onto a leaf of a characteristic foliation on $S$ with an exception that some sides could be mapped onto the union of two incoming or two outgoing separatrices of a hyperbolic point; in the latter case the corresponding interior point of the side is called a pseudo-vertex of the polygon. See Fig. 2.11.
Fig. 2.11: Legendrian polygon; $h^1, h^2, h^3$ are pseudovertices

Lemma 2.8. Suppose $\mathcal{F}$ is Morse and tight. Then among elliptic vertices and pseudovertices of a Legendrian polygon there are points of both signs.

Proof. If all singular points except hyperbolic corners on the boundary of the polygon are of the same sign then they can first be disjoined or smoothed using Lemma 2.6(ii) or Corollary 2.5, and then pairwise cancelled using Lemma 2.6(i) to get a closed non-singular leaf of the characteristic foliation. But this contradicts the tightness assumption.

Lemma 2.9. Suppose $\mathcal{F}$ has no homoclinics (i.e. separatrices connecting hyperbolic points). Then the union $\Lambda$ of stable separatrices of positive hyperbolic points is a connected tree with vertices in positive elliptic points, and edges in 1-1 correspondence with positive hyperbolic points. See Fig. 2.12.

Proof. First, observe that $\Lambda$ contains no loops, thanks to the tightness condition and Lemma 2.8. We also have $1 = e_+ - h_+ = b_0(\Gamma)$, and hence $\Gamma$ is connected.

Corollary 2.10. Under the assumptions of Lemma 2.9 there is a unique path consisting of separatrices of positive hyperbolic points connecting any 2 positive elliptic vertices.
3 Existence of allowable singularities

We prove in this section the main technical proposition about tight foliations on the 2-sphere.

3.1 Allowable singular points

Given a characteristic foliation $\mathcal{F}$, its singular point $x$ is called allowable in one of the following 4 cases:

- $x$ is a positive hyperbolic point with two incoming separatrices from different positive elliptic points;
- $x$ is a negative hyperbolic point with two incoming separatrices from the same positive elliptic point;
- $x$ is a positive embryo with the incoming separatrix from a positive elliptic point;
- $x$ is a negative embryo with all the incoming trajectories from a positive elliptic point.

Lemma 3.1. If two incoming separatrices of a positive hyperbolic point $h$ come from the same elliptic point then the characteristic foliation is overtwisted.

Indeed, two separatrices form a Legendrian polygon with only positive singular points.
### 3.2 Key technical proposition

The union $\Sigma = \Sigma(F)$ of all outgoing separatrices of all hyperbolic points and embryos is called the **skeleton of $F$**.

![Basin and semibasin](image)

Fig. 3.1: Basin and semibasin

Components of $S\setminus\Sigma$ are diffeomorphic to $\mathbb{R}^2$ and could be of two types, *basins* and *semi-basins*. A basin is the union of trajectories emanating from a positive elliptic point, called the *center* of the basin. A semi-basin is the union of trajectories emanating from a positive embryo. See Fig. 3.1.

**Proposition 3.2.** Let $F$ be a tight generalized Morse foliation. Then it has an allowable vertex.

We begin by reducing the proposition to the case of a Morse foliation.

**Lemma 3.3.** If Proposition 3.2 holds for Morse foliations then it holds for generalized Morse foliations as well.

**Proof.** We argue by induction in the number of embryos. Suppose the claim is proven when there are fewer than $k$ embryos.

Let $o$ be a positive embryo. It is not allowable if there is either an incoming homoclinic, or its incoming separatrix comes from another positive embryo.

In both cases let us resolve the embryo into a pair of positive elliptic and hyperbolic points, such that the incoming separatrix of the newly created hyperbolic point is either homoclinic or comes from an embryo, see Fig. ??.

This bifurcation does not create any new allowable vertices for the resulting new foliation $F'$, and hence, the
allowable vertice provided by the induction hypothesis for $F'$ is allowable for $F$ as well.

Assume now that $F$ has no positive embryos and let $o$ be a negative embryo. It is not allowable if and only if there is an incoming separatrix $\gamma$ either from a hyperbolic point or an embryo. Let us resolve the embryo $o$ into a pair of a negative elliptic point $e$ and hyperbolic point $h$, such that the separatrix $\gamma$ ends at $h$, see Fig. 3.2. Hence, $h$ is not allowable and, therefore, the bifurcation did not create any new allowable points, so the induction hypothesis applies.

\[\square\]

3.3 Proof of Proposition 3.2

Thanks to Lemma 3.3 we can assume that $F$ is Morse. Suppose that $F$ has no allowable negative hyperbolic points.
Lemma 3.4. Under the above assumption the boundary of any basin has no identified pseudovertices.

Indeed, any such pseudovertex has to be negative, and hence, allowable.

For the induction purposes we will be proving a slightly stronger statement. A closed embedded domain $U \subset S$ bounded by some of the trajectories in $\Sigma$ is called \textit{admissible} if it is a union of basins and none of positive pseudovertices on its boundary has an incoming homoclinic from outside of $U$, see Fig. 3.3.

Lemma 3.5. Let $F$ be a tight Morse foliation. Then any admissible domain in $U$ contains an allowable pseudovertex.

Induction. We will be proving Lemma 3.5 by induction over the total number of all singular points and homoclinics in $U$. We assume that \textit{there are no allowable singularities in $U$} and will deduce from that assumption that $F$ is overtwisted.

Induction hypothesis $I_{n,k}$. The statement holds if there are $\leq n$ singular points and $\leq k$ homoclinics.

Lemma 3.6 (Base of Induction). $I_{n,0}$ holds.
Fig. 3.4: Resolving the case with 2 incoming homoclinics

Proof. Pick any basin $T \subset U$. According to Lemma 3.4 the boundary $\partial T$ has no identified pseudovertices, and hence it is a Legendrian polygon. But then Lemma 2.8 yields a positive pseudovertex on $\partial T$ which is allowable in this case.

Suppose that $I_{m,j}$ holds for $m < n$ and all $j$, and for $m = n$ and $j < k$. Let us prove $I_{n,k}$.

**Elimination of certain configurations**

The following sequence of claims proves the induction hypothesis assuming existence of certain configurations. After each step we are adding the absence of the corresponding configuration as an additional assumption.

**Step 1.** Suppose there is a hyperbolic point $h$ such that the two incoming to $h$ separatrices are homoclinic. Then $I_{n,k}$ holds.

Proof. Resolving one of the incoming separatrices we get a foliation without any additional allowable singularities. See Fig. 3.4

Therefore, we can assume that any homoclinic appears in a T-shaped configuration with exactly 1 incoming separatrix. We call the hyperbolic point with the incoming homoclinic the **center** of the T-configuration. The basins adjacent to the homoclinic will be called **side** basins, and the third basin will be called the **base**.

**Step 2.** Suppose that the center of a homoclinic configuration is positive, and the base basin coincides with one of the side basins. Then $I_{n,k}$ holds.
3 Existence of allowable singularities

Fig. 3.5: 1 incoming homoclinics, $h$ positive, $T_1 = T_3$

![Diagram](image1)

Fig. 3.6: 1 incoming homoclinics, $h$ negative, $T_1 \neq T_3$

![Diagram](image2)

**Proof.** Assuming $T_1 = T_3$ and resolving the incoming separatrix in such a way that $h$ becomes a pseudovertex on the boundaries of $T_1$ and $T_3$, see Fig. 3.5, we get an overtwisted foliation.

**Step 3.** Suppose that any of the following two conditions holds:

- the center $h$ of a homoclinic configuration is negative, or
- $h$ is positive but the side basins coincide.

Then $I_{n,k}$ holds.

**Proof.** Suppose that $h$ is negative and $T_1 \neq T_3$. Then resolving the incoming separatrix in such a way that $h$ becomes a pseudovertex on the boundaries of $T_1$ and $T_3$ we get a foliation without any additional allowable singularities. See Fig. 3.6. Hence, we can assume that for a homoclinic configuration with a negative center all adjacent basins coincide.

Suppose now that $h$ is either negative, or positive but adjacent side basins coincide. Consider the maximal homoclinic chain $h_m, h_{m-1}, \ldots, h_1, h_0 := h$ incoming to $h$, see Fig. 3.7. If $m = 1$ then $h_1$ is a pseudovertex. Moreover, it has to be negative, because
adjacent to it basins coincide, and hence, allowable. In the general case \( h_1 \) still has to be negative because we assumed that for homoclinic configurations with positive centers all adjacent basins are pairwise distinct. Hence, arguing by induction in \( m \) we conclude that that \( h_m \) is an allowable negative pseudovertex.

**Step 4.** If the skeleton \( \Sigma \) has any end points then \( I_{n,k} \) holds.

*Proof.* The end point is a negative elliptic point \( e \), and the incoming to \( e \) separatrix comes from a hyperbolic point \( h \), see Fig. 3.8. We claim that \( h \) cannot have an incoming homoclinic. Indeed, otherwise by Step 2 it would have to be negative, which was already ruled out in Step 3. Hence, \( h \) is a negative pseudovertex, as adjacent to it basins coincide, and hence it is allowable.

**Lemma 3.7.** The boundary of any basin has a positive pseudovertex.

*Proof.* By assumption there are no negative identified pseudovertices, and no negative hyperbolic points which are centers of homoclinic configurations. On the other hand, the boundary of any basin must contain at least one pseudovertex. Indeed,
only a pseudovertex can serve as a source on the boundary of a basin. If all pseudovertices in \( \partial T \) were negative then the polygon would be injective, and hence it would be a Legendrian polygon, which contradicts Lemma 2.8.

Proof of Lemma 3.5 Suppose that there are no allowable pseudovertices. We will show that this assumption implies overtwistedness of \( F \). Consider any basin \( T \subset U \). By Lemma 3.7 there is a pseudovertex \( h \in \partial T \). It has to be positive because of Step 3. By assumption it is not allowable, and hence, there exists a homoclinic \( \gamma \) incoming from outside of \( T \) from a hyperbolic point \( h_1 \). Consider a path \( \Gamma \) from \( h \) to \( h_1 \) along \( \gamma \), and continue it counter-clockwise along the boundary of a side basin adjacent to \( \gamma \). There are two possibilities:
α) we arrive to a positive pseudovertex $h'$ with an incoming from outside homoclinic;

β) the loop closes up.

In Case α) we turn to the incoming homoclinic and continue the process going counter-clockwise around the boundary of an adjacent to $h'$ side basin. See Fig. 3.9.

In Case β) we get an admissible domain with fewer vertices. The possible configuration of closing up the loop in Case β) are shown on Fig. 3.10. Note that in all of the subcases except β₃ we get an overtwisted loop. This concludes the proof of Lemma 3.5, and with it the proof of Proposition 3.2.

4 Simple functions on $B^3$ and $S^2$

4.1 Simplicity condition

Consider a function $\Phi$ on the 3-ball $B = B^3$ without critical points, which restricts to the boundary sphere $S = \partial B$ as a generalized Morse function. We call $\Phi$ simple if components of each of its level sets are contractible.

Our goal is to characterize the restrictions of simple functions to the sphere $S = \partial B^3$.

Let $\phi : S \rightarrow \mathbb{R}$ be a generalized Morse function. Choose a gradient like vector field $X$ for $\phi$ (the property we describe does not dependent on this choice). Call an index 1 Morse critical point $p$ of $\phi$ on the level $A_a := \{ \phi(p) = a \}$ positive (resp. negative) if the stable manifold of $p$ intersects the regular level set $A \varepsilon := \{ \phi = a - \varepsilon \}$ at different (resp. the same) components of this level set, see Fig. 4.1.
Let \( h_1^+, \ldots, h_\ell^+ \) be positive hyperbolic points on a critical level \( A = \{ \phi = a \} \). Let \( A^- = \{ \phi = c - \varepsilon \} \) be a regular level for a sufficiently small \( \varepsilon > 0 \). Denote by \( \sigma_j^\pm \) stable manifolds of hyperbolic points \( h_j^\pm \).

Consider a ribbon graph \( \Gamma_a^+(\phi) \) whose vertices are components of the level \( A^- \), and edges correspond to stable separatrices \( \sigma_j^+ \) of positive hyperbolic points on the critical level \( A_a = \{ \phi = a \} \), see Fig. 4.2. The ribbon structure is given by the cyclic ordering of the end points of each \( \sigma_j^+ \) adjacent to a given component of \( A_a^- \). We assume the level set \( A_a^- \) oriented as the boundary of \( \{ \phi \leq a \} \).

A function \( \phi \) is called simple if for every critical level \( a \) the graph \( \Gamma_a^+(\phi) \) is a union of trees.

Note that the simplicity condition is open, but not necessarily closed.
Lemma 4.1.  
(i) The restriction of a simple function from the ball to its boundary
sphere is simple.

(ii) Any simple function on the boundary of the ball extends to the ball as a simple
function.

Proof. (i) Let \( \Phi : B \to \mathbb{R} \) be a simple function. Passing through a critical point
of the function \( \phi := \Phi|_{\partial B} \) either adds an index 1 handle to the level sets of \( \Phi \), or
subtracts a handle. The simplicity condition for \( \Phi \) imposes constraints only on the
handle addition, and it is equivalent to the condition that the union of components
of the level \( \{ \Phi = a - \varepsilon \} \) with all attached handles has contractible components. But
this means that positive points for \( \phi \) corresponds to handle attaching points of \( \Phi \),
and the contractibility condition is equivalent to the simplicity condition for \( \phi \).

(ii) It is sufficient to consider the case when \( \phi \) is a Morse function. Indeed, a creation
of an embryo point is a phenomenon localized near a point, and if the function is
extended to the ball \( B \) it is straightforward to perform the corresponding bifurcation
keeping the function on the ball critical point free.

Recall a handlebody presentation for a function \( \Phi \) on a manifold \( M \) with boundary
which has no critical points and which restricts to \( \partial M \) as a Morse function, see Fig.
4.3. Take two copies of \( M_1, M_2 \) of \( M \), and consider a double \( \hat{M} := M_1 \cup_{\partial M_1 = \partial M_2} M_2 \),
the canonical involution \( j : \hat{M} \to \hat{M} \) and a the map \( s : \hat{M} \to M \) with the fold on
\( \Sigma := \partial M_1 = \partial M_2 \) which is a diffeomorphism on \( \Sigma \) and interiors of the two copies.
Given a function \( \Phi : M \to \mathbb{R} \) which has no critical points and which restricts to
\( \partial M \) as a Morse function \( \phi : \partial M \to \mathbb{R} \), the function \( \hat{\Phi} = \Phi \circ s \) has Morse critical
points in 1-1 correspondence with the critical points of \( \phi \). Consider the handlebody
presentation \( \hat{M} = \hat{H}_0 \cup \cdots \cup \hat{H}_m \) of \( \hat{M} \) corresponding to the function \( \hat{\Phi} \). The
involution \( j \) descends to a handle \( \hat{H}_k = D^k \times D^{n-k} \) as the reflection either on the
first, or the second factor. In the latter case the corresponding critical point \( p \) on
the critical level \( a \) has the same index for \( \phi \) and \( \hat{\Phi} \) and a “half-handle” \( D^k \times D^{n-k} \)
is attached to the sublevel set \( \{ \Phi \leq a - \varepsilon \} \) along a tubular neighborhood of \( \partial D^k \times 0 \)
in \( \{ \Phi = a - \varepsilon \} \). In the former case the critical point \( p \) has a smaller index \( k - 1 \) for
\( \hat{\Phi} \) and a “half-handle” \( D^k_+ \times D^{n-k} \) is attached to the sublevel set \( \{ \Phi \leq a - \varepsilon \} \) along
a tubular neighborhood of \( \partial_-(D^k_+) \times 0 \) in \( \{ \Phi = a - \varepsilon \} \). Here we denoted by \( D^k_+ \) the
upper-half disc \( D^k \cap \{ x_k \geq 0 \} \) in \( \mathbb{R}^k \), and by \( (\partial D^k_+)_- \) the part \( \partial D^k_+ \cap \{ x_k = 0 \} \) of its
boundary.

We will be building the ball \( B \) together with the function \( \Phi \) inductively over critical
values of \( \phi \).
Let $a_m$ be one of the critical values $a_0 < \cdots < a_k$ of $\phi$. Suppose that we already constructed a 3-manifold $B_m$ with boundary with corners, $\partial B_m = \partial_- B_m \cup \partial_+ B_m$, a diffeomorphism $g: \{ \phi \leq a - \varepsilon \} \to \partial_- B_m$, and a function $\Phi_m: B_m \to \mathbb{R}$ without critical points such that

- each component of $\partial_+ B_m$ is a 2-disc and $\Phi_m|_{S^2} = a - \varepsilon$;
- each component of $B_m$ is homeomorphic to a 3-ball;
- $\Phi_m \circ g = \phi$.

For each negative hyperbolic point $h_j^- \in A_a$ its stable manifolds $\sigma_j^-$ have end points on the same component $C$ of $A^- = \{ \phi = a - \varepsilon \}$. Moreover, for any two negative hyperbolic point $h_j^- \in A_a$ and $h_i^- \in A_a$ with the end points of $\sigma_j^-$ and $\sigma_i^-$ on $C$, these end points are not interlinked because $S$ is a sphere.

Let $D_C$ denote the component of $\partial_+ B_m$ bounded by a component $C$ of $A^-$. For all hyperbolic points $h_j^-, \ldots, h_s^- \in A_a$ with end points on $C$ there are disjoint embeddings $\psi_{ji}: \sigma_j^- \to D_C$ with end points equal to the end points $\partial \sigma_{ji}^- \subset C$. The manifold $B_{m+1}$ is the result of attaching index 2 half-handles $D^2_+ \times D^1_-$ to $B_m$ along $\tilde{\sigma}_{ji} := \psi_j(\sigma_j)$ and for each positive hyperbolic point $h_j^+$ whose stable manifold $\sigma_j^+$ have end points on components $C, C' \subset A^-$ we attach an index 1-handle half-handle $D_1 \times D^2_+$ along $\partial \sigma_j^+$. For an index 0 point we add a component $D^3$, and for an index 2 point we glue $D^2_+$ to the disc $D_C$ bounded by the corresponding component $C$ of $A_a$.

The simplicity condition for $\phi$ ensures that each component of $B_{m+1}$ is homeomorphic to the 3-ball, and that the canonical extension of $\Phi$ to $B_{m+1}$ is simple.

\[\square\]

**Corollary 4.2.** If $\phi$ is simple then $-\phi$ is simple as well.

Indeed, the above property obviously holds for simple functions on the 3-ball.
4.2 The parametric case

Denote by Simple$_3$ and Simple$_2$ the spaces of simple functions on the 3-ball and the 2-sphere, respectively.

**Proposition 4.3.** The restriction map \( r : \text{Simple}_3 \rightarrow \text{Simple}_2 \) is a Serre fibration with contractible fiber. In particular, any family of simple functions on \( S \) extends to a family of simple functions on the 3-ball \( B \).

Note that Lemma 4.1(ii) asserts that the fibers of \( r \) are non-empty.

It is convenient to use the notion of a micro-fibration introduced by M. Gromov in [12]. A map \( p : X \rightarrow Y \) is called a micro-fibration if for any map \( F : D^k \rightarrow X \) and a homotopy \( f_t : D^k \rightarrow Y \) starting at \( f_0 := p \circ F \) there exists \( \varepsilon > 0 \), and a covering homotopy \( F_t \) with \( p \circ F_t = f_t \) for \( t \in [0, \varepsilon] \). An example in Section 1.4.2 (see also the first exercise in Section 3.3.1) of Gromov’s book [12] states that any microfibration with non-empty contractible fibers is a Serre fibration, and in particular a homotopy equivalence. The details of the proof are provided by M. Weiss in [20].

**Lemma 4.4.** For any \( \phi \in \text{Simple}_2 \) the fiber \( r^{-1}(\phi) \subset \text{Simple}_3 \) is contractible.

This is an immediate corollary of Hatcher’s theorem [15].

**Proof of Proposition 4.3.** It remains to check the microfibration property. Let \( \Phi_\lambda : B \rightarrow \mathbb{R}^3, \lambda \in \Lambda \), be any family of simple functions parameterized by a compact set of parameters \( \Lambda \). The family \( \Phi_\lambda \) extends to a ball \( \hat{B} \supset B \) of a slightly larger radius. Let \( v_\lambda \) be a family of vector fields on \( \hat{B} \) such that \( d\Phi_\lambda(v_\lambda) = 1 \) (e.g. \( v_\lambda = \frac{\nabla \Phi_\lambda}{\|\nabla \Phi_\lambda\|} \)). By integrating this vector field we define for a sufficiently small \( \varepsilon \) an isotopy \( \theta_{\lambda,t} : B \rightarrow \hat{B}, \lambda \in \Lambda, t \in [0, \varepsilon] \), beginning with the inclusion \( B \hookrightarrow \hat{B} \) as \( \theta_{\lambda,0} \). For any function \( \delta : B \rightarrow [0, \varepsilon) \) we denote by \( g_{\lambda,\delta} \) an embedding \( B \hookrightarrow \hat{B} \) defined by the formula

\[
g_{\lambda,\delta}(x) := \theta_{\lambda,\delta(x)}.
\]

Note that \( \Phi_\lambda \circ g_{\lambda,\delta}(x) = \Phi_\lambda + \delta \).

Consider now a deformation \( \phi_{\lambda,t}, \lambda \in \Lambda, t \in [0,1] \), of the family \( \phi_{\lambda,0} := \Phi_\lambda|_{\partial B} \). For a sufficiently small \( t \) we have \( |\phi_{\lambda,t} - \phi_{\lambda,0}| < \varepsilon \). Consider a family of functions \( \delta_{\lambda,t} : B \rightarrow \mathbb{R} \) such that for \( x \in \partial B \) we have \( \delta_{\lambda,t}(x) := \phi_{\lambda,t}(x) - \phi_{\lambda,0}(x) \). Then \( \Phi_\lambda \circ g_{\lambda,\delta_{\lambda,t}}(x) = \phi_{\lambda,t}(x) \) for \( x \in \partial B \), and hence, the family of functions \( \Phi_{\lambda,t} := \Phi_\lambda \circ g_{\lambda,t} \) provides the required extension of the family \( \phi_{\lambda,t} \) to the ball \( B \). The simplicity is an open property. Hence, the functions \( \Phi_{\lambda,t} \) are simple if \( \varepsilon \) is chosen small enough. \( \square \)
Remark 4.5. Let us point out that the space Simple$_2$ (and hence, Simple$_3$) is not contractible. In fact, it is not even simply connected, as one can exhibit a non-contractible loop of simple functions on the 2-sphere with one minimum, one saddle point and two maxima.

5 Taming functions and their properties

5.1 Lyapunov functions

Recall that $\phi$ is called a Lyapunov function for a vector field $Z$ on a compact manifold $X$ if $d\phi(Z) \geq c_1|Z|^2 + c_2|\nabla \phi|^2$ for a positive constants $c_1, c_2$. The property is independent of the choice of an ambient metric. Equivalently one says that $Z$ is gradient-like for $\phi$.

Lemma 5.1. (i) Let $Z^*$ be the space of germs of Liouville vector fields on $(\mathbb{R}^2, 0)$ with an isolated non-degenerate zero at the origin, and $\mathcal{L}$ be the space of pairs $(Z, F)$ where $Z \in Z^*$ and $F$ is a germ of a Lyapunov function for $Z$. Then the projection $\pi : \mathcal{L} \rightarrow Z^*$ is a Serre fibration with a contractible fiber.

(ii) Let $\Lambda$ be a compact parameter space and $\Lambda_0 \subset \Lambda$ its closed subset. Consider a family $Z_\lambda$, $\lambda \in \Lambda$, of Liouville fields on $\mathcal{O}_p \mathbb{R}^2 \subset \mathbb{R}^2$ such that $Z_\lambda$ has an embryo point at 0 for $\lambda \in \Lambda_0$. Then there exist a neighborhood $U \ni 0$ in $\mathbb{R}^2$, a neighborhood $\Omega \ni \Lambda_0$, and a family of Lyapunov functions $\phi_\lambda : U \rightarrow \mathbb{R}$, $\lambda \in \Omega$, for $Z_\lambda|_U$.

Proof. (i) Let us first show that $\pi^{-1}(Z) \neq \emptyset$ for any $Z \in Z$. If $A_\lambda = d_0 Z_\lambda$ has real eigenvalues and diagonalizable in a basis $v_1, v_2$ with eigenvalues $\lambda_1, \lambda_2$ then the function $\lambda_1 x_1^2 + \lambda_2 x_2^2$, where $(x_1, x_2)$ are for coordinates in that basis, is Lyapunov for $Z_\lambda$. If $A_\lambda$ has a Jordan form $\begin{pmatrix} a_\lambda & 1 \\ 0 & a_\lambda \end{pmatrix}$, $a_\lambda = \frac{1}{2} \text{Tr} A_\lambda$, then the function $x_1^2 + cx_2^2$ is Lyapunov provided that $c > 4a_\lambda^2$. If $A_\lambda$ has complex eigenvalues $\frac{a_\lambda}{2} \pm a_\lambda i$ with eigenvectors $v_1^\lambda \pm iv_2^\lambda$ are the corresponding eigenvectors, and $B$ is the matrix made of columns $v_1^\lambda, v_2^\lambda$ then the quadratic form $||Bx||^2$ is Lyapunov for $Z_\lambda$.

Next, we observe that the fiber $\pi^{-1}(Z)$ is a convex subset in the space of functions, and hence, contractible. Hence, it is sufficient to check a microfibration property. But this is straightforward because a Lyapunov function for $Z$ serves also as a Lyapunov function for any $Z'$ which is $C^2$-close to $Z$ and has an isolated singularity at the origin.
(ii) According to Lemma 2.1 and Proposition 2.3, the family $Z_\lambda$ for $\lambda \in \Omega := O p \Lambda_0$ is orbitally equivalent to a family of vector fields $x \frac{\partial}{\partial x} + F(y, \lambda) \frac{\partial}{\partial y}$, $f(0) \neq 0$, $F(y, \lambda) = y^2 f(y)$ for $\lambda \in \Lambda_0$. Define $\psi_{\lambda, s} = \frac{x^2}{2} + \int_0^y F(u, \lambda) du$. Then $Z_\lambda$ is a gradient vector field for $\psi_\lambda$ for the standard Euclidean metric on $\mathbb{R}^2$.

**Lemma 5.2.** Let $Z$ be a vector field generating a characteristic foliation $\mathcal{F}$.

a) Let $e_1, e_2$ be positive elliptic zeroes of $Z$, and $h$ a hyperbolic zero. Suppose that the incoming separatrices $\gamma_1, \gamma_2$ of $h$ terminate at $e_1$ and $e_2$. Let $\phi$ be a Lyapunov function for $Z$ on $O p \{e_1, e_2, h\}$ with $\phi(h) > \max(\phi(e_1), \phi(e_2))$. Then there exists an arbitrarily small neighborhood $U$ of $\gamma_1 \cup \gamma_2$ and a Lyapunov function $\Phi$ on $U$ which is constant on $\partial U$ and coincides with $\phi$ in a neighborhood of critical points. See Fig. 5.1a).

![Fig. 5.1: Lyapunov function on a neighborhood of separatrices](image)

b) Suppose that $e, h$ are positive elliptic and hyperbolic zeroes of $Z$. Suppose that the incoming separatrices $\gamma_1, \gamma_2$ of $h$ terminate at $e$. Let $\phi$ be a Lyapunov function for $Z$ on $O p \{e, h\}$ with $\phi(h) > \phi(e)$. Then there exists an arbitrarily small neighborhood $U$ of $\gamma_1 \cup \gamma_2$ and a Lyapunov function $\Phi$ on $U$ which is constant on $\partial U$ and coincides with $\phi$ in a neighborhood of critical points. See Fig. 5.1b).
c) Suppose that $e$ is positive elliptic and $o$ is positive embryo points of $Z$. Suppose that the incoming separatrix $\gamma$ of $o$ ends at $o$. Let $\phi$ be a Lyapunov function for $Z$ on $\mathcal{O}p\{e, o\}$ with $\phi(o) > \max\phi(e)$. Then there exists an arbitrarily small neighborhood $U$ of $\gamma$ and a Lyapunov function $\Phi$ on $U$ which is constant on $\partial U$ and coincides with $\phi$ in a neighborhood of critical points. See Fig. 5.1c).

d) Suppose that $e$ is positive elliptic and $o$ is a negative embryo. Suppose all incoming to $o$ trajectories originate at $e$, and $D$ is the union of these trajectories. Let $\phi$ be a Lyapunov function for $Z$ on $\mathcal{O}p\{e, o\}$ with $\phi(o) > \max\phi(e)$. Then there exists an arbitrary small neighborhood $U \supset D$ and a Lyapunov function $\Phi : U \to \mathbb{R}$ which is constant on $\partial U$ and coincides with $\phi$ in a neighborhood of critical points. See Fig. 5.1d).

Proof. We consider only the case a); all other cases are similar. The construction mimics the smooth surrounding Lemma 9.20 in [2]. Denote $\Gamma := \gamma_1 \cup \gamma_2$. First, we can assume that $\phi$ is equal to 0 at elliptic points and to 1 at the hyperbolic one, and then extend $\phi$ to $\mathcal{O}p(\Gamma)$ as increasing along $\gamma_j$ when going from $e_j$ to $h$, $j = 0, 1$. Choose an arbitrary small neighborhood $U \supset \Gamma$ on which $\phi$ is already defined.

Our next goal is to construct a disk $D \subset U, \text{Int} D \supset \Gamma$ such that $\partial D$ is transverse to $Z$. Choose a sufficiently small $\varepsilon > 0$ such that the level set $\{\phi \leq \varepsilon\}$ consists of 2 domains $\Delta_1 \ni e_1, \Delta_2 \ni e_2$ which are diffeomorphic to the 2-disc. There exists a tubular neighborhood $\Omega \supset (\Gamma_\varepsilon := \Gamma \\\{\phi \leq \varepsilon\})$ and its splitting $\Omega = \Gamma_\varepsilon \times \Delta, \Delta = (\varepsilon, \varepsilon)$ such that

- the outgoing separatrices of $h$ form the fiber $h \times \Delta$;
- $\partial \Delta_1 \cap \Omega$ and $\partial \Delta_2 \cap \Omega$ are fibers over the end points of the interval $\Gamma_\varepsilon$, and
- the field $Z$ is transverse to the fibers elsewhere.

Hence, there are local coordinates $(s, u) \in [-1, 1] \times \Delta$ in $U$ and a function $f : [-1, 1] \times \Delta \setminus \{s = 0, u \neq 0\} \to \mathbb{R}$ such that $\Gamma_\varepsilon = [-1, 1], h = (0, 0)$, and the line field spanned by the vector field $Z$ can be given by the differential equation

$$\frac{du}{ds} = f(s, u), s \in [-1, 1], s \neq 0, u \in \Delta.$$ 

The function $f$ satisfies $f(s, 0) = 0$, and there exists a sufficiently small $\sigma > 0$ such that $f(s, u) > 0$ for $|s| < \sigma, s \neq 0, u > 0$ and $f(s, u) > 0$ for $|s| < \sigma, s \neq 0, u < 0$. In fact,
we have \( \lim_{s \to 0, u \neq 0} f(s, u) \to \pm \infty \). Choose a smooth function \( \tilde{f} : [-1, 1] \times \Delta \to \mathbb{R} \) with the following properties:

- \( \tilde{f}(s, 0) = 0, \ s \in [-1, 1] \);
- \( \tilde{f}(s, u) = 0, \ s \in [-\sigma, \sigma], u \in \Delta \);
- \( \tilde{f}(s, u) < f(s, u), \ s \in [-1, 1], u \in (0, \delta) \);
- \( \tilde{f}(s, u) > f(s, u), \ s \in [-1, 1], u \in (-\delta, 0) \);

Consider a differential equation

\[
\frac{du}{ds} = \tilde{f}(s, u), \ s \in [-1, 1], u \in \Delta, \quad \text{See Fig. 5.2}
\]

For a sufficiently small \( \theta > 0 \) the equation has a solution \( u = \psi_t(s) \) with the initial condition \( \psi_t(-1) = t \) if \( |t| \leq \theta \), and the vector field \( Z \) is outward transverse to the domain \( U_\theta \subset \Omega \) bounded by graphs \( u = \psi_{\pm\theta}(s), s \in [-1, 1] \). Moreover, the graphs
Lemma 5.3. Let \( \xi \) be a contact structure on \( \mathbb{R}^3 \) such that the characteristic foliation \( \mathcal{F} \) on \( \mathbb{R}^2 = \{ x_3 = 0 \} \subset \mathbb{R}^3 \) has an isolated singularity at 0. Let \( \phi \) be a Lyapunov function for \( \mathcal{F} \) on \( \mathbb{R}^2 \). Let \( \mathcal{F}_{\text{loc}} \) be any other characteristic foliation on \( \mathbb{R}^2 \) with an isolated at 0 of the same type with a local Lyapunov function \( \phi_{\text{loc}} \). Suppose that \( \phi(0) = \phi_{\text{loc}}(0) = 0 \). Then for any neighborhood \( U \ni 0, U \subset \mathbb{R}^2 \), there is a supported in \( U \) arbitrary \( C^1 \)-small 2-parametric isotopy \( j_{s,t} : \mathbb{R}^2 \to \mathbb{R}^3 \), \( s \in [-1,1], t \in [0,1] \), such that

1. \( j_{s,0} : \mathbb{R}^2 \to \mathbb{R}^3 \) is the inclusion for all \( s \in [0,1] \);
2. \( j_{1,t}(\mathbb{R}^2) \subset \mathbb{R}^3_+ = \{ x_3 \geq 0 \}, \ j_{-1,t}(\mathbb{R}^2) \subset \mathbb{R}^3_- = \{ x_3 \leq 0 \} \);
3. the induced characteristic foliation \((j_{s,1})^*\xi\) has a unique singularity at 0, equals to \( \mathcal{F}_{\text{loc}} \) near 0, and admits a Lyapunov function \( \phi \) which is equal to \( \phi \) near \( \partial U \) and to \( \phi_{\text{loc}} \) near 0.

Proof. We can write the contact form \( \beta \) for the contact structure \( \xi \) as \( dx_3 + \alpha \), where \( \alpha \) is a Liouville form on \( \mathbb{R}^2 \). Let \( \alpha_{\text{loc}} \) be the form on \( \mathbb{R}^2 \) which defines \( \mathcal{F}_{\text{loc}} \). The pairs \((\alpha, \phi)\) and \((\alpha_{\text{loc}}, \phi_{\text{loc}})\) are two local Weinstein structures on \( \mathbb{R}^2 \) with isolated singularities of the same type at the origin. Using Proposition 12.12 from [2] we can construct a Weinstein structure \((\hat{\alpha}, \hat{\phi})\) which coincides with \((\alpha, \phi)\) outside of a neighborhood \( U' \subset U, U' \ni 0 \) and with \((\alpha_{\text{loc}}, \phi_{\text{loc}})\) on a neighborhood \( U_{\text{loc}} \subset U', U_{\text{loc}} \ni 0 \). Moreover, using Darboux’s theorem we can arrange that the symplectic forms \( d\alpha \) and \( d\hat{\alpha} \) coincide, and hence, \( \hat{\alpha} = \alpha + dH \), where \( H(0) = 0 \) and \( d_0 H = 0 \). By shrinking the neighborhood \( U_{\text{loc}} \) we can make the function \( H \) arbitrarily \( C^1 \)-small.

Let \( \theta : U \to [0,1] \) be a cut-off function supported in \( U \) which is equal to 1 on \( U' \). There exists \( \sigma > 0 \) such that the function \( \hat{\phi} \) is Lyapunov for the foliation on \( U \setminus U' \) defined by a Pfaffian equation \( \alpha + s d\theta = 0 \) for any \( |s| < \sigma \). Hence, the function \( \hat{\phi} + s d\theta \) is Lyapunov for \( \hat{\mathcal{F}} \) defined by a Pfaffian equation \( \{ \hat{\alpha} + s d\theta = 0 \} \), \( |s| < \sigma \). Indeed, outside of \( U' \) we have \( \hat{\alpha} = \alpha \), and on \( U' \) we have \( d\theta = 0 \). Note that if \( |H| < \sigma \)
then the function \( H - \sigma \theta \leq 0 \leq H + \sigma \theta \). We claim that the family of isotopies \( j_{s,t} \) 
defined by the formula.

\[
j_{s,t}(x_1, x_2) := (x_1, x_2, t(H + s\sigma \theta)), s \in [-1, 1], t \in [0, 1],
\]

has the required properties. Indeed, we have \( j_{1,t}(\mathbb{R}^2) \subset \mathbb{R}^3_+, j_{-1,t}(\mathbb{R}^2) \subset \mathbb{R}^3_- \) for all \( t \in [0, 1] \), \( j_{s,1}^t \beta = \hat{\alpha} + s\sigma d\theta \), and the function \( \phi \) is Lyapunov for the characteristic foliation \( \mathcal{F}_s = \{(j_{s,1})^* \beta = 0\} \).

\[\square\]

**Remark 5.4.** Lemma 5.3 holds also in the relative parametric form.

### 5.2 Taming functions for characteristic foliations

A function \( \phi : S \to \mathbb{R} \) is said to **tame** the characteristic foliation \( \mathcal{F} \) if the foliation can be generated by a vector field \( Z \) which is gradient-like for \( \phi \) and such that its positive and negative hyperbolic points are, respectively, positive and negative zeroes of the characteristic foliation.

Note that local minima of a taming functions are automatically positive, and local maxima are negative elliptic points of \( \mathcal{F} \), while and positive (resp. negative) embryos of \( \mathcal{F} \) are embryos the function \( \phi \) of index \( \frac{1}{2} \) (resp. \( \frac{3}{2} \)) of the function \( \phi \), i.e. they split into index 0 and 1 (resp. index 1 and 2) critical points under a small perturbation, see Fig. 5.3.

![Fig. 5.3: Embryos o₁ of index \( \frac{1}{2} \) and o₂ of index \( \frac{3}{2} \)](image)

**Lemma 5.5.** Any connected component of a regular sublevel set of a taming function has \( d_+ = 1 \).

**Proof.** Arguing by induction, suppose the claim is true for sublevel sets \( \{ \phi \leq a \} \) for \( a < c \), where \( c \) is a critical value of \( \phi \). The stable separatrices of a positive
hyperbolic point \( h \in A_c \) end at two different components of \( A^-_c \), and hence two different components of \( \{ \phi \leq c - \varepsilon \} \). Hence, passing through \( h \) connect sums two components with \( d_+ = 1 \) into 1 component with \( d_+ = 1 \). The stable separatrices of a negative hyperbolic point \( h \in A_c \) end at the same component of \( \{ \phi \leq c - \varepsilon \} \) and hence, passing through \( h \) does not change \( d_+ \) of the corresponding component.

The following lemma is straightforward.

**Lemma 5.6.** Suppose \( \phi \) tames a Morse characteristic foliation \( \mathcal{F} \). Then if \( \mathcal{F}' \) is sufficiently \( C^2 \)-close to the given one and has the same singular points then \( \phi \) tames \( \mathcal{F}' \) as well.

**Lemma 5.7.** Suppose a function \( \phi \) tames a tight foliation \( \mathcal{F} \) without homoclinic connections between hyperbolic points or embryos. Given any two positive elliptic points \( e_+, e'_+ \), let \( \gamma(e_+, e'_+) \) be the unique Legendrian path \( e_+h_1e'_+ \ldots h_ke'_+ \) provided by Corollary 2.10. Denote

\[
c(e_+, e'_+; \phi) = \min \{ \phi(h^+_1), h^+_i \in \gamma(e_+, e'_+) \}.
\]

We set \( c(e_+, e'_+; \phi) = 0 \) (assuming that \( \phi = 0 \) at positive elliptic points). For each hyperbolic point we denote by \( E(h) = \{ e_+, e'_+ \} \) the positive elliptic end points of its stable separatrices. Then for any negative hyperbolic point \( h_- \) of \( \mathcal{F} \) we have

\[
\phi(h_-) > c(e_+, e'_+; \phi), \quad \text{where} \quad E(h_-) = \{ e_+, e'_+ \}.
\]  

**Remark 5.8.** Note that if \( \phi \) tames a tight foliation \( \mathcal{F} \) with homoclinic connections, then it also tames a tight foliation \( \mathcal{F}' \) without homoclinic connections, because both properties, taming and tightness, are open.

**Proof.** Let \( E(h_-) = \{ e_+, e'_+ \} \). If \( e_+ = e'_+ \) then the statement is vacuous, so we suppose that \( e_+ \neq e'_+ \). For any regular value \( c \in (0, \phi(h_-)) \) denote by \( C_c, C'_c \) components of the level set \( A_c \) intersecting separatrices of \( h_- \) originated at \( e_+ \) and \( e'_+ \), respectively. For \( c \) close to \( \phi(h_-) \) we have \( C_c = C'_c \) while for \( c < c(e_+, e'_+) \) we must have \( C_c \neq C'_c \).

**Lemma 5.9.** Let \( \mathcal{F} \) be a tight characteristic foliation without homoclinic connections. Then any Lyapunov function for \( \mathcal{F} \) which satisfies condition (1) is taming.

**Proof.** Let us check that condition (1) implies that the two notions of positivity coincide for all hyperbolic points. Suppose that \( h \) is negative for the function, i.e. both incoming separatrices of \( h \) intersect the same component \( C \subset A^-_c \), and hence
the same component $U$ of $\{ \phi \leq c - \varepsilon \}$. Arguing by induction, we can assume that the claim is already proven for hyperbolic points in $\{ \phi < c \}$, and hence Lemma 5.5 implies that $d_+ (U) = 1$ and hence, we can use Lemma 2.7(ii) to eliminate all critical points in $U$ except 1 positive elliptic point. Then both separatrices of $h$ will be originating at the same positive elliptic point which by Lemma 2.9 implies that $h$ is negative for $\mathcal{F}$. If $h$ is positive for $\phi$ then incoming separatrices of $h$ intersect different components of $A^\circ_c$, and hence, taking into account that $S$ is the sphere, different components $U, U'$ of $\{ \phi \leq c - \varepsilon \}$. Hence, generically the stable separatrices of $h$ originate at positive elliptic points $e \in U$ and $e' \in U'$. Let $\gamma(e, e')$ be the path provided by Corollary 2.10. If $h$ were negative for $\mathcal{F}$ then we would have $\phi(h) \leq c(e, e')$, contradicting condition (1). Hence, $h$ is positive for $\mathcal{F}$.

Lemma 5.10. Any taming function is simple.

Proof. Suppose there is a loop in the graph $\Gamma^+_c$ for a critical value $c$. Using Lemmas 5.5 and 2.7(ii) we deform $\mathcal{F}$ on components $\{ \phi \leq c - \varepsilon \}$ to leave exactly one positive elliptic point in each component. But then the separatrices of positive hyperbolic points forming the cycle could be continued to these elliptic points to create a Legendrian polygon with only positive vertices, thus contradicting Lemma 2.9.

5.3 Existence of taming functions

Proposition 5.11. Let $(S, \mathcal{F})$ be a generalized Morse tight foliation on a 2-sphere. Then $S$ admits a taming function.

Remark 5.12. As we will see in Proposition 7.12 below existence of a taming function for a characteristic foliation is not only necessary, but also sufficient for tightness.

Lemma 5.13. Let $\mathcal{F}$ be a tight foliation on $S$. Suppose that $\mathcal{F}$ has non-elliptic critical points. Then there exists a function $\phi : S \to \mathbb{R}$ which has the following properties:

- $\phi$ attains its minimal value 0 at all positive elliptic points;
- $\phi$ tames $\mathcal{F}$ on $V := \{ \phi \leq 1 \} \subset S$. There is a unique critical point $c$ in $\{ 0 < \phi \leq 1 \}$;
- $\phi(c) = \frac{1}{2}$ and $c$ is either hyperbolic or embryo.
Moreover, we can arbitrarily up to additive constants prescribe the Lyapunov function \( \phi \) near the critical points of \( \phi \) in \( \{ \phi \leq 1 \} \).

Proof. Lemma 5.13 is essentially a reformulation of the Key Proposition 3.2. First, let us construct the taming function with the critical value 0 on neighborhoods of positive elliptic points, see Lemma 5.1. According to Proposition 3.2 there exists an allowable vertex, which is either

a) a positive hyperbolic point with two incoming separatrices from different positive elliptic points, or

b) a negative hyperbolic point with two incoming separatrices from the same positive elliptic point, or

c) a positive embryo with the incoming separatrix from a positive elliptic point, or

d) a negative embryo with all the incoming trajectories from a positive elliptic point.

In case a) let \( h \) be a positive hyperbolic point with the incoming separatrices from two positive elliptic points. Then according to Lemma 5.2a) there exists a function on a neighborhood \( U \) of the union of both separatrices which has the elliptic points as their minima and the hyperbolic point as the saddle point and which is constant on the boundary \( \partial U \). Similarly, in case b) let \( h \) be a negative hyperbolic point with the incoming separatrices from the same positive elliptic point. Then Lemma 5.2b) yields a taming function on a neighborhood \( U \) of the union of both separatrices which has the elliptic point as its minimum and the hyperbolic point as the saddle point and which is constant on the boundary \( \partial U \). In both cases the simplicity condition is satisfied. Cases c) and d), of a positive and negative embryos follow from Lemma 5.2c) and d). In all the above cases we can prescribe \( \phi \) to be equal (up to an additive constant) to any Lyapunov function.

Proof of Proposition 5.11. We prove Proposition 5.11 by induction in the number of critical points. Suppose the proposition is already proved for less than \( k \) hyperbolic points or embryos. Consider the function \( \phi \) provided by Lemma 5.13. If \( c \) is a positive hyperbolic point then the simplicity condition implies that each component of \( V = \{ \phi \leq 1 \} \) is a disc with \( d_+ = 1 \). The same obviously holds if \( c \) is an embryo. Hence, Lemma 2.7(ii) allows us to kill all critical points in these discs except 1 positive elliptic. By induction hypothesis the new tight foliation \( \mathcal{F}' \) which coincides
with $\mathcal{F}$ on $\{\phi \geq 1\}$ admits a taming function $\psi$. We can assume that $\{\psi \leq 1\} = V$. Hence, concatenating $\phi|_V$ with $\psi|_{S,V}$ we get the required taming function for $\mathcal{F}$. If $c$ is a negative hyperbolic point, then one of the components of $V$ is an annulus $A$. The complement of $A$ is the union of two discs, $S\setminus \text{Int} A = D_1 \cup D_2$. For each of the complementary discs, $D_j' := S\setminus \text{Int} D_j$, $j = 1, 2$ we have $d_+(D_j') = 1$, according to Lemma 2.7(iii). Hence, we can deform away all singular points but 1 positive elliptic point on $D_1'$, and using the induction hypothesis construct a function $\psi_1$ which tames the foliation $\mathcal{F}_1$ on $S$ which is equal to $\mathcal{F}$ on $D_1$ and to the foliation with 1 elliptic singularity on $D_1'$. Similarly, we construct a function $\psi_2$ which tames the foliation $\mathcal{F}_2$ on $S$ which is equal to $\mathcal{F}$ on $D_2$ and to the foliation with 1 elliptic singularity on $D_2'$. We can assume that $\psi_1|_{\partial D_1} = \psi_2|_{\partial D_2} = 1$. Concatenating functions $\psi_1|_{D_1}, \psi_2|_{D_2}$ and $\phi|_A$ we get the required taming function for $\mathcal{F}$.

5.4 The parametric case

Lemma 5.14. The space of taming functions for a tight foliation is a convex set $C^\infty(S)$.

Proof. Convex combination of Lyapunov functions for the same characteristic foliation $\mathcal{F}$ is again a Lyapunov function. Note that according to Remark 5.8 we can assume that the tight foliation $\mathcal{F}$ has no homoclinic conditions. Hence, we can apply the simplicity condition (1) which survives taking a convex combination.

Let us denote by $\mathcal{T}$ the space of tight generalized Morse foliations on the sphere $S = S^2$ and by $\mathcal{P}$ the space of pairs $(\phi, \mathcal{F})$, where $\mathcal{F} \in \mathcal{T}$ and $\phi$ is its taming function.

Proposition 5.15. Let $\Lambda$ be a compact manifold with boundary. Then for any maps $h : \Lambda \to \mathcal{T}$ and $H : \mathcal{O}_p \partial \Lambda \to \mathcal{P}$ such that $\pi \circ H = h|_{\mathcal{O}_p \partial \Lambda}$ there exists a map $\tilde{H} : \Lambda \to \mathcal{P}$ such that $\tilde{H}|_{\mathcal{O}_p \partial \Lambda} = H$ and $\pi \circ \tilde{H} = h$.

Proof. We will deduce the statement by proving a slightly weaker version of the claim that $\pi : \mathcal{P} \to \mathcal{T}$ is a Serre fibration with a contractible fiber. The fiber is indeed contractible, as Lemma 5.14 shows. Let us show the following modified Serre covering homotopy property: given a map $f : D^k \times [0, 1] \to \mathcal{T}$ and a covering map $F : D^k \times [0, \varepsilon) \to \mathcal{P}$, $\pi \circ F = f|_{D^k \times [0, \varepsilon)}$ there exists a lift $\tilde{F} : D^k \times [0, 1] \to \mathcal{P}$ such that $\pi \circ \tilde{F} = f$ and $\tilde{F} = F$ on $D^k \times 0$. We will need the following
Lemma 5.16. Given any map $h : D^k(1) \to \mathcal{F}$ there exists $\varepsilon \in (0, 1]$ and a map $H : D^k(\varepsilon) \to X$ such that $p \circ H = h$.

The notation $D^k(r)$ stands here for the centered at 0 disk of radius $r$ in $R^k$.

Proof. Denote $\mathcal{F}_s := h(s), s \in D^k(\varepsilon)$. According to Proposition 5.11 it admits a taming function $\phi$.

(i) Suppose first that $\mathcal{F}_0$ has no embryos. Then for a sufficiently small $\varepsilon > 0$ and $|s| \leq \varepsilon$ there exists a family of diffeomorphisms $\alpha_s$, $\alpha_0 = \text{Id}$, such that the foliation $(\alpha_s)_* \mathcal{F}_s$ has the same critical points as $\mathcal{F}_0$, and hence according to Lemma 5.6 the functions $\phi_s \circ \alpha_s$ tame $\mathcal{F}_s$ for $s \in D^k(\varepsilon)$ if $\varepsilon$ is chosen small enough. Hence, we can set $H(s) = (\mathcal{F}_s, \phi_s)$ for $|s| \leq \varepsilon$.

(ii) If $\mathcal{F}_0$ has embryo points then, according to Lemma 5.1ii), there exists $\varepsilon > 0$ and a family of functions $\psi_s$, $|s| < \varepsilon$, on a neighborhood $U$ of embryos of $\mathcal{F}_0$ such that $\psi_0 = \phi|_U$, and $\psi_s$ is Lyapunov for $\mathcal{F}_s$. Take a smaller neighborhood $U' \subset U$ of the embryo locus and consider a cut-off function $\theta : U \to [0, 1]$ which is supported in $U$ and equals 1 on $U'$, and define $\phi_s|_U := \theta \psi_s + (1 - \theta)\phi$. Then for a small enough $\varepsilon$ and $|s| < \varepsilon$ the function $\phi_s$ is Lyapunov for $\mathcal{F}_s|_U$, and hence, applying the argument in (i) to $\mathcal{F}_s|_{S|U}$ we can extend the family $\phi_s$ to $S$.

Now we can conclude the proof of the modified Serre covering homotopy property. Lemma 5.16 allows us to construct a finite covering

$$\bigcup_{i=1}^{N} U_j \supset D^k \times [\varepsilon/2, 1], \quad U_j \subset D^k \times (\varepsilon/3, 1),$$

and maps $F_j : U_j \to \mathcal{P}$ covering $f|_{U_j}$. Consider a partition of unity $\theta_j, j = 0, \ldots, N$, subordinated to the covering $U_0 := D^k \times (0, \varepsilon), U_1, \ldots, U_N$. Set $F_0 := F$. Denote $\mathcal{F}_s = f(s), s \in D^k \times [0, 1]$, $F_j(s) = (\mathcal{F}_s, \phi^j_s), j = 0, \ldots, N$ for $s \in U_j$, we can define the required lift $\hat{F}$ by the formula $\hat{F}(s) = (\mathcal{F}_s, \sum \theta_j \phi^j_s), s \in D^k \times [0, 1]$.  

6 Igusa type theorem

We prove in this section an analog of Igusa’s theorem about functions with moderate singularities for normalization of singularities of characteristic foliations on a family
of embedded surfaces. We follow the general scheme of our papers \cite{7,9}, while specializing the argument for the specific context of the current paper.

Given a family $\xi_\lambda$ of contact structures, or more generally codimension 1 distributions on a manifold $U$, we will view it as a fiberwise distribution $\xi = \{\xi_\lambda\}_{\lambda \in \Lambda}$ on the trivial fibration $\Lambda \times U \to \Lambda$. The notation $\zeta_0$ stands for the standard contact structure on $\mathbb{R}^3$. The parameter space $\Lambda$ will be assumed to be a compact manifold with boundary $\partial \Lambda$.

**Proposition 6.1.** Let $\xi = \{\xi_\lambda\}_{\lambda \in \Lambda}$ be a fiberwise contact structure on $\Lambda \times U$, where $U = O_p \mathbb{R}^3 D$ is a neighborhood of the 2-disc $D \subset \mathbb{R}^2 = \{x_3 = 0\} \subset \mathbb{R}^3$. Suppose that $\xi_\lambda$ is transverse to $D$ for $\lambda$ in $\partial \Lambda$. Then there exists a $C^0$-small fiberwise isotopy $\phi^t : D \times \Lambda \to O_p D \times \Lambda, t \in [0, 1]$, such that

a) $\phi^0 = \text{Id}$ on $O_p (\partial \Lambda \times D \cup \Lambda \times \partial D), t \in [0, 1]$; and

b) $\phi^t (\Lambda \times D)$ has fiberwise generalized Morse tangencies to $\xi$.

Denote $W := \Lambda \times D, \hat{W} := \Lambda \times U$, where $U$ is a neighborhood of $D$ in $\mathbb{R}^3$. Denote by $\text{Vert}^3$ the rank 3 vector bundle over $\hat{W}$ tangent to the fibers of the projection $\pi : \hat{W} \to \Lambda$, and by $\text{Vert}^2$ the rank 2 bundle over $W$ tangent to the fibers $W \to \Lambda$. We have $\partial W = \partial \Lambda \times D \cup \Lambda \times \partial D$.

We begin the proof with two lemmas. A codimension 2 submanifold $V \subset W$ is said to be in an **admissible folded position** if the restriction $\pi|_V : V \to \Lambda$ has only fold singularities, the fold $\Sigma$ divides $V$ into $V = V_+ \cup V_-, V_+ \cap V_- = \Sigma$, and the line bundle $\text{Ker} (d\pi|_TV)$ over $\Sigma$ extends to $V_-$ as a trivial subbundle $\mu$ of $\text{Vert}^3|_{V_-}$. We say that its fold $\Sigma = \Sigma^1(\pi|_V)$ is small if it is contained in a union of balls in $V$.

**Lemma 6.2.** Under an assumption of Proposition 6.1 there exists a fixed on $O_p (\partial W)$ fiberwise homotopy $\xi^t, t \in [0, 1], \xi^t$ such that the tangency locus $V = \{(\lambda, x) \in \Lambda \times D, \xi^t_\lambda(x) = T_x D\}$ is a codimension 2 submanifold of $W$ which is in an admissible folded position with a small fold.

**Proof.** We trivialize $\text{Vert}^3$ by an orthonormal frame $e_1, e_2, e_3$ such that $e_1, e_2 \in TD$, i.e. $e_1, e_2$ generate $\text{Vert}^2$. Denote by $U$ the associated to $\text{Vert}^3$ unit sphere bundle. The trivialization of $\text{Vert}^3$ yields a splitting $U = \hat{W} \times S^2$. A fiberwise orthogonal to $\xi$ unit vector field $\nu$ yields a map $F := F_\xi : W \to S^2$, and $V := F^{-1}(e_3) \cup F^{-1}(-e_3) \subset W$ is the tangency locus of $D$ to $\xi = \{\xi_\lambda\}$. Perturbing, if necessary, $\xi$ we can assume that $\pm e_3$ are regular values of $F$, and hence $V$ is a framed codimension 2 submanifold of $\hat{W}$.
The triviality of the normal bundle to $V$ in $W$ implies, see e.g. Theorem 1.2 in [7], that there exists a $C^0$-small diffeotopy $\alpha_t : W \to W$, which is fixed over $\partial \Lambda$, and such that $\alpha_1(V)$ is in an admissible folded position with a small fold. Therefore, the homotopy of plane distributions orthogonal to the homotopy of vector fields $\nu \circ \alpha_t^{-1} \in \text{Vert}^3$ has the required properties.

\[\square\]

**Lemma 6.3.** (i) Suppose that $V = ^1V \cup ^-V \subset W$ is in an admissible folded position with small folds. Then there exists a fiberwise contact structure $\zeta$ on $\mathcal{O}_p V \subset \hat{W}$ such that $F_{\zeta}^{-1}(\pm e_3) = \pm V$.

(ii) Given a framing $F$ of $V$ one can find a $C^0$-small isotopy $\alpha_t : V \to W$ and a fiberwise contact structure $\zeta : \mathcal{O}_p \alpha_1(V)$ such that

- the isotopy $\alpha_t : V \to W$ is supported away from $\Sigma$;
- $\alpha_1(V)$ is in an admissible folded position, and
- the push-forward framing $(\alpha_1)_* F$ is homotopic to the framing $F_{\zeta}$.

**Proof.** (i) Denote $\pm \Sigma := \pm V \cap \Sigma$. Choose coordinates $(x, y, z, s, u)$ on a tubular neighborhood $\pm N$ of $\pm \Sigma$ in $\hat{W}$ such that the projection $\pi$ is given by $(x, y, z, s, u) \mapsto \lambda = (s, u)$, the coordinate $z$ is equal to $x_3$ (so that $W = \{z = 0\}) V = \{s = y^2\} \subset W$, $\Sigma = \{s = 0\} \subset V$, and the vector field $\frac{\partial}{\partial y}$ along $\Sigma$ is inward transverse to the boundary of $V_+ = -V_+ \cup +V_+$.

Define the fiberwise contact structure $\zeta$ on $\pm N$ by the 1-form $\hat{\alpha} := \pm dz + xdy + (s - y^2)dx$. Let us change the variable $\tilde{y} = s - y^2$ on $\pm N \cap \mathcal{O}_p \pm V_-$, so that we have $\pm V_- = \{x = \tilde{y} = 0\}$.

Note that on $\pm N \cap \mathcal{O}_p \pm V_-$ we have $\hat{\alpha} := \pm dz + \tilde{y}dx + \frac{x}{2\sqrt{s-y}}dy$. Choosing a smaller neighborhood $\pm \hat{N} \subset \pm N$ we can find a contact form $\alpha$ on $\pm \hat{N} \cup (\pm N \cap \mathcal{O}_p \pm V_-)$ which is equal to $\hat{\alpha}$ on $\pm \hat{N}$ and equal to $\pm dz + 2xd\tilde{y} + \tilde{y}dx$ near $(\pm N) \cap \mathcal{O}_p (\pm V_-)$. Extend the coordinates $(x, \tilde{y})$ to $\mathcal{O}_p (\pm V_-)$ in such a way that $\frac{\partial}{\partial \tilde{y}}$ spans $\mu$ over $\Sigma$, and extend $\alpha$ to $\mathcal{O}_p (\pm V_-)$ as $\pm dz + 2xd\tilde{y} + \tilde{y}dx$ to $\mathcal{O}_p (\pm V_-) \setminus \pm N$.

Similarly, choosing a coordinate $\tilde{y} = y^2 - s$ on $\pm N \cap \mathcal{O}_p (\pm V_+)$ we get $\hat{\alpha} := \pm dz - \tilde{y}dx + \frac{x}{2\sqrt{y+s}}dy$. Hence, we can find a contact form $\alpha = \pm \hat{N} \cup (\pm N \cap \mathcal{O}_p (\pm V_+)$ which is equal to $\hat{\alpha}$ on $\pm \hat{N}$ and equal to $\pm dz + x\tilde{y} - \tilde{y}dx = \pm dz + r^2d\phi$ near $(\pm N) \cap \mathcal{O}_p (\pm V_+)$, where $r, \phi$ are the corresponding polar coordinates. Finally we extend $\alpha$ to the rest of $\mathcal{O}_p V_+$ as equal to $dz + r^2d\phi$.  


(ii) The required framing $F$ differs from the framing $\tilde{F}$ defined by the constructed structure $\zeta$ by a homotopy class of a map $h : V \to S^1$. Let $\Delta \subset V \setminus B$ be a hypersurface dual to the cohomology class $h^\ast \mu$, where $\mu$ is a fundamental class of $S^1$. If $\Delta$ is contained in $V_-$ then by twisting the coordinate system along fibers of the normal bundle to $\Delta$ we can make the framing $\tilde{F}$ homotopic to $F$. Otherwise, by a $C^0$-small isotopy supported in $\mathcal{O}_p \Delta$ we create an additional double fold bounding a new component $\Delta \times (\varepsilon, \varepsilon)$ of $V_-$, see Fig. 6.1. By appropriately twisting the line bundle $\lambda$ across this component, see Fig. 6.2, we can change the homotopy class of the framing to make it coincide with $F$.

Fig. 6.1: Creating a double fold along $\Delta$

Fig. 6.2: Twisting the line bundle $\lambda$ across $\Delta$
Proof of Proposition 6.1. Let $\xi$ be a fiberwise distribution on $\tilde{W}$ provided by Lemma 6.2. Using Lemma 6.3 we can modify $\xi$ to make it contact on $Op \tilde{V}$ with the fiberwise generalized Morse tangencies to the fibers of the fibration $W \to \Lambda$. There exists a fiberwise diffeotopy $\delta_t : Op \tilde{V} \to Op \tilde{V}$ such that

- $\delta_t(V) = V$;
- $(\delta_1)_*\xi = \tilde{\xi}$;

the homotopy $(\delta_t)_*\xi|_V$ extends to a homotopy of distributions $\tilde{\xi}_t$ over $\tilde{W}$ connecting $\xi$ and $\tilde{\xi}$.

Note that two fiberwise contact structures $(\delta_1)^*\tilde{\xi}$ and $\xi|_V$ coincide along $V$, and hence, the fiberwise Darboux theorem implies that there exists a fiberwise isotopy $\alpha_t : Op \tilde{V} \to Op \tilde{V}$ such that $\alpha_t|_V = \text{Id}$, $(\alpha_1)_*\xi = (\delta_1)^*\tilde{\xi}$. Let $\beta_t$ be the concatenation of the isotopies $\alpha_t$ and $\delta_t$. The homotopy of homomorphisms $d\beta_t : \text{Vert}^3 \to \text{Vert}^3$ over $Op \tilde{V}$ can be extended to $Op \tilde{W}$ as a homotopy $\Phi_t$ of injective homomorphisms.

Let us consider the fiberwise tangential homotopy $\tau_t = (\Phi_1 \circ \Phi_t^{-1})_{TD}$. Using [8] it can be integrated to a wrinkled isotopy $w_t$, $C^0$-close to $\tau_t$ and which coincide with $\alpha_1 \circ \alpha_t^{-1}$ on $Op \tilde{V}$. Hence, $w_1$ is (after the smoothing) the required embedding. Note the the details of this construction in the case of non-integrable distributions is provided in the Honors Bachelor dissertation of Ying Hong Tham, see [19].

7 Extension of contact structures to the 3-ball

A characteristic foliation on the sphere $S$ is called normalized if it has a fixed positive and negative elliptic points at its poles, and a fixed arc $\gamma$ connecting the poles and transverse to $F$.

A normalized characteristic foliation $(F, \gamma)$ is called standard if it is diffeomorphic to the characteristic foliation on the round sphere $S_R = \{x_1^2 + y_1^2 + x_2^2 = R^2\} \subset (\mathbb{R}^3, \{dz + r^2d\phi = 0\})$ with a meridian connecting the poles as gamma. The space of standard foliations is contractible, and hence any standard foliation has a homotopically canonical extension as a tight contact structure on the ball $B$.

The goal of this section is the following proposition.

Proposition 7.1. Let $F_\lambda$ be a family of tight normalized generalized Morse foliations on the sphere $S = \partial B$, parameterized by a compact manifold $\Lambda$. Suppose that $F_\lambda$ is standard for $\lambda \in \partial \Lambda$. Then $F_\lambda$ extends to a family of tight contact structures $\xi_\lambda$ on the ball $B$, where the extension is standard for $\lambda \in \partial \Lambda$. 


7.1 Special simple taming functions

Let \( \phi : S \to \mathbb{R} \) be a simple taming function for a generalized Morse characteristic foliation \( \mathcal{F} \). We call \( \phi \) special if there exists a neighborhood \( U \subset S \) of the critical point locus of \( \mathcal{F} \), a complex structure \( i \) on \( U \) and a smooth function \( h : U \to \mathbb{R} \) such that \( \mathcal{F}|_{\partial_0} \) is given by a Pfaffian equation

\[
d^C \phi + h \phi = 0, \quad \text{where } d^C := d \circ i. \tag{2}
\]

We will refer to \((i, h)\) as an enrichment of the special simple taming function \( \phi \). We will always assume the complex structure \( i \) compatible with the orientation of \( S \) near positive points and opposite to it near the negative ones. This assumption automatically makes the function \( \phi \) subharmonic on a sufficiently small neighborhood of singular points.

Given a triple \((\phi, i, h)\) we denote by \( \mathcal{F}(\phi, i, h) \) the foliation defined by the Pfaffian equation \(2\).

**Lemma 7.2.** The function \( \phi \) serves as a Lyapunov function for the foliation \( \mathcal{F}(\phi, i, h) \).

**Proof.** In local holomorphic coordinate \( z = x + iy \) we have \( d^C \phi = -\phi_y dx + \phi_x dy \), \( d^C \phi = (\phi_y + h \phi_x)dx + (\phi_x + h \phi_y)dy \), and hence, \( \mathcal{F}(\phi, i, h) \) is directed by the vector field \( Z = (\phi_x + h \phi_y) \frac{\partial}{\partial x} + (\phi_y - h \phi_x) \frac{\partial}{\partial y} \). Hence, we have

\[
d(\phi(Z)) = \phi_x^2 + \phi_y^2 = ||\phi||^2 \geq \frac{||Z||^2}{1 + ||F||^2}.
\]

An isolated singularity of a foliation \( \mathcal{F} \) is said to be of complex geometric type, if it has the form \( \mathcal{F}(\phi, i, h) \) for some \( \phi, i, h \).

**Example 7.3.** The following are examples of singularities of complex geometric type:

(i) \( \mathcal{F} \) is directed by a linear vector field \( Z(u) = Au \) on \( \mathbb{R}^2 \), where \( A \) is non-degenerate;

(ii) an embryo singularity.

**Proof.** (i) Suppose first that that \( A \) is diagonalizable in a real basis, and hence, we can assume it is diagonal, \( A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \). Then the tangent to \( \mathcal{F} \) line field is defined by the 1-form \( d^C \phi \) for \( \phi = \frac{1}{2}(\lambda x^2 + \mu y^2) \) and the standard complex structure. If \( A \) is elliptic
and has complex eigenvalues then we can assume that \( A = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}, a, b \neq 0 \). Then the 1-form annihilating \( Z \) is proportional to \( \alpha = -(x + ay)dx + (-y + ax)dy \). Take 
\[
\phi = \frac{a}{2}(x^2 + y^2)
\]
and choose \( h = -1 \). Then \( dC\phi - f\phi = \alpha \).

Finally, suppose that \( A \) is in a Jordan form 
\[
A = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}.
\]
One can check that it corresponds to the singularity \( F(i, \phi, f) \), where \( \phi = ax_1^2 + bx_1y_1 + cy_1^2 \) with \( a = 2\mu^3, b = 2\mu^2, c = \mu(1 + 2\mu^2) \), and \( h = \frac{1+4\mu^2}{2\mu} \).

(ii) As it was already stated in Section 2 a result of F. Takens in [18] implies that in suitable coordinate system the directing vector field \( Z \) can be written as 
\[
Z = x\frac{\partial}{\partial x} + y^2 f(y)\frac{\partial}{\partial y},
\]
see Section 2, and hence, it corresponds to \( F(\phi, i, 0) \) with 
\[
\phi = \frac{1}{2}x^2 + \int_0^y u^2 f(u)du.
\]

**Remark 7.4.** It follows from a smooth version of Poincaré-Dulac theorem, see [3, 17], that a Liouville vector field with a non-degenerate zero is orbitally equivalent to its linear part, provided the non-resonance condition for its eigenvalues, i.e. that there are no non-negative integers \( n_1, n_2 \) with \( n_1 + n_2 \geq 2 \) such that \( \lambda_j = n_1\lambda_1 + n_2\lambda_2, j = 1, 2 \). Hence, most (and possibly all) generalized Morse singularities are of complex geometric type.

The following lemma, whose proof is straightforward, clarifies the geometric meaning of singularities of complex geometric type.

**Lemma 7.5.** Consider a function \( \phi : (\mathbb{C}, 0) \to (\mathbb{R}, 0) \) with an isolated critical point at 0. In \( \mathbb{C}^2 \) with coordinates \((z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)\) denote 
\[
\mathbb{R}^3 := \{y_2 = 0\}, \mathbb{C} := \{x_2 = 0\} \subset \mathbb{R}^3, \Gamma_\phi := \{x_2 = \phi(x_1, y_1)\} \subset \mathbb{R}^3.
\]
Consider a field of hyperplanes \( \eta = \{dx_2 - d\phi - hdy_2 = 0\} \subset T(\mathbb{C}^2) \) along the graph \( \Gamma_\phi \). Then the line field \( i\eta(u) \cap T_u\Gamma_\phi \subset T_u\Gamma_\phi, \ u \in \Gamma_\phi, \) generates the foliation \( F(\phi, i, h) \).

We observe that the field \( \eta \) can be always realized as a field of planes tangent to a hypersurface \( \Sigma \supset \Gamma_\phi \) along \( \Gamma_\phi \). The hypersurface \( \Sigma \) is automatically transverse to \( \mathbb{R}^3 \). The hypersurface \( \Sigma \) is strictly pseudoconvex in a neighborhood of 0 if and only if the function \( \phi : \mathbb{C} \to \mathbb{R} \) is subharmonic, and in that case the foliation \( F(\phi, i, h) \) coincides with the characteristic foliation induced by the contact structure on \( \Sigma \) formed by complex tangencies \( T\Sigma \cap iT\Sigma \).
Let $\mathcal{F}$ be a characteristic foliation with singularities of complex geometric type and $\phi$ its special simple taming function with its enrichment $(j, h)$. An immersion $f : S \rightarrow \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ is called compatible with $(\mathcal{F}, \phi, j, h)$ if $f$ has the form $g \times \phi$, where $g : S \rightarrow \mathbb{C}$ is a smooth map, and the complex structure $j$ on a neighborhood of a singular point $p$ of $\mathcal{F}$ coincides with $g^*i$ if $p$ is positive, and with $-g^*i$ if $p$ is negative.

**Lemma 7.6.** Let $\mathcal{F}$ be a characteristic foliation on $S$, $\phi$ its special taming function with an enrichment $(j, h)$, and $f : S \rightarrow \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ a compatible immersion. Let us identify $\mathbb{R}^3$ with the subspace $\{y_2 = 0\}$ in $\mathbb{C}^2$ with coordinates $(x_1 + iy_1, x_2 + iy_2)$. Then $f$ extends to an immersion $F : S \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{C}^2$ such that

(i) $F$ is transverse to $\mathbb{R}^3$;

(ii) $\Sigma := F(S \times (-\varepsilon, \varepsilon))$ is strictly pseudoconvex;

(iii) $\mathcal{F}$ is the characteristic foliation on $S$ induced by the contact structure $\xi$ on $\Sigma$ defined by the field of complex tangencies $T\Sigma \cap iT\Sigma$.

**Proof.** The construction is local, and hence, to simplify the notation we will be assuming that $f$ is an embedding.

We claim that the vector field $\frac{\partial}{\partial y_2}$ over $f(U)$ uniquely up to scaling extends to $f(S)$ as a transverse to $\mathbb{R}^3$ vector field which satisfies the following property:

- for any regular point $u \in S$ of $\mathcal{F}$ we have

$$d_u f^{-1}(i\text{Span}(v(f(u))), d_u f(T_u S)) = T_u \mathcal{F}.$$ 

Existence of such a vector field $v$ is guaranteed by the following lemma from Linear Algebra.

**Lemma 7.7.** Consider $\mathbb{C}^2 \supset \mathbb{R}^3 = \{y_2 = 0\} \supset \mathbb{C} = \{x_2, y_2 = 0\}$. Let $L \subset \mathbb{R}^3$ be a 2-dimensional subspace transverse to $\mathbb{C}$. Let $\ell$ be a line in $L$ transverse to $L \cap \mathbb{C}$. Then there is a unique real 3-dimensional subspace $P \subset \mathbb{C}^2$ which is transverse to $\mathbb{R}^3$ and such that $P \supset L$ and $iP \cap L = \ell$.

**Proof.** It is sufficient to consider the case

$$L = \{x_1 = kx_2, y_2 = 0\},$$

$$\ell = \{y_1 = mx_2\} \cap L.$$
The equation for $P$ can be written in the form

$$x_1 - kx_2 = cy_2,$$

and hence

$$iP := y_1 - ky_2 = -cx_2.$$  

Then

$$iP \cap L = \{ y_1 = -cx_2, x_1 = kx_2, \}.$$  

i.e. $c = -m$.  

Continuing the proof of Lemma 7.6, take a normal vector field $\nu$ to $f(S)$ in $\mathbb{R}^3$, chosen to coincide with $\frac{\partial}{\partial x_2}$ at positive singular points of $\mathcal{F}$. Consider a map $\hat{F} : S \times \mathbb{R}^2 \to \mathbb{C}^2$ given by the formula $\hat{F}(u, t_1, t_2) = f(u) + t_1 v(f(u)) + t_2 \nu(f(u))$, and define a map $F : S \times [-\varepsilon, \varepsilon] \to \mathbb{C}^2$ as $F(u, t) = \hat{F}(u, t, Ct^2)$. We claim that $F$ is the required immersion if $C > 0$ is sufficiently large and $\varepsilon > 0$ is sufficiently small. Near the critical point locus of $\mathcal{F}$ this follows from Lemma 7.5 for any $C$, even negative. Away from the critical locus property (iii) holds by construction, while (ii) can be achieved by choosing a sufficiently large $C$, see [2] for the details.

### 7.2 Families of special taming functions

Considering families of special taming functions we need their enrichments to depend continuously on the parameter. More precisely, let $\{\mathcal{F}_\lambda, \phi_\lambda\}_{\lambda \in \Lambda}$ be a family of characteristic foliations and their simple taming functions. Let us view $\{\mathcal{F}_\lambda\}$ as a fiberwise foliation on $W = \Lambda \times S$. Let $V$ be its fiberwise singular locus. The family $\{\phi_\lambda\}$ is called special if there is a fiberwise complex structure $I$ and function $H$ on a neighborhood $U \supset V$ in $W$ such that, $\mathcal{F}_\lambda|_{U_\lambda} = \mathcal{F}(\phi_\lambda|_{U_\lambda}, I|_{U_\lambda}, H|_{U_\lambda})$, where $U_\lambda = U \cap (\lambda \times S)$, $\lambda \in \Lambda$.

**Proposition 7.8.** Let $\{\mathcal{F}_\lambda, \phi_\lambda\}_{\lambda \in \Lambda}$ be a family of normalized generalized Morse foliations together with their special simple taming functions. Then there exists a family of immersions $f_\lambda : B \to \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ such that $F_\lambda|_{S=0} : B$ is compatible with $(\mathcal{F}_\lambda, \phi_\lambda)$.

**Proof.** According to Proposition 4.3 the family $\{\phi_\lambda\}$ extends to a family $\{\Phi_\lambda\}$ of simple functions on the ball $B$. Consider the map $\Phi : \Lambda \times B \to \Lambda \times \mathbb{R}$, given by $\Phi(\lambda, u) = (\lambda, \Phi_\lambda(u)), \lambda \in \Lambda, u \in B$. Slightly enlarging $B$ to an open ball $\tilde{B} \supset B$
and extending there the map $\Phi$, consider the foliation $\mathcal{H}$ of $\Lambda \times \tilde{B}$ by the level sets $\Phi^{-1}(\lambda, t), t \in \mathbb{R}, \lambda \in \Lambda$. Recall that for each critical point $p_\lambda$ of the function $\phi_\lambda$ there is a neighborhood $U_\lambda \subset S$ with a complex structure $i_\lambda$. Choose a collar $A = \partial B \times [-\varepsilon, \varepsilon] \subset \tilde{B}$. The projection $\pi : A \to \partial B$ allows us to identify leaves of $\mathcal{H}_\lambda$ near each singular point $p_\lambda$, and also to define a leafwise complex structure on leaves of $\mathcal{H}$ in a neighborhood of $p_\lambda$ by inducing it from $i_\lambda$. We will continue using the notation $i_\lambda$ for the induced local complex structure on leaves of $\mathcal{H}$. The tangent bundle to the foliation $\mathcal{H}$ is trivial, thanks to the normalization condition. This allows us to apply Hirsch-Smale parametric $h$-principle for immersions of open manifolds to construct a leafwise immersion $G : \Lambda \times \tilde{B} \to \mathbb{C}$. Moreover, we can arrange that on a neighborhood of each singular points $p_\lambda$ the map is constant on fibers of the projection $\pi$. Recall that $i_\lambda$ compatible with the orientation of $S$ near positive points and opposite to it near negative ones. Thanks to the simplicity condition for the function $\Phi_\lambda$ the complex orientation is compatible with the orientation of $\mathcal{H}$. This allows us to choose the leafwise immersion $G$ leafwise $(i_\lambda, i)$-holomorphic near positive points, and anti-holomorphic near negative ones. Then the map $F := G \times \Phi : \Lambda \times \tilde{B} \to \mathbb{C} \times \mathbb{R}$ yields a family of immersions $f_\lambda : B \to \mathbb{R}^3, f_\lambda(u) = (G(\lambda, u), \Phi_\lambda(u))$, whose restrictions to the sphere $S$ are compatible with $(\mathcal{F}_\lambda, \phi_\lambda)$. \qed

### 7.3 Extension of contact structures

**Proposition 7.9.** Consider a family $\{(\mathcal{F}_\lambda, \phi_\lambda)\}_{\lambda \in \Lambda}$ of normalized generalized Morse foliations together with their special simple taming functions. Let $f_\lambda : B \to \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ be a family of immersions compatible with $(\mathcal{F}_\lambda, \phi_\lambda)$. Then there exists a family of contact structures $\xi_\lambda$ on $B$ which induce the characteristic foliation $\mathcal{F}_\lambda$ on $S$. If $\mathcal{F}_\lambda$ is standard then $\xi_\lambda$ is standard as well.

**Proof.** According to Lemma 7.6 the family of immersions $f_\lambda|_S$ extends to a family of immersions $F_\lambda : S \times (-\varepsilon, \varepsilon) \to \mathbb{C}^2$ such that

- $F_\lambda$ is transverse to $\mathbb{R}^3$;
- $\Sigma_\lambda := F_\lambda(S \times (-\varepsilon, \varepsilon))$ is strictly pseudoconvex;
- $\mathcal{F}_\lambda$ is the characteristic foliation on $S$ induced by the contact structure $\xi_\lambda$ on $\Sigma_\lambda$ defined by the field of complex tangencies.

Consider the family $C_s := \{x_1^2 + y_1^2 + x_2^2 + (y_2 - \frac{1}{s} + \sqrt{s})^2 \leq \frac{1}{s^2}, s > 0\}$. Fix a sufficiently small $s$ such that the sphere $\partial C_s$ intersects transversely $\Sigma_\lambda$ along a closed submanifold
for all $\lambda \in \Lambda$. There is a family of immersions $\overline{F}_\lambda : B \times [-\varepsilon, \varepsilon] \to \mathbb{C}^2$, $\lambda \in \Lambda$, such that $\overline{F}|_{B \times 0} = f_\lambda$ and $\overline{F}_\lambda|_{S \times [-\varepsilon, \varepsilon]} = F_\lambda$. Let $j_\lambda$ be the induced complex structure $\overline{F}_\lambda^*i$ on $B \times [-\varepsilon, \varepsilon]$. Set $T_\lambda := \overline{F}_\lambda^{-1}(\partial C_\lambda)$. Smoothing the corner along $\partial T_\lambda$ of the piecewise smooth 3-ball $T_\lambda \cup (S \times [0, \varepsilon])$ we get a strictly pseudoconvex ball $\hat{B}_\lambda$ bounded by $S$. The corresponding contact structure $\xi_\lambda$ defined by complex tangencies is the required extension of the characteristic foliation $F_\lambda$.

Note that if $F_\lambda$ is standard then the above constructed contact extension $\xi_\lambda$ is standard as well.

\textbf{Remark 7.10.} The contact extension $\xi_\lambda$ provided by the above proposition is tight. Indeed, we could similarly construct a strictly pseudoconvex $\Sigma_\lambda \supset \hat{B}_\lambda$ which bounds a complex 4-ball. Hence, the contact structure induced by the field of complex tangencies on $\Sigma$ is holomorphically fillable, and therefore, tight.

While this observation is not needed for the proof of Proposition 7.11 and hence, for the main result of this paper, we will use in the proof of Corollary 7.12.

\textbf{Proposition 7.11.} Let $\{\xi_\lambda\}_{\lambda \in \Lambda}$ be a family of contact structures on the spherical annulus $A := S \times [-\varepsilon, \varepsilon]$, and $F_\lambda$, $\lambda \in \Lambda$, a characteristic foliation induced by $\xi_\lambda$ on the sphere $S = S \times 0$. Let $\{\phi_\lambda\}_{\lambda \in \Lambda}$ be a family of simple taming functions for $\{F_\lambda\}_{\lambda \in \Lambda}$. Denote $\widehat{W} := \Lambda \times A$, $W := \Lambda \times S = \Lambda \times (S \times 0) \subset \widehat{W}$, and let $V \subset W$ will be the fiberwise singular locus of the fiberwise foliation $F := \{F_\lambda\}$. Then there exists a fiberwise isotopy $J_{s,t} := \{J_{s,t,\lambda}\}_{\lambda \in \Lambda} : W \to \widehat{W}$, $s \in [-1, 1], t \in [0, 1]$, supported in arbitrary small neighborhood $\Omega \supset V$, and a family of functions $\Phi_s : W \to \mathbb{R}$, $s \in [-1, 1]$, with the following properties

- $J_{s,0}$ is the inclusion $W \hookrightarrow \widehat{W}$ for all $s \in [-1, 1]$;
- $J_{1,t}(W) \subset \widehat{W}_+ := \Lambda \times (S \times [0, \varepsilon])$, $J_{-1,t}(W) \subset \widehat{W}_- := \Lambda \times (S \times [-\varepsilon, 0])$ for $t \in [0, 1]$;
- $\phi_{\lambda,s} := \Phi_s|_{\Lambda \times S}$ is a simple taming function for the characteristic foliations $F_{\lambda,s} := j_{\lambda,s,1}^* \xi_\lambda$, $\lambda \in \Lambda, s \in [-1, 1]$;
- the characteristic foliations $F_{\lambda,s}$ and $F_\lambda$ have the same singular locus $V_\lambda := V \cap \lambda \times S$ for any $\lambda \in \Lambda, s \in [-1, 1]$;
- there exist a fiberwise complex structure $I := \{i_\lambda\}$ on $OpV$, $\lambda \in \Lambda$, and functions $H := \{h_\lambda\}, \hat{\Phi} := \{\hat{\phi}_\lambda\} : OpV \to \mathbb{R}$, such that $\Phi_s = \hat{\Phi}$ on $OpV$, $\Phi_s = \Phi$
on $W \setminus \Omega$, and $\mathcal{F}_{\lambda,s}|_{\Omega} = \mathcal{F}(\hat{\phi}_s, i_s, h_s)$, $\lambda \in \Lambda, s \in [-1, 1]$. In other words, $\{\hat{\phi}_\lambda\}_{\lambda \in \Lambda}$ serves as a family of special simple taming functions for the family $\{\mathcal{F}_{\lambda,s}\}_{\lambda \in \Lambda}$ for any $s \in [-1, 1]$.

Proof. Denote by $\Sigma \subset V$ the set of fiberwise embryo points. According to Proposition 2.3 there are fiberwise local coordinates $(x, y)$ on a sufficiently small neighborhood $U \supset \Sigma$ in $W$ such that the fiberwise foliation $\mathbf{F}|_U$ is generated by a fiberwise Liouville field $Z = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$. Hence, $\mathcal{F}_\lambda|_{\mathcal{O}_p \Sigma \setminus \Sigma} = \mathcal{F}(\hat{\phi}_\lambda, i_\lambda, 0)$, where $i_\lambda$ is the fiberwise coordinate system generated by a fiberwise Liouville field $\mathbf{Z} = \{\hat{Z}_\lambda\}_{\lambda \in \Lambda}$ on $U$ by the formula

$$\hat{Z}_\lambda := x \frac{\partial}{\partial x} + \left(\sigma \frac{\partial f}{\partial y}(y_0, \lambda) + 1 - \sigma\right)(y - y_0) + \sigma h(y, \lambda).$$

Choose a small neighborhood $U_1 \Subset U, U_1 \subset \Sigma$, and consider a cut-off function $\sigma : U \to [0, 1]$ supported in $U$ and equal to 1 on $U_1$. For each point $w = (0, y_0, \lambda) \in (V \setminus \Sigma) \cap U$ we have $Z = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$. Hence, the fiberwise linearization of $Z$ can be extended to $\mathcal{O}_p H$ in such a way that the vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ along $H$ are eigenvectors of the fiberwise linearization of $Z$. Hence, we can extend $\hat{Z}$ to $\mathcal{O}_p H$ as $x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$. Choosing a fiberwise metric on $\mathcal{O}_p E$ which extends the metric $dx^2 + dy^2$ from $U$ we introduce fiberwise polar coordinates $(r, \theta)$ in the tubular coordinates of $E$, so that the vector field $\hat{Z}$ on $\mathcal{O}_p \partial U \cap \mathcal{O}_p E$ is equal to $r \frac{\partial}{\partial r}$. Hence it can be extended as $r \frac{\partial}{\partial r}$ to the rest of $\mathcal{O}_p E$. Let $\hat{\mathbf{F}}_\lambda$ be the foliation generated $\hat{Z}_\lambda$ on $U_\lambda$. The fiberwise potential $\hat{\phi}_\lambda$ and its enrichment $(i_\lambda, h_\lambda)$ extend to $\mathcal{O}_p V$ in an obvious way, see Example 7.3 to satisfy $\hat{\mathbf{F}}_\lambda = \mathcal{F}(\hat{\phi}_\lambda, i_\lambda, h_\lambda)$ on $U_\lambda$.

It remains to apply Lemma 5.3 to construct a 2-parametric fiberwise isotopy $J_{s,t} = \{j_{\lambda,s,t}\}_{\lambda \in \Lambda} : W \to \hat{W}, s \in [-1, 1]$, which is supported in $\mathcal{O}_p V$ away from $\mathcal{O}_p \Sigma$, and which satisfies the following properties

- $J_{s,0}$ is the inclusion $V \hookrightarrow W$;
- $J_{-1,t}(W) \subset \hat{W}_- := \Lambda \times S \times [-\varepsilon, 0]$, $J_{1,t}(W) \subset \hat{W}_+ := \Lambda \times S \times [0, \varepsilon]$. 

- the induced characteristic foliation $\mathcal{F}_{\lambda,s} := j_{\lambda,s,1}^* \xi_\lambda$, $\lambda \in \Lambda, s \in [-1, 1]$, coincides with $\tilde{\mathcal{F}}_\lambda$ on $Op V \cap (\lambda \times S)$, has $V \cap (\lambda \times S)$ as its singular locus, and admits a simple taming function $\phi_{\lambda,s}, \lambda \in \Lambda, s \in [-1, 1]$, which coincides with $\phi_\lambda$ near its singular locus, and with $\phi_\lambda$ outside a larger neighborhood of $V$.

Thus, $\phi_{\lambda,s}$ is a special simple taming function for $\mathcal{F}_{\lambda,s}$ with an enrichment $(i_\lambda, h_\lambda)f$ or each $s \in [-1, 1]$.

Now we are ready to prove the main proposition of this section.

**Proof of Proposition 7.1.** Using Proposition 5.15 we can construct a family of simple taming functions $\phi_\lambda, \lambda \in \Lambda,$ for $\mathcal{F}_\lambda$.

Choose an extension of the family $\mathcal{F}_\lambda$ as a family of tight contact structures $\zeta_\lambda$ on a collar $U := S \times [-\varepsilon, \varepsilon]$. Using Proposition 7.11 we can construct a family of embeddings $j_{\lambda,s} : S \to U, s$, and a family of functions $\phi_{\lambda,s}, \lambda \in \Lambda, s \in [-1, 1]$, such that

- the characteristic foliation $\mathcal{F}_{\lambda,s} := j_{\lambda,s}^* \zeta_\lambda$ is normalized and $\phi_{\lambda,s}$ serves as its special simple taming function;

- $j_{\lambda,1}(S) \subset S \times [0, \varepsilon]$; $j_{\lambda,-1}(S) \subset S \times [-\varepsilon, 0]$.

Let $\tilde{B} := B \cup U$, be a larger ball bounded by $S \times \varepsilon$. Let us extend $j_{\lambda,s}$ to a family of embeddings $\tilde{j}_{\lambda,s} : B \to \tilde{B}$. Denote $B_\lambda := \tilde{j}_{\lambda,1}(B) \subset \tilde{B}$. Using Proposition 4.3 let us extend $\phi_{\lambda,s}$ to a family of simple functions $\Phi_{\lambda,s}$ to $B$, and then using Proposition 7.8 construct a family of compatible with $(\mathcal{F}_{\lambda,s}, \Phi_{\lambda,s})$ of immersions $f_{\lambda,s} : B \to \mathbb{R}^3$. Applying Proposition 7.9 we extend the foliations $\mathcal{F}_{\lambda,s}$ as tight contact structures $\xi_{\lambda,s}$ to $B$.

For each $\lambda \in \Lambda$ and $s \in [-1, 1]$ let us consider a contact structure $\eta_{\lambda,s}$ on $B_\lambda$ which is equal to $(J_{\lambda,s})^* \xi_{\lambda,s}$ on $J_{\lambda,s}(B) \subset \tilde{B}$ and equal to $\zeta_\lambda$ elsewhere. Note that $\eta_{\lambda,-1}$ induces the characteristic foliation $\mathcal{F}_\lambda$ on $S = \partial B$. The contact structure $\eta_{\lambda,1} = (j_{\lambda,1})^* \xi_{\lambda,1}$ is tight, and hence, by Gray’s stability all contact structures $\eta_{\lambda,s}$ on $B_\lambda$ are tight. Therefore, $\eta_{\lambda,-1}|_B$ is the required tight extension of the characteristic foliation $\mathcal{F}_\lambda$.

**Proposition 7.12.** Any generalized Morse foliation $\mathcal{F}$ on the sphere $S$ which admits a simple taming function is tight.
Proof. If $\mathcal{F}$ admits a special simple taming function then the claim follows from Proposition 7.9 and Remark 7.10. In the general case, we extend $\mathcal{F}$ to a contact structure $\xi$ on an annulus $S \times [-\varepsilon, \varepsilon]$ and find an isotopy $j_s : S \to S \times [-\varepsilon, \varepsilon], s \in [-1, 1]$, such that $j_1(S) \subset S \times [0, \varepsilon], j_{-1}(S) \subset S \times [-\varepsilon, 0]$ and the family of foliations $\{\mathcal{F}_s := j_s^* \xi\}$ admits a family of special taming functions. Arguing as in the proof of Proposition 7.1 we conclude that $\mathcal{F}$ embeds into a tight contact ball bounded by $\mathcal{F}_1$, and therefore it is tight itself.

Remark 7.13. Note that the simplicity assumption for a taming function $\phi$ was used in the proof twice. First, in the proof of Lemma 4.1(ii) in order to extend the taming function $\phi$ to the ball, and second time in the proof of Proposition 7.8 to construct a compatible immersion to $\mathbb{R}^3$.

8 Proof of the main theorem

Let us choose the contact form $dz + r^2 d\phi$ on $\mathbb{R}^3$ for the standard contact structure $\zeta_0$. Let $\xi_\lambda, \lambda \in \Lambda$, be a family of tight contact structures on $\mathbb{R}^3$ which coincides with $\zeta_0$ outside of a compact set $K$. We can assume that $K$ is a ball of radius 1 centered at a point with cylindrical coordinates $z = 0, \phi = 0, r = 3$. Consider the family $B_r$ of balls of radius $r$ centered at 0, $r \in [1, 5]$, so that $B_5 \supseteq K$ and $B_1 \subset \mathbb{R}^3 \setminus K$. Choose a family of meridians $\gamma_r \subset \partial B_r$ connecting the poles. Note that the characteristic foliations $(\mathcal{F}_\lambda, \gamma_r)$ induced by $\xi_\lambda$ on $\partial B_r$ are normalized.

Applying Proposition 6.1 to the complements of neighborhoods of $\gamma_r \subset \partial B_r$, we can arrange that the characteristic foliations $\mathcal{F}_\lambda, \gamma_r$ are generalized Morse. Hence, we can use Proposition 7.1 to find extensions of $\mathcal{F}_\lambda, \gamma_r$ to $B_r$ as tight contact structures $\zeta_\lambda, \gamma_r$ on $B^r, r \in [1, 5]$. Moreover, for $r = 1, 5$ and all $\lambda$ and for $\lambda \in \Lambda_0$ and all $r \in [1, 5]$ the foliations $\mathcal{F}_\lambda, \gamma_r$ are standard. Therefore, their extensions $\zeta_\lambda, \gamma_r$ are standard as well.

Denote by $\eta_\lambda, \gamma_r$ the contact structure which is equal to $\xi_\lambda$ on $\mathbb{R}^3 \setminus B_r$, and equal to $\zeta_\lambda, \gamma_r$ on $B_r$. Then $\eta_{\lambda, 1} = \xi_\lambda$ while $\eta_{\lambda, 5} = \zeta_0$. This concludes the proof of Theorem 1.1

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