A TAMED 3D NAVIER-STOKES EQUATION IN DOMAINS

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Abstract. In this paper, we analyze a tamed 3D Navier-Stokes equation in uniform $C^2$-domains (not necessarily bounded), which obeys the scaling invariance principle, and prove the existence and uniqueness of strong solutions to this tamed equation. In particular, if there exists a bounded solution to the classical 3D Navier-Stokes equation, then this solution satisfies our tamed equation. Moreover, the existence of a global attractor for the tamed equation in bounded domains is also proved. As simple applications, some well known results for the classical Navier-Stokes equations in unbounded domains are covered.

1. Introduction

The motion of a viscous incompressible fluid in a domain $\Omega \subset \mathbb{R}^3$ is described by the Navier-Stokes equation (NSE) as follows (with homogeneous boundary):

\begin{align}
\begin{cases}
\partial_t u = \nu \Delta u - (u \cdot \nabla) u + \nabla P + f, \\
\text{div}(u) = 0, \quad (t, x) \in [0, \infty) \times \Omega, \\
u > 0 \text{ is the kinematic viscosity constant, } u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x)) \text{ represents the velocity field, } P = P(t, x) \text{ is the pressure (an unknown scalar function), } f \text{ is a known external force.}
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The study of 3D NSEs has a long history. In their pioneering works, Leray [11] and Hopf [9] proved the existence of a weak solution to equation (1). Since then, there are many papers devoted to the study of regularities of Leray-Hopf weak solutions (cf. [10, 19, 17, etc.]). Up to now, one knows that the singular set of the Leray-Hopf weak solutions has Lebesgue measure zero (cf. [11, 8, 7]). Moreover, a deep result obtained by Scheffer [16] and Caffarelli, Kohn and Nirenberg [3] says that the singular set for a class of weak solutions (satisfying a generalized energy inequality) has one dimensional Hausdorff measure zero (see also [12]). However, the uniqueness and regularity of Leray-Hopf weak solutions are still big open problems.

Most of the source of difficulties to solve equation (1) comes from the nonlinear term $(u \cdot \nabla) u$ (cf. [7]). In order to counteract this term, the authors in [15] analyzed the following modified (called tamed therein) 3D NSE in $\Omega = \mathbb{R}^3$:

\begin{align}
\begin{cases}
\partial_t u = \nu \Delta u - (u \cdot \nabla) u + \nabla P - g^\nu(|u|^2)u + f, \\
\text{div}(u) = 0, \quad (t, x) \in \mathbb{R}^3, \quad u(0) = u_0,
\end{cases}
\end{align}

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where \( |u|^2 := \sum_{j=1}^{3} |u_j|^2 \) and for \( N > 0 \)
\[
g_N^{\nu}(r) := (r - N) \cdot 1_{\{r \geq N\}} / \nu. \tag{3}\]

The existence of a unique smooth solution to equation (2) was proved in [15] when
the initial velocity is smooth (in Sobolev spaces). The main feature of
equation (2) is that if there exists a bounded solution (say bounded by \( \sqrt{N} \) for some
large \( N \)) to the classical NSE, then this solution must satisfy equation (2).
Therein, the property that the Leray
projection operator onto the space of divergence free vector fields commutes with
the derivatives plays a key role. But, when we consider NSE (1) in a domain, this
property does not hold in general (cf. [13, p.83-85]).

In order to deal with the Dirichlet boundary problem and keep the same feature as
equation (2), in the present paper, we consider the following globally tamed scheme (assuming
\( f = 0 \) for simplicity):
\[
\partial_t u = \nu \Delta u - (u \cdot \nabla) u + \nabla P - g_N^{\nu,\kappa}(\|u - U\|_\infty^2)(u - U), \tag{4}\]
where \( \|u\|_\infty := \sup_{x \in \Omega} |u(x)| \), \( U \) is a reference velocity field and for \( \kappa, N \geq 1 \)
\[
g_N^{\nu,\kappa}(r) := \kappa \cdot (r - N) 1_{\{r \geq N\}} / \nu. \]
Here, \( \kappa \geq 1 \) is a dimensionless constant and \( \sqrt{N} \) has the velocity dimension.

Let \((u_N, U, P_N, U)\) be a solution pair of equation (4). Simple calculations show that
\((u_N, U, P_N, U)\) has the following properties:

(A) (Galilean invariance): for any constant velocity vector \( v \in \mathbb{R}^3 \)
\[
\begin{align*}
\text{u}_N^v(t, x) &:= u_{N,U+v}(t, x - vt) + v, \\
\text{P}_N^v(t, x) &:= P_{N,U+v}(t, x - vt)
\end{align*}
\]
is also a solution pair of equation (4).

(B) (Rotation symmetry): for any orthogonal matrix \( Q \) (i.e. \( QQ^T = I \))
\[
\begin{align*}
\text{u}_N^Q(t, x) &:= Q' u_{N,Q'U}(t, Qx), \\
\text{P}_N^Q(t, x) &:= P_{N,Q'U}(t, Qx)
\end{align*}
\]
is also a solution pair of equation (4).

(C) (Scale invariance): for any \( \lambda > 0 \)
\[
\begin{align*}
\text{u}_N^\lambda(t, x) &:= \lambda u_{\lambda^{-2}N,\lambda^{-1}U}(\lambda^2 t, \lambda x), \\
\text{P}_N^\lambda(t, x) &:= \lambda^2 P_{\lambda^{-2}N,\lambda^{-1}U}(\lambda^2 t, \lambda x)
\end{align*}
\]
is also a solution pair of equation (4).

These three properties are exhibited by the classical Navier-Stokes equations (cf. [2]).

Intuitively, when the maximum of the fluid velocity is larger than \( \sqrt{N} \),
the dissipative term \( g_N^{\nu,\kappa}(\|u\|_\infty^2)u \) (regarded as some extra force)
will enter into the equation and restrain the flux of the liquid. In this sense, the value of \( N \) plays the role of a valve. On the other
hand, when we realize equation (4) on a computer, the value of \( N \) can be reset as
an arbitrarily large number along with the process of calculations as long as there is no explosion. So, the term involving \( g_N^{\nu,\kappa} \)
plays the role of some kind of adjustment. The parameter \( \kappa \) can be understood as the extent of the extra dissipative force, and will be
used to give a better estimate for \( \|u\|_\infty \) in terms of \( N \) (see part (III) of
Theorem 2.4).

In contrast with equation (2), the tamed equation (4) in domain \( \Omega \) is global since
\( g_N^{\nu,\kappa}(\|u\|_\infty^2) \) depends on all values of \( u \) in \( \Omega \). But, better than (2), it is easy to write down
the vorticity equation: Let \( \omega = \text{curl} u = \nabla \wedge u \). Then
\[
\partial_t \omega = \nu \Delta \omega + (\omega \cdot \nabla) u - (u \cdot \nabla) \omega - g_N^\nu (|u|) \omega.
\]

We remark that in [4], Caraballo, Real and Kloeden studied the following globally modified NSE in a bounded regular domain \( \Omega \):
\[
\begin{aligned}
\partial_t u(t) &= \Delta u - \min\{1, N/\|\nabla u\|_{L^2}\}(u \cdot \nabla) u + \nabla P, \\
\text{div}(u) &= 0, \quad (t, x) \in [0, \infty) \times \Omega, \\
u(t, x) &= 0, \quad t \geq 0, \quad x \in \partial \Omega, \quad u(0) = u_0,
\end{aligned}
\]
(5)
and they proved the existence of a unique strong solution to this modified equation as well as the existence of a global attractor. Nevertheless, equation (5) does not enjoy the above properties (A)-(C).

This paper is organized as follows: In Section 2, all main results are announced. In Section 3, we prepare some necessary lemmas for later use. In the remaining sections, we shall give the proofs of main results. We want to emphasize that for the proof of existence of strong solutions (see Section 4), not using the usual Galerkin approximation, we only use the linearized equations and simple Picard’s iteration. Moreover, the semigroup method used in Fujita-Kato [6] (cf. [17]) will be used to improve the regularity of strong solutions (see Section 5). The existence of a global attractor for the evolution semigroup determined by equation (4) will follow by proving some asymptotic compactness (cf. [4, 20, etc.]).

2. ANNOUNCEMENT OF MAIN RESULTS

Throughout this paper, all \( \mathbb{R}^3 \)-valued functions and spaces of such functions will be denoted by boldfaced letters, and we use the following convention: the letter \( C \) with or without subscripts will denote a positive constant whose value may change in different occasions.

Let \( \Omega \) be a uniform \( C^2 \)-regular domain of \( \mathbb{R}^3 \) (see [11] p.84 for the definition of regular domains). Let \( C^\infty_0(\Omega) \) denote the set of all smooth functions from \( \Omega \) to \( \mathbb{R}^3 \) with compact supports in \( \Omega \), and \( C^\infty_{0,\sigma}(\Omega) \subset C^\infty_0(\Omega) \) the set of all smooth vector fields of divergence free. For \( p > 1 \), let \( L^p(\Omega) \) be the usual \( \mathbb{R}^3 \)-valued \( L^p \)-space with the norm denoted by \( \| \cdot \|_{L^p(\Omega)} = \| \cdot \|_{L^p} \), and \( L^2_{2}(\Omega) \) the closure of \( C^\infty_{0,\sigma}(\Omega) \) in \( L^p(\Omega) \). For \( k \in \mathbb{N} \) and \( p > 1 \), let \( W^{k,p}(\Omega) \) be the space of \( \mathbb{R}^3 \)-valued functions with finite norm:
\[
\| u \|_{W^{k,p}(\Omega)} := \left( \sum_{j=0}^{k} \int_{\Omega} |\nabla^j u(x)|^p dx \right)^{\frac{1}{p}} < +\infty,
\]
where \( \nabla^j \) denotes the \( j \)-th order generalized derivative operator. The space \( W^{1,2}_{0}(\Omega) \) (resp. \( W^{1,2}_{0,\sigma}(\Omega) \)) denotes the completion of \( C^\infty_{0,\sigma}(\Omega) \) (resp. \( C^\infty_{0,\sigma}(\Omega) \)) with respect to the above norm with \( k = 1 \) and \( q = 2 \).

Let \( \mathcal{P} \) be the orthogonal projection from \( L^2_{2}(\Omega) \) to \( L^2_{0}(\Omega) \). By \( A \) (called the Stokes operator) we denote the self-adjoint operator in \( L^2_{0}(\Omega) \) formally given by
\[
A := -\mathcal{P} \Delta.
\]
More precisely, \( u \in \mathcal{D}(A) \) if and only if for some \( w \in L^2_{0}(\Omega) \) (written as \( Au = w \)), it holds that
\[
\langle \nabla u, \nabla v \rangle_{L^2} = \langle w, v \rangle_{L^2}, \quad \forall v \in W^{1,2}_{0,\sigma}(\Omega).
\]
In particular, \( \mathcal{D}(A^{\frac{1}{2}}) = W^{1,2}_{0,\sigma}(\Omega) \) and
\[
\| A^{\frac{1}{2}} u \|_{L^2} = \| \nabla u \|_{L^2}, \quad u \in W^{1,2}_{0,\sigma}(\Omega).
\]
Moreover, it is well known that (cf. [17, p.129])
\[ \mathcal{D}(A) = W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega). \]
Since \( A \) is a positive self-adjoint operator in \( L^2_{\sigma}(\Omega) \), for \( \alpha \in (-1, 1) \), the fractional power \( A^\alpha \) is well defined via the spectral representation. For \( \beta \in [0, 2] \), define the Hilbert space
\[ H^\beta(\Omega) := H^\beta := \mathcal{D}(A^{\beta/2}) \]
with the norm \( \| \cdot \|_{H^\beta} \) generated by inner product
\[ \langle u, v \rangle_{H^\beta} := \langle u, v \rangle_{L^2} + \langle A^{\beta/2}u, A^{\beta/2}v \rangle_{L^2}. \]

We introduce the following bilinear form \( B \) on \( W^{1,2}_{0,\sigma}(\Omega) = H^1 \):
\[ B(v, u) := -\mathcal{D}(v \cdot \nabla u). \] (7)
Using \( \mathcal{D} \) to act on both sides of equation \( (\mathcal{R}) \), we can and shall consider the following equivalent abstract equation
\[ \partial_t u = -\nu Au + B(u, u) - g_N^{\nu,\kappa}(\|u\|_2^2)u, \quad u(0) = u_0. \] (8)

We give the following definition of strong solutions to the above equation.

**Definition 2.1.** Let \( u_0 \in H^1 \). A continuous function
\[ \mathbb{R}_+ \ni t \mapsto u(t) \in H^1 \]
is called a strong solution of equation \( (\mathcal{R}) \) if \( u \in L^2_{\text{loc}}(\mathbb{R}_+; H^2) \) and for all \( t \geq 0 \)
\[ u(t) = u_0 - \nu \int_0^t Au \, ds + \int_0^t B(u, u) \, ds - \int_0^t g_N^{\nu,\kappa}(\|u\|_2^2)u \, ds \quad \text{in} \quad L^2_\sigma(\Omega). \] (9)

Our first main result is stated as follows:

**Theorem 2.2.** Let \( \Omega \) be a uniform \( C^2 \)-domain of \( \mathbb{R}^3 \). For any \( u_0 \in H^1 \), there exists a unique strong solution \( u(t) = u_N(t) \) to equation \( (\mathcal{R}) \) in the sense of Definition 2.1 which satisfies that for any \( t \geq 0 \)
\[ \|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u\|_{L^2}^2 \, dt \leq \|u_0\|_{L^2}^2, \] (10)
\[ \|\nabla u(t)\|_{L^2}^2 + \nu \int_0^t \|Au\|_{L^2}^2 \, dt \leq \frac{\kappa N}{\nu^2} \|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2, \] (11)
and for some \( T^* = T^*(\nu, \Omega, \|u_0\|_{L^2}) \) and \( C = C(\nu, \Omega, \|u_0\|_{L^2}) \)
\[ \|\nabla u(t)\|_{L^2} \leq C/\sqrt{t}, \quad \forall t \geq T^*. \] (12)
Moreover, letting \( u_N(t) \) (resp. \( v_M(t) \)) be the solution of equation \( (\mathcal{R}) \) with initial value \( u_0 \in H^1 \) (resp. \( v_0 \in H^1 \)) and taming function \( g_N^{\nu,\kappa} \) (resp. \( g_M^{\nu,\kappa} \)), we have for any \( T > 0 \)
\[ \sup_{t \in [0, T]} \|u_N(t) - v_M(t)\|_{H^1}^2 + \int_0^T \|u_N - v_M\|_{H^1}^2 \, ds \]
\[ \leq C(\nu, N, M, \|u_0\|_{H^1}, \|v_0\|_{H^1}, T) \cdot (|N - M|^2 + \|u_0 - v_0\|_{H^1}), \] (13)
where the constant \( C(\nu, N, M, \|u_0\|_{H^1}, \|v_0\|_{H^1}, T) \) continuously depends on its parameters.
Remark 2.3. For $T > 0$ and $N \geq 1$, define

$$\mathcal{T}_N^T := \{ t \in [0, T] : \|u(t)\|_{\infty} \geq \sqrt{N} \}.$$ 

By (10), (11) and (21) below, we have

$$\lambda(\mathcal{T}_N^T) \leq \frac{1}{N} \int_0^T \|u\|_{H^2}^2 \, ds \leq \frac{C\Omega}{N} \int_0^T \|u\|_{H^2} \cdot \|\nabla u\|_{L^2} \, ds \leq \frac{C\Omega}{N} \left( \int_0^T \|u\|_{H^2}^2 \, ds \right)^{1/2} \cdot \left( \int_0^T \|\nabla u\|_{L^2}^2 \, ds \right)^{1/2} \leq \frac{C\Omega \cdot \|u_0\|_{L^2} \cdot (2\nu N^2)\|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \nu T\|u_0\|_{L^2}^2)^{1/2}}{\sqrt{2\nu N}},$$

where $\lambda(\mathcal{T}_N^T)$ denotes the Lebesgue measure of $\mathcal{T}_N^T$. This gives an estimate of the length of the time for which $u_N$ does not satisfy equation (1). In particular,

$$\lim_{N \to \infty} \lambda(\mathcal{T}_N^T) = 0,$$

which shows that as $N$ goes to infinity, $u_N$ satisfies equation (1) at “almost all” times.

We are now interested in the estimation of $\|u\|_{\infty}$ in terms of $N$ and prove the following result.

Theorem 2.4. Let $\Omega$ be a uniform $C^2$-domain and $u_0 \in H^2$. Let $u_N^\alpha$ be the unique strong solution in Theorem 2.2. We have the following conclusions:

(I) There exist two continuous function $K_1 : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $K_2 : \mathbb{R}^3_+ \to \mathbb{R}_+$ such that for all $t \geq 0$ and $N \geq 1$

$$\|u_N^\alpha(t)\|_{\infty} \leq K_1(t, \|u_0\|_{H^2}) + K_2(t, \nu, \|u_0\|_{L^2}) \cdot N^3,$$

where $K_1(t, r), K_2(t, \nu, r) \to 0$ as $t \to 0$ or $\nu \to \infty$ or $r \to 0$. In particular, for $T > 0$, if one of the following conditions is satisfied, then there is a unique strong solution in $[0, T]$ for equation (1):

(i) $T$ is small; (ii) $\|u_0\|_{H^2}$ is small; (iii) $\nu$ is large.

(II) Let $\Omega = \mathbb{R}^3$ or be a bounded uniform $C^4$-domain and $u = u_N^\alpha$. Then

$$u \in C((0, \infty) \times \Omega; \mathbb{R}^3)$$

and for $i, j = 1, 2, 3$

$$\partial_i u, \partial_i \partial_j u, \partial_i \partial_3 u \in C((0, \infty) \times \Omega; \mathbb{R}^3).$$

Moreover, for some $P \in C((0, \infty) \times \bar{\Omega}; \mathbb{R})$ (with $\int_{\Omega} P(x) \, dx = 0$), it holds that

$$\partial_i u = \Delta u - (u \cdot \nabla) u + \nabla P - g_N(\|u\|_{L^\infty}^2)u, \forall (t, x) \in (0, \infty) \times \Omega.$$  \hspace{1cm} (15)

(III) Let $\Omega = \mathbb{R}^3$ and $\nu > 0$. For any $\alpha > \frac{1}{2}$, there exist $\kappa > 0$ and two functions $K_{1, \alpha, \kappa}, K_{2, \alpha, \kappa}$ as in (I) such that for all $t \geq 0$ and $N \geq 1$

$$\|u_N^\alpha(t)\|_{\infty} \leq K_{1, \alpha, \kappa}(t, \|u_0\|_{H^2}) + K_{2, \alpha, \kappa}(t, \nu, \|u_0\|_{L^2}) \cdot N^\alpha.$$  \hspace{1cm} (16)

Remark 2.5. We do not know whether the $\alpha$ in (III) can be smaller than $1/2$. If this can be proven, then (1) will have a classical solution. In fact, even for $\alpha = 1/2$, it seems also hard to prove (16).
Remark 2.6. Fix $T > 0$ and $N_1 \geq 1$. Define a sequence of real numbers recursively as follows:

$$N_{k+1} := \sup_{t \in [0, T]} \|u_{N_k}(t)\|_\infty^2, \ k \in \mathbb{N}.$$ 

It is easy to see that equation (1) has a explosion solution in $[0, T]$ if and only if

$$N_1 < N_2 < N_3 < \cdots < N_k \to \infty.$$ 

The strict monotonicity is clear. Assume that $\lim_{k \to \infty} N_k = N_\infty < \infty$. By the continuous dependence of $u_{N_k}$ with respect to $N$ (see (13)), we have

$$\lim_{k \to \infty} \sup_{t \in [0, T]} \|u_{N_k}(t) - u_{N_\infty}(t)\|_\infty^2 = 0.$$ 

Therefore,

$$N_\infty = \sup_{t \in [0, T]} \|u_{N_\infty}(t)\|_\infty^2 < \infty,$$

which implies that $u_{N_\infty}(t)$ satisfies (1), no explosion.

For $u_0 \in H^1$, let $\{u(t; u_0); t \geq 0\}$ be the unique strong solution of equation (8), which defines a nonlinear evolution semigroup:

$$S(t)u_0 := u(t; u_0) : H^1 \to H^1. \quad (17)$$

By Theorem 2.2, $\{S(t); t \geq 0\}$ has the following properties:

(i) $S(0) = I$ identity map on $H^1$;
(ii) $S(t + s) = S(t)S(s)$ for any $t, s \geq 0$;
(iii) $[0, \infty) \times H^1 \ni (t, u_0) \mapsto S(t)u_0 \in H^1$ is continuous.

Definition 2.7. A compact subset $A \subset H^1$ is called a global attractor of the evolution semigroup $\{S(t); t \geq 0\}$ if

(i) $A$ is invariant under $S(t)$, i.e., for any $t > 0$, $S(t)A = A$;
(ii) $A$ attracts all bounded set $U \subset H^1$, i.e.,

$$\lim_{t \to \infty} \rho(S(t)U, A) = 0,$$

where $\rho(A_1, A_2) := \sup_{u \in A_1} \inf_{v \in A_2} \|u - v\|_{H^1}$.

We have the following existence of global attractors of $\{S(t), t \geq 0\}$.

Theorem 2.8. Let $\Omega$ be a bounded uniform $C^2$-domain of $\mathbb{R}^3$. Then there exists a global attractor $A \subset H^1$ to $\{S(t); t \geq 0\}$ defined by (17).

3. Preliminaries

In this section, we collect some necessary materials for later use. The following lemma is from [8, Lemma 6].

Lemma 3.1. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an absolute continuous function and $g : \mathbb{R}_+ \to \mathbb{R}_+$ a locally Lipschitz continuous function. Suppose that $\Lambda := \int_0^\infty \phi(t)dt < +\infty$ and $g(\phi) \leq \alpha \phi^2$ for $\phi \leq \beta$, where $\alpha, \beta > 0$. If

$$\phi'(t) \leq g(\phi(t)), \ \forall t \geq 0,$$

then for $t \geq (\Lambda/\beta) \exp(\alpha \Lambda)$

$$\phi(t) \leq (e^{\alpha \Lambda} - 1)/(\alpha t).$$
Let \( \{E_\lambda, \lambda > 0\} \) be the spectrum decomposition of \( A \) in \( L^2_0(\Omega) \). The Stokes semigroup is then defined by

\[
e^{-tA} := \int_0^\infty e^{-t\lambda} dE\lambda,
\]

and for \( \alpha \in [-1, 1] \), \( A^\alpha \) is given by

\[
A^\alpha := \int_0^\infty \lambda^\alpha dE\lambda.
\]

The following lemma is easily derived from the above representations (cf. [17]).

**Lemma 3.2.** (i) For any \( \alpha \in [0, 1] \) and \( u \in L^2_0(\Omega) \), we have \( e^{-tA}u \in \mathcal{D}(A^\alpha) \) and

\[
\|A^\alpha e^{-tA}u\|_{L^2} \leq t^{-\alpha}\|u\|_{L^2}, \quad \forall t > 0.
\]

(ii) For all \( u \in \mathcal{D}(A^\alpha) \) and \( t \geq 0 \)

\[
A^\alpha e^{-tA}u = e^{-tA}A^\alpha u, \quad \|e^{-tA}u - u\|_{L^2} \leq C_\alpha t^\alpha\|A^\alpha u\|_{L^2}.
\]

(iii) For any \( 0 \leq \alpha < \gamma < \beta \leq 1 \) and \( u \in \mathcal{D}(A^\beta) \)

\[
\|A^\gamma u\|_{L^2} \leq \|A^\beta u\|_{L^2}^{\frac{\beta - \gamma}{\beta - \alpha}} \cdot \|A^\alpha u\|_{L^2}^{\frac{\gamma - \alpha}{\beta - \alpha}} \leq \frac{\gamma - \alpha}{\beta - \alpha}\|A^\beta u\|_{L^2} + \frac{\beta - \gamma}{\beta - \alpha}\|A^\alpha u\|_{L^2}.
\]

We recall the following well known results (cf. [17] Lemma 2.4.2 (p.142), Lemma 2.5.2 (p.152) and Lemma 2.4.3 (p.143)).

**Lemma 3.3.** (i) For \( \alpha \in [0, 1/2] \) and \( q = \frac{6}{3-4\alpha} \), there exists a constant \( C = C(\alpha, q) > 0 \) such that for any \( u \in H^{2\alpha} \)

\[
\|u\|_{L^q} \leq C\|A^\alpha u\|_{L^2}.
\]

(ii) For \( \alpha \in [0, 1/2] \) and \( q = \frac{6}{3+4\alpha} \), there exists a constant \( C = C(\alpha, q) > 0 \) such that for any \( u \in L^q(\Omega) \)

\[
\|A^{-\alpha}u\|_{L^2} \leq C\|u\|_{L^q}.
\]

(iii) For \( \alpha \in [1/2, 1] \) and \( q = \frac{6}{5-4\alpha} \), there exists a constant \( C = C(\Omega, \alpha, q) > 0 \) such that for any \( u \in H^{2\alpha} \)

\[
\|u\|_{W^{1,q}} \leq C(\|A^\alpha u\|_{L^2} + \|u\|_{L^2}).
\]

This lemma has the following conclusions.

**Lemma 3.4.** Let \( \Omega \) be a uniform \( C^2 \)-domain. For some \( C_\Omega > 0 \) and any \( u \in H^2(\Omega) \)

\[
\|u\|_{L^\infty(\Omega)} \leq C_\Omega \cdot \|u\|_{H^2(\Omega)} \cdot \|\nabla u\|_{L^2(\Omega)},
\]

and for \( \frac{3}{4} < \alpha < 1 \), some \( C_{\alpha, \Omega} > 0 \) and any \( u \in H^{2\alpha}(\Omega) \)

\[
\|u\|_{L^\infty(\Omega)} \leq C_{\alpha, \Omega} \cdot (\|A^\alpha u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).
\]

**Proof.** Since \( \Omega \) is a uniform \( C^2 \)-domain, by [11] p. 154, Theorem 5.24 there exists a bounded linear operator \( E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^3) \) such that \( Eu = u \text{ a.e. on } \Omega \). Recall the following Gagliardo-Nirenberg inequality (cf. [3] p.24, Theorem 9.3]): Let \( 1 \leq p, q \leq \infty \) and \( \alpha \in [0, 1] \) with \( p \neq 3 \) and

\[
\frac{1}{r} = \alpha \left(\frac{1}{p} - \frac{1}{3}\right) + (1 - \alpha) \frac{1}{q}.
\]

Then, for some \( C = C(r, p, q) \) and all \( u \in W^{1,p}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) \)

\[
\|u\|_{L^r(\mathbb{R}^3)} \leq C\|u\|_{L^p(\mathbb{R}^3)}^{\alpha} \cdot \|u\|_{L^q(\mathbb{R}^3)}^{1-\alpha}.
\]
Thus, by (18) we have
\[ ||u||_{L^\infty(\Omega)}^2 \leq ||Eu||_{L^\infty(\mathbb{R}^3)}^2 \leq C||Eu||_{W^{2,2}(\mathbb{R}^3)} \cdot ||Eu||_{L^6(\mathbb{R}^3)} \leq C_\Omega ||u||_{W^{2,2}(\Omega)} \cdot ||u||_{L^6(\Omega)} \leq C_\Omega ||u||_{H^2(\Omega)} \cdot ||A^2u||_{L^2(\Omega)} \]
and for \( q = \frac{6}{5-4\epsilon} > 3 \)
\[ ||u||_{L^\infty(\Omega)} \leq ||Eu||_{L^\infty(\mathbb{R}^3)} \leq C_q ||Eu||_{W^{1,q}(\mathbb{R}^3)} \leq C_{q,\Omega} ||u||_{W^{1,q}(\Omega)} \leq C_{\Omega} \cdot (||A^0u||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}). \]
The proof is complete. \( \Box \)

The contents below in this section are only used in Section 5.

**Lemma 3.5.** For some \( C, C_\Omega > 0 \) and all \( u \in H^2 = \mathbb{D}(A) \), we have
\[ ||A^{-\frac{1}{2}}B(u, u)||_{L^2} \leq C ||A^{\frac{1}{2}}u||_{L^2} \]
and
\[ ||B(u, u)||_{L^2} \leq C_\Omega (||A^2u||_{L^2} + ||u||_{L^2}). \]

**Proof.** By Hölder’s inequality, we have
\[ ||A^{-\frac{1}{2}}B(u, u)||_{L^2} \leq C ||(u \cdot \nabla)u||_{L^{2/3}} \leq C ||u||_{L^6} \cdot ||\nabla u||_{L^2} \leq C ||A^2u||_{L^2} \]
and
\[ ||B(u, u)||_{L^2} \leq ||(u \cdot \nabla)u||_{L^2} \leq ||u||_{L^4} \cdot ||u||_{W^{1,12/5}} \leq C_\Omega ||u||_{W^{1,12/5}} \leq C_\Omega (||A^2u||_{L^2} + ||u||_{L^2}), \]
where the third inequality is due to \( W^{1,12/5}(\Omega) \subset L^{12}(\Omega) \). \( \Box \)

**Lemma 3.6.** For any \( \frac{3}{4} < \gamma < \beta \leq 1 \), there are three positive continuous functions \( F_1, F_2 : \mathbb{R}_+^2 \to \mathbb{R}_+ \) and \( F_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for all \( u, v \in H^{2\beta} \)
\[ ||B(u, u) - B(v, v)||_{L^2} + ||g_N^{\epsilon,\kappa}(||u||_{L^\infty})u - g_N^{\epsilon,\kappa}(||v||_{L^\infty})v||_{L^2} \leq F_1(||u||_{H^{2\beta}}, ||v||_{H^{2\beta}}) \cdot ||A^\beta(u - v)||_L^\gamma \cdot ||u - v||_{L^2}^{\frac{d-2}{L^2}} \]
\[ + F_2(||u||_{H^{2\beta}}, ||v||_{H^{2\beta}}) \cdot ||A^\beta(u - v)||_L^\gamma \cdot ||u - v||_{L^2}^{1 - \frac{d-2}{L^2}} \]
\[ + F_3(||u||_{H^{2\beta}}, ||v||_{H^{2\beta}}) \cdot ||u - v||_{L^2}. \]

**Proof.** Note that by (iii) of Lemma 3.2
\[ ||\nabla(u - v)||_{L^2} = ||A^\gamma(u - v)||_{L^2} \leq ||A^\beta(u - v)||_L^\gamma \cdot ||u - v||_{L^2}^{1 - \frac{d-2}{L^2}} \]
and
\[ ||u - v||_{L^\infty} \leq C_\Omega (||A^\gamma(u - v)||_{L^2} + ||u - v||_{L^2}) \leq C_\Omega (||A^\beta(u - v)||_L^\gamma \cdot ||u - v||_{L^2}^{\frac{d-2}{L^2}} + ||u - v||_{L^2}). \]
The result now follows from
\[ ||B(u, u) - B(v, v)||_{L^2} \leq ||u||_{L^\infty} \cdot ||\nabla(u - v)||_{L^2} + ||u - v||_{L^\infty} \cdot ||\nabla v||_{L^2} \]
and
\[ ||g_N^{\epsilon,\kappa}(||u||_{L^\infty})u - g_N^{\epsilon,\kappa}(||v||_{L^\infty})v||_{L^2} \leq \frac{\kappa}{8} ||u - v||_{L^\infty} \cdot (||u||_{L^\infty} + ||v||_{L^\infty}) \cdot ||v||_{L^2}. \]
\[ + \frac{K}{\nu} \|u\|_{\infty}^2 \cdot \|u - v\|_{L^2}. \]

We introduce some notations. Let \( I \) be a closed interval of \( t \), and let \( X \) be a Banach space. By \( C(I; X) \) we denote the set of all continuous \( X \)-valued functions defined on \( I \). For \( 0 < \theta < 1 \), \( C^\theta(I; X) \) means the set of all functions which are strongly Hölder continuous with the exponent \( \theta \). If \( I \) is not closed, \( v \in C^\theta(I; X) \) means that \( v \in C^\theta(I_1; X) \) for any closed interval \( I_1 \) contained in \( I \).

The following lemma is easily deduced from Lemma 3.2 (cf. [6, 14]).

**Lemma 3.7.** For \( T > 0 \), let \( f : [0, T] \mapsto H^0 = L^2_\sigma(\Omega) \) be continuous and consider

\[
\mathbf{w}(t) := \int_0^t e^{-(t-s)A}f(s)ds.
\]

(i) For any \( 0 < \alpha < \theta < 1 \)

\[
A^\alpha \mathbf{w} \in C^{1-\theta}([0, T], H^0), \quad \|A^\alpha \mathbf{w}(t)\|_{L^2} \leq C_\alpha \cdot t^{1-\alpha} \cdot \sup_{s \in [0, T]} \|f(s)\|_{L^2}.
\]

(ii) If \( f \in C([0, T], H^0) \cap C^\alpha((0, T], H^0) \) for some \( \alpha \in (0, 1) \), then for any \( 0 < \theta < \alpha \)

\[
Aw \in C^\theta((0, T], H^0), \quad \partial_t \mathbf{w} \in C((0, T], H^{2\theta}).
\]

Moreover, \((0, T] \) can be replaced by \([0, T] \) in the above condition and conclusions.

**Proof.** The first conclusion is direct from Lemma 3.2. For the second, fixing \( \delta \in (0, T) \), we write

\[
\mathbf{w}(t) = e^{-(t-\delta)A}\mathbf{w}(\delta) + \int_\delta^t e^{-(t-s)A}f(s)ds =: \Psi_\delta(t) + \Phi_\delta(t).
\]

It is easy to see that

\[
\Psi_\delta(\cdot) \in C^{\infty}((\delta, T], H^0)
\]

and

\[
\partial_t \Phi_\delta(t) = e^{-(t-\delta)A}f(t) - \int_\delta^t e^{-(t-s)A}(f(s) - f(t))ds
\]

\[
= -A\Phi_\delta(t) + f(t), \quad \delta \leq t \leq T.
\]

(ii) now follows from Lemma 3.2. \( \square \)

For \( \alpha \in [0, 1] \), let \( W^{k+\alpha, 2}(\Omega) \) be the complex interpolation space between \( W^{k, 2}(\Omega) \) and \( W^{k+1, 2}(\Omega) \). The following lemma is easily derived by [19, p.23, Proposition 2.2] and the interpolation theorem (cf. [21]).

**Lemma 3.8.** Let \( k \in \mathbb{N} \cup \{0\} \) and \( \Omega \subset \mathbb{R}^3 \) be a bounded domain of class \( C^{k+2} \). For any \( f \in W^{\alpha, 2}(\Omega) \), \( 0 \leq \alpha \leq k \), there exist unique functions \( u \in W^{2+\alpha, 2}(\Omega) \) and \( P \in W^{1+\alpha, 2}(\Omega) \) (with \( \int_\Omega Pdx = 0 \)), which solve the following Stokes problem in the distribution sense:

\[
\nu \Delta u = \nabla P + f, \quad \text{div}(u) = 0, \quad u|_{\partial \Omega} = 0.
\]

Moreover, there exists a constant \( C_{\alpha, \nu} > 0 \) such that

\[
\|u\|_{W^{2+\alpha, 2}(\Omega)} + \|P\|_{W^{1+\alpha, 2}(\Omega)} \leq C_{\alpha, \nu} \cdot \|f\|_{W^{\alpha, 2}(\Omega)}.
\]
4. Proof of Theorem 2.2

In this section, we use the following equivalent norm in $H^β$ ($β \in [0, 2]$)

$$\|u\|_{H^β} = \|(I + A)^{β/2} u\|_{L^2}, \quad u \in H^β.$$  

We first prove:

**Lemma 4.1.** For any $u, v, u', v' \in H^2$ we have

$$\langle B(u', u) - g_N^{κ, r}(u')^2 u - B(v', v) + g_N^{κ, r}(v')^2 v, (I + A)(u - v) \rangle_{L^2} \quad (26)$$

$$\leq \frac{3ν}{4} \|u - v\|^2_{H^2} + \frac{ν}{8} \|u' - v'\|^2_{H^2} + \frac{κN}{ν} \|w\|^2_{H^1}$$

$$+ \frac{C_Ω}{ν^2} \|u' - v'\|^2_{H^1} \cdot ((1 + 4κ)\|v\|^2_{H^1} + κ\|u\|^2_{H^1}).$$

*Proof.* Set

$$w = u - v, \quad w' = u' - v',$$

and write (26) as the following four terms’ sum

$$I_1 := \langle B(u', u), (I + A)w \rangle_{L^2},$$

$$I_2 := \langle B(w', v), (I + A)w \rangle_{L^2},$$

$$I_3 := -\langle g_N^{κ, r}(u')^2 w, (I + A)w \rangle_{L^2},$$

$$I_4 := -\langle g_N^{κ, r}(v')^2 v - g_N^{κ, r}(u')^2 w, (I + A)w \rangle_{L^2}.$$  

By $ab \leq \frac{a^2 + b^2}{2}$, we have

$$I_1 \leq \|B(u', w)\|_{L^2} \cdot \|(I + A)w\|_{L^2}$$

$$\leq \frac{1}{2ν} \|(u' \cdot \nabla)w\|^2_{L^2} + \frac{ν}{2} \|w\|^2_{H^2}$$

$$\leq \frac{1}{2ν} \|u'\|^2_{L^2} \|\nabla w\|^2_{L^2} + \frac{ν}{2} \|w\|^2_{H^2}$$

and similarly,

$$I_2 \leq \frac{1}{ν} \|w'\|^2_{L^2} \|\nabla w\|^2_{L^2} + \frac{ν}{4} \|w\|^2_{H^2}.$$  

For $I_3$, by $g_N^{κ, r}(r) \geq \frac{κ}{ν} (r - N)$ we have

$$I_3 = -\langle g_N^{κ, r}(u')^2 w, (I + A)^{1/2} w \rangle_{L^2}$$

$$\leq -\frac{κ}{ν} \|u'\|^2_{H^1} \cdot \|w\|^2_{H^1} + \frac{κN}{ν} \|w\|^2_{H^1}.$$  

For $I_4$, by $|g_N^{κ, r}(r) - g_N^{κ, r}(r)| \leq \frac{κ}{ν} |r - r'|$ we have

$$I_4 \leq \frac{κ}{ν} \|w'\|_{∞} (\|u'\|_{∞} + \|v'\|_{∞}) \cdot \|v\|_{H^1} \cdot \|w\|_{H^1}$$

$$\leq \frac{κ}{ν} \|w'\|_{∞} \cdot (\|u'\|_{∞} + \|w'\|_{∞}) \cdot \|v\|_{H^1} \cdot \|w\|_{H^1}$$

$$= \frac{2κ}{ν} (\|u'\|_{∞} \cdot \|w\|_{H^1}) \cdot (\|w'\|_{∞} \cdot \|v\|_{H^1})$$

$$\leq \frac{κ}{2ν} \|u'\|^2_{∞} \cdot \|w\|^2_{H^1} + \frac{κ}{ν} \|w'\|^2_{∞} \cdot \|v\|_{H^1}$$

$$+ \frac{κ}{ν} \|w'\|^2_{∞} \cdot \|v\|_{H^1} \cdot (\|u\|_{H^1} + \|v\|_{H^1}).$$
Combining the above calculations, we obtain
\[
I_1 + I_2 + I_3 + I_4 \leq \frac{3\nu}{4} \| w \|_{H^2}^2 + \frac{\kappa N}{\nu} \cdot \| w \|_{H^1}^2 + \frac{(1 - \kappa)}{2\nu} \| u' \|_\infty^2 \cdot \| w \|_{H^1}^2 \\
+ \frac{1}{\nu} \| w' \|_{\infty} \cdot ((1 + 3\kappa)\| v \|_{H^1}^2 + \kappa \| v \|_{H^1} \cdot \| u \|_{H^1})
\]
which produces the desired estimate. □

4.1. **Proof of Existence.** Let \( v \in C([0, \infty); H^1) \cap L^2_{loc}(\mathbb{R}^+; H^2) \). We first consider the following linearized equation:
\[
\partial_t u = -\nu A u + B(v, u) - g_N^{\nu}\kappa(\| v \|_{H^1}^2)u, \quad u(0) = u_0 \in H^1.
\]
By the standard theory of PDE, there is a unique strong solution \( u \) to above equation with
\[
u \in C([0, \infty); H^1) \cap L^2_{loc}(\mathbb{R}^+; H^2).
\]
Let us construct the approximation sequence of equation (8) as follows: Set \( u_1(t) \equiv 0 \). For \( k = 2, 3, \cdots \), let
\[
u_k(t) \in C([0, \infty); H^1) \cap L^2_{loc}(\mathbb{R}^+; H^2)
\]
solve the following equation
\[
\partial_t \nu_k = -\nu A \nu_k + B(\nu_{k-1}, \nu_k) - g_N^{\nu}\kappa(\| \nu_{k-1} \|_{H^1}^2)\nu_k, \quad \nu_k(0) = \nu_0.
\]
Firstly, note that
\[
\langle A \nu_k, \nu_k \rangle_{L^2} = \| \nabla \nu_k \|_{L^2}^2
\]
and
\[
\langle B(\nu_{k-1}, \nu_k), \nu_k \rangle_{L^2} = -\langle (\nu_{k-1}, \nabla) \nu_k, \nu_k \rangle_{L^2} = -\frac{1}{2} \langle \text{div} \nu_{k-1}, \| \nu_k \|_{L^2}^2 \rangle = 0.
\]
By the chain rule, we have from (28) that
\[
d\| \nu_k \|_{L^2}^2/dt = -2\nu \| \nabla \nu_k \|_{L^2}^2 - 2g_N^{\nu}\kappa(\| \nu_{k-1} \|_{H^1}^2)\| \nu_k \|_{L^2}^2 \leq -2\nu \| \nabla \nu_k \|_{L^2}^2.
\]
Integrating both sides of (29) yields that
\[
\| \nu_k(t) \|_{L^2}^2 \leq \| \nu_0 \|_{L^2}^2, \quad \forall t \geq 0.
\]
Secondly, for any \( T > 0 \) we have
\[
\int_0^T g_N^{\nu}\kappa(\| \nu_{k-1} \|_{H^1}^2) \cdot \| \nu_k \|_{L^2}^2 dt \leq \frac{\kappa}{\nu} \sup_{t \in [0, T]} \| \nu_k(t) \|_{L^2}^2 \int_0^T \| \nu_k \|_{H^1}^4 dt
\]
\[
\leq C \cdot \sup_{t \in [0, T]} \| \nu_k(t) \|_{L^2} \cdot \sup_{t \in [0, T]} \| \nu_{k-1}(t) \|_{H^1} \int_0^T \| \nu_{k-1} \|_{H^1}^2 dt \leq +\infty
\]
11
and
\[
\int_0^T \|B(u_{k-1}, u_k)\|_{L^2}^2 dt \leq C \int_0^T \|u_{k-1}\|_\infty^2 \|\nabla u_k\|_{L^2}^2 dt
\]
\[
\leq C \int_0^T \|u_{k-1}\|_{H^2}^2 \cdot \|\nabla u_k\|_{L^2}^2 dt \leq +\infty. \tag{31}
\]

Thus, recalling \(H^0 = L^2_\sigma(\Omega)\), from \(\text{[28]}\) one has
\[
\partial_t u_k \in L^2_{\text{loc}}(\mathbb{R}^*_+; H^0).
\]

Consider the evolution triple
\(H^2 \subset H^1 \subset H^0\).

By the chain rule (cf. \cite{19} p.176, Lemma 1.2)) and Young’s inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u_k\|_{L^2}^2 = -\nu \|A u_k\|_{L^2}^2 + \langle B(u_{k-1}, u_k), A u_k \rangle_{L^2} - g_N^{\nu,\kappa}(\|u_{k-1}\|_{L^2}^2) \|\nabla u_k\|_{L^2}^2
\]
\[
\leq -\frac{\nu}{2} \|A u_k\|_{L^2}^2 + \frac{1}{2\nu} \|u_{k-1}\|_{L^2}^2 \|\nabla u_k\|_{L^2}^2 - g_N^{\nu,\kappa}(\|u_{k-1}\|_{L^2}^2) \|\nabla u_k\|_{L^2}^2
\]
\[
\leq -\frac{\nu}{2} \|A u_k\|_{L^2}^2 + \frac{1}{2\nu} \|u_{k-1}\|_{L^2}^2 \|\nabla u_k\|_{L^2}^2 + \frac{\kappa N}{2\nu} \|\nabla u_k\|_{L^2}^2, \tag{32}
\]
where the last step is due to
\[
g_N^{\nu,\kappa}(r) \geq \frac{\kappa}{\nu}(r - N).
\]

Integrating both sides of \(\text{[32]}\) and using \(\text{[30]}\) and \(\kappa \geq 1\), we obtain
\[
\|\nabla u_k(t)\|_{L^2}^2 + \nu \int_0^t \|A u_k\|_{L^2}^2 ds \leq \frac{\kappa N}{\nu^2} \|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2, \quad \forall t \geq 0. \tag{33}
\]

Now set
\[
w_{k,m}(t) := u_k(t) - u_m(t).
\]

Then
\[
\partial_t w_{k,m} = -A w_{k,m} + B(u_{k-1}, w_{k,m}) + B(w_{k-1,m-1}, v_m)
\]
\[
-g_N^{\nu,\kappa}(\|u_{k-1}\|_{L^2}^2) w_{k,m} - \left[ g_N^{\nu,\kappa}(\|u_{k-1}\|_{L^2}^2) - g_N^{\nu,\kappa}(\|v_{m-1}\|_{L^2}^2) \right] v_m.
\]

Again, by \cite{19} p.176, Lemma 1.2] and Lemma \(\text{[11]}\) we have
\[
\frac{1}{2} \frac{d}{dt} \|w_{k,m}\|_{H^1}^2 = -\nu \|w_{k,m}\|_{H^2}^2 + \nu \|w_{k,m}\|_{H^1}^2
\]
\[
+ \langle B(u_{k-1}, w_{k,m}), (I + A)w_{k,m} \rangle_{L^2} + \langle B(w_{k-1,m-1}, v_m), (I + A)w_{k,m} \rangle_{L^2}
\]
\[
- g_N^{\nu,\kappa}(\|u_{k-1}\|_{L^2}^2) \langle w_{k,m}, (I + A)w_{k,m} \rangle_{L^2}
\]
\[
- \left[ g_N^{\nu,\kappa}(\|u_{k-1}\|_{L^2}^2) - g_N^{\nu,\kappa}(\|v_{m-1}\|_{L^2}^2) \right] \langle v_m, w_{k,m} \rangle_{H^1}
\]
\[
\leq -\frac{\nu}{4} \|w_{k,m}\|_{H^2}^2 + \frac{\nu}{8} \|w_{k-1,m-1}\|_{H^2}^2 + \left( \nu + \frac{\kappa N}{\nu} \right) \|w_{k,m}\|_{H^1}^2
\]
\[
+ \frac{C_N}{\nu^2} \|w_{k-1,m-1}\|_{H^1}^2 \cdot (1 + 4\kappa) \|u_k\|_{H^1}^2 + \|u_m\|_{H^1}^2)^2. \tag{35}
\]

Integrating this inequality and using \(\text{[30]}\) and \(\text{[33]}\), we get
\[
\|w_{k,m}(t)\|_{H^1}^2 + \frac{\nu}{2} \int_0^t \|w_{k,m}\|_{H^2}^2 ds
\]

Thus, there exists a function $u$ then by (30), (33) and Fatou’s lemma, we have

$$\int_0^t \|w_{k-1,m-1}\|_{H^2} ds + C_{\nu,N} \int_0^t \|w_{k,m}\|_{H^1} ds$$

$$+ C_{\nu,N} \|u_0\|_{H^1} \cdot \int_0^t \|w_{k-1,m-1}\|_{H^1} ds.$$

Set

$$h(t) := \limsup_{k,m \to \infty} \sup_{s \in [0,t]} \|w_{k,m}(s)\|_{H^1}$$

and

$$f(t) := \limsup_{k,m \to \infty} \int_0^t \|w_{k,m}\|_{H^2} ds.$$

Then by (30), (33) and Fatou’s lemma, we have

$$\frac{\nu}{2} f(t) \leq \frac{\nu}{4} f(t) + C_{\nu,N} \|u_0\|_{H^1} \int_0^t h(s) ds$$

and

$$h(t) \leq \frac{\nu}{4} f(t) + C_{\nu,N} \|u_0\|_{H^1} \int_0^t h(s) ds \leq 2C_{\nu,N} \|u_0\|_{H^1} \int_0^t g(s) ds.$$

Hence, by Gronwall’s inequality we have

$$h(t) = f(t) = 0, \quad \forall t \geq 0.$$

Thus, there exists a function $u \in C([0, \infty); H^1) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2)$ such that for any $T > 0$

$$\limsup_{k \to \infty} \sup_{s \in [0,T]} \|u_k - u\|_{H^1} + \lim_{k \to \infty} \int_0^T \|u_k(s) - u(s)\|_{H^2} ds = 0.$$

Lastly, taking limits $k \to \infty$ for

$$u_k(t) = u_0 - \int_0^t A u_k ds + \int_0^t B(u_{k-1}, u_k) ds - \int_0^t g(\|u_{k-1}\|_{L^\infty}^2) u_k ds$$

and inequalities (30) and (33), we can see that $u(t)$ satisfies (9), (10) and (11).

### 4.2. Proof of Decay Estimate (12). Following the method of Heywood [8], by the chain rule and [8, p.649 (14)], we have

$$\frac{d}{dt} \|\nabla u\|_{L^2}^2 = -2\nu \|A u\|_{L^2}^2 + 2 \langle B(u, u), A u \rangle_{L^2} - 2g_N' (\|u\|_{L^\infty}) \|\nabla u\|_{L^2}^2$$

$$= -\nu \|A u\|_{L^2}^2 + C_{\nu,\Omega} \|\nabla u\|_{L^2}^4 + C_{\nu,\Omega}' \|\nabla u\|_{L^2}^6.$$ 

Note that

$$\int_0^\infty \|\nabla u_k\|_{L^2}^2 dt \leq \frac{\|u_0\|_{L^2}^2}{2\nu} =: \Lambda.$$

In Lemma 3.1 if we take $\phi(t) = \|\nabla u(t)\|_{L^2}^2$, $\beta = 1/(C_{\nu,\Omega}')$ and $\alpha = C_{\nu,\Omega} + 1/\Lambda$, then

$$\|\nabla u(t)\|_{L^2}^2 \leq \frac{e^{C_{\nu,\Omega} A + 1} - 1}{(C_{\nu,\Omega} + 1/\Lambda) t} \leq C_3 t^{-1}$$

for $t \geq (C_{\nu,\Omega}')^{-1} e^{C_{\nu,\Omega} A + 1} = T^*$. Thus, (12) follows.
4.3. Proof of Continuous Dependence \((13)\). Set
\[ w_{N,M}(t) := u_N(t) - v_M(t). \]

Once again, by the chain rule (cf. [19] p.176, Lemma 1.2]) we have
\[
\frac{1}{2} \frac{d}{dt} \|w_{N,M}\|_{H^1}^2 = -\|w_{N,M}\|_{H^2}^2 + \|w_{N,M}\|_{H^1}^2 + \langle B(w_{N,M}) , w_{N,M} \rangle_{H^1} + \langle B(w_{N,M}) , w_{N,M} \rangle_{H^1} - g_{N,M}^{\nu,\kappa}(\|u_N\|_{\infty}^2)\|w_{N,M}\|_{H^1}^2 + \frac{\nu}{8} \|w_{N,M}\|_{H^2}^2 + \frac{\nu}{8} \|w_{N,M}\|_{H^1}^2 + \frac{\nu}{8} \|w_{N,M}\|_{H^1}^2 + \frac{\nu}{8} \|w_{N,M}\|_{H^1}^2 + \frac{\nu}{8} \|w_{N,M}\|_{H^1}^2.
\]

Noting that
\[
|g_{N,M}^{\nu,\kappa}(r) - g_{M}(r)| \leq \frac{\nu}{8} |N - M|, \quad \forall r \geq 0, \quad N, M \geq 1,
\]
as in the proof of existence, by Lemma 4.1 and Young’s inequality we find that
\[
\frac{1}{2} \frac{d}{dt} \|w_{N,M}\|_{H^1}^2 \leq -\nu \|w_{N,M}\|_{H^2}^2 + \nu \|w_{N,M}\|_{H^1}^2 + C_{\nu} \|w_{N,M}\|_{H^1}^2 + C_{\nu, v_0} |N - M|^2,
\]
where \(C_{\nu, v_0} = 4\nu^2 \|v_0\|_{L^2}^2 / \nu^3\),
\[
C_{\nu, N, u_0, v_0} = \nu \|w_{N,M}\|_{H^2}^2 + C_{\nu} \|w_{N,M}\|_{H^1}^2 + \nu \|w_{N,M}\|_{H^1}^2 + \nu \|w_{N,M}\|_{H^1}^2.
\]

The estimate \((13)\) now follows by Gronwall’s inequality.

5. PROOF OF THEOREM 2.4

5.1. Proof of Part (I). Let \(u(t)\) be the unique strong solution of equation \((5)\). By Duhamel’s formula, we may write
\[
u(t) = e^{At}u_0 + \int_0^t e^{-(t-s)A}B(u, u)ds - \int_0^t e^{-(t-s)A}g_{N,M}^{\nu,\kappa}(\|u\|_{\infty})u ds
\]
\[ =: w_1(t) + w_2(t) + w_3(t). \quad (36)
\]

First of all, it is clear that \(w_1 \in C^\infty((0, T]; H^2)\) and
\[
|Aw_1(t)|_{L^2} \leq |Au_0|_{L^2} \quad (37)
\]

For \(w_2(t)\), by (i) of Lemma 3.2 we have
\[
\|A^{\frac{1}{2}}w_2(t)\|_{L^2} \leq \int_0^t \|A^{\frac{1}{2}}e^{-(t-s)A}A^{-\frac{1}{2}}B(u, u)\|_{L^2}ds
\]
\[
\leq \int_0^t \|A^{-\frac{1}{2}}B(u, u)\|_{L^2}^2 (t-s)^{\frac{1}{2}} ds \leq C_\Omega \cdot \int_0^t \|A^{\frac{1}{2}}u\|_{L^2}^2 (t-s)^{\frac{1}{2}} ds
\]
\[
\leq C_\Omega K_{\nu, N, u_0} \cdot t^{\frac{1}{2}}, \quad (38)
\]
where

\[ K_{\nu,N,u_0} := \frac{kN}{\nu^2} \| u_0 \|^2 + \| \nabla u_0 \|^2. \]

For \( w_3(t) \), recalling (6) and by Lemma 3.2 we have for \( \alpha \in [1/2, 1) \)

\[ \| A^\alpha w_3(t) \|_{L^2} \leq \int_0^t g_N^{\nu,N}(\| u \|_\infty) \cdot \| A^\alpha e^{-(t-s)} A u \|_{L^2} ds \]

\[ \leq \int_0^t \| u \|_\infty^2 \cdot \frac{1}{(t-s)^{\alpha - \frac{1}{2}}} \cdot \| A_2^\frac{1}{2} u \|_{L^2} ds \]

\[ \leq C_{\Omega} \int_0^t \| u \|_{H^2} \cdot \| \nabla u \|_{L^2} \cdot \frac{1}{(t-s)^{\alpha - \frac{1}{2}}} ds \]

By (10), (11) and Hölder’s inequality we have

\[ I_1 \leq C_{\Omega} K_{\nu,N,u_0} t^{1-\alpha} \left( \int_0^t \| A u(s) \|_{L^2}^2 ds \right)^{1/2} \leq C_{\Omega} K_{\nu,N,u_0}^{3/2} \cdot t^{1-\alpha} \]

and

\[ I_2 \leq C_{\Omega} \| u_0 \|_{L^2} K_{\nu,N,u_0}^{1/2} t^{1-\alpha} \left( \int_0^t \| \nabla u(s) \|_{L^2}^2 ds \right)^{1/2} \leq C_{\Omega} \| u_0 \|_{L^2}^{1/2} K_{\nu,N,u_0}^{1/2} \cdot t^{1-\alpha}. \]

Hence

\[ \| A^\alpha w_3(t) \|_{L^2} \leq C_{\Omega} \left( \| u_0 \|_{L^2}^2 K_{\nu,N,u_0}^{1/2} + K_{\nu,N,u_0}^{3/2} \right) \cdot t^{1-\alpha}. \]

Combining (36), (37), (38) and (39), we find that

\[ \| A^{\frac{5}{2}} u(t) \|_{L^2} \leq \| A^{\frac{5}{2}} u_0 \|_{L^2} + C_{\Omega} K_{\nu,N,u_0} \cdot t^{\frac{3}{4}} + C_{\Omega} \left( \| u_0 \|_{L^2}^{1/2} K_{\nu,N,u_0}^{1/2} + K_{\nu,N,u_0}^{3/2} \right) \cdot t^{3/8} \]

\[ =: M_0(t, \nu, N, u_0) \]

and by (25) and (10)

\[ \| B(u(t), u(t)) \|_{L^2} \leq C_{\Omega} \left( \| A^{\frac{5}{2}} u(t) \|_{L^2}^2 + \| u(t) \|_{L^2}^2 \right) \leq C_{\Omega} \cdot (M_0(t, \nu, N, u_0)^2 + \| u_0 \|_{L^2}^2). \]

By (1) of Lemma 3.7 we have for any \( \frac{3}{4} < \gamma < 1 \)

\[ \| A^\gamma u(t) \|_{L^2} \leq \| A^\gamma u_0 \|_{L^2} + C_{\Omega} \cdot (M_0(t, \nu, N, u_0)^2 + \| u_0 \|_{L^2}^2) \cdot t^{1-\gamma} \]

\[ + C_{\Omega} \left( \| u_0 \|_{L^2}^{2} K_{\nu,N,u_0}^{1/2} + K_{\nu,N,u_0}^{3/2} \right) \cdot t^{1-\gamma}, \]

which then yields the estimate (14) by (22).
5.2. Proof of Part (II). In this subsection, we assume $\Omega = \mathbb{R}^3$ or $\Omega$ is a bounded uniform $C^4$-domain. Our proof is concentrated on the case of bounded domain. Clearly, it also works for $\Omega = \mathbb{R}^3$.

Below, fix $T > 0$ and set
\[ f(s) := B(u(s), u(s)) - g_{\nu, \kappa}^\nu N(\|u(s)\|_\infty^2)u(s). \]

Then by Lemma 3.6 and (22), (10)
\[ [0, T] \ni s \mapsto f(s) \in H^0 = L_2^2(\Omega) \quad \text{is continuous.} \]

By (i) of Lemma 3.7, we have for any $0 < \beta < 1$
\[ u \in C^\theta([0, T], H^0) \cap C([0, T]; H^{2\beta}). \] (41)

Thus, by Lemma 3.6 and (22), for any $\frac{3}{4} < \gamma < \beta \leq 1$, there are constants $C_1, C_2, C_3 > 0$ such that for all $t, s \in [0, T]$
\[ \|f(t) - f(s)\|_{L^2} \leq C_1 \cdot \|A^\beta(u(t) - u(s))\|_{L^2}^{\frac{3}{4}} \cdot \|u(t) - u(s)\|_{L^2}^{1 - \frac{3}{4}} + C_2 \cdot \|A^\beta(u(t) - u(s))\|_{L^2}^{\frac{3}{4}} \cdot \|u(t) - u(s)\|_{L^2}^{1 - \frac{3}{4}} + C_3 \cdot \|u(t) - u(s)\|_{L^2}. \] (42)

Choosing $\beta$ close to 1 and $\gamma$ close to $\frac{3}{4}$ and using (10) and (11), we find that for any $0 < \alpha < \frac{1}{4}$
\[ f \in C^\alpha([0, T], H^0). \]

Thus, by (ii) of Lemma 3.7 and (36) we have, for any $0 < \alpha < \frac{1}{4}$
\[ Au \in C^\alpha((0, T], H^0), \quad \partial_t u \in C((0, T], H^{2\alpha}). \]

Using induction and (12) with $\beta = 1$ as well as (11), one finds that for any $n \in \mathbb{N}$ and $0 < \alpha < 1 - \left(\frac{3}{4}\right)^{n+1}$
\[ f \in C^\alpha((0, T], H^0) \]
and
\[ Au \in C^\alpha((0, T], H^0), \quad \partial_t u \in C((0, T], H^{2\alpha}). \]

In particular, by $H^\alpha \subset W^{\alpha, 2}(\Omega)$ for $\alpha \in [0, 2]$ we have
\[ u \in C^\alpha((0, T], W^{2,2}(\Omega)), \quad \partial_t u \in C((0, T], W^{9/5,2}(\Omega)). \] (43)

Set
\[ b(t) := (u(t) \cdot \nabla)u(t) + g_{\nu, \kappa}^\nu N(\|u(t)\|_\infty^2)u(t). \]

As in the proof of Lemma 3.6, it is not hard to verify by (43) that
\[ b(t) \in C((0, T]; W^{1,2}(\Omega)). \] (44)

Consider the Stokes equation:
\[ \begin{cases} \nu \Delta u + \nabla P = \partial_t u + b \quad \text{in } \Omega, \\ du = 0 \quad \text{in } \Omega, \quad u_{|\partial \Omega} = 0. \end{cases} \]

By (13), (44) and Lemma 3.8 with $\alpha = 1$, we have
\[ u(t) \in C((0, T], W^{3,2}(\Omega)). \]

As above, a simple calculation shows that
\[ b(t) \in C((0, T]; W^{2,2}(\Omega)). \] (45)
By (13), (45) and Lemma 3.3 with $\alpha = \frac{9}{5}$ again, we further have
\[ \mathbf{u}(t) \in C((0, T], W^{\frac{19}{5}, 2}(\Omega)) \] (46)
and
\[ P \in C((0, T], W^{\frac{15}{2}, 2}(\Omega)). \] (47)
By (13), (46), (47) and the Sobolev embedding theorem (cf. [21 Theorem 4.6.1]), we finally obtain that
\[ \partial_t \mathbf{u}, \partial_i \mathbf{u}, \partial_i \partial_j \mathbf{u}, P \in C((0, T] \times \bar{\Omega}; \mathbb{R}^3) \]
and (15) holds.

5.3. Proof of Part (III). In this subsection, we assume $\Omega = \mathbb{R}^3$.

**Lemma 5.1.** For fixed $q \geq 2$ and $r \geq 1$, there exists $\kappa := \kappa(q) := Cq^4$, where $C$ is a universal constant, such that for any $N \geq 1$ and $t \geq 0$
\[ \|\mathbf{u}_N^r(t)\|_{L^q} \leq \|\mathbf{u}_0\|_{L^q} + \frac{r\kappa}{\nu} \cdot N \int_0^t \|\mathbf{u}_N^r\|_{L^q} ds \] (48)
and
\[ \int_0^t \|\mathbf{u}_N^r\|_{L^q}^2 \cdot \|\mathbf{u}_N^r\|_{L^r} ds \leq \frac{2\nu}{r\kappa} \|\mathbf{u}_0\|_{L^q}^2 + 2N \int_0^t \|\mathbf{u}_N^r\|_{L^q}^2 ds. \] (49)

**Proof.** Let $\mathbf{u} := \mathbf{u}_N^r$. Taking the scalar product for both sides of equation (15) with $q\mathbf{u}^{q-2}\mathbf{u}$, and then integrating over $\mathbb{R}^3$, we find by the integration by parts formula
\[ \frac{d\|\mathbf{u}\|_{L^q}^2}{dt} = -q\nu\|\nabla \mathbf{u}\|_{L^q}^2 - \frac{4(q - 2)\nu}{q}\|\nabla \mathbf{u}^{q/2}\|_{L^2}^2 + q\langle \nabla P, \mathbf{u}^{q-2}\mathbf{u}\rangle_{L^2} - q \cdot g_{g}^{\kappa}(\|\mathbf{u}\|_{L^q}^2)\|\mathbf{u}\|_{L^q}^2, \]
where we have used that
\[ q\langle (\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{u}^{q-2}\mathbf{u}\rangle_{L^2} = \langle \mathbf{u}, \nabla \mathbf{u}^{q/2}\rangle_{L^2} = 0. \]
Let $f$ be an increasing smooth function on $[0, \infty)$. We further have
\[ \frac{df(\|\mathbf{u}\|_{L^q}^2)}{dt} = f'(\|\mathbf{u}\|_{L^q}^2) \left[ -q\nu\|\nabla \mathbf{u}\|_{L^q}^2 - \frac{4(q - 2)\nu}{q}\|\nabla \mathbf{u}^{q/2}\|_{L^2}^2 + q\langle \nabla P, \mathbf{u}^{q-2}\mathbf{u}\rangle_{L^2} - q \cdot g_{g}^{\kappa}(\|\mathbf{u}\|_{L^q}^2)\|\mathbf{u}\|_{L^q}^2 \right]. \] (50)
On the other hand, taking the divergence for equation (15) we have
\[ \Delta P = \text{div}[(\mathbf{u} \cdot \nabla) \mathbf{u}], \]
which gives
\[ P = -(-\Delta)^{-1}\text{div}[(\mathbf{u} \cdot \nabla) \mathbf{u}] = -(-\Delta)^{-1}\partial_i \partial_j (u^i \cdot u^j). \]
So, by the Calderón-Zygmund inequality we get for any $\gamma \geq 2$ (cf. [18])
\[ \|P\|_{L^\gamma} \leq C_1 \cdot \gamma \cdot \|\mathbf{u}\|_{L^2}^2, \] (51)
Here and below, $C_i, i = 1, 2, 3$ are universal constants. Thus, by Young’s inequality and Hölder’s inequality we have
\[ q\langle \nabla P, \mathbf{u}^{q-2}\mathbf{u}\rangle_{L^2} = q\langle P, \mathbf{u}^{q-2}\text{div}\mathbf{u}\rangle_{L^2} + q\langle P, \nabla \mathbf{u}^{q-2} \cdot \mathbf{u}\rangle_{L^2} \]
\[ \leq q\nu\|\nabla \mathbf{u}\|_{L^2} \cdot \|\mathbf{u}^{(q-2)/2}\|_{L^2}^2 + \frac{C_2g^2}{\nu} \|P\| \cdot \|\mathbf{u}^{(q-2)/2}\|_{L^2}^2. \]
where \( k \) is a universal constant. Then we have
\[
\int \quad \text{and} \quad \text{Substituting this estimate into (50), we get}
\]
\[f(t) = \left(\frac{C_q^5}{\nu} \cdot \|u\|_{L^q}^2 \cdot \|u\|_{L^q}^2 - q \cdot g_{N}^{\nu, \kappa}(\|u\|_{L^q}^2)\right).\]

Now noticing that
\[g_{N}^{\nu, \kappa}(r) \geq \frac{\kappa(r - N)}{\nu}, \quad r \geq 0,
\]
we find that if
\[\kappa = 2C_3q^4, \quad (\text{52})\]
then
\[
\frac{d}{dt}\left(\|u\|_{L^q}^2\right) + \frac{4(q - 2)\nu}{q} f'(\|u\|_{L^q}^2) \cdot \|\nabla u\|_{L^q}^2
\leq f'(\|u\|_{L^q}^2) \cdot \left[\frac{C_3q^5}{\nu} \cdot \|u\|_{L^q}^2 \cdot \|u\|_{L^q}^2 - q \cdot g_{N}^{\nu, \kappa}(\|u\|_{L^q}^2)\right]. \quad (\text{53})
\]
Lastly, taking \( f_k(x) := (\epsilon + x)^{r/q} \) in (\text{53}), then integrating with respect to \( t \) and letting \( \epsilon \downarrow 0 \) yield (48) and (49).

**Lemma 5.2.** Fix \( r_0 \geq 1 \) and \( q_0 \geq 2 \). Let \( u_0 \in H^2 \) and set \( N_0 := C\|u_0\|_{H^2}^2 \) for some universal constant \( C \). There exists \( n_0 := n_0(\nu, N_0, q_0, r_0) \) large enough such that for all \( n \geq n_0, N \geq N_0 \) and \( t \geq 0 \)
\[
\|u_{N}^{\kappa_n}(t)\|_{L^{q_0+2\nu n}r_0} \leq 2N^{\frac{\kappa_n}{2\nu+1}} + 2N^{\frac{\frac{\kappa_n}{2\nu+1}}{q_0}} \left[\int_{0}^{t} \|u_{N}^{\kappa_n}\|_{L^{q_0}r_0}^{\frac{1}{2\nu+1}} ds\right]^{\frac{1}{\nu+1}}, \quad (\text{54})
\]
where \( \kappa_n = 2C_3 \cdot [(2n + r_0)q_0/r_0]^4 \). In particular, there is an \( n_0 := n_0(\nu, N_0) \) large enough such that for all \( n \geq n_0, N \geq N_0 \) and \( t \geq 0 \)
\[
\|u_{N}^{\kappa_n}(t)\|_{L^{q_0(n+1)}} \leq 3N^{\frac{1}{4}}. \quad (\text{55})
\]
**Proof.** Let \( u := u_{N}^{\kappa_n} \). First of all, by the Gagliado-Nireberg inequality (23), there is a universal constant \( C_0 \geq 1 \) such that for any \( q \geq 2 \)
\[
\|u_0\|_{L^q} \leq \|u_0\|_{L^q}^{1-\frac{2}{q}} \cdot \|u_0\|_{L^q}^{\frac{1}{q} / 2} \leq C_0\|u_0\|_{H^2} =: N_0^{1/2}. \quad (\text{56})
\]
Define
\[q_n := q_{n-1} + 2q_0/r_0 = (2n + r_0)q_0/r_0 \quad \text{and} \quad r_n := r_0q_n/q_0 = 2n + r_0.
\]
Then we have
\[
\int_{0}^{t} \|u\|_{L^{q_0+2\nu n}r_0}^{r_{n+1}} ds \leq \int_{0}^{t} \|u\|_{L^{q_0}r_0}^{2\nu} \cdot \|u\|_{L^{q_0}r_0}^{r_{n}} ds
\]
Hence, by (48) 

\[ n \leq 2N \int_0^t \|u\|_{L^{n+1}}^n \, ds \]

(by \( r_n \kappa(q_n) \geq 2 \)) \[ \leq \nu \|u_0\|_{L^{n+1}}^n + 2N \int_0^t \|u\|_{L^{n+1}}^n \, ds \] 

(by iterating) \[ \leq \nu \sum_{k=0}^n (2N)^k \|u_0\|_{L^{n-k}}^n + (2N)^{n+1} \int_0^t \|u\|_{L^{n+1}}^n \, ds \]

(by (56)) \[ \leq \nu \sum_{k=0}^n (2N)^k \cdot N_0^{n-k/2} + (2N)^{n+1} \int_0^t \|u\|_{L^{n+1}}^n \, ds \]

(by \( r_n = 2n + r_0 \)) \[ = \frac{\nu((2N)^{n+1} N_0^{r_0/2} - N_0^{n+1+r_0/2})}{2N - N_0} \]

\[ + (2N)^{n+1} \int_0^t \|u\|_{L^{n+1}}^n \, ds \]

(by \( N \geq N_0 \lor 1 \)) \[ \leq \frac{\nu N_0^{r_0/2}(2N)^{n+1}}{N} + (2N)^{n+1} \int_0^t \|u\|_{L^{n+1}}^n \, ds. \]

Hence, by (48)

\[ \|u(t)\|_{L^{n+1}}^n \leq \|u_0\|_{L^{n+1}}^n + r_n \kappa(q_n) \cdot N_0^{r_0/2} (2N)^n \]

\[ + r_n \kappa(q_n) N \nu \cdot (2N)^{n-1} \int_0^t \|u\|_{L^{n+1}}^n \, ds. \]

Now taking the root \( 1/r_n \) and noting that

\[ \lim_{n \to \infty} (r_n \kappa(q_n))^{1/r_n} = \lim_{n \to \infty} (2C_3 r_n q_0^{4})^{1/r_n} = 1, \]

we obtain the desired estimate (54).

As for (55), it follows by taking \( r_0 = 2 \) and \( q_0 = 6 \) in (54) and noting that

\[ \int_0^t \|u\|_{L^2}^2 \, ds \leq C \int_0^t \|\nabla u\|_{L^2}^2 \, ds \leq \frac{C}{2\nu} \|u_0\|_{L^2}^2. \]

The proof is complete.

We are now in a position to give

**Proof of (16):** By (20), (40) and (10) we have

\[ \|u_N(t)\|_{W^{1,4}} \leq C(\|A^{7/8} u_N^*(t)\|_{L^2} + \|u_N(t)\|_{L^2}) \]

\[ \leq C \|A^{7/8} u_0\|_{L^2} + C \cdot (M_0(t, \nu, N, u_0)^2 + \|u_0\|_{L^2}^2) \cdot t^{1/8} \]

\[ + C \cdot \left( \|u_0\|_{L^2}^2 K_{\nu, N, u_0}^{1/2} + K_{\nu, N, u_0}^{3/2} \right) \cdot t^{1/8} + C \|u_0\|_{L^2}. \]

By the Gagliardo-Niremberg inequality (23) and (55), we have

\[ \|u_N^*(t)\|_{\infty} \leq C \|u_N^*(t)\|_{W^{1,4}} \cdot \|u_N^*(t)\|_{L^{(n+1)}}. \]

Letting \( n \) be large enough, the estimate (16) follows from (55) and (58).
6. Proof of Theorem [2.8]

We need the following simple lemma. For the reader’s convenience, a short proof is provided here.

**Lemma 6.1.** Let \((\mathbb{X}, \| \cdot \|_\mathbb{X})\) be a uniformly convex Banach space and \(K \subset \mathbb{X}\). Then \(K\) is relatively compact in \(\mathbb{X}\) if and only if there exists a family of finite dimensional subspaces \(\{\mathbb{X}_n, n \in \mathbb{N}\}\) of \(\mathbb{X}\) such that

\[
\sup_{n \in \mathbb{N}} \sup_{x \in K} \|\Pi_n x\|_\mathbb{X} < +\infty
\]

and

\[
\lim_{n \to -\infty} \sup_{x \in K} \| (I - \Pi_n) x \|_\mathbb{X} = 0,
\]

where \(\Pi_n\) is the projection operator from \(\mathbb{X}\) to \(\mathbb{X}_n\), i.e., \(\Pi_n x \in \mathbb{X}_n\) is the unique element such that

\[
\| x - \Pi_n x \|_\mathbb{X} = \inf_{y \in \mathbb{X}_n} \| x - y \|_\mathbb{X}.
\]

**Proof.** ("Only if"). Let \(K\) be relatively compact in \(\mathbb{X}\). For any \(n \in \mathbb{N}\), there are finite points \(\{x_1, \ldots, x_m\} \subset K\) such that

\[
K \subset \bigcup_{i=1}^{m} B_{1/n}(x_i),
\]

where \(B_{1/n}(x_k)\) denotes the ball in \(\mathbb{X}\) with center \(x_k\) and radius \(1/n\). Now put

\[
\mathbb{X}_n := \text{span}\{x_1, \ldots, x_m\}.
\]

It is easy to see that the corresponding \(\Pi_n\) satisfy (59) and (60).

("If"): Fix any sequence \(\{x_k, k \in \mathbb{N}\} \subset K\). It suffices to prove that there is a subsequence \(x_{k_l}\) such that \(x_{k_l}\) converges to some point \(x \in \mathbb{X}\). For any \(n \in \mathbb{N}\), since \(\mathbb{X}_n\) is finite dimensional, by (59) there is a subsequence \(x_{k_l(n)}\) and \(y_n \in \mathbb{X}_n\) such that \(\Pi_n x_{k_l(n)}\) converges to \(y_n\) as \(l \to \infty\). By the diagonalization method, one can find a common subsequence \(x_{k_l}\) such that for any \(n \in \mathbb{N}\)

\[
\lim_{l \to \infty} \| \Pi_n x_{k_l} - y_n \|_\mathbb{X} = 0.
\]

Noting that

\[
\|y_n - y_m\|_\mathbb{X} \leq \|\Pi_n x_{k_l} - y_n\|_\mathbb{X} + \|\Pi_n x_{k_l} - y_n\|_\mathbb{X} + \|\Pi_n x_{k_l} - P_m y_n\|_\mathbb{X},
\]

we have by (60) that \(\{y_n, n \in \mathbb{N}\}\) is a Cauchy sequence in \(\mathbb{X}\). So, there is an \(x \in \mathbb{X}\) such that \(y_n\) converges to \(x\) in \(\mathbb{X}\). By (60) again, it is easy to find that \(x_{k_l}\) converges to \(x\) in \(\mathbb{X}\). The proof is complete. \(\square\)

Since we have assumed that \(\Omega\) is a bounded domain in Theorem [2.8], \(H^1 = W^{1,2}_{0,\sigma}(\Omega)\) is compactly embedded in \(H^0 = L_2^0(\Omega)\). Let \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to \infty\) be the eigenvalues of \(A\), and \(\mathcal{E} := \{e_k; k \in \mathbb{N}\}\) the corresponding orthonormal eigenvectors, i.e.,

\[
A e_k = \lambda_k e_k, \quad \langle e_k, e_j \rangle_{L^2} = \delta_{kj}.
\]

From this, one knows that the following Poincare inequality holds:

\[
\sqrt{\lambda_1} \cdot \| u \|_{L^2} \leq \| A^{1/2} u \|_{L^2}, \quad \forall u \in H^1.
\]

Moreover, by (21) and (62) we have

\[
\| u \|_{L^2} \leq C_0 \cdot \| Au \|_{L^2} \cdot \| \nabla u \|_{L^2}.
\]

We have:

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Lemma 6.2. For $\epsilon > 0$, let $\mathcal{B}_\epsilon := \{ v \in H^1 : \| v \|_{H^1} \leq \epsilon \}$. Then $\mathcal{B}_\epsilon$ is an absorbing set of $\{ S(t); t \geq 0 \}$, i.e., for any bounded set $\mathcal{U} \subset H^1$, there exists $t_\mathcal{U} > 0$ such that for any $t > t_\mathcal{U}$

$$S(t)\mathcal{U} \subset \mathcal{B}_\epsilon.$$ 

Proof. By the chain rule and (62), we have

$$d \| u \|_{L^2}^2 / dt = -2\nu \| \nabla u \|_{L^2}^2 - 2g_N^{\nu,\kappa}(\| u \|_{L^\infty}) \| u \|_{L^2}^2 \leq -2\nu \lambda_1 \| u \|_{L^2}^2,$$

which implies

$$\| u(t) \|_{L^2}^2 \leq \| u_0 \|_{L^2}^2 e^{-2\nu \lambda_1 t}.$$ (64)

As the calculation of (32), by Young’s inequality we have

$$d \| A^{1/2} u \|_{L^2}^2 / dt \leq -\nu \| Au \|_{L^2}^2 + \frac{2\kappa N}{\nu} \| A^{1/2} u \|_{L^2}^2$$

(63)

$$\leq -\nu \| Au \|_{L^2}^2 + C_{\nu,\kappa} \cdot N \cdot \| Au \|_{L^2} \cdot \| u \|_{L^2}$$

$$\leq -\nu \lambda_1 / 2 \cdot \| A^{1/2} u \|_{L^2}^2 + C_{\nu,\kappa} \cdot N^2 \cdot \| u \|_{L^2}^2$$

(62)

$$\leq -\nu \lambda_1 \cdot \| A^{1/2} u \|_{L^2}^2 + C_{\nu,\kappa} \cdot N^2 \cdot \| u_0 \|_{L^2}^2 \cdot e^{-2\nu \lambda_1 t}.$$ Integrating this differential inequality yields that

$$\| A^{1/2} u(t) \|_{L^2}^2 \leq e^{-\nu \lambda_1 t/2} \left[ \| A^{1/2} u_0 \|_{L^2}^2 + C_{\nu,\kappa} \cdot N^2 \cdot \| u_0 \|_{L^2}^2 \cdot (1 - e^{-3\nu \lambda_1 t/2}) / (\nu \lambda_1) \right].$$ (65)

Hence, for any $u_0 \in H^1$

$$\lim_{t \to \infty} \| S(t)u_0 \|_{H^1}^2 = \lim_{t \to \infty} \| u(t) \|_{H^1}^2 = 0.$$ The result follows. 

We now use Lemma 6.1 to prove the following compactness result.

Lemma 6.3. For any $t > 0$, $S(t)$ is a compact operator from $H^1$ to $H^1$, i.e., maps a bounded set in $H^1$ into a relatively compact in $H^1$.

Proof. Let $\mathcal{U} \subset H^1$ be a bounded set. Let $\Pi_n$ be the projection operator from $H^1$ to span$\{ e_k : k = 1, \cdots, n \}$, i.e.,

$$\Pi_n v := \sum_{k=1}^n \langle v, e_k \rangle_{L^2} e_k.$$ (66)

First of all, by (63) we have

$$\sup_{n \in \mathbb{N}} \sup_{u_0 \in \mathcal{U}} \| \Pi_n S(t)u_0 \|_{H^1} \leq \sup_{u_0 \in \mathcal{U}} \| S(t)u_0 \|_{H^1} < +\infty.$$ (67)

Write

$$\Pi_n := I - \Pi_n$$

By (61) and (66) we have

$$\Pi_n^c A = A \Pi_n^c.$$ 

Thus, from (9) we get

$$\Pi_n^c u(t) = \Pi_n^c u_0 - \nu \int_0^t A \Pi_n^c uds + \int_0^t \Pi_n^c B(u, u)ds - \int_0^t g_N^{\nu,\kappa}(\| u \|_{L^\infty}^2) \Pi_n^c uds.$$
By the chain rule (cf. [19, p.176, Lemma 1.2]) we have

$$\frac{d}{dt} \|\Pi_n^c u(t)\|_{H^1}^2 = -2\nu \|A\Pi_n^c u\|_{L^2}^2 + 2\langle \Pi_n^c B(u, u), \Pi_n^c u \rangle_{H^1}$$

$$-2g_{\nu, \kappa}(\|u\|_{L^\infty}^2) \cdot \|\Pi_n^c u\|_{H^1}^2$$

$$\leq -\nu \|A\Pi_n^c u\|_{L^2}^2 + \frac{1}{\nu} \|\Pi_n^c B(u, u)\|_{L^2}^2.$$

Noting that

$$\|\Pi_n^c u\|_{H^1}^2 = \|A \frac{1}{2} \Pi_n^c u\|_{L^2}^2 \leq \frac{1}{\lambda_n} \|A\Pi_n^c u\|_{L^2}^2$$

and

$$\|\Pi_n^c B(u, u)\|_{L^2}^2 \leq \|u\|_{L^\infty}^2 \cdot \|\nabla u\|_{L^2}^2 \leq C \cdot \|Au\|_{L^2} \cdot \|\nabla u\|_{L^2}^2 =: h(t),$$

we have

$$\frac{d}{dt} \|\Pi_n^c u(t)\|_{H^1}^2 + \nu \lambda_n \|\Pi_n^c u\|_{H^1}^2 \leq \frac{h(t)}{\nu}.$$

Solving this differential inequality yields that

$$\|\Pi_n^c u(t)\|_{H^1}^2 \leq e^{-\nu \lambda_n t} \left( \|\Pi_n^c u_0\|_{H^1}^2 + \int_0^t \frac{1}{\nu} e^{\nu \lambda_n s} h(s) ds \right)$$

$$\leq e^{-\nu \lambda_n t} \left( \|u_0\|_{H^1}^2 + \frac{1}{\nu} \left( \int_0^t e^{2\nu \lambda_n s} ds \right)^{1/2} \left( \int_0^t h(s)^2 ds \right)^{1/2} \right)$$

$$\leq e^{-\nu \lambda_n t} \|u_0\|_{H^1}^2 + \frac{1}{\nu \sqrt{2\nu \lambda_n}} \left( \int_0^t h(s)^2 ds \right)^{1/2}.$$  

On the other hand, by (11) we have

$$\int_0^t h(s)^2 ds \leq \sup_{s \in [0,t]} \|\nabla u(s)\|_{L^2}^2 \int_0^t \|Au\|_{L^2}^2 ds \leq \frac{1}{\nu} \left( \frac{\kappa N}{\nu^2} \|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 \right)^4.$$

Hence, by $\lambda_n \uparrow \infty$ we obtain

$$\lim_{n \to \infty} \sup_{u_0 \in U} \|\Pi_n^c S(t)u_0\|_{H^1}^2 = \lim_{n \to \infty} \sup_{u_0 \in U} \|\Pi_n^c u(t)\|_{H^1}^2 = 0,$$

which combined with (67) yields by Lemma 6.1 that $S(t)U$ is relatively compact in $H^1$. □

**Proof of Theorem 2.8** It follows from [20, p. 23 Theorem 1.1 and (1.12')] and Lemmas 6.2 and 6.3.

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