On the invariant method for the time-dependent non-Hermitian Hamiltonians

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\textbf{Abstract}

We propose a scheme to deal with certain time-dependent non-Hermitian Hamiltonian operators $H(t)$ that generate a real phase in their time-evolution. This involves the use of invariant operators $I_{PH}(t)$ that are pseudo-Hermitian with respect to the time-dependent metric operator, which implies that the dynamics is governed by unitary time evolution. Furthermore, $H(t)$ is generally not quasi-Hermitian and does not define an observable of the system but $I_{PH}(t)$ obeys a quasi-hermiticity transformation as in the completely time-independent Hamiltonian systems case. The harmonic oscillator with a time-dependent frequency under the action of a complex time-dependent linear potential is considered as an illustrative example.

PACS: 03.65.Ca, 03.65.-w

1 Introduction

In Quantum Mechanics, one of the fundamental requirements is that the Hamiltonian should be Hermitian. Imposing $H^\dagger = H$ ensures that the eigenvalue spectrum is real, the inner products of state vectors in Hilbert space have a positive norm and that the time evolution operator is unitary. However, it has been found that not only Hermitian Hamiltonians satisfy these conditions. Specifically, Bender has shown that a non-Hermitian Hamiltonian which is invariant under $PT$-symmetry satisfies all physical axioms of quantum theory \[1, 2, 3, 4, 5\]. Parity $P$ has the effect of changing the sign of the momentum operator $p$ and the position operator $x$. The anti-linear operator $T$ has the effect of changing the sign of the momentum operator $p$ and the pure imaginary complex number $i$. The reality of spectrum was attributed to an unbroken $PT$-symmetry of $H$.  

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The generalisation of the PT-symmetry concept (i.e. systems with real spectra) to pseudo-Hermiticity was formulated by Mostafazadeh [6, 7, 8]: all Hamiltonian $H$ with a real spectrum is pseudo-Hermitian if

$$H^\dagger = \eta H \eta^{-1},$$

(1)

where the operator $\eta = \rho^\dagger \rho$ (\(\rho\) is a bounded linear invertible operator, with bounded inverse) being linear, Hermitian, invertible on the vector space spanned by the eigenstates $|\phi^H_n\rangle$ of $H$. We note that Eq. (1) is equivalent to the requirement that $H$ is Hermitian with respect to the inner product $\langle \cdot, \cdot \rangle_\eta = \langle \eta | \cdot \rangle \langle \cdot | \eta \rangle$ defined as

$$\langle \phi^H_m | \phi^H_n \rangle_\eta = \delta_{mn}.$$ 

(2)

In particular, the formalism developed by Mostafazadeh, building on earlier work by Scholtz et al [9], showed that the Hamiltonian $H$ is related by a similarity transformation to an equivalent Hermitian Hamiltonian $h$ by

$$h = \rho H \rho^{-1},$$

(3)

the Hermitian Hamiltonian $h$ is equivalent to $H$ in that it has the same eigenvalue spectrum. Thus, although the eigenvalue spectra of $h$ and $H$ are identical, relations between their eigenvectors will differ

$$|\psi^h_n\rangle = \rho |\phi^H_n\rangle.$$ 

(4)

All these efforts have been devoted to study time-independent non-Hermitian systems. In contrast, time-dependent non-Hermitian systems are far less well investigated [10, 11, 12, 13, 14, 16, 17, 18, 19, 20] and it appears that so far no consensus has been reached about a number of central issues. Unexpectedly, a number of conceptual difficulties have been encountered. Serious problems have arisen, first of all, in connection with the probabilistic and unitary-evolution interpretation of the generalized models. The treatment for systems with time-dependent non-Hermitian Hamiltonians with time-dependent metric operators is still controversially discussed and was the center of an interesting debate between Mostafazadeh and Znojil [10, 11, 12].

In conventional quantum mechanics, the spectral problem for a Hamiltonian or energy operator is approached by the stationary Schrödinger equation. The general equation of motion is given by the time-dependent Schrödinger equation that describes how a quantum system evolves with time. In this work, we consider the most general non-Hermitian time-dependent Hamiltonian $H(t)$ and its associated time-dependent metric operator $\eta(t)$.

The main assumption to be made is that the two time-dependent Schrödinger equations still holds

$$H(t) \left| \Phi^H(t) \right\rangle = i\hbar \partial_t \left| \Phi^H(t) \right\rangle,$$

(5)

$$h(t) \left| \Psi^h(t) \right\rangle = i\hbar \partial_t \left| \Psi^h(t) \right\rangle,$$

(6)

both Hamiltonians involved are explicitly time dependent, with $H(t)$ being to be non-Hermitian whereas $h(t)$ is taken Hermitian, i.e., $H(t) \neq H^\dagger(t)$ and $h(t) = h^\dagger(t)$. Next, we assume that the two solutions $\left| \Phi^H(t) \right\rangle$ and $\left| \Psi^h(t) \right\rangle$ of Eqs. (5) – (6) are related by a time-dependent invertible operator $\rho(t)$ as

$$\left| \Psi^h(t) \right\rangle = \rho(t) \left| \Phi^H(t) \right\rangle,$$

(7)
it then follows immediately by direct substitution of (7) into Eqs. (5) and (6) that the two Hamiltonians are allied to each other as
\[ h(t) = \rho(t) H(t) \rho^{-1}(t) + i\hbar \dot{\rho}(t) \rho^{-1}(t), \] (8)
The key feature in this equation is the fact that \( H(t) \) is no longer quasi-Hermitian, i.e. related to \( h(t) \) by means of a similarity transformation, due to the presence of the last term. Thus \( H(t) \) is not a self-adjointed operator and therefore not observable \[10, 19, 20\]. From the relation (8) and using the Hermiticity of \( h(t) \), we deduce a relation between \( H(t) \) and its Hermitian conjugate \( H^\dagger(t) \)
\[ H^\dagger(t) = \eta(t) H(t) \eta^{-1}(t) + i\hbar \dot{\eta}(t) \eta^{-1}(t), \] (9)
the relation (9) between the Hamiltonian \( H(t) \) and its Hermitian conjugate \( H^\dagger(t) \) generalizes the well known standard quasi-Hermiticity relation (1) in the context time-independent non-Hermitian quantum mechanics \[10, 19, 20\]. Quasi-Hermitian operators are very special class of pseudo-Hermitian operators. Their importance in physics was emphasized by Scholtz et al in \[9\].

In conventional pseudo-Hermitian (or \( PT \)-symmetric) theory, when the spectrum of a non-Hermitian Hamiltonian is purely real the Hamiltonian operator determines this spectrum through the stationary Schrödinger equation, and when a non-Hermitian Hamiltonian is time-dependent we show that the phases obtained during the evolution fit the bill.

This work proceeds to investigate in detail the main frames of time-dependent non-Hermitian systems and goes on to examine how the reality of their phases can be established. Finally, the original contribution is based on the definition of pseudo-Hermitian invariant operators, demonstrating a method to calculate how a quantum system evolves in time with a real phase.

To further elaborate our theoretical proposal, we revisit in Section 2 the Lewis and Riesenfeld invariant theory problem for an Hermitian harmonic oscillator systems \[21\] and we investigate a proper mapping between conventional invariant theory and pseudo-invariant theory. In Section 3, we illustrate our time-dependent pseudo-invariant theory by adopting a simple example: a harmonic oscillator with a time-dependent frequency under the action of a time-dependent imaginary linear potential.

### 2 Invariant operator method

Here we discuss the advantages of using Lewis and Riesenfeld invariant operator method in explicitly time-dependent quantum systems by giving a brief review \[21\]. We consider a system whose Hamiltonian \( h(t) \) is Hermitian and explicitly time dependent. A Hermitian operator \( I_h(t) \) is called an invariant for the system if it satisfies
\[
\frac{dI_h(t)}{dt} = \frac{\partial I_h(t)}{\partial t} - \frac{i}{\hbar} [I_h(t), h(t)] = 0. \tag{10}
\]
The eigenvalue equation of \( I_h(t) \) can be written as
\[
I_h(t) \left| \psi_n^h(t) \right> = \lambda_n \left| \psi_n^h(t) \right> . \tag{11}
\]
With the help of Eq. (10), it is easy to show that the real eigenvalues \( \lambda_n \) are time-independent. The Schrödinger equation (6) for the system has particular solutions \( \left| \Psi_n^h(t) \right> \) different from
\[ |\psi_n^h(t)\rangle \] in Eq. (11) only by a phase factor \( e^{i\varepsilon_n(t)} \) where the phase \( \varepsilon_n(t) \) is given by

\[
\hbar \frac{d}{dt} \varepsilon_n(t) = \langle \psi_n^h(t) | i\hbar \frac{\partial}{\partial t} - h(t) | \psi_n^h(t) \rangle. \quad (12)
\]

The first special physical system to which Lewis and Riesenfeld [21] have applied their general result is that of a time-dependent harmonic oscillator for which the frequency parameter is allowed to vary with time

\[ h_{osc}(t) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2(t) x^2. \quad (13) \]

They derive an exact invariant for this system by means of the equation (10), that is

\[ I_{hosc}(t) = \sigma^2(t) p^2 - m \sigma(t) \dot{\sigma}(t) [px + xp] + \frac{1}{\sigma^2(t)} \left[ 1 + m^2 \sigma^2(t) \dot{\sigma}(t) \right] x^2, \quad (14) \]

where \( \sigma(t) \) satisfies the non-linear auxiliary equation

\[
\ddot{\sigma}(t) + \sigma(t) \omega^2(t) = \frac{1}{m^2 \sigma^3(t)}. \quad (15)
\]

Then, the eigenstates and eigenvalues of this invariant are [21][22]

\[
\psi_n^{hosc}(x,t) = \left[ \frac{1}{n! 2^n \sigma \sqrt{\pi \hbar}} \right]^\frac{3}{2} \exp \left[ \frac{im}{2\hbar} \left( \frac{\dot{\sigma}}{\sigma} + \frac{i}{m \sigma^2} \right) x^2 \right] H_n \left[ \left( \frac{1}{\hbar} \right)^\frac{3}{2} \frac{x}{\sigma} \right], \quad \lambda_n = \hbar(n + \frac{1}{2}) \quad (16)
\]

where \( H_n \) is the usual Hermite polynomial of order \( n \), and the appropriate time-dependent phase factor that make the eigenstates solutions of the Schrödinger equation is

\[
\varepsilon_n(t) = - \left( n + \frac{1}{2} \right) \int_0^t \frac{1}{m \sigma^2(t')} dt'. \quad (17)
\]

Now we proceed introducing and analyzing the spectral properties of pseudo-Hermitian invariant operator \( I_{PH}(t) \). Particular attention is given to the special subset of quasi-Hermitian operators. We start by considering a non-Hermitian quantum mechanics in its most general form by studying time-dependent Hamiltonian operators \( H(t) \) and also time-dependent metric operator \( \eta(t) = \rho(t) \) associated with \( H(t) \). In the study of the time evolution problem, let us admit that a time-dependence occurs in all the operators. A non-Hermitian operator \( I_{PH}(t) \) is said to be a pseudo-Hermitian operator if it satisfies

\[
I_{PH}(t) = \eta(t) I_{PH}(t) \eta^{-1}(t) \Leftrightarrow I_h(t) = \rho(t) I_{PH}(t) \rho^{-1}(t) = I_{h}^\dagger(t). \quad (18)
\]

The virtue of such a conjugate pair \( I_h(t) \) and \( I_{PH}(t) \) is that they possess an identical eigenvalue spectrum because the invariants lie in the same similarity class. The reality of the spectrum is guaranteed, since one of the invariants involved, i.e. \( I_h(t) \), is Hermitian. It means that any self-adjointed invariant operator \( I_h(t) \), i.e. observable, in the Hermitian system possesses an invariant counterpart \( I_{PH}(t) \) in the non-Hermitian system given by \( I_{PH}(t) = \rho^{-1}(t) I_h(t) \rho(t) \) in complete analogy to the time-independent scenario for any self-adjoint operator.
The corresponding eigenvalue equations are then simply
\[ I_h(t) \left\langle \psi_n^h(t) \right| = \lambda_n \left| \psi_n^h(t) \right\rangle, \] \[ I_{PH}(t) \left| \phi_{n}^{PH}(t) \right\rangle = \lambda_n \left| \phi_{n}^{PH}(t) \right\rangle, \] (19)
where the eigenfunctions \( \left| \psi_n^h(t) \right\rangle \) and \( \left| \phi_{n}^{PH}(t) \right\rangle \) are related as
\[ \left| \psi_n^h(t) \right\rangle = \rho(t) \left| \phi_{n}^{PH}(t) \right\rangle. \] (20)
The eigenstates and eigenvalues of the invariant operator \( I_{PH}(t) \) satisfies
\[ \langle \phi_{m}^{PH}(t) | \phi_{n}^{PH}(t) \rangle \eta = \langle \phi_{m}^{PH}(t) | \eta \rangle | \phi_{n}^{PH}(t) \rangle = \delta_{mn}. \] (21)
It is easy to verify by direct computation that the \( I_{PH}(t) \) defined by Eq. (18) satisfies
\[ \frac{\partial I_{PH}(t)}{\partial t} = \frac{i}{\hbar} [I_{PH}(t), H(t)], \] (22)
with a non-Hermitian Hamiltonian \( H(t) \), which govern the time evolution of Schrödinger equation, given by Eq. (5). The eigenstates and eigenvalues of the invariant operator \( I_{PH}(t) \) may be found by the same technique completely analogous to the method introduced above for the Hermitian case. It is, of course, natural to calculate the solution of the non-Hermitian time-dependent Schrödinger equation (5) as in the time-dependent Hermitian case.
The Schrödinger equation (5) for the system has particular solutions \( \left| \Phi_{n}^{H}(t) \right\rangle \) different from \( \left| \phi_{n}^{PH}(t) \right\rangle \) in Eq. (19) only by a phase factor \( e^{i\varepsilon_{n}^{PH}(t)} \) where the phase \( \varepsilon_{n}^{PH}(t) \) is given by
\[ \hbar \frac{d}{dt} \varepsilon_{n}^{PH}(t) = \langle \phi_{n}^{PH}(t) | \eta(t) \left[ i\hbar \frac{\partial}{\partial t} - H(t) \right] | \phi_{n}^{PH}(t) \rangle. \] (23)
In Eq. (23), the first term is parallel to a familiar non adiabatic geometrical phase and the second term represents the dynamical effect. It is the sum of these two terms that can ensure a real total phase \( \varepsilon_{n}^{PH}(t) \).
In the end, it is important to note that the Schrodinger equation for explicitly time-dependent Hamiltonian cannot be written in the form of an eigenvalue equation and therefore, in this case, nothing can be said about the spectrum of the Hamiltonian, and consequently, we are interested in its mean value. However, the invariant operator satisfies an eigenvalue equation with a real time-independent spectrum.

3 Time-dependent Harmonic oscillator with a complex time-dependent potential
As an application, we study an oscillator with time-dependent frequency under the action of a time-dependent imaginary linear potential, and we compare our results with those obtained in Ref. [13]
\[ H(t) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2(t) x^2 + i\lambda(t) x, \] (24)
where \( \lambda(t) \) is a real time-dependent function.
Let us take the metric operator in the form
\[ \eta^{-1} (t) = \exp \frac{1}{\hbar} [\alpha (t) p + \beta (t) x], \tag{25} \]
where \( \alpha (t) \) and \( \beta (t) \) are unknown real time-dependent functions.

Using Eq. (25), we obtain
\begin{align*}
\dot{\alpha} (t) + \frac{\beta (t)}{m} &= 0, \\
m \omega^2 \alpha (t) + 2 \lambda (t) - \dot{\beta} (t) &= 0. 
\end{align*} \tag{26}

From the first equation \( \beta (t) = -m \dot{\alpha} (t) \), the second equation can be reduced to
\[ m \ddot{\alpha} (t) + m \omega^2 \alpha (t) + 2 \lambda (t) = 0. \tag{27} \]

Then the time-dependents metric operators \( \eta (t) \) is given by
\[ \eta (t) = \exp \left[ -\frac{\alpha (t)}{\hbar} p + \frac{m \dot{\alpha} (t)}{\hbar} x \right], \]
using the relation \( \eta (t) = \rho^\dagger (t) \rho (t) \) where \( \rho (t) \) is not unique and can be taken as a real operator
\[ \rho (t) = \exp \left[ -\frac{\alpha (t)}{2\hbar} p + \frac{m \dot{\alpha} (t)}{2\hbar} x \right]. \tag{28} \]

It can easily be shown that under the transformation \( \rho (t) \) defined in Eq. (28), the coordinate and momentum operators change according to
\[ \rho^{-1} (t) x \rho (t) = x - i \frac{\alpha}{2}, \quad \rho^{-1} (t) p \rho (t) = p - i \frac{m \dot{\alpha}}{2}. \tag{29} \]

An important property of the transformation \( \rho^{-1} (t) \), the action of which on a wave function in the \( x \)-representation reads
\[ \rho^{-1} G(x) = \exp \left[ \frac{\hbar}{i \alpha} \right] \exp \left[ -\frac{m \dot{\alpha} (t)}{2} x \right] G(x - i \frac{\alpha}{2}). \tag{30} \]

To affect the evaluation of the phase \( \phi_n^{PH} (t) \), we need to calculate the diagonal matrix elements of the operators \( H (t) \) and \( i \hbar \frac{\partial}{\partial t} \). That is
\begin{align*}
\langle \phi_n^{PH} (t) \mid \eta (t) \left[ i \hbar \frac{\partial}{\partial t} - H (t) \right] \mid \phi_n^{PH} (t) \rangle &= \langle \phi_n^{PH} (t) \mid \rho^\dagger (t) \rho (t) \left[ i \hbar \frac{\partial}{\partial t} - H (t) \right] \rho^{-1} (t) \rho (t) \mid \phi_n^{PH} (t) \rangle \\
&= \langle \phi_n^{PH} (t) \mid \rho^\dagger (t) \left( i \hbar \frac{\partial}{\partial t} - \rho (t) H (t) \rho^{-1} (t) \right) \rho (t) \mid \phi_n^{PH} (t) \rangle \\
&= \langle \phi_n^{PH} (t) \mid \rho^\dagger (t) \left( -i \hbar \rho (t) \frac{\partial}{\partial t} \rho^{-1} (t) \right) \rho (t) \mid \phi_n^{PH} (t) \rangle \tag{31} \end{align*}
using Eq. (29) we express \( \rho H (t) \rho^{-1} \) as

6
\[
\rho(t)H(t)\rho^{-1}(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(t)x^2 + i\left(\lambda + \frac{m\alpha\omega^2(t)}{2}\right)x \\
+ i\frac{\dot{\alpha}}{2}p - \left(m\alpha^2\frac{\omega^2(t)}{8} + m\alpha\omega^2(t) + \alpha\lambda\right),
\]

and taking the partial time derivative of \(\rho^{-1}\) we obtain the appropriate product

\[
i\hbar\rho(t)\dot{\rho}^{-1}(t) = i\frac{\dot{\alpha}}{2}p - im\frac{\dot{\alpha}}{2}x - m\frac{\alpha}{8}\left(\alpha^2 - \alpha\dot{\alpha}\right).
\]

The diagonal matrix elements of the operator \((i\hbar\frac{\partial}{\partial t} - H)\) can be simplified by using Eq. \((27)\)

\[
\langle \phi_n^{PH}(t) | \eta(t) \left[ i\hbar\frac{\partial}{\partial t} - H(t) \right] | \phi_n^{PH}(t) \rangle = \langle \phi_n^{PH}(t) | \rho(t) \left[ -i\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2(t)x^2 - \frac{\alpha\lambda}{4}\right) \right] \\
+ i\hbar\frac{\partial}{\partial t} \rangle \rho(t) | \phi_n^{PH}(t) \rangle.
\]

The key feature in this equation is the fact that the eigenstates of \(I_{PH}(t)\) are related to those of the Hermitian Invariant \(I_h(t)\) by \(|\phi_n^{PH}(t)\rangle = \rho^{-1}(t)|\psi_n^h(t)\rangle\), because \(I_{PH}(t)\) is quasi-Hermitian, i.e. related to \(I_h(t)\) by means of the similarity transformation \((18)\). Therefore, the time-dependent c-number \(\hbar\alpha(t)\lambda(t)/4\) can be removed by a time-dependent unitary transformation namely \(\exp\left[i\int_0^t \alpha(t)\lambda(t')dt'\right]\) to give the time derivative of the phase

\[
\langle \phi_n^{PH}(t) | \eta(t) \left[ i\hbar\frac{\partial}{\partial t} - H(t) \right] \left| \phi_n^{PH}(t) \right\rangle = \hbar\frac{d\epsilon_n(t)}{dt} = \langle \psi_n^h(t) | i\frac{\hbar}{2m} - \frac{1}{2}m\omega^2(t)x^2 \right| \psi_n^h(t) \rangle.
\]

We recognize the phase associated to the time-dependent one-dimensional harmonic oscillator system whose Hamiltonian operator is given in Section 2 by Eq. \((13)\) and therefore the associated Hermitian invariant operator \(I_h^{osc}\), its eigenstates \(\psi_n^{osc}(x,t)\) and the phase \(\epsilon_n\) are given by the equations \((13)\), \((16)\) and \((17)\) respectively.

By using the quasi-Hermiticity equation \((18)\), the pseudo-Hermitian invariant associated to the non-Hermitian Hamiltonian \(H(t)\) can be easily obtained

\[
I_{PH} = \sigma^2(t)\left(p - i\frac{m\alpha}{2}\right)^2 - m\sigma(t)\dot{\sigma}(t)\left(p - i\frac{m\alpha}{2}\right)\left(x - i\frac{\alpha}{2}\right) + \left(x - i\frac{\alpha}{2}\right)\left(p - i\frac{m\alpha}{2}\right) \\
+ \frac{1}{\sigma^2(t)}\left[1 + m^2\sigma^2(t)\dot{\sigma}^2(t)\right] - \left(x - i\frac{\alpha}{2}\right)^2.
\]

Thus, the phase of evolved state \(|\Phi_n^{PH}(t)\rangle\) are real and can be obtained with the help of Eqs. \((17)\) and \((35)\)

\[
\epsilon_n^{PH}(t) = -\int_0^t \left[n + 1\right] \frac{1}{m\sigma^2(t')} - \frac{\lambda(t')\alpha(t')}{4\hbar} \] \(dt'\).
However, the general solution for the time-dependent Schrödinger equation of the non-Hermitian Hamiltonian (24) is given by

$$\Phi^H_n(x,t) = \exp \left[ i \epsilon_n^H(t) \right] \psi_n^PH(x,t) = \exp \left[ -i \int_0^t \left( n + \frac{1}{2} \right) \frac{1}{m \sigma^2(t')} - \frac{\lambda(t') \alpha(t')}{4 \hbar} \right] dt' \psi_n^PH(x,t),$$

(38)

where

$$\psi_n^PH(x,t) = \rho^{-1}(t) \psi_{nosc}^H(x,t) = \exp \left[ \frac{i}{8} \frac{m \alpha(t)}{\hbar} \right] \exp \left[ - \frac{m \dot{\alpha}(t)}{2} x \right] \psi_{nosc}^H(x - i \frac{\alpha(t)}{2}, t),$$

(39)

are eigenfunctions of $I_{PH}(t)$ obtained by the inverse transformation on eigenfunctions (16) of $I_{osc}(t)$.

Before concluding let us make a few remarks about the nature of the solution in certain special cases. A particular example is a harmonic oscillator with a time-dependent $PT$-violating linear potential [13] where the frequency $\omega = \omega_0$ is constant and $\lambda(t) = at$. Here, the equations (15) and (27) for $\sigma(t)$ and $\alpha(t)$ can be explicitly solved to yield

$$\frac{1}{m \sigma^2} = \omega_0, \quad \alpha(t) = - \frac{2at}{m \omega_0^2}.$$  

(40)

Then, the phase (37) can be determined as

$$\epsilon_n(t) = - \left( n + \frac{1}{2} \right) \omega_0 t - \frac{a^2 t^3}{6 \hbar m \omega_0^3},$$

(41)

and our new wave function (38) reduces to those obtained in Ref. [13]

$$\Phi^H_n(x,t) = \exp \left[ i \epsilon_n(t) \right] \exp \left[ - \frac{1}{2 m \omega_0^2} \right] \exp \left[ \frac{1}{n! 2^n \sqrt{\pi \hbar}} \right] \exp \left[ - \left( \frac{m \omega_0}{2 \hbar} \right) \left( x + i \frac{at}{m \omega_0^2} \right)^2 \right] H_n \left[ \left( \frac{m \omega_0}{\hbar} \right)^{\frac{1}{2}} \left( x + i \frac{at}{m \omega_0^2} \right) \right],$$

(42)

where the phase functions $\epsilon_n(t)$ are given by Eq. (41).

4 Conclusion

In this work, we studied a class of general explicitly time-dependent non-Hermitian problems in quantum mechanics, e.g., those with a time-dependent pseudo-Hermitian invariant operator and a time-dependent metric $\eta(t)$ which have raised a controversy [10, 11, 12]. Because a non-Hermitian time-dependent Hamiltonian $H(t)$ whose associated Schrödinger equation (54) is mapped, by means of the time-dependent operator $\rho(t)$, into the Schrödinger equation (6), where the corresponding wave functions are transformed as $\langle \Phi^H(t) \rangle = \rho^{-1}(t) \langle \Psi^H(t) \rangle$ and the Hamiltonians are related by means of the time-dependent relation (8). Thus $H(t)$ and $h(t)$ are no longer related by a quasi-hermiticity transformation as in the completely time-independent case [23] or the time-dependent case with time-independent metric [14, 15], but instead their
mutual dependence involves an additional time-dependent term $-i\hbar \rho^{-1}(t) \dot{\rho}(t)$. The authors of [19, 20] refer to Eq. (8) as the time-dependent quasi-Hermiticity relation and of course the non-Hermitian Hamiltonian $H(t)$ does not belong to the set of observables in this system as it is not related to $\hbar(t)$ by a similarity transformation. In our circumstance, it is evident that the self-adjoint invariant operator $I_h(t)$ associated with the Hermitian Hamiltonian $h(t)$, i.e., an observable, in the Hermitian system has an invariant observable counterpart $I_{PH}(t)$ associated with a non-Hermitian Hamiltonian $H(t)$ in the non-Hermitian system related to each other as $I_h(t) = \rho(t) I_{PH}(t) \rho^{-1}(t)$, since

$$
\langle \phi^{H}_m | I_{PH} \phi^{H}_n \rangle = \langle I_{PH} \phi^{H}_m | \phi^{H}_n \rangle = \langle \psi^{h}_m | I_h \psi^{h}_n \rangle = \langle I_h \psi^{h}_m | \psi^{h}_n \rangle = \lambda_n \delta_{mn}
$$

both invariants $I_{PH}(t)$ and $I_h(t)$ possess an identical eigenvalue spectrum because the invariants lie in the same similarity class. The reality of the spectrum is guaranteed, since one of the invariants involved, i.e. $I_h(t)$, is Hermitian. We have shown that the evolved state of a time-dependent non-Hermitian quantum systems acquires a real phase during its evolution. Therefore, the Lewis and Riesenfeld phase is invariant under the transformation $\rho(t)$

$$
\hbar \frac{d}{dt} \varepsilon^{PH}(t) = \langle \phi^{PH}_m | \eta(t) \left[ i\hbar \frac{\partial}{\partial t} - H(t) \right] | \phi^{PH}_n \rangle = \langle \psi^{h}_n(t) \left[ i\hbar \frac{\partial}{\partial t} - h(t) \right] | \psi^{h}_n(t) \rangle = \hbar \frac{d}{dt} \varepsilon(t)
$$

(44)

This is due essentially to the derivation, for a pseudo-Hermitian invariant, of the Liouville equation (22) which is exactly similar to the Hermitian case (10) where $I_h(t)$ and $h(t)$ are replaced by $I_{PH}(t)$ and $H(t)$.

Finally, our formalism has been applied to find the solution of the harmonic oscillator with time dependent frequency under the action of a complex time-dependent linear potential. At this point, we are able to calculate the expectation value of the Hamiltonian of the system. For this we calculate the mean value of the Hamiltonian in a closed form, as usual, using the above result in Eq.(32)

$$
\langle H(t) \rangle_\eta = \langle \phi^{PH}_m | \eta(t) H(t) | \phi^{PH}_n \rangle = \langle \psi^{h}_n(t) | \rho(t) H(t) \rho^{-1}(t) | \psi^{h}_n(t) \rangle = \frac{\hbar}{2} \left( n + \frac{1}{2} \right) \left( \frac{\sigma^2}{m} + m \omega^2(t) \sigma^2 + \frac{1}{m \sigma^2} \right) - \left( m \frac{\alpha^2}{8} + m \alpha \omega^2(t) \sigma^2 + \frac{\alpha \lambda}{2} \right),
$$

(45)

where we have used the following mean values

$$
\langle \psi^{h}_n(t) | x | \psi^{h}_n(t) \rangle = \langle \psi^{h}_n(t) | p | \psi^{h}_n(t) \rangle = 0
$$

and (21)

$$
\langle \psi^{h}_n(t) \left| \frac{p^2}{2m} + \frac{1}{2} m \omega^2(t) x^2 \right| \psi^{h}_n(t) \rangle = \frac{\hbar}{2} \left( n + \frac{1}{2} \right) \left( \frac{\sigma^2}{m} + m \omega^2(t) \sigma^2 + \frac{1}{m \sigma^2} \right).
$$

For the particular case $\omega = \omega_0$ and $\lambda(t) = at$, Eq. (45) is reduced to

$$
\langle H(t) \rangle_\eta = \hbar \omega_0 \left( n + \frac{1}{2} \right) + \frac{a^2 \ell^2}{2 m \omega_0^2} \sigma - \frac{a^2}{2 m \omega_0^2},
$$

9
which coincides with the result obtained in Ref. [13] (for the choice $\dot{\eta} = 0$ in Eq. (28) in [13]). Then we have proven that the reality of the time-dependent mean value $\langle H(t) \rangle_{\eta}$ is maintained despite the non-hermiticity of the Hamiltonian.

Acknowledgments Two of the authors (A. B and M. M) would like to thank Professor Andreas Fring for the interesting discussions on the notion of the time-dependent quasi-Hermiticity during the PHHQP16 workshop.

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