COMPLETE INTEGRABILITY OF COHOMOGENEITY-ONE STRINGS IN $\mathbb{R}^{n,1}$ AND CANONICAL FORM OF KILLING VECTOR ALGEBRA

DAISUKE IDA

Abstract. The equation of motion for cohomogeneity-one Nambu-Goto strings in flat space $\mathbb{R}^{n,1}$ has been investigated. We first classify possible forms of the Killing vector fields in $\mathbb{R}^{n,1}$ after appropriate action of the Poincaré group. Then, all possible forms of the Hamiltonian for the cohomogeneity-one Nambu-Goto strings are determined. It has been shown that the system always has the maximum number of functionally independent, pair-wise commuting conserved quantities, i.e. it is completely integrable.

We have also determined all the possible coordinate forms of the Killing vector basis for the 2-dimensional non-commutative Lie algebra.

1. Introduction

According to the standard models of elementary particles, the quantum vacuum of certain scalar field depends on the environmental temperature. As a result, the scalar field suffers the phase transition with the cosmic temperature dropping. Such a phase transition does not occur uniformly in the universe, but different phases of quantum vacua form a domain structure. Hence, cosmic membranes and strings unavoidably appear at the borders of locally homogeneous regions as topological defects in quantum-vacuum configuration. These topological defects would play an important role in the formation of various inhomogeneous structures in the universe. With this background, the dynamics of relativistic extended objects such as strings or membranes has been investigated from various aspects.

In a certain situations, the dynamics of relativistic strings or membranes can approximately be described by the Nambu-Goto action, which is the main subject of the present paper. The Nambu-Goto strings and membranes are just the zero-mean-curvature time-like surface in the spacetime, so that they might be interesting objects of study also from mathematical viewpoint.

Since it is a system with the infinite degrees of freedom, the dynamics of the Nambu-Goto string or membrane is much more complicated as compared with the particle dynamics. However, the simplification of the equation of motion occurs when the background spacetime has a symmetry, and the string or membrane respects it \cite{1,2,3}. In particular, when the spacetime admits a Killing vector field, we can consider Nambu-Goto strings those world-sheets are invariant under the action of the isometry. Such strings are called cohomogeneity-one strings.

It is well-known that general form of the Nambu-Goto strings in flat spacetime $\mathbb{R}^{n,1}$ are just given by the $n + 1$ harmonic functions. This is because the equation of motion for the string is reduced to $n + 1$ linear wave equations when isothermal coordinates are taken as
the world-sheet coordinates. This approach however would not be suitable for finding the cohomogeneity-one strings.

Our purpose here is to understand the entity of the cohomogeneity-one strings in $\mathbb{R}^{n,1}$. Since there are maximum number of Killing vector fields in $\mathbb{R}^{n,1}$, a systematic classification of a Killing vector field is required for this problem. The classification of the cohomogeneity-one strings in $\mathbb{R}^{3,1}$ has been performed by [4]. In the flat spacetime, a Killing vector field can always be written in inertial coordinate system as $\xi_\nu = F_{\mu\nu}x^\mu + f_\nu$, in terms of a constant alternative matrix $F_{\mu\nu}$ and a constant vector $f_\nu$. Any pair $(F_{\mu\nu}, f_\nu)$ corresponds to a Killing vector field in a certain inertial coordinate system, and it undergoes the action of Poincaré group under the change of the inertial coordinate system. The action of Poincaré group defines the equivalence relation in the space of pairs $(F_{\mu\nu}, f_\nu)$. We first enumerate the equivalence classes of pairs $(F_{\mu\nu}, f_\nu)$, which give the canonical classification of Killing vector fields in $\mathbb{R}^{n,1}$.

It has also been shown in [4] that the equation of motion for cohomogeneity-one strings in $\mathbb{R}^{3,1}$ are all completely integrable, so that every solution to it is given by quadrature. We will extend this result to the case of $\mathbb{R}^{n,1}$.

In the course of the analysis of the Killing vector fields in $\mathbb{R}^{n,1}$, we also obtained the canonical classification of the 2-dimensional non-abelian Lie algebra of Killing vector fields, which will also be reported here.

The organization of this paper is as follows: In Sec. 2, the general properties of the Killing vector fields in $\mathbb{R}^{n,1}$ is described. In Sec. 3, the constant 2-form $F_{\mu\nu}$, which characterizes the Killing vector field, in $\mathbb{R}^{n,1}$ under the action of the Lorentz group are investigated. In Sec. 4, the canonical classification of the Killing vector field in $\mathbb{R}^{n,1}$ is given. In Sec. 5, the complete integrability of the cohomogeneity-one Nambu-Goto strings is shown. In Sec. 6, the canonical form of the representation of the 2-dimensional noncommutative Lie algebra of Killing vector fields in $\mathbb{R}^{n,1}$ is obtained. In Sec. 7, we summerize our work.

2. **Killing vector fields in Minkowski spacetime**

We first review elementary properties of Killing vectors in the Minkowski spacetime.

We denote the $(n+1)$-dimensional Minkowski spacetime by $(\mathbb{R}^{n,1}, \eta)$. Let $(x^\mu)_{\mu=0,1,...,n}$ be the inertial coordinate system for $\mathbb{R}^{n,1}$, where the metric takes the form

$$\eta = -(dx^0)^2 + \sum_{i=1}^{n} (dx^i)^2.$$  

A Killing vector field on $\mathbb{R}^{n,1}$, which is subject to the partial differential equation

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0,$$

is generally written as

$$\xi = F_{\mu\nu}x^\mu dx^\nu + f_\nu dx^\nu,$$

in terms of the constant alternating matrix $F_{\mu\nu}$ and the constant vector $f_\nu$.

Our first task is to classify all the canonical form of $\xi$, under the action of the Poincaré group (or the inhomogeneous Lorentz group) $IO_{n,1}$.
The Poincaré transformation is given by the coordinate transformation

\[ x'^\mu = \Lambda^\mu_\nu x^\nu + c^\mu \]

between inertial coordinate systems. Accordingly, the Killing vector field transforms as

\[ \xi' = F'_{\mu\nu} x'^\mu dx'^\nu + f'_\nu dx'^\nu, \]
\[ F'_{\mu\nu} = \Lambda^\lambda_\mu \Lambda^\rho_\nu F_{\lambda\rho}, \]
\[ f'_\nu = \Lambda^\lambda_\nu f_\lambda - F'_{\mu\nu} c^\mu. \]

This can be interpreted as that the element \( g(\Lambda, c) \in IO_{n,1} \) of the Poincaré group acts on \( F_{\mu\nu} \) and \( f_\nu \) as

\[ g(\Lambda, c) : (F_{\mu\nu}, f_\nu) \mapsto \left( \Lambda^\lambda_\mu \Lambda^\rho_\nu F_{\lambda\rho}, \Lambda^\lambda_\nu f_\lambda - \Lambda^\lambda_\mu \Lambda^\rho_\nu F_{\lambda\rho} c^\mu \right), \]

which gives a representation of \( IO_{n,1} \). Hence, in particular, \( F_{\mu\nu} \) transforms as a Lorentz tensor.

3. Classification of constant 2-forms

Each Killing vector is characterized by a constant alternative matrix \( F_{\mu\nu} \) and a constant vector \( f_\nu \), and these undergo the Poincaré transformation. We want to enumerate the Killing vectors in the Minkowski spacetime \( \mathbb{R}^{n,1} \) up to the Poincaré transformation. In particular, the constant alternative matrix \( F_{\mu\nu} \) transforms as a 2-form. In the following, we give a canonical form of \( F_{\mu\nu} \) under the action of the homogeneous Lorentz transformation.

We first show the following useful fact.

**Proposition 1.** Every constant 2-form \( F \) in \( \mathbb{R}^{n,1} \) can be made into the form

\[ F = \sum_{k=0}^{n-1} u_k dx^k dx^{k+1}, \]

under the action of the special orthogonal group \( SO_n \), which preserves \( \sum_{i=1}^n (x^i)^2 \).

**Proof.** A constant 2-form \( F \) can be written as

\[ F = dx^0 \wedge \omega_1 + F_1, \]
\[ \omega_1 = \sum_{i=1}^n (\omega_1)_i dx^i, \]
\[ F_1 = \sum_{1 \leq i < j \leq n} (F_1)_{ij} dx^i dx^j. \]

There is an \( SO_n \) action preserving \( \sum_{i=1}^n (x^i)^2 \), such that \( \omega_1 \) takes the form

\[ \omega_1 = u_0 dx^1. \]

Under such a transformation, \( F \) becomes

\[ F = u_0 dx^0 dx^1 + F_1, \]

where \( F_1 \) is generated only by \( dx^1, \ldots, dx^n \).
In a similar manner, $F_1$ can be written, under the appropriate $SO_{n-1}$ transformation preserving $\sum_{k=2}^{n}(x^k)^2$, in the form

$$F_1 = u_1dx^1dx^2 + F_2,$$

where $F_2$ is generated by $dx^3, \ldots, dx^n$.

We can continue recursively until we finally get

$$F = u_0dx^0dx^1 + u_1dx^1dx^2 + \cdots + u_{n-1}dx^{n-1}dx^n.$$

□

Hence all the information of a constant 2-form is encoded in the array $(u_0, \ldots, u_{n-1})$, however, it is still not canonical in the sense that different arrays may give equivalent 2-forms. In the following, we remove such redundancy along similar lines as in the anti-de Sitter case [5, 7, 6].

We denote by $\sharp F$ the linear operator on the complexified vector space $\mathbb{C}^{n,1} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{n,1}$, or explicitly $(\sharp F v)^\mu = F^\mu_v v^\nu$ for $v \in \mathbb{C}^{n,1}$.

The following Lemmas are useful for the classification of the constant 2-form in $\mathbb{R}^{n,1}$. The argument here is mostly taken from Ruse [8].

**Lemma 1.** If $\lambda$ is an eigenvalue of the linear operator $\sharp F$, which corresponds to the constant 2-form $F$, then $-\lambda$, $\lambda$, and $\overline{-\lambda}$ are eigenvalues of $\sharp F$.

**Proof.** By regarding the 2-form $F$ as a real, alternating matrix, the statement of the Lemma follows from

$$0 = \det(\lambda\eta - F) = \det(\lambda\eta + F)$$

$$= \overline{\det(\overline{\lambda} \eta - F)}$$

$$= \det(\overline{\lambda} \eta + F).$$

□

**Lemma 2.** Let $\lambda$ and $\mu$ be eigenvalues of $\sharp F$, such that $\lambda + \mu \neq 0$, and let $v, w$ be eigenvector corresponding to $\lambda$, $\mu$, respectively. Then, $v$ and $w$ are mutually orthogonal with respect to the bilinear form $\eta$.

**Proof.** The statement of the Lemma follows from

$$\mu\eta(v, w) = \eta(v, \sharp F w) = \eta((\sharp F)^Tv, w) = -\eta(\sharp F v, w) = -\lambda\eta(v, w)$$

$$\therefore (\lambda + \mu)\eta(v, w) = 0.$$

□

**Corollary 1.** An eigenvector corresponding to a nonzero eigenvalue of $\sharp F$ is a null vector.

Note that the null eigenvector here may be a complex null vector such as $\partial_1 + i\partial_2$.

**Lemma 3.** Each eigenvalue of $\sharp F$ is real or pure imaginary.
Proof. Suppose that $\sharp F$ has an eigenvalue $\lambda = a + ib$ ($a, b \neq 0$). Let $v$ be an eigenvector corresponding to $\lambda$. By Corollary, $v$ is a null vector. By reality of $\sharp F$, $\overline{v}$ is an eigenvector corresponding to the eigenvalue $\overline{\lambda}$. Since $\lambda + \overline{\lambda} \neq 0$, $v$ and $\overline{v}$ are mutually orthogonal by Lemma. This is possible only when $v$ is a real null vector.

However, it is impossible since, for the real eigenvector $v$, $v$ should simultaneously be an eigenvector corresponding to $\lambda$ and $\overline{\lambda}$, which leads to a contradiction. □

Note that this Lemma also is specific to the Lorentzian signature case. When the underlying space is $\mathbb{R}^{n,m}$ ($n, m \geq 2$), $\sharp F$ may have general complex eigenvalues (compare with the anti-de Sitter case [6]).

**Lemma 4.** Every eigenvector corresponding to a nonzero real eigenvalue of $\sharp F$ is a real null vector.

**Proof.** Let $\lambda$ be a nonzero real eigenvalue of $\sharp F$, and let $v$ be an eigenvector corresponding to $\lambda$. By reality of $\sharp F$, $\overline{v}$ also is an eigenvector corresponding to $\lambda$. By Lemma, $v$ and $\overline{v}$ are mutually orthogonal with respect to $\eta$. This occurs only when $v$ is a real null vector. □

**Lemma 5.** If $\sharp F$ has a nonzero real eigenvalue $a$, $\sharp F$ does not have nonzero real eigenvalues other than $a$ or $-a$.

**Proof.** Let $a$ and $a'$ ($a \neq a'$) be nonzero real eigenvalues of $\sharp F$, and let $v$ and $w$ be eigenvectors corresponding to $a$ and $a'$, respectively. By Lemma, $v$ and $w$ are real null vectors. This implies that $a' = -a$, since otherwise, by Lemma, $v$ and $w$ are mutually orthogonal with respect to $\eta$, which is impossible for linearly independent real null vectors $v$ and $w$. □

**Lemma 6.** An eigenvector corresponding to a nonzero pure imaginary eigenvalue of $\sharp F$ is a complex null vector.

**Proof.** Let $ib \neq 0$ be a pure imaginary eigenvalue of $\sharp F$, and let $u$ be an corresponding eigenvector. By Corollary, $u$ is a null vector. Since $\overline{u}$ is an eigenvector corresponding to the different eigenvalue $-ib$ of $\sharp F$, $u$ and $\overline{u}$ should be linearly independent. Hence $u$ must be a complex null vector. □

The canonical forms of the constant 2-form $F$ can be obtained by choosing coordinate basis vectors respecting eigenvectors of $\sharp F$. When $\sharp F$ has a nonzero real eigenvalue, this procedure is particularly simple.

**Lemma 7.** If $\sharp F$ has a nonzero real eigenvalue $a$, $F$ can be brought into the form

$$F = adx^0 dx^1,$$

or

$$F = adx^0 dx^1 + \sum_{k=1}^{r} b_k dx^2 dx^{2k+1}, \quad (b_1 \geq b_2 \geq \cdots \geq b_r > 0)$$

by an action of the Lorentz group $O_{n,1}$. 
Proof. Let $v$ be an eigenvector corresponding to the nonzero eigenvalue $a$. By Corollary 1, $v$ is a real null vector. We can choose an inertial coordinate system, under which $v$ takes the form

$$v = \frac{1}{\sqrt{2}}(\partial_0 - \partial_1).$$

The 2-form $F$ generally takes the form

$$F = f(dx^0 + dx^1)dx^i + \sum_{2 \leq i < j \leq n} H_{ij}dx^idx^j.$$

However, since $v$ is an eigenvector corresponding to $a$, $f = a$ and $h_i = 0$ ($i = 2, 3, \ldots, n$) holds, so that we have

$$F = adx^0dx^1 + \sum_{i=2}^{n} g_i(dx^0 + dx^1)dx^i + \sum_{2 \leq i < j \leq n} H_{ij}dx^idx^j.$$

Under the Lorentz transformation

$$x^0 = \left(1 + \frac{\beta^2}{2}\right)x^0 + \frac{\beta^2}{2}x^1 + \sum_{k=2}^{n} \beta_k x^k,$$

$$x^1 = -\frac{\beta^2}{2}x^0 + \left(1 - \frac{\beta^2}{2}\right)x^1 - \sum_{k=2}^{n} \beta_k x^k,$$

$$x^i = x^i + \beta_i(x^0 + x^2), \quad (i = 2, 3, \ldots, n)$$

where $\beta^2 = \sum_{k=2}^{n}(\beta_k)^2$, $F$ transforms as

$$F \mapsto adx^0dx^1 + \sum_{2 \leq i < j \leq n} H_{ij}dx^idx^j + \sum_{i=2}^{n} \left[\sum_{j=2}^{n} (a\delta_{ij} + H_{ij})\beta_i + g_i\right](dx^0 + dx^1)dx^i.$$

Noting that $(a\delta_{ij} + H_{ij})$ constitutes a regular matrix, the third term above can be set to zero, by appropriately choosing $\beta_j$.

Finally, by an $O_{n-1}$ transformation preserving $\sum_{d=2}^{n}(x^d)^2$, the second term above can be made into the block diagonal form as

$$\sum_{2 \leq i < j \leq n} H_{ij}dx^idx^j \mapsto \sum_{i=1}^{r} b_idx^{2i}dx^{2i+1},$$

such that $b_1 \geq b_2 \geq \cdots \geq b_r > 0$ holds, whenever it is nonzero. \hfill $\Box$

Then, the following Lemma immediately follows.

Lemma 8. If $\sharp F \in \text{End}(\mathbb{C}^{n,1})$ is surjective, then $n$ is odd and $F$ can be written as

$$F = adx^0dx^1 + \sum_{i=1}^{(n-1)/2} b_idx^{2i}dx^{2i+1},$$

in an appropriate inertial coordinate system.

Proof. Since $\ker \sharp F = \{0\}$, every eigenvalue of $\sharp F$ is nonzero, real or pure imaginary number by Lemma 3.
Consider the characteristic polynomial
\[ p(x) = \det(xI - \sharp F). \]
Since this is a real polynomial, the number of pure imaginary roots counting multiplicity is even. According to Lemma 5, the number of nonzero real eigenvalues is 0 or 2. Hence the degree of \( p(x) \), which is \( n + 1 \), must be even.

If \( p(x) \) does not have no real roots, it is written as
\[ p(x) = \prod_{i=1}^{(n+1)/2} \left( x^2 + b_i^2 \right), \]
where every \( b_i \) is real. However, it is impossible since
\[ p(0) = \prod_{i=1}^{(n+1)/2} b_i^2 > 0, \]
while
\[ p(0) = \det \sharp F = - \det F \leq 0. \]
Hence, \( \sharp F \) has nonzero real roots. Then, the statement of the Lemma immediately follows from Lemma 7.

Now we are in a position to classify the canonical forms of the constant 2-form \( \sharp F \), according to the causal nature of \( \ker \sharp F \). Since \( \sharp F \) is real, when \( u \in \ker \sharp F \), its complex conjugate also belongs to \( \ker \sharp F \). This implies that \( \ker \sharp F \) is the complexification of a subspace of \( \mathbb{R}^{n,1} \). We first classify a subspace of \( \mathbb{R}^{n,1} \) according to its causal nature.

**Definition 1** (subspace of \( \mathbb{R}^{n,1} \)). A nonzero subspace of \( \mathbb{R}^{n,1} \) is classified into the following types.

1. A subspace generated by mutually orthogonal, a real timelike vector and \( m \) real spacelike vectors is called an \((m + 1)\)-dimensional timelike subspace, where \( m = 0, 1, \ldots, n \).
2. A subspace generated by mutually orthogonal, \( m \) real spacelike vectors is called an \( m \)-dimensional spacelike subspace, where \( m = 1, 2, \ldots, n \).
3. A subspace generated by mutually orthogonal, a real null vector and \( m \) real spacelike vectors is called an \((m + 1)\)-dimensional null subspace, where \( m = 0, 1, \ldots, n-1 \).

According to the causal nature of \( \ker \sharp F \), the constant 2-form \( F \) can be classified as follows.

**Proposition 2.** If \( \ker \sharp F \) is a complexified spacelike subspace of \( \mathbb{R}^{n,1} \), a constant 2-form \( F \) is brought into the form
\[ F = adx^0dx^1 \quad (a > 0) \]
or
\[ F = adx^0dx^1 + \sum_{i=1}^{r} b_i dx^2_i dx^{2i+1} \quad (a > 0, b_1 \geq b_2 \geq \cdots \geq b_r > 0) \]
under an action of the Lorentz group \( O_{n,1} \).
Proof. We can take an inertial coordinate system for $\mathbb{R}^{n,1}$, such that $\partial_{m+2}, \partial_{m+3}, \ldots, \partial_n$ generate ker $\sharp F$. Then, $\partial_0, \partial_1, \ldots, \partial_{m+1}$ generates $\text{Im} \, \sharp F$, which is isomorphic to the quotient vector space $\mathbb{C}^{n,1}/\text{ker} \, \sharp F$, and $\text{Im} \, \sharp F$ is identified with a complexified vector space $\mathbb{C}^{m,1}$. In this coordinate system, $F$ takes the form

$$F = \sum_{0 \leq i, j \leq m+1} F_{ij} dx^i dx^j,$$

and $\sharp F$ is regarded as a surjective linear operator on $\text{Im} \, \sharp F$. By Lemma 8, the complex dimension of $\text{Im} \, \sharp F$ should be even, or $m = 2r$ ($r = 0, 1, \ldots$), and $F$ can be brought into the form

$$F = a dx^0 dx^1,$$

when $r = 0$, or otherwise

$$F = ad x^0 dx^1 + \sum_{i=1}^r b_i dx^{2i-1} dx^{2i}, \quad (b_1 \geq b_2 \geq \cdots \geq b_r > 0)$$

by an $O_{2r+1,1}$ transformation. Furthermore, we can always make $a$ positive. □

Proposition 3. If ker $\sharp F$ is a complexified timelike subspace of $\mathbb{R}^{n,1}$, a constant 2-form $F$ is brought into the form

$$F = \sum_{1 \leq \mu < \nu \leq m} b_{\mu \nu} dx^\mu dx^\nu,$$

under an action of the Lorentz group $O_{n,1}$.

Proof. We can always choose an inertial coordinate system such that $\partial_0, \partial_{m+1}, \partial_{m+2}, \ldots, \partial_n$ generate ker $\sharp F$ (of course, we take ker $\sharp F = \mathbb{C} \partial_0$ if the kernel is 1-dimensional). Then, $F$ takes the form

$$F = \sum_{1 \leq \mu < \nu \leq m} F_{\mu \nu} dx^\mu dx^\nu.$$

Hence it can be brought into the form

$$F = \sum_{i=1}^r b_i dx^{2i-1} dx^{2i}, \quad (b_1 \geq b_2 \geq \cdots \geq b_r > 0)$$

by an $O_m$ transformation. □

Proposition 4. If ker $\sharp F$ is a complexified null subspace of $\mathbb{R}^{n,1}$, a constant 2-form $F$ is brought into the form

$$F = dx^0 dx^1 + dx^1 dx^2$$

or

$$F = dx^0 dx^1 + dx^1 dx^2 + \sum_{i=1}^r b_i dx^{2i-1} dx^{2i+2}, \quad (b_1 \geq b_2 \geq \cdots \geq b_r > 0)$$

under an action of the Lorentz group $O_{n,1}$. 
Proof. Since $\ker \hat{F}$ is a complexified null subspace of $\mathbb{R}^{n,1}$, $\hat{F}$ has an real null eigenvector corresponding to the eigenvalue 0. We choose an inertial coordinate system for $\mathbb{R}^{n,1}$, such that $\ell = \partial_0 + \partial_2$ is the null eigenvector.

In this coordinate system, $F$ generally takes the form

$$F = \sum_{i \neq 0,2} g_i (dx^0 - dx^2) dx^i + \frac{1}{2} \sum_{i,j \neq 0,2} H_{ij} dx^i dx^j,$$

where $h_{ij} = -h_{ji}$. This is the general form of the 2-form, such that $\ell \in \ker \hat{F}$. Furthermore, in order for $\ker \hat{F}$ to be the complexified null subspace of $\mathbb{R}^{n,1}$, the 1-form $g = \sum_{k \neq 0,2} g_k dx^k$ should not be contained in $\text{Im} H$, where $H = (1/2) \sum_{i,j = 0,2} H_{ij} dx^i dx^j$ is regarded as a linear operator from the tangent vector space into the space of 1-forms. The reason is as follows.

Suppose that there is a vector $v = \sum_{k \neq 0,2} v^k \partial_k$ such that $g_i = \sum_{j \neq 0,2} H_{ij} v^j$ holds for $i \neq 0,2$. Then, it is easily confirmed that $u = \partial_0 + \sum_{k \neq 0,2} v^k \partial_k$ belongs to $\ker \hat{F}$. On the other hand, since $\ker \hat{F} \cap \mathbb{R}^{n,1}$ is a null hypersurface in $\mathbb{R}^{n,1}$, every pair of vectors in $\ker \hat{F}$ should be mutually orthogonal. Hence it leads to a contradiction, since $\ell, u \in \ker \hat{F}$, while $\eta(\ell, u) = -1 \neq 0$.

In particular, $H$ is not surjective as a linear operator on the $(x^1, x^3, \ldots, x^n)$-plane, so that by $O_{n-1}$ transformation on this plane, it may be brought into the form

$$H = \frac{1}{2} \sum_{i=1}^r b_i dx^{2i+1} dx^{2i+2}, \quad (b_1 \geq b_2 \geq \cdots \geq b_r > 0)$$

whenever it is not zero.

Under the Lorentz transformation

$$\begin{align*}
x'^0 &= \left(1 + \frac{\beta_2^2}{2}\right) x^0 - \frac{\beta_2^2}{2} x^2 + \sum_{i \neq 0,2} \beta_i x^i, \\
x'^2 &= \frac{\beta_2^2}{2} x^0 + \left(1 - \frac{\beta_2^2}{2}\right) x^2 + \sum_{i \neq 0,2} \beta_i x^i, \\
x'^i &= x^i + \beta_i (x^0 - x^2), \quad (i \neq 0, 2)
\end{align*}$$

where $\beta^2 = \sum_{k \neq 0,2} (\beta_k)^2$, $H$ is invariant, while $g$ transforms as

$$g \mapsto g = \sum_{i = 0,2} g_i \left( g_{i} - \sum_{j \neq 0,2} H_{ij} \beta_j \right) dx^i.$$ 

In other words, $g$ can be shifted by any vector in $\text{Im} H$. Hence by appropriately choosing $\beta_i$, $g$ can be brought into the form

$$g = g_1 dx^1 + \sum_{i = 2r+3}^n g_i dx^i,$$

where the second term in the right hand side may be absent when $2r + 2 = n$. Note that this is not zero, since otherwise $g \in \text{Im} H$. Then, by an $SO_{n-2r-1}$ (or $\mathbb{Z}_2$ if $2r + 2 = n$) transformation preserving $(x^1)^2 + \sum_{i = 2r+3}^n (x^i)^2$, it becomes

$$g \mapsto ||g|| x^1, \quad \left(||g|| = \sqrt{(g_1)^2 + \sum_{i = 2r+3}^n (g_i)^2}\right).$$
At present, the 2-form $F$ can be written in the form

$$F = ||g||(dx^0 - dx^2)dx^1 + \sum_{i=1}^{r} b_{i}dx^{2i+1}dx^{2i+2}.$$  

Finally, by the Lorentz transformation

$$x^0 - x^2 = ||g||(x^0 - x^2),$$

$$x^0 + x^2 = ||g||^{-1}(x^0 + x^2),$$

$$x^i = x^i, \quad (i \neq 0, 2)$$

$||g||$ in $F$ can be made 1. So, finally we get

$$F = dx^0dx^1 - dx^1dx^2,$$

when $H_{ij} = 0$, or otherwise

$$F = dx^0dx^1 - dx^1dx^2 + \sum_{i=1}^{r} b_{i}dx^{2i+1}dx^{2i+2}. \quad (b_1 \geq b_2 \geq \cdots \geq b_r > 0)$$

□

From Propositions 2, 3, and 4, we obtain the classification Theorem for constant 2-forms.

**Theorem 1.** Let $F$ be a constant 2-form on $\mathbb{R}^{n,1}$. Under an action of the Lorentz group $O_{n,1}$, $F$ can be brought into one of following forms

(a) $u = (0, \ldots, 0)$

$$F = 0$$

(b) $u = (0, b_1, 0, b_2, \ldots, 0, b_r, 0, 0, \ldots, 0)$

$$F = \sum_{i=1}^{r} b_{i}dx^{2i+1}dx^{2i} \quad (b_1 \geq b_2 \geq \cdots \geq b_r > 0)$$

(c) $u = (a, 0, 0, \ldots, 0)$

$$F = adx^0dx^1 \quad (a > 0)$$

(d) $u = (a, 0, b_1, 0, b_2, 0, \ldots, b_r, 0, 0, \ldots, 0)$

$$F = adx^0dx^1 + \sum_{i=1}^{r} b_{i}dx^{2i+1}dx^{2i+1} \quad (a > 0, b_1 \geq b_2 \geq \cdots \geq b_r > 0)$$

(e) $u = (1, 1, 0, 0, \ldots, 0)$

$$F = dx^0dx^1 + dx^1dx^2$$

(f) $u = (1, 1, 0, b_1, 0, b_2, \ldots, 0, b_r, 0, 0, \ldots, 0)$

$$F = dx^0dx^1 + dx^1dx^2 + \sum_{i=1}^{r} b_{i}dx^{2i+1}dx^{2i} \quad (b_1 \geq b_2 \geq \cdots \geq b_r > 0)$$

and it gives a canonical classification of constant 2-forms.
4. Canonical classification of Killing vector field

Here, we enumerate all canonical forms of the Killing vector field in $\mathbb{R}^{n,1}$. The result seems to be already known by some authors, see e.g. [9], whereas the complete classification and its reasoning do not seem to appear in the literature. Here, we give the canonical classification for self-containedness.

A Killing vector field can be brought into the form

$$\xi = F_{\mu\nu}x^\mu dx^\nu + f_\nu dx^\nu,$$

where $F_{\mu\nu}$ is one of canonical forms in Theorem 1. In order to find the canonical form of $\xi$, we study the transformation of $f_\nu$ under the subgroup $G_F$ of the Poincaré group $IO_{n,1}$ that preserves $F_{\mu\nu}$.

**Definition 2.** For given canonical 2-form $F$, we denote by $G_F$ the subgroup of $IO_{n,1}$ that leaves $F$ invariant, i.e.,

$$G_F = \left\{ g(\Lambda, c) \in IO_{n,1} | A_\mu^\nu A_\nu^\rho F_{\rho\mu} = F_{\mu\nu} \right\}.$$

Under the action of $g(\Lambda, c) \in G_F$, the constant 1-form $f$ transforms as

$$g(\Lambda, c)(f) = (A_\mu^\nu f_\nu + F_{\mu\nu} c_\nu) dx^\mu.$$

In particular, we may always shift $f$ by a 1-form $Fc$ belonging to $\text{Im } F$, where $F$ is regarded as a linear operator from the real tangent space to the space of real 1-forms.

By Theorem 1 the matrix representation of $F$ in canonical form is a block diagonal matrix consisting of the following types of blocks:

- **$S$-block**

  $$S = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix},$$

- **$N$-block**

  $$N = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

- **$O$-block**

  $$O = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $s > 0$.

For given $F$, let $W_S$ be the image of the $S$-block part, i.e., if

$$F = adx^0 dx^1 + b_1 dx^1 dx^2 + \ldots + b_{2r} dx^{2r} dx^{2r+1},$$

then

$$W_S = \bigoplus_{i=0}^{2r+1} \mathbb{R} dx^i.$$
and if
\[ F = dx^0dx^1 + dx^1dx^2 + b_1dx^3dx^4 + \cdots + b_dx^{2r+1}dx^{2r+2}, \]
then
\[ W_s = \bigoplus_{i=3}^{2r+2} \mathbb{R}dx^i, \]
and so on, and let \( W_N \) be its orthogonal complement of \( W_s \) with respect to \( \eta \). Hence any 1-form \( f \) can be written as \( f = f_S + f_N, (W_S \in W_s, f_N \in W_N) \) according to this direct sum decomposition.

Under the translation \( x'^\mu = x^\mu + e^\mu \), the constant 1-form \( f \) transforms as \( f \mapsto f + Fc \), so that \( f_N \) can always be made zero. Hence the problem reduce to find out all the canonical forms of \( f_N \) and if \( N \) can be made zero. Hence the problem reduce to find out all the canonical forms of \( f_N \) under the action of \( G_F \).

The \( N \)-block and \( O \)-block part of \( F \) can be regard as the linear operator \( F_N : V_N \rightarrow W_N \), where
\[ V_N = \eta^{-1}(W_N) = \{ v \in \mathbb{R}^{n-1} | \eta(v) \in W_N \}, \]
and \( F_N \) is the restriction of \( F \) on \( V_N \). Then, the following cases appear.

- **\( V_N \) is \((x^s, x^{s+1}, \ldots, x^n)\) space, where \( s = 2, 3, \ldots, n \):**
  \( F_N \) is zero on \( V_N \), and \( f_N \) can be brought into the form \( f_0dx^0 (f_n \geq 0) \) by an \( O_{n-1} \) transformation.

- **\( V_N \) is \((x^0, x^1, x^{s+1}, \ldots, x^n)\) space, where \( s = 1, 3, 4, \ldots, n \), and \( F_N = O \):**
  By a Lorentz transformation over \( W_N \), \( f_n \) can be made into the form \( f_0dx^0 (f_0 \leq 0), f_0dx^0 (f_0 > 0), \) or \( -dx^0 + dx^n \), according to its causal property.

- **\( V_N \) is \((x^0, x^1, x^2)\) space and \( F_N = N \):**
  \( f_N \) can be made \( f_0dx^0 \) by a translation. Then, \( f_0 \) can be made nonpositive by \( x^0 \mapsto -x^0 \), if necessary.

- **\( V_N \) is \((x^0, x^1, x^2, x^3, x^{s+1}, \ldots, x^n)\) space, where \( s = 3, 4, \ldots, n \), and \( F_N = N \oplus O \):**
  Firstly, \( f_N \) can be brought into the form \( f_0dx^0 + f_1dx^1 + f_2dx^2 + f_ndx^n (f_n \geq 0) \) by \( O_{n-2} \) transformation over \((x^s, \ldots, x^n)\) space. Next, \( f_N \) can be made into the form \( f_0dx^0 + f_0dx^n \) by a translation. If \( f_0 \neq 0, f_n \) can be made zero, by a Lorentz transformation over \((x^0, x^1, x^2)\) space. So that the final form of \( f_N \) is \( f_0dx^0 (f_0 \leq 0) \) or \( f_0dx^0 (f_0 > 0) \).

We are now in a position to state the classification Theorem for Killing vector fields.

**Theorem 2.** Each Killing vector field \( \xi \) in \( \mathbb{R}^{n,1} \) is brought into one of the following forms by the \( IO_{n,1} \) transformation.

(a) \( n \geq 1 \)
\[ \xi = f_0dx^0, f_0 < 0 \]
(b) \( n \geq 1 \)
\[ \xi = f_ndx^n, f_n > 0 \]
(c) \( n \geq 1 \)
\[ \xi = -dx^0 + dx^n, \]
(d) $n \geq 1$

$$\xi = a(x^0 dx^1 - x^1 dx^0), \quad a > 0$$

(e) $n \geq 2$, $r \leq n/2$

$$\xi = \sum_{i=1}^{r} b_i (x^{2i-1} dx^{2i} - x^{2i} dx^{2i-1}) + f_0 dx^0,$$

$$b_1 \geq b_2 \geq \cdots \geq b_r > 0, \quad f_0 \leq 0$$

(f) $n \geq 2$

$$\xi = a(x^0 dx^1 - x^1 dx^0) + f_n dx^n, \quad a > 0, \quad f_n > 0$$

(g) $n \geq 2$

$$\xi = (x^0 - x^2) dx^1 - x^1 (dx^0 - dx^2) + f_0 dx^0, \quad f_0 \leq 0$$

(h) $n \geq 3$, $r < n/2$

$$\xi = \sum_{i=1}^{r} b_i (x^{2i-1} dx^{2i} - x^{2i} dx^{2i-1}) + f_n dx^n,$$

$$b_1 \geq b_2 \geq \cdots \geq b_r > 0, \quad f_n > 0$$

(i) $n \geq 3$, $r < n/2$

$$\xi = \sum_{i=1}^{r} b_i (x^{2i-1} dx^{2i} - x^{2i} dx^{2i-1}) - dx^0 + dx^n,$$

$$b_1 \geq b_2 \geq \cdots \geq b_r > 0$$

(j) $n \geq 3$, $r \leq (n-1)/2$

$$\xi = a(x^0 dx^1 - x^1 dx^0) + \sum_{i=1}^{r} b_i (x^{2i} dx^{2i+1} - x^{2i+1} dx^{2i}),$$

$$a > 0, \quad b_1 \geq b_2 \geq \cdots \geq b_r > 0$$

(k) $n \geq 3$

$$\xi = (x^0 - x^2) dx^1 - x^1 (dx^0 - dx^2) + f_n dx^n, \quad f_n > 0$$

(l) $n \geq 4$, $r < (n-1)/2$

$$\xi = a(x^0 dx^1 - x^1 dx^0) + \sum_{i=1}^{r} b_i (x^{2i} dx^{2i+1} - x^{2i+1} dx^{2i}) + f_n dx^n,$$

$$a > 0, \quad b_1 \geq b_2 \geq \cdots \geq b_r > 0, \quad f_n > 0$$

(m) $n \geq 4$, $r \leq (n/2) - 1$

$$\xi = (x^0 - x^2) dx^1 - x^1 (dx^0 - dx^2) + \sum_{i=1}^{r} b_i (x^{2i+1} dx^{2i+2} - x^{2i+2} dx^{2i+1}) + f_0 dx^0,$$

$$b_1 \geq b_2 \geq \cdots \geq b_r > 0, \quad f_0 \leq 0$$
5. Complete integrability of cohomogeneity-1 string system

As an application of Theorem 2, we show the complete integrability of the cohomogeneity-1 Nambu-Goto string system in flat spacetime.

We first describe the cohomogeneity-1 Nambu-Goto system. The world sheet of a string is a 2-dimensional timelike surface in $\mathbb{R}^{n,1}$, which is written as

$$ x^\mu = \varphi^\mu(s^0, s^1), \quad (\mu = 0, 1, \ldots, n) $$

where $\varphi^\mu$'s are smooth functions, and the parameters $(s^0, s^1)$ are called world-sheet coordinates. If the world-sheet of the string has the zero mean curvature, it is called the Nambu-Goto string.

The induced metric on the world-sheet

$$ G_{AB} = \eta_{\mu\nu} \frac{\partial \varphi^\mu}{\partial s^A} \frac{\partial \varphi^\mu}{\partial s^B} $$

is called the world-sheet metric. The equation of motion for the Nambu-Goto strings is given by

$$ \frac{\partial}{\partial s^A} \left( \sqrt{-\det GG_{AB} \frac{\partial \varphi^\mu}{\partial s^B}} \right) = 0, \quad (1) $$

which is just the requirement that $\varphi^\mu$'s are harmonic functions on the world-sheet.

Choosing a thermal coordinate on the world-sheet, this reduces to the wave equation in $\mathbb{R}^{1,1}$, so that we have ordinary plane-wave solutions for this system.

However, we here consider the cohomogeneity-1 string, that respects the spacetime symmetry in $\mathbb{R}^{n,1}$ generated by a Killing vector field. In this case, the equation of motion reduce to a geodesic equation on a certain Riemannian, or Lorentzian $n$-manifold that is the space of the Killing orbits.

In order to find the ignorable coordinate associated with the Killing vector field $\xi$, we first solve the coupled ordinary differential equations

$$ \frac{dx^\mu(\tau)}{d\tau} = \xi^\mu(x(\tau)). $$

The solution may be obtained in the form

$$ x^\mu = x^\mu(\tau, y^1, y^2, \ldots, y^n), $$

where $(y^a)_{a=1,\ldots,n}$ parametrize integral curves of $\xi^a(\cdot = \xi^a \partial_a)$. This just gives the coordinate transormation. In the $(\tau, y^a)$ coordinate system, the Killing vector field is simply given by

$$ \xi^a = \frac{\partial}{\partial \tau}. $$

Hereafter, we denote by $g$, instead of $\eta$, the spacetime metric on $\mathbb{R}^{n,1}$, when we are concerned with noninertial coordinates.
The metric in this coordinate system is written as
\[ g = X \left( ds - \sum_{a=1}^{n} W_a dy^a \right)^2 + \sum_{a,b=1}^{n} h_{ab} dy^a dy^b, \]
where \( X, W_a, \) and \( h_{ab} \) depend only on \((y^a)_{a=1,...,n}\). In particular, \( X \) is the square of \( \xi^2 \), i.e., \( X = g(\xi^2, \xi^2) \), and \( h \) is regarded as the natural metric on the space of Killing orbits, which acquires the structure of a differentiable \( n \)-manifold. We denote by \( \mathcal{M} \) the space of Killing orbits.

The cohomogeneity-1 string is described as
\[ s = \tau, \quad y^a = \varphi^a(\sigma), \quad (a = 1, 2, \ldots, n) \]
in terms of the world-sheet coordinate \((\tau, \sigma)\). Then, it is known \[1\] that the equation of motion \[1\] becomes the geodesic equation for \( \varphi^a(\sigma) \) on the Riemannian, or the Lorentzian \( n \)-manifold \((\mathcal{M}, \gamma)\), where the metric \( \gamma \) is given by
\[ \gamma_{ab} = |X|h_{ab}. \]

We show the complete integrability of this system utilizing the conformal trick developed in \[4, 10\], which is brilliantly useful for finding pair-wise Poisson commuting conserved quantities for the geodesic Hamiltonian system on \((\mathcal{M}, \gamma)\).

The Lagrangian for geodesic in \((\mathcal{M}, \gamma)\) is given by
\[ L(y, \dot{y}) = -\sqrt{\pm \gamma_{ab}\dot{y}^a\dot{y}^b}, \]
where the double-sign in the square root takes plus-sign if \((\mathcal{M}, \gamma)\) is Riemannian (i.e. if \( X < 0 \)), otherwise minus-sign if it is Lorentzian \((X > 0)\). This is a singular Lagrangian system, since it leads to the constraint
\[ \Phi := \gamma^{ab} p_a p_b \mp 1 \approx 0. \]
Then the Hamiltonian becomes zero, and the system is described by the total Hamiltonian
\[ H(y, p, \lambda) = \lambda \Phi. \]
It immediately follows that \( \Phi \) is a first-class constraint, and the Lagrange multiplier \( \lambda \) remains undetermined. This reflects the invariance of the system under the reparametrization of the Hamiltonian time parameter, which in the present case is \( \sigma \).

We can fix this ambiguity of time-parametrization simply by putting
\[ \lambda = \frac{|X|}{2}, \]
that corresponds to the time-reparametrization \(|X|d\sigma' = \lambda d\sigma \). After this gauge fixation, the Hamiltonian becomes
\[ H(y, p) = \frac{1}{2} (h^{ab} p_a p_b + X), \]
that takes the same form as the particle system with potential function \( X/2 \).
However, $h^{ab}$ is the inverse matrix of $h_{ab}$, so that we can also write as

$$ H(y, p) = \frac{1}{2} \left( \sum_{\mu, \nu=1}^{n} g^{\mu\nu} p_{\mu} p_{\nu} + X \right). $$

This Hamiltonian system is equivalent with the geodesic Hamiltonian

$$ H(s, y, p_s, p) = \frac{1}{2} \left( \sum_{\mu, \nu=0}^{n} g^{\mu\nu} p_{\mu} p_{\nu} + X \right), $$

with the initial condition $p_s = 0$. Since $p_s = \xi^\mu p_\mu$ is a conserved quantity, we can simply put $p_s = 0$ at the Hamiltonian level.

Further simplification occurs when we perform the point transformation from $(y^\mu)_{\mu=0,...,n} = (s, y^a)$ to the inertial coordinate $(x^\mu)_{\mu=0,...,n}$, that leads to

$$ H(x, P) = \frac{1}{2} \left( \eta^{\mu\nu} P_{\mu} P_{\nu} + X \right), $$

where the point transformation of the momentum is given by

$$ P_{\mu} = \frac{\partial y^\nu}{\partial x^\mu} p_\nu, $$

that is just the transformation law for the cotangent vector. In particular, the restriction $p_s = 0$ is simply written as

$$ \xi^\mu P_\mu = 0. $$

Thus, the geodesic Hamiltonian system in $(\mathcal{M}, \gamma)$ has been translated into the particle system in $(\mathbb{R}^{n,1}, \eta)$ with the potential function $X/2$ with the initial condition $\xi^\mu P_\mu = 0$.

In order to show the complete integrability of the geodesic Hamiltonian system defined by (2), we have only to find the $n+1$ pair-wise Poisson commuting conserved quantities for the Hamiltonian system (3) those are functionally independent. Among such, the Hamiltonian $H$ and $p_s = \xi^\mu p_\mu$ are already in hand. Hence our task is to find $n-1$ more pair-wise Poisson commuting conserved quantities those commute with $p_s$.

From Theorem 2 a Killing vector in $\mathbb{R}^{n,1}$ can be written as

$$ \xi = F_{\mu\nu} x^\nu dx^\mu + f_\nu dx^\nu, $$

where the matrix representation of $F_{\mu\nu}$ is a direct sum of $N$, $S$, $O$-blocks. This globally determines the orthogonal direct sum decomposition of the space of differential 1-forms on $\mathbb{R}^{n,1}$

$$ \Omega(\mathbb{R}^{n,1}) = \Omega_N \oplus \Omega_S \oplus \Omega_O, $$

in an obvious manner. Explicitly, the orthogonal direct sum decomposition of $\xi$,

$$ \xi = \xi_N + \xi_S + \xi_O, \xi_N \in \Omega_N, \xi_S \in \Omega_S, \xi_O \in \Omega_O. $$
can generally be written as
\[ \xi_N = (f_0 - x^1)dx^0 + (x^0 - x^2)dx^1 + x^1dx^2, \]
\[ \xi_S = \sum_{i=1}^r b_i(x^{2i+1}dx^{2i+2} - x^{2i+2}dx^{2i+1}), \]
\[ \xi_O = f_n dx^n, \]
when \( N \) is present,
\[ \xi_N = 0, \]
\[ \xi_S = a(x^0dx^1 - x^1dx^0) + \sum_{i=1}^r b_i(x^{2i}dx^{2i+1} - x^{2i+1}dx^{2i}), \]
\[ \xi_O = f_n dx^n, \]
when \( N \) is absent and \( \Omega_S \) is Lorentzian, or
\[ \xi_N = 0, \]
\[ \xi_S = \sum_{i=1}^r b_i(x^{2i}dx^{2i+1} - x^{2i}dx^{2i+1}), \]
\[ \xi_O = f_0 dx^0 + f_n dx^n, \]
when \( N \) is absent and \( \Omega_S \) is Riemannian. The canonical momentum 1-form \( P = P_\mu dx^\mu \) is also decomposed as
\[ P = P_N + P_S + P_O, (P_N \in \Omega_N, P_S \in \Omega_S, P_O \in \Omega_O) \]
Then, the Hamiltonian (3) is accordingly decomposed into three parts as
\[ H = H_N + H_S + H_O. \]
Here, the \( N \)-Hamiltonian is defined by
\[ H_N := \frac{1}{2} \left[ \eta^{\mu\nu}(P_N)_\mu(P_N)_\nu + \frac{1}{4} \xi_N \right] \]
\[ = \frac{1}{2} \left[ -(P_0)^2 + (P_1)^2 + (P_2)^2 - (f_0 - x^1)^2 + (x^0 - x^2)^2 + (x^1)^2 \right], \]
if \( N \)-block is present in the matrix representation of \( F_{\mu\nu} \), and otherwise \( H_N = 0. \)
The \( S \)-Hamiltonian is similarly defined as
\[ H_S := \frac{1}{2} \left[ \eta^{\mu\nu}(P_S)_\mu(P_S)_\nu + \frac{1}{4} \xi_S \right]. \]
It is
\[ H_S = \frac{1}{2} \left[ -(P_0)^2 + (P_1)^2 + a^2(x^0)^2 - a^2(x^1)^2 \right] \]
\[ + \frac{1}{2} \sum_{i} \left[ (P_2)^2 + (P_{2i+1})^2 + (b_i)^2(x^{2i})^2 + (b_i)^2(x^{2i+1})^2 \right]. \]
when \( \Omega_S \) is Lorentzian, and
\[ H_S = \frac{1}{2} \sum_{i} \left[ (P_{2i-1})^2 + (P_2)^2 + (b_i)^2(x^{2i-1})^2 + (b_i)^2(x^{2i})^2 \right], \]
when $\mathcal{Q}_s$ is Riemannian.

Finally, the $O$-Hamiltonian is defined as

$$H_O := \frac{1}{2} \left[ \eta^{\mu\nu}(P_O)_{\mu}(P_O)_{\nu} + g(\xi_O, \xi_O) \right],$$

and it generally has the form

$$H_O = \frac{1}{2} \left[ \sum_{\mu} \eta^{\mu\nu}(P_{\mu})^2 - \left( f_0 \right)^2 + \left( f_n \right)^2 \right].$$

These $N, S, O$-Hamiltonians describe mutually independent dynamical systems. With this observation, the complete integrability of our Hamiltonian system is readily understood.

**Theorem 3.** The Hamiltonian system defined by (3) is completely integrable.

**Proof.** We show that $N, S, O$-Hamiltonian has as many conserved quantities as the size of $N, S, O$-block of $F_{\mu\nu}$, respectively.

Firstly, the $N$-Hamiltonian has three conserved quantities

$$C_N := H_N, \quad D_N := (x^1 - f_0)P_0 + (x^0 - x^2)P_1 + x^1P_2, \quad E_N := P_0 + P_2.$$  

Secondly, the $S$-Hamiltonian has as many conserved quantities as the size of $S$-block, of the following forms

$$C^0_S := \frac{1}{2} \left[ -(P_0)^2 + (P_1)^2 + a^2(x^0)^2 - a^2(x^1)^2 \right], \quad D^0_S := x^0P_1 + x^1P_0,$$

$$C^i_S := \frac{1}{2} \left[ (P_i)^2 + (P_{i+1})^2 + (b_i)^2(x^i)^2 + (b_{i+1})^2(x^{i+1})^2 \right], \quad D^i_S := x^iP_{i+1} - x^{i+1}P_i.$$  

Finally, the $O$-Hamiltonian has as many conserved quantities as the size of $O$-block, which are simply

$$F^i_O := P_i.$$  

These constitute $n + 1$ functionally independent, mutually Poisson commutative conserved quantities. □

In fact, the Hamiltonian system (3) can generally be solved in terms of elementary functions. Then, the solution to the geodesic Hamiltonian system (2) is obtained simply the point transformation of the solution subject to $\xi^\mu P_{\mu} = 0$, that is just the coordinate transformation into the stationary coordinate system.

6. **Canonical representation of noncommutative algebra of Killing field**

Here we consider the minimal noncommutative Lie algebra of Killing vector fields given by

$$[\xi^\mu, \eta^\nu] = \xi^\nu.$$
We show that this algebra is realized only by a limited class of Killing vector pairs. The result shown in the following will be useful e.g. in analysis of Nambu-Goto membranes those are invariant under the action of a 2-dimensional isometry group.

Let $\xi$ and $\eta$ be written as

$$
\begin{align*}
\xi &= F_{\mu\nu}dx^\mu dx^\nu + f_2dx^2, \\
\eta &= G_{\mu\nu}dx^\mu dx^\nu + g^2dx^2,
\end{align*}
$$

respectively. Then, the condition $[\xi^\theta, \eta^\theta] = \xi^\theta$ is equivalent to

$$
\begin{align*}
[\xi^\theta F, \eta^\theta G] := \xi^\theta F^\theta G - \eta^\theta G^\theta F = \xi^\theta F, \\
(\xi^\theta G + 1)\eta^\theta = \xi^\theta G^\theta.
\end{align*}
$$

Let us introduce a several notions for a special type of 2-forms, which turns out to be useful.

**Definition 3** (superdiagonal array). For the following form of the constant 2-form

$$
F = u_0dx^0dx^1 + u_1dx^1dx^2 + \cdots + u_{n-1}dx^{n-1}dx^n,
$$

the array

$$(u_0, u_1, \ldots, u_{n-1})$$

is called the superdiagonal array of $F$.

**Definition 4** (isolated nonzero element). A nonzero element of the superdiagonal array of $F$ is called an isolated nonzero element, if it is not next to a nonzero element.

With these terminology, we can state the following Lemma.

**Lemma 9.** Let a pair of constant 2-forms $F$, $G$ be subject to the commutation relation

$$
[\xi^\theta F, \eta^\theta G] = \xi^\theta F,
$$

and let $F$ be written by a superdiagonal array. Then, the superdiagonal array of $F$ does not have an isolated nonzero element.

**Proof.** Let $F$ be written in terms of a superdiagonal array ($u_0, u_1, \ldots, u_{n-1}$).

Assume that $u_0$ is an isolated nonzero element, i.e., that $u_0 \neq 0$ and $u_1 = 0$ hold. Then, (0, 1) component of $[\xi^\theta F, \eta^\theta G]$ becomes

$$
[\xi^\theta F, \eta^\theta G]^0_1 = -G^0_0F^2_1 = G^0_0u_1 = 0,
$$

while $(\eta^\theta F)^0_1 = -u_0 \neq 0$. Hence, $[\xi^\theta F, \eta^\theta G] = \xi^\theta F$ cannot hold.

Next, assume that $u_i$ is an isolated nonzero element for some $i$ ($2 \leq i \leq n-2$), i.e., that $u_i \neq 0$ and $u_{i\pm 1} = 0$ hold. Then, it holds

$$
[\xi^\theta F, \eta^\theta G]^{i+1}_i = F^{i+1}_iG^{i-1}_{i+1} - G^{i+1}_{i+2}F^{i+2}_{i+1} = -u_{i-1}G^{i-1}_{i+1} + u_{i+1}G^{i+1}_{i+2} = 0,
$$

while $(\eta^\theta F)^{i+1}_i = u_i \neq 0$ holds. Hence, $[\xi^\theta F, \eta^\theta G] = \xi^\theta F$ does not hold again.

Finally, assume that $u_{n-1}$ is an isolated nonzero element, i.e., $u_{n-2} = 0$ and $u_{n-1} \neq 0$ hold. Then, it holds

$$
[\xi^\theta F, \eta^\theta G]^{n-1}_n = F^{n-1}_{n-2}G^{n-2}_n = -u_{n-2}G^{n-2}_n = 0,
$$

and so forth.
while $(\mathbf{F})^{n-1}_n = u_{n-1} \neq 0$. Hence, $[\mathbf{F}, \mathbf{G}] = \mathbf{F}$ does not hold.

These show that the superdiagonal array of $\mathbf{F}$ does not contain an isolated nonzero element, otherwise $[\mathbf{F}, \mathbf{G}] = \mathbf{F}$ is not fulfilled.

The Theorem combined with this Lemma leads to the following Proposition.

**Proposition 5.** Let $\xi$ and $\eta$ be Killing fields subject to $[\xi^\flat, \eta^\flat] = \xi^\flat$, and let $\xi$ be written as $\xi = F_{\mu\nu} x^\mu dx^\nu + f_i dx^i$.

Then, $\mathbf{F}$ is a nilpotent matrix.

**Proof.** By a Lorentz transformation, $\mathbf{F}$ can be brought into one of standard forms shown in Theorem which are all written by superdiagonal arrays. By Lemma, the superdiagonal array of $\mathbf{F}$ does not have an isolated nonzero element. Hence, the only possibility is (a) $F = 0$, or (e) $F = dx^0 dx^1 + dx^3 dx^2$. In both cases, the matrix $\mathbf{F}$ is nilpotent. □

This Proposition greatly restricts the form of the Killing vectors $\xi$ and $\eta$. The next Lemma determines the form of $d\eta$ in the case that $d\xi \neq 0$.

**Lemma 10.** A constant 2-form $G$ that satisfies $[\mathbf{F}, \mathbf{G}] = \mathbf{F}$ for

$$F = dx^0 dx^1 + dx^3 dx^2$$

can be brought into the form

$$G = dx^0 dx^2,$$

or

$$G = dx^0 dx^2 + \sum_{i=1}^{3} b_{2i+1} dx^{2i+1} dx^{2i+2} \quad (b_3 \geq b_5 \geq \cdots \geq b_{2r+1} > 0)$$

by a Lorentz transformation that preserves the form of $\mathbf{F}$.

**Proof.** The general form of a constant 2-form $G$ subject to $[\mathbf{F}, \mathbf{G}] = \mathbf{F}$ is given by

$$G = dx^0 dx^2 + h_{01}(dx^0 - dx^2)dx^1$$

$$+ \sum_{k=3}^{n} h_{0k}(dx^0 - dx^2)dx^k + \sum_{3 \leq i < j \leq n} h_{ij} dx^i dx^j.$$  

By a Lorentz transformation

$$x^0 = \left(1 + \frac{\alpha^2}{2}\right)x^0 - \alpha x^1 - \frac{\alpha^2}{2} x^2,$$

$$x^1 = x^1 - \alpha(x^0 - x^2),$$

$$x^2 = \frac{\alpha^2}{2} x^0 - \alpha x^1 + \left(1 - \frac{\alpha^2}{2}\right)x^2,$$

$$x^k = x^k, \quad (k = 3, 4, \ldots, n)$$
that leaves $F$ invariant, the 2-form $G$ transforms to

$$G = dx^0 dx^2 + (h_{01} - \alpha)(dx^0 - dx^2)dx^1$$
$$+ \sum_{k=3}^{n} h_{0k}(dx^0 - dx^2)dx^k + \sum_{3 \leq i < j \leq n} h_{ij} dx^i dx^j.$$ 

By taking $\alpha = h_{01}$, it becomes

$$G = dx^0 dx^2 + \sum_{k=3}^{n} h_{0k}(dx^0 - dx^2)dx^k + \sum_{3 \leq i < j \leq n} h_{ij} dx^i dx^j.$$ 

By a further Lorentz transformation

$$x^0 = \left(1 + \frac{\beta^2}{2}\right)x^0 - \frac{\beta^2}{2}x^2 - \sum_{k=3}^{n} \beta_k x^k,$$
$$x^1 = x^1,$$
$$x^2 = \frac{\beta^2}{2}x^0 + \left(1 - \frac{\beta^2}{2}\right)x^2 - \sum_{k=3}^{n} \beta_k x^k,$$
$$x^k = x^k - \beta_k(x^0 - x^2), \quad (k = 3, 4, \ldots, n)$$

that preserves $F$, the 2-form $G$ becomes

$$G = dx^0 dx^2 + \sum_{3 \leq i < j \leq n} h_{ij} dx^i dx^j$$
$$- \sum_{i=3}^{n} [(\delta_{ij} - h_{ij})\beta_j - h_{0j}](dx^0 - dx^2)dx^i,$$

where $h_{ij}$ is regarded as constituting an alternating matrix.

Since $(\delta_{ij} - h_{ij})$’s constitute a invertible matrix, the last term can be made zero by appropriately choosing $\beta_j$.

When $h_{ij} = 0$, $G$ is written as $G = dx^0 dx^2$. Otherwise, by an $O_{n-1}$ transformation that preserves $(x^3)^2 + \cdots + (x^m)^2$, the second term can be brought into the form

$$h_{ij} dx^i dx^j = \sum_{i=1}^{2r} b_{2i+1} dx^{n+2i+1} dx^{n+2i+2}, \quad (b_3 \geq b_5 \geq \cdots \geq b_{2r+1} > 0)$$

where $2r$ is the matrix rank of $h_{ij}$. \hfill \Box

The following Lemma gives a general form of the noncommutative Killing vector pair, when $F = dx^0 dx^1 + dx^1 dx^2$.

**Lemma 11.** Let $\xi$ and $\eta$ be Killing vector fields subject to $[\xi^\mu, \eta^\nu] = \xi^\mu$, which are written as

$$\xi_{\nu} = F_{\nu\mu} dx^\mu + f_{\nu} dx^\nu,$$
$$\eta_{\nu} = G_{\nu\mu} dx^\mu + g_{\nu} dx^\nu,$$

in terms of

$$F = dx^0 dx^1 + dx^1 dx^2.$$
and
\[ G = dx^0 dx^2, \]
or
\[ G = dx^0 dx^2 + \sum_{i=1}^{r} b_{2i+1} dx^{2i+1} dx^{2i+2}, \]
where \( b_{2i+1} \neq 0 \) (\( i = 1, \ldots, r \)). Then, by a Poincaré transformation that simultaneously leaves \( F \) and \( G \) invariant, we can set
\[ f = f_r dx^r = 0, \]
\[ g = g_r dx^r = q dx^a. \]

**Proof.** Recall that \( g \) transforms as
\[ g'_{\nu} = g_{\nu} - G_{\mu\nu} c^\mu, \]
according to the translation
\[ x'^\mu = x^\mu + c^\mu. \]
Hence, by taking
\[ (c^0, c^1, c^2, c^3, c^4, \ldots, c^{2r+1}, c^{2r+2}, c^{2r+3}, \ldots, c^n) \]
\[ = \left( g_2, 0, -g_0 \frac{g_4}{b_3}, \frac{g_3}{b_3}, \ldots, \frac{g_{2r+1}}{b_{2r+1}}, \frac{g_{2r+2}}{b_{2r+1}}, 0, \ldots, 0 \right), \]
g becomes
\[ g' = g'_{\nu} dx'^\nu = g_1 dx^1 + \sum_{k=2r+3}^{n} g_k dx^k. \]
Here and in what follows, the case \( G = dx^0 dx^2 \) is regarded as the \( r = 0 \) case.

Next, by \( O_{n-2r-2} \) transformation preserving \((x^{2r+3})^2 + \cdots + (x^n)^2\), we can set
\[ g = g_1 dx^1 + q dx^a, \]
where \( q = \sqrt{(g_{2r+3})^2 + \cdots + (g_n)^2}. \)
The condition \( \left[ \xi^\mu, \eta^\nu \right] = \xi^\mu \) leads to
\[ (\check{G} + I) f^{\mu} = \check{F} g^\mu. \]
Since its covariant components are
\[ (G_{\mu\nu} + \eta_{\mu\nu}) f^{\nu} dx^\mu = (-f^0 + f^2)(dx^0 + dx^2) + f^1 dx^1 \]
\[ + \sum_{i=1}^{r} \left[ (f^{2i+1} + b_{2i+1} f^{2i+2}) dx^{2i+1} + (f^{2i+2} - b_{2i+1} f^{2i+1}) dx^{2i+2} \right] \]
\[ + \sum_{k=2r+3}^{n} f^{k} dx^k, \]
\[ F_{\nu\mu} g^{\nu} dx^\mu = g^1 (dx^0 - dx^2), \]
they should hold
\[ f^2 = f^0, \]
\[ f^\mu = 0, \quad (\mu = 1, 3, 4, \ldots, n) \]
\[ g^1 = 0. \]

Moreover, by the translation
\[ x'\mu = x^\mu + f^0 \delta^\mu_0, \]
we can set
\[ f^0 = 0. \]

Hence the final forms of \( f \) and \( g \) are
\[ f = 0, \]
\[ g = qdx^n. \]

The other possibility \( F = 0 \) is covered by the following Lemmas.

**Lemma 12.** Let \( \xi = f_\mu dx^\mu (\neq 0) \) be a translation Killing field. If \( \xi \) is a timelike or spacelike 1-form, there is no Killing field \( \eta \) that satisfies \( \left[ \xi^\sharp, \eta^\sharp \right] = \xi^\sharp \).

**Proof.** The equation \( \left[ \xi^\sharp, \eta^\sharp \right] = \xi^\sharp \) is equivalent to
\[ ^\sharp G f^\sharp = f^\sharp. \]
By Corollary 1, \( f^\sharp \) should be a null vector. \( \square \)

**Lemma 13.** Let \( \xi = -dx^0 + dx^1 \) be a null translation Killing field. Then, every Killing field \( \eta \) that satisfies \( \left[ \xi^\sharp, \eta^\sharp \right] = \xi^\sharp \) can be brought into the form
\[ \eta = x^0 dx^1 - x^1 dx^0 + qdx^n, \]
or
\[ \eta = x^0 dx^1 - x^1 dx^0 + \sum_{i=1}^r b_2 (x^2 dx^{i+1} - x^{2i+1} dx^2) + qdx^n, \quad (b_2 \geq b_4 \geq \cdots \geq b_{2r} > 0) \]
by a Poincaré transformation that leaves \( \xi \) invariant.

**Proof.** For Killing fields
\[ \xi = -dx^0 + dx^1, \]
\[ \eta = G_{\mu\nu} x^\mu dx^\nu + g_s dx^s, \]
the commutation relation \( \left[ \xi^\sharp, \eta^\sharp \right] = \xi^\sharp \) is equivalent to
\[ (G_{\mu\nu} + \eta_{\mu\nu}) \xi^\nu = G_{\mu0} + G_{\mu1} + \eta_{\mu0} + \eta_{\mu1} = 0. \]
This leads to
\[ G_{01} = 1, \]
\[ G_{k0} + G_{k1} = 0. \quad (k = 2, 3, \ldots, n) \]
Hence the 2-form $G$ must be in the form
\[ G = dx^0dx^1 + \sum_{k=2}^{n} h_k(dx^0 - dx^1)dx^k + \sum_{2 \leq i < j \leq n} G_{ij}dx^idx^j. \]

By a Lorentz transformation
\[
\begin{align*}
    x^0 &= \left(1 + \frac{\beta^2}{2}\right)x'^0 - \frac{\beta^2}{2} x'^1 - \sum_{k=2}^{n} \beta_k x'^k, \\
    x^1 &= \frac{\beta^2}{2} x'^0 + \left(1 - \frac{\beta^2}{2}\right) x'^1 - \sum_{k=2}^{n} \beta_k x'^k, \\
    x^k &= x'^k - \beta_k (x'^0 - x'^1), \\
    \beta^2 &= \sum_{k=2}^{n} (\beta_k)^2,
\end{align*}
\]
that leaves the form of $\xi$ invariant, $G$ becomes
\[
G = dx^0dx^1 - \sum_{k=2}^{n} \sum_{l=2}^{n} (\delta_{kl} - G_{kl})\beta_l - h_k dx^0 - dx^1)dx^k
+ \sum_{2 \leq i < j \leq n} G_{ij}dx^idx^j.
\]
Noting that $(\delta_{kl} - G_{kl})$'s constitute a invertible matrix, the second term can be made zero by appropriately choosing $\beta_i$.

Hence, by an $O_{n-1}$ transformation that preserves $(x'^2)^2 + \cdots + (x'^n)^2$, it can be made into the form
\[ G = dx^0dx^1, \]
or
\[ G = dx^0dx^1 + \sum_{j=1}^{r} b_{2j}dx^{2j}dx^{2j+1}. \quad (b_2 \geq b_4 \geq \cdots \geq b_{2r} > 0) \]

Then, by a translation, we can made
\[ g_0 = g_1 = \cdots = g_{2r+1} = 0, \]
where the case $G = x^0dx^1$ corresponds to $r = 0$.

By further $O_{n-2r-1}$ transformation that leaves $(x^{2+2})^2 + \cdots + (x^n)^2$ invariant, we can set
\[ g_i dx^i = q dx^n. \]

Now we completely grasp all the possibilities for the noncommutative pair of Killing fields from Lemmas 10, 11, 12, and 13 which are summerized as follows.
Theorem 4. Let \( \xi \) and \( \eta \) be Killing fields those are subject to \([\xi^2, \eta^2] = \xi^2\). Then, by a Poincaré transformation, these can be brought into either of the forms

\[
\xi = (x^0 - x^2)dx^1 - x^1(dx^0 - dx^2),
\]

\[
\eta = x^0dx^2 - x^2dx^0 + \sum_{i=1}^{r} b_{2i+1}(x^{2i+1}dx^{2i+2} - x^{2i+2}dx^{2i+1}) + qdx^n,
\]

or

\[
\xi = -dx^0 + dx^1,
\]

\[
\eta = x^0dx^1 - x^1dx^0 + \sum_{i=1}^{r} b_{2i}(x^{2i}dx^{2i+1} - x^{2i+1}dx^{2i}) + qdx^n.
\]

7. Conclusion

We have shown that the equation of motion for cohomogeneity-one Nambu-Goto strings in \( \mathbb{R}^{n,1} \) is completely integrable for \( n \geq 1 \), which generalize the result of Koike et al. for \( n = 3 \). In order to enumerate the possible types of those equation of motion, we have classified the canonical form of the Killing vector field in \( \mathbb{R}^{n,1} \) under the action of the Poincaré group. It has been shown that there are 4 types for \( n = 1 \), 7 types for \( n = 2 \), 11 types for \( n = 3 \), 13 types for \( n = 4 \), and 14 types for \( n \geq 5 \) of the Killing vector fields (Theorem 2).

The equation of motion for a cohomogeneity-one Nambu-Goto string in \( \mathbb{R}^{n,1} \) reduces to the geodesic equation in the \( n \)-dimensional Killing orbit space. The Hamilton-Jacobi equations for these geodesic equations are generally non-separable. Hence, at this point, the integrability issue of this system is not self-evident.

By a conformal trick [4], the geodesic system in the curved orbit space is converted into the problem of particle motions in \( \mathbb{R}^{n,1} \) with the potential function determined by the amplitude of the Killing vector field. Then, we have found that all the canonical forms for the Killing vector fields defines a simple Hamiltonian function for the particle system, enough to be able to show that the system is completely integrable.

We have also enumerate the possible forms of the basis of the 2-dimensional Lie algebra of Killing vector fields subject to \([\xi^2, \eta^2] = \xi^2\). It has been found that there only 2-types for such pair of Killing vector fields (Theorem 4). This result would be useful to classify the Nambu-Goto membranes in \( \mathbb{R}^{n,1} \) those respects spacetime isometry isomorphic to a 2-dimensional subgroup of the Poincaré group.

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Department of Physics, Gakushuin University, Tokyo 171-8588, Japan