A MASS BOUND FOR SPHERICALLY SYMMETRIC BLACK HOLE SPACETIMES

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Abstract

Requiring that the matter fields are subject to the dominant energy condition, we establish the lower bound \((4\pi)^{-1}\kappa A\) for the total mass \(M\) of a static, spherically symmetric black hole spacetime. \((A\) and \(\kappa\) denote the area and the surface gravity of the horizon, respectively.) Together with the fact that the Komar integral provides a simple relation between \(M - (4\pi)^{-1}\kappa A\) and the strong energy condition, this enables us to prove that the Schwarzschild metric represents the only static, spherically symmetric black hole solution of a selfgravitating matter model satisfying the dominant, but violating the strong energy condition for the timelike Killing field \(K\) at every point, that is, \(R(K, K) \leq 0\). Applying this result to scalar fields, we recover the fact that the only black hole configuration of the spherically symmetric Einstein-Higgs model with arbitrary non-negative potential is the Schwarzschild spacetime with constant Higgs field. In the presence of electromagnetic fields, we also derive a stronger bound for the total mass, involving the electromagnetic potentials and charges. Again, this estimate provides a simple tool to prove a "no-hair" theorem for matter fields violating the strong energy condition.
1 Introduction

In 1979 Schoen and Yau succeeded in proving the positive mass theorem by the means of a variational technique [1]. Subsequently, Witten presented a different proof of the long standing problem, based on the Lichnerowicz identity for spinor fields [2], [3]. Later on, Gibbons et al. have generalized Witten’s approach to black hole spacetimes, establishing \( M \geq 0 \) for a spacelike hypersurface which is regular outside an apparent horizon [4].

According to Arnowitt, Deser and Misner (ADM), the total energy (four-momentum) of an asymptotically flat manifold can be defined in terms of a surface integral at spacelike infinity, involving only the asymptotic behavior of the metric [5]. Provided that spacetime admits a stationary symmetry, Komar has given another expression for the total mass in terms of the asymptotically timelike Killing field [6]. With the help of Ricci’s identity it is not difficult to see that the positivity of the Komar mass is a direct consequence of the strong energy condition (SEC). Obviously this provides no alternative proof of the positive mass theorem for stationary spacetimes, since the SEC is more restrictive than the dominant energy condition (DEC) on which the positivity of the ADM mass is based. However, if the matter model under consideration satisfies the DEC but violates the SEC for the timelike Killing field at every point of the domain, \( R(K, K) \leq 0 \), then the Komar expression for the mass \( M \) is non-positive, contradicting the positive mass theorem, unless \( M = 0 \). Hence, matter models of this kind admit only trivial selfgravitating soliton solutions.

The question arises, whether this reasoning can be extended to black hole spacetimes. In this case, the Komar integral yields the upper bound \( M \leq \frac{1}{4\pi \kappa} A \) if \( R(K, K) \leq 0 \), whereas the generalization of the positive mass theorem still gives \( M \geq 0 \) as a consequence of the DEC [4]. Hence, in order to establish “no-hair” results by applying the above argument, it is desirable to improve this bound and to conjecture that for static black holes the DEC implies \( M \geq \frac{1}{4\pi \kappa} A \).

Until now, we did not succeed in deriving this bound in the general static case. However, restricting our attention to spherically symmetric configurations, we present a simple proof of the above inequality in this paper.

As an application, we show that the only spherically symmetric black hole solution of a static selfgravitating harmonic map with non-negative potential and Riemannian target manifold is the Schwarzschild metric [7], [8].
This generalizes the well-known result due to Bekenstein \cite{9} to non-convex potentials. As is clear from the above comments, the corresponding conclusion can be drawn for \textit{soliton} solutions without any additional symmetry requirements.

In the presence of electromagnetic fields, stronger estimates are needed in order to take over the above reasoning, since the electromagnetic part $T^{(em)}$ of the energy momentum tensor $T$ does not violate the SEC. Taking advantage of Maxwell’s equations and expressing the total electric and magnetic charge ($Q$ and $P$) in terms of volume integrals, it is not difficult to show that $M \leq \frac{1}{4\pi} \kappa A - Q \Delta \phi - P \Delta \psi$, provided that $T := T - T^{(em)}$ violates the SEC for $K$ at every point of the domain (where $\phi$ and $\psi$ denote the electric and the magnetic potential, respectively). Hence, in order to gain “no-hair” results in this case, one has to establish the converse inequality on the basis of the DEC. We shall conclude this paper by doing so for the spherically symmetric case.

\section{Komar Mass and Strong Energy Condition}

Throughout this article, we shall restrict ourselves to strictly stationary, asymptotically flat spacetimes containing a non-rotating black hole. Hence, we assume that there exists a Killing field $K$, timelike in all of the domain of outer communications and coinciding with the null-generator of the Killing horizon $H$ (see \cite{10} or \cite{11}) for details. Taking advantage of the Komar integral and Ricci’s identity, the goal of this section is to establish the following expression for the total mass in terms of the $KK$-component of the Ricci tensor $R$,

\begin{equation}
M = \frac{1}{4\pi} \kappa A + \frac{1}{4\pi} \int_{\Sigma} \frac{R(K,K)}{V} \eta_{\Sigma} - \frac{1}{2\pi} \int_{\Sigma} \frac{(\omega,\omega)}{V^2} \eta_{\Sigma},
\end{equation}

where $V$ and $\omega$ denote the norm of the Killing field and its twist 1-form, respectively,

\begin{equation}
V := -(K|K) \geq 0, \quad \omega := \frac{1}{2} * (K \wedge dK).
\end{equation}

(Here and in the following we use the symbol $K$ for both, the Killing field and its assigned 1-form.) $\eta_{\Sigma} := *K = i_K \eta$ denotes the induced volume form
on the spacelike hypersurface \( \Sigma \) extending from spacelike infinity to the inner 2-dimensional boundary \( \mathcal{H} = H \cap \Sigma \) (i.e., to the “horizon at time \( \Sigma \))

As is seen from the above mass formula, Einstein’s equations together with the SEC,

\[
T(\hat{k}, \hat{k}) + \frac{1}{2} \text{tr}(T) \geq 0,
\]

(3)

for the unit timelike field \( \hat{k} := K/\sqrt{V} \), implies the positivity of the quantity \( M - (4\pi)^{-1}\kappa A \), provided that spacetime is static, i.e., that the twist-form vanishes identically. However, if the staticity requirement is dropped, we are still able to conclude that the violation of the SEC for the Killing field \( \hat{k} \) at every point of the domain implies the converse inequality, \( M \leq (4\pi)^{-1}\kappa A \).

(Note that in a strictly stationary domain \( \omega \) is nowhere timelike and hence \( (\hat{k}|\omega) \geq 0 \)).

In order to derive eq. (1) we recall that according to Komar [6] the mass of a stationary, asymptotically flat spacetime can be expressed in terms of the asymptotically timelike Killing field by the surface integral

\[
M = -\frac{1}{8\pi} \int_{S^\infty} *dK = -\frac{1}{8\pi} \int_{\mathcal{H}} *dK - \frac{1}{8\pi} \int_{\Sigma} d* dK,
\]

(4)

where we have used Stokes’ theorem in the second equality. In the non-rotating case, to which we restrict our attention, \( K \) coincides with the null-generator Killing field of the horizon [10].

Denoting the second future directed null vector orthogonal to the horizon with \( n \) (normalized such that \( (K|n) = -1 \)), the boundary integral over \( \mathcal{H} \) can be evaluated in the well known manner,

\[
\int_{\mathcal{H}} *dK = \int_{\mathcal{H}} (K \wedge n|dK) dA = \int_{\mathcal{H}} (n|dV) dA = -2\kappa A,
\]

(5)

where we have used the definition \( dV = 2\kappa K \) of the surface gravity \( \kappa \) and the fact that the latter is constant over the horizon (cf. eg. [12]).

In order to evaluate the volume integral in eq. (4) we take advantage of the Ricci identity for Killing fields,

\[
d* dK = 2 * R(K),
\]

(6)

where the components of the Ricci 1-form \( R(K) \) are \( R(K)_\mu = R_{\mu\nu\alpha}K^\alpha \). This yields the well-known formula for the Komar Mass in terms of the horizon
quantities and the Ricci 1-form,
\[ M = \frac{1}{4\pi} \kappa A - \frac{1}{4\pi} \int_{\Sigma} \ast R(K). \]  \hspace{1cm} (7)

In order to proceed, we first recall the identity \[ * d\omega = - K \wedge R(K), \]  \hspace{1cm} (8)
which is obtained from the definition (2) of \( \omega \), the Ricci identity (6) and the fact that \( d^\dagger (K \wedge \Omega) = - K \wedge d^\dagger \Omega \) for an arbitrary \( p \)-form \( \Omega \) with vanishing Lie derivative with respect to the Killing field \( K \). Making also use of the codifferential, \( d^\dagger = \ast d \ast \), and \( \ast 2 \Omega = (-1)^{p+1} \Omega \), we have \( *d\omega = (1/2)d^\dagger (K \wedge dK) = -(1/2)K \wedge d^\dagger dK = -K \wedge R(K) \), completing the derivation of eq. (8).

Applying the inner derivation \( i_K \) on the above identity and recalling that \( i_K \ast \Omega \equiv (-1)^{p} (K \wedge \Omega) \), we immediately obtain
\[ K \wedge d\omega = V \ast R(K) + R(K, K) \ast K, \]  \hspace{1cm} (9)
which enables us to express the integrand in eq. (7) in terms of \( R(K, K) \):
\[ M = \frac{1}{4\pi} \kappa A + \frac{1}{4\pi} \int_{\Sigma} \frac{R(K, K)}{V} \ast K - \frac{1}{4\pi} \int_{\Sigma} \frac{K \wedge d\omega}{V}. \]  \hspace{1cm} (10)

This already proves eq. (1) in the static case. However, if \( \omega \neq 0 \), the last term in the above equation needs further transformations. Noting that the definition (2) of the twist-form implies \( d(K/V) = -(2/V^2)i_K \ast \omega \), it is easy to derive the useful identity
\[ d\left( \frac{K \wedge \omega}{V} \right) = \frac{2}{V^2}(\omega|\omega) \ast K - \frac{1}{V}K \wedge d\omega. \]  \hspace{1cm} (11)
Integrating this identity over \( \Sigma \) and making use of Stokes’ theorem and the fact that the integrand \( (K \wedge \omega)/V \) vanishes on both components of the boundary, \( S_{\infty}^2 \) and \( \mathcal{H} \), we have
\[ \int_{\Sigma} \frac{K \wedge d\omega}{V} = 2 \int_{\Sigma} \frac{(\omega|\omega)}{V^2} \ast K, \]  \hspace{1cm} (12)
which completes the proof of the mass formula (1).
It is also worth noticing that the identity (11) yields a very direct proof of the fact that a strictly stationary spacetime is static, if it is Ricci static: First of all, eq. (8) shows that $R(K) \wedge K = 0$ (Ricci-staticity) is equivalent to $d\omega = 0$. The integrated version (12) of the identity (11) then implies that $\omega$ vanishes itself. The only non-trivial task is to establish that the boundary term does not contribute on the horizon [13], which can be concluded from the general properties of Killing horizons [11]. (We point out that the validity of this simple staticity proof is restricted to the case where the Killing field is timelike in all of the domain (strict stationarity). In order to overcome this difficulty, one takes advantage of a recent theorem by Chruściel and Wald [14], establishing the existence of a maximal slice.)

To summarize, we note that in a strictly stationary, asymptotically flat black hole spacetime the violation of the SEC for the unit Killing field at every point $\hat{k} = K/\sqrt{V}$ implies an upper bound for the total mass in terms of the horizon quantities $\kappa$ and $A$, that is,

$$M \leq \frac{1}{4\pi}\kappa A,$$

if $T(\hat{k}, \hat{k}) + \frac{1}{2} tr(T) \leq 0$, \hspace{1cm} (13)

whereas the converse inequality holds if the SEC is fulfilled and the domain is static.

3 Dominant Energy Condition and Spherical Symmetry

The violation of the SEC for the timelike Killing field implies that the total mass cannot exceed the value $\frac{1}{4\pi}\kappa A$ of the Komar integral at the horizon. Considering static spacetimes containing no black hole region, we immediately obtain $M \leq 0$, contradicting the positive energy theorem which is based on the DEC. This enables us to conclude that there are no non-trivial self-gravitating soliton solutions for matter models obeying the DEC but violating the SEC for the Killing field $\hat{k}$ at every point.

The extension of the argument to spacetimes containing a black hole would require a proof of the inequality $M \geq \frac{1}{(4\pi)}\kappa A$ as a consequence of the DEC. However, to our knowledge, a conjecture of this kind has not been established until now. In the general case, the strongest result consists in the
generalization of Witten’s proof of the positive mass theorem [2] (cf. also [3]) by Gibbons et. al. [4], establishing $M \geq 0$ for black hole spacetimes as well.

As we shall demonstrate in this section it is, however, not hard to derive the desired bound in the spherical symmetric case. As a consequence, we conclude that all static, spherically symmetric black hole solutions with matter satisfying the DEC but violating the SEC for $\hat{k}$ coincide with the Schwarzschild metric.

A static and spherically symmetric metric is parametrized by two functions depending on the radial coordinate $r$ only. Using the quantities $m(r)$ and $\delta(r)$ introduced in [7] we write

$$g = -NS^2dt^2 + N^{-1}dr^2 + r^2d\Omega^2, \quad (14)$$

$$N(r) = 1 - \frac{2m(r)}{r}, \quad S(r) = e^{-\delta(r)}. \quad (15)$$

(In terms of the familiar Schwarzschild parametrization one has $\delta = -(a+b)$ and $N = e^{-2b}$.) Tensor components will refer to the orthonormal frame $\{\theta^\mu\}$ of 1-forms,

$$\theta^0 = \sqrt{NS}dt, \quad \theta^1 = \frac{1}{\sqrt{N}}dr, \quad \theta^2 = r d\vartheta, \quad \theta^3 = r \sin \vartheta d\varphi, \quad (16)$$

where $S$ is positive and so is $N$, except on the horizon, $r = r_H$, where $N$ vanishes by definition. In terms of these quantities the Killing 1-form becomes $K = -NS^2dt = -\sqrt{NS}\theta^0$ and thus, since $*(\theta^0 \wedge \theta^1) = -\theta^2 \wedge \theta^3$, we have $*dK = S^{-1}\frac{d(NS^2)}{dr}*(\theta^0 \wedge \theta^1) = -r^2S^{-1}(NS^2) d\Omega$. Defining the “local mass” $M(r)$ by the Komar integral over a 2-sphere with coordinate radius $r$ now yields

$$M(r) = -\frac{1}{8\pi} \int_{S^2} *dK = \frac{r^2}{2S}(NS^2)' = mS + NS'r^2 - m'Sr. \quad (17)$$

As we have argued in the previous section, $M(r)$ is a decreasing function if $R(\hat{k}, \hat{k})$ is negative, i.e., if the SEC is violated for the Killing field $\hat{k}$. However, as a consequence of the DEC, we shall now establish that $\lim_{r \to \infty} M(r) \geq M_H$, independently of whether or not the SEC holds.

Computing the components of the Einstein tensor in the orthonormal frame (13), one finds

$$G_{00} = \frac{2}{r^2} m', \quad G_{00} + G_{11} = \frac{2N}{r} S^{-1} S'. \quad (18)$$
Requiring that $T(X)$ is future directed timelike or null for all future directed timelike vectors $X$, Einstein’s equations imply that the quantities $G_{00}$ and $G_{00} + G_{11}$ are not negative. Hence, provided that matter is subject to the DEC, both $m(r)$ and $S(r)$ are increasing functions,

$$m'(r) \geq 0, \quad S'(r) \geq 0 \quad \text{for} \quad r \geq r_H. \quad (19)$$

In order to gain estimates for $M(r)$ at infinity and at the horizon, it remains to note that the last term in eq. (17) does not contribute as $r \to \infty$, whereas the second term vanishes at the horizon. We obtain thus an upper bound for $M_H$ and a lower bound for $M := \lim_{r \to \infty} M(r)$. More precisely, asymptotic flatness implies the existence and finiteness of $m_\infty := \lim_{r \to \infty} m(r)$, $S_\infty := \lim_{r \to \infty} S(r)$ and $\lim_{r \to \infty} (r^2 S')$. Since, in addition, $\lim_{r \to \infty} (rm')$ vanishes and the second term in eq. (17) is non-negative, we have $M \geq m_\infty S_\infty$. For $r = r_H$, $N(r)$ vanishes which, together with the fact that $m'Sr$ is non-negative yields the estimate $M_H \leq m_H S_H$ for the Komar integral at the horizon. Hence, taking again advantage of the circumstance that $m(r)$ and $S(r)$ are increasing, we find

$$M_H \leq m_H S_H \leq m_\infty S_\infty \leq M, \quad (20)$$

which establishes the fact that the difference of the Komar integrals is not negative if the matter model is subject to the DEC,

$$- \frac{1}{8\pi} \int_{\partial \Sigma} *dK = M - M_H = M - \frac{1}{4\pi} \kappa A \geq 0, \quad (21)$$

where $\partial \Sigma = S_\infty^2 - H$.

As a consequence, we conclude that both functions, $m(r)$ and $S(r)$, have to assume constant values if the DEC holds and the SEC is violated for the timelike Killing field. In this case, the metric (14) coincides with the Schwarzschild metric and the energy momentum tensor thus vanishes.

It is probably worth noting that the DEC and eq. (17) together provide a simple estimate for the surface gravity of a spherically symmetric black hole: Evaluating the formula for $M(r)$ at the horizon, and using the fact that $M_H = \frac{1}{4\pi} \kappa A = \kappa r_H^2$, immediately yields

$$\kappa = \frac{S_H}{2r_H} (1 - 2m'_H). \quad (22)$$
Since the DEC implies that $m'(r) \geq 0$ and $S_H \leq S_\infty = 1$, we recover the fact [15], that the Hawking temperature $T_H = \frac{\kappa \hbar}{2\pi k}$ of a non-degenerate, spherically symmetric black hole with matter satisfying the DEC is bounded from above by the Hawking temperature $T_H^{(\text{vac})}$ of a Schwarzschild black hole (with the same area),

$$T_H = \frac{\hbar}{2\pi k} \frac{S_H}{2r_H} (1 - 2m'_H) \leq \frac{\hbar}{2\pi k} \frac{1}{2r_H} = T_H^{(\text{vac})}. \quad (23)$$

4 Application to Higgs Fields and Harmonic Maps

As already mentioned, the above result provides a very simple proof of the "no-hair" theorem for black hole solutions of self-gravitating spherically symmetric Higgs fields with arbitrary non-negative potentials. This result was already derived by Straumann and the author by means of scaling arguments [4], [8]. A different proof, taking advantage of the existence of a monotonic function was recently given by Sudarsky [16]. As a matter of fact, it is exactly the violation of the SEC which renders possible the construction of a Liapunov function, since eq. (6) becomes for scalar fields with potential $P[\phi]$

$$d * dK = 16\pi P[\phi] * K \Rightarrow \left[ \rho_s S^{-1} (NS^2) \right]' = 16\pi S P[\phi] \geq 0. \quad (24)$$

In order to apply the results obtained above, it only remains to verify that in a static domain, a selfgravitating harmonic map (non-linear $\sigma$-model) with an additional non-negative potential term satisfies the DEC and violates the SEC for the timelike Killing field at every point. We recall that a mapping $\phi$ from spacetime $(\mathcal{M}, g)$ into a Riemannian manifold $(\mathcal{N}, G)$ is said to be harmonic, if it is a solution to the variational equations for the matter Lagrangian

$$\frac{1}{2} G_{AB}(d\phi^A | d\phi^B) = \frac{1}{2} G_{AB}[\phi(x)] g^{\mu\nu}(x) \partial_\mu \phi^A \partial_\nu \phi^B, \quad (25)$$

where $G[\phi]$ denotes the Riemannian metric of the target manifold $\mathcal{N}$. In the simplest case, that is, if the target manifold is assumed to be a vector space, this reduces to the ordinary scalar field Lagrangian, describing a set
of Higgs fields, provided that an additional potential is taken into account as well.

Hence, we consider the matter Lagrangian

\[ \mathcal{L} = \frac{1}{2} G_{AB} (d\phi^A d\phi^B) + P[\phi], \]

with arbitrary Riemannian target metric \( G[\phi] \) and arbitrary non-negative potential \( P[\phi] \). The energy momentum tensor becomes

\[ T = G_{AB} \phi^A \otimes d\phi^B - g \mathcal{L}. \]

Together with the requirement that \( \phi \) is static, i.e., \( L_k \phi^A = 0 \), we immediately have \( T(\hat{k}, \hat{k}) = \mathcal{L} \) and \( tr(T) = -2 \mathcal{L} - 2P[\phi] \), implying the violation of the SEC for \( \hat{k} \) at every point of the domain,

\[ T(\hat{k}, \hat{k}) + \frac{1}{2} tr(T) = -P[\phi] \leq 0. \]  

On the other hand, the energy momentum tensor \( (27) \) clearly fulfills the DEC, since

\[ T_{00} = \mathcal{L}, \quad T_{00} + T_{11} = G_{AB} (d\phi^A)_1 (d\phi^B)_1 = G_{AB} N (\phi^A)' (\phi^B)'. \]  

Since the violation of the SEC for \( \hat{k} \) implies that \( M \leq \frac{1}{4 \pi \kappa A} \), whereas the DEC implies the converse inequality, we obtain \( M = \frac{1}{4 \pi \kappa A} \) and thus \( m' = S' = 0 \) and \( \phi' = 0 \), \( P[\phi] = 0 \). Hence, the metric is the Schwarzschild metric and the scalar fields have to assume a vacuum configuration.

## 5 Electromagnetic Fields

Let us now consider the case where a part of the matter consists of electromagnetic fields. Let \( T_{\mu\nu} = T_{\mu\nu}^{(em)} + T_{\mu\nu} \), where

\[ T_{\mu\nu}^{(em)} = \frac{1}{4\pi} \left[ F_\mu^\sigma F_{\nu\sigma} - \frac{1}{2} g_{\mu\nu} (F F) \right] \]

and \( T_{\mu\nu} \) denotes the energy-momentum tensor of the remaining matter fields. It is easy to see that

\[ 8\pi T^{(em)}(K, K) = (E|E) + (B|B), \]
where the 1-forms \( E \) and \( B \) are defined with respect to the Killing field, \( E := -i_K F, B := i_K * F \), also implying the identity \( V (F|F) = (B|B) - (E|E) \). Restricting ourselves to static configurations (the generalization to \( \omega \neq 0 \) is straightforward), eq. (31) becomes

\[
M = \frac{1}{4\pi}\kappa A + \frac{1}{4\pi} \int_{\Sigma} \frac{(E|E) + (B|B)}{V} \eta_{\Sigma} + 2 \int_{\Sigma} [T(\hat{k},\hat{k}) + \frac{1}{2} tr(T)] \eta_{\Sigma}, \tag{32}
\]

which shows that \( T(\hat{k},\hat{k}) + \frac{1}{2} tr(T) \leq 0 \) does no longer imply \( M \leq \frac{1}{4\pi}\kappa A \). However, as is well known, the electromagnetic contributions can be transformed into surface terms. In order to do this, we recall that the static Maxwell equations assume the simple form (cf. eg. [18])

\[
dE = dB = 0, \quad d^\dagger (E/V) = d^\dagger (B/V) = 0, \tag{33}
\]

which also implies the existence of the two potentials \( \phi \) and \( \psi \), with \( E = d\phi \) and \( B = d\psi \). (Here we assume that \( \Sigma \) is simply connected, i.e., that each component of the horizon at time \( \Sigma \) is a topological 2-sphere (cf. [10], and [19] for new results).)

Integrating \( d \ast F = 0 \) over \( \Sigma \), one immediately finds that the surface integrals of \( \ast F \) over the horizon and over \( S^2_{\infty} \) are equal, the latter being \( -4\pi \) times the electric charge \( Q \) by definition. Taking also advantage of the fact that in the static case \( E \wedge B = 0 \), we have \( d(\phi \ast F) = E \wedge \ast F = V^{-1}[E \wedge \ast (K \wedge E)] = V^{-1}(E|E) \ast K \), which yields

\[
-4\pi Q \Delta \phi = \int_{\partial \Sigma} \phi \ast F = \int_{\Sigma} \frac{(E|E)}{V} \ast K, \tag{34}
\]

where \( \Delta \phi := \phi_{\infty} - \phi_H \) and where we have also used the fact that the electric potential assumes a constant value over the horizon (cf. eg. [13]). Since similar considerations apply to the magnetic field, one obtains the generalized Smarr formula

\[
M - \frac{1}{4\pi}\kappa A + Q \Delta \phi + P \Delta \psi = 2 \int_{\Sigma} [T(\hat{k},\hat{k}) + \frac{1}{2} tr(T)] \eta_{\Sigma}, \tag{35}
\]

Hence, the combined Komar integral \( \tilde{I} \),

\[
\tilde{I} := -\frac{1}{4\pi} \int_{\partial \Sigma} \ast d\tilde{K}, \quad \text{where} \quad d\tilde{K} = dK + 2\phi F - 2\psi \ast F \tag{36}
\]
is non-positive whenever the additional matter fields violate the SEC for the timelike Killing field at every point of the domain. (Note that $2\phi F - 2\psi * F$ is closed, which justifies the introduction of the 1-form $\tilde{K}$.) It is also instructive to compute the differential of $\ast d\tilde{K}$,

$$d \ast d\tilde{K} = d \ast dK + 2(E \wedge \ast F + B \wedge F)$$

$$= 2 \ast [R(K) - 8\pi T^{(em)}(K)] = 16\pi \ast [T(K) - \frac{1}{2} tr(T)K],$$

which provides a second derivation of the mass formula (35), since for static configurations $T(K, K) * K = -V * T(K)$. Hence, in the presence of electromagnetic fields, the estimate (33) given at the end of the second section generalizes to

$$M \leq \frac{1}{4\pi} \kappa A - Q \Delta \phi - P \Delta \psi,$$

if $T(\hat{k},\hat{k}) + \frac{1}{2} tr(T) \leq 0$. (37)

The objective is now to establish the converse inequality on the basis of the DEC. As in the third section, we are able to reach this goal only under the additional assumption of spherical symmetry. With respect to the orthonormal tetrad introduced in eq. (16) we have

$$T^{(em)}_{00} = -T^{(em)}_{11} = \frac{1}{8\pi} S^{-2} (\phi'^2 + \psi'^2).$$

(38)

Hence, Einstein’s equations, together with the general identities (38) and the dominant energy condition for $T$ imply the inequalities

$$2r^{-2} m' - S^{-2} (\phi'^2 + \psi'^2) \geq 0, \quad N r^{-1} S^{-1} S' \geq 0.$$ 

(39)

In addition, in the spherically symmetric case, Maxwell’s equations for the potentials $\phi$ and $\psi$ simply reduce to

$$Q = -r^2 S^{-1} \phi', \quad P = -r^2 S^{-1} \psi'.$$

(40)

Our task is now to show that these relations imply the converse estimate for $M$ than the one given in eq. (37), that is, we have to establish that the combined Komar integral $\tilde{I}$ is non-negative. Using the expression (37), we obtain

$$\tilde{I} = \Delta(mS) + \Delta(NS'^2) - \Delta(m'Sr) + Q \Delta \phi + P \Delta \psi,$$

(41)
\( \Delta \) denoting the difference of the quantities between \( S^2_\infty \) and the horizon. Since 
\[ \lim_{\infty}(Sr) = 0, \quad S' \geq 0 \quad \text{and} \quad N_H = 0, \]
we have \( \Delta(NS'r^2 - m'Sr) \geq m'_H S_H r_H \), which we use to write
\[
\tilde{I} \geq S_\infty(\Delta m + m_H) - S_H(m_H - m'_H r_H) + Q\Delta \phi + P\Delta \psi.
\] (42)

In order to proceed, we need estimates for \( Q\Delta \phi, P\Delta \psi \) and \( \Delta m \). First of all, 
integrating Maxwell’s equations (40) and using the fact that \( 0 \leq S \leq S_\infty \) immediately yields
\[
Q\Delta \phi \geq -S_\infty \frac{Q^2}{r_H}, \quad P\Delta \psi \geq -S_\infty \frac{P^2}{r_H}.
\] (43)

Secondly, inserting Maxwell’s equations into the inequality (39) for \( m' \) and
integrating again, we also have
\[
m' \geq \frac{Q^2 + P^2}{2r_H}, \quad \Delta m \geq \frac{Q^2 + P^2}{2r_H}.
\] (44)

Taking advantage of these relations, we finally obtain the following lower bound for the combined Komar integral \( \tilde{I} \)
\[
\tilde{I} \geq S_\infty(\frac{Q^2 + P^2}{2r_H} + m_H) + S_H(\frac{Q^2 + P^2}{2r_H} - m_H) - S_\infty \frac{Q^2 + P^2}{r_H}
\] (45)
\[
= \Delta S(m_H - \frac{Q^2 + P^2}{4m_H}) \geq 0.
\]

Here we have used the regularity property of the horizon to perform the last step: As a matter of fact, taking advantage of the estimate (39) for \( m' \) as well as of the expression (17) at \( r = r_H \), we obtain
\[
m_H - \frac{Q^2 + P^2}{4m_H} \geq S_H^{-1}(m_H S_H - m'_H S_H r_H) = S_H^{-1} M_H = \frac{1}{4\pi S_H} \kappa A \geq 0.
\]

This completes the demonstration that spherically symmetric black hole configurations with electromagnetic fields and matter satisfying the DEC are subject to the inequality
\[
M \geq \frac{1}{4\pi} \kappa A - Q\Delta \phi - P\Delta \psi.
\] (46)
Hence, if $\mathcal{T}$ violates the SEC at every point for the timelike Killing, the relations (37) and (46) imply that the inequalities (39) become equalities. Computing the metric functions $S$ and $m$ and using the definition $N(r) = 1 - 2m(r)/r$ then yields the result

$$S = \text{const}, \quad N = 1 - \frac{2m_\infty}{r} + \frac{Q^2 + P^2}{r^2}, \quad \phi = \frac{Q}{r}, \quad \psi = \frac{P}{r},$$

(47)
i.e., the Reissner-Nordström solution, where the remaining matter fields have to assume a vacuum configuration, $\mathcal{T}_{\mu\nu} = 0$. As an application, this enables us to generalize the “no-hair” result for selfgravitating scalar fields discussed in the previous section to the case where electromagnetic fields are present as well.

We finally note that the estimate (22) for the Hawking temperature together with the inequality (44) for $m'$ now yields the stronger bound (48)

$$T_H = \frac{\hbar}{2\pi k} \frac{1}{2r_H} (1 - \frac{Q^2 + P^2}{r_H^2}) = T_{H}^{(\text{evac})},$$

(48)

$T_{H}^{(\text{evac})}$ denoting the Hawking temperature of the Reissner-Nordström black hole.

6 Conclusion

We have established the lower bounds (21) and (46) for the total mass of a spherically symmetric black hole spacetime with matter satisfying the DEC. On the other hand, for a static (strictly stationary) configuration the Komar expressions for the mass and electromagnetic monopole charges imply the converse estimates, provided that $T - T^{(\text{em})}$ violates the SEC for the timelike Killing field $K$ at every point of the domain. This yields a simple criterion which, for instance, enables one to exclude black hole solution with scalar hair.

As already mentioned, this “no-hair” theorem has been established some time ago by Bekenstein [4] and generalized to arbitrary non-negative potentials by different means. However, in contrast to our earlier attempts, [7], [8] the reasoning presented here enables one to exclude non-trivial solutions by simply verifying the energy conditions. Unfortunately, the second part
of the argument, i.e., the proof that the mass can be estimated from below by the Komar integral over the horizon, is heavily based on the requirement of spherical symmetry. We think that it might be interesting to investigate whether this bound can be derived on the basis of staticity alone, that is, without assuming spherical symmetry.

If no potential is taken into account, we were able to establish the general “no-hair” theorem for selfgravitating static harmonic maps [20] by adapting the uniqueness proof for static vacuum black holes [22]. It is also possible to handle the case of a convex potential, by taking advantage of Stoke’s theorem [9] (see also [21]). However, without imposing spherical symmetry, the problem still remains open for arbitrary non-negative potentials.

We finally point out that all uniqueness results for scalar maps rely on the *harmonic* structure of the Lagrangian. This is a necessary restriction, as is reflected by the fact that there do exist black hole solutions with scalar hair if more general actions are admitted [23], [24].

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