ON THE TANNAKA GROUP ATTACHED TO THE THETA DIVISOR OF A GENERIC PRINCIPALLY POLICIZED
ABELIAN VARIETY

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Abstract. To any closed subvariety $Y$ of a complex abelian variety one can attach a reductive algebraic group $G$ which is determined by the decomposition of the convolution powers of $Y$ via a certain Tannakian formalism. For a theta divisor $Y$ on a principally polarized abelian variety, this group $G$ provides a new invariant that naturally endows the moduli space $A_g$ of principally polarized abelian varieties of dimension $g$ with a finite constructible stratification. We determine $G$ for a generic principally polarized abelian variety, and for $g = 4$ we show that the stratification detects the locus of Jacobian varieties inside the moduli space of abelian varieties.

1. Introduction

Some important moduli problems in algebraic geometry are related to principally polarized abelian varieties (ppav’s) and their theta divisors. For instance, the Torelli theorem allows to reconstruct any smooth projective curve from the theta divisor on its Jacobian variety [31] [8]. Jacobian varieties have many special features, and the search for characterizations of Jacobians among all ppav’s, i.e. the classical Schottky problem, has a long history [4] [10] [17]. In this context the singularities of the theta divisor play an important role, observed already by Andreotti and Mayer and others [2] [7] [18]. Since from a cohomological point of view the study of singular varieties naturally leads to the theory of perverse sheaves, one may ask whether these can also give some insight on theta divisors. We discuss an approach along these lines that has been suggested in [34].

To any complex ppav $X$ with an irreducible theta divisor $\Theta \subset X$ we attach a complex reductive algebraic group $\tilde{G} = G(\delta_\Theta)$ via the Tannakian formalism of perverse sheaves given in [26]. This group is a new invariant of ppav’s which subtly encodes information on the singularities of the theta divisor. By construction, the theta divisor corresponds to a finite-dimensional irreducible linear representation $V$ of $G$ whose tensor powers $V^\otimes r$ describe the convolution powers of the theta divisor. The group $G$ is almost connected in the sense that $V$ is an irreducible representation of the connected component $G^0 \subset G$ and the quotient $\pi_0 = G/G^0$ is a finite abelian group. Note that $G$ depends on the theta divisor and not just on the polarization; the latter determines the former.
only up to a translation by a point in $X(\mathbb{C})$. In what follows we always choose the theta divisor $\Theta \subset X$ to be symmetric in the sense that it is stable under the morphism $-id_X$. Recall that the moduli space $A_g$ of complex ppav’s of dimension $g$ admits a finite cover

$$B_g \rightarrow A_g \quad \text{of degree } \quad 2^{2g}$$

where $B_g$ denotes the moduli space of ppav’s with a theta level structure as defined in [15, sect. I.6 and IV.7]. Each point in $B_g$ corresponds to a ppav in $A_g$ together with a symmetric theta divisor. Since the general linear group of any given rank has only finitely many connected reductive subgroups up to conjugacy, it follows that the open dense locus $B^o_g \subset B_g$ of indecomposable ppav’s is a disjoint union

$$B^o_g = \bigcup_{\alpha=1}^n S_\alpha$$

of finitely many strata such that up to isomorphism, the group $G(\delta_\Theta) = G_\alpha$ only depends on the stratum $S_\alpha$ in which the ppav $X$ with its symmetric theta divisor $\Theta$ lies. This stratification is constructible for the Zariski topology. Indeed, the results in the appendix imply that the groups $G_\alpha$ behave in a lower semicontinuous way in the sense that for any stratum $S_\beta$ in the closure of $S_\alpha$ we have a non-canonical embedding $G_\beta \hookrightarrow G_\alpha$. We also remark that the connected component of $G(\delta_\Theta)$ does not depend on the chosen symmetric theta divisor, and hence defines a finite constructible stratification of the moduli space $A_g$ whose pull-back to $B_g$ has the previous stratification as a refinement. It is an interesting question to ask what the above stratifications look like and what Tannaka groups can occur. Not much is known in this direction. The goal of this paper is to show that for a generic ppav one has $G(\delta_\Theta) = Sp_g(\mathbb{C})$ or $G(\delta_\Theta) = SO_g(\mathbb{C})$ depending on whether $g$ is even or odd. Note that by the above semicontinuity property this also provides an upper bound on the groups for non-generic ppav’s. By the dimension formula recalled below, it furthermore gives non-trivial information on the Euler characteristic of perverse sheaves that appear in the convolution powers of theta divisors.

Before we discuss our results in more detail, let us briefly recall the Tannakian description for categories of perverse sheaves. Let $X$ be a complex abelian variety. By definition, the category $P = \text{Perv}(X, \mathbb{C})$ of perverse sheaves is the core of the middle perverse $t$-structure on the derived category $D = D^b_c(X, \mathbb{C})$ of bounded constructible complexes of $\mathbb{C}$-sheaves [20, ch. 8]. In particular, it is a full abelian subcategory of the derived category. The group law $a : X \times X \rightarrow X$ gives rise to a convolution product

$$K_1 * K_2 = R\alpha_*(K_1 \boxtimes K_2) \quad \text{for} \quad K_1, K_2 \in D,$$

and although the subcategory $P \subset D$ is not stable under this product, it has been shown in [20, lemma 13.1] that the perverse sheaves of Euler characteristic zero form a Serre subcategory $N \subset P$, such that on the abelian quotient category $\overline{P} = P/N$ we obtain a well-defined convolution product $* : \overline{P} \times \overline{P} \rightarrow \overline{P}$ with similar formal properties as the tensor product of vector spaces. For any perverse sheaf $P \in \overline{P}$ the subquotients of the convolution powers of $P$ form an abelian subcategory $\langle P \rangle \subset \overline{P}$ stable under the convolution product, and by corollary 13.3 in loc. cit. there is a complex algebraic group $G = G(P)$, unique up to non-canonical isomorphism, such
that we have an equivalence of abelian categories
\[ \omega : \langle P \rangle \rightarrow \text{Rep}_C(G) \]
under which the convolution product on the left hand side corresponds to the usual tensor product on the category \( \text{Rep}_C(G) \) of finite-dimensional complex algebraic representations of \( G \). Furthermore, the dimension of the representation \( \omega(Q) \) is given by the Euler characteristic
\[ \dim_\mathbb{C}(\omega(Q)) = \chi(Q) = \sum_{n \in \mathbb{Z}} (-1)^n \dim(H^n(X, Q)) \]
for any \( Q \in \langle P \rangle \), and this is well-defined because in our quotient category we only divided by perverse sheaves of Euler characteristic zero. The above in particular applies to the perverse intersection cohomology sheaf \( P = \delta_Y \) supported on a closed subvariety \( i : Y \hookrightarrow X \) of dimension \( d \). Recall that by definition
\[ \delta_Y = \text{IC}_Y[d] = i_* j_! \mathbb{C}_U[d] \]
where \( j : U \hookrightarrow Y \) is the embedding of the smooth locus. Since \( \delta_Y \) is a semisimple perverse sheaf, the group \( G(\delta_Y) \) is reductive.

We are interested in the case where \( Y = \Theta \) is a symmetric theta divisor on a ppav \( X \) of dimension \( g > 1 \). For a generic ppav we will see later that the choice of the symmetric theta divisor does not affect the group \( G(\delta_\Theta) \). But let us start with the following example [32, p. 124 and th. 14] to emphasize that this group behaves rather differently from classical motivic invariants like Mumford-Tate groups.

**Theorem 1.1.** For the Jacobian variety \( X = \text{JC} \) of a smooth complex projective curve \( C \) of genus \( g > 1 \) the Tannaka group is
\[ G(\delta_\Theta) = \begin{cases} \text{Sp}_{2g-2}(\mathbb{C})/\mu_{g-1} & \text{if } C \text{ is hyperelliptic,} \\
\text{SL}_{2g-2}(\mathbb{C})/\mu_{g-1} & \text{otherwise,} \end{cases} \]
where \( \mu_{g-1} \) denotes the central subgroup of \((g-1)^{st}\) roots of unity and \( \epsilon_{g-1} \subseteq \{\pm 1\} \) is its intersection with the center of the symplectic group.

The proof in loc. cit. uses that the theta divisor is the image of the Abel-Jacobi morphism \( C^{g-1} \rightarrow X \) and also shows that \( \omega(\delta_\Theta) \) is the \((g - 1)^{st}\) fundamental representation of the symplectic resp. special linear group in this case. As in [34] one may pose the question whether among all ppav’s of dimension \( g \) only Jacobian varieties have the above Tannaka groups. More generally, one may ask for the stratification which the connected component of the Tannaka group \( G(\delta_\Theta) \) defines on the moduli space \( \mathcal{A}_g = \mathcal{H}_g/\text{Sp}_{2g}(\mathbb{Z}) \) of ppav’s, where \( \mathcal{H}_g \) denotes the Siegel upper half space. Since we normalized our theta divisors to be symmetric, the general constructibility results in the appendix imply that this a finite algebraic stratification. For the open stratum the following guess seems rather plausible.

**Conjecture 1.2.** Every complex ppav \( X \) of dimension \( g > 1 \) with a smooth theta divisor has the Tannaka group
\[ G(\delta_\Theta) = \begin{cases} \text{SO}_{g!}(\mathbb{C}) & \text{for } g \text{ odd,} \\
\text{Sp}_{g!}(\mathbb{C}) & \text{for } g \text{ even,} \end{cases} \]
and \( \omega(\delta_\Theta) \) is the standard representation of this classical group.
Note that for \( g = 2 \) this is true by theorem 1.1, since in this case any ppav with a smooth theta divisor is the Jacobian of a smooth hyperelliptic curve. For \( g = 3 \) any ppav with a smooth theta divisor is the Jacobian of a smooth non-hyperelliptic curve, and once again theorem 1.1 applies (the second fundamental representation of \( SL_4(\mathbb{C})/\mu_2 \) is identified with the standard representation of \( SO_6(\mathbb{C}) \) under the exceptional isomorphism between these two groups). So, the first open case occurs in dimension \( g = 4 \) where the realm of Jacobian varieties is left.

In what follows, we put \( G(\delta_{\Theta}) = SO_g(\mathbb{C}) \) if \( g \) is odd, but \( G(\delta_{\Theta}) = Sp_g(\mathbb{C}) \) if \( g \) is even. Let us say that a statement holds for a general ppav if it holds for every ppav in a Zariski-open dense subset of \( A_g \). Thus, the theta divisor of a general ppav is smooth. The main goal of this paper is to establish the following generic version of the above conjecture.

**Theorem 1.3.** For a general complex ppav \( X \) of dimension \( g > 1 \) the theta divisor has the Tannaka group
\[
G(\delta_{\Theta}) = G(g),
\]
and \( \omega(\delta_{\Theta}) \) is the standard representation of dimension \( g! \) of this group.

In the first relevant case \( g = 4 \) we have \( G(\delta_{\Theta}) = Sp_{24}(\mathbb{C}) \), and in this case we can actually say more. Recall from [9] that the Andreotti-Mayer locus \( N_g \subset A_g \) of ppav’s with a singular theta divisor is for \( g \geq 4 \) itself a divisor with two irreducible components. One component is the locus \( \theta_{null,4} \) of ppav’s with a vanishing theta null, the other component contains the closure \( J_4 \) of the locus of Jacobian varieties and for \( g = 4 \) is equal to it [9].

**Theorem 1.4.** For \( g = 4 \) the locus in \( A_4 \) of all ppav’s with \( G(\delta_{\Theta}) \neq G(g) \) is a Zariski-closed subset whose only components of codimension one in the moduli space \( A_4 \) are the Jacobian locus \( J_4 \) and the theta null locus \( \theta_{null,4} \).

The proof of this will be given along with our proof of theorem 1.3, using that every divisor on \( A_4 \) intersects the locus of Jacobian varieties. Assuming theorem 1.4 we easily recover the following result of [25] showing for \( g = 4 \) that our Tannaka groups are closely related to the Schottky problem.

**Corollary 1.5.** For \( g = 4 \) the invariant \( G = G(\delta_{\Theta}) \) determines both the Jacobian locus and the theta null locus
\[
J_4 \subset A_4 \quad \text{and} \quad \theta_{null,4} \subset A_4.
\]

**Proof.** We know from theorem 1.1 that a general ppav in \( J_4 \) has the Tannaka group \( G = SL_6(\mathbb{C})/\mu_3 \). On the other hand, for a general ppav with a vanishing theta null the perverse sheaf \( \delta_{\Theta} \) has Euler characteristic \( \chi(\delta_{\Theta}) = g! - 2 = 22 \) by part (2) of proposition 4.2 below. Hence by the dimension estimates in [10] such a ppav cannot have the Tannaka group \( SL_6(\mathbb{C})/\mu_3 \). In fact, with similar arguments as in the proof of theorem 1.1 a degeneration into a Jacobian variety shows that a general ppav in \( \theta_{null,4} \) has the Tannaka group \( G = Sp_{22}(\mathbb{C}) \).

For the proof of theorem 1.3 and 1.4 we proceed as follows. In section 2, we observe that by self-duality one always has an embedding \( G(\delta_{\Theta}) \hookrightarrow G(g) \). Thus the main task will be to find sufficiently large lower bounds on the Tannaka group of a general ppav. By constructibility (see the appendix), it suffices to do this for the generic fibre of a suitable family of ppav’s. So we can use degeneration arguments,
since the Tannaka group of the generic fibre contains the Tannaka groups of the perverse sheaves of nearby cycles on the special fibres [26 lemma 14.1]. In section 3 we gather some general properties of the monodromy filtration that allow to get hold on the nearby cycles in terms of weights. In section 4 we apply this to three degenerations of a generic ppav where the degenerate fibre has a vanishing theta null, or splits as a product of ppav’s or is a Jacobian variety. In each case, the nearby cycles contain a large irreducible constituent described geometrically. This will reduce the proof of our theorem to standard arguments in the representation theory of reductive groups, see section 5. We also include in section 6 an alternative argument, not requiring the degeneration into the theta null locus but instead embarking on more detailed information about the other two degenerations. This alternative approach has the benefit that it introduces motivic techniques that may be useful for other applications as well. Finally, the appendix in section 7 collects some general constructibility properties of the Tannaka groups defined in [26].

2. An upper bound on the Tannaka group

In this section we show that the symplectic resp. special orthogonal group $G(g)$ in conjecture 1.2 is the biggest group that possibly can occur. Let $X$ be a complex ppav of dimension $g > 1$ and $\Theta \subset X$ an irreducible symmetric theta divisor. The irreducibility implies that the theta divisor is normal [14]. Furthermore $\delta_\Theta$ is a simple perverse sheaf, and by symmetry of the theta divisor it is isomorphic to its adjoint dual $(\delta_\Theta)^\vee$ in the sense of [26 sect. 4]. On the Tannakian side, if

$$\omega: \langle \delta_\Theta \rangle \xrightarrow{\sim} \text{Rep}_C(G(\delta_\Theta))$$

is a fibre functor as in the introduction, it follows that $V = \omega(\delta_\Theta)$ is an irreducible self-dual representation of $G(\delta_\Theta)$. Any such representation is either orthogonal or symplectic, depending on whether the trivial representation 1 lies in the alternating square $\Lambda^2(V)$ or in the symmetric square $S^2(V)$. To decide which of the two cases occurs, consider the commutativity constraint $S = S_{\delta_\Theta, \delta_\Theta} : \delta_\Theta \ast \delta_\Theta \xrightarrow{\sim} \delta_\Theta \ast \delta_\Theta$ as defined in [22 sect. 2.1]. We claim that this commutativity constraint $S$ acts by $(-1)^{g-1}$ on the stalk cohomology group $H^0(\delta_\Theta \ast \delta_\Theta)_0$. Indeed $\delta_\Theta = IC_{\Theta}[g - 1]$, so our claim amounts to the statement that the constraint $S$ acts trivially on the group $H^{2g-2}(IC_\Theta \ast IC_\Theta)_0$. For this latter statement one can use that, since by assumption the theta divisor is normal, we have a natural isomorphism

$$H^{2g-2}(IC_\Theta \ast IC_\Theta)_0 \cong H^{2g-2}(C_\Theta \ast C_\Theta)_0$$

which allows to replace the intersection cohomology sheaf $IC_\Theta$ with the constant sheaf $C_\Theta$. Then our claim easily follows via base change.

The trivial representation $1 = \omega(\delta_0)$ corresponds on the geometric side to the skyscraper sheaf $\delta_0$ of rank one supported in the origin. Hence, it follows from the above considerations that the representation $V = \omega(\delta_\Theta)$ is orthogonal for $g$ even and symplectic for $g$ odd. This leads to the following upper bound.

**Lemma 2.1.** For any smooth symmetric theta divisor $\Theta \subset X$ on a complex ppav of dimension $g$ there exists an embedding

$$G(\delta_\Theta) \hookrightarrow G(g)$$

such that $V = \omega(\delta_\Theta)$ is the restriction of the standard representation of $G(g)$.
Proof. We have \( \dim_C(V) = \chi(\delta_\Theta) = g! \) by the Gauss-Bonnet formula because the theta divisor is smooth. So, by the above remarks our claim follows with the full orthogonal group in place of the special orthogonal group. It remains to show that the determinant character \( \det(V) \) is trivial. If it were not, then by [26, prop. 10.1] this character would correspond to a skyscraper sheaf \( \delta_x \) supported in a non-zero point \( x \in X(\mathbb{C}) \) which would be a 2-torsion point by self-duality. Note that, since the symmetric theta divisor \( \Theta \subset X \) is determined uniquely up to a translation by a 2-torsion point, all even convolution powers \( (\delta_\Theta)^{2n} \) with \( n \in \mathbb{N} \) are intrinsically defined. In particular, since \( g! \) is an even number for \( g > 1 \), the non-zero 2-torsion point \( x \in X(\mathbb{C}) \) with \( \omega(\delta_x) = \det(V) \) is intrinsically defined and does not depend on our choice of the symmetric theta divisor. But for monodromy reasons it is impossible to select such a point naturally on every ppav on a Zariski-open dense subset of the moduli space \( \mathcal{A}_g \). So we can finish the proof by a specialization argument, using the semicontinuity lemma 7.2 in the appendix. \( \square \)

Remark 2.2. For an indecomposable ppav \( X \) with a symmetric but not necessarily smooth theta divisor \( \Theta \subset X \) the specialization argument in the above proof still implies that the Tannaka group \( G(\delta_\Theta) \) is a subquotient of \( G(g) \).

3. Local Monodromy

For convenience of the reader we collect in this section some general facts that allow to control degenerations of perverse sheaves in terms of weights. Throughout we consider the following local algebraic setting. Let \( S \) be the spectrum of a strictly Henselian discrete valuation ring (in our case it will be the strict Henselization of a smooth complex algebraic curve in the moduli space of ppav’s, but with other applications in mind we include the case of mixed or positive characteristic). In the subsequent considerations we denote by \( s \) the closed point of \( S \) and by \( \bar{\eta} \) a geometric point over the generic point \( \eta \) of \( S \). For a separated \( S \)-scheme of finite type \( f : Y \to S \) we consider the functor of nearby cycles [22, sect. 4]

\[
\Psi : \mathbf{P}(Y_\eta) = \text{Perv}(Y_\eta, \Lambda) \longrightarrow \mathbf{P}(Y_s) = \text{Perv}(Y_s, \Lambda)
\]

on perverse sheaves with coefficients in \( \Lambda = \overline{\mathbb{Q}}_l \), where \( l \) is some fixed prime which is invertible on \( S \). For any \( \delta \in \mathbf{P}(Y_\eta) \) the local monodromy group \( G = \text{Gal}(\bar{\eta}/\eta) \) acts naturally on \( \Psi(\delta) \), see [13, exp. XIII]. Since we are working over a strictly Henselian base, this local monodromy group is equal to the inertia group.

If \( p \geq 0 \) denotes the residue characteristic of the point \( s \), the inertia group sits in an exact sequence \( 1 \to P \to G \to \prod_{l' \neq p} \mathbb{Z}_{l'}(1) \to 1 \), where \( P \) is the wild inertia group and where \( l' \) runs through the set of all primes different from \( p \). In what follows we always assume that the nearby cycles \( \Psi(\delta) \) are tame in the sense that \( P \) acts trivially on them, a condition which of course is void for \( p = 0 \). The perverse sheaf \( \Psi(\delta) \) is then equipped with a natural action of the quotient \( \mathbb{Z}_l(1) \) of the tame inertia group. We denote by \( T : \Psi(\delta) \to \Psi(\delta) \) the endomorphism induced by a topological generator \( 2\pi i \) of the group \( \mathbb{Z}_l(1) \). In the abelian category \( \text{Perv}(Y_s, \Lambda) \) we have as in [29] lemma 1.1] a Jordan decomposition

\[
\Psi(\delta) = \Psi_1(\delta) \oplus \Psi_{\neq 1}(\delta) \quad \text{with} \quad \Psi_{\neq 1}(\delta) = \bigoplus_{\alpha \neq 1} \Psi_\alpha(\delta),
\]
where for each $\alpha \in \Lambda$ the perverse subsheaf $\Psi_\alpha(\delta)$ is stable under the action of $T - \alpha \cdot \text{id}$. In what follows, we will be particularly interested in the perverse intersection cohomology sheaf $\delta = \delta_{Y\eta}$.

**Remark 3.1.** Suppose the nearby cycles $\Psi(\delta_{Y\eta})$ are tame. If $f: Y \to S$ is proper and if the geometric generic fibre $Y_{\eta}$ is smooth, then after replacing $f$ by its base change $f': Y' \to S'$ under a finite branched cover $S' \to S$ we can assume

$$H^*(Y'_s, \Psi_{\neq 1}(\delta_{Y'\eta})) = 0.$$ 

Proof. Let $S' \to S$ be the normalization of $S$ in a finite extension of the residue field of $\eta$ with generic point $\eta' \to \eta$, and denote by $\overline{S} \to S$ the normalization in the residue field of $\eta$. For the base changes $Y' = Y \times_S S'$ and $\overline{Y} = Y \times_S \overline{S}$ we then have a commutative diagram

\[
\begin{array}{ccc}
Y_s & \to & Y_{\eta} \\
\downarrow & & \downarrow \\
Y'_s & \to & Y'_{\eta} \\
\downarrow & & \downarrow \\
Y_s & \to & Y_{\eta}
\end{array}
\]

where the vertical identifications on the left hand side hold, since we are working over a strictly Henselian base. So, as an object of $\text{Perv}(Y_s, \Lambda) = \text{Perv}(Y'_s, \Lambda)$, the nearby cycles $\Psi(\delta_{Y_{\eta}})$ remain unchanged if we replace our original family $Y \to S$ by the base change $Y' \to S'$. But the local monodromy operation, and hence in the tame case the Jordan decomposition is modified under this replacement.

Now, for each $\alpha \in \Lambda$ some power of $T - \alpha \cdot \text{id}$ acts trivially on $\Psi_{\alpha}(\delta_{Y_{\eta}})$, and hence also on its hypercohomology. So the Jordan decomposition

$$H^*(Y_s, \Psi(\delta_{Y_{\eta}})) = \bigoplus_{\alpha} H^*(Y_s, \Psi_\alpha(\delta_{Y_{\eta}}))$$

shows that $H^*(Y_s, \Psi_{\neq 1}(\delta_{Y_{\eta}})) = 0$ iff $T$ acts unipotently on $H^*(Y_s, \Psi(\delta_{Y_{\eta}}))$. The latter can be achieved after a finite branched base change $S' \to S$ of the form considered above: Proper base change shows $H^*(Y_s, \Psi(\delta_{Y_{\eta}})) = H^*(Y_{\eta}', \delta_{Y_{\eta}'})$, and since the geometric generic fibre $Y_{\eta}$ is smooth, Grothendieck’s local monodromy theorem [22, th. 1.4] says that on these cohomology groups $T$ is quasi-unipotent. □

Returning to an arbitrary perverse sheaf $\delta \in \text{P}(Y_{\eta})$ with tame nearby cycles, to get hold on $\Psi_1(\delta)$ we consider the nilpotent operator

$$N = \frac{1}{2\pi i} \log(T): \quad \Psi_1(\delta) \to \Psi_1(\delta)(-1),$$

where $\frac{1}{2\pi i} \in \mathbb{Z}_l(-1)$ is the dual of the generator $2\pi i \in \mathbb{Z}_l(1)$. By [11, sect. 1.6], there exists a unique finite increasing filtration $F_*(\Psi_1(\delta))$ on the perverse sheaf $\Psi_1(\delta)$ such that $N(F_1(\Psi_1(\delta))) \subseteq F_{-2}(\Psi_1(\delta))(-1)$ and such that each iterate $N^i$ induces an isomorphism $\text{Gr}_i(\Psi_1(\delta)) \cong \text{Gr}_{-i}(\Psi_1(\delta))(-i)$ of the graded pieces with respect to the filtration. In the following, we denote by $P_{-i}(\delta)$ the kernel of $N$ on $\text{Gr}_{-i}(\Psi_1(\delta))$ for $i \geq 0$. Then

$$\text{Gr}_{-i}(\Psi_1(\delta)) \cong \bigoplus_{k \geq 0} P_{-i - 2k}(-k)$$
by loc. cit. We represent this situation by the next diagram. There each horizontal line of the triangle contains the composition factors of the graded piece shown on the left (for the morphisms $N$ the Tate twists must be ignored).

\[
\begin{array}{c c c}
\vdots & \vdots & \vdots \\
Gr_2(\Psi_1(\delta)) & P_{-2}(\delta)(-2) & \\
Gr_1(\Psi_1(\delta)) & P_{-1}(\delta)(-1) & N \\
Gr_0(\Psi_1(\delta)) & P_0(\delta) & N \\
Gr_{-1}(\Psi_1(\delta)) & P_{-1}(\delta) & N \\
Gr_{-2}(\Psi_1(\delta)) & P_{-2}(\delta) & \\
\vdots & \vdots & \vdots \\
\end{array}
\]

The lower boundary entries $P_0(\delta), P_{-1}(\delta), P_{-2}(\delta), \ldots$ of the triangle are the graded pieces of the specialization

\[sp(\delta) = \ker(N : \Psi_1(\delta) \to \Psi_1(\delta)(-1)),\]

with $P_0(\delta)$ as the top quotient. So, the graded pieces of $sp(\delta)$ determine those of all the $Gr_r(\Psi_1(\delta))$. For mixed perverse sheaves of geometric origin \cite[chapt. 6]{mix} we have the following result of Gabber \cite[th 5.1.2]{mix}.

**Remark 3.2.** If $\delta$ is pure of weight $w$, each $Gr_r(\Psi_1(\delta))$ is pure of weight $w+i$ so that the monodromy filtration coincides with the weight filtration up to a shift.

Returning to the general case, let $j : Y_\eta \to Y$ and $i : Y_s \to Y$ be the geometric generic resp. special fibre. Recall \cite[p. 48]{mix} that the perverse $t$-structure on $D(Y)$ is defined in terms of the perverse $t$-structures on these fibres by

\[
K \in \mathcal{P}D^{\leq 0}(Y) \iff i^*K \in \mathcal{P}D^{\leq 0}(Y_s) \quad \text{and} \quad j^*K \in \mathcal{P}D^{\leq -1}(Y_\eta),
\]

\[
K \in \mathcal{P}D^{\geq 0}(Y) \iff i^!K \in \mathcal{P}D^{\geq 0}(Y_s) \quad \text{and} \quad j^!K \in \mathcal{P}D^{\geq -1}(Y_\eta).
\]

We denote by $\mathcal{P}(Y)$ the abelian category of perverse sheaves defined as the core of this perverse $t$-structure. By abuse of notation, for a perverse sheaf $\delta \in \mathcal{P}(Y_\eta)$ we also write $\delta$ for the pull-back to $\mathcal{P}(Y_s)$. Then $Rj_!(\delta[1])$ and $Rj_*(\delta[1])$ are perverse and $j_*(\delta[1]) \in \mathcal{P}(Y)$ is by definition the image of the natural morphism between them in the abelian category of perverse sheaves, see loc. cit.

**Lemma 3.3.** With notations as above,

\[sp(\delta) = i^*(j_*(\delta[1]))[1].\]

**Proof.** We first claim that in the triangulated category $D(Y_s) = D^b_c(Y_s, \Lambda)$, the cone of the morphism $N$ is given by

\[\text{Cone}(\Psi_1(\delta) \xrightarrow{N} \Psi_1(\delta)(-1)) = i^*Rj_*(\delta[1]).\]

Indeed, if we forget about weights, the cone of $N$ on $\Psi_1(\delta)$ is isomorphic to the cone of $T - 1$ on $\Psi(\delta)$, because $T - 1$ is an isomorphism on $\Psi_{\neq 1}(\delta)$ and on $\Psi_1(\delta)$ its kernel and cokernel are isomorphic to those of $N$. Hence \cite[eq. (3.6.2) and thereafter]{mix}, using that the wild inertia group $P$ acts trivially on the nearby cycles. Now if for $n \in \{0, 1\}$ we consider the perverse
cohomology sheaves $sp^n(\delta) = pH^n(i^*R^j_*(\delta))$, we get from \cite{[24]} an exact sequence of perverse sheaves

$$0 \rightarrow sp^0(\delta) \rightarrow \Psi_1(\delta) \xrightarrow{N} \Psi_1(\delta)(-1) \rightarrow sp^1(\delta) \rightarrow 0$$

which in particular implies $sp(\delta) = sp^0(\delta)$. To finish the proof, note that by the perverse truncation property of the middle perverse extension \cite{[24]} sect. III.5.1 we have $i^*(j_*(\delta[1])) = p_{\tau<0}i^*(R^j_*(\delta[1]))$ and that by \cite{[24]} the complex $i^*(R^j_*(\delta[1]))$ is concentrated in perverse cohomology degrees $-1$ and $0$. $\square$

**Back to the complex analytic case.** To translate the above local results to a more global setting which is closer to our applications, it will be convenient to reset our notation. For this let $S$ be a smooth complex algebraic curve and $f : Y \rightarrow S$ a morphism of complex algebraic varieties, smooth over the complement of some given point $s \in S(\mathbb{C})$. Consider the strict Henselization $\tilde{S} = \text{Spec}(\mathcal{O}_{\tilde{S},s}) \rightarrow S$ of $S$ at the point $s$, and let $\eta$ be its generic point. Then, referring to the base change $\tilde{f} : \tilde{Y} = Y \times_S \tilde{S} \rightarrow \tilde{S}$ of the morphism $f$ under this strict Henselization, we can consider the nearby cycles and specialization functors

$$\mathbf{P}(\tilde{Y}_\eta) \xrightarrow{\psi_{\mathbb{C}}} \mathbf{P}(\tilde{Y}_s)$$

where $\tilde{Y}_\eta$ and $\tilde{Y}_s$ denote the generic resp. special fibres of $\tilde{f}$. Note that $\tilde{Y}_s$ is naturally identified with the fibre $f^{-1}(s) \subset Y$. For the perverse intersection cohomology sheaf the passage between the global and the local picture is given by the following result, where $i : \tilde{Y}_s = f^{-1}(s) \hookrightarrow Y$ denotes the embedding.

**Corollary 3.4.** In the above situation,

$$sp(\delta_{\tilde{Y}_\eta}) = i^*(\delta_Y[-1]).$$

**Proof.** Our smoothness assumption on $f$ implies that the generic fibre $Y_\eta$ is smooth, so that the perverse intersection cohomology sheaf $\delta = \delta_{\tilde{Y}_\eta}$ is the constant sheaf up to a degree shift. It follows that the middle perverse extension $j_*(\delta[1])$ in lemma 3.3 arises from the perverse intersection cohomology sheaf $\delta_Y$ on the total space $Y$ via the Henselization morphism $\tilde{Y} \rightarrow Y$, and we are done. $\square$

4. Degenerations of theta divisors

To construct degenerations of theta divisors, we fix an integer $n \geq 3$ and consider the moduli space $\mathcal{A}_{g,n}$ of ppav’s of dimension $g$ with level $n$ structure. Recall that this moduli space is represented by a quasi-projective and smooth variety over $\mathbb{Q}$ by \cite{[27]} chapt. 7.3. Analytically it is the quotient

$$\mathcal{A}_{g,n}(\mathbb{C}) = \mathcal{H}_g/\Gamma_g(n)$$

of the Siegel upper half space $\mathcal{H}_g$ by the free action of the principal congruence subgroup $\Gamma_g(n) = \text{ker}(Sp_{2g}(\mathbb{Z}) \rightarrow Sp_{2g}(\mathbb{Z}/n\mathbb{Z}))$. Let $p : \mathcal{X} \rightarrow \mathcal{A}_{g,n}$ be the universal abelian scheme. In what follows, we assume $n$ is divisible by a sufficiently high power of 2. Then the zero locus of the Riemann theta function $\vartheta(\tau, z)$ on the universal covering $\mathcal{H}_g \times \mathbb{C}^g$ descends to a relative divisor $\Theta \subset \mathcal{X}$ by the theta transformation formula \cite{[21]} cor. on p. 85]. For any morphism $S \rightarrow \mathcal{A}_{g,n}$ from a variety $S$ we denote by $X_S = \mathcal{X} \times_{\mathcal{A}_{g,n}} S \rightarrow S$ the associated principally polarized abelian scheme. By construction it carries a relative theta divisor $\Theta_S = \Theta \times_{\mathcal{A}_{g,n}} S \subset X_S$. 


Lemma 4.1. For any point \( s \in \mathcal{A}_{g,n}(\mathbb{C}) \) there exists a smooth quasi-projective complex curve \( S \hookrightarrow \mathcal{A}_{g,n} \) which passes through \( s \) along a general tangent direction and which has the property that

(a) the generic fibre of the family \( \Theta_S \to S \) is smooth,
(b) the singular loci of the total space and of the special fibre satisfy

\[
\text{Sing}(\Theta_S) \subseteq \text{Sing}(\Theta_s).
\]

If the theta divisor \( \Theta_s \) contains a singular point of precise multiplicity two, then the inclusion in (b) is strict for a suitable choice of \( S \).

Proof. Since \( \mathcal{A}_{g,n} \) is smooth and quasi-projective, for any point \( s \in \mathcal{A}_{g,n}(\mathbb{C}) \) we can find a smooth quasi-projective curve \( S \hookrightarrow \mathcal{A}_{g,n} \) passing through the point \( s \) in a general tangent direction. We can assume that our general curve \( S \) is not contained in the locus of ppav’s with singular theta divisor. So after shrinking \( S \), we can assume that for all \( t \in S(\mathbb{C}) \setminus \{ s \} \) the theta divisor \( \Theta_t \) is smooth. Then part (a) is clear and (b) follows from the observation that the total space \( \Theta_S \) is given locally in the smooth variety \( X_S \) as the zero locus of a single analytic function.

Explicitly, let \( \Delta \subset S \) be an analytic coordinate disk with coordinate \( w \) centered at \( s \), and consider a local lift \( h : \Delta \to \mathcal{H}_g \) of \( \Delta \subset S \hookrightarrow \mathcal{A}_{g,n} \). On the universal covering the divisor \( \Theta_S \) is described as the locus

\[
\{ (w, z) \in \Delta \times \mathbb{C}^g \mid F(w, z) = 0 \} \subset \Delta \times \mathbb{C}^g
\]

where the analytic function \( F(w, z) = \partial(h(w), z) \) vanishes. If a point \( (w, z) \) on this locus defines a singular point of the relative theta divisor \( \Theta_s \), then the gradient of \( F \) must vanish at this point. For the gradient in the variable \( z \) this implies

\[
0 = (\nabla_z F)(w, z) = (\nabla_z \theta)(\tau, z) \quad \text{for} \quad \tau = h(w).
\]

Hence \((\tau, z)\) defines a singular point of the fibre \( \Theta_t \) where \( t \in S(\mathbb{C}) \) denotes the image of the point \( \tau \). By our choice of the curve \( S \), the only singular fibre \( \Theta_t \) is the one over the point \( t = s \), hence claim (b) follows.

This being said, if some point \((\tau, z)\) in \( \mathcal{H}_g \times \mathbb{C}^g \) defines a singular point of the theta divisor \( \Theta_s \) with precise multiplicity two, then by definition \((\partial^2 \theta / \partial z_\alpha \partial z_\beta)(\tau, z) \neq 0 \) for some \( \alpha, \beta \in \{1, 2, \ldots, g\} \). Then the heat equation

\[
\frac{\partial \theta}{\partial r_{\alpha \beta}}(\tau, z) = 2\pi i \cdot (1 + \delta_{\alpha \beta}) \cdot \frac{\partial^2 \theta}{\partial z_\alpha \partial z_\beta}(\tau, z)
\]

implies for the gradient with respect to the variable \( \tau \) that \((\nabla_\tau \theta)(\tau, z) \neq 0 \). Hence, taking \( S \) to be a curve which passes through the point \( s \) in a sufficiently general tangent direction \( u \), we obtain with notations as above that

\[
(\partial F / \partial u)(0, z) = (u \cdot \nabla_\tau \theta)(\tau, z) \neq 0
\]

where \( F(w, z) = \partial(h(w), z) \) is defined for \( u \) in some coordinate disk \( \Delta \subset S \) by a suitable local lift \( h : \Delta \to \mathcal{H}_g \) with \( h(0) = \tau \). In particular, since the gradient of \( F \) does not vanish at the considered point, it follows that \((\tau, z)\) defines a smooth point of the total space \( \Theta_S \) of our family. Thus, the inclusion in claim (b) is strict. \( \square \)

We will now use the above construction to obtain information about the Tannaka groups \( G(\Theta_s) \) by varying the point \( \tau \in \mathcal{A}_{g,n}(\mathbb{C}) \). To apply the results about the monodromy filtration in section \( \S \) we consider perverse sheaves with coefficients
in $\Lambda = \mathbb{Q}_\ell$ throughout, but the outcome may as well be read in the larger category of analytic perverse sheaves with coefficients in $\Lambda = \mathbb{C}$.

Let $U \subset A_{g,n}$ be the Zariski-open dense locus of all ppav’s with a smooth theta divisor. By the constructibility property in proposition 7.4 of the appendix, there exist finitely many reductive groups $G_1, \ldots, G_m$ over $\Lambda$ and a stratification of $U$ into locally closed subsets

$$U = \bigsqcup_{i=0}^m U_i \text{ with } G(\delta_{\Theta_\tau}) \cong G_i \text{ for all geometric points } \tau \text{ in } U_i.$$ 

Let us denote by $U_0 \subseteq U$ the open dense stratum, so $G_0 \subseteq G(g)$ is the Tannaka group of the theta divisor on a generic ppav. With the notations of lemma 2.1, the perverse intersection cohomology sheaf of such a theta divisor corresponds to the restriction $V|_{G_0}$ of the standard representation $V$ of the group $G(g)$.

**Proposition 4.2.** With notations as above, the following properties hold for the Tannaka groups of smooth theta divisors.

1. If $g = g_1 + g_2$ and if theorem 7.3 holds for ppav’s of dimension $g_1$ and $g_2$, then the generic group $G_0$ has a subquotient isogenous to $G(g_1) \times G(g_2)$.

2. There is a connected subgroup $H \hookrightarrow G_0$ and an irreducible representation $W$ of $H$ such that

$$V|_H = \begin{cases} W \oplus 1 \oplus 1 & \text{if } g \text{ is even,} \\ W \oplus 1 & \text{if } g \text{ is odd,} \end{cases}$$

where 1 denotes the one-dimensional trivial representation.

3. For $g \geq 4$ and any stratum $U_i$ of codimension at most one in $U$ there is a homomorphism $f : SL_{2g-2}(\Lambda) \rightarrow G_i$ such that $f^*(V)$ contains the $(g-1)^{st}$ fundamental representation of the special linear group as a summand.

**Proof.** Consider a morphism from a smooth irreducible quasi-projective curve $S$ to the moduli space $A_{g,n}$ such that the generic point of $S$ is mapped into some stratum $U_i$ in our stratification, and fix a geometric generic point $\eta$ of $S$. Passing to the strict Henselization of $S$ at a point $s \in S(\mathbb{C})$ we can consider the perverse sheaf $\Psi(\delta_{\Theta_\eta}) \in \text{Perv}(X_s, \mathbb{C})$ of nearby cycles. By [28] lemma 14.1 there exists a closed embedding

$$G(\Psi(\delta_{\Theta_\eta})) \hookrightarrow G(\delta_{\Theta_\eta}) = G_i \hookrightarrow G(g)$$

such that on the Tannakian side the perverse sheaf $\Psi(\delta_{\Theta_\eta})$ corresponds to the restriction of $V$ to the subgroup $G(\Psi(\delta_{\Theta_\eta}))$. We will apply this as follows.

For part (1) take $S \hookrightarrow A_{g,n}$ to be an embedding of a smooth curve which meets the locus of decomposable ppav’s in a single point $s \in S(\mathbb{C})$ but which is otherwise contained in the open dense stratum $U_0$ and has the properties in lemma 4.1. We choose the point $s$ such that

$$X_s = X_1 \times X_2 \text{ and } \Theta_s = (\Theta_1 \times X_2) \cup (X_1 \times \Theta_2),$$

where for $\alpha \in \{1, 2\}$ the $X_\alpha$ are general complex ppav’s of dimension $g_\alpha$ with a symmetric theta divisor $\Theta_\alpha$. Like for any divisor with two components which intersect each other transversally along a smooth subvariety, we have an exact sequence

$$0 \rightarrow \delta_{\Theta_1 \times \Theta_2} \rightarrow \Lambda_{\Theta_s}[g-1] \rightarrow \delta_{\Theta_1 \times X_2} \oplus \delta_{X_1 \times \Theta_2} \rightarrow 0$$

of perverse
sheaves. Restricting this sequence to the open dense subset $Y = X_s \setminus \text{Sing}(\Theta_s)$ we get a monomorphism

$$\delta_{\Theta_1 \times \Theta_2}|_Y \hookrightarrow \Lambda_{\Theta_1}[g-1]|_Y = sp(\delta_{\Theta_s})|_Y$$

of perverse sheaves, where the last equality holds by corollary 3.4. Now by the last statement in lemma 4.1 we can assume that the singular locus $\text{Sing}(\Theta_s)$ is a proper closed subset of $\text{Sing}(\Theta_s) = \Theta_1 \times \Theta_2$, and in this case the open dense subset $Y$ will have non-empty intersection with $\Theta_1 \times \Theta_2$. Then, via middle perverse extension it follows from the above that the semisimplification of $sp(\delta_{\Theta_s})$ contains a direct summand $\delta_{\Theta_1 \times \Theta_2}$. By the properties of the monodromy filtration in section 3 the same then a fortiori holds for the semisimplification of the perverse sheaf of nearby cycles $\Psi(\delta_{\Theta_s})$. This being said, our claim follows from the elementary observation that the Tambara group $G(\delta_{\Theta_1 \times \Theta_2})$ is isogenous to $G(\delta_{\Theta_1}) \times G(\delta_{\Theta_2})$.

For part (2) let $s \in A_{g,n}(\mathbb{C})$ be a point corresponding to a general ppav $X_s$ with a vanishing theta null, and consider a general curve $S \hookrightarrow A_{g,n}$ which passes through $s$ but is otherwise contained in $U_0$ and has the properties in lemma 4.1. For the special fibre with a vanishing theta null we know from [9], with the correction in [19, rem. 4.5], or from theorem 4.2 of loc. cit. that $\Theta_1$ is the irreducible representation of $G$ such that the semisimplification of the nearby cycles must have the form

$$\text{degeneration coincide with the unipotent nearby cycles. Then corollary 3.4 implies that the semisimplification of the perverse sheaf of nearby cycles $\Psi(\delta_{\Theta_s})$ remains irreducible, so our claim follows.}$$

$$\chi(\Theta_s) = \chi(\Theta_{\eta}) + (-1)^g = (-1)^{g-1} \cdot (g! - 1)$$

because the generic fibre $\Theta_{\eta}$ is smooth of dimension $g - 1$ with $\chi(\Theta_{\eta}) = (-1)^{g-1} \cdot g!$ by the Gauss-Bonnet theorem. The ordinary double point singularity of the special fibre is resolved by the blowup $\pi : \tilde{\Theta}_s \rightarrow \Theta_s$ in the point $e$, with a smooth quadric as the exceptional divisor. The stalk cohomology of the direct image $R\pi_* \delta_{\tilde{\Theta}_s}$ can be computed using [13, exp. XII, th. 3.3], and via the decomposition theorem this leads to an exact sequence of perverse sheaves $0 \rightarrow \kappa_g \rightarrow \Lambda_{\Theta_s}[g-1] \rightarrow \delta_{\Theta_s} \rightarrow 0$ for the skyscraper sheaf

$$\kappa_g = \begin{cases} \delta_{\kappa}(-\frac{g-2}{2}) & \text{if } g \text{ is even}, \\ 0 & \text{if } g \text{ is odd.} \end{cases}$$

Hence

$$\chi(\delta_{\Theta_s}) = \chi(\Lambda_{\Theta_s}[g-1]) - \chi(\kappa_g) = \begin{cases} g! - 2 & \text{if } g \text{ is even}, \\ g! - 1 & \text{if } g \text{ is odd.} \end{cases}$$

On the other hand, by remark 3.1 we can assume that the nearby cycles for our degeneration coincide with the unipotent nearby cycles. Then corollary 3.4 implies that the semisimplification of the nearby cycles must have the form

$$\Psi(\delta_{\Theta_s})^{ss} = \delta_{\Theta_s} \oplus \gamma$$

for some perverse skyscraper sheaf $\gamma$ supported on the singular point $e$ of the special fibre. A look at the Euler characteristics shows that $\gamma$ has rank two if $g$ is even, and rank one if $g$ is odd. Now take a splitting $G_s = G(\delta_{\Theta_s}) \hookrightarrow G(\Psi(\delta_{\Theta_s}))$ and let $W$ be the irreducible representation of $G_s$ which corresponds to the simple perverse sheaf $\delta_{\Theta_s}$. In lemma 5.1 below we will see, as a general fact about irreducible theta divisors, that the restriction of $W$ to the connected component $H = G_0^1 \subseteq G_s$ remains irreducible, so our claim follows.
5. Proof of the main theorem

We now explain how to deduce theorem 1.3 from proposition 4.2. For a complex ppav $X$ of dimension $g$ with a symmetric theta divisor $\Theta \subset X$ let $G = G(\delta_\Theta)$ be the corresponding Tannaka group, and consider the representation

$$V = \omega(\delta_\Theta) \in \text{Rep}_C(G)$$

where $\omega : \langle \delta_\Theta \rangle \to \text{Rep}_C(G)$ is a fibre functor. A priori the Tannaka group $G$ does not have to be connected, but we always have the following

**Lemma 5.1.** If the theta divisor $\Theta$ is irreducible, then the restriction $V|_{G^0}$ is an irreducible representation of the connected component $G^0 \subseteq G$.

**Proof.** The irreducibility of the theta divisor implies that $\delta_\Theta$ is a simple perverse sheaf, and hence $V$ is an irreducible representation of the Tannaka group $G$. But by the fundamental result of [33] the group $G/G^0$ is abelian. Hence if the claim of our lemma were not true, then a look at the invariants under $G^0$ in $V \otimes V^*$ would imply that $V \otimes \chi \cong V$ for some non-trivial character $\chi : G \to C^*$ of finite order. Now by [26] prop. 10.1 any such character $\chi$ corresponds to a skyscraper sheaf $\delta_x$ of rank one, supported in some non-zero torsion point $x \in X(C)$. Hence the above isomorphism would geometrically correspond to an isomorphism $\delta_\Theta \ast \delta_x \cong \delta_\Theta$, meaning that $t_x(\Theta) = \Theta$ for the translation $t_x : X \to X, y \mapsto y + x$. But this is impossible for $x \neq 0$, indeed the morphism $X \to \text{Pic}^0(X), x \mapsto \mathcal{O}_X(\Theta - t_x(\Theta))$ is an isomorphism by the definition of a principal polarizations.

Now suppose that $X$ is a general ppav. By lemma 2.3 the corresponding Tannaka group admits an embedding $G = G(\delta_\Theta) \to G(g)$ and $V = \omega(\delta_\Theta)$ is the restriction of the standard representation of the classical group on the right hand side.

**Lemma 5.2.** For a general ppav $X$ with Tannaka group $G = G(\delta_\Theta)$, the connected component $G^0 \subseteq G$ is simple modulo its center.

**Proof.** For any reductive algebraic group $H$ the derived group $H^0_{\text{der}} = [H^0, H^0]$ is a connected semisimple group, and we denote by $\tilde{H} \to H^0_{\text{der}}$ its simply connected cover. The covering group $\tilde{H}$ is a product of simply connected covers of simple
algebraic groups. By the theory of reductive groups [30, cor. 8.1.6] the connected component \( H^0 \) is the product (with finite intersection) of its derived group and its center, so any irreducible representation of \( H^0 \) remains also irreducible as a representation of the covering group \( \tilde{H} \).

Now consider the group \( G = G(\delta_0) \) attached to a general ppav. If the lemma were not true, then the simply connected cover of \( G \) could be written as \( \tilde{G} = G_1 \times G_2 \) for some positive-dimensional simply connected groups \( G_1 \) and \( G_2 \). Lemma 5.1 shows that \( G^0 \) acts irreducibly on \( V = \omega(\delta_0) \), hence by the above the same holds for the covering group \( \tilde{G} = G_1 \times G_2 \). Hence

\[
V|_{\tilde{G}} = V_1 \otimes V_2
\]

for certain irreducible representations \( V_i \in \text{Rep}_C(G_i) \). Note that since \( V \) is a faithful representation of \( G \), it follows from the definition of the simply connected covering group that the representations \( V_i \) are non-trivial and \( \dim(V_i) > 1 \). Now let \( H \hookrightarrow G \) be a connected subgroup as in part (2) of proposition 4.2 such that

\[
V|_H = W \oplus \begin{cases} 1 \oplus 1 & \text{if } g \text{ is even,} \\ 1 & \text{if } g \text{ is odd,} \end{cases}
\]

where \( W \) is an irreducible representation of \( H \). Via the commutative diagram

\[
\begin{array}{ccc}
\tilde{H} & \longrightarrow & \tilde{G} \\
\downarrow & & \downarrow \\
H & \longleftarrow & G \\
\end{array}
\]

\[
\begin{array}{ccc}
\longrightarrow & \longrightarrow & Gl(V_1) \times Gl(V_2) \\
\downarrow & & \downarrow \\
Gl(V) & \longleftarrow & Gl(V_1 \otimes V_2)
\end{array}
\]

we can consider the restrictions of \( V_1 \) and \( V_2 \) to the covering group \( \tilde{H} \), and by construction we have

\[
V_1|_{\tilde{H}} \otimes V_2|_{\tilde{H}} = V|_{\tilde{H}} = W|_{\tilde{H}} \oplus \begin{cases} 1 \oplus 1 & \text{if } g \text{ is even,} \\ 1 & \text{if } g \text{ is odd,} \end{cases}
\]

where \( W|_{\tilde{H}} \) is irreducible by the general remarks above. Therefore both \( V_i|_{\tilde{H}} \) are irreducible, since otherwise more than one non-trivial direct summand would occur in their tensor product. But then, since \( \text{Hom}_{\tilde{H}}(1, V_1|_{\tilde{H}} \otimes V_2|_{\tilde{H}}) \neq 0 \), adjunction shows that \( V_1|_{\tilde{H}} \) and \( V_2|_{\tilde{H}} \) are dual to each other. In particular then \( V_1 \) and \( V_2 \) have the same dimension. This is impossible, because \( \dim(V) = g! \) is not the square of a natural number for any \( g > 1 \).

For the case \( g = 4 \) we can alternatively use the following argument, only requiring part (3) of proposition 4.2. Here both \( V_1 \) and \( V_2 \) would have dimension at most 12, since their dimensions must be non-trivial divisors of \( \dim(V) = 24 \). Now take \( H \hookrightarrow G \) to be a subgroup isogenous to \( SL_{2g-2}(\mathbb{C}) = SL_6(\mathbb{C}) \) as in part (3) of the proposition. The classification in [11] of low-dimensional representations shows that each \( V_i|_{\tilde{H}} \) can contain only trivial representations and 6-dimensional standard representations as irreducible constituents. In particular, looking at \( V_1|_{\tilde{H}} \otimes V_2|_{\tilde{H}} \) one sees that the third fundamental representation of \( SL_6(\mathbb{C}) \) cannot occur as a constituent of \( V|_{\tilde{H}} \), and this contradicts our choice of the subgroup \( H \).

To prove theorem 1.3 for the Tannaka group \( G = G(\delta_0) \to G(g) \) of a general ppav, we want to show that its connected component cannot be a proper subgroup of \( G(g) \). By lemma 5.1 this connected component \( G^0 \) is an irreducible subgroup in
the sense that for the standard representation $V \in \text{Rep}_c(G(g))$ the restriction $V|_{G^0}$ is irreducible. So we are in the situation of the following general lemma.

**Lemma 5.3.** Let $H$ be an irreducible connected subgroup of $G(g)$ which is simple modulo its center. Then $\dim(H) \leq g!$ or $H = G(g)$.

**Proof.** Let $V$ denote the standard representation of the classical group $G(g)$, and suppose that $\dim(H) > g! = \dim(V)$. Then the restriction $V|_H$ must be one of the low-dimensional representations listed in [1, table 1]. Since $\dim(V) = g!$ is the factorial of a natural number, we obtain that one of the following two cases must occur for some natural number $r \in \mathbb{N}$.

(a) The group $H$ is of type $A_r$, $C_r$ or $D_r$ and acts on $V$ via its standard representation.

(b) The group $H$ is of type $A_r$ and acts on $V$ via the symmetric or via the alternating square of its standard representation.

Case (a) can only occur for $H = G(g)$, since $G(g)$ is itself the symplectic or special orthogonal group with $V$ as its standard representation. To deal with case (b), note that by [16, th. 3.2.14] the symmetric square of the standard representation of $A_r$ is self-dual only if $r = 1$, but then it has dimension $3 \neq \dim(V)$. The alternating square of the standard representation of $A_r$ is self-dual of dimension $g!$ only in the case $(r, g) = (3, 3)$, but then $H = G(g) = SO_6(\mathbb{C})$ for dimension reasons. □

To complete the proof of theorem 1.3 we must show that for a general ppav $X$ of dimension $g$ with theta divisor $\Theta$ we have

$$\dim(G) > g!$$

for the Tannaka group $G = G(\delta_\Theta)$. This follows by induction on $g$. Indeed, by the remarks after conjecture 1.2 we can assume $g \geq 4$. For $g = 4$ we have $\dim(G) \geq \dim(SL_{2g-2}(\mathbb{C})) = 35 > g! = 24$ by part (3) of proposition 4.2. The induction step is then provided by part (1) of the same proposition using $\dim(G(g-1)) > g!$ for $g \geq 5$. Since for the case $g = 4$ we have only used part (3) of the proposition, theorem 1.4 follows as well.

### 6. An alternative approach

One of the main steps in finding the Tannaka group of a generic theta divisor has been to show that this group is simple modulo its center, see lemma 5.2. In this section we discuss another proof for this statement which introduces motivic techniques, and that may be of independent interest.

Let $X$ be a complex abelian variety. As in section 3 we fix a prime number $l$ and consider the categories $\mathbf{P}(X) = \text{Perv}(X, \Lambda)$ and $\mathbf{D}(X) = D_{b}(X, \Lambda)$ with coefficients in $\Lambda = \mathbb{Q}_l$. Any perverse sheaf $P \in \mathbf{P}(X)$ of geometric origin is defined over some finitely generated field $k$, and working over the algebraic closure $\overline{k}$ of $k$ we obtain a representation of $\text{Gal}(\overline{k}/k)$ on

$$H^\bullet(X, \Lambda) \oplus H^\bullet(X, P).$$

We define the *motivic group* $M(P)$ to be the derived group of the connected component of the Zariski closure of the image of this Galois representation. One easily checks that this motivic group does not depend on the choice of $k$ and that for any $P, Q \in \mathbf{P}(X)$ of geometric origin we have an epimorphism $M(P \oplus Q) \rightarrow M(Q)$ which is compatible with the action on $H^\bullet(X, Q)$. 

Let us say that a complex \( N \in D(X) \) is *negligible*, if all perverse constituents of its perverse cohomology sheaves have Euler characteristic zero. A perverse sheaf without negligible subquotients is called *clean*. By [26] we have for all semisimple perverse sheaves \( P_1, P_2 \in P(X) \) a decomposition \( P_1 \ast P_2 = Q \oplus N \) into a clean perverse sheaf \( Q \) and a negligible complex \( N \).

**Lemma 6.1.** Let \( X \) be the Jacobian variety of a general curve of genus \( g \) with theta divisor \( \Theta \), and let \( P \in P(X) \) be a semisimple clean perverse sheaf of geometric origin. Then the maximal clean summand \( Q \) in \( \delta_\Theta \ast P \) satisfies

\[
\dim(\text{Supp}(Q)) \geq g - 1.
\]

The same holds for the generic fibre \( X \) of a connected family of ppav’s which admits a general Jacobian variety as a special fibre.

**Proof.** Suppose first that \( X \) is the generic fibre of a connected family of ppav’s which admits a general Jacobian variety \( X_0 \) with theta divisor \( \Theta_0 \subset X \) as a special fibre. The nearby cycles functor \( \Psi : D(X) \to D(X_0) \) commutes with convolution products [26, sect. 14], preserves the property of being negligible and does not increase the support dimension. Since \( \delta_\Theta \) is a subquotient of \( \Psi(\delta_\Theta) \), the lemma for the ppav \( X \) will therefore follow from the lemma for \( X_0 \).

So it suffices to deal with the case where \( X \) is the Jacobian variety of a general curve of genus \( g \). We argue by contradiction. Suppose to the contrary that for some negligible \( N \) we have a decomposition

\[
\delta_\Theta \ast P = Q \oplus N \quad \text{with} \quad \dim(\text{Supp}(Q)) \leq g - 2.
\]

Since by the decomposition theorem \( N \) is a direct sum of degree shifts of semisimple perverse sheaves, up to summands with vanishing hypercohomology the negligible part \( N \) has the form \( V^* \otimes \delta_X \) for some complex \( V^* \) of Galois representations, using the classification of negligible complexes on a simple abelian variety [26, prop. 10.1]. So by the Künneth formula

\[
H^*(X, \delta_\Theta) \otimes H^*(X, P) = H^*(X, Q) \oplus \left(V^* \otimes H^*(X, \delta_X)\right).
\]

Our assumption \( \dim(\text{Supp}(Q)) \leq g - 2 \) furthermore implies that \( H^i(X, Q) = 0 \) for \( |i| \geq g - 1 \), and hence

\[
\bigoplus_{i \geq 0} H^i(\Theta, \Lambda) \otimes H^{d-i-1}(X, P) = \bigoplus_{n \geq 0} V^{d-n} \otimes H^n(X, \Lambda) \quad \text{for} \quad d \leq 1
\]

for the intersection cohomology \( IH^*(\Theta, \Lambda) = H^*(X, \delta_\Theta)[1 - g] \). Now \( P \) being clean implies \( H^{-g}(X, P) = 0 \), so in the above displayed equation only terms with \( i < g \) occur. In these degrees we have \( IH^i(\Theta, \Lambda) = \bigoplus_{k \geq 0} H^{i-k}(X, \Lambda) \) by [32, cor. 13] and hence

\[
\bigoplus_{n \geq 0} \bigoplus_{k \geq 0} H^{d-1-2k-n}(X, P) \otimes H^n(X, \Lambda) = \bigoplus_{n \geq 0} V^{d-n} \otimes H^n(X, \Lambda) \quad \text{for} \quad d \leq 1.
\]

Proceeding recursively, by upward induction on \( d \) starting at \( d = \min\{i \mid V^i \neq 0\} \), we get

\[
\bigoplus_{k \geq 0} H^{d-1-2k}(X, P) = V^d \quad \text{for all} \quad d \leq 1.
\]

Now for the proof of the lemma we can clearly assume that \( P \) is isomorphic to its Verdier dual. Then \( H^i(X, P) \cong H^{-i}(X, P) \) for all \( i \in \mathbb{Z} \) implies that any irreducible
representation of the motivic group $\mathcal{M} = \mathcal{M}(\delta_{\Theta} \oplus P)$ occurring in $H^*(X, P)$ must occur in $V^*$ as well.

For a general Jacobian variety $X$ we have $\mathcal{M}(\delta_X) = \text{Sp}_{2g}(\Lambda)$. Furthermore the surjection $\mathcal{M} \twoheadrightarrow \mathcal{M}(\delta_X)$ admits a non-canonical splitting, and via this splitting we consider all the above hypercohomology groups as representations of the symplectic group. We can assume $V^* \neq 0$, because otherwise a comparison of degrees in the Künneth formula would lead to a contradiction. Let $\rho$ denote the half-sum of all positive roots of $\text{Sp}_{2g}(\mathbb{C})$. Among all highest weights of the representation on $V^*$ pick $\beta$ so that the scalar product $(\rho, \beta)$ is maximal (possibly $\beta = 0$). Let $\beta_1, \ldots, \beta_g$ denote the fundamental weights of $\text{Sp}_{2g}(\mathbb{C})$. Since the highest weight $\beta_g$ occurs in $H^9(X, \Lambda)$, we know that $\beta + \beta_g$ occurs as a highest weight in

$$V^* \otimes H^*(X, \Lambda) \subset IH^*(\Theta, \Lambda)[1] \otimes H^*(X, P).$$

As $IH^*(\Theta, \Lambda)$ only contains highest weights $\beta_i$ with $i < g$, it follows that $\beta + \beta_g$ is dominated by some weight $\gamma + \beta_i$ where $i < g$ and where $\gamma$ is a highest weight occurring in $H^*(X, P)$. Therefore

$$(\rho, \beta + \beta_g) \leq (\rho, \gamma + \beta_i).$$

But by the result of the previous paragraph the weight $\gamma$ occurring in $H^*(X, P)$ must occur in $V^*$ as well, so $(\rho, \gamma) \leq (\rho, \beta)$ by our choice of $\beta$. Both inequalities together imply $(\rho, \beta_g) \leq (\rho, \beta_1)$. Since $i < g$, this gives the desired contradiction. \qed

By the above lemma we are now able to refine parts (1) and (3) of proposition 4.2 as follows, for $\mathcal{G}(g-1)$ denoting the universal cover of the group $G(g-1)$.

**Proposition 6.2.** Let $G = G(\delta_{\Theta})$ be the Tannaka group of a generic $ppav$ $X$ of dimension $g$ with a symmetric theta divisor $\Theta \subset X$, and $V = \omega(\delta_{\Theta}) \in \text{Rep}_\Lambda(G)$ the defining representation. Then there is

1. a homomorphism $f : \text{Sl}_{2g-2}(\Lambda) \to G$ such that $f^*(V)$ contains the $(g-1)^{st}$ fundamental representation precisely once,
2. a homomorphism $h : \mathcal{G}(g-1) \to G$ such that $h^*(V)$ contains the standard representation precisely twice.

**Proof.** For part (1) consider a degeneration of $X$ into the Jacobian variety $X_0$ of a general curve. Then for the degenerate theta divisor $\Theta_0 \subset X_0$, the perverse sheaf $\delta_{\Theta_0}$ with $G(\delta_{\Theta_0}) = \text{Sl}_{2g-2}(\Lambda)$ is a subquotient of the nearby cycles $\Psi(\delta_{\Theta})$, so for some complementary group $H$ we have homomorphisms

$$\text{Sl}_{2g-2}(\Lambda) \times H \longrightarrow G(\Psi(\delta_{\Theta})) \longrightarrow G(\delta_{\Theta}) = G$$

where the first arrow is the universal covering and the second one comes from the tensor functoriality of the nearby cycles [26 lemma 14.1]. Let $f$ be the restriction to $\text{Sl}_{2g-2}(\Lambda) \times \{1\}$ of the composite homomorphism. As in part (3) of proposition 4.2 the semisimplification of the nearby cycles has the form

$$(\Psi(\delta_{\Theta}))^{ss} = \delta_{\Theta_0} \oplus R \text{ with } R \in \mathcal{P}(X_0) \text{ and } \text{Supp}(R) \subseteq \text{Sing}(\Theta_0).$$

We know from [32 cor. 13] that $\omega(\delta_{\Theta_0}) = \beta_{g-1} \boxtimes 1 \in \text{Rep}_\Lambda(\text{Sl}_{2g-2}(\Lambda) \times H)$ is the $(g-1)^{st}$ fundamental representation of the special linear group. If claim (1) were not true, then we could find a direct summand $Q \subseteq R$ such that $\omega(Q) = \beta_{g-1} \boxtimes W$ for some $W \in \text{Rep}_\Lambda(H)$. By a rigidity argument then $1 \boxtimes W$ also corresponds to some perverse sheaf $P \in \mathcal{P}(X_0)$, and this gives a decomposition $\delta_{\Theta_0} \ast P = Q \oplus N$
with $N$ negligible. Hence lemma 6.1 implies that $\dim(\text{Supp}(Q)) \geq g - 1$, which is impossible for the direct summand $Q$ of $R$.

The proof of (2) is similar. Let $X$ degenerate into a product $X_1 \times X_2$ of a generic ppav of dimension $g - 1$ with an elliptic curve. Then the degenerate theta divisor is $Y = (\Theta_1 \times X_2) \cup (X_1 \times \{0\})$. As in part (1) of proposition 4.2 we have

$$(\Psi(\delta_\Theta))^{ss} = \delta_Y \oplus 2 \cdot \delta_{\Theta_1 \times \{0\}} \oplus R \quad \text{with} \quad \text{Supp}(R) \subseteq \Theta_1 \times \{0\}.$$ 

Here $\delta_Y$ is negligible, and $\delta_{\Theta_1 \times \{0\}}$ enters with multiplicity two because it comes from the second step of the weight filtration of $sp(\delta_\Theta)$. If claim (2) fails, then as above one finds a direct summand $Q \subseteq R$, a perverse sheaf $P$ and a negligible complex $N$ with $\delta_{\Theta_1 \times \{0\}} \ast P = Q \oplus N$. After a character twist on the elliptic curve we can assume by the vanishing theorem of [26] that for the projection $p : X_1 \times X_2 \rightarrow X_1$ the direct image $P_1 = Rp_\ast(P)$ is perverse. Then we get

$$\delta_{\Theta_1} \ast P_1 = Q_1 \oplus N_1$$

where $Q_1 = Rp_\ast(Q)$ is supported on a strict subset of $\Theta_1$ and where $N_1 = Rp_\ast(N)$ is negligible. Hence again lemma 6.1 leads to a contradiction. 

Alternative proof of lemma 5.2, using the above results. Let $G = G(\delta_\Theta)$ be the Tannaka group of a generic ppav of dimension $g$. Suppose that the universal covering group is a product $\tilde{G} = G_1 \times G_2$. The representation $V = \omega(\delta_\Theta)$ is then a tensor product

$$V|\tilde{G} \cong V_1 \boxtimes V_2 \quad \text{for certain} \quad V_i \in \text{Rep}_\Lambda(G_i) \quad \text{with} \quad \dim(V_i) > 1.$$ 

Choosing a lift of the homomorphism $h$ in proposition 6.2 we get a representation of the group $H = \tilde{G}(g - 1)$ on $V_1$ and $V_2$. A dimension estimate shows that $H$ must act trivially on one of these vector spaces. Indeed, for $g > 4$ any non-trivial representation of $H$ has dimension at least $(g - 1)!$ by [1, table 1]. So we may assume that $H$ acts trivially on $V_2$. Then part (2) of proposition 6.2 implies that $\dim(V_2) = 2$. By choosing a lift of the homomorphism $f$ in the proposition, we get a representation of the group $F = Sl_{2g - 2}(\Lambda)$ on $V_1$ and $V_2$. Since for $g \geq 3$ this group does not have non-trivial representations of dimension two, it must act trivially on $V_2$. Then $V_1 \cong (V_1|\rho) \boxtimes 2$ which contradicts part (1) of the proposition.

Finally we remark, in the situation of proposition 6.2 (2) a similar argument shows that the restriction $h^\ast(V)$ of the standard representation $V$ of $G$ to the group $\tilde{G}(g - 1)$ becomes isomorphic to the direct sum of a trivial representation and two copies of the standard representation of $\tilde{G}(g - 1)$.

7. Appendix: Constructibility

In this appendix we study how the Tannaka groups from [26] vary in families of perverse sheaves. We work in an algebraic setting, so as above we put $\Lambda = \mathbb{Q}_l$ for some prime $l$ and write $P(-) = \text{Perv}(-, \Lambda)$ and $D(-) = D^b_{\text{ct}}(-, \Lambda)$.

Let $S$ be an algebraic variety over an algebraically closed field $k$ of characteristic zero, and let $X \rightarrow S$ be an abelian scheme over $S$. If $Y \rightarrow X$ is a closed subvariety that maps surjectively onto $S$, we can consider for any geometric point $\bar{s}$ in $S$ the perverse intersection cohomology sheaf $\delta_{Y_\bar{s}} \in P(X_\bar{s})$ on the fibre $Y_\bar{s} \hookrightarrow X_\bar{s}$. The following example illustrates that the corresponding Tannaka groups in general do not depend on $\bar{s}$ in a constructible way.
Example 7.1. If $S = E$ is an elliptic curve and $X = E \times S \to S$ is the constant family, then for the diagonal $\mathcal{Y} = \{(e,e) \mid e \in E\}$ we have

$$G(\delta_{\mathcal{Y}}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } \eta \text{ is a torsion point in } E \text{ of precise order } n, \\ G_m & \text{if } \eta \text{ is a point of infinite order in } E. \end{cases}$$

In general we only have the following semicontinuity property.

Lemma 7.2. Let $\mathcal{Y} \hookrightarrow X$ be a closed subvariety which is smooth over $S$. Let $\eta \in S$ be a scheme-theoretic point, $s \in [\eta]$ a point in its closure, and choose geometric points $\eta$ and $s$ above them. Then there exists an embedding $G(\delta_{\mathcal{Y}}) \hookrightarrow G(\delta_{\mathcal{Y}})$.

Proof. After base change to the reduced closed subscheme $\{\eta\} \to S$ we can assume that $\eta$ is the generic point of $S$. One then easily reduces our claim to the local situation of [26, lemma 14.1], and for the nearby cycles we have $\Psi(\delta_{\mathcal{Y}}) = \delta_{\mathcal{Y}}$ because the morphism $\mathcal{Y} \to S$ is smooth by assumption.

To get a constructibility statement for the stratifications defined by the Tannaka groups, we need to impose some finiteness conditions. To simplify the notation, let us temporarily assume that $X = X$ is an abelian variety over $S = \text{Spec}(k)$. For every $P \in \mathbf{P}(X)$ we have a fibre functor

$$\omega : (P) \rightarrow \text{Rep}_\Lambda(G(P)).$$

Like any character of the Tannaka group, the determinant character $\det(\omega(P))$ corresponds by [26, prop. 10.1] to a skyscraper sheaf $\delta_x \in (P)$ supported on some point $x \in X(k)$. By abuse of notation, in what follows we write $\det(P)^n = 1$ to indicate that the determinant character has order dividing $n$, or equivalently that the above point $x$ is an $n$-torsion point. For example, in many applications $P$ is isomorphic to its adjoint dual $P^\vee$, and then $\det(P)^2 = 1$ holds by self-duality.

Lemma 7.3. Let $d, n \in \mathbb{N}$. Then there are finitely many subgroups of $\text{GL}_d(\Lambda)$ such that for any simple perverse sheaf $P \in \mathbf{P}(X)$ with $\chi(P) = d$ and $\det(P)^n = 1$ the Tannaka group $G = G(P)$ is isomorphic to one of these finitely many groups.

Proof. Let $H = G^0$ be the connected component. The determinant condition gives an upper bound on the group $\pi = G/H$ of components. By [33] the group $\pi$ is abelian, and the geometric description of the restriction and induction functors between $\text{Rep}_\Lambda(G)$ and $\text{Rep}_\Lambda(H)$ in loc. cit. shows that on $H/[H,H]$ the adjoint action of $\pi$ is trivial. Using this one reduces the proof to the fact that $\text{GL}_d(\Lambda)$ has only finitely many connected reductive subgroups up to isomorphism.

Let us now return to an abelian scheme $p : X \to S$ whose base scheme is any variety $S$ over $k$. For $K \in \mathbf{D}(X)$ and geometric points $\eta$ of $S$ we write $K_\eta = i^*_x(K)$ for the pull-back to the geometric fibre $i_x : X_\eta \to X$.

Proposition 7.4. Let $n \in \mathbb{N}$ and $P \in \mathbf{D}(X)$ be such that for any geometric point $\eta$ the pull-back $P_\eta$ is a simple perverse sheaf with $\det(P_\eta)^n = 1$. Then there are reductive algebraic groups $G_1, \ldots, G_m$ and a stratification into locally closed subsets

$$S = \bigsqcup_{i=0}^m S_i$$

such that $G(P_\eta) = G_i$ for all geometric points $\eta$ in $S_i$. 
Proof. If $V$ is a finite-dimensional vector space over $\Lambda$, then every reductive algebraic subgroup of $GL_\Lambda(V)$ is determined uniquely by its invariants in the tensor powers $W^\otimes r$ of the representation $W = V \oplus V^\vee$ \cite[prop. 3.1(c)]{12}. Furthermore, if we only want to distinguish between finitely many given reductive subgroups up to conjugacy, then it suffices to consider only finitely many exponents $r \in \mathbb{N}$.

In the case at hand, it follows via lemma 7.3 that the Tambara group $G(P_s)$ is determined by the collection of all direct summands $\delta_0$ inside the convolution powers $(P_s \oplus P_s^\vee)^{*r}$ for finitely many $r$. To see that these direct summands depend on $s$ in a constructible way, one can use the general fact \cite[prop. 1.1.7]{23} that for any morphism $p : X \to S$ of varieties with smooth target $S$ and any $P \in D(X)$ there is an open dense subset $S' \to S$ such that the formation of the relative Verdier dual $R\mathcal{H}om(P, p!\Lambda_S)$ commutes with any base change that factors over $S'$.

\[QED\]

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