Abstract: The main purposes of this article are to extend our previous results on homogeneous sprays [13] to arbitrary (generalized) sprays, to show that locally diffeomorphic exponential maps can be defined for any (generalized) spray, and to give a (possibly nonlinear) covariant derivative for any (possibly nonlinear) connection. In the process, we introduce vertically homogeneous connections. Unlike homogeneous connections, these allow us to include Finsler spaces among the applications.

We provide significant support for the prospect of studying nonlinear connections via (generalized) sprays. One of the most important is our generalized APS correspondence.

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1 Introduction

An important class of systems of second order differential equations can be represented as (generalized) sprays on a manifold $M$ with tangent bundle $TM \to M$. So far only quadratic sprays are well understood, and they correspond to linear connections. But nonlinear connections are of real interest, especially in some newer applications [2, 3, 4, 24, 25].

In Riemannian geometry, the (usual) geodesic spray, whose integral curves are the geodesics of the Levi-Civita connection, has played an important rôle; see, for example, [10, 8]. In Finsler geometry, four main connections have been used, none of them linear: those of Cartan, Berwald, Hasiguchi, and Chern [5]. Riemannian geometry has been a main thread of mathematics over the last century, and Finsler geometry has recently undergone a great revival. Applications of it now include modeling the singular sets of Monge-Ampère equations [24], studying the manifold of Hamiltonians [9, 22], and modeling river flows and mountain slopes [3].

One of our motivations for this work was the desire to make a comprehensive theory of sprays and nonlinear connections which would include Riemannian and Finsler spaces as examples. We have recovered enough of the Riemannian results to be assured of the correctness of our approach; comparing the results for Finsler spaces of our methods with those of other methods will be the subject of future study.

Section 2 contains our notation, conventions, and a summary of our earlier article [13]. In Section 3 we present the new exponential maps defined by (generalized) sprays. Section 4 describes some relations between (possibly nonlinear) connections and (generalized) sprays and the associated (possibly nonlinear) covariant derivatives and geodesics. Section 5 begins with the extension of the main results of [7] to (generalized) sprays, using our new construction of (generalized) exponential maps. It also includes the extension of the main stability result of [6, 13] to all (generalized) sprays.

Throughout, all manifolds are smooth (meaning $C^\infty$), connected, paracompact, and Hausdorff.

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2 Review and definitions

A (general) spray on a manifold \( M \) is defined as a projectable section of the second-order tangent bundle \( TTM \rightarrow TM \). This is precisely the condition needed to define a second-order differential equation [10, 8]. Recall that an integral curve of a vector field on \( TM \) is the canonical lift of its projection if and only if the vector field is projectable. For any curve \( c \) in \( M \) with tangent vector field \( \dot{c} \), this \( \dot{c} \) is the canonical lift of \( c \) to \( TM \) and \( \ddot{c} \) is the canonical lift of \( \dot{c} \) to \( TTM \). Then each projectable vector field \( S \) on \( TM \) determines a second-order differential equation on \( M \) by \( \ddot{c} = S \circ \dot{c} \), and any such curve with \( \ddot{c}(s_0) = v_0 \in T_{c(s_0)}M \) is a solution with initial condition \( v_0 \). Solutions are preserved under translations of parameter, they exist for all initial conditions by the Cauchy theorem, and, as our manifolds are assumed to be Hausdorff, each solution will be unique provided we take it to have maximal domain; i.e., to be inextendible [10, 12, 19].

Let \( J \) be the canonical involution on \( TTM \) and \( C \) the Euler (or Liouville) vector field. We recall that in local coordinates, \( J(x, y, X, Y) = (x, X, y, Y) \) and \( C : (x, y) \mapsto (x, y, 0, y) \).

**Definition 2.1** A section \( S \) of \( TTM \) over \( TM \) is a spray when \( JS = S \); that is, when it can be expressed locally as \( S : (x, y) \mapsto (x, y, y, S(x, y)) \).

Before commenting on this definition, we must briefly digress to consider the notion of homogeneity for functions.

Consider the equation \( f(ax) = a^m f(x) \). In projective geometry, for example, one usually requires this to hold only for \( a \neq 0 \). We shall call this homogeneous of degree \( m \). In other areas, such as Euler’s Theorem in analysis, one further restricts to \( a > 0 \). We shall call this positively homogeneous of degree \( m \). Finally, in order that homogeneity of degree 1 coincide with linearity, one must allow any scalar \( a \in \mathbb{R} \) (including zero). We shall call this completely homogeneous of degree \( m \) and denote it by \( h(m) \).

The difference between homogeneity and complete homogeneity is minor; essentially, it is just the difference between working on \( TM - 0 \) and on \( TM \). The difference between positive homogeneity and the other two is more significant. For example, the inward-going and outward-going radial geodesics of the Finsler-Poincaré plane in [5] have different arclengths.

Now we are ready to consider homogeneity for sprays.

**Definition 2.2** We say that a spray \( S \) is homogeneous of degree \( m \) when the functions \( S(x, y) \) are completely homogeneous (respectively, homogeneous)
of degree $m$ in the vertical component in some induced local coordinates:
\[ S(x, ay) = a^m S(x, y) \] for some $m \geq 2$ (respectively, $m < 2$) and all scalars $a \in \mathbb{R}$ (respectively, $a \neq 0$).

The break comes at $m = 2$ because an $h(m)$ spray is to be associated with a connection whose homogeneity formula will contain $a^{-2}$; cf. (4.3). In the distinguished induced local coordinates, $S : (x, ay) \mapsto (x, ay, ay, a^m S(x, y))$. Only induced local coordinates $(x', y', X', Y')$ related to this $(x, y, X, Y)$ by a block-diagonal transition matrix
\[
\begin{bmatrix}
\frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\
\frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x'}{\partial x} & 0 \\
0 & \frac{\partial y'}{\partial y}
\end{bmatrix}
\]
will preserve the form of such an $S$. Other induced local coordinates preserve the correct degree of homogeneity in the vertical component $Y$, but may change the degree of homogeneity in the “horizontal” component $X$. Thus from now on, we shall use only these admissible atlases on $TM$ when studying homogeneous sprays et relata; cf. after Theorem 4.4.

**Remark 2.3** In the extant literature [13, 17, 18, 20, 21], one finds homogeneous vector fields of degree $m$ defined by $[C, S] = (m-1)S$. In any (not just admissible) local coordinates, $S : (x, ay) \mapsto (x, ay, a^{m-1} y, a^m S(x, y))$. It follows that a spray in our theory can be a homogeneous vector field only for $m = 2$.

Hereinafter we shall call $h(2)$ sprays quadratic sprays, in agreement with [18, 20, 21]. (Note that complete homogeneity is required for our quadratic sprays to coincide with the usual spray of [1].) We denote the set of our sprays on $M$ by $\text{Spray}(M)$ and those which are $h(m)$ by $\text{Spray}_m(M)$. It has been usual to consider only (positive) integral degrees of homogeneity, but we make no such restriction.

Previously [20], projectable vector fields on $TM$ were called semisprays and the name sprays used for those that were homogeneous. We will associate one of our (general) sprays to each (possibly nonlinear) connection as its geodesic spray (see Theorems 4.1 and 4.10), so we are using the name “sprays” to reflect this new, extended rôle. We do, however, explicitly consider only sprays defined on the entire tangent bundle $TM$; others [4, 5, 20] have used the reduced tangent bundle with the 0-section removed, which is appropriate when considering $h(m)$ sprays when $m < 2$ (including $m < 0$). For $0 \leq m < 2$, one usually requires sprays to be $C^0$ across the zero-section;
e.g., for Finsler spaces. Most of our results are easily seen to hold *mutatis mutandis* in these cases as well; any unobvious exceptions will be noted specifically.

In fact, the desire to make a comprehensive theory including Finsler spaces was one of our motivations. What *should* be the Finsler-geodesic spray associated with a Finsler metric tensor is *not* a homogeneous vector field, but an $h(1)$ spray in our theory; cf. [11] for related results. However, what is frequently used as the Finsler-geodesic spray has both quadratic and $h(1)$ parts; cf. [5], for example. We plan to address these peculiarities of the existing theory in subsequent work.

Several important results concerning quadratic sprays [1, 10, 16, 20] rely on the facts that each such spray $S$ determines a unique torsion-free linear connection $\Gamma$, and conversely, every quadratic spray $S$ arises from a linear connection $\Gamma$ the torsion of which can be assigned arbitrarily. The solution curves of the differential equation $\ddot{c} = S_\Gamma \circ \dot{c}$ for a connection-induced spray are precisely the geodesics of that (linear) connection. These solution curves are not only preserved under translations, as is true in general, but also under affine transformations of the parameter $s \mapsto as + b$ for constants $a, b$ with $a \neq 0$. Note that, with our definition, the latter also holds for homogeneous sprays.

In the general case, a (possibly nonlinear) connection $\Gamma$ gives rise to a spray $S$ (see Proposition 4.1), but the correspondence has not been well studied before. We shall extend most of the preceding features of the quadratic spray—linear connection correspondence to the general setting. One of our ultimate goals is to determine just how well nonlinear connections can be studied *via* sprays.

We continue with the principal definitions. Let $S$ be a (generalized) spray on $M$.

**Definition 2.4** We say that a curve $c : (a, b) \to M$ is a geodesic of $S$ or an $S$-geodesic if and only if the natural lifting $\dot{c}$ of $c$ to $TM$ is an integral curve of $S$.

This means that if $\ddot{c}$ is the natural lifting of $\dot{c}$ to $TTM$, then $\ddot{c} = S(\dot{c})$.

**Definition 2.5** We say that $S$ is pseudoconvex if and only if for each compact $K \subseteq M$ there exists a compact $K' \subseteq M$ such that each $S$-geodesic segment with both endpoints in $K$ lies entirely within $K'$.

If we wish to work directly with the integral curves of $S$, we merely replace “in” and “within” by “over”.

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Definition 2.6 We say that $S$ is disprisoning if and only if no inextendible $S$-geodesic is contained in (or lies over) a compact set of $M$.

In relativity theory, such inextendible geodesics are said to be imprisoned in compact sets; hence our name for the negation of this property.

Following this definition, we make a convention: all $S$-geodesics are always to be regarded as extended to the maximal parameter intervals (i.e., to be inextendible) unless specifically noted otherwise. When the spray $S$ is clear from context, we refer simply to geodesics. Also, we shall frequently consider noncompact manifolds because no spray can be disprisoning on a compact manifold. However, Corollary 5.2 may be used to obtain results about compact manifolds for which the universal covering is noncompact.

We refer to [13] for motivation, further general results, and results specific to homogeneous sprays, and to [14] for more examples. Note that the sprays in [13] were positively homogeneous; the extension of those results to complete homogeneity is straightforward, once the definition of homogeneous spray there is corrected to the one here.

3 Exponential maps

Let $S$ be a spray on $M$. We define the generalized exponential maps (plural!) $\exp^\varepsilon$ of $S$ as follows.

First let $p \in M$, $v \in T_p M$, and $c$ be the unique $S$-geodesic such that

\[
\begin{align*}
\dot{c} &= S(\dot{c}) \\
c(0) &= p \\
\dot{c}(0) &= v
\end{align*}
\]

Define

\[
\exp_p^\varepsilon(v) = c(\varepsilon)
\]

for all $v \in T_p M$ for which this makes sense. From the existence of flows (e.g., [19, p. 175]), it follows that this is well defined for all $\varepsilon$ in some open interval $(-\varepsilon_p, \varepsilon_p)$, which in general depends on $p$, and for all $v$ in some open neighborhood $U_p$ of $0 \in T_p M$, which in general depends on the choice of $\varepsilon \in (-\varepsilon_p, \varepsilon_p)$. This defines $\exp_p^\varepsilon$ at each $p \in M$.

Next, choose a smooth function $\varepsilon : M \to \mathbb{R}$ such that $\varepsilon(p) \in (-\varepsilon_p, \varepsilon_p)$ for every $p \in M$. (The smoothness of $\varepsilon$ is for our later convenience: we want $\exp_p^\varepsilon$ to be smooth in $\varepsilon$ as well as in all other parameters.) Then the global
map \exp^\varepsilon is defined pointwise by \((\exp^\varepsilon)_p = \exp^{\varepsilon(p)}_p\). The domain of \exp^\varepsilon is a tubular neighborhood of the 0-section in \(TM\) and the graph of \(\varepsilon\) lies in a tubular neighborhood of the 0-section in the trivial line bundle \(\mathbb{R} \times M\).

We have an example, given to us by J. Hebda, to show that it is possible that \(\varepsilon_p < 1\) for every open neighborhood of \(0 \in T_p M\) if the spray is inhomogeneous.

**Example 3.1** Consider the spray on \(\mathbb{R}\) given by

\[
\ddot{x}(t) = \pi (1 + \dot{x}(t)^2).
\]

To integrate, we rewrite this as

\[
\frac{d\dot{x}}{1 + \dot{x}^2} = \pi \, dt
\]

and obtain

\[
\arctan \dot{x} = \pi t + C_1.
\]

Thus

\[
\dot{x}(t) = \tan (\pi t + C_1), \quad \dot{x}(0) = \tan C_1
\]

so

\[
x(t) = \log|\sec (\pi t + C_1)| + C_2.
\]

For \(C_1 > 0\), \(x\) cannot be continued beyond

\[
\pi t + C_1 = \frac{\pi}{2},
\]

\[
t = \frac{1}{2} - \frac{C_1}{\pi} < 1.
\]

Therefore the usual exponential map of this spray is not defined (i.e., at \(t = \varepsilon = 1\)) for any \(C_1 > 0\).

The closer the graph of \(\varepsilon\) gets to the 0-section of \(\mathbb{R} \times M\), the larger the tubular neighborhood of the 0-section in \(TM\) gets.

**Proposition 3.2** For \(\varepsilon_1 \leq \varepsilon_2\), we have \(\text{dom}(\exp^{\varepsilon_2}) \supseteq \text{dom}(\exp^{\varepsilon_1})\), attaining all of \(TM\) for \(\varepsilon = 0\) when \(\exp^0 = \pi\).

This puts the bundle projection \(TM \to M\) in the interesting position of being a member of a one-parameter family of maps, all of whose other members are local diffeomorphisms. (This is reminiscent of singular perturbations.)
Figure 1: curves exp^ε_p(av) — Each black curve is a geodesic with 0 < ε < 3 and a and v fixed. From shortest to longest in each plume, a steps in increments of 0.05 from 0.05 to 1. In each plume, v is constant. There are three implicit a-parameter curves readily located, one along the endpoints of each of the three plumes.

**Theorem 3.3** For every ε such that 0 < |ε| < ε_p, the generalized exponential map exp^ε_p is a diffeomorphism of an open neighborhood of 0 ∈ T_pM with an open neighborhood of p ∈ M.

**Proof:** This follows from the flow theorems in ODE (e.g., [19, pp. 175, 302]). □

Note that for v ∈ T_pM, exp^ε_p v = πΦ(ε, v) where Φ is the local flow of S.

For reference, we record the following obvious result.

**Lemma 3.4** ε is a geodesic parameter; i.e., the curve obtained by fixing v and varying ε is a geodesic through p. □

Now consider another parameter a as in

exp^ε_p(av).

In general, a will not be a geodesic parameter; i.e., the curve obtained by fixing ε and v and varying a is not a geodesic through p. See Figures 1 and 2 for a comparison.

**Proposition 3.5** If S is homogeneous, then a as above is a geodesic parameter.
Figure 2: curves $\exp_p^\varepsilon(a v)$ — This is one plume from Figure 1. Each black curve is a geodesic and each gray curve is an $a$-parameter curve. The new Jacobi fields are along the black curves but tangent to the gray curves.

**Proof:** When $S$ is homogeneous, we can take $\varepsilon = 1$ and recover the usual exponential map, and then $a$ is the usual geodesic parameter.

The $a$-parameter curves are interesting: they are the integral curves for our new Jacobi vector fields. These were mentioned in [14] and will be studied in more detail elsewhere. For now, we have the following example.

**Example 3.6** In $\mathbb{R}^2$, consider the spray given by $S^i(x, y) = x^i$ for $i = 1, 2$. The geodesics are easily found to be $c(t) = v e^t + p$ where $v$ is the initial velocity and $p$ is the initial position. We can use the usual exponential map since these curves are always defined for $t = 1$. Thus we obtain $\exp_p(v) = c(1) = v e + p$, regarding both $v$ and $p$ as vectors in $\mathbb{R}^2$.

For the $a$-curves, we have $\exp_p(a v) = a v e + p$, showing the difference between the two types quite clearly: the geodesics have exponential growth in velocity, while the $a$-curves have only linear growth.

Finally, note that we could just as well define exponential-like maps based on the $a$-curves and they would share most of the properties of our new exponential maps.

### 4 Connections and sprays

In general, a *connection* on a manifold $M$ is a subbundle $\mathcal{H}$ of the second tangent bundle $\pi_T : TTM \to TM$ which is complementary to the vertical
bundle $\mathcal{V} = \ker(\pi_*: TTM \to TM)$, so

$$TTM = \mathcal{H} \oplus \mathcal{V}.$$  \hfill (4.1)

We note there are two vector bundle structures on $TTM$ over $TM$, denoted here by $\pi_T$ and $\pi_*$. While $\mathcal{V}$ is always a subbundle with respect to both [23, pp. 18, 20], $\mathcal{H}$ is a subbundle with respect to $\pi_*$ if and only if the connection is linear [8, p. 32].

Recall that quadratic sprays correspond to linear connections. In terms of the horizontal bundle $\mathcal{H}$, linearity is expressed as

$$\mathcal{H}_av = a_\pi \mathcal{H}_v$$

for $a \in \mathbb{R}$ and $v \in TM$. Thus one has

$$\mathcal{H}_av = a_\pi a^{m-1} \mathcal{H}_v$$ \hfill (4.2)

as the second defining equation, together with (4.1), of a connection that is $h(m)$.

Note that for an $h(m)$ semispray $S$ with integral $m$, Grifone’s [18] associated (generalized) Christoffel symbols $\Gamma$ are $h(m-1)$, appropriately. See (4.6) below for our version, which allows for nonlinear, including inhomogeneous, connections.

Here is the spray induced by a connection. We shall call it the geodesic spray associated to the connection and its geodesics the geodesics of the connection.

**Theorem 4.1** For each connection $\mathcal{H}$, there is an induced spray $S$ given by

$$S(v) = \pi_*|_{\mathcal{H}}^{-1}(v),$$

where $\pi : TM \to M$ is the natural projection and $v \in TM$. We write $\mathcal{H} \vdash S$ to denote this relationship.

**Proof:** As in the first paragraph of Poor’s proof of 2.93 [23, p. 95], it is easily verified that $S$ so defined is a spray. Indeed, $S$ is a section of $\pi_*$ by construction, and $S$ is a section of $\pi_T$ because $\mathcal{H}$ is a subbundle with respect to $\pi_T$. \hfill $\square$

It is clear that this spray is horizontal, so compatible with the given connection.
Unfortunately, when the connection is \( h(m - 1) \) this spray is not homogeneous as a spray; it is only an \( h(m) \) vector field on \( TM \). In order to avoid this problem, we must consider a new type of (partial) homogeneity for connections.

**Definition 4.2** A connection \( \mathcal{H} \) on \( TM \) is vertically homogeneous of degree \( m \), denoted by \( vh(\mathcal{H}) \), if and only if

\[
\mathcal{H}_{av} = a* a_v^{m-1} \mathcal{H}_v \tag{4.3}
\]

where \( a_v^n \) denotes scalar multiplication by \( a^n \) in the vertical component and \( a \) in the horizontal component in some (hence any) admissible local coordinates.

More explicitly, \( a_v^m(x, y, X, Y) = (x, a y, a X, a^m Y) \) in admissible coordinates. Note that \( h(m) \) and \( vh(m) \) coincide only for \( m = 1 \), the linear connections.

**Proposition 4.3** If \( \mathcal{H} \) is a connection with geodesic spray \( S \), then \( S \) is \( h(m) \) if and only if \( \mathcal{H} \) is \( vh(m - 1) \).

**Proof:** That \( S \) is \( h(m) \) if \( \mathcal{H} \) is \( vh(m - 1) \) follows as in the second paragraph of Poor’s proof of 2.93 [23, p. 95], mutatis mutandis; the converse results from a similar calculation. \( \Box \)

Connections may also be seen as sections of the bundle \( G_H(TTM) \) of all possible horizontal spaces, a subbundle of the Grassmannian bundle \( G_n(TTM) \). To see what structure \( G_H(TTM) \) has, consider \( \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n \) as the model fiber of \( TTM \) and regard the first summand as horizontal, the second as vertical. With \( GL_{2n} \) as the structure group of \( TTM \), we want the subgroup \( A_H \) that preserves the vertical space and maps any one horizontal space into another. This can be conceived as occurring in two steps. First, we may apply any automorphisms of the vertical and horizontal spaces separately. Second, we may add vertical components to horizontal vectors to obtain the new horizontal space.

\[
\begin{bmatrix}
I & 0 \\
gl_{n} & I
\end{bmatrix}
\begin{bmatrix}
GL_n & 0 \\
0 & GL_n
\end{bmatrix}
\]

Our group \( A_H \) is thus found to be a semidirect product entirely analogous to an affine group. The action is transitive and the right-hand factor is
the isotropy group of any fixed horizontal space, so the model fiber for $G_H(TTM)$ is the resulting homogeneous space. The induced operation on representatives being given by
\[
\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ A+B & I \end{bmatrix},
\]
it follows that $G_H(TTM)$ is an affine bundle (bundle of affine spaces, vs. vector spaces). Thus a connection, being a section of this bundle, provides a choice of distinguished point in each fiber, hence a vector bundle structure on this affine bundle.

If we wish to consider only those connections compatible with a given spray, we just replace arbitrary elements of $\mathfrak{gl}_n$ with those having a first column comprised entirely of zeros. Note that this yields an affine subbundle $G^S_H(TTM)$ of $G_H(TTM)$, with fibers being pencils of possible horizontal spaces.

**Theorem 4.4** Given a spray $S$ on $M$, there exists a compatible connection $\mathcal{H}$ in $TTM$.

Since the fibers of $G^S_H(TTM)$ are contractible, this is an easy exercise in obstruction theory [15, Ch. 8]; however, an explicit construction is desirable.

**Proof:** Let $\Phi$ denote the local flow of $S$ and $\gamma$ an integral curve of $S$ with $\gamma(0) = v \in T_pM$. The basic idea is to use $S$ and $\Phi$ to define notions of horizontal and parallel which will coincide with the usual ones along $\gamma$ for any $\mathcal{H} \vdash S$. This is essentially the same as the usual construction [23].

The problem is that for inhomogeneous $S$, the ray $\{tv\}$ in $T_pM$ does not exponentiate to a geodesic in $M$.

To remedy this, we proceed as follows. For each $v \in T_pM$, choose $\varepsilon_v$ so that $\exp_{p,v}^{\varepsilon_v} v$ is defined. Such $\varepsilon_v$ exist by Proposition 3.2. For $0 \leq t \leq \varepsilon_v$, define
\[
\alpha_v(t) = \left(\exp_{p,v}^{\varepsilon_v}\right)^{-1} \exp_{p,v}^{t\varepsilon_v} v.
\]
Then $\alpha_v(0) = 0$, $\alpha_v(\varepsilon_v) = v \in T_pM$, and $\alpha_v$ exponentiates to the geodesic with initial condition $v$ at $p$. Note that if $S$ is homogeneous, then $\alpha_v(t) = tv$.

We have a vector bundle map $\mathcal{J} : \pi^*TM \to V$ which is an isomorphism on fibers. It is one version of canonical parallel translation on a vector space, identifying the tangent space at each point with the vector space itself. Now, for each $w \in T_pM$ define
\[
\mathcal{H}_w = \left\{ \frac{d}{dt} \bigg|_{t=0} \pi_* \Phi_{t,\ast} \mathcal{J}_{\alpha_v(t)} w \bigg| v \in T_pM \right\}.
\]
Clearly, this does not depend on the choices of $\varepsilon_v$ made earlier. (Note we are evaluating at 0.) If $S$ is quadratic, it is easy to check that this coincides with the usual construction as found in [23, pp.96–97], since in that case $\exp_p tv = \pi \Phi(t, v)$ for $v \in T_p M$. The proof that $\mathcal{H}$ so defined is a connection and that $\mathcal{H} \mid S$ follows Poor’s proof of 2.98 [23, pp.97–99] *mutatis mutandis.*

These connections will be our “standard”—our generalization of torsion-free linear connections; *cf.* equation (4.10), Definition 4.16 and after.

We further note that admissible atlases correspond to certain reductions of the structure group of $TTM$ from $GL_{2n}$ to $GL_n \oplus GL_n$, those which in turn correspond to direct-sum decompositions of $TTM$ in which one of the summands is the vertical bundle $\mathcal{V}$ (and the other is perform a horizontal bundle), hence to connections in $TTM$. Thus any homogeneous spray $S$ comes with a particular associated compatible connection $\mathcal{H}$, the one corresponding to the associated admissible atlas; *cf.* after Definition 2.2. Note, however, that it may not be the one naturally associated by the preceding construction.

Here is an alternative, axiomatic characterization of a connection in terms of the horizontal projection $H$.

\begin{itemize}
  \item[C1] $H$ is a smooth section of $\text{End}(TTM)$ over $TM$.
  \item[C2] $H^2 = H$.
  \item[C3] $\ker H = \mathcal{V}$.
\end{itemize}

Then $\mathcal{H} = \text{im} \ H$ is the horizontal bundle. Vertical homogeneity is expressed with an optional axiom.

\begin{itemize}
  \item[Ch] $H$ is $vh(m)$ if and only if $H_{av}a_* = a_*a_v^{m-1}H_o$ for all $v \in TM$ and $a \in \mathbb{R}$ ($v \in TM - 0$ and $a \neq 0$ for $m < 1$).
\end{itemize}

Homogeneous connections may be similarly axiomatized.

There is another natural vector bundle map $K : \mathcal{V} \to TM$ respecting $\pi_T$ which is an isomorphism on fibers, another version of canonical parallel translation of a vector space. Using this, we define a connection map or connector for an arbitrary connection and thence a covariant derivative.

**Definition 4.5** For a connection $\mathcal{H}$, define the associated connector $\kappa : TTM \to TM : z \mapsto K(z - H_v z)$ for $z \in T_v TM$.  

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Proposition 4.6 The connector $\kappa$ is a vector bundle map respecting $\pi_T$ but not $\pi_*$ in general. It respects $\pi_*$ if and only if the connection is linear.

Proof: As in Poor [23, p. 72f], mutatis mutandis. \hfill $\square$

According to Besse [8, p. 32f], a symmetric connector (connection) is invariant under the natural involution $J$ of $TTM$. Clearly this is possible only for linear connections.

Now we are ready for the main event.

Definition 4.7 The covariant derivative associated to the connection $\mathcal{H}$ is the operator defined by

$$\nabla_u v = \kappa(v_* u) = K(v_* u - H_v v_* u)$$

and is tensorial in $u$ but nonlinear (in general) in $v$.

This last comes from the general lack of respect for the $\pi_*$ structure by $\mathcal{H}$, $H$, and $\kappa$.

Example 4.8 We always have $\nabla_0 v = 0$. For any $vh(m)$ connection, $\nabla_u av = K(a_* v_* u - H_a v_* u) = aK(v_* u - a_v a_* v_* u)$, and similarly for homogeneous ones. So (vertically) homogeneous connections do not differ significantly from linear ones. In particular, $\nabla_u 0 = 0$ for all $u$ for all (vertically) homogeneous connections; in fact, they all have the same horizontal spaces along the 0-section of $TM$, namely the subspaces tangent to it (i.e., those in the image of $0_* : TM \to TTM$). We call all such connections sharing this property $0$-preserving; they differ minimally from (vertically) homogeneous (including linear) connections. In contrast, connections with $\nabla_u 0 \neq 0$ for even some $u$ are much farther from linear; we call them strongly nonlinear. See Figure 3 for a schematic view.

As usual, $X$ denotes the vector fields on $M$. We recall the vector bundle map $\beta : \pi^* TM \to V$ which is an isomorphism on fibers and a version of canonical parallel translation on a vector space.

Theorem 4.9 There is a bijective correspondence between (possibly nonlinear) connections $\mathcal{H}$ and our (possibly nonlinear) covariant derivatives $\nabla$ on $TM$. 

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Figure 3: Each set of connections is closed with empty interior in the next: linear in homogeneous, linear in vertically homogeneous, linear and homogeneous in 0-preserving, linear and vertically homogeneous in 0-preserving, linear and homogeneous in 0-preserving, 0-preserving in the whole. The strongly nonlinear connections may be visualized as a 3-d cloud containing the 0-preserving ones.

**Proof:** It suffices to show that we can reconstruct $\mathcal{H}$ from its associated covariant derivative $\nabla$. For each $u \in T_p M$, define

$$\mathcal{H}_u = \{ U \ast v - \mathcal{J}_u \nabla_v U \mid U \in \mathfrak{X}, U_p = u, v \in T_p M \}$$

and form the subbundle $\mathcal{H}$ in $TTM$ in the obvious way. It is easy to see that $\mathcal{H}$ is complementary to $\mathcal{V}$ as required, hence a connection. That $\mathcal{H}$ is smooth is straightforward. Finally, $\mathcal{H} = \mathcal{H}$ from this construction and the construction of $\nabla$ from $\mathcal{H}$. □

Compare [23, p. 77, proof of 2.58]. Thus as usual, we may refer indifferently to $\mathcal{H}$ or its associated $\nabla$ as the connection.

Generalized Christoffel symbols may be introduced through

$$(KH_v u)_k = \Gamma^k_i(v)u^i, \quad (4.6)$$

making manifest the tensoriality in $u$. Here are some examples of their use. Observe that $(Kv, u)^k = u^i \partial_i v^k$ so that

$$(\nabla_v u)_k^i = u^i \partial_i v^k - \Gamma^k_i(v)u^i \quad (4.7)$$

is the covariant derivative. The geodesic equation is

$$\ddot{c}^k = \Gamma^k_i(\dot{c})\dot{c}^i, \quad (4.8)$$
which means that
\[ S^k(\dot{c}) = \Gamma^k_i(\dot{c})\dot{c}^i \]  
(4.9)
for \( S \) the spray induced by the connection \( \nabla \). Note that we can write any \( S^k(v) \) in the form
\[ S^k(v) = \Gamma^k_i(v)v^i, \]  
(4.10)
although \( \Gamma \) may be less well-behaved than \( S \). In this way we can obtain the standard (“torsion-free”) connection associated to \( S \) by our generalized APS construction (proof of Theorem 4.4).

We obtain the usual relation between two notions of geodesic.

**Theorem 4.10** A curve \( c \) is a geodesic of \( \mathcal{H} \) if and only if \( \nabla_\epsilon \dot{c} = 0 \).

**Proof:** \( \nabla_\epsilon \dot{c} = \kappa(\dot{c}, \dot{c}) = K(\dot{c}, \dot{c} - H_\epsilon \dot{c}, \dot{c}) = K(\dot{c}, \dot{c} - S(\dot{c})) \) by the construction of \( S \) in Proposition 4.1. Now all we have to do is identify \( \dot{c}, \dot{c} \) as \( \ddot{c} \) and recall that \( K \) is an isomorphism on fibers. \( \square \)

Curvature is readily handled. Let \( \mathcal{H} \) be a connection on \( M \). The horizontal lift of a vector field \( U \) on \( M \) is defined as usual and denoted by \( \bar{U} \).

**Definition 4.11** Given vector fields \( U \) and \( V \) on \( M \), the curvature operator \( R(U, V) : TM \to TM \) is defined by
\[ R(U, V)w = \kappa([\bar{V}, \bar{U}]_w) \]
for all \( w \in TM \). It is tensorial in the first two arguments, but nonlinear (in general) in the third.

The arguments are reversed on the right in order to obtain the usual formula in terms of the associated covariant derivative,
\[ R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U, V]} W, \]
as one may verify readily. It is also easy to check that this curvature vanishes if and only if \( \mathcal{H} \) is integrable, thus justifying our definition.

Torsion is considerably more obscure. Consider two (possibly nonlinear) connections \( \mathcal{H} \) and \( \mathcal{H} \) on \( TM \) with corresponding (possibly nonlinear) covariant derivatives \( \nabla \) and \( \nabla \).

**Definition 4.12** Given two covariant derivatives \( \nabla \) and \( \nabla \), define the difference operator \( D = \nabla - \nabla \).
We think of $D$ as having two arguments, $D(u, v) = \bar{\nabla}_u v - \nabla_u v$. It is always tensorial in $u$, but is nonlinear (in general) in $v$. Alternative notations include $D_u v$ and $D(u)v$ when one wishes to emphasize certain aspects.

We define the covariant differential as usual via $(\nabla v)u = \nabla u v$. As an operator, $\nabla v$ is still linear in its argument $u$.

**Proposition 4.13** For all $v \in TM$, $\mathcal{H}_v = \{ z - J_v \mathcal{D}(\pi^* z, v) | z \in \mathcal{H}_v \}$.

**Proof:** Let $v \in T_p M$, $z \in \mathcal{H}_v$, $V \in \mathfrak{X}$ such that $(\nabla V)_p = 0$ and $V_p = v$. Thus if $u = \pi_* z \in T_p M$, then $z = V_* u \in \mathcal{H}_v$. Now

$$\bar{\kappa} V_* u = \bar{\nabla}_u V = \nabla_u V + \mathcal{D}(u, v) = \mathcal{D}(u, v) = \bar{\kappa} \mathcal{D}(u, v),$$

so $\bar{\kappa} (z - J_v \mathcal{D}(u, v)) = 0$ and $z - J_v \mathcal{D}(u, v) \in \mathcal{H}_v$.

Since $\pi_*$ is an isomorphism of the horizontal spaces $\mathcal{H}_v$ and $\mathcal{H}_v$ with $T_p M$ and $\pi_* z = \pi_* (z - J_v \mathcal{D}(u, v))$, this yields all of $\mathcal{H}_v$. □

**Proposition 4.14** Two connections on $TM$ have the same geodesic spray if and only if their associated difference operator is alternating (vanishes on the diagonal of $TM \oplus TM$).

**Proof:** For each $v \in TM$, $S_v = \pi_* |_{\mathcal{H}_v}^{-1}(v)$ while $\tilde{S}_v = \pi_* |_{\mathcal{H}_v}^{-1}(v) = \pi_* |_{\mathcal{H}_v}^{-1}(v) - J_v \mathcal{D}(v, v)$. Therefore $\tilde{S} = S$ if and only if $\mathcal{D}(v, v) = 0$ for all $v \in TM$. □

For linear connections, $D$ is bilinear and alternating is equivalent to anti-symmetric (or, skewsymmetric). In general, of course, this does not hold.

The familiar formula for torsion $T(u, v) = \nabla_u v - \nabla_v u - [u, v]$ is not linear (let alone tensorial) in either argument. Thus the usual trick to get a torsion-free linear connection, replacing $\nabla$ by $\nabla = \nabla - \frac{1}{2} T$, will not work for our nonlinear connections. Indeed, $\nabla$ and $\tilde{\nabla}$ seem to have the same geodesics and $\tilde{\nabla}$ is formally torsion-free, but the new $\tilde{\nabla}$ is not one of our nonlinear covariant derivatives: $\tilde{\nabla}_u v$ is not tensorial in $u$.

**Definition 4.15** Denote by $\text{Op}TM$ the smooth maps $TM \to TM$ which preserve fibers (i.e., commute with the projection onto $M$). We shall write $\text{Op}_m TM$ when the maps are $h(m)$ on each fiber.

Note they are smooth on fibers, but not necessarily linear. When they are fiberwise linear, we have $\text{End}TM$ as in the usual formulation. As is $\text{End}TM$, $\text{Op}TM$ is a vector bundle over $M$ and a Lie algebra. (Addition,
scalar multiplication, and composition—thus commutators—are well defined and preserve fibers.) By analogy to the linear theory, we usually think of a covariant derivative $\nabla$ as a section (over $M$) of $\text{Hom}(TM, \text{Op}TM)$ and $D$ as an $\text{Op}TM$-valued 1-form on $M$.

A replacement $T$ for torsion must also be an element of $A^1(M, \text{Op}TM)$ in order for it to play the same role in general that torsion does for linear connections. For then, given any $T \in A^1(M, \text{Op}TM)$, $\bar{\nabla} = \nabla + T$ is another nonlinear covariant derivative of our type. We want to choose $T$ so that $\bar{\nabla}$ has the same geodesics as $\nabla$ but is as analogous to a torsion-free linear connection as possible.

As we noted immediately after the proof of Theorem 4.4, what we shall do is one of the classic mathematical gambits: turn a theorem into a definition.

**Definition 4.16** We define the connections constructed in the proof of Theorem 4.4 to be the (generalized) torsion-free connections.

(We refer to Poor [23, pp. 101–102] for the relation to the classic Ambrose-Palais-Singer correspondence.) Equivalently, we are regarding the usual torsion formula as derived from the difference operator (difference tensor in the linear case) construction; cf. [23, pp. 99–100].

Now we may construct the (generalized) torsion of a (possibly nonlinear) connection $\mathcal{H}$ with corresponding (possibly nonlinear) covariant derivative $\nabla$. By Theorem 4.1, $\mathcal{H}$ induces a (unique, generalized) spray $S$. Use the proof of Theorem 4.4 to construct the connection $\hat{\mathcal{H}}$ from $S$. By Theorem 4.9 there is a unique covariant derivative $\hat{\nabla}$ corresponding to $\hat{\mathcal{H}}$. Let $D = \nabla - \hat{\nabla}$ be the difference operator, so $\hat{\nabla} = \nabla - D$ is (generalized) torsion-free.

**Definition 4.17** Using the preceding notations, the (generalized) torsion of $\nabla$ is defined by $T = 2D = 2\left(\nabla - \hat{\nabla}\right)$.

The factor of two here and the subtraction order make verification that this reduces to classical torsion in the linear case immediate, and preserves the traditional formula $\hat{\nabla} = \nabla - \frac{1}{2}T$ for the associated torsion-free connection.

### 5 Geodesic connectivity and stability

In [13], we defined a spray to be LD if and only if its usual exponential map is a local diffeomorphism. For some results there, we used the fact that the geodesics of such sprays give normal starlike neighborhoods of each point in $M$. (In fact, the $a$-curves also give such neighborhoods, as is easily seen.)
These results now immediately extend to all sprays. For convenience, we state them here.

**Proposition 5.1** Let $M$ be a manifold with a pseudoconvex and disprisoning spray $S$. If $S$ has no conjugate points, then $M$ is geodesically connected.

Let $M$ be a manifold with a spray $S$ and let $\tilde{M}$ be a covering manifold. If $\phi : \tilde{M} \to M$ is the covering map, then it is a local diffeomorphism. Thus $\tilde{S} = (\phi_\ast)^* S$ is the unique spray on $\tilde{M}$ which covers $S$, geodesics of $\tilde{S}$ project to geodesics of $S$, and geodesics of $S$ lift to geodesics of $\tilde{S}$. Also, $S$ has no conjugate points if and only if $\tilde{S}$ has none. The fundamental group is simpler, and $\tilde{S}$ may be both pseudoconvex and disprisoning even if $S$ is neither.

**Corollary 5.2** Let $M$ be a manifold with a pseudoconvex and disprisoning spray $S$ and let $\tilde{M}$ be a covering manifold with covering spray $\tilde{S}$. If $\tilde{S}$ has no conjugate points, then both $\tilde{M}$ and $M$ are geodesically connected.

**Theorem 5.3** Let $S$ be a pseudoconvex and disprisoning spray on $M$. If $S$ has no conjugate points, then for each $p \in M$ the exponential maps of $S$ at $p$ are diffeomorphisms.

We remark that none of these results require (geodesic) completeness of the spray $S$.

We now consider the joint stability of pseudoconvexity and disprisonment for (general) sprays in the fine topology. Because each linear connection determines a (quadratic) spray, Examples 2.1 and 2.2 of [6] show that neither condition is separately stable. (Although [6] is written in terms of principal symbols of pseudodifferential operators, the cited examples are actually metric tensors). We shall obtain $C^0$-fine stability, rather than $C^1$-fine stability as in [6], due to our effective shift from potentials to fields as the basic objects. The proof requires some modifications of that in [6]; we shall concentrate on the changes here and refer to [6] for an outline and additional details.

Rather than considering $r$-jets of functions, we now take $r$-jets of sections in defining the Whitney or $C^r$-fine topology as in Section 2 of [6]. Let $h$ be an auxiliary complete Riemannian metric on $M$. Thus we look at the $C^r$-fine topology on the sections of $TTM$ over $TM$.

If $\gamma_1$ and $\gamma_2$ are two integral curves of a spray $S$ with $\gamma_1(0) = (x, v)$ and $\gamma_2(0) = (x, lv)$ for some positive constant $l$, then the inextendible geodesics
\[ \pi \circ \gamma_1 \text{ and } \pi \circ \gamma_2 \text{ no longer differ only by a reparametrization. Thus, in contrast to [6], we must now consider an integral curve for each non-zero tangent vector at each point of } M. \text{ Note this also means that we can no longer use the } h\text{-unit sphere bundle to obtain compact sets in } TM \text{ covering compact sets in } M. \]

Observe that the equations of geodesics involve no derivatives of \( S \). Thus if \( \gamma : [0, a] \to TM \) is a fixed integral curve of \( S \) in \( TM \) with \( \gamma(0) = v_0 \in TM \), then \( d_h(\pi \circ \gamma(t), \pi \circ \gamma'(t)) < 1 \) for \( 0 \leq t \leq a \) provided that \( v \) is sufficiently close to \( v_0 \) and \( S' \) is sufficiently close to \( S \) in the \( C^0 \)-fine topology. This and the \( \sigma \)-compactness of \( TK_1 \) when \( K_1 \) is compact yield the following result.

**Lemma 5.4** Assume \( K_1 \) is a compact set contained in the interior of the compact set \( K_2 \), \( V \) is an open neighborhood of \( K_2 \), \( S \) is a disprisoning spray, and let \( \epsilon > 0 \). There exist countable sets \( \{ v_i \} \subseteq TK_1 \) of tangent vectors and \( \{ \delta_i \} \) and \( \{ a_i \} \) of positive constants such that if \( S' \) is in a \( C^0 \)-fine \( \epsilon \)-neighborhood of \( S \) over \( V \), then the following hold:

1. if \( c \) is an inextendible \( S \)-geodesic with \( c(0) \) in a \( \delta_i \)-neighborhood of \( v_i \), then \( c[0, a_i] \subset V \) and \( c(a_i) \in V - K_2 \);
2. If \( c' \) is an inextendible \( S' \)-geodesic with \( c'(0) \) in a \( \delta_i \)-neighborhood if \( v_i \), then \( c'[0, a_i] \subset V \) and \( c'(a_i) \in V - K_2 \);
3. Two inextendible geodesics, \( c \) of \( S \) and \( c' \) of \( S' \) with \( c(0) \) and \( c'(0) \) in a \( \delta_i \)-neighborhood of \( v_i \), remain uniformly close together for \( 0 \leq t \leq a_i \);
4. The union of all the \( \delta_i \)-neighborhoods of the \( v_i \) covers \( TK_1 \). \( \square \)

Continuing to follow [6], we construct the increasing sequence of compact sets \( \{ A_n \} \) which exhausts \( M \) and the monotonically nonincreasing sequence of positive constants \( \{ \epsilon_n \} \). The only additional changes from [6, p. 17f] are to use integral curves of \( S \) in \( TM \) instead of bicharacteristic strips in \( T^*M \). No other additional changes are required for the proof of the next result either.

**Lemma 5.5** Let \( S \) be a pseudoconvex and disprisoning spray and let \( S' \) be \( \delta \)-near to \( S \) on \( M \). If \( c' : (a, b) \to M \) is an inextendible \( S' \)-geodesic, then there do not exist values \( a < t_1 < t_2 < t_3 < b \) with \( c'(t_1) \in A_n \), \( c'(t_3) \in A_n \), and \( c'(t_2) \in A_{n+4} - A_{n+3} \). \( \square \)
Now we establish the stability of pseudoconvex and disprisoning sprays by showing that the set of all sprays in Spray($M$) which is pseudoconvex and disprisoning is an open set in the $C^0$-fine topology. The only changes needed from the proof of Theorem 3.3 in [6, p. 19] are replacing principal symbols by sprays, bicharacteristic strips by integral curves, $S^*A_n$ by $TA_n$, and references to Lemma 3.2 there by references to Lemma 5.5 here.

**Theorem 5.6** If $S \in \text{Spray}(M)$ is a pseudoconvex and disprisoning spray, then there is some $C^0$-fine neighborhood $W(S)$ in Spray($M$) such that each $S' \in W(S)$ is both pseudoconvex and disprisoning.

**Corollary 5.7** If $M$ is a pseudoconvex and disprisoning pseudoriemannian manifold, then any (possibly nonlinear) connection on $M$ which is sufficiently close to the Levi-Civita connection is also pseudoconvex and disprisoning.

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