Abstract

Network data are often sampled with auxiliary information or collected through the observation of a complex system over time, leading to multiple network snapshots indexed by a continuous variable. Many methods in statistical network analysis are traditionally designed for a single network, and can be applied to an aggregated network in this setting, but that approach can miss important functional structure. Here we develop an approach to estimating the expected network explicitly as a function of a continuous index, be it time or another indexing variable. We parameterize the network expectation through low dimensional latent processes, whose components we represent with a fixed, finite-dimensional functional basis. We derive a gradient descent estimation algorithm, establish theoretical guarantees for recovery of the low dimensional structure, compare our method to competitors, and apply it to a data set of international political interactions over time, showing our proposed method to adapt well to data, outperform competitors, and provide interpretable and meaningful results.

Keywords: Latent space model; Multilayer; Multiplex; Functional network; Dynamic network; B-spline

1 Introduction

Modern data are collected in a much greater variety of forms than classical statistics considered, and require novel models and methods. Networks are one important example of a complex data structure which has received recent interest in many fields of application. In general, network data on \( n \) statistical units, or nodes, describe connections, or edges, between pairs of those units. This information is stored in the \( n \times n \) adjacency matrix \( A \), where each entry \( \{ A_{ij} : i, j = 1, \ldots, n \} \) describes the connection from node \( i \) to node \( j \), which could be binary or real-valued. Much of the statistical network analysis literature deals with a single network, sometimes with auxiliary information, but increasingly samples of networks are also studied. Samples of networks arise in diverse applications: for instance, in neuroimaging we observe a brain connectivity network for each study subject, and in political science, we observe relations between countries over many years. In this paper, we focus on functional network data, the setting where edges are indexed by a continuous
auxiliary variable. This variable is commonly time, but it can be anything else that has a natural ordering.

Often, functional network data are collected as repeated, indexed snapshots of a single evolving network. In this regime, suppose that we observe a network on a common set of \( n \) nodes, at \( m \) distinct indices \( \{x_k\}_{k=1}^m \) in a compact set \( \mathcal{X} \subseteq \mathbb{R} \). The complete data is made up of a collection of indexed adjacency matrix snapshots, \( \{A_k\}_{k=1}^m \subseteq \mathbb{R}^{n \times n} \), where each matrix entry \( [A_k]_{ij} \) gives the value of edge \((i, j)\) from node \( i \) to node \( j \) for the snapshot corresponding to index \( x_k \). These adjacency matrices may have binary edges with values in \( \{0, 1\} \), or weighted edges taking any real value, and may be either undirected or directed. Examples of functional network data collected as snapshots include dynamic friendship networks [35], or dynamic networks of international conflict (cf. Section 6).

It is also common that functional network data are collected in the form of time-stamped records of interactions between pairs of nodes, or more generally indexed in some continuous space. Examples of functional network data collected in this manner include e-mail networks, bike share networks [26], or animal interaction networks [28]. We can construct snapshots from such event data by partitioning the index set into \( m \) contiguous intervals, and constructing weighted adjacency matrices which count the number or total weight of events between nodes \( i \) and \( j \) in a given interval.

Another way to view adjacency matrix snapshots is to treat them as a multiplex network [18], a multilayer network object with \( m \) layers corresponding to the snapshots. However, the ordering of the layers, inherited from the ordering of the indexing variable, gives them additional structure not usually present in a multiplex network.

Much of the existing literature focuses on the dynamic network setting, with the snapshots indexed by time, rather than the general functional setting. We briefly review recent related work for dynamic networks; see [17] for a more detailed review. Approaches based on the Stochastic Block Model (SBM) include [27], which assumes node community memberships follow a Markov chain and allows connection probabilities to vary; [9], which assumes community memberships are fixed but allows connection probabilities to vary; and [29], which allows both community memberships and connection probabilities to vary smoothly in time. Latent space or latent position models for networks have also been popular since the seminal work of [16]. Latent space approaches to dynamic networks go back to [32], which models latent positions in discrete time with independent Gaussian random walks. A Bayesian latent space approach with similar random walk transitions on latent positions in discrete time was proposed by [34], and extended to continuous time by [11]. Both [22] and [31] consider extensions of the random dot product graph (RDPG) to the dynamic setting. In all of these papers, the single network latent space framework is extended by mapping each node to a sequence or continuum of positions in the latent space, with network snapshots which are conditionally independent given the positions. More recently, [8] consider a similar functional RDPG model, and make a valuable contribution towards the statistical interpretation of a sequence of embeddings of functional network snapshots, in order to reveal the overall changes in network structure over time. In contrast, our work contributes
new, improved methodology for producing such a sequence.

Direct modeling of indexed dyadic connection events has been studied by [30] and [20], among others, again in the dynamic case. These two papers model edge event intensities using time-varying edge covariates, with constant and time-varying coefficients, respectively. A variational Bayes approach to fitting edge event intensities according to a latent community structure was proposed by [26]. A similar community-based model was proposed by [5], focusing on the effect of self-excitation on community detection.

In this paper, we propose a new model for functional network data: a continuously indexed inner product latent position model, which we will call a latent process model. In contrast to many previous approaches for functional latent position models, which treat the positions as random variables, we treat the latent processes as function-valued parameters. As a result, our approach does not rely on any assumptions of an explicit, simple, and discrete transition model for the latent positions between snapshots [27, 32, 34]. Instead, it allows for arbitrary, continuously varying latent processes, with estimation efficiency depending primarily on their functional smoothness. This means that our approach can easily handle irregularly spaced snapshot indices and missing edge entries, with faster computation time than similarly flexible Bayesian methods [11].

Our key contribution is to make this function estimation problem tractable by modeling latent processes using a finite, prespecified function basis. By adaptively selecting the basis based on the data, we are able to share information to efficiently estimate network structure which is shared locally between snapshots, but need not assume that that any part of that structure is common to all snapshots [9, 6]. In contrast to approaches which first smooth network snapshots and then estimate latent network structure [29], our estimation approach performs both tasks simultaneously, hence it can adapt to smoothness in the latent structure that may not be directly detectable from the network edges. Conversely, compared to approaches which estimate latent network structure for each snapshot, then summarize or smooth the output [8, 31], our approach can accurately estimate the latent processes as the edge variance increases (see Figure 1), as its performance does not rely directly on the signal-to-noise ratio of the individual network snapshots.

The rest of this paper is organized as follows. In Section 2, we define the latent process network model. In Section 3, we develop an estimation algorithm using gradient descent on coordinates in a function basis. Section 4 provides theoretical guarantees for recovery of the latent processes. Section 5 investigates the proposed methods in simulation studies, and Section 6 applies them to a data set of international political interactions. Finally, Section 7 contains brief discussion and future research directions. Proofs and supplemental information are provided in appendices A to E.
2 Latent Process Network Models

2.1 Notation

Throughout this paper, \(\|\cdot\|_F\) will denote the matrix Frobenius norm, \(\|\cdot\|_2\) the matrix \(\ell_2\) operator norm, and \(\langle \cdot, \cdot \rangle\) the Frobenius inner product. When applied to a vector, these coincide with the standard vector \(\ell_2\) norm and inner product. The notation \([\cdot]_{ij}\) denotes the \((i,j)\)-th entry of a matrix. We use \(\lambda_{\max}(\cdot)\) and \(\lambda_{\min}(\cdot)\) to denote the maximum and minimum eigenvalues of a symmetric matrix; The condition number is defined as the ratio of the maximum and minimum singular values of a matrix, it is bounded between 0 and 1; and \(\mathcal{O}_d\) will denote the set of \(d \times d\) orthogonal transformation matrices, which includes rotations, reflections, and combinations of the two.

Capital calligraphic letters are generally reserved for 3-mode tensors, and notation for tensor operations will follow [19]. The slices of a 3-mode tensor are constructed by fixing the first, second or third index respectively, and varying the other two. The fibers of a 3-mode tensor are constructed by fixing two indices and varying the third. For \(m = 1, 2, 3\), tensor-matrix multiplication in the \(m\)th mode is defined by the operator \(\times_m\) and matrix multiplies each \(m\)th mode slice of the first tensor argument by the second matrix argument. Similarly, for \(m = 1, 2, 3\), tensor-vector multiplication in the \(m\)th mode is defined by the operator \(\bar{\times}_m\) and matrix-vector multiplies \(m\)th mode slice of the first tensor argument by the second vector argument. Note that tensor-vector multiplication in the \(m\)th mode results in a matrix with dimensions corresponding to the other two modes of the original tensor argument.

2.2 Latent Functional Parameterization

We parameterize functional networks through a matrix-valued network mean function \(\Theta : \mathcal{X} \to \mathbb{R}^{n \times n}\), such that \(\Theta_{ij}(x_k) = \mathbb{E}([A_k]_{ij})\) for all \(i, j\) and \(k = 1, \ldots, m\). We assume independent edges, that is, \([A_k]_{ij}\) are independent for all \(i \leq j\) and \(k\). Formally, suppose \(q(\cdot; \theta, \phi)\) is the edge distribution, parameterized by its mean \(\theta\) and some possible nuisance parameters \(\phi\). Then for \(1 \leq i, j \leq n\) and \(k = 1, \ldots, m\),

\[
[A_k]_{ij} \overset{\text{ind}}{\sim} q(\cdot; \Theta_{ij}(x_k), \phi).
\]

For instance, we could model edges with \(q(\cdot; \theta, \sigma) = \mathcal{N}(\theta, \sigma^2)\), in which case the network is fully parameterized by \(\Theta\) and the nuisance edge variance \(\sigma^2\). For binary edge networks, we could model \(q(\cdot; \theta, \phi) = \text{Bernoulli}(\theta)\), in which case the model is fully parameterized by \(\Theta\). In making the independence assumption, we follow both the single network latent space literature, which typically assumes edge independence conditional on latent positions [7], and multilayer network latent space models that also make the assumption of independence across layers [25]. While the independence assumption is likely not exactly correct, it is a common and useful analysis tool for estimating the network structure, and, in this setting, the functional trends in this structure.
For a fixed node pair \((i, j)\), we may also view the sequence of random variables \(\{[A_k]_{ij}\}_{k=1}^m\) as a univariate functional response with inputs \(x_k\). The independence of edges over \(k\) implies that these responses have a mean which is a function of the continuous index \(x\), and independent errors. Hence the focus of this work is on modeling functional mean structure with no temporal dependence. In general functional settings, for instance where network snapshots represent brain scans of different patients indexed by a continuous task performance score, independence across snapshots is a reasonable working assumption. In dynamic networks, independence across snapshots is not guaranteed, but still commonly assumed conditional on latent structure [e.g., 34]. In future work, we may consider allowing within-edge autoregressive errors, or other forms of dependence. However, accurate nonparametric estimation of the underlying trend component, which we develop in this paper, will be of primary interest in many applications, and such an estimator is necessary to make individual edge sequences stationary before applying time series modeling approaches to the residuals [13].

Extending the latent space modeling approach, we assume that the parameter of interest \(\Theta\) is determined by the trajectories of each node in a \(d\)-dimensional latent space, denoted by \(Z(i) : \mathcal{X} \to \mathbb{R}^d\) for \(i = 1, \ldots, n\). We denote the component functions by \(Z^{(i)} = (z_{i,1}, \ldots, z_{i,d})^\top\). Throughout this paper, we assume an inner product similarity function, namely that \(\Theta_{ij}(x) = (Z^{(i)}(x))^\top Z^{(j)}(x)\) for any \(1 \leq i, j \leq n\) and \(x \in \mathcal{X}\). We refer to this as an inner product latent process network model. Collecting all the latent processes \(Z^{(i)}\) into rows of an \(n \times d\) matrix-valued function \(Z\), we can write \(\Theta(x) = Z(x)Z(x)^\top\). For each \(x\), \(Z(x)Z(x)^\top\) is a rank \(d\) matrix. The appeal of the inner product similarity is a parameterization of the network mean function which is low rank for any \(x \in \mathcal{X}\).

Evaluating the latent processes at the snapshot indices, we see that our function-valued parameterization produces a tensor decomposition of the \(n \times n \times m\) tensor \(E(A)\) with \(n \times n\) slices \(E(A_k)\) for \(k = 1, \ldots, m\), similar to the Tucker or canonical polyadic (CP) decompositions [19]. However, our formulation does not force the latent structure to factorize in the third mode. Consider the simple case where \(d = 1\), and compare our parameterization

\[
E(A_k) = Z(x_k)Z(x_k)^\top
\]

to a representation of \(E(A)\) by a CP decomposition which is symmetric in the first two modes. The latent process representation cannot in general be reproduced by a rank 1 CP decomposition, which would require that \(E(A_k) = w_k zz^\top\) for an \(n\)-vector \(z\) and scalars \(w_k\) for \(k = 1, \ldots, m\). Under the rank 1 CP decomposition, the expected value of every network snapshot would share a common eigenvector. To capture the structure of the latent process representation would require a rank \(m\) CP decomposition with different eigenvectors, and in general no dimension reduction or information sharing across snapshot means.

Rather than putting linear restrictions on the low rank representations of each snapshot mean, we treat them as functions of the index, and propose methodology with good theoretical and empirical properties when these representations are smooth in \(x\). In other words, we do not force our functional network models to rely on tensor-valued extensions of matrix algebra procedures, recognizing that node modes should be treated differently.
from the index mode. The two node modes are assumed to have low rank or latent position
structure, a highly successful and popular approach for network model; while the third index
mode is only assumed to have smooth structure as a function of the index, as in classical
nonparametric regression.

2.3 Identifiability

Latent space models with inner product link functions are well known to be non-identifiable
due to their invariance to orthogonal transformations of the latent positions, since $E(A) =
XX^\top = XO(XO)^\top$ for any $n \times d$ matrix $X$ and $d$-dimensional orthogonal transformation
$O \in O_d$. This non-identifiability also extends to continuous time: for any orthogonal matrix-
valued function $Q: \mathcal{X} \to O_d$, we have

$$Z(x)Z(x)^\top = Z(x)Q(x)\{Z(x)Q(x)\}^\top. \quad (1)$$

Thus, $Z$ is identifiable only up to uncountably many orthogonal transformations. In partic-
ular, for a given $Z$, we define the unidentified class of targets $T(Z)$ by

$$T(Z) = \{Z(x)Q(x) : Q : \mathcal{X} \to O_d\}. \quad (2)$$

Our goal is to take advantage of smoothness in $Z$ to share information across network snap-
shots. The non-identifiability could mask the smoothness, because even if $Z(x)$ is a smooth
function of $x$, $Z(x)Q(x)$ may not be. Although all elements of $T(Z)$ lead to identically dis-
tributed network snapshots, an estimation algorithm which targets the class representative
which is “maximally smooth” will have the greatest potential to share information across
snapshots and improve estimation efficiency. With additional conditions on $Z$, discussed in
Section 4.2, we can construct an estimate which targets a special orthogonalized representa-
tive of $T(Z)$. However, in general, our theory will instead consider the distance between
an estimate $\hat{Z}$ and an unknown representative of $T(Z)$.

3 Estimation With Gradient Descent

3.1 Approximation With Finite Dimensional Function Bases

The latent process assumption reduces the parameter space from $n^2$ to $nd$ function-valued
parameters. To further simplify estimation, we will restrict to a finite dimensional parameter
space by assuming that each component function of each latent process, $z_{i,r}$, $i = 1, \ldots, n,$
$r = 1, \ldots, d$ is well approximated by the span of a common $q$-dimensional function basis.
That is, suppose that $B = (B_1, \ldots, B_q)^\top$ is a $q$-dimensional basis of functions each mapping
from $\mathcal{X}$ to $\mathbb{R}$. We assume that every component function $z_{i,r}(x)$ is close to $w_{i,r}^\top B(x)$ for
some $q$-dimensional coordinate vector $w_{i,r}$. For each $r$, we collect $w_{i,r}$ as the rows of an
$n \times q$ matrix $W_r$. Let $W = \{W_r\}_{r=1}^d$ denote the $n \times q \times d$ tensor containing all the basis
coordinates for all nodes in all latent dimensions.

The basis $B$ could be, for instance, a $B$-spline basis on $\mathcal{X}$, or any other similar functional
basis. A $B$-spline basis of order $D \geq 0$ is defined by an increasing sequence of $K$ internal knots, as well as boundary knots. The dimension of a $B$-spline basis is $q = K + D + 1$. The span of a given $B$-spline basis is a collection of piecewise polynomial functions which are $(D - 1)$-times differentiable at the internal knots and smooth elsewhere. For additional mathematical properties, we refer readers to [33]. In this paper, we will use order 3, or cubic $B$-spline bases, although the algorithms to follow would proceed similarly for any function basis $B$, including orthonormal function bases such as the Fourier basis.

For simplicity, we assume a common basis $B$ for all $i = 1, \ldots, n$ and all $r = 1, \ldots, d$, although this could also be relaxed. In practice, the underlying latent processes may not exactly belong to span$(B)$. However, if the latent processes are smooth in $x$, we will be able to approximate them effectively with functions in span$(B)$. Our estimation procedure will specify this function basis in the course of tuning.

We begin by defining a nonconvex least squares optimization problem equivalent to maximizing a Gaussian likelihood. Estimation can proceed similarly for other edge distributions, in particular the binary edge model with $q(\theta; \phi) = \text{Bernoulli}(\theta)$. Denote

$$\ell(W) = \sum_{k=1}^{m} \|A_k - \sum_{r=1}^{d} W_r B(x_k) B(x_k)^\top W_r^\top \|_F^2.$$  \hspace{1cm} (3)

Fixing the latent space dimension $d$, and a $q$-dimensional function basis $B$, we can apply gradient descent over the $n \times q \times d$ tensor-valued argument $W$ with $n \times q$ slices $W_r \in \mathbb{R}^{n \times q}$ for $r = 1, \ldots, d$. The following proposition derives the gradient of $\ell$ with respect to each $n \times q$ slice of $W$. The proof is provided in Appendix A.

**Proposition 1.** Define $\ell(W)$ as in (3). Then

$$\frac{\partial \ell}{\partial W_r}(W) \propto -\sum_{k=1}^{m} \left\{ A_k - \sum_{r'=1}^{d} W_{r'} B(x_k) B(x_k)^\top W_{r'}^\top \right\} W_r B(x_k) B(x_k)^\top \in \mathbb{R}^{n \times q}$$

for $r = 1, \ldots, d$.

Since (3) is a nonconvex objective, gradient descent will not necessarily converge to the global optimum, and the result depends on the starting value. In Section 4, we will directly prove results for the output of two gradient descent algorithms proposed below, rather than for the global minimizer of (3). We shall also see in Section 4 that the starting value for gradient descent may affect the target of estimation among the unidentified class $T(Z)$ of latent processes defined by (2).

We propose two different gradient descent schemes for estimating $W$. In the first scheme, we estimate the dimensions sequentially for $r = 1, \ldots, d$. As inputs, the sequential gradient descent algorithm takes some set of initial coordinates

$$\widehat{W}_0 = (\widehat{W}_r^0)_{r=1}^d;$$

step sizes $\eta_{h,r} > 0$, which may depend on the iteration number $h \geq 0$; the latent dimension $d$; and a maximum iteration number $H$. In practice, for both gradient descent schemes we
Algorithm 1: Sequential gradient descent algorithm.

Set $\tilde{W}^0 = 0_{n \times q \times d}$

For $r = 1$ to $r = d$

$\tilde{W}^0_r \leftarrow \hat{W}^0_r$

For $h = 1$ to $h = H$

$\tilde{W}^h_r \leftarrow \tilde{W}^{h-1}_r - \eta_{h-1,r} \frac{\partial \ell}{\partial \tilde{W}_r} (\tilde{W}^{h-1}_r)$

$\tilde{W}^H_r \leftarrow \tilde{W}^H_r$

Output $\hat{W}^H = \{\tilde{W}^H_r\}_{r=1}^d$

Algorithm 2: Concurrent gradient descent algorithm.

For $h = 1$ to $h = H$

For $r = 1$ to $r = d$

$\tilde{W}^h_r \leftarrow \tilde{W}^{h-1}_r - \eta_h \frac{\partial \ell}{\partial \tilde{W}_r} (\tilde{W}^{h-1}_r)$

Output $\hat{W}^H = \{\tilde{W}^H_r\}_{r=1}^d$

Algorithm 1 is an $n \times q \times d$ tensor-valued coordinate estimator $\hat{W}^H$. Algorithm 1 populates the final coordinate estimator $\hat{W}^H$ one $n \times q$ slice at a time by estimating $d$ one dimensional latent process models in sequence. Once a slice is estimated, it is fixed for the remainder of the algorithm, and its contribution is subtracted from the network structure. On the other hand, slices which have not yet been estimated have all entries fixed at 0. We will show in Section 4 that this sequential scheme can overcome some identifiability issues, and recover an orthogonalized representative of $T(Z)$, if the true processes have singular values which are uniformly well separated in $x$.

The second gradient descent scheme estimates the latent dimensions concurrently. Similar to the sequential algorithm, the concurrent gradient descent algorithm takes initial coordinates $\hat{W}^0 = \{\tilde{W}^0_r\}_{r=1}^d$, step sizes $\eta_h > 0$, and a maximum number of iterations $H$ as inputs. For the concurrent scheme, the step sizes will only depend on the iteration number $h \geq 0$. The output of Algorithm 2 is also an $n \times q \times d$ tensor-valued coordinate estimator $\hat{W}^H$. In contrast to Algorithm 1, each gradient descent step in Algorithm 2 updates all $d$ slices of the estimated coordinate tensor at the same time. We will show in Section 4 that compared to the sequential scheme, the concurrent scheme can provide a good estimator with weaker conditions on
However, rather than estimating an orthogonalized representative of $T(Z)$, we can only prove that the output of Algorithm 2 is close to some unknown representative of the same class.

Based on an estimator $\hat{W}^H$ found using either gradient descent scheme, we can use the function basis $B$ to convert back to an estimate of the unknown latent processes. For $x \in \mathcal{X}$, define the $n \times d$ matrix

$$\hat{Z}^H(x) = \hat{W}^H \bar{x}_2 B(x).$$

We will refer to the estimator $\hat{Z}^H$, or simply $\hat{Z}$, as a functional adjacency spectral embedding (FASE).

FASE estimation using Algorithm 2, which is used for all the simulations and real data analysis to follow, is implemented in a development R package `fase` available at github.com/peterwmacd/fase.

### 3.2 Initializing Gradient Descent

To initialize gradient descent, we propose a family of kernel smoothed embedding algorithms. The basic idea is to get local estimates of $Z(x)$ at a user-specified grid of indices. Then each component function is projected into the span($B$) to recover initial estimates for the basis coordinates. As in Section 3.1, we will work with prespecified $d$, and $q$-dimensional basis $B$.

We start with the formal definition of adjacency spectral embedding (ASE) from [36]. For an $n \times n$ symmetric matrix $P$ with eigendecomposition $Y \Lambda Y^\top$, define

$$\text{ASE}_d(P) = Y_d \Lambda_d^{1/2} \in \mathbb{R}^{n \times d},$$

where $Y_d \in \mathbb{R}^{n \times d}$ and $\Lambda_d \in \mathbb{R}^{d \times d}$ correspond to the $d$ largest eigenvectors and eigenvalues of $P$. If the diagonal entries of $\Lambda$ are distinct, then the ASE is uniquely defined up to sign flips of each column.

Our initialization algorithm takes the network snapshots $\{A_k\}_{k=1}^m$ and snapshot indices $\{x_k\}_{k=1}^m$ as input, and further depends on a user specified kernel $K(y)$, and grid of indices $\{\tilde{x}_1, \ldots, \tilde{x}_{q'}\} \subset \mathcal{X}$ for some $q'$ satisfying $q \leq q' \leq m$. We assume that $K(y)$ is nonnegative and normalized with respect to the snapshot indices, so that, for any $y \in \mathcal{X}$,

$$\sum_{k=1}^m K(y - x_k) = 1.$$

Moreover, define

$$B_{\text{init}} = \begin{pmatrix} B(\tilde{x}_1) & \cdots & B(\tilde{x}_{q'}) \end{pmatrix}^\top,$$

and assume that the indices in the grid are well spaced such that $B_{\text{init}}^\top B_{\text{init}}$ is invertible. With this notation in hand, we may state our initialization algorithm. Algorithm 3 includes an alignment step which orthogonally transforms the columns of each embedding to minimize the discrepancies between consecutive embeddings. This helps to produce $Z_{\text{init}}$ which is close to a relatively smooth in $x$ representative of $T(Z)$, increasing the potential to share information across network snapshots. The optimal transformations each solve a so-called
Algorithm 3: Kernel smoothed embedding initialization algorithm.

For $\ell = 1$ to $\ell = q'$

Set $\hat{Z}_{\text{init}}(\tilde{x}_\ell) = \text{ASE}_d(\sum_{k=1}^{m} K(\tilde{x}_\ell - x_k)A_k)$

If $\ell > 1$ then

Set $O_\ell = \arg\min_{O \in O_d} \left\| \hat{Z}_{\text{init}}(\tilde{x}_\ell)O - \hat{Z}_{\text{init}}(\tilde{x}_{\ell-1}) \right\|^2_2$

$\hat{Z}_{\text{init}}(\tilde{x}_\ell) \leftarrow \hat{Z}_{\text{init}}(\tilde{x}_\ell)O_\ell$

Set $Z_{\text{init}} = \left\{ \hat{Z}_{\text{init}}(\tilde{x}_\ell) \right\}_{\ell=1}^{q'} \in \mathbb{R}^{n \times q' \times d}$

Set $\hat{W}^0 = Z_{\text{init}} \times_2 (B_{\text{init}}^TB_{\text{init}})^{-1} B_{\text{init}}^T$

Output $\hat{W}^0$

Procrustes problem, which has a closed form solution [10].

In our implementation of Algorithm 3 in Sections 5 and 6, we choose $q' = q$ and use an equally spaced grid $\{\tilde{x}_1, \ldots, \tilde{x}_q\} \subset \mathcal{X}$, and a normalized rectangular kernel. With equally spaced snapshot indices on $\mathcal{X}$, this implies that each local embedding is the $d$-dimensional ASE of the local average of the closest snapshots to a given grid point.

3.3 Parameter Tuning

Up to this point, we have treated both the latent space dimension $d$, and function basis $B$ as fixed. In practice these will have to be selected based on data. To simplify this tuning problem, we consider only cubic $B$-spline bases with equally spaced knots on $\mathcal{X}$. Thus, basis selection is fully determined by an integer $q$ which is a function of the number of knots. The parameter $d$ controls static complexity: flexibility of the network mean structure for each fixed $x \in \mathcal{X}$, while the parameter $q$ controls functional complexity: flexibility of each latent process as a function of the index.

To select these two tuning parameters, we derive a penalized least squares criterion based on the total squared error (3) for the observed network snapshots, and the number of parameters $(nqd)$ used to define the latent process model. The details of this derivation are given in Appendix B.

In short, the objective (3) can be decomposed as the sum of squared residuals for $2n$ linear regression problems, comprising the incoming and outgoing edge values for each node. Each of these problems is based on $nm/2$ independent observations, and $qd$ unknown coefficients. We evaluate the generalized cross validation (GCV) criterion for each problem and take the mean, resulting in an overall network GCV (NGCV) criterion equivalent to

$$\text{NGCV}(q, d) = \log \left( \frac{\ell(\hat{W})}{mn^2} \right) - 2 \log \left( 1 - \frac{2qd}{nm} \right)$$ (4)
for an estimated $n \times q \times d$ tensor $\hat{W}$ of basis coordinates. Then, parameter tuning will proceed by minimizing NGCV over a grid \{(q,d) : q_{\text{min}} \leq q \leq q_{\text{max}}, \ 1 \leq d \leq d_{\text{max}}\}, with user specified lower and upper bounds on each parameter.

While minimization of NGCV only requires the model to be fit once for each $(q,d)$ pair, this can still be computationally costly for large $m$ and $n$. For comparison, we also propose a more efficient heuristic approach to optimizing over the grid using coordinate descent. In the coordinate descent scheme, we initialize $d = 1$, and perform alternating minimization for $q$ and $d$ by evaluating NGCV on a grid and treating the other as fixed.

We evaluate our tuning approach on synthetic data in Section 5.2. It appears to consistently recover the ground truth parameters when the signal-to-noise ratio is sufficiently high. In other cases, it still tends to select parameters with good performance in terms of recovery of the latent processes.

4 Theoretical Guarantees

4.1 Preliminaries

In this section, we establish nonasymptotic consistency results, on average over nodes and snapshots, of the gradient descent estimators for orthogonal transformations of the true latent processes. All proofs are provided in Appendix A. As in Section 2, denote the true latent processes by $Z$, an $n \times d$ matrix-valued function of $X$. The latent space dimension $d$ will be treated as fixed. We will use $Z_r$ for $r = 1, \ldots, d$ to denote the $n$ vector-valued function given by the $r$th column of $Z$. Suppose that after centering, the edge distribution $q(\cdot; \theta, \phi)$ is sub-Gaussian with parameter $\sigma$, and for simplicity assume that the networks are undirected and self loops are allowed. Throughout this section, constants may depend on $d$, but will always be free of $n$, $m$, $q$, and $\sigma$, which will be tracked in the final nonasymptotic bounds.

Estimation will proceed by either Algorithm 1 or Algorithm 2 on $\ell(W)$ with fixed $q$-dimensional $B$-spline basis $B$. Recall that compared to Algorithm 2, Algorithm 1 can recover a better recovered, orthogonalized representative of the class $T(Z)$ defined by (2), subject to stronger assumptions on $Z$.

Let

$$B = \left(B(x_1) \ \cdots \ \ B(x_m)\right)^\top,$$

an $m \times q$ matrix which we will refer to as the $B$-spline design matrix. Throughout the section, we will make the following assumptions on the basis and associated design matrix.

Assumption 1.

(A) Assume that for each $k$, $B(x_k) \geq 0$ element-wise, $\|B(x_k)\|_1 = 1$, and $B(x_k)$ has at most $2D + 1$ nonzero entries.
(B) Assume that $B$ satisfies
\[
\frac{c_B m}{q} \leq \lambda_{\min}(B^T B) \leq \lambda_{\max}(B^T B) \leq \frac{C_B m}{q}
\]
for constants $C_B > c_B > 0$.

Part (A) is satisfied by $B$-spline bases of fixed order $D \geq 0$. Moreover, because $B$-splines are locally supported, if the snapshot indices and basis knots are evenly spread across $\mathcal{X}$, it follows that $\text{tr}(B^T B) \sim m$. Thus part (B) simply requires that $B^T B$ has a condition number of a constant order, a condition which holds for reasonably evenly spaced snapshot indices.

4.2 Consistency for Algorithm 1

In this subsection, we will establish a result for the recovery of the orthogonalized true processes, based on a FASE estimator found using Algorithm 1.

We begin by formally defining the orthogonalized true processes. For static latent position models, [7] show that when the $n \times d$ latent position matrix has singular value decomposition $X = USV^T$, and the singular values of $X$ are well separated, the rows of the $d$-dimensional ASE of the network will be close to $US$, with orthogonal columns of nonincreasing magnitude. In the functional setting, we can apply the same transformation, based on singular value decomposition, pointwise over $x \in \mathcal{X}$, written as $Z(x) = U(x)S(x)V(x)^T$. We refer to the orthogonalized target by $Z^{\text{orth}} \in \mathcal{T}(Z)$, defined by $Z^{\text{orth}}(x) = Z(x)V(x)$ for $x \in \mathcal{X}$.

Unfortunately, without additional conditions on singular value separation, $Z^{\text{orth}}$ may not inherit any continuity or differentiability in $x$ from the original $Z$. In particular, $Z^{\text{orth}}$ may be discontinuous at any $x$ for which $S(x)$ has nondistinct diagonal entries [4]. In order to get an estimator which is consistent for $Z^{\text{orth}}$, we require an analogous singular value separation condition to [7], but one which ensures that the singular values of the true processes $Z$ are uniformly separated for all $x \in \mathcal{X}$. This is formalized in part (A) of in Assumption 2.

A suitable target for gradient descent is a collection of $(n \times q)$-dimensional target coordinate matrices $W_r^*$ for each $r = 1, \ldots, d$ which minimize
\[
\varepsilon_r = \frac{1}{m} \sum_{k=1}^m \|W_r^* B(x_k) - Z^{\text{orth}}_r(x_k)\|_2^2.
\]
Each row of each $W_r^*$ solves a least squares problem with $m$ observations and $q$ unknowns. In particular, the fitted values are the projection of the true processes evaluated at each snapshot index onto the column space of $B$, a $q$-dimensional subspace of $\mathbb{R}^m$. Before proving our main result, we make the following technical assumptions, which we separate into deterministic and random statements.

Assumption 2.
(A) For $r = 1, \ldots, d$, define
\[ \gamma_r^2 = \min_{k=1,\ldots,m} \left\{ \| Z_{r}^{\text{orth}}(x_k) \|_2^2 \right\}. \]
Suppose that
\[ \| W^*_r \|_2^2 \leq c_r \gamma_r^2 \]  
for a constant $c_r > 0$, and that
\[ \gamma_r^2 + 1 \leq c_{\text{sep}} \gamma_r^2 \]  
for a sufficiently small constant $c_{\text{sep}}$, where $\gamma_d+1 = 0$.

(B) Suppose
\[ \sum_{s=1}^{r} \frac{\gamma_s^2 \varepsilon_s}{\gamma_r} \leq c_{\text{approx}} \frac{\gamma_r^2}{q}, \]  
and
\[ \max_{k=1,\ldots,m} \| W^*_r B(x_k) - Z_{r}^{\text{orth}}(x_k) \|_2^2 \leq c_{\text{approx}} \gamma_r^2, \]  
for $r = 1, \ldots, d$, and for a sufficiently small constant $c_{\text{approx}}$.

(C) Suppose
\[ \frac{\gamma_d^2}{\sigma} \geq c_{\text{SNR}} \left( \frac{q^7 n \log q}{m} \right)^{1/2} \]  
for a sufficiently large constant $c_{\text{SNR}}$.

Part (A) is a scaling condition which relates the singular values of the true latent processes to those of the target coordinates. It is easiest to understand in the special case where the corresponding $\varepsilon_r = 0$. In this case (5) puts an upper bound on the largest singular value of the target coordinates in terms of the corresponding minimum column norm of $Z_{r}^{\text{orth}}(x)$ over $x$, or equivalently the singular values of $Z(x)$. Along with Assumption 1, a sufficient condition for (5) is that $W^*_r$ is well conditioned for a finite set of directions:
\[ \min_{k=1,\ldots,m} \left\{ \frac{B(x_k)}{\| B(x_k) \|_2} \right\}_r (W^*_r)^\dagger W^*_r \left\{ \frac{B(x_k)}{\| B(x_k) \|_2} \right\} \geq \kappa_r \| W^*_r \|_2^2 \]  
for a constant $\kappa_r > 0$ which, along with $D$ in Assumption 1, part (A), determines a sufficiently large value for $c_r$. Then, (6) ensures uniform separation of the singular values of the target coordinates so that the output of Algorithm 1 correctly orders the columns. In addition to being orthogonal and identifying $Z_{r}^{\text{orth}}$, the columns need to be well separated so that the smaller dimensions in terms of the magnitude of $W^*_r$ do not interfere with the estimation of the larger dimensions.

Parts (B) and (C) are both conditions related to the eigengap, so that each $W^*_r$ is a meaningful target for gradient descent. Part (B) puts a requirement on the approximation error. In the simplest case with $q = 1$ and setting $r = 1$, the latent processes are modeled as constant in $x$. Then, the left-hand side of (7) is the total sample variance of the true processes around their means, and we require that this variation is small compared to the squared magnitude of the process. For larger $q$, we need the additional parameters to
proporionally reduce this variation. The particular summation on the left hand side of
(7) appears in the final error bound, and will be explained in more detail following the
statement of Theorem 1. Part (C) puts a requirement on the signal to noise ratio. Each \( \gamma_r \)
for \( r = 1, \ldots, d \) will be tracked in the nonasymptotic bounds to follow, and in general part
(C) will at least require that they increase with \( n \) for the FASE estimator to be consistent.

**Assumption 3.**

(A) For \( r = 1, \ldots, d \), suppose that each initializer \( \hat{W}_r^0 \) satisfies
\[
\| \hat{W}_r^0 - W_r^* \|_F^2 \leq c_{\text{init}} \gamma_r^2
\]
for a sufficiently small constant \( c_{\text{init}} \).

This assumption provides a sufficient condition for the initialization error of each latent
dimension. With these assumptions, we may state the main result of this section.

**Theorem 1.** Suppose \( \{A_k\}_{k=1}^m \) are generated from a latent process network model, with
independent sub-Gaussian edges with parameter \( \sigma \). Suppose we compute a FASE estimator
using Algorithm 1 with fixed \( q \)-dimensional \( B \)-spline basis \( B \) and fixed step size \( \eta_{h,r} = \eta'/m\gamma_r^2 \)
for a sufficiently small constant \( \eta' \). Suppose Assumptions 1 and 2 hold, Assumption 3
holds with probability at least \( 1 - \delta \), and that \( n, q \) are such that \( nq \log q \geq nq \log 5 + (n + q) \log 9 \). Then with probability greater than \( 1 - 2 \exp(-n/2) - \delta \), and for a sufficiently large
\( H \geq 0 \),
\[
\frac{1}{m} \sum_{k=1}^m \| \hat{Z}_r^H(x_k) - Z_{r}^{\text{orth}}(x_k) \|_2^2 \leq C_1 \left( \sigma^2 q^4 n \log q \gamma_r^2 \gamma_r^2 m \sum_{s=1}^r \gamma_s^2 \epsilon_s \right)
\]
for each \( r = 1, \ldots, d \) and a constant \( C_1 > 0 \).

We refer to the two terms on the right hand side of (8) as statistical error and approximation bias, respectively. As the vectors on the left hand side are of dimension \( n \), the average statistical error over the nodes decreases in both \( m \) and \( \gamma_r \) as the size of the functional
network data increases. The approximation bias term for a given \( r = 1, \ldots, d \) contains a
summation over all the dimensions fit to that point in the sequential algorithm, as these
errors will compound as we fit additional dimensions. Recalling the precise form of each \( \epsilon_r \),
we see that they represent intrinsic approximation bias due to the chosen \( B \)-spline basis,
and would be unavoidable even with access to a noiseless version of \( Z_{r}^{\text{orth}} \).

If we make a parametric assumption that each orthogonalized latent process is in \( \text{span}(B) \),
we have \( \epsilon_r = 0 \) for all \( r = 1, \ldots, d \), and get a consistent estimator for the orthogonalized
latent processes on average over the nodes and snapshot indices. In the corresponding
nonparametric setting, suppose that for fixed \( r \), each component function \( z_{i,r}^{\text{orth}} \) is twice
differentiable. Then by approximation results for cubic \( B \)-splines [33], there exists \( w \in \mathbb{R}^q \)
such that
\[
\sup_{x \in \mathcal{X}} \left| z_{i,r}^{\text{orth}}(x) - w^T B(x) \right| \lesssim \frac{1}{q^2} \sup_{x \in \mathcal{X}} \left| \frac{\partial^2 z_{i,r}^{\text{orth}}(x)}{\partial x^2} \right|.
\]

Thus, if the orthogonalized true processes have bounded second derivatives, there exists a
theoretically optimal \( q \) which grows with \( m \) and \( n \) such that \( \hat{Z}_r^H \) is consistent for \( Z_r^{\text{orth}} \) as \( n \)
and \( m \) grow, on average over the nodes and snapshot indices.
4.3 Consistency for Algorithm 2

In this subsection, we will establish a result for the recovery of some unknown orthogonal transformation of the true processes $Z$ based on a FASE estimator found using Algorithm 2. Recall that in general we can only identify an unknown representative of $T(Z)$. Thus, as in single layer latent space approaches [24], for a given iteration $h$, we pick a target from $T(Z)$ by minimizing a Frobenius norm error over the unidentified class. However, to account for the fact that we have many network snapshots, we calculate this error in terms of coordinates in our prespecified function basis.

In the following, we will evaluate the error only at the prespecified snapshot indices, so we will only require the true processes evaluated at those indices, which we store in an $n \times m \times d$ tensor $Z$ with $n \times d$ slices given by $Z(x_k)$ for $k = 1, \ldots, m$. Then for a collection of $m$ orthogonal transformations $Q = (Q_1, \ldots, Q_m) \in O_m^d$, define a transformed $n \times m \times d$ tensor $ZQ$ with new slices $Z(x_k)Q_k$ for $k = 1, \ldots, m$. Now, define a linear operator $\mathcal{P}_B : \mathbb{R}^{n \times m \times d} \to \mathbb{R}^{n \times q \times d}$ by

$$\mathcal{P}_B(Y) = Y \times_2 (B^T B)^{-1} B^T.$$

In words, $\mathcal{P}_B$ takes each $m$-dimensional fiber of its argument and finds the best fitting coordinates in the column space of $B$ according to a least squares criterion. For a given iteration $\hat{W}^h$ of basis coordinates, define the optimal orthogonal alignment

$$Q^{*,h} = (Q_1^{*,h}, \ldots, Q_m^{*,h}) = \arg\min_{Q \in O_m^d} \|\hat{W}^h - \mathcal{P}_B(ZQ)\|_F^2.$$

We also define the target in the coordinate space at each iteration by

$$W^{*,h} = \mathcal{P}_B(ZQ^{*,h}).$$

While in the previous section, the target coordinates were deterministic, fixed at every iteration, and depended only on $B$ and the true processes $Z$; here the target coordinates are random and vary across iterations of gradient descent. With these definitions in place, we again define two sets of technical conditions, one set which is deterministic and the other which is random.

**Assumption 4.**

(A) Define

$$\gamma_Z^2 = \min_{k=1,\ldots,m} \left[ \lambda_{\min} \left\{ Z(x_k)^T Z(x_k) \right\} \right].$$

Suppose that

$$\frac{\gamma_Z^2}{\sigma} \geq c_{\text{SNR}} \left( \frac{q^5 n \log q}{m} \right)^{1/2}$$

for a sufficiently large constant $c_{\text{SNR}}$.

In contrast to the previous section, only the signal to noise ratio condition is deterministic, as the target coordinates now depend on random and iteration specific orthogonal
transformations.

**Assumption 5.**

(A) Suppose that 
\[ \|W^*_r, h\|_2^2 \leq c_* \gamma_Z^2 \]
uniformly over \( r = 1, \ldots, d \) and \( h \geq 0 \) for some constant \( c_* \).

(B) Assume that 
\[ \varepsilon^{(h)} = \frac{1}{m} \sum_{k=1}^{m} \|W^*_h x_k B(x_k) - Z(x_k) Q^*_k \|_F^2 \leq c_{\text{approx}} \gamma_Z^2, \]
\[ \max_{k=1, \ldots, m} \|W^*_h x_k B(x_k) - Z(x_k) Q^*_k \|_F^2 \leq c_{\text{approx}} \gamma_Z^2, \]
uniformly over \( h \geq 0 \) for a sufficiently small constant \( c_{\text{approx}} \).

(C) Assume that the initializer \( \hat{W}^0 \) satisfies 
\[ \|\hat{W}^0 - W^*, 0\|_F^2 \leq c_{\text{init}} \gamma_Z^2 \]
for a sufficiently small constant \( c_{\text{init}} \).

These assumptions are all analogous to assumptions made in Section 4.2. Part (A) puts a scaling assumption on the singular values of the target coordinates, and the latent processes \( Z \), now made as one inequality, and without conditions on uniform separation of singular values for different \( r = 1, \ldots, d \). Part (B) is now random, as it depends on the iteration specific target, leading to an approximation error term \( \varepsilon^{(h)} \) also depending on \( h \). Since Algorithm 2 estimates all the latent dimensions at the same time, part (B) no longer contains a summation over previous dimensions, like the analogous bound in Section 4.2.

With these assumptions, we are ready to state the main result of this section.

**Theorem 2.** Suppose \( \{A_k\}_{k=1}^{m} \) are generated from a latent process network model, with independent sub-Gaussian edges with parameter \( \sigma \). Suppose we compute a FASE estimator using Algorithm 2 with fixed \( q \)-dimensional B-spline basis \( B \) and fixed step size \( \eta_h = \eta' q / m \gamma_Z^2 \) for a sufficiently small constant \( \eta' \). Suppose Assumptions 1 and 4 hold, Assumption 5 holds with probability at least \( 1 - \delta \), and that \( n, q \) are such that 
\[ nq \log q \geq nq \log 5 + (n + q) \log 9. \]
Then with probability greater than \( 1 - 2 \exp(-n/2) - \delta \), and for sufficiently large \( H \geq 0 \),
\[ \frac{1}{m} \sum_{k=1}^{m} \|\hat{Z}^{H}(x_k) - Z(x_k) Q^*_k \|_F^2 \leq C_2 \left( \frac{\sigma^2 q^4 n \log q}{\gamma_Z^2 m} + \limsup_{h \to \infty} \varepsilon^{(h)} \right) \]
for a constant \( C_2 > 0 \).

The right hand side of (9) is similar to the bound derived in Theorem 1, with a statistical error term and an approximation bias term. In this setting, even if we make a parametric assumption that each latent process is in \( \text{span}(B) \), this may no longer hold after applying the unknown orthogonal transformation, and we may still have nonzero approximation bias. Thus, estimation will always revert to the corresponding nonparametric setting, with consistency if we can approximate the transformed processes well by functions in \( \text{span}(B) \).
While we make no assumptions on the convergence of $\varepsilon^{(h)}$ as $h \to \infty$, by Assumption 5, it is uniformly bounded and hence has a convergent subsequence. Thus for a sufficiently large number of iterations $H$, the approximation bias may be bounded above by $\limsup_{h \to \infty} \varepsilon^{(h)}$, plus a difference which is negligible compared to the statistical error term.

5 Evaluation on Synthetic Networks

5.1 Latent Process Recovery

We compare our FASE estimator to existing methods for similar inner product latent space models for both weighted and binary edge networks. We will implement FASE with both an oracle and an adaptive NGCV tuning scheme, as well as three ASE-based approaches that have been applied in the past to functional, in particular dynamic, network data [31]. First, we apply the usual $d$-dimensional ASE to each of the $m$ adjacency matrix snapshots. Second, we apply the omnibus ASE (OMNI) [23], which finds a $d$-dimensional embedding at each snapshot index based on the ASE of the so-called omnibus matrix given by

$$
\begin{pmatrix}
A_1 & \frac{A_1 + A_2}{2} & \cdots & \frac{A_1 + A_m}{2} \\
\frac{A_1 + A_2}{2} & A_2 & \cdots & \\
\vdots & \ddots & \ddots & \\
\frac{A_1 + A_m}{2} & \cdots & & A_m
\end{pmatrix}
$$

Third, we apply the multiple ASE (COSIE) [6], which assumes that the expectations of the adjacency matrix snapshots share a common invariant subspace. For these baseline estimators, we assume oracle knowledge of $d$.

We generate instances of the latent process network model under three scenarios.

(i) Parametric Gaussian network with $B$-spline processes. In this scenario we generate each component process $z_{i,r}$ for $i = 1, \ldots, n$ and $r = 1, \ldots, d$ from a cubic $B$-spline basis $B$ on $[0, 1]$, with equally spaced knots and dimension 10. In particular we generate $w_{i,r} \sim \mathcal{N}_{10}(0, I_{10})$, and then define $z_{i,r}(x) = w_{i,r}^\top B(x)$ for $x \in [0, 1]$. Then for equally spaced snapshot indices $0 = x_1 < \cdots < x_m = 1$, we set

$$A_k = Z(x_k)Z(x_k)^\top + E_k,$$

where each $E_k$ is a symmetric matrix of independent Gaussian random variables with variance $\sigma^2$.

(ii) Nonparametric Gaussian network with sinusoidal processes. In this scenario we generate each component process as

$$z_{i,r}(x) = \frac{3 \sin[2\pi(2x - U_{i,r})]}{1 + 5[x + B_{i,r}(1 - 2x)]} + G_{i,r}$$

where $U_{i,r} \overset{iid}{\sim} \text{Unif}[0, 1]$, $B_{i,r} \overset{iid}{\sim} \text{Bernoulli}(1/2)$, and $G_{i,r} \overset{iid}{\sim} \mathcal{N}(0, 1/4)$. Each process
is a shifted (according to \( G \)) sine function which goes through 2 full cycles from a random starting point (controlled by \( U \)), with amplitude either increasing or decreasing (controlled by \( B \)) from 3 to \( 1/2 \) or \( 1/2 \) to 3. Then for equally spaced snapshot indices \( 0 = x_1 < \cdots < x_m = 1 \), we set

\[
A_k = Z(x_k)Z(x_k)^\top + E_k,
\]

where each \( E_k \) is a symmetric matrix of independent Gaussian random variables with variance \( \sigma^2 \).

(iii) Parametric RDPG network with \( B \)-spline processes. In this scenario we generate each component process \( z_{i,r} \) for \( i = 1, \ldots, n \) and \( r = 1, \ldots, d \) from a cubic \( B \)-spline basis \( B \) on \([0,1]\), with equally spaced knots and dimension 10. We generate each \( d \)-dimensional fiber of the full \( n \times 10 \times d \) coordinate tensor \( W \) as an independent Dirichlet random variable with parameter \([0.1 \cdots 0.1]^\top\). The coordinates are then rescaled to control the overall network density. Then for \( i \leq j \), we generate

\[
[A_k]_{ij} \sim \text{Bernoulli} \left\{ \sum_{r=1}^{d} z_{i,r}(x_k)z_{j,r}(x_k) \right\}
\]

and set \([A_k]_{ji}\) to make each \( A_k \) symmetric.

To compare the performance of FASE against the baseline estimators, we evaluate error for recovery of the latent processes up to orthogonal transformation, averaged over the snapshot indices:

\[
\text{Err}_{Z}^Z = \left\{ \frac{1}{n/dm} \sum_{k=1}^{m} \min_{Q_k \in O_d} \| \hat{Z}(x_k) - Z(x_k)Q_k \|_F^2 \right\}^{1/2}
\]

This error metric is similar to the error bounded in the conclusion of Theorem 2. If our adaptive implementation of FASE selects \( d \) incorrectly, then either the true or estimated latent processes are given additional columns of all zeros so that the dimensions match. We report two errors for the FASE estimator found using Algorithm 2, one for the adaptive version which selects \( d \) and \( q \) using a grid search and the NGCV criterion defined in Section 3.3 (FASE (NGCV)), and another oracle FASE estimator which fits the model with ground truth \( d \), and \( q \) selected using oracle knowledge of the true latent processes \( Z \) (FASE (ORC)).

In Figures 1 and 2, we report results for scenario (i) generated with \( \sigma \in \{2, 4, 6, 8\} \). In Figure 1 we vary the number of snapshots \( m \in \{20, 40, \ldots, 200\} \) for fixed \( n = 100 \) and \( d = 2 \), and in Figure 2 we vary the number of nodes \( n \in \{80, 120, \ldots, 400\} \) for fixed \( m = 80 \) and \( d = 2 \). In all settings, \( \text{Err}_Z \) is averaged over 50 independent replications. In Figure 1, in all four panels, none of the baseline ASE estimators show an improvement with increasing \( m \), while FASE does. In Figure 2, only FASE and ASE show improvement with increasing \( n \). While ASE improves at a faster rate than FASE, it never outperforms it, even for the largest values of \( n \) considered. In almost all settings, FASE performs the best of all methods. This
is true regardless of whether we have oracle knowledge of the tuning parameters \( d \) and \( q \), as the errors for FASE (NGCV) and FASE (ORC) are nearly indistinguishable in these plots. Among the baselines, the errors for ASE, which is unbiased, are by far the most sensitive to \( \sigma \). On the other hand, COSIE and OMNI, which share information globally, incur a lot of bias in this setting, where the latent processes are only similar locally in the index variable.

In Figure 1 for \( \sigma = 8 \) and \( m = 20 \), the adaptive FASE estimator is able to outperform the oracle on average. This phenomenon, which we discuss in more detail in Section 5.2, can occur when the signal is low and the adaptive estimator selects a value of \( d \) which achieves better error than the ground truth \( d \).

In Figures 3 and 4, we report results for scenario (ii) generated with \( \sigma \in \{2, 4, 6, 8\} \). In Figure 3 we vary the number of snapshots \( m \in \{20, 40, \ldots, 200\} \) for fixed \( n = 100 \) and \( d = 2 \), and in Figure 4 we vary the number of nodes \( n \in \{80, 120, \ldots, 400\} \) for fixed \( m = 80 \) and \( d = 2 \). In all settings, \( \text{Err}_Z \) is averaged over 50 independent replications. Results in these plots look similar to those in scenario (i), confirming that FASE is not relying on any parametric assumptions made on the true latent processes. In fact, even in the low signal to noise parameter settings for scenario (i) where OMNI outperformed FASE, FASE now outperforms all of its competitors. In Figure 4, for \( \sigma = 2 \) the errors for ASE are very close to those for FASE, but this is only because of the very high signal to noise ratio, and even in relative terms, the performance of ASE is comparatively worse as \( \sigma \) increases.

In Figures 5 and 6, we report results for scenario (iii) generated with edge densities 0.1, 0.25 and 0.5. In Figure 5 we vary the number of snapshots \( m \in \{20, 40, \ldots, 200\} \) for fixed \( n = 100 \) and \( d = 2 \), and in Figure 6 we vary the number of nodes \( n \in \{80, 120, \ldots, 400\} \) for
Figure 2: Mean of $\text{Err}_Z$, varying $n$, the number of nodes. Scenario (i), parametric Gaussian networks. Plots are labeled by edge standard deviation $\sigma$.

Figure 3: Mean of $\text{Err}_Z$, varying $m$, the number of snapshots. Scenario (ii), nonparametric Gaussian networks. Plots are labeled by edge standard deviation $\sigma$. 

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fixed \( m = 80 \) and \( d = 2 \). In all settings, \( \text{Err}_Z \) is averaged over 50 independent replications. Once again, none of the baseline ASE estimators show an improvement with increasing \( m \), while FASE does, and only ASE and FASE improve with increasing \( n \). In all settings, FASE performs the best of all methods. For these RDPG networks, we see that while the more conservative COSIE and OMNI approaches improve as the density decreases, the unbiased ASE approach gets substantially worse, as the signal is decreasing. Similarly, FASE gets slightly worse for decreasing density, but still always outperforms COSIE and OMNI.

Taken together, we see that among spectral embedding approaches for functional network data, FASE shows state of the art performance for recovery of the underlying latent process structure, up to unknown rotations. As suggested by Theorem 2, in both RDPG and Gaussian settings, we do not see the errors for FASE vanishing to zero, instead they appear to be bounded below by an approximation error term. Even in scenarios (i) and (iii), when we generate the true latent processes from a \( B \)-spline basis, since we cannot necessarily remove the unknown orthogonal transformation, we essentially revert to a nonparametric regime, and benefit from the fact that FASE only requires smoothness of the latent processes for efficient recovery.

### 5.2 Tuning With the NGCV Criterion

To evaluate our new NGCV model selection criterion, we evaluate its performance on latent process network models generated from scenarios (i), (ii), and (iii) as defined in Section 5.1. We evaluate the quality of selection and the quality of the eventual fitted model compared...
Figure 5: Mean of \( \text{Err}_Z \), varying \( m \), the number of snapshots. Scenario (iii), parametric RDPG networks. Plots are labeled by edge density.

Figure 6: Mean of \( \text{Err}_Z \), varying \( n \), the number of nodes. Scenario (iii), parametric RDPG networks. Plots are labeled by edge density.
Table 1: Parameter tuning results for scenario (i).

| $d$ | $\sigma$ | $d$-Prop (grid) | Prop (grid) | Ratio (grid) | $d$-Prop (CD) | Prop (CD) | Ratio (CD) |
|-----|----------|-----------------|-------------|--------------|----------------|-----------|------------|
| 2   | 2        | 1.00            | 1.00        | 1.000        | 1.00           | 1.00      | 1.000      |
| 2   | 4        | 1.00            | 1.00        | 1.000        | 1.00           | 1.00      | 1.000      |
| 2   | 6        | 1.00            | 0.42        | 1.009        | 1.00           | 0.42      | 1.009      |
| 2   | 8        | 1.00            | 0.88        | 1.003        | 1.00           | 0.88      | 1.003      |
| 4   | 2        | 1.00            | 1.00        | 1.000        | 1.00           | 1.00      | 1.000      |
| 4   | 4        | 1.00            | 1.00        | 1.000        | 1.00           | 1.00      | 1.000      |
| 4   | 6        | 1.00            | 0.58        | 1.007        | 1.00           | 0.58      | 1.007      |
| 4   | 8        | 0.88            | 0.86        | 0.998        | 0.84           | 0.82      | 1.002      |

to an oracle.

To evaluate quality of selection of $d$, we see if our selected model matches the ground truth used to generate the model. To evaluate the quality of selection of $q$, as there is not always a ground truth parameter, we see if our selected model matches the oracle $q$, denoted by $q_{ORC}$, and which minimizes the process recovery error up to orthogonal transformation:

$$q_{ORC} = \arg\min_{q_{\text{min}} \leq q' \leq q_{\text{max}}} \text{Err}_Z(\hat{Z}(q')),$$

where $\hat{Z}(q')$ is a FASE estimator fit with latent space dimension $d$ and basis dimension $q'$. In Tables 1-3, we report the proportion of replications in which the NGCV selection matches the ground truth value for $d$ (d-Prop) and the proportion of replications in which it matches both values in the best pair ($q_{ORC}, d$) (Prop). To evaluate the quality of the fitted model, Tables 1-3 also display the ratio between $\text{Err}_Z$ for the FASE estimator fit with $\hat{q}, \hat{d}$ selected according to NGCV, and the FASE estimator fit with the best pair ($q_{ORC}, d$) (Ratio).

As in Section 5.1, we consider three scenarios: (i) parametric Gaussian networks with $B$-spline latent processes, (ii) nonparametric Gaussian networks with sinusoidal latent processes, and (iii) parametric RDPG networks with $B$-spline latent processes. In all scenarios, we search for $(q, d)$ pairs over a $6 \times 6$ grid with $q = 6, 8, \ldots, 16$ and $d = 1, 2, \ldots, 6$. We will perform selection either by fitting models over the entire grid, or by coordinate descent (CD), as described in Section 3.3. Typically, coordinate descent converges in 3 or 4 univariate searches, meaning that it fits around $1/2$ to $2/3$ as many models compared to the full grid search over the $6 \times 6$ grid. The computational improvement of coordinate descent over a full grid search will be more pronounced for larger grids.

For scenario (i), we fix $n = 100$, $m = 80$, $q = 10$, and vary $\sigma = 2, 4, 6, 8$ and $d = 2, 4$. The results, averaged over 50 replications, are given in Table 1. With both grid selection and coordinate descent, even when the selected parameters do not match the oracle, the average error for the selected model is at most $1\%$ greater than the average oracle error. Moreover, despite fitting fewer models, the coordinate descent approach almost always agrees with the full grid selection. In all settings, selection of both $d$ and $q$ becomes more challenging for large values of $\sigma$. As the noise level increases, both the oracle and NGCV tend to select
smaller values of $q$. However, NGCV is more conservative in this respect: its choice of $q$ has already decreased for $\sigma = 6$, while the oracle choice does not decrease until $\sigma = 8$, which explains why selection of $q$ is better for $\sigma = 8$ compared to $\sigma = 6$. Finally, for $\sigma = 6$, selection is comparable or better with $d = 4$ than $d = 2$. This is because while selection of $d$ is equally hard in both settings, selection of $q$ is more stable when $d$ is larger: incrementing $q$ has a greater effect on the overall number of model parameters for greater values of $d$. Finally, when $d$ is selected incorrectly, it is possible to fit a model which has smaller error than the oracle, as the oracle must use the true $d$. This explains why the error ratio is smaller than 1 for grid selection with $d = 4$ and $\sigma = 8$, where NGCV sometimes selects $\hat{d} = 3$ due to a lack of signal.

For scenario (ii) we fix $n = 100$, $m = 80$, and vary $\sigma = 2, 4, 6, 8$ and $d = 2, 4$. The results, averaged over 50 replications, are given in Table 2. With both grid selection and coordinate descent, even when the selected parameters do not match the oracle, the average error for the selected model is at most 17% greater than the average oracle error. Compared to scenario (i), selection of $d$ is much more difficult in this setting. When incorrectly selected, $d$ is typically chosen to be larger than the true value, likely due to the unknown orthogonal rotations. Wrong selection of $d$ has a large relative effect on the error, especially with low $\sigma$, as the true $Z$ must be padded with zeros to match the dimensions of the two objects. However, we can see that especially for grid selection, if we restrict to cases where the ground truth $d$ is selected according to NGCV with grid selection, it is likely that it will also correctly select $q_{ORC}$. Conversely, when $d$ is chosen to be larger than the truth, the NGCV criterion tends to compensate by choosing $q$ smaller than $q_{ORC}$.

For scenario (iii) we fix $n = 100$, $m = 80$, $q = 10$, and vary the edge density in 0.5, 0.25, 0.1 and $d = 2, 4$. Under the Dirichlet-based simulation scheme described in the previous section, we cannot generate RDPG networks with $d = 4$ and density 0.5, so this combination is omitted. In brief, note that the coordinates for different nodes are generated independently from a Dirichlet distribution on the $d$-dimensional probability simplex, centered at $(1/d \cdots 1/d)^\top$. For two such variables $X$ and $Y$, $\mathbb{E}(X'Y) = 1/d$. To reduce the overall network density, we can rescale the positions by a constant $0 < \rho \leq 1$, however $\rho > 1$ will produce many pairs of positions with inner product greater than 1, outside the parameter

| $d$ | $\sigma$ | $d$-Prop (grid) | Prop (grid) | Ratio (grid) | $d$-Prop (CD) | Prop (CD) | Ratio (CD) |
|-----|----------|----------------|-------------|--------------|---------------|------------|------------|
| 2   | 2        | 1.00           | 0.96        | 1.001        | 1.00          | 0.96       | 1.001      |
| 2   | 4        | 0.98           | 0.96        | 1.019        | 1.00          | 0.98       | 1.001      |
| 2   | 6        | 0.44           | 0.32        | 1.170        | 0.54          | 0.34       | 1.154      |
| 2   | 8        | 0.46           | 0.34        | 1.106        | 0.56          | 0.42       | 1.080      |
| 4   | 2        | 0.98           | 0.86        | 1.022        | 1.00          | 0.88       | 1.002      |
| 4   | 4        | 0.76           | 0.62        | 1.123        | 0.92          | 0.74       | 1.048      |
| 4   | 6        | 0.30           | 0.26        | 1.140        | 0.42          | 0.36       | 1.118      |
| 4   | 8        | 0.36           | 0.30        | 1.083        | 0.46          | 0.36       | 1.070      |

Table 2: Parameter tuning results for scenario (ii).
space of the Bernoulli edge distribution. The results, averaged over 50 replications, are
given in Table 3. In this scenario, both grid selection and coordinate descent give particu-
larly good selection performance for $d$, even for smaller edge densities with weaker signal.
Although NGCV typically chooses $q$ smaller than $q_{ORC}$ for $d = 2$ and density $1/10$, we see
that this has a small relative effect on the error, as the average selected model error is at
most about 8% greater compared to the average oracle error.

Table 3: Parameter tuning results for scenario (iii).

| $d$ | Density (grid) | $d$-Prop (grid) | Prop (grid) | Ratio (grid) | $d$-Prop (CD) | Prop (CD) | Ratio (CD) |
|-----|----------------|-----------------|-------------|--------------|---------------|-----------|------------|
| 2   | 1/2            | 1.00            | 1.00        | 1.000        | 1.00          | 1.00      | 1.000      |
| 2   | 1/4            | 1.00            | 0.88        | 1.09         | 1.00          | 0.88      | 1.000      |
| 2   | 1/10           | 1.00            | 0.34        | 1.061        | 0.98          | 0.32      | 1.082      |
| 4   | 1/4            | 1.00            | 1.00        | 1.000        | 1.00          | 1.00      | 1.000      |
| 4   | 1/10           | 1.00            | 0.86        | 1.010        | 1.00          | 0.86      | 1.010      |

6 Analysis of International Political Interactions

As an application to real functional network data, we apply FASE to data collected by
the Integrated Crisis Early Warning System (ICEWS) [21]. In this aggregated data set
of international political interactions, we have $m = 108$ monthly snapshots of interaction
networks on the $n = 50$ most active countries from January 2005 to December 2013 in
terms of total absolute edge weight. An undirected edge $[A_k]_{ij}$ describes the total “weight”
of bilateral interaction between country $i$ and country $j$ in month $k$. “Weight” is a signed
measure of the intensity and nature of interactions, calculated by the ICEWS. Weights can
be both positive, corresponding to cooperative interactions such as giving aid; or negative,
corresponding to hostile interactions such as military action. To calculate a weight, the
ICEWS automatically scrapes and assigns signed weights to news articles, with edge weights
calculated by summing all the news articles for a given month.

As the distribution of edge weights is highly skewed, we apply FASE after a log trans-
formation given by

\[
\text{sign}([A_k]_{ij}) \log(1 + |[A_k]_{ij}|)
\]

for $k = 1, \ldots, m$ and $1 \leq i < j \leq n$. We use a cubic $B$-spline basis with equally spaced knots,
and select $\hat{q} = 5$ and $\hat{d} = 8$ using NGCV. For interpretability of plots, as a post-processing
step we perform a Procrustes alignment of each embedded snapshot to the previous snap-
shot’s embedding. The resulting plotted latent processes are still in the unidentified class $\mathcal{T}(\hat{Z})$. In Figures 7 and 8, we show an exploratory plot of the FASE at four time points for a
subset of the latent dimensions. The remaining dimensions are plotted in Appendix D. Fig-
ure 7 plots the first latent dimension against the second latent dimension, and Figure 8 plots
the third latent dimension against the fourth latent dimension. In Figure 7, most countries
have positive coordinates in the first latent dimension, corresponding to the total weight and
Figure 7: First (horizontal axis) and second (vertical axis) dimensions of FASE evaluated at four times: January 2006, January 2008, January 2010, and January 2012. Points are colored by geographical region. Purple: Africa, Red: Asia-Pacific, Blue: Europe, Cyan: Middle East, Orange: North America, Green: South America.
Figure 8: Third (horizontal axis) and fourth (vertical axis) dimensions of FASE evaluated at four times: January 2006, January 2008, January 2010, and January 2012. Points are colored by geographical region. Purple: Africa, Red: Asia-Pacific, Blue: Europe, Cyan: Middle East, Orange: North America, Green: South America.
sign of interactions. Countries like the USA, China and Russia, have large positive values in both dimensions at all four of the plotted times. Most Asian countries, plotted in red, have positive coordinates in the second latent dimension, while most European countries, plotted in blue, have negative coordinates. The four Asian nations with large negative coordinates in January 2006, January 2008, and January 2010 are Armenia, Azerbaijan, Georgia and Kazakhstan, four former Soviet republics which are geographically Asian but have more political ties with Europe than with East Asia [12]. We see some dynamic behavior in these plots as well. The first latent coordinate for Syria moves substantially between January 2010 and January 2012, possibly a consequence of the Syrian civil war, which began in late 2011 [2]. There is also consistent movement in the first latent coordinate among countries in the European Union. In January 2006, their mean first latent coordinate is about 0.92, while in January 2012 it is 0.53, reflecting an overall decrease in cooperative relationships during this time period.

In Figure 8, we again see a regional split between Asian countries with mostly positive third latent coordinates; and European countries with mostly negative coordinates. The top left quadrant and the bottom right quadrant separate countries with respect to the Israel-Palestine conflict, which accounts for the largest magnitude negative edges in this network. We see that this conflict appears to pit Israel against most other Middle Eastern nations, while Europe and the USA tend towards the Israeli side of the conflict. Again, there are key dynamic shifts in these plots. In January 2006, the top right and bottom left quadrants appear to separate countries with respect to conflicts between the USA and Iraq, and between the USA and Afghanistan. However, by January 2012 all three countries' latent coordinates are again much closer to the bulk of the cloud.

To further evaluate the dynamic behavior in this network, for each node we calculate the total distance traversed by its latent process in the 8-dimensional latent space. In Figure 9, we show the 20 countries with the greatest distance traversed. Due to boundary effects around the beginning and end of the time interval, we restrict to distance traversed between January 2006 and December 2012.

Taking a closer look at countries with the most dynamic behavior, we see that the civil war in Syria appeared to have implications for its relations with many countries, and its latent position changes substantially in the first, as well as the fifth and seventh latent dimensions during this time period. Afghanistan and Iraq’s coordinates both move in the third latent dimension, as well as the seventh, apparently as a result of improving relations with the USA during this time period. Afghanistan’s position also moves in the sixth latent dimension, possibly in response to border skirmishes with both Iran [1] and Pakistan [3] during this time period. The latent position for North Korea moves substantially in the first and third latent dimensions, in both cases reaching a local minimum around January 2010. As a result, the latent processes for Syria, Afghanistan, Iraq, and North Korea move the most of the countries in the network, despite not being extremely active in terms of total absolute edge weight. They are the 21st, 15th, 22nd, and 14th most active countries, respectively. These findings are consistent with the major world events of that time period.
Figure 9: Distance traversed by the estimated latent processes, restricted to the 20 countries with the greatest distance traversed. Bars are colored by geographical region. Red: Asia-Pacific, Blue: Europe, Cyan: Middle East, Orange: North America.
7 Discussion

In this paper, we have introduced a new latent process network model for functional network data collected as either adjacency matrix snapshots or aggregated indexed events. We provide a fitting algorithm using $B$-spline approximation and gradient descent, leading to the FASE estimator. We give theoretical guarantees and demonstrate the efficacy of our method on simulated and real data with both weighted and binary edges, comparing it to existing ASE-based approaches from the literature.

Identifiability remains a challenge for latent process network models of this type. Without strong conditions on eigenvalue separation, even if the orthogonalized latent processes truly belong to $\text{span}(B)$, because of the unknown orthogonal transformation, we cannot take advantage of their parametric form in estimation. Despite this, we have provided theoretical guarantees up to orthogonal transformation, and demonstrated that for smooth latent processes, sharing of information across network snapshots can still lead to more efficient recovery of the underlying network structure.

Future directions include extending the model to accommodate some dependence, in particular autoregressive edge variables, and theory for more general basis functions, including periodic bases which can be used to model seasonality in dynamic networks. Another important direction is developing inference for $Z$ or $W$, with a view towards finding confidence bands for the latent processes, or testing whether a latent dimension is homogeneous across index values.

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A Technical proofs

A.1 Proof of Proposition 1

Proof of Proposition 1. To find the gradient for a given latent dimension \( r = 1, \ldots, d \), we can rewrite the objective as

\[
\sum_{k=1}^{m} \left\| A_k - \sum_{r' \neq r} W_{r'} B(x_k) B(x_k)^\top W_{r'}^\top \right\|_F^2 - W_r B(x_k) B(x_k)^\top W_r^\top \tag{10}
\]

where the matrix in braces is free of \( W_r \). Thus it is sufficient to analyze (10) in the special case \( d = 1 \), where the objective can be written as

\[
\min_{W} \left\{ \sum_{k=1}^{m} \| A_k - W B(x_k) B(x_k)^\top W^\top \|_F^2 \right\}
\]

\[
= \sum_{k=1}^{m} \text{tr} \left( [A_k - W B(x_k) B(x_k)^\top W^\top]^2 \right)
\]

\[
= \sum_{k=1}^{m} \text{tr} \left( A_k^2 - A_k W B(x_k) B(x_k)^\top W^\top - \text{tr} (W B(x_k) B(x_k)^\top W^\top A_k) \right.
\]

\[
+ \left. W B(x_k) B(x_k)^\top W^\top W B(x_k) B(x_k)^\top W^\top \right)
\]

\[
\propto \sum_{k=1}^{m} \left\{ - \text{tr} (A_k W B(x_k) B(x_k)^\top W^\top) - \text{tr} (W B(x_k) B(x_k)^\top W^\top A_k) \right.
\]

\[
+ \text{tr} (W B(x_k) B(x_k)^\top W^\top W B(x_k) B(x_k)^\top W^\top) \right\}
\]

where in the final expression we drop the term not depending on \( W \). Now take a derivative of each term with respect to \( W \). First,

\[
\frac{\partial}{\partial W} \text{tr} (A_k W B(x_k) B(x_k)^\top W^\top) = 2A_k W B(x_k) B(x_k)^\top
\]

and the other cross term is the same. Then,

\[
\frac{\partial}{\partial W} \text{tr} (W B(x_k) B(x_k)^\top W^\top W B(x_k) B(x_k)^\top W^\top) = 4W B(x_k) B(x_k)^\top W^\top W B(x_k) B(x_k)^\top.
\]

The entire gradient with respect to \( W \) is

\[
-4 \sum_{k=1}^{m} (A_k - W B(x_k) B(x_k)^\top W^\top) W B(x_k) B(x_k)^\top.
\]
Thus for the general case with \( d > 1 \) we see that the gradient with respect to \( W_r \) is the desired
\[
-4 \sum_{k=1}^{m} \left\{ A_k - \sum_{r'=1}^{d} W_{r'} B(x_k) B(x_k)^\top W_{r'}^\top \right\} W_r B(x_k) B(x_k)^\top.
\]

### A.2 Preliminaries for proofs of Theorems

In this section we will introduce notation as well as some preliminary results which we will use in the proofs of Theorems 1 and 2.

We continue to use matrix and tensor notation introduced in Section 2. With some abuse of notation we write the Frobenius norm and inner product of 3-mode tensors for the standard vector \( \ell_2 \) norm and inner product of the vectorized tensors. Auxiliary results will typically be referenced below mathematical displays in which they are used. Some well known matrix algebra results will be stated with references, including the submultiplicative property of matrix norms and norm duality for the Frobenius inner product. We also prove two basic matrix algebra lemmas, and one probability lemma, both of which will be used in the proofs to follow.

**Lemma 1.** Suppose \( M \) is an \( n \times n \) symmetric matrix, and \( X, Y \) are \( n \times d \) matrices. Then
\[
\langle MX, X - Y \rangle = \frac{1}{2} \langle M, XX^\top - YY^\top \rangle + \frac{1}{2} \langle M, (X - Y)(X - Y)^\top \rangle
\]

**Proof of Lemma 1.**
\[
\langle MX, X - Y \rangle = \langle M, XX^\top - YY^\top \rangle = \left\langle M, \frac{1}{2}(XX^\top - YY^\top) + \frac{1}{2}(XX^\top + YY^\top) - YY^\top \right\rangle = \frac{1}{2} \langle M, XX^\top - YY^\top \rangle + \frac{1}{2} \langle M, XX^\top + YY^\top - YX^\top - XY^\top \rangle = \frac{1}{2} \langle M, XX^\top - YY^\top \rangle + \frac{1}{2} \langle M, (X - Y)(X - Y)^\top \rangle
\]
where the second to last equality uses the symmetry of \( M \). \qed

**Lemma 2.** Suppose an \( n \times q \) matrix \( W \) satisfies \( \|W\|_2 \leq \gamma \), and \( B(x) \) and \( B \) are defined as in Section 4. Then under Assumption 1, the \( q \times nm \) block matrix
\[
M = \begin{pmatrix} B(t_1)B(t_1)^\top W^\top & \cdots & B(t_m)B(t_m)^\top W^\top \end{pmatrix}
\]
satisfies \( \|M\|_2 \leq \gamma(C_B m/q)^{1/2} \).

**Proof of Lemma 2.** Rewrite
\[
M = B \begin{pmatrix} B(t_1)^\top W^\top & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 0 & B(t_m)^\top W^\top \end{pmatrix},
\]

34
where the second factor is an $m \times nm$ block diagonal matrix. Then by Assumption 1, part (B), $\|B\|_2 \leq (C_B m/q)^{1/2}$, and $\|WB(x_k)\|_2 \leq \gamma$ by assumption.

**Lemma 3.** Suppose $\{A_k\}_{k=1}^m$ are generated from a latent process network model, with independent sub-Gaussian edges with parameter at most $\sigma$, $B(x)$ satisfies Assumption 1, and $n$, $q$ are such that $nq \log q \geq nq \log 5 + (n + q) \log 9$. Then there is a constant $c_{\text{prob}}$ such that the event

$$\bigcap_{\{W : \|W\|_2 \leq \gamma\}} \left\{ \left\| \frac{1}{m} \sum_{k=1}^m \left\{ A_k - \sum_{r=1}^d Z_r(x_k)Z_r(x_k)^T \right\} WB(x_k)B(x_k)^T \right\|_2 \leq c_{\text{prob}} \gamma \left( \frac{\sigma^2 q^2 n \log q}{m} \right)^{1/2} \right\}$$

denoted by $\mathcal{E}$, satisfies $\mathbb{P}(\mathcal{E}) \geq 1 - 2 \exp(-n/2)$.

**Proof of Lemma 3.** Define the set

$$\mathcal{B} = \{W : \|W\|_2 \leq \gamma\} \subseteq \mathbb{R}^{n \times q}.$$

We will prove a high probability bound for

$$\sup_{W \in \mathcal{B}} \|M(W)\|_2,$$

where $M(W) = \sum_{k=1}^m \left\{ A_k - \sum_{r=1}^d Z_r(x_k)Z_r(x_k)^T \right\} WB(x_k)B(x_k)^T$. Define

$$\mathcal{B}^+ = \{W : \|W\|_F \leq q^{1/2} \gamma\} \supseteq \mathcal{B}.$$

Note that $\mathcal{B}^+$ is equivalent to a closed Euclidean ball in $\mathbb{R}^{nq}$ of radius $q^{1/2} \gamma$.

By usual covering results [37, Proposition 4.2.12], we can find a $(\gamma/2)$-net for $\mathcal{B}^+$ (under Frobenius metric), denoted by $\mathcal{L}$, satisfying

$$|\mathcal{L}| \leq (4q^{1/2} + 1)^n q \leq \left( 5q^{1/2} \right)^n.$$

Every element $W$ of $\mathcal{B}$ can be written as $W' + E$, where $W' \in \mathcal{L}$, and $\|E\|_2 \leq \gamma/2$. Fix $W \in \mathcal{B}$. Then

$$\|M(W)\|_2 \leq \|M(W')\|_2 + \|M(E)\|_2 \leq \max_{W' \in \mathcal{L}} \|M(W')\|_2 + \frac{1}{2} \sup_{W \in \mathcal{B}} \|M(W)\|_2 \leq \max_{W' \in \mathcal{L}} \left\{ 2 \max_{x \in \mathcal{N}, y \in \mathcal{M}} x^T M(W') y \right\} + \frac{1}{2} \sup_{W \in B} \|M(W)\|_2.$$

where $\mathcal{N}$ and $\mathcal{M}$ are $(1/4)$-nets for $\mathcal{S}^{n-1}$ and $\mathcal{S}^{q-1}$ of cardinalities $9^n$ and $9^q$, respectively [37, Theorem 4.4.5]. Taking a supremum on the left hand side and rearranging,

$$\sup_{W \in \mathcal{B}} \|M(W)\|_2 \leq 4 \max_{W' \in \mathcal{L}, x \in \mathcal{N}, y \in \mathcal{M}} x^T M(W') y. \quad (11)$$

For fixed $x$, $y$, and $W'$, concentration follows directly from [37, Theorem 4.4.5], and the
fact that \( \|B(x_k)\|_2 \leq 1 \) for all \( k \), resulting in the sub-Gaussian tail bound

\[
P \{ x^T M(W') y \geq t \} \leq 2 \exp \left( \frac{-c'_{\text{prob}} t^2}{\sigma^2 m q \gamma^2} \right)
\]

for a constant \( c'_{\text{prob}} \). We now take a union bound over the elements in the net.

\[
P \left[ \max_{W' \in \mathcal{L}, x \in \mathcal{N}, y \in \mathcal{M}} \{ x^T M(W') y \} \geq t \right] \leq 2 \cdot \left( 5 q^{1/2} \right)^{nq} \left( 9^{n+q} \right) \exp \left( \frac{-c'_{\text{prob}} t^2}{\sigma^2 m q \gamma^2} \right).
\]

By assumption, \( n \) and \( q \) satisfy \( nq \log q \geq nq \log 5 + (n + q) \log 9 \). Choose a constant

\[
c_{\text{prob}} \geq 4 \left( \frac{2}{c'_{\text{prob}}} \right)^{1/2},
\]

and set \( t^* = c_{\text{prob}} \sigma \gamma (nmq^2 \log q)^{1/2} \). Then

\[
P \left[ \max_{W' \in \mathcal{L}, x \in \mathcal{N}, y \in \mathcal{M}} \{ x^T M(W') y \} \geq t^* \right] \leq 2 \exp \left( -\frac{n}{2} \right).
\]

In combination with (11), this completes the proof. \( \square \)

A.3 Proof of Theorem 1

To simplify notation, in this section we will assume that \( Z^{\text{orth}} \) already coincides with the true \( Z \). Without loss of generality, we will prove Theorem 1 in the special case of \( d = 2 \). The proof will consist of two parts. First, we will prove an error bound for recovery of the first latent dimension, where the second latent dimension is not yet estimated and controlled using the singular value separation assumption (Assumption 2, part (A)). Second, we will prove an error bound for recovery of the second dimension, where we assume the first latent dimension has already been estimated with error controlled by the first part of the proof.

For general \( d \), a proof for recovery of dimension \( r = 1, \ldots, d \) would (up to constants), control the error of dimensions \( r' > r \) with an approach similar to the first part, while controlling errors of dimensions \( r' < r \) based on previous errors like in the second part.

Each part of the proof of Theorem 1, as well as the proof of Theorem 2 will follow the same general approach, similar to the approach taken in [24]. Throughout, we will invoke Lemma 3 to control the operator norm error for the gradient. First, we will prove an auxiliary lemma which controls the operator norm of the current estimate of \( \mathcal{W} \). Second, we will prove the key lemma of each proof, a one-step error bound which controls the error of the \( (h + 1) \)th gradient descent iterate based on the error of the \( h \)th iterate. Finally, we will complete the proof by induction on \( h \), and show that the corresponding results hold for sufficiently large \( H \).

We now proceed to prove Theorem 1 (for \( d = 2 \)) in two parts.
A.4 Error bound for first latent dimension

To prove (8) for the first latent dimension \((r = 1)\), we state two lemmas. Lemma 4 is given without proof, while we provide a proof for Lemma 5 immediately following its statement.

**Lemma 4.** Suppose the assumptions of Theorem 1 hold. Fix \(h \geq 0\) and let
\[
\alpha_{\text{prev}} = \|\hat{W}_h - W_1^*\|_F^2.
\]
Suppose
\[
\alpha_{\text{prev}} \leq c_{\text{prev}} \gamma_1^2
\]
for a sufficiently small constant \(c_{\text{prev}}\). Then
\[
\|\hat{W}_h\|_2 \leq c_W \gamma_1
\]
for a constant \(c_W \geq (c_1^{1/2} + c_{\text{prev}}^{1/2})\).

**Lemma 5.** Suppose the assumptions of Theorem 1 hold and that \(\mathcal{E}\) and Assumption 3 occur. Fix \(h \geq 0\), define \(\alpha_{\text{prev}}\) as in Lemma 4, and suppose (12) holds for sufficiently small \(c_{\text{prev}}\). Then for positive constants \(\rho\) and \(c_{\text{step}}\),
\[
\|\hat{W}_h^{h+1} - W_1^*\|_F^2 \leq \left(1 - \frac{\eta' \rho}{q}\right) \|\hat{W}_h^h - W_1^*\|_F^2 + c_{\text{step}} \eta' \left(\frac{\sigma^2 q^4 \log q n \gamma_2 m}{\gamma_1^2} + \epsilon_1\right).
\]
Proof of Lemma 5. We define the following terms which we will see later in the proof:

\[
T_{\text{mean}} = \sum_{k=1}^{m} \left\| \tilde{W}_1^h B(x_k) B(x_k)^\top (\tilde{W}_1^h)^\top - Z_1(x_k) Z_1(x_k)^\top \right\|_F^2,
\]

\[
T_{\text{ww}} = \sum_{k=1}^{m} \left\langle \tilde{W}_1^h B(x_k) B(x_k)^\top (\tilde{W}_1^h)^\top - Z_1(x_k) Z_1(x_k)^\top, (\tilde{W}_1^h - W_1^h) B(x_k) B(x_k)^\top (\tilde{W}_1^h - W_1^h)^\top \right\rangle,
\]

\[
T_{\text{op}} = 2 \left\langle \sum_{k=1}^{m} \left\{ A_k - \sum_{r=1}^{2} Z_r(x_k) Z_r(x_k)^\top \right\} \tilde{W}_1^h B(x_k) B(x_k)^\top, \tilde{W}_1^h - W_1^h \right\rangle,
\]

\[
T_{\text{approx}} = 2 \left\| \sum_{k=1}^{m} \left\{ \tilde{W}_1^h B(x_k) B(x_k)^\top (\tilde{W}_1^h)^\top - Z_1(x_k) Z_1(x_k)^\top, W_1^h B(x_k) B(x_k)^\top (W_1^h)^\top - Z_1(x_k) Z_1(x_k)^\top \right\} \right\|_F,
\]

\[
T_{\text{orth}} = 2 \left\langle \sum_{k=1}^{m} Z_2(x_k) Z_2(x_k)^\top \tilde{W}_1^h B(x_k) B(x_k)^\top, \tilde{W}_1^h - W_1^h \right\rangle,
\]

\[
T_{\text{quad.mean}}^2 = 4 \left\| \sum_{k=1}^{m} \left\{ \tilde{W}_1^h B(x_k) B(x_k)^\top (\tilde{W}_1^h)^\top - Z_1(x_k) Z_1(x_k)^\top \right\} \tilde{W}_1^h B(x_k) B(x_k)^\top \right\|_F^2,
\]

\[
T_{\text{quad.op}}^2 = 4 \left\| \sum_{k=1}^{m} \left\{ A_k - \sum_{r=1}^{2} Z_r(x_k) Z_r(x_k)^\top \right\} \tilde{W}_1^h B(x_k) B(x_k)^\top \right\|_F^2,
\]

\[
T_{\text{quad.orth}}^2 = 2 \left\| \sum_{k=1}^{m} Z_2(x_k) Z_2(x_k)^\top \tilde{W}_1^h B(x_k) B(x_k)^\top \right\|_F^2.
\]  (13)

Then we have

\[
\| \tilde{W}_1^{h+1} - W_1^* \|_F^2
\]

\[
= \| \tilde{W}_1^h - W_1^* + \eta_{h,1} \sum_{k=1}^{m} \left\{ A_k - \tilde{W}_1^h B(x_k) B(x_k)^\top (\tilde{W}_1^h)^\top \right\} \tilde{W}_1^h B(x_k) B(x_k)^\top \|_F^2
\]

\[
\leq \| \tilde{W}_1^h - W_1^* \|_F^2 + \eta_{h,1}^2 T_{\text{quad.mean}}^2 + \eta_{h,1}^2 T_{\text{quad.op}}^2 + \eta_{h,1}^2 T_{\text{quad.orth}}^2
\]

\[
+ 2\eta_{h,1} \left\langle \sum_{k=1}^{m} \left\{ A_k - \tilde{W}_1^h B(x_k) B(x_k)^\top (\tilde{W}_1^h)^\top \right\} \tilde{W}_1^h B(x_k) B(x_k)^\top, \tilde{W}_1^h - W_1^h \right\rangle
\]

\[
\leq \| \tilde{W}_1^h - W_1^* \|_F^2 + \eta_{h,1}^2 T_{\text{quad.mean}}^2 + \eta_{h,1}^2 T_{\text{quad.op}}^2 + \eta_{h,1}^2 T_{\text{quad.orth}}^2 + \eta_{h,1} T_{\text{op}}
\]

\[
- 2\eta_{h,1} \left\langle \sum_{k=1}^{m} \left\{ \tilde{W}_1^h B(x_k) B(x_k)^\top (\tilde{W}_1^h)^\top - \sum_{r=1}^{2} Z_r(x_k) Z_r(x_k)^\top \right\} \tilde{W}_1^h B(x_k) B(x_k)^\top, \tilde{W}_1^h - W_1^h \right\rangle
\]

\[
\leq \| \tilde{W}_1^h - W_1^* \|_F^2 + \eta_{h,1}^2 T_{\text{quad.mean}}^2 + \eta_{h,1}^2 T_{\text{quad.op}}^2 + \eta_{h,1}^2 T_{\text{quad.orth}}^2
\]

\[
+ \eta_{h,1} T_{\text{op}} + \eta_{h,1} T_{\text{approx}} + \eta_{h,1} T_{\text{orth}} + \eta_{h,1} T_{\text{ww}} - \eta_{h,1} T_{\text{mean}},
\]  (14)

where the final inequality uses Lemma 1. We now bound each of these terms in sequence.
For $T_{\text{quad.mean}}$,

$$T_{\text{quad.mean}} = 2 \left\| \begin{pmatrix} \hat{W}_1^h B(t_1) & (\hat{W}_1^h)^\top - Z_1(t_1) Z_1(t_1)^\top \\ \vdots & \vdots \\ \hat{W}_1^h B(t_m) & (\hat{W}_1^h)^\top - Z_1(t_m) Z_1(t_m)^\top \end{pmatrix} \right\|_F \leq 2 \left\| \begin{pmatrix} \hat{W}_1^h B(t_1) & (\hat{W}_1^h)^\top - Z_1(t_1) Z_1(t_1)^\top \\ \vdots & \vdots \\ \hat{W}_1^h B(t_m) & (\hat{W}_1^h)^\top - Z_1(t_m) Z_1(t_m)^\top \end{pmatrix} \right\|_F \leq 2T_{\text{mean}}^{1/2}C_{W1} \left( \frac{C_{BM}}{q} \right)^{1/2} \tag{15}$$

where the final inequality uses Lemma 2.

For $T_{\text{quad.op}}$,

$$T_{\text{quad.op}} \leq 2m \left\| \frac{1}{m} \sum_{k=1}^m \left\{ A_k - Z_1(x_k) Z_1(x_k)^\top \right\} \hat{W}_1^h B(x_k) B(x_k)^\top \right\|_F \leq 2mq^{1/2} \left\| \frac{1}{m} \sum_{k=1}^m \left\{ A_k - Z_1(x_k) Z_1(x_k)^\top \right\} \hat{W}_1^h B(x_k) B(x_k)^\top \right\|_2 \leq 2mq^{1/2}C_{\text{prob}}C_{W1} \left( \frac{\sigma^2 q^2 n \log q}{m} \right)^{1/2} \tag{16}$$

where the final inequality uses Lemma 3.

For $T_{\text{quad.orth}}$,

$$T_{\text{quad.orth}} = 2^{1/2} \left\| \begin{pmatrix} Z_2(t_1) Z_2(t_1)^\top \\ \vdots \\ Z_2(t_m) Z_2(t_m)^\top \end{pmatrix} \right\|_F \leq (2m)^{1/2}C_{\gamma_2} \left\| \begin{pmatrix} \hat{W}_1^h B(t_1) - W_1 B(t_1) + W_1^* B(t_1) - Z(t_1) \} B(t_1)^\top \\ \vdots \\ \hat{W}_1^h B(t_m) - W_1 B(t_m) + W_1^* B(t_m) - Z(t_m) \} B(t_m)^\top \end{pmatrix} \right\|_2 \leq (2m)^{1/2}C_{\gamma_2} \left\| \begin{pmatrix} \hat{W}_1^h - W_1^* \} B(t_1)^\top \\ \vdots \\ \hat{W}_1^h - W_1^* \} B(t_m)^\top \end{pmatrix} \right\|_2 + (2m)^{1/2}C_{\gamma_2} \left\| \begin{pmatrix} W_1^* B(t_1) - Z(t_1) \} B(t_1)^\top \\ \vdots \\ W_1^* B(t_m) - Z(t_m) \} B(t_m)^\top \end{pmatrix} \right\|_F \leq C_{\gamma_1} C_{\gamma_1} \left( \frac{2C_B}{\sqrt{q}} \right)^{1/2} \left\| \hat{W}_1^h - W_1^* \right\|_F + 2^{1/2}C_{\gamma_2} \left( \frac{2C_B}{\sqrt{q}} \right)^{1/2} \geq 1 \tag{17}$$

where the last inequality uses Assumption 2 part (B), and Lemma 2.
For $T_{\text{ww}}$,

$$T_{\text{ww}} \leq T_{\text{mean}}^{1/2} \left\| \begin{pmatrix} (\hat{W}^h_1 - W^*_1)B(t_1)B(t_1)^T(\hat{W}^h_1 - W^*_1)^T \\ \vdots \\ (\hat{W}^h_1 - W^*_1)B(t_m)B(t_m)^T(\hat{W}^h_1 - W^*_1)^T \end{pmatrix} \right\|_F$$

$$\leq T_{\text{mean}}^{1/2} \left\| \hat{W}^h_1 - W^*_1 \right\|_F $$

$$\leq \left( \frac{C_BmT_{\text{mean}}}{q} \right)^{1/2} \left\| \hat{W}^h_1 - W^*_1 \right\|_{F_{\alpha_{\text{prev}}}}$$

$$\leq c_{\text{ww}}T_{\text{mean}} + \frac{c_{\text{prev}}C_Bm\gamma_1^2}{4c_{\text{ww}}q^2} \left\| \hat{W}^h_1 - W^*_1 \right\|_F^2 \quad (18)$$

for an arbitrary positive constant $c_{\text{ww}}$ to be specified later. The second to last inequality uses Lemma 2 and the last inequality uses (12).

For $T_{\text{op}}$,

$$T_{\text{op}} \leq 2m\left\| \frac{1}{m} \sum_{k=1}^m (A_k - Z_1(x_k)Z_1(x_k)^T)\hat{W}^h_1B(x_k)B(x_k)^T \right\|_2 \left\| \hat{W}^h_1 - W^*_1 \right\|_*$$

$$\leq mq^{1/2} \left\{ c_{\text{prob}}c_W\gamma_1 \left( \frac{\sigma^2 q^3 n \log q}{m} \right)^{1/2} \right\} \left\| \hat{W}^h_1 - W^*_1 \right\|_F$$

$$\leq c_{\text{prob}}^2 c_W^2 q^2 n \log q + \frac{c_{\text{op}}m\gamma_1^2}{q} \left\| \hat{W}^h_1 - W^*_1 \right\|_F^2$$

$$\quad (19)$$

for an arbitrary positive constant $c_{\text{op}}$ to be specified later. The second inequality uses Lemma 3.

For $T_{\text{approx}}$, we require one auxiliary result:

$$\sum_{k=1}^m \left\| W^*_1 B(x_k)B(x_k)^T W^*_1 \right\|_F - Z_1(x_k)Z_1(x_k)^T \right\|_F^2 \leq 9c_1\gamma_1^2 \sum_{k=1}^m \left\| Z_1(x_k) - W^*_1 B(x_k) \right\|_2^2$$

$$\leq 9c_1m\gamma_1^2 \varepsilon_1$$

$$\quad (20)$$

where the first inequality uses Cauchy-Schwarz and [24], Lemma 29, as long as

$$\left\| W^*_1 B(x_k) - Z_1(x_k) \right\|_2 \leq \left\| W^*_1 B(x_k) \right\|_2,$$
which holds by Assumption 2, part (B) for sufficiently small $c_{\text{approx}}$. Then we have

$$T_{\text{approx}} = 2 \left| \sum_{k=1}^{m} \left( \hat{W}_1^h B(x_k)B(x_k)^\top \hat{W}_1^h \right)^\top - Z_1(x_k)Z_1(x_k)^\top, \hat{W}_1^* B(x_k)B(x_k)^\top \hat{W}_1^* \right| - Z_1(x_k)Z_1(x_k)^\top \right|$$

$$\leq T_{\text{mean}}^{1/2} \left( \sum_{k=1}^{m} \| \hat{W}_1^* B(x_k)B(x_k)^\top \hat{W}_1^* \|_F^2 \right)^{1/2}$$

$$\leq 3 \sqrt{c_1 m \gamma_1} T_{\text{mean}}^{1/2} \frac{\epsilon_1}{\epsilon_1}$$

$$\leq c'_{\text{approx}} T_{\text{mean}} + \frac{9 c_1 m \gamma_1^2}{4 c'_{\text{approx}}} \epsilon_1$$

(21)

for an arbitrary positive constant $c'_{\text{approx}}$ to be specified later. The third line uses (20).

For $T_{\text{orth}}$,

$$T_{\text{orth}} = 2 \left| \sum_{k=1}^{m} \left( Z_2(x_k)Z_2(x_k)^\top \begin{bmatrix} \{ \hat{W}_1^h B(x_k) - W_1^* B(x_k) \} B(x_k)^\top, \hat{W}_1^h - W_1^* \end{bmatrix} \right) \right|$$

where the first equality uses the orthogonality assumption. We factor based on the braces into parts (I) and (II).

For (I),

$$= 2 \left| \sum_{k=1}^{m} \left( Z_2(x_k)Z_2(x_k)^\top \left\{ \hat{W}_1^h B(x_k) - W_1^* B(x_k) \right\} B(x_k)^\top, \hat{W}_1^h - W_1^* \right) \right|$$

$$= 2 \left| \sum_{k=1}^{m} \left( \left( \hat{W}_1^h - W_1^* \right)^\top Z_2(x_k), B(x_k)B(x_k)^\top \left( \hat{W}_1^h - W_1^* \right)^\top Z_2(x_k) \right) \right|$$

$$\leq 2 \left\| \left( \hat{W}_1^h - W_1^* \right)^\top Z_2(t_1) \right\| \left\| \left( B(t_1)B(t_1)^\top \left( \hat{W}_1^h - W_1^* \right)^\top Z_2(t_1) \right) \right\|$$

$$\vdots$$

$$\leq 2 \left\| \left( \hat{W}_1^h - W_1^* \right)^\top Z_2(t_m) \right\| \left\| \left( B(t_m)B(t_m)^\top \left( \hat{W}_1^h - W_1^* \right)^\top Z_2(t_m) \right) \right\|$$
\[
\leq 2 \left\| \left( (\tilde{W}_1^h - W_1^*)^T Z_2(t_1) \cdots (\tilde{W}_1^h - W_1^*)^T Z_2(t_m) \right) \right\|_F \\
\leq 2 \left\| \left( (B(t_1)B(t_1)^T 0 \right) \right\|_{\infty} \left\| \left( (\tilde{W}_1^h - W_1^*)^T Z_2(t_1) \cdots (\tilde{W}_1^h - W_1^*)^T Z_2(t_m) \right) \right\|_F \\
\leq 2 \sum_{k=1}^{m} \| (\tilde{W}_1^h - W_1^*)^T Z_2(x_k) \|_2^2 \\
\leq 4 \sum_{k=1}^{m} \| (\tilde{W}_1^h - W_1^*)^T W_2 B(x_k) \|_F^2 + 4 \sum_{k=1}^{m} \| (\tilde{W}_1^h - W_1^*)^T (Z_2(x_k) - W_2^* B(x_k)) \|_F^2 \\
\leq \left( 4c_2\gamma_2^2 \frac{C_B m}{q} + 4m\varepsilon_2 \right) \| \tilde{W}_1^h - W_1^* \|_F^2 \\
\leq 4(c_2 C_B + c_{approx}) c_{sep} \frac{m\gamma_2^2}{q} \| \tilde{W}_1^h - W_1^* \|_F^2 \\
\]
We show that for each \( k = 1, \ldots, m \), the optimal (1-dimensional) orthogonal transformation that aligns \( Z_1(x_k) \) and \( \hat{W}_1^h B(x_k) \) is the identity transformation. By [10] this transformation has a closed form given by

\[
\text{sign}\left\{ Z_1(x_k)^T \hat{W}_1^h B(x_k) \right\}.
\]

Then

\[
Z_1(x_k)^T \hat{W}_1^h B(x_k) = Z_1(x_k)^T Z_1(x_k) - Z_1(x_k)^T \left\{ Z_1(x_k) - W_1^* B(x_k) + W_1^* B(x_k) - \hat{W}_1^h B(x_k) \right\}
\]

\[
\geq \gamma_1^2 - c_1^{1/2} \gamma_1 \| Z_1(x_k) - W_1^* B(x_k) \|_2 - c_1^{1/2} \gamma_1 \| \hat{W}_1^h B(x_k) - W_1 B(x_k) \|_2
\]

\[
\geq \gamma_1^2 - c_1^{1/2} \gamma_1 \| Z_1(x_k) - W_1^* B(x_k) \|_2 - c_1^{1/2} \gamma_1 \| \hat{W}_1^h - W_1 \|_F
\]

\[
\geq \left\{ 1 - c_{\text{sep}} (c_1 c_{\text{approx}})^{1/2} - (c_1 c_{\text{prev}})^{1/2} \right\} \gamma_1^2 > 0 \quad (23)
\]

where the final inequality follows by (12) and Assumption 2, part (B), as long as \( c_{\text{prev}}, c_{\text{sep}} \) and \( c_{\text{approx}} \) are sufficiently small.

Then we have

\[
\| \hat{W}_1^h - W_1 \|_F^2 \leq \| (\hat{W}_1^h - W_1) B (B^T B)^{-1} \|_F^2
\]

\[
\leq \frac{q}{c_B m} \sum_{k=1}^m \| \hat{W}_r^h B(x_k) - W_1^* B(x_k) \|_2^2
\]

\[
\leq \frac{q}{c_B m} \sum_{k=1}^m \| \hat{W}_r^h B(x_k) - Z_1(x_k) \|_2^2
\]

\[
\leq \frac{q}{c_B m 2(2^{1/2} - 1) \gamma_1^2} T_{\text{mean}}, \quad (24)
\]

where the third equality follows from a projection argument (the norm of a projection is at most the norm of the original vector) and the final inequality uses [24, Lemma 28], noting that by (23), the optimal orthogonal transformation is the identity. Then (24) implies that

\[
\frac{1}{2} T_{\text{mean}} \geq \frac{c_B (2^{1/2} - 1) m \gamma_1^2}{q} \| \hat{W}_1^h - W_1 \|_F^2.
\]
Substituting all these inequalities into (14), we have that

\[
\|\hat{W}^{h+1} - W^h\|_F^2 \leq \|\hat{W}^h - W^h\|_F^2 + \eta_{h,1}^2 T_{\text{quad.mean}}^2 + \eta_{h,1}^2 T_{\text{quad.orth}}^2 + \eta_{h,1}^2 T_{\text{quad.op}}^2 + \eta_{h,1} T_{\text{op}} + \eta_{h,1} T_{\text{approx}} + \eta_{h,1} T_{\text{mean}} - \eta_{h,1} T_{\text{orth}}
\]

\[
\leq \left[ 1 + \eta_{h,1} \left( c_{\text{op}} + \frac{c_{\text{prev}}}{4c_{\text{ww}}} + 4(c_2 C_B + c_{\text{approx}}) c_{\text{sep}}^2 \right)
\right] \|\hat{W}_1^h - W_1^h\|_F^2 + \eta_{h,1} \left( c_{\text{ww}} + c_{\text{approx}} + \eta_{h,1}^2 C_B \frac{m^2 \gamma_1}{q} - \frac{1}{2} \right) T_{\text{mean}}
\]

\[
+ \eta_{h,1}^2 \frac{c_{\text{prob}}^2 c_{\text{ww}}^2 q^2 n \log q}{4c_{\text{op}}} + \eta_{h,1}^2 C_B \frac{c_{\text{sep}}^2}{c_{\text{orth}}} \frac{m^2 \gamma_1}{q} \log q
\]

\[
+ \eta_{h,1} \left( \frac{9c_1}{4c_{\text{approx}}} + \frac{c_{\text{sep}}^2 c_{\text{ww}}}{c_{\text{orth}}} + \eta_{h,1}^2 c_{\text{sep}}^2 m^2 \gamma_1 \right) m^2 \gamma_1 \epsilon_1 \tag{25}
\]

We consider the coefficients of the first two terms of (25) separately. For the second term, expanding \( \eta_{h,1} \equiv \eta'/m^2 \gamma_1^2 \), we have coefficient

\[
\left( \frac{\eta'}{m^2 \gamma_1} \right) \left( c_{\text{ww}} + c_{\text{approx}}' + 4\eta' C_B m^2 \gamma_1 \right).
\]

If we choose \( c_{\text{ww}}, c_{\text{approx}}', \) and \( \eta' \) sufficiently small, we can upper bound this term by zero.

For the first term, again expanding the definition of \( \eta_{h,1} \), we have coefficient

\[
\left[ 1 - \frac{\eta'}{q} \left( c_B(2^{1/2} - 1) + c_{\text{op}} + \frac{c_{\text{prev}}}{4c_{\text{ww}}} + 4(c_2 C_B + c_{\text{approx}}) c_{\text{sep}}^2 + c_{\text{orth}} + 4\eta' C_b \frac{c_{\text{sep}}^2}{q} \right) \right] .
\]

If we choose \( c_{\text{op}} \) and \( c_{\text{orth}} \) sufficiently small, and since \( c_{\text{prev}}, c_{\text{approx}}, \) and \( c_{\text{sep}} \) are sufficiently small by assumption, we lower bound the quantity inside the square brackets by a small constant \( \rho > 0 \). Then, expanding the definition of \( \eta_{h,1} \) in the final three terms of (25), and choosing a constant

\[
c_{\text{step}} \geq \max \left\{ \frac{c_{\text{prob}}^2 c_{\text{ww}}^2}{4c_{\text{op}}}, \frac{4\eta' c_{\text{prob}}^2 c_{\text{ww}}^2}{4c_{\text{approx}}'}, \frac{9c_1}{4c_{\text{approx}}'} + \frac{c_{\text{sep}}^2 C_B}{c_{\text{orth}}} + 4\eta' C_b \frac{c_{\text{sep}}^2}{q} \right\}
\]

we complete the proof.

We now complete the proof of the error bound for the first latent dimension. We begin by showing by induction that if \( \mathcal{E} \) and Assumption 3 occur, which happens with probability greater than \( 1 - \delta - 2 \exp(-n/2) \), (12) holds for all \( h \geq 0 \). The base case holds by Assumption 3. Suppose (12) holds for all \( 0 \leq h' \leq h \). Then by repeated application of Lemma 5,

\[
\|\hat{W}_{h+1}^* - W_{h+1}^*\|_F^2 \leq \|\hat{W}_1^0 - W_1^0\|_F^2 + \frac{c_{\text{step}}}{\rho} \left( \frac{\sigma^2 q^5 n \log q}{\gamma_1^2 m} + q \epsilon_1 \right).
\]
Then Assumption 2, parts (B) and (C), and Assumption 3, we have
\[
\|\hat{W}_1^{h+1} - W_1^*\|_F^2 \leq \left( c_{\text{init}} + \frac{c_{\text{step}}}{\rho c_{\text{SNR}}} + \frac{c_{\text{approx}} c_{\text{sep}}}{\rho} \right) \gamma_1^2,
\]
which implies that (12) holds for \( h + 1 \) as long as \( c_{\text{init}} \), \( c_{\text{approx}} \), and \( c_{\text{sep}} \) are sufficiently small, and \( c_{\text{SNR}} \) is sufficiently large.

Now, since (12) holds for all \( h \geq 0 \), we can repeatedly apply Lemma 5 to conclude that for any \( h \geq 0 \),
\[
\|\hat{W}_1^h - W_1^*\|_F^2 \leq \left( 1 - \frac{\eta' \rho}{q} \right)^h \|\hat{W}_1^0 - W_1^*\|_F^2 + c_{\text{step}} \eta' \left( \frac{\sigma^2 q^4 n \log q}{\gamma_1^2 m} + \varepsilon_1 \right) \sum_{j=0}^{h} \left( 1 - \frac{\eta' \rho}{q} \right)^{-j}.
\]
Then by a geometric series argument, and applying Assumption 3,
\[
\|\hat{W}_1^h - W_1^*\|_F^2 \leq \left( 1 - \frac{\eta' \rho}{q} \right)^h \frac{c_{\text{init}} \gamma_1^2}{q} + \frac{c_{\text{step}}}{\rho} \left( \frac{\sigma^2 q^5 n \log q}{\gamma_1^2 m} + q \varepsilon_1 \right).
\]
To complete the proof we note that for sufficiently large (but finite) number of iterations \( H \), and a constant \( C_1' = c_{\text{step}}/\rho + 1 \).
\[
\|\hat{W}_1^h - W_1^*\|_F^2 \leq C_1' \left( \frac{\sigma^2 q^5 n \log q}{\gamma_1^2 m} + q \varepsilon_1 \right).
\]
Then to complete the proof of (8) for the first dimension,
\[
\frac{1}{m} \sum_{k=1}^{m} \|\tilde{Z}_1^H(x_k) - Z_1(x_k)\|_2^2 = m^{-1} \sum_{k=1}^{m} \|\hat{W}_1^H B(x_k) - W_1^* B(x_k) + W_1^* B(x_k) - Z_1(x_k)\|_2^2
\leq 2m^{-1} \|\hat{W}_1^H - W_1^*\|_F^2 + 2 \varepsilon_1
\leq 2 C_B q^{-1} \left[ \frac{\sigma^2 q^5 n \log q}{\gamma_1^2 m} + q \varepsilon_1 \right] + 2 \varepsilon_1
\leq C_1 \left( \frac{\sigma^2 q^4 n \log q}{\gamma_1^2 m} + \varepsilon_1 \right),
\]
for a constant \( C_1 = 2 C_B C_1' + 2 \), as desired.

### A.5 Error bound for second latent dimension

The proof for the second dimension is similar to the proof for the first dimension, with some additional terms. To reduce notation, analogous quantities defined in Appendix A.4 will be redefined here with the same symbols for the second dimension. Moreover, where appropriate, we will omit detailed calculations if they are similar to the corresponding calculations in the proof of the error bound for the first dimension. As above, we state two lemmas. Lemma 6 is given without proof, while we provide a proof sketch for Lemma 7 immediately after its statement.
Lemma 6. Suppose the assumptions of Theorem 1 hold. Fix $h \geq 0$ and let

$$\alpha_{\text{prev}} = \| \hat{W}_2^h - W_2^* \|_F^2.$$ 

Suppose

$$\alpha_{\text{prev}} \leq c_{\text{prev}} \gamma_2^2 \quad (27)$$

for a sufficiently small constant $c_{\text{prev}}$. Then

$$\| \hat{W}_2^h \|_2 \leq c_W \gamma_2$$

for a constant $c_W \geq (c_1^{1/2} + c_{\text{prev}}^{1/2})$.

Lemma 7. Suppose the assumptions of Theorem 1 hold and that $\mathcal{E}$ and Assumption 3 occur. Fix $h \geq 0$, define $\alpha_{\text{prev}}$ as in Lemma 6, and suppose (27) holds for sufficiently small $c_{\text{prev}}$. Define the first dimension error by

$$\alpha_{\text{first}} = \frac{1}{m} \sum_{k=1}^{m} \| \hat{W}_1 B(x_k) - Z_1(x_k) \|_2^2$$

where $\hat{W}_1$ was estimated in the first step of the sequential algorithm.

Then for positive constants $\rho$ and $c_{\text{step}},$

$$\| \hat{W}_2^{h+1} - W_2^* \|_F^2 \leq \left( 1 - \frac{\eta h}{q} \right) \| \hat{W}_2^h - W_2^* \|_F^2 + c_{\text{step}} \eta h \left( \frac{\sigma^4 q^4 \log q n}{\gamma_2^2 m} + \varepsilon_2 + \frac{\gamma_2^2}{\gamma_2^2} \alpha_{\text{first}} \right).$$

Proof sketch of Lemma 7. In the remainder, We redefine the terms in (13) swapping subscripts 1 and 2 for latent dimension, and define two additional terms

$$T_{\text{first}} = 2 \left| \sum_{k=1}^{m} \left\{ \hat{W}_1 B(x_k) B(x_k)^\top \hat{W}_1^\top - Z_1(x_k) Z_1(x_k)^\top \right\} \hat{W}_2^h B(x_k) B(x_k)^\top, \hat{W}_2^h - W_2^* \right|,$$

$$T_{\text{quad.first}}^2 = 2 \left| \sum_{k=1}^{m} \left\{ \hat{W}_1 B(x_k) B(x_k)^\top \hat{W}_1^\top - Z_1(x_k) Z_1(x_k)^\top \right\} \hat{W}_2^h B(x_k) B(x_k)^\top \right|_F^2.$$

Then based on calculations analogous to (14),

$$\| \hat{W}_2^{h+1} - W_2^* \|_F^2 \leq \| \hat{W}_2^h - W_2^* \|_F^2 + \eta_{h,2}^2 T_{\text{quad.mean}}^2 + \eta_{h,2}^2 T_{\text{quad.op}}^2 + \eta_{h,2}^2 T_{\text{quad.first}}^2$$

$$+ \eta_{h,2} T_{\text{op}} + \eta_{h,2} T_{\text{approx}} + \eta_{h,2} T_{\text{first}} + \eta_{h,2} T_{\text{ww}} - \eta_{h,2} T_{\text{mean}}, \quad (28)$$

where $T_{\text{orth}}$ and $T_{\text{quad.orth}}$ are no longer present as there are no additional orthogonal dimensions. We now bound each of these terms in sequence. The terms $T_{\text{quad.mean}}, T_{\text{quad.mean}}, T_{\text{quad.mean}}, T_{\text{quad.mean}}, T_{\text{quad.mean}},$ and $T_{\text{quad.mean}}$ are bounded as in Appendix A.4, with $\gamma_1, c_1,$ and $\varepsilon_1$ replaced by $\gamma_2, c_2,$ and $\varepsilon_2$ respectively.
For $T_{\text{first}}$,
\[
T_{\text{first}} = 2 \left\| \sum_{k=1}^{m} \left\{ \tilde{W}_1 B(x_k) B(x_k)^\top \tilde{W}_1^\top - Z_1(x_k) Z_1(x_k)^\top \right\} \tilde{W}_2^h B(x_k) B(x_k)^\top, \tilde{W}_2^h - \tilde{W}_2^* \right\|_F \\
= 2 \left\| \sum_{k=1}^{m} \left\{ \tilde{W}_1 B(x_k) B(x_k)^\top \tilde{W}_1^\top - Z_1(x_k) Z_1(x_k)^\top, (\tilde{W}_2^h - \tilde{W}_2^*) B(x_k) B(x_k)^\top \tilde{W}_2^h \right\} \right\|_F^{1/2} \left\{ \sum_{k=1}^{m} \left\| (\tilde{W}_2^h - \tilde{W}_2^*) B(x_k) B(x_k)^\top \tilde{W}_2^h \right\|_F^{1/2} \right\}^{1/2} \\
\leq \left\{ \sum_{k=1}^{m} \left\| \tilde{W}_1 B(x_k) B(x_k)^\top \tilde{W}_1^\top - Z_1(x_k) Z_1(x_k)^\top \right\|_F^{2} \right\}^{1/2} \left\{ \sum_{k=1}^{m} \left\| (\tilde{W}_2^h - \tilde{W}_2^*) B(x_k) B(x_k)^\top \tilde{W}_2^h \right\|_F \right\}^{1/2} \\
\leq \left\{ 4m(c_1 + c_{\text{approx}}) \gamma_1^2 \alpha_{\text{first}} \right\}^{1/2} \left( \frac{C_B m}{q} \right)^{1/2} \left\| \tilde{W}_2^h - \tilde{W}_2^* \right\|_F \\
\leq c_{\text{first}} \frac{m \gamma_2^2}{q} \left\| \tilde{W}_2^h - \tilde{W}_2^* \right\|_F^2 + \frac{C_B^2(c_1 + c_{\text{approx}}) \gamma_1^2 m \gamma_2^2}{c_{\text{first}}} \alpha_{\text{first}}
\]
for an arbitrary positive constant $c_{\text{first}}$ to be specified later. The fourth line follows from [24], Lemma 29, Assumption 2 parts (B) and (C), as well as the bound on $\alpha_{\text{first}}$ proven in Appendix A.4.

For $T_{\text{quad.first}}$,
\[
T_{\text{quad.first}} = 2^{1/2} \left\{ \left\| \left( \begin{array}{c} \tilde{W}_1^h B(t_1) B(t_1)^\top \tilde{W}_1^h - Z_1(t_1) Z_1(t_1)^\top \\
\vdots \\
\tilde{W}_1^h B(t_m) B(t_m)^\top \tilde{W}_1^h - Z_1(t_m) Z_1(t_m)^\top \\
\end{array} \right) \left( \begin{array}{c} \tilde{W}_2^h B(t_1) B(t_1)^\top \\
\vdots \\
\tilde{W}_2^h B(t_m) B(t_m)^\top \\
\end{array} \right) \right\}_F \\
\leq 2^{1/2} \left\{ \left\| \left( \begin{array}{c} \tilde{W}_1^h B(t_1) B(t_1)^\top \tilde{W}_1^h - Z_1(t_1) Z_1(t_1)^\top \\
\vdots \\
\tilde{W}_1^h B(t_m) B(t_m)^\top \tilde{W}_1^h - Z_1(t_m) Z_1(t_m)^\top \\
\end{array} \right) \right\}_F^{1/2} \left\{ \sum_{k=1}^{m} \left\| \tilde{W}_1 B(x_k) B(x_k)^\top \tilde{W}_1^\top - Z_1(x_k) Z_1(x_k)^\top \right\|_F^2 \right\}^{1/2} \\
\leq 2^{1/2} \left\{ \left( \frac{C_B m}{q} \right)^{1/2} \sum_{k=1}^{m} \left\| \tilde{W}_1 B(x_k) B(x_k)^\top \tilde{W}_1^\top - Z_1(x_k) Z_1(x_k)^\top \right\|_F \right\}^{1/2} \\
\leq 2^{1/2} \gamma_2 \left( \frac{C_B m}{q} \right)^{1/2} \left\{ 4m(c_1 + c_{\text{approx}}) \gamma_1^2 \alpha_{\text{first}} \right\}^{1/2}
\]
so that
\[
T_{\text{quad.first}}^2 \leq 8C_B(c_1 + c_{\text{approx}}) \gamma_1^2 \gamma_2^2 m \frac{\gamma_2^2}{q} \alpha_{\text{first}}.
\]
The arguments used for this term are similar to those used for $T_{\text{first}}$.

Plugging in these bounds, collecting like terms as in (25), and expanding $\eta_{h,2} = \eta/m \gamma_2^2$ completes the proof of the lemma, subject to requirements on the constants $c_{\text{prev}}$ and $c_{\text{approx}}$.

To prove the final error bound for the second dimension, we follow the same arguments from Appendix A.4, carrying through the additional term involving $\alpha_{\text{first}}$. Plugging in (26) for $\alpha_{\text{first}}$ completes the proof, subject to requirements on $c_{\text{init}}, c_{\text{approx}}$ and $c_{\text{SNR}}$. 

\[\square\]
A.6 Proof of Theorem 2

We will prove Theorem 2 for general $d$, following the same approach taken in the previous sections. Recall throughout this section that the Frobenius norm and inner product applied to tensors will be used to denote the Euclidean norm and inner product of the vectorized tensor (analogous to their definitions for matrices).

As in Appendix A.5, analogous quantities defined in Appendix A.4 will be redefined here with the same symbols, and where appropriate, we will omit detailed calculations if they are similar to the corresponding calculations in the proof of Theorem 1. As above, we state two lemmas. Lemma 8 is given without proof, while we provide a proof of Lemma 9 immediately after its statement.

**Lemma 8.** Suppose the assumptions of Theorem 2 hold. Fix $h \geq 0$ and let

$$\alpha_{\text{prev}} = \| \hat{W}^h - W^{*,h} \|_F^2.$$  

Suppose

$$\alpha_{\text{prev}} \leq c_{\text{prev}} \gamma_Z^2$$  

(29)

for a sufficiently small constant $c_{\text{prev}}$. Then

$$\| \hat{W}_r^h \|_2 \leq c_W \gamma_Z$$

uniformly over $r = 1, \ldots, d$ for a constant $c_W \geq (c_* + c_{\text{prev}}^{1/2})$.

**Lemma 9.** Suppose the assumptions of Theorem 2 hold, and that $\mathcal{E}$ and Assumption 5 occur. Fix $h \geq 0$, define $\alpha_{\text{prev}}$ as in Lemma 8, and suppose (29) holds. Then for positive constants $\rho$ and $c_{\text{step}},$

$$\| \hat{W}^{h+1} - W^{*,h+1} \|_F^2 \leq (1 - \eta' \rho) \| \hat{W}^h - W^{*,h} \|_F^2 + c_{\text{step}} \eta' \left( \frac{\sigma^2 q^5 \log q n}{\gamma_Z^2 m} + q \varepsilon(h) \right).$$

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Proof of Lemma 9. We define the following terms which we will see later in the proof:

\[
T_{\text{mean}} = \sum_{k=1}^{m} \left\| \sum_{r=1}^{d} \left( \sum_{k'=1}^{d} \hat{W}_r^h B(x_k) B(x_k)^\top (\hat{W}_{r'}^h) \right) - Z(x_k) Z(x_k)^\top \right\|_F^2,
\]

\[
T_{\text{ww}} = \left\| \sum_{r=1}^{d} \sum_{k=1}^{m} \left\{ \sum_{r'=1}^{d} \hat{W}_r^h B(x_k) B(x_k)^\top (\hat{W}_{r'}^h) \right\} - Z(x_k) Z(x_k)^\top, \right\|_F
\]

\[
\left( \hat{W}_r^h - \hat{W}_{r'}^{*,h} \right) B(x_k) B(x_k)^\top (\hat{W}_r^h - \hat{W}_{r'}^{*,h}) \right\|, \]

\[
T_{\text{op}} = 2 \left\| \sum_{r=1}^{d} \left\{ \sum_{k=1}^{m} \{A_k - Z(x_k) Z(x_k)^\top\} \hat{W}_r^h B(x_k) B(x_k)^\top, \hat{W}_r^h - \hat{W}_{r'}^{*,h}\right\} \right\|_F
\]

\[
T_{\text{approx}} = 2 \left\| \sum_{k=1}^{m} \left\{ \sum_{r=1}^{d} \hat{W}_r^h B(x_k) B(x_k)^\top (\hat{W}_{r'}^h) \right\} - Z(x_k) Z(x_k)^\top, \right\|_F \left\{ \sum_{r'=1}^{d} \hat{W}_{r'}^{*,h} B(x_k) B(x_k)^\top (\hat{W}_{r'}^{*,h}) \right\} - Z(x_k) Z(x_k)^\top \right\|_F
\]

\[
T_{\text{quad.mean}}^2 = 2 \sum_{r=1}^{d} \left\| \sum_{k=1}^{m} \left\{ \sum_{r'=1}^{d} \hat{W}_r^h B(x_k) B(x_k)^\top (\hat{W}_{r'}^h) \right\} - Z(x_k) Z(x_k)^\top \right\|_F \left\{ \sum_{r'=1}^{d} \hat{W}_{r'}^{*,h} B(x_k) B(x_k)^\top (\hat{W}_{r'}^{*,h}) \right\} - Z(x_k) Z(x_k)^\top \right\|_F
\]

\[
T_{\text{quad.op}}^2 = 2 \sum_{r=1}^{d} \left\| \sum_{k=1}^{m} \{A_k - Z(x_k) Z(x_k)^\top\} \hat{W}_r^h B(x_k) B(x_k)^\top \right\|_F^2.
\]

Then by calculations similar to (14)

\[
\left\| \hat{W}_{r+1}^h - \hat{W}_{r+1}^{*,h} \right\|_F^2 \\
\leq \left\| \hat{W}_r^h - \hat{W}_r^{*,h} \right\|_F^2 + \eta_r^2 T_{\text{quad.mean}}^2 + \eta_r^2 T_{\text{quad.op}}^2 + \eta_r T_{\text{op}} + \eta_r T_{\text{approx}} + \eta_r T_{\text{ww}} - \eta_r T_{\text{mean}}.
\]

(30)

We will bound each of these terms in sequence.

For \(T_{\text{quad.mean}}\), we fix \(r\) and bound each term as in (15), to conclude

\[
T_{\text{quad.mean}}^2 \leq 2cW d^\frac{m\gamma^2}{2} T_{\text{mean}}
\]

For \(T_{\text{quad.op}}\), we fix \(r\) and bound each term as in (16), to conclude

\[
T_{\text{quad.op}}^2 \leq 2c^2 \sigma^2 d^2 q^3 m^2 \gamma^2 n \log q
\]
For $T_{ww}$, we fix $r$ and bound each term as in (18), to conclude
\[
T_{ww} \leq \sum_{r=1}^{d} \left\{ \left( \frac{C_B m T_{\text{mean}}}{q} \right)^{1/2} \| \tilde{W}_r^h - W_{r,*}^h \|_F^2 \right\}
\]
\[
= \left( \frac{C_B m T_{\text{mean}}}{q} \right)^{1/2} \| \tilde{W}^h - W_{*}^h \|_F^2
\]
\[
\leq c_{ww} T_{\text{mean}} + \frac{c_{\text{prev}} C_B m \gamma Z}{4c_{ww} q} \| \tilde{W}^h - W_{*}^h \|_F^2
\]
for a positive constant $c_{ww}$ to be specified later.

For $T_{op}$, applying Lemma 3 as in (19), we get
\[
T_{op} \leq \frac{c_{\text{prob}} c_{op} \alpha q^{5/2} \ln q}{4c_{op}} + \frac{c_{op} m \gamma_1^2}{q} \| \tilde{W}^h - W_{*}^h \|_F^2.
\]
for a positive constant $c_{op}$ to be specified later.

For $T_{\text{approx}}$, we require one auxiliary result:
\[
\sum_{k=1}^{m} \left\{ \sum_{r=1}^{d} W_{r,*}^h B(x_k) B(x_k)^T (W_{r,*}^h)^T \right\} - Z(x_k) Z(x_k)^T \|_F^2 \leq 9 c_s \gamma_2^2 \sum_{k=1}^{m} \| W_{*}^h x_2 B(x_k) - Z(x_k) Q_{k}^h \|_F^2
\]
\[
\leq 9 c_s \gamma_2^2 \| Z \|_2
\]
where first inequality uses Cauchy-Schwarz and [24], Lemma 29, as long as
\[
\| W_{*}^h x_2 B(x_k) - Z(x_k) Q_{k}^h \|_F \leq \| Z(x_k) \|_2
\]
for all $k = 1, \ldots, m$, which holds by Assumption 5, parts (A) and (B) for sufficiently small $c_{\text{approx}}$.

For $T_{\text{approx}}$,
\[
T_{\text{approx}} = 2 \left\{ \sum_{k=1}^{m} \left\{ \sum_{r=1}^{d} \tilde{W}_r^h B(x_k) B(x_k)^T (\tilde{W}_r^h)^T \right\} - Z(x_k) Z(x_k)^T, \right\}
\]
\[
\leq T_{\text{mean}}^{1/2} \left\{ \sum_{k=1}^{m} \left\{ \sum_{r'=1}^{d} W_{r',*}^h B(x_k) B(x_k)^T (W_{r',*}^h)^T \right\} - Z(x_k) Z(x_k)^T \right\}^{1/2}
\]
\[
\leq 3 \sqrt{c_s m \gamma Z T_{\text{mean}}} (\varepsilon(\gamma))^{1/2}
\]
\[
\leq c'_{\text{approx}} T_{\text{mean}} + \frac{9 c_s \gamma_2^2}{4c'_{\text{approx}}} \varepsilon(\gamma),
\]
where $c'_{\text{approx}}$ is an arbitrary constant to be specified later. The third line uses (31).

Finally, for $T_{\text{mean}}$, define an $m$-tuple of orthogonal transformations, $Q_{\text{Proc},*}^h$ such that
for each \( k = 1, \ldots, m \),

\[
Q_{k}^{\text{Proc},h} = \arg\min_{Q \in \mathcal{O}_{d}} \| \tilde{W}^{h} \times_{2} B(x_{k}) - Z(x_{k})Q \|_{F}^{2}
\]

which has a closed form expression \([10]\). Then,

\[
\| \tilde{W}^{h} - W^{*,h} \|_{F}^{2} \leq \| \tilde{W}^{h} - P_B(ZQ^{\text{Proc},h}) \|_{F}^{2}
\]

\[
= \frac{q}{c_B m} \| \tilde{W}^{h} \times_{2} B - P_B(ZQ^{\text{Proc},h}) \times_{2} B \|_{F}^{2}
\]

\[
\leq \frac{q}{c_B m} \sum_{k=1}^{m} \| \tilde{W}^{h} \times_{2} B(x_{k}) - Z(x_{k})Q_{k}^{\text{Proc},h} \|_{2}^{2}
\]

\[
\leq \frac{q}{c_B m} 2(2^{1/2} - 1) \gamma_{Z}^{2} T_{\text{mean}}.
\]

The first inequality follows from the choice of \( W^{*,h} \), the second inequality follows from Assumption 1, the third inequality follows from a projection argument, and the final inequality follows from \([24]\), Lemma 28.

This display implies that

\[
\frac{1}{2} T_{\text{mean}} \geq \frac{c_B (\sqrt{2} - 1) m \gamma_{Z}^{2}}{q} \| \tilde{W}^{h} - W^{*,h} \|_{F}^{2}.
\]

Substituting all these inequalities into (30), we have that

\[
\| \tilde{W}^{h+1} - W^{*,h+1} \|_{F}^{2} \leq \| \tilde{W}^{h} - W^{*,h} \|_{F}^{2} + \eta_{h} T_{\text{quad,mean}}^{2} + \eta_{h}^{2} T_{\text{quad,op}}^{2} + \eta_{h} T_{\text{top}}
\]

\[
+ \eta_{h} T_{\text{approx}} + \eta_{h} T_{\text{ww}} - \eta_{h} T_{\text{mean}}
\]

\[
\leq \left( 1 + \frac{\eta_{h} c_{\text{op}} m \gamma_{Z}^{2}}{q} + \frac{\eta_{h} c_{\text{prev}} C_B m \gamma_{Z}^{2}}{4 c_{\text{ww}} q} - \frac{\eta_{h} c_{B}(2^{1/2} - 1) m \gamma_{Z}^{2}}{q} \right) \| \tilde{W}^{h} - W^{*,h} \|_{F}^{2}
\]

\[
+ \left( \eta_{h} c_{\text{ww}} + \eta_{h} c'_{\text{approx}} + 2 \eta_{h}^{2} c_{W} d^{2} \gamma_{Z}^{2} - \frac{1}{2} \eta_{h} \right) T_{\text{mean}}
\]

\[
+ \frac{\eta_{h} c_{\text{prob}}^{2} \sigma_{d}^{2} q^{4} n \log q}{4 c_{\text{op}}} + \frac{2 \eta_{h}^{2} c_{W} d^{2} \gamma_{Z}^{2} q^{2} m c_{w} \gamma_{Z}^{2} n \log q}{4 c_{\text{approx}}}
\]

\[
+ \eta_{h} \frac{9 c_{w} m \gamma_{Z}^{2} q^{2}}{4} \varepsilon^{(h)}.
\]

We consider the coefficients of the first two terms of (32) separately. For the second term, expanding \( \eta_{h} \equiv \eta' q / m \gamma_{Z}^{2} \), we have coefficient

\[
\left( \frac{\eta' q}{\gamma_{Z}^{2} m} \right) \left( c_{\text{ww}} + c'_{\text{approx}} + 2 \eta' c_{W} d - \frac{1}{2} \right).
\]

If we choose \( c_{\text{ww}}, c'_{\text{approx}}, \) and \( \eta' \) sufficiently small, we can upper bound this term by zero.
For the first term, again expanding the definition of $\eta_h$, we have coefficient

$$\left(1 - \eta'\left[c_B(2^{1/2} - 1) + c_{\text{op}} + \frac{c_{\text{prev}}C_B}{4c_{\text{ww}}}\right]\right).$$

If we choose $c_{\text{op}}$ sufficiently small, and since $c_{\text{prev}}$ is assumed to be sufficiently small, we lower bound the quantity inside the square brackets by a small constant $\rho > 0$. Then, expanding the definition of $\eta_h$ in the final three terms of (32), and choosing a constant

$$c_{\text{step}} \geq \max\left\{\frac{c^2_{\text{prob}}^2c_wd}{4c_{\text{op}}} + 2\eta'c^2_{\text{prob}}c^2_wd, \frac{9c_{\text{w}}}{4\epsilon'_{\text{approx}}}\right\},$$

we complete the proof.

To complete the proof of Theorem 2, we begin by showing by induction that if $E$ and Assumption 5 occurs, which happens with probability greater than $1 - \delta - 2\exp(-n/2)$, (29) holds for all $h \geq 0$. The base case holds by Assumption 5, part (C). Suppose (29) holds for all $0 \leq h' \leq h$. Then by repeated application of Lemma 9,

$$\|\hat{W}^{h+1} - W^{*,h+1}\|_{F}^2 \leq \|\hat{W}^0 - W^{*,0}\|_{F}^2 + \frac{c_{\text{step}}}{\rho} \left(\frac{\sigma^2q^5n\log q}{\gamma^2_1m} + q \cdot \max_{0 \leq h' \leq h} \epsilon^{(h')}\right).$$

Then Assumption 4, and Assumption 5, parts (B) and (C), we have

$$\|\hat{W}^{h+1} - W^{*,1}\|_{F}^2 \leq \left(c_{\text{init}} + \frac{c_{\text{step}}}{\rho c_{\text{SNR}}} + \frac{c_{\text{step}}c_{\text{approx}}}{\rho}\right)\gamma^2_Z,$$

which implies that (29) holds for $h + 1$ as long as $c_{\text{init}}$ and $c_{\text{approx}}$ are sufficiently small, and $c_{\text{SNR}}$ is sufficiently large.

Now, since (29) holds for all $h \geq 0$, we can repeatedly apply Lemma 9 to conclude that for any $h \geq 0$,

$$\|\hat{W}^h - W^{*,h}\|_{F}^2 \leq \left(1 - \eta'\rho\right)^h \|\hat{W}^0 - W^{*,0}\|_{F}^2 + c_{\text{step}}\eta'\frac{\sigma^2q^5n\log q}{\gamma^2_Zm}\sum_{j=0}^{h} (1 - \eta'\rho)^j + c_{\text{step}}\eta'\sum_{j=0}^{h} \epsilon^{(h-j)}(1 - \eta'\rho)^j.$$

For the first term, we note that it is bounded above by

$$\frac{\sigma^2q^5n\log q}{\gamma^2_Zm}$$

for sufficiently large $h$. The second term is bounded above by

$$\frac{c_{\text{step}}}{\rho} \left(\frac{\sigma^2q^5n\log q}{\gamma^2_Zm}\right)$$
by a geometric series argument. For the final term, we first note that for even \( h \),
\[
\sum_{j=0}^{h} (1 - \eta' \rho)^j \leq \frac{\text{cmax} \gamma Z^2 q}{\eta' \rho q} \left(1 - \eta' \rho\right)^{h/2} \sum_{j=0}^{h/2} \left(1 - \eta' \rho \right)^j + \left(\sup_{j>h/2} \varepsilon(j)\right) \sum_{j=0}^{h/2} \left(1 - \eta' \rho \right)^j
\]
\[
\leq \frac{\text{cmax} \gamma Z^2 q}{\eta' \rho q} \left(1 - \eta' \rho\right)^{h/2} + \frac{\text{cmax} \gamma Z^2 q}{\eta' \rho q} \left(1 - \eta' \rho\right)^{h/2} \sup_{j>h/2} \varepsilon(j),
\]
Then we can choose \( H \) sufficiently large such that the first term of (33) is less than
\[
\frac{\sigma^2 q^4 n \log q}{2 \gamma Z m}
\]
Moreover, since the sequence \( \varepsilon(j) \) is bounded above, \( \sup_{j>h/2} \varepsilon(j) \) converges to a finite limit, given by \( \limsup_{h \to \infty} \varepsilon(h) \). Thus, we may also choose \( H \) sufficiently large such that the second term of (33) is bounded above by
\[
\limsup_{h \to \infty} \varepsilon(h) + \frac{\sigma^2 q^4 n \log q}{2 \gamma Z m}
\]
Thus for sufficiently large \( H \) and constant \( C_2' = c_{\text{step}} / \rho + 2 \),
\[
\|\hat{W}^H - W^{*H}\|^2_F \leq C_2' \left(\frac{\sigma^2 q^4 n \log q}{\gamma Z m} + q \limsup_{h \to \infty} \varepsilon(h)\right).
\]
Then to complete the proof of (9),
\[
\frac{1}{m} \sum_{k=1}^{m} \|\hat{Z}_k^H - Z(x_k)Q_k^{*H}\|^2_F
\]
\[
= \frac{1}{m} \sum_{k=1}^{m} \|\hat{W}^H \times_2 B(x_k) - W^{*H} \times_2 B(x_k) + W^{*H} \times_2 B(x_k) - Z(x_k)Q_k^{*H}\|^2_F
\]
\[
\leq \frac{2}{m} \|\hat{W}^H - W^{*H}\|_F \times 2 B\|_F + 2 \sup_{j>H/2} \varepsilon(j)
\]
\[
\leq \frac{2 C_B C_2'}{q} \left[\frac{\sigma^2 q^4 n \log q}{\gamma Z m} + q \limsup_{h \to \infty} \varepsilon(h)\right] + 2 \eta' \rho \left(\limsup_{h \to \infty} \varepsilon(h) + \frac{\sigma^2 q^4 n \log q}{2 \gamma Z m}\right)
\]
\[
\leq C_2 \left[\frac{\sigma^2 q^4 n \log q}{\gamma Z m} + \limsup_{h \to \infty} \varepsilon(h)\right]
\]
for a constant \( C_2 = 2 C_B C_2' + 2 \eta' \rho \), as desired.

**B Derivation of NGCV criterion**

In this appendix we will derive the specific form of the NGCV criterion (4). First, recall the standard generalized cross validation (GCV) criterion for linear regression, which is derived based on leave one out cross validation [14]. Suppose we have univariate responses \( y_i \) and
\(p\)-dimensional predictors \(x_i\) for \(i = 1, \ldots, n\), with \(n \times p\) design matrix \(X\). The least squares regression coefficients for this problem are \(\hat{\beta} = (X^\top X)^{-1}X^\top y\) and the hat matrix is given by \(H = X(X^\top X)^{-1}X^\top\). Then the generalized cross validation criterion \([14]\) is

\[
\sum_{i=1}^{n} \left( \frac{y_i - x_i^\top \hat{\beta}}{1 - [H]_{ii}} \right)^2.
\]

A common approximation is to replace each of the diagonal elements of \(H\) with their mean, resulting in the criterion

\[
\left\{ 1 - \frac{\text{tr}(H)}{n} \right\}^{-2} \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - x_i^\top \hat{\beta})^2 \right\}.
\]

In least squares, \(\text{tr}(H) = p\), so we get the further simplification

\[
\left(1 - \frac{p}{n}\right)^{-2} \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - x_i^\top \hat{\beta})^2 \right\}.
\]

which can be calculated directly from the mean of squared residuals and the dimensions of the linear regression problem.

This simplified GCV criterion will be the building block of the NGCV criterion. Recall from (3) the form of \(\ell(W)\), which we will expand in the following way in terms of individual basis coordinates \(w_{i,r}\) for \(i = 1, \ldots, n\) and \(r = 1, \ldots, d\):

\[
\ell(W) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} \left\{ [A_k]_{ij} - \sum_{r=1}^{d} w_{i,r}^\top B(x_k)B(x_k)^\top w_{j,r} \right\}^2.
\]

Note that we can view this as a summation of \(2n\) least squares objectives, where the first \(n\) fix \(i\), the snapshot row, and the second \(n\) fix \(j\), the snapshot column, and sum over the other two indices. However, to avoid double counting matrix entries, we suppose that each \([A_k]_{ij}\) is assigned in a balanced way to either the row \(i\) problem or the column \(j\) problem. As a result, each problem is a least squares problem with \(nm/2\) observations.

In truth, the basis coordinates are shared between these problems and each problem involves the coordinates for all the nodes. Without loss of generality, consider the row 1 problem. For simplicity, suppose the problem only optimizes over the \(qd\) total coordinates \(\{w_{1,r}\}_{r=1}^{d}\), treating the others as fixed. Thus, plugging in the fitted coordinates \(\hat{W}\) and summing over the \(nm/2\) observations for this problem, we can calculate the simplified GCV criterion as

\[
\left(1 - \frac{2qd}{nm}\right)^{-2} \left[ \frac{2}{nm} \sum_{j} \sum_{k} \left\{ [A_k]_{1j} - \sum_{r=1}^{d} \hat{w}_{1,r}^\top B(x_k)B(x_k)^\top \hat{w}_{j,r} \right\}^2 \right].
\]

Each of the component squared errors in \(\ell(W)\) appears in exactly one of the \(2n\) problems.
Thus taking a mean of GCV’s over these problems we get the overall criterion
\[
\left(1 - \frac{2qd}{nm}\right)^{-2} \left\{ \frac{1}{mn^2} \ell(\hat{W}) \right\},
\]
which is equivalent to the NGCV criterion (4).

C Additional evaluation on synthetic networks

In this appendix we evaluate FASE on synthetic functional network data in terms of latent process recovery up to a single unknown orthogonal transformation. As additional post-processing, we perform a sequential Procrustes alignment for a collection of snapshot process estimates, similar to the alignment procedure used in Algorithm 3, and used as post-processing for the FASE estimate in Section 6. We will use this procedure to unambiguously select representatives of the unidentified classes \(T(Z)\) for the ground truth latent processes, and \(T(\hat{Z})\) for the FASE estimator. Formally, suppose we have latent processes \(\tilde{Z}\), evaluated at indices \(\{y_1, \ldots, y_m\} \subset \mathcal{X}\) and stored in an \(n \times m' \times d\) tensor. Then the sequential Procrustes alignment procedure \(\text{Proc}_{m'}\) sets \(\tilde{O}_1 = I_d\), then for \(k = 2, \ldots, m'\), replaces the \(k\)th \(n \times d\) slice \(\tilde{Z}(y_k)\) by \(\tilde{Z}(y_k)\tilde{O}_k\), where
\[
\tilde{O}_k = \arg\min_{O \in O_d} \| \tilde{Z}(y_k)O - \tilde{Z}(y_{k-1})\tilde{O}_{k-1} \|_F^2.
\]
For simplicity, we will set \(m' = m\) and compute \(\text{Proc}_m\) using the same snapshot times used to generate the data. As the FASE estimator is well-defined for any \(x \in \mathcal{X}\), we can evaluate this same sequential Procrustes alignment for arbitrarily fine grids. We stress that this alignment procedure is completely internal to its argument \(\tilde{Z}\), and does not require oracle knowledge of any ground truth \(Z\).

We will compare FASE to the same baseline approaches for the same scenarios and settings as in Section 5, but with a new error metric given by
\[
\text{Err}_{\hat{Z}}^*(\hat{Z}) = \min_{Q_0 \in O_d} \left\{ \frac{1}{ndm} \sum_{k=1}^m \| \text{Proc}_m(\tilde{Z})(x_k) - \text{Proc}_m(Z)(x_k)Q_0 \|_F^2 \right\}^{1/2}.
\]
In contrast to \(\text{Err}_Z\) defined in Section 5, this metric only requires optimization over a single orthogonal transformation-valued argument. We also plot \(\text{Err}_Z\) for the oracle version of FASE (FASE (ORC, \(\text{Err}_Z\))) as an achievable lower bound for \(\text{Err}_{\hat{Z}}^*\) for the same oracle FASE estimator.

Many of the overall conclusions from these plots are the same as in Section 5, although we will highlight some key differences in performance seen as a result of switching error metrics.

In Figures 10 and 11, we report results for scenario (i). In all settings, FASE shows a modest difference between \(\text{Err}_{\hat{Z}}^*\) and the lower bound \(\text{Err}_Z\). The difference is most pronounced for small \(m\) and \(n\), where the larger estimation error is magnified by the sequential Procrustes alignment, and for large \(m\), as the domain of the optimization in the definition
Figure 10: Mean of $\text{Err}_Z^*$, varying $m$, the number of snapshots. Scenario (i), parametric Gaussian networks. Plots are labeled by edge standard deviation $\sigma$.

of $\text{Err}_Z$ grows relative to the analogous domain in the definition of $\text{Err}_Z^*$.

In Figure 12, we show the relationship between the two error metrics for scenario (i) and the setting with $\sigma = 2$, $n = 100$ and $m = 200$. We can see that in the majority of cases for FASE (left panel), both metrics perform well, with comparable error that exceeds the best results of ASE (right panel). Moreover, and as expected, in terms of $\text{Err}_Z^*$, FASE always outperforms ASE. However, if $\text{Err}_Z$ for FASE exceeds a threshold around $1/10$, then the corresponding $\text{Err}_Z^*$ can be much larger, a phenomenon that occurs much less frequently for ASE. In this case the larger values for both metrics imply that the FASE estimate has attempted to “smooth out” a discontinuity in the sequence of aligning orthogonal transformations used to evaluate $\text{Err}_Z$, and converged to a local minimum of the objective which cannot take full advantage of the parametric form of the true processes. Despite this phenomenon, FASE still outperforms its competitors on average in terms of $\text{Err}_Z^*$ for sufficiently large values of $m$.

In Figures 13 and 14, we report results for scenario (ii). In this nonparametric scenario, switching error metrics has a more substantial effect on the performance of FASE. While we still see a decrease in $\text{Err}_Z^*$ when increasing either $m$ or $n$, there are now high signal settings for $\sigma \leq 4$ in which FASE does not clearly dominate ASE, even for relatively large values of $m$ and $n$.

In Figure 15, we report results for scenario (iii). In this scenario, for sufficiently large $n$, and most values of $m$, there appears to be very little difference in the two error metrics. Especially for functional networks with edge density $1/2$, $\text{Err}_Z^*$ can increase as $m$ increases. This phenomenon is considered above for scenario (i), see Figure 12 and the related discus-
Figure 11: Mean of $\text{Err}_Z^*$, varying $n$, the number of nodes. Scenario (i), parametric Gaussian networks. Plots are labeled by edge standard deviation $\sigma$.

Figure 12: Scatter plot of $\text{Err}_Z$ against $\text{Err}_Z^*$ for FASE (ORC) (left panel) and ASE (right panel). Scenario (i), $\sigma = 2$, $n = 100$, $m = 200$. Dotted lines denote $x = y$. 
Figure 13: Mean of \( \text{Err}_Z^* \), varying \( m \), the number of snapshots. Scenario (ii), nonparametric Gaussian networks. Plots are labeled by edge standard deviation \( \sigma \).

Figure 14: Mean of \( \text{Err}_Z^* \), varying \( n \), the number of nodes. Scenario (ii), nonparametric Gaussian networks. Plots are labeled by edge standard deviation \( \sigma \).

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Figure 15: Mean of $\text{Err}_Z^*$, varying $m$, the number of snapshots (left column), and $n$, the number of nodes (right column). Scenario (iii), parametric RDPG networks. Plots are labeled by edge density.

D Additional analyses of international relations

In this appendix we include some additional details of the analysis of international political interactions described in Section 6. As described briefly in Section 6, we tune the model parameters $d$ and $q$ by finding a FASE estimate for each pair in a grid, and evaluating NGCV. In particular, we vary $d$ between 1 and 10, incrementing by 1, and vary $q$ between 4 and 12, incrementing by 1. The results are shown in Figure 16. Importantly, we note that the NGCV criterion reaches a minimum on the interior of the grid, supporting the use of a functional embedding on this data, rather than one which is constant over time.

After selecting $\hat{d} = 8$ and $\hat{q} = 5$, we calculate our final estimator $\hat{Z}$ and apply the sequential Procrustes alignment procedure described in Appendix C. To unambiguously
label the latent dimensions from 1 to 8, we evaluate an average magnitude

\[
\frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} \left\{ \hat{z}_{i,r}(t_k) \right\}^2
\]

for each \( r = 1, \ldots, 8 \). The largest average magnitude (dimension 1) is about 46.3, dimensions 2 – 4 have smaller average magnitude between 12.9 and 14.8, and the remaining dimensions 5 – 8 have average magnitudes between 6.7 and 11.1.

The first four estimated latent dimensions are plotted in Section 6, and we plot the remaining four here, and make some brief remarks on the embeddings. Figure 17 plots the fifth latent dimension against the sixth latent dimension, and Figure 18 plots the seventh latent dimension against the eighth latent dimension.

In Figure 17, the USA and Venezuela are separated at extremes in the fifth dimension, while for much of the time period, the top right and bottom left quadrants separate countries with respect to a conflict between Israel and Iran. As noted in Section 6, the fifth latent coordinate for Syria moves substantially, from the positive to negative half-plane between January 2010 and January 2012 [2]. We also see movement, mostly in the sixth latent dimension of Afghanistan and India, reflecting worsening relations with Pakistan during this period [3].

In Figure 18, we again see a regional clusters formed by countries from Europe and Asia, although these begin to merge by the end of the time period. The seventh dimension separates the USA and Iraq in January 2006 and January 2008, but similar to the conclusion from Figure 8, this conflict appears to have fully dissipated by January 2012, as the two
Figure 17: Fifth (horizontal axis) and sixth (vertical axis) dimensions of FASE evaluated at four times: January 2006, January 2008, January 2010, and January 2012. Points are colored by geographical region. Purple: Africa, Red: Asia-Pacific, Blue: Europe, Cyan: Middle East, Orange: North America, Green: South America.
Figure 18: Seventh (horizontal axis) and eighth (vertical axis) dimensions of FASE evaluated at four times: January 2006, January 2008, January 2010, and January 2012. Points are colored by geographical region. Purple: Africa, Red: Asia-Pacific, Blue: Europe, Cyan: Middle East, Orange: North America, Green: South America.
countries have similar seventh latent coordinates, with the same sign.

E  FASE with smoothing splines

In this appendix we will report preliminary results using an extension of the FASE methodology to smoothing splines, in which we select a maximal natural spline basis and optimize a penalized objective function.

Briefly, recall that the optimization problem introduced in Section 3 minimizes a loss function $\ell(W)$ over coordinate tensors $W \in \mathbb{R}^{n \times q \times d}$. The vector-valued function $B(x) \in \mathbb{R}^q$ contains the $B$-spline basis for a $q$-dimensional cubic spline space. We can rewrite this in a functional way if we let $\mathcal{S}^{n \times d}_q$ denote the space of functions from $\mathcal{X}$ to $\mathbb{R}^{n \times d}$ with components in $\text{span}(B)$. Then we can rewrite (3) as

$$\min_{Z \in \mathcal{S}^{n \times d}_q} \left\{ \sum_{k=1}^m \|A_k - Z(x_k)Z(x_k)^T\|^2_F \right\}.$$ 

Following the usual development for smoothing splines, suppose we instead optimize over $Z$’s with components in the Sobolev space $\text{Sob}^{n \times d}_{2,2}$ of twice-differentiable functions and add a penalty term

$$\text{Pen}(Z) = \sum_{i=1}^n \sum_{r=1}^d \int_{\mathcal{X}} \left\{ \frac{n}{2} z_{ir}(x) \right\}^2 dt$$

scaled by a penalty parameter $\lambda \geq 0$. That is we solve the smoothing spline optimization problem

$$\min_{Z \in \text{Sob}^{n \times d}_{2,2}} \left\{ \sum_{k=1}^m \|A_k - Z(x_k)Z(x_k)^T\|^2_F + \lambda \text{Pen}(Z) \right\}.$$ 

Classical results on smoothing splines [15] can be used to justify that this is equivalent to solving

$$\min_{Z \in \mathcal{N}^{n \times d}_m} \left\{ \sum_{k=1}^m \|A_k - Z(x_k)Z(x_k)^T\|^2_F + \lambda \text{Pen}(Z) \right\} \quad (34)$$

where $\mathcal{N}^{n \times d}_m$ is the natural cubic spline with knots at the $x_k$’s for $k = 1, \ldots, m$. Hence, we can easily adapt FASE to these settings, solving (34) with gradient descent. The penalty term can be evaluated in terms of integrals of the second derivatives of the natural spline basis functions, and rewritten as a quadratic function of the basis coordinates, hence its gradient is easy to calculate.

We compare this smoothing spline version of FASE (FASE (SS)) to the B-spline version developed in the body of the paper (FASE (BS)) through a small simulation study. In particular, we generate functional networks as in scenario (ii) in Section 5, fixing $n = 100$ and varying $m$ from 20 to 200 and $\sigma \in \{2, 4, 6, 8\}$. We perform 50 replications for each setting, and evaluate the mean of $\text{Err}_Z$ (see Section 5). We select the nonparametric scenario (ii) as it should favor the fully nonparametric smoothing spline version of FASE.

For each of FASE (SS) and FASE (BS), we fit an oracle version with $d$ fixed at the ground
truth value and λ and q respectively selected from a grid to minimize Err\(_Z\). In about 98% of replications, the grids contain a local minimum of Err\(_Z\). We try two initialization routines. First, the usual initializer introduced in Appendix 3.2. Second, an oracle initialization from the ground truth processes \(Z\), or the closest approximation to each component in the \(B\)-spline space (ORC-init). The results are shown in Figure 19.

In Figure 19, we see that with data-driven initialization, FASE (BS) outperforms FASE (SS) in terms of Err\(_Z\) except in the setting where \(m = 20\). This ordering is reversed with oracle initialization. In fact, the performance of FASE (SS) is insensitive to \(\sigma\) for large \(m\), implying that gradient descent is converging to a local minimum very close to the starting point. This provides evidence that for large \(m\) and \(n\), as FASE (SS) must optimize far more parameters than FASE (BS), gradient descent becomes unreliable and highly dependent on the starting value. While this does not preclude the existence of an efficient implementation of FASE (SS) which can overcome these optimization issues, it is not clear that such an implementation would substantially outperform FASE (BS).

Figure 19: Mean of Err\(_Z\), varying \(m\), the number of snapshots. Scenario (ii), nonparametric Gaussian networks. Plots are labeled by edge standard deviation \(\sigma\).