Hyper Fuzzy AT-ideals of AT-algebra

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Abstract. The aim of this paper is to introduce the notion of hyper fuzzy AT-ideals on hyper AT-algebra. Also, hyper fuzzy AT-subalgebras and fuzzy hyper AT-ideal of hyper AT-algebras are studied. We study on the fuzzy theory of hyper AT-subalgebras and hyper AT-ideal of hyper AT-algebras. Furthermore, the fuzzy set theory of the (weak, strong, s-weak) hyper fuzzy AT-ideals in hyper AT-algebras are applied and the relations among them are obtained.

Keywords. AT-algebra, hyper AT-algebra, hyper AT-subalgebra, hyper AT-ideal, hyper fuzzy AT-subalgebra, hyper fuzzy AT-ideal, weak hyper fuzzy AT-ideal, strong hyper fuzzy AT-ideals s-weak hyper fuzzy AT-ideal.

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1. Introduction
The concept of a fuzzy set, was introduced by L.A. Zadeh [10]. A. T. Hameed [4] introduced KUS-algebra, they have studied a few properties of these algebras, the notion of KUS-ideals on KUS-algebras was formulated and some of its properties are investigated. She made an extension of the concept of fuzzy set to fuzzy KUS-subalgebras and fuzzy KUS-ideals on KUS-algebras. In [3], A.T. Hameed introduced a new algebraic structure, called AT-algebra, in [5,6], A.T. Hameed and et al have studied a few properties of AT-algebras. In [2], A.T. Hameed and et al applied the hyper structures to AT-algebras and introduced the concept of a hyper AT-algebra which is a generalization of AT-algebra, and investigated some related properties. They also introduced the notions of a hyper AT-subalgebra, a weak hyper AT-subalgebra, hyper AT-ideal, a weak hyper AT-ideal and gave relations between these. In this paper, we applied the fuzzy hyper structures to hyper AT-algebras and introduced the concept of a hyper fuzzy AT-ideal on hyper AT-algebra which is a generalization of AT-algebra, and investigated some related properties. They also introduced the notions of a hyper fuzzy AT-subalgebra, a weak hyper fuzzy AT-subalgebra, hyper fuzzy AT-ideal, a weak hyper fuzzy AT-ideal, a strong hyper fuzzy AT-ideal, a s-weak hyper fuzzy AT-ideal and gave relations between these, more properties of hyper homomorphism fuzzy hyper structure theory to hyper AT-algebras are given and the fuzzy set theory to the fuzzy (weak, strong, s-weak) hyper AT-ideals in hyper AT-algebras are applied and discussed.
2. Preliminaries

This section gives some basic definitions and preliminaries lemmas of hyper AT-subalgebra and hyper AT-ideal of hyper AT-algebra.

Definition 2.1[3]. An AT-algebra is a nonempty set $X$ with a constant (0) and a binary operation ($*$) satisfying the following axioms: for all $x, y, z \in X$,

(i) $((x * y) * ((y * z) * (x * z))) = 0$,
(ii) $0 * x = x$,
(iii) $x * 0 = 0$.

In $X$, we can define a binary relation ($\leq$) by $x \leq y$ if and only if, $y * x = 0$.

In AT-algebra $(X; *, 0)$, the following properties are satisfied: for all $x, y, z \in X$,

(i') $(y * z) * (x * z) \leq (x * y)$,
(ii') $0 \leq x$.

Definition 2.2[3]. A nonempty subset $S$ of an AT-algebra $X$ is called an AT-subalgebra of $X$ if $x * y \in S$, whenever $x, y \in S$.

Definition 2.3[5]. A nonempty subset $I$ of an AT-algebra $X$ is called an AT-ideal of $X$ if it satisfies the following conditions: for all $x, y, z \in X$,

AT1) $0 \in I$;
AT2) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$.

Proposition 2.4. Every AT-ideal of AT-algebra $X$ is an AT-subalgebra of $X$, and the convers is not true.

Definition 2.5[5]. Let $X$ be an AT-algebra. A fuzzy set $\mu$ in $X$ is called a fuzzy AT-subalgebra of $X$ if, for all $x, y \in X$, $\mu (x * y) = \min \{ \mu (x), \mu (y) \}$.

Definition 2.6[2,5]. Let $X$ be an AT-algebra. A fuzzy set $\mu$ in $X$ is called a fuzzy AT-ideal of $X$ if it satisfies the following conditions: for all $x, y, z \in X$,

FAT1) $\mu (0) = \mu (x)$.
FAT2) $\mu (x * z) = \min \{ \mu (x * (y * z)), \mu (y) \}$.

Proposition 2.7. Every fuzzy AT-ideal of AT-algebra $X$ is a fuzzy AT-subalgebra of $X$, and the convers is not true.

Definition 2.8[5]. Let $(X; *, 0)$ and $(Y; *, 0')$ be two AT-algebras, the mapping $f: (X; *, 0) \rightarrow (Y; *, 0')$ is called a homomorphism if it satisfies: for all $x, y \in X$ $f (x * y) = f (x) *' f (y)$.

Remark 2.9[2,5]. Let $f: (X; *, 0) \rightarrow (Y; *, 0')$ be a homomorphism from an AT-algebra $X$ into an AT-algebra $Y$. A is a nonempty subset of $X$ and $B$ is a nonempty subset of $Y$. The image of $A$ of $X$ under $f$ is $\{ f (a) : a \in A \}$, and the inverse image of $B$ of $Y$ under $f$ is $f^{-1} (B) = \{ y \in Y : y = f(x) \in B, x \in X \}$.

Theorem 2.10[5]. Let $f: (X; *, 0) \rightarrow (Y; *, 0')$ be into homomorphism of AT-algebras, then:

A) $f (0) = 0'$.
B) $x \leq y$ implies $f (x) \leq f (y)$.

Theorem 2.11[2,5]. Let $f: (X; *, 0) \rightarrow (Y; *, 0')$ be into homomorphism of AT-algebras, then:

(F1) If $S$ is an AT-subalgebra of $X$, then $f (S)$ is an AT-subalgebra of $Y$, where $f$ is onto.
(F2) If $I$ is an AT-ideal of $X$, then $f (I)$ is an AT-ideal in $Y$, where $f$ is onto.
(F3) If $D$ is an AT-subalgebra of $Y$, then $f^{-1} (D)$ is an AT-subalgebra of $X$.
(F4) If $J$ is an AT-ideal in $Y$, then $f^{-1} (J)$ is an AT-ideal in $X$.

Remake 2.12[7].
Let $H$ be a nonempty set and $P'(H) = P(H) \setminus \{\emptyset\}$ the family of the nonempty subsets of $H$. A multi valued operation (said also hyper operation) "\(\circ\)" on $H$ is a function, which associates with every pair \((x, y) \in H \times H = H^2\) a nonempty subset of $H$ denoted $x \circ y$.

An algebraic hyper structure or simply a hyper structure is a nonempty set $H$ endowed with one or more hyper operations, see ([8,9]).

**Definition 2.13([3]).**

Let $H$ be a nonempty set and "\(\circ\)" a hyper operation on $H$, such that $\circ : H \times H \to P'(H)$. Then $H$ is called a hyper AT-algebra if it contains a constant "0" and satisfies the following axioms: for all $x, y, z \in H$

\[
(HAT_1) \quad ((y \circ z) \circ (x \circ z)) \ll x \circ y,
\]

\[
(HAT_2) \quad 0 \circ x = \{x\},
\]

\[
(HAT_3) \quad x \circ 0 = \{0\}.
\]

Where $x \ll y$ is defined by $0 \in y \circ x$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A \exists b \in B$ such that $a \ll b$. In such case, we call "\(\ll\)" the hyper order in $H$. We shall use the $x \circ y$ instead of $x \circ \{y\}, \{x\} \circ y$ or $\{x\} \circ \{y\}$.

**Note that:** if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{a \circ b \in B} a \circ b$ of $H$.

**Definition 2.14([3]).**

Let $S$ be a non-empty subset of a hyper AT-algebra $(H, \circ, 0)$. Then $S$ is said to be a **hyper AT-subalgebra** of $H$ if $x \circ y \ll S$, $\forall x, y \in S$.

**Definition 2.15([3]).**

Let $A$ be a nonempty subset of a hyper AT-algebra $(H, \circ, 0)$. Then $A$ is said to be a **hyper AT-ideal** of $H$ if for all $x, y, z \in H$,

\[
(HI_1) \quad 0 \in A,
\]

\[
(HI_2) \quad x \circ (y \circ z) \ll A \quad \text{and} \quad y \in A \implies x \circ z \ll A.
\]

**Proposition 2.16([3]).** In hyper AT-algebra $(H, \circ, 0),$ the following properties are satisfied: for all $x, y, z \in H$,

\[
(1) \quad (y \circ z) \circ (x \circ z) \ll x \circ y,
\]

\[
(2) \quad 0 \ll x,
\]

\[
(3) \quad x \circ x \ll x.
\]

**Proposition 2.17([3]).**

Every hyper AT-ideal of hyper AT-algebra $H$ is a hyper AT-subalgebra of $X$, and the convers is not true.

**Definition 2.18([3]).**

Let $A$ be a nonempty subset of a hyper AT-algebra $(H, \circ, 0)$ and $0 \in A$. Then for all $x, y, z \in H$,

\[
(1) \quad A \text{ is said to be a weak hyper AT-ideal of } H \text{ if } x \circ (y \circ z) \subseteq A \quad \text{and} \quad y \in A \implies x \circ z \in A.
\]

\[
(2) \quad A \text{ is said a strong hyper AT-ideal} \text{ of } H \text{ if } x \circ (y \circ z) \cap A \neq \emptyset \quad \text{and} \quad y \in A \implies x \circ z \in A.
\]

**Definition 2.19([3]).**

Let $A$ be a nonempty subset of a hyper AT-algebra $(H, \circ, 0)$ and $0 \in A$, then for all $x, y, z \in H$,

\[
(1) \quad A \text{ is said to be a weak hyper ideal of } H \text{ if } x \circ y \subseteq A \quad \text{and} \quad x \in A \implies y \in A.
\]

\[
(2) \quad A \text{ is said a strong hyper ideal} \text{ of } H \text{ if } (x \circ y) \cap A \neq \emptyset \quad \text{and} \quad x \in A \implies y \in A.
\]

**Proposition 2.20([3]).**
Every (weak, strong) hyper AT-ideal is a (weak, strong) hyper ideal, and the converse is not true.

**Definition 2.21 ([3]).**

Let \((H; °, 0)\) and \((H'; °', 0')\) be hyper AT-algebras. A mapping \(f: (H; °, 0) \rightarrow (H'; °', 0')\) is called a **hyper homomorphism** if

\[
(\text{HH}_1) \quad f(0) = 0',
\]

\[
(\text{HH}_2) \quad f(x ° y) = f(x) °' f(y).
\]

**Theorem 2.22 ([3]).**

Let \((H; °, 0)\) and \((H'; °', 0')\) be hyper AT-algebras and \(f: (H; °, 0) \rightarrow (H'; °', 0')\) be a hyper homomorphism of hyper AT-algebras.

1. If \(x << y\) in \(H\), then \(f(y) << f(x)\) in \(H'\).
2. If \(A << B\) in \(H\), then \(f(A) << f(B)\) in \(H'\).

**Proposition 2.23 ([3]).**

Let \((H; °, 0)\) and \((H'; °', 0')\) be hyper AT-algebras and \(f: (H; °, 0) \rightarrow (H'; °', 0')\) be a hyper homomorphism of hyper AT-algebras.

1. If \(A\) is a hyper ideal of \(H\), then \(f(A)\) is a hyper ideal of \(H'\), where \(f\) is onto.
2. If \(B\) is a hyper ideal of \(H'\), then \(f^{-1}(B)\) is a hyper ideal of \(H\).
3. If \(A\) is a hyper AT-ideal of \(H\), then \(f(A)\) is a hyper AT-ideal of \(H'\), where \(f\) is onto.
4. If \(B\) is a hyper AT-ideal of \(H'\), then \(f^{-1}(B)\) is a hyper AT-ideal of \(H\).

3. **Hyper Fuzzy AT-subalgebras on Hyper AT-algebra**

In this section, we will discuss a new notion called a hyper fuzzy AT-subalgebras of hyper AT-algebra and we study several basic properties which are related to fuzzy hyper AT-ideals.

**Remark 3.1.**

Some fuzzy logic concepts are reviewed. A fuzzy set \(\mu\) in a set \((H; °, 0)\) is a mapping \(\mu: H \rightarrow [0,1]\). A fuzzy set \(\mu\) in a set \(H\) is said to satisfy the inf property (resp. sup property) if for any subset \(T\) of \(H\) there exists \(x_0 \in T\) such that \(\mu(x_0) = \inf_{x \in T} \mu(x)\) (resp. \(\mu(x_0) = \sup_{x \in T} \mu(x)\)).

**Definition 3.2.**

For a fuzzy set \(\mu\) in \(X\) and \(t \in [0,1]\) the set \(\mu_t = \{x \in H, \mu(x) \geq t\}\), which is called a level set of \(\mu\).

**Definition 3.3.**

A fuzzy set \(\mu\) in hyper AT-algebra \((H; °, 0)\) is said to be a hyper fuzzy AT-subalgebra of \(H\) if it satisfies the inequality: \(\inf_{x \in H} \mu(x) \geq \min\{\mu(x), \mu(y)\}\), \(\forall x, y, z \in H\).

**Proposition 3.4.**

Let \(\mu\) be a hyper fuzzy AT-subalgebra of \((H; °, 0)\), then \(\mu(0) \geq \mu(x)\), for all \(x \in H\).

**Proof:**

We see that \(0 \in x \circ x\), for all \(x \in H\). Hence \(\inf_{0 \leq x < x} \mu(0) \geq \min\{\mu(x), \mu(x)\} = \mu(x)\). \(\blacksquare\)

**Example 3.5.**

Let \(H = \{0, a, b\}\) be a set. Define hyper operation \(\circ\) on \(H\) as follows:
Then \((H, \circ, 0)\) is a hyper AT-algebra. Define a fuzzy set \(\nu : H \to [0, 1]\) defined by \(\nu(0) = 0.7\), \(\nu(a) = 0.5\) and \(\nu(b) = 0.2\) is also a hyper fuzzy AT-subalgebra of \(H\).

Theorem 3.6.

Let \(\mu\) be a fuzzy subset of AT-algebra \((H, \circ, 0)\). If \(\mu\) is a hyper fuzzy AT-subalgebra of \(H\) if and only if, for every \(t \in [0,1]\), \(\mu_t\) is hyper AT-subalgebras of \(H\), when \(\mu_t \neq \emptyset\).

Proof:

Assume that \(\mu\) is a hyper fuzzy AT-subalgebra of hyper AT-algebra \(H\), let \(x, y \in H\) be such that \(x \in \mu_t\) and \(y \in \mu_t\), then \(\mu(x) \geq t\) and \(\mu(y) \geq t\). Since \(\mu\) is a hyper fuzzy AT-subalgebra, it follows that \(\inf_{z \in x,y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \geq t\) and that \(x \circ y \ll \mu_t\). Hence \(\mu\) is a hyper AT-subalgebra of \(H\).

Conversely, assume \(\inf_{z \in x,y} \mu(z) \geq \min\{\mu(x), \mu(y)\}\) is false, there exist \(x'\) and \(y' \in H\) such that, \(\inf_{z \in x,y} \mu(z) < \min\{\mu(x'), \mu(y')\}\). Putting

\[
t' = \left( \inf_{z \in x,y} \mu(z') + \min\{\mu(x'), \mu(y')\} \right) / n, n \in N - \{1\}
\]

\(\mu(z') < t'\), and \(0 \leq t' < \min\{\mu(x'), \mu(y')\} \leq 1\), hence \(\mu(x') > t'\) and \(\mu(y') > t'\), which imply that \(x' \in \mu_t\) and \(y' \in \mu_t\), since \(\mu_t\) is a hyper AT-subalgebra, it follows that \(x' \circ y' \ll \mu_t\) and that \(\inf_{z \in x,y} \mu(z) \geq t'\), this is also a contradiction. Then \(\inf_{z \in x,y} \mu(z) \geq \min\{\mu(x), \mu(y)\}\).

Hence \(\mu\) is a hyper fuzzy AT-subalgebra of \(H\).

Proposition 3.7.

Let \(\{\mu_i | i \in \Lambda\}\) be a family of hyper fuzzy AT-subalgebras on \((H, \circ, 0)\), then \(\bigcap_{i \in \Lambda} \mu_i\) is a hyper fuzzy AT-subalgebra of \(H\).

Proof:

Since \(\{\mu_i | i \in \Lambda\}\) be a family of hyper fuzzy AT-subalgebras of \(H\), for any \(x, y \in H\), suppose \(x \in \mu_i\) and \(y \in \mu_i\), for all \(i \in \Lambda\), but \(\mu_i\) is a hyper fuzzy AT-subalgebra on \(H\), for all \(i \in \Lambda\). Then \(x \circ y \ll (\mu_i)\), for all \(i \in \Lambda\). Hence \(\inf_{z \in x,y} \mu_i(z) \geq \min\{\mu_i(x), \mu_i(y)\}\), \(\forall x, y, z \in H\), therefore, \(\inf_{z \in x,y} \bigcap_{i \in \Lambda} \mu_i(z) \geq \min\{\bigcap_{i \in \Lambda} \mu_i(x), \bigcap_{i \in \Lambda} \mu_i(y)\}\).

Hence \(\bigcap_{i \in \Lambda} \mu_i\) is hyper fuzzy AT-subalgebra of \(H\).

Proposition 3.8.

Let \(\{\mu_i | i \in \Lambda\}\) be a family of hyper fuzzy AT-subalgebras on \((H, \circ, 0)\), then \(\bigcup_{i \in \Lambda} \mu_i\) is a hyper fuzzy AT-subalgebra of \(H\), where its chain.

Proof:

Since \(\{\mu_i | i \in \Lambda\}\) be a family of hyper fuzzy AT-subalgebras of \(H\), for any \(x, y \in H\), suppose \(x \in \mu_i\) and \(y \in \mu_i\), for some \(i \in \Lambda\), but \(\mu_i\) is a hyper fuzzy AT-subalgebra on \(H\), for some \(i \in \Lambda\). Then

|   | 0   | a   | b   |
|---|-----|-----|-----|
| 0 | {0} | {a} | {b} |
| a | {0} | {0,a} | {a,b} |
| b | {0} | {0,a} | {0,a,b} |

Table 1.
$x \circ y \ll (\mu_i)$, for some $i \in \Lambda$. Hence $\inf_{i \in \Lambda} \mu_i(z) \geq \min\{\mu_i(x), \mu_i(y)\}, \quad \forall x, y, z \in H$, therefore,

\[
\inf_{i \in \Lambda} \bigcup_{i \in \Lambda} \mu_i(z) \geq \min\{\bigcup_{i \in \Lambda} \mu_i(x), \bigcup_{i \in \Lambda} \mu_i(y)\}.
\]

Hence $\bigcup_{i \in \Lambda} \mu_i$ is hyper fuzzy AT-subalgebra of $H$. $\blacksquare$

4. Hyper Fuzzy AT-ideals on Hyper AT-algebra

In this section, we will discuss a new notion called a hyper fuzzy AT-ideal of hyper AT-algebra and we study several basic properties which are related to hyper fuzzy AT-ideals. We will discuss the formalization of some types of hyper fuzzy AT-ideals on hyper AT-algebras and we prove some related properties.

**Definition 4.1.**

Let $(H, \circ, 0)$ be a hyper AT-algebra, the map $\mu : H \to [0, 1]$ is a fuzzy subset of $H$ and, then $\mu$ is said to be a hyper fuzzy AT-ideal of $H$, if for all $x, y, z, u \in H$,

1) $\mu(0) \geq \mu(x)$,
2) $x \ll y$ implies $\mu(y) \geq \mu(x)$
3) $\mu(x \circ z) \geq \min\{\inf_{u \in \{x \circ y\}} \mu(u), \mu(y)\}$.

**Example 4.2.** Let $H = \{0, a, b\}$ be a set. Define hyper operation $\circ$ on $H$ as follows:

| $\circ$ | 0   | a   | b   |
|--------|-----|-----|-----|
| 0     | {0} | {a} | {b} |
| a     | {0} | {0, a} | {a, b} |
| b     | {0} | {0, a} | {0, a, b} |

Then $(H, \circ, 0)$ is a hyper AT-algebra. Define a fuzzy set $\mu : H \to [0, 1]$ by $\mu(0) = \mu(a) = \alpha_1 > \alpha_2 = \mu(b)$, where $\alpha_1, \alpha_2 \in (0, 1]$, thus $\mu$ is a hyper fuzzy AT-ideal of $H$.

A fuzzy set $\nu : H \to [0, 1]$ defined by $\nu(0) = 0.7$, $\nu(a) = 0.5$ and $\nu(b) = 0.2$ is also a hyper fuzzy AT-ideal of $H$.

**Theorem 4.3.**

Let $\mu$ be a fuzzy subset of hyper AT-algebra $(H, \circ, 0)$. $\mu$ is a hyper fuzzy AT-ideal of $H$ if and only if, for every $t \in [0, 1]$, $\mu_t$ is a hyper AT-ideal of $H$, when $\mu \neq \phi$.

**Proof:**

Assume that $\mu$ is a hyper fuzzy AT-ideal of $H$, we have for all $x, y \in H$, $x \ll y$ implies $0 \in y \circ x$ implies $\inf_{i \in \Lambda} \mu(0) \geq \mu(y) \geq \mu(x)$, then $\mu(y) \geq \mu(x)$, therefore $\mu(0) \geq \mu(x) \geq t$, for $x \in \mu_t$, and so $0 \in \mu_t$.

Let $x, y, z, u \in H$ be such that $(x \circ (y \circ z)) \ll \mu_t$ and $y \in \mu_t$, then $\mu(x \circ (y \circ z)) \geq t$, and $\mu(y) \geq t$, since $\mu$ is a hyper fuzzy AT-ideal, it follows that $\mu(x \circ z) \geq \min\{\inf_{u \in \{x \circ y\}} \mu(u), \mu(y)\} \geq t$ and we have that $x \ll y \ll \mu_t$. Hence $\mu_t$ is a hyper AT-ideal of $H$. Conversely, if $\mu_t$ is a hyper AT-ideal of $H$. Suppose that if $x \ll y$ implies $\mu(y) \geq \mu(x)$ is false, then there exist $x', y' \in H$ such that $\mu(y') < \mu(x')$. If we take $t' = (\mu(x') + \mu(y'))/2$, then $\mu(0) < t'$ and $0 \leq t' \leq \mu(x') \leq 1$, then $x' \in \mu_t$ and $\mu_t \neq \phi$. As $\mu_t$ is a hyper AT-ideal of $H$, we have $0 \in \mu_t$, and $\inf_{i \in \Lambda} \mu(0) \geq t'$. This is a contradiction. Then $x \ll y$ implies $\mu(y) \geq \mu(x)$ is true.
Now, assume \( \mu(x \circ z) \geq \min \{ \inf_{u \in \{x \circ (y \circ z)\}} \mu(u), \mu(y) \} \) is false, then there exist \( x', y', z', u' \in H \) such that \( \mu(x' \circ z') < \min \{ \inf_{u \in \{x' \circ (y' \circ z')\}} \mu(u'), \mu(y') \} \). Putting \( t' = (\mu(x' \circ z')) + \min \{ \inf_{u \in \{x' \circ (y' \circ z')\}} \mu(u'), \mu(y') \} / 2 \), then 
\[
\mu(x' \circ z') < t' \quad \text{and} \quad 0 \leq t' < \min \{ \mu(x'(y' \circ z')), \mu(y') \} \leq 1 ,
\]
hence 
\[
\mu(x'(y' \circ z')) > t' \quad \text{and} \quad \mu(y') > t' ,
\]
which imply that \( x'(y' \circ z') \in \mu_r \) and \( y' \in \mu_r \). Since \( \mu_r \) is a hyper AT-ideal, it follows that \( x' \circ z' \in \mu_r \) and by \( \mu(x' \circ z') \geq t' \), this is also a contradiction.

Then \( (\mu(x \circ z) \geq \min \{ \inf_{u \in \{x \circ (y \circ z)\}} \mu(u), \mu(y) \}) \) is true. Hence \( \mu \) is a hyper fuzzy AT-ideal of \( H \). \( \blacksquare \)

**Proposition 4.4.**
Every hyper fuzzy AT-ideal of hyper AT-algebra is a hyper fuzzy AT-subalgebra.

**Proof:**
Since \( \mu \) is hyper fuzzy AT-ideal of \( H \), then by Theorem (4.3), for every \( t \in [0,1], \mu_t \) is a hyper AT-ideal of \( X \). By Proposition (2.17), for every \( t \in [0,1], \mu_t \) a hyper AT-subalgebra of \( H \). Hence \( \mu \) is a hyper fuzzy AT-subalgebra of \( H \) by Theorem (3.6). \( \blacksquare \)

**Remark 4.5.**
The converse of Proposition (4.4) is not true as shows in the Example (4.2), It is easy to show that \( (H; \ast, 0) \) hyper fuzzy AT-ideal. Define a fuzzy subset \( \mu : X \rightarrow [0,1] \) by:
\[
\mu(x) = \begin{cases} 
0.7 & \text{if } x \in \{0,2\} \\
0.3 & \text{otherwise}
\end{cases}
\]
Routine calculations give that \( \mu \) is a hyper fuzzy AT-subalgebra of \( H \), but not hyper fuzzy AT-ideal of \( H \).

**Proposition 4.6.**
Let \( \mu \) be a hyper fuzzy AT-ideal of \( H \) and let \( x, y, z \in H \), then
(i) \( \mu(0) \geq \mu(x) \),
(ii) \( x < y \) implies \( \mu(x) \geq \mu(y) \).

**Proof:**
(i) Since \( 0 \in x \circ x \ \forall x \in H \), we have \( \mu(0) \geq \inf_{a,x} \mu(a) \geq \mu(x) \), Which proves (i).

(ii) Let \( x, y \in H \) be such that \( x < y \). Then \( 0 \in y \circ x \ \forall x, y \in H \) and so \( \sup_{b(a,x)} \mu(b) \geq \mu(0) \) It follows from (i) that \( \mu(x) \geq \min \{ \sup_{a \in \{0\}} \mu(a) \}, \mu(y) \} \geq \min \{ \mu(0), \mu(y) \} = \mu(y) \). \( \blacksquare \)

**Proposition 4.7.**
Let \( \{ \mu_i \}_{i \in I} \) be a family of hyper fuzzy AT-ideals on \( (H; \ast, 0) \), then \( \bigcap_{i \in I} \mu_i \) is a hyper fuzzy AT-ideal of \( H \).

**Proof:**
Since \( \{ \mu_i \}_{i \in I} \) be a family of hyper fuzzy AT-ideals of \( H \), then for all \( i \in I, 0 < x \implies \mu_i(0) \geq \mu_i(x) \). Thus \( 0 < x \implies \bigcap_{i \in I} \mu_i(0) \geq \bigcap_{i \in I} \mu_i(x) \).
\[
x < y \implies \mu_i(y) \geq \mu_i(x) , \text{ for all } i \in I ,
\]

\[
x < y \implies \bigcap_{i \in I} \mu_i(y) \geq \bigcap_{i \in I} \mu_i(x)
\]
For any \( x, y, z \in H \), suppose \( x \circ (y \circ z) \ll \mu \), and \( y \in \mu_\i \), for all \( i \in \Lambda \), but \( \mu_\i \) is a hyper fuzzy AT-ideal on \( H \), for all \( i \in \Lambda \). Then \( x \circ z \ll \mu_\i \), for all \( i \in \Lambda \), therefore, \( x \circ z \ll \bigcap_{i \in \Lambda} \mu_\i \). Hence \( \bigcap_{i \in \Lambda} \mu_\i \) is hyper fuzzy AT-ideal of \( H \). ■

**Proposition 4.8.** Let \( \{ \mu_\i \mid i \in \Lambda \} \) be a family of hyper fuzzy AT-ideals on \((H, \circ, 0)\). Then
\[
\bigcup_{i \in \Lambda} \mu_\i \text{ is a hyper fuzzy AT-ideal of } H.
\]

**Proof:** Since \( \{ \mu_\i \mid i \in \Lambda \} \) be a family of hyper fuzzy AT-ideals of \( H \), then for some \( i \in \Lambda \),
\[
0 \ll x \text{ implies } \mu_\i(0) \geq \mu_\i(x).
\]
\[
x \ll y \text{ implies } \mu_\i(y) \geq \mu_\i(x), \text{ for some } i \in \Lambda,
\]
\[
x \ll y \text{ implies } \bigcup_{i \in \Lambda} \mu_\i(y) \geq \bigcup_{i \in \Lambda} \mu_\i(x).
\]

For any \( x, y, z \in H \), suppose \( x \circ (y \circ z) \ll \mu \), and \( y \in \mu_\i \), for some \( i \in \Lambda \), but \( \mu_\i \) is a hyper fuzzy AT-ideal on \( H \), for some \( i \in \Lambda \). Then \( x \circ z \ll \mu_\i \), for some \( i \in \Lambda \), therefore, \( x \circ z \ll \bigcup_{i \in \Lambda} \mu_\i \). Hence

\[
\bigcup_{i \in \Lambda} \mu_\i \text{ is hyper fuzzy AT-ideal of } H. \quad ■
\]

**Definition 4.9.**
Let \((H, \circ, 0)\) be a hyper AT-algebra, the map \( \mu : H \rightarrow [0, 1] \) is a fuzzy subset, then

1. \( \mu \) is said to be a **weak hyper fuzzy AT-ideal** of \( H \) if for all \( x, y, z, u \in H \), \( \mu(0) \geq \mu(x) \)
\[
\mu(x \circ z) \geq \min\{ \inf_{a \in (x,y)} \mu(u), \mu(y) \}. \tag{1}
\]

2. \( \mu \) is said to be a **strong hyper fuzzy AT-ideal** of \( H \) if for all \( x, y, z, a \in H \), \( \mu(0) \geq \mu(x) \)
\[
\inf_{a \in (x,y)} \mu(a) \geq \mu(x), \text{ and } \mu(x \circ z) \geq \min\{ \sup_{a \in (x,y)} \mu(u), \mu(y) \}. \tag{2}
\]

**Definition 4.10.**
Let \((H, \circ, 0)\) be a hyper AT-algebra, the map \( \mu : H \rightarrow [0, 1] \) is a fuzzy subset, then

1. \( \mu \) is said to be a **hyper fuzzy ideal** of \( H \) if \( \mu(0) \geq \mu(x) \)
\[
\text{and } \mu(y) \geq \min\{ \inf_{a \in (x,y)} \mu(z), \mu(x) \}. \tag{3}
\]

2. \( \mu \) is said to be a **weak hyper fuzzy ideal** of \( H \) if for all \( x, y, z \in H \), \( \mu(0) \geq \mu(x) \)
\[
\mu(y) \geq \min\{ \inf_{a \in (x,y)} \mu(z), \mu(x) \}. \tag{4}
\]

3. \( \mu \) is said to be a **strong hyper fuzzy ideal** of \( H \) if for all \( x, y, z \in H \), \( \mu(0) \geq \mu(x) \)
\[
\inf_{a \in (x,y)} \mu(a) \geq \mu(y) \geq \min\{ \sup_{a \in (x,y)} \mu(z), \mu(x) \}. \tag{5}
\]

**Proposition 4.11.**
Let \( \mu \) be a fuzzy subset of hyper AT-algebra \((H, \circ, 0)\). \( \mu \) is a hyper fuzzy ideal of \( H \) if and only if, for every \( t \in [0, 1] \), \( \mu_\i \) is a hyper ideal of \( H \), when \( \mu_\i \neq \phi \).

**Proof:**
Put \( x = 0 \), we get it by Theorem (4.3) and Proposition (2.20). ■

**Proposition 4.12.**
In any hyper AT-algebra \((H, \circ, 0)\),
1. Strong hyper fuzzy ideal is a hyper fuzzy ideal.
2. Hyper fuzzy ideal is a weak hyper fuzzy ideal.
3. Strong fuzzy hyper ideal is a weak hyper fuzzy ideal.

**Proof:** Straightforward. ■
Proposition 4.13.
Every (weak, strong) hyper fuzzy AT-ideal is a (weak, strong) hyper fuzzy ideal.

Proof:
Let $\mu$ be a hyper fuzzy AT-ideal of $H$, we get for any $x, y, z \in H$

$\mu(x \circ z) \geq \min \left\{ \inf_{u \in (x \circ y)} \mu(u), \mu(y) \right\}.$

Put $x = 0$, we get

$\mu(0 \circ z) \geq \min \left\{ \inf_{u \in (0 \circ y)} \mu(u), \mu(y) \right\},$ which gives, $
\mu(z) \geq \min \left\{ \inf_{u \in (0 \circ y)} \mu(u), \mu(y) \right\}.$

Ending the proof.

Generally, every (weak, strong) hyper fuzzy ideal is not a (weak, strong) hyper fuzzy AT-ideal. It can be observed with the help of examples given below:

Example 4.14.
(1) Let $H = \{0, 1, 2, 3\}$ be a set with the following table.

| $\circ$ | 0   | 1   | 2   | 3   |
|--------|-----|-----|-----|-----|
| 0      | $\{0\}$ | $\{1\}$ | $\{2\}$ | $\{3\}$ |
| 1      | $\{0\}$ | $\{0,1\}$ | $\{0,2\}$ | $\{2,3\}$ |
| 2      | $\{0\}$ | $\{0,2\}$ | $\{0\}$ | $\{1\}$ |
| 3      | $\{0\}$ | $\{0,2\}$ | $\{0\}$ | $\{0,1\}$ |

Then $(H; \circ, 0)$ is a hyper AT-algebra. Define a fuzzy subset $\mu : X \rightarrow [0,1]$ by:

$\mu(x) \begin{cases} 
0.7 & \text{if } x \in \{0,2\} \\
0.3 & \text{otherwise}
\end{cases}$

Routine calculations give that $\mu$ is a weak hyper fuzzy ideal of $H$, but not weak hyper fuzzy AT-ideal of $H$.

(2) Let $H = \{0, a, b\}$ be a set with the following table.

| $\circ$ | 0     | a     | b     |
|--------|-------|-------|-------|
| 0      | $\{0\}$ | $\{a\}$ | $\{b\}$ |
| 1      | $\{0\}$ | $\{0,a\}$ | $\{b\}$ |
| 2      | $\{0\}$ | $\{b\}$ | $\{0,b\}$ |

Then, $(H; \circ, 0)$ is a hyper AT-algebra. Define a fuzzy subset $\mu : H \rightarrow [0,1]$ by:

$\mu(x) \begin{cases} 
0.7 & \text{if } x \in \{0,b\} \\
0.3 & \text{otherwise}
\end{cases}$

Routine calculations give that $\mu$ is a strong hyper fuzzy ideal of $X$, but not strong hyper fuzzy AT-ideal of $H$.

Proposition 4.15.
Every hyper fuzzy AT-ideal of hyper AT-algebra is a hyper fuzzy ideal.

Proof:
Since $\mu$ is hyper fuzzy AT-ideal of $H$, then by Theorem (4.3), for every $t \in [0,1]$, $\mu_t$ is a hyper AT-ideal of $X$. By Proposition (2.20), for every $t \in [0,1]$, $\mu_t$ a hyper ideal of $H$. Hence $\mu$ is a hyper fuzzy ideal of $H$ by Proposition (4.11).

Proposition 4.16.
If $\mu$ is a weak hyper fuzzy AT-ideal of $H$, then the set $\mu_t \neq \phi$, for $t \in [0,1]$ is a weak hyper AT-ideal of $H$.

**Proof:**
Let $\mu$ be a weak hyper fuzzy AT-ideal of $H$ and $\mu_t \neq \phi$, for $t \in [0,1]$. Then there $a \in \mu_t$ and so $\mu(a) \geq t$. By Definition (4.9 (i)), $\mu(0) \geq \mu(a) \geq t$ and so $0 \in \mu_t$.

Let $x, y, z \in H$ such that $x \circ (y \circ z) \cap \mu_t \neq \phi$ and $y \in \mu_t$. Then there exist $a_0 \in x \circ (y \circ z) \subseteq \mu_t$ and hence $\mu(a_0) \geq t$. By definition (4.1), we have

$$\mu(x \circ z) \geq \min \left\{ \sup_{a \in x \circ (y \circ z)} \mu(a), \mu(y) \right\} \geq \min \left\{ \mu(a), \mu(y) \right\} \geq \min \{t, t\} = t,$$

and so $(x \circ z) \in \mu_t$. It follows that $\mu_t$ is a weak hyper AT-ideal of $H$.

**Corollary 4.17.**
If $\mu$ is a weak hyper fuzzy ideal of $H$, then the set $[1,0]$, $\mu_t$ for $t \in [0,1]$ is a weak hyper ideal of $H$.

**Proof:**
By Proposition (4.16) and Proposition (4.13).

**Proposition 4.18.**
Let $\mu$ be a fuzzy set in $H$ satisfying the sup property. If the set $[1,0]$, $\mu_t$ for $t \in [0,1]$ is a weak hyper AT-ideal of $H$ for all $t \in [0,1]$, then $\mu_t$ is a weak hyper fuzzy AT-ideal of $H$.

**Proof:**
Assume that $\mu_t \neq \phi$ is a weak hyper AT-ideal of $H$, for all $t \in [0,1]$. Then there $x \in \mu_t$ and hence $x \circ x \subseteq x \in \mu_t$. Using Proposition (2.16), we have $x \circ x \subseteq \mu_t$. Thus for $a \in x \circ x$, we have $a \in \mu_t$ and hence $\mu(u) \geq t$. It follows that $\inf_{a \in x \circ x} \mu(a) \geq t = \mu(x)$.

Moreover, let $x, y, z, u, a_0 \in H$ and put $\Omega = \min \left\{ \sup_{a \in x \circ (y \circ z)} \mu(u), \mu(y) \right\}$. By hypothesis $\mu_t$ is a weak hyper AT-ideal of $H$.

Since $\mu$ satisfies the sup property there is $a_0 \in x \circ (y \circ z)$ such that $\mu(a_0) = \sup_{a \in x \circ (y \circ z)} \mu(u)$. Thus

$$\mu(a_0) = \sup_{a \in x \circ (y \circ z)} \mu(u) \geq \min \left\{ \sup_{a \in x \circ (y \circ z)} \mu(u), \mu(y) \right\} = \Omega$$

and $a_0 \in \mu_\Omega$. This shows that $a_0 \in x \circ (y \circ z) \subseteq \mu_\Omega$ and hence $x \circ (y \circ z) \subseteq \mu_\Omega$. Combining $y \in \mu_\Omega$ and noticing that $\mu_\Omega$ is a weak hyper AT-ideal of $H$, we get $x \circ z \in \mu_\Omega$. Hence $\mu(x \circ z) \geq \Omega = \min \left\{ \sup_{a \in x \circ (y \circ z)} \mu(u), \mu(y) \right\}$. Therefore $\mu_t$ is a weak hyper fuzzy AT-ideal of $H$.

**Corollary 4.19.**
Let $\mu_t$ be a fuzzy set in $H$ satisfying the sup property. If the set $\mu_t \neq \phi$, for $t \in [0,1]$ is a weak hyper ideal of $H$ for all $t \in [0,1]$, then $\mu_t$ is a weak hyper fuzzy ideal of $H$.

**Proof:**
By Proposition (4.18) and Proposition (4.13).

**Proposition 4.20.**
If $\mu_t$ is a strong hyper fuzzy AT-ideal of $H$, then the set $\mu_t \neq \phi$, for $t \in [0,1]$ is a strong hyper AT-ideal of $H$.
Proof:
Let $\mu$ be a strong hyper fuzzy AT-ideal of $H$ and $\mu_t \neq \phi$, for $t \in [0,1]$. Then there $a \in \mu_t$ and so $\mu(a) \geq t$. By Definition (4.9 (ii)), $\mu(0) \geq \mu(a) \geq t$ and so $0 \in \mu_t$.

Let $x, y, z, u, a_0 \in H$ such that $x \circ (y \circ z) \cap \mu_t \neq \phi$ and $y \in \mu_t$. Then there exist $a_0 \in x \circ (y \circ z) \cap \mu_t$ and hence $\mu(a_0) \geq t$. By definition (4.1), we have

$$\mu(x \circ z) \geq \min \left\{ \sup_{u \in x(y \circ z)} \mu(u), \mu(y) \right\} \geq \min\{\mu(a_0), \mu(y)\} \geq \min\{t, t\} = t,$$

and so $t \circ x \circ z \in \mu_t$. It follows that $\mu_t$ is a strong hyper AT-ideal of $H$.

Corollary 4.21.
If $\mu$ is a strong hyper fuzzy ideal of $H$, then the set $\mu_t \neq \phi$, for $t \in [0,1]$ is a strong hyper ideal of $H$.

Proof:
By Proposition (4.20) and Proposition (4.13).

Proposition 4.22.
Let $\mu$ be a fuzzy set in $H$ satisfying the sup property. If the set $\mu_t \neq \phi$, for $t \in [0,1]$ is a strong hyper AT-ideal of $H$ for all $t \in [0,1]$, then $\mu$ is a strong hyper fuzzy AT-ideal of $H$.

Assume that $\mu_t \neq \phi$ is a strong hyper AT-ideal of $H$ for all $t \in [0,1]$. Then there $x \in \mu_t$ and hence $x \circ x \ll x \in \mu_t$. Using Proposition (2.16), we have $x \circ x \subseteq \mu_t$. Thus for $a \in x \circ x$, we have $a \in \mu_t$ and hence $\mu(a) \geq t$. It follows that $\inf_{u \in x(y \circ z)} \mu(a) \geq t = \mu(x)$.

Moreover, let $x, y, z, u \in H$ and put $\Omega = \min \left\{ \sup_{u \in x(y \circ z)} \mu(u), \mu(y) \right\}$. By hypothesis $\mu_t$ is a strong hyper AT-ideal of $H$.

Since $\mu$ satisfies the sup property there is $a_0 \in x \circ (y \circ z)$ such that $\mu(a_0) = \sup_{u \in x(y \circ z)} \mu(u)$. Thus

$$\mu(a_0) = \sup_{u \in x(y \circ z)} \mu(u) \geq \min \left\{ \sup_{u \in x(y \circ z)} \mu(u), \mu(y) \right\} = \Omega$$

and $a_0 \in \mu_t$. This shows that $a_0 \in x \circ (y \circ z) \cap \mu_t$ and hence $x \circ (y \circ z) \cap \mu_t \neq \phi$. Combining $y \in \mu_t$ and noticing that $\mu_t$ is a strong hyper AT-ideal of $H$, we get $x \circ z \in \mu_t$. Hence $\mu(x \circ z) \geq \Omega = \min \left\{ \sup_{u \in x(y \circ z)} \mu(u), \mu(y) \right\}$.

Therefore $\mu$ is a strong hyper fuzzy AT-ideal of $H$.

Corollary 4.23.
Let $\mu$ be a fuzzy set in $H$ satisfying the sup property. If the set $\mu_t \neq \phi$, for $t \in [0,1]$ is a strong hyper ideal of $H$ for all $t \in [0,1]$, then $\mu$ is a strong hyper fuzzy ideal of $H$.

Proof:
By Proposition (4.22) and Proposition (4.13).

5. $s$-weak Hyper Fuzzy AT-ideals on Hyper AT-algebra
In this section, we will discuss a new notion called a $s$-weak hyper fuzzy AT-ideals of hyper AT-algebra and we study several basic properties which are related to hyper fuzzy AT-ideals.

Definition 5.1.
A fuzzy set $\mu$ in $H$ is called a $s$-weak hyper fuzzy AT-ideal of $H$ if
(i) For every $x \in H$, $\mu(0) \geq \mu(x)$,

(ii) For every $x, y, z, u \in H$, there exists $u \in x \circ (y \circ z)$ such that $\mu(x \circ z) \geq \min\{\mu(u), \mu(y)\}$

**Proposition 5.2.**

Let $\mu$ be a weak hyper fuzzy AT-ideal of $H$. If $\mu$ satisfies the inf property, then $\mu$ is a s-weak hyper fuzzy AT-ideal of $H$.

**Proof:**

Since $\mu$ satisfies the inf property, let $x, y, z, u, a_0 \in H$, there exists $a_0 \in x \circ (y \circ z)$, such that $\mu(a_0) = \inf_{x \circ (y \circ z)} \mu(u)$. It follows that $\mu(x \circ z) \geq \min\{\inf_{x \circ (y \circ z)} \mu(u), \mu(y)\}$.

Ending the proof. ■

**Remark 5.3.**

In a finite hyper AT-algebra, every fuzzy set satisfies inf property (also sup property). Hence the concept of weak hyper fuzzy AT-ideals and s-weak hyper fuzzy AT-ideals coincide in a finite hyper AT-algebra.

**Proposition 5.4.**

Let $\mu$ be a strong hyper fuzzy AT-ideal of $H$ and let $x, y, z, a \in H$, then $\mu(x \circ z) \geq \min\{\mu(a), \mu(y)\}, \forall a \in x \circ (y \circ z)$.

**Proof:**

$$\mu(x \circ z) \geq \min\left\{\sup_{x \circ (y \circ z)} \mu(a), \mu(y)\right\}, \forall a \in x \circ (y \circ z),$$

ending the proof. ■

**Proposition 5.5.**

In any hyper AT-algebra $(H, \circ, 0)$,

1- Strong hyper fuzzy AT-ideal is a hyper fuzzy AT-ideal.
2- Hyper fuzzy AT-ideal is a weak hyper fuzzy AT-ideal.
3- Strong hyper fuzzy AT-ideal is a weak hyper fuzzy AT-ideal.
4- Strong hyper fuzzy AT-ideal is a s-weak hyper fuzzy AT-ideal.

**Proof:** Straightforward. ■

**Proposition 5.6.**

Let $\mu$ be a hyper fuzzy AT-ideal of $H$ and let $x, y, z \in H$, if $\mu$ satisfies the inf property, then $\mu(x \circ z) \geq \min\{\mu(x), \mu(y)\}, \forall a \in x \circ (y \circ z)$.

**Proof:**

Since $\mu$ satisfies the property, there is $a_0 \in x \circ (y \circ z)$, such that $\mu(a_0) = \inf_{a \in x \circ (y \circ z)} \mu(a)$. Hence $\mu(x \circ z) \geq \min\{\inf_{a \in x \circ (y \circ z)} \mu(a), \mu(y)\} = \min\{\mu(x), \mu(y)\}$, which implies that is true. ■

**Proposition 5.7.**

(i) Every hyper fuzzy AT-ideal of $H$ is a weak hyper fuzzy AT-ideal of $H$.
(ii) If $\mu$ is a hyper fuzzy AT-ideal of $H$ satisfying inf property, then $\mu$ is a s-weak hyper fuzzy AT-ideal of $H$.

**Proof:** Straightforward. ■

The following example shows that the converse of Corollary (5.7) may not be true.

**Example 5.8.**

Consider the hyper AT-algebra $H$ in Example (3.5). Define a fuzzy set $\mu$ in $H$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x = b \\ 0 & \text{if } x = a \end{cases}$$

Then $\mu$ is a weak hyper fuzzy AT-ideal of $H$ but it is not a hyper fuzzy AT-ideal of $H$. 

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since \( a << b \), but \( \mu(a) \geq \mu(b) \).

**Example 5.9.**

Consider the hyper AT-algebra \( H \)

|   | 0   | 1   | 2   |
|---|-----|-----|-----|
| 0 | \{0\} | \{1\} | \{2\} |
| 1 | \{0\} | \{0,1\} | \{1,2\} |
| 2 | \{0\} | \{0,1\} | \{0,1,2\} |

Define a fuzzy set \( \mu \) in \( H \) by:

\[
\mu(x) = \begin{cases} 
1 & \text{if } x = 0 \\
1/2 & \text{if } x = 1 \\
0 & \text{if } x = 2 
\end{cases}
\]

Then we can see that \( \mu \) is a hyper fuzzy AT-ideal of \( H \). and hence it is also a weak hyper fuzzy AT-ideal of \( H \). But \( \mu \) is not a strong hyper fuzzy AT-ideal of \( H \) since

\[
\min \{ \sup_{x \in \{0,1,2\}} \mu(a), \mu(y) \} \geq \min \{ \mu(1), \mu(1) \} = 1/2 \geq 0 = \mu(2), \forall a \in \{0 \circ (1 \circ 2) \}.
\]

**6. A Homomorphism on hyper AT-algebra**

In this section, we study the homomorphic images and inverse images of (weak, strong, s-weak) hyper fuzzy AT-ideals become (weak, strong, s-weak) hyper fuzzy AT-ideals of hyper AT-algebra is studied as well.

**Proposition 6.1.**

Let \( f: (H; \circ, 0) \rightarrow (H'; \circ', 0') \) be a hyper homomorphism of hyper AT-algebras. If \( \mu \) is a hyper fuzzy ideal of \( H' \), then \( f^{-1}(\mu) \) is a hyper fuzzy ideal of \( H \).

**Proof:**

Clearly \( \mu(0') \geq \mu(x') \), then \( f^{-1}(\mu)(0) \geq f^{-1}(\mu)(x) \). Let \( x', y' \in H' \) such that \( x, y \in H, f(x) = x' \) and \( f(y) = y' \) be such that \( x << y \) implies \( \mu(y') \geq \mu(x') \), then

\[
f^{-1}(\mu)(y) \geq \mu(0') \geq \mu(x') \geq \mu(y') \geq f^{-1}(\mu)(y).
\]

Then we can see that \( \mu \) is a hyper fuzzy AT-ideal of \( H' \). But \( \mu \) is not a strong hyper fuzzy AT-ideal of \( H \) since

\[
\min \{ \sup_{x \in \{0,1,2\}} \mu(a), \mu(y) \} \geq \min \{ \mu(1), \mu(1) \} = 1/2 \geq 0 = \mu(2), \forall a \in \{0 \circ (1 \circ 2) \}.
\]

**Corollary 6.2.**

Let \( f: (H; \circ, 0) \rightarrow (H'; \circ', 0') \) be a hyper homomorphism of hyper AT-algebras. If \( \mu \) is a (weak, strong, s-weak) hyper fuzzy ideal of \( H' \), then \( f^{-1}(\mu) \) is a (weak, strong, s-weak) hyper fuzzy ideal of \( H \).

**Proof:**

By Proposition (6.1) and Proposition (4.13).
\( x \ll y \) implies \( f^{-1}(\mu)(y) \geq f^{-1}(\mu)(x) \). And
\( u \in x^0 (y^0 z) \ll f^i(\mu) \) and \( y \in f^i(\mu) \), then \( f(u) \in \mu \) and \( f(y) \in \mu \), where \( u \in x^0 (y^0 z) \). But \( \mu \) is hyper fuzzy AT-ideal of \( H' \), then \( \mu(x^0 z^0) \geq \min \{ \inf_{u \in (x^0 (y^0 z))} \mu(u'), \mu(y') \} \), implies that
\( f^{-1}(\mu)((x \circ z)) \geq \min \{ \inf_{z \in (x \circ z)} f^{-1}(\mu)(u), f^{-1}(\mu)(y) \} \). By Definition (4.9(1)). Hence \( f^i(\mu) \) is a hyper fuzzy AT-ideal of \( H' \). ■

**Corollary 6.4.**
Let \( f : (H; \circ, 0) \rightarrow (H'; \circ', 0') \) be a hyper homomorphism of hyper AT-algebras. If \( \mu \) is a (weak, strong, s-weak) hyper fuzzy AT-ideal of \( H' \), then \( f^i(\mu) \) is a (weak, strong, s-weak) hyper fuzzy AT-ideal of \( H' \).

**Proof:**
By Proposition (6.3) and Proposition (4.13). ■

**Proposition 6.5.**
Let \( f : (H; \circ, 0) \rightarrow (H'; \circ', 0') \) be an onto hyper homomorphism of hyper AT-algebras. If \( \mu \) is a hyper fuzzy ideal of \( H \) satisfying inf property, then \( f(\mu) \) is a hyper fuzzy ideal of \( H' \).

**Proof:**
Clearly \( \mu(0') \geq \mu(x') \), then \( f(\mu)(0') \geq f(\mu)(x') \). Let \( x', y', z', u' \in H' \) such that \( x, y, u \in H \), \( f(x) = x' \) and \( f(y) = y' \) be such that \( x' \ll y' \) implies \( \mu(y') \geq \mu(x') \), then
\( x \ll y \) implies \( f(\mu)(y) \geq f(\mu)(x) \). And \( u \in 0^0 (x^0 y) = x^0 y \ll f(\mu) \) and \( x \in f(\mu) \), then
\( f^i(\mu) \) is a hyper fuzzy AT-ideal of \( H' \), then \( \mu(y) \geq \min \{ \inf_{u \in (x \circ y)} \mu(u), \mu(x) \} \), implies that \( f(\mu)(y) \geq \min \{ \inf_{u \in (x \circ y)} f(\mu)(u), f(\mu)(x) \} \). By Definition (4.10(1)). Hence \( f(\mu) \) is a hyper fuzzy ideal of \( H' \). ■

**Corollary 6.6.**
Let \( f : (H; \circ, 0) \rightarrow (H'; \circ', 0') \) be a hyper homomorphism of hyper AT-algebras. If \( \mu \) is a (weak, strong, s-weak) hyper fuzzy AT-ideal of \( H' \), then \( f^i(\mu) \) is a (weak, strong, s-weak) hyper fuzzy AT-ideal of \( H' \).

**Proof:**
By Proposition (6.5) and Proposition (4.13). ■

**Proposition 6.7.**
Let \( f : (H; \circ, 0) \rightarrow (H'; \circ', 0') \) be an onto hyper homomorphism of hyper AT-algebras. If \( \mu \) is a hyper fuzzy AT-ideal of \( H \) with inf property, then \( f(\mu) \) is a hyper fuzzy AT-ideal of \( H' \).

**Proof:**
Clearly \( \mu(0') \geq \mu(x') \), then \( f(\mu)(0') \geq f(\mu)(x') \). Let \( x', y', z', u' \in H' \) such that \( x, y, z, u \in H \), \( f(x) = x' \) and \( f(y) = y' \) be such that \( x' \ll y' \) implies \( \mu(y') \geq \mu(x') \), then
\( x \ll y \) implies \( f(\mu)(y) \geq f(\mu)(x) \). And \( u \in x^0 (y^0 z) \ll f(\mu) \) and \( y \in f(\mu) \), then \( f^i(\mu) \) is a hyper fuzzy AT-ideal of \( H' \), then
\( f(\mu)(y) \geq \min \{ \inf_{u \in (x \circ y)} \mu(u), \mu(x) \} \), implies that \( f(\mu)(y) \geq \min \{ \inf_{u \in (x \circ y)} f(\mu)(u), f(\mu)(x) \} \). By Definition (4.10(1)). Hence \( f(\mu) \) is a hyper fuzzy AT-ideal of \( H' \). ■

**Corollary 6.8.**
Let \( f : (H; \circ, 0) \rightarrow (H'; \circ', 0') \) be a hyper homomorphism of hyper AT-algebras. If \( \mu \) is a (weak, strong, s-weak) hyper fuzzy ideal of \( H' \), then \( f(\mu) \) is a (weak, strong, s-weak) hyper fuzzy ideal of \( H' \).

**Proof:**
By Proposition (6.7) and Proposition (4.13). ■

**Theorem 6.9.**
Let \( f : (H; \circ, 0) \to (H'; \circ', 0') \) and \( g : (H; \circ, 0) \to (H''; \circ'', 0'') \) be two homomorphisms of hyper AT-algebras such that \( f \) is onto and \( \ker(f) \subseteq \ker(g) \), then there exists a homomorphism \( h : (H'; \circ', 0') \to (H''; \circ'', 0'') \) such that \( h \circ f = g \).

**Proof:**

Let \( y \in H' \). Since \( f \) is onto, there exists \( x \in H \) such that \( y = f(x) \). Define \( h : H' \rightarrow H'' \) by \( h(y) = g(x) \), for all \( y \in H' \).

Now, we show that \( h \) is well-defined. Let \( y_1, y_2 \in H' \) and \( y_1 = y_2 \). Since \( f \) is onto, then there are \( x_1, x_2 \in H \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \).

Hence \( f(x_1) = f(x_2) \) and \( 0 \in f(x_1) \circ f(x_2) = f(x_1 \circ x_2) \).

It follows that there exists \( z \in x_1 \circ x_2 \) such that \( f(z) = 0 \). Thus \( z \in \ker(f) \subseteq \ker(g) \) and \( g(z) = 0 \). Since \( z \in x_1 \circ x_2 \), then \( 0 = g(z) = g(x_1 \circ x_2) \).

On the other hand since \( 0 \in f(x_1) \circ f(x_2) = f(x_1 \circ x_2) \), similarly we can conclude that \( 0 \in g(x_2 \circ x_1) = g(x_1 \circ x_2) \).

Thus \( g(x_1) \ll g(x_2) \), which shows that \( h \) is well-defined. Clearly \( h \circ f = g \).

Finally, we show that \( h \) is a homomorphism. Let \( y_1, y_2 \in H' \). Since \( f \) is onto, there are \( x_1, x_2 \in H \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \), then

\[
\begin{align*}
\text{h}(y_1 \circ y_2) &= h(f(x_1) \circ f(x_2)) = h(f(x_1 \circ x_2)) \\
&= (h \circ f)(x_1 \circ x_2) = g(x_1 \circ x_2) = g(x_1) \circ^" g(x_2) \\
&= (h \circ f)(x_1) \circ^" (h \circ f)(x_2) \\
&= h(f(x_1)) \circ^" h(f(x_2)) = h(y_1) \circ^" h(y_2).
\end{align*}
\]

Moreover, since \( f(0) = 0 \) and \( g(0) = 0 \), then \( h(0) = h(f(0)) = (h \circ f)(0) = g(0) = 0 \).

Thus, \( h \) is a homomorphism. \( \Box \)

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