On current algebras, generalised fluxes and non-geometry

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Abstract

A Hamiltonian formulation of the classical world-sheet theory in a generic, geometric or non-geometric, NSNS background is proposed. The essence of this formulation is a deformed current algebra, which is solely characterised by the generalised fluxes describing such a background. The construction extends to backgrounds for which there is no Lagrangian description – namely magnetically charged backgrounds or those violating the strong constraint of double field theory – at the cost of violating the Jacobi identity of the current algebra.

The known non-commutative and non-associative interpretation of non-geometric flux backgrounds is reproduced by means of the deformed current algebra. Furthermore, the provided framework is used to suggest a generalisation of Poisson-Lie T-duality to generic models with constant generalised fluxes. As a side note, the relation between Lie and Courant algebroid structures of the string current algebra is clarified.
1 Introduction

Non-linear $\sigma$-models [1] have been of great importance in particle physics and gravity. In particular two dimensional ones play an important role in the study of integrable models and string theory. In the latter they are descriptions of a string in a curved target space. The Lagrangian of such a $\sigma$-model is characterised by a metric $G$ and a 2-form gauge field $B$ on the target space.$^1$ Studying the physical properties of these models reveals geometrical structure. Two prominent examples are the equations of motion and the 1-loop $\beta$-functions. Former is given by the geodesic equation to a torsionful connection – the sum of the Levi-Cività connection to the metric $G$ and a torsion term determined by the $H$-flux, $H = dB$. The 1-loop $\beta$-function to the coupling on the other hand is given by the (generalised) Ricci tensor to this torsionful connection [2–4].

$^1$We are be interested in the classical properties of the model and ignored the term proportional to the world-sheet Ricci scalar containing the dilaton, as it is not relevant to the classical (world-sheet) theory.
As long as the background is globally geometric – meaning only diffeomorphisms and B-field gauge transformations are necessary for gluing coordinate patches – metric and B-field are globally well-defined and seem to be an appropriate description of the background. But not all backgrounds in string theory can be described as such. So called non-geometric backgrounds have been shown to arise naturally as T-duals of geometric backgrounds \cite{5}. The ones we consider here can be understood as T-folds \cite{6–8}, meaning that we allow for patching with T-duality transformations as well. They are expected to make up a big part of the landscape of string theory \cite{9–13}, this includes not only duals of geometric backgrounds but also genuinely non-geometric backgrounds. These backgrounds can be described in terms of generalised geometry \cite{14–16} or the generalised fluxes. These fluxes arise as parameters in gauged supergravities \cite{17,18}, are the basis of a formulation of double field theory \cite{13,19–23} and have been shown to be related to the non-commutative and non-associative interpretations of these backgrounds \cite{24–31}.

The aim of this article is to present a convenient formulation of the world-sheet theory which highlights the role of these generalised fluxes, making the non-geometric features more apparent than the not generally globally defined Lagrangian data $G$ and $B$. The key result of this article is that a Hamiltonian description in terms of non-canonical coordinates on the string phase space achieves this objective. This description consists of a ‘free’ Hamiltonian. All the physical information about the background is encoded in a deformation of the Poisson structure

$$\Pi^{\text{def}} = \Pi^\eta + \Pi^\text{bdy} + \Pi^\text{flux}. \quad (1.1)$$

The canonical Poisson structure consists of an $O(d,d)$-invariant part $\Pi^\eta$ and a boundary contribution $\Pi^\text{bdy}$, relevant for open strings and winding along compact directions. $\Pi^\text{flux}$ is characterised exactly by the generalised fluxes. This perspective was already studied in \cite{32} for geometrical H-flux backgrounds. On the other hand non-geometric fluxes were already introduced as generalised WZ-terms in first order Lagrangians \cite{33,34}, but only for a certain choice of generalised vielbein. Other perspectives on the connection of $\sigma$-models, current algebras and generalised geometry include \cite{35–40}.

Before we state the main results and outline the structure of the paper, let us motivate the approach to our paper in two ways. The first one is a review of the Hamiltonian description of an electrically charged point particle in a magnetically charged background in electromagnetism. The second point is a collection of examples from the integrable models literature. Both points share the feature of a possible description by a free Hamiltonian and a deformed Poisson resp. symplectic structure.

### 1.1 Point particle in an electromagnetic background

As a motivational example, that shares many features with the string in NSNS backgrounds, let us consider a relativistic point particle with mass $m$ and electric charge $q$ in an arbitrary electromagnetic field \cite{41,42}. At first we define by an electric potential $A = A_\mu dx^\mu$ with field strength $F = dA = F_{\mu\nu}dx^\mu \wedge dx^\nu$. A convenient choice\footnote{The free Hamiltonian $H_{\text{free}} = \frac{p^2}{2m}$ with 4-momentum $p$ is obtained via a Polyakov trick with the einbein $e$ so that $H_{\text{free}}$ is indeed the constraint corresponding to time reparameterisation invariance in this case. After gauge choice $e = 1$ and minimal substitution we are left with above Hamiltonian.} of Hamiltonian, $H = \frac{1}{2m}(p - qA)^2$, together with the canonical Poisson structure gives the
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equations of motion

\[ \dot{x}^\mu = \frac{1}{m} \pi^\mu = \frac{1}{m} (p^\mu - q A^\mu) \quad \text{and} \quad \dot{\pi}_\mu = \frac{q}{m} F_{\mu\nu} \pi^\nu. \]  

(1.2)

Alternatively this problem can be phrased in terms of new coordinates on the phase space \((x^i, \pi_i)\) with the kinematic momentum \(\pi_\mu\). Let us note a few important characteristics of this formulation, which will also be key points in the string discussion:

- **Preferred non-canonical phase space coordinates.** In terms of kinematic momentum \(\pi^\mu\) the Hamiltonian is \(H = \frac{\pi_\mu \pi^\mu}{2m}\), so we have a 'free' Hamiltonian. All background data - the coupling to the electromagnetic field - is encoded in the deformed Poisson brackets

\[
\{x^\mu, x^\nu\} = 0, \quad \{x^\mu, \pi_\nu\} = \delta^\mu_\nu, \quad \{\pi_\mu, \pi_\nu\} = q F_{\mu\nu},
\]

(1.3)

resp. a the deformed symplectic structure \(\omega = \omega_0 + qF\). The Jacobi identity of the Poisson bracket resp. the closedness of \(\omega\) is equivalent to the Bianchi identity in the standard Maxwell equations:

\[ d\omega = 0 \iff dF = 0. \]  

(1.4)

The field equations for \(F\) can also be phrased conveniently in terms of the symplectic structure: \(\partial^\mu \omega_{\mu\nu} = 4\pi j^{(e)}\).

- **Generalisation to magnetically charged backgrounds.** In this formulation there is no need to refer to the potential \(A\), it is phrased only in terms of the field strength \(F\). So it is well suited for generalisations to magnetically charged backgrounds with \(\ast dF = 4\pi j^{(m)}\).

Alternatively one could take another point of view, namely to consider this as a free particle in non-commutative or, in case \(dF \neq 0\), even non-associative momentum space. This fact is basis for a example and toy model for the treatment of non-associative phase spaces [43–46]. Recently it has been shown that such a non-associative, or almost symplectic, phase space can be realised in a higher dimensional symplectic one [47].

- **Charge algebra.** This coordinate change in phase space (a symplectomorphism in the case without magnetic sources\(^3\)) is simply the local field redefinition from canonical to kinematic momenta

\[ \omega = -d\theta = d(p_\mu dx^\mu) = d(\pi_\mu + q A_\mu) \wedge dx^\mu = d \pi_\mu \wedge dx^\mu + qF. \]  

(1.5)

As a consequence the Poisson algebra of charges (functions on the phase space) does not change.

\(^3\)To make this problem symmetric in electric and magnetic terms we could consider a dyon \((q, g)\) in an electromagnetic background \(F\), e.g. a particle with Lorentz force \(\pi_\mu = \frac{1}{2} (q F_{\mu\nu} + g \tilde{F}_{\mu\nu}) \pi^\nu\), thus corresponding to the deformed symplectic structure is \(\omega = \omega_0 + qF + g \ast F\), which is not symplectic anymore, as soon as we have any electric or magnetic sources for \(F\). For the dyon then there is no (local) field redefinition anymore connecting the two formulations.
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1.2 Integrable models and deformations of current algebras

The principal chiral model, the theory of the embedding of a classical string into a group manifold \( G \), is one of the most important toy models for the study of integrable \( \sigma \)-models. It can be defined by a Hamiltonian

\[
H = \frac{1}{2} \int d\sigma \left( \kappa^{ab} j_{0,a} j_{0,b} + \kappa_{ab} j_{1,a} j_{1,b} \right)
\]  

and the following Poisson structure, the current algebra, 

\[
\begin{align*}
\{ j_{0,a}(\sigma), j_{0,b}(\sigma') \} &= -f^c_{ab} j_{0,c}(\sigma) \delta(\sigma - \sigma') \\
\{ j_{0,a}(\sigma), j_{1,b}^{\sigma}(\sigma') \} &= -f^{b}_{ca} j_{1}^{a}(\sigma) \delta(\sigma - \sigma') - \delta^{b}_{c} \partial_{\sigma} \delta(\sigma - \sigma') \\
\{ j_{1}^{a}(\sigma), j_{1}^{b}(\sigma') \} &= 0.
\end{align*}
\]  

where we chose some coordinates \( x \) on \( G \).

The principal chiral model possesses many deformations which preserve one its most interesting properties: its classical integrability. Although integrability will not be the main focus of this paper, one detail of these integrable deformations motivates our approach - the deformations can be understood as deformations of the current algebra (1.7) instead of the deformation of a Hamiltonian or Lagrangian.

- The introduction of a WZ-term in the Lagrangian can be accounted for by a change of the \( j_{0} j_{0} \) Poisson bracket in comparison to (1.7)

\[
\{ j_{0,a}(\sigma), j_{0,b}(\sigma') \}_{\text{WZW}} = -f^c_{ab} j_{0,c}(\sigma) + k f_{abc} j_{1}^{a}(\sigma) \delta(\sigma - \sigma').
\]  

Classically \( k \) can be considered as a deformation parameter. See for example the standard textbook [48] for more details on the Hamiltonian treatment of the WZW-model.

- The \( \sigma \)-model Lagrangian of the \( \eta \)-deformation was discovered in [49, 50] and its target space interpretation as a \( q \)-deformation of the original group manifold was given in [51]. It can also be represented by a modification of the current algebra, as such it arose already in [52]. Compared to (1.7) the Poisson bracket between the \( j_{1} \) is:

\[
\{ j_{1,a}^{\sigma}(\sigma), j_{1,b}^{\sigma}(\sigma') \}_{\eta} = \eta^2 \left( \frac{\eta^2}{1 - \eta^2} f_{abc} j_{0,c}(\sigma) \delta(\sigma - \sigma') \right).
\]
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• The $\lambda$-deformation was introduced directly in terms of a deformation of the current algebra, originally for $G = SU(2)$ in [53] and later generalised to arbitrary groups in [54], accompanied with a Lagrangian derivation. It can be completed to supergravity solutions, corresponds to certain $q$-deformations of the original group and has been argued to be equivalent via Poisson-Lie $T$-duality and analytic continuation of the deformation parameter $\eta \leftrightarrow \pm i \lambda$ to the $\eta$-deformation [55–58].

Again after some rescaling of the currents compared to the original articles the $\lambda$-deformation corresponds only to a change in the $j_1$-currents compared to the original articles the $\lambda$-deformation corresponds only to a change in the $j_1$-Poisson bracket:

$$\{ f^1_\lambda(\sigma), f^\delta_1(\sigma') \}_\lambda = -\frac{\lambda^2}{1 + \lambda^2} f^{abc} j_{0,c}(\sigma) \delta(\sigma - \sigma'). \quad (1.10)$$

Phrased like this in the Hamiltonian formalism and compared to (1.9), we see directly that $\lambda$- and $\eta$-deformations are equivalent via analytic continuation $\eta \leftrightarrow \pm i \lambda$.

With this short survey we have motivated that in the Hamilton formulation deformations of the current algebra are a convenient playground. In fact we will see that every bosonic string $\sigma$-model can be represented by the free Hamiltonian and a modified current algebra.

A related discussion of the $SU(2)$ principal chiral model aimed on the features connected the generalised geometry can be found in [59].

1.3 Main results and overview

Strings in arbitrary $\sigma$-model backgrounds  Let us define the Hamiltonian theory of a string in an NSNS background characterised by the (geometric and globally non-geometric) generalised fluxes $F_{ABC}$. In terms of phase space variables $E_A(\sigma)$ the Hamiltonian takes the form of a ‘free’ Hamiltonian

$$H = \frac{1}{2} \oint d\sigma \delta^{AB} E_A(\sigma) E_B(\sigma), \quad (1.11)$$

whereas the background data, namely the generalised fluxes $F_{ABC}$ is encoded in the deformed current algebra

$$\{ E_A(\sigma_1), E_B(\sigma_2) \} = \frac{1}{2} \eta^{AB} (\partial_1 - \partial_2) \delta(\sigma_1 - \sigma_2) - F^C_{AB}(\sigma) E_C(\sigma) \delta(\sigma_1 - \sigma_2)
+ \text{boundary term}, \quad (1.12)$$

where $\eta$ is the $O(d,d)$ metric. The $E_A$ are a priori abstract, but from a Lagrangian perspective they are connected to Darboux coordinates ($x^i, p_i$) via $E_A = E_A^I E_I = E_A^I (p_i, \partial x^i)$. $E_A^I$ is a generalised vielbein. The Hamiltonian equations of motion take the convenient form of a Maurer-Cartan equation pulled back to the world-sheet.

$$d\mathcal{E}^A + \frac{1}{2} F_{ABC} \mathcal{E}^B \wedge \mathcal{E}^C = 0 \quad \text{and} \quad \mathcal{E}_A = \delta_{AB} \ast \mathcal{E}^B. \quad (1.13)$$

The connection to a $\sigma$-model Lagrangian respectively a choice of ‘Darboux coordinates’ is given by a generalised vielbein $E_A^I$, s.t.

$$F_{ABC} = \left( D_{[A} E_{B]}^I \right) E_{C]|I. \quad (1.14)$$
Such a frame exists if the Jacobi identity of the current algebra (1.12), which is equivalent to the Bianchi identity of generalised fluxes

\[ \partial[A F_{BCD}] - \frac{3}{4} F^{E} [A B F_{CD}]_{E} = 0, \]  

(1.15)
is fulfilled.

**Brackets on the phase space** The unspecified boundary term in (1.12) stems from the second term in canonical current algebra

\[ \{ E_{I}(\sigma_{1}), E_{I}(\sigma_{2}) \} = \frac{1}{2} \eta_{IJ} (\partial_{1} - \partial_{2}) \delta(\sigma_{2} - \sigma_{1}) + \frac{1}{2} \omega_{IJ} \int d\sigma \delta(\sigma - \sigma_{1}) \delta(\sigma - \sigma_{2}), \]  

(1.16)
phrased in an O\((d, d)\)-covariant way. Here we have that

\[ E_{I}(\sigma) = (p_{I}(\sigma), \partial x^{I}(\sigma)) \quad \text{and} \quad \omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

For the open string it gives a boundary contribution but also for closed strings it might give a contribution, e.g. from winding along compact directions. Of the three properties – skew-symmetry, Jacobi identity and O\((d, d)\)-invariance – Lie, Courant and Dorfman bracket satisfy two each and the third one up to such a total derivative term under the \(\sigma\)-integral. E.g. the above form (1.16) is the Lie bracket on sections of \((T \oplus T^{*})LM\), where \(M\) is the target space and \(LM\) denotes the configuration space of the string in \(M\). Without the second term (1.16) would be a Courant bracket, which is O\((d, d)\)-invariant and skew-symmetric but violates the Jacobi identity by such a total derivative term under the \(\sigma\)-integral.

This second term has been neglected in previous literature but becomes crucial for the non-geometric interpretation of the current algebra. For example only when it is considered the current algebra of the locally geometric pure \(Q\)-flux background is associative as expected. This is shown in section 5.

**Generalisations to magnetically charged and double field theory backgrounds**

The world-sheet theory as a Lagrangian \(\sigma\)-model is only defined in ‘electric’ backgrounds. I.e. these are those that fulfil (1.15), and are locally geometric.

Instead the Hamiltonian formulation in the generalised flux frame extend straightforwardly to magnetically charged and locally non-geometric backgrounds. If we have a magnetically charged background the Bianchi identity of generalised fluxes (1.15) is not fulfilled. This means we cannot find a generalised vielbein that will connect the deformed current algebra (1.12) to the canonical one (1.16). Analogously to the case of the point particle in an magnetic monopole background the violation of the Bianchi identity corresponds to a violation of the Jacobi identity of the current algebra.

In order to study double field theory we do not need to double the phase space. The dual field \(\tilde{x}\) is not independent and given by \(p_{I}(\sigma) = \partial x^{I}(\sigma)\). Allowing for a dependence of the generalised vielbein on the original as well as the dual coordinates we will get an additional strong constraint violating term in the deformed current algebra (1.12). This term will be non-local in general. The full current algebra including this term will still satisfy the Jacobi identity. This might be surprising. Quite analogously to the boundary
term discussed above, we will get a violation of the Jacobi identity corresponding to violation of the Bianchi identity (1.15) if we neglect that strong constraint violating term. So as in the original discussion of the generalised flux formulation of double field theory, such background might be suited to describe magnetically charged backgrounds via a generalised vielbein. See section 4.3 for more details.

Non-geometric interpretation Let us summarise the different local forms of canonical (Poisson) brackets, which we will discuss in that article.

current bracket : \( \{ E_i(\sigma_1), E_j(\sigma_2) \} = \frac{1}{2} \eta_{ij} (\partial_1 - \partial_2) \delta(\sigma_2 - \sigma_1) + \frac{1}{2} \omega_{ij} \int d\sigma \partial(\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)) \)

Poisson bracket : \( \{ x^i(\sigma_1), p_j(\sigma_2) \} = \delta_i^j \delta(\sigma_1 - \sigma_2) = 0 \) (1.17)

DFT bracket : \( \{ X^I(\sigma_1), X^J(\sigma_2) \} = \eta^{IJ} \bar{\Theta}(\sigma_1 - \sigma_2) \) with \( X^I = (\tilde{x}^i, x^i) \) and \( \bar{\Theta}(\sigma) = \frac{1}{2} \text{sign}(\sigma) \)

Going to the generalised flux frame is very convenient in the current algebra. Given such a deformed current algebra, we decompose \( E_A \) to \((e_0, a(\sigma)), e_1(\sigma))\) and define \( \partial y^a = e_1^a \). These coordinates \( y^a \) are the ones which show a potentially non-geometric, e.g. non-commutative or non-associative, behaviour. We obtain their Poisson brackets simply by integrating the deformed current algebra (1.16).

This is in contrast to many previous derivations of the non-geometric nature of the backgrounds which relied on finding mode expansions first. In section 5 we show that we reproduce the known results on open strings in a constant \( B \)-field background and closed strings in a constant \( Q \)-flux background.

Classical generalised T-dualities The framework easily realises abelian T-duality. For Poisson-Lie T-dualisable resp. the \( E \)-models \([57,60]\) the current algebra is exactly of the kind (1.12) with

\[ F^c_{ab} = f^c_{ab}, \quad F_c^{ab} = \tilde{f}^c_{ab} \quad \text{and} \quad F_{abc} = F^{abc} = 0, \]

where the constants \( f^c_{ab} \) and \( \tilde{f}^c_{ab} \) are structure constants to a Lie bialgebra \([61,62]\). The duality transformations are linearly realised in that basis. We show that for certain parameterisations of \( F_{ABC} \) there exists an extension of Poisson-Lie T-duality, which we call Roytenberg duality. It nevertheless relies on the same trick as Poisson-Lie T-duality, namely that a Poisson-bivector on a group manifold realises a constant generalised flux background.

We discuss the distinction between canonical transformation and these (generalised) T-dualities. The generating functions for these can be implicitly defined by

\[ \mathcal{F}_{\Omega_{AB}} = -\eta^{AB} \Omega_{AB} \]

where the \( \Omega_{AB} \) are \( o(d,d) \)-charges on the phase space fulfilling \( \{ \Omega_{[AB]}, E_C \} = \eta_{C[AB]} E_{B]} \).

Structure of the paper Section 2.1 sets conventions and aims to clarify our interpretation of the generalised flux backgrounds. The review of the string in an \( H \)-flux background in section 2.2 sets the basis for further discussion and introduces the \( H \)-flux
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as a twist of symplectic structure. Based on observations in that section we distinguish the algebraic structures of the phase space in section 3.

The central result – the formulation of string theory in an arbitrary generalised flux background by a free Hamiltonian but a deformed current algebra – is derived in section 4.1. This includes a discussion on the general form of the equations of motion, the Virasoro constraints, the boundary term and the generalisation to magnetically charged backgrounds. In section 4.2 and 4.3 the investigation on (generalised) T-dualities and motivated by this a brief extension of the previous discussion to double field theory backgrounds follow.

In section 5 we propose a direct non-geometric interpretation of the deformed current algebra and confirm that it reproduces the standard results of an open string in a constant $B$-field backgrounds and of a closed string in a constant $Q$-flux background. We close with some outlooks and potential further directions.

2 Preliminaries

2.1 Generalised fluxes and non-geometric backgrounds

In this section we collect and review well-known material about generalised geometry and generalised fluxes in order to clarify our conventions and prepare later discussions. We refer to standard reviews of double field theory [13, 19–22], generalised geometry [15, 16] and the generalised flux formulation [23] for more details.

**Generalised metric and generalised vielbeins** We define $O(d, d)$-transformations to be $2d \times 2d$-matrices, which leave the $O(d, d)$-metric

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(2.1)

invariant. The action of an $M \in O(d, d)$ on a (bosonic) string background, characterised by a metric $G$ and a 2-form gauge field $B$, or equivalently by the generalised metric

$$\mathcal{H}(G, B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix},$$

(2.2)

is given by $\mathcal{H}(G', B') = M\mathcal{H}(G, B)M^T$. A **generalised vielbein** or frame $E_A^I(x)$ is defined to be any (local) $O(d, d)$-transformation in the component connected to the identity that diagonalises and trivialises the generalised metric, i.e.

$$E_A^I E_B^J \eta_{IJ} = \eta_{AB} \quad \text{and} \quad E_A^I E_B^J \mathcal{H}_{IJ} = \gamma_{AB} := \begin{pmatrix} \gamma_{ab} & 0 \\ 0 & \gamma_{ab} \end{pmatrix},$$

(2.3)

where $\gamma$ is a flat metric in the signature of the target space and is used to raise and lower indices $a, b, \ldots = 1, \ldots, d$. Indices $A, B, \ldots = 1, \ldots, 2d$ denote the ‘flat’ indices and are raised and lowered by $\eta_{AB}$.

Unless stated otherwise we will assume that the generalised vielbeins are (local) functions on the original target space with coordinates $x^i$. With this assumption we restrict to **locally geometric** backgrounds, but below and in section 4.3 we will also discuss the generalisation to **locally non-geometric** backgrounds.
All generalised vielbeins can be generated by successively performing

\[ B\text{-shifts: } E^{(B)} = \left( \begin{array}{cc} 1 & B \\ 0 & 1 \end{array} \right), \quad GL\text{-transformations } E^{(e)} = \left( \begin{array}{cc} e & 0 \\ 0 & (e^{-1})^T \end{array} \right) \]

\[ \beta\text{-shifts: } E^{(\beta)} = \left( \begin{array}{cc} 1 & 0 \\ \beta & 1 \end{array} \right), \quad \text{factorised dualities: } E^{(T)} = \left( \begin{array}{ccc} 1 & -\delta_i & \delta_i \\ -\delta_i & 1 & -\delta_i \\ \delta_i & -\delta_i & 1 \end{array} \right) \]

for skewsymmetric \( d \times d \)-matrices \( B \) and \( \beta \), an invertible matrix \( e \) (a \( d \)-dimensional vielbein) and \( (\delta_i)_{jk} = \delta_{ij}\delta_{ik} \).

**Weitzenböck connection and generalised fluxes** The generalised Weitzenböck connection of such a generalised flux frame is defined by

\[ \Omega_{C,AB} = D_C E_A \Omega_{EBI} \quad \text{with} \quad \partial_A := E_A^I \partial_I, \]

fulfilling \( \Omega_{C,AB} = -\Omega_{C,BA} \) due to (2.3). \( \partial_I = (\partial_i, \partial^i) \), where \( \partial^i \) denotes the derivative w.r.t. to dual coordinates \( \tilde{x}_i \), which vanishes for locally geometric backgrounds. In fact only the totally skewsymmetric combination\(^4\) will be relevant for us: the generalised flux

\[ F_{ABC} := \Omega_{[C,AB]} = \left( D_A E_B^I \right) E_{CJ}. \]

It includes the four fluxes – \( H, f, Q \) and \( R \) – for different choices of the indices on the \( F_{ABC} \)

\[ H_{abc} \equiv F_{abc}, \quad f^c_{ab} \equiv F^c_{ab} = F_{b a}^c = F_{ab}^c \]

\[ R^{abc} \equiv F^{abc}, \quad Q_{c}^{ab} \equiv F_{c}^{ab} = F_{a}^{bc} = F_{c}^{ab} \]

In a generalised flux frame (2.3) all the information about the background is stored inside the generalised fluxes, instead of the generalised metric. The generalised metric will be trivial in that frame.

**Bianchi identities** Generalised fluxes, given as above in terms of a generalised vielbein, cannot be chosen arbitrarily but have to fulfil the (dynamical) Bianchi identity \([23, 63–65]\)

\[ \partial_A F_{BCD} - \frac{3}{4} F^E_{[AB} F_{CD]E} = 0, \quad (2.7) \]

or in the decomposition into the \( d \)-dimensional fluxes

\[
\begin{align*}
0 &= \partial_{[a} H_{b]cd} - \frac{3}{2} H_{k[ab} f^k_{cd]} = (dH)_{abcd} \\
0 &= \partial^a H_{bcd} + \partial_{[b} f^c_{cd]} - f^k_{[a} f^k_{bc]} - H_{k(bc Q_d)_{ak}} \\
0 &= \partial_{[a} f^c_{bd]} + \partial_{[a} Q_{bd]} - f^k_{[a} Q_{bc]} + f^k_{[a} Q_{bd]} b_{dk} - H_{abk} R_{cd} \\
0 &= \partial_{[a} R_{bcd]} + \partial_{[a} Q_{bcd]} - Q_{a}^{k[a} Q_{bc]} + R_{kbc}^k_{d[a} - R_{kbc}^k_{d[a} \\
0 &= \partial_{[a} R_{bcd]} + \frac{3}{2} R_{k[a} Q_{bc]}.
\end{align*}
\]

\(^4\)We use the conventions:

\[ v_{[a w_b v_c w_d] = v_a v_b v_c v_d + \text{cyclic perm.} \]

\[ u_{[a v_b w_c z_d] = u_a v_b w_c z_d + (-1)^{\text{sign}} \times \text{all permutations} \]
If the fluxes violate this condition they cannot be written in terms of a generalised vielbein via (2.6). In the following we call the corresponding backgrounds \textit{magnetically charged}.

**The locally geometric T-duality chain and the non-geometric fluxes**  The starting point in the T-duality chain is the flat torus with \( h \) units of H-flux, i.e.

\[ H = h dx^1 \wedge dx^2 \wedge dx^3. \]  

(2.9)

A choice of B-field for this H-flux is \( B = x^3 dx^1 \wedge dx^2 \), such that the two commuting isometries of the background are manifest. After a T-duality along the isometry \( x^1 \) the Buscher rules \cite{66} produce a pure metric background. This background turns out to be parallelisable, e.g. there is a globally defined frame field \( e_a^i \). The only non-vanishing component of the generalised flux (2.6) is

\[ f_{123}^i = h \quad \text{with} \quad f_{ab} = e_j^i \epsilon_{[a} \partial_i e_{b]} \]  

(2.10)

The interpretation of the locally geometric pure f-flux is that it is the totally skewsymmetric combination of the spin connection of a \( d \)-dimensional vielbein.

Performing a second T-duality along \( x_2 \) we arrive at the background

\[ G = \frac{1}{1 + q(x^3)^2} \left( (dx^1)^2 + (dx^2)^2 \right) + (dx^3)^2, \]  

(2.11)

\[ H = - \frac{q}{(1 + q(x^3)^2)^2} (1 - q(x^3)^2) dx^1 \wedge dx^2 \wedge dx^3 \]

with identifications \( x^i \sim x^i + 1 \). At \( x^3 + 1 \sim x^3 \) it is not possible to patch geometrically. Instead we can describe this background by the generalised vielbein

\[ E(Q) = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & h x^3 \\ -h x^3 & 0 \end{pmatrix} \quad \Rightarrow \quad Q_{312} = h. \]  

(2.12)

So a constant \( \beta \)-shift by \( h dx^1 \wedge dx^2 \) can be used to patch at \( x^3 + 1 \sim x^3 \). In other words, a 3-torus solely together with a constant Q-flux is characterised by a non-trivial monodromy of \( \beta \). The background has the interpretation of a non-commutative spacetime with \( \{x^1, x^2\} \sim w^3 \), where \( w^3 \) is the winding around the \( x^3 \)-cycle.

Let us summarise this in the scheme \cite{5}

\[ H_{123} \leftrightarrow T_1, \quad f_{123}^{23} \leftrightarrow T_2, \quad Q_1^{23} \leftrightarrow T_3, \quad R^{123}. \]  

(2.13)

The first two steps can be realised via standard abelian T-duality, whereas the last step cannot because background (either described by a generalised metric or generalised vielbeine) does not possess a corresponding isometry for \( x^3 \). More details on the non-geometric interpretation of these backgrounds can be found in \cite{13,24–31}.
Local non-geometry and the locally non-geometric T-duality chain In order to allow for such T-dualities along non-isometric direction, we need to allow for the dependence on dual coordinates $\tilde{x}_i$. Derivatives with respect to them are included into $\partial_I = (\partial_i, \partial^i)$. The dependence of functions on the 2d coordinates $X^I = (x^i, \tilde{x}_i)$ is normally restricted by constraints

$$0 = \partial_I f(X) \cdot \partial^I g(X),$$

$$0 = \partial_I \partial^I f(X)$$

for all functions $f, g$. The strong constraint is typically considered a consistency condition of the gauge algebra of double field theory - as such a consistency or simplifying condition it will also appear in section 4.3 - whereas the weak constraint corresponds to the level matching condition acting as an operator. Now we can understand T-duality simply as the exchange $x^i \leftrightarrow \tilde{x}_i$.

A background is called locally non-geometric, if the generalised metric resp. the vielbein depends on the dual coordinates as well. So for example the only choice of a generalised vielbein reproducing a pure R-flux with $R^{123} = h$ is

$$E_{(R)} = \left( \begin{array}{cc} 1 & 0 \\ \beta & 1 \end{array} \right), \quad \beta = \left( \begin{array}{cc} 0 & \tilde{h} \tilde{x}^3 \\ -\tilde{h} \tilde{x}^3 & 0 \end{array} \right).$$

For such a background we cannot write down a $\sigma$-model Lagrangian in the usual fashion, as metric and $B$-field do not depend on the coordinates alone.

But we can also choose locally non-geometric generalised vielbeins for a pure $f$- or $Q$-flux background, e.g.

$$\tilde{E}_{(f)} = \left( \begin{array}{cc} 1 & B \\ 0 & 1 \end{array} \right), \quad B = \left( \begin{array}{cc} 0 & \tilde{h} \tilde{x}^3 \\ -\tilde{h} \tilde{x}^3 & 0 \end{array} \right), \quad \tilde{E}_{(Q)} = \left( \begin{array}{cc} e & 0 \\ 0 & (e^T)^{-1} \end{array} \right), \quad e = \left( \begin{array}{ccc} 1 & 0 & 0 \\ -\tilde{h} \tilde{x}^3 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

demonstrating some level of arbitrariness in our notion of local (non-)geometry as seen by a closed string. There seems to be some restriction is: it is impossible to write down a locally non-geometric generalised vielbein for a pure $H$-flux background, or locally geometric one for a pure $R$-flux background. The above examples show that local non-geometry is not restricted to pure $R$-flux backgrounds.

Examples and Lagrangians In the following we will include some explicit examples of such generalised flux frames. Besides setting conventions for later discussion, we would like to emphasise here that in our definition as components of $F_{ABC}$ the physical interpretation of the fluxes $H, f, Q, R$ is frame dependent. By this we mean in a generic generalised frame the $Q$-flux might not correspond to a monodromy of $\beta$ or closed string non-commutativity or the $R$-flux not to local non-geometry.

- Geometric frame. This is the standard frame of a Lagrangian $\sigma$-model given by a metric and a $B$-field. Only the $H$-flux and the geometric $f$ are non-vanishing

$$H_{abc} = \partial_{[a} F_{b]c} + f_{[ab} B_{c]d}$$

$$f^a_{\,bc} = f^c_{\,ab} = e^c_{\,f} e_{[a} \partial_{b]} f^f$$

$$Q_{c}^{\,ab} = 0 = R^{abc}$$

(2.17)
The corresponding vielbein is a composition of a $d$-dimensional tetrad rotation and a $B$-shift: $E = E^{(\beta)} E^{(e)}$.

- **Non-geometric frame** (resp. open string variables). $\sigma$-models like

\[
S = -\frac{1}{2} \int d^2\sigma \left( \frac{1}{\gamma - \Pi(x)} \right) e_i^a e_j^b \partial_+ x^i \partial_- x^j. \tag{2.18}\]

are described by the vielbein $E = E^{(\beta)} E^{(e)}$, with $E^{(\beta)}$ denoting a $\beta$-shift by a bivector $\Pi$. This results in the generalised fluxes

\[
\begin{align*}
H_{abc} &= 0, \\
\mathbf{f}_{ab}^{\ \ c} &= f_{ab}^{\ \ c}, \\
Q_a^{\ ab} &= Q_c^{\ abc} = \partial_c \Pi^{ab} + f_d^{\ [a}_{dc} \Pi^{b]\ ]d, \\
R^{abc} &= R^{abc} = \Pi^{[a}_{de} \partial_d \Pi^{bc]} + f_f^{\ [a}_{de} \Pi^{b]d} \Pi^{c]e}. 
\end{align*} \tag{2.19}\]

This is an important class of backgrounds as this parameterisation is relevant for open strings in NSNS-backgrounds. Also non-abelian $T$-duals, Poisson-Lie $\sigma$-models are of this form.

- **The $e$-$B$-$\Pi$-frame.** The next logical step is to introduce a frame in which all the fluxes are non-vanishing. The nearly exclusively used choice in the literature is the generalised flux frame for the $\sigma$-model

\[
S = -\frac{1}{2} \int d^2\sigma \left( \frac{1}{\gamma - B(x)} - \Pi(x) \right) e_i^a e_j^b \partial_+ x^i \partial_- x^j. \tag{2.20}\]

The corresponding generalised vielbein is of the type $E = E^{(\beta)} E^{(e)} E^{(\gamma)}$ and the resulting generalised fluxes are

\[
\begin{align*}
H_{abc} &= \partial_{[a} B_{bc]} + f_d^{\ [a}_{bc]} d, \\
\mathbf{f}_{ab}^{\ \ c} &= f_{ab}^{\ \ c} + H_{abc} \Pi^{de}, \\
Q_a^{\ ab} &= Q_c^{\ abc} + H_{cde} \Pi^{ad} \Pi^{bc} = \partial_a \Pi^{ab} + f_f^{\ [a}_{de} \Pi^{b]d} + H_{cde} \Pi^{ad} \Pi^{be}, \\
R^{abc} &= R^{abc} + H_{def} \Pi^{ad} \Pi^{bc} \Pi^{cf} = \Pi^{[a}_{de} \partial_d \Pi^{bc]} + f_f^{\ [a}_{de} \Pi^{b]d} \Pi^{c]e} + H_{cde} \Pi^{ad} \Pi^{ac}. 
\end{align*} \tag{2.21}\]

This has been derived several times in the literature \cite{33,65,67}.

- **The $e$-$\Pi$-$B$-frame.** The previous choice was not the only possible one. For example $E = E^{(\beta)} E^{(\Pi)} E^{(e)}$ is a valid parameterisation for which generically all the components of $F_{ABC}$ might be non-vanishing. The generalised fluxes are

\[
\begin{align*}
H_{abc} &= \partial_{[a} B_{bc]} + f_d^{\ [a}_{bc]} d + [B, B]^{K,S}_{abc} + B_{[a\delta} B_{bc}\Pi_{\delta]}^{de} + B_{ad} B_{bc} B_{de} R^{def}, \\
\mathbf{f}_{ab}^{\ \ c} &= f_{ab}^{\ \ c} + Q_{[a\delta} B_{bc]c} + \Pi^{cd} \partial_d B_{ab} + B_{ab} B_{de} R^{abc}, \\
Q_a^{\ ab} &= Q_c^{\ abc} + R^{abc} B^{ac} = \partial_c \Pi^{ab} + f_f^{\ [a}_{de} \Pi^{b]d} + R^{abc} B^{ac}, \\
R^{abc} &= \Pi^{[a}_{de} \partial_d \Pi^{bc]} + f_f^{\ [a}_{de} \Pi^{b]d} \Pi^{c]e}. 
\end{align*} \tag{2.22}\]
We recognise the (dual) Koszul derivative $\partial^c = \Pi^d \partial_d$, which is used to define the Koszul-Schouten bracket $[,]^{K.S}$ of forms analogously to the usual Schouten bracket of multivector fields. This vielbein corresponds to the $\sigma$-model

$$ S = \frac{-1}{2} \int d^2 \sigma \left( \frac{1}{\Pi(x)} - B(x) \right) e^a e^b \partial_+ x^a \partial_- x^b. $$

(2.23)

• The completely general expression for $F_{ABC}$ in terms of a completely generic generalised vielbein can be found in [13], also including vielbein which might violate the strong constraint.

In contrast to the case in the $T$-duality chain, where always only one of the fluxes $H, f, Q, R$ was turned on, the single components have no general interpretation. E.g. here can be $R$-flux in a locally geometric background, if other fluxes are turned on as well.

**Global non-geometry** Metric and $B$-field, encoded in the generalised metric, are only defined locally. If the patching involves only $B$-field gauge transformations and $d$-dimensional diffeomorphisms we call the background *globally geometric*. On the other hand for a generic non-geometric background we can patch as

$$ H_{IJ}(G(x), B'(x)) = M^{KL}(x) H_{KL}(G(x), B(x)) M_{IJ}^{-1}(x) $$

(2.24)

for an $M_{KL}(x) \in \text{O}(d,d)$. In a corresponding generalised flux frame (2.3) we have that the 'internal' generalised metric $\mathcal{H}_{AB} = E_A^I \mathcal{H}_{IJ} E_B^J$ is trivial and globally well-defined. The generalised vielbein will in general be defined only patch-wise and patched via $E_A^I(x) = M_I^J(x) E_A^J(x)$. The generalised fluxes transform according to

$$ \tilde{F}_{ABC} = F_{ABC} + E_A^I E_B^J (\partial C_I) M_{IK}. $$

(2.25)

'O($d,d$) gauge transformations' are those $M(x)$ for which the second term vanishes such that, as expected, the generalised fluxes are globally defined description in a non-geometric backgrounds. Such $O(d,d)$ gauge transformations include for example

• in the *geometric frame*: geometric gauge transformations, i.e. $B$-field gauge transformations and $d$-dimensional diffeomorphisms.

• in the *geometric frame with $H = 0$*: certain (coordinate dependent) $\beta$-shifts in non-holonomic coordinates, s.t. both $Q^{ab} = 0$ and $R^{abc} = 0$. Such $\beta$-shifts exist, homogeneous Yang-Baxter deformations of group manifolds are of this kind for example [68]. It has been shown that these correspond to a *non-local* field redefinition in the Lagrangian [69].

• frame independent: all constant $O(d,d)$ transformations, including factorised dualities. For example the constant $Q$-flux background in the $T$-duality chain is of this type, where $M$ is a constant $\beta$-shift.

As (2.25) shows the allowed $M_I^J(x)$ depend on the generalised frame $E_A^I$ under investigation. Not all of these necessarily have to be interpretable as standard abelian $T$-duality, for example they might also correspond to non-abelian $T$-duality transformations [70].
Finding such a generalised flux frame for some given generalised metric $\mathcal{H}$ is non-trivial and not unique, as we have a huge gauge freedom. There is in general no preferred frame, except if we have can find a globally well-defined generalised vielbein (this case is called a generalised parallelisable manifold - see e.g. [16]). We are only concerned with local properties of the target space in the following, so all statements involving the generalised vielbeins $E_A^I$ are to be understood in a single patch.

In section 4 we strive for a Hamiltonian formulation of classical string theory given directly in terms of these globally well-defined generalised fluxes $F_{ABC}$. This formulation will only hide the fact that in principle we still need to work in the different coordinates patches in which the $E_A^I$ are defined. Steps towards a more rigorous discussion of global issues have been taken in [39, 40] in the present context of current algebras and loop groups as phase space.

### 2.2 String in an $H$-flux background

The generalisation of the point particle in a electromagnetic field (section 1.1) to strings in a geometric $H$-flux background was achieved in [32]. We review this result here to set a basis for later discussion. Consider the $\sigma$-model of a (classical) string in a geometric background, defined by metric $G$ and Kalb-Ramond field $B$.

$$S = -\frac{1}{2} \int dx^i \wedge (G_{ij}(x) \ast + B_{ij}) \, dx^j$$

**Free loop space** The configuration space of a closed string moving in a manifold $M$ is the (free) loop space $LM$:

$$LM = \left\{ x : S^1 \to M, \sigma \mapsto x(\sigma) \right\}.$$ 

We denote elements of $LM$ by $x$ or $x^i(\sigma)$, working in a coordinate patch of $M$. We take $\sigma$ to have values between 0 and 1 and in a slight abuse of nomenclature for $LM$ also sometimes discuss open strings, by discussion of different boundary conditions on the $x(\sigma)$.

The class of smooth functions on $LM$, that we will consider most often, are (multi-local) functionals on $M$

$$F : LM \to \mathbb{R}, F[x] = \int d\sigma_1 \ldots d\sigma_n f(x(\sigma_1), \ldots, x(\sigma_n))$$

induced by smooth functions $f : M \times \ldots \times M \to \mathbb{R}$ - in particular this includes all the background fields and fluxes. We assume no explicit $\sigma$-dependence required by independence under $\sigma$-reparameterisations.

The tangent space $T(LM)$ is spanned by variational derivatives and consists of elements

$$V[x] = \int d\sigma V^i(x(\sigma)) \frac{\delta}{\delta x^i(\sigma)} \in T(LM).$$

---

5Condition (2.3) fixes only the gauge for the flat internal indices, the gauge freedom corresponds to the gauge freedom of the original $H_{ij}$. 

---
For simplicity of the notation, we will write \( V^i(\sigma) \equiv V^i(x(\sigma)) \). These \( V^i(\sigma) \) are also only implicit functions of \( \sigma \), i.e.
\[
\delta V^i(\sigma) \equiv \int d\sigma' \partial x^j(\sigma') \frac{\partial}{\partial x^i(\sigma)} V^i(\sigma),
\]
where \( \delta = \oint d\sigma \delta x^i(\sigma) \). The de Rham differential on \( LM \) is the de Rham differential on \( LM \) and we use the notation \( \partial \equiv \partial_\sigma \).

Not all functions on \( LM \) are multilocal functionals of well-defined functions on \( M \). E.g.
\[
w = \oint d\sigma \partial x^i(\sigma),
\]
where \( x(\sigma) = x_0 + w \sigma + \text{osc.} \), is a total derivative under the integral over the closed circle, but the coordinate \( x \) is not a smooth function on the circle.

Twisted symplectic structure
Following the same steps as in section 1.1, we express the symplectic structure in terms of the kinematic momenta \( \pi_i := p_i + B_{ij}(x) \partial x^j \)
\[
\omega = \int d\sigma \delta p_i(\sigma) \wedge \delta x^i(\sigma)
= \int d\sigma \left( \delta \pi_i \wedge \delta x^i - \frac{1}{2} H_{ijk}(x) \partial x^k \delta x^i \wedge \delta x^j + \frac{1}{2} \left[ B_{ij}(x) \delta x^i \wedge \delta x^j \right] \right).
\]

Up to the total derivative term, the symplectic structure is twisted in a \( B \)-field gauge independent way, by the \( H \)-flux, similarly to the electromagnetic case (1.5). Imposing that the symplectic form (2.28) is closed,
\[
\delta \omega = \frac{1}{12} \oint d\sigma \partial |H|_{ijkl}(x) \partial x^i \delta x^j \wedge \delta x^k \wedge \delta x^l = 0,
\]
requires that \( H \) is a closed 3-form on \( M \). If we instead neglect the boundary contribution in the symplectic two-form (2.29), we get such a contribution for the closure of the symplectic form
\[
\delta \omega_{\text{bdy}} = \partial \left( H_{ijk}(x) \delta x^i \wedge \delta x^j \wedge \delta x^k \right)
\]
up to a total derivative term. So together with the Hamiltonian
\[
H = \frac{1}{2} \oint d\sigma \left( G^{ij}(x) \pi_i \pi_j + G_{ij}(x) \partial x^i \partial x^j \right)
\]
this defines a world-sheet theory in backgrounds which are magnetically charged under the NSNS flux, e.g. the NS5-brane - in particular in and near these magnetic sources, but requiring that the phase space there is only almost symplectic.

The total derivative terms in (2.28) resp. (2.29), which naively vanish for closed strings, are for example relevant for

- **open strings ending on D-branes.** A contribution to the symplectic structure from a (potentially pure-gauge) \( B \)-field on the brane is the well known source for the fact, that we find non-commutative gauge theories on the brane. In the present context of deformations of the symplectic/Poisson structure this has been discussed in [32], in particular closedness of the symplectic structure requires \( H \big|_{D-\text{brane}} = 0 \) if we neglect the boundary term in the current algebra. For the some of the models motivating this article D-branes have been discussed, i.e. Poisson-Lie \( \sigma \)-models [71] or \( \lambda \)-deformations [72].
3 Current algebra and algebroid structures

- **winding strings.** As discussed above the winding number \( w = \oint d\sigma \, \partial x(\sigma) \) along a compact direction is such an integral over a total derivative. In section 5 we show that such winding contributions need to be considered so that the current algebra still satisfies the Jacobi identity.

- **globally non-geometric backgrounds.** E.g. consider the Q-flux background obtained from the standard T-duality chain of \( T^3 \) with \( q \) units of H-flux, expressed in terms of a metric \( G \) and the H-flux (2.11). We expect a contribution of a monodromy \( H(1) - H(0) \). But let us note that also the Hamiltonian (2.31) is not well-defined at \( x^3 + 1 \sim x^3 \) in the geometric frame.

Choosing the generalised flux frame instead – here in particular the one for the pure Q-flux background – should give a globally well-defined description of the background and be used to twist the symplectic structure. Hence this is the route we want to take in the following, in particular in section 4, and the main results of this article.

It turns out that these twists by the generalised fluxes are more conveniently defined in terms of the variables \( p_i(\sigma) \) and \( \partial x^i(\sigma) \) and their Poisson structure, the current algebra\(^6\):

\[
\{ \partial x^i(\sigma), \partial x^j(\sigma') \} = \{ p_i(\sigma), p_j(\sigma') \} = 0, \quad \{ \partial x^i(\sigma), p_j(\sigma') \} = \delta^i_j \partial_\sigma (\sigma - \sigma').
\] (2.34)

Let us make a the connection between what follows in the next sections and the above twisting of the symplectic structure by the H-flux. Going to kinematic momentum \( \pi_i \) the Poisson brackets are

\[
\{ \partial x^i(\sigma_1), \partial x^j(\sigma_2) \} = 0, \quad \{ \partial x^i(\sigma_1), \pi_j(\sigma_2) \} = \delta^i_j \partial_\sigma (\sigma_1 - \sigma_2).
\] (2.35)

\[
\{ \pi_i(\sigma_1), \pi_j(\sigma_2) \} = -H_{ijk}(\sigma_1) \partial x^k(\sigma_1) \delta(\sigma_1 - \sigma_2) + \int d\sigma \partial (B_{ij}(\sigma) \delta(\sigma - \sigma_1) \delta(\sigma - \sigma_2))
\]

The Jacobi identity imposes, of course equivalently to (2.29) \( \partial_i H_{jk} = 0 \) and in case we neglect the total derivative term in (2.35) \( H\big|_{D-brane} = 0 \).

3 Current algebra and algebroid structures

This sections aims to clarify the relation between the standard current algebra, derived from the canonical Poisson structure, and the Courant algebroid structure first discussed in [32]. Different versions of \( O(d, d) \)-invariant and -covariant current algebras exist in the literature, all of these differ by total derivative terms \( \int d\sigma \partial (...) \).

\(^6\)The current algebra and its deformations could in principle also be phrased in terms of a symplectic structure. But in case of such a Poisson structure containing so-called *ultralocal* terms, the symplectic structure will be non- resp. bi-local:

\[
\omega_{\text{current}} = \int d\sigma_1 d\sigma_2 \Theta(\sigma_1 - \sigma_2) \delta p_i(\sigma_1) \wedge \delta(\partial x^i)(\sigma_2),
\] (2.32)

where \( \Theta \) is the step function with \( \delta p_i \Theta = \delta(\sigma) \). Trying to invert the H-twisted Poisson current algebra (2.28) to obtain a twisted \( \omega_{\text{current}} \) we get:

\[
\omega_{ij}(\sigma_1, \sigma_2) = \begin{pmatrix}
A_{ij}(\sigma_1, \sigma_2) & \delta^j_i \Theta(\sigma_1 - \sigma_2) \\
-\delta^i_j \Theta(\sigma_1 - \sigma_2) & 0
\end{pmatrix}
\] with \( \int d\sigma_2 A_{ij}(\sigma_1, \sigma_2) = -H_{ijk} \partial x^k(\sigma_1) \) (2.33)

neglecting boundary terms.
3.1 Definitions

Let us first define our notation and collect some well-known facts about the algebroid structures relevant to us [13–15, 38, 67, 73–77].

**Lie algebroid** A vector bundle $E \to M$ over a manifold $M$ with a Lie bracket $[\cdot , \cdot ]_L$, i.e. skew-symmetric and satisfying the Jacobi identity, on the space of sections $\Gamma(E)$ and an anchor $\rho : E \to TM$, is called a Lie algebroid (over $M$), iff $[\cdot , \cdot ]_L$ together with the anchor $\rho$ satisfies the Leibniz rule

$$\left[\phi_1, f\phi_2\right]_L = (\rho(\phi_1)f)\phi_2 + f\left[\phi_1, \phi_2\right]_L, \quad \text{for } \phi_1, \phi_2 \in \Gamma(E), \ f \in C^\infty(M).$$

$[\cdot , \cdot ]$ is the Lie bracket on $TM$ and the fact that $\rho$ is a homomorphism of Lie brackets

$$\rho([\phi_1, \phi_2]_L) = [\rho(\phi_1), \rho(\phi_2)],$$

follows from the Leibniz rule.

**Courant algebroid** A Courant algebroid over a manifold $M$ is a vector bundle $E \to M$, together with a bracket $[\cdot , \cdot ]_D$ on $\Gamma(E)$, a fibre-wise non-degenerate symmetric bilinear form $\langle \cdot , \cdot \rangle_E$ and an anchor $\rho : E \to TM$, satisfying the following axioms:

$$[\phi_1, [\phi_2, \phi_3]_D]_D = [[\phi_1, \phi_2]_D, \phi_3]_D + [\phi_2, [\phi_1, \phi_3]_D]_D,$$

$$[\phi_1, f\phi_2]_D = (\rho(\phi_1)f)\phi_2 + f[\phi_1, \phi_2]_D,$$

$$[\phi, \phi]_D = \frac{1}{2}\mathcal{D}(\phi_1, \phi_2),$$

$$\rho(\phi_1)[\phi_2, \phi_3] = (\langle [\phi_1, \phi_2]_D, \phi_3 \rangle + \langle \phi_2, [\phi_1, \phi_3]_D \rangle).$$

for $\phi_i \in \Gamma(E), \ f \in C^\infty(M)$ and the derivation $\mathcal{D} : C^\infty(M) \to E)$:

$$\langle \mathcal{D}f, \phi \rangle = \rho(\phi)f.$$

We call $[\cdot , \cdot ]_D$ Dorfman bracket in the following, it is also called generalised Lie derivative in the literature. From the first two axioms follows that $\rho$ is homomorphism of brackets. The third axiom implies that $[\cdot , \cdot ]_D$ is not skew-symmetric, the first axiom describes a certain Jacobi identity for this non skew-symmetric bracket.

**Skew-symmetric realisation** A Courant algebroid as defined above possesses an equivalent representation via a skew-symmetric bracket

$$[\phi_1, \phi_2]_C = \frac{1}{2}([\phi_1, \phi_2]_D - [\phi_2, \phi_1]_D) = [\phi_1, \phi_2]_D - \frac{1}{2}\mathcal{D}(\phi_1, \phi_2),$$

which we call Courant bracket. It satisfies modified axioms - in particular, the Jacobi identity only holds up to a total derivation by $\mathcal{D}$

$$[\phi_1, [\phi_2, \phi_3]_C] + \text{c.p.} = \mathcal{D}\left(\frac{1}{3}\langle [\phi_1, \phi_2]_C, \phi_3 \rangle + \text{c.p.}\right).$$ (3.1)
The standard Courant algebroid on \( TM \oplus T^*M \) The Courant bracket for sections \( \phi = v + \zeta \in TM \oplus T^*M \) is given by

\[
[\phi_1, \phi_2]_C = [v_1, v_2] + \mathcal{L}_{v_1} \zeta_2 - \mathcal{L}_{v_2} \zeta_1 - \frac{1}{2} d (\zeta_2(v_1) - \zeta_1(v_2))
\]

In the following we use the notation \( \phi = \phi^I \partial_I \) with \( \partial_I = (\partial_i, dx^l) \) where the action of \( dx^l \) on functions is \( dx^l f = 0 \). Then the coordinate expression for Courant resp. Dorfman bracket is:

\[
[\phi_1, \phi_2]_C^I = \phi_1^I \partial_I \phi_2^I + \frac{1}{2} \eta_{IJK} \phi_1^I \partial^J \phi_2^K, \quad [\phi_1, \phi_2]_D^I = \phi_1^I \partial_I \phi_2^I + \eta_{IJK} \phi_1^I \partial^J \phi_2^K,
\]

where \( \eta \) is the \( O(d,d) \) metric, which raises indices \( I, J = 1, \ldots, 2d \). The anchor is simply projection to \( TM: v + \zeta \mapsto v \).

One motivation for the Courant bracket from the point of view of the study of \( T \)-dualities is, that it possesses an invariance under global \( O(d,d) \)-transformations (manifest through the index structure in (3.3)) and also under the geometric subgroup of local \( O(d,d) \)-transformations, namely diffeomorphisms and \( B \)-field gauge transformations via \( \partial_i \rightarrow \partial_i + B_{ij}(x) dx^j \) with \( dB = 0 \).

### 3.2 Current algebra as Lie and Courant algebroids

The current algebra derived from the canonical Poisson structure is given by (2.34). We write it in an \( O(d,d) \)-covariant way, defining \( E_I(\sigma) = (p_i(\sigma), \partial x^l(\sigma)) \),

\[
\{E_I(\sigma_1), E_J(\sigma_2)\}[\sigma] = \frac{1}{2} \eta_{IJ}(\partial_1 - \partial_2) (\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)) + \frac{1}{2} \omega_{IJK} \partial (\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2))
\]

without neglecting the second total derivative term and where we employ the notation:

\[
\{\cdot, \cdot\} = \int d\sigma \{\cdot, \cdot\}[\sigma].
\]

The second term containing \( \omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) is a total derivative under the \( \sigma \)-integrable and not invariant under \( O(d,d) \)-transformations. It is the boundary term that was already discussed in section 2.2.

### 3.2.1 Algebroids over \( LM \)

To compare it to the definitions of the previous sections, let us compute the algebra of arbitrary multilocal 'charges'. A section \( \phi \in \Gamma(E) \) is given by

\[
\phi = \phi[x] = \int d\sigma \phi^I(\sigma) E_I(\sigma)
\]

Here and later in the text we make use of the distributional identities

\[
(\partial_1 + \ldots + \partial_n) (\delta(\sigma - \sigma_1) \ldots \delta(\sigma - \sigma_n)) = \delta(\delta(\sigma - \sigma_1) \ldots \delta(\sigma - \sigma_n))
\]

for arbitrary (matrix-valued) functions \( \epsilon \) and \( \eta \), which hold without any additional boundary terms.
The Poisson bracket between these sections $\phi$ is
\[
\{\phi_1, \phi_2\} = \int \frac{d\sigma_1 d\sigma_2}{2} E_i(\sigma_1) \left( \phi^I_1(\sigma_2) \frac{\delta}{\delta X^I(\sigma_2)} \phi^j_2(\sigma_1) + \frac{1}{2} \phi^I_1(\sigma_2) \frac{\delta}{\delta X^I(\sigma_1)} \phi^j_2(\sigma_2) \right) \\
+ \frac{\delta}{\delta X^I(\sigma_1)} \left( \omega_{KL} \phi^K_1(\sigma_2) \phi^L_2(\sigma_2) \right) \tag{3.7}
\]
with $\frac{\delta}{\delta X^I(\sigma)} := \left( \frac{\partial}{\partial \sigma^I(\sigma)}, 0 \right)$. Also, we have a natural anchor map $\rho : E \to T(LM)$ defined via the Poisson bracket
\[
\phi \in \Gamma(E) \mapsto \rho(\phi) = \{ \cdot, \phi \} = \int d\sigma \phi^I(\sigma) \frac{\delta}{\delta \sigma^I(\sigma)} \in \Gamma(T(LM)). \tag{3.8}
\]
The Leibniz rule follows from the properties of the fundamental Poisson brackets. Also the Jacobi identity
\[
\{\phi_1, \{\phi_2, \phi_3\}\} + \text{ c. p.} = 0 \tag{3.9}
\]
holds identically, i.e. without any total derivative terms under the $\sigma$-integrals. For this we have to use $\frac{\delta}{\delta X^I(\sigma)} F \frac{\delta}{\delta X^J(\sigma)} G$ for arbitrary functions $F,G$ on $LM$, which is the strong constraint of double field theory on $LM$ and follows here from our definition of $\frac{\delta}{\delta X^I(\sigma)}$. Resultantly the full (multilocal) charge algebra is not only a Lie algebra as expected, but also a Lie algebroid $(E, \{ \cdot, \cdot \}, \rho)$ over the free loop space $LM$. It is something which could be called standard Lie algebroid of the generalised tangent bundle $(T \oplus T^*)(LM)$, for which for an arbitrary manifold the Lie bracket is the semi-direct product of $TM$, with the Lie bracket and $T^*M$.
\[
[L_1, L_2] = [v_1, v_2] + \mathcal{L}_{v_1} \xi_2 - \mathcal{L}_{v_2} \xi_1. \tag{3.10}
\]
From (3.7), which is written in an $O(d,d)$-covariant way, we see that the Lie algebroid bracket is not invariant under $O(d,d)$-transformations due to the presence of the last term containing $\omega$.

There is a natural non-degenerate inner product on $E \to LM$ induced by the $O(d,d)$-metric $\eta$ on $(T \oplus T^*)M$:
\[
\langle \phi_1, \phi_2 \rangle = \int d\sigma \eta_{IJ} \phi^I_1(\sigma) \phi^J_2(\sigma) \tag{3.11}
\]
This product is the canonical bilinear form on $(T \oplus T^*)LM$. Following the definition $\mathcal{D}$, $\langle DF, \phi \rangle = \rho(\phi) F$, we find the derivation
\[
\mathcal{D} F[x] = \int d\sigma \ E_i(\sigma) \frac{\delta}{\delta X^I(\sigma)} F[x] = \int d\sigma_\sigma \phi^I(\sigma) \frac{\delta}{\delta \sigma^I(\sigma)} F[x] \tag{3.12}
\]
\[
= \int d\sigma_1 \ldots d\sigma_n (\partial_1 + \ldots + \partial_n) f(x(\sigma_1), \ldots, x(\sigma_n)),
\]
With the help of these objects we can define the standard Courant algebroid on $(T \oplus ...
In this paragraph we want to tackle two questions

3.2.2 Algebroids over $M$

For convenience of the reader we also give all the relevant brackets in a local form in terms of the basis $E_I(\sigma)$ of the current algebra:

- **Brackets:**
  
  \[
  \text{Lie : } \{E_I(\sigma_1), E_J(\sigma_2)\} [\sigma] = \frac{1}{2} \eta_{IJ} (\partial_1 - \partial_2) (\delta(\sigma - \sigma_1) \delta(\sigma - \sigma_2)) \\
  + \frac{1}{2} \omega_{IJ} \delta(\sigma - \sigma_1) \delta(\sigma - \sigma_2)
  \]  

  \[
  \text{Courant : } \{E_I(\sigma_1), E_J(\sigma_2)\} [\sigma] = \frac{1}{2} \eta_{IJ} (\delta(\sigma - \sigma_1) \delta(\sigma - \sigma_2))
  \]  

  \[
  \text{Dorfman : } \{E_I(\sigma_1), E_J(\sigma_2)\} [\sigma] = \eta_{IJ} \delta(\sigma - \sigma_1) \delta(\sigma - \sigma_2)
  \]

- **Non-degenerate $O(d,d)$-invariant inner product:**
  
  \[
  \{E_I(\sigma_1), E_J(\sigma_2)\} [\sigma] = \eta_{IJ} \delta(\sigma - \sigma_1) \delta(\sigma - \sigma_2)
  \]
Can we find bundle maps $e_\ast$, such that
\[
T^*M \xrightarrow{\epsilon^T} T^*(LM) \xrightarrow{\rho^T} E \xrightarrow{\phi} T(LM) \xrightarrow{\epsilon^T} TM
\] (3.20)
is a (non-exact) Courant algebroid over $M$ with anchor $e_\ast \circ \rho$?

Does such a bundle map $e_\ast$ also extend to a homomorphism of Lie resp. Courant algebroids? What happens to the total derivation terms?

In general these questions seem to go beyond the scope of this article, both for reasons of mathematical rigor - which seems to be required if we consider bundle maps which keep track of more of the 'non-local' structure of the full current algebra - and also for physical reasons - we work in a fully generic background so far, so no mode expansion of the basis $E_1(\sigma)$ is available. A stringy expansion of the full current algebra could be an interesting question for further study. A more rigorous study of the current algebra and loop space structure of the phase space can be found in previous literature \[39, 40\].

Nevertheless we can find a simple example. Let us consider the bundle map
\[
e^0_\ast : v = \int d\sigma_1 \ldots d\sigma_n v^i(x(\sigma_1), \ldots, x(\sigma_n)) \frac{\delta}{\delta x^i(\sigma_1)} \mapsto v^i(x) \equiv v^i(x, \ldots, x) \partial_i,
\] (3.21)
which is something like the push-forward of the evaluation map of the loop space, $e^0_\ast : LM \to M$, $x(\sigma) \mapsto x \equiv x(\sigma_0)$ for some $\sigma_0$. This bundle map is simply the projection from the loop space phase space to the phase space associated to a point of the string. It induces an anchor $e^0_\ast \circ \rho$, as we can show that $e^0_\ast$ is an Lie algebra homomorphism. This can be used to view the current algebra as an algebroid over $M$, not only over $LM$.

It extends easily to a complete (Lie resp. Courant) algebroid homomorphism $E \to (T \oplus T^*)M$. Consider the generic bracket on $E$,
\[
\{\phi_1, \phi_2\}_{a,b} = \int d\sigma_1 d\sigma_2 E_1(\sigma_1) \left( \phi_1^i(\sigma_2) \frac{\delta}{\delta x^i(\sigma_1)} \phi_2^i(\sigma_1) + 2 \frac{\delta}{\delta x^i(\sigma_1)} \phi_2^i(\sigma_1) + \frac{\delta}{\delta x^i(\sigma_1)} \phi_2^i(\sigma_2) \right)
\]
(3.22)
for some $a, b \in \mathbb{R}$, which incorporates all brackets discussed in the previous section. $e^0_\ast$ defines a bracket on $(T \oplus T^*)M$
\[
e^0_\ast \{\phi_1, \phi_2\} = \{e^0_\ast \phi_1, e^0_\ast \phi_2\} = \phi_1^l \phi_2^l + \frac{1}{2} \phi_1^l \phi_2^l + \frac{1}{2} \partial^l (a \omega_{KL} + b \eta_{KL}) \phi^K_1 \phi^L_2
\] (3.23)
and is a true Courant algebroid homomorphism, but the brackets (3.22) differ only by total derivative term under the integral, so they might be argued to be equivalent for sufficiently nice charges for closed strings - but their projections to points are truly inequivalent. This issue is quite logical because the map $e^0_\ast$ does not really correspond to a point-particle limit the string\(^9\), but to a restriction of the total phase space (the current algebra) to a local phase space associated to one point on the string. Total derivative terms correspond to a kind of flux on the string, which adds up to zero for closed strings without winding.

\(^9\)An obvious candidate for this would seems to be a bundle map associated to the zero mode projection
\[
\bar{\epsilon} : LM \to M, x(\sigma) = x_0 + \bar{\epsilon}(\sigma) \mapsto x_0 \equiv d\sigma x(\sigma) = x_0.
\] (3.24)
4 The Hamiltonian realisation of the generalised flux frame

In the last section we only had a very generic look on aspects of current algebras, valid for arbitrary backgrounds - we did not introduce any dynamics. This section aims to show how the Hamiltonian world-sheet theory in any generalised flux background can be defined by a Hamiltonian of the form of the one of the free string. All the background information is encoded in a deformation of the Poisson structure. This deformation of the current algebra will be accounted for by the generalised (geometric and non-geometric) NSNS fluxes, in perfect analogy to the point particle in an electromagnetic field. This generalises the result of [32], reviewed in section 2.2. Many aspects of this were discussed already in [33, 34] from a Lagrangian point of view and for a certain parameterisation of generalised vielbeins reviewed in section 2.1.

4.1 Hamilton formalism for string $\sigma$-models

Let us consider a generic string $\sigma$-model coupled to metric and $B$-field of a $d$-dimensional target space

$$S = \frac{-1}{2} \int \left( G_{ij}(x) \, dx^i \wedge \star dx^j + B_{ij}(x) \, dx^i \wedge dx^j \right).$$

Choosing conformal gauge, we find the Hamiltonian to be

$$H = \frac{1}{2} \oint d\sigma \mathcal{H}_{IJ}(\sigma) E^I(\sigma) E^J(\sigma)$$

where $\mathcal{H}_{IJ}(\sigma)$ is the generalised metric (2.2), which depends on $\sigma$ via the coordinate dependence of $G$ and $B$. $E_I(\sigma) = (p_i(\sigma), \partial x_i(\sigma))$, where $p_i(\sigma)$ is the canonical momentum, fulfils the canonical current Poisson brackets (3.7).

Generalised fluxes in Hamiltonian formalism Assume we have a generalised flux frame describing our background, e.g. a generalised vielbein $E_A^I(x)$ with

$$E_A^I(x)E_B^J(x)\mathcal{H}_{IJ}(G(x),B(x)) = \gamma^{AB} = \left( \begin{array}{cc} \gamma^{ab} & 0 \\ 0 & \gamma_{ab} \end{array} \right),$$

where $\gamma^{ab}$ is some convenient flat metric in the signature of the target space, e.g. $\gamma^{ab} = \delta^{ab}$. We could be tempted to phrase the Hamiltonian world-sheet theory also in terms of a new basis of the current algebra: $E_A = E_A^I E_I$. The Hamiltonian is again of the form of a 'free' Hamiltonian:

$$H_0 = \frac{1}{2} \int d\sigma \, \gamma^{AB} E_A(\sigma) E_B(\sigma)$$

But the push-forward bundle homomorphism,

$$\tilde{e}_* : \phi : \int d\sigma \phi'(\sigma) \frac{\delta}{\delta \phi'(\sigma)} \mapsto \left( d\sigma \phi'(\sigma) \right),$$

turns out have several issues. Written as such

- It is conceptually ill-defined, because we add vectors of tangent spaces at different points. We would need to transport them back to $x_0$ before summing them up.
- It will not be an algebra homomorphism.
Thus all the information is expected to be encoded in the current algebra. The redefinition \( E_A = E_A \Gamma_4 E_I \) of (3.4) results in the twisted current algebra

\[
\{E_A(\sigma^1), E_B(\sigma^2)\} \propto \delta = \frac{1}{2} \eta_{AB} (\partial_1 - \partial_2) (\delta(\sigma^1 - \sigma) \delta(\sigma^2 - \sigma) + \frac{1}{2} \delta(\omega_{AB}(\sigma) \delta(\sigma^1 - \sigma) \delta(\sigma^2 - \sigma))
- F^C_{AB}(\sigma) E_C(\sigma) \delta(\sigma^1 - \sigma) \delta(\sigma^2 - \sigma),
\]

with \( F_{ABC} = (D_{\{E_B\}})_C I, \) or decomposed into the four components \( H, f, Q \) and \( R: \)

\[
\begin{align*}
\{e_{0,a}(\sigma), e_{0,b}(\sigma')\} &= - (f_{ab}(\sigma) e_{0,c}(\sigma) + H_{abc}(\sigma) e_{1}^a(\sigma)) \delta(\sigma - \sigma') \\
\{e_{0,a}(\sigma), e_{1}^b(\sigma')\} &= - \left( f_{cb}(\sigma) e_{1}^c(\sigma) + Q_{a}^{b,c}(\sigma) e_{0,c}(\sigma) \right) \delta(\sigma - \sigma') - \delta_0^b \delta_{\sigma'}(\sigma - \sigma') \\
\{e_{1}^a(\sigma), e_{1}^b(\sigma')\} &= - \left( Q_{a b}^{c}(\sigma) e_{1}^c(\sigma) + R_{a b}^{c,d}(\sigma) e_{0,c}(\sigma) \right) \delta(\sigma - \sigma')
\end{align*}
\]

with \( E_A(\sigma) = (e_{0,a}(\sigma), e_{0,b}(\sigma)) \). In contrast to the \( \eta \)-term the total derivative term containing \( \omega_{AB}(\sigma) \) is not invariant under this change of basis as

\[
\omega_{AB}(\sigma) = E_A^I(\sigma) E_B^J(\sigma) \omega_{IJ} \neq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{AB} \quad (4.7)
\]

in general. \( c \)-transformations leave the \( \omega_{IJ} \)-term invariant compared to (3.4), whereas for example a \( B \)- resp. a \( \beta \)-shift leads to

\[
\omega^{(B)} = \begin{pmatrix} 2B & -1 \\ 1 & 0 \end{pmatrix} \quad \text{resp.} \quad \omega^{(\beta)} = \begin{pmatrix} 0 & -1 \\ 1 & -2\beta \end{pmatrix} \quad (4.8)
\]

**Equations of motion** The Hamilton equations of motion are

\[
\begin{align*}
\delta \mathbf{e}^c + \frac{1}{2} \left( Q_{a b}^{c} + H_{c m n} \gamma^{m a} \gamma^{n b} \right) e_a \wedge e_b + \frac{1}{2} f^{(a} \gamma_{b)} c d e_a \wedge *e_b &= 0 \quad (4.9) \\
\delta \mathbf{e}^e + \frac{1}{2} \left( f_{a b} + R_{a b}^{c} \gamma_{m n} \right) e_a \wedge e_b + \frac{1}{2} Q_{(a}^{b} \gamma_{b)} c d e_a \wedge *e_b &= 0 \quad (4.10)
\end{align*}
\]

with one-forms \( e^c = e_a \gamma^c d \sigma^a \). In terms of a Lagrangian formulation these correspond to an equation of motion and a world-sheet Bianchi identity. The Hamiltonian formalism does not distinguish between these two ‘types’ of equations of motion, showing that it is a convenient framework to study dualities.

The equations of motion of the string in an arbitrary locally geometric background can be encoded very conveniently into the \( O(d, d) \)-covariant form

\[
d \mathcal{E}^A + \frac{1}{2} F_{B C}^{A} \mathcal{E}^B \wedge \mathcal{E}^C = 0, \quad \text{with} \quad \mathcal{E}^A := \{ e^a, *e_a \} \quad \text{resp.} \quad \mathcal{E}^A = \gamma^{A B} \star \mathcal{E}_B. \quad (4.11)
\]

In this form the equations of motion are nothing else than the pullback \( \mathcal{E} = x^* E \) of a structure equation for frame fields \( E^A_I(x) dX^I \) together with the constraint \( \star E^A = \gamma^{A B} E_B \).
Virasoro constraints To complete the description of a string theory in a generalised flux background we give the Virasoro constraints and their properties. There is, of course, nothing new to expect - they are a consequence of world-sheet reparameterisation invariance and hold identically. Similarly to the Hamiltonian, the constraints and their properties take the same form as the ones for the string in flat space. This relies solely on the fact that the $F_{ABC}$ are totally skew-symmetric. The conservation of the energy-momentum tensor additionally requires the equation of motion as usual. So we can phrase the whole dynamics of a string solely in terms of the generalised fluxes without referring to the generalised vielbeins.

With the definition $T_{\alpha\beta} = \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{\alpha\beta}}$ and choosing a generalised flux frame $E_A$ as before, these constraints take the form (in flat gauge on the world-sheet)

$$T_{00}(\sigma) = T_{11}(\sigma) = -\frac{1}{2} \gamma^{AB} E_A(\sigma) E_B(\sigma) = 0,$$

$$T_{01}(\sigma) = T_{10}(\sigma) = -\frac{1}{2} \eta^{AB} E_A(\sigma) E_B(\sigma) = 0.$$  \hfill (4.12)

Moreover, their respective zero modes $H$ and $P$ correspond to world-sheet derivatives $\partial_+ = \{\cdot, H\}$ and $\partial_- = \{\cdot, P\}$. Even if we consider the current algebra with all boundary contributions (3.13), we get the standard Virasoro algebra

$$\{T_{\pm\pm}(\sigma_1), T_{\pm\pm}(\sigma_2)\}[\sigma] = \mp 2 \{T_{\pm\pm}(\sigma_1) + T_{\pm\pm}(\sigma_2)\} \gamma_{\pm\pm}^\pm(\sigma - \sigma) = 0.$$  \hfill (4.13)

Conservation of the energy momentum tensor holds on-shell (4.11) and for totally skew-symmetric $F_{ABC}$

$$\partial_- T_{-+}(\sigma) \pm \partial_+ T_{+-}(\sigma) = \mp \frac{1}{4} F_{ABC}(\sigma) \gamma^{CD} E^A(\sigma) E^B(\sigma) E^C(\sigma) = 0.$$  \hfill (4.14)

Deformation of current algebra structure and generalised fluxes The approach taken above shows a generalisation of the previously known statement, demonstrated in sections 1.1 and 2.2 for point particles in Maxwell background or strings in H-flux backgrounds, that the coupling to these background fields can be encoded in a deformation of the symplectic structure of the phase space, instead of introducing interaction terms in the Lagrangian or the Hamiltonian. So locally the world-sheet theory in any generalised flux background is characterised by

$$\{E_A(\sigma_1), E_B(\sigma_2)\} = \eta_{AB} \partial_1 \delta(\sigma_1 - \sigma_2) - F_{ABC}(\sigma) E_C(\sigma_1) \delta(\sigma_1 - \sigma_2)$$  \hfill (4.15)

in terms of the generalised fluxes $F_{ABC}$, neglecting total derivative terms, together with a 'free Hamiltonian $H_0 = \frac{1}{2} \int d\sigma \gamma^0_{AB} E_A(\sigma) E_B(\sigma)$ (and similarly the full set of Virasoro constraints).\footnote{From this point of view, we could imagine to generalise to a current algebra twisted by the Weitzenböck connection $\Omega^{\alpha\beta}$ (2.5)

$$\{E_A(\sigma_1), E_B(\sigma_2)\} = \eta_{AB} \partial_1 \delta(\sigma_1 - \sigma_2) - \Omega^C_{AB}(\sigma) E_C(\sigma_1) \delta(\sigma_1 - \sigma_2).$$  \hfill (4.16)

This however seems to be a substantial change in the theory, as the Virasoro algebra (4.13) and the conservation of the energy momentum tensor (4.14) rely on the total skewsymmetry of $F_{ABC}$.}
This formulation focuses on the physical content of a background, namely the globally well-defined fluxes opposed to the potentially not globally well-defined objects in the generalised metric formulation. In the case of the point particle in an electromagnetic background or the string in H-flux background, this formulation also seemed to be gauge invariant under A- resp. B-field gauge transformation. Indeed, all the objects in the twisted current algebra (4.15) transform as a tensor under $O(d,d)$ gauge transformations $E_A' \rightarrow E_A'^A E_A$. With $O(d,d)$ gauge transformation, we mean as defined in section 2.1 precisely those $E_A'$, under which $F_{ABC}$ transforms as a tensor. So all results are expected to take a gauge covariant form, as is usual in the generalised flux formulation of double field theory [23]. The Bianchi identity, which will be discussed in the next paragraph, will serve as an example for that.

If we wanted to define the Hamiltonian theory only by means of (4.15) and a free Hamiltonian $H_0$, we need to specify the Poisson brackets between the $E_A$ and functions of the phase space as well:

$$\{E_A(\sigma), f(x(\sigma'))\} = \partial_A f(x(\sigma))\delta(\sigma - \sigma')$$ \hspace{1cm} (4.17)

with $\partial_A = E_A \partial_j$ and $\partial_j = (\partial_j, 0)$ as before.

**Bianchi identities and magnetically charged backgrounds**  In analogy to the examples in section 2, let us show what kind of consistency condition the Jacobi identity of the deformed Poisson brackets implies

$$0 = \left\{E_A(\sigma_1), \left\{E_B(\sigma_2), E_C(\sigma_2)\right\}\right\}[\sigma] + \text{c. p.}$$ \hspace{1cm} (4.18)

$$= \left(\frac{1}{4}E_{[A}F_{BC]}D(\sigma) + F_{D[AB}F_{C]E}(\sigma)\right)E^{D}(\sigma) + F_{ABC}(\sigma)E^{D}(\sigma)\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\delta(\sigma - \sigma_3)$$

$$= \left(\frac{1}{4}E_{[A}F_{BC]}E(\sigma)\right)E^{D}(\sigma)\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\delta(\sigma - \sigma_3).$$

We recognise the Bianchi identity of generalised fluxes (2.7) in the last line which takes the form of a covariant derivative [23]

$$\partial_{[A}F_{BC]} - \frac{3}{4}F_{[AB}F_{CD]}E = \nabla_{[A}F_{BC]} = 0.$$ \hspace{1cm} (4.19)

This calculation holds exactly, meaning without neglecting total derivative terms, if we start with the full form of (4.5) including the total derivative term there. Instead, we could simplify (4.5) to (4.15) neglecting the total derivative term as previously done in the H-flux case, see section 2.2 or [32].\footnote{On reason for doing this is that the equations of motion for an open Dirichlet string for example, considering all the boundary terms coming from (3.4), take the inconvenient form}

$$dE^A(\sigma) + \frac{1}{2}F_{A[BC}(\sigma)E^B(\sigma)\wedge E^C(\sigma) = \frac{1}{2}e^{AB + \omega^{AB}}_{\gamma BC}E^C(\sigma)\delta(\sigma - \sigma_1)|_{\gamma_1 = 1/2} = 0.$$ \hspace{1cm} (4.20)
So, e.g. in the geometric frame and for an open Dirichlet string we have
\[
\frac{1}{2} \partial_j \left( \omega (B) \right)_{bc} \big|_{\text{D-brane}} = H_{abc} \big|_{\text{D-brane}} = 0 \tag{4.22}
\]
and all components vanishing as a sufficient condition for associativity of the phase space. This reproduces the boundary contribution to open strings in an H-flux background in the Jacobi identity section as expected in section 2.2 resp. ref. [32].

In full analogy to the point particle in magnetic monopole backgrounds, we expect violations of this Bianchi identity and thus of the Jacobi identity of our current algebra for magnetically charged backgrounds. Such backgrounds like NS5-branes and its T-duals have been studied in [23,78,79] in the generalised flux formulation.\textsuperscript{12} They would source the Bianchi identity like.

\[
\partial_j [A F_{BCD}] - \frac{3}{4} F^E |_{AB} F_{CD} |_E = J_{ABCD}. \tag{4.23}
\]

In principle this implies that inside the magnetic sources the background cannot be described anymore by a generalised vielbeins that gives the generalised flux $F_{ABC}$ (2.6). This means that in this case we cannot untwist the current algebra and that it is not possible to find a Lagrangian description of the world-sheet theory.\textsuperscript{13} Working in the Hamiltonian formalism we still have to specify a generalised vielbein resp. frame, in which all the objects are phrased, although this vielbein will not account for the whole amount of $F_{ABC}$.

### 4.2 Classical T-dualities

The discovery and examination of (generalised) T-dualities followed the path of constructions on the Lagrangian level. A classical proof of a duality is finding that such a construction corresponds to a canonical transformation.\textsuperscript{14}

\textsuperscript{12}In [23] also the following Bianchi identities/potential source terms have been discussed:

\[
J = D^A F_A - \frac{1}{2} F^A F_A + \frac{1}{12} F^{ABC} F_{ABC},
J_{AB} = D^C F_{CAB} + 2D[A F_B] - F^C F_{CAB}
\]

with $F_A = \Omega_B^{AB} + 2D_A d$, where $d$ is the generalised dilaton. We do not expect an appearance of these terms in the classical world-sheet theory, as they do explicitly contain the dilaton an the Weitzenböck connection. Thus we will not consider them in the following. From the side of gauged supergravity both $F_{ABC}$ as well as $F_A$ are known to correspond to electric gauging parameters [17, 18].

\textsuperscript{13}In even other terms this can be phrased as, that there are no Darboux coordinates to this problem, as the canonical Poisson bracket cannot be used to represent the then non-associative phase space.

\textsuperscript{14}Demanding that the equations of motions take the same form is not enough. Otherwise, for example, the principal chiral model and the WZW model would be the same, as both equations of motion, as well as Bianchi identities can be arranged to be

\[
d + \frac{1}{2} j \wedge j = 0, \quad d * j = 0 \tag{4.24}
\]

The difference lies in the meaning of $j$, which in the language of the Hamiltonian formalism corresponds to different Poisson brackets of the component $j$, see e.g. [48] for details. So only if the equations of motion are the same and the transformation leaves the (canonical) Poisson structure invariant, thus is a canonical one, we can say that the two models are dual to each other.
In this section we want to pinpoint peculiarities on $T$-duality from the point of view of the Hamiltonian formalism in the generalised flux frame. We reverse the logic and construct canonical transformations that can be interpreted as (classical) $T$-dualities between different $\sigma$-model Lagrangians.

4.2.1 (Generalised) $T$-duality

In a Hamiltonian formulation there is no notion of duality, only the more general notion of canonical transformations. Let us give some criteria from point of view of the generalised flux frame.

On canonical transformations and dualities

- A generalised flux $F_{ABC}$ and a generalised metric $H$ do not yet define a string $\sigma$-model Lagrangian. We need to specify a corresponding generalised vielbein $E_A^I$ or in other words Darboux coordinates of our deformed current algebra.
  
  This choice of generalised vielbein might not be unique. Different generalised vielbeins for a given generalised flux background correspond to dual $\sigma$-models Lagrangians.

- The framework, that we choose to study dualities, are models with constant generalised fluxes $F_{ABC}$. In slight contrast to earlier in this section we define a generic string model in the generalised flux frame by a Hamiltonian defined by a constant generalised metric $H(G_0, B_0)$.\footnote{If we do not relax the condition $H = 1$ on the generalised flux frame, the component connected to the identity of $O(d,d)$ will generically lead out of this condition: $MHM^T \neq 1$.}

  The duality group is realised linearly. I.e. a group element $M_A^B$ leads to a dual model defined by
  
  \[
  H_{A'B'}(G'_{0}, B'_0) = M_A^CM_B^DM_{CD}(G_0, B_0), \quad F'_{A'B'C'} = M_A^DM_B^FM_C^F\]

  Given that we find generalised vielbeins, $E_A^I$ resp. $E_A'^I(x)$ to original generalised fluxes $F_{ABC}$ resp. the dual generalised fluxes $F'_{A'B'C'}$, this defines two $\sigma$-model Lagrangians with equivalent Hamiltonian dynamics.

- The $M_A^B$ are $O(d,d)$-matrices, in order to keep the current algebra (4.5) form-invariant\footnote{We ignore the non $O(d,d)$-invariant $\omega$-term in (3.4) for our considerations in this section.}.

  We take them to be constant such that the dual generalised metric and fluxes stay constant.

- Canonical transformations are normally characterised by generating functions. Our approach instead motivates directly that

  \[
  M_I^J(x) = E_I^{A'}(x)M_A^B E_B^J(x)
  \]

  corresponds a canonical transformation, i.e. leaves the canonical Poisson brackets of the $E_I$ (3.4) invariant. In the next section we will motivate the existence of generating functions which would generate exactly to the linearly realised factorised dualities and construct closely related charges on the phase space that generate the component connected to the identity of $O(d,d)$.\footnote{If we do not relax the condition $H = 1$ on the generalised flux frame, the component connected to the identity of $O(d,d)$ will generically lead out of this condition: $MHM^T \neq 1$.}
From the Hamiltonian point of view a (constant) basis change of the $E_A$ does not seem to make any difference on the first sight. The point is that we keep the role of the $(e_0,e_1^a)$ resp. $(p_i,\partial x^i)$ fixed. So e.g. the $f$- and $H$-flux always describe the $e_0$-bracket and so on. Rotating the generalised fluxes around and finding new generalised vielbeins which may depend on the same coordinates $x$ is, what we define to be a duality here.\footnote{It is here where the pure $R$-flux background fails to exist purely geometrically, as we do not find such generalised vielbein only depending on the $x$.}

We could have taken the other perspective of rotating our choices of Darboux coordinates, i.e. what of the $E_I$ correspond to $p_i$ or $\partial x^i$. In the language of double field theory these would be different solutions to the strong constraint. Both perspectives are of course equivalent.

These duality transformations resp. canonical transformations should not be realisable by purely local field redefinitions in the $\sigma$-model Lagrangian, otherwise we would call them symmetries.

For the remainder of this section we will discuss standard (abelian) $T$-duality and Poisson-Lie $T$-duality from this point of view. Also we propose a generalisation, which we call Roytenberg duality, for the case of frames with generic constant generalised fluxes.

### Abelian $T$-duality

The framework for the study of abelian $T$-duality is a background with commuting isometries. Let us choose coordinates, such that the isometries are manifest and ignore the spectator coordinates that do not correspond to isometries.

Such a model is defined by the Hamiltonian

$$H = \frac{1}{2} \int d\sigma \, \mathcal{H}^{IJ}(G_0,B_0)E_I(\sigma)E_J(\sigma)$$

with constant $G_0, B_0$ and – neglecting the total derivative term from (3.4) –

$$\{E_I(\sigma_1),E_J(\sigma_2)\}[\sigma] = \eta_{IJ} \partial(\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)).$$

Abelian $T$-duality acts via $O(d,d)$-matrices $M$ as $M_I^J E_J$, leaving the current algebra invariant, but generating new Hamiltonians. Thus the space of dual models is given by the coset $O(d,d)\times O(d,d)/O(d)$. This can be seen by going to the model with $H = 1$, where $O(d)\times O(d)$-matrices leave the Hamiltonian as well as the canonical current algebra invariant.

### Poisson-Lie $T$-duality

The case of Poisson-Lie $T$-duality \cite{60,80,81}, and included in there also non-abelian $T$-duality \cite{82,83,84}, is the one with

$$H = R = 0, \quad f^c_{ab} = f^c_{ab}, \quad Q_c^{ab} = T_c^{ab}. \quad (4.29)$$

The Bianchi identities of generalised fluxes (2.8) reduce to Jacobi identities of the $f$- and $\overline{f}$-structure constants and a mixed Jacobi identity. The algebraic setting is that the generalised fluxes $F_C^{AB}$ correspond to structure constant of a Lie bialgebra $\mathfrak{d}$. A Lie bialgebra is a 2$d$-dimensional Lie algebra, with a non-degenerate symmetric bilinear
form $\langle \cdot , \cdot \rangle$ on $\mathfrak{d}$ given by the $O(d,d)$-metric $\eta$ and two (maximally) isotropic\footnote{meaning $\langle g, g \rangle = 0$.} subalgebras $\mathfrak{g}$ and $\mathfrak{g}^*$, of which $f$ resp. $\mathcal{F}$ are the structure constants. Together with the Hamiltonian corresponding to an arbitrary constant generalised metric $\mathcal{H}(G_0, B_0)$ this model is also known under the name $\mathcal{E}$-model in the literature\footnote{named after the operator $\mathcal{E}_A^B = \mathcal{H} \eta^{CB}$ fulfilling $\mathcal{E}^2 = 1$.}.

It is well-known how the corresponding generalised vielbein looks like: It is of the type $E^a = E^0 (g) E^a (\beta) E^b (\eta) E^c (\varepsilon)$ as discussed in section 2.1. The $d$-dimensional vielbein $e$ is given by the components of the Maurer-Cartan forms to the Lie group $G$ associated to the structure constants $f^c_{ab}$, $e^a_i = (g^{-1} \partial_i g)^a$ where $g$ are $G$-valued fields. $\beta$ is the homogeneous Poisson bivector $\Pi$ on $G$ defined by the dual structure constants $f^c_{ab}$, fulfilling $\Pi(e) = 0$, $\partial_i \Pi^{ab}(g) = f^c_{ab} + f^d_{ca} \Pi^{bd}$. (4.30)

This bivector $\Pi$ is uniquely determined by such a Lie bialgebra structure. The corresponding $\sigma$-model has the form

\[ S \sim \int d^2 \sigma \left( \frac{1}{e^{-1} + b_0 - \Pi(g)} \right) (g^{-1} \partial_+ g)^a (g^{-1} \partial_- g)^b. \] (4.31)

Poisson-Lie $T$-duality acts linearly on the associated deformed current algebra (4.29). This was discovered already in [61] and discussed in present form already in [57, 60, 85]. The total factorised duality simply corresponds to $f \leftrightarrow Q$, respectively $\mathfrak{g} \leftrightarrow \mathfrak{g}^*$. The full duality group, which maintains the structure of the generalised fluxes (4.29) of the $\mathcal{E}$-model is discussed in detail in [68]. It is the group of different Manin triple decompositions of the Lie bialgebra $\mathfrak{d}$.

At the Lagrangian level, the duality can be realised by considering a 'doubled' $\sigma$-model with target being the Drinfel’d double $\mathcal{D}$, and then integrating out d.o.f.s corresponding to different (isotropic) subalgebras $\mathfrak{g}^*$ of the Lie bialgebra $\mathfrak{d}$ [60, 86, 87]. Other approaches to Poisson-Lie $T$-duality via double field theory and generalised geometry include [88–91].

**Roytenberg duality** Let us consider the generic case: arbitrary constant generalised fluxes and a Hamiltonian corresponding to an arbitrary constant generalised metric $\mathcal{H}(G_0, B_0)$. Let us call this case Roytenberg model, as a configuration with a generic fluxes with non-vanishing $H$, $f$, $Q$ and $R$ was first considered in [67]. It is not clear, in contrast to the Poisson-Lie $\sigma$-model, how to find a generalised vielbein for a generic choice of constant generalised fluxes. In section (2.1) we introduced two choices of generalised vielbeins which generically turn on all of the four generalised fluxes. We consider choices of generalised vielbeins which build upon these two and the one of the Poisson-Lie $\sigma$-model:

- $E_1 = E^{(B)}_{\beta} E^{(\beta)}_{\beta_0} E^{(\beta)}_{\eta} E^{(\epsilon)}$
- $E_2 = E^{(B)}_{\beta} E^{(B)}_{\beta_0} E^{(\beta)}_{\eta} E^{(\epsilon)}$


as before (and want to have constant generalised fluxes). We take \( b \) and \( \beta_0 \) to be constant, \( e \) the vielbein of a Lie group \( G \) (corresponding to Lie algebra structure constants \( f_{a b c} \)) and \( \Pi(g) \) to be again a homogeneous Poisson bivector on \( G \), associated to dual structure constants \( \tilde{T}^a_{bc} \). This choice of \( \beta = \beta_0 + \Pi(g) \) in \( E_1 \) ensures that the resulting Q- and R-flux are constant as wished. The choice of \( \beta = \beta_0 + \Pi(g) \) arose as well, if we go to the complete generalised flux frame of the Poisson-Lie model, i.e. \( H = 1 \), see [68]. A generalised version of Poisson-Lie T-duality, called affine Poisson-Lie T-duality, taking into account exactly such constant \( \beta_0 \)’s and mapping between different dual choices of \( \beta_0 \) and \( \Pi(g) \) for \( B = 0 \) was considered in [92]. In the language of the Poisson-Lie T-duality group studied in [68] these were ‘non-abelian \( \beta \)-shifts’. Let us give the corresponding fluxes and \( \sigma \)-model Lagrangians for \( E_1 \),

\[
\begin{align*}
H_{abc} &= b_{[ab} b_{c]} T_{de} \tilde{T}^{de} - b_{[a} f_{bc]} - b_{[ab} b_{c]e} \beta_0^{ef} \tilde{T}^{ef} \tilde{T}^{de} + b_{ad} b_{be} b_{cf} R^{def} \\
f_{ab} &= f_{ab} - b_{[a} f_{bc]} - b_{[ab} e_{c]e} + b_{ad} b_{be} R^{def} \\
Q_{abc} &= \tilde{T}^{ab} - \beta_0^{[a} f^{bc]} + b_{cd} R^{abd} \\
R^{abc} &= \beta_0^{[a} f^{bc]} - \beta_0^{[a} f^{bc]} \\
S_1 &\sim \int d^2\sigma \left( \frac{1}{c_0 + \beta_0 - \Pi(g) - b} \right)^{-1} (g^{-1} \partial + g)^a (g^{-1} \partial - g)^b,
\end{align*}
\]

and for \( E_2 \),

\[
\begin{align*}
H_{abc} &= b_{[ab} b_{c]} T_{de} - b_{[a} f_{bc]} \\
f_{ab} &= f_{ab} - b_{[a} f_{bc]} - b_{[ab} e_{c]} + \beta_0^{[a} f^{bc]} - \beta_0^{[a} f^{bc]} + \beta_0^{[a} \beta_0^{bc} H^{cd} \\
Q_{abc} &= \tilde{T}^{ab} - \beta_0^{[a} f^{bc]} + \beta_0^{[a} f^{bc]} + \beta_0^{[a} \beta_0^{bc} H^{cd} \\
R^{abc} &= \beta_0^{[a} f^{bc]} - \beta_0^{[a} f^{bc]} \\
S_2 &\sim \int d^2\sigma \left( \frac{1}{c_0 + \beta_0 - \Pi(g) - b} \right)^{-1} (g^{-1} \partial + g)^a (g^{-1} \partial - g)^b.
\end{align*}
\]

So by construction the identifications

\[
b \leftrightarrow \beta_0 \quad \text{and} \quad f \leftrightarrow \tilde{T}
\]

(4.34)



\[
H \leftrightarrow R \quad \text{and} \quad f \leftrightarrow Q.
\]

(4.35)

This would be what we call the (factorised) Roytenberg duality in the terminology of (4.25). At the Lagrangian level, the two \( \sigma \)-models \( S_1 \) and \( S_2 \) are (classically) dual to each other with the identifications

\[
G^{(1)}_0 + B_0^{(1)} = \frac{1}{G^{(2)}_0 + B_0^{(2)}}, \quad \beta^{(1)}_0 = b^{(1)} \quad \text{and} \quad \tilde{T}^{(2)} = \tilde{T}^{(1)},
\]

(4.36)

(4.37)

The coefficients \( G^{(1)}_0 + B_0^{(1)} \) are used to ensure the dualising choice of \( \Pi(g) \) and \( \beta_0 \) in the particular Lagrangians (4.33) and (4.34), as before.
where the superscript \((i)\) denotes the quantities in \(S_i\) and we raised and lowered the indices appropriately.

Using the two generalised vielbeins \(E_1\) and \(E_2\) to describe these backgrounds, the Roytenberg duality simply seems to be an extension of the Poisson-Lie \(T\)-duality group. But these vielbeins are probably not the most general description of constant generalised fluxes, so the above example might give just a vague idea, of what a Roytenberg duality is in general and what kind of \(\sigma\)-model Lagrangians are connected by it.

The Roytenberg duality group is the full \(O(d,d)\times O(d,d)\) rotating the generalised fluxes and is an interesting object of further study. A Lagrangian derivation of this duality might or might not exist. But still the Hamiltonian theory is well-defined as long as the constant generalised fluxes fulfil the Bianchi identities (2.8).

Let us close this section with the following remark. There seems to be no difference between abelian and generalised \(T\)-dualities from the Hamiltonian point of view. We could have viewed the standard \(T\)-duality chain of section 2.1 in same fashion—again the true problem continues to be whether we can find appropriate vielbeins to the new fluxes.

### 4.2.2 Realisation in the Poisson algebra

In this section we want to construct the charges that generate infinitesimal \(O(d,d)\)-transformation in different generalised flux frames. These will show the need for isometries and are closely related to generating functions of the factorised dualities, not only for abelian \(T\)-duality but also the generalised version discussed above.

**Infinitesimal \(o(d,d)\)-transformations via charges** Let us define the non-local charges\(^{21}\)

\[
\Omega_{[IJ]} = \frac{1}{2} \oint d\sigma \int_{\sigma_0}^{\sigma} d\sigma' E_I(\sigma') E_J(\sigma)
\]

which generate \(o(d,d)\)-transformations on the \(E_K(\sigma)\):

\[
\left\{ \Omega_{[IJ]}, E_K(\sigma) \right\} = \eta_{IK} E_J(\sigma)
\]

From this and only with help of the Jacobi identity for the \(E_I(\sigma)\)-current algebra it is easy to show that these charges fulfil the \(O(d,d)\) Lorentz algebra

\[
\left\{ \Omega_{[IJ]}, \Omega_{KL} \right\} = \eta_{IK} \Omega_{JL} + \text{permutations}.
\]

A general infinitesimal \(O(d,d)\)-transformation

\[
M_{IJ} = 1 + m_{IJ}, \quad m \in o(d,d),
\]

\(^{20}\)The only difference is that it includes one non-isometric spectator coordinate.

\(^{21}\)We use \(\int^\prime\) as a formal expression denoting the antiderivative. More precisely we the following procedure

\[
\left\{ \Omega_{IJ}, F(\sigma) \right\} = \frac{1}{4} \lim_{\epsilon_0 \to 0} \left( \int d\sigma' \int_{\sigma_0}^{\sigma' \epsilon_0} d\sigma'' \left\{ E_I(\sigma'') E_J(\sigma'), F(\sigma) \right\} \right)
\]

where it is only important that \(c_0 \neq \sigma\). We will come across similar ambiguities later as well, where we will define doubled coordinates \(X^I = (x^i, \tilde{x}_i)\) as fundamental fields in the phase space, with \(E^I = \partial X^I(\sigma)\).
The action of these charges on functions of the original world-sheet phase space (functions of \( x^i(\sigma) \) and \( p_j(\sigma) \)) is non-local in general. In particular the action of the \( \beta \)-transformations acts non-locally on functions on the original manifold

\[
\{ \Omega_{ij}, f(x(\sigma)) \} = -\tilde{x}_i \partial_j f(x(\sigma)), \quad \text{with} \quad \tilde{x}_i(\sigma) = \int^\sigma \text{d}\sigma' p_i. \tag{4.41}
\]

So far \( \tilde{x}(\sigma) \) here is a non-local variable on the phase space. With the definitions \( \partial_I = (\partial_i, \bar{\partial}) \) and \( X^I(\sigma) = (x^i(\sigma), \tilde{x}_i(\sigma)) \) we have for functions in terms of this non-local variable \( \tilde{x} \)

\[
\{ \Omega_{IJ}, f(X(\sigma)) \} = -X_{[I}(\sigma) \partial_{J]} f(X(\sigma)) \tag{4.42}
\]

in an \( O(d,d) \)-covariant way. If we instead considered (multi-)local function(al)s on the current phase space, spanned by the \( E_i(\sigma) \), everything stays (multi-)local

\[
\{ \Omega_{IJ}, f(E_K(\sigma)) \} = -E_{[I}(\sigma) \frac{\partial}{\partial E_{J]} f(\sigma). \tag{4.43}
\]

These charges are (in general) not conserved - they do not commute with the Hamiltonian. Instead they generate infinitesimal \( O(d,d) \)-transformations of the generalised metric as wished, if the generalised metric is constant (again neglecting spectator coordinates). So the charges \( \Omega_{IJ} \) generate abelian \( T \)-dualities.

**'Non-abelian' \( o(d,d) \)-transformations** \( \Omega_{IJ} \) is a tensor under constant \( O(d,d) \)-transformations, but not under local ones (due to the integral). So instead we claim that we have natural charges \( \Omega_{AB} \) w.r.t. to some generalised vielbein \( E^I_A(\sigma) \) by the relation

\[
\left\{ \Omega_{[AB]}, E_C(\sigma) \right\} = \eta_{C[A} E_{B]}(\sigma). \tag{4.44}
\]

Such a \( \Omega_{AB} \) exists. An implicit realisation for infinitesimal fluxes would be

\[
\Omega^0_{[AB]} = \frac{1}{2} \int \text{d}\sigma \int_{\sigma_0}^\sigma \text{d}\sigma' \left( E^I_A(\sigma') E_B(\sigma) + \bar{\delta}^0_{AB} \right)
\]

with \( \delta\Omega_{AB} = \int \text{d}\sigma \left( \delta E^C(\sigma) \right) \Omega^0_{C[D]}(\sigma) \int_{\sigma_0}^\sigma \text{d}\sigma' (M^{-1})_{D[E} \Omega_{E[A}(\sigma') E_B](\sigma'), \)

where \( M(\sigma) = \exp \left( -\int_{\sigma_0}^\sigma \text{d}\sigma' \Omega(\sigma') \right) \) and \( \Omega_{AB} = F^C_{AB} E_C \). Finding an integrated form of this expression for a generic background seems highly non-trivial. Nevertheless, assuming that the Poisson brackets of the \( E_A(\sigma) \) fulfil the Jacobi identity, the Lorentz algebra follows directly from (4.44).

What this means is, that for every choice of generalised vielbein \( E_A(\sigma) \) modulo global \( O(d,d) \) transformation, there is a representation of \( O(d,d) \) acting on the phase space.

These are simply the linearly realised (infinitesimal) \( O(d,d) \)-transformations in the generalised flux frame \( E_A(\sigma) \) of the previous section. The action of constant but infinitesimal \( O(d,d) \)-matrix \( M^{AB} = \mathbb{1} + m^{AB} \), the corresponding \( O(d,d) \) transformation is

\footnote{The defining relation (4.44) is an ODE in \( \sigma \).}
generated by \( m^{AB} \Omega_{AB} \) as in the abelian case and similarly the \( \beta \)-shifts act non-locally (in momentum) on any function \( f(x) \). Again these charges are not conserved but generate \( O(d,d) \)-transformations on a constant generalised metric defining the Hamiltonian.\(^\text{23}\)

**Generating function of factorised dualities**

Factorised dualities are canonical transformations generated by generating functions of type \( F[q, Q] \) \([61, 93]\). For this type of generating function we have

\[
\frac{\delta F}{\delta q} = p \quad \text{and} \quad \frac{\delta F}{\delta Q} = -P.
\]  

The generating function for abelian \( T \)-duality is \([93]\)

\[
F[x, \tilde{x}] = -\frac{1}{2} \oint d\sigma \ (\tilde{x} \partial x - x \partial \tilde{x}),
\]

leading as wished to the identifications

\[
p = \partial \tilde{x} \quad \text{and} \quad \tilde{p} = \partial x.
\]

Using the notation of the previous paragraphs this generating function can be written as

\[
F_{\Omega_{II}}[x, \tilde{x}] = -\eta^{II} \Omega_{IJ}
\]

The generating function and even the associated canonical transformations seem to exist for any background and for any generalised flux frame, independent of whether the background possesses (generalised) isometries or not. The problem of the canonical transformations in non-isometric backgrounds (such that generalised metric or generalised fluxes are functions of \( x \) therein) again is, that the dual fields become functions of \( \tilde{x} \) non-local functions of the canonical momenta.

\[\text{4.3 Strings in double field theory backgrounds}\]

In the previous section we saw that the action of \( T \)-duality is not well-defined without assuming some isometry. The obstruction was that the dual backgrounds would become functions of new dual coordinates \( \tilde{x}_i(\sigma) \). These, being antiderivatives of the canonical momentum densities \( p_i(\sigma) \), are not uniquely defined.

\(\text{\textsuperscript{23}}\)We assumed that the algebra of the \( E_A(\sigma) \) fulfils the Jacobi identity. There might be problems if the background is magnetically charged or, as we will see later, violates the strong constraint.
A natural approach to this problem is to define \( X_I = (\bar{x}_\mu, x^i) \) to be the fundamental fields of the phase space. In particular then the generators of infinitesimal \( o(d,d) \)-transformations \( \Omega_{IJ} \) are then simply generators of rotations

\[
\Omega_{[IJ]} = \frac{1}{4} \int d\sigma X^I[\sigma] E_J[\sigma].
\]

This is the approach to double field theory.

Remarkably it seems, from the point of view of the Hamiltonian formalism, we would not need to 'double' phase space but instead allow a dependence of the background on the momenta in this very peculiar non-local way, namely via \( \bar{x}_i = \int^\sigma d\sigma' p_i(\sigma) \).

**Poisson brackets on doubled space** We are looking for a skewsymmetric Poisson bracket \( \{X_I(\sigma), X_J(\sigma')\} \), such that \( \partial_\sigma \{X_I(\sigma), X_J(\sigma')\} = \{E_I(\sigma), X_J(\sigma')\} = \eta_{IJ} \delta(\sigma - \sigma') \). The solution is

\[
\{X_I(\sigma), X_J(\sigma')\} = \eta_{IJ} \bar{\Theta}(\sigma - \sigma')
\]

with \( \bar{\Theta}(\sigma) = 1/2 \text{sign}(\sigma), \text{s.t.} \partial_\sigma \bar{\Theta}(\sigma) = \delta(\sigma) \). For the remainder of the section this will be the fundamental 'canonical' Poisson bracket.

Going to a generalised flux frame \( E_A^I(X) \), allowing for a generic dependence on the doubled space, we get the Poisson current algebra

\[
\{E_A(\sigma), E_B(\sigma')\} = \eta_{AB} \partial_\sigma \delta(\sigma - \sigma') - F^C_{AB}(\sigma) E_C(\sigma) \partial_\sigma' \delta(\sigma - \sigma') - G^C_{AB}(\sigma, \sigma') E_C(\sigma) E_D(\sigma') \bar{\Theta}(\sigma - \sigma')
\]

The last term is given in terms of the Weitzenböck connection (2.5)

\[
G^C_{AB}(\sigma, \sigma') = \eta^{KL} \left( \partial_k E_A^I(\sigma) \right) \left( \partial_l E_B^J(\sigma') \right) E^C_I(\sigma) E^D_J(\sigma') = \eta^{KL} \Omega_{K,A}^C(\sigma) \Omega_{L,B}^D(\sigma').
\]

If \( F \) and \( G \) are given in terms of generalised vielbeins, then (4.52) is a Poisson bracket because (4.51) is. The equation of motion of the string in a DFT background would be also modified by some \( G \)-terms.

**Constraints** The second term in (4.52) is non-local and also leads to non-local equations of motion. Trying to eliminate this unwished term we get a form of the *strong constraint* of double field theory:

\[
\eta^{KL} \partial_k E_A^I(X) \partial_l E_B^J(Y) = 0, \quad \text{for all } X = (x, \bar{x}) \text{ and } Y = (y, \bar{y}).
\]

Also the *weak constraint* of double field theory on any function \( f \) on the doubled space can be phrased in this language

\[
D^A D_A f(X(\sigma)) = \eta^{AB} \int d\sigma'' \int d\sigma' \left\{ E_A(\sigma''), \left\{ E_B(\sigma'), f(X(\sigma)) \right\} \right\} = 0
\]

but in this form there is no motivation from the view on the classical theory. The weak constraint arises as consequence of the level matching condition for massless states, so as a consequence of the constraint \( P \approx 0 \) on the quantum level.

So, similarly to target space considerations we have that the strong constraint is a simplifying condition – here: locality of Poisson structure and equations of motion (or in the target space: closure of algebra of generalised diffeomorphisms). The weak constraint on the other hand is a true constraint of the world-sheet theory.
**Bianchi identity**  As in section 4, we calculate the Bianchi identity of the objects $F$ and $G$ by imposing the Jacobi identity on (4.52)

$$D_{[A}F_{BCD]} - \frac{3}{4}F^{E}_{[AB}F_{CD]E} = \text{strong constraint violating terms.}$$

Thus one way to account for a violation of the Bianchi identities of generalised fluxes (4.10), e.g. in order to describe magnetically charged backgrounds, is to consider violations of the strong constraint, but only if we trade off (manifest) locality of the equations of motion for it.

Let us emphasise the full current algebra (4.52) is always associative by construction, even for backgrounds violation the strong constraint. The non-associativity appears if we neglect the non-local and strong-constraint violating $G$-terms in (4.51). This is in some sense similar to the observation that neglecting the total derivative terms in (3.4) leads to a violation of the Jacobi identity, being also a total derivative. The manifestation of non-geometry in the generalised flux frame of the current algebra is further discussed in the next section.

### 5  Non-geometry and the deformed current algebra

In this section we want to demonstrate the approach taken in the last section and clarify it by studying examples. We do so by reproducing standard results for constant $B$-field in case of the open string, and constant $H$, $f$, $Q$- or $R$-flux for the closed string. The key points are

- **Leaving magnetic or locally non-geometric backgrounds aside, there should be 'Darboux coordinates' $(x^i(\sigma), p_i(\sigma))$ fulfilling the canonical Poisson brackets. The question is where the well-known non-geometric nature of the backgrounds is 'hidden', meaning their non-commutative and non-associative behaviour.**

  In section 4 we saw that beside Darboux coordinates $x^i(\sigma), p_i(\sigma)$, the generalised flux frame of a given background gives rise to a second preferred set of coordinates for the current algebra $E_A(\sigma)$. We define 'non-geometric coordinates' $y^a$ and 'non-geometric momenta' $\pi_a$ by $\partial y^a = E^a(\sigma)$ and $\pi_a(\sigma) = E_a(\sigma)$. In the spirit of section 1.1 we dub them 'kinematic'. Their Poisson brackets agree with the known ones usually associated to non-geometric backgrounds.

  With this we can generalise the non-geometric interpretation to more complicated generalised flux backgrounds. Also, we do not need to know the mode expansions of the fields $y^a(\sigma)$ or even impose the equations of motion to study the non-geometric behaviour of the background.

- **In the spirit of generalised geometry and double field theory, we demonstrate in the language of the current algebra, how $T$-dualities can be reproduced by choosing different solutions to the strong constraint.**

- **The significance of the non $O(d,d)$-invariant boundary term in (3.4) or (4.5) lies**
  - reproducing non-commutativity for the endpoints of open strings.
  - ensuring associativity of the charge algebra for the (locally geometric and non-commutative) constant $Q$-flux backgrounds.
5.1 Open string non-commutativity

In this section we review the classic result of [94, 95] and are interested in the worldsheet dynamics of an open string in a constant $B$-field background. It can be expressed the open string variables resp. the non-geometric frame with flat metric and $\beta$

$$\beta^{ij} = \left( \frac{1}{G + \mathcal{F}} \right)^{ik} \mathcal{F}^{kl} \left( \frac{1}{G - \mathcal{F}} \right)^{lj}, \quad \mathcal{F} = B - dA = B - F,$$

(5.1)

where $G$ is the flat Minkowski metric and $F$ is the constant field strength of a Maxwell field. The current algebra in the generalised flux basis (the non-geometric frame) in which we have the 'free' Hamiltonian is

$$\{ e_{0,i} (\sigma_1), e_{0,j} (\sigma_2) \} = 0$$
$$\{ e_{0,i} (\sigma_1), e'_{1,j} (\sigma_2) \} = -\delta^i_1 \partial_2 \delta (\sigma_1 - \sigma_2)$$
$$\{ e'_{1,i} (\sigma_1), e'_{1,j} (\sigma_2) \} = \beta^{ij} \int d\sigma \partial (\delta (\sigma - \sigma_1) \delta (\sigma - \sigma_2)).$$

(5.2)

Now we associate new 'non-geometric coordinates' to this new basis: meaning $e_{1} = \partial y_a (\sigma)$. Simply integrating both sides of the last line of (5.2) gives the result:

$$\{ y^i (\sigma_1), y^j (\sigma_2) \} = \begin{cases} +\beta^{ij}, & \sigma_1 = \sigma_2 = 1 \\ -\beta^{ij}, & \sigma_1 = \sigma_2 = 0 \\ 0 & \text{else}. \end{cases}$$

(5.3)

This is exactly the result of [94], derived without any reference to a mode expansion. Let us note that the total derivative $\omega$-term in the last line of (5.2) was crucial for this result.

5.2 Closed string non-commutativity and non-associativity

Next let us demonstrate the logic explicitly for the well-known standard example of the $T$-duality chain of the 3-torus with constant $H$-flux. Assuming that we have a background with constant fluxes we saw above that at the level of the current algebra $O(d, d)$-duality transformations are linearly realised and the fluxes can be considered to be the fundamental description of the current algebra.

Let us consider the $Q$-flux background $Q_{3}^{12} = h$, all other components being zero, which is described by the generalised vielbein

$$E_{(Q)} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}, \quad \beta^{12} = h x^3.$$

(5.4)

The corresponding current algebra including boundary terms is

$$\{ e_{0,a} (\sigma_1), e_{0,b} (\sigma_2) \} = 0$$
$$\{ e_{0,a} (\sigma_1), e'_{1,b} (\sigma_2) \} = -\delta^a_1 \partial_2 \delta (\sigma_1 - \sigma_2) - Q_a^{bc} e_{0,c} (\sigma_1) \delta (\sigma_1 - \sigma_2)$$
$$\{ e'_{1,a} (\sigma_1), e'_{1,b} (\sigma_2) \} = -Q_a^{ab} e_{1} (\sigma_1) \delta (\sigma_1 - \sigma_2) + \int d\sigma \partial (\beta^{ab} (\sigma) \delta (\sigma - \sigma_1) \delta (\sigma - \sigma_2)).$$

(5.5)
Let us consider the zero modes of the 'kinematic coordinates' associated to this generalised flux frame

\begin{align*}
p_a &= \oint d\sigma \, p_a(\sigma) = \oint d\sigma \, e_{0,a}(\sigma) \\
w^a &= \oint d\sigma \, \partial y^a(\sigma) = \oint d\sigma \, e^a_1(\sigma) \\
y^a &= \oint d\sigma' \int^{\sigma'} d\sigma' \partial y^a(\sigma) \\
y_a &= \oint d\sigma' \int^{\sigma'} d\sigma \, p_a(\sigma).
\end{align*}

These modes have a priori nothing to do with the original target space interpretation. This seems particular confusing in case of the winding number. But cases like this exist in the literature, there it is sometimes called 'twisted boundary conditions', see e.g. in the context of $\beta$-deformations of $AdS_5 \times S^5$ [96]. In the present case we have

\begin{equation}
\partial y^a(\sigma) \equiv e^a_1(\sigma) = \delta^a_0 \partial x^i(\sigma) + \beta^{ab}_i \delta^i_j p_j(\sigma) \tag{5.6}
\end{equation}

and

\begin{equation}
w^3 = w^3_\chi \quad \text{and} \quad w^{1/2} = w^{1/2}_\chi \pm h \oint d\sigma x^2 p_{2/1}. \tag{5.7}
\end{equation}

The winding along the $y^3$ direction coincides with the actual one along the $x^3$ direction as also $y^3$ coincides with $x^3$ up to a constant. Now we can integrate the current algebra (5.5). We use two assumptions:

- We use a schematic mode expansion of the kinematic coordinates

\begin{equation}
y^a(\sigma) = y^a + w^a \sigma + y^a_{osc}(\sigma) \tag{5.8}
\end{equation}

with $y^a_{osc}(\sigma) = y^a_{osc}(\sigma + 1)$ denoting oscillator terms, of which we will not keep track explicitly as we are interested in the zero modes.

- There is an ambiguity in defining integrals over the occurring $\delta$-functions, when the integral is not over the whole circle. We resolve it in a similar fashion as in deriving (4.52) and define expressions like $\int_{v_0}^{v_2} \delta(\sigma - \sigma_1)$ to be $\tilde{\Theta}(v_2 - \sigma) = \Theta(v_2 - \sigma) - \frac{1}{2}$, where $\Theta$ is the Heavyside function.\textsuperscript{24}

The non-vanishing Poisson brackets of the zero modes are:

\begin{align*}
\{y^1, y^2\} &\sim -hw^3 + \text{osc.}, & \{w^1, w^2\} &\sim -hw^3 \\
\{\tilde{y}_3, y^1\} &\sim -hp_2 + \text{osc.}, & \{\tilde{y}_3, y^2\} &\sim hp_1 + \text{osc.} \\
\{p_3, w^1\} & = -hp_2, & \{p_3, w^2\} & = hp_1 \\
\{y^a, p_b\} & = \delta^a_b + \text{osc.}, & \{\tilde{y}_a, w^b\} & = \delta^b_a + \text{osc.} \\
\{y^1, w^2\} & = \{y^2, w^1\} = -h \left( y^3 + \frac{1}{2} w^3 + \text{osc.} \right),
\end{align*}

\textsuperscript{24}Alternatively we could always write all the occurring integrals, which do not go over the full circle as $\int_{v_0}^{v_2}$ and averaging over $v_0$. This gives the same result.
reproducing the known non-commutative interpretation of the pure $Q$-flux background. The underlined terms only stem from the boundary term and $\sim$ denotes some neglected constant factors. Let us note, that if we would not neglect the oscillators, the canonical position-momentum Poisson bracket would be modified as well. Also let us emphasise again that our assumptions do not imply anything about a mode expansion apart from (5.8). So we can discuss the non-geometric structure without solving the theory first.

**Non-associativity** There are non-trivial Jacobi identities of the zero mode Poisson brackets:

\[
\{\{w^1, w^2\}, \tilde{y}_3\} + \text{c.p.} \sim \{\{y^1, y^2\}, \tilde{y}_3\} + \text{c.p.} \sim -h = -Q_3^{12} \tag{5.10}
\]

neglecting oscillator terms. The zero mode part of the second line vanishes due to the boundary term contribution (the underlined term in (5.9)). The first line is a non-associativity coming from a potential violation of the strong constraint. This will not be relevant if we only 'probe' the phase space with functions

\[
f(y^1; E_A, \partial E_A, \ldots) \text{ resp. } f(x^1; E_I, \partial E_I, \ldots). \tag{5.11}
\]

As $\tilde{y}_3$ is not an argument of these functions the $Q$-flux background given by the current algebra (5.5) is *associative* and thus locally geometric. But there are other different choices of solutions of the strong constraints\(^{25}\) which correspond to the $T$-dual backgrounds of the $T$-duality chain (see section 2.1):

- $f(y^1, y^2, y^3; \ldots)$ locally geometric $Q$-flux background,
- $f(\tilde{y}_1, y^2, y^3; \ldots)$ or $f(y^1, \tilde{y}_2, y^3; \ldots)$ locally geometric $f$-flux backgrounds,
- $f(\tilde{y}_1, \tilde{y}_2, y^3; \ldots)$ locally geometric $H$-flux background.

In addition there are of course also the continuous $O(2,2)$-transformations on the $y_1, y_2$. The solutions of the strong constraint containing $\tilde{y}_3$ give non-associative phase spaces, corresponding to the locally non-geometric backgrounds:

- $f(y^1, y^2, \tilde{y}_3; \ldots)$ locally non-geometric $R$-flux background,
- $f(\tilde{y}_1, y^2, \tilde{y}_3; \ldots)$ or $f(y^1, \tilde{y}_2, \tilde{y}_3; \ldots)$ locally non-geometric $Q$-flux backgrounds,
- $f(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3; \ldots)$ locally non-geometric $f$-flux background.

These are all locally non-geometric as the generalised vielbein depends via $\beta$ on $x^3 = y^3$, which is the origin of the non-associativity.

Overall we reproduce the well-known zero mode brackets [13, 24–31] of the considered (non-geometric) backgrounds without imposing a mode expansion or the equations of motion.

\(^{25}\)We phrase them in the phase space variables of the $Q$-flux background. To get the standard picture, e.g. of the $H$-flux we make the identifications $y^1 \leftrightarrow \tilde{y}_1$ and $y^2 \leftrightarrow \tilde{y}_2$. 
6 Discussion

6.1 Summary

The central result of this paper was introduced in section 4.1. The world-sheet theory in a generic NSNS background, including non-geometric ones, can be defined in the following way. In terms of some phase space variables $E_A(\sigma)$ there is a Hamiltonian in a background independent form $H \sim \int d\sigma \delta^{AB} E_A(\sigma) E_B(\sigma)$, and similarly for the Virasoro constraints. Instead the information about the background is encoded in the Poisson structure. This is most conveniently formulated in terms of the current algebra (the algebra of the $E_A(\sigma)$)

$$\{E_A(\sigma_1), E_B(\sigma_2)\} = \Pi^\theta_{AB}(\sigma_1, \sigma_2) + \Pi^{\text{bdy}}_{AB}(\sigma_1, \sigma_2) + \Pi^{\text{flux}}_{AB}(\sigma_1, \sigma_2)$$

(6.1)

$\Pi^\theta$ is the $O(d,d)$-invariant part of the canonical current algebra (3.4), whereas

$$\Pi^{\text{flux}}_{AB}(\sigma_1, \sigma_2) = -F_{ABC}(\sigma_1) E^C(\sigma_1) \delta(\sigma_2 - \sigma_1)$$

is characterised solely by the generalised flux $F_{ABC}$, building on known results in the literature [32–34]. This formulation seems to be the world-sheet version of the generalised flux formulation of generalised geometry resp. double field theory [16, 23].

In case of an electric and locally geometric background, meaning the Bianchi identity (2.7) is fulfilled, there is a connection to Darboux coordinates $(x^i, p_i)$ on the phase space resp. a Lagrangian formulation. This connection is given by a choice of generalised vielbein $E_A^I(c)$, s.t. $E_A(\sigma) = E_A^I(x(\sigma))(p_1(\sigma), \partial x^i(s_\sigma))$ and $F_{ABC} = (D_A E^B_I) E^C_{IJ}$.

In the cases of a magnetically charged NSNS background (like an NS5-brane) or a locally non-geometric background the Hamiltonian world-sheet theory as defined above is still defined. But there are some obstructions in either case. In former the Bianchi identity of generalised fluxes is sourced. Resultantly the associated current algebra violates the Jacobi identity and thus there cannot be Darboux coordinates on the phase space. In the non-geometric case there will be a strong constraint violating bi-local contribution to (6.1). Also the generalised vielbein depends on dual coordinates $\tilde{x}(\sigma) = \int^\sigma d\sigma \ p(\sigma)$, generically inducing a non-local dependence on the momenta of a potential Lagrangian. See section 4.3 for more details.

One difference to previous discussions in the literature is the consideration of the total derivative term,

$$\Pi^{\text{bdy}}_{AB}(\sigma_1, \sigma_2) = \int d\sigma \delta(\omega_{AB}(\sigma) \delta(\sigma - \sigma_1) \delta(\sigma - \sigma_2))$$

This occurs in this form as a non-$O(d,d)$-invariant boundary contribution from the canonical current algebra (3.4). Terms like this in the current algebra itself or its Jacobi identity make the difference between a Lie or a Courant algebroid structure of the phase space $(T \oplus T^*) LM$. This was discussed in detail in section 3. For open strings they lead to the known constraint of $H |_{D\text{-brane}} = 0$ [32] and the non-commutativity at the ends of the open string [94]. For closed strings a winding contribution from this term is necessary such that the standard Q-flux background is an associative background.

We discussed two applications of this formulation of the world-sheet theory. The first one is the observation that (generalised) $T$-dualities act linearly on the variables in
the generalised flux frame. This lead to the proposal of a generalisation of Poisson-Lie $T$-duality to Roytenberg duality, applicable to models with constant generalised fluxes. This was shown using a certain parameterisation of the constant generalised flux based on the ones of Poisson-Lie $\sigma$-models in section 4.2.

The second application is a direct derivation of the well-known non-commutative and non-associative behaviour of some generalised flux backgrounds from the deformed current algebra in section 5. This interpretation does not rely on a mode expansion or even on imposing the equations of motion, it is purely kinematic. Also it extends straightforwardly to any generalised flux background.

6.2 Potential applications and future directions

Part of the original motivation was the study of integrable deformation, as these can be conveniently represented as deformations of the current algebra – see section 1.2. The discussion in this paper connecting the possible deformations of the current algebra for string $\sigma$-models to generalised fluxes, hints at a connection of generalised geometry to the Hamilton formulation of integrable $\sigma$-models. From a purely technical side there is also an argument to maybe expect a connection to integrability. The currents $e_a$, used here to write down the equations of motion (4.9), are the ones which are used to calculate the Lax pair in all the examples – principal chiral model, $\eta$-deformation, $\lambda$-deformation, Yang-Baxter deformation.

There are two generalisations of this article's approach which come to mind immediately. The first one is the generalisation to the Green-Schwarz superstring and in particular introducing $RR$-fluxes into this formalism. From the point of view of the target space our treatment is purely kinematic. For the Green-Schwarz superstring a complete kinematic description includes $\kappa$-symmetry, which on other hand is also closely connected to the supergravity equations [97] and thus dynamics of the background. The fact that this formalism relies on a flat internal space might be useful to define spacetime fermions in a background independent way and a formulation of the Green-Schwarz superstring, that is not only valid in very symmetric spacetimes.

Another generalisation would be to the Hamiltonian treatment of membrane $\sigma$-models. There has been a lot of work on topological membrane $\sigma$-models. An approach similar to the one discussed here could be useful to understand the kinetic term of membrane $\sigma$-models better. Also the appearance and the interpretation of expected higher brackets in the phase space of a membrane seems interesting to study. Very recently a generalisation of Poisson-Lie $T$-duality to higher gauge theories was proposed [98], it would be interesting to investigate whether such dualities are realised in a membrane current algebra in a similarly simple fashion as (generalised) $T$-dualities here.

As demonstrated in [34] it is not advantageous to parameterise the background by the generalised fluxes in order to calculate the 1-loop $\beta$-function and check the quantum conformality like this. But this formulation might be potentially a good framework to quantise the string canonically. In particular for constant generalised fluxes the equations of motion (4.11) take the form of a (constrained) Maurer-Cartan structure equations of a $2d$-dimensional (non-compact) Lie group. If it would be possible to construct a mode expansion, it seems possible to quantise the bosonic theory directly as also the Virasoro constraint take a simple form in the generalised flux frame.

It was mentioned before that apart from the fact that we assumed our generalised
fluxes to be globally well-defined tensors we only discussed local properties of our globally non-geometric backgrounds. Previous work discussing current algebras, loop algebras and their global properties is [39,40]. Connecting these approaches and the generalised flux formulation of non-geometric background seems to be an important step for future work.

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