A functional-type a posteriori error estimate of approximate solutions for Reissner-Mindlin plates and its implementation

Maxim Frolov, Olga Chistiakova
Department of Applied Mathematics, Institute of Applied Mathematics and Mechanics, Peter the Great St. Petersburg Polytechnic University, 195251, Polytechnicheskaya st. 29, St. Petersburg, Russia
E-mail: frolov_me@spbstu.ru

Abstract. Paper is devoted to a numerical justification of the recent a posteriori error estimate for Reissner-Mindlin plates. This majorant provides a reliable control of accuracy of any conforming approximate solution of the problem including solutions obtained with commercial software for mechanical engineering. The estimate is developed on the basis of the functional approach and is applicable to several types of boundary conditions. To verify the approach, numerical examples with mesh refinements are provided.

1. Introduction
Different approaches to adaptive solution of boundary-value problems based on a posteriori error estimation are rapidly developing and form one of the modern research areas. Although this field is of interest for more than 30 years and the corresponding literature is vast, there is a very restricted set of papers on error estimation in the theory of plates. Most likely, the first paper on this subject appeared in 1999 [1] – authors used the constitutive relation method to propose an error estimate.

Nowadays, classical approaches (such as residual-based or postprocessing-based) dominate in the literature [2]–[18], but all these methods are essentially based on the assumption that Galerkin approximation is considered.

The functional approach of Repin and co-workers [19]–[23] is fully reliable even for commercial tools, which behave as a “black box”, due to its ability to control accuracy of non-Galerkin approximations both with and without locking. The first reliable error estimate for Kirchhoff-Love plates based on this approach [24] was implemented numerically in [25]. In comparison to all functional-type error majorants obtained earlier for Kirchhoff-Love and Reissner-Mindlin plates (see [26]–[29]), the main advantage of the new result [30] consists in less restrictive sets of admissible fields.

2. Reissner-Mindlin plates
In case of linear problems the respective mathematical model describes the bending of linearly elastic plates of small to moderate thickness in terms of pair of variables in $\Omega \subset \mathbb{R}^2$: $u$ — scalar-valued function (displacement) and $\theta$ — vector-valued function (rotations). The equilibrium
equations in this case are as follows:

\[
\begin{aligned}
\begin{cases}
- \text{Div} \left( C \varepsilon(\theta) \right) = \gamma & \text{in } \Omega, \\
- \text{div} \gamma = g & \text{in } \Omega, \\
\gamma = \lambda t^{-2} \left( \nabla v - \theta \right) & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

where \( t \) — thickness of a plate; \( \lambda = \frac{E}{2(1+\nu)} \); \( \varepsilon(\theta) = \frac{1}{2} (\nabla \theta + (\nabla \theta)^T) \); \( g t^2 \) represents the transverse loading; \( C \) — tensor of bending moduli; \( E \) and \( \nu \) — material constants; \( k \) — correction factor.

Boundary conditions of two types are considered. Let \( \Gamma_D \) and \( \Gamma_S \) be two non-intersecting parts of the boundary \( \Gamma \). \( \Gamma_D \) is a hard clamped part \((u = 0, \theta \cdot n = 0, \theta \cdot s = 0 \text{ on } \Gamma_D)\), \( \Gamma_S \) is a free part \((\partial u / \partial n = \theta \cdot n, n \cdot C \varepsilon(\theta)n = 0, s \cdot C \varepsilon(\theta)n = 0 \text{ on } \Gamma_S)\), where \( n \) is the outward unit normal vector to the boundary, \( s \) is the counterclockwise unit tangent vector. If \( \Gamma_S \) is a soft simply supported part, one has \( u = 0 \) on \( \Gamma_S \) instead of \( \partial u / \partial n = \theta \cdot n \) on \( \Gamma_S \).

Weak formulation of this problem has the following form: find a triple \((u, \theta, \gamma) \in U \times \Theta \times Q\)

\[
\begin{aligned}
\int \varepsilon(\theta) : \varepsilon(\varphi) \, d\Omega - \int \gamma \cdot \varphi \, d\Omega = 0, & \quad \forall \varphi \in \Theta := \{ \varphi \in \mathcal{W}^1_2(\Omega, \mathbb{R}^2) \mid \varphi = 0 \text{ on } \Gamma_D \}, \\
\int \gamma \cdot \nabla w \, d\Omega = \int g w \, d\Omega, & \quad \forall w \in U := \{ w \in \mathcal{W}^1_2(\Omega) \mid w = 0 \text{ on } \Gamma_D \}, \\
\int (\lambda^{-1} t^2 \gamma - (\nabla u - \theta)) \cdot \tau \, d\Omega = 0, & \quad \forall \tau \in Q := L^2(\Omega, \mathbb{R}^2),
\end{aligned}
\]

where \( \mathcal{W}^1_2 \) and \( L^2(\Omega) \) are standard denotations for Sobolev and Lebesgue spaces respectively. The domain \( \Omega \) is assumed to be bounded and connected with Lipschitz-continuous boundary, \( g \in L^2(\Omega) \). \( C \) is symmetric and there exist real constants \( \xi_1 \) and \( \xi_2 \) such that \( \xi_1 |\varepsilon| \leq C \varepsilon : \varepsilon \leq \xi_2 |\varepsilon|^2 \forall \varepsilon \in \mathcal{M}_{sym}^{2 \times 2} \), \( |\varepsilon|^2 = \varepsilon : \varepsilon \), where \( \mathcal{M}_{sym}^{2 \times 2} \) — space of symmetric tensors of 2nd rank. For numerical tests the plate material is assumed to be isotropic, which implies \( C \varepsilon = \frac{E}{12(1-\nu^2)} ((1-\nu) \varepsilon + \nu \text{tr} \varepsilon \mathbb{I}) \), where \( \mathbb{I} \) — identity tensor of 2nd rank.

3. A reliable a posteriori error estimate

To obtain an error estimate which is robust to some hidden details of commercial software and even possible bugs of a program code, it is necessary to consider an arbitrary pair of conforming approximations \((\tilde{u}, \tilde{\theta})\) without any additional assumptions or restrictions. Therefore, the following natural error measures are introduced: \( e_a = u - \tilde{u}, \ e_b = \theta - \tilde{\theta}, \ e_\gamma = \gamma - \tilde{\gamma} \).

Functional approach is based on the introduction of additional variables with appropriate mechanical (physical) meaning. In case of Reissner-Mindlin plates they could be a vector \( \tilde{y} \) and a symmetric tensor \( \tilde{\varepsilon} \). The estimate proposed in [30] is exact for \( \tilde{\varepsilon} = C \varepsilon(\theta) \) and \( \tilde{y} = \gamma \). Recent numerical results of the first author obtained for classical and Cosserat elasticity show that explicit symmetry condition for tensor variable with standard finite elements yields unsatisfactory results, but implicit symmetry condition with \( \tilde{\varepsilon} \) split it into parts \([\tilde{\varepsilon}^1, \tilde{\varepsilon}^2]\) and finite element approximations for mixed methods provides reliable estimates with efficient implementation. If so, one has the same functional space for computations:

\[
\tilde{y}, \tilde{\varepsilon}^1, \tilde{\varepsilon}^2 \in \mathbb{H}(\Omega, \text{div}) := \{ y \in L^2(\Omega, \mathbb{R}^2) \mid \text{div} y \in L^2(\Omega) \}.
\]

Main result of [30] is the following inequality:

\[
\| e_a \|^2 + \lambda^{-1} t^2 \| e_\gamma \|^2 \leq \hat{a}^2 + \lambda^{-1} t^2 \hat{b}^2,
\]

\[
\hat{a} = \| C^{-1} \text{sym}(\tilde{\varepsilon}) - \varepsilon(\tilde{\theta}) \| + \epsilon \| \text{skew}(\tilde{\varepsilon}) \| \Omega^+ + \\
+ \epsilon |\Omega| \| g + \text{div} \tilde{y} \|^2 \| \Omega^+ + |\Gamma_S| \| \tilde{y} \cdot n \|^2 |\Gamma_S| + \\
+ \epsilon |\Omega| \| \tilde{y} + [\text{div} \tilde{\varepsilon}^1, \text{div} \tilde{\varepsilon}^2] \|^2 \| \Omega^+ + |\Gamma_S| \| [\tilde{\varepsilon}^1 \cdot n, \tilde{\varepsilon}^2 \cdot n] \|^2 |\Gamma_S|.
\]
\[
\hat{b} = \|\tilde{y} - \tilde{\gamma}\|_\Omega + \epsilon_{III} \sqrt{\|\Omega\|} \|\tilde{g} + \text{div}\tilde{\gamma}\|_\Omega^2 + |\Gamma_S| \|\tilde{y} \cdot n\|_{\Gamma_S}^2,
\]

where
\[
\|\tilde{e}_\gamma\|_\Omega^2 := \int_{\Omega} |\tilde{e}_\gamma|^2 \, dx, \quad \|\tilde{e}_\theta\|_\Omega^2 := \int_{\Omega} C \varepsilon(\tilde{e}_\theta) : \varepsilon(\tilde{e}_\theta) \, dx
\]

and the auxiliary constants are mesh-independent and come from the following inequalities:
\[
\|\nabla \varphi\|_\Omega^2 \leq c_{\text{II}}^2 \|\varphi\|_\Omega^2, \quad \|\varphi\|_\Omega^2 \leq c_{\text{II}}^2 \|\varphi\|_\Omega^2,
\]
\[
\frac{1}{|\Omega|} \|w\|_\Omega^2 + \frac{1}{|\Gamma_S|} \|w\|_{\Gamma_S}^2 \leq c_{\text{III}}^2 \|\nabla w\|_\Omega^2, \quad \frac{1}{|\Omega|} \|\varphi\|_\Omega^2 + \frac{1}{|\Gamma_S|} \|\varphi\|_{\Gamma_S}^2 \leq c_{\text{IV}}^2 \|\varphi\|_\Omega^2.
\]

4. Implementation and numerical results

Well-known Raviart–Thomas approximation was introduced in [31] on a basis of mapping of the reference square $\hat{K} = (-1, 1) \times (-1, 1)$ to arbitrary quadrilateral (degrees of freedom are normal components of a vector-field in the midpoints): $RT_0(\hat{K}) = P_{1,0}(\hat{K}) \times P_{0,1}(\hat{K})$, where $P_{i,j}(\hat{K})$ – space of polynomials over $\hat{K}$ power of $i$ or less on $\hat{x}_1$ and $j$ – on $\hat{x}_2$. Here $\hat{x}_1$ and $\hat{x}_2$ are coordinates in a local element coordinate system. In 2005 a new type of approximation was proposed by Arnold, Boffi and Falk [32]. It is based on $ABF_0(\hat{K}) = P_{2,0}(\hat{K}) \times P_{0,2}(\hat{K})$ – a wider space that includes the original degrees of freedom on edges and two additional internal degrees of freedom on every element.

![Figure 1. Example 1: ANSYS solution (auxiliary mesh, displacement and rotations).](image-url)
well as for Galerkin ones. We suggest functional-type error estimates as such a tool and provide some numerical experiments to verify this.

![Figure 1](image1.png)

**Figure 2.** Example 1: components of \( \tilde{\kappa} \) marked by KxxLS, KyxLS, etc. and components of \( \tilde{\gamma} \) marked by YTxLS and YTylS, where “LS” means that fields come from solving of the corresponding linear system of algebraic equations.

### 4.1. Example 1. Circular plate.

As a first numerical example a circular plate is considered. For material and shape parameters ANSYS Verification Manual (VM138, Timoshenko S.) was used. Radius of the plate is 0.25 m, thickness — 0.0025 m, \( E = 2 \times 10^5 \text{ N/m}^2 \), \( \nu = 0.3 \). The plate is uniformly loaded (6585.175 N/m²).

ANSYS solution of this problem (see figure 1, where \( x, y, z \) are spatial coordinates) was obtained using SHELL181-element, and the results of error estimation are set out in table 1. In figure 1 auxiliary splitting of a quadrilateral mesh into triangles is used for output. From the results collected in table 1 one can conclude that overestimation of the true error (square root of the left-hand side of (4)) by the majorant (square root of the right-hand side of (4)) is a stable non-increasing constant. Corresponding free variables are presented in figure 2. If Galerkin approximations are used for this problem, locking effect takes place — the displacement is underestimated over than 5 times.

| DOFs  | 123  | 435  | 1635 | 6339 |
|-------|------|------|------|------|
| Error | 0.689e5 | 0.359e5 | 0.181e5 | 0.842e4 |
| Estimate | 0.101e6 | 0.527e5 | 0.266e5 | 0.133e5 |
| Overestimation | 1.5 | 1.5 | 1.5 | 1.5 |

As a posteriori error estimates are usually represented as a sum of local errors, they may be used for the construction of adaptive algorithms of mesh refinement. The main idea is to reduce...
computational costs by refining only areas with a high estimated error instead of the whole mesh. Thus, a reliable error indicator is necessary for any adaptation algorithm. Indicator based on the majorant from (4) and an adapted mesh are shown in figure 3. Method proposed in [35] was used in mesh refinements for quadrilaterals.

Figure 3. Example 1: local error indicator for a coarse mesh and an adaptive refinement.

Table 2. Example 2: error estimation for subsequent refinements.

| DOFs  | 195   | 675   | 2499  | 9603  |
|-------|-------|-------|-------|-------|
| Error | 0.175e3 | 0.953e2 | 0.487e2 | 0.242e2 |
| Estimate | 0.399e3 | 0.211e3 | 0.108e3 | 0.559e2 |
| Overestimation | 2.3 | 2.2 | 2.2 | 2.3 |

Figure 4. Example 2: ANSYS solution (auxiliary mesh, displacement and rotations).
4.2. Example 2. L-shaped plate.

Next example is L-shaped plate with moderate thickness and artificial material constants selected to provide a moderate locking. The side of the plate is 2 m, thickness — 0.1 m, \( E = 1 \text{ N/m}^2 \), \( \nu = 0.3 \). The plate was uniformly loaded (1 N/m\(^2\)). ANSYS solution is depicted in figure 4, the results of error estimation by (4) are presented in table 2, while corresponding free variables are shown in figure 5. All previous conclusions are also valid for this case: error overestimation does not increase with the number of DOFs. Adaptive algorithm was also implied for this example: see figure 6 for initial functional-type error indicator and obtained refined mesh.

**Figure 5.** Example 2: components of \( \tilde{\zeta} \) marked by \( \text{KxxLS, KyxLS, etc.} \) and components of \( \tilde{\eta} \) marked by \( \text{YTxLS and YTyLS, where “LS” means that fields come from solving of the corresponding linear system of algebraic equations.} \)

**Figure 6.** Example 2: local error indicator for a coarse mesh and an adaptive refinement.

5. Conclusions

From the presented results we conclude that the functional approach to error estimation of approximate solutions for Reissner-Mindlin plates is suitable even for coarse meshes. Arnold-Boffi-Falk approximation provides an efficient implementation of obtained majorant. Therefore,
the approach forms a good independent error control tool, which is applicable to commercial software.

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