Multi-loop soliton solutions and their interaction in the Degasperis–Procesi equation

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Abstract
In this paper, we construct loop soliton solutions and mixed soliton–loop soliton solutions for the Degasperis–Procesi equation. To explore these solutions, we adopt the procedure given by Matsuno (2005 Inverse Problems 21 1553). Appropriately modifying the $\tau$-function given in the above paper we derive these solutions. We present the explicit form of one- and two-loop soliton solutions and mixed soliton–loop soliton solutions and investigate the interaction between (i) two-loop soliton solutions in different parametric regimes and (ii) a loop soliton with a conventional soliton in detail.

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1. Introduction

In this paper, we investigate loop soliton solutions and their dynamics in the Degasperis–Procesi (DP) equation [2],

$$u_t + 3\kappa^3 u_x - u_{xxx} + 4uu_x = 3u_x u_{xx} + uu_{xxx},$$

where subscripts denote partial derivatives and $\kappa$ is a positive parameter. The integrability of the DP equation was proved by constructing a Lax pair, deriving an infinite sequence of conservation laws and the existence of a bi-Hamiltonian structure [3]. Equation (1) arises in a hydrodynamical context [4]. Interestingly, when $\kappa = 0$, equation (1) admits peakon solutions which are of the form $u = c e^{-|x-ct|}$, where $c$ is the velocity of the peakon. Consequently, the $N$-peakon solutions of the DP equation were constructed and the dynamics of these special solutions were also studied by Lundmark and Szmigielski [5]. The prolongation algebra and the Hamiltonian operator of this equation were reported in [6]. Vakhnenko and Parkes have studied various travelling wave solutions of the DP equation including hump-like, loop-like and coshoidal periodic-wave solutions [7]. Qiao [8] has come up with three new types of soliton solutions to this model, namely M-shape peakons, dehisced solitons, double dehisced one-peak solitons.

Here we are interested in exploring loop soliton solutions exhibited by equation (1). The motivation comes from the contemporary interest in studying loop soliton solutions in integrable nonlinear evolutionary equations [9–19]. To our knowledge periodic inverted loop solutions and one-loop soliton solutions were reported for the DP equation [7]. The explicit form of two-loop soliton or higher-order loop soliton solutions is not yet reported. Moreover, the collision dynamics between the loop soliton solutions is also yet to be studied for this equation.

To explore loop soliton solutions in equation (1) we follow the procedure given by Matsuno [1]. The author has derived $N$-soliton solutions for the DP equation from the modified version of the Kaup equation since the $\tau$-function of the latter is already known. By carefully examining the $\tau$-function we observe that one has the freedom in changing the sign of the coefficients of the exponential functions so that even after the sign changes the resultant $\tau$-function satisfies the bilinear identities. By taking this advantage, we reshape the $\tau$-function appropriately and derive loop soliton solutions in a systematic manner. In the case of a one-soliton solution, there are three new types of soliton solutions to this model, namely M-shape peakons, dehisced solitons, double dehisced one-peak solitons.
two exponential functions present in the $\tau$-function one has two choices in fixing the coefficients of these exponential functions. Either the coefficients of both the exponential functions are negative or one coefficient is negative and the other one is positive.

For the first choice we obtain two inverted loop solitons and for the second choice we obtain a mixed loop soliton–smooth soliton type solution. We consider both solutions and study their interaction properties. To begin with, we allow two-loop solitary waves (among the two, one is longer and the other is shorter) to travel in a particular direction. As expected the taller loop wave travels faster than the smaller one and crosses the shorter loop wave in a finite time. When it approaches the smaller loop they start to interact. As a consequence the amplitude of the taller loop becomes shorter while the amplitude of the shorter loop becomes larger. This continues until the amplitudes of both the taller and shorter loop solutions become equal.

In the second parametric regime, when the loop solitary waves cross each other the smaller and larger loop solitary waves overlap each other besides changing their amplitudes. In the third parametric regime, we show that while the larger loop solitary wave overtakes the smaller one, the smaller loop solitary wave revolves circularly (in a clockwise direction) inside the larger loop solitary wave.

In the second case (mixed loop soliton–smooth soliton), we allow the loop solitary wave to interact with the smooth solitary wave. Here, we bring out a totally different kind of interaction. The smaller loop solitary wave travels along the surface of the smooth soliton. All these results are new.

The plan of the paper is as follows. In section 2, we recall the method of finding soliton solutions to this model. We present the method of constructing loop soliton solutions to this equation and derive the explicit expressions of one- and two-loop soliton solutions in section 3. We investigate the collision dynamics in the case of two-loop solitons in detail. In section 4, we derive mixed soliton–loop soliton solutions and study their interaction properties. We present our conclusions in section 5.

2. Method of finding soliton solutions [1]

In this section, we briefly recall the method of constructing $N$-soliton solutions of (1). The DP equation can be written in a compact form, $q_t + q u_t = 0$, first by defining a new variable $q$ with

$$q^3 = u - u_{11} + \kappa^3,$$

and then introducing a reciprocal transformation, $dy = q dx - q u dt$, $d\bar{t} = dt$, in the resultant equation. Rewriting relation (2) one can express the old variable ($u$) in terms of new variable ($q$) as

$$u = -q (\ln q)_t + q^3 - \kappa^3,$$

where we have dropped the tilde above ‘$t$’ for simplicity. From (3) we can find $u_{11}$. Substituting this derivative into the expression $q_t + q^2 u_{11} = 0$ and differentiating the resultant equation with respect to $y$ we obtain a third-order ordinary differential equation in $q$. By successive differentiation this third-order equation can be transformed to the first negative flow in the KP hierarchy (for more details one may refer to [1]). The soliton solutions of this member can be identified from the soliton solutions of the modified version of the KP equation. From the known $\tau$-function of the KP equation one can go back and construct $N$-soliton solutions for the DP equation. To do this first one should express $q$ in terms of the $\tau$-function of the KP equation. Doing so we obtain

$$q^2 = -(\ln f)_y + \kappa^2, \quad f = \text{det} A,$$

where $A = (a_{jk})$ is a $2N \times 2N$ matrix with elements

$$a_{jk} = \left(1 + e^{q_j}\right) \delta_{jk} + \frac{\tilde{p}_j - \tilde{q}_j}{\tilde{p}_j - \tilde{q}_k} (1 - \delta_{jk}), \quad j, k = 1, 2, \ldots, N; \quad (5a)$$

$$\tilde{\xi}_{2j-1} = \tilde{\xi}_{2j} = k_j (y + \tilde{c}_j t - \gamma_j y_0) + \ln a_j, \quad j = 1, 2, \ldots, N; \quad (5b)$$

$$\tilde{p}_{2j-1} = q_j, \quad \tilde{q}_{2j-1} = -p_j, \quad \tilde{p}_{2j} = p_j, \quad \tilde{q}_{2j} = -q_j, \quad j = 1, 2, \ldots, N; \quad (5c)$$

with the soliton velocity defined by

$$c_j = \frac{3k_j^2}{\kappa^2 k_j^2 - 1}, \quad a_j = \sqrt{\frac{1 - \kappa^2 k_j^2}{1 - \kappa^2 k_j^2}}, \quad j = 1, 2, \ldots, N. \quad (6)$$

In the above $k_j$ are wave parameters and the amplitude parameters $p_j$ and $q_j$ are found to be

$$p_j = \frac{k_j}{2} \left[1 + \frac{2}{\kappa k_j} \left(\frac{1}{3} \left(1 - \frac{\kappa^2 k_j^2}{4}\right)\right)\right],$$

$$q_j = \frac{k_j}{2} \left[1 - \frac{2}{\kappa k_j} \left(\frac{1}{3} \left(1 - \frac{\kappa^2 k_j^2}{4}\right)\right)\right].$$

In a nutshell, to construct the solutions, one should first obtain $q$ by substituting the explicit form of $f$ (vide equation (5a)) into (4). By plugging this $q$ in (3) one can obtain an implicit form of $u(y, t)$. Finally, substituting the expression $q$ into the mapping function,

$$x = \frac{y}{\kappa} + \int_{-\infty}^{y} \left(\frac{1}{q} - \frac{1}{\kappa}\right) dy + d,$$

where $d$ is an integration constant, and integrating it one can get a relationship between $x$ and $y$. We note that the function $q(y, t)$ should not vanish for any $y$ and $t$. Otherwise, a simple zero of $q$ would yield a logarithmic singularity of $x(y, t)$. Expressions (3) and (8) constitute the solution for the DP equation in parametric form. As $q$ involves the variable $y$, after integration, it is often difficult to express $y$ in terms of $x$ explicitly. In [1], using the above procedure, the author has explicitly derived one and two soliton solutions of the DP equation (1).
3. Method of finding loop solitons

To explore $N$-loop soliton solutions of the DP equation we modify the bilinear solution (5a) as

$$a_{jk} = (1 - e^{\delta_j})\delta_{jk} + \frac{\tilde{p}_j - \tilde{q}_j}{\tilde{p}_j - \tilde{q}_k}(1 - \delta_{jk}), \quad j, k = 1, 2, \ldots, N,$$

with all other parameters $\tilde{p}_j$, $\tilde{q}_j$, and $\tilde{q}_k$ as given in equations (5b) and (5c). One can unambiguously prove that the resultant $\tau$-function which comes out from the modified matrix elements also satisfies the bilinear identities. A simple sign change in the $\tau$-function leads to a new class of soliton solutions as we see below.

3.1. One-loop soliton solution

Let us take $N = 1$ in (4) with the matrix elements defined in (9). Expanding the determinant (4) we find the $\tau$-function modified to

$$f = a_1^2 \left(1 - \frac{2}{a_1} e^{\xi_1} + e^{2\xi_1}\right), \quad (9)$$

where

$$\xi_1 = k_1 \left(y + \frac{3k_1^2}{k_1^2 - 1} - t - y_10\right), \quad a_1 = \frac{1 - k_1^2/4}{1 - k_1^2/4},$$

$$(\kappa k_1 > 2).$$

(11)

One may observe that the sign in front of the second exponential in $f$ is positive in the case of one smooth soliton solution [1]. However, this change of sign has other impacts on the solution. For example, in the regime $\kappa k_1 < 2$, $q$ has a simple zero and $u$ has a simple pole and the variable change $y$ has a logarithmic singularity, as we see below.

Substituting (10) into (4) and carrying out the derivatives and simplifying the resultant equation, we find

$$q = \kappa \frac{\cosh \xi_1 - 2a_1 + 1/a_1}{\cosh \xi_1 - 1/a_1}.$$  

(12)

Now, plugging $q$ and its derivatives in (3) and simplifying the resultant expressions to a compact form we arrive at

$$u = -\frac{8k_3 (a_1^2 - 1) (a_1^2 - 1/4)}{a_1} \cosh \xi_1 - 2a_1 + 1/a_1.$$  

(13)

Substituting $q(y, t)$ into equation (8) and performing integration we obtain the coordinate transformation between $x$ and $y$ in the following implicit form:

$$x = \frac{\kappa}{\kappa} \ln \left[\frac{1 + a_1 + (1 - a_1)e^{\xi_1}}{1 - a_1 + (1 + a_1)e^{\xi_1}}\right],$$  

(14)

where

$$a_1 = \frac{(2a_1 - 1)(a_1 + 1)}{(2a_1 + 1)(a_1 - 1)}.$$  

(15)

Since the variable $y$ depends on $\xi_1$ one cannot express $y$ in terms of $x$ explicitly.

Due to coordinate transformation and the consequence of the change of sign in front of the exponential in expression (10), the smooth solitary wave form morphed into a multivalued inverted loop-like wave form with the condition given in equation (11). We present the nature of the obtained solutions in figure 1. To begin with, we draw the solution (13) in the variable $y$ where we get a smooth inverted solitary wave (figure 1(a)). While we depict the solution in terms of the original variable $x$ we obtain only the loop solitary wave form (figure 1(b)). To understand the geometrical connection between $x$ and $y$ we also draw a graph between these variables separately (figure 1(c)).

3.2. Two-loop soliton solutions

The $\tau$-function for the two-loop soliton solution follows from equation (4) by restricting to $N = 2$. The associated determinant of $f$ now reads,

$$f = \left|\begin{array}{cccc}
1 - a_1 e^{\xi_1} & a_1 e^{\xi_1} &\ldots & a_1 e^{\xi_1} \\
\frac{p_1 q_{i1}}{2p_{i1}} & 1 - a_1 e^{\xi_1} & \ldots & a_1 e^{\xi_1} \\
\frac{q_{i1}}{p_{i1}} & \frac{p_{i1}}{q_{i1}} & \ldots & a_1 e^{2\xi_1} \\
\frac{q_{i1}}{p_{i1}} & \frac{p_{i1}}{q_{i1}} & \ldots & 1 - a_2 e^{\xi_2} \\
\frac{p_{i1} q_{i1}}{p_{i2} q_{i1}} & \frac{p_{i1} q_{i1}}{p_{i1} q_{i2}} & \ldots & 1 - a_2 e^{2\xi_2}
\end{array}\right|.$$  

(16)

When compared with two smooth soliton solutions, the signs in front of the parameters $a_1$ and $a_2$ are different [1].
Substituting the expressions of \( p_j \) and \( q_j \), \( j = 1, 2 \), given in (7), into the above determinant and expanding it we obtain the explicit form of the \( \tau \)-function as

\[
f = (a_1a_2)^2 \left( \frac{2\delta}{a_1} \frac{e^{\xi_1}}{a_2} + \frac{2\delta}{a_2} e^{\xi_2} + \frac{2v}{a_1a_2} e^{\xi_1+\xi_2} \right),
\]

where

\[
\delta = \frac{(k_1-k_2)^2[k^2(k_1^2-k_1k_2+k_2^2)-3]}{(k_1+k_2)^2[k^2(k_1^2+k_1k_2+k_2^2)-3]},
\]

\[
v = \frac{(2k_1^2-k_1^2k_2^2+2k_2^2)k^2-6(k_1^2+k_2^2)}{(k_1+k_2)^2[k^2(k_1^2+k_1k_2+k_2^2)-3]}.
\]

Substituting (17) and (18) into equation (4) we obtain a compact expression for \( q(y,t) \), that is

\[
q(y,t) = \frac{g_1g_2}{f'(a_1a_2)^2} = \kappa \frac{g_1g_2}{f'(a_1a_2)^2}.
\]

The explicit forms of \( g_1 \) and \( g_2 \) are given by

\[
g_1 = \delta \frac{2-kk_1}{2a_1(1+kk_1)} e^{\xi_1} + \frac{2-kk_2}{2a_2(1+kk_2)} e^{\xi_2} + \frac{(2-kk_1)(2-kk_2)}{4a_1a_2(1+kk_1)(1+kk_2)} e^{\xi_1+\xi_2},
\]

\[
g_2 = \delta \frac{2+kk_1}{2a_1(1+kk_1)} e^{\xi_1} + \frac{2+kk_2}{2a_2(1+kk_2)} e^{\xi_2} + \frac{(2+kk_1)(2+kk_2)}{4a_1a_2(1+kk_1)(1+kk_2)} e^{\xi_1+\xi_2}.
\]

We note here that the right-hand side of equation (4) turns out to be a perfect square and the numerator in the resultant expression is factorized into the product of two functions, that is \( g = g_1g_2 \), with \( g_1 \) and \( g_2 \) being polynomials of \( e^{\xi_1} \) and \( e^{\xi_2} \).

Substituting (19) into (3) and performing differentiation we find the solution \( u(y,t) \) to be of the form

\[
u = \kappa^2 \frac{h}{g},
\]

with

\[
h = -\frac{9\delta k^2 k_2^2}{a_1(1-k^2k_1^2)^2} e^{\xi_1} + \frac{9\delta k^2 k_2^2}{a_2(1-k^2k_2^2)^2} e^{\xi_2} + \frac{9\delta k_1^2 k_2^2}{a_1(1-k^2k_1^2)^2} e^{\xi_1+\xi_2} + \frac{9\delta k_1^2 k_2^2}{a_2(1-k^2k_2^2)^2} e^{2\xi_1+\xi_2}.
\]

We observe that the numerator of \( u \) is of the same form as for the two soliton solutions with the only difference being an over all sign change. The coordinate transformation between \( x \) and \( y \) is given by

\[
x(y,t) = \frac{y}{\kappa} + \ln \left( \frac{g_1}{g_2} \right) + d,
\]

where \( g_1 \) and \( g_2 \) are given in (20) and (21).

3.3. Two-loop soliton interactions

Expressions (22) and (24) provide a complete description of the two-loop soliton solution in the form of a parametric representation.

It describes the two-loop solitary wave troughs. Figure 2 shows the interaction between two-loop solitary wave troughs. Here, we plot the solution \( u \) in terms of \( x \) with the parameter values \( \kappa = 1.5, k_1 = 3.2, k_2 = 3.8 \) and \( d = 0 \). Both the solitary waves propagate towards the negative \( x \)-direction. Initially, at \( t = -10 \), the larger loop solitary wave is well separated from the smaller one (figure 2(a)). Since the larger loop soliton travels faster than the smaller one it starts to cross the smaller one in a finite time. Now, the larger wave loses energy to the smaller wave. As a consequence, the amplitude of the larger loop decreases and the amplitude of the smaller one increases. At \( t = -0.8 \), these two-loop solitons have equal amplitude or energy (figure 2(b)). The two-loop solitons do not superpose into a single wave as one can see from (figure 2(b)). After interaction (elastic interaction) these two solitons re-emerge and travel in their original direction by keeping their original amplitudes (figure 2(c)).

Now, we investigate the two-loop soliton interaction in a different parametric regime, say for example, \( \kappa = 2.5, k_1 = 3.4, k_2 = 4.8 \) and \( d = 0 \). To begin with \( t = -5 \) the two-loop solitons are well separated (figure 3(a)). As in the previous case both the larger and smaller loop waves move towards the left. At \( t = -1 \) the larger loop solitary wave approaches the smaller one (figure 3(b)). Figure 3(c) shows the change in their...
amplitudes when they start to interact with the amplitude of the taller one becoming shorter and the smaller one becoming larger. Interestingly, at $t = -0.25$, the amplitudes of both the taller and smaller waves become equal and overlap each other (figure 3(d)). As time progresses the two-loop solitons emerge out from each other in the original direction by changing their heights (figure 3(e)). At $t = 5$ the loops are completely separated and attain their original amplitudes.

Another interesting loop soliton interaction can be seen by fixing the parametric values as $\kappa = 3.5$, $k_1 = 10.4$, $k_2 = 4.2$ and $d = 0$. We draw the interaction picture in figure 4. The two-loop solitons are well separated from each other in the beginning, see figure 4(a). When time moves on the larger loop meets the smaller loop (figure 4(b)) and they start to interact (figure 4(c)). In distinction to the previous two cases the smaller loop revolves inside the larger loop in clockwise direction (figures 4(c)–(e)) and finally emerges out from the larger loop. Later, both the loops attain their original amplitudes.

4. Mixed loop soliton–soliton solution of the DP equation

In the previous section, while deriving the two-loop soliton solutions we changed the sign of the coefficients of both exponential functions in the $\tau$-function to negative values, see equation (16). In this section, we consider another possibility, we only change the sign of one exponential function

$$f = \frac{1 - a_2 e^{z_1}}{1 + a_2 e^{z_1}} \left| \begin{array}{ccc} p_{1+q_2} & p_{1+q_2} & p_{1+q_2} \\ -p_2 & -p_2 & -p_2 \\ p_2 & p_2 & p_2 \end{array} \right| \frac{p_{1+q_1}}{2p_2} \left| \begin{array}{ccc} p_{1+q_1} & p_{1+q_1} & p_{1+q_1} \\ p_{1+q_2} & p_{1+q_2} & p_{1+q_2} \\ 1 + a_2 e^{z_1} \end{array} \right| .$$

(25)

For the opposite sign choice we would obtain inverted solutions. By performing the same procedure as in section 3.1, we obtain the following expression for the functions $f$, $g$ and $u$ respectively, that is,

$$f = (a_1 a_2)^2 \left( 2 \frac{a_1}{a_2} e^{z_1} + \frac{a_1}{a_2} e^{2z_1} + e^{z_2} + e^{2z_2} - \frac{2y}{a_1 a_2} e^{z_1+z_2} + \frac{2}{a_2} e^{2z_1+z_2} + e^{2z_1+z_2} \right),$$

(26)

$$g(y, t) = \kappa \frac{g_1 g_2}{f(a_1 a_2)^2},$$

(27)

$$u = \kappa^3 \frac{h}{g},$$

(28)

where,

$$g_1 = \delta - \frac{2 - \kappa k_1}{2a_1(1 + \kappa k_1)} e^{z_1} + \frac{2 - \kappa k_2}{2a_2(1 + \kappa k_2)} e^{z_2} - \frac{(2 - \kappa k_1)(2 - \kappa k_2)}{4a_1 a_2(1 + \kappa k_1)(1 + \kappa k_2)} e^{z_1+z_2},$$

(29)

$$g_2 = \delta - \frac{2 + \kappa k_1}{2a_1(1 + \kappa k_1)} e^{z_1} + \frac{2 + \kappa k_2}{2a_2(1 + \kappa k_2)} e^{z_2} - \frac{(2 + \kappa k_1)(2 + \kappa k_2)}{4a_1 a_2(1 + \kappa k_1)(1 + \kappa k_2)} e^{z_1+z_2},$$

(30)

$$h = -\frac{9 \kappa^2 k_1^2}{a_1(1 - \kappa k_1^2)} e^{z_1} + \frac{9 \kappa^2 k_2^2}{a_2(1 - \kappa k_2^2)} e^{z_2} - \frac{9 \kappa^2 k_1^2}{a_1(1 - \kappa k_1^2)} e^{z_1+z_2} + \frac{9 \kappa^2 k_2^2}{a_2(1 - \kappa k_2^2)} e^{z_1+z_2}.$$
Equation (28) is a mixed soliton–loop soliton solution of the DP equation with coordinate transformation given in equation (24).

4.1. Mixed soliton–loop soliton interaction

In this sub-section we investigate the head on collision between a loop soliton with a smooth soliton. Equation (28) describes a smooth solitary wave form travelling towards the right and a loop wave trough travelling towards the left. These two waves propagate in opposite directions and collide with each other. To investigate the outcome we draw the solution (28) in figure 5 with the parameter values $\kappa = 0.91$, $k_1 = 2.6$ and $k_2 = 0.91$. In this case, we observe that the anti-loop soliton creates a secondary wave crest. As time goes on the secondary pulse grows in amplitude while the primary wave loses its momentum. One can observe a double-peaked smooth wave when the loop wave reaches the top of the smooth solitary wave. After a certain time the smaller loop starts sliding in the left direction. This interaction process continues until the two waves separate from each other.

5. Conclusion

In this paper, we have focused our attention on obtaining loop soliton solutions of the DP equation (1). We have
recovered these solutions directly from the τ-function of the modified version of the Kaup equation. We have given parametric representations for both the pure loop solitons and mixed loop–smooth solitons and studied their wave dynamics. We have investigated the loop–loop soliton and mixed loop–smooth soliton interactions in detail. In the case of the loop soliton interaction, when the amplitudes of the loop solitons are dissimilar, we observed that the smaller loop travels across the larger one in three different fashions before being shifted. We have shown the formation of double peaked soliton waves in the case of mixed loop–soliton interaction. Currently, we are formulating \( N \)-loop soliton solutions for the DP equation.

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References

[1] Matsuno Y 2005 Multisoliton solutions of the Degasperis–Procesi equation and their peakon limit Inverse Problems \textbf{21} 1553

[2] Degasperis A and Procesi M 1999 Symmetry and Perturbation Theory (Singapore: World Scientific) p 22

[3] Degasperis A, Hone A N W and Holm D D 2002 A new integrable equation with peakon solutions Theor. Math. Phys. \textbf{133} 1463

Constantin A, Ivanov R I and Lenells J 2010 Inverse scattering transform for the Degasperis–Procesi equation Nonlinearity \textbf{23} 2559

[4] Johnson R S 2002 Camassa–Holm, Korteweg–de Vries and related models for water waves \textit{J. Fluid Mech.} \textbf{455} 63

Constantin A and Lannes D 2009 The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations Arch. Ration. Mech. Anal. \textbf{192} 165

[5] Lundmark H and Szmigielski J 2003 Multi-peakon solutions of the Degasperis–Procesi equation Inverse Problems \textbf{19} 1241

[6] Hone A N W and Wang J P 2003 Prolongation algebras and Hamiltonian operators for peakon equations Inverse Problems \textbf{19} 129

[7] Vakhnenko V O and Parkes E J 2004 Periodic and solitary-wave solutions of the Degasperis–Procesi equation Chaos Soliton Fractals \textbf{20} 1059

Linells J 2005 Traveling wave solutions of the Degasperis–Procesi equation J. Math. Anal. Appl. \textbf{306} 72

[8] Qiao Z 2008 M-shape peakons, dehisced solitons, cuspons and new 1-peak solitons for the Degasperis–Procesi equation Chaos Soliton Fractals \textbf{37} 501

[9] Vakhnenko V A 1992 Solitons in a nonlinear model medium J. Phys. A: Math. Gen. \textbf{25} 4181

[10] Vakhnenko V O and Parkes E J 1998 The two loop soliton solution of the Vakhnenko equation Nonlinearity \textbf{11} 1457

[11] Morrison A J and Parkes E J 2001 The \( N \)-soliton solution of a generalised Vakhnenko equation Glasgow Math. J. \textbf{43A} 65

[12] Morrison A J and Parkes E J 2003 The \( N \)-soliton solution of the modified generalised Vakhnenko equation (a new nonlinear evolution equation) Chaos Solitons Fractals \textbf{16} 13

[13] Sakovich A and Sakovich S 2006 Solitary wave solutions of the short pulse equation J. Phys. A: Math. Gen. \textbf{39} L361

[14] Matsuno Y 2008 Periodic solutions of the short pulse model equation J. Math. Phys. \textbf{49} 073508

[15] Feng B F, Maruno K I and Ohta Y 2010 Integrable discretizations of the short pulse equation J. Phys. A: Math. Theor. \textbf{43} 085203

[16] Lin J, Ren B, Li H M and Li Y S 2008 Soliton solutions for two nonlinear partial differential equations using a Darboux transformation of the Lax pairs Phys. Rev. E \textbf{77} 036605

[17] Rogers C, Schief W K and Szereszewski A 2010 Loop soliton interaction in an integrable nonlinear telegraphy model: reciprocal and Bäcklund transformations J. Phys. A: Math. Theor. \textbf{43} 385210

[18] Matsuno Y 2010 A direct method for solving the generalized sine-Gordon equation J. Phys. A: Math. Theor. \textbf{43} 105204

[19] Matsuno Y 2006 Cusp and loop soliton solutions of short-wave models for the Camassa–Holm and Degasperis–Procesi equations Phys. Lett. A \textbf{359} 451