Super Vector Bundles

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Abstract. We review the basic aspects of the theory of smooth super vector bundles. In particular we give the precise link existing between the geometrical and the sheaf theoretical approaches.

1. Introduction
In [1, § 3.2] it is stated very concisely that, if $M$ is a (smooth) supermanifold, a (smooth) super vector bundle of rank $m|n$ over $M$ can be defined in two equivalent ways as:

(i) a super fiber bundle $E$ over $M$, with typical fiber $\mathbb{R}^{m|n}$ and structural group $\text{GL}(m|n)$ (geometrical approach);
(ii) a sheaf of $\mathcal{O}_M$-modules $\mathcal{V}$, locally free of dimension $m|n$ (algebraic approach).

In this paper we are going to examine how this objects can be defined, how the two approaches are linked and how it is possible to recover the ideas of fiber and section. The main problem is that a super vector bundle cannot be described as a family of super vector spaces parameterized by the points of $\tilde{M}$. Indeed, as usual, working pointwise, one loses information about the nilpotent part of the sheaf. So we always have to consider neighbourhoods of the points.

2. Supermanifolds
A (smooth) supermanifold $M$ of dimension $p|q$ is a locally compact, second countable, Hausdorff topological space $|M|$ endowed with a sheaf $\mathcal{O}_M$ of superalgebras, locally isomorphic to $C^\infty(\mathbb{R}^p) \otimes \Lambda(\vartheta_1, \ldots, \vartheta_q)$. A morphism $\psi: M \rightarrow N$ between supermanifolds is a pair of morphisms $(|\psi|, \psi^*)$ where $|\psi|: |M| \rightarrow |N|$ is a continuous map and $\psi^*: \mathcal{O}_N \rightarrow \mathcal{O}_M$ is a sheaf morphism above $|\psi|$. We will consider only smooth supermanifolds. It can be proved that in this category a morphism of supermanifolds is determined once we know the corresponding morphism on the superalgebra of global sections (see, for example, [2] and [3]). In other words, a morphism $\psi: M \rightarrow N$ can be identified with a superalgebra map $\psi^*: \mathcal{O}_N(|N|) \rightarrow \mathcal{O}_M(|M|)$. We will tacitly use this fact several times. Moreover, in the following, we will denote with $\mathcal{O}(M)$ the superalgebra of global sections $\mathcal{O}_M(|M|)$.

Remark 2.1. Suppose now $U$ is an open subset of $|M|$ and let $\mathcal{J}_M(U)$ be the ideal of the nilpotent elements of $\mathcal{O}_M(U)$. It is possible to prove that $\mathcal{O}_M/\mathcal{J}_M$ defines a sheaf of purely even algebras over $|M|$. 

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locally isomorphic to $C^\infty(R^p)$. Therefore $\tilde{M} := (|M|, \mathcal{O}_M/\mathcal{J}_M)$ defines a classical manifold, called the reduced manifold associated to $M$. The projection $s \mapsto \tilde{s} := s + \mathcal{J}_M(U)$, with $s \in \mathcal{O}_M(U)$, is the pullback of the embedding $j : \tilde{M} \to M$. In the following we denote with $ev_x(s) := \tilde{s}(x)$ the evaluation of $s$ at $x \in U$. It is also possible to check that, given a morphism $\psi : M \to N$, $|\psi|^* (\tilde{s}) = \psi^* (s)$, so that the map $|\psi|$ is automatically smooth. As a consequence, it is possible to associate a reduced map $\tilde{\psi} = |\psi| : \tilde{M} \to \tilde{N}$ to each supermanifold morphism $\psi$.

An important and very used tool in working with supermanifolds is the functor of points. Given a supermanifold $M$ one can construct the functor

$$M(\cdot) : \text{SMan}^{\text{op}} \to \text{Set}$$

from the opposite of the category of supermanifolds to the category of sets defined by $S \mapsto M(S) := \text{Hom}(S, M)$ and called the functor of points of $M$. In particular, for example, $M(R^{0|0}) \cong |M|$ as sets. Each supermanifold morphism $\psi : M \to N$ defines the natural transformation $\psi(\cdot) : M(\cdot) \to N(\cdot)$ given by $[\psi(S)](x) := \psi \circ x$. Due to Yoneda’s lemma, each natural transformation between $M(\cdot)$ and $N(\cdot)$ arises from a unique morphism of supermanifolds in the way just described. The category of supermanifolds can thus be embedded onto a full subcategory of the category $[\text{SMan}^{\text{op}}, \text{Set}]$ of functors from the opposite of the category of supermanifolds to the category of sets. Let

$$\mathcal{Y} : \text{SMan} \to [\text{SMan}^{\text{op}}, \text{Set}]$$

$$M \mapsto M(\cdot)$$

denote such embedding. It is a fact that the image of $\text{SMan}$ under $\mathcal{Y}$ is strictly smaller than $[\text{SMan}^{\text{op}}, \text{Set}]$. The elements of $[\text{SMan}^{\text{op}}, \text{Set}]$ isomorphic to elements in the image of $\mathcal{Y}$ are called representable. Supermanifolds can thus be thought as the representable functors in $[\text{SMan}^{\text{op}}, \text{Set}]$. For all the details we refer to [1, 2, 4, 5, 6].

**Example 2.2.** $R^{p|q}$ is the supermanifold whose reduced manifold is $R^p$, and with the sheaf of sections given by the restriction of $C^\infty(R^p) \otimes \Lambda(R^q)$. Notice that $R^{p|q}$ denotes also the super vector space $R^{p|q} = R^p \oplus R^q$. Using the functor of points one can prove that the two concepts can be identified (in the sheaf-theoretical approach, the linear structure of $R^{p|q}$ is encoded by the linear sections $(R^{p|q})^*)$.

### 3. Constant rank morphisms and Subsupermanifolds

The definition of immersion and submersion does not present particular problems.

**Definition 3.1.** Let $\psi : M \to N$ be a supermanifold morphism and $x \in \tilde{M}$. $\psi$ is an immersion (resp. a submersion) at $x$ if $(d\psi)_x$ is injective (resp. surjective). $\psi$ is an immersion (resp. a submersion), if this happens at each point. $\psi$ is an injective immersion (resp. a surjective submersion), if, in addition, $\tilde{\psi}$ is injective (resp. surjective).

**Remark 3.2.** If $\psi : M \to N$ is a submersion and $U$ is an open subsupermanifold of $N$, then $\psi^{-1}(U)$ is defined as the open subsupermanifold of $M$ on $\tilde{\psi}^{-1}(U)$.

In the super context the concept of constant rank is more subtle (see, in particular, [4, § 2.3]). We recall that, if $M$ is a supermanifold, $\text{Aut}_{\text{CM}}(\mathcal{O}(M)^{p|q})$ denotes the group of the even invertible $(p|q) \times (p|q)$ matrices $\begin{pmatrix} s_{11,j_1} & t_{k_1,l_1} \\ t_{k_2,l_2} & s_{12,j_2} \end{pmatrix}$ with entries $s_{i,r,j,r} \in \mathcal{O}(M)_0$ and $t_{k,r,j,r} \in \mathcal{O}(M)_1$. 
Definition 3.3. Let $U$ be a superdomain. If $J = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$ is an even $(p|q) \times (m|n)$ matrix with entries in $\mathcal{O}(U)$, then we say that $J$ has constant rank $r$ if there exist $G_1 \in \text{Aut}_{\mathcal{O}(U)}(\mathcal{O}(U)^{p|q})$ and $G_2 \in \text{Aut}_{\mathcal{O}(U)}(\mathcal{O}(U)^{m|n})$ such that $G_1JG_2$ has the form $\left( \begin{array}{c} A' & 0 \\ 0 & D' \end{array} \right)$ with $A' = \left( \begin{array}{c} I_p \varepsilon \end{array} \right)$ and $D' = \left( \begin{array}{c} 0 \varepsilon \end{array} \right)$. If $\psi: M \to N$ is a morphism between supermanifolds, we say that $\psi$ has constant rank $r$ at $x \in \tilde{M}$ if there exists a coordinate neighborhood of $x$ such that the super Jacobian $J_\psi$ has constant rank $r$ at every point.

Proposition 3.4. Let $\psi: M \to N$ be a supermanifold morphism and $x \in \tilde{N}$, such that $\psi$ has constant rank $r$ at each point of $\tilde{\psi}^{-1}(x)$ (see definition 3.3). If $\mathcal{J}_x := \{ s \in \mathcal{O}(N) \mid \tilde{s}(x) = 0 \}$ and $\mathcal{J}_x^{\psi} := \mathcal{O}(M)\psi^*(\mathcal{J}_x)$ is the ideal in $\mathcal{O}(M)$ generated by $\psi^*(\mathcal{J}_x)$, then there exists a unique closed subsupermanifold $S$ of $M$ such that $\mathcal{J}_x^{\psi} = \mathcal{J}_S$. $S$ has dimension $\dim M - r$.

Definition 3.5. $\psi^{-1}(x)$ denotes the closed subsupermanifold $S$ in proposition 3.4.

4. Super Vector Bundles

4.1. Geometrical approach

In this paragraph we discuss the geometrical formulation of the concept of super vector bundle. We begin with the definition of super fiber bundle.

Before we proceed, let us fix some notation. We recall that, if $M$ is a supermanifold, $\tilde{M}$ denotes the associated reduced manifold defined in the previous section. In order to avoid possible confusions, we distinguish between an open subset $\tilde{U}$ of $\tilde{M}$ and the corresponding open subsupermanifold $U := (\tilde{U}, \mathcal{O}_M|\tilde{U})$ of $M$ (see also section 3).

As usual, $!_M$ (or simply $!$) denotes the unique map from some supermanifold $M$ to the terminal object $R^{0|0}$. If $x \in \tilde{M}$, $\hat{x}$ denotes the constant map from some supermanifold to $M$ whose image is $x$ or, in other words, $\hat{x}$ is the morphism whose pullback is the evaluation $ev_x$.

Definition 4.1. Let $E$, $M$, and $N$ be supermanifolds and let

$$\pi: E \rightarrow M$$

be a surjective submersion. $E$ is called super fiber bundle with base $M$ and typical fiber $N$ if, for all $x \in \tilde{M}$, there is an open neighborhood $\tilde{U}$ of $x$ and an isomorphism

$$t: \pi^{-1}(U) \rightarrow U \times N$$

such that $p_U \circ t = \pi$ (here and often in the following the necessary restrictions are understood). The morphism $\pi$ is called projection and a pair $(U, t)$ local trivialization.

Notice that, according to remark 3.2,

$$\pi^{-1}(U) = (\tilde{\pi}^{-1}(\tilde{U}), \mathcal{O}_E|\tilde{\pi}^{-1}(\tilde{U}))$$

Since $\pi$ is a submersion, it is a constant rank morphism. The following definition is hence well posed.

Definition 4.2. Let $x \in \tilde{M}$. The subsupermanifold $\pi^{-1}(x)$ of $E$ specified by definition 3.5 is called fiber of $E$ at $x$. We denote the embedding $\pi^{-1}(x) \rightarrow E$ by $i_x$. 

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The next proposition shows that the above definition is coherent.

**Proposition 4.3.** Let \( \pi^{-1}(x) \) be as in the foregoing definition. Then

(i) \( \pi^{-1}(x) \) is the pullback of \( \xrightarrow{\pi} E \xrightarrow{\delta} M \); \( \pi^{-1}(x) \) is (not canonically) isomorphic to typical fiber \( N \).

(ii) \( \pi^{-1}(x) \) is (not canonically) isomorphic to typical fiber \( N \).

**Example 4.4.** \( E := M \times N \) with \( \pi := \text{pr}_M \) is a super fiber bundle (trivial super fiber bundle) with typical fiber \( N \).

**Definition 4.5.** We say that \( (E, \pi, M, V) \) is a super vector bundle if

(i) \( V = V_0 \oplus V_1 \) is a finite-dimensional super vector space;

(ii) \( (E, \pi, M, V) \) is a fiber bundle with typical fiber \( V \);

(iii) there exists a family of local trivializations \( \{ (U_i, t_i) \} \) such that

(a) \( \bigcup_i \tilde{U}_i = \tilde{M} \);

(b) if \( \tilde{U}_{i,j} := \tilde{U}_i \cap \tilde{U}_j \), there exists a map (called transition function)

\[
\varphi_{i,j} : \tilde{U}_{i,j} \longrightarrow \text{GL}(V)
\]

such that the isomorphism \( t_i \circ t_j^{-1} \) is of the form

\[
\begin{align*}
\tilde{U}_{i,j} \times V & \xrightarrow{t_i \circ t_j^{-1}} \tilde{U}_{i,j} \times V \\
(\tilde{x}, v) & \xrightarrow{1 \times \alpha} \tilde{U}_{i,j} \times \text{GL}(V) \times V
\end{align*}
\]

where \( \alpha \) the action of \( \text{GL}(V) \) on \( V \).

**Remark 4.6.** The condition expressed by diagram (1) is stronger than the request that changes of trivialization are fiberwise linear. Example 4.15 will further clarify this aspect.

**Proposition 4.7.** If \( (E, \pi, M, V) \) is a super vector bundle, then, for all \( x \in \tilde{M} \), \( \pi^{-1}(x) \cong V \) (in not canonical way) as super vector spaces.

**Proof.** From proposition 4.3, \( \pi^{-1}(x) \cong V \) as supermanifolds and so \( \pi^{-1}(x) \) inherits a super vector space structure from \( V \). The choice of another trivialization involves an even linear isomorphism that does not alter this structure. Indeed, if \( (U, t) \) and \( (U, t') \) are local trivializations in a neighborhood of \( x \), \( \varphi : U \rightarrow \text{GL}(V) \) is the transition function between these, and \( u, u' : \pi^{-1}(x) \rightarrow V \) are the isomorphisms induced by \( t \) and \( t' \), we have

\[
u \circ u'^{-1} = \alpha \circ (\varphi \times 1_V) \circ (\tilde{x}, 1_V) : V \longrightarrow V
\]

that is an even linear isomorphism since \( \tilde{\varphi}(x) \in \tilde{\text{GL}}(V) = \text{GL}(V_0) \times \text{GL}(V_1) \).

Let now \( M \) be a supermanifold, \( V \) a finite-dimensional super vector space, and \( \{ \tilde{U}_i \} \) an open covering of \( \tilde{M} \).
Definition 4.8. A family \( \{ \varphi_{i,j} \} \) of morphisms

\[
\varphi_{i,j} : U_{i,j} \to \mathrm{GL}(V)
\]

is said to satisfy the cocycle condition if, for each \( i,j,k \), \( \varphi_{i,j} \cdot \varphi_{j,k} = \varphi_{i,k} \) in \( \mathrm{GL}(V)(U_{i,j,k}) \), where \( U_{i,j,k} \) denotes the open subsupermanifold on \( \tilde{U}_i \cap \tilde{U}_j \cap \tilde{U}_k \).

Proposition 4.9. Transition functions of a super vector bundle \((E, \pi, M, V)\) satisfy the cocycle conditions. Vice-versa if \( M, V \), and \( \{ \tilde{U}_i \} \) are as above, and a family of functions \( \{ \varphi_{i,j} \} \) is given satisfying the cocycle conditions, then there exists a unique super vector bundle \((E, \pi, M, V)\) (up to isomorphism) with typical fiber \( V \) and \( \{ \varphi_{i,j} \} \) as transition functions.

Proof. First statement is immediate using functor of points. Indeed if \( \{ \varphi_{i,j} \} \) is the family of transition functions of \( E \), we have\(^1\)

\[
(U_{i,j,k} \times V)(S) \xrightarrow{\varphi_{i,j,k}^{-1}} (U_{i,j,k} \times V)(S) \xrightarrow{\varphi_{j,k}^{-1}} (U_{i,j,k} \times V)(S)
\]

\[
(x, v) \mapsto (x, (\varphi_{i,j,k}(x), v)) = (x, (\varphi_{i,j}(x) \cdot \varphi_{j,k}(x), v))
\]

for each supermanifold \( T \), \( x : T \to U_{i,j,k} \), and \( v : T \to V \); taking \( x = 1_{U_{i,j,k}} \) and comparing with eq. (1), we have \( \varphi_{i,j} \cdot \varphi_{j,k} = \varphi_{i,k} \).

Vice-versa, define \( E_i := U_i \times V \) and

\[
\chi_{i,j} : E_i \to E_{i,j} \times V
\]

as \( \chi_{i,j} := (1_{U_{i,j}} \times 1) \circ ((1_{U_{i,j}} \times \varphi_{i,j}) \times 1_V) \). Now we can glue supermanifolds \( \{ E_i \} \) into a super vector bundle \( E \). Indeed, by a similar classical result (see [7, prop. 2.2.5]), reduced manifolds \( \{ \tilde{E}_i \} \) can be glued into a classical vector bundle \( \tilde{E} \) on \( \tilde{M} \) with \( \{ \tilde{\varphi}_{i,j} \} \) as transition functions. Sheaves on them can also be glued by \( \{ \chi_{i,j} \} \) into a supermanifold sheaf on \( \tilde{E} \) (using, as above, the functor of points language, it is easy to check that the \( \{ \chi_{i,j} \} \) satisfy cocycle condition stated in the proposition); finally we define \( \pi : E \to M \) as the map such that \( \pi|_{E_i} = \text{pr}_{U_i} \) for all \( U_i \) (this makes sense since \( \text{pr}_{U_i} \circ \chi_{i,j} = \text{pr}_{U_j} \)). For how \((E, \pi, M, V)\) has been constructed, it is the required super vector bundle. \( \square \)

4.2. Morphisms

Definition 4.10. Let \((E, \pi_E, M)\) and \((F, \pi_F, N)\) be two fiber bundles. A morphism \( \psi : E \to F \) is a morphism of super fiber bundles if there exists \( \psi_b : M \to N \) such that

\[
\begin{array}{ccc}
E & \xrightarrow{\psi} & F \\
\pi_E & \downarrow & \pi_F \\
M & \xrightarrow{\psi_b} & N
\end{array}
\]

commutes.

\(^1\) Here we write formally \( \varphi(x) \) for \( \varphi \circ x \) (and so on).
Since \( \pi_E \) is an epimorphism, if \( \psi_\flat \) exists, it is unique.

If \( x \in \tilde{M} \) and \( y = \tilde{\psi}(x) \), \( \psi \) induces a map

\[
\psi_x : \pi_E^{-1}(x) \longrightarrow \pi_F^{-1}(y)
\]

that is defined as the unique morphism that makes the diagram

\[
\begin{array}{ccc}
\pi_E^{-1}(x) & \xrightarrow{\psi_x} & \pi_F^{-1}(y) \\
\uparrow & & \uparrow \\
\hat{x} & \xrightarrow{\pi_E} & \hat{y} \\
\downarrow & & \downarrow \\
\mathbb{R}^{0|0} & \xrightarrow{\psi_\flat} & \mathbb{R}^{0|0} \\
\end{array}
\]

commuting. Such morphism exists and it is unique since \( \pi_F^{-1}(y) \) is a pullback and

\[
\pi_F \circ \psi \circ i_x = \psi_\flat \circ \pi_E \circ i_x = \psi_\flat \circ \hat{x} \circ i = \hat{y} \circ i.
\]

**Remark 4.11.** In example 2.2 we saw that a super vector space \( V \) can be thought as a supermanifold whose functor of points is

\[
S \longrightarrow (\mathcal{O}(S) \otimes V)_0
\]

If \( V \) and \( W \) are super vector spaces, and \( \text{Hom}(V, W) \) is the super vector space of internal homomorphisms between them, we can define

\[
\beta : \text{Hom}(V, W) \times V \longrightarrow W
\]

through the natural transformation

\[
(\mathcal{O}(S) \otimes \text{Hom}(V, W))_0 \times (\mathcal{O}(S) \otimes V)_0 \longrightarrow (\mathcal{O}(S) \otimes W)_0 \\
(s \otimes K, t \otimes v) \longmapsto (-1)^{|K||s|}st \otimes K(v)
\]

where \( S \) is a generic supermanifold.

**Definition 4.12.** Let \( E \) and \( F \) be super vector bundles with typical fiber \( V \) and \( W \) respectively, and \( \beta : \text{Hom}(V, W) \times V \rightarrow W \) as in remark 4.11. A **morphism of super vector bundles** from \( E \) to \( F \) is a morphism \( \psi \) of super fiber bundles such that for all \( x \in \tilde{M} \) there exists an open neighborhood \( \tilde{U} \) with the following properties:

(i) there are two local trivializations \((U, t_E)\) on \( E \) and \((U', t_F)\) on \( F \) with \( \tilde{\psi}_\flat(\tilde{U}) \subseteq \tilde{U}' \);

(ii) there is a map \( \xi : U \rightarrow \text{Hom}(V, W) \) such that

\[
\begin{array}{ccc}
\pi_E^{-1}(U) & \xrightarrow{\psi_{|\pi_E^{-1}(U)}} & \pi_F^{-1}(U') \\
\downarrow t_E & & \downarrow t_F \\
U \times V & \xrightarrow{(1_U, \xi) \times 1_V} & U \times \text{Hom}(V, W) \times V \\
\psi_\flat \times \beta & & \psi_\flat \times \beta \\
& & U' \times W \\
\end{array}
\]

commutes.
It is easy to see that the above definition does not depend on the chosen trivializations. From definition 4.12, it follows immediately:

**Proposition 4.13.** If $\psi : E \to F$ is a morphism of super vector bundles, the map

$$\psi_x : \pi_F^{-1}(x) \to \pi_F^{-1}(\tilde{\psi}_x(x))$$

is a morphism of super vector spaces.

**Remark 4.14.** Also in this case, saying that a morphism of super fiber bundles $\psi : E \to F$ is a morphism of super vector bundles it is not sufficient that $\psi_x$ is linear for all $x$ (see the next example).

**Example 4.15.** Let $M$ be a supermanifold, $E := M \times \mathbb{R}^{1|1}$, and $\psi : E \to E$ a morphism of super vector bundles with $\tilde{\psi} = \mathbb{1}_M$. Then there is $\xi : M \to \text{Hom}(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$ such that

$$\psi = (\mathbb{1}_M \times \beta) \circ (\mathbb{1}_M, \xi) \times \mathbb{1}_{\mathbb{R}^{1|1}}$$

As usual $\xi$ can be identified with a matrix $\left(\begin{array}{cc} f_1 & g_1 \\ f_2 & g_2 \end{array}\right)$ with $f_i \in \mathcal{O}(M)_0$ and $g_j \in \mathcal{O}(M)_1$, which is the matrix of pullbacks of canonical coordinates of $\text{Hom}(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$ under $\xi^*$. So, if $\{x_i, \vartheta_j\}$ are coordinates on $M$ and $\{y, \eta\}$ on $\mathbb{R}^{1|1}$, $\psi$ is defined by

$$\psi^* : \mathcal{O}(M \times \mathbb{R}^{1|1}) \to \mathcal{O}(M \times \mathbb{R}^{1|1})$$

$$x_i, \vartheta_j \mapsto x_i, \vartheta_j$$

$$y \mapsto f_1 y + g_1 \eta$$

$$\eta \mapsto g_2 y + f_2 \eta$$

If $x \in \tilde{M}$, we have

$$\tilde{\psi}^* : \mathcal{O}(\mathbb{R}^{1|1}) \to \mathcal{O}(\mathbb{R}^{1|1})$$

$$y \mapsto \tilde{f}_1(x) y$$

$$\eta \mapsto \tilde{f}_2(x) \eta$$

Another morphism of super fiber bundles $\psi : E \to E$ with, for example, $\psi^*(\eta) = g_2 y^2 + f_2 \eta$ gives the same map on fibers, but it is not a morphism of super vector bundles.

**Proposition 4.16.** Let $(F, \pi, N, W)$ be a super fiber bundle, $M$ a supermanifold and $\zeta : M \to N$ a morphism. Then the pullback

$$\zeta^*(F) \xrightarrow{\tilde{\zeta}} F$$

exists and $\zeta^*(F)$ is a super fiber bundle on $M$ with projection $\tilde{\pi}$ and typical fiber $W$. If $F$ is a super vector bundle with transition functions $\{\varphi_{i,j}\}$, then $\zeta^*(F)$ is a super vector bundle with transition functions $\{\varphi_{i,j} \circ \zeta\}$ and $(\tilde{\zeta}, \zeta)$ is a morphism of super vector bundles.

**Proof.** Let $(U, t)$ be a local trivialization of $F$ and $\tilde{U}' = \tilde{\zeta}^{-1}(\tilde{U})$. The diagram below is a pullback diagram:

$$\begin{array}{ccc}
U' \times W & \xrightarrow{\zeta \times 1} & U \times W \\
\downarrow \text{pr}_{U'} & & \downarrow \text{pr}_U \\
U' & \xrightarrow{\zeta} & U
\end{array}$$
This fact can be checked directly in the category of sets and then be extended to each category with products by means of Yoneda’s lemma. From the uniqueness of pullbacks, it follows that \( \zeta^*(F) \) is locally isomorphic to \( U' \times W \); each local trivialization \( t: \pi^{-1}(U) \to U \times W \) induces a local trivialization \( t': \tilde{\pi}^{-1}(U') \to U' \times W \).

If \( F \) is a super vector bundle, \( t_1 \) and \( t_2 \) are two local trivializations of \( U \), and \( \varphi_{1,2} \) is the relative transition function, we have

\[
\begin{array}{ccc}
U' \times W & \xrightarrow{\zeta \times 1_W} & U \times W \\
\downarrow{t' \circ t_2^{-1}} & & \downarrow{(1_U, \varphi_{1,2}) \times 1_W} \\
U' \times W & \xrightarrow{\zeta \times 1_W} & U \times W
\end{array}
\]

where \( t'_1 \) and \( t'_2 \) are the local trivializations of \( \zeta^*(F) \) induced by \( t_1 \) and \( t_2 \); using functor of points one can see that \( \varphi_{1,2} \circ \zeta \) is the relative transition function. The fact that \( \tilde{\zeta} \) is locally of the form \( \zeta \times 1 \) proves the last statement.

\[\square\]

**Corollary 4.17.** Every morphism \( (\psi, \psi_0): (E, \pi_E, M) \to (F, \pi_F, N) \) splits into the two super fiber bundle morphisms

\[
\begin{array}{c}
E \xrightarrow{\psi} F \\
\downarrow{\pi_E} & \downarrow{\pi_F} \\
M \xrightarrow{1_M} & \xrightarrow{\psi_0} N
\end{array}
\]

one on the identity and the other fixed by \( \psi_0 \).

4.3. **Sections**

Let \( (E, \pi, M, V) \) be a super vector bundle with \( \dim V = m|n \). It is natural to define sections, in analogy with the classical case, as morphisms

\[\sigma: M \to E\]

such that \( \pi \circ \sigma = 1_M \). We denote the set of such maps by \( \Gamma_E(M) \) and we call them **geometrical sections** on \( M \). Clearly it is possible to define local geometrical sections

\[\Gamma_E(U) := \{ \sigma: U \to \pi^{-1}(U) \mid \pi \circ \sigma = 1_U \}\]

for each open subsupermanifold \( U \) of \( M \).

It is easy to check that \( \Gamma_E \) is a locally free sheaf of \( \mathcal{O}_{M,0} \)-modules on \( \tilde{M} \). The problem is that \( \Gamma_E(M) \) is an ungraded object, and it cannot encode the information contained in the super vector bundle \( E \). We want to construct a super vector space \( \Gamma_E(M) \) with \( \Gamma_E(M)_0 \cong \Gamma_E(M) \) that we will call the **full sections** of \( E \).

As usual, let \( \tilde{\pi}_* \mathcal{O}_E \) be the push-forward sheaf given by

\[\tilde{\pi}_* \mathcal{O}_E: W \mapsto \mathcal{O}_E(\tilde{\pi}^{-1}(W))\]

for each open subset \( W \) of \( \tilde{M} \). We want to define a subsheaf of \( \tilde{\pi}_* \mathcal{O}_E \) that could be called the sheaf of sections that are linear on the fibers. If \( E := M \times V \) is a trivial super vector bundle, the natural candidate is \( \mathcal{O}(M) \otimes V^* \). In general, we can give the following definition.
Definition 4.18. If \( W \) is an open subset of \( \wtilde{M} \), let \( \mathcal{O}^\text{lin}_{E}(W) \) be the set of \( s \in \mathcal{O}_{E}(\wtilde{\pi}^{-1}(W)) \) such that for all \( x \in W \) there exists a local trivialization \((U, t)\) with \( x \in \wtilde{U} \subseteq W \) and
\[
(t^{-1})^{*}(s|_{\wtilde{\pi}^{-1}(\wtilde{U})}) \in \mathcal{O}(U) \otimes V^{*}
\] (2)

It is easy to check that \( \mathcal{O}^\text{lin}_{E} \) is a subsheaf of \( \wtilde{\pi}^{*}\mathcal{O}_{E} \) (notice that \( \mathcal{O}^\text{lin}_{E} \) is a sheaf on \( \wtilde{M} \)), and that the above definition makes sense.

The sheaf \( \wtilde{\pi}^{*}\mathcal{O}_{E} \) is a sheaf of \( \mathcal{O}_{M} \)-modules on \( \wtilde{M} \). For each \( W \subseteq \wtilde{M} \), \( f \in \mathcal{O}_{M}(W) \), and \( s \in \mathcal{O}_{E}(\wtilde{\pi}^{-1}(W)) \), we define
\[
f : s := \pi^{*}(f)s
\]
\( \mathcal{O}^\text{lin}_{E} \) is a locally free sheaf of \( \mathcal{O}_{M} \)-modules of dimension \( \dim V = m|n \).

So we can define the sheaf of full sections of \( E \) by duality.

Definition 4.19. For each open subset \( W \) of \( \wtilde{M} \) let
\[
\Gamma_{E}(W) := \text{Hom}_{\mathcal{O}_{M}(W)}(\mathcal{O}^\text{lin}_{E}(W), \mathcal{O}_{M}(W))
\]
i. e. \( \Gamma_{E}(W) \) is the set of all \( \mathcal{O}_{M}(W) \)-linear maps \( \mathcal{O}^\text{lin}_{E}(W) \rightarrow \mathcal{O}_{M}(W) \). We call \( \Gamma_{E}(W) \) the space of full sections of \( E \) on \( W \).

Since we are in the smooth category and sheaf morphisms are determined by maps of global sections, it is not hard to define restriction maps for \( \Gamma_{E} \) and to check that it defines a sheaf on \( \wtilde{M} \).

For each \( W \), \( \Gamma_{E}(W) \) is a graded \( \mathcal{O}_{M}(W) \)-module and, if \((U,t)\) is a local trivialization,
\[
\Gamma_{E}(U) \cong \text{Hom}_{\mathcal{O}(U)}(\mathcal{O}(U) \otimes V^{*}, \mathcal{O}(U)) \cong \mathcal{O}(U) \otimes V
\]
Then also \( \Gamma_{E} \) is a locally free sheaf of \( \mathcal{O}_{M} \)-modules of dimension \( m|n \).

In the classical context, \( \mathcal{O}^\text{lin}_{E} \) is the sheaf of sections of the dual vector bundle. Using proposition 4.9, one could prove a similar result in the supergeometrical setting.

Proposition 4.20. \( \mathcal{O}^\text{lin}_{E} \) is the sheaf of full sections of a super vector bundle with typical fiber \( V^{*} \) and transition functions \((\varphi_{i,j})^{-1}\) that we can call dual vector bundle of \( E \).

Moreover it is possible to prove the following (non surprising) result.

Proposition 4.21. \( \Gamma_{M}(M) \cong \Gamma_{E}(M)_{0} \).

4.4. Algebraic approach
If \( V \) and \( W \) are two (algebraic) super vector bundles on \( M \), a morphism of super vector bundles
\[
\rho : V \rightarrow W
\]
is a \( \mathcal{O}_{M} \)-module morphism.

We end this section examining the relation between the geometrical definition of super vector bundle and the algebraic one (for the classical case see [8, 7]).

Proposition 4.22. The category of geometrical super vector bundles on a supermanifold \( M \) whose morphisms are of the form \((\psi, \mathbf{I}_{M})\) (see definition 4.12) is equivalent to the above described category of algebraic super vector bundles on \( M \).
Sketch of Proof. If \((E, \pi, M, V)\) is a geometrical super vector bundle, \(\Gamma_E\) is a locally free sheaf of \(\mathcal{O}_M\)-modules of dimension \(\dim V\). \(E \to \Gamma_E\) gives the equivalence functor on the objects.

Vice-versa, let \(V\) an algebraic super vector bundle on \(M\) of dimension \(m|n\). Since \(V\) is locally free, for all \(x \in M\) there exists an open neighborhood \(\tilde{U}\) of \(x\) such that \(V(\tilde{U}) \cong \mathcal{O}_M(\tilde{U})^{m|n} = \mathcal{O}_M(\tilde{U}) \otimes \mathbb{R}^{m|n}\). Let \(\{\tilde{U}_i\}\) be an open covering of \(\tilde{M}\) of such neighborhoods and let \(t_i: V(\tilde{U}_i) \to \mathcal{O}_M(\tilde{U}_i)^{m|n}\) be the corresponding isomorphisms. If \(\tilde{U}_{ij} := \tilde{U}_i \cap \tilde{U}_j\), we have an automorphism \(\varphi_{ij} = t_i \circ t_j^{-1}\) of \(\mathcal{O}_M(\tilde{U}_{ij})^{m|n}\) that is an \((m + n) \times (m + n)\) invertible matrix \((f_{k,l} g_{r,s}^{-1})_{k,l,r,s} \in \mathcal{O}_M(\tilde{U}_{ij})_0\) and \(g_{r,s}, g_{r',s'} \in \mathcal{O}_M(\tilde{U}_{ij})_1\). As usual, this matrix identifies a morphism \(U_{ij} \to \text{GL}(m|n)\), which we denote equally by \(\varphi_{ij}\). Restricting to \(\tilde{U}_{ijk} := \tilde{U}_i \cap \tilde{U}_j \cap \tilde{U}_k\), the composition of such module automorphisms is the usual matrix product. So one can see that the morphisms \(\varphi_{ij}: U_{ij} \to \text{GL}(m|n)\) satisfy cocycle condition (definition 4.8) and, according to proposition 4.9, a geometrical super vector bundle can be constructed with typical fiber \(\mathbb{R}^{m|n}\) and \(\{\varphi_{ij}\}\) as transition functions.

Now we have to construct the equivalence functor on morphisms. Let \((\psi, \mathbf{1}_M): (E, \pi_E, M, V) \to (F, \pi_F, M, W)\) be a morphism of geometrical super vector bundles, and let \(\tilde{U} \subseteq \tilde{M}\) and \(\xi: U \to \text{Hom}(V, W)\) be as in definition 4.12. We know that \(\xi\) is identified with an element

\[
\sum_i s_i \otimes K_i \in (\mathcal{O}_M(\tilde{U}) \otimes \text{Hom}(V, W))_0
\]

(compare example 2.2 and remark 4.11) and then it determines an even map of \(\mathcal{O}_M(\tilde{U})\)-modules

\[
\mathcal{O}_M(\tilde{U}) \otimes V \longrightarrow \mathcal{O}_M(\tilde{U}) \otimes W
\]

\[
t \otimes v \longmapsto \sum_i (-1)^{|K_i| |t|} s_i t \otimes K_i(v)
\]

(notice that, since \(|s_i| = |K_i|, (s_i \otimes K_i)(t \otimes v) = (t \otimes 1)(s_i \otimes K_i)(1 \otimes v)\)).

Vice-versa an even map

\[
\mathcal{O}_M(\tilde{U}) \otimes V \longrightarrow \mathcal{O}_M(\tilde{U}) \otimes W
\]

of \(\mathcal{O}_M(\tilde{U})\)-modules is an even matrix with entries in \(\mathcal{O}_M(\tilde{U})\) and it identifies a morphism \(U \to \text{Hom}(V, W)\). Therefore, proceeding locally, we can construct a geometrical morphism from an algebraic one.

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