Inverse scattering for Schrödinger operators with Miura potentials: I. Unique Riccati representatives and ZS-AKNS systems

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Abstract
This is the first in a series of papers on scattering theory for one-dimensional Schrödinger operators with highly singular potentials \( q \in H^{-1}_{loc}(\mathbb{R}) \). In this paper, we study Miura potentials \( q \) associated with positive Schrödinger operators that admit a Riccati representation \( q = u' + u^2 \) for a unique \( u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Such potentials have a well-defined reflection coefficient \( r(k) \) that satisfies \( |r(k)| < 1 \) and determines \( u \) uniquely. We show that the scattering map \( S : u \mapsto r \) is real analytic with real-analytic inverse. To do so, we exploit a natural complexification of the scattering map associated with the ZS-AKNS system. In subsequent papers, we will consider larger classes of potentials including singular potentials with bound states.

1. Introduction
This is the first of a series of papers on scattering theory for one-dimensional Schrödinger operators

\[ S := -\frac{d^2}{dx^2} + q \]

on the line with highly singular potentials \( q \). Our goal is to extend the inverse scattering method as discussed e.g. in [13, 17, 29–31] in order to study initial value problems for completely integrable dispersive equations with highly singular initial data.
It is natural to begin with the case where the Schrödinger operator has no bound states; the corresponding potentials then admit a Riccati representation given by the Miura map [32]. Recall that the Miura map is the nonlinear mapping:

\[
B : L^2_{\text{loc}}(\mathbb{R}) \rightarrow H_{\text{loc}}^{-1}(\mathbb{R}),
\]

\[u \mapsto u' + u^2.
\]

It is not difficult to see [32] that if \(u(x, t)\) is a smooth solution of the mKdV equation, then

\[(Bu)(x, t) = \frac{\partial u}{\partial x}(x, t) + u^2(x, t),
\]

is a smooth solution of the KdV equation. For this reason, the Miura map has played a fundamental role in the study of existence and well-posedness questions for these two equations.

The range of the Miura map may be characterized as follows. For \(q \in H_{\text{loc}}^{-1}(\mathbb{R})\) real valued and \(\varphi \in C^\infty_0(\mathbb{R})\), consider the Schrödinger form

\[s(\varphi) := \int |\varphi'(x)|^2 \, dx + \langle q, |\varphi|^2 \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) is the pairing between \(H_{\text{loc}}^{-1}(\mathbb{R})\) and \(H^1_{\text{comp}}(\mathbb{R})\). In [26] it was shown that if \(q\) is any real-valued distribution in \(H^{-1}_{\text{loc}}(\mathbb{R})\) for which the Schrödinger form \(s\) is nonnegative, then \(q\) may be presented as \(q = Bu\) for a function \(u \in L^2_{\text{loc}}(\mathbb{R})\) that need not be unique. We will call such a potential \(q\) a Miura potential, and we will call any function \(u\) with \(q = Bu\) a Riccati representative for \(q\). It is not difficult to see that two Riccati representatives for a given \(q\) differ by a continuous function. Any Riccati representative \(u\) is the logarithmic derivative of a positive distributional solution to the zero-energy Schrödinger equation for \(q\).

In this paper, we will study inverse scattering for the subset of the Miura potentials which admit a Riccati representative \(u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\). In this case, such a Riccati representative is unique, so we may parameterize the potentials by their Riccati representatives. In paper II of this series [24], we consider Miura potentials \(q\) which do not have a unique Miura representative but instead possess the following property. There exist Riccati representatives \(u_+\) and \(u_-\) which belong to \(L^2(\mathbb{R})\) and, in addition, \(u_-\) is integrable on \((−\infty, 0)\), while \(u_+\) is integrable on \((0, \infty)\). It will turn out that a potential \(q\) in this class is uniquely determined by the data \(u_-|_{(−\infty, 0)}\), \(u_+|_{(0, \infty)}\), and the ‘jump’ \((u_+ − u_-)(0)\). This class includes the Faddeev–Marchenko class of potentials \(q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\) as well as many highly oscillatory and distributional potentials. Finally, in paper III [25], we will show how to add bound states. The class of potentials covered in the present paper is ‘non-generic’ in the sense that, even for regular short-range potentials, the Riccati representative is generically non-unique; on the other hand, the ideas used here lay the groundwork for the analysis of ‘generic’ singular potentials in papers II and III.

The inverse scattering problem for Miura potentials with \(u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\) was worked out in detail from the point of view of Schrödinger scattering in the thesis of the first author [20]. In this paper, we take a somewhat different point of view and exploit a natural complexification of the problem that leads to the ZS-AKNS system (see Zakharov and Shabat [41] and Ablowitz, Kaup, Newell and Segur [1]) associated with the defocusing nonlinear Schrödinger equation. Elsewhere [10], we will apply the refined continuity results obtained here to study solutions to the defocusing NLS equation with singular initial data.

Let us first describe the connection between the ZS-AKNS system and scattering with Miura potentials. For a real-valued \(u\) in the Schwartz class \(S(\mathbb{R})\), suppose that \(\Psi\) is a

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6 See subsection 2.1 for explicit definitions of all the function spaces appearing in the paper.
matrix-valued solution of the ZS-AKNS system\textsuperscript{7}

\[ \Psi' = i \sigma \Psi + Q(x) \Psi, \]

where

\[ \sigma = \frac{1}{2} \sigma_3 := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & u(x) \\ u(x) & 0 \end{pmatrix}. \]

An easy calculation shows that if \( \Psi \) solves (1.3) for a real-valued \( u \in S(\mathbb{R}) \), then the row vector

\[ \chi := (1 1) \Psi \]

solves the Schrödinger equation

\[ -\chi'' + q(x) \chi = \frac{z^2}{4} \chi, \]

where

\[ q(x) = u'(x) + u^2(x) \]

is the image of \( u \) under the Miura map (1.1). Hence, in order to study the direct and inverse scattering maps for the Schrödinger equation with a Miura potential, it suffices to study the analogous maps for the ZS-AKNS system.

To state the connection between the scattering maps more precisely, suppose again that \( u \in S(\mathbb{R}) \) is real valued and \( s \in \mathbb{R} \). As is well known (see for example [5], where more general problems under weaker regularity and decay assumptions are studied), there exist solutions \( \Psi_{\pm}(x, s) \) of the ZS-AKNS system with respective asymptotics

\[ \lim_{x \to \pm \infty} |\Psi_{\pm}(x, s) - \exp(i s x \sigma)| = 0 \]

and a unique matrix \( A(s) \) of the form

\[ A(s) = \begin{pmatrix} a(s) & \overline{b(s)} \\ b(s) & \overline{a(s)} \end{pmatrix} \]

with

\[ |a(s)|^2 - |b(s)|^2 = 1 \]

such that

\[ \Psi_+(x, s) = \Psi_-(x, s) A(s). \]

If \( u \in S(\mathbb{R}) \), we have \( a - 1 \in S(\mathbb{R}) \) and \( b \in S(\mathbb{R}) \). The function \( a(s) \) extends to a bounded analytic function \( a(z) \) on \( \text{Im} \, z > 0 \) with \( |a(z)| \to 1 \) as \( |z| \to \infty \).

The left reflection coefficient is given by

\[ r_-(s) := b(s)/a(s), \]

and \( |r_-(s)| < 1 \) strictly if \( u \in S(\mathbb{R}) \). Similarly, the right reflection coefficient is given by

\[ r_+(s) := -\overline{b(s)}/a(s). \]

\textsuperscript{7} Our equation (1.3) is actually a special case of the ZS-AKNS system since in general one takes

\[ Q(x) = \begin{pmatrix} 0 & u_1(x) \\ u_2(x) & 0 \end{pmatrix} \]

for complex-valued \( u_1 \) and \( u_2 \).
These reflection coefficients are identical (up to the change $s \mapsto 2s$ in the spectral parameter) to the reflection coefficients in the Schrödinger problem for the Miura potential (1.4). Indeed, if

$$\chi_{\pm} := (1 \, 1) \Psi_{\pm},$$

and $\chi_{\pm} = (\chi_{1}^{\pm}, \chi_{2}^{\pm})$, we can use (1.5) and (1.6) to conclude that

$$\chi_{1}^{+}(x, s) \sim \begin{cases} a(s) e^{i sx/2} + b(s) e^{-i sx/2}, & x \to -\infty, \\ a(s) e^{i sx/2}, & x \to +\infty, \end{cases}$$

and

$$\chi_{2}^{-}(x, s) \sim \begin{cases} e^{-i sx/2}, & x \to -\infty, \\ (1 - a(s)) e^{-i sx/2} - b(s) e^{i sx/2}, & x \to +\infty. \end{cases}$$

Thus, up to the reparametrization, $r_{-}(s)$ is the usual Schrödinger reflection coefficient for scattering from the left, and $r_{+}(s)$ is the usual Schrödinger reflection coefficient for scattering from the right. The asymptotic relations (1.9) and (1.10) show that, up to reparametrization, $\chi_{1}^{+}(\cdot, s)$ and $\chi_{2}^{-}(\cdot, s)$ are the usual Jost solutions at $+\infty$ and $-\infty$ respectively.

These calculations motivate us to consider the inverse problem for the ZS-AKNS system, now for $u \in L^{1}(\mathbb{R}) \cap L^{2}(\mathbb{R})$. This class of $u$ is chosen so that the corresponding Miura potentials lie in $H^{-1}(\mathbb{R})$ and define self-adjoint Schrödinger operators $S$ [23, 26]; moreover, they have sufficient decay to allow a well-defined scattering theory. It will pose no essential difficulty to consider the inverse scattering problem for the system (1.3) with

$$Q(x) = \begin{pmatrix} 0 & u(x) \\ \overline{u(x)} & 0 \end{pmatrix}$$

and $u$ complex valued; the scattering map for the Schrödinger problem with the Miura potential $q = u' + u^2$ and $u$ real valued will be the restriction of the resulting scattering maps to real-valued $u$.

For this more general scattering problem, there still exist matrix-valued Jost solutions $\Psi_{\pm}$ and a matrix $A$ with the above properties. In particular, the reflection coefficients $r_{\pm}$ are well defined, and we introduce the direct scattering maps $S_{\pm}$ via

$$S_{+} : u \rightarrow r_{+}, \quad S_{-} : u \rightarrow r_{-}.$$ 

Define the Fourier transforms $\mathcal{F}_{+}$ and $\mathcal{F}_{-}$ via

$$(\mathcal{F}_{+} f)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{2\pi i s x} f(s) ds,$$

$$(\mathcal{F}_{-} f)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\pi i s x} f(s) ds;$$

it will be convenient to work with

$$F_{+}(x) := (\mathcal{F}_{+} r_{+})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{2\pi i s x} r_{+}(s) ds,$$

$$F_{-}(x) := (\mathcal{F}_{-} r_{-})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\pi i s x} r_{-}(s) ds$$

and to analyze the mappings $u \mapsto F_{+}$ and $u \mapsto F_{-}$ given by $\mathcal{F}_{+} \circ S_{+}$ and $\mathcal{F}_{-} \circ S_{-}$ respectively.

To state our results, let us define the Banach space

$$X := L^{1}(\mathbb{R}) \cap L^{2}(\mathbb{R})$$

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with the norm
\[ \| F \|_X := \| F \|_{L^1} + \| F \|_{L^2}. \] (1.15)

Let
\[ X_1 := \{ F \in X : \| \hat{F} \|_{\infty} < 1 \} \]
with the norm induced from \( X \), where
\[ \hat{F}(s) := \int_{-\infty}^{\infty} e^{-i sx} F(x) \, dx = (F_+^{-1} F)(s). \] (1.16)

Our first result is

**Theorem 1.1.** The mappings \( \mathcal{F}_+ \circ S_+ \) and \( \mathcal{F}_- \circ S_- \) are continuous bijections between \( X \) and \( X_1 \). The inverse mappings take the form
\[ (\mathcal{F}_\pm \circ S_\pm)^{-1} = I + \Phi_\pm \]
where \( I \) is the identity map and
\[ \Phi_\pm : X_1 \to L^1(\mathbb{R}) \cap C(\mathbb{R}) \]
are continuous maps. Moreover, both the direct maps \( \mathcal{F}_\pm \circ S_\pm \) and their inverses are analytic in the sense of definition 2.1 below.

We note that the mappings of the above theorem are not analytic in the usual sense as, for example, the direct maps depend on \( u \) and \( \overline{u} \) and the inverse maps depend on \( F \) and \( \overline{F} \).

Now let \( X_\mathbb{R} \) be the restriction of \( X \) to real-valued functions, and let \( (X_\mathbb{R})_1 \) be the analogous restriction of \( X_1 \). If \( u \) is real valued, the reflection coefficients obey the additional constraint
\[ r_\pm(s) = r_\pm(-s), \] (1.17)
which implies that \( F_+ \) and \( F_- \) are real valued. Thus, the maps \( \mathcal{F}_\pm \circ S_\pm \) restrict to maps from \( X_\mathbb{R} \) to \( (X_\mathbb{R})_1 \). From theorem 1.1 and the analyticity properties of the direct and inverse maps, we obtain our main result on inverse scattering for Schrödinger operators with Miura potentials:

**Corollary 1.2.** The maps \( \mathcal{F}_+ \circ S_+ \) and \( \mathcal{F}_- \circ S_- \) restrict to real-analytic bijections between \( X_\mathbb{R} \) and \( (X_\mathbb{R})_1 \). Moreover,
\[ (\mathcal{F}_\pm \circ S_\pm)^{-1} = I + \Phi_\pm \]
where \( I \) is the identity map and
\[ \Phi_\pm : (X_\mathbb{R})_1 \to L^1(\mathbb{R}) \cap C(\mathbb{R}). \]

**Remark 1.3.** The class of Miura potentials with a Riccati representative \( u \in X_\mathbb{R} \) is ‘non-generic’ among potentials generating nonnegative Schrödinger operators: even for regular potentials, the generic case is that \( r(0) = -1 \) (see, e.g. [13], theorem 1, p 146 and remark 9, pp 152–3, and example 1.7 below). In the generic case, the scattering problem can be parametrized by left and right Riccati representatives in \( L^2(\mathbb{R}) \) having additional integrability on left and right half-lines [24].

In the proof of theorem 1.1, we reconstruct \( u \) on a half-line \((c, \infty)\) using \( r_+ \) and a ‘right’ Gelfand–Levitan–Marchenko equation, and we reconstruct \( u \) on a half-line \((-\infty, c)\) using a ‘left’ Gelfand–Levitan–Marchenko equation, and show that these reconstructions give continuous maps from \( X_1 \) into the respective spaces \( X^+_c := L^1(c, \infty) \cap L^2(c, \infty) \) and \( X^-_c := L^1(-\infty, c) \cap L^2(-\infty, c) \). We then exploit the existence of an involution \( T \) that intertwines \( S_+ \) and \( S_- \) to obtain continuity of the inverse maps into the full space \( X \).
Inverse scattering for the AKNS system has been extensively studied: see for example the original papers [41] and [1], Shabat’s solution of the inverse problem for potentials in $L^1(\mathbb{R})$ [38, 39], the works of Beals and Coifman [5–7], Cohen and Kappeler [11], and the monographs of Faddeev and Takhtajan [18], and Beals, Deift and Tomei [8]. Aktosun, Klaus and van der Mee [3] investigated direct and inverse scattering for AKNS systems on the line with integrable matrix-valued potentials and, in particular, derived partial characterization of the scattering data. Earlier in [2], they extended the classical scattering theory for Schrödinger operators by treating a class of singular potentials constructed via the Darboux transformations that lead to ambiguities in the inverse scattering; see also the related paper by Degasperis and Sabatier [12]. Recently, Demontis and van der Mee [15] studied Darboux transformations that lead to ambiguities in the inverse scattering; see also the related theory for Schrödinger operators by treating a class of singular potentials constructed via the Fourier transform. In a similar way, theorem 1.1 can be interpreted as saying that the singularities of $u$ are recovered modulo $C(\mathbb{R})$ from the Fourier transform of the reflection coefficient. Xin Zhou [42] used Riemann–Hilbert techniques to show that the direct scattering maps $S_{\pm} : u \mapsto r_{\pm}$ are bijections from $H^{1,k}(\mathbb{R})$ to $H^{1,k}_{\pm}(\mathbb{R})$ where $H^{1,k}(\mathbb{R})$ is the $L^2$ weighted Sobolev space of functions with $j$ derivatives belonging to $L^2(\mathbb{R}, (1+|x|^2)^k\,dx)$, and

$$H^{1,k}_{\pm}(\mathbb{R}) = \{ r \in H^{1,k}(\mathbb{R}) : \|r\|_{\infty} < 1 \}.$$

The $H^{1,k}$ classes are very natural in that certain such classes are preserved under the completely integrable flows for the NLS equation, the mKdV equation, etc. On the other hand, it is of interest to see how far these can be relaxed and still obtain a bijective map. While our class is not preserved under KdV or mKdV, the mappings $\Phi_{\pm}$ extend to the space

$$(L^2(\mathbb{R}))_1 = \{ F \in L^2(\mathbb{R}) : \hat{F} \in L^\infty(\mathbb{R}), \|\hat{F}\|_{\infty} < 1 \}.$$

This space is preserved under maps of the form $\hat{F} \mapsto e^{iu\tau}\hat{F}$. In a separate paper [10], we will use this extension to construct Strichartz solutions to the defocusing cubic NLS equation and study their asymptotic behavior.

We close this introduction with several examples of Miura potentials.

**Example 1.4.** Let $u$ be an even function that for $x > 0$ equals $x^{-\alpha} \sin x^\beta$. Assume that $\alpha > 1$ and $\beta > \alpha + 1$. Then $u$ belongs to $X$ and the corresponding Miura potential $q = u' + u^2$ is of the form

$$q(x) = \beta \text{sign}(x) |x|^{\beta-\alpha-1} \cos |x|^\beta + \tilde{q}(x)$$

for some bounded function $\tilde{q}$. Thus, $q$ is unbounded and oscillatory; nevertheless, the corresponding Schrödinger operator possesses only absolutely continuous spectrum filling out the positive semi-axis and the scattering and inverse scattering on such a potential are well defined.

**Example 1.5.** Assume that $\phi \in C^\infty_0(\mathbb{R})$ is such that $\phi \equiv 1$ on $(-1, 1)$. Take a function $u(x) = \alpha \phi(x) \log |x|$ with $\alpha > 0$. Then $u \in X$; moreover, since the distributional derivative of $\log |x|$ is the distribution $P.v. 1/x$, the corresponding Miura potential $q$ is smooth outside the origin and has there a Coulomb-type singularity. See e.g. [9, 19, 27] and the references therein for discussion and rigorous treatment of Schrödinger operators with Coulomb potentials.

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Example 1.6 (Frayer [20]). The Riccati representative \( u = \alpha \chi_{[-1,1]} \) with \( \alpha \) a nonzero real constant and \( \chi_{\Delta} \) the indicator function of a set \( \Delta \), corresponds to the Miura potential

\[
q = \alpha \delta(x + 1) - \alpha \delta(x - 1) + \alpha^2 \chi_{[-1,1]},
\]

\( \delta \) being the Dirac delta-function centered at the origin. One can explicitly calculate the Jost solutions and hence the reflection coefficients by matching solutions in the regions \( x < -1 \), \( -1 < x < 1 \) and \( x > 1 \), at \( x = -1 \) and \( x = +1 \), however, the formula is rather involved. We omit details.

Example 1.7. For the potential \( q(x) = \alpha \delta(x) \), with \( \alpha \) a nonzero real constant, the extremal solutions \( \varphi_{\pm} \) of the zero-energy Schrödinger equation (see section 2.2) are different and equal:

\[
\varphi_{+}(x) = \begin{cases} 1 & \text{for } x > 0, \\ 1 - \alpha x & \text{for } x < 0, \end{cases}
\]

and

\[
\varphi_{-}(x) = \begin{cases} 1 + \alpha x & \text{for } x > 0, \\ 1 & \text{for } x < 0, \end{cases}
\]

respectively (see also [26, appendix A] and [28]). Therefore, such \( q \) does not belong to the class of singular potentials treated in this paper.

For this potential, the reflection coefficient satisfies \( r(0) = -1 \). This is in fact a generic situation even for potentials \( q \) in the Faddeev–Marchenko class \( L^1(\mathbb{R}, (1 + |x|) \, dx) \), for which the standard scattering theory is well understood. Therefore, a generic Faddeev–Marchenko potential is not a Miura potential in the sense of this paper.

As mentioned above, we shall extend the scattering theory developed here to a wider class of potentials including delta functions, Faddeev–Marchenko potentials and many other in the papers [24] and [25].

This paper is organized as follows. In section 2 we introduce necessary notation and function spaces and recall some basic results about the Miura map. We review the direct scattering problem in section 3 and prove theorem 1.1 in section 4. In an appendix, we review the Wiener and associated algebras used to analyze the direct and inverse scattering maps.

2. Preliminaries

2.1. Notation

In what follows, \( M_2(\mathbb{C}) \) will stand for the \( 2 \times 2 \) complex matrices with the Euclidean operator norm \( |\cdot| \) and

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

is the usual Pauli matrix. Upright capital Roman or Greek letters (e.g. A, I, \( \Psi \), etc) will usually denote elements of \( M_2(\mathbb{C}) \); in particular, I is the \( 2 \times 2 \) identity matrix.

Next, \( L^1(\mathbb{R}) \) and \( L^2(\mathbb{R}) \) are the standard Lebesgue function spaces on the real line \( \mathbb{R} \), and \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \) will stand for the respective \( L^1 \)- and \( L^2 \)-norms of scalar, vector or matrix-valued functions on \( \mathbb{R} \). Recalling the Banach space \( X \) (see (1.14) and (1.15)), we define, for any \( c \in \mathbb{R} \),

\[
X^+_c = L^1(c, \infty) \cap L^2(c, \infty)
\]

and

\[
X^-_c = L^1(-\infty, c) \cap L^2(-\infty, c).
\]
As usual, \( C_0^\infty (\mathbb{R}) \) denotes the linear space of all functions of compact support that are infinitely often differentiable. The Schwartz class \( \mathcal{S} (\mathbb{R}) \) consists of all infinitely often differentiable functions \( f \) such that \( x^k f^{(l)} \) are bounded on \( \mathbb{R} \) for every natural \( k \) and \( l \). \( H^1 (\mathbb{R}) \) is the space of functions in \( L^2 (\mathbb{R}) \) whose distributional derivative also belongs to \( L^2 (\mathbb{R}) \), the norm being
\[
\| f \|_{H^1} = (\| f \|_2^2 + \| f' \|_2^2)^{1/2}.
\]
The space \( H^{-1} (\mathbb{R}) \) is a dual space to \( H^1 (\mathbb{R}) \), i.e. the space of all distributions that are bounded functionals on \( H^1 (\mathbb{R}) \). If \( Y \) is any of the above spaces, then \( Y_{\text{loc}} \) is defined as the space of all functions (or distributions) \( f \) such that \( f \phi \in Y \) for every \( \phi \in C_0^\infty (\mathbb{R}) \).

Finally, we denote by \( H^2_\pm (\mathbb{R}) \) the Hardy spaces of functions on the real line having respective representations of the form
\[
f_\pm(s) = \int_0^\infty e^{is\zeta} g_\pm(\zeta) \, d\zeta, \quad f_-(s) = \int_{-\infty}^0 e^{is\zeta} g_-(\zeta) \, d\zeta
\]
for \( g_+ \in L^2 (0, \infty) \) and \( g_- \in L^2 (-\infty, 0) \). Clearly, \( L^2 (\mathbb{R}) = H^2_+(\mathbb{R}) \oplus H^2_- (\mathbb{R}) \). For \( f \in L^2 (\mathbb{R}) \), we denote by \( C \) the Cauchy integral operator
\[
(Cf)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{s - z} f(s) \, ds, \quad z \in \mathbb{C}\setminus\mathbb{R},
\]
and by \( C_\pm \) the operators
\[
(C_\pm f)(s) = \lim_{\varepsilon \downarrow 0} (Cf)(s \pm i\varepsilon), \quad s \in \mathbb{R},
\]
(2.1)
where the limit is taken in \( L^2 (\mathbb{R}) \). The operators \( C_+ \) and \( -C_- \) are orthogonal projections onto \( H^2_+ (\mathbb{R}) \) and \( H^2_- (\mathbb{R}) \) respectively, and \( C_+ - C_- = I \) as operators on \( L^2 (\mathbb{R}) \). We have the formulas
\[
(C_\pm f)(s) = \pm \mathcal{F}_\pm^{-1} \chi_\pm \mathcal{F}_+ f,
\]
(2.2)
where \( \chi_+ \) (resp. \( \chi_- \)) is the indicator function of \((0, \infty)\) (resp. of \((-\infty, 0)\)).

Both the direct and inverse scattering maps are obtained as solutions of integral equations that naturally lead to series representations. To capture their analyticity properties, we make the following rather nonstandard definition (see [16, 35] and [37, appendix A] for discussion of analytic mappings between Banach spaces).

**Definition 2.1.** Let \( X \) be a Banach space of complex-valued functions, let \( U \) be an open subset of \( X \) containing along with every \( f \) its complex conjugate \( \overline{f} \), and let \( \Psi \) be a nonlinear mapping from \( U \) into a Banach space \( Y \). We say that \( \Psi \) is analytic in variables \( f \) and \( \overline{f} \) if \( \Psi(f) = \Psi_2(f, \overline{f}) \), where \( \Psi_2 : U \times U \rightarrow Y \) is analytic.

**Remark 2.2.** If \( \Psi : X \rightarrow Y \) is analytic in \( f \) and \( \overline{f} \), then the restriction of \( \Psi \) to the subspace \( X_\mathbb{R} \) of real-valued functions is a real-analytic map from \( X_\mathbb{R} \) to \( Y \).

We also note that if \( \Psi \) is analytic in \( f \) and \( \overline{f} \) and \( \phi \) is an analytic function of a complex variable, then the point-wise composition \( \phi \circ \Psi \) often turns out to be analytic in \( f \) and \( \overline{f} \), see lemma A.3.

### 2.2. The Miura map and Riccati representatives

In this subsection we briefly recall some results of [26] relating the Miura map and the set of positive solutions of the Schrödinger equation at zero energy. We begin by considering
distributional solutions of the Schrödinger equation with a real-valued potential \( q \in H^1_{\text{loc}}(\mathbb{R}) \). For such \( q \), we define the quadratic form \( s \) via (1.2) and let
\[
\lambda_0(q) := \inf \{ s(\psi) : \psi \in C_0^\infty(\mathbb{R}), \|\psi\|_2 = 1 \}.
\]
The quadratic form \( s \) is nonnegative if \( \lambda_0(q) \geq 0 \). A function \( y \in H^1_{\text{loc}}(\mathbb{R}) \) is a (distributional) solution of \( -y'' + qy = 0 \) if for every \( \psi \in C_0^\infty(\mathbb{R}) \), \( s(y, \psi) = 0 \), with \( s(y, \psi) \) denoting the value of the associated sesquilinear form. It is not difficult to see that any solution \( y \) may only have isolated zeros where the solution \( y \) changes sign; in particular, if \( y \) is nonnegative, it is either identically zero or strictly positive. If \( \lambda_0(q) \geq 0 \) and \( y \) is a solution with either \( y \in L^2(\mathbb{R}^+) \) or \( y \in L^2(\mathbb{R}^-) \), then \( y \) has a single sign or is identically zero: if \( y \) is a zero, then there is a half-line Dirichlet problem operator \( -d^2/dx^2 + q \) with a negative eigenvalue, contradicting the assumption that \( \lambda_0(q) \geq 0 \). Finally, \( \lambda_0(q) \geq 0 \) if and only if the set
\[
\text{Pos}(q) = \{ \psi \in H^1_{\text{loc}}(\mathbb{R}) : \psi > 0, \psi(0) = 1, \forall \varphi \in C_0^\infty(\mathbb{R}) \, s(\varphi, \psi) = 0 \}
\]
of positive distributional solutions to the equation \( -\psi'' + q\psi = 0 \) is nonempty. If \( \psi \in \text{Pos}(q) \), then \( q = u' + u^2 \) for \( u = \psi'/\psi \). We initially consider the Miura map \( B : u \mapsto u' + u^2 \) acting from \( L^2_{\text{loc}}(\mathbb{R}) \) into \( H^1_{\text{loc}}(\mathbb{R}) \), by interpreting \( u' \) as the distribution derivative of \( u \). If \( q \) lies in the range \( \text{Ran} \, B \), then any function \( u \in B^{-1}(q) \) is called a Riccati representative of \( q \). The range of the Miura map consists of those \( q \in H^1_{\text{loc}}(\mathbb{R}) \), for which \( \lambda_0(q) \geq 0 \), or, equivalently, for which the set \( \text{Pos}(q) \) is nonempty. The set \( \text{Ran} \, B \) is nowhere dense in \( H^1_{\text{loc}}(\mathbb{R}) \).

It is not difficult to see that the mappings
\[
u(s) = \exp \left( \int_0^s u(s) \, ds \right)
\]
and
\[
\varphi \mapsto \frac{\varphi'}{\varphi}
\]
take \( B^{-1}(q) \) into \( \text{Pos}(q) \) and vice versa, so that these two sets are homeomorphic respectively as subsets of \( L^2_{\text{loc}}(\mathbb{R}) \) and \( H^1_{\text{loc}}(\mathbb{R}) \). Given any \( y_1 \in \text{Pos}(q) \), any solution of \( -y'' + qy = 0 \) with \( y(0) = 1 \) (whether positive or not) takes the form
\[
y(x) = y_1(x) \left( 1 + c \int_0^x y_1(s)^{-2} \, ds \right)
\]
for some \( c \) (see [22, chapter IX.2 (ix)]). Using this fact, it is not difficult to see that the set \( \text{Pos}(q) \) is the set of convex combinations of extremal solutions \( \varphi_+ \) and \( \varphi_- \) characterized by the respective divergence of \( \int_0^\infty \varphi_+^{-2}(s) \, ds \) and \( \int_{-\infty}^0 \varphi_-^{-2}(s) \, ds \). As we will see, the extremal solutions \( \varphi_+ \) and \( \varphi_- \) coincide up to constant factors with the usual right and left Jost solutions at zero energy whenever the latter are defined. The extremal solutions \( \varphi_+ \) and \( \varphi_- \) either coincide or are linearly independent, so that \( B^{-1}(q) \) (with the topology induced from \( L^2_{\text{loc}}(\mathbb{R}) \)) is homeomorphic either to a point or to a line segment.

Here we are concerned with the restriction of \( B \) to \( L^2(\mathbb{R}) \) so that the range of \( B \) is contained in \( H^{-1}(\mathbb{R}) \). A distribution \( q \in H^{-1}(\mathbb{R}) \) lies in the range of \( B \) if and only if the positivity condition \( \lambda_0(q) \geq 0 \) holds and \( q \) can be presented as \( f' + g \) for \( f \in L^2(\mathbb{R}) \) and \( g \in L^1(\mathbb{R}) \). The following result is a consequence of the proof of lemma 4.1 in [26].

**Lemma 2.3.** Suppose that \( q \in H^{-1}(\mathbb{R}) \) and that there exists a Riccati representative \( u \) of \( q \) with \( u \in L^2(\mathbb{R}^+) \). Then the same is true for every Riccati representative of \( q \). A similar statement holds with \( L^2(\mathbb{R}^+) \) replaced by \( L^2(\mathbb{R}^-) \).

The following simple uniqueness result is very important for the present paper.


Lemma 2.4. Suppose that \( q \in H^{-1}(\mathbb{R}) \) and that \( u_1 \) and \( u_2 \) are Riccati representatives of \( q \) belonging to \( X_0^+ \). Then \( u_1 = u_2 \). Similarly, if \( u_1 \) and \( u_2 \) are Riccati representatives of \( q \) belonging to \( X_0^- \), then \( u_1 = u_2 \).

Proof. If \( u \in X_0^+ = L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \) belongs to \( B^{-1}(q) \), then \( \phi(x) := \exp \left( \int_0^x u(s) \, ds \right) \) is bounded above and below by fixed constants for \( x > 0 \), so \( \int_0^\infty \phi(s)^{-2} \, ds \) diverges. Since \( \phi \) solves the equation \( -y'' + qy = 0 \), we conclude that \( \phi = \phi_+ \) is an extremal positive solution, and \( u = \phi_+ / \phi_+ \). Hence, \( u_1 = u_2 = \phi_+ / \phi_+ \). The other proof is similar. \( \square \)

Recall that \( X = L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). We immediately obtain from lemma 2.4 the following uniqueness result.

Corollary 2.5. Every \( q \in H^{-1}(\mathbb{R}) \) has at most one Riccati representative \( u \) belonging to \( X \).

3. Direct scattering

Although direct scattering for the ZS-AKNS system is well understood (see, for example, Beals and Coifman [5], Zhou [42], and the monographs of Faddeev and Takhtajan [18] and Beals, Deift, and Tomei [8]), we give a brief synopsis here. We will study the direct scattering map, derive representation formulas for the scattering solutions and finally derive the Gelfand–Levitan–Marchenko equations and the reconstruction formula.

It is useful to factor out the leading behavior of solutions to (1.3) with \( Q \) of (1.11) by setting

\[
\Psi(x, z) = M(x, z) \exp(izx\sigma),
\]

where \( M \) takes values in \( 2 \times 2 \) complex matrices, so that

\[
M'_x = iz \text{ad}_\sigma(M) + Q(x)M,
\]

(3.1)

where

\[
\text{ad}_A(B) := [A, B] = AB - BA.
\]

To study scattering asymptotics, we remove the linear term from (3.1) by setting

\[
M(x, z) = \exp(izx\text{ad}_\sigma)N(x, z),
\]

(3.2)

where

\[
\exp(u \text{ad}_\sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & e^ub \\ e^{-u}c & d \end{pmatrix}
\]

is an automorphism of the Lie group

\[
SU(1, 1) := \{ Y \in M_2(\mathbb{C}) : Y^*\sigma Y = \sigma, \det Y = 1 \}.
\]

Then \( N(x, z) \) obeys the differential equation

\[
N'_x(x, z) = G(x, z)N(x, z),
\]

(3.3)

where

\[
G(x, z) := \exp(-izx\text{ad}_\sigma)Q(x) = \begin{pmatrix} 0 & e^{-izx}u(x) \\ e^{izx}u(x) & 0 \end{pmatrix}.
\]

(3.4)

Since \( G \) is traceless and \( \sigma G + G\sigma = 0 \), it is not difficult to see that \( \det N \) and \( N^*\sigma N \) are independent of \( x \) for any solution of (3.3).
Now consider the singular initial value problem
\begin{align}
N'(x, z) &= G(x, z) N(x, z) \\
N(\infty, z) &= I. 
\end{align}
\tag{3.5}

Reformulating this problem as an integral equation
\begin{align}
N(x, z) &= I + \int_{\infty}^{x} G(y, z) N(y, z) \, dy, 
\end{align}
\tag{3.6}

it is not difficult to see that if $u \in L^1(\mathbb{R})$ and $z = s$ is real, there exists a unique solution which is a continuous curve in $SU(1, 1)$ with a well-defined limit as $x \to -\infty$. Tracing through the definitions and comparing with (1.6), we see that
$$A(s) = N(-\infty, s).$$

Since $N(-\infty, s) \in SU(1, 1)$, we may write
\begin{align}
N(-\infty, s) &= \begin{pmatrix} a(s) & \overline{b(s)} \\ b(s) & \overline{a(s)} \end{pmatrix}, \tag{3.7}
\end{align}

In what follows, we will study the direct scattering map by studying the singular initial value problem (3.5). We will then return to (3.1) in order to study the scattering solutions and their analyticity properties. From these properties we can deduce the Gelfand–Levitan–Marchenko equations and reconstruction formulas for $u$.

3.1. The direct scattering map

The basic objects of direct scattering are the reflection coefficients (1.7) and (1.8) computed from the asymptotic behavior of solutions to (1.3). It is simplest to study the map
$$u \mapsto \begin{pmatrix} a(s) \\ b(s) \end{pmatrix} \begin{pmatrix} \overline{b(s)} \\ \overline{a(s)} \end{pmatrix},$$
using (3.5) and (3.7) and deduce properties of the reflection coefficients from those of $a$ and $b$.

The first result is

Lemma 3.1. Suppose that $u \in X$. Then the representations
\begin{align}
a(s) &= 1 + \int_{0}^{\infty} e^{\alpha \xi} A(\xi) \, d\xi, \\
b(s) &= \int_{-\infty}^{\infty} e^{i \xi s} B(\xi) \, d\xi 
\end{align}
\tag{3.8, 3.9}

hold, where $A, B \in X$ with
\begin{align}
\|A\|_1, \|B\|_1 &\leq \exp(\|u\|_1) - 1, \\
\|A\|_2, \|B\|_2 &\leq \exp(\|u\|_1) \|u\|_2. 
\end{align}
\tag{3.10, 3.11}

Moreover, the maps $u \mapsto A$ and $u \mapsto B$ are analytic in $u$ and $\overline{u}$ in the sense of definition 2.1.

Proof. To obtain representations (3.8), (3.9) and the estimates (3.10), (3.11), we study the Volterra series for (3.6). A straightforward computation with (3.6) using (3.4) shows that the entries $n_{11}$ and $n_{21}$ of $N$ obey the system of integral equations
\[ n_{11}(x, z) = 1 + \int_{-\infty}^{x} e^{-iy_1u(y)n_{21}(y, z)} dy, \]  
(3.12)

\[ n_{21}(x, z) = \int_{-\infty}^{x} e^{iy_1\overline{\mu}(y)n_{11}(y, z)} dy. \]  
(3.13)

We can iterate equations (3.12) and (3.13) to obtain Volterra series representations, and then obtain the required formulas for \( a \) and \( b \) since, by (3.7), we have \( a = n_{11}(-\infty, \cdot) \) and \( b = n_{21}(-\infty, \cdot) \). First, we have

\[ n_{11}(x, s) = 1 + \sum_{n=1}^{\infty} T_{2n}(x, s), \]  
(3.14)

\[ n_{21}(x, s) = \sum_{n=0}^{\infty} T_{2n+1}(x, s). \]  
(3.15)

Here

\[ T_n(x, s) := \int_{y: y_1 \leq \ldots \leq y_n} \exp(iz\varphi_n(y))U_n(y) dy, \]

where \( y := (y_1, \ldots, y_n) \),

\[ \varphi_n(y) := \sum_{j=0}^{n-1} (-1)^j y_n - j, \]

\[ U_{2n}(y) := u(y_1)\overline{\mu}(y_2) \cdots u(y_{2n-1})\overline{\mu}(y_{2n}) \]

and

\[ U_{2n+1}(y) := -\overline{\mu}(y_1)u(y_2)\overline{\mu}(y_2) \cdots u(y_{2n})\overline{\mu}(y_{2n+1}). \]

Note that \( \varphi_{2n}(y) \geq 0 \) if \( y_1 \leq \ldots \leq y_{2n} \), whereas \( \varphi_{2n-1}(y) \geq x \) for \( y \) in the range of integration for \( T_{2n-1} \). We may write

\[ T_{2n}(x, s) = \int_{0}^{\infty} e^{i\xi t} A_{2n}(x, \xi) d\xi, \]

\[ T_{2n+1}(x, s) = \int_{x}^{\infty} e^{i\xi t} B_{2n+1}(x, \xi) d\xi, \]

where

\[ A_{2n}(x, \xi) := \int_{\varphi_{2n}(y) = \xi} U_{2n}(y) dS_{2n}, \]

\[ B_{2n+1}(x, \xi) := \int_{\varphi_{2n+1}(y) = \xi} U_{2n+1}(y) dS_{2n+1} \]

are multilinear functions of \( u \) and \( \overline{\mu} \) respectively. Here \( dS_n \) is the surface measure on the hypersurface \( \{ y \in \mathbb{R}^n : \varphi_n(y) = \xi \} \). Using the fact that \( dy = dS_n d\xi \) and letting \( \eta \) and \( \gamma \) be the bounded monotone functions

\[ \eta(x) := \int_{x}^{\infty} |u(t)| dt, \]

\[ \gamma(x) := \left( \int_{x}^{\infty} |u(t)|^2 dt \right)^{1/2}, \]
we can easily derive the estimates
\[
\| A_{2n}(x, \cdot) \|_1 \leq \frac{\eta(x)^{2n}}{(2n)!},
\]
\[
\| B_{2n+1}(x, \cdot) \|_1 \leq \frac{\eta(x)^{2n+1}}{(2n+1)!},
\]
\[
\| A_{2n}(x, \cdot) \|_2 \leq \frac{\eta(x)^{2n-1}}{(2n-1)!} \gamma(x),
\]
\[
\| B_{2n+1}(x, \cdot) \|_2 \leq \frac{\eta(x)^{2n}}{(2n)!} \gamma(x).
\]

These estimates show that the Volterra series (3.14)–(3.15) converge and define bounded absolutely continuous functions of \( x \) having finite limits as \( x \to -\infty \) for each \( s \). By taking limits in (3.14) and (3.15), it is then easy to deduce that \( a \) and \( b \) respectively have the series representations
\[
a(s) = 1 + \sum_{n=1}^{\infty} T_{2n}(s),
\]
\[
b(s) = \sum_{n=1}^{\infty} T_{2n-1}(s),
\]
where
\[
T_n(s) := \lim_{x \to -\infty} T_n(x, s)
\]
is a bounded continuous function of \( s \) for each \( n \). Setting \( A_{2n}(\xi) = A_{2n}(-\infty, \xi) \) and \( B_{2n-1}(\xi) = B_{2n-1}(-\infty, \xi) \) and defining
\[
A(\xi) := \sum_{n=1}^{\infty} A_{2n}(\xi),
\]
\[
B(\xi) := \sum_{n=0}^{\infty} B_{2n+1}(\xi),
\]
we obtain the required representations. The continuity follows from the estimates
\[
\| A_{2n} - \tilde{A}_{2n} \|_p \leq C^{2n-1} \frac{\eta(x)^{2n-1}}{(2n-1)!} \| u - \tilde{u} \|_p,
\]
\[
\| B_{2n+1} - \tilde{B}_{2n+1} \|_p \leq C^{2n} \frac{\eta(x)^{2n}}{(2n)!} \| u - \tilde{u} \|_p
\]
for \( p = 1, 2 \) where \( C := \| u \|_1 + \| \tilde{u} \|_1 \). □

Since \( A(s) \in SU(1, 1) \) for every \( s \in \mathbb{R} \), we have the identity
\[
|a(s)|^2 - |b(s)|^2 = 1,
\]
and thus \( a \) does not vanish on the real line. Moreover, the integral representation (3.8) shows that the function \( a \) is the boundary value of a function of the complex variable \( z \) (also denoted by \( a \)) that is defined in the closed upper half-plane \( \mathbb{C}^+ \) by the formula
\[
a(z) := 1 + \int_{0}^{\infty} e^{iz\xi} A(\xi) \, d\xi.
\]
The function $a$ so defined is continuous in $\mathbb{C}^+$, analytic in $\mathbb{C}^+$ and has no zeros there (see, e.g. [1], [14, section 3]).

**Remark 3.2.** Since $a$ does not vanish on the real line and tends to 1 at infinity in view of (3.8) and the Riemann–Lebesgue lemma, it follows from lemma A.2 that $a$ is an invertible element of the Banach algebra $1 + \hat{X}$ (see appendix A). The function $a^{-1}$ can be extended to the upper half-plane $\mathbb{C}^+$ as an analytic function $1/a(z)$; thus $a^{-1}$ belongs to the Hardy space $H_2$ and

$$a^{-1}(z) = 1 + \int_0^\infty C(\zeta) e^{i\zeta} \, d\zeta$$

for a function $C \in X_0^+$. Moreover, $C$ is analytic in $u$ and $v$ in the sense of definition 2.1, as follows from lemma A.3 by setting $\phi(z) := 1/z$.

Recalling formulas (1.7) and (1.8), the constraint $|a(s)|^2 - |b(s)|^2 = 1$ and the bound $\|a\|_\infty \leq 1 + \|A\|_1$, it is easy to see that the reflection coefficients $r_+$ and $r_-$ are continuous with $\|r_\pm\|_\infty < 1$. By remark 3.2, $r_\pm$ are also elements of the Banach algebra $\hat{X}$ (i.e. Fourier transforms of functions in $X$) which are analytic in $u$ and $v$. Thus

**Lemma 3.3.** The maps $F_+ \circ S_+$ and $F_- \circ S_-$ are continuous maps of $X$ into $X_1$, analytic in the sense of definition 2.1.

It will be important to formulate the exact relationship between $r_+$ and $r_-$. It follows from definitions (1.7) and (1.8) that

$$r_-(s) = -r_+(s) \frac{\overline{a(s)}}{a(s)},$$

Relation (3.16) yields for $s \in \mathbb{R}$ the identity

$$|a(s)|^{-2} = 1 - |r(s)|^2,$$

where $r = r_+$ or $r = r_-$, so that $|a|$ is determined by $r_\pm$ on the real line $\mathbb{R}$. Since the function $a$ is bounded and continuous in $\mathbb{C}^+$, analytic in $\mathbb{C}^+$ and has no zeros in $\mathbb{C}^+$, the function $\log a$ is well defined in $\mathbb{C}^+$ and analytic in $\mathbb{C}^+$. Observe that

$$\text{Re} \log a(s) = \log |a(s)| = -\frac{1}{2} \log(1 - |r(s)|^2);$$

thus, we can use the Schwarz formula to reconstruct the function $\log a$ from its real part on $\mathbb{R}$. Explicitly, we get

$$a(z) = \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\xi - z} \log(1 - |r(\xi)|^2) \, d\xi \right],$$

where $r = r_+$ or $r_-; \text{ the boundary value is}$

$$a(s) = \exp[C_\ast \log(1 - |r(\cdot)|^2)](s), \quad (3.17)$$

where $C_\ast$ is the Cauchy operator (2.1), which is bounded from $L^2(\mathbb{R})$ to itself. We also observe that the problem of reconstruction of the function $a$ in $\mathbb{C}^+$ from the values of $|a|$ on the real line could be reduced to the classical Riemann–Hilbert problem, cf [14, 42].

Let us denote by $\hat{X}_1$ the image of $X_1$ under the Fourier transform (1.16). If $r \in \hat{X}_1$, it follows from the Banach algebra properties of $\hat{X}$ (see appendix A) and lemma A.2 that, also, $\log(1 - |r(\xi)|^2) \in \hat{X}$. By lemma A.3, $\log(1 - |r(\xi)|^2)$ is analytic in $r$ and $\overline{r}$ in the sense of definition 2.1. From (2.2) we deduce that the function

$$g(s) = C_\ast \log(1 - |r(\cdot)|^2)](s)$$
also belongs to \( \hat{X} \) and is analytic in \( r \) and \( \tau \) in the sense of definition 2.1. It now follows from lemma A.2 that
\[
a(s) - 1 = \exp[g(s)] - 1
\]
begins to \( \hat{X} \) as well and is also analytic in the same sense. Hence, \( a \) of (3.17) is an invertible element of \( 1 + \hat{X} \) whenever \( r \in \hat{X}_1 \).

Now we define a nonlinear map
\[
\mathcal{I} : \hat{X}_1 \to \hat{X}_1
\]
by
\[
(\mathcal{I}r)(s) = -r(s)\frac{\pi(s)}{a(s)},
\]
where \( a \) is given by (3.17). Note that in the Schrödinger case, where (1.17) holds and, also, \( a(-s) = \bar{a}(s) \), the map \( \mathcal{I} \) is given by
\[
r \mapsto -r(-s)\frac{a(-s)}{a(s)}
\]
(cf for example [13], section 2.2, theorem 1).

Lemma 3.4. The map \( \mathcal{I} \) is a continuous involution of \( \hat{X}_1 \), analytic in \( r \) and \( \tau \) in the sense of definition 2.1.

Proof. The map \( r \mapsto a \) defined by the solution of the Schwarz problem above is continuous from \( \hat{X}_1 \) to \( 1 + \hat{X} \) and \( a \mapsto a/\bar{a} \) is continuous from \( 1 + \hat{X} \) to itself. Multiplication by \( r \) is continuous from \( 1 + \hat{X} \) to \( \hat{X} \), so we conclude that \( \mathcal{I} \) is continuous as claimed. To see that \( \mathcal{I} \) is an involution, we note that \( |(\mathcal{I}r)(s)| = |r(s)| \) and deduce from (3.17) that \( \mathcal{I}^2 r = r \). Finally, the analyticity follows from analyticity of the map \( r \mapsto a \) and invertibility of \( a \) in \( 1 + \hat{X} \). \( \square \)

It now follows that the identities
\[
S_\pm = \mathcal{I} \circ S_\pm
\]
and
\[
S_\pm = \mathcal{I} \circ S_\pm
\]
hold.

3.2. Scattering solutions

Recalling the scattering solutions \( \Psi_{\pm} \) of (1.3) with
\[
\lim_{x \to \pm \infty} |\Psi_{\pm}(x, z) - \exp(izx\sigma)| = 0,
\]
let us write
\[
\Psi_{\pm}(x, z) = M_{\pm}(x, z) \exp(izx\sigma),
\]
so that \( M_{\pm} \) solve (3.1) with
\[
\lim_{x \to \pm \infty} |M_{\pm}(x, z) - I| = 0.
\]
These singular initial value problems can be reformulated as Volterra integral equations
\[
M_{\pm}(x, z) = I + \int_{\pm \infty}^x \exp(iz(x - y) \, ad_{\sigma})[Q(y)M(y, z)] \, dy. \tag{3.18}
\]
These equations have a unique absolutely continuous solution for each real $z$ provided that $u \in L^1(\mathbb{R})$. The map $x \mapsto M_\pm(x, z)$ is a continuous curve in $SU(1, 1)$ for each fixed $z$. Letting $m_{ij}^\pm$ denote the entries of $M_\pm$, it is not difficult to see that

$$m_{11}^+(x, z) = 1 + \int_{-\infty}^x u(y)m_{21}^+(y, z) \, dy,$$

$$m_{21}^+(x, z) = \int_{-\infty}^x e^{-iz(x-y)} \Pi(y)m_{11}^+(y, z) \, dy$$

(and similar formulas with all + signs replaced by − signs), while

$$m_{12}^+(x, z) = m_{22}^+(x, z),$$

$$m_{22}^-(x, z) = m_{11}^-(x, z),$$

(and similar formulas with + replaced by −) since $M_\pm$ take values in $SU(1, 1)$. Thus, the matrix-valued function $M_+$ is determined by solutions of the system of two integral equations (3.19) and (3.20), and $M_-$ is determined by solutions of a similar system for $m_{11}^-$ and $m_{21}^-$. The integral equations (3.19) and (3.20) can be analyzed in the same way as equations (3.12) and (3.13) in the proof of lemma 3.1, or we can note from (3.2) that

$$m_{11}^+(x, z) = n_{11}^+(x, z),$$

$$m_{21}^+(x, z) = e^{-izx} n_{21}^+(x, z).$$

It follows directly from representations (3.14) and (3.15) already established for $n_{11}^+$ and $n_{21}^+$ that

$$m_{11}^+(x, z) = 1 + \int_0^\infty e^{ic\zeta} \Gamma_1^+(x, \zeta) \, d\zeta,$$

$$m_{21}^+(x, z) = \int_0^\infty e^{ic\zeta} \Gamma_2^+(x, \zeta) \, d\zeta,$$

where the estimates

$$\| \Gamma_1^+(x, \cdot) \|_1 \leq \exp \{ \eta(x) \} - 1,$$

$$\| \Gamma_1^+(x, \cdot) \|_2 \leq \exp \{ \eta(x) \} \gamma(x)$$

hold (note that in the series for $m_{21}^+$, the factor $\exp(−izx)$ in (3.24) changes the phase function to $\varphi_{n+1}(y) − x$; this phase function is nonnegative for any $x$).

The explicit multilinear expressions for $\Gamma_1^+(x, \cdot)$ and $\Gamma_2^+(x, \cdot)$ show that they are analytic functions of $u$ and $\Pi$ in the sense of definition 2.1, taking values in the space $X_0^+$ for each real $x$. Using the symmetry properties (3.21) and (3.22), we recover the full matrix-valued function $\Gamma_+$. We can make an analogous analysis for $M_-$, and obtain the following representations. In what follows, $X_0^+ \otimes M_2(\mathbb{C})$ denotes the space of matrix-valued functions over $\mathbb{R}^+$ with entries belonging to $X_0^+$.

**Proposition 3.5.** Suppose that $u \in X$ and that $M_\pm$ are the unique solutions of the integral equations (3.18). Then there exist continuous functions

$$\mathbb{R} \ni x \mapsto \Gamma_+(x, \cdot) \in X_0^+ \otimes M_2(\mathbb{C})$$
and
\[ \mathbb{R} \ni x \mapsto \Gamma_-(x, \cdot) \in X_0^- \otimes M_2(\mathbb{C}) \]
with
\[ \Gamma_+(x, \zeta) = \left( \begin{array}{c} \Gamma_{11}^+(x, \zeta) \\ \Gamma_{21}^+(x, \zeta) \end{array} \right) \]
and
\[ \Gamma_-(x, \zeta) = \left( \begin{array}{c} \Gamma_{11}^-(x, \zeta) \\ \Gamma_{21}^-(x, \zeta) \end{array} \right), \]
so that the representations
\[ M_+(x, z) = I + \int_0^\infty \Gamma_+(x, \zeta) \exp(i\zeta \sigma_3) \, d\zeta, \quad (3.25) \]
\[ M_-(x, z) = I + \int_{-\infty}^0 \Gamma_-(x, \zeta) \exp(i\zeta \sigma_3) \, d\zeta \quad (3.26) \]
hold. Moreover, the estimates
\[ \|\Gamma_+(x, \cdot)\|_1 \leq \exp[\eta_+(x)] - 1, \]
\[ \|\Gamma_-(x, \cdot)\|_1 \leq \exp[\eta_-(x)] - 1, \]
\[ \|\Gamma_+(x, \cdot)\|_2 \leq \exp[\eta_+(x)] \gamma_+(x), \]
\[ \|\Gamma_-(x, \cdot)\|_2 \leq \exp[\eta_-(x)] \gamma_-(x) \]
hold, where
\[ \eta_\pm(x) = \pm \int_{-\infty}^{\pm \infty} |u(s)| \, ds, \]
\[ \gamma_\pm(x) = \left( \pm \int_{-\infty}^{\pm \infty} |u(s)|^2 \, ds \right)^{1/2}. \]

Finally, for every fixed \( x \in \mathbb{R} \), the maps \( u \mapsto \Gamma_\pm(x, \cdot) \) from \( X \) into \( X_0^\pm \otimes M_2(\mathbb{C}) \) are analytic in the sense of definition 2.1.

**Remark 3.6.** The function \( \Gamma_+(x, \zeta) \) depends only on values of \( u(y) \) with \( y > x \), while the function \( \Gamma_-(x, \zeta) \) depends only on values of \( u(y) \) with \( y < x \). We will derive integral equations for \( \Gamma_\pm \) given the functions \( F_\pm \) of (1.12) and (1.13); the equation for \( \Gamma_+ \) will provide a stable reconstruction of \( u(x) \) on any half-line \( (c, \infty) \), while the equation for \( \Gamma_- \) will provide a stable reconstruction for any half-line \( (-\infty, c) \).

If we write \( M_+ = (m_1^+, m_2^+) \), where \( m_1^+ \) and \( m_2^+ \) are column vectors, and similarly write \( M_- = (m_1^-, m_2^-) \), then \( (m_1^+, m_2^+) \) extends to an analytic matrix-valued function of \( z \) in \( \text{Im} \, z > 0 \), while \( (m_1^-, m_2^-) \) extends to an analytic matrix-valued function of \( z \) in \( \text{Im} \, z < 0 \). Indeed, as an immediate corollary of proposition 3.5, we have

**Proposition 3.7.** For every fixed \( x \in \mathbb{R} \), the functions
\[ m_1^+(x, \cdot) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad m_2^-(x, \cdot) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
belong to the Hardy space \( H^2_+(\mathbb{R}, \mathbb{C}^2) \).

These Hardy space properties are the starting point for the derivation of the Gelfand–Levitan–Marchenko equations.
3.3. The Gelfand–Levitan–Marchenko equations

We now derive integral equations for $\Gamma_\pm$ in terms of $F_\pm$ and also derive a reconstruction formula for $u$ from $\Gamma_\pm$. In the next section we will solve these equations and prove consistency of the two reconstructions. Given functions $F_\pm$ of (1.12) and (1.13), let us define

$$
\Omega_-(x) := \begin{pmatrix} 0 & F_-(x) \\ F_-(x) & 0 \end{pmatrix},
$$

$$
\Omega_+(x) := \begin{pmatrix} 0 & F_+(x) \\ F_+(x) & 0 \end{pmatrix}.
$$

We will prove

**Proposition 3.8.** Suppose that $u \in X$; then the kernels $\Gamma_\pm$ and $\Gamma_-$ obey the equations

$$
\Gamma_-(x, \xi) + \Omega_-(x + \xi) + \int_{-\infty}^{0} \Gamma_- (x, t) \Omega_-(x + t + \xi) \, dt = 0 \quad \text{for a.e.} \quad \xi < 0, \tag{3.27}
$$

$$
\Gamma_+(x, \xi) + \Omega_+(x + \xi) + \int_{0}^{\infty} \Gamma_+ (x, t) \Omega_+(x + t + \xi) \, dt = 0 \quad \text{for a.e.} \quad \xi > 0. \tag{3.28}
$$

Equations (3.27) and (3.28) are the left and right Gelfand–Levitan–Marchenko equations respectively.

**Proof.** We will derive (3.27); the derivation of (3.28) is similar and will be omitted. We begin with the identity

$$
\Psi_+(x, z) = \Psi_-(x, z) \begin{pmatrix} a(z) & b(z) \\ \overline{b(z)} & \overline{a(z)} \end{pmatrix},
$$

for a fixed $x$ and real $z$, which implies that

$$
\frac{m_1^+}{a} = m_1^- + r_- e^{-iz} m_2^-,
$$

$$
\frac{m_2^+}{a} = m_2^- + r_- e^{iz} m_1^-.
$$

From proposition 3.7 and remark 3.2, the left-hand side of (3.29) satisfies

$$
\frac{m_1^+}{a} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in H^2_b(\mathbb{R}, \mathbb{C}^2)
$$

for each $x$. On the other hand, it follows from proposition 3.5 and (1.13) that

$$
\frac{m_1^- + r_- e^{-iz} m_2^-}{a} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

is represented as

$$
\int_{-\infty}^{\infty} \left[ \Gamma_1^- (x, \xi) + F_- (x + \xi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] e^{iz} \, d\xi
$$

$$
+ \int_{-\infty}^{\infty} \int_{-\infty}^{0} \left[ \int_{t}^{\infty} \Gamma_2^- (x, t) F_- (x + t + \xi) \, dt \right] e^{iz} \, d\xi,
$$

for $a \cdot \xi < 0$.
where we regard the columns $\Gamma_1^\pm(x, \cdot)$ and $\Gamma_2^\pm(x, \cdot)$ of $\Gamma_\pm(x, \cdot)$ as vector-valued functions on the line which vanish for $\zeta > 0$. Since (3.31) should give a function in $H^2_0(\mathbb{R}, C^2)$, we get

$$\Gamma_1^-(x, \zeta) + F_-(x + \zeta) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \int_{-\infty}^{0} \Gamma_2^-(x, t) F_-(x + t + \zeta) \, dt = 0$$

(3.32)

for almost every $\zeta < 0$. A similar calculation with (3.30) shows that

$$\Gamma_2^-(x, \zeta) + F_-(x + \zeta) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \int_{-\infty}^{0} \Gamma_1^-(x, t) F_-(x + t + \zeta) \, dt = 0$$

(3.33)

for almost every $\zeta < 0$. Equations (3.32) and (3.33) together imply (3.27).

Finally, we recall the formulas for reconstructing $u$ from $\Gamma_\pm$, which play a crucial role in the inverse theory.

**Proposition 3.9.** Suppose that $u \in S(\mathbb{R})$. Then

$$u(x) = -\Gamma_{12}^+(x, 0)$$

(3.34)

and

$$u(x) = \Gamma_{12}^-(x, 0).$$

(3.35)

**Proof.** If $u \in S(\mathbb{R})$, it is easy to see that the kernels $\Gamma_\pm(x, \zeta)$ are smooth functions of $x$ and $\zeta$, and that the derivatives are integrable in $\zeta$. Using the integral representations for $M_\pm$ together with the integration by parts formula

$$\int_0^\infty f(\zeta) \exp(i\zeta \sigma_3) \, d\zeta = \sum_{j=1}^k \left( \frac{-1}{i\zeta} \right)^j f^{(j-1)}(0) + O \left( \frac{1}{\zeta^{j+1}} \right),$$

which is true for $f \in C^{k+1}([0, \infty); M_2(\mathbb{C}))$ with $\int |f^{(j)}(\zeta)| \, d\zeta < \infty$ for $0 \leq j \leq k + 1$, it is easy to see from the integral representations (3.25) and (3.26) that $M_+$ and $M_-$ have differentiable asymptotic expansions of the form

$$M_\pm(x, z) \sim I + \frac{M_{\pm,1}(x)}{z} + O \left( \frac{1}{z^2} \right)$$

(3.36)

for $u \in S(\mathbb{R})$, where

$$M_{\pm,1}(x) = \pm i \Gamma_\pm(x, 0) \sigma_3.$$

Substituting (3.36) into (3.1), we conclude that

$$Q(x) = -i \alpha(x) M_{\pm,1}(x)$$

$$= \pm \frac{1}{2} [\sigma_3 \Gamma_\pm(x, 0) \sigma_3 - \Gamma_\pm(x, 0)].$$

Formulas (3.34) and (3.35) are an immediate consequence.

4. **Inverse scattering**

We now suppose given $(F_+, F_-) \in X_1 \times X_1$ with $F_- = \mathcal{F}_-(\mathcal{I} \mathcal{F}_+)$ and show that equations (3.27) and (3.28) admit unique solutions $\Gamma_\pm$ depending continuously on $F_\pm \in X_1$. We then argue by continuity and density that the reconstruction formulas (3.34) and (3.35), known from the classical theory for $F_\pm$ in the Schwartz class $S(\mathbb{R})$, continue to hold also for $F_\pm \in X_1$. We remark that equations (3.27) and (3.28) can also be analyzed using the factorization theory in operator algebras as developed, e.g. in [21, chapter IV]; see also [33, 34] for some extensions and [4] for an example of application in inverse problems for Dirac operators in the AKNS form.
We give the details for recovering \( u \) from \( F_+ \) and \( F_- \) since the recovery of \( u \) from \( F_+ \) and \( F_- \) is closely analogous. By (3.34), it will suffice to solve (3.28) for \( \Gamma L_1 \) and show that the solution is sufficiently regular so that \( u(x) = -\Gamma L_1(x, 0) \) is well defined and gives a continuous map from \( X_1 \) to \( X_1^* \) for any \( c \in \mathbb{R} \).

First we prove the existence of a unique solution. From (3.28) we have (writing \( F \) for \( F_+ \))

\[
F(x + \zeta) + \Gamma L_1(x, \zeta) + \int_0^\infty \Gamma L_1(x, t) F(x + t + \zeta) \, dt = 0,
\]

Iterating these equations and writing \( \Gamma L_1(x, \zeta) \) exists as a bounded operator from \( X \) to \( X \), we see that

\[
\Gamma(x, \zeta) + F(x + \zeta) - \int_0^\infty \Gamma(x, t_2) F(x + t_2 + t_1) F(x + t_1 + \zeta) \, dt_2 \, dt_1 = 0.
\]

To solve this equation, we consider the scalar operators \( T_F(x) \) and \( T_T(x) \) on \( L^1(\mathbb{R}^+) \) given by

\[
[T_F(x)(\psi)](\zeta) = \int_0^\infty \psi(t) F(x + t + \zeta) \, dt,
\]

\[
[T_T(x)(\psi)](\zeta) = \int_0^\infty \psi(t) F(x + t + \zeta) \, dt
\]

and regard (4.1) as an equation in \( L^1(\mathbb{R}^+) \) for each fixed \( x \):

\[
\Gamma(x, \cdot) + F(x + \cdot) - (T_F(x) \circ T_T(x)) \Gamma(x, \cdot) = 0.
\] (4.2)

The operators \( T_F(x) \) and \( T_T(x) \) satisfy the estimates

\[
\|T_F(x)\|_{L^1 \to L^1}, \quad \|T_T(x)\|_{L^1 \to L^1} \leq \|F\|_1,
\]

\[
\|T_F(x)\|_{L^1 \to L^1}, \quad \|T_T(x)\|_{L^1 \to L^1} \leq \|F\|_X.
\] (4.3)

where \( r = \hat{F} \). Moreover, the map

\[
F \mapsto T_F
\]

is a continuous map from \( X \) into the bounded operators from \( L^1(\mathbb{R}^+) \) to itself, from \( L^1(\mathbb{R}^+) \) to \( L^2(\mathbb{R}^+) \) and from \( L^2(\mathbb{R}^+) \) to itself. Since \( T_F \) is compact as an operator from \( L^p(\mathbb{R}^+) \) to itself for \( F \in C_0^\infty(\mathbb{R}) \) and \( p = 1, 2 \), it follows by density that the same is true for any \( F \in X \).

The \( L^2 \) estimates imply that the operator

\[
(I - T_F(x) \circ T_T(x))^{-1}
\]

exists as a bounded operator from \( L^2(\mathbb{R}^+) \) to itself given by a convergent Neumann series. In particular, \( \ker_{L^2(\mathbb{R}^+)}(I - T_F(x) \circ T_T(x)) \) is trivial. On the other hand, any solution \( \psi \in L^1(\mathbb{R}^+) \) of the equation

\[
(I - T_F(x) \circ T_T(x))\psi = 0
\]

actually belongs to \( L^2(\mathbb{R}^+) \) since \( T_F(x) \circ T_T(x) \) maps \( L^1(\mathbb{R}^+) \) into \( L^2(\mathbb{R}^+) \). It follows that \( \ker_{L^2(\mathbb{R}^+)}(I - T_F(x) \circ T_T(x)) \) is also trivial. Since \( T_F \) and \( T_T \) are compact, we conclude that \( (I - T_F(x) \circ T_T(x))^{-1} \) also exists as a bounded operator from \( L^1(\mathbb{R}^+) \) to \( L^1(\mathbb{R}^+) \).

The solution to (4.2) is given by

\[
\Gamma(x, \cdot) = -(I - T_F(x) \circ T_T(x))^{-1} F \xi = -F \xi - G(x, \cdot),
\] (4.4)

where \( F \xi (\cdot) := F(x + \cdot) \) and

\[
G(x, \cdot) := T_F(x)(I - T_T(x) \circ T_F(x))^{-1}[T_T(x)F \xi].
\]
The right-hand side of (4.4) defines for each \( x \in \mathbb{R} \) a function \( \Gamma(x, \cdot) \in X_0^+ \).

To study continuity in \( x \) and \( \zeta \), set
\[
H(x, \cdot) := (I - T_F(x) \circ T_F(x))^{-1}[T_F(x)F_\zeta].
\] (4.5)

The estimate
\[
\|H(x, \cdot)\|_{L^2(\mathbb{R}^+)} \leq \left(1 - \|r\|_\infty^2\right)^{-1}\|r\|_\infty \|F\|_2
\]
holds, and \( x \mapsto H(x, \cdot) \) is a continuous mapping from \( \mathbb{R} \) into \( L^2(\mathbb{R}^+) \), as follows from the continuity of \( T_F(x) \) as an operator-valued function of \( x \), the continuity of \( x \mapsto F_\zeta \) as a mapping from \( \mathbb{R} \) into \( L^2(\mathbb{R}^+) \) and the uniform bounds on the resolvent. From the formula
\[
G(x, \zeta) = \int_0^\infty H(x, t)F(x + t + \zeta) \, dt,
\]
the Schwarz inequality and the continuity of \( H(x, \cdot) \) as a mapping into \( L^2(\mathbb{R}^+) \), we deduce that \( G(x, \zeta) \) is jointly continuous in \( x \) and \( \zeta \) and is uniformly bounded. Thus

**Lemma 4.1.** Suppose that \( F \in X_1 \). For each \( x \in \mathbb{R} \), there exists a unique solution \( \Gamma(x, \cdot) \) of (4.1) belonging to \( X_0^+ \). Moreover,
\[
\Gamma(x, \zeta) = -F(x + \zeta) - G(x, \zeta),
\]
where \( G \) is a bounded, jointly continuous function of \( x \) and \( \zeta \).

We can now obtain a limit of \( \Gamma(x, \zeta) \) as \( \zeta \to 0 \) and compute a putative reconstruction
\[
u_+(x) := -\Gamma(x, 0) = F(x) + w(x),
\] (4.6)
where
\[
w(x) = \int_0^\infty H(x, t)F(x + t) \, dt.
\] (4.7)

Recall that \( X^+_c = L^1(c, \infty) \cap L^2(c, \infty) \). We will now show that, for any fixed \( c \in \mathbb{R} \), the map
\[
G_c : X_1 \to X^+_c,
\]
\[
F \mapsto \nu_+
\]
defined by (4.6) is bounded and continuous. We will also show that
\[
G_c(F) = F + \sum_{n=1}^\infty G^+_n(F, \overline{F})
\] (4.8)
for multilinear maps \( G^+_n : X^n_1 \to X^+_c \). Indeed, set
\[
H_{2n}(x, \cdot) := (T_F(x) \circ T_F(x))^{n-1}[T_F(x)F_\zeta]
\]
and
\[
G^+_n(F, \overline{F}) = \int_0^\infty F(x + t)H_{2n}(x, t) \, dt.
\]

Then the claimed representation follows from (4.5), (4.6) and (4.7) provided that the series (4.8) converges in \( X^+_c \).

To prove convergence, suppose that \( F \in X_1 \) and choose \( \rho < 1 \) such that \( \|\bar{F}\|_\infty < \rho \). First, from (4.3) we have
\[
\|H_{2n}(x, \cdot)\|_2 \leq \rho^{2n-1}\|F\|_2,
\]
so that by the Schwarz inequality
\[ \|G_{2n+1}^+(F, \overline{F})\|_\infty \leq \rho^{2n-1}\|F\|_2^2. \]
(4.9)

It follows that \( \sum_{n=1}^{\infty} G_{2n+1}^+(F, \overline{F}) \) converges in \( L^\infty(\mathbb{R}) \). On the other hand, we have the explicit formula
\[ G_{2n+1}^+(F, \overline{F})(x) = \int_{\mathbb{R}^2_+} F(x + t_1) \overline{F}(x + t_1 + t_2) \cdots \overline{F}(x + t_{2n-1} + t_{2n}) F(x + t_{2n}) \, dt, \]
with \( t := (t_1, \ldots, t_{2n}) \in \mathbb{R}^{2n}_+ \). Fix \( F \) and choose \( x_0 \) so that
\[ \int_{x_0} ^{\infty} |F(s)| \, ds < \rho. \]

Note that the same condition holds for some relatively open set of \( F \in X_1 \). For any \( F \) in such an open set we then have
\[ \|G_{2n+1}^+(F, \overline{F})\|_{L^1(x_0, \infty)} \leq \rho^{2n+1}. \]
(4.10)

Thus, combining (4.9) and (4.10), we have for any \( c \in \mathbb{R} \) the estimate
\[ \|G_{2n+1}^+(F, \overline{F})\|_{L^1(c, \infty)} \leq (\max|x_0 - c, 0|)\|F\|_2^2 + \rho^2 \rho^{2n-1}, \]
(4.11)
which shows that \( \sum_{n=1}^{\infty} G_{2n+1}^+(F, \overline{F}) \) converges in \( L^1(c, \infty) \) for any fixed \( c \in \mathbb{R} \). It now follows from (4.9) and (4.11) that the series (4.8) converges in \( L^2(c, \infty) \), and hence in \( X_n^+ \) as claimed.

We have shown

**Lemma 4.2.** For any fixed \( c \in \mathbb{R} \), the mapping \( G_n^+ : X_1 \to X_n^+ \) is continuous and admits representation (4.8). Moreover, \( G_n^+-I \) is a continuous map of \( X_1 \) into \( C(\mathbb{R}) \).

Similarly, we can study the ‘left’ Gelfand–Levitan–Marchenko equation (3.27) to define a map \( G_n^- : F_1 \mapsto u_- \) via the reconstruction formula (3.35). The analogue of (4.1) (where now \( \Gamma \) denotes \( \Gamma_{12} \) and \( F \) denotes \( F_- \)) is
\[ \Gamma(x, \xi) + F(x + \xi) - \int_{-\infty}^{0} \int_{-\infty}^{0} \Gamma(x, t_2) \overline{F}(x + t_2 + t_1) F(x + t_1 + \xi) \, dt_2 \, dt_1 = 0, \]
and we set
\[ u_-(x) = \lim_{\gamma \to 0} \Gamma(x, y) \]
where the limit is taken in \( X_n^- \). In an analogous way we obtain a representation
\[ u_- = G_n^-(F) := -F + \sum_{n=1}^{\infty} G_{2n+1}^-(F, \overline{F}), \]
(4.12)
where \( G_n^- \) are multilinear functions acting from \( (X_1)^n \) to \( X_n^- \). An analogous argument shows

**Lemma 4.3.** For any fixed \( c \in \mathbb{R} \), the mapping \( G_n^- : X_1 \to X_n^- \) is continuous and admits representation (4.12). Moreover, \( G_n^- - I \) is a continuous map of \( X_1 \) into \( C(\mathbb{R}) \).

With these continuity statements, we can prove theorem 1.1.

**Proof of theorem 1.1.** We will discuss only the mapping \( F \circ S_n \). To show it is injective, assume that \( F_1 \in X_1 \) corresponds to \( u \in X \). Then, solving the Marchenko equation (3.28), we find the kernel \( \Gamma^* \) that by (3.25) determines uniquely the Jost solution \( \Psi_+ \), which, in turn, gives the matrix potential \( Q \) of (1.11). Therefore, different \( u \) are mapped into different \( F \), so that there is at most one potential \( u \in X \) with a given right reflection coefficient.

We next construct the inverse of \( F \circ S_n \). Given \( F \) in \( X_1 \), we set \( u_+ = G_n^+ F \) and \( u_- = G_n^-(F_-(\widehat{F})) \). We need to show the following facts:
1. \( u_+ = u_- \), and
2. the potential \( u = u_+ = u_- \) has reflection coefficients \( r_+ = \widehat{F} \) and \( r_- = \widehat{F} \).


Then continuity of the mapping $F \mapsto u$ follows from those of $G_\pm$ and (1). Analyticity of this map follows similarly from (1), analyticity of $G_-$ on $X_1^-$ and $G_+$ on $X_1^+$, and the representation $u = \xi u_+ + (1 - \xi)u_-$ for a function $\xi \in C^\infty$ with $\xi(t) = 1$ for $t < -1$ and $\xi(t) = 0$ for $t > 1$.

Fact (2) shows that the range of $F_+ \circ S_+$ is $X_1$.

We will appeal to standard results for the Gelfand–Levitan–Marchenko equations when $F \in S(\mathbb{R})$ (see for example section 2.4 of [18]) and the continuity of the direct and inverse scattering maps established above. Suppose given $X = \mathbb{R}$. Then continuity of the mapping $\xi u_+ + (1 - \xi)u_-$ for a function $\xi \in C^\infty$. By the continuity of the direct scattering maps of lemma 3.3, we can also conclude that $u$ has reflection coefficients $r_x(s) = \hat{F}_x(s)$ and $r_x(-s) = \hat{F}_x(-s) = (\mathcal{IF})(s)$.

Now it is easy to deduce corollary 1.2.

**Proof of corollary 1.2.** We only need to prove that the maps are real analytic. Real analyticity of the direct maps follows immediately from lemma 3.3 and remark 2.2. Real analyticity of the inverse map follows similarly.

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Appendix

Let us denote by $\hat{L}^1(\mathbb{R})$ the Wiener algebra of Fourier transforms (1.16) of functions in $L^1(\mathbb{R})$ with norm $\|f\|_{\hat{L}^1} := \|\hat{f}\|_{L^1}$, and by $\hat{X}$ the Banach algebra that is the image of $X = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ under the Fourier transform, equipped with the norm $\|\hat{f}\|_{\hat{X}} := \|\hat{f}\|_{L^1}$. We also denote by $1 + \hat{X}$ the unital extension of $\hat{X}$ obtained by adding the constant functions and norming $1 + \hat{X}$ with the norm $\|c + \hat{f}\|_{1 + \hat{X}} = |c| + \|\hat{f}\|_{\hat{X}}$.

We similarly define $1 + \hat{L}^1(\mathbb{R})$. The Fourier transform extends to $1 + \hat{X}$ by mapping the constant 1 into the convolution identity $\delta$.

We will need the following results.

**Lemma A.1.** Suppose that $f = \alpha + \hat{h} \in 1 + \hat{X}$. Then $f$ is invertible in the Banach algebra $1 + \hat{X}$ if and only if $f$ is non-vanishing on $\mathbb{R}$ and $\alpha \neq 0$.

**Proof.** If $f$ is invertible in $1 + \hat{X}$ it is also invertible in $1 + \hat{L}^1(\mathbb{R})$, so the condition is necessary by the Wiener theorem. If, on the other hand, $f$ does not vanish on $\mathbb{R}$ and $\alpha \neq 0$, then

$$\|f\|_{1 + \hat{X}} = |\alpha| + \|\hat{h}\|_{\hat{X}} > 0,$$

so $f$ is invertible in $1 + \hat{X}$.
then \( f \) is invertible in \( 1 + L^1(\mathbb{R}) \) with \( f^{-1} = \alpha^{-1} + g \) for \( g \in L^1(\mathbb{R}) \). We need to check that \( g \in L^2(\mathbb{R}) \). Without loss we take \( \alpha = 1 \) and compute that \( \hat{g} = -(1 + \hat{h})^{-1} \hat{h} \), which shows that \( g \in L^2(\mathbb{R}) \) as required.

We now have an analogue of the Wiener–Levi theorem for \( 1 + \hat{X} \).

**Lemma A.2.** Assume that \( f \in 1 + \hat{X} \) and that \( \phi \) is a function that is analytic in an open neighborhood \( \Omega \) of the closure of the range of \( f \). Then \( \phi \circ f \in 1 + \hat{X} \) and, moreover, the map

\[
\phi \mapsto \phi \circ f
\]

is an analytic map from \( 1 + \hat{X} \) into itself when restricted to functions with range contained in a fixed compact subset of \( \Omega \).

**Proof.** It suffices to note that, according to the above, the spectrum of \( f \) in \( 1 + \hat{X} \) coincides with the closure of its range. Then the standard functional calculus for Banach algebras applies, thus yielding the result.

**Lemma A.3.** Assume that \( U \) is an open subset of \( 1 + \hat{X} \) and that \( \Psi : U \to 1 + \hat{X} \) is analytic in \( f \) and \( \bar{f} \) in the sense of definition 2.1. Let also \( \phi \) be a complex-valued function that is analytic in a neighborhood of the closure of the set \( \bigcup_{f \in U} \text{Ran} \psi(f) \). Then the map \( f \mapsto \phi \circ \psi(f) \) from \( U \) into \( 1 + \hat{X} \) is analytic in \( f \) and \( \bar{f} \).

**Proof.** By lemma A.2, \( g \mapsto \phi \circ g \) is an analytic mapping in the Banach algebra \( 1 + \hat{X} \) defined on \( \psi(U) \). It is therefore analytic in \( 1 + \hat{X} \) considered as a Banach space. Thus the mapping \( f \mapsto \phi \circ \psi(f) \) is analytic in \( f \) and \( \bar{f} \) as a composition of analytic mappings between Banach spaces.

**References**

[1] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 The inverse scattering transform-Fourier analysis for nonlinear problems Stud. Appl. Math. 53 249–315

[2] Akhtosun T, Klaus M and van der Mee C 1993 On the Riemann–Hilbert problem for the one-dimensional Schrödinger equation J. Math. Phys. 34 2651–90

[3] Akhtosun T, Klaus M and van der Mee C 2000 Direct and inverse scattering for selfadjoint Hamiltonian systems on the line Integ. Equ. Oper. Theory 38 129–71

[4] Albeverio S, Hryniv R and Mykytyuk Ya V 2005 Inverse spectral problems for Dirac operators with summable potentials Russ. J. Math. Phys. 12 406–23

[5] Beals R and Coifman R R 1984 Scattering and inverse scattering for first order systems Commun. Pure Appl. Math. 37 39–90

[6] Beals R and Coifman R R 1985 Inverse scattering and evolution equations Commun. Pure Appl. Math. 38 29–42

[7] Beals R and Coifman R R 1987 Scattering and inverse scattering for first-order systems II Inverse Problems 3 577–93

[8] Beals R, Deift P and Tomei C 1988 Direct and Inverse Scattering on the Line (Mathematical Surveys and Monographs vol 28) (Providence, RI: American Mathematical Society)

[9] Bodenstorfer B., Dijksma A and Langer H 2000 Dissipative eigenvalue problems for a Sturm–Liouville operator with a singular potential Proc. R. Soc. Edinb. A 130 1237–57

[10] Brown R, Hryniv R and Perry P Solutions of the nonlinear Schrödinger equation with singular initial data by the method of inverse scattering (in preparation)

[11] Cohen A and Kappeler T 1992 Solutions to the cubic Schrödinger equation by the inverse scattering method SIAM J. Math. Anal. 23 900–22

[12] Degasperis A and Sabatier P C 1987 Extension of the one-dimensional scattering theory, and ambiguities Inverse Problems 3 73–109

[13] Deift P and Trubowitz E 1979 Inverse scattering on the line Commun. Pure Appl. Math. 32 121–251

[14] Deift P and Zhou X 2003 Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space Commun. Pure Appl. Math. 56 1029–77
[15] Demontis F and van der Mee C 2008 Scattering operators for matrix Zakharov–Shabat systems Integ. Equ. Oper. Theory 62 517–40

[16] Dineen S 1981 Complex Analysis in Locally Convex Spaces (Amsterdam: North-Holland)

[17] Faddeev L D 1959 The inverse problem in the quantum theory of scattering (Russian) Usp. Mat. Nauk. 14 57–119

Faddeev L D 1963 The inverse problem in the quantum theory of scattering J. Math. Phys. 4 72–104 (Engl. transl.)

[18] Faddeev L D and Takhtajan L A 2007 Hamiltonian Methods in the Theory of Solitons (Translated from the 1986 Russian original by Alexey G Reyman; Reprint of the 1987 English edition Classics in Mathematics) (Berlin: Springer)

[19] Fischer W, Leschke H and Müller P 1995 The functional-analytic versus the functional–integral approach to quantum Hamiltonians: the one-dimensional hydrogen atom J. Math. Phys. 36 2313–23

[20] Frayer C 2008 Doctoral thesis University of Kentucky

[21] Gohberg I C and Kreĭin M G 1969 Theory and Applications of Volterra Operators in Hilbert Space (Translated from the 1967 Russian original by A Feinstein) (Translations of Mathematical Monographs vol 24) (Providence, RI: American Mathematical Society)

[22] Hartman P 1964 Ordinary Differential Equations (New York–London–Sydney: Wiley)

[23] Hryniv R and Mykytyuk Ya V 2001 I-D Schrödinger operators with periodic singular potentials Methods Funct. Anal. Topology 7 31–42

Hryniv R, Mykytyuk Ya V and Perry P Inverse scattering for Schrödinger operators with Miura potentials: II. Different Riccati representatives arXiv:0910.0639

[24] Hryniv R, Mykytyuk Ya V and Perry P Inverse scattering for Schrödinger operators with Miura potentials: III. Bound states (in preparation)

Kappeler T, Perry P, Shubin M and Topalov P 2005 The Miura map on the line Int. Math. Res. Not. 3091–133

[25] Kurasov P 1996 On the Coulomb potential in one dimension J. Phys. A: Math. Gen. 29 1767–71

[26] Levitan B M 1979 Sufficient conditions for the solvability of the inverse problem of scattering theory on the entire line (Russian) Mat. Sb. (N.S.) 108 350–7

[27] Nachbin L 1969 Topology on Spaces of Holomorphic Mappings (Ergebnisse der Mathematik und ihrer Grenzgebiete vol 47) (Berlin: Springer)

Novikov R G 1997 Inverse scattering up to smooth functions for the Dirac–ZS-AKNS system Sel. Math. (N.S.) 3 245–302

[28] Pöschel J and Trubowitz E 1987 Inverse Spectral Theory (Pure and Applied Mathematics vol 130) (Orlando, FL: Academic)

[29] Shabat A B 1975 Inverse scattering problem for a system of differential equations Funkt. Anal. i Pril. 9 75–8 (Russian)

Shabat A B 1975 Inverse scattering problem for a system of differential equations Funkt. Anal. Appl. 9 244–7 (English transl.)

[30] Shabat A B 1979 An inverse scattering problem Diff. Uravn. 15 1824–34 (Russian)

Shabat A B 1980 An inverse scattering problem Diff. Eqs. 15 1299–307 (English transl.)

Wadati M 1973 The modified Korteweg-de Vries equation J. Phys. Soc. Japan 32 1289–96

[31] Zakharov V E and Shabat A B 1971 Exact theory of two-dimensional self-focussing and one-dimensional self-modulation of waves in nonlinear media Zh. Eksp. Teor. Fiz. 61 118–34 (Russian)

Zakharov V E and Shabat A B 1972 Exact theory of two-dimensional self-focussing and one-dimensional self-modulation of waves in nonlinear media Sov. Phys.—JETP 34 62–9 (English transl.)

Zhou X 1998 $L^2$-Sobolev space bijectivity of the scattering and inverse scattering transforms Commun. Pure Appl. Math. 51 697–731