Refutation of Hilbert Space Fundamentalism

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According to the “Hilbert Space Fundamentalism” Thesis, all features of a physical system, including the 3D-space, a preferred basis, and factorization into subsystems, uniquely emerge from the state vector and the Hamiltonian alone. I give a simplified account of the proof from arXiv:2102.08620 showing that such emerging structures cannot be both unique and physically relevant.

Hilbert Space Fundamentalism

Quantum Mechanics (QM) represents the state of any closed system, which may even be the entire universe, as a vector $|\psi(t)\rangle$ in a complex Hilbert space $\mathcal{H}$. The Hamiltonian operator $\hat{H}$ fully specifies its evolution equation,

$$|\psi(t)\rangle = \hat{U}_{t,t_0}|\psi(t_0)\rangle,$$

where $\hat{U}_{t,t_0} := e^{-i\hat{H}(t-t_0)}\hat{H}$ is the time evolution operator.

Since QM is invariant to unitary symmetries, this seems to justify the following:

**HSF.** The Hilbert-space fundamentalism Thesis (HSF): everything about a physical system, including the 3D-space, a preferred basis, a preferred factorization of the Hilbert space (needed to represent subsystems, e.g. particles), emerge uniquely from the triple $(\mathcal{H}, \hat{H}, |\psi\rangle)$.

**MQS.** We call $(\mathcal{H}, \hat{H}, |\psi\rangle)$ minimalist quantum structure.

The HSF Thesis is sometimes assumed more or less explicitly in some versions of various approaches to QM, like information-theoretic decoherence, and Everett’s Interpretation. The sufficiency of the MQS is claimed, perhaps most explicitly, by Carroll and Singh [3], p. 95:

“Everything else—including space and fields propagating on it—is emergent from these minimal elements.

But this goes beyond Everettianism. We read in [2]

The laws of physics are determined solely by the energy eigenspectrum of the Hamiltonian.

Scott Aaronson states in “The Zen Anti-Interpretation of Quantum Mechanics” [1] that a quantum state is a unit vector of complex numbers [...] which encodes everything there is to know about a physical system.

I confess that I think the HSF Thesis makes sense, if we take seriously the unitary symmetry of QM. Why would the position basis of the space $\mathcal{H}$ be fundamental, when it’s like other bases of $\mathcal{H}$? Just like the reference frames provide convenient descriptions of the 3D-space with no physical reality, why would $|x\rangle$ and $|p\rangle$ be fundamental among all of the bases of $\mathcal{H}$? And if they play a privileged role, shouldn’t this role emerge from the MQS alone? Unfortunately, we will see that this cannot happen.

Refutation of Hilbert Space Fundamentalism

In [4] I gave a fully general proof that the HSF Thesis is not true: for any MQS, no emerging structure can be both unique and physically relevant. Here I will give a less abstract and less technical version of that proof.

Denote a preferred structure expected to uniquely emerge from the MQS by $\mathcal{S}^{(\psi)}$, to express its dependence on $\hat{H}$ and $|\psi\rangle$. Any such structure should be invariant, so it can be defined in terms of tensor objects from $\otimes^s \mathcal{H} \otimes \otimes^s \mathcal{H}^\ast$ [5].

For a structure to be considered “preferred”, it has to consist of tensors that satisfy specific conditions, expressed as invariant tensor equations or inequations. For example, a preferred basis can be given by a set of operators $\hat{A}_\alpha \in \mathcal{H} \otimes \mathcal{H}^\ast$, satisfying the relations $\hat{A}_\alpha \hat{A}_\alpha^\dagger - \hat{A}_\alpha^\dagger \hat{A}_\alpha = 0$, $\hat{I}_\mathcal{H} - \sum_\alpha \hat{A}_\alpha = 0$, and tr $\hat{A}_\alpha = 1$.

This is generally true, and justifies the following

**Definition 1.** The kind $\mathcal{K}$ of a structure is given by the types of its tensors and the defining tensor (in)equations specific to that structure (more details in [4]).

In every case of interest, a set of Hermitian operators $\mathcal{S}^{(\psi)}_R = (\hat{A}_\alpha^{(\psi)})_{\alpha \in A}$ suffices to specify the candidate preferred structure. We already saw this for a preferred basis. The 3D-space can be given by the number operators for particles of each type $P$ at each point $x$ in space, $\hat{a}_P^\dagger(x)\hat{a}_P(x)$. Tensor product structures of $\mathcal{H}$ can be specified in terms of local algebras of Hermitian operators. Therefore, we can limit to Hermitian operators only, avoiding much of the mathematical framework developed in [4] to include any kind of tensor objects.

Two state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ are physically equivalent, $|\psi_1\rangle \sim |\psi_2\rangle$ if they represent the same physical state, i.e. if they are related by gauge symmetry, space isometries etc. There is a group $G_P$ of unitary or anti-unitary transformations of $\mathcal{H}$, so that $|\psi_1\rangle \sim |\psi_2\rangle$ iff $|\psi_2\rangle = \hat{g}|\psi_1\rangle$ for some $\hat{g} \in G_P$. This equivalence extends uniquely to tensors on $\mathcal{H}$. The main claim of the HSF Thesis is:
Theorem 1. (“Essentially unique”-ness). For any two \( \mathcal{K} \)-structures \( \hat{S}^{(\psi)}_H = (\hat{A}^{(\psi)}_\alpha)_{\alpha \in A} \) and \( \hat{S}'^{(\psi)}_H = (\hat{A}'^{(\psi)}_\alpha)_{\alpha \in A} \), there is a symmetry transformation \( \hat{g} \in G_P \) so that
\[
(\hat{A}'^{(\psi)}_\alpha)_{\alpha \in A} = (\hat{g} \hat{A}^{(\psi)}_\alpha \hat{g}^{-1})_{\alpha \in A}.
\] (3)

Often the symmetry transformation from eq. (3) simply permutes the operators \( \hat{A}'^{(\psi)}_\alpha \). For example, space isometries permute the position operators.

To be physically relevant, a 3-space should be able to distinguish among physically distinct states, e.g., since the density \( \langle \psi | \hat{g}^*_\alpha(x) \hat{g}_\alpha(x) | \psi' \rangle \) can be different for different values of \( |\psi'\rangle \), the same should be true for any candidate preferred 3-space. In particular, it should detect changes that the Hamiltonian cannot detect (the Hamiltonian cannot distinguish \( |\psi\rangle \) from \( |\psi'\rangle \) iff there is a unitary \( \hat{S} \) so that \( [\hat{H}, \hat{S}] = 0 \) and \( |\psi'\rangle = \hat{S} |\psi\rangle \), for example \( \hat{S} = \hat{U}_{t_1,t_2} \), but space should be able to distinguish them). The same applies to the components of \( |\psi\rangle \) in a preferred basis, and to tensor factors of \( \mathcal{K} \). Otherwise, such structures would have no physical relevance.

Physical equivalence requires that the structure has to distinguish not merely the unit vectors \( |\psi\rangle \sim |\psi'\rangle \in \mathcal{H} \), but any other state vectors \( \hat{g} |\psi\rangle \) and \( \hat{g}' |\psi'\rangle \in \mathcal{H} \) representing the same physical states, for any \( \hat{g}, \hat{g}' \in G_P \).

Condition 2 (Physical relevance). There exist at least two unit vectors \( |\psi\rangle \sim |\psi'\rangle \in \mathcal{H} \) representing distinct physical states not distinguished by the Hamiltonian, so that for any symmetry transformations \( \hat{g}, \hat{g}' \in G_P \),
\[
(\langle \psi | \hat{g}_\alpha^{\dagger} \hat{S}^{(\psi)}_H \hat{g} | \psi' \rangle)_{\alpha \in A} \neq (\langle \psi' | \hat{g}'_\alpha^{\dagger} \hat{S}^{(\psi)}_H \hat{g}' | \psi' \rangle)_{\alpha \in A}.
\] (4)

Proof. Fig. 2 illustrates the following construction. The \( \mathcal{K} \)-structure \( \hat{S}^{(\psi)}_H = (\hat{A}_\alpha^{(\psi)})_\alpha \) being a tensor object, its defining conditions are invariant to any unitary transformation \( \hat{S} \) that commutes with \( \hat{H} \). Then \( \hat{S}^{-1} \) transforms \( \hat{S}^{(\psi)}_H \) into a \( \mathcal{K} \)-structure \( \hat{S}^{(\psi)}_H = \hat{S}^{-1} \left[ \hat{S}^{(\psi)} \right]_H \) for \( |\psi\rangle \),
\[
\hat{S}^{(\psi)}_H := \left( \hat{S}^{-1} \hat{A}_\alpha^{(\psi)} \hat{S} \right)_\alpha.
\] (5)

From the uniqueness condition (3), (5) implies that
\[
(\langle \psi | \hat{g}_\alpha^{\dagger} \hat{A}_\alpha^{(\psi)} \hat{g} | \psi' \rangle)_{\alpha \in A} \neq (\langle \psi | \hat{S}^{-1} \hat{A}_\alpha^{(\psi)} \hat{S} | \psi' \rangle)_{\alpha \in A}.
\] (6)

for some transformation \( \hat{g} \in G_P \). From eq. (6)
\[
(\langle \psi | \hat{g}_\alpha^{\dagger} \hat{A}_\alpha^{(\psi)} \hat{g} | \psi' \rangle)_{\alpha \in A} = (\langle \psi | \hat{S}^{-1} \hat{A}_\alpha^{(\psi)} \hat{S} | \psi' \rangle)_{\alpha \in A}.
\] (7)

This contradicts Condition 2.

\[ \square \]

Example 1 gives an infinite family of physically distinct ways to choose the 3-space. But in fact there is such an infinite family for each generator commuting with \( \hat{H} \).

Theorem 1 was applied explicitly to various constructions assumed by the HSF Thesis to emerge from the MQS, and showed that, if they are physically relevant, they are not unique [4]. In particular, uniqueness fails for the generalized preferred basis (including for subsystems), the tensor product structure, emergent 3D-space, emergent classicality etc. Such structures cannot emerge uniquely from the MQS, a choice is always required.

A question stands: what breaks the symmetry of QM?

References

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