Shortcuts to Adiabatic Classical Spin Dynamics Mimicking Quantum Annealing

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We propose a simple construction of shortcuts to adiabaticity tracking instantaneous stationary states in classical spin systems without knowing tracked stationary states. In our construction, control fields of the counter-diabatic driving are constituted by state-dependent magnetic fields, which can be easily determined with an aid of numerical calculations. Easiness of our construction is a remarkable feature since it is usually a hard task to determine the explicit form of the required counter-diabatic terms in many-body systems. We also argue that our method can be applied to solve combinatorial optimization problems by considering classical spin dynamics under a time-dependent Hamiltonian, which mimics the procedure of quantum annealing.

Introduction.— Experimental techniques precisely tailoring quantum systems have been developed in these decades. They have opened up new worlds of quantum science and technology, especially quantum information processing [1]. The adiabatic control technique is one of the key concepts to harness quantum systems. Adiabatic control schemes have been used for implementing adiabatic quantum computations [2, 3], solving combinatorial optimization problems by using quantum annealing [4], generating highly entangled states [5–7], and optimizing quantum heat engines [8]. One of the main drawbacks of the adiabatic control is the long evolution time required by the adiabatic theorem [9, 10], which ensures that unitary time evolution under a time-dependent Hamiltonian tracks the instantaneous energy eigenstates when the Hamiltonian varies slowly enough in time.

Theory of shortcuts to adiabaticity (STA) has been developed as a strategy to realize such adiabatic time evolution within a short time [11–16]. STA enables us to realize the same time evolution without requiring slow change of a Hamiltonian by applying the counter-diabatic terms instead, which are constructed by using the energy eigenstates of the original Hamiltonian [11, 12].

It is of great interest to apply STA to above adiabatic control schemes [2–8]. Recently STA has been applied, for examples, to improve the performance of quantum heat engines [17–19], to create highly entangled states that can be used as resources of quantum metrology and quantum computation [20–22], to accelerate primitive processes of adiabatic quantum computation [23, 24], and to speedup quantum annealing in a simple model [25]. However, application of STA to quantum many-body systems is limited due to the requirements of knowing instantaneous energy eigenstates and of implementing non-local and many-body control Hamiltonians. In particular, the requirement of knowing instantaneous energy eigenstates makes difficult to apply STA to quantum annealing because what we have to know is nothing but what we want to know.

STA for classical systems has also been formulated, in which the counter-diabatic terms are constructed to conserve the volume of phase space enclosed by the equal energy surface, i.e., the adiabatic invariant [26, 27]. Application of classical STA to many-body systems is also difficult because calculation of the equal energy surface for many-body systems is hardly possible except for some special cases. However, the correspondence between quantum and classical STA [28] encourages us to investigate classical STA in detail.

In this Letter, we propose a simple construction of STA that tracks the instantaneous stationary solutions of the classical spin dynamics. In this construction, we do not need to know tracked stationary states. The control fields of the counter-diabatic driving in our construction are given by state-dependent magnetic fields, which can be easily obtained by numerical calculations. Our result also suggests easy implementation of STA in experiments. Moreover, our method offers an efficient classical algorithm for solving combinatorial optimization problems by considering classical spin dynamics mimicking quantum annealing, which enables us to speedup each annealing process.

Note that there is a method to construct approximate counter-diabatic terms for quantum systems without knowing instantaneous energy eigenstates based on the variational approach [29] (see also [30]). In contrast, our method can obtain exact counter-diabatic terms for classical systems without knowing instantaneous stationary states.

Classical spin dynamics.— We consider a classical spin system consisting of N spins expressed by three-dimensional unit vectors $\mathbf{m}_i = (m^x_i, m^y_i, m^z_i)$, $|m_i| = 1$, $i = 1, 2, \ldots, N$. Suppose that the system is described by a time-dependent Hamiltonian $\mathcal{H}_i(\{m_i\})$. The classical equations of motion are given by

$$\dot{m}_i(t) = 2m_i(t) \times \mathbf{h}^{\text{eff}}_i(t), \quad (1)$$

where $\mathbf{h}^{\text{eff}}_i(t)$ denotes an effective field at the $i$th spin, which is given by

$$\mathbf{h}^{\text{eff}}_i(t) = \frac{\partial \mathcal{H}_i}{\partial m_i} \quad (2)$$

Note that the classical equations of motion (1) can be

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viewed as the Hamiltonian dynamics for canonical variables \( \{q_i, p_i\}_{i=1}^N \) defined by

\[
\begin{align*}
    m_i^x &= \sqrt{1 - (2q_i)^2} \cos p_i, \\
    m_i^y &= \sqrt{1 - (2q_i)^2} \sin p_i, \\
    m_i^z &= 2q_i,
\end{align*}
\]

i.e., a set of the equations of motion (1) is equivalent to that of the Hamilton equations

\[
    \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.
\]  

An instantaneous stationary state at time \( t \), which is defined by \( \{m_i\} \) satisfying \( m_i \times \dot{h}^{\text{cd}}(t) = 0 \) for all \( i \) at a given time \( t \), is specified by a minimum of \( H \) as a function of \( z = \{q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N\} \), i.e.,

\[
    \frac{\partial^2 H}{\partial z^2} = 0.
\]

The instantaneous stationary state corresponding to the global minimum of the Hamiltonian is called the instantaneous ground state. We say that the stationary state \( \{m_i\} \) at the point \( z \) is critical if the determinant of the Hessian matrix at this point is also zero,

\[
    \det \left( \frac{\partial^2 H}{\partial z^2} \right) = 0.
\]

Below we specify the Hamiltonian in the form

\[
    H_t(\{m_i\}) = -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij}(t)m_i^z m_j^z - \sum_{i=1}^{N} h_i(t) \cdot m_i,
\]

but our method is not restricted to this specific form. It should be noted that the equations of motion (1) under the Hamiltonian (7) are the classical counterparts of the Heisenberg equations \( i\hbar \frac{d}{dt} \hat{\sigma}_i(t)/dt = [\hat{\sigma}_i(t), H_t] \) under the corresponding quantum Hamiltonian

\[
    \hat{H}(t) = -\frac{1}{2} \sum_{i,j=1}^{N} \hat{J}_{ij}(t)\hat{\sigma}_i^z \hat{\sigma}_j^z - \sum_{i=1}^{N} \hat{h}_i(t) \cdot \hat{\sigma}_i,
\]

where \( \hat{\sigma}_i = (\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z) \) denotes the Pauli matrices describing the \( i \)th spin. Here and hereafter we put \( \hbar = 1 \).

**Shortcuts to adiabaticity.** — Now we introduce STA. First we consider a generic quantum system described by a time-dependent Hamiltonian

\[
    \hat{H}(t) = \sum_n E_n(t) |n(t)\rangle \langle n(t)|,
\]

where \( |n(t)\rangle \) is the eigenstate of the Hamiltonian (9) corresponding to the energy eigenvalue \( E_n(t) \). In STA, diabatic transitions due to time dependence of the Hamiltonian (9) are canceled out by applying an extra control Hamiltonian

\[
    \hat{H}^{\text{cd}}(t) = \frac{i}{\hbar} \sum_n (1 - |n(t)\rangle \langle n(t)|) \partial_t |n(t)\rangle \langle n(t)|,
\]

which is called the counter-diabatic terms. We can show that the solution \( |\Psi(t)\rangle \) of the Schrödinger equation

\[
    i\hbar \frac{d}{dt} |\Psi(t)\rangle = [\hat{H}(t) + \hat{H}^{\text{cd}}(t)] |\Psi(t)\rangle
\]

coincides with the adiabatic dynamics under the Hamiltonian (9).

For a two-level system

\[
    \hat{H}(t) = -\hbar \hat{\sigma} \cdot \hat{\sigma},
\]

the counter-diabatic Hamiltonian is known to be given by

\[
    \hat{H}^{\text{cd}}(t) = \hat{f}(t) \cdot \hat{\sigma},
\]

where

\[
    \hat{f}(t) = \frac{\hbar(t) \times \dot{\hbar}(t)}{2|\hbar(t)|^2}.
\]

The total Hamiltonian \( \hat{H}^{\text{tot}}(t) = \hat{H}(t) + \hat{H}^{\text{cd}}(t) \) is thus given by

\[
    \hat{H}^{\text{tot}}(t) = -[\hbar(t) - \hat{f}(t)] \cdot \hat{\sigma}.
\]

**Method.** — We point out that the above counter-diabatic Hamiltonian for a two-level system can be used to construct STA tracking instantaneous stationary states in classical spin systems. This is because classical spin systems can be described by using product states of two-level systems. Our counter-diabatic terms for the classical spin system with the Hamiltonian \( H_t \) is given by

\[
    \hat{H}_t^{\text{cd}} = \sum_{i=1}^{N} \hat{f}_i(t) \cdot \hat{\sigma}_i,
\]

where

\[
    \hat{f}_i(t) = \frac{\hbar^{\text{eff}}(t) \times \dot{\hbar}^{\text{eff}}(t)}{2|\hbar^{\text{eff}}(t)|^2}.
\]

This counter-diabatic Hamiltonian is obtained by just replacing \( \hbar(t) \rightarrow \hbar^{\text{eff}}(t) \) and \( \hat{\sigma} \rightarrow \hat{\sigma}_i \), and by taking summation over \( i \) in Eqs. (13) and (14). We can show that the solution of the classical equations of motion

\[
    \dot{m}_i(t) = 2m_i(t) \times [\hbar^{\text{eff}}(t) - \hat{f}_i(t)]
\]

tracks the instantaneous stationary states of \( H_t \), i.e., the solution \( \{m_i(t)\} \) satisfies \( m_i(t) \times \dot{h}^{\text{cd}}(t) = 0 \) for all \( i \), if the initial state is stationary, i.e., \( m_i(0) \times \hbar^{\text{eff}}(0) = 0 \) for all \( i \). Note that STA fails when the instantaneous stationary states undergo some criticality, i.e., Eq. (6), because it leads to divergence of the counter-diabatic fields \( f_i(t) \).

For some applications, we want to track the instantaneous ground state, i.e., the stationary state with the
minimum energy. In such a case, the presence of a first order transition (a discontinuous jump of the ground state) also matters because the ground state becomes a metastable state (a stationary state with a higher energy) when a first order transition takes place. Thus, our method succeeds in obtaining the target ground state if there is neither a criticality nor a first order transition.

The counter-diabatic field \( \mathbf{f}(t) \) depends on \( \{ \mathbf{m}_j \} \) and \( \{ \mathbf{m}_i \} \), which is of a mean-field character. Indeed, we can derive Eqs. (16) and (17) for the classical Hamiltonian (7) as a result of the mean-field approximation for the quantum Hamiltonian (8) [31]. Because of this mean-field feature, the equations of motion (18) can be regarded as the self-consistent equations for \( \{ \mathbf{m}_i \} \). Since the set of equations (18) is linear in \( \{ \mathbf{m}_i \} \), it is not hard to solve these self-consistent equations.

In this way, we can easily perform STA in classical spin systems by just applying additional magnetic fields (17), which enables us to simulate adiabatic tracking of instantaneous stationary states. It is in stark contrast to quantum many-body systems, in which it is in general a hard task to obtain the explicit form of the counter-diabatic Hamiltonian since it depends on energy eigenstates of the many-body Hamiltonian (see Eq. (10)). Even if we could obtain an expression of the counter-diabatic Hamiltonian in a quantum many-body system, it would also be very hard to implement it in experiments because the counter-diabatic Hamiltonian contains non-local many-body interactions among spins.

**Demonstration in a simple model.**— Now we demonstrate our method by using the following paradigmatic model

\[
\mathcal{H}_t = -\frac{J}{2N} \sum_{i,j=1}^{N} m_i^z m_j^z - h^z(t) \sum_{i=1}^{N} m_i^z - h^x(t) \sum_{i=1}^{N} m_i^x,
\]

where the coupling strength \( J \) is a positive constant. We show how our method can find the exact ground state and how transitions and criticality affect stationary state tracking by STA. In this model, first order transitions take place in the ground state when the system undergoes the transition line \( h^x(t) = 0 \) and \( h^x(t) \in (-J, J) \), which is represented by a dotted line in the inset of Fig. 1, and stationary states show criticality when the condition \( J^{2/3} = (h^z(t))^{2/3} + (h^x(t))^{2/3} \) is satisfied, which is also represented by a dashed curve in the inset of Fig. 1. Especially, the ground state shows criticality when the condition is \( J = |h^z(t)| \) and \( h^x(t) = 0 \), which is specified by a black point in the inset of Fig. 1, and otherwise the metastable state shows it.

We simulate the following three cases: (i) no transition takes place, (ii) a first order transition takes place, and (iii) the system undergoes criticality after a first order transition. We assume that the magnetic fields are given by \( h^z(t) = J \cos(\pi t/\tau)/2 \) and \( h^x(t) = h_0 \sin(\pi t/\tau) \), where \( h_0 (> 0) \) enables us to change the path in parameter space and \( \tau \) is the operation time. In this setup, we can test above three cases with the same initial Hamiltonian and with the same final Hamiltonian. That is the parameters of the initial Hamiltonian are \( (h^z(0), h^x(0)) = (0, J/2) \) and those of the final Hamiltonian are \( (h^z(\tau), h^x(\tau)) = (0, -J/2) \), and the system undergoes a first order transition when \( 0 < h_0/J < 1 \) and shows criticality when \( 1/2 < h_0/J < 1 \). Note that the ground state of the final Hamiltonian is given by all spin-down state and the metastable state of that is given by all spin-up state. We perform numerical simulations with the parameters (i) \( h_0/J = 5/4 \) (purple curves), (ii) \( h_0/J = 1/4 \) (green curves), and (iii) \( h_0/J = 3/4 \) (cyan curves), and depict \( m^z(t) \equiv \sum_{i=1}^{N} m_i^z(t)/N \) in Fig. 1. Note that the results in Fig. 1 do not depend on \( N, \tau, \) and \( J \).

**Mimicking quantum annealing.**— Next we consider to solve combinatorial optimization problems, which can be formulated as a problem to find the ground state of the Ising Hamiltonian

\[
\hat{\mathcal{H}}_T = \frac{1}{2} \sum_{i,j=1}^{N} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z - \sum_{i=1}^{N} h_i^z \hat{\sigma}_i^z,
\]

called the target Hamiltonian. In the quantum annealing, we utilize the transverse field Hamiltonian \( \hat{\mathcal{V}} = -\sum_{i=1}^{N} \hat{\sigma}_i^z \) as a source of quantum fluctuations to find the ground state of the target Hamiltonian \( \hat{\mathcal{H}}_{T} \) [2–4, 32–35]. We change the Hamiltonian as

\[
\hat{\mathcal{H}}(t) = g(t/\tau)\hat{\mathcal{H}}_{T} + [1 - g(t/\tau)]\hat{\mathcal{V}},
\]

where \( g(t/\tau) \) is a continuous function of time satisfying \( g(0) = 0 \) and \( g(1) = 1 \), and \( \tau \) is the annealing time. In this letter, we assume \( g(t/\tau) = [1 - \cos(\pi t/\tau)]/2 \). The
initial state is prepared as the ground state of the initial Hamiltonian \( \hat{H}(0) = \mathcal{V} \), and then the adiabatic theorem ensures that the system remains in the instantaneous ground state and finally reaches the ground state of the final Hamiltonian \( \hat{H}(\tau) = \mathcal{H}_T \) if the annealing time \( \tau \) is sufficiently large. It means that we can solve combinatorial optimization problems.

Nowadays, we can implement the quantum annealing by using quantum annealers, e.g., the D-Wave machine [36]. Recent argument about quantumness of the D-Wave machine yielded new algorithms using classical spin dynamics that mimics the methodology of the quantum annealing [37–41]. Strong correlations between performance of the D-Wave machine and that of those classical algorithms have been reported [37, 40, 41].

Now we consider a classical analog of the quantum annealing Hamiltonian (21) expressed by the time-dependent classical Hamiltonian

\[
\mathcal{H}_t = g(t/\tau)\mathcal{H}_T + [1 - g(t/\tau)]\mathcal{V},
\]

where \( \mathcal{H}_T \) is the classical target Hamiltonian

\[
\mathcal{H}_T = -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij} m_i^z m_j^z - \sum_{i=1}^{N} h_i^z m_i^z,
\]

and \( \mathcal{V} \) is the classical transverse field Hamiltonian \( \mathcal{V} = -\sum_{i=1}^{N} m_i^x \). Starting from the ground state of the initial Hamiltonian \( \mathcal{H}_0 = \mathcal{V} \), we expect to reach the ground state of the final Hamiltonian \( \mathcal{H}_t = \mathcal{H}_T \) in the limit of \( \tau \to \infty \) or by using our method. However, as demonstrated by using the model (19), classical algorithms relying on deterministic classical dynamics result in failure to obtain the exact ground state when a stationary state undergoes transitions and/or criticality. It is known that first order transitions are sometimes resolved when we apply the inhomogeneous driving [42, 43]. Here, we consider the random transverse-field Hamiltonian

\[
\mathcal{V}' = -\sum_{i=1}^{N} h_i^z m_i^x,
\]

instead of \( \mathcal{V} \) in Eq. (22).

We numerically test our method by using the random field Ising model on the \( L \times L \) square lattice, i.e., \( J_{ij} = 1 \) for the neighboring pairs and \( J_{ij} = 0 \) otherwise, and \( \{h_i^z\} \) are random variables taking \( h_i^z = \pm 0.3 \). The number of spins is given by \( N = L^2 \). The ground state of this model can be exactly obtained by using the max-flow-min-cut algorithm (see, e.g., [44]). For a given realization of \( \{h_i^z\} \), we perform our method mimicking the annealing procedure with \( \tau = 1 \) by solving Eq. (18) for \( M \) realizations of the random transverse fields \( \{h_i^z\} \). The minimum energy among these \( M \) realizations, which is denoted by \( E_{\text{est}} \) is compared to the exact ground state energy \( E_g \) obtained by using the max-flow-min-cut algorithm. The result is regarded as a failure if the difference of the energies measured by \( \Delta = |(E_{\text{est}} - E_g)/E_g| \) is greater than 0.01. Note that the total computation time is proportional to \( M \) since we repeat the annealing procedure \( M \) times under different realizations of \( \{h_i^z\} \).

In Fig. 2, the system-size dependence of the failure probability for 3,456 realizations of \( \{h_i^z\} \) is plotted in the case of the uniform transverse field \( (h_i^z = 1 \text{ for all } i \text{ and } M = 1) \). For small system sizes \( L \leq 8 \) \( (N \leq 64) \), the failure probability is less than 1%, but it grows rapidly as the system size increases. The failure is due to the occurrence of first order transitions or criticality in the ground state.

Next, we repeat \( M \) different realizations of the random transverse fields \( \{h_i^z\} \) for each realization of \( \{h_i^z\} \). We choose \( h_i^z \) uniformly from the interval \( h_i^z \in [1, 2] \) for each \( i \). The repetition dependence of the failure probability is depicted in Fig. 3. The plot shows that for a fixed system size \( N \) the failure probability decreases as \( P_f \sim M^{-\gamma} \) with an exponent \( \gamma \). Thus, we can avoid the occurrence of the transition and the criticality by increasing \( M \). However, as shown in Fig. 4, the exponent \( \gamma \) decreases in the exponential way \( \gamma \sim e^{-O(N)} \), and thus the inhomogeneous driving based on uniform random numbers \( \{h_i^z\} \) is not so efficient for large system sizes even if STA is applied.

**Conclusion.**— In this Letter, we proposed a simple construction of STA for classical spin systems, which tracks instantaneous stationary states. Our construction does not require the knowledge of tracked instantaneous stationary states. In contrast, in order to construct the counter-diabatic terms, the energy eigenstates are required in quantum cases and the volume of phase space is required in classical cases in previous works. Starting from a stationary state of the initial Hamiltonian, our method results in one of the stationary states of the final Hamiltonian if there is no criticality.

Our method can be used to solve combinatorial op-
optimization problems by mimicking the procedure of the quantum annealing. In this algorithm, we aim to track the instantaneous ground state of a classical spin system, which mimics the quantum annealing, within a short time. In this case, not only criticalities, but also first order transitions matter because the ground state becomes a metastable state at a transition point. It has been shown that we can speedup each annealing process, whereas we need to repeat the annealing procedure for a lot of realizations of $\{h_i^c\}$ in order to obtain the exact ground state by avoiding criticalities and first order transitions. It is a future problem to find more efficient methods to obtain the exact ground state within the classical algorithm.

Note that some of the combinatorial optimization problems have $p$-body interactions with $p > 2$, which are usually decomposed into one- and two-body interactions by adding ancilla spins in order to implement in quantum annealers. It should be stressed that our method can be applied to systems which include $p$-body interactions with $p > 2$, and thus we do not need any ancilla spin to solve such problems. It is advantageous to deal with problems described by a large number of spins with many-body interactions.

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge Series on Information and the Natural Sciences (Cambridge University Press, 2000).

[2] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, arXiv preprint quant-ph/0001106 (2000).

[3] E. Farhi, J. Goldstone, S. Gutmann, J. Lapan, A. Lundgren, and D. Preda, *Science* 292, 472 (2001).

[4] T. Kadowaki and H. Nishimori, *Phys. Rev. E* 58, 5355 (1998).

[5] J. I. Cirac, M. Lewenstein, K. Mølmer, and P. Zoller, *Phys. Rev. A* 57, 1208 (1998).

[6] A. S. Sørensen and K. Mølmer, *Phys. Rev. Lett.* 86, 4431 (2001).

[7] T. H. Kyaw, Y. Li, and L.-C. Kwek, *Phys. Rev. Lett.* 113, 180501 (2014).

[8] A. E. Allahverdyan and T. M. Nieuwenhuizen, *Phys. Rev. E* 71, 046107 (2005).

[9] M. Born and V. Fock, *Z. Phys.* 51, 165 (1928).

[10] T. Kato, *J. Phys. Soc. Jpn.* 5, 435 (1950).

[11] M. Demirplak and S. A. Rice, *J. Phys. Chem. A* 107, 9937 (2003); *J. Phys. Chem. B* 109, 6838 (2005); *J. Chem. Phys.* 129, 154111 (2008).

[12] M. V. Berry, *J. Phys. A: Math. Theor.* 42, 365303 (2009).

[13] X. Chen, A. Ruschhaupt, S. Schmidt, A. del Campo, D. Guéry-Odelin, and J. G. Muga, *Phys. Rev. Lett.* 104, 063002 (2010).

[14] A. del Campo, M. M. Rams, and W. H. Zurek, *Phys. Rev. Lett.* 109, 115703 (2012).

[15] A. del Campo, *Phys. Rev. Lett.* 111, 100502 (2013).

[16] E. Torrontegui, S. Ibáñez, S. Martinez-Garaot, M. Modugno, A. del Campo, D. Guéry-Odelin, A. Ruschhaupt, X. Chen, and J. G. Muga, *Adv. At. Mol. Opt. Phys.* 62, 117 (2013).

[17] J. Deng, Q.-h. Wang, Z. Liu, P. Hänggi, and J. Gong, *Phys. Rev. E* 88, 062122 (2013).

[18] A. del Campo, J. Goold, and M. Paternostro, *Sci. Rep.*
We can construct exact counter-diabatic terms if all possible operators are taken into account for trial counter-diabatic terms. However, it is usually difficult for many-body systems.