COUNTEREXAMPLES TO HYPERKÄHLER KIRWAN SURJECTIVITY

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Abstract. Suppose that $M$ is a complete hyperkähler manifold with a compact Lie group $K$ acting via hyperkähler isometries and with hyperkähler moment map $(\mu_C, \mu_R) : M \to \mathfrak{t}^* \otimes \text{Im}(\mathbb{H})$. It is a long-standing problem to determine when the hyperkähler Kirwan map

$$H_K^*(M, Q) \to H^*(M/\!/K, Q)$$

is surjective. We show that for each $n \geq 2$, the natural $U(n)$-action on $T^*(SL_n \times \mathbb{C}^n)$ admits a hyperkähler quotient for which the hyperkähler Kirwan map fails to be surjective. As a tool, we establish a “Kähler = GIT quotient” assertion for products of cotangent bundles of reductive groups, equipped with the Kronheimer metric, and representations.

1. Introduction

Suppose that $M$ is a complete hyperkähler manifold with a compact Lie group $K$ acting via hyperkähler isometries and with hyperkähler moment map $(\mu_C, \mu_R) : M \to \mathfrak{t}^* \otimes \text{Im}(\mathbb{H})$. Choosing $\xi \in Z(\mathfrak{t}^*)$ sufficiently generic, we assume that $K$ acts quasi-freely (i.e., with finite stabilizers) on $\mu_C^{-1}(0) \cap \mu_R^{-1}(\xi)$ making $M/\!/K := \mu_C^{-1}(0) \cap \mu_R^{-1}(\xi)/K$ a smooth hyperkähler orbifold. The hyperkähler Kirwan map is the restriction map,

$$H_K^*(M, Q) \to H^*(M/\!/K, Q).$$

It is a long-standing question when (1.1) is surjective. The map (1.1) is known (cf. [Ko, JKK, FR], among others) to be surjective for some classes of examples including quiver varieties [MN]; it is known not to be surjective for some hyperkähler quotients of infinite-dimensional vector spaces (cf. [Hi, Hal, DW, CNS]).

The present paper exhibits an infinite list of examples for which $M$ is a (finite-dimensional) affine algebraic variety (in complex structure $I$), $G$ is a unitary group, and (1.1) is not surjective. Our counterexamples are built via familiar methods in complex algebraic geometry, and indeed are closely related to manifolds for which (1.1) is known (via [V]) to be surjective.

In fact, the paper combines two largely independent parts to achieve this goal.

First, let $G_1, \ldots, G_n$ be a finite collection of complex reductive groups and let $G = \prod_i G_i$. Then $G^2$ acts on $G$ via the product of left and right actions (which we call the “left-right action”); write $K \subset G$ for a choice of maximal compact subgroup. Let $G \subseteq G^2$ be a reductive subgroup acting on $G$ via the left-right actions and with maximal compact subgroup $K = K \cap G$. Let $V$ be a finite-dimensional representation of $G$ with $K$-invariant Hermitian metric and let $M = T^*G \times T^*V$. Thanks to work of Kronheimer [Kr] (see also [DS] for a clear exposition), it is known that $T^*G$ admits a complete, $K$-invariant hyperkähler metric (which we call the “Kronheimer metric”), and hence $K$ acts on $M$ by hyperkähler isometries. Because $M$ is a cotangent bundle, we obtain canonical
complex and real moment maps \((\mu_C, \mu_R) : M \to g^* \times \mathfrak{t}^*\). Choose a character \(\chi : G \to \mathbb{G}_m = \mathbb{C}^*\) and let \(\lambda = d\chi : \mathfrak{g} \to \mathbb{C}\), yielding \(-i\lambda \in \text{Hom}(\mathfrak{t}, \mathbb{R}) = \mathfrak{t}^*\).

**Theorem 1.1** (Theorem 2.6). The GIT and hyperkähler quotients of \(M\) are isomorphic: that is,
\[
\mu_C^{-1}(0) \cap \mu_R^{-1}(-i\lambda)/K \cong \mu_C^{-1}(0)/\chi G,
\]
where the right-hand side is the GIT quotient of \(\mu_C^{-1}(0)\) with respect to the character \(\chi\).

Such “Kähler quotient = GIT quotient” assertions are ubiquitous in the literature and there is a standard approach to proving them: we note the relevant references [KN, KB, AL, S], and especially the recent papers [Ho, Ma, Ta, NT]. In particular, it was known to some experts that one could prove (some version of) Theorem 1.1 using the standard approach (see [Ta] for a closely related situation the recent papers [Ho, Ma, Ta, NT]. In particular, it was known to some experts that one could prove a convergence condition on downward Morse flows that does not seem to have been documented as well as Theorem 2.1 and Section 3.3 of [NT]). But the standard approach still requires checking (some version of) Theorem 1.1 using the standard approach (see [Ta] for a closely related situation). In any case, their availability beyond a circle of experts was uncertain. In any case, we note the relevant references [KN, Kir, AL, Sj], and especially (\(\mu\) Tor equivalently (by Theorem 1.1) GIT quotient, becomes (Lemma 4.1)

\[\text{coulomb branches associated to } 3\text{D } N = 4 \text{ gauge theories; he also explained that there should probably be many more examples of failure of hyperkähler Kirwan surjectivity to be found among branches of moduli spaces of vacua of such theories: see [NT] for extensive treatment of Coulomb branches of affine quiver gauge theories from the viewpoint of Cherkis bow varieties.}

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4In particular, our method (using the action of the center \(Z(SL_n) = \mu_n\) on cohomology) will be recognized by the reader familiar with the Higgs bundle context (cf. [DW3], [CNS]).
Counterexamples to Hyperkähler Kirwan Surjectivity

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Convention 1.3. All algebraic varieties and algebraic groups are defined over \( \mathbb{C} \).

2. Construction of the Quotient

2.1. Complex Group Construction. We begin with a finite list \( G_1, \ldots, G_n \) of complex reductive groups, and let \( G = \prod_i G_i \). Each factor \( G_i \) inherits the left and right actions of \( G_i \) by \((g_t, g_r) \cdot g = g_t g g_r^{-1}\); we call the resulting action of \( G_i \times G_j \) the left-right action. Then \( G \) inherits a left-right action of \( G^2 \), inducing an action on \( T^*G \) as well. For a reductive subgroup \( G \subset G^2 \), we may choose a representation \( V \) of \( G \) and obtain a product \( M = T^*G \times T^*V \) with an induced \( G \)-action. As we explain below, \( T^*G \) admits a hyperkähler metric, constructed by Kronheimer; if \( K \subset G \) is a maximal compact subgroup, the Kronheimer metric is \( K \)-equivariant.

2.2. Special Case. In Section 2.1 below, we will consider the group

\[
G := GL_n(\mathbb{C}) \text{ with maximal compact subgroup } K := U(n);
\]

\( G \) acts on itself by the adjoint action, preserving the subgroup \( SL_n \). The maximal compact subgroup \( K \) thus acts compatibly on \( SL_n \) and \( GL_n \), with induced actions on their cotangent bundles.

Remark 2.1. Henceforth, we \( GL_n \)-equivariantly identify \( \mathfrak{gl}_n \) with \( \mathfrak{gl}_n^\ast \) via the pairing \((a, b) \mapsto \text{Tr}(ab)\).

Consider the vector representation \( \mathbb{C}^n \) of \( GL_n \), and the induced \( GL_n \)-action on \( T^*\mathbb{C}^n \). Write

\[
M_n := T^*SL_n \times T^*\mathbb{C}^n = SL_n \times \mathfrak{sl}_n^\ast \times T^*\mathbb{C}^n
\]

with its induced \( GL_n \)-action.

Writing \( \mathfrak{gl}_n = \mathfrak{sl}_n \times \mathbb{C} \) as \( GL_n \)-representations, we obtain a closed immersion \( T^*SL_n \hookrightarrow T^*GL_n \). Via the trace identification, \( T^*SL_n \) is identified with the fiber \((\det \times \text{tr})^{-1}(1, 0)\) of the map

\[
GL_n \times \mathfrak{gl}_n \xrightarrow{(\det \times \text{tr})} \mathbb{C}^* \times \mathbb{C}.
\]

One checks that the canonical complex moment map for the \( GL_n \)-action on \( T^*(GL_n \times \mathbb{C}^n) \) is given, under the trace identification, by

\[
\pi_C(X, Y, i, j) = XYX^{-1} - Y + ij \quad \text{for} \quad (X, Y, i, j) \in GL_n \times \mathfrak{gl}_n \times \mathbb{C}^n \times (\mathbb{C}^n)^*.
\]

Via the identification of \( \mathfrak{gl}_n^\ast \cong \mathfrak{gl}_n \) and the resulting identification \( \mathfrak{sl}_n^\ast \cong \mathfrak{sl}_n \), we find that the restriction of \( \pi_C \) to \( T^*SL_n \times T^*\mathbb{C}^n \) is identified with the complex moment map \( \mu_C \) for the latter.

2.3. Hyperkähler Structure. For this section we fix a complex reductive group \( G \).

We now consider the space \( \mathcal{A} = C^\infty([0, 1], \mathfrak{t} \otimes \mathbb{H}) \) of smooth maps from the interval \([0, 1]\) to the quaternionic Lie algebra \( \mathfrak{t} \otimes \mathbb{H} \). Write \( T = (T_0, T_1, T_2, T_3) \) for an element of \( \mathcal{A} \). The gauge group \( \mathcal{G} = C^\infty_0([0, 1], K) \) of smooth maps \( f : [0, 1] \to K \) that satisfy \( f(0) = e = f(1) \), where \( e \in K \) is the identity element, acts on \( \mathcal{A} \) by \( f \cdot (T_0, T_1, T_2, T_3) = (fT_0f^{-1} + \frac{df}{dt}f^{-1}, fT_1f^{-1}, fT_2f^{-1}, fT_3f^{-1}) \).
As explained by Kronheimer [Kr], and subsequently explored and clearly exposed in [DS], the zero pre-image $Z$ of the natural associated infinite-dimensional hyperkähler moment map consists of those $(T_0, T_1, T_2, T_3)$ satisfying Nahm’s equations,

$$\frac{dT_i}{dt} + [T_0, T_i] = [T_j, T_k],$$

where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$. Kronheimer shows that $Z/G \cong T^*G$. It follows from the construction that $T^*G$ inherits a complete hyperkähler metric that is $K \times K$-invariant under the left-right action.

One can more easily see the complex structure $I$, the standard complex structure, on $T^*G$ from the complex Nahm equation. More precisely, write $(\alpha, \beta) = (T_0 + iT_1, T_2 + iT_3)$; the complex Nahm equation is $\frac{d\beta}{dt} + [\alpha, \beta] = 0$. Write $Z_C \subset A_C = \{((\alpha, \beta))\}$ for its space of solutions, the zero preimage of a complex moment map for the action on $A_C$ of the complex group $G_C = C^\infty(\{0, 1\}, G)$ via

$$g(t) \cdot (\alpha(t), \beta(t)) = (\text{Ad}_{g(t)}(\alpha(t)) - \frac{dg}{dt}(t)^{-1}, \text{Ad}_{g(t)}(\beta(t))).$$

and one gets a diffeomorphism $Z/G \rightarrow Z_C/G_C$ defined by $(T_0, T_1, T_2, T_3) \mapsto (T_0 - iT_1, T_2 + iT_3)$.

Under Kronheimer’s identification of $Z_C/G_C$ with $T^*G$, the left-right action of $e^Y \in G \times G$ for $Y \in \mathfrak{g} \times \mathfrak{g}$ takes on a particularly simple form. Namely, write $Y = (Y_e, Y_r)$, and consider the functions $Y : [0, 1] \rightarrow \mathfrak{g}$ and $g : [0, 1] \rightarrow G$ defined by

$$(2.3) \quad Y(t) = (1 - t)Y_e + tY_r \text{ and } g(t) = e^{Y(t)}.$$

Then $g(t)$ naturally acts on $A = A_C$ preserving $Z_C$. It follows from the discussion at the end of Section 2 of [DS] that this action is identified with the left-right action of $e^Y$ on $T^*G$.

### 2.4. Kähler Potential

It is known that there is a (global) Kähler potential $\mu_K$ for the Kronheimer metric on $T^*G$ for $G$ any complex reductive group.

More precisely, [HKLR] show that if one forms the hyperkähler reduction of $a$, possibly infinite-dimensional, hyperkähler manifold by an appropriate group of hyperkähler isometries; and if, in addition, the reduced manifold comes equipped with an $S^1$-action that rotates complex structures $I$ and $J$ and fixes complex structure $K$, then the moment map $\mu_K$ for this $S^1$-action is a global potential for the hyperkähler metric.

It is computed in [DS] that $\mu_K(T_0, T_1, T_2, T_3) = \int_0^1 |T_2|^2 + |T_2|^2$. Mayrand proves:

**Proposition 2.2** (Mayrand [Ma]). The Kähler potential $\mu_K$ is proper and bounded below.

In order to more easily analyze its growth farther on, we wish to modify the Kähler potential $\mu_K$ as follows. Recall that the inner product on $\mathfrak{t}$ is given by $\langle a, b \rangle = \text{Tr}(ab^\dagger)$. We use repeatedly that for $T \in u(n)$, $\overline{T} = -T$. Then $\beta = T_2 + iT_3 = -T_2 + iT_3$. Thus $\beta - \overline{\beta} = 2T_2$. Now

$$(2.4) \quad \int_0^1 |T_2|^2 = \int_0^1 T_2T_2 = \frac{1}{4} \int_0^1 (\beta - \overline{\beta})(\beta - \overline{\beta})^\dagger = -\frac{1}{4} \int_0^1 \text{Tr}(\beta - \overline{\beta})^2.$$

It is clear from the above description of complex structure $I$ that the function

$$h(\beta) = \frac{1}{4} \int_0^1 \text{Tr} \beta^2$$

defines a holomorphic function on $T^*G$ (it is holomorphic by the above, and it is clearly invariant under the complex gauge group, hence descends to $T^*G$ in complex structure $I$). It follows:

**Lemma 2.3.** The function $\mu_K + h + \overline{h}$ is also a Kähler potential for the Kronheimer metric.
We have
\[
\overline{h(\beta)} = \frac{1}{4} \int_0^1 \text{Tr} \beta^2 = \frac{1}{4} \int_0^1 \text{Tr} \left( \overline{\beta} \right)^2.
\]
Combining (2.4) and (2.6) gives:
\[
(\mu_K + h + \overline{h})(\alpha, \beta) = \int_0^1 |T_1|^2 + \frac{1}{4} \int_0^1 \text{Tr} (\beta \overline{\beta} + \overline{\beta} \beta).
\]
Using that \(\text{Tr}(AB) = \text{Tr}(BA)\) and that \(\overline{T_i} = -T_i\) for \(T_i \in \mathfrak{k}\) gives
\[
\text{Tr} \beta \overline{\beta} = \text{Tr} \overline{\beta} \beta = \text{Tr} \left[ (T_2 + iT_3)(T_2 - iT_3) \right] = |T_2|^2 + |T_3|^2, \text{ yielding}
\]
\[
(\mu_K + h + \overline{h})(\alpha, \beta) = \int_0^1 |T_1|^2 + \frac{1}{2} |T_2|^2 + \frac{1}{2} |T_3|^2 = \int_0^1 |\text{Im}(\alpha)|^2 + \frac{1}{2} |\beta|^2.
\]
In particular, it is immediate that
\[
\mu_K + h + \overline{h} \geq \frac{1}{2} \mu_K.
\]
Recall that, under the identification of the moduli space of solutions of Nahm’s equations with \(T^* G\), the left-right action of \(K \times K\) is identified with the action of \(C^\infty([0,1], K)\). It is immediate that:

**Lemma 2.4.** The potential (2.7) is invariant under the left-right action of \(K \times K\).

Suppose given a finite collection \(G_i, i = 1, \ldots, n\) of complex reductive groups and a representation \(V\) of \(G := \prod_i G_i\). Choose a Hermitian inner product on \(V\) that is \(K\)-invariant for \(K = \prod_i K_i\), \(K_i\) a maximal compact subgroup of \(G_i\). We now equip \(M = \prod_i T^* G_i \times T^* V\) with the product hyperkähler metric, where each \(T^* G_i\) has the Kronheimer metric and \(T^* V\) is given the flat metric with Kähler potential \(|\cdot|^2\). We thus obtain a Kähler potential \(F_1(x, y) = \mu_K(x) + h(x) + \overline{h}(x) + |y|^2\) on \(T^* G \times T^* V\). The following is immediate from Proposition 2.2 and the inequality (2.8).

**Corollary 2.5.** The Kähler potential \(F_1 : M \to \mathbb{R}\) is proper and bounded below.

### 2.5. The GIT/Hyperkähler Quotient Construction

Give a finite collection \(G_i, i = 1, \ldots, n\) of complex reductive groups as above, choose and fix a complex reductive subgroup \(G \subset G \times G\), acting on \(T^* G\) via the left-right action: write \(K \subset G\) for a choice of maximal compact subgroup. Choose a representation \(V\) of \(G\), and let \(M = T^* (G \times V)\). Choose a character \(\chi : G \to \mathbb{C}_m\) and write \(\lambda = d\chi : g \to \mathbb{C}\).

**Theorem 2.6.** The map
\[
\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(-i\lambda) \hookrightarrow \mu_{\mathbb{C}}^{-1}(0)
\]
takes values in \((\mu_{\mathbb{C}}^{-1}(0))^{\chi-ss}\) and induces a homeomorphism of topological stacks
\[
(\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(-i\lambda))/K \simeq \mu_{\mathbb{C}}^{-1}(0)^{\chi-ss}/G
\]
and a complex-analytic isomorphism
\[
(\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(-i\lambda))/K = M\|_{(0,-i\lambda)}K \xrightarrow{\cong} \mu_{\mathbb{C}}^{-1}(0)/\chi G.
\]

**Remark 2.7.** As in the introduction, we remark that the (closely related) theorem was known to experts (cf. Sections 2.2 and 3.3 of [NPT]). Also as remarked in the introduction, the proof follows the usual approach: standard methods and results (see [S]) immediately reduce the proof to showing a certain assertion about limits of downward Morse flows, which is the main content of Section 3 and which does not seem to have been previously documented in the literature.
It follows that we get a commutative diagram of equivariant cohomology groups
\[(2.9)\]
\[
\begin{array}{ccc}
H^*_G(M, \mathbb{Q}) & \xrightarrow{\cong} & H^*_K(M, \mathbb{Q}) \\
\downarrow & & \downarrow \\
H^*_G(\mu^{-1}_G(0)|\sim) & \xrightarrow{\cong} & H^*_K(\mu^{-1}_G(0) \cap \mu^{-1}_K(\xi), \mathbb{Q}).
\end{array}
\]

We use the identifications in diagram (2.9) to reduce Theorem 1.2 to the corresponding assertion about the algebraic-symplectic Kirwan map, which we analyze in Section 4. The proof of Theorem 2.6 appears in Section 3.

3. Proof of Theorem 2.6

As explained above, we equip \(M = T^* (G \times V)\) with the product of the Kronheimer hyperkähler metrics on \(T^* G\) and the flat hyperkähler metric on \(T^* C^n\).

3.1. Kähler Potential and Moment Map Pre-Image. As above, we write \(F_1 : T^* G \times T^* V \rightarrow \mathbb{R}\) for the Kähler potential on \(M\) given by \(F_1(x, y) = \mu_K(x) + h(x) + \overline{h}(x) + |y|^2\). We further equip \(\mathbb{C}\) with the singular Kähler metric (the “lifted Fubini-Study metric”) with Kähler potential \(F_2(z) = \frac{1}{2} \log |z|^2\). We define \(F : T^* G \times T^* V \times \mathbb{C} \rightarrow \mathbb{R}\),

\[
(3.1) \quad F(x, y, z) = F_1(x, y) + F_2(z) = (\mu_K(x) + h(x) + \overline{h}(x)) + |y|^2 + \frac{1}{2} \log |z|^2.
\]

We let \(G \subset G \times G\) act on \(\mathbb{C}\) by \(g \cdot z = \chi(g)z\), and give \(T^* G \times T^* V \times \mathbb{C}\) the product action of \(G\). Then \(K\) acts preserving the Kähler metric, and the Kähler potentials \(F_1\) and \(F\) are evidently \(K\)-invariant. As above, write \(\lambda = d\chi : \mathfrak{g} \rightarrow \mathbb{C}\).

Choose \((x, y, z) \in T^* G \times T^* V \times \mathbb{C}\) (notation \(\mathbb{C} \setminus \{0\}\)), and let \(O = G \cdot (x, y, z)\) be its \(G\)-orbit. We use:

**Theorem 3.1** ([Mo], Theorem 4.1). Let \(H \subset K\) be a closed subgroup of a compact connected Lie group \(K\). Then there is an \(H\)-invariant subspace \(m \subset \mathfrak{k}\) for which the map \((k, v) \mapsto ke^{iv}\) induces a diffeomorphism \(K \times_H m \rightarrow K_C/H_C\).

Assume that the orbit \(O\) is closed in \(T^* G \times T^* V \times \mathbb{C}\): in other words, that the point \((x, y) \in T^* G \times T^* V\) is a \(\chi\)-semistable point of \(T^* G \times T^* V\) under the action of \(G\). Then the orbit \(O\) is affine, and it follows that the stabilizer \(G_{(x, y, z)}\) is a reductive subgroup of \(G\). Choosing an appropriate \((x, y, z) \in O\), we may assume chosen, without loss of generality, a maximal compact subgroup \(H\) of \(G_{(x, y, z)}\) that is also a subgroup of \(K\). Mostow’s Theorem [14] yields a diffeomorphism

\[
(3.2) \quad K \times_H m \xrightarrow{\cong} O, \quad (k, v) \mapsto ke^{iv} \cdot (x, y, z).
\]

For \(X \in \mathfrak{g}\), write \(\tilde{X}\) for the induced vector field on \(O\).

**Lemma 3.2.** For any \(q = g \cdot (x, y) \in M\), \(p = g \cdot (x, y, z) \in O\) and \(X \in \mathfrak{f}\), we have

\[
dF_p(i\tilde{X}) = \langle \mu_{\mathbb{R}}(q), X \rangle + \lambda(iX).
\]

**Proof.** Using \(dF = dF_1 + \frac{1}{2} d \log |z|^2\), we get \((dF_1)_q(i\tilde{X}) = \langle \mu_{\mathbb{R}}(q), X \rangle\) from [Mo, Proposition 4.1]. Now observe that since \(X \in \mathfrak{f}\), we have \(\chi(e^{isX}) = e^{\lambda(s)X} \forall s \in \mathbb{R}\). Then \(\frac{d}{ds} \log |\chi(e^{isX})|^2|_{s=0} = 2\lambda(iX)\), yielding the desired result.

**Proposition 3.3.**

1. A point \(p \in O\) is a critical point of \(F|_O\) if and only if \(p \in \mu_{\mathbb{R}}^{-1}(-i\lambda)\).
2. Any critical point of \(F|_O\) is a minimum.
3. The critical locus of \(F|_O\) is either empty, or consists of a single \(K\)-orbit.
Proof. Assertion (1) is immediate from Lemma 3.2. Assertion (2) is immediate from Lemma 1 of [AL]. Assertion (3) follows from the Proposition of [AL]. □

Proposition 3.4. Suppose that for every closed orbit \( \mathcal{O} = G \cdot (x, y, z) \) for \( z \neq 0 \), the function \( F|_\mathcal{O} \) is proper and bounded below. Then the map \( \mu_C^{-1}(0) \cap \mu_R^{-1}(-i\lambda) \to \mu_C^{-1}(0) \) induces a diffeomorphism

\[
(\mu_C^{-1}(0) \cap \mu_R^{-1}(-i\lambda))/K \to \mu_C^{-1}(0)/G.
\]

Proof. It follows from the hypothesis that each closed orbit \( G \cdot (x, y, 1) \) of \( T^*G \times T^*V \times C \) contains a critical point. Proposition 3.3 then implies that each closed orbit contains a unique \( K \)-orbit of critical points. It thus follows by Proposition 3.3 that a \( \chi \)-semistable orbit \( G \cdot (x, y) \subset T^*G \times T^*V \) contains a unique \( K \)-orbit in \( \mu_R^{-1}(-i\lambda) \). The map (3.3) is thus a \( C^\infty \) bijection. That it is a diffeomorphism is immediate by standard arguments. □

Unfortunately, we do not know a general result that would establish the hypothesis of Proposition 3.4 without some specific information about the Kähler potential—though one might hope that a general result should exist. Thus, in Section 3.4 we prove by an explicit analysis of the growth of the Kähler potential that when \( \mathcal{O} \) is a closed orbit, then \( F|_\mathcal{O} \) is indeed proper and bounded below.

3.2. Action of Semisimple Elements of \( \mathfrak{g} \). We will say that a function \( G(s) \) of a real variable \( s \geq 0 \) grows exponentially in \( s \) if there exist constants \( c > 0, C > 0 \), and \( C_0 \) such that \( G(s) \geq C e^{cs} + C_0 \) for all \( s \geq 0 \). We say \( G(s) \) grows quadratically in \( s \) if there is a real polynomial \( a_2 s^2 + a_1 s + a_0 \) with \( a_2 > 0 \) such that \( G(s) \geq a_2 s^2 + a_1 s + a_0 \) for all \( s \geq 0 \).

Suppose that \( V \) is a finite-dimensional complex vector space with Hermitian inner product \( \langle \cdot, \cdot \rangle \), and that \( X \in \text{End}(V) \) is a semisimple endomorphism that is skew-Hermitian. That is, \( \langle Xv, w \rangle + \langle v, Xw \rangle = 0 \) for all \( v, w \in V \); then \( X \) has imaginary eigenvalues. Since \( X \) is semisimple and skew-Hermitian, \( iX \) is semisimple with real eigenvalues. Decompose \( V = \oplus_\eta V_\eta \), where \( V_\eta \) is a direct sum of eigenspaces for \( X \) with eigenvalue \( \eta \). Furthermore, the subspaces \( V_\eta, V_{\eta'} \) are orthogonal for distinct \( \eta, \eta' \), since they are orthogonal for \( X \). Then:

Lemma 3.5. For any \( v = \sum \eta v_\eta \in V \), with \( v_\eta \in V_\eta \), we have, for \( s \in \mathbb{R} \),

\[
e^iX \cdot v = \sum \eta v_\eta \quad \text{and} \quad |e^{iX} \cdot v|^2 = \sum |v_\eta|^2 = \sum e^{2s|\eta|} |v_\eta|^2.
\]

Continuing with the above hypotheses, suppose next that \( V \) comes equipped with a real structure, i.e., a complex antilinear involution \( \sigma : V \to V \). The imaginary part of a vector \( v \) is then \( \text{Im}(v) = \frac{v + \sigma(v)}{2i} \). If \( X \) is a real endomorphism, so \( \sigma \circ X = X \circ \sigma \), and if \( v \in V_\eta \) is an eigenvector for \( iX \) with eigenvalue \( \eta \), then \( iX \sigma(v) = \sigma(-iXv) = \sigma(-\eta v) = -\eta \sigma(v) \), so \( \sigma(v) \in V_{-\eta} \).

3.3. Action of Certain Gauge Transformations. Fixing \( X_\ell, X_r \in u(V) \) and taking \( Y_\ell = iX_\ell, Y_r = iX_r \) and defining \( Y(t) \) as in Formula (2.3), there is a finite subset \( \mathcal{S} = \{s_0, s_1, \ldots, s_d\} \subset [0, 1] \) so that on each interval \( (s_{k-1}, s_k) \), the number of positive eigenvalues, and the multiplicity of each, of \( Y(t) \) is constant. It follows from Proposition 5.1 that we can find an orthogonal eigenbasis for \( Y(t) \), \( v_i(t) : \mathbb{R} \to V, \) \( i = 1, \ldots, n, \) varying continuously in \( t \in \mathbb{R} \), with continuously varying eigenvalues \( \eta_i(t) \).

It follows that, fixing an interval \( (s_{k-1}, s_k) \), there are index sets \( J_+, J_- \) and continuous functions \( \nu_j(t), j \in J_+ \cup J_- \) so that if \( j \neq j' \) then \( \nu_j(t) \neq \nu_{j'}(t) \) for all \( t \in (s_{k-1}, s_k) \), and for each \( i \in I_+ \) and \( i \in J_- \), there exists a unique \( j \in J_+ \), respectively a unique \( j \in J_- \), with \( \eta_i(t) = \nu_j(t) \) on \( [s_{k-1}, s_k] \). We assume the index sets \( J_+, J_- \) have been chosen so that \( \eta_{-j}(t) = -\eta_j(t) \) for all \( j \).

A continuous map \( v(t) : [s_{k-1}, s_k] \to V \) then admits a unique expression \( v(t) = \sum_{j \in J_+} v_j(t) + v_0 + \sum_{j \in J_-} v_j(t), \) with \( Y(t)v_j(t) = \nu_j(t)v_j(t) \) for \( j \in J_+ \cup J_- \) and \( Y(t)v_0(t) = 0 \).
Lemma 3.6. For any continuous \(v(t) : [s_{k-1}, s_k] \to V\), either \(\int_{s_{k-1}}^{s_k} |\text{Im}(e^{Y(t)} \cdot v(t))|^2 dt\) is bounded as a function of \(s \geq 0\) or it grows exponentially in \(s\).

Proof. We have

\[
\text{Im}(e^{Y(t)}v(t)) = \text{Im}(v_0(t)) + \frac{1}{2t} \sum_{j \in J_t} \left[ e^{s\eta_j(t)}v_j(t) + \sigma(e^{s\eta_j(t)}v_j(t)) + e^{-s\eta_j(t)}v_{-j}(t) + \sigma\left(e^{-s\eta_j(t)}\sigma(v_{-j}(t))\right)\right]
\]

\[
= \text{Im}(v_0(t)) + \frac{1}{2t} \sum_{j \in J_t} \left[ \left(e^{s\eta_j(t)}(v_j(t) + \sigma(v_j(t))) + e^{-s\eta_j(t)}(v_{-j}(t) + \sigma(v_{-j}(t)))\right)\right].
\]

Since \(\text{Im}(v_0(t))\) and \(\sum_{j \in J_t} e^{-s\eta_j(t)}(v_{-j}(t) + \sigma(v_{-j}(t)))\) are continuous functions of \(t\), uniformly bounded in norm for \(s \geq 0\), we find that \(\int_{s_{k-1}}^{s_k} |\text{Im}(e^{Y(t)} \cdot v)|^2 dt\) is unbounded as a function of \(s \geq 0\) if and only if

\[
\frac{1}{4} \int_{s_{k-1}}^{s_k} \left| \sum_{j \in J_t} (e^{s\eta_j(t)}(v_j(t) + \sigma(v_j(t))))^2 \right| dt = \frac{1}{4} \sum_{j \in J_t} \int_{s_{k-1}}^{s_k} e^{2s\eta_j(t)} \left| v_j(t)^2 + |\sigma(v_j(t))|^2 \right| dt
\]

is unbounded as a function of \(s\). This happens if and only if \(|v_j(t)|^2 + |\sigma(v_j(t)|^2 > 0\) for some \(j\) and some \(t \in [s_{k-1}, s_k]\): in that case, by continuity we can find some closed interval \([a, b] \subset (s_{k-1}, s_k)\) (with \(b > a\) and \(\epsilon > 0\) for which \(|v_j(t)|^2 + |\sigma(v_j(t)|^2 > \epsilon\) on \([a, b]\); and then there exists a \(C > 0\) with \(\eta_j(t) > C\) on \([a, b]\), showing that the right-hand side of (3.4) is bounded below by \(\frac{(b-a)t}{4}e^{2Cs}\), which grows exponentially in \(s\).

Note that, taking \(g(t) = e^{Y(t)}\), we get \(\frac{da}{dt} g(t)^{-1} = s \frac{d\sigma}{dt} = s(Y_r - Y_t)\). The modified Kähler potential \(\mu_K + h + \overline{h}\) of (2.7) thus satisfies

\[
(\mu_K + h + \overline{h})(\text{Ad}_{e^{Y(t)}}(\alpha, \beta)) = \int_0^1 |\text{Im}(\text{Ad}_{e^{Y(t)}}\alpha(t)) + s\text{Im}(Y_r - Y_t)|^2 dt + \frac{1}{2} \int_0^1 |\text{Ad}_{e^{Y(t)}}\beta(t)|^2 dt.
\]

Proposition 3.7. Let \((Y_r, Y_t) \in g \times g\) and let \(Y(t) = (1 - t)Y_t + tY_r\). If \((\mu_K + h + \overline{h})(\text{Ad}_{e^{Y(t)}}(\alpha, \beta))\) is unbounded as a function of \(s\) then \((\mu_K + h + \overline{h})(\text{Ad}_{e^{Y(t)}}(\alpha, \beta))\) grows at least quadratically in \(s\).

Proof. Suppose first that the term \(\frac{1}{2} \int_0^1 |\text{Ad}_{e^{Y(t)}}\beta(t)|^2 dt\) is unbounded as a function of \(s\). Combining Proposition 5.1 with Lemma 5.5 we get

\[
\frac{1}{2} \int_0^1 |\text{Ad}_{e^{Y(t)}}\beta(t)|^2 dt = \frac{1}{2} \sum_i \int_0^1 e^{2s\eta_i(t)}|\beta_i(t)|^2 dt.
\]

Either each \(\beta_i(t) \leq 0\) whenever \(\eta_i(t) > 0\), in which case the above expression is bounded as a function of \(s\); or there is some closed interval \([a, b]\) and \(\epsilon > 0\), \(C > 0\) on which \(|\beta_i(t)| > \epsilon\) and \(\eta_i(t) > 0\), showing that the above expression is bounded below by \(\frac{(b-a)t}{2}e^{2Cs}\) and thus grows exponentially in \(s\). We conclude that \((\mu_K + h + \overline{h})(\text{Ad}_{e^{Y(t)}}(\alpha, \beta))\) grows exponentially as a function of \(s\).

Suppose next that \(\int_0^1 |\text{Im}(\text{Ad}_{e^{Y(t)}}\alpha(t))|^2 dt\) is unbounded as a function of \(s\). Applying Lemma 3.6 to \(\int_0^1 |\text{Im}(\text{Ad}_{e^{Y(t)}}\alpha(t))|^2 dt\), we see that either it is bounded in \(s\) or grows exponentially in \(s\).

Since \(|s \text{Im}(Y_r - Y_t)|^2\) is bounded above by a quadratic polynomial in \(s\), if \(\int_0^1 |\text{Im}(\text{Ad}_{e^{Y(t)}}\alpha(t))|^2 dt\)
Case 1. We consider the cases separately:

For each \( 3.4. \) implying that \((\mu_K + h + \overline{h})(Ad_{e^{X(t)}}(\alpha, \beta))\) grows exponentially in \( s \).

Finally, suppose that \( \int_0^1 |\text{Im}(Ad_{e^{X(t)}}(\alpha(t)))|^2 \, dt \) is bounded as a function of \( s \). Then either \( \text{Im}(Y_r - Y_t) = 0 \), in which case the entire integral \( \int_0^1 |\text{Im}(Ad_{e^{X(t)}}(\alpha(t))) + s \text{Im}(Y_r - Y_t)|^2 \, dt \) is bounded as a function of \( s \), or \( \text{Im}(Y_r - Y_t) \neq 0 \), in which case the entire integral \( \int_0^1 |\text{Im}(Ad_{e^{X(t)}}(\alpha(t))) + s \text{Im}(Y_r - Y_t)|^2 \, dt \) grows quadratically in \( s \).

Combining the conclusions of the previous three paragraphs yields the conclusion. \( \square \)

**Remark 3.8.** The proofs of Lemma 3.6 and Proposition 3.7 show the following. Let \( X(z) \) be an element of \( u(V) \) depending continuously on \( z \in B \) where \( B \) is a small ball around \( 0 \in \mathbb{R}^N \), and associate the function \( Y(t, z) \) via Formula (2.3). If \((\mu_K + h + \overline{h})(Ad_{e^{X(t)}}(\alpha, \beta))\) is unbounded as a function of \( s \), then there are a small ball \( B \) around \( 0 \in \mathbb{R}^N \) and a choice of \( a_2 > 0 \) such that

\[
(\mu_K + h + \overline{h})(Ad_{e^{X(t,z)}}(\alpha, \beta)) \geq a_2 s^2 + a_1 s + a_0
\]

for all \( s \geq 0 \) and \( z \in B \).

### 3.4. Properness of \( F|_O \).

We are now ready to prove:

**Proposition 3.9.** For each closed orbit \( O = G \cdot (x, y, z) \subset T^*G \times T^*V \times \mathbb{C} \) with \( z \neq 0 \), the function \( F|_O \) is proper and bounded below.

We will make extensive use of Mostow’s coordinates \( 3.2 \) on \( O \). Since \( F \) is \( K \)-invariant, it suffices to prove that the composite \( F \circ \phi \), with \( \phi : m \to O \) defined by \( \phi(X) = e^{iX} \cdot (x, y, z) \), is proper and bounded below.

**Assumption 3.10.** We assume without loss of generality that \( z = 1 \).

Note that \( \log |\chi(e^{iX})|^2 = 2\lambda(iX)s \) with \( \lambda(iX) \in \mathbb{R} \). Then:

**Proposition 3.11.** For each \( X \in m \), the function \( G(s) = F(e^{iX} \cdot (x, y, z)) \) satisfies one of:

\begin{enumerate}
  \item \( \lambda(iX) > 0 \) and \( F(e^{iX} \cdot (x, y, z)) \geq F_1(e^{iX} \cdot (x, y)) \) for all \( s \geq 0 \).
  \item \( \lambda(iX) \leq 0 \) and \( F(e^{iX} \cdot (x, y, z)) \) grows at least quadratically in \( s \).
\end{enumerate}

**Proof.** We consider the cases separately:

**Case 1.** \( \lambda(iX) \geq 0 \). Then \( G(s) = F_1(e^{iX} \cdot (x, y)) + \frac{1}{2} \log |\chi(e^{iX})|^2 \geq F_1(e^{iX} \cdot (x, y)) \).

**Case 2.** \( \lambda(iX) \leq 0 \). If \( F_1(e^{iX} \cdot (x, y)) \) is bounded as a function of \( s \), then by Corollary 2.10 the trajectory \( \{e^{iX} \cdot (x, y)\} \) lies in a compact subset of \( T^*G \times T^*V \). Thus, there exists an unbounded sequence \( s_n \) in \( \mathbb{R}_{\geq 0} \) for which the sequence \( e^{is_nX} \cdot (x, y) \) converges in \( T^*G \times T^*V \), say to \((x_0, y_0)\).

If \( \lambda(iX) < 0 \), it follows from the previous paragraph that \( \lim_{n \to \infty} e^{is_nX} \cdot (x, y) = (x_0, y_0, 0) \) lies in the closure of \( O \) in \( T^*G \times T^*V \times \mathbb{C} \); since its third coordinate is 0 it cannot lie in \( O \), a contradiction since \( O \) was assumed closed. Thus \( F_1(e^{iX} \cdot (x, y)) \) is unbounded as a function of \( s \). On the other hand, if \( \lambda(iX) = 0 \), then \( \lim_{n \to \infty} e^{is_nX} \cdot (x, y) = (x_0, y_0, 1) \); since the orbit \( O \) is assumed closed, we have \((x_0, y_0, 1) = ke^{ix'} \cdot (x, y, 1) \) for some \( k \in K \) and \( x' \in m \). But Theorem 3.1 then implies that \( (1, 0, 0) \to (k, x') \) in \( K \times m \) as \( n \to \infty \), which is obviously false. Thus again \( F_1(e^{iX} \cdot (x, y)) \) is unbounded as a function of \( s \).

Now choose a solution \( (\alpha, \beta) \) of the complex Nahm equation representing \( x \in T^*G \). By the conclusion of the previous paragraph, either \( (\mu_K + h + \overline{h})(Ad_{e^{iX}}(\alpha, \beta)) \) is unbounded as a function of
s, or $|e^{ixY}y|^2$ is unbounded as a function of $s$. In the first case, identifying $iX = (Y_t, Y_\ell)$ with $Y(t)$ in the gauge group via Formula (2.3), we conclude from Proposition 3.7 that $(\mu_K + h + \ell)(\Ad_{iX}(\alpha, \beta))$ grows at least quadratically as a function of $s$; while in the second case, we conclude from Lemma 3.5 that $|e^{ixX}y|^2$ grows exponentially as a function of $s$. In either case, adding the linear function $\lambda(iX)s$ we still obtain that $F(e^{ixX} \cdot (x, y, z))$ grows at least quadratically as a function of $s$.

Proof of Proposition 2.9. Let $S \subset \mathfrak{m}$ denote the unit sphere in $\mathfrak{m}$ in the $K$-invariant inner product induced from $\mathfrak{k}$. We then get a proper surjective map $\mathbb{R}_{\geq 0} \times S \to \mathfrak{m}$, $(t, X) \mapsto tX$: it suffices to show that the composite map $\mathbb{R}_{\geq 0} \times S \to \mathbb{R}$, $(s, X) \mapsto F(e^{ixX} \cdot (x, y, z))$ is proper and bounded below.

Consider the function $e : S \to \mathbb{R}$ defined by $e(X) = \lambda(iX)$; write

$$S_+ = e^{-1}(\mathbb{R}_{>0}), \quad S_0 = e^{-1}(0), \quad S_- = e^{-1}(\mathbb{R}_{<0}).$$

By Proposition 3.11 and Remark 3.8, for every point $X \in S_0 \cup S_0$ there are an open neighborhood $U_X$ of $X$ in $S_0 \cup S_0$ and $\alpha_2 s^2 + \alpha_1 s + a_0$ with $\alpha_2 > 0$ such that $F(e^{ixX} \cdot (x, y, z)) \geq \alpha_2 s^2 + \alpha_1 s + a_0$ for all $X' \in U_X$. Since $S_0 \cup S_0$ is compact, it follows that there exists a single choice of $\alpha_2 s^2 + \alpha_1 s + a_0$ such that $F(e^{ixX} \cdot (x, y, z)) \geq \alpha_2 s^2 + \alpha_1 s + a_0$ for all $X' \in S_0 \cup S_0$. Thus the restriction of $(s, X) \mapsto F(e^{ixX} \cdot (x, y, z))$ to $\mathbb{R}_{\geq 0} \times (S_0 \cup S_0)$ is proper and bounded below.

Now, Proposition 3.11 and Remark 3.8 together imply that the restriction of $(s, X) \mapsto F(e^{ixX} \cdot (x, y, z))$ to a neighborhood of $\mathbb{R}_{\geq 0} \times S_0$ in $\mathbb{R}_{\geq 0} \times (S_0 \cup S_0)$ grows at least quadratically in $s$; moreover, the proposition immediately implies that $(s, X) \mapsto F(e^{ixX} \cdot (x, y, z))$ grows at least linearly in $s$, with a lower bound on the slope, on the complement of that neighborhood in $\mathbb{R}_{\geq 0} \times (S_0 \cup S_0)$. It follows that the restriction of $(s, X) \mapsto F(e^{ixX} \cdot (x, y, z))$ to $\mathbb{R}_{\geq 0} \times (S_0 \cup S_0)$ is proper and bounded below.

Combining the conclusions of the previous two paragraphs yields Proposition 3.9.

Proof of Theorem 2.6. The hypothesis of Proposition 3.4 is supplied by Proposition 3.9. Then Proposition 3.4 immediately yields the conclusion.

4. Hilbert Schemes and Subvarieties

We now turn to the situation of Theorem 1.2 of the introduction. Thus, we return to the notation of Section 2.2.

Consider $M = M_n = T^*SL_n \times T^*C^n$. Applying Theorem 2.6 to the $GL_n = SL_n \times_{\mu_n} \mathbb{G}_m$-action induced from the adjoint action on $T^*SL_n$ and the obvious action on $C^n$ shows that, for $\chi = \det : GL_n \to \mathbb{G}_m$ and $\xi = -i \det$, we have $\mu^{-1}_C(0)\parallel_{\chi G} \cong M_{\parallel(0, \xi)}K$. It follows that the hyperkähler Kirwan map is identified with the map

$$H^*_{CGL_n}(SL_n) \cong H^*_{GL_n}(T^*SL_n \times T^*C^n) \xrightarrow{\sim} H^*_{GL_n}(\mu^{-1}(0) \parallel_{\det} \mathbb{G}_m) = H^*(\mu^{-1}_C(0)\parallel_{\det} GL_n).$$

As in Section 2.2 above, the image of the natural embedding $T^*SL_n \times T^*C^n \to T^*GL_n \times T^*C^n$ is the preimage of $(1, 0)$ under the map

$$T^*GL_n \times T^*C^n \cong GL_n \times \mathfrak{gl}_n \times C^n \times (\mathbb{C}^n)^* \ni (X, Y, i, j) \mapsto (\det(X), \text{tr}(Y)) \in \mathbb{C}^* \times \mathbb{C}.$$ 

We write $\det \times \text{tr}$ for the map. The map is clearly $GL_n$-invariant and thus descends to a map $\Pi^*_{C^{-1}(0)}\parallel_{\det} GL_n$.

Lemma 4.1. The Hamiltonian reduction $\Pi^{-1}_C(0)\parallel_{\det} GL_n$ of $T^*GL_n \times T^*C^n$ is isomorphic to the Hilbert scheme of points $(\mathbb{C}^* \times \mathbb{C})[n]$. Under this isomorphism, the function $\det \times \text{tr}$ is identified with the product, respectively sum, of the coordinates of the $n$ points.

Proof. The subset $\Pi_C^{-1}(0)$ consists of $(X, Y, i, j)$ with $X$ invertible and $XYX^{-1} - Y + ij = 0$ is identified with the set of $(X, Y, i, j) \in T^*(\mathfrak{gl}_n \times C^n)$ satisfying $[X, Y] + ij = 0$ and with $X$ invertible via $(X, Y, i, j) \mapsto (X, Y, i, j) = (X, Y, i, jX)$. One easily sees that $\det$-stability corresponds. The result is thus immediate from $[N_3]$.
We obtain a commutative diagram

\begin{equation}
H^*_{GL_n}(GL_n, \mathbb{Q}) \xrightarrow{\kappa} H^*_{GL_n}(SL_n, \mathbb{Q})
\end{equation}

\begin{equation}
H^*((\mathbb{C}^* \times \mathbb{C})^{|n|}, \mathbb{Q}) \xrightarrow{\kappa} H^*(\mathcal{M} \parallel K, \mathbb{Q}).
\end{equation}

For later reference, we note one easy topological fact. Consider the natural map $H^*_{GL_n}(GL_n, \mathbb{Q}) \rightarrow H^*_{GL_n}(SL_n, \mathbb{Q})$ of equivariant cohomology groups associated to the adjoint-equivariant inclusion $SL_n \rightarrow GL_n$.

**Proposition 4.2.** The homomorphism of Ad-equivariant cohomology

\[ H^*_{GL_n}(GL_n, \mathbb{Q}) \rightarrow H^*_{GL_n}(SL_n, \mathbb{Q}) \]

is surjective.

**Proof.** We use the map $SL_n \twoheadrightarrow PGL_n$. Over $\mathbb{C}$, $H^*(PGL_n, \mathbb{C}) \rightarrow H^*(SL_n, \mathbb{C})$ is an isomorphism since both are identified with the cohomology of their common Lie algebra; hence $H^*(PGL_n, \mathbb{Q}) \rightarrow H^*(SL_n, \mathbb{Q})$ is also an isomorphism. This yields an isomorphism of $E_2$ pages for the Leray spectral sequences abutting to $p^*: H^*_{GL_n}(PGL_n, \mathbb{Q}) \rightarrow H^*_{GL_n}(SL_n, \mathbb{Q})$, showing that $p^*$ is an isomorphism. Since $p^*$ factors through $H^*_{GL_n}(GL_n, \mathbb{Q}) \rightarrow H^*_{GL_n}(SL_n, \mathbb{Q})$, the conclusion follows. \qed

The right-hand vertical map $\kappa$ in (4.1) is the hyperkähler Kirwan map (1.1) for our manifold $\mathcal{M}$. Since the top horizontal map is surjective, if the hyperkähler Kirwan map for $\mathcal{M}$ were surjective then the bottom horizontal arrow would be surjective. We will show that the map

\[ H^*((\mathbb{C}^* \times \mathbb{C})^{|n|}, \mathbb{Q}) \rightarrow H^*(\mathcal{M} \parallel K, \mathbb{Q}) = H^*((\det \times \text{tr})^{-1}(1, 0), \mathbb{Q}) \]

is not surjective.

To do this, we consider the $\mathbb{C}^*$-action on $(\mathbb{C}^* \times \mathbb{C})^{|n|}$ defined by scaling in the $\mathbb{C}$-factor, as in [Gro] or [Na]. We use Chapter 7 of [Na] as our reference. This action is elliptic in the sense used in [BDMN]: that is, all downward flows converge, and thus one obtains a Białynicki-Birula (henceforth, BB) decomposition.

More precisely, we abbreviate $N = (\mathbb{C}^* \times \mathbb{C})^{|n|}$. Recall that $\text{Sym}^m(\mathbb{C}^*) \cong \mathbb{C}^* \times \mathbb{C}^{m-1}$: the first coordinate is the product of the $m$ elements of $\mathbb{C}^*$, and the remaining coordinates are (up to signs) the remaining elementary symmetric functions of the $m$ scalars.

For a partition $\lambda = 1^{\lambda_1}2^{\lambda_2}3^{\lambda_3} \ldots$, we have the symmetric product

\[ S^\lambda(\mathbb{C}^*) = S^{\lambda_1}(\mathbb{C}^*) \times S^{\lambda_2}(\mathbb{C}^*) \times \ldots \cong \coprod_{\lambda_i > 0} (\mathbb{C}^* \times \mathbb{C}^{\lambda_i-1}). \]

Then the fixed locus $N^{\mathbb{C}^*}$ is isomorphic to the disjoint union,

\[ N^{\mathbb{C}^*} \cong \bigsqcup_{\lambda: |\lambda| = n} S^\lambda(\mathbb{C}^*), \]

and writing

\[ S_\lambda = \{ x \in N \mid \lim_{t \to 0} t \cdot x \in S^\lambda(\mathbb{C}^*) \}, \]

we get $N = \bigsqcup_{\lambda} S_\lambda$. Then

\[ H^*(N, \mathbb{Q}) \cong \bigoplus_{\lambda} H^*(S_\lambda) \]

(a BB decomposition; we ignore grading shifts). If $\Gamma$ is any finite group acting by automorphisms of $N$ commuting with the $\mathbb{C}^*$-action, then $\Gamma$ acts naturally on the left-hand and right-hand sides of...
and the splittings involved in choosing a BB decomposition can be chosen \( \Gamma \)-equivariantly to make \( \lambda \) an isomorphism of \( \Gamma \)-representations.

The \( \mathbb{C}^* \)-component of the “center-of-mass” map, i.e., \( \det : N \to \mathbb{C}^* \times \mathbb{C} \overset{\pi_1}{\to} \mathbb{C}^* \), restricts to \( S^\lambda(\mathbb{C}^*) \) as the projection on the product \( \prod_{i=1}^n \mathbb{C}^* \) of \( \mathbb{C}^* \)-factors of \( \det \) followed by the map

\[
\prod_{i=1}^n \mathbb{C}^* \to \mathbb{C}^*, \quad (x_1, \ldots, x_n) \mapsto x_1 \cdot x_2^2 \cdot x_3^3 \cdots x_n^n.
\]

Now consider the action of the group \( \Gamma = \mu_n \) of \( n \)th roots of unity, identified with the center of \( SL_n \), on \( N \) by left multiplication on \( SL_n \). This action extends to an action of the connected group \( \mathbb{C}^* \) on \( GL_n \), hence on \( N \), which thus acts trivially on \( H^*(N) \). Considering the \( \Gamma \)-action on \( M/\mathbb{R} \), we find that, for \( \lambda = (n) = (1^0 2^1 \ldots n^1) \), we have that \( \det^{-1}(1) \cap S^\lambda(\mathbb{C}^*) \) is in natural bijection with \( \Gamma \). In other words:

**Proposition 4.3.** The set of length \( n \) subschemes that are \( \mathbb{C}^* \)-fixed and have the form \( \{ \xi \} \times \text{Spec} \mathbb{C}[t]/(t^n) \subset \mathbb{C}^* \times \mathbb{C} \) for some \( \xi \in \Gamma \), form a collection of connected components of \( \left( (\det \times \text{tr})^{-1}(1, 0) \right)^{\mathbb{C}^*} \).

Since the action of \( \Gamma \) on this set of components obviously freely cyclically permutes them, we find that the regular representation of \( \Gamma \) appears as a subrepresentation of \( H^*(M/\mathbb{R}, \mathbb{Q}) \), thus completing the proof of Theorem 1.2.

**Remark 4.4.** As in Remark 7.6 of [CNS], the proof above actually shows (as asserted in Theorem 1.2) that the regular representation \( \mathbb{Q}[\Gamma] \) actually appears in the pure cohomology (in the Hodge-theoretic sense) \( \bigoplus_k W_k H^k(M/\mathbb{R}, \mathbb{Q}) \): the Białynicki-Birula decomposition is compatible with Hodge weights, and we have identified the regular representation in the pure part of the cohomology of the \( \mathbb{C}^* \)-fixed locus.

## 5. Appendix: Some Hermitian Linear Algebra

This section proves an elementary result about families (over the interval \([0, 1] \subset \mathbb{R}\)) of self-adjoint operators on a finite-dimensional Hermitian vector space needed in the proof of Theorem 1.1. While much stronger results are available in the literature, we include a proof of what we need, to emphasize the more algebro-geometrically inclined reader that no sophisticated real analysis is needed.

Fix a complex vector space \( V \) of dimension \( n \) with Hermitian metric \( \langle \cdot, \cdot \rangle \). Let \( L \in \mathfrak{gl}(V)[t] \) be a polynomial map \( \mathbb{A}^1_\mathbb{C} \to \mathfrak{gl}(V) \) for which \( L(\mathbb{R}) \subseteq \mathfrak{u}(V) \), the space of self-adjoint operators on \( V \).

**Proposition 5.1.** There exist continuous maps \( v_i(t) : \mathbb{R} \to V, i = 1, \ldots, n \), and continuous functions \( \eta_i(t) : \mathbb{R} \to \mathbb{R} \) such that:

1. The vectors \( v_1(t), \ldots, v_n(t) \) form a \( \mathbb{C} \)-linear basis of \( V \) for each \( t \in \mathbb{R} \), orthogonal with respect to the Hermitian inner product.
2. \( L(t)v_i(t) = \eta_i(t)v_i(t) \) for all \( i \) and \( t \).

*Proof.* Write \( C = \mathbb{A}^1_\mathbb{C} \) for the domain of the morphism \( L \), with coordinate \( t \). Taking the characteristic polynomial defines a polynomial map \( \text{char}(L) : C \to \text{Sym}^n(\mathbb{A}^1_\mathbb{C}) \cong \mathbb{A}^n_\mathbb{C} \).

**Remark 5.2.** The ramified covering \( c : \mathbb{A}^n_\mathbb{C} \to \text{Sym}^n(\mathbb{A}^1_\mathbb{C}) \), as well as its restriction to every intersection of reflection hyperplanes, is defined over \( \mathbb{R} \).

There exists a finite subset \( S \subset C \) such that \( L|_{C \smallsetminus S} \) has a constant number of distinct (generalized) eigenvalues and the set of their multiplicities is constant. We now pass to a finite covering \( \mathbb{C} = \mathbb{A}^1_\mathbb{C} \overset{\pi}{\to} \mathbb{A}^n_\mathbb{C} = C, \pi(u) = t \), obtained by pulling back \( c \) along \( \text{char}(L) \). The covering \( \pi \) is ramified at most
over $S$, and there are polynomials $\eta_i(u) : \tilde{C} \to C$, $i = 1, \ldots, n$ giving the generalized eigenvalues of $L(u) := L(\pi(u))$. Let $D = \prod_{p \in S} (u - p)$ and let $R = C[u][D^{-1}]$.

Each linear operator $L_i = (L(u) - \eta_i(u) \text{Id})^n$ has constant rank $r_i$ as a function of $u \in \tilde{C} \setminus \tilde{S}$, and thus $K_i := \ker (L_i : V \otimes R \to V \otimes R)$ is a projective $R$-submodule of $V \otimes R$ of rank $n - r_i$. Such a submodule is of the form $K_i = \frac{R}{R} \otimes_{C[u]} R$ for a submodule $K_i \subset V[u]$ uniquely determined by the properties that $K_i = \frac{R}{R} \otimes_{C[u]} R$ and that $V[u]/K_i$ is torsion-free. By the classification of modules over a PID, we may choose an isomorphism $\frac{R}{R} \cong C[u]^{n-r_i}$, thus yielding a basis of $K_i$; since every element $b(u)$ of this basis satisfies $(L(u) - \eta_i(u))^n b(u) = 0$ for $u \in \tilde{C} \setminus \tilde{S}$, $b(u)$ is a generalized $\eta_i(u)$-eigenvector over all of $\tilde{C}$.

Repeating the previous paragraph for all $\eta_i(u)$, we thus get a basis $w_1(u), \ldots, w_n(u) \in V[u]$ so that $(L(u) - \eta_i(u))^n w_i(u) = 0$ for $i = 1, \ldots, n$. Apply Gram-Schmidt to the basis $\{w_i(u)\}$ to obtain an orthogonal basis that depends polynomially on $u$ and $R$, we write $\{v_i(u, R)\}$ for this basis, and $\{v_i(u)\}$ for the basis restricted to $u \in \mathbb{R}$, which depends polynomially on $u \in \mathbb{R}$. Since $L(u)$ is self-adjoint for $u \in \mathbb{R}$, we have that $L(u)v_i(u) = \eta_i(u)v_i(u)$ for all $u \in \mathbb{R}$.

Finally, the ramified cover $\tilde{C} \to C$, restricted to the real curve $\text{char}(L(t), t \in \mathbb{R})$, has a continuous section; pulling back the $v_i(u)$ and $\eta_i(u)$ gives the claimed assertion.

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