A Wiener-Type Condition for Boundary Continuity of Quasi-Minima of Variational Integrals

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Abstract

A Wiener-type condition for the continuity at the boundary points of Q-minima, is established, in terms of the divergence of a suitable Wiener integral \( (1.8) \) and Theorem 1.1.

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1 Introduction

Let \( E \) be a bounded, open subset of \( \mathbb{R}^N \) and let \( f : E \times \mathbb{R}^{N+1} \to \mathbb{R} \) be a Carathéodory function satisfying

\[
C_0 |Du|^p \leq f(x, u, Du) \leq C_1 |Du|^p,
\]

for constants \( 0 < C_0 \leq C_1 \), and some fixed \( p > 1 \). A function \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \) is a Q-sub(super)minimum for the functional

\[
J(u) = \int_E f(x, u, Du)dx
\]

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The relative capacity $\delta$ for $u$ with respect to the ball $B_p$ is defined by
$$\delta_y(\rho) = \frac{c_p[E^c \cap \bar{B}_p(y)]}{\rho^{N-p}}, \quad (1 < p < N).$$

If $p = N$, and for $0 < \rho < 1$, the $N$-capacity of the compact set $E^c \cap \bar{B}_p(y)$, with respect to the ball $B_2(y)$, is defined by
$$c_N[E^c \cap \bar{B}_p(y)] = \inf_{\psi \in W^{1,N}_0(B_2(y)) \cap C_c(B_2(y))} \int_{B_2(y)} |D\psi|^N dx.$$

The relative capacity $\delta_y(\rho)$ can be formally defined by $c_N[E^c \cap \bar{B}_p(y)]$, for all $1 < p \leq N$: for $p = N$, $\delta_y(\rho) = c_N[E^c \cap \bar{B}_p(y)]$, as defined by (1.7).

For a positive parameter $\epsilon$ denote by $I_{p,\epsilon}(y, \rho)$ the Wiener integral of $\partial E$ at $y \in \partial E$, i.e.,
$$I_{p,\epsilon}(y, \rho) = \int_0^1 [\delta_y(t)]^{\frac{1}{\epsilon}} \frac{dt}{t}.$$

The main result of this note is:

**Theorem 1.1** Let $u$ be a $Q$-minimum for the functional $J(u)$, for $1 < p \leq N$. Assume that $u$ takes a continuous datum $u = g$ on $\partial E$ in the sense of (1.4). There exists $\epsilon \in (0, 1)$, and $\gamma > 1$, that can be determined apriori, quantitatively only in terms of $N$, $p$, and $Q$, such that for all $y \in \partial E$, and all $\rho \in (0, 1)

$$\text{ess osc } u \leq \gamma \max \left\{ \text{osc}_{E \cap \bar{B}_\rho(y)} g; \left( \text{osc}_{E \cap \bar{B}_\rho(y)} u \right) \exp \left( -I_{p,\epsilon}(y, \rho) \right) \right\}.$$  

Thus, when $1 < p \leq N$, a $Q$-minimum $u$, when given continuous boundary data $g$ on $\partial E$, is continuous up to $y \in \partial E$, if the Wiener integral $I_{p,\epsilon}(y, \rho)$ diverges as $\rho \to 0$. If $p > N$ the continuity of $u$, is insured by the Sobolev embedding theorem.
1.1 Novelty and Significance

The celebrated Wiener criterion states that a harmonic function in $E$ is continuous up to $y \in \partial E$ if and only if the Wiener integral $I_2(y, \rho)$ diverges as $\rho \to 0$ \cite{9}. Next, for a given $g \in W^{1,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ consider the boundary value problem

$$
\begin{align*}
u - g &\in W^{1,p}_a(E), \quad \text{for } p > 1, \\
\text{div } a(x, u, Du) &\equiv 0, \quad \text{weakly in } E, \quad (1.10)
\end{align*}
$$

where, the vector field $a$ is subject to the structure conditions

$$
\begin{align*}
|a(x, u, Du)| &\leq C_1 |Du|^{p-1} \\
|a(x, u, Du) \cdot Du| &\geq C_0 |Du|^p
\end{align*}
$$

for constants $0 < C_0 \leq C_1$, and some fixed $p > 1$. The prototype is

$$
\begin{align*}
u - g &\in W^{1,p}_a(E), \quad \text{for } p > 1, \\
\text{div } |Du|^{p-2}Du &\equiv 0 \text{ weakly in } E. \quad (1.12)
\end{align*}
$$

For solutions of (1.12) Theorem 1.1 is due to Maz’ja \cite{6}, with the optimal value of the parameter $\epsilon = (p - 1)$. The proof is based on the comparison principle and the Harnack inequality. For solutions of (1.10)–(1.11) the result is due to Gariepy and Ziemer \cite{3}, still for optimal value of the parameter $\epsilon = (p - 1)$. For these quasi-linear equations there is not, in general, a maximum principle. Their proof is based on the Moser’s logarithmic estimates \cite{7} leading to the Harnack inequality for some proper convex functions of the solutions, near the boundary point $y \in \partial E$. In their approach, the structure of the p.d.e.’ in (1.10)–(1.11) is crucial.

Each such quasi-linear equation is the Euler equation of a functional $J$, for a suitable integrand $f(x, u, Du) \cite{4}$. The notion of Q-minimum is considerably more general as it includes almost minimisers, or even minimisers of functionals $J(u)$ which do not admit a Euler equation due to the possible lack of Gateaux differentiability of $J$.

Nevertheless Q-minima share several crucial properties of solutions of quasi-linear equations of the type (1.10)–(1.11). For example they are locally bounded and locally Hölder continuous in $E$. Their interior continuity carries at those boundary points where $\partial E$ has positive geometric density \cite{4}. Moreover non-negative Q-minima satisfy the Harnack inequality \cite{2}. However Q-minima are not known to satisfy a maximum principle, nor Harnack inequalities near $\partial E$.

The significance of a Wiener condition for Q-minima, is that the structure of $\partial E$ near a boundary point $y \in \partial E$, for $u$ to be continuous up to $y$, hinges on minimizing a functional, rather than solving an elliptic p.d.e.

The only result, to date, in this direction, states that a Q-minimum $u$, with continuous boundary data $g \in C(\partial E)$, is continuous up to a boundary point $y \in \partial E$ if (1.10)

$$
\int_0^1 \exp \left( - \frac{1}{\delta_y(t)^{\frac{p-1}{p}}} \right) \frac{dt}{t} \to \infty \quad \text{as } \rho \to 0. \quad (1.13)
$$
Ziemer’s proof follows from a standard DeGiorgi iteration technique. The novelty of our Theorem 1.1 is in replacing the exponential decay (1.13) in the Wiener integral with a power-like decay. The technical novelty is in extending a weak Harnack inequality for quasi minima ([2]), to hold near the boundary, coupled with proper choices of test functions in (1.3) as indicated by Tolksdorf ([8]). The optimal value of the parameter \( \epsilon = (p - 1) \), remains elusive.

2 Main Tools in the Proof of Theorem 1.1

2.1 Q-Subminima and Test Functions

**Proposition 2.1** Let \( y \in \partial E \) and let \( u \) be a non-negative \( Q \)-subminimum for \( J \), in \( B_\rho(y) \cap \bar{E} \), such that \( u = 0 \) on \( B_\rho(y) \cap \partial E \). There is a positive constant \( \gamma_o \) that can be determined apriori only in terms of \( N, p, Q \), such that

\[
\int_{B_\rho(y) \cap \bar{E}} |Du|^p |\varphi|^p dx \leq \gamma_o \int_{B_\rho(y) \cap \bar{E}} u^p |D\varphi|^p dx, \tag{2.1}
\]

for all non-negative \( \varphi \in W^{1,p}_o(B_\rho(y)) \).

Note that \( \varphi \) is not required to vanish on \( B_\rho(y) \cap \partial E \). The proof results from a minor variant of an argument of Tolksdorf [8]. From the property (1.1) of \( f \) and the definition (1.2)–(1.3) of \( Q \)-subminimum,

\[
\int_{B_\rho(y) \cap \bar{E}} |Du|^p dx \leq Q \frac{C_1}{C_o} \int_{B_\rho(y) \cap \bar{E}} |D(u - w\varphi)|^p dx, \tag{2.2}
\]

for all non-negative \( \varphi \in W^{1,p}_o(B_\rho(y)) \). The new observation here is that since \( u \) vanishes on \( B_\rho(y) \cap \partial E \), the test function \( w\varphi \) is admissible in (1.3) even if \( \varphi \) does not vanish on \( B_\rho(y) \cap \partial E \), provided it does vanish on \( \partial B_\rho(y) \). The remaining arguments leading to (2.1) starting from (2.2) are identical to those in [8].

**Corollary 2.1** Let \( u \) satisfy the same assumptions as Proposition 2.1. Then for all constants \( h > 0 \)

\[
\int_{B_\rho(y) \cap \bar{E}} |D(u + h\varphi)|^p dx \leq \gamma_o \int_{B_\rho(y) \cap \bar{E}} (u + h\varphi)^p |D\varphi|^p dx, \tag{2.3}
\]

for all non-negative \( \varphi \in W^{1,p}_o(B_\rho(y)) \). The constant \( \gamma_o \) is the same as in (2.1) and is independent of \( h \).

2.2 Q-Superminima and the Weak Harnack Inequality

**Proposition 2.2** Let \( y \in \partial E \) and let \( v \in W^{1,p}(B_{2\rho}(y)) \) be non-negative and satisfying

\[
\int_{B_{\rho}(\cdot)} |D(v - k)|^p dx \leq \frac{2_1}{p} \int_{B_{2\rho}(\cdot)} (v - k)^p dx \tag{2.4}
\]
for all balls $B_{2r}(z) \subset B_{2r}(y)$ and all $k > 0$, for a constant $\gamma_1$ independent of $k$, $z$ and $r$. Then, there exist constants $C > 1$ and $\epsilon \in (0, 1)$, that can be determined apriori only in terms of $N$, $p$, and the constants $\gamma_0$ and $\gamma_1$ in (2.1) and (2.4), such that

\[
\left( \frac{1}{|B_{\rho}(y)|} \int_{B_{\rho}(y)} v^{+} dx \right)^{\frac{1}{p}} \leq C \text{ess inf} v. \quad (2.5)
\]

The weak Harnack inequality (2.5) is a sole consequence of the family of inequalities (2.4), and as such, disconnected from the notion of Q-superminimum ([2]). However, if $v$ is a Q-superminimum in $E$, for balls $B_{2\rho}(y) \subset E$, inequalities (2.4) are satisfied by $v$ ([4]).

3 Proof of Theorem 1.1

3.1 Estimating the Oscillation About a Point $y \in \partial E$ by the Weak Harnack Inequality

Having fixed $y \in \partial E$ assume without loss of generality that $y = 0$ and write $B_{\rho}(0) = B_{\rho}$, and continue to denote by $g$ the boundary datum of $u$, in the sense of (1.4). We may assume that at least one of the following two inequalities holds true:

\[
\begin{align*}
\text{ess sup}_{B_{2r} \cap E} u - \frac{1}{4} \text{ess osc}_{B_{2r} \cap E} u & > \text{ess sup}_{B_{2r} \cap \partial E} g; \\
\text{ess inf}_{B_{2r} \cap E} u - \frac{1}{4} \text{ess osc}_{B_{2r} \cap E} u & < \text{ess inf}_{B_{2r} \cap \partial E} g.
\end{align*}
\]

Indeed if both are violated one has

\[
\text{ess osc}_{B_{2r} \cap E} u \leq 2 \text{ess osc}_{B_{2r} \cap \partial E} g,
\]

and the assertion of the theorem follows. Assuming then that the first holds, the function

\[
\left( u - \left( \text{ess sup}_{B_{2r} \cap E} u - \frac{1}{4} \text{ess osc}_{B_{2r} \cap E} u \right) - (1 - k)\frac{1}{4} \text{ess osc}_{B_{2r} \cap E} u \right)^{+}
\]

is a non-negative Q-subminimum, for $J$, in $B_{2\rho} \cap \bar{E}$, for all $0 < k \leq 1$, vanishing on $B_{2\rho} \cap \partial E$. As such it satisfies (2.4) of Proposition 2.1 over $B_{2\rho}$, which we rewrite as

\[
\begin{align*}
\int_{B_{2r} \cap E} |D(w - (1 - k))^{+}|^p |\varphi|^p dx \\
\leq \gamma_0 \int_{B_{2r} \cap E} (w - (1 - k))^\frac{p}{2} |D\varphi|^p dx,
\end{align*}
\]

(3.1)
for all non-negative $\varphi \in W^{1,p}_0(B_{2\rho})$, where

$$w \overset{\text{def}}{=} \left(\frac{u - \left(\text{ess sup}_{B_{2\rho} \cap E} u - \frac{1}{4} \text{ess osc}_{B_{2\rho} \cap E} u\right)}{\frac{1}{4} \text{ess osc}_{B_{2\rho} \cap E} u}\right)_+,$$

for all $0 < k \leq 1$. From the definitions one verifies that $0 \leq w \leq 1$, and it vanishes on $B_{2\rho} \cap \partial E$. We continue to denote by $w$ and $(w - (1 - k))_+$ their extensions with zero on $B_{2\rho} \cap E^c$. By Corollary 2.1, inequalities (3.1) continue to hold for all $k \geq 0$. Set $v = 1 - w$ and rewrite (3.1) in the form

$$\int_{B_{2\rho}} |D(v - k)_-|^p |\varphi|^p dx \leq \gamma_o \int_{B_{2\rho}} (v - k)^p |D\varphi|^p dx,$$

(3.2)

for all non-negative $\varphi \in W^{1,p}_0(B_{2\rho})$, and for all $k \geq 0$. In what follows we denote by $\gamma$ a generic, positive constant that can be quantitatively determined apriori only in terms of $Q, N, p$.

For a ball $B_{2\rho}(z) \subset B_{2\rho}$, in (3.2) choose $\varphi$ as the standard, non-negative cutoff function in $B_{2\rho}(z)$ which equals 1 on $B(r)$ and such that $|D\varphi| \leq r^{-1}$. For such a choice $(v - k)_-$ satisfies the assumptions of Proposition 2.2. Hence there exists $\gamma > 1$ and $\epsilon \in (0, 1)$ that can be determined apriori only in terms of $N, p$, such that

$$\int_{B_{2\rho}} v^\epsilon dx \leq \gamma^\epsilon \left(\frac{\text{ess sup}_{B_{2\rho} \cap E} u - \text{ess sup}_{B_{2\rho} \cap E} u}{\frac{1}{4} \text{ess osc}_{B_{2\rho} \cap E} u}\right)^\epsilon$$

(3.3)

Remark 3.1 Whence the parameter $\epsilon$ has been identified, inequality (3.3) continues to hold for smaller $\epsilon$, with the same constant $\gamma$.

### 3.2 Estimating the Oscillation About a Point $y \in \partial E$ by the Capacity of $E^c \cap \bar{B}_\rho(y)$

Continue to assume $y = 0$ and write $B_\rho(0) = B_\rho$.

Proposition 3.1 There exists $p_o \in (1, p)$, that depends only on the $Q, N, p$, such that for all $p_o \leq q < p$, and for all non-negative $\zeta \in W^{1,p}_0(B_{2\rho})$, there holds

$$\int_{B_{2\rho}} v^{-q} |Dv|^p |\zeta|^p dx \leq \gamma \int_{B_{2\rho}} v^{p-q} |D\zeta|^p dx$$

(3.4)

for a constant $\gamma > 1$ that depends only on $N, p, Q, q, p_o$.

Proof: Using an idea of [5], set $\varphi = v^\sigma \zeta$ in (3.2) where $\sigma \in (0, 1)$ is a parameter to be chosen and $\zeta \in W^{1,p}_0(B_{2\rho})$ is non-negative. For such choices (3.2) yields

$$\int_{B_{2\rho}} |D(v - k)_-|^p v^\sigma |\zeta|^p dx \leq \gamma \int_{B_{2\rho}} (v - k)^p v^{(\sigma-1)p} |Dv|^p |\zeta|^p + v^\sigma |D\zeta|^p$$

(3.3)
Choose $\sigma > 0$ and $1 < q < p$ so that $(1 - \sigma)p < q$, multiply both sides of this inequality by $k^{-\sigma p - q - 1}$ and integrate in $dk$ over $(0, \infty)$. Interchanging the order of integration with the aid of Fubini’s theorem, the left-hand side equals

$$\int_0^\infty \int_{B_{2^p}} |D(v - k)|_p ^p v^p \zeta^p k^{-\sigma p - q - 1} dxdk = \frac{1}{\sigma p + q} \int_{B_{2^p}} |Dv|^p v^{-q} \zeta^p dx.$$

The right-hand side is transformed and estimated by

$$\int_0^\infty \int_{B_{2^p}} (v - k)^p \left[ \sigma^p v^{(\sigma - 1)p} |Dv|^p \zeta^p + v^{\sigma p} |D\zeta|^p \right] k^{-\sigma p - q - 1} dxdk = \sigma^p \int_{B_{2^p}} v^{(\sigma - 1)p} |Dv|^p \zeta^p \left( \int_0^\infty k^{(1 - \sigma)p - q - 1} dk \right) dx + \int_{B_{2^p}} v^{\sigma p} |D\zeta|^p \left( \int_0^\infty k^{(1 - \sigma)p - q - 1} dk \right) dx = \frac{1}{q - (1 - \sigma)p} \int_{B_{2^p}} v^{(1 - \sigma)p - q} \left[ \sigma^p v^{-(1 - \sigma)p} |Dv|^p \zeta^p + v^{\sigma p} |D\zeta|^p \right] dx.$$

Combining these estimates yields

$$\int_{B_{2^p}} v^{-q} |Dv|^p \zeta^p dx \leq \gamma \frac{\sigma p + q}{q - (1 - \sigma)p} \int_{B_{2^p}} v^{-q} |Dv|^p \zeta^p + \gamma \frac{\sigma p + q}{q - (1 - \sigma)p} \int_{B_{2^p}} v^{p - q} |D\zeta|^p dx.$$

To conclude the proof choose $\sigma \in (0, 1)$ such that

$$\gamma \frac{\sigma p + q}{q - (1 - \sigma)p} \frac{\sigma p}{\sigma p} = \frac{1}{2}, \quad \text{and} \quad (1 - \sigma)p < q < p.$$

One may first choose $p_o = (1 - \sigma^2)p \leq q < p$ and then $\sigma$ so small that the first of these inequalities is in force.

We now conclude the proof of the Theorem, still following \[5\]. Fix $p_o \leq q < p$ where $p_o$ is the parameter claimed in Proposition \[3\] and rewrite it as $q = p - \epsilon$. By virtue of Remark \[4\] this value of $\epsilon$ can be taken equal to the analogous in \[3\]. For such a choice, \[5\] gives

$$\int_{B_{2^p}} |D(v^\epsilon \varphi)|^p dx \leq \gamma(\epsilon) \int_{B_{2^p}} v^\epsilon |D\varphi|^p dx.$$

Next choose $\varphi \in W^{1,p}_o(B_{2p})$ to be the standard, non-negative cutoff function in $B_{2p}$ which equals 1 on $B_\rho$ and such that $|D\varphi| \leq \rho^{-1}$. For such a choice and $\rho$ sufficiently small $v^\varphi = 1$ on $B_\rho \cap E^c$ and therefore,

$$c_p [E^c \cap B_\rho] \leq \frac{\gamma(\epsilon)}{\rho^p} \int_{B_{2p}} v^\epsilon dx.$$
Dividing by $\rho^{N-p}$ and combining the resulting inequality with (5.3) gives

$$
\delta_\rho^+(\rho) \leq \frac{\text{ess sup } u - \text{ess sup } u}{\frac{1}{2} \text{ess osc } u}\frac{\text{ess sup } u - \text{ess sup } u}{B_{2\rho} \cap E}.
$$

This in turn implies

$$
\text{ess osc } u \leq \left(1 - \frac{1}{4\gamma} \delta_\rho(\rho)\right) \text{ess osc } u
$$

Iteration of this inequality over a sequence of balls of dyadic radii $\rho_{-n} = 2^{-n}\rho$ yields the Theorem.

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