Existence of a non-trivial solution for fourth-order elastic beam equations involving Lipschitz non-linearity

Mohsen Khaleghi Moghadam

Abstract: In this paper, we consider the existence of a non-trivial solution for a class of fourth-order elastic beam equations involving Lipschitz non-linearity with Navier boundary value condition. The technical approach is essentially based on a critical point theorem. As an application, an example is presented.

Subjects: Advanced Mathematics; Analysis - Mathematics; Applied Mathematics; Differential Calculus; Differential Equations; Mathematical Physics; Mathematics & Statistics; Pure Mathematics; Science

Keywords: fourth-order equations; critical point; non-trivial solution

2000 mathematics subject classifications: 35J35; 34B15; 58E05

1. Introduction

The aim of this paper is to study the existence of non-trivial solution for the following boundary value problem:

\begin{equation}
\begin{aligned}
\quad u^{(iv)} + Au'' + Bu + g(u) &= f(t, u), \\
\quad u(0) &= u(1) = 0, \\
\quad u''(0) &= u''(1) = 0,
\end{aligned}
\end{equation}

where \( A, B \) are real constants, \( f: [0, 1] \times \mathbb{R} \to \mathbb{R} \) is an \( L^1 \)-Carathéodory function, \( g: \mathbb{R} \to \mathbb{R} \) is a Lipschitz continuous function with Lipschitzian constant \( L \geq 0 \) and \( g(0) = 0, \lambda > 0 \) is a parameter.

In recent years, the fourth-order boundary value problems have been extensively considered by many authors, for instance, see Bai and Wang (2002), Cabada, Cid, and Sanchez (2007), Grossinho, Sanchez, and Tersian (2005), Liu and Li (2007a), Peletier, Troy, and Van der Vorst (1995) and references therein. These kind of problems arising in real-world phenomena play a fundamental role in different fields of research, such as mechanical engineering, control systems, economics, computer

ABOUT THE AUTHOR

Mohsen Khaleghi Moghadam is an assistant professor of Mathematics, Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, Sari, Iran.

The author’s key research activities are: (1) Nonlinear Analysis: Variational Principles, Critical Point Theory, Variational Inequalities; (2) Partial Differential Equations: Semilinear and Quasilinear, Elliptic Boundary Value Problems.

This paper relates to wider projects that the author will work.

PUBLIC INTEREST STATEMENT

In the paper, the author, using variational methods, established the existence of a non-trivial solution for a class of fourth-order elastic beam equations involving Lipschitz non-linearity with Navier boundary value condition. Using a critical points theorem, the author has ensured the exact collections of the parameters in which the problem possesses at least a non-trivial solution, and has presented two examples to illustrate the results.
science, physics, biology and many others. For example, fourth-order BVPs describe the deformations of an elastic beam in an equilibrium state whose both ends are simply supported. Also, a non-linear fourth-order equation describes traveling waves in suspension bridges. So, for this reason, there are wide papers about these problems which authors have investigated by different methods such as fixed point theorems (Liu & Li, 2007b), lower and upper solutions method (Cabada et al., 2007), critical point theory (Bonanno, 2012), Morse theory (Han & Xu, 2007) and mountain-pass theorem (Gyulov & Morosanu, 2010) that for more references, we refer the reader to Afrouzi, Heidarkhani and O’Regan (2011), Bai (2010), Bonanno and Di Bella (2008), Bonanno and Di Bell (2010), Bonanno, Di Bella and O’Regan (2011), Chai (2007), Li (2007) and references therein. In Khalkhali, Heidarkhani and Razani (2012), (1.1) have studied the existence of infinitely many solutions. The authors in Heidarkhani, Ferrara, Salari and Caristi (2016) established the existence and multiplicity results by variational methods and critical point theory for the following fourth-order Navier boundary value problem

\[
\begin{align*}
&\Delta^2_{p,u} u = \lambda f(t,u), \quad \text{in } \Omega, \\
&u = \Delta u = 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

in a non-empty bounded open set \( \Omega \subset \mathbb{R}^N (N > 1) \) with a sufficient smooth boundary \( \partial \Omega \) where, \( \Delta^2_{p,u} u = \Delta (|\Delta u|^{p-2} \Delta u) \) is the \( p \)-biharmonic operator of fourth order with \( p(\cdot) \in C^0(\Omega) \) and \( \lambda \in (0, \infty) \) and \( f: \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is an \( L^2 \)-Carathéodory function.

There are so many equations in engineering, physics and mathematics that are studied through Navier boundary conditions. For the use of Navier boundary conditions in the mathematical literature, see Busuioc and Ratiu (2003) and the related references.

Very recently, some researchers have studied the existence and multiplicity of solutions for impulsive fourth-order elastic beam equation; we refer the reader to Heidarkhani, Afrouzi, Ferrara and Moradi (2016), Heidarkhani, Ferrara and Khademloo (2016) and references therein.

In the present paper, based on a local minimum theorem (Theorem 2.1) due to Bonanno (2012), we ensure an exact interval of parameters, in which the problem (1.1) admits at least a non-trivial solution. We also refer the interested reader to the papers (Bonanno, Di Bella, & O’Regan, 2011; Heidarkhani, 2012a, 2012b, 2014; Heidarkhani et al., 2016; Khaleghi Moghadam & Heidarkhani, 2014) in which Theorem 2.1 has been successfully employed to the existence of at least one non-trivial solution for some boundary value problems.

Our main result, Theorem 3.2, and its consequence, Theorem 3.3, ensure the existence of a non-trivial solution to problem (1.1). Moreover, when \( f \) has separable variables, Theorem 3.6 points out some relevant consequences of the main result.

As an example of our results, a special case of Theorem 3.2 is presented here (the proof of Theorem 1.1 comes in Remark 3.4).

**Theorem 1.1** Let \( g: \mathbb{R} \rightarrow \mathbb{R} \) is a Lipschitz continuous function with Lipschitzian constant \( L = \frac{c}{T} \) and \( g(0) = 0 \). Assume \( f: (0,1) \times \mathbb{R} \rightarrow \mathbb{R} \) be a non-negative and continuous function, such that

\[
\int_0^1 \int_0^{\frac{1}{2}} f(t,s) ds dt < \frac{4}{3} \int_0^{\frac{1}{2}} f(t,s) ds dt.
\]

Then, for every

\[
\lambda \in \Lambda = \left\{ \frac{2048 \pi^2}{5 \int_0^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} f(t,s) ds dt}, \frac{2048 \pi^2}{15 \left( \int_0^{\frac{1}{2}} f(t,s) ds dt - \frac{1}{15} \left( \int_0^{\frac{1}{2}} f(t,s) ds dt \right)^2 \right) \right\},
\]

\[
\int_0^1 \int_0^{\frac{1}{2}} f(t,s) ds dt.
\]
the problem
\[
\begin{aligned}
\begin{cases}
\dot{u}(t) + g(u) &= f(t, u), \\
    u(0) &= u(1) = 0, \\
    u''(0) &= u''(1) = 0,
\end{cases}
\end{aligned}
\tag{1.4}
\]
has at least one non-trivial generalized solution \( u_0 \) such that
\[
0 < \frac{1}{2} \| u_0'' \|_{L^2([0,1])}^2 - \int_0^1 r_0^0 g(s) ds < \frac{2048 \pi^2}{5}.
\]

The rest of this paper is arranged as follows. In Section 2, we recall some basic definitions and the main tool (Theorem 2.1) and in Section 3, we provide our main result that contains several theorems and finally, we illustrate the results by giving several examples as applications of our results.

2. Preliminaries
First, we here recall for the reader's convenience (Bonanno, 2012, Theorem 5.1) (see also Bonanno, 2012, Proposition 2.1) which is our main tool. For a given non-empty set \( X \), and two functionals \( \Phi, \Psi : X \to \mathbb{R} \), we define the following functions
\[
\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}(r_1, r_2)} \sup_{u \in \Phi^{-1}(r_1, r_2)} \frac{\Psi(u) - \Psi(v)}{r_2 - \Phi(v)},
\tag{2.1}
\]
\[
\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(r_1, r_2)} \Psi(v) - \sup_{u \in \Phi^{-1}(r_1, r_2)} \Psi(u),
\tag{2.2}
\]
for all \( r_1, r_2 \in \mathbb{R} \), with \( r_1 < r_2 \).

**Theorem 2.1** (Bonanno, 2012, Theorem 5.1) Let \( X \) be a reflexive real Banach space, \( \Phi : X \to \mathbb{R} \) a sequentially weakly lower semi-continuous coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on \( X^* \) and \( \Psi : X \to \mathbb{R} \) a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put \( I_{\lambda} = \Phi - \lambda \Psi \) and assume that there are \( r_1, r_2 \in \mathbb{R} \), \( r_1 < r_2 \) such that
\[
\beta(r_1, r_2) < \rho(r_1, r_2).
\]
Then, for each \( \lambda \in \Lambda = \left\{ \frac{1}{\rho(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)} \right\} \) there is \( u_{\lambda, \beta} \in \Phi^{-1}(r_1, r_2) \) such that \( I_{\lambda}(u_{\lambda, \beta}) \leq I_{\lambda}(u) \) for all \( u \in \Phi^{-1}(r_1, r_2) \) and \( I'_{\lambda}(u_{\lambda, \beta}) = 0 \).

Let us introduce some notations that will be used later. Assume that \( A \) and \( B \) be two real constants such that
\[
\max \left\{ \frac{A}{r^2}, \frac{-B}{r^2}, \frac{A}{r^2} - \frac{B}{r^2} \right\} < 1,
\tag{2.3}
\]
(for instance, if \( A \leq 0 \) and \( B \geq 0 \), then (2.3) holds). Also, set
\[
\sigma := \max \left\{ \frac{A}{r^2}, \frac{-B}{r^2}, \frac{A}{r^2} - \frac{B}{r^2}, 0 \right\},
\]
\[
\delta := \sqrt{1 - \sigma},
\]
and
\[
k := 2 \delta^2 \pi^2 \left( \frac{2048}{27} - \frac{32}{9} A + \frac{13}{40} B \right)^{-1}.
\tag{2.4}
\]
Clearly, 0 < k < \frac{1}{2}. Assume that the Lipshitzian constant L be such that L < \delta^2 \pi^4. Let \( X = H^2_0([0,1]) \cap H^4([0,1]) \) be the Sobolev space endowed with the usual norm.

It is well-known poincaré-type inequalities (see, for instance, Peletier et al. (1995, Lemma 2.3))

\[
\|u'\|_{L^2([0,1])} \leq \frac{1}{\pi} \|u''\|_{L^2([0,1])},
\]

and

\[
\|u\|_{L^2([0,1])} \leq \frac{1}{\pi^2} \|u''\|_{L^2([0,1])},
\]

for all \( u \in X \). By (2.3)–(2.6), one can show that the following norm

\[
\|u\|_X = \left( \int_0^1 |u''(x)|^2 - A|u'(x)|^2 + B|u(x)|^2 \, dx \right)^{\frac{1}{2}},
\]

is equivalent to the usual one and in particular one has (see, for instance Proposition 2.1 in Bonanno and Di Bella (2008))

\[
\|u\|_{L^2([0,1])} \leq \frac{1}{\pi} \|u''\|_{L^2([0,1])} \leq \frac{1}{\pi^2} \|u''\|_{L^2([0,1])} \leq \frac{1}{\delta^2 \pi^4} \|u\|_X,
\]

and

\[
\|u\|_\infty \leq \frac{1}{2 \pi \delta} \|u\|_X.
\]

Recall that function \( f: [0,1] \times \mathbb{R} \to \mathbb{R} \) is said to be an \( L^1 \)-Carathédory function if the function \( t \to f(t,x) \) is measurable for every \( x \in \mathbb{R} \) and the function \( x \to f(t,x) \) is continuous for almost every \( t \in [0,1] \) and for every \( \mu > 0 \), there exists a function \( f_\mu \in L^1([0,1]) \) such that \( \sup_{|x| \leq \mu} |f(t,x)| \leq f_\mu(t) \) for almost every \( t \in [0,1] \).

Put \( F(t, \xi) = \int_0^1 f(t,s) \, ds \) and \( G(\xi) = \int_0^1 g(s) \, ds \) for each \( (t, \xi) \in [0,1] \times \mathbb{R} \). By a weak solution of (1.1) we mean any \( u \in X \) such that

\[
\int_0^1 [u'' \nu'' - Au' \nu' + Buv] \, dt + \int_0^1 [g(u) - \lambda f(t,u)] \, dt = 0,
\]

for every \( v \in X \). Moreover, a function \( u: [0,1] \to \mathbb{R} \) is said to be a generalized solution to problem (1.1), if \( u \in C^1([0,1], \mathbb{R}) \) and \( u'' \in AC([0,1], \mathbb{R}) \), \( u(0) = u(1) = u''(0) = u''(1) = 0 \) and \( u'' + Au' + Bu + g(u) = \lambda f(t,u) \) for almost every \( t \in [0,1] \). If \( f \) and \( g \) are continuous functions in \([0,1] \times \mathbb{R}, \mathbb{R}\), then each generalized solution \( u \) is a classical solution. The assumptions on \( f \) and \( g \) imply that a weak solution to problem (1.1) is a generalized one (see Bonanno & Di Bella, 2008, Proposition 2.2).

**Remark 2.2** If \( f(t,x)x < 0 \) and \( g(x)x > 0 \) for every \( t \in [0,1] \) and \( x \neq 0 \), then by (2.7) and (2.10), one can see that the problem (1.1) has only the trivial solution.

**Remark 2.3** If \( u(t) \) be a solution of problem (1.1) that the values of the parameters \( A \) and \( B \) satisfy \( A^2 - 4B \geq 0 \) and the non-linear functions \( g(x) \) and \( f(t, x) \) satisfy \( f(t, u(x)) \geq g(u(t)) \) for very \( t \in [0,1] \), then using Proposition 2.3 of Bonanno and Di Bella (2008), \( u(t) \geq 0 \) for every \( t \in [0,1] \).
3. Main results

We firstly introduce the functionals \( \Phi, \Psi : X \to \mathbb{R} \) as follows

\[
\Phi(u) = \frac{1}{2} \| u \|^2_X - \int_0^1 G(u(t)) \, dt,
\]

(3.1)

and

\[
\Psi(u) = \int_0^1 F(t, u(t)) \, dt,
\]

(3.2)

for each \( u \in X \). We need the following lemma in the proof of our main results.

**Lemma 3.1** The functional \( \Phi : X \to \mathbb{R} \) be a sequentially weakly lower continuous and coercive and continuously Gâteaux differentiable function on \( X \) with

\[
\Phi'(u)[v] = \int_0^1 u''v'' -Au''v' + Buv \, dt, \quad \forall v \in X.
\]

(3.3)

The functional \( \Psi : X \to \mathbb{R} \) be a continuously Gâteaux differentiable function on \( X \) with

\[
\Psi'(u)[v] = \int_0^1 f(t, u) v \, dt, \quad \forall v \in X,
\]

(3.4)

such that \( \Phi' : X \to X^* \) admits a continuous inverse \( X^* \) and \( \Psi' : X \to X^* \) is a compact operator.

**Proof** Put \( J(u) = \int_0^1 G(u) \, dt \) for \( u \in X \) and let \( u_n \rightharpoonup u \) since \( J \) is sequentially weakly continuous functional on \( X \) and sequentially weakly semi-continuity of \( \| \cdot \|_X \), one has

\[
\liminf_{n \to \infty} \Phi(u_n) = \liminf_{n \to \infty} \frac{\| u_n \|^2}{2} - \lim_{n \to \infty} J(u_n) \geq \frac{1}{2} \| u \|^2_X - J(u) = \Phi(u);
\]

hence, \( \Phi \) is a sequentially weakly lower continuous. To prove coercivity of \( \Phi \), one has

\[
\Phi(u) \geq \frac{1}{2} \| u \|^2 \left( 1 - \frac{4}{\pi^2 \delta^2} \right) \to \infty, \quad \text{as} \quad \| u \|_X \to \infty.
\]

(3.5)

Indeed, by Lipschitz continuity \( g \), with the Lipschitsian constant \( L < \delta^2 x^4 \), (2.6) and (2.8), one can conclude that

\[
\Phi(u) \geq \frac{1}{2} \| u \|^2 - \int_0^1 |G(u)| \, dt \geq \frac{1}{2} \| u \|^2 - \int_0^1 \left( \int_0^u |g(s)| \, ds \right) \, dt
\]

\[
\geq \frac{1}{2} \| u \|^2 - \frac{L}{2} \int_0^1 |u|^2 \, dt \geq \frac{1}{2} \| u \|^2 - \frac{L}{2\pi} \| u'' \|^2
\]

\[
\geq \frac{1}{2} \| u \|^2 - \frac{L}{2\pi} \| u'' \|^2 \| u \|^2 = \frac{1}{2} \| u \|^2 \left( 1 - \frac{L}{\pi \delta^2} \right) \to \infty, \quad \text{as} \quad \| u \|_X \to \infty.
\]

We let \( u, v \in X \) and \( s \neq 0 \); then, by the Mean Value Theorem for integrals

\[
\left| \frac{J(u + sv) - J(u)}{s} - \int_0^1 g(u) \, v \, dt \right| \leq \int_0^1 \left| \frac{G(u + sv) - G(u)}{s} - g(u)v \right| \, dt
\]

\[
= \int_0^1 |g(u + sv) - g(u)| \, |v| \, dt
\]

\[
\leq L |s| \| v \|^2 \to 0 \quad \text{as} \quad s \to 0,
\]

in which \( 0 < \eta < 1 \). Hence, \( J'(u)[v] = \int_0^1 g(u) \, v \, dt \) for every \( v \in X \). Also, by a routine argument on the first term of \( \Phi(u) \), and similar argument on the \( \Psi(u) \) and using Lebesgue Convergent Theorem, one can follow (3.3) and (3.4).
Now, we show that \( \Phi': X \to X' \) admits a continuous inverse on \( X' \). Indeed, we need to show that \( \Phi' \) is a Lipschitzian and strongly monotone operator, i.e. for every \( u, v \in X \), there are two positive constants \( K_1 \) and \( L_1 \) such that

\[
|\Phi'(u) - \Phi'(v)|_{X'} \leq L_1 \|u - v\|_{X'},
\]

and

\[
\langle \Phi'(u) - \Phi'(v), u - v \rangle \geq K_1 \|u - v\|_{X'}^2.
\]

(3.6)

(3.7)

So by Zeidler (1985, Theorem 26.A(d)) \( \Phi' \) admits a Lipschitzian continuous inverse.

By the Hölder’s inequality and the inequalities in (2.8), we have

\[
|\Phi'(u) - \Phi'(v)|_{X'} = \sup_{\|w\|_{L^1} \leq 1} |\Phi'(u) - \Phi'(v), w|
\]

\[
\leq \sup_{\|w\|_{L^1} \leq 1} \left\{ \int_0^1 \|u'' - v''\|_{L^1} \|w\| dt + A \int_0^1 \|u' - v'\| \|w\| dt + B \int_0^1 \|u - v\| \|w\| dt \right\}
\]

\[
+ \sup_{\|w\|_{L^1} \leq 1} \left\{ \int_0^1 |g(u) - g(v)| \|w\| dt \right\}
\]

\[
\leq \sup_{\|w\|_{L^1} \leq 1} \left\{ \int_0^1 \|u'' - v''\|_{L^1} \|w\|_{L^1} \right\}
\]

\[
+ \|A\| \|u' - v'\|_{L^1} \|w\|_{L^1} + \|\mu - \nu\|_{L^1} \|w\|_{L^1}
\]

\[
+ A \sup_{\|w\|_{L^1} \leq 1} \left\{ \int_0^1 \|u - v\| \|w\|_{L^1} \right\}
\]

\[
+ \frac{1}{\delta^2 \pi^2} \sup_{\|w\|_{L^1} \leq 1} \left\{ \int_0^1 \|u - v\| \|w\|_{L^1} \right\}
\]

\[
\leq \frac{1}{\delta^2 \pi^2} \|u - v\| + \frac{\|A\| \|u - v\| \|w\|_{L^1} + \|\mu - \nu\| \|w\|_{L^1}}{\delta^2 \pi^2} + \frac{1}{\delta^2 \pi^2} \|u - v\|_{L^1}
\]

\[
\leq \|u - v\| \left( \frac{1}{\delta^2 \pi^2} + \frac{\|A\| \|u - v\| + \|\mu - \nu\| \|w\|_{L^1}}{\delta^2 \pi^2} + \frac{1}{\delta^2 \pi^2} \right) =: L_2 \|u - v\|_{X'}
\]

in which \( L_2 > 0 \). Also

\[
\langle \Phi'(u) - \Phi'(v), u - v \rangle \geq \|u - v\|_{X'}^2 - \|u - v\|_{L^1}^2 \]

\[
\geq \left( 1 - \frac{L_2}{\delta^2 \pi^2} \right) \|u - v\|_{X'}^2 = K_2 \|u - v\|_{X'}^2
\]

where \( K_2 > 0 \) (Because of \( L < \delta^2 \pi^2 \)). It is clear that \( \Psi' \) is a compact operator. \( \square \)

Fourth-order differential equations such as (1.1) arise in the study of deflections of elastic beams on non-linear elastic foundations in engineering and physical sciences. Therefore, for its importance and contribution to its area, we consider the following theorem as a main result of this paper.

**Theorem 3.2**  Let \( k, \delta \) and \( L \) be the real constants as defined above. Assume that there exist three non-negative constants \( c_1, c_2 \) and \( d \) such that \( c_1 = \frac{d}{\sqrt{k}} \), \( c_2 = \frac{d}{\sqrt{k}} \sqrt{\frac{2d^2 \pi^4}{\delta^2 \pi^2}} \) and

(i) \( F(t, \xi) \geq 0 \)  \( \forall (t, \xi) \in \left( [0, \frac{1}{8}] \cup \left[ \frac{5}{8}, 1 \right] \times [0, d] \right) \)


(ii) \( \int_0^1 \max_{\|x\| \leq k} F(t, x) dt + \int_0^1 \max_{\|x\| \leq k} F(t, x) dt < 2 \int_0^1 F(t, d) dt. \)

Then, for any

\[
\lambda \in \left[ \frac{2d^2(\delta^2 \pi^4 - L)}{\pi^2 k}, \frac{2d^2(\delta^2 \pi^4 - L)}{\pi^2 k} \right],
\]

the problem (1.1) has at least one non-trivial generalized solution \( u_0 \in X \) such that

\[
\frac{2d^2(\delta^2 \pi^4 - L)}{\pi^2 k} < \frac{1}{2} \|u_0\|_X^2 - \int_0^1 G(u_0) dt < \frac{4d^2(\delta^2 \pi^4 + L)}{\pi^2 k}.
\]

**Proof** Our goal is to use Theorem 2.1 to our problem. To this end, take \( \Phi \) and \( \Psi \) as given in (3.1) and (3.2), respectively. From Lemma 3.1, we observe that the regularity assumptions on \( \Phi \) and \( \Psi \) are verified. Put

\[
\hat{\nu}(t) = \begin{cases} 
\frac{-\delta t}{\pi} \left(t^2 - \frac{1}{4} t\right), & t \in [0, \frac{3}{4}], \\
\delta, & t \in [\frac{3}{4}, \frac{7}{8}], \\
\frac{-\delta t}{\pi} \left(t^2 - \frac{1}{4} t + \frac{1}{4}\right), & t \in [\frac{7}{8}, 1]. 
\end{cases}
\]

Clearly, \( \hat{\nu} \in X \); moreover, it is easy to verify that

\[
\|\hat{\nu}\|_X^2 = 2 \left( \frac{20\delta^4}{27} - \frac{32}{9} A + \frac{13}{40} B \right) = \frac{d^2\delta^2 + \delta^2}{2} \text{ and}
\]

\[
\Phi(\hat{\nu}) \leq \frac{1}{2} \|\hat{\nu}\|_X^2 + \left( \int_0^1 G(\hat{\nu}) dt \right) \leq \frac{2d^2\delta^2 + \delta^2}{k} - \left( \int_0^1 \nu \|g(s)\| ds \right) dt \leq \frac{2d^2\delta^2 + \delta^2}{k} - \frac{1}{2} \|\hat{\nu}\|_X^2,
\]

\[
\Phi(\hat{\nu}) \leq \frac{2d^2(\delta^2 \pi^4 + L)}{\pi^2 k}.
\]

Also by similar argument, one has

\[
\Phi(\hat{\nu}) \geq \frac{1}{2} \|\hat{\nu}\|_X^2 - \left( \int_0^1 G(\hat{\nu}) dt \right) \geq \frac{2d^2\delta^2 + \delta^2}{k} - \left( \int_0^1 \nu \|g(s)\| ds \right) dt \geq \frac{2d^2(\delta^2 \pi^4 + L)}{\pi^2 k}.
\]

Put \( r_1 = \frac{2d^2(\delta^2 \pi^4 - L)}{\pi^2 k} \) and \( r_2 = \frac{2d^2(\delta^2 \pi^4 + L)}{\pi^2 k} \); hence, \( r_1 < \Phi(\hat{\nu}) < \frac{r_2}{2} \), thus,

\[
\frac{1}{2} r_2 - \Phi(\hat{\nu}) < \frac{1}{2d^2(\delta^2 \pi^4 - L)} < \frac{1}{\Phi(\hat{\nu}) - r_1}.
\]

Taking (2.9) and (3.5) into account, when \( \Phi(u) < r, i = 1, 2 \), one has \( \max_{\|x\| \leq k} |u(t)| \leq c_i, i = 1, 2 \), hence,

\[
\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u) = \sup_{u \in \Phi^{-1}(-\infty, r)} \int_0^1 F(t, u(t)) dt \leq \int_0^1 \max_{\|x\| \leq k} F(t, u(t)) dt, \quad i = 1, 2.
\]
On the other hand, from (i), one has

\[
\Psi(\tilde{v}) = \int_0^1 F(t, \tilde{v}(t)) \, dt \geq \int_0^1 \frac{1}{\beta} F(t, \tilde{v}(t)) \, dt = \int_0^1 \frac{1}{\beta} F(t, v) \, dt.
\]

Thus,

\[
0 \leq \rho(r_1, r_2) \leq \frac{\sup_{u \in \Phi} \Psi(u) - \Psi(\tilde{v})}{r_2 - \Phi(\tilde{v})} \left[ \int_0^1 \max_{\mathcal{H}(\xi)} F(t, \xi) \, dt - \int_0^1 \frac{1}{\beta} F(t, v) \, dt \right] \leq \frac{\pi^2 k}{2d^4(\delta^2 - \lambda)} \left\{ \int_0^1 \max_{\mathcal{H}(\xi)} F(t, \xi) \, dt - \int_0^1 \frac{1}{\beta} F(t, v) \, dt \right\}.
\]

Also by arguing before, one has

\[
\rho(r_1, r_2) \geq \frac{\Psi(\tilde{v}) - \sup_{u \in \Phi \setminus \{v\}} \Psi(u)}{r_2 - \Phi(\tilde{v})} \left[ \int_0^1 \frac{1}{\beta} F(t, v) \, dt - \int_0^1 \max_{\mathcal{H}(\xi)} F(t, \xi) \, dt \right] \geq \frac{\pi^2 k}{2d^4(\delta^2 - \lambda)} \left\{ \int_0^1 \frac{1}{\beta} F(t, v) \, dt - \int_0^1 \max_{\mathcal{H}(\xi)} F(t, \xi) \, dt \right\}.
\]

Taking (ii) into account, we get \( \beta(r_1, r_2) < \rho(r_1, r_2) \).

Hence, Theorem 2.1 follows that for each

\[
\lambda \in \left\{ \frac{2\lambda^2\delta^4\pi^4}{\pi^4} \right\},
\]

the problem (1.1) has at least one non-trivial generalized solution \( u_0 \in X \) such that

\[
r_1 < \frac{1}{\beta} \|u_0\|_X - \int_0^1 G(u_0) \, dt < r_2.
\]

As a simple consequence of Theorem 2.1, we point out the following corollary.

**Corollary 3.3** Let \( k, \delta \) and \( L \) be real constants as defined above. Assume that there exist two non-negative constants \( d \) and \( c \) with \( c = \frac{d}{\lambda^2 \sqrt{\frac{2\pi^2}{\lambda^2} - 1}} \) such that

\[
(i) \quad F(t, \xi) \geq 0 \quad \forall (t, \xi) \in \left( \left[ 0, \frac{\pi}{2} \right] \cup \left[ \frac{\pi}{2}, 1 \right] \times [0, d] \right)
\]

\[
(ii) \quad \int_0^1 \max_{\mathcal{H}(t)} F(t, \xi) \, dt < \left( \frac{2\pi^2}{4 \lambda^2 \pi^4} \right) \int_0^1 \frac{1}{\beta} F(t, v) \, dt.
\]

Then, for every

\[
\lambda \in \Lambda : \quad \left\{ \frac{4\lambda^2(\delta^2 \pi^4 + L)}{\pi^4 k}, \frac{2\lambda^2(\delta^2 \pi^4 - L)}{\pi^4 k} \right\}
\]

the problem (1.1) has at least one non-trivial generalized solution \( u_0 \) such that

\[
0 < \frac{1}{\beta} \|u_0\|_X - \int_0^1 G(u_0) \, dt < \frac{4\lambda^2(\delta^2 \pi^4 + L)}{k \lambda^2}.
\]
In Corollary 3.3, taking Theorem 3.2 into account, it is enough to take \( c_1 = 0 \) and \( c_2 = c \).

**Remark 3.4** Theorem 1.1 follows immediately from Theorem 3.3 taking into account that \( A = B = 0 \) and \( d = \frac{3\pi}{2} \).

**Example 3.5** Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a Lipschitz continuous function with Lipschitzian constant \( L = \frac{\pi}{2} \) and \( g(0) = 0 \), and \( f(t, s) = f_0(t)f_1(s) \) where,

\[
f_1(s) = \begin{cases} 
2^{1-n} \min(s - n, n + 1 - s), & \text{if } s \in \bigcup_{n=0}^{\infty} [n, n+1), \\
0, & \text{if } s \in ] - \infty, 0[,
\end{cases}
\]

and

\[
f_0(t) = \begin{cases} 
10^{-3}, & \text{if } 0 \leq t < \frac{3}{8}, \\
1, & \text{if } \frac{3}{8} \leq t < \frac{5}{8}, \\
10^{-3}, & \text{if } \frac{5}{8} \leq t \leq 1.
\end{cases}
\]

and, by simple calculations, we obtain

\[
\int_0^1 \int_0^4 f(t, s)dsdt = \int_0^1 f_0(t)dt \int_0^4 f_1(s)ds = \int_0^1 f_0(t)dt \sum_{n=0}^{15} \int_n^{n+1} f_1(s)ds
\]

\[
= \int_0^1 f_0(t)dt \sum_{n=0}^{15} \frac{1}{2^{n+1}} = \int_0^1 f_0(t)dt \left(1 - \frac{1}{2^{15}}\right)
\]

\[
= \frac{1003}{4000} \left(1 - \frac{1}{2^{15}}\right) \approx 0.251
\]

\[
\frac{4}{3} \int_0^4 \int_0^2 f(t, s)dsdt = \frac{4}{3} \int_0^2 f_0(t)dt \left(\int_0^4 f_1(s)ds + \int_0^{2} f_1(s)ds\right)
\]

\[
= \frac{4}{3} \int_0^2 f_0(t)dt \left(\sum_{n=0}^{3} \int_n^{n+1} f_1(s)ds + \int_0^{2} f_1(s)ds\right)
\]

\[
= \frac{1}{3} \left(1 - \frac{1}{2^4} + \int_0^{2} f_1(s)ds\right)
\]

\[
= \frac{1}{3} \left(1 - \frac{1}{2^4} + \frac{1}{2^6} - \frac{1}{2^5} \left(5 - \frac{3\pi}{2}\right)\right) \approx 0.321
\]

Thus, we ensure the inequality \( (1.3) \) holds. Hence, owing to Theorem 1.1, for every

\[
\lambda \in \Lambda = \left\{ \begin{array}{c}
\frac{2048x^2}{\left(1 - \frac{1}{2^4} - \frac{1}{2^6} \left(5 - \frac{3\pi}{2}\right)\right)^2} - \frac{2048x^2}{15 \left(1003 \left(1 - \frac{1}{2^{15}}\right) - \frac{1}{2^4} \left(1 - \frac{1}{2^4} - \frac{1}{2^5} \left(5 - \frac{3\pi}{2}\right)\right)\right)}
\end{array} \right\}
\]

where, \( \Lambda \subset [15759, 136789] \), the problem

\[
\begin{align*}
&u^{(iv)} + g(u) = \lambda f_0(t)f_1(u); \text{ in } [0, 1], \\
u(0) = u(1) = 0; \\
u''(0) = u''(1) = 0,
\end{align*}
\]
has at least one non-trivial generalized solution \( u_0 \) that 
\[ 0 < \| u_0 \|^2_2 - 2 \int_{0}^{1} G(u_0) dt < \frac{2048r^2}{5}. \]

We now point out a consequence of Corollary 3.3, in which the function \( f(t, u) \) has separable variables.

**Theorem 3.6** Let \( f_0 : [0, 1] \to \mathbb{R} \) be a non-negative, non-zero and essentially bounded function and \( f_1 : \mathbb{R} \to \mathbb{R} \) be a non-negative and continuous function and \( F_1(\xi) = \int_{0}^{\xi} f_1(x) dx \) for every \( \xi \in \mathbb{R} \). Assume that there exists a positive constant \( d \) such that \( F_1(d) > C_f F_1(c) \) where \( c = \frac{\alpha}{2\pi} \sqrt{\frac{2(\delta^2 + \lambda^2 - L)}{\delta^2}} \) and

\[ C_f = \left( \frac{2\delta^2\lambda^4 + 2L}{3\delta^2\lambda^4 + L} \right) \| f_0 \|_{L^1([0,1])}, \]

then, for every

\[ \lambda \in \left[ \frac{4d^2(\delta^2 - \lambda^4 + L)}{\pi^2kF_1(d)} \left\{ \int_{\frac{1}{2}}^{\frac{1}{4}} f_0(t) dt \right\} \frac{2d^2(\delta^2 - \lambda^4)}{\pi^2kF_1(d)} \right\} \],

the problem

\[ \begin{cases} u^{(iv)} + Au'' + Bu + g(u) = f_0(t)f_1(u) \\ u(0) = u(1) = 0 \\ u''(0) = u''(1) = 0 \end{cases} \]  \hspace{1cm} (3.8)

has at least one non-trivial generalized solution \( u_0 \) such that 
\[ 0 < \frac{1}{2\pi} \| u_0 \|^2_2 - \int_{0}^{1} G(u_0) dt < \frac{4d(\delta^2 - \lambda^4)}{\pi^2}. \]

Finally, we present an example to illustrate the results of Theorem 3.6.

**Example 3.7** Let \( A = 9 \) and \( B = 2 \). Clearly, \( \delta = \frac{\sqrt{2r} - 3}{r} \approx 0.297 \) and \( k = \frac{1080r^2 - 9}{24031} \approx 0.0391 \). Put \( L = 1 \) and

\[ f_1(s) = \left\{ \begin{array}{ll} 2^{1-n} \min(s - n, n + 1 - s), & \text{if } s \in \bigcup_{n=0}^{\infty} [n, n + 1), \\ 0, & \text{if } s \in ] - \infty, 0[. \end{array} \right. \]

and

\[ f_0(t) = \left\{ \begin{array}{ll} 1, & \text{if } 0 \leq t < \frac{3}{8}, \\ 10^{-3}, & \text{if } \frac{3}{8} \leq t < \frac{5}{8}, \\ 1, & \text{if } \frac{5}{8} \leq t \leq 1, \end{array} \right. \]

and

\[ C_f = \left( \frac{2\left(1 - \frac{2}{r^4}\right)r^4 + 2}{3\left(1 - \frac{9}{r^4}\right)r^4 + 1} \right)^{\frac{3}{4}} \frac{1}{4} \left( \frac{10^4}{4} \right) \approx 0.714 \]

Put \( d = 25 \). By simple calculations, we obtain \( c = 25 \sqrt{\frac{24031r^2(\delta^2 - \lambda^4)}{540r^2(\delta^2 - \lambda^4) - 1}} \approx 178.84 \) and

\[ F_1(d) = \int_{0}^{25} f_1(s) ds = 2^{24} \sum_{n=0}^{24} \int_{n}^{n+1} f_1(s) ds = 2^{24} \sum_{n=0}^{24} \frac{1}{2n+1} = 1 - \frac{1}{25}, \]

\[ F_1(c) = \int_{0}^{178.84} f_1(s) ds = 1. \]
Hence, owing to Theorem 3.6, for every \( \lambda \in [249, 32763] \cap \Lambda_\rho \), the problem

\[
\begin{align*}
\begin{cases} 
  u^{(iv)} + 9u'' + 2u + g(u) = f_\lambda(t)f_\lambda(u); & \text{in } [0, 1], \\
  u(0) = u(1) = 0; & \\
  u''(0) = u''(1) = 0,
\end{cases}
\end{align*}
\]

has at least one non-trivial generalized solution \( u_0 \) that 0 < \( \|u_0\|^2_{L^2} - 2 \int_0^1 G(u_0)dt \) < 62109.

4. Conclusion
In the paper, the author has established the existence of a non-trivial solution for a class of fourth-order elastic beam equations involving Lipschitz non-linearity with Navier boundary value condition. Using a critical points theorem, the author has ensured the exact collections of the parameters in which the problem possesses at least a non-trivial solution on one-dimensional space. On N-dimensional spaces \( N > 1 \), the same discussions can be used for future ideas based on the results of this paper. Also the existence of three solutions or infinitely many solutions can be used for future ideas based on the other tool theorems.

Acknowledgements
The author express his gratitude to referees for their useful suggestions.

Funding
This work was supported by Sari agricultural sciences and natural resources university [grant number 01-1395-04].

Author details
Mohsen Khaleghi Moghadam
E-mails: mohsen.khaleghi@rocketmail.com, m.khaleghi@snnu.ac.ir
1 Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, 875 Sari, Iran.

Citation information
Cite this article as: Existence of a non-trivial solution for fourth-order elastic beam equations involving Lipschitz non-linearity, Mohsen Khaleghi Moghadam, Cogent Mathematics (2016), 3: 1226040.

References
Afrouzi, G. A., Heidarkhani, S., & O’Regan, D. (2011). Existence of three solutions for a doubly eigenvalue fourth-order boundary value problem. Taiwanese Journal of Mathematics, 15, 201–210.
Bai, Z. (2010). Positive solutions of some nonlocal fourth-order boundary value problem. Applied Mathematics and Computation, 215, 4191–4197.
Bai, Z., & Wang, H. (2002). On positive solutions of some nonlinear fourth-order boundary value problem. Applied Mathematics and Computation, 215, 4191–4197.
Bonanno, G. (2012). Critical point theorem via the Ekeland variational principle. Nonlinear Analysis, 75, 2992–3007.
Bonanno, G., & Di Bella, B. (2008). A boundary value problem for fourth-order elastic beam equations. Journal of Mathematical Analysis and Applications, 343, 1166–1176.
Bonanno, G., & Di Bella, B. (2010). A fourth-order boundary value problem for a Sturm-Liouville type equation. Applied Mathematics and Computation, 217, 3635–3640.
Bonanno, G., Di Bella, B., & O’Regan, D. (2011). Non-trivial solutions for nonlinear fourth-order elastic beam equations. Computers and Mathematics with Applications, 62, 1862–1869.
Busuioc, A., & Ratiu, T. (2003). The second grade fluid and averaged Euler equations with Navier-slip boundary conditions. Nonlinearity, 16, 1119–1149.
Cabada, A., Cid, J. A., & Sanchez, L. (2007). Positive and lower and upper solutions for fourth order boundary value problems. Nonlinear Analysis, 67, 1599–1612.
Chai, G. (2007). Existence of positive solutions for fourth-order boundary value problem with variable parameters. Nonlinear Analysis, 66, 870–880.
Grossinho, M. R., Sanchez, L., & Tersian, S. A. (2005). On the solvability of a boundary value problem for a fourth-order ordinary differential equation. Applied Mathematics Letters, 18, 439–444.
Gyulov, T., & Morosanu, G. (2010). On a class of boundary value problems involving the p-biharmonic operator. Journal of Mathematical Analysis and Applications, 367, 43–57.
Han, G., & Xu, Z. (2007). Multiple solutions of some nonlinear fourth-order beam equations. Nonlinear Analysis, 68, 3646–3656.
Heidarkhani, S. (2012a). Existence of solutions for a two-point boundary-value problem of a fourth-order Sturm-Liouville type. Electronic Journal of Differential Equations, 84, 1–15.
Heidarkhani, S. (2012b). Non-trivial solutions for a class of \( p_1, ..., p_n \)-biharmonic systems with Navier boundary conditions. Annales Polonici Mathematici, 105, 65–76.
Heidarkhani, S. (2014). Existence of non-trivial solutions for systems of \( n \) fourth order partial differential equations. Mathematica Slovaca, 645, 1249–1262.
Heidarkhani, S., Afrouzi, G. A., Ferrara, M., & Moradi, S. (2016). Variational approaches to impulsive elastic beam equations of Kirchhoff type. Complex Variables and Elliptic Equations. doi:10.1080/17476933.2015.1131681
Heidarkhani, S., Ferrara, M., & Khademloo, S. (2016). Nontrivial solutions for one-dimensional fourth-order Kirchhoff-type equations. Mediterranean Journal of Mathematics, 13, 217–236.
Heidarkhani, S., Ferrara, M., Solari, A., & Caristi, G. (2016). Multiplicity results for \( p(x) \)-biharmonic equations with Navier boundary. Complex Variables and Elliptic Equations, 61, 1494–1516.
Khaleghi Moghadam, M., & Heidarkhani, S. (2014). Existence of a Non-trivial solution for nonlinear difference equations. Differential Equations & Applications, 6, 517–525.
Khalkhali, S. M., Heidarkhani, S., & Razani, A. (2012). Infinitely many solutions for a fourth-order boundary-value problem. *Electronic Journal of Differential Equations*, 164, 1–14.

Li, Y. (2007). On the existence of positive solutions for the bending elastic beam equations. *Applied Mathematics and Computation*, 189, 821–827.

Liu, X.-L., & Li, W.-T. (2007a). Existence and multiplicity of solutions for fourth-order boundary value problems with three parameters. *Mathematical and Computer Modelling*, 46, 525–534.

Liu, X.-L., & Li, W.-T. (2007b). Existence and multiplicity of solutions for fourth-order boundary value problems with parameters. *Journal of Mathematical Analysis and Applications*, 327, 362–375.

Peletier, L. A., Troy, W. C., & Van der Vorst, R. C. A. M. (1995). Stationary solutions of a fourth order nonlinear diffusion equation. (V. V. Kurt., Trans.). *Differentsialnye Uravneniya* (Differential Equations), 31, 301–314.

Zeidler, E. (1985). *Nonlinear functional analysis and its applications* (Vol. II/B). New York, NY: Berlin-Heidelberg.