An optimal result for global existence and boundedness in a three-dimensional Keller-Segel-Stokes system with nonlinear diffusion

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Abstract

This paper investigates the following Keller-Segel-Stokes system with nonlinear diffusion

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n \nabla c), & x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, t > 0, \\
  u_t + \nabla P &= \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, t > 0
\end{align*}
\]

(KSF)

under homogeneous boundary conditions of Neumann type for \( n \) and \( c \), and of Dirichlet type for \( u \) in a three-dimensional bounded domains \( \Omega \subseteq \mathbb{R}^3 \) with smooth boundary, where \( \phi \in W^{1,\infty}(\Omega), m > 0 \). It is proved that if \( m > \frac{4}{3} \), then for any sufficiently regular nonnegative initial data there exists at least one global boundedness solution for system (KSF), which in view of the known results for the fluid-free system mentioned below (see Introduction) is an optimal restriction on \( m \).

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1 Introduction

Many phenomena, which appear in natural science, especially, biology and physics, support animals’ lives (see [16, 44, 32, 8]). Chemotaxis is the biological phenomenon of oriented movement of cells under influence of a chemical signal substance (see Keller and Segel [15]). A classical mathematical model for this type of processes was proposed by Keller and Segel in [15] as follows:

$$\begin{align*}
  n_t &= \Delta n - \chi \nabla \cdot (n \nabla c), \quad x \in \Omega, \ t > 0, \\
  c_t &= \Delta c - c + n, \quad x \in \Omega, \ t > 0,
\end{align*}$$

where $\chi > 0$ is called chemotactic sensitivity, $n$ and $c$ denote the density of the cell population and the concentration of the attracting chemical substance, respectively. Starting from the pioneering work of Keller and Segel (see Keller and Segel [15]), an extensive mathematical literature has grown on the Keller-Segel model and its variants (see e.g. [1, 11, 13, 12]). To prevent any chemotactic collapse in (1.1), the following variant has also been widely investigated

$$\begin{align*}
  n_t &= \Delta n^m - \chi \nabla \cdot (n \nabla c), \\
  c_t &= \Delta c - c + n.
\end{align*}$$

(1.2)

The main issue of the investigation was whether the solutions of the models are bounded or blow-up. In fact, all solutions are global and uniformly bounded if $m > 2 - \frac{2}{N}$ (see Tao and Winkler [27, 37]), whereas if $m < 2 - \frac{2}{N}$, (1.2) possess some solutions which blow up in finite time (see Winkler et. al. [3, 37]). Therefore,

$$m = 2 - \frac{2}{N}$$

(1.3)

is the critical blow-up exponent, which is related to the presence of a so-called volume-filling effect. For a more detailed discussion on this issue and on a parabolic-elliptic version of Keller-Segel system and its variants we refer readers to see Winkler et al. [5, 43, 38, 37, 35], Zheng et al. [57, 47, 50, 48, 53, 58].

In various situations, however, the migration of bacteria is furthermore substantially affected by changes in their environment (see [22, 30]). Since the bacteria consume the chemical instead of producing it, Tuval et al. (30) proposed the following (quasilinear)
chemotaxis(-Navier)-Stokes system

\[
\begin{aligned}
\frac{\partial n}{\partial t} + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \mathbf{S}(x,n,c) \nabla c), \quad x \in \Omega, t > 0, \\
\frac{\partial c}{\partial t} + u \cdot \nabla c &= \Delta c - \rho(c), \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial t} + \kappa(u \cdot \nabla)u + \nabla P &= \Delta u + n \nabla \phi, \quad x \in \Omega, t > 0, \\
\nabla \cdot u &= 0, \quad x \in \Omega, t > 0,
\end{aligned}
\]  

(1.4)

in a bounded domain \(\Omega \subset \mathbb{R}^3\) with smooth boundary, where \(f(c)\) is the consumption rate of the oxygen by the cells and \(\mathbf{S}(x,n,c)\) is a tensor-valued function or a scalar function which satisfies

\[
|\mathbf{S}(x,n,c)| \leq C_S(1 + n)^{-\alpha} \quad \text{for all} \quad (x,n,c) \in \Omega \times [0,\infty)^2
\]  

(1.5)

with some \(C_S > 0\) and \(\alpha > 0\). Here \(n\) and \(c\) are defined as before, \(u, P, \phi\) and \(\kappa \in \mathbb{R}\) denote, respectively, the velocity field, the associated pressure of the fluid, the potential of the gravitational field and the strength of nonlinear fluid convection. In recent years, approaches have been developed based on a natural energy functional, in the past several years there have been numerous analytical approaches that addressed issues of the solvability result for system (1.4) with \(\mathbf{S}(x,n,c) := \mathbf{S}(n)\) is a scalar function (see e.g. Chae et al. [2], Duan et al. [6], Liu and Lorz [18, 21], Tao and Winkler [28, 39, 40, 42], Zhang et. al. [46, 45] and references therein). On the other hand, for general \(S\) is a chemotactic sensitivity tensor, one can see Winkler (41) and Zheng (55) and the references therein for details.

Concerning the framework where the chemical is produced by the cells instead of consumed, then corresponding chemotaxis–fluid model is then the quasilinear Keller-Segel-Stokes system of the form

\[
\begin{aligned}
\frac{\partial n}{\partial t} + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \mathbf{S}(x,n,c) \nabla c), \quad x \in \Omega, t > 0, \\
\frac{\partial c}{\partial t} + u \cdot \nabla c &= \Delta c - c + n, \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial t} + \kappa(u \cdot \nabla)u + \nabla P &= \Delta u + n \nabla \phi, \quad x \in \Omega, t > 0, \\
\nabla \cdot u &= 0, \quad x \in \Omega, t > 0,
\end{aligned}
\]  

(1.6)

which describe chemotaxis-fluid interaction in cases when the evolution of the chemoattractant is essentially dominated by production through cells ([11]).
Compared to the classical Keller-Segel chemotaxis and chemotaxis(-Navier)-Stokes system (1.4), the mathematical analysis of Keller-Segel-Stokes system (1.6) has to cope with considerable additional challenges. (see Wang, Xiang et. al. [24, 33, 34, 31], Zheng [55, 54]). Xiang et. al. ([17]) established the global existence and boundedness of the 2D system (1.6) under the assumption of (1.5) with $\alpha = 0$ for any $m > 1$. In a three-dimensional setup involving linear diffusion ($m = 1$ in (1.6)) and tensor-valued sensitivity $S$ satisfying (1.5) global weak solutions have been shown to exists for $\alpha > \frac{3}{7}$ (see [20]) and $\alpha > \frac{1}{3}$ (see [54] and also [31]), respectively. If the bacteria diffuses in a porous medium ($m \neq 1$) and sensitivity $S(x, n, c) \equiv 1$, the global weak solutions for (1.6) whenever $m > 2$ ([52]), which most probably is not optimal in the sense of $m + \alpha > 2 - \frac{2}{N}$ ([37]). For the more related works in this direction, we mention that a corresponding quasilinear version or the logistic damping has been deeply investigated by Zheng [55, 49], Wang and Liu [19], Tao and Winkler [28], Wang et. al. [33, 34].

Motivated by the above works, the aim of the present paper is to study the following Keller-Segel-Stokes system with nonlinear diffusion

\begin{equation}
\begin{aligned}
    n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n \nabla c), \quad x \in \Omega, t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - c + n, \quad x \in \Omega, t > 0, \\
    u_t + \nabla P &= \Delta u + n \nabla \phi, \quad x \in \Omega, t > 0, \\
    \nabla \cdot u &= 0, \quad x \in \Omega, t > 0, \\
    \nabla n \cdot \nu = \nabla c \cdot \nu &= 0, u = 0, \quad x \in \partial \Omega, t > 0, \\
    n(x, 0) &= n_0(x), c(x, 0) = c_0(x), u(x, 0) = u_0(x), \quad x \in \Omega.
\end{aligned}
\end{equation}

This paper is organized as follows. In Section 2, we state the main results, give an approximate problem and some basic properties. In Section 3, we derive an upper bound for regularized problems of (1.7) by using the Maximal Sobolev regularity and a Moser-type iteration. Finally, in Section 4 we prove our main results by passage to the limit in the approximate problem via estimates from Section 3.
2 Preliminaries and main results

In this section, we give some notations and recall some basic facts which will be frequently used throughout the paper. To formulate the main result, let us suppose that

\[ \phi \in W^{1,\infty}(\Omega) \]  

(2.1)

and the initial data \((n_0, c_0, u_0)\) fulfills

\[
\begin{aligned}
n_0 &\in C^{\kappa}(\bar{\Omega}) \text{ for certain } \kappa > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega, \\
c_0 &\in W^{2,\infty}(\Omega) \text{ with } c_0 \geq 0 \text{ in } \bar{\Omega}, \\
u_0 &\in D(A_r^\gamma) \text{ for some } \gamma \in \left(\frac{3}{4}, 1\right) \text{ and any } r \in (1, \infty),
\end{aligned}
\]

(2.2)

where \(A_r\) denotes the Stokes operator with domain \(D(A_r) := W^{2,r}(\Omega) \cap W^{1,0}_0(\Omega) \cap L^r(\Omega)\), and \(L^r(\Omega) := \{ \phi \in L^r(\Omega) | \nabla \cdot \phi = 0 \} \) for \(r \in (1, \infty)\) ([26]).

**Theorem 2.1.** Let \(\Omega \subset \mathbb{R}^3\) be a bounded domain with smooth boundary, with smooth boundary. Suppose that the assumptions (2.1) and (2.2) hold. If

\[ m > \frac{4}{3}, \]

(2.3)

then the problem (1.7) possesses at least one global weak solution \((n, c, u, P)\) in the sense of Definition 4.1. Moreover, this solution is bounded in \(\Omega \times (0, \infty)\) in the sense that

\[ \|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t > 0. \]

(2.4)

Furthermore, \(c\) and \(u\) are continuous in \(\bar{\Omega} \times [0, \infty)\) and

\[ n \in C^0_{\omega-}(0, \infty; L^\infty(\Omega)). \]

(2.5)

**Remark 2.1.** (i) From Theorem 2.1, we conclude that if the exponent \(m\) of nonlinear diffusion is large than \(\frac{4}{3}\), then model (1.7) exists a global (weak) bounded solution, which yields to the nonlinear diffusion term benefits the global of solutions.

(ii) In comparison to the result for the corresponding fluid-free system [1, 27, 37], it is easy to see that the restriction on \(m\) here is optimal.
(iii) Obviously, $2 > \frac{4}{3}$, Theorem 2.1 seems to partly improve the results of Zheng ([52]), who showed the global weak existence of solutions for (1.6) in the cases $S(x, n, c) \equiv 1$ with $m > 2$.

(iv) If $\alpha = 0$, then $\max\{2 - 2\alpha, \frac{3}{4}\} = 2 > \frac{4}{3}$; so that, Theorem 2.1 also (partly) improve the results of Peng and Xiang ([24]), who showed the global weak existence of solutions for (1.6) in the cases $S(x, n, c)$ satisfying (1.5) with $m > \max\{2 - 2\alpha, \frac{3}{4}\}$.

(v) We should pointed that the idea of this paper can be also solved with other types of models, e.g. an attraction-repulsion chemotaxis fluid model with nonlinear diffusion (see [56]).

In order to construct solutions of (1.7) through an appropriate approximation, we then need to consider the approximate system

$$\begin{cases}
 n_{\epsilon t} + u_{\epsilon} \cdot \nabla n_{\epsilon} = \Delta (n_{\epsilon} + \epsilon)^m - \nabla \cdot (n_{\epsilon} \nabla c_{\epsilon}), & x \in \Omega, t > 0, \\
 c_{\epsilon t} + u_{\epsilon} \cdot \nabla c_{\epsilon} = \Delta c_{\epsilon} - c_{\epsilon} + n_{\epsilon}, & x \in \Omega, t > 0, \\
 u_{\epsilon t} + \nabla P_{\epsilon} = \Delta u_{\epsilon} + n_{\epsilon} \nabla \phi, & x \in \Omega, t > 0, \\
 \nabla \cdot u_{\epsilon} = 0, & x \in \Omega, t > 0, \\
 \nabla n_{\epsilon} \cdot \nu = \nabla c_{\epsilon} \cdot \nu = 0, u_{\epsilon} = 0, & x \in \partial \Omega, t > 0, \\
 n_{\epsilon}(x, 0) = n_0(x), c_{\epsilon}(x, 0) = c_0(x), u_{\epsilon}(x, 0) = u_0(x), & x \in \Omega.
\end{cases}$$

(2.6)

Next, we will provide some results which will be used later. To this end, by an adaptation of well-established fixed point arguments, one can readily verify local existence theory for (2.6) (see [41], Lemma 2.1 of [23] and Lemma 2.1 of [42]).

**Lemma 2.1.** Assume that $\epsilon \in (0, 1)$. Then there exist $T_{\text{max}} \in (0, \infty]$ and a classical solution $(n_{\epsilon}, c_{\epsilon}, u_{\epsilon}, P_{\epsilon})$ of (2.6) in $\Omega \times (0, T_{\text{max}})$ such that

$$\begin{cases}
 n_{\epsilon} \in C^0(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
 c_{\epsilon} \in C^0(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
 u_{\epsilon} \in C^0(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
 P_{\epsilon} \in C^{1,0}(\bar{\Omega} \times (0, T_{\text{max}})),
\end{cases}$$

classically solving (2.6) in $\Omega \times [0, T_{\text{max}})$. Moreover, $n_{\epsilon}$ and $c_{\epsilon}$ are nonnegative in $\Omega \times (0, T_{\text{max}})$,
\[ \|n_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} + \|c_{\varepsilon}(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} \to \infty \quad \text{as} \quad t \to T_{\text{max}}, \quad (2.8) \]

where \(\gamma\) is given by (2.2).

Given all \(s_0 \in (0, T_{\text{max}})\), from the regularity properties asserted by Lemma 2.1 we know that there exists \(\beta > 0\) such that
\[ \|n_{\varepsilon}(\tau)\|_{L^\infty(\Omega)} \leq \beta \quad \text{and} \quad \|c_{\varepsilon}(\tau)\|_{W^{2,\infty}(\Omega)} \leq \beta \quad \text{for all} \quad \tau \in [0, s_0]. \quad (2.9) \]

**Lemma 2.2.** ([44]) Let \(l \in [1, +\infty)\) and \(r \in [1, +\infty]\) be such that
\[
\begin{aligned}
&l < \frac{3r}{3-r} \quad \text{if} \quad r \leq 3, \\
&l \leq \infty \quad \text{if} \quad r > 3.
\end{aligned}
\quad (2.10)
\]

Then for all \(K > 0\) there exists \(C = C(l, r, K)\) such that
\[ \|n(\cdot, t)\|_{L^r(\Omega)} \leq K \quad \text{for all} \quad t \in (0, T_{\text{max}}), \quad (2.11) \]

then
\[ \|D\alpha(\cdot, t)\|_{L^l(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}}). \quad (2.12) \]

**Lemma 2.3.** ([9, 77, 53]) Suppose \(\gamma \in (1, +\infty)\), \(g \in L^\gamma((0, T); L^\gamma(\Omega))\) and \(v_0 \in W^{2,\gamma}(\Omega)\) such that \(\frac{\partial v_0}{\partial \nu} = 0\). Let \(v\) be a solution of the following initial boundary value
\[
\begin{aligned}
&v_t - \Delta v = g, \quad (x, t) \in \Omega \times (0, T), \\
&\frac{\partial v}{\partial \nu} = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
&v(x, 0) = v_0(x), \quad (x, t) \in \Omega.
\end{aligned}
\quad (2.13)
\]

Then there exists a positive constant \(\delta_0\) such that
\[
\begin{aligned}
&\int_0^T \|v(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma dt + \int_0^T \|v_t(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma dt + \int_0^T \|\Delta v(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma dt \\
&\leq \delta_0 \left( \int_0^T \|g(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma dt + \|v_0(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma + \|\Delta v_0(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma \right).
\end{aligned}
\quad (2.14)
\]

On the other hand, assuming \(v\) is a solution of the following initial boundary value
\[
\begin{aligned}
&v_t - \Delta v + v = g, \quad (x, t) \in \Omega \times (0, T), \\
&\frac{\partial v}{\partial \nu} = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
&v(x, 0) = v_0(x), \quad (x, t) \in \Omega.
\end{aligned}
\quad (2.15)
\]
Then there exists a positive constant $C_{\gamma} := C_{\gamma, |\Omega|}$ such that if $s_0 \in [0,T)$, $v(\cdot, s_0) \in W^{2,\gamma}(\Omega)(\gamma > N)$ with $\partial v(\cdot, s_0) = 0$, then

$$
\int_{s_0}^{T} e^{\gamma s} (\|v(\cdot, t)\|_{L^\gamma(\Omega)} + \|\Delta v(\cdot, t)\|_{L^\gamma(\Omega)}) ds \\
\leq C_{\gamma} \left( \int_{s_0}^{T} e^{\gamma s} (\|g(\cdot, s)\|_{L^\gamma(\Omega)} + \|v_0(\cdot, s_0)\|_{L^\gamma(\Omega)} + \|\Delta v_0(\cdot, s_0)\|_{L^\gamma(\Omega)}) ds \right). 
$$

(2.16)

3 A priori estimates

In this section, we proceed to derive $\varepsilon$-independent estimates for the approximate solutions constructed above. The iteration depends on a series of a priori estimate. To this end, throughout this section, for any $\varepsilon \in (0,1)$, we let $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ be the global solution of problem (2.6).

The following estimates of $n_\varepsilon$ and $c_\varepsilon$ are basic but important in the proof of our result.

**Lemma 3.1.** There exists $\lambda > 0$ independent of $\varepsilon$ such that the solution of (2.6) satisfies

$$
\int_{\Omega} n_\varepsilon + \int_{\Omega} c_\varepsilon \leq \lambda \text{ for all } t \in (0, T_{\text{max}}).
$$

(3.1)

We next show the following lemma which holds a key for the proof of Theorem 2.1. Employing the same arguments as in the proof of Lemma 3.3 in [54] (see also [55]), we derive the following Lemma:

**Lemma 3.2.** Let $m > \frac{4}{3}$. Then there exists $C > 0$ independent of $\varepsilon$ such that the solution of (2.6) satisfies

$$
\int_{\Omega} (n_\varepsilon + \varepsilon)^{m-1} + \int_{\Omega} c_\varepsilon^2 + \int_{\Omega} |u_\varepsilon|^2 \leq C \text{ for all } t \in (0, T_{\text{max}}).
$$

(3.2)

Moreover, for $T \in (0, T_{\text{max}})$, it holds that one can find a constant $C > 0$ independent of $\varepsilon$ such that

$$
\int_{0}^{T} \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + |\nabla c_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \leq C. 
$$

(3.3)

In the following, we shall derive an important inequality, which plays a key role in the proof of our main result.
Lemma 3.3. Let \( p = \frac{25}{16} \) and \( \theta = \frac{8}{7} \). Then there exists a positive constant \( \tilde{l}_0 \in \left( \frac{1737}{682}, 3 \right) \) such that
\[
\frac{\frac{5}{6} - \frac{1}{\theta'(p+1)}}{\frac{7}{6} - \frac{1}{p+1}} + \frac{\frac{1}{l_0} - \frac{1}{\theta'(p+1)}}{\frac{1}{l_0} + \frac{2}{3} - \frac{1}{p+1}} < 1, \tag{3.4}
\]
where \( \theta' = \frac{\theta}{\theta - 1} = 8. \)

Proof. It can readily be verified that
\[
\frac{1}{p + 1} = \frac{16}{41}
\]
and
\[
1 - \frac{\frac{1}{l_0} - \frac{1}{\theta'(p+1)}}{\frac{2}{3} - \frac{1}{p+1}} = \frac{\frac{2}{3} - \frac{1}{\theta'(p+1)}}{\frac{1}{l_0} + \frac{2}{3} - \frac{1}{p+1}},
\]
due to our assumption \( p = \frac{25}{16} \) and \( \theta = \frac{8}{7} \). These together with some basic calculation yield to (3.4). \( \square \)

The following estimates are crucial to prove our main results, which are based on the Maximal Sobolev regularity (see Hieber and Prüss [9]).

Lemma 3.4. If
\[
m > \frac{4}{3}, \tag{3.5}
\]
then there exists a positive constant \( p_0 > \frac{3}{2} \) independent of \( \varepsilon \) such that the solution of (1.7) from Lemma 2.1 satisfies
\[
\int_\Omega n_\varepsilon^{p_0}(x,t)dx \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}}). \tag{3.6}
\]

Proof. Let \( p = \frac{25}{16} \). Multiplying the first equation of (1.7) by \( (n_\varepsilon + \varepsilon)^{p-1} \) and using \( \nabla \cdot u_\varepsilon = 0 \), we derive that
\[
\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + m(p - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3} |\nabla n_\varepsilon|^2
\]
\[
= - \int_\Omega (n_\varepsilon + \varepsilon)^{p-1} \nabla \cdot (n_\varepsilon \nabla c_\varepsilon) \tag{3.7}
\]
\[
= (p - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{p-2} n_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\]
which yields to
\[
\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + m(p - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3} |\nabla n_\varepsilon|^2 \\
\leq -\frac{p+1}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p + (p - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{p-2} n_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \frac{p+1}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p
\]  
(3.8)
for all \( t \in (0, T_{\text{max}}) \).

Here, according to the Young inequality, it reads that
\[
\frac{p+1}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p \leq \varepsilon_1 \int_\Omega n_\varepsilon + \varepsilon)^{p+\frac{3}{2}} + C_1(\varepsilon_1, p),
\]  
(3.9)
where
\[
C_1(\varepsilon_1, p) = \frac{m - \frac{1}{3}}{p + m - \frac{1}{3}} \left( \varepsilon_1 \frac{p + m - \frac{1}{3}}{p} \right)^{-\frac{p}{m-\frac{1}{3}}} \left( \frac{p+1}{p} \right)^{\frac{m-\frac{1}{3}}{p}} |\Omega|.
\]

Once more integrating by parts, combine with (1.5), we also find that
\[
(p - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{p-2} n_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
= (p - 1) \int_\Omega \nabla \int_0^{n_\varepsilon} (\tau + \varepsilon)^{p-2} \tau d\tau \cdot \nabla c_\varepsilon \\
= -(p - 1) \int_\Omega \int_0^{\varepsilon_0} (\tau + \varepsilon)^{p-2} \tau d\tau \Delta c_\varepsilon \\
\leq \frac{(p - 1)}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p |\Delta c_\varepsilon|.
\]  
(3.10)
Utilizing the Young inequality to the term on the right side of (3.10) leads to
\[
\frac{(p - 1)}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p |\Delta c_\varepsilon| \\
\leq \int_\Omega (n_\varepsilon + \varepsilon)^{p+1} + \frac{1}{p+1} \left[ \frac{p+1}{p} \right]^{-p} \left( \frac{(p-1)}{p} \right)^{p+1} \int_\Omega |\Delta c_\varepsilon|^{p+1} \\
= \int_\Omega (n_\varepsilon + \varepsilon)^{p+1} + A_1 \int_\Omega |\Delta c_\varepsilon|^{p+1},
\]  
(3.11)
where
\[
A_1 := \frac{1}{p+1} \left[ \frac{p+1}{p} \right]^{-p} \left( \frac{(p-1)}{p} \right)^{p+1}.
\]
Hence (3.8), (3.9) and (3.11) results in
\[
\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + m(p - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3} |\nabla n_\varepsilon|^2 \\
\leq \varepsilon_1 \int_\Omega (n_\varepsilon + \varepsilon)^{p+\frac{3}{2}} + \int_\Omega (n_\varepsilon + \varepsilon)^{p+1} - \frac{p+1}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p \\
+ A_1 \int_\Omega |\Delta c_\varepsilon|^{p+1} + C_1(\varepsilon_1, p) \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\]
Since, \( m > \frac{4}{3} \), yields to \( p + 1 < p + m - \frac{1}{3} \), therefore, by the Young inequality, we conclude that
\[
\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + \frac{4m(p - 1)}{(m + p - 1)^2} \| \nabla (n_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}} \|_{L^2(\Omega)}^2 \\
\leq 2\varepsilon_1 \int_\Omega (n_\varepsilon + \varepsilon)^{p + m - \frac{1}{3}} - \frac{p + 1}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p \\
+ A_1 \int_\Omega |\nabla \varepsilon|^p + C_2(\varepsilon_1, p),
\] (3.12)

where
\[
C_2(\varepsilon_1, p) = \frac{m - \frac{4}{3}}{p + m - \frac{1}{3}} \left( \varepsilon_1 \frac{p + m - \frac{1}{3}}{p + 1} \right)^{- \frac{p + 1}{m - \frac{3}{3}}} \left( \frac{p + 1}{p} \right)^{\frac{p + m - \frac{1}{3}}{m - \frac{3}{3}}} |\Omega|.
\]

On the other hand, by the Gagliardo–Nirenberg inequality and (3.1), one can get there exist positive constants \( \lambda_0 \) and \( \lambda_1 \) such that
\[
\int_\Omega (n_\varepsilon + \varepsilon)^{p + m - \frac{1}{3}} = \| (n_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}} \|_{L^{\frac{2(p+m-\frac{1}{3})}{m+p-1}}(\Omega)} \|
\leq \lambda_0 \| \nabla (n_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}} \|_{L^{\frac{2(p+m-\frac{1}{3})}{m+p-1}}(\Omega)}^2 + \| (n_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}} \|_{L^{\frac{2(p+m-\frac{1}{3})}{m+p-1}}(\Omega)}^2 \\
\leq \lambda_1 \| \nabla (n_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}} \|_{L^{\frac{2(p+m-\frac{1}{3})}{m+p-1}}(\Omega)}^2.
\] (3.13)

In combination with (3.12) and (3.13), this shows that
\[
\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p \\
\leq (2\varepsilon_1 - \frac{4m(p - 1)}{(m + p - 1)^2} \lambda_1) \int_\Omega (n_\varepsilon + \varepsilon)^{p + m - \frac{1}{3}} - \frac{p + 1}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p \\
\leq \frac{4m(p - 1)}{(m + p - 1)^2} \lambda_1, \quad \text{for all } t \in (0, T_{max}),
\] (3.14)

where
\[
C_3(\varepsilon_1, p) = C_2(\varepsilon_1, p) + \frac{4m(p - 1)}{(m + p - 1)^2}.
\]

For any \( t \in (s_0, T_{max}) \), employing the variation-of-constants formula to (3.14) and using
\[\varepsilon < 1,\] we obtain

\[
\frac{1}{p} \| n_\varepsilon(t) + \varepsilon \|^p_{L^p(\Omega)} 
\leq \frac{1}{p} e^{-(p+1)(t-s_0)} \| n_\varepsilon(s_0) \|_{L^p(\Omega)} + (2\varepsilon_1 - \frac{4m(p-1)}{m+p-1}) \int_{s_0}^t e^{-(p+1)(t-s)} \int_\Omega (n_\varepsilon + \varepsilon)^{p+m-\frac{4}{n}} dx ds \\
+ A_1 \int_{s_0}^t e^{-(p+1)(t-s)} \int_\Omega |\Delta c_\varepsilon|^{p+1} dx ds + C_3(\varepsilon_1, p) \int_{s_0}^t e^{-(p+1)(t-s)} ds \\
\leq (2\varepsilon_1 - \frac{4m(p-1)}{m+p-1}) \int_{s_0}^t e^{-(p+1)(t-s)} \int_\Omega (n_\varepsilon + \varepsilon)^{p+m-\frac{1}{2}} dx ds + A_1 \int_{s_0}^t e^{-(p+1)(t-s)} \int_\Omega |\Delta c_\varepsilon|^{p+1} dx ds \\
+ C_4(\varepsilon_1, p)
\]

with

\[
C_4 := C_4(\varepsilon_1, p) = \frac{1}{p} e^{-(p+1)(t-s_0)} \| n_\varepsilon(s_0) \|_{L^p(\Omega)} + C_3(\varepsilon_1, p) \int_{s_0}^t e^{-(p+1)(t-s)} ds.
\]

Due to (3.1), in view of Lemma 2.2, we derive that

\[
\| D u_\varepsilon(\cdot, t) \|_{L^l(\Omega)} \leq C_5 \text{ for all } t \in (0, T_{max}) \text{ and for any } l \leq \frac{3}{2}.
\]

Here employing the three-dimensional Sobolev inequality, we can find

\[
\| u_\varepsilon(\cdot, t) \|_{L^0(\Omega)} \leq C_6 \text{ for all } t \in (0, T_{max}) \text{ and for any } l_0 < 3.
\]

Now, due to Lemma 2.3 and the second equation of (1.7) and using the Hölder inequality, we have

\[
A_1 \int_{s_0}^t e^{-(p+1)(t-s)} \int_\Omega |\Delta c_\varepsilon|^{p+1} ds \\
= A_1 e^{-(p+1)t} \int_{s_0}^t e^{(p+1)s} \int_\Omega |\Delta c_\varepsilon|^{p+1} ds \\
\leq 2^{p+1} A_1 e^{-(p+1)t} C_{p+1} \left( \int_{s_0}^t \int_\Omega e^{(p+1)s} (|u_\varepsilon \cdot \nabla c_\varepsilon|^{p+1} + n_\varepsilon^{p+1}) ds + e^{(p+1)s_0} \| c_\varepsilon(s_0, t) \|_{W^{p+1}_2(\Omega)} \right) \\
\leq 2^{p+1} A_1 e^{-(p+1)t} C_{p+1} \int_{s_0}^t e^{(p+1)s} (\| u_\varepsilon \|_{L^{p+1}(\Omega)}^{p+1} \| \nabla c_\varepsilon \|_{L^{p+1}(\Omega)}^{p+1} + n_\varepsilon^{p+1}) ds + C_7
\]

for all \( t \in (s_0, T_{max}) \), where \( \theta = \frac{8}{7}, \theta' = \frac{\theta}{\theta-1} = 8, C_7 = A_1 e^{-(p+1)t} C_{p+1} 2^{p+1} e^{(p+1)s_0} \| c_\varepsilon(s_0, t) \|_{W^{p+1}_2(\Omega)}^{p+1} \).

Next, an application of the Gagliardo–Nirenberg inequality and (3.2) infers that

\[
\| \nabla c_\varepsilon \|_{L^{p+1}(\Omega)}^{p+1} \\
\leq C_8 \| \Delta c_\varepsilon \|_{L^{p+1}(\Omega)}^{(1-\theta)(p+1)} \| c_\varepsilon \|_{L^2(\Omega)}^{(1-\theta)(p+1)} + C_8 \| c_\varepsilon \|_{L^2(\Omega)}^{p+1} \\
\leq C_9 \| \Delta c_\varepsilon \|_{L^{p+1}(\Omega)}^{(p+1)} + C_9
\]
with some constants $C_8 > 0$ and $C_9 > 0$, where
\[
  a = \frac{\frac{5}{6} - \frac{1}{\sigma(p+1)}}{\frac{2}{6} - \frac{1}{p+1}} \in (0, 1).
\]

We derive from the Young inequality that for any $\delta \in (0, 1)$,
\[
\|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} \|\nabla c_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} \\
\leq C_9 \|\Delta c_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{q(p+1)} \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} + C_9 \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} \\
\leq \delta \|\Delta c_\varepsilon\|_{L^{p+1}(\Omega)}^{p+1} + C_{10} \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} + C_9 \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1},
\]
where $C_{10} = (1 - a) \left( \delta \times \frac{1}{a} \right)^{-\frac{1}{1-a}} C_9^{\frac{1}{1-a}}$.

Substituting (3.20) into (3.18) yields that
\[
A_1 \int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} |\Delta c_\varepsilon|^{p+1} ds \\
\leq 2^{p+1} A_1 e^{-(p+1)t} C_{p+1} \delta \int_{s_0}^{t} e^{(p+1)s} \|\Delta c_\varepsilon\|_{L^{p+1}(\Omega)}^{p+1} ds \\
+ 2^{p+1} A_1 e^{-(p+1)t} C_{p+1} \int_{s_0}^{t} e^{(p+1)s} [C_{10} \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} + C_9 \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1}] ds \\
+ 2^{p+1} A_1 e^{-(p+1)t} C_{p+1} \int_{s_0}^{t} e^{(p+1)s} n_\varepsilon^{p+1} ds + C_7
\]
for all $t \in (s_0, T_{max})$. Therefore, choosing $\delta = \frac{1}{2} \times 2^{p+1} A_1 C_{p+1}$ yields to
\[
A_1 \int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} |\Delta c_\varepsilon|^{p+1} ds \\
\leq 2^{p+2} A_1 e^{-(p+1)t} C_{p+1} C_{10} \int_{s_0}^{t} e^{(p+1)s} \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} ds \\
+ 2^{p+2} A_1 e^{-(p+1)t} C_{p+1} C_9 \int_{s_0}^{t} \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} ds \\
+ 2^{p+2} A_1 e^{-(p+1)t} C_{p+1} \int_{s_0}^{t} e^{(p+1)s} n_\varepsilon^{p+1} ds + 2C_7.
\]

On the other hand, by Lemma 3.3, we may choose $\frac{1277}{852} < \tilde{t}_0 < 3$ such that
\[
\frac{\frac{5}{6} - \frac{1}{\sigma(p+1)}}{\frac{2}{6} - \frac{1}{p+1}} + \frac{\frac{1}{l_0} - \frac{1}{\sigma(p+1)}}{\frac{1}{l_0} + \frac{2}{3} - \frac{1}{p+1}} < 1.
\]

Therefore, it follows from the Gagliardo–Nirenberg inequality, (3.17) and the Young inequality that there exist constants $C_{11} = C_{11}(p) > 0$ and $C_{12} = C_{12}(p) > 0$ such that
\[
\|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} \\
\leq A u_\varepsilon \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} C_{11} \\
\leq A u_\varepsilon \|u_\varepsilon\|_{L^{q(p+1)}(\Omega)}^{p+1} + C_{12}
\]
with
\[
\tilde{a} = \frac{1}{l_0} - \frac{1}{\theta(p+1)} \in (0, 1).
\]

Here we have use the fact that \(\frac{p+1}{1-a} = (p+1) \left(\frac{1}{l_0} + \frac{1}{\theta(p+1)}\right) < p+1\) by (3.23). In light of \(\frac{1}{1-a} > 1\), similarly, we derive that
\[
\|u_\varepsilon\|_{L^{p+1}(\Omega)} < 2^{14}C_{p+1} e^{(p+1)t} ds
\]
(3.25)

Then along with (3.22), (3.24) and (3.25), we have
\[
A_1 \int_{s_0}^t \text{e}^{-(p+1)(t-s)} \int_\Omega |\Delta C_\varepsilon|^{p+1} ds
\leq 2^{p+2} A_1 \text{e}^{-(p+1)t} C_{p+1} \int_{s_0}^t \text{e}^{(p+1)s} \left[\|Au_\varepsilon\|_{L^{p+1}(\Omega)} + C_{12}\right] ds
+ 2^{p+2} A_1 \text{e}^{-(p+1)t} C_{p+1} \int_{s_0}^t \text{e}^{(p+1)s} \left[\|Au_\varepsilon\|_{L^{p+1}(\Omega)} + C_{13}\right] ds
+ 2^{p+2} A_1 \text{e}^{-(p+1)t} C_{p+1} \int_{s_0}^t \text{e}^{(p+1)s} n_\varepsilon^{p+1} ds + 2C_7
\]
(3.26)

where \(C_{14} = 2^{p+2} A_1 \text{e}^{-(p+1)t} C_{p+1} (C_{10}C_{12} + C_9C_{13}) + 2C_7\). Putting \(\tilde{u}_\varepsilon(\cdot, s) := e^s u_\varepsilon(\cdot, s), s \in (s_0, t)\), we obtain from the third equation in (1.7) that
\[
\tilde{u}_\varepsilon = \Delta \tilde{u}_\varepsilon + \tilde{u}_\varepsilon + e^s n_\varepsilon \nabla \phi + e^s \nabla P_\varepsilon
\]
(3.27)

which derives
\[
\tilde{u}_\varepsilon + A\tilde{u}_\varepsilon = P(\tilde{u}_\varepsilon + e^s n_\varepsilon \nabla \phi + e^s \nabla P_\varepsilon)
\]
(3.28)

where \(P\) denotes the Helmholtz projection mapping \(L^2(\Omega)\) onto its subspace \(L_\sigma^2(\Omega)\) of all solenoidal vector field. Thus by \(p < 2\) and (3.17), we derive from Lemma 2.3 (see also Theorem 2.7 of [7]) that there exist positive constants \(C_{15}, C_{16}, C_{17}\) and \(C_{18}\) such that
\[
\int_{s_0}^t \text{e}^{(p+1)s} \|Au_\varepsilon(\cdot, t)\|^{p+1}_{L^{p+1}(\Omega)} ds
\leq C_{15} \left(\int_{s_0}^t \text{e}^{(p+1)s} \left[\|u_\varepsilon(\cdot, s)\|^{p+1}_{L^{p+1}(\Omega)} + \|n_\varepsilon(\cdot, s)\|^{p+1}_{L^{p+1}(\Omega)}\right] ds + \text{e}^{(p+1)t} + 1\right)
\leq C_{16} \left(\int_{s_0}^t \text{e}^{(p+1)s} \left[\|u_\varepsilon(\cdot, s)\|^{p+1}_{L^{p+1}(\Omega)} + \|n_\varepsilon(\cdot, s)\|^{p+1}_{L^{p+1}(\Omega)}\right] \epsilon + \|n_\varepsilon(\cdot, s)\|^{p+1}_{L^{p+1}(\Omega)} ds + \text{e}^{(p+1)t} + 1\right)
\leq C_{17} \int_{s_0}^t \text{e}^{(p+1)s} \|n_\varepsilon(\cdot, s)\|^{p+1}_{L^{p+1}(\Omega)} ds + (1 + C_{18}) \text{e}^{(p+1)t}.
\]
(3.29)
By virtue of (3.29) and (3.30), we can see
\[
A_1 \int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} |\Delta_c \varepsilon|^{p+1} ds \leq 2^{p+2} A_1 e^{-(p+1)t} C_{p+1} [C_{10} + C_9] \left( C_{17} \int_{s_0}^{t} e^{(p+1)s} \|n_\varepsilon(\cdot, s)\|^{p+1}_{L^{p+1}(\Omega)} ds + (1 + C_{18}) e^{(p+1)t} \right) + 2^{p+2} A_1 e^{-(p+1)t} C_{p+1} \int_{s_0}^{t} \int_{\Omega} e^{(p+1)s} n_\varepsilon^{p+1} ds + C_{14} \leq C_{19} \int_{s_0}^{t} e^{(p+1)s} \|n_\varepsilon(\cdot, s)\|^{p+1}_{L^{p+1}(\Omega)} ds + C_{20},
\]
(3.30)

where \( C_{19} = 2^{p+2} A_1 e^{-(p+1)t} C_{p+1} [C_{10} + C_9] C_{17} + 2^{p+2} A_1 e^{-(p+1)t} C_{p+1} \) and
\[
C_{20} := C_{20}(p) = 2^{p+2} A_1 e^{-(p+1)t} C_{p+1} [C_{10} + C_9] (1 + C_{18}) e^{(p+1)t} + C_{14}.
\]

Collecting (3.15) and (3.30), applying Lemma 3.3 and the Young inequality, we derive that
\[
\frac{1}{p} \|n_\varepsilon(t) + \varepsilon\|_{L^p(\Omega)}^p \leq \left( 2 \varepsilon_1 - \frac{4m(p-1)}{(m+p-1)^2 \lambda_1} \right) \int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} (n_\varepsilon + \varepsilon)^{p+m-\frac{4}{p}} ds + C_{19} \int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} n_\varepsilon^{p+1} ds + C_{21} \leq \left( 2 \varepsilon_1 - \frac{4m(p-1)}{(m+p-1)^2 \lambda_1} \right) \int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} (n_\varepsilon + \varepsilon)^{p+m-\frac{4}{p}} ds + C_{21} \leq \left( 3 \varepsilon_1 - \frac{4m(p-1)}{(m+p-1)^2 \lambda_1} \right) \int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} (n_\varepsilon + \varepsilon)^{p+m-\frac{4}{p}} ds + C_{22}
\]
with \( C_{21} = C_{20} + C_4(\varepsilon_1, p) \) and \( C_{22} = \frac{m-\frac{4}{p}}{p+m-\frac{4}{p}} \left( \varepsilon_1 \frac{p+m-\frac{1}{2}}{p+1} \right) \frac{p+1}{m-\frac{4}{p}} (C_{19}) \frac{p+m-\frac{4}{p}}{m-\frac{4}{p}} + C_{21} \). Thus, choosing \( \varepsilon_1 \) small enough (e.g. \( \varepsilon_1 < \frac{(p-1)}{(m+p-1)^2 \lambda_1} \)) in (3.31), using (2.9), the Hölder inequality and \( \varepsilon < 1 \), we derive that there exists a positive constant \( p_0 > \frac{3}{2} \) such that
\[
\int_{\Omega} n_\varepsilon^{p_0}(x, t) dx \leq C_{23} \text{ for all } t \in (0, T_{max}).
\]
(3.32)

The proof of Lemma 3.4 is completed. \( \square \)

Underlying the estimates established above, we can derive the following higher integrability properties by applying arguments which are essentially standard in the analysis of the heat as well as the Stokes equations and a Moser-type iteration.
Lemma 3.5. Let $m > \frac{4}{3}$ and $\gamma$ be as in (2.2). Then one can find a positive constant $C$ independent of $\varepsilon$ such that

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}})$$

(3.33)

and

$$\|c_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}})$$

(3.34)

as well as

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}).$$

(3.35)

Moreover, we also have

$$\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}).$$

(3.36)

Proof. In what follows, let $C, C_i$ denote some different constants, which are independent of $\varepsilon$, and if no special explanation, they depend at most on $\Omega, \phi, m, n_0, c_0$ and $u_0$.

**Step 1. The boundedness of $\|c_\varepsilon(\cdot, t)\|_{L^4(\Omega)}^4$ for all $t \in (0, T_{\text{max}})$**

Firstly, taking $c_2^3 \varepsilon$ as the test function for the second equation of (1.7) and using $\nabla \cdot u_\varepsilon = 0$, the Hölder inequality and (3.6) yields that

$$\frac{1}{4} \frac{d}{dt}\|c_\varepsilon\|_{L^4(\Omega)}^4 + 3 \int_\Omega c_\varepsilon^2 |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^4$$

$$= \int_\Omega n_\varepsilon c_\varepsilon^3$$

$$\leq \left( \int_\Omega n_\varepsilon^\frac{2}{3} \right)^\frac{3}{4} \left( \int_\Omega c_\varepsilon^9 \right)^\frac{1}{3}$$

(3.37)

$$\leq C_1 \left( \int_\Omega c_\varepsilon^9 \right)^\frac{1}{3} \text{ for all } t \in (0, T_{\text{max}}).$$

Now, due to (3.2), in light of the Gagliardo–Nirenberg inequality and the Young inequality, we derive that

$$\left( \int_\Omega c_\varepsilon^9 \right)^\frac{1}{3} = \|c_\varepsilon^2\|_{L^\frac{9}{2}(\Omega)}^\frac{3}{2}$$

$$\leq C_2 \left( \|\nabla c_\varepsilon^2\|_{L^2(\Omega)} \|c_\varepsilon^2\|_{L^1(\Omega)} + \|c_\varepsilon^2\|_{L^1(\Omega)} \right)$$

(3.38)

$$\leq C_3 (\|\nabla c_\varepsilon^2\|_{L^2(\Omega)} + 1)$$

$$\leq \frac{1}{4} \|\nabla c_\varepsilon^2\|_{L^2(\Omega)}^2 + C_4 \text{ for all } t \in (0, T_{\text{max}}).$$
Collecting (3.38) into (3.37), in view of an ODE comparison argument entails
\[
\int_\Omega c_\varepsilon^4 \leq C_5 \quad \text{for all } t \in (0, T_{\text{max}}).
\] (3.39)

**Step 2. The boundedness of** \( \|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \) and \( \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \) **for all** \( t \in (0, T_{\text{max}}) \)

On the basis of the variation-of-constants formula for the projected version of the third equation in (2.6), we derive that
\[
u_\varepsilon(\cdot, t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}\mathcal{P}(n_\varepsilon(\cdot, t)\nabla \phi)d\tau \quad \text{for all } t \in (0, T_{\text{max}}).
\]

Therefore, according to standard smoothing properties of the Stokes semigroup we see that
\[
\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq \|A^\gamma e^{-tA}u_0\|_{L^2(\Omega)} + \int_0^t \|A^\gamma e^{-(t-\tau)A}h_\varepsilon(\cdot, \tau)\|_{L^2(\Omega)}d\tau
\]
\[
\leq \|A^\gamma u_0\|_{L^2(\Omega)} + C_6 \int_0^t (t-\tau)^{-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{2})}e^{-\lambda(t-\tau)}\|h_\varepsilon(\cdot, \tau)\|_{L^{p_0}(\Omega)}d\tau
\]
\[
\leq C_7 \quad \text{for all } t \in (0, T_{\text{max}}),
\] (3.40)

where \( \gamma \in \left(\frac{3}{4}, 1\right) \), \( h_\varepsilon = \mathcal{P}(n_\varepsilon \nabla \phi) \) and \( p_0 \) is the same as Lemma 3.4. Here we have used the fact that
\[
\|h_\varepsilon(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}})
\]
as well as
\[
\int_0^t (t-\tau)^{-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{2})}e^{-\lambda(t-\tau)}d\tau \leq \int_0^\infty \sigma^{-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{2})}e^{-\lambda\sigma}d\sigma < +\infty.
\]

Observe that \( D(A^\gamma) \) is continuously embedded into \( L^\infty(\Omega) \) by \( \gamma > \frac{3}{4} \), so that, (3.40) yields to
\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_8 \quad \text{for all } t \in (0, T_{\text{max}}).
\] (3.41)

**Step 3. The boundedness of** \( \|\nabla c_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \) **for all** \( t \in (0, T_{\text{max}}) \)

An application of the variation of constants formula for \( c_\varepsilon \) leads to
\[
\|\nabla c_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq \|\nabla e^{t(\Delta-1)c_0}\|_{L^2(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)}(n_\varepsilon(s)\nabla \phi)\|_{L^2(\Omega)}ds
\]
\[
+ \int_0^t \|\nabla e^{(t-s)(\Delta-1)}\nabla \cdot (u_\varepsilon(s)c_\varepsilon(s))\|_{L^2(\Omega)}ds,
\] (3.42)
To estimate the terms on the right of (3.42), we use the $L^p$-$L^q$ estimates associated heat semigroup to get that

$$\|\nabla e^{t(\Delta^{-1})}c_0\|_{L^\infty(\Omega)} \leq C_9 \text{ for all } t \in (0, T_{\text{max}}) \quad (3.43)$$

as well as

$$\int_0^t \|\nabla e^{(t-s)(\Delta^{-1})}n_\varepsilon(s)\|_{L^\infty(\Omega)} \, ds$$

$$\leq C_{10} \int_0^t [1 + (t-s)^{-\frac{1}{2} - \frac{3}{2(p_0 - \frac{3}{7})}]e^{-(t-s)}\|n_\varepsilon(s)\|_{L^{p_0}(\Omega)} \, ds \quad (3.44)$$

$$\leq C_{11} \text{ for all } t \in (0, T_{\text{max}})$$

and

$$\int_0^t \|\nabla e^{(t-s)(\Delta^{-1})}\nabla \cdot (u_\varepsilon(s)c_\varepsilon(s))\|_{L^\infty(\Omega)} \, ds$$

$$\leq C_{12} \int_0^t \|(-\Delta + 1)^{\varepsilon}(t-s)(\Delta^{-1})\nabla \cdot (u_\varepsilon(s)c_\varepsilon(s))\|_{L^4(\Omega)} \, ds$$

$$\leq C_{13} \int_0^t (t-s)^{-\frac{1}{2} - \frac{3}{2} - \lambda(t-s)}\|u_\varepsilon(s)c_\varepsilon(s)\|_{L^4(\Omega)} \, ds \quad (3.45)$$

$$\leq C_{14} \int_0^t (t-s)^{-\frac{1}{2} - \frac{3}{p_0} - \lambda(t-s)}\|u_\varepsilon(s)\|_{L^\infty(\Omega)}\|c_\varepsilon(s)\|_{L^4(\Omega)} \, ds$$

$$\leq C_{15} \text{ for all } t \in (0, T_{\text{max}}).$$

where $t = \frac{13}{28}$, $\tilde{\kappa} = \frac{1}{56}$. Here we have use the fact that (2.2), (3.6) as well as $\frac{1}{2} + \frac{3}{2}(\frac{1}{4} - \frac{3}{7}) < \frac{1}{2}$ and $\min\{-\frac{1}{2} - \frac{3}{2} - \frac{3}{2}(\frac{1}{p_0} - \frac{3}{7})\} > -1$. Combined with (3.42)–(3.45), we derive that

$$\|\nabla c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{16} \text{ for all } t \in (0, T_{\text{max}}). \quad (3.46)$$

**Step 4. The boundedness of $\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)}$ for all $p > 2 + m$ and $t \in (0, T_{\text{max}})$**

Taking $(n_\varepsilon + \varepsilon)^{p-1}$ as the test function for the first equation of (2.6) and combining with the second equation, using $\nabla \cdot u_\varepsilon = 0$ and the Young inequality, in view of the Hölder inequality and (3.46), we obtain

$$\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + \frac{m(p-1)}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3} |\nabla n_\varepsilon|^2$$

$$\leq \frac{(p-1)}{2m} \int_\Omega (n_\varepsilon + \varepsilon)^{p+1-m} |\nabla c_\varepsilon|^2$$

$$\leq \frac{(p-1)}{2m} \left( \left( \int_\Omega (n_\varepsilon + \varepsilon)^{3(p+1-m)} \right)^\frac{1}{3} \left( |\nabla c_\varepsilon|^3 \right)^\frac{2}{3} \right)$$

$$\leq C_{17} \left( \int_\Omega (n_\varepsilon + \varepsilon)^{3(p+1-m)} \right)^\frac{1}{3} \text{ for all } t \in (0, T_{\text{max}}). \quad (3.47)$$
On the other hand, in view of $m > \frac{4}{3} > 1$ and $p > 2 + m$, and hence, due to the Gagliardo–Nirenberg inequality and the Young inequality, we derive that

\[
C_{18} \| (n_\varepsilon + \varepsilon)^{p + m - 1} \|_{L^{\frac{2 + m}{p + m - 1}}(\Omega)} \leq C_{19} \| \nabla (n_\varepsilon + \varepsilon)^{p + m - 1} \|_{L^{2}(\Omega)} \leq C_{20} \left( \| \nabla (n_\varepsilon + \varepsilon)^{p + m - 1} \|_{L^{2}(\Omega)} + 1 \right) \leq \frac{m(p - 1)}{4} \int_{\Omega} (n_\varepsilon + \varepsilon)^{m + p - 3} |\nabla n_\varepsilon|^2 + C_{21} \text{ for all } t \in (0, T_{\max}),
\]

(3.48)

which together with (3.47) and an ODE comparison argument entails that

\[
\| n_\varepsilon(\cdot, t) \|_{L^{p}(\Omega)} \leq C_{22} \text{ for all } t \in (0, T_{\max}) \quad \text{and } p > 2 + m.
\]

(3.49)

**Step 5.** The boundedness of $\| c_\varepsilon(\cdot, t) \|_{W^{1,\infty}(\Omega)}$ for all $t \in (\tau, T_{\max})$ with $\tau \in (0, T_{\max})$

Choosing $\theta \in \left( \frac{1}{2} + \frac{3}{7}, 1 \right)$, then the domain of the fractional power $D((-\Delta + 1)^{\theta}) \hookrightarrow W^{1,\infty}(\Omega)$. Hence, again using the $L^{p}$-$L^{q}$ estimates associated heat semigroup,

\[
\| c_\varepsilon(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq C_{23} \| (-\Delta + 1)^{\theta} c_\varepsilon(\cdot, t) \|_{L^{2}(\Omega)} \leq C_{24} t^{-\theta} e^{-\lambda t} \| c_\varepsilon(\cdot, t) \|_{L^{2}(\Omega)} + C_{24} \int_{0}^{t} (t - s)^{-\theta} e^{-\lambda(t-s)} \| (n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon)(s) \|_{L^{2}(\Omega)} ds \leq C_{25} + C_{25} \int_{0}^{t} (t - s)^{-\theta} e^{-\lambda(t-s)} \| n_\varepsilon(s) \|_{L^{2}(\Omega)} + \| u_\varepsilon(s) \|_{L^{\infty}(\Omega)} \| \nabla c_\varepsilon(s) \|_{L^{2}(\Omega)} ds \leq C_{26} \text{ for all } t \in (\tau, T_{\max})
\]

(3.50)

with $\tau \in (0, T_{\max})$, where we have used (3.49), (3.46), (3.49) as well as the Hölder inequality and

\[
\int_{0}^{t} (t - s)^{-\theta} e^{-\lambda(t-s)} \leq \int_{0}^{\infty} \sigma^{-\theta} e^{-\lambda\sigma} d\sigma < +\infty.
\]

**Step 6.** The boundedness of $\| n_\varepsilon(\cdot, t) \|_{L^{\infty}(\Omega)}$ for all $t \in (0, T_{\max})$ for all $t \in (\tau, T_{\max})$ with $\tau \in (0, T_{\max})$

In view of (3.50) and using the outcome of (3.47) with suitably large $p$ as a starting point, which by means of a Moser-type iteration (see e.g. Lemma A.1 of [27]) applied to the first equation of (2.6) to get that

\[
\| n_\varepsilon(\cdot, t) \|_{L^{\infty}(\Omega)} \leq C_{27} \text{ for all } t \in (\tau, T_{\max})
\]

(3.51)
with \( \tau \in (0, T_{\text{max}}) \).

**Step 7. The boundedness of** \( \|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \) **and** \( \|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \) **for all** \( t \in (0, T_{\text{max}}) \)

In light of (2.9), (3.50) and (3.51), we conclude that

\[
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{28} \text{ for all } t \in (0, T_{\text{max}})
\]

and

\[
\|\nabla c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{29} \text{ for all } t \in (0, T_{\text{max}}).
\]

The proof is complete.

By virtue of (2.8) and Lemma 3.5, the local-in-time solution can be extended to the global-in-time solution.

**Proposition 3.1.** Let \((n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)_{\varepsilon \in (0, 1)}\) be classical solutions of (2.6) constructed in Lemma 2.1 on \([0, T_{\text{max}}]\). Then the solution is global on \([0, \infty)\). Moreover, one can find \( C > 0 \) independent of \( \varepsilon \in (0, 1) \) such that

\[
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, \infty)
\]

as well as

\[
\|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, \infty)
\]

and

\[
\|u_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, \infty).
\]

Moreover, we also have

\[
\|A^\varepsilon u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C \text{ for all } t \in (0, \infty).
\]

**Lemma 3.6.** Let \( m > \frac{4}{3} \). Then one can find \( \mu \in (0, 1) \) such that for some \( C > 0 \)

\[
\|c_\varepsilon(\cdot, t)\|_{C^m(\Omega \times [t, t+1])} \leq C \text{ for all } t \in (0, \infty)
\]

as well as

\[
\|u_\varepsilon(\cdot, t)\|_{C^m(\Omega \times [t, t+1])} \leq C \text{ for all } t \in (0, \infty),
\]

and such that for any \( \tau > 0 \) there exists \( C(\tau) > 0 \) fulfilling

\[
\|\nabla c_\varepsilon(\cdot, t)\|_{C^m(\Omega \times [t, t+1])} \leq C \text{ for all } t \in (\tau, \infty).
\]
Proof. Firstly, let \( g_\varepsilon(x,t) := -c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon \). Then by Proposition 3.1, we derive that \( g_\varepsilon \) is bounded in \( L^\infty(\Omega \times (0,T)) \) for any \( \varepsilon \in (0,1) \), we may invoke the standard parabolic regularity theory to the second equation of (2.6) and infer that (3.58) and (3.60) holds. With the help of the Proposition 3.1 again, performing standard semigroup estimation techniques to the third equation of (2.6), we can finally get (3.59).

4 Passing to the limit

To prepare our subsequent compactness properties of \( (n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon) \) by means of the Aubin-Lions lemma (See e.g. Simon [25]), employing almost exactly the same arguments as in the proof of Lemma 5.1 in [49] (see also Lemmas 3.22–3.23 of [41]), and taking advantage of Proposition 3.1 we conclude the following regularity property with respect to the time variable.

Lemma 4.1. Let \( m > \frac{4}{3} \). Then one can find \( \varepsilon \in (0,1) \) such that for some \( C > 0 \)

\[
\| \partial_t n_\varepsilon(\cdot,t) \|_{(W^{2,2}_0(\Omega))^*} \leq C \quad \text{for all} \quad t \in (0,\infty).
\]

(4.1)

Moreover, let \( \varsigma > \max\{m,2(m-1)\} \). Then for all \( T > 0 \) and \( \varepsilon \in (0,1) \) there exists \( C(T) > 0 \) such that

\[
\int_0^T \| \partial_t n_\varepsilon^\varsigma(\cdot,t) \|_{(W^{3,2}_0(\Omega))^*} dt \leq C(T).
\]

(4.2)

Based on above lemmas and by extracting suitable subsequences in a standard way, we could see the solution of (1.7) is indeed globally solvable. To this end, from the idea of [49] (see also [41] and [19]), we state the solution conception as follows.

Definition 4.1. Let \( T > 0 \) and \((n_0, c_0, u_0)\) fulfills (2.2). Then a triple of functions \((n, c, u)\) is called a weak solution of (1.7) if the following conditions are satisfied

\[
\begin{align*}
n &\in L^1_{\text{loc}}(\bar{\Omega} \times [0,T)), \\
c &\in L^1_{\text{loc}}([0,T); W^{1,1}(\Omega)), \\
u &\in L^1_{\text{loc}}([0,T); W^{1,1}(\Omega)),
\end{align*}
\]

(4.3)
where \( n \geq 0 \) and \( c \geq 0 \) in \( \Omega \times (0,T) \) as well as \( \nabla \cdot u = 0 \) in the distributional sense in \( \Omega \times (0,T) \), moreover,

\[
n^n \text{ belong to } L^1_{\text{loc}}(\bar{\Omega} \times [0,\infty)),
\]

\[
cu, \; nu \; \text{and} \; n\nabla c \text{ belong to } L^1_{\text{loc}}(\bar{\Omega} \times [0,\infty; \mathbb{R}^3])
\]

and

\[
- \int_0^T \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot,0) = \int_0^T \int_\Omega n^m \Delta \varphi + \int_0^T \int_\Omega n\nabla c \cdot \nabla \varphi + \int_0^T \int_\Omega nu \cdot \nabla \varphi
\]

for any \( \varphi \in C^\infty_0(\bar{\Omega} \times [0,T]) \) satisfying \( \frac{\partial \varphi}{\partial \nu} = 0 \) on \( \partial \Omega \times (0,T) \) as well as

\[
- \int_0^T \int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) = - \int_0^T \int_\Omega \nabla c \cdot \nabla \varphi - \int_0^T \int_\Omega n\varphi + \int_0^T \int_\Omega cu \cdot \nabla \varphi
\]

for any \( \varphi \in C^\infty_0(\bar{\Omega} \times [0,T]) \) and

\[
- \int_0^T \int_\Omega w\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = - \int_0^T \int_\Omega \nabla u \cdot \nabla \varphi - \int_0^T \int_\Omega n\nabla \varphi \cdot \varphi
\]

for any \( \varphi \in C^\infty_0(\bar{\Omega} \times [0,T]; \mathbb{R}^3) \) fulfilling \( \nabla \varphi \equiv 0 \) in \( \Omega \times (0,T) \). If \( \Omega \times (0,\infty) \rightarrow \mathbb{R}^5 \) is a weak solution of (1.7) in \( \Omega \times (0,T) \) for all \( T > 0 \), then we call \((n,c,u)\) a global weak solution of (1.7).

With the uniform bounds from Proposition 3.1 and Lemma 3.6 we are now in the position to obtain limit functions \( n, c \) and \( u \), which at least fulfill the regularity assumptions required in Definition 4.1.

**Lemma 4.2.** Assume that \( m > \frac{4}{3} \). Then there exists \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1)\) such that \( \varepsilon_j \rightarrow 0 \) as \( j \rightarrow \infty \) and that

\[
n_\varepsilon \rightarrow n \; \text{a.e. in} \; \Omega \times (0,\infty),
\]

\[
n_\varepsilon \rightharpoonup n \; \text{weakly star in} \; L^\infty(\Omega \times (0,\infty)),
\]

\[
n_\varepsilon \rightarrow n \; \text{in} \; C^0_{\text{loc}}([0,\infty); (W^{2,2}_0(\Omega))^*)
\]

\[
c_\varepsilon \rightarrow c \; \text{in} \; C^0_{\text{loc}}(\bar{\Omega} \times [0,\infty)),
\]

\[
\nabla c_\varepsilon \rightarrow \nabla c \; \text{in} \; C^0_{\text{loc}}(\bar{\Omega} \times [0,\infty))
\]
\[ \nabla c_\varepsilon \to \nabla c \text{ in } L^\infty(\Omega \times (0, \infty)), \quad (4.13) \]
\[ u_\varepsilon \to u \text{ in } C^0_{\text{loc}}(\Omega \times (0, \infty)), \quad (4.14) \]

and
\[ Du_\varepsilon \rightharpoonup Du \text{ weakly in } L^\infty(\Omega \times [0, \infty)) \quad (4.15) \]

with some triple \((n, c, u)\) which is a global weak solution of \((1.7)\) in the sense of Definition 4.1. Moreover, \(n\) satisfies
\[ n \in C^0_{\omega-\varepsilon}([0, \infty); L^\infty(\Omega)). \quad (4.16) \]

**Proof.** Firstly, Proposition 3.1 warrants that for certain \(n \in L^\infty(\Omega \times (0, \infty)), \) \((4.9)\) is valid. Next, the bounds featured in Proposition 3.1, we derive from (3.47) that there exists a positive constant \(C_1 := C_1(T)\) such that
\[ \int_0^T \int_\Omega n^{m+p-3}_\varepsilon |\nabla n_\varepsilon|^2 \leq C_1 \quad (4.17) \]
for any \(p > 1.\) In particular, we choose \(p := 2\zeta - m + 1,\) where \(\zeta > \max\{m, 2(m - 1)\}\) is the same as Lemma 4.1. Therefore, \((4.17)\) asserts that for each \(T > 0, (n^{\varepsilon})_{\varepsilon \in (0,1)}\) is bounded in \(L^2((0,T); W^{1,2}(\Omega)),\) so that combined \((4.2)\) with the Aubin-Lions lemma (see e.g. [29]), we derive that \(n^{\varepsilon}_\varepsilon \to z^\varepsilon\) for some nonnegative measurable \(z : \Omega \times (0, \Omega) \to \mathbb{R}.\) Thus, \((4.9)\) and the Egorov theorem yields to \(z = n\) necessarily, and thereby \((4.8)\) holds. Finally, employing the same arguments as in the proof of Lemma 4.1 in [41] (see also [49]), taking advantage of Proposition 3.1, we can conclude \((4.10)-(4.16),\) where the required equicontinuity property used in the proof, is implied by \((4.1).\) The proof of Lemma 4.2 is completed. \(\Box\)

**Proof of Theorem 2.1** The conclusion in Theorem 2.1 follows from Lemma 4.2 and Proposition 3.1.

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