REGULARITY OF EXTREMAL SOLUTIONS OF SEMILINEAR FOURTH-ORDER ELLIPTIC PROBLEMS WITH GENERAL NONLINEARITIES

A. Aghajani and S. F. Mottaghi

School of Mathematics, Iran University of Science and Technology
Narmak, Tehran, 16846-13114, Iran

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Abstract. We consider the fourth order problem \( \Delta^2 u = \lambda f(u) \) on a general bounded domain \( \Omega \) in \( \mathbb{R}^n \) with the Navier boundary condition \( u = \Delta u = 0 \) on \( \partial \Omega \). Here, \( \lambda \) is a positive parameter and \( f : [0, a_f) \to \mathbb{R}_+ (0 < a_f \leq \infty) \) is a smooth, increasing, convex nonlinearity such that \( f(0) > 0 \) and which blows up at \( a_f \). Let

\[
0 < \tau_\omega := \liminf_{t \to a_f} \frac{f(t)f''(t)}{f'(t)^2} \leq \tau_\omega := \limsup_{t \to a_f} \frac{f(t)f''(t)}{f'(t)^2} < 2.
\]

We show that if \( u_m \) is a sequence of semistable solutions correspond to \( \lambda_m \) satisfy the stability inequality

\[
\sqrt{\lambda_m} \int_\Omega \sqrt{f'(u_m)} \phi^2 dx \leq \int_\Omega |\nabla \phi|^2 dx, \text{ for all } \phi \in H^1_0(\Omega),
\]

then \( \sup_m \|u_m\|_{L^\infty(\Omega)} < a_f \) for \( n < \frac{4\alpha^*(2-\tau_\omega)+2\tau_\omega}{\tau_\omega} \max\{1, \tau_\omega\} \), where \( \alpha^* \) is the largest root of the equation

\[
(2 - \tau_\omega)^2 \alpha^4 - 8(2 - \tau_\omega) \alpha^2 + 4(4 - 3\tau_\omega) \alpha - 4(1 - \tau_\omega) = 0.
\]

In particular, if \( \tau_\omega = \tau_\omega := \tau \), then \( \sup_m \|u_m\|_{L^\infty(\Omega)} < a_f \) for \( n \leq 12 \) when \( \tau \leq 1 \), and for \( n \leq 7 \) when \( \tau \leq 1.57863 \). These estimates lead to the regularity of the corresponding extremal solution \( u^*(x) = \lim_{\lambda \to \lambda^*} u_\lambda(x) \), where \( \lambda^* \) is the extremal parameter of the eigenvalue problem.

1. Introduction. In this article, we consider the problem

\[
\begin{cases}
\Delta^2 u = \lambda f(u) & x \in \Omega, \\
u = \Delta u = 0 & x \in \partial \Omega,
\end{cases}
\tag{N_\lambda}
\]

where \( \Omega \subset \mathbb{R}^n \) is a smooth bounded domain, \( n \geq 1 \), \( \lambda > 0 \) is a real parameter, and the nonlinearity \( f \) satisfies

(H) \( f : [0, a_f) \to \mathbb{R}_+ (0 < a_f \leq \infty) \) is a smooth, increasing, convex function such that \( f(0) > 0 \) and \( \lim_{t \to a_f} f(t) = \infty \). Also, when \( a_f = \infty \) we assume that \( f \) is superlinear, i.e., \( \lim_{t \to \infty} \frac{f(t)}{t} = \infty \).

We call the nonlinearity \( f \) regular if \( a_f = \infty \) and singular when \( a_f < \infty \).

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* Corresponding author: A. Aghajani.
By a semistable solution of $N_\lambda$ we mean a solution $u$ satisfies
\begin{equation}
\int_\Omega (\Delta \varphi)^2 - \int_\Omega \lambda f(u) \varphi^2 \geq 0, \quad \varphi \in H^2(\Omega) \cap H^3_0(\Omega). \tag{1}
\end{equation}
Also, we say that a smooth solution $u$ of $N_\lambda$ is minimal provided $u \leq v$ a.e. in $\Omega$ for any solution $v$ of $N_\lambda$ (see [10, 11]).

When $f$ satisfies (H) is a regular, or $f(t) = (1 - t)^{-p}$ ($p > 1$), it is well known [4, 7, 22] that there exists a finite positive extremal parameter $\lambda^* > 0$ depending on $f$ and $\Omega$ such that for any $0 < \lambda < \lambda^*$, problem $(N_\lambda)$ has a minimal smooth solution $u_\lambda$, which is semistable and unique among the semistable solutions, while no solution exists for $\lambda \geq \lambda^*$. The function $\lambda \to u_\lambda$ is strictly increasing on $(0, \lambda^*)$, the increasing pointwise limit $u^*(x) = \lim_{\lambda \to \lambda^*} u_\lambda(x)$ is called the extremal solution. For $0 < \lambda < \lambda^*$, the minimal solution $u_\lambda$ of problem $(N_\lambda)$ satisfies the following stability inequality, for the proof see Corollary 1 in [11] or Lemma 6.1 in [15],
\begin{equation}
\sqrt{\lambda} \int_\Omega \sqrt{f(u_\lambda)} \phi^2 dx \leq \int_\Omega |\nabla \phi|^2 dx, \tag{2}
\end{equation}
for all $\phi \in H^3_0(\Omega)$.

The regularity and properties of the extremal solutions have been studied extensively in the literature [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 16, 20] and it is shown that it depends strongly on the dimension $n$, domain $\Omega$ and nonlinearity $f$.

Cowan, Esposito and Ghoussoub in [10] showed that for general nonlinearity $f$ satisfies (H), $u^*$ is bounded for $n \leq 5$. When $f(u) = e^u$, in [10] it is shown that $u^*$ is bounded for $n \leq 8$. This result improved by Cowan and Ghoussoub to $n \leq 10$ in [11], and by Dupaigne, Ghergu and Warnault in [15] to $n \leq 12$ which is the optimal dimension as we know on the unit ball $u^*$ is bounded if and only if $n \leq 12$. As we shall see, in this paper we prove the same for a large class of nonlinearities including $e^u$. When $f(u) = (1 + u)^p$ ($p > 1$) in [10] it is proved that $u^*$ is bounded if $n < \frac{8p}{p+1}$ that improved in [11] for to $n < 4h(p) > \frac{8p}{p+1}$ (for the definition of $h(p)$ which is a decreasing function on $(1, \infty)$ see [11]) with $\lim_{p \to \infty} 4h(p) \approx 10.718$. Recently, Hajlaoui, Harrabi and Ye in [23] improved this result by showing that $u^*$ is bounded for any $p > 1$ and $n \leq 12$.

For the singular nonlinearity $f(u) = (1 - u)^{-p}$ ($p > 1$), in [10] it is proved that $\sup_\Omega u^* < 1$ if $n \leq \frac{8p}{p+1}$. In particular, when $p = 2$, $u^*$ is bounded away from 1 for $n \leq 5$. The later result (and also the general case $1 < p \neq 3$) is improved in [11] to $n \leq 6$, and further improved by Guo and Wei in [20] to $n \leq 7$. However, for $p = 2$ the expected optimal dimension is $n = 8$, holds on the ball, see [24].

By imposing extra assumptions on the general nonlinearity $f$ satisfies (H), the authors in [10] obtained more regularity results in higher dimensions on general domains. Let $f$ satisfy (H) and define
\begin{equation}
\tau_- := \liminf_{t \to a^-} \frac{f(t) f''(t)}{f'(t)^2} \leq \tau_+ := \limsup_{t \to a^+} \frac{f(t) f''(t)}{f'(t)^2}. \tag{3}
\end{equation}
In [10] the authors also show that for a regular and superlinear nonlinearity $f$ with $\tau_- > 0$, $u^*$ is bounded for $n \leq 7$ (see [10], Theorem 4.1). As we shall see here in Corollary 2.4, with a minor change in their proof, the same holds with a weaker condition. Also, they showed that if $\tau_+ < \infty$ then $u^*$ is bounded for $n < \frac{8}{\tau_+}$, see Theorem 5.1 in [10].
It is worth mentioning here that the second order analog of \((N_\lambda)\) with Dirichlet boundary conditions, i.e., \(-\Delta u = \lambda f(u)\) in \(\Omega\), with \(u = 0\) on \(\partial\Omega\), is by now quite well understood whenever \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^n\) and \(f\) is a nonlinearity of type (H). See, for instance, [1, 2, 5, 6, 17, 19, 25, 26, 27, 29, 33].

The main results of this paper are as follows.

2. Main results.

**Theorem 2.1.** Let \(f\) satisfy (H) with \(0 < \tau_- \leq \tau_+ < 2\), and \(\Omega\) an arbitrary bounded smooth domain. Also, let \(u_m\) be a sequence of semistable solutions of \((N_{\lambda_m})\) satisfy the stability inequality (2). Then \(\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f\) for

\[
\alpha_*, 1 < \text{denotes the largest root of the polynomial}
\]

\[
P_f(\alpha, \tau_-, \tau_+) := (2 - \tau_-)^2 \alpha^4 - 8(2 - \tau_+) \alpha^2 + 4(4 - 3\tau_+)\alpha - 4(1 - \tau_+).
\]

As a consequence, if \(\tau_- = \tau_+ = \tau\), then \(\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f\) for \(n \leq 12\) when \(\tau \leq 1\), and for \(n \leq 7\) when \(\tau \leq 1.57863\).

**Corollary 1.** Let \(f\) satisfy (H) be a regular nonlinearity with \(0 < \tau_- \leq \tau_+ < 2\) and \(\Omega\) an arbitrary bounded smooth domain. Let \(u^*\) be the extremal solution of problem \((N_\lambda)\). Then \(u^* \in L^\infty(\Omega)\) for

\[
n < \frac{4\alpha_*(2 - \tau_+) + 2\tau_+}{\tau_+} \max\{1, \tau_+\}.
\]

In particular, if \(\tau_- = \tau_+\) then \(u^* \in L^\infty(\Omega)\) for \(n \leq 12\).

For example consider problem \((N_\lambda)\) with \(f(u) = e^u\) or \(e^{\alpha u}\) (\(\alpha > 0\)), then \(\tau_+ = \tau_- = 1\), hence by Theorem 2.1, \(u^* \in L^\infty(\Omega)\) for \(n \leq 12\). The same is true for \(f(u) = (1 + u)^p\) \((p > 1)\) as in this case we have \(\tau_+ = \tau_- = \frac{p+1}{p}\). More precisely we have \(u^* \in L^\infty(\Omega)\) for \(n < \frac{4(p+1)}{p-1}\alpha_* + 2\) where \(\alpha_*\) denotes the largest root of the polynomial

\[
P_f(\alpha) := (p + 1)^2 \alpha^4 - 8p(p + 1)\alpha^2 + 4p(p + 3)\alpha - 4p.
\]

This is exactly the same as the result obtained by Hajlaoui-Harrabi-Ye in [23].

Now consider problem \((N_\lambda)\) with the singular nonlinearity \(f(u) = (1 - u)^{-p}\) \((p > 1)\) and \(\Omega\) an arbitrary bounded smooth domain. Then from the fact that \(\tau_+ = \tau_- = \frac{p+1}{p}\) and Theorem 2.1, we get \(\|u^*\|_{L^\infty(\Omega)} < 1\) for \(n < 4\alpha_* + 2\), where \(\alpha_*\) denotes the largest root of the polynomial

\[
P_f(\alpha) := \alpha^4 - 8 \frac{p(p-1)}{(p+1)^2} \alpha^2 + 4p(p-1)(p-3)\frac{\alpha}{(p+1)^3} + 4 \frac{p(p-1)^2}{(p+1)^5}.
\]

This results coincides with that of Guo-Wei [22]. In particular, when \(p > 1.72822\) then \(\|u^*\|_{L^\infty(\Omega)} < 1\) for \(n \leq 7\). Also, when \(p > 2.2609\) the same is true for \(n \leq 8\).
3. Preliminaries and auxiliary results. The following standard regularity result is taken from [12], for the proof see Theorem 3 of [28].

**Proposition 1.** Let \( u \in H^1_0(\Omega) \) be a weak solution of
\[
\begin{cases}
\Delta u + c(x)u = g(x) & x \in \Omega, \\
u = 0 & x \in \partial \Omega,
\end{cases}
\]
with \( c, g \in L^q(\Omega) \) for some \( q > \frac{n}{2} \). Then there exists a positive constant \( C \) independent of \( u \) such that:
\[
||u||_{L^\infty(\Omega)} \leq C(||u||_{L^1(\Omega)} + ||g||_{L^p(\Omega)}).
\]

Consider problem \((N_\lambda)\). By the elliptic regularity we know that, if for some \( q \geq 1 \) we have \( ||f(u_\lambda)||_{L^q(\Omega)} \leq C \), where \( C \) is a constant independent of \( \lambda \), then \( u^* \) is bounded, (hence smooth when \( f \) is regular), whenever \( n < 4q \). Using the above proposition we show that, a similar result holds (for regular or singular nonlinearity) if \( f'(u_\lambda) \) is uniformly bounded in \( L^q(\Omega) \). For the proof we need the following two lemmas, the first one gives pointwise estimate on \( \Delta u \) for a solution \( u \) of problem \((N_\lambda)\), for the proof see [10].

Define the functions \( F, g, \tilde{f} : [0, a_f) \rightarrow \mathbb{R} \) as
\[
F(t) = \int_0^t f(s)ds, \quad g(t) = \sqrt{2}(F(t) - t)^{\frac{1}{2}} \quad \text{and} \quad \tilde{f}(t) = f(t) - f(0), \quad 0 \leq t < a_f.
\]

**Lemma 3.1 ([10]).** Let \( u \) be a solution of problem \((N_\lambda)\). Then
\[\Delta u \geq \sqrt{\lambda}g(u), \quad \text{in} \ \Omega.\]

**Lemma 3.2.** Let \( u \) be a semistable solution of problem problem \((N_\lambda)\) with \( f \) satisfy (H). Then
\[
\int_{\Omega} -\Delta u dx < C,
\]
where \( C \) is a constant independent of \( u \).

**Proof.** Let \( \psi \) be the unique positive smooth function such that
\[
\begin{cases}
-\Delta \psi = 1 & x \in \Omega, \\
\psi = 0 & x \in \partial \Omega,
\end{cases}
\]
Let \( u \) be a semistable solution of problem \((N_\lambda)\). By multiplying the equation \( \Delta^2 u = \lambda f(u) \) in \( \psi \) and then an integration we get (using Green’s formula)
\[
\lambda \int_{\Omega} \psi(x)f(u)dx = \int_{\Omega} \psi(x)\Delta^2 u dx = \int_{\Omega} \Delta \psi(x)\Delta u dx = \int_{\Omega} -\Delta u dx.
\]
This gives that
\[
\int_{\Omega} -\Delta u dx \leq \lambda \max_{\Omega} \psi(x) \int_{\Omega} f(u)dx.
\]
The inequality above and the uniform \( L^1(\Omega) \) boundedness of \( f(u) \) for semistable solutions (proved in Lemma 3.5 in [10]) gives the desired result.

In the sequel we will frequently use the following simple lemma.

**Lemma 3.3.** Let \( g_1, g_2 : [0, a_f) \rightarrow [0, \infty) \) be continuous functions such that for some \( T \in (0, a_f) \) we have \( g_2(t) \leq g_1(t), \quad T \leq t < a_f \). If for a sequence \( u_m \) of solutions of problem \( N_{\lambda m} \) we have
\[
\int_{\Omega} g_1(u_m)dx \leq C,
\]
where \( C \) is a constant independent of \( u_m \), then the same holds for \( \int_\Omega g_2(u_m)dx \).

**Proof.** Indeed, we have
\[
\int_\Omega g_2(u_m)dx = \int_{u_m \leq T} g_2(u_m)dx + \int_{u_m > T} g_2(u_m)dx \leq M|\Omega| + \int_{\Omega} g_1(u_m)dx \leq M|\Omega| + C, \text{ where } M := \sup_{[0,T]} g_2(t).
\]
\[
\]

**Proposition 2.** Let \( f \) satisfy (H) (when \( f \) is singular we additionally assume that \( \lim_{t \to a_f} F(t) = \infty \)). Let \( u_m \) be a sequence of semistable solutions of problem \((N_{\lambda_m})\).

If \( \sup_m \|\frac{f(u_m)}{\sqrt{F(u_m)}}\|_{L^q(\Omega)} < \infty \), for some \( q \geq 1 \), then
\[
\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f, \quad (11)
\]
for \( n < 2q \). In particular, if \( \sup_m \|f'(u_m)\|_{L^q(\Omega)} < \infty \), then (11) holds for \( n < 4q \).

**Proof.** Take \( v_m := -\Delta u_m \), then from \((N_\lambda)\), \( v_m \) satisfies
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\Delta v_m + \lambda_m f(u_m) = 0 & \quad x \in \Omega, \\
v_m = 0 & \quad x \in \partial \Omega.
\end{array} \right.
\end{aligned}
\]
(12)

We rewrite problem (12) as \( \Delta v_m + c(x)v_m = -\lambda_m f(0) \) where \( c_m(x) := \lambda_m \hat{f}(u_m) \).

By the pointwise estimate in Lemma 3.1 we have
\[
0 \leq c_m(x) = \lambda_m \hat{f}(u_m) / v_m \leq \sqrt{\lambda_m \hat{f}(u_m) / g(u_m)}.
\]

Now using the inequality
\[
0 \leq \frac{\hat{f}(t)}{g(t)} \leq \sqrt{\frac{2}{\sqrt{F(t)}}}, \quad t > T, \text{ for some } T < a_f,
\]
which comes from the fact that \( \lim_{t \to a_f} \frac{g(t)}{\sqrt{F(t)}} = 1 \), and the assumptions, we get \( \sup_m \|c_m(x)\|_{L^q(\Omega)} < \infty \). Thus, by the assumptions, Lemma 3.2 and Proposition 1, \( \|v_m\|_{L^\infty(\Omega)} \leq C \), and hence \( \|F(u_m)\|_{L^\infty(\Omega)} \leq C \) (by Lemma 3.1), where \( C \) is a constant independent of \( m \), for \( n < 2q \). Now the fact that \( \lim_{t \to a_f} F(t) = \infty \) gives the first part. To prove the second part, it suffices to use Lemma 3.3 and note that by the convexity of \( f \), we have
\[
\frac{\hat{f}(t)}{\sqrt{F(t)}} \leq 2\sqrt{f'(t)}, \text{ for } t \text{ sufficiently close to } a_f.
\]
(13)

Indeed, \( f' \) is a nondecreasing function by the convexity of \( f \), thus we have, for \( 0 < t < a_f \)
\[
f'(t)F(t) = f'(t) \int_0^t f(s)ds \geq \int_0^t f'(s)f(s)ds = \frac{f(t)^2}{2} - \frac{f(0)^2}{2},
\]
now the fact that \( f(t) \to \infty \) as \( t \to a_f \) gives (13).

**Remark 1.** The condition \( \lim_{t \to a_f} F(t) = \infty \) in the above proposition, which is needed for a singular nonlinearity \( f \), is satisfied by the extra assumption that \( \tau_+ < 2 \). Indeed, for a \( \tau \in (\tau_+, 2) \) there exists \( T \in (0, a_f) \) such that \( \frac{f'(t)}{F(t)} \leq \)
\[ \tau f'(t) \] for \( t \in (T, a_f) \), thus by an integration we get \( f(t) \leq C f(t)^\tau \) or equivalently \( f'(t)f(t)^{1-\tau} \leq C f(t) \) for \( t \in (T, a_f) \). Again an integration gives

\[
\frac{f(t)^{2-\tau}}{2-\tau} - \frac{f(T)^{2-\tau}}{2-\tau} \leq C(F(t) - F(T)), \quad t \in (T, a_f). 
\]

Now the facts that \( \lim_{t \to a_f} f(t) = \infty \) and \( \tau < 2 \) imply that \( \lim_{t \to a_f} F(t) = \infty \).

For example, take the singular nonlinearity \( f(t) = (1 - t)^{-p} \) \( (p > 1) \) on \( [0, 1) \). We have \( \tau = \frac{p+1}{p} \in (0, 2) \) and

\[
F(t) = \frac{1}{p-1} \left( \frac{1}{(1-t)^{p-1}} - 1 \right) \to \infty, \quad t \to 1.
\]

Then, as a corollary of Proposition 2, we have the next regularity result for problem \((N_\lambda)\). It is proved in [10, 11] by a different proof with the restriction that \( p \neq 3 \).

**Proposition 3.** Let \( f(u) = (1 - u)^{-p} \) \( (p > 1) \) and \( u_m \) be a sequence of semistable solutions of problem \((P_{\lambda m})\), such that for some \( q > 1 \) and \( q \geq \frac{(p+1)n}{4p} \) so that \( \sup_m ||f(u_m)||_{L^q(\Omega)} < \infty \). Then \( \sup_m ||u_m||_{L^\infty(\Omega)} < 1 \).

**Proof.** Notice that we have

\[
f'(t) = p(1-t)^{(p+1)} = f(t)^{\frac{p+1}{p}}, \quad t \in [0, 1).
\]

Hence, by the assumption \( \sup_m ||f'(u_m)||_{L^p(\Omega)} < \infty \). \( \square \)

As an application of Proposition 2, consider problem \((N_\lambda)\) with a convex nonlinearity \( f \) satisfies \((H)\) such that \( f(t) = t \ln t \) for \( t \) large. Then, for every \( \epsilon > 0 \) there exist \( T_\epsilon, C_\epsilon > 0 \) such that \( f'(t) \leq f(t)^\epsilon \) for \( t \geq T_\epsilon \).

\[
f'(t) \leq f(t)^\epsilon \quad \text{for} \quad t \geq T_\epsilon. \tag{14}
\]

Now if \( u \geq 0 \) is a semistable solution of problem \((N_\lambda)\), from Lemma 3.5 [10] we have \( \int_{\Omega} f(u)dx \leq C \) with \( C \) independent of \( \lambda \) and \( u \). This together (14) and Lemma 3.3 give \( f'(t) \in L^\frac{1}{\epsilon}(\Omega) \) uniformly, hence by Proposition 2, \( u^* \) is bounded for \( n < \frac{4}{\epsilon} \), and since \( \epsilon > 0 \) was arbitrary, \( u^* \) is bounded in every dimension \( n \). Indeed, the same result is true for every regular nonlinearity \( f \) satisfies \((H)\) with \( \tau_+ = 0 \) or equivalently

\[
\lim_{t \to \infty} \frac{f(t)f''(t)}{f'(t)^2} = 0. \tag{15}
\]

Indeed, (15) implies (14) and we can proceed as above.

The following lemma is a special case of an interesting result of [10].

**Lemma 3.4.** Let \( u \) be a semistable solution of problem \((N_\lambda)\). If \( H(t) := \int_0^t f''(s)\sqrt{F(s)}ds \) for \( t \geq 0 \). Then

\[
\int_{\Omega} \sqrt{F(u)}H(u)dx \leq C, \quad \tag{16}
\]

where \( C \) is a constant independent of \( \lambda \) and \( u \).

When \( f \) is regular, in [10] the authors used the above lemma to prove that \( u^* \) is bounded for \( n < \frac{2}{\tau_+} \). In a completely similar manner and using Proposition 2, we can prove a similar result when \( f \) is singular.
Lemma 3.5. Let $f$ satisfy (H) be a singular nonlinearity with $0 < \tau_+ < 2$, and $u_m$ be a sequence of semistable solutions of problem $(N_{\lambda_m})$. Then
\[
\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f, \tag{17}
\]
for $n < \frac{8}{\tau_+}$.

Proof. Take an arbitrary number $\tau > \tau_+$, then from the definition of $\tau_+$ there exists a $T_1 \in (0, a_f)$ such that $\frac{f(t)f''(t)}{f'(t)^2} \leq \tau$, $T_1 \leq t < a_f$, which is equivalent to \( \frac{d}{dt}(\frac{f'(t)}{f(t)^{\frac{3}{2}}}) \leq 0 \) for $T_1 \leq t < a_f$. This gives $f'(t) \leq C_0 f(t)^{\tau}$ for $T_1 \leq t < a_f$. Hence, using the inequality (13), $F(t) \geq C_1 f'(t)^{\frac{2}{3}} - 1$, $T_1 \leq t < a_f$, for some $T_2 \in (T_1, a_f)$. Thus, for a $T > T_2$ sufficiently close to $a_f$ we have
\[
\sqrt{F(t)}H(t) \geq C_2 f'(t)^{\frac{2}{3}} - 1 \int_{T_2}^{t} f''(s)f'(s)^{\frac{2}{3}} - \frac{2}{3} ds
\]
\[
\geq C_3 f'(t)^{\frac{2}{3}}, \text{ for } t > T \text{ sufficiently close to } a_f.
\]
Using the inequality above, Lemma 3.3 and Lemma 3.4, we have \( \|f'(u_m)\|_{L^\frac{2}{\tau} (\Omega)} \leq C \). Hence by Remark 1 and Proposition 2, \( \sup_m \|u_m\|_{L^\infty(\Omega)} < a_f \) for $n < \frac{8}{\tau_+}$, and since $\tau > \tau_+$ was arbitrary we get (17).

As we have mentioned before, another main result of [10] is that if $\tau_- > 0$ then $u^*$ is bounded for $n \leq 7$. Using the same proof of this in [10] we can prove it by a weaker assumption as follows:

Corollary 2. Consider problem $(N_{\lambda})$ with a regular nonlinearity $f$ satisfies (H) such that for some $0 \leq \epsilon < 1$
\[
\liminf_{t \to \infty} \frac{f(t)^{\frac{1}{2} + \epsilon} f''(t)}{f'(t)^2} > 0. \tag{18}
\]
Then $u^*$ is bounded for $n \leq 7$.

Proof. From (18) we have $f(t)^{\frac{1}{2} + \epsilon} f''(t) \geq C_0 f'(t)^2$, $t \geq T$, for some $T > 0$. Hence, using the inequality (13) and the fact that $F$ is a nondecreasing function we get, for a $T' > T$ sufficiently large,
\[
\sqrt{F(t)}H(t) \geq C_1 \int_{0}^{t} f''(s)F(s)ds \geq C_2 \int_{T}^{t} f''(s)f(s)^{\frac{2}{3}} \geq C_3 \int_{T}^{t} f'(s)f(s)^{\frac{1}{3} - \frac{2}{3}} \geq C_4 f'(t)^{2 - \frac{\epsilon}{2}} \text{ for } t > T'.
\]
Thus, from Lemmas 3.3 and 3.4 we have \( \|f(u)\|_{L^\frac{2}{\tau_-} (\Omega)} < C \) where $C$ is independent of $u$. Now the elliptic regularity implies $u^*$ is bounded for $n \leq 8 - \epsilon > 7$, that gives the desired result.

4. Proof of the main results. Following the idea of Dupaigne, Ghergu and War- nault in [11], we prove the following lemma which is crucial for the proof of the main results.

Lemma 4.1. Let $u$ be a positive smooth solution of $(N_{\lambda})$ satisfy the stability inequality (2), $\theta : [0, a_f) \to [0, \infty)$ a $C^1$ positive function with $\theta(0) = 0$, and $\Theta(t) := \int_{0}^{t} \theta'(s)^2 ds$, for $0 \leq t < a_f$. Then for every $\alpha > \frac{1}{2}$ we have
\[
\int_{\Omega} \sqrt{f(u)} \theta(u)^2 dx \leq \alpha^2 \int_{\Omega} f(u)^{\frac{2a}{\alpha}} dx \left( \int_{\Omega} \frac{\theta(u)^{\frac{2a}{\alpha} - \frac{2}{3}}}{f(u)^{\frac{2a}{\alpha} - \frac{2}{3}}} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \frac{\Theta(u)^{\frac{2a}{\alpha} - \frac{2}{3}}}{f(u)^{\frac{2a}{\alpha} - \frac{2}{3}}} dx \right)^{\frac{1}{2} - \frac{1}{2a}}. \tag{19}
\]
Proof. Let \( u \) be a positive smooth solution of \((N_\lambda)\) satisfy \((2)\) and set \( v := -\Delta u \). Up to rescaling, we may assume that \( \lambda = 1 \). Take \( \phi = \theta(u) \) as a test function in the stability inequality \((2)\). Then we get
\[
\int_\Omega \sqrt{f'(u)} \theta(u)^2 dx \leq \int_\Omega v \Theta(u) dx. \tag{20}
\]
Also, taking \( \phi = v^\alpha \ (\alpha > \frac{1}{2}) \) as a test function in the stability inequality \((2)\), we get
\[
\int_\Omega \sqrt{f'(u)} v^{2\alpha} dx \leq \frac{\alpha^2}{2\alpha - 1} \int_\Omega f(u) v^{2\alpha - 1} dx. \tag{21}
\]
Using Hölder inequality (with two conjugate numbers \(2\alpha\) and \(\frac{2\alpha}{2\alpha - 1}\)) on the right-hand side of inequality \((20)\) we get
\[
\int_\Omega \sqrt{f'(u)} \theta(u)^2 dx \leq \left( \int_\Omega f'(u) v^2 dx \right)^{\frac{2\alpha}{2\alpha - 1}} \left( \int_\Omega \Theta(u) f^{\frac{2\alpha}{2\alpha - 1}} dx \right)^{\frac{2\alpha - 1}{2\alpha - 1}}. \tag{22}
\]
Similarly, from \((21)\) and Hölder inequality we get
\[
\int_\Omega \sqrt{f'(u)} v^{2\alpha} dx \leq \frac{\alpha^2}{2\alpha - 1} \left( \int_\Omega f'(u) v^{2\alpha} dx \right)^{\frac{2\alpha - 1}{2\alpha - 1}} \left( \int_\Omega f(u) v^{\alpha - \frac{1}{2}} dx \right)^{\frac{1}{\alpha}}, \tag{23}
\]
that gives
\[
\int_\Omega \sqrt{f'(u)} \theta(u)^2 dx \leq \left( \frac{\alpha^2}{2\alpha - 1} \right)^{2\alpha} \int_\Omega f(u)^{2\alpha - \frac{1}{2}} dx.
\]
Plugging \((23)\) in \((22)\) we arrive at
\[
\int_\Omega \sqrt{f'(u)} \theta(u)^2 dx \leq \frac{\alpha^2}{2\alpha - 1} \left( \int_\Omega f(u)^{2\alpha} dx \right)^{\frac{1}{2\alpha}} \left( \int_\Omega \Theta(u) f^{\frac{2\alpha}{2\alpha - 1}} dx \right)^{\frac{2\alpha - 1}{2\alpha - 1}},
\]
which is the desired result. \(\square\)

**Proof of Theorem 2.1.** Fix an \( \alpha > 1 \) such that \( P_f(\alpha, \tau_-, \tau_+) < 0 \). Such an \( \alpha \) exists since we have \( P_f(1, \tau_-, \tau_+) = (2 - \tau_-)^2 - 4 < 0 \) and \( P_f(+\infty, \tau_-, \tau_+) = +\infty \). Now take positive numbers \( \tau_1 \in (0, \tau_-) \) and \( \tau_2 \in (\tau_+, 2) \) such that
\[
P_f(\alpha, \tau_1, \tau_2) < 0. \tag{24}
\]
We claim that
\[
I_m := \int_\Omega \frac{\dot{f}(u_m)^{2\alpha}}{f'(u_m)^{\alpha - \frac{1}{2}}} dx < C, \tag{25}
\]
where \( C \) is independent of \( m \). To this end, take \( \theta(t) = \frac{\dot{f}(t)^{\alpha}}{f'(t)^{\frac{\alpha}{2}}} \) in the inequality \((19)\).

First we estimate the function \( \Theta(t) = \int_0^t \dot{\theta}'(s)^2 ds \) as follows. We have
\[
\Theta(t) = \alpha^2 \int_0^t \dot{f}(s)^{2\alpha - 2} f'(s)^{2 - \alpha} \left( 1 - \frac{\dot{f}(s) f''(s)}{2f'(s)^2} \right)^2 ds. \tag{26}
\]
By the definitions of \( \tau_\pm \) there exists a \( T < a_f \) such that \( \tau_1 \leq \frac{\dot{f}(t) f''(t)}{f'(t)^2} \leq \tau_2 \) for \( T \leq t < a_f \) that also gives
\[
0 < 1 - \frac{\tau_1}{2} \leq \frac{\dot{f}(t) f''(t)}{2f'(t)^2} \leq 1 - \frac{\tau_2}{2}, \text{ for } T \leq t < a_f. \tag{27}
\]
Using (27) in (26) we get
\[ \Theta(t) \leq \Theta(T) + \alpha^2(1 - \frac{\tau_1}{2})^2 \int_T^t \tilde{f}(s)^{2\alpha-2} f'(s)^{2-\alpha} ds, \text{ for } T \leq t < a_f. \] (28)

Now, notice that taking \( h(t) := \hat{f}(t)^{2\alpha-1} f'(t)^{1-\alpha} \) for \( 0 \leq t < a_f \), then
\[ h'(t) = (2\alpha - 1) \hat{f}(t)^{2\alpha-2} f'(t)^{2-\alpha} \left( 1 - \frac{\alpha - 1}{2\alpha - 1} \frac{\hat{f}(s)^{\alpha}(s)}{f'(s)^2} \right) \]
\[ \geq (2\alpha - 1) (1 - \frac{\alpha - 1}{2\alpha - 1} \tau_2) \hat{f}(t)^{2\alpha-2} f'(t)^{2-\alpha}, \text{ for } T \leq t < a_f. \]

Using the above inequality in (28) we obtain
\[ \Theta(t) \leq C + A \hat{f}(t)^{2\alpha-1} f'(t)^{1-\alpha}, \] (29)
where \( A := \frac{\alpha^2}{(2\alpha - 1)(1 - \frac{\alpha - 1}{2\alpha - 1} \tau_2)} \) and \( C := \Theta(T) - Ah(T) \). Note that in the above we also used that \( 1 - \frac{\alpha - 1}{2\alpha - 1} \tau_2 > 0 \) which holds since \( \tau_2 < 2 \). Now, the fact that
the inequality \( \frac{\hat{f}(t)f'(t)}{\hat{f}(t)^{\alpha} f'(t)^{\alpha}} \leq \tau_2 \) for \( T \leq t < a_f \) is equivalent to \( \frac{d}{dt}(\hat{f}(t)^{\alpha} f'(t)^{\alpha}) \leq 0 \) for \( T \leq t < a_f \) gives
\[ f'(t) \leq C_1 \hat{f}(t)^{\tau_2} \text{ for } T \leq t < a_f. \] (30)

Using this we obtain, for \( T \leq t < a_f \)
\[ \hat{f}(t)^{2\alpha-1} f'(t)^{1-\alpha} \geq f'(t)^{2\alpha-1-(\alpha-1)} \to \infty, \text{ as } t \to a_f. \]

Now take an \( \epsilon > 0 \). From the inequality above and (29), there exists an \( M_\epsilon \in [T, a_f) \) such that
\[ \Theta(t) \leq (A + \epsilon) \hat{f}(t)^{2\alpha-1} f'(t)^{1-\alpha}, \text{ for } t \in [M_\epsilon, a_f). \] (31)

Hence,
\[ \frac{\Theta(t)}{f'(t)^{2\alpha-1}} \leq (A + \epsilon) \frac{2\alpha}{\tau_2} \frac{\hat{f}(t)^{2\alpha}}{f'(t)^{\alpha}}, \text{ for } t \in [M_\epsilon, a_f). \] (32)

Also, we can find an \( M'_\epsilon > 0 \) such that
\[ f(t) \leq (1 + \epsilon) \hat{f}(t), \text{ for } t \in [M'_\epsilon, a_f). \] (33)

Now, taking \( M''_\epsilon := \max\{M_\epsilon, M'_\epsilon\} \), then plugging (33), (32) in (19) we arrive at
\[ I_m = \int_\Omega \frac{\hat{f}(u_m)^{2\alpha}}{f'(u_m)^{\alpha-\frac{1}{2}}} dx \]
\[ \leq \frac{\alpha^2}{2\alpha - 1} \left( C_{\epsilon,m} + (1 + \epsilon)^{2\alpha} \int_{u_m \geq M''_\epsilon} \frac{\hat{f}(u_m)^{2\alpha}}{f'(u_m)^{\alpha-\frac{1}{2}}} dx \right)^{\frac{1}{2\alpha}} \]
\[ \left( C'_{\epsilon,m} + (A + \epsilon) \frac{2\alpha}{\tau_2} \int_{u_m \geq M''_\epsilon} \frac{\hat{f}(u_m)^{2\alpha}}{f'(u_m)^{\alpha-\frac{1}{2}}} dx \right)^{\frac{2\alpha-1}{2\alpha}}, \]
where
\[ C_{\epsilon,m} := \int_{u < M'_\epsilon} \frac{\hat{f}(u_m)^{2\alpha}}{f'(u_m)^{\alpha-\frac{1}{2}}} dx, \text{ and } C'_{\epsilon,m} := \int_{u_m < M''_\epsilon} \frac{\Theta(u_m)^{2\alpha}}{f'(u_m)^{2\alpha-1}} dx. \]
Note that $C_{c,m}$ and $C'_{c,m}$ are bounded by a constant independent of $m$. Replacing the integrals on the right-hand side of the above inequality with integrals over the full region $\Omega$ we get

$$I_m \leq \frac{\alpha^2}{2\alpha - 1} \left( C_{c,m} + (1 + \epsilon)^{2\alpha} I_m \right)^{\frac{1}{2\alpha}} \left( C'_{c,m} + (A + \epsilon) \frac{2\alpha - 1}{2\alpha - 1} I_m \right)^{\frac{2\alpha - 1}{2\alpha}}.$$  \hfill (34)

Now if (25) does not hold then, $I_m \to \infty$ as $m \to \infty$. Hence, dividing two sides of (34) by $I_m$ and letting $m \to \infty$, we must have

$$1 \leq \frac{\alpha^2}{2\alpha - 1} (1 + \epsilon)(A + \epsilon),$$

and since $\epsilon > 0$ was arbitrary we get

$$1 \leq \frac{\alpha^2}{2\alpha - 1} A = \frac{\alpha^4}{(2\alpha - 1)^2} \frac{(1 - \frac{7}{2})^2}{1 - (\frac{7}{2\alpha - 1})^2},$$

which is equivalent to $P_f(\alpha, \tau_1, \tau_2) \geq 0$, a contradiction, that proves (25). Now, using inequality (30) we have

$$\frac{\hat{f}(t)^{2\alpha}}{f'(t)^{\alpha - \frac{\alpha}{2}}} \geq C_2 f'(t)^{\alpha(\frac{2}{\tau_2} - 1) + \frac{\alpha}{2}}, \text{ for } T \leq t < a_f,$$

hence, thanks to Lemma 3.3 and (25) we get

$$\|f(u_m)\|_{L^{q_1}(\Omega)} \leq C, \text{ where } q_1 := \alpha(\frac{2}{\tau_2} - 1) + \frac{1}{2},$$

and $C$ is a constant independent of $m$. Now Proposition 2 implies that

$$\sup_m \|u_m\|_{L^{q_1}(\Omega)} < a_f, \text{ for } n < 4\alpha(\frac{2}{\tau_2} - 1) + 2.$$  \hfill (35)

Again from inequality (30) we have

$$\frac{\hat{f}(t)^{2\alpha}}{f'(t)^{\alpha - \frac{\alpha}{2}}} \geq C_3 f(t)^{\alpha(2 - \tau_2) + \frac{\alpha}{2}}, \text{ for } T \leq t < a_f,$$

and using the above inequality, Lemma 3.3 and (25) we get

$$\|f(u_m)\|_{L^{q_2}(\Omega)} \leq C, \text{ where } q_2 := \alpha(2 - \tau_2) + \frac{\tau_2}{2},$$

and $C$ is a constant independent of $m$. Hence, in the case when $f$ is regular, by the elliptic regularity theory

$$\sup_m \|u_m\|_{L^{q_2}(\Omega)} < \infty, \text{ for } n < 4\alpha(2 - \tau_2) + 2\tau_2.$$  \hfill (36)

Now, since we can choose $\tau_2$ arbitrary close to $\tau_+ \alpha$ near to the largest root of the polynomial $P_f$, then (35) and (36) complete the proof of the first part.

To see the second part, suppose that $\tau_\ast := \tau_\ast > 0$. If $\tau < \frac{7}{2}$ then from Lemma 3.5, $\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f$ for $n \leq \frac{5}{2} > 12$, so we need to prove it for the case $\frac{5}{2} \leq \tau \leq 1$. It is not hard to see (for example by using a computing device) that, for $\alpha = \frac{5\tau}{2(2 - \tau)}$ we have $P_f(\alpha, \tau, \tau) < 0$ on the interval $[\frac{57}{3}, 1]$, hence $\alpha^* > \frac{5\tau}{2(2 - \tau)}$ that gives

$$\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f, \text{ for } n < 4\alpha^*(\frac{2}{\tau} - 1) + 2 > 12.$$
Also, when $1 \leq \tau \leq 1.57863$ then for $\alpha = \frac{5\tau}{4(2-\tau)}$ we have $P_f(\alpha, \tau, \tau) < 0$ on the interval $\left[\frac{3}{4}, 1\right]$, hence $\alpha^* > \frac{5\tau}{4(2-\tau)}$ that gives
\[
\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f, \text{ for } n < 4\alpha^*(\frac{2}{\tau} - 1) + 2 > 7,
\]
and now the proof is complete. \( \square \)

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*E-mail address*: aghajani@iust.ac.ir
*E-mail address*: mottaghi@mathdep.iust.ac.ir