Siegel disk for complexified Hénon map

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Abstract

It is shown that critical phenomena associated with Siegel disk, intrinsic to 1D complex analytical maps, survives in 2D complex invertible dissipative Hénon map. Special numerical method of estimation of the Siegel disk scaling center position (for 1D maps it corresponds to extremum) for multi-dimensional invertible maps are developed.

It is known that complexification of real 1D logistic map

\[ z_{n+1} = f(z_n) = \lambda - z_n^2, \]  

(1)

where \( \lambda, z \in \mathbb{C} \) leads to the origination of the Mandelbrot set at the complex parameter \( \lambda \) plane \([1]\) and a number of other accompanying phenomena.

Opportunity for realization of the phenomena, characteristic for the dynamics of complex maps (Mandelbrot and Julia sets etc.) at the physical systems seems to be interesting problem \([2, 3, 4, 5, 6]\). In the context of this problem the following question is meaning: Does phenomena of dynamics of the 1D complex maps (like classic Mandelbrot map (1)) survive for the more realistic (from the point of view of possible physical applications) model – two-dimensional maps invertible in time. For example, more realistic model rather than logistic map, is the Hénon map

\[ z_{n+1} = f(z_n, w_n) = \lambda - z_n^2 - d \cdot w_n, \quad w_{n+1} = g(z_n, w_n) = z_n. \]  

(2)

In the real variable case the system (2) is 2D invertible map and, hence, can be realized as Poincare cross-section of flow system with three dimensional phase space – minimal dimension, providing opportunity of nontrivial dynamics and chaos. Hénon map is suitable for modelling of the chaotic dynamics of the generator with non-inertial nonlinearity, dissipative oscillator and rotator with periodic impulse internal force etc. \([7]\). Moreover, Hénon map expresses the principal properties of large class of differential systems.

Let us complexify the map (2) in a such way that \( z, w, \lambda \in \mathbb{C}, d \in \mathbb{R} \). According to the work \([3, 5]\), complexified Hénon map can be reduced to the two symmetrically coupled real Hénon maps and can be realized at the physical experiment \([4]\).
Let us remark that with $|d| < 1$ the Hénon map is dissipative system, with $|d| \to 1$ – it is area-preserving map, and with $d \to 0$ it corresponds to complex 1D quadratic map (1).

In previous work [8] we have investigated intrinsic to 1D complex maps special scenario of transition to chaos through the period multiplication (for example period-tripling) bifurcation cascades and found that accumulation points of these bifurcations survives for the Hénon map with $1 > |d| \neq 0$.

In present work we aim to be convinced of existence of one more interesting critical phenomena – so-called Siegel disk for the Hénon map. This special type of critical dynamics of the one-dimensional complex analytical maps corresponds to a stability loss of the fixed point in case of an irrational winding number, that is an irrational phase of a complex multiplier with unit modulo $\mu = e^{2\pi i \varphi}$ (for example equal to "golden mean" $\varphi = (\sqrt{5} - 1)/2$). Siegel disk – is a domain inside Julia set at the phase plane, filled with the invariant quasiperiodic trajectories rotating around of a neutral fixed point.

Implementation of domain of rotation about the fixed point is connected with existence of local variable change $z = h(\omega)$ – conjugacy function, which conjugate $f$ with irrational rotation $\omega \to \omega \mu$ near to the fixed point so, that

$$f(h(\omega)) = h(\mu \omega). \quad (3)$$

This equation, known as Schröder equation, means, that each iteration of $f$ is equivalent to rotation of new variable $\omega$ on an angle $2\pi \varphi$. Conjugacy function (smooth differentiable) exists not at every value of complex variable. The domain of its existence corresponds to Siegel disk. The boundary of this disk is described by the special fractal quasiperiodic trajectory started from an extremum of map (images and pre-images of an extremum are distributed on this boundary dense everywhere).

Really, let us take derivative of the Schröder equation with regard for $h(\omega) = z$

$$f'(z)h'(\omega) = \mu h'(\mu \omega). \quad (4)$$

Let’s consider a critical value of the variable $f'(z_c) = 0$. From last expression it is evident, that for this value (and also for its images and pre-images) conjugacy function should have a singularity – its derivatives should turn to zero or infinity.. Besides it is necessary to note, that the extremum is the point of a Siegel disk mostly distant from the fixed point.

At Fig 1 the phase plane for map (1) with a value of parameter

$$\lambda_{\text{Siegel}} = 0.3905409 + 0.5867879i, \quad (5)$$

relevant to existence of an irrational neutral fixed point with a "golden mean" winding number is represented. Several invariant curves and boundary of a Siegel disk are shown. In a neighbourhood of the fixed point invariant curves are close to circles. Invariant curves more distant from the fixed point are more distorted. The boundary of a disk, defined by a trajectory, started from an extremum is a fractal curve.
Renormalization group analysis of dynamics at the Siegel disk boundary has been developed by M.Widom [9] (see also paper of N.S. Manton and M. Nauenberg [10]). According to this works, two types of scaling are possible for quasiperiodic trajectories at Siegel disk:

1) So-called simple scaling

\[ \frac{f_{F_{N+1}}(z) - z}{f_{F_N}(z) - z} = -\varphi, \quad (6) \]

which is valid for any point \( z \) inside a disk (Here \( F_N \) are Fibonacci numbers, and \( \varphi \) – “golden mean” winding number);

2) Boundary of Siegel disk nontrivial scaling law

\[ \frac{f_{F_{N+1}}(z_c) - z_c}{f_{F_N}(z_c) - z_c} = \begin{cases} \alpha, & N \text{ even}, \\ \alpha^*, & N \text{ odd}, \end{cases} \quad (7) \]

with universal scaling factor

\[ \alpha = -0.22026597 - 0.70848172i. \quad (8) \]

Let’s consider now complex Hénon map. The fixed points of map (2) are

\[ z_{1,2} = w_{1,2} = \frac{-(1 + d) \pm \sqrt{(1 + d)^2 + 4\lambda}}{2}. \quad (9) \]

There are two complex multipliers \( \mu_{1,2} \) for map (2), which can be found as eigenvalues of a Jacobi matrix

\[ J(z, w) = \begin{pmatrix} \frac{\partial f(z, w)}{\partial z} & \frac{\partial g(z, w)}{\partial z} \\ \frac{\partial f(z, w)}{\partial w} & \frac{\partial g(z, w)}{\partial w} \end{pmatrix} = \begin{pmatrix} -2z & -d \\ 1 & 0 \end{pmatrix}. \quad (10) \]

Thus, multipliers satisfy to the equations

\[ \mu_1 \mu_2 = DetJ = d, \quad \mu_1 + \mu_2 = TrJ = -2z. \quad (11) \]

One can found the value of parameter \( \lambda \), at which one of the fixed points (9) is neutral

\[ \lambda = \frac{1}{4}((e^{2\pi i \varphi} + de^{-2\pi i \varphi})^2 - 2(1 + d)(e^{2\pi i \varphi} + de^{-2\pi i \varphi})). \quad (12) \]

Whereas \( \mu_1 \mu_2 = d \) (\( |d| < 1 \)), the fixed points with neutral multiplier should have the second multiplier with modulo less than unity, i.e. the fixed points should be attractive in directions, orthogonal to surface in 4D phase space, at which Siegel disk can occur.

With \( d = -0.3 \) and \( \varphi = (\sqrt{5} - 1)/2 \) value of parameter corresponded to a Siegel disk is

\[ \lambda = 0.05447888511456006 + 0.5339769956857415i. \quad (13) \]

The fixed point is placed in a phase space at

\[ z_0 = w_0 = 0.25807910732741196 + 0.4390686912699904i. \quad (14) \]

Nontrivial problem is the determination of the scaling center of Siegel disk of Hénon map, which corresponds to an extremum in the case of one-dimensional map. For this purpose the special
original numerical method based on universal scaling properties of a Siegel disk (6) and (7), has been developed.

The main content of a method is following. One should start from any initial point in a multidimensional phase space close enough to the fixed point (14). After several time iterations of map (2) trajectory of this point attracts due to a dissipation to one of smooth quasiperiodic invariant curves of Siegel disk. Then one should determine the point \((z^{(1)}, w^{(1)})\) at this invariant curve, which is farthest from rotation center \((z_0, w_0)\). One should prolong iterations, until images (of the Fibonacci numbers orders) of this point obey with sufficient accuracy to a law of simple scaling.

\[
\frac{f^{F_N+2}(z^{(1)}, w^{(1)}) - z^{(1)}}{f^{F_N}(z^{(1)}, w^{(1)}) - z^{(1)}} = \varphi^2, \quad \frac{g^{F_N+2}(z^{(1)}, w^{(1)}) - w^{(1)}}{g^{F_N}(z^{(1)}, w^{(1)}) - w^{(1)}} = \varphi^2. \tag{15}
\]

According to the nontrivial scaling law of a Siegel disk boundary one can make small step in the direction of scaling center and find its approximate location as

\[
z^{(2)} = \frac{1}{\alpha^2 - 1} \left( \alpha^2 f^{F_N}(z^{(1)}, w^{(1)}) - f^{F_N+2}(z^{(1)}, w^{(1)}) \right),
\]

\[
w^{(2)} = \frac{1}{\alpha^2 - 1} \left( \alpha^2 g^{F_N}(z^{(1)}, w^{(1)}) - g^{F_N+2}(z^{(1)}, w^{(1)}) \right). \tag{16}
\]

The point \((z^{(2)}, w^{(2)})\) belongs to a new invariant curve, which is farther from the fixed point and closer to the boundary of disk. Then one should repeat procedure: 1) find farthest from rotation center point of the invariant curve; 2) determine the scale at which this curve is smooth (invariant curves more closer to the disk boundary are more distorted), i.e. determine order \(F_N\), for which simple scaling implements; 3) calculate new approximation of the boundary scaling center using position of \(F_N\) and \(F_{N+2}\)-th iterations of farthest point.

Repeating procedure more and more times the position of scaling center can be calculated more and more precisely \(((z^{(n)}, w^{(n)}) \rightarrow (z_c, w_c) \text{ with } n \rightarrow \infty)\). As a result the following position of scaling center have been found

\[
z_c = 0.45756301999 + 0.30965877i,
\]

\[
w_c = 0.22028610776 + 0.64312280i. \tag{17}
\]

At Fig. 2 the projection of Siegel disk to three-dimensional space is represented. Siegel disk looks as smooth surface with fractal boundary. Several invariant curves (concentric lines) on a disk are represented. Radial lines at figure correspond to images of a curve, which is a locus of invariant curves points mostly distant from the fixed point. Scaling center of Siegel disk is designated.

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Figure 1: Phase plane of complex map (1). By gray colors the areas of starting values of variable, which iterations escape to infinity by different time are marked. Black color corresponds to restricted in a phase space dynamics. Curves represent quasiperiodic trajectories around of a neutral fixed point (designated by a obelisk). The fractal trajectory started from an extremum of map (designated by a circle) represents boundary of a Siegel disk.

Figure 2: A projection of a Siegel disk for the complexified Hénon map (2) to a three-dimensional phase space. Concentric lines on a disk represent quasiperiodic trajectories. Radial lines correspond to images of a curve (bold line with arrow), which is a locus of invariant curves points mostly distant from the fixed point. Scaling center of Siegel disk (the point being analog of an extremum) is designated as \((z_c, w_c)\).