The Memory of Initial Conditions in Isolated Quantum Systems

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Since the Fermi-Pasta-Ulam paradox [1] the question of the memory of the initial conditions in a dynamical system has been crucial for our understanding of the emergence of thermal behavior. The authors’ a priori expectation has been that for any perturbation away from integrability, no matter how weak, the time evolution in any dynamical system would erase the memory of the initial values of the integrals of motion, and, as a result, would thermalize in a relaxation process. However, this was not what they observed in their numerical experiment. It has been later discovered that the transition between the integrable and chaotic systems is not at all abrupt; the ability to forget the initial conditions and to thermalize requires that the strength of the non-integrable perturbation exceeds a certain threshold value [2, 3]. In the intermediate range of the interaction strength, dynamical systems are able to partially retain the memory; a quantitative treatment of this effect in the classical case is provided by the famous Kolmogorov-Arnold-Moser theorem (KAM) [4–7]. In this article, we present a simple formula that quantifies the effect of the memory of initial conditions in a quantum system, weakly perturbed away from the integrable point.

Two other regimes—strong deviation from integrability and the exact integrable point—are already well understood.

The question on how well observables of an isolated quantum-chaotic system can converge to their thermal expectations by the end of a relaxation from a highly excited initial state goes back to the classic 1974 paper by Shnirelman [8] and the body of work in mathematics and mathematical physics that immediately followed [9–13]. Figuratively speaking, here thermalization happens at the level of individual eigenstates of the perturbed system, which are represented by large representative quasi-random superpositions of the eigenstates of the underlying integrable system. Under this scenario, an individual eigenstate is able to emulate the microcanonical averaging of the observables over the unperturbed eigenstates. The typical deviation between the mean of an observable over an individual eigenstate and its microcanonical value has been linked to physically relevant autocorrelation function [14] and studied in many examples [15–19]. The vanishing of this deviation in the thermodynamics limit (the effect otherwise known as Eigenstate Thermalization) is believed to lie at the very foundation of quantum thermodynamics [20–22]. Indeed, it determines the maximal possible deviation of the infinite time average of the quantum mean of the observable of interest from its microcanonical average. A direct study of the relaxation process away from integrability, both theoretical [22–31] and experimental [32, 33] confirms the ability of isolated quantum systems to thermalize.

On the contrary, completely integrable quantum systems have been shown to retain a strong memory of the initial conditions [28, 29, 34–37]. Here, relaxation is contained within individual degrees of freedom leading to a steady state that is far from thermal.

However, only a few results exist on how the memory of the initial conditions diminishes as one gradually moves from a completely integrable regime towards a well-developed quantum chaos. These include, first the experimental results [37] on the one-dimensional Lieb-Liniger gas [38] perturbed by finite-range corrections to the interatomic interaction potential and by a tunneling between individual one-dimensional systems, and second, the numerical study of the one-dimensional integrable bosonic and fermionic lattice models with added integrability-breaking next-to-the-nearest-neighbor perturbation [39, 40]. Our article is an attempt to give a quantitative description to the effect. The final formula smoothly covers the range of strengths of the nonintegrable perturbation from the completely integrable limit through the collision-less regime where the frequencies governing the individual degrees of freedom are still higher than the relaxation rate. The formula also assumes the absence of selection rules for the perturbation.

As a concrete example, we consider two short-range-interacting bosons in a circular, transversally harmonic waveguide [41–43] far above the transverse excitation threshold. In many respects the system is similar to the so-called Šeba billiard [44, 45]—a two-dimensional rectangular box with a point scatterer in the middle. In our view, systems of this type are ideally suited for the studies on quantum nonequilibrium dynamics close to the integrable point. They do not possess any integrals of motion besides the energy, while allowing for an exact analytic solution [44, 46]. The integrability-breaking perturbation here is a simple one-parametric family of singular potentials with no selection rules. It is known that Šeba-type systems do show some signatures of the quantum-chaotic behavior: a gap in the level spacing distribution and Gaussian statistics for the wavefunction in particular. However, at the same time, the tail of the level statistics and the momentum distributions in individual eigenstates exhibit substantial deviations from the quantum chaos predictions [46–50].

The results of our study of the waveguide system stand in very good agreement with the our general predictions for the memory of initial conditions in weakly perturbed integrable systems.

Statement of problem: memory of initial conditions in quantum systems weakly perturbed away from integrability.—Consider the Hamiltonian of an integrable quantum system
under a nonintegrable perturbation $\hat V$: 
\[ \hat H = H_0(\tilde n) + \hat V, \]
where $\tilde n = \{\tilde n_1, \tilde n_2, \ldots, \tilde n_N\}$ is a complete set of the integrals of motion, and $N$ is the number of the degrees of freedom. Prepare the system in an initial state $|\psi_0\rangle \equiv |\psi(t = 0)\rangle$ and let it evolve. For a generic observable $\hat A$, the infinite-time average of the quantum-mechanical mean is
\[ A_{\text{relax.}} \equiv \lim_{t_{\text{max}} \to \infty} \frac{1}{t_{\text{max}}} \int_0^{t_{\text{max}}} dt \langle \psi(t) | \hat A | \psi(t) \rangle \]
\[ = \sum_\alpha |\langle \alpha | \psi_0 \rangle|^2 \langle \alpha | \hat A | \alpha \rangle \]
\[ = \sum_\alpha \sum_{\tilde n, \tilde n', \tilde n''} \langle \alpha | \tilde n \rangle \langle \tilde n | \alpha \rangle \langle \tilde n | \tilde n' \rangle \langle \tilde n' | \alpha \rangle \times (1) \]
\[ \langle \tilde n | \psi_0 \rangle \langle \tilde n | \tilde n'' \rangle \langle \tilde n'' | \alpha \rangle \]

In particular, it describes the mean outcome of the measurement of the observable $\hat A$ performed at a random time after relaxation. Here and below, $\langle \tilde n |$ and $| \alpha \rangle$ are the eigenstates of the unperturbed and perturbed systems, respectively:
\[ H_0(\tilde n) | \tilde n \rangle = E_\tilde n | \tilde n \rangle \]
\[ \hat H | \alpha \rangle = E_\alpha | \alpha \rangle \]
and $E_{\tilde n}$ and $E_\alpha$ are the corresponding eigenenergies.

The goal of this article is to assess quantitatively the correlation between the final value of the observable after relaxation and its initial value.

**Assumptions.** Three important energy scales are involved in the problem: the range of the unperturbed energies $(\Delta E)_{\alpha}$ a typical eigenstate $| \alpha \rangle$ of the perturbed system involves,
\[ (\langle \tilde n | | \alpha \rangle \neq 0) \Rightarrow |E_{\tilde n} - E_\alpha| \lesssim (\Delta E)_{\alpha}; \]
the typical energy distance $(\Delta E)_A$ between the states coupled by the observable $\hat A$,
\[ (\langle \tilde n | \hat A | \tilde n'' \rangle \neq 0) \Rightarrow |E_{\tilde n} - E_{\tilde n''}| \gtrsim (\Delta E)_A; \]
the typical energy range $(\Delta E)_{|\psi_0\rangle}$ covered by the initial state $| \psi_0 \rangle$,
\[ |\langle \tilde n | \psi_0 \rangle|^2 \neq 0 \Rightarrow |E_{\tilde n} - E_{\psi_0}| \lesssim (\Delta E)_{|\psi_0\rangle}, \]
where
\[ E_{\psi_0} \equiv \langle \psi_0 | H_0(\tilde n) | \psi_0 \rangle \]
is the energy of the initial state. Note that the first energy width $(\Delta E)_{\alpha}$ is usually associated of the relaxation rate $\Gamma$ [24]:
\[ (\Delta E)_{\alpha} \sim \Gamma. \]

In what follows, we will impose the following assumptions:

(a) The observable $\hat A$ acts on only one (out of $N$) degrees of freedom. In this case, the energy differences $(\Delta E)_A$ between different states coupled by $\hat A$ are much greater than the inverse density of states. Consistently, we further assume that the energy distance between two eigenstates of the underlying integrable system coupled by $\hat A$ is typically greater than the range of energies spanned by a typical eigenstate $| \alpha \rangle$ of the full system,
\[ (\Delta E)_{\alpha} \ll (\Delta E)_A. \]

A system whose parameters obey the above inequality is usually said to be in the collisionless regime where the dephasing of individual degrees of freedom is faster than the relaxation rate $\Gamma \sim (\Delta E)_{\alpha}$.

(b) The sequence $\langle \tilde n | \alpha \rangle$ of the coefficients of the decomposition of a given perturbed eigenstate $| \alpha \rangle$ over the unperturbed eigenstates $| \tilde n \rangle$ show quasi-random behavior. Traditionally, this property is modeled using ensembles of random matrices [51]. In this spirit, we assume that the matrix of the matrix elements $\langle \tilde n | \hat V | \tilde n' \rangle$ of the perturbation $\hat V$ is realizations of a Gaussian ensemble of random matrices [52], where the second moments
\[ \left( \langle \tilde n | \hat V | \tilde n' \rangle \right)^2 = g \left( \langle \tilde n | \hat V | \tilde n' \neq \tilde n \rangle \right)^2 = V \]
do not depend on the states $\tilde n$ and $\tilde n'$. Here, $g = 2$ ($g = 1$) for the Gaussian Orthogonal (Unitary) Ensemble.

Under the above assumptions, the average value of the observable after relaxation (1) becomes
\[ A_{\text{relax.}} \approx \sum_\tilde n \sum_{\tilde n' \neq \tilde n} \tilde F(E_{\tilde n}, E_{\tilde n'}) \left| \langle \tilde n | \psi_0 \rangle \right|^2 \langle \tilde n | \hat A | \tilde n' \rangle + \sum_\tilde n \eta(E_{\tilde n}) \left| \langle \tilde n | \psi_0 \rangle \right|^2 \langle \tilde n | \hat A | \tilde n \rangle. \]

(See Eq. (5) for the rationale behind the singling out of the diagonal contribution.) Above, the terms with $\tilde n'' \neq \tilde n$ are eliminated from the right hand side of the Eq. (1) under the assumption (a). The assumption (b) eliminates the $\tilde n'' \neq \tilde n'$. Here
\[ \tilde F(E_{\tilde n}, E_{\tilde n'}) \equiv \sum_\alpha \left| \langle \tilde n | \alpha \rangle \right|^2 \left| \langle \tilde n' | \alpha \rangle \right|^2 \]
is the so-called overlap function similar to the one appearing in the treatment of the embedded Gaussian models of EGOE(1+2) type [24, 51], and
\[ \eta(E_{\tilde n}) \equiv \sum_\alpha \left| \langle \tilde n | \alpha \rangle \right|^4 \]
is the so-called inverse participation ratio, the inverse of which is often called the number of the principal components (see [25] for a review). Note that usually the object of interest is the inverse participation of the perturbed eigenstates over
the unperturbed ones: $\sum_{\vec{n}} |\langle \vec{n} | \alpha \rangle|^2$; it has the same spectral average as $\eta(E_\vec{n})$. Note that both $\tilde{F}(E_{\vec{n}_r}, E_{\vec{n}'}_r)$ and $\eta(E_{\vec{n}_r})$ above are functions of energies only and are insensitive to the properties of the states $|\vec{n}\rangle$. It can be shown that this feature guaranteed by the assumption (b).

We will further assume that

(c) $\tilde{F}(E_{\vec{n}_r}, E_{\vec{n}'_r})$ is a slow function of the two energies that does not change substantially on the scale of a typical energy spacing between the levels.

For example, for the EGOE(1+2)-type models and in the case of a weak perturbation, the overlap function has the form

$$\tilde{F}(E, E') = \frac{1}{\pi} \frac{\Gamma/\rho_E}{(E - E')^2 + \Gamma^2}$$

(see [24]). Here, $\rho_E$ is the density of states, and $\Gamma \sim (\Delta E)_{\alpha}$ is the so-called spreading width.

Next, we are going to assume that

(d) the energy range spanned by the initial state $(\Delta E)_{\psi_0}$ is much narrower than any energy scale over which any macroscopic property of the unperturbed system can change.

(e) within a microcanonical window $E_{\psi_0} - \Delta E \leq E \leq E_{\psi_0} + \Delta E$ (that includes the energy range $(\Delta E)_{\psi_0}$) the energy-ordered sequence of the unperturbed eigenstates $\{|\vec{n}_s\rangle\}$ (with $s' > s \Rightarrow E_{\vec{n}_r,s} > E_{\vec{n}_r,s'}$) can be modeled by a random sequence of the eigenstates, $\{|\vec{n}'_s\rangle\}$ with one realization being statistically independent of another:

$$\text{Prob}[|\vec{n}_s = \vec{n}\rangle \& |\vec{n}'_s = \vec{n}'\rangle] \approx \text{Prob}[|\vec{n}_s = \vec{n}\rangle] \text{Prob}[|\vec{n}'_s = \vec{n}'\rangle].$$

The probabilities of the individual events are numerically equal to the diagonal matrix elements of the microcanonical density matrix:

$$\text{Prob}[|\vec{n}_s = \vec{n}\rangle] = \rhoMC, E_{\psi_0}(\vec{n})\|\vec{n}\.\$$

Here,

$$\rhoMC, E_{\psi_0}(\vec{n}) = \begin{cases} \rho & \text{for } E_{\vec{n}} \in [E_{\psi_0} - \Delta E, E_{\psi_0} + \Delta E] \\ 0 & \text{otherwise} \end{cases},$$

where $\Delta E$ defines a suitable microcanonical window $W = [E_{\psi_0} - \Delta E, E_{\psi_0} + \Delta E]; \rho = N^{-1}; N = \sum_{\vec{n} \in W} 1$ is the number of states inside the window; $\sum_{\vec{n} \in W} \ldots$ denotes a sum over the window, $\sum_{\vec{n} \in W} E_{\vec{n}} \ldots$. The assumption (e) is rooted in the observation that for integrable systems with incommensurate frequencies, neighboring eigenstates typically show no correlation between the respective sets of quantum numbers.

Consider now the first sum on the right hand side of Eq. (2). We factorize each initial state component $|\langle \vec{n}'_s | \psi_0 \rangle|^2$ into a slow envelope $\sigma(E_{\vec{n}'_s})$ of width $(\Delta E)_{\psi_0}$ and a fluctuating part $\xi(\vec{n}')$ whose statistical properties are approximately uniform over the microcanonical window $W$:

$$|\langle \vec{n}'_s | \psi_0 \rangle|^2 = \sigma(E_{\vec{n}'_s}) \xi(\vec{n}')$$

with $\sum_{\vec{n}'} \sigma(E_{\vec{n}'_s}) = 1$

and $\xi(\vec{n}') = 1$.

Now, according to the assumption (e), we replace the indices $\vec{n}_s$ in the fluctuating, energy-independent functions of $\vec{n}_s$, $\xi(\vec{n}_s)$ and $(\vec{n}_s | \hat{A} | \vec{n}_s)$, by events in a random sequence $\{\vec{n}'_s\}$:

$$\sum_{\vec{n}} \sum_{\vec{n}' \neq \vec{n}} \tilde{F}(E_{\vec{n}}, E_{\vec{n}'}) |\langle \vec{n}'_s | \psi_0 \rangle|^2 \langle \vec{n} | \hat{A} | \vec{n} \rangle$$

(4)

(Notice that according to the assumption (c), $\tilde{F}(E_{\vec{n}_r}, E_{\vec{n}'_r})$ is treated as a slow function of two energies.) Averaging over many realizations of the sequences $\{|\vec{n}'_s\rangle\}$ yields

$$\sum_{\vec{n}} \sum_{\vec{n}' \neq \vec{n}} \tilde{F}(E_{\vec{n}}, E_{\vec{n}'}) \sigma(E_{\vec{n}'}) \xi(\vec{n}') \langle \vec{n}'_s | \hat{A} | \vec{n}_s \rangle$$

$$= \sum_{\vec{n}} \sum_{\vec{n}' \neq \vec{n}} \tilde{F}(E_{\vec{n}_r}, E_{\vec{n}'_r}) \sigma(E_{\vec{n}'_r}) \xi(\vec{n}') \langle \vec{n}'_s | \hat{A} | \vec{n}_s \rangle$$

$$= \sum_{\vec{n}} \sum_{\vec{n}' \neq \vec{n}} \tilde{F}(E_{\vec{n}_r}, E_{\vec{n}'_r}) \sigma(E_{\vec{n}'_r}) \langle \vec{n}'_s | \hat{A} | \vec{n}_s \rangle$$

$$\approx \sum_{\vec{n}} \sum_{\vec{n}' \neq \vec{n}} \tilde{F}(E_{\vec{n}_r}, E_{\vec{n}'_r}) \sigma(E_{\vec{n}'_r}) \hat{A}_{therm}$$

(5)

$$= \langle 1 - \eta' \rangle \hat{A}_{therm}.$$

where

$$\hat{A}_{therm} = \frac{1}{N} \sum_{\vec{n}} \langle \vec{n} | \hat{A} | \vec{n} \rangle$$

(6)

is the microcanonical expectation value of the observable,

$$\eta' = \sum_{\vec{n}''} \sigma(E_{\vec{n}''}) \eta(E_{\vec{n}''})$$

(7)

is the average of the inverse participation ratio over the slow envelope $\sigma(E_{\vec{n''})}$, and in the last step in the chain (5) the corrections appearing when $\vec{n}$ is close to the edge of the microcanonical window are neglected. The product $\xi(\vec{n}') \langle \vec{n}'_s | \hat{A} | \vec{n}_s \rangle$
is factorized according to the assumption (e). Note that such a factorization is justified only when the $s$-th event \( \vec{n}_s \) and the \( s' \)-th event \( \vec{n}_{s'} \) are two separate events, i.e. when \( s \neq s' \). This distinction justifies the singling out of the diagonal terms in Eq. (2).

The first sum in the Eq. (4) constitutes an unbiased single-realization approximant for the final expression in the chain (5). Accordingly, the Eq. (2) becomes

$$\overline{A}_{\text{relax.}} - \overline{A}_{\text{therm., approx.}} \approx \sum_{\vec{n}} \eta(E_{\vec{n}}) |\langle \vec{n}|\psi_0 \rangle|^2 \langle \vec{n}|\hat{A}|\vec{n} \rangle - \eta' \overline{A}_{\text{therm., approx.}},$$

where

$$\overline{A}_{\text{therm., approx.}} = \sum_{\vec{n}} K_{\vec{n}} \langle \vec{n}|\hat{A}|\vec{n} \rangle$$

is an unbiased single-realization approximant for the microcanonical average (6) and

$$K_{\vec{n}} = \frac{1}{1 - \eta'} \sum_{\vec{n}' \neq \vec{n}} \hat{F}(E_{\vec{n}'}, E_{\vec{n}}) |\langle \vec{n}'|\psi_0 \rangle|^2$$

where

$$\sum_{\vec{n}} K_{\vec{n}} = 1.$$

For a sufficiently large window spanned by the initial state \( (\Delta E)_{\psi_0} \) the discrepancy between the approximant \( \overline{A}_{\text{therm., approx.}} \) and the true thermal average \( \overline{A}_{\text{therm.}} \) is small. It can be shown that the discrepancy typically scales as the inverse square root of the width of the window: \( |\overline{A}_{\text{therm., approx.}} - \overline{A}_{\text{therm.}}| \propto 1/\sqrt{(\Delta E)_{\psi_0}} \).

Let us now introduce the final assumption stating that

(f) the eigenstate-by-eigenstate fluctuations of the inverse participation ratio \( \eta(E_{\vec{n}}) \) are much lower than its running average, and its running average does not change substantially inside the energy interval covered by the initial state, \( (\Delta E)_{\psi_0} \). Consequently, the inverse participation ratio \( \eta(E_{\vec{n}}) \) entering the relationship (8) and the definition (7) can be approximately replaced by its microcanonical average \( \overline{\eta}^{\psi_0} \):

$$\eta(E) \approx \overline{\eta}^{\psi_0},$$

where

$$\overline{\eta}^{\psi_0} = \frac{1}{N} \sum_{\vec{n}} \eta(E_{\vec{n}})$$

is the microcanonical average of the inverse participation ratio. The assumption (f) brings us to the central result of this article.

The central result and its interpretation.— We arrive at the central result of this article:

$$\overline{A}_{\text{relax.}} - \overline{A}_{\text{therm., approx.}} \approx \overline{\eta}^{\psi_0} \langle \overline{A}_{\text{init.}} \rangle_{\text{NID}} - \overline{\eta}^{\psi_0} \overline{A}_{\text{therm., approx.}},$$

where the approximant \( \overline{A}_{\text{therm., approx.}} \) to the microcanonical average (6) is given by the expression (9), the average inverse participation ratio is represented by the expression (10), and

$$\langle \overline{A}_{\text{init.}} \rangle_{\text{NID}} \equiv \sum_{\vec{n}} |\langle \vec{n}|\psi_0 \rangle|^2 \langle \vec{n}|\hat{A}|\vec{n} \rangle$$

is the noninteracting diagonal average of the observable over the initial state; it equals to the time average of the quantum-mechanical mean of the observable in a (fictitious) time evolution governed by the unperturbed Hamiltonian alone. It determines the temporal average for the observable at the initial, integrable stage of the relaxation process.

The physical meaning of the formula (11) is transparent. For a weak deviation from integrability, the loss of memory of initial conditions is a two-stage process. As it has been mentioned above, at the initial stage of the relaxation the loss of memory is not related to the perturbation at all: it is merely a result of the equilibration within each degree of freedom separately. This interpretation is consistent with the assumption (a) above: for example, in the EGOE(1+2)-type models the energy width \( (\Delta E)_{\alpha_0} \) is associated with the relaxation rate \( \Gamma \) [24] and thus the assumption (a) simply states that the relaxation time is much longer than the period of the initial integrable equilibration. At the later stages, each unperturbed component \( |\vec{n}' \rangle \) of the initial state gets transformed into an incoherent superposition of the perturbed eigenstates \( |\alpha \rangle \). A given component \( |\vec{n}' \rangle \) of \( |\psi_0 \rangle \) has a weight of the order of \( \eta \) in a typical perturbed eigenstate \( |\alpha \rangle \); the rest of the components are indiscriminately drawn from a microcanonical shell and do not participate in the memory retention (see Fig. 1). The combination of the two effects leads to the expression (11).

A particular example: two bosons in a multi-mode waveguide.— Below, we are going to test our predictions on a simple model. Consider the relative motion of two short-range-interacting bosons in a circular, transversally harmonic waveguide. The analogous model involving a linear waveguide has been previously studied in [41]. The Hamiltonian of the system

$$\hat{H} = -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial z^2} - \frac{\hbar^2}{2\mu} \Delta_\varphi + \hat{U} + \hat{V} - \hbar \omega_\perp,$$

is a sum of the longitudinal and transverse kinetic energies, the transverse trapping energy \( \hat{U} = \mu \omega^2 \varphi^2/2 \), and the Fermi-Huang-type interaction that couples the transverse and longitudinal degrees of freedom, \( \hat{V} = (2\pi \hbar^2 a_s/\mu) \delta_\varphi(\varphi/\partial \varphi)(r) \). Here \( \omega_\perp \) is the transverse frequency, \( \Delta_\varphi \) is the transverse two-dimensional Laplacian, \( \mu = m/2 \) is the reduced mass, \( m \) is the atomic mass, and \( a_s \) is the three-dimensional s-wave scattering length. We will assume periodic boundary conditions along \( z \), with a period \( L \). In what follows, we will restrict the Hilbert space to the states of zero \( z \)-component of the angular momentum that are also even under the \( z \leftrightarrow -z \) reflection. Note that the zero-range interaction has no effect on the rest of the Hilbert space. The non-interacting eigenstates are products of
the transverse two-dimensional zero-angular-momentum harmonic wavefunctions, labeled by the quantum number $n \geq 0$, and the symmetric plane waves $\cos(2mlz/L)$, $l \geq 0$. The unperturbed spectrum is therefore given by

$$E_{nl} = 2\hbar\omega_\perp n + \hbar^2(2ml/L)^2/(2\mu) \quad (13)$$

Even though the perturbation $\hat{V}$ destroys the complete set of integrals of motion of the unperturbed system, the separable nature of the interaction allows to solve the problem exactly. The details of our calculations will be published elsewhere [46]. Solutions of this type were first obtained in Refs. [44, 45] for the Šeba billiard—a two-dimensional rectangular box with a point scatterer in the middle—and its generalizations.

Our primary observable of interest is the transverse trapping energy $\hat{U}$. For a given energy $E_{\psi_0}$ of the initial state the allowed values of $\hat{U}$ span the range between 0 and $E_{\psi_0}$. The noninteracting diagonal average over the initial state is more restricted,

$$0 \leq \langle U_{\text{init.}} \rangle_{\text{NID}} \leq E_{\psi_0}/2 \quad (15)$$

due to the equipartition between the transverse kinetic and transverse trapping energies in the noninteracting system. In thermal equilibrium one would expect an equipartition between the three parts of the Hamiltonian (12), leading to

$$U_{\text{therm.}} \approx E_{\psi_0}/3 \quad (16).$$

At rational values of the parameter $(\pi L/a_\perp)^2$ the energy spectrum (13) shows degeneracies. To avoid the degeneracies we fix it to a highly irrational number: $(\pi L/a_\perp)^2 = 2\phi_{gr}\pi^5 \approx 990.3$, where $\phi_{gr} = (1 + \sqrt{5})/2$ is the golden ratio.

In all calculations, the energy of the initial state was fixed to $E_{\psi_0} \approx 205\hbar\omega_\perp$.

We have investigated two distinct paths in the space of initial states. In both cases the initial wavepacket is a product of a transverse and a longitudinal state. In the first case, the transverse state is a transverse eigenstate $|n_0\rangle$, while the longitudinal one is represented by two wave packets moving in mutually opposite directions and initially localized at the point...
opposite to the scatterer:

\[
\langle z | \psi_0 \rangle = \text{const} \times |n_\rho \rangle \times \cos(2p_0(z - L/2)/\hbar) \times \\
\left\{ \begin{array}{ll}
\cos({\pi}z/(dL)), & |z - L/2| \leq dL \\
0, & \text{otherwise}
\end{array} \right.
\]

(14)

The surface of constant energy \( E_{\psi_0} \) was explored along the line of constant width of the wavepackets fixed to \( d = 0.1 \), while the parameters \( n_0 \) and \( p_0 \) were scanned. In the second case, the distribution of the transverse integral of motion \( n_\rho \) is chosen to be broad. The initial state had a form

\[
\langle \rho, z | \psi_0 \rangle = \text{const} \times \exp(-\kappa \rho^2/a^2_\perp) \times \\
\left\{ \begin{array}{ll}
\cos({\pi}z/(dL)), & |z - L/2| \leq dL \\
0, & \text{otherwise}
\end{array} \right.
\]

(15)

This time, we were scanning the widths of the transverse and longitudinal wavepackets, \( 1/\kappa \) and \( d \).

Figure 2 shows a good agreement with the general prediction in Eq. (11), for both strong and weak interactions. Unlike in the general model, no ensemble average over different perturbations was performed.

Notice also a peculiarity of the systems with separable interaction, including the present model and the Šeba-type billiards: even for infinitely strong interactions, the inverse participation ratio \( \eta \) never falls below approximately \( 1/3 \). Using the prediction in Eq. (11) one can say that in these systems the initial deviation from the thermal equilibrium can never be healed completely; the observable will stay away from the thermal prediction by at least \( 1/3 \) of the initial deviation.

**Summary and outlook.—** In this article we suggest a simple law governing the effect of the loss of memory of initial conditions in an integrable system weakly perturbed away from the integrable point. The law assumes the “collisionless regime” where the period of oscillation for individual degrees of freedom is much shorter than the relaxation time. Another assumption was that the perturbation is not constrained by any selection rules. Our findings are confirmed by the results of an *ab initio* study of two short-range-interacting bosonic particles in a circular, transversally harmonic waveguide in the multi-mode regime.

We believe that our results can be extended beyond the collisionless regime. In this case, the noninteracting diagonal average over the initial state in Eq. (11) is likely to be replaced by the true initial value of the observable. On the other hand, we think that the weakest point of our model is most likely the assumption of the absence of selection rules, which are present in any many-body system. It remains to be proven that the selection rules do not introduce a systematic bias in the approximant of Eq. (9) for the thermal expectation values of the one-degree-of-freedom observables of interest.

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[25] Flambaum, V. V. & Izrailev, F. M. Entropy production and wave packet dynamics in the fock space of closed chaotic many-body systems. *Phys. Rev. E* **64**, 036220 (2001).

[26] Kollath, C., Läuchli, A. M. & Altman, E. Quench dynamics and nonequilibrium phase diagram of the bose-hubbard model. *Phys. Rev. Lett.* **98**, 180601 (2007).

[27] Manmana, S. R., Wessel, S., Noack, R. M. & Muramatsu, A. Strongly correlated fermions after a quantum quench. *Phys. Rev. A* **69**, 053616 (2004).

[28] Rigol, M., Muramatsu, A. & Olshanii, M. Fermionization in an expanding 1d gas of hard-core bosons. *Phys. Rev. Lett.* **94**, 240403 (2005).

[29] Calabrese, P. & Cardy, J. Quantum quenches in extended systems. *J. Stat. Mech.* P06008 (2007).

[30] Cazalilla, M. A. Effect of suddenly turning on interactions in the luttinger model. *Phys. Rev. Lett.* **97**, 156403 (2006).

[31] Kinoshita, T., Wenger, T. & Weiss, D. S. A quantum Newton’s cradle. *Nature* **440**, 900–903 (2006).

[32] Lieb, E. H. & Liniger, W. Exact analysis of an interacting Bose gas. i. the general solution and the ground state. *Phys. Rev.* **130**, 1605 (1963).

[33] Rigol, M. Breakdown of thermalization in finite one-dimensional systems. *Phys. Rev. Lett.* **103**, 100403 (2009).

[34] Rigol, M. Quantum quenches and thermalization in one-dimensional fermion systems. *Phys. Rev. A* **80**, 053607 (2009).

[35] Olshanii, M. Atomic scattering in the presence of an external confinement and a gas of impenetrable bosons. *Phys. Rev. Lett.* **81**, 938–941 (1998).

[36] Moore, M. G., Bergeman, T. & Olshanii, M. Scattering in tight atom waveguides. *J. Phys. IV (France)* **116**, 69–86 (2004).

[37] Yurovsky, V. A., Olshanii, M. & Weiss, D. S. Collisions, correlations, and integrability in atom waveguides. In Adv. At. Mol. Opt. Phys., vol. 55, 61–138 (Elsevier Academic Press, New York, 2008).

[38] Šeba, P. Wave chaos in singular quantum billiard. *Phys. Rev. Lett.* **64**, 156403 (1990).

[39] Albeverio, S. & Šeba, P. Wave chaos in quantum-systems with point interaction. *J. Stat. Phys.* **64**, 369–383 (1991).

[40] Yurovsky, V. A. & Olshanii, M. Restricted thermalization for two interacting atoms in a multi-mode waveguide (2009). Unpublished.

[41] Bogomolny, E., Gerland, U. & Schmit, C. Singular statistics. *Phys. Rev. E* **63**, 036206 (2001).

[42] Bogomolny, E., Giraud, O. & Schmit, C. Nearest-neighbor distribution for singular billiards. *Phys. Rev. E* **65**, 056214 (2002).

[43] Berkolaiko, G., Keating, J. P. & Winn, B. Intermediate wave function statistics. *Phys. Rev. Lett.* **91**, 134103 (2003).

[44] Stone, C., Aoud, Y. A. E., Yurovsky, V. A. & Olshanii, M. Two simple systems with cold atoms: quantum chaos tests and nonequilibrium dynamics (2009). Unpublished.