Interaction of boundaries with heterogeneous matter states in matrix models

Masahiro Anazawa, Atushi Ishikawa and Hirokazu Tanaka

Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606-01, Japan

Abstract

We study disk amplitudes whose boundary conditions on matter configurations are not restricted to homogeneous ones. They are examined in the two-matrix model as well as in the three-matrix model for the case of the tricritical Ising model. Comparing these amplitudes, we demonstrate relations between degrees of freedom of matter states in the two models. We also show that they have a simple geometrical interpretation in terms of interactions of the boundaries. It plays an important role that two parts of a boundary with different matter states stick each other. We also find two closed sets of Schwinger-Dyson equations which determine disk amplitudes in the three-matrix model.

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*Supported by JSPS. E-mail address: anazawa@yukawa.kyoto-u.ac.jp.
† Supported by JSPS. E-mail address: ishikawa@yukawa.kyoto-u.ac.jp.
‡E-mail address: hirokazu@yukawa.kyoto-u.ac.jp.
1 Introduction

It is well known that $(m, m+1)$ unitary conformal model can be constructed microscopically as a lattice statistical model\cite{1}. At each site this statistical model has local degrees of freedom labeled by the points of $A_{m-1}$ Dynkin diagram. The $(m, m+1)$ model coupled to 2d gravity can be described by matrix model. The $(m, m+1)$ model has $(m-1)$ microscopic degrees of freedom and the $(m-1)$-matrix chain model\cite{2} naturally corresponds to the $(m, m+1)$ model coupled to gravity. On the other hand the two-matrix model\cite{3, 4, 5, 6} can also describe this system near an appropriate critical point though there are only two matrices. In this paper we address the correspondence between the degrees of freedom described by the matrices in the $(m-1)$-matrix chain model and those in the two-matrix model\cite{7}.

As a non-trivial simplest case, we study the $(4, 5)$ tricritical Ising model coupled to 2d gravity. As actions of the two- and the three-matrix models which realize this system, we take

$$S(A, B) = \frac{N}{A} tr \left\{ U(A) + U(B) - AB \right\},$$  \hspace{1cm} (1.1)

$$S(A, B, C) = \frac{N}{A} tr \left\{ U_1(A) + U_2(B) + U_1(C) - AB - BC \right\}. \hspace{1cm} (1.2)$$

An amplitude of loops which have homogeneous matter configuration corresponds to an expectation value of $tr(A^n)$ and so on. The integration of the matrices is to be separated into one over eigenvalues and one over angular variables. We can integrate the angular variables first and reduce the original integral to that in terms of the eigenvalues. Then through the orthogonal polynomial method we can show that the loop composed of the matrix $A(B)$ in the two-matrix model corresponds to that composed of the matrix $A(C)$ in the three-matrix model. On the other hand, an amplitude of loops with heterogeneous matter configurations corresponds to an expectation value of $tr(A^nB^k \cdots)$ and so on. We cannot integrate the angular variables first in this case. Then the argument in the case of homogeneous loops can not be applied, and the correspondence between the matrices in the two models is not so trivial in this case. For simplicity we restrict our attention to disk amplitudes. One of the purposes of the paper is to calculate disk amplitudes with heterogeneous loops in the two models using Schwinger-Dyson technique\cite{8, 9, 10, 11}, and to study the correspondence between the matrices in the two- and three-matrix models.

Studying these disk amplitudes is also a very interesting problem by itself. We are forced to study heterogeneous loop when merging different homogeneous ones into a single one. The amplitudes involving heterogeneous loops have not been studied deeply\cite{1}.  

1 In the case of Ising model these amplitudes were studied in \cite{3, 4}.  

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We will obtain an interesting geometrical picture on the amplitudes involving heterogeneous loops in sect.4. If two parts of a loop have different matter states, these are forced to stick each other and the original heterogeneous loop reduces to homogeneous ones. We obtain this picture through the study of the disk amplitudes. We believe that this picture can naturally be extended to cases with many loops or cases with many handles because the sticking of boundaries of loops is a local phenomenon and does not depend on the global nature of surfaces.

The paper is organized as follows. In sect.2 we compute heterogeneous disk amplitudes whose loops are composed of two arcs using the Schwinger-Dyson technique in the two-matrix model. In sect.3 we compute similar amplitudes in the three-matrix model. In the process of our calculation we find two closed sets of Schwinger-Dyson equations which are composed of seven and ten equations respectively. The success of our calculation is due to these findings. In sect.4 we discuss the correspondence between the matrices in the two- and the three-matrix models, and provide a geometrical picture on these amplitudes. Sect.5 includes a summary and discussion.

2 Two-matrix model case

As a critical potential which realizes the \((4, 5)\) model in the two-matrix model, we take

\[
U(\phi) = 8\phi + 2\phi^2 + \frac{8}{3}\phi^3 + \frac{1}{4}\phi^4, \tag{2.1}
\]

which can be determined by the method of orthogonal polynomial [6] (see appendix). In this section, we would like to calculate the disk amplitude

\[
W_{AB}(p, q) = \frac{\Lambda}{N} \langle tr \left( \frac{1}{p - A} \frac{1}{q - B} \right) \rangle = \sum_{n,m=0}^{\infty} \frac{\Lambda}{N} \langle tr(A^n B^m) \rangle p^{-n-1} q^{-m-1}, \tag{2.2}
\]

and its continuum universal part \(w_{AB}(\zeta_A, \zeta_B, t)\) in the large \(N\) limit by means of the Schwinger-Dyson technique. The boundary is consisted of two parts which have different matter states.

Let us work out the disk amplitude

\[
W_A(B^m)(p) = \sum_{n=0}^{\infty} \frac{\Lambda}{N} \langle tr(A^n B^m) \rangle p^{-n-1}, \tag{2.3}
\]

which will be necessary for the calculation of the amplitude (2.2). Consider the Schwinger-Dyson equations

\[
0 = \sum_a \int [dAdB] \frac{\partial}{\partial A^a} \left\{ tr \left( A^n t^a B^k \right) e^{-S(A,B)} \right\},
\]

\[
0 = \sum_a \int [dAdB] \frac{\partial}{\partial B^a} \left\{ tr \left( A^n t^a \right) e^{-S(A,B)} \right\}, \tag{2.4}
\]
where we decomposed the matrices as $A = \sum_{a=1}^{N^2} A^a t^a$ etc. by introducing the basis $\{t^a\}$ of the hermitian matrix. Using a notation $[A^n B^k] = \frac{1}{N}(\text{tr}(A^n B^k))$, we can rewrite eqs.(2.4) as

$$0 = \sum_{l=0}^{n-1} [A^l][A^{n-1-l} B^k] - 8[A^n B^k] - 4[A^{n+1} B^k] - 8[A^{n+2} B^k] - [A^{n+3} B^k] + [A^n B^{k+1}],$$

$$0 = 8[A^n] + 4[A^n B] + 8[A^n B^2] + [A^n B^3] - [A^{n+1}],$$

(2.5)

in the large $N$ limit. It is convenient to use the resolvent representation eq.(2.3) and we obtain

$$0 = \{W_A(p) - x(p)\} W_A^{(B^k)}(p) + W_A^{(B^{k+1})}(p) + a^{(B^k)}(p),$$

(2.6)

$$0 = (8 - p)W_A(p) + 4W_A^{(B)}(p) + 8W_A^{(B^2)}(p) + W_A^{(B^3)}(p) + \Lambda,$$

(2.7)

where

$$x(p) = 8 + 4p + 8p^2 + p^3,$$

$$a^{(B^k)}(p) = (4 + 8p + p^2)[B^k] + (8 + p)[AB^k] + [A^2 B^k].$$

(2.8)

Note that $[1] = \Lambda$ and we used the $Z_2$ symmetry. One can easily find that eqs.(2.6) for $k = 0, 1, 2$ and eq.(2.7) make a closed set of equations [8, 9]. We can eliminate $W_A^{(B)}(p)$, $W_A^{(B^2)}(p)$ and $W_A^{(B^3)}(p)$, and have the following fourth order equation of $W_A(p)$:

$$V_A(p)^4 + \alpha_3(p)V_A(p)^3 + \alpha_2(p)V_A(p)^2 + \alpha_1(p)V_A(p) + \alpha_0(p) = 0,$$

(2.9)

where

$$V_A(p) = W_A(p) - x(p),$$

$$\alpha_3(p) = x(p) - 8,$$

$$\alpha_2(p) = 4 - 8x(p) + a^{(1)}(p),$$

$$\alpha_1(p) = p - 8 + 4x(p) - 8a^{(1)}(p) - a^{(B)}(p),$$

$$\alpha_0(p) = -\Lambda + (p - 8)x(p) + 4a^{(1)}(p) + 8a^{(B)}(p) + a^{(B^2)}(p).$$

(2.10)

We must provide the amplitudes $[A]$, $[A^2]$, $[AB]$, $[A^2 B]$ and $[A^2 B^2]$ in order to solve eq.(2.3). These can be determined by the method of orthogonal polynomial (see appendix). The continuum limit can be carried out by the renormalization $\Lambda = 70 - 10a^2 t$ and $p = 2a\zeta_A$ with the lattice spacing $a$ [12]. Here 70 is a critical value of $\Lambda$. Assuming the scaling form of $V_A(p)$ as

$$V_A(p) = c_0 + c_1\zeta_A a + c_2 w_A(\zeta_A, t)a^{5/4} + O(a^{6/4}),$$

(2.11)
and substituting this form into eq. (2.13), we have the equation of \( w_A(\zeta_A, t) \),

\[
    w_A(\zeta_A, t)^4 - 4t^{5/4}w_A(\zeta_A, t)^2 + 2(t^{5/2} - 5t^2\zeta_A + 20t\zeta_A^3 - 16\zeta_A^5) = 0 ,
\]

and find \((c_0, c_1, c_2) = (0, -2, \pm 2)\). By solving eq. (2.12), we can obtain the continuum universal disk amplitude \( w_A(\zeta_A, t) \) as

\[
    w_A(\zeta_A, t) = \left( \zeta_A + \sqrt{\zeta_A^2 - t} \right)^{5/4} + \left( \zeta_A - \sqrt{\zeta_A^2 - t} \right)^{5/4} .
\]

We examine next the amplitude \( W_{AB}(p, q) \) which is the prime interest. Observing that \( W_{AB}(p, q) = \sum_{k=0}^{\infty} W_A^{(B^k)}(p)q^{-k} \), from eqs. (2.6), (2.7) we can find the following equation,

\[
    W_{AB}(p, q) = \frac{(4 + 8p + p^2)W_A(q) + (8 + p)W_A^{(B)}(q) + W_A^{(B^2)}(q) - W_A(p)}{x(p) - q - W_A(p)}. \tag{2.14}
\]

A careful consideration is needed to extract the universal amplitude \( w_{AB}(\zeta_A, \zeta_B, t) \) from this equation. For example let us consider \( W_A^{(B^m)}(p) \) for finite \( m \). The boundary of the corresponding disk involves a part of finite lattice length composed of the matrix \( B \). Then the contribution from such a part in \( W_{AB}(p, q) \) turns out to be non-universal. In general, any polynomials of \( \zeta_A \) and \( \zeta_B \) multiplied by \( W_A^{(B^m)}(p) \) or \( W_B^{(A^m)}(q) \) are non-universal quantities of \( W_{AB}(p, q) \). Polynomials of \( \zeta_A \) and \( \zeta_B \) are also non-universal. We should drop these quantities appropriately to extract a universal part of \( W_{AB}(p, q) \). Using the expansion of \( V_A(p) \) (2.11) and a similar expansion of \( V_B(q) \), we can find

\[
    W_{AB}(p, q) - 2(4 + a\zeta_A + a\zeta_B)(W_A(p) + W_B(q)) - \left( W_A^{(B)}(p) + W_B^{(A)}(q) \right)
    = -7 - 32(\zeta_A + \zeta_B)a - 4(\zeta_A^2 + 4\zeta_A\zeta_B + \zeta_B^2)a^2
    + 4 \left( 4t^{5/4} + w_A(\zeta_A, t)w_B(\zeta_B, t) \right) a^{5/2} + O(a^{11/4}) . \tag{2.15}
\]

In the left hand side, we appropriately subtracted some non-universal quantities in advance. Moreover we should drop any terms which are analytic in both \( \zeta_A \) and \( \zeta_B \) from the right hand side. Therefore we can read the continuum universal part of \( W_{AB}(p, q) \) as

\[
    w_{AB}(\zeta_A, \zeta_B, t) = w_A(\zeta_A, t)w_B(\zeta_B, t) , \tag{2.16}
\]

where \( w_B(\zeta, t) = w_A(\zeta, t) \) from the \( Z_2 \) symmetry. It should be noted that the terms with order higher than \( a^{5/4} \) in \( V_A(p) \) (2.11) do not appear in the right hand side of eq. (2.16), so that \( w_{AB}(\zeta_A, \zeta_B, t) \) can be expressed only in terms of \( w_A(\zeta_A, t) \) and \( w_B(\zeta_B, t) \). We will discuss the implication of this fact in sect.4.
3 Three-matrix model case

In this section, we will investigate the disk amplitudes

$$W_{AB}(p, q) = \sum_{n,m=0}^{\infty} A \frac{\Lambda}{N} \langle tr(A^n B^m) \rangle p^{-n-1} q^{-m-1},$$

(3.1)

$$W_{AC}(p, r) = \sum_{n,m=0}^{\infty} A \frac{\Lambda}{N} \langle tr(A^n C^m) \rangle p^{-n-1} r^{-m-1},$$

(3.2)

and their continuum universal parts $\mathcal{W}_{AB}(\zeta_A, \zeta_B), \mathcal{W}_{AC}(\zeta_A, \zeta_C)$ in the three-matrix model. As potentials which describe the $(4, 5)$ model, we take

$$U_1(\phi) = \frac{111}{16} \phi - \frac{9}{4} \phi^2 - \frac{1}{3} \phi^3,$$

$$U_2(\phi) = -\frac{3}{4} \phi^2 - \frac{1}{12} \phi^3.$$  

These can be found by the orthogonal polynomial method (see appendix). In order to obtain the amplitudes $\langle 3.1 \rangle$, $\langle 3.2 \rangle$, we have to calculate $W^{(B^m C^k)}_A(p) = \sum_{n=0}^{\infty} [A^n B^m C^k] p^{-(n+1)}$ and $W^{(A^n C^k)}_B(q) = \sum_{m=0}^{\infty} [A^n B^m C^k] q^{-(m+1)}$, where $[A^n B^m C^k] = \frac{\Lambda}{N} \langle tr(A^n B^m C^k) \rangle$.

Let us examine $W^{(B^m C^k)}_A(p)$ first. Consider the Schwinger-Dyson equations

$$0 = \sum_a \int [dAdBdC] \frac{\partial}{\partial A^a} \left\{ tr \left( A^n t^a B^m C^k \right) e^{-S(A,B,C)} \right\},$$

$$0 = \sum_a \int [dAdBdC] \frac{\partial}{\partial B^a} \left\{ tr \left( A^n t^a C^k \right) e^{-S(A,B,C)} \right\},$$

(3.4)

$$0 = \sum_a \int [dAdBdC] \frac{\partial}{\partial C^a} \left\{ tr \left( A^n B^m t^a \right) e^{-S(A,B,C)} \right\}.$$

We may write them in the resolvent representation,

$$0 = \left\{ W_A(p) - y(p) \right\} W^{(B^m C^k)}_A(p) + W^{(B^{m+1} C^k)}_A(p) + a^{(B^m C^k)}(p),$$

(3.5)

$$0 = -\frac{3}{2} W^{(B^m C^k)}_A(p) - \frac{9}{4} W^{(B^{m+1} C^k)}_A(p) - p \bar{W}^{(C^k)}_A(p) + [A^k],$$

(3.6)

$$0 = \frac{111}{16} W^{(B^m C^k)}_A(p) - \frac{9}{2} W^{(B^{m+1} C^k)}_A(p) - \bar{W}^{(B^m C^k)}_A(p) - \bar{W}^{(B^m C^{k+1})}_A(p),$$

(3.7)

where

$$y(p) = \frac{111}{16} - \frac{9}{2} p - p^2,$$

$$a^{(B^m C^k)}(p) = -\frac{9}{2} p + [B^m C^k] - [A B^m C^k].$$

Here $[1] = \Lambda$ and we used the $Z_2$ symmetry. One can find that eqs.$\langle 3.5 \rangle$ for $(m, k) = (0, 0), (0, 1), (1, 0), (1, 1)$, eqs.$\langle 3.6 \rangle$ for $k = 0, 1$ and eq.$\langle 3.7 \rangle$ for $m = 0$ are independent and
Here we consider the problem of making a closed set of equations. By eliminating \( \overline{A}(p) \), \( \overline{B}(p) \), \( \overline{C}(p) \), and \( \overline{D}(p) \), we obtain the following fifth order equation of \( \overline{A}(p) \),

\[
U_A(p)^5 + \overline{A}_4(p)U_A(p)^4 + \overline{A}_3(p)U_A(p)^3 + \overline{A}_2(p)U_A(p)^2 + \overline{A}_1(p)U_A(p) + \overline{A}_0(p) = 0. \tag{3.8}
\]

Here

\[
\begin{align*}
U_A(p) &= \overline{A}(p) - y(p), \\
\overline{A}_4(p) &= -12 + y(p), \\
\overline{A}_3(p) &= 18 + 8p + a^{(1)}(p) - 12y(p), \\
\overline{A}_2(p) &= 92 - 48p - 12a^{(1)}(p) - a^{(B)}(p) - 4\Lambda + (18 + 8p)y(p), \\
\overline{A}_1(p) &= -111 - 72p + 16p^2 + (18 + 4p)a^{(1)}(p) + 6a^{(B)}(p) - 4a^{(C)}(p) \\
&\quad + 24\Lambda + (92 - 48p)y(p), \\
\overline{A}_0(p) &= (92 - 24p)a^{(1)}(p) + (18 - 4p)a^{(B)}(p) + 24a^{(C)}(p) + 4a^{(BC)}(p) \\
&\quad + (72 - 16p)\Lambda + 16[A] + (-111 - 72p + 16p^2)y(p).
\end{align*}
\] (3.9)

The amplitudes \( [A], [B], [AB], [AC] \) and \( [ABC] \) are determined by the orthogonal polynomial method (see appendix). With the renormalization, \( \Lambda = 35 - \frac{5}{2}a^2t \) and \( p = \frac{3}{2}a\zeta, \) we assume the scaling behavior of \( U_A(p) \) as

\[
U_A(p) = \bar{c}_0 + \bar{c}_1\zeta_Aa + \bar{c}_2\overline{A}(\zeta_A, t)a^{5/4} + O(a^{6/4}). \tag{3.10}
\]

Substituting eq. (3.10) into eq. (3.8) and after similar calculation in sect. 2, we find that \((\bar{c}_0, \bar{c}_1, \bar{c}_2) = (0, 2, \pm 2/3)\) and

\[
\overline{A}(\zeta_A, t) = \left( \zeta_A + \frac{\sqrt{\Lambda^2}}{2} - t \right)^{5/4} + \left( \zeta_A - \frac{\sqrt{\Lambda^2}}{2} - t \right)^{5/4}. \tag{3.11}
\]

As expected, this coincides with the result for \( w_A(\zeta_A, t) \) in the two-matrix model, eq. (2.13).

Next let us examine \( \overline{W}_2(q) \). In this case, we found that ten Schwinger-Dyson equations are needed. For example let us consider the following ten equations:

\[
\begin{align*}
0 &= \sum_a \int [dAdBdc] \frac{\partial}{\partial B^a} \left\{ tr \left( t^a B^n \right) e^{-S(A,B,C)} \right\}, \\
0 &= \sum_a \int [dAdBdc] \frac{\partial}{\partial B^a} \left\{ tr \left( A t^a B^n \right) e^{-S(A,B,C)} \right\}, \\
0 &= \sum_a \int [dAdBdc] \frac{\partial}{\partial B^a} \left\{ tr \left( C t^a AB^n \right) e^{-S(A,B,C)} \right\}, \\
0 &= \sum_a \int [dAdBdc] \frac{\partial}{\partial B^a} \left\{ tr \left( CAt^a B^n \right) e^{-S(A,B,C)} \right\},
\end{align*}
\]
We can find that these are independent and make a closed set of equations. The fourth order equation which determines $W_B(q)$ is obtained as:

$$U_B(q)^4 + \mathcal{P}_2(q)U_B(q)^2 + \mathcal{P}_0(q) = 0,$$  

(3.15)
where
\[ U_B(q) = \nabla B(q) - z(q) - \frac{9}{2}. \]

The coefficients of this equation are given by
\[ \bar{\beta}_0(q) = -\frac{273}{4} - 3\Lambda - \frac{[B]}{2} + \left( \frac{35}{2} - \frac{\Lambda}{2} \right) q - \frac{3}{4} q^3 - \frac{1}{16} q^4, \]
\[ \bar{\beta}_2(q) = \frac{135}{8} [A] - \frac{27}{4} [AC] + \frac{3}{4} \Lambda [B] - 6 [AC] + \frac{[B]^2}{16} - 9 [AB] - [ABC] \]
\[ + \left\{ -729 + \frac{81}{16} \Lambda + \frac{3}{4} \Lambda^2 - [AC] + \frac{19}{8} [B] + \frac{1}{8} \Lambda [B] + \frac{3}{2} [AB] \right\} q \]
\[ + \left\{ 54 + \frac{19}{4} \Lambda + \frac{\Lambda^2}{16} + 3 [A] + \frac{9}{16} [B] + \frac{[AB]}{4} \right\} q^2 \]
\[ + \left\{ 36 + \frac{9}{16} \Lambda + \frac{[A]}{4} \right\} q^3 - \frac{q^5}{4}. \]  

With the renormalization \( q = 2a\zeta_B \), solving eq. (3.15) directly, we can find the disk amplitude \( U_B(q) \) as
\[ U_B(q) = \pm \nabla_B(\zeta_B, t) a^{5/4} + \mathcal{O}(a^{6/4}), \]  

where \( \nabla_B(\zeta_B, t) \) coincides with \( \nabla_A(\zeta_B, t) \).

Now let us turn to the calculation of \( \nabla_{AB}(p, q), \nabla_{AC}(p, r) \). From eq. (3.13) for \( k = 0 \), one can obtain the relation
\[ \nabla_{AB}(p, q) = \left( \frac{q}{2} + p \right) \nabla_B(q) + \nabla_A(q) + \nabla_A(p) \frac{\nabla_B(p) - y(p) + q}{\nabla_A(p)}. \]  

By combining eq. (3.18) and the first equation of (3.13), \( \nabla_{AB}(p, q) \) can be expressed in terms of \( \nabla_A(p) \) and \( \nabla_B(q) \). In order to extract a universal part, we must drop polynomials of \( \zeta_A \) and \( \zeta_B \) multiplied by \( \nabla_A(B_mC^k)(p) \) or \( \nabla_B(A_nC^k)(q) \) as well as polynomials of both \( \zeta_A \) and \( \zeta_B \) appropriately, because of the same reason as stated in sect.2. Using the expressions (3.10) and (3.17), we can find
\[ \nabla_{AB}(p, q) - \frac{9}{16} \nabla_A(p) - \frac{3}{4} \nabla_B(q) \]
\[ = \frac{-1607}{256} + \frac{9}{64} (11 \zeta_A + 8 \zeta_B) a \]
\[ - \frac{1}{8} \frac{\nabla_A(\zeta_A, t)^2 + 2 \nabla_A(\zeta_A, t) \nabla_B(\zeta_B, t) + 2 \nabla_B(\zeta_B, t)^2 - 2 t^{5/4}}{\zeta_A + \zeta_B} a^{3/2} + \mathcal{O}(a^{7/4}). \]  

In the left hand side, we subtracted some non-universal quantities in advance appropriately. Moreover we should drop first and second terms in the right hand side, because
they are polynomials of both $\zeta_A$ and $\zeta_B$. From this equation, we can find the continuum universal disk amplitude $\overline{w}_{AB}(\zeta_A, \zeta_B, t)$ as

$$
\overline{w}_{AB}(\zeta_A, \zeta_B, t) = \frac{\overline{w}_A(\zeta_A, t)^2 + 2\overline{w}_A(\zeta_A, t)\overline{w}_B(\zeta_B, t) + 2\overline{w}_B(\zeta_B, t)^2 - 2t^{5/4}}{\zeta_A + \zeta_B}.
$$

We can observe that terms with order higher than $a^{5/4}$ in $\overline{W}_A(p)$ and $\overline{W}_B(q)$ do not appear in the right hand side of eq.\((3.20)\). Thus $\overline{w}_{AB}(\zeta_A, \zeta_B, t)$ is expressed only in terms of $\overline{w}_A(\zeta_A, t)$ and $\overline{w}_B(\zeta_B, t)$.

Next let us consider $\overline{W}_{AC}(p, r)$. From eqs.\((3.5)\) for $m = 0, 1$ and eq.\((3.6)\), we can obtain the equations

$$
\overline{W}_{AC}(p, r) = \frac{\left(\frac{9}{2} + p\right)\overline{W}_C(r) + \overline{W}_C^{(A)}(r) - \overline{W}_{AC}(p, r)}{\overline{W}_A(p) - y(p)},
$$

$$
\overline{W}_{AC}^{(B)}(p, 0, r) = \frac{\left(\frac{9}{2} + p\right)\overline{W}_C^{(B)}(r) + \overline{W}_C^{(AB)}(r) - \overline{W}_{AC}^{(B)}(p, r)}{\overline{W}_A(p) - y(p)},
$$

$$
\frac{3}{2} \overline{W}_{AC}^{(B)}(p, r) + \frac{1}{4} \overline{W}_{AC}^{(B)}(p, r) + (p + r)\overline{W}_{AC}(p, r) - \overline{W}_A(p) - \overline{W}_C(r) = 0,
$$

respectively. By combining these, we can express $\overline{W}_{AC}(p, r)$ in terms of $\overline{W}_A(p)$ and $\overline{W}_C(r)$. Note that $\overline{W}_C(r) = \overline{W}_A(r)$ because of the $Z_2$ symmetry. Using the expression \((3.10)\) and \((3.17)\), we find

$$
\begin{align*}
\overline{W}_{AC}(p, r) &= \frac{41}{5} \left(\overline{W}_A^{(C^2)}(p) + \overline{W}_C^{(A^2)}(r)\right) - \frac{801}{112} \left(\overline{W}_A^{(C)}(p) + \overline{W}_C^{(C)}(r)\right) \\
&\quad - \left(\frac{90}{7} - \frac{11}{32} a\zeta_C\right) \overline{W}_A(p) - \left(\frac{90}{7} - \frac{11}{32} a\zeta_A\right) \overline{W}_C(r) \\
&= -\frac{44649}{7168} + \frac{99}{32} (\zeta_A + \zeta_C)a - \frac{1}{3584} \left(-23200 t + 1377\zeta_A^2 + 8120\zeta_A\zeta_C + 1377\zeta_C^2\right)a^2 \\
&\quad + \left(-\frac{73}{28} t^{5/4} + \frac{1}{9} \overline{W}_A(\zeta_A, t)\overline{W}_C(\zeta_C, t)\right)a^{5/2} + \mathcal{O}(a^{11/4})
\end{align*}
$$

Here we subtracted some non-universal quantities in advance appropriately from $\overline{W}_{AC}(p, r)$. The first, second, third and $t^{5/4}$ terms in the right hand side should be dropped, because they are polynomials of both $\zeta_A$ and $\zeta_B$. We can read, therefore, the continuum universal disk amplitude $\overline{w}_{AC}(\zeta_A, \zeta_C, t)$ as

$$
\overline{w}_{AC}(\zeta_A, \zeta_C, t) = \overline{w}_A(\zeta_A, t)\overline{w}_C(\zeta_C, t).
$$
4 Comparison and interpretation

In the previous two sections, we obtained the disk amplitudes with heterogeneous boundaries \( w_{AB}(\zeta_A, \zeta_B, t) \), \( w_A^B(\zeta_A, \zeta_B, t) \) and \( w_{AC}(\zeta_A, \zeta_C, t) \) (eqs. (2.16), (3.20) and (3.25) respectively). In this section, we will compare them and provide a geometrical interpretation of these amplitudes. In this and the next sections, we denote the matrices \( A \), \( B \) and \( C \) in the three matrix model as \( \bar{A} \), \( \bar{B} \) and \( \bar{C} \) respectively, in order to distinguish from those in the two-matrix model. We will refer to a part of boundary which is composed of the matrix \( A \) as “boundary \( A \)” and so on.

From eqs. (2.16) and (3.25), we observe that \( w_{AB} \) and \( w_{\bar{A}\bar{C}} \) have exactly the same form. We can consider that boundaries \( A \) and \( B \) correspond to boundaries \( \bar{A} \) and \( \bar{C} \) respectively. In the case of loops with homogeneous matter states, this correspondence is natural from the viewpoint of the orthogonal polynomial method. In the case of heterogeneous boundaries, however, the method of the orthogonal polynomial cannot be applied and this correspondence is not so trivial.

From eqs. (3.20) and (3.25), we can observe that \( w_{\bar{A}\bar{B}} \) and \( w_{\bar{A}\bar{C}} \) have quite different forms. But \( w_{\bar{A}\bar{B}} \) obtained by eq. (3.20) has a very similar form to the disk amplitude \( w_{AB}^{(I)} \) in the case of the Ising model realized by the two-matrix model [10, 11]:

\[
    w_{AB}^{(I)}(\zeta_A, \zeta_B, t) = w_{AB}^{(I)}(\zeta_A, t)^2 + w_{AB}^{(I)}(\zeta_B, t)^2 + \frac{3t^{4/3}}{\zeta_A + \zeta_B}.
\]

Here \( w_{AB}^{(I)} \) is given by

\[
    w_{AB}^{(I)}(\zeta, t) = \left( \zeta + \sqrt{\zeta^2 - t} \right)^{4/3} + \left( \zeta - \sqrt{\zeta^2 - t} \right)^{4/3}.
\]

Note that, however, there is no symmetry under interchange of \( \bar{A} \) and \( \bar{B} \) in eq. (3.20).

For the sake of discussing why \( w_{AB} \) and \( w_{\bar{A}\bar{C}} \) have so different forms, it is useful to consider the inverse Laplace transformed amplitudes. Let us denote the inverse Laplace transformed amplitudes of \( w_A(\zeta_A, t) \) as \( W_A(\ell_A) \) etc. For example, \( w_{AB}(\zeta_A, \zeta_B, t) \) and \( W_{AB}(\ell_A, \ell_B) \) are related by the equation

\[
    w_{AB}(\zeta_A, \zeta_B, t) = \mathcal{L}_A \mathcal{L}_B [W_{AB}(\ell_A, \ell_B)] = \int_0^\infty d\ell_A \int_0^\infty d\ell_B e^{-\ell_A \zeta_A - \ell_B \zeta_B} W_{AB}(\ell_A, \ell_B). \tag{4.3}
\]

Here \( W_{AB}(\ell_A, \ell_B) \) represents a disk amplitude where length of each part of the boundary is fixed. First we easily obtain the relations:

\[
    W_{AB}(\ell_A, \ell_B) = W_A(\ell_A) W_B(\ell_B) \tag{4.4}
\]

\[
    W_{\bar{A}\bar{C}}(\ell_A, \ell_C) = W_{\bar{A}}(\ell_A) W_{\bar{C}}(\ell_C) \tag{4.5}
\]
As for $\overline{W}_{AB}$, we use the following formulas of the inverse Laplace transformation:

$$\mathcal{L}^{-1}\left[ \frac{1}{\zeta + a} \right] = e^{-at},$$

(4.6)

$$\mathcal{L}^{-1}\left[ e^{-\zeta_a F(\zeta)} \right] = \theta(\ell - a) f(\ell - a),$$

(4.7)

$$\mathcal{L}^{-1}\left[ \frac{F(\zeta)}{\zeta + a} \right] = e^{-\zeta a} \int_0^\ell dx e^{ax} f(x),$$

(4.8)

where $F(\zeta)$ denotes the image of the Laplace transformation of $f(\ell)$. Using the formulas (4.6) and (4.7), we obtain the relation:

$$\mathcal{L}_A^{-1} \mathcal{L}_B^{-1} \left[ \frac{\overline{w}_A(\zeta_A, t)}{\zeta_A + \zeta_B} \right] = \mathcal{L}_A^{-1} \left[ e^{-\zeta_a\ell_B \overline{w}_A(\zeta_A, t)^2} \right]$$

$$= \theta(\ell_A - \ell_B) \mathcal{L}_A^{-1} \left[ \overline{w}_A(\zeta_A, t)^2 \right]$$

$$= \theta(\ell_A - \ell_B) \int_0^{\ell_A - \ell_B} d\ell \overline{W}_A(\ell) \overline{W}_A(\ell_A - \ell_B - \ell).$$

(4.9)

We also have

$$\mathcal{L}_A^{-1} \mathcal{L}_B^{-1} \left[ \frac{\overline{w}_A(\zeta_A, t) \overline{w}_B(\zeta_A, t)}{\zeta_A + \zeta_B} \right] = \mathcal{L}_B^{-1} \left[ \int_0^{\ell_A} d\ell e^{-\zeta_B(\ell_A + \ell)} \overline{W}_A(\ell) \overline{W}_B(\zeta_B, t) \right]$$

$$= \int_0^{\ell_A} d\ell \overline{W}_A(\ell) \theta(\ell_B - \ell_A + \ell) \overline{W}_B(\ell_B - \ell_A + \ell),$$

(4.10)

by using the formulas (4.8) and (4.7). From (4.6) we have

$$\mathcal{L}_A^{-1} \mathcal{L}_B^{-1} \left[ \frac{1}{\zeta_A + \zeta_B} \right] = \mathcal{L}_B^{-1} \left[ e^{-\zeta_a \ell_B} \right] = \delta(\ell_A - \ell_B).$$

(4.11)

Collecting eqs. (4.9) - (4.11) together, we obtain the expression for $\overline{W}_{AB}$,

$$\overline{W}_{AB}(\ell_A, \ell_B) = \theta(\ell_A - \ell_B) \int_0^{\ell_A - \ell_B} d\ell \overline{W}_A(\ell) \overline{W}_A(\ell_A - \ell_B - \ell)$$

$$+ 2 \int_0^{\min(\ell_A, \ell_B)} d\ell \overline{W}_A(\ell_A - \ell) \overline{W}_B(\ell_B - \ell)$$

$$+ 2 \theta(\ell_B - \ell_A) \int_0^{\ell_B - \ell_A} d\ell \overline{W}_B(\ell) \overline{W}_B(\ell_B - \ell_A - \ell)$$

$$- 2\ell^{5/4} \delta(\ell_A - \ell_B).$$

(4.12)

Now let us consider the geometrical meaning of eqs. (4.3) and (4.12). As for eq. (4.5), it is easy to understand that a loop composed of boundary $\bar{A}$ and $\bar{C}$ splits into two loops with different homogeneous matter states (see fig.1). On the other hand, the first term in the right hand side of eq. (4.12) represents the configuration depicted in fig.2(a).
Figure 1: The original loop composed of two different parts of boundary splits into two loops each of which has homogeneous matter configurations.

Figure 2: Due to the sticking of two different kinds of boundaries, the original loop splits into two loops with homogeneous matter configurations.
region of the boundary $\bar{B}$ is stuck to the boundary $\bar{A}$, and the original loop also splits into two loops with homogeneous matter states. Likewise the second term in eq. (4.12) corresponds to the case in fig.2(b). Parts of boundaries $\bar{A}$ and $\bar{B}$ are stuck each other, so that the original loop splits into two loops with homogeneous matter states. The fourth term represents the contribution from the case where the boundaries $\bar{A}$ and $\bar{B}$ are stuck completely. From this geometrical picture, we can conclude that the original loop, in fact, splits into two loops with homogeneous matter states.

Next we consider the reason why there is a difference between $W_{\bar{A}\bar{B}}$ and $W_{\bar{A}\bar{C}}$ from this geometrical point of view. In the case of $W_{\bar{A}\bar{B}}$ the boundaries $\bar{A}$ and $\bar{B}$ stick each other. On the other hand, the boundaries $\bar{A}$ and $\bar{C}$ are connected at only one point in $W_{\bar{A}\bar{C}}$. This difference can be accounted for as follows. The $(4, 5)$ minimal conformal model has three degrees of matter freedom labeled by the points of the $A_3$ Dynkin diagram. We can interpret that the matrices $\bar{A}$ and $\bar{C}$ correspond to the ends of the diagram and $\bar{B}$ to the middle point. The boundaries $\bar{A}$ and $\bar{B}$ stick each other, because the corresponding states of $\bar{A}$ and $\bar{B}$ interact directly as opposed to $\bar{A}$ and $\bar{C}$. In the case of the two-matrix model eq. (4.4), the matrices $A$ and $B$ correspond to the ends of the Dynkin diagram. The boundaries $A$ and $B$, therefore, do not stick each other.

5 Summary

In this paper, we have considered the $(4, 5)$ minimal model coupled to 2d gravity described by both the two- and the three-matrix models. We have calculated the disk amplitudes with non-trivial boundary conditions for the matter configurations, and have shown explicitly the relation among the matrices of these two models. A geometrical interpretation of the resulting amplitudes have also been obtained. In the process of our calculation in the three-matrix model, we found that seven and ten Schwinger-Dyson equations make two closed sets. These two sets of equations determine the disk amplitudes $W_A(p)$ and $W_B(q)$ respectively.

We obtained the universal disk amplitudes $w_{AB}$, $w_{\bar{A}\bar{B}}$ and $w_{\bar{A}\bar{C}}$, whose boundaries composed of two arcs of finite length with different matter states, as eqs.(2.16), (3.20) and (3.25) respectively. We learned that the matrices $A$ and $B$ in the two-matrix model correspond to the matrices $\bar{A}$ and $\bar{C}$ in the three-matrix model.

The geometrical meaning of these results is that the loop of the disk $W_{\bar{A}\bar{B}}$ or $W_{\bar{A}\bar{C}}$ splits into two loops with homogeneous matter states. Only these configurations contribute to the amplitudes.

In this paper, we only studied disk amplitudes with two arcs. It is straightforward to generalize our calculation to disk amplitudes with more than two arcs. Using the
Schwinger-Dyson equations, we can compute recursively the disk amplitudes with $n$ arcs from those with smaller numbers of $n$ arcs.

What extent can we apply the technique in this paper to other matrix models? The point is whether we can construct closed sets of equations which determine the disk amplitudes with homogeneous boundaries (ex. $\mathcal{W}_A(p)$ and $\mathcal{W}_B(q)$). In the three-matrix model of closed chain type, we can find that this is the case. We believe that the Schwinger-Dyson technique could be applied successfully to the multi-matrix models of closed chain type as well as open ones.

We believe that the geometrical picture in this paper can be extended to the cases of 2d surface with many loops and handles. This expectation is natural because the interaction of boundaries is a local phenomenon and does not depend on the global nature of surfaces. This picture must be applied also to the case of general unitary minimal matter. Let us consider the $(n + 1, n + 2)$ unitary model coupled to 2d gravity realized by the $n$-matrix chain model. The matrices naturally correspond to the matter degrees of freedom labelled by the points of $A_n$ Dynkin diagram. We expect that two parts of the loops stick each other if the two corresponding matter states are adjacent in the Dynkin diagram. Due to this mechanism, heterogeneous loops must reduce to homogeneous ones. This phenomenon may be related to the formulation of the non-critical string field theory\cite{13, 10}, which is constructed in the limited space of loops with simple matter configurations.

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Appendix A Orthogonal Polynomial Method

In this appendix, we derive the critical potentials (2.1), (3.3) and critical values of $\Lambda$ by following ref.\cite{8}. We also show how to evaluate $\frac{4}{N}\langle trA^4 \rangle$ etc., which are necessary to solve the Schwinger-Dyson equations in the text. We show the details of the calculation by restricting our attention to the tricritical Ising model; $(p, q) = (4, 5)$.

\footnote{For the action $S = \frac{4}{N}\langle tr\{V(A) + V(B) + V(C) - AB - BC - CA\} \rangle$ with third order potential $V$, we can find that eight Schwinger-Dyson equations make a closed set. These leads to a fourth order equation.}
First, we consider the two-matrix model. The potential $U(\phi)$ in the action (1.1) is an arbitrary polynomial:

$$U(\phi) = \sum_{k=1}^{p} \frac{g_k}{k} \phi^k. \quad (A.1)$$

The integral over matrix elements can be converted into the one over the eigenvalues:

$$Z(\Lambda) = \int \prod_{i=1}^{N} dx_i dy_i \Delta(x) \Delta(y) e^{-\sum_{i=1}^{N} S(x_i,y_i)}, \quad (A.2)$$

$$S(x_i,y_i) = \frac{N}{\Lambda} \{ U(x_i) + U(y_i) - x_i y_i \},$$

$$\Delta(x) = \prod_{i<j} (x_i - x_j)^2.$$ 

By introducing the orthogonal polynomials $\Pi_n(x)$ which satisfy

$$\langle m|n \rangle \equiv \int dx dy e^{-S(x,y)} \Pi_m(y) \Pi_n(x) = \delta_{mn}, \quad (A.3)$$

we denote the matrix elements as

$$X_{mn} = \langle m|x|n \rangle, \quad P_{mn} = \langle m|\frac{\Lambda}{N} \frac{\partial}{\partial x}|n \rangle. \quad (A.4)$$

We can also derive the equation of motion

$$P_{mn} = \langle m|U'(x)|n \rangle - X_{mn}. \quad (A.5)$$

In the large $N$ limit, the matrices $X$ and $P$ are replaced with the classical functions $X(z,\Lambda)$ and $P(z,\Lambda)$ respectively:

$$P(z,\Lambda) = U'(X(z,\Lambda)) - X(1/z,\Lambda). \quad (A.6)$$

Let us determine the critical potential which realizes the $(4,5)$ model. We know that $X$ will be a fourth order differential operator in the continuum limit. At the critical point, therefore, we can set $X(z,\Lambda = \Lambda_c) = (1-z)^4/z$ and $P(z,\Lambda = \Lambda_c) = \Lambda_c z^+ (\text{higher powers of } z)$. After substituting these into the equation of motion, we can find the critical potential

$$U'(\phi) = 8 + 4\phi + 8\phi^2 + \phi^3, \quad (A.7)$$

and the critical value of the cosmological constant as $\Lambda_c = 70$.

Off the critical point, we set the classical functions $X(z,\Lambda)$ and $P(z,\Lambda)$ as

$$X(z,\Lambda) = \sqrt{R(\Lambda)/z + a(\Lambda) + b(\Lambda)z + c(\Lambda)z^2 + d(\Lambda)z^3},$$

$$P(z,\Lambda) = \Lambda z/(\sqrt{R(\Lambda)} + (\text{higher powers of } z)).$$
Here $R(\Lambda)$ is called the specific heat function. Expanding the equation of motion in powers of $z$, we obtain the third order equation of $a(\Lambda)$:

$$a(\Lambda)^3 + \ell(R, \Lambda)a(\Lambda)^2 + m(R, \Lambda)a(\Lambda) + n(R, \Lambda) = 0,$$

where

$$\ell(R, \Lambda) = 8,$$
$$m(R, \Lambda) = \frac{3 + 271R(\Lambda) + 9R(\Lambda)^2 - 27R(\Lambda)^3}{1 + 15R(\Lambda)},$$
$$n(R, \Lambda) = \frac{8 + 40R(\Lambda) + 24R(\Lambda)^2 - 72R(\Lambda)^3}{1 + 15R(\Lambda)}.$$

Other functions are given in terms of $a(\Lambda)$ as follows

$$b(\Lambda) = \sqrt{\frac{R(\Lambda)}{1 - 3R(\Lambda)}} \left\{ 4 + 16a(\Lambda) + 3a(\Lambda)^2 \right\},$$
$$c(\Lambda) = R(\Lambda) \{ 8 + 3a(\Lambda) \}, \quad d(\Lambda) = R(\Lambda)^{\frac{3}{2}},$$
$$\Lambda = 3R(\Lambda)^3 + \left\{ 18a(\Lambda)^2 + 96a(\Lambda) + 128 \right\} R(\Lambda)^2 + \left\{ 3b(\Lambda)^2 - 1 \right\} R(\Lambda) + \left\{ 3a(\Lambda)^2 + 16a(\Lambda) + 4 \right\} b(\Lambda) \sqrt{R(\Lambda)}.$$

The third order equation (A.8) has three possible solutions. Generally, the $(p, q)$ model has a relation between the cosmological constant $\Lambda$ and the specific heat $R(\Lambda)$:

$$R(\Lambda) - 1 \sim (\Lambda - \Lambda_c)^{2/(p+q-1)}.$$  

We should take the solution that satisfies this relation for the case of the $(p, q) = (4, 5)$ model.

The exact expression of $X(z, \Lambda)$ can determine the expectation value $\frac{1}{N}\langle trA^n \rangle$. In the large $N$ limit, the summation is replaced with the integration:

$$\frac{\Lambda}{N}\langle trA^n \rangle = \frac{\Lambda}{N} \sum_{i=1}^{N} \langle i | x^n | i \rangle \simeq \int_0^\Lambda d\lambda \oint \frac{dz}{2\pi i z} [X(z, \lambda)]^n.$$

\hspace{1cm} (A.10)

Next, let us consider the three-matrix model. As in the case of the two-matrix model, we introduce the orthogonal polynomials $\tilde{\Pi}(x)$ which satisfy

$$\langle m | n \rangle \equiv \int dx dy dz e^{-S(x, y, z)} \tilde{\Pi}(z) \tilde{\Pi}(x) = \delta_{mn}.$$  

\hspace{1cm} (A.11)
It is useful to introduce matrices $X_1$, $X_2$, $X_3$ and $P_1$:

$$
[X_1]_{mn} = \langle m| x |n \rangle,
[X_2]_{mn} = \langle m| y |n \rangle = [X_2]_{nm},
[X_3]_{mn} = \langle m| z |n \rangle = [X_1]_{nm},
[P_1]_{mn} = \langle m| \frac{\Lambda}{N} \cdot \frac{\partial}{\partial x} |n \rangle.
$$

With these matrices, the equations of motion are expressed as

\begin{align}
[P_1]_{mn} &= \langle m| U'_1(x) |n \rangle - [X_2]_{mn}, \\
\langle m| U'_2(y) |n \rangle &= [X_1]_{mn} + [X_1]_{nm}.
\end{align}

Introducing classical functions, in the large $N$ limit, (A.12) can be rewritten as

$$
P(z, \Lambda) = U'_1(X_1(z, \Lambda)) - X_2(z, \Lambda),
U'_2(X_2(z, \Lambda)) = X_1(z, \Lambda) + X_1(1/z, \Lambda).
$$

Remark that the classical function satisfies $X_2(z, \Lambda) = X_2(1/z, \Lambda)$, because the matrix $X_2$ is symmetric under the transposition. Now let us determine the critical potentials which realize the (4, 5) model. The classical functions are now

$$
X_1(z, \Lambda_c) = \frac{(1 - z/4)(1 - z)^4}{z},
X_2(z, \Lambda_c) = -\frac{(1 - z)^4}{z^2}.
$$

By using these critical behaviors and the equations of motion (A.13), (A.14), we obtain the critical potentials and the critical value of the cosmological constant,

$$
U'_1(\phi) = \frac{111}{16} - \frac{9}{2} \phi - \phi^2,
U'_2(\phi) = -\frac{3}{2} \phi - \frac{1}{4} \phi^2,
\Lambda_c = 35.
$$

The critical potentials determine the classical functions $X_i(z, \Lambda)$. We expand them in terms of $z$:

$$
X_1(z, \Lambda) = \sqrt{R(\Lambda)}/z + \bar{a}(\Lambda) + \bar{b}(\Lambda)z + \bar{c}(\Lambda)z^2 + \bar{d}(\Lambda)z^3 + \bar{e}(\Lambda)z^4,
X_2(z, \Lambda) = \bar{f}(\Lambda)/z^2 + \bar{g}(\Lambda)/z + \bar{h}(\Lambda) + \bar{g}(\Lambda)z + \bar{f}(\Lambda)z^2,
P(z, \Lambda) = \Lambda z/\sqrt{R(\Lambda)} + \text{(higher powers of } z)\).
After substituting these into \((A.13)\) and \((A.14)\), we get the fourth order equation of \(\tilde{a}(\Lambda)\):

\[
\tilde{a}(\Lambda)^4 + \tilde{k}(R, \Lambda)\tilde{a}(\Lambda)^3 + \tilde{\ell}(R, \Lambda)\tilde{a}(\Lambda)^2 + \tilde{m}(R, \Lambda)\tilde{a}(\Lambda) + \tilde{n}(R, \Lambda) = 0,
\]

(A.16)

where

\[
\tilde{k}(R, \Lambda) = \frac{288R(\Lambda)^3 + 60R(\Lambda)^2 + 9}{32R(\Lambda)^3 + 1},
\]

\[
\tilde{\ell}(R, \Lambda) = \frac{12R(\Lambda)^4 + 972R(\Lambda)^3 + 297R(\Lambda)^2 + 36R(\Lambda) + \frac{8}{3}}{32R(\Lambda)^3 + 1},
\]

\[
\tilde{m}(R, \Lambda) = \frac{54R(\Lambda)^4 + 1474R(\Lambda)^3 + \frac{1941}{4}R(\Lambda)^2 + 54R(\Lambda) - \frac{1303}{10}}{32R(\Lambda)^3 + 1},
\]

\[
\tilde{n}(R, \Lambda) = \frac{243R(\Lambda)^4 + \frac{6849}{8}R(\Lambda)^3 + \frac{4443}{16}R(\Lambda)^2 - \frac{3}{4}R(\Lambda) + \frac{22977}{256}}{32R(\Lambda)^3 + 1}.
\]

The solution \(\tilde{a}(\Lambda)\) has four possible branches. As in the case of the two-matrix model, we require the relation, \(R(z, \Lambda) - 1 \sim (\Lambda - \Lambda_c)^{2/(4+5-1)}\), which determines \(\tilde{a}(\Lambda)\) uniquely. By using \(\tilde{a}(\Lambda)\), the other functions are given as

\[
\tilde{b}(\Lambda) = \frac{R(\Lambda)^{1/2} \{1367 - 12\tilde{a}(\Lambda) - 432\tilde{a}(\Lambda)^2 - 64\tilde{a}(\Lambda)^3 - (144 + 64\tilde{a}(\Lambda))R(\Lambda)\}}{32 \{2 + 9R(\Lambda) + 4\tilde{a}(\Lambda)R(\Lambda)\}},
\]

\[
\tilde{c}(\Lambda) = -\left(\frac{3}{2}\tilde{a}(\Lambda)^2 + \frac{27}{4}\tilde{a}(\Lambda) + \frac{3}{32}\right) R(\Lambda) - \tilde{b}(\Lambda)R(\Lambda)^{3/2},
\]

\[
\tilde{d}(\Lambda) = -\frac{R(\Lambda)^{3/2}}{2} \left\{2\tilde{a}(\Lambda) + \frac{9}{2}\right\}, \quad \tilde{e}(\Lambda) = -\frac{R(\Lambda)^2}{4}, \quad \tilde{f}(\Lambda) = -R(\Lambda),
\]

\[
\tilde{g}(\Lambda) = -\left\{2\tilde{a}(\Lambda) + \frac{9}{2}\right\} R(\Lambda)^{1/2},
\]

\[
\tilde{h}(\Lambda) = -\tilde{a}(\Lambda)^2 - \frac{9}{2}\tilde{a}(\Lambda) + \frac{1111}{16} - 2\tilde{b}(\Lambda)R(\Lambda)^{1/2},
\]

\[
\Lambda = -\left\{2\tilde{a}(\Lambda)\tilde{b}(\Lambda) + \frac{9}{2}\tilde{b}(\Lambda)\right\} R(\Lambda)^{1/2} + \left\{2\tilde{a}(\Lambda) + \frac{9}{2}\right\} R(\Lambda)
\]

\[
+ \left\{3\tilde{a}(\Lambda)^2 + \frac{27}{2}\tilde{a}(\Lambda) + \frac{3}{16}\right\} R(\Lambda)^2 + 2\tilde{b}(\Lambda)R(\Lambda)^{5/2}.
\]

(A.17)

The classical functions \(X_1(z, \Lambda)\) and \(X_2(z, \Lambda)\) enable us to evaluate the expectation values \(\frac{A}{N}\langle trA^n \rangle\) (= \(\frac{A}{N}\langle trC^n \rangle\)) and \(\frac{A}{N}\langle trB^n \rangle\). For example, \(\frac{A}{N}\langle trB^n \rangle\) can be determined by

\[
\frac{A}{N}\langle B^n \rangle \simeq \int_0^\Lambda \int_0 \frac{dz}{2\pi iz}[X_2(z, \Lambda)]^n.
\]

It is hard to calculate the expectation values \(\frac{A}{N}\langle tr(A^k B^\ell) \rangle\) and \(\frac{A}{N}\langle tr(A^k B^\ell C^m) \rangle\) directly. These expectation values, however, can be reduced to \(\frac{A}{N}\langle trA^n \rangle\) by using some kinds of the Schwinger-Dyson equations.
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