TRIHARMONIC ISOMETRIC IMMERSIONS INTO A MANIFOLD OF NON-POSITIVELY CONSTANT CURVATURE

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Abstract. A triharmonic map is a critical point of the 3-energy in the space of smooth maps between two Riemannian manifold. We study the generalized Chen’s conjecture for a triharmonic isometric immersion \( \phi \) into a space form of non-positively constant curvature. We show that if the domain is complete and both the 4-energy of \( \phi \), and the \( L^4 \)-norm of the tension field \( \tau(\phi) \), are finite, then such an immersion \( \phi \) is minimal.

1. INTRODUCTION

Harmonic maps play a central role in geometry; they are critical points of the energy functional \( E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g \) for smooth maps \( \phi \) of \((M, g)\) into \((N, h)\). The Euler-Lagrange equations are given by the vanishing of the tension field \( \tau(\phi) \).

In 1983, J. Eells and L. Lemaire [7] extended the notion of harmonic map to \( k \)-harmonic map, which are, by definition, critical points of the \( k \)-energy functional

\[
E_k(\phi) = \frac{1}{2} \int_M |(d + \delta)^k \phi|^2 v_g.
\]

For \( k = 1 \), \( E_1 = E \), and for \( k = 2 \), after G.Y. Jiang [15] studied the first and second variation formulas of \( E_2 \) (\( k = 2 \)), extensive studies in this area have been done (for instance, see [14], [1], [16], [27], [29], [21], [19], [2], [12], [13], etc.). Notice that harmonic maps are always biharmonic by definition.

For harmonic maps, it is well known that:

If a domain manifold \((M, g)\) is complete and has non-negative Ricci curvature, and the sectional curvature of a target manifold \((N, h)\) is non-positive, then every energy finite harmonic map is a constant map (cf. [30]).

Therefore, it is a natural question to consider \( k \)-harmonic maps into a Riemannian manifold of non-positive curvature. In this connection, Baird, Fardoun and Ouakkas (cf. [2]) showed that:

If a non-compact Riemannian manifold \((M, g)\) is complete and has non-negative Ricci curvature and \((N, h)\) has non-positive sectional curvature, then every bienergy finite biharmonic map of \((M, g)\) into \((N, h)\) is harmonic.

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In our previous paper ([25]), we showed without the Ricci curvature condition of \((M,g)\), that

**Theorem 1.1.** Under only the assumptions of completeness of \((M,g)\) and non-positivity of curvature of \((N,h)\),

1. every biharmonic map \(\varphi : (M,g) \to (N,h)\) with finite energy and finite bienergy must be harmonic.
2. In the case \(\text{Vol}(M,g) = \infty\), every biharmonic map \(\varphi : (M,g) \to (N,h)\) with finite bienergy is harmonic.

We obtained (cf. [23], [24], [25])

**Theorem 1.2.** Assume that \((M,g)\) is a complete Riemannian manifold, and let \(\varphi : (M,g) \to (N,h)\) is an isometric immersion, and the sectional curvature of \((N,h)\) is non-positive. If \(\varphi : (M,g) \to (N,h)\) is biharmonic and \(\int_M |\mathbf{H}|^2 v_g < \infty\), then it is minimal. Here, \(\mathbf{H}\) is the mean curvature normal vector field of the isometric immersion \(\varphi\).

The above theorems gave an affirmative answer to the generalized B.Y. Chen’s conjecture (cf. [4]) under natural conditions:

“Every biharmonic isometric immersion into a Riemannian manifold of non-positive curvature must be harmonic.”

Now in this paper, we will discuss the 3-energy \(E_3\), \((k = 3)\) and a triharmonic map which is a critical point of the 3-energy \(E_3\) in the space of smooth maps of \(M\) into \(N\). We first show (cf. Theorem 2.1) the first variational formula of triharmonic maps which is of simple form in the case of an isometric immersion into the Riemannian manifold of constant curvature (cf. Corollary 2.3). Then, we want to show that the generalized Chen’s conjecture is true for a triharmonic isometric immersion into a Riemannian manifold of non-positive curvature. More precisely, we will show that

**Theorem 1.3** (cf. Theorem 2.4 and [4,11]). Assume that \(\varphi : (M,g) \to N(c)\) is an isometric immersion of a complete Riemannian manifold \((M,g)\) into another Riemannian manifold \(N(c)\) of non-positively constant curvature \(c\). In the case \(c < 0\), if \(\varphi\) is triharmonic and both the extended 4-energy \(\tilde{E}_4(\varphi) = \frac{1}{2} \int_M |\Delta \tau(\varphi)|^2 v_g\) and the \(L^4\)-norm \(\int_M |\tau(\varphi)|^4 v_g\) are finite, then \(\varphi\) is minimal.

In the case \(c = 0\), the same conclusion holds if we assume more \(E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g < \infty\) or \(E_3(\varphi) = \frac{1}{2} \int_M |\nabla \tau(\varphi)|^2 v_g < \infty\).

2. Preliminaries and statement of main theorem

In this section, we prepare materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map \(\varphi : (M,g) \to (N,h)\), of a compact Riemannian manifold \((M,g)\) into another Riemannian manifold \((N,h)\), which is an extremal of the energy functional defined by

\[
E(\varphi) = \int_M e(\varphi) v_g,
\]
where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of $\varphi$. That is, for any variation $\{\varphi_t\}$ of $\varphi$ with $\varphi_0 = \varphi$,

$$
\left. \frac{d}{dt} \right|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V)v_9 = 0,
$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along $\varphi$ which is given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi_t(x)}N, (x \in M)$, and the tension field is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined frame field on $(M, g)$, and $B(\varphi)$ is the second fundamental form of $\varphi$ defined by

$$
B(\varphi)(X, Y) = (\overline{\nabla} d\varphi)(X, Y)
= (\overline{\nabla}_X d\varphi)(Y)
= \overline{\nabla}_X (d\varphi(Y)) - d\varphi(\nabla_X Y),
$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, $\nabla$, and $\nabla^N$, are Levi-Civita connections on $TM$, $TN$ of $(M, g)$, $(N, h)$, respectively, and $\nabla$, and $\overline{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2.1), $\varphi$ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that $\varphi$ is harmonic. Then,

$$
\left. \frac{d^2}{dt^2} \right|_{t=0} E(\varphi_t) = \int_M h(J(V), V)v_9,
$$

where $J$ is an elliptic differential operator, called the Jacobi operator acting on $\Gamma(\varphi^{-1}TN)$ given by

$$
J(V) = \overline{\Delta} V - \mathcal{R}(V),
$$

where $\overline{\Delta} V = \overline{\nabla} \overline{\nabla} V = -\sum_{i=1}^m \{e_i, \overline{\nabla} e_i, V - \overline{\nabla} e_i, e_i, V\}$ is the rough Laplacian and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}(V) = \sum_{i=1}^m R^N(V, d\varphi(e_i))d\varphi(e_i)$, and $R^N$ is the curvature tensor of $(N, h)$ given by $R^N(U, V) = \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U, V]}$ for $U, V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire [7] proposed polyharmonic ($k$-harmonic) maps and Jiang [15] studied the first and second variation formulas of biharmonic maps. Let us consider the $k$-energy defined by

$$
E_k(\varphi) = \frac{1}{2} \int_M |(d + \delta^k)\varphi|^2 v_9
$$

for a smooth map $\varphi$ from $M$ into $N$, and $\varphi$ is called $k$-harmonic if it is a critical point of $E_k$ ($k = 1, 2, 3, \cdots$). Here, we also define the extended $k$-energy $\widetilde{E}_k$ is given (cf. [13], p.270) as follows:

$$
\widetilde{E}_k(\varphi) = \begin{cases} 
\int_M |W^\ell_\varphi|^2 v_9 & (k = 2\ell), \\
\int_M |\nabla W^\ell_\varphi|^2 v_9 & (k = 2\ell + 1)
\end{cases}
$$

where $W^\ell_\varphi$ is given by

$$
W^\ell_\varphi = \overline{\Delta} \cdots \overline{\Delta} \tau(\varphi)
$$
if \( \ell \geq 1 \). Notice that \( E_k(\varphi) = \tilde{E}_k(\varphi) \) \((k = 1, 2, 3)\), but for \( k = 4 \), it holds that

\[
E_4(\varphi) = \frac{1}{2} \int_M |(d + \delta)(d + \delta)\tau(\varphi)|^2 v_g
\]

\[
= \frac{1}{2} \int_M |d d \tau(\varphi)|^2 v_g + \frac{1}{2} \int_M |\Delta \tau(\varphi)|^2 v_g
\]

\[
= \frac{1}{2} \int_M |d d \tau(\varphi)|^2 v_g + \tilde{E}_4(\varphi).
\]

(2.8)

If \( \ell = 0 \), we put \( W_0 = \varphi \) and \( E_1(\varphi) = \frac{1}{2} \int_M |\nabla \varphi|^2 v_g = \frac{1}{2} \int_M |d \varphi|^2 v_g \). For \( k = 1 \), \( E_1 = \tilde{E}_1 = E \), and for \( k = 2 \), the bienergy functional \( E_2 \) is given by

\[
E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g,
\]

where \( \sqrt{\varphi} = h(V, V), V \in \Gamma(\varphi^{-1}TN) \). Then, the first variation formula of the bienergy functional is given by

\[
\frac{d}{dt} \Big|_{t=0} E_2(\varphi_t) = - \int_M h(\tau_2(\varphi), V) v_g.
\]

Here,

\[
\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - R(\tau(\varphi)),
\]

which is called the bitension field of \( \varphi \), and \( J \) is given in (2.4). A smooth map \( \varphi \) of \((M, g)\) into \((N, h)\) is said to be biharmonic if \( \tau_2(\varphi) = 0 \).

For \( k = 3 \), the first variation formula of the trienergy \( E_3 \) given by

\[
E_3(\psi) = \frac{1}{2} \int_M |(d + \delta)(d + \delta)\psi|^2 v_g
\]

is given as follows:

**Theorem 2.1.** The first variational formula of \( E_3 \) is given by

\[
\frac{d}{dt} \Big|_{t=0} E_3(\varphi_t) = - \int_M \langle \tau_3(\varphi), V \rangle v_g,
\]

\[
\tau_3(\varphi) = J(\overline{\Delta}(\tau(\varphi))) - \sum_{i=1}^{m} R^N(\nabla_{e_i} \tau(\varphi), \tau(\varphi)) d \varphi(e_i).
\]

Here, \( \tau_3(\varphi) \) is called the tritension field of \( \varphi \).

For completeness, we give a proof. The proof is standard.

For \( V \in \Gamma(\varphi^{-1}TN) \), let \( \varphi_t (-\epsilon < t < \epsilon) \) of \( \varphi \) be a \( C^\infty \) variation of \( V \) with \( \varphi_0 = \varphi \), \( V(x) = \frac{d}{dt} \big|_{t=0} \varphi_t(x) \) \((x \in M)\). Let us define a \( C^\infty \) map \( F : (-\epsilon, \epsilon) \times M \to N \), in such a way that

\[
\begin{cases}
F(0, x) = \varphi(x), & x \in M, \\
F(t, x) = \varphi_t(x), & -\epsilon < t < \epsilon, \ x \in M.
\end{cases}
\]

We need the following lemma.
Lemma 2.2. For every smooth vector field $X$ in $M$,
\[
\left.\nabla_{\frac{\partial}{\partial t}} \nabla_X \tau(F)\right|_{t=0} = -\nabla_X (\Delta V) + \sum_{j=1}^m \nabla_X \left(R^N(V,d\varphi(e_j))d\varphi(e_j)\right) + \sum_{j=1}^m \nabla_X \left(R^N(V,d\varphi(e_j))d\varphi(e_j)\right) + R^N(V,d\varphi(X))\tau(\varphi),
\]
(2.15)
where $\{e_j\}_{j=1}^m$ is a locally defined orthonormal frame field on $(M,g)$.

Proof. Since $[\frac{\partial}{\partial t}, X] = 0$, we have
\[
\left.\nabla_{\frac{\partial}{\partial t}} (\nabla_X \tau(F))\right|_{t=0} = \nabla_X (\nabla_{\frac{\partial}{\partial t}} \tau(F)) + R^N\left(\frac{\partial}{\partial t}, dF(X)\right) \tau(F).
\]
(2.16)
Due to (23) and (22) in Jiang’s paper ([15], English version, p. 214), we have
\[
\left.\nabla_{\frac{\partial}{\partial t}} \tau(F)\right|_{t=0} = \nabla_X \left(\sum_{j=1}^m \nabla_X (d\varphi(e_j))\right) \bigg|_{t=0}
\]
(2.17)
\[
= -\Delta V + \sum_{j=1}^m R^N(V,d\varphi(e_j))d\varphi(e_j).
\]
(2.18)

By substituting (2.17) into (2.16), we have (2.15). □

Proof of Theorem 2.1. By definition of $E_3$, we have for every $C^\infty$ map $\varphi: M \to N$,
\[
E_3(\varphi) = \frac{1}{2} \int_M \langle d(\delta d\varphi), d(\delta d\varphi) \rangle v_g
\]
(2.18)
\[
= \frac{1}{2} \int_M \sum_{i=1}^m \langle \nabla_{e_i}(\tau(\varphi)), \nabla_{e_i}(\tau(\varphi)) \rangle v_g.
\]

By Lemma 2.2, we have for a $C^\infty$ variation $\varphi_t$ of $V$ with $\varphi_0 = \varphi$,
\[
\frac{d}{dt} \bigg|_{t=0} E_3(\varphi_t) = \int_M \sum_{i=1}^m \langle \nabla_{\frac{\partial}{\partial t}} (\nabla_{e_i}(\tau(F)), \nabla_{e_i}(\tau(F)) \rangle v_g \bigg|_{t=0}
\]
(2.19)
\[
= \int_M \sum_{i=1}^m \left( -\nabla_{e_i} (\Delta V) + \sum_{j=1}^m \nabla_{e_i} (R^N(V,d\varphi(e_j))d\varphi(e_j)) + R^N(V,d\varphi(e_i))\tau(\varphi), \nabla_{e_i}(\tau(\varphi)) \right) v_g.
\]

Here, by using formula for every $\omega_j \in \Gamma(\varphi^{-1}TN)$, $(j = 1, 2)$,
\[
\int_M \sum_{i=1}^m \langle \nabla_{e_i}(\omega_1), \nabla_{e_i}(\omega_2) \rangle v_g = \int_M \langle \Delta \omega_1, \omega_2 \rangle v_g,
\]
(2.20)
we have
\[
\frac{d}{dt} \Big|_{t=0} E_3(\varphi_t) = \int_M \langle V, -\Delta^2 \tau(\varphi) \rangle \, v_g \\
+ \int_M \sum_{j=1}^{m} (R^N(V, d\varphi(e_j))d\varphi(e_j), \Delta \tau(\varphi)) \, v_g \\
+ \int_M \sum_{j=1}^{m} (R^N(V, d\varphi(e_j))\tau(\varphi), \nabla_{e_j} \tau(\varphi)) \, v_g \\
= \int_M \left\langle V, -\Delta^2 \tau(\varphi) + \sum_{j=1}^{m} R^N(\Delta \tau(\varphi), d\varphi(e_j))d\varphi(e_j) \\
+ \sum_{j=1}^{m} R^N(\nabla_{e_j} \tau(\varphi), \tau(\varphi))d\varphi(e_j) \right\rangle v_g,
\]

in which we used the property \( \langle R^N(v_3, v_2), v_1 \rangle = \langle R^N(v_1, v_2), v_3 \rangle \). We have Theorem 2.1. \( \square \)

**Corollary 2.3.** Assume that \((N, h)\) is an \(n\)-dimensional Riemannian manifold \((N(c), h)\) of constant curvature \(c\), and let \(\varphi : (M, g) \to N(c)\) be an isometric immersion. Then, we have
\[
\tau_3(\varphi) = \Delta^2 \tau(\varphi) - \sum_{j=1}^{m} R^N(\Delta \tau(\varphi), d\varphi(e_j))d\varphi(e_j) \\
- c h(\tau(\varphi), \tau(\varphi)) \tau(\varphi).
\]

\(2.21\)

**Proof.** Since \(R^N(X, Y)Z = c \{ h(Y, Z)X - h(X, Z)Y \} \), \((X, Y, Z \in \mathfrak{X}(N))\), we have
\[
\sum_{j=1}^{m} R^N(\nabla_{e_j} \tau(\varphi), \tau(\varphi))d\varphi(e_j) = \sum_{j=1}^{m} c \{ h(\tau(\varphi), d\varphi(e_j)) \nabla_{e_j} \tau(\varphi) \\
- h(\nabla_{e_j} \tau(\varphi), d\varphi(e_j)) \tau(\varphi) \} \\
= -c \sum_{j=1}^{m} h(\nabla_{e_j} \tau(\varphi), d\varphi(e_j)) \tau(\varphi).
\]

\(2.22\)

Because the tension field \(\tau(\varphi)\) is orthogonal to the subspace \(d\varphi(T_x M) \ (x \in M)\) since \(\varphi : (M, g) \to N(c)\) is an isometric immersion. Then,
\[
\sum_{j=1}^{m} h(\nabla_{e_j} \tau(\varphi), d\varphi(e_j)) = \sum_{j=1}^{m} \{ e_j(h(\tau(\varphi), d\varphi(e_j)) - h(\tau(\varphi), \nabla_{e_j} (d\varphi(e_j))) \}) \\
= -h(\tau(\varphi), \sum_{j=1}^{m} \nabla_{e_j} (d\varphi(e_j))) \\
= -h(\tau(\varphi), \tau(\varphi) + \sum_{j=1}^{m} d\varphi(\nabla_{e_j} e_j)) \\
= -h(\tau(\varphi), \tau(\varphi))
\]

\(2.23\)
since $\tau(\varphi) = \sum_{j=1}^{m} \{ \nabla_{e_j} (d\varphi(e_j)) - d\varphi(\nabla_{e_j} e_j) \}$. By substituting (2.23) into (2.22), (2.22) is equal to $c h(\tau(\varphi), \tau(\varphi)) \tau(\varphi)$. Then, the right hand side of (2.14) is equal to $J(\Delta(\tau(\varphi))) - c h(\tau(\varphi), \tau(\varphi)) \tau(\varphi)$. We obtain Corollary 2.3.

Then, we can state our main theorem.

**Theorem 2.4.** Let $\varphi: (M,g) \to N(c)$ be an isometric immersion of a complete Riemannian manifold $(M,g)$ into a Riemannian manifold $N(c)$ of non-positively constant curvature $c$.

1. In the case of $c < 0$, if $\varphi$ is triharmonic and both the extended 4-energy $\tilde{E}_4(\varphi) = \frac{1}{2} \int_{M} |\Delta \tau(\varphi)|^2 v_g$ and the $L^4$-norm $\int_{M} |\tau(\varphi)|^4 v_g$ are finite, then $\varphi$ is harmonic, i.e., minimal.

2. In the case of $c = 0$, and $\text{Vol}(M,g) = \infty$, if $\varphi$ is triharmonic, and $\tilde{E}_4(\varphi) = \frac{1}{2} \int_{M} |\Delta \tau(\varphi)|^2 v_g < \infty$, $E_2(\varphi) = \frac{1}{2} \int_{M} |\tau(\varphi)|^2 v_g < \infty$ and $\int_{M} |\tau(\varphi)|^4 v_g < \infty$, then $\varphi$ is harmonic, i.e., minimal.

3. In the case of $c = 0$ and $\text{Vol}(M,g) < \infty$, if $\varphi$ is triharmonic, and $\tilde{E}_4(\varphi) = \frac{1}{2} \int_{M} |\Delta \tau(\varphi)|^2 v_g < \infty$, $E_3(\varphi) = \frac{1}{2} \int_{M} |\nabla \tau(\varphi)|^2 v_g < \infty$ and $\int_{M} |\tau(\varphi)|^4 v_g < \infty$, then $\varphi$ is harmonic, i.e., minimal.

3. **Proof of Theorem 2.4**

In this section, we will give a proof of Theorem 2.4 which consists of eight steps.

**Proof of Theorem 2.4.** (The first step) For a fixed point $x_0 \in M$, and for every $0 < r < \infty$, we first take a cut-off $C^\infty$ function $\eta$ on $M$ (for instance, see [10]) satisfying that

\[
\begin{cases}
0 \leq \eta(x) \leq 1 & (x \in M), \\
\eta(x) = 1 & (x \in B_r(x_0)), \\
\eta(x) = 0 & (x \notin B_{2r}(x_0)), \\
|\nabla \eta| \leq \frac{2}{r} & (x \in M).
\end{cases}
\]

(3.1)

For a triharmonic map $\varphi: (M,g) \to N(c)$, the tritension field is given as

\[
\tau_3(\varphi) = \Delta^2(\tau(\varphi)) - \sum_{i=1}^{m} R^N(\Delta \tau(\varphi), d\varphi(e_i)) d\varphi(e_i) \nonumber \\
- c h(\tau(\varphi), \tau(\varphi)) \tau(\varphi)
\]

(3.2)
\[ \Delta = - \sum_{i=1}^{m} \{( \nabla \epsilon_i, \nabla \epsilon_i - \nabla \epsilon_i ) \}. \] By taking the inner product of (3.2) and \( \bar{\Delta} \tau(\varphi) \eta^2 \) and integrate over \( M \), so we have
\[
\int_M \langle \Delta^2 (\tau(\varphi)), \bar{\Delta} \tau(\varphi) \eta^2 \rangle v_g \\
- \int_M \sum_{i=1}^{m} \langle R^N(\Delta \tau(\varphi), d\varphi(e_i)) d\varphi(e_i), \bar{\Delta} \tau(\varphi) \rangle \eta^2 v_g \\
- \epsilon \int_M \langle \tau(\varphi), \tau(\varphi) \rangle \langle \tau(\varphi), \bar{\Delta} \tau(\varphi) \rangle \eta^2 v_g \\
= 0. \tag{3.3}
\]

(The second step) For the first term of the left hand side of (3.3),
\[
\int_M \langle \Delta^2 (\tau(\varphi)), \bar{\Delta} \tau(\varphi) \eta^2 \rangle v_g = \int_M \sum_{i=1}^{m} \langle \nabla \epsilon_i(\Delta \tau(\varphi)), \nabla \epsilon_i(\bar{\Delta} \tau(\varphi) \eta^2) \rangle v_g. \tag{3.4}
\]
Here by using that \( e_i(\eta^2) = 2\eta e_i(\eta) \) and
\[
\nabla \epsilon_i(\bar{\Delta} \tau(\varphi) \eta^2) = \nabla \epsilon_i(\bar{\Delta} \tau(\varphi)) \eta^2 + 2\eta \nabla \epsilon_i \bar{\Delta} \tau(\varphi),
\]
(3.4) coincides with
\[
\int_M |\nabla \Delta \tau(\varphi)|^2 \eta^2 v_g \\
+ 2 \int_M \sum_{i=1}^{m} \langle \eta \nabla \epsilon_i(\bar{\Delta} \tau(\varphi)), \nabla \epsilon_i \bar{\Delta} \tau(\varphi) \rangle v_g, \tag{3.5}
\]
where we used
\[
|\nabla \Delta \tau(\varphi)|^2 = \sum_{i=1}^{m} \langle \nabla \epsilon_i(\bar{\Delta} \tau(\varphi)), \nabla \epsilon_i(\bar{\Delta} \tau(\varphi)) \rangle.
\]
For the second term of (3.5), put \( S_i := \eta \nabla \epsilon_i(\bar{\Delta} \tau(\varphi)) \), and \( T_i := \nabla \epsilon_i \bar{\Delta} \tau(\varphi) \) \((i = 1 \cdots, m)\), and recall the Young’s inequality: for every \( \epsilon > 0 \),
\[
\pm 2 \langle S_i, T_i \rangle \leq \epsilon |S_i|^2 + \frac{1}{\epsilon} |T_i|^2,
\]
because of the inequality \( 0 \leq |\sqrt{\epsilon} S_i \pm \frac{1}{\sqrt{\epsilon}} T_i|^2 \). Therefore, (3.5) is bigger than or equal to
\[
\int_M |\nabla \Delta \tau(\varphi)|^2 \eta^2 v_g \\
- \left\{ \epsilon \int_M \eta^2 |\nabla \Delta \tau(\varphi)|^2 \eta^2 v_g + \frac{1}{\epsilon} \int_M |\nabla \eta|^2 |\Delta \tau(\varphi)|^2 v_g \right\} \\
= (1 - \epsilon) \int_M |\nabla \Delta \tau(\varphi)|^2 \eta^2 v_g - \frac{1}{\epsilon} \int_M |\Delta \tau(\varphi)|^2 |\nabla \eta|^2 v_g. \tag{3.6}
\]
(The third step) For the second term of the left hand side of (3.3),

\[- \int_M \sum_{i=1}^m (R^N(\Delta \tau(\varphi), d\varphi(e_i)) d\varphi(e_i), \Delta \tau(\varphi)) \eta^2 \nu_g \]

\[= \int_M \eta^2 \sum_{i=1}^m \langle R^N(d\varphi(e_i), \Delta \tau(\varphi)) d\varphi(e_i), \Delta \tau(\varphi) \rangle \nu_g \]

\[= c \int_M \eta^2 \sum_{i=1}^m \left\{ (\Delta \tau(\varphi), d\varphi(e_i))^2 - (d\varphi(e_i), d\varphi(e_i)) \langle \Delta \tau(\varphi), \Delta \tau(\varphi) \rangle \right\} \nu_g \]

\[= c \int_M \eta^2 \left\{ (\Delta \tau(\varphi), d\varphi)^2 - |\Delta \tau(\varphi)|^2 |d\varphi|^2 \right\} \nu_g \]

(3.7) \[\geq 0\]

since \(c \leq 0\).

(The fourth step) For the third term of the left hand side of (3.3), since

\[\langle \tau(\varphi), \Delta \tau(\varphi) \rangle = \frac{1}{2} \Delta |\tau(\varphi)|^2 + |\nabla \tau(\varphi)|^2,\]

and

\[|\tau(\varphi)|^2 \Delta |\tau(\varphi)|^2 = \frac{1}{2} \Delta |\tau(\varphi)|^4 + |\nabla |\tau(\varphi)|^2|^2,\]

we have

\[-c \int_M \langle \tau(\varphi), \tau(\varphi) \rangle \langle \tau(\varphi), \Delta \tau(\varphi) \rangle \eta^2 \nu_g \]

\[= -c \int_M |\tau(\varphi)|^2 \Delta |\tau(\varphi)|^2 \eta^2 \nu_g \]

\[= -c \int_M |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta^2 \nu_g \]

\[= -c \int_M \Delta |\tau(\varphi)|^4 \eta^2 \nu_g - \frac{c}{2} \int_M |\nabla |\tau(\varphi)|^2|^2 \eta^2 \nu_g \]

\[= -\frac{c}{4} \int_M |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta^2 \nu_g \]

\[= -c \int_M \sum_{i=1}^m \langle \nabla e_i, |\tau(\varphi)|^4, \nabla e_i, \eta^2 \rangle \nu_g - c \int_M |\nabla |\tau(\varphi)|^2|^2 \eta^2 \nu_g \]

\[= -c \int_M |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta^2 \nu_g \]

\[= -c \int_M \eta \sum_{i=1}^m \langle \nabla e_i, |\tau(\varphi)|^4, \nabla e_i, \eta \rangle \nu_g - \frac{c}{2} \int_M |\nabla |\tau(\varphi)|^2|^2 \eta^2 \nu_g \]

\[= -c \int_M \eta \sum_{i=1}^m \langle \nabla e_i, |\tau(\varphi)|^4, \nabla e_i, \eta \rangle \nu_g - \frac{c}{2} \int_M |\nabla |\tau(\varphi)|^2|^2 \eta^2 \nu_g \]

(3.8) \[-c \int_M |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta^2 \nu_g.]
(The fifth step) Here, we have
\[
\sum_{i=1}^{m} |\langle \nabla e_i, |\tau(\varphi)|^4, \nabla e_i, \eta \rangle| \leq \sum_{i=1}^{m} |\nabla e_i| |\tau(\varphi)|^4 |\nabla \eta|
\]
(3.9)
\[
= |\nabla |\tau(\varphi)|^4| |\nabla \eta|,
\]
so that we have, since \(c \leq 0\),
\[
(3.8) \geq \frac{c}{2} \int_{M} |\nabla |\tau(\varphi)|^4| \eta |\nabla \eta| v_{g}
- \frac{c}{2} \int_{M} |\nabla |\tau(\varphi)|^2| \eta^2 \nabla \eta| v_{g}
- c \int_{M} |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta v_{g}.
\]
(3.10)
Since we have
\[
|\nabla |\tau(\varphi)|^4| = 2 |\tau(\varphi)|^2 \cdot |\nabla |\tau(\varphi)|^2|,
\]
the right hand side of (3.10) is equal to
\[
c \int_{M} |\nabla |\tau(\varphi)|^2| \eta \cdot |\tau(\varphi)|^2 |\nabla \eta| v_{g}
- \frac{c}{2} \int_{M} |\nabla |\tau(\varphi)|^2| ^2 \eta^2 v_{g}
- c \int_{M} |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta ^2 \nabla \eta| v_{g}.
\]
(3.11)
Now applying again for \(A := |\nabla |\tau(\varphi)|^2| \eta\) and \(B := |\tau(\varphi)|^2 |\nabla \eta|\), the Young’s inequality: for every positive number \(\delta > 0\),
\[
\pm 2 \langle A, B \rangle \leq \delta |A|^2 + \frac{1}{\delta} |B|^2,
\]
we obtain because of \(c \leq 0\),
\[
(3.11) \geq \frac{c \delta}{2} \int_{M} |\nabla |\tau(\varphi)|^2| ^2 \eta^2 v_{g} + \frac{c}{2 \delta} \int_{M} |\tau(\varphi)|^4 |\nabla \eta|^2 v_{g}
- \frac{c}{2} \int_{M} |\nabla |\tau(\varphi)|^2| ^2 \eta^2 v_{g}
- c \int_{M} |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta ^2 \nabla \eta| v_{g}
= - \frac{c}{2} (1 - \delta) \int_{M} |\nabla |\tau(\varphi)|^2| ^2 \eta^2 v_{g}
+ \frac{c}{2 \delta} \int_{M} |\tau(\varphi)|^4 |\nabla \eta|^2 v_{g}
- c \int_{M} |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta ^2 v_{g}.
\]
(3.12)
By putting $\delta = \frac{1}{2}$, we obtain

$$
(3.12) = -\frac{c}{4} \int_M |\nabla |\tau| \phi| \eta^2 v_g + c \int_M |\tau| \phi| |\nabla \eta|^2 v_g
$$

$$
(3.13) = -c \int_M |\tau| |\nabla \tau| \phi| \eta^2 v_g.
$$

(The sixth step) All together the above, we obtain

$$
0 \geq (1 - \epsilon) \int_M |\nabla \nabla \tau| \phi| \eta^2 v_g - \frac{1}{\epsilon} \int_M |\nabla \nabla \tau| \phi| |\nabla \eta|^2 v_g
$$

$$
- c \int_M |\nabla |\tau| \phi| |\nabla \eta|^2 v_g + c \int_M |\tau| \phi| |\nabla \eta|^2 v_g
$$

$$
(3.14) = -c \int_M |\tau| |\nabla \tau| \phi| \eta^2 v_g
$$

which is equivalent to that

$$
\frac{1}{\epsilon} \int_M |\nabla \nabla \tau| \phi| \eta^2 v_g - c \int_M |\tau| \phi| |\nabla \eta|^2 v_g
$$

$$
\geq (1 - \epsilon) \int_M |\nabla \nabla \tau| \phi| \eta^2 v_g - \frac{c}{4} \int_M |\nabla |\tau| \phi| |\nabla \eta|^2 v_g
$$

$$
(3.15) = -c \int_M |\tau| |\nabla \tau| \phi| \eta^2 v_g.
$$

Here, we put $\epsilon = \frac{1}{2}$ in (3.15), and notice that $\eta = 1$ on $B_r(x_0)$, and $|\nabla \eta| \leq \frac{2}{r}$. Then, we have

$$
\frac{8}{r^2} \int_M |\nabla \nabla \tau| \phi| \eta^2 v_g - \frac{4c}{r^2} \int_M |\tau| |\nabla \eta|^2 v_g
$$

$$
\geq \frac{1}{2} \int_{B_r(x_0)} |\nabla \nabla \tau| \phi| \eta^2 v_g - \frac{c}{4} \int_{B_r(x_0)} |\nabla |\tau| \phi| |\nabla \eta|^2 v_g
$$

$$
- c \int_{B_r(x_0)} |\tau| |\nabla \tau| \phi| \eta^2 v_g.
$$

(The seventh step) By virtue of our assumptions that $\int_M |\nabla \tau| \phi| \eta^2 v_g < \infty$ and $\int_M |\tau| |\nabla \eta|^2 v_g < \infty$, and $B_r(x_0)$ goes to $M$ if $r \to \infty$ because of completeness of $(M, g)$, the left hand side of (3.16) goes to zero if $r \to \infty$. We obtain

$$
\frac{1}{2} \int_M |\nabla \nabla \tau| \phi| \eta^2 v_g - \frac{c}{4} \int_M |\nabla |\tau| \phi| |\nabla \eta|^2 v_g
$$

$$
- c \int_M |\tau| |\nabla \tau| \phi| \eta^2 v_g
$$

$$
(3.17) \leq 0.
$$

Since $c \leq 0$, all the terms of (3.17) are non-negative and we have

$$
(3.18) \nabla \nabla \tau = 0.
$$
In the case \( c < 0 \), we have

\[
\begin{cases}
\nabla \Delta \tau (\varphi) = 0, \\
\n\nabla |\tau (\varphi)|^2 = 0, \\
\n|\tau (\varphi)|^2 |\nabla \tau (\varphi)|^2 = 0.
\end{cases}
\]

(3.19)

Notice that by (3.18), \( \Delta \tau (\varphi) \) is constant, say \( c_0 \). Because, for every \( C^\infty \) vector field \( X \) on \( M \), by (3.18),

\[
X |\Delta \tau (\varphi)|^2 = 2 \langle \nabla_X \Delta \tau (\varphi), \Delta \tau (\varphi) \rangle = 0.
\]

(The eighth step) In the case that \( c < 0 \), by the second equation of (3.19), \( |\tau (\varphi)|^2 \) is constant. Therefore, by the last equation of (3.19), it holds that \( |\tau (\varphi)|^2 = 0 \) or \( |\nabla \tau (\varphi)|^2 = 0 \), i.e., \( \nabla \tau (\varphi) \equiv 0 \). We have \( \Delta \tau (\varphi) \equiv 0 \). Since \( \varphi : (M, g) \to N(c) \) is triharmonic, (3.2) holds. Substituting \( \Delta \tau (\varphi) \equiv 0 \) in (3.2), we obtain \( \tau (\varphi) \equiv 0 \) in the case of \( c < 0 \).

In the case that \( c = 0 \) and \( \text{Vol}(M, g) = \infty \), we have

\[
\infty > 2 \int_M |\Delta \tau (\varphi)|^2 v_g = c_0 \text{Vol}(M, g).
\]

Thus, we obtain \( c_0 = 0 \), i.e., \( \Delta \tau (\varphi) \equiv 0 \), i.e., \( \varphi \) is a biharmonic map of \((M, g)\) into the Euclidean space \( N(0) \). Then, applying (2) in Theorem 2.1 of [25], or Theorem 3.1 in [20], \( \varphi \) is harmonic, i.e., minimal, by virtue of the assumption that the bienergy \( E_2(\varphi) = \frac{1}{2} \int_M |\tau (\varphi)|^2 \) is finite.

In the case that \( c = 0 \) and \( \text{Vol}(M, g) < \infty \), we have \( E(\varphi) = \frac{m}{2} \text{Vol}(M, g) < \infty \) since \( \varphi \) is an isometric immersion. Furthermore, we have that

\[
E_2(\varphi) = \frac{1}{2} \int_M |\tau (\varphi)|^2 v_g \leq \frac{1}{2} \left( \int_M 1 v_g \right)^{1/2} \left( \int_M |\tau (\varphi)|^4 v_g \right)^{1/2} < \infty.
\]

By virtue of the assumption that the 4-energy \( E_4(\varphi) \) and the 3-energy \( E_3(\varphi) = \frac{1}{2} \int_M |\nabla \tau (\varphi)|^2 v_g \) are finite, we can apply again Theorem 3.1 in [20], and then we also obtain \( \tau (\varphi) = 0 \), i.e., \( \varphi \) is minimal.

We have Theorem 2.4.

\[\Box\]

4. Triharmonic isometric immersions with the constant mean curvature

In the case that \( |\tau (\varphi)| \) is constant, that is, the mean curvature is constant, the finiteness of \( \int_M |\Delta \tau (\varphi)|^2 v_g \) in Theorem 2.4 can be replaced into the weaker condition as follows.

**Theorem 4.1.** Let \( \varphi : (M, g) \to N(c) \) be an isometric immersion with the constant mean curvature from a complete Riemannian manifold \((M, g)\) into a Riemannian manifold \( N(c) \) of negatively constant curvature \( c \). If \( \varphi \) is triharmonic and \( \frac{1}{2} \int_M |\Delta \tau (\varphi)|^p v_g < \infty \) (for some \( 2 \leq p < \infty \)), then \( \varphi \) is harmonic, i.e., minimal.

Before mentioning the proof of Theorem 4.1, we shall show the following lemma.

**Lemma 4.2.** Assume that \( \alpha \in \Gamma(\varphi^{-1}TN) \) satisfies that

\[
|\alpha|^q |\nabla \alpha|^2 = 0 \quad \text{(for some} \ q > 0)\].
Then, (1) $|\alpha|$ is constant everywhere on $M$, and then
(2) either $\alpha = 0$ or $\nabla \alpha = 0$.

Proof. We first notice that for every $\alpha \in \Gamma(\varphi^{-1}T)$, it holds that
\begin{equation}
|\nabla |\alpha|| \leq |\nabla \alpha| \quad \text{(everywhere on the set } \{x \in M| |\alpha x| \neq 0\}).
\end{equation}

Because
\begin{align*}
|\alpha||\nabla |\alpha|| &= \frac{1}{2}|\nabla |\alpha|^2 |
\quad = \frac{1}{2}|\nabla h(\alpha, \alpha)|
\quad = |h(\nabla \alpha, \alpha)|
\quad \leq |\nabla \alpha||\alpha|,
\end{align*}
so we obtain (4.1) due to (4.2).

Therefore, we have
\begin{equation}
0 \leq |\alpha|^q|\nabla |\alpha||^2 \leq |\alpha|^q|\nabla \alpha|^2 = 0
\end{equation}
everywhere on $M$. Thus, we have
\begin{equation}
|\alpha|^q|\nabla |\alpha||^2 = 0.
\end{equation}

By (4.4), we have
\begin{equation}
\left(\frac{2}{q+2}\right)^2|\nabla |\alpha|^{q+1}|^2 = \left(\frac{2}{q+2}\right)^2\left(\frac{q}{2} + 1\right)|\alpha|^{q/2} |\nabla |\alpha||^2
\quad = |\alpha|^q|\nabla |\alpha||^2
\quad = 0.
\end{equation}

We have
\begin{equation}
\nabla |\alpha|^{q+1} = 0,
\end{equation}
which implies that $|\alpha|^{q/2+1}$ is constant, i.e., $|\alpha|$ is a constant, say $C_0$. Then,
(1) in the case that $C_0 = 0$, we have $\alpha = 0$. (2) In the case that $C_0 \neq 0$, we have
\begin{equation}
C_0^q |\nabla \alpha|^2 = |\alpha|^q |\nabla \alpha|^2
\quad = 0
\end{equation}
by virtue of the assumption of Lemma 4.2. We obtain $\nabla \alpha = 0$. \hfill \Box

By using Lemma 4.2, we shall show Theorem 4.1.

Proof of Theorem 4.1. We will use an argument similar to the proof of Theorem 2.4.
We take the cut-off function in the first step of the proof of Theorem 2.4. By taking the inner product of (3.2) and $|\Delta \tau(\varphi)|^{p-2} \Delta \tau(\varphi) \eta^2$ and integrate over $M$,
so we have

\[
\int_M (\nabla^2 (\tau(\varphi)), |\nabla \tau(\varphi)|^{p-2} \nabla \tau(\varphi) \eta^2) v_g
- \int_M \sum_{i=1}^m (R^N (\Delta \tau(\varphi), d\varphi(e_i)) d\varphi(e_i), |\nabla \tau(\varphi)|^{p-2} \nabla \tau(\varphi)) \eta^2 v_g
- c \int_M \langle \tau(\varphi), \tau(\varphi) \rangle \langle \tau(\varphi), |\nabla \tau(\varphi)|^{p-2} \nabla \tau(\varphi) \rangle \eta^2 v_g
= 0.
\]

(4.8)

For the first term of the left hand side of (4.8),

\[
\int_M (\nabla^2 (\tau(\varphi)), |\nabla \tau(\varphi)|^{p-2} \nabla \tau(\varphi) \eta^2) v_g
= \int_M \sum_{i=1}^m \langle \nabla_{e_i}(\nabla \tau(\varphi)), \nabla_{e_i}(|\nabla \tau(\varphi)|^{p-2} \nabla \tau(\varphi) \eta^2) \rangle v_g.
\]

(4.9)

Here we use

\[
\nabla_{e_i} |\nabla \tau(\varphi)|^{p-2} = (p-2) |\nabla \tau(\varphi)|^{p-4} \langle \nabla_{e_i} \nabla \tau(\varphi), \nabla \tau(\varphi) \rangle,
\]

and then the right hand side of (4.9) coincides with

\[
\int_M |\nabla \tau(\varphi)|^{p-2} |\nabla \nabla \tau(\varphi)|^2 \eta^2 v_g
+ 2 \int_M \sum_{i=1}^m \langle \eta |\nabla \tau(\varphi)|^{p-4} \nabla_{e_i} \nabla \tau(\varphi), \nabla_{e_i} \eta |\nabla \tau(\varphi)|^{p-4} \nabla \tau(\varphi) \rangle v_g
+ (p-2) \int_M \sum_{i=1}^m |\nabla \tau(\varphi)|^{p-4} \langle \nabla_{e_i} |\nabla \tau(\varphi)|^{p-4} \nabla \tau(\varphi) \rangle \eta^2 v_g.
\]

(4.10)

For the second term of (4.10), put \( S_i := \eta |\nabla \tau(\varphi)|^{p-4} \nabla_{e_i} |\nabla \tau(\varphi)|^{p-4} \nabla \tau(\varphi) \), and \( T_i := \nabla_{e_i} \eta |\nabla \tau(\varphi)|^{p-4} \nabla \tau(\varphi) \) \((i = 1 \cdots, m)\), and recall the Young’s inequality: for every \( \epsilon > 0 \),

\[
\pm 2 \langle S_i, T_i \rangle \leq \epsilon |S_i|^2 + \frac{1}{\epsilon} |T_i|^2,
\]

because of the inequality \( 0 \leq |\sqrt{\epsilon} S_i + \frac{1}{\sqrt{\epsilon}} T_i|^2 \). Therefore, (4.10) is bigger than or equal to

\[
\int_M |\nabla \tau(\varphi)|^{p-2} |\nabla \nabla \tau(\varphi)|^2 \eta^2 v_g
- \left\{ \epsilon \int_M \eta^2 |\nabla \tau(\varphi)|^{p-2} |\nabla \nabla \tau(\varphi)|^2 v_g + \frac{1}{\epsilon} \int_M |\nabla \eta|^2 |\nabla \tau(\varphi)|^p v_g \right\}
= (1-\epsilon) \int_M \eta^2 |\nabla \tau(\varphi)|^{p-2} |\nabla \nabla \tau(\varphi)|^2 v_g - \frac{1}{\epsilon} \int_M |\nabla \eta|^2 |\nabla \tau(\varphi)|^p v_g.
\]

(4.11)
For the second term of the left hand side of (4.8), by the same reason of the third step of Theorem 2.4 we have

\[ -\int_M \sum_{i=1}^m \langle R^N(\overline{\Delta} \tau(\varphi), d\varphi(e_i)) d\varphi(e_i), \overline{\Delta} \tau(\varphi) \rangle |\overline{\Delta} \tau(\varphi)|^{p-2} \eta^2 \geq 0. \]  

(4.12)

For the third term of the left hand side of (4.8), since the mean curvature is constant,

\[ \langle \tau(\varphi), \overline{\Delta} \tau(\varphi) \rangle = |\nabla \tau(\varphi)|^2. \]

By using this, the third term of the left hand side of (4.8) is equal to

\[ -c \int_M \langle \tau(\varphi), \tau(\varphi) \rangle \langle \tau(\varphi), \overline{\Delta} \tau(\varphi) \rangle |\overline{\Delta} \tau(\varphi)|^{p-2} \eta^2 \varepsilon_g \]

\[ = -c \int_M |\overline{\Delta} \tau(\varphi)|^{p-2} |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta^2 \varepsilon_g. \]

(4.13)

Combining (4.11), (4.12) and (4.13), and noticing that \( \eta = 1 \) on \( B_r(x_0) \), we have

\[ (1 - \epsilon) \int_{B_r(x_0)} \eta^2 |\overline{\Delta} \tau(\varphi)|^{p-2} |\nabla \overline{\Delta} \tau(\varphi)|^2 \varepsilon_g \]

\[ -c \int_{B_r(x_0)} |\overline{\Delta} \tau(\varphi)|^{p-2} |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \eta^2 \varepsilon_g \]

\[ \leq \frac{1}{\epsilon} \int_M |\nabla \eta|^2 |\overline{\Delta} \tau(\varphi)|^p \varepsilon_g \]

\[ \leq \frac{1}{\epsilon} \frac{4}{r^2} \int_M |\overline{\Delta} \tau(\varphi)|^p \varepsilon_g, \]

(4.14)

where the last inequality follows from \( |\nabla \eta| \leq \frac{2}{r} \). By virtue of our assumptions that \( \int_M |\overline{\Delta} \tau(\varphi)|^p \varepsilon_g < \infty \) and \( B_r(x_0) \) goes to \( M \) if \( r \to \infty \) because of completeness of \( (M, g) \), the right hand side of (4.14) goes to zero if \( r \to \infty \). We obtain

\[ (1 - \epsilon) \int_M |\overline{\Delta} \tau(\varphi)|^{p-2} |\nabla \overline{\Delta} \tau(\varphi)|^2 \varepsilon_g - c \int_M |\overline{\Delta} \tau(\varphi)|^{p-2} |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 \varepsilon_g \leq 0. \]

(4.15)

Since we can take that \( 0 < \epsilon < 1 \), and the assumption \( c < 0 \), all the terms of (4.15) are non-negative and we have

\[ \left\{ \begin{array}{c} |\overline{\Delta} \tau(\varphi)|^{p-2} |\nabla \overline{\Delta} \tau(\varphi)|^2 = 0, \\ |\overline{\Delta} \tau(\varphi)|^{p-2} |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 = 0. \end{array} \right. \]

(4.16)

If we put \( \alpha := \overline{\Delta} \tau(\varphi) \), (4.16) is equivalent to that

\[ |\alpha|^{p-2} |\nabla \alpha|^2 = 0, \]

(4.17)

and

\[ |\alpha|^{p-2} |\tau(\varphi)|^2 |\nabla \tau(\varphi)|^2 = 0. \]

(4.18)

Applying Lemma (4.2) for putting \( q := p - 2 > 0 \), by (4.17), we have \( |\overline{\Delta} \tau(\varphi)| = |\alpha| \) is a constant, say \( C_1 \).

In the Case (I): \( C_1 = 0 \), we have

\[ \overline{\Delta} \tau(\varphi) = \alpha = 0 \quad \text{(everywhere on } M). \]
By vanishing of the tritension field (3.2), we have $\tau(\varphi) = 0$ on $M$.

In the Case (II): $C_1 \neq 0$, we have by (4.18),

$$(4.19) \quad |\tau(\varphi)|^2 |\nabla\tau(\varphi)|^2 = 0.$$ 

By applying Lemma 4.2 for $\alpha := \tau(\varphi)$ and $q = 2$, we have $|\tau(\varphi)|$ is a constant, say $C_2$. In the case (II-i) $C_2 = 0$, we clearly have $\tau(\varphi) = 0$ on $M$. In the case (II-ii) $C_2 \neq 0$, we have $\nabla\tau(\varphi) = 0$ on $M$ by virtue of (4.19). Then, we have $\Delta\tau(\varphi) = 0$ on $M$ which contradicts that $C_1 \neq 0$. This case (II-ii) does not occur. □

**Remark 4.3.** (1) Let $\varphi : (M, g) \to N(0) = \mathbb{E}^n$ be an isometric immersion of a complete Riemannian manifold $(M, g)$ into the Euclidean space $N(0)$ with $|\tau(\varphi)|$ is constant. In the case that $\text{Vol}(M, g) < \infty$, if $\varphi$ is triharmonic and both $\int_M |\tau(\varphi)|^p v_g < \infty$ (for some $2 \leq p < \infty$) and the 3-energy is finite, then $\varphi$ is harmonic, i.e., minimal (cf. [22]).

(2) In the case that $\text{Vol}(M, g) = \infty$, if $\varphi : (M, g) \to (N, h)$ is a smooth map from a Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ with $|\tau(\varphi)|$ is constant and $\int_M |\tau(\varphi)|^p v_g < \infty$ (for some $0 < p < \infty$), then $\varphi$ is harmonic.

**References**

[1] P. Baird and J. Eells, *A conservation law for harmonic maps*, Lecture Notes in Math., Springer, 894 (1981), 1–25.

[2] P. Baird, A. Fardoun and S. Ouakkas, *Liouville-type theorems for biharmonic maps between Riemannian manifolds*, Adv. Calc. Var., 3 (2010), 421–445.

[3] P. Baird and J. Wood, *Harmonic Morphisms Between Riemannian Manifolds*, Oxford Science Publication, 2003, Oxford.

[4] R. Caddeo, S. Montaldo, P. Piu, *On biharmonic maps*, Contemp. Math., 288 (2001), 286–290.

[5] B.Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math., 17 (1991), 169–188.

[6] J. Eells, L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc., 10 (1978), 1–68.

[7] J. Eells, L. Lemaire, *Selected topics in harmonic maps*, CBMS, 50, Amer. Math. Soc, 1983.

[8] J. Eells, L. Lemaire, *Another Report on Harmonic Maps*, Bull. London Math. Soc., 20 (1988), 385–524.

[9] J. Eells and J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., 86 (1964), 109–160.

[10] M.P. Gaffney, *A special Stokes’ theorem for complete Riemannian manifold*, Ann. Math., 60 (1954), 140–145.

[11] S. Gudmundsson, *The Bibliography of Harmonic Morphisms*, \url{http://matematik.lu.se/matematiklu/personal/sigma/harmonic/bibliography.html}

[12] T. Ichihara, J. Inoguchi, H. Urakawa, *Biharmonic maps and bi-Yang-Mills fields*, Note di Matematica, 28, (2009), 233–275.

[13] T. Ichihara, J. Inoguchi, H. Urakawa, *Classifications and isolation phenomena of biharmonic maps and bi-Yang-Mills fields*, Note di Matematica, 30, (2010), 15–48.

[14] S. Ishihara, S. Ishikawa, *Notes on relatively harmonic immersions*, Hokkaido Math. J., 4 (1975), 234–246.

[15] G.Y. Jiang, *2-harmonic maps and their first and second variational formula*, Chinese Ann. Math., 7A (1986), 388–402; Note di Matematica, 28 (2009), 209–232.

[16] A. Kasue, *Riemannian Geometry*, in Japanese, Baidu-kan, Tokyo, 2001.

[17] T. Lamm, *Biharmonic map heat flow into manifolds of nonpositive curvature*, Calc. Var., 22 (2005), 421–445.

[18] E. Loubeau, C. Oniciuc, *The index of biharmonic maps in spheres*, Compositio Math., 141 (2005), 729–745.

[19] E. Loubeau and C. Oniciuc, *On the biharmonic and harmonic indices of the Hopf map*, Trans. Amer. Math. Soc., 359 (2007), 5239–5256.
[20] E. Loubeau and Y-L. Ou, Biharmonic maps and morphisms from conformal mappings, Tohoku Math. J., 62 (2010), 55–73.
[21] S. Montaldo, C. Oniciuc, A short survey on biharmonic maps between Riemannian manifolds, Rev. Un. Mat. Argentina 47 (2006), 1–22.
[22] S. Maeta, Polyharmonic maps of order k with finite $L^p$ $k$-energy into Euclidean spaces, arXiv:1305.7065v4.
[23] N. Nakauchi and H. Urakawa, Biharmonic hypersurfaces in a Riemannian manifold with non-positive Ricci curvature, Ann. Global Anal. Geom., 40 (2011), 125–131.
[24] N. Nakauchi and H. Urakawa, Biharmonic submanifolds in a Riemannian manifold with non-positive curvature, Results Math. 63 (2013), 467–474.
[25] N. Nakauchi, H. Urakawa and S. Gudmundsson, Biharmonic maps into a Riemannian manifold of non-positive curvature, to appear in Geometriae Dedicata, arXiv: 1201.6457v4.
[26] N. Nakauchi and H. Urakawa, Polyharmonic maps into the Euclidean space, arXiv: 1307.5089v2.
[27] C. Oniciuc, On the second variation formula for biharmonic maps to a sphere, Publ. Math. Debrecen., 67 (2005), 285–303.
[28] Ye-Lin Ou and Liang Tang, The generalized Chen’s conjecture on biharmonic submanifolds is false, arXiv: 1006.1838v1.
[29] T. Sasahara, Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors, Publ. Math. Debrecen, 67 (2005), 285–303.
[30] R. Schoen and S.T. Yau, Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature, Comment. Math. Helv. 51 (1976), 333–341.
[31] S.T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, Indiana Univ. Math. J., 25 (1976), 659–670.
[32] Z-P Wang and Y-L Ou, Biharmonic Riemannian submersions from 3-manifolds, Math. Z., 269 (2011), 917–925, arXiv: 1002.4439v1.

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