Nonlinear partial differential equations with delay: linear stability/instability of solutions, numerical integration

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Abstract. The paper deals with partial differential equations of parabolic and hyperbolic types that contain a nonlinear kinetic function with delay. Conditions for the linear stability and instability of stationary solutions of reaction-diffusion equations and conditions for the instability of solutions of more complicated equations with delay are formulated. A nonstationary solution of the model initial-boundary value problem with delay and quadratic nonlinearity is investigated for stability/instability. A numerical solution of the test problem in the domain of stability is obtained by the method of lines.

Keywords: partial differential equations with delay, delay reaction-diffusion equations, exact solutions, stability and instability of solutions, numerical integration

1. Introduction

Differential equations with delay arise in biology, biophysics, biochemistry, chemistry, medicine, control, climate model theory, fluid mechanics, heat conduction and other areas [1–5]. It is noteworthy that these equations occur in the mathematical theory of artificial neural networks, whose results are used for signal and image processing and in image recognition problems [6, 7].

Differential equations with delay are used to describe phenomena and processes, the state of which depends not only on the current time $t$, but also on some moment $t - \tau$ in the past. The initial conditions of the corresponding initial-boundary value problems must be specified on the interval $-\tau \leq t \leq 0$ (or $0 \leq t \leq \tau$); the boundary conditions are given in the same way as for problems without delay.

Many methods of numerical integration of partial differential equations without delay after certain modification can be used to solve initial-boundary value problems of the corresponding equations with delay (see article [9] and references therein). The modification, development and use of numerical methods must be carried out taking into account the qualitative features of the equations with delay, associated primarily with the possible instability of solutions.

The most obvious and very effective way to evaluate the accuracy of numerical methods is a direct comparison of numerical and exact solutions of test problems. Many exact solutions of nonlinear partial differential equations with delay (as well as systems of such equations) are given in [4, 5, 10–19, 21]. These equations and their exact solutions contain a number of...
free parameters (which can be varied) and can be used to formulate test problems that allow constructive investigation of the region of applicability of various numerical methods.

In this paper we discuss questions related to the linear instability of solutions of a certain class of nonlinear partial differential equations with delay; we give an example of numerical integration of the equation of this class by the method of lines.

2. Linear stability (instability) of stationary solutions of problems with delay

Let us consider a fairly general class of nonlinear partial differential equations with delay:

\[ \varepsilon u_{tt} + \sigma(u, w) u_t = a u_{xx} + f(u, w), \quad w = u(x, t - \tau), \quad (1) \]

where \( a > 0, \varepsilon \geq 0, \sigma = \sigma(u, w) \geq 0 (\varepsilon + \sigma > 0), \tau > 0. \) The values \( \varepsilon = 0, \sigma = 1 \) define a subclass of reaction-diffusion equations with delay; the values \( \varepsilon = 1, \sigma = 0 \) define a subclass of nonlinear equations of Klein-Gordon type.

Let the constant \( u = u_0 \) be a solution of Eq. (1), i.e. we have the equality

\[ f(u_0, u_0) = 0. \quad (2) \]

The constant \( u_0 \) is also a solution of the problem for Eq. (1) on the interval \( 0 \leq x \leq h \) with the following initial data and Dirichlet boundary conditions:

\[ u(x, t) = u_0, \quad u_t(x, t) = 0 \quad \text{if} \quad -\tau \leq t \leq 0, \quad (3) \]
\[ u(0, t) = u(h, t) = u_0 \quad \text{if} \quad t > 0. \quad (4) \]

(For \( \varepsilon = 0 \), the second initial condition for the derivative \( u_t \) in (3) should be omitted.)

We seek perturbed solutions of the form

\[ u = u_0 + \delta e^{-\lambda t} \sin(\pi n x / h), \quad n = 1, 2, \ldots, \quad (5) \]

where \( \delta \) is a small parameter, \( \lambda \) is the spectral parameter (note that solution (5) satisfies the boundary conditions (4)). Substituting (5) into the considered equation (1) and neglecting terms of the order of smallness \( \delta^2 \) and higher, we obtain the dispersion (characteristic) equation for determining the spectral parameter \( \lambda \):

\[ Q(\lambda) = -\varepsilon \lambda^2 + \sigma(u_0, u_0) \lambda - a (\pi n / h)^2 + f_u(u_0, u_0) + f_w(u_0, u_0) e^{\lambda \tau} = 0, \quad (6) \]

where \( n = 1, 2, \ldots \). We note that the function \( Q(\lambda) \) for \( f_w(u_0, u_0) \neq 0 \) is a quasi-polynomial of the special form [20] and has an infinite set of zeros.

If the real part of at least one root of the transcendental equation (6) is negative, then the solution \( u_0 = \text{const} \) of the initial-boundary value problem for Eq. (1) is unstable (since \( \lim_{t \to \infty} \bar{u} = \infty \) at any point \( x_* \) satisfying condition \( \sin(\pi n x_*) \neq 0 \). For the linear stability of this problem, the real parts of all the roots of the transcendental equation (6) must be positive.

We specify a simple (but rather coarse) criterion for the instability of a stationary solution \( u_0 = \text{const} \). To do this, consider the function \( Q(\lambda) \), which defines the dispersion equation (6). We have \( Q(\lambda) \to -\infty \) for \( \lambda \to -\infty \). If \( Q(0) > 0 \), then there exists a real value \( \lambda = \lambda_* < 0 \), for which \( Q(\lambda_*) = 0 \). This value yields the unstable perturbed solution (5). The described sufficient criterion for the linear instability of a stationary solution can be represented in the form of inequality

\[ f_u(u_0, u_0) + f_w(u_0, u_0) > a \pi^2 / h^2, \quad (7) \]

where the minimum value \( n = 1 \) was taken.
Let us consider in more detail the reaction-diffusion equation
\[ u_t = au_{xx} + f(u, w), \quad w = u(x, t - \tau), \tag{8} \]
which is obtained by substituting the values \( \varepsilon = 0, \sigma = 1 \) into Eq. (1). In this case the dispersion equation (6) can be represented in the form
\[ p - \zeta + qe^{-\zeta} = 0, \tag{9} \]
where we use the notation:
\[ p = \tau f_u(u_0, u_0) - a\tau(\pi n/h)^2, \quad q = \tau f_w(u_0, u_0), \quad \zeta = -\lambda \tau. \tag{10} \]

For what follows we need the results of the theorem formulated below.

*Hayes theorem* [20]. All the roots of Eq. (9) with real coefficients \( p \) and \( q \) have negative real parts (\( \text{Re} \zeta < 0 \)) if and only if the following three inequalities are simultaneously satisfied:

(i) \( p < 1 \),
(ii) \( p + q < 0 \),
(iii) \( q + \sqrt{p^2 + \mu^2} > 0 \),

where \( \mu \) is the root of the transcendental equation \( \mu = p \tan \mu \) that satisfies the condition \( 0 < \mu < \pi \). For \( p = 0 \), we must take \( \mu = \pi/2 \).

If inequalities (11) with \( p \) and \( q \) defined in (10) are fulfilled, then the stationary solution \( u = u_0 \) of problem (1), (3), (4) for \( \varepsilon = 0, \sigma = 1 \) under condition (2) will be stable in the linear approximation.

Substituting (10) into (11), we can formulate the stability conditions of the stationary solution in terms of partial derivatives of the kinetic function. To analyze the arising inequalities, it is convenient to introduce the notation:
\[ U = \tau f_u(u_0, u_0), \quad W = \tau f_w(u_0, u_0), \quad s = \pi^2 ah^{-2} \tau, \tag{12} \]
where \( s \) is the characteristic dimensionless parameter of the problem, \( \tau > 0 \). As a result, conditions (11) are transformed to the form

(i) \( U < sn^2 + 1 \),
(ii) \( U + W < sn^2 \),
(iii) \( W + \sqrt{(U - sn^2)^2 + \mu^2} > 0 \),

where \( \mu \) is the root of the transcendental equation \( \mu = (U - sn^2) \tan \mu \) that satisfies the condition \( 0 < \mu < \pi \).

The stationary solution \( u_0 = \text{const} \) of the problem under consideration will be stable if three inequalities (13) are satisfied simultaneously for all integer values \( n = 1, 2, \ldots \).

The equation of the curve
\[ W + \sqrt{(U - sn^2)^2 + \mu^2} = 0, \tag{14} \]
which describes part of the boundary of the stability region in the plane \((U, W)\) and is obtained by replacing the sign > by the sign = in the last condition (13), is conveniently represented in the parametric form
\[ U = sn^2 + \mu \cot \mu, \quad W = -\mu/\sin \mu \quad (0 < \mu < \pi). \tag{15} \]
For a given $n$, the curve (15) passes through the point $(sn^2, -\pi/2)$ of the plane $(U, W)$. In addition, we have $U \to sn^2 + 1$, $W \to -1$ as $\mu \to 0$, and $U \to -\infty$, $W \simeq U - sn^2 \to -\infty$ as $\mu \to \pi$.

The analysis shows that the condition of simultaneous fulfillment of three inequalities (13) for all integer values $n = 1, 2, \ldots$ reduces to a simpler condition of fulfillment of three inequalities (13) for $n = 1$.

Example. Let us consider the reaction-diffusion equation with delay

$$u_t = au_{xx} + bu - b[ru + (1 - r)w]^\beta, \quad w = u(x, t - \tau),$$  \hspace{1cm} (16)$$

which is a special case of Eqs. (1) and (8).

Eq. (16) admits two stationary solutions $u_0 = 0$ and $u_0 = 1$ for any admissible values of the defining parameters $b, r, \beta > 0, \tau > 0$. Using the results of Section 2, we analyze the stability/instability of the solution $u_0 = 1$ of the initial boundary value problem, which is described by Eq. (16), the first initial condition (3) and the boundary conditions (4). For concreteness, we now set $a = h = 1$. Then the stability conditions (13) for Eq. (16) will have the form:

(i) $b(1 - \beta r) < \pi^2 + \tau^{-1}$,
(ii) $b(1 - \beta) < \pi^2$,
(iii) $-b\beta(1 - r) + \sqrt{[b(1 - \beta r) - \pi^2]^2 + \mu^2 \tau^{-2}} > 0$,

where $\mu$ is the root of the transcendental equation $\mu = [b(1 - \beta r) - \pi^2]r \tan \mu$ that satisfies the condition $0 < \mu < \pi$.

Remark 1. All mentioned above in Section 2 is also applicable to the analysis of the linear stability/instability of nonstationary solutions of a broader class of nonlinear initial boundary value problems that are described by Eq. (1) with initial and boundary conditions of the general form:

$$u(x, t) = \varphi_1(x, t), \quad u_t(x, t) = \varphi_2(x, t) \quad \text{if} \quad -\tau \leq t \leq 0,$$
$$u(0, t) = \psi_1(t), \quad u(h, t) = \psi_2(t) \quad \text{if} \quad t > 0$$ \hspace{1cm} (17)

(for $\varepsilon = 0$, the second initial condition for the derivative $u_t$ in (17) should be omitted), if the functions $\psi_1(t)$ and $\psi_2(t)$, appearing in the boundary conditions (18), have asymptotic properties

$$\lim_{t \to \infty} \psi_1(t) = \lim_{t \to \infty} \psi_2(t) = u_0,$$ \hspace{1cm} (19)

where the constant $u_0$ satisfies condition (2).

3. Stability/instability analysis of the nonstationary solution of a nonlinear test problem with delay

Consider a nonlinear reaction-diffusion equation with delay

$$u_t = u_{xx} + bu - b\left(\frac{u - kw}{1 - k}\right)^2, \quad w = u(x, t - \tau), \quad b = \frac{\ln k}{\tau} - 1, \quad k > 0, \quad k \neq 1,$$ \hspace{1cm} (20)$$

which is a particular case of Eq. (16) for $r = (1 - k)^{-1}$ and $\beta = 2$ and in degenerate cases $k = 0$ or $\tau = 0$ goes into the non-normalized Fisher’s equation [22].
Eq. (20) admits an exact solution

\[ u(x,t) = U_1(x,t) \equiv 1 + \frac{e^{ct+1}}{e^x - 1}(e^x - e^{-x}), \quad c = \frac{\ln k}{\tau}. \]  

(21)

Remark 2. The parameters \( b \) and \( c = b + 1 \) in Eq. (20) and solution (21) depend singularly on the delay time \( \tau \), since \( \lim_{\tau \to 0} |b| = \infty \).

It is not difficult to show that the function \( u = U_1(x,t) \) is also a solution of the problem for Eq. (20) in the domain \( 0 \leq x \leq 1, \ t > 0 \) with the following initial data and boundary conditions:

\[ \begin{align*}
 u(x,t) &= U_1(x,t), \text{ if } -\tau \leq t \leq 0; \\
 u(0,t) &= 1, \quad \text{if } u(1,t) = 1 + e^{ct}, \ t > 0.
\end{align*} \]  

(22)

For \( 0 < k < 1 \), solution (21) and functions from (22) defined on the boundaries tend asymptotically to the stationary solution \( u_0 = 1 \), which makes it possible to analyze the linear stability/instability of the nonstationary solution (21) (see Remark 1).

The stability conditions (13) for the stationary solution \( u_0 = 1 \) of Eq. (20) with the Dirichlet boundary conditions (4) for \( a = h = n = 1 \ (k > 0, k \neq 1) \) take the form

\[ \begin{align*}
 (i) \quad \tau &> \zeta(k) \text{ if } k < \frac{\pi^2 - 1}{\pi^2 + 1} \text{ and } k > 1; \quad \tau < \zeta(k) \text{ if } \frac{\pi^2 - 1}{\pi^2 + 1} < k < 1, \\
 (ii) \quad \tau &> \frac{\ln k - \pi^2}{1 - \pi^2}, \\
 (iii) \quad \theta(k,\tau) &> 0,
\end{align*} \]  

(23)

where

\[ \begin{align*}
 \zeta(k) &= \left(\frac{k + 1}{k - 1} \ln k - 1\right)\left(\frac{\pi^2 + k + 1}{k - 1}\right)^{-1}, \\
 \theta(k,\tau) &= \frac{2k}{1 - k^2} \ln k - \tau + \sqrt{\left(\frac{k + 1}{k - 1} \ln k - \pi^2\tau\right)^2 + \mu^2},
\end{align*} \]

and \( \mu \) is the root of the transcendental equation \( \mu \cot \mu = \frac{k + 1}{k - 1} \ln k - \pi^2\tau \) that satisfies the condition \( 0 < \mu < \pi \).

In particular, the following stability conditions follow from inequalities (23): \( \tau > 0.455059 \) for \( k = 0.5 \), \( \tau > 0.205634 \) for \( k = 2 \).

In Fig. 1 the regions of the plane \((k,\tau)\) highlighted in gray correspond to the case when the stationary solution \( u_0 = 1 \) of Eq. (20) is stable (in the linear approximation), and the regions of instability of this solution are not painted; the stability/instability region of the nonstationary solution (21) of the problem under consideration is determined by the values \( 0 < k < 1 \). It is clear that for \( k = 0.5 \), the small delay times \( \tau < 0.455 \) lie in the instability region of the nonstationary solution (21) (see point \( B(0.5,0.455) \)).

4. Numerical integration of a nonlinear test problem with delay by the method of lines

The essence of the method of lines is in reducing the partial differential equation to a system of ordinary differential equations by approximating the spatial derivatives by finite differences [23,24].

Let us set the spatial grid \( x_m = m\Delta x \), where \( m = 0, 1, \ldots, M \), \( \Delta x = 1/M \) is the space step. Let us carry out a discretization of the initial-boundary value problem and obtain a system,
Figure 1. Regions of stability (shaded) and instability of the stationary solution $u_0 = 1$ of problem (20)–(22). Point $A(0.239, 0.161)$ denotes the intersection point of the three curves. Point $B(0.5, 0.455)$ denotes the boundary of the stable region for the problem under consideration for a fixed $k = 0.5$. The curves are indicated by numbers: 1 — $\tau = \zeta(k)$, 2 — $\tau = \frac{\ln k}{1-\pi^2}$, 3 — $\theta(k, \tau) = 0$, 4 — $k = \frac{\pi^2-1}{\pi^2+1}$, 5 — $k = 1$ consisting of $M - 1$ ordinary differential equations and two boundary conditions. The resulting problem, which also contains the initial conditions, can be solved numerically, for example, by implicit Runge-Kutta methods.

The resulting discretized system for test problem (20)–(22) has the form

$$
(u_m)_t = a\delta_{xx}u_m + bu_m - b(1-k)^{-2}(u_m - ku_m)^2, \quad m = 1, \ldots, M-1, \quad 0 \leq t \leq T;
$$

$$
u_0(t) = 1, \quad u_M(t) = 1 + e^{ct}, \quad 0 \leq t \leq T;
$$

$$
u_m(t) = 1 + \frac{e^{ct+1}}{e^2-1}(e^x - e^{-x}), \quad m = 0, 1, \ldots, M, \quad -\tau \leq t \leq 0,
$$

where $\delta_{xx}$ is the difference operator, $\delta_{xx}u_m = (u_{m+1} - 2u_m + u_{m-1})/(\Delta x)^2$, $T$ is the time period of calculations.

A numerical experiment was conducted in the computing system Mathematica (for a brief description, see [25]) by means of a combination of the method of lines with the Runge — Kutta method and the BDF method (Gear’s method).

At moderate and large delay times (the values $\tau = 0.5, 1, 5$ were tested), both methods demonstrate a good approximation of the exact solution of problem (20)–(22) for $a = 1$, $k = 0.5$ on the whole interval $0 \leq t \leq T = 50\tau$. The relative errors are of the order $10^{-5}$ for $M = 10$ and $10^{-7}$ for $M = 100$. In Fig. 2 the exact solutions of the problem for $a = 1$, $k = \tau = 0.5$ are represented by solid lines, and the numerical solutions for $M = 100$, obtained by a combination of the method of lines and the Runge — Kutta method, are represented by circles.

At sufficiently small delay times $\tau = 0.05$ and $\tau = 0.1$ the numerical solution of problem (20)–(22) for $a = 1$, $k = 0.5$ begins to deviate strongly from the exact one after reaching the stationary mode. Then execution of the program is interrupted with an error. This is due to the instability for small $\tau$ of the stationary solution $u = 1$, to which the solution of the considered test problem (20)–(22) tends as $t \to \infty$ (see Section 3).

We note that for $\tau = 0.1$ the Runge — Kutta method and the BDF method provide a sufficiently accurate approximation of the desired solution over a fairly large time interval.
\[ u_x = 0.85 \]
\[ x = 0.5 \]
\[ x = 0.15 \]

0 \leq t \leq 1.7 (practically reaching the stationary mode \( u_0 = 1 \)), and the BDF method has a slightly larger range of applicability for \( t \).

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