Quantum-based solution of complex time-dependent Riccati equations

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We use BCH-like relations to solve the time evolution operator of quantum systems described by time-dependent (TD) hamiltonians given in terms of the generators of the $\mathfrak{su}(1,1)$, $\mathfrak{su}(2)$ and $\mathfrak{so}(2,1)$ Lie algebras and establish the necessary and sufficient conditions for its unitarity in the factorized representation. Then, we use this solution to solve a TD complex Riccati equation (CRE) which is derived from the Schrödinger equation using the Wei-Norman theory. This solution is given as continued generalized fractions, being optimal for numerical implementations, and covers the wide branch of the CRE that heritate the symmetries corresponding to the aforementioned Lie algebras. Moreover, our formalism allows to straightforwardly obtain effective quantum hamiltonians for systems described by this CRE, as we show for the Bloch-Riccati equation (BRE) whose hamiltonian can be realized with a TD-qubit, for instance. Finally, as an application but also as a consistency test, we compare our solution with the analytical one for the BRE with the Rabi frequency driven by a complex hyperbolic secant pulse that generates spin inversion, showing an excellent agreement.

Symmetries always had an important place in physics, and they became mainstays since Emmy Noether’s theorem [1], where they were formally connected with conserved quantities. This theorem arises from the study of a lagrangian under the action of groups of infinitesimals transformations known as Lie groups [2], which are of special interest in physics because they are continuous groups with the structure of a differential manifold [3]. Lie groups can be introduced through their corresponding Lie algebras [4] with the group structures identified from the commutation relations satisfied by the generators of the algebra. A paradigmatic example of algebraic methods, i.e., methods that use the algebraic structure to describe and solve physical systems, can be found in one of the many ways of solving the quantum harmonic oscillator, where ladder operators are introduced to diagonalize the hamiltonian allowing a precise and elegant way of finding the corresponding energy levels and energy eigenfunctions [5]. Algebraic methods are important not just in the obtaining of the energy spectrum of physical systems [6], but also in the computation of dynamical properties as the time evolution operator (TEO), Feynman propagators or Green functions [7, 8]. Moreover, these methods can be used in the treatment of physical systems described by time-dependent (TD) hamiltonians, which are natural scenarios for describing interactions with external agents. As a remarkable example, the so-called Wei-Norman theory [9, 10] allows to find the TEO of these systems when their hamiltonians can be written as a linear combination of time-independent generators of a finite Lie algebra. Using this method, the Schrödinger equation is mapped on a set of coupled non-linear differential equations from which the TEO can be calculated as a factorized element (that is, as a product of exponentials each containing only one generator of the algebra) of the correspondent Lie group. It is worth emphasizing that, in most cases, such solutions must be calculated numerically. A different algebraic approach, based on Baker-Campbell-Haussdorf (BCH)-like relations obtained recently [11], provides a simple recursive way of calculating the TEO of physical systems described by TD hamiltonians which are written as linear combinations of the generators of the $\mathfrak{su}(1,1)$, $\mathfrak{su}(2)$ and $\mathfrak{so}(2,1)$ Lie algebras. Notably, its numerical implementation is easy and limited only by computational capacity, and such approach has proven to be efficient in the study of the time evolution of a TD-quantum harmonic oscillator [12, 13] and a system of two coupled TD-qubits [14].

We start in Section I by setting the mathematical scenario for the simultaneous treatment of TD-quantum hermitian systems of the $\mathfrak{su}(1,1)$, $\mathfrak{su}(2)$ or $\mathfrak{so}(2,1)$ Lie algebras. Moreover, we obtain the complete solution for the TEO in the factorized representation together with the unitarity criteria. To the best of the author’s knowledge, this is the first time that all the necessary and sufficient conditions for the unitarity of the elements of the groups under consideration in this representation are listed. In Section II, we use the Wei-Norman theory to show that obtaining the TEO of any of these systems is equivalent to solving a CRE. Consequently, we solve directly this CRE as generalized continued fractions (GCF) and develop a rule for obtaining its analytical $n$-th time derivative. Furthermore, using our formalism we can associate effective quantum hamiltonians to systems described by CREs, as we show mapping the so-called Bloch-Riccati equation (BRE), which is of fundamental importance in the theory of nuclear magnetic resonance [15, 16], into an effective quantum hamiltonian of the $\mathfrak{su}(2)$ Lie algebra that can be realized by a TD-qubit. As an application but also as a consistency test, we apply our method and calculate numerically the solution of the BRE for a complex hyperbolic secant pulse in the parameter domain where

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spin inversion phenomenon is generated. Comparison with the analytical results shows excellent agreement. Section III is left for conclusions and final comments.

I. TIME EVOLUTION OF QUANTUM SYSTEMS

In this section we set the mathematical scenario for the simultaneous treatment of quantum systems described by time-dependent (TD) hermitian hamiltonians that are written as linear combinations of the generators of the three Lie algebras mentioned previously. We solve the time evolution operator (TEO) in the factorized representation and unveil the necessary and sufficient criteria for its unitarity. Consider the following hamiltonian:

\[ \hat{H}(t) = \eta_+(t)\hat{T}_+ + \eta_c(t)\hat{T}_c + \eta_-(t)\hat{T}_-, \]  

where the \( \eta \)-coefficients are in principle arbitrary scalar functions of time and the \( \hat{T} \)'s are time-independent operators satisfying

\[ [\hat{T}_-, \hat{T}_+] = 2\hat{T}_c \quad \text{and} \quad [\hat{T}_c, \hat{T}_\pm] = \pm \delta \hat{T}_\pm. \]  

The parameters \( \epsilon \) and \( \delta \) allows us to identify the operators \( \hat{T} \) as the generators of the \( su(1,1) \), \( su(2) \) or \( so(2,1) \) Lie algebras, as indicated in the following table:

| Lie Algebra | \( \epsilon \) | \( \delta \) |
|-------------|----------------|----------------|
| \( su(1,1) \) | 1 | 1 |
| \( su(2) \) | -1 | 1 |
| \( so(2,1) \) | \( i/2 \) | \( i \) |

Table I. Relations between the Lie algebras under consideration and parameters \( \epsilon \) and \( \delta \).

Let us assume \( \hat{T}_+ = \hat{T}_+^\dagger \). Therefore, from Eqs. (2) \( \hat{T}_c \) is anti-hermitian for the \( so(2,1) \) algebra or hermitian for the other two. From the above and considering the hamiltonian as hermitian, it follows that \( \eta_c(t) = \eta^*_c(t) \), with * denoting complex conjugation, while \( \eta_c(t) \) is either pure imaginary for the \( so(2,1) \) algebra or real for the other two. Accordingly, three independent real-valued functions are needed to define completely the hamiltonian, namely, two for \( \eta_+(t) \) and one for \( \eta_c(t) \). The hamiltonian can be thus written as

\[ \hat{H}(t) = \eta_+(t)\hat{T}_+ + \eta_c(t)\hat{T}_c + \eta_-(t)\hat{T}_-. \]

Once the algebra is fixed, i.e., the values of epsilon and delta are fixed, the functions \( \eta_+ \) and \( \eta_- \) allow to identify the physical system under consideration. Let us write them in compact way, henceforth, as \( \eta = (\eta_+, \eta_c, \eta_-^\ast) \). Notice, in the latter expressions we have omitted the temporal dependence in the argument of the functions for simplicity of notation. We shall do that along the text whenever there is not risk of confusion. The vector state of a quantum system \( |\psi(t)\rangle \) obeys the Schrödinger equation \( i\frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \) \((h = 1)\) [17], and the corresponding TEO is defined by \( |\psi(t)\rangle = \hat{U}(t,0) |\psi(0)\rangle \), where we set the initial time at \( t = 0 \). Thereupon, the TEO fulfils the initial condition \( \hat{U}(0,0) = \mathbb{1} \), obeys the differential equation

\[ i\frac{\partial}{\partial t} \hat{U}(t,0) = \hat{H}(t)\hat{U}(t,0), \]  

and satisfies the composition property \( \hat{U}(t,0) = \hat{U}(t,t_N)\hat{U}(t_{N-1},t_{N-2}) \cdots \hat{U}(t_2,t_1)\hat{U}(t_1,0) \). Although there is no general method to find the TEO for an arbitrary TD hamiltonian in the above equation, the formal solution can be written as a Dyson series [17] and its computation can be extremely difficult. However, a simple recursive solution corresponding to the hamiltonian in Eq. (1) have been recently developed in Ref. [11]. There, the authors considered time-splitting in \( N \) intervals of equally small enough size \( \tau = t/N \), such that the \( \eta \)-functions, and therefore the hamiltonian itself, can be regarded constant in each \( j \)-th time-interval (\( j = 1, 2, ..., N \)). Formally, this implies that the present solution will coincide with the exact one only in the limit \( N \to \infty \). Nevertheless, for numerical implementation of this method, it is enough to choose \( \tau \) to be much smaller than the typical time scale of the variation of the \( \eta \)-functions. For our hamiltonian in Eq. (3), without loss of generality we define the \( j \)-th value of the \( \eta \)-functions as \( \eta_j \equiv \eta(T = j\tau) \), where \( \eta_j = (\eta_{j+}, \eta_{j-}, \eta_{j+}^\ast) \). The correspondent \( j \)-th TEO can be thus written as a Lie group element in the unfactorized representation

\[ \hat{U}_j = \exp \left\{ \lambda_+ \hat{T}_+ + \lambda_c \hat{T}_c + \lambda_- \hat{T}_- \right\}, \]

where \( \lambda_j \equiv (\lambda_{j+}, \lambda_{j-}, \lambda_{j+}^\ast) \). Using BCH-like relations [11] each \( \hat{U}_j \) can be re-expressed in the (usual order) factorized representation \( \hat{U}_j = e^{\Lambda_{j+} \hat{T}_+} e^{\Lambda_c \hat{T}_c} e^{\Lambda_{j-} \hat{T}_-} \) (see Appendix A for details), with the coefficients given by

\[ \Lambda_{jc} = \left( \frac{\cosh(\nu_j) - \frac{\delta \lambda_{jc}}{2\nu_j} \sinh(\nu_j)}{\nu_j} \right)^{-\frac{2}{\delta}}, \]

\[ \Lambda_{j+} = \frac{2\lambda_{j+} \sinh(\nu_j)}{\nu_j \cosh(\nu_j)} - \delta \lambda_{jc} \sinh(\nu_j), \]

and

\[ \nu_j^2 = \left( \frac{\delta \lambda_{jc}}{2} \right)^2 - \delta \epsilon \lambda_{j+} \lambda_{j-}. \]

Notice that the above factorization does not depend on the unitarity of \( \hat{U}_j \), but just on the commutation relations of the generators of the algebra given by Eqs. (2). Also notice that there are \( 3! \) different but equivalent ways of ordering the exponentials of the generators, each arrangement with a different set of coefficients. Since each \( \hat{U}_j \) is a Lie group element and the TEO is given by the composition of them, then the total TEO must also be an element of the Lie group, so that it can be written in the form

\[ \hat{U}(t,0) = e^{\alpha N \hat{T}_+} e^{\Omega N \hat{T}_c} e^{\gamma N \hat{T}_-}. \]

Using the BCH-like relations shown in Appendix B, the coeffi-
coefficients in the above equation can be written recursively as

\[
\alpha_j = \Lambda_{j+} - \frac{(\Lambda_{j\epsilon})^\delta}{1 - \epsilon \delta \Lambda_{j\epsilon}^{-1}}, \quad (9)
\]
\[
\beta_j = \frac{\Lambda_{j\epsilon}^{-1} \Lambda_{j\epsilon}}{(1 - \epsilon \delta \Lambda_{j\epsilon}^{-1})^2}, \quad (10)
\]
\[
\gamma_j = \gamma_{j-(1)} + \frac{\Lambda_{j\epsilon}^{-1} \Lambda_{j\epsilon}}{1 - \epsilon \delta \Lambda_{j\epsilon}^{-1} \Lambda_{j\epsilon}}, \quad (11)
\]

with \(\alpha_1 = \Lambda_{1+}, \beta_1 = \Lambda_{1\epsilon}, \gamma_1 = \Lambda_{1-}\) and \(j = 1, 2, \ldots, N\).

As a particular case, we can realize that \(j = 1\) corresponds to the exact analytic solution for a sudden change (or just a jump) in the Hamiltonian coefficients at \(t = 0\). Moreover, Eqs. (9)-(11) represent the exact analytic solution for a sequence of \(N\)-jumps equally spaced in time of the \(\eta\)-functions.

For just one jump, the Hamiltonian coefficients can be written as \(\eta(t) = \eta^0 + (\eta^1 - \eta^0)\Theta(t)\), where \(\eta^0 = \eta(t < 0)\), \(\eta^1 = \eta(t \geq 0)\), and \(\Theta\) is the usual Heaviside step function.

Accordingly, the corresponding TEO is given by \(\hat{U}_1(t, 0) = e^{\Lambda_{1+} T_+} e^{\ln(\Lambda_{1\epsilon}) T_\tau} e^{\Lambda_{1-} T_-}\), with the \(\Lambda\)-functions given by Eqs. (5) and (6) evaluated for \(\lambda_1 = -i(\eta^0, \eta^1, \eta^0)\).

Note that the BCH-like relations used to calculate Eqs. (9) to (11) are essentially the composition rule for the elements of the groups corresponding to the algebras under consideration (see Appendix B). In other words, this is the first time that the above relations are formally calculated for the algebras under consideration written as \(G = e^{\alpha \epsilon^\sigma T_+} e^{\ln(\beta \epsilon^\tau) T_\tau} e^{\gamma \epsilon^\sigma T_-}\), the first constraint is

\[
|\alpha| = |\gamma|, \quad (13)
\]

independently of the group. For the \(so(2,1)\) Lie algebra the other two constraints are

\[
e^{-x} = 1 + \frac{|\alpha|}{2}^2 \quad \text{and} \quad \ln |\beta| = \theta + \phi \pm n\pi, \quad (14)
\]

with \(n = 1, 2, \ldots\). On the other hand, for the \(su(1,1)\) and \(su(2)\) Lie algebras the other two constraints are

\[
|\beta| + |\epsilon| = 1 \quad \text{and} \quad x = \theta + \phi \pm n\pi, \quad (15)
\]

with \(n = 1, 2, \ldots\).

To the best of the author’s knowledge, this is the first time that the above relations are formally calculated for the algebras under consideration. Nevertheless, in an important paper of Truax [25] he showed that, starting with an unfactorized representation for an unitary element of the \(SU(1,1)\) and \(SU(2)\) Lie groups, the factorized representation remains unitary. It is important to note that the above constraints can be used as a fundamental test for numerical implementations, once the \(\Lambda\)-functions in Eqs. (5) and (6) must satisfy them at any time (for any \(j\)), and the same is true for Eqs. (9) to (11).

II. FROM SCHROEDINGER TO RICCATI

The Wei-Norman theory [9, 10] ensures that, whether a Hamiltonian can be expressed as a linear combination of time-independent generators of a finite Lie algebra, therefore the TEO can be written as an element of the correspondent Lie group, expressed as a product of exponentials of the algebra generators [26]. Accordingly, for our Hamiltonian in Eq. (3) we are allowed to consider, from the beginning, the TEO already factorized in the following convenient arrangement:

\[
\hat{U}(t) = e^{\alpha(t) T_+} e^{\ln(\beta(t)) T_\tau} e^{\gamma(t) T_-}, \quad (16)
\]

where we suppressed the initial time in our notation and the choice of the coefficients is not coincidental. Indeed, comparing Eqs. (8) and (16) it is clear that the solution for \(\alpha\) is the same i.e., the one given in Eq. (9). Substituting the above equation together with Eq. (3) in the Schrödinger equation (4), and with the aid of ordering techniques [27] (similarly to those found in Appendix A), we obtain the following set of
The coupled differential equations
\[ \dot{\alpha} - \frac{\delta}{\beta} \alpha + \epsilon \frac{\gamma}{\beta \delta} \alpha^2 + i \eta_+ = 0, \]
\[ \frac{\dot{\beta}}{\beta} - 2 \epsilon \frac{\gamma}{\beta \delta} \alpha + i \eta_c = 0, \]
\[ \frac{\dot{\gamma}}{\beta} + i \eta^*_+ = 0, \]
\[ (17) \]
satisfying the initial conditions \( \alpha = 0 \) and \( \beta = 1 \) at \( t = 0 \), and where the overdot indicates time derivative. The decoupling of the above equations leads to
\[ \dot{\alpha} + \epsilon (i \eta^*_+) \alpha^2 + \delta (i \eta_+) \alpha + i \eta_+ = 0, \]
\[ (18) \]
which is a TD complex Riccati equation (CRE) in \( \alpha \). Specifically, the above equation represents three families of CREs, each associated to one of the algebras at issue. Note that \( \eta_+ \) is the parameter associated to the non-linearity of the Eq. (18), and the solution for \( \eta_+ = 0 \) with the above mentioned initial condition for \( \alpha \) is the trivial one, namely, \( \alpha(t) = 0 \). Once the equation for \( \alpha \) is solved then one can find \( \beta \) from Eqs. (17) as
\[ \beta(t) = \exp \left\{ -2 i \epsilon \int_0^t \eta^*_+(t') \alpha(t') dt' - i \int_0^t \eta_c(t') dt' \right\}, \]
while \( \gamma \) can be calculated as
\[ \gamma(t) = -i \int_0^t \eta^*_+(t') \beta(t') dt'. \]
\[ (19) \]
We can conclude, therefore, that the solution of the Schrödinger equation for the TD Hamiltonian in Eq. (3) is equivalent to solve the CRE of Eq. (18) and, as we mentioned before, the solution of the latter is given by Eq. (9) as well. In addition, we now identify that the differential equation that \( \alpha \) satisfies is the CRE, from which we can calculate exactly its \( n-th \) time-derivative: just isolate its first derivative in Eq. (18) and then derive it as many times as desired. In other words, we have shown that the solution for the CRE in Eq. (18) is given by the GCF of Eq. (12) and, at the same time, we also obtained a manner to calculate the derivatives of such -families of GCFs as they satisfy the CRE. Notice that the CRE is of great importance, e.g., in mathematics [28], physics [29] and optimal control theory [30]. Recall, the CRE has some known analytical solutions but, in general, it must be solved numerically [31]. Therefore, our solution is very convenient, as GCFs are optimal for numerical implementations. One important non trivial example of a CRE with a known analytical solution is the so-called Bloch-Riccati equation (BRE) [15], which we shall use in the following subsection to do a consistency test of our results. Hence, it is appropriate to look at the problem from a reverse perspective and ask, when can we use our results if we start with a generic CRE as
\[ \dot{\alpha} + b_0 \alpha^2 + b_1 \alpha + b_2 = 0, \]
\[ (21) \]
with \( b_0 \), \( b_1 \) and \( b_2 \) arbitrary complex functions of time? Comparison of the above equation with Eq. (18) allows us to conclude that, to apply our solution, \( b_1 \) must be a pure imaginary function of time, \( b_2 \) arbitrary and \( b_0 \) depend on the algebra as \( b_0 = b_0^{\frac{1}{2}} - \epsilon \mathbf{su}(2,1) \), \( b_0 = -b_2 \mathbf{su}(1,1) \) or \( b_0 = b_2 \mathbf{su}(2) \). Importantly, using the above prescription it is possible to relate directly CREs with quantum hamiltonians of the mentioned algebras, as we shall show in the following subsection for a specific case.

A. Effective Hamiltonian and Solution of the Bloch Equations

Since its publication in 1946 [32], the Bloch equations became of fundamental importance in the realm of nuclear magnetic resonance (NMR). They provide a quantitative description of any NMR experiment that involves radio frequency pulses, which are at the heart of all modern NMR experiments [16]. Following Ref. [15], the Bloch equations in the rotating frame neglecting relaxation terms can be written as
\[ \dot{M} + i \Delta \omega M + M_2 \Omega(t) = 0, \]
\[ M_z - \frac{i}{2} (M \Omega^*(t) - M^* \Omega(t)) = 0, \]
\[ (22) \]
where \( M \) is the complex magnetization in the \( x-y \) plane, \( M_z \) the longitudinal magnetization and \( \Omega(t) = -\gamma (B_{1x} + i B_{1y}) \) is the complex TD driving function. Using the following definition
\[ f = \frac{M}{M_0 + M_z}, \]
\[ (23) \]
where \( M_0 \) is the equilibrium magnetization, Eqs. (22) can be transformed into the Bloch-Riccati equation (BRE) [15], namely,
\[ \dot{f} - \frac{i}{2} \Omega^*(t) f^2 + i \Delta \omega f + \frac{i}{2} \Omega(t) = 0. \]
\[ (24) \]
Notice that the above equation is identical to Eq. (18) for the particular case of the Lie algebra \( \mathbf{su}(2) \) (\( \delta = 1 \) and \( \epsilon = -1 \)), with the identifications \( \eta_+ = \Omega(t)/2 \) and \( \eta_c = \Delta \omega \). Therefore, the corresponding effective hamiltonian is simply given by
\[ \dot{H}(t) = \frac{1}{2} \Omega(t) \hat{T}_+ + \Delta \omega \hat{T}_x + \frac{1}{2} \Omega^*(t) \hat{T}_-. \]
\[ (25) \]
Furthermore, if we consider a realization of this algebra for the Pauli operators \( \sigma_+ \), \( \sigma_- \) and \( \sigma_z \), with \( \sigma_{\pm} = \sigma_x \pm i \sigma_y \), we note that the above hamiltonian will be exactly the same of a TD-qubit [11, 33] with \( \Omega(t) \) the Rabi frequency and \( \Delta \omega \) an effective detuning. We can conclude, therefore, that the time evolution of a TD-qubit is, among several other possible TD systems of the \( \mathbf{su}(2) \) Lie algebra, a set up equivalent to RMN described by the Bloch equations in the rotation frame and with relaxation terms neglected, a result that is in agreement with the well-known connection between quantum optics and RMN [16]. Note that although there are some exact solutions for a TD-qubit with certain driving terms (see e.g., Ref. [34]), our general approach allows us to consider arbitrary complex TD functions for the Rabi frequency as well as real-valued functions for the detuning.
A non trivial emblematic example with analytical solution for the BRE and with relevant experimental applications in NMR comes from the use of a complex hyperbolic se-cant pulse as the driven function for the Rabi frequency \cite{15}. Moreover, choosing appropriately the domain of the involved functions, spin inversion phenomenon can be achieved. As a consistency test, we now shall recover the results for such driving. Let us consider the following family of functions for the Rabi frequency:

$$\Omega(t) = \Omega_o \left( \text{sech} \left( \chi(t - t_o) \right) \right)^{1+i\mu} ,$$  

(26)

where \(\mu\) is a real constant and \(\Omega_o\) is the pulse amplitude. Using the above driving, the BRE (24) can be transformed in a hypergeometric equation with known solutions. After taking into account the initial conditions, the authors in Ref. \cite{15} found for the stationary solution of the magnitude of \(f(t)\) the following simple expression

$$|f|_{t \rightarrow \infty}^2 = \frac{\cosh^2 \left( \frac{\pi \mu}{2} \right) - \cos^2 \left( \frac{\pi y}{2} \right)}{\cosh^2 \left( \pi \Delta \omega / 2 \chi \right) - \sin^2 \left( \frac{\pi y}{2} \right)} ,$$  

(27)

with

$$y = \left\{ \left( \frac{\Omega_o}{2 \chi} \right)^2 - \left( \frac{\mu}{2} \right)^2 \right\}^{1/2} ,$$  

(28)

and where they considered solutions for \(y\) real and \(2y \neq 1, 2, 3, ...\). The quantity that serves to predict the spin inversion is given by \(M_z / M_o\). Using Eq. (27), it can be shown that

$$\frac{M_z}{M_o} = \tanh \varphi_1 \tanh \varphi_2 + \cos \varphi_3 \text{sech} \varphi_1 \text{sech} \varphi_2 .$$  

(29)

where

$$\varphi_1 = \pi \left\{ \frac{\Delta \omega}{2 \chi} + \frac{\mu}{2} \right\} , \quad \varphi_2 = \pi \left\{ \frac{\Delta \omega}{2 \chi} - \frac{\mu}{2} \right\}$$

and

$$\varphi_3 = \pi \left\{ \left( \frac{\Omega_o}{2 \chi} \right)^2 - \left( \frac{\mu}{2} \right)^2 \right\}^{1/2} .$$  

(30)

Spin inversion is achieved if the above quantity changes from \(M_z = 1\) to \(M_z = -1\) (and vice versa). Moreover, notice from Eq. (29) that without phase modulation (\(\mu = 0\)) and whenever that \(\Omega_o = 2n\chi\) with \(n = 1, 2, ...\) there is an excursion of the magnetization (\(M_z = M_o\)). On the other hand, in the limit of \(\mu \rightarrow \infty\), \(\chi \rightarrow 0\), with \(\mu \chi \rightarrow C\) a constant, and \(\Omega_o \geq C\), magnetization is inverted over all frequencies (\(M_z = -M_o\)). To capture this phenomenon and taking into account the considered analytical solutions, we chose for the numerical calculations the following values of the parameters in arbitrary units: \(\Omega_o = 10\), \(\chi = \Omega_o / 2\mu\) and for the phase modulation parameter \(\mu = \{1.4, 2, 4\}\), with fixed values of \(\Delta \omega\) sweeping the interval \(-15 \leq \Delta \omega \leq 15\). We also chose a time interval for the analysis of \(t \in [0, 40]\). In Fig. 1, we plot in solid lines the analytic solution of Eq. (29) for the chosen values of the parameters. Recall that the intermediary numerical calculations of Eqs. (5)-(7) allow for the testing of the code using the unitary relations given in Eqs. (13) and (15) \((\epsilon = -1)\). Now, considering the time-splitting in \(N = 8 \times 10^4\) intervals for each value (point) of \(\Delta \omega\), we numerically calculate a total of 300 points within the effective detuning interval to evaluate the same quantity with our formalism. The calculations took less than 20 minutes in a laptop machine and are plotted as the dot patterns in Fig. 1. As it can be noted, the matching is excellent, allowing us to validate our results. Recall that our formalism allows one to fully calculate the TEO corresponding to the RMN effective hamiltonian obtained in Eq. (25) i.e., the \(\beta\) and \(\gamma\) functions in Eq. (16), and consequently, to evolve any initial state. Nevertheless, the only necessary parameter to describe spin inversion is \(\alpha\). This is because the Bloch equations (22) consider intrinsically the initial state of the system as the fundamental, where \(\alpha\) is the protagonist of the dynamics. Therefore, we can expect that in RMN analysis using arbitrary drivings and initial states, our results will be useful.

### III. CONCLUSIONS

In the first part of this work we developed a formalism based in recently developed BCH-like relations to solve the time evolution operator (TEO) of hermitian hamiltonians of the \(su(1, 1)\), \(su(2)\) and \(so(2, 1)\) Lie algebras and we obtained all the necessary and sufficient conditions for the unitarity of the elements of the correspondent Lie groups given in the factorized representation. We expressed these conditions through specific constraints connecting the coefficients of the TEO. Then, in the second part of this work we derived a TD com-
plex Riccati equation (CRE) from the Schrödinger equation using the Wei-Norman theory and used our solution obtained for the TEO to solve it recursively as generalized continued fractions (GCF), which are optimal for numerical implementations. Moreover, we developed a rule to calculate the n-th time derivative of such GCF. The formalism we developed also serves to associate effective hamiltonians directly to CREs, as we showed for the so-called Bloch-Riccati equation (BRE), mapping it in an effective quantum hamiltonian of the su(2) Lie algebra. As an application but also as a consistency test we numerically calculated the solution of the BRE for a complex hyperbolic secant pulse generating spin inversion and compare it to analytical results, showing excellent agreement. Our results are quite general and can be used not just to solve the TEO of any TD system of the algebras at issue and its related CRE, but also CREs of the Lie algebras under consideration that not need to be related to quantum systems. For instance, Newton’s laws can be put in the form of Riccati equations under certain conditions [35] and therefore quantum hamiltonians can be associated.

Our results can be straightforwardly extended for other nonlinear differential equations that can be derived from the CRE, e.g., the (dissipative) Ermakov equation [36–38] with its respective invariant, paving the way for new possibilities in the quantum-classical connection [29]. Furthermore, our results can be also extended for other algebras with a higher number of generators [39], arriving to different sets of differential equations that could be solved in terms of GCFs, also amplifying the possibilities of investigating more complex systems as coupled TD-quantum harmonic oscillators [40, 41] or coupled TD-qubits [14]. Our results related to GCFs could be also useful in the analysis of quantum systems using differential Galois theory [42], as well as in extensions for non-hermitian time-dependent systems [43]. Finally, we can expect applications of our results in several branches of physics, as they encompasses the solution of the TEO of important fundamental systems as TD-qubits and TD-quantum harmonic oscillators, being useful in the control of quantum systems [44, 45], in the design of shortcuts to adiabaticity [46], in the harnessing of nonadiabatic excitations promoted by quantum critical points [47], in the description of ion traps dynamics [48], in quantum thermodynamics [49–52], or in quantum interference of levitated nano rotors [18], among others.

ACKNOWLEDGMENTS

D.M.T. thanks J. C. Correa, R. C. L. Bruni and L. Pires for enlightening discussions. The authors thank the Brazilian agencies for scientific and technological research CAPES, CNPq and FAPERJ for partial financial support. This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, 310365/2018-0 (C.F) and Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro - FAPERJ, 2021042663 (D.M.T.).

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Appendix A: BCH-like relations

In this appendix we shall show that an arbitrary element of the Lie groups at issue and given in the unfactorized representation, namely,
\[ G(\lambda) = \exp\left\{ \lambda_+ \hat{T}_+ + \lambda_c \hat{T}_c + \lambda_- \hat{T}_- \right\}, \]  
(A1)
can be factorized in the usual order as indicated in Eqs. (5) to (7), or as:
\[ \hat{G}(\Sigma) = e^{\Sigma_- \hat{T}_-} e^{\ln(\Sigma_+)} e^{\Sigma_+ \hat{T}_+}, \]  
(A2)
where
\[ \Sigma_+ = \frac{2 \lambda_+ \sinh(\nu)}{2 \nu \cosh(\nu) + \delta \lambda_+ \sinh(\nu)}, \]  
(A3)
\[ \Sigma_- = \frac{2 \lambda_- \sinh(\nu)}{2 \nu \cosh(\nu) + \delta \lambda_- \sinh(\nu)}, \]  
(A4)
with
\[ \nu^2 = \left( \frac{\delta \lambda_c}{2} \right)^2 - \epsilon \delta \lambda_+ \lambda_. \]  
(A5)

Firstly, let us re-define Eq. (A1) as the special case \( \rho = 1 \) of the operator
\[ \hat{F}_1(\rho) = e^\rho(\lambda_+ \hat{T}_+ + \lambda_c \hat{T}_c + \lambda_- \hat{T}_-), \]  
(A6)
The basic idea is to find an equivalent expression in the form
\[ \hat{F}_1(\rho) = e^{\Sigma_- (\rho) \hat{T}_-} e^{\ln(\Sigma_+ (\rho))} e^{\Sigma_+ (\rho) \hat{T}_+}, \]  
(A7)
and therefore, to write the functions \( \Sigma_+, \Sigma_- \) and \( \Sigma_c \) in terms of the small lambdas, so that the last two equations are equal.
The relations between these two set of functions \( (\Sigma \text{ and } \lambda) \) are known as BCH-like relations [27]. Accordingly, we derive both expressions with respect to parameter \( \rho \) and impose the derivatives to be equal. The derivative of Eq. (A6) is direct, and given by
\[ \hat{F}_1' = \left( \lambda_+ \hat{T}_+ + \lambda_c \hat{T}_c + \lambda_- \hat{T}_- \right) \hat{F}_1, \]  
(A8)
where the prime indicates derivative with respect to \( \rho \). On the other hand, the derivative of Eq. (A7) can be written as
\[ \hat{F}_1' = \left( \Sigma_- \hat{T}_- + \Sigma'_c \hat{I}_1 + \Sigma_+ \hat{I}_2 \right) \hat{F}_1, \]  
(A9)
with
\[ \hat{I}_1 = e^{\Sigma_- \hat{T}_-} \hat{T}_c e^{-\Sigma_- \hat{T}_-}, \]  
\[ \hat{I}_2 = e^{\Sigma_- \hat{T}_-} e^{\ln(\Sigma_+)} \hat{T}_c e^{-\ln(\Sigma_+)} e^{-\Sigma_- \hat{T}_-}. \]  

Using the following BCH relation [27]
\[ e^A B e^{-A} = \hat{B} + \left[ \hat{A}, \hat{B} \right] + \frac{1}{2!} \left[ \hat{A}, \left[ \hat{A}, \hat{B} \right] \right] + \frac{1}{3!} \left[ \hat{A}, \left[ \hat{A}, \left[ \hat{A}, \hat{B} \right] \right] \right] + \ldots, \]  
(A11)
together with the commutation relations given in Table I we can solve Eqs. (A10), and then equalize Eqs. (A8) and (A9) to obtain the following set of coupled differential equations:
\[ \Sigma_- + \frac{\delta \Sigma_-}{\Sigma_c} \frac{\Sigma'}{\Sigma_c} + \epsilon \delta(\Sigma_-)^2 \frac{\delta \Sigma_+}{\Sigma_+} = \lambda_-, \]  
(A12)
\[ \frac{\Sigma'}{\Sigma_c} + 2 \epsilon \Sigma_- (\Sigma_-) \frac{\delta \Sigma_+}{\Sigma_+} = \lambda_c, \]  
(A13)
\[ (\Sigma_+)^2 \frac{\delta \Sigma_+}{\Sigma_+} = \lambda_+. \]  
(A14)
Substitution of Eq. (A14) in Eq. (A13) leads to
\[ \frac{\Sigma'}{\Sigma_c} = \lambda_c - 2 \epsilon \lambda_+ \lambda_-, \]  
(A15)
and we obtain the differential equation for \( \Sigma_- \) substituting the above equation together with Eq. (A14) into Eq. (A12):
\[ \Sigma'_- - \epsilon \delta \lambda_+ (\Sigma_-)^2 + \delta \lambda_- \Sigma_- - \lambda_- = 0. \]  
(A16)
This is a first order, quadratic and non-homogeneous ordinary differential equation known as the (time-independent) complex Riccati equation. It has unique solution and can be transformed into an ordinary, homogeneous and second order differential equation with the aid of the well-known transformation
\[ \Sigma_- = -\frac{1}{\epsilon \delta \lambda_+} u', \]  
(A17)
leading to
\[ u'' + \Gamma u' + \varsigma^2 u = 0, \]  
(A18)
where we defined \( \varsigma^2 = \epsilon \delta \lambda_- \lambda_+ \) and \( \Gamma = \delta \lambda_c \) in order to identify it as the classical equation of a damped harmonic oscillator with natural frequency \( \varsigma \) and damped coefficient \( \Gamma \). Its general solution is given by
\[ u(\rho) = e^{-\frac{\varsigma}{2} \rho} \left( A e^{\varsigma \rho} + B e^{-\varsigma \rho} \right), \]  
(A19)
where \( \nu \) is given by Eq. (A5) and constants \( A \) and \( B \) are determined from the initial condition \( \Sigma_-(\rho = 0) = 0 \). Using the above results in Eq. (A17) we obtain
\[ \Sigma_-(\rho) = \frac{2 \lambda_- \sinh(\nu \rho)}{2 \nu \cosh(\nu \rho) + \delta \lambda_+ \sinh(\nu \rho)}, \]  
which leads to the desired expression written in Eq. (A4) if we take \( \rho = 1 \). Now, using Eq. (A15) and the above result together with the initial condition \( \Sigma_c(\rho = 0) = 1 \), we can calculate
\[ \Sigma_c = \left( \cosh(\nu \rho) + \frac{\delta \lambda_+}{2 \nu} \sinh(\nu \rho) \right)^\frac{2}{3}, \]  
which after taking \( \rho = 1 \) leads to the desired result of equation in (A3). To find \( \Sigma_+ \) we replace the above equation in Eq. (A14) and take into account the initial condition \( \Sigma_+(\rho = 0) = 0 \), obtaining
\[ \Sigma_+ = \frac{2 \lambda_+ \sinh(\nu \rho)}{2 \nu \cosh(\nu \rho) + \delta \lambda_+ \sinh(\nu \rho)} . \]
which leads to the desired result in Eq. (A4) with $\rho = 1$. To finish this section we shall factorize expression (A1) in the usual order:

$$\hat{G}(\Lambda) = e^{\Lambda_+ T_+ e^{\ln(\lambda_+)} T_+ e^{\Lambda_- T_-}}. \quad \text{(A20)}$$

Following a similar process, the correspondent set of coupled differential equations is given by

$$\begin{align*}
\Lambda'_+ - \delta \Lambda_+ & = \lambda_+ + \epsilon (\Lambda_+)^2 \Lambda'_- = \lambda_+, \quad \text{(A21)} \\
\Lambda'_- - 2\epsilon \Lambda_+ (\Lambda_-)^{-\delta} \Lambda'_- = \lambda_-, \quad \text{(A22)} \\
(\Lambda_-)^{-\delta} \Lambda'_- = \lambda_- \quad \text{(A23)}
\end{align*}$$

Then, solving the above system it is straightforward to show that

$$\begin{align*}
\Lambda_c &= \left( \cosh(\nu) - \frac{\delta \lambda_c}{2\nu} \sinh(\nu) \right)^{-\frac{\delta}{2}}, \quad \text{(A24)} \\
\Lambda_\pm &= \frac{2\lambda_\pm \sinh(\nu)}{2\nu \cosh(\nu) - \delta \lambda_c \sinh(\nu)}, \quad \text{(A25)}
\end{align*}$$

which are equivalent to Eqs. (5) and (6) [11]. The latter results can be used to check Eqs. (A3) and (A4) as follows. First, notice that the inverse of Eq. (A1) can be easily calculated as $\hat{G}^{-1} = \exp\{ -\Lambda_+ T_+ - \Lambda_- T_- \}$, i.e., is equivalent to the change $\Lambda \rightarrow -\Lambda$. Using this condition in Eqs. (A3) and (A4), it is straightforward to show that $\Sigma_c \rightarrow (\Lambda)^{-1}$ and $\Sigma_\pm \rightarrow -\Lambda_\pm$. Therefore, from Eq. (A21) it follows that $\hat{G}^{-1} = e^{-\Lambda_+ T_+} e^{-\ln(\lambda_+)} T_+ e^{\Lambda_- T_-}$, which is consistent with Eq. (A20) since $\hat{G} \hat{G}^{-1} = \hat{G}^1 \hat{G} = \hat{I}$.

**Appendix B: BCH-like relations and unitary criteria**

The BCH-like relations that we use to calculate the coefficients of the TEO in Eq. (8) are, essentially, the composition rule for the elements of the groups corresponding to the algebras under consideration, once they are written in the factorized representation of Eq. (A20). More specifically, given two arbitrary elements $\hat{G}_2 = e^{\alpha T_+ e^{\ln(\beta)} T_+ e^{\gamma T_-} - \alpha T_-}$ and $\hat{G}_1 = e^{\delta T_+ e^{\ln(\beta)} T_+ e^{\gamma T_-} - \delta T_-}$, their product is another element of the group [11], namely

$$\hat{G} = \hat{G}_2 \hat{G}_1 = e^{\zeta T_+ e^{\ln(\zeta)} T_+ e^{\zeta T_-}}, \quad \text{(B1)}$$

where

$$\begin{align*}
\zeta &= \alpha + \frac{\tilde{\alpha} \beta^\delta}{1 - \epsilon \delta \alpha \gamma}, \quad \zeta_+ = \frac{\tilde{\beta}^\gamma}{1 - \epsilon \delta \alpha \gamma}. \\
\zeta_- &= \tilde{\gamma} + \gamma (\tilde{\beta}^\gamma), \quad \text{and} \quad \zeta_- = \tilde{\gamma} + \frac{\gamma (\tilde{\beta}^\gamma)}{1 - \epsilon \delta \alpha \gamma}. \quad \text{(B2)}
\end{align*}$$

Using the above equations, it can be shown that the composition of $N$ elements is given by Eqs. (9) and (11) [11]. Notice, the inverse product i.e., $\hat{G}_1 \hat{G}_2$, leads to identical relations as in Eqs. (B2) but interchanging the letters having tilde with those that do not. Suppose that we know $\hat{G}_2$ and we desire to obtain its inverse, namely $\hat{G}_1$. Accordingly, their product must equals the identity, i.e., $\hat{G}_2 \hat{G}_1 = \hat{G}_1 \hat{G}_2 = \hat{I}$. In Eq. (B2) this implies $\zeta_+ = 0$, $\zeta_- = 0$, from which we obtain $\tilde{\alpha} = -\frac{\alpha}{\gamma}$, $\tilde{\beta} = \frac{\beta}{\gamma}$, and $\tilde{\gamma} = -\frac{\gamma}{\gamma}$, where we defined $l \equiv \beta^\gamma - \epsilon \delta \alpha \gamma$. It can also be proven that for the inverse product the above relations holds true, and therefore $\hat{G}_2$ is the inverse of $\hat{G}_1$. Let us now define

$$\alpha = |\alpha| e^{i\theta}, \quad \beta = |\beta| e^{i\theta} \quad \text{and} \quad \gamma = |\gamma| e^{i\beta}. \quad \text{(B3)}$$

The unitary condition demands $\hat{G}^{-1} = \hat{G}^1$, where $\hat{G}^1 = e^{\gamma T_+ e^{\ln(\beta)} T_+ e^{\gamma T_-} - \alpha T_-}$, leading to

$$\gamma^* = -\frac{\alpha}{\gamma}, \quad \ln(\beta^*) T_+ = \ln\left(\frac{\beta}{\gamma}\right) T_+ \quad \text{and} \quad \alpha^* = -\frac{\gamma}{\gamma}. \quad \text{(B4)}$$

From the left and right hand side equations above and Eqs. (B3) we obtain the following results valid for all the algebras at issue. First, the constraint

$$|\alpha| = |\gamma|. \quad \text{(B5)}$$

Second, that $l$ is just a phase once $|l| = 1$. And third, that $l^2 = \epsilon 2 \ln(\theta + \phi)$. Recall that, by construction, $T_\nu$ is anti-hermitian for the $so(2,1)$ algebra and hermitian for the other two. Accordingly, for the $so(2,1)$ algebra, the middle equation in Eqs. (B4) implies that $|\beta|^2 = l^2$, with $l \equiv \beta^\gamma + \frac{\alpha^\gamma}{\gamma}$ (see Table I). Using the above results together with Eqs. (B3) and (B4), it is straightforward to show that

$$e^{-x} = 1 + \frac{|\alpha|^2}{2} \quad \text{and} \quad \ln|\beta| = \theta + \phi \pm n\pi, \quad \text{(B6)}$$

with $n = 1, 2, \ldots$. On the other hand, for the $su(1,1)$ and $su(2)$ algebras the middle equation in Eqs. (B4) implies that $\frac{\beta^\gamma}{\gamma} = l^2$ with $l = \beta - \epsilon \alpha \gamma$, leading to

$$|\beta| + \epsilon |\alpha|^2 = 1 \quad \text{and} \quad x = \theta + \phi \pm n\pi, \quad \text{(B7)}$$

with $n = 1, 2, \ldots$. Using the above results it can be shown that, for all the algebras at issue, $l = -\epsilon e^{i(\theta + \phi)}$. Finally, notice that the arbitrary composition of squeeze or rotation operators [11], among other unitary operators of the groups corresponding to the algebras at issue [18], can be calculated using these BCH-like relations.