The Diagonal Distribution for the Invariant Measure
of a Unitary Type Symmetric Space

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Abstract. Let Θ denote an involution for a simply connected compact Lie group U, let K denote the fixed point set, and let µ denote the U-invariant probability measure on U/K. Consider the geodesic embedding φ : U/K → U : u → uu−Θ of Cartan. In this paper we compute the Fourier transform of the diagonal distribution for φ∗µ, relative to a compatible triangular decomposition of G, the complexification of U. This boils down to a Duistermaat-Heckman exact stationary phase calculation, involving a Poisson structure on the dual symmetric space G0/K discovered by Evens and Lu.

§0. Introduction.

Suppose that K is a simply connected compact Lie group, and let G denote the complexification. Given a triangular decomposition g = n− ⊕ h ⊕ n+, a generic g ∈ K has a unique “LDU decomposition”, g = lmau, where l ∈ N− (lower triangular), u ∈ N+ (upper triangular), and ma ∈ H, with m ∈ H ∩ K (unitary) and a ∈ exp(hR = h ∩ iR) (positive). A formula of Harish-Chandra (essentially equivalent to the Weyl dimension formula) asserts that for λ ∈ h∗R

\[ \int_K a^{-i\lambda} = c(2\delta - i\lambda) = \prod_{\alpha > 0} \frac{\langle 2\delta, \alpha \rangle}{\langle 2\delta - i\lambda, \alpha \rangle}, \]

(0.1)

where the integral is with respect to normalized Haar measure, the product is over positive complex roots, and 2\delta is the sum of the positive complex roots (there are various ways in which the c-function arises, and this formula has many extensions and interpretations; see e.g. [H2], especially §5-6 of chapter IV).

The purpose of this paper is to present a generalization of this formula, and some of the related geometry, in which K is replaced by a compact symmetric space.
Suppose that $X$ is a simply connected compact symmetric space with a fixed basepoint. From this we obtain (1) a diagram of groups,

\[
\begin{array}{ccc}
G & \\ & \nearrow U & \\ G_0 & \\ & \nwarrow K
\end{array}
\]

where $U$ is the universal covering of the identity component of $\text{Aut}(X)$, $X \cong U/K$, $G$ is the complexification of $U$, and $G_0/K$ is the noncompact type symmetric space dual to $X$; and (2) a diagram of equivariant totally geodesic (Cartan) embeddings of symmetric spaces:

\[
\begin{array}{ccc}
U/K & \xrightarrow{\phi} & U \\
\downarrow & & \downarrow \\
G/G_0 & \xrightarrow{\phi} & G \\
\uparrow & & \uparrow \\
G_0 & \xleftarrow{\psi} & G_0/K
\end{array}
\]

We also consider one additional ingredient: a triangular decomposition of $g$, $g = n^- \oplus h \oplus n^+$, which is $\Theta$-stable and for which $t_0 = h \cap k$ is maximal abelian in $k$, where $\Theta$ is the involution corresponding to the pair $(U,K)$.

Given this triangular structure, a generic element of $\phi(U/K)$ can be written as $g = lwa_ml^*\Theta$, where $l$, $a_\phi$, $m$ are roughly as before, and $w \in T_0^{(2)}$ (elements of order two); the possible $w$ index the connected components of the set of generic elements. In the special case in which $\Theta$ is an inner automorphism, the generalization of (0.1) which we consider is of the form

\[
\int_{\phi(U/K)} a_{\phi}^{-i\lambda} = \frac{|W(K)|}{|W(U)|} \prod_w \frac{\langle \delta, \alpha \rangle}{\langle \delta - i\lambda, \alpha \rangle}
\]

where, given $w$, the product is over positive roots $\alpha$ which are of noncompact type for the involution $\text{Ad}(w)\Theta$, and $|W(\cdot)|$ denotes the order of the Weyl group.

The plan of the paper is the following. In §1 we compute the intersections of the $\phi$-images in (0.3) with the triangular decomposition of $G$. A notable qualitative fact is that just as the map $U/K \to G/G_0$ in (0.3) is a homotopy equivalence, so also are the intersections with the triangular decomposition of $G$. Possibly everything in this section is known; it can certainly be generalized and packaged in various ways (the canonical source is [Wolf]).
In §2 the general formulation and a proof of (0.4) is presented. It turns out that, in the inner case for example, the \( w = 1 \) term in (0.4) equals
\[
\int_{G_0/K} a(g_0)^{-2\delta - 2(\delta - i\lambda)} dV(g_0K) = \int_{\psi(G_0/K)} a^{-\delta - (\delta - i\lambda)} \psi(g_0K)
\]
where \( g_0 = lau \) is an Iwasawa decomposition in \( G \), \( \psi(g_0K) = g_0 g_0^* = la\psi l^* \) is an LDU decomposition, and the integrals are with respect to a \( G_0 \)-invariant measure. It is remarkable, although not a surprise, that \( a(g_0)^{-2\delta} dV(g_0K) \) is the volume element for a symplectic form having a momentum map \( \log(a(g_0K)) \). Hence (0.5) can be evaluated using (a noncompact version of) the Duistermaat-Heckman exact stationary phase method. The symplectic structure was discovered by Evens and Lu, in a general setting ([EL]); the relevance of this structure was pointed out to me by Foth and Otto ([FO]), to whom I am grateful.

It is natural to consider the more general integral
\[
\Psi(g) = \int_{G_0} a(g_0g)^{-2\delta - 2(\delta - i\lambda)} dg_0
\]
for \( g \in G_0 \backslash G/U \). This is an eigenfunction for \( G \)-invariant differential operators on \( G/U \). This can also be evaluated exactly, by the same method.

In [Pi1,2] I have discussed conjectural generalizations of (0.1) and (0.4) to loop spaces, and other kinds of infinite symmetric spaces. The localization argument applies in a heuristic way. In appendix A there is a proof of (0.1), involving an explicit factorization of the integral, which has elements that seem useful in the loop space context.

Notation. \( \langle \cdot, \cdot \rangle \) will denote the Killing form for \( g \). For an automorphism \( \theta \) of \( g \), we will often write \( \theta(x) = x^\theta \), and more briefly, \( Ad(g)(x) = x^g \). We will write \( x = x_- + x_h + x_+ \) for the triangular decomposition of \( x \in g \), and \( x = x_t + x_p \) for the Cartan decomposition of \( x \in g_0 \).

§1. Symmetric Spaces and Triangular Decomposition.

Throughout the remainder of this paper, \( U \) will denote a simply connected compact Lie group, \( \Theta \) will denote an involution of \( U \), with fixed point set \( K \), and \( X \) will denote the quotient, \( U/K \). This implies that \( K \) is connected and \( X \) is simply connected (Theorem 8.2 of [H1]).
Corresponding to the diagram of groups in (0.1), there is a Lie algebra diagram

\[ g = u \oplus iu \]
\[ g_0 = k \oplus p \]
\[ u = k \oplus ip \]

(1.1)

where \( \Theta \), acting on the Lie algebra level and extended complex linearly to \( g \), is +1 on \( k \) and −1 on \( p \). We let \((\cdot)^{-*}\) denote the Cartan involution for the pair \((G,U)\). The Cartan involution for the pair \((G,G_0)\) is given by

\[ \sigma(g) = g^{*\Theta} \]

Since \( *, \Theta, \sigma, \) and \((\cdot)^{-1}\) commute, our practice of writing \( g^\Theta \) for \( \Theta(g) \), etc, should not cause any confusion.

We have natural maps

\[ K \to U \to U/K \]
\[ \downarrow \quad \downarrow \quad \downarrow \]
\[ G_0 \to G \to G/G_0 \]

(1.2)

The vertical arrows (given by inclusion) are homotopy equivalences; more precisely, there are diffeomorphisms (polar or Cartan decompositions)

\[ K \times p \to G_0, \quad U \times iu \to G, \quad U \times_K i\mathfrak{k} \to G/G_0, \]

(1.3)

in each case given by the formula \((g, X) \to g \exp(X) \mod G_0\) in the last case. In turn there are totally geodesic embeddings of symmetric spaces

\[ U/K \xrightarrow{\phi} U : gK \to gg^{\Theta}, \]
\[ \downarrow \quad \downarrow \]
\[ G/G_0 \xrightarrow{\phi} G : gg_0 \to gg^{*\Theta} = gg^{\sigma}, \]

(1.4)

where the symmetric space structures are derived from the Killing form.

A group element of the form \( g = g_1g_1^{-\sigma} \) satisfies the equation \( g^* = g^\Theta \) (i.e. \( gg^{\sigma} = 1 \)); \( g^* = g^\Theta \) implies that \( Ad(g) \circ \sigma \) is an antilinear involution; and if \( g = g_1g_1^{-\sigma} \), then \( Ad(g) \circ \sigma = Ad(g_1) \circ \sigma \circ Ad(g_1^{-1}) \), hence \( \sigma \) and \( Ad(g) \circ \sigma \) are inner conjugate. These considerations lead to the following well-known

(1.5) Proposition. (a) In terms of \( g \in G \),

\[ \phi(U/K) = \{ g^{-1} = g^* = g^\Theta \} \to U = \{ g^{-1} = g^* \} \]
\[ \phi(G/G_0) = \{ g^* = g^\Theta \} \to G \]
where \{·\}_0 denotes the connected component containing the identity.

(b) The connected components of \{g^{-1} = g^* = g^\Theta\} are determined by the map which sends \(g\) to the inner conjugacy class of the involution \(\eta = \text{Ad}(g) \circ \Theta\), subject to the constraint that \(\eta\) equals \(\Theta\) in \(\text{Out}(U) = \text{Ad}(U) \setminus \text{Aut}(U)\). A similar statement applies to \(\{g^* = g^\Theta\}\), with \(\sigma\) and antilinear automorphisms of \(G\) in place of \(\Theta\) and involutions of \(K\).

**Proof of (1.5).** We first recall why \(\{gg^\sigma = 1\}\) is smooth.

Consider the map \(\psi : G \to G : g \mapsto gg^\sigma\). If we use right translation to identify the tangent space at any point of \(G\) with \(g\), the derivative at \(g\) is given by \(x \mapsto x + \text{Ad}(g)[\sigma(x)]\). Thus \(\ker(d\psi|_g)\) is identified with the \(-1\) eigenspace of \(\text{Ad}(g) \circ \sigma\) acting on \(g\).

Now suppose \(gg^\sigma = 1\). Since \(\text{Ad}(g) \circ \sigma\) is an involution, the spectrum of \(\text{Ad}(g) \circ \sigma\) is fixed. Thus the dimension of the \(-1\) eigenspace of \(\text{Ad}(g) \circ \sigma\) is constant on \(\{gg^\sigma = 1\}\). It follows that \(\psi\) has constant rank on the connected components of \(\psi^{-1}(1)\). Since \(\psi\) is an algebraic map, this implies that \(\{g^* = g^\Theta\}\) is an embedded submanifold. A similar argument applies to the intersection with \(U\).

The action

\[
G \times \{gg^\sigma = 1\} \to \{gg^\sigma = 1\} : g, g_1 \to gg_1g^\sigma
\]

is isometric (for the symmetric space structure). The constancy of the rank of \(\psi\) on connected components is equivalent to the statement that the dimension of the isotropy subgroup for the action of \(G\) is constant on connected components of \(\{gg^\sigma = 1\}\) (in fact this dimension is the same on all components). Hence the action of \(G\) must be transitive on connected components. The same applies to the same action of \(U\) on \(\{g \in U : gg^\Theta = 1\}\). This implies (a).

For the first part of (b), note that in fact the map

\[
\{g \in U : g^{-1} = g^\Theta\} \to \{\eta \in \text{Aut}(U)^{(2)} : \text{Out}(\eta) = \text{Out}(\Theta)\} : g \to \text{Ad}(g) \circ \Theta
\]

is a universal covering for each connected component (For the identity component this covering is understood more intellibly by identifying the total space with \(U/K\):

\[
\mathcal{C}_U(K)/\mathcal{C}(U) \to U/K \xrightarrow{q} \text{Ad}(U) \cdot \Theta
\]

where \(q(g_1K) = \text{Ad}(g_1) \circ \Theta \circ \text{Ad}(g_1)^{-1}\); to obtain a similar picture for another component, we replace \(\Theta\) by \(\text{Ad}(g) \circ \Theta\), for some \(g\) in the component).
The second part of \((b)\) is similar (We could also note that the inclusion \(\{g^{-1} = g^* = g^\Theta\} \to \{g^* = g^\Theta\}\) is a homotopy equivalence, since we know this is true for the identity component, and we are free to change \(\Theta\) to \(Ad(g) \circ \Theta\); the fact that the \(\pi_0\)'s are the same is a reflection of the fact that classifying \(\Theta\)'s and classifying \(\sigma\)'s are canonically isomorphic problems (see e.g. 2. of §6, chapter 10 of [H])). \(\square\)

Remarks (1.8). (a) I do not know of a uniform way to define an invariant for the class of an involution \(\eta\) as in \((b)\). However it is a simple matter to produce an invariant in a case by case manner from the classification of symmetric spaces (see Table V of [H1]).

(b) The groups and maps in (1.4) exist for any automorphism \(\Theta\) of \(K\). However it seems that there is a linear characterization of the \(\phi\)-images (up to connectedness issues), and \(G_0\) is a real form, only in the symmetric case, \(\Theta^2 = 1\).

Fix a maximal abelian subalgebra \(\mathfrak{t}_0 \subset \mathfrak{k}\). We then obtain \(\Theta\)-stable Cartan subalgebras

\[
\mathfrak{h}_0 = \mathcal{Z}_{\g_0}(\mathfrak{t}_0) = \mathfrak{t}_0 \oplus \mathfrak{a}_0, \quad \mathfrak{t} = \mathfrak{t}_0 \oplus i\mathfrak{a}_0, \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}_0^C
\]  

(1.9)

for \(\g_0, \mathfrak{u}, \) and \(\g\), respectively, where \(\mathfrak{a}_0 \subset \mathfrak{p}\) (see (6.60) of [Kn]). We let \(T_0\) and \(T\) denote the maximal tori in \(K\) and \(U\) corresponding to \(\mathfrak{t}_0\) and \(\mathfrak{t}\), respectively.

Let \(\Delta\) denote the roots for \(\mathfrak{h}\) acting on \(\g\); \(\Delta \subset \mathfrak{h}^*_R\), where \(\mathfrak{h}^*_R = \mathfrak{a}_0 \oplus i\mathfrak{t}_0\). We choose a Weyl chamber \(C^+\) which is \(\Theta\)-stable (to prove that \(C^+\) exists, we must show that \(i\mathfrak{t}_0\), the \(+1\) eigenspace of \(\Theta\) acting on \(\mathfrak{h}^*_R\), contains a regular element of \(\g\); this is equivalent to the fact that \(\mathfrak{h}_0\) in (1.8) is a Cartan subalgebra). Since \(\sigma = -(\cdot)^\Theta\) and \((\cdot)^*\) is the identity on \(\mathfrak{h}^*_R\), \(\sigma(C^+) = -C^+\).

Given our choice of \(C^+\), we obtain a \(\Theta\)-stable triangular decomposition \(\g = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+\), so that \(\sigma(\mathfrak{n}^\pm) = \mathfrak{n}^\mp\). Let \(N^\pm = \exp(\mathfrak{n}^\pm), H = \exp(\mathfrak{h})\), and \(B^\pm = H N^\pm\). We also let \(W = W(G, T)\) denote the Weyl group, \(W = N_U(T)/T \simeq N_G(H)/H\).

At the group level we have the Birkhoff or triangular or LDU decomposition for \(G\),

\[
G = \bigsqcup_W \Sigma^G_w, \quad \Sigma^G_w = N^- w H N^+,
\]  

(1.10)

where \(\Sigma^G_w\) is diffeomorphic to \((N^- \cap wN^- w^{-1}) \times H \times N^+\). When we intersect this decomposition with \(\phi(G/G_0)\), and the other spaces in (b) of (1.5), we obtain various decompositions. We will first determine the structure of the pieces in the \(\{g^* = g^\Theta\}\) case (thus we initially ignore connectedness issues).
(1.11) Proposition. Fix \( w \in W \).

(a) The intersection \( \{g^* = g^\Theta\} \cap \Sigma^G_w \) is nonempty if and only if there exists \( w \in w \subset N_U(T) \), such that \( w^* = w \); \( w \) is unique modulo the action

\[
T \times \{w \in N_U(T) : w^* = w\} \to \{w^* = w\} : \lambda, w \to \lambda w \lambda^\Theta.
\]

(b) For the action \( B^- \times \{g^* = g^\Theta\} \to \{g^* = g^\Theta\} : b, g \to bg^\Theta \), the stability subgroup is given by

\[
B_w^- = \{b : w^{-1}bw = \sigma(b)\} \simeq \{l \in N^- : w^{-1}lw = \sigma(l) \in N^+\} \times \{h \in H : h^{-1} = \sigma(h)\}.
\]

(c) The orbits of \( B^- \) in \( \{g^* = g^\Theta\} \cap \Sigma^G_w \) are open and indexed by

\[
\pi_0(\{w \in w : w^* = w\}) \simeq \{w \in w : w^* = w\}/T,
\]
where \( T \) acts as in part (a).

(d) The map

\[
N^- \cap N^w \times \{h \in H, L \in N^- \cap N^w : h^{-1} = h^*, \sigma(L)^w = L^{-1}\} \to \{g^* = g^\Theta\} \cap \Sigma^G_w
\]

given by \( l, h, L \to lL^{-1}wh(lL^{-1})^\Theta \) is a diffeomorphism onto the connected component containing \( w \). This component is homotopic to the torus \( \exp(\{\text{Ad}(w^{-1})\}l_l = -1)\).

(e) In particular for \( w = 1 \), the map

\[
N^- \times (T_0^{(2)} \times \exp(i\alpha_0)^{(2)} \exp(i\alpha_0)) \times \exp(i\alpha_0) \to \{g^* = g^\Theta\} \cap \Sigma^G_1
\]

\[
l, [w, m], a_{\phi} \to g = l\exp(m)^{\alpha_{\phi}}l^{\alpha_{\phi}}
\]
is a diffeomorphism, so that the connected components for \( \{g^* = g^\Theta\} \cap \Sigma^G_1 \) are indexed by \( T_0^{(2)} / \exp(i\alpha_0)^{(2)} \).

Proof of (1.11). Suppose that \( g \in \Sigma^G_w \). We write \( g = lwhu \), for some \( l \in N^- \), \( w \in w \subset N_U(T) \), \( h \in \exp(h_{\mathbb{R}}) \), \( u \in N^+ \). If we additionally require that \( l \in N^- \cap (N^-)^w \), then this decomposition is unique, but we will not require this at the outset.

We have \( g = g^* \) if and only if

\[
lwhu = u^*(wh)^*l^\Theta
\]
if and only if

\[
(wh)^* = (u^l)(wh)(ul^*)
\]
\[ (u^\sigma l)_{\pm} (u^\sigma l)_+ (wh) (ul^\sigma) \]
\[ = (u^\sigma l)_{\pm} (wh) \{(u^\sigma l)^{\pm (wh)^{-1}} ul^\sigma \} \]  
(1.13)

where \( L = L_- L_+ \) denotes the decomposition induced by the diffeomorphism

\[ N^- \cap (N^-)^w \times N^- \cap (N^+)^w \rightarrow N^- : L_-, L_+ \rightarrow L = L_- L_+ \]  
(1.14)

Thus (1.13) holds if and only if

\[ (u^\sigma l)_- = 1 = (u^\sigma l)_+^{(wh)^{-1}} ul^\sigma \]  
(1.15)

and \((wh)^* \Theta = wh\), or, using the fact that \( h \) is real,

\[ h^\Theta = h^w \quad \text{and} \quad ww^\Theta = 1. \]  
(1.16)

Consider part (a). If \( g \) is in the intersection, then we have just seen that \( w \) must satisfy \( w^* \Theta = w \). Conversely, given a unitary representative \( w \) for \( u \) satisfying \( w^* \Theta = w \), the intersection contains \( w \) and hence is nonempty. This proves (a).

Part (b) is straightforward.

Now consider (c). We first write \( g \in \{ g^* = g^\Theta \} \cap \Sigma_w^G \) uniquely as \( l\omega u \), where \( l \in N^- \cap wN^- w^{-1} \), \( \omega = wh \), and \( u \in N^+ \). For the first part of (c) we must prove that we can relax the constraint on \( l \) to arrange for \( u = l^* \Theta \). We can write

\[ g = ll^{-1} \omega \{ (\omega L \omega^{-1}) u \}, \]  
(1.17)

where \( L \in N^- \cap wN^+ w^{-1} \) is arbitrary. We must prove the existence of \( L \) such that \( ll^{-1} = u^* \Theta \omega \sigma (L^{-1}) \omega^{-1} \), or

\[ u^\sigma l = \omega L^{-\sigma} \omega^{-1} L. \]  
(1.18)

The basic fact is that this equation has a unique solution \( L \in N^- \cap N^+ w \) satisfying \( \omega L^\sigma \omega^{-1} = L^{-1} \), namely \( L = (u^\sigma l)^{1/2} \) (square root has an unambiguous meaning in a simply connected nilpotent Lie group). To see this simply plug such an \( L \) into (1.18). We obtain the equation \( L^2 = u^\sigma l \). The fact that \( u^\sigma l \), and its square root, satisfy \( \omega L^\sigma \omega^{-1} = L^{-1} \) follows from (1.15), and uniqueness of the square root.

As we remarked previously, the existence of a solution \( L \) proves that \( B^- \) has open orbits. The rest of part (c) is relatively straightforward, using (1.16).

The uniqueness of the solution \( L \), subject to the constraint we imposed, implies the first part of (d). The second statement in (d) follows routinely from the first part.
For part (e), to clarify the statement, observe that
\[ \exp(i a_0)^{(2)} = K \cap \exp(i a_0) = T_0^{(2)} \cap \exp(i a_0). \] (1.19)

Now suppose that \( w = 1 \). In this case \( w \) is in the kernel of the homomorphism \( T \rightarrow T : w \mapsto w w^\Theta \), and this equals the subgroup generated by \( T_0^{(2)} \) and \( \exp(i a_0) \). We can modify \( w \) by multiplying by something in the image of the homomorphism \( T \rightarrow T : \lambda \mapsto \lambda \lambda^\Theta \). This image is \( \exp(i a_0) \). Therefore we can choose \( w \in T_0^{(2)} \), but this choice is unique only modulo the intersection of \( T_0^{(2)} \) and \( \exp(i a_0) \). This proves (e). \( \square \)

Example (1.20). In the group case, \( U = K \times K \), where \( K \) embeds diagonally. The image of \( t_0 \) inside \( u \) is \( \{(x, x) : x \in t_0\} \), while \( i a_0 = \{(x, -x) : x \in t_0\} \). This implies that the quotient \( T_0^{(2)} / \exp(i a_0)^{(2)} \) is trivial. Thus in this group case, the set of generic elements (considered in part (e)) is connected, as we already know.

(1.22) Notation. Given \( w \) as in (c) of (1.11), we let \( \Sigma_w^{(g^* = g^\Theta)} \) denote the corresponding connected component of \( \{g^* = g^\Theta\} \cap \Sigma_w^G \) (the \( B^- \)-orbit of \( w \), in the sense of (c) of (1.11)). If \( w \in \phi(G/G_0) \), then we will write \( \Sigma_{w}^{\phi(G/G_0)} \) for this component. We also set \( \Sigma_{w}^{\phi(U/K)} = \phi(U/K) \cap \Sigma_{w}^{\phi(G/G_0)} \).

Having understood the intersection of \( \{g^* = g^\Theta\} \) with the triangular decomposition, we now want to specialize this to the identity component. In an abstract way this is answered by (1.5b) (and Remark (1.8a)). Concerning open orbits, we have the following.

(1.23) Proposition. Suppose that \( w \in N_U(T) \) satisfies \( w^{-\Theta} = w \). The following are equivalent:

(a) \( \Sigma_{w}^{(g^* = g^\Theta)} \) is an open \( B^- \)-orbit in the identity component, \( \phi(G/G_0) \).

(b) There exists \( w_1 \in N_U(T_0) \) such that \( \phi(w_1 K) = w \).

Hence the open orbits can be parameterized by either \( N_U(T_0)/N_K(T_0) \) (the intrinsic point of view), or the set of \( w \in T_0^{(2)} / \exp(i a_0)^{(2)} \) such that \( \text{Ad}(w) \circ \Theta \) is equivalent to \( \Theta \) in the sense of (1.5b) (the nonintrinsic point of view, as in (1.11e)).

In addition, the \( w_1 K \) are exactly the \( T_0 \) fixed points in \( U/K \).

Proof of (1.23). Determining the possible (open) \( B^- \) orbits in \( G/G_0 \) is equivalent to determining the possible (open) \( G_0 \) orbits in \( B^- \backslash G \). Thus the equivalence of (a) and (b) follows from Theorem 4.6 and its Corollaries in [Wolf]. The other statements are obvious. \( \square \)
In general it apparently remains an open question to systematically obtain representatives for all $B^{-}$ orbits in $G/G_0$, from the intrinsic point of view (see [WZ] for the Hermitian symmetric case). In this regard the nonintrinsic point of view of (1.11) seems to have some utility.

Let $q : G \to G/B^+$ denote the quotient map. The map $q$ applied to the decomposition (1.9) induces the (more conventional) triangular stratification for the flag space,

$$U/T \simeq G/B^+ = \bigsqcup_{w} \Sigma_w, \quad \Sigma_w = N^- \cdot wB^+, \quad (1.24)$$

where each $\Sigma_w$ is a cell ($\simeq N^- \cap wN^-w^{-1}$). As a consequence, for the pieces of the induced decomposition for $U$, there are diffeomorphisms

$$\Sigma^U_w = U \cap \Sigma^G_w \simeq \Sigma_w \times T. \quad (1.25)$$

The inclusions $\Sigma^U_w \to \Sigma^G_w$ are homotopy equivalences, because $T$ is homotopy equivalent to $B^+$:

$$\begin{align*}
T & \to \Sigma^U_w q \to \Sigma_w \\
\downarrow & \downarrow \parallel \\
B^+ & \to \Sigma^G_w q \to \Sigma_w
\end{align*} \quad (1.26)$$

The main point of this section is now to describe the generalization of this to $U/K \to G/G_0$. We consider the Iwasawa decomposition for $G$, which we write as

$$G \simeq N^- \times A \times U : g = l(g)a(g)u(g), \quad (1.27)$$

where $A = \exp(\mathfrak{h}_R)$. There is an induced right action

$$U \times T \times G_0 \to U : (u, t, g_0) \to t^{-1}u(ug_0) \quad (1.28)$$

arising from the identification of $U$ with $N^-A\backslash G$.

**Proposition.** Suppose that $w \in N_U(T)$ satisfies $w^{-\Theta} = w$ and $\Sigma^*_w = g^* = g^{\Theta} \subset \phi(G/G_0)$. Fix a choice of $w_1 \in U$ such that $w_1^{-\Theta} = w$.

(a) The map

$$T \times G_0 \to \Sigma^\phi(U/K) : (t, g_0) \to \phi(t^{-1}u(w_1g_0)) \quad (1.30)$$

is surjective, and induces a diffeomorphism

$$T \times \exp(\{Ad(w)^{\Theta}w_1 : w \in G\}) \to \Sigma^\phi(U/K) \quad (1.31)$$
where \( R = (N^{-1}A)^{w^{-1}} \cap G_0 \) is a contractible subgroup of \( G_0 \), and \( \lambda \in \exp(\{ Ad(w) \Theta | t = 1 \}) \) is identified with a pair \( (\lambda, \lambda^{w^{-1}}) \).

(b) The inclusion

\[
\Sigma^\phi(U/K) \to \Sigma^\phi(G/G_0).
\]

is a homotopy equivalence; each is homotopic to the torus \( \exp(\{ Ad(w^{-1}) \Theta | t = -1 \}) \).

(c) The connected components of \( \phi(U/K) \) intersected with \( \Sigma^U \) are indexed by \( w \in T_0^{(2)}/\exp(ia_0)^{(2)} \) such that \( w = w_1w_1^{-\Theta} \), for some \( w_1 \in N_U(T_0) \); for such a \( w \), and choice of \( w_1 \), the diffeomorphism in (a) simplifies to

\[
\exp(ia_0)/\exp(ia_0)^{(2)} \times A_0 \setminus G_0/K \to \Sigma^\phi(U/K),
\]

where \( A_0 = \exp(a_0) \).

Proof of (1.29). In proving the first part of (a), it is convenient to work with \( U/K \) instead of \( \phi(U/K) \). Thus we consider a point in the intersection of \( U/K \) and the \( B^- \)-orbit of \( w_1G_0 \in G/G_0 \). This point can be represented by a \( u \in U \) such that \( u = b^-w_1g_0 \), for some \( b^- \in B^- \) and \( g_0 \in G_0 \). This immediately implies that \( u \) is in the \( T \times G_0 \)-orbit of \( w_1 \), and this proves surjectivity of the first map in (a).

For the second part of (a), we first calculate the stabilizer for the action (1.28) at the point \( w_1 \). Suppose that \( t \in T \) and \( g_0 \in G_0 \) satisfy \( tu(w_1g_0) = w_1 \). This is equivalent to

\[
w_1g_0w_1^{-1} = l(w_1g_0)a(w_1g_0)t.
\]

This implies that \( g_0 \in G_0 \cap (B^-)^{w_1} \), and \( t = T(g_0^{w_1}) \). Conversely if \( g_0 \in G_0 \cap (B^-)^{w_1} \), then (1.34) holds with \( t = T(g_0^{w_1}) \). Thus the stabilizer is isomorphic to

\[
\{(T(g_0^{w_1}), g_0) : g_0 \in G_0 \cap (B^-)^{w_1} \} \subset T \times G_0.
\]

The group \( G_0 \cap (B^-)^{w_1} \) is connected and solvable. The torus part is isomorphic to \( \{ \lambda \in T : \lambda w^{-1} \in G_0 \} \). This condition on \( \lambda \) is equivalent to \( (\lambda w^{-1})^T = \lambda w^{-1} \), or \( \lambda \in \exp(\{ Ad(w) \Theta | t = 1 \}) \). This implies (1.31).

From (a), since \( R \setminus G_0/K \) is contractible, it follows that the double coset space in (1.31) is homotopic to \( \exp(\{ Ad(w) \Theta | t = -1 \}) \), modulo elements of order 2. A \( t \) in this torus is mapped in (1.30) to \( t^{-1}wt^\Theta = w(t^{-w^{-1}}t^\Theta) \). It is straightforward to check that \( t^{-w^{-1}}t^\Theta \) belongs to \( \exp(\{ Ad(w^{-1}) \Theta | t = -1 \}) \). Together with (d) of (1.11), this implies (b).

Part (c) follows from (a). □
We want to explain how this proposition is related to familiar facts in special cases. First consider the group case $X = K$. We have already explained why the generic set is connected; see Example (1.20). In the group case, (1.1) has the form

$\{(x, y) \in \mathfrak{k}^C \oplus \mathfrak{k}^C \} \leftrightarrow \{(x, -x^*) : x \in \mathfrak{k}^C \} \leftrightarrow \{(x, x) : x \in \mathfrak{k} \}$ \hspace{1cm} (1.36)

Given $g_0 \in G_0 = K^C$, $g_0$ maps to $(g_0, g_0^{-*}) \in G = K^C \times K^C$. Given an arbitrary triangular decomposition $\mathfrak{k}^C = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, we obtain a $\Theta$-stable triangular decomposition for $g$ by defining $n_{\pm} = \mathfrak{n}_{\pm} \times \mathfrak{n}_{\pm}$, and $\mathfrak{h} = \tilde{\mathfrak{h}} \times \tilde{\mathfrak{h}}$. The Iwasawa factorization (1.27) is equivalent to the two Iwasawa factorizations

$g_0 = l_1 a_1 k_1, \quad g_0^{-*} = l_2 a_2 k_2, \quad \text{ (1.37)}$

where $a_i \in \exp(\tilde{\mathfrak{h}} R)$, $l_i \in \tilde{N}^-$, $k_i \in K$. We have an equivariant isomorphism $U/K \to K : (g, h) \to gh^{-1}$. The map in (c) of (1.29) (using $(\cdot)^{-}$ in place of inverse) is given by

$T/T^{(2)} \times A \backslash K^C/K \to \Sigma_1 \mathfrak{K} : t, g_0 \mathfrak{K} \to t a_1^{-1} l_1^{-1} l_2^- a_2^* t. \quad \text{ (1.38)}$

So $a_\phi = a_1^{-1} a_2^{-*} = (a_1 a_2)^{-1}$. Thus in (1.29) we are using $\exp(\tilde{\mathfrak{h}} R) g_0 \mathfrak{K} \in \exp(\tilde{\mathfrak{h}} R) \backslash K^C/K$ as coordinate, which is completely equivalent to using $l_1$ or $l_2 \in \tilde{N}^-$. From this point of view, $l_1$ is a horocycle coordinate.

Now suppose that $U/K$ is Hermitian symmetric. In this case $\Theta$ is inner, $P = K^C B^+$ is a parabolic subgroup of $G$, and the natural map $i : U/K \to G/P$ is a $U$-equivariant isomorphism. The natural map $\eta : G_0/K \to G/P$ is an open holomorphic embedding, and the image is contained in $i((U/K)_r)$, the regular set (see ch VIII of [H1], especially Prop 7.14).

There is a commutative diagram

$$
\begin{array}{ccc}
G_0/K & \xrightarrow{u} & \Sigma_1^{U/K} : g_0 \mathfrak{K} \\
\downarrow I & & \downarrow i \\
G_0/K & \xrightarrow{\eta} & G/P
\end{array}
$$

where the top arrow $u$ is a diffeomorphism, by (b) of (1.29), and the map $I$ is defined in the following way: given $g_0 \mathfrak{K} \in G_0/K$, we can write $u(g_0) = \exp(ix) k$ uniquely, where $x \in \mathfrak{p}$, $\exp(itx) k$ is a geodesic of minimal length joining the basepoint to
\( u(g_0)K \), and \( k \in K \); we set \( I(g_0 K) = (g_0 k^{-1})^{-1} K \). To see that the diagram is commutative, note that because \( g_0 k^{-1} \in G_0 \), \((g_0 k^{-1})^{-1} = (g_0 k^{-1})^* \Theta = \exp(ix) a^* \Theta l^* \Theta \), and \( l^* \Theta \in N^+ \); thus \( I(g_0 K) \) equals \( u(g_0) \mod P \).

Thus in the Hermitian symmetric case, \( \Sigma U/K \) is the usual model of \( G_0/K \) inside \( U/K \), but the parameterization in \((b)\) of (1.29) is related to the natural holomorphic map \( \eta \) in a clumsy way.

\section{Diagonal Distribution}

Suppose that \( w \in T_0^{(2)} = T \cap \{ g^* = g^\Theta \} \). Then \( Ad(w)\Theta \) is an involution, and \( n^- \oplus h \oplus n^+ \) is \( Ad(w)\Theta \)-stable. We also suppose that \( w \in \phi(U/K) \).

Given \( g \in \Sigma_{\phi(U/K)} \), we factor \( g \) as in (e) of (1.11),

\[
g = lwma_\phi l^{*\Theta},
\]

where \( l \in N^-, a_\phi \in \exp(it_0) \), and \([w, m] \in T_0^{(2)} \times_{\exp(it_0)^{(2)}} \exp(it_0)\).

\begin{align}
(2.2) \text{Theorem.} & \quad \text{For } \lambda \in (it_0)^*, \\
\int_{\Sigma_{\phi(U/K)}} a_\phi(g)^{-i\lambda} = \frac{1}{M} \prod w \frac{\langle \delta, \alpha \rangle}{\langle \delta - i\lambda, \alpha \rangle} \tag{2.3}
\end{align}

where the product is over pairs \((\alpha, Ad(w)\Theta(\alpha))\) of positive complex roots which are not of compact type for \( Ad(w)\Theta \), and \( M = |N_U(T_0)/N_K(T_0)| \). Hence

\[
\int_{\phi(U/K)} a_\phi(g)^{-i\lambda} = \frac{1}{M} \sum w \frac{\langle \delta, \alpha \rangle}{\langle \delta - i\lambda, \alpha \rangle} \tag{2.4}
\]

where the sum is over representatives \( w \) for the connected components of \( \phi(U/K) \cap \Sigma_1^U \).

Note that it does not matter whether we take \( \alpha \) or \( Ad(w)\Theta(\alpha) \) in the product (2.3), because \( \delta \) and \( \lambda \) are fixed by \( Ad(w)\Theta \). In the case in which \( \Theta \) is inner, i.e. \( a_0 = 0 \), all roots are either of compact or noncompact type. Hence in this case the product in (2.3) is over the noncompact type roots.

There is a more intrinsic way to write (2.4). The right hand side can be expressed as a sum over \( w_1 \in N_U(T_0)/N_K(T_0) \), using \( Ad(w)\Theta = Ad(w_1)\Theta Ad(w_1^{-1}) \) (see (1.23)).

To prove (2.2) we will first note that we can reduce to the case \( w = 1 \) in (2.3). We will then need several Lemmas.
To see that it suffices to prove (2.3) in the case $w = 1$, observe that left translation by $w \in T_0^{(2)}$, which is an isometric map for the Riemannian structure of $U$, maps the $w = 1$ component to the $w$-component:

$$L_w : \Sigma^\phi_{1}(U/K) \to \Sigma^\phi_w(U/K).$$  \hfill (2.5)

This map interchanges the canonical factorization from that relative to $\Theta$ to the one relative to $Ad(w)\Theta$: if $g$ has the unique decomposition $g = lma_\phi l^*\Theta$, then $wg$ has the unique decomposition $wg = l^w wma_\phi(l^w)^*Ad(w)\Theta$. Since $a_\phi$ is unchanged, the integral is evaluated in the same way, except the meaning of the roots changes.

We henceforth suppose $w = 1$. Consider the parameterization

$$\Phi : exp(i a_0) \times A_0 \backslash G_0 / K \to \Sigma^\phi_{1}(U/K) : (t, A_0 g_0 K) \to \phi(t^{-1}u(g_0)).$$  \hfill (2.6)

from (1.33). Note that $a_\phi = a(g_0)^{-1}a(g_0)^\tau \in exp(it_0)$.

(2.7) Lemma. We have

$$\Phi^*(dV_{U/K}) = a_\phi^{2s} (dV_{exp(i a_0)} \times dV_{A_0 \backslash G_0 / K})$$

where $dV_{A_0 \backslash G_0 / K}$ is obtained by integrating a $G_0$-invariant measure on $A_0 \backslash G_0$ over $K$. Thus (2.3) equals

$$\int a_\phi(u(g_0))^{2s-\i \lambda} dV_{A_0 \backslash G_0 / K}. \hfill (2.8)$$

Proof of (2.7). We will consider a slight reformulation of the problem. We identify $U / K$ with $\phi(U / K)$. Let $S$ denote the inverse image of $\Sigma^U_{1}$ in $U$, with respect to the projection $U \to U / K$. Consider the lift

$$\Psi : exp(i a_0) \times A_0 \backslash G_0 \to S : (t, A_0 g_0) \to t^{-1}u(g_0)$$  \hfill (2.9)

of $\Phi$ in (2.6). We must show that the Jacobian for the mapping $\Psi$, with respect to the Riemannian structures induced by the Killing form, is equal to a constant times $a_\phi^{2s}$. To do this we identify $i a_0$, $g_0 \oplus a_0$, and $u$ with the tangent spaces to $exp(i a_0)$, $A_0 \backslash G_0$, and $U$, respectively, using the exponential map and right translation (we use right translation because $A_0$ appears on the left). Let $P : g \to u$ denote the projection with kernel $n^- \oplus h_\mathbb{R}$. We compute

$$d\Psi_{|_{(t, A_0 g_0)}} : i a_0 \oplus (g_0 \oplus a_0) \to u : (\chi, x) \to \frac{d}{d\epsilon}|_{\epsilon=0}(te^{\epsilon x})^{-1}u(e^{\epsilon x}g_0)u(g_0)^{-1}t$$
\[ = \text{Ad}(t^{-1})\{ -\chi + P(\text{Ad}(a^{-1}1^{-1}))(x) \}. \tag{2.10} \]

The operator \( \text{Ad}(t^{-1}) \) preserves \( u \)-volume, so it can be ignored.

Write \( a = a_1a_0 \), relative to the decomposition \( A = \exp(it_0)A_0 \). Since \( a_0 \in G_0 \), \( \text{Ad}(a_0) \) will preserve \( g_0 \)-volume. Thus the determinant of (2.10) equals the determinant of the map

\[ g_0 \oplus a_0 \to u \oplus ia_0 : x \to P_1(\text{Ad}(\Lambda')\text{Ad}(a_1^{-1}))(x), \tag{2.11} \]

where \( \Lambda' = a_1^{-1}la_1 \in N^- \) and \( P_1 \) is \( P \) followed by the projection to \( u \oplus ia_0 \).

Given \( x \in g \), if \( x = x_+ + x_+ + x_+ \) is its triangular decomposition, then

\[ P(x) = -x^*_+ + \frac{1}{2}(x_+ - x^*_+) + x_+. \tag{2.12} \]

If \( x \in g_0 \), then \( x_- = x_+^a \), and \( x_+ = x_{t_0} \in t_0 \). Because \( \Lambda'a^{-1}_1 \) maps \( n_- \) into itself, (2.11) is given by

\[ P_1(\text{Ad}(\Lambda')a^{-1}_1)(x)) = -[(x_+^{\Lambda'a^{-1}_1})_+^* + (x_{t_0} + (x_+^{\Lambda'a^{-1}_1})_0) + (x_+^{\Lambda'a^{-1}_1})_+]. \tag{2.13} \]

Thus the determinant of (2.10) is the same as the (real) determinant of the map \( x_+ \to (x_+^{\Lambda'a^{-1}_1})_+ \). Because of the unipotence of \( \text{Ad}(\Lambda') \), this is equal to

\[ \prod_{\alpha > 0} |a_1^{-\alpha}|^2 = a_1^{-4\delta} = a_1^{-2\phi}. \tag{2.14} \]

\[ \square \]

We will now show that the integral (2.8) can be computed using a Duistermaat-Heckman exact stationary phase calculation. The relevant Poisson structure was discovered by Evens and Lu in a very general setting ([EL]). We will introduce this structure directly, but to understand why it is natural the reader will need to consult the original paper.

To do calculations we will use the isomorphism of vector bundles

\[ G_0 \times_K p \to T(G_0/K) : [g_0, x] \to \frac{d}{dt}|_{t=0}(g_0e^{tx}K), \]

and we will use the Killing form to identify \( p^* \) with \( p \).

Consider the \( \text{Ad}(T_0) \) and \( \text{Ad}(A_0) \)-stable decomposition of \( g \) as a direct sum of subalgebras:

\[ g = g_0 \oplus (n^- \oplus ih_0). \tag{2.15} \]
Let \( pr_{\mathfrak{g}_0} \) denote the projection \( \mathfrak{g} \to \mathfrak{g}_0 \) along this decomposition. Given \( x \in \mathfrak{g} \), with triangular decomposition \( x = x_- + x_0 + x_+ \),

\[
pr_{\mathfrak{g}_0}(x) = (x_+^\sigma + (x_0)_{h_0} + x_+).
\]  

(2.16)

The Evens-Lu Poisson bivector is given by

\[
\Pi([g_0, x] \land [g_0, y]) = \langle \Omega(g_0)(x), y \rangle,
\]  

(2.17)

where \( \Omega(g_0) : \mathfrak{p} \to \mathfrak{p} \) is given by

\[
\Omega(g_0)(x) = \{(pr_{g_0}(ix g_0))_{g_0^{-1}}\}_p.
\]  

(2.18)

The operator \( \Omega \) satisfies the equivariance condition

\[
\Omega(a_0g_0k) = Ad(k)^{-1}\Omega(g_0)Ad(k).
\]  

(2.19)

To understand \( \Omega \), it is useful to consider the augmented operator \( \tilde{\Omega} : \mathfrak{g}_0 \to \mathfrak{g}_0 \) given by

\[
\tilde{\Omega}(g_0)(x_t + x_p) = \{(pr_{g_0}((x_t + ix_p)g_0))_{g_0^{-1}}\}.
\]  

(2.20)

Relative to the decomposition \( \mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p} \),

\[
\tilde{\Omega} = \begin{pmatrix}
1 & * \\
0 & \Omega
\end{pmatrix}.
\]  

(2.21)

This augmented operator can be factored as the composition of four operators

\[
\mathfrak{g}_0 \xrightarrow{I} \mathfrak{u} \xrightarrow{Ad(u(g_0))} \mathfrak{u} \xrightarrow{T} \mathfrak{g}_0 \xrightarrow{Ad(a_0^{-1}g_0)^{-1}} \mathfrak{g}_0
\]  

(2.22)

where the first operator is given by \( I(x_t + x_p) = x_t + ix_p \), and \( T(g_0) \) maps \( x = -x_+^* + (x_{t_0} + x_{ia_0}) + x_+ \) to

\[
T(g_0)(x) = pr_{g_0}(x^{\mathfrak{l'}}a_1(g_0)) = 
\]

\[
[(x_+^{\mathfrak{l'}}a_1)^+)^\sigma + (x_{t_0} + (x_+^{\mathfrak{l'}}a_1)t_0 + a_0) + (x_+^{\mathfrak{l'}}a_1)^+,
\]  

(2.23)

where \( \mathfrak{l'} = a_0l_{a_0}^{-1} \), and the last equality depends upon the fact that conjugation by \( l'a_1(g_0) \) maps \( \mathfrak{n}^- \) into itself, and that \( \mathfrak{n}^- \) terms disappear when we use (2.16).
(2.24) Lemma. (a) \( \Omega \in so(p) \); the Schouten bracket \([\Pi, \Pi]\) vanishes, so that \((G_0/K, \Pi)\) is a Poisson manifold.

(b) \( \ker(\Omega(g_0)) = \{[g_0, (a_0^{u(g_0)^{-1}})_p]\} \)

(c) Pfaffian\( (\Omega(g_0)|_{\ker(\Omega)^\perp}) = a_1(g_0)^{2\delta} \).

Proof of (2.24). For (a) let \( X = x^{g_0}, Y = y^{g_0}, x, y \in p \). Then

\[
\langle \Omega(g_0)x, y \rangle = \langle pr_{g_0}(iX), Y \rangle
\]

\[
= \langle -iX_+^\sigma + iX_+^\sigma + Y_0 + Y_+ \rangle = 2\langle iX_+, Y_+^\sigma \rangle. \tag{2.25}
\]

This is clearly skew-symmetric in \( X \) and \( Y \), because \( \sigma \) preserves the Killing form and it is complex antilinear. For the second part of (a) we refer to [EL] (or see §3 of [FO] for an exposition specific to this case).

For (b), note that (2.23) implies the kernel of \( T \) is \( i a_0 \). Thus (2.22) implies

\[
\ker(\hat{\Omega}(g_0)) = \{[g_0, x] : (x_t + ix_p) \in i a_0^{u(g_0)^{-1}} \} \tag{2.26}
\]

This, together with (2.21), implies (b).

For (c), note that in (2.22) the first, second and fourth operators preserve volume determined by the Killing form. The determinant of \( T \) (relative to the Killing form volumes) is the same as the determinant of the operator on \( n_+ \) which maps \( x_+ \) to \( (x_+^{a_1})_+ \). This determinant equals

\[
\prod_{\alpha > 0} a_1^{2\alpha} = a_1^{4\delta} \tag{2.27}
\]

Thus the Pfaffian is \( a_1^{2\delta} \). □

By (b) the tangent directions in \( G_0/K \) determining the symplectic leaves are given by \( [g_0, x] \) such that \( x^u \perp a_0 \). This is clearly \( A_0 \)-invariant, because \( u(a_0 g_0) = u(g_0) \). Thus the left action of \( A_0 \) permutes the symplectic leaves. The symplectic form is given by the formula

\[
\omega([g_0, x], [g_0, y]) = \langle \Omega(g_0)|_{\ker(\Omega)^\perp} \rangle^{-1}(x), y \rangle. \tag{2.28}
\]

This form does not in general descend to a form on the quotient \( A_0 \setminus G_0/K \). However (c) of the preceding Lemma does imply that the volume form descends.
(2.29) Proposition. (a) The action of $T_0$ is Hamiltonian with momentum map
\[
\mu : G_0/K \to (t_0)^* : g_0K \to \langle i\log(a_1(g_0)), \cdot \rangle,
\]
This momentum map is proper, and it is semibounded.

(b) The symplectic measure is
\[
\frac{\omega^d}{d!} = a_1(g_0K)^{-2\delta} dV_{A_0\backslash G_0/K}(A_0g_0K)
\]
(where the invariant measure is suitably normalized).

Proof of (2.29). Part (a) is proven in [FO] (Lemma 3.3, which in turn refers to a result of Van Den Ban). Part (b) follows from (c) of (2.24). □

We can now apply the Duistermaat-Heckman exact stationary phase method, as generalized to noncompact manifolds in [PW]. For definiteness we will consider the symplectic leaf through the basepoint of $G_0/K$.

We must first find the fixed points of the $T_0$ action. Suppose that $g_0K$ is fixed by $T_0$. If we choose $\lambda \in T_0$ which generates a dense subgroup of $T_0$, this is equivalent to $g_0^{-1}\lambda g_0 \in K$. Since $T_0$ is maximal abelian in $K$, we can assume (by multiplying $g_0$ on the right by $k \in K$ if necessary) that $g_0^{-1}\lambda g_0 \in T_0$. Since $N_{G_0}(T_0) = N_K(T_0)exp(h_0)$, $g_0K = a_0K$ for some $a_0 \in A_0$. Thus each symplectic leaf has exactly one $T_0$ fixed point. Since we are considering the leaf through the basepoint, there is just one $T_0$ fixed point, the basepoint.

If $X$ denotes the element of $t_0$ corresponding to $\delta + \Lambda$ ($\Lambda = -i\lambda$), then the Pfaffian of the infinitesimal action of $X$ at the basepoint equals
\[
Pf(ad(X)|_p) = \prod \langle \delta + \Lambda, \alpha \rangle,
\]
where the product is over pairs of positive roots $(\alpha, \Theta(\alpha))$ which are not of compact type. The Duistermaat-Heckman formula now implies (2.3) in the case $w = 1$. This concludes the proof of (2.2).

We end this section with two brief remarks. First, it is interesting to consider the integral
\[
\psi_\Lambda(g) = \int_{A_0\backslash G_0} a_1(g_0g)^{-2\delta-2(\Lambda+\delta)}dg_0,
\]
for $g \in G$, which generalizes (2.8). When this is well-defined, (1) this is a function of $g \in G_0\backslash G/U$, (2) this is a $G_0$-invariant eigenfunction for $G$-invariant differential
operators on $G/U$; see Lemma 5.15 of ch2 section 5 of [H2] (one is usually interested in $U$-invariant eigenfunctions, i.e. spherical functions).

To explicitly evaluate (2.31), first note that $G_0 \exp(it_0) U = G$ (this is existence of polar decomposition for the non-Riemannian symmetric space $G_0 \backslash G$). Thus we can suppose that $g = a \in \exp(it_0)$. In this case (2.31) is an integral over $A_0 \backslash G_0/C_K(a)$. One can define a Poisson structure on $G_0/C_K(a)$, using the Evens-Lu method, as above (see §3 of [FO]). As in (2.29), the moment map can be identified with $\log(a_1(ga))$, and the symplectic volume of a symplectic leaf can be identified with the form $a_1(g_0a)^{-2\delta}dV_{A_0}\backslash G_0/C_K(a)$, via the projection to the double coset space. The fixed points for the $T_0$ action are of the form $a_0wC_K(a)$, where $a_0 \in A_0$ and $w \in W(K,T_0)$. Thus (2.31) equals

$$\sum_{\{w\}} a_1(wa)^{-2(\delta+\Lambda)} \prod_{w} \langle \delta + \Lambda, \alpha \rangle$$

where given $w \in W(K,T_0)/W(C_K(a),T_0)$, the product is over (1) pairs of positive roots $(\alpha, \Theta(\alpha))$ which are not of compact type (relative to $\Theta$), and (2) positive compact type roots which vanish on $C_K(a)$.

The second remark is that there is a kind of “dual” Poisson structure, on all of $U/K$, which can be used so that the sum (2.4) has the structure of an exact stationary phase calculation. In the terminology of the paper [EL], in this section we used the Lagrangian splitting (2.15), to obtain a Poisson structure on $G_0/K$; the “dual” is the (Iwasawa) Lagrangian splitting $g = u \oplus (h_R + n^-)$, which induces a Poisson structure on $U/K$. This will hopefully be taken up elsewhere.

Appendix. Special Features of the Group Case.

In this appendix we will present a proof of (0.1), using facts about Bott-Samelson resolutions of Schubert varieties. One rationale for including this appendix is that many of the arguments are valid in the more general context of Kac-Moody Lie algebras and groups. Throughout this appendix, we will use the notation and basic results in [Kac].

We start with the following data: $A$ is an irreducible symmetrizable generalized Cartan matrix; $g = g(A)$ is the corresponding Kac-Moody Lie algebra, realized via its standard (Chevalley-Serre) presentation; $g = n^- \oplus h \oplus n^+$ is the triangular decomposition; $b = h \oplus n^+$ the upper Borel subalgebra; $G = G(A)$ is the algebraic group associated to $A$ by Kac-Peterson; $H, N^\pm$ and $B$ are the subgroups of $G$. 
corresponding to \( \mathfrak{h}, \mathfrak{n}^{\pm}, \) and \( \mathfrak{b}, \) respectively; \( K \) is the “unitary form” of \( G; T = K \cap H \) the maximal torus; and \( W = N_K(T)/T \cong N_G(H)/H \) is the Weyl group.

A basic fact is that \((G, B, N_G(H))\) with Weyl group \( W \) is an abstract Tits system. This yields a complete determination of all the (parabolic) subgroups between \( B \) and \( G \). They are described as follows.

Let \( \Phi \) be a fixed subset of the simple roots. The subgroup of \( W \) generated by the simple reflections corresponding to roots in \( \Phi \) will be denoted by \( W(\Phi) \). The parabolic subgroup corresponding to \( \Phi \), \( P = P(\Phi) \), is given by \( P = BW(\Phi)B \).

Given \( w \in N_K(T) \), we will denote its image in \( W/W(\Phi) \) by \( \bar{w} \).

The basic structural features of \( G/P \) which we will need are the Birkhoff and Bruhat decompositions

\[
G/P = \bigsqcup \Sigma_{\bar{w}}, \quad \Sigma_{\bar{w}} = N^- w P \tag{A.1}
\]

\[
G/P = \bigsqcup \Sigma_{\bar{w}}, \quad \Sigma_{\bar{w}} = N^- w P \tag{A.2}
\]

respectively, where the indexing set is \( W/W(\Phi) \) in both cases. The strata \( \Sigma_{\bar{w}} \) are infinite dimensional if \( g \) is infinite dimensional, while the cells \( C_{\bar{w}} \) are always finite dimensional. Our main interest lies in the Schubert variety \( \bar{C}_{\bar{w}} \), the closure of the cell.

Fix \( \bar{w} \in W/W(\Phi) \). We choose a representative \( w \in N(T) \) of minimal length \( n \); for definiteness we will always take \( w \) of the form

\[
w = r_n \cdots r_1 \tag{A.3}
\]

where \( r_i = i_{\alpha_i}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \), and \( i_{\alpha_i} : SL_2 \to G \) is the canonical homomorphism of \( SL_2 \) onto the root subgroup corresponding to the simple root \( \alpha_i \).

\textbf{(A.4) Proposition.} For \( w \) as in \( (A.3) \), the map

\[
r_n \exp(g_{-\alpha_n}) \times \ldots \times r_1 \exp(g_{-\alpha_1}) \to G/P : (p_j) \to p_n \cdots p_1 P
\]

is a complex analytic isomorphism onto \( \bar{C}_{\bar{w}} \).

This result is essentially (5) of [Kac] together with Tits’s theory. We will sketch a proof for completeness.

\textit{Proof of (A.4).} Let \( \Delta^+ \) denote the positive roots, \( \Delta^+(\Phi) \) the positive roots which are combinations of elements from \( \Phi \). The “Lie algebra of \( P \)” is \( \mathfrak{p} = \Sigma \mathfrak{g}_{-\beta} \oplus \mathfrak{b} \)
where the sum is over $\beta \in \Delta^+(\Phi)$; this is the Lie algebra of $P$ in the sense that it is the subalgebra generated by the root spaces $g_{\gamma}$ for which $\exp : g_{\gamma} \to G$ is defined and have image contained in $P$. The subgroups $\exp(g_{\gamma})$ generate $P$. We also let $p^-$ denote the subalgebra opposite $p$: $p^- = \sum g_{-\gamma}$, where the sum is over $\gamma \in \Delta^+ \setminus \Delta^+(\Phi)$. The corresponding group will be denoted by $P^-$. The cell $C_w$ is the image of the map $N^+ \to G/P : u \to uwP$. The stability subgroup at $wP$ is $N^+ \cap wPw^{-1}$.  

At the Lie algebra level we have the splitting

\[ n^+ = n^+ \cap \text{Ad}(w)(p) \oplus n^+ \cap \text{Ad}(w)(p^-). \tag{A.5} \]

The second summand equals

\[ n^+_w = \oplus g_{\beta} \tag{A.6} \]

where the sum is over roots $\beta > 0$ with $w^{-1}\beta \in - (\Delta^+ \setminus \Delta^+(\Phi))$. These roots $\beta$ are necessarily real, so that $\exp : n^+_w \to N^+_w \subseteq N^+$ is well-defined.

For $q \in \mathbb{Z}^+$ let $N^+_q$ denote the subgroup corresponding to $n^+_q = \text{span}\{g_{\beta} : \text{height}(\beta) \geq q\}$. Then $N^+/N^+_q$ is a finite dimensional nilpotent Lie group, and it is also simply connected. By taking $q$ sufficiently large and considering the splitting (A.5) modulo $n^+_q$, we conclude by finite dimensional considerations that each element in $N^+$ has a unique factorization $n = n_1n_2$, where $n_1 \in N^+_w$ and $n_2 \in N^+ \cap wPw^{-1}$:

\[ N^+ \simeq N^+_w \times (n^+ \cap wPw^{-1}). \tag{A.7} \]

The important point here is that modulo $N^+_q$ we can control $N^+ \cap wPw^{-1}$ by the exponential map.

We now recall the following standard

(A.8) **Lemma.** In terms of the minimal factorization $w = r_n \cdots r_1$, the roots $\beta > 0$ with $w^{-1}\beta < 0$ are given by

\[ \beta_j = r_{n} \cdots r_{j+1}(\alpha_j) = r_{n} \cdots r_{j}(-\alpha_j), \quad 1 \leq j \leq n. \]

Because $w$ is a representative of $\bar{w} \in W/W(\Phi)$ of minimal length, all of these $\beta_j$ satisfy $w^{-1}\beta_j \in - (\Delta^+ \setminus \Delta^+(\Phi))$. Otherwise, if say $w^{-1}\beta_j \in -\Delta^+(\Phi)$, then

\[ w^{-1}r_{\beta_j}w = r_1 \cdots r_{j-1}r_jr_{j-1} \cdots r_1 \in N(T) \cap P \tag{A.9} \]
and \( w' = w(w^{-1}r_{\beta_j}w) = r_n \cdots r_j \cdots r_1 \) would be a representative of \( \bar{w} \) of length \( < n \) (here we have used the fact that \( W(\Phi) = N(T) \cap P/T \), which follows from the Bruhat decomposition). For future reference we note this proves that

\[
N_w^+ = N^+ \cap (N^-)^w = N^+ \cap (P^-)^w
\]  

(A.10)

and (2.4) shows that

\[
N_w^+ \times w \cong C_{\bar{w}}.
\]  

(A.11)

Now for any \( 1 \leq p \leq q \leq n \), \( \bigoplus_{p \leq j \leq p} g_{\beta_j} \) is a subalgebra of \( n_w^+ \). Thus by (2.7)

\[
\exp(g_{\beta_n}) \times \cdots \times \exp(g_{\beta_1}) \times w \cong C_{\bar{w}}.
\]  

(A.12)

This yields (A.4) when we write

\[
\exp(g_{\beta_j}) = r_n \cdots r_j \exp(g_{\alpha_j}) r_j \cdots r_n.
\]  

(A.13)

\[ \square \]

We now note several important corollaries of (A.4).

For each \( i \), let \( P_i \) denote the parabolic subgroup \( i_{\alpha_i}(SL_2)B \). Let

\[
\Gamma_w = P_n \times_B \cdots \times_B P_1 / B
\]  

(A.14)

where

\[
P_n \times \cdots \times P_1 \times B \times \cdots \times B \to P_n \times \cdots \times P_1
\]  

(A.15)

is given by

\[
(p_j) \times (b_j) \to (p_nb_n, b_n^{-1}p_{n-1}b_{n-1}, \ldots, b_2^{-1}p_1b_1).
\]  

(A.16)

We have written “\( \Gamma_w \)” instead of “\( \bar{\Gamma}_w \)” to indicate that this compact complex manifold depends upon the factorization.

(A.17) Corollary. The map

\[
\Gamma_w \to \bar{C}_{\bar{w}}: (p_j) \to p_n \cdots p_1 P
\]

is a desingularization of \( \bar{C}_{\bar{w}} \).

Let

\[
SL'_2 = \{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) : a \neq 0 \}.
\]  

(A.17)
(A.18) Corollary. Let \( \phi \) denote the surjective map

\[
SL_2 \times \cdots \times SL_2 \to \tilde{C}_{\bar{w}} : (g_j) \mapsto r_n i_{\alpha_n} (g_n) \cdots r_1 i_{\alpha_1} (g_1) P.
\]

The inverse image of \( C_{\bar{w}} \) under \( \phi \) is \( SL_2' \times \cdots \times SL_2' \).

Proof of (A.18). Let \( \sigma = r_{n-1} \cdots r_1 \). It suffices to show that for the natural actions

\[
r_n i_{\alpha_n} (SL_2) \times C_{\bar{\sigma}} \to C_{\bar{w}}, \tag{A.19}
\]

\[
r_n i_{\alpha_n} (SL_2 \setminus SL_2') \times C_{\bar{\sigma}} \to \tilde{C}_{\bar{w}}, \tag{A.20}
\]

and

\[
r_n i_{\alpha_n} (SL_2) \times (C_{\bar{\sigma}} \setminus C_{\bar{\sigma}}) \to \tilde{C}_{\bar{w}} \setminus C_{\bar{w}}. \tag{A.21}
\]

The first line, (A.19), follows from (A.4) since \( i_{\alpha_n} (SL_2') \subseteq \exp(-g_{-\alpha_n}) B \) and \( B \times C_{\bar{\sigma}} \subseteq C_{\bar{\sigma}} \). The second line follows from

\[
r_n i_{\alpha_n} \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \cdot C_{\bar{\sigma}} = i_{\alpha_n} \begin{pmatrix} c & b \\ 0 & d \end{pmatrix} \cdot C_{\bar{\sigma}} \subseteq C_{\bar{\sigma}}. \tag{A.22}
\]

For the third line it’s clear that the image of the left hand side is a union of cells, since we can replace \( r_n i_{\alpha_n} (SL_2) \) by \( P_n \). This image is at most \( n - 1 \) dimensional. Therefore it must have null intersection with \( C_{\bar{w}} \). \( \square \)

Fix an integral functional \( \lambda \in h^* \) which is antidominant. Denote the (algebraic) lowest weight module corresponding to \( \lambda \) by \( L(\lambda) \), and a lowest weight vector by \( \sigma_\lambda \). Let \( \Phi \) denote the simple roots \( \alpha \) for which \( \lambda (h_\alpha) = 0 \), where \( h_\alpha \) is the coroot, \( P = P(\Phi) \) the corresponding parabolic subgroup. The Borel-Weil theorem in this context realizes \( L(\lambda) \) as the space of strongly regular functions on \( G \) satisfying

\[
f(gp) = f(g) \lambda(p)^{-1} \tag{A.23}
\]

for all \( g \in G \) and \( p \in P \), where we have implicitly identified \( \lambda \) with the character of \( P \) given by

\[
\lambda(u_1 w \exp(x) u_2) = \exp \lambda(x) \tag{A.24}
\]

for \( x \in h \), \( u_1, u_2 \in N^+ \), \( w \in W(\Phi) \). Thus we can view \( L(\lambda) \) as a space of sections of the line bundle

\[
L_\lambda = G \times_{\lambda} C \to G/P. \tag{A.25}
\]

If \( g \) is of finite type, then \( L(\lambda) = H^0(L_\lambda) \); if \( g \) is affine (and untwisted), then \( L(\lambda) \) consists of the holomorphic sections of finite energy, as in [PS]. Normalize \( \sigma_\lambda \) by \( \sigma_\lambda (1) = 1 \).
(A.26) Proposition. Let \( \bar{w} \in W/W(\Phi) \), and let \( w = r_n \cdots r_1 \) be a representative of minimal length \( n \). The positive roots mapped to negative roots by \( w \) are given by

\[
\tau_j = r_1 \cdots r_{j-1}(\alpha_j), \quad 1 \leq j \leq n;
\]

let \( \lambda_j = -\lambda(h_{\tau_j}) \), where \( h_{\tau} \) is the coroot corresponding to \( \tau \). Then

\[
\sigma_{\lambda}^w(r_n i_{\alpha_n}(g_n) \cdots r_1 i_{\alpha_1}(g_1)) = \prod_{j=1}^{n} a_{\lambda_j}^j
\]

where \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2 \).

Proof of (A.26). The claim about the \( \tau_j \) is easily derived from (2.5). None of these roots lie in \( \Delta^+(\Phi) \), by the same argument as follows (2.5). Thus each \( \lambda_j > 0 \). It follows that \( \prod_{j} a_{\lambda_j}^j \) is nonzero precisely on the set \( SL'_2 \times \cdots \times SL'_2 \).

Now \( \sigma_{\lambda}^w \), viewed as a section of \( L_{\lambda} \rightarrow G/P \), is nonzero precisely on the \( w \)-translate of the largest strata,

\[
w \Sigma_0 = wP^{-} P = (P^{-})^w wP. \tag{A.27}
\]

We claim the intersection of this with \( \bar{C}_w \) is \( C_{\bar{w}} \). In one direction

\[
C_{\bar{w}} = (N^+ \cap (P^{-})^w) wP \subseteq (P^{-})^w wP \tag{A.28}
\]

by (2.6). On the other hand \( (N^+ \cap (P^{-})^w) \) is a closed finite dimensional subgroup of \( (P^{-})^w \). Since \( (P^{-})^w \) is topologically equivalent to \( w \Sigma_0 \), the limit points of \( C_{\bar{w}} \) must be in the complement of \( w \Sigma_0 \). This establishes the other direction.

It now follows from (A.4) that \( \sigma_{\lambda}^w \) is also nonzero precisely on \( SL'_2 \times \cdots \times SL'_2 \), viewed as a function of \( (g_n, \cdots, g_1) \).

We now calculate that

\[
\sigma_{\lambda}^w(r_n i_{\alpha_n}(g_n) \cdots r_1 i_{\alpha_1}(g_1)) = \sigma_{\lambda}(w^{-1} r_n i_{\alpha_n}(g_n) \cdots r_1 i_{\alpha_1}(g_1))
\]

\[
= \sigma_{\lambda} \left( \omega_n^{-1} i_{\alpha_n}(g_n) \cdots \omega_1^{-1} i_{\alpha_1}(g_1) \right)
\]

\[
= \sigma_{\lambda} \left( i_{\omega_n}(g_n) i_{\omega_{n-1}}(g_{n-1}) \cdots i_{\omega_1}(g_1) \right) \tag{A.29}
\]

where we have set \( \omega_i = r_1 \cdots r_i, \ 0 \leq i < n \), and we have used

\[
\omega_{i-1}(\alpha_i) = r_1 \cdots r_{i-1}(\alpha_i) > 0 \tag{A.30}
\]
to conclude that \( \omega_{i-1} \alpha_i (g) \omega_{i-1}^{-1} = i_{\tau_i} (g) \).

The map

\[
SL_2 \times \cdots \times SL_2 \rightarrow w^{-1} \bar{C}_\omega : (g_j) \rightarrow i_{\tau_n} (g_n) \cdots i_{\tau_1} (g_1) P
\]

is surjective and the inverse image of \( \Sigma_0 \cap w^{-1} \bar{C}_\omega \) is precisely \( SL'_2 \times \cdots \times SL'_2 \).

For \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL'_2 \), write \( g = LDU \), where

\[
L = \begin{pmatrix} 1 & 0 \\ c a^{-1} & 1 \end{pmatrix}, \quad D = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & a^{-1} b \\ 0 & 1 \end{pmatrix}.
\]

Then for \( (g_j) \in SL'_2 \times \cdots \times SL'_2 \), (3.2) equals

\[
\sigma_\lambda (i_{\tau_n} (L_n D_n U_n) \cdots i_{\tau_1} (L_1 D_1 U_1))
= \sigma_\lambda (i_{\tau_n} (L_n U'_n) i_{\tau_{n-1}} (L'_{n-1} U'_{n-1}) \cdots i_{\tau_1} (L'_1 U'_1) i_{\tau_n} (U_n) \cdots i_{\tau_1} (U_1))
= \sigma_\lambda (i_{\tau_n} (L_n U'_n) \cdots i_{\tau_1} (L'_1 U'_1)) \Pi_{j \lambda}^1
\]

where each \( L'_j \) (\( U'_j \)) has the same form as \( L_j \) (\( U_j \), respectively). This follows from the fact that \( H \) normalizes each \( exp(\{ \pm r \}) \).

Now each \( L'_j U'_j \in SL'_2 \), so that \( i_{\tau_n} (L_n U'_n) \cdots i_{\tau_1} (L'_1 U'_1) \) is in \( \Sigma_0 \). We now conclude that

\[
\sigma_\lambda (i_{\tau_n} (L_n U'_n) \cdots i_{\tau_1} (L'_1 U'_1)) = 1,
\]

by the fundamental theorem of algebra, since this is polynomial and never vanishes. \( \square \)

**(A.35) Proposition.** Suppose that \( \mathfrak{g} \) is finite dimensional. Given \( g \in K \) such that \( gT \in \Sigma_1 \), we can write \( g \) uniquely as \( g = lmau \), where \( l \in N^- \), \( m \in T \), \( a \in exp(\mathfrak{h}_R) \), and \( u \in N^+ \). Then

\[
\int_K a(g)^{-i \lambda} = \prod_{\alpha > 0} \frac{\langle 2 \delta, \alpha \rangle}{\langle 2 \delta - i \lambda, \alpha \rangle},
\]

where the integral is with respect to the normalized Haar measure of \( K \), and \( 2 \delta \) denotes the sum of the positive complex roots.

**Proof of (A.35).** Let \( \{ \Lambda_j \} \) denote the set of basic dominant integral functionals.

We apply (A.26) to \( w = w_0 \). We write \( w_0 = r_n \cdots r_1 \) as in (A.3). Then

\[
gT = i_{\tau_n} (g_n) i_{\tau_{n-1}} (g_{n-1}) \cdots i_{\tau_1} (g_1) T \tag{A.36}
\]
\[ a(g) = \prod_{j=1}^{l} |\sigma_{\Lambda_j}(g)|^{h_j} = \prod_{j=1}^{l} \left( \prod_{k=1}^{n} |a_k|^{\Lambda_j(h_{\tau_k})} \right)^{h_j} = \prod_{k=1}^{n} |a_k|^{h_{\tau_k}}, \quad (A.37) \]

since the \( \Lambda_j \) are dual to the \( h_j \). Therefore

\[ a(g)^{-i\lambda} = \prod_{k=1}^{n} |a_k|^{-i\lambda(h_{\tau_k})}. \quad (A.38) \]

Also, in terms of the coordinates \( a_k \), the invariant measure is given by

\[ a(g)^{2\delta} \prod_{k} |a_k|^{-2} dm(a_k), \quad (A.39) \]

up to a normalization factor.

The roots \( \tau_k \) range over all the positive complex roots. Thus by (A.26),

\[ \int_{K} a(g)^{-i\lambda} = \int_{\Sigma_1} a(gT)^{-i\lambda} \]

\[ Z^{-1} \prod_{\alpha > 0} \int_{SU(2)} |a|^{(2\delta - i\lambda)(h_{\alpha})} |a|^{-2} = Z^{-1} \prod_{\alpha > 0} \int_{0}^{1} r^{(2\delta - i\lambda)(h_{\alpha}) - 1} dr \]

\[ = Z^{-1} \prod_{\alpha > 0} \frac{1}{(2\delta - i\lambda)(h_{\alpha})} = \prod_{\alpha > 0} \frac{\langle 2\delta, \alpha \rangle}{\langle 2\delta - i\lambda, \alpha \rangle}. \]

\( \square \)

This proof was given a Poisson-theoretic interpretation in [Lu].

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