GETZLER-KAPRANOV COMPLEXES AND MODULI STACKS OF CURVES

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ABSTRACT. In this paper, we study so-called Getzler-Kapranov complexes and their relation to the cohomology of moduli stacks of curves.

1. INTRODUCTION

1.1. Introduction. We investigate an interplay between the cohomology of graph complexes and the cohomology $H^*_c(M_{g,n}, \mathbb{Q})$ of moduli stacks $M_{g,n}$. This subject was initiated by S. Merkulov and T. Willwacher in [MW15] and later investigated by M. Chan, S. Galatius, and S. Payne [CGP21a, CGP21b] (see also [AWŽ20] and [AZ20]). Recall that for any $g$ and $n$ such that $2g-2+n>0$ $M_{g,n}$ is the separated, smooth and non-proper Deligne-Mumford stack $[DM69]$, $[Kn83]$. According to P. Deligne [Del71, Del74] the rational compactly supported cohomology $H^i_c(M_{g,n}, \mathbb{Q})$ of $M_{g,n}$ carries a weight filtration:

$$W^0_0 H^i_c(M_{g,n}, \mathbb{Q}) \subset \cdots \subset W^i_c H^i_c(M_{g,n}, \mathbb{Q}) \subset H^i_c(M_{g,n}, \mathbb{Q})$$

The weight filtration reflects geometric and topological properties of moduli stacks $M_{g,n}$. A starting point of our study is a fascinating result by M. Chan, S. Galatius, and S. Payne [CGP21a, CGP21b]. In ibid., it was shown that the weight zero quotient of $H^i_c(M_{g,n}, \mathbb{Q})$ is identified with the cohomology of the $g$-loop part of an $n$-hairy (marked) graph complex:

$$W_0 H^i_c(M_{g,n}, \mathbb{Q}) \cong H^i(B_{g,n} GC, \mathbb{Q}).$$

Here $B_{g,n} GC, \mathbb{Q}$ is the combinatorial complex with cochains generated by at least trivalent graphs of genus $g$ with $n$-markings and a certain orientation. A differential $\delta$ is defined by splitting a vertex [KWŽ17], in particular when $n=0$ we recover a definition of the famous M. Kontsevich graph complex $B_{g,n} GC, \mathbb{Q}$ from [Wil15].

1.2. Results. We extend the result of Chan-Galatius-Payne to higher weights. For every $g, n$ such that $2g-2+n>0$ we introduce a decorated graph complex $W_k GK_{g,n}$ which computes the weight $k$-quotient of $H^i_c(M_{g,n}, \mathbb{Q})$. These complexes are quasi-isomorphic to certain subcomplexes of a Getzler-Kapranov complex introduced in [AWŽ20]. We call $W_k GK_{g,n}$ a Getzler-Kapranov complex of the weight $k$. Our first main result is:

**Theorem 1.** For $g, n$ such that $2g-2+n>0$ we have the following description of the weight quotients on the compactly supported cohomology of $M_{g,n}$:

$$Gr^W_k H^{i+k}_c(M_{g,n}, \mathbb{Q}) \cong H^i(W_k GK_{g,n}).$$

In ibid. this complex was defined as a covariant Feynman transform of the modular cooperad $C^*(\overline{\mathcal{M}}, \mathbb{Q})$ from [KGO95].
In particular, we show that the weight zero Getzler-Kapranov complex $W_0 \mathbf{GK}_{g,n}^*$ is quasi-isomorphic to $B_\delta H_n \mathbf{GC}^*(\delta)$ and we recover the result of Chan-Galatius-Payne. Further applying explicit computations of the rational cohomology of $\mathcal{M}_{g,n}$ in low degrees from $\text{[AC98]}$ we show that the weight quotients of $\mathcal{M}_{g,n}$ in degrees $1, 3, 5$ vanish and the first nontrivial weight quotient (after the weight zero) may appear in the weight two. Explicit computations, in this case, were made in the recent beautiful work of S. Payne and T. Willwacher $\text{[PW21]}$.

The second object of our study is a hairy Getzler-Kapranov complex $\mathbf{H}_{\geq 1} \mathbf{GK}_g^*$. Before we give a definition let us recall a construction from hairy graph complexes. Following $\text{[TW17]}$ we consider a hairy graph complex $B_\delta H_{\geq 1} \mathbf{GC}^*(\delta + \chi)$. As a mere graded vector space this complex is defined by the rule:

$$B_\delta H_{\geq 1} \mathbf{GC}^* := \prod_{n=1}^{\infty} (B_\delta H_n \mathbf{GC}^* \otimes_{\Sigma_n} \text{sgn}_n)^{\Sigma_n}$$

A differential is defined as a sum $\delta + \chi$, where $\chi$ is a differential acting by an inserting a hair in all possible ways:

$$\chi : \Gamma \mapsto \sum_{v \in V(\Gamma)} h_v(\Gamma)$$

The result from $\text{ibid.}$ identifies the cohomology of this complex with the shifted cohomology of the M. Kontsevich graph complex:

$$B_\delta H_{\geq 1} \mathbf{GC}^*(\delta + \chi) \cong B_\delta \mathbf{GC}^*-1(\delta)$$

A definition of the hairy Getzler-Kapranov $\mathbf{H}_{\geq 1} \mathbf{GK}_g^*$ is reminiscent of the definition above. Cohains are collections of isomorphism classes of $C^*(\mathcal{M}_{g',n'}, \mathbb{Q})$-decorated stable graphs of genus $g$ with skew-symmetrised hairs (markings). A differential is defined as a sum of two differentials. The first differential splits a vertex and acts on decorations by a pullback along the clutching morphism and the second differential is defined by adding a decorated hair. Our first result about the hairy Getzler-Kapranov complexes (also proved in $\text{[AWZ20]}$) is the following:

**Theorem 2.** For every $g \geq 2$ the hairy Getzler-Kapranov complex is quasi-isomorphic to the shifted compactly supported cochains of $\mathcal{M}_g$ with trivial coefficients:

$$C_c^*(\mathcal{M}_g, \mathbb{Q})[-1] \xrightarrow{\sim} H_{\geq 1} \mathbf{GK}_g^*$$

Our second result (Conjecture 31 from $\text{[AWZ20]}$) is a realisation of the hairy Getzler-Kapranov complex $\mathbf{H}_{\geq 1} \mathbf{GK}_g^*$ as the total DG-vector space associated with the certain double complex:

**Theorem 3.** Let $g \geq 0$ then:

(i) There exists a well-defined complex in the derived category of vector spaces:

$$C_c^*(\mathcal{M}_{g,1}, \mathbb{Q}) \xrightarrow{\nabla_1} \cdots \xrightarrow{\nabla_1} (C_c^*(\mathcal{M}_{g,n}, \mathbb{Q}) \otimes_{\Sigma_n} \text{sgn}_n)^{\Sigma_n} \xrightarrow{\nabla_1} \cdots$$

Here $\nabla_1$ is a so-called Willwacher differential:

$$\nabla_1 : (C_c^*(\mathcal{M}_{g,n}, \mathbb{Q}) \otimes_{\Sigma_n} \text{sgn}_n)^{\Sigma_n} \longrightarrow (C_c^*(\mathcal{M}_{g,n+1}, \mathbb{Q}) \otimes_{\Sigma_{n+1}} \text{sgn}_{n+1})^{\Sigma_{n+1}}.$$
(ii) The induced morphism preserves the weight quotients on the compactly supported cohomology:

\[ \nabla_1 : \left( \text{Gr}_k^W H_i^c(\mathcal{M}_{g,n}, \mathbb{Q}) \otimes \Sigma_n \text{sgn}_n \right)^{\Sigma_n} \rightarrow \left( \text{Gr}_k^W H_i^c(\mathcal{M}_{g,n+1}, \mathbb{Q}) \otimes \Sigma_{n+1} \text{sgn}_{n+1} \right)^{\Sigma_{n+1}}. \]

Moreover, under the Chan-Galatius-Payne equivalence, the morphism \( \nabla_1|_{W_0} \) coincides with the differential \( \chi \).

(iii) For \( g \geq 2 \) the total DG-vector space of the complex above is quasi-isomorphic to the hairy Getzler-Kapranov complex \( H_{\geq 1}^c \mathcal{G}K_g^* \).

Theorem 2 and Theorem 3 imply the following result:

**Theorem 4.** For every \( g \geq 2 \) the total DG-vector space associated with the double complex:

\[ C^c_c(\mathcal{M}_{g,1}, \mathbb{Q}) \xrightarrow{\nabla_1} \cdots \xrightarrow{\nabla_1} \left( C^c_c(\mathcal{M}_{g,n}, \mathbb{Q}) \otimes \Sigma_n \text{sgn}_n \right)^{\Sigma_n} \xrightarrow{\nabla_1} \cdots \]

is quasi-isomorphic to \( C^c_c(\mathcal{M}_g, \mathbb{Q})[-1] \).

Applying explicit computations of the cohomology of \( \mathcal{M}_{1,n} \) from [CF06], [Pet12] we get.

**Theorem 5.** For \( g = 1 \) the cohomology of the total DG-vector space associated with the double complex:

\[ C^c_c(\mathcal{M}_{1,1}, \mathbb{Q}) \xrightarrow{\nabla_1} \cdots \xrightarrow{\nabla_1} \left( C^c_c(\mathcal{M}_{1,n}, \mathbb{Q}) \otimes \Sigma_n \text{sgn}_n \right)^{\Sigma_n} \xrightarrow{\nabla_1} \cdots \]

is isomorphic to:

\[ \prod_{n=3}^{\infty} (S_{n+1} \oplus S_{n+1} \oplus Eis_{n+1})[2n] \oplus \mathbb{Q}[3]. \]

Here \( S_k \) (resp. \( \overline{S}_k \)) is a vector space of holomorphic (resp. antiholomorphic) cusp forms of the weight \( k \) and \( Eis_k \) is a vector space of the Eisenstein series of the weight \( k \).

1.3. **Methods.** The main technical tools in this paper are P. Deligne’s theory of mixed Hodge structures and constructible sheaves on moduli spaces of tropical curves. P. Deligne’s theory is already classical and few words need to be said. Constructible sheaves on moduli spaces of tropical curves can be considered as a geometric avatar of Getzler-Kapranov’s theory of modular operads (cf. [KG94]) and all our constructions can be painlessly translated to operadic language. We choose to work with constructible sheaves on moduli spaces of tropical curves to make our techniques more flexible (for example the differential of \( H_{\geq 1}^c \mathcal{G}K_g^* \) is not covered by the Getzler-Kapranov formalism) and also to make a relation to moduli spaces of tropical curves (originated in [CGP21a], [CGP21b]) more transparent.

\[ ^2 \text{This complex is closely related to the cochain complex of an Artin stack } \mathcal{M}_1; \text{ [Tae15], however, we were unable to find a precise connection.} \]
1.4. Structure of the paper. In Section 2 we recollect some facts about constructible sheaves on diagrams and mixed Hodge structures. In Section 3 we recall some basic facts from the theory of moduli stacks $\mathcal{M}_{g,n}$ and the moduli spaces of tropical curves. In this section, we introduce the Getzler-Kapranov complex and reprove an original result of E. Getzler and M. Kapranov concerning the Feynman transform of the modular cooperad $C^* (\mathcal{M}, \mathbb{Q})$. Also in this section, we prove Theorem 1 and further discuss the cohomology of the Getzler-Kapranov complexes $W_k \mathbf{GK}_{g,n}$ for small values of $k$. In Section 4, we define the hairy Getzler-Kapranov complex and prove Theorem 2 and Theorem 3. Further, we obtain Theorem 4 as an immediate corollary and prove Theorem 5.

1.5. Acknowledgments. I would like to thank Nikita Markarian, Sergei Merkulov, and Thomas Willwacher for fruitful discussions. Special thanks are due to Thomas Willwacher for introducing the author to almost all problems attacked in this paper and for his input on this work and to Sergei Merkulov for his constant support and for bringing my attention to the beautiful paper [MW15]. Finally, I would like to thank Sergey Shadrin for various important suggestions and comments which helped the author to improve this text. This work was supported by the FNR project number: PRIDE 15/10949314/GSM

2. Preliminaries

2.1. Notation. For a natural number $n \in \mathbb{N}_+$ we will denote by $[n]$ a finite set such that $[n] := \{1, 2, \ldots, n\}$. Let $I$ be a finite set, by $\text{Aut}(I)$ we will denote a group of automorphisms of this set, in the case when $I = [n]$ we will use a notation $\Sigma_n := \text{Aut}([n])$ for a symmetric group on $n$-letters. We usually work over the field of rational number $\mathbb{Q}$. For a $\mathbb{Q}$-linear representation $V$ of the finite group $G$, we will denote by $V^G$ (resp. $V_G$) the space of $G$-invariants (resp. $G$-coinvariants). Since the characteristic of the field $\mathbb{Q}$ is zero we have a canonical isomorphism $V_G \xrightarrow{\sim} V^G$ and hence we will freely switch between invariants and coinvariants. For a finite $S$ we will denote by $V(S)$ a free $\mathbb{Q}$-vector space $V(S)$ generated by the set $S$, by $\det(S) := \bigwedge^{\dim V(S)} V(S)$ we denote a determinant of $V(S)$. By $\text{Top}$ we denote a category of locally compact topological spaces equipped with a stratification and by $\text{Cat}$ we denote a 2-category of all categories. For categories $A$ and $B$ we denote by $\text{Fun}(A, B)$ the corresponding category of functors.

2.2. Combinatorial sheaves. Let $X$ be a topological space equipped with a Whitney stratification $\mathcal{S} := \{S_\alpha\}$. Then we have the corresponding category of $\mathcal{S}$-smooth combinatorial sheaves on $X$ denoted by $\text{Sh}_c(X, \mathcal{S})$. This is a full subcategory of the category of sheaves of finite-dimensional $\mathbb{Q}$-vector spaces on $X$. By $\text{Ch}_c(X, \mathcal{S})$ we denote the category of complexes of sheaves with constructible cohomology i.e. $H^*(\mathcal{X}) \in \text{Sh}_c(X, \mathcal{S})$. The corresponding triangulated category will be denoted by $D_c(X, \mathcal{S})$. According to Proposition 1.9 from [KS16] a natural comparison functor:

\[
D(\text{Sh}_c(X, \mathcal{S})) \longrightarrow D_c(X, \mathcal{S})
\]

is the equivalence of triangulated categories. Objects of the category $D_c(X, \mathcal{S})$ will be called $\mathcal{S}$-smooth combinatorial DG-sheaves on $X$.

Denote by $\mathbb{S}$ a poset associated with the stratified space $(X, \mathcal{S})$. By a definition, an object $\alpha$ in $\mathbb{S}$ corresponds to a stratum $S_\alpha$ in $\mathcal{S}$ and for two objects $\alpha$ and $\beta$ in
S we define the arrow \( \alpha \to \beta \) if \( S_\alpha \subset \overline{S}_\beta \). Hence we construct a functor:

\[
(2) \quad \text{Real}: \text{Sh}_c(\mathcal{X}, S) \to \text{Fun}(\mathcal{S}, \text{Vect}_{\mathcal{Q}}^f),
\]

which maps a combinatorial sheaf \( \mathcal{K} \) to the functor \( K \in \text{Fun}(\mathcal{S}, \text{Vect}_{\mathcal{Q}}^f) \). This functor is defined by the following rule. A value of the functor \( K \) at \( \alpha \) equals \( K_\alpha := H^0(\mathcal{S}_\alpha, \mathcal{K}) \), and for every pair of adjusted strata \( S_\alpha \subset \overline{S}_\beta \) we have the corresponding variation operator:

\[
\text{var}_{\alpha, \beta}: K_\alpha \to K_\beta.
\]

It is easy to see that \( \text{Real} \) is the equivalence of abelian categories and hence from \([1]\) we get the following:

**Proposition 2.2.1.** The composition of the functors above induces the equivalence of the triangulated categories:

\[
\text{D}_c(\mathcal{X}, S) \xrightarrow{\sim} \text{D}(\text{Fun}(\mathcal{S}, \text{Vect}_{\mathcal{Q}}^f))
\]

We have a derived functor of the global sections with compact support:

\[
\text{RI}_c(\mathcal{X}, -): \text{D}_c(\mathcal{X}, S) \to \text{D}(\text{Vect}_{\mathcal{Q}}^f).
\]

In terms of \([2]\) the derived functor of global sections with compact support can be realised by the following complex \([\text{KG91}]\):

**Proposition 2.2.2.** The DG-vector space \( \text{RI}_c(\mathcal{X}, \mathcal{K}) \) is naturally quasi-isomorphic to (the total DG-vector space arising from) the complex:

\[
\bigoplus_{\dim S_\alpha = 0} K_\alpha \otimes \text{or}_\alpha \to \bigoplus_{\dim S_\alpha = 1} K_\alpha \otimes \text{or}_\alpha \to \ldots
\]

Here \( \text{or}_\alpha := H_c^{\dim S_\alpha}(\mathcal{S}_\alpha, \mathbb{Q}) \) is a one-dimensional space of orientations and the sum over strata of dimension \( m \) is placed in degree \( m \).

Let \( \mathcal{K} \) be a combinatorial DG-sheaf. For \( k \in \mathbb{Z} \) we denote by \( H^k(\mathcal{K}) \in \text{Sh}_c(\mathcal{X}, \mathcal{S}) \) the corresponding \( k \)-cohomology of the combinatorial DG-sheaf.

### 2.3. Sheaves on Diagrams.

Let \( \mathcal{C} \) be a category, in this paper by a \( \mathcal{C} \)-diagram \( \mathcal{X}_C \) we will understand a functor \( \mathcal{X}_C: \mathcal{C} \to \text{Top} \), such that for every \( i \in \mathcal{C} \) a slice category \( \mathcal{C}_i \) defines the stratification on \( X_i \), denoted by \( S_i \). Following P. Deligne \([\text{SD72}]\) (Exposé \( \text{V}^{\text{bis}} \)) we have a notion of a sheaf on such objects. Let \( X_C \) be a diagram represented by a system \( \{X_i\}_{i \in \mathcal{C}} \), where \( f_{ij}: X_i \to X_j \). Then using the construction from \textit{ibid.}, one may define a 2-functor \( \mathcal{C} \to \text{Cat} \) by sending each object \( i \in \mathcal{C} \) to the category of \( S_i \)-smooth combinatorial sheaves \( \text{Sh}_c(X_i, S_i) \) and every morphism \( f: i \to j \) to a pullback functor \( f^*: \text{Sh}_c(X_j, S_j) \to \text{Sh}_c(X_i, S_i) \). We can consider a category of lax sections of the corresponding Grothendieck fibration denoted by \( \text{Sh}_c^{\text{lax}}(X_C, S) \) \([\text{GR71}]\). This is an abelian category, with the corresponding category of cocartesian sections will be denoted by \( \text{Sh}_c(X_C, S) \). Objects of the category \( \text{Sh}_c^{\text{lax}}(X_C, S) \) (resp. \( \text{Sh}_c(X_C, S) \)) will be called \( \mathcal{S} \)-smooth lax combinatorial sheaves on the \( \mathcal{C} \)-diagram \( X_C \) (resp. \( \mathcal{S} \)-smooth combinatorial sheaves on the \( \mathcal{C} \)-diagram \( X_C \)). For the reader’s convenience we will unpack these definitions:

**Definition 2.3.1.** An \( \mathcal{S} \)-smooth lax combinatorial sheaf \( \mathcal{P} \) on the \( \mathcal{C} \)-diagram \( X_C \) is a combinatorial sheaf \( \mathcal{P}_i \) on \( X_i \) for every \( i \in \mathcal{C} \), such that for every \( m: i \to j \) we have a morphism:

\[
\vartheta: m^* \mathcal{P}_j \to \mathcal{P}_i
\]
which is compatible with compositions in $C$.

We say that $P$ is an $S$-smooth combinatorial sheaf on the $C$-diagram $X_C$ if for all $m: i \rightarrow j$ a morphism $\vartheta$ is the isomorphism.

Mutatis mutandis we define a 2-functor $C \rightarrow \text{Cat}$ which takes every object $i$ to the category of all sheaves of finite-dimensional vector spaces $\text{Sh}(X_i)$. The corresponding category of lax sections will be denoted by $\text{Sh}(X_C)$. We denote by $D_{\text{lax}}(X_C)$ the associated derived category and by $D_c(X_C, S)$ the triangulated subcategory with the cohomology in $\text{Sh}_c(X_C, S)$. Objects of this category will be called $S$-smooth combinatorial DG-sheaves on the $C$-diagram $X_C$. Once again for the reader’s convenience, we will unpack this definition:

**Definition 2.3.2.** An $S$-smooth DG-combinatorial sheaf $P$ on $X_C$ is a complex of lax sheaves on $X_C$ such that for every $i \in C$ the cohomology of these sheaves are combinatorial and for every $m: i \rightarrow j$ we have a quasi-isomorphism:

$$\vartheta: m^*P_j \sim \rightarrow P_i$$

which is compatible with compositions in $C$.

Note that by the definition of the $C$-diagram $X_C$ for any morphism of stratified spaces $f_{ij}: (X_i, S_i) \rightarrow (X_j, S_j)$ we have the induced functor between posets $F_{ij}: S_i \rightarrow S_j$. Analogous to the definitions above one may consider a 2-functor $C \rightarrow \text{Cat}$ which sends every $i \in C$ to the category $\text{Fun}(S_i, \text{Vect}_Q^f)$ and every morphism $i \rightarrow j$ to a restriction functor:

$$\text{Res}(F_{ij}): \text{Fun}(S_j, \text{Vect}_Q^f) \rightarrow \text{Fun}(S_i, \text{Vect}_Q^f).$$

The corresponding category of lax sections will be denoted by $\text{Sec}(C, \text{Fun})$. By $D_{\text{cocart}}(\text{Sec}(C, \text{Fun}))$ we will denote the triangulated category with the cocartesian cohomology. Then we have the following:

**Proposition 2.3.3.** The functor $\text{Real}$ induces an equivalence of triangulated categories:

$$D_c(X_C, S) \xrightarrow{\sim} D_{\text{cocart}}(\text{Sec}(C, \text{Fun}))$$

**Proof.** The proof boils down to checking that under (2) the pullback functor $f_{ij}^*$ corresponds to the restriction functor $\text{Res}(F_{ij})$ which is a direct verification.

Denote by $\tilde{X}_C := \text{colim}_C X_C$ the corresponding colimit in the category of topological spaces. For every $i \in C$ we denote by $v_i: X_i \rightarrow \tilde{X}_C$ a canonical morphism. By $D(\tilde{X}_C)$ we will denote a derived category of sheaves of vector spaces on the topological space $\tilde{X}$. We have a natural comparison functor:

$$Rv_*: D_c(X_C, S) \rightarrow D(\tilde{X}_C).$$

defined on objects by the following rule:

$$Rv_*: P \mapsto \text{holim}_{i \in C} Rv_i_* P_i, \quad P \in D_c(X_C, S).$$

Note that the homotopy limit above is well defined since the category of sheaves is complete. One may show that (4) induces the fully faithful embedding of triangulated categories. By a constant sheaf on the $C$-diagram $X_C$ we understand an
object $\underline{Q}_{X_C}$ in the category $D_c(X_C, S)$, defined as follows: for every $i \in C$ the value of $\underline{Q}_{X_C}$ at $i$ is a one-dimensional constant sheaf on $X_i$, and connecting morphisms are identical.

By a morphism of $C$-diagrams $P: X_C \to Y_C$ we understand a natural transformation between functors $X_C$ and $Y_C$, such that every square of the natural transformation is fibered. If every morphism $P_i$ in the natural transformation $P$ is closed (resp. open) inclusion we call the morphism $P$ closed (resp. open) inclusion. For an inclusion $P: X_C \to Y_C$ of $C$-diagrams we denote by $(Y \setminus X)_{C}$ the corresponding complement $C$-diagram defined by the obvious rule. Note that if $P$ is open (resp. closed) the corresponding inclusion $(Y \setminus X)_C \to Y_C$ will be closed (resp. open). Due to the proper base change theorem, we have a functor of !-extension along open or (resp. closed) morphisms. The corresponding standard restriction functors are also well defined. Therefore if we have a sequence of morphisms of $C$-diagrams $j: U_C \to X_C$, $\Delta: Z_C \to X_C$, such that $j$ is an open inclusion and $\Delta$ is an inclusion of the corresponding closed complement we can apply the standard technique of Gysin triangles (cf. [Beh03]). That means that with a sequence of functors:

$$
(5) \quad j!: D_c(U_C, S) \to D_c(X_C, S): j^*, \quad \Delta^*: D_c(X_C, S) \to D_c(Z_C, S): \Delta_*,
$$

we associate the standard distinguished triangle in the category $D_c(X_C, S)$:

$$
\xymatrix{ j_* j^* \ar[r] & \text{Id} \ar[r] & \Delta_* \Delta^* \ar[r] & j_* j^*[1] }.
$$

We define a derived functor of the global sections with compact support:

$$
(6) \quad R\Gamma_c(X_C, -): D_c(X_C, S) \to D(\text{Vect}_Q),
$$

by the following rule:

$$
R\Gamma_c(X_C, P) := \text{holim}_{i \in C} R\Gamma_c(X_i, P_i),
$$

where $P \in D_c(X_C, S)$ and connecting homomorphisms are defined by composition with derived direct image functors $Rf_{ij*}$ (these morphisms are well defined since $f_{ij}$ are closed morphisms and therefore a !-extension can be identified with the standard extension functor). By construction we have a canonical morphism:

$$
(7) \quad R\Gamma_c(X_C, -) \to R\Gamma_c(X_C, -).
$$

2.4. P. Deligne’s resolution. In this subsection, we recall some facts about the mixed Hodge structures that we will need.

Let $\mathcal{X}$ be a smooth and separated Deligne-Mumford stack equipped with a compactification $\mathcal{X} \to \mathcal{W}$, such that the complement $\mathcal{W} \setminus \mathcal{X} := D$ is a divisor with normal crossings. Note that the divisor $D$ is naturally stratified $\{D_p\}_{p \in \mathbb{N}}$, where a stratum $D_p$ contains points of multiplicity at least $p$. By $\overline{D}_p \to D_p$ we will denote the corresponding (smooth) normalised stack. One may think about an element $z$ in $\overline{D}_p$ as a point $x \in D_p$ and $p$ components of $D$ through $z$. A set of these components will be denoted by $E_p(z)$. We also set $\overline{D}_0 = D_0 = \mathcal{W}$. Let $\overline{D}_{p-1,p}$ be space whose points are pairs $(y, L)$, where $y = (x, E_p(y)) \in \overline{D}_p$ and $L \in E_p(y)$. Note that
we have a natural correspondence:

\[
\begin{align*}
\tilde{D}_{p-1,p} \xrightarrow{\xi_{p-1,p}} & \tilde{D}_{p-1} \\
\pi_{p-1,p} & \downarrow \\
\tilde{D}_p
\end{align*}
\]  

Most of the time we will omit indices in the notation of morphisms above. Note that by varying point \( z \) in \( \tilde{D}_p \) we obtain a local system \( E_p \) on \( \tilde{D}_p \) with the corresponding determinant local system \( \varepsilon_p := \det(E_p) \). We have the following isomorphism:

\[
\pi! \varepsilon_p \cong \pi^* \varepsilon_p \otimes \text{or}_z \cong \xi^* \varepsilon_{p-1}.
\]

Where by \( \text{or}_z := \pi^! Q\tilde{D}_p \) we have denoted a relative dualising (orientation) sheaf. Therefore we can define a pull-push homomorphism:

\[
\begin{align*}
C^q(\tilde{D}_{p-1,p}, \varepsilon_{p-1}) & \xrightarrow{\xi^*} C^q(\tilde{D}_{p-1,p}, \xi^* \varepsilon_{p-1}) \cong C^q(\tilde{D}_{p-1,p}, \pi^! \varepsilon_p) \xrightarrow{\pi!} C^q(\tilde{D}_p, \varepsilon_p)
\end{align*}
\]

Complexes that compute the cohomology of \( \tilde{D}_p \) with coefficients in \( \varepsilon_p \) naturally form a DG-vector space:

\[
\begin{align*}
C^* (\mathcal{W}, \mathbb{Q}) \xrightarrow{\pi^! \xi^*} C^* (\tilde{D}_1, \varepsilon_1) \xrightarrow{\pi^! \xi^*} \cdots \xrightarrow{\pi^! \xi^*} C^* (\tilde{D}_p, \varepsilon_p) \xrightarrow{\pi^! \xi^*} \cdots
\end{align*}
\]

The following Proposition should be well known to experts:

**Proposition 2.4.1.** The total DG-vector space of the complex above is quasi-isomorphic to \( C^*_c(\mathcal{X}, \mathbb{Q}) \).

**Proof.** This proposition is Poincaré-Verdier dual to the original result of P. Deligne for the cohomology of \( \mathcal{W} \). For the details, we refer to P. Deligne’s beautiful paper [Del71].

\[\square\]

According to P. Deligne, the compactly supported cohomology \( H^k_c(\mathcal{X}, \mathbb{Q}) \) of a smooth Deligne-Mumford stack \( \mathcal{X} \) carries a canonical *mixed Hodge structure* and hence the increasing *weight filtration*:

\[
W_0 H^k_c(\mathcal{X}, \mathbb{Q}) \subset \cdots \subset W_{k-1} H^k_c(\mathcal{X}, \mathbb{Q}) \subset H^k_c(\mathcal{X}, \mathbb{Q})
\]

Following P. Deligne one can use the DG-vector space above to define the weight filtration on \( H^*_c(\mathcal{X}, \mathbb{Q}) \). For \( j \in \mathbb{N} \) denote by \( C^*_j \) the following cochain complex:

\[
\begin{align*}
H^j (\mathcal{W}, \mathbb{Q}) \xrightarrow{\pi^! \xi^*} H^j (\tilde{D}_1, \varepsilon_1) \xrightarrow{\pi^! \xi^*} \cdots \xrightarrow{\pi^! \xi^*} H^j (\tilde{D}_p, \varepsilon_p) \xrightarrow{\pi^! \xi^*} \cdots
\end{align*}
\]

Then we have:

**Proposition 2.4.2.** For every smooth and separated Deligne-Mumford stack \( \mathcal{X} \), compactified by the divisor \( \mathcal{D} \) with normal crossings we have the following description of the weight filtration on \( H^*_c(\mathcal{X}, \mathbb{Q}) \) :

\[
H^j (C^*_j) \cong \text{Gr}^W_j H^*_c(\mathcal{X}, \mathbb{Q}).
\]

**Proof.** Once again we refer to [Del71].

\[\square\]

---

\(^3\)The construction in *ibid.* was given for complex varieties but can be generalised to the case of Deligne-Mumford stacks.
3. Sheaves on moduli spaces of tropical curves

3.1. Tropical curves. By a graph $\Gamma$ we understand a connected graph with loops and parallel edges allowed. More precisely a (nonnecessary connected) graph $\Gamma$ is a triple $(H(\Gamma), \sigma_1, \sim)$ where $H(\Gamma)$ is a finite set called the set of half-edges of a graph $\Gamma$, a fixed point free involution $\sigma_1 : H(\Gamma) \to H(\Gamma)$ with a set of orbits $E(\Gamma) := H(\Gamma)/\sigma_1$ called the set of edges of a graph $\Gamma$ and an equivalence relation $\sim$ on $H(\Gamma)$ with the corresponding set of equivalence classes denoted by $V(\Gamma) := H(\Gamma)/\sim$ and called the set of vertices of $\Gamma$. For a vertex $v \in V(\Gamma)$ we will denote by $H_v(\Gamma) := p^{-1}(v)$, a set of half-edges attached to the vertex $v$, where $p : H(\Gamma) \to V(\Gamma)$ is a canonical projection. A number of elements in $H(v)$ will be called the valency of the vertex $v$ and denoted by $val(v)$. It is possible to consider $\Gamma$ as a one-dimensional CW complex and hence we will call $\Gamma$ a connected graph if it is connected as a one-dimensional CW complex, further we assume that all graphs are connected. By a weighted $n$-marked graph we understand a triple $(\Gamma, w, m)$ where $\Gamma$ is a graph together with function $w : V(\Gamma) \to \mathbb{N}$ called the weight (genus) function and the function $m : \{0, \ldots, n\} \to V(\Gamma)$ called $n$-marking. The genus of a weighted $n$-marked graph $(\Gamma, w, m)$ is defined by the standard formula:

$$g := b_1(\Gamma) + \sum_{v \in V(\Gamma)} w(v),$$

where $b_1(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$ is the first Betti number of $\Gamma$, considered as a one dimensional CW complex. For a vertex $v$ in a weighted marked graph we will use the following notation $n_v := val(v) + |m^{-1}(v)|$. We say that a graph is stable if for every vertex $v$ we have the following equality $2w(v) - 2 + n_v > 0$. Sometimes it will be convenient to consider markings $m^{-1}(v)$ attached to a vertex as marked hairs (legs) attached to $v$. We will freely switch between these two notation.

Stable weighted $n$-marked graphs of genus $g$ naturally form a category denoted by $J_{g,n}$. Objects of this category are stable weighted $n$-marked graphs and morphisms are given by compositions of contractions of edges and isomorphisms which preserve markings and genus labelings (for details see [KG98, CGP21b] and [CGP21a]).

For a stable $n$-marked graph $\Gamma$ we will denote by $\text{Aut}(\Gamma) := \text{Isom}_J(\Gamma, \Gamma)$ a group of automorphisms of $\Gamma$. For a stable $n$-marked graph $\Gamma$ and $e \in E(\Gamma)$ we will denote by $\Gamma_e$ a graph that is obtained from $\Gamma$ by contracting an edge $e$. The category $J_{g,n}$ has the terminal object denoted by $\bullet_{g,n}$. By $J^k_{g,n}$ we will denote a set of stable $n$-marked graphs $\Gamma$ with $E(\Gamma) = k$.

Definition 3.1.1. Fix $g, n \geq 0$ with a condition $2g - 2 + n > 0$. We define a $J_{g,n}$-diagram $M^{trop}_{g,n}$ called a moduli diagram of tropical curves of genus $g$ with $n$-punctures:

$$M^{trop}_{g,n} : J^0_{g,n} \to \text{Top}$$

By the following rule:

For every object $(\Gamma, w, m) \in J_{g,n}$ we set:

$$M^{trop}_{g,n} : (\Gamma, w, m) \mapsto \mathbb{R}^{E(\Gamma)}_{\geq 0} := \sigma_\Gamma.$$
For a morphism \( f: (\Gamma, w, m) \to (\Gamma', w', m') \) we set:
\[
\mathcal{M}^{trop}_{g,n}: f \mapsto \sigma f,
\]
where \( \sigma: \Gamma' \to \Gamma \) is a map that sends an \( e' \)-coordinate of \( \mathbb{R}^E(\Gamma') \geq 0 \) to an \( e \)-coordinate of \( \mathbb{R}^E(\Gamma) \geq 0 \) if \( f \) sends the edge \( e \) to the edge \( e' \) and zero otherwise.

**Remark 3.1.2.** Note that the colimit of the \( J^{g,n} \)-diagram above:
\[
\mathcal{M}^{trop}_{g,n} := \text{colim}_{J^{g,n}} \mathcal{M}^{trop}_{g,n}.
\]
is usually called a moduli space of tropical curves of genus \( g \) with \( n \) punctures \([CGP21a, CGP21b]\). Each space \( \sigma \Gamma \) carries a natural stratification by strata \( S_{\Gamma'} : = \mathbb{R}^E(\Gamma) \geq 0 \), where \( \Gamma' \) is a graph that is defined by contracting some edges of a graph \( \Gamma \) (such stratum is of dimension \(|E(\Gamma)|\)). Moreover:
\[
S_{\Gamma'} \subset S_{\Gamma''} \iff \Gamma'' \to \Gamma' \to \Gamma'.
\]
We will denote this stratification by \( S_{\Gamma} \) and the resulting stratification on \( \mathcal{M}^{trop}_{g,n} \) will be denoted by \( S \).

### 3.2. Sheaves on \( \mathcal{M}^{trop}_{g,n} \)
In this subsection, we define and study combinatorial DG-sheaves \( \mathcal{D}\mathcal{M}_{g,n} \) on \( J^{g,n} \)-diagram \( \mathcal{M}^{trop}_{g,n} \). To do it let us recollect some well-known facts about moduli stacks of curves:

Let \( I \) be a finite set. We denote by \( \mathcal{M}_{g,I} \) the **moduli stack of stable genus \( g \) curves with \(|I|-marked points labeled by the set \( I \). When \( 2g + |I| - 2 > 0 \) this is the smooth and proper Deligne-Mumford stack with the corresponding coarse moduli space denoted by \( \overline{M}_{g,I} \) [DM69, KM83]. In the special case when \( I = \{1, \ldots, n\} \) we use a notation \( \overline{M}_{g,n} \). We shall write \( \mathcal{M}_{g,I} \) (resp. \( \overline{M}_{g,I} \)) to indicate the open substack (resp. open coarse moduli space) parameterising smooth curves. We will also use a special notation \( \overline{M}_g \) for a moduli stack of curves marked by the empty set.

For a weighted graph \( \Gamma \in J_{g,n} \) we can associate the following stack:
\[
\overline{\mathcal{M}}_{\Gamma} := \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{w(v), n_v}
\]
By \( \mathcal{M}_{\Gamma} \) we will denote a product of the corresponding open substacks. Following [KM83] for a morphism \( \Gamma \to \Gamma' \) in the category \( J_{g,n} \) we associate **clutching morphisms**:
\[
\xi_{\Gamma' \to \Gamma}: \overline{\mathcal{M}}_{\Gamma'} \to \overline{\mathcal{M}}_{\Gamma},
\]
which are defined by gluing stable curves at marked points. An important property of these maps is that they satisfy associativity conditions:
\[
\xi_{\Gamma'' \to \Gamma} \circ \xi_{\Gamma' \to \Gamma'} = \xi_{\Gamma' \to \Gamma}, \quad \Gamma'' \to \Gamma' \to \Gamma.
\]
Denote by \( \partial \overline{\mathcal{M}}_{g,I} := \overline{\mathcal{M}}_{g,I} \setminus \mathcal{M}_{g,I} \) the complement to the substack parameterising smooth \( I \)-marked curves. It is known that \( \partial \overline{\mathcal{M}}_{g,I} \) is a normal crossing divisor. We have the following diagram of stacks:
\[
\begin{array}{ccc}
\mathcal{M}_{g,I} & \xrightarrow{j} & \overline{\mathcal{M}}_{g,I} \\
\partial \mathcal{M}_{g,I} & \xrightarrow{i} & \partial \overline{\mathcal{M}}_{g,I}
\end{array}
\]
where a morphism $j$ is open and a morphism $i$ is closed.

For any $\Gamma \in J_{g,n}$ denote by $D_{\Gamma}$ a substack in the moduli stack $\overline{M}_{g,I}$, defined as a closure of the locus of stable curves with the dual graph given by $\Gamma$. The boundary divisor admits the following decomposition:

$$ \partial \overline{M}_{g,I} = \bigsqcup_{\Gamma \in J_{g,n}} D_{\Gamma} $$

Denote by $\overline{D}_{\Gamma} \rightarrow D_{\Gamma}$ the corresponding normalised stack. Note that $\overline{D}_{\Gamma}$ is the smooth stack. We have the following equivalence of stacks:

$$ \overline{M}_{\Gamma}/\text{Aut}(\Gamma) \cong \overline{D}_{\Gamma}, $$

Let $\Gamma$ and $\Gamma'$ be objects of the category $J_{g,n}$ such that $\Gamma$ has exactly $k$ edges such that if one contracts one of them the resulting graph is isomorphic to $\Gamma'$. Denote by $F$ a set of these edges. Then a group $\text{Aut}(\Gamma)$ acts on $M_{\Gamma} \times F$ and we set $\overline{D}_{\Gamma',\Gamma} := \overline{M}_{\Gamma} \times F/\text{Aut}(\Gamma)$. We have the following correspondence:

$$ \overline{D}_{\Gamma',\Gamma} \xrightarrow{\xi_{\Gamma',\Gamma}} \overline{D}_{\Gamma'} $$

Where $\pi_{\Gamma',\Gamma}$ is a standard projection map, and $\xi_{\Gamma',\Gamma}$ is defined by the following rule: with an element in $\overline{M}_{\Gamma}$ and an edge $e \in F$, the clutching along $e$ produces an element that is well defined up to an automorphism of $\Gamma'$. This construction gives us a morphism $\overline{M}_{\Gamma} \times F \rightarrow \overline{D}_{\Gamma'}$ which factors through $\overline{D}_{\Gamma',\Gamma}$.

For a stratified space $\sigma_{\Gamma}$ associated with a stable weighted $n$-marked graph $\Gamma \in J_{g,n}$ we define an $\mathcal{S}_{\Gamma}$-smooth combinatorial DG-sheaf $\mathcal{D}M_{\Gamma} \in \mathcal{Sh}_{c}(\sigma_{\Gamma}, \mathcal{S})$ by the rule:

- For every graph $\Gamma' \in J_{g,n}^{\circ}$ we set:
  $$ R^\Gamma(S_{\Gamma'}, \mathcal{D}M_{\Gamma}) := \bigotimes_{v \in V(\Gamma')} C^*(\overline{M}_{w(v), n_v}, \mathbb{Q}) $$

- For every inclusion of strata $S_{\Gamma'} \subset \overline{S}_{\Gamma''}$ we define a variation map:
  $$ \text{var}_{\Gamma',\Gamma''} : R^\Gamma(S_{\Gamma''}, \mathcal{D}M_{\Gamma}) \rightarrow R^\Gamma(S_{\Gamma''}, \mathcal{D}M_{\Gamma}), \quad \text{var}_{\Gamma',\Gamma''} := \xi_{\Gamma',\Gamma''} $$

as being induced by a pullback along the clutching morphism.

Since clutching maps satisfy the associativity condition we have the well-defined combinatorial DG-sheaf on $\sigma_{\Gamma}$.

**Definition 3.2.1.** For every $g, n \geq 0$ with a condition $2g - 2 + n > 0$ we define an $\mathcal{S}$-smooth combinatorial DG-sheaf $\mathcal{D}M_{g,n} := \{ \mathcal{D}M_{\Gamma} \}_{\Gamma \in J_{g,n}}$ on the $J_{g,n}^\circ$-diagram $\mathcal{M}_{g,n}^{\text{top}}$, called a Deligne-Mumford DG-sheaf with connecting quasi-isomorphisms:

$$ \alpha(m) : m^* \mathcal{D}M_{\Gamma} \xrightarrow{\sim} \mathcal{D}M_{\Gamma'}, \quad m : \Gamma \rightarrow \Gamma' $$

Where $\alpha(m)$ is a natural quasi-isomorphism if $m$ is an edge contraction and $\alpha(m)$ is a permutation morphism if $m$ is an automorphism. It is easy to see that all properties from Definition 2.3.2 are satisfied.
Recall that for any complex $C$ in an abelian category a standard truncation functors $\tau_{\leq k}$ define an increasing filtration:

$$\cdots \subset \tau_{\leq k}C \subset \tau_{\leq k+1}C \subset \cdots \subset C.$$ 

By applying these truncation functors to the definition of Deligne-Mumford DG-sheaves and using (3) we get a sequence of DG-combinatorial sheaves on $M_{g,n}^{\text{trop}}$:

$$\mathcal{W}_0\mathcal{D}M_{g,n} \to \mathcal{W}_1\mathcal{D}M_{g,n} \to \cdots \to \mathcal{D}M_{g,n}.$$ 

**Definition 3.2.2.** For every $k \geq 0$ and $g,n$ such that $2g - 2 + n > 0$ we define an $S$-smooth DG-combinatorial sheaf $\mathcal{G}_{k}^W\mathcal{D}M_{g,n}$ on the $J_{g,n}$-diagram $M_{g,n}^{\text{trop}}$ called a Deligne-Mumford sheaf of the weight $k$ by the rule:

$$\mathcal{G}_{k}^W\mathcal{D}M_{g,n} := \text{Cone}(\mathcal{W}_{k-1}\mathcal{D}M_{g,n} \to \mathcal{W}_k\mathcal{D}M_{g,n}).$$

Recall that for a complex $C$ in an abelian category we have a natural projection morphism $\tau_{\geq k}C/\tau_{\geq k-1}C \to H^k(C)[-k]$ which induces the isomorphism on cohomology. Hence for every $k$, we have the quasi-isomorphism of combinatorial DG-sheaves:

$$\mathcal{G}_{k}^W\mathcal{D}M_{g,n} \sim \to H^k(\mathcal{D}M_{g,n})[-k].$$

**Remark 3.2.3.** Denote by $\text{ModOp}_D$ a category of $D$-twisted DG-modular cooperads in the sense E. Getzler and M. Kapranov [KG98]. Then using the description of combinatorial sheaves on $M_{g,n}^{\text{trop}}$ one can show that equivalence (3) defines the functor:

$$\text{Ho}(\text{ModOp}_D) \to D_c(M_{g,n}^{\text{trop}}, S).$$

Under this functor, the combinatorial DG-sheaf $\mathcal{D}M_{g,n}$ corresponds to the (non-twisted) DG-modular cooperad $C^*((\overline{M}, Q))$ from [KG98].

### 3.3. Getzler-Kapranov complexes

In this subsection, we compute the cohomology with compact support of combinatorial DG-sheaves $\mathcal{D}M_{g,n}$. To do it we give the following:

**Definition 3.3.1.** The Getzler-Kapranov complex $\mathcal{GK}_{g,n}$ is defined as the decorated graph complex which computes the compactly supported cohomology with coefficients in the Deligne-Mumford sheaf:

$$\mathcal{GK}_{g,n} := R\Gamma_c(M_{g,n}^{\text{trop}}, \mathcal{D}M_{g,n}).$$

Following [KG98] we compute the cohomology of the Getzler Kapranov complex:

**Proposition 3.3.2.** We have the following quasi-isomorphism of DG-vector spaces:

$$C^*_c(M_{g,n}, \mathbb{Q}) \sim \to \mathcal{GK}_{g,n}.$$

**Proof.** For any $\Gamma \in J_{g,n}$ by Proposition 2.2.2 a complex which computes the cohomology of $R\Gamma_c(\sigma_{\Gamma}, \mathcal{D}M_{\Gamma})$ can be represented as the total complex of the following DG-vector space:

$$C^*((\overline{M}_{\Gamma}), \mathbb{Q}) \to \cdots \to \bigoplus_{\Gamma' \in J_{g,n}^+} \bigotimes_{\Gamma'' \in J_{g,n}^+} C^*((\overline{M}_{w(v), n_v, \mathbb{Q}}) \otimes \det(\Gamma')) \to \cdots$$

(18)
Here \( \det(\Gamma') := \det(E(\Gamma')) \cong \text{or}_{G_n} \) is the determinant on the set of edges of \( \Gamma' \) and graphs with \( k \) edges are placed in degree \( k \). A differential \( D \) is defined by the rule:

\[
D = \sum_{\Gamma \in J^k_{g,n}, \Gamma' \in J^{k-1}_{g,n}} \xi_{\Gamma', \Gamma}^*.
\]

To compute the homotopy limit over \( J^o_{g,n} \) we do the following trick:

The canonical functor to the terminal category \( a_{J^o_{g,n}} : J^o_{g,n} \to 1 \) can be decomposed as \( J^o_{g,n} \to N \to 1 \), where \( N \) is a category associated with the poset of natural numbers and the first functor sends every graph \( \Gamma \) with \( k \) edges to the natural number \( k \).

Hence:

\[
\lim_{J^o_{g,n}} \cong \lim_{N} \text{LKan}_P.
\]

Thus since every graph \( \Gamma \) has a finite automorphism group \( \text{Aut}(\Gamma) \) and we work over \( \mathbb{Q} \) it is easy to see that the left Kan extension functor \( \text{LKan}_P \) along the functor \( P \) is the exact functor. Thus applying this construction to the DG-combinatorial sheaf \( \mathcal{D}M_{g,n} \) we get the functor from \( N \), with a value at the number \( n \) is given by the total complex of the following DG-vector space:

\[
C^* \left( \mathcal{M}_{g,n}, \mathbb{Q} \right) \to \cdots \to \bigoplus_{\Gamma \in J^k_{g,n}} \left( \bigotimes_{v \in V(\Gamma)} C^* \left( \mathcal{M}_{w(v), n_v}, \mathbb{Q} \right) \otimes \det(\Gamma) \right)^{\text{Aut}(\Gamma)} \to \cdots
\]

Where in the last non-zero entry, we place graphs \( \Gamma \) with \( E(\Gamma) = n \). A differential is defined by the following rule:

\[
\tilde{D} = \sum_{\Gamma \in J^k_{g,n}, \Gamma' \in J^{k-1}_{g,n}} \pi_{\Gamma', \Gamma} \circ \xi_{\Gamma', \Gamma}^*.
\]

This functor satisfies a generalized Mittag-Leffler condition and hence the limit of this functor over \( N \) is exact. Hence the homotopy limit \( \text{holim}_{J^o_{g,n}} \mathcal{R} \Gamma_*(\sigma, \mathcal{D}M_\Gamma) \) is exact and therefore \( \mathcal{R} \Gamma_*(\mathcal{M}_{g,n}^{\text{trop}}, \mathcal{D}M_{g,n}) \) is quasi-isomorphic to the total DG-vector space of the following complex:

\[
C^* \left( \mathcal{M}_{g,n}, \mathbb{Q} \right) \to \cdots \to \bigoplus_{\Gamma \in J^k_{g,n}} \left( \bigotimes_{v \in V(\Gamma)} C^* \left( \mathcal{M}_{w(v), n_v}, \mathbb{Q} \right) \otimes \text{det}(\Gamma) \right)^{\text{Aut}(\Gamma)} \to \cdots
\]

By a Künneth formula, the functor of the compactly supported cochains is monoidal and also it commutes with homotopy colimits. Hence we get the following quasi-isomorphism:

\[
\bigoplus_{\Gamma \in J^k_{g,n}} \left( \bigotimes_{v \in V(\Gamma)} C^* \left( \mathcal{M}_{w(v), n_v}, \mathbb{Q} \right) \otimes \text{det}(\Gamma) \right)^{\text{Aut}(\Gamma)} \cong
\]

\[
C^* \left( \prod_{\Gamma \in J^k_{g,n}} \prod_{v \in V(\Gamma)} \mathcal{M}_{w(v), n_v} / \text{Aut}(\Gamma), \varepsilon_k \right).
\]
Where $\varepsilon$ is a local system defined by a sign representation of a group that permutes the edges of a stable graph. By (15) the latter stack is equivalent to the stack $\tilde{D}_k$, which is a normalisation of the stack of nodal curves with at least $k$ nodes. A differential is given by the pull-push along diagram (16). Therefore by Proposition 2.4.1 we have the desired result.

Remark 3.3.3. Note that using a notion of a constructible DG-cosheaf on the $J^t_{g,n}$-diagram $M^t_{g,n}$ similarly to Remark 3.2.3 one can encode a notion of a $D$-twisted modular operad:

$$\text{ModOp} : \longrightarrow \mathcal{D}^c(M^t_{g,n}, S).$$

Moreover one can show that the following diagram is commutative:

$$\begin{array}{c}
\text{Ho}(\text{ModCoop}_D) \\ \downarrow \quad \downarrow
\end{array} \xrightarrow{\mathcal{F}_D} \begin{array}{c}
\text{Ho}(\text{ModOp}_D) \\
\text{D}_c(M^t_{g,n}, S) \\
\downarrow \quad \downarrow
\end{array} \xrightarrow{\mathcal{V}} \begin{array}{c}
\text{D}_c^c(M^t_{g,n}, S) \\
\text{D}_c^c(M^t_{g,n}, S)
\end{array}$$

Where $\mathcal{F}_D$ is a covariant $D$-Feynman transform [KG98] and $\mathcal{V}$ is a covariant Verdier duality functor [GL14]. Under functor (24) a DG-cosheaf $\mathcal{V}(DM_{g,n})$ corresponds to the twisted modular operad $\text{Grav}$. By Poincaré-Verdier duality we have:

$$H^*_c(M^t_{g,n}, \mathcal{V}(DM_{g,n})) \cong H^*_c(M^t_{g,n}, DM_{g,n})$$

All these facts together with a formality of the DG-modular cooperad $\bar{C}^*(\overline{M}, \mathbb{Q})$ [GSNPR05] imply an equivalence between Definition 3.3.1 and the definition from [AWZ20].

3.4. The weight filtration on $H^*_c(M_{g,n}, \mathbb{Q})$. Note that since $M_{g,n}$ is the smooth Deligne-Mumford stack for every $k$ the corresponding cohomology with compact support $H^*_c(M_{g,n}, \mathbb{Q})$ has weights in a region $\{0, \ldots, k\}$ i.e. a priori only the corresponding graded quotients are non zero. From Proposition 2.4.2 we obtain a description of the weight graded quotients of $H^*_c(M_{g,n}, \mathbb{Q})$ in terms of decorated graph complexes.

Definition 3.4.1. For $2g - 2 + n > 0$ we denote by $W_k GK_{g,n}$ a decorated graph complex which computes the cohomology with compact support of the $J^t_{g,n}$-diagram $M^t_{g,n}$ with coefficients in the DG-combinatorial sheaf $\text{Gr}_k W DM_{g,n} := \text{Gr}_k W_k GD_{g,n}$:

$$R\Gamma_c(M^t_{g,n}, \text{Gr}_k W_{DM_{g,n}}) := W_k GK_{g,n}$$

This DG-vector space will be called the Getzler-Kapranov complex of the weight $k$.

Our main result in this section is:

Theorem 3.4.2. We have the following description of the weight filtration on the compactly supported cohomology of $M_{g,n}$:

$$\text{Gr}_k W H^*_c(M_{g,n}) \sim \rightarrow H^*_c(W_k GK_{g,n})$$

5By $H^*_c$ we understand a (cosheaf) pushforward functor [GL14] along the morphism to a point.
Proof. Let us compute the complex:

\[(26) \quad R\Gamma_c(M_{g,n}^{\text{trop}}, Gr^W_k DM_{g,n})\]

Applying the quasi-isomorphism \(Gr^W_k DM_{g,n} \xrightarrow{\sim} H^k(\text{DM}_{g,n})[k]\) of combinatorial DG-sheaves on \(M_{g,n}^{\text{trop}}\) and acting like in the proof of Proposition \ref{prop:3.3.2} we obtain that \(26\) is quasi-isomorphic to the total DG-vector space of the complex:

\[(27) \quad H^k(M_{g,n}, \mathbb{Q}) \xrightarrow{\sim} \tilde{H}^k(\tilde{D}_1, \varepsilon_1) \to \ldots \to \tilde{H}^k(\tilde{D}_p, \varepsilon_p) \to \ldots,\]

where \(\tilde{D}\) is defined by \(20\). By Proposition \ref{prop:2.4.2} the \(i\)-cohomology of the complex above coincides with \(Gr^W_k \tilde{H}^{i+k}(M_{g,n}, \mathbb{Q})\).

\[\square\]

3.5. Low weights. Denote by \(H_n GC^\ast(\delta)\), the \(n\)-labelled hairy graph complex in the sense [AWZ20] and [CGP21a]. Elements of this complex are stable weighted \(n\)-marked graphs without loops and with at least trivalent vertices. The differential \(\delta\) is defined as the signed sum of vertex splitting operators. In the special case when \(n\) is zero this complex coincides with \(GC^\ast(\delta)\) the M. Kontsevich graph complex [Wil15].

A cohomological grading on \(H_n GC^\ast(\delta)\) is defined by a number of edges.

Since the differential in \(H_n GC^\ast(\delta)\) does not change the genus of a graph, complex \(H_n GC^\ast(\delta)\) can be decomposed as:

\[H_n GC^\ast(\delta) := \prod_{g=0}^{\infty} B_g H_n GC^\ast(\delta),\]

where \(B_g H_n GC_0^\ast(\delta)\), is the subcomplex that consists of stable \(n\)-marked graphs of genus \(g\).

Corollary 3.5.1. We have the following quasi-isomorphism of complexes:

\[B_3 H_n GC^\ast(\delta) \cong W_0 GK^\ast_{g,n}\]

Proof. Consider the weight zero combinatorial DG-sheaf \(W_0 DM_{g,n}\). From the irreducibility of the moduli stack of curves of genus \(g\) with \(n\)-punctures [DM69] we have an isomorphism of sheaves:

\[W_0 DM_{g,n} \cong \mathcal{Q}_{M_{g,n}^{\text{trop}}}.\]

Note that in our case \(\mathcal{Q}\) is the quasi-isomorphism (an automorphism group of a stable graph acts freely). Hence we have:

\[R\Gamma_c(M_{g,n}^{\text{trop}}, \mathcal{Q}_{M_{g,n}^{\text{trop}}}) := \text{holim}_{\Delta_{g,n}} R\Gamma_c(\sigma_{\Delta}, \mathcal{Q}_{\Delta}) \cong R\Gamma_c(M_{g,n}^{\text{trop}}, \mathcal{Q}).\]

Recall (for details see [CGP21b] and [CGP21a]) that a moduli space of volume 1 tropical curves of genus \(g\) with \(n\) punctures \(\Delta_{g,n}\) is a certain symmetric \(\Delta\)-complex with the geometric realisation \(|\Delta_{g,n}|\) identified with the complement to a link in \(M_{g,n}^{\text{trop}}\), hence:

\[H_c^k(M_{g,n}^{\text{trop}}, \mathcal{Q}) \cong \tilde{H}^{k-1}(|\Delta_{g,n}|, \mathcal{Q}).\]

Further using \textit{ibid.} one may decompose:

\[C^\ast(|\Delta_{g,n}|, \mathcal{Q}) = C^\ast(|\Delta_{g,n}^{\text{lw}}|, \mathcal{Q}) \oplus C^\ast(|\Delta_{g,n}|, |\Delta_{g,n}^{\text{lw}}|, \mathcal{Q}).\]

Where \(\Delta_{g,n}^{\text{lw}}\) is a sub symmetric \(\Delta\)-complex which consists of graphs with loops and vertices of positive weights. Applying Proposition [CGP21a] it can be shown that
\( \Delta^{lw}_{g,n} \) is contractible therefore by the definition of the hairy \( n \)-marked complex we get:
\[
C^\ast(\vert \Delta_{g,n} \vert, \vert \Delta^{lw}_{g,n} \vert, \mathbb{Q}) := B_g H_n GC^\ast(\delta).
\]
Hence we get the desired quasi-isomorphism.

\[\square\]

**Remark 3.5.2.** For \( g, n \) with \( 2g - 2 + n > 0 \) consider the canonical morphism of combinatorial DG-sheaves

\[
W_0 DM_{g,n} \longrightarrow DM_{g,n}
\]

Applying the functor of the compactly supported cohomology we get the morphism:

\[
cgp: H^\ast(B_g H_n GC) \longrightarrow H^\ast_c(M_{g,n}, \mathbb{Q})
\]

From the result above we see that this morphism is injective and tautologically coincides with a morphism constructed in [CGP21b] and [CGP21a]. We can also describe (29) from a little bit different perspective. The correspondence from Remark 3.3.3 and the formality result [GSNPR05] applied to (28) produce the morphism of graded modular cooperads [AWŽ20]:

\[
Comm \longrightarrow C^\ast(\mathcal{M}, \mathbb{Q}),
\]

where \( Comm \) is a modular envelope of the cyclic cocommutative cooperad. Note that the compactly supported cochains with coefficients in \( W_0 DM_{g,n} \) and \( DM_{g,n} \) can be identified with the DG-vector spaces over \( \bullet_{g,n} \), of the Feynman transforms of the corresponding modular cooperads. Hence the morphism \( cgp \) coincides with the morphism from [AWŽ20].

Recall that in [AC98] (Theorem 2.1) it was proved that \( H^k(\mathcal{M}_{g,n}, \mathbb{Q}) = 0 \) for \( k = 1, 3, 5 \) and all \( g, n \) such that \( 2g - 2 + n > 0 \). Thus by quasi-isomorphism (17) and by the Künneth formula, we have that the cohomology of the Deligne-Mumford sheaves \( Gr^W_k DM_{g,n} \) vanishes when \( k = 1, 3, 5 \). We have the following:

**Corollary 3.5.3.** We have the vanishing of the \( k \)-associated weight quotients of \( H^{i+k}_c(M_{g,n}, \mathbb{Q}) \) in the case when \( k = 1, 3, 5 \):

\[
Gr^W_k H^{i+k}_c(M_{g,n}, \mathbb{Q}) = 0, \quad k = 1, 3, 5.
\]

**Remark 3.5.4.** (i) The morphism \( cgp \) allows producing previously unknown classes in moduli stacks of curves. In particular when \( n = 0 \) by Theorems of [Wil15] and [Bro12] one gets an injection:

\[
\tilde{Free}_{L_{kx}}(\sigma_3, \ldots, \sigma_{2k+1}, \ldots) \hookrightarrow \prod_{g=3}^{\infty} H^2_c(M_{g}, \mathbb{Q}).
\]

Where \( \tilde{Free}_{L_{kx}}(\sigma_3, \ldots, \sigma_{2k+1}, \ldots) \) is a (completed) free Lie algebra on odd generators, see a discussion in [CGP21a].

(ii) In the recent interesting work [PW21] the cohomology of the weight two Getzler-Kapranov complex was computed. It would be very interesting to calculate the cohomology of \( W_4 GK^\ast_{g,n} \).
4. Sheaves on moduli spaces of hairy tropical curves

4.1. Moduli spaces of hairy tropical curves. For every $g \geq 2$ we consider a category $HJ_g$ with objects defined by stable weighted $n$-marked graphs $\Gamma$ of genus $g$, where the marking function:

$$m : \{1, \ldots, n\} \rightarrow V(\Gamma)$$

is considered to be defined for an arbitrary value of $n$ i.e. $n \in \mathbb{N}$. Morphisms in this category are defined as compositions of contractions of edges, morphisms that forget markings, and isomorphisms of graphs that may not preserve labelings of markings. Similar to the case of the category $J_{g,n}$ it is convenient to represent an object in the category $HJ_g$ as a stable graph of genus $g$ with any finite number of unlabelled hairs attached to vertices (such that the stability condition holds for each vertex). For $\Gamma \in HJ_g$ we will denote by $\text{Aut}^h(\Gamma) := \text{Isom}_{HJ_g}(\Gamma, \Gamma)$ a group of automorphisms of a graph $\Gamma$ in the category $HJ_g$. It is important to distinguish it from the group of automorphisms of $\Gamma$ considered as an object of $J_{g,n}$ (for example $\text{Aut}^h(\bullet_{g,n}) \cong \Sigma_n$, while $\text{Aut}((\Gamma_{g,n})$ is trivial). For a graph $\Gamma \in HJ_g$ and $h \in m^{-1}(V(\Gamma))$ we denote by $\Gamma_h$ an element in $HJ_g$ with a hair $h$ being contracted. By $HJ_{g,h}^2$ we denote a collection of graphs in $HJ_g$ with exactly $n$-edges and $k$-hairs. Analogous to the definition of the $J_{g,n}$-diagram $\mathcal{M}_{g,n}^{trop}$ we give:

**Definition 4.1.1.** For every $g \geq 2$ we define a $HJ_g$-diagram $\mathcal{H} \mathcal{M}_{g}^{trop}$ called a moduli diagram of tropical curves of genus $g$ with an arbitrary number of hairs:

$$\mathcal{H} \mathcal{M}_{g}^{trop} : HJ_g \rightarrow \text{Top}$$

by the following rule:

For every object $(\Gamma, w, m) \in HJ_g$ we set:

$$\mathcal{H} \mathcal{M}_{g}^{trop} : (\Gamma, w, m) \mapsto \sigma^\text{hair}_g := \mathbb{R}_{\geq 0}^{E(\Gamma) \cup m^{-1}(V(\Gamma))}$$

For a morphism $f : (\Gamma, w, m) \rightarrow (\Gamma', w', m')$ in $HJ_g$ we set:

$$\mathcal{H} \mathcal{M}_{g}^{trop} : f \mapsto \sigma^\text{hair}_f,$$

where $\sigma^\text{hair}_f : \sigma^\text{hair}_{\Gamma'} \rightarrow \sigma^\text{hair}_{\Gamma}$ is a map that sends an $e'$-coordinate (resp. an $h'$-coordinate) of a topological space $\mathbb{R}_{\geq 0}^{E(\Gamma') \cup m^{-1}(V(\Gamma'))}$ to an $e$-coordinate (resp. an $h$-coordinate) of space $\mathbb{R}_{\geq 0}^{E(\Gamma) \cup m^{-1}(V(\Gamma))}$ if $f$ sends the edge $e$ (resp. the hair $h$) to the edge $e'$ (resp. the hair $h'$) and zero otherwise.

Analogous to the case of moduli diagrams $\mathcal{M}_{g,n}^{trop}$ each topological space $\sigma^\text{hair}_g$ is naturally stratified by graphs $\Gamma' \in HJ_{g,1}$. We will denote this stratification by $\mathcal{S}^\text{hair}_g$ and the resulting stratification on the $HJ_g$-diagram $\mathcal{H} \mathcal{M}_{g}^{trop}$ by $\mathcal{S}^\text{hair}_g$.

We also consider a natural $HJ_g$-diagram $\mathcal{H} \mathcal{M}_{g,0}^{trop}$ defined by sending each stable weighted marked graph $\Gamma$ to the topological space $\sigma_{\Gamma}$. We have a canonical closed morphism between diagrams $\nu_g : \mathcal{H} \mathcal{M}_{g,0}^{trop} \hookrightarrow \mathcal{H} \mathcal{M}_{g}^{trop}$. The complement $HJ_g^s$-diagram will be denoted by $\mathcal{H} \mathcal{M}_{g}^{trop}$.

---

6We consider this operation when the resulting graph $\Gamma_h$ is stable.
at a stable graph $\Gamma$ by $h_{\geq 1}^{\text{hair}}$. For every $g \geq 2$ we have a sequence of morphisms of $HJ_g$-diagrams:

\[
\mathcal{H}_{\geq 1}M_g^{\text{trop}} \xrightarrow{\nu \geq 1} \mathcal{H}M_g^{\text{trop}} \xrightarrow{\nu 0} \mathcal{H}_0M_g^{\text{trop}}
\]

4.2. Sheaves on $\mathcal{H}M_g^{\text{trop}}$. We will define an analog of the combinatorial Deligne-Mumford DG-sheaves $\mathcal{D}\mathcal{M}_{g,n}$ on the diagram $\mathcal{H}M_g^{\text{trop}}$. To do it let us recall that for every finite set $I$ with a subset $J \subset I$ there is a surjective morphism of stacks:

\[
\pi_J: \mathcal{M}_{g,I} \longrightarrow \mathcal{M}_{g,I} \setminus J,
\]

defined by forgetting $J$-labelled marked points and stabilising the resulting nodal curve $[\text{Knu83}]$. In particular, we will use a notation $\pi_I$ when $J = I$.

For a stratified space $\sigma^{\text{hair}}_\Gamma$ associated with the stable marked graph $\Gamma \in HJ_g$ we define an $S^{\text{hair}}_\Gamma$-smooth combinatorial DG-sheaf $\mathcal{D}\mathcal{M}_n^{\text{hair}}$ on $\sigma^{\text{hair}}_\Gamma$ by the following rule:

- For every graph $\Gamma' \in HJ_g$ we set:

\[
\mathcal{R}(S_{\Gamma'}, \mathcal{D}\mathcal{M}_n^{\text{hair}}) := \bigotimes_{v \in V(\Gamma')} C^*(\mathcal{M}_{w(v), n_v}, \mathcal{Q})
\]

- For every inclusion of strata $S_{\Gamma''} \subset S_{\Gamma'}$ we define a variation morphism:

\[
\text{var}_{\Gamma'', \Gamma'}: \mathcal{R}(S_{\Gamma'}, \mathcal{D}\mathcal{M}_n^{\text{hair}}) \longrightarrow \mathcal{R}(S_{\Gamma''}, \mathcal{D}\mathcal{M}_n^{\text{hair}})
\]

by the following rule:

(i) If the inclusion of strata is induced by a contraction of edges we define the variation operator as it was defined in the case of the DG-sheaf $\mathcal{D}\mathcal{M}_{g,n}$.

(ii) If $\Gamma''$ is obtained from $\Gamma'$ by a contraction of hairs $h_{i,v_j}$ at a vertex $v_j$ we define $\text{var}_{\Gamma'', \Gamma'}$ as the composition of morphisms:

\[
\text{id} \otimes \cdots \otimes \pi^*_h_{i,v_j} \otimes \cdots \otimes \text{id},
\]

where a morphism:

\[
\pi^*_h_{i,v_j}: C^*(\mathcal{M}_{w(v_j), H_v \cup m^{-1}(v_j)} \cup h_{i,v_j}, \mathcal{Q}) \longrightarrow C^*(\mathcal{M}_{w(v_j), H_v \cup m^{-1}(v_j)} \cup h_{i,v_j}, \mathcal{Q})
\]

is placed at the vertex $v_j$.

The direct check shows that for every $\Gamma \in HJ_g$ the combinatorial DG-sheaf $\mathcal{D}\mathcal{M}_n^{\text{hair}}$ is well defined.

**Definition 4.2.1.** For every $g \geq 2$ we define an $S^{\text{hair}}_\Gamma$-smooth combinatorial DG-sheaf $\mathcal{D}\mathcal{M}_n^{\text{hair}} := \{\mathcal{D}\mathcal{M}_n^{\text{hair}}\}_{\Gamma \in HJ_g}$ on the $HJ_g$-diagram $\mathcal{H}M_g^{\text{trop}}$, called a hairy Deligne-Mumford DG-sheaf with the connecting quasi-isomorphisms:

\[
\alpha(m): m^* \mathcal{D}\mathcal{M}_n^{\text{hair}} \sim \mathcal{D}\mathcal{M}_n^{\text{hair}}, \quad m: \Gamma \longrightarrow \Gamma'.
\]

Where $\alpha(m)$ is a natural quasi-isomorphism if $m$ is an edge contraction or a hair contraction and $\alpha(m)$ is a permutation morphism if $m$ is an automorphism of a graph. It is easy to see that all properties from Definition 2.3.2 are satisfied.
4.3. The hairy Getzler-Kapranov complex. Analogous to Definition 3.3.1 we give:

**Definition 4.3.1.** For every \( g \geq 2 \) we define the *hairy Getzler-Kapranov complex* \( \text{HGK}^*_g \) to be the complex that computes the compactly supported cohomology of the \( HJ^g \)-diagram \( \mathcal{HM}^\text{trop}_g \) with coefficients in the hairy Deligne-Mumford sheaf \( \mathcal{DM}^\text{hair}_g \):

\[
\text{HGK}^*_g := R\Gamma_c(\mathcal{HM}^\text{trop}_g, \mathcal{DM}^\text{hair}_g)
\]

**Remark 4.3.2.** The definition of the hairy Getzler-Kapranov complex was also presented in [AWŽ20]. In *ibid.*, it is defined as the total complex associated with the skew symmetrisation of the Feynman transform of the modular cooperad \( Hq(M, Q) \) with an additional differential defined by adding a decorated hair. Note that methods of [GSNPR05] do not directly imply a formality quasi-isomorphism in this case. However, the methods of [CH20] do imply the formality quasi-isomorphism and therefore Definition 4.3.1 of the hairy Getzler-Kapranov complex coincides with one given in [AWŽ20]. We thank Dan Petersen for pointing out this reference.

Our first result (also proved in [AWŽ20] (Theorem 30)) concerning the cohomology of the hairy Getzler-Kapranov complex is the following:

**Proposition 4.3.3.** The hairy Getzler-Kapranov complex \( \text{HGK}^*_g \) is acyclic.

**Proof.** To prove this result, we realise the hairy Deligne-Mumford DG-sheaf as the total object of the Cousin complex defined by a number of internal edges in stable graphs.

For every \( k \geq 0 \) we introduce the following \( HJ^g \)-diagram:

\[
F_{\geq k} \mathcal{HM}^\text{trop}_g : HJ^g \rightarrow \text{Top}.
\]

Defined by a rule: a stratified topological space \( \sigma^\text{hair}_\Gamma \) carries the following filtration:

\[
\sigma^\text{hair}_\Gamma := e_0 \sigma^\text{hair} \supset \cdots \supset e_{|E(\Gamma)|} \sigma^\text{hair},
\]

where \( e_{\geq k} \sigma^\text{hair}_\Gamma \) is a subspace where at least \( k \) "internal edge coordinates" are non-zero. Hence we define \( F_{\geq k} \mathcal{M}^\text{trop}_g \) by sending a stable graph \( \Gamma \) to \( e_{\geq k} \sigma^\text{hair}_\Gamma \) if \( k \leq |E(\Gamma)| \) and otherwise to the empty set. We get the following decreasing filtration:

\[
\mathcal{HM}^\text{trop}_g := F_{\geq 0} \mathcal{HM}^\text{trop}_g \leftarrow F_{\geq 1} \mathcal{HM}^\text{trop}_g \leftarrow \ldots,
\]

where the morphism \( j_{\geq k} : F_{\geq k} \mathcal{HM}^\text{trop}_g \hookrightarrow \mathcal{HM}^\text{trop}_g \) is closed. We have the associated sequence of DG-combinatorial sheaves:

\[
\mathcal{DM}^\text{hair}_g = j_{\geq 0} j_{\geq 0}^* \mathcal{DM}^\text{hair}_g \leftarrow \ldots \leftarrow j_{\geq k} j_{\geq k}^* \mathcal{DM}^\text{hair}_g \leftarrow \ldots
\]

We have the Postnikov system associated with [KS16] (cf. [KS16] and [KG94]). Hence for every \( g \geq 2 \) we realise \( \mathcal{DM}^\text{hair}_g \) as the convolution object of the following complex of DG-constructible sheaves:

\[
j_{\geq 0} j_{\geq 0}^* \mathcal{DM}^\text{hair}_g \rightarrow \ldots \rightarrow j_{\geq k} j_{\geq k}^* \mathcal{DM}^\text{hair}_g \rightarrow \ldots,
\]

where \( j_{k} : F_{k} \mathcal{HM}^\text{trop}_g \rightarrow \mathcal{HM}^\text{trop}_g \) is an open inclusion of diagrams. Here \( F_{k} \mathcal{HM}^\text{trop}_g \) is the open complement to \( F_{k+1} \mathcal{HM}^\text{trop}_g \). We will show that the for every \( k \) the following complex:

\[
\text{HGK}^*_{g,n} := R\Gamma_c(\mathcal{HM}^\text{trop}_g, j_{n} j_{n}^* \mathcal{DM}^\text{hair}_g)[n]
\]
is contractible. Acting like in the proof of Theorem 3.3.2 the complex $\text{HGK}^*_{g,k}$ can be realised as the total DG-vector space of a complex:

\[
\bigoplus_{\Gamma \in HJ^k_g,0} \left( \bigotimes_{v \in V(\Gamma)} C^*(\overline{M}_{w(v),n_v}, \mathbb{Q}) \otimes \det^h(\Gamma) \right)_{\text{Aut}^h(\Gamma)} \rightarrow \ldots
\]

\[
\ldots \rightarrow \bigoplus_{\Gamma \in HJ^k_g,n} \left( \bigotimes_{v \in V(\Gamma)} C^*(\overline{M}_{w(v),n_v}, \mathbb{Q}) \otimes \det^h(\Gamma) \right)_{\text{Aut}^h(\Gamma)} \rightarrow \ldots
\]

Here $\det^h(\Gamma) := \text{or}_{\text{ Bri}} \cong \det(E(\Gamma)) \otimes \det(m^{-1}(V(\Gamma)))$. Note that:

\[
C^*(\overline{M}_\Gamma / \text{Aut}^h(\Gamma), \varepsilon_p \otimes \epsilon_n) \cong \left( \bigotimes_{v \in V(\Gamma)} C^*(\overline{M}_{w(v),n_v}, \mathbb{Q}) \otimes \det^h(\Gamma) \right)_{\text{Aut}^h(\Gamma)}.
\]

Where by $\epsilon_n$ we have denoted a sign local system on the moduli stack $\overline{M}_\Gamma / \text{Aut}^h(\Gamma)$ with a monodromy defined by a sign representation that acts on hairs of a stable graph $\Gamma$. A differential in (34) is defined by the following rule:

Suppose that $\Gamma$ and $\Gamma'$ are two objects of $HJ_g$ such that $\Gamma$ has exactly $k$-hairs with the following property: if one contracts one of them the resulting graph will be isomorphic to $\Gamma'$. We will denote this set of hairs by $K$. Consider the following correspondence:

\[
(M_\Gamma \times K) / \text{Aut}^h(\Gamma) \xrightarrow{\pi_{\Gamma',\Gamma}} M_{\Gamma'} / \text{Aut}^h(\Gamma')
\]

Where a morphism $\pi_{\Gamma',\Gamma}$ is given by applying a pullback along the morphism $\pi_h$ at the hair $h \in K$ and $\pi_{\Gamma',\Gamma}$ is the projection morphism. Hence we get a differential defined by the rule:

\[
\tilde{\nabla}_1 := \sum_{\Gamma \in HJ^k_g, \Gamma' \in HJ^k_{g,n-1} \Gamma \rightarrow \Gamma'} \pi_{\Gamma',\Gamma} \circ \tilde{\pi}_h \circ \pi_{\Gamma',\Gamma}.
\]

Note that on a component $C^*(\overline{M}_{w(v),n_v}, \mathbb{Q})$ corresponding to the vertex $v \in V(\Gamma)$ with $m^{-1}(v) \neq \emptyset$ the differential $\tilde{\nabla}_1$ acts by the following rule:

\[
\tilde{\nabla}_1 : \omega \mapsto \sum_{h \in m^{-1}(v)} (-1)^{h-1} \pi_h^* \omega
\]

We shall construct an explicit homotopy and get the desired result. For every $k \geq 1$ we define the morphism

\[
H : \text{HGK}^*_{g,k} \longrightarrow \text{HGK}^*_{g,k}[-1]
\]

by the following rule:

(i) Suppose that $m^{-1}(v) = \emptyset$, then we define this morphism to be zero on this component.
Consider the Gysin triangle associated with diagram (30):

\[ \text{id} : \Lambda \to \Lambda_{m^{-1}(v)} \]

\[ H : \omega \mapsto \sum_{h \in m^{-1}(v)} \pi_{m_h}(\omega \mapsto \varphi_h) \]

Where:

\[ \pi_{m_h} : C^*((\Lambda_{m(v)}, v)_n, Q) \to C^{*-2}(\Lambda_{m(v)}, v)_n, Q) \]

is a Gysin pushforward morphism along the forgetful morphism \( \pi_{m} \) and:

\[ \varphi_h := \frac{1}{2w(v) - 2 + n_v} \psi \in H^2(\Lambda_{m(v)}, v)_n, Q) \]

is a normalised psi-class at \( h \). Further applying (37) together with a projection formula and the fact that the psi-class \( \psi_h \) can be identified with the Euler class of the fibration \( \pi_h \) one can explicitly compute (for details see Theorem 30 in [AWZ20]):

\[ (|m^{-1}(v)| + 1)\omega = H(\varphi_1(\omega)) + \varphi_1(H(\omega)), \quad \omega \in C^*((\Lambda_{m(v)}, v)_n, Q). \]

Denote by \( H_{\geq 1, \text{DK}_g} \) a decorated graph complex that computes the compactly supported cohomology of the \( HJ_g^c \)-diagram \( H_{\geq 1, \text{DK}_g} \) with coefficients in the combinatorial DG-vector space \( \mathbb{D}M_{g, \geq 1}^\text{hair} := v_{\geq 1}^* \mathbb{D}M_{g}^\text{hair} :\)

\[ H_{\geq 1, \text{DK}_g} := R\Gamma_c(H_{\geq 1, \text{DK}_g}^\text{trg}, \mathbb{D}M_{g, \geq 1}^\text{hair}) \]

(38)

We will call this complex a hairy Getzler-Kapranov complex with at least one hair or just the hairy Getzler-Kapranov complex if it will not lead to confusion. We have the following immediate:

**Corollary 4.3.4.** For \( g > 1 \) the hairy Getzler-Kapranov complex \( H_{\geq 1, \text{DK}_g} \) is quasi-isomorphic to \( C_c^*(\mathcal{M}_g, Q)[-1] \).

**Proof.** Consider the Gysin triangle associated with diagram (30):

\[ v_{\geq 1}^* \mathbb{D}M_{g}^\text{hair} \to \mathbb{D}M_{g}^\text{hair} \to v_{\geq 1}^* \mathbb{D}M_{g}^\text{hair} \to v_{\geq 1}^* \mathbb{D}M_{g}^\text{hair} \]

By applying the compactly support cohomology functor and using Proposition 4.3.3 we obtain that \( R\Gamma_c(H_{\geq 1, \text{DK}_g}^\text{trg}, v_{\geq 1}^* \mathbb{D}M_{g}^\text{hair}) \) is quasi-isomorphic to a complex:

\[ R\Gamma_c(H_{\geq 1, \text{DK}_g}^\text{trg}, v_{\geq 1}^* \mathbb{D}M_{g}^\text{hair})[-1]. \]

The latter complex can be identified with the shifted Getzler-Kapranov complex \( \text{DK}_g[1] \) and hence by Proposition 3.3.2 with the DG-vector space \( C_c^*(\mathcal{M}_g, Q)[-1] \).

**Remark 4.3.5.** It is possible to extend the definition of the hairy Getzler-Kapranov complex \( H_{\geq 1, \text{DK}_g} \) to the case when \( g = 1 \). One can consider a category \( H_{\geq 1, J_g} \) with objects being stable marked graphs of genus \( g \) with at least one marking. Note that we have a natural inclusion of categories \( H_{\geq 1, J_g} \to HJ_g \). The category \( H_{\geq 1, J_g} \) is well defined for \( g = 1 \) and one can consider a \( HJ_g^c \)-diagram \( H_{\geq 1, \text{DK}_g}^\text{trg} \). For \( g = 1 \) we can define the hairy Getzler-Kapranov complex \( H_{\geq 1, \text{DK}_1} \) verbatim to (38). Analogous to the proof of Proposition 4.3.3 one can compute the cohomology of \( H_{1} \) (see Theorem 4.4.7).
4.4. The Willwacher differential $\nabla_1$. In this subsection, we will relate the hairy Getzler-Kapranov complex $H_{\geq 1}^*GK_g$ to the certain double complex which comes from varying a number of markings on the moduli stacks $\mathcal{M}_{g,n}$. To do it let us recall some definitions:

For $g > 0$ and a finite set $I$ such that $2g + |I| - 2 > 0$ we denote by $\mathcal{M}_{g,I}^t$ the moduli stack of $I$-marked curves of genus $g$ with rational tails. For $g \geq 2$ this stack can be defined as a fibered product:\n
\begin{equation}
\mathcal{M}_{g,I}^t := \mathcal{M}_{g,I} \times_{\mathcal{M}_g} \mathcal{M}_g
\end{equation}

For $g = 1$ the stack $\mathcal{M}_{g,I}^t$ can be defined as the moduli stack of curves with compact Jacobian $\mathcal{M}_{g,1}$ i.e. it consists of nodal curves with a dual graph being a tree. By the definition we have $\mathcal{M}_{g,1}^t \cong \mathcal{M}_{g,1}$. By the construction for every $x$ we have a proper morphism of stacks:

\begin{equation}
\mu_x: \mathcal{M}_{g,I\cup\{x\}}^t \longrightarrow \mathcal{M}_{g,I}^t.
\end{equation}

We obtain the following symmetric $\Delta$-stack $\mathcal{M}_{g,I}^t$:\n
\begin{equation}
\mathcal{M}_{g,1}^{t} \xrightarrow{\mu_{g,1}} \mathcal{M}_{g,2}^{t} \xrightarrow{\mu_{g,2}} \mathcal{M}_{g,3}^{t} \ldots
\end{equation}

Applying the functor of the cochains with compact support we obtain the symmetric $\Delta$-cochain complex. By a version of the Dold-Puppe construction we construct the following DG-vector space:

\begin{equation}
C^*_{\chi}(\mathcal{M}_{g,I}^t, \mathbb{Q}) \xrightarrow{\nabla_1^{t}} \ldots \xrightarrow{\nabla_1^{t}} (C^*_{\chi}(\mathcal{M}_{g,n}^t, \mathbb{Q}) \otimes_{\Sigma_n} \text{sgn}_n \Sigma_n) \xrightarrow{\nabla_1^{t}} \ldots
\end{equation}

Where a morphism $\nabla_1^{t}$ is defined by the following rule. For every $n \geq 1$ denote by $\mathcal{M}_{g,n}^{t}$ the following stack:

$$\mathcal{M}_{g,n}^{t} := \mathcal{M}_{g,n}/\text{Aut}^h(\bullet_g)$$

Then one can consider the following correspondence:

$$\mathcal{M}_{g,n}^{t} \xrightarrow{\bar{\mu}_{g,n}(\bullet_g)} \mathcal{M}_{g,n}^{t}$$

$$\pi_{\bullet_g,n}(\bullet_{g+1}) : \mathcal{M}_{g,n+1} \longrightarrow \mathcal{M}_{g,n}^{t}$$

Where a stack $\mathcal{M}_{g,n}^{t}$ is defined as $(\mathcal{M}_{g,n+1}^{t} \times m^{-1}(V(\bullet_g)))/\Sigma_{n+1}$ and $\pi_{\bullet_g,n}(\bullet_{g+1})$ is a projection morphism. $\bar{\mu}_{g,n}(\bullet_g)$ is defined as a morphism that factors the morphism $(\mathcal{M}_{g,n+1}^{t} \times m^{-1}(V(\bullet_g))) \longrightarrow \mathcal{M}_{g,n}^{t}$. The later morphism is defined by applying the morphism $\mu_i$ along the $i$-hair of $\bullet_{g+1}$. We denote by $\epsilon_1$ the sign local system on the stack $\mathcal{M}_{g,n}^{t}$ with the monodromy defined by $^{7}$Morris in the corresponding fibered square are given by the canonical open inclusion $j_I$ and the forgetful morphism $\pi: \mathcal{M}_{g,I} \longrightarrow \mathcal{M}_g$. $^{8}$By the symmetric $\Delta$-stack we understand a functor from a category of symmetric semi-simplicial sets $I^{op}$ to the category of Deligne-Mumford stack with proper morphisms.
the sign representation of $\text{Aut}^h(g,n) := \Sigma_n$. We define the following pull-push morphism of DG-vector spaces (cf. (36)):

$$\nabla_1^{rt} : C^*_c(M_{g,n}^{rt}, \epsilon_n) \to C^*_c(M_{g,n+1}^{rt}, \epsilon_{n+1})$$

By the rule:

$$\nabla_1^{rt} := \pi_{g,n} \cdot \mu_{g,n+1} : M_{g,n} \to M_{g,n+1}$$

Note the cochains with compact support of $M_{g,n}^{rt}$ with coefficients in the sign local system can be naturally identified with the DG-vector space $(C^*_c(M_{g,n}^{rt}, \mathbb{Q}) \otimes \Sigma_n \text{sgn})_{\Sigma n}$ and hence we get the desired operator.

The moduli stack $M_{g,n}^{rt}$ is equipped with a natural inclusion $j^{rt} : M_{g,n} \hookrightarrow M_{g,n}^{rt}$. Then we have the following:

**Proposition 4.4.1.** For $2g + n - 2 > 0$ an !-extension functor along $j^{rt}$ induces the quasi-isomorphism of complexes which computes the compactly supported cohomology with coefficients in the sign local systems:

$$j^{rt} : C^*_c(M_{g,n}/\Sigma_n, \epsilon_n) \xrightarrow{\sim} C^*_c(M_{g,n}^{rt}/\Sigma_n, \epsilon_n)$$

**Proof.** Denote by $D$ the complement to $M_{g,I}$ in $M_{g,n}^{rt}$. Then $D$ is a divisor with the following decomposition:

$$D = \bigsqcup D_T,$$

where $T \in HJ_g$ has a combinatorial type of a tree with a unique vertex of genus $g$ and all other vertices are marked by genus 0. We claim that a complex $C^*_c(D/\Sigma_n, \epsilon_n)$ which computes the compactly supported cohomology with coefficients in the sign local system is acyclic:

A stable tree $T$ always has a vertex $v$ with the associated moduli space $\overline{M}_{w(v), n_v} = \overline{M}_{0, |m^{-1}(v)| + 1}$, where $2 \leq |m^{-1}(v)| \leq n$ is a number of hairs attached to $v$ and 1 represents a unique root (an internal half-edge) attached to $v$. Hence the claim will follow from the following:

**Lemma 4.4.2.** For every $n > 1$ we have an equivalence:

$$H^*(\overline{M}_{0,n+1}/\Sigma_n, \epsilon_n) = 0$$

**Proof.** We will prove this statement by induction. Assume that the assertion is true for $k$. We will prove it for $k+1$:

Diagram (14) induces the Gysin long exact sequence:

$$\cdots \to H^i(\partial M_{0,k+1}/\Sigma_k, \epsilon_k) \to H^i_c(M_{0,k+1}/\Sigma_k, \epsilon_k) \to H^i_c(\overline{M}_{0,k+1}/\Sigma_k, \epsilon_k) \to \cdots$$

First, we will show that the compactly supported cohomology of the smooth locus vanishes:

$$H^*_c(M_{0,k+1}/\Sigma_k, \epsilon_k) = 0$$

Namely due to Vaknin we know that $H^*_c(M_{0,k}/\Sigma_k, \epsilon_k) = 0$. We consider the fibration:

$$\pi_1 : M_{0,k+1}/\Sigma_k \to M_{0,k}/\Sigma_k$$

But the Serre spectral sequence we get the vanishing of (45).
Hence it is enough to prove that \( H^*(\partial \mathcal{M}_{0,k+1}/\Sigma_k, \epsilon_k) \) vanishes as well. Consider the Deligne complex which computes \( H^*(\partial \mathcal{M}_{0,k+1}/\Sigma_k, \epsilon_k) \). The \( p \)-cochains of this complex are defined by the following rule:

\[
\bigoplus_{T \in H \mathcal{J}^{k+1}_{0,j}} \left( \bigotimes_{v \in V(T)} C^*(\mathcal{M}_{\mathcal{W}(v), n_u}, \mathbb{Q}) \otimes \det^h(T) \right)^{\text{Aut}^h(T)}.
\]

Where \( T \) is a stable tree with \( p \) edges and \( k+1 \) leaves and \( \text{Aut}^h(T) \) is a group of automorphisms of a tree that fix one marking. This group of automorphisms consists of permutations of hairs of a tree. Each tree \( T \) has a vertex \( v' \) such that all hairs \( m^{-1}(v') \) attached to this vertex are unfrozen i.e. all hairs are allowed to be permuted. Denote by \( \text{Aut}(T)v' \) the subgroup of \( \text{Aut}^h(T) \) which permute these hairs. Note that \( |m^{-1}(v')| < k \) and hence by the assumption of induction we have:

\[
H^*(\mathcal{M}_{g,|m^{-1}(v')|+1}/\Sigma_{|m^{-1}(v')|}, \epsilon_{|m^{-1}(v')|}) = 0.
\]

Since the moduli stack of stable rational curves is formal and each element in (46) must be in particular skew-invariant under \( \text{Aut}^h(T) \) and we get the vanishing of (46).

\[\square\]

From the standard Gysin long exact sequence:

\[
\cdots \to H^{i-1}_c(D/\Sigma_n, \epsilon_n) \to H^i_c(\mathcal{M}_{g,n}/\Sigma_n, \epsilon_n) \to H^i_c(\mathcal{M}_{g,n+1}/\Sigma_{n+1}, \epsilon_n) \to \cdots
\]

we obtain the desired quasi-isomorphism.

\[\square\]

Remark 4.4.3 (V. Dotsenko). To prove Lemma 4.4.2 one can use the following argument. Consider the hypercommutative operad \( \mathcal{H}_{\text{com}} = \{ H^*(\mathcal{M}_{0,k+1}) \}_{k \geq 2} \). It is the quotient of the free operad generated by elements \( \{ m_k \}_{k \geq 2} \) (fundamental classes) modulo the quadratic ideal. A sign representation does not appear in the decomposition of the free operad into irreducible representations and thus operadic ideal is this subrepresentation, hence we do not get any new irreducible representations after passing to the quotient.

Definition 4.4.4. We define the Willwacher differential \( \nabla^1 \) as a unique zigzag morphism:

\[
\nabla^1 : C_c^*(\mathcal{M}_{g,n}/\Sigma_n, \epsilon_n) \longrightarrow C_c^*(\mathcal{M}_{g,n+1}/\Sigma_{n+1}, \epsilon_{n+1}),
\]

making the following diagram commutes:

\[
\begin{array}{ccc}
C_c^*(\mathcal{M}_{g,n}^r, \epsilon_n) & \xrightarrow{\nabla^1} & C_c^*(\mathcal{M}_{g,n+1}^r, \epsilon_{n+1}) \\
\downarrow j_{tt}^r & & \downarrow j_{tt}^r \\
C_c^*(\mathcal{M}_{g,n}/\Sigma_n, \epsilon_n) & \xrightarrow{\nabla^1} & C_c^*(\mathcal{M}_{g,n+1}/\Sigma_{n+1}, \epsilon_{n+1})
\end{array}
\]

\[^9\text{We named this object in honour of Thomas Willwacher who predicted its existence [Wil19].}\]
Thus one may consider the following cochain complex in the derived category of vector spaces:

\[(48) \quad C^*_c(M_{g,1}, \mathbb{Q}) \xrightarrow{\nabla_1} \cdots \xrightarrow{\nabla_1} \left( C^*_c(M_{g,n}, \mathbb{Q}) \otimes_{\Sigma_n} \text{sgn}_n \right) \Sigma_n \xrightarrow{\nabla_1} \cdots \]

Now we can relate the cohomology of the hairy Getzler-Kapranov complex to the cohomology of the total DG-vector space of the complex above:

**Proposition 4.4.5.** For every \( g \geq 1 \) the hairy Getzler-Kapranov complex \( H_{g} \mathcal{GK}_{g} \) is a quasi-isomorphic to the total DG-vector space of \( \mathcal{GK}_{g} \).

**Proof.** We are going to use the Cousin resolution for the hairy Deligne-Mumford DG-sheaf \( \mathcal{D}M_{g, \geq 1}^{\text{hair}} \) (a filtration is defined by a number of non-internal edges (hairs) in the stable graph):

For every \( g \geq 1 \) consider the following sequence of \( H_{g}J_g^{\circ} \)-diagrams:

\[ H_{g} \mathcal{M}_{g}^{\text{trop}} \leftarrow \cdots \leftarrow H_{k} \mathcal{M}_{g}^{\text{trop}} \leftarrow \cdots \]

Here \( H_{k} \mathcal{M}_{g}^{\text{trop}} \) is an \( H_{g}J_g^{\circ} \)-diagram defined by the following rule. A stratified topological space \( h_{g} \sigma^{\text{hair}} \) carries the following filtration:

\[ h_{g} \sigma^{\text{hair}} \supset \cdots \supset h_{g,|m^{-1}(V(\Gamma))|}^{\text{hair}}, \]

where \( h_{g} \sigma^{\text{hair}} \) is a subspace where at least \( k \) "hair coordinates" are non-zero. Hence \( H_{k} \mathcal{M}_{g}^{\text{trop}} \) sends a stable graph \( \Gamma \) to \( h_{g} \sigma^{\text{hair}} \) if \( k \leq |m^{-1}(V(\Gamma))| \) and to the empty set otherwise. We have the associated sequence of morphisms of constructible DG-sheaves:

\[(49) \quad \mathcal{D}M_{g, \geq 1}^{\text{hair}} \longrightarrow \cdots \longrightarrow i_{g} \mathcal{D}M_{g, \geq 1}^{\text{hair}} \longrightarrow \cdots \]

Where \( i_{g} : H_{g} \mathcal{M}_{g}^{\text{trop}} \leftarrow H_{g} \mathcal{M}_{g}^{\text{trop}} \) is a closed inclusion. We have the standard Postnikov system associated with the sequence above. Hence we can realise the combinatorial DG-sheaf \( \mathcal{D}M_{g, \geq 1}^{\text{hair}} \) as the total object of the following complex of constructible DG-sheaves:

\[(50) \quad i_{g,1} \mathcal{D}M_{g, \geq 1}^{\text{hair}} \longrightarrow \cdots \longrightarrow i_{g,1} \mathcal{D}M_{g, \geq 1}^{\text{hair}}[k - 1] \longrightarrow \cdots \]

Where \( i_{g} : H_{g} \mathcal{M}_{g}^{\text{trop}} \leftarrow H_{g} \mathcal{M}_{g}^{\text{trop}} \) is the open inclusion and \( H_{g} \mathcal{M}_{g}^{\text{trop}} \) is the complement to \( H_{g+1} \mathcal{M}_{g}^{\text{trop}} \). Passing to the compactly supported cochains we get the following DG-vector space:

\[(51) \quad H_{1} \mathcal{GK}_{g} \xrightarrow{\nabla_1} \cdots \xrightarrow{\nabla_1} H_{k} \mathcal{GK}_{g} \xrightarrow{\nabla_1} \cdots \]

Where we have used the following notation:

\[(52) \quad H_{k} \mathcal{GK}_{g} := R\Gamma_{c}(H_{g} \mathcal{M}_{g}^{\text{trop}}, i_{g,1} \mathcal{D}M_{g, \geq 1}^{\text{hair}}[k - 1]) \]

Note that \( \mathcal{GK}_{g} \) is a decorated graph complex that consists of genus \( g \) decorated stable graphs with exactly \( k \) hairs and \( \nabla_1 \) is a differential defined by \( \nabla_1 := \nabla_{1,1} \)
Acting like in the proof of Proposition 3.3.2 for every $n$ we can explicitly realise the complex $H_n \mathbb{G}^\alpha$ as a total DG-vector space associated with double complex:

\[ C^* (M_{g,n}, \mathbb{Q}) \otimes \text{det}^h (\bullet, g, n) \rightarrow \ldots \]

(54) \[ \ldots \rightarrow \bigoplus_{\Gamma \in H J^*_n} \left( \bigotimes_{v \in V(\Gamma)} C^* (\mathcal{M}_{w(v), n_v}, \mathbb{Q}) \otimes \text{det}^h (\Gamma) \right) \overset{\text{Aut}^h (\Gamma)}{\rightarrow} \ldots , \]

The differential $\tilde{D}$ in the complex above is defined by (20). By Proposition 2.4.1 we obtain that the complex $H_n \mathbb{G}^\alpha$ is quasi-isomorphic to $C^c (M_{g,n}/\Sigma_n, \epsilon_n)$. In terms of P. Deligne’s complexes morphism (43) corresponds to pullbacks along natural inclusions of the corresponding strata. This follows from the base change for the compactly supported cochains. Hence the following diagram commutes:

\[ C^c (M_{g,n}/\Sigma_n, \epsilon_n) \rightarrow C^c (M_{g,n}/\Sigma_n, \epsilon_n) \]

By the definition of the Willwacher differential, the morphism $\tilde{\nabla}_1$ coincides with $\nabla_1$ and the desired result follows.

From Proposition 4.4.5 and Theorem 3.4.2 we immediately get:

**Corollary 4.4.6.** For any $g, n$ such that $2g + n - 2 > 0$ the induced Willwacher differential:

\[ \nabla_1 : (H^*_c (M_{g,n}, \mathbb{Q}) \otimes \Sigma_n \text{sgn}_n)^{\Sigma_n} \rightarrow (H^*_c (M_{g,n+1}, \mathbb{Q}) \otimes \Sigma_{n+1} \text{sgn}_{n+1})^{\Sigma_{n+1}} \]

preserves the canonical weight quotients.

Our main result in this section is the following theorem which was originally conjectured in [AWŽ20] (Conjecture 31):

**Theorem 4.4.7.** Consider the following complex:

(55) \[ C^* (M_{g,1}, \mathbb{Q}) \rightarrow \ldots \rightarrow (C^* (M_{g,n}, \mathbb{Q}) \otimes \Sigma_n \text{sgn}_n) \rightarrow \ldots \]

(i) For $g \geq 2$ (55) is quasi-isomorphic to $C^c (M_g, \mathbb{Q})$.

(ii) For $g = 1$ the cohomology of (55) is given by:

\[ \prod_{n=3}^{\infty} (S_{n+1} \oplus S_{n+1} \oplus E i s_{n+1}) [2n] \oplus \mathbb{Q}[3] \]

\[ \text{The compactly supported cochains of the moduli stack } M_{g,n}/\Sigma_n \text{ can be realised by P. Deligne’s complex (we assume that this moduli stack is compactified by the stack } \overline{M}_{g,n}/\Sigma_n, \text{ with the corresponding complement } D^{rt}). \]
(iii) For \( g = 0 \) \([54]\) is acyclic.

**Proof.**

(i) For \( g \geq 2 \) the proof immediately follows from Proposition \([4,4.8]\) and Corollary \([4.3.3]\).

(ii) Consider a local system \( V \) on \( \mathcal{M}_{1,1} \) which is defined by the rule \( V := R^1\pi_1^*\mathbb{Q} \). By \( V_k \) we denote the \( k \)-symmetric power of this local system. Note that for every \( k \) the local system \( V_k \) underlies the certain VPHS of the weight \( k \), which will be denoted by the same symbol. Due to the Eichler-Shimura theory, the weight filtration \( W \) on the cohomology of the VPHS has the following description:

\[
W_0 H^1_*(\mathcal{M}_{1,1}, V_k) \subset W_{k+1} H^1_*(\mathcal{M}_{1,1}, V_k) := H^1_*(\mathcal{M}_{1,1}, V_k),
\]

with the following graded quotients:

\[
W_0 H^1_*(\mathcal{M}_{1,1}, V_k) \cong \text{Eis}_{k+2},
\]

\[
\text{Gr}_{k+1} H^1_*(\mathcal{M}_{1,1}, V_k) \cong S_{k+2} \oplus \overline{S}_{k+2}.
\]

Here \( S_{k+2} \) is the vector space of cusp forms of the weight \( k+2 \) and \( \overline{S}_{k+2} \) is the vector space of the antiholomorphic cusp forms of weight \( k+2 \) and \( \text{Eis}_{k+2} \) is the vector space of the Eisenstein series of weight \( k+2 \).

From Proposition 1 from \([CF06]\) (see also \([Ger13]\) and \([Pet12]\)) one can show that for any \( n > 1 \) there is an isomorphism of graded vector spaces:

\[
(\text{H}^\ast_*(\mathcal{M}_{0,n}, \mathbb{Q}) \otimes \text{sgn}_n)^{\Sigma_n} = H^1_*(\mathcal{M}_{0,1}, V_{n-1})[n]
\]

Further notice that the "fundamental cohomology class" \( \omega \in H^2_*(\mathcal{M}_{1,1}, \mathbb{Q}) \) goes to zero under the differential \( \nabla_1 \) (due to the lack of non-zero modular forms for odd weight). Hence the cohomology of \([55]\) is isomorphic to (cf. \([Tac15]\) \([Beh03]\)):

\[
\prod_{n=3}^{\infty} (S_{n+1} \oplus \overline{S}_{n+1} \oplus \text{Eis}_{n+1})[2n] \oplus \mathbb{Q}[3]
\]

(iii) Following \([Vas88]\) we have \((\text{H}^\ast_*(\mathcal{M}_{0,n}, \mathbb{Q}) \otimes \text{sgn}_n)^{\Sigma_n} = 0\), hence the result follows.

\[\square\]

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