The Laplace-Beltrami operator on surfaces with axial symmetry

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Abstract

A solution for the mathematical problem of functional calculus with Laplace-Beltrami operator on surfaces with axial symmetry is found. A quantitative analysis of the spectrum is presented.

1 Introduction

The physical situation which has initiated this research is that of a dielectric particle with electric charges on its surface, placed in electric field. Here, the diffusion equation of the charges is coupled with the Maxwell equations. There is an analytical solution of this system of equation which involves some functional calculus with operators, in particular with Laplace-Beltrami operator defined on the surface of that particle. We can imagine many other physical situations described by a complicated system of equations where the Laplace-Beltrami operator is implicated (e.g. that of the acoustic wave scattering on particles with membrane, etc.). As before, one can find a compact solution by using functional calculus. However, these solutions are not complete because, at this level, all is formal. We must have an effective procedure to calculate the expressions which involve operators. One can try
to compute the matrices of those operators in some orthonormal basis and to transform the problem in to algebraic one. The practical problem is that one can compute only a finite number of matrix elements and this can leads to serious problem when unbounded operators are implicated. If we choose an inappropriate basis, it is possible that the expressions, calculated with truncated matrices, to not converge at the correct result.

In this paper we will find an orthonormal basis in the space of square integrable functions defined on a surfaces with axial symmetry such that the truncated matrices of Laplace-Beltrami operator converge in the norm resolvent sense. Then, according to [3] we can use these truncated matrices in functional calculus.

2 The result

Let M be a $C^\infty$ closed 2-dimensional surface which in the spherical coordinates $\{r, \theta, \phi\}$ relative to a 3-orthogonal system of axes is defined by the equation $r = r(\theta)$. We consider that all necessary conditions to have a $C^\infty$ surface are fulfilled. Let this surface be equipped with the metric which is induced by the embedding in $\mathbb{R}^3$ and let $x_0 \in M$ be the point defined by $\theta = 0$. Relative to this point, the normal coordinates $\{\lambda, \varphi\}$ are defined by

$$\begin{cases}
\lambda(x(\theta, \phi)) = d(x(\theta, \phi), x_0) = \int_0^\theta \sqrt{r(t)^2 + r'(t)^2} dt \\
\varphi = \phi
\end{cases}$$

which parameterize the entire surface, without the points $\theta = 0, \pi$. We define $R = \lambda(x(\pi))/\pi$ and the new coordinates: $\{\vartheta = \lambda/R, \varphi\}$. In these coordinates, the metric form is

$$g(\vartheta, \varphi) = \begin{pmatrix} R^2 & 0 \\ 0 & r(\theta(\vartheta))^2 \sin(\theta(\vartheta))^2 \end{pmatrix}.$$

**Proposition 1** The set of $C^\infty$ functions:

$$\mathcal{Y}_{lm} : M \to \mathbb{C}, \mathcal{Y}_{lm}(\vartheta, \varphi) = \sqrt{\frac{R \sin \vartheta}{r(\theta(\vartheta)) \sin \theta(\vartheta)}} \frac{Y_{lm}(\vartheta, \varphi)}{R}, m \in \mathbb{Z}, l \geq |m|$$

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is an orthonormal basis in $L_2(M, \mu_g)$, where $Y_{lm}$ represent the spherical harmonics and $\mu_g$ is the measure induced on $M$ by the metric $g$.

**Proof.** The orthonormality:

$$\langle Y_{lm}, Y_{l'm'} \rangle = \int_0^\pi \int_0^{2\pi} \sqrt{\det g} \cdot Y_{lm}(\vartheta, \varphi) Y^*_{l'm'}(\vartheta, \varphi) =$$

$$\int_0^\pi \int_0^{2\pi} Rr(\theta) \sin \theta \cdot \frac{R \sin \vartheta Y_{lm}(\vartheta, \varphi) Y^*_{l'm'}(\vartheta, \varphi)}{r \sin \theta} =$$

$$\int_0^\pi \int_0^{2\pi} \sin \vartheta \cdot Y_{lm}(\vartheta, \varphi) Y^*_{l'm'}(\vartheta, \varphi) = \delta_{ll'} \delta_{mm'}$$

The completeness:

$$\int_0^\pi \int_0^{2\pi} \sqrt{\det g} \cdot \sum_{l,m} Y_{lm}(\vartheta, \varphi) Y^*_{l'm'}(\vartheta, \varphi') f(\vartheta', \varphi') =$$

$$\int_0^\pi \int_0^{2\pi} Rr(\theta) \sin \theta \cdot \sum_{l,m} \sqrt{\frac{R \sin \vartheta}{r(\theta) \sin \theta}} \sqrt{\frac{R \sin \vartheta'}{r(\theta') \sin \theta'}} \times$$

$$\frac{Y_{lm}(\vartheta, \varphi) Y^*_{l'm'}(\vartheta', \varphi')} {R} f(\vartheta', \varphi') = \sqrt{\frac{R \sin \vartheta}{r(\theta) \sin \theta}} \times$$

$$\int_0^\pi \int_0^{2\pi} \sum_{l,m} Y_{lm}(\vartheta, \varphi) Y^*_{l'm'}(\vartheta', \varphi') \sqrt{\frac{r(\theta') \sin \theta'}{R \sin (\theta')}} f(\vartheta', \varphi') = f(\vartheta, \varphi),$$

because $\sqrt{\frac{r(\theta') \sin \theta'}{R \sin (\theta')}} f(\vartheta', \varphi')$ is in $L_2(M, \mu_g)$ if $f \in L_2(M, \mu_g)$. □

For a fixed $m$, let $S_m$ be the Hilbert subspace spanned by $\{Y_{lm}\}_{l \geq |m|}$, which is invariance by the Laplace Beltrami operator. In the following, we
will consider the restriction of this operator at a $S_m$ subspace, $\Delta^{(m)} = \Delta |_{S_m}$.

Let $P_k^{(m)}$, $k \geq |m|$, be the projection on the subspace spanned by the vectors $Y_{|m|}, \ldots, Y_{km}$. Our main result is:

**Theorem 2** The sequence of operators

$$\left\{ P_k^{(m)} \left[ P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - z \right]^{-1} \right\}_{k \geq |m|}$$

converges in norm topology at the operator $[\Delta^{(m)} - z]^{-1}$, for any $z \in \mathbb{C}$ with $\text{Im} \, z \neq 0$.

**Proof.** We have successively:

$$P_k^{(m)} \frac{1}{P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - z} - \frac{1}{\Delta^{(m)} - z} =$$

$$P_k^{(m)} \left[ \frac{1}{P_k^{(m)} \Delta^{(m)} P_k^{(m)} - z} - \frac{1}{\Delta^{(m)} - z} \right] - \left( I - P_k^{(m)} \right) \frac{1}{\Delta^{(m)} - z} =$$

$$\frac{1}{P_k^{(m)} \Delta^{(m)} P_k^{(m)} - z} P_k^{(m)} \Delta^{(m)} \left( I - P_k^{(m)} \right) \frac{1}{\Delta^{(m)} - z} - \left( I - P_k^{(m)} \right) \frac{1}{\Delta^{(m)} - z} =$$

$$\frac{z}{P_k^{(m)} \Delta^{(m)} P_k^{(m)} - z} \left[ I + \frac{1}{z} P_k^{(m)} \Delta^{(m)} \left( I - P_k^{(m)} \right) \right] \left( I - P_k^{(m)} \right) \frac{1}{\Delta^{(m)} - z}.$$

Without loss of generality we can choose $z = i\omega$, $\omega \in \mathbb{R}$, $\omega \neq 0$. Thus:

$$\left\| P_k^{(m)} \frac{1}{P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - z} - \frac{1}{\Delta^{(m)} - z} \right\| \leq$$

$$\left( 1 + \frac{1}{\omega} \left\| P_k^{(m)} \Delta^{(m)} \left( I - P_k^{(m)} \right) \right\| \right) \cdot \left\| (I - P_k^{(m)}) \frac{1}{\Delta^{(m)} - i\omega} \right\|.$$
Lemma 3 For $l, k \in \mathbb{N}$, $l, k \geq |m|$, with $l \neq k$:

$$\langle Y_{lm}, \Delta^{(m)} Y_{km} \rangle = \langle Y_{lm}, h \cdot Y_{km} \rangle,$$

where $h$ is at least a $C^0$ function on $M$. The operators $P_k^{(m)} \circ \Delta^{(m)} \circ (I - P_k^{(m)})$ are bounded and their norms satisfy:

$$\left\| P_k^{(m)} \circ \Delta^{(m)} \circ (I - P_k^{(m)}) \right\| \leq \|h\|_\infty.$$

It follows that

$$\left\| \frac{1}{P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - z} - \frac{1}{\Delta^{(m)} - z} \right\| \leq \left(1 + \frac{\|h\|_\infty}{\omega}\right) \left\| (I - P_k^{(m)}) \frac{1}{\Delta^{(m)} - i\omega} \right\|.$$

To evaluate the last norm, we use the following

Lemma 4 Let $s(\vartheta)$ be the quantity $\sqrt{\frac{r \sin \theta}{R \sin \vartheta}}$. If $\lambda^{(m)}_{|m|}, \ldots, \lambda^{(m)}_n, \ldots$ are the ordered eigenvalues of $\Delta^{(m)}$ and $v^{(m)}_{|m|}, \ldots, v^{(m)}_n, \ldots$ are the corresponding eigenvectors, then for any $m \in \mathbb{Z}$ and $l \geq |m|$, $l \geq 1$ and $n \geq |m|$

$$\left| \langle Y_{lm} | v^{(m)}_n \rangle \right| \leq Rc \left[ \|ds\|_\infty + c \sqrt{\lambda^{(m)}_n} \right] \sqrt{l(l+1)},$$

and for any $m \in \mathbb{Z}$, $l \geq |m|$, and $n \geq |m|$, $n \geq 1$

$$\left| \langle Y_{lm} | v^{(m)}_n \rangle \right| \leq c \left[ \|ds^{-1}\|_\infty + \frac{c}{R} \sqrt{l(l+1)} \right] \sqrt{\lambda^{(m)}_n}.$$

Moreover:

$$\frac{1}{c^2} \frac{l(l+1)}{R^2} \leq \lambda^{(m)}_l \leq c^2 \frac{l(l+1)}{R^2},$$

where $c = \max \{\|s^2\|_\infty, \|s^{-2}\|_\infty\}^{1/2}$. 

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Now, let $v \in L^2(M, \mu_g)$, $v = \sum_{n \geq |m|} a_n \cdot \nu_n^{(m)}$. Then

$$\left\| \left( I - P_k^{(m)} \right) \frac{1}{\Delta_n - i\omega} v \right\|^2 = \left\| \sum_{l \geq k+1} \sum_{n \geq |m|} a_n \cdot \frac{\langle \mathcal{Y}_n^{(m)} \rangle}{\lambda_n^{(m)} - i\omega} \right\|^2 =$$

$$\sum_{l \geq k+1} \sum_{n \geq |m|} a_n^2 \cdot \left\| \frac{\langle \mathcal{Y}_n^{(m)} \rangle}{\lambda_n^{(m)} - i\omega} \right\|^2 \leq$$

$$\sum_{l \geq k+1} \sum_{n \geq |m|} \left\| \frac{\langle \mathcal{Y}_n^{(m)} \rangle}{\lambda_n^{(m)} - i\omega} \right\|^2 \leq$$

$$\|v\|^2 \sum_{l \geq k+1} \sum_{n \geq |m|} \frac{cR \left[ \|d_s\|_\infty + c\sqrt{\lambda_n^{(m)}} \right]^2}{\sqrt{l(l+1)} \left( \lambda_n^{(m)} - i\omega \right)} =$$

$$(cR)^2 \frac{\|v\|^2}{k+1} \sum_{n \geq |m|} \left\| \frac{\langle \mathcal{Y}_n^{(m)} \rangle}{\lambda_n^{(m)} - i\omega} \right\|^2 .$$

For $|m| > 0$,

$$\left\| \left( I - P_k^{(m)} \right) \frac{1}{\Delta_n - i\omega} v \right\|^2 \leq$$

$$(cR)^2 \frac{\|v\|^2}{k+1} \sum_{n \geq |m|} \frac{\left[ c + \frac{\|d_s\|_\infty}{\sqrt{\lambda_n^{(m)}}} \right]^2}{\lambda_n^{(m)} - 1 + \frac{\omega^2}{\lambda_n^{(m)}}} \leq$$

$$(cR)^2 \frac{\|v\|^2}{k+1} \left[ c + \frac{\|d_s\|_\infty}{\sqrt{\lambda_n^{(m)}}} \right]^2 \sum_{n \geq |m|} \frac{1}{\lambda_n^{(m)}} .$$

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Finally
\[
\left\| \left( I - P_k^{(m)} \right) \frac{1}{\Delta^{(m)} - i\omega} \right\| \leq \frac{(cR)^2}{\sqrt{(k + 1)} |m|} \left[ c + \frac{cR \|ds\|_{\infty}}{\sqrt{|m|(|m| + 1)}} \right].
\]

For \( |m| = 0, \)
\[
\left\| \left( I - P_k^{(0)} \right) \frac{1}{\Delta^{(0)} - i\omega} v \right\|^2 \leq \frac{(cR)^2 \|ds\|_{\infty}^2}{\omega^2} + \sum_{n \geq 1} (cR)^2 \frac{1}{\chi_n^{(0)}} \left[ c + \frac{\|ds\|_{\infty}}{\sqrt{\chi_n^{(0)}}} \right]^2 \leq (cR)^2 \frac{k + 1}{k + 1} \left( \frac{\|ds\|_{\infty}^2}{\omega^2} + (cR)^2 \left[ c + \frac{cR \|ds\|_{\infty}}{\sqrt{2}} \right]^2 \right)
\]

thus:
\[
\left\| \left( I - P_k^{(0)} \right) \frac{1}{\Delta^{(0)} - i\omega} \right\| \leq \frac{(cR)^2}{\sqrt{(k + 1)}} \sqrt{\|ds\|_{\infty}^2 + \left[ c + \frac{cR \|ds\|_{\infty}}{\sqrt{2}} \right]^2}.
\]

Having that \( \|ds\|_{\infty} = \frac{1}{R} \frac{\|s'\|_{\infty}}{\omega} = \frac{1}{R} \|s'\|_{\infty} \) we can conclude:
\[
\left\| P_k^{(m)} P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - i\omega \right\| \leq (1 + \frac{\|h\|_{\infty}}{\omega}) \frac{(cR)^2}{\sqrt{(k + 1)}} \sqrt{\frac{1}{R} \frac{\|s'\|_{\infty}}{\omega}^2 + c^2 \left[ 1 + \frac{\|s'\|_{\infty}}{\sqrt{2}} \right]^2}, \text{ for } m = 0
\]
\[
\left\{ \begin{array}{l}
(1 + \frac{\|h\|_{\infty}}{\omega}) \frac{c^3 R^2}{\sqrt{(k + 1)} |m|} \left[ 1 + \frac{\|s'\|_{\infty}}{\sqrt{|m|(|m| + 1)}} \right], \text{ for } |m| \geq 1
\end{array} \right.
\]

Proof of Lemma 3. We have successively
\[
\langle Y_{lm}, \Delta^{(m)} Y_{km} \rangle = \langle dY_{lm}, dY_{km} \rangle =
\]
\[
\begin{align*}
\int_0^{2\pi} d\varphi \int_0^{2\pi} d\vartheta R^2 \sin \vartheta s^2 \left[ \frac{1}{R^2} \frac{\partial}{\partial \vartheta} \left( \frac{Y_{lm}^*}{s R} \right) \frac{\partial}{\partial \vartheta} \left( \frac{Y_{km}}{s R} \right) + \frac{m^2 s^{-6}}{R^2 \sin \vartheta} \frac{Y_{lm}^* Y_{km}}{R} \right] = \\
\int_0^{2\pi} d\varphi \int_0^{2\pi} d\vartheta \frac{\sin \vartheta}{R^2} \left[ \frac{\partial}{\partial \vartheta} Y_{lm}^* \frac{\partial}{\partial \vartheta} Y_{km} - \frac{\partial \ln s}{\partial \vartheta} \frac{\partial}{\partial \vartheta} (Y_{lm}^* Y_{km}) + \left( \frac{\partial \ln s}{\partial \vartheta} \right)^2 Y_{lm}^* Y_{km} \right] + \\
\frac{1}{R^2} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\vartheta \frac{m^2 s^{-4}}{R^2 \sin^2 \vartheta} Y_{lm}^* Y_{km} - \\
\frac{1}{R^2} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\vartheta \frac{\sin \vartheta}{R^2} \left[ \frac{\partial}{\partial \vartheta} Y_{lm}^* \frac{\partial}{\partial \vartheta} Y_{km} + \frac{m^2 s^{-4}}{R^2 \sin^2 \vartheta} Y_{lm}^* Y_{km} \right] + \\
\frac{1}{R^2} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\vartheta \frac{\sin \vartheta}{R^2} \left[ \left( \frac{\partial \ln s}{\partial \vartheta} \right)^2 - \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial \ln s}{\partial \vartheta} \right) \right] Y_{lm}^* Y_{km} = \\
\int_0^{2\pi} d\varphi \int_0^{2\pi} d\vartheta \frac{\sin \vartheta}{R^2} \left[ m^2 \frac{s^{-4} - 1}{R^2 \sin^2 \vartheta} + \left( \frac{\partial \ln s}{\partial \vartheta} \right)^2 - \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial \ln s}{\partial \vartheta} \right) \right] Y_{lm}^* Y_{km}.
\end{align*}
\]

It is easy to check that \( s(\vartheta) \) is at least of \( C^2 \) class, so the function \( h : [0, \pi] \to \mathbb{R} \),

\[
h(\vartheta) = m^2 \frac{s^{-4} - 1}{R^2 \sin^2 \vartheta} + \left( \frac{\partial \ln s}{\partial \vartheta} \right)^2 - \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial \ln s}{\partial \vartheta} \right)
\]

is at least of \( C^0 \) class. Finally

\[
\langle \mathcal{Y}_{lm}, \Delta^{(m)} \mathcal{Y}_{km} \rangle = \frac{1}{R^2} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\vartheta \sin \vartheta \cdot h(\vartheta) \cdot Y_{lm}^* Y_{km} = \\
\int_0^{2\pi} d\varphi \int_0^{2\pi} d\vartheta \sin(\vartheta) s^2 \cdot h(\vartheta) \cdot \frac{Y_{lm}^* Y_{km}}{R s} \frac{R s}{R s} = \langle \mathcal{Y}_{lm}, h \cdot \mathcal{Y}_{km} \rangle.
\]

For the second part

\[
\left| \left\langle v \left| P_k^{(m)} \circ \Delta^{(m)} \circ (I - P_k^{(m)}) \right| u \right\rangle \right| = \left| \left\langle P_k^{(m)} v \left| \Delta^{(m)} \right| (I - P_k^{(m)}) u \right\rangle \right| =
\]

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\[
\left\langle P_k^{(m)} v, h \cdot \left( I - P_k^{(m)} \right) u \right\rangle \leq \left\| P_k^{(m)} v \right\| \cdot \left\| h \cdot \left( I - P_k^{(m)} \right) u \right\| \leq \| h \|_\infty \| v \| \cdot \| u \|
\]

Proof of Lemma 4.

**Proposition 5** The application \( \tilde{g} : (0, \pi) \times [0, 2\pi] \rightarrow M (2 \times 2) \)

\[
\tilde{g}(\vartheta, \varphi) = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \sin(\vartheta)^2 \end{pmatrix},
\]

defines a metric on \( M \). Moreover, \( \sqrt{\frac{\det g}{\det \tilde{g}}} = s^2 \).

If we consider the spaces of the squared integrable functions with the measures induced by the two metrics, \( L_2(M, \mu_g) \) and \( L_2(M, \mu_{\tilde{g}}) \), and the spaces of one-differential forms with the standard scalar products, \( A^{(1)}(M, \mu_g) \) and \( A^{(1)}(M, \mu_{\tilde{g}}) \), then:

**Proposition 6** The spaces \( L_2(M, \mu_g) \) and \( L_2(M, \mu_{\tilde{g}}) \) coincide, \( A^{(1)}(M, \mu_g) \) and \( A^{(1)}(M, \mu_{\tilde{g}}) \) coincide too.

**Proof.** For \( f \in L_2(M, \mu_g) \) we have:

\[
\left\| f \right\|_{\tilde{g}} = \int_0^\pi \int_0^{2\pi} \vartheta \varphi \sqrt{\det \tilde{g}(\vartheta)} |f(\vartheta, \varphi)|^2 \leq \left\| \frac{\sqrt{\det \tilde{g}(\vartheta)}}{\sqrt{\det g(\vartheta)}} \right\|_\infty \cdot \left\| f \right\|^2 \leq \infty,
\]

thus: \( f \in L_2(M, \mu_{\tilde{g}}) \). Analogous, for \( f \in L_2(M, \mu_{\tilde{g}}) \) results:

\[
\left\| f \right\|_{g} = \int_0^\pi \int_0^{2\pi} \vartheta \varphi \sqrt{\det g(\vartheta)} |f(\vartheta, \varphi)|^2 \leq \left\| \frac{\sqrt{\det g(\vartheta)}}{\sqrt{\det \tilde{g}(\vartheta)}} \right\|_\infty \cdot \left\| f \right\|^2 \leq \infty,
\]

thus \( f \in L_2(M, \mu_g) \).
Let $\omega \in A^{(1)}(\mathbb{M}, \mu_g)$, $\omega = \omega_\theta d\theta + \omega_\varphi d\varphi$. We will follow
\[
\int_0^\pi d\theta \int_0^{2\pi} d\varphi \sqrt{\operatorname{det} \tilde{g}(\varphi)} g(\varphi, \omega) = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sqrt{\operatorname{det} \tilde{g}(\varphi)} \left[ |\omega_\varphi|^2 + \frac{|\omega_\varphi|^2}{\operatorname{det} \tilde{g}} \right] \leq 
\]
\[
\leq \max \left\{ \sqrt{\frac{\operatorname{det} \tilde{g}}{\operatorname{det} g}}, \sqrt{\frac{\operatorname{det} g}{\operatorname{det} \tilde{g}}} \right\} \cdot \|\omega\|^2_g \leq \infty,
\]
thus $\omega \in A^{(1)}(\mathbb{M}, \mu_g)$. The same steps can be followed to show that $\omega \in A^{(1)}(\mathbb{M}, \tilde{\mu}_\tilde{g}) \Rightarrow \omega \in A^{(1)}(\mathbb{M}, \mu_g)$. Denoting
\[
c = \sqrt{\max \{\|s^2\|_\infty, \|s^{-2}\|_\infty\}},
\]
we have on $L_2(\mathbb{M}, \mu_g) \equiv L_2(\mathbb{M}, \tilde{\mu}_\tilde{g})$:
\[
\frac{1}{c} \|\tilde{g}\| \leq \|g\| \leq c \|\tilde{g}\|,
\]
and, on $A^{(1)}(\mathbb{M}, \mu_g) \equiv A^{(1)}(\mathbb{M}, \mu_g)$:
\[
\frac{1}{c} \|\tilde{g}\| \leq \|g\| \leq c \|\tilde{g}\|.
\]

Now, we have successively
\[
\left\langle \mathcal{Y}_{lm} | v_n^{(m)} \right\rangle_g = \left\langle \mathcal{Y}_{lm} \big| sR v_n^{(m)} \right\rangle_g = \left\langle \mathcal{Y}_{lm} \big| s^{-1} \cdot v_n^{(m)} \right\rangle_g = \left\langle \mathcal{Y}_{lm} \big| s^{-1} \cdot v_n^{(m)} \right\rangle_{\tilde{g}}.
\]
\[
\frac{l(l+1)}{R^2} \leq \frac{l(l+1)}{R^2} \leq \frac{l(l+1)}{R^2} \leq \frac{cR}{\sqrt{l(l+1)}} \left[ \|ds\|_\infty \|v_n^{(m)}\|_{\tilde{g}} + \|s\|_\infty \|dv_n^{(m)}\|_{\tilde{g}} \right] \leq \frac{cR}{\sqrt{l(l+1)}} \left[ \|ds\|_\infty + c \sqrt{\lambda_n} \right].
\]

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For the second set of inequalities:

\[
\left| \langle Y_{lm} | v_n^{(m)} \rangle \right| = \frac{1}{\lambda_n^{(m)}} \left| \langle Y_{lm} | \Delta v_n^{(m)} \rangle \right| \leq \frac{1}{\lambda_n^{(m)}} \| dY_{lm} \|_g \| dv_n^{(m)} \|_g \leq
\]

\[
c \frac{\| dY_{lm} \|_g}{\sqrt{\lambda_n^{(m)}}} \leq c \left[ \| ds^{-1} \|_\infty + \frac{\sigma}{R} \sqrt{l(l+1)} \right].
\]

For the last set of inequalities of lemma 4, once we have the results of the last proposition we can follow the way of \[2\], or that presented in \[3\].

### 3 Numerical application

In general, the spectrum of the truncated matrices does not converge at the exact spectrum. Without additional results, one knows that only the lowest eigenvalue of the truncated matrices converges at the exact value. About these facts, one can consult \[4\]. The results of the last section have another important consequence: in the proposed basis, the spectrum of the truncated matrices converges at the exact spectrum. Moreover, because the matrix of the Laplace-Beltrami operator in the \(Y\) basis is “cvasidiagonal” in the sense that all nondiagonal elements are bounded by \(\| h \|_\infty\) and the diagonal elements increase approximatively as \(l(l+1)/R^2\), it is to be expected that the spectrum of these truncated matrices to be very stable. That means, that even for low dimensions these matrices give us a good approximation of the exact spectrum. Let us choose the following particular surfaces for our numerical application: \(r(\theta) = 1 + 1.2 \cos(\theta) + 3 \cos^2(\theta)\), presented in figure 1. The eigenvalues for different truncated matrices and \(m = 0\) are presented in figures 2-6.

Now, let us choose an orthonormal basis for which the affirmation of Lemma 4 is not true. If \(d\mu_g(\theta, \phi) = \sigma(\theta, \phi) \sin \theta \cdot d\theta d\phi\) is the measure induced by the metric \(g\) in the coordinates \(\{\theta, \phi\}\), then:

**Proposition 7** The set \(\tilde{Y}\) of functions:

\[
\tilde{Y}_{lm}(\theta, \phi) = \frac{Y_{lm}(\theta, \phi)}{\sqrt{\sigma(\theta, \phi)}}, m = 0, 1, ..., l = |m|, |m| + 1, ...
\]
is an orthonormal basis in $L_2(M, \mu)$. 

The proof of this proposition is analogous with that of Proposition 1. The eigenvalues of different truncated matrices, calculated in this basis and for the case $m = 0$, are presented in figures 7-11. The numerical application shows that in this case the spectrum of the truncated matrices are very unstable. This instability can be considered as an indicator of the fact that for the $\tilde{\mathcal{Y}}$ basis the affirmation of our theorem is not true.

4 Conclusion

This paper has shown how to construct an orthonormal basis in the space of square integrable functions defined on a $C^\infty$ surfaces with axial symmetry, basis which is appropriate for the problems which involve the Laplace-Beltrami operator. The procedure is standard, in the sense that it can be applied following the same steps for any $C^\infty$ surface with axial symmetry. The stability of the truncated matrices spectrum was theoretically anticipated and numerically verified. By practical point of view, this allow us to use truncated matrices with small number of rows and columns.

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Fig. 1 The particular surface chosen for our numerical application: \( r (\theta) = 1 + 1.2 \cos (\theta) + 3 \cos (\theta)^2 \).

Fig. 2 The eigenvalues of the 15\( \times \)15 truncated matrix of Laplace-Beltrami operator in \( Y \) orthonormal basis.

Fig. 3 The eigenvalues of the 20\( \times \)20 truncated matrix of Laplace-Beltrami operator in \( Y \) orthonormal basis.

Fig. 4 The eigenvalues of the 25\( \times \)25 truncated matrix of Laplace-Beltrami operator in \( Y \) orthonormal basis.

Fig. 5 The eigenvalues of the 30\( \times \)30 truncated matrix of Laplace-Beltrami operator in \( Y \) orthonormal basis.

Fig. 6 The superposition of figures 2-5.

Fig. 7 The eigenvalues of the 15\( \times \)15 truncated matrix of Laplace-Beltrami operator in \( \tilde{Y} \) orthonormal basis.

Fig. 8 The eigenvalues of the 20\( \times \)20 truncated matrix of Laplace-Beltrami operator in \( \tilde{Y} \) orthonormal basis.

Fig. 9 The eigenvalues of the 25\( \times \)25 truncated matrix of Laplace-Beltrami operator in \( \tilde{Y} \) orthonormal basis.

Fig. 10 The eigenvalues of the 30\( \times \)30 truncated matrix of Laplace-Beltrami operator in \( \tilde{Y} \) orthonormal basis.

Fig. 11 The superposition of figures 7-10.

Note: the horizontal coordinate in figures 2-11 is just an ordering index which puts the eigenvalues of the truncated matrices in an increasing order.
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