Anti-rotating terms in resonance optics and Lindbladian operators for relaxation

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Abstract. The Lindbladian master equation has been derived for atom and photons of cavity within the theory of open quantum systems based on the effective Hamiltonian and stochastic differential equations. The result obtained differs from the phenomenological one with the emergence of a new additional relaxation operator describing interference processes.

1. Introduction
Lindblad [1] has derived a general form of relaxation operator for quantum open systems in Markovian approximation and many authors are using it without any appropriate derivation in their investigations. Such phenomenological approaches for studying the relaxation influence on various physical processes in open quantum systems are widespread. Nevertheless, in spite of its general view, the allowance for the Lindbladian relaxation operator obviously implies certain approximations of other terms in the initial Hamiltonian for the problem [2]. Therefore, its allowance with no appropriate approximations is not quite correct, which can lead to erroneous results. This is particularly evident in optical problems where various resonant conditions are satisfied. First of all, to obtain a Lindbladian relaxation operator in optical problems, it is necessary to pass from an initial Hamiltonian to the effective Hamiltonian by means of the unitary transformation of the Hamiltonian of the whole system. Optical problems have a natural small parameter – the ratio of interaction energy to quantum energy and/or energy of quantum transitions. Hence, to obtain a Lindblad operator it is reasonable to use the algebraic perturbation theory based on such unitary transformations and Baker-Hausdorff formula [2]. The principle of term separation of the effective Hamiltonian in resonance optics states that rapidly changing in time terms should be missing in the Dirac picture (interaction representation). This principle was first clearly formulated in works [3, 4], though the similar perturbation theory had been mentioned in the book by Heitler [5], and in the resonance optics some elements of the effective Hamiltonian method were found in work [6]. In work [7], the theory of two-photon and Raman optical resonances is built on the method of the effective Hamiltonian. Work [8] has shown that the principle of absence of rapidly changing in time terms in the effective Hamiltonian provides the natural presentation of any broadband environment of an open system, e.g. a quantized vacuum electromagnetic field in the form of independent quantum noise sources, each of which defines its own Lindbladian relaxation operators of an open system.

Let us demonstrate the statement mentioned above for an open system described in works [9,10]. Let us consider the interaction of a two-level atom with the mode of high-Q cavity. As in [9, 10], the
system loss is supposed to be missing, so both photon and atomic subsystems have only dephasing described by the Lindblad operators. The master equation for the density matrix $\rho^S$ of an open system in Markovian approximation takes the form

$$\frac{d\rho^S}{dt} = \frac{i}{\hbar}[\rho^S, H^{\text{Eff-S}}] - \sum_n \Gamma_n \rho^S, \quad \hat{\Gamma}_n \rho^S = -\frac{i}{\hbar}[\rho^S, H^{\text{Shift-S}}_n] + \frac{1}{2} \left( \hat{L}_n^+ \hat{L}_n \rho^S + \frac{1}{2} \rho^S \hat{L}_n^+ \hat{L}_n - \hat{L}_n^+ \rho^S \hat{L}_n \right).$$

Here, the effective Hamiltonian of an open system $H^{\text{Eff-S}}$ is separated from the relaxation operators $\hat{\Gamma}_n$ defined by Lindblad operators $H^{\text{Shift-S}}_n$ and $L_n$ [2]. However, works [9,10] used the initial Hamiltonian $V_{\text{int}}$ of photon-atom interaction (instead of the effective Hamiltonian $H^{\text{Eff-S}}$) and Lindbladian dephasing operators $\hat{\Gamma}_c$ and $\hat{\Gamma}_a$ for photon and atomic subsystems. Also, the works [9,10] discussed the allowance for anti-rotating terms of $V_{\text{int}}$. I would like to emphasize that not only the atom-photon interaction operator $V_{\text{int}}$ has anti-rotating terms, but the interaction operators of subsystems with an environment of an open system also have anti-rotating terms in the initial correct problem formulation. Later in the paper I will prove that the allowance for other anti-rotating terms according to the theory [2] provides additional Lindbladian relaxation operators which have not been taken into account in case of the phenomenological approaches as in works [9,10].

2. The simple model for dephasing processes

Let us suppose that there is an atom with two energy levels $|1>\,$ and $|2>$ with the transition frequency $\omega_0$ between them in the one-mode high-Q cavity with the mode frequency $\omega_c$. To take into account atom and photon dephasing, it is necessary to consider some interaction model of cavity photons and atom with a certain broadband source. The same Lindbladian relaxation operator can be of different origin, but in spite of this the following statement is true - if the Lindblad operator has been accounted correctly, then there occurs a transition from the initial Hamiltonian to the effective Hamiltonian [2]. This leads to the conclusion that one can’t take into account anti-rotating terms additionally since they have already been accounted in the effective Hamiltonian. Illustratively, let us take the following model of an open system and its environment, that gives rise to Lindbladian photon dephasing considered in works [9,10].

The objects of an open system will be described by means of the following operators:

1. Cavity photon annihilation operator $c$ obeying the algebra $cc^* - c^*c = 1, [N,c] = -c, [N,c^+] = c^+$, where $N = c^*c$ is the operator of the number of photons;
2. Operators $R_\pm$ and $R_3$ describing transitions between atomic levels obey commutation relations of SU(2) algebra: $[R_3, R_\pm] = \pm R_\pm, [R_+, R_-] = 2R_3, R_+ |1> \sim |2> = |R_+^> , R_- = \frac{1}{2}( |2> < 2| - |1> < 1|) .
3. Atom-photon interaction operator $V_{\text{int}} = g(c^+ + c)(R_+ + R_-)$ with the coupling constant $g$.

The environment is to be regarded as being composed of non-resonant atoms (“thermostat” atoms) populating the ground energy state, whose parameters have randomly been dispersed in some way. The Hamiltonian of environment atoms is to be designated as $H_b$ and the interaction operator of cavity photons with environment atoms as $V_b$, and

$$H_b = \sum_{b,j} E^b_j |E^b_j><E^b_j|, \quad V_b = (c^+ + c) \sum_{b,j} d^b_j |E^b_j><E^b_j|,$$

the upper index $b$ numbers environment atoms and atom parameters.

Thus, the initial Hamiltonian of an open system and environment is as follows:

$$H_{\text{total}} = H_a + H_c + H_b + V_{\text{int}} + V_b.$$
According to the general theory of open quantum systems [2], the initial Hamiltonian is subject to unitary transformation
\[
\tilde{H}_{\text{total}} = U H_{\text{total}} U^\dagger - i \hbar U \frac{d}{dt} U^\dagger = \sum_{k,m=0,1,2,...} \tilde{H}^{k,m}, \quad U = e^{-iS}, \quad S = \sum_{k,m=0,1,2,...} S^{k,m} - S^{(0,0)},
\]
where upper indices indicate the order of term of the transformed Hamiltonian at expansion in terms of coupling constants of cavity mode with a microcavity atom and environment atoms, respectively.

The unique condition for the term selection is the demand for the absence of rapidly changing terms in the interaction representation. If (argument \( t \) indicates the interaction representation):
\[
\tilde{H}^{(1,0)}(t) = \hbar dS^{(1,0)}(t)/dt + V_{\text{int}}(t), \quad \tilde{H}^{(0,1)}(t) = \hbar dS^{(0,1)}(t)/dt + V_b(t), \quad \tilde{H}^{(1,1)}(t) = \hbar dS^{(1,1)}(t)/dt - \frac{i}{2} [S^{(0,1)}(t), V_{\text{int}}(t)] - \frac{i}{2} [S^{(1,0)}(t), V_b(t)] - \frac{i}{2} [S^{(0,0)}(t), \tilde{H}^{(0,1)}(t)],
\]

then
\[
\tilde{H}^{(0,0)}(t) = H_a + H_c + H_b, \quad \tilde{H}^{(1,0)} = g(c^\dagger R_c + cR_c), \quad \tilde{H}^{(1,1)} = 0,
\]
and we have the following operators defining the unitary transformation in the interaction representation:
\[
S^{(0)}(t) = -i g(c R_c \frac{e^{-i(\omega_0 + \omega_c)t}}{\hbar(\omega_0 + \omega_c)} - c^\dagger R_c \frac{e^{-i(\omega_c - \omega_0)t}}{\hbar(\omega_0 + \omega_c)}),
\]
\[
S^{(1)}(t) = i c^\dagger \sum_{b,j} E_b^j <E_b^j | \frac{e^{i(\omega_0 + \omega_j)t} d_{bj}^b}{\hbar(\omega_0 + \omega_j)} - i c \sum_{b,j} E_b^j <E_b^j | \frac{e^{i(\omega_c - \omega_0)t} d_{bj}^b}{\hbar(\omega_c - \omega_0)}.
\]
It is important to stress that all these terms owe to their appearance to anti-rotating terms in all initial interaction operators. Here, \( \omega_0^b = (E_c^b - E_c^b)/\hbar \) for particulars see [3,4].

The effective Hamiltonian of an open system of zero- and first order over coupling constants does not have terms providing phase relaxation of the open-system.

In case of the second order we have terms as in \( H_a \) and \( H_c \) describing frequency shifts of Lamb type. The former ones are involved into renormalized frequencies \( \omega_0 \) and \( \omega_c \) and are not considered further in the paper. Also, we obtain \( \tilde{H}^{(1,1)} = 0 \). There are terms remaining which describe the so-called Stark level shift (alternatively, the Stark interaction [2-4]):
\[
\tilde{H}^{(2,0)} = \eta_S c^\dagger c R_c, \quad \tilde{H}^{(2,0)} = c^\dagger c \sum_{k,b} \Pi^k_b(\omega_k) | E_k^b > | E_k^b >,
\]
\[
\eta_S = \frac{g^2}{\hbar \omega_c}, \quad \Pi^k_b(\omega_k) = \sum_j \left| d_{kj}^b \right|^2 \frac{1}{\hbar} \left( \frac{1}{\omega_{kj}^b + \omega_k} + \frac{1}{\omega_k^b - \omega_k} \right).
\]
Parameter \( \eta_S \) describes the Stark shift for atom levels and makes allowance for \( \omega_c \approx \omega_0 \). In case of a two-level atom and classical field, this shift is known as the Bloch-Siegert shift [3,4].

### 3. The master equation

Now, to obtain relaxation operators, we need a random process model for \( \tilde{H}^{(0,2)} \). One can substitute the operator \( \sum_{k,b} \Pi^k_b(\omega_k) | E_k^b > | E_k^b > dt \) by the increment \( dW(t) \) or \( dQ(t) \) of classical or quantum random processes [2]. So, using the approach of Ref.[2] for classical and quantum random processes we have derived the following Lindblad operators

\[
\eta_S = \frac{g^2}{\hbar \omega_c}, \quad \Pi^k_b(\omega_k) = \sum_j \left| d_{kj}^b \right|^2 \frac{1}{\hbar} \left( \frac{1}{\omega_{kj}^b + \omega_k} + \frac{1}{\omega_k^b - \omega_k} \right).
\]
\[ L^S = \sqrt{2\gamma_c} c^+ c, \quad H_{c^{\text{shift-S}}} = 0, \]

differing only by the value of relaxation constant \( \gamma_c \).

It is to be noted that we considered above the term \( \widetilde{H}^{(2,0)} \) of the effective Hamiltonian of the second order over the coupling constant of cavity photons and atom. On the one hand, this has proved the allowance for anti-rotating terms in the initial operator of interaction of photons and atom. On the other hand, it is necessary to take into account other “combinational terms”. The additional terms making contribution into relaxation determined by the interaction of photons with “thermostat” atoms are as follows

\[
\widetilde{H}^{(1,2)}(t) = \hbar dS^{(1,2)}(t)/dt - \frac{i}{2}[S^{(1,1)}(t), V_b(t)] - \frac{i}{2}[S^{(0,2)}(t), V_{\text{int}}(t)] - \frac{i}{2}[S^{(1,0)}(t), [S^{(0,1)}(t), V_b(t)] - \\
- \frac{i}{2}[S^{(0,1)}(t), [S^{(1,0)}(t), V_b(t)]] - \frac{i}{12}[S^{(0,3)}(t), [S^{(0,1)}(t), V_{\text{int}}(t)]] + \frac{i}{12}[S^{(0,1)}(t), [S^{(0,1)}(t), \widetilde{H}^{(1,0)}(t)]]
\]

This term of the effective Hamiltonian defines an additional relaxation operator \( \Gamma_{ca}^{\text{shift-S}} \) with Lindblad operators

\[ L^S_{ca} = \sqrt{2\gamma_{ca}} (c^+ R_c + c R_c), \quad H_{ca}^{\text{shift-S}} = 0. \]

It is important to stress that in [9,10] the unaccounted relaxation parameter \( \gamma_{ca} \) is of the same order as accounted parameter \( \gamma_c \) in [9,10] in terms of the expansion in powers of coupling constant \( g \).

Similarly, consideration can be given to the model of forming phase relaxation of atomic subsystem of an open system and account can be taken of appropriate «combinational terms» in the effective Hamiltonian of an open system, which will give rise to an additional phase relaxation operator. However, the case of atomic relaxation from the viewpoint of invalid allowance for anti-rotating terms in the phenomenological approach has been considered by the author in other works [11,12], which indicated the impropriety of adiabatic following approximation of the work [13] where the relaxation operator was introduced phenomenologically.

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4. References

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