Late cosmology in massive conformal gravity

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Abstract
In this paper we find the cosmological solutions of the massive conformal gravity field equations in the presence of matter fields. In particular, we show that the solution of negative curvature is in good agreement with the late universe.

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Introduction

The massive conformal gravity (MCG) is a conformally invariant theory of gravity in which the gravitational action is the sum of the Weyl action with the Einstein-Hilbert action conformally coupled to a scalar field \([1]\). At the classical level, it has been shown that the theory is free of the vDVZ discontinuity \([2]\) and can reproduce the orbit of binaries by the emission of gravitational waves \([3]\). Furthermore, MCG is a renormalizable and unitary quantum theory of gravity \([4, 5, 6]\).

Despite the promising results of MCG obtained so far, it is very important to test the theory with cosmological observations. The most accepted cosmological model to explain the dynamics of the universe is the ΛCDM model. However, this model suffers from important problems such as the cosmological constant problem \([7, 8]\) and the Hubble tension between the early and late universe observational data \([9, 10]\). Here we want to see if MCG explains the late universe without the cosmological constant problem.

This paper is organized as follows. In Sec. 2 we describe the MCG cosmological field equations. In Sec. 3 we find the MCG cosmological solutions. In Sec. 4 we compare the cosmological solutions of MCG with cosmological observations. Finally, in Sec. 5 we present our conclusions.

2 Cosmological field equations

The total MCG action is given by\([1, 2]\)

\[
S = \int d^4x \sqrt{-g} \left[ \varphi^2 R + 6 \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2 \alpha^2} C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} \right] + \int d^4x \mathcal{L}_m, \tag{1}
\]

where \(\varphi\) is a scalar field called dilatonic, \(\alpha\) is a dimensionless constant,

\[
C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} = R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - 4 R^{\mu\nu} R_{\mu\nu} + R^2 + 2 \left( R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right) \tag{2}
\]

is the Weyl tensor squared, \(R^{\alpha\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\mu\nu} + \cdots\) is the Riemann tensor, \(R_{\mu\nu} = R^a_{\mu\alpha\nu} \) is the Ricci tensor, \(R = g^{\mu\nu} R_{\mu\nu}\) is the scalar curvature, and

\[1\]This action is obtained from the action of Ref. \([2]\) by rescaling \(\varphi \rightarrow \left(\sqrt{\frac{32}{3}} \pi G / \alpha \right) \varphi\) and considering \(m = \sqrt{3} / 64 \pi G \alpha\).
\( \mathcal{L}_m = \mathcal{L}_m(g_{\mu\nu}, \Psi) \) is the Lagrangian density of the matter field \( \Psi \). It is worth noting that the action (1) is invariant under the conformal transformations

\[
\tilde{g}_{\mu\nu} = e^{2\theta(x)} g_{\mu\nu}, \quad \tilde{\varphi} = e^{-\theta(x)} \varphi, \quad \tilde{\mathcal{L}}_m = \mathcal{L}_m,
\]

where \( \theta(x) \) is an arbitrary function of the spacetime coordinates.

The variation of (1) with respect to \( g_{\mu\nu} \) and \( \varphi \) gives the MCG field equations

\[
\varphi^2 G_{\mu\nu} + 6 \partial_\mu \varphi \partial_\nu \varphi - 3g_{\mu\nu} \partial^\rho \varphi \partial_\rho \varphi + g_{\mu\nu} \nabla^\rho \nabla_\rho \varphi^2 - \nabla_\mu \nabla_\nu \varphi^2 - \alpha^{-2} W_{\mu\nu} = \frac{1}{2} T_{\mu\nu},
\]

(4)

\[
\left( \nabla^\mu \nabla_\mu - \frac{1}{6} R \right) \varphi = 0,
\]

(5)

where

\[
W_{\mu\nu} = \nabla^\alpha \nabla^\beta C_{\mu\alpha\nu\beta} - \frac{1}{2} R^{\alpha\beta} C_{\mu\alpha\nu\beta}
\]

is the Bach tensor,

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R
\]

(7)

is the Einstein tensor,

\[
\nabla^\rho \nabla_\rho \varphi = \frac{1}{\sqrt{-g}} \partial^\rho \left( \sqrt{-g} \partial_\rho \varphi \right)
\]

(8)

is the generally covariant d’Alembertian for a scalar field, and

\[
T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}}
\]

(9)

is the matter energy-momentum tensor.

At scales below the Planck scale, the dilaton field acquires a spontaneously broken constant vacuum expectation value \( \varphi_0 \). In this case, the field equations (4) and (5) become

\[
\varphi_0^2 G_{\mu\nu} - \alpha^{-2} W_{\mu\nu} = \frac{1}{2} T_{\mu\nu},
\]

(10)

\[
R = 0.
\]

(11)

In addition, for \( \varphi = \varphi_0 \), the MCG line element \( ds^2 = (\varphi/\varphi_0)^2 g_{\mu\nu} dx^\mu dx^\nu \) reduces to

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu.
\]

(12)
The full cosmological content of MCG can be obtained from (10)-(12) without loss of generality.

In order to find the MCG matter energy-momentum tensor, we consider the conformally invariant matter Lagrangian density \[ \mathcal{L}_m = -\sqrt{-g} \left[ S^2 R + 6 \partial^\mu S \partial_\mu S + \lambda S^4 + \frac{i}{2} \left( \overline{\psi} \gamma^\mu D_\mu \psi - D_\mu \overline{\psi} \gamma^\mu \psi \right) + \mu S \overline{\psi} \psi \right], \]

where \( S \) is a scalar Higgs field, \( \lambda \) and \( \mu \) are dimensionless coupling constants, \( \overline{\psi} = \psi^\dagger \gamma^0 \) is the adjoint fermion field, \( D_\mu = \partial_\mu + [\gamma^\nu, \partial_\mu \gamma^\nu]/8 - [\gamma^\nu, \gamma^\lambda] \Gamma^\lambda_{\mu \nu}/8 \) (\( \Gamma^\lambda_{\mu \nu} \) is the Levi-Civita connection), and \( \gamma^\mu \) are the general relativistic Dirac matrices, which satisfy the anticommutation relation \( \{\gamma^\mu, \gamma^\nu\} = 2 g^\mu\nu \). The variation of (13) with respect to \( S \), \( \overline{\psi} \) and \( \psi \) gives the field equations

\[
\begin{align*}
(12 \nabla^\mu \nabla_\mu - 2 R) S - 4 \lambda S^3 - \mu \overline{\psi} \psi &= 0, \\
i \gamma^\mu D_\mu \psi + \mu S \psi &= 0, \\
i D_\mu \overline{\psi} \gamma^\mu - \mu S \overline{\psi} &= 0.
\end{align*}
\]

Substituting (13) into (9), and using (14)-(16), we obtain the energy-momentum tensor

\[
T_{\mu \nu} = 2 g_{\mu \nu} \nabla^\rho S \nabla_\rho S - 8 \nabla_\mu S \nabla_\nu S + 4 S \nabla_\mu \nabla_\nu S - g_{\mu \nu} \nabla^\rho \nabla_\rho S
+ 2 S^2 \left( R_{\mu \nu} - \frac{1}{4} g_{\mu \nu} R \right) + T^f_{\mu \nu},
\]

where

\[
T^f_{\mu \nu} = \frac{i}{4} \left( \overline{\psi} \gamma_\mu D_\nu \psi - D_\nu \overline{\psi} \gamma_\mu \psi + \overline{\psi} \gamma_\mu D_\nu \psi - D_\nu \overline{\psi} \gamma_\mu \psi \right) + \frac{1}{4} g_{\mu \nu} \mu S \overline{\psi} \psi
\]

is the fermion energy-momentum tensor.

Considering that, at scales below the electroweak scale, the Higgs field acquires a spontaneously broken constant vacuum expectation value \( S_0 \), and taking an incoherent average of \( T^f_{\mu \nu} \) over all the fermionic modes propagating in a Robertson-Walker background, we find that (17) becomes the energy-momentum tensor of a dynamical perfect fluid

\[
T_{\mu \nu} = 2 S_0^2 \left( R_{\mu \nu} - \frac{1}{4} g_{\mu \nu} R \right) + \left( \rho + \frac{p}{c^2} \right) u_\mu u_\nu + g_{\mu \nu} p - g_{\mu \nu} c^2 \rho \Lambda,
\]
where \( \rho \) is the mass density of the usual kinematic perfect fluid, \( p \) is the pressure of the kinematic perfect fluid, \( u^\mu \) is the four-velocity of the kinematic perfect fluid, which is normalized to \( u^\mu u_\mu = -c^2 \), and \( c^2 \rho_\Lambda \) is the vacuum energy (dark energy) density.

Taking the trace of (19) and substituting into the trace of (10), we find

\[
-\varphi_0^2 R = \frac{1}{2} \left( -c^2 \rho + 3p - 4c^2 \rho_\Lambda \right). \tag{20}
\]

The additional use of (11) then gives the relation

\[
c^2 \rho_\Lambda = \frac{1}{4} (3p - c^2 \rho). \tag{21}
\]

Substituting this relation back into (19), we obtain

\[
T_{\mu\nu} = 2S_0^2 \left( R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) + \left( \rho + \frac{p}{c^2} \right) u_\mu u_\nu + \frac{1}{4} g_{\mu\nu} \left( c^2 \rho + p \right). \tag{22}
\]

According to (22) the vacuum energy density does not contribute to the dynamics of the MCG universe. This makes the theory free from the cosmological constant problem.

### 3 Cosmological solutions

By substituting (22) into (10), and considering (11), we find

\[
2 \left( \varphi_0^2 - S_0^2 \right) R_{\mu\nu} - 2\alpha^{-2} W_{\mu\nu} = \left( \rho + \frac{p}{c^2} \right) u_\mu u_\nu + \frac{1}{4} g_{\mu\nu} \left( c^2 \rho + p \right). \tag{23}
\]

Then, using the Friedmann–Lemaître–Robertson–Walker (FLRW) line element

\[
d s^2 = -c^2 dt^2 + a(t)^2 \left( \frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \tag{24}
\]

and the fluid four-velocity \( u^\mu = (c, 0, 0, 0) \), the 00 and 11 components of (23) gives

\[
\frac{\ddot{a}}{a} = -\frac{8\pi G_{\text{eff}}}{3c^2} \left( c^2 \rho + p \right), \tag{25}
\]

\footnote{It is worth noting that \( \rho_\Lambda \) carries the fermion four-momentum and thus it evolves over time. This means that the relation (21) does not limit the matter content of the MCG universe as it would be in the case of a constant \( \rho_\Lambda \).}
\[
\frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{K}{a^2} = \frac{8\pi G \text{eff}}{3c^2} \left( c^2 \rho + p \right),
\]

where the dot denotes \(d/dt\), \(a = a(t)\) is the scale factor, \(K = -1, 0\) or \(1\) is the spatial curvature, and

\[
G_{\text{eff}} = \frac{3c^2}{64\pi (\varphi_0^2 - S_0^2)}
\]

is an effective gravitational constant.

Subtracting (25) from (26), we obtain

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{K}{a^2} = \frac{8\pi G \text{eff}}{3c^2} \left( c^2 \rho + p \right).
\]

The combination of this equation with (25) gives the energy continuity equation

\[
\frac{d}{dt} \left[ (c^2 \rho + p) a^4 \right] = 0,
\]

from which follows that

\[
c^2 \rho(t) + p(t) = (c^2 \rho_0 + p_0) \left( \frac{a_0}{a} \right)^4,
\]

where, from now on, the subscript \(0\) denotes values at the present time \(t_0\).

We can write (28) in the usual form

\[
\Omega + \Omega_K = 1,
\]

where

\[
\Omega = \frac{8\pi G \text{eff}}{3c^2 H^2} \left( c^2 \rho + p \right), \quad \Omega_K = -\frac{K}{a^2 H^2},
\]

are dimensionless density parameters, and

\[
H = \frac{\dot{a}}{a}
\]

is the Hubble constant. By using (30) in (31), we arrive at

\[
\left( \frac{\dot{a}}{a_0 H_0} \right)^2 = \Omega_0 \left( \frac{a_0}{a} \right)^2 + \Omega_{0K},
\]

where

\[
\Omega_0 = \frac{8\pi G \text{eff}}{3c^2 H_0^2} \left( c^2 \rho_0 + p_0 \right), \quad \Omega_{0K} = -\frac{K}{a_0^2 H_0^2}.
\]
The combination of (31) and (34) gives

$$dt = \frac{dx}{H_0(1 - \Omega_0 + \Omega_0 x^{-2})^{1/2}},$$

(36)

where $x = a/a_0$. It follows from (36) that the time at which light emitted from a cosmological source reaches the earth with redshift $z$ is given by

$$t = \frac{1}{H_0} \int_0^{1/(1+z)} \frac{dx}{(1 - \Omega_0 + \Omega_0 x^{-2})^{1/2}},$$

(37)

where we considered that the zero of time corresponds to an infinite redshift, and

$$1 + z = \frac{a_0}{a}.$$  

(38)

By considering the redshift equal to zero in (37), we find the present age of the MCG universe

$$t_0 = \left(\frac{1 - \sqrt{\Omega_0}}{1 - \Omega_0}\right) \frac{1}{H_0}.$$  

(39)

We can see from this equation that

$$0 < t_0 < \frac{1}{2H_0}$$  

(40)

for a closed universe $(\Omega_0 > 1)$,

$$t_0 = \frac{1}{2H_0}$$  

(41)

for a flat universe $(\Omega_0 = 1)$, and

$$\frac{1}{2H_0} < t_0 < \frac{1}{H_0}$$  

(42)

for an open universe $(\Omega_0 < 1)$, which means that only the open MCG universe is consistent with the available observational values of $H_0$ and $t_0$.

Assuming that the MCG universe is open $(K = -1)$, we can write (34) in the form

$$\dot{a}^2 = \frac{a_0^3 H_0^2 \Omega_0}{a^2} + 1,$$

(43)
whose solution is given by

\[ a(t) = \left[ \left( 2a_0^2 H_0 \sqrt{\Omega_0} \right) t + t^2 \right]^{1/2}. \]  

(44)

Substituting (44) into the deceleration parameter

\[ q = -\frac{\ddot{a}a}{a^2}, \]

(45)

we find

\[ q(t) = \frac{a_0^4 H_0^2 \Omega_0}{(a_0^2 H_0 \sqrt{\Omega_0} + t)^2}. \]

(46)

According to (44) and (46) the MCG universe begins with a big bang at \( t = 0 \) and continues to expand decelerated forever, becoming flat as \( t \to \infty \). Despite this being the same behavior of a pure radiation universe with negative curvature, the evolution of the MCG universe is ruled by \( G_{\text{eff}} \) in the place of \( G \). Noting that the vacuum expectation values of \( S \) and \( \varphi \) vanish above the electroweak and Planck scales, respectively, we have that \( G_{\text{eff}} = 3c^2/64\pi\varphi_0^2 \) between the electroweak and Planck scales, and \( G_{\text{eff}} = \infty \) above the Planck scale. This behavior of the effective gravitational constant plays an essential role in the evolution of the early MCG universe.

It is worth noting that MCG has a massive spin-2 field with negative energy, which can lead to instabilities in the classical solutions of the theory. The stability analysis of the MCG cosmological solution requires the linearization of (43) about the perturbed scale factor

\[ a'(t) = a(t)[1 + \varepsilon(t)], \]

(47)

where \( a(t) \) is given by (44) and \( |\varepsilon(t)| \ll 1 \). This linearization gives the perturbation equation

\[ \dot{\varepsilon}(\ddot{a}a^3) + \varepsilon\left[ a^2 c^2 + 2 \left( a_0^2 H_0 \sqrt{\Omega_0} \right)^2 \right] = 0. \]

(48)

The solution of (48), which is given by

\[ \varepsilon(t) = \frac{a_0^2 H_0 \sqrt{\Omega_0} + t}{(2a_0^2 H_0 \sqrt{\Omega_0}) t + t^2}; \]

(49)

do not grow unboundedly with time. This implies that the MCG cosmological solution is stable.
4 Cosmological experimental tests

In order to compare the cosmological solution of MCG with cosmological observations, we must find the luminosity distance

\[ d_L(z) = a_0 r(z) (1 + z), \]  

(50)

where \( r(z) \) is the radial distance of a cosmological light source that is observed now on earth with redshift \( z \).

In an open universe such as the MCG universe, the radial distance is determined by the equation of the radial worldline of a light ray

\[ \int_0^{r(z)} \frac{dr}{\sqrt{1 + r^2}} = \int_{t(z)}^{t_0} \frac{c \, dt}{a(t)}. \]  

(51)

By using (36) in (51), integrating both sides, substituting the result into (50), and making some algebra, we find

\[ \frac{H_0 d_L}{c} = \frac{(1 + \sqrt{1 - \Omega_0})^2 (1 + z)^2 - (\sqrt{1 - \Omega_0} + \sqrt{1 + 2\Omega_0 z + \Omega_0 z^2})^2}{2 (\sqrt{1 - \Omega_0} + \sqrt{1 + 2\Omega_0 z + \Omega_0 z^2}) (1 + \sqrt{1 - \Omega_0}) (\sqrt{1 - \Omega_0})}. \]  

(52)

In order to estimate the values of \( H_0 \) and \( \Omega_0 \), we use the Pantheon compilation with 1048 Type Ia supernovae (SNIa) data \[13\]. The procedure consists in compare, for each SNIa at redshift \( z_i \), the observed distance modulus \( \mu_{\text{obs}}(z_i) \) \( \equiv m - M \), where \( m \) is the apparent magnitude of the SNIa and \( M = -19.3 \) its absolute magnitude) with the theoretical distance modulus \( \mu_{\text{th}}(z) \) defined as

\[ \mu_{\text{th}}(z) = 5 \log_{10} d_L(z) + 25, \]  

(53)

where \( d_L \) is measured in Mpc. The best-fit values of \( H_0 \) and \( \Omega_0 \) are determined by an iterative minimization of the function\[3\]

\[ \chi^2(\Omega_0, H_0) = \sum_{i=1}^{1048} \frac{[\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i)]^2}{\sigma_i^2}, \]  

(54)

\[3\] Although the \( \chi^2 \) procedure has a degeneracy between the Hubble constant and the absolute magnitude, it is useful to estimate the current values of the cosmological parameters with a reasonable confidence level. In order to find more accurate estimates, we must use model-independent procedures such as the Bayesian approach. However, due to the greater complexity of these procedures, we will leave them for future works.
where $\sigma_i$ is the uncertainty on $\mu_{\text{obs}}(z_i)$.

The Hubble diagram with the 1048 SNIa from the Pantheon compilation and the best-fit MCG model is shown in FIG. 1. We find that the best-fit values of the MCG parameters are given by

$$H_0 = 69.05 \pm 0.14 \text{ km s}^{-1}\text{Mpc}^{-1},$$

$$\Omega_0 = 10^{-17},$$

with $\chi^2_{\text{min}}/\text{dof} = 0.99$, where dof is the logogram of degree of freedom.

![Figure 1: Hubble diagram for the Pantheon compilation with 1048 SNIa data. The red line represents the best-fitted MCG model.](image)

We can consider that the current MCG universe is dominated by baryonic and dark matters with combined mass densities $\rho_0 \approx 2.67 \times 10^{-27} \text{ Kg m}^{-3}$ and pressures $p_0 \approx 0$. Using these values, (55), and (56) in (35), we find the effective gravitational constant

$$G_{\text{eff}} \approx 2.23 \times 10^{-27} \text{ m}^3\text{kg}^{-1}\text{s}^{-2},$$

and the current scale factor

$$a_0 \approx 4.47 \times 10^{17} \text{ s}.$$

Finally, the substitution of (55) and (56) into (39) gives the current age of the MCG universe

$$t_0 = 14.19 \pm 0.03 \text{ Gyr},$$
which is consistent with the 14 Gyr estimated from old globular clusters [14]. Further analysis is needed to see if the MCG universe accommodates the age of old quasars such as the APM 08279+5255.

It is worth noting that despite the extremely low values in (56) and (57), MCG fits to the SNIa data as well as the ΛCDM model. To see if MCG also fits well to overlapping SNIa, cosmic microwave background (CMB) and baryon acoustic oscillations (BAO) data, it is necessary the development of a theory for the growth of inhomogeneities in the model, which would be different from the one used in the ΛCDM model due to the contribution of the Bach tensor. Since this theory has not yet been developed, this is another topic that we will leave for future works.

5 Final remarks

We have shown in this paper that the negative curvature cosmological solution of MCG is compatible with SNIa data without presenting the cosmological constant problem. The early MCG cosmology, in particular CMB production and nucleosynthesis, and the compatibility of the current age of the MCG universe with the age of old high redshift objects will be investigated in the future.

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The age determination of old high redshift objects such as APM 08279+5255 requires a rigorous statistical analysis involving many astrophysical constraints, which is beyond the scope of this paper.
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