MAGNUS-TYPE INTEGRATOR FOR THE FINITE ELEMENT DISCRETIZATION OF SEMILINEAR PARABOLIC NON-AUTONOMOUS SPDES DRIVEN BY ADDITIVE NOISE

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Abstract. In this paper, we investigate a numerical approximation of a general second order semilinear parabolic non-autonomous stochastic partial differential equation (SPDE) driven by additive noise. Numerical approximations for autonomous SPDEs are thoroughly investigated in the literature while the non-autonomous case is not yet well understood. We discretize the non-autonomous SPDE in space by the finite element method and in time by the Magnus-type integrator. We provide a strong convergence proof of the fully discrete scheme toward the mild solution in the root-mean-square $L^2$ norm. Appropriate assumptions on the drift term and the noise allow to achieve optimal convergence order in time greater than $1/2$, without any logarithmic reduction of convergence order in time. In particular, for trace class noise, we achieve optimal convergence orders $O\left(h^{2-\epsilon} + \Delta t\right)$, where $\epsilon$ is a positive number small enough. Numerical simulations are provided to illustrate our theoretical results.

Résumé. Dans ce papier, nous investigions l'approximation numérique d'équations aux dérivées partielles (EDP) stochastique semilinéaire et non autonome avec un bruit additif. L'approximation numérique d'EDP stochastique autonome est largement étudiée dans la littérature scientifique, tandis que le cas non autonome reste encore très peu connu. Le but de ce papier est d'investiguer le cas non autonome avec un bruit additif. L'EDP stochastique est discrétisée en espace par la méthode des éléments finis et en temps par un schema exponentiel de type Magnus. En plus, sous des hypothèses appropriés, nous obtenons un ordre de convergence en temps supérieur à $1/2$, sans aucune réduction logarithmique. En particulier, pour un bruit de trace fini, nous obtenons une convergence de la forme $O\left(h^{2-\epsilon} + \Delta t\right)$, où $\epsilon$ est un nombre réel positif et suffisamment petit. Les simulations numériques pour illustrer les résultats théoriques sont aussi faites.

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INTRODUCTION

We consider the numerical approximations of the following semilinear parabolic non-autonomous SPDE driven by additive noise
\[ dX(t) = [A(t)X(t) + F(t, X(t))]dt + dW(t), \quad X(0) = X_0, \quad t \in (0, T), \]
in the Hilbert space \( L^2(\Lambda) \), where \( \Lambda \) is a bounded domain of \( \mathbb{R}^d \), \( d = 1, 2, 3 \) and \( T > 0 \). The family of the unbounded linear operators \( A(t) \) are not necessarily self-adjoint. Each \( A(t) \) is assumed to generate an analytic semigroup \( S_t(s) := e^{A(t)s} \). Precise assumptions on \( A(t) \) and \( F \) to ensure the existence of the unique mild solution of (1) are given in the next section. The random initial data is denoted by \( X_0 \). We denote by \( (\Omega, \mathcal{F}, \mathbb{P}) \) a probability space with a filtration \( (\mathcal{F}_t)_{t \in [0, T]} \subset \mathcal{F} \) that fulfills the usual conditions, see e.g., [34] Definition 2.1.11. The noise term \( W(t) \) is assumed to be a \( Q \)-Wiener process defined on a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]}) \), where the covariance operator \( Q : H \rightarrow H \) is assumed to be linear, self-adjoint and positive definite. It is well known (see e.g., [34]) that the noise can be represented as
\[ W(t, x) = \sum_{i=0}^{\infty} \sqrt{\beta_i} \epsilon_i(x) \beta_i(t), \]
where \( (q, \epsilon_i)_{i \in \mathbb{N}} \) are the eigenvalues and eigenfunctions of the covariance operator \( Q \), and \( (\beta_i)_{i \in \mathbb{N}} \) are independent and identically distributed standard Brownian motion. Autonomous systems are not realistic to model phenomena in many fields such as quantum fields theory, electromagnetism, nuclear physics, see e.g., [3] Section 7] and references therein. Numerical solutions of (1) based on implicit, explicit Euler methods and exponential integrators with \( A(t) = A \), where \( A \) is self-adjoint are thoroughly investigated in the literature, see e.g., [15,19,26,44]. If we turn our attention to the case of \( A(t) = A \), with \( A \) not necessary self-adjoint, the list of references become remarkably short, see e.g., [25,30]. The numerical approximation in time of the deterministic counter part of (1) with time dependent coefficient \( A(t) \) was investigated in [6,12,13,38], where Magnus-type integrator [25] was used in [6,12,13,38]; and a new exponential integrator was used in [12]. Numerical approximation for non-autonomous SPDE (1) is not yet well understood due to the complexity of the linear operator \( A(t) \) and its semigroup \( S_t(s) := e^{A(t)s} \). Recently numerical scheme for stochastic model (1) driven by multiplicative noise with time dependent linear operator \( A(t) \) was investigated in [39], where the time discretization was done using the Magnus-type integrator. The optimal convergence order in time in [39] was 1/2. This is the optimal convergence order when dealing with multiplicative noise with schemes based on Euler approximations (namely explicit Euler method, linear implicit Euler method, exponential Euler, exponential Rosenbrock-Euler). In fact even for stochastic ordinary differential equation (SODE) driven by multiplicative noise, the Euler type method achieves optimal order 1/2, see e.g., [2], whereas when dealing with SODE driven by additive noise the optimal convergence order is 1, see e.g. [18]. In this paper, we extend that result to the SPDE (1) and prove that the Magnus-type integrator applied to SPDE (1) achieves an optimal order 1 in time. The price to pay is that we require additional assumptions on the nonlinear function \( F \) than the only standard Lipschitz condition. An important ingredient to achieve that optimal convergence order is the application of Taylor’s formula in Banach space to the drift function, see Section 2.2.2. It is worth to mention that such approach and assumptions on the nonlinear drift function \( F \) were also used in [17,26,30,44] for exponential integrators and semi-implicit Euler method for autonomous SPDES driven by additive noise to achieve optimal convergence order 1 in time. Due to the complexity of the linear operator and the corresponding semi discrete linear operator after space discretisation, novel additional technical estimates are provided on the terms involving the noise to achieve higher convergence order, see e.g., Lemma 2.4.14 and Section 2.2.8. The result indicates how the convergence orders depend on the regularity of the initial data and the noise. More precisely, the fully discrete scheme achieves convergence order \( O(h^\beta + \Delta^{1/2}) \), where \( \beta \) is defined in Assumption 1.1. We emphasize that comparing with results for autonomous SPDES with not necessary self-adjoint, here we achieve optimal convergence order 1 in time for the border case \( \beta = 2 \), instead of sub-optimal convergence order \( 1 - \epsilon \) obtained in [15,30].
optimal convergence orders achieved in \cite{[19]35,14}, where due to sharp integral estimates and optimal regularity estimates in \cite{[20]}. Note that key ingredient to achieve optimal regularity estimates in \cite{[20]} is the spectral decomposition of the linear operator $A$. This cannot directly applied to the case of time dependent and not necessarily self-adjoint operator $A(t)$ due to its complexity and its associated semigroup $S_A(t) = e^{A(t)\frac{\alpha}{2}}$. In this paper, Lemmas 2.7 and 2.8 provide appropriate ingredients to fill the gap.

The rest of this paper is organised as follows. Section 1 provides the general setting, the numerical scheme and the main result. In Section 2 we provide some preparatory results and present the proof of the main results. Section 3 provides some numerical experiments to sustain our theoretical results.

1. Mathematical setting, numerical scheme and main results

1.1. Notations and main assumptions

Let $(H, \langle ., . \rangle_H, ||.||)$ be an separable Hilbert space. For all $p \geq 2$ and for a Banach space $U$, we denote by $L^p(\Omega, U)$ the Banach space of all equivalence classes of $p$ integrable $U$-valued random variables. Let $L(U,H)$ be the space of bounded linear mappings from $U$ to $H$ endowed with the usual operator norm $||.||_{L(U,H)}$. By $L_2(U,H) := HS(U,H)$, we denote the space of Hilbert-Schmidt operators from $U$ to $H$ equipped with the norm $||l||_{L_2(U,H)} := \sum_{i=1}^{\infty} ||l\psi_i||^2$, where $(\psi_i)_{i=1}^{\infty}$ is an orthonormal basis of $U$. Note that this definition is independent of the orthonormal basis of $U$. For simplicity, we use the notations $L(U,U) =: L(U)$ and $L_2(U,U) =: L_2(U)$. For all $l \in L(U,H)$ and $l_1 \in L_2(U,H)$ we have $l_1 \in L_2(U,H)$ and

$$||l_1||_{L_2(U,H)} \leq ||l||_{L(U,H)} ||l_1||_{L_2(U)},$$

see e.g., \cite{[3]}. The covariance operator $Q : H \rightarrow H$ is assumed to be positive and self-adjoint. Throughout this paper $W(t)$ is a Q-wiener process. The space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to $H$ is denoted by $L_2(Q^{1/2}(H), H) = HS(Q^{1/2}(H), H)$. As usual, $L_2^0$ is equipped with the norm

$$||l||_{L_2^0} := ||lQ^{1/2}||_{HS} = \left( \sum_{i=1}^{\infty} ||lQ^{1/2}e_i||^2 \right)^{1/2}, \quad l \in L_2^0,$$

where $(e_i)_{i=1}^{\infty}$ is an orthonormal basis of $H$. This definition is independent of the orthonormal basis of $H$. For an $L_2^0$ predictable stochastic process $\phi : [0, T] \times \Lambda \rightarrow L_2^0$ such that

$$\int_0^t E \left\| \phi(s)Q^{1/2} \right\|_{L_2(H)}^2 \, ds < \infty, \quad t \in [0, T],$$

the following relation called Itô’s isometry property holds

$$E \left\| \int_0^t \phi(s)dW(s) \right\|_{\mathcal{L}_2(H)}^2 = \int_0^t E \left\| \phi(s) \right\|_{L_2^0}^2 \, ds = \int_0^t E \left\| \phi(s)Q^{1/2} \right\|_{L_2(H)}^2 \, ds, \quad t \in [0, T],$$

see e.g., \cite{[3] 25} Step 2 in Section 2.3.2 or \cite{[34]} Proposition 2.3.5.

In the rest of this paper, we consider $H = L^2(\Lambda)$. To guarantee the existence of a unique mild solution of (1) and for the purpose of the convergence analysis, we make the following assumptions.

Assumption 1.1. The initial data $X_0 : \Omega \rightarrow H$ is assumed to be measurable and $X_0 \in L^4(\Omega, \mathcal{D}\left((-A(0))^{3/2}\right))$, $0 \leq \beta \leq 2$.

We equip $V_{\alpha}(t) := \mathcal{D}\left((-A(t))^{\beta/2}\right)$, $\alpha \in \mathbb{R}$ with the norm $||u||_{\alpha,t} := \|(A(t))^{\alpha/2}u\|$. Due to (14), (15) and for the seek of ease notations, we simply write $V_{\alpha}$ and $||.||_{\alpha}$. We follow \cite{[30]} and assume that the nonlinear operator $F$ satisfies the following Lipschitz condition.
Assumption 1.2. The nonlinear operator $F : [0, T] \times H \rightarrow H$ is assumed to be $\beta/2$-Hölder continuous with respect to the first variable and Lipschitz continuous with respect to the second variable, i.e. there exists a positive constant $K_3$ such that

$$
\|F(s, 0)\| \leq K_3, \quad \|F(t, u) - F(s, v)\| \leq K_3 \left(|t - s|^{\beta/2} + \|u - v\|\right), \quad s, t \in [0, T], \quad u, v \in H.
$$

We also assume the drift function to be twice differentiable with bounded derivative, i.e. there exists a constant $K_1 > 0$ such that

$$
\|F'(t, v)\|_{L(H)} \leq K_1, \quad \forall v \in H, \quad t \in [0, T] \quad (8)
$$

$$
\|F''(t, u)(v_1, v_2)\|_{\eta} \leq K_1\|v_1\|\|v_2\|, \quad u, v_1, v_2 \in H, \quad \text{for some } \eta \in [1, 2], \quad t \in [0, T], \quad (9)
$$

where the Fréchet first and second order derivatives are taken respect to the second variable.

Assumption 1.3. We assume the covariance operator $Q : H \rightarrow H$ to satisfy

$$
\left\|(-A(0))^{\frac{\beta}{2}}Q^{\frac{1}{2}}\right\|_{\mathcal{L}_2(H)} < \infty, \quad (10)
$$

where $\beta$ is defined in Assumption 1.1.

As in [9,10,12], we make the following assumptions on the family of linear operator $A(t)$.

Assumption 1.4. (i) We assume that $\mathcal{D}(A(t)) = D$, $0 \leq t \leq T$ and the family of linear operators $A(t) : D \subset H \rightarrow H$ to be uniformly sectorial on $0 \leq t \leq T$, i.e. there exist constants $c > 0$ and $\theta \in \left(\frac{1}{2}, \pi\right)$ such that

$$
\left\|\left(\lambda I - A(t)\right)^{-1}\right\|_{L(L^2(\Lambda))} \leq \frac{c}{|\lambda|}, \quad \lambda \in S_\theta, \quad (11)
$$

where $S_\theta := \{\lambda \in \mathbb{C} : \lambda = re^{i\phi}, r > 0, 0 \leq |\phi| \leq \theta\}$. As in [12], by a standard scaling argument, we assume $-A(t)$ to be invertible with bounded inverse.

(ii) We require the following Lipschitz conditions respect to the time

$$
\left\|\left(A(t) - A(s)\right)\left(-A(0)\right)^{-1}\right\|_{L(H)} \leq K_1|t - s|, \quad s, t \in [0, T], \quad (12)
$$

$$
\left\|\left(-A(0)\right)^{-1}\left(A(t) - A(s)\right)\right\|_{L(D, H)} \leq K_1|t - s|, \quad s, t \in [0, T]. \quad (13)
$$

(iii) As we are dealing with non smooth data, we follow [36] and assume that

$$
\mathcal{D}\left((-A(t))^\alpha\right) = \mathcal{D}\left((-A(0))^\alpha\right), \quad 0 \leq t \leq T, \quad 0 \leq \alpha \leq 1 \quad (14)
$$

and there exists a positive constant $K_2$ such that the following estimate holds

$$
K_2^{-1}\|(-A(0))^\alpha u\| \leq \|(-A(t))^\alpha u\| \leq K_2\|(-A(0))^\alpha u\|, \quad t \in [0, T], \quad u \in \mathcal{D}\left((-A(0))^\alpha\right). \quad (15)
$$

Remark 1.5. As a consequence of Assumption 1.4, for all $\alpha \geq 0$ and $\gamma \in [0, 1]$, there exists a constant $C_1 \geq 0$ such that the following estimate holds uniformly in $t \in [0, T]$

$$
\left\|\left(-A(t)\right)^\alpha e^{A(t)s}\right\|_{L(H)} \leq C_1 s^{-\alpha}, \quad s > 0, \quad (16)
$$

$$
\left\|\left(-A(t)\right)^{-\gamma} \left(\mathbf{I} - e^{A(t)s}\right)\right\|_{L(H)} \leq C_1 s^\gamma, \quad s \geq 0. \quad (17)
$$
**Proposition 1.6.** Let $\Delta(T) := \{(t,s) : 0 \leq s \leq t \leq T\}$. Under Assumption [1-4], there exists a unique evolution system [22, Definition 5.3, Chapter 5] $U: \Delta(T) \rightarrow L(H)$ such that

(i) There exists a positive constant $K_0$ such that

$$\|U(t,s)\|_{L(H)} \leq K_0, \quad 0 \leq s \leq t \leq T.$$ (18)

(ii) $U(.,s) \in C^1([s,T];L(H)), \ 0 \leq s \leq T,$

$$\frac{\partial U}{\partial t}(t,s) = -A(t)U(t,s) \quad \text{and} \quad \|A(t)U(t,s)\|_{L(H)} \leq \frac{K_0}{t-s}, \quad 0 \leq s < t \leq T.$$ (19)

(iii) $U(t,.x) \in C^1([0,t];H), \ 0 < t \leq T, \ x \in D(A(0))$ and

$$\frac{\partial U}{\partial s}(t,s) = -U(t,s)A(s)x \quad \text{and} \quad \|A(t)U(t,s)A(s)^{-1}\|_{L(H)} \leq K_0, \quad 0 \leq s \leq t \leq T.$$ (20)

**Proof.** See [22, Theorem 6.1, Chapter 5]. \hfill \Box

**Theorem 1.7.** Let Assumptions [1-1], [1-2] and [1-4] (i)-(ii) be fulfilled. Then the non-autonomous problem [1] has a unique mild solution $X(t)$, which takes the following form

$$X(t) = U(t,0)X_0 + \int_0^t U(t,s)F(s,X(s))ds + \int_0^t U(t,s)dW(s),$$ (21)

where $U(t,s)$ is the evolution system of Proposition 1.6. Moreover, there exists a positive constant $K_4$ such that

$$\sup_{0 \leq t \leq T} \|X(t)\|_{L^2(\Omega,D((\mathbb{A}(0))^\beta/2))} \leq K_4 \left(1 + \|X_0\|_{L^2(\Omega,D((\mathbb{A}(0))^\beta/2))}\right).$$ (22)

**Proof.** See [20, Theorem 1.3]. \hfill \Box

### 1.2. Fully discrete scheme and main result

In the rest of this paper, we consider the family of linear operators $A(t)$ to be of second order of the following form

$$A(t)u = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(q_{ij}(x,t) \frac{\partial u}{\partial x_j}\right) - \sum_{j=1}^{d} q_j(x,t) \frac{\partial u}{\partial x_j}.$$ (23)

We require the coefficients $q_{ij}$ and $q_j$ to be smooth functions on the variable $x \in \overline{\Lambda}$ and Hölder-continuous with respect to $t \in [0,T]$. We further assume that there exists a positive constant $c_1$ such that the following ellipticity condition holds

$$\sum_{i,j=1}^{d} q_{ij}(x,t)\xi_i \xi_j \geq c_1 |\xi|^2, \quad (x,t) \in \overline{\Lambda} \times [0,T].$$ (24)

Under the above assumptions on $q_{ij}$ and $q_j$, it is well known that the family of linear operators defined in [23] fulfills Assumption [1-4] (i)-(ii) with $D = H^2(\mathbb{A}) \cap H_0^1(\mathbb{A})$, see [22, Section 7.6] or [11, Section 5.2]. The above assumptions on $q_{ij}$ and $q_j$ also imply that Assumption [1-4] (iii) is fulfilled, see e.g., [36, Example 6.1] or [1, 35].
As in [8, 25], we introduce two spaces $H$ and $V$, such that $H \subset V$, that depend on the boundary conditions for the domain of the operator $-A(t)$ and the corresponding bilinear form. For example, for Dirichlet boundary conditions we take

$$V = H = H^1_0(\Lambda) = \{ v \in H^1(\Lambda) : v = 0 \text{ on } \partial \Lambda \}. \quad (25)$$

For Robin boundary condition and Neumann boundary condition, which is a special case of Robin boundary condition ($\alpha_0 = 0$), we take $V = H^1(\Lambda)$ and

$$H = \{ v \in H^2(\Lambda) : \partial v/\partial v_A + \alpha_0 v = 0, \text{ on } \partial \Lambda \}, \quad \alpha_0 \in \mathbb{R}. \quad (26)$$

Using Green’s formula and the boundary conditions, we obtain the corresponding bilinear form associated to $-A(t)$

$$a(t)(u, v) = \int_\Lambda \left( \sum_{i,j=1}^d q_{ij}(x, t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^d q_i(x, t) \frac{\partial u}{\partial x_i} v \right) dx, \quad u, v \in V, \quad (27)$$

for Dirichlet boundary conditions and

$$a(t)(u, v) = \int_\Lambda \left( \sum_{i,j=1}^d q_{ij}(x, t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^d q_i(x, t) \frac{\partial u}{\partial x_i} v \right) dx + \int_{\partial \Lambda} \alpha_0 uv ds, \quad u, v \in V. \quad (28)$$

for Robin and Neumann boundary conditions. Using Gårding’s inequality, it holds that there exist two constants $\lambda_0$ and $c_0$ such that

$$a(t)(v, v) \geq \lambda_0 \|v\|^2 - c_0 \|v\|^2, \quad v \in V, \quad t \in [0, T]. \quad (29)$$

By adding and subtracting $c_0 u$ on the right hand side of (1), we obtain a new family of linear operators that we still denote by $A(t)$. Therefore the new corresponding bilinear form associated to $-A(t)$ still denoted by $a(t)$ satisfies the following coercivity property

$$a(t)(v, v) \geq \lambda_0 \|v\|^2, \quad v \in V, \quad t \in [0, T]. \quad (30)$$

Note that the expression of the nonlinear term $F$ has changed as we have included the term $-c_0u$ in the new nonlinear term that we still denote by $F$.

The coercivity property (30) implies that $A(t)$ and $A_h(t)$ are sectorial on $L^2(\Lambda)$ (uniformly in $h$), see e.g., [29]. Therefore $A_h(t)$ and $A(t)$ generate analytic semigroups denoted respectively by $S_{h,t}(s) := e^{sA_h(t)}$ and $S_t(s) := e^{sA(t)}$ on $L^2(\Lambda)$ such that [11]

$$S_t(s) = e^{sA(t)} = \frac{1}{2\pi i} \int_C e^{s\lambda}(\lambda I - A(t))^{-1} d\lambda, \quad s > 0, \quad (31)$$

where $C$ denotes a path that surrounds the spectrum of $A(t)$. The coercivity property (30) also implies that $-A(t)$ is a positive operator and its fractional powers are well defined and for any $\alpha > 0$, we have

$$\left\{ \begin{array}{ll}
(-A(t))^{-\alpha} &= \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{sA(t)} ds,
\end{array} \right. \quad (32)$$

\[ 1 \text{ Defined in 44} \]
where $\Gamma(\alpha)$ is the Gamma function \cite{11}. The domain of $(-A(t))^{\alpha/2}$ are characterized in \cite{6,8,23} for $1 \leq \alpha \leq 2$ with equivalence of norms as follows

\[
\mathcal{D}((-A(t))^{\alpha/2}) = H_0^1(\Lambda) \cap H^\alpha(\Lambda) \quad \text{(for Dirichlet boundary condition)}
\]
\[
\mathcal{D}(-A(t)) = \mathbb{H}, \quad \mathcal{D}((-A(t))^{1/2}) = H^1(\Lambda) \quad \text{(for Robin boundary condition)}
\]
\[
\|v\|_{H^\alpha(\Lambda)} \equiv \|((-A(t))^{\alpha/2} v\| := \|v\|_\alpha, \quad \forall v \in \mathcal{D}((-A(t))^{\alpha/2}).
\]

The characterization of $\mathcal{D}((-A(t))^{\alpha/2})$ for $0 \leq \alpha < 1$ can be found in \cite{31} Theorem 2.1 & Theorem 2.2.

Now, we turn our attention to the discretization of the problem \cite{11}. We start by splitting the domain $\Lambda$ in finite triangles. Let $T_h$ be the triangulation with maximal length $h$ satisfying the usual regularity assumptions, and $V_h \subset V$ be the space of continuous functions that are piecewise linear over the triangulation $T_h$. We consider the projection $P_h$ from $H = L^2(\Lambda)$ to $V_h$ defined for every $u \in H$ by

\[
\langle P_h u, \chi \rangle_H = \langle u, \chi \rangle_H, \quad \phi, \chi \in V_h.
\]

For all $t \in [0, T]$, the discrete operator $A_h(t) : V_h \rightarrow V_h$ is defined by

\[
\langle A_h(t) \phi, \chi \rangle_H = \langle A(t) \phi, \chi \rangle_H = -a(t)(\phi, \chi), \quad \phi, \chi \in V_h.
\]

The coercivity property \cite{30} implies that there exist constants $C_2 > 0$ and $\theta \in (\frac{1}{4} \pi, \pi)$ such that

\[
\|((\mathbf{I} - A_h(t))^{-1}\|_{L(H)} \leq \frac{C_2}{|\lambda|} \quad \lambda \in S_\theta
\]

holds uniformly for $h > 0$ and $t \in [0, T]$. See e.g., \cite{23} (2.9) or \cite{8,11}. The coercivity property \cite{30} also implies that the smooth properties \cite{10} and \cite{17} hold for $A_h$ uniformly on $h > 0$ and $t \in [0, T]$, i.e. for all $\alpha \geq 0$ and $\gamma \in [0, 1]$, there exist a positive constant $C_3$ such that the following estimates hold uniformly on $h > 0$ and $t \in [0, T]$, see e.g. \cite{8,11}

\[
\left\|(-A_h(t))^{\alpha} e^{sA_h(t)} \right\|_{L(H)} \leq C_3 s^{-\alpha}, \quad s > 0,
\]
\[
\left\|(-A_h(t))^{-\gamma} (\mathbf{I} - e^{sA_h(t)}) \right\|_{L(H)} \leq C_3 s^\gamma, \quad s \geq 0.
\]

The semi-discrete version of \cite{11} consists of finding $X^h(t) \in V_h$, $t \in [0, T]$ such that

\[
dX^h(t) = [A_h(t)X^h(t) + P_hF(t, X^h(t))] \, dt + P_h dW(t), \quad t \in (0, T], \quad X^h(0) = P_hX_0.
\]

Throughout this paper we take $t_m = m\Delta t \in [0, T]$, where $T = M\Delta t$ for $m, M \in \mathbb{N}$, $m \leq M$. Following \cite{39}, we have the following fully discrete scheme for \cite{11}, called stochastic Magnus-type integrator (SMTI) for SPDEs

\[
X^h_{m+1} = e^{\Delta A_h,m} X^h_m + \Delta t\varphi_1(\Delta t A_h,m) P_h F(t_m, X^h_m) + e^{\Delta t A_h,m} P_h \Delta W_m, \quad m \geq 0, \quad X^h_0 = P_hX_0,
\]

where $\Delta W_m := W_{(m+1)\Delta t} - W_{m\Delta t}$, $A_h,m := A_h(t_m)$ and the linear operator $\varphi_1(\Delta t A_h,m)$ is given by

\[
\varphi_1(\Delta t A_h,m) := \frac{1}{\Delta t} \int_0^{\Delta t} e^{(\Delta t - s)A_h,m} ds. \quad (40)
\]

Note that the numerical scheme \cite{39} can be written in the following integral form, useful for the error analysis

\[
X^h_{m+1} = e^{\Delta t A_h,m} X^h_m + \int_{t_m}^{t_{m+1}} e^{(t_{m+1} - s)A_h,m} P_h F(t_m, X^h_m) \, ds + \int_{t_m}^{t_{m+1}} e^{\Delta t A_h,m} P_h dW(s). \quad (41)
\]
MAGNUS-TYPE INTEGRATOR FOR NON AUTONOMOUS SPDES DRIVEN BY ADDITIVE NOISE

Theorem 1.10. The main result of this work.

Remark 1.11. The numerical method being built, we can now state its strong convergence result toward the mild solution, which holds and 2.8 are keys ingredients to achieve optimal convergence orders with no reduction.

integral estimate [20]. In the case of non-autonomous and non necessarily self adjoint operator, Lemmas 2.7 also achieved optimal convergence orders. Note that these optimal convergence orders were due to the sharp coincides with the assumptions made in [19, 20, 43] on the constant self-adjoint operator

\[ C(\text{corresponding to the eigenvectors } A_{ij}) = \text{a generic constant that may change from one place to another}. \]

The following assumption will be needed in our convergence estimate to achieve optimal convergence order in time without any logarithmic reduction.

Assumption 1.8. Let \( A(t) = A^s(t) + A^{ns}(t) \), where \( A^s(t) \) and \( A^{ns}(t) \) are respectively the self-adjoint and the non self-adjoint parts of \( A(t) \). We assume that the family \( (\lambda_n(t))_{n \in \mathbb{N}} \) of positive eigenvalues of \(-A^s(t)\) corresponding to the eigenvectors \( (e_n(t))_{n \in \mathbb{N}} \) are such that for \( x \in H \)

\[
\sup_{0 \leq t \leq T} \lambda_n(t) < C(n), \quad \sup_{0 \leq t \leq T} (e_n(t), x) < C_1(x, n). \quad (43)
\]

where \( C(n) \) and \( C_1 = C_1(x, n) \) are two positive constants.

Remark 1.9. Typical examples which fulfilled Assumption 1.8 are linear operators \( A(t) \) defined in [23] with bounded coefficients such that \( q_{ij}(x, t) > 0 \) and \( q_{ij}(x, t) = 0, i \neq j \) with [13]. Note that Assumption 1.8 coincides with the assumptions made in [12][20][43] on the constant self-adjoint operator \( A \), where the authors also achieved optimal convergence orders. Note that these optimal convergence orders were due to the sharp integral estimate [20]. In the case of non-autonomous and non necessarily self adjoint operator, Lemmas 2.7 and 2.8 are keys ingredients to achieve optimal convergence orders with no reduction.

In the rest of this paper \( C \) denotes a generic constant that may change from one place to another. The numerical method being built, we can now state its strong convergence result toward the mild solution, which is the main result of this work.

Theorem 1.10. [Main result] Let Assumptions 1.1-1.4 and 1.8 be fulfilled. Then the following error estimate holds

(i) If \( 0 \leq \beta < 2 \) then

\[
(\mathbb{E}\|X(t_m) - X^h_m\|^2)^{1/2} \leq C \left( h^\beta + \Delta t^{\beta/2} \right). \quad (44)
\]

(ii) If \( \beta = 2 \) then

\[
(\mathbb{E}\|X(t_m) - X^h_m\|^2)^{1/2} \leq C \left[ h^2 \left( 1 + \max \left( \ln \left( \frac{t_m}{h^2} \right), 0 \right) \right) + \Delta t \right]. \quad (45)
\]

Remark 1.12. Note that as in [15][25][44], we can use the following approximation

\[
\int_{t_m-1}^{t_m} e^{Ah_{m}(t_m-s)} P_h F(s, X^h(s)) ds \approx \int_{t_m-1}^{t_m} e^{Ah_{m}\Delta t} P_h F(t_m, X^h) ds = \Delta t e^{Ah_{m}\Delta t} P_h F(t_m, X^h). \quad (48)
\]
This yields the following numerical Magnus-type integrator scheme

\[ Y_{m+1}^h = e^{\Delta t A_{h,m}} \left[ Y_m^h + \Delta t P_h F(t_m, Y_m^h) + P_h \Delta W_m \right], \quad Y_0^h = P_h X_0. \]  

(49)

Note that the convergence result in Theorem 1.10 also holds for the numerical scheme (49). The proof is similar to that of Theorem 1.10.

2. Proof of the main result

The proof of the main result needs some preparatory results.

2.1. Preparatory results

The following lemma will be useful in our convergence proof. Its proof can be found in [38].

Lemma 2.1. For any \( \gamma \in [0, 1] \), the following equivalence of norms holds uniformly in \( h > 0 \) and \( t \in [0, T] \).

\[
    K^{-1}\|(-A_h(0))^{-\gamma} v\| \leq \|((-A_h(t))^{-\gamma} v\| \leq K\|(-A_h(0))^{-\gamma} v\|, \quad v \in V_h, \tag{50}
\]

\[
    K^{-1}\|(-A_h(t))^\gamma v\| \leq \|((-A_h(t))^\gamma v\| \leq K\|(-A_h(0))^\gamma v\|, \quad v \in V_h. \tag{51}
\]

Lemma 2.2. Under Assumptions 1.3 and 1.4 (iii), the following estimate holds

\[
    \left\| (-A_h(t))^{\beta - 1/2} P_h Q_\frac{1}{2} \right\|_{L^2(H)} < C, \quad t \in [0, T], \quad h > 0. \tag{52}
\]

Proof. For \( 0 \leq \beta \leq 1 \), it follows from [40, Proposition 4.1] that

\[
    \left\| (-A_h(0))^{\beta - 1/2} P_h Q_\frac{1}{2} \right\|_{L^2(H)} < C. \tag{53}
\]

Therefore using (53) and Lemma 2.1 it follows that

\[
    \left\| (-A_h(t))^{\beta - 1/2} P_h Q_\frac{1}{2} \right\|_{L^2(H)} < C, \quad t \in [0, T], \quad \beta \in [0, 1]. \tag{54}
\]

Let us recall the following estimate [39, Lemma 1]

\[
    \|(-A_h(0))^\alpha P_h v\| \leq C\|(-A(0))^\alpha v\|, \quad 0 \leq \alpha \leq 1/2, \quad v \in D((-A(0))^\alpha). \tag{55}
\]

For \( 1 \leq \beta \leq 2 \), applying (55) with \( \alpha = \frac{\beta - 1}{2} \) and using Assumption 1.3 yields

\[
    \left\| (-A_h(0))^{\beta - 1/2} P_h Q_\frac{1}{2} \right\|_{L^2(H)} = \sum_{i=1}^\infty \left\| (-A_h(0))^{\beta - 1/2} P_h Q_\frac{1}{2} e_i \right\| \\
    \leq C \sum_{i=1}^\infty \left\| (-A(0))^{\beta - 1/2} Q_\frac{1}{2} e_i \right\| \\
    = C \left\| (-A(0))^{\beta - 1/2} Q_\frac{1}{2} \right\|_{L^2(H)} \leq C, \tag{56}
\]

Therefore, it follows from (56) by using (51) that

\[
    \left\| (-A_h(t))^{\beta - 1/2} P_h Q_\frac{1}{2} \right\|_{L^2(H)} \leq C, \quad t \in [0, T]. \tag{57}
\]
The proof of the following lemma can be found in [35].

**Lemma 2.3.** Under Assumption 1.4, the following estimates hold

\[
\|(A_h(t) - A_h(s))(-A_h(r))^{-1}u^h\| \leq C|t-s||u^h|, \quad r, s, t \in [0, T], \quad u^h \in V_h, \tag{58}
\]

\[
\|(-A_h(r))^{-1}(A_h(s) - A_h(t))u^h\| \leq C|s-t||u^h|, \quad r, s, t \in [0, T], \quad u^h \in V_h. \tag{59}
\]

**Remark 2.4.** From Lemma 2.3 and using the fact that \(\mathcal{D}(A_h(t)) = \mathcal{D}(A_h(0))\), it follows from [52] Theorem 6.1, Chapter 5] that there exists a unique evolution system \(U_h : \Delta(T) \rightarrow L(H)\), satisfying [52] (6.3), Page 149.

**Lemma 2.5.** Let Assumption 1.4 be fulfilled.

(i) The following estimate holds

\[
\|U_h(t,s)\|_{L(H)} \leq C, \quad 0 \leq s \leq t \leq T. \tag{60}
\]

(ii) For any \(0 \leq \alpha \leq 1, 0 \leq \gamma \leq 1\) and \(0 \leq s \leq t \leq T\), the following estimates hold

\[
\|(-A_h(r))^\alpha U_h(t,s)\|_{L(H)} \leq C(t-s)^{-\alpha}, \quad r \in [0, T], \tag{61}
\]

\[
\|U_h(t,s)(-A_h(r))^\alpha\|_{L(H)} \leq C(t-s)^{-\alpha}, \quad r \in [0, T], \tag{62}
\]

\[
\|(-A_h(r))^\gamma U_h(t,s)(-A_h(s))^{-\gamma}\|_{L(H)} \leq C(t-s)^{\gamma-\alpha}, \quad r \in [0, T]. \tag{63}
\]

(iii) For any \(0 \leq s \leq t \leq T\), the following useful estimate holds

\[
\| (U_h(t,s) - I)(-A_h(s))^{-\gamma} \|_{L(H)} \leq C(t-s)^\gamma, \quad 0 \leq \gamma \leq 1, \tag{64}
\]

\[
\| (-A_h(r))^{-\gamma}(U_h(t,s) - I) \|_{L(H)} \leq C(t-s)^\gamma, \quad 0 \leq \gamma \leq 1. \tag{65}
\]

**Proof.** The proof can be found in [35].

**Remark 2.6.** For relatively smooth coefficients \((q_j \in C^1(\Lambda))\), the formal adjoint of \(A(t)\) denoted by \(A^*(t)\) is given by [2] Section 6.2.3]

\[
A^*(t) = \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( q_{ij}(x,t) \frac{\partial}{\partial x_i} \right) + \sum_{j=1}^d q_j(x,t) \frac{\partial}{\partial x_j} + \left( \sum_{j=1}^d \frac{\partial q_j}{\partial x_j}(x,t) \right) I, \quad t \in [0, T]. \tag{66}
\]

Therefore the self-adjoint part of \(A(t)\) is given by

\[
A^*(t) = \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( q_{ij}(x,t) \frac{\partial}{\partial x_i} \right), \quad t \in [0, T]. \tag{67}
\]

The bilinear operator associated to \(A^*(t)\) is given by

\[
a^*(t)(u,v) = \sum_{i,j=1}^d \int_{\Lambda} q_{ij}(x,t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx, \quad u, v \in V, \quad t \in [0, T]. \tag{68}
\]

The discrete version \(A_h^*(t)\) of \(A^*(t)\) is therefore given by \(A_h^*(t) : V_h \rightarrow V_h\) such that

\[
\langle A_h^*(t)\phi, \chi \rangle_H = \langle A^*(t)\phi, \chi \rangle_H = -a^*(t)(\phi, \chi), \quad \phi, \chi \in V_h. \tag{69}
\]
Hence $A_h^x(t)$ satisfies also Assumption [13,8] and

$$A_h(t) = A_h^s(t) + A_h^{ns}(t),$$  

where $A_h^{ns}(t)$ is the non self adjoint part of $A_h(t)$.

The following sharp integral estimate will be useful in our convergence analysis to avoid suboptimal convergence order and is a key ingredient to achieve optimal convergence order in time. It is an analogue of [20, Lemma 3.2 (iii)] for evolution system.

**Lemma 2.7.** Let Assumptions [13,4] and [10,8] be fulfilled and let $0 \leq \rho \leq 1$. Then the following estimate holds

$$\int_{\tau_1}^{\tau_2} \left\| (-A_h(r))^{\rho/2} S^h_r(\tau_2 - r) \right\|^2_{L(H)} dr \leq C(\tau_2 - \tau_1)^{1-\rho}, \quad 0 \leq \tau_1 \leq \tau_2 \leq T, \quad \tau \in [0, T],$$  

(71)

$$\int_{\tau_1}^{\tau_2} \left\| (-A_h(r))^{\rho/2} U_h(\tau_2, r) \right\|^2_{L(H)} dr \leq C(\tau_2 - \tau_1)^{1-\rho}, \quad 0 \leq \tau_1 \leq \tau_2 \leq T, \quad \tau \in [0, T],$$  

(72)

$$\int_{\tau_1}^{\tau_2} \left\| U_h(\tau_2, r)(-A_h(r))^{\rho/2} \right\|^2_{L(H)} dr \leq C(\tau_2 - \tau_1)^{1-\rho}, \quad 0 \leq \tau_1 \leq \tau_2 \leq T, \quad \tau \in [0, T].$$  

(73)

**Proof.** We start with the estimate of (71). Let us recall that $A_h(r) = A_h^s(r) + A_h^{ns}(r)$, where $A_h^s(r)$ and $A_h^{ns}(r)$ are respectively the self adjoint and the non self adjoint parts of $A_h(r)$. As in [20], we use the Zassenhaus formula [28,37] to decompose the semigroup $S^h_r(t)$ as follows

$$S^h_r(t) = e^{A_h^s(r)t} = e^{(A_h^s(r) + A_h^{ns}(r))t} = e^{A_h^s(r)t}e^{A_h^{ns}(r)t} \prod_{k=2}^{\infty} e^{C^h_k(r,t)},$$  

(74)

where the $C^h_k(r,t)$ are called Zassenhaus exponents [37]. Let us set

$$T^h_r(t) := e^{A_h^{ns}(r)t} \prod_{k=2}^{\infty} e^{C^h_k(r,t)}.$$  

(75)

Therefore

$$S^h_r(t) = S^h_r(r,t)T^h_r(t),$$  

(76)

where $S^h(r,t) := e^{A_h^s(r)t}$ is the semigroup generated by $A_h^s(r)$. Using the Baker-Campbell-Hausdorff representation [4,28,37], it is well known that for non-commuting quantities $x$ and $y$, we have

$$e^x e^y = e^{x \oplus y},$$  

(77)

where the exponent $x \oplus y$ is given by an infinite Baker-Campbell-Hausdorff series of multiple commutators with rational coefficients (see e.g., [28, (1.1)] or [37, (1.2)]) and converges to $\log(e^x e^y)$. Using (77), by recurrence, there exists $Z = Z(t, A_h^s(r), A_h^{ns}(r))$ such that the operator $T^h_r(t)$ can be written as

$$T^h_r(t) = e^Z.$$

(78)

Therefore, $T^h_r(t)$ is uniformly bounded. Note that $\mathcal{D}(-A_h(r))) = \mathcal{D}(-A_h^s(r))$, with the equivalence of norms, see e.g., [8,22]. So by [24, (3.3)] and using Assumption [1,4] and Lemma 2.1, we have $\mathcal{D}((-A_h(r)^\sigma)) = \mathcal{D}((A_h^s(r)^\sigma))$. 

\[ \text{MAGNUS-TYPE INTEGRATOR FOR NON AUTONOMOUS SPDES DRIVEN BY ADDITIVE NOISE} \]
\[ D((-A_h(r))^\alpha), \quad 0 \leq \alpha \leq 1, \text{ with the equivalence of norms. Therefore, using (77) and the boundedness of } T^h_t(t) \text{ yields} \]

\[ \int_{\tau_1}^{\tau_2} \left\| (-A_h(t))^{\alpha/2} S^h_t(\tau_2 - r) \right\|^2_{L(H)} \, dr \leq C \int_{\tau_1}^{\tau_2} \left\| (-A_h(t))^{\alpha/2} S^h_t(\tau_2 - r) \right\|^2_{L(H)} \, dr \]

\[ \leq C \int_{\tau_1}^{\tau_2} \left\| (-A_h(t))^{\alpha/2} S^h_t(\tau_2 - r) T^h_t(\tau_2 - r) \right\|^2_{L(H)} \, dr \]

\[ = C \int_{\tau_1}^{\tau_2} \left\| (-A_h(t))^{\alpha/2} S^h_t(\tau_2 - r) P_h T^h_t(\tau_2 - r) \right\|^2_{L(H)} \, dr \]

\[ \leq C \int_{\tau_1}^{\tau_2} \left\| (-A_h(t))^{\alpha/2} S^h_t(\tau_2 - r) P_h \right\|^2_{L(H)} \, dr \]

\[ \leq C \int_{\tau_1}^{\tau_2} \left\| (-A^h_t(r))^{\alpha/2} S^h_t(\tau_2 - r) P_h \right\|^2_{L(H)} \, dr. \quad (79) \]

From Assumption 1.8 it follows that there exists an increasing sequence of real numbers \((\lambda^h_n(r))_{n=1}^{N_h}\) and eigenfunctions \((e^h_n(r))_{n=1}^{N_h}\) in \(H\) such that \(A_h(r)e^h_n(r) = \lambda^h_n(r)e^h_n(r)\) and

\[ 0 < \lambda^h_1(r) \leq \lambda^h_2(r) \leq \cdots \leq \lambda^h_{N_h}(r), \quad (80) \]

where \(\dim(V_h) = N_h\). From the coercivity (30), there exists \(C > 0\) such that

\[ \inf_{0 \leq t \leq T} \lambda^h_n(t) < C. \quad (81) \]

However as \(e^h_n(t)\) tends to \(e_n(t)\) when \(h \to 0\), from (18) we also have

\[ \sup_{0 \leq t \leq T} \lambda^h_n(t) < C(n), \quad \text{and} \quad \sup_{0 \leq t \leq T} (x, e^h_n(t)) < C_1(x, n), \quad n = 1, \cdots, N_h, \quad x \in H. \quad (82) \]
Like in the proof of [24 Lemma 3.2 (iii)], using the expansion of $P_h x$ (with $x \in H$), in terms of the eigenbasis $(e_n^h(r))_{n=1}^{N_h}$ of the operator $A_h(r)$ and careful estimates yields

$$\int_{\tau_1}^{T_2} \left\| (-A_h^1(r))^{\rho/2} S_h^1(r, \tau_2 - r) P_h x \right\|^2 \, dr$$

$$= \int_{\tau_1}^{T_2} \left\| \sum_{n=1}^{N_h} (-A_h^1(r))^{\rho/2} S_h^1(r, \tau_2 - r) (x, e_n^h(r)) e_n^h(r) \right\|^2 \, dr$$

$$= \sum_{n=1}^{N_h} \int_{\tau_1}^{T_2} (x, e_n^h(r))^2 \left( \lambda_n^h(r) \right)^\rho e^{-2\lambda_n^h(r)(\tau_2 - r)} \, dr$$

$$\leq \sum_{n=1}^{N_h} \int_{\tau_1}^{T_2} \sup_{0 \leq r \leq T} (x, e_n^h(r))^2 \sup_{0 \leq r \leq T} \left( \lambda_n^h(r) \right)^\rho \int_{\tau_1}^{T_2} e^{-2(\tau_2 - r) \inf_{0 \leq r \leq T} \lambda_n^h(r)} \, dr$$

$$= \sum_{n=1}^{N_h} \sup_{0 \leq t \leq T} (x, e_n^h(t))^2 \sup_{0 \leq t \leq T} \left( \lambda_n^h(t) \right)^\rho \int_{\tau_1}^{T_2} e^{-2(\tau_2 - r) \inf_{0 \leq r \leq T} \lambda_n^h(r)} \, dr$$

$$= \frac{1}{2} \sum_{n=1}^{N_h} \sup_{0 \leq t \leq T} (x, e_n^h(t))^2 \sup_{0 \leq t \leq T} \left( \lambda_n^h(t) \right)^\rho \sup_{0 \leq t \leq T} \left( \inf_{0 \leq r \leq T} \lambda_n^h(r) \right)^{-1} \left( 1 - e^{-2 \inf_{0 \leq r \leq T} \lambda_n^h(r)(\tau_2 - \tau_1)} \right)$$

$$= \frac{1}{2} \sum_{n=1}^{N_h} \sup_{0 \leq t \leq T} (x, e_n^h(t))^2 \sup_{0 \leq t \leq T} \left( \lambda_n^h(t) \right)^\rho \left( \inf_{0 \leq r \leq T} \lambda_n^h(t) \right)^{-1} \left( 1 - e^{-2 \inf_{0 \leq r \leq T} \lambda_n^h(r)(\tau_2 - \tau_1)} \right)$$

$$= \frac{1}{2} \sum_{n=1}^{N_h} \sup_{0 \leq t \leq T} (x, e_n^h(t))^2 \sup_{0 \leq t \leq T} \left( \lambda_n^h(t) \right)^\rho \left( \lambda_n^h(t) \right)^{-1} \left( 1 - e^{-2 \inf_{0 \leq r \leq T} \lambda_n^h(r)(\tau_2 - \tau_1)} \right)$$

$$= \frac{1}{2} \sum_{n=1}^{N_h} \sup_{0 \leq t \leq T} (x, e_n^h(t))^2 \left( \inf_{0 \leq r \leq T} \lambda_n^h(t) \right)^{-1} \left( 1 - e^{-2 \inf_{0 \leq r \leq T} \lambda_n^h(r)(\tau_2 - \tau_1)} \right)$$

$$= \frac{1}{2} \sum_{n=1}^{N_h} \sup_{0 \leq t \leq T} (x, e_n^h(t))^2 \left( \inf_{0 \leq r \leq T} \lambda_n^h(t) \right)^{-1} \left( 1 - e^{-2 \inf_{0 \leq r \leq T} \lambda_n^h(r)(\tau_2 - \tau_1)} \right)$$

Using the boundedness of the function $x \mapsto x^{\rho-1} (1 - e^{-x})$ for $x \in [0, \infty)$, $\rho \in [0, 1]$ and the Parseval’s identity, it follows from (83) that

$$\int_{\tau_1}^{T_2} \left\| (-A_h^1(r))^{\rho/2} S_h^1(r, \tau_2 - r) P_h x \right\|^2 \, dr \leq C(\tau_2 - \tau_1)^{1-\rho} \sum_{n=1}^{N_h} \sup_{0 \leq t \leq T} (x, e_n^h(t))^2$$

$$= C(\tau_2 - \tau_1)^{1-\rho} \sum_{n=1}^{N_h} (x, e_n^h(t))^2$$

$$\leq C(\tau_2 - \tau_1)^{1-\rho} \sup_{0 \leq t \leq T} \|x\|^2 = C(\tau_2 - \tau_1)^{1-\rho} \|x\|^2. \quad (84)$$

Substituting (84) in (83) completes the proof of (71). Let us now prove (72). Note that for $0 \leq \rho < 1$ the estimate (72) follows from Lemma 2.3. The crucial case is when $\rho = 1$. Note that the evolution parameter $U_h(\tau_2, r)$ satisfies the following integral equation, see e.g., [32 Chapter 5].

$$U_h(\tau_2, r) = S^h_\sigma(\tau_2 - r) + \int_{\tau_1}^{T_2} S^h_\sigma(\tau_2 - r) R^h(\sigma, r) d\sigma, \quad (85)$$
where $R^h(\sigma, r)$ is defined as follows, see \[32\] Chapter 5

\[ R^h(\sigma, r) = \sum_{n=1}^{\infty} R^h_n(\sigma, r), \quad (86) \]

where $R^h_n(\sigma, r)$ satisfies the following recurrence relation, see e.g., \[32\] Chapter 5

\[ R^h_1(\sigma, r) = (A_h(r) - A_h(\sigma)) S^h_{\sigma}(\sigma - r), \quad R^h_{n+1}(\sigma, r) = \int_{r}^{\sigma} R^h_n(\sigma, r) R^h_n(\gamma, r) d\gamma, \quad n \geq 1. \quad (87) \]

Using \[85\], the triangle inequality, the estimate $(a + b)^2 \leq 2a^2 + 2b^2$, $a, b \in \mathbb{R}^+$ and \[84\] yields

\[ \int_{\tau_1}^{T} \|(-A_h(\tau))^{\rho/2} U_h(\tau_2, r)\|_{L(H)}^2 d\tau \leq 2 \int_{\tau_1}^{T} \|(-A_h(\tau))^{\rho/2} S^h_{\sigma}(\tau_2 - r)\|_{L(H)}^2 d\tau \]

\[ + 2 \int_{\tau_1}^{T} \left\| \int_{r}^{\tau_2} (-A_h(\tau))^{\rho/2} S^h_{\sigma}(\tau_2 - \sigma) R^h(\sigma, r) d\sigma \right\|_{L(H)}^2 d\tau \]

\[ =: C(\tau_2 - \tau_1)^{1-\rho} + J_1. \quad (88) \]

Using Lemma 2.5 and \[30\] yields

\[ J_1 \leq \int_{\tau_1}^{T} \left( \int_{r}^{\tau_2} (-A_h(\tau))^{\rho/4} S^h_{\sigma}(\tau_2 - \sigma) \|(-A_h(\tau))^{\rho/4} R^h(\sigma, r)\|_{L(H)} d\sigma \right)^2 d\tau \]

\[ \leq C \int_{\tau_1}^{T} \left( \int_{r}^{\tau_2} (\tau_2 - \sigma)^{-\rho/4} (\sigma - r)^{-\rho/4} d\sigma \right)^2 d\tau \]

\[ \leq C \int_{\tau_1}^{T} \left( \int_{r}^{\tau_2} (\tau_2 - \sigma)^{-\rho/4} (\sigma - r)^{-\rho/4} d\sigma \right)^2 d\tau + C \int_{\tau_1}^{T} \left( \int_{r}^{\tau_2} (\tau_2 - \sigma)^{-\rho/4} (\sigma - r)^{-\rho/4} d\sigma \right)^2 d\tau \]

\[ \leq C \int_{\tau_1}^{T} (\tau_2 - r)^{-\rho/2} \left( \int_{r}^{\tau_2} (\sigma - r)^{-\rho/4} d\sigma \right)^2 d\tau + C \int_{\tau_1}^{T} (\tau_2 - r)^{-\rho/2} \left( \int_{r}^{\tau_2} (\tau_2 - \sigma)^{-\rho/4} d\sigma \right)^2 d\tau \]

\[ \leq C \int_{\tau_1}^{T} (\tau_2 - r)^{2-\rho} d\tau \]

\[ \leq C(\tau_2 - \tau_1)^{3-\rho}. \quad (89) \]

Substituting \[89\] in \[88\] yields

\[ \int_{\tau_1}^{T} \|(-A_h(\tau))^{\rho/2} U_h(\tau_2, r)\|_{L(H)}^2 d\tau \leq C(\tau_2 - \tau_1)^{1-\rho}. \quad (90) \]

This completes the proof of (72). The proof of (73) is similar to that of (72). \[\square\]

**Lemma 2.8.** Let $\rho \in [0, 1]$. Under Assumptions 1.4 and 1.8 the following estimates hold

\[ \int_{\tau_1}^{T} \|(-A_h(\tau))^{\rho} U_h(\tau_2, r)\|_{L(H)} d\tau \leq C(\tau_2 - \tau_1)^{1-\rho}, \quad 0 \leq \tau_1 \leq \tau_2 \leq T, \quad \tau \in [0, T], \quad (91) \]

\[ \int_{\tau_1}^{T} \|U_h(\tau_2, r)(-A_h(\tau))^{\rho}\|_{L(H)} d\tau \leq C(\tau_2 - \tau_1)^{1-\rho}, \quad 0 \leq \tau_1 \leq \tau_2 \leq T, \quad \tau \in [0, T]. \quad (92) \]
Proof. We only prove (91) since the proof of (92) is similar. Using (83) and triangle inequality yields
\[
\int_{\tau_1}^{\tau_2} \|(-A_h(\tau))^{\gamma} U_h(\tau_2, r)\|_{L(H)} \, dr \leq \int_{\tau_1}^{\tau_2} \|(-A_h(\tau_2 - \sigma))^{\gamma} S^h_r(\tau_2 - \sigma)\|_{L(H)} \, dr \\
+ \int_{\tau_1}^{\tau_2} \int_{\tau_2}^{\tau_2} \|(-A_h(\tau_2 - \sigma))^{\gamma} S^h_\sigma(\tau_2 - \sigma) R^h(\sigma, r)\|_{L(H)} \, d\sigma \, dr
\]
\[
= J_2 + J_3.
\]
Using Lemma 2.5 Hölder inequality and (71) yields
\[
J_2 \leq C \int_{\tau_1}^{\tau_2} \|(-A_h(r))^{\gamma} S^h_r(\tau_2 - r)\|_{L(H)} \, dr \\
= C \int_{\tau_1}^{\tau_2} \left\|(-A_h(r))^{\gamma/2} S^h_r \left( \frac{\tau_2 - r}{2} \right) (-A_h(r))^{\gamma/2} S^h_r \left( \frac{\tau_2 - r}{2} \right) \right\|_{L(H)} \, dr \\
\leq C \int_{\tau_1}^{\tau_2} \left(\frac{\tau_2 - r}{2}\right)^2 \left\|(-A_h(r))^{\gamma/2} S^h_r \left( \frac{\tau_2 - r}{2} \right) \right\|_{L(H)} \, dr \\
\leq C(\tau_2 - \tau_1)^{1-\gamma}.
\]
Using Lemma 2.5 yields
\[
J_3 \leq \int_{\tau_1}^{\tau_2} \int_{\tau_2}^{\tau_2} \left\|(-A_h(\tau_2 - \sigma))^{\gamma/2} S^h_\sigma(\tau_2 - \sigma)\right\|_{L(H)} \left\|(-A_h(\tau_2 - \sigma))^{\gamma/2} R^h(\sigma, r)\right\|_{L(H)} \, d\sigma \, dr \\
\leq C \int_{\tau_1}^{\tau_2} \int_{\tau_2}^{\tau_2} (\tau_2 - \sigma)^{-\gamma/2}(\sigma - r)^{-\gamma/2} \, d\sigma \, dr.
\]
Splitting the second integral of (95) in two parts as in the estimate of $J_1$ (89) and integrating yields
\[
J_3 \leq C(\tau_2 - \tau_1)^{2-\gamma}.
\]
Substituting (90) and (95) in (94) completes the proof of (91).

The following space and time regularity hold for the semi-discrete problem (38), and will be useful in our convergence analysis.

Lemma 2.9. Let Assumptions 1.4 (i) and (ii), 1.2 and 1.3 be fulfilled with the corresponding $0 \leq \beta < 2$. If $X_0 \in L^p(\Omega, D((-A(0))^{\beta/2}))$, $p \geq 2$, then for all $\gamma \in [0, \beta]$ the following regularity estimates hold
\[
\|(-A_h(0))^{\gamma/2} X^h(t)\|_{L^p(\Omega, H)} \leq C, \quad 0 \leq t \leq T, \tag{97}
\]
\[
\|X^h(t_2) - X^h(t_1)\|_{L^p(\Omega, H)} \leq C(t_2 - t_1)^{\min(\beta, 1)/2}, \quad 0 \leq t_1 \leq t_2 \leq T. \tag{98}
\]
Moreover, if Assumption 1.5 is fulfilled then the regularity estimates (97) and (98) still hold for $\beta = 2$.

Proof. The proof follows the same lines as that in (39) for multiplicative noise. Note that in the case of additive noise, (97) shows that we can have a spatial regularity estimate for $\gamma \in [0, 2)$. Note that in the case of multiplicative noise (39), we can only take $\gamma \in [0, 1)$; Note that the proof of Lemma 2.9 for $\beta = 2$ makes use of Lemmas 2.7 and 2.8. Note also that the optimal case (97) and (98) with $\beta = 2$ are crucial to achieve optimal convergence order in time, which corresponds to an analogue of the optimal regularity results in [20], for time independent self-adjoint operator $A$. □
For non commutative operators $H_j$ on a Banach space, we introduce the following notation

$$
\prod_{j=1}^{k} H_j = \begin{cases} 
H_k H_{k-1} \cdots H_l, & \text{if } k \geq l, \\
I, & \text{if } k < l.
\end{cases}
$$

(99)

The following lemma will be useful in our convergence proof.

**Lemma 2.10.** Let Assumption [1.4] be fulfilled. Then the following estimate holds

$$
\left\| \left( \prod_{j<l} e^{\Delta t A_{h,j}} \right) (-A_{h,l})^{\gamma_l} \left( \prod_{j=l}^{m} e^{\Delta t A_{h,j}} \right) (-A_{h,l})^{-\gamma_l} \right\|_{L(H)} \leq Ct_{m-l}^{-\gamma_l}, \quad 0 \leq l < m, \quad 0 \leq \gamma_l < 1,
$$

(100)

and

$$
\left\| \left( \prod_{j<l} e^{\Delta t A_{h,j}} \right) (-A_{h,l})^{\gamma_l} \left( \prod_{j=1}^{l-1} e^{\Delta t A_{h,j}} \right) (-A_{h,l})^{-\gamma_l} \right\|_{L(H)} \leq Ct_{m-l}^{-\gamma_l}, \quad 0 \leq l < m, \quad 0 \leq k \leq M
$$

(101)

$0 \leq \gamma_1 \leq 1, \quad 0 < \gamma_2 \leq 1.$

**Proof.** The proof can be found in [38].

Let us consider the following deterministic problem: find $u \in V$ such that

$$
\frac{du}{dt} = A(t)u, \quad u(\tau) = v, \quad t \in (\tau, T].
$$

(102)

The corresponding semi-discrete problem in space consists of finding $u^h \in V_h$ such that

$$
\frac{du^h}{dt} = A_h(t)u^h, \quad u^h(\tau) = P_h v.
$$

(103)

**Lemma 2.11.** Let Assumption [1.4] be fulfilled. For $v \in D((\cdot A(0))^{r/2})$, the following error estimate holds for the semi-discrete approximation of (102)

$$
\|u(t) - u^h(t)\|_{L_2(\Omega, H)} \leq Ch^r (t - \tau)^{-r-\alpha/2} \|v\|_{\alpha}, \quad 0 \leq \alpha \leq r, \quad r \in [0, 2].
$$

(104)

**Proof.** The proof can be found in [38].

**Lemma 2.12.** Let Assumptions [1.1-1.4] be fulfilled. Let $X(t)$ be the mild solution of (1) and the $X^h(t)$ the mild solution of (38).

(i) If $0 \leq \beta < 2$, then the following error estimate holds

$$
\|X(t) - X^h(t)\|_{L^{2}(\Omega, H)} \leq Ch^\beta, \quad 0 \leq t \leq T.
$$

(105)

(ii) If $\beta = 2$, then the following error estimate holds

$$
\|X(t) - X^h(t)\|_{L^{2}(\Omega, H)} \leq Ch^2 \left[ 1 + \max \left( \ln \left( \frac{t}{h^2} \right), 0 \right) \right], \quad 0 < t \leq T.
$$

(106)

**Proof.** The proof follows the same lines as the one in [39] for multiplicative noise.

The proof of the following lemma can be found in [39].
Lemma 2.13. Let Assumption 1.4 be fulfilled. For any $\alpha \in [-1, 1]$, the following estimate holds

$$
\| (U_h(t_j, t_{j-1}) - e^{\Delta t A_h, j-1}) (-A_{h,j-1})^\alpha \|_{L(H)} \leq C \Delta t^{1-\alpha}.
$$

The following lemma can be found in [23] or [39].

Lemma 2.14. For all $\alpha_1, \alpha_2 > 0$ and $\alpha \in [0, 1)$, there exist two positive constants $C_{\alpha_1 \alpha_2}$ and $C_{\alpha, \alpha_2}$ such that

$$
\Delta t \sum_{j=1}^{m} t_{m-j+1}^{-1+\alpha_1} t_j^{-1+\alpha_2} \leq C_{\alpha_1 \alpha_2} t_m^{-1+\alpha_1+\alpha_2}, \quad \Delta t \sum_{j=1}^{m} t_{m-j+1}^{-\alpha} t_j^{-1+\alpha_2} \leq C_{\alpha, \alpha_2} t_m^{-\alpha+\alpha_2}.
$$

Lemma 2.15. Let Assumption 1.4 be fulfilled.

(i) The following estimate holds for $1 \leq i \leq m$

$$
\left\| \left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i-1}^{m-1} e^{\Delta t A_h,j} \right) \right\|_{L(H)} \leq C \Delta t^{1-\epsilon}.
$$

(ii) The following estimate holds for $1 \leq i \leq m$

$$
\left\| \left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i-1}^{m-1} e^{\Delta t A_h,j} \right) (A_h(0))^{-\epsilon} \right\|_{L(H)} \leq C \Delta t,
$$

(iii) Then for all $1 \leq i \leq m \leq M$. For all $\alpha \in [0, 1)$, the following estimate holds

$$
\left\| \left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i-1}^{m-1} e^{\Delta t A_h,j} \right) (-A_{h,i-1})^\alpha \right\|_{L(H)} \leq C \Delta t^{1-\alpha-\epsilon} t_{m-i+1}^{-\alpha+\epsilon},
$$

for an arbitrarily small $\epsilon > 0$.

Proof. The proof of (i)-(ii) can be found in [39]. The estimate (111) is crucial to achieve higher convergence order in time on the terms involving the noise. Using the telescopic identity yields

$$
\left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m-1} e^{\Delta t A_h,j-1} \right) = \left( \prod_{j=i+1}^{m} U_h(t_j, t_{j-1}) \right) \left( U_h(t_i, t_{i-1}) - e^{\Delta t A_h,i-1} \right)
$$

$$
+ \sum_{k=2}^{m-i+1} \left( \prod_{j=i+k}^{m} U_h(t_j, t_{j-1}) \right) \left( U_h(t_{i+k-1}, t_{i+k-2}) - e^{\Delta t A_h,i+k-2} \right) \left( \prod_{j=i}^{i+k-2} e^{\Delta t A_h,j-1} \right),
$$

(112)
Taking the norm in both sides of (112), using Lemmas 2.5, 2.13, 2.10 and 2.14 yields
\[
\left\| \left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) - \left( \prod_{j=i}^{m} e^{\Delta t A_{h, j-1}} \right) (-A_{h, i-1})^\alpha \right) \right\|_{L(H)} 
\leq \left\| U_h(t_m, t_{i+k-1}) \right\|_{L(H)} \left\| (U_h(t_i, t_{i-1}) - e^{\Delta t A_{h, i-1}}) (-A_{h, i-1})^\alpha \right\|_{L(H)} 
+ \sum_{k=2}^{m-i+1} \left\| U_h(t_m, t_{i+k-1}) (-A_{h, i}) \right\|_{L(H)} \left\| (-A_{h, i})^{-1} (U_h(t_{i+k-1}, t_{i+k-2}) - e^{\Delta t A_{h, i+k-2}}) \right\|_{L(H)} 
\times \left\| \left( \prod_{j=i}^{i+k-2} e^{\Delta t A_{h, j-1}} \right) (-A_{h, i-1})^\alpha \right\|_{L(H)} 
\leq C \Delta t^{1-\alpha} + C \sum_{k=2}^{m-i+1} t_{m-i-k}^{-1} \Delta t^2 t_{k-1}^{-\alpha} 
\leq C \Delta t^{1-\alpha} + C \Delta t^{1-\alpha-\epsilon} \sum_{k=2}^{m-i+1} \Delta t t_{m-i-k}^{-1} t_{k-1}^{-\alpha} 
\leq C \Delta t^{1-\alpha-\epsilon} t_{m-i+1}^{-\alpha} 
\]  
\tag{113}
\]

Lemma 2.16. Under Assumption 1.2, the following estimates hold
\[
\| P_h F'(t, u)(v) \| \leq C \| v \|, \quad u, v \in H, \quad t \in [0, T], \tag{114}
\]
\[
\| (-A_h(\tau))^{-\tau} P_h F''(t, u)(v_1, v_2) \| \leq C \| v_1 \|, \| v_2 \|, \quad u, v_1, v_2 \in H, \quad t, \tau \in [0, T], \tag{115}
\]
where \( \eta \in [1, 2) \) comes from Assumption 1.3. Note that the first and second order derivatives are taken respect the second variable.

Proof. The proof is a combination of Lemma 2.4 and [40, Proposition 4.1].

After the above preparatory results, we can now prove our main result.

2.2. Proof of Theorem 1.10

Using triangle inequality, we split the fully discrete error in two parts as follows.
\[
\| X(t_m) - X_m^h \|_{L^2(\Omega, H)} \leq \| X(t_m) - X^h(t_m) \|_{L^2(\Omega, H)} + \| X^h(t_m) - X_m^h \|_{L^2(\Omega, H)} =: I + II. \tag{116}
\]

The space error \( I \) is estimated in Lemma 2.12. It remains to estimate the time error \( II \). Note that the mild solution of (58) can be written as follows.
\[
X^h(t_m) = U_h(t_m, t_{m-1}) X^h(t_{m-1}) + \int_{t_{m-1}}^{t_m} U_h(t_m, s) P_h F(s, X^h(s)) ds + \int_{t_{m-1}}^{t_m} U_h(t_m, s) P_h dW(s). \tag{117}
\]
Iterating the mild solution (117) yields

\[
X^h(t_m) = \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) \right) P_h X_0 + \int_{t_{m-1}}^{t_m} U_h(t_m, s) P_h F(s, X^h(s)) \, ds + \int_{t_{m-1}}^{t_m} U_h(t_m, s) P_h dW(s)
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) P_h F(s, X^h(s)) \, ds
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) P_h dW(s).
\]

(118)

Iterating the numerical scheme (111) by substituting \(X^h_j\), \(j = m - 1, \cdots, 1\) only in the first term of (111) by their expressions yields

\[
X^h_m = \left( \prod_{j=0}^{m-1} e^{\Delta t A_{h,j}} \right) X^h_0 + \int_{t_{m-1}}^{t_m} e^{(t_{m-1} - s) A_{h,m-1}} P_h F(t_{m-1}, X^h_{m-1}(s)) \, ds + \int_{t_{m-1}}^{t_m} e^{\Delta t A_{h,m-1}} P_h dW(s)
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{(t_{m-k-1} - s) A_{h,m-k-1}} P_h F(t_{m-k-1}, X^h_{m-k-1}(s)) \, ds
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{\Delta t A_{h,m-k-1}} P_h dW(s).
\]

(119)

Substracting (119) from (118) yields

\[
X^h(t_m) - X^h_m = \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) \right) P_h X_0 - \left( \prod_{j=0}^{m-1} e^{\Delta t A_{h,j}} \right) P_h X_0 + \int_{t_{m-1}}^{t_m} U_h(t_m, s) P_h F(s, X^h(s)) \, ds
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) P_h F(s, X^h(s)) \, ds
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} e^{\Delta t A_{h,j}} \right) e^{(t_{m-k-1} - s) A_{h,m-k-1}} P_h F(t_{m-k-1}, X^h_{m-k-1}(s)) \, ds
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} e^{\Delta t A_{h,j}} \right) e^{\Delta t A_{h,m-k-1}} P_h dW(s)
\]

\[=: H_1 + H_2 + H_3 + H_4 + H_5.
\]

(120)

Taking the norm in both sides of (120) yields

\[
\|X^h(t_m) - X^h_m\|_{L^2(\Omega, H)}^2 \leq 25 \sum_{i=1}^{5} \|H_i\|_{L^2(\Omega, H)}^2.
\]

(121)
We estimate separately $\|II_i\|_{L^2(\Omega, H)}$, $i = 1, \cdots, 5$.

2.2.1. Estimate of $II_1$, $II_2$ and $II_3$

Using Lemma 2.15 (ii), it holds that

$$\|II_1\|_{L^2(\Omega, H)} \leq \left\| \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) - \left( \prod_{j=0}^{m-1} e^{\Delta t A_{h,j}} \right) \right) (-A_{h}(0))^{-\beta/2} \right\|_{L(H)} \|(-A_{h}(0))^{\beta/2} X_0\|_{L^2(\Omega, H)}$$

$$\leq C\Delta t \| - A(0)^{\beta/2} X_0\|_{L^2(\Omega, H)} \leq C\Delta t.$$  \hspace{1cm} (122)

Similarly to [39], we have the following estimate

$$\|II_2\|_{L^2(\Omega, H)} \leq C\Delta t + C\Delta t \|X^h(t_{m-1}) - X^h_{m-1}\|_{L^2(\Omega, H)}.$$  \hspace{1cm} (123)

To estimate $II_3$, we split it in two terms as follows

$$II_3 \,:= \, II_{31} + II_{32}.$$  \hspace{1cm} (124)

Applying the Itô-isometry property, using Lemmas 2.2 and 2.3 yields

$$\|II_{31}\|^2_{L^2(\Omega, H)} \,:= \, \int_{t_{m-1}}^{t_m} \mathbb{E} \left\| (U_h(t_m, s) - U_h(t_m, t_{m-1})) P_h Q^\frac{1}{2} \right\|^2_{L(H)} ds$$

$$\leq \int_{t_{m-1}}^{t_m} \left\| U_h(t_m, s) \left( I - U_h(s, t_{m-1}) \right) (-A_{h,m})^{\beta/2} \right\|^2_{L(H)} \left\| (-A_{h,m})^{-\beta/2} P_h Q^\frac{1}{2} \right\|^2_{L(H)} ds$$

$$\leq C \int_{t_{m-1}}^{t_m} \left\| U_h(t_m, s) \right\|^2_{L(H)} \left\| (-A_{h,m})^{\beta/2} \left( I - U_h(s, t_{m-1}) \right) (-A_{h,m})^{-\beta/2} \right\|^2_{L(H)} ds$$

$$\leq C \int_{t_{m-1}}^{t_m} (t_m - s)^{-\beta} \left( s - t_{m-1} \right)^{-\beta/\epsilon} ds$$

$$\leq C\Delta t^{\beta - \epsilon} \int_{t_{m-1}}^{t_m} (t_m - s)^{-1+\epsilon} ds \leq C\Delta t^\beta.$$  \hspace{1cm} (125)

Applying the Itô-isometry property, using Lemmas 2.2, 2.3 (ii) (if $\beta \geq 1$), Lemma 2.15 (iii) with $\alpha = \frac{1-\beta}{2}$ (if $\beta < 1$) yields

$$\|II_{32}\|^2_{L^2(\Omega, H)} \,:= \, \int_{t_{m-1}}^{t_m} \mathbb{E} \left\| (U_h(t_m, t_{m-1}) - e^{\Delta t A_{h,m-1}}) P_h Q^\frac{1}{2} \right\|^2_{L(H)} ds$$

$$\leq \int_{t_{m-1}}^{t_m} \left\| U_h(t_m, t_{m-1}) \right\|^2_{L(H)} \left\| (-A_{h,m-1})^{\beta/2} \right\|^2_{L(H)} ds$$

$$\leq C \int_{t_{m-1}}^{t_m} \Delta t^{1+\beta} ds \leq C\Delta t^{2+\beta}.$$  \hspace{1cm} (126)

Substituting (126) and (125) in (124) yields

$$\|II_3\|^2_{L^2(\Omega, H)} \leq C\Delta t^\beta.$$  \hspace{1cm} (127)
2.2.2. Estimate of $I_{44}$

To estimate $I_{44}$, we split it in five terms as follows.

\[
I_{44} = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) \left[ P_h F \left( s, X_h(s) \right) - P_h F \left( t_{m-k-1}, X_h(t_{m-k-1}) \right) \right] ds
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) \left[ U_h(t_{m-k}, s) - U_h(t_{m-k}, t_{m-k-1}) \right] P_h F \left( t_{m-k-1}, X_h(t_{m-k-1}) \right) ds
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left[ \prod_{j=m-k}^{m} U_h(t_j, t_{j-1}) \right] - \left( \prod_{j=m-k-1}^{m-1} e^{\Delta t A_{h,j}} \right) P_h F \left( t_{m-k-1}, X_h(t_{m-k-1}) \right) ds
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{(t_{m-k}-s)A_{h,m-k-1}} \left[ P_h F \left( t_{m-k-1}, X_h(t_{m-k-1}) \right) - P_h F \left( t_{m-k-1}, X_h(t_{m-k-1}) \right) \right] ds
\]

\[=: I_{441} + I_{442} + I_{443} + I_{444} + I_{445}.
\]

Similarly to [39], we have the following estimate

\[
\|I_{442}\|_{L^2(\Omega, H)} + \|I_{443}\|_{L^2(\Omega, H)} + \|I_{444}\|_{L^2(\Omega, H)} \leq C\Delta t.
\]

It remains to estimate $\|I_{441}\|_{L^2(\Omega, H)}$ and $\|I_{445}\|_{L^2(\Omega, H)}$. Let us start with the estimate of $\|I_{441}\|_{L^2(\Omega, H)}$ as it is easy. Using Lemma 2.10 and Assumption 1.2 yields

\[
\|I_{445}\|_{L^2(\Omega, H)} \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \|X_h(t_{m-k-1}) - X_h(t_{m-k-1})\|_{L^2(\Omega, H)} \leq C\Delta t \sum_{k=0}^{m-1} \|X_h(t_k) - X_h(t_k)\|_{L^2(\Omega, H)}.
\]

To estimate $I_{441}$, we decompose it in two terms as follows.

\[
I_{441} = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} U_h(t_{m-k}, s) \left[ P_h F \left( s, X_h(s) \right) - P_h F \left( t_{m-k-1}, X_h(t_{m-k-1}) \right) \right] ds
\]

\[+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} U_h(t_{m-k}, s) \left[ P_h F \left( t_{m-k-1}, X_h(t_{m-k-1}) \right) - P_h F \left( t_{m-k-1}, X_h(t_{m-k-1}) \right) \right] ds
\]

\[=: I_{4411} + I_{4412}.
\]

Using Assumption 1.2 and Lemma 2.5 yields

\[
\|I_{4411}\|_{L^2(\Omega, H)} \leq C\Delta t^{3/2}.
\]
To achieve higher order in $II_{412}$, we apply Taylor’s formula in Banach space to $F$. This yields

$$
II_{412} = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} U_h(t_m, s) P_h F'(t_{m-k-1}, X^h(t_{m-k-1})) (U_h(s, t_{m-k-1}) - I) X^h(t_{m-k-1}) ds \\
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} U_h(t_m, s) P_h F'(t_{m-k-1}, X^h(t_{m-k-1})) \int_{t_{m-k-1}}^{s} U_h(s, \sigma) P_h F(\sigma, X^h(\sigma)) d\sigma ds \\
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} U_h(t_m, s) P_h F'(t_{m-k-1}, X^h(t_{m-k-1})) \int_{t_{m-k-1}}^{s} U_h(s, \sigma) P_h dW(\sigma) ds \\
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} U_h(t_m, s) \chi^h ds \\
=: II_{412}^{(1)} + II_{412}^{(2)} + II_{412}^{(3)} + II_{412}^{(4)}.
$$

(133)

where

$$
\chi^h = \int_0^1 P_h F''(t_{m-k-1}, X^h(t_{m-k-1}) + \lambda (X^h(s) - X^h(t_{m-k-1}))) \\
\cdot (X^h(s) - X^h(t_{m-k-1}), X^h(s) - X^h(t_{m-k-1})) (1 - \lambda) d\lambda.
$$

(134)

Using Lemmas 2.5, 2.16, 2.9 and 2.8 yields

$$
\|II_{412}^{(1)}\|_{L^2(\Omega, H)} \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| (U_h(s, t_{m-k-1}) - I) (-A_{h,m-k-1})^{\beta/2} \right\|_{L(H)} ds \\
\times \left\| (-A_{h,m-k-1})^{\beta/2} X^h(t_{m-k-1}) \right\|_{L^2(\Omega, H)} ds \\
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} (s - t_{m-k-1})^{\beta/2} ds \leq C \Delta t^{\beta/2}.
$$

(135)

Using Lemma 2.5, Assumption 1.2, Lemmas 2.2 and 2.16 yields

$$
\|II_{412}^{(2)}\|_{L^2(\Omega, H)} \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \int_{t_{m-k-1}}^{s} d\sigma ds \leq C \Delta t.
$$

(136)
Applying the Itô-isometry property, using the fact that the expectation of the cross-product vanishes, Hölder inequality, Lemmas 2.16, 2.2 and 2.5 yields

\[
\|II_{412}\|_{L^2(\Omega, H)}^2 = \sum_{k=1}^{m-1} \mathbb{E} \left[ \left\| U_h(t_m, s) P_h F' \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \int_{t_{m-k-1}}^s U_h(s, \sigma) P_h dW(\sigma) d\sigma \right\|^2 \right]
\leq \Delta t \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \int_{t_{m-k-1}}^s \mathbb{E} \left[ \left\| U_h(t_m, s) P_h F' \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \right\|^2_{L(H)} \right]
\times \left\| U_h(s, \sigma) P_h Q^2 \right\|_{L^2(H)}^2 d\sigma ds
\leq C \Delta t \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \int_{t_{m-k-1}}^s \left\| U_h(s, \sigma) (-A_h(\sigma))^{\frac{1-\beta}{2}} \right\|^2_{L(H)} \left\| (-A_h(\sigma))^{\frac{\beta-1}{2}} P_h Q^2 \right\|^2_{L^2(H)} d\sigma ds
\leq C \Delta t \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \int_{t_{m-k-1}}^s (s-\sigma)^{\min(-1+\beta, 0)} d\sigma ds \leq C \Delta t^{\min(1+\beta, 2)}.
\]

(137)

Using Lemmas 2.16 and 2.9 it follows from (134) that

\[
\left\| (-A_{h,m-k-1})^{-\frac{\beta}{2}} X^h \right\|_{L^2(\Omega, H)} \leq C \left\| X^h(s) - X^h(t_{m-k-1}) \right\|_{L^2(\Omega, H)} \leq C \Delta t^{\min(\beta, 1)}.
\]

Hence, from (133), using (138), Lemma 2.5 we have

\[
\|II_{412}\|_{L^2(\Omega, H)} \leq \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| U_h(t_m, s) \left( -A_{h,m-k-1} \right)^{-\frac{\beta}{2}} \right\|_{L(H)} \left\| \left( -A_{h,m-k-1} \right)^{-\frac{\beta}{2}} X^h \right\|_{L^2(\Omega, H)} \, ds
\leq C \Delta t^{\min(\beta, 1)} \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} (t_m - s)^{\frac{\beta}{2}} ds
\leq C \Delta t^{\min(\beta, 1)} \int_{0}^{t_m} (t_m - s)^{-\frac{\beta}{2}} ds \leq C \Delta t^{\min(\beta, 1)}.
\]

(139)

Substituting (139), (137), (136) and (135) in (133) yields

\[
\|II_{412}\|_{L^2(\Omega, H)} \leq C \Delta t^{\beta/2}.
\]

(140)

Substituting (130) and (132) in (131) yields

\[
\|II_{41}\|_{L^2(\Omega, H)} \leq C \Delta t^{\beta/2}.
\]

(141)

Substituting (141), (129) and (130) in (128) yields

\[
\|II_{4}\|_{L^2(\Omega, H)} \leq C \Delta t^{\beta/2} + C \Delta t \sum_{k=0}^{m-1} \| X^h(t_k) - X^h_k \|_{L^2(\Omega, H)}.
\]

(142)
2.2.3. Estimate of $I_5$

To estimate $I_5$, we split it into two terms as follows

$$
I_5 = \sum_{k=1}^{m-1} \int_{t_m-k}^{t_m-k-1} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) \left[ U_h(t_m-k, s) - U_h(t_{m-k}, t_{m-k-1}) \right] P_h dW(s)
+ \sum_{k=1}^{m-1} \int_{t_m-k}^{t_m-k-1} \left( \prod_{j=m-k}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=m-k-1}^{m-1} e^{\Delta t(A_{h,j})} \right) P_h dW(s)
=: VI_{51} + VI_{52}.
$$

Applying the Itô-isometry property, using Lemmas 2.13 (iii) and 2.2 yields

$$
\left\| I_{51} \right\|_{L^2(\Omega, H)}^2 = \sum_{k=1}^{m-1} \int_{t_m-k}^{t_m-k-1} \mathbb{E} \left( \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_m-k, s) (1 - U_h(s, t_{m-k-1})) P_h Q^\frac{1}{2} \right)^2 ds
\leq \sum_{k=1}^{m-1} \int_{t_m-k}^{t_m-k-1} \left\| U_h(t_m, s)(-A_{h,m-k-1})^{\frac{1}{2}} \right\|_{L(H)}^2
\times \left\| (-A_{h,m-k-1})^{\frac{1}{2}} (1 - U_h(s, t_{m-k-1})) (-A_{h,m-k-1})^{\frac{1}{2}} P_h Q^\frac{1}{2} \right\|_{L^2(H)}^2 ds
\leq C \sum_{k=1}^{m-1} \int_{t_m-k}^{t_m-k-1} \left\| U_h(t_m, s)(-A_{h,0})^{\frac{1}{2}} \right\|_{L(H)}^2 (s - t_{m-k-1})^\beta ds
\leq C \Delta t^\beta \sum_{k=1}^{m-1} \int_{t_m-k}^{t_m-k-1} \left\| U_h(t_m, s)(-A_{h,0})^{\frac{1}{2}} \right\|_{L(H)}^2 ds
\leq C \Delta t^\beta.
$$

Applying the Itô-isometry property, using Lemmas 2.13 (iii) and 2.2 yields

$$
\left\| I_{52} \right\|_{L^2(\Omega,H)}^2 = \sum_{k=1}^{m-1} \int_{t_m-k}^{t_m-k-1} \left\| \left( \prod_{j=m-k}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=m-k-1}^{m-1} e^{\Delta t(A_{h,j})} \right) \right\|_{L^2(H)}^2 ds
\leq \sum_{k=1}^{m-1} \int_{t_m-k}^{t_m-k-1} \left\| \left( \prod_{j=m-k}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=m-k-1}^{m-1} e^{\Delta t(A_{h,j})} \right) (-A_{h,m-k-1})^{\frac{1}{2} + \frac{1}{2}} \right\|_{L(H)}^2 ds
\times \left\| (-A_{h,m-k-1})^{\frac{1}{2} + \frac{1}{2}} P_h Q^\frac{1}{2} \right\|_{L^2(H)}^2 ds
\leq C \Delta t^{1+\beta} \sum_{k=2}^{m-1} \int_{t_m-k}^{t_m-k-1} t_k^{1+\beta + \epsilon} ds \leq C \Delta t^\beta.
$$
Substituting (145) and (141) in (143) yields
\[ \left\| II_h \right\|^2_{L^2(\Omega,H)} \leq C\Delta t^\beta. \] (146)

Substituting (146), (142), (122) and (123) in (120) yields
\[ \left\| X^h(t_m) - X^h_m \right\|^2_{L^2(\Omega,H)} \leq C\Delta t^\beta + C\Delta t \sum_{k=0}^{m-1} \left\| X^h(t_k) - X^h_k \right\|^2_{L^2(\Omega,H)}. \] (147)

Applying the discrete Gronwall’s lemma to (147) completes the proof of Theorem 1.10

3. Numerical experiments

We consider the reaction diffusion equation
\[ dX = [D(t)\Delta X - k(t)X]dt + dW \quad \text{given} \quad X(0) = X_0 = 0, \] (148)

in the time interval \([0,T]\) with diffusion coefficient \(D(t) = (1/10)(1 + e^{-t})\) and reaction rate \(k(t) = 1\) on homogeneous Neumann boundary conditions on the domain \(\Lambda = [0,L] \times [0,L]\). We take \(L_1 = L_2 = 1\). Our function \(F(t,u) = k(t)u\) is linear and obviously satisfies Assumption 1.2. Since \(F(t,u)\) is linear on the second variable, it holds that \(F(t,u)v = k(t)v\) for all \(u,v \in L^2(\Lambda)\), where \(F\) stands for the differential with respect to the second variable. Therefore \(\|F(t,u)\|_{L(H)} = |k(t)| = 1\) for all \(u \in L^2(\Lambda)\). Obviously we have \(F''(t,u) = 0\), for all \(u : L^2(\Lambda)\). In general, we are interested in nonlinear \(F\) however for this linear system we can find a good approximation of the exact solution to compare our numerics to. The eigenfunctions \(\left\{e_i^{(1)}, e_j^{(2)}\right\}_{i,j \geq 0}\) of the operator \(-\Delta\) here are given by
\[ e_i^{(l)} = \sqrt{\frac{1}{L^l}} \lambda_i^{(l)} = 0, \quad e_i^{(l)} = \sqrt{\frac{2}{L^l}} \cos(\lambda_i^{(l)}x), \quad \lambda_i^{(l)} = \frac{i\pi}{L^l}, \] (149)

where \(l \in \{1,2\}\) and \(i \in \{1,2,3,\ldots\}\) with the corresponding eigenvalues \(\left\{\lambda_{i,j}\right\}_{i,j \geq 0}\) given by \(\lambda_{i,j} = (\lambda_i^{(1)})^2 + (\lambda_j^{(2)})^2\). The linear operator is \(A(t) = D(t)\Delta\) and has the same eigenfunctions as \(\Delta\), but with eigenvalues \(\left\{D(t)\lambda_{i,j}\right\}_{i,j \geq 0}\). Clearly we have \(D(A(t)) = H^2(\Lambda)\) and \(D((A(t))^\alpha) = D((A(0))^\alpha)\) for all \(t \in [0,T]\), and \(0 \leq \alpha \leq 1\). Since \(D(t)\) is bounded below by \((1/10)(1 + e^{-T})\), it follows that the ellipticity condition (24) and therefore as a consequence of the analysis in Section 2, it follows that \(A(t)\) are uniformly sectorial. Obviously Assumption 1.4 and (43) are also fulfilled. We also used
\[ q_{i,j} = (i^2 + j^2)^{-(\beta+\delta)}, \quad \beta > 0, \] (150)
in the representation (150) for some small \(\delta > 0\). Here the noise and the linear operator are supposed to have the same eigenfunctions. We obviously have
\[ \sum_{(i,j) \in \mathbb{N}^2} \lambda_{i,j}^{-\beta-1} q_{i,j} < \pi^2 \sum_{(i,j) \in \mathbb{N}^2} (i^2 + j^2)^{-(1+\delta)} < \infty, \] (151)

thus Assumption 1.3 is satisfied. In our simulations, we take \(\beta \in \{1.5,2\}\), with \(\delta = 0.001\). The close form of the exact solution of (148) is known. Indeed using the representation of the noise in (150), the decomposition of (148) in each eigenvector node yields the following Ornstein-Uhlenbeck process
\[ dX_i = -(D(t)\lambda_i + k(t))X_i dt + \sqrt{q_i} d\beta_i(t) \quad i \in \mathbb{N}^2. \] (152)
This is a Gaussian process with the mild solution

\[ X_i(t) = e^{-\int_0^t b_i(s) ds} \left[ X_i(0) + \sqrt{q_i} \int_0^t e^{\int_0^s b_i(y) dy} d\beta_i(s) \right], \quad b_i(t) = D(t) \lambda_i + k(t). \]  

Applying the Ito isometry yields the following variance of \( X_i(t) \)

\[ \text{Var}(X_i(t)) = q_i e^{-\int_0^t 2 b_i(s) ds} \left( \int_0^t e^{\int_0^s 2 b_i(y) dy} ds \right). \]

During simulation, we compute the exact solution recurrently as

\[ X_i^{m+1} = e^{-\int_{t_m}^{t_{m+1}} b_i(s) ds} X_i^m + \left( q_i e^{-\int_{t_m}^{t_{m+1}} 2 b_i(s) ds} \left( \int_{t_m}^{t_{m+1}} e^{\int_{t_m}^s 2 b_i(y) dy} ds \right) \right)^{1/2} R_{i,m}, \]

where \( R_{i,m} \) are independent, standard normally distributed random variables with mean 0 and variance 1. Note that the integrals involved in (155) are computed exactly for the first integral and accurately approximated for the second integral.

![Figure 1](image-url)  

**Figure 1.** Convergence of the stochastic Magnus scheme for \( \beta = 1.5 \) and \( \beta = 2 \) in (150). The order of convergence in time is 0.995 for \( \beta = 2 \), and 0.7561 for \( \beta = 1.5 \). The total number of samples used is 100.

In Figure 1 we can observe the convergence of the stochastic Magnus scheme for two noise's parameters. Indeed the order of convergence in time is 0.995 for \( \beta = 2 \), and 0.7561 for \( \beta = 1.5 \). These orders are close to the theoretical orders 1 and 0.75 obtained in Theorem 1.10 for \( \beta = 2 \) and \( \beta = 1.5 \) respectively.

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