Nonlocal constants of motion
in Lagrangian Dynamics of any order

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Abstract
We describe a recipe to generate “nonlocal” constants of motion for ODE
Lagrangian systems. As a sample application, we recall a nonlocal constant of
motion for dissipative mechanical systems, from which we can deduce global
existence and estimates of solutions under fairly general assumptions. Then
we review a generalization to Euler-Lagrange ODEs of order higher than two,
leading to first integrals for the Pais-Uhlenbeck oscillator and other systems.
Future developments may include adaptations of the theory to Euler-Lagrange
PDEs.

Keywords: Higher-order Lagrangians, nonlocal constants, first integrals, dissi-
pative mechanical systems, Pais-Uhlenbeck oscillator.

Dedicated to prof. Chaudry Masood Khalique
on the occasion of his retirement

1 Introduction

We are interested in constants of motion for the Euler-Lagrange equation
\[
\frac{d}{dt} \partial_q L(t, q(t), \dot{q}(t)) - \partial_t L(t, q(t), \dot{q}(t)) = 0,
\]
where $L(t, q, \dot{q})$ is a smooth scalar valued Lagrangian function, $t \in \mathbb{R}$, $q, \dot{q} \in \mathbb{R}^n$. In the paper [5] the first and the last author revisited Noether’s Theorem, which links first integrals with symmetries of the Lagrangian $L$. Leaving aside asynchronous perturbations and boundary terms and other issues, here we single out the following simple result. (Notation: the central dot is the scalar product in $\mathbb{R}^n$).

**Theorem 1.1.** Let $q(t)$ be a solution to the Euler-Lagrange equation and let $q_\lambda(t)$, $\lambda \in \mathbb{R}$, be a smooth family of perturbed motions, such that $q_0(t) \equiv q(t)$. Then the following function of $t$ is constant

$$\partial_t L(t, q(t), \dot{q}(t)) \cdot \partial_\lambda q_\lambda(t)|_{\lambda=0} - \int_{t_0}^t \partial_\lambda L(s, q_\lambda(s), \dot{q}_\lambda(s))|_{\lambda=0} ds.$$  \hspace{1cm} (1.2)

The proof is straightforward: we just take the derivative of the function in (1.2) and use the Euler-Lagrange equation and reverse the order of a double derivative.

We call (1.2) the constant of motion associated to the family $q_\lambda(t)$. For a random family, we may expect the constant of motion to be trivial or inconsequential. In general it is nonlocal, which means that its value at a time $t$ depends not only on the current state $(t, q(t), \dot{q}(t))$ at time $t$, but also on the whole history between $t_0$ and $t$.

In the original spirit of Noether’s theorem we can concentrate the attention to families $q_\lambda(t)$ which make the integrand in (1.2) vanish whenever $L$ enjoys an invariance property. For instance, for the Lagrangian of a particle in the plane under a central force field

$$L(t, q, \dot{q}) := \frac{1}{2} m \|\dot{q}\|^2 - U(t, \|q\|), \quad q = (q_1, q_2) \in \mathbb{R}^2,$$  \hspace{1cm} (1.3)

the rotation family

$$q_\lambda(t) := \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}$$  \hspace{1cm} (1.4)

exploits the invariance of $L$ under rotations when plugged into (1.2), and leads to the conservation of the angular momentum $\det(q(t), \dot{q}(t))$, which is definitely a local first integral.

In a series of papers we have used Theorem 1.1 to find numerous nonlocal constants of motion which are useful. Here is a partial list.

- Consider a particle moving in a time-independent potential field $U(q), q \in \mathbb{R}^n$ and viscous, i.e., linear, fluid resistance. We see at once that we have global existence in the future of the solutions to the Euler-Lagrange equation. What about the existence in the past? For a quite natural choice of the family $q_\lambda(t)$, after an integration by parts the integrand in (1.2) becomes negative, provided we assume $U \geq 0$. Under these conditions the first term in (1.2) is an increasing function of time, which permits to prove the global existence in the past. The paper [6] also obtains other estimates for the solutions for this system and for some Lane-Emden equation for which global existence is proved too. We have picked this example for a detailed exposition in Section 2 below, as an illustration of one way of using our nonlocal constants of motion.
• General time-independent homogeneous potentials of degree $-2$, for instance Calogero’s potential for the one dimensional $n$-body problem with inversely quadratic pair potentials, taken from [5], for which an explicit formula of the time-dependence of the distance from the origin is proved, even though we do not know the shape of the orbit.

• The Maxwell-Bloch equations for conservative laser dynamics taken from [7], where (1.2) by derivation permits to separate the equations (in a nonstandard way) into a system exhibiting “fish dynamics” and a system with central force.

Theorem 1.1 can be generalized to nonvariational Lagrange equations, as we do in [8], among other results on Killing-type equations, and applications for:

• A particle under a time-independent potential field $U(q)$, $q \in \mathbb{R}^n$, and hydraulic, i.e. quadratic, fluid resistance, a result taken from Gorni-Zampieri [10], for which the result in dynamics is: if $0 \leq U \leq U_{\text{sup}} < +\infty$, all the solutions for which the initial kinetic energy is strictly greater than $U_{\text{sup}}$ explode in the past in finite time.

• The dissipative Maxwell-Bloch equations for laser dynamics, taken from Gorni-Residori-Zampieri [9], for which a quite natural family $q_{\lambda}(t)$ yields a constant of motion (1.2) which, for a special choice of the parameters, turns out to be a genuine first integral $N(t, q, \dot{q})$ since the integral term vanishes. The first integral permits some kind of separation of variables.

The paper [10] presents the result on hydraulic fluid resistance and gives a survey on all other applications we mentioned till now. The novelty in the present survey is the introduction of Scomparin’s generalization of Theorem 1.1 to higher-order Lagrangian systems, which recently appeared in [18].

For $N = 1, 2, \ldots$ consider the higher-order Euler-Lagrange equation

$$\sum_{k=0}^{N} (-1)^k \frac{d^k}{dt^k} \partial_{q^{(k)}} L(t, q, \ldots, q^{(N)}) = 0,$$  \hspace{1cm} (1.5)

where the $N^{th}$-order Lagrangian $L$ is a smooth function with $t \in \mathbb{R}$, and $q, \ldots, q^{(N)} \in \mathbb{R}^n$. We use $q^{(k)} \equiv d^k q/dt^k$.

Within the higher-order framework, a first integral is a smooth function

$$K(t, q, q^{(1)}, q^{(2)}, \ldots)$$

that is constant along all solutions of the Euler-Lagrange equation (1.5). The celebrated Noether’s Theorem establishes a relation between invariance properties of a Lagrangian and its first integrals [2, 4].

Scomparin’s paper[18] generalizes Theorem 1.1 to $N^{th}$-order Lagrangians:

**Theorem 1.2.** Let $t \mapsto q(t)$ be a solution to the Euler-Lagrange equation for smooth $L(t, q, \ldots, q^{(N)})$, and let $q_{\lambda}(t)$, $\lambda \in \mathbb{R}$, be a smooth family of perturbed motions, such
that \(q_0(t) \equiv q(t)\). Then the following function of \(t\) is constant:

\[
\sum_{j=1}^{N} \sum_{k=0}^{j-1} (-1)^k \frac{d^k}{dt^k} \partial_{q^{(j)}} L(t, q, \ldots, q^{(N)}) \cdot \partial_{q^{(j-k-1)}} |_{\lambda=0} - \int_{t_0}^{t} \frac{d}{\partial \lambda} L(s, q_{\lambda}, \ldots, q^{(N)}_{\lambda}) |_{\lambda=0} ds. \tag{1.6}
\]

A basic higher-order mechanical system is the *Pais-Uhlenbeck oscillator* [17], whose Lagrangian function is

\[
L_{PU} = \frac{1}{2} q^{(2)^2} - \frac{1}{2} (w_1^2 + w_2^2) q^{(1)^2} + \frac{1}{2} w_1^2 w_2^2 q^2, \tag{1.7}
\]

with \(w_1, w_2 > 0\). Its Euler-Lagrange equation is

\[
q^{(4)} + (w_1^2 + w_2^2) q^{(2)} + w_1^2 w_2^2 q = 0. \tag{1.8}
\]

Higher-order Lagrangians provide a very large class of models for modified gravity theories [11], quantum-loop cosmologies [12], and string theories [14]. Approaching higher-order mechanics from a new nonlocal point of view provides new perspectives to identify novel first integrals without necessarily requiring invariance proprieties on the already difficult to investigate structure of higher-order Lagrangians. Hopefully Theorem 1.2 will provide a valuable tool to give a novel insight into stability proprieties of higher-order models and boundedness of related solutions as is done for second order equations in Kaparulin [16, 15] and in the papers mentioned above.

Section 3 below summarizes the main results in Scomparin’s paper [18].

### 2 An application to dissipative dynamics

The results in this section are taken from Gorni-Zampieri [6].

Let us consider the first integral of *energy* which is generally derived in Noether’s framework by the use of asynchronous perturbations but can also be treated by means of Theorem 1.1. For a time independent Lagrangian function \(L(t, q, \dot{q}) = \mathcal{L}(q, \dot{q})\), and the *time-shift* family \(q_{\lambda}(t) = q(t + \lambda)\) we have

\[
\partial_{\lambda} L(t, q_{\lambda}(t), \dot{q}_{\lambda}(t)) |_{\lambda=0} = \partial_{q} \mathcal{L} \cdot \dot{q}(t) + \partial_{\dot{q}} \mathcal{L} \cdot \ddot{q}(t) = \frac{d}{dt} \mathcal{L}(q(t), \dot{q}(t)). \tag{2.1}
\]

Thus the constant of motion (1.2) is

\[
\partial_{q} \mathcal{L} \cdot \dot{q}(t) - \int_{t_0}^{t} \frac{d}{ds} \mathcal{L}(q(s), \dot{q}(s)) ds = \partial_{q} \mathcal{L}(q(t), \dot{q}(t)) \cdot \dot{q}(t) - \mathcal{L}(q(t), \dot{q}(t)) + \mathcal{L}(q(t_0), \dot{q}(t_0))
\]

which is energy

\[
E(q, \dot{q}) = \partial_{q} \mathcal{L}(q, \dot{q}) \cdot \dot{q} - \mathcal{L}(q, \dot{q}) \tag{2.2}
\]

up to the trivial additive constant \(\mathcal{L}(q(t_0), \dot{q}(t_0))\).
Now, we turn to dissipation. Consider $k > 0$, a smooth potential function $U : \mathbb{R}^n \to \mathbb{R}$ defined on the whole space. The equation of motion of a particle of mass $m > 0$ under this potential and viscous dissipation is

$$m\ddot{q} = -k\dot{q} - \nabla U(q), \quad q \in \mathbb{R}^n. \quad (2.3)$$

The energy first integral (2.2) for $k = 0$ is

$$E(q, \dot{q}) = \frac{1}{2}m\|\dot{q}\|^2 + U(q). \quad (2.4)$$

For $k > 0$ this function decreases along solutions

$$\dot{E} = m\ddot{q} \cdot \frac{1}{m}(-k\dot{q} - \nabla U(q)) + \nabla U(q) \cdot \dot{q} = -k\|\dot{q}\|^2 \leq 0.$$ 

In the sequel we assume that the potential is bounded from below, say $U \geq 0$. Then, for any solution $q(t)$ the velocity $\dot{q}(t)$ is bounded in the future:

$$\frac{1}{2}m\|\dot{q}(t)\|^2 \leq \frac{1}{2}m\|\dot{q}(t_0)\|^2 + U(q(t)) \leq \frac{1}{2}m\|\dot{q}(t_0)\|^2 + U(q(t_0)), \quad t \geq t_0,$$

so $q(t)$ is bounded for bounded intervals of time and we get global existence in the future. What about the past?

Notice that for $k = 0$, with no dissipation, we have global existence since the above argument holds in the past too.

Our (2.3) can be seen as the Euler-Lagrange equation for the Lagrangian

$$L = e^{kt/m} \left( \frac{1}{2}m\|\dot{q}\|^2 - U(q) \right). \quad (2.5)$$

It is quite natural to consider the family $q_\lambda(t) := q(t + \lambda e^{at})$ with $a \in \mathbb{R}$ new parameter and $q(t)$ solution. Indeed for $a = 0$ the family reduces to the time-shift used for energy conservation as $k = 0$ and the exponential function is easily inspired by the one in (2.5).

Then

$$\frac{\partial}{\partial \lambda} L(t, q_\lambda(t), \dot{q}_\lambda(t)) \bigg|_{\lambda=0} =$$

$$= \frac{d}{dt} \left( -2e^{(a+\frac{k}{m})t}U(q(t)) + e^{(a+\frac{k}{m})t} \left( (a - \frac{k}{m})m\|\dot{q}(t)\|^2 + 2(a + \frac{k}{m})U(q(t)) \right) \right)$$

where we eliminated $\ddot{q}(t)$ using the differential equation. For $a = k/m$ it simplifies and we have the constant of motion

$$t \mapsto \partial_q L(t, q(t), \dot{q}(t)) \cdot \partial_\lambda q_\lambda(t) \bigg|_{\lambda=0} - \int_{t_0}^t \frac{\partial}{\partial \lambda} L(s, q_\lambda(s), \dot{q}_\lambda(s)) \bigg|_{\lambda=0} \; ds =$$

$$= e^{2kt/m} \left( m\|\dot{q}(t)\|^2 + 2U(q(t)) \right) + 4 \frac{k}{m} \int_{t_0}^t e^{2ks/m} U(q(s)) \; ds. \quad (2.6)$$

Since we assumed $U \geq 0$, the last integral decreases for $t \leq t_0$ and the function

$$t \mapsto e^{2kt/m} \left( m\|\dot{q}(t)\|^2 + 2U(q(t)) \right) \quad (2.7)$$
increases with $t$ for all $t \leq t_0$.

Finally, we have the estimate for $t \leq t_0$:

$$m\|\dot{q}(t)\|^2 \leq e^{2k(t_0-t)/m}\left(m\|\dot{q}(t_0)\|^2 + 2U(q(t_0))\right).$$

(2.8)

In a bounded interval $(t_1, t_0]$ the velocity $\dot{q}(t)$ is bounded, so also $q(t)$ and we have global existence of solutions. Summing up:

**Theorem 2.1.** If $k > 0$ and $U$ is a smooth potential on $\mathbb{R}^n$ which is bounded from below, then all solutions of the dissipative equation $\ddot{q} = -k\dot{q} - \nabla U(q)$ are defined for all $t \in \mathbb{R}$.

## 3 First integrals for higher-order Lagrangians

Generally, we cannot expect that Theorem 1.2 yields true first integrals for a random choice of the family $q_\lambda(t)$. However, few and precious Lagrangians make our machinery work. The simplest example is for autonomous Lagrangians:

**Theorem 3.1.** Let $t \mapsto q(t)$ be a solution to the Euler-Lagrange equation for smooth time-independent $L(q, \ldots, q^{(N)})$. Then the following function is a first integral:

$$K_1(q, \ldots, q^{(2N-1)}) = \sum_{i=1}^N \sum_{k=0}^{i-1} (-1)^k \frac{d^k}{dt^k} \partial_{q(i)} L(q, \ldots, q^{(N)}) \cdot q^{(i-k)} - L(q, \ldots, q^{(N)}).$$

(3.1)

It is important to notice that Theorem 3.1 recovers the Noetherian result of [13] for $N = 2$ Lagrangians.

Since the Pais-Uhlenbeck Lagrangian $L_{PU}$ of formula (1.7) is time-independent, we deduce the following first integral:

$$2K_1^{PU} = q^{(2)} - (w_1^2 + w_2^2)q^{(1)}^2 - 2q^{(3)}q^{(1)} - w_1^2w_2^2q^2.$$  

(3.2)

In [5] the authors proved energy conservation for the canonical harmonic oscillator starting from nonlocal space-changes. Consequently, using Theorem 1.2, we deduce that first integrals are easy to be found if $\partial_{q(i)} L \propto d^i \partial_q L/dt^i$ for all $j = 1, \ldots, N$:

**Theorem 3.2.** Consider a smooth Lagrangian $L(t, q, \ldots, q^{(N)})$ for which there exists a set of constant parameters $\rho_1 \ldots \rho_N \in \mathbb{R}$ such that

$$\partial_{q(i)} L = \rho_i \frac{d^i}{dt^i} \partial_q L \quad \text{for all motions and } i \in \{1, \ldots, N\}.$$  

(3.3)

Let $t \mapsto q(t)$ be a solution to the Euler-Lagrange equation, and define $F^{(\ell)} = \sum_{j=1}^N (-1)^{j+1} d^{\ell-j} \partial_{q(i)} L/dt^{j-1}$. Then $F^{(\ell)} = d^{\ell-1} \partial_q L/dt^{\ell-1}$ with $\ell \in \{1, \ldots, 2N\}$, and the following function is a first integral:

$$K_2(t, q, \ldots, q^{(3N-1)}) = \sum_{i=1}^N \rho_i \left[ \sum_{k=0}^{i-1} (-1)^k F^{(i+k+1)} \cdot F^{(i-k-1)} - \frac{1}{2} \|F^{(i)}\|^2 \right] - \frac{1}{2} \|F^{(\ell)}\|^2.$$  

(3.4)
Notice that $L^{PU}$ satisfies Theorem 3.2 with $\rho_1 = -(w_1^2 + w_2^2)w_1^2w_2^2$ and $\rho_2 = w_1^{-2}w_2^{-2}$. Hence, $F^{(0)} = -(w_1^2 + w_2^2)q^{(1)} - q^{(3)}$ and $F^{(\ell)} = w_1^2w_2^2q^{(\ell-1)}$ ($\ell = 0, 1, 2$) give

$$2K_2^{PU} = (w_1^4 + w_2^2w_1^2 + w_2^2)q^{(1)^2} + q^{(3)^2} + 2w_1^2w_2^2\dot{q}\ddot{q} + (w_1^2 + w_2^2)(2q^{(3)}q^{(1)} + w_1^2w_2^2q^2)$$  \hspace{1cm} (3.5)

If combined, $K_1^{PU}$ and $K_2^{PU}$ recover the two first integrals recently proposed by [15].

Another interesting situation generating first integrals arises when the Lagrangian, evaluated on a perturbed motion $q_\lambda(t)$, has constant derivative at $\lambda = 0$.

**Theorem 3.3.** Consider a smooth Lagrangian, and suppose that for a given family of perturbed motions $q_\lambda$ there exists a constant $\mu \in \mathbb{R}$ such that $\partial_\lambda L(t, q_\lambda, \ldots, q_\lambda^{(N)})|_{\lambda=0} = \mu$. Then, the following function is a first integral:

$$K_3(t, q, \ldots, q^{(2N-1)}) =$$

$$= \sum_{i=1}^{N} \sum_{k=0}^{i-1} (-1)^k \frac{d^k \partial q^{(i)}}{dt^k} L(t, q, \ldots, q^{(N)}) \cdot \partial q^{(i-k-1)}\lambda|_{\lambda=0} - \mu t . \hspace{1cm} (3.6)$$

Let $q = (q_1, q_2) \in \mathbb{R}^2$ and consider the rotation family $q_\lambda(t)$ in (1.4). When evaluated on $q_\lambda(t)$, $L^{PU}$ does not depend on $\lambda$, hence Theorem 3.3 gives the new angular-momentum-like first integral

$$K_3^{PU} = (w_1^2 + w_2^2) \det(q^{(1)}, q) + \det(q^{(1)}, q^{(2)}) + \det(q^{(3)}, q). \hspace{1cm} (3.7)$$

In Scomparin [18], the machinery is also applied to analyze a higher-order generalization of the Pais-Uhlenbeck oscillator [1, 3], and a simple Degenerate Higher-Order model of Scalar-Tensor (DHOST) theory that, in these last years, inspired many modified gravity theories [11]. Again, the full consistency of the machinery is confirmed.

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