Misspecified Phase Retrieval with Generative Priors

Zhaoqiang Liu
National University of Singapore
dcslizha@nus.edu.sg

Xinshao Wang
University of Oxford
xinshao.wang@eng.ox.ac.uk

Jiulong Liu
Chinese Academy of Sciences
jiulongliu@lsec.cc.ac.cn

Abstract

In this paper, we study phase retrieval under model misspecification and generative priors. In particular, we aim to estimate an $n$-dimensional signal $x$ from $m$ i.i.d. realizations of the single index model $y = f(a^T x)$, where $f$ is an unknown and possibly random nonlinear link function and $a \in \mathbb{R}^n$ is a standard Gaussian vector. We make the assumption $\text{Cov}[y, (a^T x)^2] \neq 0$, which corresponds to the misspecified phase retrieval problem. In addition, the underlying signal $x$ is assumed to lie in the range of an $L$-Lipschitz continuous generative model with bounded $k$-dimensional inputs. We propose a two-step approach, for which the first step plays the role of spectral initialization and the second step refines the estimated vector produced by the first step iteratively. We show that both steps enjoy a statistical rate of order $\sqrt{\binom{k}{2} \log L} \cdot (\log m)/m$ under suitable conditions. Experiments on image datasets are performed to demonstrate that our approach performs on par with or even significantly outperforms several competing methods.

1 Introduction

Compressed sensing (CS) is perhaps the most popular instance of high-dimensional inverse problems, for which one has the linear measurement model

$$ y = a^T x + \eta, \quad (1) $$

where $a \in \mathbb{R}^n$ is the sensing vector, $x \in \mathbb{R}^n$ is the sparse signal to estimate, and $\eta$ represents additive noise. It has been well-known for CS that roughly $m = O(s \log(n/s))$ i.i.d. Gaussian measurements are sufficient to ensure the accurate recovery of a signal with $s$ non-zero entries $\textbf{[88, 1, 23, 11, 77]}$.

Phase retrieval (PR) arises in numerous scientific areas including X-ray crystallography, acoustics, astronomy, microscopy, optics, wireless communications, and quantum information $\textbf{[13]}$, where one cannot measure $a^T x$ directly, and can only record its magnitude. For example, the following noisy magnitude-only measurement model has been adopted in various prior works on sparse (real-valued) PR $\textbf{[92, 97, 23, 36, 9]}$:

$$ y = |a^T x| + \eta, \quad (2) $$

where the signal $x$ is assumed to be sparse, and the sensing vector $a$ is assumed to be a standard Gaussian vector.

However, both the linear and magnitude-only measure models in $\textbf{[1]}$ and $\textbf{2}$ are idealized views of the data generating process. To make the setup more general, one can utilize the following semi-parametric single index model (SIM) for general nonlinear models:

$$ y = f(a^T x), \quad (3) $$

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where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is an unknown (possibly random) nonlinear link function, and \( a \) is typically assumed to be Gaussian. In addition, since the norm of \( x \in \mathbb{R}^n \) is absorbed into the SIM, for brevity, it is common to assume that \( x \) is a unit vector, i.e., \( \|x\|_2 = 1 \). SIMs have long been studied in the conventional setting where the number of measurements \( m > n \) \cite{30,81,45}. In recent years, they have also been analyzed in the high-dimensional setting where \( m \ll n \), mainly under the assumption that the underlying signal is sparse. Relevant works include but are not limited to \cite{72,65,26,73,68}. For all these works, it is crucial to impose the following assumption on the SIM:

\[
\text{Cov}[y, a^T x] \neq 0. \tag{4}
\]

The pivotal condition in (4) is fairly generic and encompasses notable special examples such as noisy 1-bit measurements and general (non-binary) quantization schemes. However, it is not satisfied by PR models including \( y = |a^T x + \eta| \), \( y = (a^T x)^2 + \eta \), \( \eta \) refers to zero-mean random noise that is independent of \( a \). In order to formalize a class of SIMs that encompass the above-mentioned PR models (and more general models, see the discussion in Section 2.2) as special cases, misspecified phase retrieval (MPR) has been studied in \cite{64,98}, with the condition (4) being replaced by the assumption

\[
\text{Cov}[y, (a^T x)^2] \neq 0. \tag{6}
\]

It is worth mentioning that another motivation behind studying MPR is that the theoretical analysis for PR typically relies on the correct model specification that the data points are indeed generated by the correct model, and the MPR model enables theoretical analysis under statistical model misspecification.

In both works \cite{64,98}, the signal \( x \) is assumed to be sparse. Recently, motivated by tremendous successful applications of deep generative models and following the seminal work \cite{6} on generative model based linear CS, it has been popular to study high-dimensional inverse problems under generative priors \cite{76,31,32,34,96,41,2,67,95,60,40,65}. In particular, instead of being sparse, the underlying signal is assumed to be contained in (or lie near) the range of a generative model. It has been empirically demonstrated in \cite{6} and its various follow-up works (e.g., see \cite{19,85,48} and a literature review in \cite{78}) that compared to sparsity based methods, corresponding generative model based algorithms require significantly fewer samples to recover the signal up to a given accuracy.

In this paper, we study the MPR problem under generative modeling assumptions.

1.1 Related Work

In this subsection, we summarize some relevant works on PR and SIM, for both cases with or without generative priors.

**PR and SIM without generative priors:** There is a large amount of literature providing practical algorithms for PR, including convex methods \cite{66,14,12,90,4,29} and empirically more competitive non-convex methods \cite{27,21,62,70}. In particular, in the seminal work \cite{62} and a variety of its follow-up works \cite{13,10,15,92,91,99,39,82,8}, whether it is for general PR (with no priors on the signal) or sparse PR, two-step approaches have been proposed with provable guarantees. The first step consists of a spectral initialization method, and the second step is typically an iterative (such as alternating minimization and gradient descent) algorithm that further refines the initial guess of the first step. For general PR, the spectral initialization step turns out to be unnecessary, and optimal sample complexity guarantees can be established even when using random initialization \cite{83,16}. However, to the best of our knowledge, it is not the case for sparse PR. More specifically, all theoretically-guaranteed non-convex algorithms for sparse PR require spectral initialization, and this typically results in a sub-optimal sample complexity of \( O(s^2 \log n) \) (instead of \( O(s \log n) \)), where \( s \) refers to the number of non-zero entries of the signal to estimate.

MPR is also closely related to SIMs, which have been extensively studied in the conventional setting where \( m > n \), see, e.g., \cite{30,35,59}. High-dimensional SIMs have received a lot of attention in recent years, with theoretical guarantees for signal estimation and support recovery \cite{22,24,74,84,55,72,73,17,97,28,93,69,20}. In particular, motivated by the idea that under the assumption (4), a SIM can be converted into a scaled linear measurement model with unconventional noise, the authors
Wirtinger flow method \cite{10}, is proposed in \cite{98}. In the first step, identical to that in \cite{10}, the initial works \cite{94, 51}, the recovery guarantees are with respect to globally optimal solutions of typically works, making the assumption on the initial vector therein a bit stringent. On the other hand, the As mentioned earlier, MPR with sparse priors has been studied in \cite{64, 98}. The work \cite{64} implements Throughout this paper, we assume that the signal vector is calculated by a thresholded spectral method that first estimates the support of the sparse generative model with bounded \ell_1 regularized least square is solved to obtain a refined estimator. Such a procedure suffers from high computational costs. A more efficient two-step approach, which is a simple variant of the thresholded Wirtinger flow method \cite{10}, is proposed in \cite{98}. In the first step, identical to that in \cite{10}, the initial vector is calculated by a thresholded spectral method that first estimates the support of the sparse signal by thresholding and then performs the classic power method over the submatrix corresponding to the estimated support. In the second step, a thresholded gradient descent algorithm is employed. Both approaches in \cite{64, 98} can attain the optimal statistical rate of order $\sqrt{(s \log n) / m}$, provided that the sensing vector is standard Gaussian and the number of samples $m = \Omega(s^2 \log n)$. In addition, the second step of the approach proposed in \cite{98} is shown to achieve a linear convergence rate.

**PR and SIM with generative priors:** PR with generative priors has been studied in \cite{31, 37, 38, 80, 5}. More specifically, an approximate message passing algorithm is proposed in \cite{31}. The authors of \cite{31, 80} minimize the objective over the latent space in $\mathbb{R}^k$, using gradient descent, where $k$ is the latent dimension of the generative prior. Corresponding algorithms may easily get stuck in local minima since the explorable solution space is limited. Recovery guarantees for projected gradient descent algorithms over the ambient space in $\mathbb{R}^n$ for noiseless PR with pre-trained or untrained neural network priors have been proposed in \cite{37, 38}. No initialization methods have been proposed in these works, making the assumption on the initial vector therein a bit stringent. On the other hand, the authors of \cite{47} propose a spectral initialization method for PR with generative priors and provide recovery guarantees with respect to globally optimal solutions of a corresponding optimization problem. The optimization problem is non-convex, and a projected power method is proposed in \cite{50} to approximately find an optimal solution.

Generative model based SIMs have been studied in \cite{51, 49, 46, 94}. The authors of \cite{51, 49, 46} provide optimal sample complexity upper bounds under the assumption of Gaussian sensing vectors. But their results rely on the assumption \cite{3}, which is not satisfied by widely adopted PR models. The SIM studied in \cite{94} encompasses certain PR models as special cases, and the sensing vector can be non-Gaussian. However, the nonlinear link function $f$ is assumed to be differentiable, making it not directly applicable to the PR model in \cite{51}. Moreover, it is worth mentioning that in both works \cite{94, 51}, the recovery guarantees are with respect to globally optimal solutions of typically highly non-convex optimization problems. Attaining these optimal solutions is practically difficult.

### 1.2 Contributions

Throughout this paper, we assume that the signal $x$ lies in the range of an $L$-Lipschitz continuous generative model with bounded $k$-dimensional inputs. Our main contributions are as follows:

- We propose a two-step approach for MPR with generative priors. In particular, in the first step, we make use of the projected power method proposed in \cite{50} to obtain a good initial vector for the iterative algorithm used in the second step.
- We show that under appropriate initialization, both steps attain a statistical rate of order $\sqrt{(k \log L) \cdot (\log m) / m}$, and the second step achieves a linear convergence rate. The initialization condition for the first step is mild in the sense that it allows the inner product between the starting point and the signal $x$ to be a sufficiently small positive constant. In contrary, the initialization condition for the second step is more restrictive, making the first step necessary. Notably, unlike for the works on MPR with sparse priors \cite{64, 98} that require the sub-optimal $O(s^2 \log n)$ sample complexity, the sample complexity requirement for our recovery guarantees is $O((k \log L) \cdot (\log m))$, which is naturally conjectured to be near-optimal \cite{52, 42}. We perform numerical experiments on image datasets to corroborate our theoretical results. In particular, for the noisy magnitude-only measurement model \cite{2}, we observe that our approach gives reconstructions that are competitive with those of the alternating phase projected gradient descent (APPGD) algorithm proposed in \cite{37}, which is the corresponding
state-of-the-art method, though we do not utilize the knowledge of the link function and allow for model misspecification. Moreover, for several closely related measurement models that satisfy the condition (6), our approach leads to superior reconstruction performance compared to all other competing methods, including APPGD.

2 Problem Formulation

In this section, we overview some important assumptions that we adopt. Before proceeding, we present the notation used in this paper.

2.1 Notation

We use upper and lower case boldface letters to denote matrices and vectors respectively. For any positive integer \( N \), we use the shorthand notation \([N] = \{1, 2, \cdots, N\}\), and \( I_N \) represents the identity matrix in \( \mathbb{R}^{N \times N} \). The support (set) of a vector is the index set of its non-zero entries. We use \( \|X\|_{2 \rightarrow 2} \) to represent the spectral norm of \( X \). We use \( B^k(r) \) to denote the radius-\( r \) in \( \mathbb{R}^k \), i.e., \( B^k(r) := \{z \in \mathbb{R}^k : \|z\|_2 \leq r\} \), and \( S^{n-1} \) represents the unit sphere in \( \mathbb{R}^n \), i.e., \( S^{n-1} := \{s \in \mathbb{R}^n : \|s\|_2 = 1\} \). We use \( G \) to denote a pre-trained \( L \)-Lipschitz continuous generative model from \( B^k(r) \) to \( \mathbb{R}^n \). We focus on the setting where \( k < n \). For any set \( S \subseteq B^k(r) \), we write \( G(S) = \{G(z) : z \in S\} \). Our goal is to estimate the signal \( x \in \text{Range}(G) = G(B^k(r)) \) from realizations of the sensing vector \( a \) and the observation \( y \) (generated according to the SIM in (3)). For any two sequences of real values \( \{a_j\} \) and \( \{b_j\} \), we write \( a_j = O(b_j) \) if there exist an absolute constant \( C_1 \) and a positive integer \( j_1 \) such that for any \( j > j_1 \), \( |a_j| \leq C_1 b_j \), \( a_j = \Omega(b_j) \) if there exist an absolute constant \( C_2 \) and a positive integer \( j_2 \) such that for any \( j > j_2 \), \( |a_j| \geq C_2 b_j \). We use the generic notations \( C \) and \( C' \) to denote large positive constants, and we use \( c \) to denote a small positive constant; their values may differ from line to line.

2.2 Setup

First, the following are the standard definitions of a sub-exponential random variable and the associated sub-exponential norm.

**Definition 1.** A random variable \( X \) is said to be sub-exponential if there exists a positive constant \( C \) such that \((\mathbb{E}[|X|^p])^{1/p} \leq Cp\) for all \( p \geq 1 \), or equivalently, if there exists a positive constant \( C' \) such that \( \mathbb{P}(|X| > u) \leq \exp(1 - u/C') \) for all \( u \geq 0 \). The sub-exponential norm of \( X \) is defined as \( \|X\|_{\psi_1} := \sup_{p \geq 1} p^{-1} (\mathbb{E}[|X|^p])^{1/p} \).

We will focus on the following settings except where stated otherwise:

- The observations are independently generated according to the SIM \([3]\), where \( f \) is the link function that is unknown and possibly random.
- We have an \( L \)-Lipschitz continuous generative model \( G : B^k(r) \rightarrow \mathbb{R}^n \). For convenience, similarly to \([48][50]\), we assume that the generative model is normalized such that \( \text{Range}(G) \subseteq S^{n-1} \). For a general (unnormalized) generative model, we may essentially consider its normalized version. See, e.g., \([50\text{ Remark 1}]\).
- The signal \( x \) is contained in the range of \( G \), i.e., \( x \in \text{Range}(G) \subseteq S^{n-1} \).
- The sensing vector \( a \in \mathbb{R}^n \) is standard Gaussian, i.e., \( a \sim \mathcal{N}(0, I_n) \).
- The random variable \( y = f(a^T x) \) is sub-exponential with the sub-exponential norm being denoted by \( K_y \), i.e., \( K_y := \|y\|_{\psi_1} \). In addition, we use \( M_y := \mathbb{E}[y] \) to denote the expectation of \( y \), i.e., \( M_y := \mathbb{E}[y] \).

**Remark 1.** \( y \) will be sub-exponential when \( f(x) \) comprises of \( x^c \) plus lower order terms with \( c \leq 2 \) (since the product of two sub-Gaussian random variables is sub-exponential), and therefore we will see that the \( y \) corresponding to all the measurement models presented in our paper is sub-exponential. We remark that the assumption of sub-exponential \( y \) is not essential and it can be easily relaxed. For example, when \( y = x^c \) with \( c \) being an even integer that is larger than 2, there will be only a minor change in the order of the log \( m \) term in the sample complexity and statistical rate. However, for brevity, we follow \([64][98]\) and make the assumption of sub-exponential \( y \) to avoid non-essential complications.
• We consider MPR and assume that\footnote{The case that $\nu < 0$ can be similarly handled by considering $-y$.}
\begin{equation}
\nu := \text{Cov} \left[ y, (\alpha^T x)^2 \right] > 0, \tag{7}
\end{equation}

or equivalently,
\begin{equation}
\nu := \text{Cov} \left[ f(y), g^2 \right] > 0, \tag{8}
\end{equation}

where $g \sim \mathcal{N}(0, 1)$ is a standard normal random variable. Note that the assumption (8) is only with respect to the nonlinear link function $f$. The condition in (7) is satisfied by PR models described in \cite{2} and \cite{5}. It is also satisfied by relevant models such as \cite{98}
\begin{equation}
\begin{aligned}
y &= |a^T x| + 2 \tanh(|a^T x|) + \eta, \\
y &= 2(a^T x)^2 + 3 \sin(|a^T x|) + \eta. \tag{9}
\end{aligned}
\end{equation}

See \cite{64} Proposition 3 and Remark 4 for more general examples.

### 3 Algorithm

In this section, we describe our two-step algorithm devised for MPR with generative priors. Suppose that we have $m$ i.i.d. realizations of $a$ and $y$, namely $a_1, \ldots, a_m$ and $y_1, \ldots, y_m$. To estimate the signal $x$, we consider the following two-step approach:

1. We perform $T_1$ iterations in the first step. In particular, let
\begin{equation}
V = \frac{1}{m} \sum_{i=1}^{m} y_i \left( a_i a_i^T - I_m \right). \tag{10}
\end{equation}

We perform the projected power method proposed in \cite{50}: For $t = 0, 1, \ldots, T_1 - 1$, let
\begin{equation}
w^{(t+1)} = \mathcal{P}_G \left( V w^{(t)} \right), \tag{11}
\end{equation}

where $\mathcal{P}_G(\cdot)$ denotes the projection function onto $\text{Range}(G)$ \footnote{That is, for any $s \in \mathbb{R}^n$, $\mathcal{P}_G(s) := \arg \min_{a \in \mathbb{R}^n} \|G(a) - s\|_2$.} and we obtain $x^{(0)} := w^{(T_1)}$. Similarly to \cite{43}, we set the starting point $w^{(0)}$ as the column of $\frac{1}{m} \sum_{i=1}^{m} y_i a_i a_i^T$ (i.e., a shifted version of $V$) that corresponds to the largest diagonal entry. Note that it is easy to calculate that $\mathbb{E}[V] = \nu x x^T$ (see, e.g., \cite{47} Lemma 8), for which each column is a scalar product of $x$. This motivates the use of a shifted version of $V$ to get the initialization vector.

2. We perform $T_2$ iterations in the second step. In particular, let
\begin{equation}
\bar{y} = \frac{1}{m} \sum_{i=1}^{m} y_i \tag{12}
\end{equation}

be the empirical mean of the observations. We perform the following iterative procedure: For $t = 0, 1, 2, \ldots, T_2 - 1$, let
\begin{equation}
\hat{\nu}^{(t)} = \frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot \left( a_i^T x^{(t)} \right)^2 \tag{13}
\end{equation}

\begin{equation}
\hat{y}_i^{(t)} = (y_i - \bar{y}) \cdot \left( a_i^T x^{(t)} \right), \quad i = 1, 2, \ldots, m. \tag{14}
\end{equation}

\begin{equation}
\bar{x}^{(t+1)} = x^{(t)} - \frac{\zeta}{m} \sum_{i=1}^{m} \left( \hat{\nu}^{(t)} \cdot \left( a_i^T x^{(t)} \right) - \hat{y}_i^{(t)} \right) a_i, \tag{15}
\end{equation}

\begin{equation}
x^{(t+1)} = \mathcal{P}_G \left( \bar{x}^{(t+1)} \right), \tag{16}
\end{equation}

where $\zeta > 0$ is a tuning parameter, and $\hat{\nu}^{(t)}$ can be thought of as an approximation of $\nu$ defined in (7). The idea behind calculating $\hat{y}_i^{(t)}$ is that by comparing (4) and (6), we observe (see, e.g., \cite{47} Lemma 8), for which each column is a scalar product of $x$. This motivates the use of a shifted version of $V$ to get the initialization vector.
Algorithm 1: A two-step approach for misspecified phase retrieval with generative priors

Input: \( \{(a_i, y_i)\}_{i=1}^{m} \), step size \( \zeta > 0 \), number of iterations \( T_1 \) for the first step, number of iterations \( T_2 \) for the second step, pre-trained generative model \( G \), initial vector \( w^{(0)} \)

First step:
1: for \( t = 0, 1, \ldots, T_1 - 1 \) do
2: \( w^{(t+1)} = \mathcal{P}_G(V w^{(t)}) \)
3: end for

Second step:
Let \( x^{(0)} := w^{(T_1)} \)
1: for \( t = 0, 1, \ldots, T_2 - 1 \) do
2: Calculate \( \hat{\nu}^{(t)}, \hat{\eta}^{(t)} \), \( \hat{x}^{(t+1)} \) and \( x^{(t+1)} \) according to (13), (14), (15), and (16), respectively
3: end for

Output: \( \hat{x} := x^{(T_2)} \)

For convenience, we summarize the details in Algorithm 1.

4 Theoretical Results

The following theorem establishes recovery guarantees for the first step of Algorithm 1. The proof is given in the supplementary material. Note that \( K_y := \|y\|_{\psi_1} \) (cf. Section 2.2) is considered a fixed constant and is omitted in the \( O(\cdot) \) notation.

Theorem 1. Assume that there exists a positive integer \( t_0 \) such that \( x^T w^{(t_0)} \geq c_0 \), where \( c_0 \) is a sufficiently small positive constant. Suppose that \( m = \Omega\left( (k \log(n L r)) \cdot (\log m) \right) \) with a large enough implied constant. Then, we have that with probability \( 1 - O(1/m) \), it holds for all \( t > t_0 \) that

\[
\left\| w^{(t)} - x \right\|_2 \leq \frac{C K_y}{c_0} \frac{\sqrt{(k \log(n L r)) \cdot (\log m)}}{m} = O\left( \sqrt{\frac{(k \log(n L r)) \cdot (\log m)}{m}} \right). \tag{17}
\]

Since a \( d \)-layer feedforward neural network generative model from \( B^k(r) \) to \( \mathbb{R}^n \) is typically \( L \)-Lipschitz continuous with \( L = n^{\Theta(d)} \) and we may set \( r = n^{\Theta(d)} \), the upper bound in (17) is of order \( \sqrt{\frac{(k \log L) \cdot (\log m)}{m}} \). Such a statistical rate is naturally conjectured to be near-optimal according to information-theoretic lower bounds established for MPR with sparse priors [64] and generative model based principal component analysis [50]. Therefore, Theorem 1 reveals that the first step of Algorithm 1 attains the near-optimal statistical rate under appropriate initialization and exact projections. The accurate projection assumption is perhaps the major caveat to Theorem 1. However, it is a standard assumption in relevant works including [79] [37] [38] [71] [50]. In practice, both gradient- and GAN-based projection methods [79] [75] have been shown to be highly effective in approximating the projection step.

Remark 2. Spectral initialization steps in relevant works on sparsity based PR [92] [39] [25] [36] [9] [7] or MPR [64] [58] require the sub-optimal sample complexity \( O(s^2 \log n) \), where \( s \) refers to the number of non-zero entries. In contrary, according to Theorem 1, our spectral initialization step only requires the near-optimal \( O((k \log L) \cdot (\log m)) \) sample complexity (with a linear rather than a quadratic dependence on \( k \)). However, we note that such an advantage of our spectral initialization step comes at a price. In particular, we require the initialization condition \( x^T w^{(t_0)} \geq c_0 \), which is not required by spectral initialization steps in the above-mentioned works on sparse PR/MPR.

Remark 3. For some applications, we may assume that the dataset contains only vectors whose elements are all non-negative. For example, this is a natural assumption for image datasets. During pre-training, we can easily set the activation function of the last layer of the neural network generative model to be a certain non-negative function such as ReLU or sigmoid, and the range of such a
The generative model is contained in the non-negative orthant. Therefore, the assumption that \( x^T w^{(t)} \geq c_0 \) for a sufficiently small positive constant \( c_0 \) is also mild. Similar assumptions have been made in relevant works including [18, 50] where it is not appropriate to assume that \(-x\) is also contained in the structured set (such as a closed convex cone or the range of a deep generative model). As a result, we provide an upper bound on \( \| w^{(t)} - x \|_2 \), instead of \( \min \{ \| w^{(t)} - x \|_2, \| w^{(t)} + x \|_2 \} \), which is a commonly adopted distance measure in relevant literature on real-valued PR.

Moreover, although the projected power iterations in the first step of Algorithm 1 can attain the near-optimal statistical rate under appropriate conditions, it is evident in a large body of literature on PR (see, e.g., [62, 13, 10, 99, 98, 64]) that such a spectral method better serves as the initialization of a subsequent iterative approach. This motivates us to propose the second step of Algorithm 1, and in our numerical experiments, we clearly observe the benefit of the second step. More specifically, compared to simply performing the projected power method, performing both steps of Algorithm 1 leads to significantly better reconstructed images when the total number of iterations is fixed to be the same, namely \( T_1 + T_2 \).

Next, we present the following theorem, which is proved in the supplementary material. This theorem shows that under appropriate initialization and the assumption of exact projections, the iterative algorithm in the second step of Algorithm 1 converges linearly to a point achieving the near-optimal statistical rate under appropriate conditions.

**Theorem 2.** Assume that the step size \( \nu \) and the initial vector for the second step of Algorithm 1 satisfy

\[
2 \cdot (1 - \zeta \nu) + 5 \zeta \nu \cdot \| x^{(0)} - x \|_2 + \beta_1 = 1 - \beta_2, \tag{18}
\]

where both \( \beta_1 \) and \( \beta_2 \) are positive constants.

Suppose that \( m = \Omega((k \log(nLr)) \cdot (\log m)) \) with a large enough implied constant. Then, we have with probability \( 1 - O(1/m) \), the following event occurs: There exists a positive integer \( T_0 = O(\log (m \log(nLr)) \cdot (\log m)) \), such that the sequence \( \{ \| x^{(t)} - x \|_2 \}_{t \in [0,T_0]} \) is monotonically decreasing, with the following inequality holds for all \( t \leq T_0 \):

\[
\| x^{(t)} - x \|_2 < (1 - \beta_2) t \cdot \| x^{(0)} - x \|_2 + \frac{CK_y}{\beta_2} \sqrt{(k \log(nLr)) \cdot (\log m)}.
\tag{19}
\]

In addition, we have for all \( t \geq T_0 \) that

\[
\| x^{(t)} - x \|_2 \leq \frac{CK_y}{\beta_2} \sqrt{(k \log(nLr)) \cdot (\log m)} = O \left( \sqrt{(k \log(nLr)) \cdot (\log m)} \right).
\tag{20}
\]

**Remark 4.** In our analysis, we need to impose the assumption [18] on the step size \( \zeta \) and initial vector \( x^{(0)} \). This makes the first step of Algorithm 1 necessary since when \( x^{(0)} - x \|_2 \) is not small, say \( x^{(0)} - x \|_2 = 1 \), the condition [18] cannot be satisfied. In comparison, to attain the near-optimal statistical rate \( \Omega(\sqrt{(k \log L) \cdot (\log m)} / m) \), the initialization condition of the first step of Algorithm 1 is milder, and \( x^T w^{(t_0)} \) (see Theorem 2) only needs to be lower bounded by a sufficiently small positive constant (thus \( |w^{(t_0)} - x|_2 \) can be close to \( \sqrt{2} \)). However, although the second step of Algorithm 1 requires a more restrictive initialization condition, we observe from our experimental results that it clearly refines the estimate of the first step. Such a phenomenon is also observed in various works related to PR, including [62, 13, 10, 99, 98, 64].

**Remark 5.** In Remark 4, we have briefly discussed the comparison of the initialization condition \( x^T w^{(t_0)} \geq c_0 \) in Theorem 1 and the typical initialization condition \( \| x - w^{(t_0)} \|_2 < \delta \| x \|_2 \). In the following, we provide a more detailed discussion: When both \( x \) and \( w^{(t_0)} \) are unit vectors (this is the setting of our Theorem 1), the typical initialization requirement \( \| x - w^{(t_0)} \|_2 < \delta \| x \|_2 \) can be reduced to \( 2(1 - x^T w^{(t_0)}) < \delta^2 \), or equivalently, \( x^T w^{(t_0)} > \delta^2 / 2 \). Note that \( \delta \) is typically a small positive constant (e.g., \( \delta = \frac{1}{10} \) in [10] and \( \delta = \frac{1}{4} \) in [12]), and thus the typical initialization condition requires \( x^T w^{(t_0)} \) to be larger than some positive constant that is close to 1. This is stronger than the assumption \( x^T w^{(t_0)} \geq c_0 \), where \( c_0 \) is a sufficiently small positive constant.

\( ^3 \beta_1 \) is sufficiently small, and it is used to absorb a certain \( o(1) \) term.

\( ^4 \) It basically assumes the weak recovery of the signal, see, e.g., [61].
Remark 6. The condition in (18) requires $|1 - \zeta| < \frac{1}{2}$. This reveals that we should choose $\zeta$ such that $\zeta \in \left( \frac{1}{2}, \frac{3}{4} \right)$. A good choice of $\zeta$ is $\zeta = \frac{1}{3}$ (for this case, the condition (18) reduces to $\|x^{(0)} - x\|_2 < \frac{1}{2}$). Since the knowledge of the link function $f$ and thus the knowledge of $\nu$, which is dependent on $f$; See (5) cannot be assumed, in our experiments, we use $c(\nu)$ to approximate $\nu$. That is, $\zeta$ is set to be $\frac{1}{3}$ in the $t$-th iteration of the second step of Algorithm 1 (though it is slightly varying instead of being fixed).

5 Numerical Results

In this section, we demonstrate the empirical performance of our Algorithm 1 (denoted by MPRG). We remark that these numerical experiments are proof-of-concept rather than seeking to be comprehensive since our contributions are primarily theoretical. We present some numerical results for the MNIST [44] dataset in the main document. Additional results for the MNIST dataset and some experimental results for the CelebA [53] dataset are presented in the supplementary material.

The MNIST dataset contains 60,000 images of handwritten digits. The size of each image is $28 \times 28$, and thus the dimension of the image vector is $n = 784$. For the MNIST dataset, the generative model $G$ is set to be (the normalized version of) a pre-trained variational autoencoder (VAE) model with the latent dimension being $k = 20$. We make use of the VAE model pre-trained by the authors of [6] directly. For this VAE model, both the encoder and decoder are set to be fully connected neural networks with two hidden layers, and the architecture is $20 - 500 - 500 - 784$. The VAE model is trained by the Adam optimizer with a mini-batch size 100 and a learning rate of 0.001, and is trained from the images in the training set. The projection step $P_G(\cdot)$ (cf. (11)) is approximated by the Adam optimizer with a learning rate of $0.1$ and 120 steps.

We report the results on 10 testing images that are selected from the test set. Note that these images are unseen by the pre-trained generative model. We perform 10 random restarts. For reconstructed images, we choose the best among these 10 random restarts to reduce the impact of local minima. We also provide quantitative comparisons with respect to the reconstruction error $\|x - \hat{x}\|_2$, where $x$ is the underlying signal and $\hat{x}$ refers to the estimated (normalized) vector produced by each of the methods described below. The reconstruction error is averaged over both the 10 testing images and 10 random restarts. All experiments are run using Python 3.6 and TensorFlow 1.5.0, with a NVIDIA GeForce GTX 1080 Ti 11GB GPU.

For Algorithm 1, we set $T_1 = 20$ and $T_2 = 30$. As mentioned in Section 3, the starting point $w^{(0)}$ is set to be the column of $\frac{1}{m} \sum_{i=1}^{m} y_i a_i a_i^T$ (i.e., a shifted version of $V$ defined in (10)) that corresponds to the largest diagonal entry. In addition, as mentioned in Remark 6, we set the step size $\zeta$ as $\zeta = \frac{1}{2m}$ (cf. (13)) in the $t$-th iteration of the second step of Algorithm 1. We compare our Algorithm 1 (denoted by MPRG) with the following methods: 1) The method proposed in [98], which is for misspecified phase retrieval with sparse priors and is denoted by MPRS. All the involved parameters are set to be the same as those in [98]. 2) Simply performing the first step of Algorithm 1 for $T_1 + T_2 = 50$ iterations. The corresponding method is denoted by PPower. The purpose of comparing to PPower is to verify the benefit of the second step of Algorithm 1. 3) Simply performing the second step of Algorithm 1 for $T_1 + T_2 = 50$ iterations. The corresponding method is denoted by Step2. The purpose of comparing to Step2 is to check whether the first step of Algorithm 1 is practically useful. 4) The Alternating Phase Projected Gradient Descent (denoted by APPGD) algorithm proposed in [37]. This algorithm is specifically designed for phase retrieval with magnitude-only measurements (cf. (2)) and generative priors, and the corresponding iterative procedure is

$$x^{(t+1)} = P_G \left( x^{(t)} - \frac{\tau}{m} \sum_{i=1}^{m} \left( a_i^T x^{(t)} \right) - y_i \cdot \text{sign} \left( a_i^T x^{(t)} \right) a_i \right),$$

where $\tau > 0$ is the step size. We follow [37] to set $\tau = 0.9$. For a fair comparison, we use the vector produced after $T_1 = 20$ iterations of the first step of Algorithm 1 as the initialization vector of APPGD, and then we run APPGD for $T_2 = 30$ iterations.

We first consider the noisy magnitude-only measurement models for $i \in [m],$

$$y_i = |a_i^T x| + \eta_i,$$

$$y_i = |a_i^T x + \eta_i|,$$
where $\eta_i$ are i.i.d. realizations of an $\mathcal{N}(0, \sigma^2)$ random variable. For such a measurement model, the corresponding random nonlinear link function $f$ is $f(x) = |x| + \eta$ or $f(x) = |x + \eta|$, where $\eta \sim \mathcal{N}(0, \sigma^2)$. The numerical results for (22) and (23) are demonstrated in Figures 1 and 2. From these figures, we observe that the sparsity based method MPRS attains poor reconstructions, and the results of PPower are not desirable. The three methods Step2, APPGD, MPRG lead to similar reconstruction error, but the reconstructed images of APPGD and MPRG are better than those of Step2. In particular, our method MPRG leads to mostly accurate reconstructions that are competitive compared to those of APPGD, even if we do not make use of the knowledge of the link function $f$ and MPRG is not specifically designed for the magnitude-only measurement models.

![Figure 1: Examples of reconstructed MNIST images for the measurement models (22) and (23).](image)

![Figure 2: Quantitative comparisons of the performance for the measurement model (22) and (23).](image)

Next, we consider the following two measurement models:

\[ y_i = |a_i^T x| + 2 \tanh(|a_i^T x|) + \eta_i, \]
\[ y_i = 2(a_i^T x)^2 + 3 \sin(|a_i^T x|) + \eta_i, \]

where again $\eta_i$ are i.i.d. realizations of an $\mathcal{N}(0, \sigma^2)$ random variable. For both models in (24) and (25), the corresponding link functions satisfy the condition (8) for MPR [98]. The numerical results are presented in Figures 3 and 4. We observe from these figures that for the measurement models (24) and (25), our method MPRG achieves the best reconstructions. In particular, it outperforms APPGD in terms of recovery quality and/or reconstruction error.

### 6 Conclusion and Future Work

We have proposed a two-step approach for phase retrieval under model misspecification and generative priors. We show that under suitable conditions, both steps of our approach obtain estimated vectors that achieve the near-optimal statistical rate of order $\sqrt{k \log L \cdot (\log m)/m}$, where $k$ is the latent dimension and $L$ is the Lipschitz constant of the generative model respectively, and $m$ refers to the number of samples.
We assume accurate projections in our analysis and use a gradient-based method to approximate the projection step $P_G(\cdot)$ in our experiments. Although the exact projection assumption is commonly made in relevant works [79, 37, 38, 71, 50], it is a very interesting future research direction to design provably-guaranteed efficient methods for the projection step.

In addition, we focus on real Gaussian measurements. While we believe that based on the technical results in [62, 13] (which study complex Gaussian measurements), it is straightforward to extend our work to the complex case, the extension to more practical non-Gaussian measurement models such as sub-sampled Fourier measurements is a very interesting future direction. Another direction is to use different preprocessing functions to enhance the performance of our spectral initialization method [56, 54, 57]. Moreover, if one has the access to the nonlinear link function $f$, the Bayes-optimal performances can characterized using message-passing algorithms [5, 58, 3]. It would be interesting to connect or compare our results with the corresponding Bayes-optimal rate.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] The limitations are relevant to our future work, described in Section 6.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes]

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] The code is included in the supplementary material.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes]

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
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5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]
A Auxiliary Results

In this section, we first provide some useful auxiliary results that are general, and then some that are specific to our setup.

A.1 General Auxiliary Results

First, we state the following standard definitions for a sub-Gaussian random variable and the associated sub-Gaussian norm.

**Definition 2.** A random variable $X$ is said to be sub-Gaussian if there exists a positive constant $C$ such that $(\mathbb{E} [|X|^p])^{1/p} \leq C \sqrt{\frac{p}{\alpha}}$ for all $p \geq 1$. The sub-Gaussian norm of a sub-Gaussian random variable $X$ is defined as $\|X\|_{\psi_2} := \sup_{p \geq 1} \frac{p^{-1/2}}{C} (\mathbb{E} [|X|^p])^{1/p}$.

Recall that the definitions of a sub-exponential random variable and the associated sub-exponential norm have been provided in Definition 1 in the main document. The following lemma states that the product of two sub-Gaussian random variables is sub-exponential.

**Lemma 1.** ([87] Lemma 2.7.7) Let $X$ and $Y$ be sub-Gaussian random variables (not necessarily independent). Then $XY$ is sub-exponential, and satisfies

$$\|XY\|_{\psi_1} \leq \|X\|_{\psi_2} \cdot \|Y\|_{\psi_2}. \quad (26)$$

We consider sub-Weibull random variables that generalize sub-Gaussian and sub-exponential random variables.

**Definition 3.** For any $\alpha > 0$, a random variable $X$ is said to be sub-Weibull of order $\alpha$ if it has a bounded $\psi_{\alpha}$-norm, where the $\psi_{\alpha}$-norm of $X$ is defined as

$$\|X\|_{\psi_{\alpha}} := \inf \{ K \in (0, \infty) : \mathbb{E} \{ \exp (|X|/K^{\alpha}) \} \leq 2 \}. \quad (27)$$

In particular, when $\alpha = 2$ or 1, sub-Weibull random variables reduce to sub-Gaussian or sub-exponential random variables respectively. The smaller the $\alpha$ is, the heavier tail a sub-Weibull random variable has. Moreover, it follows readily from Definition 3 that $X$ is sub-exponential if and only if $|X|^{1/\alpha}$ is sub-Weibull of order $\alpha$. We have the following concentration inequality for the sum of independent sub-Weibull random variables.

**Lemma 2.** ([33] Theorem 3.1) Suppose that $X_1, X_2, \ldots, X_N$ are independent sub-Weibull random variables that are of order $\alpha$, and $K = \max_{i} \|X_i\|_{\psi_{\alpha}}$. Then, there exists a positive constant $C_{\alpha}$ only depending on $\alpha$ such that for any $b = [b_1, b_2, \ldots, b_N]^T \in \mathbb{R}^N$ and $u > 2$, with probability at least $1 - e^{-u}$, it holds that

$$\left| \sum_{i=1}^{N} b_i X_i - \mathbb{E} \left[ \sum_{i=1}^{N} b_i X_i \right] \right| \leq C_{\alpha} K \left( \|b\|_2 \cdot \sqrt{u} + \|b\|_\infty \cdot u^{1/\alpha} \right). \quad (28)$$

In addition, we have the following lemma concerning the Two-sided Set-Restricted Eigenvalue Condition (TS-REC).

**Lemma 3.** ([51] Lemma 2) Let $G : B^k(r) \rightarrow \mathbb{R}^n$ be $L$-Lipschitz continuous and $a_1, \ldots, a_m$ be i.i.d. realizations of $N(0, I_n)$. For $\epsilon \in (0, 1)$ and $\delta > 0$, if $m = \Omega \left( \frac{k^2 \log \frac{L}{\delta}}{\delta^2} \right)$, then with probability $1 - e^{-\Omega(\epsilon^2 m)}$, the following holds for all $x_1, x_2 \in G(B^k(r))$:

$$1 - \epsilon \|x_1 - x_2\|_2 - \delta \leq \frac{1}{\sqrt{m}} \sqrt{\sum_{i=1}^{m} (a_i^T (x_1 - x_2))^2} \leq (1 + \epsilon) \|x_1 - x_2\|_2 + \delta. \quad (29)$$
A.2 Auxiliary Results for Our Setup

From Chebyshev’s inequality and the definition of a sub-exponential random variable (cf. Definition 1), as well as Lemma 2 we obtain the following lemma. Here and in subsequent results where it is clear from the context, for simplicity of presentation, we think of \(a_i\) and \(y_i\) as random variables, instead of realizations of corresponding random variables.

**Lemma 4.** When \(m \geq \log^4 m\), the event

\[
E : \max_{i \in [m]} |y_i| \leq 5K_y \cdot \log m, \quad \frac{1}{m} \sum_{i=1}^{m} y_i^2 \leq 8K_y^2, \quad \frac{1}{m} \sum_{i=1}^{m} y_i^2 (a_i^T x)^2 \leq 32\sqrt{3}K_y^2,
\]

\[
\left| \frac{1}{m} \sum_{i=1}^{m} (a_i^T x)^2 - 1 \right| \leq C \sqrt{\frac{\log m}{m}}, \quad \frac{1}{m} \sum_{i=1}^{m} y_i - M_y \leq CK_y \cdot \sqrt{\frac{\log m}{m}},
\]

\[
\left| \frac{1}{m} \sum_{i=1}^{m} y_i (a_i^T x)^2 - (\nu + M_y) \right| \leq CK_y \cdot \sqrt{\frac{\log m}{m}},
\]

occurs with probability \(1 - O(1/m)\), where \(\nu := \text{Cov}(y, (a^T x)^2)\) (cf. 7), \(M_y := \mathbb{E}[y]\), \(K_y := \|y\|_{\psi_1}\) (cf. Section 2.2), and \(C\) is an absolute constant.

**Proof.** Since \(y_i\) is assumed to be sub-exponential with the sub-exponential norm being \(K_y\), from the definition of a sub-exponential random variable, we obtain for any \(i \in [m]\) and \(u > 0\) that

\[
\mathbb{P}(|y_i| > u) \leq \exp(1 - u/K_y).
\]

Then, setting \(u = 5K_y \cdot \log m\) and taking a union bound over \(i \in [m]\), we obtain with probability at least \(1 - \frac{2}{m^{2}}\) that

\[
\max_{i \in [m]} |y_i| \leq 5K_y \cdot \log m.
\]

Note that from

\[
K_y = \|y\|_{\psi_1} = \sup_{p \geq 1} \frac{1}{p} (\mathbb{E}[|y|^p])^{1/p},
\]

we obtain

\[
|M_y| \leq \mathbb{E}[|y|] \leq K_y, \quad \mathbb{E}[y^2] \leq (2K_y)^2, \quad \mathbb{E}[y^4] \leq (4K_y)^4, \quad \mathbb{E}[y^8] \leq (8K_y)^8.
\]

In addition, from Chebyshev’s inequality, for any \(\epsilon > 0\), we have

\[
\mathbb{P} \left( \left| \frac{1}{m} \sum_{i=1}^{m} y_i^2 - \mathbb{E}[y^2] \right| \geq \epsilon \right) \leq \frac{\mathbb{Var}[y^2]}{m\epsilon^2}.
\]

Setting \(\epsilon = 4K_y^2\), we obtain

\[
\mathbb{P} \left( \frac{1}{m} \sum_{i=1}^{m} y_i^2 \geq 8K_y^2 \right) \leq \mathbb{P} \left( \frac{1}{m} \sum_{i=1}^{m} y_i^2 \geq \mathbb{E}[y^2] + 4K_y^2 \right) \leq \frac{\mathbb{Var}[y^2]}{16mK_y^4} \leq \frac{\mathbb{E}[y^4]}{16mK_y^4} \leq \frac{16}{m},
\]

where we use \(34\) in the first and last inequalities. Moreover, from

\[
\mathbb{E}[y^2 (a_i^T x)^2] \leq \left( \mathbb{E}[y_i^4] \right)^{1/2} \left( \mathbb{E}[(a_i^T x)^4] \right)^{1/2} = \left( \mathbb{E}[y_i^4] \right)^{1/2} \left( \mathbb{E}[g^4] \right)^{1/2} \leq (4K_y)^2 \cdot \sqrt{3} = 16\sqrt{3}K_y^2,
\]

where \(g \sim \mathcal{N}(0, 1)\) represents a standard normal random variable, similarly to \(56\), we obtain

\[
\mathbb{P} \left( \frac{1}{m} \sum_{i=1}^{m} y_i^2 (a_i^T x)^2 \geq 32\sqrt{3}K_y^2 \right) \leq \mathbb{P} \left( \frac{1}{m} \sum_{i=1}^{m} y_i^2 (a_i^T x)^2 \geq \mathbb{E}[y^2 (a_i^T x)^2] + 16\sqrt{3}K_y^2 \right)
\]

\[
\leq \frac{\mathbb{Var}[y^2 (a_i^T x)^2]}{768mK_y^4} \leq \frac{\mathbb{E}[y^4 (a_i^T x)^4]}{768mK_y^4}
\]

\[
\leq \frac{\left( \mathbb{E}[y_i^4] \right)^{1/2} \cdot \left( \mathbb{E}[g^8] \right)^{1/2}}{768mK_y^4}
\]

\[
\leq \frac{(8K_y)^4 \cdot \sqrt{105}}{768mK_y^4} = \frac{16\sqrt{105}}{3m},
\]

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where \( g \sim \mathcal{N}(0, 1) \) and (41) follows from (34). Furthermore, since \( \mathbb{E}[y(a^T x)^2] = \text{Cov}[y, (a^T x)^2] + \mathbb{E}[y] \cdot \mathbb{E}[(a^T x)^2] = \nu + M_y \) and \( y(a^T x)^2 \) is sub-Weibull of order \( \alpha = 1/2 \) with the corresponding constant \( C_\alpha \leq CK_y \) from Lemma 2 we obtain that for any \( u > 2 \), with probability at least \( 1 - e^{-u} \),

\[
\left| \frac{1}{m} \sum_{i=1}^{m} y_i (a_i^T x)^2 - (\nu + M_y) \right| \leq C' K_y \left( \sqrt{\frac{u}{m}} + \frac{u^2}{m} \right). \tag{42}
\]

Setting \( u = \log m \), we obtain that when \( m \geq \log^3 m \), with probability at least \( 1 - 1/m \),

\[
\left| \frac{1}{m} \sum_{i=1}^{m} y_i (a_i^T x)^2 - (\nu + M_y) \right| \leq 2C' K_y \cdot \sqrt{\frac{\log m}{m}}. \tag{43}
\]

Since \( y \) is sub-exponential with \( \| y \|_{\psi_1} = K_y \) and \((a^T x)^2 \) is sub-exponential with the sub-exponential norm being upper bounded by \( C \), similarly to (43), we have with probability at least \( 1 - 2/m \) that

\[
\left| \frac{1}{m} \sum_{i=1}^{m} y_i - M_y \right| \leq CK_y \cdot \sqrt{\frac{\log m}{m}}, \tag{44}
\]

\[
\left| \frac{1}{m} \sum_{i=1}^{m} (a_i^T x)^2 - 1 \right| \leq C \sqrt{\frac{\log m}{m}}. \tag{45}
\]

\[ \square \]

**B Proof of Theorem [1]** (Guarantees for the First Step of Algorithm [1])

Before proving the theorem, we provide some additional auxiliary results.

**B.1 Useful Lemmas for Theorem [1]**

Recall that \( V \) is defined in (10) and \( \nu \) is defined in (7). First, we present the following useful lemma.

**Lemma 5.** Let \( E = V - \nu xx^T \). For any \( u > 2 \) satisfying \( m = \Omega(u \cdot \log m) \), conditioned on the event \( E \) (cf. (30)), we have for any fixed \( s_1, s_2 \in \mathbb{R}^n \), with probability \( 1 - O(e^{-u}) \) that

\[
| s_1^T Es_2 | = O \left( K_y \sqrt{\frac{u \cdot \log m}{m}} \right) \cdot \| s_1 \|_2 \cdot \| s_2 \|_2. \tag{46}
\]

**Proof.** First, it is easy to calculate that \( \mathbb{E}[V] = \nu xx^T \) (see, e.g., [47] Lemma 8)) and thus \( \mathbb{E}[E] = 0 \). Without loss of generality, we assume that \( \| s_1 \|_2 = \| s_2 \|_2 = 1 \), and we also assume that \( s_1 \neq x \) and \( s_2 \neq x \). From the definition of \( V \) in (10), we have

\[
s_1^T Es_2 = s_1^T (V - \nu xx^T) s_2 = \frac{1}{m} \sum_{i=1}^{m} y_i (a_i^T s_1)(a_i^T s_2) - s_1^T s_2 - \nu (s_1^T x)(s_2^T x). \tag{47}
\]

We focus on dealing with the term

\[
I := \frac{1}{m} \sum_{i=1}^{m} y_i (a_i^T s_1)(a_i^T s_2). \tag{48}
\]

We decompose \( s_1 \) as

\[
s_1 = (s_1^T x)x + \sqrt{1 - (s_1^T x)^2} \cdot t_1, \tag{49}
\]

\[ \text{Since} \ y \ \text{is sub-exponential with} \ \| y \|_{\psi_1} = K_y, \ \text{from} \ (27), \ \text{we obtain that} \ \sqrt{|y|} \ \text{is sub-Gaussian with the sub-Gaussian norm being} \ \sqrt{K_y}. \ \text{From Lemma} \ 1, \ \sqrt{|y| \cdot (a^T x)} \ \text{is sub-exponential with the sub-exponential norm being upper bounded by} \ \sqrt{C K_y}, \ \text{where} \ C \ \text{is an absolute constant. Again from} \ (27), \ \text{we obtain} \ |y| \cdot (a^T x)^2 \ \text{is sub-Weibull of order} \ \alpha = 1/2 \ \text{with the corresponding constant} \ C_\alpha \leq CK_y.
\]

\[ \text{We will see from the proof that the case that} \ s_1 = x \ \text{or} \ s_2 = x \ \text{is easier to handle.} \]
where $||t_1||_2 = 1$ and $t_1^T x = 0$. Similarly, letting \( w_{12} = \sqrt{1 - (s_T x)^2 - (s_T^2 t_1)^2} \), \( s_2 \) can be written as
\[
s_2 = (s_T x) x + (s_T^2 t_1) t_1 + w_{12} t_2,
\]
where $||t_2||_2 = 1$ and $t_2^T x = t_2^2 t_1 = 0$. Note that from (49) and (50), we obtain
\[
s_1^T s_2 = (s_T x)(s_T^2 x) + \sqrt{1 - (s_T x)^2} \cdot (s_T^2 t_1).
\]
Let \( g_i = a_i^T x \), \( h_{i,1} = a_i^T t_1 \), and \( h_{i,2} = a_i^T t_2 \); the three are independent standard normal random variables. From (49) and (50), \( I \) in (48) can be written as
\[
I = \frac{1}{m} \sum_{i=1}^{m} y_i(a_i^T s_1)(a_i^T s_2)
= \frac{1}{m} \sum_{i=1}^{m} y_i((s_T x) g_i + \sqrt{1 - (s_T x)^2} \cdot h_{i,1}((s_T x) g_i + (s_T^2 t_1) h_{i,1} + w_{12} h_{i,2})
\]
\[
= (s_T x)(s_T^2 x) \cdot \frac{1}{m} \sum_{i=1}^{m} y_i g_i^2 + \sqrt{1 - (s_T x)^2} \cdot (s_T^2 t_1) \cdot \frac{1}{m} \sum_{i=1}^{m} y_i h_{i,1}^2
\]
\[
+ \sqrt{1 - (s_T x)^2} \cdot w_{12} \cdot \frac{1}{m} \sum_{i=1}^{m} y_i h_{i,1} h_{i,2}
\]
\[
+ ((s_T x)(s_T^2 t_1) + \sqrt{1 - (s_T x)^2} \cdot (s_T^2 x)) \cdot \frac{1}{m} \sum_{i=1}^{m} y_i g_i h_{i,1} + (s_T^2 x) w_{12} \cdot \frac{1}{m} \sum_{i=1}^{m} y_i g_i h_{i,2}.
\]

In the following, we deal with the five terms in (53) separately.

- The first term \((s_T x)(s_T^2 x) \cdot \frac{1}{m} \sum_{i=1}^{m} y_i g_i^2\): From Lemma 4 we have conditioned on the event \( \mathcal{E} \) that
\[
\left| \frac{1}{m} \sum_{i=1}^{m} y_i g_i^2 - (\nu + M_y) \right| < C K_y \sqrt{\frac{\log m}{m}},
\]
which gives
\[
\left| (s_T x)(s_T^2 x) \cdot \frac{1}{m} \sum_{i=1}^{m} y_i g_i^2 - (\nu + M_y) \right| < \frac{1}{m} \sum_{i=1}^{m} y_i g_i^2 \cdot C K_y \sqrt{\frac{\log m}{m}} \leq C K_y \sqrt{\frac{\log m}{m}}.
\]

- The second term \( \sqrt{1 - (s_T x)^2} \cdot (s_T^2 t_1) \cdot \frac{1}{m} \sum_{i=1}^{m} y_i h_{i,1}^2\): Since \( y_i \) are independent of \( h_{i,1} \), \( \mathbb{E}[h_{i,1}^2] = 1 \), and \( h_{i,1}^2 \) are sub-exponential with the sub-exponential norm being upper bounded by an absolute constant \( C \), from Lemma 2 we obtain that for any \( u > 2 \), with probability at least \( 1 - e^{-u} \),
\[
\left| \frac{1}{m} \sum_{i=1}^{m} y_i (h_{i,1}^2 - 1) \right| \leq C \left( \frac{\sqrt{u} \cdot \sqrt{\sum_{i=1}^{m} y_i^2 / m}}{\sqrt{m}} + \frac{u \cdot \max_{i \in [m]} |y_i|}{m} \right).
\]

From Lemma 4 we obtain that when \( m = \Omega(u \cdot \log m) \), conditioned on the event \( \mathcal{E} \),
\[
\left| \frac{1}{m} \sum_{i=1}^{m} y_i (h_{i,1}^2 - 1) \right| \leq C K_y \cdot \sqrt{\frac{u \cdot \log m}{m}}.
\]

Then,
\[
\sqrt{1 - (s_T x)^2} \cdot (s_T^2 t_1) \cdot \frac{1}{m} \sum_{i=1}^{m} y_i h_{i,1}^2 - M_y \leq \left| \frac{1}{m} \sum_{i=1}^{m} y_i (h_{i,1}^2 - 1) \right| + \frac{1}{m} \sum_{i=1}^{m} y_i - M_y
\]
\[
\leq C K_y \cdot \sqrt{u \cdot \log m} \cdot \sqrt{m}.
\]
where (66) follows from 51. Then, from (47), we have where (69) follows from (67) and the definite of the event Based on Lemma 5, we obtain the following lemma.

Combining (53) with (55), (59), (62) and (65), we obtain that when \[ m = \Omega(u \cdot \log m), \]
conditioned on the event \( E \), we have with probability at least \( 1 - e^{-u} \),
\[
\sqrt{1 - \left( s_1^T x \right)^2} \cdot w_{12} \cdot \frac{1}{m} \sum_{i=1}^{m} y_i h_{i,1} h_{i,2} \leq C \frac{2}{m} \sum_{i=1}^{m} y_i g_i h_{i,1} \leq \frac{1}{m} \sum_{i=1}^{m} y_i g_i h_{i,1} \leq C K_y \sqrt{\frac{u \cdot \log m}{m}}.
\]

The fourth to fifth terms in (53) can be controlled in a same way. For example, for the fourth term \( (s_2^T x)(s_1^T t_1) + \sqrt{1 - (s_1^T x)^2} \cdot (s_2^T x) \cdot \frac{1}{m} \sum_{i=1}^{m} y_i g_i h_{i,1} \): We have for fixed \( y_i \) that
\[
\frac{1}{m} \sum_{i=1}^{m} y_i g_i h_{i,1} \sim \mathcal{N} \left( 0, \sum_{i=1}^{m} y_i^2 g_i^2 / m^2 \right).
\]

Then, from the standard Gaussian concentration [89, Example 2.1], we obtain that for any \( u > 0 \), with probability at least \( 1 - 2e^{-u} \),
\[
\left| \left( s_2^T x \right)(s_1^T t_1) + \sqrt{1 - (s_1^T x)^2} \cdot (s_2^T x) \cdot \frac{1}{m} \sum_{i=1}^{m} y_i g_i h_{i,1} \right| \leq \frac{2}{m} \sum_{i=1}^{m} y_i g_i h_{i,1} \leq \frac{\sqrt{8m \cdot \sum_{i=1}^{m} y_i^2 g_i^2 / m}}{m} = O \left( K_y \sqrt{\frac{u}{m}} \right),
\]
where the last inequality follows from Lemma 4. We have a similar result for the fifth term.

Combining (53) with (55), (59), (62) and (65), we obtain that when \( m = \Omega(u \cdot \log m) \) and conditioned on the event \( E \), with probability \( 1 - O(e^{-u}) \),
\[
\left| \frac{1}{m} \sum_{i=1}^{m} y_i \left( a_i^T s_1 \right)(a_i^T s_2) - \left( s_1^T x \right)(s_2^T x) \nu - (s_1^T s_2) M_y \right|
\]
\[
= \left| \frac{1}{m} \sum_{i=1}^{m} y_i \left( a_i^T s_1 \right)(a_i^T s_2) - \left( s_1^T x \right)(s_2^T x) \nu + M_y \right| \leq \frac{C K_y \cdot \sqrt{u \cdot \log m}}{\sqrt{m}},
\]
where (66) follows from (51). Then, from (47), we have
\[
\left| s_1^T E s_2 \right| = \left| \frac{1}{m} \sum_{i=1}^{m} \left( y_i (a_i^T s_1) (a_i^T s_2) - \left( s_1^T x \right)(s_2^T x) \nu - (s_1^T s_2) M_y \right) + \frac{(s_1^T s_2)}{m} \sum_{i=1}^{m} (M_y - y_i) \right|
\]
\[
\leq \frac{C K_y \cdot \sqrt{u \cdot \log m}}{\sqrt{m}},
\]
where (69) follows from (67) and the definite of the event \( E \) in (30). For general \( s_1 \) and \( s_2 \) (beyond unit vectors), when considering \( K_y \) as a fixed positive constant, we obtain (66) as desired.

Based on Lemma 5, we obtain the following lemma.
In addition, we have that \( \| E \|_{2 \rightarrow 2} = O((K_y \cdot n \log m)/m) \) with probability \( 1 - O(1/m) \).

**Proof.** Note that from Lemma 4\(^8\) the event \( \mathcal{E} \) occurs with probability \( 1 - O(1/m) \). Setting \( u = \log((|S_1| \cdot |S_2|)) \) in Lemma 5, and taking a union bound over all \( s_1 \in S_1 \) and \( s_2 \in S_2 \), we obtain \( \mathcal{E} \). In addition, according to \([86, \text{Lemma 5.2}]\), we have

\[
\| E \|_{2 \rightarrow 2} = \| V - \nu xx^T \|_{2 \rightarrow 2} = \sup_{r \in S^{n-1}} |r^T (V - \nu xx^T) r| \leq 2 \sup_{r \in C_{1/4}} |r^T (V - \nu xx^T) r|,
\]

where \( C_{1/4} \) is a \((1/4)\)-net of the unit sphere \( S^{n-1} \). In addition, according to \([86, \text{Lemma 5.2}]\), we have \( |C_{1/4}| \leq 9^n \). Similarly to Lemma 5\(^7\) we obtain that for any \( u \in S^{n-1} \) and any \( u > 2 \) satisfying \( u = \Omega(m) \), with probability at least \( 1 - e^{-u} \),

\[
|r^T E r| \leq C K_y \cdot \frac{u \cdot \log m}{m}.
\]

Setting \( u = Cn \) \((72)\) and taking a union bound over all \( r \in C_{1/4} \), we obtain that with probability \( 1 - e^{-\Omega(n)} - O(1/m) \), \( \| E \|_{2 \rightarrow 2} = O((K_y \cdot n \log m)/m) \). \( \square \)

In addition, we have the following lemma according to \([50, \text{Theorem 2}]^6\)

**Lemma 7.** (Adapted from \([50, \text{Theorem 2}]\)) Suppose that the data matrix \( V \in \mathbb{R}^{n \times n} \) can be written as \( V = V + E \) with \( V \) being a positive definite rank-one matrix and \( E \) satisfying the following conditions: 1) For any two finite sets \( S_1, S_2 \) in \( \mathbb{R}^n \) satisfying \( m = \Omega((|S_1| \cdot |S_2|)) \cdot (\log m) \), we have for all \( s_1 \in S_1 \) and \( s_2 \in S_2 \) that

\[
\| s_1^T E s_2 \| \leq C K_y \sqrt{\frac{(\log(|S_1| \cdot |S_2|)) \cdot (\log m)}{m}} \cdot \| s_1 \|_2 \cdot \| s_2 \|_2;
\]

2) \( \| E \|_{2 \rightarrow 2} = O((K_y \cdot n \log m)/m) \). Then, if there exists \( t_0 \in \mathbb{N} \) such that \( x^T w^{(t_0)} \geq c_0 \) with \( c_0 \) being a sufficiently small positive constant and \( m = \Omega((k \log(nLr)) \cdot (\log m)) \) with a large enough implied constant, we have that after one projected power iteration in the first step of Algorithm \( \square \)

\( \text{(beyond } t_0) \),

\[
\| w^{(t)} - x \|_2 \leq \frac{C K_y}{c_0} \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}},
\]

i.e., this equation holds for all \( t > t_0 \).

### B.2 Proof of Theorem 1\(^1\)

Combining the results of Lemmas 6\(^6\) and 7\(^7\), we obtain the desired result of Theorem 1\(^1\)

### C Proof of Theorem 2\(^2\)(Guarantees for the Second Step of Algorithm 1\(^1\))

Before presenting the proof of the theorem, we provide some useful lemmas.

#### C.1 Useful Lemmas for Theorem 2\(^2\)

Recall that \( \bar{y}, \bar{\nu}^{(t)} \) are defined in \((12)\) and \((13)\) respectively and \( M_y := E[y] \) and \( K_y := \| y \|_{\psi^1} \) (cf. Section 2.2). First, we have the following lemma.

**Lemma 8.** Conditioned on the event \( \mathcal{E} \) (cf. \([39]\)), when \( m = \Omega((k \log(nLr)) \cdot (\log m)) \), for any \( t \in \{0, 1, \ldots, T_2 - 1\} \), we have with probability \( 1 - e^{-\Omega(k \log(nLr))} \) that

\[
\| \bar{\nu}^{(t)} - \nu \| \leq C K_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + \| x^{(t)} - x \|_2^2 \cdot \nu.
\]

\( ^{\text{More precisely, only } (57) \text{ (and thus } (59) \text{) and } (62) \text{ need to be modified accordingly.}}\)

\( ^{\text{We consider the } \delta = 0 \text{ case therein.}}\)
Proof. For any $\delta \in (0, 1)$, let $M$ be a $(\delta/L)$-net of $B^k(r)$. According to \textsuperscript{86} Lemma 5.2, there exists such a net with the cardinality satisfies
\[
\log |M| \leq k \log \frac{4Lr}{\delta}.
\] (75)
Due to the $L$-Lipschitz continuity of $G$, $G(M)$ is a $\delta$-net of $G(B^k(r))$. Then, since $X(t) \in \text{Range}(G) = G(B^k(r))$, it can be written as
\[
X(t) = S(t) + E(t),
\] (76)
where $S(t) \in G(M)$ and $\|E(t)\|_2 \leq \delta$. We obtain
\[
|\hat{\nu}(t) - \nu| = \left| \frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot \left( a_i^T \frac{X(t)}{\|X(t)\|} \right)^2 - \nu \right|
\] (77)
\[
= \left| \frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot \left( a_i^T \left( S(t) + E(t) \right) \right)^2 - \nu \right|
\] (78)
\[
\leq \left| \frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot \left( a_i^T S(t) \right)^2 - \nu \right| + \left| \frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot \left( a_i^T E(t) \right)^2 \right|
\] (79)
\[
\leq \left| \frac{2}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot \left( a_i^T S(t) \right) \cdot \left( a_i^T E(t) \right) \right|
\]
From Lemma \textsuperscript{2} and by taking a union bound over $[m]$, we obtain with probability $1 - me^{-\Omega(n)}$ that
\[
\max_{i \in [m]} \|a_i\|_2 \leq \sqrt{2n}.
\] (80)
In addition, we have
\[
\frac{1}{m} \sum_{i=1}^{m} |y_i - \bar{y}| \leq \sqrt{\frac{\sum_{i=1}^{m} (y_i - \bar{y})^2}{m}} \leq \sqrt{\frac{\sum_{i=1}^{m} y_i^2 - m\bar{y}^2}{m}} \leq \sqrt{\frac{\sum_{i=1}^{m} y_i^2}{m}}.
\] (81)
(82)
(83)
Then, from the definition of the event $\mathcal{E}$ in Lemma \textsuperscript{4}, we obtain
\[
\frac{1}{m} \sum_{i=1}^{m} |y_i - \bar{y}| \leq \sqrt{\frac{\sum_{i=1}^{m} y_i^2}{m}} \leq 2\sqrt{2}K_y.
\] (84)
Therefore, since $\|E(t)\|_2 \leq \delta$, conditioned on the event in \textsuperscript{80} and event $\mathcal{E}$, we have
\[
\left| \frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot \left( a_i^T E(t) \right)^2 \right| \leq \frac{1}{m} \sum_{i=1}^{m} |y_i - \bar{y}| \cdot 2n\delta^2 \leq 2\sqrt{2}K_y \cdot 2n\delta^2.
\] (85)
(86)
In addition, since $S(t) \in G(M) \subseteq S^{n-1}$, similarly to \textsuperscript{86}, we obtain
\[
\left| \frac{2}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot \left( a_i^T S(t) \right) \cdot \left( a_i^T E(t) \right) \right| \leq 4\sqrt{2}K_y \cdot 2n\delta.
\] (87)
Then, it remains to control the first term in (79), namely \( \frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot (a_i^T s)^2 - \nu \). For any fixed \( s \in S^{n-1} \), we obtain

\[
\left| \frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot (a_i^T s)^2 - \nu \right| \leq \left| \frac{1}{m} \sum_{i=1}^{m} (y_i - M_y) \cdot (a_i^T s)^2 - \nu \right| + \left| \frac{1}{m} \sum_{i=1}^{m} (M_y - \bar{y}) \cdot (a_i^T s)^2 \right| \quad (88)
\]

\[
\leq \left| \frac{1}{m} \sum_{i=1}^{m} (y_i - M_y) \cdot (a_i^T s)^2 - (x^T s)^2 \nu \right| + \left| 1 - (x^T s)^2 \right| \cdot \nu + \left| \frac{M_y - \bar{y}}{m} \sum_{i=1}^{m} (a_i^T s)^2 \right| \quad (89)
\]

\[
\leq \left| \frac{1}{m} \sum_{i=1}^{m} (y_i - M_y) \cdot (a_i^T s)^2 - (x^T s)^2 \nu \right| + \left| \frac{M_y - \bar{y}}{m} \sum_{i=1}^{m} (a_i^T s)^2 \right| + \|x - s\|^2 \cdot \nu, \quad (90)
\]

where we use \( \left| 1 - (x^T s)^2 \right| = (1 + x^T s) \cdot (1 - x^T s) \leq 2(1 - x^T s) = \|x - s\|^2 \) in (90). Since \( s \) is a unit vector, from Lemma 2, we obtain with probability \( 1 - e^{-\Omega(m)} \) that

\[
\frac{1}{m} \sum_{i=1}^{m} (a_i^T s)^2 \leq 2. \quad (91)
\]

In addition, conditioned on the event \( \mathcal{E} \), we have

\[
\|\bar{y} - M_y\| \leq C K_y \sqrt{\frac{\log m}{m}}. \quad (92)
\]

Then, we only need to focus on the first term of (90). We write \( s \in S^{n-1} \) as

\[
s = (s^T x) x + \sqrt{1 - (s^T x)^2} \cdot t, \quad (93)
\]

where \( x^T t = 0 \) and \( \|t\|_2 = 1 \). Hence,

\[
(a_i^T s)^2 = (s^T x)^2 (a_i^T x)^2 + \left(1 - (x^T s)^2\right) \cdot (a_i^T t)^2 + 2 (x^T s) \cdot \sqrt{1 - (x^T s)^2} \cdot (a_i^T x) \cdot (a_i^T t). \quad (94)
\]

Then, the first term of (90) can be upper bounded as

\[
\left| \frac{1}{m} \sum_{i=1}^{m} (y_i - M_y) \cdot (a_i^T s)^2 - (x^T s)^2 \nu \right| \leq \left| \frac{(x^T s)^2}{m} \cdot \sum_{i=1}^{m} (y_i - M_y) (a_i^T x)^2 - \nu \right| + \left| \frac{1}{m} (x^T s)^2 \right| + \left| \frac{m}{m} \sum_{i=1}^{m} (y_i - M_y) (a_i^T t)^2 \right| + \left| \frac{m}{m} \sum_{i=1}^{m} (y_i - M_y) \cdot (a_i^T x) \cdot (a_i^T t) \right|. \quad (95)
\]

Conditioned on the event \( \mathcal{E} \), the first term in (95) can be upper bounded by

\[
\left| \frac{(x^T s)^2}{m} \sum_{i=m+1}^{2m} (y_i - M_y) (a_i^T x)^2 - \nu \right| \leq C K_y \sqrt{\frac{\log m}{m}}. \quad (96)
\]

In addition, since \( y = f(a^T x) \) is independent of \( a^T t \), from Lemma 2 for any \( u > 2 \) and fixed \( y_i \), we have with probability at least \( 1 - e^{-u} \) that

\[
\frac{1}{m} \sum_{i=1}^{m} (y_i - M_y) (a_i^T t)^2 \leq \frac{1}{m} \sum_{i=1}^{m} (y_i - M_y) ((a_i^T t)^2 - 1) + \frac{1}{m} \sum_{i=1}^{m} (y_i - M_y) \quad (97)
\]

\[
\leq C \left( \sqrt{u} \cdot \sqrt{\frac{\sum_{i=1}^{m} (y_i - M_y)^2}{m}} + u \cdot \max \left\{ \frac{\sum_{i=1}^{m} (y_i - M_y)}{m} \right\} + \frac{1}{m} \sum_{i=1}^{m} (y_i - M_y \right). \quad (98)
\]

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Conditioned on the event $\mathcal{E}$ and using the inequality $|M_y| \leq K_y$ (cf. (44)), we obtain that when $m = \Omega(u \cdot \log m)$, with probability at least $1 - e^{-u}$, it holds that

$$\frac{1 - (x^T s)^2}{m} \cdot \left| \sum_{i=1}^{m} (y_i - M_y)(a_i^T t)^2 \right| \leq CK_y \cdot \sqrt{\frac{u \cdot \log m}{m}}. \tag{99}$$

Moreover, similarly to (65), we obtain that conditioned on the event $\mathcal{E}$, with probability at least $1 - 2e^{-u}$, the third term in (99) is upper bounded as

$$\frac{2 (x^T s) \cdot \sqrt{1 - (x^T s)^2}}{m} \cdot \left| \sum_{i=1}^{m} (y_i - M_y) \cdot (a_i^T x) \cdot (a_i^T t) \right| = O \left( K_y \cdot \sqrt{\frac{u \cdot \log m}{m}} \right). \tag{100}$$

Combining (90), (91), (92), (93), (99) and (100), we obtain that conditioned on the event $\mathcal{E}$, for any fixed $s \in S^{n-1}$ and $u > 2$, when $m = \Omega(u \cdot \log m)$, with probability $1 - e^{-\Omega(u)}$,

$$\frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot (a_i^T s)^2 - \nu \leq CK_y \cdot \sqrt{\frac{u \cdot \log m}{m}} + \|x - s\|_2^2 \cdot \nu. \tag{101}$$

Taking a union bound for all $s \in G(M)$ and setting $u = k \log \frac{Lr}{\delta}$, we obtain that when conditioned on the event $\mathcal{E}$ and $m = \Omega((k \log \frac{Lr}{\delta}) \cdot (\log m))$, with probability $1 - e^{-\Omega(k \log \frac{Lr}{\delta})}$, for all $s \in G(M)$,

$$\frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y}) \cdot (a_i^T s)^2 - \nu \leq CK_y \cdot \sqrt{\frac{(k \log \frac{Lr}{\delta}) \cdot (\log m)}{m}} + \|x - s\|_2^2 \cdot \nu \tag{102}$$

and this gives an upper bound for the first term of (79) by substituting $s^{(t)}$ for $s$. Combining (79), (86), (87) and (102), we obtain that when $m = \Omega((k \log \frac{Lr}{\delta}) \cdot (\log m))$, with probability $1 - e^{-\Omega(k \log \frac{Lr}{\delta})}$,

$$\hat{\nu}^{(t)} - \nu \leq 4\sqrt{2}K_y n \delta(2 + \delta) + CK_y \cdot \sqrt{\frac{(k \log \frac{Lr}{\delta}) \cdot (\log m)}{m}} + \|x - s^{(t)}\|_2^2 \cdot \nu \tag{103}$$

$$\leq CK_y \left( n \delta + \sqrt{\frac{(k \log \frac{Lr}{\delta}) \cdot (\log m)}{m}} + \left( \|x - x^{(t)}\|_2 + \delta \right)^2 \cdot \nu \right), \tag{104}$$

where (104) follows from (76). Setting $\delta = \frac{1}{\sqrt{m}}$ and using $n = \Omega(m)$, we obtain

$$CK_y \left( n \delta + \sqrt{\frac{(k \log \frac{Lr}{\delta}) \cdot (\log m)}{m}} + \left( \|x - x^{(t)}\|_2 + \delta \right)^2 \cdot \nu \right)$$

$$\leq CK_y \sqrt{\frac{(k \log \frac{Lr}{\delta}) \cdot (\log m)}{m}} + \left( \|x - x^{(t)}\|_2 + 2\delta \cdot \|x - x^{(t)}\|_2 + \delta^2 \right) \cdot \nu \tag{105}$$

$$\leq CK_y \sqrt{\frac{(k \log \frac{Lr}{\delta}) \cdot (\log m)}{m}} + \|x - x^{(t)}\|_2^2 \cdot \nu \tag{106}$$

where (106) follows from $\|x - x^{(t)}\|_2 \leq 2$ and $\nu > 0$ is a fixed constant (recall that the value of $C$ may differ from line to line).

Next, we present the following useful lemma.

**Lemma 9.** For any $u > 2$ satisfying $m = \Omega(u \cdot \log m)$, conditioned on the event $\mathcal{E}$ (cf. (30)), we have that for any $s_1 \in S^{n-1}$ and $s_2 \in \mathbb{R}^n$, with probability $1 - e^{-\Omega(u)}$,

$$\frac{1}{m} \sum_{i=1}^{m} ((y_i - \bar{y}) \cdot (a_i^T s_1) - \nu(a_i^T x)) \cdot (a_i^T s_2) \leq \left( CK_y \sqrt{\frac{u \cdot \log m}{m}} + \|s_1 - x\|_2^2 \cdot \nu \right) \cdot \|s_2\|_2. \tag{107}$$
Proof. Without loss of generality, we assume that \( \|s_2\|_2 = 1 \). We have
\[
\frac{1}{m} \sum_{i=1}^{m} \left( (y_i - \bar{y}) \cdot (a_i^T s_1) - \nu (a_i^T x) \right) \cdot (a_i^T s_2)
\]
\[
= \frac{1}{m} \sum_{i=1}^{m} \left( (y_i - M_y) \cdot (a_i^T s_1) - \nu (a_i^T x) \right) \cdot (a_i^T s_2) + \frac{1}{m} \sum_{i=1}^{m} (M_y - \bar{y}) \cdot (a_i^T s_1) \cdot (a_i^T s_2). \tag{108}
\]
In the second term of (108), we observe that from Lemma 1, \((a_i^T s_1) \cdot (a_i^T s_2)\) are i.i.d. sub-exponential random variables with mean \(s_1^T s_2\) and the sub-exponential norm being upper bounded by \(C\). From Lemma 2, we have with probability \(1 - e^{-u}\) that
\[
\left| \frac{1}{m} \sum_{i=1}^{m} (a_i^T s_1) \cdot (a_i^T s_2) \right| \leq C \sqrt{\frac{u}{m}}. \tag{109}
\]
Then, we obtain
\[
\left| \frac{1}{m} \sum_{i=1}^{m} (M_y - \bar{y}) \cdot (a_i^T s_1) \cdot (a_i^T s_2) \right| \leq |M_y - \bar{y}| \cdot \left| \frac{1}{m} \sum_{i=1}^{m} (a_i^T s_1) \cdot (a_i^T s_2) \right| \tag{110}
\]
\[
\leq CK_y \sqrt{\log \frac{m}{m}}, \tag{111}
\]
\[
\leq CK_y \sqrt{\frac{m}{m}} \left( 1 + C \sqrt{\frac{u}{m}} \right) \tag{112}
\]
\[
\leq C' K_y \sqrt{\frac{m}{m}}, \tag{113}
\]
where (111) follows from the definition of the event \(E\) in Lemma 3, (112) follows from (109) and \(\|s_1^T s_2\| \leq \|s_1\|_2 \cdot \|s_2\|_2 = 1\), and (113) follows from the condition \(m = \Omega(u \cdot \log m)\). Then, it remains to control the first term of (108). Similarly to that in the proof of Lemma 4, we decompose \(s_1\) as (cf. 49)
\[
s_1 = (s_1^T x)x + \sqrt{1 - (s_1^T x)^2} \cdot t_1, \tag{114}
\]
where \(\|t_1\|_2 = 1\) and \(t_1^T x = 0\). Similarly, letting \(w_{12} = \sqrt{1 - (s_2^T x)^2} - (s_2^T t_1)^2\), \(s_2\) can be written as (cf. 50)
\[
s_2 = (s_2^T x)x + (s_2^T t_1) t_1 + w_{12} t_2, \tag{115}
\]
where \(\|t_2\|_2 = 1\) and \(t_2^T x = t_2^T t_1 = 0\). Let \(g_i = a_i^T x \sim \mathcal{N}(0, 1)\), \(h_{i,1} = a_i^T t_1 \sim \mathcal{N}(0, 1)\), and \(h_{i,2} = a_i^T t_2 \sim \mathcal{N}(0, 1)\); the three are independent. Therefore, we obtain
\[
\frac{1}{m} \sum_{i=1}^{m} ((y_i - M_y) \cdot (a_i^T s_1) - \nu (a_i^T x)) \cdot (a_i^T s_2) = \frac{1}{m} \sum_{i=1}^{m} ((y_i - M_y) \cdot (s_1^T x) - \nu) \cdot (s_2^T x) g_i^2
\]
\[
+ \frac{1}{m} \sum_{i=1}^{m} \left( (y_i - M_y) \cdot \sqrt{1 - (s_1^T x)^2} \cdot (s_2^T t_1) \right) h_{i,1}^2
\]
\[
+ \frac{1}{m} \sum_{i=1}^{m} \left( (y_i - M_y) \cdot \sqrt{1 - (s_1^T x)^2} \right) \cdot w_{12} h_{i,1} h_{i,2}
\]
\[
+ \frac{1}{m} \sum_{i=1}^{m} \left( (y_i - M_y) \cdot \sqrt{1 - (s_1^T x)^2} \cdot (s_2^T x) + (y_i - M_y) \cdot (s_1^T x) \cdot (s_2^T t_1) - \nu (s_2^T t_1) \right) \cdot g_i h_{i,1}
\]
\[
+ \frac{1}{m} \sum_{i=1}^{m} ((y_i - M_y) \cdot (s_1^T x) - \nu) \cdot w_{12} g_i h_{i,2}. \tag{116}
\]
The equality (116) is similar to (53), with the major difference being that \(y_i\) is replaced by \(y_i - M_y\), which has zero mean. In the following, we focus on bounding the first term in (116), and other terms
can be similarly bounded as those in the proof of Lemma 5. In particular, for the first term in (116), we have
\[
\frac{1}{m} \sum_{i=1}^{m} ((y_i - M_y) \cdot (s_1^T x) - \nu) \cdot (s_2^T x) g_i^2 \\
= \left( \frac{s_1^T x}{m} \right) \left( \frac{s_2^T x}{m} \right) \sum_{i=1}^{m} (y_i - M_y) \cdot (s_2^T x) g_i^2 + \frac{1}{m} \sum_{i=1}^{m} \nu (s_1^T x) ((s_1^T x) - 1) g_i^2
\]
(117)
\[
= \left( \frac{s_1^T x}{m} \right) \left( \frac{s_2^T x}{m} \right) \sum_{i=1}^{m} (y_i g_i^2 - M_y - \nu) + \left( \frac{s_1^T x}{m} \right) \left( \frac{s_2^T x}{m} \right) \sum_{i=1}^{m} (\nu + M_y) (1 - g_i^2)
\]
\[+ \frac{1}{m} \sum_{i=1}^{m} \nu (s_1^T x) ((s_1^T x) - 1) g_i^2. \]
(118)
By Lemma 4 and similarly to (109), as well as using \(|M_y| \leq K_y \text{ (cf. (34))}\) and \(\nu < CK_y \text{ (cf. (67))}\), we obtain
\[
\left| \frac{1}{m} \sum_{i=1}^{m} ((y_i - M_y) \cdot (s_1^T x) - \nu) \cdot (s_2^T x) g_i^2 \right| \leq C K_y \sqrt{\frac{\log m}{m}} + C' \|s_1 - x\|^2 \cdot \nu,
\]
(119)
where \(C' > 0\) can be chosen to be slightly larger than \(\frac{1}{2}\). For the last four terms in (116), similarly to (67), we obtain that when \(m = \Omega(u \cdot \log m)\), with probability \(1 - O(e^{-u})\), the sum of the absolute value of these four terms can be upper bounded by
\[
\frac{C K_y \sqrt{u \cdot \log m}}{\sqrt{m}}.
\]
(120)
Combining (108), (113), (116), (119), and (120), we obtain the desired result. \(\square\)

C.2 Proof of Theorem 2
Since \(x^{(t+1)} = P_G(x^{(t+1)})\) and \(x \in \text{Range}(G)\), we obtain
\[
\left\| \hat{x}^{(t+1)} - x^{(t+1)} \right\|_2 \leq \left\| \hat{x}^{(t+1)} - x \right\|_2,
\]
(121)
or equivalently,
\[
\left\| \hat{x}^{(t+1)} + (x - x^{(t+1)}) \right\|_2 \leq \left\| \hat{x}^{(t+1)} - x \right\|_2,
\]
(122)
which gives
\[
\left\| x^{(t+1)} - x \right\|_2^2 \leq 2\left\| \hat{x}^{(t+1)} - x, x^{(t+1)} - x \right\|
\]
(123)
\[
= 2 \left\langle x^{(t)} - \frac{\zeta}{m} \cdot \sum_{i=1}^{m} \left( \hat{\mu}_i \left( a_i^T x^{(t)} \right) - \hat{y}_i \right) a_i, x^{(t+1)} - x \right\rangle
\]
(124)
\[
= 2 \left\langle x^{(t)} - \frac{\zeta}{m} \cdot \sum_{i=1}^{m} \left( \nu \left( a_i^T x^{(t)} \right) - \hat{y}_i \right) a_i, x^{(t+1)} - x \right\rangle
\]
\[+ 2 \left\langle \frac{\zeta}{m} \cdot \sum_{i=1}^{m} \left( \hat{y}_i \right) a_i, x^{(t+1)} - x \right\rangle
\]
\[+ 2 \left\langle \left( I_n - \frac{\zeta}{m} \cdot \sum_{i=1}^{m} a_i a_i^T \right) \left( x^{(t)} - x \right), x^{(t+1)} - x \right\rangle
\]
(125)
\footnote{Recall that we will assume that \(\nu > 0\) (cf. (7)). We have \(\nu = \mathbb{E}[(y(a^T x))^2] - M_y \leq (\mathbb{E}[y^2])^{1/2} \cdot (\mathbb{E}[(a^T x)^4])^{1/2} + |M_y|.\) Note that \(a^T x \sim \mathcal{N}(0, 1)\). From (54), we have \(|M_y| \leq K_y\) and \((\mathbb{E}[y^2])^{1/2} \leq 2K_y.\) Then, we obtain \(\nu \leq (2\sqrt{3} + 1)K_y.\)}
where (124) follows from the setting of $\bar{x}^{(t+1)}$ in (15), (125) follows from the decomposition $\hat{\nu}^{(t+1)} = \nu + (\hat{\nu}^{(t)} - \nu)$, (126) follows from the fact that the first term in (125) can be written as the sum of the first two terms in (126), and (127) follows from the setting of $\tilde{y}_{i}^{(t)}$ in (14). For any $\delta \in (0, 1)$, let $M$ be a $(\delta/L)$-net of $B^{2}(r)$. Then, similarly to that in the proof of Lemma 8, we have $\log |M| \leq k \log \frac{4e}{\delta}$ (cf. (75)) and $G(M)$ is a $\delta$-net of $G(B^{2}(r))$. For any $t \in \{0, \ldots, T_{2}\}$, we write

$$x^{(t)} = s^{(t)} + e^{(t)},$$

(128)

where $s^{(t)} \in G(M)$ and $\|e^{(t)}\|_{2} \leq \delta$. Next, we provide upper bounds for the three terms in (127) separately. Throughout the following, we assume the occurrence of the event $E$ (cf. (30)) and the relevant probabilities are all conditioned accordingly.

The first term of (127): From (128), we have

$$2 \left\langle \frac{\zeta}{m} \sum_{i=1}^{m} \left( (y_{i} - \bar{y}) \cdot (a_{i}^{T} x^{(t)}) - \nu (a_{i}^{T} x^{(t)}) \right) a_{i}, x^{(t+1)} - x \right\rangle$$

$$= 2 \left\langle \frac{\zeta}{m} \sum_{i=1}^{m} \left( (y_{i} - \bar{y}) \cdot (a_{i}^{T} x^{(t)}) - \nu (a_{i}^{T} x^{(t)}) \right) a_{i}, s^{(t+1)} - x \right\rangle$$

$$+ 2 \left\langle \frac{\zeta}{m} \sum_{i=1}^{m} \left( (y_{i} - \bar{y}) \cdot (a_{i}^{T} x^{(t)}) - \nu (a_{i}^{T} x^{(t)}) \right) a_{i}, e^{(t+1)} \right\rangle$$

(129)

$$= 2 \left\langle \frac{\zeta}{m} \sum_{i=1}^{m} \left( (y_{i} - \bar{y}) \cdot (a_{i}^{T} s^{(t)}) - \nu (a_{i}^{T} x^{(t)}) \right) a_{i}, s^{(t+1)} - x \right\rangle$$

$$+ 2 \left\langle \frac{\zeta}{m} \sum_{i=1}^{m} \left( (y_{i} - \bar{y}) \cdot (a_{i}^{T} e^{(t)}) \right) a_{i}, s^{(t+1)} - x \right\rangle$$

$$+ 2 \left\langle \frac{\zeta}{m} \sum_{i=1}^{m} \left( (y_{i} - \bar{y}) \cdot (a_{i}^{T} x^{(t)}) - \nu (a_{i}^{T} x^{(t)}) \right) a_{i}, e^{(t+1)} \right\rangle.$$  

(130)

Recall that in (80), we obtain with probability $1 - me^{-\Omega(n)}$ that $\max_{i \in [m]} \|a_{i}\|_{2} \leq \sqrt{2n}$. In addition, according to (84), we have $\frac{1}{m} \sum_{i=1}^{m} |y_{i} - \bar{y}| \leq 2 \sqrt{2} K_{y}$. Then, for the second term in (130), since $\|e^{(t)}\|_{2} \leq \delta$, similarly to (86), we obtain

$$2 \left\langle \frac{\zeta}{m} \sum_{i=1}^{m} (y_{i} - \bar{y}) \cdot (a_{i}^{T} e^{(t)}) a_{i}, s^{(t+1)} - x \right\rangle \leq 2 \zeta \cdot 2n \delta \cdot \|s^{(t+1)} - x\|_{2} \cdot \frac{1}{m} \sum_{i=1}^{m} |y_{i} - \bar{y}|$$

$$\leq 8 \sqrt{2} \zeta K_{y} n \delta \cdot \|s^{(t+1)} - x\|_{2}$$

(131)

$$\leq 8 \sqrt{2} \zeta K_{y} n \delta \cdot \left( \|x^{(t+1)} - x\|_{2} + \delta \right).$$

(132)

(133)
Similarly, since both $\mathbf{x}$ and $\mathbf{x}^{(t)}$ are unit vectors and $\|\mathbf{e}^{(t+1)}\|_2 \leq \delta$, for the third term in (130),

$$
2 \left( \frac{\zeta}{m} \cdot \sum_{i=1}^{m} \left( (y_i - \bar{y}) \cdot (a_i^T \mathbf{x}^{(t)}) - \nu(a_i^T \mathbf{x}) \right) a_i, \mathbf{e}^{(t+1)} \right)
$$

$$
\leq 2\zeta \cdot 2n\delta \cdot \frac{1}{m} \sum_{i=1}^{m} |y_i - \bar{y}| + 2\zeta \cdot 2n\delta \cdot \nu
$$

$$
\leq 4\zeta n\delta \cdot (2\sqrt{2K_y} + \nu). \quad (134)
$$

It remains to control the first term of (130). In order to do this, we make use of Lemma 9 with taking a union bound over all $(\mathbf{s}_1, \mathbf{s}_2) \in G(M) \times (G(M) - \mathbf{x})$ and setting $u = k \log \frac{Lr}{\delta}$, and we obtain that when $m = \Omega \left( \left( \frac{k \log \frac{Lr}{\delta}}{\delta} \right) \cdot (\log m) \right)$, with probability $1 - e^{-\Omega \left( \frac{k \log \frac{Lr}{\delta}}{\delta} \right)}$,

$$
\left| 2 \left( \frac{\zeta}{m} \cdot \sum_{i=1}^{m} \left( (y_i - \bar{y}) \cdot (a_i^T \mathbf{s}^{(t)}) - \nu(a_i^T \mathbf{x}) \right) a_i, \mathbf{s}^{(t+1)} - \mathbf{x} \right) \right|
$$

$$
\leq 2\zeta \cdot \left( CK_y \sqrt{\frac{(k \log \frac{Lr}{\delta}) \cdot (\log m)}{m}} + \nu \left\| \mathbf{s}^{(t)} - \mathbf{x} \right\|_2^2 \right) \cdot \left\| \mathbf{s}^{(t+1)} - \mathbf{x} \right\|_2
$$

$$
\leq 2\zeta \cdot \left( CK_y \sqrt{\frac{(k \log \frac{Lr}{\delta}) \cdot (\log m)}{m}} + \nu \left( \left\| \mathbf{x}^{(t)} - \mathbf{x} \right\|_2 + \delta \right)^2 \right) \cdot \left( \left\| \mathbf{x}^{(t+1)} - \mathbf{x} \right\|_2 + \delta \right). \quad (136)
$$

The second term of (127): By (128), we have

$$
2 \left( \left( \mathbf{I}_n - \frac{\zeta\nu}{m} \cdot \sum_{i=1}^{m} a_i a_i^T \right) (\mathbf{x}^{(t)} - \mathbf{x}), \mathbf{x}^{(t+1)} - \mathbf{x} \right)
$$

$$
= 2 \left( \left( \mathbf{I}_n - \frac{\zeta\nu}{m} \cdot \sum_{i=1}^{m} a_i a_i^T \right) (\mathbf{x}^{(t)} - \mathbf{x}), \mathbf{e}^{(t+1)} \right)
$$

$$
+ 2 \left( \left( \mathbf{I}_n - \frac{\zeta\nu}{m} \cdot \sum_{i=1}^{m} a_i a_i^T \right) \mathbf{e}^{(t)}, \mathbf{s}^{(t+1)} - \mathbf{x} \right)
$$

$$
+ 2 \left( \left( \mathbf{I}_n - \frac{\zeta\nu}{m} \cdot \sum_{i=1}^{m} a_i a_i^T \right) \mathbf{s}^{(t)} - \mathbf{x}, \mathbf{s}^{(t+1)} - \mathbf{x} \right). \quad (138)
$$

Since from (80), we have with probability $1 - m \cdot e^{-\Omega(n)}$ that $\max_{i \in [m]} \|a_i\|_2 \leq \sqrt{2n}$. Similarly to (86), we obtain

$$
2 \left( \left( \mathbf{I}_n - \frac{\zeta\nu}{m} \cdot \sum_{i=1}^{m} a_i a_i^T \right) (\mathbf{x}^{(t)} - \mathbf{x}), \mathbf{e}^{(t+1)} \right) \leq 2 \left( 1 + \frac{2n\zeta\nu}{m} \right) \cdot \delta \cdot \left\| \mathbf{x}^{(t)} - \mathbf{x} \right\|_2 \quad (139)
$$

and

$$
2 \left( \left( \mathbf{I}_n - \frac{\zeta\nu}{m} \cdot \sum_{i=1}^{m} a_i a_i^T \right) \mathbf{e}^{(t)}, \mathbf{s}^{(t+1)} - \mathbf{x} \right) \leq 2 \left( 1 + \frac{2n\zeta\nu}{m} \right) \cdot \delta \cdot \left\| \mathbf{s}^{(t+1)} - \mathbf{x} \right\|_2 \quad (140)
$$

$$
\leq 2 \left( 1 + \frac{2n\zeta\nu}{m} \right) \cdot \delta \cdot \left( \left\| \mathbf{x}^{(t+1)} - \mathbf{x} \right\|_2 + \delta \right). \quad (141)
$$

Fix any pair of $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^n$. From Lemma 2 and similarly to (109), we obtain that for any $u > 2$, with probability at least $1 - e^{-u}$,

$$
\zeta\nu(\mathbf{s}_1^T \mathbf{s}_2) - C\zeta\nu \sqrt{\frac{u}{m}} \cdot \|\mathbf{s}_1\|_2 \cdot \|\mathbf{s}_2\|_2 \leq \zeta\nu \sum_{i=1}^{m} a_i a_i^T \mathbf{s}_1, \mathbf{s}_2 \leq \zeta\nu(\mathbf{s}_1^T \mathbf{s}_2) + C\zeta\nu \sqrt{\frac{u}{m}} \cdot \|\mathbf{s}_1\|_2 \cdot \|\mathbf{s}_2\|_2. \quad (142)
$$
Then, we obtain
\[
\left\| \left( I_n - \frac{\zeta \nu}{m} \sum_{i=1}^{m} a_i a_i^T \right) s_1, s_2 \right\| - (1 - \zeta \nu)(s_1^T s_2) \right\| \leq C\zeta \nu \sqrt{\frac{m}{m}} \cdot \|s_1\|_2 \cdot \|s_2\|_2. \quad (143)
\]

Taking a union bound for all pairs \((s_1, s_2) \in (G(M) - x) \times (G(M) - x)\) and setting \(u = \frac{m}{c m \log m}\), as well as considering \(\zeta, \nu\) as fixed positive constants, we obtain that when \(m = \Omega\left((k \log \frac{Lr}{\delta}) \cdot (\log m)\right)\), with probability \(1 - e^{-\Omega(k \log \frac{Lr}{\delta})}\),
\[
\left\| 2 \left( I_n - \frac{\zeta \nu}{m} \sum_{i=1}^{m} a_i a_i^T \right) \left( s^{(t)} - x \right), s^{(t+1)} - x \right\| \leq 2 \max\{1 - \zeta \nu(1 - 1/\sqrt{\log m}), \zeta \nu(1 + 1/\sqrt{\log m}) - 1\} \cdot \|s^{(t)} - x\|_2 \cdot \|s^{(t+1)} - x\|_2 \quad (144)
\]
\[
\leq (2 \cdot |1 - \zeta \nu| + 1/\sqrt{\log m}) \cdot \|s^{(t)} - x\|_2 \cdot \|s^{(t+1)} - x\|_2 \quad (145)
\]
\[
\leq (2 \cdot |1 - \zeta \nu| + 1/\sqrt{\log m}) \cdot \left( \|x^{(t)} - x\|_2 + \delta \right) \cdot \left( \|x^{(t+1)} - x\|_2 + \delta \right). \quad (146)
\]

The third term of (127). By the TS-REC in Lemma 3, we obtain that for any \(\delta \in (0, 1)\), when \(m = \Omega(k \log \frac{Lr}{\delta})\), with probability \(1 - e^{-\Omega(m)}\),
\[
\frac{1}{\sqrt{m}} \sqrt{\sum_{i=1}^{m} \left( a_i^T (x^{(t+1)} - x) \right)^2} \leq \left( (1 + c)\|x^{(t+1)} - x\|_2 + \delta \right), \quad (147)
\]
where \(c\) is a sufficiently small positive constant. By (128) and similarly to the derivation of the TS-REC, we have that when \(m = \Omega(k \log \frac{Lr}{\delta})\), with probability \(1 - e^{-\Omega(m)}\),
\[
\frac{1}{\sqrt{m}} \sqrt{\sum_{i=1}^{m} \left( a_i^T x^{(t)} \right)^2} \leq (1 + c + \delta). \quad (148)
\]

Then, from the Cauchy-Schwarz inequality, the third term in (127) can be upper bounded as
\[
\left\| x^{(t+1)} - x \right\|_2 \leq Cn\zeta K_y \delta
\]
\[
+ 2\zeta \left( CK_y \sqrt{\frac{(k \log \frac{Lr}{\delta}) \cdot (\log m)}{m}} + \sqrt{\left( \|x^{(t)} - x\|_2 + \delta \right)^2} \cdot \left( \|x^{(t+1)} - x\|_2 + \delta \right)
\]
\[
+ (2 \cdot |1 - \zeta \nu| + 1/\sqrt{\log m}) \cdot \left( \|x^{(t)} - x\|_2 + \delta \right) \cdot \left( \|x^{(t+1)} - x\|_2 + \delta \right)
\]
\[
+ 2\zeta (1 + c + \delta) \cdot \|x^{(t+1)} - x\|_2 \cdot \left( (1 + c)\|x^{(t+1)} - x\|_2 + \delta \right). \quad (151)
\]
Considering \( \zeta \) as a positive constant and setting \( \delta = \frac{K_y}{mn} \), as well as using \( n = \Omega(m) \), similarly to (106), we obtain
\[
\left\| x^{(t+1)} - x \right\|_2^2 \leq \frac{CK_y^2}{m} + \left( \left\| x^{(t+1)} - x \right\|_2 + \frac{K_y}{mn} \right)
\times \left( CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + 2 \zeta \nu \left\| x^{(t)} - x \right\|_2 \right)
+ 2 \zeta \left( 1 + c + \frac{K_y}{mn} \right) \cdot \nu \cdot \nu^{(t)} \cdot \left( (1 + c) \left\| x^{(t+1)} - x \right\|_2 + \frac{K_y}{mn} \right).
\tag{152}
\]
From Lemma \[8\] we have
\[
\left| \nu - \nu^{(t)} \right| \leq CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + \left\| x^{(t)} - x \right\|_2 \cdot \nu.
\tag{153}
\]
Then, since \( c > 0 \) is sufficiently small, we obtain
\[
2 \zeta \left( 1 + c + \frac{K_y}{mn} \right) \cdot \nu \cdot \nu^{(t)} \cdot \left( (1 + c) \left\| x^{(t+1)} - x \right\|_2 + \delta \right)
\leq \left( CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + 3 \zeta \nu \left\| x^{(t)} - x \right\|_2 \right) \cdot \left( \left\| x^{(t+1)} - x \right\|_2 + \frac{K_y}{mn} \right).
\tag{154}
\]
Combining (152) and (154), we obtain
\[
\left\| x^{(t+1)} - x \right\|_2^2 \leq C \frac{m}{m} + \left( \left\| x^{(t+1)} - x \right\|_2 + \frac{K_y}{mn} \right)
\times \left( CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + 5 \zeta \nu \left\| x^{(t)} - x \right\|_2 \right)
+ (2 \cdot \left\| x^{(t+1)} - x \right\|_2 + 2 \cdot \left\| x^{(t)} - x \right\|_2 + \frac{K_y}{mn} \).
\tag{155}
\]
Therefore, from the quadratic formula, we obtain
\[
\left\| x^{(t+1)} - x \right\|_2 \leq CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}}
+ \left( 5 \zeta \nu \left\| x^{(t)} - x \right\|_2 + (2 \cdot \left\| x^{(t+1)} - x \right\|_2 + 2 \cdot \left\| x^{(t)} - x \right\|_2 + \frac{K_y}{mn} \right) \right) \cdot \left\| x^{(t)} - x \right\|_2.
\tag{156}
\]
Then, since \( \frac{1}{\sqrt{\log m}} = o(1) \), if (18) holds for \( t = 0 \), i.e., \( 5 \zeta \nu \cdot \left\| x^{(0)} - x \right\|_2 + 2 \cdot \left\| x^{(0)} - x \right\|_2 + \beta_1 = 1 - \beta_2 \), we obtain from (156) that
\[
\left\| x^{(1)} - x \right\|_2 \leq CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + (1 - \beta_2) \cdot \left\| x^{(0)} - x \right\|_2.
\tag{157}
\]
When
\[
\left\| x^{(0)} - x \right\|_2 > CK_y \frac{1}{\beta_2} \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}},
\tag{158}
\]
from (157), we obtain
\[
\left\| x^{(1)} - x \right\|_2 < \left\| x^{(0)} - x \right\|_2.
\tag{159}
\]
This in turn leads to \( 5 \zeta \nu \cdot \left\| x^{(1)} - x \right\|_2 + 2 \cdot \left\| x^{(1)} - x \right\|_2 + \beta_1 < 1 - \beta_2 \), and thus from (156), we obtain
\[
\left\| x^{(2)} - x \right\|_2 \leq CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + (1 - \beta_2) \cdot \left\| x^{(1)} - x \right\|_2.
\tag{160}
\]
Therefore, if letting \( T_0 \in \mathbb{N} \) be the smallest integer such that the inequality
\[
\left\| x^{(i)} - x \right\|_2 > CK_y \frac{1}{\beta_2} \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}},
\tag{161}
\]

is violated, by induction, we obtain that the sequence $\{\|x^{(t)} - x\|_2\}_{t \in [0, T_0]}$ is monotonically decreasing, and for all $t \leq T_0$, we have

$$\|x^{(t+1)} - x\|_2 \leq CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + (1 - \beta_2) \cdot \|x^{(t)} - x\|_2,$$

which gives

$$\|x^{(t+1)} - x\|_2 \leq (1 + (1 - \beta_2) + \ldots + (1 - \beta_2)^t) \cdot CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} \quad (163)$$

$$+ (1 - \beta_2)^{t+1} \|x^{(0)} - x\|_2 \quad (164)$$

$$< \frac{CK_y}{\beta_2} \cdot \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + (1 - \beta_2)^{t+1} \cdot \|x^{(0)} - x\|_2. \quad (165)$$

In addition, since $T_0 \in \mathbb{N}$ is the smallest integer such that $(161)$ is violated, we have

$$\|x^{(T_0)} - x\|_2 \leq CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} \quad (166)$$

Then, by the linear convergence rate in $(165)$ and $\|x^{(0)} - x\|_2 \leq 2$, we obtain $T_0 = O\left(\log \left(\frac{m}{(k \log(nLr)) \cdot (\log m)}\right)\right)$. Moreover,

$$\|x^{(T_0 + 1)} - x\|_2 \leq CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + (1 - \beta_2) \cdot \|x^{(T_0)} - x\|_2 \quad (167)$$

$$\leq CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} + \frac{1 - \beta_2}{\beta_2} \cdot CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}} \quad (168)$$

$$= CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}}. \quad (169)$$

Then, by induction, we obtain for all $t \geq T_0$ that

$$\|x^{(t)} - x\|_2 \leq CK_y \sqrt{\frac{(k \log(nLr)) \cdot (\log m)}{m}}, \quad (170)$$

which completes the proof.

## D Additional Numerical Results for the MNIST Dataset

In this section, we present some additional numerical results for the MNIST dataset. We first present a figure to illustrate how quickly does Step 2 of Algorithm 1 converge in Appendix D.1, then we present how the reconstruction error of Algorithm 1 varies with respect to $1/\sqrt{m}$ in Appendix D.2. In Appendix D.3, we empirically illustrate the effect of the scale factor $\beta^{(t)}$ used in (15). In Appendix D.4, we compare with the method proposed in [80]. A simple numerical comparison with the approximate message passing (AMP) algorithm proposed in [3] is provided in Appendix D.5.

### D.1 The Convergence Rate of Step 2 of Algorithm 1

For the noisy magnitude-only measurement model (22), we fix $m = 400$ and $\sigma = 0.1$ to see how the reconstruction error decays with respect to the number of iterations of Step 2 of Algorithm 1. The results are illustrated in Figure 5. From Figure 5, we observe that the logarithm of the reconstruction error decays almost linearly during the first 20 iterations.

### D.2 The Reconstruction Error against $1/\sqrt{m}$

According to Theorem 2, for a fixed generative model, the final reconstruction error of our Algorithm 1 should scale as $O\left(1/\sqrt{m}\right)$ (ignoring the $\log m$ term). This is numerically verified in Figure 6 for which we consider the noisy magnitude-only measurement model (22) with $\sigma = 0$ or $\sigma = 0.1$.  

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Figure 5: A plot of the log reconstruction error against the number of iterations.

Figure 6: Plots of reconstruction error of MPRG against $1/\sqrt{m}$ for the measurement model (22).

D.3 The Effect of the Scale Factor $\hat{\nu}(t)$ used in (15)

In Algorithm 1, the scale factor $\hat{\nu}(t)$ is calculated according to (13) and is used in (15). We remark that this scale factor plays an important role. To illustrate this, we compare Algorithm 1 (recall that it is denoted by MPRG) with the case that using a fixed $\hat{\nu}(0)$ in (15) (i.e., it is not varying with respect to $t$) during the iterations of the second step of Algorithm 1, which is denoted by MPRGf. For the measurement models (24) and (25), the numerical results are presented in Figures 7 and 8. From these figures, we observe that by using a varying scale factor $\hat{\nu}(t)$, we obtain better reconstructed images and smaller reconstruction errors.

D.4 Comparison with the Method proposed in [80]

In this subsection, we compare with the method proposed in [80], which we denote by prGAN. We focus on the noiseless measurement model $y_i = |a_i^T x|$ for $i \in [m]$, and the numerical results are presented in Figures 9 and 10. We observe from these figures that the three methods prGAN, APPGD,
and MPRG lead to competitive reconstruction error. However, prGAN is not as stable as APPGD and MPRG, and it may result in reconstructed images that are not desired.

**D.5 Comparison with the AMP Algorithm proposed in [3]**

We follow the setting in [3] to use a ReLU neural network generative model with no offsets and zero-mean random Gaussian weights. The architecture of this neural network generative model $G$ is the same as the decoder of the VAE model used for the MNIST dataset, i.e., the latent dimension $k = 20$, there are two hidden layers with 500 neurons, and the output dimension is 784. We randomly
choose 10 latent vectors $\mathbf{z} \in \mathbb{R}^k$ and use the corresponding $G(\mathbf{z})$ as the signal. Under this ReLU neural network generative model and the noiseless measurement model $y_i = |a_i^T x|$, we compare our Algorithm 1 with the AMP algorithm used in [3]. The reconstruction error is averaged over the 10 signals and 10 random restarts. The results are presented in Figures 11 and 12. We can observe from these figures that MPRG leads to better reconstruction performance compared to AMP.

**E Empirical Results for the CelebA Dataset**

The CelebA dataset contains more than 200,000 face images of celebrities. Each input image is cropped into a 64 x 64 RGB image and thus $n = 64 \times 64 \times 3 = 12288$. The generative model $G$ is set to be (the normalized version of) the Deep Convolutional Generative Adversarial Networks (DCGAN) model pre-trained by the authors of [6], with the latent dimension being $k = 100$. The projection operator $\mathcal{P}_G(\cdot)$ is approximated by the Adam optimizer with a learning rate of 0.1 and 100 steps. To reduce the impact of local minima, we present the best reconstructed images among 6 random restarts. Other involved parameters are set to be the same as those for the MNIST dataset.

For the CelebA dataset, we do not compare with the sparsity-based method MPRS since the face images are clearly not sparse in the natural domain and we have observed from the results for the MNIST dataset that MPRS leads to poor reconstructions. Recall that we have performed numerical experiments for several measurement models (cf. (22), (23), (24) and (25)) for the MNIST dataset. In this section, we only present some proof-of-concept experimental results for the following measurement model:

$$y_i = |a_i^T x + \eta_i| + 5 \tanh(|a_i^T x|), \quad i = 1, 2, \ldots m,$$

(171)
where $\eta_i$ are i.i.d. realizations of an $\mathcal{N}(0, \sigma^2)$ random variable. The model in (171) can be thought of as a misspecified version of the measurement model $y_i = |a_i^T x + \eta_i|$. The reconstructed images are presented in Figure 13, from which we observe that our method MPRG leads to the best reconstructed images compared to those of PPower and APPGD. Quantitative comparisons according to the reconstruction error are provided in Figure 14. From this figure, we observe that when $m > 4000$, MPRG gives smallest reconstruction error. We expect the advantage of our method to be more significant as the level of misspecification increases, i.e., the multiplier of the tanh term in (171) becomes larger.
Figure 14: Quantitative comparisons of the performance for the CelebA dataset and measurement model \cite{171} with $\sigma = 0.4$ and varying $m$. 