A GROMOV HYPERBOLIC METRIC VS THE HYPERBOLIC AND OTHER RELATED METRICS

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ABSTRACT. We mainly consider two metrics: a Gromov hyperbolic metric and a scale invariant Cassinian metric. We compare these two metrics and obtain their relationship with certain well-known hyperbolic-type metrics, leading to several inclusion relations between the associated metric balls.

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1. Introduction

Comparison of hyperbolic-type metrics, defined over proper subdomains of $\mathbb{R}^n$, is a significant part of geometric function theory as it reveals the geometric property of the domain. For example, uniform domains are defined by comparing the quasihyperbolic metric [8] and the distance ratio metric [7]. In recent years, many authors have contributed to the study of hyperbolic-type metrics. Some of the familiar hyperbolic-type metrics are the Apollonian metric [3, 4, 12, 15], the half-Apollonian metric [14], the Seittenranta metric [23], the Cassinian metric [10, 16, 21, 22], the triangular ratio metric [6], etc. These metrics are also referred as the relative metrics since they are defined in proper subdomains of the Euclidean space $\mathbb{R}^n$, $n \geq 2$, relative to domain boundaries. A general form of some of these relative metrics has been considered by P. Hästö in [11, Lemma 6.1], in a different context.

Very recently, a scale invariant version of the Cassinian metric has been studied by Ibragimov in [19] which is defined by

$$\tilde{\tau}_D(x, y) = \log \left(1 + \sup_{p \in \partial D} \frac{|x - y|}{\sqrt{|x - p||p - y|}}\right), \quad x, y \in D \subset \mathbb{R}^n.$$  

The interesting part of this metric is that many properties in arbitrary domains are revealed in the setting of once-punctured spaces. For example, $\tilde{\tau}_D$ is a metric in an arbitrary domain $D \subset \mathbb{R}^n$ if it is a metric on once-punctured spaces. The $\tilde{\tau}_D$-metric is comparable with the Vuorinen’s distance ratio metric [25] in arbitrary domains $D \subset \mathbb{R}^n$ if they are comparable in the punctured spaces (see [19]). It is appropriate here to recall that the $\tilde{\tau}_D$-metric satisfies the domain monotonicity property, i.e. if $D_1, D_2 \subset \mathbb{R}^n$ with $D_1 \subset D_2$, then $\tilde{\tau}_{D_2}(x, y) \leq \tilde{\tau}_{D_1}(x, y)$ for all $x, y \in D_1$ [19, p. 2].

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Gromov in 1987 introduced the notion of an abstract hyperbolic space [9]. Let \((D, d)\) be a metric space and \(x, y, z \in D\). The Gromov product of \(x\) and \(y\) with respect to \(z\) is defined by the formula
\[
(x|y)_z = \frac{1}{2} [d(x, z) + d(y, z) - d(x, y)].
\]
The metric space \((D, d)\) is said to be Gromov hyperbolic if there exists \(\beta \geq 0\) such that
\[
(x|y)_w \geq (x|z)_w \wedge (z|y)_w - \beta
\]
for all \(x, y, z, w \in D\). We also say that \(D\) is \(\beta\)-hyperbolic. Equivalently, the metric space \((D, d)\) is called Gromov hyperbolic if and only if there exist a constant \(\beta > 0\) such that
\[
d(x, z) + d(y, w) \leq (d(x, w) + d(y, z)) \vee (d(x, y) + d(z, w)) + 2\beta
\]
for all points \(x, y, z, w \in D\). Note that Gromov hyperbolicity is preserved under quasi-isometries. That means it is preserved under the mappings \(f : (D, d_1) \rightarrow (f(D), d_2)\) satisfying
\[
\frac{1}{\lambda} \, d_1(x, y) - k \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + k, \quad x, y \in D,
\]
where \(\lambda \geq 1\) and \(k \geq 0\). Literature on Gromov hyperbolicity are available in [5, 9, 13, 17, 18, 24]. One natural question was to investigate whether a metric space is hyperbolic in the sense of Gromov or not? Ibragimov in [17] introduced a metric, \(u_Z\), which hyperbolizes (in the sense of Gromov) the locally compact non-complete metric space \((Z, d)\) without changing its quasiconformal geometry, by
\[
u_Z(x, y) = 2 \log \frac{d(x, y) + \max\{\operatorname{dist}(x, \partial Z), \operatorname{dist}(y, \partial Z)\}}{\sqrt{\operatorname{dist}(x, \partial Z) \operatorname{dist}(y, \partial Z)}}, \quad x, y \in Z.
\]
For a domain \(D \subseteq \mathbb{R}^n\) equipped with the Euclidean metric, the \(u_D\)-metric is defined by
\[
u_D(x, y) = 2 \log \frac{|x - y| + \max\{\operatorname{dist}(x, \partial D), \operatorname{dist}(y, \partial D)\}}{\sqrt{\operatorname{dist}(x, \partial D) \operatorname{dist}(y, \partial D)}}, \quad x, y \in D.
\]
Note that the \(u_D\)-metric does not satisfy the domain monotonicity property and it coincides with the Vuorinen’s distance ratio metric in punctured spaces \(\mathbb{R}^n \setminus \{p\}\), for \(p \in \mathbb{R}^n\). Though comparisons of the \(u_D\)-metric with some hyperbolic-type metrics are studied in [17], in this paper, we further compare the \(u_D\)-metric with the hyperbolic metric and other related metrics which were not considered in [17].

It is well known that the geometric structure of a metric space can be viewed from the geometric structure of the fundamental element, namely, the metric balls. Hence it is reasonable to study metric balls and their inclusion relations with other metric balls by fixing the centre common to each pair of metric balls. In this regard we study inclusion relations associated with the metric balls.

The paper is organized as follows: Section 2 is devoted to the preliminaries for the upcoming sections. In Section 3, we compare the \(u_D\)-metric with the hyperbolic metric of
the unit ball and upper half space. In Section 4 we compare the $u_D$-metric with the $\tilde{\tau}_D$-metric. Comparisons of the $u_D$-metric and the $\tilde{\tau}_D$-metric with other metrics are discussed in Section 5.

2. Preliminaries

Throughout the paper, $D$ denotes an arbitrary, proper subdomain of the Euclidean space $\mathbb{R}^n$. Symbolically, we write $D \subset \mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $r > 0$, the open ball centered at $x$ and of radius $r$ is denoted by $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Set $\mathbb{B} = B(0, 1)$. We denote by $D_p$ and $D_{p,q}$, the punctured spaces $\mathbb{R}^n \setminus \{p\}$ and $\mathbb{R}^n \setminus \{p,q\}$ respectively. For a given $x \in D$, we set $d(x) := \text{dist}(x, \partial D)$. For real numbers $r$ and $s$, we set $r \lor s = \max\{r, s\}$ and $r \land s = \min\{r, s\}$.

The distance ratio metric, $\tilde{j}_D$, is defined by

$$
\tilde{j}_D(x, y) = \log \left(1 + \frac{|x - y|}{d(x) \land d(y)}\right), \quad x, y \in D.
$$

The above form of the metric $\tilde{j}_D$, which was first considered in [25], is a slight modification of the original distance ratio metric, $j_D$, of Gehring and Osgood [7], defined by

$$
\frac{1}{2} \log \left(1 + \frac{|x - y|}{d(x)}\right) \left(1 + \frac{|x - y|}{d(y)}\right), \quad x, y \in D.
$$

The $\tilde{j}_D$-metric has been widely studied in the literature; see, for instance, [26]. These two distance ratio metrics are related by: $\tilde{j}_D(x, y)/2 \leq j_D(x, y) \leq \tilde{j}_D(x, y)$; see [13, 23].

The hyperbolic metric, $\rho_\mathbb{B}$, of the unit ball $\mathbb{B} = B(0, 1)$ is given by

$$
\rho_\mathbb{B}(x, y) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1 - |z|^2},
$$

where the infimum is taken over all rectifiable curves $\gamma \subset \mathbb{B}$ joining $x$ and $y$.

Now, we define the $d$-metric ball (metric ball with respect to the metric $d$) as follows: let $(D, d)$ be a metric space. Then the set

$$
B_d(x, R) = \{z \in D : d(x, z) < R\}
$$

is called the $d$-metric ball of the domain $D$.

3. Comparison of the $u_D$-metric with the hyperbolic metric

This section is devoted to the comparison of the $u_D$-metric and the $\rho_D$-metric when $D = \mathbb{B}^n$ and $D = \mathbb{H}^n := \{(x_1, x_2, \ldots, x_n) : x_n > 0\}$. The hyperbolic metric of the unit ball $\mathbb{B}^n$, $\rho_\mathbb{B}$, can be computed by the following formula (see [21, p. 40]).

$$
\sinh \left(\frac{\rho_\mathbb{B}(x, y)}{2}\right) = \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}}.
$$
We first establish the relationship between the $u_{B^n}$-metric and the $\rho_{B^n}$-metric. In this setting, the following lemma is useful which yields a relationship between the $u_D$-metric and the $j_D$-metric in arbitrary subdomains of $\mathbb{R}^n$.

**Lemma 3.1.** Let $D \subset \mathbb{R}^n$ be arbitrary. Then for $x, y \in D$ we have

$$2j_D(x, y) \leq u_D(x, y) \leq 4j_D(x, y).$$

The first inequality becomes equality when $d(x) = d(y)$.

**Proof.** The first inequality is proved in [17, Theorem 3.1]. From the definitions of the $j_D$-metric and the $u_D$-metric, it follows that $2j_D(x, y) = u_D(x, y)$ whenever $d(x) = d(y)$.

Now, we shall prove the second inequality. Without loss of generality we assume that $d(x) \geq d(y)$ for $x, y \in D \subset \mathbb{R}^n$. To show our claim, it is enough to prove that

$$\frac{|x - y| + d(x)}{\sqrt{d(x)d(y)}} \leq \left(1 + \frac{|x - y|}{d(x)}\right) \left(1 + \frac{|x - y|}{d(y)}\right),$$

or, equivalently,

$$d(x)d(y) \leq (d(y) + |x - y|)^2$$

which is true by the triangle inequality. The proof is complete. □

**Theorem 3.2.** For all $x, y \in B^n$ we have

$$\frac{1}{2}\rho_{B^n}(x, y) \leq u_{B^n}(x, y) \leq 4\rho_{B^n}(x, y).$$

**Proof.** From [1, p. 151] and from [26, p. 29] we have respectively the following two inequalities:

$$(3.2) \quad \tilde{j}_{B^n}(x, y) \leq \rho_{B^n}(x, y) \leq 2\tilde{j}_{B^n}(x, y)$$

and

$$\frac{1}{2}j_D(x, y) \leq j_D(x, y) \leq \tilde{j}_D(x, y).$$

Now the proof of our theorem follows from Lemma 3.1. □

Observe that for the choice of points $x, y \in B^n$ with $y = -x$,

$$u_{B^n}(x, -x) = 2\log \left(\frac{1 + |x|}{1 - |x|}\right) = 2\rho_{B^n}(0, x) = \rho_{B^n}(x, -x).$$

This observation leads to the following conjecture.

**Conjecture 3.3.** For $x, y \in B^n$ we have $\rho_{B^n}(x, y) \leq u_{B^n}(x, y) \leq 2\rho_{B^n}(x, y)$.

As a consequence of Theorem 3.2 we have the following inclusion relation.

**Corollary 3.4.** Let $x \in B^n$ and $t > 0$. Then

$$B_{\rho_{B^n}}(x, r) \subseteq B_{u_{B^n}}(x, t) \subseteq B_{\rho_{B^n}}(x, R),$$

where $r = t/4$ and $R = 2t$. 


Proof. The proof follows directly from Theorem 3.2. □

Next theorem shows that the $u_{B^n}$-metric and the $\rho_{B^n}$-metric are satisfying the quasi-isometry property.

**Theorem 3.5.** For all $x, y \in B^n$, we have

$$\rho_{B^n}(x, y) - 2 \log 2 \leq u_{B^n}(x, y) \leq 2 \rho_{B^n}(x, y) + 2 \log 2.$$  

Proof. The right hand side inequality easily follows from (3.2) and [17, Theorem 3.1]. For the left hand side inequality we assume that $x, y \in B^n$ with $|x| \leq |y|$. It is clear that $(1 - |x|)(1 - |y|) \leq (1 - |x|^2)(1 - |y|^2)$. Now with the help of the formula (3.1) we have

$$u_{B^n}(x, y) = 2 \log \left( \frac{|x - y| + 1 - |x|}{\sqrt{(1 - |x|)(1 - |y|)}} \right) \geq 2 \log \left( 1 + \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}} \right) \geq \rho_{B^n}(x, y) - 2 \log 2,$$

where the first inequality follows from the fact that $(1 - |x|)/(1 - |y|) \geq 1$ and the second inequality follows from the fact that $1 + \sinh(r) \geq e^r/2, r \geq 0$. The proof is complete. □

Now, we compare the $u_{\mathbb{H}^n}$-metric with the $\rho_{\mathbb{H}^n}$-metric. Note that for $x, y \in \mathbb{H}^n$, the $\rho_{\mathbb{H}^n}$-metric can be computed by the formula (see [2, p. 35])

$$2 \sinh \left( \frac{\rho_{\mathbb{H}^n}(x, y)}{2} \right) = \frac{|x - y|}{\sqrt{x_n y_n}}.$$  

(3.3)

**Theorem 3.6.** For $x, y \in \mathbb{H}^n$ we have

$$\rho_{\mathbb{H}^n}(x, y) \leq u_{\mathbb{H}^n}(x, y).$$

The inequality is sharp.

Proof. Suppose that $x, y \in \mathbb{H}^n$. Without loss of generality we assume that $x_n \geq y_n$. Now,

$$u_{\mathbb{H}^n} = 2 \log \left( \frac{|x - y| + x_n}{\sqrt{x_n y_n}} \right) \geq 2 \log \left( 2 \sinh \left( \frac{\rho_{\mathbb{H}^n}(x, y)}{2} \right) + 1 \right) \geq \rho_{\mathbb{H}^n}(x, y),$$

where the first inequality follows from (3.3) and the hypothesis. However, the second inequality follows from the fact that $1 + 2 \sinh(r) = 1 + e^r - e^{-r} \geq 2e^r$ for $r \geq 0$. For sharpness, consider the points $x = te_2$ and $y = (1/t)e_2$ with $t > 1$. Then

$$\rho_{\mathbb{H}^n}(te_2, (1/t)e_2) = 2 \sinh^{-1} \left( \frac{t^2 - 1}{2t} \right) = 2 \log t \quad \text{and} \quad u_{\mathbb{H}^n}(te_2, (1/t)e_2) = 2 \log \left( \frac{2t^2 - 1}{t} \right).$$
Now taking the limits as \( t \to \infty \) we get
\[
\lim_{t \to \infty} \rho_{H^n}(te_2, (1/t)e_2) = \lim_{t \to \infty} \frac{\log t}{\log \left( \frac{2t^2 - 1}{t} \right)} = 1.
\]

Hence completing the proof. \( \square \)

4. **Comparison of the \( u_D \)-metric and the \( \tilde{\tau}_D \)-metric**

We begin this section with the proof of the comparisons stated in Table 1:

| \( j_D \) | Comparison with \( \tilde{\tau}_D \) | Comparison with \( u_D \) |
|-----------|-------------------------------|-----------------|
| \( \frac{1}{2}j_D \leq \tilde{\tau}_D \leq j_D \) | \( 2j_D \leq u_D \leq 4j_D \) |
| \( \text{[19] Theorems 5.1,5.4] (Right hand side inequality is sharp) \) | \( \text{[Lemma 3.1]} \) |

| \( \tilde{j}_D \) | \( \frac{1}{2}\tilde{j}_D \leq \tilde{\tau}_D \leq \tilde{j}_D \) | \( \tilde{j}_D \leq u_D \leq 2\tilde{j}_D \) |
|----------------|-----------------------------|-----------------------------|
| \( \text{[19] Lemma 4.1, Theorem 4.3]} \) (Both the inequalities are sharp) | \( \text{[Theorem 4.8]} \) (Left hand side inequality is sharp) |

| \( \tilde{\tau}_D \) | - | \( 2\tilde{\tau}_D \leq u_D \leq 4\tilde{\tau}_D \) |
|----------------|---------|-----------------------------|
| \| \( \text{[Theorem 4.5]} \) (Both the inequalities are sharp) |

**Table 1.** Comparison of \( \tilde{\tau}_D \)-metric and \( u_D \)-metric with other hyperbolic-type metrics

First, we compare the \( u_D \)-metric with the \( \tilde{\tau}_D \)-metric in arbitrary domains \( D \subset \mathbb{R}^n \). Ibragimov in [19], proved that the \( \tilde{\tau}_D \)-metric is Gromov-hyperbolic (\( \eta \)-hyperbolic) with the constant \( \eta = \log 3 \) by comparing this with the \( j_D \)-metric. Comparison of the \( u_D \)-metric and the \( \tilde{\tau}_D \)-metric leads to an improvement in the constant of Gromov hyperbolicity of the \( \tilde{\tau}_D \)-metric from \( \log 3 \) to \( \log 2 \). In light of [17] Theorem 3.1 and [19] Theorem 3.7, it can easily be seen that

\[
u_D(x,y) \leq 4 \tilde{\tau}_D(x,y) + 2 \log 2.
\]

holds for \( x,y \in D \subset \mathbb{R}^n \); however, this has been improved in Theorem 4.5.

Next theorem proves the comparison of both the \( \tilde{\tau}_D \)-metric and \( u_D \)-metric in the other way, i.e., the \( \tilde{\tau}_D \)-metric as a lower bound to the \( u_D \)-metric.
Theorem 4.1. Let $D \subseteq \mathbb{R}^n$ be any domain with $\partial D \neq \emptyset$ and $x, y \in D$. Then
\[2\tilde{\tau}_D(x, y) \leq u_D(x, y)\]
Equality holds whenever $d(x) = |x - p| = |y - p| = d(y)$ for some $p \in \partial D$. Moreover, there exists no constant $k \geq 0$ such that
\[u_D(x, y) \leq 2\tilde{\tau}_D(x, y) + k\]
for all $x, y \in D$ unless $D$ is a once-punctured space. If $D$ is a once-punctured space, then
\[u_D(x, y) \leq 2\tilde{\tau}_D(x, y) + 2 \log 2\]
and the inequality is sharp.

Proof. Without loss of generality we assume that $d(x) \geq d(y)$. For $x, y \in D$, the relation $|x - p||y - p| \geq d(x)d(y)$ clearly holds for all $p \in \partial D$. Then we have
\[u_D(x, y) = 2 \log \frac{|x - y| + d(x)}{\sqrt{d(x)d(y)}} \geq 2 \log \left(1 + \frac{|x - y|}{\sqrt{d(x)d(y)}}\right)\]
\[\geq 2 \log \left(1 + \sup_{p \in \partial D} \frac{|x - y|}{\sqrt{|x - p||p - y|}}\right) = 2\tilde{\tau}_D(x, y),\]
where the first inequality follows from the fact that $d(x)/d(y) \geq 1$. It is clear that if $d(x) = |x - p| = |y - p| = d(y)$ for some $p \in \partial D$, then both the above inequalities turn into an equality and hence the sharpness part is proved.

To prove the second part, suppose that $D$ has more than one boundary point and $k \geq 0$ such that $u_D(x, y) \leq 2\tilde{\tau}_D(x, y) + k$ for all $x, y \in D$. Since $u_D$ is $\delta$-hyperbolic in $D$ and $2\tilde{\tau}_D(x, y) \leq u_D(x, y) \leq 2\tilde{\tau}_D(x, y) + k$, we conclude that $\tilde{\tau}_D$ is $\delta$-hyperbolic in $D$, contradicting \[19\] Remark 4.4.

The translation invariance of the $\tilde{\tau}_D$-metric and the $u_D$-metric allows us to take the punctured space to be $D_0$ without any loss to generality. Again we assume that $|x| \geq |y|$. To show the third part, it is sufficient to show
\[\frac{|x - y| + |x|}{\sqrt{|x||y|}} \leq \frac{2}{1 + \frac{|x - y|}{\sqrt{|x||y|}}},\]
or, equivalently,
\[\frac{|x| - |x - y|}{\sqrt{|x||y|}} \leq 2.\]
The hypothesis $|x| \geq |y|$ along with the triangle inequality yields
\[\frac{|x| - |x - y|}{\sqrt{|x||y|}} \leq \frac{|y|}{\sqrt{|x||y|}} \leq 2.\]
To prove the sharpness, let \( y = e_1 \) and \( x = te_1, t > 1 \). Then
\[
\lim_{t \to \infty} u_D(x, y) - 2\tau_D(x, y) = \lim_{t \to \infty} 2 \log \frac{2t - 1}{t + \sqrt{t} - 1} = 2 \log 2.
\]
Hence the proof is complete.

The following inclusion relation holds true between the \( u_D \)-metric ball and the \( \tau_D \)-metric ball.

**Corollary 4.2.** Let \( D \subseteq \mathbb{R}^n \) be any arbitrary domain and \( x \in D \). Then
\[
B_{u_D}(x, r) \subseteq B_{\tau_D}(x, t),
\]
where \( r = 2t \) and the inclusion is sharp.

**Proof.** Suppose that \( y \in B_{u_D}(x, 2t) \). Then \( u_D(x, y) < 2t \). Now it follows from Theorem 4.1 that \( \tau_D(x, y) < t \) and hence the proof is complete. For the sharpness, consider the domain \( D = \mathbb{R}^n \setminus \{0\} \) and \( x \in D \). Choose the point \( y = \partial B_{u_D}(x, 2t) \cap \partial B(0, |x|) \). Now, \( u_D(x, y) = 2t \) implies \( \tau_D(x, y) = t \). This proves our corollary.

Next, we aim to prove the other way of comparison. That is, to find a constant \( k \) such that \( u_D \leq k \tau_D \). First, we prove this result in once-punctured spaces and then we extend this to arbitrary proper subdomains of \( \mathbb{R}^n \). Next result shows that in punctured spaces the constant \( k = 4 \).

**Lemma 4.3.** Let \( x, y \in D_0 \). Then
\[
u_{D_0}(x, y) \leq 4\tau_{D_0}(x, y).
\]

**Proof.** Without loss of generality, we assume that \( |x| \geq |y| \). To prove the required inequality, it suffices to show that
\[
\frac{|x - y| + |x|}{\sqrt{|x||y|}} \leq \left(1 + \frac{|x - y|}{\sqrt{|x||y|}}\right)^2
\]
or, equivalently,
\[
\frac{|x| - |x - y|}{\sqrt{|x||y|}} \leq 1 + \frac{|x - y|^2}{|x||y|}.
\]
This holds true, because
\[
\frac{|x| - |x - y|}{\sqrt{|x||y|}} \leq \frac{|y|}{\sqrt{|x||y|}} \leq 1 \leq 1 + \frac{|x - y|^2}{|x||y|},
\]
completing the proof of our lemma.

We now prove that the conclusion of Lemma 4.3 still holds if we replace the once-punctured space by twice-punctured spaces.
Lemma 4.4. Let \( x, y \in D_{p,q} \). Then

\[
u_{D_{p,q}}(x, y) \leq 4 \tau_{D_{p,q}}(x, y).
\]

Proof. Suppose that \( x, y \in D_{p,q} \). If \( d(x) = |x - p| \) and \( d(y) = |y - p| \) (or \( d(x) = |x - q| \) and \( d(y) = |y - q| \)), then the proof follows from Theorem 4.3. Hence, without loss of generality, we assume \( d(x) = |x - p| \), \( d(y) = |y - q| \), and \( |x - p| \geq |y - q| \). Note that \( \tau_{D_{p,q}}(x, y) = \tau_{D_p}(x, y) \lor \tau_{D_q}(x, y) \). Hence, to prove our claim, it is enough to establish the inequality \( \nu_{D_{p,q}}(x, y) \leq 4 \tau_{D_q}(x, y) \). That is, to prove the inequality

\[
\frac{|x - y| + |x - p|}{\sqrt{|x - p||y - q|}} \leq \left( 1 + \frac{|x - y|}{\sqrt{|x - q||y - q|}} \right)^2.
\]

Let \( |x - y| = a|y - q| \), where \( a > 0 \). From the assumption we know that

\[
(4.1) \quad |x - p| \leq |x - q| \leq |x - y| + |y - q|.
\]

Now

\[
(4.2) \quad \frac{|x - y| + |x - p|}{\sqrt{|x - p||y - q|}} = \frac{|x - y|}{\sqrt{|x - p||y - q|}} + \frac{|x - p|}{\sqrt{|x - p||y - q|}} \\
\leq \frac{|x - y|}{|y - q|} + \sqrt{\frac{|x - p|}{|y - q|}} \\
\leq \frac{|x - y|}{|y - q|} + \sqrt{\frac{|x - y|}{|y - q|} + 1} \\
\leq a + \sqrt{a + 1}.
\]

We obtain from (4.1) that

\[
(4.3) \quad \frac{2|x - y|}{\sqrt{|x - q||y - q|}} \geq \frac{2|x - y|}{\sqrt{1 + a|y - q|}} = \frac{2a}{\sqrt{1 + a}}
\]

Next we divide the proof into two cases.

Case 1. \( a < 1 \).

It follows from (4.2) and (4.3) that

\[
\left( 1 + \frac{|x - y|}{\sqrt{|x - q||y - q|}} \right)^2 - \frac{|x - y| + |x - p|}{\sqrt{|x - p||y - q|}} > \frac{2|x - y|}{\sqrt{|x - q||y - q|}} + 1 - \frac{|x - y| + |x - p|}{\sqrt{|x - p||y - q|}} \\
\geq \frac{2a}{\sqrt{1 + a}} + 1 - a - \sqrt{a + 1} > 0
\]

Case 2. \( a \geq 1 \).
From (4.1) we have
\[
|\begin{array}{c}
|y - q| \\
|y - q|
\end{array} \geq \frac{|x - y|^2}{(|x - y| + |y - q|)|y - q|} = a^2 + \frac{a}{1 + a}.
\]
Then it follows from (4.2), (4.3) and (4.4) that
\[
\left(1 + \frac{|x - y|}{|x - q||y - q|}\right)^2 - \frac{|x - y| + |x - p|}{\sqrt{|x - p||y - q|}} \geq \frac{2a}{\sqrt{1 + a}} + \frac{a^2}{1 + a} + 1 - a - \sqrt{a + 1}
\]
\[
\geq 0.
\]
This completes the proof of our lemma. □

Combining Lemma 4.1 and Lemma 4.4 we obtain Theorem 4.5. Let \(x, y \in D \subseteq \mathbb{R}^n\). Then
\[
2\bar{\tau}_D(x, y) \leq u_D(x, y) \leq 4\bar{\tau}_D(x, y).
\]
Both the inequalities are sharp.

**Proof.** The first inequality is proved in Theorem 4.1. Now, we prove the second inequality. Suppose that \(p, q \in \partial D\) such that \(d(x) = |x - p|\) and \(d(y) = |y - q|\). Clearly, \(D \subset D_{p,q}\) and \(u_D(x, y) = u_{D_{p,q}}(x, y)\). Now,
\[
u_D(x, y) = u_{D_{p,q}}(x, y) \leq 4\bar{\tau}_{D_{p,q}}(x, y) \leq 4\bar{\tau}_D(x, y),
\]
where the first inequality follows from Lemma 4.4 and the second inequality follows from the monotone property of \(\bar{\tau}_D\) [19, p. 2].

The sharpness of the first inequality is given in Theorem 4.1. For the sharpness of the second inequality we consider the unit ball \(B^n\). Choose the points \(x\) and \(y\) such that \(y = -x\). Now,
\[
u_{B^n}(x, -x) = 2\log\left(\frac{1 + |x|}{1 - |x|}\right)\text{ and } \bar{\tau}_{B^n}(x, -x) = \log\left(1 + \frac{2|x|}{\sqrt{1 - |x|^2}}\right).
\]
It follows that
\[
\lim_{|x| \to 1} \frac{\nu_{B^n}(x, -x)}{4\bar{\tau}_{B^n}(x, -x)} = \lim_{|x| \to 1} \frac{(2|x| + \sqrt{1 - |x|^2})^2}{(1 + |x|)^2} = 1.
\]
Hence the proof is complete. □

An immediate consequence of Theorem 4.5 is the following inclusion relation.

**Corollary 4.6.** Let \(x \in D \subseteq \mathbb{R}^n\) and \(t > 0\). Then we have
\[
B_{u_D}(x, r) \subseteq B_{\bar{\tau}_D}(x, t) \subseteq B_{u_D}(x, R),
\]
where \(r = 2t\) and \(R = 4t\). The radii \(r\) and \(R\) are the best possible.
Proof. Let $y \in B_{u_D}(x, r)$, $r = 2t$ and $R = 4t$. Then by Theorem 4.5 we have $\tilde{\tau}_D(x, y) < t$. So, $B_{u_D}(x, r) \subseteq B_{\tilde{\tau}_D}(x, t)$. Conversely, if $y \in B_{\tilde{\tau}_D}(x, t)$, then also by Theorem 4.5 we have $y \in B_{u_D}(x, R)$. So, $B_{\tilde{\tau}_D}(x, t) \subseteq B_{u_D}(x, R)$ and hence the inclusion follows. Next, we need to prove the sharpness part.

First we consider the domain $D = D_0$ and let $x \in D_0$. Now choose $y \in B(0, |x|) \cap \partial B_{\tilde{\tau}_D}(x, t)$. Then

$$\tilde{\tau}_{D_0}(x, y) = t = \log \left(1 + \frac{|x - y|}{|x|}\right) = \frac{u_{D_0}(x, y)}{2},$$

which proves the sharpness of the first inclusion. Secondly, consider $D = \mathbb{B}^n$ and let $x \in \mathbb{B}^n$ be arbitrary. Choose $y \in \mathbb{B}^n$ such that $x$ and $y$ lie on a diameter of $\mathbb{B}^n$ with $0$ lying in-between and $|y| \leq |x|$. Now,

$$u_{\mathbb{B}^n}(x, y) = 2 \log \left(\frac{|x - y + 1 - |y||}{\sqrt{(1 - |x|)(1 - |y|)}}\right) \quad \text{and} \quad \tilde{\tau}_{\mathbb{B}^n}(x, y) = \log \left(1 + \frac{|x - y|}{\sqrt{(1 - |x|)(1 + |y|)}}\right).$$

It follows that

$$\lim_{x \to e_1} \frac{u_{\mathbb{B}^n}(x, y)}{\tilde{\tau}_{\mathbb{B}^n}(x, y)} = \lim_{x \to e_1} \frac{(|x - y + 1 - |y||)(1 - |x|)(1 + |y|)}{\sqrt{(1 - |x|)(1 - |y|)(1 - |y| + \sqrt{(1 - |x|)(1 + |y|)})^2}} = \begin{cases} 1, & \text{if } y = -x, \\ 0, & \text{otherwise}. \end{cases}$$

Hence we conclude that for each $x \in \mathbb{B}^n$ with $|x| \to 1$ and $t > 0$, there exist $y = -x$ such that $y \in \partial B_{\tilde{\tau}_D}(x, t)$ and $u_D(x, y) = 4t$. This proves the sharpness of the second inclusion relation and hence the proof is complete. \hfill \Box

Recall that

$$\frac{1}{2} j_D(x, y) \leq \tilde{\tau}_D(x, y) \leq j_D(x, y) \quad (4.5)$$

holds true for $D \subset \mathbb{R}^n$ (see [19, Theorem 4.2, 4.3]). Both the inequalities are sharp. The proof of the sharpness part of the left hand side inequality is done by the method of contradiction in [19]. Here we give a precise example to prove the sharpness part of the left hand side inequality.

Consider the unit ball $\mathbb{B}^n$ and $x, y \in \mathbb{B}^n$ with $y = -x$. Now we see that

$$\tilde{\tau}_{\mathbb{B}^n}(x, -x) = \log \left(1 + \frac{2|x|}{\sqrt{1 - |x|^2}}\right) \quad \text{and} \quad \tilde{j}_{\mathbb{B}^n}(x, -x) = \log \left(1 + \frac{2|x|}{1 - |x|}\right).$$

It follows that

$$\lim_{|x| \to 1} \frac{\tilde{j}_{\mathbb{B}^n}(x, -x)}{2\tilde{\tau}_{\mathbb{B}^n}(x, -x)} = \lim_{|x| \to 1} \frac{2|x| + \sqrt{1 - |x|^2}}{2} = 1. \quad (4.6)$$
Now, we establish inclusion relation between the \( \tilde{j}_D \)-metric and the \( \tilde{\tau}_D \)-metric balls.

**Theorem 4.7.** Let \( D \subseteq \mathbb{R}^n \) and \( x \in D \) and \( t > 0 \). Then the following inclusion property holds true:

\[
B_{\tilde{j}_D}(x, r) \subseteq B_{\tilde{\tau}_D}(x, t) \subseteq B_{\tilde{j}_D}(x, R).
\]

Here \( r = t \) and \( R = 2t \). The radii \( r \) and \( R \) are the best possible.

**Proof.** The proof follows from (4.5). To show that the radius \( r \) is the best possible, consider the domain \( D = D_0 = \mathbb{R}^n \setminus \{0\} \) and \( x \in D \). Choose \( y \in \partial B(0, |x|) \cap \partial B_{\tilde{\tau}_D}(x, t) \). Now clearly, \( \tilde{j}_D(x, y) = \tilde{\tau}_D(x, y) = t \).

To show \( R \) is the best possible, consider the domain \( D = B^n \). With the similar argument given in the proof of Corollary 4.6, for the second inclusion property, we can show that for each \( x \in B^n \) with \( |x| \to 1 \) and \( t > 0 \), there exist \( y(= -x) \) such that \( y \in \partial B_{\tilde{\tau}_D}(x, t) \) and \( \tilde{j}_{B^n}(x, y) = 2t \). This completes the proof of our theorem. \( \square \)

By Theorem 4.3 and (4.5) we have

\[
\tilde{j}_D(x, y) \leq 2\tilde{\tau}_D(x, y) \leq u_D(x, y)
\]

and also

\[
u_D(x, y) \leq 4\tilde{\tau}_D(x, y) \leq 4\tilde{j}_D(x, y).
\]

Hence we have the following relationship between the \( \tilde{j}_D \)-metric and the \( u_D \)-metric.

**Theorem 4.8.** For \( D \subseteq \mathbb{R}^n \) we have

\[
\tilde{j}_D(x, y) \leq u_D(x, y) \leq 4\tilde{j}_D(x, y).
\]

The first inequality is sharp.

**Proof.** For the sharpness part, consider the domain \( D = \mathbb{R}^n \setminus \{-e_1, e_1\} \). Choose \( x = 0 \) and \( y = te_2, t > 1 \). Then \( \tilde{j}_D(0, te_2) = \log(1 + t) \) and

\[
u_D(0, te_2) = 2 \log \left( \frac{t + \sqrt{1 + t^2}}{(1 + t^2)^{1/4}} \right) = \log \left( \frac{1 + 2t^2 + 2t\sqrt{1 + t^2}}{\sqrt{1 + t^2}} \right).
\]

Now we see that

\[
\lim_{t \to \infty} \frac{\tilde{j}_D(0, te_2)}{u_D(0, te_2)} = \lim_{t \to \infty} \frac{\log(1 + t)}{\log \left( \frac{1 + 2t^2 + 2t\sqrt{1 + t^2}}{\sqrt{1 + t^2}} \right)} = \lim_{t \to \infty} \frac{(1 + t^2)(1 + 2t^2 + 2t\sqrt{1 + t^2})}{(1 + t)(4t(1 + t^2) + 2(1 + t^2)^{3/2} - t - 2t^3)} = 1.
\]

This completes the proof of our theorem. \( \square \)
Remark 4.9. The constant 4 in the right hand side inequality of Lemma 4.8 can’t be replaced by 2 due to the fact that
\[ u_D(x, y) \leq 2\tilde{J}_D(x, y) \iff |x - y|^2 \geq d(x)d(y) \]
for every \( x, y \in D \), which is not true in general.

As an immediate consequence of Theorem 4.8 we have the following inclusion relation.

Corollary 4.10. Let \( D \subseteq \mathbb{R}^n \), \( x \in D \), and \( t > 0 \). Then
\[ B_{u_D}(x, r) \subseteq B_{\tilde{J}_D}(x, t) \subseteq B_{u_D}(x, R) \]
where \( r = t \) and \( R = 4t \). The radius \( r \) is best possible.

Proof. Proof follows from Theorem 4.8.

The next result shows that the \( \tilde{\tau}_D \)-metric balls can be written as the intersection of \( \tilde{\tau} \)-metric balls in punctured spaces over the boundary points of \( D \).

Proposition 4.11. Let \( D \subseteq \mathbb{R}^n \), \( x \in D \), and \( r > 0 \). Then
\[ B_{\tilde{\tau}_D}(x, r) = \bigcap_{p \in \partial D} B_{\tilde{\tau}_{\mathbb{R}^n \setminus \{p\}}}(x, r) \]
Proof. Suppose that \( y \in \bigcap_{p \in \partial D} B_{\tilde{\tau}_{\mathbb{R}^n \setminus \{p\}}}(x, r) \). Then \( \tilde{\tau}_{\mathbb{R}^n \setminus \{p\}}(x, y) < r \) for all \( p \in \partial D \). In particular,
\[ \tilde{\tau}_D(x, y) = \sup_{p \in \partial D} \tilde{\tau}_{\mathbb{R}^n \setminus \{p\}}(x, y) < r \]
So, \( \bigcap_{p \in \partial D} B_{\tilde{\tau}_{\mathbb{R}^n \setminus \{p\}}}(x, r) \subseteq B_{\tilde{\tau}_D}(x, r) \). Conversely, suppose that \( y \in B_{\tilde{\tau}_D}(x, r) \) and let \( p \in \partial D \). Then
\[ B_{\tilde{\tau}_D}(x, r) \subseteq B_{\tilde{\tau}_{\mathbb{R}^n \setminus \{p\}}}(x, r) \]
by the monotone property of the \( \tilde{\tau}_D \)-metric. Hence, \( B_{\tilde{\tau}_D}(x, r) \subseteq \bigcap_{p \in \partial D} B_{\tilde{\tau}_{\mathbb{R}^n \setminus \{p\}}}(x, r) \) and the proof is complete.

5. Comparison with other related metrics

In this section, we consider the Cassinian \[16\], the Seittenranta \[23\], the triangular ratio \[6\] and the half-Apollonian \[14\] metrics, and compare them with the \( \tilde{\tau}_D \)-metric and the \( u_D \)-metric. Main results of this section are stated in Table 2.
Comparison with $\tilde{\tau}_D$

$$\frac{1}{4} \delta_D(x, y) \leq \tilde{\tau}_D(x, y) \leq \delta_D(x, y)$$

[Theorem 5.3]

(Right hand side inequality is sharp)

Comparison with $u_D$

$$\frac{\delta_D}{2} \leq u_D \leq 4 j_D$$

[Corollary 5.3]

| $\delta_D$ | $c_D(x, y) \leq \frac{\exp(4\tilde{\tau}_D(x, y)) - 1}{d(y)}$ | $c_D(x, y) \leq \frac{\exp(2u_D(x, y)) - 1}{d(y)}$ |
|-----------|----------------|----------------|
|           | [Corollary 5.4] | [Corollary 5.7] |

| $s_D$ | $(\log 3) s_D \leq \tilde{\tau}_D$ | $(\log 9) s_D \leq u_D$ |
|-------|-------------------------------|-----------------|
|       | [Theorem 5.8]                  | [Corollary 5.11] |

The inequality is sharp

| $\eta_D$ | $\frac{1}{2} \eta_D \leq \tilde{\tau}_D \leq \log(2 + e^{\eta_D})$ | $\eta_D \leq u_D \leq 4 \log(2 + e^{\eta_D})$ |
|-----------|--------------------------------|--------------------------------|
|           | [20]                           | [Lemma 5.14] |

(Both inequalities are sharp)

**Table 2. Comparisons with other hyperbolic-type metrics**

We begin by comparing the Cassinian metric with the Seittenranta metric. The Cassinian metric, $c_D$, of the domain $D \subset \mathbb{R}^n$ is defined as

$$c_D(x, y) = \sup_{p \in \partial D} \frac{|x - y|}{|x - p||p - y|}.$$  

This metric was first introduced and studied in [16] and subsequently studied in [10, 21, 22]. Geometrically, the $c_D$-metric can be defined by taking the maximal Cassinian oval in $D$ with foci at $x$ and $y$ (see, [16]). Clearly, the supremum in the definition is attained at some point $p \in \partial D$.

The Seittenranta metric, $\delta_D$, introduced in [23], is defined by

$$\delta_D(x, y) = \log(1 + m_D(x, y))$$

where

$$m_D(x, y) = \sup_{a, b \in \partial D} \frac{|x - y||a - b|}{|x - a||y - b|}.$$  

Note that the quantity $m_D(x, y)$ does not define a metric. The Seittenranta metric is Möbius invariant and coincides with the hyperbolic metric of the unit ball $\mathbb{B}^n$.

The $c_D$-metric and the $\delta_D$-metric are exponentially related, which is stated in the following theorem.
**Theorem 5.1.** Let $D \subseteq \mathbb{R}^n$ be any domain. Then

$$c_D(x, y) \leq \frac{e^{\delta_D(x,y)} - 1}{d(y)}.$$ 

The inequality is sharp.

**Proof.** Let $p \in \partial D$ such that

$$c_D(x, y) = \frac{|x - y|}{|x - p||p - y|}.$$ 

Choose $q \in \partial D$ such that $|p - q| \geq |y - q|$. Now,

$$c_D(x, y) = \frac{|x - y|}{|x - p||p - y|} = \frac{|x - y||p - q| |y - q|}{|x - p||y - q||p - q||p - y|} \leq \frac{m_D(x, y)}{d(y)}.$$ 

Hence we get

$$\delta_D(x, y) = \log(1 + m_D(x, y)) \geq \log(1 + d(y) \cdot c_D(x, y))$$

and the proof is complete. For the sharpness, we consider the punctured space $D_p$. Let $x, y \in D_p$ with $|x - p| \leq |y - p|$. It is clear that $\delta_D(x, y) = \tilde{j}_{D_p}(x, y) = \log(1 + |x - y|/|x - p|)$ and hence the sharpness follows. \hfill \Box

An immediate corollary to Theorem 5.1 is the following inclusion relation.

**Corollary 5.2.** Let $D \subseteq \mathbb{R}^n$, $x \in D$, and $t > 0$. Then

$$B_{\delta_D}(x, t) \subseteq B_{c_D}(x, R),$$

where $R = (e^t - 1)/d(y)$. The inclusion is sharp.

**Proof.** If $\delta_D(x, y) < y$, then by Theorem 5.1, we have $c_D(x, y) < (e^t - 1)/d(y)$. For the sharpness, choose a point $y \in \partial B_{\delta_D}(x, t) \cap L$, where $L$ is the line passing through $0$ and $x$ with $|x| < |y|$. Then

$$\delta_D(x, y) = t = \log \left(1 + \frac{|x - y|}{|x|}\right).$$

Now,

$$c_D(x, y) = \frac{|x - y|}{|x||y|} = \frac{1}{|y|}(e^t - 1)$$

and hence the proof is complete. \hfill \Box
Again the $\delta_D$-metric is bilipschitz equivalent to the $\tilde{j}_D$-metric. Indeed, we have
\begin{equation}
\tilde{j}_D \leq \delta_D \leq 2\tilde{j}_D,
\end{equation}
see, for instance [23, p. 525]. Hence [4.5] along with (5.1) yield the following inequality between the $\delta_D$-metric and the $\tilde{\tau}_D$-metric.

**Theorem 5.3.** Let $x, y \in D \subseteq \mathbb{R}^n$. Then the following holds true:
\[
\frac{1}{4} \delta_D(x, y) \leq \tilde{\tau}_D(x, y) \leq \delta_D(x, y).
\]
The second inequality is sharp.

**Proof.** The proof of the inequality follows directly from (4.5) and (5.1). For the sharpness of the second inequality, consider the domain $D_0$ and choose $x, y \in D_0$ with $y = -x$. Then $\tilde{\tau}_{D_0}(x, -x) = \log 3 = \delta_{D_0}(x, -x)$. \hfill \Box

The following inclusion relation holds true.

**Corollary 5.4.** Let $x \in D \subseteq \mathbb{R}^n$ and $t > 0$. Then
\[
B_{\delta_D}(x, r) \subseteq B_{\tilde{\tau}_D}(x, t) \subseteq B_{\delta_D}(x, R),
\]
where $r = t$ and $R = 4t$.

**Proof.** Proof follows from Theorem 5.3. \hfill \Box

Theorem 4.5 and Theorem 5.3 together yield the following:

**Corollary 5.5.** Let $x, y \in D \subseteq \mathbb{R}^n$. Then we have
\[
\frac{1}{2} \delta_D(x, y) \leq u_D(x, y) \leq 4\delta_D(x, y).
\]
Corollary 5.5 leads to the following inclusion relation.

**Corollary 5.6.** Let $x \in D \subseteq \mathbb{R}^n$ and $t > 0$. Then
\[
B_{\delta_D}(x, r) \subseteq B_{u_D}(x, t) \subseteq B_{\delta_D}(x, R),
\]
where $r = t/4$ and $R = 2t$.

Hence, as a consequence to Theorem 5.1, we have

**Corollary 5.7.** Let $x, y \in D \subseteq \mathbb{R}^n$. Then we have
\[
c_D(x, y) \leq \frac{e^{4\tilde{\tau}_D(x, y)} - 1}{d(y)} \quad \text{and} \quad c_D(x, y) \leq \frac{e^{2u_D(x, y)} - 1}{d(y)}.
\]

**Proof.** The first inequality follows from Theorem 5.1 and Lemma 5.3 whereas the second inequality follows from Theorem 5.1 and Corollary 5.5. \hfill \Box
Now, we compare the $\tilde{\tau}_D$-metric with the triangular ratio metric, $s_D$, defined in a proper subdomain $D$ of $\mathbb{R}^n$ by

$$s_D(x, y) = \sup_{p \in \partial D} \frac{|x - y|}{|x - p| + |p - y|}, \quad x, y \in D.$$ 

Geometrically, the triangular ratio metric can be viewed by taking the maximal ellipse in $D$ with foci at $x$ and $y$ in the similar fashion as in the geometric definition the Apollonian metric [12]. For more details on $s_D(x, y)$ we refer [6].

**Theorem 5.8.** Let $D \subseteq \mathbb{R}^n$ and $x, y \in D$. Then

$$\tilde{\tau}_D(x, y) \geq (\log 3)s_D(x, y).$$

The inequality is sharp.

**Proof.** From AM-GM inequality, it follows that

$$\frac{1}{\sqrt{|x - p||y - p|}} \geq \frac{2}{|x - p| + |y - p|}.$$ 

Now,

$$\tilde{\tau}_D(x, y) \geq \log \left(1 + \frac{|x - y|}{\sqrt{|x - p||y - p|}}\right) \geq \log \left(1 + \frac{2|x - y|}{|x - p| + |y - p|}\right) \geq \frac{|x - y|}{|x - p| + |y - p|} \log 3$$

holds for all $p \in \partial D$. Here the last inequality follows from the well known Bernoulli’s inequality:

$$\log(1 + ax) \geq a \log(1 + x) \text{ for } a \in (0, 1), x > 0.$$ 

In particular, we have $\tilde{\tau}_D(x, y) \geq (\log 3)s_D(x, y)$. To prove the sharpness, consider the domain $D_0$ and $x, y \in D_0$ with $y = -x$. Then $\tilde{\tau}_D(x, -x) = \log 3$ and $s_D(x, -x) = 1$. \hfill \Box

**Remark 5.9.** Combining Theorem 5.8 and [19, Theorem 4.3], one can obtain Theorem 3.3 of [6].

Theorem 5.8 leads to the following inclusion relation.

**Corollary 5.10.** Let $x \in D \subseteq \mathbb{R}^n$ and $t > 0$. Then

$$B_{\tilde{\tau}_D}(x, t) \subseteq B_{s_D}(x, R),$$ 

where $R = t/ \log 3$. The inclusion is sharp.
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Proof. It follows from Theorem 5.8 that for \( \bar{\tau}_D(x,y) < t \), \( s_D(x,y) < t / \log 3 \). Hence, \( B_{\bar{\tau}_D}(x,t) \subseteq B_{s_D}(x,R) \) with \( R = t / \log 3 \).

To prove the sharpness part, consider the domain \( D = D_0 \) and \( t = \log 3 \). Then we have \( R = 1 \) and

\[
\bar{\tau}_{D_0}(x,y) = \log 3 \iff |x - y| = 2\sqrt{|x||y|}
\]

and hence

\[
s_{D_0}(x,y) = \frac{2\sqrt{|x||y|}}{|x| + |y|}.
\]

To show \( s_{D_0}(x,y) = 1 \), we need to choose points \( x \) and \( y \) such that \( |x| = |y| \). This implies \( x \) and \( y \) are co-linear. i.e. \( y = -x \). From the definition of the \( \bar{\tau}_D \) metric it is clear that the point \(-x\) lies on the sphere \( \partial B_{\bar{\tau}_{D_0}}(x,\log 3) \). Now, for any \( x \in D_0 \), choose \( y \in \partial B_{\bar{\tau}_{D_0}}(x,\log 3) \cap L \), where \( L \) is the line passing through 0 and \( x \) with \( |x| = |y| \). Then \( \bar{\tau}_{D_0}(x,y) = \log 3 \iff s_{D_0}(x,y) = 1 \).

Hence, the proof is complete. \( \square \)

Another consequence of Theorem 5.8 leads to the following corollary.

Corollary 5.11. Let \( D \not\subseteq \mathbb{R}^n \). Then for all \( x, y \in D \) we have

\[
s_D(x,y) \leq \frac{1}{\log 9} u_D(x,y).
\]

Proof. The proof follows from Theorem 4.5 and Theorem 5.8. \( \square \)

Next, we discuss the inclusion properties associated with the \( \bar{\tau}_D \)-metric and the half-Apollonian metric balls. The half-Apollonian metric, \( \eta_D \), of a domain \( D \not\subseteq \mathbb{R}^n \) is defined by

\[
\eta_D(x,y) = \sup_{p \in \partial D} \left| \log \frac{|x-p|}{|y-p|} \right|, \quad x, y \in D.
\]

The \( \eta_D \)-metric was introduced by Hästö and Linden in [14]. It is now appropriate to recall the following result by Seittenranta.

Lemma 5.12. [23, Theorem 3.11] Let \( D \subseteq \mathbb{R}^n \) be an open set with \( \text{card } \partial D \geq 2 \). Then

\[
\alpha_D(x,y) \leq \delta_D(x,y) \leq \log(e^{\alpha_D} + 2).
\]

The inequalities give the best possible bounds for \( \delta_D \) expressed in terms of \( \alpha_D \) only.

Here the quantity \( \alpha_D \) represents the Apollonian metric, introduced by Beardon in [3]. As a special case of Lemma 5.12, the following result holds true, which is proved in [20].

Lemma 5.13. Let \( D \not\subseteq \mathbb{R}^n \) and \( x, y \in D \). Then

\[
\frac{1}{2} \eta_D(x,y) \leq \bar{\tau}_D(x,y) \leq \log(2 + e^{\eta_D(x,y)}).
\]

Both the inequalities are sharp.
Lemma 5.13 obtains the following inclusion property.

**Corollary 5.14.** Let \( D \subseteq \mathbb{R}^n \) and \( x \in D \) and \( t > 0 \). Then the following inclusion property holds true:

\[
B_{\eta_D}(x, r) \subseteq B_{\tilde{\tau}_D}(x, t) \subseteq B_{\eta_D}(x, R).
\]

Here \( r = \log(e^t - 2) \) and \( R = 2t \). The radii \( r \) and \( R \) are best possible.

**Proof.** Let \( y \in B_{\tilde{\tau}_D}(x, t) \), i.e. \( \tilde{\tau}_D(x, y) < t \). From the left hand side inequality of Theorem 5.13 we have \( \eta_D(x, y) < 2t(= R) \). On the other hand, if \( \eta_D(x, y) < \log(e^t - 2)(= r) \), then from the right hand side inequality of Theorem 5.13 we have \( \tilde{\tau}_D(x, y) < t \). With the similar argument given in the proof for the sharpness part of second inclusion relation in Corollary 4.6, we can show that the radius \( R \) is the best possible in the punctured space \( D_0 \) with \( t = \log 3 \). \( \square \)

Comparison of Theorem 4.5 and Lemma 5.13 together lead to the following relation between the \( \eta_D \)-metric and the \( \tilde{\tau}_D \)-metric.

**Lemma 5.15.** Let \( D \subseteq \mathbb{R}^n \) and \( x, y \in D \). Then

\[
\eta_D(x, y) \leq u_D(x, y) \leq 4 \log(2 + e^{\eta_D(x, y)}).
\]

Subsequently, we have the following trivial inclusion relation.

**Corollary 5.16.** Let \( x \in D \subseteq \mathbb{R}^n \) and \( t > 0 \). Then

\[
B_{\eta_D}(x, r) \subseteq B_{\tilde{\tau}_D}(x, t) \subseteq B_{\eta_D}(x, R),
\]

where \( r = \log(e^{t/4} - 2) \) and \( R = t \).

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