Fresh-Variable Automata for Service Composition

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Abstract. To model Web services handling data from an infinite domain, or with multiple sessions, we introduce fresh-variable automata, a simple extension of finite-state automata in which some transitions are labeled with variables that can be refreshed in some specified states. We prove several closure properties for this class of automata and study their decision problems. We then introduce a notion of simulation that enables us to reduce the Web service composition problem to the construction of a simulation of a target service by the asynchronous product of existing services, and prove that this construction is computable.

1 Introduction

Service Oriented Architectures (SOA) consider services as platform-independent elementary components that can be published, invoked over a network and loosely-coupled with other services through standardized XML protocols in order to dynamically build complex distributed applications [1]. This flexible ability to compose applications can be viewed as a motto for SOA.

Service composition has been addressed in many works (e.g. [16,14,11,9,3]). One of the most successful approaches to composition amounts to abstract services as finite-state automata (FA) and apply available tools from automata theory to synthesize a new service satisfying the given client requests from an existing community of services [5,13].

However FA models are too abstract for handling data values ranging over unbounded domains, such as integer parameters of procedures or strings attached to XML documents leaves. This limitation has motivated several extensions of automata for dealing with infinite alphabets. A noticeable one is finite-memory automata (FMA) proposed by Kaminski and Francez [12], studied and compared with pebble automata in [14]. FMA have been extended to data automata (e.g. [8,7,17]) that have better connections with logic while keeping good decidability properties. Basically FMA can only remember a bounded number of previously read symbols. For instance, they can recognize the language of words where some data value occurs an even number of times. Our work is related to variable automata a simple extension of FA introduced by [10]. In this approach some automata transitions are labelled by variables that can get values from an infinite alphabet. The model in [10] allows one to keep a natural definition for runs and to obtain simple procedures for membership and non-emptiness.
However it is not obvious whether the automata-based approach to service composition (e.g. [5,13]) can still be applied with infinite alphabets. Our objective is to define a class of automata on infinite alphabets which is well-adapted to specification and composition of services and to study its properties.

Contributions. In this paper we consider the service composition problem as stated in [6]: given a client and a community of available services, synthesize a composition, i.e. a suitable function that delegates actions requested by the client to the available services in the community. This problem amounts to show (6,13) that there exists a simulation relation between the target service (specifying an expected service behaviour for satisfying the client requests) and the asynchronous product of the available services. If a simulation relation exists then it can be easily used to generate an orchestrator, that is a function that selects at each step an available service for executing an action requested by the client. In order to head for real-world applications where service actions are parameterized by terms built with data taken from infinite alphabets (identifiers, codes, addresses . . . ), we introduce an extension of FA called Fresh-Variable Automata (FVA) where some transitions are labelled by variables that can be assigned the read letter. A variable binding can be released at some states: in that case we say that the variable is refreshed. This mechanism is natural to express iteration processes, for instance when a service has to scan a list of item identifiers, or sessions. Note that our freshness notion differs from the one in [18]. We have established closure properties of FVA for union, intersection, concatenation and Kleene operator. We have shown that universality is decidable. Our main result is the decidability of the service composition problem for FVA. This gives a non-trivial extension of [6] that we illustrate with a natural example.

Related work. The related formalism of variable automata [10] was proposed as another simple extension of FA to infinite alphabets. The variables of variable automata are assigned at most once a value in a run, except for a special free variable that can get a value that is different from the other variables. This is not convenient to model services where several variables are reused in each session. [10] investigates closure properties of variable automata but do not consider simulation relations. In fact, FVAs and variable automata are incomparable. A well established model to handle infinite alphabets is FMA [12]. Although our model is less expressive than FMAs, we believe that FVAs are simpler to handle and to visualize, and they enjoy more decidable properties such as universality.

Several works deal with the problems of service composition and orchestration in different settings. In the Roman model [5], service composition was considered where the services are finite automata with no access to data. A logic-based approach was devised in [15] to solve this problem where the agents have access to infinite data. The client and the services exhibit infinite-state behavior: the transitions are labeled with guards over an infinite domain. In [2] the communication actions are performed through channels. Guards/conditions and constraints on the transitions have been introduced as well, e.g. [15]. Orchestration was studied
in [9] for services with linear behavior in presence of security constraints and where the communication actions are arbitrary terms over a given signature.

Paper organisation. Sec. 2 recalls standard notions. Sec. 3 introduces the new class of FVAs. Sec. 4 studies FVAs and shows in particular closure properties and decidability of universality. Sec. 5 defines communicating FVAs, or CFVAs for short, and introduces the notion of $\mathcal{G}$-simulation. Sec. 6 shows that $\mathcal{G}$-simulation is decidable for CFVAs. Sec. 7 applies the results to service synthesis problems. Final remarks and future works are given in Sec. 8.

2 Preliminaries

Let $X$ be a finite set of variables, $\Sigma$ an infinite alphabet of letters. A substitution is an idempotent mapping $\{x_1 \mapsto \alpha_1, \ldots, x_n \mapsto \alpha_n\} \cup \bigcup_{a \in \Sigma} \{a \mapsto a\}$ with variables $x_1, \ldots, x_n$ in $X$ and $\alpha_1, \ldots, \alpha_n$ in $X \cup \Sigma$. We call $\{x_1, \ldots, x_n\}$ its proper domain, and denote it by $\text{dom}(\sigma)$. We denote by $\text{Dom}(\sigma)$ the set $\text{dom}(\sigma) \cup \Sigma$.

If all the $\alpha_i, i = 1, \ldots, n$ are letters then we say that $\sigma$ is ground. The empty substitution (i.e., with an empty proper domain) is denoted by $\emptyset$. The set of the substitutions from $X \cup \Sigma$ to a set $A$ is denoted by $\zeta_{X,A}$, or by $\zeta_X$ if there is no ambiguity. If $\sigma_1$ and $\sigma_2$ are substitutions that coincide on the domain $\text{dom}(\sigma_1) \cap \text{dom}(\sigma_2)$, then $\sigma_1 \cup \sigma_2$ denotes their union in the usual sense. We define the function $V : \Sigma \cup X \rightarrow \mathcal{P}(X)$ by $V(\alpha) = \{\alpha\}$ if $\alpha \in X$, and $V(\alpha) = \emptyset$, otherwise. For a function $F : A \rightarrow B$, and $A' \subseteq A$, the restriction of $F$ on $A'$ is denoted by $F|_{A'}$.

A two-players game is a tuple $\langle \text{Pos}_E, \text{Pos}_A, M, p^* \rangle$, where $\text{Pos}_E, \text{Pos}_A$ are disjoint sets of positions: Eloise’s positions and Abelard’s positions. $M \subseteq (\text{Pos}_E \cup \text{Pos}_A) \times (\text{Pos}_E \cup \text{Pos}_A)$ is a set of moves, and $p^*$ is the starting position. A strategy for the player Eloise is a function $\rho : \text{Pos}_E \rightarrow \text{Pos}_E \cup \text{Pos}_A$, such that $(\varphi, \rho(\varphi)) \in M$ for all $\varphi \in \text{Pos}_E$. A (possibly infinite) play $\pi = \langle \varphi_1, \varphi_2, \ldots \rangle$ follows a strategy $\rho$ for player Eloise iff $\varphi_{i+1} = \rho(\varphi_i)$ for all $i \in \mathbb{N}$ such that $\varphi_i \in \text{Pos}_E$. Let $W$ be a (possibly infinite) set of plays. A strategy $\rho$ is winning for Eloise from a set $S \subseteq \text{Pos}_E \cup \text{Pos}_A$ according to $W$ iff every play starting from a position in $S$ and following $\rho$ belongs to $W$.

3 Fresh-variable automata

In this section we introduce the class of FVAs and illustrate it through simple examples. This formalism extends finite-state automata with two features. Firstly, the transitions labels consist of letters and variables that can be assigned a value from an infinite alphabet domain. Secondly, at each state some of the variables are freed from their assignments: they can receive other values.

A motivating example. We first motivate fresh-variable automata through an example that illustrates a service composition problem. We have an e-commerce Web site allowing customers to create shopping carts, search for items from
an infinite domain and add them to a shopping cart, see Figure 3. The main issue is that the three agents CLIENT, CART and SEARCH exhibit an infinite-state behavior involving sending and receiving messages ranging over a possibly infinite set of terms. We emphasize that variable $y$ is refreshed (i.e. freed to get a new value) when passing through the state $p_0$. In the same way variable $x$ is refreshed at $p_1$, $z$ at $q_0$ and $u$ at $q_1$, and $w$ at $r_0$ respectively.

![Diagram of CLIENT, CART, and SEARCH automata]

Fig. 1. The CART example

In this example, we ask whether the requests made by the client can be answered by combining the services CART and SEARCH. In this section we consider only automata in which the transitions are labeled by letters or variables. We introduce the communication symbols $!$, $?$ for defining a simulation in Sec. 5.

**Definition 1.** A FVA is a tuple $A = \langle \Sigma, X, Q, Q_0, \delta, F, \kappa \rangle$ where $\Sigma$ is a infinite set of letters, $X$ is a finite set of variables, $Q$ is a finite set of states, $Q_0 \subseteq Q$ is a set of initial states, $\delta = Q \times (\Sigma \cup X) \rightarrow 2^Q$ is a transition function with finite domain, $F \subseteq Q$ is a set of accepting states, and $\kappa : X \rightarrow 2^Q$ is the refreshing function that associates to every variable the (possibly empty) set of states where it is refreshed.

For a FVA $A$, we shall denote by $\Sigma_A$ the finite set of letters that appear in the transition function of $A$. Variables in a FVA are considered up to renaming, and we always assume that two FVAs have disjoint sets of variables.

The formal definition of configuration, run and recognized language follows.

**Definition 2.** Let $A = \langle \Sigma, X, Q, Q_0, \delta, F, \kappa \rangle$ be a FVA. A configuration is a pair $(q, M)$ where $q \in Q$ and $M : X \rightarrow \Sigma$ is a substitution. We define a transition relation over the configurations as follows: $(q_1, M_1) \rightarrow (q_2, M_2)$, where $a \in \Sigma$, iff there exists a label $\alpha \in \Sigma \cup X$ such that $q_2 \in \delta(q_1, \alpha)$, and either (i) $\alpha \in \text{Dom}(M_1)$, $M_1(\alpha) = a$ and $M_2 = M_1 \upharpoonright D$, with $D = \text{Dom}(M_1) \setminus \kappa^{-1}(q_2)$ or (ii) $\alpha \in (X \setminus \text{Dom}(M_1))$ and $M_2 = (M_1 \cup \{\alpha \mapsto a\}) \upharpoonright D$, with $D = (\text{Dom}(M_1) \cup \{\alpha\}) \setminus \kappa^{-1}(q_2)$. 

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A finite word \( w = w_1 w_2 \ldots w_n \in \Sigma^* \) is recognized by \( A \) iff there exists a run \((q_0, M_0) \xrightarrow{w_1} (q_1, M_1) \xrightarrow{w_2} \ldots \xrightarrow{w_n} (q_n, M_n)\), such that \( M_0 = \emptyset \), \( q_0 \in Q_0 \) and \( q_n \in F \). The set of words recognized by \( A \) is denoted by \( L(A) \).

We could define FVAs with \( \varepsilon \)-transitions too. We show in the Appendix that FVAs with \( \varepsilon \)-transitions are equivalent to FVAs.

**Example 1.** Let \( A_1 \) and \( A_2 \) be the FVAs depicted on the right, where \( \kappa(x) = \{p_0\} \) and \( \kappa(z) = \{q_0, q_1\} \). Then, \( L(A_1) \) is the set of words \( a_1 a_1 a_2 a_2 \ldots \) for \( n \geq 0 \) and \( a_i \in \Sigma \), and \( L(A_2) \) is the set of words in \( \Sigma^* \), where some letter appears at least twice. We notice that \( L(A_1) \) cannot be recognized by a variable automata [10].

**4 Properties of FVAs**

We study in this section properties of FVAs and some basic decision problems.

**4.1 Closure properties**

**FVAs with multiple labels.** To prove the closure under intersection, we first introduce a generalization of FVAs called \( n \)-FVAs where \( n \) is an integer. An \( n \)-FVA has transitions labeled with \( n \)-tuple of labels. In this general setting 1-FVAs are FVAs. We show next that \( n \)-FVAs and FVAs recognize the same languages.

**Definition 3.** An \( n \)-FVA, where \( n \in \mathbb{N}^* \), is a tuple \( A = \langle \Sigma, \mathcal{X}, Q, Q_0, \delta, F, \kappa \rangle \) which is defined like a FVA but for the transition function \( \delta : Q \times (\Sigma \cup \mathcal{X})^n \rightarrow 2^Q \).

The configurations and runs of \( n \)-FVAs are defined as for FVAs, except that the currently read letter \( u \in \Sigma \) should match simultaneously with the \( n \) components of its \( n \)-label for this transition to be fired, see Appendix E.1.

**Theorem 1.** For all \( n \geq 1 \), \( n \)-FVAs and FVAs are equivalent.

**Proof.** We sketch a proof of the non-trivial direction in the case \( n = 2 \). The general case follows directly by induction on \( n \). Let \( A = \langle \Sigma, \mathcal{X}, Q, Q_0, \delta, F, \kappa \rangle \) be a 2-FVA, and let us introduce \( n_X = |\mathcal{X}| \), and \( n_\Sigma = |\Sigma| \), and assume \( \Sigma_A = \{a_1, \ldots, a_{n_\Sigma}\} \). Let \( \Psi \subset \{1, \ldots, n_X + n_\Sigma\}^{\Sigma_A \cup \mathcal{X}} \) be the set of functions from \( \Sigma_A \cup \mathcal{X} \) to \( \{1, \ldots, n_X + n_\Sigma\} \) such that for every \( \psi \in \Psi \) we have \( \psi(a_k) = k \). Furthermore, given \( D \subseteq \mathcal{X} \) and \( \psi \in \Psi \), we let \( \psi^D \) be the subset of \( \Psi \) of functions equal to \( \psi \) on \( (\Sigma_A \cup \mathcal{X}) \setminus D \). Finally, given a substitution \( M \in \mathcal{X} \cup \Sigma \) we let \( \Psi_M \) be the subset of \( \Psi \) of functions \( \psi \) such that, for all \( x, y \in \Sigma_A \cup \text{dom}(M) \), we have...
\[ M(x) = M(y) \text{ iff } \psi(x) = \psi(y). \] Let \( A' = (\Sigma, X, Q \times \Psi, Q_0 \times \Psi, F \times \Psi, \delta', \kappa') \) where the transition function \( \delta' \) is defined as follows: for all \( (q_0, \psi_0) \in Q' \) and \( \alpha, \beta \in \Sigma_A \cup X \), \( \delta'((q_0, \psi_0), (\alpha, \beta)) = \{(q_1, \psi_1) \mid q_1 \in \delta(q_0, (\alpha, \beta)) \text{ and } \psi_0(\alpha) = \psi_1(\beta) \text{ and } \psi_1 \in \psi_0^{-1}(\{q_1\})\}; \) Finally for \( x \in X \), we define \( \kappa' \) \( x = \kappa(x) \times \Psi. \) We can prove that there exists a run \( q_0, M_0 \xrightarrow{(\alpha_1, \beta_1)} \cdots \xrightarrow{(\alpha_n, \beta_n)} q_n, M_n \) in \( A \) iff for all \( \psi_n \in \Psi_{M_n} \) there exists a run \( (q_0, \psi_0), M_0 \xrightarrow{(\alpha_1, \beta_1)} \cdots \xrightarrow{(\alpha_n, \beta_n)} (q_n, \psi_n), M_n \) in \( A' \). Thus, \( A \) and \( A' \) recognize the same language \( L \). Finally, a 1-FVA \( B \) recognizing the same language \( L \) is constructed from \( A' \) by mapping each integer in \( \psi(X \cup \Sigma_A) \) to a variable or a constant.

As shown by a language \( L = \{a\} \), with \( a \in \Sigma \), the complement of a FVA-recognizable language is not necessarily FVA-recognizable. Note also that [10] has neither considered Kleene operator nor the concatenation. The closure under union is straightforward since we just take the disjoint union of the two FVAs. The closure under intersection is an immediate consequence of the fact that FVAs with \( \varepsilon \)-transitions and FVAs are equivalent (Lemma 2 in the Appendix). The closure under intersection is an immediate consequence of Theorem 1 since the intersection of two FVAs amounts to computing their Cartesian product, which is a 2-FVA. Thus we have the following theorem.

**Theorem 2.** FVAs are closed under union, concatenation, Kleene operator and intersection.

### 4.2 Decision procedures for FVAs

We study the decidability and complexity of classical decision problems: Nonemptiness (given \( A \), is \( L(A) \neq \emptyset \)?), Membership (given a word \( w \) and \( A \), is \( w \in L(A) \)?), Universality (given \( A \), is \( L(A) = \Sigma^* \)?), and Containment (given \( A_1 \) and \( A_2 \), is \( L(A_1) \subseteq L(A_2) \)?).

**Theorem 3.** For FVAs, Nonemptiness is NL-complete, Membership is NP-complete, and Universality is decidable.

Proof for Universality. We say a variable \( x \) is free in a configuration \( q, M \) if \( x \notin \text{dom}(M) \). Out of \( A \) we construct a FVA \( A' \) such that for every reachable configuration \( q', M \) on \( A' \) every transition out of \( q' \) is labeled with a variable free in \( q' \).

**Claim 1.** If \( A \) is universal then for every \( n \geq 0 \) there exists a path of length \( n \) from an initial state to a final state in which every transition is labeled with a variable which is free in the source state of this transition.

**Proof of the claim.** By contradiction assume \( A \) is universal but there exists \( n \geq 0 \) such that every path of length \( n \) from an initial state to a final state has at least one transition over either a letter or an already bound variable. We note that the word \( w_1 \ldots w_n \in \Sigma^* \), in which \( w_i \neq w_j \) for all \( i \neq j \) and \( w_i \notin \Sigma_A \), is not recognized by \( A \). This contradicts the universality of \( A \). \( \square \)
Assume $\mathcal{A} = (\Sigma, \mathcal{X}, Q, Q_0, F, \delta, \kappa)$ and let $\mathcal{A}' = (\Sigma, \mathcal{X}, Q', Q'_0, F', \delta', \kappa')$ where:

$$\begin{align*}
Q' &= \{(q, X) \mid q \in Q \text{ and } X \subseteq \mathcal{X}\} \\
Q'_0 &= \{(q, X) \mid q \in Q_0\} \\
F' &= \{(q, X) \mid q \in F \text{ and } X \subseteq \mathcal{X}\}
\end{align*}$$

and $(q', X') \in \delta'((q, X), x)$ if, and only if, $x \in X$ and $X' = (X \setminus \{x\}) \cup \kappa^{-1}(q')$, and $\kappa'(x) = \{(q, X) \mid q \in \kappa(x)\}$.

**Claim 2.** There exists a run $q_0, M_0 \xrightarrow{x_1} \ldots \xrightarrow{x_n} q_n, M_n$ in $\mathcal{A}$ in which for all $1 \leq i \leq n$ we have $x_i \notin \text{Dom}(M_{i-1})$ if, and only if, there exists a run $(q_0, \mathcal{X}), M_0 \xrightarrow{\beta_1} \ldots \xrightarrow{\beta_r} (q_n, \mathcal{X}), M_n$ in $\mathcal{A}'$ with $\mathcal{X} = \mathcal{X} \setminus \text{Dom}(M_n)$.

**Proof of the Claim.** By induction on $n$. Since $\text{dom}(M_0) = \emptyset$ the case $n = 0$ is trivial. Assume the claim holds up to $n$. Let us prove the equivalence for $n + 1$.

$\Leftarrow$ Since $(q_{n+1}, X_{n+1}) \in \delta'((q_n, X_n), x_{n+1})$ by induction $x_{n+1} \notin \text{Dom}(M_n)$. Thus $q_{n+1} \in \delta(q_n, x_{n+1})$ and $x_{n+1}$ is free at the state $q_n$ of the run. The substitution $M_{n+1}$ obtained is as expected.

$\Rightarrow$ Assume a transition $q_n, M_n \xrightarrow{\beta} q_{n+1}, M_{n+1}$ is labeled with $x_{n+1} \notin \text{Dom}(M_n)$. By induction $x_{n+1} \in X_n$, and thus $(q_{n+1}, X_{n+1}) \in \delta'((q_n, X_n), x_{n+1})$. Thus, for every run starting from an initial state and reaching a configuration $(q, X), M$ the couple $(\text{dom}(M), X)$ is a partition of $\mathcal{X}$. Consequently each transition of $\mathcal{A}'$ is labeled with a variable which is free in every run reaching its source state. Thus it suffices to prove that in $\mathcal{A}'$, for every $n \geq 0$, there exists a path from an initial state to a final state of length $n$. We reduce this problem to the universality of the FA $A''$ on a unary alphabet $\{a\}$ obtained by replacing every transition $q_1 \xrightarrow{a} q_2$ of $\mathcal{A}'$ by the transition $q_1 \xrightarrow{a} q_2$, where $a$ is an arbitrary letter in $\Sigma$. That is, we check whether $L(A'') = a^*$. 

We cannot check $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$ by intersecting $L(\mathcal{A}_1)$ with $\Sigma^* \setminus L(\mathcal{A}_2)$ since the latter is not necessarily a FVA language even when $\mathcal{A}_2$ is a FA. However containment is decidable if one of the FVAs is a finite automaton, since in this case the intersection of the languages is regular (Lemma 5 in the Appendix).

**Theorem 4.** The containment problems between a FVA and a FA are decidable.

## 5 Games for the simulation of communicating FVAs

To deal with service composition problems we need first to extend FVAs to the communicating FVAs, or CFVA for short, where labels (letters or variables) are prefixed by a communication symbol "!" or "?". Then we generalize the standard FA simulation relation to a FVA simulation in order to formalize that a client can be satisfied by an available service (when both are specified by a CFVA). A client transition labeled by !$x$, where $x$ is not bound, should be simulated by a
service transition which is labeled by $?y$, where $y$ is not bound as well, since the service should handle all instances of $x$. On the other hand, a client transition labeled by $?x$, where $x$ is not bound, can be simulated by a service transition labeled by any $!\alpha$. Hence, in order to define properly the simulation we should take into account the refreshing of variables.

**Definition of CFVAs.** Formally, a CFVA is defined exactly like a FVA but for the transition function $\delta = Q \times (\Sigma \cup X)^{\tau} \to 2^Q$, where for a set $S$, $S^{\tau}$ denotes the set $\{s, ?s \mid s \in S\}$. To simplify the presentation from now we shall only consider CFVAs in which there is a unique initial state and all the states are accepting. The definition of the simulation game for CFVAs follows.

**Definition 4.** Let $A_1 = (\Sigma, X_1, Q_1, q_1^0, \delta_1, F_1, \kappa_1)$ and $A_2 = (\Sigma, X_2, Q_2, q_2^0, \delta_2, F_2, \kappa_2)$ be two CFVAs where $X_1 \cap X_2 = \emptyset$. Let $Pos$ be the set of positions reachable from $p^* = ((0, q_0^1), (0, q_0^2))_A$ by the set of moves $M = M_A' \cup M_A'' \cup M_E' \cup M_E''$, where:

$$M_A' = \{((\sigma_1, q_1), q_2)_A \to ((\sigma_1 | D, q_1'), q_2, (\sigma_1, ?\alpha))_E \mid q_1' \in \delta_1(q_1, ?\alpha) \text{ and } D = \text{Dom}(\sigma_1) \setminus \kappa_1^{-1}(q_1')\}$$

$$M_A'' = \{((\sigma_1, q_1), q_2)_A \to ((\sigma_1 \uplus \gamma | D, q_1'), q_2, (\gamma \uplus \sigma_1, !\alpha))_E \mid q_1' \in \delta_1(q_1, !\alpha) \text{ and } D = \text{Dom}(\sigma_1 \uplus \gamma) \setminus \kappa_1^{-1}(q_1')\}$$

$$M_E' = \{q_1, (\sigma_2, q_2), (\sigma_1, !\alpha))_E \to q_1, ((\sigma_2 \uplus \sigma) | D, q_2')_A \mid q_2' \in \delta_2(q_2, ?\beta)$$

$$\text{and } D = \text{Dom}(\sigma_2 \uplus \sigma) \setminus \kappa_2^{-1}(q_2')$$

$$\text{and } \sigma(\sigma_2(\beta)) = \sigma_1(\alpha)\}$$

$$M_E'' = \{((\sigma_1, q_1), (\sigma_2, q_2), (\sigma_1', ?\alpha))_E \to (((\sigma_1 \uplus \sigma) | D_1, q_1), ((\sigma_2 \uplus \gamma) | D_2, q_2'))_A \mid q_2' \in \delta_2(q_2, !\beta)$$

$$\text{and } D_1 = \text{Dom}(\sigma_1 \uplus \sigma) \setminus \kappa_1^{-1}(q_1)$$

$$\text{and } D_2 = \text{Dom}(\sigma_2 \uplus \gamma) \setminus \kappa_2^{-1}(q_2')$$

$$\text{and } \sigma(\sigma_2(\beta)) = \gamma(\sigma_2(\beta))$$

$$\text{and } \gamma : \text{Dom}(\sigma_2(\beta)) \to \Sigma\}$$

where the moves in $M_A' \cup M_A''$ are wrt any possible substitution $\sigma$.

We let $\text{Pos}_E = \text{Pos} \cap ((\xi X_1 \times Q_1) \times (\xi X_2 \times Q_2) \times (\xi X_1 \times (\Sigma \cup X)^{\tau})$ and $\text{Pos}_A = \text{Pos} \cap ((\xi X_1 \times Q_1) \times (\xi X_2 \times Q_2)$. The simulation game of $A_1$ by $A_2$, denoted by $G(A_1, A_2)$, is the two-players game $\langle \text{Pos}_E, \text{Pos}_A, M, p^* \rangle$. As usual, any infinite play is winning for Eloise, and any finite play is losing for the player who cannot move.

**Definition 5.** Let $A_1 = (\Sigma, X_1, Q_1, q_1^0, \delta_1, F_1, \kappa_1)$ and $A_2 = (\Sigma, X_2, Q_2, q_2^0, \delta_2, F_2, \kappa_2)$ be two CFVAs. There is a $9$-simulation of $A_1$ by $A_2$ if Eloise has a winning strategy in the game $G(A_1, A_2)$, and we shall write $A_1 \preceq A_2$.

**Explanations of the rules of the game.** The simulation game $G(A_1, A_2)$ is played between two players: Abelard ($\forall$ or attacker) and Eloise ($\exists$ or defender). Its
positions are either of the form $((\sigma_1, q_1), (\sigma_2, q_2))_A$ or $((\sigma_1, q_1), (\sigma_2, q_2), (\sigma, \alpha))_E$, where $\sigma_1, \sigma_2, \sigma$ are ground substitutions, $q_1$ (resp. of $q_2$) is a state of $A_1$ (resp. $A_2$), and $\alpha$ is a message in $\Sigma \cup X$'s. They correspond to Abelard positions (A) or Eloise positions (E). The moves $M'_A$ state that Abelard chooses a transition $q_1 ? \sigma \rightarrow q'_1$ in $A_1$ and asks Eloise to match it. Consequently, all the variables that must be refreshed in the resulting state $q'_1$ are released. The moves $M'_A$ are the same as $M'_A$ apart that they deal with a sending message $!\alpha$. In this case, Abelard must first instantiate the variable in $!\alpha$ (if any) with a letter by a ground substitution $\gamma$, then asks Eloise to match the message $\alpha!(\alpha)$. The moves $M'_E$ state that Eloise chooses a transition $q_2 ? \sigma_2 \rightarrow q'_2$ in $A_2$ to match the message $\sigma_1!(\alpha)$. Indeed, she matches $\sigma_2(\beta)$ with $\sigma_1(\alpha)$ where $\sigma_2$ represents the value of the variables in the state $q_2$. The resulting substitution $\sigma$ is stored in the resulting state $q'_2$, and all the variables that must be refreshed at $q'_2$ are released. The moves $M'_E$ are like $M'_E$ except that Eloise must first instantiate the possible variable of the sending message $\sigma_2(\beta)$ with a ground substitution $\gamma$.

Notice that for every Eloise position $((\sigma_1, q_1), (\sigma_2, q_2), (\sigma, \alpha))_E \in \text{Pos}_E$, the substitutions $\sigma_1$ and $\sigma$ coincide on $\text{dom}(\sigma_1) \cap \text{dom}(\sigma)$. Notice also that the simulation game might be infinite with possibly infinite branching since $\Sigma$ is infinite.

The $\mathcal{G}$-simulation problem for CFVAs is the following: given two CFVAs $A_1$ and $A_2$, is $A_1 \preceq A_2$?

Example 2. Let $A$ and $B$ the CFVA depicted in the Figure[2] where $\kappa(x) = \{p_1\}$ and $\kappa(y) = \{p_0\}$. One can show that $A \preceq B$.

![Fig. 2. CFVAs $A$ and $B$ with $A \preceq B$, where $\kappa(x) = \{p_1\}$ and $\kappa(y) = \{p_0\}.$](image)

6 On the decidability of the $\mathcal{G}$-simulation problem

In this section we show that the problem of $\mathcal{G}$-simulation is decidable. The idea is that this problem can be reduced to a $\mathcal{G}$-simulation problem over the same CFVAs in which the two players instantiate the variables from a finite set of letters, as proven in Proposition 4.

Definition 6. Let $A_1 = (\Sigma, \mathcal{X}_1, Q_1, q_1^0, \delta_1, F_1, \kappa_1)$ and $A_2 = (\Sigma, \mathcal{X}_2, Q_2, q_2^0, \delta_2, F_2, \kappa_2)$ be two CFVAs. We define $\overline{G}(A_1, A_2)$ to be the game obtained by restricting the codomain of $\gamma$ to $C_0$ in the rules of Eloise $M'_E$ and Abelard $M'_A$, in Def. 4 where $C_0 = \Sigma A_1 \cup \Sigma A_2 \cup (\mathcal{X}_1 \times \mathcal{X}_2) \cup (\mathcal{X}_2 \times \mathcal{X}_1)$.

The following Lemma states an immediate property of the game $\overline{G}$.

Lemma 1. Let $A_1, A_2$ be two CFVAs. Then, the game $\overline{G}(A_1, A_2)$ is finite.
In order to prove Proposition 1 we need to introduce the notion of coherence between substitutions and between game positions.

**Definition 7.** Let $C$ be a finite subset of $\Sigma$. The coherence relation $\mathcal{K}_C \subseteq \zeta \times \zeta$ between substitutions is defined by $\bar{\sigma} \mathcal{K}_C \sigma$ iff the three following conditions hold:

1. $\text{dom}(\bar{\sigma}) = \text{dom}(\sigma)$,
2. If $\bar{\sigma}(x) \in C$ then $\bar{\sigma}(x) = \sigma(x)$, and if $\sigma(x) \in C$, then $\bar{\sigma}(x) = \sigma(x)$, for any variable $x \in \text{dom}(\sigma)$, and
3. for any variables $x, y \in \text{dom}(\sigma)$, $\bar{\sigma}(x) = \bar{\sigma}(y)$ iff $\sigma(x) = \sigma(y)$.

The definition of the coherence between game positions, still denoted by $\mathcal{K}_C$, follows.

**Definition 8.** Let $C$ be a finite subset of $\Sigma$. Let $A_1 = (\Sigma, X_1, Q_1, q_0^1, \delta_1, F_1, \kappa_1)$ and $A_2 = (\Sigma, X_2, Q_2, q_0^2, \delta_2, F_2, \kappa_2)$ be two CFVAs s.t. $X_1 \cap X_2 = \emptyset$. Let $\text{Pos}_E$ (resp. $\text{Pos}_A$) be the set of Eloise’s (resp. Abelard’s) positions in the game $G(A_1, A_2)$. Then we define the relation: $\mathcal{K}_C \subseteq \text{Pos}_A \times \text{Pos}_A \times \text{Pos}_E \times \text{Pos}_E$ by:

- For any $\sigma, \sigma_i$ of proper domain included in $X_i$ ($i = 1, 2$) we have:
  $$((\bar{\sigma}, q_1), (\bar{\sigma}, q_2)) \mathcal{K}_C (\sigma, \sigma_i)$$
  iff $\bar{\sigma}_1 \uplus \bar{\sigma}_2 \mathcal{K}_C (\sigma_1 \uplus \sigma_2)$.

- For any $\sigma, \bar{\sigma}, \sigma_i$ of proper domain included in $X_i$ ($i = 1, 2$), for any substitutions $\sigma, \bar{\sigma}$ with proper domain included in $X_1$, we have:
  $$((\bar{\sigma} \uplus \bar{\sigma}_2, \bar{\sigma}_1 \uplus \sigma_2)) \mathcal{K}_C ((\bar{\sigma} \uplus \bar{\sigma}_2) \uplus \sigma_2)$$
  iff $\mathcal{K}_C ((\bar{\sigma} \uplus \bar{\sigma}_2, (\bar{\sigma}, \alpha_i))$.

Now we are ready to show that the games $G$ and $\mathcal{G}$ are equivalent in the following sense:

**Proposition 1.** Let $A_1 = (\Sigma, X_1, Q_1, q_0^1, \delta_1, F_1, \kappa_1)$ and $A_2 = (\Sigma, X_2, Q_2, q_0^2, \delta_2, F_2, \kappa_2)$ be two CFVAs. Then, Eloise has a winning strategy in $G(A_1, A_2)$ iff she has a winning strategy in $\mathcal{G}(A_1, A_2)$.

**Proof.** Up to renaming of variables, we can assume that $X_1 \cap X_2 = \emptyset$. For the direction $\Rightarrow$ we show that out of a winning strategy of Eloise in $G(A_1, A_2)$ we construct a winning strategy for her in $\mathcal{G}(A_1, A_2)$. For this purpose, we show that each move of Abelard in $\mathcal{G}(A_1, A_2)$ can be mapped to an Abelard move in $G(A_1, A_2)$, and that Eloise response in $G(A_1, A_2)$ can be actually mapped to an Eloise move in $\mathcal{G}(A_1, A_2)$. This mapping defines a relation $\mathcal{R}$ between the positions of $\mathcal{G}(A_1, A_2)$ and the positions of $G(A_1, A_2)$ as follows: $\mathcal{R} \subseteq \text{Pos}_E(\mathcal{G}(A_1, A_2)) \times \text{Pos}_E(G(A_1, A_2)) \cup \text{Pos}_A(\mathcal{G}(A_1, A_2)) \times \text{Pos}_A(G(A_1, A_2))$, such that if $(\bar{\varphi}, \varphi) \in \mathcal{R}$, and the move $\bar{\varphi} \rightarrow \bar{\varphi}'$ in $\mathcal{G}(A_1, A_2)$ is mapped to $\varphi \rightarrow \varphi'$ in $G(A_1, A_2)$, or $\varphi \rightarrow \varphi'$ in $G(A_1, A_2)$ is mapped to $\bar{\varphi} \rightarrow \bar{\varphi}'$ in $\mathcal{G}(A_1, A_2)$, then $(\bar{\varphi}', \varphi') \in \mathcal{R}$. Furthermore, we impose that the following invariant (Inv-$\mathcal{K}$) holds: If $(\bar{\varphi}, \varphi) \in \mathcal{R}$ then $\bar{\varphi} \mathcal{K}_C \varphi$, where $C = \Sigma_{A_1} \cup \Sigma_{A_2}$. We recall that the variables in $\mathcal{G}(A_1, A_2)$ are instantiated from the set of letters $C_0 = \Sigma_{A_1} \cup \Sigma_{A_2} \cup (X_1 \times X_2) \cup (X_2 \times X_1)$. The main part of the proof consists in
finding the right way to relate the instantiation of the variables of the sending messages in $G(A_1, A_2)$ and $G(A_1, A_2)$. More precisely, we distinguish three cases: when $\text{Abelard}$ in $G(A_1, A_2)$ instantiates a variable with a letter in $\Sigma_{A_1} \cup \Sigma_{A_2}$, then $\text{Abelard}$ in $G(A_1, A_2)$ must instantiate the same variable with the same letter. When $\text{Abelard}$ in $G(A_1, A_2)$ instantiates a variable with a fresh letter that belongs to $C_0 \setminus (\Sigma_{A_1} \cup \Sigma_{A_2})$—by fresh we mean it does not appear in the current position of $G(A_1, A_2)$—then $\text{Abelard}$ in $G(A_1, A_2)$ must instantiate the same variable with a fresh letter in $\Sigma$. Finally, when $\text{Abelard}$ in $G(A_1, A_2)$ instantiates a variable with a non fresh letter, say $\bar{\alpha}_0$, i.e. $\bar{\alpha}_0$ appears in the current position, then $\text{Abelard}$ in $G(A_1, A_2)$ must instantiate the same variable with the letter $\alpha_0$ related to $\bar{\alpha}_0$, i.e. in a previous step the choice of $\bar{\alpha}_0$ corresponds to the choice of $\alpha_0$.

For the other direction, i.e. $\text{Eloise}$ instantiation of the variables in $G(A_1, A_2)$ from $\Sigma$ is related to $\text{Eloise}$ instantiation of the variables in $G(A_1, A_2)$ from $C_0$ by following the same principle. Following this construction, we ensure that the invariant (Inv-$\bowtie$) is always maintained.

The proof of the direction ($\Leftarrow$) is similar to the one of ($\Rightarrow$); we follow the same instantiation principle and keep the same definition of the $\bowtie$-coherence. □

It follows from Lemma [1] and Proposition [1].

**Theorem 5.** The problem of $\bowtie$-simulation is decidable for CFVAs.

Given two CFVAs $A_1, A_2$, deciding whether $A_1 \preceq A_2$ simply amounts to construct the finite game $G(A_1, A_2)$ and compute a winning strategy for $\text{Eloise}$.

## 7 Service composition

To carry on the CART example and real-world service applications, we need to extend CFVAs and $\bowtie$-simulation so that transitions labels can be of type $! t$ or $? t$, with $t$ an arbitrary term over a first-order signature. This extended model (ECFVA) is detailed in Appendix [D]. $\bowtie$-simulation problem remains decidable for the subclass of ECFVAs in which the terms labeling the transitions are either constants or of the form $f(\alpha_1, \ldots, \alpha_n)$ where $f$ is a functional symbol and $\alpha_i$ is either a variable or a constant, as is the case for the CART example.

*Composition synthesis.* We consider the same composition synthesis problem as in [13] besides the modelling of the client goal and each service as an ECFVA. We adapt the construction of the asynchronous product $\otimes$ on FAs [13] for ECFVAs to obtain an ECFVA modelling the community of available services. Finding a simulation then amounts to constructing a winning strategy for $\text{Eloise}$ in the simulation game. In the case of the CART example, one strategy can be computed in the game $G(\text{CLIENT}, \text{CART} \otimes \text{SEARCH})$, and thus the client requests can be satisfied. Notice that this problem is EXPTIME-hard as a direct consequence of [13], where this lower bound obtained for the composition synthesis of deterministic finite automata is established.
8 Conclusion

In future works we plan to investigate the complexity of the universality and $\delta$-simulation of CFVAs and to find other classes of ECFVAs for which the $\delta$-simulation can be decided. It would be important to consider security constraints that the composition of services must fulfill as in [9]. For this purpose, suitable model-checking techniques have to be devised for FVAs.

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Appendices

A On the comparison with other models

FVAs are incomparable with variable automata [10]. On the one hand the language $L = \{a_1a_1a_2a_2 \cdots a_na_n, n \geq 0, a_i \in \Sigma\}$ cannot be recognized by a variable automaton as shown in [10]. However, it is recognized by the FVA $A_1$ of Example 1. On the other hand, the language of all the words in which the last letter is different from all the other letters can be recognized by a variable automaton but not by a FVA, since there is no way to express in FVAs that a variable is distinct from other variables. Besides, the subclass of variable automata without free variables coincides with the subclass of FVAs without fresh variables.

FVAs are weaker than FMAs [12]. The language of words in which some letter appears exactly twice can be recognized by a FMA [12] but not by a FVA.

B Appendix for Section 4

Before establishing the proofs of the claims of Section 4 we first give the formal definition of configuration and run for $n$-BFVAs since it is required thereafter.

B.1 Run and configuration for $n$-BFVAs

Definition 9. Let $A = (\Sigma, X, Q, Q_0, \delta, F, \kappa)$ be an $n$-FVA. A configuration is a pair $(q, M)$ where $q \in Q$ and $M : X \rightarrow \Sigma$ is a substitution. We define a transition relation over the configurations as follows: $(q_1, M_1) \xrightarrow{u} (q_2, M_2)$, where $u \in \Sigma$, iff there exists an $n$-label $l_n = (l_1, \ldots, l_n) \in (\Sigma \cup X)^n$, such that $q_2 \in \delta(q_1, l_n)$, and a substitution $\sigma : X \rightarrow \Sigma$ such that $\sigma(M_1(l_i)) = u$, for all $i \in \{1, \ldots n\}$, so that $M_2 = (M_1 \uplus \sigma)_D$, where $D = \text{Dom}(M_1 \uplus \sigma) \setminus \kappa^{-1}(q_2)$. A finite word $u = u_1u_2 \cdots u_m \in \Sigma^*$ is recognized by $A$ iff there exists a run $(q_0, M_0) \xrightarrow{u_1} (q_1, M_1) \xrightarrow{u_2} \cdots \xrightarrow{u_m} (q_m, M_m)$, such that $M_0 = \emptyset$, $q_0 \in Q_0$ and $q_m \in F$. The set of words recognized by $A$ is still denoted by $L(A)$.

Example 3. Let $A$ be the 2-FVA depicted below where $\kappa(y) = \{q_0, q_1\}$ and $\kappa(x) = \emptyset$. It is clear that $L(A) = \{(az)^n \mid z \in \Sigma, n \geq 1\}$.

Fig. 3. A 2-FVA.
B.2 Closure under basic operations

The class of FVAs with \( \varepsilon \)-transitions will be denoted by \( \varepsilon \text{-FVAs} \).

**Lemma 2.** For a \( \varepsilon \text{-FVA} \( A \) there exists a FVA \( \mathcal{A} \) (without \( \varepsilon \)-transitions) satisfying \( L(\mathcal{A}) = L(A) \).

**Proof.** The construction of a FVA out of a \( \varepsilon \)-FVA is more subtle than the construction known for FAs since we need to take into account the refreshing of the variables. We define an operator \( \Theta \) that transforms a \( \varepsilon \)-FVA to an equivalent \( \varepsilon \)-FVA with strictly less \( \varepsilon \)-transitions. Thus the desired FVA without \( \varepsilon \)-transitions is the least fixed-point of \( \Theta \). Intuitively, the operator \( \Theta \) eliminates all the \( \varepsilon \)-transitions which are preceded by a non-\( \varepsilon \)-transition.

Assume \( A = (\Sigma, \mathcal{X}, Q^\varepsilon, Q^\varepsilon_0, F^\varepsilon, \delta^\varepsilon, \kappa^\varepsilon) \). Let \( \Upsilon(q) \) be the set of states that are reachable from state \( q \) by following an \( \varepsilon \)-transition and let \( \Upsilon(Q') = \{ \Upsilon(q) \mid q \in Q' \} \), for \( Q' \subseteq Q^\varepsilon \). Let \( \Theta(A) = (\Sigma, \mathcal{X}, Q, Q_0, F, \delta, \kappa) \) where:

- \( Q = Q^\varepsilon \cup (Q^\varepsilon \times Q^\varepsilon) \)
- \( \pi_1 : \mathcal{P}(Q) \to \mathcal{P}(Q^\varepsilon) \)
  \( Q' \mapsto \{ p \mid (p, q) \in Q' \} \)
- \( \pi_2 : \mathcal{P}(Q) \to \mathcal{P}(Q^\varepsilon) \)
  \( Q' \mapsto \{ q \mid (p, q) \in Q' \} \)
- \( Q_0 = Q^\varepsilon_0 \cup \Upsilon(Q^\varepsilon_0) \cup \pi_1^{-1}(Q^\varepsilon_0) \)
- \( F = F^\varepsilon \cup \Upsilon^{-1}(F^\varepsilon) \cup \pi_2^{-1}(F^\varepsilon) \)
- \( \delta = \{ (q, q) \in \delta^\varepsilon \mid \alpha \neq \varepsilon \} \cup \{ q_1 \xrightarrow{\alpha} (q_2, q_3) \mid q_1 \xrightarrow{\varepsilon} q_2 \xrightarrow{\varepsilon} q_3 \in \delta^\varepsilon \mid \alpha \neq \varepsilon \} \cup \{ (q_1, q_2) \xrightarrow{\alpha} q_3 \mid q_2 \xrightarrow{\varepsilon} q_3 \in \delta^\varepsilon \} \cup \{ (q_1 \xrightarrow{\varepsilon} q_2) \mid \exists q_0 \xrightarrow{\alpha} q_1 \text{ s.t. } \alpha \neq \varepsilon \} \)
- \( \kappa = \kappa^\varepsilon \cup (\pi_1^{-1} \circ \kappa^\varepsilon) \cup (\pi_2^{-1} \circ \kappa^\varepsilon) \)

In order to prove that \( L(\Theta(A)) = L(A) \), it suffices to prove the following three Claims, the first one is straightforward:

**Claim 1.** Every accepting run in \( A \) that does not follow any \( \varepsilon \)-transition is still an accepting run in \( \Theta(A) \). Conversely, every accepting run in \( \Theta(A) \) that passes only through states in \( Q^\varepsilon \) is still an accepting run in \( A \).

**Claim 2.** There exists a run

\[ q_0, M_0 \xrightarrow{\alpha} q_1, M_1 \xrightarrow{\varepsilon} q_2, M_2 \]

in \( A \) with \( \alpha \neq \varepsilon \) iff there exists a run

\[ q_0, M_0 \xrightarrow{\alpha} (q_1, q_2), M'_2 \]

in \( \Theta(A) \) such that \( M_2 = M'_2 \).

**Proof of the Claim.**
⇒) From the definition of \( Q \) and \( \delta \), it follows that \((q_0, q_1) \in \delta(q_0, \alpha)\), and it remains to show that \( M_2 = M'_2 \). We only discuss the case when \( \alpha \) is a letter in \( \Sigma \), the case when \( \alpha \) is a variable can be handled similarly. On the one hand, \( M_2 = M_{1|D_2} \) where \( D_2 = \text{Dom}(M_1) \setminus (\kappa^\ast)^{-1}(q_2) \), and \( M_1 = M_{2|D_1} \) where \( D_1 = \text{Dom}(M_0) \setminus \kappa^{-1}(q_1) \). Hence \( M_2 = M_{0|D} \) where \( D = \text{Dom}(M_0) \setminus ((\kappa^\ast)^{-1}(q_1) \cup (\kappa^\ast)^{-1}(q_2)) \). On the other hand, we have \( M'_2 = M_0 \setminus D' \), where \( D' = \text{Dom}(M_0) \setminus \kappa^{-1}((q_1, q_2)) \). It follows from the definition of \( \kappa \), the refreshing function of \( \Theta(A') \), that \( \kappa^{-1}((q_1, q_2)) = (\kappa^\ast)^{-1}(q_1) \cup (\kappa^\ast)^{-1}(q_2) \). Hence, \( D = D' \) and \( M_2 = M'_2 \).

\( \Leftarrow \) This direction is proved by following the same reasoning made in the direction \( \Rightarrow \) on the refreshing function.

This ends the proof of Claim 2.

\[ \square \]

**Claim 3.** Let \( q_1 \in Q^\ast \) and \((q_0, q_1) \in Q \). There exists a run

\[ q_1, M_1 \xrightarrow{\alpha} q_2, M_2 \]

in \( A' \) iff there exists a run

\[ (q_0, q_1), M_1 \xrightarrow{\alpha} q_2, M_2 \]

in \( \Theta(A') \).

**Proof of the Claim.** By checking the transition function \( \delta \). \[ \square \]

To accomplish the proof, it remains to notice that if \( q \in Q \) is such that \( q \notin \pi_\ast^{-1}(Q^\ast) \), then the outgoing transitions from \( q \) in \( A' \) are exactly the outgoing transitions from \( q \) in \( \Theta(A') \).

\[ \square \]

**Lemma 3.** 2-FVAs and FVAs are equivalent (i.e. recognize the same languages).

**Proof.** First it is trivial that any language recognized by a FVA \( A \) is also recognized by the 2-FVA \( A' \), a copy of \( A \) in which transitions are indexed by couples \((x, x)\) instead of a variable or constant \( x \).

Now let \( L \) be a language recognized by a 2-FVA \( A \), and let us construct a FVA \( B \) that recognizes \( L \). It suffices to prove that for any word \( \omega \) there is a run of \( A \) that ends in a final state if, and only if, there is a run of \( B \) that also ends in a final state. In order to construct \( B \) we first construct from \( A \) another 2-FVA \( A' \) that recognizes the same language, and such that the translation of \( A' \) into a 1-FVA is trivial. In order to simplify notations, we assume in this proof that the assignment \( M \) on the variables of an automaton is extended by the identity function on the set \( \Sigma_A \) of letters occurring in the 2-FVA.

**Definition of \( \Psi \).** Let \( A = \langle \Sigma, \mathcal{X}, Q, Q_0, \delta, F, \kappa \rangle \), and let \( n_X = |\mathcal{X}| \) and \( n_\Sigma = |\Sigma_A| \), and assume \( \Sigma_A = \{a_1, \ldots, a_n\} \). Let \( \Psi \subseteq \{1, \ldots, n_X + n_\Sigma\}^{\Sigma_A \cup \mathcal{X}} \) be the set of functions from \( \Sigma_A \cup \mathcal{X} \) to \( \{1, \ldots, n_X + n_\Sigma\} \) such that for every \( \psi \in \Psi \) we have \( \psi(a_k) = k \). Furthermore, given \( D \subseteq X \) and \( \psi \in \Psi \), we let \( \Psi^D \) be the subset of \( \Psi \) of functions equal to \( \psi \) on \( \Sigma_A \cup \mathcal{X} \setminus D \). Finally, given a substitution \( M \) on \( \Sigma_A \cup \mathcal{X} \) we let \( \Psi_M \) be the subset of \( \Psi \) of functions \( \psi \) such that, for all \( x, y \in \Sigma_A \cup \text{dom}(M) \), we have \( M(x) = M(y) \text{ iff } \psi(x) = \psi(y) \).
Construction of $\mathcal{A'}$. We let $\mathcal{A'}$ be the 2-FVA automaton $(\Sigma, \mathcal{X}, Q', Q'_0, \delta', F', \kappa')$ where:

$$
\begin{align*}
Q' &= Q \times \Psi \\
\pi : P(Q') &\to P(Q) \\
Q'' &\to \{ q \mid (q, \psi) \in Q'' \}
\end{align*}
$$

and $Q'_0 = \pi^{-1}(Q_0)$, $F' = \pi^{-1}(F)$, and $\kappa' = \pi^{-1} \circ \kappa$. The transition relation $\delta'$ is defined as follows for all $(q_0, \psi_0) \in Q'$ and $\alpha, \beta \in \Sigma_A \cup \mathcal{X}$:

$$
\delta((q_0, \psi_0), (\alpha, \beta)) = \{(q_1, \psi_1) \mid q_1 \in \delta(q_0, (\alpha, \beta)) \text{ and } \psi_0(\alpha) = \psi_0(\beta) \text{ and } \psi_1 \in \psi_0^{\kappa^{-1}}(q_1)\}
$$

Claim. There exists a run $q_0, M_0 \xrightarrow{(\alpha_1, \beta_1)} q_1, M_1 \to \ldots \xrightarrow{(\alpha_n, \beta_n)} q_n, M_n$ in $\mathcal{A}$ iff for all $\psi_n \in \Psi_{M_n}$ there exists a run $(q_0, \psi_0), M_0 \xrightarrow{(\alpha_1, \beta_1)} (q_1, \psi_1), M_1 \to \ldots \xrightarrow{(\alpha_n, \beta_n)} (q_n, \psi_n), M_n$ in $\mathcal{A'}$.

Proof of the claim. We prove the two implications by induction on $n$. The case $n = 0$ is trivial in both cases, so let us focus on the induction step in each direction.

$\Leftarrow$ We note that since $\psi_{M_n}$ is never empty, it suffices to prove the existence of the run in $\mathcal{A}$ for one run in $\mathcal{A'}$. We leave to the reader this verification given the definition of the transition function.

$\Rightarrow$ Assume that for every run of length $n$ in $\mathcal{A}$ and for every possible $\psi_n$ there exists a run as prescribed in $\mathcal{A'}$. Using the above notations, let us extend a run of length $n$ with a transition to $q_{n+1} \in \delta(q_n, (\alpha_{n+1}, \beta_{n+1}))$, and let $M_{n+1}$ be the assignment to variables in $q_{n+1}$. It suffices to prove that for every function $\psi_{n+1} \in \Psi_{M_{n+1}}$ there exists a function $\psi_n \in \Psi_{M_n}$ such that $(q_{n+1}, \psi_{n+1}) \in \delta((q_n, \psi_n), (\alpha_{n+1}, \beta_{n+1}))$.

First let us prove that the subset of functions $\psi_n$ such that there is a transition from $(q_n, \psi_n)$ with the pair $(\alpha_{n+1}, \beta_{n+1})$ is not empty. This set contains all the functions $\psi_n$ such that:

$$
\begin{align*}
x, y \in \Sigma_A \cup \text{dom}(M_n), \psi_n(x) &= \psi_n(y) \iff M_n(x) = M_n(y) \\
\psi_n(\alpha_{n+1}) &= \psi_n(\beta_{n+1})
\end{align*}
$$

Since the transition is feasible on $q_n$ we note that if both $\alpha_{n+1}$ and $\beta_{n+1}$ are in $\Sigma_A \cup \text{dom}(M_n)$ we must have $M_n(\alpha_{n+1}) = M_n(\beta_{n+1})$, and thus the second condition is satisfied. Otherwise, say if $\alpha_{n+1}$ is not in $\Sigma_A \cup \text{dom}(M_n)$, any value is possible for $\psi_n(\alpha_{n+1})$, including the value $\psi_n(\beta_{n+1})$. Thus, there exists some states $(q_{n+1}, \psi_{n+1}) \in \delta'(\alpha_{n}, \beta_{n+1})$ for some $\psi_n$.

Second, let us prove that for every $\psi_{n+1}$ such that for every $\psi_n \in \Psi_{M_{n+1}}$ there exists a $\psi_n$ as above such that $(q_{n+1}, \psi_{n+1}) \in \delta'(\alpha_{n}, \beta_{n+1})$. On the one hand, if a variable $x$ is refreshed and by definition of the transition relation on $\mathcal{A'}$, if $(q_{n+1}, \psi_{n+1})$ is reached then for every $l \in \{1, \ldots, n_\Sigma + n_\mathcal{X}\}$ there exists $\psi'_{n+1}$ equal to $\psi_{n+1}$ but on $x$, where $\psi'_{n+1}(x) = l$. On the other hand, if $x$ is not refreshed, then all the possible values of $\psi_{n+1}(x)$ are also all the possible values of $\psi_n(x)$ for the $\psi_n$ on which the transition is possible. This is easily proved by considering the three cases $x \in \text{dom}(M_{n+1}) \cap \ldots$
Lemma 3 and keeping the remaining $n$ concatenation is a direct consequence of the fact that FVAs with $\varepsilon$ the disjoint union of the two FVAs. The closure under Kleene operation and concatenation of two FVAs that do not share variables.

Proof. A product, which is a 2-FVA. Formally, let $A \times A$ be two FVAs, where $A$ denoting either $A_1 \times A_2$, $A_1 = \langle \Sigma_1, Q_1, q_0^1, F_1, \delta_1, \alpha_1, \beta_1 \rangle$ and $A_2 = \langle \Sigma_2, Q_2, q_0^2, F_2, \delta_2, \alpha_2, \beta_2 \rangle$ be two FVAs, where $X_1 \cap X_2 = \emptyset$. The 2-FVA $A_1 \times A_2$ is defined by:

$$\langle \Sigma_1 \cup \Sigma_2, Q_1 \times Q_2, q_0^1 \times q_0^2, \delta, F_1 \times F_2, \kappa \rangle,$$

where $\delta$ and $\kappa$ are defined by:

$$\begin{cases} (q_1', q_2') \in \delta((q_1, q_2), (\alpha_1, \alpha_2)) & \text{iff } q_1' \in \delta_1(q_1, \alpha_1) \text{ and } q_2' \in \delta_2(q_2, \alpha_2). \\ (q_1, q_2) \in \kappa(x) & \text{iff } q_1 \in \kappa_1(x) \text{ or } q_2 \in \kappa_2(x). \end{cases}$$

The closure under intersection for FVAs follows from Lemma 3 and the following Fact:
Fact 3 Let $A_1$ and $A_2$ be FVAs. Then, $L(A_1) \cap L(A_2) = L(A_1 \times A_2)$.

This ends the proof of Theorem 2. \hfill \box

Lemma 4. FVAs are not closed under complementation.

Proof. As a counter example we consider the language $L = \{a\}$, with $a \in \Sigma$. The complement of $L$ is the language $L_2 \uplus L_1$ where $L_2$ consists of all the words of length greater (or equal) than 2 and $L_1$ consists of all the words of length 1 in which the letter differs from $a$, i.e. $L_2 = \{a_1a_2\ldots a_n | a_i \in \Sigma, n \geq 2\}$ and $L_1 = \{a_1 | a_1 \in \Sigma \setminus \{a\}\}$. The language $L_2$ can be recognized by a FVA. In order to show that $L_1 \uplus L_2$ is not FVA-recognizable, it suffices to show that $L_1$ is not FVA-recognizable. Towards a contradiction: assume that $L_1$ can be recognized by a FVA $B$ without $\varepsilon$-transitions. Hence $B$ must contain transitions of length 1 linking an initial state to an accepting state. On the one hand, each transition of $B$ can not be labeled by a variable, otherwise $B$ could accept words not in $L_2$. On the other hand, all the transitions of $B$ must be labeled by letters in $\Sigma \setminus \{a\}$, but this is impossible since $\Sigma$ is infinite. \hfill \box

B.3 Nonemptiness and membership

Theorem 3. For FVAs, Nonemptiness is NL-complete and Membership is NP-complete.

Proof. For Nonemptyness, let $A$ be a FVA and let $F(A)$ be FA obtained from $A$ by considering all the variables as letters. Notice that $F(A)$ is nonempty iff $A$ is nonempty. The complexity follows from the fact that FA nonemptiness is NL-complete. \hfill \box

For Membership, consider a FVA $A$ and a word $w$. For the upper bound, a non deterministic polynomial algorithm guesses a path in $A$ of length $|w|$ such that the final state is accepting, then checks wether the corresponding run on $w$ is possible. The lower bound is shown by a reduction from the Hamiltonian cycle problem for digraphs as in the extended version of [10]. \hfill \box

B.4 Containment

Lemma 5. Let $A$ be a FVA and $F$ be a FA. Then, $L(A) \cap L(F)$ is regular. If $L(A) = L(F)$ then all the paths of $A$ linking an initial state to a final state are labeled with letters.

Proof. The first claim follows from the proof of Theorem 2 the construction of $A \cap F$ yields a FVA in which all the transitions are labeled with letters.

For the second claim, assume that the regular language $L(F)$ is over a finite alphabet $\Sigma_f$. Towards a contradiction: Let $q_1 \xrightarrow{a_1} \ldots \xrightarrow{a_m} q_m \xrightarrow{x} \ldots q_k$ be a path in $A$ such that $q_1$ (resp. $q_k$) is an initial (resp. final) state, $x$ is a variable, and for every $i \leq m$, $a_i$ is a letter. Indeed, this path recognizes a word $w = w_1 \ldots w_k$ that does not belong to $L(F)$, e.g. by choosing $w_{m+1} \notin \Sigma_f$. This is a contradiction. \hfill \box
Theorem 4. The containment problems between a FVA and a FA are decidable.

Proof. Let \( A \) be FVA and \( F \) be a FA.

For the inclusion \( L(F) \subseteq L(A) \), we check whether \( L(F) \cap L(A) = L(F) \). From Lemma \( \ref{lem:inclusion} \) it follows that the language \( L(F) \cap L(A) \) is regular and the FA recognizing it can be constructed. Hence, the inclusion above amounts to checking the inclusion of two FAs, which is decidable.

For the inclusion \( L(A) \subseteq L(F) \), we check whether \( L(A) \cap L(F) = L(A) \). On the one hand, it follows from Lemma \( \ref{lem:inclusion} \) that \( L(A) \cap L(F) \) is regular. On the other hand, it follows from Lemma \( \ref{lem:inclusion} \) that all the (accessible) transitions of \( A \) must be labeled with letters, since \( L(A) \) is regular. Hence, the inclusion above amounts to checking the inclusion of two FAs. \( \square \)

C Appendix for Section \( \ref{sec:appendix} \)

The claims in the following remark are not hard to prove.

Remark 1. Let \( C \subseteq \Sigma \) be a finite set of letters, \( \bar{\sigma} \) and \( \sigma \) two substitutions, \( x \) a variable, and \( a \) a letter in \( C \). The following hold. If \( \bar{\sigma} \cong_C \sigma \) then \( |\text{dom}(\bar{\sigma})| = |\text{dom}(\sigma)| \) and \( \bar{\sigma}_{|D} \cong_C \sigma_{|D} \), where \( D \subseteq \text{Dom}(\sigma) \). Consequently, if \( (\bar{\sigma}_1 \circ \bar{\sigma}_2) \cong (\sigma_1 \circ \sigma_2) \) with \( \text{dom}(\bar{\sigma}_i) = \text{dom}(\sigma_i) \) for \( i = 1, 2 \).

Proposition 1. Let \( A_1 = (\Sigma_1, X_1, Q_1, q_0^1, \delta_1, F_1, \kappa_1) \) and \( A_2 = (\Sigma_2, X_2, Q_2, q_0^2, \delta_2, F_2, \kappa_2) \) be two FVAs. Then Eloise has a winning strategy in \( \mathcal{G}(A_1, A_2) \) iff she has a winning strategy in \( \mathcal{G}(A_1, A_2) \).

Proof. Up to variables renaming, we can assume that \( X_1 \cap X_2 = \emptyset \). For the direction "\( \Rightarrow \)" we show that out of a winning strategy of Eloise in \( \mathcal{G}(A_1, A_2) \) we construct a winning strategy for her in \( \mathcal{G}(A_1, A_2) \). For this purpose, we shall show that each move of Abelard in \( \mathcal{G}(A_1, A_2) \) can be mapped to an Abelard move in \( \mathcal{G}(A_1, A_2) \), and Eloise response in \( \mathcal{G}(A_1, A_2) \) can be actually mapped to an Eloise move in \( \mathcal{G}(A_1, A_2) \). This mapping defines a relation \( R \) between the positions of \( \mathcal{G}(A_1, A_2) \) and the positions of \( \mathcal{G}(A_1, A_2) \) as follows:

\[
R \subseteq \text{Pos}_E(\mathcal{G}(A_1, A_2)) \times \text{Pos}_E(\mathcal{G}(A_1, A_2)) \cup \text{Pos}_A(\mathcal{G}(A_1, A_2)) \times \text{Pos}_A(\mathcal{G}(A_1, A_2))
\]

Furthermore, we impose that the following invariant holds:

If \( (\bar{\nu}, \nu) \in R \) then \( \bar{\nu} \cong_C \nu \), \( (\text{Inv-}C) \)

where \( C = \Sigma_{A_1} \cup \Sigma_{A_2} \). In this proof, we shall simply write "\( \cong \)" instead of "\( \cong_C \)". We recall that the variables in \( \mathcal{G}(A_1, A_2) \) are instantiated from the set of letters \( C_0 = \Sigma_{A_1} \cup \Sigma_{A_2} \cup (X_1 \times X_2) \cup (X_2 \times X_1) \). The proof is by induction on \( n \).

More precisely, if \( (\bar{\nu}, \nu) \in R \), and the move \( \bar{\nu} \xrightarrow{\delta} \bar{\nu}' \) is mapped to \( \nu \xrightarrow{\bar{\delta}} \nu' \), or \( \nu \xrightarrow{\delta} \nu' \) is mapped to \( \bar{\nu} \xrightarrow{\bar{\delta}} \bar{\nu}' \), then \( (\bar{\nu}', \nu') \in R \)
Let $\bar{m}$ be the number of the moves made in $\mathcal{G}(A_1, A_2)$ plus the number of moves made in $\mathcal{G}(A_1, A_2)$. The base case, i.e. when $n = 0$, trivially holds since the starting position of $\mathcal{G}(A_1, A_2)$ and of $\mathcal{G}(A_1, A_2)$ is $(q_0', q_0')$.

For the induction case let $(\bar{\phi}_n, \bar{\varphi}_n) \in \mathcal{R}$. We consider two possibilities: when $\bar{\phi}_n$ and $\bar{\varphi}_n$ are both Abelard positions and when they are both Eloise positions. Consider the first possibility and an Abelard move $\bar{m} = \bar{\phi}_n \rightarrow \bar{\phi}_{n+1}$ in $\mathcal{G}(A_1, A_2)$. We distinguish two cases depending on $\bar{m}$.

**Case (i).** If $\bar{m} \in M^1_A$, then $\bar{m}$ is of the form:

$$\bar{m} = ((\bar{\sigma}_1, q_1), (\bar{\sigma}_2, q_2)) \rightarrow ((\bar{\sigma}_{1\dagger}, q'_1), (\bar{\sigma}_2, q_2), (\bar{\sigma}_1, ?\alpha))_\Sigma$$

where $q'_1 \in \delta_1(q_1, ?\alpha)$ and $D = \text{Dom}(\bar{\sigma}_1) \setminus \kappa_1^{-1}(q'_1)$

From the induction hypothesis we have $\bar{\phi}_n \equiv \bar{\varphi}_n$, hence $\bar{\varphi}_n = ((\sigma_1, q_1), (\sigma_2, q_2))_A$ such that $(\bar{\sigma}_1 \equiv \bar{\sigma}_2) \equiv (\sigma_1 \equiv \sigma_2)$. Thus Abelard move in $\mathcal{G}(A_1, A_2)$ is

$$((\sigma_1, q_1), (\sigma_2, q_2))_A \rightarrow ((\sigma_{1\dagger}, q'_1), (\sigma_2, q_2), (\sigma_1, ?\alpha))_\Sigma$$

and the invariant $\text{Inv-$\text{x}$}$ is maintained.

**Case (ii).** If $\bar{m} \in M^1_A$, then $\bar{m}$ is of the form:

$$((\bar{\sigma}_1, q_1), (\bar{\sigma}_2, q_2))_A \rightarrow (((\bar{\sigma}_1 \equiv \bar{\gamma}), q'_1), (\bar{\sigma}_2, q_2), (\bar{\gamma} \equiv \bar{\sigma}_1, !\alpha))_\Sigma$$

where $q'_1 \in \delta_1(q_1, !\alpha)$, $D = \text{Dom}(\bar{\sigma}_1 \equiv \bar{\gamma}) \setminus \kappa_1^{-1}(q'_1)$

and $\bar{\gamma} : \mathcal{V}(\bar{\sigma}_1(\alpha)) \rightarrow C_0$

The only relevant situation is when $\bar{\sigma}_1(\alpha)$ is a variable, say $x_1 \in X_1$. The situation when it is a letter is similar to the previous case since $\bar{\gamma} = \emptyset$. From the induction hypothesis we have that $\bar{\phi}_n \equiv \bar{\varphi}_n$, and hence $\bar{\varphi}_n = ((\sigma_1, q_1), (\sigma_2, q_2))_A$ such that $(\bar{\sigma}_1 \equiv \bar{\sigma}_2) \equiv (\sigma_1 \equiv \sigma_2)$. Therefore the corresponding Abelard move in $\mathcal{G}(A_1, A_2)$ is

$$((\sigma_1, q_1), (\sigma_2, q_2))_A \rightarrow (((\sigma_1 \equiv \gamma), q'_1), (\sigma_2, q_2), (\gamma \equiv \sigma_1, !\alpha))_\Sigma$$

where $\gamma : \mathcal{V}(\sigma_1(\alpha)) \rightarrow \Sigma$ is a (ground) substitution that will be defined next. Since $\bar{\sigma}_1 \equiv \bar{\sigma}_3$, and $\bar{\sigma}_3(\alpha)$ is the variable $x_1$, then it follows that $\bar{\sigma}_1(\alpha) = \sigma_1(\alpha) = \alpha = x_1$. Abelard choice of $\gamma$ depends on the nature of $\bar{\gamma}(x_1)$.

- If $\bar{\gamma}(x_1) \in \Sigma_{A_1} \cup \Sigma_{A_2}$, then in this case we let $\gamma := \bar{\gamma}$, and hence the invariant $\text{Inv-$\text{x}$}$ is maintained, i.e. $(\bar{\sigma}_1 \equiv \bar{\sigma}_2 \equiv \bar{\gamma}) \equiv (\sigma_1 \equiv \sigma_2 \equiv \gamma)$.
- If $\bar{\gamma}(x_1)$ appears in the current position, i.e.

$$\bar{\gamma}(x_1) \in (\text{codom}(\bar{\sigma}_1 \equiv \bar{\sigma}_2)) \setminus (\Sigma_{A_1} \cup \Sigma_{A_2}),$$

then there is a variable $y \in \text{dom}(\bar{\sigma}_1 \equiv \bar{\sigma}_2)$ such that $(y \mapsto \bar{\gamma}(x_1)) \in \bar{\sigma}_1 \equiv \bar{\sigma}_2$. Since $(\bar{\sigma}_1 \equiv \bar{\sigma}_2) \equiv (\sigma_1 \equiv \sigma_2)$, then it follows that there is a letter $y_0 \in \Sigma_0$ such that $(y \mapsto y_0) \in \sigma_1 \equiv \sigma_2$. Thus we let $\gamma := \{x_1 \mapsto y_0\}$ and the invariant $\text{Inv-$\text{x}$}$ is maintained, i.e. $(\bar{\sigma}_1 \equiv \bar{\sigma}_2 \equiv \bar{\gamma}) \equiv (\sigma_1 \equiv \sigma_2 \equiv \gamma)$.  

– Otherwise, i.e. \( \bar{\gamma}(x_1) \) is a new letter that does not appear in the current position, then we take \( \gamma(x_1) \) as a new letter from \( \Sigma_0 \), and hence the invariant \( \text{Inv-\textbullet} \) is maintained.

Secondly, we consider the possibility when both \( \bar{\varphi}_n \) and \( \varphi_n \) are Eloise positions. We consider an Eloise move \( m = \varphi_n \rightarrow \varphi_{n+1} \) in \( G(A_1, A_2) \), and we describe the corresponding Eloise move in \( \mathfrak{T}(A_1, A_2) \). We distinguish two cases depending on \( m \).

Case (i). If \( m \in M'_E \), then \( m \) is of the form:

\[
((\sigma_1, q_1), (\sigma_2, q_2), (\sigma_3, !\alpha)) \rightarrow ((\sigma_1, q_1), ((\sigma_2 \uplus \sigma) \uplus \alpha) \uplus q_2')\]

where \( q_2' \in \delta_2(q_2, ?\beta) \),

\[
D = \text{Dom}(\sigma_2 \uplus \sigma) \setminus \kappa_2^{-1}(q_2'), \quad \text{and}
\]

\[
\sigma(\sigma_2(\beta)) = \sigma_3(\alpha), \quad \text{for a substitution} \ \sigma
\]

Recall that \( \sigma_3(\alpha) \) is a letter. From the induction hypothesis we have that \( \bar{\varphi}_n \uplus \varphi_n \), therefore \( \bar{\varphi}_n = ((\bar{\sigma}_1, q_1), (\bar{\sigma}_2, q_2), (\bar{\sigma}_3, !\alpha)) \) such that

\[
((\bar{\sigma}_1 \uplus \bar{\sigma}_3) \uplus \bar{\sigma}_2) \uplus ((\sigma_1 \uplus \sigma_3) \uplus \sigma_2).
\]

The corresponding move \( \bar{m} \) in \( \mathfrak{T}(A_1, A_2) \) is:

\[
((\bar{\sigma}_1, q_1), (\bar{\sigma}_2, q_2), (\bar{\sigma}_3, !\alpha)) \rightarrow ((\bar{\sigma}_1, q_1), ((\bar{\sigma}_2 \uplus \bar{\sigma}) \uplus \alpha) \uplus q_2')\]

where \( \mathfrak{T} \) is a (possibly trivial) substitution such that \( \bar{\sigma}(\sigma_2(\beta)) = \sigma_3(\alpha) \). But we show that such a substitution exists and that the invariant \( \text{Inv-\textbullet} \) is maintained.

Notice that \( \sigma_2(\beta) \) is a variable iff \( \sigma_2(\beta) \) is a variable, and if so then \( \bar{\sigma}_2(\beta) = \sigma_2(\beta) \), since \( \bar{\sigma}_2 \uplus \sigma_2 \). Hence, we shall show that the invariant is maintained only when \( \sigma_2(\beta) \) and \( \bar{\sigma}_2(\beta) \) are variables. We distinguish two cases according to the nature of \( \sigma_2(\beta) \):

– If \( \sigma_2(\beta) \) is a variable, say \( x_2 \in \mathcal{X}_2 \), (i.e. \( x_2 \notin \text{dom}(\sigma_2) \)), then \( \bar{\sigma}_2(\beta) = \sigma_2(\beta) = \beta = x_2 \). We must show that ((\bar{\sigma}_1 \uplus \{x_2 \mapsto \bar{\sigma}_3(\alpha)\}) \uplus \bar{\sigma}_3) \uplus (\sigma_1 \uplus \{x_2 \mapsto \sigma_3(\alpha)\} \uplus \sigma_2). Since we already know that ((\bar{\sigma}_1 \uplus \bar{\sigma}_3) \uplus \bar{\sigma}_2 \uplus (\sigma_1 \uplus \sigma_3) \uplus \sigma_2) then the claim follows from the following fact:

**Fact 5** Let \( \bar{\sigma} \) and \( \sigma \) be two substitutions. If \( \bar{\sigma} \uplus \sigma \), and \( x \in \text{dom}(\sigma) \) and \( z \notin \text{dom}(\sigma) \), then \( \bar{\sigma}[z := x] \uplus \sigma[z := x] \) stands for the replacement of \( x \) by \( y \) in \( \sigma \).

– If \( \sigma_2(\beta) \) is a letter, then \( \sigma_2(\beta) = \sigma_3(\alpha) \). We distinguish two cases depending on \( \sigma_3(\alpha) \):

  • If \( \sigma_3(\alpha) \in \Sigma_{\Lambda_1} \cup \Sigma_{\Lambda_2} \) (and so \( \sigma_2(\beta) \)), then on the one hand, \( \bar{\sigma}_3(\alpha) = \sigma_3(\alpha) \), since \( \bar{\sigma}_3 \uplus \sigma_3 \), and on the other hand, \( \bar{\sigma}_2(\beta) = \sigma_2(\beta) \) since \( \bar{\sigma}_2 \uplus \sigma_2 \). Therefore \( \bar{\sigma}_3(\alpha) = \sigma_2(\beta) \), and we are done.

  • If \( \sigma_3(\alpha) \in \Sigma \setminus \{\Sigma_{\Lambda_1} \cup \Sigma_{\Lambda_2}\} \), then \( \alpha \) must be a variable, say \( x_1 \in \mathcal{X}_1 \). In this case \( \beta \) is also a variable, say \( x_2 \in \mathcal{X}_2 \), since \( \sigma_2(\beta) = \sigma_3(\alpha) \). Notice that, on the one hand, \( \{x_1 \mapsto \sigma_3(\alpha), x_2 \mapsto \sigma_3(\alpha)\} \) appears in the position
\( \varphi_n \), i.e. \( \{x_1 \mapsto \sigma_3(\alpha), x_2 \mapsto \sigma_3(\alpha)\} \subseteq \sigma_1 \cup \sigma_2 \cup \sigma_3 \). On the other hand, \( \{x_1 \mapsto \bar{\sigma}_3(\alpha), x_2 \mapsto \bar{\sigma}_2(\beta)\} \) also appears in \( \bar{\varphi}_n \), i.e. \( \{x_1 \mapsto \bar{\sigma}_3(\alpha), x_2 \mapsto \bar{\sigma}_3(\beta)\} \subseteq \bar{\sigma}_1 \cup \bar{\sigma}_2 \cup \bar{\sigma}_3 \). Therefore \( \bar{\sigma}_2(\alpha) = \bar{\sigma}_3(\beta) \), since \( (\bar{\sigma}_1 \cup \bar{\sigma}_2 \cup \bar{\sigma}_3) \not\equiv (\sigma_1 \cup \sigma_2 \cup \sigma_3) \).

**Case (ii).** If \( m \in M_2^E \), then in this case this move is of the form

\[
\begin{align*}
((\sigma_1, q_1), (\sigma_2, q_2), (\sigma_3, ?\alpha))_e & \rightarrow (((\sigma_1 \uplus \sigma)|_{D_1}, q_1), ((\sigma_2 \uplus \gamma)|_{D_2}, q_2'))_A \\
\text{where } q_2' & \in \bar{\delta}_2(q_2, \bar{\beta}), \\
D_1 & = \text{Dom}(\sigma_1 \uplus \sigma) \setminus \kappa_1^{-1}(q_1), \\
D_2 & = \text{Dom}(\sigma_2 \uplus \gamma) \setminus \kappa_2^{-1}(q_2'), \\
\sigma(\sigma_3(\alpha)) & = \gamma(\sigma_2(\beta)), \text{ and} \\
\gamma : \mathcal{V}(\sigma_2(\beta)) & \rightarrow \Sigma.
\end{align*}
\]

From the induction hypothesis we have that \( \bar{\varphi}_n \not\equiv \bar{\varphi}_n \), therefore \( \bar{\varphi}_n = \{(\bar{\sigma}_1, q_1), (\bar{\sigma}_2, q_2), (\bar{\sigma}_3, ?\alpha)\}_e \) such that \( (\sigma_1 \cup \sigma_3) \uplus \bar{\sigma}_2 \not\equiv (\sigma_1 \cup \sigma_3) \uplus \sigma_2 \). The corresponding Elise move in \( \mathcal{G}(A_1, A_2) \) is:

\[
\begin{align*}
((\bar{\sigma}_1, q_1), (\bar{\sigma}_2, q_2), (\bar{\sigma}_3, ?\alpha))_e & \rightarrow (((\bar{\sigma}_1 \uplus \bar{\sigma})|_{D_1}, q_1), ((\bar{\sigma}_2 \uplus \bar{\gamma})|_{D_2}, q_2'))_A \\
\text{where } \bar{\sigma}(\bar{\sigma}_3(\alpha)) & = \bar{\gamma}(\bar{\sigma}_2(\beta))
\end{align*}
\]

and the (ground) substitution \( \bar{\gamma} : \mathcal{V}(\bar{\sigma}_3(\alpha)) \rightarrow C_0 \) by Elise will be defined next, provided that the invariant \( \text{inv}\)-\( \mathcal{V} \) is maintained. Notice that maintaining this invariant makes sense only when \( \sigma_3(\alpha) \) or \( \sigma_2(\beta) \) is a variable. The choice of \( \bar{\gamma} \) depends on \( \sigma_3(\alpha) \).

- If \( \sigma_3(\alpha) \in \Sigma_{A_1} \cup \Sigma_{A_2} \), then this case is straightforward.
- If \( \sigma_3(\alpha) \in \Sigma \setminus (\Sigma_{A_1} \cup \Sigma_{A_2}) \), then \( \alpha \) must be a variable, say \( y_1 \in X_1 \). We distinguish two cases depending on \( \sigma_2(\beta) \).
  - If \( \sigma_2(\beta) \) is a letter then in this case \( \sigma_2(\beta) = \sigma_3(\alpha) \), and hence \( \gamma = \sigma = \emptyset \). Thus we take \( \bar{\gamma} = \sigma = \emptyset \) and we must show next \( \bar{\sigma}_3(\alpha) = \bar{\sigma}_2(\beta) \). Notice that \( \beta \) must be a variable, say \( y_2 \in X_2 \). Since \( \{y_1 \mapsto \sigma_3(\alpha), y_2 \mapsto \sigma_2(\beta)\} \) (resp. \( \{y_1 \mapsto \bar{\sigma}_3(\alpha), y_2 \mapsto \bar{\sigma}_2(\beta)\} \)) appears in the position \( \varphi_n \) (resp. \( \bar{\varphi}_n \)), and \( \sigma_3(\alpha) = \sigma_2(\beta) \) then \( \bar{\sigma}_3(\alpha) = \bar{\sigma}_2(\beta) \), since \( \bar{\varphi}_n \not\equiv \varphi_n \).
  - If \( \sigma_2(\beta) \) is a variable, say \( y_2 \in X_2 \), then \( \bar{\sigma}_2(\beta) = \sigma_2(\beta) = \beta = y_2 \), since \( \bar{\sigma}_2 \not\equiv \sigma_2 \). In this case we have \( \gamma = \{y_2 \mapsto \sigma_3(\alpha)\} \) and \( \sigma = \emptyset \). Thus we take \( \bar{\gamma} = \{y_2 \mapsto \bar{\sigma}_3(\alpha)\} \). And the invariant \( \text{inv}\)-\( \mathcal{V} \) is maintained.

- If \( \sigma_3(\alpha) \) is a variable, say \( x_1 \in X_1 \), then \( \bar{\sigma}_3(\alpha) = \sigma_3(\alpha) = \alpha = x_1 \). We distinguish two cases depending on the nature of \( \sigma_2(\beta) \).
  - If \( \sigma_2(\beta) \) is a letter then \( \bar{\sigma}_2(\beta) \) is a letter as well since \( \bar{\sigma}_2 \not\equiv \sigma_2 \). In this case \( \gamma = \emptyset \) and \( \sigma = \{x_1 \mapsto \sigma_2(\beta)\} \). Therefore we take \( \bar{\gamma} = \emptyset \) and \( \bar{\sigma} = \{x_1 \mapsto \bar{\sigma}_2(\beta)\} \).
  - If \( \sigma_2(\beta) \) is a variable, say \( y_2 \in X_2 \), then \( \bar{\sigma}_2(\beta) = \sigma_2(\beta) = \beta = y_2 \) since \( \bar{\sigma}_2 \not\equiv \sigma_2 \). Assume that \( \gamma = \{y_2 \mapsto y_0\} \), where \( y_0 \in \Sigma \) is a letter. In this case we take \( \bar{\gamma} = \{y_2 \mapsto \bar{y}_0\} \), where the choice of the letter \( \bar{y}_0 \in C_0 \) depends on \( y_0 \).

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* If \( y_0 \in \Sigma_{A_1} \cup \Sigma_{A_2} \) then we let \( \bar{y}_0 := y_0 \).

* If \( y_0 \in \text{codom}(\sigma_1 \uplus \sigma_2) \setminus (\Sigma_{A_1} \cup \Sigma_{A_2}) \) then there must exist a variable \( z \in X_1 \cup X_2 \) and a letter \( z_0 \in C_0 \) such that \((z \mapsto y_0) \in \sigma_1 \uplus \sigma_2 \) and \((z \mapsto z_0) \in (\bar{\sigma}_1 \uplus \bar{\sigma}_2) \). We let \( \bar{y}_0 := z_0 \).

* Otherwise, i.e. \( y_0 \) is a fresh letter that does not appear in \( \varphi_n \), then \( \bar{y}_0 \) must be a fresh letter as well. Since

\[
|\text{codom}(\bar{\sigma}_1 \uplus \bar{\sigma}_2)| \leq |X_1| + |X_2| - 1 < |C_0 \setminus (\Sigma_{A_1} \cup \Sigma_{A_2})|
\]

then \( \text{codom}(\bar{\sigma}_1 \uplus \bar{\sigma}_2) \subseteq C_0 \setminus (\Sigma_{A_1} \cup \Sigma_{A_2}) \). Hence we take \( \bar{y}_0 \) as an arbitrary element of the non empty set

\[
C_0 \setminus (\Sigma_{A_1} \cup \Sigma_{A_2} \cup \text{codom}(\bar{\sigma}_1 \uplus \bar{\sigma}_2))
\]

The proof of the direction \( "\Rightarrow" \) is dual w.r.t. the proof of the direction \( "\Leftarrow" \). That is, it can be obtained by replacing Eloise by Abelard, and Abelard by Eloise and keeping the same instantiation strategy and the definition of the \( \kappa \)-coherence. This ends the proof of the Proposition. \( \square \)

**D Appendix for Section 7**

We extend CFVAs so that the transitions are labeled with arbitrary terms over a first-order signature, besides the communication symbols indeed. This extended model is called ECFVA.

Let \( X \) be a finite set of variables, \( \Sigma \) a set of function symbols. Let \( T(\Sigma, X) \) denote the set of terms built out of the symbols in \( \Sigma \) and the variables in \( X \). We shall denote by \( T(\Sigma, X) \) the set \( \{!, ?\} \times T(\Sigma, X) \), where \( \{!, ?\} \cap (\Sigma \cup X) = \emptyset \).

If \( t \in T(\Sigma, X) \) then \( !t \) (resp. \(?t\)) denotes sending (receiving) the message \( t \). A matching problem of a term \( t \) by a term \( u \), denoted by \( t \lessgtr u \), is solvable iff there is a substitution \( \sigma \) such that \( \sigma(t) = u \). The set of solutions of \( t \lessgtr u \) is denoted by \( t \lessgtr u \).

The definition of ECFVAs follows.

**Definition 10.** A ECFVA is a tuple \( A = \langle \Sigma, X, Q, Q_0, \delta, F, \kappa \rangle \) where \( \Sigma \) is a denumerable set of functional symbols, \( X \) is a finite set of variables, \( Q \) is a finite set of states, \( Q_0 \subseteq Q \) is a set of initial states, \( \delta = Q \times T(\Sigma, X) \rightarrow 2^Q \) is a transition function, \( F \subseteq Q \) is a set of accepting states, and \( \kappa : X \rightarrow 2^Q \) is the refreshing function that associates to every variable the (possibly empty) set of states where it is refreshed.

We define the mirror of a word \( w = l_1 \cdot l_2 \cdot l_n \ldots \) as the word \( \bar{w} = l_n \cdot l_{n-1} \cdot \ldots \).

The definition of configuration and run for ECFVAs follows.

**Definition 11.** Let \( A = \langle \Sigma, X, Q, Q_0, \delta, F, \kappa \rangle \) be a ECFVA. A configuration is a pair \( (q, M) \) where \( q \in Q \) and \( M : X \rightarrow \Sigma \) is a partial function. We define a transition relation over the configurations as follows: \( (q_1, M_1) \rightarrow^u (q_2, M_2) \),
where \( u \in \mathcal{T}(\Sigma) \), iff there exist a term \( t \in \mathcal{T}(\Sigma, \mathcal{X}) \), such that \( q_2 \in \delta(q_1, t) \), and a substitution \( \sigma = (M_1(t) \ll \tilde{u}) \) so that \( M_2 = (M_1 \uplus \sigma)|_D \), where \( D = \text{dom}(M_1 \uplus \sigma) \setminus \kappa^{-1}(q_2) \). A finite word \( u = u_1u_2 \ldots u_n \in \mathcal{T}(\Sigma)^* \) is recognized by \( \mathcal{A} \) iff there exists a run \((q_0, M_0) \xrightarrow{u_1} (q_1, M_1) \xrightarrow{u_2} \ldots \xrightarrow{u_n} (q_n, M_n)\), such that \( M_0 = \emptyset \), \( q_0 \in Q_0 \) and \( q_n \in F \). The set of words recognized by \( \mathcal{A} \) is denoted by \( L(\mathcal{A}) \).

**Definition 12.** The asynchronous product \(*\) of \( n \) ECFVAs \( \mathcal{A}_i = (\Sigma_i, \mathcal{X}_i, Q_i, Q_0^i, \delta_i, F_i, \kappa_i) \) is an ECFVA: \( \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n = (\Sigma, \mathcal{X}, Q, \delta, F, \kappa) \), where:

- \( \Sigma = \cup_{i=1,\ldots,n} \Sigma_i \),
- \( \mathcal{X} = \cup_{i=1,\ldots,n} \mathcal{X}_i \),
- \( Q = Q_1 \times \cdots \times Q_n \),
- \( Q_0 = Q_0^1 \times \cdots \times Q_0^n \), \( F = F_1 \times \cdots \times F_n \),
- \( \delta \) is defined by: \( q \in \delta(p, t) \) iff for some \( i \), \( \pi_i(q) \in \delta_i(\pi_i(p), t) \), and for all \( j \neq i \) we have that \( \pi_j(q) = \pi_j(p) \), where \( \pi_i \) denotes the projection along the \( i^{th} \)-component, and
- \( \kappa \) is defined by: \( p \in \kappa(x) \) iff for some \( i \), \( \pi_i(p) \in \kappa_i(x) \).

### D.1 Undecidability of the \( \mathcal{G} \)-simulation problem for ECFVAs

**Theorem 6.** The \( \mathcal{G} \)-simulation is undecidable for ECFVAs in which the labels are terms over a signature containing a unary symbol.

We reduce the halting problem of 2 counter machines to the simulation problem for ECFVAs. Let us consider a deterministic 2-counter machine \( M \) with set of states \( Q \) and such that \( q_0 \) is the initial state and \( q_f \) the final one (from where no transition is possible). A configuration of the machine can be represented by a term \( q(s^n(0), s'^n(0)) \) where \( q \) is the state, and \( n \) (resp. \( m \)) the value of the first (resp. second) counter. The initial configuration of \( M \) is \( q_0(s^0(0), s'^0(0)) \) We encode every transition \( t : q(u, v) \rightarrow q'(u', v') \) of the machine by a (deterministic) ECFVAs \( \mathcal{A}_t \) as follows (we consider only the cases when the first counter is incremented, decremented or tested, the cases for the second counter are analogous): \( \Sigma_t = \{ q, q' \} \cup \{ s, 0 \}, \mathcal{X}_t \) is a finite set of variables, \( Q_t = \{ p_0^t, p_1^t, p_2^t, p_3^t, p_4^t, p_5^t \} \) and the set of transitions \( \delta_t \) (where \( u, v \in \mathcal{X}_t \)):

| Instruction \( l \) | Set of transitions \( \delta_t \) |
|-------------------|-------------------|
| \( q(u, v) \rightarrow q'(s(u), v) \) | \( \{ p_0^t \rightarrow \tilde{a}, p_1^t \rightarrow \tilde{b}, p_2^t \rightarrow \tilde{c}, p_3^t \rightarrow \tilde{d}, p_4^t \rightarrow \tilde{e}, p_5^t \rightarrow \tilde{f}, p_6^t \rightarrow \tilde{g} \} \) |
| \( q(s(u), v) \rightarrow q'(u, v) \) | \( \{ p_0^t \rightarrow \tilde{a}, p_1^t \rightarrow \tilde{b}, p_2^t \rightarrow \tilde{c}, p_3^t \rightarrow \tilde{d}, p_4^t \rightarrow \tilde{e}, p_5^t \rightarrow \tilde{f}, p_6^t \rightarrow \tilde{g} \} \) |
| \( q(0, v) \rightarrow q'(u, v) \) | \( \{ p_0^t \rightarrow \tilde{a}, p_1^t \rightarrow \tilde{b}, p_2^t \rightarrow \tilde{c}, p_3^t \rightarrow \tilde{d}, p_4^t \rightarrow \tilde{e}, p_5^t \rightarrow \tilde{f}, p_6^t \rightarrow \tilde{g} \} \) |

\(^4\) Up to variable renaming, we assume that \( \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \), for all \( i \neq j \).
Now we build a client automata \( C_M \) such that \( \Sigma = Q \cup \{ s, 0 \} \), \( X \) is a finite set of variables, the set of states is \( Q_M = \{ I, F, c^0, c^1, c^2, c^3, c^4, c^5 \} \), \( I \) is the unique initial state and all states are accepting.

The set of transitions of \( C \) is the union of the following ones (where \( u, v \in X \)):

\[
\text{Initial sequence: } \{ I \xrightarrow{1^0} I', I' \xrightarrow{1^x(0)} I^v, I^v \xrightarrow{1^x(0)} c^0 \}
\]

For all \( q \in Q \):

\[
\{ c^0 \xrightarrow{\gamma_q} c^1, c^1 \xrightarrow{\gamma_u} c^2, c^2 \xrightarrow{\gamma_v} c^3, c^3 \xrightarrow{\lambda_q} c^4, c^4 \xrightarrow{1_u} c^5, c^5 \xrightarrow{1_v} c^0 \}
\]

Final loop:

\[
\{ c^0 \xrightarrow{\lambda_q} F, F \xrightarrow{\lambda_q} F \}
\]

The Client automata starts by sending the initial configuration of \( M \), then she simply sends back the configurations she receives till she receives \( q_f \) the final state of \( M \). If this happens \( C_M \) enters a loops by keep on sending back \( q_f \). Since no transitions from \( q_f \) exists in \( M \) there is no service automaton that can accept the message \( q_f \). Hence the 2-counter automata halts iff \( C_M \) cannot be simulated by the asynchronous product of automata \( A_1 \).

E Further results on FVAs

For convincing the reader, we present here further results which have not been presented in the core of the paper.

We provide a fine comparison between FVAs and \( n \)-FVAs, then we define deterministic FVAs and study some of their properties.

E.1 The \( n \)-FVAs and their expressiveness

To compare \( n \)-FVAs and FVAs, the definition of the relation of simulation and bisimulation for FVAs is needed.

**Definition 13.** Let \( A_1 = (\Sigma, X_1, Q_1, q_{10}^1, \delta_1, F_1, \kappa_1) \) and \( A_2 = (\Sigma, X_2, Q_2, q_{20}^2, \delta_2, F_2, \kappa_2) \) be two FVAs where \( X_1 \cap X_2 = \emptyset \). A simulation of \( A_1 \) by \( A_2 \) is a relation \( \leq \subseteq (\Sigma \times Q_1) \times (\Sigma \times Q_2) \) such that

\[ (\sigma_1, q_1) \leq (\sigma_2, q_2) \text{ if } \sigma_2[1] = \kappa_1^{-1}(q_1) \text{ and } \] for a variable \( x_1 \in X_1 \), and \( \gamma_1: \mathcal{V}(\sigma_1(x_1)) \to \Sigma \) is a substitution and

\[ (\sigma_1, q_1) \xrightarrow{\gamma_1(\sigma_1(x_1))} (\sigma_1 \cup \{(x_1, a)\}) \xrightarrow{\delta_1(q_1), q_1'}, \]

where \( D_1 = \text{dom}(\sigma_2) \setminus \kappa_1^{-1}(q_1') \), then there exist a variable \( x_2 \in X_2 \) and a transition \( q_2' \in \delta_2(q_2, x_2) \) and a substitution \( \gamma_2: \mathcal{V}(\sigma_2(x_2)) \to \Sigma \) such that \( \sigma_1(x_1) = \sigma_2(x_2) \) and

\[ (\sigma_2, q_2) \xrightarrow{\gamma_2(\sigma_2(x_2))} (\sigma_2 \cup \{(x_2, a)\}) \xrightarrow{\delta_2(q_2), q_2}, \]

where \( (\sigma_1, q_1) \leq (\sigma_2, q_2) \) and \( D_2 = \text{dom}(\sigma_2) \setminus \kappa_2^{-1}(q_2') \).
The cases when \( \mathcal{A}_1 \) performs a transition labeled by a letter and \( \mathcal{A}_2 \) replies by a transition labeled by either a letter or a free variable are handled in the usual way.

- \((\emptyset, q^1_0) \leq (\emptyset, q^2_0)\).
- If \((\sigma_1, q_1) \leq (\sigma_2, q_2)\) with \(q_1 \in F_1\) then \(q_2 \in F_2\).

**Lemma 6.** The simulation relation \(\leq\) of FVAs enjoys the following properties:

1. It is a preorder, i.e., reflexive and transitive,
2. It implies language inclusion, i.e., if \(\mathcal{A} \leq \mathcal{B}\) then \(L(\mathcal{A}) \subseteq L(\mathcal{B})\), for two FVAs \(\mathcal{A}\) and \(\mathcal{B}\), and
3. It is decidable.

**Proof.** Items 1 and 2 are immediate. For the Item 3, the same technique used in the proof that the \(\mathcal{S}\)-simulation is decidable (Theorem 5) can be reused: there is a finite set \(C\) of letters such that there is a \(\mathcal{S}\)-simulation where the variables are instantiated from the infinite set \(\Sigma\) iff there is a simulation where the variables are instantiated from \(C\).

The relation of bisimulation for FVAs, denoted hereby \(\approx\), can be defined in the same fashion as the relation of simulation.

Although \(n\)-FVAs and FVAs recognize the same languages, \(n\)-FVAs are stronger than \((n-1)\)-FVAs in the following sense:

**Theorem 7.** For every \(n \geq 2\), there is an \(n\)-FVA \(\mathcal{H}_n\) so that there is no \(n'\)-FVA \(\mathcal{H}_{n'}\) such that \(\mathcal{H}_n\) and \(\mathcal{H}_{n'}\) are bisimilar and \(n' < n\).

**Proof.** Let \(\mathcal{H}_n = (\Sigma, \mathcal{X}, Q, q_0, \delta, F, \kappa)\) be the \(n\)-FVA depicted below and defined by

\[
\mathcal{X} = \{x_1, \ldots, x_n\},
\]

\[
Q = \{q_0, \ldots, q_n\} \cup \{q^1_i, i = 1, \ldots, n\} \cup \{q^2_i, i = 1, \ldots, n\},
\]

\[
F = Q
\]

\[
\delta = \{q_0 \rightarrow q_0\} \cup \{q_i \rightarrow q_{i+1}, i = 0, \ldots, n-1\} \cup \{q_i \rightarrow q^1_{i+1}, i = 1, \ldots, n-1\} \cup \{q_i \rightarrow q^2_{i}, i = 2, \ldots, n-1\}
\]

\[
dom(\kappa) = \emptyset,
\]

where \(b \in \Sigma\). We show that there is no \((n-1)\)-FVA \(\mathcal{B}_{n-1}\) that \(\mathcal{H}_n \approx \mathcal{B}_{n-1}\). Towards a contradiction: assume the existence of such \(\mathcal{B}_{n-1} = (\Sigma, \mathcal{X}', Q', Q'_0, \delta', F', \kappa')\). There exist two substitutions \(\sigma_{n-1} : \mathcal{X} \rightarrow \Sigma\) and \(\sigma'_{n-1} : \mathcal{X}' \rightarrow \Sigma\), and a state \(q'_{n-1} \in Q'\) such that \((\sigma_{n-1}, q_{n-1}) \approx (\sigma'_{n-1}, q'_{n-1})\). Notice that \(dom(\sigma_{n-1}) = \{x_1, \ldots, x_{n-1}\}\). We argue next that the transition \(q_{n-1} \rightarrow q_{n-1,1}\) of \(\mathcal{H}_n\) cannot be simulated by any transition of \(\mathcal{B}_{n-1}\) outgoing from \(q'_{n-1}\). Each transitions outgoing from \(q'_{n-1}\) is labeled by a letter or an \((n-1)\)-labels of variables \((x'_1, \ldots, x'_{n-1})\). Notice that when there exist \(i, j\) such that \(\sigma'(x'_i) \neq \sigma'(x'_j)\), then one of the outgoing transitions from \(q'_{n-1}\) is possible, but this transition
must be matched by the transition \( q_{n-1} \xrightarrow{x_1} q_n \) in \( \mathcal{H}_n \). And the \( b \)-transition of \( \mathcal{B}_{n-1} \) cannot be matched by any transition in \( \mathcal{H}_n \) since there is no outgoing transition from \( q_n \).

\[
\begin{array}{cccccc}
  q_{n-1} & \xrightarrow{x_1} & q_0 & \xrightarrow{x_1} & q_1 & \xrightarrow{x_2} & q_2 & \ldots & q_{n-1} & \xrightarrow{x_n} & q_n \\
  & & & x_1, x_2 & x_1, x_2, x_3 & & & & & & \\
  & & & q_1 & q_2 & & & & & & \\
  & & & & b & & & & & & \\
  & & & q_1 & q_2 & & & & & & \\
  & & & & b & & & & & & \\
  & & & q_1 & q_2 & & & & & & \\
  & & & & b & & & & & & \\
  & & & q_1 & q_2 & & & & & & \\
  & & & & b & & & & & & \\
\end{array}
\]

The FVA \( \mathcal{H}_n \)

\[\Box\]

### E.2 Deterministic FVAs.

We define deterministic FVAs, (DFVAs, for short) in terms of runs. Then we give a syntactic characterization of them.

**Definition 14.** A FVA \( \mathcal{A} \) is deterministic if for every word \( w \in \Sigma^* \) there exists at most one run of \( \mathcal{A} \) on \( w \).

**Theorem 8.** Let \( \mathcal{A} \) be a FVA. Then \( \mathcal{A} \) is not deterministic iff there exists an accessible state \( q \) with two outgoing transitions satisfying one of the conditions:

1. the transitions are labeled with the same letter;
2. one of the transitions is labeled by a variable.

It is clear that the above conditions are sufficient and necessary.

**Proposition 2.** There is a FVA \( \mathcal{A} \) such that no DFVA \( \mathcal{D} \) satisfies \( L(\mathcal{A}) = L(\mathcal{D}) \).

**Proof.** Let \( a, b \) be two letters in \( \Sigma \), and let \( L = \{ z \mid z \in \Sigma \} \cup \{ ab \} \). Indeed the language \( L \) is FVA-recognizable. Towards a contradiction: assume the existence of a DFVA \( \mathcal{D} \) such that \( L(\mathcal{D}) = L \). Let \( q_0 \) be the initial state of \( \mathcal{D} \). By following the syntactic characterization of DFVAs given in Theorem\[8\] we have that either (i.) all the transitions outgoing from \( q_0 \) are labeled with letters, and in this case the language \( \{ z \mid z \in \Sigma \} \) cannot be recognized by \( \mathcal{D} \) since \( \Sigma \) is infinite, which is a contradiction, or (ii.) there is only one transition outgoing from \( q_0 \) and labeled with a variable. Let \( q_0 \xrightarrow{b} q_1 \) be such transition. In this case, there must be a transition \( q_1 \xrightarrow{b} q_f \) in \( \mathcal{D} \) where \( q_f \) is a final state. This means that the set of words \( az \), where \( z \in \Sigma \) is recognized by \( \mathcal{D} \). This is a contradiction. \[\Box\]

**Corollary 1.** Deciding if a FVA is deterministic is NL-Complete.
Proof. The upper bound follows from the fact that we can guess a condition and check whether it is violated. On the other hand, NL is closed under complementation. The lower bound follows from a standard reduction from the reachability for digraphs.

Proposition 3. For DFVAs, the membership and the universality problems are in PTIME.

Proof. We only discuss the complexity of the universality since the membership problem is straightforward. Let \( A \) be a DFVA. Recall that to check whether \( A \) is universal we first construct an equivalent FVA \( A' \) in which all the transitions are labeled with free variables, see the proof of Theorem 3. To construct \( A' \) one may first eliminate all the transitions of \( A \) labeled with letters. This yields a DVFA \( A_c \) whose structure is a tail-cycle in which all the transitions are labeled with variables. Hence, the universality of \( A_c \) can be done in polynomial time.

Proposition 4. The containment problem \( L(A) \subseteq L(D) \) for two FVAs \( A \) and \( D \) where \( D \) is deterministic, is decidable.

Proof. We shall show that \( L(A) \subseteq L(D) \) iff \( D \) simulates \( A \). The direction \(( \Rightarrow )\) has been proven in item 2 of Lemma 6. The direction \(( \Leftarrow )\) follows from the fact that every accepting run by \( A \) over a word \( w \in \Sigma^* \) can be simulated by a unique run by \( D \) over \( w \).