TWO OBSERVATIONS ABOUT NORMAL FUNCTIONS

CHRISTIAN SCHNELL

Abstract. Two simple observations are made: (1) If the normal function associated to a Hodge class has a zero locus of positive dimension, then it has a singularity. (2) The intersection cohomology of the dual variety contains the cohomology of the original variety, if the degree of the embedding is large.

This brief note contains two elementary observations about normal functions and their singularities that arose from a conversation with Pearlstein. The proofs are very simple, and it is quite possible that both statements are known; nevertheless, it seemed useful to me to have them written down. Throughout, $X$ will be a smooth projective variety of dimension $2n$, and $\eta$ a primitive Hodge class of weight $2n$ on $X$, say with integer coefficients. We shall also let $\pi : X \to P$ be the universal hypersurface section of $X$ by some very ample class $H$, with discriminant locus $X^\vee \subset P$.

1. The zero locus of a normal function

Here we show that if the zero locus of the normal function associated to a Hodge class $\eta$ contains an algebraic curve, then the normal function must be singular at one of the points of intersection between $X^\vee$ and the closure of the curve.

Proposition 1. Let $\nu$ be the normal function on $P \setminus X^\vee$, associated to a non-torsion Hodge class $\eta \in H^{2n}(X,\mathbb{Z}) \cap H^{n,n}(X)$. Assume that the zero locus of $\nu$ contains an algebraic curve, and that the divisor $H$ is sufficiently ample. Then $\nu$ is singular at one of the points where the closure of the curve meets $X^\vee$.

Before giving the proof, we briefly recall some definitions. In general, a normal function for a variation of Hodge structure of odd weight on a complex manifold $Y_0$ has an associated cohomology class $[\nu]$. If $H_Z$ is the local system underlying the variation, then a normal function $\nu$ determines an extension of local systems

$$0 \longrightarrow H_Z \longrightarrow H'_Z \longrightarrow \mathbb{Z} \longrightarrow 0. \quad (1)$$

The cohomology class $[\nu] \in H^1(Y_0, H_Z)$ of the normal function is the image of $1 \in H^0(Y_0, \mathbb{Z})$ under the connecting homomorphism for the extension.

In particular, the normal function $\nu$ associated to a Hodge class $\eta$ determines a cohomology class $[\eta] \in H^1(P \setminus X^\vee, R^{2n-1}\pi_*\mathbb{Z})$. With rational coefficients, that class can also be obtained directly from $\eta$ through the Leray spectral sequence

$$E^{p,q}_2 = H^p(P \setminus X^\vee, R^q\pi_*\mathbb{Q}) \implies H^{p+q}(P \times X \setminus \pi^{-1}(X^\vee), \mathbb{Q}).$$

To wit, the pullback of $\eta$ to $P \times X \setminus \pi^{-1}(X^\vee)$ goes to zero in $E^{0,2n}_2$ because $\eta$ is primitive, and thus gives an element of $E^{1,2n-1}_2$; this element is precisely $[\eta]$.

2000 Mathematics Subject Classification. 14D05.

Key words and phrases. Normal function, Singularity of normal function, Intersection cohomology, Dual variety.
Lemma 2. Let $C_0 \to P \setminus X^v$ be a smooth affine curve mapping into the zero locus of $ν$, and let $W_0$ be the pullback of the family $X$ to $C_0$. Then the image of the Hodge class $η$ in $H^{2n}(W_0, \mathbb{Q})$ is zero.

Proof. Let $ψ: W_0 \to C_0$ be the obvious map. The pullback of $R^{2n-1}τ_∗\mathbb{Q}$ to $C_0$ is naturally isomorphic to $R^{2n-1}ψ_∗\mathbb{Q}$; moreover, when $ν$ is restricted to $C_0$, its class is simply the image of $[η]$ in $H^1(C_0, R^{2n-1}ψ_∗\mathbb{Q})$. That image has to be zero, because $C_0$ maps into the zero locus of $ν$.

Now let $η_0 ∈ H^{2n}(W_0, \mathbb{Q})$ be the image of the Hodge class $η$. The Leray spectral sequence for the map $ψ$ gives a short exact sequence

$$0 \to H^1(C_0, R^{2n-1}ψ_∗\mathbb{Q}) \to H^{2n}(W_0, \mathbb{Q}) \to H^0(C_0, R^{2n}ψ_∗\mathbb{Q}) \to 0,$$

and as before, $η_0$ actually lies in $H^1(C_0, R^{2n-1}ψ_∗\mathbb{Q})$. Because the spectral sequences for $ψ$ and $π$ are compatible, $η_0$ is equal to the image of $[η]$; but we have already seen that this is zero.

Returning to our review of general definitions, let $ν$ be a normal function on $Y_0$. When $Y_0 \subseteq Y$ is an open subset of a bigger complex manifold, one can look at the behavior of $ν$ near points of $Y \setminus Y_0$. The singularity of $ν$ at a point $y ∈ Y \setminus Y_0$ is by definition the image of $[ν]$ in the group

$$\lim_{U∋y} H^1(U \cap Y_0, H_2),$$

the limit being over all analytic open neighborhoods of the point [11 p. 1]. If the singularity is non-torsion, $ν$ is said to be singular at the point $y$.

When $ν$ is the normal function associated to a non-torsion primitive Hodge class $η ∈ H^{2n}(X, \mathbb{Z})$, Brosnan, Fang, Nie, and Pearlstein [11 Theorem 1.3], and independently de Cataldo and Migliorini [2 Proposition 3.7], have proved the following result: Provided the vanishing cohomology of the smooth fibers of $π$ is nontrivial, $ν$ is singular at a point $p ∈ X^v$ if, and only if, the image of $η$ in $H^{2n}(π^{-1}(p), \mathbb{Q})$ is nonzero. By recent work of Dimca and Saito [3 Theorem 6], it suffices to take $H = dA$, with $A$ very ample and $d ≥ 3$.

Proof of Proposition 7. Let $C$ be the normalization of the closure of the curve in the zero locus. Pulling back the universal family $X \to P$ to $C$ and resolving singularities, we obtain a smooth projective $2n$-fold $W$, together with the following two maps:

$$W \xrightarrow{λ} X \xrightarrow{ψ} C$$

This may be done in such a way that the general fiber of $ψ$ is a smooth hypersurface section of $X$; let $C_0 \subseteq C$ be the open subset where this holds, and $W_0 = ψ^{-1}(C_0)$ its preimage. Assume in addition that, for each $t ∈ C \setminus C_0$, the fiber $E_t$ is a divisor with simple normal crossing support. The map $λ$ is generically finite, and we let $d$ be its degree.

Let $η_W = λ^∗(η)$ be the pullback of the Hodge class to $W$. By Lemma 2, the restriction of $η_W$ to $W_0$ is zero. Consider now the exact sequence

$$H^{2n}(W, W_0, \mathbb{Q}) \xrightarrow{i} H^{2n}(W, \mathbb{Q}) \to H^{2n}(W_0, \mathbb{Q}).$$

By what we have just observed, $\eta_W$ belongs to the image of the map $i$, say $\eta_W = i(\alpha)$. Under the nondegenerate pairing (given by Poincaré duality)

$$H^{2n}(W, W_0, \mathbb{Q}) \otimes \bigoplus_{t \in C \setminus C_0} H^{2n}(E_t, \mathbb{Q}) \rightarrow \mathbb{Q},$$

and the intersection pairing on $W$, the map $i$ is dual to the restriction map

$$H^{2n}(W, \mathbb{Q}) \rightarrow \bigoplus_{t \in C \setminus C_0} H^{2n}(E_t \mathbb{Q}),$$

and so we get that

$$d \cdot \int_X \eta \cup \eta = \int_W \eta_W \cup \eta_W = \langle i(\alpha), \eta_W \rangle = \sum_{t \in C \setminus C_0} \langle \alpha, i^*_t(\eta_W) \rangle$$

where $i_t : E_t \rightarrow W$ is the inclusion. But the first integral is nonzero, because the intersection pairing on $X$ is definite on the subspace of primitive $(n, n)$-classes. We conclude that the pullback of $\eta$ to at least one of the $E_t$ has to be nonzero.

By construction, $E_t$ maps into one of the singular fibers of $\pi$, say to $\pi^{-1}(p)$, where $p$ belongs to the intersection of $X^\vee$ with the closure of the curve. Thus $\eta$ has nonzero image in $H^{2n}(\pi^{-1}(p), \mathbb{Q})$; by the result of [12] mentioned above, $\nu$ has to be singular at $p$, concluding the proof. □

I do not know whether a “converse” to Proposition [1] is true; that is to say, whether the normal function associated to an algebraic cycle on $X$ has to have a zero locus of positive dimension for sufficiently ample $H$. If it was, this would give one more equivalent formulation of the Hodge conjecture.

2. Cohomology of the discriminant locus

Pearlstein pointed out that the singularities of the discriminant locus should be complicated enough to capture all the primitive cohomology of the original variety, once $H = dA$ is a sufficiently big multiple of a very ample class. In this section, we give an elementary proof of this fact for $d \geq 3$.

To do this, we need a simple lemma, used to estimate the codimension of loci in $X^\vee$ where the fibers of $\pi$ have a singular set of positive dimension. Let

$$V_d = H^0(X, \mathcal{O}_X(dA))$$

be the space of sections of $dA$, for $A$ very ample.

**Lemma 3.** Let $Z \subseteq X$ be a closed subvariety of positive dimension $k > 0$. Write $V_d(Z)$ for the subspace of sections that vanish along $Z$. Then

$$\text{codim}(V_d(Z), V_d) \geq \binom{d + k}{k}.$$  

**Proof.** Since $A$ is very ample, we may find $(k + 1)$ points $P_0, P_1, \ldots, P_k$ on $Z$, together with $(k + 1)$ sections $s_0, s_1, \ldots, s_k \in V_1$, such that each $s_i$ vanishes at all points $P_j$ with $j \neq i$, but does not vanish at $P_i$. Then all the sections

$$s_0^{i_0} \otimes s_1^{i_1} \otimes \cdots \otimes s_k^{i_k} \in V_d,$$

for $i_0 + i_1 + \cdots + i_k = d$, are easily seen to be linearly independent on $Z$. The lower bound on the codimension follows immediately. □
We now use the estimate to make Pearlstein’s suggestion precise. As one further bit of notation, let $X^\text{sing} \subseteq X$ stand for the union of all the singular points in the fibers of $\pi$. It is well-known that $X^\text{sing}$ is a projective space bundle over $X$, and in particular smooth.

**Proposition 4.** Let $H = dA$ for a very ample class $A$. If $d \geq 3$, then the map $\phi: X^\text{sing} \to X^\vee$ is a small resolution of singularities, and therefore

$$IH^*(X^\vee, \mathbb{Q}) \simeq H^*(X^\text{sing}, \mathbb{Q}).$$

In particular, $H^*(X, \mathbb{Q})$ is a direct summand of $IH^*(X^\vee, \mathbb{Q})$ once $d \geq 3$.

**Proof.** By [3, p. Theorem 6], the discriminant locus is a divisor in $P$ once $d \geq 3$. This means that there are hyperplane sections of $X$ with exactly one ordinary double point [5, p. 317]. The map $\phi$ is then birational, and therefore a resolution of singularities of $X^\vee$. To prove that it is a small resolution, take a Whitney stratification of $P$ in which $X^\vee$ is a union of strata, and such that over each stratum, the maps $\pi$ and $\phi$ are topologically trivial. The dimension of the singular set of the fibers of $\pi$ is then constant along each stratum.

Let $S \subseteq X^\vee$ be an arbitrary stratum along which the singular set of the fiber has dimension $k > 0$. At a general point $t \in S$, there then has to be an irreducible component $Z$ in the singular locus of $\pi^{-1}(t)$ that remains singular to first order along $S$. Now a tangent vector to $S$ may be represented by a section $s$ of $dA$, and the condition that $Z$ remain singular to first order is that $s$ should vanish along $Z$. By Lemma [3], the space of such sections has codimension at least $\binom{d+k}{k}$, and a moment’s thought shows that, therefore,

$$\text{codim}(S, X^\vee) \geq \binom{d+k}{k} - 1.$$

This quantity is evidently a lower bound for the codimension of the locus where the fibers of $\phi$ have dimension $k$. In order for $\phi$ to be a small resolution, it is thus sufficient that

$$\binom{d+k}{k} - 1 > 2k$$

for all $k > 0$. Now one easily sees that this condition is satisfied provided that $d \geq 3$. This proves the first assertion; the second one is a general fact about intersection cohomology. Finally, $H^*(X, \mathbb{Q})$ is a direct summand of $H^*(X^\text{sing}, \mathbb{Q})$ because $X^\text{sing}$ is a projective space bundle over $X$, and the third assertion follows. □

The proof shows that, as in the theorem by Dimca and Saito, $d \geq 2$ is sufficient in most cases. A related result, showing the effect of taking $H$ sufficiently ample, was obtained by Clemens; he noticed that, as a consequence of Nori’s connectivity theorem, one has an isomorphism

$$H^{2n}(X, \mathbb{Q})_{\text{prim}} \simeq H^1(P \setminus X, (R^{2n-1} \pi_* \mathbb{Q})_{\text{van}}),$$

once $H$ is sufficiently ample [4, Proposition 7 on p. 11].

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Department of Mathematics, Statistics & Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607

E-mail address: cschnell@math.uic.edu