RATIONALITY OF CYCLES ON FUNCTION FIELD OF
EXCEPTIONAL PROJECTIVE HOMOGENEOUS VARIETIES

RAPHAEL FINO

Abstract. In this article we prove a result comparing rationality of algebraic cycles over
the function field of a projective homogeneous variety under a linear algebraic group of
type $F_4$ or $E_8$ and over the base field, which can be of any characteristic.

Keywords: Chow groups and motives, exceptional algebraic groups, projective homo-
genous varieties.

1. Introduction

Let $G$ be a linear algebraic group of type $F_4$ or $E_8$ over a field $F$ and let $X$ be a
projective homogeneous $G$-variety. We write $Ch$ for the Chow group with coefficient in
$\mathbb{Z}/p\mathbb{Z}$, with $p = 3$ when $G$ is of type $F_4$ and $p = 5$ when $G$ is of type $E_8$. The purpose of
this note is to prove the following theorem dealing with rationality of algebraic cycles on
function field of such a projective homogeneous $G$-variety.

Theorem 1.1. For any equidimensional variety $Y$, the change of field homomorphism

$$Ch(Y) \rightarrow Ch(Y_{F(X)})$$

is surjective in codimension $< p + 1$. It is also surjective in codimension $p + 1$ for a given
$Y$ provided that $1 \notin \deg Ch_0(X_{F(\zeta)})$ for each generic point $\zeta \in Y$.

The proof is given in section 3.

In previous papers ([2], [3], after the so-called Main Tool Lemma by A. Vishik, cf
[10], [17]), similar issues about rationality of cycles, with quadrics instead of exceptional
projective homogeneous varieties, have been treated. The above statement is to put in
relation with [10] Theorem 4.3, where generic splitting varieties have been considered.
Also, Theorem 1.1 is contained in [10] Theorem 4.3 if char$(F) = 0$.

On the one hand, our method of proof is basically the method used to prove [10] Theo-
rem 4.3. On the other hand, our method mainly relies on a motivic decomposition result
for projective homogeneous varieties due to V. Petrov, N. Semenov and K. Zainoulline (cf
[14] Theorem 5.17). It also relies on a linkage between the $\gamma$-filtration and Chow groups,
in the spirit of [5]. Our method works in any characteristic and is particularly suitable
for groups of type $F_4$ and $E_8$ mainly because the latter have an opportune $J$-invariant.

In the aftermath of Theorem 1.1, we get the following statement dealing with integral
Chow groups (see [10] Theorem 4.5]).

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Corollary 1.2. If \( p \in \deg CH_0(X) \) then for any equidimensional variety \( Y \), the change of field homomorphism \( CH(Y) \to CH(Y_{F(X)}) \) is surjective in codimension \( < p + 1 \). It is also surjective in codimension \( p + 1 \) for a given \( Y \) provided that \( 1 \notin \deg CH_0(X_{F(\zeta)}) \) for each generic point \( \zeta \in Y \).

Remark 1.3. Our method of proof for Theorem 1.1 works for groups of type \( G_2 \) as well (with \( p=2 \)). However, the case of \( G_2 \) can be treated in a more elementary way if \( \text{char}(F) = 0 \).

Indeed, it is known that to each group \( G \) of type \( G_2 \) one can associate a 3-fold Pfister quadratic form \( \rho \) such that, by denoting \( X_\rho \) the Pfister quadric associated with \( \rho \), the variety \( X \) has a rational point over \( F(X_\rho) \) and vice-versa. Thus, for any equidimensional variety \( Y \), one has the commutative diagram

\[
\begin{array}{ccc}
Ch(Y) & \to & Ch(Y_{F(X)}) \\
\downarrow & & \downarrow \\
Ch(Y_{F(X_\rho)}) & \to & Ch(Y_{F(X_\rho \times X)})
\end{array}
\]

where the right and the bottom maps are isomorphisms. Furthermore, as suggested in [17, Remark on Page 665] (where the assumption \( \text{char}(F) = 0 \) is required), the change of field homomorphism \( Ch(Y) \to Ch(Y_{F(Q)}) \) is surjective in codimension \( < 3 \).

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2. Filtrations on projective homogeneous varieties

In this section, we prove two propositions which play a crucial role in the proof of Theorem 1.1.

First of all, we recall that for any smooth projective variety \( X \) over a field \( E \), one can consider two particular filtrations on the Grothendieck ring \( K(X) \) (see [5, §1.A]), i.e the \( \gamma \)-filtration and the topological filtration, whose respective terms of codimension \( i \) are given by

\[
\gamma^i(X) = \langle c_{n_1}(a_1) \cdots c_{n_m}(a_m) \mid n_1 + \cdots + n_m \geq i \text{ and } a_1, \ldots, a_m \in K(X) \rangle
\]

and

\[
\tau^i(X) = \langle [O_Z] \mid Z \hookrightarrow X \text{ and codim}(Z) \geq i \rangle,
\]

where \( c_n \) is the \( n \)-th Chern Class with values in \( K(X) \) and \( [O_Z] \) is the class of the structure sheaf of a closed subvariety \( Z \). We write \( \gamma^{i/i+1}(X) \) and \( \tau^{i/i+1}(X) \) for the respective quotients. For any \( i \), one has \( \gamma^i(X) \subset \tau^i(X) \) and one even has \( \gamma^i(X) = \tau^i(X) \) for \( i \leq 2 \). We denote by \( pr \) the canonical surjection

\[
CH^i(X) \twoheadrightarrow \tau^{i/i+1}(X)
\]

\[
[Z] \mapsto [O_Z],
\]

where \( CH \) stands for the integral Chow group.
The method of proof of the following proposition is largely inspired by the proof of [9, Theorem 6.4 (2)].

**Proposition 2.1.** Let $G_0$ be a split semisimple linear algebraic group over a field $F$ and let $B$ be a Borel subgroup of $G_0$. There exist an extension $E/F$ and a cocycle $\xi \in H^1(E, G_0)$ such that the topological filtration and the $\gamma$-filtration coincide on $K(\xi(G_0/B))$.

**Proof.** Let $n$ be an integer such that $G_0 \subset \text{GL}_n$ and let us set $S := \text{GL}_n$ and $E := F(S/G_0)$. We denote by $T$ the $E$-variety $S \times_{S/G_0} \text{Spec}(E)$ given by the generic fiber of the projection $S \to S/G_0$. Note that since $T$ is clearly a $G_0$-torsor over $E$, there exists a cocycle $\xi \in H^1(E, G_0)$ such that the smooth projective variety $X := T/B_E$ is isomorphic to $\xi(G_0/B)$. We claim that the Chow ring $CH(X)$ is generated by Chern classes. Indeed, the morphism $h : X \to S/B$ induced by the canonical $G_0$-equivariant morphism $T \to S$ being a localisation, the associated pull-back

$$h^* : CH(S/B) \to CH(X)$$

is surjective. Furthermore, the ring $CH(S/B)$ itself is generated by Chern classes: by [9, §6,7] there exist a morphism

$$(2.2) \quad S(T^*) \to CH(S/B),$$

(where $S(T^*)$ is the symmetric algebra of the group of characters $T^*$ of a split maximal torus $T \subset B$) with its image generated by Chern classes. Moreover, the morphism (2.2) is surjective by [9, Proposition 6.2]. Since $h^*$ is surjective and Chern classes commute with pull-backs, the claim is proved.

We show now that the two filtrations coincide on $K(X)$ by induction on dimension. Let $i \geq 0$ and assume that $\tau^{i+1}(X) = \gamma^{i+1}(X)$. Since for any $j \geq 0$, one has $\gamma^j(X) \subset \tau^j(X)$, the induction hypothesis implies that

$$\gamma^{i/i+1}(X) \subset \tau^{i/i+1}(X).$$

Thus, the ring $CH(X)$ being generated by Chern classes, one has $\gamma^{i+i+1}(X) = \tau^{i/i+1}(X)$ by [6, Lemma 2.16]. Therefore one has $\tau^i(X) = \gamma^i(X)$ and the proposition is proved. □

Note that this result remains true when one consider a special parabolique subgroup $P$ instead of $B$.

Now, we prove a result which will be used in section 3 to get the second conclusion of Theorem 1.1.

We recall that for any smooth projective variety $X$ over a field and for any $i < p+1$, the canonical surjection $pr : Ch^i(X) \to \tau^{i+1}(X)$ with $\mathbb{Z}/p\mathbb{Z}$-coefficient is an isomorphism (cf [5, §1.A] for example). The following proposition extends this fact to $i = p + 1$ provided that $X$ is a projective homogeneous variety under a linear algebraic group $G$ of type $F_4$ or $E_8$.

**Proposition 2.3.** Let $X$ be a projective homogeneous variety under a group $G$ of type $F_4$ or $E_8$, then the canonical surjection

$$pr : Ch^{p+1}(X) \to \tau^{p+1/p+2}(X)$$

is injective.
The epimorphism \( pr : Ch^{p+1}(X) \to \tau^{p+1/p+2}(X) \) coincides with the edge homomorphism of the spectral Brown-Gersten-Quillen structure \( E_2^{p+1,p-1}(X) \Rightarrow K(X) \), i.e \( E_r^{p+1,p-1}(X) \) stabilizes for \( r > 0 \) with \( E_\infty^{p+1,p-1}(X) = \tau^{p+1/p+2}(X) \), and for any \( r \geq 2 \) the differential \( E_r^{p+1,p-1}(X) \to E_r^{p+1+r,p-r}(X) \) is zero, so that the epimorphism \( pr \) coincides with the composition

\[
Ch^{p+1}(X) \simeq E_2^{p+1,p-1}(X) \to E_3^{p+1,p-1}(X) \to \cdots \to E_\infty^{p+1,p-1}(X) = \tau^{p+1/p+2}(X).
\]

Now, it is equivalent in order to prove the proposition to prove that for any \( r \geq 2 \), the differential \( E_r^{p+1-r,-p-2+r}(X) \to E_r^{p+1,p-1}(X) \) is zero.

First of all, since we work with \( \mathbb{Z}/p\mathbb{Z} \)-coefficient, by [12, Theorem 3.6], the differential \( E_r^{p+1-r,-p-2+r}(X) \to E_r^{p+1,p-1}(X) \) is zero for any \( r \geq 2 \) with \( r \neq p \). Hence, one only has to show that the differential \( E_\infty^{1,-2}(X) \to E_\infty^{p+1,p-1}(X) \) is zero.

Let us consider the following composition given by the BGQ-structure

\[
E_\infty^{1,-2}(X) \hookrightarrow \cdots \hookrightarrow E_3^{1,-2}(X) \hookrightarrow E_2^{1,-2}(X).
\]

Note that one has \( E_\infty^{1,-2}(X) \simeq E_2^{1,-2}(X) \) if and only if for any \( r \geq 2 \) the differential \( E_r^{1,-2}(X) \to E_r^{1+r,-2+r+1}(X) \) is zero. Therefore it is sufficient to prove that \( E_\infty^{1,-2}(X) \simeq E_2^{1,-2}(X) \) to get that the differential \( E_\infty^{1,-2}(X) \to E_\infty^{p+1,p-1}(X) \) is zero.

On the one hand, by the very definition, the group \( E_\infty^{1,-2}(X) \) of the topological filtration on \( K_1(X) \). On the other hand, one has \( E_2^{1,-2}(X) \simeq H^1(X, K_2) \) (for any integers \( p \) and \( q \), one has \( E_2^{p,q}(X) \simeq H^p(X, K_{-q}) \)).

Let us now consider the commutative diagram (cf [7, §4])

\[
\begin{array}{ccc}
K_1^{1/2}(X) & \longrightarrow & H^1(X, K_2) \\
\downarrow & & \downarrow \\
H^0(X, K_1) \otimes Ch^1(X) & \longrightarrow & H^1(X, K_2)
\end{array}
\]

We claim that the natural map \( H^0(X, K_1) \otimes Ch^1(X) \to H^1(X, K_2) \) is an isomorphism. Indeed since \( G \) is of type \( F_4 \) or \( E_8 \), it has only trivial Tits algebras, and therefore, by [11, Theorem], one has

\[
H^1(X, K_2) \simeq H^1(X_{\text{sep}}, K_2)^F,
\]

where \( \Gamma \) is the absolute Galois group of \( F \). Moreover, since the variety \( X_{\text{sep}} \) is cellular, by [11, Proposition 1], one has

\[
H^1(X_{\text{sep}}, K_2) \simeq K_1 F \otimes Ch^1(X_{\text{sep}}).
\]

Thus, since the Picard group of any homogeneous projective variety under a group of type \( F_4 \) or \( E_8 \) is rational (cf [15, Example 4.1.1]) and since \( (K_1 F) = K_1 F = H^0(X, K_1) \), one has

\[
H^1(X, K_2) \simeq K_1 F \otimes Ch^1(X) \simeq H^0(X, K_1) \otimes Ch^1(X),
\]

and the claim is proved. Therefore, one has \( E_\infty^{1,-2}(X) \simeq E_2^{1,-2}(X) \) and the proposition is proved. \( \square \)
Remark 2.4. Assume that $G_0$ of strongly inner type (e.g $F_4$ and $E_8$) and consider an extension $E/F$ and a cocycle $\xi \in H^1(E, G_0)$. By [13 Theorem 2.2.(2)], the change of field homomorphism

$$K(\xi(G_0/B)_E) \to K(\xi(G_0/B)_{\overline{E}}) \simeq K(G_0/B)$$

is an isomorphism, where $\overline{E}$ denotes an algebraic closure of $E$. Therefore, since the $\gamma$-filtration is defined in terms of Chern classes and the latter commute with pull-backs, the quotients of the $\gamma$-filtration on $K(\xi(G_0/B)_E)$ do not depend on the extension $E/F$ neither on the choice of $\xi \in H^1(E, G_0)$.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

First of all, note that the $F$-variety $X$ is $A$-trivial in the sense of [10 Definition 2.3] (see [10 Example 2.5]), i.e for any extension $L/F$ with $X(L) \neq \emptyset$, the degree homomorphism $\deg: Ch_0(X_L) \to \mathbb{Z}/p\mathbb{Z}$ is an isomorphism. Therefore, by [10 Lemma 2.9], the change of field homomorphism $Ch(Y) \to Ch(Y_{F(X)})$ is an isomorphism (in any codimension) if $1 \in \deg Ch_0(X)$. Hence, one can assume that $1 \notin \deg Ch_0(X)$.

Now, we know from [14 Table 4.13] that the $J$-invariant $J_p(G)$ of $G$ is equal to $(1)$ or $(0)$. However, the assumption $J_p(G) = (0)$ implies that there exists a splitting field $K/F$ of degree coprime to $p$ (see [14 Corollary 6.7]), and in that case one has $Ch_0(X) \simeq Ch_0(X_K)$ and $1 \in \deg Ch_0(X_K)$ by $A$-triviality of $X$. Thus, under the assumption $1 \notin \deg Ch_0(X)$, one necessarily has $J_p(G) = (1)$ and that is why we can assume $J_p(G) = (1)$ in the sequel.

Since $X$ is $A$-trivial, one can use the following proposition (cf [10 Proposition 2.8]).

Proposition 3.1 (Karpenko, Merkurjev). Given an equidimensional $F$-variety $Y$ and an integer $m$ such that for any $i$ and any point $y \in Y$ of codimension $i$ the change of field homomorphism

$$Ch^m(X) \to Ch^m(X_{F(y)})$$

is surjective, the change of field homomorphism

$$Ch^m(Y) \to Ch^m(Y_{F(X)})$$

is also surjective.

Consequently, it is sufficient in order to prove the first conclusion of Theorem 1.1 to show that for any extension $L/F$, the change of field homomorphism

$$(3.2) \quad Ch(X) \to Ch(X_L)$$

is surjective in codimension $< p + 1$.

Moreover, the $F$-variety being generically split (see [14 Example 3.6]), one can apply the motivic decomposition result [14 Theorem 5.17] to $X$ and get that the motive $\mathcal{M}(X, \mathbb{Z}/p\mathbb{Z})$ decomposes as a sum of twists of an indecomposable motive $\mathcal{R}_p(G)$ (in the same way as (3.5)). Note that the quantity and the value of those twists do not depend on the base field. In particular, we get that for any extension $L/F$ and any integer $k$, the group $Ch^k(X_L)$ is isomorphic to a direct sum of groups $Ch^{k-i}(\mathcal{R}_p(G)_L)$ with $0 \leq i \leq k$. 


Therefore, the surjectivity of (3.2) in codimension \( < p + 1 \) is a consequence of the following proposition.

**Proposition 3.3.** For any extension \( L/F \), the change of field
\[
(3.4) \quad Ch(\mathcal{R}_p(G)) \twoheadrightarrow Ch(\mathcal{R}_p(G)_L)
\]
is surjective in codimension \( < p + 1 \).

**Proof.** Let \( G_0 \) be a split linear algebraic group of the same type of the type of \( G \) and let \( \xi \in H^1(F, G_0) \) be a cocycle such that \( G \) is isogenic to the twisted form \( \xi G_0 \). We write \( \mathfrak{B} \) for the Borel variety of \( G \) (i.e \( \mathfrak{B} = \xi(G_0/B) \), where \( B \) is a Borel subgroup of \( G_0 \)).

By [14, Theorem 5.17], one has the motivic decomposition
\[
(3.5) \quad \mathcal{M}(\mathfrak{B}, \mathbb{Z}/p\mathbb{Z}) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\Sigma a_i},
\]
where \( \Sigma_{i \geq 0} a_i t^i = P(CH(\mathfrak{B}), t)/P(CH(\mathcal{R}_p(G)), t) \), with \( P(\cdot, t) \) the Poincaré polynomial. Thus, for any integer \( k \), we get the following decomposition concerning Chow groups
\[
(3.6) \quad Ch^k(\mathfrak{B}_L) \simeq \bigoplus_{i \geq 0} Ch^{k-i}(\mathcal{R}_p(G)_L)^{\Sigma a_i}.
\]

First of all, the homomorphism (3.4) is clearly surjective in codimension 0 since one has \( Ch^0(\mathcal{R}_p(G)_L) = \mathbb{Z}/p\mathbb{Z} \) for any extension \( L/F \). Then, \( Ch^1(\mathfrak{B}) \) is identified with the Picard group \( Pic(\mathfrak{B}) \) and is rational (see [15, Example 4.1.1]). Furthermore, thanks to the Solomon Theorem for example (see [15, §2.5]), one can compute the coefficients \( a_i \)’s: we get \( a_0 = 1 \) and \( a_1 = \text{rank}(G) = \text{rank}(Ch^1(\mathfrak{B})) \). Thus, the isomorphism (3.6) implies that \( Ch^1(\mathcal{R}_p(G)_L) = 0 \) for any extension \( L/F \).

We have already shown that the homomorphism (3.4) is surjective in codimension 0 and 1. The following lemma implies the surjectivity in codimension 2 and 3 (and therefore proves the first conclusion of Theorem 1.1 if \( G \) is of type \( F_4 \)).

**Lemma 3.7.** Under the assumption \( J_p(G) = (1) \), one has
\[
Ch^2(\mathcal{R}_p(G)) = \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad Ch^3(\mathcal{R}_p(G)) = 0
\]

**Proof.** Since \( J_p(G) = (1) \), by [6 Example 5.3], the cocycle \( \xi \in H^1(F, G_0) \) match with a generic \( G_0 \)-torsor in the sense of [6]. Thus, by [5] Proposition 3.2 and [4] pp. 31, 133], one has \( \text{Tor}_p CH^2(\mathfrak{B}) \neq 0 \) (note that since an algebraic group of type \( F_4 \) or \( E_8 \) is simply connected, it is of strictly inner type, and we can use material from [5] §3]). The conclusion is given by [5] Proposition 5.4.

Let us fix an extension \( L/F \). We now prove the surjectivity of (3.4) in codimension 2 and 3. By [14] Example 4.7], one has \( J_p(G_L) = (0) \) or \( J_p(G_L) = (1) \).

If \( J_p(G_L) = (0) \) then one has \( \mathcal{R}_p(G_L) = \mathbb{Z}/p\mathbb{Z} \) by [14] Corollary 6.7], and on the other hand the motivic decomposition given in [14] Proposition 5.18 (i)] implies the following
decomposition on Chow groups for any integer $k$

\[(3.8)\quad Ch^k(\mathcal{R}_p(G)_L) \simeq \bigoplus_{i=0}^{p-1} Ch^{k-i(p+1)}(\mathcal{R}_p(G)_L)).\]

In particular, one has $Ch^k(\mathcal{R}_p(G)_L) = 0$ for $k = 2$ or $3$ and the conclusion follows.

If $J_p(G_L) = (1)$ then by Lemma 3.7 one has $Ch^2(\mathcal{R}_p(G)_L) = \mathbb{Z}/p\mathbb{Z}$ and $Ch^3(\mathcal{R}_p(G)_L) = 0$. Moreover, since $J_p(G_L) = J_p(G)$, one has $\mathcal{R}_p(G_L) \simeq \mathcal{R}_p(G)_L$ (see [14, Proposition 5.18 (i)]). Therefore, the homomorphism $(3.4)$ is clearly surjective in codimension 3.

We claim that it is also surjective in codimension 2. By $(3.6)$ it suffices to show that the change of field $Ch^2(\mathcal{B}) \to Ch^2(\mathcal{B}_L)$ is an isomorphism. We use material and notation introduced in section 2. Since $J_p(G) = J_p(G_L) = (1)$, the cocycles $\xi$ and $\xi_L$ match with generic $G_0$-torsors and one consequently has $\gamma^3(\mathcal{B}) = \tau^3(\mathcal{B})$ and $\gamma^3(\mathcal{B}_L) = \tau^3(\mathcal{B}_L)$ (see [5, Theorem 3.1(ii)]). It follows that

\[\gamma^{2/3}(\mathcal{B}) = \tau^{2/3}(\mathcal{B}) \quad \text{and} \quad \gamma^{2/3}(\mathcal{B}_L) = \tau^{2/3}(\mathcal{B}_L).\]

Therefore, since $2 < p + 1$, the homomorphism $Ch^2(\mathcal{B}) \to Ch^2(\mathcal{B}_L)$ coincides with.

\[Ch^2(\mathcal{B}) \simeq \gamma^{2/3}(\mathcal{B}) \to \gamma^{2/3}(\mathcal{B}_L) \simeq Ch^2(\mathcal{B}_L)\]

and the center arrow is an isomorphism by Remark 2.4.

The surjectivity of $(3.4)$ in codimension 4 and 5 is a direct consequence of the following statement, where $G$ is of type $E_8$ and $p = 5$. Consequently, Lemma 3.9 completes the proof of the first conclusion of Theorem 1.1 for $G$ of type $E_8$.

**Lemma 3.9.** For any extension $L/F$, one has

\[Ch^4(\mathcal{R}_5(G)_L) = 0 \quad \text{and} \quad Ch^5(\mathcal{R}_5(G)_L) = 0\]

**Proof.** Since $J_5(G) = (1)$, we know that $J_5(G_L) = (1)$ or $(0)$. If $J_5(G_L) = (0)$ then one has $R_5(G_L) = \mathbb{Z}/5\mathbb{Z}$ and the isomorphism $(3.8)$ implies that $Ch^4(\mathcal{R}_5(G)_L) = Ch^5(\mathcal{R}_5(G)_L) = 0$. Thus, one can assume $L = F$ and we have to prove that $Ch^4(\mathcal{R}_5(G)) = Ch^5(\mathcal{R}_5(G)) = 0$.

By Proposition 2.1 there exist an extension $E/F$ and a cocycle $\xi' \in H^1(E, G_0)$ such that the topological filtration and the $\gamma$-filtration coincide on $K(\mathcal{B}')$, with $\mathcal{B}' = \xi'(G_0/B)$. Let us denote $G'$ the variety $\xi'G_0$.

We claim that $J_5(G') = (1)$. Indeed, assume that $J_5(G') = (0)$. In that case, one has $R_5(G') = \mathbb{Z}/5\mathbb{Z}$ and the isomorphism $(3.6)$ gives that $Ch^2(\mathcal{B}') = \mathbb{Z}/5\mathbb{Z}[a_2]$. Since $2 < p + 1$, it implies that $\gamma^{2/3}(\mathcal{B}') = \mathbb{Z}/5\mathbb{Z}[a_2]$, and consequently $\gamma^{2/3}(\mathcal{B}) = \mathbb{Z}/5\mathbb{Z}[a_2]$ by Remark 2.4. However, we have $\gamma^{2/3}(\mathcal{B}) \simeq \tau^{2/3}(\mathcal{B})$ (because $\gamma^3(\mathcal{B}) \simeq \tau^3(\mathcal{B})$ since $\xi \in H^1(F, G_0)$ is generic). Thus, we have $Ch^2(\mathcal{B}) = \mathbb{Z}/5\mathbb{Z}[a_2]$, which contradicts $Ch^2(\mathcal{R}_5(G)) = \mathbb{Z}/5\mathbb{Z}$ and the claim is proved (we recall that for any $i < 6 = p + 1$, one has $\tau^{i/2+1}(X) \simeq Ch^i(X)$).

We now compute the groups $\gamma^{i/2+1}(\mathcal{B}')$ for $i = 3, 4, 5$. Note that since $K(\mathcal{B}') \simeq K(G_0/B)$ and since the description of the free group $K(G_0/B)$ in terms of generators does not depend on the characteristic char($E$) of $E$ (see [14, Lemma 13.3(4)]), we can assume that char($E$) = 0 in order to compute those groups.
In that case, since $J_5(G') \neq (0)$, the isomorphism (3.6) combined with the following theorem (adapted from [10, Theorem RM.10] to our situation)

**Theorem 3.10** (Karpenko, Merkurjev). Let $H$ be a semisimple linear algebraic group of inner type over a field of characteristic 0 and let $p$ be a torsion prime of $H$. If $J_p(H) \neq (0)$ then

$$Ch^j(\mathcal{R}_p(H)) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } j = 0 \text{ or } j = k(p+1) - p + 1, 1 \leq k \leq p - 1 \\ 0 & \text{otherwise} \end{cases}$$

gives that

$$\gamma^{i+1}(\mathcal{B}') \simeq Ch^i(\mathcal{B}') = \mathbb{Z}/5\mathbb{Z}^{\oplus(a_{i-2} + a_i)} \quad \text{for } i = 3, 4, 5$$

(where the first isomorphism is due to $i < p + 1$). Therefore, we get

$$\gamma^{i+1}(\mathcal{B}) = \mathbb{Z}/5\mathbb{Z}^{\oplus(a_{i-2} + a_i)} \quad \text{for } i = 3, 4, 5$$

(with no particular assumption on $\text{char}(F)$). Thus, since $\tau^{3/4}(\mathcal{B}) \simeq Ch^3(\mathcal{B})$, the isomorphism (3.6) for $k = 3$ gives that $\tau^{3/4}(\mathcal{B}) \simeq \gamma^{3/4}(\mathcal{B})$. Since the $\gamma$-filtration is contained in the topological one, we get

$$\tau^4(\mathcal{B}) = \gamma^4(\mathcal{B}),$$

which implies the existence of an exact sequence

$$0 \to (\tau_5(\mathcal{B})/\gamma_5(\mathcal{B})) \to \gamma^{4/5}(\mathcal{B}) \to \tau^{4/5}(\mathcal{B}) \to 0.$$ 

Thus, since $\tau^{4/5}(\mathcal{B}) \simeq Ch^4(\mathcal{B})$, by applying the isomorphism (3.6) for $k = 4$, we get a surjection

$$\mathbb{Z}/5\mathbb{Z}^{\oplus(a_2 + a_4)} \to Ch^4(\mathcal{R}_p(G)) \oplus \mathbb{Z}/5\mathbb{Z}^{\oplus(a_2 + a_4)},$$

which implies that $Ch^4(\mathcal{R}_5(G)) = 0$.

We prove that $Ch^5(\mathcal{R}_5(G)) = 0$ by proceeding in exactly the same way.

Consequently, Proposition 3.3 is proved.

Finally, we want to prove the second conclusion of Theorem 1.1 ($p = 3$ if $G$ is of type $F_4$ and $p = 5$ if $G$ is of type $E_8$). First of all, since for any generic point $\zeta$ of $Y$, one has

$$1 \notin \deg Ch_0(X_{F(\zeta)}) \iff J_p(G_{F(\zeta)}) = (1),$$

by Proposition 3.1 and in view of what has already been done, it is sufficient to prove the following lemma to get the second conclusion.

**Lemma 3.11.** Under the assumption $J_p(G) = (1)$, one has $Ch^{p+1}(\mathcal{R}_p(G)) = 0$.

**Proof.** Thanks to Proposition 2.3, one can prove the lemma by proceeding in exactly the same way Lemma 3.9 has been proved.

This concludes the proof of Theorem 1.1.
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UPMC Sorbonne Universités, Institut de Mathématiques de Jussieu, Paris, FRANCE

Web page: www.math.jussieu.fr/~fino
E-mail address: fino at math.jussieu.fr