KADOMTSEV–PETVIASHVILI HIERARCHIES OF TYPES B AND C

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This is a short review of the Kadomtsev–Petviashvili hierarchies of types B and C. The main objects are the $L$ operator, the wave operator, the auxiliary linear problems for the wave function, the bilinear identity for the wave function, and the tau function. All of them are discussed in the paper. Connections with the usual (type-A) Kadomtsev–Petviashvili hierarchy are clarified. Examples of soliton solutions and the dispersionless limit of the hierarchies are also considered.

Keywords: Kadomtsev–Petviashvili hierarchies, tau function, soliton solutions

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1. Introduction

In [1]–[3], infinite integrable hierarchies of partial differential equations with $O(\infty)$ and $Sp(\infty)$ symmetries were introduced. They can be called the Kadomtsev–Petviashvili hierarchies of type B (BKP) and C (CKP). The BKP hierarchy was also discussed in [4]–[7], and CKP, in [7]–[11].

As was pointed out in [3], the general solutions of the BKP and CKP hierarchies depend on functional parameters in two variables. In a sense to be clarified below, these hierarchies can be regarded as restrictions of the well-known Kadomtsev–Petviashvili (KP) hierarchy. In a nutshell, this can be made more precise as follows. Let $X_{KP}$ be the moduli space of solutions of the KP hierarchy (according to Segal and Wilson, this is an infinite-dimensional Grassmann manifold). The moduli spaces of solutions of the BKP and CKP hierarchies are submanifolds of $X_{KP}$: $X_{BKP} \subset X_{KP}$, $X_{CKP} \subset X_{KP}$, and the “even” time evolutions (i.e., evolutions with respect to the times $t_{2k}$, $k \geq 1$) are frozen.

This paper is a brief review of the BKP and CKP hierarchies. We discuss the main objects and notions related to them: the $L$ operator, the wave operator, the auxiliary linear problems for the wave function, the bilinear identity for the wave function, and the tau function. The tau function satisfies certain equations (the Hirota equations) that are bilinear in the BKP case and have a more complicated structure in the CKP case. The connection between tau functions of the KP, BKP, and CKP hierarchies is clarified.

As examples of solutions, we give explicit formulas for soliton solutions. The BKP and CKP $N$-solitons are specializations of $2N$-soliton solutions of the KP hierarchy. As is known, soliton solutions are degenerations of more general quasiperiodic (algebro-geometric) solutions. According to Krichever’s construction [12], any smooth algebraic curve with some additional data provides a quasiperiodic solution. Quasiperiodic solutions of the BKP hierarchy were constructed in [13] (also see [14]). A detailed discussion of
quasiperiodic solutions of the CKP hierarchy can be found in [11]. The corresponding algebraic curves should admit a holomorphic involution with two fixed points. Solutions that are double-periodic in the complex plane (elliptic solutions) were studied in [15], [16] for the BKP and in [11] for the CKP hierarchy.

We also discuss the dispersionless limit of the BKP and CKP hierarchies, which appears to be the same for both of them. In the dispersionless limit, the operator $\partial_x$ entering the pseudodifferential Lax operator is replaced with a commuting variable $p$, the Lax operator becomes a commuting function (a Laurent series), and the commutator is replaced with the Poisson bracket $\{p, x\} = 1$.

2. The KP hierarchy

Here, we briefly recall the main notions related to the KP hierarchy. The set of independent variables (“times”) is $t = \{t_1, t_2, t_3, \ldots\}$. It is convenient to set $t_1 = x + \text{const}$, such that the vector fields $\partial_{t_1}$ and $\partial_x$ are identical: $\partial_{t_1} = \partial_x$. The main object is the L operator, which is a pseudodifferential operator of the form

$$L = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \cdots,$$

(1)

with no restrictions on the coefficient functions $u_i$. The coefficient functions depend on $x$ and on all the times: $u_i = u_i(x, t)$. Together with the Lax operator, it is convenient to introduce the wave (or dressing) operator

$$W = 1 + \xi_1 \partial_x^{-1} + \xi_2 \partial_x^{-2} + \cdots$$

(2)

such that

$$L = W \partial_x W^{-1}$$

(3)

(the last equality is interpreted as the “dressing” of the operator $\partial_x$ by $W$). Clearly, there is freedom in the definition of the wave operator: it can be multiplied from the right by any pseudodifferential operator with constant coefficients.

The functions $u_i(x, 0)$ are initial conditions for the time evolution $u_i(x, 0) \rightarrow u_i(x, t)$ generated by the Lax equations of the KP hierarchy:

$$\partial_{t_k} L = [B_k, L], \quad B_k = (L^k)_+, \quad k = 1, 2, 3, \ldots,$$

(4)

where $(\cdot)_+$ means the differential part of a pseudodifferential operator (i.e., terms with nonnegative powers of $\partial_x$). In particular, $B_1 = \partial_x$ and $B_2 = \partial_x^2 + 2u_1$. Because $B_1 = \partial_x$, it follows from (4) that the evolution in the time $t_1$ is simply the shift of $x$, i.e., the solutions depend on $x + t_1$.

An equivalent formulation of the hierarchy is in terms of the zero-curvature (Zakharov–Shabat) equations

$$\partial_{t_1} B_k - \partial_{t_k} B_1 + [B_k, B_1] = 0.$$  

(5)

The equivalence of the Lax and Zakharov–Shabat formulations was proved in [17]. The famous KP equation for $u_1$ is obtained from Eq. (5) with $k = 2, l = 3$.

The Lax equations and the zero-curvature equations are the compatibility conditions of the auxiliary linear problems

$$\partial_{t_k} \psi = B_k \psi, \quad L \psi = z \psi$$

(6)

for the formal wave function

$$\psi = \psi(x, t, z) = W e^{xz + \xi(t, z)},$$

(7)

where $W$ is wave operator (2), $z$ is the spectral parameter, and

$$\xi(t, z) = \sum_{k \geq 1} t_k z^k$$

(8)
(it is understood that the operator $\partial_x^{-1}$ acts on the exponential as $\partial_x^{-1} e^{xz} = z^{-1} e^{xz}$). We can also introduce the adjoint (dual) wave function

$$\psi^\dagger = \psi^\dagger(x, t, z) = (W^\dagger)^{-1} e^{-xz - \xi(t, z)}, \quad (9)$$

where $\dagger$ denotes the formal adjoint defined by the rule $(f(x) \circ \partial_x^m)^\dagger = (-\partial_x)^m \circ f(x)$. It can be shown that the adjoint wave function satisfies the adjoint linear equations

$$-\partial_k \psi^\dagger = B_k^\dagger \psi^\dagger. \quad (10)$$

The tau function $\tau^{KP}(x, t)$ of the KP hierarchy is consistently introduced by the equations

$$\psi(x, t, z) = e^{xz + \xi(t, z)} \frac{\tau^{KP}(x, t - [z^{-1}])}{\tau^{KP}(x, t)}, \quad (11)$$

$$\psi^\dagger(x, t, z) = e^{-xz - \xi(t, z)} \frac{\tau^{KP}(x, t + [z^{-1}])}{\tau^{KP}(x, t)}, \quad (12)$$

where we use the standard notation

$$t + j[z^{-1}] = \left\{ t_1 + \frac{j}{z}, t_2 + \frac{j}{2z^2}, t_3 + \frac{j}{3z^3}, \ldots \right\}, \quad j \in \mathbb{Z}.$$ 

The wave functions satisfy the bilinear equation [4]

$$\oint_{C_\infty} \psi(x, t, z) \psi^\dagger(x, t', z) \frac{dz}{2\pi i} = 0 \quad (13)$$

for all $t$ and $t'$. Here, $C_\infty$ is a contour encircling $\infty$ (a big circle of radius $R \to \infty$). Using (11) and (12), we can rewrite (13) as a bilinear relation for the tau function,

$$\oint_{C_\infty} e^{\xi(t-t', z)} \tau^{KP}(x, t - [z^{-1}]) \tau^{KP}(x, t' + [z^{-1}]) \frac{dz}{2\pi i} = 0. \quad (14)$$

This is the generating equation for all differential equations of the KP hierarchy. A direct consequence of bilinear relation (13) is the Hirota–Miwa equation for the tau function of the KP hierarchy

$$(z_1 - z_2) \tau^{KP}(x, t - [z_1^{-1}] - [z_2^{-1}]) \tau^{KP}(x, t - [z_3^{-1}]) +$$

$$+ (z_2 - z_3) \tau^{KP}(x, t - [z_2^{-1}] - [z_3^{-1}]) \tau^{KP}(x, t - [z_1^{-1}]) +$$

$$+ (z_3 - z_1) \tau^{KP}(x, t - [z_3^{-1}] - [z_1^{-1}]) \tau^{KP}(x, t - [z_2^{-1}]) = 0. \quad (15)$$

It is a generating equation for the differential equations of the hierarchy. In the limit $z_3 \to \infty$, it becomes the equation

$$\partial_x \log \frac{\tau^{KP}(x, t + [z_1^{-1}] - [z_2^{-1}])}{\tau^{KP}(x, t)} = (z_2 - z_1) \left( \frac{\tau^{KP}(x, t + [z_1^{-1}]) \tau^{KP}(x, t - [z_2^{-1}])}{\tau^{KP}(x, t) \tau^{KP}(x, t + [z_1^{-1}] - [z_2^{-1}])} - 1 \right). \quad (16)$$

The tau function $\hat{\tau}^{KP}(x, t) = e^{\ell(x, t)} \tau^{KP}(x, t)$, where

$$\ell(x, t) = \gamma_0 + \gamma_1 x + \sum_{k \geq 1} \gamma_k t_k$$

is a linear function of the times, satisfies the same bilinear equations. We say that the tau functions that differ by a factor of the form $e^{\ell(x, t)}$ are equivalent.
3. The BKP hierarchy

Here, we present the main formulas pertaining to the BKP hierarchy in some detail. The main reference is [2] (also see [4]–[7]).

3.1. The Lax operator and the wave operator. The set of independent variables ("times") is \( t_0 = \{ t_1, t_3, t_5, \ldots \} \). They are indexed by positive odd integers. As in the KP hierarchy, we set \( t_1 = x + \text{const} \), and hence the vector fields \( \partial_{t_1} \) and \( \partial_x \) are identical: \( \partial_{t_1} = \partial_x \). The main object is the \( L \) operator, which is a pseudodifferential operator of the form

\[
L = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \cdots
\]  

(17)

with the constraint

\[
L^\dagger = -\partial_x L \partial_x^{-1}.
\]  

(18)

In contrast to the case of a reduction when only a finite number of the coefficient functions \( u_i \) remain independent, constraint (18) implies that there are infinitely many independent coefficients functions. As we see in what follows, constraint (18) is invariant under the "odd" flows \( t_1, t_3, t_5, \ldots \) of the KP hierarchy, and therefore the BKP hierarchy can be regarded as a subhierarchy (a restriction) of the KP hierarchy with the "even" times frozen.

It is instructive to reformulate constraint (18) in terms of the wave operator \( W \) in Eq. (2) such that \( L = W \partial_x W^{-1} \). Constraint (18) implies that \( W^\dagger \partial_x W \) commutes with \( \partial_x \), i.e., it is a pseudodifferential operator with constant coefficients. The freedom in the definition of the wave operator can be fixed by requiring that \( W^\dagger \partial_x W = \partial_x \), i.e.,

\[
W^\dagger = \partial_x W^{-1} \partial_x^{-1}.
\]  

(19)

The Lax equations of the hierarchy are the same as (4), but only with odd indices:

\[
\partial_{t_k} L = [B_k, L], \quad B_k = (L^k)_+, \quad k = 1, 3, 5, \ldots
\]  

(20)

Constraint (18) is equivalent to the condition that the differential operators \( B_k \) satisfy \( B_k \cdot 1 = 0 \) (for odd \( k \)) and therefore have the form

\[
B_k = \partial_x^k + \sum_{j=1}^{k-2} b_{k,j} \partial_x^j
\]

with \( b_{k,0} = 0 \). Indeed, if (18) is satisfied, then \( L^n \partial_x^{-1} = -\partial_x^{-1} L^n \partial_x^{-1} \) for odd \( n \). On the other hand, \( (L^n \partial_x^{-1})^\dagger = -\partial_x^{-1} L^n \partial_x^{-1} \), which implies that the coefficient in front of \( \partial_x^0 \) in \( L^n \) vanishes: \( b_{n,0} = 0 \) for odd \( n \). Conversely, assuming that \( b_{n,0} = 0 \) for all odd \( n \), we prove that \( R = \partial_x^{-1} L^1 + L \partial_x^{-1} = 0 \). Obviously, \( R \) is of the general form

\[
R = a \partial_x^{-m} + \text{lower-order terms},
\]

and the identity \( R^\dagger = -R \) implies that if \( a \) is not identically zero, then \( m \) is odd. Then we have (see [1])

\[
L^m \partial_x^{-1} = (R \partial_x - \partial_x^{-1} L^1 \partial_x)^m \partial_x^{-1} = -\partial_x^{-1} L^1 \partial_x^m \partial_x^{-1} + m R(\partial_x^{-1} L^1 \partial_x)^m -1 + \text{an operator of order less than } -1, =
\]

\[
= (L^n \partial_x^{-1})^\dagger + ma \partial_x^{-1} + \text{lower-order terms},
\]

which contradicts the assumption that \( a \neq 0 \).
We note that constraint (18) implies

\[(L^†_1)_{+} = -(\partial_x L^k \partial^{-1}_x)_{+} = -(L^k)_{+} - ((\partial_x L^k \partial^{-1}_x)_{+}, \]

which can be rewritten as

\[B^1_k = -\partial_x B_k \partial^{-1}_x, \quad k \text{ is odd} \quad (21)\]

(taking into account that \(b_{k,0} = 0\)). Using this relation and the Lax equations, it is straightforward to verify that constraint (18) is indeed invariant under odd flows of the KP hierarchy:

\[\partial_t (L^1 + \partial_x L \partial^{-1}_x) = 0, \quad k \text{ is odd} \quad (22)\]

Therefore, the BKP hierarchy is well defined as a subhierarchy of the KP hierarchy.

The first three differential operators \(B_k\) are as follows:

\[B_1 = \partial_x, \quad B_3 = \partial^3_x + 6u \partial_x, \quad u = \frac{1}{2} u_1, \quad B_5 = \partial^5_x + 10u \partial^3_x + 10u' \partial^2_x + v \partial_x. \quad (23)\]

An equivalent formulation of the hierarchy is through the zero-curvature equations

\[\partial_t B_k - \partial_k B_l + [B_k, B_l] = 0, \quad k, l \text{ are odd}. \quad (24)\]

The first equation of the BKP hierarchy follows from the zero-curvature equation \(\partial_t B_5 - \partial_5 B_3 + [B_5, B_3] = 0\). Calculations yield the following system of equations for the unknown functions \(u\) and \(v\):

\[3v' = 10u_5 + 20u''' + 120uu', \quad v_5 - 6u_5 = v''' - 6u'''' - 60uu'' - 60u'u'' + 6uu' - 6vu'. \quad (25)\]

We note that the variable \(v\) can be eliminated by passing to the unknown function \(U\) such that \(U' = u\).

3.2. The wave function and the tau function. The Lax equations and the zero-curvature equations are the compatibility conditions of auxiliary linear problems (6) for the formal wave function

\[\psi = \psi(x, t_0, z) = We^{xz + \xi(t_0, z)}, \quad (26)\]

where

\[\xi(t_0, z) = \sum_{k \geq 1, \text{odd}} t_k z^k. \quad (27)\]

As follows from (2), the wave function \(\psi = \psi(x, t_0, z)\) has the following expansion as \(z \to \infty\):

\[\psi(x, t_0, z) = e^{xz + \xi(t_0, z)} \left(1 + \sum_{k \geq 1} \xi_k z^{-k}\right). \quad (28)\]

As is proved in [3], the wave function satisfies the bilinear relation

\[\int_{C_\infty} \psi(x, t_0, z)\psi(x, t_0', -z) \frac{dz}{2\pi iz} = 1, \quad (29)\]
valid for all \( t_0 \) and \( t'_0 \). For completeness, we give a sketch of the proof here. By virtue of differential equations (20), the bilinear relation is equivalent to the vanishing of

\[
b_m = \partial_{x'}^m \oint_{C_{\infty}} \psi(x, t_0, z) \psi(x', t_0, -z) \frac{dz}{2\pi i z} \bigg|_{x' = x}
\]

with the additional condition that

\[
b_0 = \oint_{C_{\infty}} \psi(x, t_0, z) \psi(x, t_0, -z) \frac{dz}{2\pi i z} = 1.
\]

We have

\[
b_m = \oint_{C_{\infty}} \left( \sum_{k \geq 0} \xi_k (x) z^{-k} \right) \partial_{x'}^m \left( \sum_{l \geq 0} \xi_l (x') (-z)^{-l} \right) e^{(x-x')z} \frac{dz}{2\pi i z} \bigg|_{x' = x} = \frac{dz}{2\pi i z} = \sum_{j+k+l=m} (-1)^{m+j+l} \left( \begin{array}{c} m \\ j \end{array} \right) \xi_k \partial_x^j \xi_l.
\]

But the last expression is the coefficient of \((-1)^m \partial_x^{-m-1}\) in the operator \( W \partial_x^{-1} W^t \):

\[
W \partial_x^{-1} W^t = \partial_x^{-1} + \sum_{m \geq 1} (-1)^m b_m \partial_x^{-m-1}.
\]

Because \( W \partial_x^{-1} W^t = \partial_x^{-1} \), we conclude that \( b_m = 0 \) for all \( m \geq 1 \) and \( b_0 = 1 \).

The tau function \( \tau = \tau(x, t_0) \) of the BKP hierarchy can be consistently introduced by the formula

\[
\psi = e^{xz+\xi(t_0, z)} \frac{\tau(x, t_0 - 2[z^{-1}]_0)}{\tau(x, t_0)}, \quad (30)
\]

where

\[
t_0 + k[z^{-1}]_0 = \left\{ t_1 + \frac{k}{z}, t_3 + \frac{k}{3z^3}, t_5 + \frac{k}{5z^5}, \ldots \right\}, \quad k \in \mathbb{Z}. \quad (31)
\]

The proof of the existence of the tau function is based on the bilinear relation. We represent the wave function in the form

\[
\psi(x, t_0, z) = e^{xz+\xi(t_0, z)} w(x, t_0, z)
\]

and set \( t'_0 = t_0 - 2[a^{-1}]_0 \) in the bilinear relation. We have \( e^{\xi(t_0-t'_0, z)} = (a+z)/(a-z) \) and the residue calculus yields

\[
w(t_0, a) w(t_0 - 2[a^{-1}]_0, -a) = 1, \quad (32)
\]

where we do not indicate the dependence on \( x \) for brevity. Next, we set \( t'_0 = t_0 - 2[a^{-1}]_0 - 2[b^{-1}]_0 \) in the bilinear relation, whence \( e^{\xi(t_0-t'_0, z)} = (a+z)(b+z)/((a-z)(b-z)) \). In this case, the residue calculus yields

\[
w(t_0, a) w(t_0 - 2[a^{-1}]_0 - 2[b^{-1}]_0, -a) = w(t_0, b) w(t_0 - 2[a^{-1}]_0 - 2[b^{-1}]_0, -b). \quad (33)
\]

With the help of (32) this last relation can be rewritten as

\[
\frac{w(t_0, a) w(t_0 - 2[a^{-1}]_0, b)}{w(t_0, b) w(t_0 - 2[b^{-1}]_0, a)} = 1. \quad (34)
\]
We now show that Eq. (34) implies the existence of a function \( \tau(t_o) \) such that

\[
  w(t_o, z) = \frac{\tau(t_o - 2[z^{-1}]_o)}{\tau(t_o)}. \tag{35}
\]

To see this, we represent (35) in an equivalent form. Taking the logarithm and the \( z \)-derivative in (35), we obtain

\[
  \partial_z \log w = 2 \sum_{m \geq 1, \text{odd}} z^{-m-1} \partial_t^m \log \tau(t_o - 2[z^{-1}]_o),
\]

or, with \( \tau(t_o - 2[z^{-1}]_o) \) in the right-hand side expressed through \( w(t_o, z) \) and \( \tau(t_o) \) from (35),

\[
  \partial_z \log w = 2 \partial_{t_o} (z) \log w + 2 \partial_{t_o} (z) \log \tau, \tag{36}
\]

where \( \partial_{t_o} (z) \) is the differential operator

\[
  \partial_{t_o} (z) = \sum_{j \text{ odd}} z^{-j-1} \partial_j.
\]

In fact (36) is equivalent to (35). Indeed, writing (36) as \( (\partial_z - 2 \partial_{t_o} (z)) \log(w \tau) = 0 \), we conclude that \( w \tau = \rho \) is a function of \( t_o - 2[z^{-1}]_o \), and the normalization condition \( w(t_o, \infty) = 1 \) implies that \( \rho = \tau \), and we thus arrive at (35).

Equation (35) means that

\[
  Y_n := \text{res}_{z=\infty} \left[ z^n (\partial_z - 2 \partial_{t_o} (z)) \log w \right] = 2 \frac{\partial \log \tau}{\partial_t},
\]

where the residue is defined as \( \text{res}_{z=\infty} (z^{n-1}) = \delta_{n0} \). Therefore, the existence of the tau function is proved if we prove that \( \partial_t \partial_{t_o} Y_m(t_o) = \partial_{t_o} Y_n(t_o) \). Changing \( a \to z \) and \( b \to \zeta \) in (34) and applying the operator \( \partial_z - 2 \partial_{t_o} (z) \) to the logarithm of this equality, we rewrite it as

\[
  (\partial_z - 2 \partial_{t_o} (z)) \log w(t_o, z) - (\partial_z - 2 \partial_{t_o} (z)) \log w(t_o - 2[\zeta^{-1}]_o, z) = -2 \partial_{t_o} (z) \log w_0(t_o, \zeta),
\]

or

\[
  Y_n(t_o) - Y_n(t_o - 2[\zeta^{-1}]_o) = -2 \partial_{t_o} \log w(t_o, \zeta). \tag{37}
\]

Therefore, setting \( F_{mn} = \partial_{t_o} Y_m - \partial_{t_o} Y_n \), we see from (37) that

\[
  F_{mn}(t_o) = F_{mn}(t_o - 2[\zeta^{-1}]_o). \tag{38}
\]

This equality is valid identically in \( \zeta \). Expanding its right-hand side in a power series,

\[
  F_{mn}(t_o - 2[\zeta^{-1}]_o) = F_{mn}(t_o) - 2\zeta^{-1} \partial_{t_1} F_{mn}(t_o) - \frac{2}{3} \zeta^{-3} (\partial_{t_3} F_{mn}(t_o) + 2 \partial_{t_i}^3 F_{mn}(t_o)) + \cdots,
\]

we conclude from comparing the \( \zeta^{-1} \) terms that \( F_{mn} \) is independent of \( t_1 \). From the \( \zeta^{-3} \) terms, we see that \( F_{mn} \) is independent of \( t_3 \), and so on. In this way, we can conclude that it does not depend on \( t_k \) for all (odd) \( k \), i.e., \( F_{mn} = 2a_{mn} \), where \( a_{mn} \) are some constants such that \( a_{mn} = -a_{nm} \). Therefore, we can write

\[
  Y_n = \sum_m a_{mn} t_m + \partial_{t_n} h
\]
with some function \( h = h(t_o) \). From (37), we then have

\[
-2 \partial_{t_o} \log w(t_o, z) = \partial_{t_o} (h(t_o) - h(t_o - 2 |z^{-1}|_o)) + 2 \sum_{m \text{ odd}} \frac{a_{mn}}{m} z^{-m},
\]

or, after integration,

\[
\log w(t_o, z) = \frac{1}{2} h(t_o - 2 |z^{-1}|_o) - \frac{1}{2} h(t_o) - \sum_{m \text{ odd}} \frac{a_{mn}}{m} z^{-m} t_n + \varphi(z),
\]

where \( \varphi(z) \) is a function of only \( z \). Substituting this in (34), we conclude that \( a_{mn} = 0 \), and hence

\[
\partial_{t_o} Y = \partial_{t_o} Y_m.
\]

We now show how to easily obtain (30) up to a common \( x \)-independent factor. We apply \( \partial_{t_1} \) to (29) and set \( t'_o = t_o - 2 |a^{-1}|_o \). The residue calculus yields

\[
2a(w(t_o, a)w(t_o - 2 |a^{-1}|_o, -a) - 1) + 2w'(t_o, a)w(t_o - 2 |a^{-1}|_o, -a) +
\]

\[
+ \xi_1(t_o - 2 |a^{-1}|_o) - \xi_1(t_o) = 0.
\]

Using (32), we conclude from (39) that

\[
\partial_x \log w(t_o, a) = \frac{1}{2} (\xi_1(t_o) - \xi_1(t_o - 2 |a^{-1}|_o)).
\]

Now, setting \( \xi_1(x, t_o) = -2 \partial_x \log \tau(x, t_o) \) and integrating, we arrive at (30) up to a common \( x \)-independent factor.

The function \( u \) in (23) can also be expressed through the tau function with the help of the following argument. It is a matter of direct verification that the result of the action of \( \partial^3_x + 6u \partial_x - \partial_{t_1} \) on the wave function \( \psi \) of form (28) is \( O(z^{-1}) \) as \( z \to \infty \), i.e.,

\[
(\partial^3_x + 6u \partial_x - \partial_{t_1}) \psi = O(z^{-1})e^{xz + \xi(t_o, z)},
\]

if the conditions

\[
u = -\frac{1}{2} \xi', \quad \xi_1 \xi_1' - \xi_1'' - \xi_2' = 0
\]

are satisfied (we actually know that \( (\partial^3_x + 6u \partial_x - \partial_{t_1}) \psi = 0 \), but here we only need the weaker condition in Eq. (41)). It follows from (30) that

\[
\xi_1 = -2 \partial_x \log \tau, \quad \xi_2 = 2(\partial_x \log \tau)^2 + 2 \partial^2_x \log \tau,
\]

and therefore

\[
u = \partial^2_x \log \tau,
\]

and the second equality in (42) holds identically.

Changing the dependent variables in accordance with (43) from \( u, v \) to the tau function,

\[
v = \frac{10}{3} \partial_{t_1} \partial_x \log \tau + \frac{20}{3} \partial^4_x \log \tau + 20(\partial^2_x \log \tau)^2,
\]

makes the first equation in (25) trivial and the other bilinear [2],

\[
(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_4 D_5) \tau \cdot \tau = 0,
\]
where \( D_i \) are the Hirota operators defined by the rule
\[
P(D_1, D_3, D_5, \ldots) \tau \cdot \tau = P(\partial_{y_1}, \partial_{y_3}, \partial_{y_5}, \ldots) \tau(x, t_1 + y_1, t_3 + y_3, \ldots) \tau(x, t_1 - y_1, t_3 - y_3, \ldots)|_{y_i = 0}
\]
for any polynomial \( P(D_1, D_3, D_5, \ldots) \).

As follows from (30), the BKP hierarchy is equivalent to the following relation for the tau function:
\[
\oint_{C_\infty} e^{\xi(t_o, t'_o, z)} \tau(x, t_o - 2[z^{-1}]_o) \tau(x, t'_o + 2[z^{-1}]_o) \frac{dz}{2\pi iz} = \tau(x, t_o) \tau(x, t'_o)
\]  
(46)

for all \( t_o \) and \( t'_o \). We set \( t'_o = t_o - 2[a^{-1}]_o - 2[b^{-1}]_o - 2[c^{-1}]_o \); then
\[
e^{\xi(t_o, t'_o, z)} = \frac{(a + z)(b + z)(c + z)}{(a - z)(b - z)(c - z)}
\]
and using the residue calculus in (46) gives the equation
\[
(a + b)(a + c)(b - c)\tau[a] \tau[bc] + (b + a)(b + c)(c - a)\tau[b] \tau[ac] +
+ (c + a)(c + b)(a - b)\tau[c] \tau[ab] + (a - b)(b - c)(c - a)\tau[abc] = 0,
\]  
(47)
where \( \tau[a] = \tau(x, t_o + 2[a^{-1}]_o) \), \( \tau[ab] = \tau(x, t_o + 2[a^{-1}]_o + 2[b^{-1}]_o) \), and so on. Equation (47) must hold for all \( a, b, \) and \( c \). Taking the limit \( c \to \infty \), we obtain the equation
\[
\tau \tau[ab] \left( 1 + \frac{1}{a + b} \partial_{t_o} \log \frac{\tau}{\tau[ab]} \right) = \tau[a] \tau[b] \left( 1 + \frac{1}{a - b} \partial_{t_o} \log \frac{\tau[b]}{\tau[a]} \right).
\]  
(48)
This is the equation for the tau function of the BKP hierarchy. It must hold for all \( a \) and \( b \). The differential equations of the hierarchy are obtained by expanding it in inverse powers of \( a \) and \( b \).

### 3.3. The BKP hierarchy and the KP hierarchy.
Here, we compare the spaces of solutions of the BKP and KP hierarchies by an explicit embedding of the former into the latter on the level of tau functions.

We begin with the adjoint wave function
\[
\psi^b(x, t_o, z) = (W^1)^{-1} e^{-xz - \xi(t_o, z)} = \partial_x W \partial_{x^{-1}} e^{-xz - \xi(t_o, z)} = -z^{-1} \partial_x \psi(x, t_o, -z),
\]  
(49)
for which, as a consequence of (29), bilinear relation (13) becomes
\[
\oint_{C_\infty} \psi(x, t_o, z) \partial_x \psi(x, t'_o, -z) \frac{dz}{2\pi iz} = 0.
\]  
(50)

Using (11) and (12), we can write (49) in terms of the KP tau function:
\[
e^{-xz} \frac{\tau^{\text{KP}}(x, \dot{t} + [z^{-1}])}{\tau^{\text{KP}}(x, \dot{t})} = -z^{-1} \partial_x \left( \frac{e^{-xz} \tau^{\text{KP}}(x, \dot{t} - [-z^{-1}])}{\tau^{\text{KP}}(x, \dot{t})} \right)
\]
or
\[
\partial_x \log \frac{\tau^{\text{KP}}(x, \dot{t} + [z^{-1}])}{\tau^{\text{KP}}(x, \dot{t})} = -z \left( 1 - \frac{\tau^{\text{KP}}(x, \dot{t} - [-z^{-1}])}{\tau^{\text{KP}}(x, \dot{t} - [z^{-1}])} \right),
\]  
(51)
where we use the short-hand notation \( t = \{t_1, t_3, 0, \ldots \} \). Shifting the times \( t_3 \), we can rewrite this as

\[
\partial_x \log \frac{\tau^\text{KP}(t_1, -z^{-2}/2, t_3, -z^{-4}/4, \ldots)}{\tau^\text{KP}(t_1, t_3 - z^{-3}/3, 0, \ldots)} = -z + z \frac{\tau^\text{KP}(t_1, -z^{-2}/2, t_3, -z^{-4}/4, \ldots)}{\tau^\text{KP}(t_1, t_3 - z^{-3}/3, 0, \ldots)},
\]

or, subtracting these equalities with \( z \) and \(-z\) from each other,

\[
\partial_x \log \frac{\tau^\text{KP}(x, t_1 - z^{-1}, 0, t_3 - z^{-3}/3, 0, \ldots)}{\tau^\text{KP}(x, t_1 + z^{-1}, 0, t_3 + z^{-3}/3, 0, \ldots)} = -2z + 2z \frac{\tau^\text{KP}(x, t_1, -z^{-2}/2, t_3, -z^{-4}/4, \ldots)}{\tau^\text{KP}(x, t_1, -z^{-3}/3, 0, \ldots)}.
\]

(52)

Comparing (52) with the KP hierarchy in form (16) at \( z_2 = -z_1 = z \), we conclude that

\[
(\tau^\text{KP}(x, \dot{t} - [z^{-1}]))^2 = \tau^\text{KP}(x, \dot{t}) \tau^\text{KP}(x, \dot{t} - 2[z^{-1}]_o)
\]

(53)

(in the second tau function in the right-hand side, the even times are set equal to 0). This is the constraint that distinguishes solutions of the BKP hierarchy among all solutions of the KP hierarchy.

Another way to arrive at (53) is to note that we have two different expressions for the wave function \( \psi \) (one in terms of the KP tau function \( \tau^\text{KP} \) and the other in terms of the BKP tau function \( \tau \)), whence it follows that

\[
\frac{\tau^\text{KP}(x, \dot{t} - [z^{-1}])}{\tau^\text{KP}(x, \dot{t})} = \frac{\tau(x, t_o - 2[z^{-1}]_o)}{\tau(x, t_o)}
\]

or, after a shift of the times \( t_o \),

\[
\log \frac{\tau^\text{KP}(x, t_1, -z^{-2}/2, t_3, -z^{-4}/4, \ldots)}{\tau^\text{KP}(x, t_1 + z^{-1}, 0, t_3 + z^{-3}/3, 0, \ldots)} = \log \frac{\tau(x, t_o - [z^{-1}]_o)}{\tau(x, t_o - [-z^{-1}]_o)}.
\]

(54)

The right-hand side is an odd function of \( z \), and therefore we have

\[
\log \frac{\tau^\text{KP}(x, t_1, -z^{-2}/2, t_3, -z^{-4}/4, \ldots)}{\tau^\text{KP}(x, t_1 + z^{-1}, 0, t_3 + z^{-3}/3, 0, \ldots)} + \log \frac{\tau^\text{KP}(x, t_1, -z^{-2}/2, t_3, -z^{-4}/4, \ldots)}{\tau^\text{KP}(x, t_1 - z^{-1}, 0, t_3 - z^{-3}/3, 0, \ldots)} = 0,
\]

which is (53).

Constraint (53) must hold for all values of \( t_1, t_3, \ldots \) and \( z \). Expanding it in powers of \( z \), we can represent it as an infinite number of differential constraints, the first of which is

\[
(\partial_{t_2} + \partial_t^2) \log \tau^\text{KP} |_{t_{2k}} = 0, \quad k \geq 1.
\]

(55)

This constraint was mentioned in [14].

We now represent (53) in the form

\[
\frac{\tau^\text{KP}(x, \dot{t} - 2[z^{-1}]_o)}{\tau^\text{KP}(x, \dot{t})} = \left( \frac{\tau^\text{KP}(x, \dot{t} - [z^{-1}])}{\tau^\text{KP}(x, \dot{t})} \right)^2 = \frac{\tau^2(x, t_o - 2[z^{-1}]_o)}{\tau^2(x, t_o)}.
\]

(56)

It then follows that

\[
\tau(x, t_o) = C \sqrt{\tau^\text{KP}(x, \dot{t})}
\]

(57)

and \( \tau^\text{KP}(x, \dot{t}) \) is a full square, i.e., \( \sqrt{\tau^\text{KP}(x, \dot{t})} \) does not have square-root singularities in all the times.

3.4. Examples: soliton solutions. \( N \)-soliton solutions of the BKP hierarchy are obtained by imposing certain constraints on the parameters of \( 2N \)-soliton solutions of the KP hierarchy. The tau function of the BKP hierarchy is related to the KP tau function as \( \tau = \sqrt{\tau^\text{KP}} \), with “even” times \( t_{2k} \) set equal to zero; it is also implied that the parameters of the KP tau function \( \tau^\text{KP} \) are chosen in a special way. With this choice, \( \tau^\text{KP} \) is a full square, i.e., \( \tau \) does not have square-root singularities.
3.4.1. One-soliton solution. The tau function for one BKP soliton is the square root of a specialization of the 2-soliton tau function of the KP hierarchy:

\[ \tau_{KP}^{(1)} = 1 + \alpha(p-q)w + \frac{\alpha^2}{4}(p-q)^2w^2 = \left(1 + \frac{\alpha}{2}(p-q)w\right)^2, \quad (58) \]

where

\[ w = e^{(p+q)x+\xi(t_o,p)+\xi(t_o,q)}, \quad (59) \]

\[ \xi(t_o,z) \] is given by (27) and \( \alpha, p, \) and \( q \) are arbitrary parameters. Therefore, the tau function of the BKP hierarchy is

\[ \tau = 1 + \frac{\alpha}{2}(p-q)w. \quad (60) \]

It is an entire function of \( x. \)

We note that the extension of the tau function \( \tau_{KP}^{(n)} \) to the modified KP (mKP) hierarchy has the form

\[ \tau_{mKP}^{(n)} = 1 + \alpha \left(-q \left( \frac{p}{q} \right)^n + p \left( \frac{q}{p} \right)^n \right)w + \frac{\alpha^2}{4}(p-q)^2w^2, \quad (61) \]

where \( n \) is the integer-valued “zeroth time” (clearly, \( \tau_{mKP}^{(0)} = \tau_{KP}^{(0)} \)). We then see that the parameters of the soliton solutions are such that \( \tau_{mKP}^{(1-n)} = \tau_{mKP}^{(n)} \), which is the constraint necessary for the BKP hierarchy [4, 18].

Bilinear identity (29), which in the present case has the explicit form

\[ \oint_{C_{\infty}} e^{\xi(t_o,z)-\xi(t'_o,z)} \left(1 + \frac{\alpha}{2}(z-p)(z-q)(p-q)w\right) \left(1 + \frac{\alpha}{2}(z+p)(z-q)(p-q)w'\right) \frac{dz}{2\pi i z} = \]

\[ = \left(1 + \frac{\alpha}{2}(p-q)w\right) \left(1 + \frac{\alpha}{2}(p-q)w'\right), \]

can be verified directly by residue calculus. (Here, \( w' = e^{(p+q)x+\xi(t'_o,p)+\xi(t'_o,q)}. \))

3.4.2. Multi-soliton solutions. The general KP tau function for the \( 2N \)-soliton solution has \( 6N \) arbitrary parameters \( \alpha_i, p_i, q_i, i = 1, \ldots, 2N \), and is given by

\[ \tau_{KP} = \det \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{2N-1} & \alpha_{2N} \\ p_1, q_1 & p_2, q_2 & p_3, q_3 & \cdots & p_{2N-1}, q_{2N-1} & p_{2N}, q_{2N} \end{bmatrix} = \]

\[ = \det_{1 \leq i, k \leq 2N} \left( \delta_{ik} + \alpha_i \frac{p_i - q_i e^{(p_i-q_i)x+\xi(t,p_i)-\xi(t,q_i)}}{p_i - q_k} \right). \quad (62) \]

The \( N \)-soliton tau function of the BKP hierarchy is the square root of \( \tau_{KP} \) specialized as

\[ \tau_{KP} = \begin{bmatrix} -q_1 \alpha_1 & p_1 \alpha_1 & -q_2 \alpha_2 & p_2 \alpha_2 & \cdots & -q_N \alpha_N & p_N \alpha_N \\ p_1, -q_1 & q_1, p_1 & p_2, -q_2 & q_2, p_2 & \cdots & p_N, -q_N & q_N, p_N \end{bmatrix}, \quad (63) \]

where it is implied that evolution along even times is frozen \( (t_{2k} = 0 \text{ for all } k \geq 1). \) It can be proved that this KP tau function is a full square, i.e., \( \tau = \sqrt{\tau_{KP}} \) is an entire function of the times (no square-root singularities).

4. The CKP hierarchy

In this section, we present the main formulas related to the CKP hierarchy in some detail. The main references are [3], [7], [8].
4.1. The CKP equation and hierarchy. The set of independent variables (“times”) is \( t_o = \{t_1, t_3, t_5, \ldots \} \). As in the BKP hierarchy, they are indexed by positive odd numbers. We set \( t_1 = x + \text{const} \). The main object is the \( L \) operator in Eq. (1) with the constraint

\[
L^\dagger = -L.
\] (64)

It is instructive to reformulate this constraint in terms of the wave operator \( W \) in (2) such that

\[
L^\dagger = L.
\] (65)

The evolution equations (the Lax equations) and the zero-curvature equations have the same form (20) and (24) as in the BKP hierarchy. By applying the \( \dagger \)-operation to evolution equations (20), it is not difficult to see that they are consistent with constraint (64), i.e., \( \partial_{t_k}(L + L^\dagger) = 0 \) for odd \( k \), and hence the CKP hierarchy is well-defined.

Clearly, the differential operators \( B_k \) satisfy \( B_k^\dagger = -B_k \) (for odd \( k \)). In particular,

\[
B_1 = \partial_x, \quad B_3 = \partial_x^3 + 6u \partial_x + 3u', \quad B_5 = \partial_x^5 + 10u \partial_x^3 + 15u' \partial_x^2 + v \partial_x + \frac{1}{2}(v' - 5u''),
\] (66)

where \( u' \equiv \partial_x u \), \( u = u_1/2 \). because \( B_1 = \partial_x \), it follows from (20), similarly to the case of the KP hierarchy, that the evolution along the time \( t_1 \) is simply a shift of \( x \), i.e., the solutions depend on \( x + t_1 \).

The first equation of the CKP hierarchy follows from the zero-curvature equation \( \partial_{t_3} B_5 - \partial_{t_5} B_3 + [B_5, B_3] = 0 \) with \( B_3 \) and \( B_5 \) as in (66). Calculations yield the following system of equations for the unknown functions \( u \) and \( v \):

\[
10 \partial_{t_3} u = 3v' - 35u''' - 120uu',
6 \partial_{t_5} u - \partial_{t_3} v = \frac{57}{2}u''' + 150uu''' + 180u' u'' - \frac{5}{2}v'' + 6vu' - 6uv'.
\] (67)

We note that the variable \( v \) can be eliminated by passing to the unknown function \( U \) such that \( U' = u \).

4.2. The wave function and the tau function. Similarly to the KP and BKP hierarchies, the Lax equations and the zero-curvature equations are compatibility conditions of the auxiliary linear problems

\[
\partial_{t_o} \Psi = B_o \Psi, \quad L \Psi = z \Psi
\] (68)

for the formal wave function

\[
\Psi = \Psi(x, t_o, z) = W e^{xz + \xi(t_o, z)},
\] (69)

where \( z \) is the spectral parameter and \( \xi(t_o, z) \) is defined in (27). The wave function has the following expansion as \( z \to \infty \):

\[
\Psi(x, t_o, z) = e^{xz + \xi(t_o, z)} \left( 1 + \sum_{k \geq 1} \xi_k z^{-k} \right).
\] (70)

As is proved in [3], it satisfies the bilinear relation

\[
\oint_{C_\infty} \Psi(x, t_o, z) \Psi(x, t_o', -z) \frac{dz}{2\pi i} = 0,
\] (71)

valid for all \( t_o, t_o' \).
Here is a sketch of the proof. By virtue of differential equations (68), the bilinear relation is equivalent to the vanishing of

\[ b_m = \partial_x^m \tilde{F}_{C_{\infty}} \Psi(x, t_o, z) \Psi(x', t_o, -z) \frac{dz}{2\pi i} \bigg|_{x'=x} \quad \text{for all } m \geq 0 \]

The further calculation is similar to the one done in the BKP case. We have

\[ b_m = \int_{C_{\infty}} \left( \sum_{k \geq 0} \xi_k(x) z^{-k} \right) \partial_x^m \left( \sum_{l \geq 0} \xi_l(x') (-z)^{-l} \right) e^{(x-x')z} \frac{dz}{2\pi i} \bigg|_{x'=x} = \]

\[ = \sum_{j+k+l=m+1} (-1)^{m+j+l} \binom{m}{j} \xi_k \partial_x^j \xi_l. \]

It remains to note that this expression is the coefficient of \((-1)^m \partial_x^{-m-1}\) in the operator \(WW^\dagger\):

\[ WW^\dagger = 1 + \sum_{m \geq 0} (-1)^m b_m \partial_x^{-m-1}. \]

Because \(WW^\dagger = 1\) (see (65)), we conclude that \(b_m = 0\) for all \(m \geq 0\).

The tau function \(\tau = \tau(x, t_o)\) of the CKP hierarchy is consistently introduced by the formula [7], [8]

\[ \Psi = e^{xz + \xi(t_o, z)} G(x, t_o, z) \frac{\tau(x, t_o - 2[z]^{-1}t_o)}{\tau(x, t_o)}, \tag{72} \]

where

\[ G(x, t_o, z) = \left( 1 + z^{-1} \partial_t, \log \frac{\tau(x, t_o - 2[z]^{-1}t_o)}{\tau(x, t_o)} \right)^{1/2} \tag{73} \]

(cf. (30), where there is no factor \(G(x, t_o, z)\)). Formula (72) (and the very existence of the tau function) is based on bilinear relation (71). The proof can be found in [11]. For completeness, we give it here. We represent the wave function in the form

\[ \Psi(x, t_o, z) = e^{xz + \xi(t_o, z)} w(x, t_o, z) \]

and set \(t_o' = t_o - 2[a^{-1}]_o\) in the bilinear relation. We have \(e^{\xi(t_o-t_o', z)} = (a+z)/(a-z)\). The residue calculus in (71) yields

\[ w(t_o, a)w(t_o - 2[a^{-1}]_o, -a) = f(t, a), \tag{74} \]

where

\[ f(t_o, z) = 1 + \frac{1}{2z} (\xi_1(t_o) - \xi_1(t_o - 2[z]^{-1}t_o)) \tag{75} \]

(here and below, we do not indicate the \(x\)-dependence for brevity). Next, we set \(t_o' = t_o - 2[a^{-1}]_o - 2[b^{-1}]_o\) in the bilinear relation and the residue calculus yields

\[ \frac{a + b}{a - b} (aw(t_o, a)w(t_o - 2[a^{-1}]_o - 2[b^{-1}]_o, -a) - \]

\[ - bw(t_o, b)w(t_o - 2[a^{-1}]_o - 2[b^{-1}]_o, -b)) = \]

\[ = a + b + \frac{1}{2} (\xi_1(t_o) - \xi_1(t_o - 2[a^{-1}]_o - 2[b^{-1}]_o)). \tag{76} \]
Expressing \( w(\ldots, -a) \), \( w(\ldots, -b) \) through \( w(\ldots, a) \), \( w(\ldots, b) \) by means of relation (74), we can represent this equation in the form

\[
\frac{1}{a - b} \left( a f(t_o - 2[b^{-1}]_o, a) \frac{w(t_o, a)}{w(t_o, 2[b^{-1}]_o, a)} - b f(t_o - 2[a^{-1}]_o, b) \frac{w(t_o, b)}{w(t_o, 2[a^{-1}]_o, b)} \right) = 1 + \frac{\xi_1(t_o) - \xi_1(t_o - 2[a^{-1}]_o - 2[b^{-1}]_o)}{2(a + b)}.
\]

(77)

Shifting \( t_o \rightarrow t_o + 2[b^{-1}]_o \) here, changing the sign of \( b \) (i.e. substituting \( b \rightarrow -b \)), and then using (74) again (in the second term in the left-hand side), we arrive at the equivalent equation

\[
\frac{1}{a + b} \left( a f(t_o, a) \frac{w(t_o - 2[b^{-1}]_o, a)}{w(t_o, a)} - b f(t_o, b) \frac{w(t_o - 2[a^{-1}]_o, b)}{w(t_o, b)} \right) = 1 + \frac{\xi_1(t_o - 2[b^{-1}]_o) - \xi_1(t_o - 2[a^{-1}]_o)}{2(a - b)}.
\]

(78)

Taken together, Eqs. (77), (78) form the system of two equations

\[
\frac{1}{a - b} (a f(t_o - 2[b^{-1}]_o, a) X - b f(t_o - 2[a^{-1}]_o, b) Y) = \frac{a f(t_o, a) - b f(t_o, b)}{a + b},
\]

(79)

\[
\frac{1}{a + b} (a f(t_o, a) X - b f(t_o, b) Y) = \frac{a f(t_o, a) - b f(t_o, b)}{a - b}
\]

for the two “unknown quantities”

\[
X = \frac{w(t_o - 2[b^{-1}]_o, a)}{w(t_o, a)}, \quad Y = \frac{w(t_o - 2[a^{-1}]_o, b)}{w(t_o, b)}.
\]

(80)

The next step is to take the product of the two equations in (79). After some algebra, we obtain the simple relation

\[
Y = \frac{w(t_o, a) w(t_o - 2[a^{-1}]_o, b)}{w(t_o, b) w(t_o - 2[b^{-1}]_o, a)} (f(t_o, a) f(t_o - 2[a^{-1}]_o, b))^{1/2}.
\]

(81)

In the calculations, the identity

\[
a f(t_o, a) - a f(t_o - 2[b^{-1}]_o, a) - b f(t_o, b) + b f(t_o - 2[a^{-1}]_o, b) = 0.
\]

(82)

has been used. Introducing \( w_0(t_o, z) = w(t_o, z) f^{-1/2}(t_o, z) \), we then obtain the relation

\[
\frac{w_0(t_o, a) w_0(t_o - 2[a^{-1}]_o, b)}{w_0(t_o, b) w_0(t_o - 2[b^{-1}]_o, a)} = 1,
\]

(83)

which has the same form as (34) for the BKP hierarchy, with the change in the notation \( w \rightarrow w_0 \).

As soon as the relation of form (83) is established, the rest of the argument is the same as for the BKP hierarchy. In the same way as in Sec. 3.2, we can prove that there exists a function \( \tau(t_o) \) such that

\[
w_0(t_o, \tilde{z}) = \frac{\tau(t_o - 2[z^{-1}]_o)}{\tau(t_o)}.
\]

(84)
This function is called the tau function of the CKP hierarchy. Finally, writing \( w(t_o, z) = f^{1/2}(t_o, z)w_0(t_o, z) \) and noting that \( f(t_o, z) = 1 + O(z^{-2}) \), we see that

\[
\xi_1(t_o) = -2\partial_{t_1} \log \tau(t_o),
\]

(85)

and, recalling (75), we arrive at (72) with \( G = f^{1/2} \).

We next show that Eq. (72) can be obtained up to a common \( x \)-independent factor in the following easy way [7, 8]. We apply \( \partial_{t_1} \) to (71) and set \( t'_o = t_o - 2[a^{-1}]_o \). The residue calculus yields

\[
2a^2(1 - w(t_o, a)w(t_o - 2[a^{-1}]_o, -a) - 2aw'(t_o, a)w(t_o - 2[a^{-1}]_o, -a) + 2a(\xi_1(t_o) - \xi_1(t_o - 2[a^{-1}]_o)) + \xi_2(t_o - 2[a^{-1}]_o) + \xi_2(t_o) + \xi_1'(t_o) - \xi_1(t_o)\xi_1(t_o - 2[a^{-1}]_o) = 0,
\]

(86)

where the prime denotes the \( x \)-derivative and we again do not indicate the dependence on \( x \) explicitly. Letting \( a \to \infty \) in (86), we obtain the relation

\[
2\xi_2(t_o) = \xi_1^2(t_o) - \xi_1'(t_o)
\]

(87)

(it also directly follows from the constraint \( WW^\dagger = 1 \) for the dressing operator). Substituting this back in (86), we can rewrite that equation in the form

\[
w'(t_o, a)w(t_o - 2[a^{-1}]_o, -a) = af(t_o, a)(f(t_o, a) - 1) + \frac{1}{2} f'(t_o, a).
\]

(88)

Using (74), we conclude that

\[
\partial_x \log w(t_o, a) = a(f(t_o, a) - 1) + \frac{1}{2} \partial_x \log f(t_o, a) = \frac{1}{2}(\xi_1(t_o) - \xi_1(t_o - 2[a^{-1}]_o)) + \frac{1}{2} \partial_x \log f(t_o, a).
\]

(89)

Now, setting \( \xi_1(x, t_o) = -2\partial_x \log \tau(x, t_o) \) with some function \( \tau(x, t_o) \) and integrating, we arrive at (72) with \( G = f^{1/2} \) up to a common multiplier that does not depend on \( x \).

The function \( u \) in (66) can also be expressed through the tau function. It is a matter of direct verification that the result of the action of the operator \( \partial_x^3 + 6u \partial_x + 3u' - \partial_{t_1} \) on the wave function \( \Psi \) of form (70) is \( O(z^{-1}) \) as \( z \to \infty \), i.e.,

\[
(\partial_x^3 + 6u \partial_x + 3u' - \partial_{t_1})\Psi = O(z^{-1})e^{xz+\xi(t_o \to z)},
\]

(90)

if the conditions

\[
u = -\frac{1}{2} \xi_1', \quad u' = \xi_1 \xi_1' - \xi_1'' - \xi_2'.
\]

(91)

are satisfied. It follows from (72) that

\[
\xi_1 = -2 \partial_x \log \tau, \quad \xi_2 = 2(\partial_x \log \tau)^2 + \partial_x^2 \log \tau,
\]

and therefore

\[
u = \partial_x^2 \log \tau,
\]

(92)

and the second equality in (91) holds identically.
4.3. The equation for the tau function. As follows from (71) and (72), the CKP hierarchy is equivalent to the following relation for the tau function:

\[
\oint_{C_{\infty}} e^{\xi(t_0 - t'_0, z)} \tau(x, t_0 - 2[z^{-1}]_0) \tau(x, t'_0 + 2[z^{-1}]_0) G(x, t_0, z) G(x, t'_0, -z) \frac{dz}{2\pi i} = 0
\]  

(93)

for all \(t_0, t'_0\). We set \(t'_0 = t_0 - 2[a^{-1}]_0 - 2[b^{-1}]_0 - 2[c^{-1}]_0\); then

\[
e^{\xi(t_0 - t'_0, z)} = \frac{(a + z)(b + z)(c + z)}{(a - z)(b - z)(c - z)}
\]

and using the residue calculus in (93) gives the equation

\[
a(a + b)(a + c)(b - c)\tau^{[a]} \tau^{[bc]} G(-a) G^{[abc]}(a) +
+ b(b + a)(b + c)(c - a)\tau^{[b]} \tau^{[ac]} G(-b) G^{[abc]}(b) +
+ c(c + a)(c + b)(a - b)\tau^{[c]} \tau^{[ab]} G(-c) G^{[abc]}(c) +
+ (a + b + c)(a - b)(b - c)(c - a)\tau^{[abc]} +
+ (a - b)(b - c)(c - a)(\tau^{[abc]} \partial_{t_1} \tau - \tau \partial_{t_1} \tau^{[abc]}) = 0,
\]  

(94)

where \(\tau^{[a]} = \tau(x, t_0 + 2[a^{-1}]_0)\), \(\tau^{[ab]} = \tau(x, t_0 + 2[a^{-1}]_0 + 2[b^{-1}]_0)\), \(G^{[a]}(z) = G(x, t_0 + 2[a^{-1}]_0, z)\), and so on. Equation (94) must hold for all \(a, b, c\). We let \(c\) tend to infinity. The leading terms proportional to \(c^3\) vanish identically. The terms of the order \(c^2\) give the equation

\[
\frac{a + b}{a - b} (aG(-a)G^{[ab]}(a) - bG(-b)G^{[ab]}(b)) = \left( a + b - \partial_{t_1} \log \frac{\tau^{[ab]} \tau^{[ac]}}{\tau^{[a]} \tau^{[bc]}} \right) \tau^{[ab]} \tau^{[ac]} / \tau^{[a]} \tau^{[bc]},
\]

(95)

or, in more detail,

\[
\frac{z_1 + z_2}{z_1 - z_2} \left[ \left( z_1 - \partial_{t_1} \log \frac{\tau^{[z_1 z_2]} \tau^{[z_1]}}{\tau^{[z_2]} \tau^{[z_1]}} \right)^{1/2} \left( z_1 - \partial_{t_1} \log \frac{\tau^{[z_1]} \tau^{[z_1]}}{\tau^{[z_2]}} \right)^{1/2} - \
- \left( z_2 - \partial_{t_1} \log \frac{\tau^{[z_1 z_2]} \tau^{[z_1]}}{\tau^{[z_2]} \tau^{[z_1]}} \right)^{1/2} \left( z_2 - \partial_{t_1} \log \frac{\tau^{[z_1]} \tau^{[z_2]}}{\tau^{[z_2] \tau^{[z_1]}}} \right)^{1/2} \right] =
\]

\[
= \left( z_1 + z_2 + \partial_{t_1} \log \frac{\tau^{[z_1 z_2]} \tau^{[z_2]}}{\tau^{[z_1]} \tau^{[z_2]} \tau^{[z_1]}} \right) \tau \tau^{[z_1 z_2]} / \tau^{[z_1]} \tau^{[z_2]} \tau^{[z_1]} \tau^{[z_2]}. \]

(96)

This is the equation for the tau function of the CKP hierarchy. It must hold for all \(z_1\) and \(z_2\). In contrast to the cases of the KP and BKP hierarchies, this equation is not bilinear.

4.4. The CKP hierarchy and the KP hierarchy. Here, we give a characterization of those KP tau functions that correspond to solutions of the CKP hierarchy and show that the CKP tau function is the square root of the KP one.

Comparing (13) and (71), we conclude that the wave functions of the CKP and KP hierarchies are related as

\[
\Psi(x, t_0, z) = e^{\chi(z)} \Psi^{KP}(x, t_1, 0, t_3, 0, \ldots, z),
\]

\[
\Psi(x, t_0, -z) = e^{-\chi(z)} \Psi^{KP}(x, t_1, 0, t_3, 0, \ldots, -z).
\]

Here, \(\chi(z)\) is some function such that \(\chi(\infty) = 0\), i.e.,

\[
\Psi^{KP}(x, t_1, 0, t_3, 0, \ldots, z) = e^{2\chi(z)} \Psi^{KP}(x, t_1, 0, t_3, 0, \ldots, -z),
\]

(97)
where \( \chi_e(z) = (\chi(z) + \chi(-z))/2 \) is the even part of \( \chi(z) \). It follows from (11), (12), and (97) that the KP tau function is the extension of a solution of the CKP hierarchy if and only if the equation

\[
\tau^{\text{KP}}\left(x, t_1 + z^{-1}, \frac{1}{2}z^2, t_3 + \frac{1}{3}z^{-2}, \frac{1}{4}z^{-4}, \ldots \right) = e^{2\chi_e(z)}\tau^{\text{KP}}\left(x, t_1 + z^{-1}, -\frac{1}{2}z^2, t_3 + \frac{1}{3}z^{-2}, -\frac{1}{4}z^{-4}, \ldots \right)
\]

holds identically for all \( z, x, t_1, t_3, t_5, \ldots \). Shifting the odd times, we can rewrite this condition as

\[
\log \tau^{\text{KP}}\left(x, t_1, \frac{1}{2}z^{-2}, t_3, \frac{1}{4}z^{-4}, \ldots \right) - \log \tau^{\text{KP}}\left(x, t_1, -\frac{1}{2}z^2, t_3, -\frac{1}{4}z^{-4}, \ldots \right) = 2\chi_e(z).
\]

Comparing the coefficients at \( z^{-2} \) in the expansions of the left- and right-hand sides of (99) and passing to an equivalent tau function if necessary, we obtain the condition

\[
\partial_{t_k} \log \tau^{\text{KP}}|_{t_n=0} = 0, \quad k \geq 1.
\]

It is the CKP counterpart of condition (55) specific for the BKP hierarchy. It was proved in [11] to be a sufficient condition for the tau function \( \tau^{\text{KP}} \) to generate a solution of the CKP hierarchy. The proof is based on the technique developed in [19]–[21].

We next introduce the auxiliary wave function \( \psi \) in the same way as for the BKP hierarchy:

\[
\psi = e^{xz + \xi(t_0, z)} \frac{\tau(x, t_0 - 2[z^{-1}], z)}{\tau(x, t_0)}.
\]

Then the wave function (72) becomes

\[
\Psi = z^{-1/2} \sqrt{\partial_x \log \psi} \psi = (2z)^{-1/2} \sqrt{\partial_x \psi^2}.
\]

We now prove that the CKP and KP tau functions are related as

\[
\tau = \sqrt{\tau^{\text{KP}}},
\]

The tau function \( \tau(x, t_0) \) has square-root singularities in \( x \). However, the expression \( \partial_x \psi^2 \) turns out to be a full square for all \( z \), i.e., \( \sqrt{\partial_x \psi^2} \), and therefore \( \Psi \) is an entire function of \( x \) and \( t_0 \) (no square-root singularities). This is similar to the fact that for the BKP hierarchy, the function \( \tau^{\text{KP}} \) is a full square. Bilinear identity (71) acquires the form

\[
\oint_{C_{\infty}} \sqrt{\partial_x \psi^2(x, t_0, z)} \sqrt{\partial_x \psi^2(x, t_0, -z)} \frac{dz}{2\pi i z} = 0.
\]

To prove that \( \tau = \sqrt{\tau^{\text{KP}}} \) (see [9]), we compare two expressions for the wave function \( \Psi \) of the CKP hierarchy. The first one is in terms of the KP tau function,

\[
\Psi^{\text{KP}} = e^{xz + \xi(t_0, z)} \frac{\tau^{\text{KP}}(x, t_1 - z^{-1}, -z^{-2}/2, t_3 - z^{-3}/3, -z^{-4}/4, \ldots)}{\tau^{\text{KP}}(t_1, 0, t_3, 0, \ldots)},
\]

and the second, Eq. (72), is in terms of the CKP tau function \( \tau \). Comparing (102) and (105), we obtain the equation

\[
\frac{1}{2z} \partial_x \left( e^{2xz} \frac{\tau^{2}(x, t_0 - 2[z^{-1}], z)}{\tau^{2}(x, t_0)} \right) = e^{2xz} \left( \frac{\tau^{\text{KP}}(x, t - [z^{-1}])}{\tau^{\text{KP}}(x, t)} \right)^2,
\]
where we again use the short-hand notation \( \dot{t} = \{t_1, 0, t_3, 0, \ldots \} \). We then see that (106) is equivalent to the differential equation \( \partial_x \varphi = -2z \varphi \), where

\[
\varphi = \frac{\tau^2(x, t_o - 2[z^{-1}]_o) - \tau^{\text{KP}}(x, \dot{t} - 2[z^{-1}]_o)}{\tau^2(x, t_o)} - \tau^{\text{KP}}(x, t).
\]

(107)

for all \( z \). This is an identity for solutions of the KP/CKP hierarchies. It follows from (108) that \( \tau^{\text{KP}} = \text{const} \cdot \tau^2 \), i.e., \( \tau(x, t_o) = \sqrt{\tau^{\text{KP}}(x, \dot{t})} \) is a tau function of the CKP hierarchy.

4.5. Examples: soliton solutions.

4.5.1. One-soliton solution. We start from the simplest example of a one-soliton solution. The tau function for one CKP soliton is the square root of a specialization of the 2-soliton tau function of the KP hierarchy. The latter has the form

\[
\tau^{\text{KP}}(x, t) = c \exp \left( (p + q)x + \sum_{k \geq 1} (p^k - (-q)^k) t_k \right) +
\]

\[
+ \exp \left( (p + q)x + \sum_{k \geq 1} (q^k - (-p)^k) t_k \right) - \frac{\alpha^2 (p - q)^2}{4pq} \exp \left( 2(p + q)x + \sum_{k \geq 1} (p^k + q^k - (-p)^k - (-q)^k) t_k \right),
\]

(109)

where \( \alpha, p, \) and \( q \) are arbitrary parameters. When all even times are set equal to zero, \( t_2 = t_4 = \cdots = 0 \), this expression is simplified to

\[
\tau^{\text{KP}} = 1 + 2\omega w - \frac{\alpha^2 (p - q)^2}{4pq} w^2,
\]

(110)

where

\[
w = e^{(p + q)x + \xi(t_{e,p}) + \xi(t_{e,q})}.
\]

(111)

We see that the tau function \( \tau = \sqrt{\tau^{\text{KP}}} \) has two square-root singularities at the points \( w = w_{\pm} \),

\[
w_{\pm} = \pm \frac{2\sqrt{pq}}{\alpha(\sqrt{p} + \sqrt{q})^2}.
\]

Direct calculation shows that \( \partial_x \psi^2 \) (where \( \psi \) is given by (101)) for solution (110) is a full square for all \( z \):

\[
\frac{\partial_x \psi^2}{2} = e^{2xz + 2z\xi(t_{e,z})} \left( 1 + \frac{\alpha^2 (2z^2 - p^2 - q^2)}{(z+p)(z+q)} w - \frac{\alpha^2 (p - q)^2 (z-p)(z-q)}{4pq (z+p)(z+q)} w^2 \right)^2 = \left( \psi^{\text{KP}}(x, t_1, 0, t_3, 0, \ldots, z) \right)^2,
\]

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where $\Psi^{\text{KP}}$ is constructed from the KP tau function $\tau^{\text{KP}}$ in (109) in accordance with formula (11). Hence, $(2z)^{-1/2}\sqrt{D_z\Psi^2}$ and therefore $\Psi$ does not have square-root singularities in the variable $z$. Bilinear identity (71), which in the present case has the explicit form

$$\int_{C_\infty} e^{\xi(t_\alpha,z)-\xi(t_\alpha,z')} \left( 1 + \frac{\alpha(2z^2 - p^2 - q^2)}{z + p} \right) w - \frac{\alpha^2(p - q)^2}{4pq} (z + p)(z + q) w^2 \times \left( 1 + \frac{\alpha(2z^2 - p^2 - q^2)}{z - p} \right) w' - \frac{\alpha^2(p - q)^2}{4pq} (z - p)(z - q) w'^2 \right) \frac{dz}{2\pi i} = 0,$$

can be verified directly by the residue calculus. (Here $w' = e^{(p+q)x+\xi(t_\alpha,p)+\xi(t_\alpha,q)}$.)

4.5.2. Multi-soliton solutions. The general KP tau function for an $M$-soliton solution has $3M$ arbitrary parameters $\alpha_i, p_i, q_i, i = 1, \ldots, M$, and is given by Eq. (62). We use the notation

$$\tau^{\text{KP}} \left[ \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{2N-1} & \alpha_{2N} \\ p_1, q_1 & p_2, q_2 & p_3, q_3 & p_4, q_4 & \cdots & p_{M-1}, q_{M-1} & p_M, q_M \end{array} \right],$$

for it. The parameters $p_i$ and $q_i$ are sometimes called momenta of solitons.

The multi-soliton tau function of the CKP hierarchy is the square root of the $\tau^{\text{KP}}$ specialized as

$$\tau^{\text{KP}} \left[ \begin{array}{cccc} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_N \\ p_0, -p_0 & p_1, -q_1 & p_2, -q_2 & \cdots & p_N, -q_N \end{array} \right],$$

where it is implied that evolution along even times is suppressed ($t_{2k} = 0$ for all $k \geq 1$). Clearly, the total number of independent parameters is $3N + 2$. If $\alpha_0 = 0$, tau function (63) reduces to

$$\tau^{\text{KP}} \left[ \begin{array}{cccc} \alpha_1 & \alpha_2 & \cdots & \alpha_N \\ p_1, -q_1 & p_2, -q_2 & \cdots & p_N, -q_N \end{array} \right],$$

and it is this tau function that is usually called the $N$-soliton CKP tau function in the literature (see, e.g., [3]). It is a specialization of the $2N$-soliton KP tau function and has $3N$ free parameters.

5. Dispersionless limit of the BKP and CKP hierarchies

The dispersionless limit is the limit $\hbar \to 0$ taken after the substitutions

$$t_k \to \frac{t_k}{\hbar}, \quad \tau = e^{F/\hbar^2}.$$ 

Making these substitutions in (48) and taking the limit, we obtain the equation

$$\frac{p(z_1) - p(z_2)}{z_1 - z_2} = \frac{p(z_1) + p(z_2)}{z_1 + z_2} e^{4D^o(z_1)D^o(z_2)F},$$

where

$$p(z) = z - 2 \partial_{t_k} D^o(z)F$$

and $D^o(z)$ is the differential operator

$$D^o(z) = \sum_{k \geq 1, \text{odd}} \frac{z^{-k}}{k} \partial_{t_k}.$$
The (odd) function $p(z)$ has the expansion
\[
p(z) = z - \frac{u}{z} + \sum_{k \geq 3, \text{odd}} u_k z^{-k},
\] (117)

where $u = 2 \partial^2_t F$. Equation (114) is the generating equation for the dispersionless BKP (dBKP) hierarchy in the Hirota form [22]–[24]. It is remarkable that the dispersionless limit of CKP equation (96) is the same, and therefore the dispersionless limits of the BKP and CKP hierarchies coincide.

We now show how to represent the dispersionless hierarchy in the Lax form. Taking the logarithm of Eq. (114), differentiating with respect to $t_1$, and using definition (117), we obtain the equation
\[
2D^o(z_1)p(z_2) = \partial_{t_1} \log \frac{p(z_1) + p(z_2)}{p(z_1) - p(z_2)},
\] (118)

whence it follows that $D^o(z_1)p(z_2) = D^o(z_2)p(z_1)$ (this also follows from definition (117)). Letting $z_2 \to \infty$, we obtain
\[
\partial_{t_1} p(z) = -D^o(z)u. \tag{119}
\]

The next step is to rewrite Eq. (27) in terms of the function $z(p)$ inverse to the $p(z)$ (like $p(z)$, it is an odd function with the Laurent series expansion of the form $z(p) = p + \mathcal{O}(p^{-1})$). Using the relation
\[
\partial_{t_1} p(z) = -\frac{\partial_{t_1} z(p)}{\partial p}(p), \quad k \geq 1, \tag{120}
\]
after simple transformations we obtain
\[
2D^o(z_1) z(p) = \left\{ z(p), \log \frac{p(z_1) - p}{p(z_1) + p} \right\}, \tag{121}
\]

where
\[
\{f, g\} := \frac{\partial f}{\partial t_1} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial t_1} \frac{\partial f}{\partial p} \tag{122}
\]
is the Poisson bracket. This is the generating Lax equation for the dBKP hierarchy, $z(p)$ being the Lax function (the dispersionless limit of Lax operator (17)). Expanding Eq. (121) in powers of $z_1$, we obtain a hierarchy of Lax equations in terms of the Faber polynomials $B_k(p)$ introduced by the expansion
\[
- \log \frac{p(z) - p}{z} = \sum_{k \geq 1} \frac{z^{-k}}{k} B_k(p). \tag{123}
\]

For example, $B_1(p) = p$. It is easy to see that
\[
B_k(p) = (z^k(p))_{\geq 0}, \tag{124}
\]

where $(\ldots)_{\geq 0}$ is the polynomial part of the Laurent series in $p$ (containing only nonnegative powers of $p$). The fact that $p(z)$ is an odd function implies that $B_k(-p) = (-1)^k B_k(p)$, and we have the expansion
\[
\log \frac{p(z) + p}{p(z) - p} = 2 \sum_{k \geq 1, \text{odd}} \frac{z^{-k}}{k} B_k(p). \tag{125}
\]

The Lax equations are of the form
\[
\partial_{t_1} z(p) = \{B_k(p), z(p)\} = \{(z^k(p))_{\geq 0}, z(p)\}, \quad k \text{ is odd.} \tag{126}
\]

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