Incompressible Navier-Stokes Equation from Einstein-Maxwell and Gauss-Bonnet-Maxwell Theories

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Abstract

The dual fluid description for a general cutoff surface at radius $r = r_c$ outside the horizon in the charged AdS black brane bulk space-time is investigated, first in the Einstein-Maxwell theory. Under the non-relativistic long-wavelength expansion with parameter $\epsilon$, the coupled Einstein-Maxwell equations are solved up to $O(\epsilon^2)$. The incompressible Navier-Stokes equation with external force density is obtained as the constraint equation at the cutoff surface. For non-extremal black brane, the viscosity of the dual fluid is determined by the regularity of the metric fluctuation at the horizon, whose ratio to entropy density $\eta/s$ is independent of both the cutoff $r_c$ and the black brane charge. Then, we extend our discussion to the Gauss-Bonnet-Maxwell case, where the incompressible Navier-Stokes equation with external force density is also obtained at a general cutoff surface. In this case, it turns out that the ratio $\eta/s$ is independent of the cutoff $r_c$ but dependent on the charge density of the black brane.
I. INTRODUCTION

It has long been known that excitations of a black hole horizon dissipate like those of a fluid with viscosity \( \eta = \frac{1}{16 \pi G} \) (see also \([2-5]\)), which together with the Bekenstein-Hawking entropy density \( s = \frac{1}{4G} \) yields the dimensionless ratio \( \frac{\eta}{s} = \frac{1}{4\pi} \). Interestingly, this fact has been related to similar results \([6]\) in the AdS/CFT correspondence, which regards the change of radius \( r \) as equivalent to the renormalization group (RG) flow \([7-15]\) and the case of black hole horizon corresponds to the IR limit of this flow. Especially, it is proved that the ratio \( \frac{\eta}{s} \) does not run with the RG flow \([10, 25]\), and so the universality of this ratio in both the horizon fluid and the standard AdS/CFT follows.

According to the general holographic dictionary \([16]\), the Brown-York tensor of the bulk gravity is dual to the expectation value of the stress-energy tensor of the boundary field theory \([17]\). Under the long-wavelength limit, the boundary theory can be described by hydrodynamics, known as the gravity/fluid duality \([18, 19]\). Especially, under certain non-relativistic limit (or scaling), the boundary hydrodynamics takes the form of the standard incompressible Navier-Stokes equation \([11, 27]\). Recently, Bredberg et al give the precise definition of the boundary theory on an arbitrary cutoff surface \( r = r_c \) outside the horizon in the Rindler bulk space-time, which reduces the bulk gravitational dynamics to the incompressible Navier-Stokes equation on the boundary \([20]\). This framework is extended to the AdS black brane case \([21]\), as well as in the Gauss-Bonnet gravity \([22]\).

In this Letter, we consider an arbitrary cutoff surface \( r = r_c \) outside the horizon in the charged AdS black brane bulk space-times in both the Einstein-Maxwell and Gauss-Bonnet-Maxwell theories, further extending the framework of \([20]\) and \([22]\). Under the non-relativistic long-wavelength expansion with parameter \( \epsilon \), the coupled Einstein-Maxwell (or Gauss-Bonnet-Maxwell) equations are solved up to \( O(\epsilon^2) \). The incompressible Navier-Stokes equation with external force density is obtained as the constraint equation at the cutoff surface, with a viscosity \( \eta \) satisfying the universality relation \( \frac{\eta}{s} = \frac{1}{4\pi} \) independent of both the cutoff \( r_c \) and the black brane charge in the Einstein-Maxwell case. In the Gauss-Bonnet-Maxwell case, the incompressible Navier-Stokes equation with external force density is as well obtained at a general cutoff surface. However, it turns out that the regularity condition at the horizon determines the ratio \( \frac{\eta}{s} = \frac{1}{4\pi} \left\{ 1 - 2(n - 4)\alpha [n - 1 - (n - 3)q_h^2] \right\} \) with \( \alpha \) the Gauss-Bonnet coupling constant and \( n \) the space-time dimensionality, which is

\[\frac{\eta}{s} = \frac{1}{4\pi} \left\{ 1 - 2(n - 4)\alpha [n - 1 - (n - 3)q_h^2] \right\}\]
independent of the cutoff \( r_c \) but dependent on the charge density \( q_h \) of the black brane.

The rest of the Letter is organized as follows. In Sec. II, we present the metric and electromagnetic background and introduce the non-relativistic long-wavelength expansion, focusing on the Einstein-Maxwell case, where the bulk equations of motion are solved up to \( \mathcal{O}(\epsilon^2) \). In Sec. III, the dual fluid on the cutoff surface is analyzed with the incompressible Navier-Stokes equation obtained in the Einstein-Maxwell case. In Sec. IV, we extend the above discussion to the Gauss-Bonnet-Maxwell case.

**II. METRIC AND ELECTROMAGNETIC CONFIGURATIONS**

We consider the standard \( n \)-dimensional Einstein-Maxwell gravity with the action

\[
I = \frac{1}{16\pi G} \int d^n x \sqrt{-g} (R - 2\Lambda) - \frac{1}{4} \int d^n x \sqrt{-g} F_{\mu\nu} F^{\mu\nu},
\]

where \( \Lambda = \frac{-(n-2)(n-1)}{2l^2} \) is the negative cosmological constant. The corresponding equations of motion are

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu} = 0, \\
\nabla_\mu F^{\mu\nu} = 0,
\]

where \( T_{\mu\nu} = \frac{1}{4} g_{\mu\sigma} F^{\rho\sigma} - F_{\mu\rho} F^{\rho}_{\nu} \) is the stress-energy tensor of the electromagnetic field.

Our background is the charged black brane solution

\[
ds^2 = -f(r) dt^2 + 2 dr dt + r^2 dx^a dx^a, \quad f(r) = \frac{r^2}{l^2} - \frac{2m}{r^{n-3}} + \frac{Q^2}{r^{2n-6}}, \]

under Eddington-Finkelstein coordinates, where the index \( a \) runs from 2 to \( n-1 \), \( m \) is the mass parameter and \( Q \) the charge parameter. The corresponding electromagnetic field strength is

\[
F = \sqrt{\frac{(n-2)(n-3)}{8\pi G}} \frac{Q}{r^{n-2}} dt \wedge dr.
\]

For convenience, we take the AdS radius \( l = 1 \) hereafter.

The induced metric on the cutoff surface \( r = r_c \) outside the horizon is

\[
ds_c^2 = -f(r_c) dt^2 + r_c^2 dx^a dx^a,
\]
which is flat and kept fixed when perturbing the bulk metric. In order to introduce the fluid degrees of freedom $v^a$ (the velocity) and $P$ (the pressure), two types of diffeomorphisms that keep (3) invariant are taken:

1. Lorentz boost

$$ \begin{aligned} \sqrt{f(r_c)} t &\to \gamma (\sqrt{f(r_c)} t - \beta_a r_c x^a), \\
r_c x^a &\to (\delta^a_b - \frac{\beta_a \beta_b}{\beta^2}) r_c x^b + \gamma (\frac{\beta^a_b}{\beta^2} r_c x^b - \beta^a \sqrt{f(r_c)} t) \end{aligned} $$

with $\beta^a \equiv \frac{r_c}{\sqrt{f(r_c)}} v^a$;

2. Special rescaling

$$ r \to (1 - P) r, \quad t \to \sqrt{\frac{f(r_c)}{f((1 - P)r_c)}} t, \quad x^a \to \frac{r_c}{(1 - P)r_c} x^a $$

of $r$, $t$ and $x^a$.

Then we promote $v^a$ and $P$ to be the velocity field $v^a(t, x)$ and the pressure field $P(t, x)$, which makes the transformed bulk metric and electromagnetic field no longer be solution of the Einstein-Maxwell equations (1). We also adopt the non-relativistic long-wavelength expansion parameterized by $\epsilon \to 0$ and the scaling

$$ \partial_t \sim \epsilon^2, \quad \partial_a \sim \epsilon, \quad \partial_r \sim 1, \quad P \sim \epsilon^2, \quad v^a \sim \epsilon $$

with $\partial_a \equiv \frac{\partial}{\partial x^a}$, under which the perturbed bulk Einstein-Maxwell equations can be solved order by order. Up to $O(\epsilon^3)$, the transformed bulk metric by both types of diffeomorphisms has been given in [22] (actually for arbitrary $f(r)$) as

$$ ds^2 = -f(r) dt^2 + 2 dr dt + r^2 dx^a dx^a - 2r^2 (1 - \frac{r_c f(r)}{f(r_c)}) v^a dx^a dt - 2 \frac{r_c^2 v^a}{f(r_c)} dx^a dr $$

$$ + r^2 (1 - \frac{r_c^2 f(r)}{f(r_c)^2}) (v^2 dt^2 + \frac{r_c^2 v^a v^b}{f(r_c)} dx^a dx^b) + \frac{r_c^2 v^2}{f(r_c)} dr dt $$

$$ + f(r) (\frac{r_c f'(r)}{f(r)} - \frac{r_c^2 f'(r_c)}{f(r_c)^2}) P dt^2 + (\frac{r_c f'(r_c)}{f(r_c)} - 2) P dr dt + O(\epsilon^3). $$

The bulk electromagnetic field should also be transformed by the above two types of diffeomorphisms. After promoteing $v^a$ and $P$ to be $(t, x)$-dependent (but $r$-independent) fields

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1 Note that our $f(r)$ corresponds to $r^2 f(r)$ in [22], and we do not distinguish $v_a$ from $v^a$. 
and adopting the scaling (5), the perturbed electromagnetic field can be straightforwardly worked out as

\[
A = \sqrt{\frac{n-2}{8\pi(n-3)G}}\frac{Q}{r^{n-3}}[dt - \frac{r_c^2}{f(r_c)}v^a dx^a + \frac{r_c^2}{2f(r_c)}v^2 dt + (n-3)P dt + \frac{f'(r_c)}{2f(r_c)}r_c P dt] + \mathcal{O}(\epsilon^2),
\]

up to \(\mathcal{O}(\epsilon^2)\). It should be noted that in this charged configuration, there are gravitational degrees of freedom (DoF) and electromagnetic DoF. If the electromagnetic DoF are taken into account, there is a boundary current dual to the bulk electromagnetic field (see e.g. [25]). In our approach, we only consider the DoF induced by the above two kinds of (lifted) diffeomorphisms, which can be roughly regarded as gravitational, and do not turn on the independent electromagnetic DoF. That is, our focus is on the influence of the charged AdS black brane, as the background (compared to the uncharged one), on the properties of the dual fluid, especially the Navier-Stokes equation and the viscosity \(\eta\) (or the ratio \(\frac{\eta}{s}\)) appearing there. It turns out that the resulting dual charge density is constant at visible orders, and so the conservation law of the boundary current just coincides with the incompressibility condition of the dual fluid (see (9) below) and does not give any new equation.\(^2\)

Now we should substitute the perturbed metric (6) and electromagnetic field (7) into the Einstein-Maxwell equations (1) and see if they have already solved the equations up to \(\mathcal{O}(\epsilon^2)\). Different from the asymptotically flat case [23], the perturbed metric (together with the perturbed electromagnetic field) only solves the Einstein(-Maxwell) equations (1) up to \(\mathcal{O}(\epsilon)\), even if we imposed the \(\mathcal{O}(\epsilon^2)\) constraint equation \(\partial_a v^a = 0\) (incompressibility of the dual fluid on the cutoff surface). It is known in the case without electromagnetic field that a correction term

\[
r^2 F(r)(\partial_b v^b + \partial_b v^a) dx^a dx^b
\]

at \(\mathcal{O}(\epsilon^2)\) should be added to the metric (6) [22]. It turns out that this prescription also applies to our case, without the need of additional correction terms for the perturbed electromagnetic field (7). In fact, the Maxwell equations in (1) are automatically satisfied at \(\mathcal{O}(1)\) and \(\mathcal{O}(\epsilon)\), while at \(\mathcal{O}(\epsilon^2)\) lead to two equations

\[
\partial_a v^a = 0, \quad F'(r)\partial_a v^a = 0,
\]

\(^2\) The independent electromagnetic DoF can be turned on, e.g. by lifting the black brane charge \(Q\) to a function of \((t, x)\), which will result in a non-constant dual charge density at visible orders.
which can be solved altogether by the incompressibility condition. Then from the Einstein equations in (10) at $O(\epsilon^2)$ there is essentially only one requirement

$$r^{n-2}f(r)F''(r) + r^{n-3}[(n-2)f(r) + rf'(r)]F'(r) + (n-2)r^{n-3} = 0$$

for $F(r)$ to satisfy, which can be solved as

$$F'(r) = -\left(1 - \frac{C}{r^{n-2}}\right)\frac{1}{f(r)}.$$

Here the integration constant $C$ can be determined as $C = r_h^{n-2}$ with $r_h$ the horizon radius by the regularity condition of the perturbed metric at the horizon, provided our charged black brane is non-extremal and so $f(r)$ only has a simple zero at $r = r_h$. The explicit form of $F(r)$ is irrelevant to the succeeding discussions, where the one more integration constant can be determined such that the induced metric (3) is kept fixed to all orders in $\epsilon$.

III. DUAL FLUID ON THE CUTTOFF SURFACE

On an arbitrary cutoff surface $r = r_c$ outside the horizon, there is firstly a thermodynamic description of the (equilibrium) fluid dual to the background configuration (2). In fact, the Brown-York tensor $[24]$ $t_{ij} = \frac{1}{8\pi G}(Kg_{ij} - K_{ij} - Cg_{ij})$ (10) on the cutoff surface, with $K_{ij}$ its extrinsic curvature and $K \equiv g^{ij}K_{ij}$, is

$$t_{ij}dx^idx^j = \frac{1}{8\pi G}[-\sqrt{f(r_c)}(n-2)f(r_c)\frac{r_c}{r_c}dt^2 + \frac{r_c^2}{\sqrt{f(r_c)}}\frac{f'(r_c)}{2} + \frac{(n-3)f(r_c)}{r_c}dx^adx^a - Cds_c^2],$$

which is identified with the stress-energy tensor of the dual fluid $[25]$. On the other hand, the stress-energy tensor of a (relativistic) fluid in equilibrium is

$$t_{ij} = (\rho + p)u_iu_j + pg_{ij}$$

with $\rho$ the energy density, $p$ the pressure and $u^i = \sqrt{-g_{tt}}(1, 0, \cdots, 0)$ the normalized fluid four-velocity. Inclusion of the constant $C$ in (10) is equivalent to the replacement

$$p \rightarrow p - \frac{C}{8\pi G}, \quad \rho \rightarrow \rho + \frac{C}{8\pi G},$$

3 We have explicitly obtained the above form of equation for $4 \leq n \leq 10$. 
which leaves the combination $\rho + p$ invariant, so we can omit $C$ if we only consider this combination.\(^4\) Comparing (11) and (12), we find
\[
\rho + p = \frac{1}{8\pi G \sqrt{f(r_c)}} \frac{f'(r_c)}{2} - \frac{f'(r_c)}{r_c} = \frac{r_c^2}{16\pi G \sqrt{f(r_c)}} \frac{f'(r)}{r^2} c. \tag{13}
\]
Noticing the entropy density
\[
s_c = \frac{1}{4} r_h^{n-2} \tag{14}
\]
on the cutoff surface, the local Hawking temperature
\[
T_c = \frac{1}{\sqrt{f(r_c)}} \frac{f'(r_h)}{4\pi}, \tag{15}
\]
the explicit form of $f(r)$ in (2) and the horizon condition $f(r_h) = 0$, we have the familiar thermodynamic relation
\[
\rho + p - s_c T_c = q_c \mu_c \tag{16}
\]
with (up to some unimportant constant factor) the charge density
\[
q_c = \frac{Q}{r_c^{n-2}} \tag{17}
\]
and the corresponding chemical potential
\[
\mu_c = \frac{n - 2}{8\pi G \sqrt{f(r_c)} \left( Q r_h^{n-3} - Q r_c^{n-3} \right)}, \tag{18}
\]
which is just a (red-shifted) electric potential difference.

In the perturbed case (6), the Brown-York tensor can be worked out as
\[
8\pi G t_{ij} dx^i dx^j = -\sqrt{f(r_c)} \frac{(n - 2) f(r_c)}{r_c} dt^2 + \frac{r_c^2}{\sqrt{f(r_c)}} \left( \frac{f'(r_c)}{2} + \frac{(n - 3) f(r_c)}{r_c} \right) dx^a dx^a - C ds_c^2
\]
\[
- \frac{r_c^4}{\sqrt{f(r_c)}} (f(r) r_c^2)^c v^a dx^a dt
\]
\[
+ \frac{r_c^2}{2 \sqrt{f(r_c)}} \left[ (n - 2) f(r_c) P + r_c^2 v^2 \right] (f(r) r_c^2)^c dt^2 + \frac{r_c^6}{2 \sqrt{f(r_c)}} v^a v^b \left( \frac{f(r)}{r_c^2} \right) c dx^a dx^b
\]
\[
+ \frac{r_c^4}{2 \sqrt{f(r_c)}} \left[ \frac{r_c^3}{2 f(r_c)} (f(r) r_c^2)^2 - (n - 1) (f(r) r_c^2)^c - r_c f(r) r_c^2 \right] P dx^a dx^a
\]
\[
- \frac{r_c^2}{2 \sqrt{f(r_c)}} \left[ 1 + f(r_c) F'(r_c) \right] (\partial_a v^b + \partial_b v^a) dx^a dx^b + O(\epsilon^3), \tag{19}
\]
\(^4\) Note that $C$ is essential for the regularity of $t_{ij}$ as $r_c \to \infty$.\(^{24}\)
after imposing the incompressibility condition (9) in the $O(\epsilon^2)$ part.\footnote{Our result above, with $f''(r_c)$ presenting, is slightly different from (27) in [22], where there is no $f''(r_c)$. But in the case considered in [22] (without electromagnetic field), we have checked that these two expressions give the same result.} The transverse components of the Einstein equations give the conservation law

$$D^i t_{ij} = n^\mu T^\mu_{ij} = F_{ji} J^i, \quad J^i = -n_\mu F^{\mu i}$$

(20)

of the Brown-York tensor, where $D$ is the covariant derivative on the cutoff surface, $n$ the unit normal of the surface, $F_{ji}$ the boundary electromagnetic field and $J^i$ the boundary current dual to the bulk electromagnetic field (see e.g. [25]). In our case, the cutoff surface is always flat, so $D$ is just $\partial$. For the perturbed metric (6), it can be explicitly checked that

$$n^\mu T^\mu_{ij} = O(\epsilon^3).$$

So the leading order equation of the index $j = t$ in (20) is

$$\partial^t t_{tt} = -\frac{r_c^2}{16\pi G \sqrt{f(r_c)}} \left( \frac{f(r)}{r^2} \right)' \partial_a v^a = F_{ta} J^a = 0$$

at $O(\epsilon^2)$, which is just the incompressibility condition (9). The leading order equations of the index $j = a$ in (20) are

$$\partial^a t_{ta} = \frac{r_c^4}{16\pi G f(r_c) \sqrt{f(r_c)}} \left\{ \left( \frac{f(r)}{r^2} \right)'(\partial_t v^a + v^b \partial_b v^a + \frac{c}{r_c^2} \partial_a P) - \frac{f(r_c)}{r_c^2} [1 + f(r_c) F'(r_c)] \partial^2 v^a \right\} = f_a$$

(21)

at $O(\epsilon^3)$, where we have defined

$$c \equiv \frac{1}{2} r_c^3 \left( \frac{f(r)}{r^2} \right)'_c - (n - 1) f(r_c) - r_c f(r_c) \left( \frac{f(r)}{r^2} \right)'_c / \left( \frac{f(r_c)}{r_c^2} \right)'_c$$

(22)

and $f_a \equiv F_{ai} J^i$ as the external force density (see e.g. [5]).

Now we can read off the viscosity from the Brown-York tensor (19). In fact, the stress-energy tensor of a (relativistic) viscous fluid is

$$t_{ij} = (\rho + p) u_i u_j + p g_{ij} - 2\eta \sigma_{ij} - \zeta (g_{ij} + u_i u_j)$$

(23)

with $\sigma_{ij}$ the shear and $\theta = \partial_i u^i$ the expansion. We are only interested in the above stress-energy tensor up to $O(\epsilon^2)$ under the non-relativistic expansion. To this order we have $\theta = 0$ by the incompressibility condition (9), which renders the last term in (23) to vanish, and so

$$\sigma_{ij} = \frac{1}{2} \mathcal{P} (\partial_i u_j + \partial_j u_i),$$
where $P$ means projection to the $x^a$ directions. Comparing (23) with (19), we have

$$\eta = 1 + f(r_c)\mathcal{F}'(r_c) = \frac{1}{16\pi G} \frac{r_{h}^{n-2}}{r_{c}^{n-2}},$$

which together with (14) gives the cutoff-independent result

$$\frac{\eta}{s_c} = \frac{1}{4\pi}.$$

Under the special rescaling (4) (with constant $P$) of the background configuration (2), one finds that the transformed energy density and pressure become

$$\rho_s = \rho + \frac{(n - 2)r_c^2}{16\pi G\sqrt{f(r_c)}} \frac{f(r)}{r^2} P + O(\epsilon^3),$$

$$p_s = p + \frac{r_c^2}{16\pi G\sqrt{f(r_c)}} \left[ \frac{r_c^3}{2f(r_c)} \left( \frac{f(r)}{r^2} \right)'^2 - (n - 1) \left( \frac{f(r)}{r^2} \right)' - \frac{f(r)}{r^2} \right] P + O(\epsilon^3),$$

respectively. So the ratio of pressure (or “pressure density”) should be [22, 27]

$$P_r = \frac{p_s - p}{\rho + p} = \frac{1}{16\pi G} \left[ \frac{r_c^3}{2f(r_c)} \left( \frac{f(r)}{r^2} \right)'^2 - (n - 1) \left( \frac{f(r)}{r^2} \right)' - \frac{f(r)}{r^2} \right] = \frac{cP}{f(r_c)},$$

using (13) and (22). Introducing the coordinates

$$\tilde{t} = \sqrt{f(r_c)}t, \quad \tilde{x}^a = r_c x^a, \quad \text{(24)}$$

under which the induced metric on the cutoff surface becomes $\eta_{ij}$, one can easily recast the conservation equation (21) as the standard incompressible Navier-Stokes equation

$$\tilde{\partial}_b \beta^a + \beta^b \tilde{\partial}_b \beta_a + \tilde{\partial}_a P_r - \nu \tilde{\partial}_a \beta_a = \tilde{f}_a, \quad \tilde{f}_a = \frac{16\pi G \sqrt{f(r_c)}}{r_c^2 \left( \frac{f(r_c)}{r^2} \right)'} f_a, \quad \text{(25)}$$

with external force density $\tilde{f}_a$, where the kinematic viscosity

$$\nu = \frac{\eta}{\rho + p}. \quad \text{(26)}$$

### IV. THE GAUSS-BONNET-MAXWELL CASE

The action of the $n$-dimensional Gauss-Bonnet-Maxwell gravity is

$$I = \frac{1}{16\pi G} \int d^n x \sqrt{-g} (R - 2\Lambda + \alpha \mathcal{L}_{GB}) - \frac{1}{4} \int d^n x \sqrt{-g} F_{\mu\nu} F^{\mu\nu},$$

$$\mathcal{L}_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau},$$

where

$$\eta = 1 + f(r_c)\mathcal{F}'(r_c) = \frac{1}{16\pi G} \frac{r_{h}^{n-2}}{r_{c}^{n-2}},$$

which together with (14) gives the cutoff-independent result

$$\frac{\eta}{s_c} = \frac{1}{4\pi}.$$
with $\alpha$ the Gauss-Bonnet coupling constant. The corresponding equations of motion are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha H_{\mu\nu} + 8\pi GT_{\mu\nu} = 0,$$

$$\nabla_\mu F^{\mu\nu} = 0,$$

where

$$H_{\mu\nu} = 2(R_{\mu\sigma\tau\rho} R^{\sigma\rho\tau\nu} - 2R_{\mu\rho\sigma} R^{\rho\sigma} - 2R_{\mu\sigma} R^{\sigma} + RR_{\mu\nu}) - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{\text{GB}}. \quad (28)$$

We again take the charged black brane solution \[28\]

$$ds^2 = -f(r)dt^2 + 2drdt + r^2 dx^a dx^a,$$

$$f(r) = \frac{r^2}{2\tilde{\alpha}} \left(1 - \sqrt{1 - 4\tilde{\alpha}(1 - \frac{2m}{r^{n-1}} + \frac{Q^2}{r^{2n-4}})}\right) \quad (29)$$

as our background, where we have defined $\tilde{\alpha} = \frac{(n-3)(n-4)}{2(n-4)\alpha}$ for convenience. The electromagnetic field is of the same form as in the Einstein-Maxwell case.

It turns out that the non-relativistic long-wavelength expansion for the perturbed configurations in the Einstein-Maxwell case, i.e. the perturbed metric (6) and electromagnetic field (7) up to $O(\epsilon^2)$, as well as the form (8) of the correction term to the metric (6), can also be used in the Gauss-Bonnet-Maxwell case. Again, the Maxwell equations are automatically satisfied at $O(1)$ and $O(\epsilon)$, while at $O(\epsilon^2)$ lead to two equations

$$\partial_a v^a = 0, \quad \mathcal{F}'(r)\partial_a v^a = 0,$$

which can be solved altogether by the incompressibility condition. Then from the Einstein equations at $O(\epsilon^2)$ there is again only one requirement for $\mathcal{F}(r)$ to satisfy, which can be solved as\(^6\)

$$\mathcal{F}'(r) = -\left(1 - \frac{C}{r^{2(n-4)} - 2(n-4)\alpha[r^{n-5} f(r)]'}\right) \frac{1}{f(r)}.$$

Here the integration constant $C$ can be determined as $C = r_h^{n-2} - 2(n-4)\alpha r_h^{n-3} f'(r_h)$ with $r_h$ the horizon radius by the regularity condition of the perturbed metric at the horizon.

The Brown-York tensor on the cutoff surface $r = r_c$ in this case is \[22, 29, 30\]

$$t_{ij} = \frac{1}{8\pi G}[K g_{ij} - K_{ij} - 2\alpha(3J_{ij} - Jg_{ij}) - C g_{ij}] \quad (31)$$

with

$$J_{ij} = \frac{1}{3}(2K K_{ik} K^k_j + K_{kl} K^{kl} K_{ij} - 2K_{ik} K^{kl} K_{lj} - K^2 K_{ij}). \quad (32)$$

\(^6\) We have verified this solution for $5 \leq n \leq 10$. 

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where we have omitted terms that do not contribute for this flat cutoff surface with induced metric \([3]\). It can be explicitly worked out for the background metric \([29]\) that

\[
t_{ij}dx^i dx^j = \frac{1}{8\pi G} \left[ \sqrt{f(r_c)} \left( \frac{n-2}{r_c} f'(r_c) \left( \frac{2\tilde{W}(r_c)}{3r_c^2} - 1 \right) dt^2 + \frac{r_c^2}{\sqrt{f(r_c)}} \left( \frac{f'(r_c)}{2} + \frac{(n-3)f(r_c)}{r_c} - \tilde{\alpha} f(r_c) \left( \frac{2(n-5)f(r_c)}{3r_c^3} + \frac{f'(r_c)}{r_c^2} \right) \right) dx^a dx^a - C ds_c^2 \right].
\]

Comparing \([33]\) and \([12]\), we find

\[
\rho + p = \frac{1}{8\pi G \sqrt{f(r_c)}} \left[ (n-1) \frac{m}{r_c^{n-2}} - (n-2) \frac{Q^2}{r_c^{2n-5}} \right].
\]

The entropy density \(s_c\) and local Hawking temperature \(T_c\) on the cutoff surface are of the same form \([13]\) and \([15]\) as in the Einstein-Maxwell case, which again leads to the thermodynamic relation \([10]\) with charge density \([17]\) and chemical potential \([13]\).

For the perturbed metric \([6]\), the Brown-York tensor on the cutoff surface can be worked out as

\[
8\pi G t_{ij}dx^i dx^j = \sqrt{f(r_c)} \left( \frac{n-2}{r_c} f(r_c) \left( \frac{2\tilde{W}(r_c)}{3r_c^2} - 1 \right) dt^2 - C ds_c^2 \right. \\
+ \frac{r_c^2}{\sqrt{f(r_c)}} \left( \frac{f'(r_c)}{2} + \frac{(n-3)f(r_c)}{r_c} - \tilde{\alpha} f(r_c) \left( \frac{2(n-5)f(r_c)}{3r_c^3} + \frac{f'(r_c)}{r_c^2} \right) \right) dx^a dx^a \\
+ \left( 1 - \frac{2\tilde{\alpha} f(r_c)}{r_c^2} \right) \left( (n-2) f(r_c) P + r_c^2 v^2 \right) \frac{r_c^2}{2\sqrt{f(r_c)}} \left( \frac{f(r)}{r^2} \right)' dt^2 \\
+ \left( 1 - \frac{2\tilde{\alpha} f(r_c)}{r_c^2} \right) \frac{r_c^2}{2\sqrt{f(r_c)}} \frac{v^a v^b}{f(r_c)} \left( \frac{f(r)}{r^2} \right)' dx^a dx^b \\
+ \left( 1 - \frac{2\tilde{\alpha} f(r_c)}{r_c^2} \right) \frac{r_c^4}{2\sqrt{f(r_c)}} \frac{r_c^2}{2f(r_c)} \left( \frac{f(r)}{r^2} \right)' \left( \frac{f(r)}{r^2} \right)' \\
- \left( 1 - \frac{2\tilde{\alpha} f(r_c)}{r_c^2} \right) \frac{r_c^4}{2\sqrt{f(r_c)}} \left( \frac{f(r)}{r^2} \right)' \left( \frac{f(r)}{r^2} \right)' \\
- (n-1) \left( \frac{f(r)}{r^2} \right)' - r_c \left( \frac{f(r)}{r^2} \right)'' P dx^a dx^a \\
- \left( 1 - 2\tilde{\alpha} \left( \frac{f(r_c)}{r_c^2} + \frac{r_c}{n-3} \left( \frac{f(r)}{r^2} \right)' \right) \frac{r_c^2}{2\sqrt{f(r_c)}} \left( 1 + f(r_c) F'(r_c) \right) \right) (\partial_a v^b + \partial_b v^a) dx^a dx^b \\
+ O(\epsilon^3),
\]

(35)
after imposing the incompressibility condition (30) in the $O(\epsilon^2)$ part. So the leading order equation of the index $j = t$ in (20) is
\[
\partial^t t_{\mu t} = \left( \frac{2\tilde{\alpha} f(r_c)}{r_c^2} - 1 \right) \frac{r_c^2}{16\pi G \sqrt{f(r_c)}} \left( \frac{f(r)}{r^2} \right)_c^t \partial_a v^a = n^\mu T_{\mu t} = 0
\]
at $O(\epsilon^2)$, which is just the incompressibility condition (30). The leading order equations of the index $j = a$ in (20) are
\[
\partial^a t_{\mu a} = \frac{r_c^4}{16\pi G f(r_c) \sqrt{f(r_c)}} \left( 1 - \frac{2\tilde{\alpha} f(r_c)}{r_c^2} \right) \left( \frac{f(r)}{r^2} \right)_c^t \partial_a v^a + v^b \partial_b v^a + \frac{c}{r_c^2} \partial_a P
\]
\[
- \frac{f(r_c)}{r_c^2} \left( 1 - \frac{n}{3} \right) \left( \frac{f(r)}{r^2} \right)_c^t + \frac{r_c}{n-3} \left( \frac{f(r)}{r^2} \right)_c^t \partial^2 v^a = f_a
\]
at $O(\epsilon^3)$, where we have defined
\[
c \equiv \frac{r_c^3 (r_c^2 + 2\tilde{\alpha} f(r_c))}{2(r_c^2 - 2\tilde{\alpha} f(r_c))} \left( \frac{f(r)}{r^2} \right)_c^t - (n-1) f(r_c) - r_c f'(r_c) \left( \frac{f(r)}{r^2} \right)_c^t
\]
and $f_a \equiv n^\mu T_{\mu a} = F_{ai} J^i$ as the external force density.

Now we can read off the viscosity from the Brown-York tensor (35). Comparing (23) with (38), we have
\[
\eta = \frac{1}{16\pi G} \frac{r_c^{n-2}}{r_h^{n-2}} \left( 1 - 2\tilde{\alpha} \left( \frac{n-1}{n-3} - q_h^2 \right) \right)
\]
with the charge density of the black brane $q_h = \frac{\mathcal{Q}}{r_h}$, which together with (14) gives the cutoff-independent result
\[
\frac{\eta}{s_c} = \frac{1}{4\pi} \left( 1 - 2\tilde{\alpha} \left( \frac{n-1}{n-3} - q_h^2 \right) \right) = \frac{1}{4\pi} \left( 1 - 2(n-4)\alpha \left[ n - 1 - (n-3)q_h^2 \right] \right).
\]
This ratio agrees with the known result for infinite boundary $r_c \to \infty$ (31-34). Under the special rescaling (41) (with constant $P$) of the background configuration (29), one finds that the transformed energy density and pressure become
\[
\rho_s = \rho + \frac{(n-2)r_c^2}{16\pi G \sqrt{f(r_c)}} \left( 1 - \frac{2\tilde{\alpha} f(r_c)}{r_c^2} \right) \left( \frac{f(r)}{r^2} \right)_c^t P + O(\epsilon^3),
\]
\[
p_s = p + \frac{r_c^2}{16\pi G \sqrt{f(r_c)}} \left( \frac{f(r_c)}{r_c^2} \right)_c^t \left( \frac{r_c^3 (r_c^2 + 2\tilde{\alpha} f(r_c))}{2f(r_c)(r_c^2 - 2\tilde{\alpha} f(r_c))} \left( \frac{f(r)}{r^2} \right)_c^t \right) - (n-1) \left( \frac{f(r)}{r^2} \right)_c^t - r_c \left( \frac{f(r)}{r^2} \right)_c^t \frac{n}{r_c^2} P + O(\epsilon^3),
\]
\[
\text{Similar to the Einstein-Maxwell case, our result above is different from (57) in [22], but we have checked that these two expressions give the same result when turning off the electromagnetic field.}
respectively. So the ratio of pressure should be

\[
Pr = \frac{ps - p}{\rho + p} = \left[\frac{r_c^3(r_c^2 + 2\alpha f(r_c))}{2f(r_c)(r_c^2 - 2\alpha f(r_c))} \left(\frac{f(r)}{r^2}\right)'_c - (n - 1) - r_c\frac{f(r)}{r^2}\right]P = \frac{cP}{f(r_c)},
\]

using (34) and (37). Introducing the standard coordinates (24), one can again recast the conservation equation (36) as the standard incompressible Navier-Stokes equation (25) with external force density.

V. CONCLUDING REMARKS

Under the non-relativistic long-wavelength expansion, we have solved up to second order of the expansion parameter the bulk equations of motion for Dirichlet-like boundary conditions at an arbitrary cutoff surface outside the horizon in the charged AdS black brane space-times in both the Einstein-Maxwell and Gauss-Bonnet-Maxwell theories, without considering the independent electromagnetic DoF. The incompressible Navier-Stokes equation with external force density, as well as a cutoff-independent viscosity to entropy density ratio \( \frac{\eta_s}{s} \), for the dual fluid on the cutoff surface has been obtained in both theories, while in the Gauss-Bonnet-Maxwell case the ratio \( \frac{\eta_s}{s} = \frac{1}{4\pi}\{1 - 2(n - 4)\alpha[n - 1 - (n - 3)q_h^2]\} \) depends on the charge density of the black brane.

The framework used in our Letter seems to work well in all known cases of intrinsically flat cutoff surfaces, but for general curved cutoff surfaces it can no longer be used. For some special curved cutoff surface, there have been some discussions (35). Moreover, it is worthy to point out that very recently an alternative method has been introduced in (36) by imposing Petrov type I condition on the cutoff surface. It turns out that imposing this boundary condition is equivalent to imposing regularity condition on the horizon at least in the near horizon limit such that the Navier-Stokes equation can be derived in a much simpler way. Following this approach the generalized framework applicable to the spatially curved space-time has been presented in (37), and we expect that this strategy can be applied to our current work in future.

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