THE MAZ’YA-SHAPOSHNIKOVA LIMIT IN THE MAGNETIC SETTING

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ABSTRACT. We prove a magnetic version of the Maz’ya-Shaposhnikova singular limit of nonlocal norms with vanishing fractional parameter. This complements a general convergence result recently obtained by authors when the parameter approaches one.

1. INTRODUCTION

About fifteen years ago, V. Maz’ya and T. Shaposhnikova proved that for any \( n \geq 1 \) and \( p \in [1, \infty) \),

\[
\lim_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} \, dx \, dy = \frac{4\pi^{n/2}}{p\Gamma(n/2))} ||u||_{L_p(\mathbb{R}^n)}^p,
\]

whenever \( u \in D_0^s, p(\mathbb{R}^n) \) for some \( s \in (0, 1) \). Here \( \Gamma \) denotes the Gamma function and the space \( D_0^s, p(\mathbb{R}^n) \) is the completion of \( C_c^\infty(\mathbb{R}^n) \) with respect to the Gagliardo norm

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} \, dx \, dy.
\]

Their motivation was basically that of complementing a previous result by Bourgain-Brezis-Mironescu \cite{bib:01, bib:02} providing new characterizations for functions in the Sobolev space \( W^{1,p}(\Omega) \). Precisely, if \( \Omega \subset \mathbb{R}^n \) is a smooth bounded domain, then for any \( W^{1,p}(\Omega) \) there holds

\[
\lim_{s \searrow 1} (1 - s) \int_{\Omega} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} \, dx \, dy = Q_{p,n} \int_{\Omega} |\nabla u|^p \, dx,
\]

where \( Q_{p,n} \) is defined by

\[
Q_{p,n} = \frac{1}{p} \int_{S^{n-1}} |\omega \cdot h|^p \, dH^{n-1}(h),
\]

being \( S^{n-1} \) the unit sphere in \( \mathbb{R}^n \) and \( \omega \) an arbitrary unit vector of \( \mathbb{R}^n \). The above singular limits are natural and also admit a physical relevance in the framework of the theory of Levy processes. Also, there is a developed theory of fractional s-perimeters \cite{bib:03} and there have been several contributions concerning their asymptotic analysis in the limits \( s \searrow 1 \) and \( s \nearrow 0 \) \cite{bib:04, bib:05, bib:06}.

One of the latest generalizations of this kind of convergence results appeared recently in \cite{bib:08} in the context of magnetic Sobolev spaces \( W^{1,2}_A(\Omega) \), see \cite{bib:09}. In fact, a relevant role in the study of particles which interact with a magnetic field \( B = \nabla \times A, A : \mathbb{R}^n \to \mathbb{R}^n \), is assumed by the magnetic Laplacian \((\nabla - iA)^2\) \cite{bib:07, bib:10, bib:11}, yielding to nonlinear Schrödinger equations of the type \(- (\nabla - iA)^2 u + u = f(u)\), which have been extensively studied (see \cite{bib:09} and the references therein). The operator is defined weakly as the differential of the energy

\[
W^{1,2}_A(\Omega) \ni u \mapsto \int_{\Omega} |\nabla u - iA(x)u|^2 \, dx.
\]

If \( A : \mathbb{R}^n \to \mathbb{R}^n \) is a smooth field and \( s \in (0, 1) \), a nonlocal magnetic counterpart of the magnetic laplacian,

\[
(-\Delta)^s_A u(x) = c(n, s) \lim_{\epsilon \searrow 0} \int_{B^c_\epsilon(x)} \frac{u(x) - e^{i(x-y) \cdot A(\frac{y-x}{\epsilon})} u(y)}{|x - y|^{n+2s}} \, dy,
\]

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\]

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where \( c(n,s) \) is a normalization constant which behaves as follows

\[
\lim_{s \searrow 0} \frac{c(n,s)}{s} = \frac{\Gamma(n/2)}{\pi^{n/2}}, \quad \lim_{s \nearrow 1} \frac{c(n,s)}{1-s} = \frac{2n\Gamma(n/2)}{\pi^{n/2}},
\]

was introduced in [8, 11] for complex-valued functions, with motivations falling into the framework of the general theory of Lévy processes. Recently, the authors in [14] (see [16] for \( p = 2 \)) proved that if \( A : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^2 \) vector field, then, for any \( n \geq 1 \), \( p \in [1, \infty) \) and any Lipschitz bounded domain \( \Omega \subset \mathbb{R}^n \)

\[
\left(1 - s\right) \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A \left( \frac{c(n,s)}{\sqrt{n}} \right)} u(y)|^p}{|x-y|^{n+ps}} \, dx \, dy = Q_{p,n} \int_{\Omega} |\nabla u - iA(x)u|^p_s \, dx,
\]

for all \( u \in W_A^{1,p}(\Omega) \), where \( Q_{p,n} \) is as in (1.1) and \( |z|_p := ((|\mathbb{R}z_1, \ldots, \mathbb{R}z_n|)^p + (|\mathbb{Z}z_1, \ldots, \mathbb{Z}z_n|)^p)^{1/p} \). This has provided a new nonlocal characterization of the magnetic Sobolev spaces \( W_A^{1,p}(\Omega) \).

The main goal of this paper is to complete the picture of [14] by providing a magnetic counterpart of the convergence result by Maz’ya-Shaposhnikova for vanishing fractional orders \( s \), namely for \( s \searrow 0 \).

We consider a locally bounded vector potential field \( A : \mathbb{R}^n \to \mathbb{R}^n \) and the space of complex valued functions \( D_A^{s,p}(\mathbb{R}^n, \mathbb{C}) \) defined as the completion of \( C_c^\infty(\mathbb{R}^n, \mathbb{C}) \) with respect to the norm

\[
\|u\|_{D_A^{s,p}} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A \left( \frac{c(n,s)}{\sqrt{n}} \right)} u(y)|^p}{|x-y|^{n+ps}} \, dx \, dy \right)^{1/p}.
\]

By combining Lemma 2.1 and Lemma 2.3, we shall prove the following result.

Theorem 1.1 (Magnetic Maz’ya-Shaposhnikova). Let \( n \geq 1 \) and \( p \in [1, \infty) \). Then for every

\[
u \in \bigcup_{0 < s < 1} D_A^{s,p}(\mathbb{R}^n, \mathbb{C}),
\]

there holds

\[
\lim_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A \left( \frac{c(n,s)}{\sqrt{n}} \right)} u(y)|^p}{|x-y|^{n+ps}} \, dx \, dy = \frac{4\pi^{n/2}}{p\Gamma(n/2)} \|u\|_{L^p(\mathbb{R}^n)}^p.
\]

In particular, while the singular limit as \( s \nearrow 1 \) generates the magnetic gradient \( \nabla - iA \), the limit for vanishing \( s \) tends to destroy the magnetic effects yielding the \( L^p(\mathbb{R}^n) \)-norm of the function \( u \). We point out that, while in (1.3) the norm of complex numbers is \( |\cdot|_p \), in Theorem 1.1 we use the usual norm \( |\cdot| = |\cdot|_2 \).

In any case when \( A = 0 \) and \( u \) is real-valued the formulas are all consistent with the classical statements.

In the case \( p = 2 \), combining the asymptotic formulas in (1.2) with Theorem 1.1 implies that

\[
c(n,s) \frac{2}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A \left( \frac{c(n,s)}{\sqrt{n}} \right)} u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \approx \|u\|^2_{L^2(\mathbb{R}^n)}, \quad \text{as } s \searrow 0,
\]

for any \( u \in D_0^{s,2}(\mathbb{R}^n) \) for some \( s \in (0, 1) \). Although the magnetic setting is mainly meaningful in the framework of nonlocal Schrödinger equations, we remark that for \( E \subset \mathbb{R}^n \), if \( E^c := \mathbb{R}^n \setminus E \), the quantity

\[
P_s(E; A) := \frac{1}{2} \int_{E} \int_{E} \frac{|1 - e^{i(x-y) \cdot A \left( \frac{c(n,s)}{\sqrt{n}} \right)}|}{|x-y|^{n+ps}} \, dx \, dy + \int_{E} \int_{E^c} \frac{1}{|x-y|^{n+ps}} \, dx \, dy
\]

plays the role of a nonlocal \( s \)-perimeter of \( E \) depending on \( A \), which reduces for \( A = 0 \) to the usual notion of fractional \( s \)-perimeter of \( E \subset \mathbb{R}^n \). Then, if \( \mathcal{L}^n(E) \) denotes the \( n \)-dimensional Lebesgue measure of \( E \subset \mathbb{R}^n \), Theorem 1.1, applied with \( p = 1 \) and \( u(x) = 1_E(x) \), reads as

\[
\lim_{s \searrow 0} sP_s(E; A) = \frac{4\pi^n}{\Gamma(n/2)} \mathcal{L}^n(E),
\]

provided that \( P_{s_0}(E; A) < +\infty \), for some \( s_0 \in (0, 1) \).
2. Proof of the main result

The proof of Theorem 1.1 follows by combining Lemma 2.1 and Lemma 2.3 below.

Lemma 2.1 (Liminf inequality). Let $n \geq 1$, $p \in [1, \infty)$ and let

$$u \in \bigcup_{0 < s < 1} D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C}).$$

Then

$$\liminf_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy \geq \frac{4\pi^{n/2}}{p! (n/2)} \|u\|_{L^p(\mathbb{R}^n)}.$$  

Proof. If

$$\liminf_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy = \infty,$$

the assertion follows. Otherwise, there exists a sequence $\{s_k\} \subset (0, 1)$ with $s_k \searrow 0$ and

$$\liminf_{s \searrow 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy = \lim_{k \to \infty} s_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy,$$

the limit being finite. For a.e. $x, y \in \mathbb{R}^n$ we have the Diamagnetic inequality (cf. [8, Remark 3.2])

$$|u(x)| - |u(y)| \leq |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|.$$

In particular, since $u \in D^{s_k,p}_{A,0}(\mathbb{R}^n, \mathbb{C})$, we have $|u| \in D^{0,p}_{0}\mathbb{R}^n(\mathbb{R}^n, \mathbb{C})$ and, for any $k \geq 1,$

$$s_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} dx dy \leq s_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy.$$

Taking the limit as $k \to \infty$ on both sides and invoking [13, Theorem 3] applied to $|u|$, yields

$$\frac{4\pi^{n/2}}{p! (n/2)} \|u\|^p_{L^p(\mathbb{R}^n)} \leq \lim_{k \to \infty} s_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy,$$

which concludes the proof. \hfill $\Box$

Remark 2.2 (Magnetic Hardy inequality). By combining the pointwise Diamagnetic inequality (2.1) with the fractional Hardy inequality [10], for $n > ps$ the following magnetic Hardy inequality holds: there exists a positive constant $\mathcal{H}_{n,s,p}$ such that

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \leq \mathcal{H}_{n,s,p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy,$$

for every $u \in D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C})$. Similarly the following magnetic Sobolev inequality holds: there exists a positive constant $\mathcal{S}_{n,s,p}$ such that

$$\left( \int_{\mathbb{R}^n} |u(x)|^{\frac{np}{n-ps}} dx \right)^{\frac{n-ps}{n}} \leq \mathcal{S}_{n,s,p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy,$$

for every $u \in D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C})$.

Next we state a second lemma completing the proof of Theorem 1.1 when combined with Lemma 2.1.

Lemma 2.3 (Limsup inequality). Let $n \geq 1$, $p \in [1, \infty)$ and let

$$u \in \bigcup_{0 < s < 1} D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C}).$$

Then

$$\limsup_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{z+x}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy \leq \frac{4\pi^{n/2}}{p! (n/2)} \|u\|^p_{L^p(\mathbb{R}^n)}.$$
Proof. If \( u \notin L^p(\mathbb{R}^n) \), there is nothing to prove. Hence, we may assume that \( u \in L^p(\mathbb{R}^n) \). We observe that

\[
\begin{align*}
& s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}} \\
& = s \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| \leq 2|x|\}} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}} \\
& + s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}} \\
& = 2s \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| \leq 2|x|\}} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}} \\
& + 2s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}},
\end{align*}
\]

where the last equality follows noticing that since \( |e^{i(x-y) \cdot A(\frac{z+x}{z+y})}| = 1 \) then

\[
\begin{align*}
& \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| \leq 2|x|\}} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}} = \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}} \\
& = \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}} \\
& + \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| \leq 2|x|\}} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}}.
\end{align*}
\]

Using the triangle inequality for the \( L^p \)-norm on \( \mathbb{R}^{2n} \) and recalling that \( |e^{i(x-y) \cdot A(\frac{z+x}{z+y})}| = 1 \), yields

\[
\begin{align*}
& s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} |u(x) - e^{i(x-y) \cdot A(\frac{z+x}{z+y})} u(y)|^p \frac{dxdy}{|x-y|^{n+ps}} \\
& \leq \left\{ s \left( \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} |u(x)|^p \frac{dxdy}{|x-y|^{n+ps}} \right)^{1/p} + s \left( \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} |u(y)|^p \frac{dxdy}{|x-y|^{n+ps}} \right)^{1/p} \right\}^p.
\end{align*}
\]

We claim that

\[
\lim_{s \to 0} s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(y)|^p}{|x-y|^{n+ps}} dxdy = 0.
\]

Observe that \( 2|x-y| \geq |y| + (|y| - 2|x|) \). Then, if \( |y| \geq 2|x| \) we get \( 2|x-y| \geq |y| \). Now, if \( \mathcal{H}^{n-1} \) denotes the \((n-1)\)-dimensional Hausdorff measure, it follows that

\[
\begin{align*}
& s^{1/p} \left( \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{1/p} \\
& \leq s^{1/p} \left( \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(y)|^p}{|y|^{n+sp}} \left( \int_{|x| \leq |y|/2} dx \right) dy \right)^{1/p} \\
& = 2s \left( \frac{s}{n} \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \right)^{1/p} \left( \int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{n+sp}} dy \right)^{1/p},
\end{align*}
\]
and the last term goes to zero as \( s \searrow 0 \). Notice that \( y \mapsto |y|^{-s}u(y) \) remains bounded in \( L^p(\mathbb{R}^n) \) as \( s \searrow 0 \) by the argument indicated here below. Observe now that, if \( |y| \geq 2|x| \) we then get \( |x - y| \geq |x| \) yielding
\[
\left( s \int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} \frac{|u(x)|^p}{|x - y|^{n+sp}} dxdy \right)^{1/p} \leq \left( s \int_{\mathbb{R}^n} \int_{|x - y| \geq |x|} \frac{|u(x)|^p}{|x - y|^{n+sp}} dxdy \right)^{1/p}
\]
\[
= \left( s \int_{\mathbb{R}^n} |u(x)|^p \int_{B(0,|x|)} \frac{dz}{|z|^{n+sp}} dx \right)^{1/p} = \frac{\mathcal{H}^{-1}(S^{n-1})^{1/p}}{p^{1/p}} \left( \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \right)^{1/p}.
\]
Moreover \( |x|^{-sp} |u(x)|^p = f_s(x) + g_s(x) \), where
\[
f_s(x) := \frac{|u(x)|^p}{|x|^{sp}} \mathbf{1}_{B(0,1)}(x), \quad g_s(x) := \frac{|u(x)|^p}{|x|^{sp}} \mathbf{1}_{B(0,1)^c}(x) \leq |u(x)|^p \mathbf{1}_{B(0,1)^c}(x) \in L^1(\mathbb{R}^n),
\]
and \( s \mapsto f_s \) is decreasing and, moreover, by the Hardy inequality (2.2) and the assumption on \( u \), it follows that \( f_s \in L^1(\mathbb{R}^n) \) for some \( \tilde{s} \in (0,1) \). Hence, by monotone and dominated convergence, we conclude that
\[
\limsup_{s \searrow 0} \int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} \frac{|u(x)|^p}{|x - y|^{n+sp}} dxdy \leq \frac{\mathcal{H}^{-1}(S^{n-1})^{1/p}}{p^{1/p}} \|u\|_{L^p(\mathbb{R}^n)}^p = \frac{2\pi^{n/2}}{p^1(\frac{n}{2})} \|u\|_{L^p(\mathbb{R}^n)}^p.
\]
Then, we conclude from the above inequalities that
\[
\limsup_{s \searrow 0} 2s \int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dxdy \leq \frac{4\pi^{n/2}}{p^1(\frac{n}{2})} \|u\|_{L^p(\mathbb{R}^n)}^p.
\]
We claim that
\[
\limsup_{s \searrow 0} 2s \int_{\mathbb{R}^n} \int_{|x| \leq |y| \leq 2|x|} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dxdy = 0.
\]
By assumption let \( \tau \in (0,1) \) such that \( u \in D^p_{-\tau,A,0}(\mathbb{R}^n) \). Now let \( N \geq 1 \) and \( s < \tau \). Then
\[
2s \int_{\mathbb{R}^n} \int_{|x| \leq |y| \leq 2|x|} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dxdy
\]
\[
= 2s \int_{\mathbb{R}^n} \int_{|x| \leq |y| \leq N} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dxdy
\]
\[
+ 2s \int_{\mathbb{R}^n} \int_{|x| \leq |y| \leq 2|x|} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dxdy =: \mathcal{I} + \mathcal{II}.
\]
Let us consider \( \mathcal{I} \) first. Since \( |x - y| \leq N \), it holds that
\[
\frac{1}{|x - y|^{n+sp}} = \frac{|x - y|^{p(\tau-s)}}{|x - y|^{n+sp}} \leq \frac{N^{p(\tau-s)}}{|x - y|^{n+p}}.
\]
Therefore \( \mathcal{I} \) goes to zero as \( s \searrow \tau \), since
\[
\mathcal{I} \leq 2s N^{p(\tau-s)} \int_{\mathbb{R}^n} \int_{|x| \leq |y| \leq N} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dxdy.
\]
Let us now move to \( \mathcal{II} \). Since \( |u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p \leq 2^{p-1} (|u(x)|^p + |u(y)|^p) \), we get
\[
\mathcal{II} \leq 2^p s \int_{\mathbb{R}^n} \int_{|x| \leq |y| \leq N} \frac{|u(x)|^p}{|x - y|^{n+sp}} dxdy + 2^{p} s \int_{\mathbb{R}^n} \int_{|x| \leq |y| \leq N} \frac{|u(y)|^p}{|x - y|^{n+sp}} dxdy =: \mathcal{II}' + \mathcal{II}''.
\]
Regarding \( \mathcal{II}' \), since \( |x - y| \geq N \) and \( |y| \leq 2|x| \), it holds that
\[
N \leq |x - y| \leq |x| + |y| \leq 3|x|.
\]
which implies that $|x| \geq \frac{N}{2}$. In particular, this also implies that

$$
\mathcal{II}' \leq 2^p s \int_{|x|\geq N/3} \left( \int_{|x-y|\geq N} \frac{|u(x)|^p}{|x-y|^{n+sp}} dy \right) dx \leq C(n,p) \int_{|x|\geq N/3} |u(x)|^p dx.
$$

For $\mathcal{II''}$, since as before $|x-y| \geq N$ and $|x| \leq |y|$, we have

$$
N \leq |x-y| \leq |x| + |y| \leq 2|y|,
$$

which implies $|y| \geq \frac{N}{2} \geq \frac{N}{3}$. Therefore, we get

$$
\mathcal{II''} \leq 2^p s \int_{|y|\geq N/3} |u(y)|^p \left( \int_{|y|\geq N} \frac{1}{|z|^{n+sp}} dz \right) dy \leq C(n,p) \int_{|y|\geq N/3} |u(y)|^p dy.
$$

Combining the estimates for $\mathcal{II'}$ and $\mathcal{II''}$, we get

$$
\mathcal{II} \leq C(n,p) \int_{|x|\geq N/3} |u(x)|^p dx,
$$

which is a bound independent of $s$. Now, going back to

$$
\limsup_{s \to 0} 2s \int_{\mathbb{R}^n} \int_{|x| < |y| < 2|x|} \frac{|u(x) - e^{i(x-y) \cdot A\left(\frac{xy}{|y|^2}\right)} u(y)|^p}{|x-y|^{n+sp}} dx dy \leq 2C(n,p) \|u\|_{L^p(B(0,N/3)^c)}^p,
$$

and (2.4) follows letting $N \to \infty$, since $u \in L^p(\mathbb{R}^n)$. Collecting (2.3) and (2.4), the assertion follows. \hfill \square

References

[1] L. Ambrosio, G. De Philippis, L. Martinazzi, $\Gamma$–convergence of nonlocal perimeter functionals, Manuscripta Math. 134 (2011), 377–403.

[2] G. Arioli, A. Szulkin, A semilinear Schrödinger equation in the presence of a magnetic field, Arch. Ration. Mech. Anal. 170 (2003), 277–295.

[3] J. Avron, I. Herbst, B. Simon, Schrödinger operators with magnetic fields. I. General interactions, Duke Math. J. 45 (1978), 847–883.

[4] J. Bourgain, H. Brezis, P. Mironescu, Another look at Sobolev spaces, Schrödinger operators with magnetic fields. II. Magnetic relativistic Schrödinger operators and imaginary-time path integrals, A Volume in Honor of Professor Alain Bensoussan’s 60th Birthday (eds. J. L. Menaldi, E. Rofman and A. Sulem), IOS Press, Amsterdam, 2001, 439–455.

[5] J. Bourgain, H. Brezis, P. Mironescu, Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications, J. Anal. Math. 87 (2002), 77–101.

[6] L. Caffarelli, J.-M. Roquejoffre, O. Savin, Nonlocal minimal surfaces, Comm. Pure Appl. Math. 63 (2010), 1111–1144.

[7] L. Caffarelli, E. Valdinoci, Regularity properties of nonlocal minimal surfaces via limiting arguments, Adv. Math. 248 (2013), 843–871.

[8] P. D’Avenia, M. Squassina, Ground states for fractional magnetic operators, preprint, http://arxiv.org/abs/1601.04230.

[9] S. Dipierro, A. Figalli, G. Palatucci, E. Valdinoci, Asymptotics of the s-perimeter as s → 0, Discrete Contin. Dyn. Syst. 33 (2013), 2777–2790.

[10] R.L. Frank, R. Seiringer, Non-linear ground state representation and sharp Hardy inequalities, J. Funct. Anal. 255 (2008), 3407–3430.

[11] T. Ichinose, Magnetic relativistic Schrödinger operators and imaginary-time path integrals, Mathematical physics, spectral theory and stochastic analysis, 247–297, Oper. Theory Adv. Appl. 232, Birkhäuser/Springer, Basel, 2013.

[12] E. Lieb and M. Loss, Analysis, Graduate Studies in Mathematics 14, 2001.

[13] V. Maz’ya and T. Shaposhnikova, On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces, J. Funct. Anal. 195 (2002), 230–238.

[14] A. Pinamonti, M. Squassina, E. Vecchi, Magnetic BV functions and the Bourgain-Brezis-Mironescu formula, preprint, https://arxiv.org/abs/1609.09714.

[15] M. Reed, B. Simon, Methods of modern mathematical physics, I, Functional analysis, Academic Press, Inc., New York, 1980.

[16] M. Squassina, B. Volzone, Bourgain-Brezis-Mironescu formula for magnetic operators, C. R. Math. Acad. Sci. Paris 354 (2016), 825–831.
