A note on the Lovász-Schrijver Semidefinite Programming Relaxation for Binary Integer Programs

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Abstract

Binary Integer Programming (BIP) problems are of interest due in part to the difficulty they pose and because of their various applications, including those in graph theory, combinatorial optimization and network optimization. In this note, we explicitly state the Lovász-Schrijver Semidefinite Programming (SDP) relaxation (in primal-standard form) for a BIP problem, a relaxation that yields a tighter upper-bound than the canonical Linear Programming relaxation.

Keywords: Semidefinite programming, Integer programming, relaxation.

1 Notation

In this note, the following notational conventions are adopted:

1. \(\mathbb{R}^{1+n} := \left\{ \begin{bmatrix} x_0 \\ x \end{bmatrix} : x_0 \in \mathbb{R}, x \in \mathbb{R}^n \right\} \) and \(\{e_i\}_{i=0}^n\) denotes the canonical basis.

2. The space of real \(n \times n\) matrices is denoted by \(\mathbb{R}^{n \times n}\). The space of real, symmetric \(n \times n\) matrices is denoted by \(\mathbb{R}^{n \times n}_s\). The space of real, symmetric, positive definite (positive semidefinite) \(n \times n\) matrices is denoted by \(\mathbb{R}^{n \times n}_{>0} (\mathbb{R}^{n \times n}_{\geq 0})\).

3. The \((i,j)\) entry of a matrix \(X\) is denoted by \(x_{ij}\).

4. Positive definiteness (or positive semidefiniteness) of a matrix \(X\) is denoted by \(X > 0 (X \succeq 0)\).

5. For \(X, Y \in \mathbb{R}^{n \times n}\), \(X \bullet Y\) denotes the (Frobenius) inner product of the matrices \(X\) and \(Y\), defined by \(\text{trace}(X^T Y)\).

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6. For $X \in \mathbb{R}^{n \times n}$, $\text{vec}(X)$ denotes the column-wise vectorization of a matrix $X$.

7. For $S \subseteq \mathbb{R}^n$, $\text{conv}(S)$ denotes its convex hull.

8. $\text{Diag}(x)$ denotes the $n \times n$ diagonal matrix with the vector $x \in \mathbb{R}^n$ on its diagonal. For matrices $A_1, \ldots, A_r \in \mathbb{R}^{n_1 \times n_1}$, $\text{Diag}(A_1, \ldots, A_r) \in \mathbb{R}^d$ denotes the block-diagonal matrix with matrices $A_1, \ldots, A_r$ along its block-diagonal, where $d := \sum_{k=1}^r n_k \times \sum_{k=1}^r n_k$.

2 Lovász-Schrijver Lift-and-Project Method

2.1 Lifted Matrix Variable

Consider the binary (or 0-1) integer program

$$\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m \\
& \quad x \in \{0,1\}^n
\end{align*}$$

(BIP)

and its Linear Programming (LP) relaxation

$$\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m \\
& \quad x \in [0,1]^n
\end{align*}$$

(LPR)

Let $P$ be the polytope defined by $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ (assume $Ax \leq b$ includes the $m$ inequalities $a_i^T x \leq b_i$ and the trivial inequalities $0 \leq x \leq 1$). Let $P_t$ denote the convex hull of the 0-1 vectors belonging to $P$. Note that solving (LPR) provides an upper bound on (BIP), however this solution may not be integral and far from the actual solution. Notice that the polytope $P_t$, obtained by relaxing the condition $x \in \{0,1\}^n$ to $x \in [0,1]^n$, is an approximation of $P_t$.

Lovász and Schrijver [2] devised a method that generates nonlinear “cuts” that better approximate $P_t$ than $P$. Instead of working with $x \in \{0,1\}^n$ in (BIP), Lovász and Schrijver considered the lifted matrix variable

$$X := \begin{bmatrix}
1 \\
x \\
\end{bmatrix}
\begin{bmatrix}
1 & x^T \\
x & xx^T \\
\end{bmatrix} = \begin{bmatrix}
1 + x^T x, 0
\end{bmatrix}.$$ 

Note that $X$ has the following properties:

1. $X \in \mathbb{R}_{\geq}^{n+1 \times n+1}$. Indeed, $X$ is a symmetric, rank-one matrix with spectrum $\sigma(X) = \{1 + x^T x, 0\}$.
2. \( Xe_0 = \text{diag}(X) \), i.e., the first column of \( X \) equals the diagonal of \( X \).
This follows from \( x_{ii} = x_i^2 = x_i \) and \( x_i \in \{0, 1\} \). Moreover, following the symmetry of \( X \), we have \( Xe_0 = \text{diag}(X) = X^Te_0 \), i.e., the first column, first row and diagonal of \( X \) are equal.

### 2.2 Nonlinear Cuts

Note that for \( i = 1, \ldots, m, \ j = 1, \ldots, n \) the inequalities

\[
(b_i - a_i^T x)x_j \geq 0 \tag{1}
\]

\[
(b_i - a_i^T x)(1 - x_j) \geq 0 \tag{2}
\]

are valid for \( x \in P \).

Let \( u_i := [b_i - a_i^T]^T \). One can verify (c.f. [1]) that [1] and [2] are expressible in terms of \( X \) as

\[
u_i e_j^T \cdot X \geq 0 \tag{3}
\]

\[
u_i (e_0 - e_j)^T \cdot X \geq 0 \tag{4}
\]

Further, the condition \( Xe_0 = \text{diag}(X) \) becomes

\[
e_j (e_0 - e_j)^T \cdot X = 0. \tag{5}
\]

Finally, we require \( X_{00} = 1 \) which is expressible as

\[
e_0 e_0^T \cdot X = 1. \tag{6}
\]

Lovász and Schrijver then propose the cones

\[
M_+(P) := \{ X \in \mathbb{R}^{n+1 \times n+1} : [3] \text{--} [6] \}
\]

and

\[
N_+(P) := \{ x \in \mathbb{R}^n : [1] x^T = \text{diag}(X), X \in M_+(P) \}
\]

and establish

**Lemma 2.1** (See Lemma 1.1 in [2]). \( P_I \subseteq N_+(P) \subseteq P \).

Following Lemma 2.1, solving \( \max \{ c^T x : x \in N_+(P) \} \) produces a tighter upper-bound for [BIP] than [LPR].

### 3 Semidefinite Programming (SDP)

An SDP problem in *primal form* is given by

\[
\begin{align*}
\text{minimize} \quad & C \cdot X \\
\text{subject to} \quad & A_i \cdot X = b_i \quad i = 1, \ldots, m \\
& X \succeq 0
\end{align*}
\tag{SDPP}
\]
where \( A_i \in \mathbb{R}^{n \times n} \), \( b_i \in \mathbb{R}^n \), \( C \in \mathbb{R}^{n \times n} \) are the problem data, and \( X \in \mathbb{R}^{n \times n} \geq \) is the variable.

An SDP problem in dual form is given by

\[
\begin{aligned}
\text{maximize} & \quad b^T y \\
\text{subject to} & \quad \sum_{i=1}^m y_i A_i \preceq C
\end{aligned}
\]  

(SDPD)

where \( y \in \mathbb{R}^m \) is the variable.

Note that the linear programming problem

\[
\begin{aligned}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x = b_i \quad i = 1, \ldots, m \\
x & \geq 0
\end{aligned}
\]

becomes an SDP problem in primal form by setting \( C := \text{Diag}(c) \), \( A_i := \text{Diag}(a_i) \) and \( X := \text{Diag}(x) \) so that Semidefinite Programming is a generalization of Linear Programming.

SDP has applications in eigenvalue optimization, combinatorial optimization, and system and control theory; furthermore, there are several approximation methods for solving SDP’s. (See \([3]\) or \([4]\) for more detailed discussions concerning SDP.)

4 SDP Relaxation

Before we state the SDP primal-form problem explicitly, we prove the following lemma.

Lemma 4.1. If \( A \in \mathbb{R}^{n \times n} \), \( X \in \mathbb{R}^{n \times n} \) and \( A' := \frac{1}{2}(A + A^T) \), then \( A' \bullet X = A' \cdot X \).

Proof. Following properties of the trace and transpose operators,

\[
A' \bullet X = \text{tr} \left( \frac{1}{2}(A + A^T) X \right) = \frac{1}{2} \text{tr}(A^T X) + \frac{1}{2} \text{tr}(AX)
\]

\[
= \frac{1}{2} \text{tr}(A^T X) + \frac{1}{2} \text{tr}(XA^T)
\]

\[
= \frac{1}{2} \text{tr}(A^T X) + \frac{1}{2} \text{tr}(A^T X) = \text{tr}(A^T X) = A' \cdot X.
\]

Following Lemma 4.1 constraints \((3)-(6)\) can be written in terms of symmetric matrices (a requirement for the canonical primal- and dual-form SDP problems).

4
Let $C := c_0 [0 \quad c^T]$. Dash (c.f. [II]) demonstrated that solving $\max \{ c^T x : x \in N_+(P) \}$ is equivalent to solving the SDP (in non-canonical form)\

$\begin{align*}
\text{maximize} & \quad C \mathrel{\cdot} X \\
\text{subject to} & \quad \frac{1}{2} \left[ u_i e_j^T + \left( u_i e_j^T \right)^T \right] \mathrel{\cdot} X \geq 0 \\
& \quad \frac{1}{2} \left[ u_i (e_0 - e_j)^T + \left( u_i (e_0 - e_j)^T \right)^T \right] \mathrel{\cdot} X \geq 0 \\
& \quad \frac{1}{2} \left[ e_j (e_0 - e_j)^T + \left( e_j (e_0 - e_j)^T \right)^T \right] \mathrel{\cdot} X = 0 \\
& \quad e_0 e_0^T \mathrel{\cdot} X = 1 \\
& \quad X \succeq 0 \\
& \quad i = 1, \ldots, m \\
& \quad j = 1, \ldots, n
\end{align*}$

which, after introducing $2mn$ surplus-variables, becomes\

$\begin{align*}
\text{maximize} & \quad C \mathrel{\cdot} X \\
\text{subject to} & \quad \frac{1}{2} \left[ u_i e_j^T + \left( u_i e_j^T \right)^T \right] \mathrel{\cdot} X - s_{ij} = 0 \\
& \quad \frac{1}{2} \left[ u_i (e_0 - e_j)^T + \left( u_i (e_0 - e_j)^T \right)^T \right] \mathrel{\cdot} X - \bar{s}_{ij} = 0 \\
& \quad \frac{1}{2} \left[ e_j (e_0 - e_j)^T + \left( e_j (e_0 - e_j)^T \right)^T \right] \mathrel{\cdot} X = 0 \\
& \quad e_0 e_0^T \mathrel{\cdot} X = 1 \\
& \quad X \succeq 0 \\
& \quad i = 1, \ldots, m \\
& \quad j = 1, \ldots, n
\end{align*}$

For $i = 1, \ldots, m$, $j = 1, \ldots, n$, define\

1. $n := 2mn + n + 1$\
2. $S := [s_{ij}] \in \mathbb{R}^{m \times n}$\
3. $\bar{S} := [\bar{s}_{ij}] \in \mathbb{R}^{m \times n}$\
4. $C := \begin{bmatrix} -C & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\bar{n} \times \bar{n}}$\
5. $X := \begin{bmatrix} X \\ \text{Diag(vec}(S)) \end{bmatrix} \in \mathbb{R}^{\bar{n} \times \bar{n}}$\
6. $A_{ij} := \text{Diag} \left( \frac{1}{2} \left[ u_i e_j^T + \left( u_i e_j^T \right)^T \right], 0, \ldots, -1_{n+1+m(j-1)+i}, 0, \ldots, 0 \right) \in \mathbb{R}^{\bar{n} \times \bar{n}}$
7. $A_{ij} := \text{Diag}\left(\frac{1}{2} \left[ u_i(e_0 - e_j)^T + (u_i(e_0 - e_j)^T)^T \right], 0, \ldots, 0 \right) \in \mathbb{R}^{\bar{n} \times \bar{n}}$.

8. $\tilde{A}_{ij} := \text{Diag}\left(\frac{1}{2} \left[ e_j(e_0 - e_j)^T + (e_j(e_0 - e_j)^T)^T \right], 0_{2mn \times 2mn} \right) \in \mathbb{R}^{\bar{n} \times \bar{n}}$

9. $A := \text{Diag} \left( e_0e_0^T, 0_{2mn \times 2mn} \right) \in \mathbb{R}^{\bar{n} \times \bar{n}}$

so that the primal-form SDP relaxation of (BIP) is

\[
\begin{align*}
\text{minimize} & \quad \bar{C} \bullet \bar{X} \\
\text{subject to} & \quad A_{ij} \bullet \bar{X} = 0 \\
& \quad \tilde{A}_{ij} \bullet \bar{X} = 0 \\
& \quad A_{ij} \bullet \bar{X} = 0 \\
& \quad A \bullet \bar{X} = 1 \\
& \quad \bar{X} \succeq 0 \\
& \quad i = 1, \ldots, m \\
& \quad j = 1, \ldots, n.
\end{align*}
\]
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