THE SLICES OF QUATERNIONIC EILENBERG-MAC LANE SPECTRA

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Abstract. We compute the slices and slice spectral sequence of integral suspensions of the equivariant Eilenberg-Mac Lane spectra $H\mathbb{Z}$ for the group of equivariance $Q_8$. Along the way, we compute the Mackey functors $\pi_k \mathcal{H}_n\mathbb{Z}$. 

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Let $G$ be a finite group. The $G$-equivariant slice filtration was first defined in the context of $G$-equivariant stable homotopy theory by Dugger in [D]; it came to prominence as a result of its role in the proof of the Kervaire invariant conjecture by Hill, Hopkins, and Ravenel [HHR1]. The slice filtration is an analogue in the $G$-equivariant stable homotopy category of the classical Postnikov filtration of spectra. One can also define a $G$-equivariant Postnikov filtration; on passage to fixed points with respect to any subgroup $H \leq G$, this recovers the Postnikov filtration of the $H$-fixed point spectrum. However, there are many equivariant spectra which possess a periodicity with respect to suspension by a $G$-representation sphere, and this periodicity is not visible in the $G$-equivariant Postnikov filtration. The slice filtration was devised by Dugger in order to display this periodicity for the case of the $C_2$-spectrum $K\mathbb{R}$.

Since the groundbreaking work [HHR1], a number of authors have calculated the slice filtration, as well as the associated slice spectral sequence, for $G$-spectra of interest. A few cases are understood for an arbitrary finite group $G$. If $M$ is a $G$-Mackey functor, then the equivariant Eilenberg-Mac Lane spectrum $H_G M$ is always a 0-slice [HHR1] (in this article, we use the “regular” slice filtration, as introduced in [U]). The slice filtrations of $\Sigma^1 H_G M$ and $\Sigma^{-1} H_G M$ were described in [U]. The slices of certain suspensions of equivariant Eilenberg-Mac Lane spectra were determined for $G$ an odd cyclic $p$-group in [HHR3], [Y2] and [A], for dihedral groups of order $2p$, where $p$ is odd, in [22], and for the Klein-four group in [GY] and [S1]. We extend this list by considering in this article the case of $G = Q_8$.

Some of the most far-reaching applications of the slice filtration and associated spectral sequence have come in the case of cyclic $p$-groups of equivariance. In addition to [HHR1], this also includes [HHR2], [MSZ], [S2], and [HSWX]. In particular, in [HSWX] the authors use slice technology to understand a $C_4$-equivariant, height 4 Lubin-Tate theory at the prime 2. For each height $n$, there is a height $n$ Lubin-Tate theory that comes equipped with an action of the height $n$ (profinite) Morava stabilizer group. The homotopy fixed points with respect to this action gives a model for the $K(n)$-local sphere, a central object of study. More approachable are the homotopy fixed points with respect to finite subgroups. At height 4, the Morava stabilizer group contains a $C_4$-subgroup (in fact a $C_8$), which gives the context for [HSWX]. On the other hand, at height $2m$, where $m$ is odd, the Morava stabilizer group contains a $Q_8$-subgroup. Therefore it is possible that $Q_8$-equivariant slice techniques will eventually shed light on the $K(n)$-local sphere when $n = 2m$ and $m$ is odd.

The focus of our article is the determination of the slices of $\Sigma^n H_{Q_8} \mathbb{Z}$. We list the slices in Section 6 and describe the associated spectral sequence in Section 8. We rely heavily on the computation of the slices of $\Sigma^n H_{K_4} \mathbb{Z}$ given by the second author in [S1]. The quotient map $Q_8 \rightarrow K_4$ allows us to gain insight into the $Q_8$-equivariant slices from the $K_4$-case, as we now explain in greater generality.

Given a normal subgroup $N \leq G$, there are several constructions that will produce a $G$-spectrum from a $G/N$-spectrum. First is the ordinary pullback, or inflation, functor. If $q: G \rightarrow G/N$ is the quotient, then inflation is denoted $q^*: \text{Sp}^{G/N} \rightarrow \text{Sp}^G$; it is left adjoint to the $N$-fixed point functor. This inflation functor plays an important role. For instance $q^*(S^0_{G/N})$ is equivalent to $S^0_{G}$. However, from our point
of view, this construction has two deficiencies. First, the ordinary inflation does not interact well with the slice filtration. Secondly, the inflation of an $H_{G/N}\mathbb{Z}$-module does not have a canonical $H_{G}\mathbb{Z}$-module structure.

On the other hand, the “geometric inflation” functor ([H, Definition 4.1], [LMSM, Section II.9])
\[
\phi^*_N: \text{Sp}^{G/N} \rightarrow \text{Sp}^G,
\]
which is right adjoint to the geometric fixed points functor, interacts well with slices. Namely, if $N$ is a normal subgroup of order $d$ and $X$ is a $G/N$-spectrum, then
\[
\phi^*_N P^k_k(X) \simeq P^d_{dk}(\phi^*_N X),
\]
by [U, Corollary 4-5] (see also [H, Section 4.2]). However, in general the geometric inflation of an $H_{G/N}\mathbb{Z}$-module will not be an $H_{G}\mathbb{Z}$-module.

The third variant is the $\mathbb{Z}$-module inflation functor ([Z1, Section 3.2])
\[
\Psi^*_N: \text{Mod}_{H_{G/N}\mathbb{Z}} \rightarrow \text{Mod}_{H_{G}\mathbb{Z}}.
\]
By design, the $\mathbb{Z}$-module inflation of an $H_{G/N}\mathbb{Z}$-module has a canonical $H_{G}\mathbb{Z}$-module structure, though in general this functor does not interact well with the slice filtration.

In some cases, these constructions agree. For instance, if the underlying spectrum of the $G/N$-spectrum $X$ is contractible, then $q^* X \simeq \phi^*_N X$. If $X$ is furthermore an $H_{G/N}\mathbb{Z}$-module, then the three inflation functors coincide on $X$ (Proposition 3.18).

The above discussion applies to the slices of $\Sigma^n H_{G/N}\mathbb{Z}$: all slices, except for the bottom slice, have trivial underlying spectrum. It follows that these inflate to give many of the slices of $\Sigma^n H_{G}\mathbb{Z}$.

Our main result along these lines, Theorem 3.19, describes the higher slices of such an inflated $H_{G}\mathbb{Z}$-module. In the case of $G = Q_8$, $N = Z(Q_8)$, and $G/N = Q_8/Z \cong K_4$, it gives the following:

**Theorem 1.1.** Let $n \geq 0$. Then the nontrivial slices of $\Sigma^n H_{Q_8}\mathbb{Z}$, above level $2n$, are
\[
P^{\phi^*_N}_{\mathbb{Z}}(\Sigma^n H_{Q_8}\mathbb{Z}) \simeq \Psi^*_N P^k_k(\Sigma^n H_{K_4}\mathbb{Z}) \simeq \phi^*_N P^k_k(\Sigma^n H_{K_4}\mathbb{Z})
\]
for $k > n$. Furthermore,
\[
P^{\phi^*_N}_{\mathbb{Z}}(\Sigma^n H_{Q_8}\mathbb{Z}) \simeq \Psi^*_N P^k_{\mathbb{Z}}(\Sigma^n H_{K_4}\mathbb{Z}).
\]

As the slices of $\Sigma^n H_{K_4}\mathbb{Z}$ were determined by the second author in [S1], this immediately provides all of the slices of $\Sigma^n H_{Q_8}\mathbb{Z}$ above level $2n$. The remaining slices of $\Sigma^n H_{Q_8}\mathbb{Z}$ are then given by analyzing the slice tower of $\Psi^*_N(P^n_{\mathbb{Z}} H_{K_4}\mathbb{Z})$. We perform this analysis in Section 6.1.

1.1. **Notation.** Throughout, whenever referencing the slice filtration, we will always mean the “regular” slice filtration of [U].

We will often write simply $Q$ and $K$ to denote the quaternion group $Q_8$ and Klein four group $K_4$, respectively. We write $Z$ for the central subgroup of $Q$ of order two generated by $z = -1$. We write
\[
L = \langle i \rangle, \quad D = \langle k \rangle, \quad \text{and} \quad R = \langle j \rangle
\]
for the normal, cyclic subgroups of $Q$ of order 4. We also use the same names for the images of these subgroups in $Q/Z \cong K$. In other words, the subgroup lattices
of $Q_8$ and $K_4$ are

\[
\begin{array}{ccc}
Q_8 & & K_4 \\
L & D & R \\
Z & & e \\
e & & e
\end{array}
\]

Our nomenclature for the order 4 subgroups of $Q_8$ amounts to a choice of isomorphism $Q/Z \cong K$.

The sign representation of $C_2$ will be denoted $\sigma$, and we will write $Z^\sigma$ for the corresponding $C_2$-module.

1.2. Organization. The paper is organized as follows. In Section 2, we review the representations of $C_4$, $K_4$, and $Q_8$, as well as Mackey functors over $C_4$ and $K_4$. Then in Section 3, we introduce three inflation functors from a quotient group $G/N$ of some finite group $G$ as well as several results that will aid in the calculation of the slices of $\Sigma^n H_{Q_8} \mathbb{Z}$. The relevant $Q_8$-Mackey functors and the homology of $\Sigma^{k\rho_0} H_{Q_8} \mathbb{Z}$ are found in Section 4. Finally, we provide some examples of the slice spectral sequence for $\Sigma^n H_{C_4} \mathbb{Z}$ in Section 8.

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2. Background

2.1. Background for $C_4$. The $C_4$-sign representation $\sigma_{C_4}$ is the inflation $p^* \sigma_{C_2}$ of the $C_2$-sign representation along the surjection $C_4 \rightarrow C_2$. We will simply write $\sigma$ for $\sigma_{C_4}$. Then the regular representation for $C_4$ splits as

$$\rho_{C_4} = 1 \oplus \sigma \oplus \lambda,$$

where $\lambda$ is the irreducible 2-dimensional rotation representation of $C_4$. The $RO(C_4)$-graded homotopy Mackey functors of $H_{C_4} \mathbb{Z}$ are given in [HHR2]. More specifically, the homotopy Mackey functors of $\Sigma^{k\rho_0} H_{C_4} \mathbb{Z}$, $\Sigma^{k\lambda} H_{C_4} \mathbb{Z}$, and $\Sigma^{k\sigma} H_{C_4} \mathbb{Z}$ are given in Figures 3 and 6 of [HHR2]. Some $C_4$-Mackey functors that will appear below are displayed in Table 1.

2.2. Background for $K_4$. The Klein 4-group $K_4 = C_2 \times C_2$ has three sign representations, obtained as the inflation along the three surjections $K_4 \rightarrow C_2$. We denote these three surjections by $p_1$, $m$, and $p_2$. Then the regular representation of $K_4$ splits as

$$\rho_{K_4} \cong 1 \oplus p_1^* \sigma \oplus m^* \sigma \oplus p_2^* \sigma.$$

Some $K_4$-Mackey functors that will appear below are displayed in Table 2.
Table 1. Some $C_4$-Mackey functors

| □ = $\mathbb{Z}$ | □ = $\mathbb{Z}^*$ | $\mathbb{Z}(2,1)$ | $\circ = B(2,0)$ |
|------------------|-------------------|-------------------|------------------|
| $\mathbb{Z}$    | $\mathbb{Z}$     | $\mathbb{Z}$     | $\mathbb{Z}/4$  |
| $\mathbb{Z}$    | $\mathbb{Z}$     | $\mathbb{Z}$     | $\mathbb{Z}/4$  |
| $\mathbb{Z}$    | $\mathbb{Z}$     | $\mathbb{Z}$     | $\mathbb{Z}/2$  |
| $\mathbb{Z}$    | $\mathbb{Z}$     | $\mathbb{Z}$     | $\mathbb{Z}/2$  |

\(\ast\) = $g$ \(\bar{\ast}\) = $\phi^*f$ \(\blacklozenge\) = $\phi^*F_2$ \(\phi^*F_2^*\)

Figure 1. The homotopy Mackey functors of $\bigvee_n \Sigma^{np}H_K\mathbb{Z}$. The Mackey functor $\pi_4\Sigma^{np}H_K\mathbb{Z}$ appears in position $(k, 4n - k)$.

The homotopy Mackey functors of $\Sigma^{np}H_K\mathbb{Z}$ were computed in [S1, Section 9]. They are displayed in Figure 1. The homotopy Mackey functors of $\Sigma^{np}H_KF_2$ were computed in [GY, Section 7]. They are displayed in Figure 2.
Table 2. Some $K_4$-Mackey functors

| □ = $\mathbb{Z}$ | □ = $\mathbb{Z}^*$ | $\mathbb{Z}(2, 1)$ |
|------------------|------------------|------------------|
| ![Diagram 1]     | ![Diagram 2]     | ![Diagram 3]     |
| $\boxempty = F_2$| $\boxempty = F_2^*$| $\circ = B(2, 0)$|
| ![Diagram 4]     | ![Diagram 5]     | ![Diagram 6]     |
| $\bullet = \phi_{LDR}(F_2)$ | $\boxempty = \phi_{LDR}(F_2)^*$ | $\phi_{LDR}(f)$ |
| ![Diagram 7]     | ![Diagram 8]     | ![Diagram 9]     |
| $m$              | $m^*$            | $\ast$           |
| ![Diagram 10]    | ![Diagram 11]    |                 |
| $w$              | $w^*$            |                 |

**Diagram 1**

**Diagram 2**

**Diagram 3**

**Diagram 4**

**Diagram 5**

**Diagram 6**

**Diagram 7**

**Diagram 8**

**Diagram 9**

**Diagram 10**

**Diagram 11**
Figure 2. The homotopy Mackey functors of $\bigvee_n \Sigma^n \rho H K^4 F_2$. The Mackey functor $\pi_k \Sigma^n \rho H K^4 F_2$ appears in position $(k, 4n - k)$.

2.3. Background for $Q_8$. The regular representation of $Q$ splits as

$$\rho_Q \cong \mathbb{H} \oplus \rho_K,$$

where $\mathbb{H}$ is the 4-dimensional irreducible $Q_8$-representation given by the action of the unit quaternions on the algebra of quaternions and $\rho_K$ is the regular representation of $K$, inflated to $Q$ along the quotient.

Denoting by $C_4$ any of the subgroups $L$, $D$, or $R$ of $Q_8$, we have that

$$\downarrow_{C_4}^Q \rho_K = 2 + 2\sigma$$

and

$$\downarrow_{C_4}^Q \mathbb{H} = 2\lambda.$$

3. Inflation functors

3.1. Inflation and the projection formula. Let $N \trianglelefteq G$ be a normal subgroup and $q : G \to G/N$ the quotient map. Recall that there is an induced adjunction

$$\text{Sp}^{G/N} \xleftarrow{q^*} \text{Sp}^G$$

where the pullback functor $q^*$, called inflation, is strong symmetric monoidal. We will also need a description of the $N$-fixed points of an Eilenberg-Mac Lane $G$-spectrum. First note that there is a functor

$$(3.1) \quad \text{Mack}(G) \xrightarrow{q_*} \text{Mack}(G/N)$$

given by

$$q_*(M)(\overline{H}) = M(H),$$
where $\overline{\mathcal{F}} = H/N \leq G/N$ whenever $N \leq H$. The functor $q_*$ is denoted $\beta^!$ in [TW, Lemma 5.4]. Then the homotopy Mackey functors of the $N$-fixed points of a $G$-spectrum $X$ are given by

$$\overline{\pi}_n(X^N) \cong q_*\overline{\pi}_n(X).$$

(3.2)

In the case of an Eilenberg-Mac Lane spectrum this yields an equivalence

$$(H_GM)^N \simeq H_{G/N}(q_*M).$$

The following result will be quite useful.

**Proposition 3.3.** [HK, Lemma 2.13] [BDS, Proposition 2.15] (Projection formula)

Let $N \trianglelefteq G$ be a normal subgroup and $q: G \to G/N$ be the quotient map. Then for $X \in \text{Sp}^{G/N}$ and $Y \in \text{Sp}^G$, there is a natural equivalence of $G/N$-spectra

$$(q^*X \wedge Y)^N \simeq X \wedge Y^N.$$  

We will frequently employ this in the case that $X = S^V$ for some $G/N$-representation $V$ and $Y = H_GM$ for some $G$-Mackey functor $M$. Then the projection formula reads

$$((S^q)^* \wedge H_GM)^N \simeq S^V \wedge H_{G/N}(q_*M).$$

(3.4)

See also [Z1, Corollary 5.8]

### 3.2. Geometric fixed points.

For a normal subgroup $N \trianglelefteq G$, we define the family of subgroups $\mathcal{F}[N]$ of $G$ to consist of those subgroups that do not contain $N$. Recall that the $N$-geometric fixed points spectrum of a $G$-spectrum is defined as

$$\Phi^N(X) = (\overline{E\mathcal{F}[N]} \wedge X)^N.$$  

This notation is simultaneously used to denote the resulting $G/N$-spectrum as well as the underlying spectrum. The $N$-geometric fixed points has a right adjoint, given by the geometric inflation functor

$$\phi^*_N(Z) = \overline{E\mathcal{F}[N]} \wedge q^*Z.$$  

To sum up, we have an adjunction

$$\text{Sp}^G \xrightarrow{\Phi^N} \text{Sp}^{G/N}. \xleftarrow{\phi^*_N}$$

### 3.3. Bottleneck subgroups.

The subgroup $Z \trianglelefteq Q$ plays an important role in this article. The primary reason is that it satisfies the following property.

**Definition 3.5.** We say that $N \trianglelefteq G$ is a **bottleneck** subgroup if it is a nontrivial, proper subgroup such that, for any subgroup $H \leq G$, either $H$ contains $N$ or $N$ contains $H$.

We now demonstrate that bottleneck subgroups only occur in cyclic $p$-groups or quaternion groups. The following argument was sketched to us by Mike Geline.

**Proposition 3.6.** Let $N \trianglelefteq G$ be a bottleneck subgroup of $G$. Then $N$ is cyclic, and $G$ is either a cyclic $p$-group or a generalized quaternion group.
Proof. We will refer to a subgroup $H \leq G$ which neither contains $N$ nor is contained in $N$ as “adjacent” to $N$. The assumption that $N$ is a bottleneck subgroup means precisely that $G$ has no subgroups that are adjacent to $N$. To see that $N$ must be cyclic, note that if $g$ is not in $N$, then $N \leq \langle g \rangle$, which implies that $N$ is cyclic.

We next observe that $G$ is necessarily a $p$-group. This is because if $N$ is contained in some Sylow $p$-subgroup, then any Sylow $q$-subgroup, for a different prime $q$, would be adjacent. It follows that $N$ contains all of the Sylow subgroups and therefore is all of $G$.

Next, we recall [B, Theorem 4.3] that for a $p$-group $G$, the group contains a unique subgroup of order $p$ if and only if $G$ is either cyclic or generalized quaternion. So we will argue that $G$ contains a unique subgroup of order $p$. The first step is to note that $G$ cannot contain a subgroup isomorphic to $C_p \times C_p$. This is because such a subgroup would necessarily contain $N$. This would imply that $N \cong C_p$, and then $N$ would have a complement in $C_p \times C_p$, which would be a subgroup adjacent to $N$ in $G$.

Finally, note that the center $Z(G)$ contains a subgroup of order $p$. If $G$ has another subgroup of order $p$, these two would generate a $C_p \times C_p$, contradicting the previous step. □

Remark 3.7. It follows from Proposition 3.6 that if $N \trianglelefteq G$ is a bottleneck subgroup, then $G/N$ is either a cyclic $p$-group or a dihedral 2-group.

If $N \trianglelefteq G$ is a bottleneck subgroup, then geometric fixed points with respect to $G$ can be computed in terms of geometric fixed points with respect to the quotient group $G/N$.

Proposition 3.8. Let $N \trianglelefteq G$ be a bottleneck subgroup. Then $\Phi^GX \simeq \Phi^{G/N}X^N$ for any $X \in \text{Sp}^G$.

Proof. If $N \trianglelefteq G$ is a bottleneck subgroup, then $q^*\overline{EP}_G \simeq \overline{EP}_G$. Thus

$$\Phi^GX = (\overline{EP}_G \wedge X)^G \simeq ((q^*\overline{EP}_{G/N} \wedge X)^N)^{G/N}. $$

By the Projection Formula (Proposition 3.3), this is equivalent to

$$(\overline{EP}_{G/N} \wedge X^N)^{G/N} = \Phi^{G/N}X^N. $$

Proposition 3.8 also follows from the more general [K, Proposition 9].

3.4. Inflation for $\mathbb{Z}$-modules. Given a surjection $q: G \twoheadrightarrow G/N$, the inflation functor

$$\phi^*_N: \text{Mack}(G/N) \rightarrow \text{Mack}(G)$$

does not send $\mathbb{Z}$-modules for $G/N$ to $\mathbb{Z}$-modules for $G$. We now describe a modified inflation functor that exists at the level of $\mathbb{Z}$-modules. This functor previously appeared in [Z1, Section 3.2] and [BG, Section 3.10].

Definition 3.9. Let $\mathcal{B}_{\mathbb{Z}G} \subset \text{Mod}_{\mathbb{Z}[G]}$ denote the full subcategory of permutation $G$-modules. Recall [Z1, Proposition 2.15] that $\mathbb{Z}G$-modules correspond to additive functors $\mathcal{B}_{\mathbb{Z}G} \rightarrow \text{Ab}$. Then the $\mathbb{Z}$-module inflation functor

$$\Psi^*_N: \text{Mod}_{\mathbb{Z}G/N} \rightarrow \text{Mod}_{\mathbb{Z}G}$$

is defined to be the left Kan extension along the inflation functor $\mathcal{B}_{\mathbb{Z}G/N} \rightarrow \mathcal{B}_{\mathbb{Z}G}$.

The following is an immediate corollary of the definition as a left Kan extension.
Proposition 3.10. The functor $\Psi_N^*$ is left adjoint to the functor $q_* : \text{Mod}_{\mathbb{Z}G} \rightarrow \text{Mod}_{\mathbb{Z}G/N}$, defined as in (3.1).

Proposition 3.11 ([BG, (3.11)]). For $M \in \text{Mod}_{\mathbb{Z}G/N}$, the $\mathbb{Z}G$-module $\Psi_N^*(M)$ satisfies

1. $q_* (\Psi_N^*(M))$ is $M$ and
2. $\downarrow_N^G \Psi_N^*(M)$ is the constant Mackey functor at $M(e)$.

Note that Proposition 3.11 completely describes $\Psi_N^*(M)$ if $N$ is a bottleneck subgroup. The following result states that $\mathbb{Z}$-module inflation agrees with ordinary inflation on geometric Mackey functors.

Proposition 3.12. Let $M \in \text{Mod}_{\mathbb{Z}G/N}$, and let $N \triangleleft G$ be a bottleneck subgroup. If $M(e) = 0$, then $\Psi_N^* M \equiv \phi_N^* M$.

Proof. This follows immediately from Proposition 3.11. □

Remark 3.13. Note that Proposition 3.12 is not true without the bottleneck hypothesis. For instance, in the case $N = C_3 \triangleleft \Sigma_3$, then $\downarrow_{C_3}^\Sigma \Psi_N^* M \not\equiv M$. In particular, it is not true that $\Psi_N^* M$ is concentrated over $N = C_3$.

We now discuss the extension to equivariant spectra.

Proposition 3.14. The $N$-fixed points functor $(-)^N : \text{Mod}_{HG} \rightarrow \text{Mod}_{H_G/N}$ for $H\mathbb{Z}$-modules has a left adjoint

\[ \Psi_N^* : \text{Mod}_{H_G/N} \rightarrow \text{Mod}_{H_G}. \]

If $N \triangleleft G$ is a bottleneck subgroup, then the spectrum-level functor $\Psi_N^*$ extends the functor $\Psi_N^*$ of Definition 3.9, in the sense that

\[ \Psi_N^* H_{G/N}^* M \simeq H_G(\Psi_N^* M) \]

for $M \in \text{Mod}_{\mathbb{Z}G/N}$.

Proof. For an $H_{G/N}\mathbb{Z}$-module $X$, the inflation $q^* X$ is canonically a module over $q^* H_{G/N} \mathbb{Z}$. We then define the spectrum-level functor $\Psi_N^*$ by the formula

\[ \Psi_N^* X = H\mathbb{Z} \wedge q^* H\mathbb{Z} (q^* X). \]

We leave it to the reader to verify that this is indeed left adjoint to the $N$-fixed points functor.

To see that (3.15) holds, we show first that this holds on the indecomposable projective $\mathbb{Z}_{G/N}$-modules. These are of the form $\uparrow_{K/N}^{G/N} \mathbb{Z}$, and the diagram of commuting adjoint functors

\[ \begin{array}{ccc}
\text{Mod}_{HG/N} & \xrightarrow{\Psi_N^*} & \text{Mod}_{HG} \\
\uparrow_{K/N}^{G/N} & \xrightarrow{(-)^N} & \uparrow_{K/N}^{G/N} \\
\text{Mod}_{HK/N} & \xrightarrow{\Psi_N^*} & \text{Mod}_{HK} \\
\end{array} \]

shows that

\[ \Psi_N^* \left( H_{G/N} \uparrow_{K/N}^{G/N} \mathbb{Z} \right) \simeq \uparrow_K^G \Psi_N^* (H_{K/N} \mathbb{Z}) \simeq \uparrow_K^G H_K \mathbb{Z} \simeq H_G \uparrow_K^G \mathbb{Z} \simeq H_G \Psi_N^* \left( \uparrow_{K/N}^{G/N} \mathbb{Z} \right). \]
Since the functor $\Psi^*_N \colon \text{Mod}_{\mathbb{Z}[G/N]} \to \text{Mod}_{\mathbb{Z}}$ is exact [Z1, Lemma 3.14], it follows that if $\text{Mod}_{\mathbb{Z}[G/N]}$ has finite global projective dimension, then (3.15) will hold for any $\mathbb{Z}[G/N]$-module $M$. By [BSW, Theorem 1.7], this is the case precisely when $G/N$ is as described in Remark 3.7.

\section*{Example 3.16.}
Let $X \in \text{Sp}^{G/N}$ and $M \in \text{Mack}(G/N)$, with $M(e) = 0$. Again assume that $N$ is a bottleneck subgroup. Then Proposition 3.12 and Proposition 3.14 give that

\[ \Psi^*_N(X \wedge H_{G/N}M) \simeq q^*(X) \wedge \Psi^*_N(H_{G/N}M) \simeq q^*(X) \wedge \theta^*_N H_{G/N}M \simeq \phi^*_N(X \wedge H_{G/N}M). \]

We will employ this equivalence when $X$ is a representation sphere.

\section*{Proposition 3.17.}
Let $N \subset G$ be a bottleneck subgroup. Then for any $G/N$-representation $V$ and $\mathbb{Z}[G/N]$-module $L$, we have

\[ \pi_n(\Psi^*_N \Sigma^V H_{G/N}L) \cong \Psi^*_N \pi_n(\Sigma^V H_{G/N}L). \]

\section*{Proof.}
Let us write $X = \Psi^*_N \Sigma^V H_{G/N}L \cong \Sigma^V H_{G/N}L$. Since $N$ is a bottleneck subgroup, it is enough to describe $\psi^*_N \pi_n X$ and $q_\ast \pi_n X$. Now

\[ q_\ast \pi_n X = \pi_n \Sigma^\text{dim}_V H_{G/N}L(N/N). \]

This is a constant Mackey functor. On the other hand, by (3.2) and (3.4), we have

\[ q_\ast \pi_n X \cong \pi_n(X^N) \cong \pi_n(\Sigma^V H_{G/N}L). \]

By Proposition 3.11, this agrees with $\Psi^*_N \pi_n(\Sigma^V H_{G/N}L)$. \hfill \square

More generally, we have an extension of Proposition 3.12 to $H\mathbb{Z}$-modules:

\section*{Proposition 3.18.}
Let $X \in \text{Mod}_{H\mathbb{Z}[G/N]}$ and let $N \subset G$ be a bottleneck subgroup. If the underlying spectrum $\psi^*_N X$ is contractible, then $\Psi^*_N(X) \simeq \phi^*_N X$.

\section*{Proof.}
If the underlying spectrum of $X$ is contractible, then $X \simeq E(\widehat{G/N}) \wedge X$. The assumption that $N$ is a bottleneck subgroup implies that $E(G/N) = q^*(E(G/N))$ is the universal space for the family of subgroups of $N$, so that $E(G/N) \wedge E\mathcal{F}[N] \simeq E(G/N)$ and it follows that

\[ q^*X \simeq E(\widehat{G/N}) \wedge q^*X \simeq E(\widehat{G/N}) \wedge \phi^*_N(X) \simeq \phi^*_N X. \]

Now

\[ \Psi^*_N(X) = H_{G\mathbb{Z}} \wedge q^*_{H_{G/N}} q^*(X) \]

\[ \simeq H_{G\mathbb{Z}} \wedge q^*_{H_{G/N}} (E(\widehat{G/N}) \wedge q^*(X)). \]

Since $E(\widehat{G/N})$ is smash idempotent, this can be rewritten as

\[ \Psi^*_N(X) \simeq E(\widehat{G/N}) \wedge H_{G\mathbb{Z}} \wedge E(\widehat{G/N}) \wedge q^*_{H_{G/N}} E(\widehat{G/N}) \wedge q^*(X). \]

It remains only to show that

\[ E(\widehat{G/N}) \wedge H_{G\mathbb{Z}} \simeq E(\widehat{G/N}) \wedge q^*_{H_{G/N}} \mathbb{Z}. \]
Both sides restrict trivially to an $N$-equivariant spectrum, so it suffices to show an equivalence on $\Phi^H$, where $H$ properly contains $N$. Without loss of generality, we may suppose $H = G$. Since $\Phi^G(E(G/N)) \simeq S^0$, it suffices to show that

$$\Phi^G H G \mathbb{Z} \simeq \Phi^G q^* H G/N \mathbb{Z}.$$  

According to Proposition 3.8, the left side is $\Phi^{G/N} H G/N \mathbb{Z}$. Similarly, Proposition 3.8 and the Projection Formula (Proposition 3.3) show that the right side is

$$\Phi^G q^* H G/N \mathbb{Z} \simeq \Phi^{G/N} (H G/N \mathbb{Z} \wedge (S_G)^N) \simeq \Phi^{G/N} H G/N \mathbb{Z} \wedge \Phi^{G/N} (S_G)^N \simeq \Phi^{G/N} H G/N \mathbb{Z}.$$  

\[ \square \]

**Theorem 3.19.** Let $n \geq 0$ and let $N \trianglelefteq G$ be a bottleneck subgroup of order $p$, a prime. Let $M \in \text{Mod}_{G/N}$ such that $P^n M$ is of the form $\Sigma V H G/N L$, for some $G/N$-representation $V$ and $L \in \text{Mod}_{G/N}$. Then the nontrivial slices of the Eilenberg-Mac Lane $G$-spectrum $\Sigma^n H G(\Psi_N^* M)$, above level $pn$, are

$$P^n_k (\Sigma^n H G(\Psi_N^* M)) \simeq \Psi_N^* P^n_k (\Sigma^n H G/N M) \simeq \phi^* P^n_k (\Sigma^n H G/N M)$$

for $k > n$. Furthermore,

$$P^n_k (\Sigma^n H G(\Psi_N^* M)) \simeq \Psi_N^* P^n_k (\Sigma^n H G/N M).$$

**Proof.** Applying the functor $\Psi_N^*$ to the slice tower for $\Sigma^n H G/N M$ produces a tower of fibrations whose layers are $\Psi_N^* P^n_k (\Sigma^n H G/N M)$ for $k \geq n$. We wish to show that this is a partial slice tower for $\Sigma^n H G(\Psi_N^* M)$. For $k > n$, the $k$-slice $P^n_k (\Sigma^n H G/N M)$ has trivial underlying spectrum. It follows from Proposition 3.18 that

$$\Psi_N^* P^n_k (\Sigma^n H G/N M) \simeq \phi^* P^n_k (\Sigma^n H G/N M)$$

for $k > n$. As the geometric inflation of a $k$-slice, this is a $pk$-slice.

It remains to show that

$$\Psi_N^* P^n_k (\Sigma^n H G/N M) \simeq \Psi_N^* \Sigma^V H G/N L \simeq \Sigma^V H G \Psi_N^* L$$

has no slices above level $pn$. First, note that the restriction of $\Sigma^V H G \Psi_N^* L$ to $N$ is the $N$-spectrum $\Sigma^n H N L(N)$, where $L(N)$ is being considered as a constant $N$-Mackey functor at the value $L(G/N)$. It follows that this $N$-spectrum has no slices above dimension $|N| \cdot n = pn$. Therefore, to show that $\Sigma^V H G \Psi_N^* L$ is less than $pn$, it suffices to show that

$$[G_+ \wedge H, S^{kp H + r}, \Sigma^V H \Psi_N^* L]^G = 0$$

for any $N < H \leq G$ and integers $r \geq 0$ and $k$ such that $k|H| > pn$. Without loss of generality we consider the case $H = G$.

Denote by $U$ a complement of $\rho_{G/N}$ in $\rho_G$, so that

$$\rho_G \cong \rho_{G/N} \oplus U.$$  

We then have a cofiber sequence

$$S(k U)_+ \wedge S^{kp G} \longrightarrow S^{kp G} \longrightarrow S^{kp G}.$$
and a resulting exact sequence

$$[\Sigma^1 S(kU)_+ \land S^{k\rho_{G/N}+r}, \Sigma^V H_G \Psi_N^* L]^G \rightarrow [S^{k\rho_{G/N}+r}, \Sigma^V H_G \Psi_N^* L]^G \rightarrow [S^{k\rho_{G/N}+r}, \Sigma^V H_G \Psi_N^* L]^G = 0.$$  

We must show that the left term vanishes. Note that the $G$-action on $S(kU)$ is free, since $N$ is order $p$. Then the desired vanishing follows from the fact that $\Sigma^1 S(kU)_+ \land S^{k\rho_{G/N}+r}$ is $G$-connected, since $\dim k\rho_{G/N} > \dim V = n$. □

4. $Q_8$-Mackey functors and Bredon homology

We display a number of the $Q_8$-Mackey functors that will be relevant in Table 3. In these Lewis diagrams, we are using the subgroup lattice of $Q_8$ as displayed in Section 1.1. We will also often abuse notation and write the name for a $K_4$-Mackey functor, such as $m$ or $mg$, to denote the resulting inflated $Q_8$-Mackey functor. We will only write the symbol $\phi^*_Z$ when it is necessary to resolve an ambiguity, for instance between $\phi^*_Z F$ and $F$.

In [HHR3, Section 2.1], the authors introduce “forms of $\mathbb{Z}$” Mackey functors $\mathbb{Z}(i,j)$, where $i \geq j \geq 0$, in the case of $G = C_p^n$. From our point of view, $Q_8$ behaves very similarly to $C_8$, and we similarly write $\mathbb{Z}(i,j)$ for the Mackey functor that looks like $\mathbb{Z}^*$ between the subgroups of order $2^i$ and $2^j$ and looks like $\mathbb{Z}$ outside of this range. We will at times follow [HHR3] in denoting by $B(i,j)$ the cokernel of $\mathbb{Z}(i,j) \rightarrow \mathbb{Z}$, although we will often instead use the descriptions given in Proposition 4.1.

These Mackey functors fit together in exact sequences as follows:

**Proposition 4.1.** There are exact sequences of Mackey functors

1. $\mathbb{Z}(3, 2) \rightarrow \mathbb{Z} \rightarrow g$
2. $\mathbb{Z}(3, 1) \rightarrow \mathbb{Z} \rightarrow \phi^*_Z B(2, 0)$
3. $\mathbb{Z}(3, 1) \rightarrow \mathbb{Z}(3, 2) \rightarrow m$
4. $\mathbb{Z}(2, 1) \rightarrow \mathbb{Z} \rightarrow m$
5. $\mathbb{Z}(1, 0) \rightarrow \mathbb{Z} \rightarrow \phi^*_Z F_2$
6. $\mathbb{Z}^* \rightarrow \mathbb{Z} \rightarrow B(3, 0)$
7. $mg \rightarrow mgw \rightarrow w$.

4.1. $RO(Q_8)$-graded Mackey functor $\mathbb{Z}$-homology of a point. We will now compute the homology of $S^{k\rho_{Q}}$, with coefficients in $\mathbb{Z}$, as a Mackey functor. The starting point is that the regular representation of $Q$ splits as

$$\rho_Q \cong \mathbb{H} \oplus \rho_K,$$

where $\mathbb{H}$ is the 4-dimensional irreducible $Q$-representation given by the action of the unit quaternions on the algebra of quaternions and $\rho_K$ is the regular representation of $K$, inflated to $Q$ along the quotient. We begin by computing the homology of $S^{k\mathbb{H}}$. See also [L, Section 2] for an alternative viewpoint.

First, Proposition 3.3 and [S1, Proposition 9.1] combine to yield the following,
Table 3. Some $Q_8$-Mackey functors

| $\square = \mathbb{Z}$ | $\square = \mathbb{Z}^*$ | $\circ = \mathcal{B}(3,0)$ |
|----------------------|----------------------|----------------------|
| ![Diagram 1](image1)  | ![Diagram 2](image2)  | ![Diagram 3](image3)  |
| $\mathbb{Z}(3,2) = \Psi_2^* \mathbb{Z}(2,1)$ | $\mathbb{Z}(3,1) = \Psi_2^* \mathbb{Z}^*$ | $\mathcal{B}(3,0) = \phi_2^*(\mathcal{B}(2,0))$ |
| ![Diagram 4](image4)  | ![Diagram 5](image5)  | ![Diagram 6](image6)  |
| $\blacklozenge = \phi_2^*F_2$ | $\bigstar = \phi_2^*F_2^*$ | $\blacktriangle = mgw$ |
| ![Diagram 7](image7)  | ![Diagram 8](image8)  | ![Diagram 9](image9)  |

Proposition 4.2. For $k \geq 0$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho K}H_Q\mathbb{Z}$ are

$$\mathcal{P}_n(\Sigma^{k\rho K}H_Q\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 4k \\ \frac{mg}{2^{k(4k-n-1)}} & n = 4k - 2 \\ \frac{g_2^{k(4k-n-4)}}{2^{n-k+1}} \oplus \phi_1^*LDRF_2 & n \in [2k, 4k-3], n \text{ odd} \\ 2_{p_1} \ulcorner \mathbb{Z}^{\sigma/4} & n \in [2k, 4k-3], n \text{ even} \\ 2_{q_2} \mathbb{Z}^{\sigma/4} & n \in [k, 2k-1]. \end{cases}$$
Next, we employ the cofiber sequence
\[(4.3)\]
\[S(\mathbb{H})_+ \rightarrow S^0 \rightarrow S^\mathbb{H}\]
to obtain the homology of \(S^{pQ}\) from that of \(S^{pK}\).

\[\text{Figure 3. The 1-skeleton of } S(\mathbb{H}).\]

**Proposition 4.4.** The nontrivial homotopy Mackey functors of \(S(\mathbb{H}) \wedge H_Q\mathbb{Z}\) are
\[
\pi_n(S(\mathbb{H})_+ \wedge H_Q\mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & n = 3 \\
mgw & n = 1 \\
\mathbb{Z}^* & n = 0.
\end{cases}
\]

**Proof.** Since the action of \(Q\) on \(S(\mathbb{H})\) is free, we can write down an equivariant cell structure using only free cells. Viewing \(S(\mathbb{H})\) as the one-point compactification of \(\mathbb{R}^3\), there is a straightforward cell structure in which the subgroups \(L, D,\) and \(R\) act freely on the \(x, y,\) and \(z\)-axes, respectively. We display the 1-skeleton in **Figure 3**, and the cell structure is described by the following complex of \(\mathbb{Z}[Q]\)-modules:
\[
\mathbb{Z}[Q]^2 \xrightarrow{\begin{pmatrix} e & i \\ e & -e \\ -e & e \end{pmatrix}} \mathbb{Z}[Q]^4 \xrightarrow{\begin{pmatrix} k & e & e & k \\ -e & -e & i & i \\ i & i & -e & e \end{pmatrix}} \mathbb{Z}[Q]^3 \xrightarrow{\begin{pmatrix} 1-e & -e & k-e \end{pmatrix}} \mathbb{Z}[Q].
\]
This yields an associated complex of induced Mackey functors
\[
\mathbb{Z}[Q]^2 \rightarrow \mathbb{Z}[Q]^4 \rightarrow \mathbb{Z}[Q]^3 \rightarrow \mathbb{Z}[Q].
\]
leading to the claimed homology Mackey functors.

**Remark 4.5.** A smaller chain complex for computing the homology of \( S(\mathbb{H}) \) is given by

\[
\mathbb{Z}[Q] \xrightarrow{(e-i, e-k)} \mathbb{Z}[Q]^2 \xrightarrow{(e-i, j-e)} \mathbb{Z}[Q]^2 \xrightarrow{(i-e, j-e)} \mathbb{Z}[Q].
\]

We gave a less efficient chain complex in the proof of Proposition 4.4 for geometric reasons.

Using (4.3), this immediately yields the following.

**Corollary 4.6.** The nontrivial homotopy Mackey functors of \( \Sigma^H \mathbb{H}_Q \mathbb{Z} \) are

\[
\pi_n(\Sigma^H \mathbb{H}_Q \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & n = 4 \\
mgw & n = 2 \\
B(3,0) & n = 0.
\end{cases}
\]

We will use this to compute the homology of \( S^{\rho_2} \), using the following periodicity result.

**Proposition 4.7 ([W, Proposition 4.1]).** For any orientable representation \( V \) of dimension \( d \) and free \( Q \)-space \( X \), the orientation \( u_V \in H_d(S^V; \mathbb{Z}) \) induces an equivalence

\[
\Sigma^d X_+ \wedge H_Q \mathbb{Z} \cong \Sigma^V X_+ \wedge H_Q \mathbb{Z}
\]

We now compute the homology of \( S^{\rho_2} \).

**Proposition 4.8.** The nontrivial homotopy Mackey functors of \( \Sigma^{\rho_2} \mathbb{H}_Q \mathbb{Z} \) are

\[
\pi_n(\Sigma^{\rho_2} \mathbb{H}_Q \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & n = 8 \\
mgw & n = 6 \\
B(3,0) & n = 4 \\
mg & n = 2 \\
g & n = 1.
\end{cases}
\]

**Proof.** The representation \( \rho_K \) is orientable. For example, using the basis \( \{1, i, j, k\} \) for \( \rho_K = \mathbb{R}[K] \), the matrix \( \rho_K(i) \) is given by

\[
\rho_K(i) = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

which has determinant equal to 1. By Proposition 4.7, we have

\[
\pi_n(S(\mathbb{H})_+ \wedge \Sigma^{\rho_2} \mathbb{H}_Q \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & n = 7 \\
mgw & n = 5 \\
\mathbb{Z}^* & n = 4.
\end{cases}
\]

The result then follows from the cofiber sequence

\[
S(\mathbb{H})_+ \wedge \Sigma^{\rho_2} \mathbb{H}_Q \mathbb{Z} \rightarrow \Sigma^{\rho_2} \mathbb{H}_Q \mathbb{Z} \rightarrow \Sigma^{\rho_2} \mathbb{H}_Q \mathbb{Z}.
\]

\( \square \)

Corollary 4.6 generalizes as follows.
Proposition 4.9. The nontrivial homotopy Mackey functors of $\Sigma^k H_Q \mathbb{Z}$, for $k > 0$ are

$$\pi_n (\Sigma^k H_Q \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & n = 4k \\
mgw & 0 < n < 4k, n \equiv 2 \pmod{4} \\
B(3,0) & 0 \leq n < 4k, n \equiv 0 \pmod{4}.
\end{cases}$$

Proof. This follows by induction, using the cofiber sequence

$$S(\mathbb{H})_+ \wedge S^{(k-1)\mathbb{H}} \to S^{(k-1)\mathbb{H}} \to S^k \mathbb{H}$$

and Proposition 4.7. The latter applies since $\mathbb{H}$, and therefore also $(k-1)\mathbb{H}$, is orientable. □

Combining this with the cofiber sequence

$$S(k\mathbb{H})_+ \wedge \Sigma^k \rho_k H_Q \mathbb{Z} \to \Sigma^k \rho_k H_Q \mathbb{Z} \to \Sigma^k \rho_H \mathbb{Z}$$

and Proposition 4.7 gives the following result.

Proposition 4.10. The nontrivial homotopy Mackey functors of $\Sigma^k \rho_k H_Q \mathbb{Z}$, for $k > 0$, are

$$\pi_n (\Sigma^k \rho_k H_Q \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & n = 8k \\
mgw & 4k < n < 8k, n \equiv 2 \pmod{4} \\
B(3,0) & 4k \leq n < 8k, n \equiv 0 \pmod{4}
\end{cases}$$

where the latter Mackey functors are listed in Proposition 4.2.

The homotopy Mackey functors of $\Sigma^k \rho_k H_Q \mathbb{Z}$ are displayed in Figure 4. When $k$ is negative, the computation follows the same strategy. The initial input, which can again be computed using the chain complex given in Proposition 4.4, is that

$$H^0(S(\mathbb{H}); \mathbb{Z}) \cong \pi_{-n} (F(S(\mathbb{H})_+, H_Q \mathbb{Z})) \cong \begin{cases} 
\mathbb{Z}^* & n = 3 \\
mgw & n = 2 \\
\mathbb{Z} & n = 0.
\end{cases}$$

Using this and [S1, Proposition 9.2] leads to the following answer.

Proposition 4.12. The nontrivial homotopy Mackey functors of $\Sigma^{-k \rho_k} H_Q \mathbb{Z}$, for $k > 0$, are

$$\pi_n (\Sigma^{-k \rho_k} H_Q \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}^* & n = 8k \\
mgw & n \in [4k, 8k], n \equiv 3 \pmod{4} \\
B(3,0) & n \in [4k+5, 8k], n \equiv 1 \pmod{4} \\
\phi_n^* B(2,0) & n = 4k + 1 \\
mg^* & n = 4k - 1 \\
\phi_n^{4k+4-3} \oplus \phi_L D R^2 \mathbb{F}_2^* & n \in [2k+4, 4k-2], n \equiv 0 \pmod{2} \\
\phi_n^{4k-n-3} & n \in [2k+3, 4k-2], n \equiv 1 \pmod{2} \\
\mathbb{Z}^k & n \in [k+4, 2k+2].
\end{cases}$$

Remark 4.13. The “Gap Theorem” [HHR1, Proposition 3.20] predicts that the groups $\pi_n^{Q \Sigma^{-k}} H_Q \mathbb{Z}$ vanish for $k \geq 0$ and $n \in [-3, -1]$, as indicated in Figure 4. Actually, for $k \geq 2$ the argument there proves more. It tells us that for $k \geq 2$, the
cohomology groups $H^n_Q(S^{k_\rho}; M)$ vanish for positive $n \leq k + 1$. This is equivalent to saying that $\pi^{-k_\rho}_n H M$ vanishes, with the same conditions on $k$ and $n$.

4.2. Additional homology calculations. We will also need the following auxiliary calculations in Section 6.

**Proposition 4.14.** The nontrivial homotopy Mackey functors of $\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}$ are

$$\tilde{\pi}_n (\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}) \cong \begin{cases} \phi^*_2 \mathbb{Z} & n = 1 \\ \mathbb{Z}^* & n = 0. \end{cases}$$

**Proof.** The fiber sequence

$$\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z} \to \Sigma^{\rho K} H_Q \mathbb{Z} \to F(S(\mathbb{H}), \Sigma^{\rho K} H_Q \mathbb{Z}) \cong \Sigma^1 F(S(\mathbb{H}), H_Q \mathbb{Z})$$

yields an isomorphism $\tilde{\pi}_0 (\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}) \cong \mathbb{Z}^*$ and shows that the homotopy vanishes for $n$ outside of $[0, 2]$. Given that the restriction to any $C_4$, which is the $C_4$-spectrum $\Sigma^{2+2\sigma - 2\lambda} H_{C_4} \mathbb{Z}$, has a trivial $\pi_2$ [Z1, Theorem 6.10], the long exact sequence further shows that $\pi_2$ vanishes as well, and it implies that we have an extension

$$w \hookrightarrow \tilde{\pi}_1 (\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}) \twoheadrightarrow g.$$ 

It remains to show this is not the split extension. The fiber sequence

$$\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z} \to \Sigma^{\rho K} H_Q \mathbb{Z} \to \Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z} \to \Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}$$

shows that $\tilde{\pi}_1 (\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z})$ injects into

$$\tilde{\pi}_0 (\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}) \cong \phi^*_2 \mathbb{Z}_2.$$ 

It follows that $\tilde{\pi}_1 (\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}) \cong \phi^*_2 \mathbb{Z}_2$. □

**Proposition 4.15.** The nontrivial homotopy Mackey functors of $\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}(3, 2)$ are

$$\tilde{\pi}_n (\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}(3, 2)) \cong \begin{cases} w & n = 1 \\ \mathbb{Z}^* & n = 0. \end{cases}$$

**Proof.** The short exact sequence

$$\mathbb{Z}(3, 2) \hookrightarrow \mathbb{Z} \twoheadrightarrow g$$

gives rise to a cofiber sequence

$$\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z}(3, 2) \to \Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z} \to \Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z} \to \Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z} \cong \Sigma^1 H_Q \mathbb{Z}.$$ 

Using a naturality square, the second map factors as

$$\Sigma^{\rho K - \mathbb{H}} H_Q \mathbb{Z} \to \Sigma^{\rho K} H_Q \mathbb{Z} \to \Sigma^1 H_Q \mathbb{Z},$$

where the first map is an epimorphism on $\tilde{\pi}_1$ by the proof of Proposition 4.14 and the second is an isomorphism on $\tilde{\pi}_1$. The conclusion follows. □

**Proposition 4.16.** The nontrivial homotopy Mackey functors of $\Sigma^{\mathbb{H} - \rho K} H_Q \mathbb{Z}(2, 0)$ are

$$\tilde{\pi}_n (\Sigma^{\mathbb{H} - \rho K} H_Q \mathbb{Z}(2, 0)) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^* & n = -2. \end{cases}$$
Proof. This follows from Proposition 4.15 by duality. In more detail, Proposition 4.15 gives a fiber sequence

$$\Sigma^1 H_Q w \longrightarrow \Sigma^{\rho \kappa - H} H_Q \mathbb{Z}(3, 2) \longrightarrow H_Q \mathbb{Z}^*.$$  

Applying Anderson duality (see [S1, Section 2.2]) gives a fiber sequence

$$I(\Sigma^1 H_Q w) \longleftarrow I(\Sigma^{\rho \kappa - H} H_Q \mathbb{Z}(3, 2)) \longleftarrow I(H_Q \mathbb{Z}^*),$$

or in other words

$$\Sigma^{-1} I(H_Q w) \longleftarrow \Sigma^{\rho \kappa - H} H_Q \mathbb{Z}(2, 0) \longleftarrow H_Q \mathbb{Z}.$$  

But as the Mackey functor $w$ is torsion, the Anderson dual is the desuspension of the Brown-Comenetz dual. In other words, $I(H_Q w) \simeq \Sigma^{-1} I_{Q/\mathbb{Z}} H_Q w \simeq \Sigma^{-1} H_Q w^*$.  

5. Review of the $C_4$-slices of $\Sigma^n H\mathbb{Z}$

In this section, we review the slices of $\Sigma^n H_{C_4} \mathbb{Z}$ from [Y1]. Note that the slices as listed in [Y1] are written using the classical slice filtration, whereas we use the regular slice filtration. The only difference is a suspension by one. The Mackey functors that appear here were introduced in Table 1.

According to [Y1, Section 4.2], the $C_4$-spectrum $\Sigma^n H_{C_4} \mathbb{Z}$ is an $n$-slice for $0 \leq n \leq 4$. For $n \geq 5$, the $\Sigma^n H_{C_4} \mathbb{Z}$ has a nontrivial slice tower. Yarnall’s method for determining these slice towers is to splice together suspensions of the cofiber sequences

$$\Sigma^{-1} H_{C_4 \mathbb{Z}} \longrightarrow \Sigma^2 H_{C_4 \mathbb{Z}} \longrightarrow \Sigma^{2\sigma} H_{C_4 \mathbb{Z}}.$$
\[ \Sigma^{-1} H_{C_4} \phi_{C_2}^* \mathbb{F}_2^* \to \Sigma^2 H_{C_4} \mathbb{Z} \to \Sigma^4 H_{C_4} \mathbb{Z}(2, 1), \]
and
\[ \Sigma^{-1} H_{C_4} B(2, 0) \to \Sigma^2 H_{C_4} \mathbb{Z} \to \Sigma^4 H_{C_4} \mathbb{Z} \]
in combination with the equivalences
\[ \Sigma^2 H_{C_4} \mathbb{Z} \simeq \Sigma^{2\sigma} H_{C_4} \mathbb{Z}(2, 1) \]
and
\[ \Sigma^{-1} H_{C_4} \phi_{C_2}^* \mathbb{F}_2^* \simeq \Sigma^{-\sigma} H_{C_4} \phi_{C_2}^* f \simeq \Sigma^{1-2\sigma} H_{C_4} \phi_{C_2}^* \mathbb{F}_2. \]

We first review these slices for odd \( n \).

**Proposition 5.1.** [Y1, Theorem 4.2.6] Let \( n \geq 5 \) be odd. The bottom slice of \( \Sigma^n H_{C_4} \mathbb{Z} \) is

\[
P_n^+ (\Sigma^n H_{C_4} \mathbb{Z}) \simeq \begin{cases} 
\Sigma^{\frac{n-3}{2}} \rho^{4+\sigma} H_{C_4} \mathbb{Z} & n \equiv 1 \pmod{8} \\
\Sigma^{\frac{n-3}{2}} \rho^{4+3\sigma} H_{C_4} \mathbb{Z} & n \equiv 3 \pmod{8} \\
\Sigma^{\frac{n-3}{2}} \rho^{4+3+2\sigma} H_{C_4} \mathbb{Z} & n \equiv 5 \pmod{8} \\
\Sigma^{\frac{n-3}{2}} \rho^{4+2+\sigma} H_{C_4} \mathbb{Z} & n \equiv 7 \pmod{8}.
\end{cases}
\]

**Proposition 5.2.** [Y1, Lemma 4.2.5] Let \( n \geq 5 \) be odd. The nontrivial 4k-slices of \( \Sigma^n H_{C_4} \mathbb{Z} \) are

\[
P_{4k}^+ (\Sigma^n H_{C_4} \mathbb{Z}) \simeq \begin{cases} 
\Sigma^{k\rho} H_{C_4} B(2, 0) & 4k \in [n + 1, 2(n - 3)], k \text{ even} \\
\Sigma^{k\rho} H_{C_4} \phi^* f & 4k \in [n + 1, 2(n - 3)], k \text{ odd} \\
\Sigma^{k\rho} H_{C_4} g & 4k \in [2(n - 1), 4(n - 3)], k \text{ even}.
\end{cases}
\]

The 4k-slices can also be read off of [HHR2, Figure 3]. When \( n \) is odd, these are the only nontrivial slices of \( \Sigma^n H_{C_4} \mathbb{Z} \).

We now recall the slices of \( \Sigma^n H_{C_4} \mathbb{Z} \) for even \( n \).

**Proposition 5.3.** [Y1, Theorem 4.2.9] Let \( n \geq 6 \) be even. The bottom slice of \( \Sigma^n H_{C_4} \mathbb{Z} \) is

\[
P_n^+ (\Sigma^n H_{C_4} \mathbb{Z}) \simeq \begin{cases} 
\Sigma^{\frac{n-4}{2}} \rho^{4+\sigma} H_{C_4} \mathbb{Z} & n \equiv 0 \pmod{8} \\
\Sigma^{\frac{n-4}{2}} \rho^{4+3+3\sigma} H_{C_4} \mathbb{Z} & n \equiv 2 \pmod{8} \\
\Sigma^{\frac{n-4}{2}} \rho^{4+4+\sigma} H_{C_4} \mathbb{Z} & n \equiv 4 \pmod{8} \\
\Sigma^{\frac{n-4}{2}} \rho^{4+4+2+\sigma} H_{C_4} \mathbb{Z} & n \equiv 6 \pmod{8}.
\end{cases}
\]

**Proposition 5.4.** [Y1, Lemma 4.2.7] Let \( n \geq 6 \) be even. The nontrivial 4k-slices of \( \Sigma^n H_{C_4} \mathbb{Z} \) are

\[
P_{4k}^+ (\Sigma^n H_{C_4} \mathbb{Z}) \simeq \Sigma^k H_{C_4} g, \quad k \text{ odd}
\]
for \( 4k \) in the range \([n + 2, 4n - 12]\).

Again, the 4k-slices can also be read off of [HHR2, Figure 3].

**Proposition 5.5.** [Y1, Theorem 4.2.9] Let \( n \geq 6 \) be even. The \((4k + 2)\)-slices of \( \Sigma^n H_{C_4} \mathbb{Z} \) are

\[
P_{8k+2}^2 (\Sigma^n H_{C_4} \mathbb{Z}) \simeq \Sigma^{1+2k^2} H_{C_4} \mathbb{F}_2^2
\]
\[
P_{8k+6}^2 (\Sigma^n H_{C_4} \mathbb{Z}) \simeq \Sigma^{3+2k^2} H_{C_4} \mathbb{F}_2^2.
\]
for \( 8k + 2 \) or \( 8k + 6 \) in the range \([n + 2, 2n - 6]\)
We may also view these slices through the perspective of the \( \mathbb{Z} \)-module inflation functor. By Theorem 3.19,
\[
\Psi_{C^2}^*: \text{Mod}_{H C^4\mathbb{Z}} \longrightarrow \text{Mod}_{H C^4\mathbb{Z}}
\]
will provide all slices of \( \Sigma^n H_{C^4}\mathbb{Z} \) above level \( 2n \). Let \( r \equiv n \pmod{4} \) with \( 3 \leq r \leq 6 \).
It follows from [S1, Proposition 3.5] that the slices of \( \Sigma^n H_{C^4}\mathbb{Z} \) in level at least
\[
2n + 2r - 4
\]
are
\[
P_{4k}^* (\Sigma^n H_{C^4}\mathbb{Z}) \simeq \Psi_{C^2}^* \Sigma^k H_{C^4}\mathbb{G} \simeq \Sigma^k H_{C^4}\mathbb{G}
\]
for \( 4k \in [2n + 2r - 4, 4(n - 3)] \). The rest of the slices then follow from determining the slices of
\[
\Psi_{C^2}^* \Sigma^\frac{n+r}{2} H_{C^4}\mathbb{Z} \simeq \Sigma^\frac{n+r}{2} H_{C^4}\mathbb{Z}.
\]
The slice tower for this \( C_4 \)-spectrum can be found by splicing together the cofiber sequences listed at the start of this section.

6. \( Q_8 \)-slices

The slices of \( \Sigma^n H_{K}\mathbb{Z} \) were determined by the second author in [S1, Section 8]. As stated in Theorem 3.19, it follows that the \( \mathbb{Z} \)-module inflation functor
\[
\Psi_{2}^*: \text{Mod}_{H K\mathbb{Z}} \longrightarrow \text{Mod}_{H Q\mathbb{Z}}
\]
of Proposition 3.14 will produce all slices of \( \Sigma^n H_{Q}\mathbb{Z} \) in degree larger than \( 2n \), as the inflation of the slices of \( \Sigma^n H_{K}\mathbb{Z} \) above degree \( n \).

The remaining slices of \( \Sigma^n H_{Q}\mathbb{Z} \) will be given as the slices of \( \Psi_{2}^* (P_{n}(\Sigma^n H_{K}\mathbb{Z})) \). By [S1, Proposition 8.5], these are of the form
\[
\Psi_{2}^* \left( \Sigma^{r+j} H_{K}\mathbb{Z} \right) \simeq \Sigma^{r+j} H_{Q}\mathbb{Z},
\]
where \( r \in \{3, 4, 5\} \), if \( n \neq 2 \pmod{4} \). In the case \( n \equiv 2 \pmod{4} \), the same result states that this is
\[
\Psi_{2}^* \left( \Sigma^{2+j} H_{K}\mathbb{Z}(1, 0) \right) \simeq \Sigma^{2+j} H_{Q}\mathbb{Z}(2, 1).
\]
But the cofiber sequence (Proposition 4.1)
\[
(6.1) \quad \Sigma^{1+j} H_{Q}\mathbb{Z}(m) \longrightarrow \Sigma^{2+j} H_{Q}\mathbb{Z}(2, 1) \longrightarrow \Sigma^{2+j} H_{Q}\mathbb{Z}
\]
reduces the computation of slices of \( \Sigma^{2+j} H_{Q}\mathbb{Z}(2, 1) \) to the question of the slice tower for \( \Sigma^{2+j} H_{Q}\mathbb{Z} \), given that \( \Sigma^{1+j} H_{Q}\mathbb{Z}(m) \simeq \phi_m^* (\Sigma^{1+j} H_{K}\mathbb{Z}(m)) \) is an \( 8j+4 \)-slice [S1, Proposition 5.7]. We determine the slices of \( \Sigma^{r+j} H_{Q}\mathbb{Z} \), for \( r \in \{2, \ldots, 5\} \) in Section 6.1.

6.1. Slice towers for \( \Sigma^{r+j} H_{Q}\mathbb{Z} \). The \( K_4 \)-spectrum \( \Sigma^{r+j} H_{K}\mathbb{Z} \) is an \( n \)-slice for \( r \in \{2, \ldots, 5\} \) [S1, Proposition 7.1]. However, the inflation of this to \( Q_8 \) is no longer a slice. We here determine the slice towers of these inflations. Throughout, we will implicitly use Proposition 6.6, which does not rely on the following material.
6.1.1. \((r = 2)\). First, we observe that \(\Sigma^{2+\rho K} H_Q \mathbb{Z}\) is a 6-slice. To see this we first note that it restricts to a 6-slice at every proper subgroup by Proposition 5.3. It therefore remains only to show that it does not have any 8\(k\)-slices for \(k \geq 1\). This is equivalent to showing that \(\pi_{-2} (\Sigma^{\rho K - k\rho Q} H_Q \mathbb{Z})\) vanishes for \(k \geq 1\). In the case \(k = 1\), (4.11) shows that \(\Sigma^{-1} H_Q \mathbb{Z}\) is \((-3)\)-truncated, in the sense that it has no homotopy Mackey functors above dimension \(-3\). This remains true after further desuspending by copies of \(\rho_Q\).

Next, the tower for \(\Sigma^{2+2\rho K} H_Q \mathbb{Z}\) is given by

\[
P_{14}^{14} = \Sigma^{-1+2\rho Q} H_Q w^* \xrightarrow{} \Sigma^{2+2\rho K} H_Q \mathbb{Z}
\]

\[
P_{12}^{12} = \Sigma^{1+\rho Q} H_Q m \xrightarrow{} \Sigma^{2+\rho Q} H_Q \mathbb{Z}(2,0)
\]

\[
P_{10}^{10} = \Sigma^{2+\rho Q} H_Q \mathbb{Z}(1,0).
\]

This uses the computation (see Proposition 4.16)

\[
\Xi_n (\Sigma^{1+\rho K} H_Q \mathbb{Z}(2,0)) \cong \begin{cases} 
\mathbb{Z} & n = 0 \\
 w^* & n = -2 
\end{cases}
\]

to produce the first cofiber sequence.

Finally, for \(j \geq 3\), the tower may be obtained by recursively using

\[
P_{8j-2}^{8j-2} = \Sigma^{-1+j\rho Q} H_Q w^* \xrightarrow{} \Sigma^{2+j\rho K} H_Q \mathbb{Z}
\]

\[
P_{8j-4}^{8j-4} = \Sigma^{1+(j-1)\rho Q} H_Q m \xrightarrow{} \Sigma^{2+(j-2)\rho K + \rho Q} H_Q \mathbb{Z}(2,0)
\]

\[
P_{8j-6}^{8j-6} = \Sigma^{1+(j-1)\rho Q} H_Q \phi^* \mathbb{F}_2 \xrightarrow{} \Sigma^{2+(j-2)\rho K + \rho Q} H_Q \mathbb{Z}(1,0)
\]

\[
\Sigma^{2+(j-2)\rho K + \rho Q} H_Q \mathbb{Z}.
\]

We have proved the following result.

**Proposition 6.2.** Let \(j \geq 1\). The bottom slice of \(\Sigma^{2+j\rho K} H_Q \mathbb{Z}\) is

\[
P_{2+4j}^{2+4j} (\Sigma^{2+j\rho K} H_Q \mathbb{Z}) \cong \begin{cases} 
\Sigma^{1+\rho K + \frac{j-1}{2} \rho Q} H_Q \mathbb{Z}^* & j \text{ odd} \\
\Sigma^{2+\frac{j}{2} \rho Q} H_Q \mathbb{Z} & j \text{ even.}
\end{cases}
\]

6.1.2. \((r = 3)\). By (4.11), the cohomology of \(S^{1/2}\) is given by

\[
\tilde{H}^n (S^{1/2}; \mathbb{Z}) \cong \mathbb{Z}_{-n} (\Sigma^{-1/2} H_Q \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}^* & n = 4 \\
mw & n = 3.
\end{cases}
\]
Suspending by $3 + \rho Q$ leads to the cofiber sequence

$$
\begin{array}{c}
P^8_s = \Sigma^\rho Q H Q m g w \\
\downarrow \\
P^7_t = \Sigma^{\rho Q - 1} H Q Z^*.
\end{array}
$$

The tower for $\Sigma^{3 + j \rho K} H Q Z$, where $j \geq 2$, is then given recursively by

$$
\begin{array}{c}
P^{8j}_{8j} = \Sigma^{j \rho Q} H Q m g w \\
\downarrow \\
\Sigma^{(j - 1) \rho K + \rho Q - 1} H Q Z^*
\end{array}
\begin{array}{c}
P^{8j-4}_{8j-4} = \Sigma^{2 + (j - 1) \rho Q} H Q \phi^* Z_2 \\
\downarrow \\
\Sigma^{3 + (j - 2) \rho K + \rho Q} H Q Z(1, 0)
\end{array}
\begin{array}{c}
P^8_{8} = \Sigma^{\rho Q} H Q Z^*
\end{array}
$$

The last cofiber sequence arises from Proposition 4.1. We have proved the following result.

**Proposition 6.3.** Let $j \geq 1$. The bottom slice of $\Sigma^{3 + j \rho K} H Q Z$ is

$$
P^{3 + 4j}_{3 + 4j} (\Sigma^{3 + j \rho K} H Q Z) \simeq \begin{cases} 
\Sigma^{-1 + \frac{4j}{2j} \rho Q} H Q Z^* & j \text{ odd} \\
\Sigma^{3 + \frac{4j}{2j} \rho Q} H Q Z & j \text{ even.}
\end{cases}
$$

6.1.3. ($r = 4$). The tower for $\Sigma^{4 + \rho K} H Q Z$ is given by

$$
\begin{array}{c}
P^{12}_{12} = \Sigma^{\rho Q + 1} H Q m g \\
\downarrow \\
P^{10}_{10} = \Sigma^{\rho Q + 1} w \\
\downarrow \\
P^8_{8} = \Sigma^{\rho Q} H Q Z^*
\end{array}
$$

This uses the short exact sequence (Proposition 4.1)

$$
\mathbb{Z}(3, 1) \hookrightarrow \mathbb{Z}(3, 2) \twoheadrightarrow m^*,
$$

the equivalence $\Sigma^{\rho K} H K m^* \simeq \Sigma^2 H K m g$ ([GY, Proposition 4.8]), and the computation (see Proposition 4.15)

$$
\pi_n (\Sigma^{\rho K - \frac{n}{2}} H Q Z(3, 2)) \cong \begin{cases} 
w & n = 1 \\
Z^* & n = 0.
\end{cases}
$$
The tower for $\Sigma^{4+j\rho} H_Z$, where $j \geq 2$, may then be obtained recursively from

\[
P_{8j+4}^{8j+4} = \Sigma^{1+j\rho} H_\mathbb{Q} \mathbb{Z} \xrightarrow{m} \Sigma^{4+j\rho} H_Z \simeq \Sigma^{(j+1)\rho} H_Z(3, 1)
\]

\[
P_{8j+2}^{8j+2} = \Sigma^{1+j\rho} \mathbb{Q} \mathbb{Z} \xrightarrow{w} \Sigma^{(j+1)\rho} H_Z(3, 2)
\]

\[
\Sigma^{(j-1)\rho} H_Z^* \xrightarrow{} \Sigma^{(j-1)\rho+\rho} H_Z^*
\]

\[
P_{8j-2}^{8j-2} = \Sigma^{3+(j-1)\rho} H_\phi^* \mathbb{F}_2 \xrightarrow{} \Sigma^{4+(j-2)\rho+\rho} H_Z(1, 0)
\]

\[
\Sigma^{4+(j-2)\rho+\rho} H_Z.
\]

**Proposition 6.4.** Let $j \geq 1$. The bottom slice of $\Sigma^{4+j\rho} H_Z$ is

\[
P_{4+i}^{4+i} (\Sigma^{4+j\rho} H_Z) \simeq \begin{cases} 
\Sigma^{\frac{i+j}{2}} \rho \mathbb{Q} H_Z^* & j \text{ odd} \\
\Sigma^{\frac{i+j}{2}} \rho \mathbb{Q} H_Z^* & j \text{ even}.
\end{cases}
\]

6.1.4. ($r = 5$). Here, we start with the slice tower for $\Sigma^5 H_Z$, as this is not a slice. The short exact sequence

\[
\mathbb{Z}(3, 1) \rightarrow \mathbb{Z} \rightarrow \phi^* \mathbb{B}(2, 0)
\]

gives rise to a cofiber sequence

\[
P_8 = \Sigma^{\rho} H_\phi^* \mathbb{B}(2, 0) \rightarrow \Sigma^5 H_Z \simeq \Sigma^{1+\rho} H_Z(3, 1) \rightarrow \Sigma^{1+\rho} H_Z.
\]

Now the argument showing that $\Sigma^{2+r\rho} H_Z$ is a 6-slice, given above in Section 6.1.1, also applies to show that $\Sigma^{1+\rho} H_Z$ is a 5-slice. Thus, this cofiber sequence is the slice tower for $\Sigma^5 H_Z$.

Next, the tower for $\Sigma^{5+\rho} H_Z$ is given by

\[
P_{16} = \Sigma^{2\rho} H_\phi^* \mathbb{B}(2, 0) \rightarrow \Sigma^{5+\rho} H_Z \simeq \Sigma^{1+2\rho} H_Z(3, 1)
\]

\[
P_{12} = \Sigma^{2+\rho} H_\phi^* \mathbb{F}_2 \rightarrow \Sigma^{1+2\rho} H_Z
\]

\[
P_9 = \Sigma^{1+\rho} H_Z^*,
\]

where the bottom cofiber sequence arises from the computation (Proposition 4.14)

\[
\mathbb{Z}_n (\Sigma^{\rho} H_Z) \simeq \begin{cases} 
\phi^* \mathbb{F}_2 & n = 1 \\
\mathbb{Z}^* & n = 0.
\end{cases}
\]
The tower for $\Sigma^{5+jrk} H_Q\mathbb{Z}$, where $j \geq 2$, may then be obtained recursively from
\[
P_{5j+8}^{5j+8} = \Sigma^{(j+1)\rho q} H_Q\phi_2^8 P(2,0) \longrightarrow \Sigma^{5+jrk} H_Q\mathbb{Z} \\
\downarrow \Sigma^{1+(j+1)\rho k} H_Q\mathbb{Z}(3,1) \\
P_{5j+4}^{5j+4} = \Sigma^{2+j\rho q} H_Q\phi_2^4 P(2,0) \longrightarrow \Sigma^{1+(j+1)\rho k} H_Q\mathbb{Z} \\
\downarrow \Sigma^{1+(j+1)\rho k} H_Q\mathbb{Z} \\
P_{3j}^{3j} = \Sigma^{j\rho q} H_Q P(2,0) \longrightarrow \Sigma^{1+(j-1)\rho k+\rho q} H_Q\mathbb{Z}^* \\
\downarrow \Sigma^{1+(j-1)\rho k+\rho q} H_Q\mathbb{Z}.
\]

**Proposition 6.5.** Let $j \geq 1$. The bottom slice of $\Sigma^{5+jrk} H_Q\mathbb{Z}$ is
\[
P_{5j+4}^{5j+4} (\Sigma^{5+jrk} H_Q\mathbb{Z}) \simeq \begin{cases} 
\Sigma^{1+(j+1)\rho q} H_Q\mathbb{Z} & j \text{ odd} \\
\Sigma^{1+(j-1)\rho k+\rho q} H_Q\mathbb{Z} & j \text{ even}.
\end{cases}
\]

### 6.2. Slices of $\Sigma^n H_Q\mathbb{Z}$

In this section, we describe all slices of $\Sigma^n H_Q\mathbb{Z}$ for $n \geq 0$.

**Proposition 6.6.** The $Q_k$-spectrum $\Sigma^n H_Q\mathbb{Z}$ is an n-slice for $0 \leq n \leq 4$.

*Proof.* Since this is true after restricting to any $C_4$ (see Section 5), any higher slices would necessarily be geometric and therefore occurring in slice dimension at least 8. But we can show directly that $\Sigma^n H_Q\mathbb{Z} \neq \emptyset$ if $n \in \{0,4\}$. This follows from the vanishing of $\pi_{\rho q} \Sigma^n H_Q\mathbb{Z} \cong \pi_{-n} \Sigma^{-\rho q} H_Q\mathbb{Z}$ as displayed in Figure 4.

It remains to determine the slices of $\Sigma^n H_Q\mathbb{Z}$ when $n \geq 5$. Note that Theorem 3.19 applies by [S1, Proposition 8.5]. We first describe the bottom slice.

**Proposition 6.7 (The n-slice).** For $n \geq 5$, write $n = 8k + r$, where $r \in \{5,12\}$. Then the n-slice of $\Sigma^n H_Q\mathbb{Z}$ is
\[
P_n (\Sigma^n H_Q\mathbb{Z}) \simeq \begin{cases} 
\Sigma^{1+(j+1)\rho q} H_Q\mathbb{Z} & r = 5 \\
\Sigma^{2+(j+1)\rho q} H_Q\mathbb{Z}(3,2) & r = 6 \\
\Sigma^{-1+(k+1)\rho q} H_Q\mathbb{Z}^* & r = 7 \\
\Sigma^{(k+1)\rho q} H_Q\mathbb{Z} & r = 8 \\
\Sigma^{1+(k+1)\rho q} H_Q\mathbb{Z}^* & r = 9 \\
\Sigma^{2+(k+1)\rho q} H_Q\mathbb{Z}(1,0) & r = 10 \\
\Sigma^{3+(k+1)\rho q} H_Q\mathbb{Z} & r = 11 \\
\Sigma^{4+(k+1)\rho q} H_Q\mathbb{Z} & r = 12.
\end{cases}
\]

*Proof.* By Theorem 3.19, the n-slice of $\Sigma^n H_Q\mathbb{Z}$ is the n-slice of the $\mathbb{Z}$-module inflation of the n-slice of $\Sigma^n H_K\mathbb{Z}$ by [S1, Proposition 8.5], writing $n = 4j + r_4$ with $r_4 \in \{2,3,4,5\}$, we have
\[
\Psi_2^* P_n (\Sigma^n H_K\mathbb{Z}) \simeq \begin{cases} 
\Sigma^{2+jrk} H_Q\mathbb{Z}(2,1) & n \equiv 2 \pmod{4} \\
\Sigma^{r_4+jrk} H_Q\mathbb{Z} & \text{else}.
\end{cases}
\]
If \( n \not\equiv 2 \pmod{4} \), the slice tower was given in Section 6.1. For the case of \( n \equiv 2 \), since \( \Sigma^{1+jpk} H_{QM} \simeq \phi^*_2(\Sigma^{1+jpk} H_{Km}) \) is an 8j + 4-slice [S1, Proposition 5.7], the cofiber sequence (Proposition 4.1)

\[
(6.8) \quad \Sigma^{1+jpk} H_{QM} \to \Sigma^{2+jpk} H_{QZ}(2, 1) \to \Sigma^{2+jpk} H_{QZ}.
\]

combines with the work of Section 6.1.1 to to show that

\[
P_n^m (\Sigma^{2+jpk} H_{QZ}(2, 1)) \simeq P_n^m (\Sigma^{2+jpk} H_{QZ}).
\]

The latter is given in Proposition 6.2. \( \square \)

**Proposition 6.9** (The 8k-slices). For \( n \geq 5 \) and \( 8k > n \), the 8k-slice of \( \Sigma^n H_{QZ} \)

\[
P_{8k}^n (\Sigma^n H_{QZ}) \simeq \begin{cases} \\
\Sigma^k H_{Qg} \quad & 8k \in \{4n - 8, 8n - 32\} \\
\Sigma^{kpq} H_{Qg} \quad & 8k \in \{2n + 4, 4n - 16\} \\
\Sigma^{kpq} H_{Qg} \quad & 8k \in \{2n + 4, 4n - 12\} \\
\Sigma^{kpq} H_{Qg} \quad & 8k \in \{n + 3, 2n - 10\} \\
\Sigma^{kpq} H_{Qg} \quad & 8k \in \{n + 1, 2n\} \\
\end{cases} \\
\]

\[
\text{and } n \equiv 0 \pmod{2} \\
\text{and } n \equiv 1 \pmod{2} \\
\text{and } n \equiv 1 \pmod{2} \\
\text{and } n \equiv 3 \pmod{4}.
\]

*Proof.* This is a translation of Proposition 4.12. Alternatively, the slices above dimension 2n follow from Theorem 3.19 and [S1, Proposition 8.6]. The slices in dimensions 2n and lower follow from the towers computed in Section 6.1. \( \square \)

**Proposition 6.10** (The 8k + 4-slices). For \( n \geq 5 \) and \( 8k + 4 > n \), the 8k + 4-slices of \( \Sigma^n H_{QZ} \)

\[
P_{8k+4}^n (\Sigma^n H_{QZ}) \simeq \begin{cases} \\
\Sigma^{1+kpq} H_{Qg} \quad & 8k + 4 \in \{2n + 4, 4n - 12\}, \ n \ even \\
\Sigma^{2+kpq} H_{Qg} \quad & 8k + 4 \in \{n + 1, 2n - 4\}, \ n \ odd \\
\Sigma^{1+kpq} H_{Qg} \quad & 8k + 4 \in \{n + 2, 2n\}, \ n \equiv 2 \pmod{4} \\
\Sigma^{1+kpq} H_{Qg} \quad & 8k + 4 \in \{n + 4, 2n - 4\}, \ n \equiv 0 \pmod{4} \\
\end{cases} \\
\]

*Proof.* The first case follows from [S1, Proposition 8.7]. The remaining cases follow from (6.8) and Section 6.1. \( \square \)

**Proposition 6.11** (The 4k + 2-slices). Let \( n \geq 5 \). If \( n \) is odd, then \( \Sigma^n H_{QZ} \) has no nontrivial 4k + 2-slices if \( 4k + 2 > n \). If \( n \) is even and \( 8k + 2 > n \), then the 8k + 2-slice of \( \Sigma^n H_{QZ} \) is nontrivial only if \( 8k + 2 \in \{n + 1, 2n\} \), in which case the slice is

\[
P_{8k+2}^n (\Sigma^n H_{QZ}) \simeq \begin{cases} \\
\Sigma^{1+kpq} H_{Qg} \quad & n \equiv 0 \pmod{4} \\
\Sigma^{1+kpq} H_{Qg} \quad & n \equiv 2 \pmod{4} \\
\end{cases} \\
\]
Similarly, if $n$ is even and $8k - 2 > n$, the $8k - 2$-slice is nontrivial only if $8k - 2 \in [n + 1, 2n]$, in which case the slice is

$$P_{8k-2}^{8k-2}(\Sigma^n H\mathbb{Z}) \simeq \begin{cases} 
\Sigma^{-1+k\rho} H\phi_z^* F_{2}^* & n \equiv 0 \pmod{4} \\
\Sigma^{-1+k\rho} H\omega^* & n \equiv 2 \pmod{4}.
\end{cases}$$

**Proof.** According to [S1], the $K_4$-spectrum $\Sigma^n H\mathbb{Z}$ does not have any nontrivial slices in odd dimensions, except for the $n$-slice. By Theorem 3.19, this implies that $\Sigma^n H\mathbb{Z}$ does not have any $4k + 2$-slices above dimension $2n$. The slices in dimensions below $2n$ are given by Section 6.1. □

### 6.3. Slice towers for $\Sigma^n H\mathbb{Z}$

By Proposition 6.6, $\Sigma^n H\mathbb{Z}$ is an $n$-slice for $n \in \{0, \ldots, 4\}$. The slice tower for $\Sigma^5 H\mathbb{Z}$ was given in Section 6.1.4. We now display a few more examples of slice towers.

**Example 6.12.** The slice tower for $\Sigma^6 H\mathbb{Z}$ is

$$\begin{array}{c}
P_{16}^6 = \Sigma^2 H\mathbb{Z} \quad \longrightarrow \quad \Sigma^6 H\mathbb{Z} \\
P_{12}^3 = \Sigma^1 + \rho H\mathbb{Z} \quad \longrightarrow \quad \Sigma^{2+\rho} H\mathbb{Z}(2,1) \\
P_8^6 = \Sigma^{2+\rho} H\mathbb{Z}.
\end{array}$$

This follows immediately from combining [S1, Example 8.2], (6.8), and Section 6.1.1.

**Example 6.13.** The slice tower for $\Sigma^7 H\mathbb{Z}$ is

$$\begin{array}{c}
P_{24}^3 = \Sigma^3 H\mathbb{Z} \quad \longrightarrow \quad \Sigma^7 H\mathbb{Z} \\
P_{16}^6 = \Sigma^{2+\rho} H\mathbb{Z} \quad \longrightarrow \quad \Sigma^{3+\rho} H\mathbb{Z}(2,1) \\
P_8^8 = \Sigma^{\rho} H\mathbb{Z} \quad \longrightarrow \quad \Sigma^{3+\rho} H\mathbb{Z} \\
P_7^7 = \Sigma^{\rho-1} H\mathbb{Z}.
\end{array}$$

This follows immediately from combining [S1, Example 8.3] and Section 6.1.2.

**Example 6.14.** The slices, but not the slice tower, for $\Sigma^8 H\mathbb{Z}$ were determined in [S1, Section 8]. Let us denote by $F$ the fiber of the map $H\mathbb{Z} \rightarrow H\phi_{LDR} F_2$ induced by the map of $Q_8$-Mackey functors $\mathbb{Z} \rightarrow \phi_{LDR} F_2$ that is surjective at $L$, $D$, and $R$. Then the nontrivial homotopy Mackey functors of $F$ are $\pi_0(F) \simeq \mathbb{Z}(2,1)$. 
and $\pi_{-1}(F) \cong g^2$. The slice tower for $\Sigma^8 H_Q Z$ is

$$
\begin{array}{c}
\begin{array}{c}
P^3_{32} = \Sigma^4 Q g 
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\Sigma^8 H_Q Z \cong \Sigma^{4+\rho K} H_Q Z(3,1)
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c}
P^2_{24} = \Sigma^3 Q g^2
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\Sigma^{4+\rho K} H_Q Z(2,1)
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c}
P^2_{20} = \Sigma^{3+\rho Q} H_Q \phi_1^* LDR 
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\Sigma^{4+\rho K} F
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c}
P^2_{12} = \Sigma^{1+\rho Q} H_Q mg
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\Sigma^{4+\rho K} H_Q Z \cong \Sigma^{2\rho K} H_Q Z(3,1)
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c}
P^2_{10} = \Sigma^{\rho Q+1} w
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\Sigma^{2\rho K} H_Q Z(3,2)
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c}
P^2_{8} = \Sigma^{\rho Q} H_Q Z^*
\end{array}
\end{array}
$$

where the bottom of the tower comes from Section 6.1.3.

7. Homology calculations

In Section 6, we described the slices of $\Sigma^n H_Q Z$. In Section 8 below, we will give the corresponding slice spectral sequences. The $E_2$-pages of those spectral sequences are given by the homotopy Mackey functors of the slices. We describe those homotopy Mackey functors here.

7.1. The $n$-slice. We start with the $n$-slices in the order listed in Proposition 6.7. The homotopy Mackey functors of $\Sigma^{\rho Q} H_Q Z$ were calculated in Proposition 4.10. We use the same methods to determine the homotopy Mackey functors of $\Sigma^{\rho K + j \rho Q} H_Q Z$.

**Proposition 7.1.** For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{\rho K + j \rho Q} H_Q Z$ are

$$
\pi_i(\Sigma^{\rho K + j \rho Q} H_Q Z) \cong \begin{cases}
\mathbb{Z} & i = 8j + 4 \\
mgw & i \in [4j + 4, 8j + 3], \\
B(3,0) & i \equiv 2 \pmod{4}, \\
\phi_i^*(\Sigma^{(j+1)\rho K} H_K Z) & i \equiv 0 \pmod{4}, \\
\phi_i^*(\Sigma^{(j+1)\rho K} H_K Z) & i \in [4j + 1, 4j + 3].
\end{cases}
$$

See Proposition 4.2 or Figure 1 for the homotopy Mackey functors of $\Sigma^{(j+1)\rho K} H_K Z$.

We may now use Proposition 7.1 and the exact sequence $\mathbb{Z}(3,2) \hookrightarrow \mathbb{Z} \rightarrow \mathbb{g}$ to get the homotopy Mackey functors of $\Sigma^{\rho K + j \rho Q} H_Q Z(3,2)$. 
Proposition 7.2. For \( j \geq 1 \), the homotopy Mackey functors of \( \Sigma^{pK+j\rho} H_Q\mathbb{Z}(3,2) \) are

\[
\pi_i(\Sigma^{pK+j\rho} H_Q\mathbb{Z}(3,2)) \cong \begin{cases} 
\mathbb{Z} & i = 8j \\
m_{gw} & i \in [4j + 1, 8j - 1], \ i \equiv 2 \pmod{4} \\
B(3,0) & i \in [4j + 1, 8j - 1], \ i \equiv 0 \pmod{4} \\
\phi_2\mathbb{Z}(\Sigma^{(j-1)pK} H_K\mathbb{Z}) & i \in [j + 3, 4j - 1].
\end{cases}
\]

The key point here is that the homotopy Mackey functors of \( \Sigma^{pK+j\rho} H_Q\mathbb{Z}(3,2) \) are the same as that of \( \Sigma^{pK+j\rho} H_Q\mathbb{Z} \), except that the \( q \) in degree \( j + 1 \) has been removed.

In Proposition 4.12 we list the homotopy Mackey functors of \( \Sigma^{-j\rho} H_Q\mathbb{Z} \). Anderson duality then provides us with the homotopy Mackey functors of \( \Sigma^{j\rho} H_Q\mathbb{Z}^* \).

Proposition 7.3. For \( j \geq 1 \), the homotopy Mackey functors of \( \Sigma^{j\rho} H_Q\mathbb{Z}^* \) are

\[
\pi_i(\Sigma^{j\rho} H_Q\mathbb{Z}^*) \cong \begin{cases} 
\mathbb{Z} & i = 8j \\
m_{gw} & i \in [4j + 1, 8j - 1], \ i \equiv 2 \pmod{4} \\
B(3,0) & i \in [4j + 1, 8j - 1], \ i \equiv 0 \pmod{4} \\
\phi_2\mathbb{Z}(\Sigma^{(j-1)pK} H_K\mathbb{Z}) & i \in [j + 3, 4j - 1].
\end{cases}
\]

Finally, the homotopy Mackey functors of \( \Sigma^{j\rho} H_Q\mathbb{Z}(1,0) \) follow from the exact sequence \( \mathbb{Z}(1,0) \to \mathbb{Z} \to \phi_2^* \mathbb{F}_2 \).

Proposition 7.4. For \( j \geq 1 \), the homotopy Mackey functors of \( \Sigma^{j\rho} H_Q\mathbb{Z}(1,0) \) are

\[
\pi_i(\Sigma^{j\rho} H_Q\mathbb{Z}(1,0)) \cong \begin{cases} 
\mathbb{Z} & i = 8j \\
m_{gw} & i \in [4j + 1, 8j - 2], \ i \equiv 2 \pmod{4} \\
B(3,0) & i \in [4j + 1, 8j - 2], \ i \equiv 0 \pmod{4} \\
\phi_2\mathbb{Z}(\Sigma^{(j-1)pK} H_K\mathbb{Z}) & i \in [j + 4j - 1].
\end{cases}
\]

7.2. The 8k-slices. We now move on to the 8k-slices.

Proposition 7.5. For \( j = 1 \), the homotopy Mackey functors of \( \Sigma^{j\rho} H_Q\phi_2^* B(2,0) \) are

\[
\pi_i(\Sigma^{j\rho} H_Q\phi_2^* B(2,0)) \cong \begin{cases} 
m_{gw} & i = 2 \\
\phi_2^* LDR \mathbb{F}_2 & i = 1.
\end{cases}
\]

For \( j \geq 2 \), they are

\[
\pi_i(\Sigma^{j\rho} H_Q\phi_2^* B(2,0)) \cong \begin{cases} 
\phi_1^* LDR \mathbb{F}_2 & i = 2j \\
\phi_2^* B(2,0) & i \equiv j + 1 \\
\phi_2^* B(2,0) & i \equiv j.
\end{cases}
\]

Proof. Because \( \phi_2^* B(2,0) \) is a pullback,

\[
\Sigma^{j\rho} H_Q\phi_2^* B(2,0) \cong \Sigma^{j\rho K} H_Q\phi_2^* B(2,0).
\]
The exact sequence of $K$-Mackey functors $m^* \to B(2,0) \to g$ provides us with $\Sigma^{j\rho} H_K B(2,0) \to \Sigma^{j\rho} H_K B$. The conclusion follows from [GY, Propositions 4.8 and 7.4] and the resulting long exact sequence in homotopy. □

We may again use this strategy of reducing the calculations from $Q$ to $K$ for determining the homotopy Mackey functors of $\Sigma^{j\rho} H_Q B(3,0)$.

**Proposition 7.6.** For $j = 1$ the homotopy Mackey functors of $\Sigma^{j\rho} H_Q B(3,0)$ are

$$\pi_i(\Sigma^{j\rho} H_Q B(3,0)) \cong \begin{cases} \phi^*_2 \mathbb{F}_2 & i = 4 \\ mg & i = 2 \\ g & i = 1. \end{cases}$$

For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j\rho} H_Q B(3,0)$ are

$$\pi_i(\Sigma^{j\rho} H_Q B(3,0)) \cong \begin{cases} \phi^*_2 \mathbb{F}_2 & i = 4 \rho \\ \phi^*_L D R \mathbb{F}_2 \oplus g^{4j - 2 - i} & i \in [2j + 2, 4j - 2] \\ g^{2(k-2) + 1} & i = 2j + 1 \\ \phi^*_L D R \mathbb{F}_2 \oplus g^{2(j-3) + 1} & i = 2j \\ g^{2(j-1)} & i \in [j + 3, 2j - 1] \\ g^{j-1} & i \in [j, j + 2]. \end{cases}$$

**Proof.** Because the underlying spectrum of $H_Q B(3,0)$ is contractible, $\Sigma^{j\rho} H_Q B(3,0) \simeq \Sigma^{j\rho} H_Q B(3,0)$.

Now, we may consider $B(3,0)$ as a pullback $\phi^*_2 B := B(3,0)$, thus the calculation is reduced to one of $K$-Mackey functors. The sequence of $K$-Mackey functors $\mathbb{Z}^* \to \mathbb{Z} \to B$ provides us with

$$\Sigma^{j\rho} H_K \mathbb{Z}^* \to \Sigma^{j\rho} H_K \mathbb{Z} \to \Sigma^{j\rho} H_K B.$$ 

Except for $i = 4j - 2$, the result follows from the associated long exact sequence in homotopy. In degree $4j - 2$ we have an extension

$$mg \to \pi_{4j-2}(\Sigma^{j\rho} H_K B) \to g.$$ 

We need to show this is not the split extension. This follows from the exact sequence $B(2,0) \to B \to \mathbb{F}_2$ of $K$-Mackey functors. □

**Proposition 7.7.** For $j = 1$ and $j = 2$, the homotopy Mackey functors of $\Sigma^{j\rho} H_Q mgw$ are

$$\pi_i(\Sigma^{j\rho} H_Q mgw) \cong \begin{cases} \phi^*_2 \mathbb{F}_2 & i = 4 \\ \phi^*_2 B(2,0) & i = 2. \end{cases}$$

and

$$\pi_i(\Sigma^{2\rho Q} H_Q mgw) \cong \begin{cases} \phi^*_2 \mathbb{F}_2 & i = 8 \\ mg & i = 7 \\ \phi^*_L D R \mathbb{F}_2 & i = 6 \\ g & i = 5 \\ mg & i = 4 \\ g & i = 3. \end{cases}$$
For $j \geq 3$, the homotopy Mackey functors of $\Sigma^{j\rho}H_Qmgw$ are

$$\pi_i(\Sigma^{j\rho}H_Qmgw) \cong \begin{cases} \phi^*_ZF_2 & i = 4j \\ \text{mg} & i = 4j - 1 \\ \phi_{LDR}F_2 \oplus g^{4j-i-2} & i \in [2j + 2, 4j - 2] \\ g^{2j-3} & i = 2j + 1 \\ g^{2j-5} \oplus \phi_{LDR}F_2 & i = 2j \\ g^{2(i-j)-2} & i \in [j + 2, 2j - 1] \\ g & i = j + 1 \end{cases}$$

Proof. We first deal with the case $j = 1$. The short exact sequence of Mackey functors

$$w^* \hookrightarrow \text{mgw} \rightarrow \text{mg}$$. combines with Proposition 7.17 and Proposition 7.9 to show that the only nontrivial Mackey functors are $\phi^*_ZF_2$ in degree 4 and an extension of $\text{mg}$ by $g$ in degree 2. It remains to see that this extension is $\phi^*_ZB(2,0)$.

According to Proposition 4.12, the Postnikov tower for $\Sigma^{-\rho}H_QZ$ is

$$\Sigma^{-5}H_Q\phi^*_ZB(2,0) \rightarrow \Sigma^{-\rho}H_QZ$$

Desuspending this diagram once by $\rho_Q$ gives a tower for computing the homotopy Mackey functors of $\Sigma^{-2\rho}H_QZ$. The homotopy Mackey functors for $\Sigma^{-8-\rho}H_QZ^*$ and $\Sigma^{-5\rho}H_Q\Psi^*B(2,0)$ follow, using Anderson duality, from Proposition 4.10 and Proposition 7.5. Long exact sequences in homotopy then imply that

$$\pi_{-9}(\Sigma^{-7-\rho}H_Qmgw) \cong \phi^*_ZB(2,0)$$. Dualizing gives that $\pi_{-2}(\Sigma^{\rho}H_Qmgw)$ is $\phi^*_ZB(2,0)$.

We now have a fiber sequence

$$\Sigma^{4}H_Q\phi^*_ZF_2 \rightarrow \Sigma^{\rho}H_Qmgw \rightarrow \Sigma^{2}H_Q\phi^*_ZB(2,0).$$

Suspending this sequence by $\rho_Q$ immediately gives the homotopy Mackey functors of $\Sigma^{2\rho}H_Qmgw$. The same is true in the case $j = 3$, except that we have an extension

$$g \hookrightarrow \pi_{-3}\Sigma^{3\rho}H_Qmgw \rightarrow \phi_{LDR}F_2.$$ We claim that, more generally, any extension of $\mathbb{Z}$-modules

$$g^m \hookrightarrow E \twoheadrightarrow \phi_{LDR}F_2$$ is necessarily the split extension. To see this, first note that $\phi_{LDR}F_2$ is, by definition, the direct sum $\phi^*_D\mathbb{F}_2 \oplus \phi^*_R\mathbb{F}_2 \oplus \phi^*_F\mathbb{F}_2$. It therefore suffices to show that the only $\mathbb{Z}$-module extension of $\phi^*_D\mathbb{F}_2$ by $g^m$ is the split extension. Since any such extension will vanish at the subgroups $D$ and $R$, the $\mathbb{Z}$-module structure forces the value at $Q$ to be 2-torsion and therefore equal to $F_2^{m+1}$. Since there is a nontrivial
restriction to the subgroup \( L \), the \( \mathbb{Z} \)-module structure forces the transfer from \( L \) to vanish. Thus the extension must be the split extension.

The suspension by \((j - 1)\rho_Q\) of (7.8) gives the homotopy Mackey functors of \( \Sigma^{j\rho_Q} \mathbb{Q}mg \) in degrees \( 2j + 1 \) and higher. Now we argue by induction that the Mackey functors for \( \Sigma^{j\rho_Q} \mathbb{Q}mg \) are as claimed, for \( j \geq 3 \). For instance, since the bottom Mackey functor is\( \pi_{2j-1}(\Sigma^{(j-1)\rho_Q} \mathbb{Q}mg) \cong g \), we see by decomposing \( \Sigma^{(j-1)\rho_Q} \mathbb{Q}mg \) using the Postnikov tower that\( \pi_{2j}(\Sigma^{j\rho_Q} \mathbb{Q}mg) \cong g \) and that we have an extension of \( \mathbb{Z} \)-modules \( g \to \pi_{2j}(\Sigma^{j\rho_Q} \mathbb{Q}mg) \to \phi_{LDR} F_2 \). By the argument given above, this must be the split extension.

The homotopy Mackey functors for the remaining \( 8k \)-slices follow from \([S1, \text{Propositions 9.5, 9.8}]\).
Proposition 7.12 ([GY, Corollary 7.2]). For \( j \geq 1 \), the homotopy Mackey functors of \( \Sigma^{j \rho} H_Q \phi_Z F_2^* \) are

\[
\pi_i(\Sigma^{j \rho} H_Q \phi_Z F_2^*) \cong \begin{cases} 
\phi_Z F_2^* & i = 4j \\
mg & i = 4j - 1 \\
\phi_{LDR} F_2^* \oplus g^{4j - 2 - i} & i \in [2j, 4j - 2] \\
g^{2i} & i \in [j, 2j - 1].
\end{cases}
\]

Proposition 7.13 ([GY, Proposition 7.3]). For \( j \geq 1 \), the homotopy Mackey functors of \( \Sigma^{j \rho} H_Q m^g \) are

\[
\pi_i(\Sigma^{j \rho} H_Q m^g) \cong \begin{cases} 
\phi_{LDR} F_2^* & i = 2j \\
g^3 & i \in [j + 1, 2j - 1] \\
g & i = j.
\end{cases}
\]

Proposition 7.14 ([GY, Proposition 7.4]). For \( j \geq 1 \), the homotopy Mackey functors of \( \Sigma^{j \rho} H_Q m^{2g} \) are

\[
\pi_i(\Sigma^{j \rho} H_Q m^{2g}) \cong \begin{cases} 
\phi_{LDR} F_2^* & i = 2j \\
g^3 & i \in [j + 1, 2j - 1] \\
g & i = j.
\end{cases}
\]

7.4. The \( 4k + 2 \)-slices. The homotopy Mackey functors of the \( (4k + 2) \)-slice \( \Sigma^{i+k \rho} H_Q \phi_Z F_2^* \) are given in Proposition 7.12. The homotopy Mackey functors of the remaining \( (4k + 2) \)-slices are as follows.

Proposition 7.15 ([GY, Proposition 4.8, Corollary 7.2]). We have the equivalence \( \Sigma^{i} H_Q \phi_Z F_2^* \cong \Sigma^{i} H_Q \phi_Z F_2 \). For \( j \geq 2 \), the homotopy Mackey functors of \( \Sigma^{j \rho} H_Q \phi_Z F_2^* \) are

\[
\pi_i(\Sigma^{j \rho} H_Q \phi_Z F_2^*) \cong \begin{cases} 
\phi_Z F_2^* & i = 4j \\
mg & i = 4j - 1 \\
\phi_{LDR} F_2^* \oplus g^{4j - 2 - i} & i \in [2j + 2, 4j - 2] \\
g^{2i(j - 3)} & i \in [j + 3, 2j + 1].
\end{cases}
\]

Finally, we have the homotopy of \( \Sigma^{j \rho} H_Q w \) and \( \Sigma^{j \rho} H_Q w^* \).

Proposition 7.16. For \( j \geq 1 \), the homotopy Mackey functors of \( \Sigma^{j \rho} H_Q w \) are

\[
\pi_i(\Sigma^{j \rho} H_Q w) \cong \begin{cases} 
\phi_Z F_2^* & i = 4j \\
mg & i = 4j - 1 \\
\phi_{LDR} F_2^* \oplus g^{4j - 2 - i} & i \in [2j, 4j - 2] \\
g^{2i} & i \in [j + 1, 2j - 1].
\end{cases}
\]

Proof. The underlying spectrum of \( \Sigma^{j \rho} H_Q w \) is contractible; thus,

\[
\Sigma^{j \rho} H_Q w \simeq \Sigma^{j \rho K} H_Q w.
\]

Then, because \( w \) is a pullback over \( Z \), the calculation is essentially \( K \)-equivariant. Consider the short exact sequence of \( K \)-Mackey functors \( w \rightarrow F_2 \rightarrow g \) and the corresponding cofiber sequence \( \Sigma^{j \rho K} H_K w \rightarrow \Sigma^{j \rho K} H_K F_2 \rightarrow \Sigma^{j \rho K} H_K g \). The statement follows immediately from the resulting long exact sequence in homotopy. \( \square \)
Proposition 7.17. For \( j = 1 \), the homotopy Mackey functors of \( \Sigma^{j\rho} H_Q w^* \) are

\[
\pi_i(\Sigma^{j\rho} H_Q w^*) \cong \begin{cases} 
\phi_2^w F_2 & i = 4 \\
g & i = 2
\end{cases}.
\]

For \( j \geq 2 \), they are

\[
\pi_i(\Sigma^{j\rho} H_Q w^*) \cong \begin{cases} 
\phi_2^w F_2 & i = 4j \\
mg & i = 4j - 1 \\
g^{4j - 2 - i} & i \in [2j + 2, 4j - 2] \\
g^{2(i-j)-5} & i \in [j + 3, 2j + 1] \\
g & i = j + 1.
\end{cases}
\]

Proof. The proof is the same as that in Proposition 7.16, except that we start with the exact sequence of \( K \)-Mackey functors \( g \rightarrow F_2 \rightarrow w^* \).

\[ \square \]

8. Slice spectral sequences

Here we include the slice spectral sequences for \( \Sigma^n H_Q Z \) for several values of \( n \) between 5 and 15. In some cases, we use the restriction to the \( C_4 \)-subgroups to determine some of the slice differentials.

The grading is the same as that in [HHR1, Section 4.4.2]. The Mackey functor \( E^{t-n,t}_2 \) is \( \pi_n P^n_t(X) \). We also follow the Adams convention, where \( \pi_n P^n_t(X) \) has coordinates \( (n,t-n) \) and the differential

\[ d_r : E^{s,t}_r \rightarrow E^{s+r,t+r-1}_r \]

points left one and up \( r \).

The \( Q \)-Mackey functors that appear in these spectral sequences are listed in Table 4. We also display some companion \( C_4 \)-slice spectral sequences, and the \( C_4 \)-Mackey functors that appear are listed in Table 5.

Table 4. Symbols for \( Q \)-Mackey functors

| Symbol | Description |
|--------|-------------|
| \( \square = Z \) | \( \Diamond = \phi_2^w F_2 \) |
| \( \bullet = mg \) | \( \circ = B(3,0) \) |
| \( \blacktriangle = mgw \) | \( \blacktriangledown = g^n \) |
| \( \blacklozenge = \phi_{LDR}^w F_2 \) | \( \blacklozenge = \phi_2^w B(2,0) \) |

Table 5. Symbols for \( C_4 \)-Mackey functors

| Symbol | Description |
|--------|-------------|
| \( \square = Z \) | \( \Diamond = \phi_{C_4}^w F_2 \) |
| \( \bullet = \phi_{C_4}^w F_2 \) | \( \circ = B(2,0) \) |
| \( \bullet = g \) | \( \blacktriangle = mg \) |

Example 8.1. In the spectral sequences for \( \Sigma^5 H_Q Z, \Sigma^6 H_Q Z, \) and \( \Sigma^7 H_Q Z \), because we must be left with

\[ \pi_n(P^n_\omega \Sigma^n H_Q Z) \cong Z, \]

all differentials are forced.
Example 8.2. For $\Sigma^8H_Q\mathbb{Z}$, the pattern of differentials emanating from the Mackey functor $\mathcal{F}_8(P_8^8\Sigma^8H_Q\mathbb{Z})$ is forced; no other pattern of differentials wipes out all classes in this region. The shorter differentials clearing out the smaller region are then similarly forced.

Example 8.3. In the cases of $\Sigma^nH_Q\mathbb{Z}$ for $n = 10, 12, \text{ and } 15$, we also display the corresponding slice spectral sequence for $\Sigma^nH_{C_4}\mathbb{Z}$, where we use $C_4$ to indiscriminately refer to any of the subgroups $L, D, R \leq Q$. The slice differentials in the $C_4$-case force many of the slice differentials for the $Q$-equivariant spectra.
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