Semi-Teleparallel Theories of Gravitation

CHRISTOPHER KOHLER

Fachrichtung Theoretische Physik, Universität des Saarlandes,
Postfach 151150, D-66041 Saarbrücken, Germany

Abstract

A class of theories of gravitation that naturally incorporates preferred frames of reference is presented. The underlying space-time geometry consists of a partial parallelization of space-time and has properties of Riemann-Cartan as well as teleparallel geometry. Within this geometry, the kinematic quantities of preferred frames are associated with torsion fields. Using a variational method, it is shown in which way action functionals for this geometry can be constructed. For a special action the field equations are derived and the coupling to spinor fields is discussed.

1 Introduction

Space-time geometries with preferred frames of reference play an important role in the study of the gravitational field and its quantization. Preferred frames are often introduced in the form of preferred coordinate systems in order to simplify calculations. Such coordinate systems are, for example, Gaussian and comoving coordinates \( \text{[1]} \). Closely related to preferred coordinates is the concept of a reference medium which is used in various forms in the literature (\( \text{[2]} \) and references therein). Furthermore, it was even attempted to introduce material reference frames that couple to gravity \( \text{[3]} \).

In this article, we treat preferred frames of reference in a pure geometric way within non-Riemannian geometry. In particular, we propose a geometrical formulation of the dynamics of a preferred frame. This leads to a new class of theories of gravitation which can, in the classification of alternatives of general relativity, be placed between the Einstein-Cartan theory and teleparallel theories of gravitation. The underlying geometry consists of a partial parallelization of space-time associated with a preferred frame and will be referred to as semi-teleparallel geometry.

The introduction of this geometry is based on a consideration of the local space-time symmetries associated with preferred frames. We first define what is understood — in this article — by a preferred frame of reference within a metric space-time geometry: The primary part is a preferred timelike vector field which can be normalized and represents the tangent vectors of preferred worldlines. Along these worldlines, we consider preferred

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\( ^1 \)E-mail: c.kohler@rz.uni-sb.de
spatial triads orthogonal to the worldlines. The propagation of these triads in time should be determined dynamically. However, since space is assumed to be locally isotropic, the overall orientation of the triads along the worldlines is irrelevant from which follows that the preferred triads are given up to a constant rotation on each worldline. A preferred frame can be considered to be a reference fluid which possesses spin degrees of freedom. In terms of local symmetries, the situation is as follows: The existence of a preferred timelike vector field breaks the local Lorentz invariance down to a local $SO(3)$ symmetry of spatial rotations. The existence of preferred triads on each worldline breaks this symmetry further down to a global $SO(3)$ symmetry on each worldline. Our aim is to formulate theories of gravitation that inherently possess this symmetry.

As a guiding principle in the formulation of theories of gravitation that incorporate preferred frames we assume that the preferred frame is treated on an equal footing with the metric tensor. At the kinematical level, this means that the space-time connection parallelizes the preferred timelike vector field throughout space-time as well as the preferred triad along each worldline in the same way as a metric compatible connection parallelizes the metric tensor. At the dynamical level, it means that the preferred frames are dynamical variables like the metric tensor.

In section 2 of this article, we introduce the concept of a semi-teleparallel geometry in detail and describe its most important properties. Similar geometries have been considered previously in the context of a formulation of a spatial tensor analysis in general relativity [4]-[10]. In this article, we consider the differential geometries associated with preferred frames from the viewpoint of a symmetry breaking and develop a dynamics for this type of geometry. Accordingly, our presentation is different from the earlier approaches. In particular, we reveal the differential geometric foundation of a semi-teleparallel geometry in terms of fibre bundles.

The temporal part of the space-time torsion of a semi-teleparallel geometry is completely determined by the kinematic quantities of the preferred frame. This relation is outlined in section 3.

In section 4, we consider the formulation of a dynamics for a semi-teleparallel geometry. We describe a general method for obtaining action functionals for geometries of Riemann-Cartan type. Using this method, we propose a special action for the semi-teleparallel geometry.

Section 5 contains a discussion of the field equations and it is shown in which way matter fields can be coupled to a semi-teleparallel theory of gravitation.

In section 6, we make some comments and give an outlook on possible further studies.

## 2 Semi-Teleparallel Geometry

We assume space-time to be a four-dimensional differentiable manifold $\mathcal{M}$ endowed with a Lorentzian metric $g$ of signature $(-++++)$. In the following, we suppose that on $\mathcal{M}$ there is given a congruence of timelike worldlines being the integral curves of the tangent vector field $t$. The existence of the worldlines requires the topology of $\mathcal{M}$ to be $\Sigma \times \mathbb{R}$, where $\Sigma$ is a three-manifold. Rather than $t$, we shall use the normalized vector field $v \equiv (-g(t, t))^{-\frac{1}{2}} t$ which represents the tangent vectors of the worldlines parameterized
by their proper times. The vector field $\mathbf{v}$ satisfies $g(\mathbf{v}, \mathbf{v}) = v^a v_a = -1$ where $\mathbf{v} = v^a e_a$ is written with respect to an orthonormal basis $e_a$ ($a = 0, 1, 2, 3$) of the tangent space $T_p M$ at a point $p$ of $M$. Indices are raised and lowered with the Minkowski metric $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ or its inverse, that is, $v_a = \eta_{ab} v^b$.

The vector field $\mathbf{v}$ defines a distribution on space-time in that it singles out a three-dimensional subspace of the tangent space at each point consisting of tangent vectors orthogonal to $\mathbf{v}$. This distribution is in general not integrable, that is, there does in general not exist a foliation of $M$ by spacelike hypersurfaces orthogonal to $\mathbf{v}$. The projection tensor $h^a_b \equiv \delta^a_b + v^a v_b$ can be used to project tensors onto the distribution.

We assume that Lorentz connections are defined on $M$ which are given by their connection 1-forms $\omega_{ab}$. A connection can also be introduced by the covariant differential $D$ of tangent vectors $X$, $DX^a \equiv dX^a + \omega_{ab} X^b$. $D$ corresponds to a metric compatible connection which implies $\omega_{ab} = -\omega_{ba}$. We first consider connections $\bar{\omega}_{ab}$ adapted to the vector field $\mathbf{v}$ which are defined by the condition

$$Dv^a \equiv \bar{\omega}^a_b v^b = 0.$$ (1)

Eq. (1) says that the vector field $\mathbf{v}$ is a parallel vector field with respect to the connection $\bar{\omega}_{ab}$. This implies a partial parallelization of the manifold $M$ in that the projection of a vector onto the preferred worldlines remains parallel to the worldlines after parallel transport along arbitrary curves on $M$ while the components orthogonal to the worldlines may be rotated. It should be remarked that the connection $\bar{\omega}_{ab}$ is determined by the vector field $\mathbf{v}$ only up to a 1-form $B_{ab} = -B_{ba}$ that satisfies $B_{ab} v^b = 0$.

In order to understand the significance of connections adapted to $\mathbf{v}$, we establish the following result:

**Theorem 1** Let $\mathbf{v}$ be a vector field on $M$ with $v^a v_a = -1$. Every metric compatible connection $\omega_{ab}$ on $M$ can be uniquely decomposed as follows:

$$\omega_{ab} = \bar{\omega}_{ab} + G_{ab},$$ (2)

where $\bar{\omega}_{ab}$ is a connection adapted to $\mathbf{v}$ and $G_{ab} = -G_{ba}$ is a 1-form satisfying $h^c_a h^d_b G_{cd} = 0$.

**Proof.** Suppose that $\bar{\omega}_{ab}$ is a connection adapted to $\mathbf{v}$. From Eq. (2) follows

$$Dv_a = \bar{\omega}^b_d G_{ab} v^b = G_{ab} v^b.$$ (3)

Since every antisymmetric tensor can be decomposed as

$$G_{ab} = 2v^c_{[a} G_{bc]} v^d + h^c_a h^d_b G_{cd},$$ (4)

where the square brackets denote antisymmetrization, we obtain from Eq. (3), using $h^c_a h^d_b G_{cd} = 0$,

$$G_{ab} = v_a Dv_b - v_b Dv_a.$$ (5)

Hence, given a Lorentz connection $\omega_{ab}$, the connection $\bar{\omega}_{ab}$ is uniquely given by

$$\bar{\omega}_{ab} = \omega_{ab} - v_a Dv_b + v_b Dv_a.$$ (6)
It can be verified directly that this connection is adapted to \( v \), that is, satisfies Eq. (1).

In the case that the connection \( \omega_{ab} \) is the torsionfree Levi-Civit\'a connection \( \hat{\omega}_{ab} \) of the metric \( g \), the connection \( \hat{G}_{ab} \) defined by Eq. (6) can be physically interpreted as follows. Suppose that \( \mathcal{C} \) is a curve belonging to the congruence of preferred worldlines parameterized by the proper time \( s \). Then Eq. (6) specifies a parallel transport of a vector \( X \) along \( \mathcal{C} \) given by the vanishing of the derivative

\[
\frac{D}{ds} X^a = \hat{D} X^a - v^a \hat{D} v^b X^b + \hat{D} v^b \frac{D}{ds} X^b,
\]

where \( \hat{D} \) is the covariant derivative defined by the Levi-Civit\'a connection \( \hat{\omega}_{ab} \). This parallel transport is the well known Fermi-Walker transport along \( \mathcal{C} \) and Eq. (7) is known as the Fermi derivative [11]. Eq. (6) generalizes the Fermi-Walker transport in that torsion is allowed for and the transport may be performed in an arbitrary direction.

Theorem 1 shows that there is a one-to-one correspondence between the set of Lorentz connections \( \{ \omega_{ab} \} \) and the set of pairs \( \{ \hat{G}_{ab}, G_{ab} \} \) consisting of a connection \( \hat{G}_{ab} \) adapted to \( v \) and a 1-form \( G_{ab} \) with \( h_a^c h_b^d g_{cd} = 0 \). It should be remarked that a connection adapted to \( v \) in general possesses torsion even if the associated Lorentz connection \( \omega_{ab} \) is torsionfree.

With the help of theorem 1 we next reveal the differential geometric origin of connections adapted to \( v \) (see also [12]). For that, we identify the vector field \( v \) with the basis vector \( e_0 \), that is, \( v \) has the fixed components \( v^a = (1, 0, 0, 0) \). The remaining basis vectors \( e_i \) \((i = 1, 2, 3)\) at all points of \( M \) then form a principal fibre bundle \( P(M) \) over \( M \) with structure group \( SO(3) \). \( P(M) \) is a reduced subbundle of the bundle of orthonormal frames \( O(M) \) over \( M \) with structure group \( SO(3,1) \). The reduction is defined by the vector field \( e_0 \) which represents a section of the associated fibre bundle \( E \) over \( M \) with structure group \( SO(3,1) \) and with the coset \( SO(3,1)/SO(3) \) as standard fibre [13].

A connection on \( O(M) \) is given by an \( so(3,1) \)-valued 1-form \( \omega = \frac{1}{2} \omega_{ab} J^{ab} \) on \( M \) where \( J^{ab} \) are the generators of \( SO(3,1) \) satisfying the commutation relations

\[
[J_{ab}, J_{cd}] = 2\eta_{c[a} J_{b]d} - 2\eta_{d[a} J_{b]c}.
\]

According to theorem 1, we can decompose \( \omega \) as

\[
\omega = \frac{1}{2} \hat{G}_{ab} J^{ab} + \frac{1}{2} (v_a Dv_b - v_b Dv_a) J^{ab} = \frac{1}{2} \hat{G}_{ab} J^{ab} + Dv_i J^{i0}.
\]

Since \( Dv_i = \omega_{i0} \) in the chosen basis, it follows from Eq. (10) that \( \hat{G}_{i0} \) vanishes. Thus we obtain

\[
\omega = \frac{1}{2} \hat{G}_{ij} J^{ij} + Dv_i J^{i0}.
\]

If we require \( Dv_a = 0 \), which means that \( \omega_{ab} \) is adapted to \( v \), Eq. (11) yields

\[
\omega = \frac{1}{2} \hat{G}_{ij} J^{ij}.
\]
Since the generators $J_{ij}$ generate $SO(3)$, $\omega$ is an $SO(3)$ connection. We see that a connection adapted to $\mathbf{v}$ is a Lorentz connection that is reducible to an $SO(3)$ connection.

The essential property of a connection adapted to $\mathbf{v}$ is that it converts preferred worldlines into autoparallels by introducing torsion in a specific way. We next generalize this connection to the case that a preferred frame, as defined in the introduction, is given. To do this, we introduce spatial reference frames in the form of triads of orthonormal vectors $\mathbf{b}_{(i)} = b^a_{(i)} \mathbf{e}_a$ which are orthogonal to $\mathbf{v}$. The $\mathbf{b}_{(i)}$ together with $\mathbf{v}$ form an orthonormal basis of tangent vectors. The dual basis shall be denoted by $\{ \gamma = \gamma_a \mathbf{e}^a, \mathbf{b}^{(i)} = b^{(i)}_a \mathbf{e}^a \}$ where $\mathbf{e}^a$ is the cobasis corresponding to the basis $\mathbf{e}_a$. Then, we have the following relations:

$$
\gamma(\mathbf{v}) = \gamma_a v^a = 1, \quad \gamma(\mathbf{b}_{(i)}) = \gamma_a b^a_{(i)} = 0,
$$

$$
\mathbf{b}^{(i)}(\mathbf{v}) = b^{(i)}_a v^a = 0, \quad \mathbf{b}^{(i)}(\mathbf{b}_{(j)}) = b^{(i)}_a b^{(j)}_a = \delta^i_j. \tag{12}
$$

In the case that the triad $\mathbf{b}_{(i)}$ is Fermi-Walker transported along a worldline, the vectors $\mathbf{b}_{(i)}$ on the worldline are parallel with respect to the corresponding connection adapted to $\mathbf{v}$. If, however, there is an angular velocity of the triad, we can still consider the vectors $\mathbf{b}_{(i)}$ to be parallel at different points on the worldline if we modify the connection adapted to $\mathbf{v}$ by introducing torsion appropriately. We extend the condition (I) in the following way.

**Definition 1** Let $\{ \mathbf{v}, \mathbf{b}_{(i)} \}$ be a preferred frame of reference on $\mathcal{M}$. A connection $S_{ab}$ is called semi-teleparallel if it satisfies the conditions

$$
S_{ab} v^a = 0 \quad \text{and} \quad S_{ab} b^a_{(i)} = 0. \tag{13}
$$

In order to gain insight into the nature of these connections, we formulate the following theorem which is analogous to theorem 1.

**Theorem 2** Let $\{ \mathbf{v}, \mathbf{b}_{(i)} \}$ be an orthonormal basis on $\mathcal{M}$. Every connection $G_{ab}$ adapted to $\mathbf{v}$ can be uniquely decomposed as follows:

$$
G_{ab} = S_{ab} + F_{ab}, \tag{14}
$$

where $S_{ab}$ is a semi-teleparallel connection and the tensor $F_{ab} = -F_{ba}$ is a 1-form with $F_{ab} v^b = 0$ and $F_{ab} (\mathbf{b}_{(i)}) = 0$.

**Proof.** From Eq. (14) follows with the help of the second of Eqs. (13)

$$
\bar{G}_{ab} b^a_{(i)} = F^a_{b(\mathbf{v})} b^b_{(i)}. \tag{15}
$$

Using $F_{ab} v^b = 0$ and the fact that $\{ \mathbf{b}_{(i)}, \mathbf{v} \}$ forms an orthonormal basis, Eq. (13) can be solved for $F^a_{b(\mathbf{v})}$ yielding

$$
F^a_{b(\mathbf{v})} = -b^a_{(i)} \bar{G}_{ab} b^b_{(i)}. \tag{16}
$$

If we employ the condition $F_{ab} (\mathbf{b}_{(i)}) = 0$, we conclude that

$$
F^a_{b} = -b^a_{(i)} \bar{G}_{ab} b^b_{(i)} \gamma. \tag{17}
$$
Thus, given a connection $\bar{G}_{ab}$ adapted to $v$ and a triad $b_{(i)}$, there is a unique semi-teleparallel connection

$$\bar{S}^a_{\ b} = \bar{G}^a_{\ b} + b^a_{(i)} D_v b^i_{\ b} \gamma. \quad (18)$$

The transition from the connection $\bar{G}_{ab}$ to the connection $\bar{S}_{ab}$ has also a description in terms of fibre bundles. Given the bundle $P(M)$ of triads $b_{(i)}$ considered above, we can associate with each preferred worldline $C$ a subbundle $Q(C)$ of $P(M)$ with structure group $SO(3)$ by restricting $P(M)$ to $C$. A triad $b_{(i)}$ defines a reduction of each subbundle $Q(C)$ associated with a worldline to a fibre bundle with structure group $\{1\}$. Indeed, the triad along $C$ can be considered to be a section of the associated fibre bundle over $C$ with standard fibre $SO(3)$. Let $\omega$ be the $SO(3)$ connection adapted to $v$ given by Eq. (11).

According to Eq. (18), $\omega$ can be decomposed as

$$\omega = \frac{1}{2} \bar{G}_{ab} J^{ab} = \frac{1}{2} \bar{S}_{ab} J^{ab} - \frac{1}{2} b^a_{(k)} D_v b^k_{\ b} \gamma J^a_{\ b}. \quad (19)$$

Using the basis $e_0 = v$ and $e_i = b_{(i)}$, which implies $b^a_{(k)} = \delta^a_k$ and $b^k_{\ a} = \delta^k_a$, Eq. (19) reads

$$\omega = \frac{1}{2} \bar{G}_{ij} J^{ij} = \frac{1}{2} \bar{S}_{ij} J^{ij} - \frac{1}{2} b^i_{(k)} D_v b^k_{\ j} \gamma J^i_{\ j}. \quad (20)$$

Since $b^i_{(k)} D_v b^k_{\ j} = \bar{G}^i_{\ jv}$, it follows from Eq. (20) that $\bar{S}_{ijv}$ vanishes. This implies

$$\omega = \frac{1}{2} \bar{S}_{ijk} b^{(k)} J^{ij} - \frac{1}{2} b^i_{(k)} D_v b^k_{\ j} \gamma J^i_{\ j}. \quad (21)$$

In the case that $D_v b^k_{\ j} = 0$, that is, the connection $\omega$ is semi-teleparallel, we have

$$\omega = \frac{1}{2} \bar{S}_{ijk} b^{(k)} J^{ij}. \quad (22)$$

With respect to the bundles $Q(C)$ over the worldlines, the induced connection then is

$$\omega = 0, \quad (23)$$

which means that $\omega$ is reducible to a $\{1\}$ connection.

In summary, a semi-teleparallel connection can be defined to be a Lorentz connection that is reducible to an $SO(3)$ connection and whose induced connection on each worldline is reducible to a $\{1\}$ connection.

**Theorem 3** Let $\{v, b_{(i)}\}$ be an orthonormal basis on $M$. Every Lorentz connection $\omega_{ab}$ can be uniquely decomposed as follows:

$$\omega_{ab} = \bar{\omega}_{ab} + S_{ab}, \quad (24)$$

where $\bar{\omega}_{ab}$ is a semi-teleparallel connection associated with $\{v, b_{(i)}\}$ and the tensor field

$$S_{ab} = -S_{ba}$$

is a 1-form with $h^i_a h^j_b S_{cd}(b_{(i)}) = 0$. 

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Combining Eqs. (6) and (18), we can compute an expression for a semi-teleparallel connection $S_{\omega}^{ab}$ associated with a Lorentz connection $\omega^{ab}$:

$$
S_{\omega}^{a}_{\ b} = \omega^{a}_{\ b} - v^{a}Dv_{b} + v_{b}Db_{(i)}v^{(i)} + b_{(i)}^{a}D\gamma_{b}^{(i)}.
$$

(25)

From this equation follows that $S_{\omega}^{ab}$ is invariant under rotations $b'_{(i)} = \Lambda^{j}_{i}b_{(j)}$ of the triad for which $\partial_{s}\lambda_{j}^{i} = 0$. Hence, the connection $S_{\omega}^{ab}$ is determined by the tetrad $\{v, b_{(i)}\}$ only up to $SO(3)$ transformations that are constant along the worldlines.

In the following, we will use the cobasis $e^{a}$ as a basis of 1-forms for the connection 1-form, that is, we use the components $\omega^{abc} = \omega^{ab}_{\ mu}e^{\mu}_{\ c}$. Here, $\mu = 0, 1, 2, 3$ is a coordinate index. The special choice of basis $\{e_{0} = v, e_{i} = b_{(i)}\}$ will be called semi-teleparallel frame. Using this basis, the semi-teleparallel connection $S_{\omega}^{abc}$ is characterized by the vanishing of the following components:

$$
S_{\omega}^{0}_{ij} = 0, \quad S_{\omega}^{0i}_{0} = 0, \quad S_{\omega}^{ij0} = 0.
$$

(26)

The spatial components $S_{\omega}^{ijk}$ form an arbitrary $SO(3)$ connection.

### 3 Kinematic Quantities

If the preferred vector field $v$ is considered to be the velocity field of some form of matter, the semi-teleparallel geometry involves a new formulation of the kinematics of velocity fields. The kinematic quantities of a velocity field $v$ are defined using the covariant derivative $\overset{\circ}{D}a v_{b}$ with respect to the Levi-Civit\`{a} connection [14]. When a semi-teleparallel connection is given, this covariant derivative is, according to the first of Eqs. (13), given by

$$
\overset{\circ}{D}a v_{b} = K_{bca}v^{c},
$$

(27)

where $K_{abc} \equiv \frac{1}{2}(T_{abc} - T_{bac} - T_{cab})$ is the contortion tensor of the semi-teleparallel connection and $T_{abc}$ is the torsion tensor defined by $T^{a} = \frac{1}{2}T^{a}_{\ bc}e^{b} \wedge e^{c} = d \wedge e^{a} + S_{\omega}^{a}_{\ b} \wedge e^{b}$. Eq. (27) follows from the decomposition $S_{\omega}^{abc} = \overset{\circ}{\omega}^{abc} - K_{abc}$.

The acceleration $\dot{v}_{a}$ of $v_{a}$ is defined by $\dot{v}_{a} \equiv v^{b}\overset{\circ}{D}b v_{a}$. Eq. (27) then yields

$$
\dot{v}_{a} = K_{abc}v^{b}v^{c}.
$$

(28)

Hence, in a semi-teleparallel geometry, the acceleration is related to a part of the torsion tensor. This is also the case for the other kinematic quantities:

The vorticity is defined by

$$
\Omega_{ab} \equiv h_{a}^{c}h_{b}^{d}\overset{\circ}{D}[d v_{c}].
$$

(29)

With the help of Eq. (27), we obtain

$$
\Omega_{ab} = -K_{[c|ab]}v^{c} - v_{[a}K_{b]cd}v^{c}v^{d}.
$$

(30)
The deformation tensor is given by

\[ \theta_{ab} \equiv h^c_a h^d_b \overset{\circ}{D}_{(c} v_{d)}, \]  

where round brackets denote symmetrization. Eq. (27) yields

\[ \theta_{ab} = -K_{c(ab)} v^c + v_{(a} K_{b)cd} v^c v^d. \]  

The trace of \( \theta_{ab} \), that is, the expansion \( \theta \), is given by

\[ \theta = K^a_{\ ba} v^b. \]  

The (trace-free) shear tensor then is

\[ \sigma_{ab} = -K_{c(ab)} v^c + v_{(a} K_{b)cd} v^c v^d - \frac{1}{3} h_{ab} K^d_{\ cd} v^c. \]  

If there are spin degrees of freedom, we can additionally define a spin rotation. We assume that spatial reference frames \( b_{(i)} \) along the worldlines of the velocity field are given with respect to which the spin vector is fixed. The angular velocity \( \kappa_{ab} \) shall be given by

\[ \overset{F}{\nabla} b^a_{(i)} \equiv \kappa^a_{\ b(i)} b^b_{(i)}, \]  

where \( \overset{F}{\nabla} \) is the Fermi derivative. Using Eq. (7), it follows

\[ \kappa^a_{\ b(i)} b^b_{(i)} = \overset{\circ}{\nabla} b^a_{(i)} - v^a \overset{\circ}{\nabla} v^b_{(i)} b^b_{(i)}. \]  

On the other hand, the second of Eqs. (13) can be written as

\[ \overset{\circ}{\nabla} b^a_{(i)} = K^a_{\ bc} v^c b^b_{(i)}. \]  

Inserting Eq. (37) into Eq. (36) and solving for \( \kappa^a_{\ b(i)} \), we obtain

\[ \kappa_{ab} = K_{abc} v^c - 2v_{[a} K_{b]cd} v^c v^d. \]  

While the vorticity represents the external, orbital part of the angular velocity, the quantity \( \kappa_{ab} \) is of a pure internal nature and can be regarded as a spin rotation.

The kinematic quantities take on an especially simple form if we use a semi-teleparallel basis. The only nonvanishing components are

\[ \dot{v}_i = K_{i00}, \quad \Omega_{ij} = -K_{0[ij]}, \quad \theta_{ij} = -K_{0(ij)}, \quad \kappa_{ij} = K_{ij0}. \]  

The contortion tensor of a semi-teleparallel connection can with the help of the kinematic quantities be decomposed as follows:

\[ K_{abc} = \frac{1}{3} K_{abc} + 2\dot{v}_{[a} v_{b]} v^c + 2v_{[a} \sigma_{b]c} + \frac{2}{3} \theta v_{[a} h_{b]c} - \kappa_{ab} v^c + 2v_{[a} \Omega_{b]c}, \]  

where \( \frac{1}{3} K_{abc} = h^d_a h^e_b h^f_c K_{def} \) is the spatial part of the contortion tensor, which is not determined by kinematical quantities.
We next consider the curvature of a semi-teleparallel connection. From the definition of a semi-teleparallel connection follow the conditions

\[ v^a S R_{ab} = 0 \]  
(41)

for the curvature 2-form \( S R_{ab} \equiv d \wedge \overset{\circ}{\omega}_{ab} + \overset{\circ}{\omega}_{ac} \wedge \overset{\circ}{\omega}_{cb} \). These conditions can be expressed by the Levi-Civita connection \( \overset{\circ}{\omega}_{abc} \) and the contortion tensor which, according to Eq. (40), is related to the kinematic quantities. Eq. (41) then represents 18 equations for the kinematic quantities which coincide with the known evolution and constraint equations.

Further conditions follow from the 1st Bianchi identity \( D \wedge T^a = R^a_{\ b} \wedge e^b \). Using Eq. (41), its projection on the vector field \( v^a \) is

\[ v_a S D \wedge T^a = 0. \]  
(42)

These four conditions are, when expressed by the kinematic quantities, the evolution and constraint equations for the vorticity already contained in Eq. (41).

4 Action Functional

In order to find an action functional describing the dynamics of a semi-teleparallel space-time geometry, we use a variational method that can also be applied to other space-time geometries. This method makes use of theorem 3, that is, of the property of semi-teleparallel connections to induce a unique decomposition of Lorentz connections. This is a characteristic shared also by other geometries, for example by Riemannian geometry, by teleparallel geometry, and — in a trivial way — even by Riemann-Cartan geometry itself. The general decomposition is

\[ \omega_{abc} = \tilde{\omega}_{abc} + H_{abc}, \]  
(43)

where \( \omega_{abc} \) is a Lorentz connection, \( \tilde{\omega}_{abc} \) stands for a special connection, and \( H_{abc} = -H_{bac} \) is a tensor field which, depending on the connection \( \tilde{\omega}_{abc} \), can have restrictions. In the case that \( \tilde{\omega}_{abc} \) is a semi-teleparallel connection, the spatial part of \( H_{abc} \) is vanishing according to theorem 3. If \( \tilde{\omega}_{abc} \) is a Riemannian connection or a teleparallel connection, \( H_{abc} \) is unrestricted. If \( \tilde{\omega}_{abc} \) is a Lorentz connection, \( H_{abc} \) is zero.

To obtain an action for the connection \( \tilde{\omega}_{abc} \), we start with an action \( S_0 \) that depends on the cobasis \( e^a_\mu \), the Lorentz connection \( \omega_{abc} \), the tensor field \( H_{abc} \), and the contortion tensor \( \tilde{K}_{abc} \) of the connection \( \tilde{\omega}_{abc} \). The last two are, according to Eq. (43), uniquely given by \( \omega_{abc} \). In a next step, we replace \( \omega_{abc} \) in \( S_0 \) by the decomposition (43). Finally, we determine the stationary point of the action with respect to the tensor field \( H_{abc} \). As a result, we obtain an action that only depends on the connection \( \tilde{\omega}_{abc} \) and the cobasis \( e^a_\mu \).

The natural choice for the action \( S_0 \) is a sum of the Einstein-Cartan action and functionals quadratic in \( H_{abc} \) and \( \tilde{K}_{abc} \). We first consider the Einstein-Cartan action,

\[ S_{EC}[e^a_\mu, \omega_{abc}] = -\frac{1}{2G} \int d^4x \sqrt{-g} R(\omega_{abc}, e^a_\mu), \]  
(44)
where $R(\omega_{abc}, e^a_\mu) = R^a_{\mu b}$ is the scalar curvature corresponding to the connection $\omega_{abc}$, $G$ is the gravitational constant, and $g$ is the determinant of the metric tensor. Inserting the decomposition [13], we obtain up to a surface term

$$S_{EC}[\epsilon^a_\mu, \tilde{\omega}_{abc}, H_{abc}] = -\frac{1}{2G} \int d^4x \sqrt{-g} \left[ R(\epsilon^a_\mu, \tilde{\omega}_{abc}) + H^{abc}(H_{cba} - 2\tilde{K}_{cba}) + H^a_{ca}(H^{\lambda b} - 2\tilde{K}^{\lambda b}) \right], \quad (45)$$

In this article, we do not consider the most general additional term quadratic in $H_{abc}$ and $\tilde{K}_{abc}$. Instead, we choose a term that is quadratic in the totally antisymmetric parts of $H_{abc}$ and $\tilde{K}_{abc}$. The particular combination in which $H_{abc}$ and $\tilde{K}_{abc}$ appear in Eq. (13) suggests to use the term $H^{[abc]}(H_{[abc]} - 2\tilde{K}_{[abc]})$. Thus, we choose

$$S_0[\epsilon^a_\mu, \tilde{\omega}_{abc}, H_{abc}] = \tilde{S}_{EC} - \frac{1}{2G} \int d^4x \sqrt{-g} \left[ H^{abc}(H_{cba} - 2\tilde{K}_{cba}) + H^a_{ca}(H^{\lambda b} - 2\tilde{K}^{\lambda b}) + \lambda H^{[abc]}(H_{[abc]} - 2\tilde{K}_{[abc]}) \right], \quad (46)$$

where

$$\tilde{S}_{EC} = -\frac{1}{2G} \int d^4x \sqrt{-g} R(\epsilon^a_\mu, \tilde{\omega}_{abc})$$

and $\lambda$ is a parameter. In order to determine the stationary point of $S_0$ with respect to $H_{abc}$, we vary $S_0$ with respect to $H_{abc}$. The corresponding field equations can be solved algebraically for $H_{abc}$.

In the case that $\tilde{\omega}_{abc}$ is a Lorentz connection, $S_0$ leads to the Einstein-Cartan action since $H_{abc} = 0$. For the Riemannian and the teleparallel geometry the stationary points of $S_0$ correspond to the standard actions which are usually considered: We first assume $\lambda \neq 1$. Variation of $S_0$ with respect to $H_{abc}$ leads to the condition $H_{abc} = \tilde{K}_{abc}$. In the Riemannian case, we have $\tilde{K}_{abc} = 0$ since $\tilde{\omega}_{abc}$ is the Levi-Civita connection $\tilde{\omega}_{abc}$. Inserting $H_{abc} = 0$ into $S_0$, we obtain the Einstein-Hilbert action. In the teleparallel case, $\tilde{K}_{abc}$ is the contortion tensor of the teleparallel connection $\tilde{\omega}_{abc} = 0$ with respect to the teleparallel frame $\epsilon^a_\mu$. Since $\tilde{S}_{EC} = 0$, we obtain after insertion of $H_{abc} = \tilde{K}_{abc}$ into $S_0$ the action

$$S_T = -\frac{1}{2G} \int d^4x \sqrt{-g} \left( \hat{R} - \lambda \tilde{K}^{[abc]} \tilde{K}_{[abc]} \right), \quad (47)$$

where $\hat{R}$ is the Riemannian curvature scalar. $S_T$ is the known 1-parameter action for the teleparallel geometry [13] [15]. If $\lambda = 1$, the condition obtained by the variation of $S_0$ is $H_{a(bc)} = \tilde{K}_{a(bc)}$. The corresponding antisymmetric part of $H_{abc}$ is undetermined. However, insertion of the condition into $S_0$ yields again the Einstein-Hilbert action and the teleparallel action [17] which thus are stationary points for all values of $\lambda$.

Since the semi-teleparallel geometry interpolates — in a certain sense — the Riemann-Cartan and the teleparallel geometry, we expect that $S_0$ yields a suitable action for this geometry. The determination of the stationary point is, however, more involved. As a simplification, we will work in a semi-teleparallel basis $\epsilon^a_\mu$. Then, the condition that the spatial part of $H_{abc}$ vanishes means $H_{ijk} = 0$. We first consider the case $\lambda \neq 1$. Variation with respect to $H_{0ij}$ and $H_{ij0}$ leads to the conditions $H_{0ij} = \tilde{K}_{0ij}$ and $H_{ij0} = \tilde{K}_{ij0}$. As for the components $H_{000}$, we note that in the action $S_0$ they are contained only in the term
that is, $S_0$ is linear in $H_{0ij}$. A stationary point exists if $\tilde{K}^{ij} = 0$, which we will assume in the following. $H_{0ij}$ is then undetermined. The stationary point of $S_0$ is given by

$$S_{ST}[e^a_\mu, \tilde{\omega}_{abc}] = -\frac{1}{2G} \int d^4x \sqrt{-g} \left[ R(e^a_\mu, \tilde{\omega}_{abc}) - \tilde{K}^{0ij} \tilde{K}_{0ji} - 2\tilde{K}^{0ij} \tilde{K}_{ji0} - (\tilde{K}^{0i0})^2 \right. $$

$$\left. - 3\lambda \tilde{K}^{[0ij]} \tilde{K}_{[0ij]} \right].$$

(48)

Using the relation

$$R(e^a_\mu, \tilde{\omega}_{abc}) = \overset{\circ}{R} + \tilde{K}^{abc} \tilde{K}_{cba} + \tilde{K}^{a}c_{a}^{b} \tilde{K}^b_{ba} + 2 \overset{\circ}{D} a \tilde{K}^{ba}_{b},$$

(49)

obtained with the help of the decomposition $\tilde{\omega}_{abc} = \overset{\circ}{\omega}_{abc} - \tilde{K}_{abc}$, Eq. (48) can be written as

$$S_{ST}[e^a_\mu, \tilde{\omega}_{ijk}] = -\frac{1}{2G} \int d^4x \sqrt{-g} \left( \overset{\circ}{R} + \tilde{K}^{ijk} \tilde{K}_{kji} - 3\lambda \tilde{K}^{[0ij]} \tilde{K}_{[0ij]} \right),$$

(50)

where a surface term has been omitted. In the case $\lambda = 1$, there does not exist a unique stationary point. We obtain the condition $H_{0ij} - \tilde{K}_{0ij} = H_{ij0} - \tilde{K}_{ij0}$ which when inserted into $S_0$ leads also to the action (50).

We use the action (50) as an action for the semi-teleparallel geometry. $S_{ST}$ reflects the characteristic feature of a semi-teleparallel geometry to be a Riemann-Cartan geometry in the spatial part and a teleparallel geometry in the temporal part: The second term in the integrand corresponds to an Einstein-Cartan gravitation in the spatial projection of space-time while the third term pertains to a teleparallel gravitation in the temporal projection.

5 Field Equations and Matter Coupling

We next derive the field equations in the matter free case following from the action (50). Only the second term in the action depends on the spatial part $\omega_{ijk}$ of the connection. Through variation with respect to $\omega_{ijk}$ we obtain the vanishing of the spatial torsion, $\tilde{T}_{ijk} = 0$, that is, the spatial geometry is Riemannian. Note that the condition $\tilde{K}^{ij} = 0$ is consistent with these field equations.

Variation with respect to the semi-teleparallel frame $e^a_\mu$ leads to 16 field equations which, after contraction by $e^b_\mu$ and subtracting the trace, can be divided into the symmetric part

$$\overset{\circ}{R}_{00} - \lambda K^{ij} \tilde{K}_{[0ij]} = 0,$$

$$\overset{\circ}{R}_{0k} - \lambda \left( K^{i}_{00} \tilde{K}_{[0ik]} + \frac{1}{2} \overset{\circ}{\omega}^{ij}_{k} \tilde{K}_{[0ij]} \right) = 0,$$

$$\overset{\circ}{R}_{kl} - \lambda \left( K^{i}_{0k} \tilde{K}_{[0il]} + K^{i}_{0l} \tilde{K}_{[0ik]} \right) = 0,$$

(51)

and the antisymmetric part

$$\partial_{\mu} \left( \sqrt{-g} e^{\mu}_{i j} K_{[0ij]} \right) - \frac{3}{2} \sqrt{-g} \overset{\circ}{\omega}^{[ij]}_{k} K^{[0ij]} = 0,$$

$$\partial_{\mu} \left( \sqrt{-g} e^{\mu}_{0k} K_{[0kl]} \right) = 0,$$

(52)
where $\hat{R}_{ab}$ is the Ricci tensor of the Levi-Cività connection and we have removed the overtilde of $K_{abc}$.

The temporal part of the contortion tensor appearing in Eqs. (51) and (52) can be expressed by the kinematic quantities of the semi-teleparallel frame according to Eqs. (39). Eqs. (51) and (52), therefore, are field equations for the semi-teleparallel frame. They are similar to the field equations of the teleparallel gravitation [16], they contain, however, only the temporal part of the torsion tensor. As in the case of the teleparallel gravitation, one expects problems with the time evolution of the semi-teleparallel frame in that the frame is not uniquely determined by the initial conditions, at least for some special solutions [17][18].

Since scalar fields and gauge fields do not couple to the space-time torsion, the incorporation of these fields is unproblematic. We will now consider the incorporation of spinor fields. Simply adding a spinor action in the semi-teleparallel geometry to the action (50) leads to inconsistencies since the field equations then require $\tilde{K}^{ijj} \neq 0$ in general. Instead, we repeat the construction of the action given in the previous section beginning with the composition (43) is used. We limit ourselves to the case of a Dirac field. We add to Eq. (55) the Dirac action

$$S_D = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \left( \bar{\psi} \gamma^a \nabla_a \psi - \nabla_a \bar{\psi} \gamma^a \psi \right) - m \bar{\psi} \psi \right]$$

$$= \tilde{S}_D + \int d^4x \sqrt{-g} H_{abc} \tau^{abc},$$

where $\tilde{S}_D$ is the Dirac action with respect to a semi-teleparallel connection, $\nabla_a \equiv \partial_a + \frac{1}{2} \omega_{cb} \Sigma^{bc}$ with $\Sigma^{ab} \equiv -\frac{1}{2} \gamma^{[a} \gamma^{b]}$, and

$$\tau^{abc} \equiv \frac{i}{4} \bar{\psi} \left( \gamma^c \Sigma^{ab} + \Sigma^{ab} \gamma^c \right) \psi = -\frac{i}{4} \bar{\psi} \gamma^{[a} \gamma^b \gamma^c] \psi$$

is the totally antisymmetric spin angular momentum. We again use a semi-teleparallel frame where $H_{ijj} = 0$ and determine the stationary point of $S_0 + S_D$ with respect to the remaining components of $H_{abc}$. We obtain

$$H_{0ij} = \tilde{K}_{0ij} + \frac{G}{\lambda - 1} \tau_{0ij}, \quad H_{ij0} = \tilde{K}_{ij0} + \frac{G}{\lambda - 1} \tau_{0ij}, \quad \tilde{K}^{ij} = 0,$$

where $\lambda = 1$ is excluded. In the case $\lambda = 1$, there only exists a stationary point if $\tau_{0ij} = 0$ for which case the consideration of the previous section apply. Inserting Eq. (53) into the action $S_0 + S_D$, we obtain

$$S_{STD} = -\frac{1}{2G} \int d^4x \sqrt{-g} \left( \hat{\circ} R + \hat{K}^{ijk} \hat{K}_{kji} - 3\lambda \hat{K}^{[0ij]} \hat{K}_{[0ij]} \right)$$

$$+ \int d^4x \sqrt{-g} \left[ \frac{1}{2} \left( \bar{\psi} \gamma^a \nabla_a \psi - \nabla_a \bar{\psi} \gamma^a \psi \right) - m \bar{\psi} \psi - \hat{K}^{ijk} \tau_{ijk} - \frac{3G}{2} \frac{\tau_{0ij} \tau_{0ij}}{\lambda - 1} \right]$$

$$= S_{STD} + \tilde{S}_D - \int d^4x \sqrt{-g} \left( \hat{K}^{ijk} \tau_{ijk} - \frac{3G}{2} \frac{\tau_{0ij} \tau_{0ij}}{\lambda - 1} \right),$$

where $\tilde{S}_D$ is the Dirac action in Riemannian space-time. Only the spatial part of the torsion couples to the spin angular momentum. Furthermore, we obtain a spin-spin contact interaction which contains the temporal part of the spin angular momentum and which depends on the parameter $\lambda$.  

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6 Discussion

The theory of gravitation proposed in this article relies on a space-time geometry that is formally a constrained Riemann-Cartan geometry and can be considered to be a mixture of Riemann-Cartan and teleparallel geometry. This semi-teleparallel geometry should, however, be seen as a separate geometry since — in contrast to other geometries — it incorporates the concept of preferred frames of reference in a natural way. Accordingly, theories of gravitation based on semi-teleparallel geometry are qualitatively different from Einstein-Cartan and teleparallel gravitation since they provide a dynamics of preferred frames, which is not contained in the last two.

We have not specified in this article a physical interpretation of the preferred frames. The formalism allows for several options. For example, the preferred vector field could be a Killing vector field. Another possible interpretation of the preferred frames is in terms of the rest frame of matter. In this case, however, one encounters large temporal torsion fields, in particular those associated with the shear of the velocity field.

Although we have considered in this article the four-dimensional case, a semi-teleparallel geometry can be defined on manifolds of arbitrary dimension. Furthermore, the congruence of worldlines can be generalized to congruences of extended objects of arbitrary dimension. The fibre bundle description of the semi-teleparallel geometry given in section 2 can be easily generalized and provides then a method for introducing a semi-teleparallel geometry on manifolds with several preferred vector fields. In this context, it should be noted that the Riemann-Cartan and the teleparallel geometry correspond to generalized semi-teleparallel geometries where the extended objects are the single space-time points and space-time itself, respectively. Physical applications for the case of higher-dimensional space-times are Kaluza-Klein theories the preferred frames being specified by spacelike Killing vectors. Since these vector fields are not dynamical, the corresponding semi-teleparallel theories of gravitation are simpler than the theories given in this article. The five-dimensional case leads to the Einstein-Cartan-Maxwell theory [19].

The definition of the preferred frame of reference and of the corresponding semi-teleparallel geometry is restricted to metric-compatible geometries in this article. It is possible to extend the formalism to metric-affine geometries by allowing the spatial triad to undergo $SL(3, \mathbb{R})$ transformations.

Further possible studies are the time evolution of the semi-teleparallel theories of gravitation within the canonical formalism and the semi-teleparallel gravitation in $2+1$ dimensions.
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