Autocovariance and autocorrelation structures of the generalised autoregressive moving average (GARMA(1,3;δ,1)) model

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INTRODUCTION

A time series is a sequence of observations, recorded at successive time intervals. In time series analysis, frequency-domain methods and time-domain methods are employed to analyze time series data to extract meaningful characteristics of the data. In time series forecasting, models are used to predict future values based on previously observed values. Three such classes of models that depend linearly on previous observations or/and residuals, are the autoregressive (AR) model, the moving average (MA) models and the integrated (I) models (Gershenfeld, 1999). The various combinations of these models produce the autoregressive moving average (ARMA) and autoregressive integrated moving average (ARIMA) models.

A time series, \( \{X_t, t = 0,1, \ldots \} \) is said to be strictly stationary if the joint distributions defined by \( \{X_t, \ldots, X_n \} \) and \( \{X_{t+h}, \ldots, X_{n+h}\} \) and do not change with time. A time series is weakly stationary when mean \( \mu(t) = \mathbb{E}(X_t) \) exists and constant for all \( t \), variance \( \text{Var}(X_t) < \infty \) for all \( t \), and the autocovariance \( \gamma_X(h) = \text{Cov}(X_t + h, X_t) \) depends only on the lag, \( h \). Weak stationarity is also implied with strict stationarity together with first and second moments (Brockwell and Davis, 2002).

The ARMA(\( p,q \)) model takes the following form,

\[
X_t - \theta_1X_{t-1} - \cdots - \theta_pX_{t-p} = Z_t + \phi_1Z_{t-1} + \cdots + \phi_qZ_{t-q} \tag{1}
\]

in a more concise form,

\[
\phi(B)X_t = \theta(B)Z_t \tag{2}
\]

where \( \{Z_t\} \sim WN(0, \sigma^2) \) and the polynomials \( \phi(z) = (1 + \phi_1z - \cdots - \phi_pz^p) \) and \( \theta(z) = (1 + \theta_1z + \cdots + \theta_qz^q) \) have no common factors. \( B \) is the backward shift character, such that \( BX_t = X_{t-1} \).

ARMA models are widely known to predict the behaviour of economic and industrial data. However ARMA type models could not be used to accommodate the changing frequency behaviour of time series data as it leads to a misclassification problem. In order to solve this problem, Peiris (2003) introduced the generalised autoregressive model of order one (GAR (1)); and Peiris et.al (2004) introduced the generalised moving average order one (GMA(1)). Pillai et. al (2009, 2012) studied the generalization of the standard ARMA (1,1) model, denoted by GARMA(1,1; \( \delta_1, \delta_2 \)) and defined by

\[
(1 - \alpha B)^{\delta_1}X_t = (1 - \beta B)^{\delta_2}Z_t \tag{3}
\]

where \(-1 < \alpha, \beta < 1, \delta_1 \geq 0, \delta_2 \geq 0 \).

Shitan and Peiris (2011) studied the behaviour of the GARMA(1,1; \( \delta_1 \)) process and the model is written as

\[
(1 - \alpha B)^{\delta_1}X_t = (1 - \beta B)Z_t, \tag{4}
\]

where \(-1 < \alpha, \beta < 1, \delta \geq 0 \) [8].

Some properties of the second order of GARMA model denoted by GARMA(1,2; \( \delta_1 \)) was examined by Pillai and Shitan (2014) and the process is written as

\[
(1 - \alpha B)^{\delta_1}X_t = (1 - \beta B)Z_t, \tag{5}
\]

where \(-1 < \alpha, \beta < 1, \delta \geq 0 \) [8].
where $-1 < \alpha, \beta_1, \beta_2, \beta_3 < 1$, $\delta \geq 0$ and $\{Z_t\} \sim WN(0, \delta^2)$.

This paper focuses on the third order of the GARMA model, that is $\text{GARMA}(1, 3; \delta, 1)$ and defined as

$$(1 - \alpha B)\delta X_t = (1 - \beta_1 B - \beta_2 B^2 - \beta_3 B^3)Z_t,$$  

(6)

where $|\alpha|, |\beta_i| < 1$, $i = 1, 2, 3$ and $\delta \geq 0$. Specifically the autocovariance and autocorrelation structure of the model will be derived explicitly.

The $\text{GARMA}(1, 3; \delta, 1)$ model can be written as equation

$$X_t = \sum \phi_j Z_{t-j}$$

where $\sum_j \phi_j Z_j = \varphi(B) = (1 - \alpha B)\delta(1 - \beta_1 B - \beta_2 B^2 - \beta_3 B^3)$. This is used to obtain the spectral density of this model as given by

$$f(w) = \frac{\sigma^2}{2\pi(1 - 2\alpha \cos \omega + \alpha^2)}\delta$$

(7)

where $-\pi \leq \omega \leq \pi$. The spectral density will be used in deriving the ACVF and hence the ACF structure of the $\text{GARMA}(1, 3; \delta, 1)$ model.

This paper is organized as follows: The following section presents the derived expressions for the ACVF and ACF of the model. The next section deals with a special case of $\text{GARMA}(1, 3; \delta, 1)$ model. This is followed by some numerical studies to illustrate the behaviour of the ACVF and ACF structures of the $\text{GARMA}(1, 3; \delta, 1)$ model.

**THE ACVF AND ACF OF THE $\text{GARMA}(1, 3; \delta, 1)$ MODEL**

This section provides the autocorrelation and autocovariance structures of the $\text{GARMA}(1, 3; \delta, 1)$ model. First the variance, $\gamma_0$ for the model in Equation (6) is given as

$$\gamma(0) = \sigma^2 \left[ (1 + \beta_1^2 + \beta_2^2 + \beta_3^2)F(\delta, \delta; 1; \alpha^2) \right.$$  

$$- 2(\beta_1 - \beta_1 \beta_2)\frac{\tau(\delta)(\tau(\delta)(2))}{\tau(\delta)(\tau(2))} \right.$$  

$$- 2(\beta_2 - \beta_2 \beta_3)\frac{\tau(\delta + 2)(\tau(\delta + 3))}{\tau(\delta)(\tau(3))} \right.$$  

$$- \beta_3\frac{\tau(\delta + 3)(\tau(\delta + 4))}{\tau(\delta)(\tau(4))}.$$

(8)

where $F(\theta_1, \theta_2, \theta_3; \theta)$ is the hypergeometric function given by

$$F(\theta_1, \theta_2, \theta_3; \theta) = \sum_{j=0}^{\infty} \frac{\tau(\theta_1)(j)!\tau(\theta_2)(j)!\tau(\theta_3)(j+1)!}{\tau(\theta_1)(\tau(\theta_2)(\tau(\theta_3)(\tau(j+1))}.$$

**Proposition 1**: The variance, $\gamma_0$ for the model in Equation (6) is given as

$$\gamma_k = \sigma^2 \left[ (1 + \beta_1^2 + \beta_2^2 + \beta_3^2)F(\delta, \delta; k + 1; \alpha^2) \right.$$  

$$- 2(\beta_1 - \beta_1 \beta_2)\frac{\tau(\delta)(\tau(\delta)(2))}{\tau(\delta)(\tau(2))} \right.$$  

$$- 2(\beta_2 - \beta_2 \beta_3)\frac{\tau(\delta + 2)(\tau(\delta + 3))}{\tau(\delta)(\tau(3))} \right.$$  

$$- \beta_3\frac{\tau(\delta + 3)(\tau(\delta + 4))}{\tau(\delta)(\tau(4))} \right.$$  

(9)

for $k \geq 1$.

**Proof**: Using the spectral density given in Equation (7) and $\gamma_k = \int_{-\pi}^{\pi} e^{i\omega k} f(w) \, dw$,

$$\gamma_k = 2 \int_{-\pi}^{\pi} \cos kw f(w) \, dw$$

(10)

$$(1 - \alpha B)\delta X_t = (1 - \beta_1 B - \beta_2 B^2 - \beta_3 B^3)Z_t,$$  

(5)

where $-1 < \alpha, \beta_1, \beta_2 < 1$, $\delta \geq 0$ and $\{Z_t\} \sim WN(0, \delta^2)$. specifically the autocovariance and autocorrelation structure of the model will be derived explicitly.

This paper focuses on the third order of the GARMA model, that is $\text{GARMA}(1, 3; \delta, 1)$ and defined as

$$X_t = \sum \phi_j Z_{t-j}$$

where $|\alpha|, |\beta_i| < 1$, $i = 1, 2, 3$ and $\delta \geq 0$. Specifically the autocovariance and autocorrelation structure of the model will be derived explicitly.

The $\text{GARMA}(1, 3; \delta, 1)$ model can be written as equation

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where $\sum_j \phi_j Z_j = \varphi(B) = (1 - \alpha B)\delta(1 - \beta_1 B - \beta_2 B^2 - \beta_3 B^3)$. This is used to obtain the spectral density of this model as given by

$$f(w) = \frac{\sigma^2}{2\pi(1 - 2\alpha \cos \omega + \alpha^2)}\delta$$

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where $-\pi \leq \omega \leq \pi$. The spectral density will be used in deriving the ACVF and hence the ACF structure of the $\text{GARMA}(1, 3; \delta, 1)$ model.

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**THE ACVF AND ACF OF THE $\text{GARMA}(1, 3; \delta, 1)$ MODEL**

This section provides the autocorrelation and autocovariance structures of the $\text{GARMA}(1, 3; \delta, 1)$ model. First the variance, $\gamma_0$ is presented in the following proposition.

**Proposition 1**: The variance, $\gamma_0$ for the model in Equation (6) is given as

$$\gamma(0) = \sigma^2 \left[ (1 + \beta_1^2 + \beta_2^2 + \beta_3^2)F(\delta, \delta; 1; \alpha^2) \right.$$  

$$- 2(\beta_1 - \beta_1 \beta_2)\frac{\tau(\delta)(\tau(\delta)(2))}{\tau(\delta)(\tau(2))} \right.$$  

$$- 2(\beta_2 - \beta_2 \beta_3)\frac{\tau(\delta + 2)(\tau(\delta + 3))}{\tau(\delta)(\tau(3))} \right.$$  

$$- \beta_3\frac{\tau(\delta + 3)(\tau(\delta + 4))}{\tau(\delta)(\tau(4))} \right.$$  

(8)

where $F(\theta_1, \theta_2, \theta_3; \theta)$ is the hypergeometric function given by

$$F(\theta_1, \theta_2, \theta_3; \theta) = \sum_{j=0}^{\infty} \frac{\tau(\theta_1)(j)!\tau(\theta_2)(j)!\tau(\theta_3)(j+1)!}{\tau(\theta_1)(\tau(\theta_2)(\tau(\theta_3)(\tau(j+1))}.$$

**Proof**: The spectral density given in Equation (7) gives

$$\gamma_0 = \int_{-\pi}^{\pi} f(w) \, dw$$

and

$$\gamma_k = \sigma^2 \left[ (1 + \beta_1^2 + \beta_2^2 + \beta_3^2)F(\delta, \delta; k + 1; \alpha^2) \right.$$  

$$- 2(\beta_1 - \beta_1 \beta_2)\frac{\tau(\delta)(\tau(\delta)(2))}{\tau(\delta)(\tau(2))} \right.$$  

$$- 2(\beta_2 - \beta_2 \beta_3)\frac{\tau(\delta + 2)(\tau(\delta + 3))}{\tau(\delta)(\tau(3))} \right.$$  

$$- \beta_3\frac{\tau(\delta + 3)(\tau(\delta + 4))}{\tau(\delta)(\tau(4))} \right.$$  

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for $k \geq 1$.

**Proof**: Using the spectral density given in Equation (7) and $\gamma_k = \int_{-\pi}^{\pi} e^{i\omega k} f(w) \, dw$,

$$\gamma_k = 2 \int_{-\pi}^{\pi} \cos kw f(w) \, dw$$

(10)
Since $2 \cos(kw) \cos(w) = \cos((k+1)w) + \cos((k-1)w)$,

$$= \sigma^2 \left[ \left(1 + \beta_1^2 + \beta_2^2 + \beta_3^2 \right) \int_0^\pi \frac{\cos kw}{\left(1 - 2a \cos w + a^2\right)^{\delta}} \, dw 
- (\beta_1 - \beta_1 \beta_2) \int_0^\pi \frac{\cos(k+1)w + \cos(k-1)w}{\left(1 - 2a \cos w + a^2\right)^{\delta}} \, dw 
- (\beta_2 - \beta_1 \beta_3) \int_0^\pi \frac{\cos(k+2)w + \cos(k-2)w}{\left(1 - 2a \cos w + a^2\right)^{\delta}} \, dw 
- \beta_3 \int_0^\pi \frac{\cos(k+3)w + \cos(k-3)w}{\left(1 - 2a \cos w + a^2\right)^{\delta}} \, dw \right]$$

for $k \geq 1$. This completes the proof for $\gamma_k$.

By definition, the autocorrelation, $\rho_k$ takes the form

$$\rho_k = \frac{\gamma(k)}{\gamma(0)}. \quad (10)$$

**SPECIAL CASE**

It can be seen that when $\beta_3 = 0$, the variance expressions takes the form

$$\gamma(0) = \sigma^2 \left[ \left(1 + \beta_1^2 + \beta_2^2 + 0\right) F(\delta, \delta; 1; a^2) 
- 2(\beta_1 - \beta_1 \beta_2) \frac{\alpha \tau(1 + \delta)}{\tau(1 + \delta)^2} \right]$$

The variance structure obtained in Equation (11) is the variance structure of GARMA(1,2; $\delta$, 1) as seen in Pillai and Shitan (2014).
From Equation (9), when $\beta_3 = 0$,

$$
\gamma_k = \frac{\sigma^2}{\tau(\delta)} \left[ (1 + \beta_1^2 + \beta_2^2 + 0) \alpha \tau(k + \delta) F(\delta, k + \delta; k + 1; \alpha^2) \right. \\
- \left. (\beta_1 - \beta_1 \beta_2 - \beta_2, 0) \alpha^{k-1} \tau(k + 1 + \delta) F(\delta, k + 1 + \delta; k + 2; \alpha^2) \tau(k + 2) \\
- (\beta_1 - \beta_1 \beta_2 - \beta_2, 0) \alpha^{k-1} \tau(k - 1 + \delta) F(\delta, k - 1 + \delta; k; \alpha^2) \tau(k) \\
- (\beta_2 - \beta_1, 0) \alpha^{k+2} \tau(k + 2 + \delta) F(\delta, k + 2 + \delta; k + 3; \alpha^2) \tau(k + 3) \\
- (\beta_2 - \beta_1, 0) \alpha^{k-2} \tau(|k - 2| + \delta) F(\delta, |k - 2| + \delta; |k - 2| + 1; \alpha^2) \tau(|k - 2| + 1) \\
- \left. 0 \alpha^{k+3} \tau(k + 3 + \delta) F(\delta, k + 3 + \delta; k + 4; \alpha^2) \tau(k + 4) \\
- \left. 0 \alpha^{k+3} \tau(|k - 3| + \delta) F(\delta, |k - 3| + \delta; |k - 3| + 1; \alpha^2) \tau(|k - 3| + 1) \right]
$$

for $k \geq 1$. The autocovariance structure obtained in Equation (12) is similar to the autocovariance structure of ARMA$(1,2;\delta,1)$.

Hence, the variance and autocovariance of $\text{ARMA}(1,3;\delta,1)$ greatly reduces to the autocovariance of $\text{ARMA}(1,2;\delta,1)$ when $\beta_3 = 0$. Furthermore, when $\delta = 1$, Propositions 1 and 2 takes the form of the variance and autocovariance of ARMA$(1,3)$.

**NUMERICAL EXAMPLES**

This section presents the numerical results of the variance using Proposition 1 and the autocorrelation (ACF) using Proposition 2 of the $\text{ARMA}(1,3;\delta,1)$ model for selected values of the parameters.

Table 1 shows the computed results of the variance using Proposition 1 when $\alpha = 0.9, \beta_1 = 0.3, \beta_2 = 0.2, \beta_3 = 0.2$ and $\sigma^2 = 1$ with $\delta$ ranging from 0.2 to 1.8. It can be seen that the variance is reducing for $\delta = 0.2$ to $\delta = 0.4$ and increasing for $\delta = 0.6$ to $\delta = 1.8$.

| $\delta$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.4 |
|----------|-----|-----|-----|-----|-----|-----|
| $\gamma_k$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

Table 2 and 3 show the tabulated values of the ACF function of $\text{ARMA}(1,3;\delta,1)$ model, using Proposition 2 and $\rho_k = \gamma_k / \tau(\delta)$ with $k$ ranging from 1 to 20 and $\delta$ ranging from 0.2 to 1.8. Table 2 shows the computed results of the ACF when $\alpha = 0.9, \beta_1 = 0.3, \beta_2 = 0.2, \beta_3 = 0.2$ and $\sigma^2 = 1$, whereas Table 3 shows the computed results of the ACF when $\alpha = 0.9, \beta_1 = 0.3, \beta_2 = 0.1, \beta_3 = 0.1$ and $\sigma^2 = 1$.

| $\rho_k$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.4 |
|----------|-----|-----|-----|-----|-----|-----|
| $\rho_k$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

| $\delta$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.4 |
|----------|-----|-----|-----|-----|-----|-----|
| $\gamma_k$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

**Table 2** The values of $\rho_k$ ($\alpha = 0.9, \beta_1 = 0.3, \beta_2 = 0.2, \beta_3 = 0.2, \sigma^2 = 1$).

| $k$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.4 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\rho_k$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

**Table 3** The values of $\rho_k$ ($\alpha = 0.9, \beta_1 = 0.3, \beta_2 = 0.1, \beta_3 = 0.1, \sigma^2 = 1$).

| $k$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.4 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\rho_k$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

**Table 4** The values of $\rho_k$ ($\alpha = 0.9, \beta_1 = 0.2, \beta_2 = 0.1, \beta_3 = 0.1, \sigma^2 = 1$).

| $k$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.4 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\rho_k$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
The tabulated values in Table 2, 3 and 4 are used to plot the ACF graphs given in Figures 1, 2 and 3 respectively.

Figure 1 ACF Plot of GARMA(1,3; 𝛿,1) when 𝛼 = 0.9, 𝛽₁ = 0.3, 𝛽₂ = 0.2, 𝛽₃ = 0.2 and σ² = 1.

Figure 2 ACF Plot of GARMA(1,3; 𝛿,1) when 𝛼 = 0.9, 𝛽₁ = 0.3, 𝛽₂ = 0.1, 𝛽₃ = 0.1 and σ² = 1.

Figure 3 ACF Plot of GARMA(1,3; 𝛿,1) when 𝛼 = 0.9, 𝛽₁ = 0.2, 𝛽₂ = 0.5, 𝛽₃ = 0.1 and σ² = 1.

Figure 2 also shows that when 𝛿 values increase, the ACF values increase as well. The ACF values are negative in nature when 𝛿 is 0.2 and 0.4. When 𝛿 is 0.6, 0.8, 1.0, 1.4 and 1.8, all the ACF values are positive in nature.

Similar to Figure 1 and Figure 2, Figure 3 shows that the ACF values increase when the 𝛿 values increase. The ACF values in Figure 3 are negative when 𝛿 values are 0.2, 0.4, 0.6 and 0.8. The positive ACF values can be seen when 𝛿 values are 1.0, 1.4 and 1.8.

The observations in Figure 1, Figure 2 and Figure 3 are not seen in GARMA(1,2; 𝛿,1), as in Pillai (2012), as there were mixed positive and negative values of ACF for the same 𝛿 values. As in the case of Figure 1, Figure 2 and Figure 3 also show that when 𝛿 is 1.4 and 1.8, the ACF values are decreasing gradually. Gradual decrease of ACF in GARMA(1,2; 𝛿,1) can only be seen when 𝛿 = 1.8. In addition, when 𝛿 > 1, ACF of the corresponding models are above the standard ARMA(1,3) model. Compared with the ACF values of GARMA(1,2; 𝛿,1), the ACF values of the GARMA(1,3; 𝛿,1) model start to decrease more gradually from a lower 𝛿 value.

From the above figures, we notice that the GARMA(1,3; 𝛿,1) model can represent various structures of the ACF functions. Therefore, this model might serve as an alternative for modeling time series data.

CONCLUSION

The univariate ARMA model can be extended to a class of GARMA models. The objective of this paper was to derive the variance and the autocovariance properties of the GARMA(1,3; 𝛿,1) as was shown in the propositions. The special cases presented also shows that the GARMA(1,3; 𝛿,1) model can be reduced to the ARMA(1,3) structure. From the numerical studies, it shows that the GARMA(1,3; 𝛿,1) model can be used to represent various structures of the ACF functions. The results of this study contribute to the theory of the general order GARMA model.

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