COMPACT EMBEDDINGS AND INDEFINITE SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. Our purpose is to find positive solutions $u \in D^{1,2}(\mathbb{R}^N)$ of the semilinear elliptic problem $-\Delta u = h(x)u^{p-1}$ for $2 < p$. The function $h$ may have an indefinite sign. Key ingredients are a $h$-dependent concentration-compactness Lemma and a characterization of compact embeddings of $D^{1,2}(\mathbb{R}^N)$ into weighted Lebesgue spaces.

1. INTRODUCTION

We are interested in finding weak nonnegative solutions of Emden-Fowler type problems

$$-\Delta u - h(x)u^{p-1} = 0 \quad \text{in } \mathbb{R}^N,$$

$$0 \leq u \in E := D^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, |h|).$$

From now on we make the assumption:

$$N \geq 3, \quad p > 2 \quad \text{and} \quad h \in L^1_{\text{loc}}: \quad h^+(x) := \max(0, h(x)) \neq 0. \quad (1.2)$$

We denote by $D^{1,2}(\mathbb{R}^N)$ the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $(\int |\nabla u|^2)^{\frac{1}{2}}$ in $L^2$. Moreover, $L^p(\Omega, |h|)$ denotes the space of measurable functions $u$ satisfying $\|u\|_{L^p(\Omega, |h|)} := \int_{\Omega} |h| |u|^p = \| \chi_{\Omega} |h|^\frac{1}{p} u \|_p^p < \infty$.

$E$ is a Banach space equipped with the norm $\|u\|_E := \|\nabla u\|_2 + \|u\|_{L^p(\mathbb{R}^N, |h|)}$.

Furthermore, we assume $h$ to be symmetric with respect to some compact subgroup $G$ of $O(N)$, the group of orthogonal linear transformations in $\mathbb{R}^N$, i.e.

$$(g, h)(x) := h(g^{-1}x) = h(x) \quad \forall g \in G \quad \text{a.e. in } \mathbb{R}^N. \quad (1.3)$$

We denote by $D_G^{1,2}(\mathbb{R}^N)$ the subspace of $D^{1,2}(\mathbb{R}^N)$ consisting of all $G$-symmetric functions and define $E_G := D_G^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, |h|)$.

The basic requirements on the positive part of $h$, $h^+ := \max(0, h)$, are:

There is a $G$-symmetric $u \in C_c^\infty(\mathbb{R}^N)$: $\int h|u|^p > 0$, \quad (1.4)

$D_G^{1,2}(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N, h^+)$. \quad (1.5)

There have been many studies of the equation in (1.1), mostly for radially symmetric non-negative functions $h$. We shall mention among them the work of Ding and Ni [8], Gidas and Spruck [9], Kusano and Naito [10], Noussair and Swanson [15, 16, 17].

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Tshinanga [24] (see also [15]) proved without any symmetry assumptions for nonnegative functions $h$ the existence of a solution to (1.1) if

$$0 \neq h(x) \leq \frac{C}{(1 + |x|^2)^a} \cdot \frac{2N - 2a}{N - 2} \cdot \frac{2N - 2}{N - 2} \quad \text{for some } C > 0, \ 0 < a < 2. \quad (1.6)$$

Noussair and Swanson [17] obtained a solution of (1.1) for nonnegative $h$ if

$$2 < p < 2^*, \ 0 \neq h \in L^q \cap L^{\infty}: \ 1 < q < \frac{2^*}{2^* - p}. \quad (1.7)$$

Rother [20] solved (1.1) for sign changing, radially symmetric functions $h$ if

$$h \in L^1_{\text{loc}}, \ 0 \neq h^+(|x|) = k_1(|x|) + k_2(|x|) \text{ for some } k_1, k_2 \in L^1_{\text{loc}},$$

$$\exists f \in L^\infty: \ 0 \leq k_1(|x|) \leq f(x)|x|^{(N-2)p-2N} \quad \text{and} \quad f(x) \frac{|x|^{N-2}}{x \to 0} \to 0,$$

$$k_2(|x|) \text{ is nonnegative and } \int_0^\infty k_2(r)r^{N-1-p}N-2 dr < \infty. \quad (1.8)$$

We generalize the above results to possibly sign-changing and non-radial functions $h$. Weak solutions of (1.1) correspond to nonnegative critical points of the associated energy functional $I \in C^1(E, \mathbb{R}) \cap C^1(E, \mathbb{R})$ defined by

$$I(u) := \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p} \int h(x)|u|^p.$$

From (1.4) and (1.7) it may be concluded that

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0, \text{ where } \Gamma := \{ \gamma \in C([0,1], E, \mathbb{R}) \mid \gamma(0) = 0, I(\gamma(1)) < 0 \}.$$ 

Thus the mountain pass Theorem provides a $(PS)_c$ sequence, i.e. a sequence $(u_n)_{n \in \mathbb{N}}$ satisfying

$$I(u_n) \to c, \ I'(u_n) \to 0 \text{ as } n \to \infty.$$

We shall show that if $D^{1,2}_G(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, h^+)$ is compact, then every $(PS)_c$ sequence contains a convergent subsequence. Consequently we have

**Theorem 4.3.** Suppose (1.2), (1.3) hold. If $D^{1,2}_G(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, h^+)$ is compact, then (1.4) has a nontrivial, nonnegative weak solution.

Section 2 is devoted to the study of embeddings of $D^{1,2}(\mathbb{R}^N)$ into weighted Lebesgue spaces, e.g. we prove

**Theorem 2.1.** Suppose $k \in L^1_{\text{loc}}$ is a nonnegative function and $q \geq 2$. Then $D^{1,2}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N, k)$ if and only if the following three conditions are satisfied:

$$\sup_{x \in \mathbb{R}^N} \rho^{(1 - \frac{N}{2})q} \int_{B_\rho(x)} k < \infty, \sup_{x \in \mathbb{R}^N} \rho^{(1 - \frac{N}{2})q} \int_{B_\rho(x) \setminus B_\rho(0)} k \xrightarrow{R \to \infty} 0, \sup_{x \in \mathbb{R}^N} \rho^{(1 - \frac{N}{2})q} \int_{B_\rho(x)} k \xrightarrow{\delta \to 0} 0.$$

Theorem 2.1 and Theorem 4.3 generalize the above existence results for (1.1) obtained in [15, 20, 24], because (1.4), (1.5) and (1.8) are sufficient for the compactness of the inclusion of $D^{1,2}_G(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N, h^+)$, as it is shown in Corollary 2.2 and in [2], Lem. 6.

To deal with the non-compact case we follow the notation of Smets in [23], where the linear
If $D^{1,2}({\mathbb R}^N)$ is embedded in $L^q({\mathbb R}^N, k)$, all these quantities are bounded away from zero, however, some may be infinite, e.g. we have

**Corollary 3.2.** Suppose $k \in L^1_{loc}$ is nonnegative, $q > 2$ and $D^{1,2}({\mathbb R}^N)$ is embedded in $L^q({\mathbb R}^N, k)$. Then

$$S'_k = S'_k = \infty \quad \text{if and only if} \quad D^{1,2}({\mathbb R}^N) \hookrightarrow L^q({\mathbb R}^N, k) \quad \text{is compact.}$$

Let us introduce the compactness threshold $c_0$, defined by

$$c_0 := \left( \frac{1}{2} - \frac{1}{p} \right) \inf_{x \in {\mathbb R}^N \cup \{ \infty \}} \left\{ |G_x|(S^{x}_{h^+})^{\frac{p}{p-2}} \right\},$$

where $|G_x| = \# \{ gx \in G \}$ and $|G_\infty| := 1$. With the help of a concentration compactness Lemma, given in Section 3, it is possible to show that every $(PS)_c$-sequence contains a convergent subsequence if $c < c_0$. This is done in Section 4 and leads to

**Theorem 4.4.** Suppose $D^{1,2}({\mathbb R}^N)$ is continuously embedded in $L^p({\mathbb R}^N, h^+)$ and there is an $u \in E_G$ such that

$$\int h |u|^p > 0 \quad \text{and} \quad \max_{0 \leq t < \infty} I(tu) \leq c_0.$$

Then (1.1) is solvable.

Our approach is related to the work of Bianchi, Chabrowski and Szulkin [5], where the case $p = 2^*$ and $h \in L^\infty$ was considered. Our results for $p = 2^*$ are slight improvements of [5], because in our setting $h^-$ does not need to be bounded.

Section 5 presents some examples illustrating our results, e.g.

**Corollary 5.3.** Consider the equation

$$-\Delta u = (1 + |x|)^{-\delta}|u|^{p-2}u, \quad 0 \neq u \in D^{1,2}({\mathbb R}^N) \cap L^p({\mathbb R}^N, (1 + |x|)^{-\delta}).$$

1. (1.10) has no solution $u \in C^2({\mathbb R}^N)$ if $2 < p < 2^*$ and $\delta \leq N - \frac{p}{2}(N - 2)$.
2. (1.10) has infinitely many $C^2({\mathbb R}^N)$-solutions if $2 < p < 2^*$ and $\delta > N - \frac{p}{2}(N - 2)$. At least one solution is strictly positive in $\mathbb{R}^N$.

2. Compactness

$D^{1,2}({\mathbb R}^N)$ is embedded in $L^{2^*}({\mathbb R}^N)$ but not in $L^q({\mathbb R}^N)$ for any other $q$. However we have an embedding if we replace $L^q({\mathbb R}^N)$ by a suitable weighted Lebesgue space. Results concerning existence or compactness of such embeddings are obtained by Mazja [14], Adams [1], Berger
Consider the operator $I$. Let $D$ be outside $B_{\text{ext}}$. Because the inclusion of $D$ is compact. We use again (2.1) to get
\[ D^{1,2}(R) \hookrightarrow W^{1,2}(B_{R}(0)) \hookrightarrow L^{q}(B_{R}(0), k) \rightarrow L^{q}(R^{N}, k) \]
with compactness in the middle and where $ext$ is understood by extending the function by zero outside $B_{R}(0)$.

Consider the operator $I_{R}$ defined by $I_{R}(u) := u \cdot \eta(\frac{x}{R})$, where $R > 0$, $\eta \in C^{1}(R^{N}, [0, 1])$ with $\eta|_{B_{1}(0)} \equiv 1$ and compact support in $B_{2}(0)$. Then
\[ D^{1,2}(R^{N}) \xrightarrow{I_{R}} D^{1,2}(B_{2R}(0)) \hookrightarrow L^{q}(R^{N}, k) \]
is compact. We use again (2.1) to get
\[ \|k^{\frac{1}{q}}(u - I_{R}(u))\|_{q} \leq \|k^{\frac{1}{q}}u\|_{q} \]
\[ \leq c_{1}^{-1} \left( \sup_{x \in R^{N}} \rho^{(1 - \frac{N}{2})}k \int_{B_{\rho}(x)} k \right)^{\frac{1}{q}} \|\nabla u\|^{2} = o(1)\|\nabla u\|^{2} \]
as $R \rightarrow \infty$. Thus the inclusion of $D^{1,2}(R^{N})$ in $L^{q}(R^{N}, k)$ is compact as a limit of compact operators.

Proof (necessity). Because the inclusion of $D^{1,2}(R^{N})$ in $L^{q}(R^{N}, k)$ is bounded (2.3) holds. Let $D := \{u \in D^{1,2}(R^{N}) | \|\nabla u\|_{2} \leq 1\}$. The set $D$ is relatively compact in $L^{q}(R^{N}, k)$. Let
\( \varepsilon > 0 \). The relative compactness gives rise to
\[
\exists R > 0 : \int_{\mathbb{R}^N \setminus B_R(0)} k \cdot |u|^q \leq \varepsilon \quad \forall u \in D \tag{2.6}
\]
\[
\exists \delta > 0 : \int_{B_{\delta}(x)} k \cdot |u|^q \leq \varepsilon \quad \forall u \in D, \quad \forall x \in \mathbb{R}^N. \tag{2.7}
\]
We show only (2.7): We suppose the contrary and get a sequence \((\varepsilon > 0)\). The relative compactness gives rise to
\[
\text{Suppose Theorem 2.3.}
\]

As an easy consequence of Theorem 2.1 (see [21]) we obtain
\[
(2.8)
\]

Let \( \Omega \subset \mathbb{R}^N \) and \( \rho > 0 \). Consider \( u_{\rho,x}(y) := \eta(\frac{y-x}{\rho}) \). We obtain
\[
\int_{B_\rho(x) \setminus \Omega} k \cdot |u_{\rho,x}|^q \leq \int_{B_\rho(x) \setminus \Omega} k \frac{|u_{\rho,x}|^q}{\|\nabla u_{\rho,x}\|_2^q} \|\nabla u_{\rho,x}\|_2^q
\]
and observe that \( \frac{u_{\rho,x}}{\|\nabla u_{\rho,x}\|_2} \in D \). If we take \( B_R(0) \subset \Omega \), we get (2.4) and if we take \( \Omega = \emptyset \) and \( 0 < \rho < \delta \), we get (2.5).

As an easy consequence of Theorem 2.1 (see [21]) we obtain

**Corollary 2.2.** Suppose \( 2 < q < 2^* \) and \( k \in L^1_{\text{loc}} \) is a nonnegative function. Then \( D^{1,2}(\mathbb{R}^N) \rightarrow L^q(\mathbb{R}^N, k) \) is compact if one of the following conditions is satisfied
\[
\int k^{\frac{q}{2} - \frac{q}{q}} < \infty, \tag{2.8}
\]
\[
\exists f \in L^\infty : f(x) \xrightarrow{|x| \to \infty} 0 \quad \text{and} \quad k(x) \leq f(x)|x|^{(N-2)q-2N} \tag{2.9}
\]

The following Theorem gives some conditions ensuring that \( D^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, h) \) is compactly embedded in \( L^q(\mathbb{R}^N, k) \). Furthermore, it leads to sufficient conditions for \( D^{1,2}(\mathbb{R}^N) \) to be embedded in \( L^q(\mathbb{R}^N, k) \) for \( 1 \leq q \leq 2 \) (see Corollary 2.4 below).

**Theorem 2.3.** Suppose \( \min(p, 2^*) > q \geq 1 \) and \( h, k \) are nonnegative measurable functions which satisfy
\[
\exists R > 0 : \quad h > 0 \text{ almost everywhere in } \Omega_k \setminus B_R(0),
\]
\[
k \in L^q_{\text{loc}}(\mathbb{R}^N) \text{ and } \int_{\Omega_k \setminus B_R(0)} \left[ k \left( \frac{k}{h} \right)^{\frac{q-2}{q}} \right]^{\frac{2^*}{2(q-2)}} < \infty
\]
for some \( 0 \leq z \leq q \), where \( \Omega_k := \{ x \in \mathbb{R}^N : k(x) \neq 0 \} \).

Then \( D^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, h) \) is compactly embedded in \( L^q(\mathbb{R}^N, k) \).
Proof. We follow [3, Lem 2.3] and use the elementary upper bound for \( r > s \geq 0, \, u \in \mathbb{R}, \, k \geq 0 \) and \( h > 0 \):
\[
k|u|^s - h|u|^r \leq C(r, s) k \left( \frac{k}{h} \right)^{\frac{s-r}{r-s}}.
\] (2.10)

We use Theorem 2.1 to see that \( D^{1,2}(B_R(0)) \) is compactly embedded in \( L^q(\mathbb{R}^N, k) \). If \( q > 2 \), we may factorize the inclusion as follows
\[
D^{1,2}(B_R(0)) \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow \text{compact} \ L^q(\mathbb{R}^N, k) \hookrightarrow \text{compact} \ L^q(\mathbb{R}^N, k).
\]
If \( q \leq 2 \), then we fix \( q_1 \) between 2 and \( 2^* \) and notice that
\[
k_1 := k^{\frac{2^*-q_1}{2^*-q}} \cdot \chi_{B_R(0)} \in L^{\frac{2^*}{2^*-q_1}}.
\]
Hence we may write the inclusion of \( D^{1,2}(B_R(0)) \) in \( L^q(\mathbb{R}^N, k) \) as
\[
D^{1,2}(B_R(0)) \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow \text{compact} \ L^{q_1}(\mathbb{R}^N, k_1) \hookrightarrow \text{compact} \ L^q(\mathbb{R}^N, k).
\]
The last multiplication operator is bounded due to Hölder’s inequality.
Using again \( I_R \) as in the proof of Theorem 2.1, we see
\[
D^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, h) \overset{I_R}{\longrightarrow} D^{1,2}(B_{2R}(0)) \hookrightarrow \text{compact} \ L^q(\mathbb{R}^N, k)
\]
is also compact.
The bound in (2.10) allows to calculate for all \( \epsilon > 0 \) and \( u \in D^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, h) \) with \((\|\nabla u\|_2^2 + \|h^p u\|_p^2)^{\frac{1}{2}} = 1\):
\[
\begin{aligned}
\int_{\mathbb{R}^N \setminus B_R(0)} k|u - I_R(u)|^q &\leq \int_{\mathbb{R}^N \setminus B_R(0)} k|u|^q = \int_{\mathbb{R}^N \setminus B_R(0)} (k|u|^q - \epsilon h|u|^p) + \epsilon \int_{\mathbb{R}^N \setminus B_R(0)} h|u|^p \\
&\leq \int_{\mathbb{R}^N \setminus B_R(0)} (k|u|^{q-z} - \epsilon h|u|^{p-z})|u|^2 + \epsilon \int_{\mathbb{R}^N \setminus B_R(0)} h|u|^p \\
&\leq \epsilon^{-\frac{2-z}{p-q}} C(p, q, z) \int_{\Omega_k \setminus B_R(0)} \left[ k \left( \frac{k}{h} \right)^{\frac{q-z}{p-q}} \right] |u|^2 + \epsilon \int_{\mathbb{R}^N \setminus B_R(0)} h|u|^p \\
&\leq \epsilon^{-\frac{2-z}{p-q}} C(p, q, z) \left( \int_{\Omega_k \setminus B_R(0)} \left[ k \left( \frac{k}{h} \right)^{\frac{q-z}{p-q}} \right] \frac{2^*}{2^*-z} \right) \frac{2^*}{2^*-z} + \epsilon \int_{\mathbb{R}^N \setminus B_R(0)} h|u|^p \\
&\leq \epsilon^{-\frac{2-z}{p-q}} C(p, q, z) \left( \int_{\Omega_k \setminus B_R(0)} \left[ k \left( \frac{k}{h} \right)^{\frac{q-z}{p-q}} \right] \frac{2^*}{2^*-z} \right) \frac{2^*}{2^*-z} + \epsilon.
\end{aligned}
\]

The integral term tends to zero for \( R \to \infty \).
Hence the inclusion of \( D^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, h) \) in \( L^q(\mathbb{R}^N) \) is compact as a limit of compact operators. \( \square \)
Corollary 2.4. Suppose $1 \leq q < 2^*$ and $k \in L^2_{loc} \subseteq \mathbb{R}^N$ is a nonnegative function. Then $D^{1,2}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N, k)$ under the following condition

$$1 \leq q < p := \frac{2(N-\delta)}{N-2} \text{ and } \int k^{\frac{N}{q'}} |x|^{\frac{\delta}{q-1}} < \infty \text{ for some } \delta : 0 \leq \delta \leq 2. \quad (2.11)$$

Proof. $D^{1,2}(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N, |x|^{-\delta})$ if (see (5.1) below)

$$0 \leq \delta \leq 2 \text{ and } p = \frac{2(N-\delta)}{N-2}.$$ 

Consequently (2.11) and Theorem 2.3 with $h(x) := |x|^{-\delta}$ and $z = 0$ implies

$$D^{1,2}(\mathbb{R}^N) = D^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, |x|^{-\delta}) \hookrightarrow \text{compact } L^q(\mathbb{R}^N, k). \quad \square$$

3. A concentration compactness Lemma

The following Lemma, which is closely related to [23, Lem. 2.1] and [13, Lem. I.1], analyses the possible non-compactness of an embedding of $D^{1,2}(\mathbb{R}^N)$ in $L^q(\mathbb{R}^N, k)$ in terms of the quantities $S^x_k$ and $S^\infty_x$ defined in (1.9).

Lemma 3.1. Suppose $q > 2$ and $D^{1,2}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N, k)$ for some nonnegative $k \in L^1_{loc}$. Furthermore, let $(u_n)_{n \in \mathbb{N}}$ be bounded in $D^{1,2}(\mathbb{R}^N)$. Up to a subsequence we may assume: $u_n \rightharpoonup u$ weakly in $D^{1,2}(\mathbb{R}^N)$ and additionally $\nabla u_n - \nabla u \rightharpoonup \mu$, $|\nabla u_n|^2 \rightharpoonup k |u_n|^q \rightharpoonup \nu$ and $k |u_n - u|^q \rightharpoonup 0$ weakly in the sense of measures, where $\mu$, $\tilde{\mu}$ and $\nu$ are bounded nonnegative measures. Define

$$\mu_\infty := \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} |\nabla u_n|^2,$$

$$\nu_\infty := \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} k |u_n|^q.$$

Then

1. $\mu_\infty \geq S^\infty_x \nu^{2/q}_\infty,$

2. There exists an at most countable set $J$, a family $\{x_j \mid j \in J\}$ of distinct points in $\mathbb{R}^N$ and a family $\{\nu_j \mid j \in J\}$ of positive numbers such that

$$\nu = k |u|^q \, dx + \sum_{j \in J} \nu_j \delta_{x_j},$$

where $\delta_x$ is the Dirac measure of mass 1 concentrated at $x \in \mathbb{R}^N$.

3. There holds

$$\mu \geq |\nabla u|^2 \, dx + \sum_{j \in J} \mu_j \delta_{x_j},$$

where $\mu_j \geq S^x_k \nu^{2/q}_j$ for all $j \in J$,

4. $\limsup_{n \to \infty} \|k^{1/q} u_n\|_q^q = \|k^{1/q} u\|_q^q + \sum_{j \in J} \nu_j + \nu_\infty.$
Proof. Let \( \{x_j | j \in J\} \) be the atoms of \( \nu \) and decompose \( \nu = \nu_0 + \sum_{j \in J} \nu_j \delta_{x_j} \), where \( \nu_0 \) is nonnegative and free of atoms. Because \( \int d\tilde{\nu} < \infty \), \( J \) is at most countable. For each \( x \in \{x_j | j \in J\} \) there is a sequence (\( r_{n,k} \)) of positive numbers converging to zero such that
\[
S^x_{r_{n,k}} \geq \left\{ \begin{array}{ll}
S^x_k \frac{1}{l} & S^x_k < \infty \\
S^x_k = \infty
\end{array} \right.
\]
Letting \( \nu < C_{\infty}(\mathbb{R}) \) with \( \|\psi\|_\infty = 1 = \psi_l(x) \), then
\[
\tilde{\mu}(\{x\}) = \lim_{l \to \infty} \tilde{\mu}(\psi_l^2) = \lim_{l \to \infty} \lim_{n \to \infty} \int |\nabla (u_n - u)|^2 \psi_l^2 \\
= \lim_{l \to \infty} \lim_{n \to \infty} \int |\nabla ((u_n - u)\psi_l)|^2 \quad \text{(because } u_n \to u \text{ in } L_{\text{loc}} \text{)} \\
\geq \lim_{l \to \infty} \left\{ S^x_{r_{n,k}} \limsup_{n \to \infty} \left( \int |k|u_n - u|^q \psi_l^q \right) \right\}^{2/q} \\
= \lim_{l \to \infty} S^x_{r_{n,k}} \tilde{\nu}(\psi_l^q)^{2/q} = S^x_k \tilde{\nu}(\{x\})^{2/q}.
\]
The above calculation also shows that
\[
\left( \int |\psi|^q d\tilde{\nu} \right)^{2/q} \leq C \int |\psi|^2 d\tilde{\mu} \quad \forall \psi \in C_c(\mathbb{R}^N).
\] (3.1)
This implies that \( \nu_0 \) is absolutely continuous with respect to \( \tilde{\mu} \). By the Radon-Nikodym Theorem there is a nonnegative \( f \in L^1(\mathbb{R}^N, d\tilde{\mu}) \) such that \( d\nu_0 = fd\tilde{\mu} \) and for \( \tilde{\nu} \)-almost every \( x \in \mathbb{R}^N \)
\[
f(x) = \lim_{r \to 0} \left( \frac{\nu_0(B_r(x))}{\tilde{\mu}(B_r(x))}\right).
\]
If \( x \) is not an atom of \( \tilde{\mu} \), we use (3.1) to get
\[
f(x)^{2/q} = \lim_{r \to 0} \left( \frac{\nu_0(B_r(x))^{2/q}}{\tilde{\mu}(B_r(x))^{2/q}} \right) \leq C \lim_{r \to 0} \tilde{\mu}(B_r(x))^{\frac{2}{2q}} = 0.
\]
Because the atoms of \( \tilde{\mu} \) are at most countable and \( \nu_0 \) has no atoms, we see \( \nu_0 = 0 \). We use the inequality \( |(a - b)^2 - a^2| \leq \epsilon(a - b)^2 + c(\epsilon)b^2 \) to derive
\[
\int |\nabla(u_n - u)|^2 \psi_l^2 - \int |\nabla u_n|^2 \psi_l^2 \\
\leq \epsilon \int |\nabla(u_n - u)|^2 \psi_l^2 + c(\epsilon) \int |\nabla u|^2 \psi_l^2 \\
\leq \epsilon C + c(\epsilon)o(1)_{l \to \infty}.
\]
Letting \( l \to \infty \) we get that \( \tilde{\mu}(\{x\}) = \mu(\{x\}) \). Because of the weak lower semi-continuity we have \( \mu \geq |\nabla u|^2 dx \). Finally the Brezis-Lieb Lemma [3] implies
\[
k(|u_n|^q - |u|^q) dx = k(|u_n - u|^q) dx + o(1)_{n \to \infty}.
\]
Thus claims (2) and (3) are proved.
Let \( R > 0 \) and \( \psi_R \in C^\infty(\mathbb{R}^N) \) such that \( \psi_R \equiv 0 \) in \( B_R(0) \), \( \psi_R \equiv 1 \) in \( \mathbb{R}^N \setminus B_{R+1}(0) \) and
0 \leq \psi_R \leq 1 \text{ everywhere.}

Because \( u_n \to u \) in \( L^2_{\text{loc}} \), we have

\[
\mu_\infty = \lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \psi_R^2 \leq \lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla (u_n \psi_R)|^2
\]

\[
\geq \lim_{R \to \infty} S_{R,k} \lim_{n \to \infty} \left( \int |k| u_n |\psi_R|^q \right)^{2/q} = S_{\infty,k} \nu_\infty^{2/q}.
\]

To show (4) we use again the Brezis-Lieb Lemma and get

\[
\lim_{R \to \infty} \limsup_{n \to \infty} \int |k| u_n |(1-\psi_R)| = \lim_{R \to \infty} \limsup_{n \to \infty} \left( \int |k| u_n |u|^{q-1} (1-\psi_R) + \int |k| |u|^{q-1} \right)
\]

\[
\quad \quad \quad = \sum_{j \in J} \nu_j + \int |k| u|^q.
\]

Finally we deduce

\[
\limsup_{n \to \infty} \int |k| u_n |^q = \lim_{R \to \infty} \limsup_{n \to \infty} \left( \int |k| u_n |(1-\psi_R)| + \int |k| u_n |\psi_R| \right)
\]

\[
\quad \quad \quad = \sum_{j \in J} \nu_j + \int |k| u|^q + \nu_\infty.
\]

\[\square\]

**Corollary 3.2.** Suppose \( q > 2 \) and \( k \in L^1_{\text{loc}} \) nonnegative such that \( D^{1,2}(\mathbb{R}^N) \) is continuously embedded in \( L^q(\mathbb{R}^N, k) \). Then

\[
S_k^c = S_k^\infty = \infty \text{ if and only if } D^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, k) \text{ is compact.}
\]

**Proof (sufficiency).** Suppose \((u_n)_{n \in \mathbb{N}}\) is bounded in \( D^{1,2}(\mathbb{R}^N) \) such that \( u_n \rightharpoonup 0 \). Since \( S_k^c = S_k^\infty = \infty \), Lemma 3.2 shows that \( \int |k| u_n|^q \to 0 \) as \( n \to \infty \). \[\square\]

**Proof (necessity).** Suppose, contrary to our claim, that \( S_k^c \) for some \( x \in \mathbb{R}^N \) or \( S_k^\infty \) are finite. Hence there is a bounded sequence \((u_n)_{n \in \mathbb{N}}\) in \( D^{1,2}(\mathbb{R}^N) \) such that

\[
\int |k| u_n|^q = 1, \ u_n \in D^{1,2}(B_{1/n}(x)) \text{ or } u_n \in D^{1,2}(\mathbb{R}^N \setminus B_n(0)). \tag{3.2}
\]

Passing to a subsequence we may assume \( u_n \to u \) and \( u_n(x) \to u(x) \) for almost every \( x \in \mathbb{R}^N \). We conclude from (3.2) that \( u \equiv 0 \). The compactness of the embedding forces \( \int |k| u_n|^q \to 0 \) as \( n \to \infty \), contrary to (3.2). \[\square\]

**Remark 3.3.** Furthermore (see [21]), there are positive constants \( c_3(N,q), c_4(N,q) \) such that:

\[
\frac{c_3}{\sqrt{S_k^c}} \leq \lim_{\delta \to 0, x \in \mathbb{R}^N, 0 < \rho < \delta} \rho^{(1-\frac{N}{q})} \left( \int_{B_\rho(x)} |k| \right)^{\frac{1}{q}} \leq \frac{c_4}{\sqrt{S_k^\infty}}
\]

\[
\frac{c_3}{\sqrt{S_k^\infty}} \leq \lim_{R \to \infty, x \in \mathbb{R}^N, 0 < \rho < \delta} \rho^{(1-\frac{N}{q})} \left( \int_{B_\rho(x) \setminus B_R(0)} |k| \right)^{\frac{1}{q}} \leq \frac{c_4}{\sqrt{S_k^\infty}}
\]

Hence condition \((2.3)\) of Theorem 2.1 prevents point concentration whereas condition \((2.4)\) is related to the possible loss of mass at infinity.
In presence of a group symmetry we denote the length of the orbit containing $x \in \mathbb{R}^N$ by $|G_x| := \# \{gx \mid g \in G \}$ and $|G_\infty| := 1$.

**Corollary 3.4.** Suppose $q > 2$, $D^{1,2}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N, k)$ for some nonnegative $k \in L_{loc}^1$, $G$ is a compact subgroup of $O(N)$ and $k$ is $G$-symmetric. Then $D^{1,2}_{G}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N, k)$ if

$$\inf_{x \in \mathbb{R}^N \cup \{\infty\}} \{|G_x| S_k^x\} = \infty.$$ 

**Proof.** Suppose $(u_n)_{n \in \mathbb{N}}$ is bounded in $D^{1,2}_{G}(\mathbb{R}^N)$. We may assume $u_n \rightharpoonup 0$ in $D^{1,2}_{G}(\mathbb{R}^N)$. Because $D^{1,2}(\mathbb{R}^N) = D^{1,2}_{G}(\mathbb{R}^N) \oplus D^{1,2}_{G}(\mathbb{R}^N)^\perp$ we have $u_n \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$. Lemma 3.1 yields, that there is an at most countable set $S := \{x_j \mid j \in J\}$ of distinct points in $\mathbb{R}^N$ and a family $\{\nu_j \mid j \in J\}$ of positive numbers such that

$$k|u_n|^q \rightarrow \nu = \sum_{j \in J} \nu_j \delta_{x_j} \text{ and } |\nabla u_n|^2 \rightarrow \mu \geq \sum_{j \in J} \mu_j \delta_{x_j}$$

weakly in sense of measures $S_k^{x_j} \nu_j^{2/q} \leq \mu_j$ and $\mu_\infty \geq S_k^\infty \nu_\infty^{2/q}$. Because $S_k^\infty = \infty$ we see $\nu_\infty = 0$. Suppose $x_0 \in S \neq \emptyset$. Then the $G$-symmetry of the involved measures implies $\{gx_0 \mid g \in G\} \subset S$ and we have

$$|G_{x_0}| S_k^{x_0} \nu_0^{2/q} \leq \sum_{j \in J} \mu_j < \infty.$$ 

Thus $\nu_0 = 0$; a contradiction. Hence Lemma 3.1 (4) leads to $\int k|u_n|^q \n \rightarrow 0$. \hfill $\Box$

### 4. Palais-Smale condition

In the remainder of this section we always assume (1.2)-(1.3). Because of (1.5) we have $E_G = D^{1,2}_{G}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, h^-)$ and we may replace $\|u\|_{E_G}$ with the equivalent norm $\|\nabla u\|_2 + \|(h^-)^{1/p} u\|_p$. We still consider the functional $I : E_G \rightarrow \mathbb{R}$ defined by

$$I(u) := \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p} \int h(x)|u|^p.$$ 

Clearly $I \in C^1(E_G, \mathbb{R}) \cap C^1(E, \mathbb{R})$ and

$$I'(u) \varphi = \int \nabla u \nabla \varphi - \int h(x)|u|^{p-2} u \varphi.$$ 

Critical points of $I$ correspond to weak solutions of

$$-\Delta u - h(x)|u|^{p-2} u = 0, \quad u \in E_G.$$ 

(4.1)

**Lemma 4.1** (symmetric criticality). Let $(u_n)_{n \in \mathbb{N}}$ be a $(PS)_c$ sequence in $E_G$, i.e.

$$I(u_n) \rightarrow c, \quad I'(u_n) \rightarrow 0 \text{ in } E_G'.$$

Then $I'(u_n) \rightarrow 0$ in $E'$.

**Proof.** The $G$-symmetry of the Laplacian and of $h$ yield the $G$-symmetry of $I$. Consequently we have for $g \in G$, $u \in E_G$ and $v \in E$

$$I'(u) g_* v = \lim_{t \rightarrow 0} \frac{I(u + t g_* v) - I(u)}{t} = \lim_{t \rightarrow 0} \frac{I(g_*^{-1} u + t v) - I(g_*^{-1} u)}{t} = I'(g_*^{-1} u) v = I'(u) v.$$
Let $\mu$ denote the Haar measure corresponding to the compact group $G$, then we have

$$I'(u) \left( \int_G g_* v d\mu(g) \right) = \int_G I'(u) g_* v d\mu(g) = \int_G I'(u) v d\mu(g) = I'(u)v.$$ 

Hence

$$\sup_{\|v\|_E = 1} I'(u)v = \sup_{\|v\|_E = 1} I'(u)v,$$

and the claim follows. $\Box$

**Lemma 4.2** \((PS)_c\) condition. Every \((PS)_c\) sequence \((u_n)_{n\in\mathbb{N}}\) in $E_G$ contains a convergent subsequence if one of the following conditions is satisfied

\[
D^{1,2}_G(\mathbb{R}^N) \text{ is compactly embedded in } L^p(\mathbb{R}^N, h^+); 
\]

\[
D^{1,2}(\mathbb{R}^N) \text{ is continuously embedded in } L^p(\mathbb{R}^N, h^+); 
\]

\[
c < c_0 := \left( \frac{1}{2} - \frac{1}{p} \right) \inf_{x \in \mathbb{R}^N \cup \{\infty\}} \left\{ |G_x| \left( \frac{S_{h^+}^x}{p-2} \right) \right\}. 
\]

**Proof.** Let \((u_n)_{n\in\mathbb{N}}\) be a sequence in $E_G$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in $E'_G$. Because $\|u\|_{E_G} = \|\nabla u_n\|_2 + \|(h^-)^{1/p} u_n\|_p$ we have

$$c + o(1) \left( \|\nabla u_n\|_2^2 + \|(h^-)^{1/p} u_n\|_p^2 \right) = I(u_n) - (1/p)I'(u_n)u_n = \left( \frac{1}{2} - \frac{1}{p} \right) \|\nabla u_n\|_2^2. \quad (4.4)$$

Suppose $\|\nabla u_n\|_2 \to \infty$. Then equation (4.4) implies $\|(h^-)^{1/p} u_n\|_p \geq \|\nabla u_n\|_2^p$ for large $n$. Hence

$$c + 1 \geq I(u_n) \geq \frac{1}{2} \|\nabla u_n\|_2^2 + \frac{1}{p} \|\nabla u_n\|_2^{2p} - C\|\nabla u_n\|_2^p$$

for large $n$, which is impossible. Thus \((u_n)_{n\in\mathbb{N}}\) is bounded in $E_G$. Passing to a subsequence we may assume $u_n \rightharpoonup u$ in $E_G$.

Suppose for a moment there holds

$$u_n \to u \text{ in } L^p(\mathbb{R}^N, h^+) \text{ as } n \to \infty. \quad (4.5)$$

The fact that $u_n$ converges weakly to $u$ in $E_G$, $D^{1,2}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N, h^-)$ implies $I'(u) = 0$.

Calculating

$$0 = \lim_{n \to \infty} (I'(u_n) - I'(u))(u_n - u)$$

$$= \lim_{n \to \infty} \left( \int |\nabla u_n - \nabla u|^2 - \int h^+(|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) 
$$

$$+ \int h^- (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) \right)_{\geq 0}$$

$$\geq \lim_{n \to \infty} \left( \int |\nabla u_n - \nabla u|^2 - \int h^+(|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) 
$$

$$= \lim_{n \to \infty} \int |\nabla u_n - \nabla u|^2 + o(1)_{n \to \infty}$$

we see $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$. Finally we have

$$0 = \lim_{n \to \infty} I'(u_n)u_n - I'(u)u = \lim_{n \to \infty} h^-(|u_n|^p - \int h^- |u|^p)$$
and the uniform convexity of $L^p$ implies $u_n \to u$ in $E_G$.
What is left is to show (4.5), which immediately follows under assumption (4.2). Thus the proof is completed by showing that (4.5) holds under assumption (4.3).
By Lemma 3.1 there exist $G$-symmetric measures $\mu$ and $\nu$ satisfying (1) – (4) of Lemma 3.1 and
\[ |\nabla u_n|^2 \overset{n \to \infty}{\to} \mu, \quad h^+|u_n|^p \overset{n \to \infty}{\to} \nu. \]
Let $x_k$ be an atom of $\nu$. We take $\varphi \in C^1(\mathbb{R}^N)$ such that
\[ \varphi \cdot \chi_{B_1(0)} \equiv 1, \quad \varphi \cdot (1 - \chi_{B_2(0)}) \equiv 0, \quad |\varphi| \leq 2 \]
and define $\varphi_\varepsilon(x) := \varphi\left(\frac{x - x_k}{\varepsilon}\right)$. Lemma 4.1 implies $I'(u_n)\varphi_\varepsilon u_n \to 0$. Hence
\[ \int |\nabla u_n|^2 \varphi_\varepsilon + \int \nabla u_n \nabla \varphi_\varepsilon u + \int h^-|u_n|^q \varphi_\varepsilon - \int h^+|u_n|^q \varphi_\varepsilon n \to \infty \overset{\text{}}{\to} 0. \]
This leads to the following estimate
\[ \int \varphi d\mu - \int \varphi d\nu \leq \limsup_{n \to \infty} \int |\nabla u_n||u_n||\nabla \varphi_\varepsilon| \leq C \limsup_{n \to \infty} \left( \int |u_n|^2|\nabla \varphi_\varepsilon|^2 \right)^{1/2} \leq C \left( \int |u|^2|\nabla \varphi_\varepsilon|^2 \right)^{1/2} \leq C\|u \cdot \chi_{B_2(x_k)}\|_2^* \left( \int_{B_2(x_k)} |\nabla \varphi_\varepsilon|_N \right)^{1/N} \leq \text{const} \leq o(1)_{\varepsilon \to 0}. \]
Thus $\nu_k \geq \mu_k$ and Lemma 3.1 (3) implies:
\[ \nu_j \geq (S_{h^+}^{\infty})^{\frac{p}{p-2}}. \]
Take $\varphi_R \in C^1(\mathbb{R}^N)$ $G$-symmetric such that
\[ \varphi_R(x) = 1, \quad \forall |x| > R + 1 \quad \varphi_R(x) = 0, \quad \forall |x| < R \quad 0 \leq \varphi(x) \leq 1. \]
Then
\[ 0 = \lim_{n \to \infty} I'(u_n)\varphi_R u_n \geq \limsup_{n \to \infty} \left( \int |\nabla u_n|^2 \varphi_R - \int |\nabla u_n||u_n||\nabla \varphi_R| - \int h^+|u_n|^p \varphi_R \right). \]
As before we see
\[ \lim_{R \to \infty} \limsup_{n \to \infty} \int |\nabla u_n||u_n||\nabla \varphi_R| \leq C \lim_{R \to \infty} \|u \cdot \chi_{R < |x| < R + 1}\|_2^* = 0. \]
Hence $\nu_\infty \geq \mu_\infty$ and
\[ \nu_\infty \geq (S_{h^+}^{\infty})^{\frac{p}{p-2}}. \]
For every $\psi \in C_c^\infty(\mathbb{R}^N)$ with $0 \leq \psi(x) \leq 1$ there holds

$$c = \lim_{n \to \infty} \left\{ I(u_n) - \frac{1}{p} I'(u_n) u_n \right\} = \left( \frac{1}{2} - \frac{1}{p} \right) \lim_{n \to \infty} \int |\nabla u_n|^2$$

$$\geq \left( \frac{1}{2} - \frac{1}{p} \right) \lim_{n \to \infty} \int |\nabla u_n|^2 \psi \tag{4.8}$$

Suppose $x_k$ is an atom of $\nu$. For each $g \in G$, due to the $G$-symmetry, $gx_k$ is an atom of $\nu$ with the same mass. The $G$-symmetry of $h$ leads to $S_{h^+}^{x_k} = S_{h^+}^{gx_k}$ for all $g \in G$. Choose a $\psi$ as above with $\psi(gx_k) = 1$ for all $g \in G$. Then (4.6) and (4.8) imply

$$c \geq \left( \frac{1}{2} - \frac{1}{p} \right) |G_x| S_{h^+}^{x_k} \nu_k^{2/p} \geq \left( \frac{1}{2} - \frac{1}{p} \right) |G_x| (S_{h^+}^{x_k})^\frac{2}{p-2},$$

which is impossible. Hence $J = \emptyset$. If we use $\varphi_R$ in the estimate (4.8), we see with the help of (4.7)

$$c \geq \left( \frac{1}{2} - \frac{1}{p} \right) (S_{h^+}^\infty)^\frac{2}{p-2}$$

and get $\nu_\infty = 0$. Consequently Lemma 3.1 (4) and the uniform convexity of $L^p$ implies

$$u_n \to u \text{ in } L^p(\mathbb{R}^N, h^+) \text{ as } n \to \infty.$$  

\[\square\]

**Existence of positive solutions.** We use the mountain pass Lemma of Ambrosetti and Rabinowitz [8] and get

**Theorem 4.3.** If $D^{1,2}_G(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, h^+)$ is compact, then (1.1) has a nontrivial, non-negative weak solution.

**Theorem 4.4.** Suppose $D^{1,2}(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N, h^+)$ and there is an $u \in E_G$ such that

$$\int h|u|^p > 0 \text{ and } \max_{0 \leq t < \infty} I(tu) \leq c_0.$$

Then (1.1) is solvable.

**Proof of Theorems 4.3 and 4.4.** $D^{1,2}_G(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N, h^+)$. Consequently

$$I(u) = \frac{1}{2} \|\nabla u\|^2 + \frac{1}{p} \int h^-|u|^p - \frac{1}{p} \int h^+|u|^p \geq \frac{1}{2} \|\nabla u\|^2 + \frac{1}{p} \int h^-|u|^p - C \|\nabla u\|^2. \tag{4.9}$$

Hence

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0,$$

where $\Gamma := \{\gamma \in C([0,1], E_G) \mid \gamma(0) = 0, I(\gamma(1)) < 0\}$. The mountain pass theorem leads to a $(PS)_c$ sequence. If $c < c_0$ or (4.2) holds, then we obtain a critical point $u_0 \in E_G$ of $I$ with the help of Lemma 4.2.

If $c = c_0$, then the infimum is attained by the path $\gamma_t : t \mapsto tt_0u$ for a suitable $t_0$. Let $u_0 := tt_0u$ with $I(\gamma_0(t)) = \max I(\gamma_0(t))$. Then $I'(u_0) = 0$, because otherwise $\gamma_0$ can be deformed to a path $\gamma_1$ with $\max I(\gamma_1(t)) < c$ contradicting the definition of $c$. 

\[\square\]
In both cases we obtain a critical point $u_0$ with $I(u_0) = c > 0$. Because all the involved terms will not change their values, if we replace $u_0$ by $|u_0|$, we have

$$c = I(|u_0|) = \max_t I(t|u_0|)$$

and may deduce as above to see, that $|u_0|$ is also a critical point of $I$. 

**Corollary 4.5.** Suppose $D^{1,2}(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N, h^+)$. Then (1.4) has a solution if

$$\exists u \in E_G : \int h|u|^p > 0, \|\nabla u\|_2 = 1,$$

$$\int h|u|^p \geq \sup_{x \in \mathbb{R}^N \cup \{\infty\}} \left\{ |G_x| - \frac{p-2}{2} (S_{h^+})^{-\frac{p}{2}} \right\}. \quad (4.10)$$

**Remark 4.6.** Let $u_0$ be a positive mountain pass solution found in Theorem 4.3. Thanks to the homogeneity of the nonlinear part of $I$ it is easy to see that $u_0$ minimizes the functional $I$ on its Nehari manifold, i.e.

$$I(u_0) = \inf \{ I(u) \mid I'(u)u = 0, u \neq 0 \}.$$}

**Existence of multiple solutions.** $I$ is an even functional, i.e. $I(u) = I(-u)$. Consequently classical results leading to multiple critical points of symmetric functionals apply.

We use a version of [3], Thm. 9.12 given in [3]

**Theorem 2 in [3].** Let $E$ be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ an even functional satisfying $(PS)_c$ condition for each $c$ and $I(0) = 0$. Furthermore,

(i) there exists $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ for all $\|u\| = \rho$;

(ii) there exists an increasing sequence of subspaces $(E_n)_{n \in \mathbb{N}}$ of $E$, with $\dim E_n = n$, such that for every $n$ one can find a constant $R_n > 0$ such that $I(u) \leq 0$ for all $u \in E_n$ with $\|u\| \geq R_n$.

Then $I$ possesses a sequence of critical values $(\epsilon_n)_{n \in \mathbb{N}}$ tending to $\infty$ as $n \to \infty$.

**Theorem 4.7.** Suppose there exits a smooth $G$-symmetric domain $\emptyset \neq \Omega \subset \mathbb{R}^N$ such that $h(x) > 0$ for all $x \in \Omega$ and $D^{1,2}(\mathbb{R}^N)$ is compactly embedded in $L^p(\mathbb{R}^N, h^+)$. Then (1.4) has infinitely many $G$-symmetric solutions.

**Proof.** We apply [3], Thm. 2 with $E = E_G$. From (4.9) we obtain (i). To see that (ii) holds we choose an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of subspaces of $D^{1,2}_G(\Omega)$ with $\dim E_n = n$. We may assume $D^{1,2}_G(\Omega) \subset E_G$ by extending the functions by 0 outside $\Omega$. Since the dimension of each $E_n$ is finite, we conclude

$$\inf_{u \in E_n, \|u\| = 1} \int h(x)|u|^p =: \epsilon(n) > 0,$$

which immediately implies (ii). By Lemma 4.2 the $(PS)_c$ condition holds for all $c$, which completes the proof. 

## 5. Examples

It is known (see for instance [3], [12]) that for $N \geq 3$ and $0 \leq \delta < 2$

$$\sup_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\left( \int |x|^{-\delta} |u|^p \right)^{\frac{1}{p}}}{\|\nabla u\|_2} = K(N, \delta) < \infty, \quad (5.1)$$
where \( p = p(N, \delta) = \frac{2(N - \delta)}{N - 2} \). \( K(N, \delta) \) is attained by the function

\[
u_{\delta}(x) = (1 + |x|^{2 - \delta})^{-(N - 2)/(2 - \delta)}.
\]

Thanks to the dilatation symmetry of the quotient in (5.1) the functions

\[
u_{\delta, \sigma}(x) := \frac{u_{\delta}(x/\sigma)}{u_{\delta}(x/\sigma)^{2/2}}
\]

also maximize (5.1) for all \( \sigma > 0 \).

Suppose \( k \in C(\mathbb{R}^N) \) is a continuous nonnegative function. Then we easily get for \( N \geq 3, \)

\( 0 \leq \delta < 2 \) and \( p = p(N, \delta) \)

\[
S_{k([\cdot])|\cdot}^x = \begin{cases} 
\infty & \text{if } \delta > 0 \text{ and } x \neq 0,
\kappa(x)^{2/p}K(N, 0)^{-2} & \text{if } \delta = 0,
\kappa(0)^{-2/p}K(N, \delta)^{-2} & \text{if } \delta > 0 \text{ and } x = 0.
\end{cases}
\]

(5.2)

Furthermore, we have

\[
\left( \limsup_{|x| \to \infty} k(x) \right)^{-2/p}K(N, \delta)^{-2} \leq S_{k([\cdot])|\cdot}^x \leq \left( \liminf_{|x| \to \infty} k(x) \right)^{-2/p}K(N, \delta)^{-2}.
\]

(5.3)

**Corollary 5.1.** Suppose \( h \in L^1_{loc}, \int h|\alpha|^p > 0 \) for some \( u \in C^\infty_c(\mathbb{R}^N) \). Then (1.1) has a solution if one of the following conditions holds

\[
2 < p < 2^*, \ h^+ \leq \sum_{i \in \mathbb{N}} \alpha_i f(x - p_i)|x - p_i|^{\frac{N - 2p - N}{2}}, \text{ where } p_i \in \mathbb{R}, \ \alpha_i \in \mathbb{R},
\]

\[
\sum_{i \in \mathbb{N}} |\alpha_i| < \infty \text{ and } f \in L^\infty(\mathbb{R}^N) \text{ such that } f(x) \frac{|x| \to \infty}{x \to 0} \rightarrow 0.
\]

(5.4)

\[
p \geq 2^*, \ h \text{ is radially symmetric and satisfies (1.8)}.
\]

(5.5)

**Proof.** (5.3) implies \( D^{1,2}_{\text{O}(N)}(\mathbb{R}^N) \) is compactly embedded in \( L^p(\mathbb{R}^N, h^+) \) (see for instance [21, Lem. 6]).

Denote by \( f_i(x) \) the function \( f_i(x) := f(x - p_i)|x - p_i|^{\frac{N - 2p - N}{2}} \). By Corollary 2.2 the inclusion of \( D^{1,2}(\mathbb{R}^N) \) in \( L^p(\mathbb{R}^N, f_i) \) is compact for all \( i \in \mathbb{N} \). Hence we may estimate

\[
\sup_{x \in \mathbb{R}^N} \rho^{(1-\frac{N}{2})p} \int_{B_{\rho}(x)} \sum_{i \in \mathbb{N}} \alpha_i f_i \leq \sup_{x \in \mathbb{R}^N} \rho^{(1-\frac{N}{2})p} \int_{B_{\rho}(x)} f_1 \left( \sum_{i \in \mathbb{N}} |\alpha_i| \right) = o(1)_{\delta \to 0},
\]

\[
\sup_{x \in \mathbb{R}^N} \rho^{(1-\frac{N}{2})p} \int_{B_{\rho}(x) \setminus B_{R}(0)} \sum_{i \in \mathbb{N}} \alpha_i f_i \leq \sup_{x \in \mathbb{R}^N} \rho^{(1-\frac{N}{2})p} \left( \sum_{i \geq k} |\alpha_i| \int_{B_{\rho}(x)} f_1 + \int_{B_{\rho}(x) \setminus B_{R}(0)} \sum_{i < k} \alpha_i f_i \right) \leq o(1)_{k \to \infty} + o(1)_{R \to \infty}.
\]

Thus (5.4) implies \( D^{1,2}(\mathbb{R}^N) \) is compactly embedded in \( L^p(\mathbb{R}^N, h^+) \).

In both cases Theorem 1.3 yields the existence of a solution. \( \square \)

Pohozaev’s identity adapted to (1.1) leads to
Lemma 5.2. Suppose $2 < p$ and $h(x) = |k(x)|^{-\delta}$ for some $k \in C^1(\mathbb{R}^N)$, where $\delta = N - \frac{p}{2}(N-2)$. Then every solution $u \in C^2(\mathbb{R}^N)$ of (4.1), such that the function $<\nabla k(x), x > |x|^{-\delta}u(x)|^p \in L^1(\mathbb{R}^N)$, satisfies

$$\int <\nabla k(x), x > |x|^{-\delta}|u|^p = 0. \quad (5.6)$$

Proof. Suppose $u \in D^{1,2}(\mathbb{R}^N) \cap L^p(|h|) \cap C^2(\mathbb{R}^N)$ solves (4.1). Because $<\nabla h(x), x > = <\nabla k(x), x > |x|^{-\delta} - \delta h(x)$ we have $|<\nabla h(x), x >| |u|^p \in L^1(\mathbb{R}^N)$. We use a version of Pohozaev’s identity [18] given in [11, Thm. 29.4] and (5.7) to derive

$$\frac{N-2}{2} \int |\nabla u|^2 = \frac{N}{p} \int h(x)|u|^p + \frac{1}{p} \int <\nabla h(x), x > |u|^p \quad (5.8)$$

The fact that $I'(u)u = 0$ and (5.8) give the identity (5.6). \hfill \Box

Corollary 5.3. Consider the equation

$$-\Delta u = (1 + |x|)^{-\delta}|u|^{p-2} u, \quad 0 \neq u \in D^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, (1 + |x|)^{-\delta}). \quad (5.9)$$

(i) (5.6) has no solution $u \in C^2(\mathbb{R}^N)$ if $2 < p < 2^*$ and $\delta \leq N - \frac{p}{2}(N-2)$.

(ii) (5.6) has infinitely many $C^2(\mathbb{R}^N)$-solutions if $2 < p < 2^*$ and $\delta > N - \frac{p}{2}(N-2)$. At least one solution is strictly positive in $\mathbb{R}^N$.

Proof. From $2 < p < 2^*$, $\delta > \delta_0 : = N - \frac{p}{2}(N-2)$ and Corollary 5.1 we conclude that (5.9) possesses a nontrivial weak solution $u$. Due to Harnack’s inequality and standard regularity results (see [22, C.3]) $u$ is a strictly positive $C^2(\mathbb{R}^N)$-function. In addition by Theorem 4.7 there are infinitely many solutions of (5.9).

Considering $k(x) := (1 + |x|)^{-\delta}|x|^6_0$ it is easy to check that Lemma 5.2 yields the desired nonexistence result. \hfill \Box

Solutions of (1.1) may be obtained in a non-compact setting if it is possible to find appropriate test functions to ensure that the mountain pass value $c$ is below the compactness threshold $c_0$, i.e. we have to find a function $u \in E_0$ that satisfies (4.10).

Remark 5.4. Suppose $2 < p \leq 2^*$ and $h \in L^1_{\text{loc}}$ is $G$-symmetric such that

$$h^+(x) = k(x)|x|^{-\delta},$$

where $k \in C(\mathbb{R}^N)$ and $\delta = N - \frac{p}{2}(N-2)$. With the notation

$$k_c := \max\{ \sup_{x \in \mathbb{R}^N} \{ k(x)|G_x|^{-\frac{p-2}{2}} \}, \limsup_{|x| \to \infty} k(x)\}$$

we have

$$\sup_{x \in \mathbb{R}^N \cup \{\infty\}} \{ |G_x|^{-\frac{p-2}{2}}(S_{h^+})^{-\frac{p}{2}} \} \leq k_c K(N, \delta)^p = k_c \int |x|^{-\delta} v_{\delta, \sigma}^p.$$  

Thus (4.10) holds if we have for some $\sigma > 0$

$$\int h(x)v_{\delta, \sigma}^p > 0, \quad \int (h(x) - k_c|x|^{-\delta})v_{\delta, \sigma}^p \geq 0. \quad (5.10)$$
Under the assumptions of Remark 5.4 the space \( D^{1,2}_{G}(\mathbb{R}^N) \) is not compactly embedded in \( L^p(\mathbb{R}^N, h^+) \) if \( k(0) > 0 \) or \( \lim \inf_{|x| \to \infty} k(x) > 0 \). Corollaries 5.5 and 5.6 below yield some sufficient conditions for the existence of solutions to \((1.1)\) in the non-compact case. We leave it to the reader to verify with the help of (5.10) that the proofs given in [3, Cor. 1.2] carry over to our situation.

**Corollary 5.5.** Assuming the hypotheses of Remark 5.4 and

\[
 k(0) \geq \begin{cases} 
 \lim \sup_{|x| \to \infty} k(x) & \text{if } 2 < p < 2^*, \\
 \max\{\lim \sup_{|x| \to \infty} k(x), \sup_{x \in \mathbb{R}^N} |G_x|^{-\frac{p-2}{2}} k(x)\} & \text{if } p = 2^*
 \end{cases}
\]

\((L.A)\) has a solution if one of the following conditions is satisfied

- \( h \neq 0 \), \( h(x) \geq k(0)|x|^{-\delta} \) for all \( x \in \mathbb{R}^N \setminus \{0\} \);
- \( \exists \varepsilon, r > 0 : \int_{|x| \geq r} h^{-|x|^{2(\delta-N)} < \infty}, k(x) \geq k(0) + \varepsilon |x|^{N-\delta} \forall x \in B_r(0); \)
- \( \int |h(x) - k(0)|x|^{-\delta}|^{2(\delta-N)} < \infty \) and \( \int (h(x) - k(0)|x|^{-\delta})x|^{2(\delta-N)} > 0. \)

**Corollary 5.6.** Assuming the hypotheses of Remark 5.4 and

\[
 \lim_{|x| \to \infty} k(0) =: k(\infty) \geq \begin{cases} 
 k(0) & \text{if } 2 < p < 2^*, \\
 \sup_{x \in \mathbb{R}^N} \{|G_x|^{-\frac{p-2}{2}} k(x)\} & \text{if } p = 2^*
 \end{cases}
\]

\((L.A)\) has a solution if one of the following conditions is satisfied

- \( \exists \varepsilon, R > 0 : \int_{|x| \leq R} h^{-|x|^{2(\delta-N)} < \infty}, k(x) \geq k(\infty) + \varepsilon |x|^{-N+\delta} \forall x \in \mathbb{R}^N \setminus B_R(0); \)
- \( \int |h(x) - k(\infty)|x|^{-\delta}|^{2(\delta-N)} < \infty \) and \( \int (h(x) - k(\infty)|x|^{-\delta}) > 0. \)

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