Schrödinger equations on elliptic curves: symmetries, solutions and eigenvalue problem

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Abstract
In this paper, we study Schrödinger equations on elliptic curves called generalized Lamé equations. We suggest a method of finding integrable potentials for Schrödinger type equations. We apply this method to the Lamé equations and provide a sequence of integrable potentials for which the eigenvalue problem is solved explicitly.

Keywords Schrödinger type equations · Symmetries · Integrable potentials · Lamé equation · Eigenvalue problem

Mathematics Subject Classification 34A26 · 34A05 · 34B09

1 Introduction
The Lamé equation firstly appeared in [1] in separation of variables for the Laplace equation in elliptic coordinates. Later the Lamé equation was used in various problems of quantum mechanics, for example in theory of periodic instantons [2] and also appeared as Schrödinger equation for periodic potentials (see, for example, [3]).

The Lamé equation is also strongly related to integrability problems (see, for example, [4,5]). In [6], the Lamé equation appears as a normal variational equation for the Hénon–Heiles system, whose integrability is elaborated. Integrability properties of the Lamé equation are also addressed in [7,8], where the Lamé and Hermite solutions

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are presented, as well as Brioschi–Halphen–Crawford solutions (see also [9]). Also known are the Baldassarri solutions [10].

The present work is devoted to generalized Lamé equations and provides a method of finding analytical solutions to them, as well as solutions to the eigenvalue problem. Comparing with previous works, our approach is based on methods of geometrical theory of differential equations.

This paper has the following structure. In Sect. 2, we describe a method of finding integrable potentials and solutions for Schrödinger type equations. We recall necessary constructions from the geometrical theory of ODEs and apply its results to linear second order ODEs of Schrödinger type. We show that having a symmetry one can obtain an infinite sequence of integrable potentials and corresponding solutions. Section 3 is devoted to Schrödinger equations on elliptic curves, the so-called generalized Lamé equations. Such curves are parameterized by Weierstrass $p$-functions and we analyze the case when both potential and symmetry are polynomials in Weierstrass $p$-function and linear with respect to its first derivative. The most complete results are obtained for so-called even and odd cases, when the potential is an even function and the symmetry function is either even, or odd. We get a series of integrable potentials linear in the Weierstrass $p$-function, for which one can solve both eigenvalue problem and corresponding Schrödinger equation explicitly. Moreover, we suggest an explicit algorithm of obtaining integrable potentials and symmetries for them having known ones.

The results of this paper can be interpreted in the light of more general problem: how can one construct solutions of Schrödinger equations with potential satisfying a given ODE? The present paper provides the method for potentials satisfying $n$-th stationary KdV equations [11], and other cases are the subject of further elaboration.

2 Symmetries and integrals for Schrödinger type equations

2.1 Symmetries of ODEs

Here, we briefly describe a geometrical approach to ordinary differential equations following [12,13].

Let us consider an ordinary differential equation of order $k$. We will restrict our consideration to resolved with respect to the highest derivative ODEs of the form

$$u^{(k)} = f(x, u, u', \ldots , u^{(k-1)}),$$

which can be naturally associated with a smooth submanifold $E$ in a space of $k$-jets $J^k(\mathbb{R})$ of functions of $x$ with canonical coordinates $(x, u_0, \ldots , u_k)$ corresponding to the independent variable, unknown function and its derivatives up to order $k$:

$$E = \{ u_k = f(x, u_0, \ldots , u_{k-1}) \} \subset J^k(\mathbb{R}).$$
The space of $k$-jets $J^k(\mathbb{R})$ is equipped with the Cartan distribution

$$\mathcal{C}: J^k \ni \theta \mapsto \mathcal{C}_\theta \subset T_\theta J^k,$$

generated by vector fields $\partial_{u_k}$ and

$$\mathcal{D} = \partial_x + u_1 \partial_{u_0} + \cdots + u_k \partial_{u_{k-1}},$$

or, equivalently, by the Cartan forms

$$\omega_j = du_j - u_j + dx, \quad j = 0, k - 1.$$

The restriction $\mathcal{C}_E$ of the Cartan distribution to the submanifold $E$ is a one-dimensional distribution almost everywhere on $E$:

$$\mathcal{C}_E: E \ni \theta \mapsto \mathcal{C}_E(\theta) = T_\theta E \cap \mathcal{C}_\theta,$$

except at points $\theta \in E$ where $\mathcal{C}_\theta \subset T_\theta E$, which are called singular. The distribution $\mathcal{C}_E$ can therefore be given by a vector field

$$X_f = \mathcal{D} + \mathcal{D}(f)\partial_{u_k},$$

and an integral curve $l \subset E$ of the distribution $\mathcal{C}_E$ is said to be a solution of the equation $E$.

A transformation $\Phi: E \rightarrow E$ is called symmetry of the equation $E$ if it preserves the Cartan distribution $\mathcal{C}_E$, i.e. $\Phi_*(\mathcal{C}_E) = \mathcal{C}_E$. Infinitesimally, such a transformation is generated by a vector field $X \in D(E)$, such that $[X, \mathcal{C}_E] \subset \mathcal{C}_E$. Infinitesimal symmetries form a Lie algebra $\text{Sym}(\mathcal{C}_E)$ with respect to the commutator of vector fields. Obviously, vector fields from the distribution $\mathcal{C}_E$ are infinitesimal symmetries themselves. They are called trivial or characteristic since they transform any solution to the same solution and therefore do not give us new solutions. They form an ideal $\text{Char}(\mathcal{C}_E)$ of the Lie algebra $\text{Sym}(\mathcal{C}_E)$ and elements of a quotient algebra $\text{Shuff}(\mathcal{C}_E) = \text{Sym}(\mathcal{C}_E)/\text{Char}(\mathcal{C}_E)$ are called shuffle symmetries. Shuffle symmetries $X \in \text{Shuff}(\mathcal{C}_E)$ are defined by means of generating functions $[12, 13]$.

**Theorem 1** Shuffle symmetries of the ODE $E$ are of the form

$$X_\phi = \sum_{j=0}^{k-1} \mathcal{D}^j(\phi)\partial_{u_j},$$

where $\overline{\mathcal{D}} = \partial_x + u_1 \partial_{u_0} + \cdots + f \partial_{u_{k-1}}$ is an operator of a total derivative on $E$ and $\phi \in C^\infty(E)$ is a generating function.

The generating function is found from the Lie equation

$$\overline{\mathcal{D}}^k(\phi) - \sum_{j=0}^{k-1} f_{u_j} \mathcal{D}^j(\phi) = 0. \quad (1)$$
From now and on by a symmetry of the equation $\mathcal{E}$ we shall mean a generating function $\phi$.

The Lie algebra structure in Shuff ($\mathcal{C}_E$) induces a Lie algebra structure on a space of generating functions by the following way:

$$X_{[\phi_1, \phi_2]} = [X_{\phi_1}, X_{\phi_2}],$$

and an explicit expression for the bracket is

$$[\phi_1, \phi_2] = X_{\phi_1}(\phi_2) - X_{\phi_2}(\phi_1).$$

### 2.2 Schrödinger equations

Consider an ODE of the form

$$y'' + w(x)y = 0,$$  \hspace{1cm} (2)

where $y(x)$ is an unknown function and $w(x)$ is a potential. Equation (2) defines a smooth submanifold

$$\mathcal{E} = \{u_2 = -w(x)u_0\} \subset J^2(x, u_0, u_1, u_2).$$ \hspace{1cm} (3)

We will be interested in linear symmetries of (3):

$$\phi = a(x)u_0 + b(x)u_1,$$ \hspace{1cm} (4)

where $a(x)$ and $b(x)$ are some functions. Substituting (4) into (1) we get (see also [11])

$$\phi = cu_0 + \phi_z,$$

where $c$ is a constant,

$$\phi_z = z(x)u_1 - \frac{z'(x)u_0}{2},$$ \hspace{1cm} (5)

and the function $z(x)$ satisfies the Lie equation:

$$z''' + 4wz' + 2w'z = 0.$$ \hspace{1cm} (6)

Thus Eq. (3) has two commuting symmetries $\phi_1 = u_0$ and $\phi_2 = \phi_z$ and having a solution of (6) for a given potential $w(x)$ one can therefore integrate (3) using the Lie–Bianchi theorem [12,13]. We will call such potentials integrable. From now and on, we will call $z(x)$ symmetry function, keeping in mind that this function uniquely defines the generating function $\phi_z$ by means of (5).
Introduce the operators
\[ S_w = \partial^2 + w, \quad L_w = \partial^3 + 4w\partial + 2w' \]
corresponding to Eqs. (2) and (6) respectively, and let \( \text{Sol}(w) \) and \( L(w) \) be solution spaces of Schrödinger equation (2) and Lie equation (6) respectively, i.e.
\[ \text{Sol}(w) = \{ y \mid S_w(y) = 0 \} \]
\[ L(w) = \{ z \mid L_w(z) = 0 \}. \]

Note that there is a correspondence between \( \text{Sol}(w) \) and \( L(w) \). Namely, if \( y \in \text{Sol}(w) \), then \( z = y^2 \in L(w) \) and \( L(w) \) is therefore a symmetric square of \( \text{Sol}(w) \), \( L(w) = S^2(\text{Sol}(w)) \). Moreover, \( L(w) = sl_2(\mathbb{R}) \) with a bracket
\[ [z_1, z_2] = z_1'z_2 - z_1z_2'. \]

Indeed, if \( y_1, y_2 \in \text{Sol}(w) \) is a fundamental solution of (2), then solutions \( A = y_1^2 \), \( B = y_2^2 \), \( C = 2y_1y_2 \) of (6) satisfy \( sl_2(\mathbb{R}) \) structure equations:
\[ [A, B] = C, \quad [C, A] = -2A, \quad [C, B] = 2B. \]

Let us consider Eq. (6) as an equation for \( w(x) \). The following theorem is valid.

**Theorem 2** The symmetry function \( z(x) \) and potential \( w(x) \) are related as
\[ w(x) = \frac{c_w}{z^2} + \frac{1}{4} \left( \frac{z'}{z} \right)^2 - \frac{z''}{2z}, \tag{7} \]
where \( c_w \) is a constant,
\[ c_w = \frac{1}{4} K(z, z), \]
where \( K \) the Killing form of the Lie algebra \( L(w) \).

Let \( \hat{w}(x) \) be another potential with the same symmetry \( z(x) \). Then,
\[ \hat{w} = w + \frac{\hat{c}}{z^2}, \]
where \( \hat{c} = c_{\hat{w}} - c_w \) is a constant.

Let us now get solutions to (2) having known the symmetry \( z(x) \).

**Lemma 1** Function \( H = \phi_1 \overline{D}(\phi_2) - \phi_2 \overline{D}(\phi_1) \) is the first integral of (2) for any symmetries \( \phi_1 \) and \( \phi_2 \).
Proof Since \( \phi_1 \) and \( \phi_2 \) are symmetries, \( \mathcal{D}^2(\phi_{1,2}) = -w(x)\phi_{1,2} \) due to (1).

\[
\mathcal{D}(H) = \phi_1\mathcal{D}^2(\phi_2) - \phi_2\mathcal{D}^2(\phi_1) = 0.
\]

Applying the result of the above lemma to \( \phi_1 = \phi_z \) and \( \phi_2 = u_0 \), we get

\[
H = \frac{c_w}{z}(u_0)^2 + \frac{1}{z}(\phi_z)^2.
\]

Introducing a new variable \( v = u_0/\sqrt{|z|} \), we get \( (v')^2 = z^{-3}(\phi_z)^2 \) and

\[
H = c_wv^2 + z^2(v')^2 = H_0^2
\]

for some constant \( H_0 > 0 \). Consider three cases.

- Elliptic case, \( c_w = q_0^2 > 0 \).
  Introduce a new variable \( \psi \) by the following way:
  \[
v = \frac{H_0}{q_0} \sin \psi, \quad v' = \frac{H_0}{z} \cos \psi.
\]
  The last implies that
  \[
  \psi = \int \frac{q_0}{z} dx, \quad y = \frac{H_0}{q_0} \sqrt{|z|} \sin \left( q_0 \int \frac{dx}{z} \right).
  \]

- Hyperbolic case, \( c_w = -q_0^2 < 0 \)
  In the same way we obtain
  \[
y = \frac{H_0}{q_0} \sqrt{|z|} \sinh \left( q_0 \int \frac{dx}{z} \right).
  \]

- Parabolic case, \( c_w = 0 \)
  \[
y = H_0 \sqrt{|z|} \int \frac{dx}{z}.
  \]

Summarizing above discussion, we have the following theorem.

**Theorem 3** Let \( z(x) \) be a nonzero symmetry of (2). Then, a fundamental solution of (2) is given as

- for \( c_w = q_0^2 > 0 \)
  \[
y^{(1)}(x) = \sqrt{|z|} \sin \left( q_0 \int \frac{dx}{z} \right), \quad y^{(2)}(x) = \sqrt{|z|} \cos \left( q_0 \int \frac{dx}{z} \right).
  \]
\[
- \text{for } c_w = -q_0^2 < 0 \\
y^{(1)}(x) = \sqrt{|z|} \sinh \left( q_0 \int \frac{dx}{z} \right), \quad y^{(2)}(x) = \sqrt{|z|} \cosh \left( q_0 \int \frac{dx}{z} \right). \tag{9}
\]

\[
- \text{for } c_w = 0 \\
y^{(1)}(x) = \sqrt{|z|} \int \frac{dx}{z}, \quad y^{(2)}(x) = \sqrt{|z|}. \tag{10}
\]

A fundamental solution of (6) is given as

\[
- \text{for } c_w = q_0^2 > 0 \\
z^{(1)}(x) = z, \quad z^{(2)}(x) = z \sin \left( 2q_0 \int \frac{dx}{z} \right), \quad z^{(3)}(x) = z \cos \left( 2q_0 \int \frac{dx}{z} \right). \tag{11}
\]

\[
- \text{for } c_w = -q_0^2 < 0 \\
z^{(1)}(x) = z, \quad z^{(2)}(x) = z \sinh \left( 2q_0 \int \frac{dx}{z} \right), \quad z^{(3)}(x) = z \cosh \left( 2q_0 \int \frac{dx}{z} \right). \tag{12}
\]

\[
- \text{for } c_w = 0 \\
z^{(1)}(x) = z, \quad z^{(2)}(x) = z \left( \int \frac{dx}{z} \right)^2, \quad z^{(3)}(x) = z \int \frac{dx}{z}. \tag{13}
\]

Theorem 3 gives us a method of constructing integrable potentials by the following way.

1. Given a pair \((z, w)\)
2. Get a fundamental solution to (2) by means of (8), or (9), or (10)
3. Get a fundamental solution to (6) by means of (11), or (12), or (13)
4. Get a three-parametric family of integrable potentials (for example, in elliptic case)
   \[
   \hat{w} = w + \frac{c}{z^2} \left( \alpha_1 + \alpha_2 \sin \left( 2q_0 \int \frac{dx}{z} \right) + \alpha_3 \cos \left( 2q_0 \int \frac{dx}{z} \right) \right)^{-2}
   \]
   with new symmetries
   \[
   \hat{z} = z \left( \alpha_1 + \alpha_2 \sin \left( 2q_0 \int \frac{dx}{z} \right) + \alpha_3 \cos \left( 2q_0 \int \frac{dx}{z} \right) \right),
   \]
   where \(\alpha_1, \alpha_2, \alpha_3\) are constants.
5. Again, have a pair \((\hat{z}, \hat{w})\) and go to step 1.

It is worth to mention that symmetry \(z(x)\) not only allows to get solutions to the Schrödinger equation, but also produces an infinite hierarchy of integrable potentials and solutions for them.

### 2.2.1 Eigenvalue problem

Consider the Schrödinger equation with potential \(w(x) - \lambda\) and let

\[
y(a) = y(b) = 0, \quad a, b \in \mathbb{R}
\]  

be the Dirichlet boundary conditions.

**Theorem 4** Let potentials \(w(x) - \lambda\) be integrable and let \(z(x, \lambda)\) be their non-trivial symmetries. Then, eigenvalues \(\lambda\) of Dirichlet boundary problem (14) for Schrödinger equation (2) are solutions of the equation

\[
y^{(1)}(a, \lambda)y^{(2)}(b, \lambda) - y^{(1)}(b, \lambda)y^{(2)}(a, \lambda) = 0,
\]

where \(y^{(1)}\) and \(y^{(2)}\) are defined by (8) in elliptic case, by (9) in hyperbolic case, and by (10) in parabolic case.

**Proof** If \(z(x, \lambda)\) is a non-trivial symmetry for potential \(w(x) - \lambda\), then using Theorem 3 we get a general solution to (2) in the form

\[
y(x) = C_1y^{(1)}(x, \lambda) + C_2y^{(2)}(x, \lambda),
\]

where \(y^{(1)}(x, \lambda)\) and \(y^{(2)}(x, \lambda)\) are defined by means of (8) in elliptic case, by (9) in hyperbolic case, and by (10) in parabolic case. Boundary conditions (14) lead us to the homogeneous linear system for \(C_1\) and \(C_2\):

\[
\begin{align*}
C_1y^{(1)}(a, \lambda) + C_2y^{(2)}(a, \lambda) &= 0, \\
C_1y^{(1)}(b, \lambda) + C_2y^{(2)}(b, \lambda) &= 0.
\end{align*}
\]

Non-trivial solutions exist if the determinant of this system is equal to zero:

\[
y^{(1)}(a, \lambda)y^{(2)}(b, \lambda) - y^{(1)}(b, \lambda)y^{(2)}(a, \lambda) = 0.
\]

Applying the algorithm of generating integrable potentials described above and using results of Theorem 4, we get a series of potentials for which the eigenvalue problem admits explicit solution.
Example 1 (Mexican hat) Consider the eigenvalue problem for the so-called Mexican hat potential:

\[ w(x) = \frac{9\nu^6}{4} x^4 - 3\delta x^2, \]

where \( \nu \) and \( \delta \) are positive constants. Solving Lie equation (6) we get that the symmetry function \( z(x, \lambda) \) for this case is

\[ z(x, \lambda) = \left| \text{HeunT} \left( \frac{\lambda \nu^6 + \delta^2}{\nu^8}, 0, \frac{2\delta \nu^4}{\nu^4}, i \nu x \right) \right|^2, \tag{15} \]

where \( \text{HeunT}(\alpha, \beta, \gamma, z) \) is the Heun triconfluent function [14], \( z \in \mathbb{C} \), \( i \) is an imaginary unit.

Since \( L(w) = S^2(\text{Sol}(w)) \), the function \( z \in L(w) \) can be considered as the probability density of the particle for potential \( w(x) \). The probability density \( z(x) \) and Mexican hat potential are shown in Fig. 1.

Let us now consider the spectral problem for boundary conditions

\[ y(-2) = y(2) = 0. \]

Due to Theorems 3 and 4, we get the following equation for \( \lambda \):

\[ \text{Im} \left( \exp \left( \frac{-4i(2\nu^6 - \delta)}{\nu^3} \right) \text{HeunT}^2 \left( \frac{\lambda \nu^6 + \delta^2}{\nu^8}, 0, \frac{2\delta \nu^4}{\nu^4}, -2i\nu \right) \right) = 0, \]

where \( \text{Im} \) is an imaginary part, and the graph of its left-hand side in case of \( \nu = \delta = 1 \) is shown in Fig. 2. One can see that only negative eigenvalues are possible.
Distributions of the density for corresponding eigenvalues $\lambda_n$ are shown in Fig. 3. The next integrable potential is, for example,

$$\hat{w} = \frac{9\nu^6}{4}x^4 - 3\delta x^2 + \left| \text{HeunT}\left(\frac{\lambda \nu^6 + \delta^2}{\nu^8}, 0, 2\delta, i\nu x\right) \right|^{-4} - \lambda$$

(16)

with a symmetry given by (15).

Thus, using the symmetry function $z(x)$ we get solutions of the eigenvalue problem not only for Mexican hat potential, but also for potential (16) in quadratures.

### 3 Lamé equations

In this section, we study equations of type (2) on elliptic curves:

$$w(x) = C(p_0) + p_1 E(p_0),$$

(17)

where $p_0 = \wp(x)$, $p_1 = \wp'(x)$, $C(p_0)$ and $E(p_0)$ are polynomials and $\wp(x)$ is the Weierstrass $p$-function. Weierstrass elliptic functions satisfy an ODE

$$p_1^2 = 4p_0^3 - g_2 p_0 - g_3,$$

(18)

where $g_2$ and $g_3$ are invariants.

We will look for a symmetry $z(x)$ in the same form as potential:

$$z(x) = A(p_0) + p_1 B(p_0),$$

(19)

where $A(p_0)$ and $B(p_0)$ are polynomials.
Note that $\wp(x)$ is an even function and therefore due to (7) three cases are possible.

- **Even case**
  Here, we have both potential and symmetry as even functions, i.e.

  \[ E(p_0) = 0, \quad B(p_0) = 0. \]

- **Odd case**
  In this case, potential is even, while the symmetry is odd, i.e.

  \[ E(p_0) = 0, \quad A(p_0) = 0. \]

- **General case**
  Both potential and symmetry are neither odd, nor even.
Substituting (17) and (19) to (6) and using relation (18) we get an equation

\[ R_1(p_0) + p_1 R_2(p_0) = 0, \tag{20} \]

where

\[
R_1(p_0) = \left(4p_0^3 - g_2 p_0 - g_3\right)^2 B''' + 3\left(4p_0^3 - g_2 p_0 - g_3\right)(12p_0^2 - g_2) B'' \\
+ \left(4(4p_0^3 - g_2 p_0 - g_3)C + 300p_0^4 - 66p_0^2 g_2 - 48p_0 g_3 + \frac{3}{4} g_2^2\right) B' \\
+ 2B \left((12 p_0^2 - g_2)C + 60p_0^3 - 9 g_2 p_0 - 6 g_3\right) + E A(12p_0^2 - g_2) \\
+ 2(2EA' + BC' + AE') \left(4p_0^3 - g_2 p_0 - g_3\right),
\]

and

\[
R_2(p_0) = (4p_0^3 - g_2 p_0 - g_3) A''' + \left(18p_0^2 - \frac{3g_2}{2}\right) A'' + 4(3p_0 + C) A' \\
+ 2(4p_0^3 - g_2 p_0 - g_3)(BE' + 2 EB') + 2AC' + 3BE(12p_0^2 - g_2).
\]

Equation (20) is equivalent to the system

\[ R_1(p_0) = 0, \quad R_2(p_0) = 0. \tag{21} \]

### 3.1 Even case, \(E(p_0) = 0, B(p_0) = 0\)

In this case, the first equation in (21) is trivial and the second one is of the form

\[
(4p_0^3 - g_2 p_0 - g_3) A''' + \left(18p_0^2 - \frac{3g_2}{2}\right) A'' + 4(3p_0 + C) A' + 2AC' = 0. \tag{22}
\]

Let \(n\) and \(m\) be degrees of the polynomials \(A(p_0)\) and \(C(p_0)\) respectively, i.e. \(A(p_0) = \sum_{i=0}^{n} a_i p_0^i, C(p_0) = \sum_{i=0}^{m} c_i p_0^i,\) where \(a_n \neq 0, c_m \neq 0.\) Then, the left-hand side of (22) is a polynomial in \(p_0\) of degree \(\max(n, n + m - 1)\). If \(m \geq 2,\) then we obtain

\[ (4n + 2m)c_m a_n = 0, \]

and we get a contradictory. Therefore only cases \(m = 1\) make sense. Taking

\[
C(p_0) = c_1 p_0 + c_0, \quad A(p_0) = \sum_{i=0}^{n} a_i p_0^i,
\]
where we put $a_n = 1$ since the symmetry is defined up to a multiplicative constant, and collecting terms in $p_0$, we get the first equation in the form

$$4(n^2 + n + c_1) \left(n + \frac{1}{2}\right) = 0,$$

and hence

$$c_1 = -n(n + 1).$$

The next equations give us coefficients $a_i$ consistently.

**Theorem 5** Coefficients $a_i$ are given by the following relations

$$
\begin{align*}
    a_{n-1} &= \frac{c_0}{2n - 1}, \\
    a_{n-2} &= \frac{(8c_0^2 - n(g_2(2n - 1)^2))(n - 1)}{8(2n - 3)(2n - 1)^2}, \\
\end{align*}
$$

for $i = n - 3, 0$

$$a_i = \frac{(2i^2 + 10i + 12)a_{i+3}g_3 + (2i^2 + 7i + 6)a_{i+2}g_2 - 8c_0a_{i+1}) (i + 1)}{4(i + n + 1)(2i + 1)(i - n)},$$

and $c_0, g_2, g_3$ may be arbitrary.

The possibility for $c_0$ to be arbitrary is of great importance for is, because this fact allows us to get solutions to the eigenvalue problem for the Schrödinger operator explicitly.

**Example 2** ($n = 1$) We start with $n = 1$. In this case we have

$$w(x) = -2ϕ(x) + c_0, \quad z(x) = ϕ(x) + c_0,$$

which is the classical Lamé case. Constant $c_w$ is found from (7):

$$c_w = c_0^3 - \frac{c_0g_2 + g_3}{4}.$$

The potential and the density distribution are shown in Fig. 4.

Computing integral $\int \frac{dx}{z^{-1}(x)}$, we get (see also [11])

$$\int \frac{dx}{z(x)} = \frac{1}{\sqrt{-c_w}} \left( xϕ_σ(α) + \frac{1}{2} \ln \left( \frac{ϕ_σ(x - α)}{ϕ_σ(x + α)} \right) \right),$$
Fig. 4 Potential (point style) and
probability density (line style)
for $n = 1$

where $\wp_\zeta(x)$ and $\wp_\sigma(x)$ are Weierstrass $\zeta$- and $\sigma$-functions respectively and $\alpha$ is the root of the equation

$$\wp(\alpha) + c_0 = 0.$$ 

And therefore solution to the Lamé equation in both cases $c_w > 0$ and $c_w < 0$ is given by the same formula

$$y(x) = \sqrt{|z|} \left( D_1 \sinh(\mu(x)) + D_2 \cosh(\mu(x)) \right),$$

where

$$\mu(x) = x \wp_\zeta(\alpha) + \frac{1}{2} \ln \left( \frac{\wp_\sigma(x - \alpha)}{\wp_\sigma(x + \alpha)} \right),$$

and $D_i \in \mathbb{C}$.

Equation for eigenvalues $c_0$ in case of the Dirichlet conditions (14) is of the form

$$\sinh(\mu(a)) \cosh(\mu(b)) - \sinh(\mu(b)) \cosh(\mu(a)) = 0.$$ 

**Example 3** ($n = 2$) Potential and symmetry are

$$w(x) = -6\wp(x) + c_0, \quad z(x) = \wp^2(x) + \frac{c_0}{3} \wp(x) + \frac{c_0^2}{9} - \frac{g_2}{4},$$

The corresponding constant $c_w$ is

$$c_w = \frac{1}{324} (c_0^2 - 3g_2)(4c_0^3 - 9c_0g_2 - 27g_3).$$
Example 4 \((n = 3)\) Potential and symmetry are

\[
\begin{align*}
    w(x) &= -12 \varphi(x) + c_0, \\
    z(x) &= \varphi^3(x) + \frac{c_0}{5} \varphi^2(x) + \left( \frac{2c_0^2}{7} - \frac{g_2}{4} \right) \varphi(x) + \frac{c_0^3}{225} - \frac{c_0 g_2}{15} - \frac{g_3}{4},
\end{align*}
\]

The corresponding constant \(c_w\) is

\[
c_w = \frac{c_0^7}{50625} - \frac{7g_2c_0^5}{11250} - \frac{11g_3c_0^4}{3750} + \frac{31g_2^2c_0^3}{6000} + \frac{9g_2g_3c_0^2}{200} + \frac{(27g_3^2 - g_2^3)c_0}{240}.
\]

3.2 Odd case, \(E(p_0) = 0, A(p_0) = 0\)

Here, we get the first equation in (21) as

\[
\begin{align*}
    \left( 4p_0^3 - g_2p_0 - g_3 \right)^2 B'''' + 3 \left( 4p_0^3 - g_2p_0 - g_3 \right) (12p_0^2 - g_2) B''
    &+ \left( 4(4p_0^3 - g_2p_0 - g_3)C + 300p_0^4 - 66p_0^2g_2 - 48p_0g_3 + \frac{3}{4}g_2^2 \right) B'
    + 2B \left( (12p_0^2 - g_2)C + 60p_0^3 - 9g_2p_0 - 6g_3 \right)
    + 2BC' \left( 4p_0^3 - g_2p_0 - g_3 \right) = 0,
\end{align*}
\]

and the second one is trivial.

If \(B(p_0)\) and \(C(p_0)\) are assumed to be polynomials of degrees \(n\) and \(m\) respectively, then the left-hand side of (23) is a polynomial of degree \(\max(n + m + 2, n + 3)\) and by the same reasons as in even case only \(m = 1\) makes sense. If \(b_n = 1\) and \(C(p_0) = c_1p_0 + c_0\) then we have a system of \((n + 4)\) equations for \((n + 4)\) unknowns including \(n\) coefficients of \(B(p_0)\), 2 coefficients of \(C(p_0)\) and \(g_2, g_3\). The first equation of this system

\[
16(n + 2) \left( n^2 + 4n + c_1 + \frac{15}{4} \right) = 0
\]

implies

\[
c_1 = -\frac{15}{4} - n(n + 4).
\]

Theorem 6 Coefficients \(b_i\) are found from a recurrent relation

\[
\begin{align*}
    16b_{i-3} (i - 3) (i - 4) (i - 5) + 144b_{i-3} (i - 3) (i - 4)
    &- 8g_2b_{i-1} (i - 1) (i - 2) (i - 3) + (16c_1 + 300)b_{i-3} (i - 3)
\end{align*}
\]
One can show that constant \( c \) can be found from (25):

\[-44g_3b_1(i - 1)(i - 2) - 96g_2b_{i-1}(i - 1)(i - 2) + 8(4c_1 + 15)b_{i-3}
+ 32c_0b_{i-2}(i - 2) + 8(4c_1 + 15)g_2^2b_{i-3}b_{i+1}(i - 1)(i + 1)
+ g_2^2b_{i+1}(i - 1)(i + 1) - 36g_3b_3(i - 1) - 4(c_1 + 16)g_2b_{i-1}(i - 1)
+ 24c_0b_{i-2} + 2g_3g_2b_{i+2}(i + 1)(i + 2) + 3g_2^2b_{i+1}(i + 1)
+ (-4c_0g_2 - 4g_3(c_1 + 12))b_1i - 2(2c_1 + 9)g_2b_{i-1}
+ b_{i+3}(i + 1)(i + 2)(i + 3)g_2^2 + 2(-c_0g_2 - g_3(c_1 + 6))b_i
+ 3g_2g_3b_{i+2}(i + 1)(i + 2) + (-4c_0g_3 + \frac{3g_2^2}{4})b_{i+1}(i + 1) = 0, (25)\]

where \( i = n + 2, n + 1, \ldots, 3 \), with initial conditions \( b_{n+5} = b_{n+4} = b_{n+3} = b_{n+2} = b_{n+1} = 0, b_n = 1, \) and \( c_1 \) is given by (24).

The above theorem gives us \( b_j, j = 0, n - 1 \), as functions of \( c_0, g_2 \) and \( g_3 \). The last three equations can be considered as equations for \( c_0, g_2, g_3 \) while \( b_j \) are assumed to be found from (25):

\[6b_3g_3^2 + \left(2n^2 + 8n - \frac{9}{2}\right)b_0 - 16c_0b_1 + 24b_2g_2\]
\[= 0, \]
\[\frac{15}{2}g_2^2b_2 + \left(-6c_0b_1 + 30b_3g_3 + b_04\left(n^2 + 16n - 3\right)\left(n^2 + 16n - 3\right)\right)g_2\]
\[+ 6g_3\left(4b_4g_3 + \left(n^2 + 4n - \frac{25}{4}\right)b_1 - \frac{4}{3}c_0b_2\right) = 0, (26)\]
\[\frac{105g_2^2b_3}{4} + \left(84b_4g_3 - 10c_0b_2 + 4b_1\left(2n^2 + 8n - 13\right)\left(n^2 + 16n - 3\right)\right)g_2\]
\[+ 60b_5g_3^2 + \left(10n^2 + 40n - \frac{285}{2}\right)b_2 - 48c_0b_3\right)g_3 + 24b_0c_0 = 0.\]

One of solutions to (26) is trivial, i.e. \( g_3 = g_2 = c_0 = 0 \). In this case we have \( b_j = 0, j = 0, n - 1 \) due to (25), and \( \varphi(x) = (x + w_0)^{-2} \), where \( w_0 \) is a constant.

\[w(x) = \left(-\frac{15}{4} - n(n + 4)\right)(x + w_0)^{-2}, \quad z(x) = -2(x + w_0)^{-2n-3}. (27)\]

One can show that constant \( c_w \) for pair (27) is equal to zero and this case is therefore parabolic. General solution to the corresponding Lamé equation is

\[y(x) = \alpha_1(x + w_0)^{-3/2-n} + \alpha_2(x + w_0)^{n+5/2},\]

where \( \alpha_1 \) and \( \alpha_2 \) are constants.

Compliance with ethical standards
Conflict of interest  The authors declare that they have no conflict of interest.

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