The design learning of fraction with realistic mathematics education in elementary school

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Abstract. Fractions are one of the mathematics topics that is almost to be a problem for the elementary school. This problem arises because generally, students do not understand the concept of fraction. Fractional operations involve more procedures that must be taken than integer operations, where some of the results of the operation produce numbers that are difficult to understand logically. This research is design research designed to provide an excellent consideration of learning process of fractional operations through allegations built-in framework analysis hypothetical learning trajectory (HLT), which later tested in realistic mathematics education (RME). RME underlies all research design research activities. The results of this study are drawn from local instructional theory in the form of a high consideration of the learning process that provides general answers to a topic being taught. Design research is carried out in three stages, namely preliminary design, teaching experiment consisting of cycle 1 and cycle 2, and the third stage of retrospective analysis. The study involved using a sample of 29 grade V students at Nurul Hasanah elementary school consisting of 4 students in cycle one and 25 students in the second cycle. The results show that the use of circle contexts and bar representations trigger the learning trajectory that students pass through in understanding fraction addition operations. The stages of learning addition operations by HLT are designed, namely: understanding fraction values as part of the whole, comparing two fractions of value, looking for fractions of value, carrying out the same summing operations, and performing the summarized operations not the same.

1. Introduction
One of the mathematical topics considered difficult by elementary students is fractions [1]. Fractions involve complex problems for students, ranging from fractions, presenting fractions in various forms, to operations involving fractions. Some operations that involve fractions are not even easily understood by reason, such as a fractional division operation that produces numbers higher than the divided numbers. This reality becomes a problem that occurs repeatedly and demands changes in learning.

Fractional learning demands complex reasoning [2, 3]. In addition to being able to perform number operations, students must be able to apply the right strategies so fraction operations can be carried out. Therefore, before students do fractional operations, students must first understand the meaning and form of fractions. In the mechanistic approach, the meaning and form of the fraction become the initial cognitive burden for students where there is a new jump in learning trajectory. Students find it difficult
to do the next learning because there is a space in the meaning, then choose suitable fractions in the transformation of different fractions. It becomes the next cognitive burden in which students are faced with complex situations that are fundamentally different from the learning experiences obtained by previous students.

Judging from the complexity of learning fractional operations, the presentation in learning fractional operations requires systematic and structured learning stages. Streefland [4] once proposed five stages of fractional operations, namely: fractional recognition by the level of students, arranging the strategy for delivering fractional material, sorting rules in operating equations on fractions, operating sums independently, and performing their results following the rules for fractional operations accurately. Streefland saw a gap between the context and the meaning of each of these stages, whereas context becomes an essential part of the meaning of the concept so that students can experience gradual learning trajectories. Presentation of context allows students to deal with real problems that arise from everyday life. Thus students will be familiar with the problems presented so that the meaning of the concepts presented will be readily understood.

By the principle of learning realistic mathematics, context becomes an essential part in instilling the concept gradually. This context is a phenomenon where students can be directly involved in it. Fractional learning can be presented in various forms, such as circles, length measurements, concrete object models [5], and so on. Van Den Heuvel-Panhuizen [6] once presented fractional learning with a bar estimation context. In that learning, students can divide the flat fields into certain parts that show fractional values. For compare between fractions, these two flat fields are presented in a congruent form (both flat planes have the same area) so that the students’ interpretation is not wrong in conceptualizing fractions as a particular part of the whole. Other researchers [7, 8] use the context of organized inner circles of phenomena to introduce students to fraction sharing operations. As Streffland [9] uses the context of circles in learning fractional operations, it has advantages compared to other contexts. The context of the circle through the division of the circle jurying allows students to present fractional contexts in a consistent form. It is indeed different from other contexts, such as square or rectangular, where students have various ways to divide the square or rectangle to produce different parts. However, to create flexible learning trajectories the use of other contexts such as square or rectangular can be done as part of the progressive stages towards more formal mathematical. In this study, we use the context of a square or rectangle as a fraction plot.

The use of circle and square or rectangular contexts is the context chosen in this study. Circle context is the initial context presented to students. The aim is to remind students of the concept of fractions. The chosen learning situation is how students can split round pizza so that the same pizza parts are obtained. In order for students to understand that certain parts are the same as the other parts in different pieces, students are directed to cut back the portion of the pizza that has been cut. This cut pizza has a fractional value.

2. Method
This research was designed using design research method. While the research stage refers to the stages of design research from Gravemeijer & Cobb [10], including (1) preparing for the experiment or preliminary design, (2) implementing design experiments, or teaching experiment, and (3) analysis of data review / analysis obtained from the previous stage (retrospective analysis). By the characteristics of design research, the purpose of this study provides an excellent consideration of the learning process and a decrease in empirical learning theory. Because the theory developed is empirical, the theory constructed from the results of this study is a local instructional theory that provides general answers to a topic taught. Meanwhile, the process of obtaining local instructional theory arises from a cyclical learning design process, so that the results of HLT also remain HLT or said to be the alleged local instructional theory. The scheme of this design research stage is explicitly illustrated in the following figure.
This research was conducted in the odd semester of the 2017/2018 academic year. As a subject, there were four students VA grade, 25 students VB at Nurul Hasanah Islamic Elementary School, Tangerang City. Various sources were collected from video recordings, documentation, written data, interviews, and observations to get information about students' understanding and mastery of the broken material.

At this stage, an analysis of a hypothetical learning trajectory (HLT) on fractional topics is conducted where students can construct formal knowledge from informal knowledge built through gradual mathematical processes [12]. The analysis was sourced/obtained from the literature review (including the results of previous research), analysis of students' learning obstacle on fractional topics, initial interviews with students, discussions with teachers, and discussions with experts before designing learning activities to be conducted. The results of this HLT analysis are then stated in the HLT table as presented in Table 1.

**Table 1. Analysis of a Hypothetical Learning Trajectory**

| Activity          | Learning Objectives                                      | Description of activity                                                                 | Conjecture Student Thinking                                                                 |
|-------------------|----------------------------------------------------------|----------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------|
| Cut Pizza         | Students can specify fractions                          | ▪ Students work in groups to cut pizzas that show specific fractions                     | ▪ Some students assume that certain parts of pizza are one                                   |
|                   |                                                          | ▪ Students work in groups to describe pieces of pizza on the circumference of the divided circle | ▪ Some students assume that a piece of pizza from a specific part is a fraction of that part rather than a fraction of the whole part |
| Jurying Fractions | Students can mention fractions worth.                    | ▪ Students work in groups to compare the same two fractions through several same fractions. | ▪ Students place jurying on the central circle (fractional place) to obtain the right fractional value. |
|                   |                                                          |                                                                                        | ▪ Students mention fractions of value by multiplying the numerator and denominator by the same number |
| Coloring fractions| Students can perform the addition of two fractions.      | ▪ In the activity of the student group, coloring fraction plots that show the results of the addition of 2 fractions | ▪ Students can color the plot correctly in the sum of the fractions that are called the same |
|                   |                                                          | ▪ The teacher encourages students by pointing out that fraction plots can be combined, only if the plots are mutually compatible or congruent | ▪ Students experience a deadlock to combine the plots colored in the total number of entries that are not the same |
3. Result and Discussion

This research focuses on the development of HLT by comparing the learning trajectory that is hypothesized with the actual learning trajectory. The analysis was carried out by observing the learning videos and the results of the students' work. Several segments in the learning video are transcribed to get a picture of the dynamic situation of students, especially about the proposed conjecture. Analysis of each student learning activity is described in the following description.

3.1. Learning 1: Cutting Pizza

In Activity 1, each group of students presented a pizza of the same size. Each group is asked to cut the pizza with the same size according to the instructions in the worksheet. The problem raised is how a teacher will share a pizza equitably with his students. Some groups get the task of dividing the pizza into two equal parts (if the students are divided into two people), some other groups are asked to divide the pizza into four equal parts (if there are four students). The question asked is how many parts of the pizza each child gets.

![Figure 2. Student activity when cutting pizza (Source: Personal Documents)](image)

Figure 2 shows the activity of students cutting the pizza into several parts. These pieces of pizza are part of a whole pizza. Students then take one slice of pizza and discuss it with group members to state how many pieces of pizza are taken from the pizza slices entirely.

To see the students' thinking process in cutting pizza, the following is a conversation from fragments in activity 1.

**Dialog 1**

Researcher : How does you divide pizza for two students, so that each student gets the same share?
Student 1 : The pizza is cut into two parts, sir?
Researcher : How to cut it, try you show?
Student 1 : (Student 1 is assisted by a group friend who cuts pizza into two parts)
Researcher : Okay. Take a piece of pizza now.

(Researchers take a piece of pizza and then show the pizza slice to other students)

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**Dialog 2**

Researcher : Okay. Then, you will ask student 2. How did student 2 share this pizza with four students, so that each student got the same portion?
Student 2 : I cut it like this. (Student 2 shows researchers a piece of pizza)
Researcher : Okay Student 2. Take this one now. (Researchers take a piece of pizza, then show it to another student)
Researcher : So, how many pieces of this pizza, Student 2?
Student 2 : A quarter, sir.
Researcher : Yes. You have other pieces of pizza now, about how many parts. Try Student 3?
(Researchers take different pieces of pizza)
Student 3 : It seems like a quarter, sir.
Researcher : According to Student 4?
Student 4 : Half, isn't it, sir. However, it is bigger than a quarter.
Researcher : How can Student 4 as certain how many parts of the pieces?
Student 4 : Yes. Do you have to know how to cut it, sir?

From the above conversation, it can be seen that students have been able to mention specific fractions, especially half and quarter fractions. Student 3 and student 4 have even been able to guess the fractional value of pizza slices. This condition shows that the conjecture is not proven. Student 4 even suggested that to find out the fractional value of the piece of pizza, you must know how the pizza was cut initially. In other words, the fractional value of a particular part must be connected to the full portion. The ability of students to mention a specific fraction value is at least visible from the way students describe the pieces of pizza that represent partial fractions.

**Figure 3.** Example of students in describing pizza cut into 2 parts

In Figure 3, the half fraction value represents the pizza cut into two parts. The left picture shows a picture of pizza being cut into two parts, each part of the value is half. While the right picture shows one part of the pizza that shows half fractions. This context is critical to be understood by students where this part of the pizza is no longer reconstructed as an entire pizza whose value is 1. Similarly, if the right piece of pizza is again cut into two parts, these pieces have a value of $\frac{1}{4}$.

### 3.2. Learning 2: Jurying Fractions

In the second lesson, students are directed to learning about fractions of value. The context used is the offline-to-circle snippet of a circle that shows a particular fraction that we call fraction jurying. This context is still related to previous learning where the circle of offline circles is identical to pizza slices. In order for students to be able to specify the fractional value of a circle in a circle, students are first guided to place a circle of circles on the inner circle in a full circle which we call the base circle.
As in the first learning, students' understanding of fractional values from fractional jurying is the primary key in studying fractions of value and operations involving fractions. To find out, students can pair the fracture offline in the appropriate base circle. Students may directly mention fractions like \(\frac{1}{2}\), \(\frac{1}{4}\), \(\frac{3}{4}\), but fragments such as \(\frac{1}{3}\), \(\frac{1}{5}\), and so on. Students must first test them through the base circle. In Figure 4, students pair \(\frac{1}{5}\) fractions to make sure that the jack of the fraction is \(\frac{1}{5}\).

After students understand the fractional value of a fractional jurying, the teacher then directs students to learn about fractions that are worth. There are two stages of learning that students must go through, (1) students compare two fractions, (2) students find themselves some fractions that are worth a fraction. The dynamics of the learning process in this session can be observed from the following fragments.

**Dialog 3**

**Researcher**: Try Student 5, show partial fractions?
**Student 5**: This one, sir.
**Researcher**: Okay, please write. Now try showing a two-quarter fraction.
**Student 5**: Take from a quarter, yes sir.
**Researcher**: Yes.
**Student 5**: This one, sir.
**Researcher**: Okay, please write next to it.
**Student 5**: Sir.
**Researcher**: Okay, try Student 5 to compare whether the two fractions are the same?
**Student 5**: How, sir?
**Researcher**: Try showing half fractions and quarter fractions. Is it the same?
**Student 5**: Same, sir.
**Researcher**: Next, try Student 5 looking for other fractions that are worth half a fraction. Try that one? (Researchers refer to one-sixth fractions)
**Student 5**: This one, sir?
**Researcher**: Yes, how many pieces should you take?
**Student 5**: (Thinking for a moment) I take 3, sir.
**Researcher**: What is the fraction?
**Student 5**: Sixth, yes sir.

The conversation excerpt above shows that students can mention fractions of value by taking a few pieces of jurying so that the combination of the fractions of this fraction will be the same shape as the first fractional jurying. Initially, students have a hard time showing that those who represent specific fractions have the same fractional value. This difficulty seems to be caused by the perception of students who are still fixated on the representation of the fractional value of a particular jurying. Combining two ranges of fractions into a specific fraction value requires advanced reasoning. However, the teacher has a strategy by re-showing the offline of the two fractions that are compared. In the next stage, students are directed to find other pieces of value. In this activity students can take any fraction of jingle, then take some of the same forms. This activity allows students to manipulate some fragments into other worthless fractions. Generally, students start from fractions with numerator values 1. Students' activities in manipulating fractions that are worth can be seen from the results of the following student work.
In Figure 5, students do manipulative activities to get fractions worth half a fraction. At first, the students divided the circle into four parts. Students then put two pieces of broken pieces each worth a quarter. The value of these two juring fruits becomes $\frac{2}{4}$ because there are two parts of 4 essential circle parts. Students then match the two jings of the fraction with the half-fracture juring so that it is $\frac{1}{2}$ equal to $\frac{2}{4}$. The next activity is to divide the circle into eight parts. Students then put four pieces of broken pieces each worth one eighth. The value of the two juring fruits is the same as $\frac{4}{8}$ because there are four parts of the eight parts in the base circle. The students then match the two jings of the fraction with the half-fracture juring so that it is $\frac{1}{2}$ equal to $\frac{4}{8}$.

### 3.3. Learning 3: Coloring Fractions

In learning 3, students are faced with the coloring activities of the plots on a bar representation that has been divided (we call it a fractional plot). These colored plots show specific fractional values such as juring taken from a whole circle. The coloring context of the fraction plot is used in learning addition operations. In this case, the students merge (place) the pieces of one another with each other if these plots have the same shape. There are two stages that students go through, namely: the sum operation with the same name and the addition operation with the same name is not the same.

In the same operation, students can efficiently perform addition operations because the size of the square plots of the two fractional plots has the same shape. However, for the denominations mentioned are not the same, students experience difficulties and tend to be wrong. Some students can immediately propose a solution by stating that the fractions must be equated first. It is inspired by previous learning where the sum operation can be done directly if the denominator of the summed fractions has the same denominator.

In Figure 6 the left-hand side of the student can directly add up the fractions because the fractions of each fraction added together have a corresponding area. As in the sense of adding two fractions, moving or combining the shards of shards in the left image is easy, because the plots that are moved or combined
precisely fill the other plots. While for the denominated fraction is not the same, as, in the right picture, direct transfer or combining of plots is difficult, because the shaded patches in the first fractional plot are not appropriate to fill the other fractional plots.

For fractional operations that are not the same, students are given the freedom to divide the pieces of fractions from a full rectangular shape. Previously students were guided to get fractions worth by dividing fractions by increasing the value of the denominator gradually so that the fractions that were named were the same. Students' way of thinking in the fragment of fraction addition operations is presented in the following conversation actions.

**Dialog 4**

Researcher : Try to see if you want to see, how does student 6 get the two-thirds and a quarter fraction?
Student 6 : (After a while), like this sir?
Researcher : If the two fractions are added up, can Student 6 be able to describe the results?
Student 6 : It is difficult, sir.
Researcher : Why?
Student 6 : Because the plots are not the same.
Researcher : So that the plots are the same, how Student 6 divides these rectangles.
Student 6 : Look for worth fractions.
Researcher : How to try Student 6, describe it.

(Researchers provide the opportunity for Student 6 to get a fraction worth of two-thirds fractions and a quarter fraction)

Student 6 : I divide it into twelve plots, sir.
Researcher : Okay. What is the two-thirds fraction?
Student 6 : For a fraction of two thirds to eighth, for a fraction of a quarter to three quarters
Researcher : Good, so now Student 6 can add it up?
Student 6 : Yes sir.

In the conversation above, it can be seen that students use their previous experience to get a fraction of the value. The key to success in this segment is how students can divide the rectangles to obtain the same number of fractional plots and get valuable fractions. It is possible for students to divide the fraction plot arbitrarily, but the plots do not necessarily represent fractions that are worth the requested fraction. The key to the next success is how students choose and utilize the appropriate fractions. Some students seem to fail at this step, as the example presented below.

![Figure 7](image-url)  
**Figure 7.** Example of student failure in fraction summing operations
In figure 7, there are at least two failures experienced by students. First, the student has not received the denomination with the same denominator; both students do not take advantage of the fractions of the value obtained in the previous step. The main trigger can be because students do not get the same denominator. Also, students assume that the rectangle provided for two-thirds fractions is only three pieces. There should be four pieces because the fraction plot begins to be described for two-thirds fractions.

Cramer, Behr, Post, & Lesh [3], revealed that the initial difficulty faced by students in fractions is to internalize the symbol of a fraction that denotes a number. The presentation of the fractional symbol appears to be complicated for students where the higher the number of the denominator, the smaller the fraction value. It is very different from the number in which the more significant numbers indicate that the value of the number is higher. This problem is then solved through the pizza context. The teaching experiment results show that students can easily understand specific fraction values from pizza slices. The more pizzas are divided, then the pieces of pizza will show smaller fractions.

The use of pizza context further encourages students to experience learning trajectories. At first, students know fractions as part of the whole. Students are then directed to declare a fraction of one or several pieces of circle jurying instead of pieces of pizza. To make sure that the offline-to-jiggle piece of the circle shows a certain fraction, students place the jings on the base circle. The offline placement must be right to meet the jurying to jurying paintings with the same area as fractional jurying. Also, the fractional use of fractions also allows students to find out the equivalent fractions. Students can place a few pieces of the fraction which are then compared to the previous fraction. In the next stage, students perform fraction addition operations. The context used is fraction plots which when colored can indicate a particular fraction value. Summing up fractions means combining colored fractions so that these plots can be mapped to congruent plots. If these colored plots cannot be mapped, then the fraction addition operation cannot be done directly. Students are then given the freedom to make fractions in empty rectangles so that the plots between the first fraction with the second fraction are congruent.

Based on the stages through which students pass through the application of realistic mathematics education, the use of pizza contexts, fraction jurying, and fraction tiles trigger a comprehensive understanding of students towards fractions. It can be seen when the actual learning trajectory experienced by students can be passed step by step. Model changes to model for repeatedly encourage students to progressive mathematics in which students avoid the didactical break situation.

4. Conclusion
Based on the results of the study and the discussion that has been described, the use of circle contexts and bar representations triggers the learning trajectory that students pass through in understanding fraction addition operations. The stages of learning the addition operation by the HLT are designed, namely: understanding the fraction value as part of the whole, comparing two fractions of value, looking for fraction values, carrying out the same summing operations, and doing the same summation operations not the same.

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