Newton polyhedra and Poisson structures from certain linear Hamiltonian circle actions

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Abstract
In this paper we first describe the geometry of the Newton polyhedra of polynomials invariant under certain linear Hamiltonian circle actions. From the geometry of the polyhedra, various Poisson structures on the orbit spaces of the actions are derived and Poisson embeddings into model spaces, for the orbit spaces, are constructed. The Poisson structures, on respective source and model space, are compatible even for the minimum possible (embedding) dimension of the model spaces. This is, in particular, important since it is still an open question if, in general, there exist finite dimensional model spaces with Poisson structures compatible with the actions and the usual nondegenerate Poisson structure on the source spaces.

Contents
1 Prefatory notes 2
2 Introduction 2
3 Structure of the Newton polyhedra 3
4 Extending to affine toric varieties 5
5 Poisson embeddings and the polyhedra 8
  5.1 Faces resulting from squarefree generators 8
  5.2 Intertwining other faces 10
  5.3 Explicit structures and lifts 11
  5.4 Intertwined Poisson structures 13
  5.5 Additional squarefree generators 14
1 Prefatory notes

To understand the motivation for the subject at hand, it may be enlightening to visit the 3 examples in section 8.1 of [2], as well as the main result from that earlier paper: Namely, the rarity of the existence of Poisson structures, on the model spaces, compatible with the usual Poisson structure on the orbit spaces. In the paper at hand, we relax the strict requirements, used in [2], of starting with symplectic source spaces and instead equip the spaces with Poisson structures derived from the combinatorics observed by applying the Hamiltonian actions.

2 Introduction

In this introductory section of the paper, simple results obtained for the circle action

\[ T \times C^k \rightarrow C^k \]  

where \( T \) is the unit circle in \( C \) and \( n_1, \ldots, n_k \) are nonzero integers, called weights, are recalled. A more complete discussion is available in [2] §1-§3. A polynomial in \( C[z_1, \ldots, z_k, \bar{z}_1, \ldots, \bar{z}_k] \) is invariant under the action if and only if the exponents of each of its nonzero terms satisfy these conditions we designate them as actions, or integers, generating a minimal Hilbert basis [for the polynomials invariant under the action].

\[ \{ n_1, \ldots, n_k \} = 0. \]  

The usual basis for \( \mathbb{Z}^k \) is denoted by \( (e_1, \ldots, e_k) \), the ring of invariant polynomials is \( C[S_{n_1, \ldots, n_k}] \) and the group generated by \( S_{n_1, \ldots, n_k} \) in \( \mathbb{Z}^k \times \mathbb{Z}^k \) is denoted by \( M_{n_1, \ldots, n_k} \). It, i.e., \( M_{n_1, \ldots, n_k} \), consists of all solutions \((a, b) \in \mathbb{Z}^k \times \mathbb{Z}^k \) to \( n^*(a - b) = 0 \). The usual basis for \( \mathbb{Z}^k \times \mathbb{Z}^k \) is denoted by the elements \( e_1, \ldots, e_k, \bar{e}_1, \ldots, \bar{e}_k \).

Restricting to multiple \([k > 1]\) positive relative prime \([\gcd(n_1, \ldots n_k) = 1]\) weights \( n_1, \ldots, n_k \), let \( \iota \) be the map from the integer hyperplane \( n^* = \{ r \in \mathbb{Z}^k : r_1n_1 + \cdots + r_kn_k = 0 \} \) to the integer hyperplane \( \mathcal{I}^k = \{ t \in \mathbb{Z}^k : t_1 + \cdots + t_k = 0 \} \) given by

\[ \iota : n^* \rightarrow \mathcal{I}^k ; r \mapsto \frac{(n_1r_1, \ldots, n kr_k)}{d_1 \cdots d_k} \]

where each \( d_i \) is defined by \( d_i = \gcd(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_k) \). The following equivalences [i - iii] are observed in [2]: i) \( \iota \) is an isomorphism, ii) \( n_i = d_1 \cdots d_{i-1} d_{i+1} \cdots d_k \) for \( i = 1, \ldots, k \) and iii) the semigroup \( S_{n_1, \ldots, n_k} \) can be generated by exactly \( k^2 \) elements. If these conditions hold then the semigroup \( S_{n_1, \ldots, n_k} \) may be generated by the \( k^2 \) elements \( e_1 + \bar{e}_1, \ldots, e_k + \bar{e}_k \) and \( d_i e_i + d_j \bar{e}_j \) for \( i \neq j \). To distinguish circle actions, or integers \( n_1, \ldots, n_k \), that satisfy these conditions we designate them as actions, or integers, generating a minimal Hilbert basis [for the polynomials invariant under the action].

The group \( M_{n_1, \ldots, n_k} \) is a lattice of rank \( 2k - 1 \) and has a basis of \( 2k - 1 \) elements as follows.
Lemma 1 Assume that \( \iota \) is an isomorphism. Define elements \( l_1, \ldots, l_k \) by \( l_i = e_i + \tau_i \) and \( \eta_1, \ldots, \eta_{k-1} \) by \( \eta_i = d_i e_{i+1} + d_i \tau_i \). Then \( l_1, \ldots, l_k, \eta_1, \ldots, \eta_{k-1} \) is a basis for \( M_{n_1, \ldots, n_k} \).

Proof: Notice first that \( T^\perp \) has a \( \mathbb{Z} \)-basis \( \{ e_2 - e_1, \ldots, e_k - e_{k-1} \} \) so \( n^\perp \) has a basis containing the \( k-1 \) elements \( \iota^{-1}(e_{i+1} - e_i) = d_{i+1} e_{i+1} - d_i e_i = \eta_i - d_i l_i \) for \( i = 1, \ldots, k-1 \). Take \( x = (a, b) \in M_{n_1, \ldots, n_k} \). Let \( L = b_1 l_1 + \cdots + b_k l_k \), then \( x - L \in n^\perp = \text{span}_\mathbb{Z} \{ \eta_1 - d_1 l_1, \ldots, \eta_{k-1} - d_{k-1} l_{k-1} \} \). Conclude that \( x \) is in the \( \mathbb{Z} \)-span of the independent elements \( l_1, \ldots, l_k, \eta_1, \ldots, \eta_{k-1} \). \( \Box \)

3 Structure of the Newton polyhedra

The finitely generated semigroup \( S_{n_1, \ldots, n_k} \) spans, over \( \mathbb{R}_{\geq 0} \), a strongly convex rational polyhedral cone \( \mathbb{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k} \) in the \( 2k-1 \) dimensional vector space \( \mathbb{R} \cdot M_{n_1, \ldots, n_k} \). It is a convex polyhedral cone since it is the non-negative span of finitely many vectors in a vector space, strongly convex since it does not contain a line through the origin and rational because it is generated by elements in the lattice \( M_{n_1, \ldots, n_k} \). Assuming that the circle action generates a minimal Hilbert basis, the extreme rays [one dimensional faces or edges] of the cone are the \( k^2 \) half-lines \( \mathbb{R}_{\geq 0}(d_i e_i + d_j \tau_j) \).

Let \( \mathfrak{h} \) and \( \mathfrak{v} \) be subsets of \( \{ 1, \ldots, k \} \) and define \( \mathfrak{F}_{\mathfrak{h} \times \mathfrak{v}} \) to be the sub-cone of \( \mathbb{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k} \) given by the span

\[
\mathfrak{F}_{\mathfrak{h} \times \mathfrak{v}} = \mathbb{R}_{\geq 0} \cdot \{ d_i e_i + d_j \tau_j : (i, j) \in \mathfrak{h} \times \mathfrak{v} \}.
\]

We now prove the following.

Theorem 1 Assume that the integers \( n_1, \ldots, n_k \) generate a minimal Hilbert basis. Then the map

\[
\mathfrak{h} \times \mathfrak{v} \rightarrow \mathfrak{F}_{\mathfrak{h} \times \mathfrak{v}}
\]

defines a one-to-one relationship between subsets of \( \{ 1, \ldots, k \} \times \{ 1, \ldots, k \} \) of the form \( \mathfrak{h} \times \mathfrak{v} \) and the faces of \( \mathbb{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k} \).

In order to show this we use the following:

Lemma 2 A strongly convex polyhedral cone \( \tau \) contained as a subset in a strongly convex polyhedral cone \( \gamma \) is a face of \( \gamma \) if and only if for any \( x \) and \( y \) in \( \gamma \) with the sum \( x + y \) in \( \tau \) we have that \( x \in \tau \) and \( y \in \tau \).

Proof of lemma: First we assume that \( \tau \) is a face of \( \gamma \), say \( \tau = \gamma \cap v^\perp \) for some linear functional nonnegative on \( \gamma \). If \( x + y \in \tau \) then \( \langle v, x + y \rangle = 0 \) but also \( \langle v, x \rangle \geq 0 \) and \( \langle v, y \rangle \geq 0 \) so we must have \( \langle v, x \rangle = \langle v, y \rangle = 0 \).

Now, for the opposite direction, assume that for any \( x, y \in \gamma \) with \( x + y \in \tau \) we have that \( x, y \in \tau \). Let \( \tau' \) be a minimal face of \( \gamma \) containing \( \tau \). Let \( n = \dim(\tau') \).

Assume \( z \in \tau \cap \text{int}(\tau') \) [\( \text{int}(\tau') \) denotes the relative topological interior of \( \tau' \)]. For any \( x \) in a small \( n \)-ball in \( \tau' \) centered at \( z \) we can find \( y \) in the ball such that
Hence \( \tau \cap \operatorname{int}(\tau') \) is an open subset of \( \operatorname{int}(\tau') \), it is also closed in \( \operatorname{int}(\tau') \) since \( \tau \) is closed. The set \( \tau \cap \operatorname{int}(\tau') \) is nonempty by the minimal condition on \( \tau' \). Since \( \operatorname{int}(\tau') \) is connected we obtain that \( \operatorname{int}(\tau') \subset \tau \subset \tau' \).

Hence \( \tau' = \tau \), and in particular \( \tau \) is a face of \( \gamma \). \( \square \)

**Proof of theorem:** Using Lemma 2 it follows that each of the sets \( \tilde{\mathcal{H}}_{h \times v} \) is a face of \( \mathbb{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k} \). The map \( h \times v \to \tilde{\mathcal{H}}_{h \times v} \) is clearly injective. Assume that \( \tau \) is a face of \( \mathbb{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k} \). We can assume that \( \tau \) is spanned, over \( \mathbb{R}_{\geq 0} \), by some extreme rays. Therefore, \( \tau \) has the format \( \tau = \mathbb{R}_{\geq 0} \cdot \{ d_i e_i + d_j \tau_j : (i, j) \in i \} \) for a set of indices \( i \subset \{ 1, \ldots, k \} \times \{ 1, \ldots, k \} \). For elements \( (i', j') \) and \( (i'', j'') \) in \( i \) we have that \( (d_i e_i + d_j \tau_j) + (d_{i'} e_{i'} + d_{j'} \tau_{j'}) = (d_i e_i + d_{i'} e_{i'} + d_j \tau_j + d_{j'} \tau_{j'}) \) so by Lemma 2 we also must have \( (i', j'), (i'', j'') \in i \). From this it follows that \( \mathcal{H} \) has the cross product format \( h \times v \) for some \( h, v \subset \{ 1, \ldots, k \} \) and therefore that \( \tau = \tilde{\mathcal{H}}_{h \times v} \). \( \square \)

For a subset \( i \) of the finite \( k \)-lattice, \( i \subset \{ 1, \ldots, k \} \times \{ 1, \ldots, k \} \), we define

\[
\tilde{\mathcal{H}}_i = \tilde{\mathcal{H}}_{h \times v}
\]

where \( h \) and \( v \) are the smallest sets such that \( i \subset h \times v \). We refer to elements of \( h \) as horizontal lines and elements of \( v \) as vertical lines.

**Lemma 3** Assume that the integers \( n_1, \ldots, n_k \) generate a minimal Hilbert basis. Let \( i \) be a non-empty subset of the finite \( k \)-lattice and let \( h \) and \( v \) be minimal with respect to the inclusion \( i \subset h \times v \). Then the dimension of the face \( \tilde{\mathcal{H}}_i \) is given by

\[
\dim(\tilde{\mathcal{H}}_i) = |h| + |v| - 1.
\]

Furthermore, assume that \( v \in \mathbb{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k} \) and that

\[
v = \sum_{(i,j) \in i} c_{ij} (d_i e_i + d_j \tau_j)
\]

for positive terms \( c_{ij} \). Then \( \tilde{\mathcal{H}}_i \) is the smallest face of \( \mathbb{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k} \) containing \( v \), i.e., \( v \in \operatorname{int}(\tilde{\mathcal{H}}_i) \).

**Proof:** Let \( h \) and \( v \) be as in the Lemma. Assume that \( h \) is a proper subset of \( \{ 1, \ldots, k \} \), and fix \( j' \in v \). Adding a horizontal line \( i' \notin h \) introduces a new element \( d_{i'} e_{i'} + d_j \tau_j \) not in the vector space spanned by \( \langle d_i e_i + d_j \tau_j : (i, j) \in h \times v \rangle \). Let \( h' = h \cup \{ i' \} \), then \( \dim(\tilde{\mathcal{H}}_{h' \times v}) \geq \dim(\tilde{\mathcal{H}}_{h \times v}) + 1 \) by the above. A similar argument holds for the vertical lines. Since \( \dim(\tilde{\mathcal{H}}_{\{ 1, \ldots, k \} \times \{ 1, \ldots, k \}}) = 2k - 1 \), it follows that \( \dim(\tilde{\mathcal{H}}_i) = |h| + |v| - 1 \). The second part of the Lemma is a consequence of Lemma 2 and Theorem 1 since \( \tilde{\mathcal{H}}_i \) is the smallest face containing each of the elements \( d_i e_i + d_j \tau_j \) for \( (i, j) \in i \). \( \square \)

Using Lemma 3 one counts the number of faces of \( \mathbb{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k} \) using a simple combinatorial argument. The number of faces \( m_d \) of dimension \( d \) with
$0 \leq d \leq 2k - 1$ is given by the formula$^1$,

$$m_d = \binom{k}{1} \binom{k}{d} + \cdots + \binom{k}{d} \binom{k}{1}.$$  

In particular the number of faces of $\mathbf{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k}$ of codimension one is $m_{2k-2} = \binom{k}{k-1} \binom{k}{k} + \binom{k}{k} \binom{k}{k-1} = 2k$ and faces of codimension 2 are $m_{2k-3} = \binom{k}{k-2} \binom{k}{k} + \binom{k}{k-1} \binom{k}{k-1} + \binom{k}{k} \binom{k}{k-2} = k(2k-1)$ and so on.

**Example 1 (Visualizing the structure of $\mathbf{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k}$)**

Using Theorem 1 and Lemma 3 one can visualize the structure of the cones $\mathbf{R}_{\geq 0} \cdot S_{n_1, \ldots, n_k}$, at least for the minimal generators case mentioned in the Theorem. Figure 1 shows one way to structure the cone for $k = 3$. The extreme rays, $v_{ij} = \mathbf{R}_{\geq 0} \cdot (d_ie_i + d_je_j)$ are shown as nodes and the two-dimensional faces of $\mathbf{R}_{\geq 0} \cdot S_{n_1, n_2, n_3}$ are shown as edges connecting their generating rays/vectors.

![Figure 1: The cone $\mathbf{R}_{\geq 0} \cdot S_{n_1, n_2, n_3}$ in $\mathbf{R}^3$](image)

For the case, $k = 3$, shown on Figure 1 the number of faces is counted as follows: $m_1 = 9$ (nodes on the Figure), $m_2 = 18$ (shown as edges), $m_3 = 15$ (appear as triangles and quadrilaterals), $m_4 = 6$ (exclude a triangle from the Figure).

### 4 Extending to affine toric varieties

A maximal ideal $m$ in $\mathbf{C}[S_{n_1, \ldots, n_k}]$ determines, naturally, a ring homomorphism $\mathbf{C}[S_{n_1, \ldots, n_k}] \to \mathbf{C} = \mathbf{C}[S_{n_1, \ldots, n_k}] / m$ and therefore induces a homomorphism

$^1$Count all choices of $p$ horizontal lines and $q$ vertical lines with $p + q = d + 1$.  


between the monoids \((S_{n_1}, \ldots, n_k, 0, +)\) and \((C, 1, \cdot)\). Vice versa, a homomorphism \(u : (S_{n_1}, \ldots, n_k, 0, +) \to (C, 1, \cdot)\) extends to a ring homomorphism \(C[S_{n_1}, \ldots, n_k] \to C\), thereby determining a maximal ideal \(m = \ker(u)\) in \(C[S_{n_1}, \ldots, n_k]\). The constructions are bijective and it follows that the maximum ideals can be identified with the set of all homomorphisms \(u : (S_{n_1}, \ldots, n_k, 0, +) \to (C, 1, \cdot)\) denoted by \(\text{Hom}(S_{n_1}, \ldots, n_k, C)\) or \(\text{Hom}_{\text{e.g.}}(S_{n_1}, \ldots, n_k, C)\) here. Define the “conjugate” \(\overline{v}\) of an element \(v \in M_{n_1}, \ldots, n_k\) by interchanging \(e_i\) and \(\overline{e}_i\) for \(i = 1, \ldots, k\), e.g., \(e_i + \overline{e}_j = e_j + \overline{e}_i\). This defines a conjugate operator on the lattice such that \(\overline{S_{n_1}, \ldots, n_k} = S_{n_1}, \ldots, n_k\). Define an injection of the quotient space \(\mathbb{C}^k / \mathbb{T}\) into \(\text{Hom}(S_{n_1}, \ldots, n_k, C)\) of the circle action in the minimal generators case as follows: Take an element \((z_1, \ldots, z_k)\) from the orbit of an element \(z\) in \(\mathbb{C}^k / \mathbb{T}\) and define, using the invariant generators, \(u(z) \in \text{Hom}(S_{n_1}, \ldots, n_k, C)\) by \(u(z)(e_i + \overline{e}_i) = z^d, j = z^d, j\). The mapping \(z \mapsto u(z)\) is injective \(^2\), interestingly, it is injective independent of the assignments \(u(z)(e_i + \overline{e}_i)\). More generally, Hilbert basis resulting from linear actions of compact groups separate orbits, see for example \(\mathbb{F}\). But, as pointed out, only a simple argument is required here to show this for the orbits of the circle action at hand.

**Theorem 2** Assume that the integers \(n_1, \ldots, n_k\) generate a minimal Hilbert basis. The injection of the orbit space \(\mathbb{C}^k / \mathbb{T}\) into \(\text{Hom}(S_{n_1}, \ldots, n_k, C)\) consists of all \(x\) in \(\text{Hom}(S_{n_1}, \ldots, n_k, C)\) with \(x(\overline{v}) = x(v)\) for all \(v \in S_{n_1}, \ldots, n_k\) and \(x(e_i + \overline{e}_i) \geq 0\) for \(i = 1, \ldots, k\).

**Proof:** Denote by \(\tau\) the element in \(\text{Hom}(S_{n_1}, \ldots, n_k, C)\) defined by \(0 \mapsto 1\) and \(v \mapsto 0\) if \(v \neq 0\). For \((z_1, \ldots, z_k)\) from the orbit of some element \(z\) in \(\mathbb{C}^k / \mathbb{T}\), \(x = u(z)\) satisfies the stated conditions on the generating set \(\{e_i + \overline{e}_i\} \cup \{d_i e_i + d_i \overline{e}_i\}\) of \(S_{n_1}, \ldots, n_k\) and therefore also all of \(S_{n_1}, \ldots, n_k\). For the other direction: Take \(x \in \text{Hom}(S_{n_1}, \ldots, n_k, C)\) satisfying the conditions from the Theorem. First if \(x(e_i + \overline{e}_i) = 0\) for \(i = 1, \ldots, k\) then \(x(d_i e_i + d_i \overline{e}_i)^i = x(e_i + \overline{e}_i)^i x(e_j + \overline{e}_j)^j = 0\) for all \(i, j\), so \(x = \tau\). Now assume that \(x \neq \tau\), then for some \(i, x(e_i + \overline{e}_i) > 0\). For simplicity, assume \(x(e_i + \overline{e}_i) > 0\). Create a representative \((w_1, \ldots, w_k)\) for \(w \in \mathbb{C}^k / \mathbb{T}\) by \(w_1 = \sqrt{x(e_1 + \overline{e}_1)}\) and for \(i = 2, \ldots, k\) let \(w_i\) be any of the \(d_i\)-th roots of \(x(d_i e_i + \overline{d_i e_i})^i / w_1^i\). Then \(x(d_i e_i + \overline{d_i e_i})^i = w_i^d, j\). It follows that \(x(d_i e_i + \overline{d_i e_i}) = x(d_i e_i + \overline{d_i e_i}) x(x(d_1 e_1 + \overline{d_1 e_1}) / x(d_1 e_1 + \overline{d_1 e_1}) = x(d_i e_i + \overline{d_i e_i}) / x(e_1 + \overline{e}_1)^d = w_i^d, j\). Finally, since \(x(e_i + \overline{e}_i) \geq 0\) and \(x(e_i + \overline{e}_i)^d = x(d_i e_i + \overline{d_i e_i}) = (w_i^d, j)^d\) it follows that \(x(e_i + \overline{e}_i) = w_i\overline{w}_i\). In other words \(x = u(w)\) is in the image of \(u\) as required. \(\Box\)

By identifying \(\mathbb{C}^k / \mathbb{T}\) with its image under \(u\) one may write

\[
\mathbb{C}^k / \mathbb{T} \subset \text{Hom}_{\text{e.g.}}(S_{n_1}, \ldots, n_k, C).
\]

\(^2\)If \(u(z) = u(w)\) then either both \(z_i\) and \(w_i\) are zero or \(z_i^d w_i^{-d_i}\) is a unit constant \(c = \exp(\theta d_i)\) independent of \(i\). Assuming that all the pairs are nonzero, write \(z_i w_i^{-1} = \exp(\theta^d z_i w_i^{-d_i})\) for some integer \(q_i\) and find integers \(s_1, \ldots, s_k\) so that \(q_i + d_i s_i\) is independent of \(i\), i.e., it would be in \((q_1 + d_1 Z) \cap (q_2 + d_2 Z) \cap \cdots \cap (q_k + d_k Z)\) which is of the form \(q + d_1 \cdots d_k Z\) for some integer \(q\). Finally, for \(t = \exp(\theta^d z_i w_i^{-d_i})\) it holds that \(z_i = t^{s_i} w_i\) so \(z\) and \(w\) are in the same \(\mathbb{T}\)-orbit. The case when some of the pairs \(z_i\) and \(w_i\) are zero also follows from this one.
This identifies the orbit spaces as a nice subset of the maximal ideals, or points, of the affine toric variety \( \text{Spec}(\mathcal{C}[S_{n_1,\ldots,n_k}]) \), see Fulton [3] section 1.3 for a starting point. The description facilitates a study of the orbit space using the geometry of toric varieties.

**Example 2 (The dual cone \( S_{n_1,\ldots,n_k}^\vee \))**

Here we describe the dual lattice of \( M_{n_1,\ldots,n_k} \), which is identified with \( M_{n_1,\ldots,n_k} \) using the pairing defined below, and the dual cone of \( \mathbb{R}_{\geq 0} \cdot S_{n_1,\ldots,n_k} \). Using the basis \( l_1,\ldots,l_k,\eta_1,\ldots,\eta_{k-1} \) from Lemma 4 calculate

\[
v_{ij} = d_i e_i + d_j e_j = \left\{ \begin{array}{ll}
d_i l_i + \cdots + d_j l_j - \eta_i - \cdots - \eta_{j-1} & \text{if } i \leq j, \\
\eta_j + \cdots + \eta_{i-1} - d_{j+1} l_{j+1} - \cdots - d_{i-1} l_{i-1} & \text{otherwise.}
\end{array} \right.
\]

Using the basis, these calculations and by identifying the lattice \( M_{n_1,\ldots,n_k} \) with its dual lattice one obtains an inner product \( \langle , \rangle \) on \( M_{n_1,\ldots,n_k} \) satisfying

\[
\langle v_{ij}, l_m \rangle = \left\{ \begin{array}{ll}
+d_m & \text{if } i \leq m \leq j, \\
-d_m & \text{if } j < m < i,
0 & \text{otherwise.}
\end{array} \right. \\
\langle v_{ij}, l_m \rangle = \left\{ \begin{array}{ll}
-1 & \text{if } i \leq m < j, \\
+1 & \text{if } j \leq m < i,
0 & \text{otherwise.}
\end{array} \right.
\]

An element \( x = a_1 l_1 + \cdots + a_k l_k + b_1 \eta_1 + \cdots + b_{k-1} \eta_{k-1} \) is nonnegative on the semigroup \( S_{n_1,\ldots,n_k} \) if \( \langle v_{ij}, x \rangle \geq 0 \) always, and, by the above,

\[
\langle v_{ij}, x \rangle = a_i d_i + \cdots + a_j d_j - b_i - \cdots - b_{j-1} \quad \text{if } i \leq j
\]

\[
\langle v_{ij}, x \rangle = -a_{j+1} d_{j+1} - \cdots - a_{i-1} d_{i-1} + b_j + \cdots + b_{i-1} \quad \text{if } j < i.
\]

The set of all lattice points that are nonnegative on \( S_{n_1,\ldots,n_k} \) under the pairing is denoted by \( S_{n_1,\ldots,n_k}^\vee \); it spans a strongly convex rational polyhedral cone \( \mathbb{R}_{\geq 0} \cdot S_{n_1,\ldots,n_k}^\vee \) dual to the cone \( \mathbb{R}_{\geq 0} \cdot S_{n_1,\ldots,n_k} \). So, by the above formula, if \( x \) is in \( \mathbb{R}_{\geq 0} \cdot S_{n_1,\ldots,n_k}^\vee \) then all \( a_i \geq 0, b_j \geq 0 \), but this is not a sufficient condition for \( x \) to belong to the dual cone. The elements \( x_1,\ldots,x_k \) and \( y_1,\ldots,y_k \) given by

\[
x_1 = l_1 \quad \text{and} \quad x_2 = l_2 + d_2 \eta_1,\ldots,x_k = l_k + d_k \eta_{k-1},
\]

\[
y_1 = l_1 + d_1 \eta_1,\ldots,y_{k-1} = l_{k-1} + d_{k-1} \eta_{k-1} \quad \text{and} \quad y_k = l_k
\]

are in \( S_{n_1,\ldots,n_k}^\vee \). It follows from Lemma 2 and the above that each of these \( 2k \) elements spans a one-dimensional face of \( \mathbb{R}_{\geq 0} \cdot S_{n_1,\ldots,n_k}^\vee \). The number of such rays is equal to the number of codimensional one (facets) of the original cone \( \mathbb{R}_{\geq 0} \cdot S_{n_1,\ldots,n_k} \) which was determined to be \( n_{2k-2} = 2k \). Therefore these are all the rays.

By denoting the standard basis for \( \mathbb{Z}^{k^2} \) as \( \{e_{ij} : i,j = 1,\ldots,k\} \) define an epimorphism \( F_k : \mathbb{Z}^{k^2} \rightarrow M_{n_1,\ldots,n_k} \) given by \( e_{ij} \mapsto d_i e_i + d_j e_j \) if \( i \neq j \) and \( e_{ii} \mapsto e_i + \overline{e}_i \). This function is used extensively in the remaining sections of the paper.
Example 3 \((F_3)\)

The map \(F_3\) has matrix representation

\[
F_3 = \begin{bmatrix}
\eta_1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\
\eta_2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\
l_1 & 0 & 0 & 1 & 0 & 0 & 0 & d_1 & 0 & d_1 \\
l_2 & 0 & 0 & 0 & 1 & 0 & -d_2 & d_2 & d_2 & d_2 \\
l_3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & d_3 & d_3 
\end{bmatrix}
\]

in the bases shown and its kernel is given by the image of the matrix

\[
\begin{bmatrix}
e_{21} & 1 & -1 & 0 & -1 \\
e_{32} & 1 & 0 & -1 & -1 \\
e_{11} & 0 & d_1 & 0 & d_1 \\
e_{22} & -d_2 & d_2 & d_2 & d_2 \\
e_{33} & 0 & 0 & d_3 & d_3 \\
e_{31} & -1 & 0 & 0 & 0 \\
e_{12} & 0 & -1 & 0 & 0 \\
e_{23} & 0 & 0 & -1 & 0 \\
e_{13} & 0 & 0 & 0 & -1 
\end{bmatrix}
\]

Given a finite sequence of integers \(n_1, \ldots, n_k\) generating a minimal Hilbert basis, one may always extend it to a longer such sequence by adding the integer \(d = d_1 d_2 \cdots d_k\) repeatedly to the sequence. The resulting cones \(R_{\geq} \cdot S_{n_1, \ldots, n_k, d, \ldots, d}\) contain the original cone \(R_{\geq} \cdot S_{n_1, \ldots, n_k}\) as one of its faces according to Theorem \(\text{I}\). Similarly, the supporting lattice \(M_{n_1, \ldots, n_k, d, \ldots, d}\) contains \(M_{n_1, \ldots, n_k}\) by identifying the \(2^k - 1\) vectors \(l_1, \ldots, l_k, \eta_1, \ldots, \eta_{k-1}\) from Lemma \(\text{II}\) for both lattices. The orthogonal projection \(\pi : M_{n_1, \ldots, n_k, d, \ldots, d} \to M_{n_1, \ldots, n_k}; \sum t_i l_i + \sum s_j \eta_j \mapsto \sum_{i=1}^k t_i l_i + \sum_{j=1}^{k-1} s_j \eta_j\) is the identity map on \(S_{n_1, \ldots, n_k}\) and so maps \(S_{n_1, \ldots, n_k, d, \ldots, d}\) surjectively onto \(S_{n_1, \ldots, n_k}\). It also induces an injection \(\pi^* : \text{Hom}_{\text{s.g.}}(S_{n_1, \ldots, n_k}, C) \to \text{Hom}_{\text{s.g.}}(S_{n_1, \ldots, n_k, d, \ldots, d}, C)\).

5 Poisson embeddings and the polyhedra

5.1 Faces resulting from squarefree generators

The standard Poisson structure on \(R^{2k}\) extends to the invariant polynomials \(C[S_{n_1, \ldots, n_k}]\) as described in \(\text{II}\) and is given there by the simple bracket

\[
\{X^a, X^b\} = -2t \sum_{i=1}^{k} (a_i \bar{b}_i - \bar{a}_i b_i) X^{a+b-l},
\]
on monomials. This formula does not appear to represent the symmetry seen on Figure 12 from Example 1 very well, since, on the one hand, it favors the extremal rays generated by \( l_i (v_{ij}) \) over \( v_{ij} \), \( (i \neq j) \). On the other hand, each of the extremal rays \( v_{ij} \) is attached to a similar structure of faces: referring to Figure 12 for \( k = 3 \), this structure is always four lines (2-dim faces), two triangles and four quadrilaterals (3-dim faces), and four 4-dim facets each obtained by removing one of the other four triangles.

A family of Poisson brackets, that includes the standard Poisson algebra and is based on the structure of the polyhedral cone \( \mathbb{R}_{\geq 0}^n \) defined in the following lemma. It is assumed that the integers \( n_1, \ldots, n_k \) generate a minimal Hilbert basis and as before: \( d_i = \text{gcd}(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_k) \). The lemma connects squarefree monomials (e.g., see [5] for a discussion of Stanley-Reisner ideals) from the Hilbert basis of the \( T \) invariants and Poisson structures on the \( T \) orbit space in a simple way.

**Lemma 4** Let \( \epsilon = (\epsilon_{ij}) \) be a real \( k \times k \) matrix satisfying \( \epsilon_{ij} = 0 \) if \( d_i \neq d_j \). The bilinear antisymmetric bracket \( \{ \, , \} \) on \( \mathbb{C}[S_{n_1}, \ldots, n_k] \) determined by monomials by

\[
\{ X^a, X^b \}_\epsilon = -2i \sum_{ij} \epsilon_{ij} (a_i \overline{b}_j - \overline{a}_i b_j) X^{a+b-e_i-e_j}
\]

is a Poisson bracket on \( \mathbb{C}[S_{n_1}, \ldots, n_k] \).

**Proof:** Note first that the formula \( d_i = d_j \) always implies that \( X^{\epsilon_{ij} \overline{v}_j} \) is an invariant since either \( i = j \) or \( d_i = d_j = 1 \) if \( d_i \) and \( d_j \) are equal. If for \( a, b \in S_{n_1}, \ldots, n_k \) and \( i, j \) with \( d_i = d_j \) it holds that \( a + b - \epsilon_i - \overline{v}_j \notin S_{n_1}, \ldots, n_k \), then \( a_i + b_i - 1 < 0 \) or \( \overline{a}_i + \overline{b}_i - 1 < 0 \), but since \( a_i, b_i, \overline{a}_i \), and \( \overline{b}_i \) are all nonnegative it follows that \( a_i = b_i = 0 \) or \( \overline{a}_i = \overline{b}_i = 0 \) and therefore \( (a_i \overline{b}_j - \overline{a}_i b_j) X^{a+b-e_i-e_j} = 0 \). This shows that the bracket \( \{ \, , \} \), maps \( \mathbb{C}[S_{n_1}, \ldots, n_k] \times \mathbb{C}[S_{n_1}, \ldots, n_k] \) into \( \mathbb{C}[S_{n_1}, \ldots, n_k] \). Leibniz identity follows from verifying the formula \( \{ X^a, X^b X^c \}_\epsilon = \{ X^a, X^b \}_\epsilon X^c + \{ X^a, X^c \}_\epsilon X^b \) for \( a, b, c \in S_{n_1}, \ldots, n_k \) directly. Formally, write \( \{ X^a, X^b \}_\epsilon = \frac{1}{2} \sum_{ij} \epsilon_{ij} \partial X^a / \partial z_i \partial X^b / \partial \overline{v}_j - \partial X^a / \partial \overline{v}_i \partial X^b / \partial z_j \) is a Poisson algebra.

Using this formula one verifies Jacobi identity by extending the bracket to all polynomials in the variables \( z_i \) and \( \overline{v}_j \), i.e., \( \{ z_i, \overline{v}_j \} = -2i \epsilon_{ij} \). The identity now follows from antisymmetry and Leibniz identity. Consequently \( \mathbb{C}[S_{n_1}, \ldots, n_k] \), \( \{ \, , \} \) is a Poisson algebra.

Referring the reader to section 6 in [4] the Poisson bivector above is converted into real coordinates \( x, y \) on \( \mathbb{R}^2 \), satisfying \( x_i + y_i = z_i \) and \( x_j - y_j = \overline{v}_j \), by replacing \( \partial / \partial z_i \) with \( \frac{1}{2} (\partial / \partial x_i + i \partial / \partial y_i) \) and \( \partial / \partial \overline{v}_j \) with \( \frac{1}{2} (\partial / \partial x_i - 1 \partial / \partial y_i) \). Assuming that \( \epsilon \)

\[
\begin{align*}
\{ f, g \} = (f, g) = \{ f, h \} g \\
\{ f, g \} + \{ g, h \} f + \{ h, f \} = 0
\end{align*}
\]
is symmetric this results in the coordinate change
\[ -2i \sum_{ij} \epsilon_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} = \sum_{ij} \epsilon_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}. \]

In particular, the right hand side bivector is a true real valued, as appose to complex valued, Poisson bivector on \( \mathbb{R}^{2k} \) - as a result of the symmetry of \( \epsilon \). Its rank on \( \mathbb{R}^{2k} \) is given by the rank of the matrix
\[
\begin{pmatrix}
0 & \epsilon \\
-\epsilon & 0
\end{pmatrix}
\]

The epimorphism \( F_k : \mathbb{Z}^{k^2} \to M_{n_1,\ldots,n_k} \), defined in section 4 induces an algebra morphism \( \mathbb{C}[X_{ij}] \to \mathbb{C}[S_{n_1,\ldots,n_k}] \) by \( F_k(X_{ij}) = X F_k(\epsilon_{ij}) \). A Poisson bivector on \( \mathbb{C}[X_{ij}] \) is considered real valued, see 5, if it contains no imaginary part after being transformed into real coordinates according to the mappings:

- \( X_{ij} \mapsto R_{ij} + I_{ij} \) and \( X_{ji} \mapsto R_{ij} - I_{ij} \) for \( i < j \),
- \( \frac{\partial}{\partial x_{ij}} \mapsto \frac{1}{2}(\frac{\partial}{\partial y_{ij}} - \frac{1}{2}\frac{\partial}{\partial t_{ij}}) \) and \( \frac{\partial}{\partial y_{ij}} \mapsto \frac{1}{2}(\frac{\partial}{\partial x_{ij}} + \frac{1}{2}\frac{\partial}{\partial t_{ij}}) \) for \( i < j \).

Such a real bivector determines a Poisson algebra in the \( k^2 \) variables \( X_{ij}, R_{i<j} \) and \( I_{i<j} \) over \( \mathbb{R} \) and determines a Poisson structure on \( \mathbb{R}^{k^2} \).

**Theorem 3** A Poisson structure of the form \( \{ \ , \}_\epsilon \) on \( \mathbb{C}[S_{n_1,\ldots,n_k}] \), with \( \epsilon \) symmetric and \( \epsilon_{ij} = 0 \) unless \( d_i = d_j = 1 \), is \( F_k \) related to a real Poisson structure on \( \mathbb{C}[X_{ij}] \).

**Proof:** Define a bilinear bracket \( \{ \ , \}_\epsilon \) on \( \mathbb{C}[X_{ij}] \) using the formula
\[
\{X_{pq}, X_{st}\} = -2i(\epsilon_{pt}X_{sq} - \epsilon_{sq}X_{pt})
\]
and by extending it to all the polynomials by way of Leibniz identity and bilinearity. It is \( F_k \) related to \( \{ \ , \}_\epsilon \) since \( F_k(\{X_{pq}, X_{st}\}) = -2i(\epsilon_{pt}X F_k(\epsilon_{sq}) - \epsilon_{sq}X F_k(\epsilon_{pt})) \), as a result of the condition \( \epsilon_{ij} = 0 \) if not both \( d_i \) and \( d_j \) are equal to one. The new bracket satisfies Jacobi identity because its Jacobiator \( \mathcal{J}^6 \) maps triplets \( A, B, C \) from the set of indeterminants \( \{ X_{ij}\} \) into linear polynomials which again map to zero under \( F_k \) since \( F_k \circ \mathcal{J} = 0 \) by Jacobi identity for the original bracket \( \{ \ , \}_\epsilon \). The only linear polynomial that maps to zero under \( F_k \) is zero itself so it follows that Jacobi identity is also satisfied for the derived bracket \( \{ \ , \}_\epsilon \). The bracket \( \{ \ , \}_\epsilon \) is real since it is the unique lift under \( F_k \) of the real Poisson bracket \( \{ \ , \}_\epsilon \) to a linear Poisson structure on \( \mathbb{C}[X_{ij}] \). \( \triangleright \triangleright \triangleright \)

### 5.2 Intertwining other faces

Theorem 3 connects the \( T \) invariant Poisson structure \( \{ \ , \}_\epsilon \) on \( \mathbb{R}^{2k} \) and the special face \( \mathfrak{f}_{1\times 1} \) where \( l = \{ i : d_i = 1 \} \) of the polyhedral cone \( \mathbb{R}_{\geq 0} \cdot S_{n_1,\ldots,n_k} \).

\[ F_k(\{ f, g \}) = \{ F_k(f), F_k(g) \} \]
\[ \mathcal{J}(A, B, C) = \{ \{ A, B \}, C \} + \{ \{ B, C \}, A \} + \{ \{ C, A \}, B \} \]
In order to accommodate the other faces also, below consider a generalization denoted by \( \{ , \}^\delta \) and given by:

\[
\{ X^a, X^b \}^\delta = -2i \sum_{ij} \epsilon_{ij} (a_i \bar{b}_j - \bar{a}_j b_i) X^{a+b+\delta_{ij}F_k(e_{ij})}
\]

where \( \delta \) is taken to be an integer matrix. The bracket is extended to all of \( \mathbb{C}[z, \bar{z}] \) by the formulas \( \{ z_i, \bar{z}_j \}^\delta = -2i \epsilon_{ij} (z_i \bar{z}_j) X^{\delta_{ij}F_k(e_{ij})} \) and zero on other pairs and via linearity, antisymmetry and Leibniz identity. Consequently, it is required that \( \delta_{ij} \geq -1 \) if \( d_i = d_j \) and \( \delta_{ij} \geq 0 \) otherwise. Requiring both \( \epsilon \) and \( \delta \) to be symmetric and then converting the resulting bivector to real coordinates results in a real bracket as follows:

\[
\frac{1}{2} \sum_{i<j} \text{Re}\{z_i, \bar{z}_j\}^\delta \left( \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_j} \right) - \frac{1}{2} \sum_{ij} \text{Im}\{z_i, \bar{z}_j\}^\delta \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}.
\]

To establish what conditions are needed for Jacobi identity to hold, first note that the Jacobiator for the bracket is always zero when applied to triplets of the form \((z_p, z_q, z_r)\) and \((\bar{z}_p, \bar{z}_q, \bar{z}_r)\). For the mixed triplets \((z_p, z_q, \bar{z}_r)\) and \((\bar{z}_p, \bar{z}_q, z_r)\) it holds that

\[
\overline{J(z_p, z_q, \bar{z}_r)} = \overline{J(\bar{z}_p, \bar{z}_q, z_r)} =
\]

\[
-4\epsilon_{pr}\epsilon_{qr}(\delta_{pr}d_p^0 - \delta_{qr}d_q^0)z_pz_q\bar{z}_r X^{\delta_{pr}F_k(e_{pr}) + \delta_{qr}F_k(e_{qr})}.
\]

where the notation \( d_i^0 \) is used to denote \( d_i \) if \( i \neq j \) and \( d_i^0 = 1 \). The results are summarized as follows:

**Corollary 1** The bracket \( \{ , \}^\delta \) is Poisson on the polynomial algebra \( R[x, y] \) in 2\( k \) variables if: \( \epsilon \) and \( \delta \) are symmetric matrices, \( \epsilon \) has real coefficients, \( \delta \) has integer coefficients satisfying \( \delta_{ij} \geq -1 \) when \( d_i = d_j \) and otherwise \( \delta_{ij} \geq 0 \), and for each triplet \( p, q, r \) the equation \( \epsilon_{pr}\epsilon_{qr}(\delta_{pr}d_p^0 - \delta_{qr}d_q^0) = 0 \) holds. Furthermore, in this case, if \( f \) and \( g \) are \( T \) invariant polynomials then so is \( \{ f, g \}^\delta \).

### 5.3 Explicit structures and lifts

**Example 4** (\( \delta_{ii} = d_i \) and \( \delta_{ij} = 1 \) if \( i \neq j \))

A simple way to have \( \{ , \}^\delta \) satisfy Jacobi identity is to fix \( \delta \) by defining \( \delta_{ii} = d_i \) and \( \delta_{ij} = 1 \) if \( i \neq j \), in this case the equations \( \epsilon_{pr}\epsilon_{qr}(\delta_{pr}d_p^0 - \delta_{qr}d_q^0) = 0 \) are always satisfied. The resulting Poisson bracket may be written out as \( \{ z_i, \bar{z}_j \}^\delta = -2i\epsilon_{ij} z_i^d + \epsilon_{ij}^0 \).

**Example 5** (\( \delta = 0 \))

The bracket \( \{ z_i, \bar{z}_j \}^0 \) satisfies Jacobi identity if \( \delta \) is the zero matrix. Letting \( \delta = 0 \) results in a bracket given by \( \{ z_i, \bar{z}_j \}^0 = -2i\epsilon_{ij} z_i \bar{z}_j \). Now lift \( \{ , \}^0 \) to \( \mathbb{C}[X_{ij}] \) as follows: Define a bilinear bracket \( \{ , \} \) on \( \mathbb{C}[X_{ij}] \) using the formulas

\[
\{ X_{pq}, X_{st} \} = -2i(\epsilon_{ps}d_p^q d_t^s - \epsilon_{qs}d_q^p d_t^s) X_{pq} X_{st}
\]
and extend it to all the polynomials using Leibniz identity. The new bracket is defined in such a way that it is \( F_k \) related to \{ \cdot, \cdot \}_c. The new bracket also satisfies Jacobi identity: Calculating \{ \{X_{pq}, X_{st}\}, X_{ij}\} results in \{ \{X_{pq}, X_{st}\}, X_{ij}\} = -4E_{pq}^{ab}(E_{ij}^{ab} + E_{ij}^{ac})X_{pq}X_{st}X_{ij} \) where \( E_{pq}^{ab} = \epsilon_{ad}d_p^d \), \( d_q^d \) and \( d_c^d \) and Jacobi identity for \{ \cdot, \cdot \} now follows from calculating the other parts of the Jacobiator \( \mathfrak{J}(X_{pq}, X_{st}, X_{ij}) \) and using \( E_{ij}^{ab} = -E_{ij}^{ac} \) to cancel terms - or, even simpler, by using that \( F_k \circ \mathfrak{J} = 0 \). When real coordinates are introduced on \( C[X_{ij}] \), see discussion before Theorem 3, the resulting conjugate operator satisfies \( \mathcal{X}_{ij} = X_{ji} \) and the bracket therefore satisfies \( \{X_{pq}, X_{st}\} = \{X_{pq}, X_{st}\}_{st} \) assuming that \( \epsilon \) is symmetric. This condition guarantees that the bracket is real valued and as such restricts to a \( k^2 \) dimensional Poisson algebra on \( R[X_{ii}, R_{i<j}, I_{i<j}] \).

**Example 6** (\( \delta_{ij} = -1 \) if \( d_i = d_j = 1 \) and \( \delta_{ij} = 0 \) otherwise)

If \( \delta \) is fixed as: \( \delta_{ij} = -1 \) if \( d_i = d_j = 1 \) and \( \delta_{ij} = 0 \) otherwise, then the formulas \( \epsilon_{pq} = \epsilon_{qr} = (\delta_{pr}d_q^p - \delta_{qr}d_p^q) = 0 \) are satisfied by requiring additionally that \( \epsilon_{ij} = 0 \) whenever exactly one of \( d_i \) and \( d_j \) is equal to one. The resulting Poisson bracket is determined by

\[
\{z_i, z_j\}_\delta = -2\epsilon_{ij} if d_i = d_j = 1,
\]

\[
\{z_i, z_j\}_\delta = -2\epsilon_{ij} z_i z_j if d_i \neq 1 and d_j \neq 1.
\]

It may be lifted to an \( F_k \) related real bracket \{ \cdot, \cdot \} on \( C[X_{ij}] \) determined by \( \{X_{pq}, X_{st}\} = -2(\epsilon_{pq}P_{pq}^{st} - \epsilon_{pq}P_{pq}^{st}) \) where \( P_{pq}^{st} = X_{pq} \) if \( d_p = d_q = 1 \) and \( P_{pq}^{st} = d_p^d d_q^d X_{pq}X_{st} \) otherwise (similarly \( P_{pq}^{st} = X_{pq} \) if \( d_p = d_q = 1 \) and \( P_{pq}^{st} = d_p^d d_q^d X_{pq}X_{st} \) otherwise). This formula may also be written

\[
\{X_{pq}, X_{st}\} = -2(\epsilon_{pq}P_{pq}^{st}d_t^t \{ X_{pq}^{11} \ - \epsilon_{pq}d_t^t \}
\]

where the monomials shown are selected based on the following schema: polynomial I is used when \( d_p = d_t = 1 \) and II is used otherwise, also polynomial i is used when \( d_s = d_q = 1 \) otherwise polynomial ii is used to complete the formula. As in the previous example, this bracket is seen to be real valued since when real coordinates are introduces the resulting conjugate operator satisfies \( \{X_{pq}, X_{st}\} = \{X_{pq}, X_{st}\} \). In order to prove that the new bracket satisfies Jacobi identity, write \( \{X_{pq}, X_{st}\} = \{X_{pq}, X_{st}\}_A + \{X_{pq}, X_{st}\}_B \) where \( \{ \cdot, \cdot \}_A \) and \( \{ \cdot, \cdot \}_B \) are given by \( \{X_{pq}, X_{st}\}_A = -2(\epsilon_{pq} X_{pq} - \epsilon_{pq} X_{pt}) \) and \( \{X_{pq}, X_{st}\}_B = -2(\epsilon_{pq} X_{pq} - \epsilon_{pq} X_{pt}) \). Here \( \epsilon_{ab} \) is zero unless \( d_a = d_b = 1 \) in which case \( \epsilon_{ab} = \epsilon_{ab} \), similarly \( \epsilon_{ab} \) is zero unless \( d_a \neq 1 \) and \( d_b \neq 1 \) and then \( \epsilon_{ab} = \epsilon_{ab} \). As a result \( \epsilon_{ij} \epsilon_{ij} \epsilon_{ij} \epsilon_{ij} \) is the one used in the proof of Theorem 3 so it already satisfies Jacobi identity. The bracket \{ \cdot, \cdot \}_B is discussed in Example 6 and is shown there to satisfy Jacobi identity. It follows that the Jacobiator \( \mathfrak{J} \) for \{ \cdot, \cdot \} satisfies \( \mathfrak{J}(X_{pq}, X_{st}, X_{ij}) \) =

\[
\{\{X_{pq}, X_{st}\}_B, X_{ij}\}_A + \{\{X_{st}, X_{ij}\}_B, X_{pq}\}_A + \{\{X_{ij}, X_{pq}\}_B, X_{st}\}_A + \{\{X_{pq}, X_{st}\}_B, X_{ij}\}_A
\]
\[ \{ \{ X_{pq}, X_{st} \} , X_{ij} \} A + \{ \{ X_{st}, X_{ij} \} , X_{pq} \} B + \{ \{ X_{ij}, X_{pq} \} , X_{st} \} B. \]

Using \( \{ \{ X_{pq}, X_{st} \} , X_{ij} \} A = \)

\[-4(\epsilon_{ij} A d_p^t d_p^t) X_{pq} X_{st} + \epsilon_{ij} B d_q^t d_q^t X_{pq} X_{st}\]

\[= -A_{ij} B d_q^t d_q^t X_{pq} X_{st} - \epsilon_{ij} B d_q^t d_q^t X_{pq} X_{st}\]

and \( \{ \{ X_{pq}, X_{st} \} , X_{ij} \} B = \)

\[-4(\epsilon_{ij} A d_p^t d_p^t) X_{pq} X_{st} + \epsilon_{ij} B d_q^t d_q^t X_{pq} X_{st}\]

\[= -A_{ij} B d_q^t d_q^t X_{pq} X_{st} - \epsilon_{ij} B d_q^t d_q^t X_{pq} X_{st}\]

and similar formulas for the other terms in the expression for \( \mathcal{J}(X_{pq}, X_{st}, X_{ij}) \) one concludes that all the terms cancel when added together. It follows that the Jacobi identity for \( \{ , \} \) is satisfied.

### 5.4 Intertwined Poisson structures

The real Poisson structure \( \{ , \} \) on \( C[X_{ij}] \) defined in Example 6 will be denoted below as \( \{ , \}^\delta \) where \( \delta = \delta_{1} \times \delta_{t} \) denotes the face of \( R_{\geq 0} \cdot S_{n_{i}, \ldots, n_{t}} \) indexed by \( i = \{ i : d_i = 1 \} \). The same notation \( \{ , \}^\delta \) will be used to denote the real Poisson structure \( \{ , \}^\delta \) on \( C[z, \overline{z}] \) for \( \delta_{ij} = -1 \) if \( d_i = d_j = 1 \) and \( \delta_{ij} = 0 \) otherwise. According to the Example using this notation the brackets are \( F_k \) related by

\[ F_k(\{ f, g \}^\delta) = \{ F_k(f), F_k(g) \}^\delta \]

for polynomials \( f, g \) in \( C[X_{ij}] \). It is required that \( \epsilon \) be symmetric and \( \epsilon_{ij} \) is zero if exactly one of \( d_i \) and \( d_j \) is one. In order to simplify the notation further for \( \epsilon = id \), the identity matrix, the notation \( \{ , \}^\delta \) is reduced to \( \{ , \}^\delta_{id} \) on both spaces. The bracket \( \{ , \}^\delta \) is specified by the bivector \( \Pi^\delta \), satisfying \( \{ f, g \}^\delta = \Pi^\delta (df \wedge dg) \) under the usual pairing of bivectors and 2-forms and given by

\[ \Pi^\delta = -2i \sum_{d_i \neq 1} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial \overline{z}_i} - 2i \sum_{d_i \neq 1} z_i \overline{z}_i \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial \overline{z}_i}. \]

The more general notation \( \Pi^\delta \) is used to denote the bivector dual to the Poisson structure \( \{ , \}^\delta \) on \( C[X_{ij}] \) as well as the Poisson bivector dual to the real bracket \( \{ , \}^\delta \) on \( C[z, \overline{z}] \) where \( \epsilon \) is symmetric, \( \delta_{ij} = -1 \) if \( d_i = d_j = 1 \) and \( \delta_{ij} = 0 \) otherwise, and \( \epsilon_{ij} = 0 \) if exactly one of \( d_i \) and \( d_j \) is equal to one. The following theorem has been proven.

**Theorem 4** Assume that the positive integers \( n_1, \ldots, n_k \) generate a minimal Hilbert basis with respect to the Hamiltonian circle action \( T \times R^{2k} \rightarrow R^{2k} \) with weights \( n_1, \ldots, n_k \). Let each of \( d_i \) be the greatest common divisor of all the weights except \( n_i \) and let \( \delta \) be the face of the polyhedral cone \( R_{\geq 0}\cdot S_{n_1, \ldots, n_k} \) given
by $\mathfrak{g} = \mathfrak{g}_{1 \times 1}$ where $I = \{ i : d_i = 1 \}$. Then the Hilbert map $F_k^* = (F_k(X_{ij}))_{ij}$ restricted to $\mathbb{R}^{2k} \to \mathbb{R}^{k^2}$ projects to a Poisson embedding of the orbit space

$$(\mathbb{R}^{2k}/T, \Pi^\mathfrak{g}) \hookrightarrow (\mathbb{R}^{k^2}, \Pi^\mathfrak{g}),$$

for any symmetric real $k \times k$ matrix $\epsilon$ satisfying $\epsilon_{ij} = 0$ if $i \in I$ and $j \not\in I$.

About the Hilbert embedding: The term, “embedding”, is used since the orbit space $\mathbb{R}^{2k}/T$ may be assigned a smooth structure $C^\infty(\mathbb{R}^{2k}/T)$ of smooth $T$ invariant functions on $\mathbb{R}^{2k}$. From a theorem by Schwarz in [8] it follows that $F_k^* C^\infty(\mathbb{R}^{k^2}) = C^\infty(\mathbb{R}^{2k}/T)$ and according to Mather in [4] the Hilbert map is a proper embedding. For a more complete discussion of how to transfer differential geometry methods to the orbit spaces see [6]. Note also that in the discussion before Theorem 2 it is shown directly that the Hilbert map is injective on the orbit space since it is essentially the mapping $z \to u(z)$ from Theorem 2 even if the target space is different in that case.

The rank of the Poisson structure $\Pi^\mathfrak{g}$ on $\mathbb{R}^{2k}$ is always between $2|I|$ and $2k$. Explicitly, the rank of the Poisson structure at a point $z \in \mathbb{C}^k$ is just twice the number of indexes $i$ satisfying $z_i \neq 0$ or $i \in I$.

The discussion above and the theorem can be extended to faces $\mathfrak{F} = \mathfrak{F}_h \times \mathfrak{h}$ where $\mathfrak{h}$ is any subset of $I$. The resulting bracket on $\mathbb{R}^{2k}$ is determined, similarly as before, by

$$
\{ z_i, \overline{z}_j \}_\mathfrak{F}^\mathfrak{g} = -2\epsilon_{ij} \text{ if } i, j \in \mathfrak{h},
\{ z_i, \overline{z}_j \}_\mathfrak{F}^\mathfrak{g} = -2\epsilon_{ij} z_i \overline{z}_j \text{ otherwise}
$$

and $\epsilon_{ij} = 0$ if exactly one of $i$ or $j$ is in $\mathfrak{h}$. This Poisson structure then lifts under the Hilbert embedding to the Poisson structure on $C[X_{ij}]$ determined by:

$$
\{ X_{pq}, X_{st} \}_\mathfrak{F}^\mathfrak{g} = -2i(\epsilon_{pt} d_p d_t^{\mathfrak{F}}) \left\{ X_{sq} \right\}_{X_{pt}}^{i} - \epsilon_{sq} d_q d_s^{\mathfrak{F}} \left\{ X_{pt} \right\}_{X_{st}}^{i}
$$

where, similarly as before, polynomial I is used when $p, t \in \mathfrak{h}$ and II is used otherwise, and polynomial ii is used when $s, q \in h$ otherwise polynomial ii is used.

In particular, if $\mathfrak{h} = \emptyset$, so $\mathfrak{F}_{\mathfrak{h} \times \mathfrak{h}} = 0$, one obtains the bracket $\{ , \}_0^\mathfrak{g}$ from Example 5 which is independent of the weights $n_1, \ldots, n_k$. Even when the weights do not generate a minimal Hilbert basis, as required in the above, the structure $\{ , \}_0^\mathfrak{g}$ lifts to the target space of the Hilbert embedding to a (product) Poisson bracket related to $\Pi^0$ under the embedding.

### 5.5 Additional squarefree generators

Interestingly, the lifted Poisson structure $\Pi^\mathfrak{g}$ on $C[X_{ij}]$ for $\mathfrak{F} = \mathfrak{F}_h \times \mathfrak{h}$ where $\mathfrak{h}$ is a subset of the indexes $i$ satisfying $d_i = 1$ intertwines the simple product Poisson structure from Example 5 and the Poisson structure from Theorem 3.

To be exact, if $A$ and $B$ references the two Poisson structures this means that

$$
7 \circ \{ [a, b]_A, c \}_B + \circ \{ [a, b]_B, c \}_A = 0
$$

$$
7 \circ \{ [a, b]_A, c \}_B = \{ [a, b]_A, c \}_B + \{ [b, c]_A, a \}_B + \{ [c, a]_A, b \}_B
$$
as seen in the calculations in Example 6.

Let $n_1, \ldots, n_k$ be positive integer weights generating a minimal Hilbert basis and let $d_1, \ldots, d_k$ be as before. Define $d$ to be the product $d = d_1 \cdots d_k$. Let $T$ act on $\mathbb{C}^k$ with weights $n_1, \ldots, n_k$ and $T$ act on $\mathbb{C}^{2k}$ with weights $n_1, \ldots, n_k, d, \ldots, d$. The action on $\mathbb{C}^{2k}$ also generates a minimal Hilbert basis and to accommodate both actions the sequence $d_1, \ldots, d_k$ is extended by letting $d_{k+1} = \cdots = d_{2k} = 1$. A Hilbert basis for the invariants of the action on $\mathbb{C}^{2k}$ contains $z_1, \ldots, z_k$ and $z_{d_1}, \ldots, z_{d_k}$ with $i \neq j$ both from $\{1, \ldots, 2k\}$; the action has at least $k + k^2$ squarefree generators even if the original action on $\mathbb{C}^k$ may only produce $k$ squarefree generators. There are many ways to embed the space $\mathbb{C}^k$ into $\mathbb{C}^{2k}$ in ways compatible with the $T$ actions. Consider the following embedding

$$\mathbb{C}^k \hookrightarrow \mathbb{C}^{2k}; (z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_k, z_1^{d_1}, \ldots, z_k^{d_k}).$$

The embedding is $T$ compatible so it projects to the orbit spaces. The real Poisson structure $\Pi^T$ on $\mathbb{C}^{2k}$ derived from the cone $\mathbb{R}_{\geq 0}S_{n_1, \ldots, n_k, d, \ldots, d}$ by taking $\Sigma$ to be the face $\Sigma = \delta_{\mathbb{R}^m \times \mathbf{w}}$ with $\mathbf{w} = \{k + 1, \ldots, 2k\}$ has rank everywhere at least $2k$ and is determined by:

$$\{z_i, z_j\}^T = -2i z_i z_j \text{ and } \{z_{k+i}, z_{k+j}\}^T = -2i \text{ for } i = 1, \ldots, k.$$  

Using the Hilbert embedding $F_{2k}$ one obtains:

$$\frac{\mathbb{C}^k}{T} \hookrightarrow \left( \frac{\mathbb{C}^{2k}}{T}, \Pi^T \right) \overset{F_{2k}}{\hookrightarrow} (\mathbb{R}^{4k^2}, \Pi^\mathbf{w}).$$

It is interesting to compare the above with the Poisson embedding dimension, defined by Davis in [1] as the smallest possible dimension of the target space on which there exists a Poisson structure that pullback to the usual Poisson structure on $\mathbb{C}^k$ under the Hilbert embedding. In the previous sections, the nondegeneracy, i.e., full rank everywhere ($2k$), condition has been relaxed somewhat in order to allow Poisson embeddings of minimum embedding dimension. In this last section the embedding problem is yet again modified, requiring the target spaces to be of higher, but finite, dimension.

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That is determined by $\{x_i, y_i\} = 1$. 

15
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