Asymptotic behavior of solutions to a tumor angiogenesis model with chemotaxis–haptotaxis

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Abstract

This paper studies the following system of differential equations modeling tumor angiogenesis in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ ($N = 1, 2$):

\[
\begin{align*}
    p_t &= \Delta p - \nabla \cdot p\left(\frac{\alpha}{1+c} \nabla c + p \nabla w\right) + \lambda p(1 - p), \quad x \in \Omega, \; t > 0, \\
    c_t &= \Delta c - c - \mu pc, \quad x \in \Omega, \; t > 0, \\
    w_t &= \gamma p(1 - w), \quad x \in \Omega, \; t > 0,
\end{align*}
\]

where $\alpha, \rho, \lambda, \mu$ and $\gamma$ are positive parameters. For any reasonably regular initial data $(p_0, c_0, w_0)$, we prove the global boundedness ($L^\infty$-norm) of $p$ via an iterative method. Furthermore, we investigate the long-time behavior of solutions to the above system under an additional mild condition, and improve previously known results. In particular, in the one-dimensional case, we show that the solution $(p, c, w)$ converges to $(1, 0, 1)$ with an explicit exponential rate as time tends to infinity.

Key words: Angiogenesis, chemotaxis, haptotaxis, boundedness, asymptotic behavior

2010 Mathematics Subject Classification: 35A01, 35B35, 35K57, 35Q92, 92C17

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1 Introduction

As a physiological process, angiogenesis involves the formation of new capillary networks sprouting from a pre-existing vascular network and plays an important role in embryo development, wound healing and tumor growth. For example, it has been recognized that capillary growth through angiogenesis leads to vascularization of a tumor, providing it with its own dedicated blood supply and consequently allowing for rapid growth and metastasis.

The process of tumor angiogenesis can be divided into three main stages (which may be overlapping): (i) changes within existing blood vessels; (ii) formation of new vessels; and (iii) maturation of new vessels. Over the past decade, a lot of work has been done on the mathematical modeling of tumor growth; see, for example, [2, 3, 5, 6, 17, 28, 29] and the references cited therein. In particular, the role of angiogenesis in tumor growth has also attracted a great deal of attention; see, for example, [1, 7, 14, 24, 27] and the references cited therein. For example, in Levine et al. [14], a system of PDEs using reinforced random walks was deployed to model the first stage of angiogenesis, in which chemotactic substances from the tumor combine with the receptors on the endothelial cell wall to release proteolytic enzymes that can degrade the basal membrane of the blood vessels eventually.

In this paper we consider a variation of the model proposed in [2], namely,

\[
\begin{align*}
    p_t &= \Delta p - \nabla \cdot p(\frac{\alpha}{1 + c}\nabla c + \rho \nabla w) + \lambda p(1 - p), & x \in \Omega, t > 0, \\
    c_t &= \Delta c - c - \mu pc, & x \in \Omega, t > 0, \\
    w_t &= \gamma p(1 - w), & x \in \Omega, t > 0, \\
    \frac{\partial p}{\partial \nu} - p(\frac{\alpha}{1 + c}\frac{\partial c}{\partial \nu} + \rho \frac{\partial w}{\partial \nu}) = \frac{\partial c}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
    p(x, 0) &= p_0(x), \quad c(x, 0) = c_0(x), \quad w(x, 0) = w_0(x), & x \in \Omega,
\end{align*}
\]

in a bounded smooth domain \( \Omega \subset \mathbb{R}^N (N = 1, 2) \), where, in addition to random motion, the existing blood vessels’ endothelial cells \( p \) migrate in response to the concentration gradient of a chemical signal \( c \) (called Tumor Angiogenic Factor, or TAF) secreted by tumor cells as well as the concentration gradient of non-diffusible glycoprotein fibronectin \( w \) produced by the endothelial cells [21]. The former directed migration is a chemotatic process, whereas the latter is a haptotatic process. In this model, it is assumed that the endothelial cells
proliferate according to a logistic law, that the spatio-temporal evolution of TAF occurs through diffusion, natural decay and degradation upon binding to the endothelial cells, and that the fibronectin is produced by the endothelial cells and degrades upon binding to the endothelial cells.

For the remainder of this paper, we shall assume that the initial data satisfy the following:

\[
\begin{aligned}
(p_0, c_0, w_0) &\in \left( C^{2+\beta}(\Omega) \right)^3 \text{ is nonnegative for some } \beta \in (0, 1) \text{ with } p_0 \neq 0, \\
\frac{\partial p_0}{\partial \nu} - p_0 \left( \frac{\alpha}{1 + c_0} \frac{\partial c_0}{\partial \nu} + \rho \frac{\partial w_0}{\partial \nu} \right) - p_0 (\alpha + c_0) \frac{\partial c_0}{\partial \nu} &= 0.
\end{aligned}
\]

(1.2)

The present paper focuses on the global existence and asymptotic behavior of classical solutions to (1.1). Let us look at two subsystems contained in (1.1). The first is a Keller–Segel-type chemotaxis system with signal absorption:

\[
\begin{aligned}
p_t &= \Delta p - \nabla \cdot (p \nabla c) + \lambda p (1 - p), &\quad x \in \Omega, t > 0, \\
c_t &= \Delta c - pc, &\quad x \in \Omega, t > 0.
\end{aligned}
\]

(1.3)

It is known that, unlike the standard Keller–Segel model, (1.3) with \( \lambda = 0 \) possesses global, bounded classical solutions in two-dimensional bounded convex domains for arbitrarily large initial data; while in three spatial dimensions, it admits at least global weak solutions which eventually become smooth and bounded after some waiting time \([32]\). In the high-dimensional setting, it has been proved that global bounded classical solutions exist for suitably large \( \lambda > 0 \), while only certain weak solutions are known to exist for arbitrary \( \lambda > 0 \) \([13]\).

Another delicate subsystem of (1.1) is the haptotaxis-only system obtained by letting \( \alpha = 0 \) in (1.1):

\[
\begin{aligned}
p_t &= \Delta p - \rho \nabla \cdot (p \nabla w) + \lambda p (1 - p), &\quad x \in \Omega, t > 0, \\
w_t &= \gamma p (1 - w), &\quad x \in \Omega, t > 0.
\end{aligned}
\]

(1.4)

Here, since the quantity \( w \) satisfies an ODE without any diffusion, the smoothing effect on the spatial regularity of \( w \) during evolution cannot be expected. To the best of our knowledge, unlike the study of chemotaxis systems, the mathematical literature on haptotaxis systems is comparatively thin. Indeed, the literature provides only some results on global solvability.
in various special models, and the detailed description of qualitative properties such as long-time behaviors of solutions is available only in very particular cases (see, for example, \[8, 18, 20, 31, 35, 36, 39\]).

More recently, some results on global existence and asymptotic behavior for certain chemotaxis–haptotaxis models of cancer invasion have been obtained (see, for example, \[16, 22, 23, 30, 33, 34, 37\]). Particularly, Hillen et al. \[9\] have shown the convergence of a cancer invasion model in one-dimensional domains and the result has been subsequently extended to higher dimensions \[16, 34, 37\].

In \[21\], in two spatial dimensions, the authors showed the global existence and long-time behavior of classical solutions to (1.1) when the initial data \((p_0, c_0, w_0)\) satisfy either \(w_0 > 1\) or \(\|w_0 - 1\|_{L^\infty(\Omega)} < \delta\) for some \(\delta > 0\) (see Lemma 5.8 of \[21\]). Generalizing this result, our first main result establishes that, for any choice of reasonably regular initial data \((p_0, c_0, w_0)\), the \(L^\infty\)-norm of \(p\) is globally bounded. This is done via an iterative method.

**Theorem 1.1.** Let \(\alpha, \rho, \lambda, \mu, \) and \(\gamma\) be positive parameters. Then for any initial data \((p_0, c_0, w_0)\) satisfying (1.2), the problem (1.1) possesses a unique classical solution \((p, c, w)\) comprising nonnegative functions in \(C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))\) such that \(\|p(\cdot, t)\|_{L^\infty(\Omega)} \leq C\) for all \(t > 0\).

Next, we investigate the asymptotic behavior of solutions to (1.1). Under an additional mild condition on the initial data \(w_0\), we will show that the solution \((p, c, w)\) converges to the spatially homogeneous equilibrium \((1, 0, 1)\) as time tends to infinity.

**Theorem 1.2.** Let \(\alpha, \rho, \lambda, \mu, \) and \(\gamma\) be positive parameters, and suppose that (1.2) is satisfied and \(w_0 > 1 - \frac{1}{\rho}\). Then the solution \((p, c, w) \in C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))\) of (1.1) satisfies

\[
\lim_{t \to \infty} \|p(\cdot, t) - 1\|_{L^r(\Omega)} + \|c(\cdot, t)\|_{W^{1,2}(\Omega)} + \|w(\cdot, t) - 1\|_{L^r(\Omega)} = 0 \tag{1.5}
\]

for any \(r \geq 2\). In particular, if \(N = 1\), then for any \(\epsilon \in (0, \min\{\lambda_1, 1, \gamma, \lambda\})\) there exists \(C(\epsilon) > 0\) such that

\[
\|p(\cdot, t) - 1\|_{L^\infty(\Omega)} \leq C(\epsilon)e^{-(\min\{\lambda_1, 1, \gamma, \lambda\} - \epsilon)t}, \tag{1.6}
\]

\[
\|c(\cdot, t)\|_{W^{1,2}(\Omega)} \leq C(\epsilon)e^{-(1-\epsilon)t}, \tag{1.7}
\]
\[ \|w(\cdot, t) - 1\|_{W^{1,2}(\Omega)} \leq C(\epsilon)e^{-(\gamma-\epsilon)t}, \quad (1.8) \]

where \( \lambda_1 > 0 \) is the first nonzero eigenvalue of \(-\Delta\) in \( \Omega \) with the homogeneous Neumann boundary condition.

The main mathematical challenge of the full chemotaxis–haptotaxis system is the strong coupling between the migratory cells \( p \) and the haptotactic agent \( w \). This strong coupling has an important effect on the spatial regularity of \( p \) and \( w \). In fact, the lack of regularization effect in the spatial variable in the \( w \)-equation and the presence of \( p \) therein demand tedious estimates on the solution. The key ideas behind our results are as follows:

As pointed out in [34], the variable transformation \( z := pe^{-\rho w} \) plays an important role in the examination of global solvability for the full chemotaxis–haptotaxis model in the two- and higher-dimensional setting. However, due to the presence of the additional chemotaxis term in our model, this approach is not directly applicable to our problem. Instead, in the derivation of Theorem 1.1, we introduce the variable transformation \( q := p(c + 1)\alpha e^{-\rho w} \) as in [21], and thereby ensure that \( q(\cdot, t) \) is bounded in \( L^n(\Omega) \) for any finite \( n \) (see Lemma 2.4). It is essential to our approach to derive a bound for \( \int_\Omega q^{m+1} + \int_t^{t+\tau} \int_\Omega |\nabla q^m|^2 \) from the bound of \( \int_t^{t+\tau} \int_\Omega q^{2m} \) \( (m = 1, 2, \ldots) \) by making appropriate use of (2.3)–(2.4) in Lemma 2.3 (see (2.7) below).

The crucial idea of the proof of Theorem 1.2 is to show

\[ \frac{d}{dt} F(p(t), w(t)) + \frac{1}{2} \int_\Omega \frac{\nabla p^2}{p} + \frac{1}{2} \int_\Omega p|\nabla w|^2 \leq C \int_\Omega p|w - 1| + C \int_\Omega p|\nabla c|^2 \]

with

\[ F(p, w) = \kappa \int_\Omega |\nabla w|^2 + \int_\Omega p(\ln p - 1) + \int_\Omega p(w - 1) - \gamma \kappa \int_\Omega p(w - 1)^2 \]

for some \( C > 0 \) (see (3.10)), on the basis of the global boundedness of the \( L^\infty \)-norm of \( p \) provided by Theorem 1.1 and under the mild assumption that \( w_0 > 1 - \frac{1}{\rho} \). Furthermore, in the one-dimensional case, with the help of higher regularity estimates of \( z \) with \( z = pe^{-\rho w} \) (see (3.29)), we show that \( p(\cdot, t) \) converges to 1 uniformly in \( \Omega \) as \( t \to \infty \). From this, we derive the exponential convergence of solutions as desired.
2 Proof of Theorem 1.1

In this section, we first recall the following estimates for the heat semigroup \((e^{\tau \Delta})_{\tau \geq 0}\) in \(\Omega \subset \mathbb{R}^N\) under the Neumann boundary condition. We shall omit the proof thereof and refer the interested readers to [4, 10, 38].

**Lemma 2.1.** Let \((e^{\tau \Delta})_{\tau \geq 0}\) be the Neumann heat semigroup in \(\Omega\) and \(\lambda_1 > 0\) the first nonzero eigenvalue of \(-\Delta\) with homogeneous Neumann boundary condition. Then there exist positive constants \(k_1, k_2, k_3\) such that:

i) If \(1 \leq q \leq p \leq \infty\), then for all \(\varphi \in L^q(\Omega)\) with \(\int_{\Omega} \varphi = 0\),
\[
\|e^{\tau \Delta} \varphi\|_{L^p(\Omega)} \leq k_1 (1 + \tau^{-\frac{q}{2}(\frac{1}{q} - \frac{1}{p})}) e^{-\lambda_1 \tau} \|\varphi\|_{L^q(\Omega)};
\]

ii) If \(1 \leq q \leq p \leq \infty\), then for all \(\varphi \in L^q(\Omega)\),
\[
\|\nabla e^{\tau \Delta} \varphi\|_{L^p(\Omega)} \leq k_2 (1 + \tau^{-\frac{q}{2}(\frac{1}{q} - \frac{1}{p})}) e^{-\lambda_1 \tau} \|\varphi\|_{L^q(\Omega)};
\]

iii) If \(1 \leq q \leq p \leq \infty\), then for all \(\varphi \in C^1(\overline{\Omega}; \mathbb{R}^N)\) with \(\varphi \cdot \nu = 0\) on \(\partial \Omega\),
\[
\|e^{\tau \Delta} \nabla \cdot \varphi\|_{L^p(\Omega)} \leq k_3 (1 + \tau^{-\frac{q}{2}(\frac{1}{q} - \frac{1}{p})}) e^{-\lambda_1 \tau} \|\varphi\|_{L^q(\Omega)}.
\]

Next, we recall the following result on local existence and uniqueness of classical solutions to (1.1) as well as a convenient extensibility criterion, which follows from Theorem 3.1, Lemma 5.9 and Theorem 5.1 of [21].

**Lemma 2.2.** ([21]) Let \(\Omega \subset \mathbb{R}^N\) be a smooth bounded domain. There exists \(T_{\text{max}} \in (0, \infty]\) such that the problem (1.1) possesses a unique classical solution satisfying \((p, c, w) \in (C(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})))^3\). Moreover, for any \(s > N + 2\),
\[
\limsup_{t \searrow T_{\text{max}}} \|p(\cdot, t)\|_{W^{1, s}(\Omega)} \to \infty \tag{2.1}
\]
if \(T_{\text{max}} < +\infty\).

From now on, let \((p, c, w)\) be the local classical solution of (1.1) on \((0, T_{\text{max}})\) provided by Lemma 2.1, and \(\tau := \min\{1, \frac{T_{\text{max}}}{6}\}\).

The following basic but important properties of the solution to (1.1) can be directly obtained via standard arguments.
Lemma 2.3. ([21]) There exists a positive constant $C$ independent of time such that

$$
\int_\Omega p(\cdot, t) \leq c_1 := \max\{ \int_\Omega p_0, |\Omega| \}, \quad \int_0^t e^{-2s} \int_\Omega p^2 ds \leq C \text{ for all } t \in (0, T_{\text{max}}), \quad (2.2)
$$

$$
\int_t^{t+\tau} \int_\Omega p^2 \leq c_1 (1 + \frac{1}{\lambda}) \quad \text{for all } t \in (0, T_{\text{max}} - \tau), \quad (2.3)
$$

$$
c(t) \leq \|c_0\|_{L^\infty(\Omega)} e^{-t}, \quad \int_\Omega |\nabla c(t)|^2 + \int_0^t \int_\Omega (|\nabla c|^2 + |\Delta c|^2) \leq C \quad \text{for all } t \in (0, T_{\text{max}}), \quad (2.4)
$$

$$
0 \leq w(t) \leq \max\{ \|w_0\|_{L^\infty(\Omega)}, 1 \} \quad \text{for all } t \in (0, T_{\text{max}}). \quad (2.5)
$$

As the proof of Theorem 1.1 in the one-dimensional case is similar to that for two dimensions, henceforth in this section, we shall focus on the case $N = 2$.

First, we shall show that $p$ remains bounded in $L^n(\Omega)$ for any finite $n$. We note that the $L^n(\Omega)$-bound in Lemma 3.10 of [21] depends on the time variable.

Lemma 2.4. For any $n \in (1, \infty)$, there exists a positive constant $C(n, \tau)$ independent of $t$, such that $\|p(\cdot, t)\|_{L^n(\Omega)} \leq C(n, \tau)$ for all $t \in (0, T_{\text{max}})$.

**Proof.** Let $q := p(c+1)^{-\alpha} e^{-\rho w}$. As in the proof of Lemma 3.10 in [21], we infer that for any $m = 1, 2, \ldots$ there exist constants $c(m) > 0$ depending upon $m$ and $c_1 > 0$ such that

$$
\frac{d}{dt} \int_\Omega q^m (c+1)^{\alpha} e^{\rho w} + \int_\Omega |\nabla q^{2m-1}|^2 \leq c(m) (\int_\Omega |\Delta c|^2 + 1) \int_\Omega q^2 + c(m) (\int_\Omega q^2)^2 + c_1. \quad (2.6)
$$

Next, we use induction to show

$$
\int_\Omega q^{2m} + \int_t^{t+\tau} \int_\Omega |\nabla q^{2m-1}|^2 \leq C(m). \quad (2.7)
$$

Taking $m = 1$ in (2.6), we get

$$
\frac{d}{dt} \int_\Omega q^2 (c+1)^{\alpha} e^{\rho w} + \int_\Omega |\nabla q|^2 \leq c(1) (\int_\Omega |\Delta c|^2 + 1 + \int_\Omega q^2) \int_\Omega q^2 (c+1)^{\alpha} e^{\rho w} + c_1, \quad (2.8)
$$

which implies that for the functions $y(t) = \int_\Omega q^2 (c+1)^{\alpha} e^{\rho w}$ and $a(t) = c(1) (\int_\Omega |\Delta c|^2 + 1 + \int_\Omega q^2)$, we have

$$
\frac{dy}{dt} \leq a(t) y + c_1.
$$

On the other hand, for any given $t > \tau$, it follows from (2.3) that there exists some $t_0 \in [t - \tau, t]$ such that $y(t_0) \leq \frac{\lambda}{\tau} (1 + \frac{1}{\lambda})$. Hence by ODE comparison argument we get

$$
y(t) \leq y(t_0) e^{\int_{t_0}^t a(s) ds} + c_1 \int_{t_0}^t e^{\int_s^t a(\tau) d\tau} ds \leq c_2. \quad (2.9)
$$
In this inequality, we have taken \( t_0 = 0 \) if \( t \leq \tau \) and noticed that \( \int_{t-\tau}^t a(s) \, ds \leq c_3 \) for all \( t < T_{\text{max}} \) by Lemma 2.2. Combining (2.8) with (2.9), one can see that (2.7) is indeed valid for \( m = 1 \).

Now, suppose that (2.7) is valid for an integer \( m + 1 = k \geq 2 \), i.e.,

\[
\int_{t}^{t+\tau} \int_{\Omega} |\nabla q_k^{k-1}|^2 \leq C(k).
\] (2.10)

By the Gagliardo–Nirenberg inequality in two dimensions

\[
\|z\|_{L^4(\Omega)}^4 \leq c_3 \|\nabla z\|_{L^2(\Omega)}^2 \|z\|_{L^2(\Omega)}^2 + c_4 \|z\|_{L^4(\Omega)}^4,
\]

and hence

\[
\int_{\Omega} q_k^2 \leq c_3 \int_{\Omega} |\nabla q_k^{k-2}|^2 \int_{\Omega} q_k^{2k-1} + c_4 \int_{\Omega} q_k^{2k-1}.
\] (2.11)

Integrating (2.11) between \( t \) and \( t + \tau \) and taking (2.10) into account, we have

\[
\int_{t}^{t+\tau} \int_{\Omega} q_k^2 \leq c_5(k),
\] (2.12)

which implies that for any \( t \geq \tau \), there exists some \( t_0 \in [t - \tau, t] \) such that \( \int_{\Omega} q_k^2(t_0) \leq c_0 \).

At this point, let \( y(t) := \int_{\Omega} q_k^2 (c + 1)^{\alpha e^\rho w} \) and \( b(t) = c(k)(\int_{\Omega} |\Delta c|^2 + 1 + \int_{\Omega} q_k^2) \). Then (2.6) can be rewritten as

\[
\frac{dy}{dt} + \int_{\Omega} |\nabla q_k^{k-2}|^2 \leq b(t)y + c_1.
\]

By the argument above, one can obtain

\[
\int_{\Omega} q_k^2 + \int_{t}^{t+\tau} \int_{\Omega} |\nabla q_k^{k-2}|^2 \leq C,
\] (2.13)

and thereby conclude that (2.7) is valid for all integers \( m \geq 1 \). The proof of Lemma 2.3 is now complete in view of the boundedness of the weight \((c + 1)^{\alpha e^\rho w}\).

To establish a priori estimates of \( \|p(\cdot, t)\|_{L^\infty(\Omega)} \), we need some fundamental estimates for the solution of the following problem:

\[
\begin{aligned}
& c_t = \Delta c - c + f, \quad x \in \Omega, t > 0, \\
& \frac{\partial c}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
& c(x, 0) = c_0(x), \quad x \in \Omega.
\end{aligned}
\] (2.14)
Lemma 2.5. ([13 Lemma 2.2]) Let $T > 0$, $r \in (1, \infty)$. Then for each $c_0 \in W^{2,r}(\Omega)$ with $\frac{\partial c_0}{\partial \nu} = 0$ on $\partial \Omega$ and $f \in L^{r}(0,T;L^{r}(\Omega))$, (2.13) has a unique solution $c \in W^{1,r}(0,T;L^{r}(\Omega)) \cap L^{r}(0,T;W^{2,r}(\Omega))$ given by
\[
c(t) = e^{-t}e^{t\Delta}c_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}f(s)ds, \quad t \in [0,T],
\]
where $e^{t\Delta}$ is the semigroup generated by the Neumann Laplacian, and there is $C_r > 0$ such that
\[
\int_0^t \int_{\Omega} e^{rs} |\Delta c(x,s)|^r dx ds \leq C_r \int_0^t \int_{\Omega} e^{rs} |f(x,s)|^r dx ds + C_r \|v_0\|_{W^{2,r}(\Omega)}.
\]
(2.15)

Now applying these estimates to control the cross-diffusive flux appropriately, we can derive the boundedness of $p$ in $\Omega \times (0,T_{\text{max}})$.

Lemma 2.6. There exists a constant $C > 0$ independent of $t$ such that $\|p(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C$ for all $t \in (0,T_{\text{max}})$.

Proof. We will only give a sketch of the proof, which is similar to that of Lemma 3.13 of [21]. For $k \geq \max\{2, \|p_0\|_{L^{\infty}(\Omega)}\}$, let $q_k = \max\{q - k, 0\}$ and $\Omega_k(t) = \{x \in \Omega : q(x,t) > k\}$.

Multiplying the equation of $q$ by $q_k$, we obtain
\[
\frac{d}{dt} \int_{\Omega} q_k^2(c+1)^\alpha e^{\rho w} + 2 \int_{\Omega} |\nabla q_k|^2 + 2 \int_{\Omega} q_k^2 + 8 \int_{\Omega} q_k^2(c+1)^\alpha e^{\rho w} \\
\leq c_1 \int_{\Omega} q_k^3 + c_1 k \int_{\Omega} q_k^2 + c_1 k^2 \int_{\Omega} q_k + c_1 \int_{\Omega} (q_k^2 + kq_k)|\Delta c|
\]
(2.16)

for some $c_1 > 0$ independent of $k$. By the boundedness of $q$ in $L^n(\Omega)$ for any $n > 1$, the Gagliardo–Nirenberg inequality and Young inequality, we obtain
\[
c_1 \|q_k\|_{L^3(\Omega)}^3 \leq \frac{1}{4} \|q_k\|_{H^1(\Omega)}^2 + c_2 \|q_k\|_{L^1(\Omega)},
\]
\[
c_1 k \|q_k\|_{L^2(\Omega)}^2 \leq \frac{1}{4} \|q_k\|_{H^1(\Omega)}^2 + c_2 k^2 \|q_k\|_{L^1(\Omega)},
\]
\[
c_1 \int_{\Omega} q_k^2|\Delta c| \leq \frac{1}{4} \|q_k\|_{H^1(\Omega)}^2 + c_2 \|q_k\|_{L^2(\Omega)}^2 \|\Delta c\|_{L^2(\Omega)}^2,
\]
\[
c_1 k \int_{\Omega} q_k|\Delta c| \leq \frac{1}{4} \|q_k\|_{H^1(\Omega)}^2 + c_2 k^2 (1 + \|\Delta c\|_{L^8(\Omega)}^8)\|\Omega_k\|^{\frac{2}{3}}.
\]
Inserting the above estimates into (2.16), we have
\[
\frac{d}{dt} \int_\Omega q_k^2(c + 1)^a e^{\rho w} + \int_\Omega |\nabla q_k|^2 + \int_\Omega q_k^2 + 8 \int_\Omega q_k^2(c + 1)^a e^{\rho w} \leq c_2 \|q_k\|^2_{L^2(\Omega)} \|\Delta c\|^2_{L^2(\Omega)} + c_2 k^2 (1 + \|\Delta c\|^8_{L^8(\Omega)}) |\Omega_k|^\frac{3}{2} + (c_1 + 2c_2) k^2 \|q_k\|_{L^1(\Omega)} \leq c_2 \|q_k\|^2_{L^2(\Omega)} \|\Delta c\|^2_{L^2(\Omega)} + \frac{1}{2} \|q_k\|^2_{H^1(\Omega)} + c_3 k^4 (1 + \|\Delta c\|^8_{L^8(\Omega)}) |\Omega_k|^\frac{3}{2}.
\]

On the other hand, according to the relation between distribution functions and \(L^p\) integrals (see e.g. (2.6) of [26]), we can see that
\[
(r + 1) \int_0^\infty s^r |\Omega_k(t)| ds = \|q(t)\|^{r+1}_{L^{r+1}(\Omega)}.
\]
Hence taking into account Lemma 2.3, we get
\[
(k - 1)^{16} |\Omega_k(t)| < \int_{k-1}^k s^{16} |\Omega_k(t)| ds < \int_0^\infty s^{16} |\Omega_k(t)| ds \leq \frac{1}{17} \|q(t)\|^{17}_{L^{17}(\Omega)}
\]
and thus
\[
\frac{d}{dt} \int_\Omega q_k^2(c + 1)^a e^{\rho w} + 8 \int_\Omega q_k^2(c + 1)^a e^{\rho w} \leq c_2 \|\Delta c\|^2_{L^2(\Omega)} \int_\Omega q_k^2(c + 1)^a e^{\rho w} + c_4 (1 + \|\Delta c\|^8_{L^8(\Omega)}) |\Omega_k|^\frac{3}{2}.
\]
(2.17)

Therefore if \(h(t) = 8 - c_2 \|\Delta c\|^2_{L^2(\Omega)}\), then
\[
\int_\Omega q_k^2(c + 1)^a e^{\rho w} \leq c_4 e^{-\int_0^1 h(s) ds} \int_0^t (1 + \|\Delta c\|^8_{L^8(\Omega)}) e^{\int_0^s h(\sigma) d\sigma} |\Omega_k(s)|^{\frac{3}{2}} ds.
\]
Furthermore, since \(e^{-\int_0^1 h(s) ds} = e^{-8t} e^{c_2 \int_0^1 \|\Delta c\|^2_{L^2(\Omega)} ds} \leq c_5 e^{-8t}\) by Lemma 2.2 and \(e^{\int_0^s h(\sigma) d\sigma} \leq e^{8s}\), we get
\[
\int_\Omega q_k^2 \leq c_6 \int_0^t e^{-8(t-s)} (1 + \|\Delta c\|^8_{L^8(\Omega)}) |\Omega_k(s)|^{\frac{3}{2}} ds \leq c_6 \int_0^t e^{-8(t-s)} (1 + \|\Delta c\|^8_{L^8(\Omega)}) ds \sup_{t \geq 0} |\Omega_k(t)|^{\frac{3}{2}}.
\]
To estimate the integral term in the right-side of the above inequality, we apply Lemma 2.3 with \(r = 8\) and Lemma 2.4 to get
\[
\int_0^t e^{-8(t-s)} \|\Delta c\|^8_{L^8(\Omega)} ds \leq c_7
\]
and thus \(\int_\Omega q_k^2 \leq c_8 (\sup_{t \geq 0} |\Omega_k(t)|)^{\frac{3}{2}}\).

On the other hand, \(\int_\Omega q_k^2(t) \geq \int_{\Omega_j(t)} q_k^2(t) \geq (j - k)^2 |\Omega_j(t)|\) for \(j > k\). Consequently
\[
(j - k)^2 \sup_{t \geq 0} |\Omega_j(t)| \leq c_8 (\sup_{t \geq 0} |\Omega_k(t)|)^{\frac{3}{2}} |\Omega_k(t)|.
\]
According to Lemma B.1 of [11], there exists $k_0 < \infty$ such that $|\Omega_{k_0}(t)| = 0$ for all $t \in (0, T_{\text{max}})$. Therefore $\|q(\cdot,t)\|_{L^\infty(\Omega)} \leq k_0$ for any $t \in (0, T_{\text{max}})$ and thereby the proof is complete.

**Proof of Theorem 1.1.** By the boundedness of $p$ in $L^\infty((0, T_{\text{max}}), L^\infty(\Omega))$ from Lemma 2.6 and a bootstrap argument as in [21], we can see that the global existence of classical solutions to (1.1) is an immediate consequence of Lemma 2.2, i.e., $T_{\text{max}} = \infty$. Indeed, supposed that $T_{\text{max}} < \infty$, then by Lemma 3.15 and Lemma 3.19 of [21], we can see that for any $s > N + 2$ and $t \leq T_{\text{max}}$

$$\|c(\cdot,t)\|_{W^{1,s}(\Omega)} + \|w(\cdot,t)\|_{W^{1,s}(\Omega)} \leq C.$$ 

Further by Lemma 3.20 of [21], we have $\|p(\cdot,t)\|_{W^{1,s}(\Omega)} \leq C$ which contradicts (2.1) and thus implies that $T_{\text{max}} = \infty$. Moreover, since $\tau := \min\{1, \frac{T_{\text{max}}}{6}\} = 1$, there exists a constant $C > 0$ independent of time $t$ such that $\|p(\cdot,t)\|_{L^\infty(\Omega)} \leq C$ for all $t \geq 0$ by retracing the proofs of Lemma 2.4 and Lemma 2.6. This completes the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In this section, on the basis of the $L^\infty$-bound of $p$ provided by Theorem 1.1 we shall look at the asymptotic behavior of the solution $(p, c, w)$ of the problem (1.1).

3.1 $L^r$-convergence of solutions in two dimensions

When either $w_0 > 1$ or $\|w_0 - 1\|_{L^\infty(\Omega)} < \delta$ for some $\delta > 0$, the authors of [21] removed the time dependence of the $L^\infty$-bound of $p$ (see Lemma 5.8 of [21]) and thereby investigated the asymptotic behavior of solutions to (1.1). In this subsection, on the basis of the $L^\infty$-bound of $p$ being independent of time as provided by Theorem 1.1, we shall derive the same estimates as in Lemma 5.6 and Lemma 5.7 of [21] under the weaker assumption that $w_0 > 1 - \frac{1}{\rho}$. We shall show that the solution $(p, c, w)$ to (1.1) converges to the homogeneous steady state $(1, 0, 1)$ as $t \to \infty$. 
Before going into the details, let us first collect some useful related estimates. It should be noted that no other assumptions on the initial data \((p_0, c_0, w_0)\) are made except for reasonable regularity, i.e., (1.2).

**Lemma 3.1.** ([21, Lemmas 3.4, 5.1, 5.2, 3.8]) Let \((p, c, w)\) be the global, classical solution of (1.1). Then

\[
\int_0^\infty \int_\Omega p(1-p) \leq \max\{\int_\Omega p_0, |\Omega|\}/\lambda; \tag{3.1}
\]

\[
\int_0^\infty \int_\Omega p|w-1| \leq \|w_0-1\|_{L^1(\Omega)}; \tag{3.2}
\]

\[
\int_0^\infty \int_\Omega p|\nabla c|^2 < \infty \tag{3.3}
\]

\[
\int_\Omega |\nabla c(t)|^2 \leq e^{-2t} \left( \int_\Omega |\nabla c_0|^2 + \mu^2 \|c_0\|_{L^\infty(\Omega)}^2 \max\{\int_\Omega p_0, |\Omega|\}(t + \frac{1}{\lambda}) \right). \tag{3.4}
\]

**Lemma 3.2.** Under the assumptions of Theorem 1.1, we have

\[
\sup_{t \geq 0} \|c(t)\|_{W^{1,\infty}(\Omega)} \leq C. \tag{3.5}
\]

**Proof.** We know that \(c\) solves the linear equation

\[
c_t = \Delta c - c + f
\]

under the Neumann boundary condition with \(f := -\mu pc\). Since \(p \geq 0\), we know that \(0 \leq c(x, t) \leq \|c_0\|_{L^\infty(\Omega)} e^{-t}\) by the standard sub-super solutions method. On the other hand, by Theorem 1.1 \(\sup_{t \geq 0} \|p(t)\|_{L^\infty(\Omega)} \leq c_1\), which readily implies that \(\|f\|_{L^\infty((0, \infty); L^\infty(\Omega))} \leq c_1\).

Now upon a standard regularity argument we can deduce the desired result. For the reader’s convenience, we only give a brief sketch of the main ideas, and would like refer to the proof of Lemma 1 in [12] or Lemma 4.1 in [10] for more details. Indeed, according to the variation-of-constants formula of \(c\), we have for \(t > 2\)

\[
c(\cdot, t) = e^{(t-1)(\Delta-1)}c(\cdot, 1) + \int_1^t e^{(t-s)(\Delta-1)}f(\cdot, s)ds.
\]

So by Lemma 2(iii), we infer that

\[
\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq 2k_2\|c(\cdot, 1)\|_{L^1(\Omega)} + k_2 \int_1^t (1 + (t-s)^{-\frac{1}{2}})e^{-(t-s)(\lambda_1+1)}\|f(\cdot, s)\|_{L^\infty(\Omega)}ds
\]

\[
\leq 2k_2\|c(\cdot, 1)\|_{L^1(\Omega)} + k_2c_1\int_0^\infty (1 + \sigma^{-\frac{1}{2}})e^{-\sigma}d\sigma.
\]
Lemma 3.3. ([21 Lemma 5.4]) Let \((p, c, w)\) be the global, classical solution of \((1.1)\). Then for every \(t \geq 0\) and \(\kappa > 0\),

\[
\frac{d}{dt} F(p(t), w(t)) = G(p(t), w(t), c(t)),
\]

where

\[
F(p, w) = \kappa \int_{\Omega} |\nabla w|^2 + \int_{\Omega} p(\ln p - 1) + \int_{\Omega} p(w - 1) - \gamma \kappa \int_{\Omega} p(w - 1)^2,
\]

and

\[
G(p, w, c) = -\int_{\Omega} \frac{|\nabla p|^2}{p} + \int_{\Omega} \frac{\alpha}{1 + c} \nabla p \cdot \nabla c + \int_{\Omega} (2\alpha \gamma \kappa (1 - w) + \alpha \rho) \frac{p}{1 + c} \nabla c \cdot \nabla w
\]

\[
+ \int_{\Omega} (\rho^2 - 2\gamma \kappa + 2\rho \gamma \kappa (1 - w)) p |\nabla w|^2 + \lambda \int_{\Omega} p(1 - p) \ln p
\]

\[
+ \lambda \rho \int_{\Omega} p(1 - p)(w - 1) + \gamma \rho \int_{\Omega} p^2(1 - w)
\]

\[
+ 2\gamma^2 \kappa \int_{\Omega} p^2(w - 1)^2 - \lambda \gamma \kappa \int_{\Omega} p(1 - p)(w - 1)^2.
\]

Lemma 3.4. If \(w_0 > 1 - \frac{1}{\rho}\), then there exists \(\kappa > 0\) such that

\[
G(p, w, c) \leq -\frac{1}{2} \int_{\Omega} \frac{|\nabla p|^2}{p} - \frac{1}{2} \int_{\Omega} p |\nabla w|^2 + C \int_{\Omega} p |w - 1| + C \int_{\Omega} p |\nabla c|^2
\]

for some \(C > 0\).

Proof. By the Hölder and Young inequalities we have

\[
\int_{\Omega} \frac{\alpha}{1 + c} \nabla p \cdot \nabla c \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla p|^2}{p} + \frac{\alpha^2}{2} \int_{\Omega} p |\nabla c|^2,
\]

and

\[
\int_{\Omega} (2\alpha \gamma \kappa (1 - w) + \alpha \rho) \frac{p}{1 + c} \nabla c \cdot \nabla w \leq \frac{1}{2} \int_{\Omega} p |\nabla w|^2 + c_1 \int_{\Omega} p |\nabla c|^2
\]

for some \(c_1 > 0\).

As \(w_0 > 1 - \frac{1}{\rho}\), we can find some \(\varepsilon_1 > 0\) such that \(\rho(1 - w_0)_+ \leq 1 - \varepsilon_1\), where \((1 - w_0)_+ = \max\{0, 1 - w_0\}\). Hence from the \(w\)-equation in \((1.1)\), it follows that

\[
1 - w = (1 - w_0)e^{-\gamma \int_{0}^{t} p(s)da},
\]

(3.8)
and thus
\[ \int_\Omega (\rho^2 - 2\gamma \kappa + 2\rho \gamma \kappa (1 - w)) p |\nabla w|^2 \leq \int_\Omega (\rho^2 - 2\gamma \kappa + 2\rho \gamma \kappa (1 - w_0)_+) p |\nabla w|^2 \]
\[ \leq \int_\Omega (\rho^2 - 2\gamma \kappa \varepsilon_1) p |\nabla w|^2 \]
\[ \leq - \int p |\nabla w|^2 \]
if we pick \( \kappa > 0 \) sufficiently large such that \( \rho^2 - 2\gamma \kappa \varepsilon_1 < -1 \).

Denote the lower-order terms of \( G(p, w, c) \) by \( \theta(p, w) \), i.e.,
\[ \theta(p, w) := \lambda \int_\Omega p(1 - p) \ln p + \lambda p \int_\Omega p(1 - p)(w - 1) + \gamma p \int_\Omega p^2(1 - w) \]
\[ + 2\gamma^2 \kappa \int_\Omega p^2(w - 1)^2 - \lambda \gamma \kappa \int_\Omega p(1 - p)(w - 1)^2. \]
Since \( s(1 - s) \ln s \leq 0 \) for \( s \geq 0 \), we get
\[ \theta(p, w) \leq \lambda p \int_\Omega p(1 - p)(w - 1) + \gamma p \int_\Omega p^2(1 - w) \]
\[ + 2\gamma^2 \kappa \int_\Omega p^2(w - 1)^2 - \lambda \gamma \kappa \int_\Omega p(1 - p)(w - 1)^2, \]
which, along with \( \| p(\cdot, t) \|_{L^\infty(\Omega)} \leq C \) from Theorem 1.1 and \( \| w(\cdot, t) - 1 \|_{L^\infty(\Omega)} \leq \| w_0 - 1 \|_{L^\infty(\Omega)} \) from (3.8), yields
\[ \theta(p, w) \leq c_3 \int_\Omega p|w - 1|. \]
The desired result (3.7) then immediately follows.

**Lemma 3.5.** If \( w_0 > 1 - \frac{1}{\rho} \), then
\[ \sup_{t \geq 0} \int_\Omega |\nabla w(t)|^2 + \int_0^\infty \int_\Omega \frac{|\nabla p|^2}{p} + \int_0^\infty \int_\Omega p|\nabla w|^2 < \infty. \quad (3.9) \]

**Proof.** Combining Lemmas 3.3 and 3.4 we have
\[ \frac{d}{dt} F(p(t), w(t)) + \frac{1}{2} \int_\Omega \frac{|\nabla p|^2}{p} + \frac{1}{2} \int_\Omega p|\nabla w|^2 \leq C \int_\Omega p|w - 1| + C \int_\Omega p|\nabla c|^2. \quad (3.10) \]
Hence (3.9) follows upon integration on the time variable, and using (3.2) and (3.3).

**Lemma 3.6.** If \( w_0 > 1 - \frac{1}{\rho} \), then for any \( r \geq 2 \)
\[ \lim_{t \to \infty} \| p(\cdot, t) - \bar{p}(t) \|_{L^r(\Omega)} = 0, \quad (3.11) \]
\[
\lim_{t \to \infty} |\mathcal{P}(t) - 1| = 0, \tag{3.12}
\]
where \( \mathcal{P}(t) = \frac{1}{|\Omega|} \int_{\Omega} p(\cdot, t) \), and
\[
\lim_{t \to \infty} \|w(\cdot, t) - 1\|_{L^r(\Omega)} = 0. \tag{3.13}
\]

**Proof.** The proofs of (3.11) and (3.12) are similar to those of Lemma 5.9–5.11 of [21] respectively. However, for the reader’s convenience, we only give a brief sketch of (3.12). In fact, from (1.1) and the Poincaré–Wirtinger inequality, it follows that
\[
\mathcal{P}_t = \lambda (\mathcal{P} - \mathcal{P}^2) - \frac{1}{|\Omega|} \int_{\Omega} (p - \mathcal{P})^2 \geq \lambda \mathcal{P}(1 - \mathcal{P} - c_1 \int_{\Omega} \frac{|\nabla p|^2}{p}).
\]
Hence by (3.9), we get
\[
\mathcal{P}(t) \geq \mathcal{P}_0 \exp\{\lambda t - \lambda \int_0^t \mathcal{P}(s) ds - c_1 \lambda \int_0^t \frac{\int_{\Omega} |\nabla p|^2}{p} ds\} \geq c_2 \exp\{\lambda t - \lambda \int_0^t \mathcal{P}(s) ds\},
\]
which means that (3.12) is valid due to either \( \mathcal{P}(t) \to 1 \) or \( \mathcal{P}(t) \to 0 \) in Lemma 5.10 of [21]. Indeed, supposed that \( \mathcal{P}(t) \to 0 \), then there exists \( t_0 > 1 \) such that \( \mathcal{P}(t) \leq \frac{1}{2} \) and thus
\[
\int_0^t \mathcal{P}(s) ds \leq \frac{t}{2} + \int_0^{t_0} \mathcal{P}(s) ds \text{ for all } t \geq t_0.
\]
Therefore we arrive at \( \mathcal{P}(t) \geq c_3 e^{\frac{\lambda t}{2}} \) for all \( t \geq t_0 \), which contradicts \( \mathcal{P}(t) \to 0 \).

Now we turn to show (3.13). Invoking the Poincaré inequality in the form
\[
\int_{\Omega} |\varphi(x) - \frac{1}{|\Omega|} \int_{\Omega} \varphi(y) dy|^2 dx \leq C_p \int_{\Omega} |\nabla \varphi|^2 dx \text{ for all } \varphi \in W^{1,2}(\Omega)
\]
for some \( C_p > 0 \), one can find that for all \( j \in \mathbb{N} \)
\[
\int_j^{j+1} \|p(s) - \mathcal{P}(s)\|_{L^2(\Omega)}^2 ds \leq C_p \int_j^{j+1} \|\nabla p(s)\|_{L^2(\Omega)}^2 ds \leq C_p \sup_{t \geq 0} \|p(t)\|_{L^\infty(\Omega)} \int_j^{j+1} \int_{\Omega} \frac{|\nabla p(s)|^2}{p(s)} ds,
\]
which, along with (3.9) and Theorem 1.1, shows that
\[
\int_{\Omega} \int_j^{j+1} |p(x, s) - \mathcal{P}(s)|^2 ds dx = \int_j^{j+1} \|p(s) - \mathcal{P}(s)\|_{L^2(\Omega)}^2 ds \to 0 \tag{3.15}
\]
as \( j \to \infty \).
Now defining \( p_j(x) := \int_j^{j+1} |p(x, s) - \overline{p}(s)|^2 ds \), \( x \in \Omega, j \in \mathbb{N} \), (3.15) tells us that \( p_j \to 0 \) in \( L^1(\Omega) \) as \( j \to \infty \). There exist a certain null set \( Q \subseteq \Omega \) and a subsequence \( (j_k)_{k \in \mathbb{N}} \subseteq \mathbb{N} \) such that \( j_k \to \infty \) and \( p_{j_k}(x) \to 0 \) for every \( x \in \Omega \setminus Q \) as \( k \to \infty \). Restated in the original variable, this becomes

\[
\int_{j_k}^{j_k+1} |p(x, s) - \overline{p}(s)|^2 ds \to 0
\]

for every \( x \in \Omega \setminus Q \) as \( k \to \infty \).

Therefore, from (3.18) and \( p(x, t) \geq 0 \), it follows that for any \( x \in \Omega \setminus Q \)

\[
|w(x, t) - 1| \leq \|w_0 - 1\|_{L^\infty(\Omega)} \exp\{-\gamma \int_0^t p(x, s) ds\}
\]

\[
\leq \|w_0 - 1\|_{L^\infty(\Omega)} \exp\{-\gamma \sum_{k=0}^{m(t)} \int_{j_k}^{j_{k+1}} p(x, s) ds\}
\]

\[
\leq \|w_0 - 1\|_{L^\infty(\Omega)} \exp\{\gamma \sum_{k=0}^{m(t)} \int_{j_k}^{j_{k+1}} |p(x, s) - \overline{p}(s)| ds - \gamma \sum_{k=0}^{m(t)} \int_{j_k}^{j_{k+1}} \overline{p}(s) ds\}
\]

\[
\leq \|w_0 - 1\|_{L^\infty(\Omega)} \exp\{\gamma \sum_{k=0}^{m(t)} (\int_{j_k}^{j_{k+1}} |p(x, s) - \overline{p}(s)|^2 ds)^{\frac{1}{2}} - \gamma \sum_{k=0}^{m(t)} \int_{j_k}^{j_{k+1}} \overline{p}(s) ds\}
\]

(3.17)

where \( m(t) := \max_{k \in \mathbb{N}} \{j_k + 1, |t|\} \). Furthermore, by (3.12), there exists \( k_0 \in \mathbb{N} \) such that \( \int_{j_k}^{j_{k+1}} \overline{p}(s) ds \geq \frac{1}{2} \) for all \( k \geq k_0 \). Hence by the fact that \( m(t) \to \infty \) as \( t \to \infty \) and (3.16), we obtain that \( w(x, t) - 1 \to 0 \) almost everywhere in \( \Omega \) as \( t \to \infty \). On the other hand, as \( |w(x, t) - 1| \leq \|w_0 - 1\|_{L^\infty(\Omega)} \), the dominated convergence theorem ensures that (3.13) holds for any \( r \in (2, \infty) \).

**Remark 3.1.** 1) It is observed that since \( W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega) \) is invalid in the two-dimensional setting, \( \|p(s) - \overline{p}(s)\|^2_{L^2(\Omega)} \) in (3.14) cannot be replaced by \( \|p(s) - \overline{p}(s)\|^2_{L^\infty(\Omega)} \), and thus we cannot infer that \( \lim_{t \to \infty} \|w(\cdot, t) - 1\|_{L^\infty(\Omega)} = 0 \), even though we have established that all the related estimates of \( (p, c, w) \) in [21] continue to hold under the milder condition imposed on the initial data \( w_0 \).

2) Similarly to the remark above, we note that, even though \( \|w(\cdot, t)\|_{W^{1,n}(\Omega)} \leq C(T) \) for any \( n \geq 2 \) and \( t \leq T \), we are not able to infer the global estimate \( \sup_{t \geq 0} \int_{\Omega} |\nabla w(t)|^{2+\varepsilon} \leq C \). Otherwise, we would be able to apply regularity estimates for bounded solutions of semilinear parabolic equations (see [25] for instance) to obtain the Hölder estimates of
\( p(x,t) \) in \( \Omega \times (1, \infty) \), and thereby conclude \( \lim_{t \to \infty} \| p(\cdot, t) - 1 \|_{L^\infty(\Omega)} = 0 \). As things stand at the moment, we are only able to infer convergence in \( L^r \).

### 3.2 \( L^\infty \)-convergence of solutions with exponential rate in one dimension

It is observed that the results, in particular Lemma 3.6, in the previous subsection are still valid in the one-dimensional case. Moreover, in the one-dimensional setting, the weak convergence result in Lemma 3.6 can be improved via a bootstrap argument. In fact, we shall derive some a priori estimates of \((p, c, w)\) and thereby demonstrate that \((p, c, w)\) converges to \((1, 0, 1)\) in \(L^\infty(\Omega)\) as \(t \to \infty\). Furthermore, by a regularity argument involving the variation-of-constants formula for \(p\) and smoothing \(L^p - L^q\) type estimates for the Neumann heat semigroup, we will show that \(p(\cdot, t) - 1\) decays exponentially in \(L^\infty(\Omega)\).

As pointed out in the Introduction, the main technical difficulty in the derivation of Theorem 1.2 stems from the coupling between \(p\) and \(w\). Indeed, the lack of regularization effect in the space variable in the \(w\)-equation and the presence of \(p\) there demand tedious estimates of the solution.

The following lemma plays a crucial role in establishing the uniform convergence of \(p\) as \(t \to \infty\) (see Lemma 3.11). Thought the proof thereof only involves elementary analysis, we give a full proof here for the sake of the reader’s convenience since we could not find a precise reference covering our situation.

**Lemma 3.7.** Let \(k(t)\) be a function satisfying

\[
 k(t) \geq 0, \quad \int_0^\infty k(t)dt < \infty.
\]

If \(k'(t) \leq h(t)\) for some \(h(t) \in L^1(0, \infty)\), then \(k(t) \to 0\) as \(t \to \infty\).

**Proof.** Supposing the contrary, then we can find \(A > 0\) and a sequence \((t_j)_{j \in \mathbb{N}} \subset (1, \infty)\) such that \(t_j \geq t_{j-1} + 2, t_j \to \infty\) as \(j \to \infty\) and \(k(t_j) \geq A\) for all \(j \in \mathbb{N}\). On the other hand, by \(k'(t) \leq h(t)\), we have

\[
 k(t_j - \tau) \geq k(t_j) - \int_{t_j-\tau}^{t_j} |h(s)|ds \geq k(t_j) - \int_{t_j-1}^{t_j} |h(s)|ds \quad (3.18)
\]
for all \( \tau \in (0,1) \).

Since \( h(t) \in L^1(0,\infty) \), we have \( \int_{t_{j-1}}^{t_j} |h(s)| \, ds \to 0 \) as \( t_j \to \infty \) and thereby there exists \( j_0 \in \mathbb{N} \) such that \( \int_{t_{j-1}}^{t_j} |h(s)| \, ds \leq \frac{A}{2} \) for all \( j \geq j_0 \), which along with (5.18) implies that

\[
k(t_j - \tau) \geq k(t_j) - \frac{A}{2} \geq \frac{A}{2}
\] (3.19)

for \( j \geq j_0 \) and \( \tau \in (0,1) \). It follows that \( \int_{t_{j-1}}^{t_j} |h(s)| \, ds \to 0 \) as \( t_j \to \infty \) and thereby there exists \( j_0 \in \mathbb{N} \) such that \( \int_{t_{j-1}}^{t_j} |h(s)| \, ds \leq A^2 \) for all \( j \geq j_0 \), which along with (3.18) implies that

\[
k(t_j - \tau) \geq k(t_j) - A^2 \geq A^2
\] (3.19)

for \( j \geq j_0 \) and \( \tau \in (0,1) \). It follows that \( \int_{t_{j-1}}^{t_j} k(t) \, dt \geq \frac{A}{2} \) for all \( j \geq j_0 \), which contradicts \( \int_0^{\infty} k(t) \, dt < \infty \) and thus completes the proof of the lemma.

**Lemma 3.8.** If \( w_0 > 1 - \frac{1}{\rho} \), then there exists a constant \( C > 0 \) such that

\[
\int_0^\infty \int_{\Omega} e^{\rho w} |z_x|^2 \leq C
\] (3.20)

where \( z = pe^{-\rho w} \).

**Proof.** We know that \( z_x = e^{-\rho w} p_x - \rho zw_x \) and thus

\[
e^{\rho w} |z_x|^2 \leq 4e^{-\rho w}|p_x|^2 + 4p^2 e^{-\rho w} \rho^2 |w_x|^2.
\]

Integrating over \( \Omega \times (0,\infty) \) and taking (2.5) into account, we have

\[
\int_0^\infty \int_{\Omega} e^{\rho w} |z_x|^2 \leq 4 \int_0^\infty \int_{\Omega} |p_x|^2 + 4\rho^2 \sup_{t \geq 0} \|p(t)\|_{L^\infty(\Omega)} \int_0^\infty \int_{\Omega} p |w_x|^2
\]

\[
\leq 4(1 + \rho^2) \sup_{t \geq 0} \|p(t)\|_{L^\infty(\Omega)} (\int_0^\infty \int_{\Omega} |p_x|^2 \frac{1}{p} + \int_0^\infty \int_{\Omega} p |w_x|^2).
\]

Hence by Theorem 1.1 and Lemma 3.5 we get (3.20).

**Lemma 3.9.** If \( w_0 > 1 - \frac{1}{\rho} \), then there exists a constant \( C > 0 \) such that

\[
\frac{d}{dt} \int_{\Omega} e^{\rho w} |z_x|^2 + \frac{1}{3} \int_{\Omega} e^{\rho w} z_t^2
\]

\[
\leq C(\int_{\Omega} e^{\rho w} |z_x|^2 + \int_{\Omega} p |w_x|^2 + \int_{\Omega} |c_{xx}|^2 + \int_{\Omega} p |c_x|^2 + \int_{\Omega} p |w - 1| + \int_{\Omega} p(p - 1)^2)
\] (3.21)

with \( z = pe^{-\rho w} \).

**Proof.** Note that \( z \) satisfies

\[
z_t = e^{-\rho w}(e^{\rho w} z_x)_x - e^{-\rho w} (\frac{ze^{\rho w}}{1 + e^{\rho w}}) \nabla c)_x + \mu z(1 - ze^{\rho w}) - \rho \gamma e^{\rho w} z^2 (1 - w).
\]
Multiplying the above equation by $z_t e^{pw}$ and integrating in the spatial variable, we obtain

\[ \int_{\Omega} e^{pw} z_t^2 + \int_{\Omega} e^{pw} z_x z_{xt} = \int_{\Omega} e^{pw} z_t (\frac{\alpha}{1 + c} z_x c_x + \frac{\alpha z \rho}{1 + c} w_x c_x - \frac{\alpha z}{(1 + c)^2} |c_x|^2 + \frac{\alpha z}{1 + c} c_{xx}) \]  

(3.22)

Notice that

\[ \int_{\Omega} e^{pw} z_x z_{xt} = \frac{1}{2} \int_{\Omega} e^{pw} |z_x|^2 - \frac{\gamma \rho}{2} \int_{\Omega} e^{2pw} z(1 - w)|z_x|^2 \geq \frac{1}{2} \int_{\Omega} e^{pw} |z_x|^2, \]

\[ - \int_{\Omega} z_t \frac{\alpha z e^{pw}}{1 + c} c_x \leq \frac{1}{6} \int_{\Omega} e^{pw} z_t^2 + c_1 \sup_{t \geq 0} \|c_x(t)\|_{L^\infty(\Omega)}^2 \int_{\Omega} e^{pw} |z_x|^2, \]

\[ \int_{\Omega} \frac{z_t \alpha z e^{pw}}{1 + c} w_x c_x \leq \frac{1}{6} \int_{\Omega} e^{pw} z_t^2 + c_1 \sup_{t \geq 0} \|c_x(t)\|_{L^\infty(\Omega)}^2 \sup_{t \geq 0} \|p(t)\|_{L^\infty(\Omega)} \int_{\Omega} p|w_x|^2, \]

\[ \int_{\Omega} \frac{z_t \alpha z e^{pw}}{1 + c} c_{xx} \leq \frac{1}{6} \int_{\Omega} e^{pw} z_t^2 + c_1 \sup_{t \geq 0} \|p(t)\|_{L^\infty(\Omega)}^2 \int_{\Omega} |c_{xx}|^2, \]

\[ \int_{\Omega} e^{pw} z_t (\lambda z (1 - z e^{pw}) - \rho \gamma e^{pw} z^2 (1 - w)) \]

\[ = \lambda \int_{\Omega} z_t (1 - p) - \rho \gamma \int_{\Omega} z_t p^2 (1 - w) \leq \frac{1}{6} \int_{\Omega} e^{pw} z_t^2 + c_1 \sup_{t \geq 0} \|p(t)\|_{L^\infty(\Omega)} \int_{\Omega} p(1 - p)^2 + c_1 \sup_{t \geq 0} \|p(t)\|_{L^\infty(\Omega)}^3 \int_{\Omega} |1 - w_0|^2 \int_{\Omega} |1 - w|. \]

Applying Theorem 1.1, (3.5) and inserting the above inequalities into (3.22), we obtain (3.21).

Now we focus our attention on the decay properties of the solutions. Indeed, we will show that $p(x, t)$ converges to 1 with respect to the norm in $L^\infty(\Omega)$ as $t \to \infty$. Subsequently, we will establish the exponential decay of $\|p(\cdot, t) - 1\|_{L^\infty(\Omega)}$ with explicit rate.

**Lemma 3.10.** If $w_0 > 1 - \frac{1}{\rho}$, then

\[ \lim_{t \to \infty} \|w(\cdot, t) - 1\|_{L^\infty(\Omega)} = 0. \]  

(3.23)
Proof. From (3.8), it follows that for any \( \epsilon > 0 \)
\[
|w(x, t) - 1| \leq \| w_0 - 1 \|_{L^\infty(\Omega)} \exp\{ - \gamma \int_0^t p(s) ds \} \\
\leq \| w_0 - 1 \|_{L^\infty(\Omega)} \exp\{ \gamma \int_0^t \| p(s) - \overline{p}(s) \|_{L^\infty(\Omega)} ds - \gamma \int_0^t \overline{p}(s) ds \} \\
\leq \| w_0 - 1 \|_{L^\infty(\Omega)} \exp\{ \frac{1}{\epsilon} \int_0^t \| p(s) - \overline{p}(s) \|_{L^\infty(\Omega)}^2 ds + \epsilon \gamma t - \gamma \int_0^t \overline{p}(s) ds \},
\]
where \( \overline{p}(t) = \frac{1}{|\Omega|} \int_\Omega p(\cdot, t). \)

On the other hand, by the Poincaré–Wirtinger inequality, the Sobolev imbedding theorem in one dimension and (3.9), we have
\[
\int_0^t \| p(s) - \overline{p}(s) \|_{L^\infty(\Omega)}^2 ds \leq c_1 \sup_{t \geq 0} \| p(t) \|_{L^\infty(\Omega)} \int_0^\infty \int_\Omega \frac{|p_x(s)|^2}{p(s)} ds \\
\leq c_2
\]
for some constant \( c_2 > 0 \). Combining (3.24) with (3.25) yields
\[
\| w(t) - 1 \|_{L^\infty(\Omega)} \leq \| w_0 - 1 \|_{L^\infty(\Omega)} \exp\{ \frac{c_2 \gamma}{\epsilon} + \epsilon \gamma t - \gamma \int_0^t \overline{p}(s) ds \}
\]
for \( t \geq 0 \). The assertion now follows from the last inequality and the proof is complete.

Lemma 3.11. If \( w_0 > 1 - \frac{1}{\rho} \), then
\[
\lim_{t \to \infty} \| p(\cdot, t) - 1 \|_{L^\infty(\Omega)} = 0.
\]

Proof. We first show that
\[
\lim_{t \to \infty} \| z(\cdot, t) - \overline{z}(t) \|_{L^\infty(\Omega)} = 0
\]
where \( \overline{z}(t) = \frac{1}{|\Omega|} \int_\Omega z(\cdot, t) \). To this end, we consider the function \( k(t) \geq 0 \) defined by \( k(t) = \int_\Omega e^{\rho w} |z_x|^2 \) and prove that
\[
\lim_{t \to \infty} k(t) = 0.
\]
By Lemmas 3.7, 3.8 and 3.9, it is enough to prove that
\[
h(t) := \int_\Omega e^{\rho w} |z_x|^2 + \int_\Omega p|w_x|^2 + \int_\Omega |c_{xx}|^2 + \int_\Omega p|c_x|^2 + \int_\Omega p|w - 1| + \int_\Omega p(p - 1)^2 \in L^1(0, \infty).
\]
Noting (3.20), (3.9), (3.3), (3.2) and (2.4), it remains to estimate $\int_0^\infty \int_\Omega p(p-1)^2$. In fact, multiplying the $p$-equation in (1.1) by $p-1$, we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega (p-1)^2 = -\int_\Omega |p_x|^2 + \rho \int_\Omega p w_x p_x + \alpha \int_\Omega \frac{p}{1+c} c_x p_x - \lambda \int_\Omega p(p-1)^2$$

$$\leq -\frac{1}{2} \int_\Omega |p_x|^2 + C(\int_\Omega p^2 |w_x|^2 + \int_\Omega p^2 |c_x|^2) - \lambda \int_\Omega p(p-1)^2. \quad (3.30)$$

Hence, by the boundedness of $p$, (3.3) and (3.9), we easily infer that $\int_0^\infty \int_\Omega p(p-1)^2 \leq C$.

Furthermore, by the Poincaré–Wirtinger inequality and the Sobolev imbedding theorem in one dimension, we have

$$\|z(t) - \overline{z}(t)\|_{L^\infty(\Omega)} \leq C_p \|z_x(t)\|_{L^2(\Omega)}, \quad (3.31)$$

which along with (3.29) yields (3.28).

On the other hand, for any $\{t_j\} \subset (1, \infty)$, there exists a subsequence along which $z(\cdot, t_j) - e^{-\rho} \to 0$ a.e. in $\Omega$ as $j \to \infty$ by Lemma 3.6. We apply the dominated convergence theorem along with the uniform majorization $|z(\cdot, t_j)| \leq \sup_{j \geq 1} \|z(t_j)\|_{L^\infty(\Omega)} \leq C$ to infer that

$$\lim_{t \to \infty} |\overline{z}(t) - e^{-\rho}| = 0. \quad (3.32)$$

Hence

$$\|p(\cdot, t) - 1\|_{L^\infty(\Omega)} = \|e^{\rho w} - 1\|_{L^\infty(\Omega)}$$

$$\leq e^{\rho(1+\|w_0\|_{L^\infty(\Omega)})} (\|z(\cdot, t) - \overline{z}(t)\|_{L^\infty(\Omega)} + \|\overline{z}(t) - e^{-\rho}\|_{L^\infty(\Omega)}) + \|e^{\rho w(\cdot, t)} - e^\rho\|_{L^\infty(\Omega)}$$

$$\leq e^{\rho(1+\|w_0\|_{L^\infty(\Omega)})} (\|z(\cdot, t) - \overline{z}(t)\|_{L^\infty(\Omega)} + \|\overline{z}(t) - e^{-\rho}\| + c_1 \|w(\cdot, t) - 1\|_{L^\infty(\Omega)})$$

for some $c_1 > 0$, which, together with (3.28), (3.28) and (3.32), yields the desired result.

Having established that $p(x,t)$ converges to 1 uniformly with respect to $x \in \Omega$ as $t \to \infty$, we now go on to establish an explicit exponential convergence rate. Using (3.27), we first look into the decay of $\int_\Omega |w_x(t)|^2$.

**Lemma 3.12.** Let $w_0 > 1 - \frac{1}{\rho}$. Then for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\int_\Omega |w_x(t)|^2 \leq C(\epsilon) e^{-2\gamma(1-\epsilon)t}. \quad (3.33)$$
Taking Lemma 3.11 into account, we know that for any \( \epsilon > 0 \), there exists \( t_\epsilon > 1 \) such that \( p(x, t) > 1 - \epsilon \) for all \( x \in \Omega, t > t_\epsilon \). Therefore integrating the above inequality in the space variable yields

\[
\int_\Omega |w_x(t)|^2 \leq 4|w_{0x}|^2 e^{-2\gamma t_0}fr s^pds + 4\gamma^2 w_0 - 1|2e^{-2\gamma t_0}fr s^pds \int_0^t |p_x(s)|ds^2 \leq 4|w_{0x}|^2 e^{-2\gamma t_0}fr s^pds + 4t\gamma^2 w_0 - 1|2e^{-2\gamma t_0}fr s^pds \int_0^t |p_x(s)|^2ds.
\]

Taking Lemma 3.11 into account, we know that for any \( \epsilon > 0 \), there exists \( t_\epsilon > 1 \) such that \( p(x, t) > 1 - \epsilon \) for all \( x \in \Omega, t > t_\epsilon \). Therefore integrating the above inequality in the space variable yields

\[
\int_\Omega |w_x(t)|^2 \leq 4e^{-2\gamma(1-\epsilon)(t-t_\epsilon)} \int_\Omega |w_{0x}|^2 + 4t\gamma w_0 - 1\|L_\infty(\Omega) e^{-2\gamma(1-\epsilon)(t-t_\epsilon)} \int_0^\infty \int_\Omega |p_x|^2 \leq c_1(\epsilon)(1 + t) e^{-2\gamma(1-\epsilon)t} (\int_\Omega |w_{0x}|^2 + \|w_0 - 1\|L_\infty(\Omega) \sup_{t\geq 0} \|p(t)\|L_\infty(\Omega) \int_0^\infty \int_\Omega |p_x|^2),
\]

for all \( t > t_\epsilon \), which, along with (3.9), implies (3.33).

Now we utilize the decay properties of \( \int_\Omega |w_x(t)|^2 \), \( \int_\Omega |w_x(t)|^2 \) and the uniform convergence of \( |p(x, t) - 1| \) asserted by Lemma 3.11 to establish the decay property of \( \|p(\cdot, t) - 1\|L_2(\Omega) \).

**Lemma 3.13.** Let \( w_0 > 1 - \frac{1}{p} \). Then for any \( \epsilon \in (0, \min\{1, \gamma, \lambda\}) \), there exists \( C(\epsilon) > 0 \) such that

\[
\|p(\cdot, t) - 1\|L_2(\Omega) \leq C(\epsilon)e^{-(\min\{1, \gamma, \lambda\} - \epsilon)t}.
\]

**Proof.** By (3.27), we know that for any \( \epsilon \in (0, \min\{1, \gamma, \lambda\}) \), there exists \( t_\epsilon > 1 \) such that \( p(x, t) > 1 - \epsilon \) for all \( x \in \Omega, t > t_\epsilon \). Hence, we multiply the p-equation in (1.1) by \( p - 1 \) and integrate the result over \( \Omega \) to get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (p - 1)^2 = - \int_\Omega |p_{xx}|^2 + \rho \int_\Omega pw_xp_x + \alpha \int_\Omega \frac{p}{1 + c} p_xp_x - \lambda \int_\Omega p(p - 1)^2 \leq -\frac{1}{2} \int_\Omega |p_{xx}|^2 + c_1 \int_\Omega |w_x|^2 + \int_\Omega |c_x|^2 - \lambda(1 - \epsilon) \int_\Omega (p - 1)^2
\]

for all \( t > t_\epsilon \). Now, applying the Gronwall inequality, (3.4) and Lemma 3.12, we have

\[
\int_\Omega (p(t) - 1)^2 \leq e^{-2\lambda(1-\epsilon)(t-t_\epsilon)} \int_\Omega (p(t_\epsilon) - 1)^2 + c_1 \int_0^t e^{-2\lambda(1-\epsilon)(t-s)} \left( \int_\Omega |w_x|^2 + \int_\Omega |c_x|^2 \right) \leq c_2(\epsilon)e^{-2\lambda(1-\epsilon)t} + c_3(\epsilon) \int_0^t e^{-2\lambda(1-\epsilon)(t-s)}(e^{-2\gamma(1-\epsilon)s} + e^{-2(1-\epsilon)s}) \leq c_4(\epsilon)e^{-2\min\{\lambda, 1, \gamma\}(1-\epsilon)t},
\]

where \( c_i(\epsilon) > 0 (i = 2, 3, 4) \) are independent of time \( t \). This completes the proof.
Moving forward, on the basis of Lemma 3.13 we come to establish the exponential decay of \( \| p(\cdot, t) - \overline{p}(t) \|_{L^\infty(\Omega)} \) by means of a variation-of-constants representation of \( p \), as follows:

**Lemma 3.14.** Let \( w_0 > 1 - \frac{1}{\rho} \). Then for any \( \epsilon \in (0, \min\{\lambda_1, 1, \gamma, \lambda\}) \), there exists \( C(\epsilon) > 0 \) such that

\[
\| p(\cdot, t) - \overline{p}(t) \|_{L^\infty(\Omega)} \leq C(\epsilon)e^{-(\min\{\lambda_1, 1, \gamma, \lambda\} - \epsilon)t}. \tag{3.36}
\]

**Proof.** By noting that \( \overline{p}_t = \lambda p(1 - p)(t) \), applying the variation-of-constants formula to the \( p \)-equation in (1.1) yields

\[
p(\cdot, t) - \overline{p}(t) = e^{t\Delta}(p(\cdot, 0) - \overline{p}(0)) - \alpha \int_0^t e^{(t-s)\Delta}\left( \frac{p}{1 + c}c_x \right)_x ds - \rho \int_0^t e^{(t-s)\Delta}(pw)_x + \lambda \int_0^t e^{(t-s)\Delta}(p(1-p) - \overline{p}(1-p)).
\]

Together with (3.4) and Lemma 3.13 and Lemma 2.1, this gives

\[
\| p(\cdot, t) - \overline{p}(t) \|_{L^\infty(\Omega)} \leq \| e^{t\Delta}(p(\cdot, 0) - \overline{p}(0)) \|_{L^\infty(\Omega)} + \alpha \int_0^t \| e^{(t-s)\Delta}\left( \frac{p}{1 + c}c_x \right)_x \|_{L^\infty(\Omega)} ds + \rho \int_0^t \| e^{(t-s)\Delta}(pw)_x \|_{L^\infty(\Omega)} + \lambda \int_0^t \| e^{(t-s)\Delta}(p(1-p) - \overline{p}(1-p)) \|_{L^\infty(\Omega)} ds.
\]

\[
\leq k_1e^{-\lambda_1t}\| p(\cdot, 0) - \overline{p}(0) \|_{L^\infty(\Omega)} + c_1 \int_0^t (1 + (t-s)^{-\frac{3}{4}})e^{-\lambda_1(t-s)}\| w_x \|_{L^2(\Omega)} ds + c_1 \int_0^t (1 + (t-s)^{-\frac{3}{4}})e^{-\lambda_1(t-s)}\| p(1-p) - \overline{p}(1-p) \|_{L^2(\Omega)} ds.
\]

\[
\leq k_1e^{-\lambda_1t}\| p(\cdot, 0) - \overline{p}(0) \|_{L^\infty(\Omega)} + c_2(\epsilon) \int_0^t (1 + (t-s)^{-\frac{3}{4}})e^{-\lambda_1(t-s)}e^{-\gamma(1-\epsilon)s} ds + c_2(\epsilon) \int_0^t (1 + (t-s)^{-\frac{3}{4}})e^{-\lambda_1(t-s)}e^{-\gamma(1-\epsilon)s} ds + c_1 \int_0^t (1 + (t-s)^{-\frac{3}{4}})e^{-\lambda_1(t-s)}\| p(1-p) - \overline{p}(1-p) \|_{L^2(\Omega)} ds.
\]

It is observed that

\[
\| p(1-p) - \overline{p}(1-p) \|^2_{L^2(\Omega)} = \| p(1-p) \|^2_{L^2(\Omega)} - |\Omega|\| p(1-p) \|^2 \leq \| p(1-p) \|^2_{L^2(\Omega)}.
\]

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Hence from (3.32) and Lemma 3.13 it follows that
\[
\|p(\cdot, t) - \overline{p}(t)\|_{L^\infty(\Omega)} \leq k_1 e^{-\lambda_1 t} \|p(\cdot, 0) - \overline{p}(0)\|_{L^\infty(\Omega)} + c_2(\epsilon) \int_0^t (1 + (t - s)^{-\frac{3}{2}}) e^{-\lambda_1 (t-s) e^{-\gamma (1 - \epsilon)s}} + c_2(\epsilon) \int_0^t (1 + (t - s)^{-\frac{3}{2}}) e^{-\lambda_1 (t-s) e^{-\gamma (1 - \epsilon)s}} + c_3(\epsilon) \int_0^t (1 + (t - s)^{-\frac{3}{2}}) e^{-\lambda_1 (t-s) e^{-\min\{1, \gamma, \lambda\} (1 - \epsilon)s}} \leq c_4(\epsilon) e^{-\min\{\lambda_1, 1, \gamma, \lambda\} (1 - \epsilon)t},
\]
which implies (3.36).

**Lemma 3.15.** Let \( w_0 > 1 - \frac{1}{\rho} \). Then for any \( \epsilon \in (0, \min\{\lambda_1, 1, \gamma, \lambda\}) \), there exists \( C(\epsilon) > 0 \) such that
\[
|\overline{p}(t) - 1| \leq C(\epsilon) e^{-(\min\{2\lambda_1, 2, 2\gamma, \lambda\} - \epsilon)t}. \tag{3.39}
\]

**Proof.** We integrate the \( p \)-equation in the spatial variable over \( \Omega \) to obtain
\[
(\overline{p} - 1)_t = \lambda(\overline{p} - \overline{p}\overline{p}) - \frac{1}{|\Omega|} \int_\Omega (p - \overline{p})^2
= -\lambda \overline{p}(\overline{p} - 1) - \frac{1}{|\Omega|} \|p - \overline{p}\|_{L^2(\Omega)}^2. \tag{3.40}
\]
By (3.12), there exists \( t_\epsilon > 0 \) such that \( \overline{p}(t) \geq 1 - \epsilon \) for \( t \geq t_\epsilon \). Hence by (3.36) and (3.40), solving the differential equation entails
\[
|\overline{p}(t) - 1| \leq |\overline{p}(t_\epsilon) - 1| e^{-\lambda \int_{t_\epsilon}^t \int_\Omega ds} + \frac{\lambda}{|\Omega|} \int_{t_\epsilon}^t \int_\Omega e^{-\lambda s} \|p(s) - \overline{p}(s)\|_{L^2(\Omega)}^2 ds
\leq |\overline{p}(t_\epsilon) - 1| e^{-\lambda (1 - \epsilon)(t-t_\epsilon)} + \frac{\lambda}{|\Omega|} \int_{t_\epsilon}^t \int_\Omega e^{-\lambda(1 - \epsilon)(t-s)} \|p(s) - \overline{p}(s)\|_{L^2(\Omega)}^2 ds
\leq |\overline{p}(t_\epsilon) - 1| e^{-\lambda(1 - \epsilon)(t-t_\epsilon)} + c_1(\epsilon) \int_{t_\epsilon}^t e^{-\lambda(1 - \epsilon)(t-s)} e^{-2 \min\{\lambda_1, 1, \gamma, \lambda\} (1 - \epsilon)s} ds
\leq |\overline{p}(t_\epsilon) - 1| e^{-\lambda(1 - \epsilon)(t-t_\epsilon)} + c_2(\epsilon) e^{-\min\{2\lambda_1, 2, 2\gamma, \lambda\} (1 - \epsilon)t}
\leq c_3(\epsilon) e^{-\min\{2\lambda_1, 2, 2\gamma, \lambda\} (1 - \epsilon)t}
\]
for \( t \geq t_\epsilon \), which proves (3.39).

**Lemma 3.16.** Let \( w_0 > 1 - \frac{1}{\rho} \). Then for any \( \epsilon \in (0, \min\{\lambda_1, 1, \gamma, \lambda\}) \), there exists \( C(\epsilon) > 0 \) such that
\[
\|p(\cdot, t) - 1\|_{L^\infty(\Omega)} \leq C(\epsilon) e^{-(\min\{\lambda_1, 1, \gamma, \lambda\} - \epsilon)t}. \tag{3.41}
\]
Proof. Combining the above two lemmas, we have

$$\|p(\cdot, t) - 1\|_{L^\infty(\Omega)} \leq \|p(\cdot, t) - \overline{p}(t)\|_{L^\infty(\Omega)} + |\overline{p}(t) - 1| \leq C(\epsilon)e^{-(\min\{\lambda_1, 1, \gamma, \lambda\} - \epsilon)t}.$$ 

Proof of Theorem 1.2. (1.5) is the direct consequence of Lemma 3.6 in the previous subsection. As for (1.6)–(1.8), we only need to collect (3.4), (3.26), (3.33) and (3.41).

Remark 3.2. In comparison with (3.41), by (3.9) and (3.29), we have sup_{t \geq 0} \int_{\Omega} |w_x(t)|^2 \leq C and sup_{t \geq 0} \int_{\Omega} |z_x(t)|^2 \leq C respectively, and thus sup_{t \geq 0} \|p(t)\|_{W^{1,2}(\Omega)} \leq C. Hence an interpolation by means of the Gagliardo–Nirenberg inequality in the one-dimensional setting provides

$$\|p(\cdot, t) - 1\|_{L^\infty(\Omega)} \leq \|p(\cdot, t)\|_{W^{1,2}(\Omega)}^{\frac{1}{2}}\|p(\cdot, t) - 1\|_{L^2(\Omega)}^{\frac{1}{2}} \leq C(\epsilon)e^{-(\frac{1}{2}\min\{\lambda_1, \gamma, \lambda\} - \epsilon)t},$$

where we have used (3.34).

Acknowledgment

The authors are grateful to the referee for his illuminating comments. This work is partially supported by the NUS AcRF grant R-146-000-249-114 (PYHP) and by the NNSFC grant 11571363 (YW).

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