A completely solvable model with an infinite number of Dirac observables for a real sector of complexified Ashtekar gravity

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Abstract

We introduce a reduced model for a real sector of complexified Ashtekar gravity that does not correspond to a subset of Einstein’s gravity but for which the programme of canonical quantization can be carried out completely, both, via the reduced phase space approach or along the lines of the algebraic quantization programme.
This model stands in a certain correspondence to the frequently treated cylindrically symmetric waves.
In contrast to other models that have been looked at up to now in terms of the new variables the reduced phase space is infinite dimensional while the scalar constraint is genuinely bilinear in the momenta.
The infinite number of Dirac observables can be expressed in compact and explicit form in terms of the original phase space variables.
They turn out, as expected, to be non-local and form naturally a set of countable cardinality.

1 Introduction

Ashtekar’s variables (1) simplify the algebraic structure of the constraint equations of general relativity so tremendously that one can solve various problems of classical and quantum gravity that were simply unfeasible before in terms of the old ADM variables. In particular, a number of model systems could be solved to more extent or even completely in these new variables.
The model systems which could be solved completely up to now in terms of the new variables (e.g. 2) to our knowledge lack from not having in common at least one of the following features of general relativity:
1) the reduced phase space is infinite dimensional (i.e. the number of Dirac observables) and

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2) the scalar constraint is bilinear in the momenta.

Here we introduce a new model which is an algebraic plus Killing reduction of complexified Ashtekar gravity. In order to account for the fact that Ashtekar’s connection obeys a non-trivial reality condition, we also impose reality conditions that bring us ‘as close as possible’ to the cylindrically symmetric waves (3) which form a genuine subset of Einstein gravity whereas our model defines then a real sector of complexified gravity which is manifestly no subset of Einstein gravity.

However, our model has both features 1) and 2) and we are thus able to attack one of the major problems of general relativity, namely to find a complete set of (preferredly canonically conjugate) Dirac observables for a theory with an infinite number of degrees of freedom and non-trivial dynamics.

The results of this paper are as follows:

In section 2 we define the present model and show that it is not quite possible to formulate the cylindrically symmetric waves (which means to project our model of complex gravity to the real sector corresponding to Einstein gravity) in terms of Ashtekar’s variables after we carried out the reduction. We then impose reality conditions that bring us at least ‘as close as possible’ to the cylindrically symmetric case. In particular, the first- and the diagonal part of the second fundamental form are real while the off-diagonal part of the latter is purely imaginary.

In section 3 we carry out the symplectic reduction (4) of our model. Quite surprisingly, it is straightforward to solve the typically very complicated scalar constraint before solving the diffeomorphism constraint. As usual for Hamiltonians or scalar constraints bilinear in the momenta, there are several ‘sectors’ of the constraint surface. One of them corresponds to degenerate metrics, the other to non-degenerate metrics (the latter defines the ‘physical’ sector of our model). The most complicated step is then to find a complete but not overcomplete set of diffeomorphism invariant observables. This turns out to be feasible and the resulting functionals of the basic phase space variables can be given in explicit and compact form (the observables given in 3 are rather complicated and untractable objects due to the fact that the mode system of Bessel functions is involved).

As expected for a diffeomorphism invariant field theory, the Dirac observables come out to be naturally smeared and nonlocal. In contrast to non-diffeomorphism-invariant field theories one does not have to employ a complete system of mode functions on the hypersurface to write the Dirac observables in this form. The reduced phase space thus has naturally a countable cardinality. We argue that this is always true for a spatially diffeomorphism invariant field theory.

Section 4 is devoted to quantum theory.

From the reduced phase space point of view this is already trivial since the phase space derived in section 3 can be cast into a form so that it has the structure of a cotangent space over a real, infinite dimensional vector space.

The application of the algebraic quantization programme forces us to make use of functional integral techniques.

The two resulting Hilbert spaces turn out to be unitarily equivalent.

As to be expected for a field theory with constraints quadratic in the momenta, the final set of Dirac observables mix configuration and momentum variables with respect to the original polarization (here the Ashtekar polarization).
While it is important to derive the Dirac observables, they are typically hard to interpret. Here the framework of deparametrization ([6]) helps to define 'evolving' Dirac observables which allow for a direct interpretation in terms of metric variables and to leave the limits set by the 'frozen' formalism. We extend this formalism to field theories and apply it to our model. We thus arrive at an interpretation of the theory although the meaning of the constant Dirac observables remains veiled.

Finally, in section 5, we discuss the loop representation ([7]) of our model. First we define a set of loop variables that stand in bijection with the phase space variables underlying the connection representation. This implies that we can \textit{explicitly} express constraints, symplectic form and Dirac observables in terms of loop variables. These expressions are quite complicated compared to those obtained for the connection representation and that allows for two possible conclusions: Either it seems to be doubtful whether the loop variables are tailored for the problem to construct the reduced phase space or Killing reduced models are unsuited to simulate the situation in the full theory as far as the loop representation is concerned.

We summarize what has been learnt in dealing with the present model, valid for the full theory of Einstein gravity.

2 Definition of the model

We assume the reader to be familiar with the Ashtekar formulation of canonical gravity ([1]). In the sequel a,b,c,... will denote tensor indices and i,j,k,... SO(3) indices and we apply the abstract index formalism.

We impose the following three conditions on the phase space variables of the complexified Ashtekar formulation of gravity:

1) Killing-reduction: all fields are Lie-annihilated by a set of two commuting, hypersurface orthogonal Killing fields.

Since the Killing fields commute they are surface forming. Since they are hypersurface orthogonal these surfaces lie orthogonal to a foliating vector field. Hence we find three global independent coordinates x,y and z such that the Killing fields are given by $\partial_x$, $\partial_y$. Then all fields depend on z (and the time variable t) only.

2) algebraic reduction: We set to zero the following components of the basic fields of the Ashtekar formulation

$$ A_1^1 = A_2^2 = A_3^3 = A_y^b = A_x^a = E_1^x = E_2^y = E_3^z = E_y^x = E_x^y := 0. \quad (2.1) $$

Up to now we are on the phase space of the so-called Gowdy models ([8]). We will assume, for simplicity, the spatial topology of the initial data hypersurface to be that of the three-dimensional torus in order to avoid technical difficulties associated with boundary conditions.

One more condition brings us close to the axisymmetric spacetimes.

3) algebraic reduction: The Killing fields are orthogonal.
This last condition forces the densitized triad to be of the form
\[
(E^a_i) = \begin{pmatrix}
E^x \cos(\beta) & -E^y \sin(\beta) & 0 \\
E^y \sin(\beta) & E^y \cos(\beta) & 0 \\
0 & 0 & E^z
\end{pmatrix}
\] (2.2)

and in order to have as many configuration variables as momentum variables we require in analogy to (2.2) the Ashtekar connection to reduce to
\[
(A^i_a) = \begin{pmatrix}
A_x \cos(\alpha) & -A_y \sin(\alpha) & 0 \\
A_x \sin(\alpha) & A_y \cos(\alpha) & 0 \\
0 & 0 & A_z
\end{pmatrix}
\] (2.3)

where all variables are generally complex up to now.

As far as the triads are concerned, provided we restrict them to be purely real, we easily see that we have already a real diagonal 3-metric (compare the appendix).

Let us see whether we can also implement the reality condition
\[
\text{Re}(A^i_a - \Gamma^i_a) = 0, \quad \Gamma^i_a = \text{spin-connection} \quad (2.4)
\]

which leads to the Einstein phase space. In order to do that we first have to compute the spin-connection. The calculation is deferred to an appendix. The result is given by
\[
(\Gamma^i_a) = \begin{pmatrix}
-\Gamma_x \sin(\beta) & \Gamma_y \cos(\beta) & 0 \\
\Gamma_x \cos(\beta) & \Gamma_y \sin(\beta) & 0 \\
0 & 0 & \Gamma_z
\end{pmatrix}
\] (2.5)

where
\[
\Gamma_x = \frac{1}{2E^y} \left( \frac{E^y E^z}{E^x} \right)', \quad \Gamma_y = -\frac{1}{2E^z} \left( \frac{E^x E^z}{E^y} \right)', \quad \Gamma_z = -\beta'.
\] (2.6)

The problem is already obvious: both, \(\Gamma^i_a\) and \(A^i_a\) have five nonvanishing components, giving rise to five reality conditions induced by (2.4), but these components are coordinatized respectively by only four independent coordinates (namely \(A_x, A_y, A_z, \alpha\) and \(E^x, E^y, E^z, \beta\)). The explicit form of these reality conditions is given by the requirement that the following quantities \(i K^i_a := i K_{ab} E_b^i / \sqrt{\det(q)} = A^i_a - \Gamma^i_a\) where \(K_{ab}\) is the extrinsic curvature
\[
i K^i_x := A_x \cos(\alpha) + \Gamma_x \sin(\beta), \quad i K^2_x := A_x \sin(\alpha) - \Gamma_x \cos(\beta), \\
i K^1_y := -A_y \sin(\alpha) - \Gamma_y \cos(\beta), \quad i K^2_y := A_y \cos(\alpha) - \Gamma_y \sin(\beta), \\
i K^i_z := A_z - \Gamma_z
\] (2.7)

are imaginary. Equivalently
\[
i K^i_x E^y = E^y (A_x \sin(\alpha - \beta) - \Gamma_x), \quad i K^i_y E^x = E^x (-A_y \sin(\alpha - \beta) - \Gamma_y), \\
i K^1_x E^y = E^y A_x \cos(\alpha - \beta), \quad i K^2_y E^x = E^x A_y \cos(\alpha - \beta), \\
i K^i_z E^i = E^z (A_z - \Gamma_z)
\] (2.8)

are imaginary.

We will now show that these are five functionally independent equations whence our
reduction is incompatible with (a subset of) the real form of complexified gravity that corresponds to the Einstein phase space:

We will immediately show that the Gauss constraint of this model is given by

$$\mathcal{G} = \left( E^z \right)' - \sin(\alpha - \beta)(A_x E^x + A_y E^y) . $$

Let $\delta := \alpha - \beta$. Then, using the Gauss constraint we derive that

$$\tan(\delta) = \frac{\left( E^z \right)'}{E^x[A_x \cos(\delta)] + E^y[A_y \cos(\delta)]} \quad (2.9)$$

which is weakly imaginary since the square brackets in the denominator are according to (2.8). Assuming that the imaginary part of $\delta$ is finite, this implies unambiguously that $\delta$ itself is imaginary: $\delta = i\xi$ where $\xi$ is real. Then $\cos(\delta) = \text{ch}(\xi)$ is real whence again from (2.8) we derive that $A_x, A_y$ are imaginary. Finally we have $\gamma := A_z + \alpha' = (A_z - \Gamma_z) + \delta' = (A_z - \Gamma_z) + i\xi'$ which displays $\gamma$ and $A_z - \Gamma_z$ as (weakly) imaginary quantities.

We still have to satisfy the two first conditions in (2.8). First of all, again using the Gauss constraint, we find that

$$\Re(-A_x \sin(\delta) + \Gamma_x) = \frac{1}{E^x} \Re(-A_x E^x \sin(\delta) + E^x \Gamma_x)$$

$$= \frac{1}{E^x} \Re(\mathcal{G} + A_y E^y \sin(\delta) - \left( E^z \right)' + E^x \Gamma_x)$$

$$= \frac{E^y}{E^x} \Re\left( \frac{\mathcal{G}}{E^y} + A_y \sin(\delta) + \Gamma_y \right)$$

$$\approx \frac{E^y}{E^x} \Re(A_y \sin(\delta) + \Gamma_y) \quad (2.10)$$

where we have used the identity $\left( E^z \right)' - \Gamma_x E^x + \Gamma_y E^y = 0$ (which is due to the covariant constance of the triads) and the reality of the triads. This demonstrates that the real part of the first quantity in the first line of (2.10) vanishes if and only if the last line in (2.10) is zero, at least on the Gauss-constraint surface. However, since $A_x, A_y, \delta$ are imaginary, the imaginary part of both expressions is already zero so that at least one of the fields $A_x, A_y, \delta$ would be expressible completely in terms of the triads and therefore the symplectic structure would become degenerate. This furnishes the proof.

Since we are mainly interested in playing with a model that captures some of the features of general relativity rather than describing the cylindrically symmetric case, we take the following viewpoint:

We want to preserve at least the reality of the diagonal part of the extrinsic curvature (the extrinsic curvature is diagonal for the cylindrically symmetric waves). Since the metric is manifestly diagonal, we can impose the simple reality condition that

$$\alpha - \beta, \ A_z - \Gamma_z, \ A_x, \ A_y$$

are all purely imaginary. Now, looking at (2.8), we see that the extrinsic curvature will be symmetric upon imposition of the Gauss constraint but it will develop an

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1The cylindrically symmetric case is not of physical importance anyway because an axisymmetric matter distribution is quite improbable.
imaginary off-diagonal part in the course of evolution. The only case when it is real is when the off-diagonal part vanishes and we are then back on a genuine subset of the phase space of the cylindrically symmetric model (this subset is a fix point under evolution). The diagonal part is real. The metric is diagonal and real by construction so we managed to impose reality conditions that are 'as close' to the cylindrically symmetric case as possible. These reality conditions are also preserved under evolution as we will show shortly and that raises another problem: Due to the evolution law of full general relativity

\[ \dot{q}_{ab} = 2NK_{ab} + D(a)N_b \] (2.12)

an initially real, diagonal metric becomes complex and nondiagonal when inserting our \( K_{ab} \) above. This is apparently a contradiction! However, there is a mistake in this argument: the processes of deriving the field equations and reducing down the degrees of freedom cannot be expected to commute if the phase space of the reduced model does not lie entirely in that of the full theory as it is the case for our model. One might ask whether it is possible to recover the cylindrically symmetric case by making use of all SO(3) degrees of freedom (the three Euler angles rather than only one, \( \alpha \) and \( \beta \) for \( A^i_a \) and \( E^a_i \) respectively). However, if that was true one could always go to the gauge (2.2), (2.3) and since the reality structure of Einstein gravity is SO(3) covariant (since SO(3) is a real Lie group) one would obtain a contradiction. Thus, it is not possible to treat the cylindrically symmetric case while making use of a corresponding reduction on the Ashtekar phase space. Nevertheless, this model has all features in common with full general relativity (except for the modified reality conditions) so that it serves as an interesting new testing ground for the quantization programme of full quantum gravity as we will now show (actually this is also the viewpoint that one takes in dealing with 2+1 gravity as a toy model for 3+1 gravity: the connection of 2+1 gravity is purely real).

3 Complete symplectic reduction of the model

It will turn out that the symplectic reduction programme can be carried out completely, that is, with respect to all constraints. After integrating over the (finite) range of the coordinates \( x \) and \( y \), the reduced action becomes ([4]; we neglect a trivial prefactor coming from this integration)

\[ S = \int_R dt \int_{\mathcal{T}_1} dz \left[ -i \dot{\Lambda}^i_a E^a_i - (i \Lambda^i \mathcal{G}_i - iN^a V_a + \frac{1}{2} N^{-1} C) \right] \] (3.1)

where it is understood that we have to substitute for \( A^i_a, E^a_i \) the reduced expressions (2.2) and (2.3). Here \( \mathcal{G}_i, V_a, C \) are, respectively, the Gauss-, Vector- and scalar constraint and \( \Lambda^i, N^a, \bar{N} \) are the corresponding Lagrange multipliers, called respectively the Gauss scalar potential, shift vector and \( N := \sqrt{\det(q)} \bar{N} \) the lapse function.

By plugging the formulae (2.2) and (2.3) into the expressions for the constraint
functions we obtain

\[ G_i := \mathcal{D}_a E_i^a := \partial_a E_i^a + \epsilon_{ijk} A_j^a E_k^a \]
\[ = \delta_{ij} \left( (E^x)' - \sin(\alpha - \beta)(A_x E^x + A_y E^y) \right) =: \delta_{i3} \mathcal{G} \]
\[ V_a := F_{ab}^i E_i^b \]
\[ = \delta_a^z \left[ \cos(\alpha - \beta)((A_x)'E^x + (A_y)'E^y) - A_z(E^z)' - \alpha' \sin(\alpha - \beta)((A_x)'E^x + (A_y)'E^y) \right] \]
\[ = \delta_a^z \left[ \cos(\alpha - \beta)((A_x)'E^x + (A_y)'E^y) - \gamma(E^z)' + \alpha' \mathcal{G} \right] =: \delta_a^z \mathcal{V} \]
\[ C := \epsilon_{ijk} F_{ab}^i E_i^a E_j^b \]
\[ = 2(A_x A_y E^x E^y + ((A_x)'E^x + (A_y)'E^y)E^z \sin(\alpha - \beta)) \]
\[ + \gamma(A_x E^x + A_y E^y)E^z \cos(\alpha - \beta) \] =: 2C \quad (3.2) \]

where \( F_{ab}^i \) is the curvature of the Ashtekar connection, \( \epsilon_{ijk} \) is the Levi-Civita symbol and we have abbreviated \( \gamma := A_z + \alpha' \). Further we have absorbed the trivial factor of 2 in the last line into the Lagrange multiplier of the scalar constraint.

Now we insert the reduced form of our basic coordinates into the symplectic potential and obtain up to a total differential (note that spatial integrations by parts do not contribute boundary terms because the torus has no boundary)

\[ i\Theta[\partial_t] = \int_{\Sigma} dz \left[ \dot{A}_x E^x + \dot{A}_x E^x \cos(\alpha - \beta) \right. \]
\[ + \dot{A}_y E^y \cos(\alpha - \beta) - \dot{\alpha}(A_x E^x + A_y E^y) \sin(\alpha - \beta) \left. \right] \]
\[ = \int_{\Sigma} dz \left[ (\dot{A}_x + \dot{\alpha}') E^x + \dot{A}_x E^x \cos(\alpha - \beta) + \dot{A}_y E^y \cos(\alpha - \beta) + \dot{\alpha} \mathcal{G} \right] \]
\[ =: \int_{\Sigma} dz \left[ \dot{\gamma} \Pi + \dot{A}_x \Pi^x + \dot{A}_y \Pi^y + \dot{\alpha} \Pi^\alpha \right]. \quad (3.3) \]

Note that all configuration variables on the Gauss-reduced phase space are imaginary while the momenta are all real. This is a very nice reality structure.

We next write the constraints in the so defined canonical coordinates. For the Gauss and vector constraint this is easy:

\[ \mathcal{G} = \Pi^\alpha, \]
\[ V = (A_x)'\Pi^x + (A_y)'\Pi^y + \alpha'\Pi^\alpha - \gamma(\Pi^\gamma)' \] \quad (3.4)

while for the scalar constraint we have to do some more work. We have directly from the definitions of our canonical coordinates

\[ \tan(\alpha - \beta) = \frac{(\Pi^\gamma)' - \Pi^\alpha}{A_x \Pi^x + A_y \Pi^y} \] \quad (3.5)

so that we have managed to express the combination \( \alpha - \beta \) in terms of canonical variables.

Next, we use the trigonometrical identities

\[ \sin^2 \theta = \frac{\tan^2 \theta}{1 + \tan^2 \theta}, \quad \cos^2 \theta = \frac{1}{1 + \tan^2 \theta} \]

and have unambiguously

\[ C = A_x A_y \frac{\Pi^x \Pi^y}{\cos^2(\alpha - \beta)} + ((A_x)'\Pi^x + (A_y)'\Pi^y) \Pi^\gamma \tan(\alpha - \beta) + \gamma(A_x \Pi^x + A_y \Pi^y) \Pi^\gamma \]
By studying the transformations that these constraints generate, we obtain that 

e are scalars while 

to the momenta ! They are given by

\[ (\Pi^\gamma)' - \Pi^\alpha \]
\[ \frac{1}{A_x \Pi^x + A_y \Pi^y} + \gamma (A_x \Pi^x + A_y \Pi^y) \Pi^\gamma \]

\[ = \left( \frac{1}{A_x \Pi^x + A_y \Pi^y} \right)^2 A_x A_y \Pi^x \Pi^y (\Pi^\gamma)' - \Pi^\alpha \] + \gamma (A_x \Pi^x + A_y \Pi^y) \Pi^\gamma \]

\[ = \left( \frac{1}{A_x \Pi^x + A_y \Pi^y} \right)^2 C. \] (3.6)

We now rescale all configuration variables except for \( \alpha \) by a factor of i so that these become real too (e.g. \(-iA_x \) will be called \( A_x \) again. Then the reduced action becomes

\[ S = \int dt \int \Sigma \dot{A}_x E^x + \dot{A}_y E^y + \hat{\gamma} \Pi^\gamma + i\alpha \Pi^\alpha - [i\Lambda \Pi^\alpha + N^x V - \tilde{N} C] \] (3.7)

with manifestly real constraint functions \( G, V \) and \( C \) and Lagrange multipliers. This furnishes the proof alluded to in the previous section that the reality structure of the model is preserved under evolution.

We will assume that the prefactor \( 1/(A_x \Pi^x + A_y \Pi^y)^2 \) never diverges or vanishes so that we can absorb it into the Lagrange multiplier and thus obtain a fourth order scalar constraint for this model.

It is not trivial to check whether the constraint algebra closes. Before doing that, we will pass to the following equivalent set of constraints, obtained by subtracting a term proportional to the Gauss constraint from the scalar and vector constraint:

\[ G = \Pi^\alpha, \]
\[ V = (A_x)' \Pi^x + (A_y)' \Pi^y - \gamma (\Pi^\gamma)' \]
\[ C = A_x A_y \Pi^x \Pi^y (-((\Pi^\gamma)')^2 + (A_x \Pi^x + A_y \Pi^y)^2) + \gamma (A_x \Pi^x + A_y \Pi^y)^3 \Pi^\gamma \]

\[ -((A_x)' \Pi^x + (A_y)' \Pi^y) \Pi^\gamma (\Pi^\gamma)' [A_x \Pi^x + A_y \Pi^y] \] (3.8)

and something amazing happens when adding the term \( \Pi^\gamma (\Pi^\gamma)' [A_x \Pi^x + A_y \Pi^y] V \) proportional to the vector constraint to the scalar constraint

\[ C = A_x A_y \Pi^x \Pi^y (-((\Pi^\gamma)')^2 + (A_x \Pi^x + A_y \Pi^y)^2) \]

\[ -\gamma \Pi^\gamma (\Pi^\gamma)' [A_x \Pi^x + A_y \Pi^y] + \gamma (A_x \Pi^x + A_y \Pi^y)^3 \Pi^\gamma \]

\[ = [-(\Pi^\gamma)'^2 + (A_x \Pi^x + A_y \Pi^y)^2][A_x A_y \Pi^x \Pi^y + \gamma \Pi^\gamma (A_x \Pi^x + A_y \Pi^y)] \] (3.9)

i.e. the scalar constraint factorizes into two second order constraints with respect to the momenta! They are given by

\[ C_1 := A_x A_y \Pi^x \Pi^y + \gamma A_x \Pi^x \Pi^\gamma + \gamma A_y \Pi^y \Pi^\gamma \] and \( \gamma \Pi^\gamma \] are densities of weight one, i.e. \( G \) generates \( O(2) \)}

By studying the transformations that these constraints generate, we obtain that \( A_x, A_y, \gamma, \pi^x, \Pi^y, \Pi^\alpha, \Pi^\gamma \) are \( O(2) \)-invariant while \( \alpha \) is an \( O(2) \)-angle and that \( A_x, A_y, \alpha, \Pi^\gamma \) are scalars while \( E^x, E^y, \Pi^\alpha, \gamma \) are densities of weight one, i.e. \( G \) generates \( O(2) \).
transformations while \( V \) generates diffeomorphisms on the torus. By the way, \( \gamma \) can be interpreted as the abelian version of a connection plus the Maurer-Cartan form \( \theta_{MC} = +(dS)S^{-1} \) (which in one dimension for \( S = \exp(\alpha) \) boils down to \( \theta_{MC} = \alpha' dx \)) and therefore is an O(2)-invariant quantity.

The only nontrivial Poisson-bracket to check is then again that between two scalar constraints (all constraints are O(2) invariant and Diff(\( \Sigma \)) covariant). We have

\[
\{C[M], C[N]\} = \int_{\Sigma} dx \int_{\Sigma} dy M(x)N(y)\{C_1(x)C_2(x), C_1(y)C_2(y)\}
\]

\[
= \int_{\Sigma} dx \int_{\Sigma} dy M(x)N(y)[C_1(x)C_1(y)\{C_2(x), C_2(y)\} + C_1(x)C_2(y)\{C_2(x), C_1(y)\}
+ C_2(x)C_1(y)\{C_1(x), C_2(y)\} + C_2(x)C_2(y)\{C_1(x), C_1(y)\}]
\]

\[
= \frac{1}{2} \int_{\Sigma} dx \int_{\Sigma} dy (M(x)N(y) - M(y)N(x))\{C_1(x)C_1(y)\{C_2(x), C_2(y)\}
+ 2C_1(x)C_2(y)\{C_2(x), C_1(y)\} + C_2(x)C_2(y)\{C_1(x), C_1(y)\}
\]

\[
= \int_{\Sigma} dx (M'N - MN')C_1C_2((\Pi')^2)'(A_x\Pi^x + A_y\Pi^y)
= C[(M'N - MN')(\Pi')^2]'(A_x\Pi^x + A_y\Pi^y)
\]

(3.12)

i.e. the constraints form a first class algebra, open in the BRST-sense, and we can apply the framework of symplectic reduction [4].

We begin with the Gauss-constraint:

We just have to pull back the symplectic potential to the \( \Pi^x = 0 \) surface, the vector and scalar constraint as they stand in (3.8) and (3.9) are already reduced.

As experience with other model systems shows [9], it turns out to be much more convenient to first reduce the scalar constraint and then the vector constraint because the scalar constraint is not diffeomorphism invariant but only diffeomorphism covariant.

We have to distinguish between two possible sectors: sector I is defined by the vanishing of \( C_1 \), sector II by the vanishing of \( C_2 \).

We should mention that sector II corresponds to metrics that are degenerate in the \( z \)-direction: For our model the metric tensor takes the form

\[
q_{ab} = \text{diag}(\frac{E^\gamma E^z}{E^x}, \frac{E^z E^x}{E^y}, \frac{E^x E^y}{E^z})
\]

(3.13)

From the Gauss-constraint and the definition of \( \Pi_x, \Pi_y \) we infer

\[
(A_xE^x + A_yE^y)^2 = -[(E^z)^2 + [A_x\Pi^x + A_y\Pi^y]^2] \quad \text{and} \quad \frac{E^x}{E^y} = \frac{\Pi^x}{\Pi^y}
\]

(3.14)

so that we can solve \( E^x, E^y, E^z \) in terms of reduced coordinates

\[
(E^x, E^y, E^z) = (\pm \frac{\Pi^x}{A_x\Pi^x + A_y\Pi^y} \sqrt{C_2}, \pm \frac{\Pi^y}{A_x\Pi^x + A_y\Pi^y} \sqrt{C_2}, \Pi^z)
\]

(3.15)

\[\text{specifically, the author was looking at spherically symmetric gravity plus matter (9) }\]
so that
\[ q_{ab} = \text{diag}(\pm \frac{\Pi^\gamma \Pi^x}{\Pi^x}, \pm \frac{\Pi^y}{\Pi^y}, \pm \frac{\Pi^x \Pi^y}{[A_x \Pi^x + A_y \Pi^y]^{2 \Pi^\gamma} C_2}) \quad (3.16) \]

Note that the quotient $\Pi^x / \Pi^y$ remains finite.
This shows that sector II is the 'unphysical' one.

### 3.1 Sector I

We pass to new canonical pairs,
\[ (q_x := \ln(A_x), p^x := A_x \Pi^x; q_y := \ln(A_y), p^y := A_y \Pi^y; q := \ln(\gamma), p := \gamma \Pi^\gamma) \quad (3.17) \]

because then the scalar constraint $C_1$ adopts the simple form
\[ p^x p^y + p^p + pp^x = 0 \quad (3.18) \]

which does not even make any ordering problems any more if one would pass to the quantum theory directly, that is using the Dirac approach. Moreover, it defines trivially an abelian constraint because it consists only of momenta.

Note that we could have changed the polarization for the pair $\gamma, \Pi^\gamma$ in order that all configuration variables are scalars without destroying the simple form of the scalar constraint, but then we would leave the Ashtekar-polarization. Let us therefore proceed this way although this will equip the canonical coordinates with a somewhat awkward tensor valence.

It is clear that one will now diagonalize this constraint because then one can solve the Hamilton-Jacobi equation by separation of variables. We have
\[ C = \frac{1}{4} [(p^x + p^y + 2p)^2 - (p^x - p^y) - (2p)^2] \quad (3.19) \]

so that we define still another set of canonical pairs
\[ P^0 := \frac{1}{2}(p^x + p^y + 2p), \quad P^1 := \frac{1}{2}(p^x - p^y), \quad P^2 := p \quad (3.20) \]

and conversely
\[ p^x = P^0 + P^1 - 2P^2, \quad p^y = P^0 - P^1 - 2P^2, \quad p := P^2 \quad (3.21) \]

which allows for the determination of the new canonical configuration variables via inserting (3.21) into the symplectic potential
\[ \Theta[\partial_t] = \int_\Sigma dx \left[ \dot{q}_x (P^0 + P^1 - 2P^2) + \dot{q}_y (P^0 - P^1 - 2P^2) + \dot{q} P^2 \right] \]
\[ = \int_\Sigma dx \left[ P^0 \frac{d}{dt}(q_x + q_y) + P^1 \frac{d}{dt}(q_x - q_y) + P^2 \frac{d}{dt}(q - 2q_x - 2q_y) \right] \quad (3.22) \]

from which we read off
\[ Q_0 = q_x + q_y, \quad Q_1 = q_x - q_y, \quad Q_2 = q - 2q_x - 2q_y \quad . \quad (3.23) \]
Then the scalar constraint just says that \((P^0, P^1, P^2)\) is a null-vector in a 3-dimensional Minkowski space at every point of \(\Sigma\). There are two branches of general solutions to the associated Hamilton-Jacobi equation,

\[
S_{\pm}[p^\mu, p^\nu; Q._\mu] = \int_\Sigma dz[\pm\sqrt{(p^\mu)^2 + (p^\nu)^2}Q_0 + p^\mu Q_1 + p^\nu Q_2] \quad (3.24)
\]

where, following the second reference of [4], \(p_a = P^1, p_v = P^2\) are the new momenta and

\[
q_a := \frac{\delta S}{\delta p^a} = \pm \frac{p^a}{\sqrt{(p^\mu)^2 + (p^\nu)^2}}Q^0 + Q^1
\]

\[
q_v := \frac{\delta S}{\delta p^v} = \pm \frac{p^v}{\sqrt{(p^\mu)^2 + (p^\nu)^2}}Q^0 + Q^2 \quad (3.25)
\]

are the new invariant (under the scalar constraint) configuration variables.

One can get them also just by pulling back the symplectic structure to the scalar constraint surface via its embedding \(i\) into the phase space (compare [4]). Inserting the solutions \(P^0 = \pm\sqrt{(P^1)^2 + (P^2)^2}\) into the symplectic potential yields (as before we do not display total differentials)

\[
(i^*\Theta)[\partial_t] = \int_{\partial_S} [\pm \dot{Q}_0\sqrt{(p_a)^2 + (p_v)^2} + \dot{Q}_1p_a + \dot{Q}_2p_v]
\]

\[
= \int_{\partial_S} [\dot{p}^a(\pm \frac{1}{\sqrt{(p_a)^2 + (p_v)^2}}Q_0 + Q_1) + \dot{p}^v(\pm \frac{1}{\sqrt{(p_a)^2 + (p_v)^2}}Q_0 + Q_2)]
\]

\[
= \int_{\partial_S} \left[p^a \frac{d}{dt}(\pm \frac{p^a}{\sqrt{(p_a)^2 + (p_v)^2}}Q_0 + Q_1) + p^v \frac{d}{dt}(\pm \frac{p^v}{\sqrt{(p_a)^2 + (p_v)^2}}Q_0 + Q_2)\right] \quad (3.26)
\]

It remains to reduce the vector constraint. Resubstituting all the symplectomorphisms we have carried out up to now, we obtain (note that for any two functions \(a\) and \(b\) holds

\[
\left(\frac{a}{\sqrt{a^2 + b^2}}\right)'a + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)'b = \frac{(a^2 + b^2)'}{2\sqrt{a^2 + b^2}}' - \frac{(a^2 + b^2)'}{2(\sqrt{a^2 + b^2})^3}(a^2 + b^2) = 0
\]

\[
V = (A_x)'\Pi^x + (A_y)'\Pi^y - \gamma(P^x)'\]

\[
= (q_a)'p^a + (q_v)'p^v - \gamma(\Pi^y)'
\]

\[
= (q_a)'p^a + (q_v)'p^v + q'p - p'
\]

\[
= (Q_0)'P^0 + (Q_1)'P^1 + (Q_2)'P^2 - (P^2)'
\]

\[
= \pm(Q_0)'\sqrt{(p^\mu)^2 + (p^\nu)^2} + (Q_1)'p^\mu + (Q_2)'p^\nu - (p^\nu)'
\]

\[
= (q_a)'p^a + (q_v)'p^v - (p^\nu)'
\]

\[
= (q_v)'p^v + (\ln(q_w))'q_wp^w - (q_wp^w)'
\]

\[
= (q_a)'p^a - q_w(p^w)'
\]

(3.27)

where we have carried out one more canonical transformation

\[
q_v := \ln(q_w), \quad p^w := q_wp^w. \quad (3.28)
\]

Equation (3.28) displays \(q_a, p^w\) as scalars and \(q_w, p^a\) as densities of weight one.
3.1.1 Reality structure of sector I

From their definition (3.20) it is clear that \( p_x, p_y, p \) as products of two real quantities are all real, whence also \( p^u, p^w \) are real.

From (3.17) we have that

\[
Q_0 = \ln(A_x A_y), \quad Q_1 = \ln\left(\frac{A_x}{A_y}\right), \quad Q_2 = \ln\left(\frac{\gamma}{(A_x A_y)^2}\right)
\]  

(3.29)

and so we see that the reality structure of \( q_u, q_v \) depends on the sign of the arguments of the logarithms above. However, they are always real possibly up to an imaginary constant \( \pm i \pi/2 \) which drops out of the symplectic potential anyway if we assume the number of sign changes of the configuration variables for any instant of time to be of measure zero with respect to the measure \( dx \) on the hypersurface (hence, the set \( \{ x \in \Sigma; \lim_{s \to 0} (\text{sgn}(A_x + s) - \text{sgn}(A_x - s) \neq 0) \} \) is such that \( d/dt \ln(\text{sgn}(A_x)) \) etc. are integrable where \( \text{sgn}(x) := x/|x| \) is the sign of \( x \)). Hence we will assume in the sequel that \( Q_\mu \) is real whence this is also true for \( q_u, q_v \). This then implies that \( q_w = \exp(q_v) \) is always positive.

3.2 Sector II

Now the scalar constraint \( C_2 \) can even be written as one of two branches of a constraint linear in momentum

\[
C_{\pm} := A_x \Pi^x + A_y \Pi^y \pm (\Pi^\gamma)'.
\]  

(3.30)

Quite similar as for sector I we define new canonical pairs

\[
(q_x := \ln(A_x), p_x := A_x \Pi^x; q_y := \ln(A_y), p_y := A_y \Pi^y; q := \gamma, p := \Pi^\gamma)
\]  

(3.31)

and obtain

\[
C_{\pm} := p_x + p_y \pm p'.
\]  

(3.32)

The general solution of the Hamilton-Jacobi equation is obviously given by

\[
S_{\pm} = \int_\Sigma dx'[p^v]'q_x + (p^w)'q_y + (p^u + p^w)q = \int_\Sigma dx[p^v(\mp q - (q_x)') + p^w(\mp q - (q_y)')]
\]  

(3.33)

whence we read off for the new momenta

\[
(p^v)' = p_x, \quad (p^w)' = p_y
\]  

(3.34)

and for the new configuration variables

\[
q_v := \mp q - (q_x)', \quad q_w := \mp q - (q_y)'
\]  

(3.35)

which are both densities of weight one so that the reduced (with respect to the scalar constraint) vector constraint becomes

\[
V = -q_v(p^v)' - q_w(p^w)' = (q_u)'p^u - q_w(p^w)'
\]  

(3.36)
where we have defined still another canonical pair by

\[ q_v =: (q_u)', \quad q_u := -(p_v)' . \]  

(3.37)

That this defines indeed a symplectomorphism is obvious from the following short calculation (spatial surface integrals vanish due to our choice of the spatial topology)

\[ \int_\Sigma \dot{q}_v p_v = \int_\Sigma (\dot{q}_u)' p_v = \int_\Sigma \dot{q}_u (-p_v)' . \]  

(3.38)

3.2.1 Reality structure of sector II

Again we have that \( p^x, p^\nu, p \) are all real which then is true via (3.34) also for \( p^u, p^v \). With the same assumption as for sector I we obtain that \( q_v, q_w \) are real and finally via (3.37) also for \( q_u, q_w \). This time, however, \( q_w \) is not necessarily positive.

3.3 Reduction of the vector constraint

We come now to the main problem for a spatially diffeomorphism invariant field theory, namely to find a complete but minimal set of Dirac observables which are to be promoted to the basic set of quantum operators later.

It is always easy to find (classical) expressions that Poisson-commute (even strongly) with the diffeomorphism constraint - the recipe is the following:

1) From the transformation law of the basic field variables under diffeomorphisms derive their tensor nature.
2) Construct scalar densities of weight one.
3) Integrate them over the initial data hypersurface.

These objects are then already invariant under diffeomorphisms that are connected to the identity (in case of an asymptotically flat topology in general only if the diffeomorphism tends to the identity rapidly enough at spatial infinity). This is obvious from the transformation property of a scalar density \( \tilde{s} \) of weight one under an infinitesimal diffeomorphism (generated by a vector field \( \xi^a \)) \( \delta \tilde{s} = \partial_a (\xi^a \tilde{s}) \).

However, the scalar constraint is not at all so easy to solve since it can be viewed as a dynamical constraint. Hence, it is most important to keep the algebraic structure of the scalar constraint as simple as possible.

Now, since the scalar constraint is a density of weight two, it has a nontrivial transformation law under diffeomorphisms. Accordingly, solving the vector constraint before solving the scalar constraint will most likely complicate rather than simplify the algebraic structure of the scalar constraint because it will become a non-local object and, moreover, cannot be reduced because it cannot be written in terms of diffeomorphism invariant quantities.

The conclusion of all this is that the scalar constraint is to be solved before the diffeomorphism constraint for a field theory with a scalar constraint which is at least quadratic in the momenta. This is what we were able to do up to now.

Now we are going to derive a sequence of results which generalize in an obvious manner (although this will become combinatorically much more difficult) to higher
Lemma 3.1 The set of all integrated scalar densities constructed from the configuration variables provides for a (possibly overcomplete) system of coordinates of the diffeomorphism reduced configuration space.

Proof:
1) Any local object constructed from the configuration variables has a nontrivial transformation law with respect to infinitesimal diffeomorphisms. Accordingly, diffeomorphism-invariant objects are non-local, i.e. they can be expressed as integrals over (parts of) $\Sigma$ of local objects.
2) Were these integrals not over all of $\Sigma$ then they were not invariant because the diffeomorphisms have their support everywhere except for the boundary of $\Sigma$.
3) Only scalar densities of weight one are invariant when integrated over $\Sigma$.
4) One could consider objects of the form
$$\int_\Sigma dx_1 \ldots \int_\Sigma dx_m K(x_1, \ldots, x_m)$$
(3.39)
i.e. $K(x_1, \ldots, x_m)$ is some integral kernel. However, any such integral kernel can be approximated arbitrarily well (in the given norm) by linear combinations of tensor products $f_1(x_1) \ldots f_m(x_m)$ (for each $k$ the different functions $f_k$ will belong to a base). Thus, the integral of $K$ can be written as a sum of products of single integrals which displays $K$ as a derived object.
This proves the lemma.

Let $\{q^i\}_{i \in I}$ denote this set of Dirac observables where the labelling is with respect to an index set $I$ whose cardinality is to be obtained. Let us further assume that all these observables are algebraically independent. Any linear combination
$$S := \sum_{i \in I} p_i q^i$$
(3.40)
of these observables is again an observable where the $p_i$ are, in general, complex numbers (the reality conditions on the $p_i$ match those on the $q^i$ so that there are as many independent momenta as configuration variables). Then, provided that the set of the $q^i$ is complete in the sense that every diffeomorphism invariant object constructed from the configuration variables can be expressed by them, the above functional $S$ is the Hamilton-Jacobi functional relative to the diffeomorphism constraint and the $p_i$ play the role of the integration momenta (recall the formalism in the second paper of [4]).
We will now define an overcomplete set of diffeomorphism invariant functionals of the configuration variables for our model and then systematically obtain a minimal subset. When stripping off the technical methods that are particular for this model, one will get an insight for what is necessary for the full theory.
According to lemma (3.1) what we have to do is first is to construct all possible scalar densities $\tilde{s}$ that can be built from $q_u, q_w$. We first need two technical lemmas:
Lemma 3.2 1) In one dimension, tensors of valence \((r,s)\) are scalar densities of weight \(s-r\).

2) In one dimension, a scalar density of weight one \(\tilde{s}\) gives rise to a \(\tilde{s}\)-compatible connection \(\Gamma\) according to

\[
\Gamma := [\ln(\tilde{s})]' .
\] (3.41)

Proof:

1) This lemma is most easily proved by comparing the corresponding transformation properties of tensors \(T^{a..b..}\) under diffeomorphisms \(x^a \to \tilde{x}^a = x^a - \xi^a\)

\[
\delta T^{a..b..} = \mathcal{L}_\xi T^{a..b..} = \xi^c T^{c..b..} - T^{c..b..} \xi_c - .. + T^{a..c..} \xi^c + ..
\] (3.42)

which in one dimension reduces to

\[
\delta T = \xi T' + (s-r)\xi' T .
\] (3.43)

On the other hand, a scalar density \(\tilde{s}\) of weight \(n\) transforms as

\[
\delta \tilde{s} = \xi^a \tilde{s}_a + n \tilde{s} \xi^a
\] (3.44)

which again in one dimension reduces to

\[
\delta \tilde{s} = \xi \tilde{s}' + n \tilde{s}'
\] (3.45)

and that furnishes the proof of the first part of the lemma.

Accordingly, in one dimension tensor valences are just characterized by their density weight.

2) One simply builds the unique torsion-free connection compatible with a given metric by restricting the Christoffel formula \(\Gamma^{a..b..}_{bc} = 1/2 \tilde{q}^{ad}(q_{db,c} + q_{dc,b} - q_{bc,d})\) to one dimension. A metric in one dimension is, according to 1), just an arbitrary density of weight 2 corresponding to a symmetric tensor of valence \((0,2)\) (i.e. a metric \(\tilde{\tilde{q}}\)). Since the inverse of a metric in one dimension is just one over this metric we have for the affine connection in one dimension \(\Gamma = 1/2(\tilde{\tilde{q}}' + \tilde{\tilde{q}}' - \tilde{\tilde{q}}')/\tilde{\tilde{q}} = 1/2[\ln(\tilde{\tilde{q}})]'\) or, by defining the density \(\tilde{s}\) of weight one through the equation \(\tilde{\tilde{q}} = \tilde{s}^2\), we have finally a connection defined by an arbitrary density \(\tilde{s}\) of weight one given by

\[
\Gamma := [\ln(\tilde{s})]' .
\] (3.46)

Again, by just restricting the covariant derivative

\[
D_c T^{a..b..} = \partial_c T^{a..b..} + \Gamma^{a..b..}_{cd} T^{d..b..} + .. - \Gamma^{a..} c_{d..b} + ..
\] (3.47)

to one dimension we find that for a density \(T\) of weight \(n\) we have

\[
DT = T' - n\Gamma T
\] (3.48)

and one can explicitly check that \(DT\) is a density of weight \(n+1\), i.e. a density again with one more covariant valence.

In particular \(D\tilde{s} = \tilde{s}' - \Gamma \tilde{s} = 0\) which shows that the connection is \(\tilde{s}\)-compatible.

\(\square\)

We are now ready to prove one of the main theorems of this section.
Theorem 3.1 An (overcomplete) system of scalar densities of weight one is given by

\[ q_w f(\{(q_u^{(n)})\}_{n \in \mathcal{N}}) \]  

(3.49)

where \( f \) is an arbitrary function of the infinite number of arguments indicated, \( \mathcal{N} \) is the set of non-negative integers and for any scalar function \( s, s^{(n)} \) denotes the scalar \( d^n(s) \) where \( d \) is the operator \( d = 1/q_w \partial \) which is obviously anti-self-adjoint with respect to the measure \( d\mu = q_w dz \).

Proof:
It is clear that any scalar density \( \tilde{s} \) of weight one can be written as \( q_w s \) where \( s \) is a scalar, simply define \( s := \tilde{s}/q_w \). The proof thus boils down to showing that the above function \( f \) in (3.49) is the most general scalar that we can build.

Such a scalar will be the most general scalar constructed from \( q_w, q_u \) and all their partial derivatives up to arbitrarily high order. This can be done in a covariant way only by contraction of tensors, that is in one dimension, by multiplying densities such that their overall weight is zero, according to lemma (3.2), first part. Hence we need to construct a covariant derivative such that the derived tensors, i.e. densities, transform as tensors, i.e. densities again. This can be done by constructing an affine connection in one dimension and here part 2) of lemma (3.2) comes into play.

We are now able to take arbitrarily high covariant derivatives of \( q_w, q_u \), for example with respect to the connection derived from \( \tilde{s} := q_w \). But then \( \tilde{D}q_w = 0 \) because \( D \) is \( q_w \) compatible which shows that the covariant derivatives of \( q_w \) do not contribute in our construction and furthermore \( D^n \phi = q_w^n(1/q_w D)^n \phi = q_w^n d^n \phi \) where we have introduced the operator \( d = 1/q_w \partial \).

It remains to show that there is no loss in generality in choosing \( \tilde{s} := q_w \).

Assume that we had chosen any other density \( \tilde{s} \) instead of \( q_w \) in the construction of \( \Gamma \) then the covariant derivatives \( \tilde{D} \) of \( q_w \) would not vanish, however then \( \tilde{D}q_w = \tilde{s}\partial(q_w/\tilde{s}) = \tilde{s}D(q_w/\tilde{s}) = -q_w/\tilde{s}D\tilde{s} \) and \( \tilde{s} \) is again built only from our configuration variables and \( D \) (it must be a tensor and it cannot be constructed by using \( \tilde{D} \) because \( \tilde{D} \) is defined through \( \tilde{s} \)). Hence \( \tilde{D}q_w \) can be recast in the form of a function \( f \) of the type given in (3.49).

This furnishes the proof.
\[ \square \]

What we have achieved up to now is to obtain a sufficient number of observables \( q^i \) which are functionals of the configuration variables only.

We now ask for possible redundancies among them. Two sources of redundancy are obvious:
1) by doing integrations by parts it is possible to relate some of the \( q^i \) and
2) there might be topological identities which could arise, for example, via index theorems and thus affect the range of \( q^i \) (for example in 2 dimensions the integrated densitized curvature scalar is an object of the form we are considering here, but the Gauss-Bonnet theorem states that it depends only on the genus of the compact hypersurface without boundary).

Under the assumption that we can restrict to analytical \( f \)'s in (3.49) it is possible to get rid of the first redundancy. What we are presupposing is that there exists an
expansion of \( f \) of the type

\[
f(\{a_m\}_{m \in \mathbb{N}}) = \sum_{\{n_k\}} f_{\{n_k\}} \prod_{k=0}^{\infty} a_k^{n_k}
\]  

(3.50)

where \( a_k := d^k q_u \), the sum is over all sequences of (non-negative) occupation numbers \( n_k \) and the coefficients of the monomials

\[
M_{\{n_k\}} := \prod_{k=0}^{\infty} a_k^{n_k}
\]  

(3.51)

are real numbers because \( q_u, q_w \) are real as shown above. Hence, in our case the index set is given by the set of sequences, \( I = \{\{n_k\}\} \ni i \), and the (still overcomplete) set of \( q^i \)'s is given by the integrals

\[
q^i := \int_{\Sigma} dz q_w M_i
\]  

(3.52)

whereas the \( f_i \) are the conjugate momenta.

We will divide out the first, combinatorical redundancy by bringing the \( q^i \) into a standard form thereby introducing a terminology that one is used to from Fock-space or statistical mechanics.

First we observe that by doing an integration by parts the occupation numbers \( n_k \) of the quantum states or energy levels \( k \) for particles of type \( q_u \) may change but that the total particle number (of each particle type separately if there are more than 2 fields on the configuration space which it is not the case in our situation)

\[
N := \sum_{k=0}^{\infty} n_k
\]  

(3.53)

as well as the total energy

\[
E := \sum_{k=0}^{\infty} k n_k
\]  

(3.54)

are preserved under such a process, simply because the number of factors of a (irrespective of how often they are derived) and the number of derivations does not change. In fact, one could write the operation of doing an integration by parts in terms of annihilators and creators.

Lemma 3.3 Consider now the class of \( (q^i) \)'s with the same total particle number \( N \) and total energy \( E \). Then an algebraically independent genuine subset of \( q^i \)'s in this class is given by those with indices \( i \) such that \( m_0 = 1, N, m_1 = 0 \), the rest of the occupation numbers being arbitrary (subject to the condition that they lie in the class labelled by \( (N,E) \).

Proof:

In the first step we show that every member in the given class can be written as a linear combination of these \( q^i \)'s and in a second step we show that these \( q^i \) cannot be transformed into each other.
Consider the class $(N, E) = (M, k)$ and we may exclude the trivial case $M = 0$. Then $q^i$ is of the form

$$q^i = \int d\mu \, a_{k_1}..a_{E-(k_1+..+k_{M-1})}$$  \hspace{1cm} (3.55)$$

where we do not care about occupation numbers, i.e. some of the $k_r$ may be equal. Let, without loss of generality, $k_1$ be the lowest of the $k_r$. If $k_1 = 0$ we are done, otherwise do an integration by parts. After that, $q^i$ is a linear combination (with integer coefficients) of $q^j$’s of the required form since all these $q^j$ have $m_0 = 1$. By doing further integrations by parts if necessary for these $q^j$ (since $k(a_0)^{k-1}a_1 = d((a_0)^k)$ can be integrated by parts) one can satisfy $m_1 = 0, m_0 > 0$. This finishes the first step.

Let now one of these ‘reduced’ $q^i$’s be given. Then the set of its occupation numbers is characterizing, i.e. by doing an integration by parts, starting from $m_1 = 0$, one necessarily picks up terms that have $m_1 \neq 0$ from the differentiation of a since we already have $m_0 > 0$. This furnishes the proof.

\[\square\]

From now on we will understand the index set $I$ to be reduced, that is, we consider the ‘Mandelstam-identities’ between the original $q^i$’s originating from integrations by parts to be taken care of by lemma 3.3.

As far as the second source of redundancy is concerned, we do not have any indication that any of the $q^i$ should have a non-continuous range.

There is still a more subtle source of redundancy:

One could fix a gauge (a frame) and expand $q_u$ into a Taylor series with respect to this frame where we have assumed these variables to be smooth functions on $\Sigma$. The freedom in choosing these functions is then captured in the choice of the values of their Taylor coefficients.

However, then one could just integrate out all of our reduced $q^i$ with respect to this frame and we would get complicated rational functions of these Taylor coefficients multiplied by real numbers coming from the integration of the powers of the frame-variable $x$ over the hypersurface and depending only on the topology of $\Sigma$.

Since any such rational function can be generated from the Taylor coefficients itself, the knowledge of the latter is sufficient and provides for the minimal subset of observables.

The result of all this analysis summarized in the subsequent theorem.

**Theorem 3.2** Only the following $q^i$’s are independent variables on the diffeomorphism-reduced configuration space ($n$ is any non-negative integer):

$$q_n := \frac{1}{2} \int_\Sigma d\mu q_u d^{2n}q_u = (-1)^n \frac{1}{2} \int_\Sigma d\mu |d^n q_u|^2.$$  \hspace{1cm} (3.56)

**Proof:**

There is a one to one mapping between these functionals and the Taylor coefficients of $q_u$, possibly up to sign. For example, after fixing a gauge, e.g. $q_w(x) = g'(x) = 1, d = \partial$ where $g$ is a fixed scalar function, the $q_n$ allow to extract all the Taylor coefficients of $q_u$ (e.g. $\partial^n \phi$ starts with the $n$’th coefficient), possibly up to sign.

\[\square\]

Note that (3.56) are $|N|$ independent quantities, where $|S|$ denotes the cardinality of the set $S$, because they satisfy all the criteria of the previous theorem to be
algebraically independent and precisely for that reason we have $b_{n/2} = \Phi_{n/2} = 0$. Accordingly, the number of degrees of freedom is *naturally countable* without employing a complete system of mode functions on $\Sigma$ to express the true degrees of freedom in discrete form.

Moreover, these observables have the nice feature that

$$\frac{\delta q_n}{\delta q_u(x)} = q_w d^{2n} q_u(x),$$  \hspace{1cm} (3.57)

$$\frac{\delta q_n}{\delta q_w(x)} = -\frac{1}{2} \sum_{k=1}^{2n-1} (-1)^k (d^k q_u(x)) (d^{2n-1} q_u(x)).$$  \hspace{1cm} (3.58)

We will denote the momenta conjugate to $q_n$ by $p^n$.

The associated Hamilton-Jacobi functional is

$$S := \sum_{n=0}^{\infty} p^n q_n$$  \hspace{1cm} (3.59)

and the old momenta are the functional derivatives of $S$ with respect to the old coordinates.

For sector I we have seen that $q_w$ is positive (and therefore serves as a genuine density for a measure on $\Sigma$ in the the definition of $q_n$) whereas for sector II the range of $q_w$ is the whole real line. Since $q_u, q_w$ are both real, $(d^n q_u)^2$ is positive whence $q_n$ has half the or the full range of the real line for sector I or II respectively. $p^n$ on the other hand is generally real so that the pair $(q_n, p^n)$ coordinatizes the cotangent space over the (positive) real line.

Our, admittedly somewhat sketchy, construction generalizes in an obvious way to more dimensions, for every diffeomorphism invariant field theory.

## 4 Quantum theory and determination of the physical inner product

### 4.1 Reduced phase space approach

Quantum theory is straightforward. Keeping the polarization that we have ended up with, physical states depend only on the variables $q_n$.

It is important to see that the scalar constraint *forced* us to let the reduced configuration space (with respect to the scalar constraint) not only depend on the original configuration variables (before reducing) but also on the original momenta! This mixing of original polarizations is typical for genuinely quadratic constraints with respect to the momenta and can be viewed as a source of difficulty in the Dirac approach to select physical states.

The Dirac observables $q_n, p^n$ are both real where $p^n$ has infinite range and $q^n$ has infinite or half infinite range for sector II or I respectively. Therefore, the unique inner
product that turns \( q_n, p^n \) into self-adjoint operators is determined by the physical Hilbert spaces \( \mathcal{H}_{phys} \)

\[
L_2(\times_{n=0}^{\infty} R, \bigwedge_{n=0}^{\infty} dq_n) \text{ or } L_2(\times_{n=0}^{\infty} R^+, \bigwedge_{n=0}^{\infty} dq_n/q_n) \quad (4.1)
\]
respectively (as follows from the group theoretical approach \([10]\)).

### 4.2 Dynamics of the model - deparametrization for an infinite number of degrees of freedom

What we have obtained up to now is the complete set of Dirac observables which are time-independent. However, their interpretation is still missing. On the other hand, interpretation is usually straightforward for non-gauge invariant quantities.

In order to arrive at an interpretation there are two possible strategies available: either one expresses an object that is \( O(2) \times Diff(\Sigma) \) invariant in terms of Dirac observables plus an additional gauge function \( t \) which labels the choice of foliation of spacetime and quantizes only the Dirac observables \([11]\) or one uses the framework of time-dependent Dirac observables (deparametrization, \([6]\)).

Let us try to apply the latter one.

In the sequel we will only deal with the 'physically relevant' sector I.

We are asked to cast the scalar constraint into a form that allows to apply an extension to an infinite number of degrees of freedom of the framework of deparametrization. As is proved in \([9]\) the only change is that the time variable becomes 'many-fingered' (Tomonaga-Schwinger extension).

In particular we are interested in the time-development of the quantities \( E^x, E^y, E^z \) which determine the metric.

The starting point will be equation (3.19). We take the square root and restrict to the positive branch. Then the scalar constraint adopts the familiar form

\[
P^0(z) - H(Q_1, P^1; Q_2, P^2)(z) := P^0(z) - \sqrt{|P^1(z)|^2 + |P^2(z)|^2} \quad (4.2)
\]

which is already deparametrized, although at each point \( x \) of the hypersurface separately. In (4.2) we have defined an infinite number of 'Hamiltonians', \( H(x) \), that is, an 'energy density'. Note that, since \( H \) involves only momenta, there are no ordering problems.

Following this framework, we construct the time-evolution operator

\[
\hat{U}_t := \exp\left(-i \int_{\Sigma} dz (Q_0 - t) \hat{H}\right) \quad (4.3)
\]

and define any operator \( \hat{O} \) (to be factor-ordered appropriately) on the set of physical states \( \Psi = \hat{U}_t \psi \) (with respect to the scalar constraint) by

\[
(\hat{O}_t \psi)[Q_0, Q_1, Q_2] := \hat{U}_t \hat{O}_{t=Q_0} \psi[Q^1, Q^2] \quad . \quad (4.4)
\]

Now, in contrast to toy models for cosmology, physical states should also be diffeomorphism invariant. We will therefore take the operator to be diffeomorphism-invariant, but not necessarily invariant under the scalar constraint.
We want to apply the above framework for example to the operator for the three-volume whose classical counterpart is given by

\[ V := \int_\Sigma dz \sqrt{\det q} = \int_\Sigma dz \sqrt{E_x E_y E_z} \tag{4.5} \]

We will proceed as follows: We start from (4.2) and proceed to the Dirac quantization of the model. As for the scalar constraint, we have no operator-ordering problems and the complete solution of

\[ (-[\hat{P}^0]^2 + [\hat{P}^1]^2 + [\hat{P}^2]^2)\Psi[Q_0, Q_1, Q_2] = 0 \tag{4.6} \]

is formally given by a functional integral (positive energy solutions)

\[ \Psi[Q_0, Q_1, Q_2] = \int_{\mathbb{R}^2} [d^2k] \phi[k^1, k^2] \exp(-i \int_\Sigma dz \sqrt{(k^1)^2 + (k^2)^2} Q_0 - k^1 Q_1 - k^2 Q_2) \]

\[ = \exp(-i \int_\Sigma dz Q_0 \sqrt{(\hat{P}^1)^2 + (\hat{P}^2)^2}) \psi[Q_1, Q_2] \tag{4.7} \]

where \([d^2k]\) is an appropriate limes of a cylinder measure in the usual sense (skeletonized Lebesgue measure ([12]) with a finite number of integration variables since \(\Sigma\) is compact and the limes corresponds to the removement of the skeletonization) and \(\phi \psi[k^1, k^2]\) is a yet arbitrary functional.

Since \(\hat{U}_t\) commutes with the vector constraint (ordered with the momenta to the right), diffeomorphism invariance is obeyed if and only if for an infinitesimal density \(\xi\) of weight -1 holds

\[ \psi[Q_1 + (Q_1')\xi, Q_2 + (Q_2')\xi + \xi] = \psi[Q_1, Q_2] \tag{4.8} \]

Inserting, the integrand of the path integral becomes after doing an integration by parts in the exponential (up to first order in \(\xi\))

\[ \phi[k^1, k^2] \exp(i \int_\Sigma dx [(k^1 - (\xi k^1)'\xi)Q_1 + (k^2 - (\xi k^2)'\xi)Q_2 + \xi'k^2]) \]

We change to new variables \(K^i := k^i - (\xi k^i)'\), \(i = 1, 2\) in the functional integral and obtain (up to first order in \(\xi\)) the condition

\[ \phi[K_1 + (K^i\xi)', K^2 + (K^2\xi)'] \exp(i \int_\Sigma dz \xi'K^2) = \phi[K^1, K^2] \tag{4.9} \]

where we have used the diffeomorphism invariance of the Lebesgue measure (the Jacobian is 1 to first order in \(\xi\): denote by \(h\) the spacing of the skeletonization, \(k = k^1\) or \(k^2\), \(k_n := k(nh)\), then the Jacobian is

\[ \frac{\partial k_n}{\partial K_m} = \frac{\partial[K_n - 1/(2h)((\xi K)_{n+1} - (\xi K)_{n-1})]}{\partial K_m} = \delta_{n,m} - \frac{\xi_m}{2h}(\delta_{n,m-1} - \delta_{n,m+1}) =: J_{nm} =: (1 + A(\xi))_{nm} \tag{4.10} \]

where the matrix \(A\) is trace-free and of first order in \(\xi\), so \(\det(J) = 1 + \text{tr}(A) + .. = 1 + O(\xi^2)\).
We now split \( \phi[k_1, k^2] =: \exp(i \int_{\Sigma} dz f(K^2)) \tilde{\psi}[k_1, k^2] \) and seek to determine the function \( f \) according to the requirement that

\[
\int_{\Sigma} dx [f(K^2 + (\xi K^2)' + \xi'K^2)] = \int_{\Sigma} dx f(K^2)
\]

up to first order in \( \xi \). We expand \( f \) and perform integrations by parts until only \( \xi \) appears derivated with respect to \( z \). Then a sufficient condition for the last equation to hold is that

\[
k^2(1 + \frac{\partial f(k^2)}{\partial k^2}) - f = 0 .
\]

The solution of this ordinary first order differential equation is given by

\[
f(k^2) = [c - \ln(k^2)]k^2
\]

where \( c \) is a real integration constant.

This means that \( \tilde{\psi} \) is diffeomorphism invariant when \( k_1, k^2 \) transform as densities of weight one. Again we are therefore lead to the functionals

\[
\tilde{\psi}[k_1, k^2] = f(\{q^n\}), \text{ where } q^n = \int_{\Sigma} d\mu(d^n(k^2/k^1))^2
\]

with \( d\mu = dzk^1, \frac{d}{d^1} = 1/k^1\partial \).

Diffeomorphism invariance up to first order implies diffeomorphism invariance with respect to the component of the identity of the diffeomorphism group.

We are now in the position to define the operator \( \hat{V} \).

By making use of the definitions we can express (4.5) in terms of \( Q_\mu, P^\mu \)

\[
\sqrt{\det(q)} = \exp(-Q_0) \sqrt{(P_0 - P_2)^2 - (P_1)^2}
\]

and we face severe factor ordering problems when promoting the integral over this function to quantum theory.

The action of this operator on physical states is given by (we assume that we order all momenta to the right handside in the formal power expansions of the non-analytical terms in (4.12))

\[
\hat{V}_t \Psi[Q_0, Q_1, Q_2] = \hat{U}_t \int [d^2k] \exp(i \int_{\Sigma} dz [Q_1k^1 + Q_2k^2 + [c - \ln(k^2)]k^2]) \times
\]

\[
\int_{\Sigma} dx \exp(-t) \sqrt{\frac{(k^2)^2 - (k^1)^2}{(-2)k^2}} \sqrt{4(k^2)^2 - [(\exp(-(Q_2 + 2t))k^2)]^2}
\]

which, when acting with the evolution operator (which, by the way, is unitary for \( Q_0 - t = \text{const.} \) with respect to the inner product (4.1) : since the integrated Hamiltonian commutes with both constraints, it is physical, that is, well-defined on \( \mathcal{H}_{\text{phys}} \).
and since it is classically real it promotes to a self-adjoint operator by a suitable choice of ordering) results in a horrible object which we cannot work out explicitly. The point of this subsection was to show how deparametrization works in principle for field theories and how one can arrive at an interpretation. It also shows that most of the evolving operators which one would intuitively think of as observables are probably not well-defined on the physical Hilbert space (here in the connection representation).

5 Loop variables

The concluding section of this paper is designated to the issue of using (traces of) Wilson-loop observables for the quantization of or even for the classical treatment of gravity. Because our model system is completely integrable one can ask and answer all kinds of questions that have been raised in the literature. In particular we ask if the operators that are usually defined for the loop representation in the literature ([7]) are well defined at all, that is, if they make sense when applied to physical states. Of course, as they stand they will not leave the physical Hilbert space invariant and thus are not even expected to have a well-defined action on physical states. However, it will be interesting to see how the Dirac observables look in terms of loop-coordinates, in particular the constraints, what their Poisson structure is and how one can define a 'loop-transform' into the loop representation ([7]), from states that have been defined for the connection representation. Also, one could construct loop-operators as certain 'time-dependent' Dirac observables.

We begin by recalling that one defines the following 'basic' loop variables when dealing with the loop representation:

The $T^0$ variables are defined to be traces of the holonomies (parallel transport operators)

$$U_\sigma[A](x) := \mathcal{P} \exp(\int_\sigma A)$$

around closed curves (loops $\sigma$) with base point $x$ in the fundamental representation of SU(2) (the Ashtekar variables are complex-valued whence one they lie in the Lie algebra of complexified SU(2,C)=SL(2,C); although the gauge group of gravity is really SU(2), the loop variables are even invariant under SL(2,C))

$$T_\sigma[A] := \frac{1}{2} \text{tr}[U_\sigma[A]]$$

where $\mathcal{P}$ denotes path-ordering and $A := A_i^a dx^a \tau_i$, $\tau_i$ a basis in the Lie algebra su(2). We will choose $\tau_i := -i/2\sigma_i$, the latter being the usual Pauli matrices. Note that the trace of the holonomy is independent of the base-point on the given loop. The $T^1$-variables are defined to be

$$T_\sigma^a[A, E](x) := \text{tr}[E^a(x)U_\sigma[A](x)] .$$

As one can show, they form a closed Poisson-algebra and can thus serve as a starting point for the group-theoretical quantization scheme. Let us explore what happens
with these variables for our model.

The (pull-back under a section of the principal SU(2)-bundle of the) connection reduces to

\[ A(x, y, z) = A_x(\cos(\alpha)\tau_1 + \sin(\alpha)\tau_2)dx + A_y(-\sin(\alpha)\tau_1 + \cos(\alpha)\tau_2)dy + A_z\tau_3dz \quad (5.4) \]

and the gauge transformations generated by the Gauss-constraint for our reduced configuration variables are those under the U(1) subgroup of SU(2) generated by \( \tau_3 \) (even its complexification) :

\[ -dSS^{-1} + SAS^{-1} \text{ where } S = \exp(\Lambda\tau_3) \quad \quad (5.5) \]

can be written as in (5.4) only that \( \alpha, A_z \) are replaced by \( \alpha + \Lambda, A_z - \Lambda' \). Under a gauge transformation, the holonomy and the triad \( E = E^a_\tau\tau_\alpha \) transform covariantly

\[ E(x) \rightarrow S(x)E(x)S^{-1}(x) \text{ and } U_\sigma(x) \rightarrow S(x)U_\sigma(x)S^{-1}(x). \quad \quad (5.6) \]

whence the \( T^0, T^1 \) variables live on the Gauss-reduced phase space.

It is convenient to eliminate the angle \( \alpha \) by doing an U(1,C) rotation with gauge parameter \( \Lambda = -\alpha \) (note that \( \alpha \) is complex so this transformation is really an element of U(1,C)). Then the connection and the triad adopt the reduced structure

\[ A(x, y, z) = A_x(z)\tau_1 dx + A_y(z)\tau_2 dy + \gamma(z)\tau_3 dz \quad \quad (5.7) \]

\[ E(x, y, z) = [\Pi^x(z)\partial_x - E^y(z)\sin(\alpha - \beta)\partial_y]\tau_1 + [\Pi^y(z)\partial_y - E^x(z)\sin(\alpha - \beta)\partial_x]\tau_2 + \Pi^z(z)\tau_3\partial_z. \quad \quad (5.8) \]

Note that both quantities do not have any \( x,y \) dependence!

We will now start to find suitable loops to capture the full U(1) invariant information needed in order to actually coordinatize the Gauss-reduced phase space. Three loops are obvious: they are those lying entirely in one of the three \( S^1 \)-factors of the three-torus. We will call them \( \sigma(x), \sigma(y) \) and \( \sigma(z) \) respectively. A short calculation yields for the corresponding \( T^0, T^1 \)s (the range of the three coordinates \( x,y \) and \( z \) is taken to be the interval \([0, 1]\) with endpoints identified):

\[ T_{\sigma(x)}(z) = \cos(\frac{1}{2}A_x(z)) \quad \quad (5.9) \]
\[ T_{\sigma(y)}(z) = \cos(\frac{1}{2}A_y(z)) \quad \quad (5.10) \]
\[ T_{\sigma(z)} = \cos(\frac{1}{2}\int_0^1 dz\gamma(z)) \quad \quad (5.11) \]
\[ T_{\sigma(x)}^x(z) = -\Pi^x(z)\sin(\frac{1}{2}A_x(z)) \quad \quad (5.12) \]
\[ T_{\sigma(y)}^y(z) = -\Pi^y(z)\sin(\frac{1}{2}A_y(z)) \quad \quad (5.13) \]
\[ T_{\sigma(z)}^z(z) = -\Pi^z(z)\sin(\frac{1}{2}\int_0^1 dz\gamma(z)) \quad \quad (5.14) \]
\[ T_{\sigma(y)}^y(z) = \frac{(\Pi_x\Pi^z)(z)}{(A_x\Pi^z + A_y\Pi^y)(z)}\sin(\frac{1}{2}A_y(z)) \quad \quad (5.15) \]
\[ T_{\sigma(x)}^y(z) = \frac{(\Pi_y\Pi^z)(z)}{(A_x\Pi^z + A_y\Pi^y)(z)}\sin(\frac{1}{2}A_x(z)) \quad \quad (5.16) \]
\[ T_{\sigma(z)}^x(z) = T_{\sigma(z)}^y(z) = T_{\sigma(x)}^z(z) = 0 \quad \quad (5.17) \]
Note that from the four quantities $T_{\sigma(\mu)}, T^\mu_{\sigma(\mu)}(z), \mu \in \{x,y\}$ we can already extract the two gauge invariant canonical pairs $A_\mu, \Pi^\mu$ locally while the two variables $T^z_{\sigma(z)}(z), T_{\sigma(z)}$ allow to recover only $\Pi^\nu$ locally whereas the integrated quantity $\int_z dz \gamma(z)$ is a global one. This has two consequences:

1) The information contained in the two ‘off-diagonal’ variables $T^\mu_{\sigma(\mu)}(z), \mu \neq \nu$ is already redundant and so we can neglect them,

2) We need a further $T^0$ variable in order to get hand on $\gamma(z)$.

Before doing that we show that the variables defined above form a closed (classical) $*$-Poisson algebra (the Loop-variables for the Ashtekar phase space are not closed under complex conjugation at least it is not obvious how to express $T^\alpha = \frac{1}{2} \text{tr}(P \exp(\int_a [2 \Gamma - A]))$ in terms of $T^0, T^1$ where $\Gamma$ is the spin-connection).

We have proved in the sections before that $A_x, A_y, \gamma$ are purely imaginary while $\Pi^x, \Pi^y, \Pi^z$ are purely real. This implies the $*$-relations

$$\overline{T_{\sigma(a)}(z)} = T_{\sigma(a)}(z) \quad \text{and} \quad \overline{T^a_{\sigma(a)}(z)} = -T^a_{\sigma(a)}(z)$$

(5.18)

i.e. the $T^0$’s are purely real, the $T^1$’s purely imaginary, and also that the range of the $T^0$ are the real numbers larger than or equal to one while the $T^1$’s have the whole imaginary axis as their range. This opens the chance for a cotangent topology of the associated phase space and therefore a group theoretical quantization scheme.

Another consequence is that the variables $T_{\sigma(\mu)}$ are not quite sufficient to recover the connection up to gauge transformations: we have the degeneracy

$$T_{\sigma(\mu)}[A] = T_{\sigma(\mu)}[-A]$$

(5.19)

which means that these loop variables are not seperating on the moduli space $\mathcal{A}/\mathcal{G}$ of SL(2,C) connections modulo gauge transformations for our model when restricting ourselves to the above defined class of loops. This is in analogy to the known degeneracies (13) of these loop variables as coordinates for $\mathcal{A}/\mathcal{G}$.

Next we compute the Poisson-structure which turns out to be quite similar to the one obtained in the first reference of (8) when restricting ourselves to the z-direction:

$$\{T_{\sigma(a)}(z), T_{\sigma(b)}(z')\} = 0$$

(5.20)

$$\{T_{\sigma(\mu)}(z), T^\nu_{\sigma(\nu)}(z')\} = \frac{1}{2} \delta_{\mu\nu} \delta(z, z') \sin^2\left(\frac{1}{2} A_\mu(z)\right) = \frac{1}{2} \delta_{\mu\nu} \delta\left(z, z'\right)\left(1 - [T_{\sigma(\mu)}(z)]^2\right)$$

(5.21)

$$\{T_{\sigma(\mu)}(z), T^z_{\sigma(z)}(z')\} = \frac{1}{2} \sin^2\left(\frac{1}{2} \int_0^1 \gamma(z)dz\right) = \frac{1}{2} \left(1 - [T_{\sigma(z)}]^2\right)$$

(5.22)

$$\{T^\mu_{\sigma(\mu)}(z), T^\nu_{\sigma(\nu)}(z')\} = \left\{T^\mu_{\sigma(\mu)}(z), T^z_{\sigma(z)}(z')\right\} = 0$$

(5.23)

$$\{T^z_{\sigma(z)}(z), T^z_{\sigma(z)}(z')\} = \frac{1}{2} [T^z_{\sigma(z)}(z) - T^z_{\sigma(z)}(z')] T_{\sigma(z)}$$

(5.24)

and we could even have written the right hand side of all the equations as linear combinations of single $T^0$’s or $T^1$’s with distribution-valued coefficients at the price of extending the so defined loop algebra to all such loops with any winding number.

\[^3\text{This result implies, for example, that any topology on } \mathcal{A}/\mathcal{G} \text{ which is based on the use of the loop variables (traces of the holonomy) displays } \mathcal{A}/\mathcal{G} \text{ as a non-Hausdorff space (14)!}\]
(with respect to an orientation on the three torus) by making use of the Mandelstam identities

\[ T_\alpha[A]T_\beta[A] = \frac{1}{2}[T_{\alpha\circ\beta} + T_{\alpha\circ\beta^{-1}}] \]  (5.25)

where \( \circ \) is the product in the loop group (with respect to a given base point). This Mandelstam identity becomes here just the trigonometrical identity

\[ \cos(\alpha) \cos(\beta) = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)] \]  (5.26)

but we refrain from introducing the winding number because we will have to deal with higher order polynomials of loop variables anyway.

Note incidentally that (5.21) implies that the conjugate variable to \( T^\mu_\sigma(\mu) \) is given by \( 2\text{arcth}(T_\sigma(\mu)) \)!

Remark:
It is clear that the Hamilton-Jacobi functional \( S := cT_\sigma(z) \) solves all constraints for the degenerate sector for any constant \( c \). Thus it turns out that the loop variables serve as 'good' coordinates only for (part of) the degenerate sector as for the full theory (16).

We now have to extract \( \gamma(z) \). One way would be to make use of the so-called area-derivative (15), whose relevant components for the already defined loops become (the area-derivative in the (ab)-plane is denoted \( \delta_{ab} \)) at the point with coordinate \( z \) (\( x,y \) are irrelevant)

\[ \delta_{xy}T_\sigma(z) = -\frac{1}{2}A_x(z)A_y(z)\sin\left(\frac{1}{2}\int_0^1 dz\gamma(z)\right) \]  (5.27)

\[ \delta_{yz}T_\sigma(x) = -\frac{1}{2}A_y(z)\gamma(z)\sin\left(\frac{1}{2}A_x(z)\right) \]  (5.28)

\[ \delta_{zx}T_\sigma(y) = -\frac{1}{2}\gamma(z)A_x(z)\sin\left(\frac{1}{2}A_y(z)\right) \]  (5.29)

but since the area-derivative is a certain limit of a difference between one of the above loop variables and an extended one (the area derivative in the (ab)-plane with respect to a given loop at one of its points is obtained by attaching an infinitesimal loop lying in the (ab)-plane to the given loop at the point of interest, computing the trace of the holonomy for this modified loop and the original one, taking the difference and dividing by the area element) we actually leave the above algebra and look at an infinite number of new loops. We would like to have only as many loop variables as reduced canonical variables. Accordingly, we are looking for a better suited object.

After some trial, one of the most simple objects seems to be an eyeglass loop (7) obtained by drawing two loops with equal orientation at \( z_1 \) and \( z_2 \) in x or y direction connected by a line in z-direction back and forth between \( z_1 \) and \( z_2 \).

We will denote this loop by \( \sigma(xz) \) or \( \sigma(yz) \). The computation of the associated \( T^0 \) reveals

\[ T_\sigma(xz)(z_1, z_2) = \cos\left(\frac{1}{2}A_x(z_1)\right)\cos\left(\frac{1}{2}A_x(z_2)\right) - \cos\left(\int_{z_1}^{z_2} dz\gamma(z)\right)\sin\left(\frac{1}{2}A_x(z_1)\right)\sin\left(\frac{1}{2}A_x(z_2)\right) \]  (5.30)
and thus allows to extract $\gamma(z)$ by taking the derivative with respect to, say, $z_2 = z$. Fix a value $z_0$ on the z-circle and write

$$T_{\sigma(xz)}(z) := T_{\sigma(xz)}(z_0, z).$$  \hspace{1cm} (5.31)$$

We observe that

$$T_{\sigma(xz)}(z_0 + 1) = [T_{\sigma(x)}(z_0)]^2 - (2[T_{\sigma(z)}]^2 - 1)(1 - [T_{\sigma(x)}(z_0)]^2)$$ \hspace{1cm} (5.32)

whence

$$T_{\sigma(z)} = \sqrt{\frac{1 - T_{\sigma(xz)}(z_0)}{2(1 - [T_{\sigma(x)}(z_0)]^2)}}$$ \hspace{1cm} (5.33)

so that the information contained in $T_{\sigma(z)}$ is also contained in $T_{\sigma(xz)}(z)$ (if one uses also $T_{\sigma(\mu)}(z)$) (note that even its sign can be determined from (5.23) as the $T_{\sigma(\mu)}(z)$ are all positive). Therefore we will consider $T_{\sigma(z)}$ as redundant in the following although we will use it as an abbreviation for (5.33). Also the analogous $T_{\sigma(yz)}$ of $T_{\sigma(xz)}$ is not needed.

We now compute the extended loop algebra : the only non-vanishing brackets of our new element with the rest are (up to a sign)

$$\{T_{\sigma(xz)}(z), T_{\sigma(z)}^x(z')\} = -\chi_{[z_0,z]}(z')\sin(\int_{z_0}^{z} \gamma(t) dt) \sin(\frac{1}{2} \int_{0}^{1} dt \gamma(t))$$

$$\sin(\frac{1}{2}A_{\sigma}(z_0))\sin(\frac{1}{2}A_{\sigma}(z)) = -\chi_{[z_0,z]}(z') \times$$

$$\sqrt{(1 - [T_{\sigma(z)}]^2)(1 - [T_{\sigma(x)}(z_0)]^2)(1 - [T_{\sigma(x)}(z)]^2) - (T_{\sigma(xz)}(z) - T_{\sigma(x)}(z_0)T_{\sigma(x)}(z))^2}$$ \hspace{1cm} (5.34)

$$\{T_{\sigma(xz)}(z), T_{\sigma(x)}^x(z')\} = \frac{1}{2} \delta(z', z_0) [-T_{\sigma(x)}^x(\zeta) + T_{\sigma(x)}^x(z_0)T_{\sigma(xz)}(z)]$$

$$+ \delta(z', z_1) [-T_{\sigma(x)}^x(z_0) + T_{\sigma(xz)}^x(z)T_{\sigma(xz)}(z)]$$ \hspace{1cm} (5.35)

Unfortunately, the right handside becomes quite complicated and it would seem to be better to work with an overcomplete set and apply the algebraic quantization programme ([7]). In our case that would boil down to the problem to find loop-variables that depend only on the configuration variables, contain $\sin(\frac{1}{2}A_{\mu}(z))$, $\sin(\int_{0}^{1} dz \gamma(z)/2)$ and yield brackets that have analytical right hand-sides for the corresponding Poisson-brackets, but that turns out not to be straightforward without regarding the whole set of loops.

We now come to set up a bijection (up to signs) between the Loop-variables defined so far and the canonical variables determined in section 3.

We have already defined the T’s in terms of $A_{\mu}, \Pi^{\mu}, \gamma, \gamma'$. Here is the local inversion :

$$A_{\mu}(z) = 2\arccos(T_{\sigma(\mu)}(z))$$  \hspace{1cm} (5.36)

$$\Pi^{\mu}(z) = \frac{T_{\sigma(\mu)}^x(z)}{\sqrt{1 - [T_{\sigma(\mu)}(z)]^2}}$$  \hspace{1cm} (5.37)

$$\gamma(z) = [\arccos(\frac{T_{\sigma(x)}(z_0)T_{\sigma(x)}(z) - T_{\sigma(xz)}(z)}{(1 - [T_{\sigma(x)}(z)]^2)(1 - [T_{\sigma(x)}(z_0)]^2)^{\prime}})]'$$ \hspace{1cm} (5.38)
\[
\Pi^\gamma(z) = -\frac{T_{\sigma(z)}^\gamma(z)}{1 - \frac{\sqrt{1 - T_{\sigma(z)}(z_0)^2}}{2(1 - [T_{\sigma(z)}(z_0)]^2)}}
\]  

(5.39)

where the prime denotes derivation with respect to the \( z \)-argument only. The \( z \)-derivation considered as a difference of Loop-variables does not lead out of the class of variables that we have defined and so we are fine.

(5.36)-(5.39) defines a (local) point transformation which is always a local symplectomorphism. Accordingly, when plugging the above formulae into the symplectic potential, we are actually able to determine the momenta conjugate to the three variables \( T_{\sigma(\mu)}, T_{\sigma(xz)} \)!

After some tedious algebraic manipulations, an integration by parts, using that \( \int_{S^1} \gamma(z) dz \) written in terms of \( T^0 \)'s is a spatial constant and making use of the fact that quantities frozen at \( z_0 \) are also spatial constants we derive the momenta conjugate to \( T_{\sigma(x)}(z), T_{\sigma(y)}(z), T_{\sigma(xz)}(z) \), the basic loop variables that we are using, respectively as

\[
\frac{[T_{\sigma(x)}^z(z)]'[T_{\sigma(x)}(z_0) - T_{\sigma(x)}(z)T_{\sigma(xz)}(z)]}{\sqrt{(1 - [T_{\sigma(z)}]^2)[(1 - [T_{\sigma(x)}(z_0)]^2)(1 - [T_{\sigma(x)}(z)]^2) - (T_{\sigma(xz)}(z) - T_{\sigma(x)}(z_0)T_{\sigma(x)}(z))^2]}} \times
\]

\[
1 - \frac{1}{1 - [T_{\sigma(x)}(z)]^2}T_{\sigma(x)}^y(z)
\]

(5.40)

\[
1 - [T_{\sigma(y)}(z)]^2.
\]

(5.41)

\[
\frac{[T_{\sigma(z)}^y(z)]'}{\sqrt{(1 - [T_{\sigma(z)}]^2)[(1 - [T_{\sigma(x)}(z_0)]^2)(1 - [T_{\sigma(x)}(z)]^2) - (T_{\sigma(xz)}(z) - T_{\sigma(x)}(z_0)T_{\sigma(x)}(z))^2]}}
\]

(5.42)

and the momentum conjugate to \( T_{\sigma(x)}(z_0) \) is the integral over \( S^1 \) of expression (5.40) only that the roles of \( T_{\sigma(x)}(z_0) \) and \( T_{\sigma(x)}(z) \) are interchanged. The reason for the appearance of this global degree of freedom is that we fixed the point \( z_0 \) to be an observable by hand.

Next we look at the constraints expressed in terms of loop-variables. From their definition it is obvious that \( T_{\sigma(\mu)} \) is a scalar, and since (5.38) shows that the density \( \gamma \) is a spatial derivative of a function of our three loop variables, we find that also \( T_{\sigma(xz)} \) must be a scalar (alternatively, this follows from its definition). \( T_{\sigma(x)}(z_0) \) is a constant and its conjugate, as an integral over a density, is also diffeomorphism invariant.

Thus, necessarily the vector constraint must be

\[
(T_{\sigma(\mu)})'S^\mu + (T_{\sigma(xz)})'S
\]

(5.43)

where we have denoted the corresponding conjugate momenta in (5.40), (5.41) by \( S^\mu, S \). Of course, it is also obvious from the definitions that \( T_{\sigma(\mu)}^\mu \) and \( T_{\sigma(z)}^z \) are densities and a scalar respectively.

It turns out to be more convenient to make direct use of the inversion formulas
\[
V = 2\left[\frac{T_\sigma(x)T_\sigma(x)}{1-(T_\sigma(x))^2} + \frac{T_\sigma(y)T_\sigma(y)}{1-(T_\sigma(y))^2}\right]
+ \left[\arccos\left(\frac{T_\sigma(x) - T_\sigma(x)}{\sqrt{(1-(T_\sigma(x))^2)(1-(T_\sigma(x))^2)}}\right)\right]' \frac{(T_\sigma(z))^y}{2 - \sqrt{1-(T_\sigma(x))^2}} \] (5.44)

\[
C = 4\left[\arccos(T_\sigma(x))T_\sigma(x) \arccos(T_\sigma(y))T_\sigma(y)\right] \frac{\arccos(T_\sigma(y))T_\sigma(y)}{\sqrt{1-(T_\sigma(x))^2}} \frac{\arccos(T_\sigma(x))T_\sigma(x)}{\sqrt{1-(T_\sigma(x))^2}}
+ \left[\frac{\arccos(T_\sigma(x))T_\sigma(x)}{\sqrt{1-(T_\sigma(x))^2}} + \frac{\arccos(T_\sigma(x))T_\sigma(x)}{\sqrt{1-(T_\sigma(x))^2}}\right] \times
\left[\arccos\left(\frac{T_\sigma(x) - T_\sigma(x)}{\sqrt{(1-(T_\sigma(x))^2)(1-(T_\sigma(x))^2)}}\right)\right]' \frac{T_\sigma(z)}{2 - \sqrt{1-(T_\sigma(x))^2}} \] (5.45)

where all expressions are to be evaluated at \(z\) except for those with a 0 superscript which are to be evaluated at \(z_0\).

The result looks much more complicated than in the connection representation. From the point of view of the loop representation one would have to take equations (5.44) and (5.45) and apply it (after a choice of ordering) to a loop state ([7]) according to the Rovelli Smolin loop representation of \(T_0, T_1\). In order to do that one would have to expand the trigonometrical formulas involved in both constraints and one would obtain an expression involving an infinite sum of loop states. This is a very discouraging result.

It is clear how to obtain the observables of the theory: simply substitute the loop variables via (5.36)-(5.39) into the expressions for the observables given in section 3. These formulas are too long and it is not worthwhile as to display them here.

6 Conclusions

Let us briefly summarize what we have learnt by studying this model system: First of all we worked out a number of ideas and methods to reduce a spatially diffeomorphism invariant field theory which might be useful to keep in mind for future work. In particular, our methods do not suffer from the problem of the existence of a bijection between a frame on the hypersurface \(\Sigma\) and a collection of scalar field to solve the diffeomorphism constraint as it occurs in the second reference of [4].

The symplectic reduction of the diffeomorphism constraint has expectedly lead to nonlocal ‘already smeared’ Dirac observables in a natural way. It was not necessary to make use of a complete system of mode functions on the hypersurface (exploiting the fact that the Hilbert space of square integrable functions on \(\Sigma\) is separable) to write the true degrees of freedom in such a countable fashion.

The solution of the scalar constraint quadratic in the momenta reveals that the
Dirac observables will in general mix momentum and configuration variables relative to the polarization of the phase space that one started with.

The formalism of deparametrization applied to field theories shows that there are even more serious problems than those that have been discussed for finite-dimensional examples in [4] at least on the technical side.

The fact that the loop-variables are so badly suited for our model is quite discouraging and scratches at the hope that many authors had when employing them for gravity, namely that they could serve as the natural ‘polarization’ to solve quantum gravity nonperturbatively. Actually, the quantum solutions that have been found in [7] are solutions for the degenerate sector only. In our model, the variables \( T^{\sigma(z)}_\sigma \) and \( T^{z(\sigma)}_\sigma \) are simple expressions and the former is easily seen to be a Dirac observable on the degenerate sector only. However, it is not the only solution of the constraints (in [7] it was speculated that the found solutions are complete). The assumption arises that it will be a general feature that the solution of the constraints expressed in terms of the loop variables will take a simple form only on the degenerate sector.

On the other hand, one should never overestimate the results obtained from a model and it could in fact also happen that our conclusions are due to an artifact of the reduction process.

A Calculation of the spin-connection

We will derive here a more general result, namely how the spin-connection looks like for the Gowdy models ([8]).

The densitized triad for the Gowdy models takes the form

\[
(\mathbf{E}_i^a) = \begin{pmatrix}
E_x^a & E_y^a & 0 \\
E_x^a & E_y^a & 0 \\
0 & 0 & E_z
\end{pmatrix}
\]

so that \( q^{ab} = \begin{pmatrix}
E_x^x & E_y^x & E_z^x & 0 \\
E_x^y & E_y^y & E_z^y & 0 \\
0 & 0 & E_z & 0 \\
(E_z^2)
\end{pmatrix} \)

is the expression for the twice densitized inverse metric \( \mathbf{E}_i^a \mathbf{E}_i^b \). We conclude that

\[
[\text{det}(q)]^2 = [\text{det}(\mathbf{E}_i^a)]^2 = (E_z E^a E^b) = (E_z E^2)^2 \quad (A.2)
\]

and can invert expression (A.1) to obtain the metric tensor

\[
(q_{ab}) = \frac{E_z}{E} \begin{pmatrix}
E_x^x & -E_y^x E_z^y & 0 \\
-E_x^y & E_y^x & 0 \\
0 & 0 & (E_z^2)
\end{pmatrix} \quad (A.3)
\]

From this we infer the expressions for the triad and its inverse, that is

\[
(e_i^a) = \frac{E_i^a}{\sqrt{\text{det}(q)}} \quad \text{and} \quad e^i_a = q_{ab} e_b^a \quad (A.4)
\]

namely

\[
(e_i^a) = \frac{1}{\sqrt{E^z E}} \begin{pmatrix}
E_x^x & E_y^y & 0 \\
E_x^y & E_y^x & 0 \\
0 & 0 & E_z
\end{pmatrix} \quad \text{and} \quad (e_i^a) = \sqrt{E_z} \begin{pmatrix}
E_y^y & -E_x^x & 0 \\
-E_y^x & E_x^y & 0 \\
0 & 0 & E_z
\end{pmatrix} \quad (A.5)
\]
which we have to insert into the formula for the spin-connection
\[ \Gamma^i_a = -\frac{1}{2} e^{ij} e^b_k (2e^a_{[i,b]} + e^c_{e_a} e^e_{c,b}). \] (A.6)

Note that \( e^i_{a,b} = (e^i_a)' \delta_{b^2} \) in the course of the calculation and that \( e^i_{a} e^j \epsilon^{ij} = e^i_{a} (e^i_a)' \epsilon^{ij} = e^i_{a} e^j = 0 \) etc. due to the symmetry properties of the triads so that we can simplify (A.6) without reference to the explicit form of the triads to
\[ \Gamma^i_a = -\frac{1}{2} e^{ij} [e^x_k ((e^{i})_a)' + e^c_{e_a} (e^l_a)' + \delta_{a2} e^b_{e_a} (e^k_a)'. \] (A.7)

More explicitly
\[ \begin{align*}
\Gamma^i_x &= \frac{1}{2} e^{ij} e^z_{e^j_a} ((e^{i})_a)' + e^c_{e_a} (e^l_a)' + e^y_{e_a} (e^l_a)' \\
\Gamma^i_y &= \frac{1}{2} e^{ij} e^z_{e^j_a} ((e^{i})_a)' + e^c_{e_a} (e^l_a)' + e^y_{e_a} (e^l_a)' \\
\Gamma^i_z &= \frac{1}{2} e^{ij} (e^x_{e^j_a} e^{y}_{e^l_a} + e^y_{e_a} (e^k_a)'). \end{align*} \] (A.8)

Now we have to go into details and have to make use of the expressions (A.1). The calculation is rather tedious. The result is given by
\[ \begin{align*}
(\Gamma^i_x) &= \frac{E^z}{2E} \left( (E^y)' E^y_E^i + \frac{1}{E} [(E^y)' E - (E^y)' E^r_{E^i} E^y_{E^i}] + (E^r)' E^y_{E^i} E^y_{E^i} \right), \\
(\Gamma^i_y) &= \frac{E^z}{2E} \left( -(E^y)' E^y_{E^i} + \frac{1}{E} [-(E^r)' E - (E^r)' E^r_{E^i} E^y_{E^i}] + (E^r)' E^r_{E^i} E^y_{E^i} \right), \\
\Gamma^i_z &= \frac{1}{2E} \left( 0, 0, E^z_{E^i} E^y_{E^i} - E^y_{E^i} (E^r)' \right). \end{align*} \] (A.9)

These are the corresponding real parts for the Ashtekar connection for the Gowdy models. Let us restrict now (A.9) to our model, that is
\[ (E^i_a) = \begin{pmatrix}
E^x \cos(\beta) & -E^y \sin(\beta) & 0 \\
E^x \sin(\beta) & E^y \cos(\beta) & 0 \\
0 & 0 & E^z
\end{pmatrix}. \] (A.10)

Then (A.9) simplifies tremendously to
\[ (\Gamma^i_a) = \begin{pmatrix}
-\Gamma_x \sin(\beta) & \Gamma_y \cos(\beta) & 0 \\
\Gamma_x \cos(\beta) & \Gamma_y \sin(\beta) & 0 \\
0 & 0 & \Gamma_z
\end{pmatrix}, \] (A.11)

where
\[ \begin{align*}
\Gamma_x &= \frac{1}{2E_y} \left( E^y E^z \right)', \ \Gamma_y = -\frac{1}{2E_z} \left( E^z E^y \right)', \ \Gamma_z = -\beta'. \end{align*} \] (A.12)
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