Research Article

Characterization, Dilation, and Perturbation of Basic Continuous Frames

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A vector-valued function is called a basic continuous frame if it is a continuous frame for its spanning space. It is shown in this article that basic continuous frames and their oblique duals can be characterized by operators with closed ranges. Furthermore, we show that any oblique dual pair of basic continuous frames for a Hilbert space can be dilated to a Type II dual pair for a larger Hilbert space. Finally, a perturbation result for basic continuous frames is given. Since the spanning spaces of two basic continuous frames for a Hilbert space are often different, the research process is more complex than the setting of general continuous frames.

1. Introduction

The concept of discrete frames was first formally introduced by Duffin and Schaeffer [1] in 1952 and popularized greatly after the significant paper [2] by Daubechies et al. in 1986. A discrete frame is an overcomplete family of countable elements in a Hilbert space which allows every element in the space to be represented as a linear combination of the frame elements. It has been widely used in many fields such as image and signal processing, approximation theory and wireless communications. Due to different applications and theoretical goals, various generalizations of discrete frames have been presented. For example, pseudoframes [3], g-frames [4], fusion frames [5] and operator-valued frames [6]. One important generalization of discrete frames is the so-called continuous frames, which is introduced by Kaiser [7] and independently by Ali, Antonie and Gazeau [8]. We refer to [9–12] for more studies on continuous frames.

Dilation and perturbation are two significant properties for discrete and continuous frames. Gabardo and Han [13] generalized the dilation theorem for dual discrete frames (cf. [14]) to dual continuous frames. Using the method considered by Casazza and Christensen (cf. [15]), they deduced a perturbation theorem for continuous frames. Kaushik et al. [16] gave some equivalent conditions of perturbation for continuous frames and obtained a sufficient condition for the stability of a continuous frame. Since the basic continuous frame is more general in theory and more freedom when the dual frame is selected, we study basic continuous frames around their dilation and perturbation properties.

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ denote Hilbert spaces over the complex field and $\mathcal{J}$ a countable index set. For an operator $T \in B(\mathcal{H}, \mathcal{K})$, we denote its range and kernel by $\text{ran } T$ and $\text{ker } T$, respectively. Let $\mathcal{U}, \mathcal{V}$ be closed subspaces of $\mathcal{H}$. We denote by $\mathcal{U} \oplus \mathcal{V}$ the orthogonal direct sum of closed subspaces and by $\mathcal{U} \ominus \mathcal{V}$ the space $\mathcal{U} \cap \mathcal{V}^\perp$. We call an operator $P \in B(\mathcal{H})$ a (oblique) projection if it satisfies $P^2 = P$. In this case, the decomposition $\mathcal{H} = \text{ran }P + \ker P$ holds. Conversely, if $\mathcal{H} = \mathcal{U} \oplus \mathcal{V}$, then we can find a unique projection $P_{\mathcal{U}, \mathcal{V}}$ satisfying $\text{ran }P_{\mathcal{U}, \mathcal{V}} = \mathcal{U}$ and $\ker P_{\mathcal{U}, \mathcal{V}} = \mathcal{V}$. Let $P_{\mathcal{U}, \mathcal{V}}^\perp = P_{\mathcal{U} \ominus \mathcal{V}}$ be the orthogonal projection onto $\mathcal{U}$ and $P_{\mathcal{U} \ominus \mathcal{V}}$ be its restriction to $\mathcal{V}$.

Let $(\Omega, \mu)$ be a measure space where $\mu$ is positive. Recall that a vector-valued function $F: \Omega \longrightarrow \mathcal{H}$ is said to be a continuous frame for $\mathcal{H}$ with respect to $(\Omega, \mu)$ or a $(\Omega, \mu)$-frame if

(i) $F$ is weakly measurable, i.e., for all $x \in \mathcal{H}, \omega \longrightarrow \langle x, F(\omega) \rangle$ is a measurable function on $\Omega$.

(ii) There exist constants $A, B > 0$ such that
\[ A\|x\|^2 \leq \int_\Omega |\langle x, F(\omega) \rangle|^2 d\mu(\omega) \leq B\|x\|^2, \forall x \in \mathcal{H}. \quad (1) \]

The numbers \( A, B \) are called frame bounds. We call \( F \) a \((\Omega, \mu)\)-Bessel mapping if only the second inequality of (1) holds, a tight \((\Omega, \mu)\)-frame if \( A = B \), and a Parseval \((\Omega, \mu)\)-frame if \( A = B = 1 \). The mapping \( F \) is called a basic \((\Omega, \mu)\)-frame or a basic continuous frame for \( \mathcal{H} \) if it is a continuous frame for the spanning space \( \overline{\text{span}}\{F(\omega)\}_{\omega \in \Omega} \). When \( \Omega = \mathbb{N} \) and \( \mu \) is the counting measure, the family \( \{F(\omega)\}_{\omega \in \Omega} \) is a discrete frame.

To a \((\Omega, \mu)\)-Bessel mapping \( F \), we associate the analysis operator \( \theta_F \) given by

\[ \theta_F: \mathcal{H} \longrightarrow L^2(\Omega, \mu), \]

\[ \langle \theta_F x(\omega) \rangle = \langle x, F(\omega) \rangle, \quad (2) \]

Note that the frame condition (1) implies that \( \sqrt{A}\|x\| \leq \|\theta_F x\| \leq \sqrt{B}\|x\| \). The adjoint \( T_F \) of \( \theta_F \) is called the synthesis operator and is given by

\[ T_F: L^2(\Omega, \mu) \longrightarrow \mathcal{H}, \]

\[ \langle T_F f, y \rangle = \int_\Omega f(\omega) \langle F(\omega), y \rangle d\mu(\omega), \forall y \in \mathcal{H}. \quad (3) \]

In this case, we usually say that the following equation holds in the weak sense:

\[ T_F f = \int_\Omega f(\omega) F(\omega) d\mu(\omega). \quad (4) \]

The frame operator \( S_F \) is defined to be \( T_F \theta_F \). When \( F \) is a continuous frame for \( \mathcal{H} \), \( S_F \) is a bounded, invertible and positive operator. A \((\Omega, \mu)\)-Bessel mapping \( G \) is said to be a dual of \( F \) if \( T_F \theta_G = I_{\mathcal{H}} \), i.e.,

\[ x = \int_\Omega \langle x, G(\omega) \rangle F(\omega) d\mu(\omega), \quad \forall x \in \mathcal{H}. \quad (5) \]

Every \((\Omega, \mu)\)-frame always has a dual \( S_F^* F \), called the canonical dual, satisfying the following reconstruction formula:

\[ x = \int_\Omega \langle x, S_F^* F(\omega) \rangle F(\omega) d\mu(\omega), \forall x \in \mathcal{H}. \quad (6) \]

If \( F \) has the unique dual, then we say \( F \) is a Riesz-type \((\Omega, \mu)\)-frame. We see from ([13], Proposition 1) that \( F \) is a Riesz-type \((\Omega, \mu)\)-frame if and only if its analysis operator is surjective. Two \((\Omega, \mu)\)-frames \( F \) and \( G \) for \( \mathcal{H} \) and \( \mathcal{K} \), respectively, are called similar if \( LF = G \) holds for an invertible operator \( L: \mathcal{H} \longrightarrow \mathcal{K} \). Let \( \mathcal{H} \) be a closed subspace of \( L^2(\Omega, \mu) \). We call \( M \) a frame range if there exist a Hilbert space \( \mathcal{K} \) and a \((\Omega, \mu)\)-frame \( F \) for \( \mathcal{H} \) whose analysis operator has range space \( \mathcal{M} \).

The paper is organized as follows. In Section 2, we present some preliminary results about basic \((\Omega, \mu)\)-frames.

In Section 3, we characterize basic \((\Omega, \mu)\)-frames and their oblique duals. Section 4 is devoted to the dilation of oblique dual pairs of basic \((\Omega, \mu)\)-frames. A perturbation theorem of basic \((\Omega, \mu)\)-frames is considered in Section 5.

### 2. Preliminaries

This section is devoted to some preliminary results about basic continuous frames and their duals. In the rest of the paper, we agree to use the following notation:

\[ \mathcal{H}_F := \overline{\text{span}}\{F(\omega)\}_{\omega \in \Omega}. \quad (7) \]

For a given vector-valued function \( F: \Omega \longrightarrow \mathcal{H} \). For a \((\Omega, \mu)\)-Bessel mapping \( F \), we always use \( \theta_F \) and \( T_F \) to denote the analysis operator and synthesis operator, respectively.

Recall that every basic continuous frame is a \((\Omega, \mu)\)-Bessel mapping. Conversely, a \((\Omega, \mu)\)-Bessel mapping \( F \) constitutes a basic continuous frame if and only if \( \text{ran} T_F \) is closed. In this case, \( \mathcal{H}_F = \text{ran} T_F \).

For a basic \((\Omega, \mu)\)-frame \( F \), the frame operator \( S_F \) is invertible when restricted to \( \overline{\text{span}}\{F(\omega)\}_{\omega \in \Omega} \). Its canonical dual is \( F = S_F^* F \), where \( S_F^* \) denotes the pseudoinverse of \( S_F \) (see [17]). We recall that

\[ \text{ran} \theta_F = \text{ran} \theta_{F^*}, \quad \text{ran} \theta_{F^*} = \text{ran} \theta_F. \quad (8) \]

Now, we give definitions on duals of basic \((\Omega, \mu)\)-frames with the following. For the discrete case, we see [18–20]

**Definition 1.** Suppose that \( F \) is a basic \((\Omega, \mu)\)-frame and that \( G \) is a \((\Omega, \mu)\)-Bessel mapping for \( \mathcal{H} \).

(i) \( G \) is a dual of \( F \) if \( T_F \theta_G = I_{\mathcal{H}_F} \), i.e.,

\[ x = \int_\Omega \langle x, \mu(\omega) \rangle F(\omega) d\mu(\omega), \quad \forall x \in \mathcal{H}_F. \]

(ii) \( G \) is a Type I dual of \( F \) if \( G \) is a dual of \( F \) and \( \text{ran} T_G \subseteq \text{ran} T_F \).

(iii) \( G \) is a Type II dual of \( F \) if \( G \) is a dual of \( F \) and \( \text{ran} \theta_G \subseteq \text{ran} \theta_F \).

(iv) \( G \) is an oblique dual of \( F \) if \( G \) is a basic \((\Omega, \mu)\)-frame that is a dual of \( F \) and \( G \) is a dual of \( F \).

Heil et al. showed in [20] that Type I or II duals of discrete frame sequences are oblique duals. They also characterized the existence of oblique duals with respect to the direct sum decomposition of \( \mathcal{H} \). It is easy to show that these facts still hold for basic \((\Omega, \mu)\)-frames. If \( G \) is a Type I dual of \( F \), then \( G \) is an oblique dual of \( F \) and \( \text{ran} T_G = \text{ran} T_F \); if \( G \) is a Type II dual of \( F \), then \( G \) is an oblique dual of \( F \) and \( \text{ran} \theta_G = \text{ran} \theta_F \). Similar to ([20], Theorem 1.4), we also have:

**Proposition 1.** Suppose that \( \mathcal{H} \) and \( \mathcal{K} \) are closed subspaces of \( \mathcal{H} \) and that \( F \) is a \((\Omega, \mu)\)-frame for \( \mathcal{H} \). Then the following are equivalent:
(i) \( \mathcal{H} = \mathcal{U} + \mathcal{V} \)
(ii) There is a \( (\Omega, \mu) \)-frame \( G \) for \( \mathcal{V} \) that is a Type II dual of \( F \)
(iii) There is a \( (\Omega, \mu) \)-frame \( G \) for \( \mathcal{V} \) that is an oblique dual of \( F \)

From the above proposition, the fact that \( G \) is an oblique dual of \( F \) shows \( \mathcal{H} = \mathcal{H}_F + \mathcal{H}_G \). For a pair of oblique duals, \( L^2(\Omega, \mu) \) also have a decomposition.

**Lemma 1.** Suppose that \( F \) is a basic \( (\Omega, \mu) \)-frame for \( \mathcal{H} \) and that \( G \) is an oblique dual of \( F \). Assume \( \mathcal{M} \subset L^2(\Omega, \mu) \) is closed containing both \( \text{ran} \theta_F \) and \( \text{ran} \theta_G \). Then,

\[
\mathcal{M} = \text{ran} \theta_F + \mathcal{M} \cap \text{ran} \theta_G = \text{ran} \theta_F + \mathcal{M} \cap \text{ran} \theta_G.
\]

**Proof.** (1) By assumption, we have

\[
T_G \theta_F |_{\mathcal{H}_G} = I |_{\mathcal{H}_G},
\]

Hence,

\[
(\theta_F T_G)^2 = \theta_F (T_G \theta_F) T_G = \theta_F T_G,
\]

which implies that \( \theta_F T_G \) is a projection. Using (9), we obtain

\[
\text{ran} \theta_F = \theta_F (\mathcal{H}_G + \mathcal{H}_F) = \theta_F \mathcal{H}_G = \text{ran} \theta_F T_G,
\]

\[
\text{ran} \theta_G = \theta_G (\mathcal{H}_G + \mathcal{H}_F) = \theta_G \mathcal{H}_F = \mathcal{H}_F,
\]

Therefore, ker \( (\theta_F T_G)^2 = (\text{ran} \theta_F T_G)^2 = (\text{ran} \theta_G)^2 \) and thus \( \theta_F T_G = P_{\text{ran} \theta_F \cap \text{ran} \theta_G} \). This implies that

\[
L^2(\Omega, \mu) = (\text{ran} \theta_F T_G) \cap \text{ker} (\theta_F T_G) = \text{ran} \theta_F + (\text{ran} \theta_G)^2.
\]

(ii). Note that (i) shows that \( \text{ran} \theta_F \cap (\mathcal{M} \cap \text{ran} \theta_G) = [0] \).

Now, the conclusion follows from [19, Lemma 3.9]

**Lemma 2** (see [21]). Let \( (\Omega, \mu) \) be a \( \sigma \)-finite, positive measure space.

(i) \( L^2(\Omega, \mu) \) itself is a frame range if and only if \( (\Omega, \mu) \) is purely atomic

(ii) Every closed subspace of a frame range is a frame range

We finish this section with the following equivalent condition about the frame range:

**Lemma 3** (see [13], Corollary 2.9). For a closed subspace \( \mathcal{M} \) of \( L^2(\Omega, \mu) \), the following are equivalent:

(i) \( \mathcal{M} \) is a frame range

(ii) There exists an orthonormal basis \( \{ \psi_j \}_{j \in J} \) for \( \mathcal{M} \) such that \( \sum_{j \in J} |\psi_j(\omega)|^2 < \infty \) for a.e. \( \omega \in \Omega \)

3. Characterization of Basic Continuous Frames and Their Oblique Duals

This section focuses on the characterization of basic continuous frames and their oblique duals. We begin with a characterization for basic parseval \( (\Omega, \mu) \)-frames.

**Lemma 4.** Suppose that \( (\Omega, \mu) \) is a measure space with positive measure \( \mu \). Then the following are equivalent:

(i) \( F \) is a basic parseval continuous frame for \( \mathcal{H} \)

(ii) \( F(\omega) = \sum_{j \in J} \overline{\psi_j(\omega)} |\theta_j \psi_j | \) for some operator \( \theta: \mathcal{H} \rightarrow L^2(\Omega, \mu) \) whose range is a frame range and some orthonormal basis \( \{ \psi_j \}_{j \in J} \) of a frame range \( \mathcal{M} \)

**Proof.** (i) \( \Rightarrow \) (ii). Let \( \theta \) be the analysis operator for \( F \) and \( \mathcal{M} = \text{ran} \theta \). Since \( F \) is a basic parseval continuous frame, \( \theta \) is a partial isometry. By Lemma 3, we can denote an orthonormal basis \( \{ \psi_j \}_{j \in J} \) for \( j \in J \) satisfying \( \sum_{j \in J} |\psi_j(\omega)|^2 < \infty \) for a.e. \( \omega \in \Omega \). For any \( x \in \mathcal{H} \) followed that

\[
\langle x, \sum_{j \in J} \overline{\psi_j(\omega)} \theta^* \psi_j \rangle = \langle \sum_{j \in J} \overline{\psi_j(\omega)} \bar{\psi}_j \psi_j \rangle (\omega) = \langle \sum_{j \in J} \overline{\psi_j(\omega)} \bar{\psi}_j \psi_j \rangle (\omega),
\]

which implies \( F(\omega) = \sum_{j \in J} \overline{\psi_j(\omega)} \theta^* \psi_j \).

(ii) \( \Rightarrow \) (iii). Write \( \mathcal{M} = \text{ran} \theta \) and \( \bar{e}_j = \theta^* \psi_j \) for every \( j \in J \). Since \( \theta \) is a partial isometry, \( \theta^* \) is an isometry restricted to \( \text{ker} \theta^* \). This implies that \( \text{ker} \theta^* \) is an orthonormal set.

(iii) \( \Rightarrow \) (i). For any \( x \in \text{span} \{ F(\omega) \}_{\omega \in \Omega} \subset \text{span} \{ \bar{e}_j \}_{j \in J} \), it follows that

\[
\int_{\Omega} |\langle x, F(\omega) \rangle|^2 d\mu(\omega) = \| x \| ^2 = \sum_{j \in J} |\langle x, \bar{e}_j \rangle \psi_j (\omega) |^2
\]

implying that \( F \) is a basic parseval continuous frame.

Similarly, we can deduce the following characterization for basic \( (\Omega, \mu) \)-frames.

**Proposition 2.** Suppose that \( (\Omega, \mu) \) is a measure space with positive measure \( \mu \). Then the following are equivalent:

(i) \( F \) is a basic \( (\Omega, \mu) \)-frame for \( \mathcal{H} \)

(ii) \( F(\omega) = \sum_{j \in J} \overline{\psi_j(\omega)} |\theta_j \psi_j | \) for some operator \( \theta: \mathcal{H} \rightarrow L^2(\Omega, \mu) \) whose range is a frame range and some orthonormal basis of \( \text{ran} \theta \)

\[
\sum_{j \in J} |\langle x, \bar{e}_j \rangle \psi_j (\omega) |^2 = \| x \| ^2,
\]

\[
[k\rangle_{i,j}\]
(iii) $F(\omega) = \sum_{j \in J} \overline{\psi_j(\omega)} e_j$ for some Riesz sequence $\{e_j\}_{j \in J}$ of $\mathcal{H}$ and some orthonormal basis $\{\psi_j\}_{j \in J}$ of a frame range $\mathcal{M}$

Proof. (i) $\Rightarrow$ (ii) follows from the proof of Lemma 4. For (ii) $\Rightarrow$ (iii), we see that $\theta^*$ is bijective restricted to $\mathcal{M} \longrightarrow \text{ran}\theta^*$. Hence, $\{\theta^* \psi_j\}_{j \in J}$ is a Riesz sequence for $\mathcal{H}$. Now, suppose (iii) holds. For any $x \in \text{span}\{F(\omega)\}_{\omega \in \Omega} \subseteq \text{span}\{e_j\}_{j \in J}$, it follows that

$$
\int_{\Omega} |\langle x, F(\omega) \rangle|^2 d\mu(\omega) = \|\langle x, F(\omega) \rangle\|^2 = \sum_{j \in J} |\langle x, e_j \rangle\|^2.
$$

(15)

implying that $\omega \mapsto \sum_{j \in J} \overline{\psi_j(\omega)} \psi_j$ forms a Parseval $(\Omega, \mu)$-frame for $\mathcal{M}$. If $F$ is a basic $(\Omega, \mu)$-frame for $\mathcal{H}$ with associated analysis operator $\theta$, then it follows from Proposition 2 that

$$
F(\omega) = \sum_{j \in J} \overline{\psi_j(\omega)} \theta^* \psi_j = \theta^* \left( \sum_{j \in J} \overline{\psi_j(\omega)} \psi_j \right).
$$

(16)

where $\theta^*$ is invertible restricted to $\mathcal{M} \longrightarrow \text{ran}\theta^*$. This means that every basic $(\Omega, \mu)$-frame is similar to a basic Parseval $(\Omega, \mu)$-frame for $L^2(\Omega, \mu)$.

Putting Lemma 2 and Proposition 2 together, we can characterize basic Riesz-type $(\Omega, \mu)$-frames: □

Corollary 1. Suppose that $(\Omega, \mu)$ is a purely atomic, positive measure space where $\mu$ is $\sigma$-finite, and that $\{\psi_j\}_{j \in J}$ is an orthonormal basis for $L^2(\Omega, \mu)$. Then, the following are equivalent:

(i) $F$ is a basic Riesz-type $(\Omega, \mu)$-frame for $\mathcal{H}$

(ii) $F(\omega) = \sum_{j \in J} \overline{\psi_j(\omega)} \theta^* \psi_j$ for some surjective operator $\theta : \mathcal{H} \longrightarrow L^2(\Omega, \mu)$

(iii) $F(\omega) = \sum_{j \in J} \overline{\psi_j(\omega)} e_j$ for some Riesz sequence $\{e_j\}_{j \in J}$ of $\mathcal{H}$

It is known that there is a bijective correspondence between the set of discrete frame sequences and all the operators with closed range. The following proposition derives a corresponding result for basic continuous frames. This is the essential difference between discrete frame sequences and basic continuous frames.

Proposition 3. Every basic $(\Omega, \mu)$-frame corresponds to an operator whose range is a frame range.

Proof. Suppose $\theta : \mathcal{H} \longrightarrow L^2(\Omega, \mu)$ is an operator whose range is a frame range. Then, by Proposition 2, $F(\omega) = \sum_{j \in J} \overline{\psi_j(\omega)} \theta^* \psi_j$ is a basic continuous frame for $\mathcal{H}$, where $\{\psi_j\}_{j \in J}$ is an orthonormal basis for $\text{ran}\theta$. Moreover, we compute

$$
\langle x, F(\omega) \rangle = \sum_{j \in J} \langle \theta x, \psi_j \rangle \psi_j(\omega) = \langle \theta x(\omega) \rangle, \forall x \in \mathcal{H},
$$

(18)

which implies that $\theta$ is the analysis operator for $F$. □

A new basic $(\Omega, \mu)$-frame can be obtained by applying a suitable operator to a basic $(\Omega, \mu)$-frame. Note that the spanning space may be changed after the action of an operator. Using Proposition 3, we can derive it in an easy way.

Corollary 2. Suppose that $F$ is a basic $(\Omega, \mu)$-frame where $\mu$ is $\sigma$-finite, and that $W$ is an operator on $\mathcal{H}$. Then, $WF$ is a basic continuous frame if and only if $W$ has a closed range restricted to $\text{span}\{F(\omega)\}_{\omega \in \Omega}$.

Proof. By assumption, $WF$ is a $(\Omega, \mu)$-Bessel mapping with associated analysis operator $\theta^* W^*$. □
Now, suppose \(WF\) is a basic continuous frame. Then, the analysis operator \(\theta_fW^*\) has a closed range and so is \(W\theta_f\). This means that \(W\) has a closed range restricted to \(\text{ran}\theta_f = \text{span}\{F(\omega)\}_{\omega \in \Omega}\).

Conversely, suppose \(W\) has a closed range. Since \(\text{ran}(\theta_fW^*) \subset \text{ran}\theta_f\) and \(\text{ran}\theta_f\) is a frame range, we see from Lemma 2 (ii) that \(\text{ran}(\theta_fW^*)\) is also a frame range. Then, by Proposition 3, \(WF\) is a basic continuous frame.

We finish this section with the following characterization for an oblique dual pair of basic \((\Omega, \mu)\)-frames.

**Proposition 4.** For \((\Omega, \mu)\)-Bessel mappings \(F\) and \(G\) of \(H\), the following are equivalent:

1. \(G\) is an oblique dual of \(F\)
2. \(T_F^*\theta_G = P_{\mathcal{F}_F, \mathcal{G}^\perp}\)
3. \(T_F\theta_G = P_{\mathcal{F}_F, \mathcal{G}^\perp}\)
4. \(T_F\theta_G|_{\mathcal{F}_F} = I_{\mathcal{F}_F}, \mathcal{H}_F = \mathcal{F}_F + \mathcal{G}_G\)
5. \(T_G\theta_F = P_{\mathcal{G}_G, \mathcal{F}^\perp}\)
6. \(\theta_F T_G = P_{\text{ran} \theta_F, \text{ran} \theta_G}\) where \(F, G\) are basic \((\Omega, \mu)\)-frames
7. \(\theta_F T_G = P_{\text{ran} \theta_F, \text{ran} \theta_G}\) where \(F, G\) are basic \((\Omega, \mu)\)-frames

**Proof.** (i) \(\Rightarrow\) (ii). By the definition of oblique duals, we get that

\[
T_F^*\theta_G|_{\mathcal{F}_F} = I_{\mathcal{F}_F},
\]

which implies that \(T_F\theta_G = P_{\mathcal{F}_F, \mathcal{G}^\perp}\).

(ii) \(\Rightarrow\) (i). Since \(T_F^*\theta_G = P_{\mathcal{F}_F, \mathcal{G}^\perp}\), we get

\[
\|x\|^2 = \langle T_F^*\theta_G x, x \rangle = \int_\Omega \langle x, G(\omega) \rangle \langle F(\omega), x \rangle d\mu(\omega), \]

\(\forall x \in \mathcal{F}_F\).

Using Cauchy Schwarz’ inequality and that \(G\) is a \((\Omega, \mu)\)-Bessel mapping, we can deduce the lower frame condition for \(F\). Therefore \(F\) is a \((\Omega, \mu)\)-frame for \(\mathcal{F}_F\) and \(G\) is a dual of \(F\). We can prove \(G\) is a \((\Omega, \mu)\)-frame and \(F\) is a dual of \(G\) in the same way.

**4. Dilation of Oblique Dual Pairs**

Gabardo and Han derived a dilation theorem ([13], Theorem 1.1) for dual pairs of continuous frames. In this section, we consider the dilation for an oblique dual pair of basic \((\Omega, \mu)\)-frames. Our approach relies strongly on the following result:

**Lemma 5.** Suppose that \(F\) is a basic \((\Omega, \mu)\)-frame for \(\mathcal{H}\) and that both \(G\) and \(E\) are oblique duals of \(F\) with \(\mathcal{H}_G = \mathcal{H}_F\). If either \(\text{ran} \theta_G \subset \text{ran} \theta_F\) or \(\text{ran} \theta_F \subset \text{ran} \theta_G\) holds, then \(G = E\).

**Proof.** Since \(G\) and \(E\) are oblique duals of \(F\), we have

\[
T_F^*\theta_G = T_F^*\theta_E = I_{\mathcal{F}_F},
\]

\[
\mathcal{H}_G = \mathcal{H}_F + \mathcal{G}_G.
\]

Note that \(\ker \theta_G = \mathcal{H}_G = \mathcal{H}_F = \ker \theta_F\), which implies \(\theta_E \mathcal{H}_F = \theta_F \mathcal{H}_F\) and \(\theta_G \mathcal{H}_F = \theta_F \mathcal{H}_F\). Now suppose \(\text{ran} \theta_G \subset \text{ran} \theta_F\). Thus \(\theta_G \mathcal{H}_F \subset \mathcal{H}_F\) and so for any \(y \in \mathcal{H}_F\), we can find \(z \in \mathcal{H}_F\) satisfying \(\theta_E z = \theta_G y\). Therefore \(y = T_F^*\theta_G y = T_F^*\theta_E z = z\), and thus \(\theta_G x = \theta_G x\) holds for all \(x \in \mathcal{H}\). This forces that \(E = G\), as required.

The preceding lemma also implies the uniqueness of the Type II dual for a fixed direct sum decomposition of \(\mathcal{H}\). One can see ([22], Proposition 3.3) for a similar assertion for discrete frame sequences. Now we state this result in the following:

**Corollary 3.** Suppose that \(F\) is a basic \((\Omega, \mu)\)-frame for \(\mathcal{H}\) and that \(\mathcal{V}\) is a subspace of \(\mathcal{H}\) such that \(\mathcal{H} = \mathcal{H}_F + \mathcal{V}^\perp\). Then there exists a unique \((\Omega, \mu)\)-frame for \(\mathcal{V}\) which is the type II dual of \(F\).

**Proof.** From Proposition 1, we can find a \((\Omega, \mu)\)-frame \(G\) for \(\mathcal{V}\) that is a Type II dual of \(F\). If there exists another
Now we state our main result of this section in the following:

**Theorem 1.** Suppose that $F$ is a basic $(\Omega,\mu)$-frame for $\mathcal{H}$ and that $G$ is an oblique dual of $F$ Assume both $\text{ran}F_1$ and $\text{ran}G_1$ are contained in a frame range $\mathcal{M}$. Then there exist a Hilbert space $\mathcal{H}_F$ and a basic $(\Omega,\mu)$-frame $F_1$ for $\mathcal{H}$ such that $PF_1 = F$, $PG_1 = G$ and $\text{ran}F_1 = \mathcal{M}$, where $G_1$ is a (unique) Type II dual of $F_1$ and $P$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_F$.

**Proof.** Denote by $\{\psi_j\}_{j \in J}$ an orthonormal basis for $\mathcal{M}$ and write $\mathcal{M}_1 := (I_{\mathcal{H}} - Q)L^2(\Omega,\mu) = \mathcal{M} \cap \text{ran}G_1$, where $Q$ and $I_{\mathcal{H}}$ denote orthogonal projections from $L^2(\Omega,\mu)$ onto $\text{ran}G_1$ and $\mathcal{M}$ respectively. By Lemma 4, we know that $F_0(x) := \sum_{j \in J} \langle \psi_j, \omega \rangle (I_{\mathcal{H}} - Q)\psi_j$ defines a parseval $(\Omega,\mu)$-frame for $\mathcal{M}_1$. Therefore, for any $\phi \in \mathcal{M}_1$,

$$\theta_{F_1}(\phi) (x) = \langle \phi, F_0(x) \rangle = \sum_{j \in J} \langle (I_{\mathcal{H}} - Q)\psi_j, \psi_j \rangle \psi_j(x) = \varphi(x),$$

which implies $\text{ran}F_0 = \mathcal{M}_1$.

Let $\mathcal{H} = \mathcal{H}_F \oplus \mathcal{M}_1$ and $F_1 = F \ominus F_0$. Clearly $F_1$ is a $(\Omega,\mu)$-Bessel mapping for $\mathcal{H}$ and $PF_1 = F$, where $P$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_F$. From (29) it follows that, for any $x \oplus \phi \in \mathcal{H}$,

$$\theta_{F_1}(x \oplus \phi)(x) = (\theta_{F_1}x)(x) + \varphi(x).$$

Then by Lemma 1, we have

$$\text{ran}F_1 = \text{ran}F_1 \oplus \mathcal{M}_1 = \mathcal{M}.$$

Since $\theta_{F_1}1_{\mathcal{H}_F}$ is injective and $\text{ran}F_1 \oplus \mathcal{M}_1 = \mathcal{M}$, it follows that $\theta_{F_1}$ is bijective from $H_1 \ominus \mathcal{M}_1$ onto $\mathcal{M}$, which means that $F_1$ is a continuous frame for $H_1 \ominus \mathcal{M}_1$ with $\text{ran}F_1 = \mathcal{M}$. Since $G$ is an oblique dual of $F$, we see from Proposition 1 that $\mathcal{H} = \mathcal{H}_F \oplus \mathcal{M}_1$, and thus we can write

$$\mathcal{H} = \mathcal{H}_F \oplus \mathcal{M}_1 = (\mathcal{H}_F \oplus \mathcal{M}_1) \oplus (\mathcal{M}_G \oplus \mathcal{M}) \ominus \mathcal{M}_1.$$

By Corollary 3, we can find a unique Type II dual $G_1$ of $F_1$ satisfying $G_1G_1 = \mathcal{M}_G \oplus \mathcal{M}_1$ and $\text{ran}G_1 = \text{ran}F_1 = \mathcal{M}$. Put $E = PG_1$. Obviously, $E$ is a continuous frame for $\mathcal{H}_G$ and thus it only remains to show $E = G$. On the one hand, we compute, for any $x \in \mathcal{H}_F \subset \mathcal{H}_F \ominus \mathcal{M}_1$,

$$x = Px = P\left(\int_{\Omega} \langle Px, G_1(\omega) \rangle F_1(\omega)d\mu(\omega) \right)^{1/2}$$

$$= \int_{\Omega} \langle x, PG_1(\omega) \rangle PF_1(\omega)d\mu(\omega)$$

$$= \int_{\Omega} \langle x, E(\omega) \rangle F(\omega)d\mu(\omega),$$

which shows that $E$ is a dual of $F$. Thus, it follows from Proposition 4 that $E$ is an oblique dual of $F$. On the other hand, we compute, for any $x \in \mathcal{H}_F$,

$$x = \int_{\Omega} \langle Px, G_1(\omega) \rangle F_1(\omega)d\mu(\omega)$$

$$= \int_{\Omega} \langle x, E(\omega) \rangle F(\omega)d\mu(\omega).$$

Putting (33) and (34) together, for any $x \in \mathcal{H}_F$, we have

$$\int_{\Omega} \langle x, E(\omega) \rangle F_0(\omega)d\mu(\omega) = \int_{\Omega} \langle x, E(\omega) \rangle F(\omega)d\mu(\omega).$$

which implies that

$$\theta_{F_1}x \in \ker T_{F_0} \cap \mathcal{M} = \text{ran}F_1_1 \cap \mathcal{M} = \mathcal{M}_1 \cap \mathcal{M} = \text{ran}G_1.$$
which means that $G_1$ is a Type I dual of $F_1$. Moreover, we see from Theorem 1 that $G_1$ is a Type II dual of $F_1$. Hence $G_1$ is simultaneously a Type I and a Type II dual of $F_1$, i.e., canonical dual of $F_1$. 

When $G$ is a Type II dual of $F$ in Theorem 1, we cannot get a dilation result similar to the above proposition.

Suppose that $F$ is a continuous frame and that $P \in B(\mathcal{H})$ is an orthogonal projection. It is easy to check that $PF$ is a continuous frame for $P\mathcal{H}$. When $P$ is an oblique projection, we also have the following result:

**Lemma 6.** Suppose that $F$ is a continuous frame for $H$ and that $P_\perp = P_{H^\perp} \in B(\mathcal{H})$ is an oblique projection, where $\mathcal{U}$ and $\mathcal{V}$ are closed subspaces of $\mathcal{H}$. Then $PF$ is a continuous frame for $\mathcal{U}$.

**Proof.** We compute for any $x \in \mathcal{H}$,
\[
(\theta_{PF}x)(\omega) = \langle x, P_{\mathcal{H}^\perp}F(\omega) \rangle = \langle P_{\mathcal{H}^\perp}x, F(\omega) \rangle = (\theta_F P_{\mathcal{H}^\perp}x)(\omega).
\]
Now assume that $\theta_P y = 0$ holds for any $y \in \mathcal{U}$. Then we see from (38) that $\theta_{PF} y = 0$. Since $\theta_F$ is injective, it follows that $P_{\mathcal{H}^\perp}x \perp \mathcal{U}$ and $y \in \mathcal{U}^\perp \cap \mathcal{U} = \{0\}$. This shows that $\theta_{PF} y$ is injective and we have the result. 

We have considered the dilation of oblique dual pairs under orthogonal projections in Theorem 1. Now we deduce a dilation result in terms of oblique projections.

**Theorem 2.** Suppose that $F$ is a basic $(\Omega, \mu)$-frame for $\mathcal{H}$ and that $G$ is an oblique dual of $F$. Assume both $\text{ran} \theta_F$ and $\text{ran} \theta_G$ are contained in a frame range $\mathcal{M}$. If $\dim(\mathcal{M} \cap \text{ran} \theta_G) = \dim \mathcal{H}_G$, then there exist a $(\Omega, \mu)$-frame $F_1$ for $\mathcal{H}$ with its canonical dual $G_1$ such that $P_{\mathcal{H}^\perp}F_1 = F$, $P_{\mathcal{H}^\perp}G_1 = G$ and $\text{ran} \theta_{F_1} = \mathcal{M}$.

**Proof.** Let $Q$ and $I$ be orthogonal projections from $L^2(\Omega, \mu)$ onto $\text{ran} \theta_G$ and $\mathcal{M}$ respectively and write $\mathcal{M}_1 = (I - Q)L^2(\Omega, \mu) = \mathcal{M} \cap \text{ran} \theta_G$. Since $\dim \mathcal{M}_1 = \dim \mathcal{H}_G$, we can find a unitary operator $W \in B(\mathcal{M}_1, \mathcal{H}_G)$. Let $\{\psi_j\}_{j \in J}$ be an orthonormal basis for $\mathcal{M}_1$ and define
\[
F_0 = \sum_{j \in J} \langle (I - Q)W^{-1}x, \psi_j \rangle \psi_j = \langle (I - Q)W^{-1}x, W^{-1}x \rangle = W^{-1}x.
\]
We see that $\theta_{F_1} \mid_{\mathcal{H}_G}$ is injective, and thus $F_0$ is a continuous frame for $\mathcal{H}_G$ with frame range $\mathcal{M}_1$. Put $F_1 = F + F_0$. Clearly $P_{\mathcal{H}^\perp}F_1 = F$ and $F_1$ is a $(\Omega, \mu)$-Bessel mapping for $\mathcal{H}$.

Since $\mathcal{H} = \mathcal{H}_F \oplus \mathcal{H}_G$, for any $y \in \mathcal{H}$, we can write $y = x_1 + z_1, x_2 + z_2 \in \mathcal{H}$, where $x_1 \in \mathcal{H}_F$, $z_1 \in \mathcal{H}_G$, $x_2 \in \mathcal{H}_G$, $z_2 \in \mathcal{H}_G^\perp$. Note that $\theta_{F_1} \mid_{\mathcal{H}^\perp} = 0$ and $\theta_{F_1} \mid_{\mathcal{H}_G} = 0$, which implies that
\[
\theta_{F_1}y = \theta_F(x_1 + z_1) + \theta_F(x_2 + z_2) = \theta_Fx_1 + \theta_Fz_2 = \theta_Fx_1 + W^{-1}z_2.
\]

Using Lemma 1, we have $\text{ran} \theta_{F_1} = \text{ran} \theta_F + \mathcal{M} = \mathcal{M}$. The fact that $F$ is a continuous frame for $\mathcal{H}_F$ shows that $\theta_{F_1} \mid \mathcal{H}_F$ is injective. Thus $\theta_{F_1}$ is injective, implying that $F_1$ is a continuous frame for $\mathcal{H}_F$. Let $G_1$ be the canonical dual of $F_1$ and $E = P_{\mathcal{H}^\perp}G_1$. It follows from Lemma 6 that $E$ is a continuous frame for $\mathcal{H}_G$ and thus it only remains to show $E = G$. For any $x \in \mathcal{H}_F$, we get
\[
x = P_{\mathcal{H}_F}x = P_{\mathcal{H}_F} \left( \int \langle x, G_1(\omega) \rangle F_1(\omega) \mu(\omega) \right) = \int \langle x, P_{\mathcal{H}_F}G_1(\omega) \rangle P_{\mathcal{H}_F}F_1(\omega) \mu(\omega) = \int \langle x, E(\omega) \rangle F(\omega) \mu(\omega),
\]
which implies that $E$ is a dual of $F$. Then by Proposition 4, it follows that $E$ is an oblique dual of $F$. For any $y \in \mathcal{H}_F$, we have
\[
y = \int \langle y, E(\omega) \rangle F(\omega) \mu(\omega) = \int \langle y, E(\omega) \rangle F(\omega) \mu(\omega) + \int \langle y, E(\omega) \rangle F_1(\omega) \mu(\omega) = y + T_{\theta_F}y
\]
Therefore $T_{\theta_F}y = 0$ for $y \in \mathcal{H}_F$, and thus,
\[
\theta_{F_1} \mid \mathcal{H}_F = 0 \quad \text{and} \quad \mathcal{H} = \mathcal{H}_F + \mathcal{H}_G,
\]
we have $\text{ran} \theta_{F_1} = 0$ and $\mathcal{H} = \mathcal{H}_F + \mathcal{H}_G$, we have $\text{ran} \theta_{F_1} = 0$ and the theorem now follows by Lemma 5. 

**5. Perturbations of Basic Continuous Frames**

The perturbation theorem ([13], Theorem 1.2), as a generalization of Casazza and Christensen’s result, stated a perturbation result for continuous frames. However, the spanning space of basic continuous frames may change when adding new elements. This makes the theorem in [13] no longer applicable to basic continuous frames. So we need to derive a new perturbation condition for basic continuous frames.

We begin with some concepts related to subspaces. Let $\mathcal{U}, \mathcal{V}$ denote closed subspaces of $\mathcal{H}$ and define two angles between $\mathcal{U}$ and $\mathcal{V}$:

\[
S(\mathcal{U}, \mathcal{V}) = \begin{cases} \sup \left\{ \frac{\|P_{\mathcal{U}}x\|}{\|x\|} : x \in \mathcal{U} \setminus \{0\} \right\}, & \text{if } \mathcal{U} \neq \{0\}, \\ 0, & \text{if } \mathcal{U} = \{0\}, \end{cases}
\]

\[
R(\mathcal{U}, \mathcal{V}) = \begin{cases} \inf \left\{ \frac{\|P_{\mathcal{U}}x\|}{\|x\|} : x \in \mathcal{U} \setminus \{0\} \right\}, & \text{if } \mathcal{U} \neq \{0\}, \\ 1, & \text{if } \mathcal{U} = \{0\}. \end{cases}
\]
It is known that
\[ S(\mathcal{U}, \mathcal{V}^*) = (1 - R(\mathcal{U}, \mathcal{V}^*))^{1/2}. \]  
(45)

The following lemma relates the angle to projections.

**Lemma 7** (see [23]). For closed subspaces \( \mathcal{U} \) and \( \mathcal{V} \) of \( \mathcal{H} \) with at least one nontrivial, the following are equivalent.

(i) \( 0 < R(\mathcal{U}, \mathcal{V}) \) and \( 0 < R(\mathcal{V}, \mathcal{U}) \)
(ii) \( 0 < R(\mathcal{U}, \mathcal{V}) = R(\mathcal{V}, \mathcal{U}) = R(\mathcal{U}^2, \mathcal{V}^2) = R(\mathcal{V}^2, \mathcal{U}^2) \)
(iii) \( P_{\mathcal{U}^c} : \mathcal{Y} \rightarrow \mathcal{V} \) is invertible
(iv) \( P_{\mathcal{V}^c} : \mathcal{Y} \rightarrow \mathcal{U} \) is invertible
(v) \( \mathcal{H} = \mathcal{U} + \mathcal{V}^2 \)
(vi) \( \mathcal{H} = \mathcal{U} + \mathcal{V}^2 \)

We also need the following classical fact in operator theory:

\[ \left\| \int_\Omega f(\omega)(F(\omega) - G(\omega))d\mu(\omega) \right\| \leq \lambda_1 \left\| \int_\Omega f(\omega)G(\omega)d\mu(\omega) \right\| + \lambda_2 \left\| \int_\Omega f(\omega)G(\omega)d\mu(\omega) \right\| + \mu \left\| f \right\|, \]
(48)

for all \( f \in L^2(\Omega, \mu) \) with \( \mu(\{ f \neq 0 \}) < \infty \), then \( G \) is a \((\Omega, \mu)\)-Bessel mapping with a Bessel bound
\[ B\left( 1 + \frac{\lambda_1 + \lambda_2 + \mu \sqrt{A}}{1 - \lambda_2} \right)^2. \]
(49)

If furthermore \( \lambda_1 + \mu \sqrt{A} < R(\mathcal{H}_G, \mathcal{H}_F) \), then \( G \) is a basic \((\Omega, \mu)\)-frame with a lower frame bound
\[ A\left( 1 - \frac{\lambda_1 + \lambda_2 + \mu \sqrt{A}}{1 + \lambda_2} \right)^2. \]
(50)

Now we give a condition for the perturbation of basic continuous frames:

**Theorem 3.** Suppose that \( F \) is a basic \((\Omega, \mu)\)-frame for \( \mathcal{H} \) with frame bounds \( A, B \) and that \( G : \Omega \rightarrow \mathcal{H} \) is a vector-valued function. If there exist constants \( \lambda_1, \mu \geq 0 \) and \( \lambda_2 \in (0, 1) \) such that
\[ \left\| Lx - x \right\| \leq \lambda_1 \left\| x \right\| + \lambda_2 \left\| Lx \right\|, \quad \forall x \in \mathcal{H}. \]
(46)

Then \( L \) is bounded and invertible. Moreover, for all \( x \in X \)
\[ \frac{(1 - \lambda_1)}{1 + \lambda_2} \left\| x \right\| \leq \left\| Lx \right\| \leq \frac{(1 + \lambda_1)}{(1 - \lambda_2)} \left\| x \right\|. \]
(47)

Hence,
\[ \| T f \| \leq \left\| T F f \right\| + \left\| T f - T f \right\| \leq (1 + \lambda_1) \left\| T F f \right\| + \lambda_2 \left\| T f \right\| + \mu \left\| f \right\| \]
(53)
\[ \leq (1 + \lambda_1) \sqrt{B} \left\| f \right\| + \lambda_2 \left\| T f \right\| + \mu \left\| f \right\|, \]
which implies that
\[ \| T f \| \leq \left( \frac{(1 + \lambda_1) \sqrt{B} + \mu}{1 - \lambda_2} \right) \left\| f \right\|. \]
(54)

Since \( \mathcal{M} \) is dense in \( L^2(\Omega, \mu) \), we can extend \( T \) uniquely to a bounded operator from \( L^2(\Omega, \mu) \) into \( \mathcal{H} \). Therefore (48) holds for every \( f \in L^2(\Omega, \mu) \) and \( G \) is a \((\Omega, \mu)\)-Bessel mapping with a Bessel bound
\[ \left( \frac{(1 + \lambda_1) \sqrt{B} + \mu}{1 - \lambda_2} \right) = B\left( 1 + \frac{\lambda_1 + \lambda_2 + \mu \sqrt{B}}{1 - \lambda_2} \right)^2. \]
(55)

Since \( G \) is a \((\Omega, \mu)\)-Bessel mapping, one can define an operator \( L \in B(\mathcal{H}) \) by
\[ Lx = \int_\Omega \langle x, S_{\mathcal{H}_F}^* F(\omega)G(\omega)d\mu(\omega) + P_{\mathcal{H}_E}^* P_{\mathcal{H}_F} x \rangle. \]
(56)

Note that \( S_{\mathcal{H}_F}^* F \) is the canonical dual of \( F \) with frame bounds \( 1/B, 1/A \). Thus for any \( y \in \mathcal{H}_F \),
\[ y = \int_{\Omega} \langle y, S'_F(\omega)F(\omega) \rangle F(\omega) d\mu(\omega). \quad (57) \]

Fix \( x \in \mathcal{H} \) and write \( x = x_1 + x_2 \), where \( x_1 \in \mathcal{H}_F \), \( x_2 \in \mathcal{H}_F^\perp \). Using (48) and (57), we obtain

\[
\|x - Lx\| \leq \|x_1 - Lx_1\| + \|x_2 - Lx_2\| \leq \int_{\Omega} \|\langle x_1, S'_F(\omega)F(\omega) - G(\omega) \rangle d\mu(\omega)\| + \left\| (P_{\mathcal{H}_F^\perp} - P_{\mathcal{H}_F^\perp}^*) x_2 \right\|
\]

\[
\leq \lambda_1 \left\| \int_{\Omega} \langle x_1, S'_F(\omega)F(\omega) \rangle d\mu(\omega) \right\| + \lambda_2 \left\| \int_{\Omega} \langle x_1, S'_F(\omega)G(\omega) \rangle d\mu(\omega) \right\|
\]

\[
+ \mu \left( \int_{\Omega} \|\langle x_1, S'_F(\omega)F(\omega) \rangle d\mu(\omega)\|^{1/2} \left\| (I - P_{\mathcal{H}_F^\perp}) P_{\mathcal{H}_F^\perp} x_2 \right\| \right)
\]

\[
\leq \lambda_1 \left\| x_1 \right\| + \lambda_2 \| Lx_1 \| + \mu \sqrt{A} \| x_1 \| + \lambda_1 \left\| x_2 \right\| \leq (\lambda_1 + \mu \sqrt{A}) \| x_1 \| + S(\mathcal{H}_F^\perp, \mathcal{H}_G)^2 \cdot \| x_2 \|^2 + \lambda_2 \| Lx \|
\]

\[
= \left( \lambda_1 + \mu \sqrt{A} \right)^2 + S(\mathcal{H}_F^\perp, \mathcal{H}_G)^2 \cdot \| x \|^2 + \lambda_2 \| Lx \|.
\]  

If furthermore \( \lambda_1 + \mu \sqrt{A} < R(\mathcal{H}_G, \mathcal{H}_F^\perp) \), then by Lemma 7 and (45),

\[
\begin{aligned}
\left( \lambda_1 + \mu \sqrt{A} \right)^2 + S(\mathcal{H}_F^\perp, \mathcal{H}_G)^2 &< R(\mathcal{H}_G, \mathcal{H}_F^\perp)^2 + S(\mathcal{H}_F^\perp, \mathcal{H}_G)^2 \\
&= R(\mathcal{H}_F^\perp, \mathcal{H}_G)^2 + S(\mathcal{H}_F^\perp, \mathcal{H}_G)^2 = 1.
\end{aligned}
\]  

Thus using Lemma 8, \( L \) is invertible and

\[
\|L^{-1}\| \leq \frac{1 + \lambda_2}{\lambda_1 + \mu \sqrt{A} + S(\mathcal{H}_F^\perp, \mathcal{H}_G)^2}.
\]  

Hence, \( L \) maps \( \mathcal{H}_F \) onto \( \mathcal{H}_G \) and \( \mathcal{H}_F^\perp \) onto \( \mathcal{H}_G^\perp \). For any \( x \in \mathcal{H}_G \), it follows that \( L^{-1} x \in \mathcal{H}_F \), and thus \( P_{\mathcal{H}_F^\perp} P_{\mathcal{H}_F^\perp} L^{-1} x = 0 \). So, we can write

\[
x = LL^{-1} x = \int_{\Omega} \langle L^{-1} x, S'_F(\omega) \rangle G(\omega) d\mu(\omega),
\]

which implies that

\[
\int_{\Omega} \|\langle x, G(\omega) \rangle \|^2 d\mu(\omega) \geq A \left( \frac{1 - \sqrt{(\lambda_1 + \mu \sqrt{A})^2 + S(\mathcal{H}_F^\perp, \mathcal{H}_G)^2}}{1 + \lambda_2} \right)^2 \|x\|^2
\]

\[
= A \left( 1 - \lambda_2 + \sqrt{(\lambda_1 + \mu \sqrt{A})^2 + S(\mathcal{H}_F^\perp, \mathcal{H}_G)^2} \right)^2 \|x\|^2,
\]

as required.

\( \square \)
6. Conclusion

In this paper, we investigate continuous frames for their spanning sets and call them basic continuous frames. The use of basic continuous frames allows more freedom when we design the optimal dual frame and the corresponding analysis-synthesis system. We first characterize basic continuous frames and their oblique duals by using operators with closed ranges. Then we show that an oblique dual pair of basic continuous frames for a Hilbert space can be dilated to a Type II dual pair for a larger Hilbert space. Also, we present a condition under which an oblique dual pair of basic continuous frames can be dilated to a dual pair of continuous frames for the same space. Finally, with the help of angles between different spanning subspaces, a perturbation condition for basic continuous frames is given.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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