HOLOMORPHIC STABILITY FOR CARLEMAN PAIRS OF FUNCTION SPACES

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ABSTRACT. We introduce a notion of holomorphic stability for pairs of function spaces on a planar domain $\Omega$. In the case of the open unit disk $\Omega = \mathbb{D}$ equipped with a radial measure $\mu$, by establishing Bourgain-Brezis type inequalities, we show that the pair $$(B^2(\mathbb{D}, \mu), h^1(\mathbb{D}))$$ of weighted harmonic Bergman space and harmonic Hardy space is holomorphically stable if and only if $\mu$ is a $(1, 2)$-Carleson measure. With some extra efforts, we also obtain an analogous result for the upper half plane equipped with horizontal-translation invariant measures.

1. Introduction

1.1. Holomorphic stability for Carleman pairs. Let $\Omega$ be a planar domain and let $\mathcal{O}(\Omega)$ be the space of holomorphic functions on $\Omega$. In what follows, we shall always consider a pair $(X, Y)$ of vector spaces both consisting of functions on $\Omega$.

Definition (Holomorphic stability). The pair $(X, Y)$ is called holomorphically stable if one of the following equivalent conditions is satisfied:

(i) $(X + \mathcal{O}(\Omega)) \cap Y \subset X$;
(ii) $(X + \mathcal{O}(\Omega)) \cap (Y \setminus X) = \emptyset$;
(iii) $(X + Y) \cap \mathcal{O}(\Omega) = X \cap \mathcal{O}(\Omega)$.

Here $X + \mathcal{O}(\Omega) = \{f + g | f \in X, g \in \mathcal{O}(\Omega)\}$ and $X + Y = \{f + g | f \in X, g \in Y\}$.

The Venn diagram in Figure 1 illustrates a holomorphically stable pair $(X, Y)$.

![Venn Diagram](image)

Figure 1. $(X + \mathcal{O}(\Omega)) \cap Y \subset X$.

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For any \( f \in X \) and \( g \in O(\Omega) \), we interpret \( f + g \) as a holomorphic perturbation of \( f \). Then the holomorphic stability of a pair \((X,Y)\) means that a holomorphic perturbation of any element in \( X \) remains in \( X \) provided that it is contained in \( Y \) after the perturbation.

Note that a holomorphically stable pair \((X,Y)\) always satisfies the condition
\[
Y \cap O(\Omega) \subset X.
\]
Such pairs will be referred as Carleman pairs. This terminology comes from the classical Carleman embeddings of some classical holomorphic-function spaces on the open unit disk \( \mathbb{D} \) in the complex plane (see [Car21, Sai79, GK89, Vuk03]). For its generalization to complex domains of arbitrary dimension, see Hörmander [Hor67]. Note that if \( Y \subset X \), then the pair \((X,Y)\) is trivially holomorphically stable. Therefore, the notion of the holomorphic stability is of interests only for the pairs \((X,Y)\) satisfying
\[
Y \not\subset X \quad \text{and} \quad Y \cap O(\Omega) \subset X.
\]

Our research is inspired by Da Lio, Rivièere and Wettstein’s very recent work [DLRW21] on Bourgain-Brezis type inequalities, where they essentially proved that the pair \((B^2(\mathbb{D}), h^1(\mathbb{D}))\) of the harmonic Bergman space \( B^2(\mathbb{D}) \) and the classical harmonic Hardy space \( h^1(\mathbb{D}) \) is holomorphically stable (the precise definitions of \( B^2(\mathbb{D}) \) and \( h^1(\mathbb{D}) \) are given in §1.2). The notion of holomorphic stability leads to many natural questions. Generalization of our work in more general planar domains (including the non-simply connected ones) is more involved and will be given in the sequel to this paper.

1.2. Main results. The harmonic Hardy space \( h^1(\mathbb{D}) \) is defined by
\[
h^1(\mathbb{D}) := \left\{ u \in O_h(\mathbb{D}) \left| \| u \|_{h^1(\mathbb{D})} = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})| d\theta < \infty \right. \right\},
\]
where \( O_h(\mathbb{D}) \) denotes the space of all harmonic functions on \( \mathbb{D} \). The Hardy space \( H^1(\mathbb{D}) \) is then defined as \( H^1(\mathbb{D}) := h^1(\mathbb{D}) \cap O(\mathbb{D}) \).

Throughout the paper, all measures are assumed to be positive measures. Given a measure \( \mu \) on \( \mathbb{D} \), the associated weighted harmonic Bergman space \( B^2(\mathbb{D}, \mu) \) and weighted Bergman space \( A^2(\mathbb{D}, \mu) \) are defined as
\[
B^2(\mathbb{D}, \mu) := L^2(\mathbb{D}, \mu) \cap O_h(\mathbb{D}) \quad \text{and} \quad A^2(\mathbb{D}, \mu) := L^2(\mathbb{D}, \mu) \cap O(\mathbb{D}),
\]
both of which inherit the norm of \( L^2(\mathbb{D}, \mu) \). If \( \mu \) is the Lebesgue measure on \( \mathbb{D} \), then we use the simplified notation \( B^2(\mathbb{D}) \) and \( A^2(\mathbb{D}) \).

A measure \( \mu \) on \( \mathbb{D} \) is called boundary-accessible if its support is not relatively compact in \( \mathbb{D} \). Note that if \( \mu \) is boundary-inaccessible, then it is trivial to verify that the pair \((B^2(\mathbb{D}, \mu), h^1(\mathbb{D}))\) is holomorphically stable. Therefore, in what follows, we always assume that \( \mu \) is boundary-accessible.

A measure \( \mu \) on \( \mathbb{D} \) is called a \((1,2)\)-Carleson measure if there exists a constant \( C > 0 \) such that
\[
(\int_{\mathbb{D}} |f(z)|^2 \mu(dz))^{1/2} \leq C\| f \|_{H^1(\mathbb{D})}, \quad \forall f \in H^1(\mathbb{D}).
\]

**Theorem 1.1.** Let \( \mu \) be a radial boundary-accessible measure on \( \mathbb{D} \). Then the pair
\[
(B^2(\mathbb{D}, \mu), h^1(\mathbb{D}))
\]
is holomorphically stable if and only if \( \mu \) is a \((1, 2)\)-Carleson measure.

A natural conjecture is

**Conjecture.** For any boundary-accessible measure \( \mu \) on \( \mathbb{D} \), the pair \((B^2(\mathbb{D}, \mu), h^1(\mathbb{D}))\) is holomorphically stable if and only if \( \mu \) is a \((1, 2)\)-Carleson measure.

For a finite measure \( \mu \) on \( \mathbb{D} \) which is not necessarily radial, a simple situation (which in general is rather different from the situation in Theorem 1.1, see Theorem 1.2 below for more details) for the holomorphic stability of \((B^2(\mathbb{D}, \mu), h^1(\mathbb{D}))\) is provided as follows. Consider the linear map \( Q_+ \) defined on the space \( \mathcal{O}_h(\mathbb{D}) \) by

\[
Q_+ \left( \sum_{n \geq 0} a_n z^n + \sum_{n \geq 1} b_n \bar{z}^n \right) := \sum_{n \geq 0} a_n z^n.
\]

Then the pair \((B^2(\mathbb{D}, \mu), h^1(\mathbb{D}))\) is holomorphically stable if both

\[
Q_+ : h^1(\mathbb{D}) \rightarrow A^2(\mathbb{D}, \mu)
\]

and

\[
Q_+ : B^2(\mathbb{D}, \mu) \rightarrow A^2(\mathbb{D}, \mu)
\]

are bounded linear operators.

It is not hard to see that the operator (1.3) is bounded if and only if

\[
\sup_{\theta \in [0, 2\pi)} \int_{\mathbb{D}} \frac{\mu(dz)}{|1 - e^{-i\theta}z|^2} < \infty.
\]

Hence the boundedness of the operator (1.3) in general fails even for a radial \((1, 2)\)-Carleson measure. On the other hand, the boundedness of (1.4) holds for all radial finite measure \( \mu \) on \( \mathbb{D} \). For general weights, a clear sufficient condition for the boundedness of the operator (1.4) is that the Bergman projection being bounded on \( L^2(\mathbb{D}, \mu) \) (which then is equivalent to the condition that \( \mu \) is a \( B_2 \)-weight à la Békollé-Bonami, see [BB78] for more details on Bergman projections). Consequently, for any \( B_2 \)-weight \( \mu \) on \( \mathbb{D} \) satisfying (1.5), the pair \((B^2(\mathbb{D}, \mu), h^1(\mathbb{D}))\) is holomorphically stable.

For a radial boundary-accessible finite measure \( \mu \) on \( \mathbb{D} \), the space \( B^2(\mathbb{D}, \mu) \) is complete and \( B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \) is a Banach space equipped with the norm:

\[
\|f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} := \inf \left\{ \|g\|_{B^2(\mathbb{D}, \mu)} + \|h\|_{h^1(\mathbb{D})} \middle| f = g + h, \ g \in B^2(\mathbb{D}, \mu) \text{ and } h \in h^1(\mathbb{D}) \right\}.
\]

Recall that a closed subspace \( B_1 \) of a Banach space \( B \) is called complemented in \( B \) if there exists a bounded linear projection from \( B \) onto \( B_1 \).

**Theorem 1.2.** Let \( \mu \) be a radial boundary-accessible \((1, 2)\)-Carleson measure on \( \mathbb{D} \). Then

\[
(B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})
\]

is a closed subspace of \( B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \). Moreover, the above subspace is complemented in \( B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \) if and only if \( \mu \) satisfies the condition

\[
\int_{\mathbb{D}} \frac{\mu(dz)}{|1 - |z|^2} < \infty.
\]

**Remark.** For radial measures on \( \mathbb{D} \), the conditions (1.5) and (1.6) are clearly equivalent.
In general, the notion of holomorphic stability for the pair \((B^2(\Omega, \mu), h^1(\Omega))\) is not conformally invariant, where \(\Omega\) is assumed to have a nice boundary and \(h^1(\Omega)\) is then defined as the set of all the Poisson convolutions of functions in \(L^1(\partial\Omega, ds)\) (where \(ds\) is the arc-length measure on \(\partial\Omega\)). In particular, our following result for the upper half plane does not seem to be a direct consequence of the result on the unit disk and its proof requires more efforts.

Let \(\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}\) denote the upper half plane. In this case, the suitable spaces for studying the holomorphic stability are the harmonic Zen-type spaces (which reduces to the ordinary harmonic Bergman space when the weight is the Lebesgue measure on \(\mathbb{H}\)), see \([\text{Har}09, \text{JPP}13]\).

A measure \(\mu\) on \(\mathbb{H}\) is called boundary-accessable if its support is not contained in \(H_\varepsilon := \{z \in \mathbb{C} | \Im(z) > \varepsilon\}\) for any \(\varepsilon > 0\). Given a horizontal translation-invariant boundary-accessable measure \(\mu\) on \(\mathbb{H}\), define the harmonic Zen-type space by

\[
B^2(\mathbb{H}, \mu) := \left\{ g \in \mathcal{O}_h(\mathbb{H}) \left| \|g\|_{B^2(\mathbb{H}, \mu)} = \sup_{L > 0} \left( \int_{\mathbb{H}} |g(z + iL)|^2 \mu(dz) \right)^{1/2} < \infty \right. \right\}.
\]

where \(\mathcal{O}_h(\mathbb{H})\) denotes the set of all harmonic functions on \(\mathbb{H}\). It is easy to see that the above space \(B^2(\mathbb{H}, \mu)\) is complete and thus is a Hilbert space.

The relation between \(B^2(\mathbb{H}, \mu)\) and the ordinary weighted harmonic Bergman space \(B^2(\mathbb{H}, \mu) := L^2(\mathbb{H}, \mu) \cap \mathcal{O}_h(\mathbb{H})\) is given as follows. For any \(g \in \mathcal{O}_h(\mathbb{H})\) and any \(y > 0\), define \(g_y : \mathbb{R} \rightarrow \mathbb{C}\) by

\[
g_y(x) := g(x + iy), \quad \forall x \in \mathbb{R}.
\]

Recall the Poisson kernel for \(\mathbb{H}\) at the point \(z = x + iy \in \mathbb{H}\):

\[
P^\mathbb{H}_z(t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}, \quad t \in \mathbb{R}.
\]

Set

\[
\text{Poi}(\mathbb{H}) := \left\{ g \in \mathcal{O}_h(\mathbb{H}) \left| g_y \in L^2(\mathbb{R}) \text{ and } g_{y+y'} = P^\mathbb{H}_{iy'} * g_y \forall y, y' > 0 \right. \right\}.
\]

One can easily check that, for any horizontal translation-invariant boundary-accessable measure \(\mu\) on \(\mathbb{H}\),

\[
B^2(\mathbb{H}, \mu) = B^2(\mathbb{H}, \mu) \cap \text{Poi}(\mathbb{H}).
\]

The harmonic Hardy space \(h^1(\mathbb{H})\) is defined by

\[
h^1(\mathbb{H}) := \left\{ u \in \mathcal{O}_h(\mathbb{H}) \left| \|u\|_{h^1(\mathbb{H})} = \sup_{y > 0} \int_{\mathbb{R}} |u(x + iy)| dx < \infty \right. \right\}
\]

and the Hardy space \(H^1(\mathbb{H})\) is defined as \(H^1(\mathbb{H}) := h^1(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})\). A measure \(\mu\) on \(\mathbb{H}\) is called a \((1,2)\)-Carleson measure if there exists a constant \(C > 0\) such that

\[
\left( \int_{\mathbb{H}} |f(z)|^2 \mu(dz) \right)^{1/2} \leq C \|f\|_{H^1(\mathbb{H})}, \quad \forall f \in H^1(\mathbb{H}).
\]
Theorem 1.3. Let $\mu$ be a horizontal translation-invariant boundary-accessible measure on $\mathbb{H}$. Then the pair

$$(\mathcal{B}^2(\mathbb{H}, \mu), h^1(\mathbb{H}))$$

is holomorphically stable if and only if $\mu$ is a $(1, 2)$-Carleson measure.

1.3. Sketch of the proof. Here we give a sketch of the proof of Theorem 1.1. The proof is based on a generalization of the following one-dimensional Bourgain-Brezis-type inequality due to Da Lio-Rivièr-Wettstein: there exists a universal constant $C > 0$ such that for any smooth function $u \in C^\infty(\mathbb{T})$ with $\int u(e^{i\theta})d\theta = 0$,

$$\|u\|_{L^2(\mathbb{T})} \leq C \left(\|(-\Delta)^{1/4}u\|_{H^{-1/2}(\mathbb{T})+L^1(\mathbb{T})} + \|\mathcal{H}(-\Delta)^{1/4}u\|_{H^{-1/2}(\mathbb{T})+L^1(\mathbb{T})}\right),$$

where the Hilbert transform of $u$ is given by

$$\mathcal{H}u(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \operatorname{sgn}(n)\hat{u}(n)e^{in\theta}$$

and the 1/4-fractional Laplace transform $(-\Delta)^{1/4}u$ is given by

$$(-\Delta)^{1/4}u(e^{i\theta}) = \sum_{n \in \mathbb{Z}} |n|^{1/2}\hat{u}(n)e^{in\theta}.$$

The inequality (1.13) implies

$$\|f\|_{B^2(\mathbb{D})} \leq C(\|f\|_{B^2(\mathbb{D})+h^1(\mathbb{D})} + \|\mathcal{H}f\|_{B^2(\mathbb{D})+h^1(\mathbb{D})}), \quad \forall f \in \mathcal{O}_h(\mathbb{D}) \cap L^\infty(\mathbb{D}),$$

where, slightly by abusing the notation, $\mathcal{H}f$ is defined by

$$\mathcal{H}f(z) = \sum_{n \geq 1} a_n z^n - \sum_{n \geq 1} b_n \bar{z}^n, \quad \text{provided that } f(z) = \sum_{n \geq 0} a_n z^n + \sum_{n \geq 1} b_n \bar{z}^n$$

In our situation, we are able to prove that, if $\mu$ is a radial boundary-accessible measure, then $\mu$ is a $(1, 2)$-Carleson measure on $\mathbb{D}$ if and only if there exists a constant $C_\mu > 0$ such that for any bounded harmonic function $f \in \mathcal{O}_h(\mathbb{D}) \cap L^\infty(\mathbb{D})$,

$$\|f\|_{B^2(\mathbb{D}, \mu)} \leq C_\mu(\|f\|_{B^2(\mathbb{D}, \mu)+h^1(\mathbb{D})} + \|\mathcal{H}f\|_{B^2(\mathbb{D}, \mu)+h^1(\mathbb{D})}).$$

The inequality (1.16) applied to holomorphic functions immediately gives the result stated in Theorem 1.1.

The proof of the inequality (1.16) relies on a weighted version of Bourgain-Brezis-type inequality obtained in Theorem 1.4 below. More precisely, define a Fourier multiplier operator by

$$A_\mu u \sim \sum_{n \in \mathbb{Z}} \left(\int_{\mathbb{D}} |z|^{2|n|} \mu(dz)\right)^{-1/2} \hat{u}(n)e^{in\theta}, \quad u \in C^\infty(\mathbb{T}),$$

where

$$\hat{u}(n) := \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})d\theta, \quad n \in \mathbb{Z}.$$

Define also a Sobolev-type space corresponding to the radial weight $\mu$ by

$$H_\mu(\mathbb{T}) := \{v \sim \sum_{n \in \mathbb{Z}} \hat{v}(n)e^{in\theta} \|v\|_{H_\mu(\mathbb{T})} = \left(\sum_{n \in \mathbb{Z}} |\hat{v}(n)|^2 \int_{\mathbb{D}} |z|^{2|n|} \mu(dz)\right)^{1/2} < \infty\}.$$
Remark. For a general $\mu$, the coefficients of a formal Fourier series $v \in H_\mu(T)$ may have non-polynomial growth and it may not represent a distribution in $D'(T)$. However, if $\mu$ is the Lebesgue measure on $\mathbb{D}$, then $H_\mu(T)$ is the Sobolev space $H^{-1/2}(T) \subset D'(T)$.

**Theorem 1.4.** Let $\mu$ be a radial boundary-accessible finite measure on $\mathbb{D}$. Then $\mu$ is a $(1, 2)$-Carleson measure if and only if there exists a universal constant $C_\mu > 0$ such that for any smooth function $u \in C^\infty(T)$,

$$
\|u\|_{L^2(T)} \leq C_\mu \left(\|A_\mu u\|_{H_\mu(T) + L^1(T)} + \|H A_\mu u\|_{H_\mu(T) + L^1(T)}\right).
$$

(1.19)

2. Preliminaries on Carleson measures

Recall that throughout the paper, all measures are assumed to be positive. We shall use the famous geometric characterization of the $(1, 2)$-Carleson measures on $\mathbb{D}$ defined in (1.1). For any interval $I \subset \mathbb{T}$, the Carleson box $S_I$ is defined by

$$
S_I = \left\{ z \in \mathbb{D} \left| \frac{z}{|z|} \in I, 1 - \frac{|I|}{2\pi} \leq |z| < 1 \right. \right\},
$$

where $|I|$ denotes the arc-length of $I$. Let $|S_I|$ denote the Lebesgue measure of $S_I$, then a measure $\mu$ on $\mathbb{D}$ is a $(1, 2)$-Carleson measure if and only if (see [Car62] and [Dur69])

$$
\sup_{I \text{ is an arc in } \mathbb{T}} \frac{\mu(S_I)}{|S_I|} < \infty.
$$

In particular, we have

**Lemma 2.1.** Let $\mu(dz) = \sigma(dr)d\theta$ be a radial measure on $\mathbb{D}$. Then $\mu$ is a $(1, 2)$-Carleson measure if and only if

$$
\sup_{0 < \delta < 1} \frac{\sigma([1 - \delta, 1])}{\delta} < \infty.
$$

(2.1)

The $(1, 2)$-Carleson measures on the upper half plane $\mathbb{H}$ is defined in (1.12) and its geometric characterization (see, e.g., [Ryd20, Thm. 2.1]) is given as follows: a positive Radon measure $\mu$ on $\mathbb{H}$ is a $(1, 2)$-Carleson measure on $\mathbb{H}$ if and only if

$$
\sup_{I \text{ is an interval in } \mathbb{R}} \frac{\mu(Q_I)}{|Q_I|} < \infty,
$$

where $|Q_I|$ is the Lebesgue measure of the Carleson box $Q_I$ defined by

$$
Q_I = \left\{ z = x + iy \in \mathbb{H} \left| x \in I, 0 < y < |I| \right. \right\},
$$

here $|I|$ denotes the Lebesgue measure of the interval $I \subset \mathbb{R}$. In particular, we have

**Lemma 2.2.** Let $\mu(dz) = dx \Pi(dy)$ be a horizontal translation-invariant measure on $\mathbb{H}$. Then $\mu$ is a $(1, 2)$-Carleson measure if and only if

$$
\sup_{y > 0} \frac{\Pi((0, y))}{y} < \infty.
$$

(2.2)

Remark. While all $(1, 2)$-Carleson measures on $\mathbb{D}$ are finite, a $(1, 2)$-Carleson measure on $\mathbb{H}$ needs not be.
3. Holomorphic stability: the disk case

This section is mainly devoted to proving Theorems 1.1 and 1.2. We shall use the following elementary observation: if \( \mu(dz) = \sigma(dr)d\theta \) is a radial boundary-accessible finite measure on \( \mathbb{D} \), then

- both \( A^2(\mathbb{D}, \mu) \) and \( B^2(\mathbb{D}, \mu) \) are closed in \( L^2(\mathbb{D}, \mu) \);
- for any \( z \in \mathbb{D} \), the evaluation map \( \text{ev}_z : B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \to \mathbb{C} \) defined by

\[
\text{ev}_z(f) = f(z)
\]

is a continuous linear functional on \( B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \);

- for any \( \rho \in (0, 1) \) and any \( k \in \mathbb{N} \),

\[
\sigma_k = \int_0^1 r^{2k}\sigma(dr) \geq \int_{[\rho, 1)} r^{2k}\sigma(dr) \geq \rho^{2k}\sigma([\rho, 1)).
\]

Recall the definition (1.18) of the space \( H_\mu(\mathbb{T}) \): for any \( v \in H_\mu(\mathbb{T}) \), we set

\[
\|v\|_{H_\mu(\mathbb{T})}^2 = \sum_{n \in \mathbb{Z}} |\hat{v}(n)|^2 \int_{\mathbb{D}} |z|^{2|n|} \mu(dz) = 2\pi \sum_{n \in \mathbb{Z}} |\hat{v}(n)|^2\sigma_n.
\]

By (3.2) and (3.3), for any \( r \in [0, 1) \), the Poisson transformation \( P^D_r * v \) of an element \( v \in H_\mu(\mathbb{T}) \) is a smooth function given by

\[
P^D_r * v(e^{i\theta}) = \sum_{n \in \mathbb{Z}} r^{|n|}\hat{v}(n)e^{in\theta}.
\]

Any \( v \in H_\mu(\mathbb{T}) \) has a natural harmonic extension (denoted again by \( v \)) on \( \mathbb{D} \):

\[
v(z) = \sum_{n \in \mathbb{Z}} \hat{v}(n)e_n(z) \quad \text{with} \quad e_n(z) = \begin{cases} z^n & \text{if } n \geq 0 \\ \bar{z}^{|n|} & \text{if } n \leq -1 \end{cases}.
\]

3.1. The derivation of Theorem 1.1 from Theorem 1.4. If \( (B^2(\mathbb{D}, \mu), h^1(\mathbb{D})) \) is a holomorphically stable pair, then we have set-theoretical inclusion

\[
H^1(\mathbb{D}) \subset A^2(\mathbb{D}, \mu).
\]

It follows that \( \mu \) is a finite measure, which, when combined with the assumption of the theorem, implies that \( \mu \) is a radial boundary-accessible finite measure on \( \mathbb{D} \). Therefore, \( A^2(\mathbb{D}, \mu) \) is complete. Hence the embedding \( H^1(\mathbb{D}) \subset A^2(\mathbb{D}, \mu) \) is continuous by the Closed Graph Theorem. In other words, \( \mu \) is a \( (1, 2) \)-Carleson measure on \( \mathbb{D} \).

Now assume that \( \mu \) is a radial boundary-accessible \( (1, 2) \)-Carleson measure \( \mu \) on \( \mathbb{D} \). To prove the holomorphic stability of the pair \( (B^2(\mathbb{D}, \mu), h^1(\mathbb{D})) \), it suffices to show that there exists a constant \( C > 0 \) such that

\[
\|f\|_{A^2(\mathbb{D}, \mu)} \leq C\|f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})}, \quad \forall f \in \mathcal{O}(\mathbb{D}).
\]

Indeed, assuming (3.5) and let \( u \in B^2(\mathbb{D}, \mu), f \in \mathcal{O}(\mathbb{D}) \) with \( u + f \in h^1(\mathbb{D}) \), we obtain

\[
\|f\|_{A^2(\mathbb{D}, \mu)} \leq C\|u - (u + f)\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \leq C\|u\|_{B^2(\mathbb{D}, \mu)} + \|u + f\|_{h^1(\mathbb{D})} < \infty
\]

and hence \( f \in A^2(\mathbb{D}, \mu) \subset B^2(\mathbb{D}, \mu) \). It follows that \( u + f \in B^2(\mathbb{D}, \mu) \) and this gives the holomorphic stability of the pair \( (B^2(\mathbb{D}, \mu), h^1(\mathbb{D})) \).
It remains to prove (3.5). For any \( f \in \mathcal{O}(\mathbb{D}) \) and any \( 0 < r < 1 \), write \( f_r(z) := f(rz) \). Then, since \( \mu \) is radial,

\[
\lim_{r \to 1^-} \|f_r\|_{A^2(\mathbb{D}, \mu)} = \|f\|_{A^2(\mathbb{D}, \mu)}
\]

and

\[
\limsup_{r \to 1^-} \|f_r\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \leq \|f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})}.
\]

Therefore, it suffices to show that (3.5) holds for all \( f \) belonging to the following class:

\[
\mathcal{O}(\mathbb{D}) = \{ f | f \text{ is holomorphic in a neighborhood of } \mathbb{D} \}.
\]

We now proceed to the derivation of the inequality (3.5) for any \( f \in \mathcal{O}(\mathbb{D}) \) from the inequality (1.19) obtained in Theorem 1.4. Observe that, any \( v \in B^2(\mathbb{D}, \mu) \) has the form

\[
v(z) = \sum_{n \in \mathbb{Z}} a_n e_n(z),
\]

where \( e_n \) is defined as in (3.4). Then, by the radial assumption on \( \mu \),

\[
(3.6) \quad \|v\|_{B^2(\mathbb{D}, \mu)}^2 = \sum_{n \in \mathbb{Z}} |a_n|^2 \int_{\mathbb{D}} |z|^{2|n|} \mu(dz).
\]

Comparing (3.6) and (1.18), we obtain a natural isometric isomorphism \( H_\mu(\mathbb{T}) \to B^2(\mathbb{D}, \mu) \) that associates any \( v \in H_\mu(\mathbb{T}) \) to its harmonic extension in \( \mathbb{D} \) defined by (3.4). Similarly, there is a natural isometric isomorphism \( L^1(\mathbb{T}) \to h^1(\mathbb{D}) \). Therefore, we get a natural identification of the Banach spaces

\[
H_\mu(\mathbb{T}) + L^1(\mathbb{T}) \simeq B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}).
\]

Hence, for any \( f \in \mathcal{O}(\mathbb{D}) \), by taking \( u = f|_\mathbb{T} \in C^\infty(\mathbb{T}) \) in (1.19), we obtain

\[
\|A_\mu^{-1}(f|_\mathbb{T})\|_{L^2(\mathbb{T})} \leq C_\mu(\|f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} + \|\mathcal{H}f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})}),
\]

with \( A_\mu \) defined in (1.17), \( \mathcal{H}f \) defined in (1.15). Since \( \mathcal{H}f = f - f(0) \) for all \( f \in \mathcal{O}(\mathbb{D}) \) and \( |f(0)| \leq \|f\|_{h^1(\mathbb{D})} \leq \|f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \), we get

\[
\|A_\mu^{-1}(f|_\mathbb{T})\|_{L^2(\mathbb{T})} \leq 3C_\mu \|f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})}, \quad \forall f \in \mathcal{O}(\mathbb{D}).
\]

Finally, notice that, for any \( f = \sum_{n \geq 0} c_n z^n \in \mathcal{O}(\mathbb{D}) \),

\[
\|A_\mu^{-1}(f|_\mathbb{T})\|_{L^2(\mathbb{T})}^2 = \sum_{n \geq 0} |c_n|^2 \int_{\mathbb{D}} |z|^{2|n|} \mu(dz) = \|f\|^2_{A^2(\mathbb{D}, \mu)}.
\]

Thus we obtain the desired inequality (3.5) for all \( f \in \mathcal{O}(\mathbb{D}) \) and complete the derivation of Theorem 1.1 from Theorem 1.4.

3.2. The proof of Theorem 1.4. A pair \((a, b)\) of sequences \( a = (a(n))_{n \in \mathbb{Z}} \) and \( b = (b(n))_{n \in \mathbb{Z}} \) is called \( \mu \)-adapted if the following conditions are satisfied:

(i) \( a(0) \neq 0 \);

(ii) for any \( n \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\} \),

\[
(3.7) \quad |a(n)|^2 b(n) \text{sgn}(n) = \sigma_n^{-1}, \quad \text{where } \sigma_n := \int_0^1 r^{2|n|} \sigma(dr) > 0;
\]
Hence, there exists a constant $C_b$ such that

\begin{equation}
0 < \frac{1}{C_b} \leq |b(n)| \leq C_b, \quad n \in \mathbb{Z}^*.
\end{equation}

Note that the condition (3.7) implies in particular that $b(n) \in \mathbb{R}$ for all $n \in \mathbb{Z}^*$.

**Proposition 3.1.** Suppose $\mu$ is a radial boundary-accessible $(1,2)$-Carleson measure on $\mathbb{D}$. Let $(a, b)$ be a $\mu$-adapted pair of sequences. Then there exists a constant $C$ such that

\begin{equation}
\|u\|_{L^2(\mathbb{T})} \leq C(\|\mathcal{T}_a u\|_{H_\mu(\mathbb{T})} + \|\mathcal{T}_b u\|_{H_\mu(\mathbb{T})} + \|\mathcal{T}_a u\|_{H_\mu(\mathbb{T})} + \|\mathcal{T}_b u\|_{H_\mu(\mathbb{T})}), \quad \forall u \in C^\infty(\mathbb{T}),
\end{equation}

where $\mathcal{T}_a$ and $\mathcal{T}_b$ are the Fourier multipliers defined by

$$
\mathcal{T}_a u(n) = a(n) \widehat{u}(n) \quad \text{and} \quad \mathcal{T}_b u(n) = b(n) \widehat{u}(n), \quad n \in \mathbb{Z}.
$$

The next criterion of radial $(1,2)$-Carleson measures will be useful for us.

**Lemma 3.2.** Let $\alpha(dr)$ be a finite measure on $[0, 1)$. Then the inequality

\begin{equation}
\sup_{0 < \delta < 1} \frac{\alpha([1 - \delta, 1])}{\delta} < \infty.
\end{equation}

We postpone the proof of Lemma 3.2 for a while and proceed to the proof of Theorem 1.4.

**Lemma 3.3.** Let $\mu(dz) = \sigma(dr) d\theta$ be a radial boundary-accessible $(1,2)$-Carleson measure on $\mathbb{D}$, then there exists a function $w_\sigma \in L^\infty(\mathbb{T})$ such that

\begin{equation}
\widehat{w}_\sigma(n) = \text{sgn}(n) \sigma_n, \quad n \in \mathbb{Z}.
\end{equation}

**Proof.** Under the assumption of the lemma, set

\begin{equation}
w_\sigma(e^{i\theta}) = 2i \int_0^1 \frac{r^2 \sin \theta}{\sqrt{r^2 - e^{-i\theta}^2}} \sigma(dr).
\end{equation}

We first show that $w_\sigma \in L^\infty(\mathbb{T})$. Indeed, by change-of-variable $r = \sqrt{s}$,

$$w_\sigma(e^{i\theta}) = 2i \int_0^1 \frac{\sin \theta}{(\cos \theta - s)^2 + \sin^2 \theta} \sigma'(ds),$$

where $\sigma'(ds) = s \sigma_s(ds)$ with $\sigma_s(ds)$ being the push-forward of the measure $\sigma$ under the map $s = r^2$. By Lemma 2.1, there exists a constant $C > 0$ such that

$$\sigma([1 - \delta, 1]) \leq C\delta, \quad \forall \delta \in (0, 1).$$

Then, by the definition of $\sigma'$, there exists a constant $C' > 0$ such that

$$\sigma'(1 - \delta, 1) \leq C'\delta, \quad \forall \delta \in (0, 1).$$

Hence, $w_\sigma \in L^\infty(\mathbb{T})$ by Lemma 3.2.

It remains to prove the equality (3.11). Since $w_\sigma \in L^\infty(\mathbb{T})$, we have

\begin{equation}
\sup_{\theta \in [0, 2\pi]} \int_0^1 \frac{r^2 \sin \theta}{\sqrt{r^2 - e^{-i\theta}^2}} \sigma(dr) = \sup_{\theta \in [0, 2\pi]} \left| \int_0^1 \frac{r^2 \sin \theta}{\sqrt{r^2 - e^{-i\theta}^2}} \sigma(dr) \right| < \infty.
\end{equation}
Therefore, for any $n \in \mathbb{Z}$, by Fubini’s Theorem,
\[
\hat{w}_\sigma(n) = \frac{1}{2\pi} \int_0^{2\pi} \left( 2i \int_0^1 \frac{r^2 \sin \theta}{|r^2 - e^{-i\theta}|^2} \sigma(dr) \right) e^{-in\theta} d\theta \\
= \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 \sin \theta}{|r^2 - e^{-i\theta}|^2} e^{-in\theta} d\theta \right) \sigma(dr).
\]
Then by the elementary identity (which converges absolutely for any fixed $0 \leq r < 1$)
\[
2i \frac{r^2 \sin \theta}{|r^2 - e^{-i\theta}|^2} = \sum_{n \in \mathbb{Z}} \text{sgn}(n) \mathcal{F}^{-1} r^{2|n|}, \quad \forall r \in [0, 1),
\]
we have
\[
\frac{1}{2\pi} \int_0^{2\pi} 2i \frac{r^2 \sin \theta}{|r^2 - e^{-i\theta}|^2} e^{-in\theta} d\theta = \text{sgn}(n) r^{2|n|}
\]
and hence
\[
\hat{w}_\sigma(n) = \text{sgn}(n) \int_0^1 r^{2|n|} \sigma(dr) = \text{sgn}(n) \sigma_n.
\]
This is the desired equality (3.11). \hfill \Box

**Proof of Proposition 3.1.** Let $u \in C^\infty(\mathbb{T})$. Note that $\| u \|_{L^2(\mathbb{T})} \leq \| u - \hat{u}(0) \|_{L^2(\mathbb{T})} + \| \hat{u}(0) \|$. Clearly, if $T_\alpha u = f + g$ with $f \in H_\mu(\mathbb{T})$, $g \in L^1(\mathbb{T})$, then $a(0) \hat{u}(0) = \hat{f}(0) + \hat{g}(0)$ and $|a(0) \hat{u}(0)| \leq |\hat{f}(0)| + |\hat{g}(0)| \leq \| f \|_{H_\mu(\mathbb{T})} + \| g \|_{L^1(\mathbb{T})}$.

It follows that $|\hat{u}(0)| \leq |a(0)|^{-1} \| T_\alpha u \|_{H_\mu(\mathbb{T}) + L^1(\mathbb{T})}$. Therefore, from now on, we may assume that $\hat{u}(0) = 0$. Take any pairs of decompositions
\[
(3.13) \quad \begin{cases} 
T_\alpha u &= f_1 + g_1, \\
T_\beta T_\alpha u &= f_2 + g_2,
\end{cases}
\]
with $f_1, f_2 \in H_\mu(\mathbb{T})$ and $g_1, g_2 \in L^1(\mathbb{T})$. Then for any $0 < r < 1$, we have (the following Poisson convolutions will be used in the proof of the equality (3.21) below)
\[
(3.14) \quad \begin{cases} 
P_r^D \ast (T_\alpha u) &= P_r^D \ast f_1 + P_r^D \ast g_1, \\
P_r^D \ast (T_\beta T_\alpha u) &= P_r^D \ast f_2 + P_r^D \ast g_2.
\end{cases}
\]
That is,
\[
(3.15) \quad \begin{cases} 
r^{2|n|} a(n) \hat{u}(n) &= r^{2|n|} \hat{f}_1(n) + r^{2|n|} \hat{g}_1(n), \quad n \in \mathbb{Z}, \\
r^{2|n|} b(n) a(n) \hat{u}(n) &= r^{2|n|} \hat{f}_2(n) + r^{2|n|} \hat{g}_2(n), \quad n \in \mathbb{Z}.
\end{cases}
\]
From (3.15), we have
\[
(3.16) \quad \| P_r^D \ast u \|_{L^2(\mathbb{T})}^2 = \sum_{n \in \mathbb{Z}^*} r^{2|n|} |\hat{u}(n)|^2 = \sum_{n \in \mathbb{Z}^*} \frac{r^{2|n|}}{a(n)} \hat{f}_1(n) r^{2|n|} \hat{u}(n) + \sum_{n \in \mathbb{Z}^*} \frac{r^{2|n|}}{a(n)} \hat{g}_1(n) r^{2|n|} \hat{u}(n).
\]
By Cauchy-Schwarz’s inequality,
\begin{equation}
|I| \leq \left( \sum_{n \in \mathbb{Z}^*} r^{2|n|} |\tilde{u}(n)|^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}^*} r^{2|n|} |\hat{f}_1(n)|^2 \right)^{1/2}
\leq \sqrt{C_b/2\pi} \|P_r^B * u\|_{L^2(\mathbb{T})} \|P_r^D * f_1\|_{H_{\mu}(\mathbb{T})},
\end{equation}
where we used the fact that if \((a, b)\) is a \(\mu\)-adapted pair of sequences, then by (3.7) and (3.8), for any \(v \in H_{\mu}(\mathbb{T})\),
\begin{equation}
(\sum_{n \in \mathbb{Z}^*} \frac{|\tilde{v}(n)|^2}{|a(n)|^2})^{1/2} = (\sum_{n \in \mathbb{Z}^*} |\tilde{v}(n)|^2 |b(n)| \sigma_n)^{1/2} \leq \sqrt{C_b/2\pi} \|v\|_{H_{\mu}(\mathbb{T})}.
\end{equation}
By (3.15),
\begin{equation}
III = \sum_{n \in \mathbb{Z}^*} \frac{r^{2|n|}(a(n)\tilde{u}(n) - \hat{f}_1(n))}{|a(n)|^2 b(n)} f_2(n) + \sum_{n \in \mathbb{Z}^*} \frac{r^{2|n|}|\hat{g}_1(n)r^{2|n|}\tilde{g}_2(n)|}{|a(n)|^2 b(n)} \text{ denoted by IV}.
\end{equation}
Then by Cauchy-Schwarz’s inequality,
\begin{align*}
|III| & \leq \left( \sum_{n \in \mathbb{Z}^*} \frac{r^{2|n|}|\tilde{u}(n)|}{|a(n)b(n)|} r^{2|n|} f_2(n) \right) + \left( \sum_{n \in \mathbb{Z}^*} \frac{r^{2|n|}|\hat{f}_1(n)|}{|a(n)|^2 b(n)} r^{2|n|} f_2(n) \right) \\
& \leq \|P_r^B * u\|_{L^2(\mathbb{T})} \left( \sum_{n \in \mathbb{Z}^*} \frac{r^{2|n|} f_2(n)}{|a(n)|^2 b(n)} \right)^{1/2} + \left( \sum_{n \in \mathbb{Z}^*} \frac{r^{2|n|}|\hat{f}_1(n)|^2}{|a(n)|^2 b(n)} \right)^{1/2} \left( \sum_{n \in \mathbb{Z}^*} \frac{r^{2|n|}|\hat{f}_2(n)|^2}{|a(n)|^2 b(n)} \right)^{1/2}.
\end{align*}
Using similar inequality as (3.18), under the conditions (3.7) and (3.8), we have
\begin{equation}
|III| \leq \sqrt{C_b/2\pi} \|P_r^B * u\|_{L^2(\mathbb{T})} \|P_r^D * f_2\|_{H_{\mu}(\mathbb{T})} + \frac{1}{2\pi} \|P_r^D * f_1\|_{H_{\mu}(\mathbb{T})} \|P_r^D * f_2\|_{H_{\mu}(\mathbb{T})}.
\end{equation}
We now proceed to the estimate of term IV in the decomposition (3.19). Note that
\begin{equation}
\tilde{g}_2(n) = \tilde{\gamma}_2(n), \text{ where } \tilde{\gamma}_2(e^{i\theta}) := g_2(e^{-i\theta}).
\end{equation}
For any \(0 < r < 1\), set
\begin{equation}
h_r = (P_r^B * g_1) * (P_r^D * \tilde{\gamma}_2).
\end{equation}
A priori, we only have \(g_1 * \tilde{\gamma}_2 \in L^1(\mathbb{T})\), but \(h_r \in L^2(\mathbb{T})\) for any \(0 < r < 1\). By (3.7),
\begin{equation}
IV = \sum_{n \in \mathbb{Z}^*} \text{sgn}(n) \sigma_n r^{2|n|} \tilde{\gamma}_1(n) r^{2|n|} \tilde{\gamma}_2(n) = \sum_{n \in \mathbb{Z}^*} \hat{h}_r(n) \text{sgn}(n) \sigma_n.
\end{equation}
By Lemma 3.3, \(w_\sigma \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})\). Then the Plancherel’s identity implies
\begin{equation}
|IV| \leq \frac{1}{2} \|h_r\|_{L^1(\mathbb{T})} \|w_\sigma\|_{L^\infty(\mathbb{T})}.
\end{equation}
By \((3.16), (3.17), (3.19), (3.20)\) and \((3.22)\), there is a constant \(C = C(a, b, \mu)\), depending only on \((a, b)\) and the measure \(\mu\) but not on \(r \in (0, 1)\), such that

\[
\|P^D_r * u\|_{L^2(T)}^2 \leq C \|P^D_r * f_1\|_{L^2(T)} \left( \|P^D_r * f_1\|_{L^2(T)}^2 + C \|P^D_r * u\|_{L^2(T)} \right) + C \|P^D_r * f_1\|_{H_\mu(T)} \left( \|P^D_r * f_1\|_{H_\mu(T)} + C \|P^D_r * f_2\|_{H_\mu(T)} \right) + C \|P^D_r * f_1\|_{H_\mu(T)} \left( \|P^D_r * f_1\|_{H_\mu(T)} + C \|P^D_r * f_2\|_{H_\mu(T)} \right).
\]

Therefore, by a standard argument, there is a constant \(C' = C'(a, b, \mu)\) such that

\[
\|P^D_r * u\|_{L^2(T)} \leq C' \left( \|P^D_r * f_1\|_{H_\mu(T)} + \|P^D_r * f_2\|_{H_\mu(T)} + \|P^D_r * f_1\|_{L^1(T)} + \|P^D_r * f_2\|_{L^1(T)} \right).
\]

where the last inequality is due to the contractive property of the Poisson convolution on both \(H_\mu(T)\) and \(L^1(T)\). Let \(r\) approach 1, then

\[
\|u\|_{L^2(T)} \leq C' \left( \|f_1\|_{H_\mu(T)} + \|g_1\|_{L^1(T)} + \|f_2\|_{H_\mu(T)} + \|g_2\|_{L^1(T)} \right).
\]

Since the decompositions \((3.13)\) are arbitrary, we obtain the desired inequality \((3.9)\). \(\square\)

**Proof of Theorem 1.4.** If \(\mu = \sigma(dr) d\theta\) is a radial boundary-accessible \((1, 2)\)-Carleson measure on \(\mathbb{D}\), then we obtain the inequality \((1.19)\) from Proposition 3.1 by taking

\[
a(n) = \left( \int_{\mathbb{D}} |z|^{2|n|} \mu(dz) \right)^{-1/2} = \frac{1}{\sqrt{2\pi n}} \quad \text{and} \quad b(n) = \text{sgn}(n).
\]

Conversely, if the inequality \((1.19)\) holds, then by the argument in the first two paragraphs of \(\S\,3.1\), the measure \(\mu\) is a \((1, 2)\)-Carleson measure on \(\mathbb{D}\). \(\square\)

It remains to prove Lemma 3.2. We shall apply a result due to Garnett about the boundary behavior of Poisson integrals on the upper half plane \(\mathbb{H}\).

**Lemma 3.4** (Garnett, see, e.g., [RUS8, pp. 210]). Let \(\nu\) be a measure on \(\mathbb{R}\) with \(\int_{\mathbb{R}} \frac{1}{1+\tau^2} \nu(dt) < \infty\). Then the following two assertions are equivalent:

1. \(\sup_{y>0} \int_{\mathbb{R}} \frac{y}{t^2+y^2} \nu(dt) < \infty\)
2. \(\sup_{L>0} \frac{\nu([-L,L])}{2L} < \infty\).

**Proof of Lemma 3.2.** Note that

\[
\sup_{\theta \in [0,2\pi)} \left| \int_0^1 \frac{\sin \theta}{(r - \cos \theta)^2 + \sin^2 \theta} \alpha(dr) \right| = \sup_{\theta \in (0,\pi)} \int_0^1 \frac{\sin \theta}{(r - \cos \theta)^2 + \sin^2 \theta} \alpha(dr).
\]

For any \(\theta \in (0, \pi)\), consider the point \(z = e^{i\theta} = \cos \theta + i \sin \theta\) and recall the Poisson kernel \(P^\mathbb{H}_z\) at the point \(z \in \mathbb{H}\) given in \((1.9)\), then

\[
\int_0^1 \frac{\sin \theta}{(r - \cos \theta)^2 + \sin^2 \theta} \sigma(dr) = \pi \int_{\mathbb{R}} P^\mathbb{H}_z(t) 1_{[0,1]}(t) \sigma(dt).
\]

Consider the Möbius transformation \(\phi\) defined by \(\phi(z) = \frac{z-1}{z+1}\). Then \(\phi\) is an automorphism of the upper half plane and

\[
P^\mathbb{H}_{\phi(z)}(\phi(t)) |\phi'(t)| = P_z^\mathbb{H}(t), \quad z \in \mathbb{H}, \; t \in \mathbb{R} \setminus \{-1\}.
\]
Note that when $\theta$ ranges over $(0, \pi)$, the image $\phi(e^{i\theta})$ ranges over $i\mathbb{R}_+$. Therefore,

$$\sup_{\theta \in (0, \pi)} \int_{\mathbb{R}} P^\mathbb{H}_{e^{i\theta}}(t)1_{[0,1]}(t)\sigma(dt) = \sup_{\theta \in (0, \pi)} \int_{\mathbb{R}} P^\mathbb{H}_{\phi(e^{i\theta})}(\phi(t))|\phi'(t)|1_{[0,1]}(t)\sigma(dt) = \sup_{y > 0} \int_{0}^{1} P^\mathbb{H}_{iy}(\phi(t))|\phi'(t)|\sigma(dt).$$

By change-of-variable $s = \phi(t),$

$$\int_{0}^{1} P^\mathbb{H}_{iy}(\phi(t))|\phi'(t)|\sigma(dt) = \int_{-1}^{0} P^\mathbb{H}_{iy}(s)(1-s)^2\sigma \circ \phi^{-1}(ds).$$

Then

$$\sup_{\theta \in (0, \pi)} \int_{0}^{1} \frac{\sin \theta}{(r - \cos \theta)^2 + \sin^2 \theta} \sigma(dr) = \frac{\pi}{2} \sup_{y > 0} \int_{\mathbb{R}} \frac{y}{y^2 + s^2} \tilde{\sigma}(ds),$$

where $\tilde{\sigma}(ds) = (1-s)^21_{(-1,0)}(s)\sigma \circ \phi^{-1}(ds)$. Clearly, $\int_{\mathbb{R}} \frac{\tilde{\sigma}(ds)}{1+s^2} < \infty$. Therefore, by Lemma 3.4, the inequality (3.10) holds if and only if

$$(3.23) \quad \sup_{L > 0} \frac{\tilde{\sigma}([-L,L])}{L} < \infty.$$ 

By the definition of $\tilde{\sigma}$, it is easy to see that $\frac{1}{2} \sigma(I_L) \leq \tilde{\sigma}([-L,L]) \leq 2\sigma(I_L)$, where $I_L$ is the open interval

$$I_L := \left(\frac{1 - \min(L,1)}{1 + \min(L,1)}, 1\right) \subset (0,1).$$

It follows that, (3.23) holds if and only if $\sup_{L > 0} \frac{\sigma(I_L)}{L} < \infty$, which in turn is equivalent to

$$\sup_{0 < \delta < 1} \frac{\sigma([1-\delta,1])}{\delta} < \infty.$$ 

By Lemma 2.1, the above inequality holds if and only if $\mu(dz) = \sigma(dr)d\theta$ is a $(1,2)$-Carleson measure on $\mathbb{D}$. This completes the whole proof. \qed

3.3. **Proof of Theorem 1.2.** Fix a radial boundary-accessible $(1,2)$-Carleson measure $\mu(dz) = \sigma(dr)d\theta$. By Theorem 1.1,

$$(3.24) \quad (B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D}) = B^2(\mathbb{D}, \mu) \cap \mathcal{O}(\mathbb{D}) = A^2(\mathbb{D}, \mu).$$

By (3.5), there exists a constant $C = C_\mu > 0$, such that for all $f \in \mathcal{O}(\mathbb{D}),$

$$(3.25) \quad \frac{1}{C} \|f\|_{A^2(\mathbb{D}, \mu)} \leq \|f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \leq \|f\|_{B^2(\mathbb{D}, \mu)} = \|f\|_{A^2(\mathbb{D}, \mu)}.$$ 

That is, the identity map

$$id : A^2(\mathbb{D}, \mu) \to (B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D}) \subset B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})$$

is an isomorphic isomorphism. Thus $(B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})$ is a closed subspace of $B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})$. 

Now suppose that the extra condition (1.6) is satisfied. We are going to show that the closed subspace \((B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})\) is complemented in \(B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})\). Indeed, under the condition (1.6), for any \(\theta \in [0, 2\pi)\), the following holomorphic function
\[
   k_\theta(z) := \frac{1}{1 - e^{-i\theta}z} = \sum_{n \geq 0} e^{-i\theta}z^n, \quad z \in \mathbb{D}
\]
belongs to \(B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})\) and
\[
   M_\mu := \sup_{\theta \in [0, 2\pi)} \|k_\theta\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \leq \sup_{\theta \in [0, 2\pi)} \|k_\theta\|_{A^2(\mathbb{D}, \mu)} = \left( \int_{\partial \mathbb{D}} \frac{\mu(dz)}{1 - |z|^2} \right)^{1/2} < \infty.
\]
Recall the definition (1.2) of \(Q_+\). Clearly, since \(\mu\) is radial, \(Q_+\) defines an orthogonal projection from \(B^2(\mathbb{D}, \mu)\) onto \(A^2(\mathbb{D}, \mu)\). Then
\[
   \|Q_+(u)\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \leq \|Q_+(u)\|_{B^2(\mathbb{D}, \mu)} \leq \|u\|_{B^2(\mathbb{D}, \mu)}, \quad \forall u \in B^2(\mathbb{D}, \mu).
\]
Note also that if \(v = \sum_{n \in \mathbb{Z}} a_n e_n \in h^1(\mathbb{D})\), that is,
\[
   \tilde{v} := \sum_{n \in \mathbb{Z}} a_n e^{i\theta} \in L^1(\mathbb{T}) \text{ and } \|v\|_{h^1(\mathbb{D})} = \|\tilde{v}\|_{L^1(\mathbb{T})},
\]
then it is easy to see that
\[
   Q_+(v) = Q_+\left( \sum_{n \in \mathbb{Z}} a_n e_n \right) = \frac{1}{2\pi} \int_0^{2\pi} k_\theta(1\theta) d\theta.
\]
Hence, by (3.26),
\[
   \|Q_+(v)\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \leq \frac{1}{2\pi} \int_0^{2\pi} \|k_\theta\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} |\tilde{v}(1\theta)| d\theta
\]
\[
   \leq M_\mu \|\tilde{v}\|_{L^1(\mathbb{T})} = M_\mu \|v\|_{h^1(\mathbb{D})}.
\]
By (3.27) and (3.28) and the definition of the norm on \(B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})\),
\[
   \|Q_+(f)\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \leq M_\mu \|f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})}, \quad \forall f \in B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}).
\]
It follows that \(Q_+\) defines a bounded linear projection from \(B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})\) onto \((B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})\).

Hence \((B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})\) is complemented in \(B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})\).

Finally, assume that the condition (1.6) is not satisfied. Then
\[
   \sum_{n \geq 0} \sigma_n = \sum_{n \geq 0} \int_0^1 r^{2n} \sigma(dr) = \int_0^1 \sigma(dr) - \frac{1}{2\pi} \int_\mathbb{D} \frac{\mu(dz)}{1 - |z|^2} = \infty.
\]
Let us show that \((B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})\) is not complemented in \(B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})\). Otherwise, there exists a bounded linear projection operator
\[
   P : B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \rightarrow B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})
\]
on onto the closed subspace \((B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})\). That is,

- \(P \circ P = P\),
- \(P(f) = f\) for all \(f \in (B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})\),
- \(P(g) \in (B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})\) for all \(g \in B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})\).
Since \( \mu \) is radial, for any \( \theta \in [0, 2\pi) \), the rotation map \( \tau_\theta \) defined by \( \tau_\theta(f)(z) = f(e^{i\theta}z) \) preserves both the norms of functions in \( B^2(\mathbb{D}, \mu) \) and the norms of functions in \( h^1(\mathbb{D}) \). Therefore, \( \tau_\theta \) preserves the norms of functions in \( B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \):

\[
\|\tau_\theta(f)\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} = \|f\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})}, \quad \forall f \in B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}).
\]

Consequently, the operator-norm of the composition operator

\[
P_\theta = \tau_\theta \circ P \circ \tau_\theta : B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \rightarrow B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})
\]

is bounded by that of \( P \):

\[
\|P_\theta\| \leq \|P\|.
\]

It can be easily checked that \( P_\theta \) is also a projection operator from \( B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \) onto \( (B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D}) \).

Define a bounded linear operator \( \mathcal{P} : B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \rightarrow B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \) via the Bochner integral (see, e.g., [Yos95, Section V.5])

\[
\mathcal{P} := \frac{1}{2\pi} \int_0^{2\pi} P_\theta d\theta.
\]

Then

\[
(3.29) \quad \|\mathcal{P}\| \leq \sup_{\theta \in [0, 2\pi)} \|P_\theta\| = \|P\| < \infty
\]

and

\[
(3.30) \quad \mathcal{P}(f) = \frac{1}{2\pi} \int_0^{2\pi} P_\theta(f) d\theta, \quad \forall f \in B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}).
\]

Since the evaluation map \( ev_z \) defined in (3.1) is a continuous linear functional on \( B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}) \) for any \( z \in \mathbb{D} \),

\[
(3.31) \quad [\mathcal{P}(f)](z) = \frac{1}{2\pi} \int_0^{2\pi} [P_\theta(f)](z) d\theta, \quad \forall f \in B^2(\mathbb{D}, \mu) + h^1(\mathbb{D}).
\]

Note that \( P_\theta(f) = f \) for any \( f \in (B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D}) \) and any \( \theta \in [0, 2\pi) \), thus

\[
\mathcal{P}(f) = f, \quad \forall f \in (B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D}).
\]

On the other hand, for any integer \( n \geq 1 \),

\[
(3.32) \quad \mathcal{P}(e_{-n}) = 0, \quad \forall n \geq 1.
\]

Indeed, for any \( \theta \in [0, 2\pi) \),

\[
(\tau_\theta(e_{-n}))(z) = (e^{i\theta}z)^n = e^{-in\theta}z^n = e^{-in\theta}e_{-n}(z).
\]

Thus \( P \circ \tau_\theta(e_{-n}) = e^{-in\theta}P(e_{-n}) \). By (3.24),

\[
P(e_{-n}) \in (B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D}) = A^2(\mathbb{D}, \mu),
\]

we can write

\[
P(e_{-n})(z) = \sum_{k=0}^{\infty} c^{(n)}_k z^k \in A^2(\mathbb{D}, \mu),
\]

with

\[
(3.33) \quad \|P(e_{-n})\|^2_{A^2(\mathbb{D}, \mu)} = 2\pi \sum_{k=0}^{\infty} |c^{(n)}_k|^2 \sigma_k < \infty.
\]
Thus, for all $z \in \mathbb{D}$,
\[
P_\theta(e^{-n})(z) = [\tau_\theta(e^{-i\theta} P(e^{-n}))(z) = e^{-in\theta}[\tau_\theta(P(e^{-n}))(z)
\]
\[
e^{-in\theta} \sum_{k=0}^{\infty} c_k^{(n)} (e^{-i\theta}z)^k = \sum_{k=0}^{\infty} c_k^{(n)} e^{-i(k+n)\theta} z^k,
\]
where the last series converges absolutely by the inequalities (3.2), (3.33) and
\[
\left(\sum_{k=0}^{\infty} |c_k^{(n)} z^k|^2\right)^{\frac{1}{2}} \leq \sum_{k=0}^{\infty} |c_k^{(n)}|^2 \frac{\sigma_k}{\sigma_k} \sum_{k=0}^{\infty} |z|^{2k} \leq \sum_{k=0}^{\infty} |c_k^{(n)}|^2 \sigma_k \sum_{k=0}^{\infty} \frac{|z|^{2k}}{\rho^{2k} \sigma'(\rho, 1)}, \quad \forall \rho \in (0, 1).
\]
Therefore, by (3.31), for all $z \in \mathbb{D}$,
\[
[P(e^{-n})](z) = \int_0^{2\pi} \sum_{k=0}^{\infty} c_k^{(n)} e^{-i(k+n)\theta} z^k \frac{d\theta}{2\pi} = \sum_{k=0}^{\infty} c_k^{(n)} e^{-i(k+n)\theta} z^k \frac{d\theta}{2\pi} = 0.
\]
This is the desired equality (3.32).

However, if we take the harmonic extension of the Féjer kernel on $\mathbb{D}$:
\[
\mathcal{F}_N(z) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N}\right) e_j(z), \quad N \geq 1,
\]
then
\[
(3.34) \quad \|\mathcal{F}_N\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \leq \|\mathcal{F}_N\|_{h^1(\mathbb{D})} = \|\mathcal{F}_N\|_{L^1(T)} = 1.
\]
Since $\mathcal{P}$ is a projection onto $(B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})) \cap \mathcal{O}(\mathbb{D})$ and satisfies (3.32), we have
\[
\mathcal{P}(\mathcal{F}_N) = \sum_{j=0}^{N} \left(1 - \frac{j}{N}\right) e_j
\]
and, by the radial assumption on $\mu$,
\[
\|\mathcal{P}(\mathcal{F}_N)\|^2_{A^2(\mathbb{D}, \mu)} = \sum_{j=0}^{N} (1 - j/N)^2 \|e_j\|^2_{A^2(\mathbb{D}, \mu)} = 2\pi \sum_{j=0}^{N} (1 - j/N)^2 \sigma_j.
\]
Then, by (3.25),
\[
\liminf_{N \to \infty} \|\mathcal{P}(\mathcal{F}_N)\|^2_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \geq \liminf_{N \to \infty} \frac{\|\mathcal{P}(\mathcal{F}_N)\|^2_{A^2(\mathbb{D}, \mu)}}{C^2} \geq \frac{2\pi}{C^2} \sum_{j=0}^{\infty} \sigma_j = \infty.
\]
This contradicts to the following inequality (which is a consequence of (3.29) and (3.34))
\[
\sup_{N \geq 1} \|\mathcal{P}(\mathcal{F}_N)\|_{B^2(\mathbb{D}, \mu) + h^1(\mathbb{D})} \leq \|P\| < \infty.
\]
Hence we complete the whole proof of the theorem.
4. Holomorphic stability: the upper half plane case

In this section, we will prove Theorem 1.3. For any Radon measure \( \Pi \) on \( \mathbb{R}_+ = (0, \infty) \) satisfying the condition (2.2), define
\[
\mathcal{L}_\Pi(\xi) := \begin{cases} 
\int_{\mathbb{R}^+} e^{-4\pi y/|\xi|} \Pi(dy) & \text{if } \xi \in \mathbb{R}^*, \\
0 & \text{if } \xi = 0.
\end{cases}
\]
Recall the following definition of the Fourier transform for \( f \in L^1(\mathbb{R}) \):
\[
\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-i2\pi x\xi}dx, \quad \xi \in \mathbb{R}.
\]

Recall the definition (1.7) of \( \mathcal{B}^2(\mathbb{H}, \mu) \). For any \( g \in \mathcal{B}^2(\mathbb{H}, \mu) \) and \( y > 0 \), recall the definition of the function \( g_y \) defined in (1.8). Note that the Fourier transform of the Poisson kernel \( P_{iy}^H \) given in (1.9) has the following form (see [Kat04, Chapter VI, p. 140]):
\[
\widehat{P}_{iy}^H(\xi) = e^{-2\pi y/|\xi|}, \quad \xi \in \mathbb{R}.
\]
Then, by (1.10) and (1.11), we have \( \widehat{g}_y(\xi) = e^{-2\pi(y-y')/|\xi|}\widehat{g}_{y'}(\xi) \) for all \( 0 < y' < y \) and hence
\[
e^{2\pi y/|\xi|}\widehat{g}_y(\xi) = e^{2\pi y'/|\xi|}\widehat{g}_{y'}(\xi), \quad \forall 0 < y' < y.
\]

**Definition.** Let \( \mu(dz) = dx\Pi(dy) \) be a boundary-accessable \((1,2)\)-Carleson measure on \( \mathbb{H} \). For any \( g \in \mathcal{B}^2(\mathbb{H}, \mu) \), define a function \( \widehat{g}_0 \) by
\[
(4.2) \quad \widehat{g}_0(\xi) := e^{2\pi y/|\xi|}\widehat{g}_y(\xi), \quad y > 0,
\]
where, by (4.1), the right hand side of the equality (4.2) is independent of \( y > 0 \).

By (4.2) and the Plancherel's identity, the norm of any \( g \in \mathcal{B}^2(\mathbb{H}, \mu) \) has the form:
\[
(4.3) \quad \|g\|_{\mathcal{B}^2(\mathbb{H}, \mu)} = \left( \int_{\mathbb{R}} |\widehat{g}_0(\xi)|^2\mathcal{L}_\Pi(\xi)d\xi \right)^{1/2}.
\]

**Remark.** If the function \( \widehat{g}_0 \) defined in (4.2) belongs to \( L^2(\mathbb{R}) \), then it is the Fourier transform of a function \( g_0 \in L^2(\mathbb{R}) \) and the equality (4.2) is equivalent to \( g_y = P_{iy}^Hg_0 \). However, the notation \( \widehat{g}_0 \) is only formal for a general \( g \in \mathcal{B}^2(\mathbb{H}, \mu) \), that is, it may not correspond to the Fourier transform of a generalized function \( g_0 \) on \( \mathbb{R} \).

**Definition.** Suppose that \( \mu(dz) = dx\Pi(dy) \) is a boundary-accessable \((1,2)\)-Carleson measure on \( \mathbb{H} \). Let \( H_\mu(\mathbb{R}) \) be the Hilbert space defined by the norm completion as follows:
\[
H_\mu(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) \middle| \|f\|_{H_\mu(\mathbb{R})} = \left( \int_{\mathbb{R}} |\hat{f}(\xi)|^2\mathcal{L}_\Pi(\xi)d\xi \right)^{1/2} < \infty \right\}.
\]

For any \( f \in L^2(\mathbb{R}) \), set
\[
\mathcal{P}^H(f)(z) := (P_{iy}^Hf)(x), \quad z = x + iy \in \mathbb{H}.
\]

Immediately from the definition of \( H_\mu(\mathbb{R}) \), we see that the map
\[
L^2(\mathbb{R}) \ni f \mapsto \mathcal{P}^H(f) \in \mathcal{B}^2(\mathbb{H}, \mu)
\]
extends to a unitary map from \( H_\mu(\mathbb{R}) \) to \( \mathcal{B}^2(\mathbb{H}, \mu) \).

Similar to the disk case, for a given measure \( \mu(dz) = dx\Pi(dy) \) on \( \mathbb{H} \), a pair \((a, b)\) of two functions on \( \mathbb{R} \) is called \( \mu \)-adapted if the following conditions are satisfied:
implies that the following function $W_{\Pi}$, define $R > \mu$.

To avoid technical issues, we first consider the truncated measures of (4.12).

Theorem, this embedding is continuous: there exists $C > R$ and the constant $u_L$ measure on $H^1$.

Proposition 4.1. Suppose that $\mu(dx) = dx\Pi(dy)$ is a boundary-accessible $(1,2)$-Carleson measure on $\mathbb{H}$ and let $(a, b)$ be a $\mu$-adapted pair of functions defined on $\mathbb{R}$. Then for any $u \in L^2(\mathbb{R})$,

$$
|a(\xi)|^2 |b(\xi)| \text{sgn}(\xi) = L_{\Pi}(\xi)^{-1};
$$

(ii) there exists a constant $C_b > 0$ such that

$$
\frac{1}{C_b} \leq |b(\xi)| \leq C_b.
$$

Given any Radon measure $\Pi$ on $\mathbb{R}_+$ satisfying (2.2), Garnett’s result stated in Lemma 3.4 implies that the following function $W^{\Pi}$ belongs to $L^\infty(\mathbb{R})$:

$$
W^{\Pi}(x) := i \int_{\mathbb{R}_+} \frac{\pi x}{y^2 + \pi^2 x^2} \Pi(dy), \quad x \in \mathbb{R}.
$$

**Proposition 4.1.** Suppose that $\mu(dx) = dx\Pi(dy)$ is a boundary-accessible $(1,2)$-Carleson measure on $\mathbb{H}$ and let $(a, b)$ be a $\mu$-adapted pair of functions defined on $\mathbb{R}$. Then for any $u \in L^2(\mathbb{R})$,

$$
\|u\|_{L^2(\mathbb{R})} \leq C(\|T_a u\|_{H^1_2(\mathbb{H})} + \|T_b u\|_{H^1_2(\mathbb{H})}),
$$

where $T_a, T_b$ are the Fourier multipliers associated to $a, b$ given by

$$
\hat{T_a u}(\xi) = a(\xi)\hat{u}(\xi), \quad \hat{T_b u}(\xi) = b(\xi)\hat{u}(\xi)
$$

and the constant $C = C(b, \Pi) > 0$ can be taken to be

$$
C(b, \Pi) = \sqrt{C_b + \|W^{\Pi}\|_{L^\infty(\mathbb{R})}} + 1 < \infty
$$

**Remark.** If either $T_a u$ or $T_b T_b u$ does not belong to $H^1_2(\mathbb{H})$, then the right hand side of (4.7) is understood as $\infty$.

### 4.1. The derivation of Theorem 1.3 from Proposition 4.1.

Let $\mu(dx) = dx\Pi(dy)$ be a boundary-accessible Radon measure on $\mathbb{H}$. If the pair $(B^2(\mathbb{H}, \mu), h^1(\mathbb{H}))$ is holomorphically stable, then $H^1(\mathbb{H}) = h^1(\mathbb{H}) \cap \mathcal{O}(\mathbb{H}) \subset B^2(\mathbb{H}, \mu)$ and by the Closed Graph Theorem, this embedding is continuous: there exists $C > 0$ such that

$$
\|f\|_{B^2(\mathbb{H}, \mu)} \leq C\|f\|_{H^1(\mathbb{H})}, \quad \forall f \in H^1(\mathbb{H}).
$$

Recall the definition (1.10) of the space $\text{Poi}(\mathbb{H})$. Since $H^1(\mathbb{H}) \subset \text{Poi}(\mathbb{H})$,

$$
\|f\|_{B^2(\mathbb{H}, \mu)}^2 = \int_{\mathbb{H}} |f(z)|^2 \mu(dz), \quad \forall f \in H^1(\mathbb{H}).
$$

The inequality (4.9) and the equality (4.10) together imply that the measure $\mu$ is a $(1,2)$-Carleson measure.

Suppose now that $\mu(dx) = dx\Pi(dy)$ is a boundary-accessible $(1,2)$-Carleson measure on $\mathbb{H}$. Assume that

$$
f = g + h \quad \text{with} \quad f \in \mathcal{O}(\mathbb{H}), \quad g \in B^2(\mathbb{H}, \mu), \quad h \in h^1(\mathbb{H}).
$$

Then, the goal is to show that $f \in B^2(\mathbb{H}, \mu)$. It suffices to show

$$
\|f\|_{B^2(\mathbb{H}, \mu)} \leq 2\sqrt{2 + \|W^{\Pi}\|_{L^\infty(\mathbb{R})} (\|g\|_{B^2(\mathbb{H}, \mu)} + \|h\|_{h^1(\mathbb{H})})}.
$$

To avoid technical issues, we first consider the truncated measures of $\mu$. That is, for any $R > 0$, define

$$
\mu_R(dx) = dx\Pi_R(dy), \quad \text{where} \quad \Pi_R(dy) = 1(y < R) \cdot \Pi(dy).
$$
Define $W_{H}^{R} \in L^{\infty}(\mathbb{R})$ in a similar way as in (4.6). Then $\|W_{H}^{R}\|_{L^{\infty}(\mathbb{R})} \leq \|W\|_{L^{\infty}(\mathbb{R})}$ for any $R > 0$. Therefore, the desired inequality (4.12) follows from

$$
\|f\|_{L^{2}(\mathbb{H}, \mu_{R})} \leq 2\sqrt{2 + \|W_{H}^{R}\|_{L^{\infty}(\mathbb{R})}(\|g\|_{L^{2}(\mathbb{H}, \mu_{R})} + \|h\|_{L^{2}(\mathbb{H})}).
$$

Now we are going to apply Proposition 4.1. For any $R > 0$, define a $\mu_{R}$-adapted pair $(a_{R}, b)$ of functions by

$$
a_{R}(\xi) := \mathcal{L}_{H}^{R}(\xi)^{-1/2} = \left(\int_{0}^{R} e^{-4\pi y|\xi|} \Pi(d\gamma)\right)^{-1/2}
$$

and $b(\xi) = \text{sgn}(\xi)$. In particular, by (2.2),

$$
\sup_{\xi \in \mathbb{R}} a_{R}(\xi)^{-1} \leq \sqrt{\Pi((0, R))} < \infty.
$$

For any $y > 0$, define $f_{y} : \mathbb{R} \to \mathbb{C}$ and $f^{y} : \mathbb{H} \to \mathbb{C}$ by

$$
f_{y}(x) = f(x + iy), \quad x \in \mathbb{R}
$$

and

$$
f^{y}(z) = f(z + iy), \quad z \in \mathbb{H}.
$$

And $g_{y}, g^{\epsilon}, h_{y}, h^{\epsilon}$ are defined similarly.

**Claim I.** For any $\epsilon > 0$, the function $f_{\epsilon}$ belongs to the classical analytic Hardy space $H^{2}(\mathbb{R})$ and hence

$$
\text{supp}(\widehat{f}_{\epsilon}) \subset [0, \infty).
$$

**Remark.** The assertion (4.15) does not follow from the fact that $f_{\epsilon}$ is the restriction onto the real line of a holomorphic function defined on a neighborhood of the closed upper-half plane. For instance, the following function

$$
K(x) := \frac{\sin(\pi x)}{\pi x}, \quad x \in \mathbb{R}
$$

belongs to $L^{2}(\mathbb{R})$ and is the restriction of an entire function on the complex plane. However, $\text{supp}(\widehat{K}) = [-1/2, 1/2] \not\subset [0, \infty)$.

Since $h \in h^{1}(\mathbb{H})$, there exists $h_{0} \in L^{1}(\mathbb{R})$ with

$$
h_{y} = P_{iy}^{H} * h_{0}, \quad \forall y > 0.
$$

Thus $h_{y} \in L^{1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}) \subset L^{2}(\mathbb{R})$. By (1.11), the assumption $g \in \mathcal{B}^{2}(\mathbb{H}, \mu)$ implies that $g_{y} \in L^{2}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$. Consequently

$$
f_{y} = g_{y} + h_{y} \in L^{2}(\mathbb{R}).
$$

Again by (4.16) and (1.11), for any $\epsilon > 0$,

$$
f_{y} = P_{ix(y-\epsilon)}^{H} * g_{\epsilon} + P_{ix(y-\epsilon)}^{H} * h_{\epsilon}, \quad \forall y \geq \epsilon.
$$

Therefore, for any $\epsilon > 0$,

$$
\sup_{y \geq \epsilon} \left(\int_{\mathbb{R}} |f(x + iy)|^{2} dx\right)^{1/2} \leq \|g_{\epsilon}\|_{L^{2}(\mathbb{R})} + \|h_{\epsilon}\|_{L^{2}(\mathbb{R})} < \infty.
$$

The above inequality combined with $f \in \mathcal{O}(\mathbb{H})$ implies that $f_{\epsilon} \in H^{2}(\mathbb{R})$. This completes the proof of Claim I.

Since $g \in \mathcal{B}^{2}(\mathbb{H}, \mu)$, for any $\epsilon > 0$, the function $g^{\epsilon}$ belongs to $\mathcal{B}^{2}(\mathbb{H}, \mu)$ and hence $g^{\epsilon} \in \mathcal{B}^{2}(\mathbb{H}, \mu_{R})$. Then by the natural unitary map between $H_{H}^{R}(\mathbb{R})$ and $\mathcal{B}^{2}(\mathbb{H}, \mu_{R})$,

$$
\|g_{\epsilon}\|_{H_{H}^{R}(\mathbb{R})} = \|g^{\epsilon}\|_{\mathcal{B}^{2}(\mathbb{H}, \mu_{R})}.
$$
Note that the equality (4.17) and the inequality (4.14) together imply that the function \( a_R(\xi)^{-1} \hat{f}_e(\xi) \) belongs to \( L^2(\mathbb{R}) \). Then there exists a unique function \( u_\varepsilon \in L^2(\mathbb{R}) \) with
\[
(4.18) \quad \hat{u}_\varepsilon(\xi) = a_R(\xi)^{-1} \hat{f}_e(\xi).
\]
Hence, by (4.15), \( \text{supp}(\hat{u}_\varepsilon) \subset [0, \infty) \). It follows that,
\[
\hat{f}_e(\xi) = a_R(\xi) \hat{u}_\varepsilon(\xi) = a_R(\xi) \text{sgn}(\xi) \hat{u}_\varepsilon(\xi) = a_R(\xi) b(\xi) \hat{u}_\varepsilon(\xi).
\]
That is, \( f_e = T_a b u_\varepsilon = T_a b u_\varepsilon \). Therefore, since \( u_\varepsilon \in L^2(\mathbb{R}) \), we may apply (4.7) and get
\[
\|u_\varepsilon\|_{L^2(\mathbb{R})} \leq 2 \sqrt{2 + \|W\|_{L^\infty(\mathbb{R})} \| \hat{f}_e\|_{H_{\mu_R}(\mathbb{R}) + L^1(\mathbb{R})}} \]
\[
\leq 2 \sqrt{2 + \|W\|_{L^\infty(\mathbb{R})} \left( \|g_\varepsilon\|_{H_{\mu_R}(\mathbb{R})} + \|h_\varepsilon\|_{L^1(\mathbb{R})} \right)} \]
\[
= 2 \sqrt{2 + \|W\|_{L^\infty(\mathbb{R})} \left( \|g\|_{B^2(\mathbb{H}, \mu_R)} + \|h\|_{H^1(\mathbb{H})} \right)},
\]
where the last inequality is due to the simple observation: for any \( \varepsilon > 0 \),
\[
\|g_\varepsilon\|_{B^2(\mathbb{H}, \mu_R)} \leq \|g\|_{B^2(\mathbb{H}, \mu_R)} \quad \text{and} \quad \|h_\varepsilon\|_{L^1(\mathbb{R})} \leq \|h_0\|_{L^1(\mathbb{R})} = \|h\|_{H^1(\mathbb{H})}.
\]
Finally, by Plancherel’s identity, the equalities (4.18) and (4.3),
\[
\|u_\varepsilon\|_{L^2(\mathbb{R})}^2 = \int_\mathbb{R} \left| \frac{\hat{f}_e(\xi)}{a_R(\xi)} \right|^2 \, d\xi = \int_\mathbb{R} |\hat{f}_e(\xi)|^2 \mathcal{L}_\mu(\xi) \, d\xi = \|f_\varepsilon\|_{B^2(\mathbb{H}, \mu_R)}^2.
\]
Thus,
\[
\|f_\varepsilon\|_{B^2(\mathbb{H}, \mu_R)} \leq 2 \sqrt{2 + \|W\|_{L^\infty(\mathbb{R})} \left( \|g\|_{B^2(\mathbb{H}, \mu_R)} + \|h\|_{H^1(\mathbb{H})} \right)}.
\]
The inequality (4.13) now follows immediately since
\[
\lim_{\varepsilon \to 0^+} \|f_\varepsilon\|_{B^2(\mathbb{H}, \mu_R)} = \|f\|_{B^2(\mathbb{H}, \mu_R)}.
\]

4.2. The proof of Proposition 4.1.

**Lemma 4.2.** Let \( \Pi \) be a Radon measure on \( \mathbb{R}_+ \) satisfying (2.2). Then there is a function \( W^\Pi \in L^\infty(\mathbb{R}) \) such that the following equality
\[
(4.19) \quad \int_\mathbb{R} u(x) W^\Pi(x) \, dx = \int_\mathbb{R} \hat{u}(\xi) \text{sgn}(\xi) \mathcal{L}_\Pi(\xi) \, d\xi
\]
holds for all \( u \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) satisfying
\[
(4.20) \quad \int_\mathbb{R} |\hat{u}(\xi)| \mathcal{L}_\Pi(\xi) \, d\xi < \infty.
\]

Remark. The equality (4.19) means that the Fourier transform, in a certain distributional sense, of the function \( W^\Pi \), is given by \( \hat{W}^\Pi(\xi) = \text{sgn}(\xi) \mathcal{L}_\Pi(\xi) \). If \( \Pi(dy) = dy \) is the Lebesgue measure on \( \mathbb{R}_+ \), then
\[
\mathcal{L}_\Pi(\xi) = \frac{1}{2|\xi|} \quad \text{and} \quad W^\Pi(x) = i\frac{\pi}{2} \text{sgn}(x).
\]
In general, the Fourier transform of \( W^\Pi \) can only be understood in a certain distributional sense and the condition (4.20) in Lemma 4.2 can not be removed.
The proof of Lemma 4.2 is postponed to the end of this section.

**Proof of Proposition 4.1.** Take \( u \in L^2(\mathbb{R}) \). Suppose that we have decompositions

\[
\mathcal{T}_a u(x) = f_1(x) + g_1(x) \quad \text{and} \quad \mathcal{T}_a \mathcal{T}_b u(x) = f_2(x) + g_2(x)
\]

with \( f_1, f_2 \in H_\mu(\mathbb{R}) \) and \( g_1, g_2 \in L^1(\mathbb{R}) \). That is,

\[
\hat{u}(\xi) a(\xi) = \hat{f}_1(\xi) + \hat{g}_1(\xi), \quad \hat{u}(\xi) a(\xi) b(\xi) = \hat{f}_2(\xi) + \hat{g}_2(\xi).
\]

For any fixed \( y > 0 \), applying the Poisson convolution to both sides of (4.21), we have

\[
P_{iy}^H * \mathcal{T}_a u = P_{iy}^H * f_1 + P_{iy}^H * g_1, \quad P_{iy}^H * (\mathcal{T}_a \mathcal{T}_b) u = P_{iy}^H * f_2 + P_{iy}^H * g_2.
\]

By Plancherel’s identity and (4.22),

\[
\|P_{iy}^H * u\|_{L^2(\mathbb{R})}^2 = \left( \int_\mathbb{R} |\hat{P}_{iy}^H(\xi)|^2 |\hat{u}(\xi)|^2 d\xi \right) = \left( \int_\mathbb{R} |\hat{P}_{iy}^H(\xi)|^2 \frac{\hat{f}_1(\xi) + \hat{g}_1(\xi)}{a(\xi)} \frac{\bar{a}(\xi) d\xi}{\bar{u}(\xi)} \right)
\]

\[
= \int_\mathbb{R} |\hat{P}_{iy}^H(\xi)|^2 \frac{\hat{f}_1(\xi) + \hat{g}_1(\xi)}{a(\xi)} \frac{\bar{a}(\xi) d\xi}{\bar{u}(\xi)} + \int_\mathbb{R} |\hat{P}_{iy}^H(\xi)|^2 \frac{\hat{g}_1(\xi)}{a(\xi)} \frac{\bar{a}(\xi) d\xi}{\bar{u}(\xi)}.
\]

Cauchy-Schwarz’s inequality and the conditions (4.4), (4.5) together imply

\[
|I_1| \leq \sqrt{C_b} \|P_{iy}^H * f_1\|_{L^2(\mathbb{R})} \|P_{iy}^H * u\|_{H_\mu(\mathbb{R})} \leq \sqrt{C_b} \|P_{iy}^H * u\|_{L^2(\mathbb{R})} \|f_1\|_{H_\mu(\mathbb{R})}.
\]

And, by (4.22) and \( b(\xi) \in \mathbb{R} \), the integral \( I_2 \) can be decomposed as

\[
I_2 = \int_\mathbb{R} \left| \hat{P}_{iy}^H(\xi) \right|^2 \frac{\hat{g}_1(\xi)}{a(\xi)} \frac{\bar{a}(\xi) d\xi}{\bar{u}(\xi)} - \left( \int_\mathbb{R} \left| \hat{P}_{iy}^H(\xi) \right|^2 \frac{\hat{f}_2(\xi)}{a(\xi)} \frac{\bar{a}(\xi) d\xi}{\bar{u}(\xi)} \right) + \int_\mathbb{R} \left| \hat{P}_{iy}^H(\xi) \right|^2 \frac{\hat{g}_1(\xi) \bar{g}_2(\xi)}{a(\xi)^2 b(\xi)} \frac{\bar{a}(\xi) d\xi}{\bar{u}(\xi)}
\]

The integral \( I_3 \) can be easily controlled. Indeed, again by Cauchy-Schwarz’s inequality and (4.4), (4.5),

\[
|I_3| \leq \left( \int_\mathbb{R} \left| \hat{P}_{iy}^H(\xi) \right|^2 d\xi \right)^{1/2} \left( \int_\mathbb{R} \left| \hat{P}_{iy}^H(\xi) \right|^2 \frac{\hat{f}_2(\xi)}{a(\xi) b(\xi)} \frac{\bar{a}(\xi) d\xi}{\bar{u}(\xi)} \right)^{1/2} + \left( \int_\mathbb{R} \left| \hat{P}_{iy}^H(\xi) \right|^2 \frac{\hat{f}_1(\xi)}{|a(\xi)^2 b(\xi)|} \frac{\bar{a}(\xi) d\xi}{\bar{u}(\xi)} \right)^{1/2} \left( \int_\mathbb{R} \left| \hat{P}_{iy}^H(\xi) \right|^2 \frac{\hat{f}_2(\xi)}{|a(\xi)^2 b(\xi)|} \frac{\bar{a}(\xi) d\xi}{\bar{u}(\xi)} \right)^{1/2}
\]

\[
\leq \sqrt{C_b} \|P_{iy}^H * u\|_{L^2(\mathbb{R})} \|P_{iy}^H * f_2\|_{H_\mu(\mathbb{R})} + \|P_{iy}^H * f_1\|_{H_\mu(\mathbb{R})} \|P_{iy}^H * f_2\|_{H_\mu(\mathbb{R})} \leq \sqrt{C_b} \|P_{iy}^H * u\|_{L^2(\mathbb{R})} \|f_2\|_{H_\mu(\mathbb{R})} + \|f_1\|_{H_\mu(\mathbb{R})} \|f_2\|_{H_\mu(\mathbb{R})}.
\]
It remains to estimate the integral $I_4$. Since $g_1, g_2 \in L^1(\mathbb{R})$, for any $y > 0$, one can define
\begin{equation}
G_y := (P_{iy}^H * g_1) * (P_{iy}^H * \tilde{g}_2) = P_{2iy}^H * (g_1 * \tilde{g}_2), \quad \text{where} \quad \tilde{g}_2(x) := g_2(-x).
\end{equation}
In particular,
\begin{equation}
\hat{G}_y(\xi) = \left| \hat{P}_{iy}^H(\xi) \right|^2 \hat{g}_1(\xi) \hat{\tilde{g}}_2(\xi) = e^{-4\pi|y|\xi} \hat{g}_1(\xi) \hat{\tilde{g}}_2(\xi).
\end{equation}

**Claim A.** For any $y > 0$, the function $G_y$ defined in (4.27) satisfies
\begin{equation}
G_y \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})
\end{equation}
and
\begin{equation}
\int_\mathbb{R} |\hat{G}_y(\xi)| \mathcal{L}_H(\xi) d\xi < \infty.
\end{equation}

Indeed, $g_1 * \tilde{g}_2 \in L^1(\mathbb{R})$ since $g_1, g_2 \in L^1(\mathbb{R})$. Therefore, (4.29) follows from the definition (4.27) and the simple observation that $P_{2iy}^H \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By (4.22),
\begin{equation}
\frac{\hat{g}_1(\xi)}{a(\xi)} = \hat{u}(\xi) - \frac{\hat{f}_1(\xi)}{a(\xi)}, \quad \frac{\hat{g}_2(\xi)}{a(\xi)b(\xi)} = \hat{u}(\xi) - \frac{\hat{f}_2(\xi)}{a(\xi)b(\xi)}.
\end{equation}
The assumptions $f_1, f_2 \in H_\mu(\mathbb{R})$, combined with the conditions (4.4), (4.5) on the pair $(a, b)$ imply that both functions $\hat{f}_1/a$ and $\hat{f}_2/(ab)$ belong to $L^2(\mathbb{R})$. Since $u \in L^2(\mathbb{R})$ and hence $\hat{u} \in L^2(\mathbb{R})$, we obtain, by using (4.4) again, that
\begin{equation}
\int_\mathbb{R} |\hat{G}_y(\xi)| \mathcal{L}_H(\xi) d\xi = \int_\mathbb{R} e^{-4\pi|y|\xi} \left| \hat{g}_1(\xi) \right| \frac{\left| \hat{g}_2(\xi) \right|}{\left| a(\xi) \right|} \frac{\left| a(\xi)b(\xi) \right|}{\left| a(\xi)b(\xi) \right|} d\xi
\end{equation}
\begin{equation}
\leq \int_\mathbb{R} \left| \hat{u}(\xi) - \frac{\hat{f}_1(\xi)}{a(\xi)} \right| \frac{\left| \hat{u}(\xi) - \frac{\hat{f}_2(\xi)}{a(\xi)b(\xi)} \right|}{\left| a(\xi)b(\xi) \right|} d\xi
\end{equation}
\begin{equation}
\leq \left\| \hat{u} - \frac{\hat{f}_1}{a} \right\|_{L^2(\mathbb{R})} \left\| \hat{u} - \frac{\hat{f}_2}{ab} \right\|_{L^2(\mathbb{R})} < \infty.
\end{equation}

By Claim A, the function $G_y$ satisfies all the required conditions of Lemma 4.2. Hence, by (4.28), (4.4) and (4.19),
\begin{equation}
I_4 = \int_\mathbb{R} \hat{G}_y(\xi) \text{sgn}(\xi) \mathcal{L}_H(\xi) d\xi = \int_\mathbb{R} G_y(x) \overline{W^H(x)} dx.
\end{equation}

It follows that
\begin{equation}
|I_4| \leq \| W^H \|_{L^\infty(\mathbb{R})} \| G_y \|_{L^1(\mathbb{R})} = \| W^H \|_{L^\infty(\mathbb{R})} \| P^H_{2iy} * (g_1 * \tilde{g}_2) \|_{L^1(\mathbb{R})}
\end{equation}
\begin{equation}
\leq \| W^H \|_{L^\infty(\mathbb{R})} \| g_1 \|_{L^1(\mathbb{R})} \| g_2 \|_{L^1(\mathbb{R})}.
\end{equation}
Combining (4.23), (4.24), (4.25), (4.26) and (4.31), we get
\begin{equation}
\| P^H_{iy} * u \|_{L^2(\mathbb{R})} \leq C_b \left( \| P^H_{iy} * u \|_{L^2(\mathbb{R})} \| f_1 \|_{H_\mu(\mathbb{R})} + \| f_2 \|_{H_\mu(\mathbb{R})} \right)
\end{equation}
\begin{equation}
+ C_b \left( \| f_1 \|_{H_\mu(\mathbb{R})} \| f_2 \|_{H_\mu(\mathbb{R})} + \| W^H \|_{L^\infty(\mathbb{R})} \| g_1 \|_{L^1(\mathbb{R})} \| g_2 \|_{L^1(\mathbb{R})} \right).
\end{equation}
Therefore, by a standard argument, there exists a constant $C > 0$ depending only on the constants $C_b$ and $\| W^H \|_{L^\infty(\mathbb{R})}$ such that
\begin{equation}
\| P^H_{iy} * u \|_{L^2(\mathbb{R})} \leq C(\| f_1 \|_{H_\mu(\mathbb{R})} + \| g_1 \|_{L^1(\mathbb{R})} + \| f_2 \|_{H_\mu(\mathbb{R})} + \| g_2 \|_{L^1(\mathbb{R})}).
\end{equation}
The constant \( C \) in the above inequality can be taken to be
\[
C = \sqrt{C_b + \|W^\Pi\|_{L^\infty(\mathbb{R})} + 1}.
\]

Since the decompositions (4.21) are arbitrary, we get
\[
\|P_{iy}^H u\|_{L^2(\mathbb{R})} \leq C(\|T_a u\|_{H^\mu(\mathbb{R}) + L^1(\mathbb{R})} + \|T_b T_a u\|_{H^\mu(\mathbb{R}) + L^1(\mathbb{R})}) + 1.
\]

Finally, by taking the limit \( y \to 0^+ \) and using
\[
\lim_{y \to 0^+} \|P_{iy}^H u\|_{L^2(\mathbb{R})} = \|u\|_{L^2(\mathbb{R})},
\]
we obtain the desired inequality (4.7) and complete the whole proof of the proposition. \( \square \)

**Proof of Lemma 4.2.** Fix a Radon measure \( \Pi \) on \( \mathbb{R}^+ \) satisfying (2.2). By Garnett’s result stated in Lemma 3.4, one can define a function \( W^\Pi \in L^\infty(\mathbb{R}) \) by (4.6).

Now we show that \( W^\Pi \) satisfies the equality (4.19). For any \( 0 < \varepsilon < R < \infty \), set
\[
W^\Pi_{\varepsilon,R}(x) := i \int_{\mathbb{R}^+} \frac{\pi x}{y^2 + \pi^2 x^2} \Pi_{\varepsilon,R}(dy), \quad \text{where} \quad \Pi_{\varepsilon,R}(dy) = 1(\varepsilon < y < R) \cdot \Pi(dy).
\]

**Claim B.** For any \( 0 < \varepsilon < R < \infty \), we have \( W^\Pi_{\varepsilon,R} \in L^2(\mathbb{R}) \) and the Fourier transform of \( W^\Pi_{\varepsilon,R} \) is given by the Bochner integral for \( L^2(\mathbb{R}) \)-vector valued function:
\[
\widehat{W^\Pi_{\varepsilon,R}}(\xi) = \int_{\varepsilon}^{R} \ell_y(\Pi(dy)) := \text{sgn}(\xi)e^{-2y|\xi|}.
\]

In particular, \( \widehat{W^\Pi_{\varepsilon,R}} \) can be identified with a \( C^\infty(\mathbb{R}^*) \)-function by the formula
\[
\widehat{W^\Pi_{\varepsilon,R}}(\xi) = \int_{\varepsilon}^{R} \text{sgn}(\xi)e^{-2y|\xi|} \Pi(dy), \quad \xi \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}.
\]

Indeed, for any \( y > 0 \), recall that the conjugate Poisson kernel (see, e.g., [Gra14, formula (4.1.16)]) of \( \mathbb{H} \) is given by
\[
Q^\Pi_{iy}(x) = \frac{\pi x}{y^2 + \pi^2 x^2}, \quad x \in \mathbb{R}.
\]

Clearly, \( Q^\Pi_{iy} \in L^2(\mathbb{R}) \) for all \( y > 0 \) and the map \( y \mapsto Q^\Pi_{iy} \) is continuous from \( \mathbb{R}^+ \) to \( L^2(\mathbb{R}) \), hence it is uniformly continuous from \( [\varepsilon, R] \) to \( L^2(\mathbb{R}) \). Consequently, using the definition (4.32) of \( W^\Pi_{\varepsilon,R} \) and the fact that \( \Pi_{\varepsilon,R} \) is a finite measure with support contained in \( [\varepsilon, R] \), we obtain that \( W^\Pi_{\varepsilon,R} \in L^2(\mathbb{R}) \) and the following equality in the sense of the Bochner integral for \( L^2(\mathbb{R}) \)-vector valued functions:
\[
\widehat{W^\Pi_{\varepsilon,R}} = i \int_{\varepsilon}^{R} Q^\Pi_{iy} \Pi(dy).
\]

Then the equality (4.33) follows immediately since (see, e.g., [Gra14, formula (4.1.33)])
\[
\widehat{Q^\Pi_{iy}}(\xi) = -i\ell_y(\xi) = -i\text{sgn}(\xi)e^{-2y|\xi|}.
\]

**Claim C.** For any \( \varphi \in L^1(\mathbb{R}) \),
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \varphi(x) \left[ \int_{\varepsilon}^{R} \frac{\pi x}{y^2 + \pi^2 x^2} \Pi(dy) \right] dx = 0.
\]
Indeed, for any $y > 0$, set $F_H(y) := \Pi((0,y])$. Then $\Pi(dy) = dF_H(y)$ and by the assumption (2.2), there exists a constant $C > 0$ such that

$$F_H(y) \leq Cy, \quad \forall y > 0. \quad (4.36)$$

By integration by parts for the absolutely continuous function $F_H$,

$$\int_0^\varepsilon \frac{\pi x}{y^2 + \pi^2x^2} \Pi(dy) = \frac{\pi x}{y^2 + \pi^2x^2} F_H(y)\bigg|_{y=\varepsilon}^0 + \int_0^\varepsilon F_H(y) \frac{2\pi xy}{(y^2 + \pi^2x^2)^2} dy. \quad (4.37)$$

In particular, if $\Pi(dy)$ is the Lebesgue measure on $\mathbb{R}_+$, then the equality (4.37) becomes

$$\arctan\left(\frac{\varepsilon}{\pi x}\right) = \int_0^\varepsilon \frac{\pi x}{y^2 + \pi^2x^2} dy = \frac{\pi x}{y^2 + \pi^2x^2} \bigg|_{y=\varepsilon}^0 + \int_0^\varepsilon \frac{2\pi xy}{(y^2 + \pi^2x^2)^2} dy. \quad (4.38)$$

Comparing (4.37) and (4.38) and using (4.36), we obtain

$$\int_0^\varepsilon \frac{\pi |x|}{y^2 + \pi^2x^2} \Pi(dy) \leq C \arctan\left(\frac{\varepsilon}{\pi |x|}\right).$$

Therefore, by dominated convergence theorem, for any $\varphi \in L^1(\mathbb{R})$,

$$\limsup_{\varepsilon \to 0^+} \left| \int_{\mathbb{R}} \varphi(x) \int_0^\varepsilon \frac{\pi x}{y^2 + \pi^2x^2} \Pi(dy) dx \right| \leq C \limsup_{\varepsilon \to 0^+} \int_{\mathbb{R}} |\varphi(x)| \arctan\left(\frac{\varepsilon}{\pi |x|}\right) dx = 0. \quad (4.39)$$

Claim D. For any $\varphi \in L^1(\mathbb{R})$,

$$\lim_{R \to \infty} \int_{\mathbb{R}} u(x)^2 W_H^\varepsilon(x) dx = \int_{\mathbb{R}} u(x)^2 W_H(x) dx. \quad (4.40)$$

Moreover, since both $u$ and $W_H$ belong to $L^2(\mathbb{R})$, the Plancherel’s identity implies

$$\int_{\mathbb{R}} u(x) \overline{W_H^\varepsilon(x)} dx = \int_{\mathbb{R}} \hat{u}(\xi) \overline{W_H^\varepsilon(\xi)} d\xi = \int_{\mathbb{R}} \hat{u}(\xi) \text{sgn}(\xi) \left[ \int_{\varepsilon}^R e^{-2y|\xi|} \Pi(dy) \right] d\xi. \quad (4.41)$$

Using the assumption (4.20), we obtain, by dominated convergence theorem,

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \hat{u}(\xi) \text{sgn}(\xi) \left[ \int_{\varepsilon}^R e^{-2y|\xi|} \Pi(dy) \right] d\xi = \int_{\mathbb{R}} \hat{u}(\xi) \text{sgn}(\xi) \mathcal{L}_H(\xi) d\xi. \quad (4.42)$$

Combining (4.40), (4.41) and (4.42), we obtain the desired equality (4.19). \(\square\)

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