GLOBAL EXISTENCE FOR SEMILINEAR DAMPED WAVE EQUATIONS IN RELATION WITH THE STRAUSS CONJECTURE

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Abstract. We study the global existence of solutions to semilinear wave equations with power-type nonlinearity and general lower order terms on $n$-dimensional nontrapping asymptotically Euclidean manifolds, when $n = 3, 4$ as well as two dimensional Euclidean space. In addition, we prove almost global existence with sharp lower bound of the lifespan for the four dimensional critical problem.

1. Introduction. Let $(\mathbb{R}^n, g)$ be a nontrapping asymptotically Euclidean (Riemannian) manifolds, with $n \geq 3$,

$$g = g^0 + g^1(r) + g^2(x),$$

where $g = g_{ij} dx^i dx^j$ and $g_{ij} = g^0_{ij} + g^1_{ij} + g^2_{ij}$, $g^0_{ij} = \delta_{ij}$, $(g^{ij}(x))$ denotes the inverse matrix of $(g_{ij}(x))$. Here and in what follows, the Einstein summation convention is performed over repeated upper and lower indices, $1 \leq i, j \leq n$. We assume the first perturbation $g^1$ is radial and for some fixed $\rho_1 > 0, \rho_2 > 1$,

$$|\nabla^a g^l_{ij}| \lesssim |a|^{-\rho_l}, l = 1, 2, \rho_1 < \rho_2, \rho = \min(\rho_1, \rho_2 - 1) > 0, |a| \leq 4,$$

where $\langle x \rangle = \sqrt{1 + x^2}$. Notice that the form of the metrics mimics that in [20]. The radial assumption on $g^1$ is primarily used to assist with controlling the commutators of rotational vector fields with the wave operator. On $(\mathbb{R}^n, g)$ with $n \geq 3$, it is known from the works of Bony-H{"a}fner [1] and Sogge-Wang [23] that we have the local energy estimates without loss and Strichartz estimates (see also Metcalfe-Sterbenz-Tataru [18] for recent work with weaker asymptotically flat assumption). The nonlinear wave equations on the nontrapping asymptotically Euclidean manifolds has received much attention in recent years. For example, Bony-H{"a}fner [1], Sogge-Wang [23] and Yang [35] studied the analogs of the John-Klainerman theorem [10] and global existence under null conditions for semilinear wave equations (see also Wang-Yu [34] and Yang [36] for quasilinear wave equations). Sogge-Wang [23],

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Wang-Yu [32], Metcalfe-Wang [20] and Wang [30] proved the analogs of the global existence of the Strauss conjecture when \( n = 3, 4 \) (see also [31] for a review of recent results and Wakasa-Yordanov [27] for the recent blow-up results with critical power). For the analogs of the Glassey conjecture, see Wang [29] and references therein.

In this paper, we are interested in the small data global existence of solutions for the Cauchy problem of the following semilinear wave equations with general lower order term, in relation with the Strauss conjecture, posed on nontrapping asymptotically Euclidean (Riemannian) manifolds

\[
\begin{cases}
    u_{tt} - \Delta g u + \mu(t, x) \partial_t u + \mu^j(t, x) \partial_j u + \mu_0(t, x) u = F_p(u), \\
    u(0, x) = u_0(x), \partial_t u(0, x) = u_1(x).
\end{cases}
\]

(3)

Here, \( \Delta g = \sqrt{|g|}^{-1} \partial_i g^{ij} \sqrt{|g|} \partial_j \) is the standard Laplace-Beltrami operator, with \( |g| = \det(\mathfrak{g}(x)) \). The nonlinearity \( F_p \) is assumed to behave like \( |u|^p \), more precisely, we assume

\[
|u| \ll 1 \Rightarrow \sum_{0 \leq i \leq 2} |u|^i |\partial^k_i F_p(u)| \lesssim |u|^p.
\]

Concerning lower order terms, we assume

\[
Y^\alpha(\mu, \mu^j) \in L^1_t(L^\infty_t \cap \dot{W}^{1, n}_t), |x|Y^\alpha \mu_0 \in L^1_t(L^{\infty}_t \cap \dot{W}^{1, n}_t), \forall |\alpha| \leq 2,
\]

(4)

where the vector fields \( Y \) are the collection of translational vector fields \( \partial_j \) and spatial rotational vector fields \( \Omega_{ij} = x_i \partial_j - x_j \partial_i, 1 \leq i < j \leq n \), that is,

\[
Y = (Y_1, \ldots, Y_{n(n+1)/2}) = \{ \nabla_x, \Omega \}.
\]

When \( \mathfrak{g} = \mathfrak{g}^0 \) and \( \mu = \mu^j = \mu_0 = 0 \), the problem has been extensively investigated and is known as the Strauss conjecture, which was initiated in the work of John [11]. It is known that, in general (with \( F_p(u) = |u|^p \)), the problem admits small data global existence only if \( p > p_c(n) \) (see John [11], Glassey [6], Sideris [21], Yordanov-Zhang [37] and Zhou [38]), where the critical power \( p_c(n) \) is the positive root of equation

\[
(n - 1)p^2 - (n + 1)p - 2 = 0.
\]

(5)

The global existence for small initial data when \( p \in (p_c(n), 1 + 4/(n - 1)) \) followed in Georgiev-Lindblad-Sogge [5] (see also Tataru [24]). See Wang-Yu [33], Wang [31] for a complete history and recent works for the problem on various space-time manifolds.

On the other hand, there are many recent works concerning the damped wave equations, \( \mathfrak{g} = \mathfrak{g}^0 \) and \( \mu^j = \mu_0 = 0 \), with typical damping term depending only on time

\[
\mu(t, x) = \frac{\mu}{(1 + t)^\beta}, \mu > 0.
\]

For the case \( \beta < 1 \), the damping term is strong enough to make the problem behaves totally different from the wave equations and the problem has been well-understood, see, e.g., [15], [25], [14], [16], [4]. There are some interesting critical phenomena happening for the scale-invariant case \( \beta = 1 \) and it appears that the critical power is \( p_c(n + \mu) \) for relatively small \( \mu > 0 \), see, e.g., [2], [13], [8], [26]. See also [12] for more discussion on the history.

For the remaining case, \( \beta > 1 \) (which is also referred as the scattering case), where the damping term is integrable, it is natural to expect that the problem behaves like the nonlinear wave equations without damping term. In a recent work
Lai and Takamura proved blow up results with $g = g^0$ and $\mu^0 = \mu_0 = 0$, for $1 < p < p_c(n)$, together with upper bound of the lifespan. In particular, it is shown that for $1 < p < p_c(n)$ with $n \geq 2$, we have

$$T_\epsilon \leq C\epsilon^{(n-1)(p-1)(\alpha+1)-\frac{2p(p-1)}{n}} ,$$

where $T_\epsilon$ denotes the lifespan and $\epsilon$ is the size of the initial data. For the critical case, $p = p_c(n)$, with $g = g^0 + g^2(x)$ and general (nonnegative) damping term $\mu = \mu(t) \in L^1$, where $|g^2_{ij}| + |\nabla g^2_{ij}| \leq Ce^{-\alpha(1+|x|)}$ for some $\alpha > 0$, at the final stage of preparation of the current manuscript, we learned that Wakasa-Yordanov [28] obtained the expected exponential upper bound of the lifespan,

$$T_\epsilon \leq \exp(Ce^{-\rho(p-1)}) .$$

In this paper, we are interested in complementing to the blow up results for the scattering case, by proving global existence results on general nontrapping asymptotically Euclidean manifolds. Moreover, in the process, we find that we could handle more general lower order perturbation terms as in (3), under the assumption (4). The first main theorem of this paper states as follows:

**Theorem 1.1.** Let $n = 3, 4$, and consider the Cauchy problem (3) with (4), posed on nontrapping asymptotically Euclidean manifolds, with (1) and (2). Then if $p > p_c(n)$, the problem (3) admits a unique global solution for any initial data which are sufficiently small, decaying and regular.

For more precise statement, see Theorem 4.1. Concerning the proof, the idea is to adapt the recent approach of using local energy and weighted Strichartz estimates, which has been very successful in the recent resolution of the Strauss conjecture on various space-times, including Schwarzschild/Kerr black-hole space-times ([3], [7], [23], [17], [20]). In particular, we revisit the proof of [20, Theorem 4.1] to extract the key weighted Strichartz estimates, Lemma 3.2, which, combined with the local energy estimates ([1], [23], [29], [18]), is good enough to treat the lower order terms in (3) in a perturbative way.

When there are no global solutions, it is also interesting to obtain sharp estimates of the lifespan. On this respect, it turns out that our argument could also be adapted to show some of the sharp results, as long as we could add some favorable terms in the desired space-time estimates, which could be used to absorb the lower order term in (3). To illustrate the argument, as an example, we prove the following lower bound estimate of the lifespan for the four-dimensional critical problem ($p_c(4) = 2$), by adding some favorable terms in [30, Lemma 5.6], which is sharp in general, comparing with (7) of [28].

**Theorem 1.2.** Let $n = 4$, $F_p = u^2$ and consider the Cauchy problem (3) with $Y^\alpha(\mu, \mu^0), |x|^\alpha \mu_0 \in L^1_t(L^\infty_x \cap \dot{W}^{1,4}), \forall |\alpha| \leq 3$, posed on nontrapping asymptotically Euclidean manifolds, with (1) and (2). Then there exists $c > 0$, such that the problem (3) admits almost global solution, up to

$$T_\epsilon = \exp(ce^{-\frac{1}{2}}) ,$$

for any initial data which are sufficiently small (of size $\epsilon \ll 1$), decaying and regular.

Actually, it is clear from the proof that it applies for any $C^3$ nonlinearity satisfying $|Y^\leq 3F_p(u)| \leq |u||Y^\leq 3u| + |Y^\leq 1u||Y^\leq 2u|$, for small $|u|$, where $|Y^\leq m u| = \sum_{|\alpha| \leq m} |Y^\alpha u|$. See Theorem 5.1 for more precise statement.
Remark 1. Theorems 1.1 and 1.2 remain valid for general asymptotically flat space-time manifolds, under the assumption of (weaker) asymptotic flatness for \( g = n^0 + g^1(t,r) + g^2(t,x) \), uniform energy estimates and (weak) local energy estimates for the associated d’Alembertian operators, where \( m^0 = -dt^2 + dx^2 \) is the standard Minkowski metric. For more precise assumptions, see Hypothesis 1 and 2 in [20]. For simplicity of presentation, we choose to state only the results for the nontrapping asymptotically Euclidean (Riemannian) manifolds.

Remark 2. It is remarkable that, in our statement, the coefficient \( \mu(t,x) \) is not required to be nonnegative, which were assumed in both [12] and [28]. Moreover, the authors believe that the nonnegative assumption there are not necessary.

Remark 3. The lower order terms that we considered in (3) are mainly concerned with the integrability in time. As comparison, there are also lower order terms allowed in [20], which are concerned with the decay rate near spatial infinity.

In addition, on the two dimensional Euclidean space, i.e., \( n = 2 \) with \( g^1 = g^2 = 0 \), the idea in this paper can be exploited to show the weighted Strichartz estimates of [3] and [7] are strong enough to yield small data global existence for (3) with \( p > p_c(2) \).

Theorem 1.3. Let \( n = 2 \) and \( p > p_c(2) \). Consider (3) with \( g = g^0 \) and

\[
Y^\alpha(\mu, \mu^j), |x|Y^\alpha \mu_0 \in \overline{L}_t^1(L_\infty^\infty \cap \dot{H}^1), \forall |\alpha| \leq 1,
\]

(10) Then the problem (3) admits a unique global solution for any initial data which are sufficiently small, decaying and regular.

See Theorem 6.1 for more precise statement.

Remark 4. Based on the existence results in previous works, as we have illustrated in our theorems, our argument could be adapted to show the following lower bounds of the lifespan for (3)

1. \( n = 2 \), \( 2 < p < p_c(2), g = g^0, T_\varepsilon \geq ce^{2p(p-1)\varepsilon^{-2}}, [9, Theorem 6.1] \).
2. \( n = 2, p = p_c(2), g = g^0, T_\varepsilon \geq \exp(ce^{-(p-1)^2/2}), [9, Theorem 6.2] \).
3. \( n = 3, 2 \leq p < p_c(3), T_\varepsilon \geq ce^{p(p-1)^2}, [30, Theorem 3.2] \).
4. \( n = 3, p = p_c(3), T_\varepsilon \geq \exp(ce^{-2(p-1)}), [30, Theorem 3.2] \).

These lower bounds, together with the upper bounds available from [12], show the sharpness of the lifespan estimates for \( n = 2 \) with \( p \in (2, p_c(2)) \), \( n = 3 \) with \( p \in [2, p_c(3)) \).

1.1. Notation. For a norm \( X \) and a nonnegative integer \( m \), we shall use the shorthand

\[
|Y^{\leq m}u| = \sum_{|\alpha| \leq m} |Y^\alpha u|, \quad \|Y^{\leq m}u\|_X = \sum_{|\alpha| \leq m} \|Y^\alpha u\|_X.
\]

Let \( L_\omega^q \) be the standard Lebesgue space on the sphere \( S^{n-1} \), we will use the following convention for mixed norms \( L_\omega^q L_\theta^r L_\xi^s \), with \( r = |x| \) and \( \omega \in \mathbb{S}^{n-1} \):

\[
\|f\|_{L_\omega^q L_\theta^r L_\xi^s} = \left( \int \|f(t,r\omega)\|_{L_\xi^s}^{q_2} r^{-n-1} dr \right)^{1/q_2}
\]

with trivial modification for the case \( q_2 = \infty \). Clearly \( L_\omega^q L_\theta^r = L_\omega^q L_\theta^r L_\xi^s \). We denote \( L_\theta^r L_\xi^s = L_\theta^r ([0,T]; L_\xi^s (\mathbb{R}^n)) \) for some \( T > 0 \). Occasionally, when the
meaning is clear, we shall omit the subscripts. As usual, we use $\| \cdot \|_{E_m}$ to denote the energy norm of order $m \geq 0$.

$$
\| u \|_E = \| u \|_{E_0} = \| \partial u \|_{L^\infty_t L^2_x}, \quad \| u \|_{E_m} = \| Y \leq m \|_E. \tag{11}
$$

We will use $\| \cdot \|_{LE}$ to denote the (strong) local energy norm

$$
\| u \|_{LE} = \| u \|_E + \| \partial u \|_{\ell^{-1/2} L^2_t L^2_x} + \| r^{-1} u \|_{\ell^{-1/2} L^2_t L^2_x} \tag{12}
$$

and $\| u \|_{LE_m} = \| Y \leq m \|_{LE}$, where we write

$$
\| u \|_{\ell_j(A)} = \| \phi_j(x) u(t,x) \|_{E(A)} = \| (2^j)^s \| \phi_j(x) u(t,x) \|_A \|_{\ell^s(j \geq 0)},
$$

for a partition of unity subordinate to the (inhomogeneous) dyadic annuli, $\Sigma_{j \geq 0} \phi_j^2 = 1$. We denote $\| u \|_{LE_T} = \| u \|_{E_T} + \| \partial u \|_{\ell^{-1/2} L^2_t L^2_x} + \| r^{-1} u \|_{\ell^{-1/2} L^2_t L^2_x}$ and $\| u \|_{E_T} = \| \partial u \|_{L^\infty_t L^2_x}$.

2. Preliminary. In this section, we collect some inequalities we shall use later.

**Lemma 2.1.** Let $s \in [0,1]$ and $n \geq 3$. If $f \in L^\infty(\mathbb{R}^n) \cap \tilde{W}^{1,n}(\mathbb{R}^n)$, $g \in \dot{H}^{s-1}$, then we have

$$
\| fg \|_{\dot{H}^{-s-1}} \lesssim \| f \|_{L^\infty \cap \tilde{W}^{1,n}} \| g \|_{\dot{H}^{s-1}}. \tag{13}
$$

**Proof.** When $s = 1$, by Hölder’s inequality

$$
\| fg \|_{L^2} \lesssim \| f \|_{L^\infty} \| g \|_{L^2}.
$$

When $s = 0$, by duality, we need only to show

$$
\| fg \|_{\dot{H}^{1}} \lesssim \| f \|_{L^\infty \cap \tilde{W}^{1,n}} \| g \|_{\dot{H}^{1}}.
$$

Since $\| fg \|_{\dot{H}^{1}} \leq \| \nabla f g \|_{L^2} + \| f \nabla g \|_{L^2}$, by Hölder’s inequality and Sobolev embedding

$$
\| \nabla f g \|_{L^2} + \| f \nabla g \|_{L^2} \lesssim \| \nabla f \|_{L^n} \| g \|_{L^{2n/n-2}} + \| f \|_{L^\infty} \| \nabla g \|_{L^2}
\lesssim \| f \|_{L^\infty \cap \tilde{W}^{1,n}} \| g \|_{\dot{H}^{1}}.
$$

By interpolation, (13) follows. \hfill \Box

**Lemma 2.2.** Let $s \in [0,1]$ and $n \geq 3$, then we have

$$
\| u/r \|_{\dot{H}^{-s-1}} \lesssim \| u \|_{\dot{H}^{s}}. \tag{14}
$$

**Proof.** When $s = 1$, it is the classical Hardy’s inequality

$$
\| u/r \|_{L^2} \lesssim \| u \|_{\dot{H}^{1}}.
$$

By duality, we get

$$
\| u/r \|_{\dot{H}^{-1}} \lesssim \| u \|_{L^2},
$$

which is the case $s = 0$. Then by interpolation, (14) follows. \hfill \Box
3. Space-time estimates. In this section, we collect various space-time estimates for linear wave equation
\[ u_{tt} - \Delta_g u = F(= F_1 + F_2) \quad (15) \]

**Lemma 3.1** (Local energy estimates). Let \( n \geq 3 \) and consider linear wave equation \((15)\) satisfying \((1), (2)\). Then we have the following higher order local energy estimates
\[ \|u\|_{L^\infty_t L^m_x} \lesssim \|\partial Y^{\leq m} u(0)\|_{L^2_x} + \|Y^{\leq m} F\|_{L^1_t L^2_x}, \quad 0 \leq m \leq 3. \] \quad (16)

**Proof.** It is proven in Wang [29, Lemma 3.5]. The result with \( g^1 = 0 \) has been proven in Bony-Häfner [1] and Sogge-Wang [23]. See also Metcalfe-Sterbenz-Tataru [18]. \( \square \)

**Lemma 3.2** (Weighted Strichartz estimates). Let \( n \geq 3 \). Consider linear wave equation \((15)\) with \((1), (2)\). Then there exists \( R > 0 \) so that if \( \psi_R \) is identically 1 on \( B^*_R \) and vanishes on \( B_R \), for \( 0 \leq m \leq 2 \), we have
\[ \|\psi_R Y^{\leq m} u\|_{L^\infty_t H^{s-1}} + \|\partial (\psi_R Y^{\leq m} u)\|_{L^\infty_t H^{s-1}} \]
\[ \lesssim \|\psi_R Y^{\leq m} u(0)\|_{H^s} + \|\psi_R Y^{\leq m} \partial_t u(0, \cdot)\|_{H^{s-1}} + \|\partial Y^{\leq m} F_1\|_{L^1_t L^2_x} + \|\psi_R Y^{\leq m} F_2\|_{L^1_t H^{s-1}} \]
for any \( p \in (2, \infty) \), \( s \in (1/2 - 1/p, 1/2) \), and \( 0 \leq \sigma < \min(s - 1/2 + 1/p, 1/2 - 1/p) \).

**Proof.** It is essentially proved in [20, Theorem 4.1], with new terms
\[ \|\partial (\psi_R Y^{\leq m} u)\|_{L^\infty_t H^{s-1}}, \quad \|\psi_R Y^{\leq m} F_2\|_{L^1_t H^{s-1}} \]
on the left and right. In fact, the main approach to obtain Theorem 4.1 in [20] is to exploit a sharper version of local energy estimates for small metric perturbation, due to Metcalfe-Tataru [19, Corollary 1]:
\[ \|\partial u\|_{L^\infty_t H^{s+} \cap X^\delta} \lesssim \|\partial u(0, \cdot)\|_{H^s} + \|F\|_{L^1_t H^{s+} \cap (X^\delta)^\prime}, \quad -1 < \delta < 0, \]
and interpolation between trace estimates (see, e.g., [20, Lemma 3.1]), where \( X^\delta \) is a microlocal version of the local energy norm, see, e.g., [20, Theorem 4.3] for its definition. To give the proof, we just follow the proof in [20, Theorem 4.1], while keeping \( \|\partial u\|_{L^\infty_t H^s} \) on the left hand and \( \|F_2\|_{L^1_t H^s} \) on the right in the process of interpolation. \( \square \)

**Lemma 3.3** (Space-time estimates). Let \( n \geq 4 \). Consider linear wave equations \((15)\) with \((1), (2)\). Then there exist a \( R \gg 1 \) so that \( \psi_R \) is identically 1 on \( B^*_R \) and vanishes on \( B_R \), for any \( T > 2 \) and \( m \in [0, 3] \), we have
\[ (\ln T)^{-1/4} \|\psi_R (r) - \frac{\alpha^2}{3} Y^{\leq m} u\|_{L^2_x L^2_t} + \|Y^{\leq m} u\|_{L^\infty T} \]
\[ + \|\psi_R Y^{\leq m} u\|_{L^2_x L^2_t} + \|\partial (\psi_R Y^{\leq m} u)\|_{L^\infty T H^{s-1}} \]
\[ \lesssim \|\partial Y^{\leq m} u(0, \cdot)\|_{L^2_t} + \|\psi_R Y^{\leq m} \partial_t u(0)\|_{H^{s-1}} + \|\psi_R Y^{-(n-3)/2} Y^{\leq m} F_1\|_{L^1_t L^1_x H^{s-1/2+}} \]
\[ + \|Y^{\leq m} (F_1, F_2)\|_{L^1_t L^2_x} + \|\psi_R Y^{\leq m} F_2\|_{L^1_t H^{s-1}} \]
**Proof.** This is essentially proved in [30, Lemma 5.6], with additional terms
\[ \|\partial (\psi_R Y^{\leq m} u)\|_{L^\infty T H^{s-1}}, \quad \|\psi_R Y^{\leq m} F_2\|_{L^1_t H^{s-1}} \]
on the left and right. The proof is based on the local energy estimates on non-trapping asymptotically Euclidean (Riemannian) manifolds, Lemma 3.1, as well as a sharper version of local energy estimates for small metric perturbation, due to Metcalfe-Tataru [19, Theorem 1].

At the end of this section, we collect the weighted Strichartz estimate for linear wave equation (15) posed on Euclidean space, i.e., $g = g^0$, which was essentially proved in [3] and [7].

**Lemma 3.4.** Let $n \geq 2$. Consider (15) with $g = g^0$. Then for $2 \leq p \leq \infty$ and $1/2 - 1/p < s < 1/2$, we have the following estimate

$$
\| r^{n\frac{1}{p} - \frac{n+1}{2} - s} Y^{\leq 1} u \|_{L^p_t L^p_x L^2_y} + \| \partial Y^{\leq 1} u \|_{\dot{H}^{-s}} 
\lesssim \| \partial Y^{\leq 1} u(0) \|_{\dot{H}^{-s}} + \| r^{-\frac{n}{2} + 1 - s} Y^{\leq 1} F_1 \|_{L^p_t L^p_x L^2_y} + \| Y^{\leq 1} F_2 \|_{L^p_t \dot{H}^{1-s}}.
$$

(18)

**Proof.** As $[\partial_t^2 - \Delta, Y] = 0$, it suffices to prove the estimate for $u$ instead of $Y^{\leq 1} u$. By [3, (1.14)] or [7, (3.8)], together with energy estimates, we have for any solutions to the homogeneous wave equations

$$
\| r^{n\frac{1}{p} - \frac{n+1}{2} - s} u \|_{L^p_t L^p_x L^2_y} + \| \partial u \|_{\dot{H}^{-s}} \lesssim \| \partial u(0) \|_{\dot{H}^{-s}}, \quad \frac{1}{2} - \frac{1}{p} < s < \frac{n}{2} - \frac{1}{p}.
$$

By Duhamel’s principle, we get for any solutions to (15),

$$
\| r^{n\frac{1}{p} - \frac{n+1}{2} - s} u \|_{L^p_t L^p_x L^2_y} + \| \partial u \|_{\dot{H}^{-s}} \lesssim \| \partial u(0) \|_{\dot{H}^{-s}} + \| F_1 \|_{L^1_t \dot{H}^{1-s}} + \| F_2 \|_{L^1_t \dot{H}^{1-s}}.
$$

For the term $\| F_1 \|_{L^1_t \dot{H}^{1-s}}$, we use the following dual estimate of trace lemma (see, e.g., [3, (1.3)])

$$
\| F_1 \|_{L^1_t \dot{H}^{1-s}} \lesssim \| r^{-\frac{n}{2} + 1 - s} F_1 \|_{L^1_t L^p_x L^2_y}, \quad \frac{1}{2} < 1 - s < \frac{n}{2}.
$$

This completes the proof. \qed

4. **Global existence for** $n = 3, 4$. In this section, we prove the global existence results, Theorem 1.1.

**Theorem 4.1.** Let $n = 3, 4$, and assume that (1), (2) hold. Consider (3) with (4) and $p > p_c$. Set $s = \frac{n}{2} - \frac{2}{p-1} \in \left(\frac{1}{2} - \frac{1}{q}, \frac{1}{2}\right)$ with $q = p$ if $p \in (p_c, 1 + 4/(n-1))$ and $q \in (p_c, 1 + 4/(n-1))$ is any fixed choice when $p \geq 1 + 4/(n-1)$. Then there exist $\epsilon_0$ sufficiently small and a $R$ sufficiently large, so that if $0 < \epsilon < \epsilon_0$ and

$$
\| \nabla Y^{\leq 2} u_0 \|_{L^\infty_x} + \| Y^{\leq 2} u_1 \|_{L^2_y} + \| \psi R Y^{\leq 2} u_1 \|_{\dot{H}^{1-s}} \leq \epsilon,
$$

(19)

then there exists a unique global solution $u \in C([0, \infty); H^3) \cap C^1([0, \infty); H^2)$.

**Proof.** Without loss of generality, we may assume $p \in (p_c, 1 + 4/(n-1))$. If not, one need only fix any $q \in (p_c, 1 + 4/(n-1))$ and apply the proof below while noting that Sobolev embeddings provide $\| u \|_{L^\infty_x} \lesssim \| \partial Y^{\leq 2} u \|_{L^p_t L^2_x}$, which suffices to handle the $p - q$ extra copies of the solution in the nonlinearity.

For $0 \leq m \leq 2$, we shall apply Lemma 3.2 with $s = \frac{n}{2} - \frac{2}{p-1}$ and note that $s \in \left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2}\right]$ precisely $p_c < p < 1 + 4/(n-1)$. We set $-\alpha = \frac{n}{2} - \frac{n+1}{p} - s = \frac{2}{p-1} - \frac{n+1}{p}$, then $-\frac{n}{2} - s = -\alpha p$. Let $\theta$ be a fixed number satisfying

$$
2 < \theta < \min\left(p, \frac{2(n-1)}{n-1 - 2 \min\left(s - \frac{1}{2} + \frac{1}{p}, \frac{1}{2} - \frac{1}{p}\right)}\right).
$$
We introduce
\[
\|u\|_{X_m} = \|r^{-\alpha}\psi R Y^{\leq m} u\|_{L^p L^p L^q} + \|\partial(\psi R Y^{\leq m} u)\|_{L^\infty H^{s-1}} \\
+ \|Y^{\leq m} u\|_{L^2 L^2 L^2} + \|\partial Y^{\leq m} u\|_{L^\infty L^2 L^2},
\]
where \( R \) is the large constant occurred in Lemma 3.2 and \( F = F_1 + F_2 \). In addition, for any \( T \in (0, \infty) \), we set \( \|u\|_{X_m}^+, \|F\|_{X_m^+} \) to denote integration about time from 0 to \( T \) in the definitions of \( X_m, N_m \).

Then by Lemma 3.1 and Lemma 3.2, for linear wave equation (15), there exist constants \( C_0, C_1 > 0 \) such that, for any \( 0 \leq m \leq 2 \) and \( T > 0 \),
\[
\|u\|_{X_m^+} \leq C_0 (\|\partial \delta^1 Y^{\leq m} u(0)\|_{L^2} + \|\partial \psi R Y^{\leq m} u(0)\|_{H^{s-1}} + \|F\|_{X_m^+}) \leq C_1 \epsilon + C_0 \|F\|_{N_m^+},
\]
where we have used (19) for the initial data.

We set \( u(0) = 0 \) and recursively define \( u^{(k+1)} \) be the solution to the linear equation
\[
\begin{cases}
\partial_t^2 u^{(k+1)} - \Delta g u^{(k+1)} + \mu \partial_t u^{(k+1)} + \mu \partial_j u^{(k+1)} + \mu_0 u^{(k+1)} = F_p(u^{(k)}), \\
u^{(k+1)}(0, x) = u_0, \partial_t u^{(k+1)}(0, x) = u_1.
\end{cases}
\]

We will prove that \( u^{(k)} \) is well defined, bounded in \( X_2 \) and convergent in \( X_0 \).

**Well defined:** It is easy to see \( u^{(1)} \in C([0, \infty); H^3) \cap C^1([0, \infty); H^2) \). We claim that \( u^{(k)} \in C([0, \infty); H^3) \cap C^1([0, \infty); H^2) \) for any \( k \geq 1 \), for which we prove by induction. Suppose we have \( u^{(k)} \in C([0, \infty); H^3) \cap C^1([0, \infty); H^2) \) for some \( k \geq 1 \). Since
\[
|\nabla^{\leq 2} F_p(u^{(k)})| \lesssim \|u^{(k)}\|_{H^1} |\nabla^{\leq 2} u^{(k)}| + |u^{(k)}| p-2 |\nabla^{\leq 1} u^{(k)}|^2,
\]
by Hölder’s inequality and Sobolev embedding, we have, for any \( t \in [0, T] \) with fixed \( T < \infty \),
\[
|\nabla^{\leq 2} F_p(u^{(k)})|_{L^2} \lesssim \|u^{(k)}\|_{H^1} |\nabla^{\leq 2} u^{(k)}|_{L^2} + \|u^{(k)}\|_{H^2} |\nabla^{\leq 1} u^{(k)}|_{L^2} \lesssim \|u^{(k)}\|_{H^1},
\]
Thus \( F_p(u^{(k)}) \in L^1([0, T]; H^2) \) for any \( T < \infty \). By standard existence theorem of linear wave equations (see, e.g., Sogge [22]), \( u^{(k+1)} \in C([0, T]; H^3) \cap C^1([0, T]; H^2) \) for any \( T < \infty \) and so \( u^{(k+1)} \in C([0, \infty); H^3) \cap C^1([0, \infty); H^2) \). Hence the iteration sequence is well defined.

**Boundedness:** For \( k = 0 \), we rewrite the equation (21) as
\[
\begin{cases}
\partial_t^2 u(0) - \Delta g u(0) = F = -\mu \partial_t u(0) - \mu \partial_j u(0) - \mu_0 u(0) \\
u(0, x) = u_0, \partial_t u(0, x) = u_1.
\end{cases}
\]
By applying (20) to (22) with \( F_1 = 0, F_2 = -\mu \partial_t u(0) - \mu \partial_j u(0) - \mu_0 u(0) \), we have
\[
\|u(0)\|_{X_2} \leq C_1 \epsilon + C_0 (\|u(0)\|_{H^{s-1}} + \|Y^{\leq 2} u(0)\|_{L^1 L^2 L^2}).
\]
By applying (13) and (14) to the last two terms, we have for any \( t > 0 \),
\[
\|
\psi R Y^{\leq 2} F_2(t)\|_{H^{s-1}} + \|Y^{\leq 2} F_2(t)\|_{L^2} \lesssim \|Y^{\leq 2} (\mu, \mu^j, r Y^{\leq 2} u_0(t))\|_{L^\infty W^{1,n}} \\
\times (\|\psi R Y^{\leq 2} u(0)(t)\|_{H^{s-1}} + \|\partial Y^{\leq 2} u(0)(t)\|_{L^2} + \|\partial (\psi R Y^{\leq 2} u(0)(t))\|_{H^{s-1}}).
\]
Notice that $\nabla_x \psi_R$ is compactly supported and so $\nabla_x \psi_R \in L^{n/(2-s)}$, then by Sobolev embedding and Hölder’s inequality, we have

\[
\|\psi_R \partial Y^{\leq 2} u\|_{\dot{H}^{s-1}} \lesssim \|\partial (\psi_R Y^{\leq 2} u)\|_{\dot{H}^{s-1}} + \|([\nabla_x \psi_R] Y^{\leq 2} u)\|_{\dot{H}^{s-1}} \\
\lesssim \|\partial (\psi_R Y^{\leq 2} u)\|_{\dot{H}^{s-1}} + \|([\nabla_x \psi_R] Y^{\leq 2} u)\|_{L^\frac{2n}{n-2}} \\
\lesssim \|\partial (\psi_R Y^{\leq 2} u)\|_{\dot{H}^{s-1}} + \| Y^{\leq 2} u \|_{L^\frac{2n}{n-2}} \|\nabla_x \psi_R\|_{L^\frac{n}{n-2}} \\
\lesssim \|\partial (\psi_R Y^{\leq 2} u)\|_{\dot{H}^{s-1}} + \| \nabla Y^{\leq 2} u \|_{L^2}.
\]

Hence we obtain from (23) and (24) that there exists some $C_2 > 0$, such that

\[
\|u^{(1)}\|_{X^2_T} \leq C_1 \epsilon + C_2 \int_0^T \| (Y^{\leq 2}(\mu, \mu^i), r Y^{\leq 2} \mu_0) (t)\|_{L^\infty_x \cap W^{1,n}} \\
\times (\|\partial (\psi_R Y^{\leq 2} u^{(1)})(t)\|_{\dot{H}^{s-1}} + \| \partial Y^{\leq 2} u^{(1)}(t)\|_{L^2}) \, dt,
\]

for any $T \in (0, \infty)$. As

\[
\|\partial (\psi_R Y^{\leq 2} u^{(1)})(t)\|_{\dot{H}^{s-1}} + \| \partial Y^{\leq 2} u^{(1)}(t)\|_{L^2} \leq \| u^{(1)} \|_{X^2_T}, \forall t > 0,
\]

by Gronwall’s inequality and (4), we obtain

\[
\|u^{(1)}\|_{X^2_T} \leq C_1 e^{C_2 \epsilon} = C_3 \epsilon,
\]

if we take possibly bigger $C_2$ such that $C_2 > \|(Y^{\leq 2}(\mu, \mu^i), r Y^{\leq 2} \mu_0)\|_{L^1_T(L^\infty_x \cap W^{1,n})}$.

Hence we obtain $\|u^{(k)}\|_{X^2_T} \leq 2C_3 \epsilon$ for some $k \geq 1$. Then we rewrite (21) as

\[
\begin{cases}
(\partial_t^2 - \Delta_g) u^{(k+1)} = F = F_p(u^{(k)}) - \mu \partial_t u^{(k+1)} - \mu^i \partial_j u^{(k+1)} - \mu_0 u^{(k+1)} \\
u^{(k+1)}(0, x) = u_0, \partial_t u^{(k+1)}(0, x) = u_1.
\end{cases}
\]

Applying (20) to (27) with $F_1 = F_p(u^{(k)})$, $F_2 = -\mu \partial_t u^{(k+1)} - \mu^i \partial_j u^{(k+1)} - \mu_0 u^{(k+1)}$, we get

\[
\|u^{(k+1)}\|_{X^2_T} \leq C_1 \epsilon + C_0 \|\psi_R Y^{\leq 2} F_2\|_{L^\frac{2}{1} \dot{H}^{s-1}} + C_0 \| Y^{\leq 2} F_2\|_{L^\frac{1}{2} L^2} \\
+ C_0 \| r^{-\alpha} \psi_R Y^{\leq 2} F_p(u^{(k)})\|_{L^\frac{1}{2} L^1 L^2} + C_0 \| Y^{\leq 2} F_p(u^{(k)})\|_{L^\frac{1}{2} L^2}. \tag{28}
\]

For the norms of $F_1 = F_p(u^{(k)})$, by [20, (5.4)] we have

\[
\|r^{-\alpha} \psi_R Y^{\leq 2} F_p(u^{(k)})\|_{L^\frac{1}{2} L^1 L^2} + \| Y^{\leq 2} F_p(u^{(k)})\|_{L^\frac{1}{2} L^1 L^2} \lesssim \| u^{(k)}\|_{X^2_T}^{\frac{3}{2}} \lesssim \epsilon^p. \tag{29}
\]

In fact, the $L^1 L^1 L^2$ norm of $r^{-\alpha} \psi_R Y^{\leq 2} F_p(u^{(k)})$ can be easily controlled by $\|u^{(k)}\|_{X^2_T}$ by applying Hölder’s inequality and the Sobolev embedding on the sphere. The $L^1 L^2$ norm is controlled for the region near spatial infinity and the remaining local part separately. For the region near spatial infinity, it is bounded by the weighted Strichartz norms by exploiting weighted Sobolev estimates. For the remaining part, it could be controlled by $L^p L^2$ norms by Sobolev estimates, which is further controlled by local energy and energy norms.

For the norms of $F_2$, we have $-\mu \partial_t u^{(k+1)} - \mu^i \partial_j u^{(k+1)} - \mu_0 u^{(k+1)}$, by the similar argument above we obtain for any $t > 0$,

\[
\|\psi_R Y^{\leq 2} F_2(t)\|_{\dot{H}^{s-1}} + \| Y^{\leq 2} F_2(t)\|_{L^2} \lesssim \|(Y^{\leq 2}(\mu, \mu^i), r Y^{\leq 2} \mu_0)(t)\|_{L^\infty_x \cap W^{1,n}} (\|\partial (\psi_R Y^{\leq 2} u^{(k+1)})(t)\|_{\dot{H}^{s-1}} + \| \partial Y^{\leq 2} u^{(k+1)}(t)\|_{L^2}) \\
\lesssim \|(Y^{\leq 2}(\mu, \mu^i), r Y^{\leq 2} \mu_0)(t)\|_{L^\infty_x \cap W^{1,n}} \| u^{(k+1)} \|_{X^2_T}.
\]
Hence by (28) we get for any $T \in (0, \infty)$,

\[
\|u^{(k+1)}\|_{X_T} \leq C_1 \epsilon + O(\epsilon^p) + C_2 \int_0^T \|(Y \leq 2(\mu, \mu^2), rY \leq 2\mu_0)(t)\|_{L^p_x \cap W^{1,n}} \|u^{(k+1)}\|_{X_T} dt.
\]

Applying Gronwall’s inequality, there exists $\epsilon_1 \ll 1$ so that we have

\[
\|u^{(k+1)}\|_{X_T} \leq (C_1 \epsilon + O(\epsilon^p)) e^{C_2 T} \leq 2C_3 \epsilon,
\]

for $0 < \epsilon < \epsilon_1$. Hence $\|u^{(k)}\|_{X_2} \leq 2C_3 \epsilon$ for any $k \geq 0$, by induction.

**Convergence of the sequence**: by (21) we have

\[
\left\{ \begin{array}{l}
(\partial_t^2 - \Delta_g)(u^{(k+1)} - u^{(k)}) = G, \\
(u^{(k+1)} - u^{(k)})(0, x) = 0, \partial_t(u^{(k+1)} - u^{(k)})(0, x) = 0,
\end{array} \right.
\]

where

\[
G = F_p(u^{(k)}) - F_p(u^{(k-1)}) - \mu \partial_t(u^{(k+1)} - u^{(k)}) - \mu^2 \partial_j(u^{(k+1)} - u^{(k)}) - \mu_0(u^{(k+1)} - u^{(k)}).
\]

By applying (20) to (31) with $G_1 = F_p(u^{(k)}) - F_p(u^{(k-1)})$ and $G_2 = -\mu \partial_t(u^{(k+1)} - u^{(k)}) - \mu^2 \partial_j(u^{(k+1)} - u^{(k)}) - \mu_0(u^{(k+1)} - u^{(k)})$, then we have

\[
\|u^{(k+1)} - u^{(k)}\|_{X_0^T} \leq C_0 \|r^{-\alpha_p} \psi R G_1\|_{L^1_t L^1_x L^2} + C_0 \|G_1\|_{L^1_t L^2_x} + C_0 \|\psi R G_2\|_{L^1_t H^{-1}} + C_0 \|G_2\|_{L^1_t L^2_x}.
\]

By [20, (5.5)], similar to (29), we have

\[
\|r^{-\alpha_p} \psi R G_1\|_{L^1_t L^1_x L^2} + \|G_1\|_{L^1_t L^2_x} \lesssim \|(u^{(k)}, u^{(k-1)})\|_{X_0^T}^p \|u^{(k)} - u^{(k-1)}\|_{X_0^T} \lesssim \epsilon^{p-1} \|u^{(k)} - u^{(k-1)}\|_{X_0^T}.
\]

For $G_2$ part, we have for any $t > 0$,

\[
\|\psi R G_2(t)\|_{H^{-1}} + \|G_2(t)\|_{L^2_x} \lesssim \|(\mu, \mu^2, r\mu_0)(t)\|_{L^\infty_x \cap W^{1,n}} \|\partial(\psi R(u^{(k+1)} - u^{(k)}))\|_{H^{-1}} + \|\partial(u^{(k+1)} - u^{(k)})\|_{L^2_x} \lesssim \|(\mu, \mu^2, r\mu_0)(t)\|_{L^\infty_x \cap W^{1,n}} \|u^{(k+1)} - u^{(k)}\|_{X_0^T}.
\]

Thus by (32), we have for any $T \in (0, \infty)$,

\[
\|u^{(k+1)} - u^{(k)}\|_{X_T} \leq C_4 \epsilon^{p-1} \|u^{(k)} - u^{(k-1)}\|_{X_0^T} + C_4 \int_0^T \|(\mu, \mu^2, r\mu_0)(t)\|_{L^\infty_x \cap W^{1,n}} \|u^{(k+1)} - u^{(k)}\|_{X_T} dt
\]

for some $C_4 > C_0 > 0$. By Gronwall’s inequality, for any $T < \infty$, we have

\[
\|u^{(k+1)} - u^{(k)}\|_{X_T} \leq C_4 \epsilon^{p-1} e^{C_4 T} \|u^{(k)} - u^{(k-1)}\|_{X_0^T} \leq \frac{1}{2} \|u^{(k)} - u^{(k-1)}\|_{X_0^T},
\]

for $0 < \epsilon < \epsilon_2$ with $\epsilon_2 \ll 1$, which yields convergence of the sequence in $X_0$. When $\epsilon_0 = \min(\epsilon_1, \epsilon_2)$ and $\epsilon \in (0, \epsilon_0)$, the limit $u \in X_2$ with $\|u\|_{X_2} \leq 2C_3 \epsilon$ is the solution we are looking for.
Theorem 5.1. Let almost global existence.

5. **Almost global existence.** In this section, we prove the almost global existence for the four-dimensional critical problem, Theorem 1.2. We set, for $T \in (0, \infty)$,

$$\|u\|_{\tilde{X}_T^3} := (\ln(2 + T))^{-1/4} \|\psi_R(r)^{-\frac{1}{2}} Y^{\leq m} u\|_{L^1_t L^2_x} + \|Y^{\leq m} u\|_{L^1_t L^2_x} + \|\partial_x Y^{\leq m} u\|_{L_T^2 H^{-1}}.$$  

Then by Lemma 3.3 with $n = 4$, for linear equation (15), there exists a constant $C_5 > 0$ such that for any $0 \leq m \leq 3$, we have

$$\|\nabla \dot{x}^3 u\|_{L^2_x} + \|Y^{\leq 3} u_1\|_{L^2_x} + \|\psi R Y^{\leq 3} u_1\|_{H^{-1}} \leq \epsilon$$

then there is a unique solution $u \in [0, T] \times \mathbb{R}^4$ with $\|u\|_{\tilde{X}_T^3} \leq \epsilon$, where $T_\epsilon = \exp(\epsilon^2)$.

**Proof.** Basically, the proof follows the similar way in Theorem 4.1. For convenience of statement, we introduce

$$\|F\|_{\tilde{X}_T^2} = \|\dot{\psi} R^{-1/2} Y^{\leq m} F_1\|_{L^1 L^2_t L^{\frac{2}{1+2}}} + \|Y^{\leq m} (F_1, F_2)\|_{L^1 L^2_t L^2} + \|\psi R Y^{\leq m} F_2\|_{L^1 L^2_t H^{-1}}.$$  

Then by Lemma 3.3 with $n = 4$, for linear equation (15), there exists a constant $C_5 > 0$ such that for any $0 \leq m \leq 3$, we have

$$\|u\|_{\tilde{X}_T^3} \leq C_5 \|\nabla \dot{x}^3 u\|_{L^2_x} + C_5 \|\psi R Y^{\leq m} \partial_x u(0)\|_{H^{-1}} + C_5 \|F\|_{\tilde{X}_T^2}.$$  

We set $u(0) = 0$ and recursively define $u^{(k+1)}$ be the solution to the linear equation

$$\begin{cases}
\partial_t^2 u^{(k+1)} - \Delta u^{(k+1)} + \mu \partial_x u^{(k+1)} + \mu^2 \partial_x u^{(k+1)} + \mu_0 u^{(k+1)} &= (u^{(k)})^2,

u^{(k+1)}(0, x) &= u_0, \partial_x u^{(k+1)}(0, x) = u_1.
\end{cases}$$

To complete the proof, we need to show that $u^{(k)}$ is well defined, bounded in $\tilde{X}_T^3$, and convergent in $\tilde{X}_T^3$.

**Well defined:** It is easy to see $u^{(1)} \in C([0, \infty); H^1) \cap C^1([0, \infty); H^3)$. If we have $u^{(k)} \in C([0, \infty); H^3) \cap C^1([0, \infty); H^4)$ for some $k \geq 1$. Since

$$|\nabla \dot{x}^3 u^{(k)}| \leq |\nabla \dot{x}^1 u^{(k)}| + |\nabla \dot{x}^2 u^{(k)}| + |u^{(k)} \nabla \dot{x}^3 u^{(k)}|,$$

by Hölder’s inequality and Sobolev embedding, we have, for any $t \in [0, T]$ with fixed $T < \infty$,

$$\|\nabla \dot{x}^3 u^{(k)}\|_{L^2_x} \lesssim \|\nabla \dot{x}^1 u^{(k)}\|_{L^4_x} + \|\nabla \dot{x}^2 u^{(k)}\|_{L^4_x} + \|u^{(k)}\|_{L^\infty_x} \|\nabla \dot{x}^3 u^{(k)}\|_{L^2_x} \lesssim \|\nabla \dot{x}^1 u^{(k)}\|_{H^1_x} + \|\nabla \dot{x}^2 u^{(k)}\|_{H^1_x} + \|u^{(k)}\|_{H^3_x} \|\nabla \dot{x}^3 u^{(k)}\|_{L^2_x} < \infty.$$  

Thus $(u^{(k)})^2 \in L^1([0, T]; H^3)$ for any $T < \infty$. By standard existence theorem of linear wave equations, we have $u^{(k+1)} \in C([0, T]; H^4) \cap C^1([0, T]; H^3)$ for any $T < \infty$ and so $u^{(k+1)} \in C([0, \infty); H^4) \cap C^1([0, \infty); H^3)$. Hence the iteration sequence is well defined.

In the following, we give the proof of boundedness and omit the similar proof of convergence.

**Boundedness in $\tilde{X}_T^3$:** As usual, we prove the boundedness by induction. Assuming $\|u^{(k)}\|_{\tilde{X}_T^3} \leq 2C_6$, for some $k \geq 0$ and $C_6$ to be determined later. Then we rewrite (36) as

$$\begin{cases}
(\partial_t^2 - \Delta_\mu) u^{(k+1)} = F = (u^{(k)})^2 - \mu \partial_x u^{(k+1)} - \mu^2 \partial_x u^{(k+1)} - \mu_0 u^{(k+1)},

u^{(k+1)}(0, x) = u_0, \partial_x u^{(k+1)}(0, x) = u_1.
\end{cases}$$

(37)
By applying (35) to (37) with $F_1 = (u^{(k)})^2$ and $F_2 = -\mu_0 \partial_t u^{(k+1)} - \mu_1 \partial_y u^{(k+1)} - \mu_0 u^{(k+1)}$, we obtain
\[
\|u^{(k+1)}\|_{\mathcal{X}_2^2} \leq C_5 \epsilon + C_5 \|F\|_{\mathcal{X}_2^2}.
\] (38)

For the norms of $F_1$ part, by [30, (5.15)], we have
\[
\|\psi_{R^{-1/2}} Y \leq 3 (u^{(k)})^2\|_{L^2_t L^1_x H^{-1/2} + \|Y \leq 3 (u^{(k)})^2\|_{L^1_t L^2_x} \lesssim (\ln(2 + T))^{1/2}\|u^{(k)}\|_{X^2_2}^2,
\]
for any $0 < T < \infty$. For the norms of $F_2$ part, by the similar argument of (24)-(25), we have for any $t > 0$,
\[
\|\psi_{R} Y \leq 3 F_2(t)\|_{H^{-1}} + \|Y \leq 3 F_2(t)\|_{L^2_x} \\
\lesssim \|Y \leq 3 (\mu, \mu^2), rY \leq 3 \mu_0(t)\|_{L^2_t \cap W^{1,4}} (\|\partial (\psi_{R} Y \leq 3 u^{(k+1)})\|_{H^{-1}} + \|\partial Y \leq 3 u^{(k+1)}\|_{L^2_x}) \\
\lesssim \|Y \leq 3 (\mu, \mu^2), rY \leq 3 \mu_0(t)\|_{L^2_t \cap W^{1,4}}\|u^{(k+1)}\|_{X^2_2}.
\]
Thus by (38) we get for any $0 < T < \infty$,
\[
\|u^{(k+1)}\|_{\mathcal{X}_2^2} \lesssim C_7 \epsilon + C_7 (\ln(2 + T))^{1/2}\|u^{(k)}\|_{X^2_2}^2 \\
+ C_7 \int_0^T \|Y \leq 3 (\mu, \mu^2), rY \leq 3 \mu_0(t)\|_{L^2_t \cap W^{1,4}}\|u^{(k+1)}\|_{X^2_2} dt,
\]
for some $C_7 > \|Y \leq 3 (\mu, \mu^2), rY \leq 3 \mu_0\|_{L^1_t \cap W^{1,4}}$ and $C_7$ is independent of $C_6$ and $T$. By Gronwall’s inequality we obtain
\[
\|u^{(k+1)}\|_{X^2_2} \leq (C_5 \epsilon + C_7 (\ln(2 + T))^{1/2}\|u^{(k)}\|_{X^2_2}^2) e^{C_7^2}.
\]
Let $C_6 = C_5 \epsilon C_7^2$, and $c, \epsilon_0 > 0$ such that
\[
8c^{1/2}C_6 C_7 e^{C_7^2} \leq 1, e^{c \epsilon_0^2} \geq 2,
\]
then for any $\epsilon \in (0, \epsilon_0)$, if we set $T = T_\epsilon = e^{c \epsilon_0^2} \geq 2$, we have
\[
\|u^{(k+1)}\|_{X^2_2} \leq (C_5 \epsilon + C_7 (\ln(2 + T))^{1/2}\|u^{(k)}\|_{X^2_2}^2) e^{C_7^2} \leq 2C_6 \epsilon.
\]
Hence $\|u^{(k)}\|_{X^2_2} \leq 2C_6 \epsilon$ for any $k \geq 0$, by induction. □

6. Global existence for $n = 2$. In this section, we prove the global existence of (3) when $n = 2$.

**Theorem 6.1.** Let $n = 2$ and $p > p_c(2)$. Consider (3) with $q = q^0$ and (10). Set $s = 1 - \frac{2}{p - 1} \in (\frac{1}{2} - \frac{1}{2}, \frac{1}{2})$ with $q = p$ if $p \in (p_c, 5)$ and $q \in (p_c, 5)$ is any fixed choice when $p \geq 5$. Then there exists $\epsilon_0 > 0$, such that (3) admits a unique solution $u \in C([0, \infty); H^{s+1}) \cap C^1([0, \infty); H^s)$ for any data satisfying
\[
\|Y \leq 1 u_0\|_{H^s} + \|Y \leq 1 u_1\|_{H^{s-1}} \leq \epsilon_0.
\] (39)

**Proof.** As in Section 4, we may assume $p \in (p_c, 5)$ without loss of generality, as we could always use the Sobolev embedding to control $p - q$ extra copies of the solution in the nonlinearity: $\|u\|_{L^\infty_t L^p x} \lesssim \|u\|_{L^p_t (H^{s+1} \cap H^s)}$.

We will apply Lemma 3.4 with $n = 2$, $s = 1 - \frac{2}{p - 1} \in (\frac{1}{2} - \frac{1}{2}, \frac{1}{2})$, as $p_c < p < 5$. Let $-\alpha = 1 - \frac{3}{p - 1} - s = \frac{2}{p - 1} - \frac{3}{p}$, then $-s = -\alpha p$. For $0 \leq m \leq 1$, we set
\[
\|u\|_{X_m} = \|r^{-\alpha} Y \leq m u\|_{L^p_t L^p_x} + \|\partial Y \leq m u\|_{L^\infty H^{s+1}},
\]
\[
\|F\|_{N_m} = \|r^{-\alpha p} Y \leq m F_1\|_{L^1_t L^1_x} + \|Y \leq m F_2\|_{L^1 H^{s+1}}.
\]
where \( F = F_1 + F_2 \). In addition, we introduce \( X^T_m, N^T_m \) to denote the integration about time from 0 to \( T \) in the definition of \( X_m, N_m \).

Then for linear equation (15), by Lemma 3.4 and (39), there exists constant \( C_8 > 0 \) such that for any \( T \in (0, \infty) \),

\[
\| u \|_{X^T_m} \leq C_8 \epsilon + C_8 \| F \|_{N^T_m},
\]

where \( \epsilon = \| Y^{\leq 1} u_0 \|_{H^s} + \| Y^{\leq 1} u_1 \|_{H^{s-1}} \). We set \( u^{(0)} = 0 \) and recursively define \( u^{(k+1)} \) be the solution to the linear equation

\[
\begin{cases}
\partial_t^2 u^{(k+1)} - \Delta u^{(k+1)} + \mu \partial_t u^{(k+1)} + \mu^2 \partial_j u^{(k+1)} + \mu_0 u^{(k+1)} = F_p(u^{(k)}), \\
u^{(k+1)}(0, x) = u_0, \partial_t u^{(k+1)}(0, x) = u_1.
\end{cases}
\]

We will prove that \( u^{(k)} \) is well defined, bounded in \( X_1 \) and convergent in \( X_0 \).

**Well defined:** It is easy to see \( u^{(1)} \in C([0, \infty); H^{s+1}) \cap C^1([0, \infty); H^s) \). We claim that \( u^{(k)} \in C([0, \infty); H^{s+1}) \cap C^1([0, \infty); H^s) \) for any \( k \geq 1 \), for which we prove by induction. Suppose we have \( u^{(k)} \in C([0, \infty); H^{s+1}) \cap C^1([0, \infty); H^s) \) for some \( k \geq 1 \). Since \( p > p_c = (3 + \sqrt{17})/2 \), we have

\[
\| F_p(u^{(k)}) \|_{H_s} \leq \| u^{(k)} \|_{L_p}^{p-1} \| u^{(k)} \|_{H^s} \leq \| u^{(k)} \|_{H^{s+1}},
\]

and so \( F_p(u^{(k)}) \in L^1_1([0, T]; H^s) \) for any \( T < \infty \). By standard existence theorem of linear wave equations (see, e.g., Sogge [22]), \( u^{(k+1)} \in C([0, T]; H^{s+1}) \cap C^1([0, T]; H^s) \) for any \( T < \infty \) and so \( u^{(k+1)} \in C([0, \infty); H^{s+1}) \cap C^1([0, \infty); H^s) \). Hence the iteration sequence is well defined.

In the following, we give the proof of boundedness and omit the similar proof of convergence.

**Boundedness in \( X_1 \):** As usual, we prove the boundedness by induction. Assuming \( \| u^{(k)} \|_{X_1} \leq 2C_{10}\epsilon \), for some \( k \geq 0 \) and \( C_{10} \) to be determined later. Then we rewrite (41) as

\[
\begin{cases}
(\partial_t^2 - \Delta) u^{(k+1)} = F = F_p(u^{(k)}) - \mu \partial_t u^{(k+1)} - \mu^2 \partial_j u^{(k+1)} - \mu_0 u^{(k+1)}, \\
u^{(k+1)}(0, x) = u_0, \partial_t u^{(k+1)}(0, x) = u_1.
\end{cases}
\]

By applying (40) to (42) with \( F_1 = F_p(u^{(k)}) \) and \( F_2 = -\mu \partial_t u^{(k+1)} - \mu^2 \partial_j u^{(k+1)} - \mu_0 u^{(k+1)} \), we obtain for any \( T \in (0, \infty) \),

\[
\| u^{(k+1)} \|_{X^T} \leq C_8 \epsilon + C_8 \| F \|_{N^T}.
\]

For the norm of \( F_1 \), since

\[
\| Y^{\leq 1} F_1 \| \lesssim \| u^{(k)} \|^{p-1} \| Y^{\leq 1} u^{(k)} \|,
\]

by Hölder’s inequality and Sobolev embedding (\( H^1(S^1) \subset L^\infty(S^1) \)), we have

\[
\| u^{(k)} \|^{p-1} \| Y^{\leq 1} u^{(k)} \|_{V^2} \lesssim \| u^{(k)} \|^{p-1} \| Y^{\leq 1} u^{(k)} \|_{L^\infty} \lesssim \| Y^{\leq 1} u^{(k)} \|_{L^\infty} \lesssim \| Y^{\leq 1} u^{(k)} \|_{L^p}^{p},
\]

which gives

\[
\| r^{-\alpha} Y^{\leq 1} F_1 \|_{L^p_{\alpha} L^2} \lesssim \| r^{-\alpha} Y^{\leq 1} u^{(k)} \|_{L^p_{\alpha} L^2} \lesssim \| u^{(k)} \|_{X^T} \lesssim (2C_{10}\epsilon)^p.
\]
For the first two terms in $F_2$, by fractional Leibniz rule [32, Lemma 2.7], we have for any $t > 0$,
\[
\|Y^{\leq 1}(\mu \partial_t u^{(k+1)} + \mu^j \partial_j u^{(k+1)})(t)\|_{\dot{H}^{s-1}} \\
\lesssim \|Y^{\leq 1}(\mu, \mu^j)(t)\|_{L^\infty \cap W^{1-\frac{1}{2}-s, 2(1-s)}_1} \|\partial Y^{\leq 1} u^{(k+1)}(t)\|_{\dot{H}^{s-1}} \\
\lesssim \|Y^{\leq 1}(\mu, \mu^j)(t)\|_{L^\infty \cap H^1} \|\partial Y^{\leq 1} u^{(k+1)}(t)\|_{\dot{H}^{s-1}},
\]
where we have used the Sobolev embedding $\dot{H}^1 \subset \dot{W}^{1-s, 2/(1-s)}$ since $s \in (0, 1/2)$.
For the term $\|Y^{\leq 1}(\mu_0 u^{(k+1)})(t)\|_{\dot{H}^{s-1}}$, by Hardy’s inequality and its dual version, we have
\[
\|Y^{\leq 1}(\mu_0 u^{(k+1)})(t)\|_{\dot{H}^{s-1}} \lesssim \|\langle Y^{\leq 1} \mu_0 \rangle(Y^{\leq 1} u^{(k+1)})(t)\|_{\dot{H}^{s-1}} \\
\lesssim \|r^{-s} \langle Y^{\leq 1} \mu_0 \rangle(Y^{\leq 1} u^{(k+1)})(t)\|_{L^2} \\
\lesssim \|r Y^{\leq 1} \mu_0(t)\|_{L^\infty} \|Y^{\leq 1} u^{(k+1)}(t)\|_{\dot{H}^{s-1}}.
\]
Hence by (43) we get for any $T \in (0, \infty)$,
\[
\|u^{(k+1)}\|_{X_T^s} \leq C_8 \epsilon + C_9 (2C_{10} \epsilon)^p \\
+ C_9 \int_0^T \|Y^{\leq 1}(\mu, \mu^j), r Y^{\leq 1} \mu_0\|_{L^\infty \cap \dot{H}^1} \|u^{(k+1)}\|_{X_s^1} \, dt,
\]
for some $C_9 > 0$. By Gronwall’s inequality and (10), we obtain for any $T \in (0, \infty)$,
\[
\|u^{(k+1)}\|_{X_T^s} \leq (C_8 \epsilon + C_9 (2C_{10} \epsilon)^p) e^{C_9^2},
\]
if we further take $C_9 > \|\langle Y^{\leq 1}(\mu, \mu^j), r Y^{\leq 1} \mu_0\|_{L^s_{\infty}(L^\infty \cap \dot{H}^1)}$.
If we take $C_{10} = C_8 \epsilon^{C_9^2}$ and $\epsilon_0$ such that
\[
C_8 \epsilon^{C_9^2} e^{C_9^2} \leq 1,
\]
then $\|u^{(k)}\|_{X_1^s} \leq 2C_{10} \epsilon$ for any $0 < \epsilon < \epsilon_0$ and $k \geq 0$, by induction. \hfill $\square$

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