COMPARATIVE STUDY OF FRACTIONAL FOKKER-PLANCK EQUATIONS WITH VARIOUS FRACTIONAL DERIVATIVE OPERATORS

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Abstract. This paper presents a comparative study of fractional Fokker-Planck equations with various fractional derivative operators such as Caputo fractional derivative, Atangana-Baleanu fractional derivative and conformable fractional derivative. The new iterative method has been successively applied for finding approximate analytical solutions of the fractional Fokker-Planck equations with various fractional derivative operators. This method gives an analytical solution in the form of a convergent series with easily computable components. The behavior of solutions and the effects of different values of fractional order are shown graphically for various fractional derivative operators. Some examples are given to show ability of the method for solving the fractional Fokker-Planck equations.

1. Introduction. Fractional calculus has been studied for over three centuries and it has numerous applications in science and engineering. The analysis and applications of fractional differential equations become an active research area due to their various applications in science and engineering. (See [12, 15, 9]).

Our aim in this paper is to examine the approximate analytical solutions of Fokker-Planck equations with various space and time fractional derivatives which are given by

\[ \frac{\partial^\alpha z}{\partial t^\alpha} = \left[ -\frac{\partial^\beta U(x, t, z)}{\partial x^\beta} + \frac{\partial^{2\beta} V(x, t, z)}{\partial x^{2\beta}} \right] z(x, t), 0 < \alpha, \beta \leq 1, \]  

subject to the initial condition \( z(x, 0) = f(x) \). \( U(x, t, z) \) and \( V(x, t, z) \) are drift and diffusion coefficients, \( \alpha \) and \( \beta \) are parameters which describe the order of time and space derivatives, respectively. In the case of \( \alpha = 1, \beta = 1 \), equation (1) reduces to classical Fokker-Planck equations.

Analytic solution of fractional order Fokker-Planck equation has been obtained by different methods such as iterative Laplace transform method [17], Adomian decomposition method [14], q-homotopy analysis transform method(q-HATM) [16] and homotopy perturbation sumudu transform method [8].

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2. Preliminaries. In this section, some basic definitions of fractional calculus are given.

**Definition 2.1.** A real valued function \( f(t), t > 0 \) is said to be in the space \( C_{\mu}, \mu \in \mathbb{R} \) if there exist a real number \( \rho(\mu) \) such that \( f(t) = t^\rho f_1(t) \), where \( f_1 \in C[0, \infty) \), and is said to be in the space \( C_{\mu}^m \) if \( f^m \in C_{\mu}, m \in \mathbb{N} \cup \{0\} \).

**Definition 2.2.** ([11]) The Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( f \in C_{\mu}, \mu \geq -1 \) is defined as:

\[
_{RL}aI_t^\alpha(f(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \, d\tau,
\]  
where \( 0 < \alpha < 1 \).

**Definition 2.3.** The Caputo fractional derivative of order \( \alpha \) of \( f, f \in C_{m-1}, m \in \mathbb{N} \cup \{0\} \) defined as([5])

\[
_{C}D_t^\alpha(f(t)) = \begin{cases} 
\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha+1-m}} \, d\tau, & \text{if } \alpha = m \\
\frac{d^m}{dt^m} f(t), & \text{if } 0 < \alpha < m, m \in \mathbb{N}\end{cases}
\]

where \( m-1 < \alpha < m, m \in \mathbb{N} \) and \( \frac{d^m}{dt^m} f(t) \) is the \( m \)-th derivative of the function \( f(t) \) with respect to \( t \).

It is important to note that([11])

\[
_{RL}0I_t^\alpha(t^\mu) = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu},
\]

and

\[
_{RL}0I_t^\alpha(_{C}D_t^\alpha(f(t))) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}.
\]  

**Definition 2.4.** ([2]) Let \( f \) be a differentiable function on \( [a, b] \) such that \( f' \in L^1(a, b) \). Then the Atangana-Baleanu fractional derivative (in Caputo sense) of order \( \alpha \) is defined as

\[
_{AB}aD_t^\alpha(f(t)) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(\tau) E_\alpha \left( -\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right) \, d\tau,
\]  
where \( 0 < \alpha < 1, a < t < b \) and Mittag-Leffler function \( E_\alpha(z) \) is defined by ([13])

\[
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}
\]

and \( B(\alpha) \) is a normalization function such that \( B(0) = B(1) = 1([6]) \).

**Definition 2.5.** ([3]) The Atangana-Baleanu fractional integral operator \( _{AB}aI_t^\alpha \) defined by

\[
_{AB}aI_t^\alpha(f(t)) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} _{RL}aI_t^\alpha(f(t)),
\]  
where \( 0 < \alpha < 1, a < t < b \) and \( f \in L^1(a, b) \).
Lemma 2.6. ([3]) The Atangana-Baleanu fractional integral (5) and the Atangana-Baleanu fractional derivative (4) satisfy the following Newton-Leibnitz formula:

\[ AB_0^t I_0^\alpha (AB_0^a D_0^\alpha f(t)) = f(t) - f(a), \]

where \( 0 < \alpha \leq 1, a < t < b \) and \( f \) is a differentiable function on \([a, b]\) such that \( f' \in L^1(a, b) \).

**Definition 2.7.** The conformable fractional derivative of a function \( f : [a, \infty) \to \mathbb{R} \) of order \( 0 < \alpha \leq 1 \) is defined by ([10])

\[ a D_t^\alpha (f(t)) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t - a)^{1-\alpha}) - f(t)}{\epsilon}. \]

**Definition 2.8.** ([1]) (conformable fractional integral) Let \( \alpha \in (0, 1] \) and \( 0 \leq a < b \). A function \( f : [a, b] \to \mathbb{R} \) is \( \alpha \)-fractional integrable on \([a, b]\) if the integral

\[ a I_t^\alpha (f(t)) = \int_a^b f(x)x^{\alpha-1} \, dx \]  

exist and finite.

The conformable fractional integral of power function \((t-a)^\mu\) of order \( 0 < \alpha \leq 1 \) is

\[ a I_t^\alpha ((t-a)^\mu)(x) = \frac{\Gamma(\alpha + \mu)}{\Gamma(\alpha + \mu + 1)}(x-a)^{\alpha+\mu}. \]

We shall use following result ([11]).

**Lemma 2.9.** Let \( f : (a, b) \to \mathbb{R} \) be differentiable and \( 0 < \alpha \leq 1 \). Then, for all \( t > a \), we have

\[ a I_t^\alpha (a D_t^\alpha f(t)) = f(t) - f(a). \]

3. **The new iterative method.** Daftardar-Gejji and Jafari [7] have consider the following nonlinear functional equation

\[ z(\bar{x}, t) = f(\bar{x}, t) + L(z(\bar{x}, t)) + N(z(\bar{x}, t)), \]

where \( L \) and \( N \) are given linear and nonlinear functions of \( u \), \( f \) is a known function and \( \bar{x} = (x_1, x_2, ..., x_n) \). Equation (9) is assumed to have a solution of the form

\[ z(\bar{x}, t) = \sum_{i=0}^{\infty} z_i(\bar{x}, t). \]

Since \( L \) is linear,

\[ L \left( \sum_{i=0}^{\infty} z_i \right) = \sum_{i=0}^{\infty} L(z_i). \]

Further define

\[ G_0 = N(z_0), \]

\[ G_m = N \left( \sum_{i=0}^{m} z_i \right) - N \left( \sum_{i=0}^{m-1} z_i \right), \]

The recurrence relation is defined as

\[ z_0 = f, \]

\[ z_1 = L(z_0) + G_0, \]

\[ z_{m+1} = L(z_m) + G_m, m = 1, 2, ... \]
hence,
\[ \sum_{i=1}^{m+1} z_i = L \left( \sum_{i=0}^{\infty} z_i \right) + N \left( \sum_{i=0}^{m} z_i \right) \]
and
\[ \sum_{i=0}^{\infty} z_i = f + L \left( \sum_{i=0}^{\infty} z_i \right) + N \left( \sum_{i=0}^{\infty} z_i \right). \]

The m-term approximate solution of (9) is given by \( z(x,t) = \sum_{k=0}^{m-1} z_k \).

We present below the condition for convergent of series \( \sum_{i=1}^{\infty} z_i \).

**Theorem 3.1.** ([4]) If \( N \) is \( C^{(\infty)} \) in a neighbourhood of the \( u_0 \) and \( \|N^{(n)}(z_0)\| \leq L \), for any \( n \) and for some real \( L > 0 \) and \( \|z_i\| \leq M < \frac{1}{L}, i = 1, 2, \ldots \), then the series \( \sum_{n=0}^{\infty} G_n \) is absolutely convergent and moreover,
\[ \|G_n\| \leq LM^n e^{n-1}(e-1), n = 1, 2, \ldots \]

**Theorem 3.2.** ([4]) If \( N \) is \( C^{(\infty)} \) and \( \|N^{(n)}(z_0)\| \leq M \leq e^{-1}, \forall n \), then the series \( \sum_{n=0}^{\infty} G_n \) is absolutely convergent.

4. **Fractional Fokker-Planck equations with various fractional derivative operators.** In this section, we solve linear and non-linear fractional Fokker-Planck equation with different fractional derivative operators using new iterative method.

**Example 4.1.** We consider the following linear time fractional Fokker-Planck equation with Caputo fractional derivative
\[ C_0D_t^\alpha(u(x,t)) = -\frac{\partial(xu)}{\partial x} + \frac{\partial^2(x^2u)}{\partial x^2}, \quad x > 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (10) \]
with initial condition
\[ u(x,0) = x. \quad (11) \]

The exact solution of the problem (10) for \( \alpha = 1 \) is given as \( u(x,t) = xe^t \). First, apply Riemann-Liouville integral operator (2) on both side of (10) and using (3) and (11), we get
\[ u(x,t) = x - RL_0 I_t^\alpha \left( \frac{\partial(xu)}{\partial x} \right) + RL_0 I_t^\alpha \left( \frac{\partial^2(x^2u)}{\partial x^2} \right). \quad (12) \]

Equation (12) has the form (9) with \( f = x, \quad L(u) = -RL_0 I_t^\alpha \left( \frac{\partial(xu)}{\partial x} \right) + RL_0 I_t^\alpha \left( \frac{\partial^2(x^2u)}{\partial x^2} \right) \) and \( N(u) = 0 \). Using the algorithm of new iterative method, we get recurrence relations
\[ u_0 = x, \quad u_1 = x, \quad u_2 = x, \quad \ldots \]

Thus solution of problem (10)-(11) is
\[ u(x,t) = x \sum_{k=0}^{\infty} \frac{t^\alpha}{\Gamma(\alpha n + 1)} = x E_\alpha(t^\alpha). \]
Example 4.2. Now, we consider the following linear time fractional Fokker-Plank equation with Atangana-Baleanu fractional derivative

$$AB_0D_t^\alpha (u(x,t)) = -\frac{\partial (xu)}{\partial x} + \frac{\partial^2 (x^2 u^2)}{\partial x^2}, \quad x > 0, \ t > 0, \ 0 < \alpha \leq 1, \quad (13)$$

with initial condition

$$u(x,0) = x. \quad (14)$$

Apply Atangana-Baleanu fractional integral operator (5) on both side of (13) and using (6) and (14), we obtain

$$u(x,t) = x - AB_0I_t^\alpha \left( \frac{\partial (xu)}{\partial x} \right) + AB_0I_t^\alpha \left( \frac{\partial^2 (x^2 u^2)}{\partial x^2} \right).$$

In view of (9), we have

$$L(u) = -AB_0I_t^\alpha \left( \frac{\partial (xu)}{\partial x} \right) + AB_0I_t^\alpha \left( \frac{\partial^2 (x^2 u^2)}{\partial x^2} \right)$$

and

$$N(u) = 0.$$

Applying new iterative method, we get

$$u_0 = x, \ u_1 = x - t^\alpha \frac{\alpha t^\alpha}{B(\alpha)} + \frac{\alpha t^\alpha}{B(\alpha) \Gamma(\alpha + 1)};$$

$$u_2 = x \left( \frac{1 - \alpha}{B(\alpha)} \right)^2 + x \frac{2\alpha(1 - \alpha)t^\alpha}{B(\alpha)2\Gamma(\alpha + 1)} + \frac{\alpha^2 t^{2\alpha}}{(B(\alpha))^2\Gamma(2\alpha + 1)};$$

Hence the three term solution of problem (13)-(14) is

$$u(x,t) = x + x \frac{1 - \alpha}{B(\alpha)} + x \frac{\alpha t^\alpha}{B(\alpha) \Gamma(\alpha + 1)} + x \left( \frac{1 - \alpha}{B(\alpha)} \right)^2 + x \frac{2\alpha(1 - \alpha)t^\alpha}{(B(\alpha))^2\Gamma(2\alpha + 1)} + \frac{x \alpha^2 t^{2\alpha}}{(B(\alpha))^2\Gamma(2\alpha + 1)}.$$

Example 4.3. Let us consider the following linear time fractional Fokker-Plank equation with conformable time fractional derivative

$$0D_t^\alpha (u(x,t)) = -\frac{\partial (xu)}{\partial x} + \frac{\partial^2 (x^2 u^2)}{\partial x^2}, \quad x > 0, \ t > 0, \ 0 < \alpha \leq 1, \quad (15)$$

with initial condition

$$u(x,0) = x. \quad (16)$$

Apply conformable fractional integral operator (7) on both side of (15) and using (8) and (11), we get

$$u(x,t) = x - 0I_t^\alpha \left( \frac{\partial (xu)}{\partial x} \right) + 0I_t^\alpha \left( \frac{\partial^2 (x^2 u^2)}{\partial x^2} \right).$$

In view of (9), we have

$$L(u) = -0I_t^\alpha \left( \frac{\partial (xu)}{\partial x} \right) + 0I_t^\alpha \left( \frac{\partial^2 (x^2 u^2)}{\partial x^2} \right)$$

and

$$N(u) = 0.$$

Applying the algorithm of new iterative method, we obtain

$$u_0 = x, \ u_1 = x \frac{t^\alpha}{\alpha}, \ u_2 = x \frac{t^{2\alpha}}{2\alpha^2}, \ u_3 = x \frac{t^{3\alpha}}{6\alpha^3};$$

Thus solution of (15)-(16) is

$$u(x,t) = x \sum_{k=0}^{\infty} \frac{\left( \frac{t^\alpha}{\alpha} \right)^n}{n!} = xe^{\frac{t^\alpha}{\alpha}}.$$
Figure 1. Behavior of $u(x,t)$ corresponding to the values $\alpha = 0.3$, $\alpha = 0.6$ and $\alpha = 0.9$ for $B(\alpha) = 1$ and $t = 5$ from left to right.

Figure 2. Behavior of $u(x,t)$ corresponding to the values $\alpha = 0.5$ for Caputo fractional derivative, Atangana-Baleanu fractional derivative and conformable fractional derivative from left to right.
Table 1. Comparison of \( u(x, t) \) with different fractional differential operators at different values of \( \alpha \) when \( x = 2, t = 3 \)

| \( \alpha \) | Caputo derivative | Atangana-Baleanu derivative | Conformable derivative |
|-------------|------------------|----------------------------|------------------------|
| 0.25        | 8.8128           | 6.6843                     | 40.2414                |
| 0.5         | 11.9088          | 8.9088                     | 20.9282                |
| 0.75        | 14.7781          | 12.6030                    | 17.3163                |
| 1           | 17               | 17                         | 17                     |

Figure 1 gives comparison of the solution \( u(x, t) \) of (10), (13) and (15) for different values of \( \alpha \). Figure 2 exhibits the behavior of solution for different fractional derivative operators for \( \alpha = 0.5 \).

**Example 4.4.** We consider the following non-linear time fractional Fokker-Plank equation with Caputo fractional derivative

\[
C_0 D_t^\alpha (u(x,t)) = -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} - \frac{xu^3}{3} \right) + \frac{\partial^2}{\partial x^2} (u^2), \quad x > 0, \ t > 0, \ 0 < \alpha \leq 1,
\]

with initial condition

\[
u(x,0) = x^2.
\]

Apply Riemann-Liouville integral operator (2) on both side of (17) and using (3) and (18), we get

\[
u(x,t) = x^2 + \int_0^t \left[ -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} - \frac{xu^3}{3} \right) \right] + \int_0^t \left[ \frac{\partial^2}{\partial x^2} (u^2) \right].
\]

Equation (19) has the form (9) with \( f = x^2 \), \( N(u) = \int_0^t \left[ -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} \right) \right] \) + \( \int_0^t \left[ \frac{\partial^2}{\partial x^2} (u^2) \right] \) \( L(u) = \int_0^t \left[ \frac{\partial}{\partial x} (\frac{xu}{3}) \right] \). Using the algorithm of new iterative method, we get recurrence relations

\[
u_0 = x^2, \ \nu_1 = x^2 + \frac{t^\alpha}{\Gamma(\alpha + 1)}, \ \nu_2 = x^2 + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \ldots
\]

Thus solution of (17)-(18) is

\[
u(x,t) = x^2 \sum_{k=0}^{\infty} \frac{(t^\alpha)^n}{\Gamma(\alpha n + 1)} = x^2 E_\alpha(t^\alpha).
\]

**Remark 1.** Setting \( \alpha = 1 \), Example 4.7 reduces to non-linear partial differential equation

\[
\frac{\partial}{\partial t} (u(x,t)) = -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} - \frac{xu^3}{3} \right) + \frac{\partial^2}{\partial x^2} (u^2), \quad x > 0, \ t > 0,
\]

with initial condition

\[
u(x,0) = x^2
\]

and a solution as

\[
u(x,t) = x^2 e^t.
\]
Example 4.5. Now, we consider the following non-linear time fractional Fokker-Plank equation with Atangana-Baleanu fractional derivative
\[
\mathcal{A} \mathcal{B}_0 D_t^\alpha (u(x,t)) = -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} - \frac{xt}{3} \right) + \frac{\partial^2}{\partial x^2} (u^2), \quad x > 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (20)
\]
with initial condition
\[
u(x,0) = x^2. \quad (21)
\]
Apply Atangana-Baleanu fractional integral operator (5) on both side of (20) and using (6) and (21), we obtain
\[
u(x,t) = x^2 + \mathcal{A} \mathcal{B}_0 I_t^\alpha \left[ -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} - \frac{xt}{3} \right) \right] + \mathcal{A} \mathcal{B}_0 I_t^\alpha \left[ \frac{\partial^2}{\partial x^2} (u^2) \right].
\]
In view of (9), we have \(N(u) = \mathcal{A} \mathcal{B}_0 I_t^\alpha \left[ -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} \right) \right] + \mathcal{A} \mathcal{B}_0 I_t^\alpha \left[ \frac{\partial^2}{\partial x^2} (u^2) \right]\) and \(L(u) = \mathcal{A} \mathcal{B}_0 I_t^\alpha \left[ \frac{\partial}{\partial x} \left( \frac{u^2}{x} \right) \right]\). Applying new iterative method, we get
\[
\begin{align*}
u_0 &= x^2, 
\nu_1 &= x^2 \left[ 1 - \frac{\alpha t}{\mathcal{B}(\alpha)} + \frac{\alpha t^\alpha}{\mathcal{B}(\alpha) \Gamma(\alpha + 1)} \right], 
\nu_2 &= x^2 \left[ \left( 1 - \frac{1 - \alpha}{\mathcal{B}(\alpha)} \right)^2 + \frac{2\alpha(1 - \alpha)}{(\mathcal{B}(\alpha))^2} \frac{t^\alpha}{\Gamma(\alpha + 1)} + \left( \frac{\alpha}{\mathcal{B}(\alpha)} \right)^2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \right] \cdot ...
\end{align*}
\]
Hence the three term solution of problem (20)-(21) is
\[
u(x,t) = x^2 + x^2 \left[ 1 - \frac{\alpha t}{\mathcal{B}(\alpha)} + \frac{\alpha t^\alpha}{\mathcal{B}(\alpha) \Gamma(\alpha + 1)} \right] + x^2 \left[ \left( 1 - \frac{1 - \alpha}{\mathcal{B}(\alpha)} \right)^2 + \frac{2\alpha(1 - \alpha)}{(\mathcal{B}(\alpha))^2} \frac{t^\alpha}{\Gamma(\alpha + 1)} + \left( \frac{\alpha}{\mathcal{B}(\alpha)} \right)^2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \right].
\]
Example 4.6. Let us consider the following non-linear time fractional Fokker-Plank equation with conformable time fractional derivative
\[
_{0}D_t^\alpha (u(x,t)) = -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} - \frac{xt}{3} \right) + \frac{\partial^2}{\partial x^2} (u^2), \quad x > 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (22)
\]
with initial condition
\[
u(x,0) = x^2. \quad (23)
\]
Apply conformable fractional integral operator (7) on both side of (22) and using (8) and (23), we get
\[
u(x,t) = x^2 + _0I_t^\alpha \left[ -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} - \frac{xt}{3} \right) \right] + _0I_t^\alpha \left[ \frac{\partial^2}{\partial x^2} (u^2) \right].
\]
In view of (9), we have \(N(u) = _0I_t^\alpha \left[ -\frac{\partial}{\partial x} \left( \frac{4u^2}{x} \right) \right] + _0I_t^\alpha \left[ \frac{\partial^2}{\partial x^2} (u^2) \right]\) and \(L(z) = _0I_t^\alpha \left[ \frac{\partial}{\partial x} \left( \frac{u^2}{x} \right) \right]\). Applying the algorithm of new iterative method, we obtain
\[
\begin{align*}
u_0 &= x^2, 
\nu_1 &= x^2 \frac{t^\alpha}{\alpha}, 
\nu_2 &= x^2 \frac{t^{2\alpha}}{2\alpha^2}, 
\nu_3 &= x^2 \frac{t^{3\alpha}}{6\alpha^3} \ldots
\end{align*}
\]
Thus solution of (22)-(23) is
\[
u(x,t) = x^2 \sum_{k=0}^{\infty} \left( \frac{t^\alpha}{\alpha} \right)^n \frac{n!}{n!} = x^2 e^{\frac{t^\alpha}{\alpha}}.
\]
Figure 3. Behavior of $u(x, t)$ corresponding to the values $\alpha = 0.3$, $\alpha = 0.6$ and $\alpha = 0.9$ for $B(\alpha) = 1$ and $t = 5$ from left to right.

Figure 4. Behavior of $u(x, t)$ corresponding to the values $\alpha = 0.3$, $\alpha = 0.6$ and $\alpha = 0.9$ for $B(\alpha) = 1$ and $t = 5$ from left to right.
Equation (26) has the form (9) with Caputo fractional derivative.

We consider the following linear space-time fractional Fokker-Plank equation with Caputo fractional derivative:

$$C_0 D_t^\alpha (u(x,t)) = -C_0 D_x^\beta \left( \frac{ux^2}{6} \right) + C_0 D_x^{2\beta} \left( \frac{ux^2}{12} \right), \quad x > 0, \; t > 0, \; 0 < \alpha, \beta \leq 1,$$

(24)

with initial condition

$$u(x,0) = x^2.$$  

(25)

Apply Riemann-Liouville integral operator (2) on both side of (24) and using (3) and (25), we get

$$u(x,t) = x^2 + RL_0 I_t^\alpha \left[-C_0 D_x^\beta \left( \frac{ux^2}{6} \right) \right] + RL_0 I_t^\alpha \left[C_0 D_x^{2\beta} \left( \frac{ux^2}{12} \right) \right].$$

(26)

Equation (26) has the form (9) with $f = x^2$, $I(u) = RL_0 I_t^\alpha \left[-C_0 D_x^\beta \left( \frac{ux^2}{6} \right) \right] + RL_0 I_t^\alpha \left[C_0 D_x^{2\beta} \left( \frac{ux^2}{12} \right) \right]$ and $N(u) = 0$. Using the algorithm of new iterative method, we get recurrence relations

$$u_0 = x^2, \; u_1 = \frac{t^{\alpha}}{\Gamma(\alpha + 1)} \left[ \frac{2x^4 - 2^\beta}{4} - \frac{x^{3-\beta}}{4} \right]$$

and

$$u_2 = \frac{x^2}{\Gamma(2\alpha + 1)} \left[ \frac{\Gamma(5-\beta)x^4 - 2^\beta}{4(5-2\beta)(4-\beta)} - \frac{\Gamma(6-2\beta)x^5 - 3^\beta}{4(6-2\beta)(5-2\beta)} - \frac{\Gamma(6-\beta)x^4 - 2\beta}{4(6-\beta)(5-2\beta)} + \frac{\Gamma(7-2\beta)x^{6-4\beta}}{64(5-2\beta)(4-\beta)(7-4\beta)} \right].$$

Thus three term solution of (24)-(25) is $u(x,t) = u_0 + u_1 + u_2$.

Remark 2. The exact solution of equation (24) for $\alpha = 1, \beta = 1$ is $u(x,t) = x^2 e^t$.

Example 4.8. Now, we consider the following linear space-time fractional Fokker-Plank equation

$$AB_0 D_t^\alpha (u(x,t)) = -C_0 D_x^\beta \left( \frac{ux^2}{6} \right) + C_0 D_x^{2\beta} \left( \frac{ux^2}{12} \right), \quad x > 0, \; t > 0, \; 0 < \alpha, \beta \leq 1,$$

(27)

with initial condition

$$u(x,0) = x^2,$$

(28)

Where $AB_0 D_t^\alpha (f(t))$ is the Atangana-Baleanu fractional derivative and $C_0 D_t^\alpha (f(t))$ is the Caputo fractional derivative of $f$.

Apply Atangana-Baleanu fractional integral operator (5) on both side of (27) and using (6) and (28), we obtain

$$u(x,t) = x^2 + AB_0 I_t^\alpha \left[-C_0 D_x^\beta \left( \frac{ux^2}{6} \right) \right] + AB_0 I_t^\alpha \left[C_0 D_x^{2\beta} \left( \frac{ux^2}{12} \right) \right].$$

### Table 2. Comparison of $u(x, t)$ with different fractional differential operators at different values of $\alpha$ when $x = 2, t = 3$

| $\alpha$ | Caputo derivative | Atangana-Baleanu derivative | Conformable derivative |
|----------|-------------------|----------------------------|------------------------|
| 0.25     | 17.6255           | 12.2796                    | 80.4828                |
| 0.5      | 23.8176           | 15.8632                    | 41.8564                |
| 0.75     | 29.5563           | 23.3458                    | 34.6326                |
| 1        | 34                | 34                         | 34                     |
In view of (9), we have $L(u) = AB_0I_t^\alpha \left[-C_0D_x^\beta \left(\frac{xu}{6}\right)\right] + AB_0I_t^\alpha \left[C_0D_x^{2\beta} \left(\frac{ux^2}{12}\right)\right]$ and $N(u) = 0$. Applying new iterative method, we get

$$u_0 = x^2, \quad u_1 = \left(1 - \frac{\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)} \frac{t^\alpha}{\Gamma(\alpha + 1)} \right) \left[\frac{2x^{4-2\beta}}{\Gamma(5 - 2\beta)} - \frac{x^{3-\beta}}{\Gamma(4 - \beta)}\right],$$

and

$$u_2 = \left(\frac{\alpha(1 - \alpha)}{(B(\alpha))^2} \frac{t^\alpha}{\Gamma(\alpha + 1)} + \left(\frac{\alpha}{B(\alpha)}\right)^2 \frac{t^{2\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)}\right)$$

$$\times \left(\frac{\Gamma(5 - \beta)x^{4-2\beta}}{6\Gamma(5 - 2\beta)\Gamma(4 - \beta)} - \frac{\Gamma(6 - 2\beta)x^{5-3\beta}}{3\Gamma(5 - 2\beta)\Gamma(6 - 3\beta)}\right)$$

$$+ \left(\frac{\alpha(1 - \alpha)}{(B(\alpha))^2} \frac{t^\alpha}{\Gamma(\alpha + 1)} + \left(\frac{\alpha}{B(\alpha)}\right)^2 \frac{t^{2\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)}\right)$$

$$\times \left(\frac{\Gamma(7 - 2\beta)x^{6-4\beta}}{6\Gamma(5 - 2\beta)\Gamma(7 - 4\beta)} - \frac{\Gamma(6 - \beta)x^{5-3\beta}}{12\Gamma(4 - \beta)\Gamma(6 - 3\beta)}\right).$$

Hence the three term solution of problem (27)-(28) is

$$u(x, t) = u_0 + u_1 + u_2.$$ 

**Example 4.9.** Let us consider the following linear space-time fractional Fokker-Plank equation

$$D_t^\alpha (u(x, t)) = -C_0D_x^\beta \left(\frac{xu}{6}\right) + C_0D_x^{2\beta} \left(\frac{ux^2}{12}\right), \quad x > 0, \ t > 0, \ 0 < \alpha, \beta \leq 1,$$

with initial condition

$$u(x, 0) = x^2,$$

(30)

Where $D_t^\alpha (f(t))$ is the conformable fractional derivative and $C_0D_t^\alpha (f(t))$ is the Caputo fractional derivative of $f$.

Apply conformable fractional integral operator (7) on both side of (29) and using (8) and (30), we obtain

$$u(x, t) = x^2 + 0I_t^\alpha \left[-C_0D_x^\beta \left(\frac{xu}{6}\right)\right] + 0I_t^\alpha \left[C_0D_x^{2\beta} \left(\frac{ux^2}{12}\right)\right].$$

(31)

In view of (9), we have $L(u) = 0I_t^\alpha \left[-C_0D_x^\beta \left(\frac{xu}{6}\right)\right] + 0I_t^\alpha \left[C_0D_x^{2\beta} \left(\frac{ux^2}{12}\right)\right]$ and $N(u) = 0$. Applying new iterative method, we get

$$u_0 = x^2, \quad u_1 = \left[\frac{2x^{4-2\beta}}{\Gamma(5 - 2\beta)} - \frac{x^{3-\beta}}{\Gamma(4 - \beta)}\right] t^\alpha,$$

and

$$u_2 = \left(\frac{\Gamma(5 - \beta)x^{4-2\beta}}{6\Gamma(5 - 2\beta)\Gamma(4 - \beta)} - \frac{\Gamma(6 - 2\beta)x^{5-3\beta}}{3\Gamma(5 - 2\beta)\Gamma(6 - 3\beta)} + \frac{\Gamma(7 - 2\beta)x^{6-4\beta}}{6\Gamma(5 - 2\beta)\Gamma(7 - 4\beta)} - \frac{\Gamma(6 - \beta)x^{5-3\beta}}{12\Gamma(4 - \beta)\Gamma(6 - 3\beta)}\right) t^{2\alpha}.$$ 

Hence the three term solution of problem (27)-(28) is

$$u(x, t) = u_0 + u_1 + u_2.$$
Figure 5. Behavior of $u(x,t)$ corresponding to the values $(\alpha = 0.3, \beta = 0.8)$, $(\alpha = 0.7, \beta = 0.4)$ and $(\alpha = 0.9, \beta = 0.9)$ for $B(\alpha) = 1$ and $t = 5$ from left to right.

Figure 6. Behavior of $u(x,t)$ corresponding to the values $\alpha = 0.5, \beta = 0.5$ for Caputo fractional derivative, Atangana-Baleanu fractional derivative and conformable fractional derivative from left to right.
Table 3. Comparison of \( u(x, t) \) with different fractional differential operators at different values of \( \alpha \) when \( x = 2, t = 3 \)

| \( \alpha, \beta \) | Caputo derivative | Atangana-Baleanu derivative | conformable derivative |
|------------------|-------------------|-----------------------------|-----------------------|
| \( \alpha = 0.9, \beta = 0.2 \) | 5.1777             | 5.0846                      | 5.2498                |
| \( \alpha = 0.7, \beta = 0.4 \) | 6.7396             | 6.0633                      | 7.5260                |
| \( \alpha = 0.5, \beta = 0.6 \) | 8.1927             | 6.5332                      | 11.8531               |
| \( \alpha = 1, \beta = 1 \) | 14.5               | 14.5                        | 14.5                  |

5. Conclusion. This paper gives a new approximate solution of space-time fractional Fokker-Planck equations with different fractional derivative operators by a new iterative method. A variety of illustrative examples including these fractional derivative operators are solved by a new iterative method. The computations are easy and suitable for computer algorithm. The accuracy is high with some computed terms of the solution which confirm that this method with the given algorithm is a powerful method for such equations.

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