Chiral Waves on the Fermi-Dirac Sea:
Quantum Superfluidity and the Axial Anomaly

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We show that massless fermions at zero temperature define a relativistic quantum superfluid. The low energy fermion/anti-fermion collective pair excitations of the Fermi-Dirac sea are propagating Chiral Density Waves (CDWs), i.e. acoustic modes characterized by a hydrodynamic action principle and the Hamiltonian of an irrotational and dissipationless perfect fluid. In $D = 2$ dimensions the chiral superfluid effective action coincides with that of the Schwinger model as $e \to 0$, with the CDW acoustic mode precisely the Schwinger boson required by the axial anomaly. Upon quantization, the canonical commutator of the gapless CDW of the superfluid also coincides with the anomalous current commutator of the bosonized Schwinger model, an identity holding as well at zero chiral density, implying that the Dirac vacuum itself may be viewed as a quantum superfluid state. We show that the gapless CDW collective boson is a $U^{ch}(1)$ phase field, following from a somewhat novel realization of the Nambu-Goldstone theorem, due to the periodic vacuum structure, and axial anomaly in all higher even dimensional spacetimes. In QED$_4$ this Goldstone mode appears as a massless pole in the axial anomaly triangle diagram, and is directly responsible for the macroscopic Chiral Magnetic and Chiral Separation Effects. The effect of electromagnetic interactions and finite fermion mass as well as possible extensions and connection of the macroscopic superfluid description to microscopic field theory are briefly discussed.

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I. INTRODUCTION

Hydrodynamics is a general continuum effective field theory description of many body physics on macroscopic distance scales, when conditions of approximate local equilibrium and conservation laws can be assumed [1]. Quantum Field Theory (QFT) is the presumed fundamental microscopic underpinning of this many body physics. Generally the distance between the two descriptions is large enough that the precise relationship between them, and specifically how macroscopic behavior emerges from microscopic physics is quite difficult to discern.

The classical fluid description is modified when the temperature goes to zero, and dissipational effects are suppressed. In such circumstances superfluid behavior may be seen, characterized experimentally by persistent flow, and theoretically by the spontaneous breaking of a continuous global phase symmetry in its ground state. The quantum phase field describes a gapless Nambu-Goldstone sound mode whose derivatives are the superfluid velocity field and whose low energy interactions are determined by the equilibrium equation of state of the system [2]. In macroscopic superfluidity at $T=0$, the close connection to the microscopic QFT underpinning is much more apparent.

Although a quantum anomaly represents an explicit breaking of symmetry, it shares some features of the spontaneously broken case, and in particular, also leads to a massless bosonic collective excitation [3, 4], whose effects can extend to macroscopic scales [5, 6]. This suggests that an effective superfluid description should apply also to anomalous global symmetries at $T=0$, at least in the limit where the interactions producing the anomaly are weak or can be neglected. If the Nambu-Goldstone theorem can be extended to the case of global symmetries broken only by quantum anomalies, it would then become possible to regard the massless boson excitation implied by the anomaly as also the gapless sound mode of a quantum superfluid state.

In this paper we demonstrate that the effective action of the axial anomaly of massless QED does imply relativistic superfluid behavior of zero temperature fermion systems. The collective boson of fermion/anti-fermion pair excitations of the filled $T=0$ Fermi-Dirac sea, required by the anomaly is a Chiral Density Wave (CDW) that can be identified with the gapless sound mode of the superfluid description. The existence of a massless bosonic collective excitation may also be demonstrated by a novel non-linear realization of Goldstone’s theorem, resulting from the anomalous current commutators, or Schwinger terms, in the QFT description. In the limit when the interaction with external electromagnetic fields is turned off, so that the axial anomaly violating the $U^{ch}(1)$ symmetry itself vanishes, the effective action, commutation relations and ground state/vacuum
structure it implies leads to the existence of a Goldstone boson, which coincides with the gapless acoustic excitation of the quantum superfluid description. Thus a clear direct relationship between the effective macroscopic superfluid and QFT microscopic descriptions is established in this case.

In $D=2$ dimensions this relationship between quantum superfluidity and massless fermion QFT becomes an identity. We demonstrate that the Schwinger model of massless QED$_2$ [7] is in fact a chiral quantum superfluid at $T=0$ and vanishingly small electric charge coupling $e \to 0$. The chiral boson of the bosonized form of the Schwinger model is both a Chiral Density Wave and Charge Density Wave perturbation on the Fermi-Dirac sea, and also the Goldstone acoustic mode of the chiral superfluid description. For non-zero chiral chemical potential $\mu_5$ and fermion chiral charge density $n_5$, this bosonic CDW is a fermion/hole pair excitation of the Fermi sea. Since the acoustic mode depends on the ratio $\mu_5/n_5 = \pi$ in $D=2$ for free fermions, this description extends also to limiting case of $\mu_5, n_5 \to 0$, whereupon the CDW becomes a fermion/anti-fermion pair excitation of the Dirac vacuum itself, which may therefore be regarded equally well as a superfluid state.

This result, dependent upon the bosonization of fermion/anti-fermion pair excitations, can be extended to $D=4$ and higher (even) dimensions in the longitudinal sector of the axial current, where the massless CDW boson resides. This shows that macroscopic superfluidity effects persist in higher dimensions as a direct result of the axial anomaly, at least at zero temperature and in a subsector of the theory when there are no other interactions. We also show that the massless boson anomaly pole of the well-known triangle $\langle J_5^\lambda J_\alpha^\sigma J_\beta^\rho \rangle$ diagram composed of massless fermions is entirely responsible for the Chiral Magnetic and Chiral Separation Effects of QED$_4$ [8–10]. This establishes an additional connection between macroscopically observable effects and the microscopic QFT underlying them, when quantum anomalies are present. In the limit of a constant and uniform magnetic field $B$, and with a parallel $E$ field independent of the transverse coordinates, the $D=4$ axial anomaly reduces to the $D=2$ case, and the $D=4$ CDW along the common $E, B$ direction becomes identical to that in $D=2$, as an explicit example of dimensional reduction.

The paper is organized as follows. In the next Section we define the Euler-Lagrange variational principal for a relativistic chiral superfluid at $T = 0$ in any dimension, taking the axial current anomaly into account, and derive the corresponding Hamiltonian, showing that its dynamics is irrotational, fully reversible and dissipationless. In Sec. III we show that this chiral superfluid effective action in $D = 2$ coincides with that of the Schwinger model of massless QED$_2$ at zero coupling. In Sec. IV we show how the anomalous current commutators (Schwinger terms) required by the axial anomaly lead to a non-linear realization of Goldstone’s theorem and a gapless Nambu-Goldstone boson excitation. We discuss also how this non-linear realization may be regarded
as a certain limit of the usual linear realization where the role of vacuum expectation value is
taken by the ratio $n_5/\mu_5$ of the fluid. In Sec. V we reconsider the axial triangle anomaly in
massless QED$_4$, give a new form of the bosonic effective action it implies, and show that this
effective action is responsible for both the Chiral Magnetic Effect (CME) and Chiral Separation
Effect (CSE), describing a massless chiral boson collective excitation, whose canonical commutator
implies the anomalous Schwinger terms in current commutators. When $E \parallel B$ and $B$ is constant
and uniform in space the axial anomaly in $D = 4$ factorizes and reduces to the $D = 2$ case.
Sec. VI contains the description of a relativistic chiral superfluid of massless Dirac fermions in
$D=4$, explicitly exhibiting its CDW Nambu-Goldstone mode which propagates at the sound speed
$v_s^2 = \frac{dp}{d\epsilon}$. Electromagnetic interactions giving rise to mixed Chiral Density and Chiral Magnetic
Waves are discussed in Sec. VII, and the effects of finite fermion masses briefly discussed in Sec. VIII.
There are three Appendices, the first on fluid energy-momentum conservation in the presence of
the anomaly and relationship to the fluid Hamiltonian, and the second collecting a number of
useful formulae for free fermions in two dimensions and their bosonization, the third clarifying the
vacuum periodicity, and fate of the Goldstone mode in the Schwinger model as $e \to 0$.

II. IDEAL HYDRODYNAMICS OF AN ANOMALOUS CHIRAL FLUID

A. Euler-Lagrange Action Principle

The action principle for relativistic ideal fluids have been discussed by several authors [11, 12].
It generalizes the corresponding Euler-Lagrange action principle for non-relativistic fluids, the key
feature of which is the introduction of a ‘dynamic velocity’ vector field $\xi_\lambda$ that can be expressed
in terms of scalar (Clebsch) potentials [13]. In its relativistic generalization the dynamic velocity
$\xi_\lambda$ is a covariant $D$-vector in $D$ spacetime dimensions that couples to the conserved current whose
hydrodynamic flow is under study. In this paper the fermionic chiral current $J_5^\lambda = \bar{\psi} \gamma^\lambda \gamma_5 \psi$ will be
our primary focus. In general several Clebsch potentials, denoted here by $\eta, \alpha, \beta, \ldots$ are necessary
to describe rotational motions of a fluid and its entropy flow at finite temperature, and the dynamic
velocity can be expressed $\xi_\lambda = -\partial_\lambda \eta + \alpha \partial_\lambda \beta + \ldots$ [11]. If the fluid is ideal, irrotational and at zero
temperature, then the dynamic velocity field can be expressed as a pure gradient of a single scalar
potential which will be denoted by $\eta$, so that $\xi_\lambda = -\partial_\lambda \eta$. In that case the minimal action for ideal
hydrodynamics of a chiral fluid in $D = d + 1$ (even) spacetime dimensions can be taken to be

$$S_{\chi\parallel} = \int d^Dx \mathcal{L}_{\chi\parallel} = \int d^Dx \left\{ (\partial_\lambda \eta + A_{5\lambda}) J_5^\lambda + \eta \mathcal{A}_D - \varepsilon(n_5) \right\}$$  \hspace{1cm} (2.1)
where \( n_5 \) is the chiral number density in the fluid rest frame, \( \varepsilon(n_5) \) is the equilibrium energy density of the fluid in its rest frame, and \( A_{5\lambda} \) is an external axial potential, such that the variation

\[
\frac{\delta S_{\chi}^{\text{sf}}}{\delta A_{5\lambda}} = J_{5}^{\lambda} = n_5 u^{\lambda}
\]  

(2.2)

is defined, with \( u^{\lambda} \) the relativistic kinetic velocity of the fluid.\(^1\) Since the relativistic velocity is normalized by \( u^{\lambda} u_{\lambda} = u^{\lambda} g_{\lambda\nu} u^{\nu} = -1 \) in units in which the speed of light \( c = 1 \), with \( g_{\lambda\nu} \) the spacetime metric, the chiral number density can be expressed

\[
n_5 \equiv \left( -J_{5}^{\lambda} J_{5\lambda} \right)^{\frac{1}{2}}
\]  

(2.3)

in a relativistically invariant form in terms of the chiral current. The \( \mathcal{A}_{D} \) term takes account of the chiral current anomaly in (2.4) below. For other discussions of relativistic hydrodynamic action of anomalous fluids and superfluids, see [14–18].

The variational principle for the fluid action (2.1) requires that the pseudoscalar Clebsch potential \( \eta \) and the axial current \( J_{5}^{\lambda} \) be varied independently, in addition to the variation (2.2). Variation with respect to \( \eta \) gives

\[
\partial_{\lambda} J_{5}^{\lambda} = \mathcal{A}_{D}
\]  

(2.4)

where \( \mathcal{A}_{D} \) is the axial anomaly for massless Dirac fermions. We discuss the \( D = 2, 4 \) cases explicitly in the following sections, and provide a basis for the hydrodynamic action (2.1) from QFT first principles, identifying \( \eta \) with a dynamical phase field of chiral symmetry breaking.

In view of (2.3), variation of (2.1) with respect to the current \( J_{5}^{\lambda} \) gives

\[
\frac{\delta}{\delta J_{5}^{\lambda}} S_{\chi}^{\text{fl}} = \partial_{\lambda} \eta + A_{5\lambda} - \left( \frac{d\varepsilon}{dn_5} \right) \left( \frac{dn_5}{dJ_{5}^{\lambda}} \right) = \partial_{\lambda} \eta + A_{5\lambda} + \frac{\mu_5}{n_5} J_{5\lambda} = 0
\]  

(2.5)

where

\[
\mu_5 \equiv \frac{d\varepsilon}{dn_5}
\]  

(2.6)

is the equilibrium chiral chemical potential. Thus using (2.2) the dynamic velocity field is

\[
\xi_{\lambda} = -\partial_{\lambda} \eta = \mu_5 u_{\lambda}
\]  

(2.7)

where we set the external axial potential \( A_{5\lambda} = 0 \) here and in the following. We also have

\[
\mu_5 = \left( -\xi_{\lambda} \xi^{\lambda} \right)^{\frac{1}{2}} = \left( -\partial^{\lambda} \eta \partial_{\lambda} \eta \right)^{\frac{1}{2}}
\]  

(2.8)

\(^1\) In an abuse of notation we use a subscript 5 for the axial potential and axial current, in any \( D \) even dimension.
analogously to (2.3).

The vorticity of the fluid involves the curl of the velocity field \( \nabla \times \mathbf{u} \), whose relativistically covariant definition involves the fully anti-symmetric tensor \( u_{[\alpha} \partial_{\beta]u_{\gamma]} \). This vanishes identically in \( D = 2 \), while in \( D = 4 \) the relativistic vorticity pseudovector is defined by

\[
\omega^\lambda \equiv \frac{1}{2} \epsilon^{\lambda\alpha\beta\gamma} u_\alpha \partial_\beta u_\gamma
\]

\[
= \frac{1}{2\mu_5} \epsilon^{\lambda\alpha\beta\gamma} u_\alpha \left[ \partial_\beta (\mu_5 u_\gamma) - u_\gamma \partial_\beta \mu_5 \right]
\]

\[
= \frac{1}{2\mu_5^2} \epsilon^{\lambda\alpha\beta\gamma} \xi_\alpha \partial_\beta \xi_\gamma = 0 \tag{2.9}
\]

which vanishes since \( \xi^\lambda \) in (2.7) is a pure gradient. This continues to hold in any number of dimensions. Thus (2.1) with (2.7) describes an irrotational fluid, provided \( \eta \) is non-singular.

If (2.1) is evaluated at the extremum (2.5), denoted by an overline

\[
\begin{align*}
S_{\chi^A} \bigg|_{A^5 = 0} &= \int p \, d^D x = \int dt \int p \, d^D x \\
\end{align*}
\]

where

\[
p(\mu_5) = \mu_5 n_5 - \varepsilon = \mu_5^2 \left( \frac{\varepsilon}{n_5} \right), \quad \frac{dp}{d\mu_5} = n_5 \tag{2.11}
\]

is the pressure of the perfect fluid, neglecting the anomaly term \( \omega_D \). Thus the fluid effective action at its extremum (2.10) is (minus) the grand canonical potential of the fluid with conserved axial current, when \( \int dt \) is replaced by \( \beta = 1/k_B T \) upon continuation to imaginary time appropriate to the finite temperature equilibrium ensemble, although in this paper we work in real time and at \( T = 0 \). The effective action of finite temperature fluids requires additional Clebsch potentials [11].

B. Canonical Hamiltonian Formulation

The dissipationless nature of the perfect chiral fluid described by (2.1) is made clear from its Hamiltonian form [12, 17, 19]. Defining

\[
\Pi_\eta \equiv \frac{\delta}{\delta \dot{\eta}} S_{\chi^A} = J_5^0 \tag{2.12}
\]

the momentum conjugate to \( \eta \), we find the Hamiltonian density

\[
\mathcal{H}_{\chi^A} = \Pi_\eta \dot{\eta} - L_{\chi^A} = -J_5^i \nabla_i \eta - \eta \omega_D + \varepsilon \tag{2.13}
\]

at \( A_{5\lambda} = 0 \). To express this in terms of the canonical pair \((\eta, \Pi_\eta)\) we first solve (2.5) for

\[
J_5^i = -\frac{n_5}{\mu_5} \nabla^i \eta \tag{2.14}
\]
again at $A_{5\lambda} = 0$, and thus from (2.11) obtain
\[ \varepsilon - J_5 \nabla_i \eta = \mu_5 n_5 - p + \frac{n_5}{\mu_5} (\nabla^i \eta) (\nabla_i \eta) = \frac{n_5}{\mu_5} \left[ \mu_5^2 + (\nabla \eta)^2 \right] - p \] (2.15)
while from (2.3), (2.12) and (2.14) we have
\[ \Pi_\eta^2 = n_5^2 + J_5 J_i^5 \eta = n_5^2 + \frac{n_5^2}{\mu_5^2} (\nabla \eta)^2 = \left( \frac{n_5}{\mu_5} \right)^2 \left[ \mu_5^2 + (\nabla \eta)^2 \right]. \] (2.16)
Therefore
\[ \frac{n_5}{\mu_5} = \frac{1}{\mu_5} \frac{dp}{d\mu_5} = \frac{\left| \Pi_\eta \right|}{\sqrt{\mu_5^2 + (\nabla \eta)^2}} \geq 0 \] (2.17)
where the positive square root is always taken. Making use of (2.15) and (2.17), (2.13) becomes
\[ H_{\chi} = \int d^d x \, H_{\chi} \] and Hamilton’s eqs. are
\[ \dot{\eta} = \frac{\delta}{\delta \Pi_\eta} H_{\chi} = \frac{\mu_5}{n_5} \Pi_\eta = \text{sgn}(\Pi_\eta) \sqrt{\mu_5^2 + (\nabla \eta)^2} \] (2.19a)
\[ \dot{\Pi}_\eta = -\frac{\delta}{\delta \eta} H_{\chi} = \nabla \cdot \left( \frac{n_5}{\mu_5} \nabla \eta \right) + \mathcal{A}_D \] (2.19b)
for the canonical pair $(\eta, \Pi_\eta)$, since the variations of the $\mu_5$ dependence drops out upon using (2.17). Eq. (2.19a) recovers the time component of (2.5), and then the second eq. (2.19b) is
\[ \frac{\partial}{\partial t} \left( \frac{n_5}{\mu_5} \frac{\partial \eta}{\partial t} \right) - \nabla \cdot \left( \frac{n_5}{\mu_5} \nabla \eta \right) = \partial_\lambda J_5^\lambda = \mathcal{A}_D \] (2.20)
which recovers the axial current anomaly (2.4), if (2.12) and (2.14) are used.

When (2.20) is applied to small perturbations of the equilibrium fluid, it describes a gapless CDW acoustical mode with $\mathcal{A}_D$ as its source. Thus the perfect chiral fluid hydrodynamics determined by (2.1) is both irrotational and dissipationless, with a time reversible Hamiltonian dynamics (2.19) and a gapless excitation. These are necessary features of a chiral superfluid [20, 21]. We shall show in Sec. IV that $\eta$ is a phase field associated with spontaneous breaking of $U^\chi(1)$ symmetry, giving rise to a Nambu-Goldstone mode, establishing this last requirement for superfluidity.

The Hamiltonian fluid acoustic mode is quantized by replacing the Poisson bracket of the canonical pair $(\eta, \Pi_\eta)$ by their equal time commutator
\[ [\eta(t, x), \Pi_\eta(t, x')] = [\eta(t, x), J_5^0(t, x')] = i \delta^d(x - x') \] (2.21)
in units where $\hbar = 1$. We shall show presently that this canonical commutator of the hydrodynamic description is in fact required by the anomalous current commutators of the underlying QFT description of a quantum chiral superfluid.
C. Chiral Superfluid Energy-Momentum Tensor

The Energy-Momentum Tensor for the chiral fluid may be found by generalizing the effective hydrodynamic action (2.1) to curved spacetime by the Equivalence Principle, given by [11, 12, 17]

\[ S_{\chi \text{sf}} = \int d^Dx \sqrt{-g} \left\{ J_{5 \lambda} \partial_{\lambda} \eta - \varepsilon(n_5) \right\} + \int d^Dx \eta \mathcal{A}_D \]  \hspace{1cm} (2.22)

where we have set \( A_{5 \lambda} = 0 \), and used the fact that the axial anomaly term \( \mathcal{A}_D \) is directly a tensor density and thus does not acquire a \( \sqrt{-g} \) factor. It follows that

\[ T^{\lambda \nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\chi \text{sf}}}{\delta g_{\lambda \nu}} = p g^{\lambda \nu} + (p + \varepsilon) u^\lambda u^\nu = p g^{\lambda \nu} + \frac{n_5}{\mu_5} \partial^\lambda \eta \partial^\nu \eta \]  \hspace{1cm} (2.23)

where in performing the metric variation in (2.23) \( \partial_{\lambda} \eta \) and \( J_{5 \lambda} \) are taken to be metric independent, while \( J_{5 \lambda} = g_{\lambda \nu} J^\nu_5 \). The anomaly term \( \mathcal{A}_D \), lacking a \( \sqrt{-g} \) factor, does not contribute to \( T^{\lambda \nu} \) in the variation (2.23). The anomaly term does give a contribution to the electromagnetic current

\[ J^\nu = \left( \frac{\delta S_{\chi \text{sf}}}{\delta A^\nu} \right) = \frac{\delta}{\delta A^\nu} \int d^Dx \eta \mathcal{A}_D \]  \hspace{1cm} (2.24)

dependent upon \( \eta \). The energy-momentum tensor (2.23) is that of a perfect chiral fluid.

The divergence of (2.23) evaluated in flat spacetime reads

\[ \partial_\nu T^{\lambda \nu} = \partial^\lambda p + \partial_\nu \left( \frac{n_5}{\mu_5} \partial^\nu \eta \right) \partial^\lambda \eta + \frac{n_5}{\mu_5} \left( \partial^\nu \eta \right) \left( \partial_\nu \partial_\lambda \eta \right) \]  \hspace{1cm} (2.25)

Using (2.8) and (2.11), the pressure gradient in this expression is

\[ \partial^\lambda p = \frac{dp}{d\mu_5} \partial^\lambda \mu_5 = -\frac{n_5}{\mu_5} \left( \partial^\nu \eta \right) \left( \partial_\nu \partial_\lambda \eta \right) \]  \hspace{1cm} (2.26)

which cancels the last term in (2.25), resulting in

\[ \partial_\nu T^{\lambda \nu} = \partial_\nu \left( \frac{n_5}{\mu_5} \partial^\nu \eta \right) \partial^\lambda \eta = -\left( \partial_\nu J^\nu_5 \right) \left( \partial^\lambda \eta \right) = -\mathcal{A}_D \partial^\lambda \eta \]  \hspace{1cm} (2.27)

where the relation (2.5) between \( J^\nu_5 \) and \( \partial^\nu \eta \) and the anomalous divergence (2.4) have been used. Eq. (2.27) together with the anomaly eq. (2.4) show that if \( \mathcal{A}_D \) vanishes, so that \( J^\nu_5 \) is conserved, then the Energy-Momentum Tensor \( T^{\lambda \nu} \) is conserved as well. In the presence of an external, i.e. non-dynamical, electromagnetic field the non-conservation of the Energy-Momentum Tensor (2.27) is expected, and should be given by

\[ \partial_\nu T^{\lambda \nu} = F^{\lambda \nu} J_\nu \]  \hspace{1cm} (2.28)

where \( J_\nu \) is the electromagnetic current (2.24) induced by the anomaly. Proof of the equality of (2.27) and (2.28), as well as the reason for the difference between the Hamiltonian (2.18) and \( T^{00} \) are given in Appendix A, as both depend upon the special properties of the axial anomaly \( \mathcal{A}_D \).
III. QUANTUM CHIRAL SUPERFLUID IN TWO DIMENSIONS

In this section we show that the bosonized form of massless free fermions in $D = 2$ in fact coincides the effective chiral fluid description of the previous section, thus justifying the quantum superfluid interpretation of (2.1), and deriving it from first principles in the $D = 2$ case.

A. The Schwinger Model and its Axial Anomaly

Electrodynamics in $1 + 1$ dimensions, QED$_2$, is defined by the classical action

$$S_{cl} = \int d^2x \bar{\psi} \left\{ i\gamma^a \left( \partial_a - iA_a \right) - m \right\} \psi - \frac{1}{4e^2} \int d^2x F_{ab}F^{ab}$$

(3.1)

and is exactly soluble for vanishing fermion mass $m = 0$ [7]. The solution relies upon the special property of the Dirac matrices $\gamma^a\gamma^5 = -\epsilon^{ab}\gamma_b$ in two dimensions. This property has the consequence that the chiral current

$$j^5_a \equiv \bar{\psi} \gamma^a \gamma^5 \psi = -\epsilon^{ab}\psi \gamma_b \psi = -\epsilon^{ab}j^b_a$$

(3.2)

is dual to the charge current $j^a \equiv \bar{\psi} \gamma^a \psi$. Both currents $j^a, j^5_a$ would appear to be conserved Noether currents, corresponding to the classical $U(1) \times U^{\text{ch}}(1)$ symmetry of the Dirac Lagrangian in (3.1) when $m = 0$. However, both $U(1)$ symmetries cannot be maintained at the quantum level, since there is an anomaly [22].

The addition of the $F_{ab}F^{ab}$ term in (3.1) and consistency of Maxwell’s eqs. $\partial_a F^{ba} = e^2j^b$ requires $\partial_a j^a = 0$, and the breaking of the $U^{\text{ch}}(1)$ chiral symmetry. Explicitly, if conservation of $j^a$ is enforced, then (3.2) and the one-loop vacuum polarization ‘diangle’ diagram of Fig. 1 implies

$$j^5_a(x) \bigg|_{m=0} = -ie^a_b \int d^2y \langle T^* j^b(x) j^c(y) \rangle_0 A_c(y)$$

$$= -e^a_b \int d^2y \Pi^{bc}(k) e^{ik\cdot(x-y)} A_c(y)$$

(3.3)

where $T^*$ denotes covariant time ordering, and

$$\Pi^{bc}(k) = \frac{1}{\pi k^2} \left( k^b k^c - g^{bc}k^2 \right)$$

(3.4)

is the polarization tensor for massless fermions in $D = 2$. Thus

$$j^5_a(x) \bigg|_{m=0} = \frac{1}{\pi} e^{ab} \int d^2y \Box^{-1}_{xy} \partial_c F^c_b(y) = \frac{1}{\pi} \partial^a_x \int d^2y \Box^{-1}_{xy} \tilde{F}(y)$$

(3.5)

2 Notation: Indices $a, b = 0, 1$; Metric $g_{ab} = \text{diag}(-1, 1) = g^{ab}$; $\epsilon_{ab} = -\epsilon_{ba}$; $\epsilon_{01} = +1 = -\epsilon_{01}$; $F_{ab} \equiv \partial_a A_b - \partial_b A_a$; $\partial_a \equiv \frac{1}{2}(\partial_a - \partial_a)$; 2 × 2 Dirac matrices $\gamma^0 = \sigma_2 = (\gamma^0)^\dagger$, $\gamma^1 = \sigma_1$, $\gamma^2 = \gamma^0\gamma^1 = \sigma_3$; with $\sigma_i, i = 1, 2, 3$ the usual Pauli matrices, $\bar{\psi} \equiv \psi^\dagger \gamma^0$, and anti-symmetization over $\bar{\psi}, \psi$ is understood. We use lower case $j^a, j^5_a$ for currents and $a, b$ for Latin indices in $D = 2$ to distinguish them from $D \geq 4$ currents $J^\nu, J^5_\lambda$ with Greek indices $\nu, \lambda$ etc.
where $\tilde{F} \equiv \frac{1}{2} e^{ab} F_{ab}$ is the pseudoscalar dual to $F_{ab}$, and $\square^{-1}_{xy} = -\frac{1}{4\pi} \ln(x-y)^2 + \text{const.}$ is the massless scalar propagator in $D=2$. Thus the chiral current (3.5) acquires the anomalous divergence

$$\partial_a j_5^a = \frac{1}{2\pi} e^{ab} F_{ab} = \frac{\tilde{F}}{\pi} = \mathcal{A}_2$$

(3.6)

at the one-loop level. This turns out to be an exact result in massless QED$_2$, because a further special property of massless QED$_2$ is that the current induced by a background gauge potential is strictly linear in $A_a$ for arbitrary $A_a$. Note also that $\tilde{F} = F_{10} = E$ is just the electric field in one spatial dimension.

The chiral anomaly (3.6) corresponds to the exact non-local 1PI effective action

$$S_{\text{anom}}^{NL}[A] = -\frac{1}{2\pi} \int d^2x \int d^2y \tilde{F}_x \square^{-1}_{xy} \tilde{F}_y$$

(3.7)

obtained by integrating out the massless fermions in the functional integral for (3.1). Variation of (3.7) with respect to $A_a$ and making use of $j_a = -\epsilon^{ab} j_b^5$ gives (3.5).

FIG. 1: One-loop fermion polarization diagram (a) and equivalent pseudoscalar tree diagram (b).

The appearance of the $1/k^2$ massless pole in (3.5) corresponding to the massless scalar propagator $(\square^{-1})_{xy}$ in the 1PI non-local effective action (3.7) signals that an effective scalar boson degree of freedom is associated with the anomaly. Indeed a (pseudo)scalar boson field $\chi$ may be introduced to rewrite the non-local action (3.7) in the local form [23]

$$S_{\text{anom}}[\chi; A] = \int d^2x \left\{ -\frac{\pi}{2} (\partial_a \chi)(\partial^a \chi) - \tilde{F} \chi \right\}$$

(3.8)

whose variation with respect to $\chi$ gives

$$\square \chi = \frac{1}{\pi} \tilde{F}.$$

(3.9)

Solving this eq. for $\chi$ by inverting $\square$, and substituting the result back into (3.8) returns (3.7).
The axial and vector currents may be expressed in terms of the effective boson field $\chi$ by

$$j^a_5 = \partial^a \chi,$$
$$j^a = \delta_A S_{\text{anom}} = -e^{ab} \partial_b \chi$$  \hspace{1cm} (3.10)$$

so that the conservation of $j^a$ becomes a topological identity and the chiral anomaly (3.6) follows from the eq. of motion for $\chi$ (3.9). This is the chiral bosonization of fermion currents in 1 + 1 dimensions [24, 25]. The chiral boson $\chi$ obeys a massless wave eq. (3.9) with $A_2$ as its source, becoming finite ranged with a mass $M = e/\sqrt{\pi}$ only if the coupling $e \neq 0$ in (3.1).

The explicit Fock space operator representation of the chiral boson in terms of fermion bilinears is given in Appendix B. These lead to the canonical equal time commutator

$$[\chi(t, x), \dot{\chi}(t, x')] = \frac{i}{\pi} \delta(x - x')$$  \hspace{1cm} (3.11)$$

which is exactly the canonical commutation relation expected by from the action functional (3.8) directly. From (B8)-(B13) the fermion current densities in the Fock space representation are

$$j^0 = \partial_x \chi = j^1_5$$  \hspace{1cm} (3.12a)$$
$$j^1 = -\partial_t \chi = j^0_5$$  \hspace{1cm} (3.12b)$$

recovering the bosonization relations (3.10) obtained in the effective action representation. These relations and the commutator (3.11) imply

$$[j^0(t, x), j^1(t, x')] = -\frac{i}{\pi} \partial_x \delta(x - x')$$  \hspace{1cm} (3.13)$$

which is the Schwinger term in the equal time commutator of the current components [26]. The normal ordering prescription on the currents and precise definition of the fermion vacuum is essential to derive these commutation relations in the fermion representation.

Thus the \textit{anomalous} Schwinger current commutator in QED$_2$ is equivalent to the \textit{canonical} commutator of the chiral boson $\chi$, which is a genuinely propagating collective degree of freedom composed of fermion bilinears, whose massless propagator in (3.5) is associated with the axial anomaly. The fermion bilinears are correlated/entangled fermion pairs in the two-fermion intermediate states of the polarization diagram Fig. 1, moving co-linearly at the speed $c = 1$, and equivalent to a massless boson according to the identifications (B7)-(B12). Because of (3.10) the propagating wave solutions of (3.9) are jointly Chiral Density Waves and Charge Density Waves, arising from massless particle-hole excitations of the Dirac sea. As we shall now show, these same Chiral Density Waves may also be regarded as acoustic modes in the superfluid hydrodynamic description at $T = 0$, and thus the superfluid hydrodynamic description applies to the filled Fermi-Dirac sea and the Dirac fermion vacuum itself.
B. Equivalence to Chiral Superfluid Hydrodynamics in $D = 2$

In two dimensions and at zero temperature the QFT description of the Schwinger model can be extended to finite chiral fermion densities, and shown to be identical to the ideal superfluid hydrodynamic description of Sec. II. At finite chiral chemical potential $\mu_5$, but $T = 0$, massless fermions in $D=2$ have the energy density and pressure (cf. Appendix B),

\[ \varepsilon = \frac{\pi}{2} n_5^2 = p = \frac{1}{2\pi} \mu_5^2 \quad \text{where} \quad \mu_5 = \pi n_5, \]

so that (2.5) gives at $A_5 \lambda = 0$

\[ \partial_a \eta = -\pi \partial_5 j_a = -\pi \partial_5 \chi \]

and $j_5^a = \partial^a \chi$ is also a pure gradient by the bosonization formulae (3.10) in $D=2$. Recalling (2.3) we obtain

\[ (\partial_a \eta) j_5^a - \varepsilon(n_5) = -\pi (\partial_a \chi)(\partial^a \chi) + \frac{\pi}{2} j_5^a j_5^a = -\frac{\pi}{2} (\partial_a \chi)(\partial^a \chi) \]

for two of the terms in (2.1). Then integrating the relation (3.15), so that $\eta = -\pi \chi + \theta/2$ where $\theta$ is a spacetime constant, and using (3.6) for the chiral anomaly $\mathcal{A}_2$ in $D = 2$, we have

\[ S_{\chi^{sf}} \bigg|_{D=2, A_5^a = 0} = \int d^2x \left\{ -\frac{\pi}{2} (\partial_a \chi)(\partial^a \chi) - \chi \tilde{F} + \frac{\theta}{2\pi} \tilde{F} \right\} \]

so that the hydrodynamic action (2.1), postulated on the basis only of macroscopic conservation laws for an irrotational and isentropic fluid, and the axial anomaly agrees precisely with the microscopic QFT effective action of the zero temperature Schwinger model (3.8) at zero coupling $e = 0$. Note that this equivalence requires the addition of the $-\varepsilon(n_5)$ energy density term in (3.16), just as prescribed by the general hydrodynamic fluid action (2.1). Addition of the spacetime constant $\theta$ term does not affect the $\chi$ dynamics.

Since

\[ [\eta(t, x), \Pi_\eta(t, x')] = \pi [\chi(t, x), \dot{\chi}(t, x')] = i \delta(x - x') \]

the canonical commutation relation (2.21) of the superfluid hydrodynamic description also coincides with the commutator (3.11) for the chiral boson of QED$_2$, which also implies the Schwinger term (3.13) in the current commutators. Also from (3.14) $\mu_5/n_5 = \pi$ is a constant in $D = 2$, so that the wave eq. of the hydrodynamic description (2.20) also coincides with that of the chiral boson of $e = 0$ electrodynamics (3.9) in two dimensions. Because of the simple relation (3.15), the CDW
solutions for $\chi$ are trivially also waves of the Clebsch potential $\eta$, which are the sound waves of the fluid. At finite equilibrium chiral density $n_5 \neq 0$, these are CDW excitations of the Fermi surface at Fermi energy $\mu_5$. The vacuum limit $\mu_5 \to 0, n_5 \to 0$, with $\mu_5/n_5 = \pi$ fixed shows that this description holds when the Fermi surface of filled positive energy single fermion states goes over to the Dirac fermion QFT vacuum where only the negative energy Dirac sea fermion states are filled.

IV. NAMBU-GOLDSTONE THEOREM FOR THE CHIRAL SUPERFLUID

A. Lorentz Invariant Case

Since apart from the anomaly term, the velocity potential $\eta$ appears in the effective action (2.1) and Hamiltonian (2.18) only under derivatives, the non-anomalous terms clearly have a constant shift symmetry under $\eta \to \eta + \eta_0$. Since $\eta$ multiplies the chiral anomaly $\mathcal{A}_D$ in $S_{\chi sf}$ and $\int d^Dx \mathcal{A}_D = 2\nu$, where $\nu$ is the Pontryagin index of the gauge field in $D$ Euclidean dimensions, which is the integral of a total derivative $\mathcal{A}_D = \partial_\lambda K^\lambda$ (with $K^\lambda$ the Chern-Simons current), the total action (2.1) is also invariant under the constant phase shift transformation up a topological term that does not affect the local dynamics, except as acting as a fixed source (for fixed $A_\lambda$) in (2.4). The $\eta$ eq. of motion (2.20) is invariant under the constant shift symmetry. Thus we should expect that the Nambu-Goldstone theorem applies to this gapless mode of the chiral superfluid described by (2.1) in a fixed (i.e. non-dynamical) $A_\lambda$ background.

Since $\nu \in \mathbb{Z}$ is an integer for sufficient rapid falloff of the gauge field $A_\lambda$ at infinity, $2\eta$ is expected to be a $2\pi$-periodic phase field whose expectation value describes spontaneous breaking of the global $U^{ch}(1)$ constant $\eta_0$ shift symmetry. The Euclidean path integral and vacuum is $2\pi$-periodic in $\theta$, and it follows from the axial anomaly (2.4) that $\Delta Q_5 = 2\nu = 2 \Delta N_{CS} = \Delta \int d^Dx K^0$ is the change in the chiral charge under topological transitions under change of the Chern-Simons winding number $\Delta N_{CS} = \nu$ by an integer.

That $\langle e^{2i\eta} \rangle$ plays the role of a complex order parameter characterizing chiral symmetry breaking is verified from the fundamental commutation relation (2.21), giving

$$[Q_5(t), \eta(t, x)] = \int d^Dx' [J_5^0(t, x'), \eta(t, x)] = -i$$

which is a c-number, whose further commutators with $Q_5$ vanish. Hence

$$e^{i\alpha Q_5(t)} \eta(t, x) e^{-i\alpha Q_5(t)} = \eta(t, x) + i\alpha [Q_5(t), \eta(t, x)] + \frac{(i\alpha)^2}{2} [Q_5(t), [Q_5(t), \eta(t, x)]] + \ldots$$

$$= \eta(t, x) + \alpha$$

(4.2)
shifts $\eta(t, \mathbf{x})$ by a constant $\alpha$ under a global chiral rotation of magnitude $\alpha$. Therefore

$$e^{i\alpha Q_5(t)} e^{2i\eta(t, \mathbf{x})} e^{-i\alpha Q_5(t)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(2\eta(t, \mathbf{x}) + 2\alpha\right)^n = e^{2i\alpha} e^{2i\eta(t, \mathbf{x})}$$  \hspace{1cm} (4.3)

as expected for a $2\pi$-periodic phase field associated with the breaking of global chiral symmetry if $\langle e^{2i\eta} \rangle \neq 0$ in the ground state.

Since the chiral anomaly breaks the chiral symmetry explicitly rather than spontaneously, it seems at first sight that Goldstone’s theorem does not apply. However when the electromagnetic coupling vanishes, and there are no background gauge fields so that $A_D = 0$ as well, the chiral current in (2.4) is conserved, and the action (2.1) is invariant under the shift symmetry (4.2) of a global chiral transformation. Then if the expectation value

$$\langle e^{2i\eta} \rangle = e^{2i\eta_0} \equiv z_0 \neq 0$$ \hspace{1cm} (4.4)

is both well-defined and non-vanishing, the necessary and sufficient condition for the existence of a massless Nambu-Goldstone boson [27–29] is satisfied.

If furthermore the ground state is both Lorentz and translationally invariant, it follows that

$$\int d^Dy \ e^{ik \cdot (x-y)} \langle T^* J_5^\lambda(x) e^{2i\eta(y)} \rangle = ik^\lambda F(k^2)$$ \hspace{1cm} (4.5)

for some Lorentz invariant function $F(k^2)$. Computing

$$\partial_\lambda \langle T^* J_5^\lambda(x) e^{2i\eta(y)} \rangle = \delta(x^0 - y^0) \langle [J_5^0(x), e^{2i\eta(y)}] \rangle = \delta^D(x - y) z_0$$ \hspace{1cm} (4.6)

from the commutation relation (2.21), and Fourier transforming, we find

$$k^2 F(k^2) = z_0 \implies F(k^2) = \frac{z_0}{k^2}$$ \hspace{1cm} (4.7)

showing the existence of the massless Nambu-Goldstone pole at $k^2 = 0$. Thus the gapless acoustic mode of (2.20), derived from the Hamiltonian and commutation relations of the canonical pair $\{\eta, \Pi_\eta\}$, may be viewed as a consequence of Goldstone’s theorem, which defines another distinguishing characteristic of superfluidity. The realization of these relations and subtleties in the $D=2$ Schwinger model is discussed in the two Appendices B and C. The gapless mode due to the axial anomaly and its relation to Goldstone’s theorem was discussed in the context of universality of transport properties in a chiral medium in Ref. [30].

**B. General Case of Superfluid Hydrodynamics**

The Goldstone mode propagates at the speed of light if and only if the ground state is Lorentz invariant as assumed in (4.4)-(4.6). This need not be the case. Indeed if the wave eq. (2.20) for
the phase field $\eta$ in the superfluid description is linearized around its equilibrium solution

$$
\eta = -\mu_5 t + \delta \eta \quad \text{(4.8a)}
$$

$$
\delta \left( \frac{n_5}{\mu_5} \right) = \frac{d}{d\mu_5} \left( \frac{n_5}{\mu_5} \right) \delta \mu_5 = -\frac{d}{d\mu_5} \left( \frac{n_5}{\mu_5} \right) \delta \dot{\eta} \quad \text{(4.8b)}
$$

where $\delta \mu_5 = -\delta \dot{\eta}$ follows from variation of (2.8) and use of (4.8a), then from the variation of (2.5) with $A_{5\lambda} = 0$ we obtain

$$
\delta J_5^0 = -\mu_5 \delta \left( \frac{n_5}{\mu_5} \right) + \frac{n_5}{\mu_5} \delta \dot{\eta} = \left[ \mu_5 \frac{d}{d\mu_5} \left( \frac{n_5}{\mu_5} \right) + \frac{n_5}{\mu_5} \right] \delta \dot{\eta} = \frac{dn_5}{d\mu_5} \delta \dot{\eta} \quad \text{(4.9a)}
$$

$$
\delta J_5^i = -n_5 \mu_5 \partial_i (\delta \eta) \quad \text{(4.9b)}
$$

where the equilibrium (spacetime independent) values for $n_5, \mu_5$ and their derivatives are to be used. Eqs. (4.9) can be combined and written in the covariant form

$$
\delta J_5^\lambda = \frac{dn_5}{d\mu_5} \left[ u^\lambda u^\nu (1 - v_s^2) - v_s^2 g^{\lambda\nu} \right] \partial_\nu (\delta \eta) \quad \text{(4.10)}
$$

by use of the kinetic velocity $u^\lambda$ which is $\delta \lambda_0$ in the fluid rest frame, and

$$
v_s^2 = \frac{n_5}{\mu_5} \frac{d\mu_5}{dn_5} = \frac{dp}{d\varepsilon} \quad \text{(4.11)}
$$

by (2.6) and (2.11). Thus the wave eq. (2.20) becomes

$$
\partial_\lambda (\delta J_5^\lambda) = \frac{dn_5}{d\mu_5} \left( \frac{\partial^2}{\partial t^2} - v_s^2 \nabla^2 \right) \delta \eta = 0 \quad \text{(4.12)}
$$

in the absence of the anomaly source, and the gapless Nambu-Goldstone mode propagates at speed $v_s \leq 1$, becoming the speed of light if and only if $\mu_5$ and $n_5$ are linearly related, and $p = \varepsilon$.

Upon quantization the relations (4.1)-(4.3) continue to hold for the variations $\delta J_5^\lambda$ and $\delta \eta$. Thus (4.5) may be reconsidered for the linearized variations away from the ground state of the general non-vanishing $\mu_5, n_5$, and

$$
\int dt e^{-i\omega(t-t')} \int d^d x \ e^{ik(x-x')} \langle \mathcal{T} \delta J_5^\lambda(t, x) e^{2i\delta \eta(t'x')} \rangle = -i \left[ u^\lambda u^\nu (1 - v_s^2) - v_s^2 g^{\lambda\nu} \right] k_\nu F(\omega, |k|) \quad \text{(4.13)}
$$

in terms of a scalar function $F(\omega, |k|)$, which follows from (4.10) and the fact that the variation of the velocity potential $\delta \eta$ and $\mu_5, n_5$ are spacetime (pseudo)scalars. Repeating the steps leading to (4.7) leads then to

$$
\left[ -(k \cdot u)^2 (1 - v_s^2) + v_s^2 k^2 \right] F(\omega, |k|) = \left( -\omega^2 + v_s^2 k^2 \right) F(\omega, |k|) = z_0 \quad \text{(4.14)}
$$

which implies

$$
F(\omega, |k|) = \frac{z_0}{-\omega^2 + v_s^2 k^2} \quad \text{(4.15)}
$$
instead, showing the gapless acoustic propagator pole in superfluid hydrodynamics for general $v_s$. An important point to notice about this derivation is that Lorentz invariance as well as axial $U^{ch}(1)$ symmetry is spontaneous broken in general by a background $\mu_5$, and this is reflected in both the time dependence of (4.8a) and the necessity of taking the variation of the magnitude $n_5/\mu_5$ of the chiral current into account in (4.9a), this variation being responsible for the sound speed $v_s$ differing in general from the speed of light.

C. Relativistic Field Theory Model of Superfluid Goldstone Mode

An additional subtlety concerning that the proof of Goldstone’s theorem in (4.7) is that the magnitude of the spontaneously broken chiral symmetry order parameter is not immediately apparent, as $e^{2i\eta_0}$ can be multiplied by any real magnitude. Both this point and the case of general sound velocity are illuminated by considering the linear representation of a scalar field exhibiting spontaneous symmetry breaking usually required for Goldstone’s theorem, as for example in the prototypical gauged $U(1)$-Higgs model

\[
S_\Phi = -\int d^Dx \left\{ (D^\lambda \Phi)^\dagger (D_\lambda \Phi) + V(\rho) \right\}
\]  

(4.16)

where

\[
D_\lambda \Phi \equiv (\partial_\lambda - igA_{5\lambda})\Phi, \quad \Phi = \frac{\rho}{\sqrt{2}} e^{-ig\eta}
\]  

(4.17)

in $(\rho, \eta)$ polar field coordinates. This model has a global $U^{ch}(1)$ phase symmetry under rotations of the angle $\eta$, and a corresponding conserved Noether current

\[
J_N^\lambda = \frac{\delta S_\Phi}{\delta A_{5\lambda}} = ig \Phi (D^\lambda \Phi)^* - ig \Phi^* (D^\lambda \Phi) = -g^2 \rho^2 (\partial^\lambda \eta + A^\lambda_5)
\]  

(4.18)

which realizes the symmetry linearly:

\[
[Q_N(t), \Phi(t, x)] = \int d^Dx' [J_N^\lambda(t, x'), \Phi(t, x')] = -g \Phi(t, x).
\]  

(4.19)

The global $U^{ch}(1)$ symmetry is promoted to a local symmetry by the introduction of the gauge field $A_{5\lambda}$. We have allowed for an arbitrary gauge coupling $g$ in order to match the canonical normalizations and phase periodicity of this model to that of the theory generated by anomalous superfluid hydrodynamics discussed previously.

Substituting the polar representation (4.17) the action (4.16) becomes

\[
S_\Phi = -\int d^Dx \left\{ \frac{1}{2} (\partial_\lambda \rho)^2 + \frac{g^2 \rho^2}{2} (\partial_\lambda \eta + A_{5\lambda})^2 + V(\rho) \right\}
\]  

(4.20)
with a corresponding Hamiltonian

\[ H = \int d^4x \left\{ \Pi_\Phi \dot{\Phi} + \Pi_\Phi^\dagger (D_\lambda \Phi)^\dagger (D_\lambda \Phi) + V(\rho) \right\} \]

\[ = \int d^4x \left\{ \frac{1}{2} \dot{\rho}^2 + \frac{1}{2} (\nabla \rho)^2 + \frac{g^2 \rho^2}{2} [\dot{\eta}^2 - A_{50}^2 + (\nabla \eta + A_5)^2] + V(\rho) \right\} \tag{4.21} \]

where \( \Pi_\Phi = (D_0 \Phi)^\dagger \) is the field momentum conjugate to \( \Phi \). Since the phase field \( \eta \) appears in the action (4.20) only in the combination \( \partial_\lambda \eta + A_{5\lambda} \), it is clear that a non-zero chemical potential \( \mu_5 \) is equivalent to a zero chemical potential but with a shift of \( \eta \rightarrow -\mu_5 t + \eta \). In the Hamiltonian framework, the gauge field Hamiltonian (4.21) with non-zero \( A_{50} = -\mu_5 \) but \( \dot{\eta} = 0 \) is equivalent to the Grand Canonical Hamiltonian \( H + \int d^4x J_5^0 \) of the ungauged theory, with \( H = H_\Phi \) evaluated at \( A_{5\lambda} = 0 \) but \( \dot{\eta} = -\mu_5 \), since \( J_5^0 = g^2 \rho^2 \dot{\eta} = -g^2 \rho^2 \mu_5 \) [21]. In either case there results from (4.21) and (4.18) the effective potential

\[ V_{eff}(\rho, \mu_5) = -\frac{1}{2} g^2 \mu_5^2 \rho^2 + V(\rho) \tag{4.22} \]

which develops a minimum at a non-zero \( \rho \) given by the ground state equilibrium condition

\[ \left. \frac{\partial V_{eff}(\rho, \mu_5)}{\partial \rho} \right|_{\rho = \bar{\rho}} = -g^2 \mu_5^2 \bar{\rho} + V'(\bar{\rho}) = 0 \tag{4.23} \]

that implicitly defines a function \( \bar{\rho}(\mu_5) \). Comparing (4.18) with (2.5), we see that in order to identify \( J_5^\lambda = J_5^\lambda \) and make contact with our anomalous superfluid hydrodynamic description, \( \bar{\rho}(\mu_5) \) must be such that

\[ g^2 \bar{\rho}^2 = \frac{n_5}{\mu_5} \tag{4.24} \]

thereby fixing the magnitude of the chiral symmetry breaking order parameter of the Nambu-Goldstone mode in the linear realization of the spontaneous breaking of chiral symmetry corresponding to the phase \( e^{2i\eta_0} \) of (4.4) of the fermionic ground state in the anomalous realization. The charge \( g = -2 \) is fixed by (4.19) and the chiral charge of the order parameter \( \Phi \sim \bar{\psi}(1 + \gamma_5)\psi \).

Since the pressure satisfies (2.11), from (4.23) and (4.24) we have

\[ p = \int n_5 d\mu_5 = \frac{1}{2} \int \frac{n_5}{\mu_5} d\mu_5^2 = \frac{1}{2} \int \mu_5 \bar{\rho}^2 d\left( \frac{\bar{\rho}}{\mu_5} \right) = \frac{1}{2} \bar{\rho} V'(\bar{\rho}) - \int \bar{\rho} V' d\bar{\rho} = \frac{1}{2} \bar{\rho} V'(\bar{\rho}) - V(\bar{\rho}) \tag{4.25} \]

where the integration constant can be absorbed into \( V \). The Lagrangian density of the action functional (4.20) evaluated at its equilibrium value is also equal to \( P \), consistent with (2.10). The equilibrium energy density may be found then from

\[ \epsilon = \mu_5 n_5 - p = \frac{n_5}{\mu_5} \mu_5^2 - p = \bar{\rho} V'(\bar{\rho}) - \frac{1}{2} \bar{\rho} V'(\bar{\rho}) + V(\bar{\rho}) = \frac{1}{2} \bar{\rho} V'(\bar{\rho}) + V(\bar{\rho}) \tag{4.26} \]
in terms of the arbitrary scalar potential \( V \) and \( \bar{\rho}(\mu_5) \) defined by (4.23).

Expressing the polar field variables as their equilibrium values plus small perturbations, \( \rho = \bar{\rho} + \delta \rho \), \( \eta = \delta \eta \) with \( A_{50} = -\mu_5 \), and expanding the action (4.20) to the second order in perturbations around the ground state, we find (cf. [21])

\[
S^{(2)}_\Phi = -\frac{1}{2} \int d^D x \left\{ \delta \rho \left( -\Box + M^2_\rho \right) \delta \rho + g^2 \bar{\rho}^2 \delta \eta \left( -\Box \right) \delta \eta - 4g^2 \mu_5 \bar{\rho} \delta \rho \delta \dot{\eta} \right\}
\]

(4.27)

where \( M^2_\rho = V''(\bar{\rho}) - g^2 \mu_5^2 = V''(\bar{\rho}) - V'(\bar{\rho})/\bar{\rho} \). Substituting the complex Fourier decomposition \( e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} \) we find the \( 2 \times 2 \) Hermitian matrix

\[
\begin{pmatrix}
-\omega^2 + k^2 + M^2_\rho & 2i\omega g^2 \mu_5 \bar{\rho} \\
-2i\omega g^2 \mu_5 \bar{\rho} & g^2 \bar{\rho}^2 (-\omega^2 + k^2)
\end{pmatrix}
\]

(4.28)

Setting the determinant of this matrix to zero yields the spectrum, consisting of one solution for \( \omega^2 \) that is gapped at the scale \( M^2_\rho \), and a second solution at

\[
\omega^2 = v_{s}^2 k^2 + O\left( \frac{k^4}{M^2_\rho} \right)
\]

(4.29)

which is a gapless acoustic Goldstone mode with speed of sound given by

\[
v_{s}^2 = \frac{\bar{\rho} V''(\bar{\rho}) - V'(\bar{\rho})}{\bar{\rho} V''(\bar{\rho}) + 3V'(\bar{\rho})} = \frac{dp}{d\varepsilon}
\]

(4.30)

and agrees with that obtained from the hydrodynamic superfluid action (2.1) at quadratic order.

This model exercise shows that the macroscopic acoustic Nambu-Goldstone mode of (4.12) may be regarded as the low energy limit of a more ultraviolet complete QFT where there are additional massive excitations that decouple in the regime of applicability of the hydrodynamic description, and identifies through (4.24) the magnitude \( \rho \) of the order parameter of spontaneously broken \( U^{\text{ch}}(1) \) symmetry corresponding to the phase \( \eta \). Note also that as in the superfluid EFT description, it is necessary to allow this magnitude \( \delta \rho \) to vary dynamically along with the phase \( \delta \eta \), in order to obtain the correct speed of sound \( v_s \leq 1 \), though the off-diagonal mixing term in the matrix (4.28). From (3.14) in the case of the Schwinger model reviewed in the last section, the ratio (4.24) is a simple pure number \( 1/\pi \), \( p = \varepsilon \) which does not vary, and \( v_{s}^2 = 1 \) in \( D=2 \) for massless fermions. In this case the ground state is Lorentz invariant, as we show in Appendix B and (4.7) applies.

The \( D=2 \) case of the Schwinger model at non-zero coupling \( e \neq 0 \), and the non-commutativity of the \( e \to 0 \) and infinite volume limit \( L \to \infty \) is discussed in Appendix C. As soon as \( e \neq 0 \), however infinitesimally small, the would-be Nambu-Goldstone \( \chi \leftrightarrow -\eta/\pi \) excitation combines
with the electric field through solution of the Gauss Law constraint: $E = e^2 \chi + E_0$ (where $E_0$ is an integration constant, with the interpretation of a spacetime constant background electric field), and $\chi$ becomes massive. In other words, the classically constrained gauge field is ‘eaten’ by the propagating would-be Goldstone boson and there is finally only a propagating massive $\chi$ boson with $M^2 = e^2/\pi$. This non-linear realization of the Stueckelberg-Higgs mechanism [31] thus avoids any conflict with general theorems forbidding true Goldstone bosons limit in $D = 2$ [32, 33]. At time and distance scales much less than $1/M \ll L$, the considerations of this section nevertheless apply and the propagating CDW $\eta$ phase wave may be understood as a Nambu-Goldstone mode.

V. EFFECTIVE ACTION OF THE CHIRAL ANOMALY IN FOUR DIMENSIONS

A. The Effective Action of the Triangle Anomaly

In $D=4$ dimensions, the $U(1)$ axial current has the well-known anomalous divergence [34–37]

$$\partial_\lambda J^\lambda_5 = \frac{\alpha}{2\pi} F^{\lambda\nu} \tilde{F}_{\lambda\nu} = \mathcal{A}_4$$ (5.1)

in the limit of massless fermions, where $\tilde{F}_{\lambda\nu} \equiv \frac{1}{2} \epsilon_{\lambda\nu\rho\sigma} F^{\rho\sigma}$ is the dual of the field strength tensor $F_{\lambda\nu}$ and $\alpha = e^2/4\pi$. If the current $J^\lambda_5$ is decomposed into its longitudinal and transverse components

$$J^\lambda_5 = J^\lambda_{5\|} + J^\lambda_{5\perp}, \quad \partial_\lambda J^\lambda_{5\perp} = 0$$ (5.2)

it is clear that only the first, longitudinal component $J^\lambda_{5\|}$ can contribute to the non-vanishing divergence (5.1). Since the longitudinal projection of $J^\lambda_5$ is

$$J^\lambda_{5\|} = \partial^\lambda (\Box^{-1} \partial_\nu J^\nu_5)$$ (5.3)

the axial anomaly (5.1) corresponds to the one-loop non-local 1PI quantum effective action [4, 23]

$$S^{NL\text{anom}}[A, A_5] = \frac{\alpha}{2\pi} \int d^4x \int d^4y [A^5_\mu \partial^\mu]_x \Box^{-1}_{xy} [F^{\lambda\nu} \tilde{F}_{\lambda\nu}]_y$$ (5.4)

where $\Box^{-1}_{xy} = \frac{1}{4\pi^2} (x - y)^{-2}$ denotes the massless scalar propagator in $D = 4$. As in $D = 2$ its appearance again signals that the anomaly is associated with a massless pseudoscalar collective excitation, here and in higher dimensions residing in the longitudinal subsector of the full theory.
The anomalous divergence (5.1) and effective action (5.4) results from the one-loop triangle diagram of Fig. 2 corresponding to the amplitude in momentum space

$$\Gamma^{\lambda\alpha\beta}(p, q) = \sum_{i=1}^{6} f_i(k^2; p^2, q^2) \tau_i^{\lambda\alpha\beta}(p, q)$$

which is expressed as a sum over six basis tensors $\tau_i^{\lambda\alpha\beta}(p, q)$ multiplied by scalar form factor functions $f_i$ of the three Lorentz invariants $k^2, p^2, q^2$. Here $k = p + q$ is the ingoing momentum at the axial vertex and $p$ and $q$ are the momenta on the outgoing external photon legs. The coefficient functions are given in Ref. [4] and for zero fermion mass are

$$f_1(k^2; p^2, q^2) = f_4(k^2; q^2, p^2) = \frac{4\alpha}{\pi} \int_0^1 dx \int_0^{1-x} dy \frac{xy}{D}$$

$$f_2(k^2; p^2, q^2) = f_5(k^2; q^2, p^2) = \frac{4\alpha}{\pi} \int_0^1 dx \int_0^{1-x} dy \frac{x(1-x)}{D}$$

$$f_3(k^2; p^2, q^2) = f_6(k^2; q^2, p^2) = 0$$

where $D = (p^2x + q^2y)(1 - x - y) + xyk^2$, in the basis where

$$\tau_1^{\lambda\alpha\beta}(p, q) = \tau_4^{\lambda\beta\alpha}(q, p) = -p \cdot q \epsilon^{\lambda\alpha\beta\gamma} p_\gamma - p^\beta \nu^{\lambda\alpha}(p, q)$$

$$\tau_2^{\lambda\alpha\beta}(p, q) = \tau_5^{\lambda\beta\alpha}(q, p) = p^2 \epsilon^{\lambda\alpha\beta\gamma} q_\gamma + p^\alpha \nu^{\lambda\beta}(p, q)$$

and

$$\nu^{\alpha\beta}(p, q) \equiv \epsilon^{\alpha\beta\rho\sigma} p_\rho q_\sigma.$$ (5.8)

The tensors $\tau_3, \tau_6$ are redundant, and not needed because of (5.6c).

The triangle amplitude (5.5) may be decomposed into its longitudinal and transverse parts

$$\Gamma^{\lambda\alpha\beta}(p, q) = \frac{2\alpha}{\pi} \frac{k^\lambda}{k^2} \nu^{\alpha\beta}(p, q) + \Gamma_{\perp}^{\lambda\alpha\beta}(p, q)$$

where the first, longitudinal part explicitly exhibits a $1/k^2$ pole, and gives the total anomaly

$$k_\lambda \Gamma^{\lambda\alpha\beta}(p, q) = \frac{2\alpha}{\pi} \nu^{\alpha\beta}(p, q)$$ (5.10)
which is (5.1) in momentum space, while the transverse part satisfies
\[ k_{\lambda} \Gamma^{\lambda \alpha \beta}_{\perp}(p, q) = 0 \] (5.11)
and does not contribute to the anomaly.

As in the \( D = 2 \) case, the massless boson degree of freedom represented by the \( 1/k^2 \) pole in (5.9) or \( \Box_{xy}^{-1} \) in (5.4) may be made explicit by expressing the non-local action (5.4) in a local form. Since the non-local action (5.4) involves the axial potential \( A_5^\lambda \) and \( [F \tilde{F}] \) asymmetrically, expressing the action in a local form apparently requires the introduction of two pseudoscalar fields \((\eta, \chi)\) and we may write

\[ S_{\chi \text{anom}}[\eta; A, A_5] = \int d^4x \left\{ (\partial_{\lambda} \eta + A_5^\lambda) \partial^\lambda \chi + \eta \mathcal{A}_4 \right\} \] (5.12)
for the minimal local form of the chiral anomaly effective action (5.4) of massless QED\(_4\) [4]. This effective action is the generating function of the longitudinal component only of the axial current

\[ \frac{\delta}{\delta A_{5\lambda}} S_{\text{anom}} = J_{5\parallel}^\lambda = \partial^\lambda \chi \] (5.13)
as only the dependence upon \( J_{5\parallel}^\lambda \) is determined by the axial anomaly (5.1), and this component can always be expressed as the pure gradient \( \partial^\lambda \chi \), cf. (5.3). Variation of \( S_{\text{anom}} \) with respect to \( \eta \) reproduces the axial anomaly

\[ \partial_{\lambda} J_{5}^\lambda = \partial_{\lambda} J_{5}^\parallel = \Box \chi = \mathcal{A}_4 = \frac{2\alpha}{\pi} \mathbf{E} \cdot \mathbf{B} \] (5.14)
which is a massless wave eq. for the pseudoscalar boson field, with the chiral anomaly as its source, just as in the \( D = 2 \) case (3.9). Analogously to \( D = 2 \) this gapless mode is a collective mode of the two-fermion intermediate state in the anomaly amplitude, if fermion masses and interactions can be neglected, and is a Chiral Density Wave (CDW).

Unlike in \( D = 2 \) (5.12) is only part of the 1PI effective action of QED\(_4\), with the dependence upon the transverse component \( J_{5\perp}^\lambda \) not fixed by the anomaly. As a result, we cannot determine any relationship between \( \eta \) and \( \chi \) from QFT first principles without further information. If \( \chi \) is varied independently of \( \eta \) a second massless wave eq. \( -\Box \eta = \partial^\lambda A_5^\lambda \) is obtained. This would imply the existence of a second independent gapless mode, which would not seem to be warranted by the triangle amplitude (5.5) itself. Indeed as we show in Sec. V C the anomalous current canonical commutators determined by the chiral anomaly imply that \( \eta \) and \( \dot{\chi} \) form a single canonical pair, as in \( D = 2 \), and hence one would expect just a single gapless effective degree of freedom.

Guided by the previous \( D = 2 \) example, and anticipating the relationship to the fluid description, we could posit the local anomaly action

\[ S_{J_{5\text{anom}}}[\eta; A, A_5] = \int d^4x \left\{ (\partial_{\lambda} \eta + A_5^\lambda) J_{5}^\lambda + \eta \mathcal{A}_4 \right\} \] (5.15)
in terms of the full chiral current $J^\lambda_\delta$, instead of (5.12), together with the additional postulate that this effective action should be stationary against variations of the full vector $J^\lambda_\delta$ (not just the pseudoscalar $\chi$ determining $J^\lambda_\delta\|\parallel$). In that case $J^\lambda_\delta$ acts a Lagrange multiplier field, enforces the constraint, $\partial_\lambda \eta + A^5_\lambda = 0$, which when solved for $\eta = -\Box^{-1}\partial^\lambda A^5_\lambda$ and substituted back into (5.15) also reproduces the required non-local action (5.4) with its massless pole. Thus the action (5.15) is just as good an effective action for the axial anomaly as (5.12). However since the requirement of stationarity under the full $J^\lambda_\delta$ variation produces a constraint $\partial_\lambda \eta + A^5_\lambda = 0$ it fixes $\eta$ up to a simple constant, unlike the eq. of motion, $\partial^\lambda(\partial_\lambda \eta + A^5_\lambda) = 0$, and hence (5.15) and its variational principle does not lead to a second independent massless mode, resulting from varying $\chi$ in (5.12).

If (5.15) and stationarity against variation of the full $J^\lambda_\delta$ are adopted for the local anomaly effective action, and one goes one step further by adding $-\varepsilon(n_5)$ to (5.15), then one arrives at the chiral fluid hydrodynamic action (2.1) of Sec. II. Adding $-\varepsilon(n_5)$ leaves the chiral current anomaly (5.1) resulting from independent variation of $\eta$ unchanged, and in the $D = 2$ case is essential to reproducing the full 1PI effective action of the Schwinger model, as shown in Sec. III B above. The pseudoscalar field $\eta$ which was introduced in [4] as an auxiliary field, needed simply to express the non-local anomaly effective action (5.4) in the local form (5.12), is recognized then as the Clebsch potential of the dynamic velocity field (2.7) of a dissipationless, irrotational chiral fluid in the hydrodynamic fluid action (2.1).

B. Chiral Magnetic and Separation Effects from the Anomaly Effective Action

Independently of the addition of $-\varepsilon(n_5)$ and any hydrodynamic fluid interpretation, the axial anomaly effective action in either of its forms (5.12) or (5.15) incorporates several chiral effects, found previously by other methods, see e.g. [38–52]. The electromagnetic current due to the axial anomaly is

$$J^\lambda = \delta S_{\text{anom}} \delta A^\lambda = \frac{\alpha}{2\pi} \frac{\delta}{\delta A^\lambda} \int d^4x \eta F^{\mu\nu} \tilde{F}_{\mu\nu} = \frac{2\alpha}{\pi} \tilde{F}^{\lambda\mu} \partial_\mu \eta$$

(5.16)

in terms of $\eta$. In components this is

$$J^0 = \rho = -\frac{2\alpha}{\pi} B \cdot \nabla \eta$$

(5.17a)

$$\mathbf{J} = \frac{2\alpha}{\pi} (B \cdot \nabla \eta - \mathbf{E} \times \nabla \eta)$$

(5.17b)
For a constant uniform $B$ field, and in the Lorentz frame where $\nabla \eta = 0$, and $\dot{\eta} = \mu_5$, consistent with the relation (2.7), (5.17b) gives

$$J = \frac{2\alpha}{\pi} \mu_5 B$$

which is the Chiral Magnetic Effect (CME).

Similarly, the anomaly action implies a longitudinal chiral current (5.13), with components

$$J_5^0 = J_{5||} = -\dot{\chi}$$

$$J_5 = J_{5||} = \nabla \chi$$

where $\chi$ satisfies (5.14), as determined by the axial anomaly. In a static, constant $B = B\hat{x}$ field and parallel static electric field $E = -\nabla \Phi = -\hat{x} \frac{d\Phi}{dx}$ in the same direction, (5.14) becomes

$$\frac{d}{dx} \left( \frac{d\chi}{dx} \right) = -\frac{2\alpha}{\pi} \frac{d\Phi}{dx} B$$

assuming also $\chi = \chi(x)$. Integrating this once and substituting into (5.19b) gives

$$J_5 = \nabla \chi = \frac{2\alpha}{\pi} \mu_5 B$$

upon taking $\Phi = -\mu$ for the charge chemical potential. In this way the Chiral Separation Effect (CSE) is also implied by and follows directly from the axial anomaly and its effective action (5.12).

### C. Anomalous Current Commutators

The momentum $\Pi_\eta$ conjugate to $\eta$ field can be defined as in $D = 2$ by (2.21), so that in $3 + 1$ dimensions their equal time commutator is

$$[\eta(t, x), \Pi_\eta(t, x')] = \frac{i}{\hbar} \delta^3(x - x')$$

upon quantization, where $\Pi_\eta = J_5^0$ is the total chiral charge density if (5.15) is used, or its longitudinal projection $-\dot{\chi}$ if (5.12) is used for the anomaly effective action. The two are equivalent if we assume that the transverse components of the chiral charge density $J_{5\perp}$ do not contribute to the commutator. Then by (5.17), we have

$$[J_5^0(t, x), J_5^0(t, x')] = -\frac{2i\alpha}{\pi} B \cdot \nabla_x \delta^3(x - x')$$

$$[J(t, x), J_5^0(t, x')] = -\frac{2i\alpha}{\pi} E \times \nabla_x \delta^3(x - x')$$

in a background electric or magnetic field [37, 53, 54]. Thus as in $D = 2$ the Schwinger terms in commutators of fermionic currents are necessarily implied by the canonical commutator (2.21) of
the bosonic effective action (5.15). This shows that quantization of the effective boson described by $\eta$ and its canonical momentum in (2.13) is necessary also in $D = 4$ to obtain the correct anomalous current commutators of QED$_4$, and this conclusion is independent of any particular relation between $\chi$ and $\eta$. Note in particular that (5.23) hold also in the vacuum independently of a chiral density $n_5$ or chemical potential $\mu_5$ [4]. As in the $D = 2$ case the commutator (2.21) shows that there is a single bona fide pseudoscalar collective degree of freedom necessarily associated with the $U_{ch}^1(1) \text{ chiral anomaly in } D = 4$.

Let us also remark in passing that the Schwinger terms (5.23) in the current commutators depend only upon the longitudinal anomalous part of the triangle diagram and are determined by the effective action (5.12). Additional Schwinger terms in the commutators and in particular the commutator

$$[J^0(t, x), J^5(t, x')]$$

which depend upon the transverse part of the amplitude $\Gamma_{\perp}^{\lambda\alpha\beta}(p, q)$, unlike (5.23) are not related to the axial anomaly and not protected by the Adler-Bardeen theorem [35]. Hence they can be cancelled by regularization scheme dependent ‘seagull’ terms, and removed entirely [37].

D. Dimensional Reduction and Bosonization in a Constant Uniform Magnetic Field

The reduction of the $D = 4$ axial anomaly to the $D = 2$ case for a constant, uniform magnetic field can also be obtained directly from the anomalous triangle amplitude (5.5) in QED$_4$ by computing $\Gamma_{\perp}^{\lambda\alpha\beta}(p, q)A_\beta(q)$ with $A_\beta(q)$ the gauge potential of the external magnetic field $B = F_{23}(0)$ at zero momentum with $A_{\beta = 2, 3}(q)$ taken to be in transverse $y$, directions. In this limit of a uniform external magnetic field only the tensors $\tau_1$ and $\tau_2$ in (5.5) which are linear in $q$ contribute, and we can neglect $q^2$ and set $k^2 = p^2$ in the denominator $D$ of (5.6), so that $D = k^2 x(1 - x)$ in this limit and the Feynman integrals in (5.6) are simply evaluated to give

$$f_2 = 2f_1 = \frac{2\alpha}{\pi k^2} \quad \text{for} \quad q^2 = 0.$$

Then taking $k_\perp = p_\perp = 0$ and noting that $k_\nu F^{\nu\lambda}(q) = 0$, we find

$$\lim_{q \to 0} \Gamma^{\lambda\alpha\beta}(k - q, q)A_\beta(q) \Bigg|_{k_\perp = 0} = \frac{2i\alpha}{\pi k^2} \left( k^2 \delta^\alpha_\nu - k^\alpha k_\nu \right) \tilde{F}^{\lambda\nu}$$

$$= \begin{cases} -2i\alpha B \epsilon^c_b \Pi^{ab}_2(k) & \text{if } \lambda = c, \alpha = a \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda = c, \alpha = a$ range only over the 0,1 subspace of the $D = 4$ spacetime and $\Pi^{ab}_2(k)$ is the $D = 2$ vacuum polarization of (3.4). Thus, the full triangle diagram contracted with a constant
uniform magnetic field reduces to the 2D anomalous diangle of Fig. 1, consistent with dimensional reduction, and the $1/k^2$ pole in the 4D triangle anomaly becomes at $k_\perp = 0$ the $1/k^2$ propagator pole of the effective boson $\chi_2$ in the 2D polarization tensor $\langle j^a j^c_5 \rangle$, under dimensional reduction in a constant uniform magnetic field.\(^3\)

Since the anomaly pole structure is preserved under this dimensional reduction it is instructive to repeat this same calculation for the longitudinal part of the triangle amplitude alone, namely

$$\Gamma_{\parallel}^{(\lambda\alpha\beta)}(p,q) = \frac{2\alpha k^{\lambda}}{\pi k^2} v^{\alpha\beta}(p,q). \quad (5.27)$$

Contracting $\Gamma_{\parallel}^{(\lambda\alpha\beta)}(p,q)$ with an external uniform magnetic field $A^\beta(q)$ and taking the same kinematic limit we find

$$\lim_{q \to 0} \bigg|_{k_\perp = p_\perp = 0} \Gamma_{\parallel}^{(\lambda\alpha\beta)}(k - q, q) A^\beta(q) = \lim_{q \to 0} \frac{2\alpha k^{\lambda}}{\pi k^2} v^{\alpha\beta}(p,q) A^\beta(q) = -i\frac{2\alpha}{\pi k^2} \tilde{F}^{\alpha\rho} k_\rho \quad (5.28)$$

where the only surviving indices are two-dimensional ranging only over $t, z$. Then $\tilde{F}^{ab} = \epsilon^{ab} B$ for the 2D subspace of the 4D spacetime and the Schouten relation for $\epsilon^{ab}$ (cf. Appendix A) can be applied to obtain

$$\lim_{q \to 0} \bigg|_{k = 0} \Gamma_{\parallel}^{(\lambda\alpha\beta)}(p,q) A^\beta(q) = \frac{2i\alpha}{\pi k^2} \left( k^2 \delta^a_b - k^a k_b \right) \epsilon^{cb} B \quad (5.29)$$

which coincides with (5.26), proving that the transverse part of the anomalous triangle diagram does not contribute in the dimensional reduction limit of a constant, uniform magnetic field, which is accounted for completely by the longitudinal anomaly action (5.12).

The effective action (5.12) for the axial anomaly, with the longitudinal part of the axial current expressed as a gradient of a bosonic field $\chi$ in (5.13), the wave equation for $\chi$ (5.14) describing CDWs, and the bosonic anomalous commutation relations (5.23) represent a partial bosonization of fermions, for the longitudinal axial current sector. This generalizes the bosonization (3.10) in 1 + 1 dimensional fermion systems, such as the massless Schwinger model, to 3 + 1 dimensions. For the case of a constant, uniform background magnetic field and an accompanying electric field

$$B = B \hat{x} \quad \text{and} \quad E = E(t, x) \hat{x} \quad (5.30)$$

which is both parallel to $B$ and independent of the transverse coordinates $y = (y, z)$, the four dimensional case reduces to the two dimensional one [55–57].

\(^3\) We use the subscript 2 in this subsection only to distinguish the quantities in $D = 2$ dimensions from those in $D = 4$, for which the subscript is omitted.
Note first that in the bosonic form if (5.14) is solved for \( \chi \) in this case of constant, uniform \( B \), and assuming (5.30),

\[
\chi(t, x; B) = \int dt' dx' d^2y' G(t - t', x - x', y - y') \frac{2\alpha}{\pi} E \cdot B \\
= \frac{2\alpha B}{\pi} \int dt' \int dx' G_2(t - t', x - x') E(t', x') \\
= 2\alpha B \chi_2(t, x)
\]

(5.31)

where \( G_2(t, x) = \int d^2y G(t, x, y) \) is the Green’s function of the \( D = 2 \) wave operator \( \Box_2 \), and \( \chi_2 \) is the collective boson of QED\(_2\), related to the axial current \( j_5^0 \) and its anomaly by (3.6). Thus the \( D = 4 \) chiral boson \( \chi \) in a constant, uniform magnetic field is simply proportional to the \( D = 2 \) chiral boson \( \chi_2 \) of the Schwinger model when \( E \cdot B = E(t, x) \). With (5.13), (5.31) implies that the longitudinal chiral charge density in \( D = 4 \) is related to the two-dimensional one

\[
J^0_5(t, x; B) = 2\alpha B j^0_5(t, x)
\]

(5.32)

independent of the transverse \( y \) coordinates when (5.30) holds. Thus the \( D = 4 \) axial anomaly effectively factorizes and reduces to the \( D = 2 \) anomaly in a constant uniform, magnetic field \( B \), provided (5.30) holds for the electric field as well.

Next we observe that we can integrate the \( D = 4 \) anomalous current commutator (5.23a) over the transverse coordinates \( \int d^2y \), and use (5.32) to obtain

\[
\left[ \int d^2y \, J^0(t, x, y; B), j^0_5(t, x') \right] = -\frac{i}{\pi} \partial_x \delta(x - x')
\]

(5.33)

after cancelling the common factor of \( 2\alpha B \) from both sides. Since \( j^0_5 = j^1 \) by (3.12), comparing (5.33) with the \( D = 2 \) anomalous current commutator (3.13) enables us to identify

\[
j^0(t, x) = \int d^2y \, J^0(t, x, y; B) = j^1_5(t, x) = \partial_x \chi_2(t, x)
\]

(5.34)

up to terms that may commute with \( j^0_5 \). Then the anomalous current relation (5.17a) gives

\[
2\alpha B \int d^2y \, \frac{\partial}{\partial x} \eta(t, x, y; B) = -\pi \frac{\partial}{\partial x} \chi_2(t, x) = \frac{\partial}{\partial x} \eta_2(t, x) = -\frac{\pi}{2\alpha B} \frac{\partial}{\partial x} \chi(t, x; B)
\]

(5.35)

so that integrating with respect to \( x \)

\[
\eta_2(t, x) = 2\alpha B \int d^2y \, \eta(t, x, y; B)
\]

(5.36)

gives the additional relation

\[
\chi(t, x; B) = -\frac{4\alpha^2 B^2}{\pi} \int d^2y \, \eta(t, x, y; B)
\]

(5.37)
between the two $D=4$ pseudoscalar potentials $(\eta, \chi)$ in the case that (5.30) holds, up to an arbitrary function of time that however also drops out of the commutator (2.21), when compared between $D=2$ and $D=4$ dimensions.

The relations (5.31) and (5.35) substituted into the anomaly effective action give

$$S_{\chi_{\text{anom}}}|_B = \int dt \, dx \left\{ \partial_a \left( \int \eta \, d^2 y \right) \partial^a (2\alpha B \chi_2) - \frac{2\alpha B}{\pi} E(t, x) \int \eta \, d^2 y \right\}_B$$

$$= \int dt \, dx \left\{ -\pi (\partial_a \chi_2)(\partial^a \chi_2) - E \chi_2 \right\}$$

which is remarkably similar, but not quite equal to the complete $D=2$ Schwinger model action (3.17). That the two do not quite agree is hardly surprising since the effective action (5.12) only accounts for the anomalous longitudinal part of the chiral current in $D=4$, and is clearly incomplete, neglecting the transverse, non-anomalous contributions to the effective action, which we have further neglected in the commutator arguments leading to (5.37) dropping terms which do not contribute to the anomalous commutators. However the relation (5.37) between the two \textit{a priori} unrelated pseudoscalar fields $(\eta, \chi)$ in $D=4$ dimensions and the similarity of (5.38) to the two-dimensional perfect chiral fluid hydrodynamic action suggest that as in the $D=2$ case one should add to (5.38) a non-anomalous contribution $-\varepsilon(n_5)$, as in (3.16), in order to render the relations (5.31)-(5.37) consistent with the eqs. of motion following from the variation of $\chi_2$ in the effective action. We know from the $D=2$ case and (3.16) that this requires completing the dimensionally reduced anomalous effective action (5.38) by adding to it the two-dimensional integral of $\varepsilon(n_5)$ which we may express in the form

$$-\frac{1}{2} \int dt \, dx \, \mu_5 n_5 = -\frac{1}{2} \int dt \, dx \, (\partial_a \eta_2)(\partial^a \chi_2) = \frac{\pi}{2} \int dt \, dx \, (\partial_a \chi_2)(\partial^a \chi_2)$$

which added to (5.38) does give exactly the $D=2$ Schwinger model action (3.17). This shows that if the energy density takes on the appropriate dimensionally reduced $D=2$ form when in an external constant, uniform magnetic field, when (5.30) holds, then the QFT effective action of massless fermions in $D=4$ reduces to that of the chiral superfluid Schwinger model at $\epsilon = 0$ as well. To show this conclusively requires evaluating the four-point square diagrams of Fig. 3 of massless QED$_4$, where two of the vertices are attached to a constant $B$ field, and the other two are attached to the parallel two-dimensional electric field $E(t, x)$.

It is interesting also to note that the anomalous current commutator expectation value can be evaluated from the discontinuity in $k^0 = \omega$ of the polarization operator

$$\langle [J^0(t, x, y), J^0_5(t, x', y')] \rangle = \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{d^2 k_\perp}{(2\pi)^2}$$
FIG. 3: The Phton-Photon Scattering or ‘Box’ Diagram

\begin{align}
&\times e^{ik(x-x') + i\mathbf{k}_\perp \cdot (y-y')} \left\{ 2 \Im \Pi_{00}^g (\omega + i\epsilon, k, \mathbf{k}_\perp) \right\}.
\end{align}

(5.40)

in the Lowest Landau Level (LLL) approximation where \[58–62]\

\begin{align}
\Pi_{ab}^{g \theta} (\omega, k, \mathbf{k}_\perp) & \bigg|_{_{LLL}} = 2\alpha B \exp \left( -\frac{k^2}{2eB} \right) \Pi_c^{ac} (\omega, k) \epsilon^b \\
\end{align}

(5.41)

in terms of the $D = 2$ polarization (3.4). Thus integrating (5.40) over $y$ sets $\mathbf{k}_\perp = 0$ in (5.41) and

\begin{align}
&\int d^2 y \langle [J^0(t, x, y), J^0(t, x', y')] \rangle_{_{LLL}} \\
&= 2\alpha B \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} e^{ik(x-x')} \left\{ 2 \Im \Pi_{2}^{01} (\omega + i\epsilon, k) \right\} \\
&= 2\alpha B \langle [j^0(t, x), j^1(t, x')] \rangle = -\frac{2i\alpha B}{\pi} \frac{\partial}{\partial x} \delta(x - x')
\end{align}

(5.42)

consistent with (5.23a) and dimensional reduction to $D=2$. The fact that the LLL approximation saturates the anomalous commutator is consistent also with the fact that only the LLL in a constant magnetic field has gapless excitations, so that if an external electric field is turned on adiabatically only fermions in the LLL can be excited, and the $D = 4$ axial anomaly factorizes into its $D = 2$ counterpart with a transverse density proportional to the magnetic field $B$ [55].

VI. CHIRAL SUPERFLUID HYDRODYNAMICS IN FOUR DIMENSIONS

The anomaly action (5.12) derived from the triangle diagram of massless fermions in QED$_4$ is similar to chiral superfluid description of the longitudinal chiral excitations of the Fermi-Dirac sea of (2.1) for $D=2$. There are two obvious differences. First, the hydrodynamic action (2.1) allows for a general axial current $J_5^\lambda$, while the anomaly action (5.12) is restricted to the longitudinal
projection of $J_5^\lambda$ only, which is expressed as the pure gradient $\partial^\lambda \chi$. Secondly, the fluid action (2.1) contains the non-anomalous energy density $\varepsilon(n_5)$, with $n_5^2 = -J_5^\lambda J_5^\lambda$ dependent upon the full axial current, while $\varepsilon(n_5)$ is absent from (5.12). Note that in $D=2$ because of the very restricted kinematics, and the fixed constant relation between $\partial_\lambda \eta$ and $\partial_\lambda \chi$, (3.15), the anomalous effective action (3.8) is identical to the fluid action, and already contains the $\varepsilon(n_5)$, as (3.16) shows. We cannot expect these special features of the $D=2$ case to hold in general $D$ dimensions.

Note in particular that the ideal hydrodynamic action (2.1) for a potential flow at zero temperature in $D=4$ implies a constraint

$$J_5^\lambda + \frac{n_5}{\mu_5} \partial^\lambda \eta = 0 \quad (6.1)$$

which cannot be satisfied by the longitudinal pure gradient part of the axial current $J_5^\lambda = \partial^\lambda \chi$ alone, unless $n_5/\mu_5$ is a constant as it is in $D=2$. Hence we face the choice of either adhering to a description of only the longitudinal modes, along the direction of $J_5^i$, which are effectively 1 + 1 dimensional excitations of the Fermi-Dirac sea, essentially dimensionally reducing $D=4$ back to the $D=2$ case, or alternatively, departing from the strictly QFT derivation of the anomaly effective action by adopting the fluid ansatz of $\varepsilon(n_5)$ describing the full 3 + 1 dimensional equilibrium energy density, including also the transverse components which have been neglected in (5.12).

If the latter course is followed, and to characterize the regime of validity of the hydrodynamic ansatz, it is instructive to consider the hydrodynamic mode(s) that appear when the theory is linearized around a non-zero background $n_5, \mu_5$. In general, a hydrodynamic system with an axial charge at zero temperature and in the absence of external electromagnetic fields is described by five conservation laws (2.4) and (2.28). From (2.27) current conservation implies energy-momentum conservation and visa versa. We have seen in Secs. II and IV that chiral current conservation implies the existence of a gapless Goldstone acoustic mode (2.20) or (4.12). If one considers the variation of the energy-momentum tensor linearized around a constant background $n_5$ and the corresponding $\mu_5$ one obtains

$$M^{\lambda\nu} \delta J_{5,\nu} = \left\{ g^{\lambda\nu} + u^\lambda u^\nu (1 - v_s^2) (u \cdot \partial) + u^\lambda \partial^\nu - v_s^2 u^\nu \partial^\lambda \right\} \delta J_{5,\nu} = 0 \quad (6.2)$$

This equation possess non-trivial solutions in general $D = d + 1$ dimensions only if the Fourier components of $\delta J_{5,\alpha}$ satisfy the relation

$$\det M = (-i\omega)^{d-1} \left[ \omega^2 - v_s^2 |k|^2 \right] = 0 \quad (6.3)$$

which is the same condition as that obtained in Sec. IV., showing that the only propagating mode in the non-dissipative superfluid is the gapless (first) sound mode.
In $D=4$ dimensions massless fermions have

$$p(\mu_5) = \frac{\mu_5^4}{12\pi^2}, \quad n_5 = \frac{dp}{d\mu_5} = \frac{\mu_5^3}{3\pi^2} \quad (6.4)$$

and the speed of sound is given by

$$v_s^2 = \frac{dp}{d\varepsilon} = \frac{n_5}{\mu_5} \frac{d\mu_5}{dn_5} = \frac{1}{3} \quad (6.5)$$

which clearly differs from the propagation speed of the collective excitation described by the pole in the perturbative triangle anomaly diagram in $D=4$. One may have expected that the velocity of the anomalous mode in the hydrodynamic description is renormalized by the inclusion of the transverse components of the axial current not protected by the anomaly, but that the Goldstone theorem continues to apply in $D > 2$ dimensions.

It is also instructive to consider the sound wave generated by the anomalous divergence (2.4) treated as a source, then the hydrodynamic action implies

$$\delta \eta = -\frac{d\mu_5}{dn_5} \frac{1}{[-\partial_t^2 + v_s^2 \nabla^2]} A_4 \quad (6.6)$$

and the corresponding axial current variation is

$$\delta J_5^0 = -\frac{\partial_t}{[-\partial_t^2 + v_s^2 \nabla^2]} A_4$$

$$\delta J_5 = \frac{v_s^2 \nabla}{[-\partial_t^2 + v_s^2 \nabla^2]} A_4. \quad (6.7)$$

Strikingly, the anomalous hydrodynamics indicates that the axial current of a sound wave sourced by the gauge invariant anomalous divergence has a pole similar to the anomalous pole in vacuum but modified by the finite speed of the CDW. If this axial current is substituted back to the action (2.1) expanded around the background the second order part of the action reads

$$S^{(2)}_{\chi} = \int d^4x \left\{ \frac{1}{2} \frac{dn_5}{d\mu_5} \left[ \delta \eta^2 - v_s^2 (\nabla \delta \eta)^2 \right] + \delta \eta A_4 \right\} \quad (6.8)$$

where the first two terms can be seen as the second order correction to the pressure $P$. This action can also be expressed in a non-local

$$S^{(2)}_{\chi} = -\frac{1}{2} \int d^4x d^4y \frac{d\mu_5}{dn_5} \left\{ A_4(x) \left[ \frac{1}{[-\partial_t^2 + v_s^2 \nabla^2]} \right]_{xy} A_4(y) \right\} \quad (6.9)$$

coupling the anomaly source at two different spacetime points by the retarded interaction with a gapless acoustic CDW.
VII. CHIRAL DENSITY WAVES AND ELECTROMAGNETIC INTERACTIONS

The CDW of the anomalous hydrodynamics described by (2.1) result in propagating waves of the electric density due to (5.16). This joint propagation of the two densities is similar to CMW – a collective excitation caused by an interplay between CME and CSE [63]. However, a wave of $\eta$ propagates in all directions in contrast to CMW, while the absence of a hydrodynamic equation for the electric density indicates that this combination of CDW and CME is, in general, some mixture of chiral waves. Note also that while the anomaly effective action (5.12) incorporates CME and CSE in background magnetic field and at non-zero densities, the collective wave involves the relation between $\chi$ and $\eta$ and can be derived only if the theory completed with the energy density $\varepsilon(n_5)$, either by a hydrodynamic ansatz for the transverse degrees of freedom or a complete QFT calculation of higher order non-anomalous processes.

In a background magnetic field CDW propagates with a general speed of sound (2.20) depending on the equation of state. In the strong field limit $|B| \gg \mu_5^2, \mu^2$ waves become ultra-relativistic $v_s \to 1$ and reduce to $D=2$ waves. Indeed, the absence of the transverse motion in a strong field background can be understood in terms of fermions which are projected to their Lowest Landau Level and propagate only along the field direction. In this limit

$$\frac{n_5}{\mu_5} = \frac{2\alpha}{\pi} B$$

(7.1)

and, for the waves propagating along the $B$-direction, we have $\Box 2\eta = 0$, with the electric and axial densities

$$J_5^0 = \frac{2\alpha}{\pi} B \dot{\eta}, \quad J_0^0 = -\frac{2\alpha}{\pi} B \cdot \nabla \eta.$$  

(7.2)

Thus, CDW propagating along a strong magnetic field reduces to the $D=2$ collective wave and coincides with the CMW in this limit.

One can also ask how the CDW dynamics is modified by dynamical electromagnetic fields. Then the anomalous hydrodynamic action (2.1) should be extended to include the kinetic term of electromagnetic fields and the eqs. of motion are completed by Maxwell equations

$$\nabla \cdot E = -\nabla^2 \Phi = J^0 = \rho = -\frac{2\alpha}{\pi} B \cdot \nabla \eta$$  

(7.3a)

$$\nabla \times B - \frac{\partial E}{\partial t} = J = \frac{2\alpha}{\pi} (B \dot{\eta} - E \times \nabla \eta)$$  

(7.3b)

where $\Phi \equiv A^0 + \nabla^{-2} \nabla \cdot \dot{A}$ is the gauge invariant electromagnetic potential, and $\rho \Phi$ is the Coulomb energy. If we consider a background spatially uniform and constant magnetic field $B$ the wave
eq. (2.20) becomes

\[-\frac{dn_5}{d\mu_5} (\partial_t^2 - v_s^2 \nabla^2) \eta = \left(\frac{2\alpha}{\pi}\right)^2 (B \cdot \nabla) \frac{1}{\nabla^2} (B \cdot \nabla) \eta + \frac{2\alpha}{\pi} B \cdot \dot{A}^\perp \] (7.4)

where (7.3a) is used to solve for \( \Phi \) and \( A^\perp \) is the transverse part of the vector potential. This term represents a coupling between usual electromagnetic radiation and CDW. Taking for simplicity the solution \( A^\perp = 0 \), we find for waves propagating along the \( B \)-direction

\[-\frac{dn_5}{d\mu_5} (\partial_t^2 - v_s^2 \nabla^2) \eta + \left(\frac{2\alpha}{\pi}\right)^2 B^2 \eta = 0 \] (7.5)

The CDWs are massless in the absence of interactions with the electromagnetic field but acquire a mass term when these interactions are considered. In the strong field limit, the underlying fermionic system is restricted to its Lowest Landau Level so that the effective squared mass of the collective CDW is \( \frac{2\alpha}{\pi} B \) and \( v_s \to 1 \). Thus, the dimensional reduction of the longitudinal CDW recalls the similar result in massless QED, in the Schwinger model with \( e \neq 0 \), and coincides with the dimensional reduction of CMW in the presence of interactions, see [63].

**VIII. THE EFFECT OF FERMION MASS**

Although the anomaly violates chiral symmetry, it does so in a very special way, affecting global properties of the ground state, that allows it to be incorporated into a low energy effective hydrodynamic framework. Indeed the massless pole implied by the anomalous chiral Ward Identity requires that this low energy excitation be accounted for in the effective hydrodynamics, where it describes a Chiral Density Wave that can be understood as a Nambu-Goldstone boson that is usually expected only for spontaneously broken global symmetries.

On the other hand, once a finite mass \( m \) for the fermions is admitted, the chiral symmetry is broken explicitly. This makes the assumption of local conservation of axial charge density and a corresponding axial chemical potential \( \mu_5 \) unjustified, undercutting the basis for a hydrodynamic description of chiral excitations, affecting the longest wavelength acoustic modes. Considering first the case of two dimensions, at \( m = 0 \) the left and right moving chiral modes \( \phi_\pm \) defined in B12 of Appendix B are decoupled. When \( m \neq 0 \) that bosonization formulae for the chiral and vector currents (3.10) continue to hold, but the massive Schwinger model is equivalent to a self-interacting sine-Gordon theory, with eq. of motion [24]

\[-\Box \chi + \frac{m^2}{2\pi} \sin (2\pi \chi) = 0 \quad (D=2) \] (8.1)
for the pseudoscalar $\chi = \phi_- - \phi_+$ field in $D=2$ dimensions, in the present notation, and with a suitable definition of normal ordering of the sinusoidal potential term. Note that this theory still possesses the discrete global symmetry $\chi \to \chi + N$ for integer $N$, which with $\eta = -\pi \chi$ in the $m \to 0$ limit is equivalent to the $2\eta \to 2\eta - 2\pi N$, $2\pi$-periodicity of $2\eta$ discussed in Sec. IV. Note also that the bosonization (3.10) guarantees that the charge current $j^a = -\epsilon^{ab} \partial_b \chi$ is automatically conserved for any smooth $\chi$ field, no matter what its eq. of motion, while the chiral current $j^a_5$ is no longer conserved if $m \neq 0$ according to (8.1).

One may ask then what becomes of the scalar mode $\varphi \equiv -(\phi_+ + \phi_-)$ whose gradient is dual to the gradient of $\chi$ at $m = 0$, and could have been treated on an equal footing to $\chi$ (save for the axial anomaly), once $m \neq 0$. The symmetric scalar $\varphi$ is naturally connected to electric charge $Q$ which remains exactly conserved, rather than chiral charge $Q_5$ which is not conserved when $m \neq 0$. Thus $\chi$ and $\varphi$ are on quite different footings in the case of non-zero fermion mass, and one cannot expect the hydrodynamic description of Sec. II any longer to apply to $j^a_5$, although it should continue to apply to the conserved charge current $j^a$, with $n$ replacing $n_5$ and the charge chemical potential $\mu$ replacing $\mu_5$ in the fluid action (2.1). In that case the fluid action is given by

$$S_{\text{fluid}} = \int d^2 x \left\{ ( - \partial_a \sigma + A_a ) j^a - \varepsilon(n) \right\}$$

(8.2)

where $\sigma$ is the scalar Clebsch potential for the charged fluid with

$$\partial_a \sigma = \mu u_a = \frac{\mu}{n} j_a$$

(8.3)

and

$$\varepsilon = \frac{1}{2} \mu n + \frac{m^2}{2\pi} \ln \left( \frac{\mu + \pi n}{m} \right)$$

$$p = \frac{1}{2} \mu n - \frac{m^2}{2\pi} \ln \left( \frac{\mu + \pi n}{m} \right)$$

$$n = \frac{1}{\pi} \sqrt{\mu^2 - m^2}$$

(8.4)

satisfying $\varepsilon + p = \mu n$ for free massive fermions in $D=2$. The scalar mode $\varphi$ thus becomes an Charge Density Wave acoustic mode with the sound speed $v_s$ given by

$$v_s = \left( \frac{dp}{d\varepsilon} \right)^{1/2} = \left( \frac{n d\mu}{\mu dn} \right)^{1/2} = \sqrt{1 - \frac{m^2}{\mu^2}}$$

(8.5)

in units in which the speed of ‘light’ is unity. Thus while the bosonized theory (8.1) is no longer a free theory at finite mass, there is still a gapless Goldstone mode at finite fermion density described by hydrodynamics with conserved electric charge, with

$$\delta j^a = \frac{dn}{d\mu} \left[ u^a u^b \left( 1 - v_s^2 \right) - v_s^2 g^{ab} \right] \partial_b (\delta \sigma)$$

(8.6)
completely consistent with the discussion of Nambu-Goldstone theorem in Sec. IV for a non-Lorentz invariant ground state in higher dimensions, leading to (4.10). The gapless mode, which because of the relations (8.3) is both a Chiral Density Wave and a Charge Density Wave, persists even at finite fermion mass and breaking of Lorentz invariance, but only in the massless limit does $v_s = 1$ in (8.5), and is the current perturbation a pure gradient of a scalar potential $\delta j_a = -\partial_a(\delta \sigma/\pi)$.

Indeed this is precisely the result obtained by Haldane for the low energy excitation of the massive Schwinger model as the limiting case of the Thirring model, described as a Luttinger quantum liquid in $D=2$ [64–66]. There is only one low energy acoustic mode at any mass, since the massive $\chi$ field in (8.1) decouples at low energies. For finite $\mu$ with $\mu \leq m$ the acoustic mode disappears from the spectrum and one is left only with the free massive fermions, alternately described by (8.1). Conversely, in the high fermion density limit, $n \rightarrow \mu/\pi \rightarrow \infty$, the fermion mass may be neglected, $\varphi$ and $\chi$ are on the same footing again and the acoustic mode of (8.2) becomes indistinguishable from the CDW of the massless chiral superfluid in Sec. III B.

Thus although the spectrum and character of low energy excitations as joint Chiral/Charge Density Waves is continuous in $m$, the description of the acoustic mode is quite different in (8.2) compared to the chiral superfluid (2.1), as the acoustic mode of (8.2) is not a Nambu-Goldstone mode of global chiral symmetry breaking but of $U(1)$ electric charge at finite $m$. The axial anomaly and its massless pseudoscalar pole at $k^2 = 0$ is screened and removed completely at any finite $m$. This is clear from the fact that in $D=2$ the scalar function in the polarization tensor of (3.4) is a function of $k^2/m^2$ which vanishes as $k^2/m^2 \rightarrow 0$ since this is equivalent to the decoupling limit $m \rightarrow \infty$ where there is no anomaly [4, 23]. This decoupling persists for a non-zero electric charge $e \ll m$ in $D=2$, where weak coupling methods can be applied [67].

The state of affairs is different for non-zero fermion mass in $D=4$ dimensions. Since there are additional invariants $p^2$ and $q^2$ associated with the photon legs in the triangle diagram of Fig. 2, finite fermion mass does not imply decoupling of the anomaly pole at $k^2 = 0$, provided $p^2, q^2 \gg m^2$. Evaluation of the amplitude (5.5) for finite fermion mass gives a finite residue of the anomaly pole

$$\lim_{k^2 \rightarrow 0} k\lambda \Gamma^{\lambda \alpha \beta}(p, q; m) A_\alpha(p)A_\beta(q) = I \left( \frac{p^2}{m^2}, \frac{q^2}{m^2} \right) \nu^{\alpha \beta}(p, q)A_\alpha(p)A_\beta(q)$$

$$= \frac{2\alpha}{\pi} \left\{ 1 - 2m^2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{(p^2 x + q^2 y)(1-x-y) + m^2} \right\} E(p) \cdot B(q)$$

which is non-vanishing in Fourier space, if either of the photon legs is off shell and their virtualities $p^2$ or $q^2$ are non-zero. For $m \rightarrow 0$ this gives the full anomaly source $\mathcal{A}_4$ of the $\chi$ field in (5.14),
while in the opposite limit $p^2 \ll m^2, q^2 \ll m^2$, (8.7) gives

$$\Box \chi = \frac{\alpha}{3\pi m^2} \tilde{F}^{\mu\nu} \Box F_{\mu\nu} + O \left( \frac{(\partial^2 F)^2}{m^4} \right)$$

in position space. Thus, the pseudoscalar field $\chi$ can be produced by an off-shell source of $\mathbf{E} \cdot \mathbf{B}$, provided its scale of spatiotemporal variation is comparable to or faster/shorter than the electron Compton wavelength $m^{-1}$. The anomaly pole persists and the $\chi$ propagates even in vacuo for $m \neq 0$ as in the massless case, but chiral hydrodynamics is no longer applicable and we lose the clear connection between $\chi$ and $\eta$. It is interesting to speculate that in this case the completion of the theory requires evaluating the $\langle J JJ JJ \rangle$ box diagram of Fig. 3, to find the relation between the periodic $\eta$ velocity potential phase field which is coupled directly to the anomaly and the longitudinal chiral current field $\chi$ in the quantum vacuum of full QED$_4$, when the hydrodynamic relation (2.5) no longer applies.

**IX. CONCLUSION AND SUMMARY**

In this paper we have given an action principle and Hamiltonian for an irrotational and dissipationless chiral superfluid at zero temperature, and demonstrated the close connection between its macroscopic hydrodynamics and the QFT of massless fermions, with the axial anomaly playing a central role. In particular, the collective chiral boson of the QFT axial anomaly effective action may be understood as the fermion/anti-fermion (or particle/ hole) pair excitations of the Fermi surface at zero temperature and finite fermion density. This excitation is a Chiral Density Wave.

In the chiral superfluid description there are two bosonic fields, $\chi$ and $\eta$, with $\chi$ defining the longitudinal part of the chiral current through $J_5^\lambda = \partial^\lambda \chi$, and $\eta$ the phase field Clebsch potential that couples to the axial anomaly. The time derivative $\dot{\chi}$ and $\eta$ form a canonical pair in the superfluid Hamiltonian so that it is clear that these two fields are closely related, their canonical commutator exactly reproducing the anomalous Schwinger terms in the current commutators of the fermion QFT. In the fluid description both spatial and temporal gradients of the two fields are related by (2.5), hence the equilibrium equation of state through the ratio $\mu_5/n_5$. In the case of $D=2$ this ratio $\mu_5/n_5 = \pi$ is fixed, and the superfluid description can be extended to zero fermion density. In this case of QED$_2$ the superfluid effective action and that of the Schwinger model at vanishing electric charge $e = 0$ actually coincide. The acoustic mode of the relativistic superfluid is precisely the massless chiral boson vacuum excitation of the Schwinger model, and the Dirac vacuum itself supporting this excitation may be viewed as a superfluid medium.
In all even dimensions the commutation relation (4.1) implies that $2\eta$ is a $2\pi$-periodic phase field that satisfies a non-linear realization of Goldstone’s theorem, when the ground state of the theory has a non-zero expectation value $\langle e^{2i\eta} \rangle$. Thus although the axial anomaly implies that fermionic chiral charge is not conserved, its effects are akin to spontaneous symmetry breaking by a bosonic field expectation value, and the massless collective chiral excitation implied by the axial anomaly pole may be understood as a Nambu-Goldstone boson. This interpretation is further supported by a standard consideration of a simple QFT $U(1)$-Higgs model of a linear realization of Goldstone’s theorem with a general $U(1)$ symmetric potential $V(\rho)$, whose low energy excitation is exactly that of the hydrodynamic description, allowing for a gapless acoustic mode with any speed of sound $0 < v_s^2 \leq 1$. The cases where $v_s^2 < 1$ correspond to the breaking of Lorentz invariance by the ground state at non-zero chemical potential.

Clear macroscopic consequences of the axial anomaly are the Chiral Magnetic and Chiral Separation Effects, following directly from the effective action of the anomaly in $D=4$. In a constant uniform magnetic field $B$ with parallel electric field independent of transverse directions, the axial anomaly factorizes and the massless boson Chiral Density Wave reduces to that of the $D=2$ Schwinger model.

Since $n_5/\mu_5$ is not a constant in $D > 2$ dimensions, and there is a transverse component in the chiral current which is not fixed by the axial anomaly, the chiral superfluid description of massless fermions applies only to a subsector of the theory, and is clearly incomplete. Nevertheless in this sector the axial anomaly pole (5.9) exists, $\chi$ still satisfies the massless wave equation (5.14) of a propagating chiral collective boson, even when the fermion mass is non-zero, and the anomalous current commutators (5.23) are non-vanishing. Moreover, expanding around non-zero chiral density and chemical potential for which $n_5/\mu_5$ is a constant the hydrodynamic approximation continues to hold, and a gapless acoustic mode is obtained, although with a velocity less than the speed of light. This shift of the velocity of the propagating gapless mode should be checked in the QFT description by evaluating the axial anomaly triangle diagram of Fig. 2 at finite fermion density. Based on the dimensional reduction in a background magnetic field, requiring the box diagram of Fig. 3, we speculate that the completion of the bosonic anomalous sector in $D=4$ dimensions QFT requires the evaluation of this $\langle JJJJ \rangle$ amplitude. The effect of finite fermion mass in full QED in $D=4$ can be studied in this way as well.

Finally, we note that the superfluid description of the fermion vacuum should admit quantized vortex line solutions described by the periodic phase $\eta$ wrapping an integer number of times around the azimuthal angle. In the abelian Higgs model of (4.16) finite energy vortex line solutions
exist, where the magnitude $\rho$ of symmetry breaking goes to the minimum $\bar{\rho}$ at infinity but $\rho \to 0$ at the center of the vortex, corresponding to the non-superconducting state. This solution depends on having an independent eq. for the magnitude $\rho$ as in the scalar model but not in the hydrodynamic fluid description of (2.1), where the only degree of freedom is $\eta$, and the magnitude of symmetry breaking $n_5/\mu_5$ is fixed by a non-linear relation with $\partial_\lambda \eta$ itself. That the fermion vacuum admits vortices in a superfluid description may also be suggestive of a superfluid picture of rotating spacetimes in general relativity [68]. These considerations and a fuller QFT treatment of the axial and gravitational anomalies in $D > 2$ dimensions remain for future work.

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Appendix A: Energy-Momentum Tensor Conservation and Comparison of $H$ and $T^{00}$

To demonstrate the equivalence of (2.27) and (2.28) expected for charged matter in a background, non-dynamical electromagnetic field requires knowledge of the general tensorial form of the axial anomaly in $D$ (even) dimensions

$$A_D = c_D \epsilon^{\alpha_1 \ldots \alpha_D} F_{\alpha_1 \alpha_2} \ldots F_{\alpha_{D-1} \alpha_D}$$

(A1)

where $c_D$ is a dimension dependent constant, and $\epsilon^{\alpha_1 \ldots \alpha_D}$ is the totally anti-symmetric $D$-dimensional Levi-Civita tensor, which satisfies the Schouten identity

$$g^{\lambda \beta} \epsilon^{\alpha_1 \ldots \alpha_D} + g^{\lambda \alpha_1} \epsilon^{\alpha_2 \ldots \alpha_D \beta} + \ldots + g^{\lambda \alpha_{D-1}} \epsilon^{\alpha_{D-1} \ldots \alpha_D} = 0.$$  

(A2)

In (A2) the sum is over all $D + 1$ cyclic permutations of the indices $(\nu, \alpha_1, \ldots, \alpha_D)$. Since the tensor on the left side of (A2) is totally anti-symmetric in all $D + 1$ indices, but no such tensor exists in $D$ dimensions, it must vanish identically. Thus, on the one hand, multiplying (A2) by $c_D F_{\alpha_1 \alpha_2} \ldots F_{\alpha_{D-1} \alpha_D}$ gives

$$g^{\lambda \beta} A_D + c_D D g^{\lambda_1 \alpha_1} \epsilon^{\alpha_2 \ldots \alpha_D \beta} F_{\alpha_1 \alpha_2} \ldots F_{\alpha_{D-1} \alpha_D} = 0$$

(A3)

where the latter $D$ terms are all the same after relabeling indices. Multiplying (A3) by $\partial_\beta \eta$ gives

$$-A_D \partial_\lambda \eta = c_D D g^{\lambda \alpha_1} \epsilon^{\alpha_2 \ldots \alpha_D \beta} F_{\alpha_1 \alpha_2} \ldots F_{\alpha_{D-1} \alpha_D} \partial_\beta \eta$$

(A4)

after another relabeling of indices. On the other hand from (2.24),

$$J^\nu = c_D D \epsilon^{\nu \alpha_2 \alpha_3 \alpha_4 \ldots \alpha_D} \partial_{\alpha_2} (\eta F_{\alpha_3 \alpha_4} \ldots F_{\alpha_{D-1} \alpha_D})$$

(A5)

by the Bianchi identity for the electromagnetic field strength $F_{\alpha \beta}$, and therefore

$$F^{\lambda \nu} J^\nu = c_D D g^{\lambda \beta} \epsilon^{\alpha_1 \alpha_2 \ldots \alpha_D} F_{\beta \alpha_1} F_{\alpha_3 \alpha_4} \ldots F_{\alpha_{D-1} \alpha_D} (\partial_{\alpha_2} \eta).$$

(A6)

which coincides with (A4). Thus the equivalence of (2.27) and (2.28) is demonstrated.

The fact that the simple perfect fluid form of the Energy-Momentum Tensor (2.23) satisfies the partial conservation law (2.28) shows that it already contains the $J^\nu A_\nu$ interaction term of the underlying fermionic theory. However, comparing its $T^{00}$ component with the canonical Hamiltonian $H$ of (2.18), one finds that they differ by the anomaly term $\eta A_D$. The reason for this difference is
again the special form of the axial anomaly (A1) being linear in the $F_{i0}$ electric component of the field strength tensor in any dimension (all of the other indices of $F_{ij}$ being necessarily spatial by anti-symmetry of the $\epsilon$ symbol).

The difference between $T^{00}$ and $H$ is best illustrated by a toy model in $0 + 1$ dimensions defined by the Lagrangian

$$L_{\text{tot}}[\chi, A] = \frac{m_1}{2} \dot{\chi}^2 + \chi \dot{A} + \frac{m_2}{2} \dot{A}^2 \equiv L_\chi + L_A$$

(A7)

where $L_\chi$ comprises the first two $\chi$ dependent terms, including the $\chi \dot{A}$ interaction linear in the velocity of $A$. This gives rise to a ‘current’ $J = \delta S_{\text{tot}}/\delta A = -\dot{\chi}$. If $A(t)$, modeling the external gauge potential, is taken to be an arbitrary non-dynamical external field, its Lagrangian $L_A$ and eq. of motion can be neglected, and we may compute the Hamiltonian for the $\chi$ field only, obtaining

$$H_\chi = \dot{\chi} \frac{\partial}{\partial \dot{\chi}} L_\chi - L_\chi = \frac{m_1}{2} \dot{\chi}^2 - \chi \dot{A} = \frac{1}{2m_1} p_\chi^2 - \chi \dot{A}$$

(A8)

which contains the time dependent interaction term, and yields the correct $\chi$ eq. of motion

$$m_1 \ddot{\chi} = \dot{A}$$

(A9)

analogous to the axial anomaly eq. (3.9) or (5.14) in the arbitrary potential $A(t)$.

On the other hand, being linear in the velocity $\dot{A}$, the interaction term will not appear in the covariant definition of the energy since the action $\int dt \chi \dot{A}$ is invariant under arbitrary reparameterizations of time. If one further calculates the total Hamiltonian corresponding to (A7)

$$H_{\text{tot}} = \dot{\chi} \frac{\partial}{\partial \dot{\chi}} L_{\text{tot}} + \dot{A} \frac{\partial}{\partial A} L_{\text{tot}} - L_{\text{tot}} = H_\chi + \chi \dot{A} + \frac{m_2}{2} \dot{A}^2$$

$$= \frac{m_1}{2} \dot{\chi}^2 + \frac{m_2}{2} \dot{A}^2$$

(A10)

the interaction term apparently cancels entirely, when $H_{\text{tot}}$ is expressed in terms of the coordinate velocities, although of course the conjugate momentum $p_A = m_2 \dot{A} + \chi = \text{const.}$, and the eqs. of motion consisting of (A9) and

$$m_2 \ddot{A} = -\dot{\chi}$$

(A11)

still reflect the presence of the interaction. Since

$$\frac{d}{dt} \left( \frac{m_1}{2} \dot{\chi}^2 \right) = \dot{\chi} \ddot{\chi} = -\frac{d}{dt} \left( \frac{m_2}{2} \dot{A}^2 \right)$$

(A12)

the total Hamiltonian is conserved $\dot{H}_{\text{tot}} = 0$ upon using both eqs. of motion. The $\dot{A} \ddot{\chi} = EJ$ partial conservation of the $m_1 \dot{\chi}^2/2$ apparently ‘free’ kinetic term of $\chi$ is analogous to the $F^{\lambda\nu} J_\nu$ term in the
partial conservation (2.28) of the covariant $T^{\lambda\nu}$, for the apparently ‘free’ matter, which is cancelled only if the Maxwell Eqs. of the full interacting theory are considered. Thus $m_1 \chi^2/2$, corresponding to $\int d^d x T^{00}$ of the covariant tensor (2.23), contains the full interaction energy, without the $\chi \dot{A}$ term, whereas the partial canonical Hamiltonian (A8), corresponding to $\int d^d x \mathcal{H}$ of (2.18) which differs from it and does contain the $\chi \dot{A}$ term is not the true energy of even the $\chi$ subsystem, although it does give the correct eq. of motion (A9) in an external potential $A(t)$.

This curious situation is clearly tied to the special nature of the interaction linear in velocity $\dot{A}$ which models the anomaly term $\mathcal{A}_D$ in the fluid effective action (2.1). If one takes $m_1 = \mu_5/n_5 = \pi$ and $m_2 = 1/e^2$ corresponding to the $D = 2$ Schwinger model, constrained to its spatially independent mode, one obtains from (A9) and (A11) oscillatory solutions at the frequency $e/\sqrt{\pi}$, corresponding to the mass of the Schwinger model boson, with the $p_A$ constant of motion proportional to the integration constant $E_0$ corresponding to constant background electric field in $E = e^2 \chi + E_0$, corresponding in turn to the $\theta$ parameter $\theta = 2\pi E_0/e^2$ of the model.

Appendix B: Massless Fermions in $D = 2$ at Finite Chiral Charge Density

The two-component Dirac field $\Psi$ for 1 + 1 dimensional massless QED can be written

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

(B1)

in terms of its chirality $\pm$ right or left moving components which satisfy

$$(i\partial_t + A_t)\psi_\pm = \mp(i\partial_x + A_x)\psi_\pm$$

(B2)

and do not mix in the absence of fermion mass. Treating first the case of vanishing gauge potential $A_\lambda = 0$, the chirality components of the Dirac field operator may be expanded in Fourier modes

$$\psi_\pm(t,x) = \frac{1}{\sqrt{L}} \sum_{q \geq \frac{1}{2}} \left( b_q^{(\pm)} e^{-ik_q (t+x)} + d_q^{(\pm)\dagger} e^{ik_q (t+x)} \right)$$

(B3)

on the finite spatial interval $x \in [0, L]$, where

$$k_q = \frac{2\pi q}{L}, \quad q = \frac{1}{2}, \frac{3}{2}, \ldots$$

(B4)

$q$ taking on positive half-integer values for anti-periodic boundary conditions. The fermion Fock space operators obey the anti-commutation relations

$$\left\{ b_q^{(\pm)}, b_{q'}^{(\pm)\dagger} \right\} = \delta_{qq'}, \quad \left\{ d_q^{(\pm)}, d_{q'}^{(\pm)\dagger} \right\} = \delta_{qq'}$$

(B5)
with all other anti-commutators vanishing. The free fermion Dirac vacuum is defined by

\[ b_q^{(\pm)}|0\rangle = a_q^{(\pm)}|0\rangle = 0, \quad \forall q \]  

while the fermion chiral densities can be expressed in the form

\[ :\psi_\pm^\dagger \psi_\pm: = \frac{1}{L} \sum_{n \in \mathbb{Z}} \rho_n^{(\pm)} e^{-ik_n t} e^{\pm ik_n x} \]  

where the colons denote normal ordering with respect to the Dirac vacuum, and

\[ \rho_n^{(\pm)} = \sum_{q=\frac{1}{2}}^{n-\frac{1}{2}} d_{n-q}^{(\pm)} b_q^{(\pm)} + \sum_{q=n+\frac{1}{2}}^{\infty} \left( b_q^{(\pm)\dagger} b_q^{(\pm)} - d_q^{(\pm)\dagger} d_q^{(\pm)} \right), \quad \rho_{-n}^{(\pm)} = \rho_n^{(\pm)\dagger} \]  

with \( n \) taking on integer values. A short calculation shows that the bosonic operators

\[ a_n^{(\pm)} \equiv -\frac{i}{\sqrt{|n|}} \rho_n^{(\pm)}, \quad n \neq 0 \]  

obey the canonical commutation relations

\[ [a_n^{(\pm)}, a_{n'}^{(\pm)\dagger}] = \delta_{n,n'}, \]  

with the other commutators vanishing. The \( n = 0 \) modes must be treated separately.

The chiral boson fields \((\chi, \eta)\) in \( D=2\) are given in terms of the bosonic operators through

\[ \chi = -\left( \phi_+ - \phi_- \right), \quad \eta = -\pi \chi = \pi \left( \phi_+ - \phi_- \right) \]  

where

\[ \phi_\pm (t,x) \equiv \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( a_n^{(\pm)} e^{-ik_n(t\mp x)} + a_n^{(\pm)\dagger} e^{ik_n(t\mp x)} \right) \]  

and \( \phi_\pm^0 \) is the contribution of the \( n = 0 \) modes. These are linear in \( t \) and \( x \) and given by

\[ \phi_\pm^0 = \frac{1}{2\pi} R_\pm + \frac{1}{L} (t \mp x) Q_\pm, \quad Q_\pm = \int_0^L dx :\psi_\pm^\dagger \psi_\pm: \]  

obeying the commutation relation

\[ [R_\pm, Q_\pm] = i \]  

with the remaining commutators \([R_\pm, Q_\mp] = 0\) vanishing.

These relations allow us to identify the \( n = 0 \) mode coordinate of the \( \eta \) field

\[ \hat{\eta}_0 \equiv \frac{1}{2} \left( R_+ - R_- \right), \quad [\hat{\eta}_0, Q_\mp] = i \]  

(B15)
with the phase coordinate that should acquire a non-zero expectation value $\eta_0$ if chiral symmetry is spontaneously broken in $D = 2$ according to the general discussion of Sec. IV. Its conjugate 'momentum' is $Q_5$, so that if $\hat{\eta}_0$ is sharply defined, $Q_5$ is maximally uncertain, and visa versa.

The operators of $\pm$ chiral charge for $\mathcal{Q}_2 = 0$ are

$$Q_\pm = \int_0^L dx \, \psi_\pm^\dagger \psi_\pm = \rho_0^{(\pm)} = \sum_{q \geq \frac{1}{2}} \left( \begin{pmatrix} b^{(\pm)}_q \psi_\pm^\dagger d^{(\pm)}_q \right)$$

(B16)

defined by normal ordering with respect to the Dirac vacuum state. The electric charge and axial charge operators given then by

$$Q = \int_0^L dx \, j^0 = Q_+ + Q_- = \int_0^L dx \, \frac{\partial}{\partial x} \chi = (\phi^0_+ - \phi^0_-) \bigg|_{x = L} \bigg|_{x = 0}$$

(B17a)

$$Q_5 = \int_0^L dx \, j^5_0 = Q_+ - Q_- = -\int_0^L dx \, \frac{\partial}{\partial t} \chi = \int_0^L dx \, (\dot{\phi}^0_+ - \dot{\phi}^0_-)$$

(B17b)

respectively, where (3.10), (B11) and (B12) have been used.

At zero temperature it is straightforward to construct a fermion state with finite chiral charge density by filling single particle positive chirality states and single anti-particle negative chirality states up to a sharp Fermi momentum $k \leq k_F = \mu_5$. Thus we define the filled Fermi level states

$$|N\rangle \equiv \begin{cases} 
\prod_{\frac{1}{2} \leq q \leq q_N} b^{(+)}_q d^{(-)}_q \big| 0 \rangle & \text{for } N = 1, 2, \ldots \\
|0\rangle & \text{for } N = 0 \\
\prod_{\frac{1}{2} \leq q < q_{N+1}} b^{(-)}_q d^{(+)}_q \big| 0 \rangle & \text{for } N = -1, -2, \ldots 
\end{cases}$$

(B18)

for every integer $N \in \mathbb{Z}$ and all real $\mu_5$. These states are eigenstates of chiral charge:

$$Q_\pm |N\rangle = \pm N |N\rangle, \quad Q |N\rangle = 0, \quad Q_5 |N\rangle = 2N |N\rangle$$

(B19)

since there are equal integer numbers of occupied single particle and anti-particle states in $|N\rangle$, and particles and anti-particles carry opposite electric charges.

The unitary operator

$$U \equiv e^{2i\tilde{\eta}_0} = \exp \left\{ i \left( R_+ - R_- \right) \right\}$$

(B20)

satisfies

$$[Q_5, U] = i \left[ Q_5, (R_+ - R_-) \right] U = 2U, \quad [Q, U] = 0$$

(B21)

so that $U$ raises the chiral charge $Q_5$ by 2 units, and its phase can be chosen so that

$$U |N\rangle = |N + 1\rangle$$

(B22)
without changing the value of $Q$ vanishing electric charge sector. From (B19) it follows that
\[ \exp(i\pi Q_5) \] is the unit operator on any $|N\rangle$ state, and hence from (4.3) that $2\eta$ is a $2\pi$-periodic
phase field [69].

From the definition of $q_N$ in (B18) we have
\[ \frac{\mu_5 L}{2\pi} + \frac{1}{2} = N + \text{fr}\left\{ \frac{\mu_5 L}{2\pi} \right\} \] (B23)
where fr\{\ldots\} $\in [0,1)$ denotes the fractional part of the quantity within the brackets. Dividing $Q_5$
in (B19) by the linear volume $L$ and passing to the infinite $L$ limit
\[ n_5 = \lim_{L \to \infty} \frac{1}{L} \langle N | Q_5 | N \rangle = \lim_{L \to \infty} \frac{1}{L} \left( \frac{\mu_5 L}{\pi} + 1 - 2 \text{fr}\left\{ \frac{\mu_5 L}{2\pi} \right\} \right) = \frac{\mu_5}{\pi} \] (B24)
is the chiral charge density in the continuum limit, since the fractional part drops out in this limit.

At exactly zero electric coupling $e = 0$, $Q_5$ is conserved and any of its eigenstates $|N\rangle$ are also
eigenstates of the free Dirac fermion Hamiltonian, which can be normal ordered as
\[ H_f = \int_0^L dx : \psi^\dagger (-i\sigma_3 \frac{\partial}{\partial x}) \psi : = \sum_{s=\pm} \sum_{q \geq \frac{1}{2}} k_q \left( b_{q}^{(s)} \dagger b_{q}^{(s)} + d_{q}^{(s)} \dagger d_{q}^{(s)} \right). \] (B25)
Thus the state (B18) is an eigenstate of $H_f$ with eigenvalue $E_N$ given by
\[ H_f | N \rangle = E_N | N \rangle = \left( 2 \sum_{q=\frac{1}{2}}^{q_N} k_q \right) | N \rangle = \frac{2\pi}{L} N^2 | N \rangle = \frac{\pi}{2L} Q_5^2 | N \rangle \] (B26)
and the energy density of this state in the infinite volume limit is
\[ \varepsilon = \lim_{L \to \infty} \frac{2}{L} \sum_{q=\frac{1}{2}}^{q_N} k_q = 2 \int_0^{\mu_5} \frac{dk}{2\pi} k = \frac{\mu_5}{2\pi} = \frac{\pi}{2} n_5^2. \] (B27)
If one evaluates the chiral symmetry breaking condensate $\langle N' | \bar{\psi} \psi | N \rangle$, then noting that its only
non-zero matrix elements are for $N' = N \pm 1$, these are given by
\[ \langle N | \bar{\psi} \psi | N + 1 \rangle = \frac{1}{L} \exp \left\{ -2\pi i \left( 2N + 1 \right) \frac{t}{L} \right\} = \left[ \langle N + 1 | \bar{\psi} \psi | N \rangle \right]^* \] (B28)
independent of $x$. Thus the condensate has vanishing expectation value in any $|N\rangle$ state.

The Grand Canonical Potential function for the fermions
\[ \Omega_f = H_f - \mu_5 Q_5 \] (B29)
may also be defined at finite $L$ and has eigenvalue
\[ \Omega_N = \frac{2\pi N}{L} \left( N - \frac{\mu_5 L}{\pi} \right) \] (B30)
in the state $|N\rangle$. This can be compared with pressure calculated directly from the definition

$$p = \langle T_{xx} \rangle$$  \hspace{1cm} (B31)$$

of the fermion energy-momentum-stress tensor which is given by

$$T_{\lambda\nu} = -\frac{i}{4} \left[ \bar{\psi}, \gamma(\lambda \hat{\partial}_\nu) \psi \right]$$  \hspace{1cm} (B32)$$

where the commutator anti-symmetrizes the fermion operators. Taking the expectation value of $T_{tt}$ gives $\varepsilon$ in (B27). For $T_{xx}$ we obtain

$$p = 2 \int \frac{dk}{2\pi} k = \frac{\mu_5^2}{2\pi} = \varepsilon$$

consistent with the tracelessness of the stress tensor $-T_{tt} + T_{xx} = 0$ for a massless fermion, and the negative of the eigenvalue of the Grand Canonical Potential $\Omega$. From (B24), (B27) and (B33) we verify the Gibbs relation

$$\varepsilon + p = \mu_5 n_5$$  \hspace{1cm} (B34)$$

for the two-dimensional massless fermion fluid in the absence of any interactions.

We can also use the energy-momentum tensor to prove that the state $|N\rangle$ with any $\mu_5$ is Lorentz invariant. The generator of Lorentz boosts is

$$\int_0^L dx : T^{01} := \sum_{q \geq \frac{1}{2}} k_q \left\{ b_q^{(-)\dagger} b_q^{(-)} + d_q^{(-)\dagger} d_q^{(-)} - b_q^{(+)} b_q^{(-)} - d_q^{(+)} d_q^{(-)} \right\}.$$  \hspace{1cm} (B35)$$

The contribution of right-handed particles enter with a sign opposite to the contribution of the left-handed particles, but the state $|N\rangle$ involves the same number of right (left) particles and left (right) anti-particles resulting in a cancellation of the momentum transfer

$$\int_0^L dx : T^{01} : |N\rangle = 0$$  \hspace{1cm} (B36)$$

proving the Lorentz invariance of the arbitrary $|N\rangle$ state, consistent with the velocity $v_s = 1$ of the CDW of the $\eta$ phase field composed of massless fermions in $D=2$.

**Appendix C: Coupling to the Gauge Field and Fate of the Goldstone Mode in $D=2$**

The $|N\rangle$ states of B18 are defined in the strictly free theory ($e \equiv 0$) in the absence of any interaction with the $U(1)$ gauge potential. Since the energy (B26) is proportional to $N^2$, hence $Q_5^2$, the ground state of the free fermion system is the $N=0$ state with $Q_5 = 0$ and exact chiral symmetry
preserved. When the non-zero chiral chemical potential is turned on, $Q_5$ is non-zero according to (B24) and chiral symmetry is broken, but in either case a sharp value for the momentum operator $Q_5$ means that the conjugate position operator $\eta_0$ of (B15) is completely uncertain and the conditions for the Nambu-Goldstone mode given in Sec. IV do not clearly apply since $\langle e^{2i\eta_0} \rangle$ is ill-defined. Indeed, it is well-known that Goldstone’s theorem breaks down in $D=2$ [32, 33]. How then should the gapless anomaly pole of Sec. III be understood?

Clarifying the situation in $D=2$ requires turning on the gauge field coupling $e \neq 0$ and considering the Schwinger model in a finite spatial volume $L$. When $e \neq 0$ the bare fermion $N=0$ vacuum state fails to satisfy the cluster decomposition property, signaling the appearance of off-diagonal long range order and spontaneous chiral symmetry breaking [69, 70]. Of course in the full Schwinger model the would-be Goldstone mode is ‘eaten’ by the gauge field by the Stuekelberg-Higgs mechanism and the $\chi$ boson becomes massive, with mass gap $M = e/\sqrt{\pi}$, but here we are interested in the limit $e \to 0$.

When $e \neq 0$ it is convenient to switch to a Hamiltonian treatment of the Schwinger model in the Schrödinger picture [71–73]. In a given Lorentz frame the spatial component of the gauge potential $A_x$ can be decomposed it into a longitudinal piece, $\partial_x \Lambda$ and a ’transverse’ piece, denoted by $A$. In one spatial dimension, ’transverse’ means $\partial_x A = 0$ so that $A = A(t)$ is independent of $x$ and a single quantum mechanical variable. Defining a field $\Phi(t,x) \equiv \dot{\Lambda} - A_t$ for the time component $A_t$, the general gauge potential can be written

$$ A_t = \dot{\Lambda} - \Phi \quad \text{(C1a)} $$
$$ A_x = \partial_x \Lambda + A \quad \text{(C1b)} $$

The usefulness of this parameterization is that under the $U(1)$ gauge transformation

$$ A_a \rightarrow A_a + \partial_a \lambda \implies \Lambda \rightarrow \Lambda + \lambda \quad \text{(C2)} $$

so that $\Lambda$ parameterizes the gauge orbit, and the remaining field $\Phi$ and single degree of freedom $A$ taken to be gauge invariant. The electric field $E_x = F^{01} = \dot{\Lambda} - \partial_x \Phi$ is clearly independent of $\Lambda$ and gauge invariant.

Constructing the canonical Hamiltonian in the standard way from (3.1) gives

$$ H = \frac{e^2}{2L} \Pi_A^2 + H_f(A) + H_c \quad \text{(C3)} $$

in terms of gauge invariant variables, where

$$ \Pi_A \equiv \frac{\delta S_{cl}}{\delta \dot{A}} = \frac{LE}{e^2} \quad \text{(C4)} $$
is the momentum conjugate to $A$, given in terms of the ‘transverse,’ i.e. spatially constant electric field $E = -\dot{A}$,

$$H_f(A) = -\int_0^L dx \Psi^\dagger(x) \sigma_3 (i\partial_x + A) \Psi(x) \quad (C5)$$

is the Dirac fermion kinetic Hamiltonian, and

$$H_c = \frac{1}{2} \int_0^L dx \Phi(x) \rho(x) = \frac{e^2}{2} \int_0^L dx \int_0^L dx' \rho(x) D_L(x, x') \rho(x') \quad (C6)$$

is the Coulomb interaction. This results from solving the Gauss Law constraint

$$\partial_x E_x = -\partial^2_x \Phi = e^2 \rho = e^2 j^0 = e^2 \Psi^\dagger \Psi \quad (C7)$$

in terms of the Green’s function inverse

$$D_L(x, x') = \frac{1}{L} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{k_n^2} \exp \{ i k_n (x - x') \} = \frac{L}{2} - \frac{|x - x'|}{2} + \frac{(x - x')^2}{2L} \quad (C8)$$

of the one-dimensional Laplacian $-\partial^2_x$ defined on the periodic interval $x, x' \in [0, L]$, with $k_n = 2\pi n / L$ and $D_L$ satisfying

$$-\partial^2_x D_L(x, x') = \frac{1}{L} \sum_{n \in \mathbb{Z}, n \neq 0} \exp \{ i k_n (x - x') \} = \delta_L(x - x') - \frac{1}{L}. \quad (C9)$$

Integrating (C7) over the spatial interval implies that the total electric charge must vanish on the physical state space with periodic boundary conditions. This insures that the $n = 0$ constant term in $D_L$ which is omitted from the sum in (C8) does contribute to $H_c$, and also accounts for the difference of (C9) from a pure $\delta_L(x - x')$ function periodic on the interval.

In order to diagonalize the full $H$ (C3) in both the zero mode ($n = 0$) and non-zero mode ($n \neq 0$) subspaces, the spatial Fourier transforms of the two chirality components

$$\psi_{\pm}(x) = \frac{1}{\sqrt{L}} \sum_{q \in \mathbb{Z} + \frac{1}{2}} c^{(\pm)}_q e^{ik_q x} \quad (C10)$$

are introduced, obeying anti-periodic boundary conditions on the interval if $q = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$ taking on all (positive or negative) half-integer values, and the anti-commutation relations

$$\left\{ c^{(\pm)}_{q'}, c^{(\mp)}_q \right\} = \delta_{qq'} \quad (C11)$$

all other anti-commutators vanishing. Substituting (C10) into (C5) gives

$$H_f(A) = \sum_{q \in \mathbb{Z} + \frac{1}{2}} \left\{ (k_q - A) c^{(+)}_q c^{(+)}_{q'} - (k_q - A) c^{(-)}_q c^{(-)}_{q'} \right\} \quad (C12)$$
for the unregularized fermion Hamiltonian. Regularization of $H_f(A)$ may be performed in a number of different ways, for example the normal ordering prescription for the Dirac vacuum

$$
c_q^{(+)}|0\rangle = c_q^{(-)\dagger}|0\rangle = 0, \quad k_q - A \geq 0 \quad (C13a)
$$

$$
c_q^{(-)}|0\rangle = c_q^{(+\dagger)}|0\rangle = 0, \quad k_q - A < 0 \quad (C13b)
$$

re-ordering and discarding the contribution of the Dirac sea. This gives

$$
:H_f(A): = \frac{2\pi}{L} \left\{ \sum_{q \geq N_{CS}} (q - N_{CS}) c_q^{(+\dagger)} c_q^{(+)} + \sum_{q < N_{CS}} (N_{CS} - q) c_q^{(+)} c_q^{(+\dagger)} \right. \\
+ \left. \sum_{q \leq N_{CS}} (N_{CS} - q) c_q^{(-\dagger)} c_q^{(-)} + \sum_{q > N_{CS}} (q - N_{CS}) c_q^{(-)} c_q^{(-\dagger)} \right\} \quad (C14)
$$

where the colons denote that normal ordering has been performed, and

$$
N_{CS} = \frac{1}{2\pi} \oint_{S^1} A_x dx = \frac{1}{2\pi} \int_0^L A_x dx = \frac{AL}{2\pi} \equiv [N_{CS}] + \text{fr}\{N_{CS}\} \quad (C15)
$$

is the gauge invariant Chern-Simons number. Here $[N_{CS}] \in \mathbb{Z}$ is an integer (the ‘floor’ of $N_{CS}$) and $\text{fr}\{N_{CS}\}$ is its fractional part. $N_{CS}$ takes on integer values for periodic windings of the $U(1)$ gauge potential $A_x$ around the spatial configuration space $[0, L]$, which has the topology of a circle $S^1$ for periodic boundary conditions. These integers describe a periodic vacuum structure, with $N_{CS} \rightarrow N_{CS} + N$ being topologically non-trivial ‘large’ gauge transformations $\Lambda(L) - \Lambda(0) = 2\pi N$ for which $e^{i\Lambda(L)} = e^{i\Lambda(0)}$ leads to a physically equivalent gauge potential. Since $A_x = A$ is an arbitrary spatial constant, so that $N_{CS}$ takes on arbitrary continuous (non-vacuum) values $N_{CS} \in (-\infty, \infty)$. Owing to the periodic vacuum structure, its range can be restricted to its fractional part $\text{fr}(N_{CS}) \in [0, 1)$ provided we sum over all equivalent vacuum sectors labelled by $N \in \mathbb{Z}$.

According to the definition of the fermion Fock vacuum state (C13), we generalize the definition of the $Q = 0$ state with unequal chiral Fermi surfaces of (B18) to arbitrary finite $A$ by [71]

$$
|N; A\rangle \equiv \begin{cases} \\
\prod_{N_{CS} - 1/2 < q \leq q_N} c_q^{(+\dagger)} c_q^{(-)}|0\rangle & \text{for } N = [N_{CS}] + 1, [N_{CS}] + 2, \ldots \\\n|0\rangle & \text{for } N = [N_{CS}] \\\n\prod_{q_{N+1} \leq q < N_{CS} + 1/2} c_q^{(+)} c_q^{(-\dagger)}|0\rangle & \text{for } N = [N_{CS}] - 1, [N_{CS}] - 2, \ldots
\end{cases} \quad (C16)
$$

with $q_N$ defined as before in (B18). For $A = 0, N_{CS} = 0$, this reduces to (B18) with the identification of $c_q^{(+)} = b_q^{(+)}$, $c_q^{(-)} = d_q^{(-\dagger)}$ for $q \geq 1/2$ positive, and $c_q^{(+)} = b_{-q}^{(+\dagger)}$, $c_q^{(-)} = d_{-q}^{(-)}$ for $q \leq -1/2$ negative.
Since the energies of the single particle occupied states are shifted: \( k_q \rightarrow k_q - A \), the normal ordering also shifts the \( Q_\pm \) operators so that only those single particle states with positive energy should be counted in the sums after the shift. Since

\[
\sum_{q \geq \frac{1}{2} + N_{CS}} q \geq \frac{1}{2} + N_{CS}
\]

the eigenvalues of \( Q_\pm \) in the state defined by (B18) but with energy levels shifted by \( A \) according to (C17) are given by

\[
Q_\pm |N; A\rangle = \sum_{q \geq \frac{1}{2} + N_{CS}} q \geq \frac{1}{2} + N_{CS} |N; A\rangle = \pm (N - N_{CS}) |N; A\rangle
\]

and

\[
Q_5 |N; A\rangle = (Q_+ - Q_-) |N; A\rangle = 2 (N - N_{CS}) |N; A\rangle.
\]

Likewise the state \( |N; A\rangle \) is also an eigenstate of the Dirac Hamiltonian with eigenvalue

\[
E_{N; A} = 2 \sum_{q \geq \frac{1}{2} + N_{CS}} (k_q - A) = 4\pi \frac{1}{L} \left( \sum_{n=1}^{N} - \sum_{n=1}^{N_{CS}} \right) \left( n - \frac{1}{2} - N_{CS} \right) = \frac{2\pi}{L} \left( N(N + 1) - N_{CS}(N_{CS} + 1) - (2N_{CS} + 1)(N - N_{CS}) \right) = \frac{2\pi}{L} (N - N_{CS})^2
\]

so that

\[
:H_f(A): |N; A\rangle = E_{N; A} |N; A\rangle = \frac{\pi}{2L} Q_5^2 |N; A\rangle
\]

and the last form of (B26) remains valid for the shifted \( Q_5 \). Note that (C20) implies

\[
\frac{dQ_5}{dt} = -2 \frac{dN_{CS}}{dt} = -\frac{\dot{A}}{\pi} L = \int_0^L dx \phi_2
\]

as required by the axial anomaly (3.6).

The fermion Hamiltonian (C5) can be decomposed into its spatially constant zero mode part \( H_0 \), and its non-zero mode part which is composed of the bosonized field operators \( \bar{\phi}_+ - \bar{\phi}_- \) of (B8), (B9) and (B12), normal ordered in \( a_n^+, a_n^- \). These describe fermion/anti-fermion pair excitations of the \( |N; A\rangle \) base states with higher energy, but which are otherwise independent of \( A \) [71, 73]. The boson Hamiltonian together with the Coulomb interaction (C6) can be diagonalized by a unitary canonical transformation \( e^{iS} \) whose net effect is to endow the boson field with a mass \( M = e/\sqrt{\pi} \),
and normal order its Hamiltonian with respect to this mass [72, 73]. This $n \neq 0$ bosonic part of $H$ commutes with the zero mode part $H_0$. Thus to find the ground state of the full Hamiltonian, one can focus only upon the zero mode subspace spanned by the $|N; A\rangle$ states, with Hamiltonian

$$H_0 = \frac{e^2}{2L} \Pi_A^2 + \frac{\pi}{2L} Q_5^2$$

from (C3) and (C22). Denoting the coordinate $q_5$ corresponding to the operator $Q_5$, we observe that $\pi dq_5 = -L dA$, so that evidently $H_0$ is a simple harmonic oscillator Hamiltonian on this one-dimensional space of states, where the momentum $\Pi_A$ can be represented as

$$\Pi_A = -i \frac{\partial}{\partial A} + \frac{L \theta}{2\pi} = \frac{L}{\pi} \left( i \frac{\partial}{\partial q_5} + \frac{\theta}{2} \right)$$

in the Schrödinger representation, where the real constant $\theta$ allows for a constant background electric field $E_0 = e^2 \theta / 2\pi$ [67]. Hence the ground state wave functional of the Schwinger model normalized by $\int_{-\infty}^{\infty} dq_5$ is the Gaussian weighted sum [72, 73]

$$|\theta\rangle = (ML)^{-\frac{1}{4}} \sum_{N=-\infty}^{\infty} \exp \left\{ -\frac{2\pi}{ML} \left( N - \text{fr}\{N_{CS}\} \right)^2 + i\theta \left( N - \text{fr}\{N_{CS}\} \right) \right\} e^{-iS} |N; A\rangle$$

$$= (ML)^{-\frac{1}{4}} \exp \left\{ -\frac{\pi q_5^2}{2ML} + i\theta q_5 \right\} e^{-iS} |q_5\rangle$$

with fr$\{N_{CS}\} \in [0, 1)$ restricted to its fractional part in the first expression, and the operator $e^{-iS}$ needed to diagonalize the Coulomb interaction term $H_c$, given explicitly in Refs. [73–75]. The zero-point energy (and negative pressure) of this $|\theta\rangle$ vacuum state is $M/2$.

The point pertinent to our discussion of Sec. IV is that this ground state (C26) parameterized by $\theta \in [0, 2\pi]$ breaks chiral symmetry. A chiral rotation of $\alpha$ acting upon it shifts $\theta$ according to

$$e^{i\alpha Q_5} |\theta\rangle = |\theta + 2\alpha\rangle$$

by (C20), and $\alpha = \pi$ the identity operator, well-defined for any finite $eL$. The expectation value of the chiral symmetry breaking condensate operator $\psi_-^\dagger \psi_+$ in the Gaussian $|\theta\rangle$ state is correspondingly well-defined for $eL$ finite, and takes on the non-zero value

$$\langle \theta | \psi_-^\dagger \psi_+ | \theta \rangle = e^{i\theta} \exp \left( -\frac{\pi}{ML} \right) B(M, L)$$

in our phase convention, where $B(M, L)$ resulting from the normal ordering transformation $e^{-iS}$ from zero to finite boson mass $M$ is given in Refs. [72–75]. $B(M, L)$ goes to the finite value $1/L$ as $ML \to 0$, so that the condensate (C28) vanishes exponentially in this limit.

It will now be evident also that dependence of the Gaussian width and overlap between successive $|N; A\rangle$ states upon the product $eL$ means that the limits $e \to 0$ ($L$ fixed) and $L \to \infty$ ($e$ fixed) do
not commute. In the first limit, the Gaussian becomes infinitely sharply peaked around \( q_5 = 0 \), and the \( \theta \) dependence drops out entirely. The state \( |\theta\rangle \) becomes non-normalizable and the spontaneous symmetry breaking condensate (C28) vanishes, as in the free fermion theory of Appendix B in this singular limit. If on the other hand we take the limit \( e \to 0, L \to \infty \) with \( eL \) fixed, the \( |\theta\rangle \) state is well-defined, (C27) remains valid, the condensate (C28) is non-vanishing, and chiral symmetry remains broken. An approximately massless Goldstone pole is also recovered over the very large range of momentum scales

\[
k \gg M \sim \frac{1}{L}
\]

or distance scales

\[
|x| \ll e^{-1} \sim L
\]

for arbitrarily large lineal spatial volume as \( e \to 0 \) and \( L \to \infty \) together. The variable \( \hat{\eta}_0 \) of (B15) conjugate to \( Q_5 \) is peaked around \( \eta_0 = \theta/2 \) consistent with (4.4) and (4.7) but with a finite root mean square width of order \( (eL)^{-\frac{1}{2}} \). This can be made arbitrarily small (but finite) if \( eL \gg 1 \), with the range (C30) for which Goldstone’s theorem applies then restricted to \( |x| \ll e^{-1} \ll L \).

This discussion is easily generalized to a state with finite chiral density at zero temperature by considering the Grand Canonical Potential in this zero mode sector, which is

\[
\Omega = H_0 - \mu_5 Q_5 = \frac{\pi^2}{2e^2 L} \left\{ e^{\frac{4}{\pi^4}} \left( -i \frac{\partial}{\partial q_5} - \frac{\theta}{2} \right)^2 + \frac{e^2}{\pi} \left( q_5 - \frac{\mu_5 L}{\pi} \right)^2 \right\} - \frac{\mu_5^2 L}{2\pi}
\]

in the zero mode \( q_5 \) coordinate representation. Hence with the shift \( q_5 \to q_5 - \frac{\mu_5 L}{\pi} \) we obtain

\[
|\theta; \mu_5\rangle = (ML)^{-\frac{1}{4}} \sum_{N = -\infty}^{\infty} \exp \left\{ -\frac{2\pi}{ML} \left( N - \text{fr\{N}_{CS} \right) - \frac{\mu_5 L}{2\pi} \right)^2 + i\theta \left( N - \text{fr\{N}_{CS} \right) - \frac{\mu_5 L}{2\pi} \right) \right\} e^{-iS}|N; A\rangle
\]

\[
= (ML)^{-\frac{1}{4}} \exp \left\{ -\frac{\pi}{2ML} \left( q_5 - \frac{\mu_5 L}{\pi} \right)^2 + \frac{i\theta}{2} \left( q_5 - \frac{\mu_5 L}{\pi} \right) \right\} e^{-iS}|q_5\rangle
\]

from (C26). The chiral symmetry is broken and the conditions for the Nambu-Goldstone theorem are again satisfied in the same sense as before for the coordinate ranges (C29)-(C30).

In the Schrödinger picture \( |\theta; \mu_5\rangle \) will be recognized as a (non-stationary) coherent Gaussian state of the harmonic oscillator displaced from its minimum, with its evolution described by

\[
e^{-iH_0 t} |\theta; \mu_5\rangle = (ML)^{-\frac{1}{4}} \exp \left\{ -\frac{\pi}{2ML} \left( q_5 - \bar{q}_5 \right)^2 + \frac{i\theta}{2} \left( q_5 - \frac{\mu_5 L}{\pi} \right) + i\bar{p}_5 \left( q_5 - \frac{\mu_5 L}{2} \right) - \frac{i}{2} Mt \right\} e^{-iS}|q_5\rangle
\]

where \( \bar{q}_5(t), \bar{p}_5(t) \) are the time dependent centroid and corresponding momentum of the coherent state Gaussian. These satisfy the classical eqs. of the oscillator [76], and in the present case with
The initial conditions set by (C32) are simply
\[
\bar{q}_5(t) = \frac{\mu_5 L}{\pi} \cos(Mt) \quad \text{(C34a)}
\]
\[
\bar{p}_5(t) = \frac{\pi}{M^2 L} \frac{d\bar{q}_5}{dt} = -\frac{\mu_5}{M} \sin(Mt) \quad \text{(C34b)}
\]

Since the chiral symmetry breaking condensate operator \( \bar{\psi}_{-}\psi_{+} \) has non-zero matrix elements between neighboring winding number sectors \( N \) and \( N+1 \), according to (B28), and its expectation value in the time independent Gaussian stationary state \( |\theta\rangle \) is given by (C28), we obtain
\[
\langle \theta; \mu_5 | e^{iH_0 t} \bar{\psi}_{-}\psi_{+} e^{-iH_0 t} | \theta; \mu_5 \rangle = \exp\left(2i \bar{p}_5(t)\right) e^{i\theta} e^{-\pi/ML} B(M, L) \quad \text{(C35)}
\]
in the time dependent state of non-zero chemical potential \( \mu_5 \). The time dependent phase
\[
\exp\left(2i \bar{p}_5(t)\right) = \exp\left\{-\frac{2i\mu_5}{M} \sin(Mt)\right\} \approx e^{-2i\mu_5 t} \quad \text{(C36)}
\]
for \( Mt \ll 1 \). This shows that the chiral phase \( 2\eta = -2\mu_5 t \) as expected for a finite condensate transforming as a \( 2\pi \)-periodic phase field corresponding to the complex scalar in the abelian Higgs model (4.19) with \( g = \pm 2 \), but only for times \( t \ll M^{-1} \), consistent with the restriction on the spatial range (C30) where the phase is well-defined. This range of time and spatial scales where the Goldstone theorem of Sec. IV applies is arbitrarily large for \( e \to 0 \), provided \( L \gg 1/e \).

Finally is easily checked that
\[
\int_0^L dx :T^{01}: |\theta; \mu_5 \rangle = 0 \quad \text{(C37)}
\]
so that the proof of Lorentz invariance of the finite chiral density states shown in Appendix B holds for the superposition \( |\theta; \mu_5 \rangle \) states as well, so that the gapless Goldstone mode propagates at the speed of ‘light’ \( v_s = c = 1 \) for all \( \mu_5 \) in \( D=2 \) dimensions, consistent with the discussion in Sec. IV, with zero fermion mass.