WEIGHTS IN A BENSON-SOLOMON BLOCK

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Abstract. To each pair consisting of a saturated fusion system over a $p$-group together with a compatible family of Külshammer-Puig cohomology classes, one can count weights in a hypothetical block algebra arising from these data. When the pair arises from a bonafide block of a finite group algebra in characteristic $p$, the number of conjugacy classes of weights is supposed to be the number of simple modules in the block. We show that there is unique such pair associated with each Benson-Solomon exotic fusion system, and that the number of weights in a hypothetical Benson-Solomon block is 12, independently of the field of definition. This is carried out in part by listing explicitly up to conjugacy all centric radical subgroups and their outer automorphism groups in these systems.

1. Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$, and let $G$ be a finite group. Associated to each block $b$ of $kG$, there is a saturated fusion system $\mathcal{F} = \mathcal{F}_S(b)$ over the defect group $S$ of the block in which the morphisms between subgroups are given by conjugation by elements of $G$ preserving the corresponding Brauer pairs \cite{AKO11, Cra11}. Several questions in the modular representation theory of finite groups concern the connection between representation theoretic properties of $kGb$ and the category $\mathcal{F}$. However, it is known that for many purposes $\mathcal{F}$ does not, in general, retain enough information about $kGb$-mod. For example, it does not determine the number of simple modules in $b$, in part because it retains too little of the $p'$-structure of $p$-local subgroups.

On the other hand, the block $b$ also determines a family of degree 2 cohomology classes $\alpha \in H^2(\text{Aut}_\mathcal{F}(Q), k^\times)$, for $Q \in \mathcal{F}^c$ an $\mathcal{F}$-centric subgroup, by work of Külshammer and Puig (see \cite[IV.5.5]{AKO11}.) This family is expected to supply the missing information away from the prime $p$. The Külshammer-Puig classes are compatible in the sense that, by \cite[Theorem 8.14.5]{Lin19}, they determine an element

$$\alpha \in \varprojlim_{[S(\mathcal{F}^c)]} \mathcal{A}_2^\mathcal{F}$$

where $[S(\mathcal{F}^c)]$ is the poset of $\mathcal{F}$-isomorphism classes of chains $\sigma = (R_0 < R_1 \cdots < R_n)$ of $\mathcal{F}$-centric subgroups, and $\mathcal{A}_2^\mathcal{F}$ is the covariant functor which sends a chain $\sigma$ to $H^2(\text{Aut}_\mathcal{F}(\sigma), k^\times)$. Here, $\text{Aut}_\mathcal{F}(\sigma) \leq \text{Aut}_\mathcal{F}(R_n)$ is the group of automorphisms in $\mathcal{F}$ of $R_n$ preserving all members $R_i$ of the chain. For example, if $b$ is the principal block of $kG$ then $\alpha$ is always the trivial class \cite[IV.5.32]{AKO11}.

Thus, by a Külshammer-Puig pair, we mean a pair $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a saturated fusion system on a $p$-group $S$ and $\alpha$ is an element of $\varprojlim_{[S(\mathcal{F}^c)]} \mathcal{A}_2^\mathcal{F}$. Given such a pair $(\mathcal{F}, \alpha)$ arising from a block

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Let $b$, the quantity
\begin{equation}
 w(\mathcal{F}, \alpha) := \sum_{Q \in \mathcal{F}^c / \mathcal{F}} z(k_{\alpha Q} \text{Out}_\mathcal{F}(Q)),
\end{equation}
counts the number of $kG_b$-weights. Here, $k_{\alpha Q} \text{Out}_\mathcal{F}(Q)$ is the algebra obtained from the group algebra $k \text{Out}_\mathcal{F}(Q)$ by twisting with $\alpha_Q$ [AKO11, IV.5.36], $z(-)$ denotes the number projective simple modules, and the sum is taken over a set of representatives for the conjugacy classes of $\mathcal{F}$-centric radical subgroups. Thus, Alperin’s Weight Conjecture says that $w(\mathcal{F}, \alpha)$ is the number of simple $kG$-modules which lie in $b$ [AKO11, IV.5.46].

There is always a natural map $H^2(\mathcal{F}^c, k^\times) \to \lim_{\mathcal{F}^c} [S(\mathcal{F}^c)] A^2_\mathcal{F}$, and the gluing problem asks whether this map is surjective (see [Lin09] and [Lib11] for further details). Linckelmann has shown that Alperin’s conjecture has a structural reformulation in terms of algebras constructed from $p$-local finite groups provided the gluing problem always has a solution. However, while the weight conjecture has relevance for actual blocks only, the gluing problem is a question about the Külshammer-Puig pair itself and can be considered (1) when $\mathcal{F}$ is the fusion system of a block, but of no block with the compatible specified family $\alpha$, and (2) when $\mathcal{F}$ is the fusion system of no block at all. Thus, we are interested in investigating such pairs disembodied from an actual block as a way of gauging the degree to which certain questions, and potential answers to those questions, are $p$-locally determined. A direct study of Külshammer-Puig pairs might reveal, for example, that there is an exotic pair as in (1) or (2) that does not satisfy the gluing problem. At this stage, such a possibility seems unlikely. On the other hand, and conversely, we would be very interested in a structural explanation why the gluing problem should hold in general, and it seems reasonable to expect that such an explanation would apply to all such pairs, exotic or not.

In this paper we consider Külshammer-Puig pairs associated with the exotic family $\text{Sol}(q)$ of Benson-Solomon 2-fusion systems [AC10, LO02]. These systems are defined for any odd prime power, but $\text{Sol}(q)$ and $\text{Sol}(q')$ are isomorphic as fusion systems if and only if $v_2(q^2-1) = v_2(q'^2-1)$, where $v_2$ is the 2-adic valuation. A Benson-Solomon system is known not to be the fusion system of any bonafide block. This is a result of Kessar for the smallest such system [Kes06], while Craven extended Kessar’s proof to the general case in [Cra11, Theorem 9.34]. Our first theorem determines the possible Külshammer-Puig classes that these fusion systems support.

**Theorem 1.1.** Let $\mathcal{F} = \text{Sol}(q)$. Then
\[
\lim_{[S(\mathcal{F}^c)]} A^2_\mathcal{F} \cong \lim_{[S(\mathcal{F}^c)]} A^2_\mathcal{F} = 0.
\]
That is, each Benson-Solomon system supports a unique Külshammer-Puig pair.

Theorem 1.1 is shown by explicitly computing the $\mathcal{F}$-conjugacy classes of centric radical subgroups along with their outer automorphism groups in $\mathcal{F}$. The results of [AC10, Section 10] go a long way towards accomplishing such a task, but more details are required for the present applications. In Section 2 we refine the results of [AC10] to prove the following.

**Theorem 1.2.** Let $\mathcal{F} = \text{Sol}(q)$. Representatives for the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric radical subgroups, together with their $\mathcal{F}$-outer automorphism groups, are listed in Tables 1 and 4.

**Theorem 1.3.** The number of weights in the unique pair of Theorem 1.1 is
\[
w(\text{Sol}(q), 0) = 12,
\]
independently of $q$. 


We prove this result in Section 4 by explicitly computing the quantity \( z(k \Out_{\mathcal{F}}(Q)) \) for each of the groups \( Q \) appearing in Tables 1 and 4 of Theorem 1.2.

Beyond the weight conjecture, and assuming its validity, we have in mind other counting questions that can be considered for Kulshammer-Puig pairs without reference to a group or a block. For example, Malle and Robinson recently conjectured that if \( b \) is a \( p \)-block associated to a finite group \( G \) then the number of simple \( kG \)-modules in \( b \) should be bounded by \( p^{s(S)} \), where \( S \) is a defect group of \( b \) and \( s(S) \) denotes the sectional rank of \( S \), namely the largest rank of an elementary abelian section \([MR17]\). Moreover, they verified their conjecture in a large number of cases where the weight conjecture holds. In Lemma 2.10, we observe that the sectional rank \( S \) is 6, and so the following conjecture, which was suggested to us by Kessar and Linckelmann, also holds easily for \( \Sol(q) \).

**Conjecture 1.4.** Let \((\mathcal{F}, \alpha)\) be a Kulshammer-Puig pair, where \( \mathcal{F} \) is a saturated fusion system on \( S \). Then \( w(\mathcal{F}, \alpha) \leq p^{s(S)} \).

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## 2. The Benson-Solomon fusion systems

### 2.1. Fusion system preliminaries.

Throughout this paper, our group-theoretic nomenclature is standard and follows \([Wil09]\), and we are usually consistent with the fusion-theoretic terminology and notation of \([AKO11]\). One exception to this is that we use exponential notation for images of subgroups and elements under a morphism in a fusion system, as described below.

A fusion system on a finite \( p \)-group \( S \) is a category with objects the subgroups of \( S \), and with morphisms injective group homomorphisms subject to two weak axioms. The standard example of a fusion system is that of a finite group \( G \) with Sylow \( p \)-subgroup \( S \), where the morphisms are the conjugation homomorphisms between subgroups of \( S \) induced by elements of the group \( G \), which is denoted \( \mathcal{F}_S(G) \). Due to the validity of Sylow’s Theorem, the standard example satisfies two additional saturation axioms, which can be found in \([AKO11]\). All fusion systems in this paper are assumed to be (or known already to be) saturated unless otherwise stated, and we will often drop that adjective and speak simply of a fusion system.

For this subsection, we let \( \mathcal{F} \) be a saturated fusion system over the \( p \)-group \( S \). We will need very few basic definitions and results from the theory of fusion systems. By analogy with the standard example, two subgroups of \( S \) are said to be \( \mathcal{F} \)-conjugate if they are isomorphic in the category \( \mathcal{F} \). For a morphism \( \varphi: P \to Q \) in \( \mathcal{F} \), we write \( P^\varphi \) for the image of \( \varphi \). Similarly, \( x^\varphi \) denotes the image of an element \( x \) under a morphism whose domain contains \( x \). This is a departure from the notation used in \([AKO11]\) and elsewhere.

**Definition 2.1.** Fix a subgroup \( P \leq S \). We say that \( P \) is

1. fully \( \mathcal{F} \)-normalized if \( |N_S(P)| \geq |N_S(Q)| \) whenever \( Q \) is \( \mathcal{F} \)-conjugate to \( P \),
2. \( \mathcal{F} \)-centric if \( C_S(Q) = Z(Q) \) for each \( \mathcal{F} \)-conjugate \( Q \) of \( P \),
3. \( \mathcal{F} \)-radical if \( O_p(\Out_{\mathcal{F}}(P)) = 1 \), and
4. \( \mathcal{F} \)-centric radical if it is both \( \mathcal{F} \)-centric and \( \mathcal{F} \)-radical.

Denote by \( \mathcal{F}^c \), \( \mathcal{F}^r \), and \( \mathcal{F}^{cr} \) the collection of \( \mathcal{F} \)-centric, \( \mathcal{F} \)-radical, and \( \mathcal{F} \)-centric radical subgroups of \( S \), respectively.

The collections \( \mathcal{F}^c \), \( \mathcal{F}^r \) and \( \mathcal{F}^{cr} \) are all closed under \( \mathcal{F} \)-conjugacy. Also, the \( \mathcal{F} \)-centric subgroups are closed under passing to overgroups.
Definition 2.2. Fix a subgroup $P \leq S$.

(a) The normalizer $N_F(P)$ of $P$ is the fusion system on $N_S(P)$ consisting of those morphisms $\varphi: Q \to R$ in $F$ for which there exists an extension $\hat{\varphi}: PQ \to PR$ of $\varphi$ in $F$ such that $P^\varphi = P$.

(b) The centralizer $C_F(P)$ of $P$ is the fusion system on $C_S(P)$ consisting of those morphisms $\varphi: Q \to R$ in $F$ for which there exists an extension $\hat{\varphi}: PQ \to PR$ of $\varphi$ in $F$ such that the restriction $\hat{\varphi}|_P$ is the identity on $P$.

(c) $P$ is normal in $F$ if $F = N_F(P)$.

These centralizer and normalizer fusion systems are not always saturated, but they are both saturated provided $P$ is fully $F$-normalized. The following basic lemma shows that each subgroup $P$ of $S$ has a fully $F$-normalized $F$-conjugate, a fact which is closely related to the saturation axioms.

Lemma 2.3. Let $P \leq S$. Then there exists a morphism $\alpha \in \text{Hom}_F(N_S(P), S)$ such that $P^\alpha$ is fully $F$-normalized.

Proof. See [AKO11] I.2.6(c)].

Finally, we require the following elementary lemma.

Lemma 2.4. Suppose that $P \leq S$ is normal in $F$. Then $P$ is contained in every $F$-centric radical subgroup.

Proof. Let $Q \in F^{ct}$. Then $\text{Aut}_{F}(Q)$ is normal in $\text{Aut}_{F}(Q)$, and so $\text{Aut}_{F}(Q) \leq \text{Inn}(Q)$ since $Q$ is radical. Then $P \leq PQ \leq QC_S(Q) = Q$ with the equality because $Q$ is centric.

2.2. Construction of $\text{Sol}(q)$. The Benson-Solomon systems were predicted to exist by Benson, and then later constructed by Levi and Oliver [LO02, LO05]. They are exotic in the sense that they are not of the form $F_S(G)$ for any finite group $G$ with Sylow 2-subgroup $S$. They are also not the fusion system of any 2-block of a finite group [Kes06, Cra11], an a priori stronger statement. However, Aschbacher and Chermak gave a different construction of the Benson-Solomon systems as the fusion system of a certain free amalgamated product of two finite groups having Sylow 2-subgroup isomorphic to Spin$_7(q)$ [AC10]. We primarily view $\text{Sol}(q)$ through the lens of [AC10], so we consider it as the 2-fusion system of an amalgamated product $G = H \ast_B K$, where $H := \text{Spin}_7(q)$ is the double cover of the simple group $\Omega_7(q)$ (see [Wil09] Section 3.7]), and where $K$ and $B$ are constructed as follows.

Consider the natural inclusion $\text{SL}_2(q) \leq \text{SL}_2(q^2)$ induced by an inclusion of fields, and define $N := N_{\text{SL}_2(q^2)}(\text{SL}_2(q))$ so that $|N : \text{SL}_2(q)| = 2$ and $N$ and $\text{SL}_2(q)$ both have generalized quaternion Sylow 2-subgroups (this is explained more fully in Subsection 2.3). Form the wreath product $W := N \wr S_3$, and let $N_0 := N_1 \times N_2 \times N_3$ and $X := S_3$ be the base and acting group respectively. Note that $O^2(N_0) \leq W$ is a direct product $L_1 \times L_2 \times L_3$ of three copies of $\text{SL}_2(q)$ permuted transitively by $X$.

Define $\hat{K} := O^2(N_0)XC_{N_0}(X)$ regarded as the group generated by the wreath product $O^2(N_0) \rtimes X$, and an element of $N_0 \setminus O^2(N_0)$ acting in the same way simultaneously on each factor $L_i$ of $O^2(N_0)$. Thus, $Z(O^2(N_0)) = Z(O^2(N_0)C_{N_0}(X)) = \langle \pm 1, \pm 1, \pm 1 \rangle$ and $Z(\hat{K}) = \langle -1, -1, -1 \rangle$. Here, we write 1 for the identity matrix. Finally, set $K := \hat{K}/Z(\hat{K})$.

We will write $[x, y, z]$ for the image $K$ of an element $(x, y, z)$ of $O^2(N_0)C_{N_0}(X)$. 

Notation 2.5. We fix the following notation for certain subgroups of $K$.

(a) $L_i \cong \text{SL}_2(q)$ for $i = 1, 2, 3$ are the images in $K$ of the subgroups of $\overline{K}$ with the same names;
(b) $L_0 := L_1L_2L_3$;
(c) $X \cong S_3$ is the image in $K$ of the subgroup with the same name;
(d) $\tau \in X$ is the permutation $(1, 2)$ on the indices of the $L_i$;
(e) $S$ is a Sylow 2-subgroup of $K$ containing $\tau$;
(f) $U = Z(L_0) = \langle \{\pm 1, \pm 1, \pm 1\} \rangle \cong C_2 \times C_2$; and
(g) $B := L_0S$.

Thus, the subgroup $B$ in Notation 2.5(g) is a subgroup of $K$ of index 3, and $B \cap X = \langle \tau \rangle$. As was shown in [LO05, AC10], there is a four subgroup $U \leq H$ such that $B \cong N_H(U)$, and a choice of injection $i : B \hookrightarrow H$ such that the amalgam $G = H *_B K$ determines a saturated fusion system over $S$ which we call $\text{Sol}(q)$. The reader should be cautioned that an incorrect choice of $i$ can lead to a fusion system which is not saturated. Such a choice was the reason for the correction in [LO05]. See [AC10, Section 5] and [LO05] for more details, but generally this subtlety will be unimportant in our computations.

2.3. Generalised quaternion subgroups of $K$. At this point, it will be helpful to fix a choice of $q$. By [COS08, Theorem 3.4], $\text{Sol}(q) \cong \text{Sol}(5^2)$ for some $l \geq 0$, so there is no loss in assuming that $q := 5^2$. Thus $\text{SL}_2(q)$ has generalised quaternion Sylow 2-subgroups of order $2^{l+3}$. The size of a Sylow 2-subgroup can be deduced from the order $q(q - 1)(q + 1)$ of $\text{SL}_2(q)$, together with the polynomial identity

$$x^{2l} - 1 = (x - 1) \cdot \prod_{i=0}^{l-1} (x^{2^i} + 1).$$

By our choice of $q$, the multiplicative group $\mathbb{F}_q^\times$ contains an element $\omega$ of order $2^{l+2}$. Thus,

$$x := \left(\begin{array}{cc} \omega & 0 \\ 0 & \omega^{-1} \end{array}\right) \quad \text{and} \quad y := \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right)$$

generate a Sylow 2-subgroup of $\text{SL}_2(q)$. Since $x$ and $y$ satisfy the relations

$$x^{2^{l+2}} = y^4 = 1, \quad x^{2^l} = y^2, \quad y^{-1}xy = x^{-1}$$

we see that $R := \langle x, y \rangle$ is a generalised quaternion group of order $2^{l+3}$.

Lemma 2.6. The following hold:

(a) each element of $R$ is of the form $x^iy^j$ with $0 \leq i \leq 2^{l+2} - 1$ and $0 \leq j \leq 1$;
(b) each element in $R\setminus\{x\}$ is of order $4$;
(c) $x^iy$ is conjugate to $x^iy$ if and only if $i \equiv j \mod 2$, where $0 \leq i, j \leq 2^{l+2} - 1$;
(d) the set $Q$ of subgroups of $R$ isomorphic with $Q_8$ is given by $\langle x^2, x^iy \rangle$ with $0 \leq i \leq 2^{l+2} - 1$;
(e) there are two conjugacy classes of subgroups each of length $2^{l-1}$ represented by $Q := \langle x^2, y \rangle$ and $Q' := \langle x^2, xy \rangle$; and
(f) $N_S(Q) = \langle Q, x^{2^{l-1}} \rangle$ and $N_S(Q') = \langle Q', x^{2^{l-1}} \rangle$, for $l \geq 1$.

Proof. Part (a) is clear, and (b) follows since

$$(x^i)^2 = x^iyx^iy = yx^{-i}x^iy = y^2$$

has order 2. A general element $x^iy^j$ as in (a) conjugates $x^iy$ to

$$y^{-\ell}x^{-j}x^iyx^jy^{\ell} = (y^{-\ell}x^{i-2j}y^j)y = \begin{cases} x^{i-2j}y, & \text{if } \ell = 0 \\ x^{2j-i}y, & \text{if } \ell = 1. \end{cases}$$
field of order $q^2$ chosen so that $q^2 = x^{-1}$ (so $c$ has order $2i+3$) and that $c^{-1}y = xy$ and $cyc^{-1} = yx$. Hence by Lemma 2.6(d),(e), $c$ fuses the two conjugacy classes of subgroups of $R$ isomorphic with $Q_8$. Finally, we will need the following lemma.

**Lemma 2.7.** Let $\mathcal{F} := \mathcal{F}_R(SL_2(q))$ be the fusion system of $SL_2(q)$. Then $\{R, Q, Q'\}$ is a complete set of conjugacy class representatives of $\mathcal{F}$-centric radical subgroups. Moreover, $N_{SL_2(q)}(Q) \cong N_{SL_2(q)}(Q') \cong GL_2(3)$ and $Out_\mathcal{F}(Q) \cong Out_\mathcal{F}(Q') \cong S_3$.

**Proof.** See [AKO11, I.3.8].

It will be helpful to introduce some more notation. Some of it follows the notation of [AC10, Section 10] in preparation for the application in Section 3 of some of the results there.

**Notation 2.8.** We fix the following additional notation for subgroups and elements of $K$.

(a) $R_i \cong Q_{2i+3}$ is a Sylow $2$-subgroup of $L_i$ for $i = 1, 2, 3$, chosen so that $X \cong S_3$ acts on the set $\{R_1, R_2, R_3\}$;

(b) $R_0 := R_1R_2R_3 \in Syl_2(L_0)$;

(c) $Q_i$ is the set of subgroups of $R_i$ isomorphic to $Q_8$ for $i = 1, 2, 3$;

(d) $Q_i, Q'_i \in Q_i$ are representatives for the two $R_i$-conjugacy classes of subgroups (see Lemma 2.6) chosen so that $X \cong S_3$ acts by permuting the sets $\{Q_1, Q_2, Q_3\}$ and $\{Q'_1, Q'_2, Q'_3\}$;

(e) $c := [c, c, c]$ where $c$ is as in Section 2.3. Thus, $c$ acts simultaneously on $L_i \cong SL_2(q)$ by conjugation in the way described there;

(f) $d := [y, y, y]c \in K$, an involution commuting with $\tau$, so that $\langle d, \tau \rangle$ is a four group complementing $R_0$ in $S$; and

(g) $\tau' = d\tau$.

Refining Notation 2.5(e), we fix the following Sylow $2$-subgroup of $K$ throughout the remainder of this section and in Section 3

$$S = R_0(d, \tau).$$

Thus, $R_0$ is normal in $S$, and $\langle d, \tau \rangle \cong C_2 \times C_2$ is a complement to $R_0$ in $S$. Finally, we define

$$\mathcal{K} := \mathcal{F}_S(K), \quad \mathcal{H} := \mathcal{F}_S(H) \quad \text{and} \quad \mathcal{F} := \mathcal{F}_S(G).$$

We note that $\mathcal{F}$ is the fusion system generated by $\mathcal{H}$ and $\mathcal{K}$ by [Sem14, Theorem 3.3], namely $\mathcal{F}$ is the smallest fusion system on $S$ containing all morphisms in $\mathcal{H}$ and $\mathcal{K}$.

**2.4. The torus of $\mathcal{F}$.** The next lemma calls attention to the $2$-power torsion subgroup $T \leq S$ in a maximal torus of $H$. Viewed as a subgroup of $K$, it may be generated by the elements $[x, 1, 1], [1, x, 1], [c, c, c]$ in the notation of Subsection 2.3, and it is inverted by the involution $d$.

**Lemma 2.9.** There is a unique normal subgroup $T \leq S$ isomorphic with $(C_{2i+2})^3$, and $S/T \cong C_2 \times D_8$. 
Proof. By [AC10, Lemma 4.9(c)], there is a unique homocyclic subgroup of $S$ of rank 3 and exponent 4. Further, $T$ is the centralizer in $S$ of that subgroup and so is the unique subgroup of $S$ of its isomorphism type. Then [AC10 Lemma 4.3(c)] shows that $S/T \cong C_2 \times D_8$, a Sylow 2-subgroup of the Weyl group of type $B_3$. \qed

2.5. The standard elementary abelian sequence in $S$. We refer to Sections 4 and 7 of [AC10] for more discussion on the following items. Set $z := [−1, −1, 1] = [1, 1, −1] \in S$. There is a sequence of elementary abelian subgroups $Z < U < E < A$ of ranks 1, 2, 3, and 4, respectively, where $Z = Z(S) = \langle z \rangle$, $U$ is the unique normal four subgroup of $S$ of Notation 2.5(f), $E = \Omega_1(T)$, and $A = E(d)$. For a member $P_n$ of the above sequence of rank $n$, $Aut_\mathcal{F}(P_n) = Out_\mathcal{F}(P_n) \cong GL_n(2)$. Also, $\mathcal{H} = C_\mathcal{F}(Z)$ and $\mathcal{K} = N_\mathcal{F}(U)$.

2.6. The sectional rank of $S$. Before continuing, we pause to record the sectional rank of $S$ using the later Proposition 3.2. Its proof uses the fact that $S$ contains an extraspecial subgroup of order $2^7$.

Lemma 2.10. The sectional rank of $S$ is 6.

Proof. By Lemma 3.2(a) below, $S$ contains an extraspecial subgroup with central quotient of rank 6, and hence $s(S) \geq 6$. On the other hand, the sectional rank of a group is at most the sum of the sectional ranks of a normal subgroup and corresponding quotient, so Lemma 2.9 shows that $s(S) \leq s(T) + s(S/T) = 3 + 3 = 6$. \qed

3. Centric radicals in $\text{Sol}(q)$

The aim of this section is to refine the description of the centric radical subgroups of a Benson-Solomon system that results from a combination of [AC10 Section 10] and [COS08 Section 2]. A starting point is the next proposition due to Aschbacher and Chermak, which allows us to work in the groups $H$ and $K$ separately. Adopt the notation from Section 2, and in particular from Notation 2.5, 2.8, and Subsection 2.5. Recall that $G$ is the Aschbacher-Chermak amalgam, and that $\mathcal{F} = \mathcal{F}_S(G)$.

Proposition 3.1. Up to $\mathcal{F}$-conjugacy, a subgroup $P \leq S$ is $\mathcal{F}$-centric radical if and only if

(a) $P = A$ is elementary abelian of order $2^4$ and $Out_\mathcal{F}(P) \cong GL_4(2)$; or

(b) $P = C_S(E)$ and $Out_\mathcal{F}(P) \cong GL_3(2)$; or

(c) Either:

(i) $N_G(P) \leq K$ and $P \in \mathcal{K}^{cr}$; or

(ii) $N_G(P) \leq H$ and $P \in \mathcal{H}^{cr}$.

Proof. See [AC10 Lemma 10.9]. \qed

For the smallest of the Benson-Solomon systems, the results of [COS08], when combined with Proposition 3.1, supply sufficiently precise information for our needs, as we make clear in Subsection 3.1. For the larger systems, Proposition 3.2 below yields a sufficiently detailed description for the centric radicals occurring in Proposition 3.1(c)(ii) whose normalizer in $G$ is not contained in $H$. In its statement, we make reference to the normalizer fusion system and to the largest normal 2-subgroup of a 2-fusion system, both of which are defined in Subsection 2.1.

Proposition 3.2. Suppose that $P \in \mathcal{H}^{cr}$ with $N_H(P) \not\leq K$. Then one of the following holds:
(a) \( P \cong D_8 \ast D_8 \ast D_8 \) and \( \text{Out}_F(P) \cong \begin{cases} A_7 & \text{if } l = 0 \\ S_7 & \text{if } l > 0 \end{cases} \)
(b) \( P \cong C_4 \ast D_8 \ast Q_8 \) and \( \text{Out}_F(P) \cong S_8 \);
(c) \( l > 0, P \cong D_8 \ast Q_8 \times D_{2l+2} \), and \( \text{Out}_F(P) \cong S_6 \);
(d) \( P = O_2(N_H(E)), |S : P| = 2 \) and \( \text{Out}_F(P) \cong S_3 \).
Moreover, there is exactly one \( H \)-conjugacy class of subgroups of \( S \) of each of the given types.

Proof. This is [AC10] Lemma 10.7 with small changes. The description of the subgroup in part (c) is discussed in the proof that result. There is a small difference between our statement (d), and point (a)(3) of [AC10]. Namely 10.7(a)(3) states in our current notation that \( P = O_2(N_H(E)) \), but the fact is \( O_2(N_H(E)) = O_2(N_H(T)) = T \) unless \( q = 3 \) or 5, and \( T \) is never radical in the fusion system \( H \). (It is a radical 2-subgroup of the group \( H \) outside the small cases, namely when \( H \) has nontrivial odd torsion in a maximal torus containing \( T \).) What is presumably meant there is that \( P \) is the preimage in \( S \) of the largest normal 2-subgroup of the Weyl group \( W \cong C_2 \times S_4 \) of \( H \), and this is the subgroup listed in (d).

Finally, we must verify the last statement, which follows from the statement of [AC10] Lemma 10.7 and its proof, although it is not explicitly stated. Set \( \overline{H} = H/Z = \Omega_7(q) \), and let \((V,q)\) be the associated seven dimensional orthogonal space over \( \mathbb{F}_q \). By the statement and proof of [AC10] Lemma 10.7, a subgroup in (a) arises as the subgroup of \( S \) preserving all members of appropriate orthogonal decomposition of \( V \), all of whose summands are one-dimensional spaces spanned by vectors \( v \) for which \( q(v) \) is a square in \( \mathbb{F}_q \). A subgroup in (b) arises by preserving a similar decomposition, but with six of the summands spanned by vectors \( v \) with \( q(v) \) a nonsquare. A subgroup in (c) arises by preserving an orthogonal decomposition \( V = V_1 \perp \cdots \perp V_5 \perp W \), where each \( V_i \) is spanned by a vector of square norm, and where \( (W,q|_W) \) is a hyperbolic line. Since any two decompositions of a given type are isometric, by Witt’s Lemma there is exactly one \( \overline{H} \)-conjugacy class in each of (a), (b), (c), and hence exactly one \( H \)-conjugacy class.

\( Q \) Notation 3.3. We denote a member of the \( \mathcal{F} \)-conjugacy class of a subgroup appearing in Proposition 3.2 parts (a),(b), and (c) by \( R, R^* \) and \( R^{**} \) respectively. We note that when \( l = 0 \), the subgroups \( R \) and \( R^* \) correspond with the subgroups with the same names constructed in [COS08] Section 2).

We next describe the centric radical subgroups arising in case (c)(i) of Proposition 3.1. Recall Notation 2.5 and Notation 2.8. In addition, for a subgroup \( Y \) of \( K \) and an integer \( 1 \leq i \leq 3 \), denote by \( Y_i \) the projection of \( Y \cap L_0 \) in \( L_i \).

Proposition 3.4. Fix \( P \leq S \). Then \( P \in K^{cr} \) if and only if
(a) \( P \cap L_0 = P_1P_2P_3 \), and for each \( i \in \{1,2,3\} \), either \( P_i \in Q_i \) or \( P_i = R_i \); and
(b) one of the following holds. Either,
(i) \( P \in \{C_S(U), S\} \), or
(ii) \( P = P_1P_2P_3 \leq R_0 \) with \( P_i \in Q_i \) for at least two indices \( i \), or
(iii) \( P = P_0(s) \) for some \( s \in P \setminus C_P(U) \) such that
   (1) \( s^2 \in P_0 \),
   (2) either \( P_3 \in Q_3 \), or \( P_3 \in Q_i \) for \( i = 1,2 \), and
   (3) if \( P_3 \in Q_3 \), then \( \text{Out}_{L_3}(P) \) is not a 2-group.

Proof. This is essentially [AC10] Lemma 10.2]. The requirement in (b)(iii) that \( s \) square into \( P_0 \) does not appear in [AC10], but it is needed for the converse to hold in general (see Remark 3.9).
More precisely, for \( l > 0 \) there is a non-radical subgroup \( P \) of \( S \) such that \( P_i \in Q_i \) for \( i = 1, 2, 3 \), \( P/P_0 \) is cyclic of order 4, and \( \text{Out}_{L_i}(P) \) is not a 2-group. \( \square \)

3.1. **The case** \( l = 0 \). An important distinguishing feature of the smallest Benson-Solomon system is that \( R_0 \) is normal in the fusion system \( K \). When \( l = 0 \), this is most naturally seen over \( F_3 \), where a \( Q_8 \) Sylow 2-subgroup is normal in \( \text{SL}_2(3) \). Over \( F_5 \), the normalizer of a quaternion Sylow 2-subgroup of \( \text{SL}_2(5) \) is \( \text{SL}_2(3) \), which still controls 2-fusion in \( \text{SL}_2(5) \). It will therefore be convenient to treat the cases \( l = 0 \) and \( l > 0 \) separately. So assume here that \( l = 0 \). We adopt the previous notation, except that we set

\[
Q := R_0
\]

in this smallest case. We first list the \( K \)-centric radicals, which does not require Proposition 3.4.

**Proposition 3.5.** Let \( l = 0 \) and \( P \in K^{cr} \). Then exactly one of the following holds.

(a) \( P = S \), and \( \text{Out}_K(P) = 1 \);

(b) \( P = Q \), and \( \text{Out}_K(P) \cong (C_3)^3 \times 1 \times (C_2 \times S_3) \);

(c) \( P = QR = Q(\tau) \), and \( \text{Out}_K(P) \cong (C_3 \times C_3, 1 \times C_2) \);

(d) \( P = QR^* = Q(\tau^*) \), and \( \text{Out}_K(P) \cong S_3 \); or

(e) \( P = C_S(U) = Q(d) \), and \( \text{Out}_K(P) \cong S_3 \).

**Proof.** As \( Q \) is a centric normal 2-subgroup of \( K \), it is contained in every member of \( K^{cr} \) by Lemma 2.4. Since the four group \( \langle d, \tau \rangle \) is a complement to \( Q \) in \( S \), there are only five possible centric radical subgroups, and no pair of them are \( K \)-conjugate. Then one computes the automorphism groups induced on these subgroups using the fact that \( \text{Out}_K(Q) \cong (C_3)^3 \times (C_2 \times S_3) \), where the \( C_2 \) factor is the class of conjugation by \( d \) and inverts the base, and where an \( S_3 \) factor is induced by \( X \) permuting coordinates. Visibly no resulting outer automorphism group has a nontrivial normal 2-subgroup, so all the candidate subgroups are \( K \)-centric radical. \( \square \)

**Proposition 3.6.** Let \( l = 0 \). Then, up to conjugacy, the \( K \)-, \( H \)- and \( F \)-centric radical subgroups of \( S \) together with their orders and automorphism groups appear in the following table, where a ‘-’ indicates that the subgroup is not centric radical in that fusion system.

**Table 1.** Sol(5)-conjugacy classes of Sol(5)-centric radical subgroups

| \( P \) | \(|P|\) | \( \text{Out}_H(P) \) | \( \text{Out}_K(P) \) | \( \text{Out}_F(P) \) |
|---|---|---|---|---|
| \( S \) | \( 2^{10} \) | 1 | 1 | 1 |
| \( Q \) | \( 2^8 \) | \( (C_3)^3 \times (C_2 \times C_2) \) | \( (C_3)^3 \times (C_2 \times S_3) \) | \( (C_3)^3 \times (C_2 \times S_3) \) |
| \( QR \) | \( 2^9 \) | \( C_3 \times C_3 \times C_2 \) | \( C_3 \times C_3 \times C_2 \) | \( C_3 \times C_3 \times C_2 \) |
| \( QR^* \) | \( 2^9 \) | \( S_3 \) | \( S_3 \) | \( S_3 \) |
| \( C_S(U) \) | \( 2^9 \) | - | \( S_3 \) | \( S_3 \) |
| \( R \) | \( 2^7 \) | \( A_7 \) | - | \( A_7 \) |
| \( R^* \) | \( 2^6 \) | \( S_6 \) | - | \( S_6 \) |
| \( RR^* \) | \( 2^9 \) | \( S_3 \) | - | \( S_3 \) |
| \( C_S(E) \) | \( 2^7 \) | - | - | \( \text{GL}_3(2) \) |
| \( A \) | \( 2^4 \) | - | - | \( \text{GL}_4(2) \) |
Table 2. $\mathcal{K}$-conjugacy classes of $\mathcal{K}$-centric radical subgroups, $l > 0$

| $P$ | $|P|$ | $\text{Out}_\mathcal{K}(P)$ |
|-----|------|------------------|
| $S$ | $2^{10+3l}$ | 1 |
| $C_5(U)$ | $2^{9+3l}$ | $S_3$ |
| $Q_1Q_2Q_3$ | $2^5$ | $S_3 \wr S_3$ |
| $Q_1Q_2Q_3'$ | $2^5$ | $(S_3 \wr C_2) \times S_3$ |
| $Q_1Q_2R_3$ | $2^{8+l}$ | $S_3 \wr C_2$ |
| $Q_1Q_2R_3'$ | $2^{8+l}$ | $S_3 \wr C_2$ |
| $Q_1Q_2Q_3(\tau)$ | $2^9$ | $S_3 \times S_3$ |
| $Q_1Q_2Q_3'(\tau)$ | $2^9$ | $S_3 \times S_3$ |
| $Q_1Q_2R_3(\tau)$ | $2^{9+l}$ | $S_3$ |
| $Q_1Q_2R_3'(\tau)$ | $2^{9+l}$ | $S_3$ |
| $R_1R_2Q_3(\tau)$ | $2^{9+2l}$ | $S_3$ |

Proof. By Proposition 3.5, the column for $\mathcal{K}$ is correct. By [COS08, Lemma 2.1], the column for $\mathcal{H}$ is correct. We work up to $\mathcal{F}$-conjugacy in what follows. Let $P \in \mathcal{F}^{cr}$. By Proposition 3.1, $P$ is listed in the last two rows of Table 1, or one of the following holds: $P \in \mathcal{K}^{cr}$ and $\text{Out}_\mathcal{F}(P) = \text{Out}_\mathcal{K}(P)$, or $N_G(P) \not\leq K$, $P \in \mathcal{H}^{cr}$ and $\text{Out}_\mathcal{F}(P) = \text{Out}_\mathcal{H}(P)$. If the former holds, then $P$ is listed in the first five rows of the table. If the former does not hold, then the latter holds, and $P$ is listed in the next three rows of the table. Finally note that no additional $\mathcal{F}$-conjugacy can occur between these subgroups because each pair of subgroups with isomorphic outer automorphism groups in $\mathcal{F}$ have center $Z$ or $U$. This means that any conjugating morphism would necessarily be contained in $\mathcal{H} = C_\mathcal{F}(Z)$ or $\mathcal{K} = N_\mathcal{F}(U)$. \qed

We end this subsection with a lemma which will be needed in Section 5.

Lemma 3.7. Assume $l = 0$. Then $Q$, $R$, and $QR$ are weakly $\mathcal{F}$-closed.

Proof. By Proposition 3.6, we may assume that the images of $Q$ and $R$ in $\Omega_\tau(q) = H/Z$ are given explicitly as a subgroup of $S$ as outlined just before Lemma 2.1 of [COS08]. Then since both $Q$ and $R$ contain $Z$ and are weakly $\mathcal{H}/Z$-closed by [COS08, Lemma 2.1], they are both weakly $\mathcal{H}$-closed. In particular, all three subgroups $Q$, $R$, and $QR$ are normal in $S$ and hence fully $\mathcal{D}$-normalized for $\mathcal{D} \in \{\mathcal{H}, \mathcal{K}, \mathcal{F}\}$.

As $Q$ is normal in $\mathcal{K}$, $Q$ is weakly $\mathcal{K}$-closed. We claim that $R$ is also weakly $\mathcal{K}$-closed. Assume to the contrary. Then by Alperin’s Fusion Theorem and Proposition 3.5, there is $\alpha \in \text{Aut}_\mathcal{K}(QR)$ such that $R^\alpha \neq R$. However, $Z(QR) = C_{\mathcal{Z}(Q)}(\tau) = Z$, and so $\alpha \in C_{\mathcal{F}}(Z) = \mathcal{H}$. Thus, $R^\alpha = R$ by the previous paragraph, a contradiction which establishes the claim. It follows that $Q$ and $R$ are weakly $\mathcal{F}$-closed since $\mathcal{F} = \langle \mathcal{H}, \mathcal{K}\rangle$, and so $QR$ is weakly $\mathcal{F}$-closed as well. \qed

3.2. The case $l > 0$. In this subsection, we determine a set of representatives for the $\mathcal{D}$-conjugacy classes of elements in $\mathcal{D}^{cr}$ in the case when $l > 0$ for $\mathcal{D} \in \{\mathcal{K}, \mathcal{H}, \mathcal{F}\}$. We begin with the case when $\mathcal{D} = \mathcal{K}$. 

Proposition 3.8. Suppose that $l > 0$. There are eleven $\mathcal{K}$-conjugacy classes of elements of $\mathcal{K}^{cr}$. Representatives of these classes together with their outer automorphism groups in $\mathcal{K}$ are listed in Table 2.
Proof. Let \( P \leq S \) be a centric radical subgroup of \( K \). We proceed through the possibilities in the description of \( K^{cr} \) given by Proposition 3.4 and refer to the labelings of the three cases given there. If \( P \) occurs in (b)(i), then \( P \) is listed in the first two rows of the table. By [HL17 Lemma 3.4(b)], \( \text{Aut}_K(S) = \text{Inn}(S) \) so that \( \text{Out}_K(S) = 1 \). Also, \( C_S(U) = R_0(d) \), so that \( \text{Out}_K(C_S(U)) \cong S_3 \) is induced by \( X \).

Consider a subgroup \( P \) in (b)(ii). First assume that \( P_i \in Q_i \) for all \( i \). Upon conjugating in \( L_0 \), we may assume that \( P_i = Q_i \) or \( Q_i' \) for each \( i \). Conjugating by \( d \), we may assume that there is at most one \( Q_i' \) among the \( P_i \)'s. Finally, we may conjugate by elements of \( X \) to see that \( P \) is one of the subgroups in rows 3 and 4 of the table. If \( P = Q_1Q_2Q_3 \), then \( N_K(P) = N_{L_1}(Q_1)N_{L_2}(Q_2)N_{L_3}(Q_3)X \), and we see that \( \text{Out}_K(P) \cong S_3 \wr S_3 \). If \( P = Q_1Q_2Q_3' \), then \( N_K(P) = N_{L_1}(Q_1)N_{L_2}(Q_2)N_{L_3}(Q_3')(\tau) \), so that \( \text{Out}_K(P) \cong (S_3 \wr C_2) \times S_3 \).

Next, assume that \( P_i \in Q_i \) for exactly two indices \( i \). Then as before, we may conjugate so that \( P = Q_1Q_2R_3 \) or \( Q_1Q_2' \) is on the table. In the former case, we have \( \text{Out}_K(P) \cong S_3 \wr C_2 \) with the class of \( \tau \) wreathing, while in the latter case we have a similar situation with the class of \( \tau' \) wreathing. This concludes the case (b)(ii).

Consider now a subgroup \( P \) in (b)(iii). Thus \( P = P_0(s) \) with \( s \in C_P(U) - U \) normalizing \( P_0 \). Set \( N = N_K(P) \) and \( M = N_{K}(P_0) \). Denote quotients modulo \( P_0 \) with bars, and set \( M^+ = \overline{M}/O_3(\overline{M}) \).

Since \( L_0 \leq K \), we see that \( P_0 = P \cap L_0 \leq N \) so that \( N \leq M \). Also, since \( \overline{s} \) is of order 2, \( N \) is the preimage in \( M \) of \( C_{\overline{M}}(\overline{s}) \). As \( P \) is radical, we must have

\[
(\overline{s}) = O_2(C_{\overline{M}}(\overline{s})).
\]

We consider separately the cases where \( P_0 \notin K^{cr} \) and where \( P_0 \in K^{cr} \). Assume first that \( P_0 \notin K^{cr} \), the easier case. Upon comparing the conditions in (b)(ii) and (b)(iii), we have by our assumption that \( P_3 \in Q_3 \) and \( P_i = R_i \) for \( i = 1, 2 \). Thus, \( \overline{M} = (\overline{\tau}) \times \overline{N}_{L_3}(P_3) \cong C_2 \times S_3 \), and so \( P = P_0(\tau) \) by (3.1). Hence, \( \text{Out}_K(P) \cong S_3 \), and \( P \) appears in the last row of the table.

Assume next that \( P_0 \in K^{cr} \), so that \( P_0 \) is conjugate to a subgroup considered in (b)(iii), rows 3-6 of the table. From our description of the normalizers in \( K \) of those 2-groups, each is fully \( K \)-normalized. Thus, by Lemma 2.3 (and since \( P_0 \leq P \)), we may replace \( P \) by a \( K \)-conjugate and assume that \( P_0 \) is on the table.

**Case 1.** Assume \( P_0 = Q_1Q_2Q_3 \) and recall Lemma 2.6(e). Here \( \overline{M} \cong S_3 \wr X \), \( O_2(M^+) = N_{R_1}(Q_1)^+N_{R_2}(Q_2)^+N_{R_3}(Q_3)^+ \cong (C_2)^3 \), and \( M^+ = O_2(M^+)X^+ \). Since \( s \notin C_P(U) \), it follows that \( s^+ \notin O_2(M^+) \). Thus, \( s^+ \in O_2(M^+)\tau^+ \), and \( C_{O_2(M^+)}(s^+) \) is of order 4 generated by \( [x^{2^l-1}, x^{2^l-1}, 1]^+ \) and \( [1, 1, x^{2^l-1}]^+ \). This implies via (3.1) that \( \overline{s} \) must centralize \( O_3(\overline{N}_{L_3}(Q_3)) \) as well as nontrivial subgroup of \( O_3(\overline{N}_{L_1}(Q_1)\overline{N}_{L_2}(Q_2)) \) (since \( \overline{s} \) has order 2, whence does not contain \( [x^{2^l-1}, x^{2^l-1}, 1] \)). We conclude that \( \overline{s} = \tau \) or \( [x^{2^l-1}, x^{2^l-1}, 1] \), \( |O_3(C_{\overline{M}}(\overline{s}))| = 9 \), and

\[
C_{\overline{M}}(\overline{s}) = \langle \overline{s} \rangle \times C_{\overline{N}_{L_1}(Q_1)\overline{N}_{L_2}(Q_2)}(\overline{s}) \times C_{\overline{N}_{L_3}(Q_3)}(\overline{s}) \cong C_2 \times S_3 \times S_3.
\]

However, since the two possibilities for \( \overline{s} \) are conjugate under \( \overline{N}_{R_1}(Q_1) \), we may take \( \overline{s} = \tau \), as desired.

**Case 2.** Assume \( P_0 = Q_1Q_2Q_3' \). In this case \( \overline{M} \cong S_3 \wr \langle \overline{\tau} \rangle \times S_3 \) and \( M^+ = N_{R_1}(Q_1)^+N_{R_2}(Q_2)^+N_{R_3}(Q_3)^+\langle \tau^+ \rangle \). Now argue as in case (1) (replacing \( Q_3 \) by \( Q_3' \) everywhere) to see that again we may take \( \overline{s} = \tau \).
Case 3. Assume $P_0 = Q_1Q_2R_3$. We have $M = S_3 \langle \tau \rangle$ and $M^+ = N_{R_1}(Q_1)^+N_{R_2}(Q_2)^+$. As in the previous cases, we may take $\bar{s} = s$ or $[x^{2^{r-1}}, x^{2^{r-1}}, 1]$, so that

$$C_M(\bar{s}) = \langle \bar{s} \rangle \times C_{N_L(Q_1)N_L(Q_2)}(\bar{s}) = C_2 \times S_3.$$ 

Again, these possibilities for $\bar{s}$ are conjugate under $N_{R_1}(Q_1)$ and we see that we may take $\bar{s} = s$, as needed.

Case 4. Assume $P_0 = Q_1Q_2^2R_3$. This time, replace $\tau$ by $\tau'$, and repeat the argument from the previous case.

\[\square\]

Remark 3.9. The minor omission in the proof of [AC10, Lemma 10.2] alluded to earlier occurs in the middle of page 953 with the claim “$|O_3(N)| = 9$”. It is possible that $|O_3(N)| = 3$ under the hypotheses there. More precisely, consider the element $\bar{s}$ in Case 1 of the above proof. Choose $\bar{s}$ to centralize $O_3(N_{R_1}(Q_3))$ and have order 4. Then $\bar{s}$ squares to $[x^{2^{r-1}}, x^{2^{r-1}}, 1]$, and $|O_3(C_{M^+}(\bar{s}))| = 3$. But in this case, $O_3(N_{M^+}(\bar{s})) \cong D_8$ while $\langle \bar{s} \rangle$ is cyclic of index 2 in this subgroup, and so $|O_3(\text{Out}_K(P))| = 2$ is generated by the image of $[x^{2^{r-1}}, 1, 1]$. Thus, $P_0(s)$ satisfies (b)(iii)(2-3), but is not $K$-radical.

We next determine the set $H^{cr}$ up to $H$-conjugacy in the case $l > 0$.

Proposition 3.10. Suppose that $l > 0$. There are eighteen $H$-conjugacy classes of elements of $H^{cr}$. Representatives for these classes together with their automorphism groups are listed in Table 3.

Proof. Let $P \in H^{cr}$. If $N_H(P) \not\leq K$ then using Proposition 3.2(a)-(d), we obtain the groups in the last 4 rows of Table 3. Hence, we may assume that $N_H(P) \leq K$. By [LO92, Lemma 3.3(a)], $P$ is $F$-centric, so that $P$ is also $K$-centric.

Suppose first that $P$ is $K$-radical, so that $P \in K^{cr}$. In this case we appeal to Proposition 3.8 to obtain the first thirteen entries in Table 3. A case-by-case check shows that for each $K$-conjugacy class $C$ of a subgroup listed in Table 2, one of the following holds: either no member of $C$ is $H$-radical, or $C$ meets exactly one $H$-radical conjugacy class, or $C$ is the class of one of the entries in rows 4 through 6 of Table 2. In this last case, $C$ meets one of two $H$-classes of $H$-radical subgroups, and corresponding representatives of these $H$-classes appear in rows 3 through 8 of Table 3.

We may therefore assume that $P$ is not $K$-radical. We claim in this case that $P$ is $H$-conjugate to $Q_1R_2R_3$, the last remaining entry of Table 3. Observe first that $P \leq C_S(U) = R_0(d)$, since otherwise $Z(P) \cap U = Z$ is $N_K(P)$-invariant, and so $N_H(P) = N_K(P)$ yields that $\text{Out}_H(P) = \text{Out}_K(P)$ has no nontrivial normal 2-subgroups, which contradicts that $P$ is not $K$-radical. Hence, $U \leq Z(P)$.

For each $i = 0, 1, 2, 3$, set $P_0 = P \cap L_0$, and let $P_i$ be the projection of $P_0$ in $R_i$. A reading of the first three paragraphs of the proof of [AC10, Lemma 10.2] reveals that the given argument applies to a centric radical $P \leq S$ of $H$ whose normalizer is contained in $K$, our current situation. We conclude that $P_i = P \cap R_i$ and that $P_i \in Q_i$ or $P_i = R_i$ for each $i = 1, 2, 3$. In particular, $P_0 = P_1P_2P_3$.

We next claim that $P = P_0$. Suppose on the contrary that $P_0 < P = C_P(U)$, and choose $d \in C_P(U) - P$. As $d$ interchanges the two $R_0$-conjugacy classes of subgroups in $Q_i$ for each $i$, and as $d$ normalizes $P_i = P \cap R_i$, we conclude that $P_i = R_i$ for each $i$. But then $P = R_0(d) = C_S(U)$ and $\text{Out}_H(P)$ is of order 2. Thus, $P$ is not $H$-radical, contrary to the original choice of $P$.

Finally, conjugating in $L_0(d) = C_B(U) \leq H$ if necessary, we have $P = Q_1R_2R_3$ or $R_1R_2Q_3$. But in the latter case, $\text{Out}_H(P) \cong C_2 \times S_3$ is induced by $\langle \tau \rangle \times N_{L_0}(Q_3)$, so again, $P$ is not $H$-radical, a
Table 3. $H$-conjugacy classes of $H$-centric radical subgroups, $l > 0$

| $P$        | $|P|$          | $\text{Out}_H(P)$ |
|------------|---------------|------------------|
| $S$        | $2^{10+3l}$   | 1                |
| $Q_1Q_2Q_3$ | $2^8$         | $S_3 \times S_3 \wr C_2$ |
| $Q'_1Q_2Q_3$ | $2^8$         | $S_3 \wr C_2 \times S_3$ |
| $Q_1Q_2R_3$ | $2^{8+l}$     | $S_3 \wr C_2$    |
| $Q_1Q'_2R_3$ | $2^{8+l}$     | $S_3 \times S_3$ |
| $Q_1R_2Q_3$ | $2^{8+l}$     | $S_3 \times S_3$ |
| $Q_1Q_2Q_3(\tau)$ | $2^9$       | $S_3 \times S_3$ |
| $Q_1Q_2Q_3'(\tau)$ | $2^9$       | $S_3 \times S_3$ |
| $Q_1Q_2R_3(\tau)$ | $2^{9+l}$     | $S_3$            |
| $Q_1Q'_2R_3(\tau')$ | $2^{9+l}$     | $S_3$            |
| $R_1R_2Q_3(\tau)$ | $2^{9+2l}$   | $S_3$            |
| $Q_1R_2R_3$ | $2^{8+2l}$    | $S_3$            |
| $R$        | $2^7$         | $S_7$            |
| $R^*$      | $2^6$         | $S_6$            |
| $R^{**}$   | $2^{7+l}$     | $S_5$            |
| $O_2(N_H(E))$ | $2^{9+3l}$   | $S_3$            |

contradiction. Thus, up to $H$-conjugacy, $P = Q_1R_2R_3$ and $\text{Out}_H(P) \cong S_3$ is induced by $N_{L_1}(Q_1)$, and this is the only remaining entry on Table 3.

Finally, we are able to describe the set of $F$-centric radical subgroups, up to $F$-conjugacy:

**Theorem 3.11.** Let $F = \text{Sol}(5^2)^l$ with $l > 0$. Representatives for the $F$-conjugacy classes of $F$-centric radical subgroups, together with their orders and automorphism groups, are listed in Table 4, where ‘−’ indicates that the subgroup is not centric radical in that fusion system.

**Proof.** This follows upon combining Propositions 3.1, 3.8 and 3.10.

\[\square\]

4. **Blocks of defect zero**

In this section we calculate the number of projective simple modules for the outer automorphism groups in the various tables of the previous section. Note that by [Ben98a, Proposition 1.8.5] any such module is the unique simple module in a block isomorphic with a matrix algebra over $k$. Moreover blocks with this property are exactly the blocks of defect zero by [Cra11, Theorem 2.39]. We thus obtain:

*for any finite group $G$, $z(kG)$ is the number of blocks of defect 0.*

Thus in order to calculate $z(kG)$ we may appeal to a result of Robinson [Rob83], which allows one to compute the number of blocks with a given defect group by way of group theoretic information.

Fix a finite group $G$ and a Sylow 2-subgroup $S$ of $G$. Recall that a conjugacy class $C$ of $G$ is said to be of 2-defect zero if $|C_G(y)|$ is odd for some (hence any) representative $y \in C$. Denote by $Y_0$ the union of the classes of defect zero, and by $Y$ a set of representatives of those classes. Set

\[\mathcal{X} = \{SxS \mid x \in Y_0 \text{ and } S \cap S^x = 1\},\]
Table 4. $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric radical subgroups, $l > 0$

| $P$        | $|P|$ | $\text{Out}_{\mathcal{K}}(P)$ | $\text{Out}_{\mathcal{H}}(P)$ | $\text{Out}_{\mathcal{F}}(P)$ |
|------------|------|-------------------------------|-------------------------------|-------------------------------|
| $S$        | $2^{10+3l}$ | 1                             | 1                             | 1                             |
| $C_S(U)$   | $2^{9+3l}$   | $S_3$                         | $S_3$                         | $S_3$                         |
| $Q_1Q_2Q_3$| $2^8$      | $S_3 \times S_3 \times C_2$   | $S_3$                         | $S_3 \times S_3 \times C_2$   |
| $Q'_1Q_2Q_3$| $2^8$         | $S_3 \times S_3 \times C_2$   | $S_3$                         | $S_3 \times S_3 \times C_2$   |
| $Q_1Q_2R_3$| $2^{9+4l}$ | $S_3 \times C_2$              | $S_3$                         | $S_3 \times C_2$              |
| $Q'_1Q_2R_3$| $2^{9+4l}$ | $S_3 \times C_2$              | $S_3$                         | $S_3 \times C_2$              |
| $Q_1Q_2Q_3(\tau)$ | $2^9$   | $S_3 \times S_3 \times S_3$ | $S_3$                         | $S_3 \times S_3 \times S_3$ |
| $Q_1Q_2Q_3(\tau')$ | $2^9$    | $S_3 \times S_3 \times S_3$ | $S_3$                         | $S_3 \times S_3 \times S_3$ |
| $Q_1Q_2R_3(\tau)$ | $2^{9+4l}$ | $S_3$                         | $S_3$                         | $S_3$                         |
| $Q_1Q_2R_3(\tau')$ | $2^{9+4l}$ | $S_3$                         | $S_3$                         | $S_3$                         |
| $R_1R_2Q_3(\tau)$ | $2^{9+2l}$ | $S_3$                         | $S_3$                         | $S_3$                         |
| $R$        | $2^7$   | $-$                           | $S_7$                         | $S_7$                         |
| $R^*$      | $2^6$   | $-$                           | $S_6$                         | $S_6$                         |
| $R^{**}$   | $2^{7+4l}$ | $-$                           | $S_5$                         | $S_5$                         |
| $O_2(N_{\mathcal{H}}(E))$ | $2^{9+3l}$ | $-$                           | $S_3$                         | $S_3$                         |
| $A$        | $2^4$   | $-$                           | $-$                           | $GL_4(2)$                     |
| $C_S(E)$   | $2^{7+3l}$ | $-$                           | $-$                           | $GL_3(2)$                     |

a collection of $S$-$S$ double cosets in $G$. Choose a function $f: \mathcal{X} \to Y_0$ such that $f(D) \in D$ for each $D \in \mathcal{X}$, and set $X = f(\mathcal{X})$. Finally, for each $y_i \in Y$ and $x_j \in X$, define a $(|Y| \times |X|)$-matrix $N$ over $\mathbb{F}_2$ by $N_{ij} = |y_i^G \cap x_j^S|$ (mod 2).

Theorem 4.1 ([Rob83]). The number of $2$-blocks of $G$ of defect zero is equal to the rank of $NN^T$. In particular, this number is at most $\min(|X|, |Y|)$.

Corollary 4.2 ([Rob83 Corollary 3]). There are at least as many blocks of defect zero for $G$ as there are defect zero classes in $O_2'(G)$.

Corollary 4.3. For a finite group $G$ with a normal 2-complement, the number of defect zero blocks equals the number of defect zero classes.

Proof. By assumption the number of defect zero classes is the number of such lying in $O_2'(G)$. So Theorem 4.1 says that there are at most this many blocks of defect zero, while Corollary 4.2 says there are at least this many.

□

Proposition 4.4. Each of the groups listed in Table 3 has the stated number of blocks of defect zero.

Proof. Let $G$ be one of the groups $S_3$, $S_3 \times S_3$, $S_3 \times S_3 \times S_3$, $S_3 \times C_2$, $(C_3 \times C_3)^{-1} \times C_2$, or $(C_3)^3 \times (C_2 \times S_3)$. Then in each case $G$ has a normal 2-complement so that the number of blocks of defect zero is equal to the number of classes of defect zero by Corollary 4.3, and the latter quantity is easily computed. Next, suppose that $G$ is one of the groups $GL_3(2)$, $GL_4(2)$, $S_6 \cong Sp_4(2)$. Then $G$ has a BN-pair at the prime 2, and so each has exactly one block of defect zero, containing the Steinberg module. Alternatively, we know the existence of the Steinberg module, and then from the Bruhat decomposition in which $B = S$, we have $G = \prod_{w \in W} SwS$, and exactly one such double coset has
WEIGHTS IN A BENSON-SOLOMON BLOCK

Table 5. The number of blocks of defect zero

| $G$                              | $z(kG)$ |
|----------------------------------|---------|
| $S_3$                            | 1       |
| $S_3 \times S_3$                 | 1       |
| $S_3 \times S_3 \times S_3$      | 1       |
| $S_3 \wr C_2$                    | 0       |
| $(C_3 \times C_3) \wr C_2$       | 4       |
| $(C_3)^3 \wr (C_2 \times S_3)$   | 1       |
| $GL_3(2)$                        | 1       |
| $GL_4(2)$                        | 1       |
| $S_6$                            | 1       |
| $S_3 \wr S_3$                    | 1       |
| $S_5$                            | 0       |
| $A_7$                            | 0       |
| $S_7$                            | 0       |

Using Tables 1, 3 and 4 we can give a count of the number of weights.

Corollary 4.5. For $D \in \{H, F\}$ and all $l \geq 0$, the number of weights associated to the Kulshammer-Puig pair $(D, 0)$ is

$$w(D, 0) = 12.$$  

Note that $w(H, 0) = 12$ is known as a consequence of results in [An93].
5. Külshammer-Puig classes

In this section, we prove Theorem 1.1. Theorem 1.3 will then follow from this and Corollary 4.5. We start by recalling some well-known results from group cohomology.

**Lemma 5.1.** Let $G$ and $H$ be finite groups, $p$ be a prime and $k$ be an algebraically closed field of characteristic 2. Write $|G| = 2^r w$ where $w$ is odd. The following hold.

(a) For any abelian group $A$ with trivial $G$-action, $H^1(G, A) = \text{Hom}(G, A)$.

(b) There is a surjective map

$$H^2(G, \mathbb{Z}/w\mathbb{Z}) \to H^2(G, k^\times),$$

which is an isomorphism if $G$ is $p$-perfect for every odd prime $p$ dividing $|G|$.

(c) If $H^2(G, \mathbb{F}_p) = 0$ for all odd primes $p$, then $H^2(G, k^\times) = 0$.

(d) If $p$ is odd and $G$ is a $p$-perfect group with cyclic Sylow $p$-subgroups then $H^2(G, \mathbb{F}_p) = 0$.

(e) If $G$ is a $p$-perfect group with an elementary abelian Sylow $p$-subgroup $V$ of order $p^2$. Then

$$H^2(G, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } \text{Aut}_G(V) \subseteq \text{SL}(V), \\ 0 & \text{otherwise.} \end{cases}$$

(f) If $G$ and $H$ are $p$-perfect, then

$$H^2(G \times H, \mathbb{F}_p) \cong H^2(G, \mathbb{F}_p) \oplus H^2(H, \mathbb{F}_p).$$

(g) If $G$ is $p$-perfect and the $p$-part $M(G)_p$ of the Schur multiplier of $G$ is of exponent at most $p$ then

$$M(G)_p = H^2(G, \mathbb{C}^\times) \otimes \mathbb{Z}_{(p)} \cong H^2(G, \mathbb{F}_p) \cong H^2(G, k^\times) \otimes \mathbb{Z}_{(p)}.$$

Here, $\mathbb{Z}_{(p)}$ denotes the $p$-local integers.

(h) (Schur) If $M(G)$ has exponent $e$, then $e^2$ divides the order of $G$.

**Proof.**

(a) By the Universal Coefficient Theorem for cohomology, there is an exact sequence

$$0 \to \text{Ext}^1(H_0(G, \mathbb{Z}), A) \to H^1(G, A) \to \text{Hom}(H_1(G, \mathbb{Z}), A) \to 0.$$

Now $H_1(G, \mathbb{Z}) \cong G/[G, G]$ (it is the abelianization of the fundamental group of $BG$), and the Ext term vanishes since $H_0(G, \mathbb{Z}) = \mathbb{Z}$ is a free abelian group. Thus, $H^1(G, A) \cong \text{Hom}(G/[G, G], A) \cong \text{Hom}(G, A)$ as desired.

(b) Fix a Sylow 2-subgroup $S$ of $G$. Since $k^\times$ has all odd roots of unity, powering by $w$ is a surjective endomorphism with kernel $\mathbb{Z}/w\mathbb{Z}$. Thus, there is an exact sequence

$$H^1(G, k^\times) \to H^2(G, \mathbb{Z}/w\mathbb{Z}) \to H^2(G, k^\times) \to H^2(G, k^\times).$$

The last map is multiplication by $w = |G:S|$, and so it factors as

$$H^2(G, k^\times) \xrightarrow{\text{res}} H^2(S, k^\times) \xrightarrow{\text{tr}} H^2(G, k^\times).$$

Since $H^2(S, k^\times) = 0$, we conclude that the last map is 0. The middle map is therefore a surjection, and since $H^1(G, k^\times) = \text{Hom}(G, k^\times)$ by (a), we see that it is an isomorphism if $G$ is $p$-perfect for every odd prime $p$ dividing $|G|$.

(c) Set $W = \mathbb{Z}/w\mathbb{Z}$. By (2), $H^2(G, W)$ surjects onto $H^2(G, k^\times)$, so it suffices to show that $H^2(G, W) = 0$ under our assumption. We prove this by induction on $w$. The case where $w = 1$ is trivial. Let $Z$ be the subgroup of $W$ of order $p$. We have an exact sequence

$$H^2(G, Z) \to H^2(G, W) \to H^2(G, W/Z)$$

and so...
Lemma 5.3. Let \( \sigma \) be a Sylow \( p \)-subgroup with \( p \) odd. Restriction induces an isomorphism \( H^*(G,F_p) \to H^*(N_G(P),F_p)^{\text{Aut}_G(P)} \). Now \( H^*(P,F_p) \cong F_p[x,y]/(x^2) \) with \( \deg x = 1 \) and \( \deg y = 2 \), and the Bockstein \( H^1(P,F_p) \to H^2(P,F_p) \) is an isomorphism of \( N_G(P) \)-modules (cf. [Ben98a, p.132, Example]). As \( N_G(P) \) has no invariants in \( H^1(P,F_p) \) by assumption it also has no invariants on \( H^2(P,F_p) \).

(e) Restriction to \( V \) identifies \( H^*(G,F_p) \) with the invariants \( H^*(V,F_p)^{\text{Aut}_G(V)} \). Now

\[
H^*(V,F_p) \cong \Lambda_{F_p}(x_1, x_2) \times F_p[y_1, y_2],
\]

with \( \deg x_i = 1 \) and \( \deg y_k = 2 \), so that

\[
H^1(V,F_p) = \langle x_1, x_2 \rangle_{F_p} \quad \text{and} \quad H^2(V,F_p) = \langle y_1, y_2, x_1 x_2 \rangle_{F_p}.
\]

Here, \( H^1(V,F_p) \) is the natural module for \( \text{Aut}(V) \cong GL(V) \), while \( H^2(V,F_p) \) is the direct sum of the natural module \( \langle y_1, y_2 \rangle_{F_p} \) and \( \langle x_1 x_2 \rangle_{F_p} \) on which \( GL(V) \) acts via the determinant map. By assumption, \( \text{Aut}_G(V) \leq GL(V) \) has no fixed points on the natural module, so \( H^2(V,F_p) \) is nontrivial generated by \( x_1 x_2 \) if and only if every element of \( \text{Aut}_G(V) \) has determinant one.

(f) This follows from the Künneth Theorem [Ben98a, Theorem 2.7.1] and the assumption.

(g) The assumptions imply that \( H^1(G,\mathbb{C}^\times) \) is a (finite abelian) \( p' \)-group by (a) and that the powering by \( p \) map \( H^2(G,\mathbb{C}^\times) \to H^2(G,\mathbb{C}^\times) \) is 0. Powering by \( p \) on \( \mathbb{C}^\times \), we have an exact sequence

\[
H^1(G,\mathbb{C}^\times) \to H^2(G,F_p) \to H^2(G,\mathbb{C}^\times) \to 0.
\]

Tensoring with \( \mathbb{Z}(p) \) we obtain an isomorphism \( H^2(G,F_p) \cong H^2(G,\mathbb{C}^\times) \otimes \mathbb{Z}(p) \). An analogous argument with \( \mathbb{C}^\times \) replaced by \( k^\times \) shows \( H^2(G,F_p) \cong H^2(G,k^\times) \otimes \mathbb{Z}(p) \).

(h) We refer to [Kar87] 2.1.5 for a proof.

\[ \square \]

**Lemma 5.2.** Let \( C \) be a saturated fusion system and let \( F: [S(C)] \to \text{Ab} \) be a covariant functor. The cohomology groups \( H^n(C,F) \), and thus the derived functors of \( \text{lim} F \), can be computed via the cochain complex \( C^*(F) \) defined as follows:

\[
C^n(F) = \bigoplus_{|\sigma|=n} F([\sigma]),
\]

whose elements are viewed as functions \( \alpha \) from isomorphism classes of chains of length \( n \), and where \( |\sigma| \) denotes the length of \( \sigma \). The coboundary map \( \delta^n: C^n(F) \to C^{n+1}(F) \) is defined by

\[
\delta^n(\alpha)([\sigma]) = \sum_{i=0}^n (-1)^i F(\iota_{[\sigma(i)]}[\sigma])(\alpha([\sigma(i)])),
\]

where \( \sigma(i) \) denotes the chain \( \sigma \) with its \( i \)th term removed, and \( \iota_{[\sigma(i)]}[\sigma] \) denotes the unique morphism from \( [\sigma(i)] \) to \( [\sigma] \).

**Proof.** This is [Lin05 Proposition 3.2], applied as in [Par10] Lemma 3.1

\[ \square \]

The purpose of the following, highly specialized lemma is to orient the reader to the way in which Lemma 5.2 will be used later in the proof of Theorem 5.5.

**Lemma 5.3.** Let \( C \) be a saturated fusion system, and let \( F: [S(C_{cr})] \to \text{Ab} \) be a covariant functor. Then \( \text{lim}_{[S(C_{cr})]} F = 0 \) under either of the following conditions.
Lemma 5.4. Fix a subgroup $P \in \mathcal{C}^{cr}$.

(a) $F([X]) = 0$ for all subgroups $X \in \mathcal{C}^{cr}$;

(b) $F$ is zero on all but two distinct chains $[X_1]$ and $[X_2]$ of length zero, and there exists a subgroup $Y \in \mathcal{C}^{cr}$ such that

(i) $X_1 < X_2 > Y$,

(ii) the maps $F([X_1]) \rightarrow F([X_1 < X_2]) \leftarrow F([X_2])$ are injective with the same image, and

(iii) $F([X_2]) \rightarrow F([Y < X_2])$ is injective.

Proof. We view $\lim_{\mathcal{S}(\mathcal{C}^{cr})} F$ as the degree 0 cohomology of the functor $F$. As such it can be computed by using the cochain complex of Lemma 5.2. The coboundary map $d: C^0(F) \rightarrow C^1(F)$ on 0-cochains is obtained by extending linearly from

$$\delta^0(\alpha)([X < Y]) = F(t_{\mathcal{S}[X < Y]})(\alpha([Y])) - F(t_{\mathcal{S}[X < Y]})(\alpha([X])).$$

With this in mind, the two parts of the lemma are simply ways of saying that the kernel of $\delta^0$, and thus $\lim_{\mathcal{S}(\mathcal{C}^{cr})} F$, is 0. This is trivial in the case of part (a). The assumption in (b) implies that $C^0(F) = F([X_1]) \oplus F([X_2])$ and then (ii) and (iii) ensure that the composite

$$C^0(F) \xrightarrow{\delta^0} C^1(F) \xrightarrow{\text{proj}} F([X_1 < X_2]) \oplus F([Y < X_2])$$

is injective. \hfill \Box

For the remainder of this section, we let $\mathcal{D}$ be one of $\mathcal{F}$, $\mathcal{H}$, or $\mathcal{K}$.

Lemma 5.4. Fix a subgroup $P \in \mathcal{D}^{cr}$. Then one of the following holds.

(a) $H^2(\text{Out}_\mathcal{D}(P), k^\times) = 0$, or

(b) $l = 0$, $H^2(\text{Out}_\mathcal{D}(P), k^\times) \cong H^2(\text{Out}_\mathcal{D}(P), \mathbb{F}_3) \cong C_3$, and either

(i) $P = QR$, or

(ii) $P = Q$ and $\mathcal{D} = \mathcal{H}$, or

(iii) $P = R$ and $\mathcal{D} = \mathcal{H}$ or $\mathcal{F}$.

Proof. We first prove the lemma for $l > 0$. Let $G$ be one of the groups in Tables 2, 3 and 4. Notice first that $G$ is $p$-perfect for all odd primes $p$. Moreover $G$ has cyclic Sylow $p$-subgroups for all $p \geq 5$, so by Lemma 5.1(d) we are concerned only with the prime 3, where we may assume that $G$ has 3-rank at least 2. Now Lemma 5.1(e) shows that $S_6$, $S_7$, $GL_4(2)$, $S_3 \times S_3$, and $S_3 \wr C_2$ all have trivial degree two $\mathbb{F}_3$-cohomology and hence trivial $k^\times$-cohomology by Lemma 5.1(c). Thus, after applying 5.1(f), we are left with $G = S_3 \wr S_3$. In this case one can apply the Lyndon-Hochschild-Serre spectral sequence with respect to the base $B$ of the wreath product. The relevant parts of the $E_2$-page are

- $H^0(S_3, H^2(B, \mathbb{F}_3)) = 0$ (the coefficients are 0);
- $H^1(S_3, H^1(B, \mathbb{F}_3)) = 0$ (since the base is 3-perfect); and
- $H^2(S_3, H^0(B, \mathbb{F}_3)) = 0$ (trivial invariants).

Hence, $H^2(G, \mathbb{F}_3) = 0$ and the result now follows from Lemma 5.1(c).

We now turn to the case $l = 0$. By inspection of Table 1 either it was shown in the previous case that $H^2(\text{Out}_\mathcal{D}(P), k^\times) = 0$, or else the subgroup $P$ is listed in (i)-(iii) of the lemma. We go through these three cases turn.

Case 1. $P = QR$ and $\text{Out}_\mathcal{D}(P) \cong (C_3 \times C_3)^{-1} (d)$: From Lemma 5.1 the exponent of $H^2(\text{Out}_\mathcal{D}(P), k^\times)$ is 3. Hence, by Lemma 5.1(e),(g) we see that $H^2(\text{Out}_\mathcal{D}(P), k^\times) \cong C_3$ in this case.
Case 2. $P = Q$: Suppose first that $D = \mathcal{H}$. Then $\text{Out}_D(P) \cong C_3^3 \times (C_2 \times C_2)$ with one factor inverting $C_3^3$ and the other swapping the first two $C_3$ factors. Now
\begin{equation}
H^2(C_3^3, \mathbb{F}_3) = \langle x, y, z, x_1 x_2, x_1 x_3, x_2 x_3 \rangle_{\mathbb{F}_3},
\end{equation}
where $x$, $y$, and $z$ are polynomial generators and the $x_i$ are exterior generators. Further, $H^2(\text{Out}_D(P), \mathbb{F}_3)$ are the invariants under $(d, \tau)$ here. We compute directly that the invariants are spanned by $x_1 x_3 + x_2 x_3$, and so have dimension 1. So by Lemma 5.1(a,g), $H^2(\text{Out}_D(P), k^\times) = C_3$.

Now suppose that $D = K$ or $\mathcal{F}$. Then $G = \text{Out}_D(P) \cong C_3^3 \times (\langle d \rangle \times S_3)$ with $d$ inverting and we have
\[ H^2(G, \mathbb{F}_3) = H^2(C_3 \wr C_3, \mathbb{F}_3)^{(d, \tau)}, \]

since a Sylow 3-subgroup is normal in $G$. Let $W \cong C_3 \wr C_3$ be the Sylow 3-subgroup of $G$, and write $W_0$ for the base subgroup of $W$. By a result of Nakaoka [Nak61, Theorem 3.3], we have
\[ H^2(C_3 \wr C_3, \mathbb{F}_3) = H^0(C_3, H^2(W_0, \mathbb{F}_3)) \oplus H^1(C_3, H^1(W_0, \mathbb{F}_3)) \oplus H^2(C_3, H^0(W_0, \mathbb{F}_3)) \]
The middle term above vanishes: as $H^1(W_0, \mathbb{F}_3) \cong W_0$ as a $W/W_0$-module, we have that $H^1(C_3, H^1(W_0, \mathbb{F}_3)) \cong H^1(C_3, W_0) = 0$, with the last equality since there is exactly one $W_0$-conjugacy class of complements to $W_0$ in $W$ (cf. [Ben98a, Proposition 3.7.2]). Hence,
\[ H^2(C_3 \wr C_3) = H^2(W_0, \mathbb{F}_3)^{C_3} \oplus H^2(C_3, \mathbb{F}_3). \]
With notation as in (5.1), the first summand is spanned by $x + y + z$ and $x_1 x_2 + x_2 x_3 + x_3 x_1$, both being negated by the action of $d \tau$. Similarly, the second summand is also negated by $d \tau$. Hence, $H^2(G, \mathbb{F}_3) = 0$, and we conclude that $H^2(G, k^\times) = 0$ by Lemma 5.1(c).

Case 3. $P = R$: Then $D = \mathcal{H}$ or $\mathcal{F}$, and $\text{Out}_D(P) \cong A_7$. The odd part of the Schur multiplier is well-known to be $C_3$. Alternatively, apply Lemma 5.1(h) to see that the exponent of the odd part of the Schur multiplier is 3, and then use Lemma 5.1(e,g).

\begin{theorem}
We have
\[ \lim_{[S(\mathcal{F}^\sigma)]} \mathcal{A}_F^2 = 0. \]
\end{theorem}

\begin{proof}
Let $(\mathcal{F}, \alpha)$ be a Külshammer-Puig pair. When $l > 0$, all minimal elements of the poset $[S(\mathcal{F}^\sigma)]$, namely the chains $\sigma = (R)$ of length one, have $\alpha_{[\sigma]} = 0$ by Lemma 5.4. Thus, the theorem holds in this case by Lemma 5.3(a).

It remains to consider the case $l = 0$. Then $H^2(\text{Out}_\mathcal{F}(P), k^\times)$ is nonzero (of order 3) if and only if $P = R$ or $QR$. Consider the chains $\sigma := (R < QR)$ and $\tau := (Q < QR)$. All three subgroups $R$, $Q$, and $QR$ are weakly $\mathcal{F}$-closed by Lemma 3.7, hence $\text{Aut}_\mathcal{F}(\sigma) = \text{Aut}_\mathcal{F}(QR) = \text{Aut}_\mathcal{F}(\tau)$ and the induced map on $\mathcal{A}^2$ is the identity in each of these cases. We next prove that the induced map
\[ H^2(\text{Aut}_\mathcal{F}(R), k^\times) \to H^2(\text{Aut}_\mathcal{F}(\sigma), k^\times) \]
is an isomorphism, hence has the same image as the induced map from $H^2(\text{Aut}_\mathcal{F}(QR), k^\times)$. Once this is done, Lemma 5.3(b) then yields that $\lim_{[S(\mathcal{F}^\sigma)]} \mathcal{A}_F^2 = 0$. Recall that $Z(R) = Z(QR) = Z(S)$ is of order 2, and that $\text{Out}_\mathcal{F}(R) = \text{Out}_\mathcal{H}(R) \cong A_7$ while $\text{Out}_\mathcal{F}(QR) = \text{Out}_\mathcal{H}(QR) \cong (C_3 \times C_3)^{-1} \cong C_2$. Therefore we may set $H = \text{Spin}_7(3)$ and work in $H = \mathcal{F}_S(H)$. In $\overline{H} = H/Z \cong \Omega_7(3)$, the subgroup $\overline{R}$ is elementary abelian of order $2^6$ and is the subgroup of $S$ fixing an orthogonal decomposition of the underlying vector space into seven one-dimensional subspaces, each of which is generated by a vector of square norm. Also, $\text{Out}_\mathcal{H}(\overline{R}) \cong A_7$ acts by permuting these subspaces. The subgroup
$QR$ contains $R$ as a normal subgroup with index 4, and the image of $QR$ in $\text{Out}_H(R)$ is a four-subgroup of $A_7$ moving four points (see [COS08 Section 2]). The restriction map $\text{Aut}_\mathcal{F}(R < QR) \to \text{Aut}_\mathcal{F}(R)$ is therefore injective and identifies modulo $\text{Inn}(R)$ with $N_{A_7}(\langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle)$, up to conjugacy in $A_7$. Since this normalizer contains a Sylow $3$-subgroup of $A_7$, we conclude that the map $H^2(\text{Aut}_\mathcal{F}(R), \mathbb{F}_3) \to H^2(\text{Aut}_\mathcal{F}(R < QR), \mathbb{F}_3)$ is injective. Thus, our claim follows from 5.1(g). This completes the proof in this case.

**Proof of Theorem 1.1.** By [LO02] there exists a centric linking system associated to $\mathcal{F}$. Thus [Lib11 Theorem 1.2] implies that

$$\lim_{[S(\mathcal{F}^e)]} A^2_{\mathcal{F}} \simeq \lim_{[S(\mathcal{F}^e)]} A^2_{\mathcal{F}^e}$$

and the result now follows from Theorem 5.5. □

**Appendix: Hasse diagrams**

**Figure 1.** Hasse diagram for $[\text{Sol}(q)^{\text{cr}}]$, $q \equiv \pm 3 \pmod{8}$

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Hasse diagram for $[\text{Sol}(q)^{\text{cr}}]$, $q \equiv \pm 3 \pmod{8}$}
\end{figure}
\end{center}

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Figure 2. Hasse diagram for $[\text{Sol}(q)^{{\tau}}]$, $q \equiv \pm 7 \pmod{16}$, i.e. for $l = 1$