RINGS WITH EACH RIGHT IDEAL AUTOMORPHISM-INVARIANT

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Abstract. In this paper, we study rings having the property that every right ideal is automorphism-invariant. Such rings are called right $a$-rings. It is shown that (1) a right $a$-ring is a direct sum of a square-full semisimple artinian ring and a right square-free ring, (2) a ring $R$ is semisimple artinian if and only if the matrix ring $M_n(R)$ for some $n > 1$ is a right $a$-ring, (3) every right $a$-ring is stably-finite, (4) a right $a$-ring is von Neumann regular if and only if it is semiprime, and (5) a prime right $a$-ring is simple artinian. We also describe the structure of an indecomposable right artinian right non-singular right $a$-ring as a triangular matrix ring of certain block matrices.

1. Introduction

The study of rings characterized by homological properties of their one-sided ideals has been an active area of research. Rings for which every right ideal is quasi-injective (known as right $q$-rings) were introduced by Jain, Mohamed and Singh in [22] and have been studied in a number of other papers ([3], [4], [5], [16]-[26], [29] and [30]) by Beidar, Byrd, Hill, Ivanov, Koehler and Mohamed. In [23] Jain, Singh and Srivastava studied rings whose each right ideal is a finite direct sum of quasi-injective right ideals and called such rings right $\Sigma$-$q$ rings. Jain, López-Permouth and Syed in [21] studied rings with each right ideal quasi-continuous and in [6] Clark and Huynh studied rings with each right ideal, a direct sum of quasi-continuous right ideals.

Recall that a module $M$ is called quasi-injective if $M$ is invariant under any endomorphism of its injective envelope; equivalently, any homomorphism from a submodule of $M$ to $M$ extends to an endomorphism of $M$. As a natural generalization of these modules Dickson and Fuller initiated study of modules which are invariant under any automorphism of their injective envelope [7]. These modules have been recently named as automorphism-invariant modules by Lee and...
Zhou in [28]. In [9] Er, Singh and Srivastava have shown that a module \( M \) is automorphism-invariant if and only if any monomorphism from a submodule of \( M \) to \( M \) extends to an endomorphism of \( M \). And in [13] Guil Asensio and Srivastava have shown that automorphism-invariant modules satisfy the full exchange property and these modules also provide a new class of clean modules. The decomposition of automorphism-invariant modules has been described in [9]. If \( M \) is an automorphism-invariant module, then \( M \) has a decomposition \( M = A \oplus B \) where \( A \) is quasi-injective and \( B \) is square-free. Recall that a module \( M \) is called \textit{square-free} if \( M \) does not contain a nonzero submodule \( N \) isomorphic to \( X \oplus X \) for some module \( X \). See [1], [12], [13], [15], [34] and [35] for more details on automorphism-invariant modules.

Rings all of whose right ideals are automorphism-invariant are called \textit{right a-rings} ([35]). Since every quasi-injective module is automorphism-invariant, the family of right \( a \)-rings includes right \( q \)-rings. The goal of this paper is to study these right \( a \)-rings. We extend the results in [22] for this new class of rings and show that

1. A right \( a \)-ring is a direct sum of a square full semisimple artinian ring and a right square-free ring (Theorem 3.4);

2. A ring \( R \) is semi-simple artinian if and only if the matrix ring \( M_n(R) \) for some \( n > 1 \) is an \( a \)-ring (Theorem 3.6);

3. If \( R \) is a right \( a \)-ring, then \( R \) is stably-finite, that is, every matrix ring over \( R \) is directly-finite (Theorem 4.3).

4. A right \( a \)-ring is von Neumann regular if and only if it is semiprime (Theorem 4.2), and a prime right \( a \)-ring is simple artinian (Theorem 4.7).

We also characterize indecomposable non-local right CS right \( a \)-rings. It is shown that

5. Let \( R \) be an indecomposable, non-local ring. Then \( R \) is a right \( q \)-ring if and only if \( R \) is right CS and a right \( a \)-ring (Theorem 4.9).

Let \( \Delta \) be a right \( q \)-ring with an essential maximal right ideal \( P \) such that \( \Delta/P \) is an injective right \( \Delta \)-module. In a right \( q \)-ring, every essential right ideal is two-sided by [22, Theorem 2.3]. Hence \( \Delta/P \) is a skew field. Let \( n \) be an integer with \( n \geq 1 \), let \( D_1, D_2, \ldots, D_n \) be skew fields and \( \Delta \) be a right \( q \)-ring, all of whose idempotents are central and the right \( \Delta \)-module \( \Delta/P \) is not embedable into \( \Delta \Delta \). Next, let \( V_i \) be \( D_i-D_{i+1} \)-bimodule such that

\[
\dim(\{V_i\}_{D_{i+1}}) = 1
\]
for all $i = 1, 2, \ldots, n-1$, and let $V_n$ be a $D_n$-$\Delta$-bimodule such that $V_nP = 0$ and $\dim(\{V_n\}_{\Delta/P}) = 1$.

We denote by $G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$, the ring of $(n+1) \times (n+1)$ matrices of the form

$$
G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n) := \begin{pmatrix}
D_1 & V_1 & & \\
& D_2 & V_2 & \\
& & D_3 & V_3 \\
& & & \ddots \\
& & & & D_n & V_n \\
& & & & & \Delta
\end{pmatrix}.
$$

Consider the ring $G(D, \Delta, V)$. In [4, Theorem 4.1], it is shown that $G(D, \Delta, V)$ is a right $q$-ring. Note that if we consider transpose then it is a left $q$-ring. In the present paper, we obtain that

1. $G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$ is a right $a$-ring all of whose idempotents are central, where $\Delta$ is a right $a$-ring, $\dim(D_i\{V_i\}) = \dim(\{V_i\}_{D_{i+1}}) = 1$ for all $i = 1, 2, \ldots, n-1$ and $\dim(D_n\{V_n\}) = \dim(\{V_n\}_{\Delta/P}) = 1$ (Theorem 5.2).

Finally, we finish our paper with a structure theorem for an indecomposable right artinian right non-singular right $a$-ring as a triangular matrix ring of certain block matrices.

Throughout this article all rings are associative rings with identity and all modules are right unital unless stated otherwise. For a submodule $N$ of $M$, we use $N \leq M$ ($N < M$) to mean that $N$ is a submodule of $M$ (respectively, proper submodule), and we write $N \leq^e M$ and $N \leq^g M$ to indicate that $N$ is an essential submodule of $M$ and $N$ is a direct summand of $M$, respectively. We denote by $Soc(M)$ and $E(M)$, the socle and the injective envelope of $M$, respectively. For any term not defined here the reader is referred to [2] and [31].

2. An Example

As already mentioned any right $q$-ring is a right $a$-ring. Recall that right $q$-rings are precisely those right self-injective rings for which every essential right ideal is a two sided ideal [22]. So, in particular, any commutative self-injective ring is a $q$-ring and hence an $a$-ring. Now we would like to present some examples of right $a$-rings that are not right $q$-rings. First, we have the following useful observation.

**Lemma 2.1.** A commutative ring is an $a$-ring if and only if it is an automorphism-invariant ring.
\textit{Proof.} Let $R$ be a commutative automorphism-invariant ring and $I$ be an ideal of $R$. There exists an ideal $U$ of $R$ such that $I \oplus U$ is essential in $R$. Then $E(R) = E(I \oplus U)$. Let $\varphi$ be an automorphism of $E(R)$. Clearly, $\varphi(1) \in R$. Now, for all $x \in I \oplus U$, we have $\varphi(x) = \varphi(1)x \in I \oplus U$. So $\varphi(I \oplus U) \leq I \oplus U$ which implies that $I \oplus U$ is an automorphism-invariant module. Since direct summand of an automorphism-invariant module is automorphism-invariant, it follows that $I$ is automorphism-invariant. This shows that $R$ is an $a$-ring. The converse is obvious. \hfill $\Box$

In view of the above, we have the following example of $a$-ring which is not a $q$-ring.

\textbf{Example 2.2.} Consider the ring $R$ consisting of all eventually constant sequences of elements from $F_2$ (see [9, Example 9]). Clearly, $R$ is a commutative automorphism-invariant ring as the only automorphism of its injective envelope is the identity automorphism. Hence $R$ is an $a$-ring by the above lemma. But $R$ is not a $q$-ring because $R$ is not self-injective.

3. Some characterizations of $a$-rings

In this section we will prove some characterizations for right $a$-rings. These equivalent characterizations will be easier to use.

\textbf{Proposition 3.1.} The following conditions are equivalent for a ring $R$:

(1) $R$ is a right $a$-ring.

(2) Every essential right ideal of $R$ is automorphism-invariant.

(3) $R$ is right automorphism-invariant and every essential right ideal of $R$ is a left $T$-module, where $T$ is a subring of $R$ generated by its unit elements.

\textit{Proof.} (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) By the hypothesis, $R$ is a right automorphism-invariant ring. Let $I$ be an essential right ideal of $R$. Then $E(I) = E(R)$. Let $T$ be a subring of $R$ generated by its units. Then $T$ is a subring of $\text{End}(E(R))$, and so $TI = I$.

(3) $\Rightarrow$ (1) Let $I$ be an essential right ideal of $R$. Then $E(I) = E(R)$. If $\varphi$ of $E(R)$ is an automorphism, then $\varphi(R) = R$ which implies that $\varphi(1)$ is a unit of $R$. By (3), we have $\varphi(1)I \leq I$ and so $\varphi(I) \leq I$. \hfill $\Box$

\textbf{Corollary 3.2.} Let $R = S \times T$ be a product of rings. Then $R$ is a right $a$-ring if and only if $S$ and $T$ are $a$-rings.
Let $M$ be a right module over a ring $R$. The singular submodule $Z(M)$ of $M$ is defined as $Z(M) = \{m \in M : \text{ann}_r(m) \text{ is an essential right ideal of } R\}$. The singular submodule of the right $R$-module $R_R$ is called the (right) singular ideal of the ring $R$ and denoted by $Z(R_R)$, that is, $Z(R_R) = \{x \in R : r_R(x) \cap H \neq 0 \text{ for every nonzero right ideal } H \text{ of } R\}$. It is well known that $Z(R_R)$ is indeed an ideal of $R$.

**Lemma 3.3.** Let $R$ be a right $a$-ring and $A, B$ right ideals of $R$ with $A \cap B = 0$ and $A \simeq B$. Then

1. $A$ and $B$ are semisimple and injective.
2. The right ideals $A$ and $B$ are nonsingular.

**Proof.** (1) Let $A$ and $B$ be right ideals of a right $a$-ring $R$ with $A \cap B = 0$ and $A \simeq B$. Let $D$ be a complement of $A \oplus B$ in $R_R$. Then $(A \oplus B) \oplus D \leq^e R_R$. It follows that $E((A \oplus B) \oplus D) \leq^e E(R_R)$. On the other hand, $E((A \oplus B) \oplus D)$ is a direct summand of $E(R_R)$ and so $E((A \oplus B) \oplus D) = E(R_R)$. We have $E((A \oplus B) \oplus D) = E(A) \oplus E(B) \oplus E(D)$. Thus $E(R_R) = E(A) \oplus E(B) \oplus E(D)$ which means that we have a decomposition $E(R_R) = E(A) \oplus E(B) \oplus C$ for some $C \leq E(R_R)$. Note that $E(A) \simeq E(B)$ and $R$ is right automorphism-invariant. By [28, Lemma 7], we get

$$R_R = (R \cap E(A)) \oplus (R \cap E(B)) \oplus (R \cap C).$$

We also have $B \cap (R \cap E(A)) = 0$ and $A \cap [(R \cap E(B)) \oplus (R \cap C)] = 0$. Since $R$ is a right $a$-ring, the modules $B \oplus [R \cap E(A)]$ and $A \oplus [(R \cap E(B)) \oplus (R \cap C)]$ are automorphism-invariant. By [28, Theorem 5], $B$ is $[R \cap E(A)]$-injective and $A$ is $[(R \cap E(B)) \oplus (R \cap C)]$-injective. Note that $A \simeq B$. Thus $A$ is $R$-injective (injective). Let $\varphi : A \to B$ be an isomorphism and $U$ be a submodule of $A$. Clearly, $U \simeq \varphi(U)$. Let $V = \varphi(U)$. Then $U \cap V = 0$ and $U \simeq V$. By a similar argument as above, we have $U$ and $V$ are injective modules. It follows that $U$ is a direct summand of $A$. Thus both $A$ and $B$ are semisimple modules.

(2) Let $a$ be an arbitrary element of $Z(A)$. Then $aR$ is an injective module since it is a direct summand of $A$. It follows that $aR = eR$ for some $e^2 = e \in R$. Therefore $e \in Z(A)$ and so $e = 0$. Thus $a = 0$ which shows $Z(A) = Z(B) = 0$.  

Recall that two modules $M$ and $N$ are said to be orthogonal if no submodule of $M$ is isomorphic to a submodule of $N$. A module $M$ is said to be a square module if there exists a right module $N$ such that $M \simeq N^2$. A submodule $N$ of a module $M$ is called square-root in $M$ if $N^2$ can be embedded in $M$. A module $M$ is called square-free if $M$ contains no non-zero square roots and $M$ is called square-full if every submodule of $M$ contains a non-zero square root in $M$. 

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As a consequence of the above lemma, we are now ready to prove a useful decomposition theorem for any right $a$-ring.

**Theorem 3.4.** A right $a$-ring is a direct sum of a square-full semisimple artinian ring and a right square-free ring.

**Proof.** By [9, Theorem 3], there exists a decomposition $R = A \oplus B \oplus C$ where $A \simeq B$ and the module $C$ is square-free which is orthogonal to $A \oplus B$. Let $X := A \oplus B$ and $Y := C$. Now $X$ is square-full. In fact, let $U$ be a non-zero arbitrary submodule of $X$. There exist either non-zero submodules $U_1$ of $U$ and $V_1$ of $A$ such that $U_1 \simeq V_1$ or non-zero submodules $U_2$ of $U$ and $V_2$ of $B$ such that $U_2 \simeq V_2$. It follows that $U_1^2$ or $U_2^2$ can be embedded in $X$. That means $U$ contains a square root in $X$.

By Lemma 3.3, $A$ and $B$ are injective semisimple modules and so $X$ is injective and semisimple. Next we show that $X$ and $Y$ are ideals of $R$. Since $X$ is semisimple which is orthogonal to $Y$, we have $\text{Hom}(X,Y) = 0$. Assume that $\varphi: Y \to X$ is a non-zero homomorphism. Then $Y/\text{Ker}(\varphi) \simeq \text{Im}(\varphi)$ is projective (since $\text{Im}(\varphi)$ is a direct summand of $X$). It follows that there exists non-zero submodule $K$ of $Y$ such that $\text{Ker}(\varphi) \cap K = 0$. So $K \simeq \varphi(K)$, a contradiction with orthogonality of $X$ and $Y$. Therefore $\text{Hom}(Y,X) = 0$.

Thus $R = X \oplus Y$, where $X$ is a square-full semisimple artinian ring and $Y$ is a right square-free ring. $\square$

**Corollary 3.5.** An indecomposable ring $R$ containing a square is a right $a$-ring if and only if $R$ is simple artinian.

By $M_n(R)$, we denote the ring of $n \times n$ matrices over the ring $R$.

**Theorem 3.6.** Let $n > 1$ be an integer. The following conditions are equivalent for a ring $R$:

1. $M_n(R)$ is a right $q$-ring for every $n > 1$.
2. $M_n(R)$ is a right $q$-ring for some $n > 1$.
3. $M_n(R)$ is a right $a$-ring for every $n > 1$.
4. $M_n(R)$ is a right $a$-ring for some $n > 1$.
5. $R$ is semisimple artinian.

**Proof.** The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) Assume that $R$ is not semi-simple artinian. Then there exists an essential right ideal, say $B$, of $R$ such that $B \neq R$. Define $E := \{ \sum a_{ij}e_{ij} : a_{1j} \in B, 1 \leq j \leq n \text{ and } e_{ij} \in R, 1 \leq i, j \leq n \}$ where $e_{ij}$ ($1 \leq i, j \leq n$) are the units of $M_n(R)$. Then clearly $E$ is an essential right ideal of $M_n(R)$. Consider the unit
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This is a contradiction by Proposition 3.1.

(5) ⇒ (1) This is obvious.

The following example shows that there exists automorphism-invariant rings which are not right \(a\)-rings.

Example 3.7. Let \(R = \mathbb{Z}_{p^n}\), where \(p\) is a prime. It is well known that \(R\) is self-injective. By [37, Theorem 8.3], \(M_m(R)\) is right self-injective (for all \(m > 1\)). Thus, for instance, \(M_m(\mathbb{Z}_{p^2})\) is a right automorphism-invariant ring but it is not a right \(a\)-ring for any \(m > 1\).

4. Special classes of right \(a\)-rings

In this section, we will consider some special classes of rings, for example, simple, semiprime, prime and CS and characterize as to when these rings are right \(a\)-rings. We begin this section with a simple observation.

Lemma 4.1. Let \(A\) and \(B\) be right ideals of a right \(a\)-ring \(R\) with \(A \cap B = 0\). Then the following conditions hold:

1. If \(\varphi : A \rightarrow B\) is a nonzero homomorphism, then
   (i) \(\varphi(A)\) is a semisimple module.
   (ii) \(\varphi(A)\) is simple if \(B\) is uniform.
2. If \(e\) is a non-trivial idempotent of \(R\) such that \(eR(1-e) \neq 0\), then \(\text{Soc}(eR) \neq 0\).

Proof. (1)(i). Let \(U\) be an arbitrary essential submodule of \(B\). Then \(E(U) = E(B)\) and \(U \oplus A\) is automorphism-invariant. It follows that \(U\) is \(A\)-injective. On the other hand, there exists a homomorphism \(\bar{\varphi} : E(A) \rightarrow E(B)\) such that \(\bar{\varphi}|_A = \varphi\). It follows that \(\bar{\varphi}(A) \leq U\) and so \(\varphi(A) \leq U\). This shows that \(\varphi(A) \leq \text{Soc}(B)\).
Recall that a ring \( R \) is called von Neumann regular if for every \( a \in R \), there exists some \( b \in R \) such that \( a = aba \). A ring \( R \) is said to be prime if the product of any two nonzero ideals of \( R \) is nonzero and a ring \( R \) is called semiprime if it has no nonzero nilpotent ideals.

**Theorem 4.2.** A right \( a \)-ring is von Neumann regular if and only if it is semiprime.

**Proof.** Let \( R \) be a right \( a \)-ring. By [13, Proposition 1], \( J(R) = Z(R_R) \).

\((\Rightarrow)\) Since \( R \) is von Neumann regular, it is well known that every ideal of \( R \) is idempotent. Hence \( R \) is semiprime.

\((\Leftarrow)\) Assume that \( R \) is semiprime. Since \( R \) is right automorphism-invariant, \( R/J(R) \) is von Neumann regular. Now we proceed to show that \( J(R) = 0 \). In fact, for any \( x \in J(R) \), there exists an essential right ideal \( E \) of \( R \) such that \( xE = 0 \). Since \( R \) is a right \( a \)-ring, \( uE \leq E \) for all units \( u \) in \( R \) by Lemma 3.1. It follows that \( (RxR)E \leq E \) and so \( (RxR)E \leq xE = 0 \), and so either \( xRxR \leq P \) or \( E \leq P \) for all prime ideal \( P \) of \( R \). Let \( \{P_i\}_{i \in I} \) and \( \{P_j\}_{j \in J} \) be families of all prime ideals of \( R \) such that \( xRxR \leq P_i \) for all \( i \in I \) and \( xRxR \leq P_j \) for all \( j \in J \). Taking \( X = \cap_{i \in I} P_i \) and \( Y = \cap_{j \in J} P_j \). Since \( R \) is semiprime, \( X \cap Y = 0 \). Moreover, we have \( E \leq Y \) and so \( y \leq e R_R \). If \( xRxR \neq 0 \), there exists \( r_1, r_2 \in \bar{R} \) such that \( xr_1xr_2 \neq 0 \). Then there is \( y \in \bar{R} \) such that \( xrx_1r_2y \neq 0 \) and \( xrx_1r_2y \in Y \), a contradiction. Thus \( xRxR = 0 \). Furthermore, as \( R \) is semiprime, we have \( x = 0 \). This completes the proof. \( \square \)

Recall that a ring \( R \) is called directly-finite if \( xy = 1 \) implies \( yx = 1 \) for all \( x, y \in \bar{R} \). Assume that \( R \) is a right \( a \)-ring. By Theorem 3.1, we have a decomposition \( R = S \times T \), where \( S \) is semi-simple artinian and \( T \) is square-free. Since \( S \) and \( T \) are directly-finite rings, one infers that the ring \( R \) is also directly-finite. Next, we will see that a right \( a \)-ring is not only directly-finite but it is stably-finite. If for a ring \( R \), every matrix ring \( M_n(R) \) is directly finite then \( R \) is called a stably-finite ring.
A ring $R$ is called right quasi-duo (left quasi-duo) if every maximal right ideal (every maximal left ideal) is two-sided. It is still an open problem whether quasi-duo rings are left-right symmetric or not.

**Theorem 4.3.** Every right $a$-ring is stably-finite.

**Proof.** Let $R$ be a right $a$-ring. Then $R = S \times T$, where $S$ is semi-simple artinian and $T$ is square-free. By [15, Theorem 15], $T$ is a right quasi-duo ring. Then $M_n(R) = M_n(S \oplus T)$. Thus $M_n(R) \cong M_n(S) \oplus M_n(T)$. Clearly, $M_n(S)$ is directly finite. Now we proceed to show that $M_n(T)$ is directly finite. Let $\{M_i\}$ be the set of maximal right ideals of the quasi-duo ring $T$. Then each $M_i$ is a two-sided ideal and $J(T) = \cap M_i$. Clearly, each $T/M_i$ is a division ring. Thus $M_n(T)/M_n(M_i) \cong M_n(T/M_i)$ is a simple artinian ring which is clearly directly finite. Consider the natural ring homomorphism $\varphi : M_n(T) \to \prod_i M_n(T/M_i)$. We have $\ker(\varphi) = M_n(J(T)) = J(M_n(T))$. Since each $M_n(T/M_i)$ is directly finite, $\prod_i M_n(T/M_i)$ is directly finite and consequently, $M_n(T)/J(M_n(T))$ is directly finite being a subring of a directly finite ring. Hence $M_n(T)$ is directly finite. Thus $M_n(R)$ is directly finite and therefore $R$ is stably-finite. □

A ring $R$ is called unit-regular if, for every element $x \in R$, there exists a unit $u \in R$ such that $x = xux$. We can now have the following result.

**Corollary 4.4.** Every von Neumann regular right $a$-ring is unit-regular.

**Corollary 4.5.** The ring of linear transformations $R := \text{End}(V_D)$ of a vector space $V$ over a division ring $D$ is a right $a$-ring if and only if the vector space is finite-dimensional.

**Proof.** If $V$ is an infinite-dimensional vector space over $D$ then End$(V_D)$ is not unit-regular. So the result follows from above corollary. □

A ring $R$ is said to be strongly regular if for every $a \in R$, there exists some $b \in R$ such that $a = a^2b$.

**Proposition 4.6.** Let $R$ be a semi-prime right $a$-ring with zero socle. Then $R$ is strongly regular.

**Proof.** Assume that $R$ is a semi-prime right $a$-ring. Clearly, $R$ is von Neumann regular. Let $e$ be an idempotent in $R$. Suppose $(1 - e)Re \neq 0$. Then $\text{Soc}((1 - e)R) \neq 0$, a contradiction. Hence $(1 - e)Re = 0$ and this shows that $e$ is a central idempotent (see [11, Lemma 2.33]). Because every idempotent of $R$ is central, $R$ is strongly regular. □
Theorem 4.7. Let R be a prime ring. Then R is a right a-ring if and only if R is a simple artinian ring.

Proof. Assume that R is a prime right a-ring. In view of Theorem 3.4, we obtain that either R is a simple artinian ring or R is a square-free ring. So, it suffices to consider the case that R is a square-free prime right a-ring. By Theorem 1.2, R is a von Neumann regular ring. Since R is square-free, all idempotents of R are central and hence R is a strongly regular ring. Now as every prime strongly regular ring is a division ring, the result follows. □

In particular, from the above theorem it follows that every simple right a-ring is artinian.

A module M is said to satisfy:

CS-condition if every submodule of M is essential in a direct summand of M.

weak CS-condition if every semisimple submodule of M is essential in a direct summand of M.

C2-condition if every submodule of M isomorphic to a direct summand of M is itself a direct summand of M.

C3-condition if whenever $M_1$ and $M_2$ are direct summands of M and $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is a direct summand of M.

A module M is called a continuous module if it satisfies CS and C2 conditions; M is called a quasi-continuous module if it satisfies CS and C3 conditions (see [31]); and M is called a CS module (weak CS module) if it satisfies the CS (weak CS) condition (see [36]).

Next, we consider right weak CS right a-rings.

Proposition 4.8. Let R be a right weak CS right a-ring. If e is a primitive idempotent of R such that $eR(1-e) \neq 0$, then $eRe$ is a division ring and $eR(1-e)$ is the only proper R-submodule of eR.

Proof. By Lemma 4.1, $Soc(eR) \neq 0$. Since R is right automorphism-invariant, $R$ is right C2 by [9]. By [10, Theorem 1.4], $eR$ is also a weak CS module. Firstly, we show that $Soc(eR)$ is a simple module which is essential in $eR$. Since $eR$ is a weak CS module, $Soc(eR)$ is essential in a direct summand of $eR$. But $eR$ is
an indecomposable module which implies that $Soc(eR)$ is essential in $eR$. For any nonzero arbitrary element $a \in Soc(eR)$, we obtain that $aR$ is essential in $eR$ (because $eR$ is an indecomposable weak CS module). It follows that $Soc(eR) \leq aR$ and so $Soc(eR) = aR$. Thus $Soc(eR)$ is a simple module. Therefore $eR$ is uniform. Since a uniform automorphism-invariant module is quasi-injective, $eR$ is quasi-injective. Thus $eRe \cong \text{End}(eR)$ is a local ring, i.e. $e$ is a local idempotent of $R$.

Next we show that $eR(1-e)$ is the only proper submodule of $eR$. Since $eR(1-e) \neq 0$, one infers $eR(1-e) \subset Soc(eR)$ by Lemma 4.1. Hence $eR(1-e) = Soc(eR)(1-e)$.

We next show that $eJ(R)e$ is a submodule of $eR$. Since $R$ is right automorphism-invariant, $J(R) = Z(R_R)$ by [13, Proposition 1] and so $J(R)Soc(eR) = 0$. Now $(eJ(R)e)Soc(eR) = eJ(R)Soc(eR) = 0$ and so $(eJ(R)e)(eR(1-e)) = 0$. On the other hand, we have

$$eJ(R)eRe = eJ(R)e(Re + R(1-e)) = eJ(R)eRe \subset eJ(R)e.$$  

Hence $eJ(R)e$ is an $R$-submodule of $eR$. Since $Soc(eR)$ is simple, we have $eJ(R)e \cap Soc(eR) = 0$ or $Soc(eR) \leq eJ(R)e$. Suppose $Soc(eR) \leq eJ(R)e$. Then $eR(1-e) = Soc(eR)(1-e) \leq eJ(R)e(1-e) = 0$, a contradiction. It follows that $eJ(R)e \cap Soc(eR) = 0$. Thus $eJ(R)e = 0$.

Let $I$ be a proper submodule of $eR$. Since $eR$ is local, $I \leq eJ(R)$ and so $Ie = 0$. On the other hand, we have $I(1-e) \leq eR(1-e)$ which implies that $I \leq eR(1-e) = Soc(eR)$. Thus $I = 0$ or $I = Soc(eR)$. In particular, we have $Soc(eR)e = 0$. Therefore $eR(1-e) = Soc(eR)(1-e) = Soc(eR)$.  

As a consequence, we have the following.

**Theorem 4.9.** Let $R$ be an indecomposable, non-local ring. The following conditions are equivalent:

1. $R$ is a right $q$-ring.
2. $R$ is a right CS and $a$-ring.

**Proof.** This follows from previous proposition and [17, Theorem 3].  

## 5. Two structure theorems

In this section we would like to describe two structures of right $a$-rings. In the case of right $q$-rings, Byrd [5] and Ivanov ([17], [18]) gave a description of right
$q$-rings but their characterizations turned out to be not complete. Finally, the structure of right $q$-rings was completely described by Beidar et al in \cite{4}.

**Theorem 5.1.** (Beidar, Fong, Ke, Jain, \cite{4}) A right $q$-ring $R$ is isomorphic to a finite direct product of right $q$-rings of the following types:

1. Semisimple artinian ring.
2. $H(n; D; \text{id}_D)$ where $\text{id}_D$ is the identity automorphism on division ring $D$.
3. $G(n; \Delta; P)$ where $\Delta$ is a right $q$-ring whose all idempotents are central.
4. A right $q$-ring whose all idempotents are central.

Here

$$H(n; D; \alpha) = \begin{bmatrix}
D & V & 0 & 0 \\
0 & D & V & 0 \\
& & D & V \\
V(\alpha) & 0 & & D
\end{bmatrix},$$

where $V$ is one-dimensional both as a left $D$-space and a right $D$-space, $V(\alpha)$ is also a one-dimensional left $D$-space as well as a right $D$-space with right scalar multiplication twisted by an automorphism $\alpha$ of $D$, i.e., $vd = v \cdot \alpha(d)$ for all $v \in V, d \in D$, and

$$G_n(n; \Delta; P) := \begin{pmatrix}
D & V \\
D & V \\
& D & V \\
& & D & V \\
& & & \Delta
\end{pmatrix},$$

where $V$ is as above and $\Delta$ is a right $q$-ring with maximal essential right ideal $P$ and hence $D = \Delta/P$ is a division ring.

Now, using the above defined notations, we give the following descriptions the structure of right $a$-rings.

**Theorem 5.2.** Let $n \geq 1$ be an integer, $D_1, D_2, \ldots, D_n$ be division rings and $\Delta$ be a right $a$-ring with all idempotents central and an essential ideal, say $P$, such that $\Delta/P$ is a division ring and the right $\Delta$-module $\Delta/P$ is not embeddable into $\Delta_{\Delta}$. Next, let $V_i$ be a $D_i$-$D_{i+1}$-bimodule such that

$$\dim(D_i\{V_i\}) = \dim(\{V_i\}_{D_{i+1}}) = 1$$
for all \( i = 1, 2, \ldots, n - 1 \), and let \( V_n \) be a \( D_n - \Delta \)-bimodule such that \( V_n P = 0 \) and
\[
\dim(D_n \{ V_n \}) = \dim(\{ V_n \}_{\Delta/P}) = 1.
\]
Then \( R := G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n) \) is a right a-ring.

**Proof.** Let \( 1 \leq i \leq n + 1 \) and \( e_i \) be the matrix whose \((i, i)\)-entry is equal to 1 and all the other ones are equal to 0. It is easy to see that \( e_j R e_{j+1} \) are minimal right ideals of \( R \) for all \( j = 1, 2, \ldots, n \). Let \( K \) be a right ideal of the ring \( \Delta \) and let \( \hat{K} \) to be the set of all matrices whose \((n + 1, n + 1)\)-entries are from \( K \) and all the other ones are equal to 0. Given \( 1 \leq i \leq n \) and a right ideal \( K \) of \( \Delta \). By the proof of Proposition 2.16 of [3], we will use the following facts in the proofs below:

**Fact 5.3.** \( e_i R \) and \( \hat{K} \) are relatively injective. Also, \( e_i R e_{i+1} \) and \( \hat{K} \) are relatively injective.

**Fact 5.4.** \( \mathrm{Hom}(e_i R, \hat{K}) = 0 = \mathrm{Hom}(e_i R/\hat{K}, \hat{K}) \).

**Fact 5.5.** \( e_i R \) and \( e_j R \) are relatively injective for all \( j \neq i \). Also, \( e_i R e_{i+1} \) and \( e_j R \) are relatively injective for all \( j \neq i \).

Let \( U \) be an essential right ideal of \( R \). Then \( e_i R e_{i+1} \leq U \) for all \( i = 1, 2, \ldots, n \). Set \( W := \sum_{i=1}^{n} e_i R e_{i+1} \). Note that \( W \) is an ideal of \( R \) and \( W \leq U \). Since the factor ring \( R/W \) is isomorphic to the ring \((\oplus_{i=1}^{n} D_i) \oplus \Delta \) and \( U/W \) is a right ideal of \( R/W \), we conclude that there exists a partition \( I, J \) of the set \( \{1, 2, \ldots, n\} \) and a right ideal \( K \) of \( \Delta \) such that \( U = (\oplus_{i \in I} e_i R) \oplus (\oplus_{j \in J} e_j R_{j+1}) \oplus \hat{K} \).

Now we deduce the following useful conclusions.

(i) \( \oplus_{j \in J} e_j R_{j+1} \) is a semisimple right \( R \)-module and so \( \oplus_{j \in J} e_j R_{j+1} \) is quasi-injective.

(ii) \( \oplus_{i \in I} e_i R \) is a quasi-injective right \( R \)-module. In fact, by Fact 5.5, we only prove that each \( e_i R \) is a quasi-injective right \( R \)-module for all \( i \in I \). Note that \( e_i R e_{i+1} \) is only proper submodule of \( e_i R \). Let \( f : e_i R e_{i+1} \to e_i R \)
\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & V_i \\
0 & 0 & \cdots & 0 \\
& \cdots & 0 & 0 \\
& & \cdots & 0 \\
& & & \cdots & 0 \\
& & & & \cdots & 0 \\
\end{pmatrix}
\]
be an \( R \)-homomorphism. Note that \( e_i R e_{i+1} = \{ V_i \}_{D_i+1} \). Then
\[
f(e_i R e_{i+1}) = e_i R e_{i+1}.
\]
Since \( \dim(D_i \{ V_i \}) = \dim(\{ V_i \}_{D_i+1}) = 1 \), there exists \( v_i \in
V_i such that \( D_i v_i = v_i D_{i+1} \). Assume that
\[
f(v_i) = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
\ddots & v_i d_{i+1} \\
\vdots & \ddots & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & \ddots & 0 & 0
\end{pmatrix}
\]
for some \( d_{i+1} \in D_{i+1} \). There exists \( d_i \in D_i \) such that \( d_i v_i = v_i d_{i+1} \). We consider the \( R \)-homomorphism \( \bar{f} : e_i R \rightarrow e_i R \) defined as left multiplication by
\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\ddots & d_i \\
\vdots & \ddots & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & \ddots & 0 & 0
\end{pmatrix}
\]
then \( \bar{f} \) is an extension of \( f \). In case of \( e_n R \), it is similar.

(iii) \( \hat{K} = \hat{K}_1 \oplus \hat{K}_2 \), where \( \hat{K}_1 \) is a quasi-injective \( R \)-module and \( \hat{K}_2 \) is a square-free automorphism-invariant \( R \)-module. In fact, by Theorem 3.4 we have a decomposition \( \Delta = \Delta_1 \times \Delta_2 \), where \( \Delta_1 \) is semi-simple artinian and \( \Delta_2 \) is square-free. It follows that there exists a quasi-injective \( \Delta \)-module \( K_1 \) and a square-free \( \Delta \)-module \( K_2 \) such that \( K = K_1 \oplus K_2 \). Thus \( \hat{K} = \hat{K}_1 \oplus \hat{K}_2 \). Since \( e_{n+1} R (1 - e_{n+1}) = 0 \), we obtain that \( \hat{K}_1 \) is quasi-injective and \( \hat{K}_2 \) is square-free by [3] Lemma 2.3(6)]. Furthermore, by the hypothesis, \( \hat{K}_2 \) is automorphism-invariant.

Let \( X = \left( \bigoplus_{i \in I} e_i R \right) \oplus \left( \bigoplus_{j \in J} e_j R e_{j+1} \right) \oplus \hat{K}_1 \) and \( Y = \hat{K}_2 \). Then \( U = X \oplus Y \). By Facts 5.3, 5.4 and 5.5, \( X \) is quasi-injective, \( Y \) is automorphism-invariant square-free which is orthogonal to \( X \), and \( X \) and \( Y \) are relatively injective. By [34], \( U \) is automorphism-invariant. This shows that each essential right ideal of \( R \) is automorphism-invariant. Now, let \( A \) be any right ideal of \( R \). Let \( C \) be a complement of \( A \) in \( R \). Then \( A \oplus C \) is an essential right ideal of \( R \). Thus, as shown above, \( A \oplus C \) is automorphism-invariant and consequently, \( A \) is automorphism-invariant. This proves that \( R \) is a right \( a \)-ring.

We finish this paper by giving by another structure theorem for indecomposable right artinian right non-singular right \( a \)-ring as a triangular matrix ring of certain block matrices.
Theorem 5.6. Any indecomposable right artinian right nonsingular right weakly
CS right a-ring \( R \) is isomorphic to

\[
\begin{pmatrix}
M_{n_1}(e_1Re_1) & M_{n_1\times n_2}(e_1Re_2) & \cdots & M_{n_1\times n_k}(e_1Re_k) \\
0 & M_{n_2}(e_2Re_2) & \cdots & M_{n_2\times n_k}(e_2Re_k) \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & M_{n_k}(e_kRe_k)
\end{pmatrix},
\]

where \( e_iRe_i \) is a division ring, \( e_iRe_i \cong e_jRe_j \) for each \( 1 \leq i, j \leq k \) and \( n_1, \ldots, n_k \) are any positive integers. Furthermore, if \( e_iRe_j \neq 0 \), then

\[
\dim(e_iRe_i(e_iRe_j)) = 1 = \dim((e_iRe_j)_{e_jRe_j}).
\]

Proof. Let \( R \) be an indecomposable right artinian right nonsingular right weakly
CS right a-ring. We first show that \( eR \) is quasi-injective for any idempotent \( e \in R \). Since \( R \) is right artinian, we have \( \text{Soc}(eR) \neq 0 \). As \( R \) is right automorphism-invariant and right weak CS, \( eR \) is also a weak CS module. Therefore \( \text{Soc}(eR) \) is a simple module which is essential in \( eR \), and so \( eR \) is uniform. Therefore \( eR \) is quasi-injective. Now rest of the proof follows from Theorem 23 in [23]. For the
sake of completeness, we give the argument below.

Choose an independent family \( \mathcal{F} = \{e_iR : 1 \leq i \leq n \} \) of indecomposable right ideals such that \( R = \bigoplus_{i=1}^{n} e_iR \). After renumbering, we may write \( R = [e_1R] \oplus [e_2R] \oplus \cdots \oplus [e_kR] \), where for \( 1 \leq i \leq k \), \([e_iR]\) denotes the direct sum of those \( e_jR \) that are isomorphic to \( e_iR \). Let \([e_iR]\) be a direct sum of \( n_i \) copies of \( e_iR \). Consider \( 1 \leq i < j \leq k \). We arrange the summands \([e_iR]\) in such a way that \( l(e_jR) \leq l(e_iR) \). Suppose \( e_jRe_i \neq 0 \). Then we have an embedding of \( e_iR \) into \( e_jR \), hence \( l(e_jR) \leq l(e_iR) \). But by assumption \( l(e_jR) \leq l(e_iR) \), so \( l(e_jR) = l(e_iR) \), we get \( e_jR \cong e_iR \), which is a contradiction. Hence \( e_jRe_i = 0 \) for \( j > i \). Thus we have

\[
R \cong \begin{pmatrix}
M_{n_1}(e_1Re_1) & M_{n_1\times n_2}(e_1Re_2) & \cdots & M_{n_1\times n_k}(e_1Re_k) \\
0 & M_{n_2}(e_2Re_2) & \cdots & M_{n_2\times n_k}(e_2Re_k) \\
0 & 0 & M_{n_3}(e_3Re_3) & \cdots & M_{n_3\times n_k}(e_3Re_k) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & M_{n_k}(e_kRe_k)
\end{pmatrix},
\]

where each \( e_iRe_i \) is a division ring, \( e_iRe_i \cong e_jRe_j \) for each \( 1 \leq i, j \leq k \) and \( n_1, \ldots, n_k \) are any positive integers. Furthermore, if \( e_iRe_j \neq 0 \), then

\[
\dim(e_iRe_i(e_iRe_j)) = 1 = \dim((e_iRe_j)_{e_jRe_j}).
\]
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