Isotypic components of left cells in type $D$

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Introduction

This paper extends the results of [9] to type $D$, showing that the rule given there for computing basis vectors of isotypic components of a Kazhdan-Lusztig left (or right) cell extends to that type.

Section 1 Operators on tableau pairs

We will continue to use the notation of the three parts of Garfinkle’s series of papers on the classification of primitive ideals in types $B$ and $C$, as maintained and supplemented in [10]. Let $W$ be a Weyl group of type $D_n$. If $\alpha, \beta$ are adjacent simple roots, now necessarily of the same length, then we use the wall-crossing operator $T_{\alpha\beta}$ as defined in [2] 2.1.10, unless $\{\alpha, \beta\} = \{\alpha'_1, \alpha_3\} = \{e_2 + e_1, e_3 - e_2\}$, in which case we use the operators $T_{\alpha'_1}^{L_{\alpha_3}}, T_{\alpha_3, a_1}^{L_{\alpha_3}}$ defined in [10] 4.3.5, alternatively denoting either of these by $T_{a\beta}^L$. These last operators do not preserve right tableaux in general; we will use them only in the cases where they do preserve this tableau, that is, either case (1) of [10] 4.3.5, or in case (2) of this definition or its analogue in case (4) if the cycle of the 3-domino is closed in $T_1$, or in case (3) or its analogue in case (4) if the cycle of the 3-domino is closed in $T_1$. As in [9], we extend the operators $T_{a\beta}$ to tableau pairs $(T_1, T_2)$ by decreeing that they act trivially on $T_2$; the operators $T_{\alpha'_1}^{L_{\alpha_3}}, T_{\alpha_3, a_1}^{L_{\alpha_3}}$ are already defined on tableau pairs.

We also use the operators $T_{\alpha A}^{L_{\alpha A}}, T_{\alpha B}^{L_{\alpha B}}$ defined in [10] 4.4.10 and their truncations $U_{\alpha A}^{L_{\alpha A}}, U_{\alpha B}^{L_{\alpha B}}$, where the value or values of the latter operators on a tableau pair $(T_1, T_2)$ consist of all pairs in $T_{\alpha A}^{L_{\alpha A}}(T_1, T_2)$ or $T_{\alpha B}^{L_{\alpha B}}(T_1, T_2)$ whose right tableau is $T_2$. As in [9], we need a family of additional operators to generate the equivalence relation among tableau pairs of having the same right tableau. By analogy with the corresponding definition in [9], we define a tableau shape to be a quasi staircase if it takes the form $\lambda_n = (2n + 1, \ldots, n + 3, n + 2, n + 2, n, n, n - 1, \ldots, 1)$ or $\mu_n = (2n + 2, 2n + 1, \ldots, n + 3, n + 2, n + 2, n, n, n - 1, \ldots, 1)$; note that the quasi staircase shapes in type $D$ are just the transposes of the quasi staircase shapes in type $C$. We then define operators $S_n$, $S_n'$, $S_n''$, and $S_n'''$ much as we did in [9], for each $n$ fixing two domino tableaux $\hat{T}_1, \hat{U}_1$ of shape $\lambda_n$ and two others $\tilde{T}_2, \tilde{U}_2$ of shape $\mu_n$, such that the two largest dominos of all of them lie vertically at the end of the two rows of length $n$ and horizontally at the end of the row just above these rows and each $\hat{U}_1$ is obtained from $\hat{T}_1$ by interchanging these dominos. Then the operator $T_n$ is defined on a tableau pair $(T_1, T_2)$ such that the first $(n + 1)^2$ dominos form the subtableau $\hat{T}_1$ or $\hat{U}_1$ by interchanging the two largest dominos in the subtableau while leaving all other dominos in $T_1$ and $T_2$
unchanged; we define $S'_n$ similarly, working with $\tilde{T}_2$ and $\tilde{U}_2$ rather than $\tilde{T}_1$ and $\tilde{U}_1$, and we define $\dagger S_n, \dagger S'_n$ to be the transposes of $S_n, S'_n$, respectively, taking the transpose of a pair on which $\pi_n$ or $\pi'_n$ is defined to the transpose of the image of that pair under $S_n$ or $S'_n$. The composition $T_{\Sigma}$ of a sequence $\Sigma$ of operators $T_{\alpha\beta}, T_{\alpha\beta}^L, U_{\alpha X}^L, U_{\alpha X}^L, S_n, S'_n, \dagger S_n, \dagger S'_n$, and $\dagger S_n'$, is defined as in [9].

Section 2 Transitivity of the action on tableau pairs

We now extend Theorem 1 of [9] to type $D$.

Theorem 1. Given two pairs $(T_1, T_2), (T'_1, T'_2)$ of domino tableaux of the same shape such that $T_2 = T'_2$, there is a sequence $\Sigma$ of operators $T_{\alpha\beta}, T_{\alpha\beta}^L, U_{\alpha X}^L, U_{\alpha X}^L, S_n, S'_n, \dagger S_n, \dagger S'_n$, and $\dagger T'_n$, such that $(T'_1, T'_2)$ is one of the pairs in $T_{\Sigma}(T_1, T_2)$.

Proof. This is proved in the same way as in [4] and [9] Theorem 1), using Lemma 4.6.8 and Theorem 4.6.2 of [10] in place of Theorem 3.2.2 of [3]. Once again the most difficult case in proving the analogue of Lemma 4.6.8 is case I and it is this case that gives rise to the operators $S_n, S'_n$ and their transposes. $\square$

We enlarge the operators $S_n, S'_n, \dagger S_n, \dagger S'_n$ to operators $T_n, T'_n, \dagger T_n, \dagger T'_n$ as in [9], so that these last operators are defined on all tableau pairs $(T_1, T_2)$ such that $T_1$ can be moved through open cycles to produce a tableau with quasi staircase or transposed quasi staircase shape and the image of a tableau pair under an operator is either one or two tableau pairs of the same shape. These operators again extend to $W$-equivariant linear maps from left cells on which they are defined to other left cells; the maps also preserve right cells. Similarly the operators $U_{\alpha X}^L, U_{\alpha X}^L$ extend to $W$-equivariant maps $T_{\alpha X}^L, T_{\alpha X}^L$ from left cells of $W$ to other left cells that also preserve right cells.

Section 3 Decomposition of left cells

The rule stated before Lemma 1 of [3] for constructing bases of isotypic components of Kazhdan-Lusztig left cells in types $B$ or $C$ continues to hold for a Weyl group $W$ of type $D$. Fix left cells $\mathcal{C}, \mathcal{R}$ of $W$ lying in the same double cell $\mathcal{D}$. Let $x$ be the unique element of $\mathcal{C} \cap \mathcal{D}$ whose left tableau $T_L(x)$ has special shape. Let $\sigma$ be either of the two partitions of $2n$ corresponding to a representation $\pi$ of $W$ occurring in both $\mathcal{C}$ and $\mathcal{R}$, if $\sigma$ is not very even; if it is very even, then $\mathcal{C}$ and $\mathcal{R}$ are both irreducible as $W$-modules and there is nothing to do. Let $e_1, \ldots, e_r$ be the extended open cycles of $T_L(x)$ relative to $T_R(x)$ such that moving $T_L(x)$ through these open cycles produces a tableau of shape $\sigma$. Given any $w \in \mathcal{C} \cap \mathcal{R}$, let $T_L(w)$ be obtained from $T_L(x)$ by moving through the extended open cycles $f_1, \ldots, f_s$ (relative to $T_R(x)$). Put $\sigma_w = \pm 1$ according as an even or odd number of $f_i$ appear among the $e_j$. Set $R_\sigma = \sum_{w \in \mathcal{C} \cap \mathcal{R}} \sigma_w C_w$.

Theorem 2. The right or left $W$-submodule generated by $R_\sigma$ is irreducible and $W$ acts on it by $\pi$.  

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\textit{Proof.} This is proved in the same way as \cite[Theorem 2]{9}, using \cite[4.4.10]{10} in place of \cite[2.3.4]{2}, to show that the formula for $R_\sigma$ is compatible with the operators $T_{a\beta}, T^L_{a\beta}, T^L_{aX}, T^L_{Xa}$, in the sense that if a particular $R_\sigma$ coincides with $R'_\sigma$ and so transforms by $\pi$, then the same will be true of the image of $R_\sigma$ under any composition $\Sigma$ of maps $T_{a\beta}, T^L_{a\beta}, T^L_{aX}, T^L_{Xa}$ that is nonzero on $R_\sigma$. We must show that this continues to hold for compositions including the operators $T_n, T'_n, T''_n$, and $T'''_n$, and for this it is enough to check that the formula for $R_\sigma$ holds for particular cell intersections arising in the definition of the $S_n$ and $S'_n$, as in \cite[9]{9}, assuming inductively that it holds in smaller rank.

Assume now that $W$ is of type $D_9$ and consider a cell intersection $I = \mathcal{C} \cap \mathcal{C}^{-1}$, where $\mathcal{C}$ is represented by an element with left tableau $T_1$ chosen as above for the shape $(5,4,4,2,2,1)$; suppose in addition that the 7-domino in $T_1$ is horizontal and lies directly above the 8-domino, while the 6-domino is horizontal and lies at the end of the first row, so that the irreducible constituents of $\mathcal{C}$ as a $W$-module are indexed by the partitions $(5,5,3,3,1,1), (5,4,4,2,1,1), (5,5,3,2,2,1)$, and $(5,4,4,3,1,1)$ of 18. Denote by $x_p$ the unique element in the intersection $I$ whose left and right tableaux have shape $p$. Then compositions of operators $T_{a\beta}, T^L_{a\beta}, U^L_{Xa}$ and $U^L_{aX}$ act transitively on tableau pairs with a fixed right tableau not of shape $(5,4,4,2,2,1)$ or its transpose $(6,5,3,3,1)$, so the argument in the proof of \cite[8]{8} applies to show that

\[
R'_{5,5,3,3,1,1,1} = x_{5,5,3,3,1,1} + x_{5,5,3,3,2,1} + x_{5,5,3,3,2,2,1} + x_{5,4,4,3,1,1} \\
R'_{5,5,3,2,2,2,1} = x_{5,5,3,3,1,1} - x_{5,4,4,2,2,1} - x_{5,5,3,2,2,1} + x_{5,4,4,3,1,1}
\]

while either

\[
R'_{5,4,4,2,2,1,1} = x_{5,5,3,3,1,1} + x_{5,4,4,2,2,1} - x_{5,5,3,2,2,1} - x_{5,4,4,3,1,1} \\
R'_{5,4,4,3,1,1} = x_{5,5,3,3,1,1} - x_{5,4,4,2,2,1} + x_{5,5,3,2,2,1} - x_{5,4,4,3,1,1}
\]

or else

\[
R'_{5,4,4,2,2,1,1} = x_{5,5,3,3,1,1} - x_{5,4,4,2,2,1} + x_{5,5,3,2,2,1} - x_{5,4,4,3,1,1} \\
R'_{5,4,4,3,1,1} = x_{5,5,3,3,1,1} + x_{5,4,4,2,2,1} - x_{5,5,3,2,2,1} - x_{5,4,4,3,1,1}
\]

But now if we take the basis elements for the Weyl group $W'$ of type $D_5$ corresponding to the tableau pairs consisting of the first five dominos of every tableau in all the pairs corresponding to elements of $I$ and label the resulting elements $y_q$ in type $D_5$ by partitions $q$ of 10 as we did the elements of $I$ by partitions
of 18, we find that $\gamma_{(3,3,1,1,1,1)} - \gamma_{(3,2,2,1,1,1)}$ transforms by the representation corresponding to $(3,2,2,1,1,1)$ of $W'$, whose truncated induction to $W$ is the direct sum of the representations corresponding to $(5,4,4,3,1,1)$ and $(5,4,4,2,2,1)$. In order to make $\gamma_{(3,2,2,1,1,1)} - \gamma_{(3,2,2,1,1,1)}$ transforms by the representation corresponding to $(3,2,2,1,1,1)$ of $W'$, whose truncated induction to $W$ is the direct sum of the representations corresponding to $(5,4,4,3,1,1)$ and $(5,4,4,2,2,1)$. In order to make $\gamma_{(5,4,4,2,2,1)} - \gamma_{(5,5,3,3,1,1)}$ transform by representations lying in this last truncated induced representation, we find that the first pair of equations for $R'_{(5,4,4,2,2,1)}$ and $R_{(5,4,4,3,1,1)}'$ must hold, as desired. It follows that the operators $T_2, T_2', T_2$, and $T_2'$, extended to linear maps between left cells regarded as $W$-modules, are indeed equivariant for the left $W$-action. Like the maps $T_{a\beta}$ and $T_{a\beta}'$, they are compatible with the formula for $R_\sigma$. Similar arguments show that the linear extensions of the other maps $T_n, T_n', T_n$, and $T_n'$ are also $W$-equivariant and compatible with the formula for $R_\sigma$. As in [9] we now have enough $W$-equivariant maps between left cells to validate the proof of [8] Theorem 1.

We then correct the statements of Theorem 4.2 in [6] and Theorems 2.1 and 2.2 in [7] as in [9], all three of these results being corrected and superseded by the following one: given any two left cells $C_1, C_2$ in a Weyl group $W$ of type $D$ that have a representation $\pi$ of $W$ in common, there is a composition $\Sigma$ of maps $T_{a\beta}, T_{a\beta}', T_{aX}, T_{aX}', T_n, T_n', T_n$, and $T_n'$ from $C_1$ to $C_2$ whose restriction to the copy of $\pi$ in $C_1$ maps it isomorphically onto the corresponding copy of $\pi$ in $C_2$.

References

[1] D. Garfinkle. On the classification of primitive ideals for complex classical Lie algebras (I), Compositio Math., 75(2):135–169, 1990.
[2] D. Garfinkle. On the classification of primitive ideals for complex classical Lie algebras (II), Compositio Math., 81(3):307–336, 1992.
[3] D. Garfinkle. On the classification of primitive ideals for complex classical Lie algebras (III), Compositio Math., 88:187–234, 1993.
[4] B. Hopkins. Domino Tableaux and Single-Valued Wall-Crossing Operators, Ph.D. dissertation, University of Washington, 1997.
[5] G. Lusztig. Leading coefficients of character values of Hecke algebras, Proc. Symp. Pure Math., 47 (2): 235–262, 1987.
[6] W. M. McGovern. Left cells and domino tableaux in classical Weyl groups, Compositio Math., 101:77–98, 1996.
[7] W. M. McGovern, Standard domino tableaux and asymptotic Hecke algebras, Compositio Math., 101:99–108, 1996.
[8] W. M. McGovern, Errata and a new result on signs, Compositio Math., 117:117–121, 1999.
[9] W. M. McGovern, A family of operators generating domino tableaux of a fixed shape and a decomposition of left cells into isotypic components, preprint, 2020.

[10] W. M. McGovern and T. Pietraho, On the classification of primitive ideals for complex classical Lie algebras (IV), preprint, 2016.