FOURTH-ORDER PROBLEMS WITH LERAY-LIONS TYPE OPERATORS IN VARIABLE EXPONENT SPACES

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Abstract. The Leray-Lions operators are versatile enough to be particularized to various elliptic operators, so they receive a lot of attention. This paper introduces to the mathematical literature Leray-Lions type operators that are appropriate for the study of the variable exponent problems of higher order. We establish some properties concerning these general operators and then we apply them to a fourth order problem with variable exponents.

1. Introduction. The variable exponent problems have numerous applications, and we can refer to those involving non-Newtonian fluids [30], elastic materials [33, 5], image restoration [8] etc. At the same time, the study of the fourth-order problems is motivated by multiple applications as well, see for example [13, 26, 27] and the references therein. In 2009 these two research directions have intersected [3, 15], but we are only aware of such problems driven by the $p(\cdot)$-biharmonic Laplace operator,

$$\Delta^2_{p(\cdot)}(u) = \Delta(|\Delta u|^{p(\cdot)} - 2\Delta u),$$

(1)

and here we are going to consider more general operators. Bearing the names of the mathematicians that introduced them to the mathematical literature [23], the Leray-Lions operators have the interesting property that they can be particularized to different well-known operators, like the Laplacian, or the mean curvature operator. Over the years, many variants of such operators appeared, each of them being adapted to a specific type of problem, see for example [9, 4, 25]. Among others, we recall the second order elliptic problems with variable exponents and Leray-Lyons type operators, see [22, 7], and the fourth-order elliptic problems with constant exponents and Leray-Lyons type operators, see [31]. Until now there has not been considered a fourth-order elliptic problem with variable exponents and Leray-Lyons type operators, so we intend to fill this gap. More precisely, we are interested in the weak solvability of the problems involving the operators

$$\Delta(a(x, \Delta u))$$

where we assume the following.

(p): $1 < \text{ess inf}_{x \in \Omega} p(x) \leq \text{ess sup}_{x \in \Omega} p(x) < \infty$ for all $x \in \Omega$ and the exponent $p$ is log-Hölder continuous, that is, there exists $c > 0$ such that

$$|p(x) - p(y)| \leq \frac{c}{-\log |x - y|}$$

for all $x, y \in \Omega$, $0 < |x - y| \leq \frac{1}{2}$.
(a0): $a : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $a(x, 0) = 0$ for a.e. $x \in \Omega$;

(a1): There exist a positive constant $\tilde{c}$ and a nonnegative function $d \in L^{p'(\cdot)}(\Omega)$ (where $1/p(x) + 1/p'(x) = 1$) such that

$$|a(x, t)| \leq \tilde{c} \left( d(x) + |t|^{p(x)-1} \right)$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$;

(a2): The monotonicity condition

$$[a(x, s) - a(x, t)](s - t) \geq 0$$

holds for a.e. $x \in \Omega$ and all $s, t \in \mathbb{R}$ with equality if and only if $s = t$;

(a3): There exists $0 < \bar{c} < 3 \min\{1, \tilde{c}\}$ such that

$$\bar{c}|t|^{p(x)} \leq \min_{\Omega}(a(x, t)t; p(x)A(x, t))$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$,

where $\tilde{c}$ is the one from (a1) and $A : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ represents the antiderivative of $a$, that is,

$$A(x, t) = \int_0^t a(x, s) \, ds.$$

Our interest goes to the Leray-Lions type operators because they are quite general. Indeed, let us take

$$a(x, t) = h(x)|t|^{p(x)-2}$$

where $p$ satisfies (p), $h \in L^\infty(\Omega)$ and there exists $h_0 > 0$ such that $h(x) \geq h_0 > 0$ for a.e. $x \in \Omega$. One can see that $a$ from (2) satisfies hypotheses (a0)-(a3) and we arrive at the operator

$$h(\cdot)\Delta(|\Delta u|^{p(\cdot)-2}\Delta u).$$

Note that, when $h \equiv 1$, the above operator becomes the operator $\Delta^2_{p(\cdot)}(u)$ from (1).

Moreover, we can make the choice

$$a(x, t) = h(x)(1 + |t|^2)^{(p(x)-2)/2}t$$

in order to obtain the operator

$$h(\cdot)\Delta \left( (1 + |\Delta u|^2)^{(p(\cdot)-2)/2}\Delta u \right),$$

where $p, h$ are taken as above.

At the same time, our operator can be viewed as an extension of the Leray-Lyons type operator from [31], where fourth-order elliptic problems with constant exponents are treated. As for the difficulties that appear when passing from a constant exponent to a variable one, it is already known that there are several properties that do not hold when the exponent varies, see for example the exhaustive work [29] and the references therein. However, even if we do not take into account the fact that we work with variable exponents, our hypotheses on the Leray-Lions type operators are more general in the sense that in [31] the authors impose in addition the condition

$$a(x, t)t \leq pA(x, t)$$

for a.e $x \in \Omega$ and all $t \in \mathbb{R}$.

Furthermore, instead of our monotonicity condition (a2) they use a more restrictive one, that is, they assume that $A$ is $p$-uniformly convex. To give an example, if we want to particularize the Leray-Lions operator to the $p$-biharmonic Laplace operator we take $a(x, t) = |t|^{p-2}t$ and the $p$-uniform convexity condition forces $p \geq 2$ (see [9, Remark 2.3]), while, in our case, we allow $p > 1$. 

Since most of the studies concerning existence and multiplicity results are based on the critical point theory, our main focus will go to the study of the functional that is generated by our Leray-Lions type operator. Thus, we devote Section 3 to the study of this functional because, in our opinion, the importance of its properties goes beyond this study. Indeed, these properties are useful for the treatment of many classes of fourth-order problems involving Leray-Lions operators in the variable exponent spaces. To bring an example, in Section 4 we briefly discuss a problem that extends [6] by generalizing the \( p(\cdot)\)-biharmonic Laplace operator \( \Delta_{p(\cdot)}^2(u) \) that appears in their problem. Moreover, at the end we indicate several problems that can be extended due to the results from Section 3.

2. Functional framework. We start by recalling some standard notation that will be used in this paper. We denote by \( X^* \) the dual of a Banach space \( X \) and by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( X^* \) and \( X \). By \( | \cdot | \) we denote the absolute value of a number, the Euclidean norm of a vector, or the Lebesgue measure of a set. For simplicity, we put

\[
h^- = \text{ess inf}_{x \in \Omega} h(x) \quad \text{and} \quad h^+ = \text{ess sup}_{x \in \Omega} h(x).
\]

Thus we can say that everywhere below we consider \( p \) to be log-Hölder continuous with \( 1 < p^- \leq p^+ < \infty \), if not otherwise stated. In addition, through all this paper \( \Omega \subset \mathbb{R}^N \ (N \geq 2) \) is a bounded domain with smooth boundary.

Now we can introduce the functional setting for our work and we briefly recall some properties. The variable exponent Lebesgue space is defined by

\[
L^{p(\cdot)}(\Omega) = \{ u : \Omega \to \mathbb{R} : \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \},
\]

and, when equipped with the Luxemburg norm,

\[
\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \mu > 0 : \int_{\Omega} \frac{|u(x)|^{p(x)}}{\mu} \, dx \leq 1 \right\},
\]

it is a separable and reflexive Banach space, see [20, Theorem 2.5, Corollary 2.7].

**Proposition 1.** (see [19, Theorem 1.4]) If \( u, u_n \in L^{p(\cdot)}(\Omega) \) for all \( n \in \mathbb{N} \), then

\[
\lim_{n \to \infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \text{ if and only if } \lim_{n \to \infty} \int_{\Omega} |u_n - u|^{p(x)} \, dx = 0. \tag{3}
\]

We denote by \( L^{p'(\cdot)}(\Omega) \) the dual of \( L^{p(\cdot)}(\Omega) \), obtained by conjugating the exponent pointwise, that is, \( 1/p(x) + 1/p'(x) = 1 \), see [20, Corollary 2.7], and we have the following Hölder type inequality:

\[
\left| \int_{\Omega} u(x)v(x) \, dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{p'(\cdot)} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}, \tag{4}
\]

for all \( u \in L^{p(\cdot)}(\Omega) \) and \( v \in L^{p'(\cdot)}(\Omega) \), see [20, Theorem 2.1]. Similarly, if \( 1/r_1(x) + 1/r_2(x) + 1/r_3(x) = 1 \), then

\[
\left| \int_{\Omega} u(x)v(x)w(x) \, dx \right| \leq \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right) \|u\|_{L^{r_1(\cdot)}(\Omega)} \|v\|_{L^{r_2(\cdot)}(\Omega)} \|w\|_{L^{r_3(\cdot)}(\Omega)}, \tag{5}
\]

for all \( u \in L^{r_1(\cdot)}(\Omega) \), \( v \in L^{r_2(\cdot)}(\Omega) \) and \( w \in L^{r_3(\cdot)}(\Omega) \), see [18, Proposition 2.5].
Also, to a Carathéodory function \( g : \Omega \times \mathbb{R}^m \to \mathbb{R}, m \in \mathbb{N} \), we can associate a Nemytsky operator \( N \) that maps an \( m \)-tuple of functions \((u_1, \ldots, u_m)\) into
\[
N_g(u_1, \ldots, u_m)(x) = g(x, u_1(x), \ldots, u_m(x)) \quad x \in \Omega.
\] (6)

**Theorem 2.1.** ([20, Theorems 4.1–4.2]) Let \( g : \Omega \times \mathbb{R}^m \to \mathbb{R}, m \in \mathbb{N} \), be a Carathéodory function and \( l_i, l_0 \in L^\infty(\Omega) \) with \( l_i, l_0 \geq 1 \) for all \( i \in \{1, 2, \ldots, m\} \). Assume that there exist a nonnegative function \( h \) and a Carathéodory function \( g \) such that
\[
|g(x, \xi)| \leq h(x) + c \sum_{i=1}^m |\xi_i|^{l_i(x)/l_0(x)}
\]
for all \( \xi \in \mathbb{R}^N \) and a.e. \( x \in \Omega \). Then the Nemytsky operator \( N \) provided by formula (6) maps \( L^{l_1}(\Omega) \times \cdots \times L^{l_m}(\Omega) \) into \( L^{l_0}(\Omega) \) and it is a continuous and bounded operator.

Moving further to the definition of the variable exponent Sobolev space and its basic properties, we recall that
\[
W^{k,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq k \},
\]
where \( D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} u \) and \( \alpha = (\alpha_1, \ldots, \alpha_N) \) is a multi-index, \( |\alpha| = \sum_{i=1}^N \alpha_i \). The space \( W^{k,p(\cdot)}(\Omega) \) equipped with the norm
\[
\|u\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)},
\]
is a separable and reflexive Banach space too, see [20, Theorem 3.1].

Next we define \( W^{1,p(\cdot)}_0(\Omega) \) as the closure of \( C_0^\infty(\Omega) \) in \( W^{1,p(\cdot)}(\Omega) \). Taking into account the log-Hölder continuity of the exponent \( p \) and the following Poincaré type inequality (see [17, Proposition 2.3]),
\[
\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \text{for all } u \in W^{1,p(\cdot)}_0(\Omega),
\]
where \( C \) is a positive constant, we are able to say that the space \( W^{1,p(\cdot)}_0(\Omega) \) can also be described by
\[
W^{1,p(\cdot)}_0(\Omega) = \left\{ u \in W^{1,p(\cdot)}(\Omega) : \left. u = 0 \right|_{\partial \Omega} \right\},
\]
and it can be equipped with the norm
\[
\|u\|_{W^{1,p(\cdot)}_0(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.
\]
The space \( \left( W^{1,p(\cdot)}_0(\Omega), \| \cdot \|_{W^{1,p(\cdot)}_0(\Omega)} \right) \) is a separable and reflexive Banach space (see [17, Proposition 2.1]).

Now that we know that both \( W^{2,p(\cdot)}(\Omega) \) and \( W^{1,p(\cdot)}_0(\Omega) \) are separable and reflexive Banach spaces, it follows that
\[
V = W^{2,p(\cdot)}(\Omega) \cap W^{1,p(\cdot)}_0(\Omega)
\]
is a separable and reflexive Banach space too, when equipped with the norm
\[
\|u\|_V = \|u\|_{W^{2,p(\cdot)}(\Omega)} + \|u\|_{W^{1,p(\cdot)}_0(\Omega)}
\]
\[
= \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \sum_{|\alpha|=2} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}.
\]
Moreover, since \( p \) verifies (\( p \)), we have the following result.
Theorem 2.2. (see [32, Theorem 4.4] and [11, Theorem 3.16, Remark 3.17, Corollary 3.18]) Assume that $\Omega$ is a bounded domain with Lipschitz boundary. The norms $\| \cdot \|_V$ and $\| \Delta(\cdot) \|_{L^p(\Omega)}$ are equivalent on $W^{2,p(\cdot)}(\Omega) \cap W^{1,p(\cdot)}_0(\Omega)$.

Notice that, if $b$ satisfies

(b): $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for a.e. $x \in \Omega$, then

$$\|u\|_b = \inf \left\{ \mu > 0 : \int_{\Omega} \left( \frac{|\Delta u(x)|^{p(x)}}{\mu} + b(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}$$

represents a norm which is equivalent to $\| \cdot \|_V$ on $V$, see for example [16, Remark 2.1]. Therefore in what follows we will consider $(V, \| \cdot \|_b)$, where $V$ is defined by (7). Furthermore, for any $b$ which satisfies (b), we consider $\Lambda : V \to \mathbb{R}$ defined by

$$\Lambda(u) = \int_{\Omega} \left( |\Delta u|^{p(x)} + b(x) |u|^{p(x)} \right) dx$$

and we can make an important connection to the norm $\| \cdot \|_b$ by proceeding as in [6].

Proposition 2. (see [6, Proposition 1]) For $u, u_n \in W^{2,p(\cdot)}(\Omega)$ we have

$$\|u\|_b < (=: 1) \Leftrightarrow \Lambda(u) < (=: 1),$$

$$\|u\|_b \leq 1 \Rightarrow \|u\|_b^{p^-} \leq \Lambda(u) \leq \|u\|_b^{p^+},$$

$$\|u\|_b \geq 1 \Rightarrow \|u\|_b^{p^-} \leq \Lambda(u) \leq \|u\|_b^{p^+},$$

$$\|u_n\|_b \to 0 \quad (\to \infty) \quad \Leftrightarrow \quad \Lambda(u_n) \to 0 \quad (\to \infty). \quad (8)$$

Taking into account the above discussion and [14, Theorem 3.7] and [11, Section 6.5.3], one can formulate the following.

Theorem 2.3. Assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with Lipschitz boundary and $p$ is log-Hölder continuous with $1 < p^- \leq p^+ < \infty$. Then $C^\infty(\overline{\Omega})$ is dense in $V$.

Finally, we recall an embedding result.

Theorem 2.4. ([3, Theorem 3.2]) Let us consider $q \in C(\overline{\Omega}; \mathbb{R})$ such that $1 < q^- \leq q^+ < \infty$ and $q(x) < p_2^*(x)$ for all $x \in \Omega$, where

$$p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } p(x) < \frac{N}{2}, \\ +\infty & \text{if } p(x) \geq \frac{N}{2}; \end{cases}$$

for any $x \in \overline{\Omega}$. Then there is a continuous embedding $V \hookrightarrow L^q(\Omega)$.\n
3. Properties of the associated functional. Let us take $I_1 : V \rightarrow \mathbb{R}$,

$$I_1(u) = \int_{\Omega} \left[ A(x, \Delta u) + \frac{b(x)}{p(x)} |u|^{p(x)} \right] \, dx,$$

(9)

where, as denoted by (7), $V = W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$.

**Proposition 3.** Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with smooth boundary, $I_1 : V \rightarrow \mathbb{R}$ be the functional defined by (9) and $a : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that hypotheses (a1), (b), (p) are fulfilled. Then $I_1$ is well defined and of class $C^1$, with the Gâteaux derivative defined by

$$\langle I_1'(u), v \rangle = \int_{\Omega} a(x, \Delta u) \Delta v \, dx + \int_{\Omega} b(x) |u|^{p(x)-2} uv \, dx.$$  

(10)

**Proof.** To show that $I_1$ is well defined, we use (b), (a1) and the Hölder type inequality (4). Thus there exists $\tilde{c} > 0$ such that

$$|I_1(u)| \leq 2\tilde{c} \|d\|_{L^{p(\cdot)}(\Omega)} \|\Delta u\|_{L^{p(\cdot)}(\Omega)} + \frac{\tilde{c}}{p} \int_{\Omega} |\Delta u|^{p(x)} \, dx$$

$$+ \|b\|_{L^\infty(\Omega)} \int_{\Omega} |u|^{p(x)} \, dx < \infty,$$

since $u \in W^{2,p(\cdot)}(\Omega)$ implies $|\Delta u|^{p(\cdot)}(\Omega) \in L^1(\Omega)$.

Next, for the Gâteaux differentiability, we consider $u, v \in V$ and we fix $x \in \Omega$ and $0 < r < 1$. By Jensen’s inequality applied to the convex function $g_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $g_0(t) = tp(x)$, $1 < p^- \leq p^+ < \infty$, we obtain that, for any $x \in \Omega$ and $u, v \in V$,

$$|u - v|^{p(x)} \leq 2p^r - 1 \left( |u|^{p(x)} + |v|^{p(x)} \right)$$

(11)

and, obviously,

$$|\Delta u - \Delta v|^{p(x)} \leq 2p^r - 1 \left( |\Delta u|^{p(x)} + |\Delta v|^{p(x)} \right).$$

(12)

By the mean value theorem, there exist $\kappa \in (0, 1)$ and $c_1, c_2, c_3 > 0$ such that

$$\frac{|A(x, \Delta u + r \Delta v) - A(x, \Delta u) + b(x)/p(x)|u + rv|^{p(x)} - |u|^{p(x)}|}{r}$$

$$= \frac{|\Delta v| |a(x, \Delta u + r\kappa \Delta v)| + b(x)|v|| |u + r\kappa v|^{p(x)}|}{r}$$

$$\leq c_1 d(x)|\Delta v| + c_2 |\Delta v| \left( |\Delta u|^{p(x)} + |\Delta v|^{p(x)} \right)^{\kappa} + c_3 |v| \left( |u|^{p(x)} - 1 + |v|^{p(x)} - 1 \right)$$

due to (a1), (b), (p), (11) and (12). The Hölder type inequality (4) helps us deduce that the right-hand side quantity is in $L^1(\Omega)$ and we can apply Lebesgue’s dominated convergence theorem to obtain (10).

To complete the proof, it remains to show the continuity of $I_1'$. Let us consider an arbitrary sequence $(u_n) \subset V$ such that $u_n \rightarrow u$ in $V$. We intend to show that

$$\sup_{v \in V : \|v\|_1 = 1} |\langle I_1'(u_n) - I_1'(u), v \rangle| = 0.$$

To this aim, we define the Nemyskty operator

$$N_a : L^{p(\cdot)}(\Omega) \rightarrow L^{p(\cdot)}(\Omega), \quad (N_a(s))(x) = a(x, s)$$

which is continuous by Theorem 2.1. Also, we recall an inequality that was adapted from the constant exponent (see e.g. [12, Lemma 2.1]) to the variable exponent $p$,.


provided that $1 < p^- \leq p^+ < \infty$:

$$\left| |\alpha|^{p(x)-2} \alpha - |\beta|^{p(x)-2} \beta \right| \leq c_{p^+}( |\alpha| + |\beta|)^{p(x)-2} |\alpha - \beta|$$

(13)

where $\alpha, \beta \in \mathbb{R}$ and the constant $c_{p^+}$ depends on $p$ but not on $x$. Using (b), (13) and the Hölder-type inequalities (4) and (5), we get

$$\|I'_{1}(u_{n}) - I'_{1}(u), v\| \leq 2\|\Delta v\|_{L^{p}(\Omega)}\|\Delta u_{n} - \Delta u\|_{L^{p^+}(\Omega)} + c\|b\|_{L^{\infty}(\Omega)}\left(\|u_{n}\|_{L^{p^+}(\Omega)}^{p^+} + c\|u\|_{L^{p^+}(\Omega)}^{p^+}\right)\|u_{n} - u\|_{L^{p^+}(\Omega)}\|v\|_{L^{p^+}(\Omega)}$$

for some $c > 0$. The continuity of the Nemytsky operator, the boundedness of $\left(\|u_{n}\|_{L^{p^+}(\Omega)}^{p^+}\right)_{n}$ and the fact that, by Theorem 2.4 there exists $\kappa_0 > 0$ such that

$$\|u_{n} - u\|_{L^{p^+}(\Omega)} \leq \kappa_0 \|u_{n} - u\|_{b} \rightarrow 0,$$

lead to the desired continuity. \hfill \Box

Two classic results that are useful for what follows are stated below.

**Proposition 4.** (see [10, Section 2, Example B]) If $\Phi : X \rightarrow \mathbb{R}$ is a convex lower semicontinuous functional on a reflexive Banach space $X$, then $\Phi$ is weakly lower semicontinuous on $X$.

**Theorem 3.1.** ([21, Theorem 6.2.1.]) Let $X$ be a reflexive Banach space and let $f_{0} : X \rightarrow \mathbb{R}$ be Gâteaux differentiable on $X$. Then the following conditions are equivalent:

(i) $f_{0}$ is convex on $X$.

(ii) We have

$$f_{0}(u) - f_{0}(v) \geq \langle f_{0}'(v), u - v \rangle_{X^{*} \times X} \quad \forall u, v \in X,$$

where $X^{*}$ denotes, as usual, the dual of the space $X$.

(iii) The Gâteaux derivative is monotone, that is,

$$\langle f_{0}'(u) - f_{0}'(v), u - v \rangle_{X^{*} \times X} \geq 0, \quad \forall u, v \in X.$$

Now we are prepared to discuss another property of $I_{1}$.

**Proposition 5.** Let $\Omega \subset \mathbb{R}^{N}$ ($N \geq 2$) be a bounded domain with smooth boundary, $I_{1} : V \rightarrow \mathbb{R}$ be the functional defined by (9) and $a : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that hypotheses (a1), (a2), (b), (p) are fulfilled. Then $I_{1}$ is (sequentially) weakly lower semicontinuous, that is, for any $u \in V$ and any subsequence $(u_{n})_{n} \subset V$ such that $u_{n} \rightharpoonup u \text{ in } V$, there holds

$$I_{1}(u) \leq \liminf_{n \rightarrow \infty} I_{1}(u_{n}).$$

**Proof.** By (10), (b) and (a2), we can write that

$$\langle I_{1}'(u) - I_{1}'(v), u - v \rangle$$

$$\geq \int_{\Omega} \left[ a(x, \Delta u) - a(x, \Delta v) \right] (\Delta u - \Delta v) \, dx$$

$$+ b_{0} \int_{\Omega} \left[ |u|^{p(x)-2} u - |v|^{p(x)-2} v \right] (u - v) \, dx$$

$$\geq 0 \quad \text{for all } u, v \in V.$$
By Theorem 3.1, $I_1$ is convex and for every $(u_n)_n$, $u \in V$ such that $u_n \rightarrow u$ in $V$, we have that

$$I_1(u) \leq I_1(u_n) + \|I'_1(u)\|_{V^*} \|u - u_n\|_b.$$ 

By passing to the inferior limit and by applying Proposition 4 we obtain that $I_1$ is weakly lower semicontinuous.

**Theorem 3.2.** Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with smooth boundary, $I_1 : V \rightarrow \mathbb{R}$ be the functional defined by (9) and $a : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that hypotheses (a1)-(a3), (b), (p) are fulfilled. Then $I'_1 : V \rightarrow V^*$ is of type (S+), that is,

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n \rightarrow \infty} I'_1(u_n)(u_n - u) \leq 0. \quad (14)$$

imply that $u_n \rightarrow u$.

**Proof.** The argumentation of the (S+) property of $I'_1$ follows the same idea as in the proof of [22, Theorem 4.1], where Le considers Leray-Lions type operators which are involving gradient terms and are described by hypotheses that are slightly different than ours. For the convenience of the reader, we present an adaptation of this proof below.

Let us take $(u_n)_n \subset V$ with the property (14). Our goal is to show that $\|u_n - u\|_b \rightarrow 0$ as $n \rightarrow \infty$. (15)

By Proposition 2 relation (8), it is enough to establish that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left[ |\Delta u_n - \Delta u|^{p(x)} + b(x)|u_n - u|^{p(x)} \right] dx = 0.$$ 

By Theorem 2.4, we have that $W^{2,p(\cdot)}(\Omega)$ is compactly embedded in $L^{p(\cdot)}(\Omega)$, thus $u_n \rightharpoonup u$ in $L^{p(\cdot)}(\Omega)$, and, by (3) and (b), we are able to write

$$\lim_{n \rightarrow \infty} \int_{\Omega} b(x)|u_n - u|^{p(x)} dx = 0.$$ 

Therefore it remains to show that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\Delta u_n - \Delta u|^{p(x)} dx = 0. \quad (16)$$

To this end, we will apply Vitali’s convergence theorem, so we first need to prove that the sequence $(|\Delta u_n - \Delta u|^{p(x)})_n$ is uniformly integrable in $\Omega$, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $H \subseteq \Omega$ is measurable with $|H| \leq \delta$, then

$$\int_H |\Delta u_n - \Delta u|^{p(x)} dx \leq \varepsilon \quad \text{for all } n \in \mathbb{N}.$$ 

Let us fix $\varepsilon > 0$. Due to the fact that $|\Delta u|^{p(\cdot)} \in L^1(\Omega)$, there exists $\delta_1 > 0$ such that if $|H| \leq \delta_1$, then

$$\int_H |\Delta u|^{p(x)} dx \leq \varepsilon 2^{-p^+}.$$ 

Consequently, by (12) it is sufficient to prove that

$$\int_H |\Delta u_n|^{p(x)} dx \leq \varepsilon 2^{-p^+}. \quad (17)$$

provided that $|H| \leq \delta_1$. Let us proceed.

By (a2) and (b),

$$\langle I'_1(u_n) - I'_1(u), u_n - u \rangle \geq 0. \quad (18)$$
In addition, $u_n \rightarrow u$ yields that
\[ \langle I'_1(u), u_n - u \rangle = 0. \]  \hfill (19)

Putting together (18), (19) and (14),
\[ 0 \leq \liminf_{n \rightarrow \infty} (I'_1(u_n) - I'_1(u), u_n - u) \leq \limsup_{n \rightarrow \infty} (I'_1(u_n), u_n - u) \leq 0 \]  \hfill (20)

and we deduce that there exists $N_\varepsilon \in \mathbb{N}$ such that
\[ \int_{\Omega} [a(x, \Delta u_n) - a(x, \Delta u)](\Delta u_n - \Delta u) \, dx \leq \frac{\varepsilon \delta^{-p+1}}{3} \quad \text{for all } n \geq N_\varepsilon, \]
where $\delta > 0$ was introduced by (a3). The above relation implies that there exists $0 < \delta_2 \leq \delta_1$ such that if $H \subset \Omega$ is measurable and $|H| \leq \delta_2$,
\[ \int_{H} [a(x, \Delta u_n) - a(x, \Delta u)](\Delta u_n - \Delta u) \, dx \leq \frac{\varepsilon \delta^{-p+1}}{3} \quad \text{for all } n \in \mathbb{N}. \]  \hfill (21)

By (a3),
\[ \delta |\Delta u_n|^{p(x)} \leq a(x, \Delta u_n)\Delta u_n \quad \text{for a.e. } x \in \Omega. \]  \hfill (22)

On the other hand, by (a1),
\[ a(x, \Delta u_n)\Delta u \leq \delta d(x)|\Delta u| + \delta |\Delta u_n|^{p(x)-1}|\Delta u| \quad \text{for a.e. } x \in \Omega. \]  \hfill (23)

At this point, we recall that there is a variant of Young’s inequality (see [22, relation 3.14]) which states that, for a $\tau \in (0, 1)$, there exists $C(\tau) > 0$ depending on $\tau$ and $p(\cdot)$, but not on $x$, such that for all $\alpha, \beta \in \mathbb{R}$ and $x \in \Omega$,
\[ \alpha \beta \leq \tau \alpha |\alpha|^{p(\cdot)} + C(\tau)|\beta|^{p(\cdot)}. \]  \hfill (24)

Since $\delta < 3 \min\{1, \delta_1\}$, we apply this inequality and we infer the existence of two positive constants, $C(\delta/3)$ and $C(\delta/(3\delta))$, such that
\[ a(x, \Delta u)\Delta u_n \leq \frac{\delta}{3} |\Delta u_n|^{p(x)} + C \left( \frac{\delta}{3} \right) |a(x, \Delta u)|^{p(x)} \quad \text{for a.e. } x \in \Omega, \]  \hfill (25)

and, by (23),
\[ a(x, \Delta u_n)\Delta u \leq \frac{\delta d(x)|\Delta u|}{3} + \frac{\delta}{3} |\Delta u_n|^{p(x)} + C \left( \frac{\delta}{3\delta} \right) |\Delta u|^{p(x)} \quad \text{for a.e. } x \in \Omega. \]  \hfill (26)

We introduce (22), (25) and (26) into (21) and we obtain
\[ \frac{\delta}{3} \int_{H} |\Delta u_n|^{p(x)} \, dx \leq \frac{\varepsilon \delta^{-p+1}}{3} + \frac{\delta}{3} \int_{H} d(x)|\Delta u| \, dx + C \left( \frac{\delta}{3\delta} \right) \int_{H} |\Delta u|^{p(x)} \, dx \]  \hfill (27)

\[ \quad \quad + C \left( \frac{\delta}{3} \right) \int_{H} |a(x, \Delta u)|^{p(x)} \, dx + \int_{H} |a(x, \Delta u)||\Delta u| \, dx. \]

We have that $|\Delta u|^{p(\cdot)} \in L^{1}(\Omega)$ and, by the H"older-type inequality (4), $d|\Delta u| \in L^{1}(\Omega)$ too. By (a1),
\[ \int_{\Omega} |a(x, \Delta u)||\Delta u| \, dx \leq \frac{\varepsilon}{3} \int_{\Omega} d(x)|\Delta u| \, dx + \frac{\varepsilon}{3} \int_{\Omega} |\Delta u|^{p(x)} \, dx \]
and, as above, $|a(\cdot, \Delta u)||\Delta u| \in L^{1}(\Omega)$. Finally, using (a1) and an inequality similar to (12),
\[ \int_{\Omega} |a(x, \Delta u)|^{p(x)} \, dx \leq \frac{\varepsilon 2^{p-1}}{3} \int_{\Omega} d(x)^{p(x)} \, dx + \frac{\varepsilon 2^{p-1}}{3} \int_{\Omega} |\Delta u|^{p(x)} \, dx \]

hence $|a(\cdot, \Delta u)|^{p(\cdot)} \in L^{1}(\Omega)$ as well. Due to all these $L^{1}(\Omega)$ terms, we can use (27) to infer that there exists $0 < \varepsilon \leq \delta_2 \leq \delta_1$ such that (17) is fulfilled. As specified
at the beginning of the proof, once we have (17), the sequence \(|\Delta u_n - \Delta u|^{p(\cdot)}\) is uniformly integrable in \(\Omega\). Thus, in order to apply Vitali’s convergence theorem, we only need to show that
\[
|\Delta u_n(x) - \Delta u(x)| \to 0 \text{ as } n \to \infty \text{ for a.e. } x \in \Omega.
\] (28)

By (20) we deduce that
\[
[a(x, \Delta u_n) - a(x, \Delta u)](\Delta u_n(x) - \Delta u) \to 0 \text{ in } L^1(\Omega).
\]

Taking into account (a2),
\[
\lim_{n \to \infty} [a(x, \Delta u_n(x)) - a(x, \Delta u(x))](\Delta u_n(x) - \Delta u(x)) = 0 \text{ for a.e. } x \in \Omega. \tag{29}
\]

The convergence provides boundedness, so we can use (a3) and (a1) such that there exist \(M, \hat{c}_0 > 0\) with the property
\[
\hat{c}_0 |\Delta u_n(x)|^{p(\cdot)} \leq a(x, \Delta u_n(x)) \Delta u_n(x) \leq M + a(x, \Delta u(x)) \Delta u_n(x) + \hat{c} \left[ d(x) + |\Delta u_n(x)|^{p(\cdot)-1} \right] \Delta u(x) + |a(x, \Delta u(x)) \Delta u(x)| \text{ for a.e. } x \in \Omega.
\]

Arguing by contradiction in the previous relation we obtain that \((\Delta u_n(x))\) is bounded in \(\mathbb{R}\). Then, passing to a subsequence, there exists \(\xi = \xi(x) \in \mathbb{R}\) fulfilling
\[
\Delta u_n_k(x) \to \xi \text{ as } k \to \infty \text{ for a.e. } x \in \Omega.
\]

Being a Carathéodory function,
\[
a(x, \Delta u_n_k(x)) \to a(x, \xi) \text{ as } k \to \infty \text{ for a.e. } x \in \Omega.
\]

Introducing this information into (29) and passing to the limit, we get
\[
[a(x, \xi - a(x, \Delta u(x)))(\xi - \Delta u(x)) = 0 \text{ for a.e. } x \in \Omega
\]
and the strict monotonicity from (a2) guarantees that
\[
\Delta u_n_k(x) \to \Delta u(x) \text{ as } k \to \infty \text{ for a.e. } x \in \Omega.
\]

Since the same arguments go for any subsequence of \((u_n)_n\), we have arrived at (28) and, by means of Vitali’s theorem, (16) takes place. Following the steps back, we deduce that (15) holds as well, so we conclude this proof.

**Remark 1.** The restriction \(\hat{c} < 3 \min\{1, \hat{c}\}\) is only needed to apply the Young-type inequality (24) to show that \(I'_1\) has the \((S+)\) property. Thus, for problems that are weakly solved by methods which do not require the \((S+)\) property (like the Weierstrass-type theorem), this restriction will not be needed. However, even when the restriction \(\hat{c} < 3 \min\{1, \hat{c}\}\) applies, our hypotheses regarding the Leray-Lions type operators are more relaxed than in the standard case when \(\hat{c} = \hat{c} = 1\).

### 4. An application.

To give an example of an elliptic problem in which the results proved in the previous section are valuable, we consider the following problem with Navier boundary condition:
\[
\begin{align*}
\begin{cases}
\Delta(a(x, \Delta u)) + b(x)|u|^{p(x)-2}u &= \lambda f(x, u) \quad \text{for } x \in \Omega, \\
u &= 0 &\text{for } x \in \partial\Omega,
\end{cases}
\end{align*}
\tag{30}
\]
where \(\Omega \subset \mathbb{R}^N \quad (N \geq 2)\) is a bounded domain with smooth boundary and \(\lambda > 0\). Everywhere below we consider that hypotheses (p), (a0)-(a3), (b) are satisfied. Moreover, we assume
(f1): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist $t_0 > 0$ and a ball $B$ with $\overline{B} \subset \Omega$ such that

$$\int_B F(x, t_0) \, dx > 0,$$

where $F$ represents the antiderivative of $f$, that is, $F(x, t) = \int_0^t f(x, s) \, ds$;

(f2): $\lim_{|t|\to \infty} \frac{f(x, t)}{|t|^{p(x)}} = 0$ uniformly with respect to $x \in \Omega$;

(f3): $\lim_{|t|\to 0} \frac{f(x, t)}{|t|^{p(x)}} = 0$ uniformly with respect to $x \in \Omega$.

Notice that we are imposing the same hypotheses as the ones considered by Boureanu, Rădulescu and Repovš [6] in their study of a $p(\cdot)$-biharmonic Laplace problem with no-flux boundary condition. This choice will be motivated in Remark 2.

Due to Theorem 2.3, assumption (a0) and to the boundary conditions of (30), we introduce the definition of a weak solution to our problem.

**Definition 4.1.** We say that $u \in V = W^{2,p(\cdot)}(\Omega) \cap W^{1,p(\cdot)}_0(\Omega)$ is a weak solution of the boundary value problem (30) if and only if

$$\int_\Omega a(x, \Delta u) \Delta v \, dx + \int_\Omega b(x)|u|^{p(x)-2}uv \, dx - \lambda \int_\Omega f(x, u)v \, dx = 0 \quad \text{for all } v \in V.$$

To problem (30) we associate $I : V \to \mathbb{R}$, $I = I_1 - \lambda I_2$, where $I_1$ was introduced by (9) and

$$I_2(u) = \int_\Omega F(x, u) \, dx.$$

Taking into account Proposition 3 and performing a similar calculus for $I_2$, we deduce that $I$ is well defined and of class $C^1$, with the Gâteaux derivative defined by

$$\langle I'(u), v \rangle = \int_\Omega a(x, \Delta u) \Delta v \, dx + \int_\Omega b(x)|u|^{p(x)-2}uv \, dx - \lambda \int_\Omega f(x, u)v \, dx.$$

It is obvious that the critical points of the functional $I$ are weak solutions to problem (30). Hence the interest switches to the critical points of $I$ and we can apply a Weierstrass-type theorem in order to obtain the following existence result.

**Theorem 4.2.** Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with smooth boundary and assume hypotheses (p), (b), (f1)-(f2), (a0)-(a3) (without restriction $\bar{c} < 3 \min\{1, \bar{c}\}$) are fulfilled. Then there exists a constant $\lambda_0 > 0$ such that problem (30) has at least a nontrivial weak solution in $V$ for every $\lambda > \lambda_0$.

The proof of Theorem 4.2 is almost identical to [6, Theorem 6], although in [6] the authors work on a different functional space $V$, which is appropriate for the investigation of the fourth order problems with no-flux boundary conditions. There are only a few distinctions between these two proofs, like the fact that we use Proposition 5, Theorem 2.4 and we rely on hypothesis (a3) for an inequality of the type

$$\bar{c}|t|^{p(x)} \leq p(x)A(x, t) \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$ 

Therefore we can skip the details of the proof of Theorem 4.2.

An interesting fact about the hypotheses imposed on $f$ is that, even though $f$ does not satisfy the Ambrosetti-Rabinowitz type condition (not to mention that $f$ is $p(\cdot) - 1$ sublinear at infinity), we can apply a mountain pass type argument in order to obtain a second nontrivial solution. For more information concerning mountain
pass theorems we refer to Pucci and Radulescu [28] and for more variational methods applicable to variable exponent problems we refer to Radulescu and Repovš [29].

**Theorem 4.3.** Let \( \Omega \subset \mathbb{R}^N (N \geq 2) \) be a bounded domain with smooth boundary and assume hypotheses (p), (b), (f1)-(f3), (a0)-(a3) are fulfilled. Then there exists \( \lambda_0 > 0 \) such that problem (30) has at least two nontrivial weak solutions in \( V \) for every \( \lambda > \lambda_0 \) (where \( \lambda_0 \) is the one from Theorem 4.2).

Again, the proof of Theorem 4.3 is similar to [6, Theorem 8], due to (a3), Theorem 3.2 and Theorem 2.4, thus we omit it.

**Remark 2.** The reader can see that one of the main differences between the study of problem (30) and the study of the problem from [6] consists in the properties established in Section 3. In our opinion, these properties are the key in generalizing many other problems that are driven by the \( p(x) \)-biharmonic Laplace operator (see for example the fourth-order problems with variable exponent from [1, 2, 3, 15, 24]) by passing them to the Leray-Lions type operator.

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