Supplementary Materials for Integrative Learning for Population of Dynamic Networks with Covariates

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1 Additional Details for EM Algorithm for Pair-wise Correlations

M-step for γ∗: Under the fused lasso prior on γ∗, the parameter estimates (γ∗h,jl1, ..., γ∗h,jlT), h = 2, ..., H, can be obtained by maximizing the following log-posterior, which we defined as log(π(Γ∗h|−)):  

\[
= \arg \min \sum_{t=1}^{T} \frac{1}{2\sigma^2_{\gamma,h}} \sum_{i=1}^{N} \hat{\psi}_{h,jlt}^{(i)}(\gamma_{jl,t}^{(i)} - \gamma_{jl,t}^{*})^2 + \lambda \sum_{t=1}^{T-1} |\gamma_{jl,t}^{*} - \gamma_{jl,t-1}^{*}|
\]

= \arg \min \sum_{t=1}^{T} \frac{1}{2\sigma^2_{\gamma,h}} \sum_{i=1}^{N} \hat{\psi}_{h,jlt}^{(i)}(\gamma_{jl,t}^{(i)} - \gamma_{jl,t}^{*} + \gamma_{jl,t}^{*} - \gamma_{jl,t}^{*})^2 + \lambda \sum_{t=1}^{T-1} |\gamma_{jl,t}^{*} - \gamma_{jl,t-1}^{*}|

= \arg \min \sum_{t=1}^{T} \frac{1}{2\sigma^2_{\gamma,h}} \sum_{i=1}^{N} \hat{\psi}_{h,jlt}^{(i)}(\gamma_{jl,t}^{(i)} - \gamma_{jl,t}^{*})^2 + 0

+ \sum_{t=1}^{T} \frac{1}{2\sigma^2_{\gamma,h}} \left( \sum_{i=1}^{N} \hat{\psi}_{h,jlt}^{(i)}(\gamma_{jl,t}^{*} - \gamma_{jl,t}^{*})^2 + \lambda \sum_{t=1}^{T-1} |\gamma_{jl,t}^{*} - \gamma_{jl,t-1}^{*}| \right)

= \arg \min \sum_{t=1}^{T} \left\{ \left( \sum_{i=1}^{N} \hat{\psi}_{h,jlt}^{(i)} \right)(\gamma_{jl,t}^{*} - \gamma_{jl,t}^{*})^2 + \lambda \sum_{t=1}^{T-1} |\gamma_{jl,t}^{*} - \gamma_{jl,t-1}^{*}| \right\}, \quad (1)

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where $\tilde{\gamma}_{h,jl,t} = \frac{1}{\sum_{i=1}^{N} \psi_{h,jl,t}^{(i)}} \sum_{i=1}^{N} \psi_{h,jl,t}^{(i)} \hat{\gamma}_{h,jl,t}^{(i)}$, and the cross-product term is zero. Note that $\sum_{i=1}^{N} \hat{\psi}_{h,jlt}^{(i)} \neq 1$. The above can be solved by adapting the GFL algorithm using the following steps. Denote $\gamma_{h,jl} = (\gamma_{h,jl,1}, \ldots, \gamma_{h,jl,T})$, $w_{h,jlt} = \frac{\sum_{i=1}^{N} \hat{\psi}_{h,jlt}^{(i)}}{2\sigma_{h}^{2}}$, denote $\tilde{\gamma}_{h,jlt}^{(w)} = \sqrt{w_{h,jlt}} \hat{\gamma}_{h,jlt}$, and denote $\eta_{h,jl,0} = \gamma_{h,jl,1}$ and $\eta_{h,jl,t-1} = \gamma_{h,jl,t} - \gamma_{h,jl,t-1}$. Then equation (1) can be re-written as

$$
\tilde{\gamma}_{h,jl}^{*} = \arg \min_{T} \sum_{t=1}^{T} \left( \sqrt{w_{h,jlt}} \tilde{\gamma}_{h,jl,t} - \sqrt{w_{h,jlt}} \gamma_{h,jl,t}^{*} \right)^{2} + \lambda \sum_{t=1}^{T-1} |\gamma_{h,jl,t}^{*} - \gamma_{h,jl,t-1}^{*}| \tag{2}
$$

where $\tilde{\eta}_{h,jl} = (\tilde{\eta}_{h,jl,0}, \tilde{\eta}_{h,jl,1}, \ldots, \tilde{\eta}_{h,jl,T-1})$, $\tilde{\eta}_{h,jl,0} = \gamma_{h,jl,1}$, $\tilde{\eta}_{h,jl,t-1} = \gamma_{h,jl,t} - \gamma_{h,jl,t-1}$, and the $T \times T$ matrix $\tilde{M}_{h,jl}$ has the following form

$$
\tilde{M}_{h,jl} = \begin{bmatrix}
\sqrt{w_{h,jl,1}} & 0 & 0 & \ldots & 0 \\
\sqrt{w_{h,jl,2}} & \sqrt{w_{h,jl,2}} & 0 & \ldots & 0 \\
\sqrt{w_{h,jl,3}} & \sqrt{w_{h,jl,3}} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sqrt{w_{h,jl,T}} & \sqrt{w_{h,jl,T}} & \sqrt{w_{h,jl,T}} & \ldots & \sqrt{w_{h,jl,T}}
\end{bmatrix}.
$$

Equation (2) can be solved using a Lasso algorithm with the penalty parameter $\lambda$ being chosen using BIC. The solutions for $\tilde{\eta}_{h,jl}$ can be directly used to recover the estimates for $\gamma_{h,jl}^{*} = (\gamma_{h,jl,1}^{*}, \ldots, \gamma_{h,jl,T}^{*})$.}
2 Additional Details for EM Algorithm for Dynamic Precision Matrix Estimation

M-step for $\omega^*$:

$$\hat{\omega}_{h,vt}^* = \arg \min \sum_{t=1}^{T} \left\{ \left( \frac{\sum_{i=1}^{N} \hat{\psi}_{h,vt}^{(i)}}{2\sigma_{h,h}^2} \right) (\hat{\omega}_{h,vt} - \omega_{h,vt}^*)' (\hat{\omega}_{h,vt} - \omega_{h,vt}^*) + \lambda \sum_{t=1}^{T-1} |\omega^*_{h,jt} - \omega^*_{h,jl,t-1}| \right\},$$

$$= ||\bar{W}_{h,v} - \text{diag}(\sqrt{w_{h,v,1}}, \ldots, \sqrt{w_{h,v,T}})\omega_{h,v,1}^* - M_{h,v} E_{h,v}||^2 + \lambda \sum_{t=1}^{T-1} |e_{h,vt}|_1,$$ (3)

where $\hat{\omega}_{h,vt} = \frac{1}{\sum_{i=1}^{N} \hat{\psi}_{h,vt}^{(i)}} \sum_{i=1}^{N} \hat{\psi}_{h,vt}^{(i)} \omega_{h,v,t}$, $w_{h,v,t} = \frac{\sum_{i=1}^{N} \hat{\psi}_{h,vt}^{(i)}}{2\sigma_{h,h}^2}$, $E_{h,v} = (e_{h,v,1}, \ldots, e_{h,v,T-1})'$ is a $(T - 1) \times (V - 1)$ matrix with the $t$-th row as $e_{h,vt} = (\omega_{h,v,t}^* - \omega_{h,v,t-1}^*)'$, $\bar{W}_{h,v} = (w_{h,v,1}\omega_{h,v,1}', \ldots, w_{h,v,T}\omega_{h,v,T}')$ is a $T \times (V - 1)$ matrix, and

$$M_{h,jt} = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
\sqrt{w_{h,jl,2}} & 0 & 0 & \ldots & 0 \\
\sqrt{w_{h,jl,3}} & \sqrt{w_{h,jl,3}} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sqrt{w_{h,jl,T}} & \sqrt{w_{h,jl,T}} & \sqrt{w_{h,jl,T}} & \ldots & 0 
\end{bmatrix}.$$}

Some additional algebra yields the expression for the estimate provided in M-step for updating $\omega^*$.

3 Computational Details for Change Point Estimation

Generalizing the approach in Tibshirani and Wang (2007) to the multivariate case, we apply lowess independently to each time-series of pair-wise correlations as a first step. Denote the smoothed fit as $\bar{\rho}_q = (\bar{\rho}_{q1}, \ldots, \bar{\rho}_{qT})$ for the $q$-th pair-wise correlation profile, $q = 1, \ldots, V(V - 1)/2$. The fraction parameter in the lowess fit which controls the smooth-
ness level, is chosen to be small so as to avoid oversmoothing which will cause difficulty in
detecting potential change points. Then for each time series, we compute the first order

\[ \delta_{qt} = \bar{\rho}_{qt} - \bar{\rho}_{q,t+1}, t = 1, \ldots, T - 1, \]

followed by the median of \((\delta_{1q}, \ldots, \delta_{q,T-1})\)
denoted as \(\mu_q, q = 1, \ldots, V(V - 1)/2\). Next, we compute the median of the absolute
deviations \(|\delta_{1q} - \mu_q|, \ldots, |\delta_{q,T-1} - \mu_q|\), and denote it as \(\Delta_q, q = 1, \ldots, V(V - 1)/2\). Finally,
we note that equation (2) can be expressed as

\[
\min_{u \in \mathbb{R}^{V(V - 1)/2}} \frac{1}{2} \sum_{t=1}^{T-1} ||\tilde{r}_t - u_t||^2 \\
\text{subject to } \sum_{t=1}^{T-1} ||u_{t+1} - u_t|| \leq s_2, \]

where \(\lambda \propto 1/s_2^2\), with a smaller value of \(s_2\) implying a lesser number
of increments. We choose the threshold parameter as

\[
s_2 = \max_{q=1, \ldots, V(V - 1)/2} \left\{ 2\Delta_q + \sum_{t=1}^{T-1} |\delta_{qt}| 1(|\delta_{qt}| > 4\Delta_q) \right\}, \tag{4}
\]

where the expression inside the parenthesis corresponds to the threshold for the \(q\)-th indi-
vidual pairwise connectivity time series and is motivated by the choice in Tibshirani and
Wang (2007), and where \(1(\cdot)\) is the indicator function. The above expression (4) specifies
the threshold for the increment term in the fused lasso as the maximum of the thresholds
for the individual pairwise connectivity time series, in order to ensure that no true change
point is omitted in the first step. Equation (4) assumes that first order differences with
absolute values greater than \(4\Delta_q\) corresponds to a change in connectivity, and the \(2\Delta_q\) term
ensures that the threshold is not very small and is able to capture all the true change points
accurately with no omissions, even at the cost of detecting spurious change points. Using
the choice of the threshold in (4), we then re-fit the fused lasso to obtain an initial estimate
of the number of change points \((K_{max})\) and their locations \(\tau^* = (t^*_1, \ldots, t^*_{K_{max}})\). The initial
estimate for the number of change points is potentially inflated due to the choice of a small
value of the lowess fraction parameter, and the manner in which \(\lambda\) was computed in (4).
This is to ensure that we do not exclude true change points in the initial fit. In the next
step we propose a screening criteria to exclude false change points in \(\tau^*\).

In particular, the screening criteria involves a post-processing step as follows. For each
given subset of \( k < K_{max} \) change-points, we approximate the signals between the successive change-points with the mean value of the points in that interval. Subsequently, we calculate the total sum of squared errors (SSE) between the set of real signals and these piecewise constant approximations to them. Though it may appear computationally intensive to do this for all subsets of \( k < K_{max} \) change-points, a dynamic programming strategy (Picard et al., 2005) enables us to compute the subset of \( k \) change points having the minimum SSE from among all possible sets of \( k < K_{max} \) change points in \( O(K_{max}^2) \) time. The dynamic programming ensures that the computation is tractable even for a large number of nodes and change points. An exhaustive search becomes impossible for large \( K_{max} \) since the number of partitions of a set with \( T \) elements into \( K_{max} \) segments is \( \binom{T-1}{K_{max}-1} \). The dynamic programming strategy reduces the computation time from \( O(T^{K_{max}}) \) to \( O(T^2) \) using the additive property of SSE. Let \( \beta_{k+1}(i, j) \) denote the SSE corresponding to the best partition of the data between the \( i \)-th and \( j \)-th time points into \( k + 1 \) segments, noting that \( \beta_{k+1}(0, n) = SSE(k + 1) \). The recursive algorithm is as follows

\[
k = 0, \quad \forall 0 \leq i < j \leq T, \quad \beta_1(i, j) = \sum_{t=i+1}^{j} ||\bar{r}_t - \bar{r}_{ij}||^2, \quad \forall k \in [1, K_{max}], \quad \beta_{k+1}(i, j) = \min_h \{\beta_{k}(1, h) + \beta_{1}(h + 1, j)\}, \quad (5)
\]

where \( \bar{r}_{ij} = \frac{1}{j-i} \sum_{t=i}^{j} \bar{r}_t \) is the mean for the vector of sample pairwise correlations between time points \( i \) and \( j \). The above dynamic programming takes advantage of the additivity of the SSE, considering that a partition of the data into \( k + 1 \) segments is a union of a partition into \( k \) optimal segments and a set containing 1 segment, which enables us to efficiently compute the best partition of the data into \( k + 1 \) segments, \( k = 1, \ldots, K_{max} \). Once these optimal partitions corresponding to \( \tau^*_k, k = 1, \ldots, K_{max} \) have been computed via the dynamic programming strategy, one can eliminate spurious change points from this set using the curvature approach described below.

We determine the optimal number of change points by examining the curvature of the
SSE curve as follows. Denote the minimum SSE obtained from the subset of all possible $k$ change points derived from the set $\tau^*$ as $\text{SSE}(k)$, and denote the corresponding locations of the change points as $\tau_k^*$. Clearly, $\text{SSE}(k+1)$, will be smaller than $\text{SSE}(k)$; however, after a certain point, adding an extra change-point will have a negligible impact on the SSE reduction. First we normalize the minimum SSE scores $\text{SSE}(1), \ldots, \text{SSE}(K_{\text{max}})$ as $J_k = \frac{\text{SSE}(K_{\text{max}})-\text{SSE}(k)}{\text{SSE}(K_{\text{max}})-\text{SSE}(1)}(K_{\text{max}}-1)+1$, where $J(1) = K_{\text{max}}$, and $J(K_{\text{max}}) = 1$. Then we compute the curvature of these normalized scores via discrete second derivatives as $\nabla_k = J_{k-1} - 2J_k + J_{k+1}$, and select the number of change points as $K = \max\{1 < k < K_{\text{max}} : \nabla_k > 0.5\}$ such that the second derivative does not rise above a certain threshold on addition of further change points. The choice of the threshold 0.5 is recommended in earlier papers on change point estimation (Picard et al., 2005), and worked adequately well for our applications. The idea behind this approach is that if the curvature of the normalized SSE scores does not change beyond a certain value by adding an extra change point, then that will imply that the SSE does not reduce significantly, suggesting that no additional change points are required. In our experience based on extensive numerical studies, this approach is able to eliminate any false change points included in the initial set of change points $\tau^*$ and produces reliable estimates.

4 Prewhitening step

Typically, the prewhitening step is applied to the pre-processed and de-meaned fMRI data (with zero mean), before applying the algorithms for estimating functional connectivity. These prewhitening steps usually first fit a ARIMA model to the fMRI data separately for each voxel/region of interest, and subsequently compute the residuals from fitting these models, which are then used for computing the brain functional connectivity. For example, a prewhitening procedure involving an AR(1) model would first fit $y_t = \phi y_{t-1} + e_t, t = 1, \ldots, T$, separately for each individual, and then use the residual $r_t = y_t - \hat{\phi}y_{t-1}, t = 1, \ldots, T$, for
computing the brain functional network, where $\hat{\phi}$ is the estimated autocorrelation coefficient. The ARIMA model used for prewhitening includes autoregressive models and moving average models as special cases, and can also accommodate certain types of non-stationary patterns. Since the true temporal autocorrelations are unknown, we use the ‘auto.arima’ function in R to compute the residuals for prewhitening, that automatically determines the model order for the ARIMA model from the time-series data, and subsequently computes the residuals based on the fitted ARIMA model. In our experience, applying a prewhitening step based on the ARIMA model was sufficient for minimizing the temporal correlations even under non-stationary time-series, such that the proposed approaches applied to the prewhitened measurements were able to accurately estimate the true dynamic network and detect underlying connectivity change points.

5 Additional Numerical Results

The standard deviations corresponding to the reported point estimates in Tables 3 and 4 of the manuscript are presented below, along with Tables that report sensitivity analysis results corresponding to our simulation scenarios. In particular, we conduct sensitivity analysis for (i) the number of mixture components $H$ as well as for the prior variance on the covariate effects $\Sigma_\beta$ for both the dynamic pairwise correlation and dynamic precision matrix approaches; (ii) the hyperparameters for $\pi(\sigma^2_{\gamma,h})$ corresponding to the dynamic pairwise correlation method; and (iii) hyperparameters for $\pi(\sigma^2_{\omega,h})$, as well as parameters $\alpha$ corresponding to the dynamic precision matrix approach.

References

1. A statistical approach for array CGH data analysis. Picard F, Robin S, Lavielle M, Vaisse C, Daudin JJ (2005). BMC Bioinformatics. 6:27.
| Results for V=40 | Network CP Edge CP MSE | Network CP Edge CP MSE |
|------------------|------------------------|------------------------|
|                  | sens FP sens FP MSE    | sens FP sens FP MSE    |
| BPMM-PC           |                        |                        |
| GGM+Erdos-Renyi   | (0.04) (1.23) (0.06) (0.15) (0.02) | (0) (0.18) (0.03) (0.12) (0.01) |
| GGM+Small-world   | (0.04) (1.41) (0.05) (0.15) (0.02) | (0.01) (0.19) (0.03) (0.13) (0.01) |
| GGM+Scale-free    | (0.05) (1.23) (0.05) (0.17) (0.02) | (0) (0.17) (0.03) (0.13) (0.01) |
| SD+GFL           |                        |                        |
| GGM+Erdos-Renyi   | (0.06) (0.22) (0.05) (0.22) (0.07) | (0.03) (0.18) (0.05) (0.31) (0.03) |
| GGM+Small-world   | (0.05) (0.25) (0.05) (0.25) (0.06) | (0.04) (0.21) (0.05) (0.31) (0.04) |
| GGM+Scale-free    | (0.05) (0.24) (0.04) (0.26) (0.06) | (0.04) (0.18) (0.04) (0.33) (0.03) |
| CCPD             |                        |                        |
| VAR+Erdos-Renyi   | (0.07) (1.31) (0.07) (0.15) (0.04) | (0.06) (1.22) (0.05) (0.12) (0.01) |
| VAR+Small-world   | (0.08) (1.26) (0.06) (0.15) (0.04) | (0.06) (1.2) (0.06) (0.11) (0.01) |
| VAR+Scale-free    | (0.06) (1.22) (0.08) (0.14) (0.05) | (0.07) (1.21) (0.05) (0.11) (0.01) |
| SD+GFL           |                        |                        |
| VAR+Erdos-Renyi   | (0.05) (1.44) (0.04) (0.25) (0.06) | (0.05) (0.16) (0.05) (0.41) (0.04) |
| VAR+Small-world   | (0.04) (1.39) (0.04) (0.23) (0.07) | (0.06) (0.2) (0.05) (0.33) (0.04) |
| VAR+Scale-free    | (0.04) (1.42) (0.05) (0.25) (0.05) | (0.06) (0.17) (0.06) (0.33) (0.04) |

| Results for V=100 | Network CP Edge CP MSE | Network CP Edge CP MSE |
|-------------------|------------------------|------------------------|
|                  | sens FP sens FP MSE    | sens FP sens FP MSE    |
| BPMM-PC           |                        |                        |
| GGM+Erdos-Renyi   | (0.03) (0.98) (0.06) (0.22) (0.02) | (0) (0.21) (0.04) (0.14) (0.01) |
| GGM+Small-world   | (0.03) (1.02) (0.05) (0.2) (0.01) | (0) (0.2) (0.05) (0.15) (0.02) |
| GGM+Scale-free    | (0.03) (0.99) (0.06) (0.18) (0.01) | (0) (0.19) (0.05) (0.14) (0.01) |
| SD+GFL           |                        |                        |
| GGM+Erdos-Renyi   | (0.05) (0.38) (0.04) (0.27) (0.06) | (0.05) (0.13) (0.06) (0.41) (0.03) |
| GGM+Small-world   | (0.05) (0.33) (0.05) (0.25) (0.06) | (0.06) (0.14) (0.06) (0.4) (0.04) |
| GGM+Scale-free    | (0.06) (0.33) (0.05) (0.24) (0.07) | (0.05) (0.18) (0.05) (0.44) (0.04) |
| CCPD             |                        |                        |
| VAR+Erdos-Renyi   | (0.07) (1.22) (0.08) (0.15) (0.03) | (0.05) (1.31) (0.05) (0.13) (0.02) |
| VAR+Small-world   | (0.08) (1.19) (0.08) (0.15) (0.03) | (0.05) (1.26) (0.06) (0.14) (0.02) |
| VAR+Scale-free    | (0.07) (1.24) (0.07) (0.14) (0.02) | (0.04) (1.28) (0.06) (0.13) (0.02) |
| SD+GFL           |                        |                        |
| VAR+Erdos-Renyi   | (0.05) (1.24) (0.04) (0.23) (0.04) | (0.05) (0.14) (0.06) (0.42) (0.04) |
| VAR+Small-world   | (0.04) (1.33) (0.04) (0.21) (0.06) | (0.06) (0.2) (0.05) (0.32) (0.04) |
| VAR+Scale-free    | (0.04) (1.32) (0.04) (0.21) (0.06) | (0.06) (0.16) (0.06) (0.29) (0.04) |

Table 1: Standard deviation for the reported metrics under the dynamic pair-wise correlation approaches for network and edge-level connectivity change-point estimation corresponding to Table 4 in the manuscript. GGM and VAR correspond to data generated from Gaussian graphical models and vector autoregressive models. Significantly improved metrics among the four approaches corresponding to the GGM data and separately for the VAR data, are highlighted in bold.
| Results for V=40 | Network CP sens | Edge CP sens | MSE | F1 | Network CP sens | Edge CP sens | MSE | F1 |
|-----------------|-----------------|--------------|-----|----|-----------------|--------------|-----|----|
|                 | BPMM-PM         | idPMAC       |     |    |                 |               |     |    |
| GGM+Erdos-Renyi | (0.05) (1.44)   | (0.06) (0.13) | (0.01) (0.05) | (0) (1.07) | (0.03) (0.14) | (0.01) (0.04) |
| GGM+Small-world | (0.04) (1.28)   | (0.06) (0.1)  | (0.01) (0.05) | (0) (0.99) | (0.03) (0.14) | (0.01) (0.05) |
| GGM+Scale-free  | (0.03) (1.16)   | (0.07) (0.12) | (0.01) (0.06) | (0.02) (1.21) | (0.04) (0.11) | (0.01) (0.05) |
|                 | DCR             | SINGLE       |     |    |                 |               |     |    |
| GGM+Erdos-Renyi | (0.03) (2.16)   | (0.05) (1.27) | (0.04) (0.04) | (0.07) (1.18) | (0.02) (0.44) | (0.02) (0.04) |
| GGM+Small-world | (0.03) (1.86)   | (0.05) (1.22) | (0.04) (0.04) | (0.06) (1.09) | (0.01) (0.52) | (0.01) (0.04) |
| GGM+Scale-free  | (0.04) (1.77)   | (0.05) (1.19) | (0.04) (0.05) | (0.07) (1.14) | (0.01) (0.41) | (0.01) (0.04) |
|                 | BPMM-PM         | idPMAC       |     |    |                 |               |     |    |
| VAR+Erdos-Renyi | (0.07) (0.89)   | (0.07) (0.16) | (0.04) (0.04) | (0.06) (0.88) | (0.05) (0.13) | (0.01) (0.04) |
| VAR+Small-world | (0.06) (0.91)   | (0.07) (0.16) | (0.04) (0.05) | (0.05) (0.81) | (0.05) (0.14) | (0.01) (0.05) |
| VAR+Scale-free  | (0.06) (0.84)   | (0.06) (0.22) | (0.04) (0.05) | (0.05) (0.83) | (0.05) (0.13) | (0.03) (0.05) |
|                 | DCR             | SINGLE       |     |    |                 |               |     |    |
| VAR+Erdos-Renyi | (0.03) (1.88)   | (0.08) (0.61) | (0.04) (0.05) | (0.09) (1.22) | (0.03) (0.41) | (0.06) (0.04) |
| VAR+Small-world | (0.03) (1.91)   | (0.07) (0.66) | (0.04) (0.06) | (0.1) (1.05) | (0.03) (0.38) | (0.06) (0.04) |
| VAR+Scale-free  | (0.03) (1.72)   | (0.07) (0.67) | (0.04) (0.05) | (0.09) (0.99) | (0.03) (0.44) | (0.05) (0.04) |

| Results for V=100 | Network CP sens | Edge CP sens | MSE | F1 | Network CP sens | Edge CP sens | MSE | F1 |
|-------------------|-----------------|--------------|-----|----|-----------------|--------------|-----|----|
|                  | BPMM-PM         | idPMAC       |     |    |                 |               |     |    |
| GGM+Erdos-Renyi  | (0.03) (1.37)   | (0.04) (0.13) | (0.01) (0.05) | (0.03) (1.02) | (0.05) (0.12) | (0.01) (0.04) |
| GGM+Small-world  | (0.04) (1.31)   | (0.05) (0.1)  | (0.01) (0.05) | (0) (0.99)   | (0.03) (0.14) | (0.01) (0.05) |
| GGM+Scale-free   | (0.04) (1.34)   | (0.05) (0.12) | (0.01) (0.05) | (0.02) (1.27) | (0.04) (0.09) | (0.01) (0.05) |
|                  | DCR             | SINGLE       |     |    |                 |               |     |    |
| GGM+Erdos-Renyi  | (0.03) (2.22)   | (0.05) (1.99) | (0.04) (0.04) | (0.07) (1.22) | (0.02) (0.44) | (0.02) (0.04) |
| GGM+Small-world  | (0.04) (2.3)    | (0.05) (2.09) | (0.04) (0.05) | (0.06) (1.19) | (0.01) (0.47) | (0.01) (0.04) |
| GGM+Scale-free   | (0.04) (2.09)   | (0.05) (1.87) | (0.04) (0.05) | (0.07) (1.31) | (0.01) (0.4)  | (0.01) (0.04) |
|                  | BPMM-PM         | idPMAC       |     |    |                 |               |     |    |
| VAR+Erdos-Renyi  | (0.11) (0.91)   | (0.07) (0.16) | (0.04) (0.06) | (0.06) (0.77) | (0.05) (0.13) | (0.01) (0.04) |
| VAR+Small-world  | (0.09) (0.89)   | (0.07) (0.16) | (0.04) (0.05) | (0.05) (0.77) | (0.05) (0.12) | (0.01) (0.05) |
| VAR+Scale-free   | (0.09) (0.89)   | (0.06) (0.22) | (0.05) (0.04) | (0.05) (0.68) | (0.05) (0.13) | (0.03) (0.07) |
|                  | DCR             | SINGLE       |     |    |                 |               |     |    |
| VAR+Erdos-Renyi  | (0.03) (1.66)   | (0.08) (0.61) | (0.04) (0.05) | (0.09) (1.2)  | (0.03) (0.44) | (0.06) (0.04) |
| VAR+Small-world  | (0.03) (1.74)   | (0.07) (0.58) | (0.04) (0.06) | (0.1) (1.05) | (0.03) (0.38) | (0.05) (0.04) |
| VAR+Scale-free   | (0.03) (1.72)   | (0.07) (0.57) | (0.04) (0.05) | (0.09) (1.04) | (0.03) (0.44) | (0.05) (0.04) |

Table 2: Standard deviation for reported metrics under the dynamic precision matrix approaches corresponding to Table 5 in the manuscript. GGM and VAR correspond to data generated from Gaussian graphical models and vector autoregressive models. Significantly improved metrics among the four approaches corresponding to the GGM data and separately for the VAR data, are highlighted in bold.
| idPAC | Clustering Error | Network CP | MSE | Clustering Error | Network CP | MSE |
|-------|------------------|------------|-----|------------------|------------|-----|
|       | CE   | VI | sens | FP | CE   | VI | sens | FP |
| V=40  |       |     |      |    |       |     |      |    |
|       | 0    | 0  | 1    | 2.77 | 0    | 0  | 1    | 2.31 |
|       | 0    | 0  | 0.99 | 2.74 | 0    | 0  | 0.99 | 2.37 |
|       | 0    | 0  | 0.99 | 2.76 | 0    | 0  | 0.99 | 2.36 |
| V=100 |       |     |      |    |       |     |      |    |
|       | 0    | 0  | 1    | 2.31 | 0    | 0  | 1    | 2.28 |
|       | 0    | 0  | 0.99 | 2.37 | 0    | 0  | 0.99 | 2.28 |
|       | 0    | 0  | 0.99 | 2.36 | 0    | 0  | 0.99 | 2.36 |
| H=6   |       |     |      |    |       |     |      |    |
|       | 0    | 0  | 1    | 2.81 | 0    | 0  | 0.98 | 2.27 |
|       | 0    | 0  | 1    | 2.77 | 0    | 0  | 0.99 | 2.28 |
|       | 0    | 0  | 0.99 | 2.68 | 0    | 0  | 0.98 | 2.3  |
| H=7   |       |     |      |    |       |     |      |    |
|       | 0    | 0  | 1    | 2.79 | 0    | 0  | 0.98 | 2.33 |
|       | 0    | 0  | 1    | 2.74 | 0    | 0  | 0.97 | 2.35 |
|       | 0    | 0  | 0.99 | 2.68 | 0    | 0  | 0.97 | 2.33 |
| H=8   |       |     |      |    |       |     |      |    |
|       | 0    | 0  | 1    | 2.79 | 0    | 0  | 1    | 2.33 |
|       | 0    | 0  | 1    | 2.74 | 0    | 0  | 0.97 | 2.35 |
|       | 0    | 0  | 0.99 | 2.68 | 0    | 0  | 0.97 | 2.33 |

Hyperparameters in $\pi(\sigma^2_{\gamma_i}), i.e. (a_{\sigma}, b_{\sigma}) = (0.1, 10)$

|       | 0.02 | 0.22 | 0.96 | 3.21 | 0.1 | 0.02 | 0.21 | 0.96 | 3.11 | 0.1 |
|       | 0    | 0   | 1    | 2.8  | 0.08 | 0.01 | 0.12 | 0.97 | 3.04 | 0.1 |
|       | 0    | 0   | 0.99 | 2.68 | 0.09 | 0    | 0   | 0.99 | 2.78 | 0.09 |

Hyperparameters in $\pi(\sigma^2_{\gamma_i}), i.e. (a_{\sigma}, b_{\sigma}) = (0.1, 25)$

|       | 0.02 | 0.2  | 1    | 2.79 | 0.09 | 0.01 | 0.14 | 0.97 | 2.98 | 0.1 |
|       | 0.01 | 0.14 | 1    | 2.77 | 0.08 | 0.02 | 0.22 | 0.97 | 3.06 | 0.11 |
|       | 0.01 | 0.13 | 0.99 | 2.7  | 0.09 | 0.01 | 0.15 | 0.96 | 2.89 | 0.09 |

$\Sigma_{\beta} = 10 * I_q$

|       | 0    | 0   | 0.96 | 2.72 | 0.09 | 0    | 0   | 0.97 | 2.48 | 0.09 |
|       | 0    | 0   | 0.99 | 2.77 | 0.09 | 0    | 0   | 0.99 | 2.32 | 0.09 |
|       | 0    | 0   | 1    | 2.68 | 0.08 | 0    | 0   | 1    | 2.29 | 0.09 |

$\Sigma_{\beta} = 200 * I_q$

|       | 0    | 0   | 1    | 2.65 | 0.08 | 0    | 0   | 0.98 | 2.32 | 0.09 |
|       | 0    | 0   | 0.94 | 2.81 | 0.09 | 0    | 0   | 0.98 | 2.33 | 0.09 |
|       | 0    | 0   | 0.97 | 2.75 | 0.08 | 0    | 0   | 0.97 | 2.39 | 0.1 |

Table 3: Sensitivity analysis for hyper-parameters of dynamic pairwise correlation method.
| idPMAC         | V=40 | V=100 |
|---------------|------|-------|
|                | CE   | VI    | sens | FP  | MSE  | CE   | VI    | sens | FP  | MSE  |
| GGM+Erdos Renyi | 0    | 0     | 0.99 | 5.17| 0.09 | 0    | 0     | 1    | 5.27| 0.09 |
| GGM+Scale Free | 0    | 0     | 1    | 5.68| 0.09 | 0.04| 0.06  | 0.98 | 5.34| 0.1  |
| GGM+Small World | 0.04| 0.07  | 0.99 | 5.81| 0.09 | 0    | 0     | 0.99 | 5.28| 0.09 |
| GGM+Erdos Renyi | 0    | 0     | 0.97 | 5.81| 0.08 | 0    | 0     | 0.98 | 5.41| 0.1  |
| GGM+Scale Free | 0    | 0     | 0.97 | 5.66| 0.08 | 0    | 0     | 0.97 | 5.33| 0.09 |
| GGM+Small World | 0    | 0     | 0.99 | 5.77| 0.09 | 0    | 0     | 0.98 | 5.3  | 0.09 |
| GGM+Erdos Renyi | 0    | 0     | 0.97 | 5.81| 0.08 | 0    | 0     | 0.97 | 5.42| 0.09 |
| GGM+Scale Free | 0    | 0     | 0.97 | 5.85| 0.09 | 0    | 0     | 1    | 5.29| 0.08 |
| GGM+Small World | 0    | 0     | 0.99 | 5.44| 0.09 | 0    | 0     | 0.97 | 5.33| 0.09 |

Hyperparameters in $\pi(\sigma^2_{\omega,h})$, i.e. $(a_\sigma, b_\sigma) = (0.1, 10)$

|                | CE   | VI    | sens | FP  | MSE  | CE   | VI    | sens | FP  | MSE  |
| GGM+Erdos Renyi | 0.02 | 0.22  | 0.96 | 5.44| 0.09 | 0.02| 0.24  | 0.95 | 5.5  | 0.1  |
| GGM+Scale Free | 0.02 | 0.22  | 0.96 | 5.64| 0.1  | 0.03| 0.3   | 0.95 | 5.48 | 0.09 |
| GGM+Small World | 0.02 | 0.31  | 0.94 | 5.81| 0.1  | 0.03| 0.27  | 0.96 | 5.28 | 0.1  |

Hyperparameters in $\pi(\sigma^2_{\omega,h})$, i.e. $(a_\sigma, b_\sigma) = (0.1, 25)$

|                | CE   | VI    | sens | FP  | MSE  | CE   | VI    | sens | FP  | MSE  |
| GGM+Erdos Renyi | 0.02 | 0.24  | 0.95 | 5.28| 0.09 | 0.02| 0.24  | 0.95 | 5.72| 0.1  |
| GGM+Scale Free | 0.03 | 0.24  | 0.93 | 4.99| 0.11 | 0.01| 0.16  | 0.98 | 5.44| 0.09 |
| GGM+Small World | 0.01 | 0.32  | 0.99 | 5.16| 0.1  | 0.03| 0.35  | 0.93 | 5.69| 0.1  |

$\alpha = 0.01$

|                | CE   | VI    | sens | FP  | MSE  | CE   | VI    | sens | FP  | MSE  |
| GGM+Erdos Renyi | 0    | 0     | 1    | 5.77| 0.09 | 0    | 0     | 0.99 | 4.91| 0.09 |
| GGM+Scale Free | 0    | 0     | 1    | 5.75| 0.09 | 0.04| 0.06  | 0.98 | 5.31| 0.09 |
| GGM+Small World | 0    | 0     | 0.99 | 5.69| 0.09 | 0    | 0     | 0.98 | 4.97| 0.09 |

$\alpha = 0.1$

|                | CE   | VI    | sens | FP  | MSE  | CE   | VI    | sens | FP  | MSE  |
| GGM+Erdos Renyi | 0    | 0     | 0.98 | 5.42| 0.09 | 0    | 0     | 0.96 | 5.08| 0.09 |
| GGM+Scale Free | 0    | 0     | 0.98 | 5.66| 0.09 | 0.04| 0.06  | 1    | 4.91| 0.09 |
| GGM+Small World | 0    | 0     | 0.99 | 5.63| 0.09 | 0    | 0     | 0.97 | 5.22| 0.09 |

$\Sigma_\beta = 10 * I_q$

|                | CE   | VI    | sens | FP  | MSE  | CE   | VI    | sens | FP  | MSE  |
| GGM+Erdos Renyi | 0    | 0     | 0.95 | 5.51| 0.1  | 0    | 0     | 0.99 | 5.18| 0.09 |
| GGM+Scale Free | 0    | 0     | 0.99 | 5.12| 0.09 | 0    | 0     | 0.98 | 5.28| 0.09 |
| GGM+Small World | 0    | 0     | 0.98 | 5.68| 0.09 | 0    | 0     | 0.98 | 5.53| 0.09 |

$\Sigma_\beta = 200 * I_q$

|                | CE   | VI    | sens | FP  | MSE  | CE   | VI    | sens | FP  | MSE  |
| GGM+Erdos Renyi | 0    | 0     | 0.98 | 5.34| 0.09 | 0    | 0     | 0.99 | 5.47| 0.09 |
| GGM+Scale Free | 0    | 0     | 0.99 | 4.99| 0.08 | 0    | 0     | 0.99 | 5.62| 0.08 |
| GGM+Small World | 0    | 0     | 1    | 5.48| 0.09 | 0    | 0     | 0.1  | 5.22| 0.09 |

Table 4: Sensitivity analysis for hyper-parameters of dynamic precision matrix method.