Methods for the Nonperturbative Approximation of Form Factors and Scattering Amplitudes$^1$

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Abstract. Methods are described for the nonperturbative calculation of wave functions and scattering amplitudes in light-cone quantization. Form factors are computed from the boost-invariant wave functions, which appear as coefficients in a Fock-state expansion of the field-theoretic eigenstate. A technique is proposed for calculating scattering amplitudes from matrix elements of a $T$ operator between such composite-particle eigenstates.

INTRODUCTION

To benefit from the recent progress on the calculation of field-theoretic bound states in light-cone quantization $[1,2]$, we explore methods by which form factors and scattering amplitudes can be extracted nonperturbatively. In the case of form factors, this is relatively straightforward; well-known formulas $[3]$ yield the form factors as overlap integrals of Fock-state wave functions. For scattering amplitudes, the way is less certain. One possible method $[4]$ is discussed briefly here. Others have been considered by Kröger $[5]$, Ji and Surya $[6]$, and Fuda $[7]$.

The formulations given are in terms of light-cone coordinates $[8,1]$, where $x^+ \equiv t + z$ plays the role of time and the conjugate variable $p^- \equiv E - p_z$ is the light-cone energy. The light-cone three-momentum is $\mathbf{p} = (p^+ \equiv E + p_z, \mathbf{p}_\perp)$. An eigenstate $|P, \sigma\rangle$ of the light-cone Hamiltonian operators $\mathcal{P}^\pm$, $\mathcal{P}_\perp$ and helicity $\sigma$ is written as a Fock-state expansion

$$|P, \sigma\rangle = \sum_n \int [dx][d^2k_\perp] \psi^{(n)}_{P, \sigma}(x, \mathbf{k}_\perp) |n : \mathbf{p}\rangle,$$

with

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\[
\int [dx][d^2k_\perp] = \int \delta(1 - \sum_i x_i) \prod_i \frac{dx_i}{\sqrt{x_i}} 16\pi^3 \delta(\sum_i k_{\perp,i}) \prod_i \frac{d^2k_{\perp,i}}{16\pi^3} \tag{2}
\]

and where the $\psi^{(n)}$ are wave functions for $n$ particles, $x_i \equiv p_i^+/P^+$ are longitudinal momentum fractions, and $k_{\perp,i} = p_{\perp,i} - x_i P_\perp$ are relative transverse momenta. Use of light-cone coordinates brings several advantages, including boost invariance of the wave functions.

The eigenvalue problem $\mathcal{P}|P,\sigma\rangle = P|P,\sigma\rangle$ for fixed $\sigma$ determines the wave functions as solutions of a coupled set of integral equations. A method frequently applied to these equations is discrete light-cone quantization (DLCQ) \cite{9,1}, which approximates the integrals by the trapezoidal rule and computes the wave functions on an equally spaced momentum grid. Any bound-state property can then, in principle, be calculated from these wave functions. The grid is parameterized by a longitudinal resolution $K$ and transverse resolution $N_\perp$, such that longitudinal momentum fractions are multiples of $1/K$ and transverse momenta have as many as $2N_\perp + 1$ values in each direction. The value of $N_\perp$ is associated with a cutoff $\Lambda^2$ on the invariant mass of each constituent and with the choice of transverse momentum scale $\pi/L_\perp$.

**FORM FACTORS**

For a spin-1/2 fermion, the two form factors can be obtained from matrix elements of the plus component of the electromagnetic current $J$

\[
F_1(Q^2) = \frac{1}{2} \langle P + Q, \sigma | J^+(0)/P^+ | P, \sigma \rangle, \tag{3}
\]

\[
- \left( \frac{Q_x - iQ_y}{2M} \right) F_2(Q^2) = \frac{1}{4\sigma} \langle P + Q, \sigma | J^+(0)/P^+ | P, -\sigma \rangle. \tag{4}
\]

These can be reduced to overlap integrals \cite{3}

\[
F_1(Q^2) = \sum_n \sum_j e_j \int [dx][d^2k_\perp] \psi^{(n)*}_{P+Q,1/2}(x,k'_\perp) \psi^{(n)}_{P,1/2}(x,k_\perp), \tag{5}
\]

\[
- \left( \frac{Q_x - iQ_y}{2M} \right) F_2(Q^2) = \sum_n \sum_j e_j \int [dx][d^2k_\perp] \psi^{(n)*}_{P+Q,1/2}(x,k'_\perp) \psi^{(n)}_{P,-1/2}(x,k_\perp), \tag{6}
\]

in the frame where the photon momentum $Q$ is written $(0,2Q \cdot P/P^+, Q_\perp)$ and

\[
k'_\perp i = \begin{cases} k_{\perp,i} - x_i Q_\perp, & i \neq j \\ k_{\perp,j} + (1 - x_j) Q_\perp, & i = j. \end{cases} \tag{7}
\]

For the model studied by Brodsky, Hiller, and McCartor \cite{2}, an explicit calculation of $F_1$ has been done \cite{10}. In this model, a bare fermion acts as a source and sink for bosons of mass $\mu$. The lowest massive eigenstate is a fermion dressed by a boson cloud. The theory is regulated by a Pauli–Villars boson \cite{11} with an
imaginary coupling, and renormalized by fits of physical quantities to "data." Because no spin-flip interactions are included, $F_2$ is zero. Results for $F_1$ are shown in Fig. 1. The large-momentum-transfer value of $F_1$ is the bare fermion probability and therefore is not zero.

**FIGURE 1.** The form factor $F_1$ for fixed longitudinal resolution $K = 9$ and transverse scale $L_\perp = 2\pi/\mu$, and for a particular set of model parameters. Various cutoffs $\Lambda^2$ are considered, with the transverse resolution $N_\perp$ ranging from 5 to 9.

**SCATTERING AMPLITUDES**

The center-of-mass cross section for two-body scattering ($A + B \rightarrow C + D$) is [12]

$$\frac{d\sigma}{d\Omega_{cm}} = \frac{1}{2E_A2E_Bv_{rel}} \frac{|\vec{p}_C||M_{fi}|^2}{16\pi^2E_{cm}},$$

where $M_{fi}$ is the invariant amplitude obtained from the $S$ matrix

$$S_{fi} = \langle f|i\rangle + (2\pi)^4\delta^{(4)}(p_f - p_i)iM_{fi} = \delta_{CD,AB} - 2\pi i \delta(s_{AB} - s_{CD})T_{LCfi},$$

with $s_{AB} = \frac{m_A^2 + p_{A/\perp}^2}{p_A^+/P^+} + \frac{m_B^2 + p_{B/\perp}^2}{p_B^+/P^+}$. The $T$ matrix for scattering of composites is given by [4,13]

$$T_{LCfi} = P^+T_{fi} = \langle C|V^\dagger_D\frac{1}{s_{AB} + i\epsilon - H_{LC}}V_B|A\rangle + \langle C|DV_B|A\rangle.$$  

Here $|A\rangle$ and $|C\rangle$ are composite-particle eigenstates of the light-cone Hamiltonian $H_{LC}$, and the operator $V_B$ is defined by
\[ V_B = [H_{LC}, B^\dagger] - \frac{m_B^2 + p_B^2}{p_B^+/P^+} B^\dagger, \]

with \( B^\dagger \) the creation operator for the \( B \) particle, \textit{i.e.} \( |B\rangle = B^\dagger|0\rangle \). This construction generalizes one presented some time ago by Wick [13]. Details can be found in Ref. [4]. Given numerical solutions for the composite-particle eigenstates, obtained with DLCQ, the most difficult remaining task is the estimation of the matrix element of \((s + i\epsilon - H_{LC})^{-1}\). For this type of matrix element, the recursion method of Haydock [14] has worked well. A nonrelativistic application is described in [4]; an application to the field-theoretic model studied in [2] is in progress.

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