Maximally Supersymmetric RG Flows and AdS Duality

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We discuss four dimensional renormalization group flows which preserve sixteen supersymmetries. In the infra-red, these can be viewed as deformations of the $\mathcal{N} = 4$ superconformal fixed points by special, irrelevant operators. It is argued that the gauge coupling beta function continues to vanish identically, for all coupling constants and energy scales, for such RG flows. In addition, the dimensions of all operators in short supersymmetry representations are constant along such flows. This is compatible with a conjectured generalization of the AdS/CFT correspondence which describes such flows, e.g. the D3 brane vacuum before taking the near-horizon limit. RG flows in three and six dimensions, preserving 16 supersymmetries, are also briefly discussed, including a discussion of generalized AdS/CFT duality for the M2 and M5 brane cases. Finally, we discuss maximally supersymmetric RG flows associated with non-commutative geometry.
1. Introduction

Four dimensional $\mathcal{N} = 4$ Super-Yang-Mills theories have 32 conserved supercharges: 16 ordinary ones, $Q^I_\alpha$ and $\overline{Q}_{\dot{\alpha} I}$, and 16 additional superconformal supercharges, $S_{\alpha I}$ and $\overline{S}_{\dot{\alpha} I}$, with $\alpha, \dot{\alpha} = 1, 2$ spinor indices and $I = 1 \ldots 4$ in the fundamental of the global $SU(4)_R$ symmetry. The 16 additional, superconformal supercharges are associated with the fact that the theory is conformally invariant, with $\beta_\tau(\tau) \equiv 0$ for arbitrary gauge coupling and theta angle $\tau \equiv \frac{\theta}{2\pi} + 4\pi ig_{YM}^2$. We will here be interested in theories which preserve the 16 ordinary supersymmetries but are not conformally invariant. These are the maximally supersymmetric renormalization group flows (introducing more supersymmetries either makes the theory conformally invariant or necessitates adding gravity and other higher spin fields).

The infrared endpoints of such RG flows are the usual $\mathcal{N} = 4$ superconformal theories with 32 supercharges. The RG flows can be viewed as these RG fixed points with additional perturbing operators, which preserve the 16 supersymmetries and become irrelevant in the infrared. The least irrelevant such operator is a dimension 8 operator of the form $\text{Tr} F^4 + \ldots$ (the $\ldots$ are terms related by the 16 supersymmetries; in terms of $\mathcal{N} = 1$ supersymmetry, it is $\int d^4 \theta \text{Tr} [W^2 \overline{W}^2] + \ldots$); this operator is an $SU(4)_R$ singlet. There is another dimension 8, $SU(4)_R$ singlet, scalar operator, which also preserves 16 supersymmetries, given by $\text{Tr} F^2 \text{Tr} F^2 + \ldots$. Because these two operators have the same quantum numbers, it is exceedingly difficult to tell them apart (see [1])—so we will not bother doing so. We will refer to either operator, or a general mixture of them, as $O_H$.

Other operators which preserve the 16 supersymmetries are of dimension $8 + n$, of the form $\text{Tr} F^4 \phi^n + \ldots$, and in the $SU(4)_R$ representation with Dynkin indices $(0, n, 0)$, for arbitrary integer $n \geq 0$. Again, we will not bother distinguishing between these and multi-trace analogs of these operators with the same quantum numbers. All of these operators are in short supersymmetry representations and are of the form $Q^7 \overline{Q}^4 O_{\text{short}}$. There are also 16 supersymmetry preserving operators in long representations of the supersymmetry algebra, whose dimensions can vary with $g_{YM}$; these operators are of the form $Q^8 \overline{Q}^8 O_{\text{long}}$ and thus have dimension larger than 8.

We will argue in the next section that certain properties known to hold for the 4d $\mathcal{N} = 4$ superconformal fixed points continue to hold along RG flows which preserve the 16 supersymmetries. In particular, the gauge coupling $g_{YM}$ and theta angle do not change with the energy scale $\mu$ along such flows, i.e. $\beta_\tau(\tau) \equiv \mu \frac{d}{d\mu} \tau = 0$ for all $\tau$. Even though
the theory is not scale invariant, there are exactly marginal operators, $O_\tau$ and $O_{\overline{\tau}}$, which deform $\tau$ and $\overline{\tau}$ respectively. This is depicted in figure 1. More generally, there are still short representations of the supersymmetry algebra and the dimensions of all operators in such representations are independent of the energy scale along these RG flows. Briefly, the argument is that the dimension of the stress tensor is not renormalized, because it is a conserved current, and that this, along with supersymmetry, is enough to fix all operator dimensions in all short multiplets to also not be renormalized.

\begin{figure}
\centering
\begin{tikzpicture}
\draw[help lines] (0,0) grid (8,6);
\draw[->] (0,0) -- (8,0) node[below] {$g$};
\draw[->] (0,0) -- (0,6) node[left] {$h$};
\end{tikzpicture}
\caption{RG flows to the IR, preserving 16 supersymmetries. Coefficients $h$ of perturbing operators flow to zero, while the gauge coupling $g$ remains constant along the flow.}
\end{figure}

To give a concrete example of a RG flow preserving 16 supersymmetries, consider deforming a $\mathcal{N} = 4$ superconformal theory by moving away from the origin of the moduli space of vacua, giving an expectation value to the adjoint scalars $\phi$, which Higgses the gauge group $G$ to a subgroup such as $H \times U(1)$. This deformation leads to a non-trivial renormalization group flow, with 16 supersymmetries preserved along the entire flow; the additional 16 superconformal symmetries emerge in the UV and IR limits of the flow. The UV limit is the $\mathcal{N} = 4$ superconformal theory with gauge group $G$ and the IR limit is the $\mathcal{N} = 4$ superconformal theory with gauge group $H \times U(1)$. The fact that the gauge coupling $\tau$ is exactly (i.e. including all quantum effects, both perturbative and non-perturbative) constant along such renormalization group flows is well known: e.g. the Seiberg-Witten curve for the $\mathcal{N} = 4$ theory gives a gauge coupling which does not depend on the coordinates of the Coulomb branch. (This is a special case of the fact that the gauge coupling does not depend on the Higgs branch moduli in $\mathcal{N} = 2$ supersymmetric theories because of the decoupling there of vector and hypermultiplets.)

Viewed from the IR, the above RG flow is the $H \times U(1)$ gauge theory, perturbed by various irrelevant operators which preserve the 16 supersymmetries, such as the dimension
8 operator $O_H$ mentioned above. We will consider more general RG flows preserving 16 supersymmetries, without worrying about the UV starting point. If necessary, we can imagine that there is a UV cutoff, which preserves the 16 supersymmetries, below the scale where the irrelevant operators might blow up.

As will be discussed in sect. 3, D-branes and $AdS$ duality [2] motivate considering RG flows preserving 16 supersymmetries. In this section we will make contact with the works [3,4]. We will discuss a conjectured extension of the duality between field theories and gravity theories away from the $AdS$ limit. For example, IIB string theory in asymptotically flat 10d spacetime (with $N$ D3 branes infinitely far away), is conjectured to be holographically dual to the UV limit of $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory with a particular choice of irrelevant operators. The value of $N$ is arbitrary.

Section 4 is devoted to 3d RG flows with 16 preserved supercharges. The IR fixed points are the 3d $\mathcal{N} = 8$ superconformal theories with 32 supercharges. Several peculiarities are noted. The flows which are associated with $M$ theory vacua containing M2 branes are discussed.

Section 5 is devoted to 6d RG flows with 16 preserved supercharges. The IR fixed points are the 6d $\mathcal{N} = (2,0)$ superconformal field theories with 32 supercharges. The flows which are associated with $M$ theory vacua containing M5 branes are discussed.

Section 6 discusses maximally supersymmetric RG flows associated with non-commutative geometry. The RG flow of fig. 1, with $g_{YM}$ constant, is verified in the string vacua which are proposed to be holographically dual to the field theory RG flows.

In the appendices we list, for convenience, results obtained in [5,6] (see also [7]) on the short representations of the superconformal group, with 32 supercharges, in 3d and 6d.

2. The power of 16 supersymmetries

Recall that the superconformal theory, with 32 supercharges, has two types of operator representations: short and long. The representations are generated by a primary operator $O_P$, along with descendent operators, related to $O_P$ by supersymmetry, of the form $Q^n \overline{Q}^m O_P$, and their conformal descendents. It should be understood that $Q^n \overline{Q}^m O_P$ is a shorthand for a nested graded commutator of the operator $O_P$ with $n$ of the supercharges $Q^I$ and $m$ of the supercharges $\overline{Q}_I$, e.g. $Q^2 \overline{Q}O_P$ should be understood as a shorthand for $[Q, \{Q, \overline{Q}, O_P\}]$. The remaining 16 superconformal supercharges, $S$ and $\overline{S}$, act on these representations as lowering operators. For the generic, long, representation, the operators
\( Q^n\overline{Q}^m O_P \) truncate at \( n \leq 8 \) and \( m \leq 8 \) by Fermi statistics. Taking \( n > 8 \) or \( m > 8 \) in \( Q^n\overline{Q}^m O_P \) gives zero, up to a total derivative. The dimensions of long operators are not constrained by supersymmetry and depend on \( g_{YM} \) as well as the gauge group.

The short representations have the defining property that they instead truncate at \( n \leq 4 \) and \( m \leq 4 \); they are the \( \mathcal{N} = 4 \) version of chiral superfields. The spectrum of short representations was found in [8] and their table, with our present field-theory operator notation, can be found reproduced in [9]. The primary operators which generates the short representations are 
\[
O_P \sim [\text{Tr}_G(\phi^p)]_{(0,p,0)},
\]
where \( \phi \) is the \( \mathcal{N} = 4 \) scalar in the adjoint of the gauge group \( G \) and the \( 6 = (0, 1, 0) \) of the \( SU(4)_R \) global symmetry and \( (0, p, 0) \) are the Dynkin labels of the \( SU(4)_R \) representation. In addition, there are multi-trace short representations with primary operator 
\[
O^{(p_i)}_P = [\prod_i \text{Tr}_G(\phi^{p_i})]_{(0,p,0)}, \quad \text{with} \quad p = \sum_i p_i.
\]
These primary operators all have dimension \( \Delta = p \), independent of \( g_{YM} \). As mentioned in the introduction, we won’t bother distinguishing between single and multi-trace operators with the same quantum numbers. The descendent \( Q^n\overline{Q}^m O_P \) has dimension \( p + \frac{1}{2}(n + m) \).

The short multiplet associated with the primary operator \( O_2 \sim [\text{Tr}_G(\phi^2)]_{(0,2,0)} \) is very special. It contains the conserved currents: the stress-tensor, \( T_{\mu\nu} \sim Q^2\overline{Q}^2 O_2 \), \( SU(4)_R \) currents, \( J^a_\mu \sim Q\overline{Q}O_2 \), and supercurrents \( j^I_\mu \sim Q^2\overline{Q}O_2 \) and \( \tilde{j}^\alpha I_\mu \sim \overline{Q}Q O_2 \).

In addition, \( O_2 \) contains the exactly marginal scalar operators \( O_\tau \sim Q^4 O_2 \) and \( \overline{O}_\tau \sim \overline{Q}^4 O_2 \). These operators can be added to the Lagrangian density without breaking any supersymmetries, as \( Q \) and \( \overline{Q} \) annihilate them up to total derivatives. Indeed, adding \( \delta \tau O_\tau \) to the Lagrangian changes the gauge coupling \( \tau \to \tau + \delta \tau \), which preserves all the supersymmetries. Adding \( O_\tau \) also exactly preserves conformal invariance, and thus the full superconformal group, as \( O_\tau \) has dimension exactly 4 and is thus exactly marginal. This corresponds to the fact that the \( \mathcal{N} = 4 \) super-Yang-Mills theory has exactly vanishing beta function for all \( \tau \), \( \beta_\tau(\tau) \equiv 0 \).

Now consider a non-conformal theory, with 16 conserved supercharges. Such a theory can be obtained by starting, for example, with the conformal Lagrangian density \( \mathcal{L}_0 \) and adding terms which are annihilated, up to total derivatives, by the supercharges \( Q \) and \( \overline{Q} \): 
\[
\mathcal{L} = \mathcal{L}_0 + \sum_p h_p Q^4\overline{Q}^4 O_P + \sum_i h_i^{(L)} Q^8\overline{Q}^8 O_i^{(L)},
\] 
(2.1)
where \( \mathcal{O}_p \) are the short primary operators, including multi-trace operators, and \( \mathcal{O}^{(L)}_i \) are generic long primary operators. The \( h_p \) and \( h^{(L)}_i \) are some real coupling constant parameters, which we will often refer to collectively as \( h \), as in figure 1. The short multiplet operator \( Q^4 \overline{Q}^4 \mathcal{O}_p \) is in the \((0, p - 4, 0)\) representation of \( SU(4)_R \) (it vanishes, up to total derivatives, for \( p < 4 \)). The operator \( \mathcal{O}_H \equiv Q^4 \overline{Q}^4 \mathcal{O}_{p=4} \) is the least irrelevant such perturbation and is an \( SU(4)_R \) singlet. Again, the short operator \( \mathcal{O}_{p=4} \) here can actually be any combination of the single trace operator \([\text{Tr} G(\phi^4)](0,4,0)\) and the double trace operator \([\text{Tr} G(\phi^2)\text{Tr} G(\phi^2)](0,4,0)\). The deformation (2.1) by the short operators \( Q^4 \overline{Q}^4 \mathcal{O}_p \) breaks \( SU(4)_R \) for \( p > 4 \). There are many long operators \( Q^8 \overline{Q}^8 \mathcal{O}^{(L)}_i \) which preserve the \( SU(4)_R \) symmetry, for example there are \( SU(4)_R \) singlets in \( Q^8 \overline{Q}^8 [\text{Tr} G(\phi^p)](0,0,0) \), as well as many which break \( SU(4)_R \).

The deformed theory (2.1) again has short multiplets, with primary operator \( \mathcal{O}_p \) in the \((0, p, 0)\) representation of \( SU(4)_R \) given by the same single or multi-trace operators involving \( \phi^p \). These short multiplets again consist of operators \( Q^n \overline{Q}^m \mathcal{O}_p \), with \( n \leq 4 \) and \( m \leq 4 \), and with \( Q \) or \( \overline{Q} \) acting on \( Q^4 \overline{Q}^4 \mathcal{O}_p \) vanishing up to total derivatives. Indeed, this is necessary for (2.1) to preserve the 16 supersymmetries (see footnote 1). This is the \( \mathcal{N} = 4 \) extension of the statement in \( \mathcal{N} = 1 \) supersymmetry that chiral operators form short supersymmetry multiplets, whether or not the theory is conformally invariant.

Let’s initially restrict our attention to theories (2.1) which preserve the \( SU(4)_R \) symmetry. It is then clear that all of the \((p + 2)^2((p + 2)^2 - 1)/12\) operators in the primary operator \( \mathcal{O}_p \) (this is the dimension of the \((0, p, 0)\) representation of \( SU(4)_R \)) must have the same dimension \( \Delta_p \). A’ priori, as the theory is not conformally invariant, \( \Delta_p \) can

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1 The notation in (2.1) is slightly deceptive: the supercharges \( Q \) and \( \overline{Q} \) will themselves depend on the parameters \( h \), so the Lagrangian is not necessarily linear in the parameters \( h \). One can find the form of the Lagrangian via an iterative procedure, first taking the \( Q \) in (2.1) to be independent of the \( h \), then find the first order in \( h \) correction to \( Q \) and plug back in (2.1) to find the next order correction to \( Q \), and so on, until one obtains a Lagrangian which is invariant under its conserved supercharges. This is similar to the situation in \( \mathcal{N} = 1 \) supersymmetry, where one adds a superpotential \( \delta L = Q^2 W + c.c. \). The above iterative procedure properly leads to a Lagrangian containing both the linear term \( W'' \psi \bar{\psi} \) and the quadratic term \( \sim |W|^2 \).

2 We can discuss the \( SU(4)_R \) transformation properties of operators even if (2.1) violates \( SU(4)_R \). We can think of the parameters \( h \) in (2.1) as expectation values of background fields which carry charge under \( SU(4)_R \) so that (2.1) is invariant, replacing explicit \( SU(4)_R \) breaking with spontaneous breaking, much as in [10]. \( SU(4)_R \) thus still constrains the theory and can lead to selection rules for the \( h \).
depend on the RG scale $\mu$, as well as the gauge group, $g_{YM}$, and the parameters $h$ in (2.4).

However, we will now argue that this is not the case.

In terms of $\mathcal{N} = 1$ supersymmetry, the real scalars $\phi$ in the $(0,1,0)$ representation of $SU(4)_R$ are the bosonic components of chiral superfields $\Phi$ in the $3_{2/3}$ of $SU(3)_F \times U(1)_R \subset SU(4)_R$ and anti-chiral superfields $\bar{\Phi}$ in the $\bar{3}_{-2/3}$. The operators $\mathcal{O}_p$ then contains the bosonic component of purely chiral superfields $\text{Tr}\Phi^p$ in the $[(p+2)(p+1)/2]_{2p/3}$ representation of $SU(3)_F \times U(1)_R$, along with the conjugate, purely anti-chiral superfields, and mixed operators, which contain both chiral and anti-chiral superfields. We now use the fact that $\mathcal{N} = 1$ supersymmetry implies that the purely chiral superfields form a “chiral ring,” with purely additive anomalous dimensions at all scales. Thus the purely chiral superfields $\text{Tr}\Phi^p$ must have dimensions $\Delta_p = p\Delta_1$ for some $\Delta_1$ which, a’ priori, might still depend on the RG scale.

By the $SU(4)_R$ symmetry, all operators in $\mathcal{O}_p$, in the $(0,p,0)$ $SU(4)_R$ representation, must then have dimension $\Delta_p = p\Delta_1$ for all RG scales and coupling constant parameters. Similarly, using the supersymmetry algebra, the supersymmetry descendents $Q^n\overline{Q}^m\mathcal{O}_p$ must have dimension $\frac{1}{2}(n+m) + p\Delta_1$ for all RG scales and coupling constant parameters.

Even though the theory is not conformally invariant, the conserved currents are given by the supersymmetry descendents of the $\mathcal{O}_2$ operators. As in the conformal theory, the conserved stress tensor is $Q^2\overline{Q}^2\mathcal{O}_2$ and the conserved supercurrents are $Q^2\overline{Q}\mathcal{O}_2$ and $Q\overline{Q}^2\mathcal{O}_2$. By the discussion above, the stress tensor $T_{\mu\nu}$ will thus have dimension $2 + 2\Delta_1$.

But, because $T_{\mu\nu}$ is a conserved current, its scaling dimension can not be renormalized: it must be exactly 4. Thus $\Delta_1 \equiv 1$ and the dimensions of all short operators are not renormalized: $Q^n\overline{Q}^m\mathcal{O}_p$ has dimension exactly given by $p + \frac{1}{2}(n+m)$ for all RG scales and coupling constant parameters.

In particular, the operator $\mathcal{O}_\tau \sim Q^4\mathcal{O}_2$, which can be added to the Lagrangian to change the gauge coupling $\tau$, has dimension exactly 4 for all RG scales and coupling constant parameters. Thus $\delta \mathcal{L} \sim \delta \tau \int d^4x \mathcal{O}_\tau$ is an exactly marginal deformation for all scales and coupling constants, which implies that the gauge coupling beta function continues to vanish identically, $\beta_\tau(\tau, h) \equiv 0$, even though the theory is not conformally invariant. The gauge coupling $\tau$ does not vary with the RG scale, as depicted in figure 1.

We now relax our restriction that the deformation (2.1) preserves the $SU(4)_R$ symmetry. One might then worry that the dimensions of the different operators in $\mathcal{O}_p$, which were previously equal as they were in the irreducible $(0,p,0)$ representation of $SU(4)_R$, could now be split. But any possible splitting must be compatible with imposing the $SU(4)_R$
symmetry which can be regarded as being unbroken provided the parameters \( h \) in (2.1) are assigned appropriate \( SU(4)_R \) transformation properties. The only natural choice is to have the possible splitting of the operator dimensions be a combination of the parameters \( h \) which is in the \((0, p, 0)\) representation of \( SU(r)_R \). Similarly, the possible splittings of the operators in \( Q\mathcal{O}_p \) should be in the same representation as this collection of operators, the \((0, p - 1, 1)\) representation of \( SU(4)_R \). But, for any operator in \( \mathcal{O}_p \) with dimension \( \Delta \), the corresponding operator in \( Q\mathcal{O}_p \) must have dimension \( \Delta + \frac{1}{2} \); this would not be compatible with having non-trivial splittings of the \( \mathcal{O}_p \) operators dimensions in the \((0, p, 0)\) of \( SU(4)_R \) and \( Q\mathcal{O}_p \) in the \((0, p - 1, 1)\). We thus conclude that all operators in \( Q^n\mathcal{O}_p \) which are related by \( SU(4)_R \) rotations must have the same operator dimensions, even if (2.1) breaks \( SU(4)_R \). The above argument that these operators all must then have unrenormalized dimensions, for all RG scales and coupling constant parameters, then goes through exactly as in the case where (2.1) respects \( SU(4)_R \).

Note that it is necessary to have 16 preserved supersymmetries for the above arguments to go through. For example, if \( \mathcal{N} = 4 \) is broken to \( \mathcal{N} = 2 \) (via adjoint masses) the gauge beta function no longer vanishes. This can happen because the operator \( \mathcal{O}_\tau \) associated with changing the gauge coupling is no longer in the same supermultiplet as the conserved stress tensor \( T_{\mu\nu} \), and can thus obtain an anomalous dimension.

Finally, the reader might wonder about any possible connections between the non-renormalization of operator dimensions discussed here and other non-renormalization theorems concerning \( \mathcal{N} = 4 \) theories, which have been of recent interest [11-14] in the context of the matrix description of \( M \) theory. The non-renormalization theorems of [11-14] concern the effective action for the pure \( \mathcal{N} = 4 \) super-Yang-Mills theory along the Coulomb branch, where the unbroken gauge group is generically the Abelian Cartan torus \( U(1)^r \) of the gauge group. The low-energy theory is then \( \mathcal{N} = 4 \), with \( U(1)^r \) gauge group and 16 supersymmetry preserving deformations, of the form (2.1), which arise via integrating out the massive gauge bosons of the original gauge group. In this situation, where the scalar expectation values are generic and the low-energy gauge group is purely abelian, the IR fixed point is a free-field theory, which is perturbed by the irrelevant operators in (2.1).

Because the IR theory is free in this case, the statement of the present paper, that certain operator dimensions are not renormalized, is essentially trivial in this context. The non-trivial non-renormalization theorems of [11-14], in the context of the IR free theories, concerns the exact expressions for the coefficients \( h \) appearing in (2.1). Perhaps it is also possible to obtain exact expressions for some of the coefficients \( h \), appearing in a more general effective action of the form (2.1), even in our present context, where the IR theory is non-Abelian, and thus not free.
3. Holography and RG flows

A motivation for considering 4d RG flows with 16 preserved supersymmetries comes from the AdS/CFT correspondence \[2\] and some conjectured extensions. Recall that type IIB string theory has vacua with metric

$$ds^2 = H^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2} \delta_{ab} dy^a dy^b,$$

(3.1)

with coordinates $x^\mu$ spanning $\mathbb{R}^{1,3}$ and $y^a$, which we often write as $\vec{y}$, spanning $\mathbb{R}^6$. $H(y)$ is an arbitrary function which satisfies the $y^a$ Laplace equation, allowing for possible delta function source terms: $\Delta_y H = -2\pi^{1/2} \kappa \sum_{i=1}^N \delta^6(\vec{y} - \vec{a}_i)$, where the $\vec{a}_i$ are arbitrary vectors in $\mathbb{R}^6$. These vacua have $F_5 \sim \kappa^{-1}(1 + \ast)(dx)^4 \wedge dH^{-1}$, with $N$ units of integrated $F_5$ flux. They generally preserve 16 supersymmetries. A point which we would especially like to emphasize is that these vacua all have constant axio-dilaton $\tau$, for arbitrary $\tau$ in the upper half plane. We will connect this with the statement in the previous section that $\beta_\tau \equiv 0$ with 16 supersymmetries.

A special case, which preserves the $SO(6) \cong SU(4)$ symmetry of rotating $\vec{y}$, is

$$H = h + \frac{R^4}{r^4},$$

(3.2)

where $h$ is an arbitrary real constant, $R^4 \equiv 4\pi g_s \alpha'^2 N \equiv N\kappa/2\pi^{5/2}$, and $r^2 = \delta_{ab} y^a y^b \equiv \vec{y} \cdot \vec{y}$. Choosing the constant $h = 1$, the metric is the D3 brane metric \[15\], which asymptotes for $r \to \infty$ to flat, 10 dimensional space-time. For $r \to 0$, there is a tube in $r$ with an $S^5$ section of radius $R$ and non-singular horizon at $r = 0$. Rather than choosing $h = 1$, one could take $h = 0$, which is the $AdS_5 \times S^5$ vacuum of IIB string theory. The two cases, $h = 1$ and $h = 0$, are clearly asymptotically the same in the near horizon limit, $r \to 0$, but differ drastically for $r \to \infty$. While the $h = 1$ case approaches flat 10d space-time in this limit, the $h = 0$ case always remains $AdS_5 \times S_5$, with the $AdS_5$ coordinate $r$ approaching a boundary which $AdS_5$ has at $r \to \infty$. It is this boundary (rather than the horizon) where operators are inserted in the prescription of \[16,17\].

The $h = 1$ case preserves 16 supersymmetries for generic $r$ and asymptotically preserves an additional 16 supersymmetries in the limits $r \to 0$ and $r \to \infty$. In the $r \to 0$ limit, the symmetries combine into the supergroup $PSU(2,2|4)$, which is the 4d $\mathcal{N} = 4$ superconformal group. The $h = 0$ case, $AdS_5 \times S^5$, identically has the $PSU(2,2|4)$ superconformal symmetry group, with the 32 supersymmetries, for all $r$. 

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The works [18,19,20] considered the $h = 1$ case and compared the asymptotic scattering of bulk waves from $r \to \infty$ to the dynamics of the world-volume gauge theory of the D3 brane. Similarly, [21] started from the $h = 1$ case and then conjectured duality with the world-volume gauge theory in the $r \to 0$ near-horizon limit.

By starting instead with the $h = 0$ case, it is not necessary to take the near horizon limit: the $h = 0$ case is conjectured to be holographically dual, for all energies and for all $r$, to $\mathcal{N} = 4$ super-Yang-Mills, with gauge group $SU(N)$. The fact that the dilaton is an arbitrary constant $\tau$ corresponds to the fact that the $\mathcal{N} = 4$ Super-Yang-Mills theory is exactly conformally invariant for arbitrary $\tau$. The fact that the $h = 0$ case has symmetry group $PSU(2,2|4)$ for all $r$ shows that the holographic dual is precisely the $\mathcal{N} = 4$ super-Yang-Mills theory and not, for example, some deformation of this theory, such as Born-Infeld, which would not be exactly conformally invariant.

On the other hand, following [3,4] we would like to interpret the case with $h \neq 0$ as being holographically dual to a deformation of $\mathcal{N} = 4$ super-Yang-Mills. Interpreting $r$ as the renormalization group energy scale parameter, this deformation flows to the $\mathcal{N} = 4$ superconformal fixed point in the infrared, $r \to 0$, as $h$ becomes irrelevant in this limit. The deformation of $\mathcal{N} = 4$ is thus by operators which become irrelevant in the IR. In [3] the deformation was regarded as the replacement of the $\mathcal{N} = 4$ Yang-Mills Lagrangian with a Born-Infeld generalization (though no such non-Abelian $\mathcal{N} = 4$ Born-Infeld action is known as of yet). A simpler possibility for the IR irrelevant deformation was conjectured in [4] on the basis of $PSU(2,2|4)$ representation theory. This will be discussed further in what follows. (We also mention the possibility that the seemingly simpler deformation actually is $\mathcal{N} = 4$ Born Infeld.)

The more general vacuum solution is given by (3.1) with

$$H(y) = h + 4\pi g_s \alpha'^2 \sum_{i=1}^{k} \frac{N_i}{|\vec{y} - \vec{a}_i|^4},$$

for some integers $N_i$ and vectors $\vec{a}_i$ in $\mathbb{R}^6$. For $h = 1$ this corresponds to separating $N = \sum_i N_i$ D3 branes, placing $N_i$ at $\vec{y}_i = \vec{a}_i$. We can choose the origin of $\vec{y}$ so that $\sum_i N_i \vec{a}_i = 0$. The general solution (3.3) does not preserve $SU(4)_R$ but, again, does preserve 16 supersymmetries. Also, we again emphasize that the dilaton is constant with the general solution (3.3).

We note that the metric (3.1), with (3.2), is invariant under the transformation:

$$x_\mu \to \lambda x_\mu, \quad r \to \lambda^{-1} r, \quad h \to \lambda^4 h, \quad d\Omega_5 \to d\Omega_5,$$

for some integer $\lambda$. This is the non-Abelian generalization of the usual $SO(6)$ rotation of the D3 branes. The asymptotic behavior of the solutions depends on the sign of $\lambda$.
with $\lambda$ an arbitrary real parameter and $g_s$ and $\alpha'$ held fixed. Similarly, the more general solution (3.3) is invariant under the transformation

$$x_\mu \rightarrow \lambda x_\mu, \quad \vec{y} \rightarrow \lambda^{-1} \vec{y}, \quad h \rightarrow \lambda^4 h, \quad \vec{a}_i \rightarrow \lambda^{-1} \vec{a}_i. \quad (3.5)$$

This transformation leaves $F_5 \sim \kappa^{-1}(1 + \ast)(dx)^4 \wedge dH^{-1}$ invariant, so it preserves the $F_5$ flux quantization condition. This transformation is a symmetry of the full, interacting, IIB supergravity, with the various fields, e.g. $F_5$, the anti-symmetric $B_{\mu\nu}$ fields and, in particular, the dilaton, all invariant.

We expect that the above transformation is an exact symmetry of the full IIB string theory in these vacua with $F_5$ flux. For $h = 0$, this symmetry is the exact gauge symmetry of the string theory vacuum under the dilatation element of the symmetry group $PSU(2,2|4)$. It is on a similar footing as the full translational invariance of the flat $\mathbb{R}^{1,9}$ vacuum of IIB string theory. In the field theory dual, the transformation (3.4) or (3.3) are interpreted as the renormalization group flow, with $\lambda \rightarrow 0$ in the UV and $\lambda \rightarrow \infty$ in the IR. The fact that $\vec{y}$ or $r$ scale oppositely from $x_\mu$ is the well-known UV-IR correspondence \cite{22}. The fact that $g_s = g_{YM}^2$ is invariant under this transformation corresponds to the fact that the Yang-Mills gauge coupling beta function vanishes identically along RG flows.

For $h = 0$ and all $\vec{a}_i = 0$, the invariance under (3.4) corresponds to the fact that the dual theory is conformally invariant and thus unchanged by RG flow. For $h \neq 0$, this symmetry is broken, but can be regarded as being restored provided $h$ transforms as in (3.4). In other words, the transformation of $h$ as in (3.4) exactly compensates for the fact that the theory is not invariant under scale transformations, with the theory unchanged under the combined transformation (3.4). This means that, at length scales $x$, the theory only depends on $h$ via the effective coupling $h_{eff} \sim h/x^4$, which is invariant under (3.4).

Similarly, for $\vec{a}_i \neq 0$, the transformation in (3.5) indicates that the theory only depends on this parameter via the invariant combination $\vec{a}_{i,eff} \sim \vec{a}_i x$.

We interpret the power of $x$ in $h_{eff} \sim h/x^4$ as showing that, in the holographic dual 4d field theory, $h$ is a coupling constant which multiplies a term of energy dimension exactly 4 in the action $S$, i.e. a term of energy dimension exactly 8 in the Lagrangian density $L$. For any fixed $h$, $h_{eff}$ indeed vanishes at long distances, $x \rightarrow \infty$, which is the statement that the theory flows to the conformal $\mathcal{N} = 4$ RG fixed point in the far IR. Similarly, the parameters $\vec{a}_i$ correspond to scalar $\vec{\phi}$ expectation, which Higgs $SU(N) \rightarrow S(\prod_i U(N_i))$. The fact that the theory only depends on $\vec{a}_{i,eff} \sim \vec{a}_i x$ is simply the statement that $\vec{\phi}$ has exact energy dimension 1.
We now argue that the theory actually only depends on $h$ via the dimensionless quantity $h_{\text{eff}} = hR^4/x^4$. Consider the scale transformation

$$x^\mu \rightarrow x^\mu, \quad g_s \rightarrow g_s, \quad h \rightarrow \lambda^{-4}h, \quad \alpha' \rightarrow \lambda^2\alpha', \quad \vec{y} \rightarrow \lambda^2\vec{y}, \quad \vec{a}_i \rightarrow \lambda^2\vec{a}_i.$$  \hspace{1cm} (3.6)

Unlike (3.4) and (3.5), the metric (3.1) is not invariant under (3.6), but rather scales as $ds^2 \rightarrow \lambda^2 ds^2$; also unlike (3.4) and (3.5), the string tension (and thus the gravitational coupling $\kappa$) scales under (3.6). Since $\alpha' \sim (\text{length})^2$ and $ds^2 \sim (\text{length})^2$, the transformation in (3.6) amounts to a rescaling of all lengths by a factor of $\lambda$. Since all lengths in string theory are measured relative to $\alpha'$, IIB string theory and all of its scattering amplitudes must be invariant under (3.6). Because the absorption probability for waves scattered from $r = \infty$ to $r = 0$ do not depend on $r$, the scaling of $r$ in (3.6) is immaterial. The essential point is that the absorption probability only depends on $h$ via the dimensionless quantity $h_{\text{eff}} = hR^4/x^4$, which is invariant under (3.4), (3.5), and (3.6). We have chosen to write $h_{\text{eff}}$ in terms of $R$ rather than $\alpha'$ because the leading order supergravity results depend on $\alpha'$, $g_s$, and $N$ via the combination $R^4 \equiv 4\pi g_s\alpha'^2N$. Away from the leading order supergravity limit, there can be additional explicit dependence on $g_s$ and $N$.

Similarly, $\vec{a}_{i,\text{eff}} = \vec{a}_i x/R^2$ is the quantity which is invariant under (3.5) and (3.6).

So the conjecture is that IIB string theory in the vacuum (3.1) with (3.2), for general $h$, is holographically dual to the 4d field theory with Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + (\text{const.}) \, hR^4 O_H,$$  \hspace{1cm} (3.7)

where $\mathcal{L}_0$ is the Lagrangian of the $\mathcal{N} = 4$ superconformal theory, with gauge group $G = SU(N)$, and $O_H$ is an operator, of dimension exactly 8, which preserves 16 supersymmetries, and which preserves the $SU(4)_R$ symmetry. $O_H$ is thus $Q^4\bar{Q}^4 O_{p=4}$, where $O_{p=4}$ can, again, be a combination of $[\text{Tr}_{G}(\phi^4)]_{(0,4,0)}$ and $[\text{Tr}_{G}(\phi^2)\text{Tr}_{G}(\phi^2)]_{(0,4,0)}$. For $h = 1$, this was originally conjectured in [4] based on the fact that $O_H$ is the unique scalar $SU(4)_R$ singlet (besides $O_\tau$ and $O_{\bar{\tau}}$) in the short representations of $PSU(2,2|4)$.

As mentioned in footnote 1, the deformation (3.7) is actually not linear in $h$, as the supercharges in $O_H = Q^4\bar{Q}^4 O_4$ get $h$ dependent corrections from (3.7). It is interesting to speculate that perhaps (3.7) actually generates the full $\mathcal{N} = 4$ Born Infeld Lagrangian.

In line with our discussion in the previous section, the gauge coupling should be an arbitrary constant, for all RG scales, with the Lagrangian (3.7). This is clearly the case in the holographic gravity dual, as the axio-dilaton is an arbitrary constant with the vacuum
(3.1) and (3.2) or (3.3). In particular, it does not vary with the coordinate $r$, which corresponds to the RG scale. Also, based on (3.4), we argued that $h$ should multiply an operator of dimension exactly 8 in the Lagrangian, and this should be the case for arbitrary $h$. This is in line with our argument in the previous section that the dimensions of operators in short representations are not modified by deformations (2.1), of which (3.7) is a special case.

While the dimension 8 operator $O_H$ is the unique short representation operator which could enter in the conjectured field theory dual (3.7) of IIB string theory with (3.1) and (3.2), there could also be long operators, as in (2.1), which preserve the 16 supersymmetries and $SU(4)_R$. Consider such an operator $Q^8\bar{Q}^8O_{\text{long}}$, of operator dimension $\Delta(g_{YM})$, which generally depends on $g_{YM}$. To preserve the invariance under (3.4), the parameter $h$ would have to multiply such a term in the Lagrangian with a power $h^{(\Delta(g_{YM})-4)/4}$. This $g_{YM}$ dependent power of $h$ would lead to a non-trivial $g_{YM}$ RG running, in contrast to the fact that the axio-dilaton remains constant. It thus seems likely that (3.7) is actually the exact\footnote{However, (3.7) leads to contact terms whose cancellation should require additional counter-terms (e.g. $\sim (hO_H)^n$); I thank G. Moore and S. Shatashvili for stressing this point to me.} holographic dual of IIB string theory with (3.2), without any additional short or long operators, for any $g_{YM}$ and $N$.

The above, conjectured, duality between the IIB string theory with (3.2) and the field theory with (3.7) is conjectured to hold for all RG scales. In particular, the UV limit of this flow corresponds to $r \to \infty$ and thus gives a field theory which is holographically dual to the asymptotically flat 10-dimensional, Minkowski space-time! Note that we are defining the UV limit of a RG flow in terms of an irrelevant perturbation of the IR fixed point, i.e. defining the UV theory via reversing the RG flow. This requires fine-tuning the coefficients of every irrelevant operator so that the Lagrangian is exactly (3.7) at all scales.

As already mentioned, the invariance (3.4) is to be interpreted as RG flow in the dual field theory. The correlation functions of operators in the dual field theory thus satisfy a corresponding Ward identity

$$\left(4h\frac{\partial}{\partial h} + \sum_{i=1}^{n} (x_i \frac{\partial}{\partial x_i} + \Delta_i) \right) \left( \prod_{i=1}^{n} O_i(x_i) \right)_h = 0, \quad (3.8)$$

where $\Delta_i$ are the operator dimensions and the $h$ subscript is a reminder that the expectation value is in the theory deformed by the parameter $h$. Using the fact that the
Lagrangian is (3.7), with the $h$ term the only part which violates scale invariance, we have, $T_\mu^\nu \sim hR^4O_H$. Thus, for arbitrary correlation functions,

$$4h \frac{\partial}{\partial h} \langle \prod_{i=1}^n O_i(x_i) \rangle_h = - \int d^4y \langle T_\mu^\nu(y) \prod_{i=1}^n O_i(x_i) \rangle_h. \quad (3.9)$$

It then follows that (3.8) is the Callan-Symanzik Ward identity for correlation functions in a theory which is not conformally invariant,

$$\sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + \Delta_i) \langle \prod_{i=1}^n O_i(x_i) \rangle_h = \int d^4y \langle T_\mu^\nu(y) \prod_{i=1}^n O_i(x_i) \rangle_h. \quad (3.10)$$

By the argument of the previous section, the dimensions $\Delta_i$ of operators in short representations are not renormalized and are independent of $h$.

In particular, (3.8) implies that 2-point functions are of the form

$$\langle O_\Delta(x)O_\Delta(0) \rangle_h = \frac{c_\Delta}{x^{2\Delta}} f_\Delta(h_{\text{eff}}, g_{YM}, N), \quad (3.11)$$

where $c_\Delta$ is a $h$-independent constant, $h_{\text{eff}} = hR^4/x^4$, and $f_\Delta(h_{\text{eff}})$ is a function which can be normalized to equal 1 for $h_{\text{eff}} = 0$, which is the far IR limit. For long operators, the dimension $\Delta$ appearing in the exponent in (3.11) could also be a function $\Delta(h_{\text{eff}}, g_{YM}, N)$. However, for short operators, as we have argued in the previous section, the dimension $\Delta$ is an unrenormalized constant, independent of $g_{YM}$, $h$, and the RG scale. For example, the 2-point function of $O_\tau$ with its conjugate operator $O_{\bar{\tau}}$ should be given by (3.11) with $\Delta \equiv 4$.

This can be compared with the calculation of [4], where the dilaton 2-point function was computed in the theory with $h = 1$ via a supergravity computation of the corresponding partial-wave absorption cross section. Restoring the $h$ dependence via the argument which follows (3.6), the result of [4] is

$$\langle O_\tau(x)O_{\bar{\tau}}(0) \rangle = \frac{3(N^2 - 1)}{\pi^4 x^8} f\left(\frac{hR^4}{x^4}\right), \quad (3.12)$$

for a function $f(t)$ which was determined in terms of solutions of Mathieu’s equation. Note, in particular, that this is indeed of the form (3.11), with the $x$ exponent, $2\Delta = 8$, an unrenormalized constant and not a non-trivial function $\Delta(h_{\text{eff}})$ of $h_{\text{eff}} = hR^4/x^4$.

---

As discussed in [8] all leading supergravity results for correlation functions are proportional to $h^{-1} \sim R^3 \kappa^{-2}_5 \sim N^2 - 1$, with the replacement of $N^2$ by $N^2 - 1$ presumably coming from a one-loop string correction to the relation between $\kappa_{10}$ and $\kappa_5$ in $S^5$ dimensional reduction.
So the result of [4] is compatible with our statement that the dimension of $O_r$ is not renormalized. A scale-dependent renormalization of the dimension of $O_r$ was obtained in [4] because the right hand side of (3.10), involving $T^\mu_i$, was omitted. Again, the fact that the dimension of $O_r$ is precisely 4, for all RG scales and $h$, agrees with the fact that the gauge coupling is an arbitrary constant, which does not change with the RG scale. This agrees with the fact that the dilaton is constant, independent of $r$ and $x_\mu$, for the vacuum (3.2).

The function $f(t)$ of (3.12) is given by [4] as

$$f(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_{n,k} t^n \left(\frac{1}{2} \log t \right)^k,$$

(3.13)

with coefficients $c_{n,k}$ which are independent of $g_{YM}$ and $N$, at least in the large $N$ limit, and given in [4] via a complicated expression. The first few terms quoted in [4] are

$$\begin{align*}
c_{0,0} &= 1, & c_{1,1} &= -320 & c_{2,2} &= 571200 \\
c_{1,0} &= -1024, & c_{2,1} &= 4408560 & c_{2,0} &= \frac{14}{3}(1422697 - 12000\pi^2).
\end{align*}$$

(3.14)

This can be compared with (3.7), which gives

$$\langle \prod_i O_i(x_i) \rangle_h = \langle e^{-(\text{const.}) \, h R^4 \int \! d^4 y \, O_H(y) \prod_i O_i(x_i)} \rangle_{h=0}.$$

(3.15)

The constant appearing in (3.7) and (3.15) can be fit so that (3.15) reproduces, say $c_{1,1}$. The rest of the $c_{n,k}$ appearing in the function $f(t)$ of (3.12) should then be completely determined by (3.15); it would be interesting to complete this check. Perhaps the equations of [4] for the function $f(t)$ can be obtained directly using (3.9) with $T^\mu_i \sim h R^4 O_H$.

The above considerations can be similarly applied for the more general vacuum (3.3), which corresponds to the theory with Lagrangian (3.7), deformed away from the origin of the moduli space of vacua, where $SU(N) \rightarrow S(\prod_i U(N_i))$. As a particular example, consider the theory with $H = 4\pi g_s \alpha'^2 ((N-1)/|\vec{y}|^4 + 1/|\vec{y} - \vec{a}|^4)$, where we shifted the $\vec{y}$ origin for convenience. This is expected to be dual to the RG flow from the $N = 4$ superconformal theory with gauge group $SU(N)$ in the UV and $SU(N-1) \times U(1)$ in the IR (large $\vec{y}$ is the UV and small $\vec{y}$ is the IR). The least irrelevant operator IR, along which the theory flows to the IR fixed point, is $O_H \sim Q^4 \bar{Q}^4 O_4$. The coefficient of this operator is $\sim 1/\nu^4$, where $\nu$ is the Higgs expectation value which breaks
SU(N) to SU(N − 1) × U(1). We can also see this from the above $H(\tilde{y})$, which is given by

$$H(y) \approx H = 4\pi g_s \alpha'^2 ((N - 1)/|\tilde{y}|^4 + 1/|\tilde{a}|^4) + \ldots$$

for small $\tilde{y}$. The IR theory thus effectively has $H(y)$ given by (3.2) with $h = R^4/|\tilde{a}|^4$. If we identify $v \sim a/R^2$, the coefficient $hR^4$ of $\mathcal{O}_H$ is indeed $1/v^4$. Using the analysis of [23] it should be possible to find the precise relation between $v$ and $a/R^2$ and, by comparing with the precise coefficient of $\mathcal{O}_H$ induced by the above Higgsing, obtain an independent derivation of the constant appearing in (3.7). It would be interesting to complete this exercise and to compare the value of the constant thus obtained with that required to reproduce (3.13) and (3.14) via (3.15).

4. Three dimensional theories with 16 or 32 supercharges

We now consider three dimensional gauge theories with 16 supercharges, which is sometimes referred to as $\mathcal{N} = 8$ supersymmetry in 3d. Useful aspects of these theories can be found in [24]. The supercharges are 8 $SO(2,1)$ spinors $Q^I_\alpha$, where $I = 1 \ldots 8$ and $\alpha = 1, 2$ is the Lorentz spinor index. The supersymmetry algebra admits an $SO(8)_R$ automorphism, with $I$ taken to reside in the $8_s$; the Yang-Mills Lagrangian is invariant only under an $SO(7)_R$ subgroup for general gauge coupling $g_{YM}$. The three dimensional theory can be regarded as the dimensional reduction of the 4d $\mathcal{N} = 4$ theory on a circle of radius $R \to 0$. The $SU(4)_R \cong SO(6)_R$ of the 4d theory then extends to the $SO(7)_R$ symmetry of the Lagrangian and the $SO(8)_R$ symmetry of the supersymmetry algebra; under $SO(6)_R \subset SO(7)_R \subset SO(8)_R$, the supercharges combine as $4 + \bar{4} \to 8_s \to 8_s$.

As opposed to the situation in 4d, in 3d the gauge coupling $g_{YM}$ is classically dimensionful, flowing to strong coupling in the IR. It is believed that the coupling flows until it reaches some fixed point value $g^*_{YM}$ where the beta function vanishes and the theory is conformally invariant and interacting. At this point, the theory has a total of 32 supercharges (the original 16 and 16 additional, superconformal ones), which combine into the superconformal group $SO(3,2|8)$.

The superconformal group $SO(3,2|8)$ again has both short and long representations, with supermultiplets generated by primary operators $\mathcal{O}_P$ via graded commutators with $Q^I_\alpha$, which we again denote by $Q^n\mathcal{O}_P$. Here $n \leq 16$ for the long multiplets and $n \leq 8$ for the short multiplets. The extremal cases $Q^{16}\mathcal{O}_{long}$ or $Q^8\mathcal{O}_{short}$ can be added to the Lagrangian without violating the 16 supersymmetries as, up to total derivatives, they are annihilated by the supercharges. These deformations of the superconformal theory are all irrelevant in the far IR.
The short multiplets of $O(3,2|8)$ were constructed in [3]. They are given by scalar primary operators $O_p \sim \text{Tr} \phi^p$ of dimension $\Delta_p = \frac{1}{2}p$ and in the $(p,0,0,0)$ representation of $SO(8)_R$, along with descendents $Q^n O_p$, for $n \leq 8$, with $\Delta = \frac{1}{2}p + \frac{1}{2}n$ and other quantum numbers as reviewed in appendix A. The representations with $p < 4$ are shorter than the generic short representation, as the operators in the table in the appendix which would otherwise have negative $SO(8)_R$ Dynkin weights actually vanish. For example, the case $p = 1$ is the singleton representation, given by scalars $O_1$, in the $8_v$ of $SO(8)_R$ with $\Delta = \frac{1}{2}$, and fermions $Q O_1$, in the $8_c$ of $SO(8)_R$ with $\Delta = 1$; acting with more powers of $Q$ on $O_1$ gives zero (up to total derivatives). This $p = 1$ multiplet is that of the gauge invariant operators in $U(1)$ gauge theory, with one of the 8 scalars in $O_1$ identified as the dualized photon: $*dA = d\phi_8$.

As in four dimensions, the $p = 2$ short multiplet contains the conserved currents. The $SO(8)_R$ currents $J^a_\mu$ (which may or may not be conserved) are the descendents $Q^2 O_2$, the supercharges are $Q^3 O_2$, and the conserved stress tensor $T_{\mu\nu}$ is $Q^4 O_2$. Because $T_{\mu\nu}$ is definitely a conserved current, its dimension must always be exactly $\Delta = 3$.

Now, as in sect. 2, we consider a deformed theory, as in (2.1), with 16 supersymmetries preserved. The dimension of $T_{\mu\nu}$ remains exactly $\Delta = 3$ and thus the dimension of $O_2$ remains exactly 1. As in 4d, the chiral ring structure of additive anomalous dimensions for chiral superfields then ensures that all short operators $Q^n O_p$ continue to have their unrenormalized dimension $\Delta = \frac{1}{2}(p + n)$, even in the deformed theory, with arbitrary deforming parameters $h$ in (2.1).

As an aside, we mention some peculiar aspects of the 3d theories:

1. In 4d, the microscopic, non-Abelian, Yang-Mills fields are in the $p = 1$ short multiplet representation of the supersymmetry algebra. The situation in 3d is different because the $p = 1$ multiplet contains dualized scalars rather than gauge fields. While 3d Abelian gauge fields can always be dualized to scalars, it is not known how to do this for non-Abelian gauge fields. Note also that the $p = 1$ multiplet in 4d contains $O_1$ descendents up to $Q^2 O_1$ and $\overline{Q}^2 O_1$ (which is why this multiplet is sometimes referred to as the “doubleton” rather than “singleton” multiplet), while the $p = 1$ multiplet in 3d only contains descendents up to $Q O_1$.

2. Unlike the situation in 4d, the $p = 2$ short multiplet in 3d does not contain an operator associated with changing the gauge coupling $g_Y$. The only candidate for such an operator would be a Lorentz scalar in $Q^4 O_2$, which is the 3d analog of the operators
$O_\tau$ and $O_\bar{\tau}$ in 4d. But this operator vanishes in 3d (up to total derivatives); this can be seen in Appendix A because the scalar in $Q^4 O_p$ is in the $(p - 4, 0, 0, 2)$ representation of $SO(8)_R$ and must vanish for $p < 4$. For $p \geq 4$ $Q^4 O_p$ is not annihilated by the supercharges (up to total derivatives), so there is no value of $p$ for which it can be added to the Lagrangian while preserving 16 supersymmetries.

The fact that there is no short multiplet operator associated with changing the gauge coupling $g_{YM}$ actually prevents a contradiction. The RG flow associated with $g_{YM}$ changing with RG scale, until it hits the RG fixed point value $g_{YM}^*$, is one which preserves 16 supersymmetries. Thus, by the argument above, the dimensions of all operators in short representations are not renormalized along this RG flow. But the dimension of the operator responsible for changing $g_{YM}$ must vary with the RG flow, such that it is relevant for $g_{YM} \ll g_{YM}^*$ and becomes irrelevant in the IR, for $g_{YM} = g_{YM}^*$ (the intuition is that the fixed point at $g_{YM}^*$ is attractive in the IR). If the operator responsible for changing $g_{YM}$ were $Q^4 O_2$, it would always be exactly marginal, which we know to be untrue even for small $g_{YM}$.

We thus expect (though with some confusion) that the operator associated with changing $g_{YM}$ is actually a long operator which preserves 16 supersymmetries.

As another peculiar aside, note that the pseudoscalar operator $Q^2 O_2 \sim \text{Tr} \psi \psi$ can be added to the Lagrangian without breaking the 16 supersymmetries. The reason is that $Q$ acts on this operator to give the spin $\frac{1}{2}$ operator in $Q^3 O_2$, which vanishes up to total derivatives. As seen in the table in appendix A, the spin $\frac{1}{2}$ operator in $Q^3 O_p$ is in the $(p - 3, 1, 1, 0)$ of $SO(8)_R$ and must thus vanish for $p < 3$. So $Q^2 O_2$ is a relevant pseudoscalar perturbation, with $\Delta \equiv 2$, which preserves 16 supersymmetries. This pseudoscalar deformation is associated with fermion masses.

We now turn to a holographic duality motivation for considering 3d RG flows with 16 preserved supercharges. It is expected that $M$ theory has exact vacua with metric [2]

$$ds^2 = H^{-2/3} dx_\mu^2 + H^{1/3} (dr^2 + r^2 d\Omega_7^2);$$

$$H = h + \frac{2 \pi^2 N l_p^6}{r^6},$$

(4.1)

where $x_\mu$ span $\mathbb{R}^{1,2}$, and there is $G_4$ field given by $G_4 \sim l_p^{-3} (dx_\mu)^\wedge 3 \wedge dH^{-1}$, with $N$ units of $M2$ brane $G$ flux. $M$ theory has no dilaton, which corresponds to the fact that the dual [21] 3d theory has no exactly marginal operator: the theory is conformally invariant for a fixed value of the gauge coupling constant $g_{YM}$. The parameter $h$ in (4.1) is again
arbitrary. Taking $h = 0$, the vacuum (4.1) is exactly $AdS_4 \times S^7$. Taking $h = 1$ gives the $M2$ brane metric, which asymptotes to $\mathbb{R}^{1,10}$ for $r \to \infty$ (far from the brane) and to $AdS_4 \times S^7$ for $r \to 0$ (near horizon limit).

In [21] the vacuum was originally taken to be the $h = 1$ case of (4.1), but then the near horizon limit, $r \to 0$, was taken, leading to $AdS_4 \times S^7$ in the limit. One could instead take $h = 0$ from the outset. The $h = 0$ case exactly preserves 32 supersymmetries, which combine with the bosonic generators to give the 3d superconformal group $SO(3,2|8)$. This theory is expected to be exactly dual to the 3d $SU(N)$ Yang-Mills theory with 16 supersymmetries at the RG fixed point value of the coupling constant, $g_{YM}^*$, where the theory is conformally invariant (and thus has an additional 16 superconformal symmetries). For $h \neq 0$, the vacua (4.1) generally preserve 16 supersymmetries, with an additional 16 supersymmetries emerging in the $r \to 0$ and $r \to \infty$ limits. This case is conjectured to be holographically dual to a field theory with a non-trivial RG flow along which 16 supersymmetries are preserved.

$M$ theory in the vacuum (4.1) is expected to be exactly invariant under

$$x_\mu \to \lambda x_\mu, \quad r \to \lambda^{-1/2} r, \quad h \to \lambda^3 h, \quad d\Omega_7 \to d\Omega_7.$$  \hspace{1cm} (4.2)

This operation preserves the metric (4.1), $G_4 \sim l_p^{-3}(dx_\mu)^3 \wedge dH^{-1}$, and the other supergravity fields. For $h = 0$, (4.2) is a symmetry which corresponds to the dilatation generator of the superconformal group $SO(3,2|8)$; this must be an exact gauge symmetry of the $M$ theory vacuum in order for the theory to be holographically dual to the exactly conformally invariant 3d field theory. For $h \neq 0$ this symmetry is broken, but can be regarded as being restored provided that $h$ transforms as in (4.2). Thus, at length scale $x$, $h$ enters only via $h_{eff} \sim h/x^3$ (here $x^3 \equiv (x_\mu x^\mu)^{3/2}$), which is invariant under (4.2).

We thus find that the operator by which the theory is deformed for $h \neq 0$ has dimension $\Delta = 6$ (dimension 3 in the action). As in the 4d case, this is twice the dimension of a marginal operator. This was also noted in [3] via an absorption calculation.

Now consider the transformation

$$x^\mu \to x'^\mu, \quad l_p \to \lambda l_p, \quad h \to \lambda^{-3} h, \quad r \to \lambda^{3/2} r,$$ \hspace{1cm} (4.3)

under which the metric (4.1) transforms as $ds^2 \to \lambda^2 ds^2$ and $G_4 \sim l_p^{-3}(dx_\mu)^3 \wedge dH^{-1}$ is invariant. Because all lengths in $M$ theory are measured relative to $l_p$, $M$ theory should
be invariant under the combined rescaling \((4.3)\) of \(l_p\) and \(ds^2\). The upshot is that \(h\) should only enter via \(h_{\text{eff}} = hl_p^3/x^3\), which is dimensionless and invariant under \((4.2)\) and \((4.3)\).

We thus propose that \(M\) theory in the vacuum \((4.1)\) is dual to the 3d \(\mathcal{N} = 8\) field theory with Lagrangian

\[
\mathcal{L} = \mathcal{L}_0 + (\text{const.}) hl_p^3 \mathcal{O}_H,
\]

where \(\mathcal{L}_0\) is the superconformal Lagrangian at \(g_{YM}^2\) and \(\mathcal{O}_H = Q^8 \mathcal{O}_4\), much as in the 4d case \((3.7)\). (Again, as in 4d, the operator \(\mathcal{O}_4\) can actually be a combination of a single and a double trace operator with the same quantum numbers.) The theory \((3.7)\) properly preserves 16 supersymmetries and the \(SO(8)_R\) symmetry, as does \((4.1)\). Also, \(h\) properly couples to an operator of dimension \(\Delta = 6\). (In \(D\) spacetime dimensions a scalar \(\phi\) has canonical dimension \(1/2 (D - 2)\), so \(Q^8 (\phi)^4\) always has dimension \(2D\).) The fact that \(Q^4 \mathcal{O}_4\) continues to have dimension \(\Delta = 6\), even in the deformed theory \((4.4)\) with \(h \neq 0\), is compatible with our argument above that the dimensions of short representation operators are not renormalized as long as 16 supersymmetries are preserved. There are also many long operators which preserve the 16 supersymmetries and are \(SO(8)_R\) singlets, but deforming by these would not be compatible with the deformation depending only on \(h_{\text{eff}} = hl_p^3/x^3\), so we do not expect them in \((4.4)\).

As in the 4d case, correlation functions in the proposed dual field theory for \(h \neq 0\) are given by

\[
\langle \prod_i \mathcal{O}_i(x_i) \rangle_h = \langle e^{-(\text{const.})} hl_p^3 \int d^3y \mathcal{O}_H(y) \prod_i \mathcal{O}_i(x_i) \rangle_{h=0}.
\]

The transformation \((4.2)\) leads to the Ward identities for correlation functions

\[
\sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + \Delta_i) \langle \prod_{i=1}^n \mathcal{O}_i(x_i) \rangle_h = -3h \frac{\partial}{\partial h} \langle \prod_{i=1}^n \mathcal{O}_i(x_i) \rangle_h = \int d^3y \langle T^\mu_\mu(y) \prod_{i=1}^n \mathcal{O}_i(x_i) \rangle_h,
\]

where \(\Delta_i\) are the dimensions of the operators in the perturbed theory. For operators in short representations, as argued above, these dimensions are not renormalized and are independent of \(h\) and the RG scale. In particular, the two-point function of the short operator \(\mathcal{O}_p\) in the perturbed theory must be of the form

\[
\langle \mathcal{O}_p(x) \mathcal{O}_p(0) \rangle = \frac{f_p hl_p^3}{(x^2)^{\frac{D}{2}}}.
\]

It would be nice to compare \((4.5)\) with a detailed partial wave absorption analysis along the lines of \([3]\).
5. Six dimensional theories with 32 supercharges and RG flows preserving 16

Much work points to the existence of interacting 6d superconformal field theories with 32 supercharges residing in the superconformal group $SO(6, 2|4)$. These supercharges are the 16 of $\mathcal{N} = (2, 0)$ supersymmetry in 6d, along with 16 superconformal partners. The superconformal group again has short and long representations. The short representations of the 6d superconformal group $SO(6, 2|4)$ were obtained in [3] and are given in appendix B for convenience. For example, the $p = 1$ multiplet is the (free) 6d $\mathcal{N} = (2, 0)$ matter multiplet, consisting of scalars $O_1$ in the $5 = (0, 1)$ of $Sp(2)$, and self-dual tensor fields $Q^2 O_2$ in the $1$ of $Sp(2)_R$. The notation $\sqrt{a_{\alpha\beta\gamma}}$, listed in the table for the Lorentz spin of $Q^2 O_2$, is to indicate that these are two-form gauge fields with self-dual field strength.

The RG fixed point theory can be deformed as in (2.1), preserving the 16 supercharges. The $p = 2$ short multiplet again contains the currents: the $SO(5)_R$ currents are the $\Delta = 5$ Lorentz vectors $Q^2 O_2$, the supercharges are the $\Delta = 5.5$ “gravitinos” $Q^3 O_2$, and the stress tensor is the $\Delta = 6$ Lorentz “graviton” $Q^4 O_2$. Again, the stress tensor remains conserved when the theory is deformed as in (2.1), and thus its operator dimension is not renormalized. The chiral ring structure of additive anomalous dimensions for chiral superfields then ensures that the dimensions of all operator in short representations are independent of the RG scale, and not renormalized, in RG flows which preserve 16 supersymmetries.

Such a 6d RG flow, preserving 16 supersymmetries, can be the holographic dual of $M$ theory vacua containing $M5$ branes. It is expected that $M$ theory has exact vacua [2]

$$
\begin{align*}
    ds^2 &= H^{-1/3} dx^2_\mu + H^{2/3} (dr^2 + r^2 d\Omega^2_4); \\
    H &= h + \frac{\pi N l_5^3}{r^3},
\end{align*}
$$

(5.1)

where $x_\mu$ span $\mathbb{R}^{1,5}$, and there is $G_4$ field given by $*G_4 \sim l_p^{-6} (dx_\mu)^\wedge 6 \wedge dH^{-1}$, with $N$ units of M5 $G$ flux. The value of the real parameter $h$ is again arbitrary. For $h = 1$, (5.1) asymptotes to $\mathbb{R}^{1,10}$ for $r \to \infty$ and to $AdS_7 \times S^4$ for $r \to 0$. For $h = 0$, (5.1) is identically $AdS_7 \times S^4$, with 32 conserved supercharges, for all $r$. The $h = 0$ case is expected to be exactly holographically dual to the 6d, $A_{N-1}$ type, $\mathcal{N} = (2, 0)$ conformal field theory. The $h = 1$ case is expected to be holographically dual to a RG flow which preserves 16 supercharges and flows in the IR limit, $r \to 0$, to the same CFT as in the $h = 0$ case.

$M$ theory in the vacuum (5.1) is expected to be exactly gauge invariant under

$$
    x_\mu \to \lambda x_\mu, \quad r \to \lambda^{-2} r, \quad h \to \lambda^6 h, \quad d\Omega_4 \to d\Omega_4.
$$

(5.2)
This operation preserves the metric (5.1), $*G_4 \sim l_p^{-6} (dx_\mu)^\wedge 6 \wedge dH^{-1}$, and the other supergravity (and $M$ theory) fields. We again interpret this symmetry as that associated with the dilatation generator of the superconformal group, which is spontaneously broken for $h \neq 0$ but can be regarded as being restored provided $h$ transforms as in (5.2). The effective parameter which is invariant under (5.2) is $h_{eff} \sim h/x^6$; this power of $x$ reveals that $h$ multiplies a term of exact dimension 12 in the Lagrangian density (dimension 6 in the action).

Now consider the transformation

$$x^\mu \to x^\mu, \quad l_p \to \lambda l_p, \quad h \to \lambda^{-6} h, \quad r \to \lambda^3 r, \quad (5.3)$$

under which the metric (4.1) transforms as $ds^2 \to \lambda^2 ds^2$ and $*G_4 \sim l_p^{-6} (dx_\mu)^\wedge 6 \wedge dH^{-1}$ is invariant. Because all lengths in $M$ theory are measured relative to $l_p$, $M$ theory should be invariant under the combined rescaling (4.3) of $l_p$ and $ds^2$. The upshot is that $h$ should only enter via $h_{eff} = lh_p^6 / x^6$, which is dimensionless and invariant under (5.2) and (5.3).

$M$ theory with vacuum (5.1) is thus expected to be holographically dual to the 6d field theory with Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + (\text{const.}) \ h l_p^6 O_H, \quad (5.4)$$

where, $\mathcal{L}_0$ represents the conformal theory and again, $O_H = Q^8 O_4$, which is an $SO(5)_R$ singlet and now has $\Delta = 12$, as required above. ((5.4) is perhaps schematic as the theory contains non-Abelian, self-dual tensor fields which do not have a known, standard Lagrangian formulation.) The fact that $O_H$ has dimension $\Delta = 12$ even in the theory with $h \neq 0$ is compatible with the above argument that the dimensions of operators in short representations are not renormalized as long as the 16 supersymmetries are preserved. The deformation of the fixed point by a dimension $\Delta = 12$ operator is compatible with the scattering results of [3].

Corresponding to (5.2), we have the Ward identities

$$\sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + \Delta_i) \left( \prod_{i=1}^n O_i(x_i) \right)_h = -6h \frac{\partial}{\partial h} \left( \prod_{i=1}^n O_i(x_i) \right)_h = \int d^6 y \langle T_\mu^\mu(y) \prod_{i=1}^n O_i(x_i) \rangle_h, \quad (5.5)$$

where $\Delta_i$ are the dimensions of the operators in the perturbed theory. For operators in short representations, as argued above, these dimensions are not renormalized and are
independent of $h$ and the RG scale. In particular, the two-point function of the short operator $O_p$ in the perturbed theory must be of the form
\[
\langle O_p(x)O_p(0) \rangle = f_p\left(\frac{h}{x^2}\right) \frac{1}{(x^2)^{2p}}.
\] (5.6)

6. Non-commutative geometry and maximally supersymmetric RG flows

There has been recent interest in gauge theories in non-commutative spaces; such theories arise in the world-volume of branes with background $B$ field (or, in $M$ theory, the $C$ field). The non-commutativity of space-time, $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, introduces a length scale via the parameter $\theta^{\mu\nu}$, which clearly has mass dimension $\Delta = -2$. Thus the maximal supersymmetry for $\theta^{\mu\nu} \neq 0$ is the 16 ordinary supercharges, without the superconformal symmetries, and there is a RG flow to the IR, where $\theta^{\mu\nu}$ becomes irrelevant. The IR fixed points are the ordinary, superconformal $\mathcal{N} = 4$ Yang-Mills theories on commutative space, with 32 supercharges. The non-commutativity thus leads to maximally supersymmetric RG flows, of precisely the type discussed in the previous sections.

Indeed, it was argued in [25] that non-commutative gauge theories can be related to ordinary gauge theories by a field-redefinition of the gauge field strength, order-by-order in the parameter $\theta^{\mu\nu}$; the explicit change of variables can be found in sect. 3.1 of [25]. Thus the non-commutative theory with 16 supercharges is equivalent to an ordinary theory with 16 supercharges with higher dimension terms in the Lagrangian, exactly as in (2.1), coming from the field redefinition; these terms are weighted by powers of $\theta^{\mu\nu}$ and become irrelevant in the IR.

Suppose e.g. that we start with a 4d (other $d$ are similar) Lagrangian which is formally the same as the $\mathcal{N} = 4$ superconformal Lagrangian, but is not conformally invariant simply because the space-time is non-commutative. Via the change of variables of [25], this should be equivalent to the ordinary gauge theory with Lagrangian
\[
\mathcal{L} = \mathcal{L}_0 + \theta^{\alpha\beta}[Q^2Q^4O_3]_{\alpha\beta} + \theta^{\dot{\alpha}\dot{\beta}}[\bar{Q}^2Q^4O_3]_{\dot{\alpha}\dot{\beta}} + (\text{const.})(\theta^{\mu\nu})^2O_H + \ldots,
\] (6.1)
with $\theta^{\alpha\beta}$ and $\theta^{\dot{\alpha}\dot{\beta}}$ the Lorentz spin $(1,0)$ and $(0,1)$, respectively, parts of $\theta^{\mu\nu}$. The operators $[Q^2Q^4O_3]_{\alpha\beta} \sim (\text{Tr} F^3)_{\alpha\beta} + \ldots$ and $[\bar{Q}^2Q^4O_3]_{\dot{\alpha}\dot{\beta}}$ are dimension $\Delta = 6$ short operators and $O_H = Q^4\bar{Q}^4O_4$ is the $\Delta = 8$ short operator; all of these operators are $SU(4)_R$ flavor singlets and annihilated by the 16 supersymmetries, so (6.1) respects the expected symmetries of
non-commutative $\mathcal{N} = 4$. In principle, there should be terms in (6.1) at higher orders of $\theta^{\mu\nu}$. However, any terms of higher order in $\theta$ in (6.1) must be long operators which preserve the 16 supersymmetries and $SU(4)_R$, as there are no other short operators which respect these symmetries. The fact that the dimensions of the short operators appearing in (6.1) are not renormalized along the RG flow of (6.1) is consistent with the expected non-renormalization of the dimension, $\Delta = -2$, of the parameter $\theta^{\mu\nu}$ appearing in $[x^\mu, x^\nu]$. On the other hand, the coefficients of possible long operators appearing in (6.1) would have to have quantum-corrected anomalous dimensions, to compensate for the anomalous dimensions of the long operators to which they couple.

The arguments of the previous sections apply directly here. The 16 supersymmetry RG flows associated with non-commutative geometry will have short operators, with non-renormalized dimensions along the entire RG flow. In the 4d case, the gauge coupling must thus remain constant along the RG flow, exactly as in fig. 1.

String (or M theory) vacua with non-zero $B$ (or $C$) field and their conjectured holographic duality to world-volume field theories were discussed in [26-28]. These vacua have 16 supersymmetries. For example, the case associated with IIB vacua with D3 brane charge (the M2 and M5 cases are similar) has [26,27], in our earlier notation, exact vacua:

$$\begin{align*}
    ds_{\text{str}}^2 &= H^{-1/2}[f_1(dx_0^2 + dx_1^2) + f_2(dx_2^2 + dx_3^2)] + H^{1/2}d\vec{y} \cdot d\vec{y}, \\
    H &= h + \frac{4\pi g_0 \alpha'^2}{\cos \theta_1 \cos \theta_2} \sum_{i=1}^{k} \frac{N_i}{|\vec{y} - \vec{a}_i|^4}, \quad f_j^{-1} = \sin^2 \theta_j H^{-1} + \cos^2 \theta_j, \\
    e^\phi &= g_0 \sqrt{f_1 f_2}, \quad 2\pi \alpha' B_{01} = \tan \theta_1 H^{-1} f_1, \quad 2\pi \alpha' B_{23} = \tan \theta_2 H^{-1} f_2.
\end{align*}$$

(6.2)

The $\vec{y}$ dependent dilaton in (6.2) seems to contradict our arguments of sect. 2 that $g_{YM}$ must be independent of the RG scale. Fortunately, it is wrong here to simply identify $e^\phi$ with $g_{YM}^2$. (Rather, $e^\phi$ is the suppression factor for non-planar diagrams [29].)

---

5 I am grateful to N. Seiberg for pointing this out to me.
The point is that the supergravity solution (6.2) should be regarded as giving the closed string quantities, whereas the worldvolume gauge theory is sensitive to open string quantities [25]. We’ll assume that we can directly apply the formulae of [25],

\[
G_{ij} = g_{ij} - (2\pi\alpha')^2 (Bg^{-1}B)_{ij}, \\
\theta^{ij} = 2\pi\alpha' \left( \frac{1}{g + 2\pi\alpha' B} \right)^{ij}_A, \\
g_{YM}^{-2} = e^{-\phi} \left( \frac{\det(g + 2\pi\alpha' B)}{\det(g - (2\pi\alpha')^2 Bg^{-1}B)} \right)^{1/2},
\]

(6.3)

to the non-flat background (6.2). Doing so, we obtain for the open string metric and \( \theta^{ij} \),

\[
dS_{open}^2 = H^{-1/2} [\sec^2 \theta_1 (dx_0^2 + dx_1^2) + \sec^2 \theta_2 (dx_2^2 + dx_3^2)] + H^{1/2} d\vec{y} \cdot d\vec{y}, \\
\theta^{01} = -\pi\alpha' \sin 2\theta_1, \quad \theta^{23} = -\pi\alpha' \sin 2\theta_2.
\]

(6.4)

The worldvolume gauge coupling obtained via (6.3) is

\[
g_{YM}^{-2} = \frac{1}{g_0 \sqrt{f_1 f_2}} \prod_{i=1,2} \left( \frac{f_i^2 (H^{-1} + H^{-2} \tan^2 \theta_i)}{(H^{-1/2} f_i + \tan^2 \theta_i H^{-3/2} f_i)^2} \right) = \frac{\cos \theta_1 \cos \theta_2}{g_0}.
\]

(6.5)

Remarkably, the non-trivial functions \( f_i \) of the AdS bulk coordinate \( \vec{y} \) appearing in (6.2) all completely cancel out of the open string quantities! Up to constant rescalings, \( dS_{open}^2 \) and \( g_{YM}^{-2} \) are unaffected by the \( B \) field. The \( \vec{y} \) independence of the \( \theta^{ij} \) in (6.4) shows that these field theory parameters are \( \Delta = -2 \) constants, which are otherwise not renormalized. This is as should have been expected from \( [x^\mu, x^\nu] = i\theta^{\mu\nu} \). The \( \vec{y} \) independence of \( g_{YM}^{-2} \) in (6.5) is in agreement with the general considerations of sect. 2: the dual field theory RG flow is indeed as in fig. 1, with \( g_{YM} \) constant along the entire RG flow.

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Appendix A. The short multiplets of the 3d $N = 8$ superconformal group (see [5])

| form   | spin/parity | $\Delta$     | $SO(8)_R$   |
|--------|-------------|--------------|-------------|
| $O_p$  | 0$_+$       | $\frac{1}{2}p$ | $(p, 0, 0, 0)$ |
| $QO_p$ | $\frac{1}{2}$ | $\frac{1}{2}p + \frac{1}{2}$ | $(p - 1, 0, 1, 0)$ |
| $Q^2O_p$ | 1$_-$      | $\frac{1}{2}p + 1$ | $(p - 2, 1, 0, 0)$ |
| $Q^3O_p$ | $\frac{3}{2}$ | $\frac{1}{2}p + \frac{3}{2}$ | $(p - 2, 0, 0, 1)$ |
| $Q^4O_p$ | 2           | $\frac{1}{2}p + 2$ | $(p - 2, 0, 0, 0)$ |
| $Q^2O_p$ | 0$_-$       | $\frac{1}{2}p + 1$ | $(p - 2, 0, 2, 0)$ |
| $Q^3O_p$ | $\frac{3}{2}$ | $\frac{1}{2}p + \frac{3}{2}$ | $(p - 3, 0, 0, 0)$ |
| $Q^4O_p$ | $\frac{1}{2}$ | $\frac{1}{2}p + 2$ | $(p - 3, 0, 1, 1)$ |
| $Q^3O_p$ | $\frac{3}{2}$ | $\frac{1}{2}p + \frac{5}{2}$ | $(p - 3, 0, 0, 1)$ |
| $Q^5O_p$ | $\frac{1}{2}$ | $\frac{1}{2}p + \frac{3}{2}$ | $(p - 4, 1, 0, 0)$ |
| $Q^7O_p$ | $\frac{1}{2}$ | $\frac{1}{2}p + \frac{3}{2}$ | $(p - 4, 0, 0, 0)$ |

Appendix B. The short multiplets of the 6d $\mathcal{N} = (2, 0)$ superconformal group (see [6])

| form   | spin   | $\Delta$     | $Sp(2)_R$   |
|--------|--------|--------------|-------------|
| $O_p$  | scalar | $2p$         | $(0, p)$    |
| $QO_p$ | spinor | $2p + \frac{1}{2}$ | $(1, p - 1)$ |
| $Q^2O_p$ | $\sqrt{a_{\alpha\beta\gamma}}$ | $2p + 1$ | $(0, p - 1)$ |
| $Q^2O_p$ | vector | $2p + 1$ | $(2, p - 2)$ |
| $Q^3O_p$ | gravitino | $2p + \frac{3}{2}$ | $(1, p - 2)$ |
| $Q^4O_p$ | graviton | $2p + 2$ | $(0, p - 2)$ |
| $Q^3O_p$ | spinor | $2p + \frac{3}{2}$ | $(3, p - 3)$ |
| $Q^4O_p$ | $a_{\alpha\beta}$ | $2p + 2$ | $(2, p - 3)$ |
| $Q^5O_p$ | gravitino | $2p + \frac{5}{2}$ | $(1, p - 3)$ |
| $Q^6O_p$ | $\sqrt{a_{\alpha\beta\gamma}}$ | $2p + 3$ | $(0, p - 3)$ |
| $Q^4O_p$ | scalar | $2p + 3$ | $(4, p - 4)$ |
| $Q^5O_p$ | spinor | $2p + \frac{5}{2}$ | $(3, p - 4)$ |
| $Q^6O_p$ | vector | $2p + 3$ | $(2, p - 4)$ |
| $Q^7O_p$ | spinor | $2p + \frac{7}{2}$ | $(1, p - 4)$ |
| $Q^8O_p$ | scalar | $2p + 4$ | $(0, p - 4)$. |
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