In this paper, we study a distributionally robust chance-constrained programming (DRCCP) under Wasserstein ambiguity set, where the uncertain constraints require to be jointly satisfied with a probability of at least a given risk level for all the probability distributions of the uncertain parameters within a chosen Wasserstein distance from an empirical distribution. Differently from the previous results concentrating mainly on the linear uncertain constraints, we consider a DRCCP involving convex non-linear uncertain constraints. We investigate an equivalent reformulation and a deterministic approximation of such optimization problem. It is shown that this approximation is essentially exact under a certain condition and it can be reformulated as a tractable convex programming for a single DRCCP. We also demonstrate that the proposed approximation is equivalent to a tractable mixed-integer convex programming when the decision variables are binary and the uncertain constraints are linear. Numerical results show that the proposed mixed-integer convex reformulation can be solved efficiently.

Keywords Distributionally Robust Optimization Problem · Chance-Constrained Programming · Wasserstein Distance · Mixed-Integer Programming

Mathematics Subject Classification (2010) 90C15 · 90C11 · 90C25

This research was supported by National Natural Science Foundation of China (Grant No. 11271243).
1 Introduction

1.1 Problem Setting

In this paper, we study a distributionally robust chance-constrained programming (DRCCP) of the form:

\[
\begin{align*}
\min_{x} & \quad g(x) \\
\text{s.t.} & \quad x \in S,
\end{align*}
\]

\[
\inf_{P \in P} \Pr \{ \xi : f_t(x, \xi) \geq 0, \forall t \in [T] \} \geq 1 - \epsilon,
\]

where \( g : \mathbb{R}^n \to \mathbb{R} \) often represents a convex cost function; \( x \in \mathbb{R}^n \) is a decision vector; the set \( S \subseteq \mathbb{R}^n \) represents deterministic constraints on \( x \); \( \xi \in \mathbb{R}^m \) represents an m-dimensional random vector supported on \( \Xi \subseteq \mathbb{R}^m \); \( P \) is termed as “ambiguity set” comprising all distributions that are compatible with the decision maker’s prior information; the mapping \( f_t(x, \xi) : \mathbb{R}^n \times \Xi \to \mathbb{R} \) for all \( t \in [T] := \{1, 2, \cdots, T\} \) represents a set of the uncertain constraints on \( x \). In addition, the distributionally robust chance constraint (DRCC) \((1c)\) requires all \( T \) uncertain constraints to be jointly satisfied for all the probability distributions from ambiguity set with a probability of at least \( 1 - \epsilon \), where \( \epsilon \in (0, 1) \) represents the risk level specified by the decision makers, and \( \epsilon \) is often chosen to be small, e.g., 0.10 or 0.05. The problem (1) is called a single or joint DRCCP if \( T = 1 \) or \( T > 1 \), respectively.

We denote the feasible region induced by \((1c)\) as

\[
Z_D := \left\{ x \in \mathbb{R}^n : \inf_{P \in P} \Pr \{ \xi : f_t(x, \xi) \geq 0, \forall t \in [T] \} \geq 1 - \epsilon \right\}
\]

\[
= \left\{ x \in \mathbb{R}^n : \sup_{P \in P} \Pr \{ \xi : f_t(x, \xi) < 0, \exists t \in [T] \} \leq \epsilon \right\}
\]

\[
= \left\{ x \in \mathbb{R}^n : \sup_{P \in P} \mathbb{E}_P \left[ I_{\{f_t(x, \xi) < 0, \exists t \in [T] \}}(\xi) \right] \leq \epsilon \right\}.
\]

Note that many tractability results in the remainder of this paper are predicated on the following assumptions.

(A1) Each function \( f_t(x, \xi) \) is convex in \( \xi \) for any fixed \( x \), and is concave in \( x \) for any fixed \( \xi \).

(A2) The random vector \( \xi \) is supported on a nonempty closed convex set \( \Xi \subseteq \mathbb{R}^m \).

In this paper, we consider a distributionally robust chance-constrained programming under Wasserstein ambiguity set \( P_W \), which is defined as

\[
P_W = \left\{ P : \Pr \{ \xi \in \Xi \} = 1, W \left( P, P_\xi \right) \leq \delta \right\},
\]
where 1-Wasserstein distance is defined as

\[
W(P_1, P_2) = \inf_Q \left\{ \int_{\Xi \times \Xi} \| \xi_1 - \xi_2 \| Q(d\xi_1, d\xi_2) : Q \text{ is a joint distributionally of } \xi_1 \text{ and } \xi_2 \right\},
\]

for all distributions \( P_1, P_2 \in \mathcal{M}(\Xi) \), where \( \mathcal{M}(\Xi) \) contains all probability distributions \( P \) supported on \( \Xi \) with \( \int_{\Xi} \| \xi \| P(d\xi) < \infty \).

1-Wasserstein distance between \( P_1 \) and \( P_2 \), equipped with an arbitrary norm \( \| \cdot \| \) on \( \mathbb{R}^m \), represents the minimum transportation cost generated by moving the probability mass from \( P_1 \) to \( P_2 \). In addition, \( P_{\bar{\zeta}} \) represents a discrete empirical distribution of \( \bar{\zeta} \) with i.i.d. samples \( Z = \{ \zeta_j \}_{j \in [N]} \subseteq \Xi \) from the true distribution \( P^\infty \), i.e., its point mass function is \( P_{\bar{\zeta}} \{ \bar{\zeta} = \zeta_j \} = \frac{1}{N} \), and \( \delta > 0 \) represents Wasserstein radius.

### 1.2 Literature Review

There are significant efforts on reformulations, approximations and convexity properties of a DRCCP under various ambiguity sets. In particular, many approaches based on moments and statistical distances are commonly used to build ambiguity sets in a DRCCP.

We now review existing works on a DRCCP with a moment-based ambiguity set [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. It is well-known that more efforts have been made to derive a tractable reformulation for a single DRCCP with the linear uncertain constraints. For instance, the authors in [1] demonstrated that with given first- and second-order moments, the set \( Z_D \) for a single DRCCP is equivalent to a tractable second-order conic representation. In [9], the authors developed a tractable semidefinite programming for a single DRCCP with given first- and second-order moments as well as the support of the uncertain parameters. In addition, the authors in [2] showed that the set \( Z_D \) for a single DRCCP is convex when \( P \) involves conic moment constraints or unimodality of \( P \). However, tractability results for a joint DRCCP with the linear uncertain constraints are very rare. The authors in [2] showed that the optimization problem over the set \( Z_D \) is NP-hard in general. Thus, much of the earlier works derived deterministic approximations of the set \( Z_D \) instead of developing its equivalent reformulations. For instance, in [7], with given first- and second-order moments, the authors converted the joint chance constraint into a single chance constraint by scaling each uncertain constraint with a positive number and for any given scaler, they were able to derive a conservative second-order conic programming approximation. In [9], with given first- and second-order moments, the authors derived a deterministic reformulation of the set \( Z_D \), but it is not convex due to bilinear terms, which is naturally hard to solve. Besides, the authors in [11] provided several sufficient conditions under which the
well-known Bonferroni approximation is exact and obtained its convex reformulation. On the other hand, there is very limited literature on a DRCCP with the non-linear uncertain constraints. For instance, in [9], with given first- and second-order moments, the authors proved that a single DRCCP can be reformulated as a tractable semidefinite programming when the constraint function $f(x, \xi)$ is either concave or quadratic in $\xi$. In addition, with any ambiguity set including convex moment constraints, the authors in [5] proposed that a single DRCCP can be expressed as a tractable convex programming when the constraint function $f(x, \xi)$ is convex in $\xi$ and is concave in $x$. In [6], the authors proved that with given first- and second-order moments, a single DRCCP is equivalent to a robust optimization problem when the constraint function $f(x, \xi)$ is quasi-convex in $\xi$ and is concave in $x$.

Recently, there are many successful developments on a DRCCP with Wasserstein ambiguity set [12, 13, 14, 15, 16, 17]. However, most existing results on the tractability of a DRCCP with Wasserstein ambiguity set are restricted to the case of the linear uncertain constraints. For instance, the authors in [12] derived exact mixed-integer conic reformulations for a single DRCCP as well as a joint DRCCP with right-hand side uncertainty. In [13], the authors showed that a joint DRCCP is mixed-integer representable once the set $Z_D$ is bounded and derived tractable outer and inner approximations. In addition, the authors in [14] provided a mixed-integer linear programming reformulation for a single DRCCP under discrete support and a mixed-integer second-order cone programming reformulation for a single DRCCP with right-hand side uncertainty under continuous support. To the best of our knowledge, the case of the non-linear uncertain constraints is largely untouched. This kind of optimization problem is of great interest, because the uncertainty is inherently non-linear in many applications. One exception that we are aware of is [15], where the authors provided a convex inner approximation for a single DRCCP when the constraint function $f(x, \xi)$ is bounded in $\xi$ and is convex in $x$. Besides, the authors in [15] presented a tractable reformulation for a single DRCCP when the constraint function $f(x, \xi)$ is the maximum of a set of functions that are affine in $\xi$ and are convex in $x$.

1.3 Contributions, Structure, and Notations

In this paper, we consider a joint DRCCP with the non-linear uncertain constraints under Wasserstein ambiguity set. Specifically, our main contributions are summarized as below.

1. We derive a deterministic approximation of the set $Z_D$, which is not convex in general and can be expressed as an optimization problem involving biconvex constraints.
2. We show that the proposed approximation is essentially exact under a certain condition and it can reduce to a tractable convex programming for a single DRCCP.

3. We demonstrate that when the decision variables are binary and the uncertain constraints are linear, the proposed approximation can be reformulated as a mixed-integer convex programming. We also present a numerical study to demonstrate that the mixed-integer convex reformulation can be solved effectively.

The remainder of the paper is organized as follows. Section 2 derives an equivalent reformulation of the set $Z_D$ according to strong duality result. Next, section 3 develops a biconvex approximation based on the worst-case CVaR constraint and this approximation is shown to be essentially exact under a certain condition. Finally, section 4 demonstrates that the proposed approximation is equivalent to a mixed-integer convex programming when the decision variables are binary and the uncertain constraints are linear. A numerical study is presented to test the effectiveness of the mixed-integer convex reformulation.

**Notation.** The following notation is used throughout the paper. We use bold-letters (e.g., $x, A$) to denote vectors or matrices, and use corresponding non-bold letters to denote their components. We use $\xi$ to denote a random vector and $\xi$ to denote a realization of $\xi$. We let $e$ be the all-ones vector. Given a positive integer $n$, we let $\mathbb{Z}^n := \{1, 2, \cdots, n\}$ and use $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, \forall i \in [n]\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n : x_i > 0, \forall i \in [n]\}$.

We denote $(t)_+ = \max\{t, 0\}$ for any given real number $t$. We define the indicator function as $I_A(\xi) = 1$, if $\xi \in A; = 0$, otherwise. Similarly, the characteristic function is defined as $\chi_\Xi(\xi) = 0$, if $\xi \in \Xi; = \infty$, otherwise. Given a norm $\|\cdot\|$ on $\mathbb{R}^m$, the dual norm $\|\cdot\|_*$ is defined by $\|z\|_* := \sup_{\|\xi\| \leq 1} z^T \xi$. Additional notation will be introduced as needed.

**2 Exact Reformulation**

In this section, we will develop a deterministic reformulation of the set $Z_D$. The derivation of the exact reformulation utilizes strong duality result which will be introduced in Theorem 2.1.

We first review the well-known strong duality result from [18], which will be applied to formulate the worst-case chance constraint into its dual form, and indeed by the proof of strong duality theorem in [18], we will present another equivalent dual reformulation in Theorem 2.1.

**Theorem 2.1 (Dual Reformulation)** (i) The uncertainty set $\Xi \subseteq \mathbb{R}^m$ is convex and closed. (ii) Let $l(x, \xi) := \max_{k \in [K]} l_k(x, \xi)$ represent a pointwise maximum function, and the negative constituent functions $-l_k, \forall k \in [K]$ are proper, convex and lower semi-continuous and assume that $l_k, \forall k \in [K]$ is not identically $-\infty$ on $\Xi$. Then, for any given $\delta \geq 0$, the worst-case expectation $\sup_{P \in \Pi^w} \mathbb{E}_P [l(x, \xi)]$ equals the optimal value of the
where $\lambda$, $s$ and $z$ are decision variables, $\lambda \geq 0$ is the dual variable for Wasserstein distance constraint $f_{\Xi \times \Xi} \| \xi - \tilde{\xi} \| \mathcal{Q} (d\xi, d\tilde{\xi}) \leq \delta$, and $\| z_{ik} \|_*$ is the dual norm.

Next, we can represent the indicator function as a pointwise maxima by Lemma 2.1 due to [14, 18].

**Lemma 2.1** The indicator function $\{f_t(x, \xi) < 0, \exists t \in [T]\} (\xi)$ can be rewritten as the pointwise maximum of a finite number of concave functions, which is defined as

$$\{f_t(x, \xi) < 0, \exists t \in [T]\} (\xi) = \max \left\{1 - \chi_{(f_t(x, \xi) < 0)} (\xi), \cdots, 1 - \chi_{(f_T(x, \xi) < 0)} (\xi), 0\right\},$$

in which for all $t \in [T]$

$$\chi_{(f_t(x, \xi) < 0)} (\xi) = \begin{cases} 0, & \text{if } f_t(x, \xi) < 0 \\ \infty, & \text{otherwise} \end{cases}$$

is the characteristic function of the open convex set defined by $f_t(x, \xi) < 0$.

We will combine Theorem 2.1 with Lemma 2.1 to obtain the equivalent reformulation of the set $Z_D$ in the next theorem.

**Theorem 2.2** The feasible set $Z_D$ is equivalent to

$$Z_D = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \lambda \delta + \frac{1}{N} \sum_{i=1}^N s_i \leq \epsilon, \\ G_{f_t} (z_{it}, \eta_t, x) + 1 - z_{it}^T \xi^i - s_i \leq 0, \forall i \in [N], \forall t \in T (x), \\ \| z_{it} \|_* - \lambda \leq 0, \forall i \in [N], \forall t \in [T], \\ \lambda \geq 0, \eta_t \geq 0, \forall i \in [N], \forall t \in T (x), \end{array} \right\}$$

where each function $G_{f_t} (z_{it}, \eta_t, x) = \sup_{\xi \in \Xi} \left[z_{it}^T \xi - \eta_t f_t (x, \xi)\right]$ and the set $T (x)$ is defined as $T (x) = \{ t \in [T] : \exists \xi \in \Xi, f_t (x, \xi) < 0 \}$. 

The finite convex program

$$\inf_{\lambda, s, z} \lambda \delta + \frac{1}{N} \sum_{i=1}^N s_i$$

s.t. $\sup_{\xi \in \Xi} \left[z_{ik}^T \xi + l_k (x, \xi)\right] - z_{ik}^T \xi^i \leq s_i, \forall i \in [N], \forall k \in [K],$$

$$\| z_{ik} \|_* \leq \lambda, \forall i \in [N], \forall k \in [K],$$

$$\lambda \geq 0,$$
Proof. Note that
\[ Z_D := \left\{ x \in \mathbb{R}^n : \sup_{p \in P} \mathbb{E}_p \left[ \mathbb{I}_{\{f_t(x, \xi) < 0, \exists \xi \in [T]\}} (\xi) \right] \leq \epsilon \right\}. \]

According to Lemma 2.1, the indicator function \( \mathbb{I}_{\{f_t(x, \xi) < 0, \exists \xi \in [T]\}} (\xi) \) can be represented as the pointwise maximum as denoted in (7).

Therefore, by Theorem 2.1, the left-hand side of the constraint defining \( Z_D \) can be rewritten as

\[
\begin{align*}
\inf_{\lambda, s, z} & \quad \lambda \delta + \frac{1}{N} \sum_{i=1}^{N} s_i \\
\text{s.t.} & \quad \sup_{\xi \in \Xi} \left[ z_{it}^T \xi + 1 - \chi_{\{f_t(x, \xi) < 0\}} (\xi) \right] - z_{it}^T \xi \leq s_i, \quad \forall i \in [N], \forall t \in T (x), \\

\|z_{it}\|_* & \leq \lambda, \quad \forall i \in [N], \forall t \in [T], \\
\lambda & \geq 0.
\end{align*}
\]

By definition of the set \( T(x) \), the optimization problem \( \sup_{\xi \in \Xi} \left[ z_{it}^T \xi + 1 - \chi_{\{f_t(x, \xi) < 0\}} (\xi) \right] \) in (9b) can be rewritten as

\[
\begin{align*}
\sup_{\xi \in \Xi} & \quad z_{it}^T \xi + 1 \quad (10a) \\
\text{s.t.} & \quad f_t(x, \xi) < 0, \quad (10b)
\end{align*}
\]

for all \( i \in [N] \) and \( t \in T (x) \).

Then, for all \( i \in [N] \) and \( t \in T (x) \), we use Lagrangian duality result to reformulate the problem (10) as

\[
\begin{align*}
\sup_{\xi \in \Xi} & \quad z_{it}^T \xi + 1 = \sup_{\xi \in \Xi} \inf_{\eta_{it} \geq 0} \left[ z_{it}^T \xi + 1 - \eta_{it} f_t (x, \xi) \right] \\
& \quad = \inf_{\eta_{it} \geq 0} \sup_{\xi \in \Xi} \left[ z_{it}^T \xi + 1 - \eta_{it} f_t (x, \xi) \right], \quad (11a)
\end{align*}
\]

where the first equality follows from the fact that for any given \( x, f_t (x, \xi) \) and the objective function \( z_{it}^T \xi + 1 \) are both continuous in \( \xi \), \( \Xi \) is a nonempty closed set, so that we can replace “<” by “\( \leq \)” without effect on the supremum.

Thus, the constraint (9b) can be rewritten as

\[
\begin{align*}
\sup_{\xi \in \Xi} \left[ z_{it}^T \xi + 1 - \eta_{it} f_t (x, \xi) \right] - z_{it}^T \zeta_i & \leq s_i,
\end{align*}
\]
for all \( i \in [N] \) and \( t \in T(x) \). By definition of \( G_{f_i}(z_{it}, \eta_{it}, x) \), the above constraint is equivalent to (8b).

**Remark 2.1** We note that each function \( G_{f_i}(z_{it}, \eta_{it}, x) \) in the constraint (8b) is equivalent to solving a concave maximization problem and such a problem is often tractable.

**Remark 2.2** The exact reformulation (8) of the set \( Z_D \) is not convex because the index set \( T(x) \) depends on \( x \) and each function \( G_{f_i}(z_{it}, \eta_{it}, x) \) in the constraint (8b) is not convex in general. Therefore, we will attempt to investigate the tractability of the set \( Z_D \) by establishing conditions under which \( T(x) \) can be replaced by \([T]\) and \( G_{f_i}(z_{it}, \eta_{it}, x) \) can be convex for all \( i \in [N] \) and \( t \in [T] \).

### 3 The Worst-Case CVaR Approximation

We first recall the definition of CVaR due to [19]. For a given measurable function \( L(\xi) : \mathbb{R}^n \to \mathbb{R} \), let \( \mathbb{P} \) be its probability distribution and the risk level \( \epsilon \in (0, 1) \), then the CVaR at level \( \epsilon \) with respect to \( \mathbb{P} \) is defined as

\[
\text{CVaR}_{1-\epsilon} (L(\xi)) := \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\epsilon} \mathbb{E}_{\mathbb{P}} \left[ (L(\xi) - \beta)_+ \right] \right\}.
\]

It is well-known that for any \( \alpha \in \mathbb{R}^T_{++} \), the joint chance constraint (1c) can be reformulated as

\[
Z_D = \left\{ x \in \mathbb{R}^n : \inf_{\mathbb{P} \in \mathcal{P}} \left\{ \xi : \max_{t \in [T]} \{ \alpha_t (-f_t(x, \xi)) \} \leq 0 \right\} \geq 1 - \epsilon \right\}.
\]

(12)

Note that (12) represents a single distributionally robust chance constraint, which can be conservatively approximated by a worst-case CVaR constraint, which due to [7, 9]. Therefore, for any \( \alpha \in \mathbb{R}^T_{++} \), we have

\[
Z_D \supseteq Z_C(\alpha) = \left\{ x \in \mathbb{R}^n : \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\epsilon} \mathbb{E}_{\mathbb{P}} \left[ \max_{t \in [T]} \{ \alpha_t (-f_t(x, \xi)) \} - \beta \right]_+ \right\} \right\} \leq 0 \right\}.
\]

Furthermore, we denote \( Z_C \) as

\[
Z_C = \bigcup_{\alpha \in \mathbb{R}^T_{++}} Z_C(\alpha) = \left\{ x \in \mathbb{R}^n : \alpha \in \mathbb{R}^T_{++}, \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\epsilon} \mathbb{E}_{\mathbb{P}} \left[ \max_{t \in [T]} \{ \alpha_t (-f_t(x, \xi)) \} - \beta \right]_+ \right\} \right\} \leq 0 \right\}.
\]

We observe that \( Z_C = \bigcup_{\alpha \in \mathbb{R}^T_{++}} Z_C(\alpha) \subseteq Z_D \), then we will show that \( Z_C \) can be reformulated as a disjunction of a convex set \( Z_{C_1} \) and a non-convex set \( Z_{C_2} \). Furthermore, we will demonstrate that for binary DRCCP, we can convert the set \( Z_{C_2} \) into a tractable mixed-integer convex reformulation, which will be discussed in section 4.
Theorem 3.1 The set \( Z_C = Z_{C_1} \cup Z_{C_2} \), where

\[
Z_{C_1} = \left\{ \mathbf{x} \in \mathbb{R}^n : \inf_{\xi \in \Xi} f_t(\mathbf{x}, \xi) \geq 0, \forall t \in [T] \right\},
\]

and

\[
Z_{C_2} = \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{aligned}
\lambda \delta + \frac{1}{N} \sum_{i=1}^N s_i &\leq \epsilon, \\
G_{f_t}(\mathbf{z}_t, \alpha_t, \mathbf{x}) + 1 - z_t^T \xi^i - s_i &\leq 0, \forall i \in [N], \forall t \in [T], \\
\|z_t\|_* - \lambda &\leq 0, \forall i \in [N], \forall t \in [T], \\
\lambda &\geq 0, \alpha_t \geq 0, \forall t \in [T],
\end{aligned} \right\}
\]

where each function \( G_{f_t}(\mathbf{z}_t, \alpha_t, \mathbf{x}) = \sup_{\xi \in \Xi} [z_t^T \xi - \alpha_t f_t(\mathbf{x}, \xi)] \).

Proof. We separate the proof into three parts.

(i) Note that

\[
Z_C = \left\{ \mathbf{x} \in \mathbb{R}^n : \alpha \in \mathbb{R}^T_+, \sup_{\mathbf{p} \in \mathcal{P}} \left\{ \inf_{\beta \in \mathbb{R}} \left( \beta + \frac{1}{\epsilon} \mathbb{E}_p \left[ \max_{t \in [T]} \{ \alpha_t (-f_t(\mathbf{x}, \xi)) \} - \beta \right]_+ \right) \right\} \leq 0 \right\}
\]

\[
= \left\{ \mathbf{x} \in \mathbb{R}^n : \alpha \in \mathbb{R}^T_+, \inf_{\beta \in \mathbb{R}} \left[ \beta + \frac{1}{\epsilon} \sup_{\mathbf{p} \in \mathcal{P}} \left[ \max_{t \in [T]} \{ \alpha_t (-f_t(\mathbf{x}, \xi)) \} - \beta \right]_+ \right] \leq 0 \right\}
\]

\[
= \left\{ \mathbf{x} \in \mathbb{R}^n : \alpha \in \mathbb{R}^T_+, \beta \in \mathbb{R}, \beta + \frac{1}{\epsilon} \sup_{\mathbf{p} \in \mathcal{P}} \left[ \max_{t \in [T]} \{ \alpha_t (-f_t(\mathbf{x}, \xi)) \} - \beta \right]_+ \leq 0 \right\}.
\]

where the second equality is due to standard minimax argument and the third equality follows from replacing infimum operator with its equivalent “existence” argument.

Then, we will prove that \( \beta \leq 0 \). Suppose \( \beta > 0 \). Since \( \max_{t \in [T]} \{ \alpha_t (-f_t(\mathbf{x}, \xi)) \} - \beta \geq 0 \) for all \( \xi \in \Xi \), we must have \( \mathbb{E}_p \left[ \max_{t \in [T]} \{ \alpha_t (-f_t(\mathbf{x}, \xi)) \} - \beta \right] \geq 0 \). Thus, the left-hand side of (17) is strictly positive, which yields a contradiction.

(ii) Now, we will show that \( Z_C \subseteq Z_{C_1} \cup Z_{C_2} \). For any \( \mathbf{x} \in Z_C \), there exists \( (\alpha, \beta) \in \mathbb{R}^T_+ \times \mathbb{R}_- \) such that

\[
\beta \epsilon + \sup_{\mathbf{p} \in \mathcal{P}} \mathbb{E}_p \left[ \max_{t \in [T]} \{ \alpha_t (-f_t(\mathbf{x}, \xi)) \} - \beta \right]_+ \leq 0.
\]

Then, we will distinguish whether \( \beta = 0 \) or \( \beta < 0 \).

Case 1. Note that if \( \beta = 0 \), then the inequality (18) implies

\[
\sup_{\mathbf{p} \in \mathcal{P}} \mathbb{E}_p \left[ \max_{t \in [T]} \{ \alpha_t (-f_t(\mathbf{x}, \xi)) \} \right]_+ \leq 0,
\]
which is equivalent to
\[
\inf_{P} \mathbb{P}[f_t(x, \xi) \geq 0, \forall t \in [T]] = 1 > 1 - \epsilon,
\]
and hence by continuity of each function \( f_t(x, \xi) \), we have \( f_t(x, \xi) \geq 0 \) for all \( \xi \in \Xi \). Thus, \( x \in Z_C \).

Case 2. On the other hand, if \( \beta < 0 \), then divide (18) by \( -\beta \) and add \( \epsilon \) on both sides, we have
\[
- \frac{1}{\beta} \sup_{P} \mathbb{E}_{P} \left[ \left( \max_{t \in [T]} \{ \alpha_t ( - f_t(x, \xi) ) \} - \beta \right)_{+} \right] \leq \epsilon.
\]
(20)

Since \( \beta < 0 \), we can redefine \( \alpha_t \) as \( \alpha_t / (-\beta) \) for all \( t \in [T] \). Then, the inequality (20) can be rewritten as
\[
\sup_{P} \mathbb{E}_{P} \left[ \left( \max_{t \in [T]} \{ \alpha_t ( - f_t(x, \xi) ) \} + 1 \right)_{+} \right] \leq \epsilon.
\]
(21)

Since
\[
\left( \max_{t \in [T]} \{ \alpha_t ( - f_t(x, \xi) ) \} + 1 \right)_{+} = \max \left\{ \max_{t \in [T]} \{ \alpha_t ( - f_t(x, \xi) ) \} + 1, 0 \right\}
\]
\[
= \max \{ \alpha_1 ( - f_1(x, \xi) ) + 1, \cdots, \alpha_T ( - f_T(x, \xi) ) + 1, 0 \}.
\]

Therefore, by Theorem 2.1, for any given \( \alpha \in \mathbb{R}_{++}^T \), the supremum in the left-hand side of (21) is equivalent to
\[
\inf_{\lambda, s, z, \alpha} \frac{1}{N} \sum_{i=1}^{N} s_i \quad \text{s.t.} \quad \sup_{\xi \in \Xi} \left[ z^T \xi - \alpha_t f_t(x, \xi) + 1 \right] - z^T \zeta_i \leq s_i, \ \forall i \in [N], \forall t \in [T],
\]
(22a)
\[
\|z_t\|_* \leq \lambda, \ \forall i \in [N], \forall t \in [T],
\]
(22b)
\[
\lambda \geq 0, \ \alpha_t > 0, \ \forall t \in [T],
\]
(22c)

which implies \( x \in Z_{C_2} \).

(iii) Now, we will show that \( Z_{C_1} \cup Z_{C_2} \subseteq Z_C \). Similarly, given \( x \in Z_{C_1} \cup Z_{C_2} \). If \( x \in Z_{C_1} \), then we can choose \( \beta = 0, \ \alpha = e \), thus \( x \in Z_C \). If \( x \in Z_{C_2} \), there exists \( (\lambda', s', z', \alpha', x) \) which satisfies the constraints in (14). According to the following Corollary 3.1, we must have \( \alpha' > 0 \), and hence let \( \beta = -1, \ \alpha = \alpha' \) in (17). Then, by strong duality result introduced in Theorem 2.1, we have \( x \in Z_C \). \( \square \)

Remark 3.1 We note that to solve the inner approximation of the problem (1) (i.e., \( \min_{x \in S \cap Z_C} g(x) \)), we can optimize \( g(x) \) over \( S \cap Z_{C_1} \) and \( S \cap Z_{C_2} \) separately, then choose the smallest value.

Remark 3.2 We observe the constraint system (14) contains biconvex terms \( \alpha_t f_t(x, \xi) \) in (14b). In par-
the constraints in (14) are convex in \((\lambda, s, z, x)\) for any given \(\alpha \in \mathbb{R}_+^T\), and also convex in \(\alpha\) for any given \((\lambda, s, z, x)\).

Next, we will prove that in the constraint system (14), \(\alpha_t\) must be strictly positive for all \(t \in [T]\).

**Corollary 3.1** For any \(x\) satisfying (14), we must have \(\alpha_t > 0\) for all \(t \in [T]\).

**Proof.** Suppose that we let \(\alpha_{t_0} = 0\) for some \(t_0 \in [T]\), then from (14b), we have

\[
G_{f_{t_0}} (z_{t_0}, 0, x) + 1 - z_{t_0}^T \zeta^i = \sup_{\xi \in \Xi} z_{t_0}^T \xi + 1 - z_{t_0}^T \zeta^i = \sup_{\xi \in \Xi} [z_{t_0}^T (\xi - \zeta^i)] + 1 \leq s_i, \ \forall i \in [N].
\]

We note that \(\|z_{t_0}\|_\ast \leq \lambda\) for all \(i \in [N]\) in (14c), we may thus conclude that

\[
\inf_{\|z_{t_0}\|_\ast \leq \lambda} [z_{t_0}^T (\xi - \zeta^i)] + 1 \leq s_i, \ \forall \xi \in \Xi, \forall i \in [N],
\]

which can be rewritten as

\[
- \sup_{\|z_{t_0}\|_\ast \leq \lambda} [-z_{t_0}^T (\xi - \zeta^i)] + 1 \leq s_i, \ \forall \xi \in \Xi, \forall i \in [N],
\]

furthermore, which is equivalent to

\[
- \lambda \|\zeta^i - \xi\| + 1 \leq s_i, \ \forall \xi \in \Xi, \forall i \in [N].
\]

Therefore, according to (14a), we have

\[
\lambda \delta - \frac{1}{N} \sum_{i=1}^{N} \lambda \|\zeta^i - \xi\| + 1 \leq \lambda \delta + \frac{1}{N} \sum_{i=1}^{N} s_i \leq \epsilon, \ \forall \xi \in \Xi,
\]

which can be rewritten as

\[
\lambda \delta - \frac{1}{N} \sum_{i=1}^{N} \lambda \inf_{\xi \in \Xi} \|\zeta^i - \xi\| + 1 \leq \lambda \delta + \frac{1}{N} \sum_{i=1}^{N} s_i \leq \epsilon. \quad (23)
\]

Note that \(\inf_{\xi \in \Xi} \|\zeta^i - \xi\| = 0\) for all \(i \in [N]\), thus the inequality (23) yields a contradiction to the fact that \(\epsilon < 1\). \(\Box\)

We will demonstrate that \(\alpha\) could be bounded, which will be useful for the study of binary DRCCP in section 4.
Corollary 3.2 If $S$ is compact and $[T] = T(x)$ for all $x \in Z_{C_2}$, then there exists an $M \in \mathbb{R}_+^T$ such that $\alpha_t \leq M_t$ for all $t \in [T]$.

Proof. We note that the statement that $[T] = T(x)$ for all $x \in Z_{C_2}$ implies given $x \in S \cap Z_{C_2}$, there exists $\xi \in \Xi$ such that $f_t(x, \xi) < 0$ for all $t \in [T]$.

Since the constraint (14b) can be rewritten as

$$\sup_{x \in \Xi} \left[ z_d^T \xi - \alpha_t f_t(x, \xi) \right] \leq - \left( 1 - z_d^T \zeta^i - s_i \right), \forall i \in [N], \forall t \in [T],$$

furthermore, which is equivalent to

$$z_d^T \xi - \alpha_t f_t(x, \xi) \leq - \left( 1 - z_d^T \zeta^i - s_i \right), \forall \xi \in \Xi, \forall i \in [N], \forall t \in [T].$$

Since $S$ is compact and $\Xi$ is closed, we have $f_t(x, \xi)$ must be finite. Therefore, when $f_t(x, \xi) < 0$, we obtain

$$\alpha_t \leq \frac{1}{f_t(x, \xi)} \left[ z_d^T \xi + 1 - z_d^T \zeta^i - s_i \right], \forall i \in [N], \forall t \in [T].$$

Thus, one can find a upper bound $M \in \mathbb{R}_+^T$ such that $\alpha_t \leq M_t$ for all $t \in [T]$.

We can easily find that the constraint system (14) is quite similar to the constraint system (8) and thus we will show that $Z_{C_2}$ is equivalent to $Z_D$ under a certain condition.

Theorem 3.2 (i) Let $C := \{x \in Z_D : \exists t \in [T], \inf_{\xi \in \Xi} f_t(x, \xi) \geq 0\} = \emptyset$. (ii) For any given $x \in Z_D$, there exists $(\lambda, s, z, \eta, x)$ which satisfies the constraints in (8) and for all $t \in [T]$, $\sup_{i \in [N]} f_t(x, \xi^{(i)}) < 0$ or $\inf_{i \in [N]} f_t(x, \xi^{(i)}) \geq 0$, where $f_t(x, \xi^{(i)})$ is satisfied with $z_d^T \xi^{(i)} - \eta_t f_t(x, \xi^{(i)}) = \sup_{i \in \Xi} \left[ z_d^T \xi - \eta_t f_t(x, \xi) \right]$.

Then, $Z_{C_2} = Z_C = Z_D$.

Proof. We observe that the statement that $C = \emptyset$ implies $T(x) \neq \emptyset$ for any $x \in Z_D$.

By Theorem 3.1, we obtain $Z_{C_2} \subseteq Z_C \subseteq Z_D$. Thus, we only need to prove that $Z_D \subseteq Z_{C_2}$. We note that $Z_{C_2}$ can be rewritten as

$$Z_{C_2} = \left\{ x \in \mathbb{R}^n : G_{f_t}(z_{it}, \alpha_t, x) + 1 - z_d^T \zeta^i - s_i \leq 0, \forall i \in [N], \forall t \in T(x), \right\}$$

(24a)

(24b)

$$\|z_{it}\|_* - \lambda \leq 0, \forall i \in [N], \forall t \in [T],$$

(24d)

$$\lambda \geq 0, \alpha_t \geq 0, \forall t \in [T],$$

(24e)
where \( [T] \setminus T(x) = \{ t \in [T] : f_t(x, \xi) \geq 0, \forall \xi \in \Xi \} \).

Note that if \( C = \emptyset \), then we can obtain that \( [T] \setminus T(x) = \emptyset \) for any \( x \in Z_D \) and the constraint system (24) is equivalent to

\[
Z_{C_2} = \left\{ x \in \mathbb{R}^n : \begin{array}{l}
\lambda \delta + \frac{1}{n} \sum_{i=1}^{N} s_i \leq \epsilon, \\
G_{f_t}(z_{it}, \alpha_t, x) + 1 - z_{it}^T \xi^i - s_i \leq 0, \forall i \in [N], \forall t \in T(x), \\
\|z_{it}\|_s - \lambda \leq 0, \forall i \in [N], \forall t \in [T], \\
\lambda \geq 0, \alpha_t \geq 0, \forall t \in [T]
\end{array} \right\}.
\] (25a, 25b, 25c, 25d)

According to the constraint system (8), we note that if for any given \( x \in Z_D \), there exists \((\lambda, s, z, \eta, x)\) which satisfies the constraints in (8) and for all \( t \in [T] \), \( \sup_{i \in [N]} f_t(x, \xi(i)) < 0 \), then we can choose \( \alpha_t = \min_{i \in [N]} \eta_t \).

Similarly, if for any given \( x \in Z_D \), there exists \((\lambda, s, z, \eta, x)\) which satisfies the constraints in (8) and for all \( t \in [T] \), \( \inf_{i \in [N]} f_t(x, \xi(i)) \geq 0 \), then we can choose \( \alpha_t = \max_{i \in [N]} \eta_t \).

Thus, we must have \( Z_D \subseteq Z_{C_2} \). \( \square \)

Despite the tightness of \( Z_C \), it is not convex in general due to the biconvex terms in the constraint system (14) defining the set \( Z_{C_2} \). Now, we will demonstrate that \( Z_C \) is a convex set when there is a single uncertain constraint.

**Theorem 3.3** When \( T = 1 \), we have

\[
Z_C = \left\{ x \in \mathbb{R}^n : \delta \|z_{11}\|_s + \frac{1}{N} \sum_{i=1}^{N} s_i + \epsilon \left[ G_{f_1}(z_{11}, 1, x) - z_{11}^T \xi^1 - s_1 \right] \leq 0, \forall i \in [N] \right\},
\] (26)

which is a convex set.

**Proof.** Note that when \( T = 1 \), by the definition of the set \( Z_C \) in (17), we have

\[
Z_C = \left\{ x \in \mathbb{R}^n : \alpha_1 > 0, \beta \in \mathbb{R}, \beta + \frac{1}{\epsilon} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[ (\alpha_1 (-f_1(x, \xi)) - \beta)_+ \right] \leq 0 \right\}.
\] (27)

Thus, by replacing \( \beta \) with \( -\beta \), we obtain that (27) can be rewritten as

\[
Z_C = \left\{ x \in \mathbb{R}^n : \alpha_1 > 0, \beta \in \mathbb{R}, \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[ (\alpha_1 (-f_1(x, \xi)) + \beta)_+ \right] \leq \beta \epsilon \right\}.
\] (28)
Then, by Theorem 2.1 and the proof of Theorem 3.1, (28) is equivalent to

\[
Z_C = \left\{ x \in \mathbb{R}^n : \begin{align*}
\lambda d + \frac{1}{N} \sum_{i=1}^N s_i & \leq \beta c, \\
G_{f_1} (z_{i1}, \alpha_1, x) + \beta - z^T_{i1} \xi - s_i & \leq 0, \ \forall i \in [N], \\
\|z_{i1}\|_* - \lambda & \leq 0, \ \forall i \in [N], \\
\lambda & \geq 0, \ \alpha_1 > 0.
\end{align*} \right. \tag{29a}
\]

Furthermore, we remove variables \( \lambda \) and \( \beta \) from the constraint system (29) by Fourier-Motzkin method, then we obtain

\[
Z_C = \left\{ x \in \mathbb{R}^n : \begin{align*}
\delta \|z_{i1}\|_* + \frac{1}{N} \sum_{i=1}^N s_i + \epsilon [G_{f_1} (z_{i1}, \alpha_1, x) - z^T_{i1} \xi - s_i] & \leq 0, \ \forall i \in [N], \\
\alpha_1 & > 0.
\end{align*} \right. \tag{30a}
\]

We observe that \( \alpha_1 \) must be positive and finite in the constraint (30a), hence by scaling it to be 1, we obtain (26).

Since \( G_{f_1} (z_{i1}, 1, x) \) is a convex function, then \( Z_C \) is a convex set. \( \square \)

4 Binary DRCCP

In this section, we will demonstrate that the tight inner approximation can be expressed as a mixed-integer convex reformulation when the decision variables are binary and the uncertain constraints are linear. Meanwhile, we will present a numerical study to illustrate that such mixed-integer convex reformulation can be solved efficiently.

**Theorem 4.1** Suppose that \( S \subseteq \{0, 1\}^n \), \( f_t (x, \xi) = (A^t x + a^t)^T \xi + B^t x + b^t \) for all \( t \in [T] \), and \( \alpha \) in (14) can be upper bounded by a vector \( M \) for any \( x \in S \). Consider a convex set

\[
S \cap \tilde{Z}_{C_2} = \left\{ x \in \{0, 1\}^n : \begin{align*}
\lambda d + \frac{1}{N} \sum_{i=1}^N s_i & \leq \epsilon, \\
G_{f_t} (z_{it}, 1, (\alpha_t, y^t)) + 1 - z^T_{it} \xi - s_i & \leq 0, \ \forall i \in [N], \ \forall t \in [T], \\
\|z_{it}\|_* - \lambda & \leq 0, \ \forall i \in [N], \ \forall t \in [T], \\
\lambda & \geq 0, \ \alpha_t \geq 0, \ \forall t \in [T],
\end{align*} \right. \tag{31a}
\]

where \( f_t (y_t, \alpha_t) = (A^t y^t + a^t \alpha_t)^T \xi + B^t y^t + b^t \alpha_t \) for all \( t \in [T] \), then

\[
S \cap \tilde{Z}_{C_2} = S \cap Z_{C_2} \subseteq S \cap Z_D.
\]
Proof. If each function \( f_t(x, \xi) = (A^t x + a^t)^T \xi + B^t x + b^t \), then for all \( t \in [T] \) and \( i \in [N] \), we have

\[
G_{f_t}(z_{it}, \alpha_t, x) = \sup_{\xi \in \Xi} \left\{ z_{it}^T \xi - \alpha_t \left[ (A^t x + a^t)^T \xi + B^t x + b^t \right] \right\}.
\]

Now, we define new variables \( y^t \) as \( y^t = \alpha_t x \) for all \( t \in [T] \).

Since \( \alpha_t \leq M_t \) for all \( t \in [T] \), and hence by McCormick inequalities due to [20], we obtain

\[
0 \leq y^t_r \leq M_t x_r, \quad \alpha_t - M_t (1 - x_r) \leq y^t_r \leq \alpha_t,
\]

which is exact for any \( x \subseteq \{0, 1\}^n \). Thus, we have \( S \cap \hat{Z}_{C_2} = S \cap Z_{C_2} \). \( \square \)

**Remark 4.1** We note that to optimize over \( S \cap Z_{C_2} \), we only need to optimize over \( S \cap \hat{Z}_{C_2} \), which is a mixed-integer convex set.

**Remark 4.2** In the above theorem, we have assumed that one can find an upper bound \( M \), and indeed Corollary 3.2 also provides a sufficient condition for the existence of the vector \( M \).

Next, we conduct a series of numerical studies to evaluate the effectiveness of the proposed reformulation in Theorem 4.1. For the evaluation purpose, we study a distributionally robust multidimensional knapsack problem (DRMKP) [5, 21, 22] with binary decision variables. The following notations are adopted for a binary DRMKP. We consider \( T \) knapsacks and \( m \) items. In addition, \( c_j \) represents the value of item \( j \) for all \( j \in [n] \), \( \xi_t = (\xi_{t1}, \ldots, \xi_{tm})^T \) represents the vector of random item weights supported on \( \Xi_t = \mathbb{R}^n \) in knapsack \( t \), and \( b_t > 0 \) represents the capacity limit of knapsack \( t \) for all \( t \in [T] \). The decision variable \( x_j \in \{0, 1\} \) represents the proportion of \( j \)th item to be picked for all \( j \in [n] \) and we let \( x \in S := \{0, 1\}^n \). Furthermore, let Wasserstein ambiguity set be defined using 1-Wasserstein distance with \( L_2 \)-norm as distance metric. With the notation above, the binary DRMKP can be formulated as

\[
\max_{x} \quad c^T x \tag{32a}
\]

s.t. \( x \in \{0, 1\}^n \), \( \tag{32b} \)

\[
\inf_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{P} \left\{ \xi_t \in \Xi_t : \xi_t^T x \leq b_t, \forall t \in [T] \right\} \right\} \geq 1 - \epsilon, \tag{32c}
\]

where the constraint (32c) is to guarantee that the worst-case probability that the capacity of each knapsack should be satisfied is at least \( 1 - \epsilon \).

According to Theorem 4.1, we note that \( S \cap Z_C = (S \cap Z_{C_1}) \cup (S \cap \hat{Z}_{C_2}) \subseteq S \cap Z_D \). Then, for the binary
DRMKP (32), we have $S \cap Z_{C_1} = \{0\}$ and $S \cap \tilde{Z}_{C_2}$ can be rewritten as

$$\begin{align*}
S \cap \tilde{Z}_{C_2} &= \left\{ x \in \{0, 1\}^n : 0 \leq y^t_i \leq M_t x_r, \alpha_t - M_t (1 - x_r) \leq y^t_i \leq \alpha_t, \forall r \in [n], \forall t \in [T], \\
&\quad \|z_{it}\|_2 - \lambda \leq 0, \forall i \in [N], \forall t \in [T], \\
&\quad \lambda \geq 0, \alpha_t \geq 0, \forall t \in [T], \right. 
\end{align*}$$

which implies that $z_{it} = -y^t_i$, and hence the constraint system (33) is equivalent to

$$\begin{align*}
S \cap \tilde{Z}_{C_2} &= \left\{ x \in \{0, 1\}^n : 0 \leq y^t_i \leq M_t x_r, \alpha_t - M_t (1 - x_r) \leq y^t_i \leq \alpha_t, \forall r \in [n], \forall t \in [T], \\
&\quad \|y\|_2 - \lambda \leq 0, \forall i \in [N], \forall t \in [T], \\
&\quad \lambda \geq 0, \alpha_t \geq 0, \forall t \in [T]. \right. 
\end{align*}$$

According to (34a) and (34d), we obtain that $\alpha_t$ can be upper bounded by $M_t = \frac{\xi}{\delta}$ for all $t \in [T]$.

Therefore, we need to solve the mixed-integer second-order cone programming (MISOCOCP) as follows.

$$\begin{align*}
\max \ & c^T x \\
\text{s.t.} \ & x \in \{0, 1\}^n, \\
& \lambda \delta + \frac{1}{N} \sum_{i=1}^{N} s_i \leq \epsilon, \\
& -\alpha_t b_t + (y^t)^T \zeta^t + 1 - s_i \leq 0, \forall i \in [N], \forall t \in [T], \\
& 0 \leq y^t_i \leq \frac{\epsilon}{\delta} x_r, \alpha_t - \frac{\epsilon}{\delta} (1 - x_r) \leq y^t_i \leq \alpha_t, \forall r \in [n], \forall t \in [T], \\
& \|y^t\|_2 - \lambda \leq 0, \forall t \in [T], \\
& \lambda \geq 0, \alpha_t \geq 0, s_i \geq 0, \forall i \in [N], \forall t \in [T]. \right. 
\end{align*}$$

The problem instances are created as follows. We generated 20 random instances with $n = 20$ and $T = 10$. For each instance, we generated $N \in \{100, 1000\}$ empirical samples $\{\zeta^j\}_{j \in [N]} \subseteq \mathbb{R}^{n \times T}$ from a uniform distribution over a box $[1, 10]^{T \times n}$. For each $j \in [n]$, we independently generated $c_j$ from the uniform distribution on the interval $[1, 10]$. In addition, we set $b_t := 50$ for each $t \in [T]$. We tested these 20 random instances with the risk level $\epsilon \in \{0.05, 0.10\}$ and Wasserstein radius $\delta \in \{0.01, 0.02\}$. All the
instances were executed on a desktop with a 4.10 GHz processor and 32GB RAM, while CPLEX 12.10 were
used with their default settings. We set the time limit of solving each instance to be 3600 seconds.

The numerical results with sample size $N = 100$ are displayed in Table 1, where we use Big-M Model and
CVaR Model denote exact reformulation proposed in [13] and inner approximation proposed in Theorem
4.1, respectively. We use “Opt.Val” to denote the optimal value $v^*$, “Value” to denote the best objective
value output from the approximation model and “Time” to denote the total running time in seconds.
Additionally, since we can solve Big-M Model to the optimality, we use GAP to denote the optimality gap
of the approximation model (35), which is computed as

$$\text{GAP} = \frac{|\text{Value} - \text{Opt.Val}|}{\text{Opt.Val}}.$$  

From Table 1, we observe that CVaR Model can be solved to the optimality within 15 seconds, while
Big-M Model often takes longer time to solve. In terms of approximation accuracy, we note that in most
instances, CVaR Model nearly finds the true optimal solutions. These results demonstrate that the proposed
CVaR Model can find near-optimal solutions.

The numerical results with sample size $N = 1000$ are displayed in Table 2, where similarly, we use Big-M
Model and CVaR Model denote exact reformulation proposed in [13] and inner approximation proposed in
Theorem 4.1, respectively. We use “LB” to denote the best lower bound found by Big-M Model or CVaR
Model, and “Time” to denote the total running time in seconds. To evaluate the effectiveness of CVaR
Model, we use Improvement to denote the percentage of differences between the lower bounds of CVaR
Model and the lower bounds of Big-M Model, which is computed as

$$\text{Improvement} = \frac{\text{LB of CVaR Model} - \text{LB of Big-M Model}}{\text{LB of Big-M Model}}.$$  

From Table 2, we observe that the total running time of CVaR Model significantly outperforms that of
Big-M Model, i.e., the majority of the instances of CVaR Model can be solved within 20 minutes, while Big-M
Model cannot be solved within the time limit. In terms of approximation accuracy, we see that CVaR Model
can find at least the same feasible solutions as Big-M Model. Additionally, in some instances, CVaR Model
can provide slightly better feasible solutions than Big-M Model. These results demonstrate the effectiveness
of CVaR Model proposed in this paper.
Table 1 Numerical results of the exact reformulation proposed in [13] and the CVaR approximation proposed in Theorem 4.1 for a binary DRMKP when sample size N = 100.

| s    | d    | Instances | Big-M Model | CVaR Model |
|------|------|-----------|-------------|------------|
|      |      |           | Opt. Val | Time | Value | GAP | Time |
| 0.05 | 0.01 | 10        | 38.81   | 4.32  | 38.81 | 0.00% | 8.68  |
|      |      | 11        | 54.96   | 3.17  | 54.96 | 0.00% | 8.17  |
|      |      | 12        | 49.95   | 2.82  | 49.95 | 0.00% | 9.14  |
|      |      | 13        | 54.41   | 2.57  | 53.86 | 1.00% | 5.34  |
|      |      | 14        | 47.29   | 2.55  | 47.29 | 0.00% | 5.31  |
|      |      | 15        | 51.64   | 3.76  | 51.09 | 1.06% | 8.36  |
|      |      | 16        | 48.41   | 8.46  | 48.41 | 0.00% | 9.49  |
|      |      | 17        | 52.30   | 37.08 | 52.30 | 0.00% | 11.60 |
|      |      | 18        | 52.61   | 4.29  | 52.64 | 0.00% | 9.40  |
|      |      | 19        | 43.08   | 1.27  | 43.08 | 0.00% | 4.87  |
|      |      | 20        | 54.98   | 3.86  | 53.93 | 1.02% | 8.15  |
| Average |      |           | 50.10   | 9.00  | 49.94 | 0.31% | 8.66  |
| 0.05 | 0.02 | 10        | 55.36   | 4.19  | 54.98 | 0.69% | 9.45  |
|      |      | 11        | 53.00   | 5.66  | 53.90 | 0.60% | 6.15  |
|      |      | 12        | 49.80   | 2.20  | 49.80 | 0.00% | 7.92  |
|      |      | 13        | 50.78   | 4.01  | 50.78 | 0.00% | 6.65  |
|      |      | 14        | 49.16   | 2.95  | 47.68 | 3.01% | 7.81  |
|      |      | 15        | 49.88   | 3.52  | 49.88 | 0.00% | 7.90  |
|      |      | 16        | 55.15   | 3.25  | 53.90 | 2.26% | 8.54  |
|      |      | 17        | 52.17   | 4.07  | 52.17 | 0.00% | 5.27  |
|      |      | 18        | 49.47   | 2.42  | 47.84 | 3.26% | 5.90  |
|      |      | 19        | 51.14   | 2.27  | 48.50 | 5.16% | 10.06 |
|      |      | 20        | 42.80   | 7.40  | 42.80 | 0.00% | 5.28  |
| Average |      |           | 50.19   | 5.53  | 49.80 | 0.76% | 7.49  |
| 0.10 | 0.01 | 10        | 52.69   | 188.16 | 52.69 | 0.00% | 11.76 |
|      |      | 11        | 50.85   | 91.39 | 50.02 | 1.62% | 12.41 |
|      |      | 12        | 51.91   | 2168.96 | 51.83 | 0.16% | 12.71 |
|      |      | 13        | 47.46   | 75.18 | 47.46 | 0.00% | 9.66  |
|      |      | 14        | 54.97   | 187.18 | 54.97 | 0.00% | 8.39  |
|      |      | 15        | 49.93   | 85.41 | 49.93 | 0.00% | 11.28 |
|      |      | 16        | 53.39   | 395.89 | 53.39 | 0.00% | 12.74 |
|      |      | 17        | 50.97   | 330.25 | 50.97 | 0.00% | 14.11 |
|      |      | 18        | 51.03   | 77.50 | 51.03 | 0.00% | 11.28 |
|      |      | 19        | 46.65   | 239.84 | 46.65 | 0.00% | 12.15 |
|      |      | 20        | 50.89   | 415.31 | 50.89 | 0.00% | 11.58 |
| Average |      |           | 50.33   | 344.38 | 50.29 | 0.99% | 11.44 |
| 0.10 | 0.02 | 10        | 47.15   | 138.42 | 47.15 | 0.00% | 12.25 |
|      |      | 11        | 44.59   | 5.02  | 43.32 | 2.85% | 7.01  |
|      |      | 12        | 49.85   | 378.80 | 49.62 | 0.46% | 12.35 |
|      |      | 13        | 55.42   | 68.98 | 55.42 | 0.00% | 9.79  |
|      |      | 14        | 50.23   | 79.51 | 50.23 | 0.00% | 12.46 |
|      |      | 15        | 50.47   | 119.03 | 50.06 | 0.80% | 14.23 |
|      |      | 16        | 47.05   | 271.61 | 47.05 | 0.00% | 15.00 |
|      |      | 17        | 52.30   | 37.53 | 51.83 | 0.89% | 8.17  |
|      |      | 18        | 51.83   | 74.08 | 51.53 | 0.69% | 9.21  |
|      |      | 19        | 54.30   | 265.85 | 54.20 | 0.18% | 14.08 |
|      |      | 20        | 50.89   | 22.53 | 50.89 | 0.00% | 8.71  |
| Average |      |           | 51.05   | 207.42 | 50.87 | 0.36% | 10.63 |
Table 2 Numerical results of the exact reformulation proposed in [13] and the CVaR approximation proposed in Theorem [14]

for a binary DRMKP when sample size $N = 1000$.

| $\epsilon$ | $\delta$ | Instances | Big-M Model | CVaR Model |
|------------|-----------|-----------|-------------|------------|
|            |           | LB Time   | LB Improvement | Time |
| 0.10       | 0.02      | 44.44     | 0.00%         | 1171.59   |
| 1          |           | 40.18     | 41.14 2.60%   | 1032.97   |
| 2          | 3         | 41.70     | 43.70 0.00%   | 476.74    |
| 4          | 5         | 43.65     | 43.65 0.00%   | 1375.80   |
| 5          | 6         | 44.15     | 44.15 0.00%   | 783.29    |
| 7          | 8         | 45.48     | 45.48 0.00%   | 637.16    |
| 10         | 11        | 47.92     | 47.92 0.00%   | 839.95    |
| 12         | 13        | 48.84     | 48.84 0.00%   | 725.06    |
| 15         | 16        | 49.00     | 47.90 2.18%   | 555.20    |
| 18         | 19        | 50.04     | 50.04 0.00%   | 555.20    |
| 20         |           | 51.15     | 51.15 0.00%   | 894.84    |

| 0.05       | 0.01      | 45.73     | 45.73 0.00%   | 986.77    |
| 10         | 11        | 44.16     | 44.16 0.00%   | 543.22    |
| 12         | 13        | 40.11     | 40.11 0.00%   | 990.35    |
| 15         | 16        | 46.67     | 46.67 0.00%   | 617.06    |
| 18         | 19        | 46.04     | 50.57 9.83%   | 1173.02   |
| 20         |           | 39.85     | 39.85 0.00%   | 427.63    |

| Average   |           | 44.13     | 44.75 1.50%   | 870.94    |

| 0.05       | 0.02      | 44.61     | 44.61 0.00%   | 504.76    |
| 10         | 11        | 45.44     | 45.44 0.00%   | 1049.89   |
| 12         | 13        | 44.44     | 44.44 0.00%   | 862.67    |
| 15         | 16        | 45.52     | 45.52 0.00%   | 924.41    |
| 18         | 19        | 46.90     | 46.90 0.00%   | 1085.63   |
| 20         |           | 44.66     | 44.66 0.00%   | 858.10    |

| Average   |           | 42.34     | 42.34 0.00%   | 907.87    |

| 0.10       | 0.01      | 52.91     | 52.91 0.00%   | 1435.84   |
| 10         | 11        | 49.08     | 49.21 0.27%   | 2074.25   |
| 12         | 13        | 50.01     | 50.81 1.60%   | 1543.52   |
| 15         | 16        | 51.03     | 51.81 0.00%   | 446.97    |
| 18         | 19        | 46.46     | 46.46 0.00%   | 1397.00   |
| 20         |           | 53.46     | 53.46 0.00%   | 958.56    |

| Average   |           | 52.23     | 52.23 0.00%   | 635.57    |

| 0.10       | 0.02      | 51.39     | 51.39 0.00%   | 813.74    |
| 10         | 11        | 51.18     | 51.18 0.00%   | 1020.88   |
| 12         | 13        | 50.01     | 50.04 0.00%   | 1286.21   |
| 15         | 16        | 47.66     | 47.66 0.00%   | 953.13    |
| 18         | 19        | 51.40     | 51.40 0.00%   | 783.22    |
| 20         |           | 52.45     | 52.45 0.00%   | 595.98    |

| Average   |           | 51.13     | 51.13 0.00%   | 1133.88   |

| 0.10       | 0.02      | 53.85     | 53.85 0.24%   | 1246.58   |
| 10         | 11        | 46.97     | 47.97 2.13%   | 867.48    |
| 12         | 13        | 46.90     | 47.90 4.00%   | 980.55    |
| 15         | 16        | 50.01     | 50.01 0.00%   | 1008.77   |
| 18         | 19        | 46.22     | 46.22 0.00%   | 746.48    |
| 20         |           | 52.48     | 52.48 0.00%   | 1115.56   |

| Average   |           | 51.15     | 51.15 0.00%   | 894.84    |

| 0.10       | 0.02      | 46.24     | 46.24 0.00%   | 882.39    |
| 10         | 11        | 50.00     | 50.00 0.00%   | 768.26    |
| 12         | 13        | 52.91     | 52.91 2.25%   | 719.44    |
| 15         | 16        | 47.99     | 48.30 0.65%   | 649.62    |
| 18         | 19        | 52.87     | 52.91 0.00%   | 843.17    |
| 20         |           | 43.44     | 44.16 1.65%   | 691.86    |

| Average   |           | 48.40     | 49.37 1.20%   | 813.81    |
5 Conclusion

In this paper, we studied a distributionally robust chance-constrained programming (DRCCP) with joint non-linear uncertain constraints under Wasserstein ambiguity set. We derived an equivalent reformulation of the set $Z_D$ and developed an inner approximation of the set $Z_D$ via a system of biconvex constraints. We also proved that this approximation is essentially exact under a certain condition and it can be converted into a convex programming for a single DRCCP. Once the decision variables are binary and the uncertain constraints are linear, we showed that this inner approximation is equivalent to a tractable mixed-integer convex programming. Numerical results demonstrated that the proposed mixed-integer convex reformulation can be solved efficiently. A future direction is to consider a DRCCP with a broader family of the non-linear uncertain mappings, for instance, when each constraint function $f(x, \xi)$ is quasi-convex in $\xi$ and is concave in $x$.

6 Reference

[1] Calafiore, G.C., ElGhaoui, L.: On distributionally robust chance-constrained linear programs. J. Optim. Theory Appl. 130(1), 1-22 (2006)

[2] Hanasusanto, G.A., Roitch, V., Kuhn, D., Wiesemann, W.: A distributionally robust perspective on uncertainty quantification and chance constrained programming. Math. Program. 151, 35-62 (2015)

[3] Hanasusanto, G.A., Roitch, V., Kuhn, D., Wiesemann, W.: Ambiguous joint chance constraints under mean and dispersion information. Oper. Res. 65(3), 751-767 (2017)

[4] Jiang, R., Guan, Y.: Data-driven chance constrained stochastic program. Math. Program. 158, 291-327 (2016)

[5] Xie, W., Ahmed, S.: On deterministic reformulations of distributionally robust joint chance constrained optimization problems. SIAM J. Optim. 28(2), 1151-1182 (2018)

[6] Yang, W., Xu, H.: Distributionally robust chance constraints for non-linear uncertainties. Math. Program. 155, 231-265 (2016)

[7] Chen, W., Sim, M., Sun, J., Teo, C.P.: From CVaR to uncertainty set: Implications in joint chance constrained optimization. Oper. Res. 58(2), 470-485 (2010)

[8] Delage, E. and Ye, Y.: Distributionally robust optimization under moment uncertainty with application to data-driven problems. Oper. Res. 58(3), 595-612 (2010)
[9] Zymler, S., Kuhn, D., Rustem, B.: Distributionally robust joint chance constraints with second-order
moment information. Math. Program. 137, 167-198 (2013)

[10] Wiesemann, W., Kuhn, D., Sim, M.: Distributionally robust convex optimization. Oper. Res. 62(6),
1358-1376 (2014)

[11] Xie, W., Ahmed, S., Jiang, R.: Optimized Bonferroni approximations of distributionally robust joint
chance constraints. Math. Program. (2019). https://doi.org/10.1007/s10107-019-01442-8

[12] Chen, Z., Kuhn, D., Wiesemann, W.: Data-driven chance constrained programs over Wasserstein
balls. arXiv preprint arXiv:1809.00210 (2018)

[13] Xie, W.: On distributionally robust chance constrained programs with Wasserstein distance. Math.
Program. 186, 115-155 (2021)

[14] Ji, R., Lejeune, M.A.: Data-driven distributionally robust chance-constrained optimization with
Wasserstein metric. J Glob Optim. (2020). https://doi.org/10.1007/s10898-020-00966-0

[15] Hota, A.R., Cherukuri, A., Lygeros, J.: Data-driven chance constrained optimization under Wasser-
stein ambiguity sets. arXiv preprint arXiv:1805.06729 (2018)

[16] Ho-Nguyen, N., Kilinc-Karzan, F., Kucukyavuz, S., Lee, D.: Distributionally robust chance-constrained
programs with right-hand side uncertainty under Wasserstein ambiguity. Math. Program. (2021). https://doi.org/10.1007/s10107-
020-01605-y

[17] Ho-Nguyen, N., Kilinc-Karzan, F., Kucukyavuz, S., Lee, D.: Strong formulations for distributionally
robust chance-constrained programs with left-hand side uncertainty under Wasserstein ambiguity. arXiv
preprint arXiv:2007.05750v2 (2020)

[18] Esfahani, P.M., Kuhn, D.: Data-driven distributionally robust optimization using the Wasserstein
metric: Performance guarantees and tractable reformulations. Math. Program. 171(1-2), 115-166 (2018)

[19] Rockafellar, R.T., Uryasev, S.: Optimization of conditional value-at-risk. J. Risk 2, 21-42 (2000)

[20] McCormick, G.P.: Computability of global solutions to factorable nonconvex programs: Part I-convex
underestimating problems. Math. Program. 10(1), 147-175 (1976)

[21] Cheng, J., Lisser, A.: Distributionally robust stochastic knapsack problem. SIAM J. Optim. 24(3),
1485-1506 (2014)

[22] Song, Y., Luedtke, J.R., Kucukyavuz, S.: Chance-constrained binary packing problems. INFORMS
J. Comput. 26(4), 735-747 (2014)