MATHEMATICAL ANALYSIS OF AN ABSTRACT MODEL AND ITS APPLICATIONS TO STRUCTURED POPULATIONS (I)

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(Communicated by Jacek Banasiak)

To the memory of the professor Ovide Arino.
He was a philanthropist and a great mathematician.

Abstract. The first part of this works deals with an integro–differential operator with boundary condition related to the interior solution. We prove that the model is governed by a strongly continuous semigroup and we precise its growth inequality. In the second part of this works, we model the proliferation-quiescence phases through a system of first order equations. We also prove that the proliferation-quiescence model is governed by a strongly continuous semigroup and we precise its growth inequality. In the last part, we give some applications in Demography and Biology.

1. Introduction. Many researchers, during the last decades, have contributed to the study of structured population dynamics (see [1, 2, 3, 4, 8, 9, 10, 14, 22, 23, 24, 28, 31] and references therein). Age is no doubt the most used structuring factor to distinguish individuals. Age is also an obvious structuring factor for demographers while many other structuring factors, in addition to age, arise for biologists, for instance: size, maturity, cycle length,...

Models of age structured populations were firstly formulated by A. Lotka and F. Sharpe ([28]) and then by A. G. McKendrick ([23]) at the beginning of last century. Nowadays they are the reference models of mathematical demography (see for instance [17, 7] and references therein).

Our goal, in this work, is to develop a mathematical theory covering models in demography, biology,... To this end, we consider abstract structured populations and we divide this work in three main parts.

The purpose of the first part deals with abstract structured populations with respect to \( x \in (a, b) \) (\( 0 < a < b < \infty \)). We assume then that the density, \( p := p(t, x) \), of all individuals having the property \( x \) at time \( t \geq 0 \) satisfies

\[
\frac{\partial p}{\partial t} + \frac{\partial (\sigma p)}{\partial x} = -\mu p + \int_a^b \eta(s, x)p(t, x)dx
\]

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where $\sigma := \sigma(x)$, $\mu := \mu(x)$ and $\eta := \eta(s,x)$ are rates with respect the studied population. We endow (1) with the following general boundary condition
\[ p(t,a) = Kp(t,) \]  
where $K$ is a given linear form on $L^1(a,b)$.

In the reference mathematical models of demography (see [23, 28] and, [17, 7] and references therein) the rate $\sigma$ is always considered to be $\sigma = 1$. However, in many studies of structured cell populations (see for instance [4, 5, 6, 19, 25] and references therein) the rate $\sigma$ is often assumed to be $\sigma \in W^{1,\infty}(a,b)$ and $\sigma(x) \geq \sigma_0 > 0 \quad x \in [a,b]$ (3)

for some $\sigma_0 > 0$. Our challenge, in this work, is to weaken (3) to
\[ \sigma \in W^{1,1}(a,b), \quad \frac{1}{\sigma} \in L^{1}(a,b) \quad \text{and} \quad \sigma(x) > 0 \quad x \in [a,b]. \]  
(4)

It is clear that our assumptions (4) are weaker than (3) and, the rate $\sigma(x) := \sqrt{b-x}$ fulfills (4) and not (3).

In the sequel, we consider our assumptions (4) and we divide the first part of this works as follows: in Section 2 we recall some well-known theoretical results that we will use throughout this work. In Section 3 we prove a trace result (Proposition 1) allowing us to make sense to (2) and so to the domains of all the unbounded linear operators considered in this work. In Section 4 we provide two different proofs to show (Lemma 8 and Proposition 3) that the unperturbed model, i.e., (1)–(with $\mu = 0$ and $\eta = 0$) and (2), is governed by $C_0$–semigroup $(T_{K,\sigma}(t))_{t \geq 0}$ on $L^1(a,b)$ satisfying
\[ \|T_{K,\sigma}(t)\psi\|_1 \leq e^{\sigma(a)\|K\|t}\|\psi\|_1, \quad \psi \in L^1(a,b), \quad t \geq 0, \]
whenever the linear form $K$ is continuous on $L^1(a,b)$. In Section 5, according to suitable assumptions on the rates $\mu$ and $\eta$, the full model (1)–(2) appears as a bounded linear perturbation of (1)–(with $\mu = 0$ and $\eta = 0$) and (2). We prove (Theorem 1) that the full model (1)–(2) is governed by a $C_0$–semigroup $(U_{K,\sigma}(t))_{t \geq 0}$ on $L^1(a,b)$ and we precise its growth inequality. We prove (Theorem 2) in particular
\[ \|U_{K,\sigma}(t)\psi\|_1 \leq e^{\sigma(a)\|K\|t}\|\psi\|_1, \quad \psi \in L^1(a,b), \quad t \geq 0, \]
whenever both rates $\mu$ and $\eta$ fulfill
\[ \int_a^b |\eta(x,y)| \, dx \leq |\mu(y)| \quad \text{a.a.} \quad y \in (a,b). \]  
(5)

This ends the first part of this work.

Actually, in many structured cell populations, not all cells are progressing to mitosis, some are in a quiescent or resting state for sometime. Quiescence phase (also called $G_0$) is the most common cell state on earth. It is the counterpart to proliferation: a reversible and nondividing state. Therefore the cell population is divided, at each moment, into two interacting subpopulations: Proliferating cells and Quiescent cells. Each cell is then: Proliferating (Active) or Quiescent (Resting). We quote for instance cells in uninjured skin, adult neuronal cells, cells of the adult mammalian heart, somatic cells,... All these cells, and so many others, are quiescent.

To show the importance of Quiescence phase we quote, in particular, Helicobacter pylori population. Nearly half of the world population are carriers of Helicobacter pylori that persists in the healthy human stomach. It is well established that Helicobacter pylori infection is the main cause of chronic gastritis and peptic ulcer...
disease. The administration of antibacterial drugs can lead not only to the emergence of resistant strains, but also contribute to the conversion of Helicobacter pylori into Quiescence phase.

We turn now to the model (1)–(2) and restrict ourselves to structured cell populations. It is readily seen that the model (1)–(2) deals only with Proliferating cells without Quiescence phase. Therefore our motivation, in the second part of this work, is to complete the model (1)–(2) by taking in account Quiescence phase. Before we formulate the Proliferation–Quiescence model, we set

Assumption 1. In Proliferation Phase (P) cells are born, grow and divide. They carry out their life processes and then they die (by mitosis or other causes).

Assumption 2. After birth cells go into Quiescence phase (Q) and remain metabolically active but do not proliferate and do not undergo any kind of division.

Assumption 3. Each cell is fully characterized by its status: Proliferating (Active) or Quiescent (Resting). Cells transit back and forth from one state to the other following the scheme

\[ \begin{align*}
& \text{Proliferation Phase (P)} \\
\rightarrow & \text{Quiescent Phase (Q)} \\
\rightarrow & \text{Proliferation Phase (P)} \\
\end{align*} \]

with rate \( \delta_1 \)

\[ \begin{align*}
& \text{Quiescent Phase (Q)} \\
\rightarrow & \text{Proliferation Phase (P)} \\
\rightarrow & \text{Quiescent Phase (Q)} \\
\end{align*} \]

with rate \( \delta_2 \)

where \( \delta_1 \) denotes the transition rate from Proliferation phase (P) to Quiescence phase (Q) while \( \delta_2 \) denotes the transition rate from Quiescence phase (Q) to Proliferation phase (P).

In the second part of this work, we consider structured cell populations with respect to \( x \in (a, b) \) \((0 < a < b < \infty)\). Let \((p, q) = (p(t, x), q(t, x))\) be, at time \( t \), the density of proliferating and quiescent cells. Taking in account the previous assumptions and following Figure 1 above we can write that

\[
\begin{align*}
\frac{\partial p}{\partial t} + \frac{\partial (\sigma p)}{\partial x} + \mu p - \int_a^b \eta(s, x)p(t, x)dx &= -\delta_1 p + \delta_2 q \quad (6) \\
\frac{\partial q}{\partial t} + \frac{\partial (\rho q)}{\partial x} &= +\delta_1 p - \delta_2 q \quad (7)
\end{align*}
\]

where \( \sigma := \sigma(x) \), \( \mu := \mu(x) \) and \( \eta := \eta(s, x) \) are rates in Proliferation phase while \( \rho := \rho(x) \) is a rate in Quiescence phase.

Proliferating cells divide (Assumption 1) following (2) while Quiescent cells do not divide (Assumption 2) and so

\[ q(t, 0) = 0. \quad (8) \]

The purpose of the second part of this work is to study the Proliferation-Quiescence Model (6), (7), (2) and (8) that is to say that
\[
\begin{align*}
    \frac{\partial p}{\partial t} &= -\frac{\partial (\sigma p)}{\partial s} - \mu p + \int_a^b \eta(s,x)p(t,x)dx - \delta_1p + \delta_2q, \quad (\text{PQ}_1) \\
    \frac{\partial q}{\partial t} &= -\frac{\partial (pq)}{\partial s} + \delta_1p - \delta_2q, \quad (\text{PQ}_2) \\
    p(t,a) &= Kp(t,\cdot), \quad (\text{PQ}_3) \\
    q(t,0) &= 0, \quad (\text{PQ}_4)
\end{align*}
\]

and we organize the rest of this work as follows: in Section 6 we prove (Theorem 3) that the full model \((\text{PQ}_1)-(\text{PQ}_4)\) is governed by a \(C_0\)-semigroup \((\mathcal{Z}_{K,p}(t))_{t \geq 0}\) on \((L^1(a,b))^2\) and we precise its growth inequality. We prove in particular (Theorem 3(2)) that

\[
\left\| \mathcal{Z}_{K,p}(t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| \leq e^{\sigma(t)\|K\|t} \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|, \quad \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in (L^1(a,b))^2, \quad t \geq 0,
\]

whenever both rates \(\mu\) and \(\eta\) fulfill (5). In Section 7 we formulate some relevant remarks about our assumptions (4) to forthcoming works.

Finally, in third and last section, we apply the previous theory to some mathematical models arising from Demography and Biology.

2. Mathematical tools. This section deals with some well-known theoretical results that we will use throughout this work. Let \(\mathcal{X}\) be a Banach space whose norm is \(\| \cdot \|\).

**Lemma 1** ([18, Cor. II.3.6]). An unbounded linear operator \((T, D(T))\) generates a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(\mathcal{X}\) satisfying

\[
\|T(t)x\| \leq e^{\omega t}\|x\|, \quad x \in \mathcal{X}, \quad t \geq 0, \quad (9)
\]

if and only if \((T, D(T))\) is closed and densely defined and, \((\omega, \infty) \subset \rho(T)\) and

\[
\| (\lambda - T)^{-1}x \| \leq (\lambda - \omega)^{-1}\|x\|, \quad x \in \mathcal{X}, \quad \lambda > \omega.
\]

**Lemma 2.** Let \((T, D(T))\) be the generator of a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(\mathcal{X}\) satisfying (9) and let \(B \in \mathcal{L}(\mathcal{X})\). Then

(1) \((T + B, D(T))\) generates a \(C_0\)-semigroup \((V(t))_{t \geq 0}\) on \(\mathcal{X}\) satisfying

\[
\|V(t)x\| \leq e^{\omega t}\|x\| \quad x \in \mathcal{X}, \quad t \geq 0. \quad (10)
\]

(2) If \(B\) is moreover dissipative then

\[
\|V(t)x\| \leq e^{\omega t}\|x\| \quad x \in \mathcal{X}, \quad t \geq 0. \quad (11)
\]

**Proof.** The point (1) follows from [18, Th. III.1.3].

(2) Let \(\overline{T} := T - \omega\). So \((\overline{T}, D(\overline{T}))\) generates the \(C_0\)-semigroup \((e^{-\omega t}T(t))_{t \geq 0}\) of contractions because of (9). Since \(B\) is dissipative and \(T\)-bounded with 0-bound, it follows from [18, Th. III.2.7] that \((\overline{T} + B, D(\overline{T}))\) is the generator of a \(C_0\)-semigroup of contractions \((\overline{V}(t))_{t \geq 0}\). Hence \(T + B = (\overline{T} + B) + \omega\) generates the unique \(C_0\)-semigroup \((V(t))_{t \geq 0} = (e^{\omega t}\overline{V}(t))_{t \geq 0}\) satisfying (11).

**Lemma 3** ([18, Th. VI.1.8]). Let \((T, D(T))\) be the infinitesimal generator of a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on a Banach lattice space \(\mathcal{X}\). Then \((T(t))_{t \geq 0}\) is positive if and only if \((\lambda - T)^{-1}\) is positive for large \(\lambda\).
3. **Trace lemma.** The aim of this section is to show a trace result which will allow us to make sense to the domain of each unbounded linear operator throughout this work. Let, until the end of this work, $a$ and $b$ be two real numbers such that

$$0 \leq a < b < \infty$$

and let $\mathcal{X}$ be the following Banach space

$$\mathcal{X} := L^1(a, b)$$

whose norm is $\|\phi\| := \int_a^b |\phi(x)| \, dx$.

Let $\sigma$ be a real function subject to the following assumption

$$\sigma \in W^{1,1}(a, b)$$

where $W^{1,1}(a, b)$ is the Sobolev space and, let $(W_x, \|\cdot\|_W)$ be such that

$$W_x := \{ \phi \in \mathcal{X} : \sigma \phi \in \mathcal{X} \text{ and } (\sigma \phi)' \in \mathcal{X} \}$$

and

$$\|\phi\|_W := \|\phi\|_1 + \|\sigma \phi\|_1 + \|\sigma \phi'\|_1$$

which can also be written in the form

$$W_x := \{ \phi \in \mathcal{X} : \sigma \phi \in W^{1,1}(a, b) \}$$

and

$$\|\phi\|_{W_x} := \|\phi\|_1 + \|\sigma \phi\|_{W^{1,1}(a, b)}.$$ 

It is readily seen that if $(\mathcal{A}_\sigma^1)$ holds then $1 \in W_x$ and therefore $W_x \neq \emptyset$.

**Lemma 4.** Assume that $(\mathcal{A}_\sigma^1)$ holds. Then $W_x$ is a Banach space satisfying

$$W^{1,1}(a, b) \subset W_x.$$  \hspace{1cm} (12)

**Proof.** Let $(\phi_n)_n$ be a Cauchy sequence in $W_x$. So $(\phi_n)_n$, $(\sigma \phi_n)_n$ and $((\sigma \phi_n)')_n$ are also Cauchy sequences in $\mathcal{X}$. Let $\phi \in \mathcal{X}$ be the limit of $(\phi_n)_n$ in $\mathcal{X}$. Since $\sigma \in W^{1,1}(a, b) \subset L^\infty(a, b)$ (see [15, Thm 8.8]) then $(\sigma \phi_n)_n$ converges to $\sigma \phi$ in $\mathcal{X}$ because of

$$\|\sigma \phi_n\|_1 = \int_a^b |\sigma(x)\phi_n(x)| \, dx \leq \|\sigma\|_\infty \|\phi_n\|_1 < \infty$$

and

$$\|\sigma \phi_n - \sigma \phi\|_1 \leq \|\sigma\|_\infty \|\phi_n - \phi\|_1 \xrightarrow{n \to \infty} 0.$$ 

Let $h \in \mathcal{X}$ be the limit of the Cauchy sequence $((\sigma \phi_n)')_n$ in $\mathcal{X}$. So

$$\int_a^b (\sigma \phi_n)'(x) \psi(x) \, dx = - \int_a^b (\sigma \phi_n)(x) \psi'(x) \, dx$$

for all $\psi \in C^\infty_c(a, b)$. Letting $n \to \infty$ yields

$$\int_a^b h(x) \psi(x) \, dx = - \int_a^b (\sigma \phi)(x) \psi'(x) \, dx = \int_a^b (\sigma \phi)'(x) \psi(x) \, dx$$

whence $(\sigma \phi)' = h \in \mathcal{X}$. Hence $\phi \in W_x$ and therefore $W_x$ is a Banach space.

Let $\phi \in W^{1,1}(a, b)$. Since $\sigma \in W^{1,1}(a, b)$ it follows that $\sigma \phi \in W^{1,1}(a, b)$ because $W^{1,1}(a, b)$ is a Banach algebra. This amounts to $\sigma \phi \in \mathcal{X}$ and $(\sigma \phi)' \in \mathcal{X}$ whence $\phi \in W_x$. Hence $W^{1,1}(a, b) \subset W_x$ which proves (12).
If $\sigma = 1$ then (12) is an equality. However (12) is strict for $\sigma(x) := \sqrt{b - x} \in W^{1,1}(a, b)$. Indeed $\phi := \frac{1}{\sqrt{b - x}} \in W_\sigma$ and $\phi \notin W^{1,1}(a, b)$. This fact can readily be inferred from the following result

**Lemma 5.** Assume that $(A_1^1)$ holds and

$$\sigma(x) \geq \sigma_0 > 0, \quad x \in [a, b]$$

(13)

for some $\sigma_0 > 0$. Then

$$W_\sigma = W^{1,1}(a, b).$$

(14)

**Proof.** Let

$$F(x) := \begin{cases} \frac{1}{|x|} & \text{if } |x| \geq \sigma_0, \\ \frac{1}{2\sigma_0^3} (3\sigma_0^2 - x^2) & \text{if } |x| \leq \sigma_0. \end{cases}$$

Since $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$ then $F \circ \sigma \in W^{1,1}(a, b)$ and therefore $\frac{1}{\sigma} \in W^{1,1}(a, b)$ because of (13).

Let $\phi \in W_\sigma$. Then $\sigma \phi \in W^{1,1}(a, b)$ and so $\phi = \frac{1}{\sigma}(\sigma \phi) \in W^{1,1}(a, b)$ because $W^{1,1}(a, b)$ is a Banach algebra. Hence $W_\sigma \subset W^{1,1}(a, b)$. This together with (12) prove (14).

As we are going to see in the sequel, the following result makes sense to the domains of all the unbounded linear operators throughout this work.

**Proposition 1.** Assume that $(A_1^1)$ hold. If $\sigma(a) \neq 0$ then

$$\phi \mapsto \phi(a)$$

(15)

is a continuous linear mapping from $W_\sigma$ into $\mathbb{R}$ satisfying for all $\phi \in W_\sigma$

$$|\phi(a)| \leq \frac{1}{|\sigma(a)|} \max \left\{ \frac{1}{b - a}, 1 \right\} \|\phi\|_{W_\sigma}.$$  

(16)

**Proof.** Let $\phi \in W_\sigma$. So $\sigma \phi \in W^{1,1}(a, b)$ which leads to

$$(\sigma \phi)(x) - (\sigma \phi)(a) = \int_a^x (\sigma \phi)'(y) dy, \quad x \in [a, b]$$

and therefore

$$|\sigma(a)\phi(a)| \leq |\sigma(x)\phi(x)| + \int_a^b |(\sigma \phi)'(y)| dy.$$

Integrating both hand sides with respect to $x \in (a, b)$ we obtain

$$|\sigma(a)\phi(a)| \leq \frac{1}{b - a} \int_a^b |\sigma(x)\phi(x)| dx + \int_a^b |(\sigma \phi)'(y)| dy$$

$$= \frac{1}{b - a} \|\sigma \phi\| + \|\sigma \phi\|_1$$

$$\leq \max \left\{ \frac{1}{b - a}, 1 \right\} \|\phi\|_{W_\sigma}$$

whence (16) (because $\sigma(a) \neq 0$) and so (15) is continuous as claimed.

**Remark 1.** Following the proof of Proposition 1 we can also prove: if $\sigma(b) \neq 0$ then $\phi \mapsto \phi(b)$ is continuous from $W_\sigma$ into $\mathbb{R}$ satisfying for $\phi \in W_\sigma$

$$|\phi(b)| \leq \frac{1}{|\sigma(b)|} \max \left\{ \frac{1}{b - a}, 1 \right\} \|\phi\|_{W_\sigma}.$$
4. The unperturbed model (1)–(2). This section is devoted to the unperturbed model (1)–(2) with $\mu = \eta = 0$ and (2). In order to prove that it is governed by a $C_0$–semigroup on $\mathcal{X}$ we assume, in addition to $(A_1^\sigma)$, that $(A_2^\sigma) \sigma(x) > 0$ a.a. $x \in [a, b) \setminus \{b\}$.

$(A_3^\sigma) \int_a^b \frac{dx}{|\sigma(x)|} < \infty.$

**Remark 2.**

1. Let for instance $\sigma(x) := \sqrt{b - x}$. Our assumptions $(A_1^\sigma)$, $(A_2^\sigma)$ and $(A_3^\sigma)$ are simultaneously fulfilled while (13) is not.

2. Note that $(A_2^\sigma)$ is weaker than (13) and so (14) may be not satisfied in the sequel. Note also that $(A_3^\sigma)$ is weaker than (13) because of $\int_a^b \sigma(x) \, dx \leq b - a \sigma_0 < \infty$.

**Lemma 6.** Assume that $(A_1^\sigma)$ and $(A_2^\sigma)$ hold and let $\theta$ be such that

$$\theta(x) = \int_a^x \frac{1}{\sigma(y)} \, dy, \quad x \in [a, b). \quad (17)$$

Then $e^{-\lambda \theta}$ ($\lambda > 0$) is almost everywhere differentiable and integrable i.e.,

$$\left( e^{-\lambda \theta} \right)' = -\lambda e^{-\lambda \theta} \sigma, \quad (18)$$

and

$$\left\| e^{-\lambda \theta} \right\|_1 \leq b - a. \quad (19)$$

Moreover, if $(A_3^\sigma)$ holds then $e^{-\lambda \theta}$ is absolutely continuous and

$$\left\| \frac{e^{-\lambda \theta}}{\sigma} \right\|_1 \leq \frac{1}{\lambda}. \quad (20)$$

**Proof.** Let $\lambda > 0$. The continuous representative of the class of $\sigma$ in $W^{1,1}(a, b)$ yields that $\theta$ is differentiable and so is $e^{-\lambda \theta} = \exp \circ (\theta \sigma)$ and therefore

$$\left( e^{-\lambda \theta} \right)' = e^{-\lambda \theta} (-\lambda \sigma') = -\lambda e^{-\lambda \theta} \sigma \quad \text{a.e.} \quad (21)$$

whence (18) as claimed. As regards (19) we have

$$\left\| e^{-\lambda \theta} \right\|_1 = \int_a^b e^{-\lambda \theta(x)} \, dx \leq \int_a^b 1 \, dx \leq b - a.$$

Suppose that $(A_3^\sigma)$ holds. Then $\theta$ is absolutely continuous and so is $e^{-\lambda \theta} = \exp \circ (\theta \sigma)$ whence $(e^{-\lambda \theta})' \in \mathcal{X}$ and

$$\int_a^b (e^{-\lambda \theta})' \, dx = e^{-\lambda \theta(b)} - e^{-\lambda \theta(a)} = e^{-\lambda \theta(b)} - 1.$$  

This together with (21) lead to

$$\left\| \frac{e^{-\lambda \theta}}{\sigma} \right\|_1 = \int_a^b \frac{e^{-\lambda \theta(x)} \, dx}{\sigma(x)} = \frac{1}{\lambda} \int_a^b \left( e^{-\lambda \theta} \right)' \, dx = \frac{1}{\lambda} \left( 1 - e^{-\lambda \theta(b)} \right) \leq \frac{1}{\lambda}$$

which proves (20) and the proof is complete. □
The rest of this section is divided in two subsections. In the first one we study the unperturbed model with homogenous boundary condition (1)–(with $\mu = \eta = 0$) and (2)–(with $K = 0$). In the second subsection we study the unperturbed model with non-homogenous boundary condition (1)–(with $\mu = \eta = 0$) and (2) which is the main subject of this section.

4.1. **Homogeneous boundary condition.** The purpose of this subsection is to prove that the unperturbed model with homogenous boundary condition (1)–(with $\mu = \eta = 0$) and (2)–(with $K = 0$) is governed by a $C_0$–semigroup on $\mathcal{X}$. So let $T_{0,\sigma}$ be the unbounded linear operator

$$T_{0,\sigma} \phi := -(\sigma \phi)'$$

on the domain

$$D_{0,\sigma} := \{ \phi \in W_\sigma : \phi(a) = 0 \}.$$  

(22)

Note that $D_{0,\sigma}$ makes a sense whenever $\sigma(a) \neq 0$ because of Proposition 1.

**Lemma 7.** Assume that $(A_1^\sigma)$, $(A_2^\sigma)$ and $(A_3^\sigma)$ hold.

1. $(0, \infty) \subset \rho(T_{0,\sigma})$ and if $\lambda > 0$ then $(\lambda - T_{0,\sigma})^{-1}$ is a bounded linear operator from $\mathcal{X}$ into itself satisfying for all $\phi \in \mathcal{X}$

$$\left\| (\lambda - T_{0,\sigma})^{-1} \phi \right\|_1 \leq \frac{1}{\lambda} \| \phi \|_1$$  

(23)

and

$$\left\| \sigma(\lambda - T_{0,\sigma})^{-1} \phi \right\|_1 \leq (b - a) \| \phi \|_1.$$  

(24)

2. $T_{0,\sigma}$ is closed and densely defined.

3. $(\lambda - T_{0,\sigma})^{-1}$ ($\lambda > 0$) is positive in $\mathcal{X}$.

**Proof.** (1). Let $\lambda > 0$ and let $\phi \in \mathcal{X}$. It is readily seen that $(\lambda - T_{0,\sigma}) \phi = g$ is a simple inhomogeneous linear ordinary differential. Hence the variation of constants method yields

$$(\lambda - T_{0,\sigma})^{-1} \phi(x) = \frac{1}{\sigma(x)} \int_a^x \exp \left( - \int_y^x \frac{\lambda}{\sigma(z)} \, dz \right) \phi(y) \, dy, \quad x \in [a, b)$$

which leads, by virtue of (17) and then (18), to

$$\left| (\lambda - T_{0,\sigma})^{-1} \phi(x) \right| \leq \frac{1}{\sigma(x)} \int_a^x e^{-\lambda(\theta(x) - \theta(y))} |\phi(y)| \, dy$$

$$= -\frac{1}{\lambda} (e^{-\lambda \theta})' (x) \int_a^x e^{\lambda \theta(y)} |\phi(y)| \, dy$$

and therefore

$$\left\| (\lambda - T_{0,\sigma})^{-1} \phi \right\|_1 \leq -\frac{1}{\lambda} \int_a^b (e^{-\lambda \theta})' (x) \left( \int_a^x e^{\lambda \theta(y)} |\phi(y)| \, dy \right) \, dx.$$  

Integration by parts yields

$$\left\| (\lambda - T_{0,\sigma})^{-1} \phi \right\|_1 \leq -\frac{1}{\lambda} e^{-\lambda \theta(b)} \int_a^b e^{\lambda \theta(y)} |\phi(y)| \, dy + \frac{1}{\lambda} \int_a^b |\phi(x)| \, dx \leq 0 + \frac{1}{\lambda} \| \phi \|_1,$$
which proves (23) as claimed and so \((\lambda - T_{o,\sigma})^{-1}\) is a bounded linear operator from \(\mathcal{X}\) into itself. Hence \((0, \infty) \subset \rho(T_{o,\sigma})\). As regards (24) we have
\[
\|\sigma(\lambda - T_{o,\sigma})^{-1}\phi\|_1 = \int_a^b |\frac{1}{\sigma(x)}| e^{-\lambda(x)} f(y) dy dx \\
\leq \int_a^b e^{-\lambda x} |\phi(y)| dy dx \\
\leq (b-a)\|\phi\|_1.
\]

(2). The boundedness of \((\lambda - T_{o,\sigma})^{-1}\) yields that \((\lambda - T_{o,\sigma})\) is closed and so is \(T_{o,\sigma} = \lambda - (\lambda - T_{o,\sigma})\). Moreover (12) yields
\[
C^\infty_c(a, b) \subset \{ \phi \in W^{1,1}(a, b) : \phi(a) = 0 \} \subset D_{o,\sigma} \subset \mathcal{X}
\]
and so \(T_{o,\sigma}\) is densely defined.

(3). The positivity of \((\lambda - T_{o,\sigma})^{-1}\) is obvious. \(\square\)

**Proposition 2.** Assume that \((A^1_a), (A^2_a)\) and \((A^3_a)\) hold. Then \(T_{o,\sigma}\) generates a \(C_0\)-semigroup \((T_{o,\sigma}(t))_{t \geq 0}\) of contractions on \(\mathcal{X}\) i.e., for all \(\phi \in \mathcal{X}\)
\[
\|T_{o,\sigma}(t)\phi\| \leq \|\phi\|, \quad t \geq 0.
\]
Furthermore \((T_{o,\sigma}(t))_{t \geq 0}\) is positive in \(\mathcal{X}\).

**Proof.** Lemma 7 yields that all the conditions of Lemma 1–(with \(\omega = 0\)) are satisfied and so \(T_{o,\sigma}\) generates a \(C_0\)-semigroup \((T_{o,\sigma}(t))_{t \geq 0}\) of contractions on \(\mathcal{X}\). Its positivity follows from that of \((\lambda - T_{o,\sigma})^{-1}\) (Lemma 7(3)) because of Lemma 3. \(\square\)

4.2. **Non-homogeneous boundary condition.** The aim of this subsection is to prove that the unperturbed model with non-homogenous boundary condition, i.e.,
\[(1)\text{–(with } \mu = \eta = 0\text{)}\text{ and } (2)\text{,}\]
is governed by a \(C_0\)-semigroup on \(\mathcal{X}\). So let \(K\) be a linear form on \(\mathcal{X}\) and let \(T_{K,\sigma}\) be the unbounded linear operator
\[
T_{K,\sigma}\psi := -(\sigma \psi)',
\]
on the domain
\[
D_{K,\sigma} := \{ \psi \in W : \psi(a) = K\psi \}.
\]

Note that the definition of the domain \(D_{K,\sigma}\) makes a sense whenever the linear form \(K\) is bounded on \(\mathcal{X}\) and \(\sigma(a) \neq 0\) because of Proposition 1.

**Lemma 8.** Assume that \((A^1_a), (A^2_a)\) and \((A^3_a)\) hold and, let \(K\) be a bounded linear form on \(\mathcal{X}\).

1. \(\sigma(a) \|K\|, \infty) \subset \rho(T_{K,\sigma})\) and if \(\lambda > \sigma(a) \|K\|\) then \((\lambda - T_{K,\sigma})^{-1}\) is a bounded linear operator from \(\mathcal{X}\) into itself satisfying for all \(\psi \in \mathcal{X}\)
\[
\|(\lambda - T_{K,\sigma})^{-1}\psi\|_1 \leq \frac{1}{\lambda - \sigma(a) \|K\|} \|\psi\|_1.
\]

2. \(T_{K,\sigma}\) is closed and densely defined.

3. If \(K\) is positive in \(\mathcal{X}\) then so is \((\lambda - T_{K,\sigma})^{-1}\) (\(\lambda > \sigma(a) \|K\|\))

We are going to give two different proofs of Lemma 8.
First Proof of Lemma 8.

(1). Let $\lambda > 0$ and let $\psi \in X$. Solving

$$ (\lambda - T_{K,\sigma})\phi = \psi $$

(27) means to find $\phi$ satisfying

$$ \lambda \phi = - (\sigma \phi)' + \psi $$

and

$$ \phi \in W_{\sigma} $$

and

$$ \phi(a) = K\phi. $$

(28)

(29)

(30)

It is readily seen that (28) is a simple inhomogeneous linear ordinary differential. Hence the variation of constants method yields

$$ \phi = C e^{-\lambda \theta} + (\lambda - T_{0,\sigma})^{-1} \psi $$

(31)

where $C$ is a constant (depending on $\psi$ and $\lambda$) and, $\theta$ and $(\lambda - T_{0,\sigma})^{-1}$ have already been studied in Lemma 6 and Lemma 7 respectively.

For convenience we divide the rest of the proof in several steps.

**Step I.** ($\phi$ satisfies (29)). Integrating (31) and then using (20) and (23) we find

$$ \|\phi\|_1 \leq |C| \left\| e^{-\lambda \theta} \sigma \right\|_1 + \left\| (\lambda - T_{0,\sigma})^{-1} \psi \right\|_1 \leq \frac{1}{\lambda} (|C| + \|\psi\|_1) < \infty $$

and by (28),

$$ \left\| (\sigma \phi)' \right\|_1 = \| -\lambda \phi + \psi \|_1 \leq \lambda \|\phi\|_1 + \|\psi\|_1 \leq |C| + 2 \|\psi\|_1 < \infty. $$

Similarly, using (19) and (24) we find

$$ \|\sigma \phi\|_1 \leq |C| \left\| e^{-\lambda \theta} \sigma \right\|_1 + \left\| \sigma (\lambda - T_{0,\sigma})^{-1} \psi \right\|_1 \leq (b - a) (|C| + \|\psi\|_1) < \infty. $$

Hence $\phi \in W_{\sigma}$.

**Step II** ($\phi$ satisfies (30)).

First let $k_{\lambda}$ be such that

$$ k_{\lambda} := \sigma(a)K \left( e^{-\lambda \theta} \sigma \right) $$

(32)

which is, by virtue of (20), finite because of

$$ |k_{\lambda}| \leq \sigma(a) \left\| K \right\| \left\| e^{-\lambda \theta} \sigma \right\|_1 \leq \frac{\sigma(a) \left\| K \right\|}{\lambda}. $$

(33)

Let $\lambda$ be such that $\lambda > \sigma(a) \|K\|$. Next, from (31) we can write

$$ \phi(a) = C e^{-\lambda \theta} \sigma(a) + (\lambda - T_{0,\sigma})^{-1} \psi(a) = C \frac{1}{\sigma(a)} + 0 = C \frac{1}{\sigma(a)} $$

and

$$ K\phi = CK \left( e^{-\lambda \theta} \sigma \right) + K(\lambda - T_{0,\sigma})^{-1} \psi = \frac{C}{\sigma(a)} K\phi + K(\lambda - T_{0,\sigma})^{-1} \psi $$

and therefore $\phi(a) = K\phi$ if and only if

$$ C = \frac{\sigma(a)}{1 - k_{\lambda}} (\lambda - T_{0,\sigma})^{-1} \psi. $$

(34)
Using (33) and then (23) it follows that
\[
\left\| \frac{\sigma(a)}{1 - k} K(\lambda - T_{o,\sigma})^{-1} \psi \right\|_1 \leq \frac{\sigma(a)}{1 - k} \left\| K \right\| \left\| (\lambda - T_{o,\sigma})^{-1} \psi \right\|_1 \\
\leq \frac{\sigma(a)}{1 - \sigma(a)K} \left\| K \right\| \frac{1}{\lambda} \left\| \psi \right\|_1 \\
= \frac{\sigma(a) \left\| K \right\|}{\lambda - \sigma(a)K} \left\| \psi \right\|_1,
\]
and so C is finite. Hence \( \phi \) satisfies (30) whenever \( \lambda > \sigma(a) \left\| K \right\| \).

**Step III** (Proof of (26)).

Let \( \lambda > \sigma(a) \left\| K \right\| \). Combining (34) and (31) yields
\[
\phi = \frac{\sigma(a)}{1 - k} K(\lambda - T_{o,\sigma})^{-1} \psi e^{-\lambda \theta} + (\lambda - T_{o,\sigma})^{-1} \psi
\]
which is the unique solution of (27) because of Steps I and II. Hence
\[
(\lambda - T_{K,\sigma})^{-1} \psi = \frac{\sigma(a)}{1 - k} K(\lambda - T_{o,\sigma})^{-1} \psi e^{-\lambda \theta} + (\lambda - T_{o,\sigma})^{-1} \psi.
\]
Moreover, from (35) and (20) we obtain
\[
\left\| \frac{\sigma(a)}{1 - k} K(\lambda - T_{o,\sigma})^{-1} \psi e^{-\lambda \theta} \right\|_1 = \left\| \frac{\sigma(a)}{1 - k} K(\lambda - T_{o,\sigma})^{-1} \psi \right\|_1 \left\| e^{-\lambda \theta} \right\|_1 \\
\leq \frac{\sigma(a) \left\| K \right\|}{\lambda - \sigma(a) \left\| K \right\|} \left\| \psi \right\|_1.
\]
This together with (23) yields
\[
\left\| (\lambda - T_{K,\sigma})^{-1} \psi \right\|_1 \leq \frac{\sigma(a) \left\| K \right\|}{\lambda - \sigma(a) \left\| K \right\|} \left\| \psi \right\|_1 + \frac{\left\| \psi \right\|_1}{\lambda} = \frac{1}{\lambda - \sigma(a) \left\| K \right\|} \left\| \psi \right\|_1,
\]
whence (26) as claimed and therefore \( (\lambda - T_{K,\sigma})^{-1} \) is a bounded linear operator from \( X \) into itself. Hence \( (\sigma(a) \left\| K \right\| , \infty) \subset \rho(T_{K,\sigma}) \).

(2). Let \( \lambda > \sigma(a) \left\| K \right\| \). The boundedness of \( (\lambda - T_{K,\sigma})^{-1} \) yields that \( (\lambda - T_{K,\sigma}) \) is closed and so is \( T_{K,\sigma} = \lambda - (\lambda - T_{K,\sigma}) \).

Let \( \psi \in X \). Since \( (\sigma(a) \left\| K \right\| , \infty) \subset \rho(T_{K,\sigma}) \) we set
\[
\psi_\lambda := \lambda(\lambda - T_{K,\sigma})^{-1} \psi \in D_{K,\sigma} \quad \lambda > \sigma(a) \left\| K \right\|.
\]

Using (37) and then (38) we find
\[
\left\| \psi_\lambda - \psi \right\|_1 \leq \lambda \left\| \frac{\sigma(a)}{1 - k} K(\lambda - T_{o,\sigma})^{-1} \psi e^{-\lambda \theta} \right\|_1 + \left\| \lambda(\lambda - T_{o,\sigma})^{-1} \psi - \psi \right\|_1 \\
\leq \frac{\sigma(a) \left\| K \right\|}{\lambda - \sigma(a) \left\| K \right\|} \left\| \psi \right\|_1 + \left\| \lambda(\lambda - T_{o,\sigma})^{-1} \psi - \psi \right\|_1,
\]
which leads to
\[
\lim_{\lambda \to \infty} \left\| \psi_\lambda - \psi \right\|_1 \leq 0 + \lim_{\lambda \to \infty} \left\| \lambda(\lambda - T_{o,\sigma})^{-1} \psi - \psi \right\|_1.
\]
Since \( T_{o,\sigma} \) is an infinitesimal generator (Proposition 2) then the right hand side tends to 0 as \( \lambda \to \infty \) and therefore \( (\psi_\lambda)_\lambda \) converges to \( \psi \) in \( X \). Hence \( T_{K,\sigma} \) is densely defined in \( X \) as claimed.
(3). Suppose \( K \) is positive in \( \mathcal{X} \) and let \( \lambda > \sigma(a) \| K \| \). From (33) we infer
\[
1 - k_1 \geq 1 - \frac{\sigma(a) \| K \|}{\lambda} > 0.
\]
If \( \psi \in \mathcal{X} \) is positive then so is \((\lambda - T_{o,\sigma})^{-1}\psi \) (Lemma 7(3)). Now all the terms involving in (37) are positive and therefore \((\lambda - T_{K,\sigma})^{-1}\psi \) is also positive. The first proof is now achieved.

Second Proof of Lemma 8.

(1). Let \( \lambda > 0 \) and let \( K_\lambda \) be such that
\[
K_\lambda := \sigma(a) \frac{e^{-\lambda \theta}}{\sigma} K
\]
where \( \theta \) is already studied in Lemma 6. Using (18) and then (19) and (20) we get, for all \( \phi \in \mathcal{X} \), that
\[
(\sigma K_\lambda \phi)' = \sigma(a) \left( e^{-\lambda \theta} \right)' K \phi = -\lambda \sigma(a) \frac{e^{-\lambda \theta}}{\sigma} K \phi = -\lambda K_\lambda \phi \quad \text{a.e.}
\]
and
\[
\| K_\lambda \phi \|_1 = \sigma(a) \left\| \frac{e^{-\lambda \theta}}{\sigma} \right\|_1 \| K \phi \| \leq \frac{\sigma(a) \| K \|}{\lambda} \| \phi \|_1
\]
and
\[
\| \sigma K_\lambda \phi \|_1 = \sigma(a) \left\| \frac{e^{-\lambda \theta}}{\sigma} \right\|_1 \| K \phi \| \leq \sigma(a)(b - a) \| K \| \| \phi \|_1.
\]
For convenience we divide the rest of the proof in several steps.

Step I. \((1 - K_\lambda) \) maps \( D_{K,\sigma} \) into \( D_{o,\sigma} \). Let \( \psi \in (1 - K_\lambda)(D_{K,\sigma}) \) where \( 1 \) denotes the identity operator in \( \mathcal{X} \). There exists \( \phi \in D_{K,\sigma} \) such that \( \psi = (1 - K_\lambda) \phi \). From (42) and (43) we obtain
\[
\| \psi \|_1 \leq \| \phi \|_1 + \| K_\lambda \phi \|_1 \leq \left(1 + \frac{\sigma(a) \| K \|}{\lambda}\right) \| \phi \|_1 < \infty
\]
and
\[
\| \sigma \psi \|_1 \leq \| \sigma \phi \|_1 + \| \sigma K_\lambda \phi \|_1 \leq \| \sigma \phi \|_1 + \sigma(a)(b - a) \| K \| \| \phi \|_1 < \infty.
\]
From (41) we infer that \((\sigma \phi)' = (\sigma \phi)' - (\sigma K_\lambda \phi)' = (\sigma \phi)' + \lambda K_\lambda \phi \) and so
\[
\| (\sigma \phi)' \|_1 \leq \| (\sigma \phi)' \|_1 + \lambda \| K_\lambda \phi \|_1 \leq \| (\sigma \phi)' \|_1 + \sigma(a) \| K \| \| \phi \|_1 < \infty.
\]
Hence \( \psi \in W_{\sigma} \). Moreover
\[
\psi(a) = \phi(a) - K_\lambda \phi(a) = \phi(a) - \sigma(a) \frac{e^{-\lambda \theta(a)}}{\sigma(a)} K \phi = 0
\]
whence \( \psi \in D_{o,\sigma} \) and so \((1 - K_\lambda)(D_{K,\sigma}) \subset D_{o,\sigma} \).

Step II (Proof of (26)).

Let \( \phi \in D_{K,\sigma} \). Since \((1 - K_\lambda) \phi \in D_{o,\sigma} \) (Step I) then
\[
(\lambda - T_{o,\sigma})(1 - K_\lambda) \phi = \lambda (1 - K_\lambda) \phi + \left( \sigma (1 - K_\lambda) \phi \right)'
= \lambda \phi - \lambda K_\lambda \phi + (\sigma \phi)' - (\sigma K_\lambda \phi)'
\]
which leads, by virtue of (41), to
\[
(\lambda - T_{o,\sigma})(1 - K_\lambda) \phi = \lambda \phi + (\sigma \phi)'
\]
and therefore
\[
(\lambda - T_{K,\sigma}) = (\lambda - T_{o,\sigma})(1 - K_\lambda).
\]
Let $\lambda > \sigma(a)\|K\|$. From (42) we infer that $(I - K_\lambda)$ is invertible from $\mathcal{X}$ into itself. This together with (23) yields that the right hand side of (44) is invertible from $\mathcal{X}$ into itself and therefore
\[(\lambda - T_{K,\lambda})^{-1} = (I - K_\lambda)^{-1}(\lambda - T_{0,\sigma})^{-1} \quad \lambda > \sigma(a)\|K\|.\] (45)
Moreover for all $\psi \in \mathcal{X}$ we have
\[\left\|(\lambda - T_{K,\lambda})^{-1}\psi\right\|_1 \leq \left\|(I - K_\lambda)^{-1}\left(\left(\lambda(\lambda - T_{0,\sigma})^{-1} - \psi\right) + K_\lambda\right)\psi\right\|_1\]
\[\leq \frac{1}{1 - \frac{\sigma(a)\|K\|}{\lambda}} \left\|\lambda(\lambda - T_{0,\sigma})^{-1} - \psi\right\|_1 + \|K_\lambda\psi\|_1\]
\[\leq \frac{1}{\lambda - \sigma(a)\|K\|} \left(\left\|\lambda(\lambda - T_{0,\sigma})^{-1} - \psi\right\|_1 + \frac{\sigma(a)\|K\|}{\lambda}\|\psi\|_1\right)\]
which proves (26) as claimed and so $(\lambda - T_{K,\lambda})^{-1}$ is a bounded linear operator from $\mathcal{X}$ into itself. Hence $(\sigma(a)\|K\|, \infty) \subset \rho(T_{K,\lambda})$.

(2). The proof of the closedness of $T_{K,\lambda}$ is the same as in the first proof.

Let $\psi \in \mathcal{X}$. Since $(\sigma(a)\|K\|, \infty) \subset \rho(T_{K,\lambda})$ we choose the sequence (39). Using (45) and then (42) we find
\[\|\psi_{\lambda} - \psi\|_1 = \left\|(I - K_\lambda)^{-1}\left(\left(\lambda(\lambda - T_{0,\sigma})^{-1} - \psi\right) + K_\lambda\right)\psi\right\|_1\]
\[\leq \frac{1}{1 - \frac{\sigma(a)\|K\|}{\lambda}} \left(\left\|\lambda(\lambda - T_{0,\sigma})^{-1} - \psi\right\|_1 + \|K_\lambda\psi\|_1\right)\]
and so
\[\lim_{\lambda \to \infty} \|\psi_{\lambda} - \psi\|_1 \leq \lim_{\lambda \to \infty} \left\|\lambda(\lambda - T_{0,\sigma})^{-1} - \psi\right\|_1\].

Since $T_{0,\sigma}$ is an infinitesimal generator (Proposition 2) then the right hand side tends to 0 as $\lambda \to \infty$ and therefore $(\psi_{\lambda})_{\lambda}$ converges to $\psi$ in $\mathcal{X}$. Hence $T_{K,\lambda}$ is densely defined in $\mathcal{X}$ as claimed.

(3). Suppose that $K$ is positive and let $\lambda > \sigma(a)\|K\|$. If $\psi \in \mathcal{X}$ is positive then (45) together with (42) and then (40) and Lemma 7(3) yields
\[(\lambda - T_{K,\lambda})^{-1}\psi = \sum_{n \geq 0} K^n_\lambda(\lambda - T_{0,\sigma})^{-1}\psi \geq (\lambda - T_{0,\sigma})^{-1}\psi \geq 0.\]

The second proof is now achieved. \(\square\)

**Remark 3.** Both (37) and (45) are the same whenever $\lambda > \sigma(a)\|K\|$. Indeed if $\lambda > \sigma(a)\|K\|$ then (32) and (40) lead to $k_\lambda K\psi = KK_\lambda K\psi$ for all $\psi \in \mathcal{X}$. By induction on the integer $n \geq 1$ we find $k_\lambda^n K\psi = KK_\lambda^n K\psi$ and so
\[\sum_{n \geq 0} k_\lambda^n K\psi = \sum_{n \geq 0} KK_\lambda^n K\psi.\]
This together with (33) and (42) yields
\[\frac{1}{I - K_\lambda} K\psi = K(I - K_\lambda)^{-1}\psi\]
which implies that
\[\frac{\sigma(a)}{I - K_\lambda} K(\lambda - T_{0,\sigma})^{-1}\psi e^{-\lambda\theta} = \sigma(a) e^{-\lambda\theta} \left[\frac{1}{I - K_\lambda} K(\lambda - T_{0,\sigma})^{-1}\psi\right]\]
Furthermore if \( X \). So let \( K \) be a linear form on \( C \) this end we prove that the full model (1)–(2) is governed by a

The full model

and so (37) and (45) are the same whenever \( \lambda > \sigma(a) \| K \| \).

**Proposition 3.** Assume that (A\(^1\))\(_1\), (A\(^2\))\(_1\) and (A\(^3\))\(_1\) hold and, let \( K \) be a bounded linear form on \( \mathcal{X} \). Then \( T_{K,\sigma} \) generates a \( C_0 \)-semigroup \((T_{K,\sigma}(t))_{t \geq 0}\) on \( \mathcal{X} \) satisfying for all \( \psi \in \mathcal{X} \)

\[
\| T_{K,\sigma}(t) \psi \|_1 \leq e^{\sigma(a)(1+|K|)} \| \psi \|_1 \quad t \geq 0. \tag{46}
\]

Moreover if \( K \) is positive in \( \mathcal{X} \) then so is \((T_{K,\sigma}(t))_{t \geq 0}\).

**Proof.** From Lemma 8 we readily infer that all the required conditions of Lemma 1–(with \( \omega = \sigma(a) \| K \| \)) are fulfilled and therefore \( T_{K,\sigma} \) generates a \( C_0 \)-semigroup \((T_{K,\sigma}(t))_{t \geq 0}\) on \( \mathcal{X} \) satisfying (46). Furthermore if \( K \) is positive then so is \((\lambda - T_{K,\sigma})^{-1}\) (Lemma 8 (3)) and therefore \((T_{K,\sigma}(t))_{t \geq 0}\) is also positive because of Lemma 3. \( \square \)

5. **The full model (1)–(2).** This section deals with the first aim of this work. To this aim we prove that the full model (1)–(2) is governed by a \( C_0 \)-semigroup on \( \mathcal{X} \). So let \( K \) be a linear form on \( \mathcal{X} \) and let \( U_{K,\sigma} \) be the unbounded linear operator

\[
U_{K,\sigma} := T_{K,\sigma} - \mu 1 + B \quad \text{on the domain} \quad D_{K,\sigma} \tag{47}
\]

where \( T_{K,\sigma} \) is defined by (25), \( 1 \) is the identity operator in \( \mathcal{X} \) and

\[
B\phi(x) := \int_a^b \eta(x,y)\phi(y)dy. \tag{48}
\]

We assume that \( \mu \) and \( \eta \) are subject to the following assumptions

\((A^1)\)

\( \mu \in L^\infty(a,b) \),

\((A^2)\)

\( \mu \) is positive,

and

\((A^3)\)

\( \int_a^b |\eta(x,\cdot)| dx \in L^\infty(a,b) \),

\((A^4)\)

\( \eta \) is positive.

The first aim of this work can now be announced as follows

**Theorem 1.** Assume that (A\(^1\))\(_1\), (A\(^2\))\(_1\) and (A\(^3\))\(_1\) hold and, let \( K \) be a bounded linear form on \( \mathcal{X} \). If both (A\(^1\))\(_2\) and (A\(^1\))\(_3\) hold then \( U_{K,\sigma} \) generates a \( C_0 \)-semigroup \((U_{K,\sigma}(t))_{t \geq 0}\) on \( \mathcal{X} \) satisfying for all \( \phi \in \mathcal{X} \)

\[
\| U_{K,\sigma}(t) \phi \|_1 \leq e^{(\sigma(a)(1+|K|))t} \| \phi \|_1 \quad t \geq 0, \tag{49}
\]

where

\[
\mu := \text{ess inf}_{x \in (a,b)} \mu(x) \quad \text{and} \quad \overline{\eta} := \text{ess sup}_{y \in (a,b)} \int_a^b |\eta(x, y)| dx. \tag{50}
\]

Furthermore if \( K \) is positive and (A\(^2\))\(_\eta\) holds then \((U_{K,\sigma}(t))_{t \geq 0}\) is positive.
Proof. For convenience we divide the proof in several steps.

**Step I.** Let \( V_{K,\sigma} \) be the unbounded linear operator
\[
V_{K,\sigma} := T_{K,\sigma} - \mu \mathbb{1} \quad \text{on the domain } \quad D_{K,\sigma}
\]  
(51)

Since the multiplication operator \((-\mu \mathbb{1})\) is bounded from \(\mathcal{X}\) into itself (Assumption \((A_1^1)\)) then \( V_{K,\sigma} \) is a bounded linear perturbation of the generator \( T_{K,\sigma} \) (Proposition 3). From Lemma 2(1) it follows that \( V_{K,\sigma} \) generates a \(C_0\)-semigroup \((V_{K,\sigma}(t))_{t \geq 0}\).

Suppose now that \( K \) is positive and let \( \phi \in \mathcal{X} \) be such that \( \phi \geq 0 \). The positivity of the \(C_0\)-semigroup \((T_{K,\sigma}(t))_{t \geq 0}\) (Proposition 3) leads to
\[
\left[ T_{K,\sigma}(\frac{t}{n}) e^{\frac{t}{n}(\mu \mathbb{1})} \right]^n \phi \geq 0 \quad n = 1, 2, 3 \ldots
\]
Letting \( n \to \infty \) yields \( V_{K,\sigma}(t)\phi \geq 0 \) because of Trotter product formula.

**Step II.** Let \( \nabla_{K,\sigma} \) be the unbounded linear operator
\[
\nabla_{K,\sigma} := T_{K,\sigma} + (\mu - \mu) \mathbb{1} \quad \text{on } \quad D_{K,\sigma}
\]  
(52)

Since the multiplication operator \((\mu - \mu) \mathbb{1}\) is bounded from \(\mathcal{X}\) into itself (Assumption \((A_1^1)\)) then \( \nabla_{K,\sigma} \) is a bounded linear perturbation of the generator \( T_{K,\sigma} \) (Proposition 3). Lemma 2(1) yields that \( \nabla_{K,\sigma} \) generates a \(C_0\)-semigroup \((\nabla_{K,\sigma}(t))_{t \geq 0}\) on \(\mathcal{X}\). Furthermore for all \( \phi \in \mathcal{X} \) we have
\[
\langle \text{sgn } \phi, (\mu - \mu) \mathbb{1} \rangle \phi = \int_a^b (\mu - \mu(x)) |\phi(x)| \, dx \leq 0
\]
whence \((\mu - \mu) \mathbb{1}\) is moreover dissipative in \(\mathcal{X}\). Thanks to Lemma 2(2) we infer, for all \( \phi \in \mathcal{X} \), that
\[
\| \nabla_{K,\sigma}(t) \phi \|_1 \leq e^{\sigma(a)} \|k\| t \| \phi \|_1 \quad t \geq 0.
\]
Next, (51) and (52) yields \( V_{K,\sigma} = \nabla_{K,\sigma} - \mu \mathbb{1} \) and so \( V_{K,\sigma} \) is a bounded linear perturbation of the generator \( \nabla_{K,\sigma} \). This together with Step I lead to \((V_{K,\sigma}(t))_{t \geq 0} = (e^{-\mu t} \nabla_{K,\sigma}(t))_{t \geq 0}\) and therefore for all \( \phi \in \mathcal{X} \)
\[
\| V_{K,\sigma}(t) \phi \|_1 = e^{-\mu t} \| \nabla_{K,\sigma}(t) \phi \|_1 \leq e^{\sigma(a) \|k\| - \mu} t \| \phi \|_1 \quad t \geq 0.
\]

**Step III.** First, for all \( \phi \in \mathcal{X} \) we have
\[
\| B\phi \|_1 \leq \int_a^b \left[ \int_a^b |\eta(x, y)| \, dy \right] |\phi(y)| \, dx \leq \eta \| \phi \|_1.
\]  
(53)

Hence \( B \) is a bounded linear operator from \(\mathcal{X}\) into itself because of \((A_1^1)\).

Next, (47) and (51) lead to \( U_{K,\sigma} = V_{K,\sigma} + B \) which is a bounded linear perturbation of the generator \( V_{K,\sigma} \) (Steps I) by \( B \) satisfying (53). Lemma 2(1) yields that \( U_{K,\sigma} \) generates a \(C_0\)-semigroup \((U_{K,\sigma}(t))_{t \geq 0}\) in \(\mathcal{X}\) satisfying (49).

Suppose now that \( K \) is positive and let \( \phi \in \mathcal{X} \) be such that \( \phi \geq 0 \). The positivity of the \(C_0\)-semigroup \((V_{K,\sigma}(t))_{t \geq 0}\) (Step I) and that of \( B \) (Assumption \((A_2^1)\)) lead to
\[
\left[ V_{K,\sigma}(\frac{t}{n}) e^{\frac{t}{n}B} \right]^n \phi \geq 0 \quad n = 1, 2, 3 \ldots
\]
At the limit $n \to \infty$ Trotter product formula yields $U_{k,\sigma}(t)\phi \geq 0$. □

The growth inequality (49) can be improved by assuming that

$$(A_{\mu-n}^1) \quad \int_a^b |\eta(x,y)| \, dx \leq |\mu(y)| \quad \text{a.a.} \quad y \in (a,b).$$

**Theorem 2.** Assume that $(A_{\sigma}^1)$, $(A_{\sigma}^2)$, $(A_{\sigma}^3)$, $(A_{\mu}^1)$ and $(A_{\mu-n}^1)$ hold and, let $K$ be a bounded linear form on $\mathcal{X}$. If both $(A_{\mu}^2)$ and $(A_{\mu-n}^2)$ hold then $U_{k,\sigma}$ generates a $C_0$-semigroup $(U_{k,\sigma}(t))_{t \geq 0}$ on $\mathcal{X}$ satisfying for all $\phi \in \mathcal{X}$

$$\|U_{k,\sigma}(t)\phi\|_1 \leq e^{\omega(a)\|K\|t}\|\phi\|_1, \quad t \geq 0. \quad (54)$$

**Proof.** $(A_{\mu}^1)$ and $(A_{\mu-n}^1)$ together with (53) yields that $(-\mu \mathbb{I} + B)$ is bounded from $\mathcal{X}$ into itself and so $U_{k,\sigma} = T_{k,\sigma} + (-\mu \mathbb{I} + B)$ is a bounded linear perturbation of the generator $T_{k,\sigma}$ (Proposition 3). From Lemma 2(1) we infer that $U_{k,\sigma}$ generates a $C_0$-semigroup $(U_{k,\sigma}(t))_{t \geq 0}$ on $\mathcal{X}$. Furthermore for all $\phi \in \mathcal{X}$ we have

$$(\text{sgn } \phi, (-\mu \mathbb{I} + B) \phi) = -\int_a^b \text{sgn } \phi(x) \mu(x) \phi(x) \, dx + \int_a^b \text{sgn } \phi(x) B \phi(x) \, dx$$

$$\leq -\int_a^b \mu(x) |\phi(x)| \, dx + \int_a^b \left[ \int_a^b |\eta(x,y)| \, dx \right] |\phi(y)| \, dy$$

$$= \int_a^b \left[ -\mu(y) + \int_a^b |\eta(x,y)| \, dx \right] |\phi(y)| \, dy$$

which leads, by virtue of $(A_{\mu-n}^2)$ together with $(A_{\mu}^2)$, to

$$(\text{sgn } \phi, (-\mu \mathbb{I} + B) \phi) \leq 0$$

whence $(-\mu \mathbb{I} + B)$ is moreover dissipative in $\mathcal{X}$. Now (54) follows from Lemma 2(2) and the proof is complete. □

Before we finish this section we consider the following assumption

$(A_{\mu-n}^2)$ \quad $\eta \leq \mu$

where $\mu$ and $\eta$ are given by (50). Note that $(A_{\mu-n}^2)$ is clearly stronger than $(A_{\mu-n}^1)$ whenever $(A_{\mu}^2)$ holds true.

**Corollary 1.** Assume that $(A_{\sigma}^1)$, $(A_{\sigma}^2)$, $(A_{\sigma}^3)$, $(A_{\mu}^1)$ and $(A_{\mu-n}^1)$ hold and, let $K$ be a bounded linear form on $\mathcal{X}$. If both $(A_{\mu}^2)$ and $(A_{\mu-n}^2)$ hold then $U_{k,\sigma}$ generates a $C_0$-semigroup $(U_{k,\sigma}(t))_{t \geq 0}$ on $\mathcal{X}$ satisfying for all $\phi \in \mathcal{X}$

$$\|U_{k,\sigma}(t)\phi\|_1 \leq e^{\omega(a)\|K\|t}\|\phi\|_1, \quad t \geq 0. \quad (55)$$

**Proof.** It suffices to apply Theorem 1. Furthermore (55) follows from (49) together with $(A_{\mu-n}^2)$. The proof can also be inferred from Theorem 2 with $(A_{\mu-n}^2)$ instead of $(A_{\mu-n}^1)$. □
6. The full model \((PQ_1)–(PQ_4)\). This section deals with the second aim of this section. To this end we prove that the full model \((PQ_1)–(PQ_4)\) is governed by a \(C_0\)-semigroup on the Banach space
\[
Z := X \times X \quad \text{whose norm is} \quad \| (\phi, \psi) \| := \| \phi \|_1 + \| \psi \|_1.
\]

Before we state the main result of this section, we first consider a real function \(\rho\) subject to the same assumptions on \(\sigma\) i.e.,

\[
(A_1^\rho) \quad \rho \in W^{1,1}(a, b)
\]

\[
(A_2^\rho) \quad \rho(x) > 0 \quad \text{a.a.} \quad x \in [a, b)
\]

\[
(A_3^\rho) \quad \int_a^b \frac{dx}{|\rho(x)|} < \infty,
\]

and let \(T_{0,\rho}\) be the unbounded linear operator
\[
T_{0,\rho} \phi := -(\rho \phi)'
\]

on the domain \(D_{0,\rho} := \{ \phi \in W_\rho : \phi(a) = 0 \}\) (56)

where \(W_\rho := \{ \psi \in X : \rho \psi \in X \text{ and } (\rho \psi)' \in X \}\)

\[
\| \psi \|_{W_\rho} := \| \psi \|_1 + \| \rho \psi \|_1 + \| (\rho \psi)' \|_1.
\]

If \((A_1^\rho), (A_2^\rho)\) and \((A_3^\rho)\) are fulfilled then all the previous results hold for \(\rho\) instead of \(\sigma\). In particular \(W_\rho\) is a Banach space (Lemma 4) and Lemma 7 and Proposition 2 hold which allows us to announce without proof

**Lemma 9.** Assume that \((A_1^\rho), (A_2^\rho)\) and \((A_3^\rho)\) hold.

1. \((0, \infty) \subset \rho(T_{0,\rho})\) and if \(\lambda > 0\) then \((\lambda - T_{0,\rho})^{-1}\) is a bounded linear operator from \(X\) into itself satisfying for all \(\psi \in X\)
\[
\| (\lambda - T_{0,\rho})^{-1} \psi \|_1 \leq \frac{1}{\lambda} \| \psi \|_1.
\] (57)

2. \(T_{0,\rho}\) is closed and densely defined.

3. \((\lambda - T_{0,\rho})^{-1}\) is positive in \(X\).

4. \(T_{0,\rho}\) generates a \(C_0\)-semigroup \((T_{0,\rho}(t))_{t \geq 0}\) of contractions on \(X\) i.e.,
\[
\| T_{0,\rho}(t) \psi \|_1 \leq \| \psi \|_1, \quad t \geq 0,
\] (58)

for all \(\psi \in X\). Furthermore \((T_{0,\rho}(t))_{t \geq 0}\) is positive in \(X\).

Next let \(P_d\) be the linear operator
\[
P_d := \begin{pmatrix} -\delta_1 \mathbb{I} & \delta_2 \mathbb{I} \\ \delta_1 \mathbb{I} & -\delta_2 \mathbb{I} \end{pmatrix}
\] (59)

where \(\delta_1\) and \(\delta_2\) are two real functions subject to the assumptions

\[
(A_1^d) \quad \delta_1 \in L^\infty(a, b) \quad \text{and} \quad \delta_2 \in L^\infty(a, b)
\]

\[
(A_2^d) \quad \delta_1 \text{ and } \delta_2 \text{ are positive.}
\]
Lemma 10. Assume that \((A_\lambda^1)\) holds. Then \(P_\lambda\) is a bounded linear operator from \(\mathcal{X}\) into itself satisfying
\[
\|P_\lambda \phi\| \leq 2 \max\{\delta_1, \delta_2\} \|\phi\| \quad \text{for all } \phi \in \mathcal{X}
\] 
(60)
where
\[
\delta_1 := \text{ess sup}_{x \in (a, b)} |\delta_1(x)| \quad \text{and} \quad \delta_2 := \text{ess sup}_{x \in (a, b)} |\delta_2(x)|.
\]
(61)
Moreover if \((A_\lambda^2)\) holds then
(1). \(P_\lambda\) is a dissipative operator in \(\mathcal{X}\).
(2). \((e^{tP_\lambda})_{t \geq 0}\) is a positive uniformly continuous semigroup in \(\mathcal{X}\).

Proof. The boundedness of \(P_\lambda\) from \(\mathcal{X}\) into itself is easy to check.

(1). Let \(\lambda > 0\) and let \(\phi \in \mathcal{X}\). Easy computation shows that
\[
(\lambda - P_\lambda)^{-1} \phi = \frac{1}{\lambda^2 + \lambda(\delta_1 + \delta_2)} \begin{pmatrix} (\lambda + \delta_2)1 & \delta_21 \\ \delta_11 & (\lambda + \delta_1)1 \end{pmatrix} \phi
\]
\[
= \frac{1}{\lambda^2 + \lambda(\delta_1 + \delta_2)} \begin{pmatrix} (\lambda + \delta_2)\phi + \delta_2\psi \\ \delta_1\phi + (\lambda + \delta_1)\psi \end{pmatrix}
\]
which leads to
\[
\|(\lambda - P_\lambda)^{-1} \phi\| = \left\| \frac{\lambda + \delta_2}{\lambda^2 + \lambda(\delta_1 + \delta_2)} \phi \right\| + \left\| \frac{\delta_1\phi + (\lambda + \delta_1)\psi}{\lambda^2 + \lambda(\delta_1 + \delta_2)} \right\|
\]
\[
\leq \int_a^b \frac{\lambda + \delta_2(|\phi(x)| + |\psi(x)|)}{\lambda^2 + \lambda(\delta_1(x) + \delta_2(x))} dx
\]
\[
+ \int_a^b \frac{\delta_1(x) + (\lambda + \delta_1(x))(|\phi(x)| + |\psi(x)|)}{\lambda^2 + \lambda(\delta_1(x) + \delta_2(x))} dx
\]
\[
= \int_a^b \frac{\lambda + \delta_1(x) + \delta_2(x)(|\phi(x)| + |\psi(x)|)}{\lambda(\lambda + \delta_1(x) + \delta_2(x))} dx
\]
\[
= \frac{1}{\lambda} \int_a^b (|\phi(x)| + |\psi(x)|) dx
\]
and therefore
\[
\|(\lambda - P_\lambda)^{-1} \phi\| \leq \frac{1}{\lambda} \left\| \phi \right\|.
\]
This is clearly equivalent to
\[
\lambda \left\| \frac{\phi}{\psi} \right\| \leq \left\| (\lambda - P_\lambda) \frac{\phi}{\psi} \right\| \quad \text{for all } (\frac{\phi}{\psi}) \in \mathcal{X}
\]
and therefore \(P_\lambda\) is a dissipative operator in \(\mathcal{X}\).

(2). From the point (1), we readily infer that \((e^{tP_\lambda})_{t \geq 0}\) is a uniformly continuous semigroup in \(\mathcal{X}\). Next, (59) yields \(P_\lambda = \hat{P}_\lambda + \tilde{P}_\lambda\) where
\[
P_\lambda := \begin{pmatrix} -\delta_11 & 0 \\ 0 & -\delta_21 \end{pmatrix} \quad \text{and} \quad \hat{P}_\lambda := \left( \begin{array}{cc} 0 & \delta_21 \\ \delta_11 & 0 \end{array} \right)
\]
which are clearly two bounded linear operators from \( \mathcal{P} \) into itself. Since \( \mathcal{P} \) is a diagonal matrix then the uniformly continuous semigroup \( (e^{\mathcal{P} t})_{t \geq 0} \) is positive in \( \mathcal{P} \). The uniformly continuous semigroup \( (e^{\mathcal{P} t})_{t \geq 0} \) is also positive in \( \mathcal{P} \) because of \( (A^2) \). Hence
\[
\left[ e^{\frac{1}{n} \mathcal{P}} e^{\frac{1}{n} \vec{\mathcal{P}}} \right]^n \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{P}^+, \quad \text{for all } \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{P}^+, \quad n = 1, 2, 3, \ldots
\]
Letting \( n \to \infty \) shows that the uniformly continuous semigroup \( (e^{\mathcal{P} t})_{t \geq 0} \) is positive in \( \mathcal{P} \) because of Trotter product formula.

Now we are able to state the second result of this work related to the full model \((PQ_1)-(PQ_4)\). Let then the unbounded linear operator
\[
Z_{K,\sigma,\rho} := \begin{pmatrix} U_{K,\sigma} - \delta_1,1 & \delta_2,1 \\ \delta_1,1 & T_{0,\rho} - \delta_2,1 \end{pmatrix}
\]
on the domain \( D_{K,\sigma} \times D_{0,\rho} \) \hspace{1cm} (62)

**Theorem 3.** Assume that \( (A^1) \), \( (A^2) \), \( (A^3) \), \( (A^4) \), \( (A^5) \), \( (A^6) \) and \( (A^7) \) hold and, let \( K \) be a bounded linear form on \( \mathcal{P} \). If \( (A^1) \) holds then \( Z_{K,\sigma,\rho} \) generates a \( C_0 \)-semigroup \( (Z_{K,\sigma,\rho}(t))_{t \geq 0} \) on \( \mathcal{P} \) satisfying
\[
\left\| Z_{K,\sigma,\rho}(t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| \leq e^{\omega(2\max(\delta_1,\delta_2))t} \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|, \quad \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| \in \mathcal{P}, \quad t \geq 0,
\]
where \( \delta_1 \) and \( \delta_2 \) are given by \((61)\) and
\[
\omega := \left\{ \sigma(a) \|K\| - \mu + \eta, \quad 0 \right\}
\]
with \( \mu \) and \( \eta \) are given by \((50)\). Moreover
\begin{enumerate}
\item if \( (A^3) \) holds then
\[
\left\| Z_{K,\sigma,\rho}(t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| \leq e^{\omega t} \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|, \quad \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| \in \mathcal{P}, \quad t \geq 0.
\]
\item if \( (A^2) \), \( (A^3) \) and \( (A^4) \) hold then
\[
\left\| Z_{K,\sigma,\rho}(t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| \leq e^{\omega (\|K\|) t} \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|, \quad \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| \in \mathcal{P}, \quad t \geq 0.
\]
\item if \( (A^2) \), \( (A^3) \) and \( (A^5) \) hold then
\[
\left\| Z_{K,\sigma,\rho}(t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| \leq e^{\omega (\|K\|) t} \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|, \quad \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| \in \mathcal{P}, \quad t \geq 0.
\]
\item if \( (A^3) \) and \( (A^3) \) hold and, \( K \) is positive then \( (Z_{K,\sigma,\rho}(t))_{t \geq 0} \) is positive.
\end{enumerate}

**Proof.** First, \((62)\) can also be written in the form
\[
Z_{K,\sigma,\rho} = S_{K,\sigma,\rho} + P_{\delta}
\]
on the domain \( D_{K,\sigma} \times D_{0,\rho} \) \hspace{1cm} (68)
where \( P_{\delta} \) is defined by \((59)\) and \( S_{K,\sigma,\rho} \) is the unbounded linear operator
\[
S_{K,\sigma,\rho} := \begin{pmatrix} U_{K,\sigma} & 0 \\ 0 & T_{0,\rho} \end{pmatrix}
\]
on the domain \( D_{K,\sigma} \times D_{0,\rho} \) \hspace{1cm} (69)
According to Theorem 1 and Lemma 9(4) we infer that $S_{K,\sigma,\rho}$ is the generator of the $C_0$–semigroup $(S_{K,\sigma,\rho}(t))_{t \geq 0}$ in $\mathcal{Z}$ defined by

$$S_{K,\sigma,\rho}(t) := \begin{pmatrix} U_{K,\sigma}(t) & 0 \\ 0 & T_{0,\rho}(t) \end{pmatrix}$$ (70)

and satisfying, by virtue of (49) and (58), for all $(\phi \psi) \in \mathcal{Z}$

$$\left\| S_{K,\sigma,\rho}(t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| = \left\| U_{K,\sigma}(t)\phi \right\|_1 + \left\| T_{0,\rho}(t)\psi \right\|_1 \\
\leq e^{\sigma(\alpha)\|K\| t - \frac{\mu + 2\eta}{2}t} \left\| \phi \right\|_1 + \left\| \psi \right\|_1 \\
\leq e^{\omega t} \left( \left\| \phi \right\|_1 + \left\| \psi \right\|_1 \right) \leq e^{\omega t} \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|$$ (71)

Hence (68) is a bounded linear perturbation of the generator $S_{K,\sigma,\rho}$ (whose generated semigroup satisfies (71)) by $P_\delta$ (Lemma 10). Lemma 2(1) yields that $Z_{K,\sigma,\rho}$ generates a $C_0$–semigroup $(Z_{K,\sigma,\rho}(t))_{t \geq 0}$ satisfying (63).

1. Assume that $(A_2^2)$ holds. So $P_\delta$ is moreover dissipative (Lemma 10(1)) and therefore (66) follows from Lemma 2(2).

2. Assume $(A_\mu^2)$ and $(A_{\mu-\eta}^1)$ hold. Both (54) and (58) imply for $(\begin{pmatrix} \phi \\ \psi \end{pmatrix}) \in \mathcal{Z}$

$$\left\| S_{K,\sigma,\rho}(t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| = \left\| U_{K,\sigma}(t)\phi \right\|_1 + \left\| T_{0,\rho}(t)\psi \right\|_1 \\
\leq e^{\sigma(\alpha)\|K\| t} \left\| \phi \right\|_1 + \left\| \psi \right\|_1 \\
\leq e^{\sigma(\alpha)\|K\| t} \left( \left\| \phi \right\|_1 + \left\| \psi \right\|_1 \right) \\
= e^{\sigma(\alpha)\|K\| t} \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|$$ (72)

Hence (68) is a bounded linear perturbation of the generator $S_{K,\sigma,\rho}$ (whose generated semigroup satisfies (72)) by $P_\delta$ (Lemma 10(1)). Thanks to Lemma 2(1) we infer that $Z_{K,\sigma,\rho}$ generates a $C_0$–semigroup $(Z_{K,\sigma,\rho}(t))_{t \geq 0}$ on $\mathcal{Z}$. Finally if $(A_2^2)$ holds then $P_\delta$ is moreover dissipative (Lemma 10(1)) and therefore (66) follows from Lemma 2(2).

3. Since (64) together with $(A_{\mu-\eta}^2)$ lead to $\omega \leq \sigma(\alpha)\|K\|$ then (67) follows from (65).

The growth inequality (67) can also be inferred from (66) because $(A_{\mu-\eta}^2)$ is stronger than $(A_{\mu-\eta}^1)$.

4. Suppose that $(A_2^2)$ and $(A_\eta^2)$ hold and, $K$ is positive. Theorem 1 and Lemma 9(4) show that the $C_0$–semigroup $(S_{K,\sigma,\rho}(t))_{t \geq 0}$ positive in $\mathcal{Z}$. This together with the positivity of the uniformly continuous semigroup $(e^{tP_\delta})_{t \geq 0}$ (Lemma 10(2)) imply that

$$\left[ S_{K,\sigma,\rho}(\frac{t}{n})e^{\frac{n}{n}P_\delta} \right]^n \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{Z}_+, \quad \text{for all} \quad \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{Z}_+, \quad n = 1, 2, 3, \cdots$$
Letting $n \to \infty$ shows that the $C_0$-semigroup $(Z_{K,\sigma}(t))_{t \geq 0}$ positive in $\mathcal{X}$ because of Trotter product formula.

7. Remarks.

Remark 4. An obvious continuation of this work is to prove that the full generated $C_0$-semigroups $(U_{K,\sigma}(t))_{t \geq 0}$ on $\mathcal{X}$ and $(Z_{K,\sigma}(t))_{t \geq 0}$ on $\mathcal{X}$ possess the AEG (Asynchronous Exponential Growth) property (see [11]).

Remark 5. Many inequalities depend on $b < \infty$, for instance (16), (19) and (24). What happens if $b = \infty$? (see [12]).

Remark 6. It is readily seen that one of the most important results of this work is Proposition 1 which makes sense (in the context of $L^1(a,b)$) to the domains of all the unbounded linear operators considered in this work. Considering it in the context of $L^p(a,b)$ ($p > 1$), our first computations show that all the results of this work depend on both terms

$\sigma_1^{\frac{1}{p} - 1}$ and $(p - 1) \inf_{x \in (a,b)} \sigma'(x)

which are respectively equal to 1 and 0 when $p = 1$! Clearly our assumptions $(A_1^\sigma)$, $(A_2^\sigma)$ and $(A_3^\sigma)$ must be improved (see [13]).

Applications. This section deals with some applications in Demography and Biology.

Application 1. The Sharpe-Lotka-McKendrick system is a basic linear model in population theory describing the evolution of an age structured population and, in particular, in mathematical demography with a considerable importance. If $p(a, t)$ denotes the age density of the population at time $t$ then

\[
\begin{align*}
\frac{\partial p}{\partial t} &= - \frac{\partial p}{\partial a} - mp, \\
p(0, t) &= \int_0^{\tau(a)} p(a, t) da, \\
p(a, 0) &= \phi(a),
\end{align*}
\]

(M1)

where $0 < a < \infty$ is the maximum age an individual may reach, $\tau(a)$ is the age specific fertility, $m(a)$ is the age specific mortality and $\phi$ is the initial age distribution.

The Sharpe-Lotka-McKendrick system was first proposed by Sharpe and Lotka in 1911 ([28]) and then by McKendrick in 1926 ([23]). After being neglected for some 30 years, it was rediscovered independently by Scherbaum and Rasch ([27]) and Von Foerster ([30]) as a model for cellular systems.

Nowadays, the Sharpe-Lotka-McKendrick system is a well-known model that has been discussed in many articles (see for instance [20, 21, 26] and references therein). Now, our Theorem 1 can readily be applied as follows

Lemma 11. Let $0 < a < \infty$ and let $\tau, m \in (L^\infty(0,\infty), +)$. The model (M1) is governed by a positive $C_0$-semigroup $(U(t))_{t \geq 0}$ in $L^1(0,\infty)$ satisfying for all $\phi \in L^1(0,a)$

\[\|U(t)\phi\|_1 \leq e^{(\tau - m)t} \|\phi\|_1, \quad t \geq 0,\] (73)

where $\tau := \sup_{a \in (0,\infty)} \tau(a)$ and $m := \inf_{a \in (0,\infty)} m(a)$. 

Proof. We first put $a = 0$ and $b = \bar{a}$ and, we suppose that $\sigma := 1$. Accordingly $(A^1_1)$, $(A^2_1)$ and $(A^3_1)$ are clearly fulfilled. Since $m \in (L^\infty(0, \bar{a}))_+$ then $\mu := m$ fulfills $(A^1_m)$ and $(A^2_m)$. $(A^1_n)$ and $(A^2_n)$ are also fulfilled for $\eta := 0$.

Next, let $K$ be such that

$$K \phi := \int_0^\pi \tau(a) \phi(a) da.$$ 

Since $\tau \in (L^\infty(0, \bar{a}))_+$ then $K$ is clearly a positive bounded linear form on $L^1(0, \bar{a})$ and $\|K\| = \bar{\tau}$. Finally (47)–(with the previous notations) yields that the model (M1) is governed by the unbounded linear operator

$$U_{K,1} := T_{K,1} - mL + 0 \quad \text{on the domain} \quad D_{K,1}.$$ 

All the required conditions of Theorem 1 are fulfilled and therefore $U_{K,1}$ generates a positive $C_0$–semigroup $(U(t))_{t \geq 0} := (U_{K,1}(t))_{t \geq 0}$ in $L^1(0, \bar{a})$. Moreover (73) follows from (49)–(with $\sigma(a) = 1$, $\|K\| = \bar{\tau}$, $\mu = m$ and $\eta = 0$).

**Application 2.** The Sinko-Streifer Model was proposed by Sinko and Streifer in 1967 ([29]). Their formulation and its generalizations have subsequently been used widely (and successfully) in biological modeling (see [5, 6, 24] and references therein). Our Theorem 1 can readily be applied as follows

**Lemma 12.** Let $k, m \in (L^\infty(0, 1))_+$ and let $g \in W^{1,\infty}(0, 1)$ be such that

$$g(s) \geq \gamma > 0 \quad \text{for all} \quad s \in [0, 1]$$

for some $\gamma > 0$. The model (M2) is governed by a positive $C_0$–semigroup $(U(t))_{t \geq 0}$ in $L^1(0, 1)$ satisfying for all $\phi \in L^1(0, 1)$

$$\|U(t)\phi\|_1 \leq e^{(\pi - m)t} \|\phi\|_1 \quad t \geq 0,$$

where $\pi := \text{ess sup}_{s \in (0, 1)} k(s)$ and $\underline{m} := \text{ess inf}_{s \in (0, 1)} m(s)$.

**Proof.** We first put $a = 0$ and $b = 1$ and, we suppose that $\sigma := g$. From $g \in W^{1,\infty}(0, 1) \subset W^{1,1}(0, 1)$ we infer that $(A^1_1)$ is fulfilled. Furthermore Remark 2 together with (74) yield that $(A^2_1)$ and $(A^3_1)$ are also fulfilled. Since $m \in (L^\infty(0, 1))_+$ then $\mu := m$ fulfills $(A^1_m)$ and $(A^2_m)$. Both $(A^1_n)$ and $(A^2_n)$ are also fulfilled for $\eta := 0$. 

...
Next, let K be such that
\[ K\phi := \frac{1}{g(0)} \int_0^1 k(a)\phi(s)ds. \]
Since \( k \in (L^\infty(0,1))_+ \) then K is clearly a positive bounded linear form on \( L^1(0,1) \) and \( \|K\| = \frac{\mu}{g(0)} \). Finally (47)–(with the previous notations) yields that the model (M2) is governed by the unbounded linear operator
\[ U_{K,g} := T_{K,g} - m\mathbb{1} + 0 \quad \text{on the domain} \quad D_{K,g}. \]
All the conditions of Theorem 1 are fulfilled and therefore \( U_{K,g} \) governs an exponentially\( C_0 \)–semigroup \( (\mathcal{U}(t))_{t \geq 0} := (U_{K,g}(t))_{t \geq 0} \) in \( L^1(0,1) \). Moreover (75) follows from (49)–(with \( \sigma(a) = g(0), \|K\| = \frac{\mu}{g(0)}, \mu = m \) and \( \bar{\eta} = 0 \)).

**Application 3.** Here we analyze a linear model of cell population dynamics structured by age with two phases. If \( p(t,a) \) and \( q(t,a) \) denote the densities of the cells in the proliferating and the quiescent phase respectively then the balance equations between both phases can be written as

\[
\begin{aligned}
\frac{\partial p}{\partial t} &= -\frac{\partial p}{\partial a} - mp - \tau_2 p + \tau_2 q, \quad 0 < a < \bar{\sigma}, \quad t > 0, \\
\frac{\partial q}{\partial t} &= -\frac{\partial q}{\partial a} + \tau_1 p - \tau_2 q, \quad 0 < a < \bar{\sigma}, \quad t > 0, \\
p(t,0) &= 2 \int_0^{\bar{\sigma}} k(a)p(t,a)da, \quad t > 0, \\
q(t,0) &= 0, \quad t > 0, \\
p(0,a) &= \phi(a), \quad 0 < a < \bar{\sigma}, \\
q(0,a) &= \psi(a), \quad 0 < a < \bar{\sigma}
\end{aligned}
\]

(M3)

where \( 0 < \bar{\sigma} < \infty \) is the maximum age a cell may reach, \( m = m(a) \) and \( k = k(a) \) are, respectively, the mortality and division rates in the proliferating phase and, \( \tau_1 \) and \( \tau_2 \) are the transition rates between both phases.

To our knowledge, the model (M3) was proposed and studied in [1]. Now, our Theorem 3 can readily be applied as follows

**Lemma 13.** Let \( 0 < \bar{\sigma} < \infty \) and let \( k, m, \tau_1, \tau_2 \in (L^\infty(0,\bar{\sigma}))_+ \). The model (M3) is governed by a positive \( C_0 \)–semigroup \( (Z(t))_{t \geq 0} \) in \( (L^1(0,\bar{\sigma}))^2 \) satisfying

\[
\begin{aligned}
\left\| Z(t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\| &\leq e^{\max\{2\bar{\sigma} - m, 0\} t} \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|, \quad \left( \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right) \in (L^1(0,\bar{\sigma}))^2, \quad t \geq 0,
\end{aligned}
\]

(76)

where

\( \bar{\sigma} := \text{ess sup}_{a \in (0,\bar{\sigma})} k(a) \quad \text{and} \quad m := \text{ess inf}_{a \in (0,\bar{\sigma})} m(a). \)

**Proof.** We first put \( a = 0 \) and \( b = \bar{\sigma} \) and, we suppose that \( \sigma := 1 \) and \( \rho := 1 \). Accordingly \( (A_1^1) \), \( (A_2^1) \), \( (A_3^1) \), \( (A_1^2) \), \( (A_2^2) \) and \( (A_3^2) \) are clearly fulfilled. Since \( m \in (L^\infty(0,\bar{\sigma}))_+ \) then \( \mu := m \) fulfills \( (A_1^1) \) and \( (A_2^1) \). Both \( (A_1^2) \) and \( (A_2^2) \) are also fulfilled for \( \eta := 0 \). Finally let \( \delta_1 := \tau_1 \) and \( \delta_2 := \tau_2 \). Since \( \tau_1, \tau_2 \in (L^\infty(0,\bar{\sigma}))_+ \) then \( (A_1^1) \) and \( (A_2^1) \) are fulfilled.
Next let $K$ be such that
\[ K\psi := 2 \int_0^\pi k(a)\psi(a)da. \]

Since $k \in (L^\infty(0, \pi))_+$ then $K$ is clearly a positive bounded linear form on $L^1(0, \pi)$ and $\|K\| = 2\pi$. Finally (62)–(with the previous notations) yields that the model (M3) is governed by the unbounded linear operator
\[ Z_{K,1,1} := \begin{pmatrix} U_{K,1} - \tau_1 I & \tau_2 I \\ \tau_1 I & T_{0,1} - \tau_2 I \end{pmatrix} \]
on the domain $D_{K,1} \times D_{0,1}$

All the required conditions of Theorem 3 are fulfilled. Hence $Z_{K,1,1}$ generates a positive $C_0$–semigroup $(Z(t))_{t \geq 0} := (Z_{K,1,1}(t))_{t \geq 0}$ in $(L^1(0, \pi))^2$. Moreover (76) follows from (65) together with (64)–(with $\sigma(a) = 1$, $\|K\| = 2\pi$, $\mu = \frac{m}{\eta}$ and $\eta = 0$).

**Application 4**. Here we analyze a linear model of cell population dynamics structured by size with two phases. If $u(t, s)$ and $v(t, s)$ denote the density of cells in the proliferating and the quiescent phase respectively then the balance equations between both phases can be written as
\[
(M4) \\begin{cases}
\frac{\partial u}{\partial t} = -\frac{\partial (\gamma_1 u)}{\partial s} - mu + \int_0^\pi r(s, y)u(t, y)dy - \tau_1 u + \tau_2 v, & 0 < s < \pi, \ t > 0, \\
\frac{\partial v}{\partial t} = -\frac{\partial (\gamma_2 v)}{\partial s} + \tau_1 u - \tau_2 v, & 0 < s < \pi, \ t > 0, \\
\gamma_1(0)u(t, 0) = 0, & t > 0, \\
\gamma_2(0)v(t, 0) = 0, & t > 0, \\
u(0, s) = \phi(s), & 0 < s < \pi, \\
v(0, s) = \psi(s), & 0 < s < \pi
\end{cases}
\]

where $0 < \pi < \infty$ is the maximum size a cell may reach, $m = m(s)$ and $r = r(s)$ are, respectively, the mortality and the division rates in the proliferating phase, $\gamma_1$ and $\gamma_2$ are the growth rates in both phases and, $\tau_1$ and $\tau_2$ are the transition rates between both phases.

The model (M4) is already studied (see [19, 25] and references therein). Now, our Theorem 3 can readily be applied as follows

**Lemma 14.** Let $0 < \pi < \infty$ and, let $m, \tau_1, \tau_2 \in (L^\infty(0, \pi))_+$ and let $\gamma_1, \gamma_2 \in W^{1,\infty}(0, \pi)$ be such that
\[
\gamma_1(s) \geq \gamma > 0 \quad \text{and} \quad \gamma_2(s) \geq \gamma > 0 \quad \text{for all} \quad s \in [0, \pi]
\]
for some $\gamma > 0$. Let $r \geq 0$ be such that
\[
\pi := \text{ess sup}_{y \in (0, \pi)} \int_0^\pi r(x, y)dx < \infty.
\]
Then the model (M4) is governed by a positive $C_0$-semigroup $(Z(t))_{t \geq 0}$ in $L^1(0, \bar{s})^2$ satisfying
\[
\left\| Z(t) \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\| \leq e^{t \max(-m + \tau, 0)} \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|, \quad \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \in (L^1(0, \bar{s}))^2, \quad t \geq 0,
\] (78)
where
\[m := \operatorname{ess} \inf_{s \in (0, \bar{s})} m(s) < \infty.\]

Moreover, if one of the following assertions holds
\[
\begin{align*}
(1) & \quad \int_0^\tau r(s, y) ds \leq m(y) \quad \text{a.a.} \quad y \in (0, \bar{s}), \\
(2) & \quad \tau \leq m,
\end{align*}
\]
then
\[
\left\| Z(t) \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\| \leq \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\| \quad t \geq 0.
\] (79)

**Proof.** We first put $a = 0$ and $b = \bar{s}$ and, we suppose that $\sigma := \gamma_1$ and $\rho := \gamma_2$. Since $\gamma_1, \gamma_2 \in W^{1, \infty}(0, \bar{s}) \subset W^{1,1}(0, \bar{s})$ we infer that $(A^1_1), (A^2_1), (A^1_2), (A^2_2)$ and $(A^3_1)$ are clearly fulfilled. Since $m \in (L^\infty(0, \bar{s}))_+$ then $\mu := m$ fulfills $(A^1_1)$ and $(A^3_1)$. Both $(A^1_2)$ and $(A^2_2)$ are also fulfilled for $\eta = \tau$. Finally let $\delta_1 := \tau_1$ and $\delta_2 := \tau_2$. Since $\tau_1, \tau_2 \in (L^\infty(0, \bar{s}))_+$ then both $(A^1_2)$ and $(A^2_2)$ are fulfilled.

Finally (62)–(with the previous notations) yields that the model (M4) is governed by the unbounded linear operator
\[
Z_{0, \gamma_1, \gamma_2} := \begin{pmatrix} U_{0, \gamma_1} - \tau_1 & \tau_2 \\ \tau_1 & T_{0, \gamma_2} - \tau_2 \end{pmatrix}
\]
on the domain $D_{0, \gamma_1} \times D_{0, \gamma_2}$.

All the conditions of Theorem 3 are fulfilled. Hence $Z_{0, \gamma_1, \gamma_2}$ generates a positive $C_0$-semigroup $(Z(t))_{t \geq 0} := (Z_{0, \gamma_1, \gamma_2}(t))_{t \geq 0}$ in $(L^1(0, \bar{s}))^2$. Furthermore (78) follows from (65) together with (64)–(with $\sigma(a) = \gamma_1(0)$, $\|K\| = 0$, $\mu = m$ and $\eta = \tau$).

Finally if (1) holds then $(A^1_{\mu-\eta})$–(with $\eta = \tau$ and $\mu = m$) holds and therefore (79) follows from (66)–(with $K = 0$).

However if (2) holds then (79) follows from (78). \qed

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**REFERENCES**

[1] O. Arino, E. Sánchez and G. F. Webb, Necessary and sufficient conditions for asynchronous exponential growth in age structured cell populations with quiescence, J. Math. Anal. Appl., 215 (1997), 499–513.

[2] J. Banasiak, W. Lamb and P. Laurenço, Analytic Methods for Coagulation-Fragmentation Models. I, CRC Press, Boca Raton, FL, 2020.

[3] J. Banasiak, W. Lamb and P. Laurenço, Analytic Methods for Coagulation-Fragmentation Models. II, CRC Press, Boca Raton, FL, 2020.

[4] J. Banasiak, K. Pichóir and R. Rudnicki, Asynchronous exponential growth of a general structured population model, Acta. Appl. Math., 119 (2012), 149–166.

[5] H. T. Banks, F. Kappel and C. Wang, Weak solutions and differentiability for size structure population models, Estimation and Control of Distributed Parameter Systems, Internat. Ser. Numer. Math., Birkhäuser, Basel, 100 (1991), 35–50.

[6] H. T. Banks and H. T. Tran, Mathematical and Experimental Modeling of Physical and Biological Processes, CRC Press, Boca Raton, FL, 2009.
[7] V. Barbu, M. Iannelli and M. Martcheva, On the controllability of the Lotka-McKendrick model of population dynamics, Journ. Math. Ana. Appl., 253 (2001), 142–165.
[8] M. Boulanouar, On a mathematical model of age-cycle length structured cell population with non-compact boundary conditions, Math. Meth. Appl. Sci., 38 (2015), 2081–2104.
[9] M. Boulanouar, A mathematical study in the theory of dynamic population, Journ. Math. Anal. Appl., 255 (2001), 230–259.
[10] M. Boulanouar, A transport equation in cell population dynamics, Diff. Int. Equa., 13 (2000), 125–144.
[11] M. Boulanouar, Mathematical analysis of an abstract model and its applications to structured populations. II, In preparation.
[12] M. Boulanouar, Mathematical analysis of an abstract model and its applications to structured populations. III, In preparation.
[13] M. Boulanouar, Mathematical analysis of an abstract model and its applications to structured populations. IV, In preparation.
[14] M. Boulanouar and L. Leboucher, A transport equation in cell population dynamics, C. R. Acad. Sci. Paris Sér. I Math., 321 (1995), 305–308.
[15] H. Brézis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011.
[16] Ph. Clément and al, One-Parameter Semigroups, North-Holland, Amsterdam, New York, 1987.
[17] R. Dilão and A. Lakmeche, On the weak solutions of the McKendrick equation: Existence of demography cycles, Math. Model. Nat. Phenom., 1 (2006), 1–32.
[18] K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, 194, Springer-Verlag, New York, 2000.
[19] J. Z. Farkas and P. Hinow, On a size-structured two-phase population model with infinite states-at-birth, Positivity, 14 (2010), 501–514.
[20] M. Iannelli, Mathematical Theory of age-structured population dynamics, Giardini Editory e Stampatori, Pisa, 1995.
[21] M. Iannelli and F. Milner, On the approximation of Lotka McKendrick equation with finite life span, Jour. Comput. Appl. Math., 136 (2001), 245–254.
[22] H. Inaba, Age-Structured Population Dynamics in Demography and Epidemiology, Springer, Singapore, 2017.
[23] A. G. McKendrick, Applications of mathematics to medical problems, Proc. Edinburgh Math. Soc., 44 (1926), 98–130.
[24] J. A. J. Metz and O. Diekmann, The dynamics of physiologically structured populations, Lecture Notes in Biomathematics, 68 (1986).
[25] M. Mokhtar-Kharroubi and Q. Richard, Spectral theory and time asymptotics of size-structured two-phase population models, Discrete Contin. Dyn. Syst. Serie B, 25 (2020), 2969–3004.
[26] G. Pelovska and M. Iannelli, Numerical methods for the Lotka McKendrick’s equation, Journ. Comp. Appl. Math., 197 (2006), 534–557.
[27] O. Scherbaum and G. Rasch, Cell size distribution and single cell growth in Tetrahymena pyriformis, GL. Arch. Pathol. Microbiol. Scand., 41 (1957), 161–182.
[28] F. R. Sharpe and A. J. Lotka, A problem in age distribution, Phil. Mag., 21 (1911), 435–438.
[29] J. W. Sinko and W. Streifer, A new model for age-size structure for a population, Ecology, 48 (1967), 910–918.
[30] H. Von Foerster, Some remarks on changing populations, The Kinetics of Cellular Proliferation (Grune and Stratton, NY), (1959), 382–407.
[31] G. F. Webb, Dynamics of structured populations with inherited properties, Comput. Math. Appl., 13 (1987), 749–757.

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