Gauge field theory without groups

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Abstract

Non-standard topics underlying a partly original approach to gauge field theory are concisely introduced, expressing ideas that were broached in several papers and, eventually, exposed in an organized form in a recently published book [1]. By proposing a change of perspective about the roles and relative importance of several notions, this approach seeks to obtain an overall clarification of foundational matters. In particular, by consistently relying on natural differential geometry, the role of groups is shown to be downgradable to secondary.

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1 Introduction

It is a fairly common opinion among mathematicians, also shared by some theoretical physicists, that the mathematical foundations of particle physics and quantum field theory are somewhat fuzzy and unclear. Moreover—partly as a consequence of the present scarcity of sound experimental evidence—one has to deal with a plethora of theories and approaches that is certainly contrary to Occam’s ‘razor principle’.

In recent years, I attempted to clear up the field by revisiting several basic notions [4–24]. Eventually, that work generated a fairly consistent view, which was exposed in a new book [1]. In this paper I sketch a few essential ideas, sticking—as in the book—to concepts that are unquestionably relevant to the foundations of present-day quantum physics. I am not really interested in topics that, up to now, seem to be only relevant to pure speculation. Actually, I am convinced that choosing fewer notions and focusing on them, rather than ever adding to them, is the sole effective route to clarity, and may even allow us to move beyond the present impasse in fundamental physics.

The main non-standard ideas considered, and briefly discussed in this presentation, are all related, in one way or another, to a preference for using a natural geometric language as far as possible. Two-spinor algebra and two-spinor bundle geometry provide an integrated, ‘minimal geometric data’ approach to Einstein-Cartan-Maxwell-Dirac fields and, more generally, to arbitrary-spin matter fields coupled with gauge fields and tetrad-affine gravity. Electroweak geometry and the ensuing field theory can be described in a completely group-free approach, too.

The ‘configuration bundle’ of gauge field theory, whose sections are both matter and gauge fields, can be introduced by intrinsic geometric constructions without any reference to structure groups and matrix formalisms. In this context, a ‘covariant differential’ formulation of gauge field theory conveniently replaces the usual jet bundle formulation, and naturally extends to tetrad-affine gravity; a first-order Lagrangian theory of fields with arbitrary spin is also exhibited. Lie derivative of all fields, including the tetrad and the spinor connection, are introduced, and lead to the notions of deformed field theory and energy tensors.

All the above concepts can be seamlessly extended to quantum bundles and quantum fields, which are introduced by precise constructions based on Frölicher’s notion of smoothness. Various aspects of QFT are exposed in a language that may appease a mathematically oriented reader who is not satisfied with usual presentations of this matter (in that case, I’d advise her to get the book).

Finally, I propose an approach to quantum particle physics which is based on quantum geometry but, actually, bypasses the use of quantum fields.

One salient characteristic of the proposed approach is its consistent reliance on natural, intrinsic geometric language, with hardly any reference to structure groups. Although these are still present in the background—as groups of automorphisms of the considered geometric structures— I maintain that their role can be quite conveniently regarded as auxiliary rather than primary. The fact that this is actually possible without losing anything essential will come as a surprise to many theoretical physicists, and deserves a thorough discussion.

1 ‘All too often in physics familiarity is a substitute for understanding’, Y. Choquet-Bruhat and C. DeWitt-Morette [2].
2 See e.g. R. Penrose [3].
2 Natural geometry and the role of groups

For historical reasons, the standard approach to gauge field theory exploits the notion of a fixed structure group via principal bundles and vector bundles associated with them. Though the abstract notion of a vector space—possibly endowed with some further algebraic structure—is not recent, many still treat vectors as elements of $\mathbb{R}^n$ (or $\mathbb{C}^n$) that ‘transform’ according to a certain law. Indeed, when a basis is chosen one gets such a representation, and a vector’s components in different bases are related by a transformation that is an element of a group of matrices. More precisely, the set $\mathcal{B}_V$ of all bases of a vector space, $V$, turns out to have the structure of a group-affine space, any two elements being related by an element of a group of matrices which is isomorphic to the group $\text{Aut} V \subset \text{End} V$ of all automorphisms of $V$ (constituted by all invertible endomorphisms). However, there is no distinguished isomorphism between these groups; rather, an arbitrary group isomorphism is singled out by the choice of a basis.

Usually one deals with vector spaces that are endowed with further algebraic structure (for example, a scalar product). This selects the subgroup $G \subset \text{Aut} V$ of all automorphisms preserving that structure, and the subset of $\mathcal{B}_V$ constituted by all ‘special’ bases (for example, orthonormal bases), whence one may recover the usual matrix groups. The actual use of matrices, however, is basically about calculations, while natural geometric notions correspond to a higher level of abstraction and are best suited to understanding the fundamental concepts. Accordingly, I feel justified in seeing the role of matrices and matrix groups as secondary.

Some further clarification will be opportune in the discussion of the meaning of ‘intrinsic’. Generally speaking, a differential geometric notion is called intrinsic if it corresponds to a well-defined geometric object that can be directly characterized in terms of natural operations allowed by the underlying geometric structure. Thus ‘intrinsic’ may be seen as roughly equivalent to the basic acceptation of ‘covariant’ in a broad sense. In the mathematics literature, several intrinsic objects and operations are often denoted in coordinate-free form, through symbols that give up indices and matrices. This economy of language is not just convenient shorthand: it corresponds to true mathematical abstraction.

Now we should distinguish between algebraic notions, related to the fibre structure of the involved bundles, and actual differential geometric notions arising from the basic concepts of (fibred) manifold and jet space. The former, including all notions derived from tensor products and scalar products of any type, can be viewed as truly intrinsic from the start: indeed, they can be introduced, and their fundamental properties established, while making no use of bases and matrices. The concept of manifold, however, originates from that of cocycle (a family of compatible charts covering an assigned topological space). Though charts need not be valued into $\mathbb{R}^n$ (an affine space is sufficient), natural notions in this context are exactly defined by the requirement that their chart expressions are ‘invariant’ with respect to transformations between compatible charts. However, once the fundamental such notions have been introduced and their main properties have been established by checking invariance, such checking need not be performed at successive stages of development: definitions and demonstrations only use the natural properties; if these are correctly implemented, then invariance is automatic and it is no longer necessary to prove it.

Thus in a general field theory the term ‘natural language’ actually refers to a superposition of two different concepts. In the standard physics literature, group invariance is mainly tied to the algebraic structure of bundle fibres in gauge field theory, while it is usually less relevant.

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3The straightforward formal definition is analogous to the usual definition of affine space modelled on a vector space.
in expositions of General Relativity. I maintain that, in both senses, the economy of language and deeper understanding provided by intrinsic language—by comparison with a language based on matrices and transformation rules—is well worth the effort. A theoretical physicist might regard the familiar group approach as more succinct, since it enables introducing a theory by specifying a group. Such evaluation, however, depends on the assumed background; if the whole machinery of principal bundles is needed, that constitutes a rather substantial assumption! Furthermore, this fixed general recipe may eventually make ‘thinking differently’ harder.

That said, it is worthwhile to point out that thoroughly giving up coordinate expressions, although possible in principle, would impose a notation too awkward to be useful in practice. In particular, one needs to deal with tensor products with several factors and many possible different contractions, symmetrizations, and antisymmetrizations. Now, the index formalism was introduced exactly for dealing with such issues; it closely reflects the intricacies arising in tensor algebra and its extensions. If a coordinate expression is intrinsically well constructed as a representation of natural operations, then checking its ‘covariance’ with respect to the appropriate group of transformations is an inessential exercise. The best route is then a middle one, in which coordinate-free expressions are reserved for the most important objects and operations, while coordinates and indices retain their role as efficient means of performing computations and demonstrations. Another way to see this is the following: tensor products represent multilinear maps; each factor can be seen as a ‘slot’ that functions either as input or output, and for efficient calculations the slots must be labelled. The index formalism provides a convenient labelling system. Thus an indexed expression need not be regarded as representing a matrix of components.

Also note that, in most cases, one need not be actually involved with the global topological aspects of the bundles under consideration; just assuming the topology allows all the needed geometric constructions is enough. Namely, one deals with local, intrinsic differential geometry; the two italicized adjectives are fully compatible, although ‘local’ is often somewhat misleadingly used in the sense of ‘coordinate-dependent’.

### 3 Configuration bundle of classical gauge field theory

In order to describe gauge fields in intrinsic terms, let us start from the observation that the vector space \( \text{End} V \cong V \otimes V^* \) of all endomorphisms of a vector space, \( V \), has a natural structure of Lie algebra determined by the ordinary commutator, and any specialized algebraic structure yields the Lie subalgebra \( \mathfrak{L} \subset \text{End} V \) of its ‘infinitesimal symmetries’. If the algebraic structure is a real or Hermitian scalar product, then \( \mathfrak{L} \) is constituted, respectively, by all antisymmetric endomorphisms and all anti-Hermitian endomorphisms (with respect to the scalar product itself). As for spinors, and everything that is associated with them, the only fundamental algebraic structure that is actually needed is that of a two-dimensional complex vector space (two-spinor space, see §3). In all cases, any references to groups can be thoroughly dropped.

In field theories the above notions can be exploited in terms of vector bundles smoothly endowed with some fibre structure. Locally, one may recover the traditional principal bundle

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4 A computer code effectively treating tensor algebra in a fully intrinsic form is certainly feasible, but its output would be hard for us to interpret. This type of difficulty, however, also applies to the internal form of even simple expressions in a symbolic language like Wolfram’s Mathematica.

5 In Ch. 2 (Vol. I) of their well-known monograph, *Spinors and space-time* [26], Penrose and Rindler discuss an ‘Abstract index notation’ aimed at resolving this duplicity of the index formalism; subsequently that notation is not extensively used, however. In the appendix of the same book they also discuss a diagrammatic approach to tensor algebra, noting that, in practice, its actual use is limited.
approach in terms of the bundle of frames that are special with respect to the considered fibre structure (a group-affine bundle). Lie-algebra bundles, on the other hand, have a definitely relevant role in gauge field theory—though the matricial formalism can be still regarded as marginal.

In a typical gauge field theory a matter field can be described as a section, \( \phi : M \rightarrow E \), of a vector bundle (usually, the base manifold \( M \) is the spacetime manifold), while a gauge field is a special connection. Thus a gauge field cannot be described as a section of some vector bundle. Indeed, an arbitrary connection is a section \( E \rightarrow J^1E \) of the first-jet prolongation bundle, so it cannot even be described as a section of any finite-dimensional bundle over \( M \). However, we may choose to select those connections which preserve the algebraic structure of a vector bundle, called linear connections\(^6\). These can indeed be regarded as sections of a finite-dimensional bundle, namely the affine sub-bundle \( L^\ast CE \subset J^1E \otimes_M E^\ast \) over \( M \) which projects onto the identity section \( \mathbb{I}_E : M \rightarrow E \otimes_M E^\ast \); its associated vector bundle is

\[
T^\ast M \otimes_{M} \text{End} E \equiv T^\ast M \otimes_{M} E \otimes_{M} E^\ast \rightarrow M .
\]

Any further algebraic fibre structure of \( E \rightarrow M \) beyond linearity selects a Lie-algebra sub-bundle, \( \mathcal{L} \subset \text{End} E \); connections that make that algebraic structure covariantly constant can be characterized as sections of an affine sub-bundle, \( K \subset L^\ast E \), whose associated vector bundle is \( T^\ast M \otimes_{M} \mathcal{L} \rightarrow M \); gauge fields are exactly sections \( \kappa : M \rightarrow K \). Groups have no role in these definitions.

Differential operations related to gauge fields can be expressed in terms of the Frölicher–Nijenhuis bracket of vector-valued forms. If \( \zeta : M \rightarrow \wedge^r \otimes_M TM \) and \( \xi : M \rightarrow \wedge^s \otimes_M TM \) are tangent-valued forms on a generic manifold, \( M \), then their Frölicher–Nijenhuis bracket is a tangent-valued form

\[
\llbracket \zeta, \xi \rrbracket : M \rightarrow \wedge^{r+s} \otimes_M TM .
\]

After replacing the generic manifold \( M \) with the total manifold of a fibred manifold, \( E \), one considers special cases such as vertical-valued forms \( E \rightarrow \wedge^r T^\ast E \otimes_E VE \) and horizontal forms \( E \rightarrow \wedge^s T^\ast E \otimes_E TE \). Because of the natural inclusion \( J^1E \subset T^\ast M \otimes_E TE \), a connection can be included in the latter. Moreover, if \( E \rightarrow M \) is a vector bundle, then we have the natural isomorphism \( VE \cong E \times_M E \), so that a matter field can be viewed as a vertical valued zero-form. Hence we get the covariant differentials

\[
d[\kappa] \phi \equiv [\kappa, \phi] : M \rightarrow T^\ast M \otimes E , \quad d[\kappa] \kappa \equiv [\kappa, \kappa] : E \rightarrow \wedge^2 T^\ast M \otimes_M E ,
\]

which can be regarded, via obvious identifications, as the covariant derivative \( \nabla[\kappa] \phi \) of the matter field and minus the curvature tensor of the gauge field. Furthermore, the latter is also linear (because of the linearity of the connection \( \kappa \)), so that it can be regarded as a section

\[
\rho \equiv -d[\kappa] \kappa : M \rightarrow \wedge^2 T^\ast M \otimes_M \mathcal{L} \subset \wedge^2 T^\ast M \otimes_M \text{End} E .
\]

Therefore, the ‘configuration bundle’ of a generic gauge field theory is \( E \times_M K \rightarrow M \), a

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\(^6\)The ‘affine’ label often attached to connections is related to the fact that, even when \( E \rightarrow M \) is a generic fibred manifold with no algebraic fibre structure, the bundle \( J^1E \rightarrow E \) turns out to be naturally affine. This fact should not generate confusion with labels used to identify connections selected by specific algebraic structures.

\(^7\)If \( E \rightarrow M \) is a vector bundle, then \( J^1E \rightarrow M \) is a vector bundle, too.

\(^8\)Characterized, via linearity, by the rule

\[
[\lambda \otimes u, \mu \otimes v] = \lambda \wedge \mu \otimes [u, v] + \lambda \wedge (L[u] \mu) \otimes v - (L[v] \lambda) \wedge \mu \otimes u + (-1)^r [v] \lambda \wedge \mu \otimes u + (-1)^r d \lambda \wedge (u[\mu]) \otimes v ,
\]

where \( \lambda : M \rightarrow \wedge^r T^\ast M \), \( \mu : M \rightarrow \wedge^s T^\ast M \), \( u, v : M \rightarrow TM \), and \([u, v]\) is the Lie bracket of \( u \) and \( v \).
field is a couple \((\phi, \kappa) : M \to E \times_M K\), and its covariant prolongation is

\[
d(\phi, \kappa) \equiv (\phi, \kappa, d[\kappa]\phi, d[\kappa]\kappa) : M \to d\left( E \times K \right) \equiv E \times K \times (T^*M \otimes E) \times (\wedge^2 T^*M \otimes \Omega^1)
\]

Of course, the above basic setting will have to be expanded and adapted to various special cases. The principal bundle formalism, though familiar to many, is comparatively fairly intricate. One may argue about the convenience of abandoning it. The short answer to this is that the effort of taking a step further in abstraction does pay off, by providing an optimized language that helps to prevent confusion or diversion by spurious ideas. On a related note, let me remark that in the literature one finds considerable confusion between the notion of observer and the notions of frames and coordinates, with the adjective ‘covariant’ often being used in the sense of ‘independent of the observer’.

The notion of ‘gauge fixing’ has various aspects, which I am not going to discuss here. Just to touch one important point: the quantum version of a classical field requires that the field be a section of a vector bundle, while a gauge field is a section of an affine bundle. A way out of this difficulty consists of fixing a gauge, namely a local ‘background’ curvature-free connection, \(\kappa_0\), to use as a reference: now any gauge field \(\kappa\) can be represented as the difference \(\kappa - \kappa_0\), a tensor field. Usually \(\kappa_0\) is seen as associated with the choice of a local frame, though there is no obligation with regard to this.

## 4 Covariant differential and Lagrangian field theory

The appropriate geometric language for a general Lagrangian field theory of arbitrary order is that of jet bundles. An actual gauge field theory related to particle physics, however, is more specific: its Lagrangian is of the first order, and depends on the fields’ derivatives only through their covariant differentials (Utiyama principle). Furthermore, the Lagrangian depends on the fields and their differentials polynomially (no arbitrary functional dependence). Hence, the derivatives of the Lagrangian with respect to the fields and their differentials are actually simple algebraic operations; this fact is essential for a seamless extension to the quantized theory (§).

Thus the Lagrangian density can be expressed as a morphism

\[
\Lambda : d\left( E \times K \right)_M \to \wedge^m T^*M , \quad m \equiv \dim M ,
\]

so that

\[
\Lambda[\phi, \kappa] \equiv \Lambda \circ d(\phi, \kappa) : M \to \wedge^m T^*M
\]

is an ordinary density on \(M\). By analogy with the momentum form associated with the Lagrangian density in the usual formulation, one introduces the sections

\[
\Pi^{(0)} \equiv \frac{\partial \Lambda[\phi, \kappa]}{\partial \phi} : M \to \wedge^m T^*M \otimes E^* ,
\]

\[
\Pi^{(1)} \equiv \frac{\partial \Lambda[\phi, \kappa]}{\partial (d[\kappa]\phi)} : M \to \wedge^{m-1} T^*M \otimes E^* ,
\]

\[
\Pi^{(2)} \equiv \frac{\partial \Lambda[\phi, \kappa]}{\partial (d[\kappa]\kappa)} : M \to \wedge^{m-2} T^*M \otimes \Omega^* ,
\]
where the derivatives are, in practice, algebraic operations, as observed above.

The reader will note how \( \Pi^{(0)} \), \( \Pi^{(1)} \), and \( \Pi^{(2)} \) can be regarded as vector-valued forms. Accordingly, it can be proved that the Euler–Lagrange equations for the couple \((\phi, \kappa)\) can be cast in the form

\[
\begin{align*}
\Pi^{(0)} - d[\kappa]\Pi^{(1)} &= 0, \\
l^*(\Pi^{(1)} \otimes \phi) - d[\kappa]\Pi^{(2)} &= 0,
\end{align*}
\]

where \( l : L \hookrightarrow \text{End}E \cong E \otimes E^\ast \) denotes the natural inclusion, and \( l^* : \text{End}E^\ast \cong E^\ast \otimes E \rightarrow L^\ast \) is its transpose morphism.

The above scheme turns out to be naturally extendable to more intricate situations in which one has several matter and gauge sector, variously interacting. Furthermore, tetrad-affine gravity can also be treated in this way (§5).

A further remark: in many cases, a matter field is actually a couple of fields \((\phi, \phi^\ast) : M \rightarrow E \times E^\ast\).

When a Hermitian fibre metric is assigned, the field equation often admit solutions in which \( \phi \) and \( \phi^\ast \) are mutually conjugate transpose fields. Nevertheless, they can (and, for clarity, should) always be viewed as mutually independent.

5 Two-spinor geometry

The ‘minimal geometric data’ approach broached in this section essentially consists of a complex bundle, \( S \rightarrow M \), with two-dimensional fibres, without any further assumptions. I will first sketch the fibre-algebraic aspect of this approach, in terms of a two-dimensional complex vector space, \( S \); the considered construction are closely related to the Penrose-Rindler formalism \(^{[25, 26]}\), though there are differences that I will not examine in detail here. Note that, although the two-spinor index formalism is useful in many computations, everything can be expressed intrinsically.

Let us start from the observation that any finite-dimensional complex vector space, \( V \), has a dual space, \( V^\ast \), and anti-dual space, \( V^\overline{\ast} \) (consisting of all anti-linear functions \( V \rightarrow \mathbb{C} \)), and a conjugate space, \( \overline{V} \equiv V^{\overline{\ast}} \); then one also gets the natural identifications \( V^\ast \cong \overline{V^\ast} \), \( V \cong V^{\overline{\ast}} \). If \( \lambda \in V^\ast \), then complex conjugation yields \( \overline{\lambda} \in V^{\overline{\ast}} \) by the rule \( \overline{\lambda(v)} \equiv \overline{\lambda}(v) \), namely anti-isomorphisms \( V^\ast \leftrightarrow \overline{V^\ast} \) and, similarly, \( V \leftrightarrow \overline{V} \). Thus, one gets the real-linear involution (Hermitian transposition)

\[
\dagger : V \otimes \overline{V} \rightarrow V \otimes \overline{V} : u \otimes \overline{v} \mapsto (u \otimes \overline{v})^\dagger \equiv v \otimes \overline{u} ,
\]

extended by linearity; this, in turn, determines the splitting

\[
V \otimes \overline{V} = \text{H}(V \otimes \overline{V}) + i\text{H}(V \otimes \overline{V})
\]

into the eigenspaces corresponding to eigenvalues \( \pm 1 \) of \( \dagger \), called the Hermitian and anti-Hermitian subspaces. When applied to the case of a two-dimensional complex vector space, \( S \), these constructions generate a rich algebraic structure.
**Complex symplectic structure and the space of length units**

The antisymmetric vector subspace $\wedge^2 S \subset S \otimes S$ is one-dimensional. Thus, the Hermitian subspace of $\wedge^2 S \otimes \wedge^2 S$ is a one-dimensional real vector space, which also turns out to have a natural orientation: the positive subspace of all elements of the type $w \otimes \bar{w}$, with $w \in \wedge^2 S$, is denoted by $L^2 \equiv L \otimes L$; the positive space $L$ will be identified with the semi-vector space of length units

The rational power of a unit space is a well-defined unit space. The two-dimensional complex vector space $U \equiv U^{1/2} \otimes S$ of ‘conformally invariant’ two-spinors has an important role. In particular, there is a distinguished Hermitian metric on $\wedge^2 U$, yielding, up to a phase factor, a unique ‘normalized complex symplectic form’ $\epsilon \in \wedge^2 U^*$. This yields the isomorphisms $\epsilon^\flat : U \to U^*$, defined by $\langle \epsilon^\flat(u), v \rangle \equiv \epsilon(u, v)$, and $\epsilon^\# \equiv -(\epsilon^\flat)^{-1} : U^* \to U$.

**Two-spinor generated Minkowski space**

Since $\epsilon \in \wedge^2 U^*$ is unique up to a phase factor, $g \equiv \epsilon \otimes \bar{\epsilon} \in \wedge^2 U \otimes \wedge^2 \bar{U}$ is a natural object, which can be regarded as the bilinear form on $U \otimes \bar{U}$ characterized by the rule

$$g(u \otimes \bar{v}, u' \otimes \bar{v}') \equiv \epsilon(u, u') \bar{\epsilon}(\bar{v}, \bar{v}') .$$

Moreover, the Hermitian subspace $H \subset U \otimes \bar{U}$ is a 4-dimensional real vector space, and the restriction of $g$ to $H$ turns out to be a Lorentz metric.

Isotropic vectors in $H$ are of the form $\pm u \otimes \bar{u}$, with $u \in U$. Thus one gets a natural time orientation of $H$.

**Dirac spinors**

Let $W \equiv U \oplus \bar{U}$. Consider the linear map $\gamma : U \otimes \bar{U} \to \text{End} W$, characterized by

$$\gamma[p \otimes \bar{q}](u, \bar{\lambda}) \equiv \sqrt{2} \left( \langle \bar{\lambda}, \bar{q} \rangle p, \epsilon(p, u) \epsilon^\flat(q) \right) .$$

Then, the restriction of $\gamma$ to the Minkowski space $H \subset U \otimes \bar{U}$ turns out to be a Clifford map (the Dirac map). The fact that $\epsilon$ is unique up to a phase factor makes $\gamma$ natural.

Thus, we are led to view the 4-dimensional complex space $W$ as the space of Dirac spinors. It is naturally endowed with a further structure, namely the isomorphism

$$\bar{W} \to W^* : (\bar{u}, \lambda) \mapsto (\lambda, \bar{u}) ;$$

this is associated with a Hermitian metric, $k$, which has the signature $(++-)$.

If $\psi \in W$, then $\bar{\psi} \in \bar{W}$ can be regarded as an element in $W^*$, the Dirac adjoint of $\psi$. It is not difficult to recover all the associated notions and identities found in standard expositions of Dirac spinors; in particular, $\gamma[y] \in \text{End} W$ is a $k$-Hermitian endomorphism for any $y \in H$.

On the other hand, the charge-conjugation anti-isomorphism

$$C_\epsilon : W \to W : (u, \bar{\lambda}) \mapsto (\epsilon^\#(\lambda), \epsilon^\flat(\bar{u}))$$

does depend on an overall phase factor.

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9For a comprehensive account of the geometry of semi-vector spaces, and its application to a rigorous mathematical treatment of physical scales, see [27]. A not-too-short introduction to this topic can also be found in the aforementioned book [1].

10If $U$ is a unit space, then we set $U^p \equiv U \otimes \cdots \otimes U$ ($p \in \mathbb{N}$ factors) The $r$-root of a unit space $U$ ($r \in \mathbb{N}$) is a unit space, $U^{1/r}$, which is characterized, up to isomorphism, by $(U^{1/r})^r = U$. Moreover $U^{-1} \equiv U^*$ (dual space), so that $U^{r/p}$ is a well-defined unit space for $p \neq 0$.

11In particular, any basis of $U$ yields a $g$-orthonormal basis of $H$, expressed in terms of the former via Pauli matrices.
Observer-related structures

The above sketched spinor algebra setting does not assume distinguished positive Hermitian metrics on the two-spinor space $U$ or the 4-spinor space $W$. Indeed, a positive Hermitian tensor $h \in U^* \otimes U^*$ can be naturally identified with a future-oriented, timelike covector in $H^*$. The assignment of such an object determines the anti-isomorphism $h^\flat : U \rightarrow U^* : u \mapsto h(u, \cdot) ;$ in turn, this yields the anti-isomorphism $W \rightarrow W^*$ which is usually denoted by $\psi \mapsto \psi^\dagger$.

A \textit{g}-normalized, future-oriented, timelike vector $\tau \in H$ is called an \textit{observer}. From the Dirac algebra identity $\gamma[\tau] \circ \gamma[\tau] = 1$ one sees that an observer determines the splitting $W = W^+ \oplus W^-$ into eigenspaces of $\gamma[\tau]$ with eigenvalues $\pm 1$ (which is related to the distinction between electrons and positrons). Furthermore, an observer yields the \textit{parity} and \textit{time-reversal} operators, and the \textit{spin operators}. The whole standard spinorial machinery can be then recovered in a very direct and natural way.

Two-spinor bundle and spacetime geometry

A two-spinor bundle $S \rightarrow M$ determines vector bundles $U \rightarrow M$, $H \rightarrow M$, $W \rightarrow M$, and so on; their fibers are smoothly endowed with the previously sketched algebraic structures.

A linear connection $\Gamma$ of $S \rightarrow M$ yields linear connections of the induced bundles; in particular, the induced connection $\Gamma$ of $H \rightarrow M$ turns out to be metric, namely one has $\nabla[\Gamma]g = 0$.

The above considerations hold independently of the base manifold $M$; now, assuming that $M$ is 4-dimensional, one defines a \textit{soldering form} (or \textit{tetrad}) as a fibred isomorphism

$$\theta : TM \rightarrow \mathbb{L} \otimes H \subset S \otimes S .$$

A tetrad can also be regarded as a section

$$\theta : M \rightarrow \mathbb{L} \otimes H \otimes T^*M .$$

The requirement that $\theta$ be non-degenerate, namely an isomorphism, is needed to recover its standard physical interpretation; but it is not actually needed to use it as a ‘sector’ in a fledged field theory (§6). A non-degenerate soldering form determines a Lorentz metric on $M$ by ‘translating’ the Lorentz metric $g$ of $H$. Moreover, the soldering form together with a spinor connection yield a linear connection, $\Gamma$, of the tangent bundle $TM \rightarrow M$, which turns out to be metric but, in general, has non-vanishing torsion. Although $g$ also yields the Levi-Civita connection, coupling with spinor fields is a source for torsion. Actually, the induced spacetime connection is just a byproduct, not a fundamental field; instead, the gravitational field the \textit{tetrad-affine} setting is represented by the couple $(\theta, \Gamma)$. One gets the identities

$$\theta_\flat T = [\Gamma, \theta] , \quad 0 = [\theta, [\Gamma, \Gamma]] \oplus [\Gamma, [\Gamma, \theta]] \oplus [\Gamma, [\theta, \Gamma]] , \quad 0 = [\Gamma, [\Gamma, \Gamma]] \equiv -[\Gamma, R] ,$$

where $T$ and $R$ are the torsion and the Riemann tensor of the induced connection $\Gamma$. The last two identities are the first and second Bianchi identities, respectively.

\textsuperscript{12}Note that the Dirac adjunction $\psi \mapsto \psi^\dagger$, which is observer-independent, is often introduced as the combination $\psi^\dagger = \psi^\dagger \gamma_0$, that is, in terms of two observer dependent operations.

\textsuperscript{13}The term ‘tetrad’ is a convenient shorthand, but is somewhat misleading as it was introduced to indicate an orthonormal spacetime frame—and it is usually still intended in that way. A soldering form, on the other hand, is a fully intrinsic notion. If one chooses an orthonormal frame of $H \rightarrow M$, then indeed recovers a ‘tetrad formalism’ which is similar to the usual one.

\textsuperscript{14}If no confusion arises, fibred tensor products will be denoted as plain tensor products.
If \( \psi : M \to W \), then \( \nabla \psi \equiv \nabla [\Gamma] \psi : M \to T^*M \otimes W \); by performing a few natural contractions in \( \Theta \otimes \gamma \otimes \nabla \psi \), where \( \Theta \) is the inverse isomorphism of \( \theta \), one gets a section

\[
\nabla \psi : M \to \mathbb{L}^{-1} \otimes W.
\]

This construction defines the Dirac operator \( \nabla \). If \( \Theta \) is degenerate, then a similar construction can be performed by replacing \( \Theta \) with \( \wedge \theta \), where \( \wedge \theta \) is the inverse isomorphism of \( \theta \), one gets a section

\[
\nabla \psi : M \to \mathbb{L}^{-1} \otimes W \otimes \Lambda^4 T^*M
\]

which fulfils \( \nabla \psi = \nabla \psi \otimes \eta \) (\( \eta \) is the metric volume form).

6 Gauge field theories and tetrad-affine gravity

In the context sketched in §5, one gets a ‘minimal geometric data’ approach to Einstein–Cartan–Maxwell–Dirac fields: all the required underlying structures are derived by natural geometric constructions from a complex bundle \( S \to M \) with two-dimensional fibres. One has the induced bundles \( \mathbb{L} \to M, \mathbb{U} \to M, \mathbb{H} \to M, \mathbb{W} \to M \), with their natural fibre structures, and considers the following fields:

- the soldering form \( \Theta : M \to \mathbb{L} \otimes T^*M \otimes H \)
- the two-spinor connection \( -\Gamma : M \to \mathbb{L} C \mathbb{U} \)
- the Dirac field \( \psi \equiv (u, \bar{\lambda}) : M \to \mathbb{L}^{-3/2} \otimes \mathbb{W} \)
- the dual Dirac field \( \bar{\psi} \equiv (\bar{u}, \lambda) : M \to \mathbb{L}^{-3/2} \otimes \mathbb{W} \)
- the Maxwell field \( F : M \to \mathbb{L}^{-2} \otimes \wedge^2 H^* \).

A further field should actually be considered, namely a dilaton field \( \mathbb{L} : M \to \mathbb{L} \). This is eliminated if we make just one a-priori hypothesis about the theory: the connection of \( \mathbb{L} \) determined by the spinor connection is flat. Then \( \mathbb{L} \) is trivial and, in practice, one deals with a fixed space \( \mathbb{L} \) of unit lengths (the notion of a coupling constant now makes sense). Note how the fields are scaled, namely, tensorialized by powers of \( \mathbb{L} \) (this is true if one uses the natural unit settings, in which \( \hbar = c = 1 \), otherwise one deals with further unit spaces).

In coordinate expressions, the scaling can be conveniently attributed to the field components, and the tensor product by a scaling factor is indicated by simple juxtaposition; in this way one gets notations close to the usual one.

It should be noticed that the spinor connection yields a Hermitian connection \( \gamma \) on \( \wedge^2 U \to M \), which can be regarded as the electromagnetic potential: its relation to the Maxwell field will be a consequence of the field equations\(^\text{15}\).

The Lagrangian density can be now expressed as \( \Lambda = \Lambda_{\text{Dir}} + \Lambda_{\text{em}} + \Lambda_{\text{grav}} \), where

\[
\begin{align*}
\Lambda_{\text{Dir}} &= \left( \frac{1}{2} \left( \langle \psi, \nabla \bar{\psi} \rangle - \langle \nabla \bar{\psi}, \psi \rangle \right) - m \langle \bar{\psi}, \nabla \psi \rangle \right) \eta, \quad m \in \mathbb{L}^{-1}, \\
\Lambda_{\text{em}} &= \frac{1}{4} \left( \mathbb{g}^\# (F \rangle, \langle \wedge^2 \Theta, d\mathbb{Y} \rangle + F \eta \right), \\
\Lambda_{\text{grav}} &= \frac{1}{4 \mathbb{G}} \langle \wedge^2 \Theta, R \rangle,
\end{align*}
\]

where \( \mathbb{G} \) is Newton’s gravitational constant. The field equation can now be straightforwardly derived with the procedure outlined in §4. Note that, in the ‘gravitational sector’, \( \Theta \) formally plays the role of the matter field and \( \Gamma \) plays the role of the gauge field. One gets: the Dirac

\(^{15}\) Thus, \( F \) and \( \gamma \) are independent fields.
The electroweak theory can be formulated within two-spinor geometry, without dealing with the Dirac current; the Einstein gravitational equation, and a further ‘torsion equation’ involving the torsion and the Dirac field.

The above scheme can be seamlessly extended to include matter fields of arbitrary spin, and gauge fields valued into non-trivial Lie algebras. The details of such extensions lie outside the scope of this summary, but let us briefly look at a few points.

- In general, the gauge Lagrangian can be expressed similarly to the e.m. case. Contraction in the fibres of the Lie algebra bundle $\mathcal{L} \to M$ are performed via the natural scalar product $(\xi, \zeta) \mapsto \text{Tr}(\xi^\dagger \circ \zeta)$, where $\xi \mapsto \xi^\dagger$ denotes adjunction with respect to the scalar product which determines $\mathcal{L} \subset \text{End} E$.
- The matter field can be a section of a bundle of the type $E \otimes_M Z \to M$, where $Z \to M$ is generic ‘internal spin’ bundle constructed from any fibre tensor products and direct products (over $M$) of $U, U^\star, U^\times, U^\times$. The vector bundle $E \to M$, instead, is unsoldered from spacetime: its sections are spin-zero fields possessing further ‘internal degrees of freedom’.
- For all types of matter fields one has the Klein–Gordon Lagrangian

$$\Lambda_{\text{KG}} \equiv \frac{1}{2}((g^\# \nabla \phi^* \otimes \nabla \phi) - m^2 (\phi^* \otimes \phi)) \eta,$$

where $g^#$ is the inverse (‘contravariant’) spacetime metric and angle brackets indicate that all possible contractions are taken.
- For fields of arbitrary spin, a first-order generalization of the Dirac equation can be introduced, though the procedure is somewhat more intricate. The action of the Dirac map can be extended to a bundle $Z$ of the above said type, but in general $Z$ is not closed with respect to such action, which generates ghost sectors $Z', Z'', \ldots$ forming a closed sequence $Z \to Z' \to Z'' \to \cdots \to Z$. A natural Lagrangian yielding a first order generalized Dirac equation can be exhibited; in flat spacetime, a plain wave solution in the main sector $Z$ determines the solution in the ghost sectors.

- The electroweak theory can be formulated within two-spinor geometry, without dealing with structure groups, by adding one main ingredient: the isospin bundle $I \to M$, a Hermitian bundle with two-dimensional complex fibres. The fermion bundle is then assumed to be the fibred direct product of a right-handed and left-handed sectors that is

$$Y \equiv Y_R \oplus Y_L \equiv (\gamma^2 I \otimes U)_M \oplus (I \otimes U^\star)_M.$$

The electroweak fields are then

- the fermion field $\Psi \equiv \Psi_R + \Psi_L : M \to \mathbb{L}^{-3/2} \otimes (Y_R \oplus Y_L)$
- the anti-fermion field $\tilde{\Psi} : M \to \mathbb{L}^{-3/2} \otimes (Y_R \oplus Y_L)^*$
- the gauge field $w : M \to \mathbb{L}^{-1} \otimes H^* \otimes \mathcal{L}_L$
- the Higgs field $\phi : M \to \mathbb{L}^{-1} \otimes \gamma^2 \mathbb{T} \otimes I \cong \mathbb{L}^{-1} \otimes I \otimes \gamma^2 I^*$
- the anti-Higgs field $\tilde{\phi} : M \to \mathbb{L}^{-1} \otimes \gamma^2 I \otimes \mathbb{T} \cong \mathbb{L}^{-1} \otimes I \otimes \gamma^2 I \otimes I^*$,

where $\mathcal{L}_L \subset I \otimes I^*$ is the anti-Hermitian Lie-subalgebra bundle. For any assigned gauge, $w$ yields a connection of $Y_L \to M$. The gauge field $\tilde{w}$ in the right-handed sector is naturally determined by $w$.

16 Besides the aforementioned book, details about this subject can be found in a dedicated paper.
17 In an even more extended theory, the fermion bundle can be written as $(E_R \otimes U) \oplus (E_L \otimes U^\star)$.
18 Here I shift from the view that a gauge field is a connection to the view that a connection and a gauge field are actually strictly related but distinct notions, with different roles. This approach is thoroughly discussed in the book, but such discussion lies outside the scope of this summary.
The full-fledged electroweak theory requires two further ingredients: a fixed, background Higgs field $H_0$, and a Weinberg angle $\theta \in (0, \pi/2)$. Together, these determine a ‘symmetry breaking’ in the geometry of $Y$, and decompositions of the fields into several sectors. One straightforwardly recovers the standard notions of e.w. theory (the complete procedure is too intricate to be reported in this summary).

A further subject which finds its natural formulation within the two-spinor setting is that of Lie derivatives of spinors and connections. Indeed, one finds that the Lie derivatives of two-spinors and Dirac spinors, as well as the Lie derivatives of the spinor connection and of the soldering form, are are strictly related to one another and, together, describe the deformations of Einstein–Cartan–Dirac fields.

Two-spinor geometry also suggests a natural extension of the Higgs sector of the electroweak theory, in which one has a natural Lagrangian such that the self-interactions of the extended Higgs field sum up to zero. In turn, this suggest a possible way for explaining the standard Higgs potential and the ‘breaking of dilatonic symmetry’.

### 7 Multi-particle algebra and operator algebra

Let $Z \rightarrow X$ be a vector bundle and denote by $Z^1$ the freely generated vector space of sections $z : X \rightarrow Z$, that is the space of all such sections which vanish outside a finite subset of $X$. Similarly, denote by $Z^{\ast}^1$ the freely generated vector space of sections $\zeta : X \rightarrow Z^{\ast}$. These spaces, which are infinite-dimensional unless the cardinality of $X$ is finite, can be regarded as mutually dual.

The space $Z^1$ is a template for the space of states of one particle of some type. The associated ‘$n$-particle state’ space

$$Z^n \equiv \mathcal{O}^n Z^1$$

is defined as either the symmetrized tensor product $\vee^n Z^1$ (bosons) or the antisymmetrized tensor product $\wedge^n Z^1$ (fermions). For $y \in Z^n$, $z \in Z^m$, $y \circ z \in Z^{n+m}$ is either $y \vee z$ or $y \wedge z$, as appropriate, and called the exterior product of $y$ and $z$.

Setting $Z^0 \equiv \mathbb{C}$ (called the vacuum), the multi-particle state space for the particle type under consideration is defined as

$$Z \equiv \bigoplus_{n=0}^{\infty} Z^n,$$

Namely, $Z$ is constituted by all formal, finite sums with arbitrarily many terms. The similarly defined space $Z^{\ast} \equiv \bigoplus_{n=0}^{\infty} Z^{\ast n}$ is regarded as its ‘dual’. By linearity, the exterior product $(y, z) \mapsto y \circ z$ can be extended to any couple of elements in $Z$. Hence $Z$ is defined similarly to the usual Fock spaces of QFT. One also has an interior product,

$$(Z \times Z^{\ast}) \rightarrow Z \cup Z^{\ast} : (\lambda, z) \mapsto \lambda \mid z,$$

which belongs to $Z$ or $Z^{\ast}$ depending on which of the two factors is of higher rank. For fermions, this is the usual interior product $\langle i| z \rangle$ of exterior algebra. For bosons it can be defined similarly, as tensor contraction with appropriate symmetrization and normalization such that it fulfils the rule

$$(\zeta \circ \lambda) \mid z = \lambda \mid (\zeta \mid z), \quad \zeta \in Z^{\ast n}, \lambda \in Z^{\ast}.$$

---

\[^{19}\text{See also the dedicated paper [19] for the details, that lie outside the scope of this presentation.}\]

\[^{20}\text{We need not be concerned with duality in the most general acceptation.}\]

\[^{21}\text{This setting suffices if one is not concerned with completions, namely with infinite sums: our multi-particle states only contain finitely many particles, though their number can be arbitrarily large.}\]
A general theory of quantum particles has several particle types; correspondingly, we consider several multi-particle state spaces $\mathcal{Z}'$, $\mathcal{Z}''$, $\mathcal{Z}'''$, and so on. The total state space is now defined as

$$\mathcal{Y} \equiv \mathcal{Z}' \otimes \mathcal{Z}'' \otimes \mathcal{Z}''' \otimes \cdots = \bigoplus_{n=0}^{\infty} \mathcal{Y}_n,$$

where $\mathcal{Y}_n$, consisting of all elements of tensor rank $n$, is the space of all states of $n$ particles of any type. Moreover, all fermionic sectors can be described by a unique overall antisymmetrized tensor algebra. A similar observation holds true for the bosonic sectors. Furthermore, the mutual ordering of fermionic and bosonic sectors is regarded as inessential.

Letting the parity (or grade) $[\phi] \in \mathbb{Z}_2$ of a monomial element $\phi \in \mathcal{Y}$ be the number of its fermion factors (mod 2), one gets a structure of ‘super-algebra’ (a $\mathbb{Z}_2$-graded algebra) on $\mathcal{Y}$. The algebra product, denoted by $\circ$, is the exterior product modulo the so-called Koszul convention, which essentially amounts to imposing anti-commutativity (or ‘super-commutativity’).

In particular, this implies the commutativity of the multiplication of any element by a bosonic factor. Similarly, one constructs a ‘dual’ space $\mathcal{Y}^* \equiv \mathcal{Z}'^* \otimes \mathcal{Z}''^* \otimes \mathcal{Z}'''^* \cdots$ and the interior product can be extended as a map $\mathcal{Y} \times \mathcal{Y}^* \rightarrow \mathcal{Y} \cup \mathcal{Y}^* : (\lambda, \psi) \mapsto \lambda \circ \psi$, where the required contractions are to be performed in the appropriate tensor factors, yielding the identities

$$(\zeta \circ \xi) \circ \psi = \xi \circ (\zeta \circ \psi),$$

$$\psi \circ \phi = (-1)^{[\phi][\psi]} \phi \circ \psi \quad \text{(anti-commutativity)},$$

$$z \circ (\phi \circ \psi) = (z \circ \phi) \circ \psi + (-1)^{[z][\phi]} \phi \circ (z \circ \psi), \quad \phi, \psi \in \mathcal{Y}, \quad \zeta, \xi \in \mathcal{Y}^1.$$

The absorption operator associated with $\zeta \in \mathcal{Y}^1$ and the emission operator associated with $z \in \mathcal{Y}^1$ are the linear maps $\mathcal{Y} \rightarrow \mathcal{Y}$ respectively defined as

$$a[\zeta] \phi \equiv \zeta \circ \phi, \quad a^*[z] \phi \equiv z \circ \phi, \quad \phi \in \mathcal{Y}.$$

Similarly, one gets operators $a[z], a^*[\zeta] : \mathcal{Y}^* \rightarrow \mathcal{Y}^*$ and gets $\lambda \circ a[\zeta] \psi = (a^*[\zeta] \lambda) \circ \psi, \forall \lambda \in \mathcal{Y}^*$, namely $a[\zeta]$ and $a^*[\zeta]$ are mutually transpose endomorphisms.

Let now $\text{op} : \mathcal{Y}^1 \oplus \mathcal{Y}^1 \rightarrow \text{End}(\mathcal{Y})$ be the linear map characterized by

$$\text{op}[v] \equiv \begin{cases} a[v], & v \in \mathcal{Y}^1, \\ a^*[v], & v \in \mathcal{Y}^1, \end{cases}$$

and consider the vector space $\mathcal{Y}^\otimes \equiv \bigotimes_{n=0}^{\infty} (\mathcal{Y}^1 \oplus \mathcal{Y}^1)$. Then one gets a natural morphism $\text{op} : \mathcal{Y}^\otimes \rightarrow \text{End}(\mathcal{Y})$ of associative algebras, defined for decomposable tensors by

$$x \otimes y \otimes \cdots \otimes z \mapsto \text{op}[x] \circ \text{op}[y] \circ \cdots \circ \text{op}[z],$$

and $\text{op}[c] \equiv c \mathbb{1}$ for a zero-rank tensor $c \in \mathbb{C}$.

Let now the grades of $a[\zeta]$ and $a^*[z]$ be $[\zeta]$ and $[z]$, respectively, and the grade of any composition the sum (mod 2) of the grades of all factors. The super-bracket (or super-commutator) of $X, Y \in \text{op}(\mathcal{Y}^\otimes)$ is then defined by

$$\{[X, Y]\} \equiv XY - (-1)^{[X][Y]}YX$$

---

22 If $\mathcal{X}$ and $\mathcal{Y}$ are any two vector spaces, then their antisymmetric tensor algebras fulfil the isomorphisms

$$\wedge^p (\mathcal{X} \oplus \mathcal{Y}) \cong \bigoplus_{h=0}^{p} (\wedge^{p-h} \mathcal{X}) \otimes (\wedge^h \mathcal{Y}), \quad (\wedge \mathcal{X}) \otimes (\wedge \mathcal{Y}) \cong \wedge (\mathcal{X} \oplus \mathcal{Y}).$$
whenever both $X$ and $Y$ have definite grade, and extended by linearity. In particular, for all $y, z ∈ V^1$ and $ζ, ξ ∈ V^1_*$ one has

$$\{[a[ζ], a[ξ]]\} = \{[a^*[y], a^*[z]]\} = 0, \quad \{[a[ζ], a^*[z]]\} = ⟨ζ, z⟩_1.$$

Because of the above super-commutation relations, the morphism $\text{op} : V^0 ⊴ → \text{End}(V)$ is not a monomorphism. On the other hand, consider the subspace

$$V^0 ≡ \bigoplus_{n=0}^∞ (V^1 ⊕ V^1_*) ⊂ V^0,$$

whose decomposable elements are Koszul products. If the product’s rules and the related identifications—that is super-commutativity—are applied to the exchange between elements in $V$ and in $V^*$ as well, then one gets the identification

$$V^0 ≅ V ⊗ V^*,$$

where, in each decomposable element, all ‘covariant’ factors are on the right of any ‘contravariant’ factors. This is called normal ordering. The image

$$\mathcal{O} ≡ \text{op}(V^0) ⊂ \text{End}(V),$$

is our fundamental operator space. A bilinear product $\mathcal{O} × \mathcal{O} → \mathcal{O}$ can be defined as composition together with super-commutative normal reordering in each decomposable term. This renders $\mathcal{O}$ a super-commutative $\mathbb{Z}_2$-graded algebra.

8 Quantum bundles and quantum fields

In my opinion, that part of elementary particle theory which is unquestionably rooted in actual physics can be actually introduced without making essential use of quantum fields. I tend to regard the notion of quantum field as auxiliary rather than fundamental, entangled with particle physics for historical reasons; and I suspect that the issue of finding a full-fledged covariant approach to quantum fields in curved spacetime might eventually turn out to be pointless. Nevertheless, since they are needed in order to cope with the literature, I studied a precise mathematical approach to quantum bundles and quantum fields, including ghosts, BRST symmetry, and the so-called ‘anti-field’ formalism. The geometry of quantum bundles and their jet prolongations can be developed in terms of F-smoothness.

As an intermediate step, we need the notion of a distributional bundle, that is, a bundle over $M$ whose fibres are distributional spaces. In general, the finite-dimensional geometric structure underlying functional bundles [30, 5, 7, 28, 29, 14] is that of a two-fibered bundle $Z → Y → X$. If $Z → Y$ is a vector bundle, then for any $x ∈ X$ one obtains the vector space $D_x(Y, Z)$ of all section-distributions [31] $Y_x → Z_x$. A smooth bundle structure on the fibred set $D(Y, Z) ≡ \bigsqcup_{x ∈ X} D_x(Y, Z) → X$ can be assigned, exploiting Frölicher’s notion of smoothness [23] by selecting, as the set $C_F$ of all F-smooth curves, the set of all local maps

\[ f ∈ FC \text{ if and only if } f ◦ c : \mathbb{R} → \mathbb{R} \text{ is smooth. Conversely, a set } \mathcal{F} \text{ of functions } X → \mathbb{R} \text{ determines a set } C\mathcal{F} \text{ of curves in } X \text{ by the same requirement. Any set of curves, or any set of functions, generates such an F-smooth structure on } X. \text{ This notion of smoothness, which was introduced by Frölicher [32], is compatible with the standard one in finite-dimensional manifolds; moreover, it behaves naturally with regard to inclusions and Cartesian products, so it yields a convenient general setting for dealing with functional spaces and functional bundles [33, 54, 35, 30, 28, 29].} \]
c : \mathbb{R} \to \mathcal{D} such that the map \langle c, u \rangle : \mathbb{R} \to \mathbb{C} : t \mapsto \langle c(t), u \rangle is smooth for any test element u (a smooth section \( Y \to Z \) with compact support).

By replacing \( Z \) with \( \wedge^n V^\ast Y \otimes Z \), where \( n \) is the fibre dimension of \( Y \to X \), one gets an F-smooth bundle of generalized densities. If the fibres of \( Y \to X \) are smoothly orientable then one may choose a positive sub-bundle \( V \equiv (\wedge^n V Y)^+ \), and gets the F-smooth bundle \( \mathcal{D}(Y, V^{-1/2} \otimes Z) \to X \) of \( Z \)-valued semi-densities. This is especially convenient as a template for bundles of quantum states, since when a fibre Hemitian structure of \( Z \to Y \) is assigned one gets a bundle of rigged Hilbert spaces \( \mathcal{D}_0 \subset \mathcal{H} \subset \mathcal{D} \), where the fibres of \( \mathcal{D}_0 \to X \) and \( \mathcal{H} \to X \) are generated, respectively, by test semi-densities and square-integrable semidensities.

We are mainly interested in the case when \( Y \equiv P_m \to M \), the future mass-shell bundle for \( m \in \mathbb{L}^{-1} \), that is the sub-bundle of \( T^\ast M \to M \) whose fibres consist of all future-oriented covectors of Lorentzian pseudo-norm \( m \).

Let us denote the ensuing F-smooth bundle of \( Z \)-valued semi-densities by \( Z^1 \). Then, the vector bundles

\[
Z^n \equiv \bigotimes^n Z^1 \to M , \quad Z \equiv \bigoplus_{n=0}^{\infty} Z^n \to M ,
\]

are constructed, fibrewise, with the same procedure as in \( \mathbb{L} \). Furthermore, one has the subbundles \( Z^n \subset Z^n, Z \subset Z \), whose fibres consist of finite linear combinations of Dirac-type semi-densities. The latter are expressable as \( \eta^{-1/2} \otimes \delta[p] \otimes z \), where

- The density \( \eta : M \to \mathcal{D}(P_m, V^{-1}) \) is the metric-induced volume form on the fibres of \( P_m \to M \).
- \( p : M \to P_m \) is a smooth section.
- \( \delta[p] : M \to \mathcal{D}(P_m, V^{-1}) \) is, at each \( x \in M \), the Dirac density \( \delta[p(x)] \) on \( (P_m)_x \).
- \( z : P_m \to Z \) is a smooth section (only its values along the image of \( p \) matter).

The bundle \( Z \to M \) is relevant under two respects. First, the inclusion \( Z \subset Z \) is dense \( \mathbb{L} \); second, the correspondence \( \eta^{-1/2} \otimes \delta[p] \otimes z \leftrightarrow z(p) \) determines a fibred isomorphism between \( Z \) and the bundle whose fibres are freely generated spaces of sections \( P_m \to Z \) (\( \mathbb{L} \)). Namely, multi-particle algebra and graded operator algebra can be readily adapted to distributional spaces and bundles. Considering several sectors, by constructions similar to \( \mathbb{L} \) we get the bundle \( V^0 \cong V \otimes V^\ast \to M \), with \( V \equiv Z^0 \otimes Z^1 \otimes Z^2 \otimes \cdots \) as well as its ‘underlined’ counterparts (finitely generated by semi-densities of Dirac type). Furthermore, we get the operator-algebra bundle \( \mathcal{O} \equiv \text{op}(V^0) \to M \); its elements, in general, are linear morphisms \( \mathcal{V}_0 \to \mathcal{V} \), where \( \mathcal{V}_0 \subset \mathcal{V} \) is the sub-bundle generated by test semi-densities (Dirac deltas cannot be contracted with arbitrary distributions), but may admit extensions.

Consider the simpler case in which the ‘internal’ bundle is actually a vector bundle \( E \to M \), so that \( Z = P_m \times_M E \). The associated quantum bundle is then defined as the vector bundle

\[
\mathcal{E} \equiv \mathcal{O} \otimes E \to M ,
\]

where \( \mathcal{O} \to M \) is the operator-algebra bundle previously described (it is not difficult to extend this ‘bundle quantization’ to the general situation of two-fibred internal bundles, which notably includes the electron bundle \( W^+ \to P_m \) and the positron bundle \( \mathcal{W}^- \to P_m \)). A Frölicher-smooth structure of \( \mathcal{E} \to M \) can be readily introduced, and all the basic differential geometric notions for classical bundles can be naturally extended to quantum bundles. A few observations are in order.

- A frame of \( E \to M \) can be regarded as a frame of \( \mathcal{E} \to M \): the components of fibre elements are elements in \( O \), which can be regarded as ‘quantum numbers’. A polynomial \( E \to \mathbb{C} \) in
the fibre coordinates determines a ‘quantum polynomial’ \( \mathcal{E} \to \mathcal{O} \); more general quantum functions are not actually needed in gauge field theory. Partial derivatives with respect to fibre coordinates are then well-defined algebraic operations.

- In order to recover the standard formalism one has to assume that \( \mathcal{O} \to M \) is a trivial bundle. A special trivialization is determined by the choice of an observer (a frame may be either associated with the observer or not). Then \( \mathcal{O} \) can be viewed as a fixed \( \mathbb{Z}_2 \)-graded algebra, and a linear connection of \( E \to M \) yields a linear connection of \( \mathcal{E} \to M \).

- In a local frame, the components of a quantum field \( \phi : M \to \mathcal{E} \) are \( \mathcal{O} \)-valued. A Lagrangian density \( J\mathcal{E} \to \mathcal{O} \otimes \wedge^4 T^* M \) can be written as \( \Lambda = \lambda \eta \), where \( \eta \) is the metric volume form of spacetime and \( \lambda \) is a quantum polynomial in the components of the field and its covariant derivatives (the gauge field itself must be quantized, so that a gauge is needed). The field equations are derived as in the classical theory, but attention to signs must be paid since \( \mathcal{O} \) is \( \mathbb{Z}_2 \)-graded.

- The existence of sections \( \phi : M \to \mathcal{E} \) fulfilling the complete field equations, with interactions, is far from guaranteed, but this is not actually an issue in perturbative particle physics.

- Given an observer and a frame adapted to it one may recover the usual notions of absorption and emission operators, free quantum fields, and so on.

In a theory with several matter sectors, each one is assumed to be either bosonic or fermionic. This means that \( \mathcal{E} \) is not obtained by tensorializing \( E \) by the whole \( \mathbb{Z}_2 \)-graded operator algebra, \( \mathcal{O} = \mathcal{O}_g \oplus \mathcal{O}_f \), but rather as either \( \mathcal{O}_g \otimes E \) or \( \mathcal{O}_f \otimes E \), respectively. Considering the exchanged parity in each sector gives rise to the notion of anti-field and to the Batalin–Vilkovisky algebra.

Furthermore, in the quantum theory derived from a classical field theory, one also has to include additional ghost fields. For example, consider the theory derived from a classical theory of a fermion field \( \psi : M \to W \otimes M E \) and a bosonic gauge field \( \lambda : M \to T^* M \otimes M \mathcal{L} \), where \( \mathcal{L} \subset \text{End } E \). Then, in addition to the independent adjoint fermion field \( \bar{\psi} : M \to W^* \otimes_M E^* \), one also has the fermionic ghost and anti-ghost fields, \( \omega : M \to \mathcal{L} \) and \( \varpi : M \to \mathcal{L}^* \), and the bosonic Nakanishi–Lautrup field \( N : M \to \mathcal{L} \).

In general, a vertical symmetry of the Lagrangian is a fibred morphism \( v : J\mathcal{E} \to V\mathcal{E} \) (a generalized vector field) fulfilling\(^{25}\) \( \delta[v] \Lambda = d_H v \), where \( \nu : J\mathcal{E} \to \mathcal{O} \otimes \wedge^3 T^* M \) and \( d_H \) denotes the horizontal differential. It turns out that the Lagrangian of the above sketched theory, besides the symmetries of the classical theory, has a further symmetry, which is a special case of the BRST symmetry\(^{26}\). In this case the components of \( v \), in each sector, have the sector’s parity. The BRST transformation, acting on horizontal forms \( \alpha : J_k \mathcal{E} \to \mathcal{O} \otimes \wedge^k T^* M \), is defined in terms of an arbitrarily chosen element \( \theta \in \mathcal{O}_{[1]} \) by

\[
\theta \circ \alpha \equiv \delta[v] \alpha, \quad \delta[v] \alpha : J_{k+1} \mathcal{E} \to \mathcal{O} \otimes \wedge^{k+1} T^* M,
\]

and turns out to be nilpotent \((s^2 = 0)\).

\(^{24}\) Here, the fermion field may have further internal structure besides spin.

\(^{25}\) The operator \( \delta[v] \) acts on horizontal forms \( \alpha : J_k \mathcal{E} \to \mathcal{O} \otimes \wedge^k T^* M \) as \( \delta[v] \alpha \equiv d\alpha \circ v : J_{k+1} \mathcal{E} \to \mathcal{O} \otimes \wedge^{k+1} T^* M \), where \( d \alpha \) is the fibre differential of \( \alpha \) and the generalized vector field \( v : J_{k+1} \mathcal{E} \to \mathcal{E} \) is the holonomic restriction of \( J_k v : J_k \mathcal{E} \to J_k V \mathcal{E} \), taking the natural isomorphism \( J_k V \mathcal{E} \cong VJ_k \mathcal{E} \) into account.

\(^{26}\) The details of the expressions of the Lagrangian and of the symmetry are not included here, as they are somewhat involved.
9 Quantum particles

In recent work, and especially in the aforementioned book [1], I proposed an approach to the physics of quantum particles that essentially dispenses with quantum fields. Indeed, although the descriptions of quantum particles and quantum fields both stem from the underlying classical bundle geometry, quantum fields can be actually sidestepped.

First, a chosen time function is needed. This requirement is common to all approaches, though it is seen as a drawback—as it ‘breaks Lorentz invariance’. In geometric terms, it can be implemented in various ways. My proposal starts from the notion of a detector, that is a one-dimensional timelike submanifold $T \subset M$. Along $T$, one gets the orthogonal splitting

$$(TM)_T = TT \oplus (TM)_T^\perp$$

into timelike and spacelike sub-bundles, and a similar splitting of the cotangent bundle. Particle spins and momenta can be Fermi-transported along $T$, while for other internal degrees of freedom, not soldered to spacetime, one needs a gauge. Accordingly, one constructs free-particle states; these, together with particle interactions, constitute the building blocks of the dynamics of many-particle quantum systems, yielding a ‘perturbative’ formalism in momentum space which can be viewed as a sort of complicated ‘clock’ carried by the detector. By choosing suitable classical frames, and considering the associated quantum generalized frames (which are Dirac-type semi-densities), one is able to recover the standard scattering matrix computations in terms of free states and the interactions among them.

Similarly, the quantum interaction is constructed from the classical interaction—which is in turn a natural byproduct of the underlying classical geometry—coupled with a ‘quantum ingredient’ which is a special generalized semi-density on the space of particle momenta.

In a curved spacetime background, the precise correspondence between momentum and position representations is lost, and I tend to view the former as more fundamental. The latter can be recovered locally, using the observation that exponentiation yields a family $\{X_t, t \in T\}$ of spacelike submanifolds orthogonal to $T$; hence, one has a time $\oplus$ space splitting in a neighbourhood of $T$.

I also want to point out to an intriguing consequence of the two-spinor formalism, associated with the idea that gauge fields and gauge particles are to be treated as distinct from connections (though the two notions are strictly related). By describing the interactions between fermions and photons in two-spinor terms, and using the free-particle relations between momentum and spin, gauge freedom can be recovered in a purely algebraic way. I view this result as supporting the idea that the relation between classical geometry and the quantum description could be inverted: the former could be obtained from the latter. Namely, the system defines the geometry [36] and reality is fundamentally discrete, while notions related to continuity should be recovered as conveniences in the description of sufficiently complex systems. Ideas of this kind have been around for some time and have inspired attempts at serious theories [36, 37].

My long-term goal is somewhat radical, since I would like to achieve a fully discrete, relational theory of spacetime and matter. I speculate that physical reality is fundamentally a network, whose nodes and edges may be called events and particles, respectively.

27 A background connection, also needed in field quantization.

28 The examples offered in the book [1] are meant to support the philosophy of my approach, with no claim of constituting a comprehensive treatment of this subject. The details do not fit into this brief summary.

29 Essentially, the Dirac delta of the sum of momenta in the usual formulation.

30 Maybe for my next reincarnation.
Approximate geometric relations among edges will emerge in a sufficiently large portion of the network; information about some edges will constrain the network’s state, allowing a probabilistic description of the missing information. In this scenario of emerging geometry we should regard smooth manifolds and bundles, as well as connections and algebraic structures (including the spacetime metric), as secondary macroscopic notions. Hence spacetime, gravity (not quantum gravity), and quantum mechanics could all emerge from a more fundamental, discrete theory.

If such a program could be fulfilled, then classical notions would take the role of ‘mean field’ background properties of physical systems. In particular, this would be true for connections of classical bundles, implying that the relation between gauge particles and connections should be thought over and need not be a precise correspondence. We may then expect gauge freedom expressed in two-spinor terms to play a relevant role. Indeed, it would be a very interesting outcome if any spacetime-related notion turned out to be founded on two-spinors.

Finally, let me observe how a possible easing of tensions among notions relevant to quantum physics could be achieved by acknowledging their diverse roles and levels of importance. We should seek no mandatory unification of everything on the same footing; above all, gravitation is not assimilable to the other interactions. Moreover, the complementarity of momentum and spatial position representations is of partial and limited scope: the former is more directly linked to internal particle structure and quantum interactions, while the latter is related to emerging geometry and the notion of a quantum field—which can be described as a section of a quantum bundle and not regarded as truly fundamental.

References

[1] D. Canarutto: *Gauge field theory in natural geometric language*, Oxford University Press, Oxford (2020).

[2] Y. Choquet-Bruhat and C. DeWitt-Morette: *Analysis, manifolds and physics*, North-Holland, Amsterdam (1982).

[3] R. Penrose: *Fashion faith and fantasy in the New Physics of the Universe*, Princeton University Press, Princeton and Oxford (2016).

[4] D. Canarutto: ‘Possibly degenerate tetrad gravity and Maxwell-Dirac fields’, *J. Math. Phys.* **39**, 4814–23 (1998).

[5] D. Canarutto: ‘Smooth bundles of generalized half-densities’, *Arch. Math. Un. Brunensis* **36**, 111–24 (2000).

[6] D. Canarutto: ‘Two-spinors, field theories and geometric optics in curved spacetime’, *Acta Appl. Math.* **62**, 187–224 (2000).

[7] D. Canarutto: ‘Generalized densities and distributional adjoints of natural operators’, *Rend. Semin. Mat. Univ. Pol. Torino* **59**, 27–36 (2001).

[8] D. Canarutto: ‘Connections on distributional bundles’, *Rend. Semin. Mat. Univ. Padova* **111**, 71–97 (2004).

[9] D. Canarutto: ‘Quantum bundles and quantum interactions’, *Int. J. Geom. Met. Mod. Phys.* **2**, 895–917 (2005).
REFERENCES

[10] D. Canarutto: “Minimal geometric data” approach to Dirac algebra, spinor groups and field theories, *Int. J. Geom. Met. Mod. Phys.* 4, 1005–40 (2007).

[11] D. Canarutto: ‘Fermi transport of spinors and free QED states in curved spacetime’, *Int. J. Geom. Met. Mod. Phys.* 6, 805–24 (2009).

[12] D. Canarutto: ‘Tetrad gravity, electroweak geometry and conformal symmetry’, *Int. J. Geom. Met. Mod. Phys.* 8, 797–819 (2011).

[13] D. Canarutto: ‘Positive spaces, generalized semi-densities and quantum interactions’, *J. Math. Phys.* 53, 032302 (2012).

[14] D. Canarutto: ‘Frölicher-smooth geometries, quantum jet bundles and BRST symmetry’, *J. Geom. Phys.* 88, 113–28 (2015).

[15] D. Canarutto: ‘Natural extensions of electroweak geometry and Higgs interactions’, *Ann. H. Poincaré* 16, 2695–711 (2015).

[16] D. Canarutto: ‘Special generalized densities and propagators: a geometric account’, *Int. J. Geom. Met. Mod. Phys.* 13, 1530004 (2016).

[17] D. Canarutto: ‘On the geometry of ghosts’, *Rep. Math. Phys.* 78, 123–56 (2016).

[18] D. Canarutto: ‘Overconnections and the energy-tensors of gauge and gravitational fields’, *J. Geom. Phys.* 106, 192–204 (2016).

[19] D. Canarutto: ‘Two-spinor tetrad and Lie derivatives of Einstein-Cartan-Dirac fields’, *Arch. Math. Un. Brunensus* 54, 205–26 (2018).

[20] D. Canarutto: ‘Covariant-differential formulation of Lagrangian field theory’, *Int. J. Geom. Met. Mod. Phys.* 15, 1530004 (2018).

[21] D. Canarutto: ‘A first-order Lagrangian theory of fields with arbitrary spin’, *Int. J. Geom. Met. Mod. Phys.* 15, 1850088 (2018).

[22] D. Canarutto: ‘On the notions of energy tensors in tetrad-affine gravity’, *Grav. Cosmol.* 24, 122–8 (2018).

[23] D. Canarutto and A. Jadczyk: ‘Fundamental geometric structures for the Dirac equation in General Relativity’, *Acta Appl. Math.* 50, 59–92 (1998).

[24] D. Canarutto and E. Minguzzi: ‘The distance formula in algebraic spacetime theories’, *J. Phys. Conf. Ser.* 1275, 012045 (2019).

[25] R. Penrose and W. Rindler: *Spinors and space-time. I: two-spinor calculus and relativistic fields*, Cambridge University Press, Cambridge (1984).

[26] R. Penrose and W. Rindler: *Spinors and space-time. II: spinor and twistor methods in space-time geometry*, Cambridge University Press, Cambridge (1988).

[27] J. Janyška, M. Modugno, R. Vitolo: ‘An algebraic approach to physical scales’, *Acta Appl. Math.* 110, 1249–76 (2010).

[28] A. Cabras, J. Janyška, I. Kolář: ‘Functorial prolongations of some functional bundles’, *Annales Acad. Paed. Cracoviensis, Stud. Math. IV* 23, 17–30 (2004).
[29] A. Cabras, J. Janyška, I. Kolář: ‘On the geometry of the variational calculus on some functional bundles’, Note di Mat. 26, 51–66 (2006).

[30] I. Kolář and M. Modugno: ‘The Frölicher–Nijenhuis bracket on some functional spaces’, Ann. Pol. Math. 68, 97–106 (1998).

[31] L. Schwartz: Théorie des distributions, Hermann, Paris (1966).

[32] A. Frölicher: Smooth structures, LNM 962, Springer-Verlag, 69–81 (1982).

[33] A. Frölicher and A. Kriegl: Linear spaces and differentiation theory, John Wiley & Sons, New York (1988).

[34] J. Janyška and M. Modugno: ‘Smooth and F–smooth systems’, arXiv:2002.11983 [math.DG].

[35] A. Kriegl and P. Michor: The convenient setting of global analysis, American Mathematical Society (1997).

[36] R. Penrose: ‘Angular momentum: an approach to combinatorial space-time’, in Quantum Theory and Beyond—essays and discussions arising from a colloquium, T. Bastin editor, Cambridge University Press, Cambridge, 151–80 (1971).

[37] E.P. Verlinde: ‘On the origin of gravity and the laws of Newton’, arXiv:1001.0785v1 (2010).

[38] D. Canarutto: ‘Nature’s software’, essay presented for the 2011 contest, ‘Is Reality Digital or Analog?’, of the Foundational Questions Institute (FQXi), http://fqxi.org/community/forum/topic/831.