REINFORCEMENT LEARNING WITH LINEAR FUNCTION APPROXIMATION AND LQ CONTROL CONVERGES*

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ABSTRACT. Reinforcement learning is commonly used with function approximation. However, very few positive results are known about the convergence of function approximation based RL control algorithms. In this paper we show that TD(0) and Sarsa(0) with linear function approximation is convergent for a simple class of problems, where the system is linear and the costs are quadratic (the LQ control problem). Furthermore, we show that for systems with Gaussian noise and non-completely observable states (the LQG problem), the mentioned RL algorithms are still convergent, if they are combined with Kalman filtering.

1. INTRODUCTION

Reinforcement learning is commonly used with function approximation. However, the technique has little theoretical performance guarantees: for example, it has been shown that even linear function approximators (LFA) can diverge with such often used algorithms as Q-learning or value iteration [1,8]. There are positive results as well: it has been shown [10,7,9] that TD(λ), Sarsa, importance-sampled Q-learning are convergent with LFA, if the policy remains constant (policy evaluation). However, to the best of our knowledge, the only result about the control problem (when we try to find the optimal policy) is the one of Gordon’s [4], who proved that TD(0) and Sarsa(0) can not diverge (although they may oscillate around the optimum, as shown in [3,1]).

In this paper, we show that RL control with linear function approximation can be convergent when it is applied to a linear system, with quadratic cost functions (known as the LQ control problem). Using the techniques of Gordon [4], we were prove that under appropriate conditions, TD(0) and Sarsa(0) converge to the optimal value function. As a consequence, Kalman filtering with RL is convergent for observable systems, too.

Although the LQ control task may seem simple, and there are numerous other methods solving it, we think that this Technical Report has some significance: (i) To our best knowledge, this is the first paper showing the convergence of an RL control algorithm using LFA. (ii) Many problems can be translated into LQ form [2].

*Last updated: 22 October 2006.
1These are results for policy iteration (e.g. [5]). However, by construction, policy iteration could be very slow in practice.
2. THE LQ CONTROL PROBLEM

Consider a linear dynamical system with state $x_t \in \mathbb{R}^n$, control $u_t \in \mathbb{R}^m$, in discrete time $t$:

$$x_{t+1} = Fx_t + Gu_t.$$ 

Executing control step $u_t$ in $x_t$ costs

$$c(x_t, u_t) := x_t^T Q x_t + u_t^T R u_t,$$

and after the $N^\text{th}$ step the controller halts and receives a final cost of $x_N^T Q_N x_N$. The task is to find a control sequence with minimum total cost.

First of all, we slightly modify the problem: the run time of the controller will not be a fixed number $N$. Instead, after each time step, the process will be stopped with some fixed probability $p$ (and then the controller incurs the final cost $c_f(x_f) := x_f^T Q_f x_f$). This modification is commonly used in the RL literature; it makes the problem more amenable to mathematical treatments.

2.1. The cost-to-go function. Let $V_t^*(x)$ be the optimal cost-to-go function at time step $t$, i.e.

$$V_t^*(x) := \inf_{u_t, u_{t+1}, \ldots} E[c(x_t, u_t) + c(x_{t+1}, u_{t+1}) + \ldots + c_f(x_f)|x_t = x].$$

Considering that the controller is stopped with probability $p$, Eq. 3 assumes the following form

$$V_t^*(x) = p \cdot c_f(x) + (1 - p) \inf_u c(x, u) + V_{t+1}^*(Fx + Gu)$$

for any state $x$. It is an easy matter to show that the optimal cost-to-go function is time-independent and it is a quadratic function of $x$. That is, the optimal cost-to-go action-value function assumes the form

$$V^*(x) = x^T \Pi^* x.$$

Our task is to estimate the optimal value functions (i.e., parameter matrix $\Pi^*$) on-line. This can be done by the method of temporal differences.

We start with an arbitrary initial cost-to-go function $V_0(x) = x^T \Pi_0 x$. After this,

1. control actions are selected according to the current value function estimate
2. the value function is updated according to the experience, and
3. these two steps are iterated.

The $t^{\text{th}}$ estimate of $V^*$ is $V_t(x) = x^T \Pi_t x$. The greedy control action according to this is given by

$$u_t = \arg\min_u \left( c(x_t, u) + V_t(Fx_t + Gu) \right)$$

$$= \arg\min_u \left( u^T R u + (Fx_t + Gu)^T \Pi_t (Fx_t + Gu) \right)$$

$$= -(R + G^T \Pi_t G)^{-1} (G^T \Pi_t F) x_t.$$

The 1-step TD error is

$$\delta_t = \begin{cases} c_f(x_t) - V_t(x_t), & \text{if } t = t^{\text{STOP}}, \\ (c(x_t, u_t) + V_t(x_{t+1})) - V_t(x_t), & \text{otherwise}. \end{cases}$$
If the written in the form $V$ and analogously to action is given as $Q$ function is defined as value function $Q^\alpha$ where subscript.

Note that $\Pi^*$ can be expressed by $\Theta^*$ using the relationship $V(x) = \min_u Q(x, u)$:

$$
\Pi^* = \Theta_{11}^* - \Theta_{12}^* (\Theta_{22}^*)^{-1} \Theta_{21}^*
$$

If the $t^{th}$ estimate of $Q^*$ is $Q_t(x, u) = [x^T, u^T]^T \Theta_t [x^T, u^T]$, then the greedy control action is given as

$$
u_t = \arg \min_u Q_t(x, u) = -\Theta_{21}^{-1} \Theta_{22}^{-1} x_t = -\Theta_{22}^{-1} \Theta_{21} x_t,
$$

where subscript $t$ of $\Theta$ has been omitted to improve readability.

The estimation error and the weight update are similar to the state-value case:

$$
\Pi_{t+1} = \Pi_t + \alpha_t \delta_t \cdot x_t x_t^T,
$$

where $\alpha_t$ is the learning rate.

The algorithm is summarized in Fig. [1]

2.2. Sarsa. The cost-to-go function is used to select control actions, so the action-value function $Q_t^*(x, u)$ is more appropriate for this purpose. The action-value function is defined as

$$
Q_t^*(x, u) := \inf_{u_1, u_2, ...} E[c(x_t, u_t) + c(x_{t+1}, u_{t+1}) + ... + c_f(x_f)|x_t = x, u_t = u],
$$

and analogously to $V_t^*$, it can be shown that it is time independent and can be written in the form

$$
Q_t(x, u) = (x^T, u^T) \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = (x^T, u^T) \Theta^* \begin{pmatrix} x \\ u \end{pmatrix}.
$$

Figure 1. TD(0) with linear function approximation for LQ control
Initialize $x_0$, $u_0$, $\Theta_0$

$z_0 = (x_0^T u_0^T)^T$

repeat

$x_{t+1} = Fx_t + Gu_t$

$\nu_{t+1} := \text{random noise}$

$u_{t+1} = - (\Theta_t)_{22} (\Theta_t)_{21} x_{t+1} + \nu_{t+1}$

$z_{t+1} = (x_{t+1}^T u_{t+1}^T)^T$

with probability $p$,

$$\delta_t = x_t^T Q f(x_t, u_t) - z_t^T \Theta_t z_t$$

STOP

else

$$\delta_t = u_t^T R u_t + z_{t+1}^T \Theta_t z_{t+1} - z_t^T \Theta_t z_t$$

$$\Theta_{t+1} = \Theta_t + \alpha_t \delta_t z_t z_t^T$$

$t = t + 1$

end

Figure 2. Sarsa(0) with linear function approximation for LQ control

3. Convergence

**Theorem 3.1.** If $\Pi_0 \geq \Pi^*$, there exists an $L$ such that $\|F + GL\| \leq 1/\sqrt{1-p}$, then there exists a series of learning rates $\alpha_t$ such that $0 < \alpha_t \leq 1/\|x_t\|^4$, $\sum_t \alpha_t = \infty$, $\sum_t \alpha_t^2 < \infty$, and it can be computed online. For all sequences of learning rates satisfying these requirements, Algorithm 1 converges to the optimal policy.

The proof of the theorem can be found in Appendix B.

The same line of thought can be carried over for the action-value function $Q(x, u) = (x^T u^T)^T \Theta(x^T u^T)$, which we do not detail here, giving only the result:

**Theorem 3.2.** If $\Theta_0 \geq \Theta^*$, there exists an $L$ such that $\|F + GL\| \leq 1/\sqrt{1-p}$, then there exists a series of learning rates $\alpha_t$ such that $0 < \alpha_t \leq 1/\|x_t\|^4$, $\sum_t \alpha_t = \infty$, $\sum_t \alpha_t^2 < \infty$, and it can be computed online. For all sequences of learning rates satisfying these requirements, Sarsa(0) with LFA (Fig. 2) converges to the optimal policy.

4. Kalman filter LQ control

Now let us examine the case when we do not know the exact states, but we have to estimate them from noisy observations. Consider a linear dynamical system with state $x_t \in \mathbb{R}^n$, control $u_t \in \mathbb{R}^m$, observation $y_t \in \mathbb{R}^k$, noises $\xi_t \in \mathbb{R}^n$ and $\zeta_t \in \mathbb{R}^k$ (which are assumed to be uncorrelated Gaussians with covariance matrix $\Omega^\xi$ and
Using the fact that $E \xi_t = 0$, respectively), in discrete time $t$:

\begin{align}
\mathbf{x}_{t+1} &= F \mathbf{x}_t + G \mathbf{u}_t + \xi_t \\
\mathbf{y}_t &= H \mathbf{x}_t + \zeta_t.
\end{align}

(14) (15)

Assume that the initial state has mean $\hat{x}_1$, and covariance $\Sigma_1$. Furthermore, assume that executing control step $\mathbf{u}_t$ in $\mathbf{x}_t$ costs

$$c(\mathbf{x}_t, \mathbf{u}_t) := x_t^T Q x_t + u_t^T R u_t,$$

(16)

After each time step, the process will be stopped with some fixed probability $\rho$, and then the controller incurs the final cost $c_f(x_f) := x_f^T Q_f x_f$.

We will show that the separation principle holds for our problem, i.e. the control law and the state filtering can be computed independently from each other. On one hand, state estimation is independent of the control selection method (in fact, the control could be anything, because it does not affect the estimation error), i.e. we can estimate the state of the system by the standard Kalman filtering equations:

\begin{align}
\hat{x}_{t+1} &= F \hat{x}_t + G \mathbf{u}_t + K_t (\mathbf{y}_t - H \hat{x}_t) \\
K_t &= F \Sigma_t H^T (H \Sigma_t H^T + \Omega_e)^{-1} \\
\Sigma_{t+1} &= \Omega_e + F \Sigma_t F^T - K_t H \Sigma_t F^T.
\end{align}

(17) (18) (19)

On the other hand, it is easy to show that the optimal control can be expressed as the function of $\hat{x}_t$. The proof (similarly to the proof of the original separation principle) is based on the fact that the noise and error terms appearing in the expressions are either linear and have zero mean or quadratic and independent of $\mathbf{u}$. In both cases they can be omitted. More precisely, let $W_t$ denote the sequence $y_1, \ldots, y_t, u_1, \ldots, u_{t-1}$, and let $e_t = x_t - \hat{x}_t$. Equation (6) for the filtered case can be formulated as

\begin{align}
\mathbf{u}_t &= \arg \min_{\mathbf{u}} E \left( c(\mathbf{x}_t, \mathbf{u}) + V_t (F \mathbf{x}_t + G \mathbf{u} + \xi_t) \big| W_t \right) \\
&= \arg \min_{\mathbf{u}} E \left( x_t^T Q x_t + u_t^T R u_t + (F \mathbf{x}_t + G \mathbf{u} + \xi_t)^T \Pi_t (F \mathbf{x}_t + G \mathbf{u} + \xi_t) \big| W_t \right).
\end{align}

Using the fact that $E(x_t^T Q x_t \big| W_t)$ and $E((\xi_t^T \Pi_t \xi_t) \big| W_t)$ are independent of $\mathbf{u}$ and that $E((F \mathbf{x}_t + G \mathbf{u})^T \Pi_t (F \mathbf{x}_t + G \mathbf{u}) \big| W_t) = 0$, furthermore that $\mathbf{x}_t = \hat{x}_t + e_t$, we get

$$\mathbf{u}_t = \arg \min_{\mathbf{u}} E \left( u_t^T R u_t + (F \mathbf{x}_t + F e_t + G \mathbf{u})^T \Pi_t (F \mathbf{x}_t + F e_t + G \mathbf{u}) \big| W_t \right)$$

Finally, we know that $E(e_t^2 \big| W_t) = 0$, because the Kalman filter is an unbiased estimator, furthermore $E(e_t^T \Pi_t e_t \big| W_t)$ is independent of $\mathbf{u}$, which yields

$$\mathbf{u}_t = \arg \min_{\mathbf{u}} E \left( u_t^T R u_t + (F \mathbf{x}_t + G \mathbf{u})^T \Pi_t (F \mathbf{x}_t + G \mathbf{u}) \big| W_t \right)$$

$$= -(R + G^T \Pi_t G)^{-1}(G^T \Pi_t F) \hat{x}_t,$$

i.e. for the computation of the greedy control action according to $V_t$ we can use the estimated state instead of the exact one. The proof of the separation principle for SARSA(0) is quite similar and therefore is omitted here.

The resulting algorithm using TD(0) is summarized in Fig. 3. The algorithm using Sarsa can be derived in a similar manner.
Initialize $x_0$, $\tilde{x}_0$, $u_0$, $\Pi_0$, $\Sigma_0$

repeat

\[ x_{t+1} = Fx_t + Gu_t + \xi_t \]
\[ y_t = Hx_t + \zeta_t \]
\[ \Sigma_{t+1} = \bar{\Sigma} + F\Sigma_t F^T - K_t H \Sigma_t F^T \]
\[ K_t = F \Sigma_t H^T (H \Sigma_t H^T + \Omega) \]
\[ \tilde{x}_{t+1} = F\tilde{x}_t + Gu_t + K_t (y_t - H\tilde{x}_t) \]
\[ \nu_{t+1} := \text{random noise} \]
\[ u_{t+1} = - (R + G^T \Pi_{t+1} G)^{-1} (G^T \Pi_{t+1} F) \tilde{x}_{t+1} + \nu_{t+1} \]
with probability $p,$
\[ \delta_t = \tilde{x}_t^T Q_f \tilde{x}_t - \tilde{x}_t^T \Pi_t \tilde{x}_t \]
STOP

else

\[ \delta_t = u_t^T R u_t + \tilde{x}_{t+1}^T \Pi_t \tilde{x}_{t+1} - \tilde{x}_t^T \Pi_t \tilde{x}_t \]
\[ \Pi_{t+1} = \Pi_t + \alpha_t \delta_t \tilde{x}_t \tilde{x}_t^T \]
\[ t = t + 1 \]

end

Figure 3. Kalman filtering with TD control

5. Acknowledgments

This work was supported by the Hungarian National Science Foundation (Grant No. T-32487). We would like to thank László Gerencsér for calling our attention to a mistake in the previous version of the convergence proof.

Appendix A. The boundedness of $\|x_t\|$ 

We need several technical lemmas to show that $\|x_t\|$ remains bounded for the linear-quadratic case, and also, $E(\|x_t\|)$ remains bounded for the Kalman filter case. The latter result implies that for the KF case, $\|x_t\|$ remains bounded with high probability.

For any positive semidefinite matrix $\Pi$ and any state $x$, we can define the action vector which minimizes the one-step-ahead value function:

\[ u_{\text{greedy}} := \arg\min_u \left( u^T R u + (Fx + Gu)^T \Pi (Fx + Gu) \right) \]

\[ = - (R + G^T \Pi G)^{-1} (G^T \Pi F) x. \]

Let

\[ L_\Pi := - (R + G^T \Pi G)^{-1} (G^T \Pi F) \]

denote the greedy control for matrix $\Pi$, and let

\[ L^* := - (R + G^T \Pi^* G)^{-1} (G^T \Pi^* F) \]

be the optimal policy, furthermore, let $q := 1/\sqrt{1-p}$.

Lemma A.1. If there exists an $L$ such that $\|F + GL\| < q$, then $\|F + GL^*\| < q$ as well.
Proof. Indirectly, suppose that \( \| F + GL^* \| \geq q \). Then for a fixed \( x_0 \), let \( x_t \) be the optimal trajectory

\[
x_{t+1} = (F + GL^*)x_t.
\]

Then

\[
V^*(x_0) = p c_f(x_0) + (1-p)c(x_0, L^*x_0)
+ (1-p)p c_f(x_1) + (1-p)^2c(x_1, L^*x_1)
+ (1-p)^2p c_f(x_2) + (1-p)^3c(x_2, L^*x_2)
+ \ldots,
\]

\[
V^*(x_0) \geq p(c_f(x_0) + (1-p)c_f(x_1) + (1-p)^2c_f(x_2) + \ldots)
= p \sum (1-p)^k x_0^T (F + GL^*)^k Q^T (F + GL^*)^k x_0.
\]

We know that \( Q^T \) is positive definite, so there exists an \( \epsilon \) such that \( x^T Q^T x \geq \epsilon \| x \|^2 \), therefore

\[
V^*(x_0) \geq \epsilon p \sum (1-p)^k \| (F + GL^*)^k x_0 \|^2.
\]

If \( x_0 \) is the eigenvector corresponding to the maximal eigenvalue of \( F + GL^* \), then \( (F + GL^*)x_0 = \| F + GL^* \| x_0 \), and so \( (F + GL^*)^k x_0 = \| F + GL^* \|^k x_0 \). Consequently,

\[
V^*(x_0) \geq \epsilon p \sum (1-p)^k \| F + GL^* \|^{2k} \| x_0 \|^2
\geq \epsilon p \sum (1-p)^k \frac{1}{(1-p)^k} \| x_0 \|^2 = \infty.
\]

On the other hand, because of \( \| F + GL \| < q \), it is easy to see that the value of following the control law \( L \) from \( x_0 \) is finite, therefore we get \( V^L(x_0) < V^*(x_0) \), which is a contradiction.

\[
\square
\]

Lemma A.2. For positive definite matrices \( A \) and \( B \), if \( A \geq B \) then \( \| A^{-1}B \| \leq 1. \)

Proof. Indirectly, suppose that \( \| A^{-1}B \| > 1 \). Let \( \lambda_{\text{max}} \) be the maximal eigenvalue of \( A^{-1}B \), and \( v \) be a corresponding eigenvector.

\[
A^{-1}Bv = \lambda_{\text{max}}v,
\]

and according to the indirect assumption,

\[
\lambda_{\text{max}} = \| A^{-1}B \| > 1.
\]

\( A \geq B \) means that for each \( x \), \( x^T Ax \geq x^T Bx \), so this holds specifically for \( x = A^{-1}Bv = \lambda_{\text{max}}v \), too. So, on one hand,

\[
(\lambda_{\text{max}}v)^T B(\lambda_{\text{max}}v) = \lambda_{\text{max}}^2 v^T Bv > v^T Bv,
\]

and on the other hand,

\[
(\lambda_{\text{max}}v)^T A(\lambda_{\text{max}}v) = (A^{-1}Bv)^T A(A^{-1}Bv) = v^T (BA^{-1}B)v,
\]

so,

\[
v^T (BA^{-1}B)v > v^T Bv.
\]

However, from \( A \geq B \), \( A^{-1} \leq B^{-1} \). Multiplying this with \( B \) from both sides, we get \( BA^{-1}B \leq B \), which is a contradiction. \( \square \)
Lemma A.3. If there exists an $L$ such that $\|F + GL\| < q$ then for any $\Pi$ such that $\Pi \geq \Pi^*$, $\|F + GL_\Pi\| < q$, too.

Proof. We will apply the Woodbury identity [6], stating that for positive definite matrices $R$ and $\Pi$,

$$(R + G^T \Pi G)^{-1} G^T \Pi = R^{-1} G^T (G R^{-1} G^T + \Pi^{-1})^{-1}$$

Consequently,

$$F + GL_\Pi = F - G(R + G^T \Pi G)^{-1}(G^T F) = F - (GR^{-1} G^T) \left((GR^{-1} G^T + \Pi^{-1})^{-1}\right) F.$$

Let

$$U_\Pi := I - (GR^{-1} G^T) \left((GR^{-1} G^T + \Pi^{-1})^{-1}\right)$$

and

$$U^* := I - (GR^{-1} G^T) \left((GR^{-1} G^T + (\Pi^*)^{-1})^{-1}\right)$$

Both matrices are positive definite, because they are the product of positive definite matrices. With these notations, $F + GL_\Pi = U_\Pi F$ and $F + GL^* = U^* F$.

It is easy to show that $U_\Pi \leq U^*$ exploiting the fact that $\Pi \geq \Pi^*$ and several well-known properties of matrix inequalities: if $A \geq B$ and $C$ is positive semidefinite, then $-A \leq -B$, $A^{-1} \leq B^{-1}$, $A + C \geq B + C$, $A \cdot C \geq B \cdot C$.

From Lemma A.3 we know that $\|U^* F\| = \|F + GL^*\| < q$, and from the previous lemma we know that $\|U_\Pi (U^*)^{-1}\| \leq 1$, so

$$\|F + GL_\Pi\| = \|U_\Pi F\| = \|U_\Pi(U^*)^{-1} U^* F\| \leq \|U_\Pi(U^*)^{-1}\| \|U^* F\| \leq 1 \cdot q$$

□

Corollary A.4. If there exists an $L$ such that $\|F + GL\| \leq q$, then the state sequence generated by the noise-free LQ equations is bounded, i.e., there exists $M \in \mathbb{R}$ such that $\|x_t\| \leq M$.

Proof. This is a simple corollary of the previous lemma: in each step we use a greedy control law $L_t$, so

$$\|x_{t+1}\| = \|(F + GL_t)x_t\| \leq q \|x_t\|$$

□

Corollary A.5. If there exists an $L$ such that $\|F + GL\| \leq q$, then the state sequence generated by the Kalman-filter equations is bounded with high probability, i.e., for any $\epsilon > 0$, there exists $M \in \mathbb{R}$ such that $\|x_t\| \leq M$ with probability $1 - \epsilon$.

Proof.

$$E \|x_{t+1}\| = E \|(F + GL_t)x_t + \xi_t\| \leq E \|G(L_t)x_t + \Omega_\xi\|$$

$$\leq \sqrt{qE \|x_t\| + \Omega_\xi},$$
so there exists a bound $M'$ such that $E \|x_t\| \leq M'$. From Markov’s inequality,

$$\Pr(\|x_t\| > M'/e) < e,$$

therefore, $M = M'/e$ satisfies our requirements. □

APPENDIX B. THE PROOF OF THE MAIN THEOREM

We will use the following lemma:

**Lemma B.1.** Let $J$ be a differentiable function, bounded from below by $J^*$, and let $\nabla J$ be Lipschitz-continuous. Suppose the weight sequence $w_t$ satisfies

$$w_{t+1} = w_t + \alpha_t b_t$$

for random vectors $b_t$ independent of $w_{t+1}, w_{t+2}, \ldots$, and $b_t$ is a descent direction for $J$, i.e. $E(b_t|w_t)^T \nabla J(w_t) \leq -\delta(t) < 0$ whenever $J(w_t) > J^* + \epsilon$. Suppose also that

$$E(\|b_t\|^2|w_t) \leq K_1 J(w_t) + K_2 E(b_t|w_t)^T \nabla J(w_t) + K_3$$

and finally that the constants $\alpha_t$ satisfy $\alpha_t > 0$, $\sum \alpha_t = \infty$, $\sum \alpha_t^2 < \infty$. Then $J(w_t) \to J^*$ with probability 1.

In our case, the weight vectors are $n \times n$ dimensional, with $w_{n+t+j} := \Pi_{ij}$. For the sake of simplicity, we denote this by $w_{(ij)}$. Let $w^*$ be the weight vector corresponding to the optimal value function, and let

$$J(w) = \frac{1}{2} \|w - w^*\|^2.$$

**Theorem B.2** (Theorem 3.1). If $\Pi_0 \geq \Pi^*$, there exists an $L$ such that $\|F + GL\| \leq q$, then there exists a series of learning rates $\alpha_t$ such that $0 < \alpha_t \leq 1/\|x_t\|^4$, $\sum \alpha_t = \infty$, $\sum \alpha_t^2 < \infty$, and it can be computed online. For all sequences of learning rates satisfying these requirements, Algorithm 1 converges to the optimal policy.

**Proof.** First of all, we prove the existence of a suitable learning rate sequence. Let $\alpha'_t$ be a sequence of learning rates that satisfy two of the requirements, $\sum \alpha_t = \infty$ and $\sum \alpha_t^2 < \infty$. Fix a probability $0 < \epsilon < 1$. By the previous lemma, there exists a bound $M$ such that $\|x_t\| \leq M$ with probability $1 - \epsilon$. The learning rates

$$\alpha_t := \min\{\alpha'_t, 1/\|x_t\|^4\}$$

will be satisfactory, and can be computed on the fly. The first and third requirements are trivially satisfied, so we only have to show that $\sum \alpha_t = \infty$. Consider the index set $H = \{t : \alpha'_t \leq 1/M^4\} \cup \{t : \alpha'_t \leq 1/\|x_t\|^4\}$. By the first condition only finitely many indices are excluded. The second condition excludes indices with $1/M^4 < \alpha'_t < 1/\|x_t\|^4$, which happens at most with probability $\epsilon$. However,

$$\sum \alpha_t \geq \sum_{t \in H} \alpha_t = \sum_{t \in H} \alpha'_t = \infty.$$

The last equality holds, because if we take a divergent sum of nonnegative terms, and exclude finitely many terms or an index set with density less than 1, then the remaining subseries will remain divergent.
Subtracting this from Eq. (22), we get

We know that the optimal value function satisfies the fixed-point equation

This yields

Clearly, \( A_t \leq 1 \). In fact, it will be one most of the time, and will damp only the samples that are too big.

We will show that

is a descent direction for every \( t \).

For the sake of simplicity, from now on we do not note the dependence on \( w_t \) explicitly.

We will show that for all \( t \), \( E(\Pi_t) > 0 \), \( E(\Pi_{t-1}) > E(\Pi_t) \) and \( E(\delta_t) \leq -p x_t^T (\Pi_t - \Pi^*) x_t \). We proceed by induction.

- \( t = 0 \), \( \Pi_0 > \Pi^* \) holds by assumption.
- Induction step part 1: \( E(\delta_t) \leq -p x_t^T (\Pi_t - \Pi^*) x_t \).

Recall that

\[
\begin{align*}
\Pi_t &= \min_{\Pi_t} \{ 1, 1/(\alpha_t^i \| x_t \|^2) \} \\
A_t &= \alpha_t / \alpha_t^i = \min\{ 1, 1/(\alpha_t^i \| x_t \|^2) \}
\end{align*}
\]

\( \Pi_t = A_t \delta_t x_t x_t^T \).
Let $\epsilon_1 = \epsilon_1(p) := 1/(1 - p) - \| F + GL^* \|^2 > 0$. Inequality (24) implies

\begin{equation}
E(\delta_t) \leq (1 - p) \left( \frac{1}{1 - p} - \epsilon_1(p) \right) x_t^T (\Pi_t - \Pi^*) x_t - x_t^T (\Pi_t - \Pi^*) x_t
\end{equation}

\begin{equation}
= -(1 - p) \epsilon_1(p) x_t^T (\Pi_t - \Pi^*) x_t,
\end{equation}

\begin{equation}
= -\epsilon_2(p) x_t^T (\Pi_t - \Pi^*) x_t,
\end{equation}

where we defined $\epsilon_2(p) = (1 - p) \epsilon_1(p)$.

- **Induction step part 2:** $E(\Pi_{t+1}) > \Pi^*$.

\begin{equation}
E(\delta_t) = p c_f(x_t) + (1 - p) \left( c(x_t, L_t x_t) + V_t(F x_t + GL_t x_t) \right) - V_t(x_t)
\end{equation}

\begin{equation}
\geq p c_f(x_t) + (1 - p) \left( c(x_t, L_t x_t) + V^*(F x_t + GL_t x_t) \right) - V_t(x_t).
\end{equation}

Subtracting eq. (24) we get

\begin{equation}
E(\delta_t) \geq (1 - p) \left( \left( c(x_t, L_t x_t) + V^*(F x_t + GL_t x_t) \right) - \left( c(x_t, L_t x_t) + V^*(F x_t + GL_t x_t) \right) \right) + V^*(x_t) - V_t(x_t)
\end{equation}

\begin{equation}
\geq V^*(x_t) - V_t(x_t) \geq -\|\Pi_t - \Pi^*\| \|x_t\|^2.
\end{equation}

Therefore

\begin{equation}
E(\Pi_{t+1}) - \Pi^* \geq \Pi_t + \alpha_t A_t E(\delta_t)x_t x_t^T - \Pi^*
\end{equation}

\begin{equation}
\geq (\Pi_t - \Pi^*) - \alpha_t \|x_t\|^4 \|\Pi_t - \Pi^*\| I
\end{equation}

\begin{equation}
\geq (\Pi_t - \Pi^*) - \|\Pi_t - \Pi^*\| I > 0.
\end{equation}

- **Induction step part 3:** $\Pi_t > E(\Pi_{t+1})$.

\begin{equation}
\Pi_t - E(\Pi_{t+1}) = -\alpha_t A_t E(\delta_t)x_t x_t^T \geq \alpha_t \epsilon_2(p) x_t^T (\Pi_t - \Pi^*) x_t \cdot x_t x_t^T,
\end{equation}

but $\alpha_t \epsilon_2(p) > 0$, $x_t^T (\Pi_t - \Pi^*) x_t > 0$ and $x_t x_t^T > 0$, so their product is positive as well.

The induction is therefore complete.

We finish the proof by showing that the assumptions of Lemma 3.1 hold:

$b_t$ is a descent direction. Clearly, if $J(w_t) \geq \epsilon$, then $\|\Pi_t - \Pi^*\| \geq \epsilon_3(\epsilon)$, but $\Pi_t - \Pi^*$ is positive definite, so $\Pi_t - \Pi^* \geq \epsilon_3(\epsilon) I$.

\begin{equation}
E(b_t|w_t)^T \nabla J(w_t) = A_t E(\delta_t|w_t) (V_t(x_t) - V^*(x_t))
\end{equation}

\begin{equation}
\leq -\epsilon_2(p) A_t x_t^T (\Pi_t - \Pi^*) x_t \cdot x_t x_t^T (\Pi_t - \Pi^*) x_t
\end{equation}

\begin{equation}
\leq -\epsilon_2^2(\Pi_t - \Pi^*) \|x_t\|^4
\end{equation}

\begin{equation}
\leq -\epsilon_2^2 \min\{|x_t|^4, 1/\alpha_t^2\}
\end{equation}
$E(\|b_t\|^2 | w_t)$ is bounded. $|E(\delta_t)| \leq |x_t^T(\Pi_t - \Pi^*)x_t|$. Therefore

$$
E(\|b_t\|^2 | w_t) \leq |A_t|^2 |E(\delta_t)|^2 \|x_t\|^2 \\
\leq \|\Pi_t - \Pi^*\|^2 \cdot \min\{1, 1/(\alpha_t^2 \|x_t\|^8)\} \cdot \|x_t\|^6 \\
\leq \|\Pi_t - \Pi^*\|^2 \cdot \min\{\|x_t\|^6, 1/(\alpha_t^2 \|x_t\|^2)\} \\
\leq K \cdot J(w_t).
$$

Consequently, The assumptions of lemma [3.1] hold, so the algorithm converges to the optimal value function with probability 1. \qed

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