Remarks on the Warped Deformed Conifold *

Christopher P. Herzog, Igor R. Klebanov and Peter Ouyang
Department of Physics, Princeton University
Princeton, NJ 08544, USA

October 30, 2018

Abstract

We assemble a few remarks on the supergravity solution of hep-th/0007191, whose UV asymptotic form was previously found in hep-th/0002159. First, by normalizing the R-R fluxes, we compare the logarithmic flow of couplings in supergravity with that in field theory, and find exact agreement. We also write the 3-form field strength $G_3 = F_3 - \tau H_3$ present in the solution in a manifestly $SO(4)$ invariant $(2, 1)$ form. In addition, we discuss various issues related to the chiral symmetry breaking and wrapped branes.

*Based on I.R.K.’s talks at the Lisbon School on Superstrings II, July 13–17, 2001 and at the Benasque Workshop “Physics in the Pyrenees: Strings, Branes and Field Theory,” July 15–27, 2001.
1 Introduction

The warped deformed conifold [1] is a solution of type IIB supergravity that is dual to a certain $\mathcal{N} = 1$ supersymmetric $SU(N + M) \times SU(N)$ gauge theory in the limit of strong ’t Hooft coupling. This solution encodes various interesting gauge theory phenomena in a dual geometrical language, such as the duality cascade in the UV and chiral symmetry breaking and confinement in the IR.

In these notes we assemble a few remarks on the solution of [1], whose UV asymptotic form was previously found in [2]. Our intention is to compromise between presenting a self-contained review of supergravity conifold solutions and their field theory duals, and presenting three or four new results that we believe to be of general interest. A summary of the new results follows.

First, by normalizing the R-R fluxes, we compare the logarithmic flow of couplings in supergravity with that in field theory, and find exact agreement.\(^1\) Next, we show that the 3-form field strength $G_3 = F_3 - \tau H_3$ present in the solution is an $SO(4)$ invariant $(2,1)$ form on the deformed conifold. (It was shown previously that $G_3$ must be $(2,1)$ in order to preserve SUSY [6], [7], [8].) Although $G_3$ has appeared explicitly in the literature before [9], the basis in which we write $G_3$ and some other differential forms important to the supergravity solutions of [2] and [1] is a particularly simple one in which many of the properties of these forms become completely obvious.

We also discuss various issues related to the chiral symmetry breaking and wrapped branes. For example, we develop the gauge field/string dictionary for this system by showing the correspondence between a certain wrapped D5-brane in supergravity and domain walls in the field theory that interpolate between inequivalent vacua. Finally, in the style of [10], we discuss various UV/IR relations for the conifold.

We now review some basic facts about the AdS/CFT correspondence first because we would like these notes to be self-contained and second because it is important to understand the normalizations here in view of the more complicated solutions ahead. The duality between $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory and the $AdS_5 \times S^5$ background of type IIB string theory [11, 12, 13] is usually motivated by considering a stack of a large number $N$ of D3-branes. The SYM theory is the low-energy limit of the gauge theory on the stack of D3-branes.

On the other hand, the curved background produced by the stack is

$$ds^2 = h^{-1/2} \left( -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + h^{1/2} \left( dr^2 + r^2 d\Omega_5^2 \right), \quad (1)$$

\(^1\)For $\mathcal{N} = 2$ supersymmetric gauge theories realized on fractional branes at orbifold singularities, the agreement of supergravity with field theory $\beta$-functions was demonstrated in [14, 15].
where $d\Omega_5^2$ is the metric of a unit 5-sphere and
\[ h(r) = 1 + \frac{L^4}{r^4}. \]  
This 10-dimensional metric may be thought of as a “warped product” of the $\mathbb{R}^{3,1}$ along the branes and the transverse space $\mathbb{R}^6$. Note that the dilaton, $\Phi = 0$, is constant, and the selfdual 5-form field strength is given by
\[ F_5 = F_5 + \star F_5, \quad \mathcal{F}_5 = 16\pi (\alpha')^2 N \text{vol}(S^5). \]

The normalization above is dictated by the quantization of D$p$-brane tension which implies
\[ \int_{S^{p-6}} \star F_{p+2} = \frac{2\kappa^2 \tau_p N}{g_s}, \]
where
\[ \tau_p = \frac{\sqrt{\pi}}{\kappa} (4\pi^2 \alpha')^{(3-p)/2} \]
and $\kappa = 8\pi^{7/2} g_s \alpha'^2$ is the 10-dimensional gravitational constant. In particular, for $p = 3$ we have
\[ \int_{S^5} F_5 = (4\pi^2 \alpha')^2 N, \]
which is consistent with (3) since the volume of a unit 5-sphere is $\text{Vol}(S^5) = \pi^3$.

Note that the 5-form field strength may also be written as
\[ g_s F_5 = d^4 x \wedge dh^{-1} - r^5 \frac{dh}{dr} \text{vol}(S^5). \]
Then it is not hard to see that the Einstein equation $R_{MN} = g_s^2 F_M^{PQRS} F_N^{PQRS}/96$ is satisfied. Since $-r^5 \frac{dh}{dr} = 4L^4$, we find by comparing with (3) that
\[ L^4 = 4\pi g_s N \alpha'^2. \]

A related way to determine the scale factor $L$ is to equate the ADM tension of the supergravity solution with $N$ times the tension of a single D3-brane [14]:
\[ \frac{2}{\kappa^2} L^4 \text{Vol}(S^5) = \frac{\sqrt{\pi}}{\kappa} N, \]
This way we find
\[ L^4 = \frac{\kappa N}{2 \pi^{5/2}} = 4\pi g_s N \alpha'^2 \]
in agreement with the preceding paragraph.
The radial coordinate $r$ is related to the scale in the dual gauge theory. The low-energy limit corresponds to $r \to 0$. In this limit the metric becomes

$$ds^2 = \frac{L^2}{z^2} (-dt^2 + dx^2 + dz^2) + L^2 d\Omega_5^2,$$  \hspace{1cm} (11)

where $z = \frac{L^2}{r}$. This describes the direct product of 5-dimensional Anti-de Sitter space, $AdS_5$, and the 5-dimensional sphere, $S^5$, with equal radii of curvature $L$.

An interesting generalization of the basic AdS/CFT correspondence [11, 12, 13] is found by studying branes at conical singularities [15, 16, 17, 18]. Consider a stack of D3-branes placed at the apex of a Ricci-flat 6-d cone $Y_6$ whose base is a 5-d Einstein manifold $X_5$. Comparing the metric with the D-brane description leads one to conjecture that type IIB string theory on $AdS_5 \times X_5$ is dual to the low-energy limit of the world volume theory on the D3-branes at the singularity. The equality of tensions now requires [19]

$$L^4 = \frac{\sqrt{\pi \kappa N}}{2 \text{Vol}(X_5)} = 4\pi g_s N \alpha'^2 \frac{\pi^3}{\text{Vol}(X_5)},$$ \hspace{1cm} (12)

an important normalization formula which we will use in the following subsection.

The simplest examples of $X_5$ are the orbifolds $S^5/\Gamma$ where $\Gamma$ is a discrete subgroup of $SO(6)$ [15]. In these cases $X_5$ has the local geometry of a 5-sphere. The dual gauge theory is the IR limit of the world volume theory on a stack of $N$ D3-branes placed at the orbifold singularity of $\mathbb{R}^6/\Gamma$. Such theories typically involve product gauge groups $SU(N)^k$ coupled to matter in bifundamental representations [20].

Constructions of the dual gauge theories for Einstein manifolds $X_5$ which are not locally equivalent to $S^5$ are also possible. The simplest example is the Romans compactification on $X_5 = T^{1,1} = (SU(2) \times SU(2))/U(1)$ [21, 17]. The dual gauge theory is the conformal limit of the world volume theory on a stack of $N$ D3-branes placed at the singularity of a Calabi-Yau manifold known as the conifold [17], which is a cone over $T^{1,1}$. Let us explain this connection in more detail.

### 1.1 D3-branes on the Conifold

The conifold may be described by the following equation in four complex variables,

$$\sum_{a=1}^{4} z_a^2 = 0.$$ \hspace{1cm} (13)

Since this equation is invariant under an overall real rescaling of the coordinates, this space is a cone. Remarkably, the base of this cone is precisely the space $T^{1,1}$ [22, 17].
In fact, the metric on the conifold may be cast in the form \[ ds_6^2 = dr^2 + r^2 ds_{T^{1,1}}^2 , \] (14)

where

\[
ds_{T^{1,1}}^2 = \frac{1}{9} \left( d\psi + \sum_{i=1}^{2} \cos \theta_i d\phi_i \right)^2 + \frac{1}{6} \sum_{i=1}^{2} \left( d\theta_i^2 + \sin^2 \theta_i d\phi_i^2 \right) . \]

(15)

is the metric on \( T^{1,1} \). Here \( \psi \) is an angular coordinate which ranges from 0 to 4\( \pi \), while \( (\theta_1, \phi_1) \) and \( (\theta_2, \phi_2) \) parametrize two \( S^2 \)'s in a standard way. Therefore, this form of the metric shows that \( T^{1,1} \) is an \( S^1 \) bundle over \( S^2 \times S^2 \).

Now placing \( N \) D3-branes at the apex of the cone we find the metric

\[
ds^2 = \left( 1 + \frac{L^4}{r^4} \right)^{-1/2} \left( -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \left( 1 + \frac{L^4}{r^4} \right)^{1/2} \left( dr^2 + r^2 ds_{T^{1,1}}^2 \right) \] (16)

whose near-horizon limit is \( AdS_5 \times T^{1,1} \). Using the metric (15) it is not hard to find that the volume of \( T^{1,1} \) is \( \frac{16\pi^3}{27} \). From (12) it then follows that

\[
L^4 = 4\pi g_s N (\alpha')^2 \frac{27}{16} = \frac{27\kappa N}{32\pi^{5/2}} . \]

(17)

The same logic that leads us to the \( \mathcal{N} = 4 \) version of the AdS/CFT correspondence now shows that the type IIB string theory on this space should be dual to the infrared limit of the field theory on \( N \) D3-branes placed at the singularity of the conifold. Since Calabi-Yau spaces preserve 1/4 of the original supersymmetries this should be an \( \mathcal{N} = 1 \) superconformal field theory. This field theory was constructed in \( [17] \): it is \( SU(N) \times SU(N) \) gauge theory coupled to two chiral superfields, \( A_i \), in the \( (\mathbf{N}, \mathbf{N}) \) representation and two chiral superfields, \( B_j \), in the \( (\overline{\mathbf{N}}, \mathbf{N}) \) representation.

In order to match the two gauge couplings to the moduli of the type IIB theory on \( AdS_5 \times T^{1,1} \), one notes that the integrals over the \( S^2 \) of \( T^{1,1} \) of the NS-NS and R-R 2-form potentials, \( B_2 \) and \( C_2 \), are moduli. In particular, the two gauge couplings are determined as follows \( [17] \) \( [18] \):^2

\[
\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi}{g_s e^\Phi} , \]

(18)

\[
\left[ \frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2} \right] g_s e^\Phi = \frac{1}{2\pi\alpha'} \left( \int_{S^2} B_2 \right) - \pi \quad (\text{mod } 2\pi) . \]

(19)

These equations are crucial for relating the SUGRA background to the field theory \( \beta \)-functions when the theory is generalized to \( SU(N+M) \times SU(N) \) \( [3] [2] \). From the

^2Exactly the same relations apply to the \( \mathcal{N} = 2 \) supersymmetric \( \mathbb{Z}_2 \) orbifold theory \( [14] [4] \).
quantization condition on $H_3$, $\frac{1}{2\pi\alpha'}(\int_{S^2} B_2)$ must be a periodic variable with period $2\pi$. This periodicity is crucial for the cascade phenomenon that we discuss in the next section.

2 The RG cascade

The addition of $M$ fractional 3-branes (wrapped D5-branes) at the singular point changes the gauge group to $SU(N + M) \times SU(N)$. Let us consider the effect on the dual supergravity background of adding $M$ wrapped D5-branes. The D5-branes serve as sources of the magnetic RR 3-form flux through the $S^3$ of $T^{1,1}$. Therefore, the supergravity dual of this field theory involves $M$ units of the 3-form flux, in addition to $N$ units of the 5-form flux:

$$\frac{1}{4\pi^2\alpha'} \int_{S^3} F_3 = M, \quad \frac{1}{(4\pi^2\alpha')^2} \int_{T^{1,1}} F_5 = N.$$ (20)

The coefficients above follow from the quantization rule (4). The warped conifold solution with such fluxes was constructed in [2]. The warped conifold solution with such fluxes was constructed in [2].

It will be useful to employ the following basis of 1-forms on the compact space [24]:

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}},$$
$$g^3 = \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}},$$
$$g^5 = e^5,$$ (21)

where

$$e^1 \equiv - \sin \theta_1 d\phi_1, \quad e^2 \equiv d\theta_1,$$
$$e^3 \equiv \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2,$$
$$e^4 \equiv \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2,$$
$$e^5 \equiv d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2.$$ (22)

In terms of this basis, the Einstein metric on $T^{1,1}$ assumes the form

$$ds_{T^{1,1}}^2 = \frac{1}{9}(g^5)^2 + \frac{1}{6} \sum_{i=1}^4 (g^i)^2.$$ (23)
Keeping track of the normalization factors, in order to be consistent with the quantization conditions (20),
\[
F_3 = \frac{M\alpha'}{2}\omega_3 , \quad B_2 = \frac{3g_s M\alpha'}{2} \omega_2 \ln(r/r_0) ,
\]
(24)
\[
H_3 = dB_2 = \frac{3g_s M\alpha'}{2r} dr \wedge \omega_2 ,
\]
(25)
where
\[
\omega_2 = \frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4) = \frac{1}{2}(\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2) ,
\]
(26)
\[
\omega_3 = \frac{1}{2}g^5 \wedge (g^1 \wedge g^2 + g^3 \wedge g^4) .
\]
(27)
In Appendix A we show that
\[
\int_{S^2} \omega_2 = 4\pi , \quad \int_{S^3} \omega_3 = 8\pi^2
\]
(28)
where the $S^2$ is parametrized by $\psi = 0$, $\theta_1 = \theta_2$ and $\phi_1 = -\phi_2$, and the $S^3$ by $\theta_2 = \phi_2 = 0$. As a result, the quantization condition for RR 3-form flux is obeyed.

Both $\omega_2$ and $\omega_3$ are closed. Note also that
\[
g_s \ast_6 F_3 = H_3 , \quad g_s F_3 = -\ast_6 H_3 ,
\]
(29)
where $\ast_6$ is the Hodge dual with respect to the metric $ds_6^2$. Thus, the complex 3-form $G_3$ satisfies the self-duality condition
\[
\ast_6 G_3 = iG_3 , \quad G_3 = F_3 - \frac{i}{g_s} H_3 .
\]
(30)
Note that the self-duality fixes the relative factor of 3 in (24) (see (14), (15)). We will see that this geometrical factor is crucial for reproducing the well-known factor of 3 in the $\mathcal{N} = 1$ beta functions.

It follows from (29) that
\[
g_s^2 F_3^2 = H_3^2 ,
\]
(31)
which implies that the dilaton is constant, $\Phi = 0$. Since $F_3^{\mu\nu\lambda}H_3^{\mu\nu\lambda} = 0$, the RR scalar vanishes as well.

The 10-d metric found in [2] has the structure of a “warped product” of $\mathbb{R}^{3,1}$ and the conifold:
\[
ds_{10}^2 = h^{-1/2}(r)dx_\mu dx_\nu + h^{1/2}(r)(dr^2 + r^2 ds_{T^{1,1}}^2) .
\]
(32)
The solution for the warp factor $h$ may be determined from the trace of the Einstein equation:

$$R = \frac{1}{24}(H_3^2 + g_s F_5^2) = \frac{1}{12}H_3^2 .$$  

(33)

This implies

$$-h^{-3/2} \frac{d}{dr}(r^5 h') = \frac{1}{6}H_3^2 .$$  

(34)

Integrating this differential equation, we find that

$$h(r) = \frac{27\pi(\alpha')^2[g_s N + a(g_s M)^2 \ln(r/r_0) + a(g_s M)^2/4]}{4r^4}$$

(35)

with $a = 3/(2\pi)$.

An important feature of this background is that $\tilde{F}_5$ acquires a radial dependence [2]. This is because

$$\tilde{F}_5 = F_5 + B_2 \wedge F_3 , \quad F_5 = dC_4 ,$$

(36)

and $\omega_2 \wedge \omega_3 = 54\text{vol}(T^{1,1})$. Thus, we may write

$$\tilde{F}_5 = \mathcal{F}_5 + \ast \mathcal{F}_5 , \quad \mathcal{F}_5 = 27\pi\alpha'^2 N_{\text{eff}}(r)\text{vol}(T^{1,1}) ,$$

(37)

and

$$N_{\text{eff}}(r) = N + \frac{3}{2\pi}g_s M^2 \ln(r/r_0) .$$

(38)

The novel phenomenon in this solution is that the 5-form flux present at the UV scale $r = r_0$ may completely disappear by the time we reach a scale where $N_{\text{eff}} = 0$. The non-conservation of the flux is due to the type IIB SUGRA equation

$$d\tilde{F}_5 = H_3 \wedge F_3 .$$

(39)

A related fact is that $\int_{S^2} B_2$ is no longer a periodic variable in the SUGRA solution once the $M$ fractional branes are introduced: as the $B_2$ flux goes through a period, $N_{\text{eff}}(r) \rightarrow N_{\text{eff}}(r) - M$ which has the effect of decreasing the 5-form flux by $M$ units. Note from [38] that for a single cascade step $N_{\text{eff}}(r) \rightarrow N_{\text{eff}}(r) - M$ the radius changes by a factor $r_2/r_1 = \exp(-2\pi/3g_s M)$, agreeing with a result of [25].

Due to the non-vanishing RHS of (39), $\frac{1}{(4\pi^2\alpha')^2} \int_{T^{1,1}} \tilde{F}_5$ is not quantized. We may identify this quantity with $N_{\text{eff}}$ defining the gauge group $SU(N_{\text{eff}} + M) \times SU(N_{\text{eff}})$ only at special radii $r_k = r_0 \exp(-2\pi k/3g_s M)$ where $k$ is an integer. Thus, $N_{\text{eff}} = N - kM$. Furthermore, we believe that the continuous logarithmic variation of $N_{\text{eff}}(r)$ is related to continuous reduction in the number of degrees of freedom as the theory flows to the IR. Some support for this claim comes from studying the high-temperature phase of this theory using black holes embedded into asymptotic KT geometry [20].
The effective number of degrees of freedom computed from the Bekenstein–Hawking entropy grows logarithmically with the temperature, in agreement with (38).

The metric (32) has a naked singularity at \( r = r_s \) where \( h(r_s) = 0 \). Writing

\[ h(r) = \frac{L^4}{r^4} \ln(r/r_s), \quad L^2 = \frac{9g_s M_0'}{2\sqrt{2}}, \]  

we find a purely logarithmic RG cascade:

\[ ds^2 = \frac{r^2}{L^2 \sqrt{\ln(r/r_s)}} dx_n dx_n + \frac{L^2 \sqrt{\ln(r/r_s)}}{r^2} dr^2 + L^2 \sqrt{\ln(r/r_s)} ds^2_{T=1,1}. \]

Since \( T_{1,1} \) expands slowly toward large \( r \), the curvatures decrease there so that corrections to the SUGRA become negligible. Therefore, even if \( g_s M \) is very small, this SUGRA solution is reliable for sufficiently large radii where \( g_s N_{\text{eff}}(r) \gg 1 \). In this regime the separation between the cascade steps is very large, so that the SUGRA calculation of the \( \beta \)-functions may be compared with \( SU(N_{\text{eff}} + M) \times SU(N_{\text{eff}}) \) gauge theory. We will work near \( r = r_0 \) where \( N_{\text{eff}} \) may be replaced by \( N \).

### 2.1 Matching of the \( \beta \)-functions

In gauge/gravity duality the 5-dimensional radial coordinate defines the RG scale of the dual gauge theory \([11, 12, 13, 27, 10]\). There are different ways of establishing the precise relation. The simplest one is to identify the field theory energy scale \( \Lambda \) with the energy of a stretched string ending on a probe brane positioned at radius \( r \). For all metrics of the form (32) this gives

\[ \Lambda \sim r. \] (42)

In this section we adopt this UV/IR relation, which typically corresponds to the Wilsonian renormalization group. We will discuss UV/IR relations in more detail in Section 5.

Now we are ready to interpret the solution of [2] in terms of RG flow in the dual \( SU(N + M) \times SU(N) \) gauge theory. The constancy of the dilaton translates into the vanishing of the \( \beta \)-function for \( \frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} \). Substituting the solution for \( B_2 \) into (19) we find

\[ \frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = 6M \ln(r/r_s) + \text{const}. \] (43)

Since \( \ln(r/r_s) = \ln(\Lambda/\mu) \), (43) implies a logarithmic running of \( \frac{1}{g_1^2} - \frac{1}{g_2^2} \) in the \( SU(N + M) \times SU(N) \) gauge theory. As we mentioned earlier, this SUGRA result is reliable...
for any value of $g_s M$ provided that $g_s N \gg 1$. We may consider $g_s M \ll 1$ so that the cascade jumps are well-separated.

Let us compare with the Shifman–Vainshtein $\beta$-functions \[28\]

\[
\frac{d}{d\log(\Lambda/\mu)} \frac{8\pi^2}{g_1^2} = 3(N + M) - 2N(1 - \gamma), \quad (44)
\]

\[
\frac{d}{d\log(\Lambda/\mu)} \frac{8\pi^2}{g_2^2} = 3N - 2(N + M)(1 - \gamma), \quad (45)
\]

where $\gamma$ is the anomalous dimension of operators $\text{Tr}A_i B_j$. The conformal invariance of the field theory for $M = 0$, and symmetry under $M \to -M$, require that $\gamma = -\frac{1}{2} + O[(M/N)^{2n}]$ where $n$ is a positive integer \[1\]. Taking the difference of the two equations in (44) we then find

\[
\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = M \ln(\Lambda/\mu)[3 + 2(1 - \gamma)] = 6M \ln(\Lambda/\mu)(1 + O[(M/N)^{2n}]). \quad (46)
\]

Remarkably, the coefficient $6M$ is in exact agreement with the result (43) found on the SUGRA side. This constitutes a geometrical explanation of a field theory $\beta$-function, including its normalization.

We may also trace the jumps in the rank of the gauge group to a well-known phenomenon in the dual $\mathcal{N} = 1$ field theory, namely, Seiberg duality \[31\]. The essential observation is that $1/g_1^2$ and $1/g_2^2$ flow in opposite directions and, according to (44), there is a scale where the $SU(N + M)$ coupling, $g_1$, diverges. To continue past this infinite coupling, we perform a $\mathcal{N} = 1$ duality transformation on this gauge group factor. The $SU(N + M)$ gauge factor has $2N$ flavors in the fundamental representation. Under a Seiberg duality transformation, this becomes an $SU(2N - [N + M]) = SU(N - M)$ gauge group. Thus we obtain an $SU(N) \times SU(N - M)$ theory which resembles closely the theory we started with \[1\].

As the theory flows to the IR, the cascade must stop, however, because negative $N$ is physically nonsensical. Thus, we should not be able to continue the solution (41) to the region where $N_{\text{eff}}$ is negative. To summarize, the fact that the solution of \[2\] is singular tells us that it has to be modified in the IR.

---

3These expressions for the $\beta$-functions differ from the standard NSVZ form \[29\] by a factor of $1/(1 - g^2 N_c/8\pi^2)$. The difference comes from the choice of normalization of the vector superfields. We choose the normalization so that the relevant kinetic term in the field theory action is $\frac{1}{2g^2} \int d^4x d^2\theta \text{Tr}(W^\alpha W_\alpha) + \text{h.c.}$; this choice is dictated by the form of the supergravity action and differs from the canonical normalization by a factor of $1/g^2$. With this convention the additional factor in the $\beta$-function does not appear. A nice review of the derivation of the exact $\beta$-functions is in \[30\].
3 Deformation of the Conifold

It was shown in [1] that, to remove the naked singularity found in [2] the conifold (13) should be replaced by the deformed conifold

\[ \sum_{i=1}^{4} \xi_i^2 = \varepsilon^2 , \]  

(47)
in which the singularity of the conifold is removed through the blowing-up of the \( S^3 \) of \( T^{1,1} \). We now review the deformed conifold in order to be able to normalize properly the field strengths and to prepare for a discussion of a new and simple \( SO(4) \) invariant way of writing the field strengths. The 10-d metric of [1] takes the following form:

\[ ds_{10}^2 = h^{-1/2}(\tau) dx_n dx_n + h^{1/2}(\tau) ds_6^2 , \]  

(48)
where \( ds_6^2 \) is the metric of the deformed conifold [19]. This is the same type of “D-brane” ansatz as [32], but with the conifold replaced by the deformed conifold as the transverse space.

The metric of the deformed conifold was discussed in some detail in [22, 24, 32]. It is diagonal in the basis (21):

\[ ds_6^2 = \frac{1}{2} \varepsilon^{4/3} K(\tau) \left[ \frac{1}{3K^3(\tau)} (d\tau^2 + (g^5)^2) + \cosh^2 \left( \frac{\tau}{2} \right) \left[ (g^3)^2 + (g^4)^2 \right] \right. \]

\[ . \]

\[ + \sinh^2 \left( \frac{\tau}{2} \right) \left[ (g^1)^2 + (g^2)^2 \right] \right] , \]  

(49)
where

\[ K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3} \sinh \tau} . \]  

(50)
For large \( \tau \) we may introduce another radial coordinate \( r \) via

\[ r^2 = \frac{3}{2^{4/3}} \varepsilon^{4/3} e^{2\tau/3} , \]  

(51)
and in terms of this radial coordinate \( ds_6^2 \rightarrow dr^2 + r^2 ds_{T^{1,1}}^2 \).

At \( \tau = 0 \) the angular metric degenerates into

\[ d\Omega_3^2 = \frac{1}{2} \varepsilon^{4/3} (2/3)^{1/3} \left[ \frac{1}{2} (g^5)^2 + (g^3)^2 + (g^4)^2 \right] , \]  

(52)
which is the metric of a round \( S^3 \) [22, 24]. The additional two directions, corresponding to the \( S^2 \) fibered over the \( S^3 \), shrink as

\[ \frac{1}{8} \varepsilon^{4/3} (2/3)^{1/3} \tau^2 [(g^1)^2 + (g^2)^2] . \]  

(53)
The simplest ansatz for the 2-form fields is

\[
F_3 = \frac{M\alpha'}{2} \{g^5 \wedge g^3 \wedge g^4 + d[F(\tau)(g^1 \wedge g^3 + g^2 \wedge g^4)]\}
\]

\[
= \frac{M\alpha'}{2} \{g^5 \wedge g^3 \wedge g^4(1 - F) + g^5 \wedge g^1 \wedge g^2 F + F'd\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\}, \quad (54)
\]

with \(F(0) = 0\) and \(F(\infty) = 1/2\), and

\[
B_2 = \frac{g_s M\alpha'}{2} [f(\tau)g^1 \wedge g^2 + k(\tau)g^3 \wedge g^4], \quad (55)
\]

\[
H_3 = dB_2 = \frac{g_s M\alpha'}{2} [d\tau \wedge (f'(\tau)g^1 \wedge g^2 + k'(\tau)g^3 \wedge g^4) + \frac{1}{2}(k - f)g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)]. \quad (56)
\]

As before, the self-dual 5-form field strength may be decomposed as \(\tilde{F}_5 = \mathcal{F}_5 + \star \mathcal{F}_5\). We have

\[
\mathcal{F}_5 = B_2 \wedge F_3 = \frac{g_s M^2(\alpha')^2}{4} \ell(\tau)g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5, \quad (57)
\]

where

\[
\ell = f(1 - F) + k F, \quad (58)
\]

and

\[
\star \mathcal{F}_5 = 4g_s M^2(\alpha')^2 \varepsilon^{-8/3} dx \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau \frac{\ell(\tau)}{K^2 h^2 \sinh^2(\tau)}. \quad (59)
\]

### 3.1 The First-Order Equations and Their Solution

In searching for BPS saturated supergravity backgrounds, the second order equations should be replaced by a system of first-order ones. Luckily, this is possible for our ansatz [1]:

\[
f' = (1 - F) \tanh^2(\tau/2),
\]

\[
k' = F \coth^2(\tau/2),
\]

\[
F' = \frac{1}{2}(k - f), \quad (60)
\]

and

\[
h' = -\alpha f(1 - F) + k F, \quad (61)
\]

where

\[
\alpha = 4(g_s M\alpha')^2 \varepsilon^{-8/3}. \quad (62)
\]

These equations follow from a superpotential for the effective radial problem [33]. Once we have the solutions to these differential equations, we can check that the
large $\tau$ limit of the properly normalized $B_2$, $F_3$ and $F_5$ field strengths agree with their simpler counterparts of section 2. Also, we can understand precisely the large and small $\tau$ behavior of the warp factor $h(\tau)$.

Note that the first three of these equations, (60), form a closed system and need to be solved first. In fact, these equations imply the self-duality of the complex 3-form with respect to the metric of the deformed conifold: $*_{6}G_3 = iG_3$. The solution is

$$F(\tau) = \frac{\sinh \tau - \tau}{2 \sinh \tau},$$
$$f(\tau) = \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau - 1),$$
$$k(\tau) = \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau + 1).$$

(63)

Now that we have solved for the 3-forms on the deformed conifold, the warp factor may be determined by integrating (61). First we note that

$$\ell(\tau) = f(1 - F) + kF = \frac{\tau \coth \tau - 1}{4 \sinh^2 \tau} (\sinh 2\tau - 2\tau).$$

(64)

This behaves as $\tau^3$ for small $\tau$. For large $\tau$ we impose, as usual, the boundary condition that $h$ vanishes. The resulting integral expression for $h$ is

$$h(\tau) = \alpha \frac{\sqrt{2/3}}{4} I(\tau) = (g_s M \alpha')^2 2^{2/3} \varepsilon^{-8/3} I(\tau),$$

(65)

where

$$I(\tau) \equiv \int_{\tau}^{\infty} dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3}.$$  

(66)

We have not succeeded in evaluating this integral in terms of elementary or well-known special functions, but it is not hard to see that

$$I(\tau \to 0) \to a_0 + O(\tau^2); \quad I(\tau \to \infty) \to 3 \cdot 2^{-1/3} \left( \tau - \frac{1}{4} \right) e^{-4\tau/3},$$

(67)

where $a_0 \approx 0.71805$. This $I(\tau)$ is nonsingular at the tip of the deformed conifold and, from (61), matches the form of the large-$\tau$ solution (10). The small $\tau$ behavior follows from the convergence of the integral (65), while at large $\tau$ the integrand becomes $\sim xe^{-4x/3}$.

Thus, for small $\tau$ the ten-dimensional geometry is approximately $\mathbb{R}^{3,1}$ times the deformed conifold:

$$ds_{10}^2 \to \frac{\varepsilon^{4/3}}{2^{1/3} a_0^{1/2} g_s M \alpha'} d\tau d\tau + a_0^{1/2} 6^{-1/3} (g_s M \alpha') \left\{ \frac{1}{2} d\tau^2 + \frac{1}{2} (g_5^2)^2 + (g_3^2)^2 + (g_4^2)^2 + \frac{1}{4} \tau^2 [(g_1^2)^2 + (g_2^2)^2] \right\}.$$  

(68)
Note that we have suppressed the $\mathcal{O}(\tau^2)$ corrections for all but the $(g^1)^2$ and $(g^2)^2$ components of the metric. This metric will be useful in section 4 where we investigate various infrared phenomenon of the gauge theory.

Very importantly, for large $g_s M$ the curvatures found in our solution are small everywhere. This is true even far in the IR, since the radius-squared of the $S^3$ at $\tau = 0$ is of order $g_s M$ in string units. This is the ‘t Hooft coupling of the gauge theory found far in the IR. As long as this is large, the curvatures are small and the SUGRA approximation is reliable.

### 3.2 $SO(4)$ invariant expressions for the 3-forms

In [7, 8] it was shown that the warped background of the previous section preserves $\mathcal{N} = 1$ SUSY if and only if $G_3$ is a $(2, 1)$ form on the CY space. Perhaps the easiest way to see the supersymmetry of the deformed conifold solution is through a T-duality. Performing a T-duality along one of the longitudinal directions, and lifting the result to M-theory maps our background to a Becker-Becker solution supported by a $G_4$ which is a $(2, 2)$ form on $T^2 \times CY$. G-flux of this type indeed produces a supersymmetric background [34].

While writing $G_3$ in terms of the angular 1-forms $g^i$ is convenient for some purposes, the $(2, 1)$ nature of the form is not manifest. That $G_3$ is indeed $(2, 1)$ was demonstrated in [9] with the help of a holomorphic basis. Below we write the $G_3$ found in [1] in terms of the obvious 1-forms on the deformed conifold: $dz^i$ and $d\bar{z}^i$, $i = 1, 2, 3, 4$:

$$G_3 = \frac{M\alpha'}{2\varepsilon^6 \sinh^4 \tau} \left\{ \frac{\sinh(2\tau) - 2\tau}{\sinh \tau} (\epsilon_{ijkl} \bar{z}_i \bar{z}_j dz^k \wedge d\bar{z}^l) \wedge (\bar{z}_m dz_m) ight.$$

$$+ 2(1 - \tau \coth \tau) (\epsilon_{ijkl} \bar{z}_i \bar{z}_j dz^k \wedge dz^l) \wedge (\bar{z}_m d\bar{z}_m) \right\}. \quad (69)$$

We also note that the NS-NS 2-form potential is an $SO(4)$ invariant $(1, 1)$ form:

$$B_2 = \frac{ig_s M\alpha'}{2\varepsilon^4} \frac{\tau \coth \tau - 1}{\sinh^2 \tau} \epsilon_{ijkl} \bar{z}_i \bar{z}_j dz^k \wedge d\bar{z}^l. \quad (70)$$

The derivation of these formulae is given in Appendix B. Our expressions for the gauge fields are manifestly $SO(4)$ invariant, and so is the metric. This completes the proof of $SO(4)$ invariance of the KS solution.
4 Infrared Physics

We have now seen that the deformation of the conifold allows the solution to be non-singular. In the following sections we point out some interesting features of the SUGRA background we have found and show how they realize the expected phenomena in the dual field theory. In particular, we will now demonstrate that there is confinement; that the theory has glueballs and baryons whose mass scale emerges through a dimensional transmutation; that there is a gluino condensate that breaks the $Z_{2M}$ chiral symmetry down to $Z_2$, and correspondingly there are domain walls separating inequivalent vacua. Other stringy approaches to infrared phenomena in $\mathcal{N} = 1$ SYM theory have recently appeared in [35, 36].

4.1 Dimensional Transmutation and Confinement

The resolution of the naked singularity via the deformation of the conifold is a supergravity realization of the dimensional transmutation. While the singular conifold has no dimensionful parameter, we saw that turning on the R-R 3-form flux produces the logarithmic warping of the KT solution. The scale necessary to define the logarithm transmutes into the the parameter $\varepsilon$ that determines the deformation of the conifold. From (51) we see that $\varepsilon^2/3$ has dimensions of length and that

$$\tau = 3 \ln(r/\varepsilon^{2/3}) + \text{const}. \quad (71)$$

Thus, the scale $r_s$ entering the UV solution (40) should be identified with $\varepsilon^{2/3}$. On the other hand, the form of the IR metric (68) makes it clear that the dynamically generated 4-d mass scale, which sets the tension of the confining flux tubes, is

$$\frac{\varepsilon^{2/3}}{\alpha' \sqrt{g_sM}}. \quad (72)$$

The reason the theory is confining is that in the metric for small $\tau$ the function multiplying $dx_n dx_n$ approaches a constant. This should be contrasted with the $AdS_5$ metric where this function vanishes at the horizon, or with the singular metric of [2] where it blows up. Consider a Wilson contour positioned at fixed $\tau$, and calculate the expectation value of the Wilson loop using the prescription [37, 38]. The minimal area surface bounded by the contour bends towards smaller $\tau$. If the contour has a very large area $A$, then most of the minimal surface will drift down into the region near $\tau = 0$. From the fact that the coefficient of $dx_n dx_n$ is finite at $\tau = 0$, we find that a fundamental string with this surface will have a finite tension, and so the resulting
Wilson loop satisfies the area law. A simple estimate shows that the string tension scales as

\[ T_s \sim \frac{\varepsilon^{4/3}}{(\alpha')^2 g_s M}. \]  (73)

The masses of glueball and Kaluza-Klein (KK) states scale as

\[ m_{\text{glueball}} \sim m_{\text{KK}} \sim \frac{\varepsilon^{2/3}}{g_s M \alpha'}. \]  (74)

Comparing with the string tension, we see that

\[ T_s \sim g_s M (m_{\text{glueball}})^2. \]  (75)

Due to the deformation, the full SUGRA background has a finite 3-cycle. We may interpret various branes wrapped over this 3-cycle in terms of the gauge theory. Note that the 3-cycle has the minimal volume near \( \tau = 0 \), hence all the wrapped branes will be localized there. A wrapped D3-brane plays the role of a baryon vertex which ties together \( M \) fundamental strings. Note that for \( M = 0 \) the D3-brane wrapped on the \( S^3 \) gave a dibaryon [23]; the connection between these two objects becomes clearer when one notes that for \( M > 0 \) the dibaryon has \( M \) uncontracted indices, and therefore joins \( M \) external charges. Studying a probe D3-brane in the background of our solution show that the mass of the baryon scales as

\[ M_b \sim M \frac{\varepsilon^{2/3}}{\alpha'}. \]  (76)

### 4.2 Chiral Symmetry Breaking and Gluino Condensation

Our \( SU(N + M) \times SU(N) \) field theory has an anomaly-free \( \mathbb{Z}_{2M} \) R-symmetry at all scales. The UV (large \( \tau \)) limit of our metric, which coincides with the solution found in [2], has a \( U(1) \) R-symmetry associated with the rotations of the angular coordinate \( \psi \). However, the background value of the R-R 2-form \( C_2 \) does not have this continuous symmetry. Although \( F_3 = dC_2 \) given in [24] is \( U(1) \) symmetric, there is no smooth global expression for \( C_2 \). Locally, we may write for large \( \tau \),

\[ C_2 \to \frac{M \alpha'}{2} \psi \omega_2. \]  (77)

Under \( \psi \to \psi + \epsilon \),

\[ C_2 \to C_2 + \frac{M \alpha'}{2} \epsilon \omega_2. \]  (78)
This modification of the asymptotic value of $C_2$ is dual to the appearance of opposite $\theta$-angles for the two gauge groups, which is a manifestation of the anomaly in the $U(1)$ R-symmetry \cite{35, 39}.

Let us show that only the discrete shifts

$$\psi \rightarrow \psi + \frac{2\pi k}{M}, \quad k = 1, 2, \ldots, M$$

are symmetries of the UV theory. To this end we may consider domain walls made of $k$ D5-branes wrapped over the finite-sized $S^3$ at $\tau = 0$, with remaining directions parallel to $\mathbb{R}^{3,1}$. Such a domain wall is obviously a stable object in the KS background and crossing it takes us from one ground state of the theory to another. Indeed, the wrapped D5-brane produces a discontinuity in $\int_B F_3$, where $B$ is the cycle dual to the $S^3$. If to the left of the domain wall $\int_B F_3 = 0$, as in the basic solution derived in the preceding sections, then to the right of the domain wall

$$\int_B F_3 = 4\pi^2 \alpha' k,$$

as follows from the quantization of the D5-brane charge. The B-cycle is bounded by a 2-sphere at $\tau = \infty$, hence $\int_B F_3 = \int_{S^2} \Delta C_2$. Therefore from (28) it is clear that to the right of the wall

$$\Delta C_2 \rightarrow \pi \alpha' k \omega_2$$

for large $\tau$. This change in $C_2$ is produced by the $\mathbb{Z}_2M$ transformation (79) on the original field configuration (77).

Recalling that $\psi$ ranges from 0 to $4\pi$, we see that the full solution, which depends on $\psi$ through $\cos \psi$ and $\sin \psi$, has the $\mathbb{Z}_2$ symmetry generated by $\psi \rightarrow \psi + 2\pi$. Therefore, a domain wall made of $M$ D5-branes returns the solution to itself. There are exactly $M$ different discrete orientations of the solution, corresponding to breaking of the $\mathbb{Z}_{2M}$ UV symmetry through the IR effects. The domain walls constructed out of the wrapped D5-branes separate these inequivalent vacua. As we expect, flux tubes can end on these domain walls \cite{41}, and baryons can dissolve in them. By studying a probe D5-brane in the metric, we find that the domain wall tension is

$$T_{wall} \sim \frac{1}{g_s \epsilon^2 (\alpha')^3} .$$

In supersymmetric gluodynamics the breaking of chiral symmetry is associated with the gluino condensate $\langle \lambda \lambda \rangle$. A holographic calculation of the condensate was carried out by Loewy and Sonnenschein in \cite{41} (see also \cite{42} for previous work on
They looked for the deviation of the complex 2-form field $C_2 - \frac{i}{g_s}B_2$ from its asymptotic large $\tau$ form that enters the KT solution:

$$
\delta \left( C_2 - \frac{i}{g_s}B_2 \right) \sim \frac{M\alpha'}{4} \tau e^{-\tau} [g_1 \wedge g_3 + g_2 \wedge g_4 - i(g_1 \wedge g_2 - g_3 \wedge g_4)] \\
\sim \frac{M\alpha' \varepsilon^2}{r^3} \ln(r/\varepsilon^{2/3}) e^{i\psi} (d\theta_1 - i \sin \theta_1 d\phi_1) \wedge (d\theta_2 - i \sin \theta_2 d\phi_2) .
$$

(83)

In a space-time that approaches $AdS_5$ a perturbation that scales as $r^{-3}$ corresponds to the expectation value of a dimension 3 operator. The presence of an extra $\ln(r/\varepsilon^{2/3})$ factor is presumably due to the fact that the asymptotic KT metric differs from $AdS_5$ by such logarithmic factors. From the angular dependence of the perturbation we see that the dual operator is $SU(2) \times SU(2)$ invariant and carries R-charge 1. These are precisely the properties of $\lambda \lambda$. Thus, the holographic calculation tells us that

$$
\langle \lambda \lambda \rangle \sim M \frac{\varepsilon^2}{(\alpha')^3} .
$$

(84)

Thus, the parameter $\varepsilon^2$ which enters the deformed conifold equation has a dual interpretation as the gluino condensate.\(^4\)

## 5 UV/IR Relations and the RG Flow

In this section we investigate some of the consequences of compactifying the conifold. If the cascade is embedded inside a compact manifold, as in [25], then the radius $\tau$ is effectively cut off at some large $\tau_c$. The radial cutoff is a scale in the theory, which becomes an ultraviolet cutoff in the boundary gauge theory. The precise relation of these distance and energy scales depends in general on the physics one is investigating. We are aware of three schemes for relating the two scales: first, one could consider the energy of a string stretched from the tip of the conifold to the regularized boundary as a function of $\tau_c$ [11]. Second, one can try to think about the warp factor $h(\tau)$ as a redshift factor which relates the energy of a probe in the bulk of $AdS$ space to its apparent energy as seen by an observer on the boundary. Third, one can consider the equations of motion for supergravity probes; this is sometimes called the holographic scheme [27]. In conformal backgrounds, the various distance/energy relations differ only by their normalization; for $AdS_5 \times S^5$, $E \propto r$ in all three schemes. However, in non-conformal backgrounds the distance/energy relations can have different functional forms [10], and we will see that this is the case for the KT and KS solutions.

\(^4\)It would be nice to understand the relative factor of $g_s M$ between $T_{wall}$ and $\langle \lambda \lambda \rangle$.  

17
One prescription for relating distance and energy scales comes from considering the energy of a string stretched from some fixed $\tau$ to the cutoff radius $\tau_c$, where it is stabilized by an external force – by a probe D-brane at $\tau_c$, for example. The energy of such a string is proportional to its worldvolume per unit time:

$$E \sim \int_0^{\tau_c} \sqrt{g_{\mu\nu}g_{\tau\tau}} d\tau \sim e^{\tau_c/3} + \text{const} \sim r_c.$$  \hspace{1cm} (85)

The energy of this string corresponds to the linearly divergent self-energy of a quark in the boundary gauge theory, and the radial cutoff of the geometry regulates the divergence. Decreasing the radial cutoff removes high energy string modes and thus corresponds to integrating out high energy gauge theory modes in the Wilsonian sense. An appealing feature of this prescription is that $\ln \frac{\Lambda}{\mu} \sim \ln(r_c) + \text{const}$, so that the difference of the couplings as predicted by supergravity agrees exactly with the gauge theory expectation, with no additional $\ln \ln$ terms.

An alternative prescription is to interpret the warp factor $h(\tau)$ as a redshift factor. An object with energy $E_\tau$ at radial position $\tau$ will appear to an observer at $\tau'$ to have energy $E_{\tau'}$, where the energies are related by $E_\tau h^{1/4}(\tau) = E_{\tau'} h^{1/4}(\tau')$. In terms of the renormalization scale, the distance/energy relation becomes

$$\Lambda \sim \mu I(\tau_c)^{-1/4} \sim \mu \left[ \tau_c^{-1/4} e^{\tau_c/3} \ldots \right].$$  \hspace{1cm} (86)

This redshift relation introduces corrections to (46) of the form $\ln \ln(\Lambda/\mu)$. They have the same form as corrections to the flow due to two-loop $\beta$-functions.

We can derive a third distance/energy relation by considering a massless supergravity probe in the KT background. For a massless scalar field the equation of motion is

$$\nabla^2 \phi = \frac{1}{L^2 \sqrt{\ln(r/r_s)}} \left[ \frac{L^4 \ln(r/r_s)}{r^2} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} + \frac{1}{r^3} \frac{\partial}{\partial r} r^5 \frac{\partial}{\partial r} \right] \phi = 0.$$  \hspace{1cm} (87)

The second term is invariant under a rescaling of the radius. Thus a solution of (87) which is wavelike on a radial slice of $AdS$ depends on the radius through the quantity $L^4 \ln(r/r_s) k^2$. For this prescription

$$\Lambda \sim \frac{r_c}{L^2 \sqrt{\ln(r_c/r_s)}}.$$  \hspace{1cm} (88)

We can obtain the same result by another physical argument which seems quite different on its surface. Let us consider instead this theory at finite temperature, as
was studied in [26]. At sufficiently high temperature, the system develops a horizon, and the Hawking temperature is related to the horizon radius \( r_H \) by

\[
T_H \sim \frac{r_H}{L^2 \sqrt{\ln(r_H/r_s)}},
\]

in the limit of high temperature. In the theory with a large radius cutoff, the maximum temperature is simply given by setting \( r_H = r_c \). Then if we identify the UV cutoff \( \Lambda \) as the maximum Hawking temperature, we recover the result (88). The agreement between these two methods for relating the RG scale to the cutoff radius is a sign that holography is at work.

Let us note that the relations we find between \( \mu \) and \( \Lambda \) are exactly of the form one finds in an asymptotically free gauge theory. To make contact with the standard dimensional transmutation formula, we have to identify \( \tau_c/3 = 8\pi^2/(\beta_0 g_0^2) \). If the beta function is

\[
\frac{dg}{d \log(\Lambda/\mu)} = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{128\pi^4} - \ldots ,
\]

then we find

\[
\mu \sim \Lambda e^{-8\pi^2/(\beta_0 g_0^2)} (8\pi^2/\beta_0 g_0^2)^{\beta_1/\beta_0^2} .
\]

To make contact with the SUGRA result (86) we have to identify \( \tau_c/3 = 8\pi^2/(\beta_0 g_0^2) \).

We can reexpress \( \tau_c \) in terms of the NS-NS flux

\[
\int_{\tau_c} \int_{S^2} H_3 = 4\pi^2 \alpha' K .
\]

One quickly finds \( \tau_c \approx 2\pi K/(g_s M) \). In order to achieve the continuum limit, we have to take \( \tau_c \to \infty \), \( \Lambda \to \infty \) while keeping the physical scale \( \mu \) fixed. The exponential factor \( e^{-\tau_c/3} = e^{-2\pi K/(3g_s M)} \), which may give rise to a large hierarchy of scales in compactified F-theory, was derived in [25]. It was observed that the type IIB supergravity picture of gluino condensation is reminiscent of the gluino condensation in the hidden sector of the heterotic string [43]. Here we note that a more precise SUGRA analysis may produce a power-law prefactor, which is analogous to the prefactor due to the 2-loop \( \beta \)-function in an asymptotically free gauge theory. With the stretched-string relation, we find \( \beta_1/\beta_0^2 = 0 \); the redshift relation gives \( \beta_1/\beta_0^2 = 1/4 \); and the holographic relation gives \( \beta_1/\beta_0^2 = 1/2 \).

For pure \( \mathcal{N} = 1 \) supersymmetric gauge theory with gauge group some simple Lie group \( G \), \( \beta_0 = 3C_2(G) \) and \( \beta_1 = 3C_2(G)^2 \) (see for example [28]). The quantity \( C_2(G) \) is the quadratic Casimir. Unfortunately, none of our distance/energy relations give the required value of 1/3. Perhaps adding the right kind of matter will fix the ratio
to the correct value. It is also possible that a different identification between the SUGRA and field theory couplings may fix the prefactor. In any case, it would be very interesting to find out if the analogy with the gluino condensation in the hidden sector of the heterotic string [25] is in fact an exact duality.

A Volume of the Two and Three Cycles

The manifold $T^{1,1}$ can be identified as the intersection of the conifold $\sum_{i=1}^{4} z_i^2 = 0$ and the sphere $\sum_{i=1}^{4} |z_i|^2 = 1$, where $z_i \in \mathbb{C}$. Dividing up $z_i = x_i + iy_i$ into real and imaginary parts, we see that $T^{1,1}$ can be thought of as the set of points satisfying $\sum x_i^2 = 1/2$ and $\sum y_i^2 = 1/2$ along with the constraint $x \cdot y = 0$. If we use this constraint to eliminate one of the $x_i$, we can see, at least in a heuristic way, that the manifold $T^{1,1}$ can be thought of as a $S^2$ defined by the remaining $x_i$ fibered over an $S^3$ base defined by the $y_i$.

We now use this observation to parametrize the two cycle $C_2$. An explicit parametrization of the whole $T^{1,1}$ is known in terms of the angles $0 \leq \psi < 4 \pi$, $0 \leq \theta_i \leq \pi$, and $0 \leq \phi_i < 2 \pi$ where $i = 1, 2$. Indeed

\[
\begin{align*}
z_1 &= \frac{e^{i\psi/2}}{\sqrt{2}} \left( \cos \left( \frac{\theta_1 + \theta_2}{2} \right) \cos \left( \frac{\phi_1 + \phi_2}{2} \right) + i \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\phi_1 + \phi_2}{2} \right) \right) \\
z_2 &= \frac{e^{i\psi/2}}{\sqrt{2}} \left( -\cos \left( \frac{\theta_1 + \theta_2}{2} \right) \sin \left( \frac{\phi_1 + \phi_2}{2} \right) + i \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \cos \left( \frac{\phi_1 + \phi_2}{2} \right) \right) \\
z_3 &= \frac{e^{i\psi/2}}{\sqrt{2}} \left( -\sin \left( \frac{\theta_1 + \theta_2}{2} \right) \cos \left( \frac{\phi_1 - \phi_2}{2} \right) + i \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\phi_1 - \phi_2}{2} \right) \right) \\
z_4 &= \frac{e^{i\psi/2}}{\sqrt{2}} \left( -\sin \left( \frac{\theta_1 + \theta_2}{2} \right) \sin \left( \frac{\phi_1 - \phi_2}{2} \right) - i \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \cos \left( \frac{\phi_1 - \phi_2}{2} \right) \right)
\end{align*}
\]

To describe the fiber, we would like to stay on one point on the base $S^3$. Thus, we want to keep the imaginary part of the $z_i$ constant while still keeping two degrees of freedom available to trace out the $S^2$ fiber. For convenience, we begin by choosing $\psi = 0$. From the parametrization, we can trace out the $S^2$ by setting $\theta_1 = \theta_2$ and $\phi_1 = -\phi_2$. Integrating using these coordinates, $\int_{C_2} \omega_2 = 4\pi$.

Next we consider the three cycle $C_3$. First recall that

\[
g_5 \wedge g^3 \wedge g^4 = \omega_3 = -\frac{1}{2} d(g^1 \wedge g^3 + g^2 \wedge g^4) . \tag{91}
\]

Moreover, $C_3$ has no boundary so

\[
\int_{C_3} \omega_3 = \int_{C_3} g_5 \wedge g^3 \wedge g^4 . \tag{92}
\]
We recall from [24] that
\[ ds^2 = \frac{1}{2} (g^5)^2 + (g^3)^2 + (g^4)^2 \] (93)
is the standard metric on a \( S^3 \) with radius \( \sqrt{2} \). Moreover \( \text{Vol}(S^3) = 2\pi^2 r^3 \). It follows then that \( \int_{C_3} \omega_3 = 8\pi^2 \).

## B Complex notation for the forms

Our strategy is to guess differential forms, written in terms of the \( z_i \), with the appropriate symmetries and properties. To refine further and check the guess, we use computer assisted algebra to rewrite the differential forms in terms of the angular coordinates on the conifold. We can then compare the guess with the differential forms given in previous sections in terms of the \( g^i \).

First, we need to review the construction of the angular coordinates on the deformed conifold. We define
\[
W = \begin{pmatrix} z_3 + iz_4 & z_1 - iz_2 \\ z_1 + iz_2 & -z_3 + iz_4 \end{pmatrix}.
\] (94)
The defining relation of the deformed conifold (47) becomes \( \det W = -\varepsilon^2 \). We introduce the angular coordinates with two \( SU(2) \) \( j = 1,2 \) matrices
\[
L_j = \begin{pmatrix} \cos \frac{\theta_j}{2} e^{i(\psi_j + \phi_j)/2} & -\sin \frac{\theta_j}{2} e^{-i(\psi_j - \phi_j)/2} \\ \sin \frac{\theta_j}{2} e^{i(\psi_j - \phi_j)/2} & \cos \frac{\theta_j}{2} e^{-i(\psi_j + \phi_j)/2} \end{pmatrix}.
\] (95)
The idea is then to take some representative point \( p \in C \) corresponding to
\[
W_0 = \begin{pmatrix} 0 & \varepsilon e^{\pi/2} \\ \varepsilon e^{-\pi/2} & 0 \end{pmatrix}.
\] (96)
We can generate all of \( C \) by acting on \( W_0 \) with \( L_1 \) and \( L_2 \)
\[
W = L_1 \cdot W_0 \cdot L_2^\dagger.
\] (97)
As the coordinates \( \psi_1 \) and \( \psi_2 \) only appear in \( W \) as \( \psi_1 + \psi_2 \), we may define a new variable \( \psi = \psi_1 + \psi_2 \). It is natural to define a radial coordinate
\[
\rho^2 \equiv \sum_{i=1}^4 z_i \bar{z}_i = \frac{1}{2} \text{Tr}(W \cdot W^\dagger).
\] (98)
With this definition, one straightforwardly obtains $\rho^2 = \varepsilon^2 \cosh \tau$. The singular case, $\varepsilon = 0$ is recovered by taking the large $\tau$ limit. Equivalently, we may start with a slightly different

$$W_0^{\text{sing}} = \begin{pmatrix} 0 & \sqrt{2}\rho \\ 0 & 0 \end{pmatrix}. \quad (99)$$

To summarize, then, the angular coordinates on the conifold are $0 \leq \psi < 4\pi$, $0 \leq \theta_j < \pi$, $0 \leq \phi_j < 2\pi$ and a radius $\rho$. In the case where $\varepsilon \neq 0$, we can substitute $\tau$ for $\rho$. In the case $\varepsilon = 0$, $\rho$ is typically expressed as $r^3 \sim \rho^2$ in order to make the conical nature of the metric evident (14).

In principle, we have an explicit coordinate transformation between the angular variables and the complex coordinates $z_i$. The goal of this appendix, to rewrite the important supergravity quantities in terms of the $z_i$, should be a straightforward task. Given any quantity written in terms of the angles, we should be able to write down the same quantity in terms of the $z_i$. In practice, this variable change is difficult for two reasons. First, and especially in the case $\varepsilon \neq 0$, the variable change is quite complicated and nearly impossible to do without some computer assisted algebra. Second, there are more $z_i$ than one needs. By choosing three out of the four $z_i$, one explicitly breaks the $\text{SO}(4)$ symmetry. The formulae involving only three $z_i$ are typically messy and uninformative. Moreover, it is usually not completely obvious how to reintroduce the fourth $z_i$ in a way that symmetrizes the quantity of interest.

**B.1 Forms on the Singular Conifold**

We begin with the easier case, the singular conifold (13). We would like to express the forms important to the KT solution [2] and discussed in section 2 in terms of the $z_i$. It is a fact that $g^5$, $\omega_2$, and $\omega_3$ all transform as singlets under the $\text{SO}(4)$ action. Another important symmetry that holds when $\varepsilon = 0$ is the scaling $z_i \rightarrow \lambda z_i$ where $\lambda \in \mathbb{C}^*$. The real part of this scaling, i.e. $\lambda \in \mathbb{R}^+$, corresponds to scaling the radius $\rho$, while the complex $U(1)$ part, i.e. $\lambda = e^{i\alpha}$, corresponds to shifting the angle $\psi$. Cursory inspection of the vielbeins and the defining relations for $\omega_2$ and $\omega_3$ (21, 26, 27) show that the forms $g^5$, $\omega_2$, and $\omega_3$ are invariant under the full scaling.

Begin with $g^5$. Using the $z_i$, there are two ways to write down singlet one forms which obey the scaling symmetries: $z_i \bar{z}_i/\rho^2$ or $\bar{z}_i z_i/\rho^2$ where summation on the indices is implied.\(^5\) Any singlet one form must be some linear combination of these.

\(^5\)Note that $z_i dz_i$ and its complex conjugate vanish by the defining relation on the conifold (14).
two, and all that need be done is fix the constants. Indeed
\[ \frac{d\rho}{\rho} + \frac{i}{2} g^5 = \frac{1}{\rho^2} \bar{z}_i dz_i \]  
(100)

and
\[ \frac{d\rho}{\rho} - \frac{i}{2} g^5 = \frac{1}{\rho^2} z_i d\bar{z}_i . \]  
(101)

Next, we consider the two form \( \omega_2 \). There are several ways of writing \( \text{SO}(4) \) invariant two forms. Indeed
\[ \eta_1 = \epsilon_{ijkl} z_i \bar{z}_j dz_k \wedge d\bar{z}_l , \quad \eta_2 = \epsilon_{ijkl} z_i \bar{z}_j dz_k \wedge dz_l , \]
\[ \eta_3 = \epsilon_{ijkl} z_i \bar{z}_j d\bar{z}_k \wedge dz_l , \quad \eta_4 = (z_i dz_i) \wedge (\bar{z}_j dz_j) , \]
\[ \eta_5 = (dz_i \wedge d\bar{z}_i) . \]  
(102)

We can eliminate \( \eta_2 \) and \( \eta_3 \) immediately because they explicitly break the \( U(1) \) symmetry \( z_i \rightarrow e^{i\alpha} z_i \). The situation is even simpler. The form \( \omega_2 \) transforms with a minus sign under the spatial inversion \( z_1 \rightarrow -z_1 \), keeping all the other \( z_i \) fixed, while the forms \( \eta_4 \) and \( \eta_5 \) are invariant under the full \( O(4) \) symmetry. Our constraints leave only \( \eta_1 \) as a candidate for \( \omega_2 \):
\[ \omega_2 = \frac{i}{\rho^4} \eta_1 . \]  
(103)

As mentioned in [44], this form is closed, as we may check explicitly:
\[ \partial \omega_2 = -\frac{i}{\rho^6} \left( 2\chi_1 + \rho^2 \chi_4 \right) , \]  
(104)

where
\[ \chi_1 = (\epsilon_{ijkl} z_i \bar{z}_j dz_k \wedge d\bar{z}_l) \wedge (\bar{z}_m dz_m) , \quad \chi_4 = \epsilon_{ijkl} \bar{z}_i dz_j \wedge dz_k \wedge d\bar{z}_l , \]  
(105)

are new \( (2,1) \) \( \text{SO}(4) \) invariant forms, labeled to conform with the notation used in the next section. With computer assisted algebra, using (13), it is easy to see that the expression on the right side of (104) vanishes. We also provide a symmetry argument for the vanishing. Because \( \chi_1 \) and \( \chi_4 \) are \( \text{SO}(4) \) and scale invariant, we are free to check the vanishing for a specific point \( p \) on the singular conifold, \( z_1 = 1, z_2 = i, z_3 = 0, \) and \( z_4 = 0 \), and then invoke the \( \text{SO}(4) \) and scaling symmetry to prove the vanishing for all points.

Finally, we turn to \( \omega_3 \), and in fact we already know the answer because \( \omega_3 = g^5 \wedge \omega_2 \). The most important form in the KT solution is not \( F_3 \) or \( H_3 \) (and hence \( \omega_3 \) or \( \omega_2 \)
independently but their combination $G_3 = F_3 - i H_3/g_s$, which needs to be a $(2,1)$ form in order to preserve supersymmetry. We may check that

$$G_3 = \frac{\alpha'}{2} M \left( g^5 - \frac{2i d\rho}{\rho} \right) \wedge \omega_2 = \frac{\alpha'}{\rho^6} M (\epsilon_{ijkl} z_i \bar{z}_j \, dz_k \wedge d\bar{z}_l) \wedge (\bar{z}_m \, dz_m)$$

which is explicitly a $(2,1)$ form, as required. Reassuringly, changing the sign of $d\rho/\rho$ in the expression above produces instead a $(1,2)$ form.

### B.2 Forms on the Deformed Conifold

The deformed conifold is more difficult, not only because the coordinate transformation is more complicated but also because the nonzero $\varepsilon$ explicitly breaks the $U(1)$ and scale invariance.

Before tackling the $(2,1)$ form $G_3 = F_3 - i H_3/g_s$, let us warm up with some simpler one and two forms. First, we check what happens to $z_i \, d\bar{z}_i$ and $\bar{z}_i \, dz_i$ when $\varepsilon$ is turned on:

$$\bar{z}_i \, dz_i = \varepsilon^2 \sinh \tau \left( d\tau + ig^5 \right).$$

Comfortingly, in the large $\tau$ limit, we recover the singular conifold result \[100\]. The result for $z_i \, d\bar{z}_i$ is, not surprisingly, the complex conjugate.

Next we consider what happens to $\omega_2$. Remember that we have broken the $U(1)$ symmetry and as a result, the forms $\eta_2$ and $\eta_3$ \[102\] may contribute. Indeed, they do as

$$\omega_2 = \frac{i \cosh \tau}{\varepsilon^4 \sinh^3 \tau} \epsilon_{ijkl} z_i \bar{z}_j \left( dz_k \wedge d\bar{z}_l - \frac{1}{2 \cosh \tau} (dz_k \wedge dz_l + d\bar{z}_k \wedge d\bar{z}_l) \right).$$

In the large $\tau$ limit, the reader may check that we recover the result for the singular conifold \[103\].

Although $\omega_2$ becomes a combination of different $U(1)$ breaking differential forms, the NS-NS potential $B_2$ is actually more closely related to the old $\omega_2$ of the singular conifold. Indeed

$$B_2 = \frac{ig_s M \alpha'}{2\varepsilon^4} \frac{\tau \coth \tau - 1}{\sinh^2 \tau} \eta_1.$$  

Again, in the large $\tau$ limit, we recover $B_2$ on the singular conifold.

Now we are ready to attack the harmonic $(2,1)$ form $G_3$. We begin by writing down all of the $SO(4)$ invariant $(2,1)$ forms, of which there are five,

$$\chi_1 = (\epsilon_{ijkl} z_i \bar{z}_j \, dz_k \wedge d\bar{z}_l) \wedge (\bar{z}_m \, dz_m),$$

\( \chi_2 = (\epsilon_{ijkl} z_i \bar{z}_j d z_k \wedge d z_l) \wedge (z_m d \bar{z}_m) \),
\( \chi_3 = (\epsilon_{ijkl} z_i d z_j \wedge d z_k \wedge d \bar{z}_l) \),
\( \chi_4 = (\epsilon_{ijkl} \bar{z}_i d z_j \wedge d z_k \wedge d \bar{z}_l) \),
\( \chi_5 = (d z_i \wedge d \bar{z}_i) \wedge (\bar{z}_j d z_j) \).

(110)

Fortunately we can eliminate \( \chi_5 \) because \( \partial (h(\rho^2) \eta_4) = h(\rho^2) \chi_5 \). Based on experience with the singular conifold, and in particular the demonstration that \( \partial \omega_2 = 0 \), one may wonder if the remaining \( \chi_i \) are linearly independent on the deformed conifold. They are not. The equation
\[
\chi \equiv \alpha \chi_1 + \beta \chi_2 + \gamma \chi_3 + \delta \chi_4 = 0
\]
is easy to satisfy. We choose a convenient point \( p \in C \), for example the point corresponding to the matrix \( W_0 \). If \( \chi \) vanishes at \( p \), it vanishes on all of \( C \) by \( SO(4) \) invariance. The two conditions that must be met for \( \chi \) to vanish are
\[
\alpha \cosh \tau + 2\beta - 2\delta / \varepsilon^2 = 0 ,
\alpha + 2\beta \cosh \tau + 2\gamma / \varepsilon^2 = 0 .
\]
(111)

We choose the ansatz for the \((2,1)\) form
\[
G_3 = \alpha \chi_1 + \beta \chi_2 + \gamma \chi_3 + \delta \chi_4 .
\]
(112)

An intensive computer assisted computation reveals
\[
\alpha \cosh \tau + 2\beta - 2\delta / \varepsilon^2 = \frac{M \alpha'}{2 \varepsilon^6 \sinh^5 \tau} [\sinh \tau (\cosh(2\tau) + 5) - 6 \tau \cosh \tau] ,
\alpha + 2\beta \cosh \tau + 2\gamma / \varepsilon^2 = \frac{M \alpha'}{\varepsilon^6 \sinh^4 \tau} [\tau (\cosh(2\tau) + 2) - 3 \sinh \tau \cosh \tau] .
\]
(113)

Because of the linear dependence of the \( \chi_i \), we are free to choose any two of the four parameters \( \alpha, \beta, \gamma, \) and \( \delta \) freely. Said another way, we can express \( G_3 \) as the sum of any two \( \chi_i, i = 1, \ldots, 4 \).

Let us choose \( \gamma = 0 \) and \( \delta = 0 \). In this case,
\[
\alpha = \frac{M \alpha' \sinh(2\tau) - 2\tau}{2 \varepsilon^6 \sinh^5 \tau}
\]
and
\[
\beta = \frac{M \alpha' 2(1 - \tau \coth \tau)}{2 \varepsilon^6 \sinh^4 \tau} .
\]

In the large \( \tau \) limit, \( \beta \) becomes vanishingly small compared to \( \alpha \). If it did not, the \( U(1) \) symmetry on the singular conifold would not be preserved! Moreover, \( \alpha \rightarrow M \alpha' / \rho^6 \), in agreement with (106).
Acknowledgements

I.R.K. is grateful to S. Gubser, N. Nekrasov, M. Strassler, A. Tseytlin and E. Witten for collaboration on parts of the material reviewed in these notes and for useful input. We also thank Sergey Frolov and John Pearson for useful discussions. This work was supported in part by the NSF grant PHY-9802484.

References

[1] I. R. Klebanov and M. Strassler, “Supergravity and a Confining Gauge Theory: Duality Cascades and $\chi$SB–Resolution of Naked Singularities,” JHEP 0008 (2000) 052, hep-th/0007191

[2] I. R. Klebanov and A. Tseytlin, “Gravity Duals of Supersymmetric $SU(N) \times SU(N + M)$ Gauge Theories,” Nucl. Phys. B574 (2000) 123, hep-th/0002159

[3] I. R. Klebanov and N. Nekrasov, “Gravity Duals of Fractional Branes and Logarithmic RG Flow,” Nucl. Phys. B574 (2000) 263, hep-th/9911096

[4] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta, I. Pesando, “Fractional D-Branes and Their Gauge Duals,” JHEP 0102 (2001) 014, hep-th/0011077

[5] J. Polchinski, “$\mathcal{N} = 2$ Gauge–Gravity Duals,” Int. J. Mod. Phys. A 16 (2001) 707, hep-th/0111193

[6] K. Dasgupta, G. Rajesh and S. Sethi, “M Theory, Orientifolds and G-Flux,” JHEP 9908 (1999) 023, hep-th/9908088

[7] M. Grana and J. Polchinski, “Supersymmetric 3-form flux perturbations on $AdS_5$,” Phys. Rev. D63 (2001) 026001, hep-th/0009211

[8] S. S. Gubser, “Supersymmetry and F-theory realization of the deformed conifold with 3-form flux,” hep-th/0010010

[9] M. Cvetić, H. Lü, and C. N. Pope, “Brane Resolution Through Transgression,” Nucl. Phys. B600 (2001) 103, hep-th/011023 M. Cvetić, G. W. Gibbons, H. Lü, and C. N. Pope, “Ricci-flat Metrics, Harmonic Forms and Brane Resolutions,” hep-th/0012011
[10] A. W. Peet and J. Polchinski, “UV/IR relations in AdS dynamics,” *Phys. Rev. D* **59**, 065011 (1999) [hep-th/9809022](http://arxiv.org/abs/hep-th/9809022).

[11] J. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231, [hep-th/9711200](http://arxiv.org/abs/hep-th/9711200).

[12] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” *Phys. Lett.* **B428** (1998) 105, [hep-th/9802109](http://arxiv.org/abs/hep-th/9802109).

[13] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253, [hep-th/9802150](http://arxiv.org/abs/hep-th/9802150).

[14] S. S. Gubser, I. R. Klebanov, and A. W. Peet, “Entropy and temperature of black 3-branes,” *Phys. Rev.* **D54** (1996) 3915, [hep-th/9602135](http://arxiv.org/abs/hep-th/9602135).

[15] S. Kachru and E. Silverstein, “4d conformal field theories and strings on orbifolds,” *Phys. Rev. Lett.* **80** (1998) 4855, [hep-th/9802183](http://arxiv.org/abs/hep-th/9802183); A. Lawrence, N. Nekrasov and C. Vafa, “On conformal field theories in four dimensions,” *Nucl. Phys.* **B533** (1998) 199, [hep-th/9803015](http://arxiv.org/abs/hep-th/9803015).

[16] A. Kehagias, “New Type IIB Vacua and Their F-Theory Interpretation,” *Phys. Lett.* **B435** (1998) 337, [hep-th/9805131](http://arxiv.org/abs/hep-th/9805131).

[17] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” *Nucl. Phys.* **B536** (1998) 199, [hep-th/9807080](http://arxiv.org/abs/hep-th/9807080).

[18] D. Morrison and R. Plesser, “Non-Spherical Horizons, I,” *Adv. Theor. Math. Phys.* **3** (1999) 1, [hep-th/9810201](http://arxiv.org/abs/hep-th/9810201).

[19] S. S. Gubser, “Einstein Manifolds and Conformal Field Theories,” *Phys. Rev. D* **59** (1999) 025006, [hep-th/9807164](http://arxiv.org/abs/hep-th/9807164).

[20] M. R. Douglas and G. Moore, “D-branes, quivers, and ALE instantons,” [hep-th/9603167](http://arxiv.org/abs/hep-th/9603167).

[21] L. Romans, “New compactifications of chiral $N = 2$, $d = 10$ supergravity,” *Phys. Lett.* **B153** (1985) 392.

[22] P. Candelas and X. de la Ossa, “Comments on Conifolds,” *Nucl. Phys.* **B342** (1990) 246.

[23] S. S. Gubser and I. R. Klebanov, “Baryons and Domain Walls in an $N = 1$ Superconformal Gauge Theory,” *Phys. Rev.* **D58** (1998) 125025, [hep-th/9808075](http://arxiv.org/abs/hep-th/9808075).
[24] R. Minasian and D. Tsimpis, “On the Geometry of Non-trivially Embedded Branes,” *Nucl. Phys.* **B572** (2000) 499, [hep-th/9911042](http://arxiv.org/abs/hep-th/9911042).

[25] S. B. Giddings, S. Kachru and J. Polchinski, “Hierarchies from fluxes in string compactifications,” [hep-th/0105097](http://arxiv.org/abs/hep-th/0105097).

[26] A. Buchel, “Finite temperature resolution of the Klebanov-Tseytlin singularity,” *Nucl. Phys.* **B600** (2001) 219, [hep-th/0011146](http://arxiv.org/abs/hep-th/0011146). A. Buchel, C. P. Herzog, I. R. Klebanov, L. Pando Zayas and A. A. Tseytlin, “Non-extremal gravity duals for fractional D3-branes on the conifold,” *JHEP* **0104** (2001) 033, [hep-th/0102105](http://arxiv.org/abs/hep-th/0102105). S. S. Gubser, C. P. Herzog, I. R. Klebanov and A. A. Tseytlin, “Restoration of chiral symmetry: A supergravity perspective,” *JHEP* **0105** (2001) 028, [hep-th/0102172](http://arxiv.org/abs/hep-th/0102172).

[27] L. Susskind and E. Witten, “The holographic bound in anti-de Sitter space,” [hep-th/9805114](http://arxiv.org/abs/hep-th/9805114).

[28] M. Shifman and A. Vainshtein, “Solution of the Anomaly Puzzle in SUSY Gauge Theories and the Wilson Operator Expansion,” *Nucl. Phys.* **B277** (1986) 456.

[29] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, *Nucl. Phys.* **B229** (1983) 381.

[30] P. C. Argyres, lecture notes, unpublished.

[31] N. Seiberg, “Electric-magnetic duality in supersymmetric nonAbelian gauge theories,” Nucl. Phys. **B435** (1995) 129, [hep-th/9411149](http://arxiv.org/abs/hep-th/9411149).

[32] K. Ohta and T. Yokono, “Deformation of Conifold and Intersecting Branes,” [hep-th/9912266](http://arxiv.org/abs/hep-th/9912266).

[33] L. Pando Zayas and A. Tseytlin, “3-branes on resolved conifold,” *JHEP* **0011** (2000) 028, [hep-th/0010088](http://arxiv.org/abs/hep-th/0010088).

[34] K. Becker and M. Becker, “M Theory on Eight-Manifolds,” *Nucl. Phys.* **B477** (1996) 155, [hep-th/9605053](http://arxiv.org/abs/hep-th/9605053).

[35] J. Maldacena and C. Nunez, “Towards the large $N$ limit of pure $\mathcal{N} = 1$ super Yang Mills,” *Phys. Rev. Lett.* **86** (2001) 588, [hep-th/0008001](http://arxiv.org/abs/hep-th/0008001).

[36] C. Vafa, “Superstrings and Topological Strings at Large $N$,” [hep-th/0008142](http://arxiv.org/abs/hep-th/0008142).
[37] J. Maldacena, “Wilson loops in large N field theories,” *Phys. Rev. Lett.* **80** (1998) 4859, [hep-th/9803002](https://arxiv.org/abs/hep-th/9803002)

[38] S. J. Rey and J. Yee, “Macroscopic Strings as Heavy Quarks of Large N Gauge Theory and Anti-de Sitter Supergravity,” [hep-th/9803001](https://arxiv.org/abs/hep-th/9803001)

[39] E. Witten, private communication.

[40] G. Dvali and M. Shifman, “Domain walls in strongly coupled theories,” *Phys. Lett.* **B396** (1997) 64, [hep-th/9612128](https://arxiv.org/abs/hep-th/9612128)

[41] A. Loewy and J. Sonnenschein, “On the holographic duals of $\mathcal{N} = 1$ gauge dynamics,” *JHEP* **0108** (2001) 007, [hep-th/0103163](https://arxiv.org/abs/hep-th/0103163)

[42] F. Bigazzi, L. Girardello and A. Zaffaroni, “A note on regular type 0 solutions and confining gauge theories,” *Nucl. Phys.* **B598** (2001) 530, [hep-th/0011041](https://arxiv.org/abs/hep-th/0011041)

[43] M. Dine, R. Rohm, N. Seiberg and E. Witten, “Gluino Condensation in Superstring Models,” *Phys. Lett.* **B156** (1985) 55.

[44] I. R. Klebanov and E. Witten, “AdS/CFT Correspondence and Symmetry Breaking,” *Nucl. Phys.* **B556** (1999) 89, [hep-th/9905104](https://arxiv.org/abs/hep-th/9905104)