CURRENT STATUS OF CONSTRAINTS ON THE ELEMENTS OF THE NEUTRINO MASS MATRIX*

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(Received October 17, 2013)

We analyse the mass matrix of the three light neutrinos in the basis where the charged-lepton mass matrix is diagonal and discuss constraints on its elements for the Majorana and the Dirac case.

DOI:10.5506/APhysPolB.44.2339
PACS numbers: 12.15.Ff, 14.60.Pq

1. Introduction

The recent enormous improvement of our knowledge of the neutrino oscillation parameters suggests a detailed investigation of the current constraints on the neutrino mass matrix. Most of these constraints depend on the assumed nature of neutrinos (Dirac or Majorana).

The structure of this paper is as follows. After a brief general discussion of the light-neutrino mass matrix in Section 2, we will investigate the implications of the currently available data on the Majorana neutrino mass matrix in Section 3. In Section 4 we will discuss constraints on the neutrino mass matrix in the Dirac case. Finally, we will conclude in Section 5.

2. The light-neutrino mass matrix

In this paper, we assume that there are exactly three light neutrino mass eigenstates with masses smaller than $\mathcal{O}(1 \text{ eV})$, i.e. we assume that there are no light sterile neutrinos. By the term “neutrino mass matrix” we thus always mean the $3 \times 3$ mass matrix of the three light neutrinos.

If neutrinos are Majorana particles, we assume that there is a (possibly effective) mass term

$$\mathcal{L} = -\frac{1}{2} \nu_L^c M \nu_L + \text{H.c.} = \frac{1}{2} \nu_L^T C^{-1} M \nu_L + \text{H.c.},$$

(1)

* Presentated at the XXXVII International Conference of Theoretical Physics “Matter to the Deepest” Ustroń, Poland, September 1–6, 2013.
where $M_\nu$ is a complex symmetric $3 \times 3$-matrix. Such a mass term directly arises from the type-II see-saw mechanism and can be effectively generated via the see-saw mechanisms of type I and III.

If neutrinos are Dirac particles, \textit{i.e.} if the total lepton number is conserved, we assume the existence of three right-handed neutrino fields $\nu_R$ leading to the mass term

$$L = -\overline{\nu_R} M_D \nu_L + \text{H.c.},$$

where $M_D$ is an arbitrary complex $3 \times 3$-matrix.

Before we can discuss any constraints on the neutrino mass matrix, we have to specify a basis in flavour space\(^1\). In models involving flavour symmetries, the chosen matrix representations of the flavour symmetry group specify the basis. Since we will at this point not assume any flavour symmetries in the lepton sector, we are free to choose a basis. For simplicity, we will always choose a basis in which the charged-lepton mass matrix is given by

$$M_\ell = \text{diag}(m_e, m_\mu, m_\tau).$$

### 3. Constraints on the Majorana neutrino mass matrix

#### 3.1. Parametrization of the Majorana neutrino mass matrix

In the basis specified by equation (3), the Majorana neutrino mass matrix has the form

$$M_\nu = U_{\text{PMNS}}^* \text{diag}(m_1, m_2, m_3) U_{\text{PMNS}}^\dagger,$$

where $U_{\text{PMNS}}$ is the lepton mixing matrix and the $m_i$ ($i = 1, 2, 3$) are the masses of the three light neutrinos. As any unitary $3 \times 3$-matrix, $U_{\text{PMNS}}$ can be parametrized by six phases and three mixing angles. We will use the parametrization

$$U_{\text{PMNS}} = D_1 V D_2$$

with

$$D_1 = \text{diag} \left( e^{i\alpha}, e^{i\beta}, e^{i\gamma} \right) \quad \text{and} \quad D_2 = \text{diag} \left( e^{i\rho}, e^{i\sigma}, 1 \right).$$

The phases $\alpha$, $\beta$ and $\gamma$ are unphysical since they may be eliminated by a suitable redefinition of the charged-lepton fields. On the contrary, $\rho$ and $\sigma$\(^1\) Since the gauge interactions are flavour-blind, we \textit{a priori} have the freedom of performing arbitrary rotations in flavour space.
are physical in the case of Majorana neutrinos and are, therefore, referred to as the Majorana phases. $V$ denotes the well-known unitary matrix

$$V = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13}e^{-i\delta} \\
0 & 1 & 0 \\
-s_{13}e^{i\delta} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} & 0 \\
0 & c_{12} & 0 \\
-s_{12} & c_{12} & 0
\end{pmatrix}, \quad (7)$$

where $s_{ij} \equiv \sin \theta_{ij}$ and $c_{ij} \equiv \cos \theta_{ij}$ are the sines and cosines of the three mixing angles, respectively. The phase $\delta$ is responsible for a possible CP violation in neutrino oscillations (also in the Dirac case) and is, therefore, frequently referred to as the Dirac CP phase.

### 3.2. Upper and lower bounds on the absolute values of the elements of the neutrino mass matrix

The fact that the neutrino masses are the singular values of $M_\nu$ allows to derive a generic upper bound on the absolute values $|(M_\nu)_{\alpha\beta}|$. From linear algebra it is known that the absolute value of an element of a matrix is smaller or equal its largest singular value. For the neutrino mass matrix, this implies [1]

$$|(M_\nu)_{\alpha\beta}| \leq \max_k m_k. \quad (8)$$

Since this bound is valid for any matrix, it holds also for Dirac neutrinos. The strongest bounds on the absolute neutrino mass scale come from cosmology, where the sum of the masses of the light neutrinos is usually constrained to be at most of the order of $\mathcal{O}(1 \text{ eV})$ — see e.g. the list of upper bounds in [2]. From this, we deduce the approximate upper bound $m_k \lesssim 0.3 \text{ eV}$ leading to

$$|(M_\nu)_{\alpha\beta}| \lesssim 0.3 \text{ eV}. \quad (9)$$

In [1] also an analytical lower bound on the $|(M_\nu)_{\alpha\beta}|$ is provided. Defining $a_k \equiv m_k|V_{\alpha k}||V_{\beta k}|$, one can show that

$$|(M_\nu)_{\alpha\beta}| \geq 2 \max_k a_k - \sum_k a_k. \quad (10)$$

Note that this lower bound is independent of the Majorana phases $\rho$ and $\sigma$. Unlike the generic upper bound discussed before, the lower bound (10) is valid only for Majorana neutrinos. Numerically evaluating this lower bound using the results of the global fits of oscillation data of [3, 4] only for two matrix elements leads to non-trivial lower bounds. The lower bounds in units of eV for these matrix elements are listed in the following table taken from [1].
TABLE I

|          | \((M_\nu)_{ee}\) (inv. spect.) | \((M_\nu)_{\tau\tau}\) (norm. spect.) |
|----------|-------------------------------|----------------------------------------|
|          | Forero et al. [3]              | Fogli et al. [4]                        |
| \(|(M_\nu)_{ee}|\) | 1.52 \times 10^{-2}            | 0                                       |
| \(|(M_\nu)_{\tau\tau}|\) | 1.62 \times 10^{-2}            | 1.86 \times 10^{-2}                     |

For both global fits the only element being bounded from below at the 3\(\sigma\)-level is \(|(M_\nu)_{ee}|\) in the case of an inverted neutrino mass spectrum, for which in both cases one finds \(|(M_\nu)_{ee}| \gtrsim 10^{-2} \text{ eV}\). Unfortunately, this bound is still far from the current upper bound stemming from searches for neutrinoless double beta decay, which is given by [5, 6]

\[
m_{\beta\beta} \lesssim 0.4 \text{ eV}.
\]

3.3. Correlations of the elements of the neutrino mass matrix

In the case of Majorana neutrinos, the absolute values of the elements of \(M_\nu\) depend on nine real parameters, namely

\[
m_0, \quad \Delta m^2_{21}, \quad \Delta m^2_{31}, \quad \theta_{12}, \quad \theta_{13}, \quad \delta, \quad \rho, \quad \sigma,
\]

where \(m_0\) denotes the mass of the lightest neutrino. Using the experimental/observational constraints on these parameters, one can create plots of the allowed ranges of the \(|(M_\nu)_{\alpha\beta}|\) versus \(m_0\), which was first done by Merle and Rodejohann in [7]. In [1] the analysis of [7] was repeated using the results of the recent global fits of oscillation data of [3, 4]. It turned out that, at the 3\(\sigma\)-level, the plots of [7] are still in good agreement with the ones of [1].

In addition, in [1] also correlation plots of the \(|(M_\nu)_{\alpha\beta}|\) were created. Since the Majorana neutrino mass matrix has six independent entries, there are 15 correlations. Taking into account the two possible neutrino mass spectra, there is a total of 30 plots. Among these 30 correlations, one finds only five which are manifest at the 3\(\sigma\)-level, namely [1]:

\[
(M_\nu)_{ee} \quad \text{vs.} \quad (M_\nu)_{\mu\mu} \quad \text{(normal spectrum)},
(M_\nu)_{ee} \quad \text{vs.} \quad (M_\nu)_{\mu\tau} \quad \text{(normal spectrum)},
(M_\nu)_{ee} \quad \text{vs.} \quad (M_\nu)_{\tau\tau} \quad \text{(normal spectrum)},
(M_\nu)_{\mu\mu} \quad \text{vs.} \quad (M_\nu)_{\mu\mu} \quad \text{(normal spectrum)},
(M_\nu)_{\mu\tau} \quad \text{vs.} \quad (M_\nu)_{\tau\tau} \quad \text{(normal spectrum)}.
\]
All of these five correlations may be subsumed as “if one matrix element is small, the other one must be large”. An example for such a correlation plot can be found in figure 1. In the case of an inverted neutrino mass spectrum, there are no correlations manifest at the $3\sigma$-level.

It is important to note that while at the $3\sigma$-level the correlation plots based on the global fits of [3] and [4] agree, this is not true at the $1\sigma$-level — for further details see [1].

![Correlation plot of $|(M_{\nu})_{ee}|$ vs. $|(M_{\nu})_{\tau\tau}|$ based on the global fit results of Forero et al. [3] assuming a normal neutrino mass spectrum and allowing $m_0$ to vary between zero and 0.3 eV. The boundaries of the allowed areas are depicted by the following symbols: best fit: *, $1\sigma$: ▲, $3\sigma$: ●.](image)

Fig. 1. Correlation plot of $|(M_{\nu})_{ee}|$ vs. $|(M_{\nu})_{\tau\tau}|$ based on the global fit results of Forero et al. [3] assuming a normal neutrino mass spectrum and allowing $m_0$ to vary between zero and 0.3 eV. The boundaries of the allowed areas are depicted by the following symbols: best fit: *, $1\sigma$: ▲, $3\sigma$: ●.

4. Constraints on the Dirac neutrino mass matrix

4.1. Parametrization of the Dirac neutrino mass matrix

In analogy to the Majorana case, we will study the $3 \times 3$ Dirac neutrino mass matrix $M_D$ in the basis where the charged-lepton mass matrix is diagonal — see equation (3). In this basis, $M_D$ takes the form

$$M_D = V_R \text{diag} (m_1, m_2, m_3) U_{\text{PMNS}}^\dagger,$$

where $V_R$ is a unitary $3 \times 3$-matrix. $V_R$ can be eliminated by considering the matrix

$$H_D \equiv M_D^\dagger M_D = U_{\text{PMNS}} \text{diag} (m_1^2, m_2^2, m_3^2) U_{\text{PMNS}}^\dagger.$$


Since all observables accessible by current experimental scrutiny are contained in \( H_D \), all matrices \( M_D \) leading to the same \( H_D \) are indistinguishable from the experimental point of view. Therefore, the nine parameters of \( V_D \) have to be treated as free parameters. Consequently, in stark contrast to the Majorana case, in the Dirac case the neutrino mass matrix has at least nine free parameters (even if the mixing matrix and the neutrino masses are known).

This freedom of choosing \( V_R \) has important consequences for the analysis of \( M_D \). Obviously it is much harder to put constraints on the elements of \( M_D \) than in the Majorana case. The freedom of choosing \( V_R \) even allows to set several elements of \( M_D \) to zero without changing the physical predictions. This directly follows from the fact that every matrix can be decomposed into a product of a unitary matrix and an upper triangular matrix\(^2\). Thus, there is a choice of \( V_R \) such that

\[
M_D = \begin{pmatrix}
m_{11} & m_{12} & m_{13} \\
0 & m_{22} & m_{23} \\
0 & 0 & m_{33}
\end{pmatrix}.
\]  

(15)

Similarly, by multiplication of \( M_D \) by one of the six \( 3 \times 3 \) permutation matrices generated by

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]  

and

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]  

(16)

from the left, one can arbitrarily permute the rows of \( M_D \) without changing any physical predictions [8].

However, it is most important to note that the freedom of choosing \( V_R \) holds only as long as we do not impose a symmetry in the lepton sector. Namely, a flavour symmetry which acts non-trivially in the neutrino sector imposes constraints on the form of the neutrino mass matrix \( M_D \), i.e. not only on the neutrino masses and \( U_{PMNS} \) but also on \( V_R \). Consequently, a choice of \( V_R \), e.g. such that \( M_D \) is upper triangular, will, in general, be incompatible with the flavour symmetry. Nevertheless, if we want to set bounds on the elements of \( M_D \) (without introducing flavour symmetries), we indeed have the freedom of arbitrarily choosing the matrix \( V_R \). Therefore, examining bounds and correlations of the elements of \( M_D \) is much less elucidating than in the Majorana case. That said, studies of \( M_D \) such as the question for the allowed cases of texture zeros in \( M_D \) are still of great interest.

\(^2\) This factorization is known as the so-called QR-decomposition.
4.2. Texture zeros in the Dirac neutrino mass matrix

In the following, we will shortly comment on the allowed cases of texture zeros in $M_D$ under the assumption that $M_\ell$ is diagonal. A detailed analysis has been done by Hagedorn and Rodejohann in [8], which provides a classification of all possible texture zeros in this framework. We repeated the analysis of [8] of the allowed cases of five, four and three texture zeros in $M_D$ based on the global fit results of [4]. Our numerical results are in perfect agreement with the analysis of [8]. However, there are some previously allowed cases of texture zeros which can be excluded due to the new data, namely precisely those which lead to a vanishing or too small value ($\lesssim 10^{-3}$) of $\sin^2 \theta_{13}$, i.e. $A, B, \tilde{B}, C, D_1-D_3, \tilde{D}_1-\tilde{D}_3, E$ (inverted spectrum) and $\tilde{E}$ (inverted spectrum) in the notation of [8]. Consequently, all cases of five texture zeros in $M_D$ are now excluded, and among the cases of four texture zeros only $E$ (normal spectrum), $\tilde{E}$ (normal spectrum) as well as $F_1-F_3$ remain valid.

5. Conclusions

In the case of Majorana neutrinos, the absolute values of the elements of the light-neutrino mass matrix $M_\nu$ can be described by nine parameters, of which seven are constrained by experiments/observations. The by now very precise knowledge of the oscillation parameters therefore allows detailed studies of the elements of $M_\nu$, including their allowed ranges and their correlations.

The situation is quite different in the case of Dirac neutrinos, where the neutrino mass matrix $M_D$ is by far not uniquely determined, even if the neutrino masses and the mixing matrix are known. Therefore, putting bounds on the elements of $M_D$ is much harder than in the Majorana case. Nevertheless studies of $M_D$ are possible, for example the analysis of texture zeros in $M_D$. We reinvestigated the allowed texture zeros of $M_D$ in the basis where the charged-lepton mass matrix is diagonal. Our results agree with the original analysis [8], the only difference being that by now we know that $\sin^2 \theta_{13} \gg 10^{-3}$, which excludes some previously viable types of texture zeros.

The author wants to thank the Organizers for the invitation to the delightful and interesting conference. This work is supported by the Austrian Science Fund (FWF), Project No. P 24161-N16.

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3 In [8] also the cases of one and two texture zeros in $M_D$ are investigated, the result being that all these cases are allowed and do not show any relations among the observables.
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