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Spin 1 particle on 4-dimensional sphere: extended helicity operator, separation of the variables, and exact solutions

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Abstract

Spin 1 particle is investigated in 3-dimensional curved space of constant positive curvature. An extended helicity operator is defined and the variables are separated in a tetrad-based 10-dimensional Duffin-Kemmer equation in quasi cylindrical coordinates. The problem is solved exactly in hypergeometric functions, the energy spectrum determined by three discrete quantum numbers is obtained. Transition to a massless case of electromagnetic field is performed.

In 3-dimensional spherical Riemann space $S_3$ will use the following system of quasi-cylindric coordinates (see [1]; the same coordinate system was used when treating Landau problem in 3-dimensional spaces of constant curvature in [4, 5, 6])

$$dS^2 = c^2 dt^2 - \rho^2 \left[ \cos^2 z (dr^2 + \sin^2 r \, d\phi^2) + dz^2 \right],$$

$$z \in [-\pi/2, +\pi/2], \quad r \in [0, +\pi], \quad \phi \in [0, 2\pi]; \quad (1)$$

a diagonal tetrad (let $x^\alpha = (t, r, \phi, z)$)

$$e^\beta_{(a)}(x) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos^{-1} z & 0 & 0 \\
0 & 0 & \cos^{-1} z \sin^{-1} r & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}; \quad (2)$$
corresponding Christofel and Ricci coefficients are

\[
\begin{align*}
\Gamma^r_{jk} &= \begin{vmatrix}
0 & 0 & -\tan z \\
0 & -\sin r \cos r & 0 \\
-\tan z & 0 & 0
\end{vmatrix}, \\
\Gamma^\phi_{jk} &= \begin{vmatrix}
0 & \cot r & 0 \\
\cot r & 0 & -\tan z \\
0 & -\tan z & 0
\end{vmatrix}, \\
\Gamma^z_{jk} &= \begin{vmatrix}
\sin z \cos z & 0 & 0 \\
0 & \sin z \cos z \sin^2 r & 0 \\
0 & 0 & 0
\end{vmatrix},
\end{align*}
\]

\[\gamma_{122} = \frac{1}{\cos z \tan r}, \quad \gamma_{311} = -\tan z, \quad \gamma_{322} = -\tan z. \quad (3)\]

Tetrad-based Duffin-Kemmer equation (the notation from [2] is used) takes the form

\[
\left\{ i\beta^0 \frac{\partial}{\partial t} + \frac{1}{\cos z} \left( i\beta^1 \frac{\partial}{\partial r} + \beta^2 i\partial_\phi + i J^1 \cos r \right) + i\beta^3 \frac{\partial}{\partial z} + i \frac{\sin z}{\cos z} (\beta^1 J^3 + \beta^2 J^2) - M \right\} \Psi = 0. \quad (4)
\]

To separate the variables, we take the substitution

\[
\Psi = e^{-i\omega t} e^{im\phi} \begin{vmatrix}
\Phi_0(r, z) \\
\Phi(r, z) \\
\bar{E}(r, z) \\
\bar{H}(r, z)
\end{vmatrix}
\]

and use a block-representation (we use so-called cyclic basis for 10 × 10
Duffin-Kemmer matrices – see in [3]

\[
\begin{bmatrix}
\epsilon \cos z & 0 & 0 & 0 & 0 & e_1 & 0 \\
0 & 0 & i & 0 & 0 & 0 & \tau_1 \\
0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\tau_1 & 0 \\
\end{bmatrix} + i
\begin{bmatrix}
0 & 0 & 0 & 0 & e_1 & 0 & \partial \frac{\partial}{\partial r} \\
0 & 0 & 0 & 0 & 0 & 0 & \tau_2 \\
0 & 0 & 0 & 0 & 0 & -\tau_2 & 0 \\
\end{bmatrix} \nu - \cos r S_3
\]

\[
\frac{-1}{\sin r}
\begin{bmatrix}
0 & 0 & e_3 & 0 \\
0 & 0 & 0 & \tau_3 \\
0 & -\tau_3 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\partial \frac{\partial}{\partial z} \\
\end{bmatrix}
\]

\[
+i \cos z
\begin{bmatrix}
0 & 0 & -2e_3 & 0 \\
0 & 0 & 0 & -\tau_3 \\
0 & 0 & 0 & 0 \\
0 & +\tau_3 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
-M \cos z \\
\end{bmatrix}
\begin{bmatrix}
\Phi_0 \\
\bar{\Phi} \\
\bar{E} \\
\bar{H} \\
\end{bmatrix}
= 0 , \quad (6)
\]

or

\[
ie_1 \partial_r \bar{E} - \frac{1}{\sin r} e_2 (m - \cos r s_3) \bar{E} +
+i (\cos z \partial_z - 2 \sin z) e_3 \bar{E} = M \cos z \Phi_0 ,
\]

\[
ie \cos z \bar{E} + i \tau_1 \partial_r \bar{H} - \frac{\tau_2}{\sin r} (m - \cos r s_3) \bar{H} +
+i (\cos z \partial_z - \sin z) \tau_3 \bar{H} = M \cos z \bar{\Phi} ,
\]

\[-ie \cos z \bar{\Phi} - ic \partial_r \Phi_0 + \frac{m}{\sin r} e_3^+ \Phi_0 - i \cos z e_3^+ \partial_z \Phi_0 = M \cos z \bar{E} ,
\]

\[-i \tau_1 \partial_r \bar{\Phi} + \frac{m - \cos r s_3}{\sin r} \tau_2 \bar{\Phi} - i (\cos z \partial_z -
\sin z) \tau_3 \bar{\Phi} = M \cos z \bar{H} . \quad (7)
\]

After calculations needed we arrive at the system

\[
\gamma \left( \frac{\partial E_1}{\partial r} - \frac{\partial E_3}{\partial r} \right) - \frac{\gamma}{\sin r} \left[ (m - \cos r) E_1 + (m + \cos r) E_3 \right] -
-(\cos z \partial_z - 2 \sin z) E_2 = M \cos z \Phi_0 , \quad (8)
\]
\[ + i \epsilon \cos z E_1 + i \gamma \frac{\partial H_2}{\partial r} + i \gamma \frac{m}{\sin r} H_2 + \\
\quad + i (\cos z \frac{\partial}{\partial z} - \sin z) H_1 = M \cos z \Phi_1 , \\
\quad + i \epsilon \cos z E_2 + i \gamma \frac{\partial H_1}{\partial r} + \frac{\partial H_3}{\partial r} \]

\[- \frac{i \gamma}{\sin r} [(m - \cos r) H_1 - (m + \cos r) H_3] = M \cos z \Phi_2 , \\
\quad + i \epsilon \cos z E_3 + i \gamma \frac{\partial H_2}{\partial r} - i \gamma \frac{m}{\sin r} H_2 - \\
\quad - i (\cos z \frac{\partial}{\partial z} - \sin z) H_3 = M \cos z \Phi_3 , \\
\quad (9) \]

\[- i \epsilon \cos z \Phi_1 + \gamma \frac{\partial \Phi_0}{\partial r} + \gamma \frac{m}{\sin r} \Phi_0 = M \cos z E_1 , \\
\quad - i \Phi_2 - \frac{\partial \Phi_0}{\partial z} = ME_2 , \\
\quad - i \epsilon \cos z \Phi_3 - \gamma \frac{\partial \Phi_0}{\partial r} + \gamma \frac{m}{\sin r} \Phi_0 = M \cos z E_3 , \\
\quad (10) \]

\[- i \gamma \frac{\partial \Phi_2}{\partial r} - i \gamma \frac{m}{\sin r} \Phi_2 - i(\cos z \frac{\partial}{\partial z} - \sin z) \Phi_1 = M \cos z H_1 , \\
\quad - i \gamma \left( \frac{\partial \Phi_1}{\partial r} + \frac{\partial \Phi_3}{\partial r} \right) + i \gamma \frac{m}{\sin r} [(m - \cos r) \Phi_1 - (m + \cos r) \Phi_3] = M \cos z H_2 , \\
\quad - i \gamma \frac{\partial \Phi_2}{\partial r} + i \gamma \frac{m}{\sin r} \Phi_2 + i(\cos z \frac{\partial}{\partial z} - \sin z) \Phi_3 = M \cos z H_3 . \\
\quad (11) \]

With the use of the notation

\[ \gamma \left( \frac{\partial}{\partial r} + \frac{m - \cos r}{\sin r} \right) = a_- , \\
\gamma \left( \frac{\partial}{\partial r} + \frac{m + \cos r}{\sin r} \right) = a_+ , \\
\gamma \left( \frac{\partial}{\partial r} + \frac{m}{\sin r} \right) = a , \\
\gamma \left( - \frac{\partial}{\partial r} + \frac{m - \cos r}{\sin r} \right) = b_- , \\
\gamma \left( - \frac{\partial}{\partial r} + \frac{m + \cos r}{\sin r} \right) = b_+ , \\
\gamma \left( - \frac{\partial}{\partial r} + \frac{m}{\sin r} \right) = b , \quad (12) \]
it reads simpler

\[- b_+ E_1 - a_+ E_3 - \cos z(\frac{\partial}{\partial z} - 2 \tan z) E_2 = M \cos z \Phi_0, \quad (13)\]

\[ia \ H_2 + i \epsilon \ \cos z E_1 + i \cos z(\frac{\partial}{\partial z} - \tan z) H_1 = M \ \cos z \Phi_1, \]

\[= i b_+ H_1 + ia_+ H_3 + i \epsilon \ \cos z E_2 = M \ \cos z \Phi_2, \]

\[- b_+ H_2 + i \epsilon \ \cos z E_3 - i(\frac{\partial}{\partial z} - \tan z) H_3 = M \ \cos z \Phi_3, \quad (14)\]

\[a \ \Phi_0 - i \epsilon \ \cos z \Phi_1 = M \ \cos z E_1, \]

\[-i \epsilon \Phi_2 - \frac{\partial}{\partial z} \Phi_0 = M \ E_2, \]

\[b \ \Phi_0 - i \epsilon \ \cos z \Phi_3 = M \ \cos z E_3, \quad (15)\]

\[- ia \ \Phi_2 - i \cos z(\frac{\partial}{\partial z} - \tan z) \Phi_1 = M \ \cos z H_1, \]

\[i b_+ \Phi_1 - ia_+ \Phi_3 = M \ \cos z H_2, \]

\[i b \ \Phi_2 + i \cos(\frac{\partial}{\partial z} - \tan z) \Phi_3 = M \ \cos H_3. \quad (16)\]

Let us employ additional operator, a generalized helicity operator – such that

\[\Sigma \Psi = \sigma \Psi, \quad \Psi = e^{-i \epsilon t} e^{iem \phi} \phi(r, z) \begin{bmatrix} \Phi_0(r, z) \\ \Phi(r, z) \\ \vec{E}(r, z) \\ \vec{H}(r, z) \end{bmatrix}, \]

\[\begin{bmatrix} 1 \\ \cos z \left( S_1 \frac{\partial}{\partial r} + i S_2 \frac{m - S_3 \cos r}{\sin r} \right) + (\frac{\partial}{\partial z} - \tan z) S_3 \end{bmatrix} \begin{bmatrix} \Phi_0 \\ \vec{E} \\ \vec{H} \end{bmatrix} = \sigma \begin{bmatrix} \Phi_0 \\ \vec{E} \\ \vec{H} \end{bmatrix}. \quad (17)\]

From (17) it follows the system of 10 equations (let \( \gamma = 1/\sqrt{2} \)):

\[0 = \sigma \ \Phi_0, \quad (18)\]
\[
\gamma \frac{\partial}{\partial r} \Phi_2 + \gamma \frac{m}{\sin r} \Phi_2 + \cos z \left( \frac{\partial}{\partial z} - \tan z \right) \Phi_1 = \sigma \cos z \Phi_1 ,
\]

\[
\gamma \frac{\partial}{\partial r} \Phi_1 + \frac{\partial}{\partial r} \Phi_3 - \frac{\gamma}{\sin r} [(m - \cos r) \Phi_1 - (m + \cos r) \Phi_3] = \sigma \cos z \Phi_2 ,
\]

\[
\gamma \frac{\partial}{\partial r} \Phi_2 - \gamma \frac{m}{\sin r} \Phi_2 - \cos z \left( \frac{\partial}{\partial z} - \tan z \right) \Phi_3 = \sigma \cos z \Phi_3 ,
\] (19)

\[
\gamma \frac{\partial}{\partial r} E_2 + \gamma \frac{m}{\sin r} E_2 + \cos z \left( \frac{\partial}{\partial z} - \tan z \right) E_1 = \sigma \cos z E_1 ,
\]

\[
\gamma \frac{\partial}{\partial r} E_1 + \frac{\partial}{\partial r} E_3 - \frac{\gamma}{\sin r} [(m - \cos r) E_1 - (m + \cos r) E_3] = \sigma \cos z E_2 ,
\]

\[
\gamma \frac{\partial}{\partial r} E_2 - \gamma \frac{m}{\sin r} E_2 - \cos z \left( \frac{\partial}{\partial z} - \tan z \right) E_3 = \sigma \cos z E_3 ,
\] (20)

\[
\gamma \frac{\partial}{\partial r} H_2 + \gamma \frac{m}{\sin r} H_2 + \cos z \left( \frac{\partial}{\partial z} - \tan z \right) H_1 = \sigma \cos z H_1 ,
\]

\[
\gamma \frac{\partial}{\partial r} H_1 + \frac{\partial}{\partial r} H_3 - \frac{\gamma}{\sin r} [(m - \cos r) H_1 - (m + \cos r) H_3] = \sigma \cos z H_2 ,
\]

\[
\gamma \frac{\partial}{\partial r} H_2 - \gamma \frac{m}{\sin r} H_2 - \cos z \left( \frac{\partial}{\partial z} - \tan z \right) H_3 = \sigma \cos z H_3 .
\] (21)

With notation (12) it reads simpler

\[
0 = \sigma \Phi_0 ,
\] (22)

\[
+ \cos z \left( \frac{\partial}{\partial z} - \tan z \right) \Phi_1 = \sigma \cos z \Phi_1 - a \Phi_2 ,
\]

\[
- b_- \Phi_1 + a_+ \Phi_3 = \sigma \cos z \Phi_2 ,
\]

\[
- \cos z \left( \frac{\partial}{\partial z} - \tan z \right) \Phi_3 = \sigma \cos z \Phi_3 + b \Phi_2 ,
\] (23)

\[
+ \cos z \left( \frac{\partial}{\partial z} - \tan z \right) E_1 = \sigma \cos z E_1 - a E_2 ,
\]

\[
- b_- E_1 + a_+ E_3 = \sigma \cos z E_2 ,
\]

\[
- \cos z \left( \frac{\partial}{\partial z} - \tan z \right) E_3 = \sigma \cos z E_3 + b E_2 ,
\] (24)
\[ + \cos z \left( \frac{\partial}{\partial z} - \tan z \right) H_1 = \sigma \cos z H_1 - a H_2 , \]
\[ -b_- H_1 + a_+ H_3 = \sigma \cos z H_2 , \]
\[ - \cos z \left( \frac{\partial}{\partial z} - \tan z \right) H_3 = \sigma \cos z H_3 + b H_2 . \]  
(25)

Taking into account eqs. (22) – (25), from (13) – (16) we get
\[ - b_- E_1 - a_+ E_3 - \cos z \left( \frac{\partial}{\partial z} - 2 \tan z \right) E_2 = M \cos z \Phi_0 , \]  
(26)

\[ i \epsilon E_1 + i \sigma H_1 = M \Phi_1 , \]
\[ i \sigma H_2 + i \epsilon E_2 = M \Phi_2 , \]
\[ i \epsilon E_3 + i \sigma H_3 = M \Phi_3 , \]  
(27)

\[ a \Phi_0 - i \epsilon \cos z \Phi_1 = M \cos z E_1 , \]
\[ - i \epsilon \Phi_2 - \frac{\partial}{\partial z} \Phi_0 = M E_2 , \]
\[ b \Phi_0 - i \epsilon \cos z \Phi_3 = M \cos z E_3 , \]  
(28)

\[ - \sigma \Phi_1 = M H_1 , \]
\[ -i \sigma \Phi_2 = M H_2 , \]
\[ -i \sigma \Phi_3 = M H_3 . \]  
(29)

Below we will need an explicit form of the Lorentz condition. Starting from its tensor form
\[ \nabla_\beta \left( e^{(b)\beta} \Phi^{cart}_{(b)} \right) = 0 = \implies \]
\[ \frac{\partial \Phi^{(b)cart}_{(b)}}{\partial x^\beta} e^{(b)\beta} + \Phi^{cart}_{(b)} \nabla_\beta e^{(b)\beta} = 0 , \]  
(30)

or
\[ \frac{\partial \Phi^{cart}_{(b)}}{\partial x^\beta} e^{(b)\beta} + \Phi^{cart}_{(b)} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} \sqrt{-g} e^{(b)\beta} = 0 , \]  
(31)

and taking into consideration (1) – (2), we transform eq. (31) to the form
\[ \frac{\partial}{\partial t} \Phi^{cart}_0 - \frac{1}{\cos z \sin r} \frac{\partial}{\partial r} \Phi^{cart}_1 - \frac{1}{\cos z \sin r} \frac{\partial}{\partial \phi} \Phi^{cart}_2 - \frac{\partial}{\partial z} \Phi^{cart}_3 - \]
\[ - \Phi^{cart}_1 \frac{1}{\cos^2 z \sin r} \frac{\partial}{\partial r} \Phi^{cart}_1 - \frac{1}{\cos z \sin r} \frac{\partial}{\partial \phi} \Phi^{cart}_2 - \frac{\partial}{\partial z} \Phi^{cart}_3 - \]
\[ \Phi^{cart}_1 \frac{1}{\cos^2 z \sin r} \frac{\partial}{\partial r} \cos^2 z \sin r \frac{1}{\cos z} - \Phi^{cart}_3 \frac{1}{\cos^2 z \sin r} \frac{\partial}{\partial z} \cos^2 z \sin r = 0 , \]  
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that is
\[
\frac{\partial}{\partial t} \Phi_0^{\text{cart}} - \frac{1}{\cos z} \left( \frac{\partial}{\partial r} + \cos r \frac{\cos r}{\sin r} \right) \Phi_1^{\text{cart}} - \frac{1}{\cos z \sin r} \frac{\partial}{\partial \phi} \Phi_2^{\text{cart}} - \left( \frac{\partial}{\partial z} - 2 \tan z \right) \Phi_3^{\text{cart}} = 0 .
\]

From whence with the substitution \((3)\) we obtain
\[
- i \epsilon \Phi_0^{\text{cart}} - \frac{1}{\cos z} \left( \frac{\partial}{\partial r} + \cos r \frac{\cos r}{\sin r} \right) \Phi_1^{\text{cart}} - \frac{im}{\cos z \sin r} \Phi_2^{\text{cart}} - \left( \frac{\partial}{\partial z} - 2 \tan z \right) \Phi_3^{\text{cart}} = 0 .
\]

To use this relation in the above equations, we should transform \((32)\) to cyclic basis:

\[
\Phi_0 = \Phi_0^{\text{cart}} , \quad \Phi_2 = \Phi_3^{\text{cart}} , \quad \Phi_3 - \Phi_1 = \sqrt{2} \Phi_1^{\text{cart}} , \quad \Phi_3 + \Phi_1 = \sqrt{2} i \Phi_2^{\text{cart}} ;
\]

thus we have
\[
- i \epsilon \Phi_0 - \frac{1}{\cos z} \left( \frac{\partial}{\partial r} + \cos r \frac{\cos r}{\sin r} \right) \frac{\Phi_3 - \Phi_1}{\sqrt{2}} - \frac{im}{\cos z \sin r} \frac{\Phi_3 + \Phi_1}{\sqrt{2} i} - \left( \frac{\partial}{\partial z} - 2 \tan z \right) \Phi_2 = 0 ,
\]

that is
\[
- i \epsilon \Phi_0 - \frac{1}{\cos z} b_- \Phi_1 - \frac{1}{\cos z} a_+ \Phi_3 - \left( \frac{\partial}{\partial z} - 2 \tan z \right) \Phi_2 = 0 .
\]

Now, let us turn to eqs. \((26) - (29)\). First, let us consider the case \(\sigma \neq 0\), when one must accept from the very beginning such a restriction \(\Phi_0 = 0\); correspondingly, equation become more simple

\[
\sigma \neq 0 , \quad \Phi_0 = 0 , \quad -b_- E_1 - a_+ E_3 - \cos z \left( \frac{\partial}{\partial z} - 2 \tan z \right) E_2 = 0 ,
\]

\[
i \epsilon E_1 + i \sigma H_1 = M \Phi_1 , \quad i \sigma H_2 + i \epsilon E_2 = M \Phi_2 , \quad i \epsilon E_3 + i \sigma H_3 = M \Phi_3 ,
\]

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\[ -i \epsilon \Phi_1 = M E_1, \]
\[ -i \epsilon \Phi_2 = M E_2, \]
\[ -i \epsilon \Phi_3 = M E_3, \] (37)

\[ -\sigma \Phi_1 = M H_1, \]
\[ -i\sigma \Phi_2 = M H_2, \]
\[ -i\sigma \Phi_3 = M H_3. \] (38)

Note that substituting (33) into (30), one gets
\[ -b_- \Phi_1 - a_+ \Phi_3 - \cos z \left( \frac{\partial}{\partial z} - 2 \tan z \right) \Phi_2 = 0, \] (39)
which coincides with the Lorentz condition (34) when \( \Phi_0 = 0 \). Condition (39) can be simplified by the following substitutions
\[ \Phi_1 = \frac{\varphi_1}{\cos z}, \quad \Phi_3 = \frac{\varphi_3}{\cos z}, \quad \Phi_2 = \frac{1}{\cos^2 z} \varphi_2, \]
which results in
\[ -b_- \varphi_1 - a_+ \varphi_3 - \frac{\partial}{\partial z} \varphi_2 = 0; \] (40)

in new variables \( b_- \varphi_1 = \bar{\varphi}_1 \), \( a_+ \varphi_3 = \bar{\varphi}_3 \) it becomes yet simpler
\[ \bar{\varphi}_1 + \bar{\varphi}_3 + \frac{\partial}{\partial z} \bar{\varphi}_2 = 0. \] (41)

Remaining algebraic relations will fixe values of \( \sigma \) and relative coefficients of various components
\[ \sigma = \pm i \sqrt{\epsilon^2 - M^2}, \quad \Phi_0 = 0, \]
\[ H_j = -i \frac{\sigma}{M} \Phi_j, \quad E_j = \frac{i \epsilon}{M} \Phi_j. \] (42)

Explicit form of the main functions \( \Phi_j \) will be found below when exploring helicity operator equations.

In massless case instead of (42) we have
\[ \sigma = \pm i \epsilon, \quad \Phi_0 = 0, \]
\[ H_j = -i \sigma \Phi_j, \quad E_j = i \epsilon \Phi_j. \] (43)
Now, let us consider the case $\sigma = 0$, when the system (26) – (29) is

$$- b_ - E_1 - a_ + E_3 - \cos z (\frac{\partial}{\partial z} - 2 \tan z) E_2 = M \cos z \Phi_0 , \quad (44)$$

$$i \epsilon E_1 = M \Phi_1 , \quad i \epsilon E_2 = M \Phi_2 , \quad i \epsilon E_3 = M \Phi_3 , \quad (45)$$

$$a \Phi_0 - i \epsilon \cos z \Phi_1 = M \cos z E_1 ,$$
$$-i \epsilon \Phi_2 - \frac{\partial}{\partial z} \Phi_0 = M E_2 ,$$
$$b \Phi_0 - i \epsilon \cos z \Phi_3 = M \cos z E_3 , \quad (46)$$

$$0 = M H_1 , \quad 0 = M H_2 , \quad 0 = M H_3 . \quad (47)$$

Note that allowing for (45), from (44) it follows

$$- b_ - \Phi_1 - a_ + \Phi_3 - \cos z (\frac{\partial}{\partial z} - 2 \tan z) \Phi_2 = i \epsilon \cos z \Phi_0 , \quad (48)$$

which coincides with the Lorentz condition.

Let us introduce substitutions

$$\Phi_1 = \frac{\varphi_1}{\cos z} , \quad \Phi_3 = \frac{\varphi_3}{\cos z} , \quad \Phi_2 = \frac{1}{\cos^2 z} \varphi_2 ,$$

$$E_1 = \frac{e_1}{\cos z} , \quad E_3 = \frac{e_3}{\cos z} , \quad E_2 = \frac{1}{\cos^2 z} e_2 ,$$

$$b_ - \varphi_1 = \bar{\varphi}_1 , \quad a_ + \varphi_3 = \bar{\varphi}_3 , \quad b_ - e_1 = \bar{e}_1 , \quad a_ + e_3 = \bar{e}_3 ,$$

then eqs. (44) – (47) read

$$\bar{\varphi}_1 + \bar{\varphi}_3 + \frac{\partial}{\partial z} \varphi_2 = -i \epsilon \cos^2 z \Phi_0 , \quad (49)$$

$$i \epsilon e_1 = M \varphi_1 , \quad i \epsilon e_2 = M \varphi_2 , \quad i \epsilon e_3 = M \varphi_3 , \quad (50)$$

$$\Delta \Phi_0 - i \epsilon \bar{\varphi}_1 = M \bar{e}_1 ,$$
$$-i \epsilon \bar{\varphi}_2 - \cos^2 z \frac{\partial}{\partial z} \Phi_0 = M \bar{e}_2 ,$$
$$\Delta \Phi_0 - i \epsilon \bar{\varphi}_3 = M \bar{e}_3 , \quad (51)$$

$$0 = H_1 , \quad 0 = H_2 , \quad 0 = H_3 . \quad (52)$$
one should take into consideration identity \( \Delta = b_+ a = a_+ b \).

Below we will show from helicity operator eigenvalue equation that when \( \sigma = 0 \) there must hold the following relationships

\[
\begin{align*}
\bar{\varphi}_1 &= \bar{\varphi}_3 = \bar{\varphi}, & \bar{e}_1 &= \bar{e}_3 = \bar{e}, & \bar{h}_1 &= \bar{h}_3 = \bar{h}, \\
\Delta \varphi_2 &= -\cos^2 z \frac{\partial}{\partial z} \varphi, & \Delta \bar{e}_2 &= -\cos^2 z \frac{\partial}{\partial z} \bar{e}, & \Delta \bar{h}_2 &= -\cos^2 z \frac{\partial}{\partial z} \bar{h};
\end{align*}
\]

(53)

so that from (49) – (52) we get

\[
-2 \bar{\varphi} - \frac{\partial}{\partial z} \varphi_2 = i \epsilon \cos^2 z \Phi_0 ,
\]

(54)

\[
\bar{e} = \frac{M}{i \epsilon} \varphi , & \quad e_2 = \frac{M}{i \epsilon} \varphi_2 , & H_j = 0 ,
\]

(55)

\[
(e^2 - M^2) \varphi_2 - i \epsilon \cos^2 z \frac{\partial}{\partial z} \Phi_0 = 0 ,
\]

\[
i \epsilon \Delta \Phi_0 + (e^2 - M^2) \bar{\varphi} = 0 .
\]

(56)

Acting on the first equation in (56) by the operator \( \partial_z \), and excluding in second equation in (55) the variable \( \bar{\varphi} \) with the help of (53) – thus we get

\[
(e^2 - M^2) \frac{\partial}{\partial z} \varphi_2 - i \epsilon \cos^2 z \frac{\partial}{\partial z} \Phi_0 = 0 ,
\]

\[
2i \epsilon \Delta \Phi_0 - (e^2 - M^2)(\frac{\partial}{\partial z} \varphi_2 + i \epsilon \cos^2 z \Phi_0) = 0 .
\]

Summing these two equations, we arrive at a second order equation for \( \Phi_0 \)

\[
- i \epsilon \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} \Phi_0 + 2i \epsilon \Delta \Phi_0 - (e^2 - M^2)i \epsilon \cos^2 z \Phi_0 = 0 ,
\]

that is

\[
\left( -2 \Delta + \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} + (e^2 - M^2) \cos^2 z \right) \Phi_0 = 0 .
\]

(57)

In eq. (57), the variables are separated straightforwardly

\[
\Phi_0(r, z) = \Phi_0(r)\Phi_0(z) , & \quad \frac{1}{\Phi_0(r)} (2\Delta)\Phi_0(r) = \Lambda ,
\]

(58)
In the same manner, with the help of (54) on can exclude the function $\Phi_0$ from second equation in (56)

$$\Delta(-2\bar{\phi} - \frac{\partial}{\partial z}\phi_2) + (\epsilon^2 - M^2) \cos^2 z \bar{\phi} = 0 ,$$

and further excluding the variable $\Delta\phi_2$ with the help of (53) we arrive at a second order equation for $\bar{\phi}$

$$\left(-2\Delta + \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} + (\epsilon^2 - M^2) \cos^2 z\right) \bar{\phi} = 0 . \quad (59)$$

In this equation, the variable are separated as well

$$\bar{\phi}(r, z) = \bar{\phi}(r)\bar{\phi}(z) ,$$

$$\frac{1}{\bar{\phi}(r)} (2\Delta)\bar{\phi}(r) = \Lambda ,$$

$$\frac{1}{\bar{\phi}(z)} \left(\frac{d}{dz} \cos^2 z\frac{d}{dz} + (\epsilon^2 - M^2) \cos^2 z \right)\bar{\phi}(z) = \Lambda . \quad (60)$$

Note, that from the first equation in (56) it follows an expression for $\phi_2$

$$\phi_2 = \frac{i\epsilon \cos^2 z}{(\epsilon^2 - M^2)} \frac{\partial}{\partial z} \Phi_0 . \quad (61)$$

One can easily verify consistency of the relations obtained. Indeed, let us act on eq. (61) by the operator $\Delta$

$$\Delta\phi_2 = \frac{i\epsilon}{(\epsilon^2 - M^2)} \Delta \cos^2 z \frac{\partial}{\partial z} \Phi_0 = 0 .$$

Further, allowing for (53) we get

$$- \cos^2 z \frac{\partial}{\partial z} \bar{\phi} = \frac{i\epsilon}{(\epsilon^2 - M^2)} \Delta \cos^2 z \frac{\partial}{\partial z} \Phi_0 = 0 ,$$

from whence it follows

$$i\epsilon \Delta \Phi_0 + (\epsilon^2 - M^2) \bar{\phi} = 0 ,$$

which is an identity

$$- \frac{\partial}{\partial z} \bar{\phi} \equiv - \frac{1}{(\epsilon^2 - M^2)}(\epsilon^2 - M^2) \frac{\partial}{\partial z} \bar{\phi} .$$
Now, let us turn to equations steaming from diagonalization of helicity operator. In (22) – (25) owe can notice three similar groups of equations. For instance, equations for $H_i$ are

\[
\begin{align*}
\alpha H_2 + \cos z \left( \frac{\partial}{\partial z} - \tan z \right) H_1 &= \sigma \cos z H_1, \\
-b_- H_1 + a_+ H_3 &= \sigma \cos z H_2, \\
-b H_2 - \cos z \left( \frac{\partial}{\partial z} - \tan z \right) H_3 &= \sigma \cos z H_3.
\end{align*}
\]

(62)

With the help of substitutions

\[
H_1 = \frac{1}{\cos z} h_1(r, z), \\
H_2 = \frac{1}{\cos^2 z} h_2(r, z), \\
H_3 = \frac{1}{\cos z} h_3(r, z),
\]

(63)

they are simplified

\[
\begin{align*}
\alpha h_2 &= \cos^2 z (+\sigma - \frac{\partial}{\partial z}) h_1, \\
-b_- h_1 + a_+ h_3 &= \sigma h_2, \\
b h_2 &= \cos^2 z (-\sigma - \frac{\partial}{\partial z}) h_3.
\end{align*}
\]

(64)

Let us introduce new variables

\[
\begin{align*}
b_- h_1 &= \bar{h}_1, \\
a_+ h_3 &= \bar{h}_3;
\end{align*}
\]

(65)

from (64) it follows

\[
\begin{align*}
b_- a h_2 &= \cos^2 z (\sigma - \frac{\partial}{\partial z}) \bar{h}_1, \\
\bar{h}_3 - \bar{h}_1 &= \sigma \bar{h}_2, \\
a_+ b h_2 &= \cos^2 z (-\sigma - \frac{\partial}{\partial z}) \bar{h}_3.
\end{align*}
\]

(66)

Note that first and third equations contain one the same second order operator

\[
b_- a = a_+ b = \frac{1}{2} \left( -\frac{\partial^2}{\partial r^2} - \frac{\cos r}{\sin r} \frac{\partial}{\partial r} + \frac{m^2}{\sin^2 r} \right) = \Delta.
\]

(67)

First, let us consider the case $\sigma \neq 0$. Equating the right-hand sides of the first and third equations in (66), we get

\[
\sigma (\bar{h}_1 + \bar{h}_3) = -\frac{\partial}{\partial z} (\bar{h}_3 - \bar{h}_1) = -\sigma \frac{\partial}{\partial z} h_2;
\]

(68)
that is

\[ \bar{h}_3 + \bar{h}_1 = -\frac{\partial}{\partial z} h_2, \quad \bar{h}_3 - \bar{h}_1 = \sigma h_2. \]

Thus, we arrive at expression for \( \bar{h}_1 \) and \( \bar{h}_3 \) through

\[ \bar{h}_3 = \frac{1}{2}(+\sigma - \frac{\partial}{\partial z})h_2, \quad \bar{h}_1 = \frac{1}{2}(-\sigma - \frac{\partial}{\partial z})h_2. \] (69)

In turn, substituting (69) into (66) we obtain one the same second order equation for \( h_2 \)

\[ b_- a \ h_2 = \cos^2 z(\sigma - \frac{\partial}{\partial z}) \frac{1}{2}(-\sigma - \frac{\partial}{\partial z})h_2, \]

\[ a_+ b \ h_2 = \cos^2 z(-\sigma - \frac{\partial}{\partial z}) \frac{1}{2}(+\sigma - \frac{\partial}{\partial z})h_2. \] (70)

The variables in (70) are separated straightforwardly

\[ h_2(r, z) = h_2(r) \ h_2(z), \]

\[ \frac{1}{h_2(r)} (2b_- a) \ h_2(r) = \frac{1}{h_2(z)} \cos^2 z(\frac{d^2}{dz^2} - \sigma^2)h_2(z) = \Lambda, \]

from whence it follows separated differential equations

\[ (2b_- a) \ h_2(r) = \Lambda \ h_2(r), \] (71)

\[ (\frac{d^2}{dz^2} - \sigma^2)h_2(z) = \frac{\Lambda}{\cos^2 z} \ h_2(z). \] (72)

Similar results are valid for functions \( e_i \) and \( \varphi_i \):

\[ (2b_- a) \ e_2(r) = \Lambda \ e_2(r), \]

\[ (\frac{d^2}{dz^2} - \sigma^2)e_2(z) = \frac{\Lambda}{\cos^2 z} \ e_2(z), \]

\[ \bar{e}_1 = \frac{1}{2}(-\sigma - \frac{\partial}{\partial z})e_2, \quad \bar{e}_3 = \frac{1}{2}(+\sigma - \frac{\partial}{\partial z})e_2; \] (73)

\[ (2b_- a) \ \varphi_2(r) = \Lambda \ \varphi_2(r), \]

\[ (\frac{d^2}{dz^2} - \sigma^2)\varphi_2(z) = \frac{\Lambda}{\cos^2 z} \ \varphi_2(z), \]

\[ \bar{\varphi}_1 = \frac{1}{2}(-\sigma - \frac{\partial}{\partial z})\varphi_2, \quad \bar{\varphi}_3 = \frac{1}{2}(+\sigma - \frac{\partial}{\partial z})\varphi_2. \] (74)
Now, let us turn to the system (66) when \( \sigma = 0 \); it gives

\[
\begin{align*}
\bar{h}_3 &= \bar{h}_1 = \bar{h} , \\
b_- a h_2 &= - \cos^2 z \frac{\partial}{\partial z} \bar{h}, \\
a_+ b h_2 &= - \cos^2 z \frac{\partial}{\partial z} \bar{h} .
\end{align*}
\]  

(75)

Just these relations were used above starting with (53).

Let us construct solutions of eqs. (74):

\[
\begin{align*}
(2b-a) \varphi_2(r) &= \Lambda \varphi_2(r) , \\
\left( \frac{d^2}{dz^2} - \sigma^2 \right) \varphi_2(z) &= \frac{\Lambda}{\cos^2 z} \varphi_2(z) , \\
\bar{\varphi}_1 &= \frac{1}{2}(-\sigma - \frac{\partial}{\partial z})\varphi_2 , \\
\bar{\varphi}_3 &= \frac{1}{2}(+\sigma - \frac{\partial}{\partial z})\varphi_2 .
\end{align*}
\]  

(76)

In the radial equation

\[
(2b-a) \varphi_2(r) = \Lambda \varphi_2(r) ,
\]

or

\[
\left( \frac{d^2}{dr^2} + \frac{\cos r}{\sin r} \frac{d}{dr} - \frac{m^2}{\sin^2 r} + \Lambda \right) \varphi_2(r) = 0 ;
\]  

(77)

let us introduce a new variable \( 1 - \cos r = 2x \), \( x \in [0, 1] \):

\[
x (1-x) \frac{d^2 \varphi_2}{dx^2} + (1-2x) \frac{d \varphi_2}{dx} + \left( \Lambda - \frac{1}{4} \frac{m^2}{x} - \frac{1}{4} \frac{m^2}{1-x} \right) \varphi_2 = 0
\]  

(78)

and make a substitution \( \varphi_2 = x^a (1-x)^b F_2 \); thus we arrive at

\[
x (1-x) \frac{d^2 F_2}{dx^2} + [2a + 1 - (2a + 2b + 2)x] \frac{dF_2}{dx} + \\
+ \left[ -(a+b)(a+b+1) + \Lambda + \frac{1}{4} \frac{4a^2 - m^2}{x} + \frac{1}{4} \frac{4b^2 - m^2}{1-x} \right] F_2 = 0 .
\]  

(79)

At \( a, b \) taken according to

\[
a = \pm \left| \frac{m}{2} \right| , \\
b = \pm \left| \frac{m}{2} \right| ,
\]  

(80)
eq. (79) becomes simpler

\[ x (1 - x) \frac{d^2 F_2}{dx^2} + [2a + 1 - (2a + 2b + 2)x] \frac{dF_2}{dx} - \\
- [(a + b)(a + b + 1) - \Lambda] F_2 = 0 \]  

(81)

it represents a hypergeometric equations \[7\] with parameters

\[ \alpha = a + b + \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4\Lambda}, \]
\[ \beta = a + b + \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\Lambda}, \]
\[ \gamma = 2a + 1. \]  

(82)

By physical reason for \( a, b \) we take positive values

\[ a = +\left| {\frac{m}{2}} \right|, \quad b = +\left| {\frac{m}{2}} \right|; \]  

(83)

so the radial function looks as

\[ \varphi_2(r) = \left( \sin \frac{r}{2} \right)^{|m|} \left( \cos \frac{r}{2} \right)^{|m|} F(\alpha, \beta, \gamma; \sin^2 \frac{r}{2}); \]  

(84)

these solutions vanish at the points \( r = 0, +\pi \). To have polynomials one should impose the known condition \( \alpha = -n_r \), so we get a quantization rule

\[ + \frac{\sqrt{1 + 4\Lambda}}{2} = n_r + |m| + \frac{1}{2}; \]  

(85)

corresponding solutions are defined according to

\[ \varphi_2 = \left( \sin \frac{r}{2} \right)^{|m|} \left( \cos \frac{r}{2} \right)^{|m|} \times \\
\times F(-n, 2 | m | +1 + n, | m | +1; -\sin^2 \frac{r}{2}). \]  

(86)

Now, let us solve equation (76) in variable \( z \)

\[ \left( \frac{d^2}{dz^2} - \sigma^2 \right) \varphi_2(z) = \frac{\Lambda}{\cos^2 z} \varphi_2(z), \quad -\sigma^2 = e^2 - M^2. \]  

(87)

A first step is to introduce a new variable (which distinguish between conjugated point \( +z \) and \( -z \) of spherical space)

\[ y = \frac{1 + i \tan z}{2}, \quad 1 - y = \frac{1 - i \tan z}{2}; \]  

(88)
if \( z \in [-\pi/2, +\pi/2] \), the variable \( y \) belongs to a vertical line in the complex plane

\[
y = \left( \frac{1}{2} - i\infty, \frac{1}{2} + i\infty \right) \tag{89}
\]

Allowing for

\[
\frac{d}{dz} = i \frac{1}{2 \cos^2 z} \frac{d}{dy} = 2iy(1-y) \frac{d}{dy},
\]

\[
\frac{\Lambda}{\cos^2 z} = 4\Lambda y(1-y) \tag{90}
\]

eq. (87) reduces to

\[
\left( y(1-y) \frac{d^2}{dy^2} + (1-2y) \frac{d}{dy} + \Lambda - \frac{\epsilon^2 - M^2}{4y(1-y)} \right) \varphi_2 = 0 \tag{91}
\]

In the region \( y \sim 0 \), eq. (91) becomes simpler

\[
\left( y \frac{d^2}{dy^2} + \frac{d}{dy} - \frac{\epsilon^2 - M^2}{4y} \right) \varphi_2 = 0, \quad \varphi_2 \sim y^a,
\]

\[
a(a-1) + a - \frac{\epsilon^2 - M^2}{4} = 0, \quad a = \pm \frac{\sqrt{\epsilon^2 - M^2}}{2} \tag{92}
\]

In the region \( y \sim 1 \), eq. (91) becomes simpler as well

\[
\left( (1-y) \frac{d^2}{dy^2} - \frac{d}{dy} - \frac{\epsilon^2 - M^2}{4(1-y)} \right) \varphi_2 = 0, \quad \varphi_2 \sim (1-y)^b,
\]

\[
b(b-1) + b - \frac{\epsilon^2 - M^2}{4} = 0, \quad b = \pm \frac{\sqrt{\epsilon^2 - M^2}}{2} \tag{93}
\]

Searching solutions in the form \( \varphi_2(y) = y^a(1-y)^b F(y) \) for \( F(y) \) we have

\[
y(1-y) F'' + F'[(2a + 1) - y(2a + 2b + 2)] F' +
+ \left[ - (a+b)(a+b+1) + \Lambda + \frac{1}{y} \left( a^2 - \frac{\epsilon^2 - M^2}{4} \right) +
+ \frac{1}{1-y} \left( a^2 - \frac{\epsilon^2 - M^2}{4} \right) \right] F = 0 \tag{94}
\]

Let it be

\[
a = \pm \frac{\sqrt{\epsilon^2 - M^2}}{2}, \quad b = \pm \frac{\sqrt{\epsilon^2 - M^2}}{2},
\]

\[
\varphi_2 = \left( \frac{1 + i \tan z}{2} \right)^a \left( \frac{1 - i \tan z}{2} \right)^b F \tag{95}
\]
there are four possibilities depending on $a, b$

\[
\begin{align*}
    a = b &= -\frac{\sqrt{\epsilon^2 - M^2}}{2}, & \varphi_2 &\sim \cos^{-2a} z \, F(z) ; \\
    a = b &= +\frac{\sqrt{\epsilon^2 - M^2}}{2}, & \varphi_2 &\sim \cos^{-2a} z \, F(z) ; \\
    a &= -b , & \varphi_2 &\sim +e^{+2iaz} \, F(z) ; \\
    a &= -b , & \varphi_2 &\sim -e^{-2iaz} \, F(z). \\
\end{align*}
\]

(96)

As relations (95) hold, eq. (94) takes the form

\[
y(1 - y) F'' + F'[\{2(a + 1) - y(2a + 2b + 2)\}] F' - \left\{ (a + b)(a + b + 1) - \Lambda \right\} F = 0 ,
\]

(97)

which can be recognized as a hypergeometric equation \[7\]

\[
y(1 - y) F + \left[ \gamma - (\alpha + \beta + 1)y \right] F' - \alpha \beta F = 0 ,
\]

\[
\gamma = (2a + 1) , \quad \alpha = a + b + \frac{1}{2} + \frac{\sqrt{4\Lambda + 1}}{2} , \\
\beta = a + b + \frac{1}{2} - \frac{\sqrt{4\Lambda + 1}}{2} .
\]

(98)

(99)

In this point we should notice that the spectrum for $\Lambda$ has been found (see (85)) from analyzing the differential equation in the variable $r$, therefore now to produce a spectrum for energy one must consider the cases with $a = b$.

There arise two possibilities.

The first:

\[
\begin{align*}
    2a = 2b &= \sqrt{\epsilon^2 - M^2} , & \beta &= -n_z , \\
    -\sqrt{\epsilon^2 - M^2} &= n_z + \frac{1}{2} - \frac{\sqrt{4\Lambda + 1}}{2} < 0 , \\
    \varphi_2 &\sim (\cos z)^{-\sqrt{\epsilon^2 - M^2}} P_n(\frac{e^{iz}}{2\cos z}) , & y &= \frac{1 + i \tan z}{2} = \frac{e^{iz}}{2\cos z} ,
\end{align*}
\]

(100)

because those solutions tends to infinity at $z = \pm \pi$ they cannot describe physical bound states.

The second:

\[
\begin{align*}
    -2a &= -2b = +\sqrt{\epsilon^2 - M^2} , & \alpha &= -n_z , \\
    +\sqrt{\epsilon^2 - M^2} &= n_z + \frac{1}{2} + \frac{\sqrt{4\Lambda + 1}}{2} > 0 , \\
    \varphi_2 &\sim (\cos z)^{+\sqrt{\epsilon^2 - M^2}} P_n(\frac{e^{iz}}{2\cos z}) , & y &= \frac{1 + i \tan z}{2} = \frac{e^{iz}}{2\cos z} .
\end{align*}
\]

(101)
These solutions are finite at the points \( z = \pm \pi/2 \) and they describe bound states.

In the formula for \( \sqrt{\epsilon^2 - M^2} \) (101) one must take into account the quantization rule \( \Lambda \) in (85) – thus we arrive at the formula determining values of energy by two discrete quantum numbers.

\[
+ \sqrt{\epsilon^2 - M^2} = n_\pi + n_r + | m | + 1 ;
\]

remember that these formulas concern the non-zero values for helicity operator \( \sigma = \pm i \sqrt{\epsilon^2 - M^2} \).

It remains to specify the energy spectrum for states with \( \sigma = 0 \) which are determined by the equations

\[
\left( -2\Delta + \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} + (\epsilon^2 - M^2) \cos^2 z \right) \tilde{\varphi} = 0 ,
\]

\[
\frac{1}{\tilde{\varphi}(r)} (2\Delta) \tilde{\varphi}(r) = \Lambda ,
\]

\[
\frac{1}{\tilde{\varphi}(z)} \left( \frac{d}{dz} \cos^2 z \frac{d}{dz} + (\epsilon^2 - M^2) \cos^2 z \right) \tilde{\varphi}(z) = \Lambda .
\]

Equation in the variable \( r \) has been solved above. The equation in \( z \) variable

\[
\left( \frac{d^2}{dz^2} - 2 \sin z \frac{d}{\cos z} \frac{d}{dz} + \epsilon^2 - M^2 - \frac{\Lambda}{\cos^2 z} \right) \varphi(z) = 0
\]

with the use of substitution \( \varphi = \frac{1}{\cos z} f(z) \) reduces to

\[
\frac{d^2 f}{dz^2} + \left( \epsilon^2 - M^2 + 1 - \frac{\Lambda}{\cos^2 z} \right) f(z) = 0 .
\]

It coincides with (87)

\[
\frac{d^2 \varphi_2}{dz^2} + \left( \epsilon^2 - M^2 - \frac{\Lambda}{\cos^2 z} \right) \varphi_2(z) = 0 ,
\]

with one formal change

\[
\epsilon^2 - M^2 \to \epsilon^2 - M^2 + 1 .
\]

Therefore, solutions of (104) are written straightforwardly

\[
f = \left( \frac{1 + i \tan z}{2} \right)^a \left( \frac{1 - i \tan z}{2} \right)^b F \left( \alpha, \beta, \gamma; \frac{1 + i \tan z}{2} \right) ,
\]

19
where $F$ stand for a hypergeometric function [7] with parameters

$$
\alpha = a + b + \frac{1}{2} + \frac{\sqrt{4\Lambda + 1}}{2}
$$

$$
\beta = a + b + \frac{1}{2} - \frac{\sqrt{4\Lambda + 1}}{2}, \quad \gamma = (2a + 1).
$$

To bound states correspond $a$, $b$ defined as

$$
a = b = -\frac{\sqrt{\epsilon^2 - M^2 + 1}}{2}.
$$

The quantization rule $\alpha = -n_z$ gives

$$
+ \sqrt{\epsilon^2 - M^2 + 1} = n_z + \frac{1}{2} + \frac{\sqrt{4\Lambda + 1}}{2} > 0.
$$

Thus, allowing for quantization for $\Lambda$ (85) we get a formulas for energy levels

$$
+ \sqrt{\epsilon^2 - M^2 + 1} = n_z + n_r + |m| + 1; \quad (105)
$$

it refers to the case of $\sigma = 0$.

**Conclusion:**

Let us summarize result.

Spin 1 particle is investigated in 3-dimensional curved space of constant positive curvature. An extended helicity operator is defined and the variables are separated in a tetrad-based 10-dimensional Duffin-Kemmer equation in quasi cylindrical coordinates. The problem is solved exactly in hypergeometric functions, the energy spectrum determined by three discrete quantum numbers is obtained. Transition to a massless case of electromagnetic field is performed.

The given problem can represent some interest as an exactly solvable model for describing composite systems (particles) of spin 1 or electromagnetic fields in the non-trivial space-time background, modeling the presence of a finite 3-dimensional box.

**References**

[1] M.N. Olevsky. Three-orthogonal coordinate systems in spaces of constant curvature, in which equation $(\Delta_2 + \lambda)U = 0$ permits the full separation of variables. Mathematical collection. Vol. 27. P. 379–426 (1950).
[2] V.M. Red’kov. The fields of the particles in a Riemannian space and the Lorentz group. Publishing House "Belarusian Science, Minsk (2009).

[3] V.M. Red’kov. Tetrad formalism, spherical symmetry and Schrödinger basis. Publishing House "Belarusian Science, Minsk (2011).

[4] A.A. Bogush, V.M. Red’kov, G.G. Krylov. Schrödinger particle in magnetic and electric fields in Lobachevsky and Riemann spaces. Nonlinear Phenomena in Complex Systems. 11, no 4, 403–416 (2008).

[5] A.A. Bogush, V.M. Red’kov, G.G. Krylov. Quantum-mechanical particle in a uniform magnetic field in spherical space $S_3$. Proceedings of the National Academy of Sciences of Belarus. Ser. fiz.-mat. 2, 57–63 (2009).

[6] V.V. Kisek, E.M. Ovsiyuk, O.V. Veko, V.M. Red’kov. Quantum mechanics for a vector particle in the magnetic field on four-dimensional sphere. Naucho-teknicheskie Vedomosti, St. Peterbourg State Pedagogical University. Ser. phys.-math. 2012. no 1 (in press).

[7] G. Bateman, A. Erdei. Higher Transcendental Functions. Vol. 1. Hypergeometric function, Legendre functions. Moscow (1973).