"Test scores correlate negatively and significantly with the irrationality measure; therefore, students with higher intellectual ability and logical thinking skills tend to be less irrational." – Educational research survey

ABSTRACT. The irrationality exponent $\mu(\alpha)$ of an irrational number $\alpha$, defined using the irrationality measure $1/q^\mu$, distinguishes among non-Liouville numbers, and is infinite for Liouville numbers. Using the irrationality measure $1/\beta^q$, we define the irrationality base $\beta(\alpha)$, which distinguishes among Liouville numbers and is 1 for non-Liouville numbers. We give some properties and examples. Assuming a condition on certain linear forms in logarithms, for which we present numerical evidence, we prove an upper bound on the irrationality base $\beta(\gamma)$ of Euler's constant, $\gamma$. If $\gamma$ is irrational and the condition turns out to be false in a certain strong sense, we prove an upper bound on the irrationality exponent $\mu(\gamma)$.

1. INTRODUCTION

The irrationality exponent $\mu(\alpha)$ of an irrational number $\alpha$ is defined in terms of the irrationality measure $1/q^\mu$ (see Section 2). Using the irrationality measure $1/\beta^q$, we introduce a weaker measure of irrationality, the irrationality base $\beta(\alpha)$, as follows. If there exists a real number $\beta \geq 1$ with the property that for any $\varepsilon > 0$ there is a positive integer $q(\varepsilon)$ such that

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{(\beta + \varepsilon)^q}$$

for all integers $p, q$ with $q \geq q(\varepsilon)$,

then we denote by $\beta(\alpha)$ the least such $\beta$, and we call $\beta(\alpha)$ the irrationality base of $\alpha$. If no such $\beta$ exists, then we call $\alpha$ a super Liouville number, and we write $\beta(\alpha) = \infty$. Since $\beta(\alpha) = 1$ if $\mu(\alpha)$ is finite (see Lemma 2), we may regard the irrationality base as a measure of irrationality for Liouville numbers. We give two examples: a super Liouville number, and a Liouville number with irrationality base 1.

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In [7] we gave criteria for irrationality of Euler’s constant,
\[
\gamma = \lim_{N \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \log N \right).
\]
The criteria involve a Beukers-type [1] double integral
\[
I_n = \int_{[0,1]^2} \frac{(x(1-x)y(1-y))^n}{(1-xy)\log xy} \, dx \, dy
\]
(defined equivalently in [8] as a "hypergeometric integral" or a Nesterenko-type [5] series) and a sequence of positive integers
\[
S_n = \prod_{m=1}^{n} \prod_{k=0}^{\min(m-1,n-m)} \prod_{j=k+1}^{n} \left( n + m \right)^{\frac{2d_{2n}}{j}},
\]
\(n = 1, 2, \ldots,\) where \(d_n\) denotes the least common multiple of the numbers \(1, 2, \ldots, n.\) In particular, we proved that \(\gamma\) is irrational if \(\frac{1}{n} \log \| \log S_n \|\) does not tend to \(-2\log(4/e) = -0.772\ldots\) as \(n \to \infty,\) where \(\| t \|\) denotes the distance from \(t\) to the nearest integer. Computations by P. Sebah [6] (see Figure 1) suggest that in fact
\[
(S) \quad \lim_{n \to \infty} \frac{1}{n} \log \| \log S_n \| = 0.
\]

The main result of the present paper is: If this limit equals zero, then Euler’s constant is irrational, but is not a super Liouville number. More precisely, we prove that condition (S) implies that \(\gamma\) has irrationality base \(\beta(\gamma) \leq 2e = 5.436\ldots,\) that is, for any \(\varepsilon > 0\) there exists \(q(\varepsilon) > 0\) such that
\[
\left| \gamma - \frac{p}{q} \right| > \frac{1}{(2e + \varepsilon)^q} \quad \text{for all integers } p, q, \text{ with } q \geq q(\varepsilon).
\]
Thus, a stronger hypothesis than is needed to guarantee irrationality of \(\gamma\) yields a stronger conclusion, namely, an irrationality measure for \(\gamma\). The proof uses a formula expressing \(d_{2n} I_n\) as a \(\mathbb{Z}\)-linear form in 1, \(\gamma\), and \(\log S_n\).
Having given an upper bound for $\beta(\gamma)$ if there does not exist a subsequence $\log S_{n_k}$ tending to zero exponentially, we then give an upper bound for the irrationality exponent $\mu(\gamma)$ if such a subsequence does exist, provided both that its convergence rate is different from that of the subsequence $d_{2n_k} I_{n_k}$ (which implies irrationality of $\gamma$), and that the sequence $n_k$ is asymptotically linear. The proof uses standard lemmas on irrationality exponents.

In Section 2, we define irrationality measures, exponents, and bases, and give some properties and examples. In Sections 3 and 4, we prove conditional upper bounds on $\beta(\gamma)$ and $\mu(\gamma)$, respectively. All the conditional bounds are effective.

In a paper in preparation, we construct a sequence of integers similar in size to $S_n$ for which the condition analogous to (S) is false. (This shows that any proof of (S) must use the explicit formula for $S_n$, and cannot rely solely on a priori lower bounds for linear forms in logarithms, such as those in Baker’s theory.) We also obtain conditional irrationality measures for $\log \pi$, and new ones for $\gamma$ derived from formulas for it in [9]. Finally, we give additional properties of the irrationality base, provide examples where it is finite but greater than 1, and fit it into Mahler’s classification of transcendental numbers.
2. IRRATIONALITY MEASURES, EXPONENTS, AND BASES

**Definition 1.** An *irrationality measure* is a function $f(q, \lambda)$ of a natural number $q$ and a positive real number $\lambda$, which takes values in the positive reals and is decreasing in both $q$ and $\lambda$. Given an irrational number $\alpha$, if there exists $\lambda > 0$ with the property that for any $\epsilon > 0$ there is a positive integer $q(\epsilon)$ such that

$$\left| \alpha - \frac{p}{q} \right| > f(q, \lambda + \epsilon) \quad \text{for all integers} \ p, \ q \ \text{with} \ q \geq q(\epsilon),$$

then we denote by $\lambda(\alpha)$ the least such $\lambda$, and we say that $\alpha$ has *irrationality measure* $f(q, \lambda(\alpha))$.

**Definition 2.** If $\alpha$ has irrationality measure $f(q, \mu) = 1/q^\mu$ for some $\mu = \mu(\alpha)$, then $\mu(\alpha) \in [2, \infty)$ is called the *irrationality exponent* (or, by abuse of terminology, the irrationality measure) of $\alpha$. Otherwise, if no such $\mu$ exists, one writes $\mu(\alpha) = \infty$ and says that $\alpha$ is a *Liouville number*.

**Definition 3.** If $\alpha$ has irrationality measure $f(q, \beta) = 1/\beta^q$, so that $\beta = \beta(\alpha)$ is the least number with the property that for any $\epsilon > 0$ there exists $q(\epsilon) > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{\beta + \epsilon} \quad \text{for all integers} \ p, \ q \ \text{with} \ q \geq q(\epsilon),$$

then we call $\beta(\alpha) \in [1, \infty)$ the *irrationality base* of $\alpha$. Otherwise, if no such $\beta$ exists, we write $\beta(\alpha) = \infty$ and we say that $\alpha$ is a *super Liouville number*.

Note that we may also write inequality (1) as

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{q^{\log(\beta + \epsilon)/\log q}}.$$

**Lemma 1.** (A characterization of super Liouville numbers)
(i). A real number $\alpha$ has $\beta(\alpha) = \infty$ if and only if, given any $\lambda > 1$, we have

$$0 < \left| \alpha - \frac{p}{q} \right| < \frac{1}{\lambda^q} \quad \text{for infinitely many integers} \ p, \ q, \ \text{with} \ q > 0.$$  

(ii). For fixed $\lambda > 1$, condition (3) holds if and only if $\alpha$ is irrational and $\beta(\alpha) \geq \lambda$.

**Proof.** We prove (ii), which implies (i). Fix $\lambda > 1$. Since $1/\lambda^q < 1/q^\lambda$ for $q$ large, (ii) follows from Definition 2, together with the fact that $\alpha$ is irrational if there exist infinitely many integers $p, q$, with $q > 0$, such that $0 < |\alpha - p/q| < 1/q^\lambda$.  

•
Example 1. (A super Liouville number) Let $T$ denote the sum of reciprocals of power towers

$$T = \sum_{n=1}^{\infty} \frac{1}{T_n} = \frac{1}{2} + \frac{1}{4^2} + \frac{1}{8^4} + \cdots,$$

where $T_1 = 2$ and $T_n = (2^n)^{T_{n-1}}$, for $n > 1$. Noting that the denominator of the $n$th partial sum $s_n$ of the series is $T_n$, we see that, given $\lambda > 1$, inequalities (3) hold with $\alpha = T$ and $p/q = s_n$, for all $n$ with $2^n \geq \lambda$. Therefore, $\beta(T) = \infty$.

By the following observation, the irrationality base measures the irrationality of Liouville numbers.

Lemma 2. (A relation between irrationality exponent and base) If $\mu(\alpha)$ is finite, then $\beta(\alpha) = 1$; equivalently, if the irrationality base of $\alpha$ exceeds 1, then $\alpha$ is a Liouville number. In particular, a super Liouville number is also a Liouville number.

Proof. This follows from Definitions 2 and 3, since with $\beta = 1$ the exponent $q \log(1 + \varepsilon)/\log q$ in (2) tends to infinity with $q$.

The converse of Lemma 2 is false, as the following example shows. (For facts used about convergents to continued fractions, see e.g. [4, Chapter 10].)

Example 2. (A Liouville number with irrationality base 1) The continued fraction

$$L = \cfrac{1}{10^1 + \cfrac{1}{10^2 + \cfrac{1}{10^3 + \cdots}}}$$

has irrationality exponent $\mu(L) = \infty$ (see [4, page 162, example (b)]). We show that $\beta(L) = 1$. Suppose on the contrary that $\beta(L) > 1$. Then for some $\varepsilon > 0$ there exist infinitely many positive integers $p, q$ such that

$$\frac{1}{(1 + \varepsilon)^q}.$$

If $q$ is large enough, this bound is less than $1/(2q^2)$, so $p/q$ must be one of the convergents $p_n/q_n$, $n = 0, 1, \ldots$. Since $q_0 = 1$, $q_1 = 10$, and $q_n = 10^n q_{n-1} + q_{n-2}$ for $n > 1$, by induction it follows that $10^{n-1} < q_n < 10^{2n}$ if $n \geq 2$. Hence, for $n$ sufficiently large,
\[
L = \frac{p_n}{q_n} > \frac{1}{q_n q_{n+2}} > \frac{1}{10^{2(n+2)(n+3)}} > \frac{1}{(1 + \epsilon)^{10^n}} > \frac{1}{(1 + \epsilon)^{q_n}},
\]

contradicting (4). Therefore, \(\beta(L) = 1\).

In Section 3, assuming a hypothesis, we deduce upper bounds for an irrationality base directly from the definition. In Section 4, however, in order to obtain conditional upper bounds for the irrationality exponent of a real number \(\alpha\), we first construct sequences of integers \(p_k, q_k\) such that

\[
\epsilon_k := q_k \alpha - p_k
\]
tends to zero exponentially. We then use one of the following two standard lemmas. (For proofs, see [2, Lemma 3.5] and [3, Remark 2.1], respectively.)

**Lemma 3.** (Chudnovsky) Suppose that

\[
\limsup_{k \to \infty} \frac{1}{k} \log |q_k| \leq \sigma, \quad \lim_{k \to \infty} \frac{1}{k} \log |\epsilon_k| = -\tau,
\]

for some positive numbers \(\sigma, \tau\). Then \(\alpha\) has irrationality exponent \(\mu(\alpha) \leq 1 + \sigma/\tau\).

**Lemma 4.** (Hata) Suppose that \(\alpha\) is irrational and that

\[
\lim_{k \to \infty} \frac{1}{k} \log |q_k| = \sigma, \quad \limsup_{k \to \infty} \frac{1}{k} \log |\epsilon_k| \leq -\tau,
\]

for some positive numbers \(\sigma, \tau\). Then \(\mu(\alpha) \leq 1 + \sigma/\tau\).

**Remark 1.** If \(q_k \sim e^{\sigma k}\) and \(\epsilon_k \sim e^{-\tau k}\) as \(k \to \infty\), then \(\alpha - p_k/q_k - q_k^{-1-\sigma/\tau}\). By Definition 2, it follows that \(\mu(\alpha) \geq 1 + \tau/\sigma\). Hence \(\sigma \geq \tau\), by Lemma 3.

**Remark 2.** It would be interesting to find analogous lemmas for the irrationality base.

3. CONDITIONAL BOUNDS ON THE IRRATIONALITY BASE OF EULER’S CONSTANT

Assuming Condition (S) or weaker conditions, we derive upper bounds on the irrationality base of \(\gamma\). Note that (S) implies that \((1/n) \log \{\log S_n\}\) also tends to zero, where \(\{t\}\) denotes the fractional part of \(t > 0\).

From [7], [8], we have the relations

\[
d_{2n} I_n = d_{2n} \left(\frac{2n}{n}\right) \gamma + \log S_n - d_{2n} A_n, \quad d_{2n} A_n \in \mathbb{Z}
\]
and the criterion

\[(6) \quad d_{2n}I_n = \{\log S_n\}, \text{ for } n \geq n_0 \iff \gamma \in \mathbb{Q}.\]

From \([7]\), Stirling's formula, and the Prime Number Theorem, we have

\[(7) \quad 4^{-2n} > I_n = 4^{-2n(1+o(1))},\]

\[(8) \quad 2^{2n} \left(\binom{2n}{n}\right) = 2^{2n(1+o(1))},\]

\[(9) \quad e^{n(1+\varepsilon)} > d_n = e^{n(1+o(1))},\]

respectively, where the first inequality in (9) holds for any \(\varepsilon > 0\) and \(n \geq n(\varepsilon)\).

Criterion (6) and asymptotics (7), (9) yield the implication

\[\gamma \in \mathbb{Q} \Rightarrow \lim_{n \to \infty} \frac{1}{n} \log \{\log S_n\} = -2\log(4/e).\]

Each of the conditions that we assume below contradicts the last equation, and therefore implies that \(\gamma \notin \mathbb{Q}\).

To find a conditional upper bound for \(\beta(\gamma)\), suppose first that \(\beta(\gamma) > 1\). It follows from Definition 3 that, given \(\lambda\) between 1 and \(\beta(\gamma)\), there exist integers \(p_k, q_k\), for \(k = 1, 2, \ldots\), such that \(0 < q_k < q_{k+1}\) and

\[(10) \quad \left| \gamma - \frac{p_k}{q_k} \right| < \frac{1}{\lambda^{q_k}}.\]

Set

\[n_k = \left\lfloor \frac{q_k}{2} \right\rfloor\]

and use (5) to write

\[(11) \quad d_{2n_k} \left( I_{n_k} - \binom{2n_k}{n_k} \left(\gamma - \frac{p_k}{q_k}\right) \right) = \log S_{n_k} - d_{2n_k} \left( A_{n_k} - \binom{2n_k}{n_k} \frac{p_k}{q_k} \right),\]

where \(d_{2n_k}A_{n_k} \in \mathbb{Z}\). Since \(q_k \leq 2n_k\), the expression following \(\log S_{n_k}\) is an integer; thus the absolute value of the left-hand side of (11), if less than 1, is \(\geq \left\lfloor \log S_{n_k} \right\rfloor\). Also, since \(q_k \geq 2n_k - 1\), inequalities (7), (8), (9), (10) imply that, for any \(\varepsilon > 0\), we have
Suppose now that $\beta(\gamma) > 2e$. Choose $\lambda$ between $2e$ and $\beta(\gamma)$, and let $\varepsilon > 0$ be so small that

\begin{equation}
(13) \quad m_{\lambda, \varepsilon} := e^{-1+\varepsilon} \min(4, \lambda/2) > 1.
\end{equation}

Then, for $k$ large, the right-hand side of (12) is less than 1. Using (11), (12), it follows that

\begin{equation}
(14) \quad \log S_{n_k} < (1 + \lambda) m_{\lambda, \varepsilon}^{-2n_k} \quad \text{for } k \geq k(\lambda, \varepsilon),
\end{equation}

which contradicts (S). This establishes our main result.

**Theorem 1.** (First conditional bound on $\beta(\gamma)$) Assume that

\[ \lim_{n \to \infty} \frac{1}{n} \log \| \log S_n \| = 0. \]

Then Euler’s constant is irrational, but is not a super Liouville number. More precisely, the irrationality base of $\gamma$ satisfies $\beta(\gamma) \leq 2e$, that is, for any $\varepsilon > 0$ there exists $q(\varepsilon) > 0$ such that

\[ \left| \gamma - \frac{p}{q} \right| > \frac{1}{(2e + \varepsilon)^q} \quad \text{for all integers } p, q \text{ with } q \geq q(\varepsilon). \]

Letting $(\lambda, \varepsilon)$ tend to $(\beta(\gamma), 0)$ in (14), and using (13), we obtain

\[ \liminf_{n \to \infty} \frac{1}{n} \log \| \log S_n \| \leq -2 \log \min(4/e, \beta(\gamma)/(2e)). \]

This yields the following generalization of Theorem 1, giving an upper bound on $\beta(\gamma)$ assuming a condition possibly less difficult to verify than (S).

**Theorem 2.** (Second conditional bound on $\beta(\gamma)$) If

\begin{equation}
(15) \quad \liminf_{n \to \infty} \frac{1}{n} \log \| \log S_n \| \geq -\delta
\end{equation}

for some non-negative number $\delta < 2 \log(4/e)$, then $\beta(\gamma) \leq 2e^{1+(\delta/2)} < 8$. 

Remark. For $1 \leq n \leq 2500$, the minimum of $(1/n) \log \| \log S_n \|$ is $-0.667\ldots$ (at $n = 5$), except for the value $-1.480\ldots$ at $n = 1$. Figure 1 suggests that (15) holds with $\delta = 0.01$, and probably also with $\delta = 0$, which is Condition (S).

Finally, suppose that $8 < \lambda < \beta(\gamma)$. We deduce from (10), (14) and asymptotics (8), (9) that, for $k$ large, the left-hand side of (11) is positive and less than 1, hence equal to $\{\log S_{n_k}\}$. Since now $m_{\lambda, \varepsilon} = 4e^{-1-\varepsilon}$, we obtain

\[
\{\log S_{n_k}\} < (1 + \lambda) (4/e^{1+\varepsilon})^{-2n_k},
\]

for $k \geq k(\lambda, \varepsilon)$. Letting $\varepsilon$ tend to zero, we obtain the following result, which assumes a weaker condition than the previous ones.

**Theorem 3.** (Third conditional bound on $\beta(\gamma)$) We have $\beta(\gamma) \leq 8$ if

\[
\liminf_{n \to \infty} \frac{1}{n} \log \{\log S_n\} > -2 \log(4/e).
\]

### 4. CONDITIONAL BOUNDS ON THE IRRATIONALITY EXPONENT OF EULER’S CONSTANT

We have shown that $\gamma$ has irrationality measure $1/\beta^q$, with $\beta = \beta(\gamma) < \infty$, if there do not exist subsequences $\log S_{n_k}$ tending to zero exponentially. We now show that $\gamma$ has irrationality measure $1/q^\mu$, with $\mu = \mu(\gamma) < \infty$, if such a subsequence does exist, provided both that its convergence rate is different from that of $d_{2n_k} I_{n_k}$, and that the sequence $n_k$ is asymptotically linear. More generally, the limit of the subsequence may be any rational number $a/b$ in the interval $[0, 1]$, and $\log S_{n_k}$ may be replaced by $\{\log S_{n_k}\}$.

**Theorem 4.** (Conditional bounds on $\mu(\gamma)$) Assume that there exists a sequence of positive integers $n_k$, $k = 1, 2, \ldots$, such that

\[
(16)_{\sigma}, (16)_{\tau} \quad \lim_{k \to \infty} \frac{n_k}{k} = \sigma, \quad \lim_{k \to \infty} \frac{1}{k} \log \{\log S_{n_k}\} - \frac{a}{b} = -\tau,
\]

for some integers $a, b$ and positive numbers $\sigma, \tau$, with $\tau \neq 2\sigma \log(4/e)$. Then $\gamma$ has irrationality exponent $\mu(\gamma) \leq \mu_{\sigma, \tau}$, where
that is, for any $\varepsilon > 0$ there exists $q(\varepsilon) > 0$ such that

$$\left| \gamma - \frac{p}{q} \right| > \frac{1}{q^{\mu_{\sigma, \tau} + \varepsilon}} \quad \text{for all integers } p, q, \text{ with } q \geq q(\varepsilon).$$

In particular, the hypothesis implies that $\gamma$ is irrational, but is not a Liouville number.

The same result holds with $\{\log S_{n_k}\}$ replaced by $\|\log S_{n_k}\|$.

**Proof.** Define integers $p_k, q_k$, for $k = 1, 2, \ldots$, by the formulas

$$p_k = b d_{2n_k} A_{n_k} - b \left\lfloor \log S_{n_k} \right\rfloor - a, \quad q_k = b d_{2n_k} \left(\binom{2n_k}{n_k}\right).$$

According to (5), we have

$$b^{-1}(q_k \gamma - p_k) = d_{2n_k} I_{n_k} + \frac{a}{b} - \{\log S_{n_k}\}.$$

Asymptotics (7), (8), (9), together with limit (16)$_{\sigma}$, imply that

$$\lim_{k \to \infty} \frac{1}{k} \log q_k = 2\sigma \log(2e)$$

and

$$\lim_{k \to \infty} \frac{1}{k} \log(d_{2n_k} I_{n_k}) = -2\sigma \log(4/e).$$

Using $\tau \neq 2\sigma \log(4/e)$, we deduce from (19), (20), (16)$_{\tau}$ that

$$\lim_{k \to \infty} \frac{1}{k} \log |q_k \gamma - p_k| = -\min(\tau, 2\sigma \log(4/e)).$$

Lemma 3 now implies the theorem, except for its last assertion, whose proof requires the following modifications. In (16)$_{\tau}$, replace $\{\log S_{n_k}\}$ by $\|\log S_{n_k}\|$. If the two numbers are equal, define $p_k$ as in (18); if they are unequal, define $p_k$ by
\[ p_k = b \, d_{2n_k} A_{n_k} - b \left\lfloor \log S_{n_k} \right\rfloor + a. \]

In both cases, define \( q_k \) as in (18). Then (19) becomes

\[ b^{-1}(q_k - p_k) = d_{2n_k} I_{n_k} \pm \left( \left\| \log S_{n_k} \right\| - \frac{a}{b} \right) \]

and the rest of the proof goes through unchanged.

Using Lemma 4 in place of Lemma 3, we can deduce the conclusions of Theorem 4.1 from a weaker hypothesis, as follows.

**Theorem 5.** Assume that there exists a sequence \( n_k \) such that (16) and

\[ \limsup_{k \to \infty} \frac{1}{k} \log \left( \left\| \log S_{n_k} \right\| - \frac{a}{b} \right) \leq -\tau \]

hold, for some integers \( a, b \) and positive numbers \( \sigma, \tau \). Assume further that

\[ -2 \sigma \log(4/e) \]

is not the limit of any subsequence of \( (1/k) \log \left| \log S_{n_k} - a/b \right| \). Then \( \gamma \) has irrationality exponent \( \mu(\gamma) \leq \mu_{\sigma, \tau} \), where \( \mu_{\sigma, \tau} \) is given by (17). The same result holds with \( \{ \log S_{n_k} \} \) replaced by \( \left\| \log S_{n_k} \right\| \).

**Proof.** We repeat the previous proof through (20). Using the subsequence condition, we deduce from (19), (20), (21) that

\[ \limsup_{k \to \infty} \frac{1}{k} \log \left( q_k - p_k \right) \leq -\min(\tau, 2 \sigma \log(4/e)). \]

By Lemma 4, it only remains to show that, under the hypothesis, \( \gamma \) is irrational. Suppose on the contrary that \( \gamma \in \mathbb{Q} \). Then, by criterion (6) and inequalities (7), (9), we have

\[ \{ \log S_{n_k} \} = d_{2n_k} I_{n_k} \to 0 \quad \text{as} \quad k \to \infty. \]

Hence \( a/b = 0 \), by (21). But then (20) and the equality in (22) contradict the subsequence hypothesis. Thus, \( \gamma \not\in \mathbb{Q} \) and the proof for \( \{ \log S_{n_k} \} \) is complete. The same modification as above works for \( \left\| \log S_{n_k} \right\| \).

**Remark.** Alternatively, one could prove Theorem 5 first and obtain Theorem 4 as a corollary. We chose the present order because Theorem 4 has a simpler statement and proof than Theorem 5.
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