The Standard Model Algebra - a summary

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Abstract. A generation of leptons and quarks and the gauge symmetries of the Standard Model can be obtained from the Clifford algebra \( \mathcal{C}_6 \). An instance of \( \mathcal{C}_6 \) is implicitly generated by the Dirac algebra combined with the electroweak symmetry, while the color symmetry gives another instance of \( \mathcal{C}_6 \) with a Witt decomposition. The minimal mathematical model proposed here results by identifying the two instances of \( \mathcal{C}_6 \). The left ideal decomposition generated by the Witt decomposition represents the leptons and quarks, and their antiparticles. The SU(3)\(_c\) and U(1)\(_{em}\) symmetries of the SM are the symmetries of this ideal decomposition. The patterns of electric charges, colors, chirality, weak isospins, and hypercharges, follow from this, without predicting additional particles or forces, or proton decay. The electroweak symmetry is present in its broken form, due to the geometry. The predicted Weinberg angle is given by \( \sin^2 \theta_W = 0.25 \).

1. Introduction
The approach [1] summarized here unifies leptons and quarks of a generation, and the gauge symmetries from the SM, in the Clifford algebra \( \mathcal{C}_6 \), whose matrix form is (1).

\[
\begin{align*}
1_w & \quad 3_c & & \bar{1}_c & & \bar{3}_c \\
\begin{array}{cccc}
\begin{array}{c}
\text{Dirac,} \\
\text{Lorentz}
\end{array} & \begin{array}{c}
\begin{array}{c}
\nu_{R1} \\
\nu_{R2} \\
\nu_{L1}
\end{array} & \begin{array}{c}
\begin{array}{c}
u_{R1}^c \\
\nu_{R2}^c \\
\nu_{L1}^c
\end{array} & \begin{array}{c}
\begin{array}{c}
u_{R1}^\prime \\
\nu_{R2}^\prime \\
\nu_{L1}^\prime
\end{array} & \begin{array}{c}
\begin{array}{c}
u_{R1}^{\prime c} \\
\nu_{R2}^{\prime c} \\
\nu_{L1}^{\prime c}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\bar{e}_{R1} \\
\bar{e}_{R2} \\
\bar{e}_{L1}
\end{array} & \begin{array}{c}
\begin{array}{c}
\bar{e}_{R1}^c \\
\bar{e}_{R2}^c \\
\bar{e}_{L1}^c
\end{array} & \begin{array}{c}
\begin{array}{c}
\bar{e}_{R1}^\prime \\
\bar{e}_{R2}^\prime \\
\bar{e}_{L1}^\prime
\end{array} & \begin{array}{c}
\begin{array}{c}
\bar{e}_{R1}^{\prime c} \\
\bar{e}_{R2}^{\prime c} \\
\bar{e}_{L1}^{\prime c}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

The group SU(3)\(_c\) permutes the columns according to the representations \( 1_c, 3_c, \bar{1}_c, \) and \( \bar{3}_c \). The rows are permuted by the SU(2)\(_L\) group according to the representations \( 1_w \) and \( 2_w \), as well as by the Lorentz group and the Dirac algebra.
In the Standard Model Algebra (SMA), the Clifford algebra $\mathcal{C}\ell(\chi^{\dagger} \oplus \chi)$ is algebraically generated by a three-dimensional Hermitian space $\chi$, and is naturally split into left ideals because of the decomposition $\chi^{\dagger} \oplus \chi$. In a basis adapted to the ideal decomposition, each column contains two 4-spinors associated to different flavors, as illustrated in (1). Each ideal is indexed with an electric charge which is multiple of $\frac{1}{2}$ partially representing the charge of the upper particle, and its color is determined by the ideal to which belongs, having thus associated the correct representation of SU(3)$_c$.

The earliest Grand Unified Theories (GUT) are based on the group SU(5) [2], and on SO(10) [3, 4]. Both these models require additional spontaneous symmetry breaking, and predict new interactions, hence new bosons, and proton decay. These predictions are not confirmed. More modern approaches either extend them with supersymmetry, or use different, larger groups. The SU(5) and SO(10) models predict a Weinberg angle $\sin^2 \theta_W = \frac{3}{8} = 0.375$. There are common ideas shared by the SMA model with previously known approaches. In previous models, particles of two distinct flavors were also combined into 8-spinor ideals: in a unified spin gauge theory of electroweak interactions and gravity based on $\mathcal{C}\ell_{1,6} \cong \mathcal{C}\ell_{1,3} \otimes \mathcal{C}\ell_{0,3}$ [5], and in the model [6, 7, 8] based on $\mathcal{C}\ell_{7} \cong \mathcal{C}\ell_{3} \otimes \mathcal{C}\ell_{4}$. This $\mathcal{C}\ell_{7}$ model uses three space dimensions, time being a scalar, and four extra dimensions related to the Higgs boson. It predicts a Weinberg angle given by $\sin^2 \theta_W = 0.375$. Remarkably, the condition to preserve the current and to leave right-handed neutrino sterile results in the full symmetries of the SM. Some of the differences are that the SMA model uses different structures, which lead to the algebra $\mathcal{C}\ell(\chi^{\dagger} \oplus \chi) \cong \mathcal{C}\ell_{6}$, it contains the Dirac algebra $\mathcal{C}\ell_{1,3} \otimes \mathbb{C}$, and $\sin^2 \theta_W = 0.25$. The ideals in the $\mathcal{C}\ell_{6}$ and $\mathcal{C}\ell_{7}$ models are obtained from primitive idempotents. The SMA is decomposed using a Witt decomposition $\chi^{\dagger} \oplus \chi$ and the exterior algebra $\wedge^2 \chi$ contained within the minimal right ideal $\mathbb{K}q\mathcal{C}\ell_{6} = \mathbb{K}q\wedge^2 \chi$, into left ideals $\mathcal{C}\ell_{6} qq^\dagger q_{\mathbb{K}}$, where $\mathbb{K} \subseteq \{1, 2, 3\}$ (notations of section 2). This emphasizes the colors and charges, and allow the coupling of minimal left ideals of the same charge and different colors into a larger ideal.

Furey uses the Witt decomposition for $\mathcal{C}\ell_{6}$ in a model based on octonions [9, 10], to represent colors and charges of up- and down-type particles by $q_{K} q q_{\mathbb{K}}$ and $q_{K} q^\dagger q$, on the minimal left ideals $\mathcal{C}\ell_{6} qq$ and $\mathcal{C}\ell_{6} q^\dagger q$. They are united into a single irreducible representation of $\mathcal{C}\ell_{6} \otimes \mathbb{C}\mathcal{C}\ell_{2}$ obtained by using the octonion algebra. To represent the complete particles, with spin and chirality, Furey proposes including the quaternion algebra, resulting in a representation of leptons and quarks as spinors of an algebra isomorphic to $\mathcal{C}\ell_{12}$. On the other hand, in the SMA model, the complete particles are contained in the $\mathcal{C}\ell_{6}$ ideals classified by the elements $qq q_{\mathbb{K}}$. However, the SU(3)$_c$ and U(1)$_{\text{em}}$ symmetries in the SMA are identical to those obtained previously by Furey [10] as the unitary spin transformations preserving the Witt decomposition of $\mathcal{C}\ell_{6}$, improving by this previous results based on octonions and Clifford algebras [11, 12, 13], and to those from [8].

2. The Standard Model Algebra

The representation of the group SU(3)$_c$ corresponding to leptons is $1_c$, the one corresponding to quarks is $3_c$, and for their antiparticles, $\bar{1}_c$ and respectively $\bar{3}_c$ (1). The group U(1)$_{\text{em}}$ acts by multiplication with $e^{i k \varphi}$, where $\varphi \in \mathbb{R}$, and $k$ is an integer satisfying $-3 \leq k \leq 3$. Thus, the representations of both symmetry groups can be seen naturally as representations on the exterior algebras $\wedge \chi$ and $\wedge \overline{\chi}$, where $\chi$ is a three-dimensional complex vector space endowed with a Hermitian inner product $h$, and $\overline{\chi}$ is its complex conjugate. Due to the presence of $h$, the following identifications hold: $\chi^{\dagger} \cong \overline{\chi}$, and also $\chi \cong \overline{\chi}^{\dagger}$.

The space $\chi^{\dagger} \oplus \chi$ is naturally endowed with the inner product (see e.g. [14])

$$\langle u_1^\dagger + u_2, u_3^\dagger + u_4 \rangle := \frac{1}{2} \left( u_1^\dagger (u_4) + u_3^\dagger (u_2) \right) \in \mathbb{C},$$

where $u_2, u_4 \in \chi$ and $u_1^\dagger, u_3^\dagger \in \chi^\dagger$. The inner product $\langle u_1^\dagger + u_2, u_3^\dagger + u_4 \rangle$ vanishes on $\chi$, so it is
not the same as the Hermitian inner product $\mathfrak{h}$.

The Standard Model Algebra is the Clifford algebra defined by the inner product (2),

$$\mathcal{A}_{\text{SM}} := \mathbb{C}\ell(\chi^\dagger \oplus \chi) \cong \mathbb{C}\ell_6 \cong \mathbb{M}_C(8),$$

with the distinguished Witt decomposition $\chi^\dagger \oplus \chi$, and endowed with the Hermitian inner product $\mathfrak{h}$ on $\chi$ and $\chi^\dagger$. Let us fix an $\mathfrak{h}$-orthonormal basis $q_1, q_2, q_3$ of $\chi$. Then, the following anticommutation relations hold

$$\{q_j, q_k\} = 0, \quad \{q_j^\dagger, q_k^\dagger\} = 0, \quad \{q_j, q_k^\dagger\} = \delta_{jk}$$

for $j, k \in \{1, 2, 3\}$ (see [15, 10]), which amounts to say that $(q_1^\dagger, q_2^\dagger, q_3^\dagger, q_1, q_2, q_3)$ is a Witt basis of $\chi^\dagger \oplus \chi$. From $(q_1^\dagger, q_2^\dagger, q_3^\dagger, q_1, q_2, q_3)$ one gets an orthonormal basis on $\chi^\dagger \oplus \chi$, by

$$\begin{cases}
\epsilon_j = q_j + q_j^\dagger \\
\bar{\epsilon}_j = i(q_j^\dagger - q_j)
\end{cases}$$

where $j \in \{1, 2, 3\}$. The basis elements satisfy $\epsilon_j^2 = 1$, $\bar{\epsilon}_j^2 = 1$, $\epsilon_j^\dagger = \epsilon_j$, and $\bar{\epsilon}_j^\dagger = \bar{\epsilon}_j$. One can go back to the Witt basis with

$$\begin{cases}
q_j = \frac{1}{2}(\epsilon_j + i\bar{\epsilon}_j) \\
q_j^\dagger = \frac{1}{2}(\epsilon_j - i\bar{\epsilon}_j).
\end{cases}$$

As a Clifford algebra, the algebra $\mathcal{A}_{\text{SM}}$ is graded, $\mathcal{A}_{\text{SM}} = \bigoplus_{k=0}^{6} \mathcal{A}_{\text{SM}}^k$. Then,

$$\mathcal{A}_{\text{SM}}^k = \mathbb{C}\ell^k(\chi^\dagger \oplus \chi) = \bigwedge^k (\chi^\dagger \oplus \chi).$$

We define

$$\begin{cases}
\epsilon := \epsilon_1\epsilon_2\epsilon_3, \\
\bar{\epsilon} = \bar{\epsilon}_1\bar{\epsilon}_2\bar{\epsilon}_3.
\end{cases}$$

It follows that $\epsilon^2 = -1$, $\bar{\epsilon}^2 = -1$, $\epsilon\bar{\epsilon} = -\bar{\epsilon}\epsilon$, and $(\epsilon\bar{\epsilon})^2 = -1$.

The elements

$$\begin{cases}
q := q_1q_2q_3, \\
q' := q_3q_2q_1,
\end{cases}$$

are nilpotent, that is, $q^2 = 0$ and $q'^2 = 0$. We denote $p := qq'$ and $p' := q'q$. Then, $p$ and $p'$ are idempotent elements, that is, $(p)^2 = p$ and $(p')^2 = p'$.

Each of the nilpotent elements $q$ and $q'$, and each of the (primitive) idempotent elements $p$ and $p'$, define minimal left and right ideals of the algebra $\mathcal{A}_{\text{SM}}$ [16, 17]. Via the representation of the Clifford algebra $\mathcal{A}_{\text{SM}}$ as an endomorphism algebra $\text{End}_C(\mathbb{C}^8) \cong \mathbb{M}_C(8)$, idempotent elements are represented as projectors.

The Clifford product is given by

$$(u^\dagger + v)\omega p = (u^\dagger \wedge \omega)p + (i_v\omega)p \in \bigwedge^\bullet \chi^\dagger p,$$

where $u^\dagger + v \in \chi^\dagger \oplus \chi$ and $\omega p \in \bigwedge^\bullet \chi^\dagger p$, and $i_v\omega$ is the interior product, defined for any $\omega \in \bigwedge^k \chi^\dagger$ by

$$(i_v\omega)(u_1, \ldots, u_k) = \begin{cases}
\omega(v, u_1, \ldots, u_{k-1}), & \text{for } k \in \{1, 2, 3\}, \\
0 & \text{for } k = 0.
\end{cases}$$
The elements $q_j$ and $q^+_j$ act like annihilation operators, and respectively creation operators on $\wedge^\bullet \chi^\dagger p$:

$$
\begin{align*}
q^+_j(\omega p) &= (q^+_j \wedge \omega)p, \\
q_j(\omega p) &= (i\sigma_j \omega)p.
\end{align*}
$$

We fix the following basis for the minimal left ideal $\wedge^\bullet \chi^\dagger p$:

$$
(1 \ p, q_{23}^\dagger \ p, q_{31}^\dagger \ p, q_{12}^\dagger \ p, q_{321}^\dagger \ p, q_1^\dagger \ p, q_2^\dagger \ p, q_3^\dagger \ p).
$$

We now determine the matrix representation of $q_j$, $q^+_j$, $e_j$, and $\tilde{e}_j$ in the basis $(13)$. In terms of the Pauli matrices $\sigma_j$, $j \in \{1,2,3\}$, and the matrices $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, $\sigma_3 = \frac{1}{2}(1 + \sigma_3)$, we obtain

$$
\begin{align*}
q_1^\dagger &= \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sigma_2 \\
-\sigma_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
q_2^\dagger &= \begin{pmatrix} 0 & 0 & 0 & \sigma_3^+ \\
0 & 0 & -\sigma_3 & 0 \\
0 & \sigma_3 & 0 & 0 \\
\sigma_3 & 0 & 0 & 0 \end{pmatrix}, \\
q_3^\dagger &= \begin{pmatrix} 0 & 0 & 0 & -\sigma_- \\
0 & -\sigma_+ & 0 & 0 \\
-\sigma_+ & 0 & 0 & 0 \\
\sigma_+ & 0 & 0 & 0 \end{pmatrix}, \\
q_4^\dagger &= \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i\sigma_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
q^\dagger &= \begin{pmatrix} 0 & 0 & \sigma_3^- & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
p &= \begin{pmatrix} \sigma_3^+ & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
p' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
$$

By $(5)$ one gets

$$
\begin{align*}
e_1 &= \begin{pmatrix} 0 & 0 & i\sigma_2 & 0 \\
0 & 0 & 0 & -i\sigma_2 \\
-i\sigma_2 & 0 & 0 & 0 \\
i\sigma_2 & 0 & 0 & 0 \end{pmatrix}, \\
e_2 &= \begin{pmatrix} 0 & 0 & 0 & 1_2 \\
0 & 0 & -1_2 & 0 \\
0 & -1_2 & 0 & 0 \\
1_2 & 0 & 0 & 0 \end{pmatrix}, \\
e_3 &= \begin{pmatrix} 0 & 0 & 0 & i\sigma_2 \\
0 & 0 & -i\sigma_2 & 0 \\
0 & i\sigma_2 & 0 & 0 \\
-i\sigma_2 & 0 & 0 & 0 \end{pmatrix}, \\
\tilde{e}_1 &= \begin{pmatrix} 0 & 0 & \sigma_2 & 0 \\
0 & 0 & 0 & \sigma_2 \\
0 & \sigma_2 & 0 & 0 \\
0 & 0 & \sigma_2 & 0 \end{pmatrix}, \\
\tilde{e}_2 &= \begin{pmatrix} 0 & 0 & 0 & -i\sigma_2 \\
0 & 0 & i\sigma_2 & 0 \\
0 & -i\sigma_2 & 0 & 0 \\
i\sigma_2 & 0 & 0 & 0 \end{pmatrix}, \\
\tilde{e}_3 &= \begin{pmatrix} 0 & 0 & 0 & -i\sigma_1 \\
0 & 0 & -i\sigma_1 & 0 \\
0 & i\sigma_1 & 0 & 0 \\
i\sigma_1 & 0 & 0 & 0 \end{pmatrix}, \\
e &= \left( \begin{array}{cc} 0_4 & 1_4 \\
-1_4 & 0_4 \end{array} \right), \\
\tilde{e} &= i\left( \begin{array}{cc} 0_4 & 1_4 \\
1_4 & 0_4 \end{array} \right), \\
\tilde{e}\tilde{e} &= i \left( \begin{array}{cc} 1_4 & 0_4 \\
0_4 & -1_4 \end{array} \right).
\end{align*}
$$

The Witt decomposition $\mathcal{A}_{SM}^\dagger = \chi^\dagger \oplus \chi$ induces the following decomposition of $\mathcal{A}_{SM}$ as a direct sum of left ideals

$$
\mathcal{A}_{SM} = \bigoplus_{k=0}^{3} \left( \wedge^k \chi \right) p \wedge^k \chi.
$$

The ideals correspond to internal degrees of freedom in $\wedge^k \chi$, similar to those of leptons and quarks.
3. The weak symmetry
Each minimal left ideal $A_{SM}p_K$, $K \subseteq \{1, 2, 3\}$, has complex dimension eight. We cannot identify these ideals with four-spinor spaces. But we can identify them with pairs of particles whose left chiral components interact weakly. Hence, the action of the Dirac algebra on each minimal left ideal is reducible. The irreducible decomposition of each ideal seen as a representation of the Dirac algebra is given by the projectors $\frac{1}{2}(1 \mp i\epsilon)$.

The reducible representation of the Dirac algebra on a minimal left ideal will be taken to be the direct sum of two irreducible representations,

$$\Gamma^\mu = \begin{pmatrix} \tilde{\gamma}^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix},$$

where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \tilde{\gamma}^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tilde{\gamma}^i = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

are the Weyl representation and a variation of it, chosen to match some notations that will appear later.

The chirality operator is

$$\Gamma^5 = -i\epsilon_1 \tilde{\epsilon}_1 = \begin{pmatrix} \Gamma^5 & 0 \\ 0 & \gamma^5 \end{pmatrix} = \begin{pmatrix} 1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & 1_2 \end{pmatrix}. \tag{23}$$

We will work in the ideal $A_{SM}p$, the others being obtained from it by multiplying at right with $q_K$.

To find on the ideal $A_{SM}p$ a basis adapted to the weak symmetry, we will use the elements

$$\omega_u = \begin{pmatrix} 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \omega_d = \begin{pmatrix} 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \omega_o = \begin{pmatrix} \sigma_+ & 0 & 0 & 0 \\ 0 & -\sigma_+ & 0 & 0 \\ 0 & 0 & -\sigma_+ & 0 \\ 0 & 0 & 0 & \sigma_+ \end{pmatrix}. \tag{24}$$

Then,

$$\left(\omega_u, \omega_d, \omega_o, \omega_u^\dagger, \omega_d^\dagger, \omega_o^\dagger\right) \tag{25}$$

is a Witt basis of the vector space $\mathcal{N}^\dagger \oplus \mathcal{N}$, where

$$\mathcal{N} := \text{span}_\mathbb{C}\left(\omega_u, \omega_d, \omega_o\right), \quad \mathcal{N}^\dagger := \text{span}_\mathbb{C}\left(\omega_u^\dagger, \omega_d^\dagger, \omega_o^\dagger\right). \tag{26}$$

We define

$$\left\{ \begin{array}{l} u_j = \omega_j + \omega_j^\dagger \\ u_j^\prime = i \left( \omega_j - \omega_j^\dagger \right) \end{array} \right. \tag{27}$$

where $j \in \{u, d, o\}$. They satisfy $u_j^2 = 1$, $u_j^\prime 2 = 1$ and have the matrix form

$$u_u = \begin{pmatrix} 0 & 1_2 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & -1_2 & 0 \end{pmatrix}, u_d = \begin{pmatrix} 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & -1_2 \\ -1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \end{pmatrix}, u_o = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & -\sigma_1 & 0 & 0 \\ 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{pmatrix}. \tag{28}$$
\[ u'_u = \begin{pmatrix} 0 & -i12 & 0 & 0 \\ i12 & 0 & 0 & 0 \\ 0 & 0 & 0 & i12 \\ 0 & 0 & -i12 & 0 \end{pmatrix}, \quad u'_d = \begin{pmatrix} 0 & 0 & i12 & 0 \\ 0 & 0 & 0 & i12 \\ -i12 & 0 & 0 & 0 \\ 0 & -i12 & 0 & 0 \end{pmatrix}, \quad u'_w = \begin{pmatrix} \sigma_2 & 0 & 0 & 0 \\ 0 & -\sigma_2 & 0 & 0 \\ 0 & 0 & -\sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \end{pmatrix}. \] (29)

We define the nilpotent elements

\[ \begin{cases} \omega := \omega_u\omega_d\omega_0, \\ \omega^\dagger := \omega^\dagger_0\omega^\dagger_d\omega^\dagger_u. \end{cases} \] (30)

Then, the idempotent \( \omega^\dagger \) is equal to \( p \), and

\[ A_{SMP} = \bigwedge \cdot M^\dagger p. \] (31)

Since (25) is a Witt basis, one can identify \( M^\dagger \) with the dual of \( M \), and obtain a Hermitian inner product \( h_M \) on \( M \) and \( M^\dagger \) similar to \( h \).

Then, \( \omega_j \) and \( \omega^\dagger_j \) act like ladder operators on this ideal \( \bigwedge \cdot M^\dagger p \), just like in (12), on the basis

\[ \left( 1, p, \omega^\dagger_0 p, \omega^\dagger_u p, \omega^\dagger_d p, \omega^\dagger_u\omega^\dagger_d p, \omega^\dagger_d\omega^\dagger_u p, \omega^\dagger_u\omega^\dagger_d\omega^\dagger_u p \right). \] (32)

While the spaces \( \bigwedge \cdot M^\dagger \) and \( \bigwedge \cdot \chi^\dagger \) are different, the ideals \( \bigwedge \cdot M^\dagger p \) and \( \bigwedge \cdot \chi^\dagger p \) are equal.

Let us define the two-dimensional Hermitian vector space \( W_w \)

\[ \left( W_w := \text{span}_C (\omega^\dagger_u, \omega^\dagger_d) \right), \quad h_w := h_M|_{W_w}. \] (33)

Then, the singlet representations of the action of SU(2)_L are \( \bigwedge^0 W_w = \text{span}_C (1) \) and \( \bigwedge^2 W_w = \text{span}_C (\omega^\dagger_u\omega^\dagger_d) \), and the doublet representation is \( \bigwedge^1 W_w = \text{span}_C (\omega^\dagger_u, \omega^\dagger_d) \). The additional element \( \omega^\dagger_0 \) is required by the fact that the chiral spaces have two dimensions.

Let us define \( W_{OR} := \text{span}_C (p, \omega^\dagger_0 p) \). Then, we define the right-handed and left-handed up singlet spaces \( W_{OR} := 1 \text{span}_C (p, \omega^\dagger_0 p) \) and \( W_{OL} := \omega^\dagger_u W_{OR} \), and the right-handed and left-handed down singlet spaces \( W_{1R} := \omega^\dagger_u \omega^\dagger_d W_{OR} \) and \( W_{1L} := \omega^\dagger_d W_{OR} \). The spin representation of the weak group SU(2)_L is given by the generators

\[ \begin{cases} \bar{T}_1 := u_u u'_d - u'_u u_d \\ \bar{T}_2 := u_u u_d + u'_u u'_d \\ \bar{T}_3 := u_u u'_u - u_d u'_d \end{cases}. \] (34)

whose matrix form in the basis (32) is

\[ \bar{T}_1 = 2i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{T}_2 = 2i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{T}_3 = 2i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (35)

The proof that their adjoint action gives exactly the desired singlet and doublet representations can be found in [1].
4. The color and electromagnetic symmetries

We will see now that the algebra \( \mathcal{A}_{SM} \), which has the symmetries of the gauge groups \( SU(3)_c \) and \( U(1)_{em} \), contains the generators of these groups. The groups \( SU(3)_c \) and \( U(1)_{em} \) are obtained as subgroups of \( SO(\chi^\dagger \oplus \chi) \), which is double-covered by the group \( Spin(\chi^\dagger \oplus \chi) \). The fact that finite dimensional Lie groups can be expressed as spin groups was explored in detail in [18].

The Lie algebra \( \mathcal{A}_{SM}^2 \), where \( [a, b] := ab - ba \) for any \( a, b \in \mathcal{A}_{SM}^2 \), is just \( \mathfrak{so}(\chi^\dagger \oplus \chi) \cong \mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{spin}(\chi^\dagger \oplus \chi) \cong \mathfrak{spin}(6, \mathbb{C}) \) corresponding to the inner product (2). The Lie algebra \( \mathfrak{su}(\chi, \mathfrak{b}) \) of the color group \( SU(3)_c \) is a Lie subalgebra of \( \mathfrak{A}_{SM}^2 \), characterized by preserving the ideal decomposition (20). A set of generators for the group \( SU(3)_c \) is

\[
\begin{align*}
\tilde{\lambda}_1 &= e_1 \tilde{e}_2 - \tilde{e}_1 e_2, \\
\tilde{\lambda}_2 &= e_1 e_2 + \tilde{e}_1 \tilde{e}_2, \\
\tilde{\lambda}_3 &= e_1 \tilde{e}_1 - e_2 \tilde{e}_2, \\
\tilde{\lambda}_4 &= e_1 \tilde{e}_3 - \tilde{e}_1 e_3, \\
\tilde{\lambda}_5 &= e_1 e_3 + \tilde{e}_1 \tilde{e}_3, \\
\tilde{\lambda}_6 &= e_2 \tilde{e}_3 - \tilde{e}_2 e_3, \\
\tilde{\lambda}_7 &= e_2 e_3 + \tilde{e}_2 \tilde{e}_3, \\
\tilde{\lambda}_8 &= \frac{1}{\sqrt{3}}(e_1 \tilde{e}_1 + e_2 \tilde{e}_2 - 2e_3 \tilde{e}_3).
\end{align*}
\]

The generators \( \tilde{\lambda}_j \) act on the ideals \( \mathcal{A}_{SM} \wedge \mathfrak{p} \wedge^k \chi \) with \( k = 0 \) and \( k = 3 \) like the representation \( 1_c \) and \( \mathbf{T}_8 \), and on those with \( k = 1 \) or \( k = 2 \) like \( 3_c \) and \( \mathbf{3}_c \) [1], similar to the Gell-Mann matrices.

The generator of the \( U(1)_{em} \) gauge transformations is

\[ Q = e_1 \tilde{e}_1 + e_2 \tilde{e}_2 + e_3 \tilde{e}_3. \]  

(37)

Its adjoint action on \( \mathfrak{p} \wedge \chi \) corresponds to the correct electric charges, given by \( \frac{e}{\sqrt{3}} \), where \( e \) is the electron charge, and \( k \in \{ \pm 0, \pm 1, \pm 2, \pm 3 \} \), with the identification \( \wedge^{-k} \chi = \wedge^k \chi^\dagger = \wedge^k \chi \).

From (36) and (37) we see that the color and electromagnetic forces are unified into an electrocolor symmetry \( U(3)_c \).

In addition to acting on \( \mathfrak{p} \wedge \chi \), the symmetry generated by (37) also transforms \( \omega_d \mathfrak{p} \). Since

\[ \omega_d \mathfrak{p} = u_d \mathfrak{p} = i \tilde{e} \mathfrak{p} = \frac{\sqrt{3}}{2} \mathfrak{p}, \]

(38)

it follows that the charge of \( \frac{\sqrt{3}}{2} \mathfrak{p} \) is equal to \(-1\). This is essential to account for the fact that in each minimal left ideal actually live two particles, whose charge differs exactly by this value.

A proof that unitary spin transformations preserving a Witt decomposition in \( \mathcal{C}_6 \) give these symmetries, and a set of generators constructed from the \( q_j \) and \( q_j^\dagger \) but equivalent to (36), was given by Furey [9]. Based on the algebra \( \mathcal{C}_7 \), [8] proposed generators of \( SU(3)_c \) are equivalent to (36) via to the isomorphisms \( \mathcal{C}_7 \cong \mathbb{M}_\mathbb{C}(8) \cong \mathbb{C}_6 \).

5. The electroweak symmetry

While in the standard electroweak theory the electromagnetic and weak interactions arise by spontaneous symmetry breaking of the electroweak symmetry, in the SMA they are already “broken” by the geometry, and appear more fundamental than the electroweak symmetry. The hypercharge \( Y \) is given by the Gell-Mann–Nishijima formula, in terms of the electric charge and the weak isospin

\[ Y = 2(Q - T_3). \]

(39)

This reversed position about the electroweak symmetry breaking may be considered to be due to the construction of the SMA. However, since the split of \( \mathbb{W}_w \mathfrak{p} \) is determined by \( i e \tilde{e} \), it follows that it is an invariant property of \( \mathcal{A}_{SM} \). Hence, the spontaneous symmetry breaking mechanism seems not to be needed, but the Higgs boson is still required to give masses to particles. Moreover, the electromagnetic and weak interactions are unified in the SMA too, but they are distinguished by the geometry. In [1] it is shown that in the SMA the Weinberg angle, which characterizes the electroweak symmetry breaking, is \( \theta_W = \frac{\pi}{6} \), being given by

\[ \sin^2 \theta_W = \frac{1}{4} = 0.25. \]

(40)
The experimental estimations range between $\sim 0.223$ and $\sim 0.24$, depending on the utilized scheme [19]. In particular, CODATA gives a value of 0.23129(5) [20]. It is hoped that by taking into account the higher order perturbative corrections, the value of 0.375 can be accommodated with the experimental value.

6. Leptons, quarks, and symmetries
The Witt decomposition $A_{\text{SM}}^1 = \chi^\dagger \oplus \chi$, together with the operator $\Gamma^5$, induce a decomposition of the algebra $A_{\text{SM}}$ into chiral spaces,

$$A_{\text{SM}} = \bigoplus_{k=0}^3 (\mathcal{W}_0 \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_1) \wedge^k \chi.$$ (41)

In the basis (13), the chiral spaces are arranged in columns, as shown in (1). Since

$$\left(\mathcal{W}_0 \oplus \mathcal{W}_1\right) \wedge^k \chi = \left(\mathcal{W}_1 \oplus \mathcal{W}_0\right) p' \wedge^{3-k} \chi,$$ (42)

the minimal left ideals contain both particles and their antiparticles, and the antiparticles have opposite chirality.

Any element of the $A_{\text{SM}}$ is a linear combination of elements of the form

$$\omega^j_a \omega^b_u q^c_p q_K,$$ (43)

where $a, b, c \in \{0, 1\}$, $(\omega^0_u)^0 = (\omega^j_u)^0 = (q^0_p)^0 = 1$ by convention, and $K \subset \{1, 2, 3\}$. The elements of the form (43) are cells in the matrix representation of the SMA. The factors present before $p$ identify the row, and those after it, the column.

The action of the group SU(3)$_c$ permutes the columns according to the representations $1_c$, $3_c$, $1_c$, and $3_c$. The color of each column is determined by the representation of SU(3)$_c$ to which it belongs. Each ideal has associated an electric charge which is multiple of $\frac{1}{3}$, but the lower half of each column has in addition to the upper half an extra charge equal to $-1$, due to the presence of the factor $q^c_p$, since $q^c_p = \omega^c_q p$.

The Dirac algebra, the Lorentz group, and the weak group SU(2)$_L$ act by permuting the rows. The action of SU(2)$_L$ corresponds to $1_w$ and $2_w$, according to the chirality.

Therefore, the leptons, quarks, their antiparticles, and the gauge symmetries of the SM are reproduced by the Standard Model Algebra.

7. Future plans
The results presented here are concerned only the spaces in which the fields associated to particles are represented at each point. Future developments need to explore the consequences of these results to the field equations and the Lagrangian. It is at that level where masses become relevant, and we see if the SMA has implications on the Higgs field, the generations, the neutrino and quark mixing matrices, and the nature of the neutrino. In particular, is the neutrino a Majorana or a Dirac spinor? Hopefully, the geometric properties of the SMA may constrain other parameters of the SM, in addition to the Weinberg angle. Also, the geometric nature of this approach, and the fact that it is based on a higher-dimensional Clifford algebra, may have something to say about the geometry of spacetime.

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