Some Results on Prime Cordial Labeling of Lilly Graphs

A. Parthiban\textsuperscript{1} and Vishally Sharma\textsuperscript{2}
Department of Mathematics, Lovely Professional University, Punjab, India
E-mail: \{mathparthi, vishuuph\}@gmail.com

Abstract.
A PCL of $G$ is a bijective map $g$ from $V$ to $\{1, 2, 3, \ldots, |V|\}$ in such a way that if an edge $st$ is given label 1 if $GCD(g(s), g(t)) = 1$ & 0 otherwise, then the edges given 0 & 1 differ by at most 1 i.e: $|e_g(0) - e_g(1)| \leq 1$. If a graph permits a PCL, then it is called a PCG. In this paper, we prove that lilly graph admits a PCL. Further, we have shown that lilly graph under some graph operations like switching of a vertex, duplication of a vertex, degree splitting graph and barycentric subdivision admits a PCL which may find its application in the development of artificial intelligence.

Keywords: Prime Cordial Labeling, Lilly Graph, Vertex Switching, Barycentric Subdivision, Degree Splitting Graph.

1. Introduction
Graph labeling is a widely used and fastest growing area in the field of mathematics and computer science. Now a days, when data security is a major area of concern, various researchers and scientist are working to develop the techniques and softwares that can resolve the issues. Graph labeling is used in many areas of science and technology. A lot of graph labeling techniques are discussed in [4], we enlist a few of them which are finding their use in different aspects of artificial intelligence [6].

1. Radio labeling is finding its application in fast communication in sensor networks.
2. The designing of fault tolerance system with particularized degree, facility graph is used.
3. The concept of chromatic number is widely used in solving many complex problems in computer which is also a type of graph labeling.
4. Mobile Adhoc Networks (MANETS) problems can also be resolved by using graph labeling.
5. Automatic routing with graph labeling is done when each network usually path, cycle, circuit, walk and connected graph represent a fixed network and labeling is done with a constant which helps routing to involuntary detect the next node in the network.
6. Behavior trees are used in robotics.

For number theory concepts, refer to [2] and for terms and terminology related to graph theory that have not been defined here, we refer to Hararay [5]. For detail survey on various

\textsuperscript{1} Corresponding author
\textsuperscript{2} Ph.D Scholar
graph labeling, refer to [4]. Cahit [3] is the introducer of cordial labeling. For the sake of simplicity, by 'PCL' we mean a prime cordial labeling and by 'PCG' we mean a prime cordial graph.

2. Main Results

2.1. Basic Definitions and Results

In this section, we discuss the PCL of lilly graphs. First we recall some basic definitions for the sake of completeness.

After Cahit [3], various researchers introduced a lot of graph labeling techniques with some type of restrictions and/or variations in the cordial theme. The notion of PCL was introduced by Sundaram, Ponraj & Somasunaram [8].

Definition 1. A PCL of $G_1$ is a map (bijective) $h$ from $V$ to $\{1, 2, 3, \ldots, |V|\}$ so that if an edge $st$ is given label 1, if $gcd(h(s), h(t)) = 1$ and 0, if $gcd(h(s), h(t)) > 1$, then the absolute difference of number of edges tagged with 0 & 1 is at most 1 i.e; $|e_h(0) - e_h(1)| \leq 1$. The graph that permits a PCL is called a PCG.

Definition 2. Duplicating a vertex $x_i$ in $G_1$ with a vertex $x_i'$ gives rise to a formation of $G'$ with $N(x_i) = N(x_i')$.

Definition 3. ([9]) Switching a vertex $s$ of a graph $G_1$ results in a formation of a new graph $G_s$, by deleting edges incident to $s$ in $G_1$ & adding the edges that are obtained by joining the vertex $s$ to the vertices which are not adjacent to $s$ in $G_1$.

Definition 4. Let $e = st$ be an edge of $G_1$. By subdivision of an edge $e$ we mean when $e$ is replaced by edges $e' = sw$ and $e'' = wt$. If every edge of graph $G_1$ is subdivided then the graph thus obtained is called barycentric subdivision of graph $G_1$.

Trees constitute an important class of graphs in graph theory. Many researchers are working on trees for different kind of graph labeling. J. Baskar Babujee in [1] proved that full binary tree admits a prime cordial labeling. Motivated by [1], [7], and [9], we attempt to contribute some results on particular type of a tree namely lilly graph.

Definition 5. ([7]) Lilly graph $I_n$, $n \geq 2$ is constructed by 2 stars $2K_{1,n}, n \geq 2$ joining 2 path graphs $2P_n, n \geq 2$ with sharing of a common vertex. i.e; $I_n = 2K_{1,n} + 2P_n$.

Result 1. $GCD(n, n + 2^k) = 1$, for an odd integer $n$ & positive integer $k$. 

![Figure 1. Lilly graph $I_4$](image-url)
2.2. PCL of Lilly Graphs

**Theorem 1.** Lilly graph $I_n$ permits a PCL.

Proof. Let $u_1, u_2, ..., u_{4n-1}$ denotes vertices of $I_n$. Construct a function (bijective) $f : V(G) \to \{1, 2, ..., 4n - 1\}$: 

Fix $f(u_{3n}) = 2, f(u_{2n+1}) = 4, f(u_i) = f(u_{i-1}) + 2; \ 2n + 2 \leq i \leq 3n - 1, f(u_{3n+1}) = 3, f(u_{3n+2}) = 9, f(u_j) = f(u_{j-1}) + 2; \ 3n + 3 \leq j \leq 4n - 1, f(u_1) = f(u_{3n-1}) + 2, f(u_i) = f(u_{i-1}) + 2; \ 2 \leq i \leq n - 1, f(u_n) = 1, f(u_{n+1}) = 5, f(u_{n+2}) = 7, f(u_{n+3}) = f(u_{4n-1}) + 2, f(u_j) = f(u_{j-1}) + 2; \ n + 4 \leq j \leq 2n. 

Observe that $\text{gcd}(f(u_{3n}), f(u_j)) \neq 1$, for $1 \leq i \leq n - 1$, $\text{gcd}(f(u_{3n+1}), f(u_{n+2})) \neq 1$, and $\text{gcd}(f(u_i), f(u_{i+1})) \neq 1$, for $2n \leq i \leq 3n - 1$.Keep above in view, these edges will contribute (bear) 0 and the rest of the edges of $I_n$ will contribute 1 (since the gcd of their end vertices is equal to 1). Evidently, $e_f(0) = e_f(1) = 2n - 1$ which shows that $I_n$ is a PCG.

![Figure 2. PCL of lilly graph $I_5$](image)

**Theorem 2.** Switching of a pendant vertex in $I_n$ admits a PCL for $n \geq 4$.

Proof. Let $u_1, u_2, ..., u_{4n-1}$ represents the vertices of $I_n$. Here, $u_1, u_2, ..., u_n, u_{n+1}, ..., u_{2n}, u_{2n+1}, u_{4n-1}$ represents the pendant vertices. Obtain $G$ by switching a pendant vertex in $I_n$ say $u_k$ where $k \in \{1, 2, ..., 2n, 2n + 1, 4n - 1\}$. Clearly, $|V(G)| = 4n - 1$ & $|E(G)| = 8n - 6$. Construct a map (bijective) $f : V(G) \to \{1, 2, ..., 4n - 1\}$ as:

Case 1: When $k \in \{1, 2, ..., 2n\}$

Set $f(u_k) = 2, f(u_{3n}) = 6$. Assign all available even labels out of $\{1, 2, ..., 4n - 1\}$ to $u_1, u_2, ..., u_{k-1}, u_{k+1}, ..., u_{2n}$ as a result of which two vertices out of $u_1, u_2, ..., u_{k-1}, u_{k+1}, ..., u_{2n}$
will not be able to be labeled, since there are exactly \( \frac{4n}{2} - 1 \) number of even labels available. For unlabeled star pendant vertices we assign the labels 3 and 9.

Next, fix \( f(u_{3n-1}) = 1, f(u_{3n+1}) = 15 \) and \( f(u_{3n+2}) \) be the largest prime \( p \leq 4n - 1 \), and assign the unused (odd) labels out of \( \{1, 2, ..., 4n - 1\} \) to unlabeled vertices in increasing order of indices with respect to increasing order of labels.

Observe \( \gcd(f(u_{3n}), f(u_i)) > 1 \), \( \forall 1 \leq i \leq 2n \), \( \gcd(f(u_{3n}), f(u_{3n+1})) > 1 \) and \( \gcd(f(u_k)) \) with all pendant vertices of star except for the those that are labeled with 3 and 9, will be greater than 1.

Clearly there are exactly \( 4n - 3 \) number of edges that will bear 0 labels and for the rest of the edges, the \( \gcd \) of their end vertices will be equal to 1.

Evidently, \( |e_f(0) - e_f(1)| \leq 1 \).

Case 2: When we switch pendant vertices of path in lilly graph. i.e: \( u_{2n+1} \) or \( u_{4n-1} \). Without loss of generality, let us switch \( u_{2n+1} \)

Fix \( f(u_{2n+1}) = 2, f(u_{3n}) = 6, f(u_1) = 4, f(u_2) = 8 \)

\( f(u_i) = f(u_{i-1}) + 2; \quad 3 \leq i \leq 2n - 3 \),

\( f(u_{2n-2}) = 3, f(u_{2n-1}) = 5, f(u_{2n}) = 9, f(u_{3n-1}) = 1 \) and \( f(u_{3n+1}) \) be the largest prime \( p \leq 4n - 1 \). Assign the unused labels out of \( \{1, 2, ..., 4n - 1\} \) in the increasing order.

Observe \( \gcd(f(u_{3n}), f(u_i)) > 1, \forall i = 1, 2, ..., 2n, 2n + 1 \) and \( i \neq 2n - 1 \), \( \gcd(f(u_{2n+1}), f(u_i)) > 1 \), for \( 1 \leq i \leq 2n - 3 \).

The edges formed using these vertices will bear 0 label which are \( 4n - 3 \) in count. The \( \gcd \) of the end vertices of the rest of the edges is qual to 1. Evidently, \( e_f(0) = e_f(1) = 4n - 3 \) which justifies \( |e_f(0) - e_f(1)| \leq 1 \). We see in both cases that lilly graph is invariant under the graph operation of switching of any pendant vertex for PCL.

**Figure 3.** Switching of \( u_1 \) in lilly graph \( I_5 \)

**Theorem 3.** Switching of an apex vertex in \( I_n \) admits a PCL.
Figure 4. Switching of $u_{11}$ in lilly graph $I_5$

Proof. Let $u_1, u_2, \ldots, u_{4n-1}$ represent the vertices of $I_n$. Obtain $G$ by switching the apex vertex of $I_n$ namely $u_{3n}$. Construct a map (bijective) $f : V(G) \rightarrow \{1, 2, \ldots, 4n-1\}$ as:

Fix $f(u_{3n}) = 1$, $f(u_{2n+1}) = 2$,

$f(u_i) = f(u_{i-1}) + 2$; $2n + 2 \leq i \leq 3n - 1$,

$f(u_{3n+1}) = f(u_{3n-1}) + 2$,

$f(u_j) = f(u_{j-1}) + 2$; $3n + 2 \leq j \leq 4n - 1$.

Assign unused labels to the remaining vertices in any order. Observe that $\gcd(f(u_3), f(u_i)) > 1$ for $2n + 1 \leq i \leq 3n - 2$ and for $3n + 1 \leq i \leq 4n - 2$, and for rest of the edges, since the $\gcd$ of their end vertices is equal to 1, therefore those edges will be labeled with label 1. Clearly, $e_f(0) = e_f(1) = 2n - 4$. Therefore $G$ is a PCG.

Theorem 4. Duplication of apex vertex in $I_n$ admits a PCL for $n \geq 3$.

Proof. Let $G_1$ be obtained by duplicating the apex vertex of $I_n$ namely $u_{3n}$, by a vertex namely $v$. $V(G) = V(I_n)U\{v\} \& E(G) = E(I_n)U\{u_i v; 1 \leq i \leq 2n, i = 3n - 1, 3n + 1\}$. Construct a map (bijective) $f : V(G) \rightarrow \{1, 2, \ldots, 4n\}$ as:

Fix $f(u_{3n}) = 2$, $f(v) = 4$, $f(u_{4n-1}) = 3$, $f(u_{4n-2}) = 6$.

Assign unused even labels to all $u_i$’s where $i \in \{1, 2, \ldots, n\} \cup \{3n + 1\} \cup \{3n + 2, \ldots, 4n - 3\}$ in any order.

Next, assign $f(u_{n+1}) = 1$, $f(u_{n+2}) = 5$,

$f(u_i) = f(u_{i-1}) + 2$; $n + 3 \leq i \leq 3n - 1$.

Observe that $\gcd(f(u_{3n}), f(u_i)) > 1$, $\forall 1 \leq i \leq n$,

$\gcd(f(u_i), f(u_{i+1})) > 1$, $\forall 3n \leq i \leq 4n - 2$,

$\gcd(f(v), f(u_i)) > 1$, $\forall 1 \leq i \leq n$, and $\gcd(f(v), f(u_{3n+1})) > 1$.

The $\gcd$ of the end vertices of the remaining edges of $G$ is equal to 1. we find that $e_f(0) = e_f(1) = 3n$ which proves that $G$ is prime cordial.
Figure 5. Switching of $u_9$ in lilly graph $I_3$

Figure 6. Duplication of $u_{3n}$ in $I_4$

**Theorem 5.** The duplication of any pendant vertex in $I_n$ for $n \geq 2$ permits a PCL.

**Proof.** Let $u_1, u_2, ..., u_{4n-1}$ represents the vertices of $I_n$, where $u_1, u_2, ..., u_{2n}, u_{2n+1}, u_{4n-1}$ are pendant vertices. Obtain $G$ by duplicating any pendant vertex of $I_n$ say $u_k$ & let the newly introduced vertex be named $v$. Construct a map (bijective) $f : V(G) \rightarrow \{1, 2, ..., 4n\}$ as:

- Fix $f(v) = 1$, $f(u_{3n}) = 2$, $f(u_1) = 4,$
- $f(u_i) = f(u_{i-1}) + 2$, for $2 \leq i \leq n,$
- $f(u_{2n+1}) = f(u_n) + 2$, $f(u_j) = f(u_{j-1}) + 2$, for $2n + 2 \leq j \leq 3n - 1,$
- $f(u_{n+1}) = 3$, $f(u_i) = f(u_{i-1}) + 2$, for $n + 2 \leq i \leq 2n$
- $f(u_{3n+1}) = f(u_{2n}) + 2$, $f(u_j) = f(u_{j-1}) + 2$, for $3n + 2 \leq j \leq 4n - 1.$

Clearly, $e_f(0) = 2n - 1$
and $e_f(1) = 2n$ which proves that $G$ is prime cordial.

\[\text{Figure 7. Duplication of } u_3 \text{ in lilly } I_4\]

**Theorem 6.** The duplication of arbitrary path vertex (except pendant and apex) in a lilly graph permits a PCL.

**Proof.** Let $u_1, u_2, \ldots, u_{4n-1}$ represents the vertices of $I_n$. Obtain $G$ by duplicating any path vertex of $I_n$ say $u_k$ with a vertex say $v$, where $k \in \{2n+2, 2n+3, \ldots, 3n-1\}U\{3n+1, 3n+2, \ldots, 4n-2\}$.

Define a (bijective) $f : V(G) \to \{1, 2, \ldots, 4n\}$ as:

- $f(u_3n) = 2, f(v) = 1, f(u_{3n+1}) = 3, f(u_{3n+2}) = 9, f(u_1) = 4,$
- $f(u_i) = f(u_{i-1}) + 2, \forall 2 \leq i \leq n,$
- $f(u_{2n+1}) = f(u_n) + 2,$
- $f(u_i) = f(u_{i-1}) + 2, \forall 2n + 2 \leq i \leq 3n - 1.$

Assign unused labels out of the available labels to the unlabeled vertices in increasing order, to the vertices beginning with $u_{3n+3}$ and heading up to $u_{4n-1}$. Next assign the unused labels to $u_i$'s where $n+1 \leq i \leq 2n$, in any order.

Observe that $gcd(f(u_3n), f(u_i)) > 1$, $\forall 1 \leq i \leq n,$

$gcd(f(u_i), f(u_{i+1})) > 1$, for $2n + 1 \leq i \leq 3n - 1$, &

$gcd(f(u_{3n+1}), f(u_{3n+2})) > 1$.

The edges due to above vertices will bear 0 label. For the rest of the edges- since the $gcd$ of their end vertices is equal to 1 therefore, those edges shall be labeled with label 1. Evidently, $e_f(1) = 2n$ and $e_f(0) = 2n$, which proves that $G$ is prime cordial.

\[\text{Theorem 7. Degree splitting graph of } I_n, n \geq 4 \text{ permits a PCL.}\]

**Proof.** Let $u_1, u_2, \ldots, u_{4n-1}$ represents the vertices of $I_n$. Let $G$ denotes the degree splitting graph of $I_n$. $V(G) = V(I_n) \cup \{v, w\}$ & $E(G) = E(I_n) \cup \{u_i v; 1 \leq i \leq 2n\} \cup \{u_{2n+1} v, u_{4n-1} v\} \cup \{u_i w; 2n + 2 \leq i \leq 4n - 2, i \neq 3n\}$. Construct a (bijective) $f : V(G) \to \{1, 2, \ldots, 4n + 1\}$ as:
The edges due to above vertices will bear label 0. For the rest of the edges, the \( \gcd \) of their end vertices is equal to 1, therefore those edges shall be labeled with label 1.

Assign unused labels out of the available labels to the unlabeled vertices in increasing order with respect to the increasing order of indices.

Observe that \( \gcd(u_{3n}), \gcd(u_i) > 1 \), \( \forall 1 \leq i \leq 2n - 2 \),
\( \gcd(u_{2n+1}), \gcd(u_{2n+3}) > 1 \), \( \gcd(u_{2n+2}), \gcd(u_{2n+3}) > 1 \),
\( \gcd(v), \gcd(u_i) > 1 \), \( \forall 1 \leq i \leq 2n - 2 \).

The edges due to above vertices will bear label 0. For the rest of the edges, the \( \gcd \) of their end vertices is equal to 1, therefore those edges shall be labeled with label 1.

Evidently, \( e_f(1) = 4n - 2 \) and \( e_f(0) = 4n - 2 \) which proves that \( G \) is prime cordial.

\[ \square \]

**Theorem 8.** Barycentric subdivision of lilly graph permits a PCL.

**Proof.** Let \( u_1, u_2, ..., u_{4n-1} \) represents the vertices of \( I_n \). Let \( G \) be formed by taking the barycentric subdivision of \( I_n \). Clearly, \( V(G) = V(I_n) \cup \{ v_1, v_2, ..., v_{2n}, v_{2n+1}, ..., v_{4n-2} \} \) and \( E(G) = \{ u_{3n}v_i; 1 \leq i \leq 2n \} \cup \{ v_1u_i; 1 \leq i \leq 2n \} \cup \{ u_iv_i; 2n+1 \leq i \leq 4n-2 \} \cup \{ v_1u_{i-1}; 2n+2 \leq i \leq 4n-2 \} \). Construct a (bijective) \( f : V(G) \to \{1, 2, ..., 8n - 3\} \) as:

\[ f(u_{3n}) = 2, f(u_2) = 3, f(v_2) = 1, f(u_{2n}) = 5, f(v_1) = 4, f(v_i) = f(v_{i-1}) + 2, \text{ for } 2 \leq i \leq 2n - 1. \]

Now Assign even labels to \( u_i \)’s for \( 1 \leq i \leq 2n - 1 \), \( i \neq 2 \), in any order.

Next, fix \( f(u_{2n+1}) = 7, f(v_{2n+1}) = 9, f(u_i) = f(u_{i-1}) + 4, \text{ for } 2n + 2 \leq i \leq 3n - 1, \)
\( f(v_i) = f(v_{i-1}) + 4, \text{ for } 2n + 2 \leq i \leq 3n - 1, \)
\( f(v_{3n}) = f(v_{3n+1}) + 2, f(u_{3n}) = f(v_{3n}) + 2, \)
\( f(v_i) = f(v_{i-1}) + 4, \text{ for } 3n + 1 \leq i \leq 4n - 2, \)
\( f(u_i) = f(u_{i-1}) + 4, \text{ for } 3n + 1 \leq i \leq 4n - 1. \)

Evidently, \( e_f(1) = 4n - 2 \) and \( e_f(0) = 4n - 2 \) which proves that \( G \) is prime cordial.  \[ \square \]
Conclusion
We have shown the PCL of lilly graph with various graph operations namely switching a vertex, duplication of vertex by a vertex, degree splitting graph and barycentric subdivision. Observe
that lilly graph is a type of tree, so it is an interesting task to investigate PCL for more tree graphs. Further there is a scope for studying the prime cordial labeling of lilly graph with some other graph operations. We believe that the concept of PCL may play a vital role in the area of robotics and artificial intelligence which is for the future work.

References

[1] Babujee B. J, Shobhna L, Prime and Prime Cordial labeling for some special graphs, *International Journal of Contemporary Mathematical Sciences*, (2010), 2347-2356.

[2] Burton David, Elementary Number Theory, *Wm. C. Brown Company Publishers, second edition* (1980).

[3] Cahit I, Cordial graphs: A weaker version of graceful and harmonious Graphs, *Ars Combinatoria*, (1987), 201-207.

[4] Gallian J.A, A Dynamic Survey of Graph Labeling, *Electronic Journal of Combinatorics* (2009), DS6.

[5] Hararay F, Graph Theory, *Addison Wesley Reading*, (1972).

[6] Prasanna N. L, Applications of Graph Labeling in Communication Networks, *Orient. J. Comp. Sci. and Technol*, (2014).

[7] Samuel Edward A and Kalaivani S, Square sum labeling for some lilly related graphs, *International Journal of Advanced Technology and Engineering Exploration*, (2017), 68-72.

[8] Sundaram M, Pouraj R, Somasundaram S, Prime Cordial Labeling of Graohs, *Journal of Indian Academy of Mathematics*, (2005), 373-390.

[9] Vaidya S.K.Shah N.H, Further Results on Divisor Cordial Labeling, *Annals of Pure and Applied Mathematics*, (2013), 150-159.