Stability of the fuzzy sphere solution from matrix model

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Abstract

We consider a matrix model depending on a parameter $\lambda$ which permits the fuzzy sphere as a classical background. By expanding the bosonic matrices around this background ones recovers a $U(1)$ ($U(n)$) noncommutative gauge theory on the fuzzy sphere. To check classical stability of this background, we look for new classical solutions of this model and find them for $\lambda < 1$, that make the fuzzy sphere solution unstable for $\lambda < \frac{1}{2}$ and stable otherwise.
1 Introduction

The notion of quantum space or pointless geometry has been inspired by quantum mechanics, since the notion of a point in a quantum phase space is meaningless because of the Heisenberg uncertainty principle.

A noncommutative space-time is defined by replacing space-time coordinates $x^i$ by the Hermitian generators $\hat{x}^i$ of a noncommutative $C^*$-algebra of "functions on space-time". Between the motivations of a noncommutative geometry there is surely the hope that the use of a pointless geometry in field theory would partially eliminate the ultraviolet divergences of quantum field theory. In practice it would be equivalent to use a fundamental length scale below which all phenomena are ignored.

While there has been a considerable amount of work for the quantum field theory and string theory defined on a quantum hyperplane [1]-[3], there has been little understanding for the possibility of defining non-commutativity for curved manifolds. As a first example in this direction, noncommutative gauge theories on a noncommutative sphere has been derived by expanding a matrix model around its classical solution [4]-[5]. The fuzzy sphere solution is considered as a classical background, and the fluctuations on the background from the matrices are the fields of noncommutative gauge theory. Being the fuzzy sphere compact [6]-[8], it is possible to study it with a matrix model at finite $N$, while the quantum plane is recovered only in the $N \rightarrow \infty$ limit [9]-[10]-[11]-[12]-[13].

Usually matrix models are obtained by the dimensional reduction from Yang-Mills theory; between them IIB matrix model is expected to give the constructive definition of type IIB superstring theory [14]-[15]-[16]-[17]-[18]-[19]. However the IIB matrix model has only flat noncommutative backgrounds as classical solutions. To describe a curved space-time we need to add a Chern-Simons term to Yang-Mills reduced model [4]

$$S = \frac{1}{g^2} Tr[-1/4[A_i, A_j][A_i, A_j] + 2/3i \rho \epsilon^{ijk} A_i A_j A_k]$$ (1.1)

where $A_i$ are three $(N + 1) \times (N + 1)$ hermitian matrices, or a mass term as in the model [3]

$$S' = -1/g^2 Tr[1/4[A_i, A_j][A_i, A_j] + \rho^2 A_i A_i]$$ (1.2)

Generally in this paper we consider an action depending on a parameter $\lambda$ which is an interpolation between the two actions (1.1) and (1.2):

$$S(\lambda) = S_0 + \lambda S_1 = -1/g^2 Tr[1/4[A_i, A_j][A_i, A_j] - 2/3i \lambda \rho \epsilon^{ijk} A_i A_j A_k + \rho^2 (1 - \lambda) A_i A_i]$$ (1.3)
and discuss stability of the fuzzy sphere solution as a function of the parameter $\lambda$.

In particular we find, with an ansatz which is exhaustive for $N = 1$, new classical solutions of this model for $\lambda < 1$ and compare them with the fuzzy sphere to establish the minimum of the action $S(\lambda)$. We find that for $\lambda < \frac{1}{2}$, the fuzzy sphere solution is unstable and that fluctuations would let the matrix model to decade into the new classical solutions. This could be an obstacle to the construction of the quantum theory of the corresponding noncommutative gauge theory. At $\lambda = \frac{1}{2}$ the two classes of solutions coincide, and this point is particularly symmetric. Our solution permits to deform with continuity the fuzzy sphere solution moving from this symmetric point. In another point, at $\lambda = -1$, our new class of solutions reduces again to a fuzzy sphere but with a different radius. Instead for $\lambda > 1$ there are no new classical solutions other than the fuzzy sphere, at least with our ansatz.

## 2 Properties of the fuzzy sphere

The fuzzy sphere \[20\] is a noncommutative manifold represented by the following algebra

\[
[\hat{x}^i, \hat{x}^j] = i\rho \epsilon^{ijk} \hat{x}^k \quad i, j, k = 1, 2, 3
\]  

(2.1)

$\hat{x}^i$ can be represented by $(N + 1) \times (N + 1)$ hermitian matrices, which can be constructed by the generators of the $(N + 1)$-dimensional irreducible representation of $SU(2)$

\[
\hat{x}^i = \rho \hat{L}^i
\]  

(2.2)

The radius of the sphere is obtained by the following condition

\[
\hat{x}^i \hat{x}^i = R^2 = \rho^2 \hat{L}^i \hat{L}^i = \rho^2 \frac{N(N + 2)}{4}
\]  

(2.3)

The commutative limit is realized by

\[
R = \text{fixed} \quad \rho \to 0 \quad (N \to \infty)
\]  

(2.4)

In this limit, $\hat{x}^i$ become the normal coordinates on the sphere $x^i$:

\[
x^1 = R sin \theta cos \phi \\
x^2 = R sin \theta sin \phi
\]
which produces the usual metric tensor of the sphere:

\[ ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\phi^2) = R^2 g_{ab} d\sigma_a d\sigma_b \]  

(2.6)

The fuzzy sphere function space is finite-dimensional, in contrast to what happens in the commutative limit. Since the coordinates \( x^i \) are substituted by hermitian matrices, the number of independent functions on the fuzzy sphere is \( (N+1)^2 \), which is exactly the number of parameters of a \( (N+1) \times (N+1) \) hermitian matrix.

This cutoff on the function space can be constructed by introducing a cutoff parameter \( N \) for the angular momentum of the spherical harmonics. An ordinary function on the sphere can be expanded in terms of the spherical harmonics:

\[ a(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\Omega) \]  

(2.7)

The \( Y_{lm}(\Omega) \) form a basis of the classical infinite-dimensional function space. By introducing a cutoff \( N \) on the number \( l \), the number of independent functions reduces to \( \sum_{l=0}^{N} (2l+1) = (N+1)^2 \). To define the noncommutative analogue of the spherical harmonics, we appeal to their classical form:

\[ Y_{lm} = R^{-l} \sum_{a} f_{a_{1},a_{2},...,a_{l}}^{(lm)} x^{a_{1}}...x^{a_{l}} \]  

(2.8)

and \( f_{a_{1},a_{2},...,a_{l}} \) is a traceless and symmetric tensor. The normalization of the spherical harmonics is fixed by

\[ \int \frac{d\Omega}{4\pi} Y_{lm}^{*} Y_{l'm'} = \delta_{ll'} \delta_{mm'} \]  

(2.9)

The corresponding noncommutative spherical harmonics \( \hat{Y}_{lm} \) are \( (N+1) \times (N+1) \) hermitian matrices,

\[ \hat{Y}_{lm} = R^{-l} \sum_{a} f_{a_{1},a_{2},...,a_{l}}^{(lm)} \hat{x}_{a_{1}}...\hat{x}_{a_{l}} \]  

(2.10)

defined by the same symmetric tensor \( f_{a_{1},a_{2},...,a_{l}}^{(lm)} \). A Weyl type ordering is implicit into this definition, due to the symmetry of the indices [2].
Normalization of the noncommutative spherical harmonics is given by

\[ \frac{1}{N+1} Tr(\hat{Y}_{l,m'}^\dagger \hat{Y}_{lm}) = \delta_{l'l} \delta_{m'm} \]  

(2.11)

An alternative definition of this cutoff on the function space can be given by introducing a star product on the fuzzy sphere analogous to the Moyal star product for the plane.

A matrix on the fuzzy sphere

\[ \hat{a} = \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm} \quad a_{lm}^* = a_{l-m} \]  

(2.12)

corresponds to an ordinary function on the commutative sphere, with a cutoff on the angular momentum:

\[ a(\Omega) = \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\Omega) \]  

(2.13)

where

\[ a(\Omega) = \frac{1}{N+1} \sum_{l=0}^{N} \sum_{m=-l}^{l} Tr(\hat{Y}_{lm}^\dagger \hat{a}) Y_{lm}(\Omega) \]  

(2.14)

and the ordinary product of matrices is mapped to the star product on the commutative sphere:

\[ \hat{a} \hat{b} \to a \ast b \]

\[ a(\Omega) \ast b(\Omega) = \frac{1}{N+1} \sum_{l=0}^{N} \sum_{m=-l}^{l} Tr(\hat{Y}_{lm}^\dagger \hat{a} \hat{b}) Y_{lm}(\Omega) \]  

(2.15)

The cutoff in the noncommutative spherical harmonics is consistent since the \( \hat{Y}_{lm} \) form a basis of the noncommutative Hilbert space of maps.

Derivative operators can be constructed by the adjoint action of \( \hat{L}_i \):

\[ Ad(\hat{L}_i) \hat{a} = \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm} [\hat{L}_i, \hat{Y}_{lm}] \]  

(2.16)

In the classical limit \( Ad(\hat{L}_i) \) tends to the Lie derivative on the sphere:
\[ Ad(\hat{L}_i) \rightarrow L_i = \frac{1}{i} \epsilon_{ijk} x_j \partial_k \] (2.17)

where we can expand the classical Lie derivative \( L_i \) in terms of the Killing vectors of the sphere

\[ L_i = -i K^a_i \partial_a \] (2.18)

In terms of \( K^a_i \) we can form the metric tensor \( g_{ab} = K^i_a K^j_b \).

In particular the analogue of the Laplacian on the fuzzy sphere is given by:

\[
\frac{1}{R^2} Ad(\hat{L}^2) \hat{a} = \frac{1}{R^2} \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm} [\hat{L}_i, [\hat{L}_i, \hat{Y}_{lm}]] = \\
= \sum_{l=0}^{N} \sum_{m=-l}^{l} \frac{l(l+1)}{R^2} a_{lm} \hat{Y}_{lm} \] (2.19)

Trace over matrices can be mapped to the integration over functions:

\[
\frac{1}{N+1} Tr(\hat{a}) \rightarrow \int \frac{d\Omega}{4\pi} a(\Omega) \] (2.20)

3 Gauge theory on the fuzzy sphere from matrix model

We now recall how to recover gauge theory on the fuzzy sphere \[22-23-24-25-26\] by expanding matrices around the classical solutions of the action. Consider firstly the following action \( S_0 \):

\[ S_0 = -\frac{1}{g^2} Tr(\frac{1}{4}[A_i, A_j][A_i, A_j] + \rho^2 A_i A_i) \] (3.1)

We expand the bosonic matrices \( A_i \) around the classical solution (2.1) as

\[ A_i = \hat{x}_i + \rho R \hat{a}_i \] (3.2)

In this way a \( U(1) \) noncommutative gauge theory on the fuzzy sphere is introduced through the fluctuation \( \hat{a}_i \). Generalizing it to \( U(N) \) gauge group is possible by changing the background classical solution:
\[ \hat{x}_i \rightarrow \hat{x}_i \otimes 1_m \] (3.3)

Therefore the fluctuations \( \hat{a}_i \) are replaced as follows:

\[ \hat{a}_i \rightarrow \sum_{a=1}^{m^2} \hat{a}_i^a \otimes T^a \] (3.4)

where \( T^a (a = 1, 2, ..., m^2) \) denote the generators of \( U(m) \).

The action (3.1) is invariant under the unitary transformation

\[ A_i \rightarrow U^{-1} A_i U \] (3.5)

Since \( A_i \) has the meaning of a covariant derivative as in (3.2), it is clear that gauge symmetry of the noncommutative gauge theories is included in the unitary transformation of the matrix model.

For an infinitesimal transformation:

\[ U \sim 1 + i \hat{\lambda} \hat{\lambda} = \sum_{l=0}^{N} \sum_{m=-l}^{l} \lambda_{lm} \hat{Y}_{lm} \] (3.6)

the fluctuations around the fixed background transforms as

\[ \hat{a}_1 \rightarrow \hat{a}_i - \frac{i}{R} [\hat{L}_i, \hat{\lambda}] + i[\hat{\lambda}, \hat{a}_i] \] (3.7)

By using the mapping from matrices to functions, we recover the local star-product gauge symmetry:

\[ a_i(\Omega) \rightarrow a_i(\Omega) - \frac{i}{R} L_i \lambda(\Omega) + i[\lambda(\Omega), a_i(\Omega)]_* \] (3.8)

(\( \cdot \))\(_*\) means that the product is to be considered as a star product.

The corresponding field strength on the sphere is given by

\[ \hat{F}_{ij} = \frac{1}{\rho^2 R^2} ([A_i, A_j] - i \rho \epsilon_{ijk} A_k) \]

\[ = \frac{\hat{L}_i}{R} \hat{a}_j - \frac{\hat{L}_j}{R} \hat{a}_i + [\hat{a}_i, \hat{a}_j] - \frac{i}{R} \epsilon_{ijk} \hat{a}_k \] (3.9)
that is mapped to the following function

\[ F_{ij}(\Omega) = \frac{1}{R} L_i a_j(\Omega) - \frac{1}{R} L_j a_i(\Omega) + [a_i(\Omega), a_j(\Omega)]_* \] (3.10)

\( F_{ij} \) is gauge covariant even in the \( U(1) \) case, as it is manifest from the viewpoint of the matrix model.

The model contains also a scalar field which belongs to the adjoint representation as the gauge field, and that can be defined as:

\[ \hat{\phi} = \frac{1}{2\rho R}(A_i A_i - \hat{x}_i \hat{x}_i) = \frac{1}{2}(\hat{x}_i \hat{a}_i + \hat{a}_i \hat{x}_i + \rho R \hat{a}_i \hat{a}_i) \] (3.11)

However, at the noncommutative level, it is impossible to disentangle the gauge field and the scalar field which are contained in the matrix model. Therefore only in the classical limit the action can be interpreted as a sum of both contributions.

### 4 Commutative limit

The action \( S_0 \) is mapped through the map (2.20) to the following field theory action as follows

\[
S_0 = -\frac{\rho^2}{4g_{YM}^2} Tr \int d\Omega (F_{ij} F_{ij}) - \frac{3i}{2g_{YM}^2} \epsilon_{ijk} Tr \int d\Omega ((L_i a_j) a_k + \frac{\rho}{3}[a_i, a_j] a_k - \frac{i}{2} \epsilon_{ijkl} a_l a_k)_* \\
- \frac{\pi}{g_{YM}^2} \frac{N(N+2)}{2R^2} (4.1)
\]

The commutative limit is realized as

\[
R = \text{fixed}, \quad g_{YM}^2 = \frac{4\pi^2 g^2}{(N+1)\rho^4 R^2} = \text{fixed} \quad N \to \infty (4.2)
\]

In the commutative limit, the star product becomes the commutative product.

In this limit, the scalar field \( \phi \) and the gauge field are separable from each other as in

\[
Ra_i(\Omega) = K_i^a b_a(\Omega) + \frac{x_i}{R} \phi(\Omega) \] (4.3)

where \( b_a \) is a gauge field on the sphere. The field strength \( F_{ij} \) can be expanded in terms of the gauge field \( b_a \) and the scalar field \( \phi \) as follows:
\[ F_{ij}(\Omega) = \frac{1}{R^2}K_i^a K_j^b F_{ab} + \frac{i}{R^2} \epsilon_{ijk} x_k \phi + \frac{1}{R^2} x_J K_i^a D_a \phi - \frac{1}{R^2} x_i K_j^a D_a \phi \quad (4.4) \]

where \( F_{ab} = -i(\partial a b - \partial b a) + [b_a, b_b] \) and \( D_a = -i\partial a + [b_a, \cdot] \).

The action \( S_0 \) is finally rewritten as:

\[
S_0 = -\frac{1}{4g_{YM}^2 R^2} Tr \int d\Omega (K_i^a K_j^b K_i^c K_j^d F_{cd} + 2i K_i^b F_{ab} \epsilon_{ijk} x_k \phi - 1 \frac{R}{R^2} \epsilon_{ijk} x_k \phi) + 2K_i^b K_j^b (D_a \phi) - 2 \phi^2 \\
- \frac{3}{2g_{YM}^2 R^2} Tr \int d\Omega (i \epsilon_{ijk} K_i^a K_j^b F_{ab} x_k \phi - \phi^2) \\
= -\frac{1}{4g_{YM}^2 R^2} Tr \int d\Omega (F_{ab} F_{ab} + 8 \frac{i}{\sqrt{g}} F_{ab} \phi + 2 (D_a \phi) (D^a \phi) - 8 \phi^2) \quad (4.5) \]

Analogously the action \( S_1 \) is mapped to the following field theory action as follows:

\[
S_1 = \frac{i}{g_{YM}} \epsilon_{ijk} Tr \int d\Omega ((L_i a_j) a_k + \frac{R}{3} [a_i, a_j] a_k - \frac{i}{2} \epsilon_{ijl} a_l a_k)_* + \frac{\pi}{3g_{YM}^2 R^2} N(N + 2) \quad (4.6) \]

In the commutative limit this becomes:

\[
S_1 = \frac{1}{g_{YM}^2 R^2} Tr \int d\Omega (i \epsilon_{ijk} K_i^a K_j^b F_{ab} x_k \phi - \phi^2) = \frac{1}{g_{YM}^2 R^2} Tr \int d\Omega \left( \frac{i \epsilon_{ab}}{\sqrt{g}} F_{ab} \phi - \phi^2 \right) \quad (4.7) \]

Let us notice that with \( \lambda = 2 \) the complete action \( S(\lambda) \) becomes

\[
S(2) = S_0 + 2S_1 = -\frac{1}{4g_{YM}^2 R^2} Tr \int d\Omega (F_{ab} F^{ab} - 2 (D_a \phi) (D^a \phi)) \quad (4.8) \]

there is no mixing term between \( \phi \) and \( F_{ab} \).

With \( \lambda = \frac{3}{2} \), the action is stable, with its minimum on the fuzzy sphere:

\[
S\left(\frac{3}{2}\right) = -\frac{1}{4g^2} Tr F_i F_i = -\frac{1}{4g^2} Tr ([A_i, A_j] - i \rho \epsilon_{ijk} A_k) ([A_i, A_j] - i \rho \epsilon_{ijk} A_k) \quad (4.9) \]
Possible instability of the fuzzy sphere solution will be analyzed in details in the next section, where it will be proved that it can appear only for $\lambda < 1$.

5 Stability of classical solutions

Let us start with the model $S_0$. The corresponding classical equations of motion are:

$$[A^j, [A^i, A^j]] + 2\rho^2 A^i = 0 \quad (5.1)$$

This type of equation of motion is considered in [27]-[28]. The first immediate consequence of this equation is that

$$Tr(A^i) = 0 \quad (5.2)$$

In the case of $N = 1$, the general solution is then given by the development

$$A^i = A^i_k \frac{\sigma^k}{2} \quad (5.3)$$

that can be generalized to arbitrary $N$ with the ansatz:

$$A^i = A^i_k \hat{L}^k \quad (5.4)$$

With this ansatz, the general solution is therefore given by three vectors $A^1_i, A^2_i, A^3_i$ satisfying:

$$A^1_i \left[ 2\rho^2 - (A^2_i)^2 - (A^3_i)^2 \right] + (A^1_i \cdot A^2_i)A^2_i + (A^1_i \cdot A^3_i)A^3_i = 0$$
$$A^2_i \left[ 2\rho^2 - (A^1_i)^2 - (A^3_i)^2 \right] + (A^1_i \cdot A^2_i)A^1_i + (A^2_i \cdot A^3_i)A^3_i = 0$$
$$A^3_i \left[ 2\rho^2 - (A^1_i)^2 - (A^2_i)^2 \right] + (A^1_i \cdot A^3_i)A^1_i + (A^2_i \cdot A^3_i)A^2_i = 0 \quad (5.5)$$

It is not difficult to find a solution to this system of equations. By multiplying the first one with $A^2_i$ and $A^3_i$ and so on we find that:

$$(A^1_i \cdot A^2_i)(2\rho^2 - (A^3_i)^2) + (A^1_i \cdot A^3_i)(A^2_i \cdot A^3_i) = 0$$
$$(A^1_i \cdot A^3_i)(2\rho^2 - (A^2_i)^2) + (A^1_i \cdot A^2_i)(A^2_i \cdot A^3_i) = 0$$
$$(A^2_i \cdot A^3_i)(2\rho^2 - (A^1_i)^2) + (A^1_i \cdot A^2_i)(A^1_i \cdot A^3_i) = 0 \quad (5.6)$$
whose solution is

\[(A^1 \cdot A^2) = (A^1 \cdot A^3) = (A^2 \cdot A^3) = 0 \implies \text{fuzzy sphere} \quad (5.7)\]

or

\[
\begin{align*}
A^1 \cdot A^2 &= -\sqrt{(2\rho^2 - (A_i^1)^2)(2\rho^2 - (A_i^2)^2)} = -\sqrt{(2\rho^2 - \alpha^2)(2\rho^2 - \beta^2)} \\
A^1 \cdot A^3 &= -\sqrt{(2\rho^2 - (A_i^1)^2)(2\rho^2 - (A_i^3)^2)} = -\sqrt{(2\rho^2 - \alpha^2)(2\rho^2 - \gamma^2)} \\
A^2 \cdot A^3 &= -\sqrt{(2\rho^2 - (A_i^2)^2)(2\rho^2 - (A_i^3)^2)} = -\sqrt{(2\rho^2 - \beta^2)(2\rho^2 - \gamma^2)} \quad (5.8)
\end{align*}
\]

Two comments are in order; firstly the argument of the square root must be definite positive, for example :

\[(2\rho^2 - \alpha^2)(2\rho^2 - \beta^2) \geq 0 \quad (5.9)\]

and therefore there are only two possibilities

\[
\begin{align*}
i) \quad & \alpha^2 \geq 2\rho^2 \quad \beta^2 \geq 2\rho^2 \quad \gamma^2 \geq 2\rho^2 \\
ii) \quad & \alpha^2 \leq 2\rho^2 \quad \beta^2 \leq 2\rho^2 \quad \gamma^2 \leq 2\rho^2 \quad (5.10)
\end{align*}
\]

Secondly the scalar products must be less than the product of the moduli of the vectors i.e.

\[(A^1 \cdot A^2)^2 \leq \alpha^2 \beta^2 \quad (5.11)\]

therefore

\[
\alpha^2 + \beta^2 \geq 2\rho^2 \quad \alpha^2 + \gamma^2 \geq 2\rho^2 \quad \beta^2 + \gamma^2 \geq 2\rho^2 \quad (5.12)
\]

By multiplying the first one of Eq. (5.5) by \(A_i^1\), the second one by \(A_i^2\) and the third one by \(A_i^3\) we find :

\[
\alpha^2 + \beta^2 + \gamma^2 = 4\rho^2 \quad (5.13)
\]
Therefore the possibility i) is impossible, and we are left with possibility ii). Moreover
ii) + (5.13) implies (5.12) automatically.

Eq. (5.13) is a two-parameter solution:

\[ \alpha^2 = 4\rho^2 \cos^2 \theta \]
\[ \beta^2 = 4\rho^2 \sin^2 \theta \cos^2 \phi \]
\[ \gamma^2 = 4\rho^2 \sin^2 \theta \sin^2 \phi \]

(5.14)

and ii) implies that

\[ \sin^2 \theta \geq \frac{1}{2}, \quad 1 - \frac{1}{2 \sin^2 \theta} \leq \sin^2 \phi \leq \frac{1}{2 \sin^2 \theta} \]

(5.15)

Till now we have found implicit consistency equations.

Let us now compute the system (5.3) component by component. By parameterizing:

\[ A^1 \cdot A^2 = \alpha \beta \cos \theta_{12} \]
\[ A^1 \cdot A^3 = \alpha \gamma \cos \theta_{13} \]
\[ A^2 \cdot A^3 = \beta \gamma \cos \theta_{23} \]

(5.16)

we can choose the first vector in a fixed direction ( \( A^i \) are invariant under the gauge
transformation \( U^{-1} A^i U \) )

\[ A^1_x = \alpha \quad A^1_y = A^1_z = 0 \]
\[ A^2_x = \beta \cos \theta_{12} \quad A^2_y = \beta \sin \theta_{12} \quad A^2_z = 0 \]
\[ A^3_x = \gamma \cos \theta_{13} \quad A^3_y = \gamma \sin \theta_{13} \sin \phi \quad A^3_z = \gamma \sin \theta_{13} \cos \phi \]

(5.17)

As a consequence

\[ \cos \theta_{23} = \cos \theta_{12} \cos \theta_{13} + \sin \theta_{12} \sin \theta_{13} \sin \phi \]

(5.18)

The component by component evaluation lead us to fix completely the solution, i.e. to
find \( \sin \phi \) and the relative signs as follows:
\[
\begin{align*}
\cos \phi &= 0 \quad \sin \phi = 1 \quad \Rightarrow \theta_{23} = \theta_{12} - \theta_{13} \\
\cos \theta_{12} &= -\frac{1}{\alpha \beta} \sqrt{(2 \rho^2 - \alpha^2)(2 \rho^2 - \beta^2)} \quad \sin \theta_{12} = \frac{1}{\alpha \beta} \sqrt{2 \rho^2 (\alpha^2 + \beta^2 - 2 \rho^2)} \\
\cos \theta_{13} &= -\frac{1}{\alpha \gamma} \sqrt{(2 \rho^2 - \alpha^2)(2 \rho^2 - \gamma^2)} \quad \sin \theta_{13} = -\frac{1}{\alpha \gamma} \sqrt{2 \rho^2 (\alpha^2 + \gamma^2 - 2 \rho^2)} \\
\cos \theta_{23} &= -\frac{1}{\beta \gamma} \sqrt{(2 \rho^2 - \beta^2)(2 \rho^2 - \gamma^2)} \quad \sin \theta_{23} = -\frac{1}{\beta \gamma} \sqrt{2 \rho^2 (\beta^2 + \gamma^2 - 2 \rho^2)} \quad (5.19)
\end{align*}
\]

Now we turn to the \( \lambda \neq 0 \) case. The equations of motion for the complete action \( S(\lambda) \) read now

\[
[A^j, [A^i, A^j]] - i \rho \lambda \epsilon^{ijk} [A^j, A^k] + 2 \rho^2 (1 - \lambda) A^i = 0 \quad (5.20)
\]

Again the condition

\[
Tr(A^i) = 0 \quad (5.21)
\]

implies the following ansatz

\[
A^i = A^i_k \hat{L}^k \quad (5.22)
\]

from which we obtain the following system

\[
\begin{align*}
(2 \rho^2 (1 - \lambda) - \beta^2 - \gamma^2) A^1_i + (A^1 \cdot A^2) A^2_i + (A^1 \cdot A^3) A^3_i + 2 \lambda \rho \epsilon_{ijk} A^j_i A^k_j &= 0 \\
(2 \rho^2 (1 - \lambda) - \alpha^2 - \gamma^2) A^2_i + (A^1 \cdot A^3) A^1_i + (A^2 \cdot A^3) A^3_i + 2 \lambda \rho \epsilon_{ijk} A^3_i A^j_k &= 0 \\
(2 \rho^2 (1 - \lambda) - \alpha^2 - \beta^2) A^3_i + (A^1 \cdot A^3) A^1_i + (A^2 \cdot A^3) A^2_i + 2 \lambda \rho \epsilon_{ijk} A^1_i A^j_k &= 0 \quad (5.23)
\end{align*}
\]

By multiplying the first equation by \( A^2_i, A^3_i \) and so on we obtain:

\[
\begin{align*}
A^1 \cdot A^2 &= -\sqrt{(2 \rho^2 (1 - \lambda) - \alpha^2)(2 \rho^2 (1 - \lambda) - \beta^2)} \\
A^1 \cdot A^3 &= -\sqrt{(2 \rho^2 (1 - \lambda) - \alpha^2)(2 \rho^2 (1 - \lambda) - \gamma^2)} \\
A^2 \cdot A^3 &= -\sqrt{(2 \rho^2 (1 - \lambda) - \beta^2)(2 \rho^2 (1 - \lambda) - \gamma^2)} \quad (5.24)
\end{align*}
\]

Again the condition that the square root is positive definite requires that:

\[
i \alpha^2 \geq 2 \rho^2 (1 - \lambda) \quad \beta^2 \geq 2 \rho^2 (1 - \lambda) \quad \gamma^2 \geq 2 \rho^2 (1 - \lambda) \quad (5.25)
\]
or

\[ ii) \alpha^2 \leq 2\rho^2(1 - \lambda) \quad \beta^2 \leq 2\rho^2(1 - \lambda) \quad \gamma^2 \leq 2\rho^2(1 - \lambda) \quad (5.26) \]

The condition that

\[ (A^1 \cdot A^2)^2 \leq \alpha^2 \beta^2 \quad (A^1 \cdot A^3)^2 \leq \alpha^2 \gamma^2 \quad (A^2 \cdot A^3)^2 \leq \beta^2 \gamma^2 \quad (5.27) \]

implies

\[ \alpha^2 + \beta^2 \geq 2\rho^2(1 - \lambda) \quad \beta^2 + \gamma^2 \geq 2\rho^2(1 - \lambda) \quad \alpha^2 + \gamma^2 \geq 2\rho^2(1 - \lambda) \quad (5.28) \]

if and only if \( \lambda \leq 1 \). For \( \lambda > 1 \) there are no solutions.

Let us verify the system of equations (5.23) with the parameterizations (5.17). In particular the equation for \( A^2_z \) implies that:

\[ \beta \gamma \cos \theta_{23} = 2 \lambda \alpha \rho \sin \phi \quad (5.29) \]

Therefore

\[ \sin^2 \phi = \frac{(2\rho^2(1 - \lambda) - \beta^2)(2\rho^2(1 - \lambda) - \gamma^2)}{(2\rho^2(1 - \lambda) - \beta^2)(2\rho^2(1 - \lambda) - \gamma^2) + 4\lambda^2 \alpha^2 \rho^2} \quad (5.30) \]

The other equations imply the following constraint (generalizing eq. (5.13))

\[ \alpha^2 + \beta^2 + \gamma^2 = 4\rho^2(1 - \lambda) + 4\lambda^2 \rho^2 \quad (5.31) \]

and fixes all the sign as follows

\[
\begin{align*}
\cos \phi &= -\frac{2\lambda \alpha \rho}{\sqrt{(2\rho^2(1 - \lambda) - \beta^2)(2\rho^2(1 - \lambda) - \gamma^2) + 4\lambda^2 \alpha^2 \rho^2}} \\
\sin \phi &= \sqrt{(2\rho^2(1 - \lambda) - \beta^2)(2\rho^2(1 - \lambda) - \gamma^2) + 4\lambda^2 \alpha^2 \rho^2} \\
\cos \theta_{12} &= -\frac{1}{\alpha \beta} \sqrt{(2\rho^2(1 - \lambda) - \alpha^2)(2\rho^2(1 - \lambda) - \beta^2)} \\
\sin \theta_{12} &= \frac{1}{\alpha \beta} \sqrt{2\rho^2(1 - \lambda)(\alpha^2 + \beta^2 - 2\rho^2(1 - \lambda))}
\end{align*}
\]
\[
\cos \theta_{13} = -\frac{1}{\alpha \gamma} \sqrt{(2\rho^2(1 - \lambda) - \alpha^2)(2\rho^2(1 - \lambda) - \gamma^2)} \\
\sin \theta_{13} = -\frac{1}{\alpha \gamma} \sqrt{2\rho^2(1 - \lambda)(\alpha^2 + \gamma^2 - 2\rho^2(1 - \lambda))} \\
\cos \theta_{23} = -\frac{1}{\beta \gamma} \sqrt{(2\rho^2(1 - \lambda) - \beta^2)(2\rho^2(1 - \lambda) - \gamma^2)} \\
\sin \theta_{23} = -\frac{1}{\beta \gamma} \sqrt{2\rho^2(1 - \lambda)(\beta^2 + \gamma^2 - 2\rho^2(1 - \lambda))}
\]

(5.32)

The position \(i\) is compatible with the constraint (5.31) if and only if:

\[
\alpha^2 + \beta^2 + \gamma^2 \geq 6\rho^2(1 - \lambda) \Rightarrow 2\lambda^2 + \lambda - 1 = 2(\lambda + 1)(\lambda - \frac{1}{2}) \geq 0
\]

(5.33)

i.e. if the following condition is met

\[
\lambda < -1 \text{ or } \lambda > \frac{1}{2}
\]

(5.34)

The position \(ii\) is compatible if instead

\[-1 < \lambda < \frac{1}{2}
\]

(5.35)

The case \(\lambda = 0\) is therefore included in the case \(ii\), as we concluded before.

The limiting cases \(\lambda = -1\) and \(\lambda = \frac{1}{2}\) are interesting. They correspond to fuzzy spheres.

For \(\lambda = -1\) we obtain

\[
\cos \phi = 1 \quad \sin \theta_{12} = 1 \quad \sin \theta_{13} = -1 \\
\alpha = \beta = \gamma = 2\rho
\]

(5.36)

and

\[
A^1 = 2\rho L_x \quad A^2 = 2\rho L_y \quad A^3 = -2\rho L_z
\]

(5.37)

For \(\lambda = \frac{1}{2}\) we obtain

\[
\cos \phi = -1 \quad \sin \theta_{12} = 1 \quad \sin \theta_{13} = -1 \\
\alpha = \beta = \gamma = \rho
\]

(5.38)

and

\[
A^i = \rho L^i
\]

(5.39)
Computation of the action

Let us compute the action on our new classical solutions. Firstly we pose

\[ [A^i, A^j] = i \epsilon^{klm} A^i_k A^j_l \hat{L}_m \] (6.1)

therefore

\[ Tr[A^i, A^j][A^i, A^j] = -\epsilon^{kln} \epsilon^{k'lm} A^i_k A^j_l A^i_k' A^j_l' Tr(\hat{L}_m \hat{L}_n) \]

\[ = -\frac{1}{3} ((A^i \cdot A^i)^2 - (A^i \cdot A^j)(A^i \cdot A^j)) Tr(\hat{L}_m \hat{L}_n) \]

\[ = -\frac{8}{3} \rho^4 (1 - \lambda) (4 \lambda^2 - \lambda + 1) Tr(\hat{L}_i \hat{L}_i) \] (6.2)

The second term is

\[ -\frac{2}{3} i \lambda \rho Tr \epsilon_{ijk} A^i A^j A^k = \frac{2}{3} \rho Tr(\hat{L}_i \hat{L}_i) \epsilon^{mnp} A^1_m A^2_n A^3_p \] (6.3)

It is not difficult to compute this triple vector product:

\[ \epsilon^{mnp} A^1_m A^2_n A^3_p = \alpha \beta \gamma \sin \theta_1 \sin \theta_3 \cos \phi = 4 \lambda (1 - \lambda) \rho^3 \] (6.4)

therefore

\[ -\frac{2}{3} i \lambda \rho Tr \epsilon_{ijk} A^i A^j A^k = \frac{8}{3} \lambda^2 (1 - \lambda) \rho^4 Tr(\hat{L}_i \hat{L}_i) \] (6.5)

Finally it is not difficult to evaluate the last term

\[ \rho^2 (1 - \lambda) Tr A^i A^i = \frac{4}{3} \rho^4 (1 - \lambda)(\lambda^2 - \lambda + 1) Tr(\hat{L}_i \hat{L}_i) \] (6.6)

We have reached the following conclusion. The evaluation of this new class of solution is independent from the two parameters \( \theta, \phi \) and depends only on \( \lambda \):

\[ S(\lambda)_{\text{new}} = S_0 + \lambda S_1 = -\frac{1}{3g^2} \rho^4 (1 - \lambda)(2 - 2\lambda + 4 \lambda^2) Tr(\hat{L}_i \hat{L}_i) \] (6.7)

Instead the action evaluated on the fuzzy sphere solution is
\[ S(\lambda)|_{\text{fuzzy sphere}} = -\frac{\rho^4}{g^2} \left( \frac{1}{2} - \frac{\lambda}{3} \right) \text{Tr}(\hat{L}_i \hat{L}_i) \] (6.8)

The condition of stability of the fuzzy sphere solution is therefore established by this equation:

\[ S(\lambda)|_{\text{new}} - S(\lambda)|_{\text{fuzzy sphere}} = \frac{4}{3g^2} \rho^4 (\lambda - \frac{1}{2})^3 \text{Tr}(\hat{L}_i \hat{L}_i) > 0 \] (6.9)

We conclude that the fuzzy sphere solution is stable for \( \lambda \geq \frac{1}{2} \), otherwise it is unstable.

A particular mention has to be devoted to the cases \( \lambda = 1/2 \) and \( \lambda = -1 \). In both cases our new classical solutions reduce to a fuzzy sphere. However only when \( \lambda = \frac{1}{2} \) both solutions coincide, instead when \( \lambda = -1 \) the fuzzy sphere coming from our solutions has double radius and a minus sign in the commutations relations as follows

\[ [A_i, A_j] = -2i\rho \epsilon_{ijk} A_k \quad \lambda = -1 \] (6.10)

The point \( \lambda = \frac{1}{2} \) is particularly symmetric, as it corresponds to the point where both solutions coincide and where the classical instability stops.

### 7 Conclusions

In this paper we have reviewed the construction of a noncommutative gauge theory on a fuzzy sphere starting from a matrix model, depending on a parameter \( \lambda \). We have then studied the classical solutions of it with an ansatz, which is exhaustive for \( N = 1 \), and found new solutions for \( \lambda < 1 \), which make unstable the fuzzy sphere background for \( \lambda < \frac{1}{2} \). These new solutions have the nice property to be a smooth deformation of the fuzzy sphere around the point \( \lambda = \frac{1}{2} \), which is particular symmetric, being the confluence of the two types of classical solutions. There is another, less symmetric point, ( \( \lambda = -1 \)), where our new class of solution reduces to a fuzzy sphere but with a different radius. It would be nice to continue this study by analyzing the corresponding quantum theory around the two different backgrounds, to establish if the quantum corrections modify our classical result on stability, and by searching for solitons solutions to the classical noncommutative gauge theory [29]–[30]–[31]–[32], inside the matrix model.

### References
[1] E. Witten, " Noncommutative geometry and string field theory ", Nucl. Phys. B268 (1986) 253.

[2] N. Seiberg and E. Witten, " String Theory and Noncommutative geometry ", JHEP 9909 (1999) 032, hep-th/9908142.

[3] A. Connes, M. R. Douglas and A. Schwarz, " Noncommutative geometry and Matrix theory: compactification on Tori ", JHEP 9802 (1998) 003, hep-th/9711162.

[4] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, " Noncommutative gauge theory on fuzzy sphere from Matrix model ", Nucl. Phys. B604 (2001) 121, hep-th/0101102.

[5] Y. Kimura, " Noncommutative gauge theories on Fuzzy sphere and Fuzzy torus from Matrix model ", Prog. Theor. Phys. 106 (2001) 445, hep-th/0103192.

[6] H. Grosse, C. Klimcik and P. Presnajder, " Towards Finite quantum field theory on noncommutative geometry ", Int. J. Theor. Phys. 35 (1996) 231, hep-th/9505175.

[7] H. Grosse, C. Klimcik and P. Presnajder, " Field theory on a supersymmetric lattice ", Comm. Math. Phys. 185 (1997) 155, hep-th/9507074.

[8] H. Grosse, C. Klimcik and P. Presnajder, " Topologically nontrivial field configurations in noncommutative geometry ", Comm. Math. Phys. 178 (1996) 507, hep-th/9510083.

[9] J. Ambjorn, Y.M. Makeenko, J. Nishimura and R. J. Szabo, " Finite N Matrix models of noncommutative gauge theory ", JHEP 9911 (1999) 029, hep-th/9911041.

[10] J. Ambjorn, Y.M. Makeenko, J. Nishimura and R. J. Szabo, " Non perturbative Dynamics of noncommutative gauge theory ", Phys. Lett. B480 (2000) 399.

[11] J. Ambjorn, Y.M. Makeenko, J. Nishimura and R. J. Szabo, " Lattice gauge fields and discrete noncommutative Yang-Mills Theory ", JHEP 0005 (2000) 023, hep-th/0004147.

[12] T. Eguchi and K. Kawai, " Reduction of dynamical degrees of freedom in the large N gauge theory ", Phys. Rev. Lett. 48 (1982) 1063.

[13] A. Gozales-Arroyo and M. Okawa, " The twisted Eguchi-Kawai model: a reduced model for large N lattice gauge theory ", Phys. Rev. D27 (1983) 2397.

[14] T. Banks, W. Fischler, S. Shenker and L. Susskind, " M theory as a matrix model: a conjecture ", Phys. Rev. D55 (1997) 5112, hep-th/9610043.
[15] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, ” A large N reduced model as superstring ”, Nucl. Phys. B498 (1997) 467, hep-th/9612115.

[16] H. Aoki, S. Iso, H. Hawai, Y. Kitazawa, T. Tada and A. Tsuchiya, ” IIB matrix model ”, Prog. Theor. Phys. Suppl. 134 (1999) 47, hep-th/9908038.

[17] H. Aoki, S. Iso, H. Hawai, Y. Kitazawa and T. Tada, ” Space-time structures from IIB Matrix model ”, Prog. Theor. Phys. 99 (1998) 713, hep-th/9802085.

[18] M. Li, ” Strings from IIB matrices ”, Nucl. Phys. B499 (1997) 149, hep-th/9612222.

[19] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, ” Noncommutative Tang-Mills in IIB Matrix model ”, Nucl. Phys. B565 (2000) 176, hep-th/9908141.

[20] J. Madore, ” The Fuzzy sphere ”, Class. Quantum Grav. 9 (1992) 69.

[21] T. Masuda, ” Normalized Weyl-type *-product on Kahler manifolds ”, Mod. Phys. Lett. A15 (2000) 2177, hep-th/001009.

[22] U. Carow-Watamura and S. Watamura, ” Noncommutative geometry and gauge theory on Fuzzy sphere ”, Comm. Math. Phys. 212 (2000) 395, hep-th/9801193.

[23] U. Carow-Watamura and S. Watamura, ” Differential calculus on fuzzy sphere and scalar field ”, Int. J. Mod. Phys. A13 (1998) 3235, q-alg/9710034.

[24] U. Carow-Watamura and S. Watamura, ” Chirality and Dirac operator on Noncommutative sphere ”, Comm. Math. Phys. 183 (1997) 365, hep-th/9605003.

[25] C. Klimcik, ” Gauge theories on the noncommutative sphere ”, Comm. Math. Phys. 199 (1998) 257, hep-th/9710153.

[26] H. Grosse and P. Presnajder, ” The Dirac operators on the fuzzy sphere ”, Lett. math. Phys. 33 (1995) 171.

[27] J. Hoppe, ” Some classical solutions of membrane matrix model equations ”, hep-th/9702169.

[28] J. Hoppe and H. Nicolai, ” Relativistic minimal surfaces ”, Phys. Lett. B196 (1987) 451.

[29] S. Baez, A.P. Balachandran, S. Vaidya, ” Monopoles and Solitons in fuzzy physics ”, Comm. Math. Phys. 208 (2000) 787, hep-th/9811169.
[30] A.P. Balachandran and S. Vaidya, ” Instantons and chiral anomaly in fuzzy physics ”, hep-th/9910129.

[31] A. P. Balachandran, X.Martin and D. O’ Connor, ” Fuzzy actions and their continuum limits ”, hep-th/0007030.

[32] P. Valtancoli, ” Projectors for the fuzzy sphere ” Mod.Phys.Lett. A16 (2001) 639, hep-th/0101189.