GODEMENT RESOLUTIONS AND SHEAF HOMOTOPY THEORY

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Abstract. The Godement cosimplicial resolution is available for a wide range of categories of sheaves. In this paper we investigate under which conditions of the Grothendieck site and the category of coefficients it can be used to obtain fibrant models and hence to do sheaf homotopy theory. For instance, for which Grothendieck sites and coefficients we can define sheaf cohomology and derived functors through it.

Contents

1. Introduction ................................................................. 2
2. Homotopical preliminaries ................................................. 5
   2.1. Descent categories .................................................. 5
   2.2. Cartan-Eilenberg categories ....................................... 7
3. Categories of sheaves .................................................... 11
   3.1. Sheaves of sets .................................................... 11
   3.2. Sheaves with general coefficients ............................... 13
   3.3. The cosimplicial Godement resolution ........................... 14
4. Cartan-Eilenberg categories of sheaves ................................ 16
   4.1. Cartan-Eilenberg fibrant sheaves ................................ 16
   4.2. The hypercohomology sheaf ....................................... 17
   4.3. Characterization .................................................. 21
   4.4. Derived functors for sheaves ..................................... 23
5. Examples ........................................................................ 25
   5.1. Bounded complexes of sheaves .................................... 25
   5.2. Unbounded complexes of sheaves .................................. 27
   5.3. Sheaves of fibrant simplicial sets ............................... 28
   5.4. Sheaves of fibrant spectra ....................................... 31
   5.5. Sheaves of filtered complexes ................................... 31
References ................................................................. 31
1. Introduction

1.0.1. Godement resolutions have been an essential tool in sheaf homotopy theory and its applications almost from the start \cite{Go} and keep cropping up in different contexts: see for instance \cite{SGA4} for abelian sheaves on a Grothendieck site, \cite{Th} for sheaves of spectra on a Grothendieck site, \cite{N} for sheaves of (filtered) dg commutative algebras over topological spaces, \cite{MV} for simplicial sheaves on a Grothendieck site, \cite{SdS} for sheaves of \(O_X\)-modules over schemes, or \cite{GL} and \cite{Ba} for sheaves of DG-categories over schemes..., to name but a few.

In particular, the great flexibility of the cosimplicial Godement resolution, together with its excellent functorial properties, appear to account for its omnipresence: in fact, in order to define it for a sheaf \(F\) on a Grothendieck site with enough points \(X\) and values in some category of coefficients \(D\), we only need \(D\) to have filtered colimits and arbitrary products. In this situation, we obtain a functor

\[ G^\bullet : \text{Sh}(X, D) \rightarrow \Delta \text{Sh}(X, D) \]

from sheaves on \(X\) with values in \(D\) to cosimplicial ones.

The question we address in this paper is the following: under which conditions for the Grothendieck site \(X\) and the category of coefficients \(D\) is the cosimplicial Godement resolution useful for doing sheaf homotopy theory? More particularly: for which \(X\) and \(D\) can it be used in order to define sheaf cohomology?

1.0.2. Let us elaborate a little further. First of all, we want to “reassemble” all the cosimplicial pieces \(G^p F\) in order to obtain a simple sheaf which might be entitled to be a “model” for \(F\). For this, we assume that our category \(D\) comes equipped with a “simple” functor \(s : \Delta D \rightarrow D\). To get anchorage for her ideas, the reader may think of \(D\) as being the category of cochain complexes of abelian groups \(C^\ast(\text{Ab})\) and \(s\) the total complex of a double complex. This simple functor naturally induces one on sheaves \(s : \text{Sh}(X, \Delta D) \rightarrow \text{Sh}(X, D)\) which comes together with a universal map \(\rho_F : F \rightarrow sG^\bullet F\). This is our candidate for a “model” of \(F\). Following Thomason and Mitchell (\cite{Th}, \cite{Mit}), we call it the hypercohomology sheaf of \(F\):

\[ \rho_F : F \rightarrow \mathbb{H}_X(F) = sG^\bullet(F) \]  \hspace{1cm} (1.0.1)

So we can rephrase our problem more precisely as: under which conditions for \(X\) and \(D\), does (1.0.1) deserve to be called a “fibrant” model for \(F\)? For instance, assume that \(X\) has a final object \(X\): when would it make sense to define sheaf cohomology of \(X\) with coefficients in \(F\) as \(\Gamma(X, \mathbb{H}_X(F))\)? More precisely, we are asking when this formula would define a right derived functor in the sense of Quillen \cite{Q}; that is, a left Kan extension.

1.0.3. In order to talk about derived functors and homotopy categories, we need to specify the class of morphisms with respect to which we localize. In all the examples we are aware of, this is the class that keeps track of the topology of \(X\), the one of local equivalences: we have a distinguished class of morphisms \(E\), or “equivalences”, in the category of coefficients \(D\), and, for a morphism of sheaves \(\varphi : F \rightarrow G\) to be called a local equivalence, we require every morphism induced on stalks \(\varphi_x : F_x \rightarrow G_x\) to be in \(E\) for all the points \(x\) in \(X\). Let us note this class
of local equivalences as $\mathcal{W}$. For instance, for $\mathcal{D} = \mathbf{C}^\ast(\mathbf{Ab})$ we could take $E$ to be the class of quasi-isomorphisms, $quis$, morphisms which induce isomorphisms in cohomology, and then $\mathcal{W}$ would be the class of sheaf morphisms inducing $quis$ stalkwise.

1.0.4. It would be tempting now to try to answer our question by introducing a Quillen model structure on $\mathbf{Sh}(\mathcal{X}, \mathcal{D})$ in such a way that $\mathcal{W}$ would be its class of weak equivalences and the hypercohomology sheaves $\mathbb{H}_X(\mathcal{F})$ a (Quillen) fibrant model for each sheaf $\mathcal{F}$.

But Quillen model structures might be too much of a good thing in our case: in addition to dealing with our prime objects of interest, sheaves $\mathbb{H}_X(\mathcal{F})$ being fibrant models for every sheaf $\mathcal{F}$, we would also have to prove the existence of a cofibrant model for every sheaf, an otherwise pointless exercise. We therefore choose to keep only the minimum amount of structure in $\mathbf{Sh}(\mathcal{X}, \mathcal{D})$ necessary to have the class $\mathcal{W}$ as “weak equivalences” and the sheaves $\mathbb{H}_X(\mathcal{F})$ as “fibrant” models.

One such minimal structure is attained with Cartan-Eilenberg structures, CE-structures, or CE-categories, for short: an approach to homotopical algebra started in [GNPR1] and further developed in [C1] and [C2]. A (right) CE-structure in a category $\mathcal{C}$ consists of two classes of distinguished morphisms, strong and weak equivalences, $S \subset W$, and a CE-fibrant model for each object (see 2.2.2 for the precise definition). The name of these structures comes from the classic book [CE], where, in modern parlance, the homotopy theory of the category of cochain complexes $\mathbf{C}^\ast(\mathbf{Ab})$ is developed around two classes of distinguished morphisms: homotopy equivalences ($S$) and $quis$ ($W$).

But CE-structures allow more freedom of choice for classes $S$ and $W$ than classical “homotopy equivalences” and “weak equivalences”. This is particularly interesting for our categories of sheaves, for which the natural choices are:

- global equivalences, as $S$: those morphisms of sheaves such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ belongs to the class of equivalences $E$ in $\mathcal{D}$ for every object (open set) $U \in \mathcal{X}$, and
- local equivalences, as $W$: already mentioned above.

Finally, it turns out that, in order to provide our categories of sheaves $\mathbf{Sh}(\mathcal{X}, \mathcal{D})$ with a CE-structure, we need very few elements in our category of coefficients $\mathcal{D}$: essentially, our needs come down to a class of equivalences $E$ and a simple functor $s : \Delta \mathcal{D} \to \mathcal{D}$. $E$ and $s$ are encapsulated in the notion of descent category (see [Rod1], [Rod2] and the second section in this paper; cf. also [GN]).

1.0.5. We can now state our answer to the question addressed. Our main result (Theorem 1.0.1) is the following:

**Theorem 1.0.1.** Let $\mathcal{X}$ be a Grothendieck site and $(\mathcal{D}, E)$ a descent category satisfying the hypotheses (4.1.1). Then, the following statements are equivalent:

1. $(\mathbf{Sh}(\mathcal{X}, \mathcal{D}), S, W)$ is a right Cartan-Eilenberg category and for every sheaf $\mathcal{F} \in \mathbf{Sh}(\mathcal{X}, \mathcal{D})$, $\rho_\mathcal{F} : \mathcal{F} \to \mathbb{H}_X(\mathcal{F})$ is a CE-fibrant model.
2. For every sheaf $\mathcal{F} \in \mathbf{Sh}(\mathcal{X}, \mathcal{D})$, $\rho_\mathcal{F} : \mathcal{F} \to \mathbb{H}_X(\mathcal{F})$ is in $\mathcal{W}$.
3. The simple functor $\rho_\mathcal{F}$ commutes weakly with stalks.
(4) For every sheaf $F \in \text{Sh}(\mathcal{X}, \mathcal{D})$, $H^X(F)$ satisfies Thomason’s descent; that is, $\rho_{\mathbb{H}_X(F)} : \mathbb{H}_X(F) \to \mathbb{H}_X^2(F)$ is in $\mathcal{S}$.

We have already proven before that sheaves $\mathbb{H}_X(F)$ are always CE-fibrant (see Proposition (4.2.5)). So this theorem shows, first, that the existence of a CE-structure on the category of sheaves $\text{Sh}(\mathcal{X}, \mathcal{D})$ boils down to the certainty that for every sheaf $F$ the universal arrow $\rho_F : F \to \mathbb{H}_X(F)$ is a local equivalence (condition (2)). Hence, we need nothing else in our CE-structure to answer our problem.

The theorem also shows that the fact of the Godement resolution being a CE-fibrant model is equivalent to Thomason’s classic descent (for sheaves of spectra [Th], condition (4); see also Corollary 4.3.5). So being CE-fibrant is quite a natural and central notion for sheaves.

Finally, the theorem gives a down-to-earth equivalent condition for all this to happen, which will be the one we will use in practice: condition (3) says that the simple functor $s$ must commute with stalks up to local equivalence. That is, for every point $x \in X$, the natural morphism

$$s(F)_x = \colim_{x \in U}s(G^\cdot(F)(U)) \to s\left(\colim_{x \in U}G^\cdot(F)(U)\right) = s(F_x),$$

must be in $E$. For instance, for bounded cochains complexes this is a consequence of the commutation of the total complex functor $\text{Tot}$ with filtered colimits.

1.0.6. Once the equivalent conditions of Theorem 4.3.2 for the existence of a CE-structure on a category of sheaves are established, we can obtain some conclusions almost for free:

1. The general criterion for deriving functors for CE-categories (Proposition 2.2.6) can be rephrased immediately for our categories of sheaves and gives the derivability of the global sections functor trivially (Corollaries 4.4.1 and 4.4.2).
2. We show that, even in the absence of an associated sheaf functor for $\mathcal{D}$, the hypercohomology sheaf can be thought of as a “homotopical” sheafification functor, and the homotopy theories of sheaves and presheaves can be seen as equivalent (Corollary 4.3.6).
3. The category of sheaves possesses well-behaved fiber sequences and homotopy limits (Section 4.3.5).

1.0.7. In Section 5 we check when the conditions of our Theorem 4.3.2 are satisfied in some important specific examples. Some are well-known: bounded complexes, unbounded complexes, fibrant simplicial sets and fibrant spectra. Others, such as filtered complexes, are not so run-of-the-mill.

In a forthcoming sequel to this paper, we will point out that the Godement cosimplicial resolution is a monoidal one, hence perfectly well-suited to tackle with multiplicative structures: those of sheaves of operads, operad algebras and filtered algebras, for instance, to which we will extend the results of the present paper.

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1.0.9. Notations. By $\Delta$ we mean the simplicial category, with objects the ordered sets $[n] = \{0, \ldots, n\}$, $n \geq 0$, and morphisms the order-preserving maps. The face and degeneracy maps are denoted by $d^i : [n-1] \to [n]$ and $s^j : [n+1] \to [n]$, respectively. The strict simplicial category $\Delta_r$ is the subcategory of $\Delta$ with its same objects and strict monotone maps as morphisms.

We denote by $\Delta \mathcal{D}$ (resp. $\Delta \Delta \mathcal{D}$, $\Delta \mathcal{W} \mathcal{D}$) the category of cosimplicial (resp. bicosimplicial, simplicial) objects in a fixed category $\mathcal{D}$.

The diagonal functor $\mathcal{D} : \Delta \Delta \mathcal{D} \longrightarrow \Delta \mathcal{D}$ is given by $\mathcal{D}(\{Z_{n,m}\}_{n,m \geq 0}) = \{Z_{n,n}\}_{n \geq 0}$. The constant simplicial object defined by $A \in \mathcal{D}$ will be denoted by $c(A)$ or by $A \times \Delta$. In this way we obtain the constant functor $c : \Delta \mathcal{D} \longrightarrow \Delta \mathcal{D}$, which is fully faithful.

If $\mathcal{D}$ has products then simplicial sets (co)act on $\Delta \mathcal{D}$ in a natural way. Namely, if $X$ is an object of $\Delta \mathcal{D}$ and $K$ is a simplicial set, then $X^K \in \Delta \mathcal{D}$ is given by

$$
(X^K)^n = \prod_{K_n} X^n.
$$

(1.0.2)

We also denote $A^K = c(A)^K$ if $A$ is an object of $\mathcal{D}$. Recall that, for $n \geq 0$, $\Delta[n]$ is the simplicial set with $\Delta[n]_m = \text{Hom}_\Delta([m],[n])$. Then, the cosimplicial path object of $X$ is just $X^{\Delta[1]}$, and it is used to define cosimplicial homotopies and cosimplicial homotopy equivalences in the usual way.

Given an object $A$ of $\mathcal{D}$ and a cosimplicial object $X$, a coaugmentation $\epsilon : A \longrightarrow X$ is just a cosimplicial morphism $\epsilon : c(A) \longrightarrow X$. This $\epsilon$ is a cosimplicial homotopy equivalence if and only if $\epsilon$ has an extra degeneracy $s^{-1}$ or $s^{n+1}$. Hence there exists $s^{-1} : X^0 \longrightarrow A$ (or $s^0 : X^0 \longrightarrow A$) and $s^{-1} : X^{n+1} \longrightarrow X^n$ (or $s^{n+1} : X^{n+1} \longrightarrow X^n$) satisfying the simplicial identities.

Given a class $E$ of morphisms of a category $\mathcal{D}$, recall that its saturation is by definition $\overline{E} = \gamma^{-1}(\text{isomorphisms})$, where $\gamma : \mathcal{D} \longrightarrow \mathcal{D}[E^{-1}]$ is the localization functor. $\overline{E}$ is the smallest saturated class containing $E$ and the class $E$ is called saturated if $E = \overline{E}$. Note that a saturated class contains all isomorphisms of $\mathcal{D}$, is closed under retracts and satisfies the 2-out-of-3 property.

Given another category $\mathcal{C}$, $\text{Fun}(\mathcal{C}, \mathcal{D})$ denotes the category of functors from $\mathcal{C}$ to $\mathcal{D}$. If $I$ is a small category, we will also write $\mathcal{D}^I$ for $\text{Fun}(I, \mathcal{D})$ and $E^I$, or just $E$, if $I$ is understood the class of natural natural transformations $\tau : F \longrightarrow G$ such that $\tau_i \in E$ for all objects $i \in I$. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ can be extended degree-wise to the categories of cosimplicial objects $F : \Delta \mathcal{C} \longrightarrow \Delta \mathcal{D}$ by $F(X^p)^p = F(X^p)$. If $I$ is any small category, the induced functor $\mathcal{C}^I \longrightarrow \mathcal{D}^I$ will be written as $F^I$, or just $F$, without further mention.

2. Homotopical preliminaries

We introduce here the definitions and results concerning descent and CE-categories necessary for our paper. The interested reader may consult $[\text{Rod1}]$, $[\text{Rod2}]$ and $[\text{GNPR}]$ for further details.
2.1. Descent categories.

2.1.1. A (cosimplicial) descent category consists, roughly, of a category \( \mathcal{D} \) endowed with a class \( E \) of `weak equivalences’ and with a `simple’ functor \( s : \Delta \mathcal{D} \to \mathcal{D} \) subject to the axioms below. These axioms ensure that \( s \) is a realization of the homotopy limit for cosimplicial objects, and that the localized category \( \mathcal{D}[E^{-1}] \) possesses a rich homotopical structure.

Definition 2.1.1. [Rod1, 1.1] A (cosimplicial) descent category is the data \( (\mathcal{D}, E, s, \mu, \lambda) \) where \( \mathcal{D} \) is a category closed under finite products and \( E \) is a saturated class of morphisms of \( \mathcal{D} \), closed under finite products, called weak equivalences. The triple \( (s, \mu, \lambda) \) is subject to the following axioms:

(S1) The simple functor \( s : \Delta \mathcal{D} \to \mathcal{D} \) commutes with finite products up to equivalence. That is, the canonical morphism \( s(X \times Y) \to s(X) \times s(Y) \) is in \( E \) for all \( X, Y \in \Delta \mathcal{D} \).

(S2) \( \mu : ss \longrightarrow sD \) is a zigzag of natural weak equivalences. Recall that \( sDZ \) denotes the simple of the diagonal of \( Z \), while \( ssZ = s(n \to s(m \to Z_{n,m})) \).

(S3) \( \lambda : \text{id}_{\mathcal{D}} \longrightarrow s(\times \Delta) \) is a zigzag of natural weak equivalences, which is assumed to be compatible with \( \mu \) in the sense of (2.1.2) below.

(S4) If \( f : X \to Y \) is a morphism in \( \Delta \mathcal{D} \) with \( f_n \in E \) for all \( n \), then \( s(f) \in E \).

(S5) The image under the simple functor of the cosimplicial map \( A^d : A^{\Delta[1]} \to A \) is a weak equivalence for each object \( A \) of \( \mathcal{D} \).

For the sake of brevity, we will also denote a descent category by \( (\mathcal{D}, E) \) if the remaining data is understood.

2.1.2. Compatibility between \( \lambda \) and \( \mu \). Given \( X \in \Delta \mathcal{D} \), denote by \( X \times \Delta, \Delta \times X \) the bicosimplicial objects with \((X \times \Delta)^{n,m} = X^n \) and \((\Delta \times X)^{n,m} = X^m \). Note that \( ss(X \times \Delta) = ss(n \to sc(X^n)) \) and \( ss(\Delta \times X) = ss(X) \). The compositions

\[
\begin{align*}
  s(X) & \xrightarrow{\mu_{X,\Delta}} scs(X) \\
  s(X) & \xrightarrow{\lambda(X)} s(X) \\
  s(X) & \xrightarrow{\mu_{X,\Delta}} ss(X) \\
  s(X) & \xrightarrow{\lambda(X)} s(X)
\end{align*}
\]

give rise to isomorphisms of \( s \) in \( Fun(\Delta \mathcal{D}, \mathcal{D})[E^{-1}] \). Then, \( \lambda \) is said to be compatible with \( \mu \) if the above isomorphisms are the identity in \( Fun(\Delta \mathcal{D}, \mathcal{D})[E^{-1}] \).

Remark 2.1.2. The presence of zigzags in the definition of descent category is needed to ensure its homotopy invariance (see [Rod1, Proposition 1.8]). However, every example used in this paper has both \( \mu \) and \( \lambda \) as actual natural transformations, and their compatibility takes place in \( Fun(\Delta \mathcal{D}, \mathcal{D}) \) rather than in \( Fun(\Delta \mathcal{D}, \mathcal{D})[E^{-1}] \) (see Examples 2.1.9 - 2.1.11).

Since this significantly simplifies exposition, we will assume that these stronger hypotheses are satisfied for the descent categories considered throughout the paper. Of course, all the results presented here may be adapted to the general case, paying attention to which assumptions must be imposed to the zigzags \( \mu \) and \( \lambda \).

We will also assume throughout the paper that simple functors preserve limits. But this is not a major restriction: it is fulfilled by all our examples of descent categories so far.
2.1.3. As mentioned before, a simple functor is a homotopy limit in the sense of Grothendieck derivators.

**Theorem 2.1.3.** [Rod1, 5.1] The simple functor $s : \Delta D[E^{-1}] \to D[E^{-1}]$ is right adjoint to the localized constant cosimplicial object functor $c : D[E^{-1}] \to (\Delta D)[E^{-1}]$.

This theorem formally implies that $s : \Delta D[E^{-1}] \to D[E^{-1}]$ is the absolute right derived functor of the limit $\lim_\Delta : \Delta D \to D$, in case where $D$ is complete. In particular, a simple functor is unique after localizing with respect to the weak equivalences. In addition, as seen in loc. cit., given an object $A \in D$ the morphism $\lambda_A : A \to \text{sc}(A)$ viewed in $D[E^{-1}]$ is precisely the counit of the adjunction $(c, s)$.

The localized category $D[E^{-1}]$ of a descent category is naturally equivalent to the homotopy category of a Brown category of fibrant objects. Therefore it supports a rich homotopical structure, including the existence of fiber sequences satisfying the usual properties ([Rod1]).

2.1.4. The following two auxiliary results will be used in the study of sheaves with coefficients in descent categories.

**Lemma 2.1.4.** [Rod1, 1.5] If $A$ is an object $A \in D$ and $c : c(A) \to X$ is a coaugmentation of $X \in \Delta D$ with an extra degeneracy, then $s(c)$ is a weak equivalence.

**Lemma 2.1.5.** [Rod1, 4.4] For a bisimplicial morphism $f^{n,m} : Z^{n,m} \to T^{n,m}$, it holds that the morphism $s(n \to s(m \to f^{n,m}))$ belongs to $E$ if and only if $s(m \to s(n \to f^{n,m}))$ does.

And the following three ones, will allow us to produce new descent structures from known ones. Their proofs are an easy exercise left to the interested reader.

**Lemma 2.1.6** (Transfer Lemma). Let $(D', E', s', \mu', \lambda')$ be a descent category. Given a functor $\psi : D \to D'$, consider in $D$ the weak equivalences $E = \psi^{-1}E'$. Assume that $D$ has finite products and is equipped with a functor $s : \Delta D \to D$, together with compatible natural weak equivalences $\mu : ss \to sD$ and $\lambda : \text{id}_D \to s(- \times \Delta)$. Then, $(D, E, s, \mu, \lambda)$ is a descent category provided the following statements hold:

1. (FD1) $\psi$ commutes with finite products up to equivalence. That is, the natural map $\psi(X \times Y) \to \psi(X) \times \psi(Y)$ is in $E$ for all $X, Y$ in $D$.
2. (FD2) There exists a natural weak equivalence $\theta : \psi s \to s' \psi$ filling the square:

$$
\begin{array}{ccc}
\Delta D & \stackrel{\psi}{\longrightarrow} & \Delta D' \\
\downarrow s & & \downarrow s' \\
D & \stackrel{\psi}{\longrightarrow} & D'
\end{array}
$$

**Lemma 2.1.7.** [Rod1, 1.6] If $(D, E, s, \mu, \lambda)$ is a descent category and $I$ is a small category, the category $D^I$ of functors from $I$ to $D$ has a natural structure $(D^I, E^I, s^I, \mu^I, \lambda^I)$ of descent category, defined objectwise. In particular, $E^I = \{ f | f(i) \in E \text{ for all } i \in I \}$ and $(s^I(X))(i) = s(X(i))$, for every $i \in I$ and $X \in \Delta D^I$.

**Lemma 2.1.8.** If $(D, E, s, \mu, \lambda)$ and $(D', E', s', \mu', \lambda')$ are descent categories then $(D \times D', E \times E', s \times s', \mu \times \mu', \lambda \times \lambda')$ has a natural structure of descent category.
2.1.5. To end with, we describe some examples of descent categories.

Example 2.1.9. Bounded complexes \([\text{Rod1}, (3.4)]\). Let \(\mathcal{A}\) be an abelian category. For a fixed integer \(b \in \mathbb{Z}\), denote by \(\mathcal{C}^{\geq b}(\mathcal{A})\) the category of uniformly bounded below cochain complexes of \(\mathcal{A}\); that is, \(A^n = 0\) for all \(n < b\) and all \(A^* \in \mathcal{C}^{\geq b}(\mathcal{A})\).

We will consider the following descent structure on \(\mathcal{C}^{\geq b}(\mathcal{A})\). The weak equivalences \(E\) are the quasi-isomorphism (\(quis\)): those maps inducing isomorphism in cohomology. The simple functor \(s : \Delta \mathcal{C}^{\geq b}(\mathcal{A}) \to \mathcal{C}^{\geq b}(\mathcal{A})\) at a given cosimplicial cochain complex \(A\) is the (product) total complex of the double complex induced by \(A\):

\[
s(A)^n = \prod_{p+q=n} A^{pq}.
\]

The boundary map \(d : s(A)^{n-1} \to s(A)^n\) is then given by the maps \(d_l^{(n-1)q} + (-1)^p d_2^{pq-1}\), where \(d_1\) is the alternate sum of the face maps of the cosimplicial object \(A^{\bullet,q}\) and \(d_2\) is the differential of the cochain complex \(A^{p,*}\). Note that since \(A\) has finite codiagonals, \(s(A)^n = \bigoplus_{p+q=n} A^{pq}\).

As for \(\mu\) and \(\lambda\): if \(Z \in \Delta \Delta \mathcal{C}^{\geq b}(\mathcal{A})\), \(\mu_Z\) is just the Alexander-Whitney map \(ssZ \to sDZ\). In degree \(n\), \((\mu_Z)^n : \prod_{i+j+k=n} Z^{i,j,k} \to \prod_{p+q=n} Z^{p,p,q}\) is the product of the maps

\[
\prod_{i+j=p} Z(d^0 \cdots d^0, d^{p} d^{p-1} \cdots d^{j+1}) : \prod_{i+j=p} Z^{i,j,q} \to Z^{p,p,q}, \quad p + q = n.
\]

And if \(X \in \mathcal{C}^{\geq b}(\mathcal{A})\), then \(s(X \times \Delta)^n = \prod_{p \geq 0} X^{n-p}\), and \(\lambda_X^n : X^n \to s(X \times \Delta)^n\) is the canonical inclusion \(X^n \to X^n \oplus \prod_{p \geq 1} X^{n-p}\).

Given an abelian category \(\mathcal{A}\), the Dold-Kan correspondence provides an equivalence of abelian categories between \(\mathcal{C}^{\geq 0}(\mathcal{A})\) and the category \(\Delta \mathcal{A}\) of cosimplicial objects in \(\mathcal{A}\). Using the Transfer Lemma, this allows to induce a descent structure on \(\Delta \mathcal{A}\) which, although equivalent to the one on \(\mathcal{C}^{\geq 0}(\mathcal{A})\), will be more convenient when dealing with multiplicative structures as we will see in a forthcoming paper.

Example 2.1.10. Unbounded complexes. The category \(\mathcal{C}^*(\mathcal{A})\) of unbounded cochain complexes of \(\mathcal{A}\) is also a descent category with weak equivalences, simple functor, \(\mu\) and \(\lambda\) defined as in the bounded case provided axiom (S4) holds. For instance, this is the case when \(\mathcal{A} = \mathcal{R}\)-modules.

Example 2.1.11. Simplicial model categories \([\text{Rod1}, \text{Theorem 3.2}]\). The subcategory of fibrant objects \(\mathcal{M}_f\) of a model category \(\mathcal{M}\) is a descent category where \(\mathcal{E}\) is the class of weak equivalences of \(\mathcal{M}\) and the simple functor is the Bousfield-Kan homotopy limit, \(\text{holim} : \Delta \mathcal{M}_f \to \mathcal{M}_f\), as defined in \([\text{Hir}]\). If \(\mathcal{M}\) is a simplicial model category, the homotopy limit of a cosimplicial object \(X\) is the end of the bifunctor \(X^{N(\Delta^\downarrow n)} : \Delta^\uparrow \times \Delta \to \mathcal{M}_f\), \((n,m) \to (X^m)^{N(\Delta^\downarrow n)}\), that is,

\[
\text{holim} X = \int_n (X^n)^{N(\Delta^\downarrow n)}.
\]

Here \(N(\Delta \downarrow n)\) denotes the nerve of the over-category \((\Delta \downarrow n)\), and \((X^m)^{N(\Delta^\downarrow n)}\) is constructed using the simplicial structure on \(\mathcal{M}\).
Morphisms \( \mu \) and \( \lambda \) are easily defined using that a functor \( F : \mathcal{B} \to \mathcal{C} \) induces a natural map
\[
\lim_{\mathcal{C}} X \to \lim_{\mathcal{B}} F^* X = \int_b X(F(b))^{N(\mathcal{B}b)}
\]
which is defined by the maps \( X(F(b))^{N(\mathcal{C}F(b))} \to X(F(b))^{N(\mathcal{B}b)} \), induced by \( F : (\mathcal{B} \downarrow b) \to (\mathcal{C} \downarrow F(b)) \). Then:

- \( \mu \) is obtained from the diagonal \( d : \Delta \to \Delta \times \Delta \), that induces for each \( Z \in \Delta \Delta \mathcal{M}_f \)
\[
\lim_{\Delta} \lim_{\Delta} Z \simeq \lim_{\Delta \times \Delta} Z \to \lim_{\Delta} d^* Z = \lim_{\Delta} DZ ,
\]
where the first isomorphism follows from the Fubini property of \( \lim \) (see \([\text{BK} \text{ X} \text{I} \text{.} \text{4} \text{.} \text{3}]\)).
- \( \lambda \) is obtained from \( l : \Delta \to * \). It induces \( \lambda_A : A \simeq \lim_{A} A \to \lim_{\Delta} (A \times \Delta) \) for each \( A \in \mathcal{M}_f \).

Two particular instances of this example are relevant when talking about sheaf cohomology theories. First, the category \( s\mathbf{S}_f \) of pointed Kan complexes, with weak equivalences the weak homotopy equivalences. Secondly, the category \( \mathbf{S}_f \) of pointed fibrant spectra, as defined in \([\text{Th} \text{ 5.2}]\). The weak equivalences for the descent structure are then the stable weak equivalences; that is, morphisms of spectra inducing bijections in all homotopy groups.

**Example 2.1.12. Filtered complexes.** Denote by \( \mathbf{FC}^{\geq b}(\mathcal{A}) \) the category of filtered complexes, with objects the pairs \( (A, F) \), where \( A \) is in \( \mathbf{C}^{\geq b}(\mathcal{A}) \) and \( F \) is a decreasing filtration of \( A \). Given \( r \geq 0 \), consider the class \( E_r \) of weak equivalences given by the \( E_r \)-quasi-isomorphisms of \( \mathbf{FC}^{\geq b}(\mathcal{A}) \), that is, morphisms of filtered complexes such that the induced morphism between the \( E_{r+1} \)-terms of the spectral sequences associated with the filtrations is an isomorphism.

It holds that \( (\mathbf{FC}^{\geq b}(\mathcal{A}), E_r) \) is a descent category with simple functor \( (s, \delta_r) : \Delta \mathbf{FC}^{\geq b}(\mathcal{A}) \to \mathbf{FC}^{\geq b}(\mathcal{A}) \) defined as \( (s, \delta_r)(A, F) = (s(A), \delta_r(F)) \) where
\[
\delta_r(F)^k(s(A)^n) = \bigoplus_{i+j=n} F^{k-ri} A^{i,j} ,
\]
and with natural transformations \( \lambda \) and \( \mu \) given at the level of complexes by those of \( \mathbf{C}^{\geq b}(\mathcal{A}) \).

If \( r = 0 \), note that an \( E_0 \)-isomorphism is the same thing as a graded quasi-isomorphism. Also, \( (s, \delta_0)(A, F) \) is just \( (s(A), s(F)) \). The fact that this is a simple functor for \( (\mathbf{FC}^{\geq b}(\mathcal{A}), E_0) \) is an easy consequence of the transfer lemma applied to the graded functor \( \text{Gr} : \mathbf{FC}^{\geq b}(\mathcal{A}) \to \mathbf{C}^{\geq b}(\mathcal{A})^\mathbb{Z} \).

To treat the general case, consider the decalage filtration functor \( \text{Dec} : \mathbf{FC}^{\geq b}(\mathcal{A}) \to \mathbf{FC}^{\geq b}(\mathcal{A}), (A, F) \mapsto (A, \text{Dec} F) \), where \( (\text{Dec} F)^k A^n = \ker\{d : F^{k+n} A^n \to F^{k+n+1} A^{n+1} / F^{k+n+1} A^{n+1}\} \). Since \( \text{Dec} (s, \delta_{r+1}) = (s, \delta_r) \text{Dec} \), by applying the transfer lemma inductively, we can conclude that \( (s, \delta_r) \) is a simple functor for \( (\mathbf{FC}^{\geq b}(\mathcal{A}), E_r) \), for each \( r \geq 0 \).

2.2. Cartan-Eilenberg categories.
2.2.1. Cartan-Eilenberg categories are a new approach to homotopical algebra developed in [GNPR1]. They use, we believe, a minimum amount of data in order to derive functors, so its conditions can be fulfilled by a wider class of categories, as we are going to show.

**Definition 2.2.1.** Let \((\mathcal{C}, \mathcal{S}, \mathcal{W})\) be a category with two classes \(\mathcal{S}\) and \(\mathcal{W}\) of distinguished morphisms, called respectively strong and weak equivalences, and such that \(\mathcal{S} \subset \mathcal{W}\). An object \(M\) of \(\mathcal{C}\) is called Cartan-Eilenberg fibrant, or \(CE\)-fibrant for short, if for each weak equivalence \(w : Y \to X \in \mathcal{W}\) and every morphism \(f \in \mathcal{C}[\mathcal{S}^{-1}]\), there is a unique morphism \(g \in \mathcal{C}[\mathcal{S}^{-1}]\) making the following triangle commutative:

\[
\begin{array}{ccc}
Y & \overset{w}{\longrightarrow} & X \\
\downarrow{f} & & \downarrow{g} \\
M & \underset{\text{unique}}{\longrightarrow} & M
\end{array}
\]

**Remark 2.2.2.** Classes \(\mathcal{S}\) and \(\mathcal{W}\) of strong and weak equivalences considered later in the study of sheaves are saturated, i.e. \(\overline{\mathcal{S}} = \mathcal{S}\) and \(\overline{\mathcal{W}} = \mathcal{W}\). In this case, Whitehead’s theorem holds: a weak equivalence between \(CE\)-fibrant objects is a strong one.

2.2.2. A right \(CE\)-fibrant model of an object \(X\) of \(\mathcal{C}\) is a morphism \(w : X \to M \in \mathcal{C}[\mathcal{S}^{-1}]\) that becomes an isomorphism in \(\mathcal{C}[\mathcal{W}^{-1}]\), and such that \(M\) is \(CE\)-fibrant. It holds that if \(X\) admits a \(CE\)-fibrant model, it is unique up to unique isomorphism of \(\mathcal{C}[\mathcal{S}^{-1}]\).

**Definition 2.2.3.** A category with strong and weak equivalences \((\mathcal{C}, \mathcal{S}, \mathcal{W})\) is called a right Cartan-Eilenberg category, or \(CE\)-category for short, if each object \(X\) of \(\mathcal{C}\) has a \(CE\)-fibrant model. In this case, we will also say that \(\mathcal{C}\) has enough \(CE\)-fibrant models.

**Example 2.2.4.** If \(\mathcal{C}\) is a Quillen model category and \(\mathcal{S}, \mathcal{W}\) are the classes of its right homotopy equivalences and weak equivalences, respectively, then \((\mathcal{C}_c, \mathcal{S}, \mathcal{W})\) is a right Cartan-Eilenberg category. Here \(\mathcal{C}_c\) is the full subcategory of Quillen cofibrant objects. In this case, every Quillen fibrant object is \(CE\)-fibrant, but the converse needs not be true: by its very definition, \(CE\)-fibrant objects are homotopically invariant, while Quillen fibrant objects are not.

**Remark 2.2.5.** So, \(CE\)-categories naturally include Quillen model ones and the inclusion is “strict” in the sense that, for instance, the class \(\mathcal{S}\) must not be any class of “homotopy equivalences”. This is particularly important for us because, in the case of sheaves, the global equivalences cannot indeed be the homotopy equivalences of any Quillen model structure, as shown in [GNPR2]. Since these global equivalences are such a natural ingredient for sheaves, this seems to be significant. Global equivalences are needed, for instance, to talk about sheaves satisfying Thomason descent, which are precisely \(CE\)-fibrant models, to close the circle.

2.2.3. Next we recall how \(CE\)-categories provide a derivability criterion of functors.

**Proposition 2.2.6.** Let \((\mathcal{C}, \mathcal{S}, \mathcal{W})\) be a Cartan-Eilenberg category and \(F : \mathcal{C} \to \mathcal{D}\) a functor such that \(F(s)\) is an isomorphism for every strong equivalence \(s \in \mathcal{S}\). Then \(F\) has a right derived functor \(RF : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}\) whose value on objects may be computed as \(RF(X) = F(M)\), where \(M\) is a fibrant model of \(X\).

**Proof.** See [GNPR1] 3.2.1. \(\square\)
2.2.4. In the CE-categories considered later on, the CE-fibrant model of an object $X$ will be functorial in the following sense.

**Definition 2.2.7.** [GNPR1, 2.5] A resolvent functor for a CE-category $(C, S, W)$ is a pair $(R, \rho)$ consisting of

1. a functor $R : C \to C$ such that $R(X)$ is CE-fibrant for each object $X \in C$, and
2. a natural transformation $\rho : \text{id} \to R$ such that $\rho_X : X \to R(X)$ is in $W$ for each object $X \in C$.

Note that a resolvent functor is in particular an idempotent functor on $C$: $R \circ R$ and $R$ agree and are isomorphisms of $C$. In this case, an object $X$ is CE-fibrant if and only if $\rho_X$ is an isomorphism of $C$. Also, a weak equivalence $w \in W$ is the same thing as an 'R-local weak equivalence', i.e., $w \in W$ if and only if $R(w) \in S$.

One of the advantages of having a resolvent functor is that, if $C_{\text{fib}}$ denotes the full subcategory of $C$ of CE-fibrant objects, there is an equivalence of categories ([GNPR1, Proposition 2.5.3(2)])

$$C_{\text{fib}}[S^{-1}] \xrightarrow{i} C[W^{-1}]$$

3. **Categories of sheaves**

We recall some general definitions and results about sheaves of sets on a Grothendieck site. Our main objective is to point out formulas (3.1.3) and (3.1.4) for stalks and skyscraper sheaves, respectively. Then we observe that these formulas still make sense for sheaves with values in any category with filtered colimits and arbitrary products, and that they do indeed form a pair of adjoint functors. The associated triple gives us the cosimplicial Godement resolution.

We also show that the category of sheaves with values in a descent category inherits a natural descent structure, which will be used repeatedly in the rest of the paper.

3.1. **Sheaves of sets.** We begin by recalling some basic concepts concerning sheaves on a Grothendieck site.

3.1.1. Let $\mathcal{X}$ be a category. A presheaf on $\mathcal{X}$ with values in $\text{Set}$, or a $\text{Set}$-valued presheaf, is a functor $\mathcal{F} : \mathcal{X}^{\text{op}} \to \text{Set}$. A morphism of presheaves is just a natural transformation of functors. Let $\text{PrSh}(\mathcal{X}, \text{Set})$ (or just $\hat{\mathcal{X}}$) denote the category with these objects and morphisms. By the Yoneda embedding, every object $U \in \mathcal{X}$ can be thought of as the representable presheaf $yU = \mathcal{X}(-, U) \in \hat{\mathcal{X}}$.

3.1.2. If $U$ is an object of $\mathcal{X}$, a sieve on $U$ is a subfunctor $S$ of the representable functor $yU$. Or, equivalently, a sieve on $U$ is a set $S$ of morphisms of $\mathcal{X}$ with codomain $U$ such that for each $f : V \to U \in S$, $fg \in S$ for any $g : W \to V$ in $\mathcal{X}$.

A Grothendieck topology on $\mathcal{X}$ is a function $J$ which assigns to each object $U \in \mathcal{X}$ a collection $J(U)$ of sieves on $U$ satisfying the axioms of [SGA4], II.1, Definition 1.1 (or [McLM], III.2,
Definition 1). A Grothendieck site is a category $\mathcal{X}$, together with a Grothendieck topology $J$. If $S \in J(U)$, we say that $S$ is a covering sieve (or just a cover) of $U$.

3.1.3. A sheaf on $\mathcal{X}$ with values in $\text{Set}$ is a presheaf $\mathcal{F}: \mathcal{X}^{\text{op}} \rightarrow \text{Set}$ such that, for every object $U \in \mathcal{X}$ and every cover $S$ of $U$, the natural map $\tilde{\mathcal{X}}(yU, \mathcal{F}) \rightarrow \tilde{\mathcal{X}}(S, \mathcal{F})$ is a bijection.

An equivalent characterization of sheaves is the following (see [McLM], page 122): a presheaf $F \in \tilde{\mathcal{X}}$ is a sheaf if and only if for every object $U \in \mathcal{X}$ and every cover $S$ of $U$, the diagram

$$
\begin{array}{ccc}
\mathcal{F}(U) & \longrightarrow & \prod_{f:V \rightarrow U \in S} \mathcal{F}(V) \\
\longrightarrow & \longrightarrow & \prod_{\substack{f:V \rightarrow U \in S \\
g:W \rightarrow V}} \mathcal{F}(W)
\end{array}
$$

(3.1.1)

is an equalizer of sets. It follows that a functor of presheaves that commutes with limits will send sheaves to sheaves.

3.1.4. Let $\tilde{\mathcal{X}} = \text{Sh}(\mathcal{X}, \text{Set})$ be the full subcategory of $\text{PrSh}(\mathcal{X}, \text{Set})$ whose objects are sheaves.

For instance, for every topological space $X$, we have an associated site, $\text{Open}(X)$, its category of open sets, and $\tilde{\text{Open}}(X) = \text{Sh}(X, \text{Set})$ is the category of $\text{Set}$-valued sheaves on $X$. If $X = \{\ast\}$ is a one-point space, $\text{Sh}(\ast, \text{Set}) = \text{Set}$.

3.1.5. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of sites; that is, a functor between the underlying categories going in the opposite direction $f^{-1}: \mathcal{Y} \rightarrow \mathcal{X}$ which is continuous. This means that for every sheaf $\mathcal{F} \in \tilde{\mathcal{X}}$, the presheaf $V \mapsto \mathcal{F}(f^{-1}(V))$, $V \in \mathcal{Y}$, is a sheaf on $\mathcal{Y}$. So, by definition, for every morphism of sites $f$, the direct image functor $f_*: \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}$, $\mathcal{F} \mapsto \mathcal{F} \circ f^{-1}$, restricts to a functor between sheaves $f_*: \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}$.

3.1.6. The inclusion $i: \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{X}}$ has a left adjoint $(-)^a: \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{X}}$ ([SGA4], exposé II, Theorem 3.4). For any presheaf $\mathcal{F} \in \tilde{\mathcal{X}}$, its associated sheaf is by definition $\mathcal{F}^a$. The associated sheaf functor commutes with finite limits ([SGA4], exposé II, Theorem 4.1). We will denote the composition of the Yoneda embedding with the associated sheaf by $e = (-)^a \circ y: \mathcal{X} \rightarrow \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{X}}$.

3.1.7. Recall that a point of a site $\mathcal{X}$ is by definition a pair of adjoint functors $x = (x^*, x_*)$

$$
\tilde{\mathcal{X}} \overset{x_*}{\underset{x^*}{\longrightarrow}} \text{Set} , \quad \text{Set}(x^* \mathcal{F}, D) = \tilde{\mathcal{X}}(\mathcal{F}, x_*D)
$$

such that $x^*$ commutes with finite limits. The right adjoint $x_*: \text{Set} \rightarrow \tilde{\mathcal{X}}$ gives for every set $D$ the so called skyscraper sheaf $x_*D$ of $D$ at the point $x$. The left adjoint $x^*: \tilde{\mathcal{X}} \rightarrow \text{Set}$ gives for every sheaf $\mathcal{F}$ the fibre or stalk $x^*\mathcal{F} = \mathcal{F}_x$ of $\mathcal{F}$ at $x$. 

3.1.8. A neighborhood of a point \( x \) in a site \( \mathcal{X} \) is a pair \((U, u)\) consisting of an object \( U \in \mathcal{X} \) and an element \( u \in (\mathbf{e}U)_x \). A morphism of neighbourhoods \( f : (U, u) \to (V, v) \) is a morphism \( f : U \to V \) of \( \mathcal{X} \) such that \((\mathbf{e}f)_x(v) = u\).

The category \( \mathbf{Nbh}(x) \) of neighbourhoods of \( x \) is a filtered one and we have a canonical and functorial isomorphism

\[ x^* \mathcal{F} = \mathcal{F}_x = \underset{(U,u)}{\text{colim}} \mathcal{F}(U), \quad (3.1.2) \]

where \((U, u)\) runs over the opposite category of neighbourhoods of \( x \) (SGA4, exposé IV, 6.8).

For a set \( D \in \mathbf{Set} \), the sheaf \( x^* D \) also admits the following description: for \( U \in \mathcal{X} \),

\[ (x^* D)(U) = \prod_{u \in x^*(\mathbf{e}U)} D_u, \quad (3.1.3) \]

where \( D_u = D \) for all \( u \in x^*(\mathbf{e}U) \).

3.1.9. A site \( \mathcal{X} \) is said to have enough points if it has a conservative family of points. That is, if there exists a set \( X \) formed by points of \( \mathcal{X} \) such that a morphism \( f \) in \( \widetilde{\mathcal{X}} \) is an isomorphism if and only if \( x^*(f) \) is a bijection for all \( x \in X \).

Given a family \( X \) of enough points of \( \mathcal{X} \), consider \( X \) as a discrete category with only identity morphisms. Then, there is an adjoint pair of functors

\[ \mathcal{X} \overset{p^*} \underset{p_*} \rightarrow \mathbf{Set}^X, \quad \mathbf{Set}^X(p^* \mathcal{F}, D) = \mathcal{X}(\mathcal{F}, p_* D) \]

defined, for \( \mathcal{F} \in \widetilde{\mathcal{X}} \) and \( D = (D_x)_{x \in X} \in \mathbf{Set}^X \), by

\[ p^* \mathcal{F} = (\mathcal{F}_x)_{x \in X} \quad \text{and} \quad p_* D = \prod_{x \in X} x_*(D_x). \quad (3.1.4) \]

3.2. Sheaves with general coefficients.

3.2.1. Let \( \mathcal{X} \) and \( \mathcal{D} \) be categories. A presheaf on \( \mathcal{X} \) with values in \( \mathcal{D} \), or a \( \mathcal{D} \)-valued presheaf, is a functor \( \mathcal{F} : \mathcal{X}^{\text{op}} \to \mathcal{D} \). A morphism of presheaves is just a natural transformation of functors. The category with these objects and morphisms is denoted by \( \mathbf{PrSh} (\mathcal{X}, \mathcal{D}) \).

Let \( \mathcal{X} \) be a Grothendieck site. A sheaf on \( \mathcal{X} \) with values in \( \mathcal{D} \), or a \( \mathcal{D} \)-valued sheaf, is a presheaf \( \mathcal{F} : \mathcal{X}^{\text{op}} \to \mathcal{D} \) such that, for every object \( D \in \mathcal{D} \), the presheaf of sets \( U \mapsto \mathcal{D}(D, \mathcal{F}(U)) \) is a sheaf. Let \( \mathbf{Sh}(\mathcal{X}, \mathcal{D}) \) denote the full subcategory of \( \mathbf{PrSh}(\mathcal{X}, \mathcal{D}) \) whose objects are sheaves. We will denote by \( i : \mathbf{Sh}(\mathcal{X}, \mathcal{D}) \to \mathbf{PrSh}(\mathcal{X}, \mathcal{D}) \) the inclusion functor. If \( \mathcal{D} \) has products, then \( \mathcal{F} \) is a sheaf if and only if, for every object \( U \) and every covering \( S \) of \( U \), diagram (3.1.1) is an equalizer in \( \mathcal{D} \) (MCL, V.4.1).

3.2.2. We can also define the direct image functor for presheaves with values in an arbitrary category \( \mathcal{D} \): if \( f : \mathcal{X} \to \mathcal{Y} \) is a morphism of sites, \( f_* : \mathbf{PrSh}(\mathcal{X}, \mathcal{D}) \to \mathbf{PrSh}(\mathcal{Y}, \mathcal{D}) \) is defined on objects \( \mathcal{F} \mapsto \mathcal{F} \circ f^{-1} \) and we have the following elementary result.

**Lemma 3.2.1.** Let \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism of sites and \( \mathcal{F} \in \mathbf{Sh}(\mathcal{X}, \mathcal{D}) \). Then \( f_* \mathcal{F} \in \mathbf{Sh}(\mathcal{Y}, \mathcal{D}) \).
Proof. By definition, $\mathcal{F} \in \text{Sh}(\mathcal{X}, \mathcal{D})$ means that for every object $D \in \mathcal{D}$, the presheaf $\mathcal{D}(D, \mathcal{F}(-))$, $U \mapsto \mathcal{D}(D, \mathcal{F}(U))$ is a sheaf of sets on $\mathcal{X}$. Hence, $f_*(\mathcal{D}(D, \mathcal{F}(-))) = \mathcal{D}(D, (\mathcal{F} \circ f^-1)(-))$ is a sheaf of sets on $\mathcal{Y}$ for every $D \in \mathcal{D}$. Thus, $f_*(\mathcal{F}) = \mathcal{F} \circ f^{-1}$ is a $\mathcal{D}$-valued sheaf on $\mathcal{Y}$.

Hence, for any morphism of sites $f : \mathcal{X} \rightarrow \mathcal{Y}$, we also have an induced direct image functor for $\mathcal{D}$-valued sheaves, $f_* : \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \text{Sh}(\mathcal{Y}, \mathcal{D})$ for any $\mathcal{D}$. In other words, the continuity of a functor for $\mathcal{D} = \text{Set}$ implies its continuity for all coefficient categories $\mathcal{D}$.

3.2.3. We have an obvious evaluation functor

$$\Gamma : \mathcal{X} \times \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{D}, \quad (U, \mathcal{F}) \mapsto \Gamma(U, \mathcal{F}) = \mathcal{F}(U).$$

If $\mathcal{X}$ has a terminal object (which we insist on noting) $\mathcal{T}$, then we have a global sections functor $\Gamma(\mathcal{X}, -) : \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{D}$, $\mathcal{F} \mapsto \Gamma(\mathcal{X}, \mathcal{F}) = \mathcal{F}(\mathcal{T})$. For instance, a topological space $\mathcal{X}$ is itself a terminal object of the associated site $\text{Open}(\mathcal{X})$.

3.2.4. Next we construct a natural, objectwise, descent structure for sheaves on a site $\mathcal{X}$ with values in a descent category $(\mathcal{D}, \mathcal{E})$.

Definition 3.2.2. Let $\mathcal{D}$ be a category with a distinguished class of morphisms $\mathcal{E}$. We will say that a morphism of (pre)sheaves $f : \mathcal{F} \rightarrow \mathcal{G}$ with values in $\mathcal{D}$ is a global equivalence if for any object $U \in \mathcal{X}$ the morphism $f(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is in $\mathcal{E}$. We will denote by $\mathcal{S}$ the class of global equivalences.

Proposition 3.2.3. Let $(\mathcal{D}, \mathcal{E})$ be a descent category which is assumed to be closed under products. If $\mathcal{X}$ is a Grothendieck site, then $\text{Sh}(\mathcal{X}, \mathcal{D})$ is a descent category with $\mathcal{s}$ defined objectwise $\mathcal{s}(\mathcal{F}^*)(U) = \mathcal{s}(\mathcal{F}^*(U))$ and the class of weak equivalences being the global ones.

Proof. Applying Lemma 2.1.7 with $I = \mathcal{X}^{\text{op}}$ we obtain that $\text{PrSh}(\mathcal{X}, \mathcal{D})$ inherits from $(\mathcal{D}, \mathcal{E})$ an objectwise descent category structure. So $\mathcal{s} : \Delta \text{PrSh}(\mathcal{X}, \mathcal{D}) \rightarrow \text{PrSh}(\mathcal{X}, \mathcal{D})$ is just $(\mathcal{s}\mathcal{F}^*)(U) = \mathcal{s}(\mathcal{F}^*(U))$ and the equivalences $\mathcal{E}^X = \mathcal{S}$ are the global ones. Since $\mathcal{s} : \Delta \mathcal{D} \rightarrow \mathcal{D}$ is assumed to preserve limits, it follows that if $\mathcal{F}^*$ is a cosimplicial sheaf, then $\mathcal{s}(\mathcal{F}^*)$ is a sheaf. Hence, $\mathcal{s}$ restricts to a functor defined between the categories of sheaves $\mathcal{s} : \Delta \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \text{Sh}(\mathcal{X}, \mathcal{D})$.

On the other hand, the subcategory $\text{Sh}(\mathcal{X}, \mathcal{D})$ is closed under products. As $\text{Sh}(\mathcal{X}, \mathcal{D})$ is a full subcategory of $\text{PrSh}(\mathcal{X}, \mathcal{D})$, then the natural transformations $\lambda$ and $\mu$ for $\text{PrSh}(\mathcal{X}, \mathcal{D})$ are also natural transformations in $\text{Sh}(\mathcal{X}, \mathcal{D})$. Thus, it is clear that the hypotheses of the Transfer Lemma 2.1.6 are verified taking as $\psi$ the inclusion functor $\mathbf{i} : \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \text{PrSh}(\mathcal{X}, \mathcal{D})$.

3.3. The cosimplicial Godement resolution. The classical cosimplicial Godement resolution makes sense for sheaves with values in categories with filtered colimits and arbitrary products.
3.3.1. If $\mathcal{D}$ is a category closed under products and filtered colimits and $x$ is a point of the site $\mathcal{X}$, then $x^* : \text{Sh}(\mathcal{X}, \mathcal{D}) \to \mathcal{D}$ and $x_* : \mathcal{D} \to \text{Sh}(\mathcal{X}, \mathcal{D})$ may be defined by the formulas (3.1.2) and (3.1.3). That is, for $\mathcal{F} \in \text{Sh}(\mathcal{X}, \mathcal{D})$ and $D \in \mathcal{D}$, write

$$x^* \mathcal{F} = \mathcal{F}_x = \colim_{(U, u)} \mathcal{F}(U) \quad \text{and} \quad (x_* D)(U) = \prod_{u \in x^*(eU)} D_u ,$$

where $(U, u)$ runs over the opposite category of neighbourhoods of $x$ and $D_u = D$ for all $u$.

**Proposition 3.3.1.** Let $\mathcal{D}$ be a category closed under products and filtered colimits, and let $x$ be a point of the site $\mathcal{X}$. Then, the functors $x_*$, $x^*$ define a pair of adjoint functors

$$\text{Sh}(\mathcal{X}, \mathcal{D}) \xrightarrow{x^*} \mathcal{D} , \quad \mathcal{D}(x^* \mathcal{F}, D) = \text{Sh}(\mathcal{X}, \mathcal{D})(\mathcal{F}, x_* D) .$$

**Proof.** First of all, let us see that $x_* D$ is indeed a $\mathcal{D}$-valued sheaf. By definition, $x_* D \in \text{Sh}(\mathcal{X}, \mathcal{D})$ means that for every object $D' \in \mathcal{D}$ the presheaf $\mathcal{D}(D', x_* D(-))$, $U \mapsto \mathcal{D}(D', x_* D(U))$ is a sheaf of sets on $\mathcal{X}$. But $\mathcal{D}(D', x_* D(-)) = (x_* \mathcal{D}(D', D))(-)$, hence it is a sheaf of sets as required. The fact that $(x_*, x^*)$ is a pair of adjoint functors may be easily proved by checking that the universal morphism $\eta^* : \text{id} \to x_* x^*$ valid for $\mathcal{D} = \text{Set}$ still works for general $\mathcal{D}$. \hfill \Box

3.3.2. Consequently, for a given set $X$ of enough points of the site $\mathcal{X}$, formulas (3.1.4) also make sense for coefficients in $\mathcal{D}$ and define a pair of adjoint functors

$$\text{Sh}(\mathcal{X}, \mathcal{D}) \xrightarrow{p^*} \mathcal{D}^X , \quad \mathcal{D}^X(p^* \mathcal{F}, D) = \text{Sh}(\mathcal{X}, \mathcal{D})(\mathcal{F}, p_* D) .$$

In case where $\mathcal{X}$ is the site associated with a topological space $X$, we may take as a set of enough points the underlying set of $X$, and the resulting adjoint pair $(p^*, p_*)$ agrees with the one induced by the continuous map $p : X_{\text{dis}} \to X$, where $X_{\text{dis}}$ is $X$ with the discrete topology.

3.3.3. Let $\mathbf{T} = (T, \eta, \nu)$ denote the triple associated with this adjoint pair of functors. Its underlying functor $T = p_* p^* : \text{Sh}(\mathcal{X}, \mathcal{D}) \to \text{Sh}(\mathcal{X}, \mathcal{D})$ is given by

$$T(\mathcal{F})(U) = \prod_{x \in X} (x_* x^* \mathcal{F})(U) = \prod_{x \in X} \prod_{u \in x^*(eU)} (x^* \mathcal{F})_u ,$$

where $(x^* \mathcal{F})_u = x^* \mathcal{F}$ for all $u \in x^*(eU)$. The natural transformation $\eta : \text{id} \to T$ is the unit of the adjunction, while $\nu : T^2 \to T$ is by definition $\nu_{\mathcal{F}} = p_*(\varepsilon_{p^* \mathcal{F}})$, where $\varepsilon : p^* p_* \to \text{id}$ is the counit of the adjunction.

The standard construction associated with the triple $\mathbf{T} = (T, \eta, \nu)$ gives a cosimplicial object $G^*_{\mathcal{F}} \in \Delta \text{Sh}(\mathcal{X}, \mathcal{D})$ with $G^p(\mathcal{F}) = T^{p+1}(\mathcal{F})$. The resulting functor

$$G^* : \text{Sh}(\mathcal{X}, \mathcal{D}) \to \Delta \text{Sh}(\mathcal{X}, \mathcal{D})$$

is nothing else than the **canonical cosimplicial resolution of Godement** (GO). For every sheaf $\mathcal{F}$, the natural transformation $\eta : \text{id} \to T$ defines a coaugmentation $\mathcal{F} \to G^*(\mathcal{F})$, that we also denote by $\eta_{\mathcal{F}}$.

We will repeatedly use the following well-known property of $G^*(\mathcal{F})$, that holds for each cosimplicial construction associated with an adjoint pair.
Lemma 3.3.2. The natural coaugmentation $\eta: \text{id} \to G^\bullet$ is such that $p^*(\eta_F)$ and $\eta_{p_*(D)}$ have an extra degeneracy for each sheaf $F \in \text{Sh}(\mathcal{X}, D)$ and each object $D \in D^X$.

4. Cartan-Eilenberg categories of sheaves

In this section we study under which conditions the Godement cosimplicial resolution deserves to be called a ‘resolution’. More precisely, we provide conditions equivalent to the fact that the Godement resolution of $F$ produces a CE-fibrant model, by means of other properties such as Thomason’s descent, or a weak commutation of the simple functor with stalks.

4.1. Cartan-Eilenberg fibrant sheaves.

4.1.1. Let $(D, E)$ be a descent category.

Definition 4.1.1. We say that filtered colimits and arbitrary products in $D$ are E-exact if:

1. for any filtered category $I$ and any natural transformation $\varphi : F \to G$ between functors $F, G : I \to D$ such that $\varphi_i \in E$ for every $i \in I$, we have $	ext{colim}_i \varphi_i \in E$; and
2. for every arbitrary family of morphisms $\{\varphi_i : F_i \to G_i\}_{i \in I}$ in $D$ such that $\varphi_i \in E$ for every $i \in I$, we have $\prod_{i \in I} \varphi_i \in E$.

In what follows, the site $\mathcal{X}$ and the descent category $(D, E)$ are assumed to verify the following hypotheses:

(G0) $\mathcal{X}$ is a Grothendieck site with a family $X$ of enough points.
(G1) $D$ is closed under filtered colimits and arbitrary products.
(G2) Filtered colimits and arbitrary products in $D$ are E-exact.

4.1.2. In sheaf theory the notion of weak equivalence that best reflects the homotopical behaviour of sheaves and the topology of the Grothendieck site is the one of local equivalence, defined stalkwise rather than objectwise.

Definition 4.1.2. A morphism of sheaves $f : F \to G$ is a local equivalence if for any point $x \in X$ the morphism $x^*f : x^*F \to x^*G$ belongs to $E$. We will denote by $W$ the class of local equivalences of $\text{Sh}(\mathcal{X}, D)$.

The following properties of the classes of global and local equivalences are immediate consequences of assumptions (4.1.1).

Lemma 4.1.3. The classes $S$ and $W$ of global and local equivalences are saturated. In addition:

1. $p_*(E) \subset S$ and $W = (p^*)^{-1}E$.
2. $T(W) \subset S \subset W$.
3. Arbitrary products of sheaves are $S$-exact.

Since every global equivalence is a local one, $\text{Sh}(\mathcal{X}, D)$ is then equipped with two classes of strong (global) and weak (local) equivalences as in Section 2.2. Therefore it makes sense to talk about CE-fibrant sheaves:
Definition 4.1.4. A sheaf $\mathcal{F}$ is Cartan-Eilenberg fibrant if for any solid diagram

\[
\begin{array}{ccc}
G & \xrightarrow{w} & G' \\
f & \downarrow & \\
\mathcal{F} & \xleftarrow{g} & \\
\end{array}
\]

with $w$ a local equivalence and $f$ a morphism of $\text{Sh}((\mathcal{X},\mathcal{D})[S^{-1}])$, there exists a unique morphism $g$ of $\text{Sh}((\mathcal{X},\mathcal{D})[S^{-1}])$ making the triangle commutative.

Example 4.1.5. It is easy to see that $T(\mathcal{F})$ is a CE-fibrant sheaf for any $\mathcal{F}$. This tells us that there is a significant number of CE-fibrant sheaves. However, the class $\{ T(\mathcal{F}) \}$ fails in general to contain enough CE-fibrant models: for the case of positive complexes of abelian sheaves on a topological space $X$, if any sheaf is locally equivalent to $T(\mathcal{F})$ for some $\mathcal{F}$ then $X$ would have cohomological dimension equal to zero.

4.2. The hypercohomology sheaf. Next we make use of the Godement cosimplicial resolution to produce a wider class of CE-fibrant sheaves: the one consisting of the hypercohomology sheaves.

4.2.1. Let $\mathcal{X}$ be a site and $(\mathcal{D},\mathcal{E})$ a descent category satisfying assumptions (4.1.1). Recall that, by Proposition 3.2.3, $(\text{Sh}(\mathcal{X},\mathcal{D}),\mathcal{S})$ is a descent category with simple functor $s$ defined objectwise. It follows from Theorem 2.1.3 that

\[
\text{Sh}(\mathcal{X},\mathcal{D})[S^{-1}] \xleftarrow{s} \text{Sh}(\mathcal{X},\mathcal{D})[S^{-1}]
\]

is an adjoint pair. In other words, $s(F^\bullet)$ is the homotopy limit of the cosimplicial diagram of sheaves $F^\bullet$. Accordingly to [Th], we make the following definition.

Definition 4.2.1. The hypercohomology sheaf of $\mathcal{F} \in \text{Sh}(\mathcal{X},\mathcal{D})$ is the image under the simple functor of the Godement cosimplicial resolution of $\mathcal{F}$,

\[
\mathbb{H}_X(\mathcal{F}) = sG^\bullet(\mathcal{F})
\]

The natural coaugmentation $\eta_\mathcal{F} : \mathcal{F} \to G^\bullet \mathcal{F}$ may be seen as a cosimplicial morphism $\eta_\mathcal{F} : c(\mathcal{F}) \to G^\bullet(\mathcal{F})$. The adjoint morphism of $\eta_\mathcal{F}$ through $(c,s)$ is then

\[
\rho_\mathcal{F} : \mathcal{F} \to \mathbb{H}_X(\mathcal{F})
\]

More explicitly, $\rho_\mathcal{F}$ is the natural morphism of sheaves obtained as the composition of $s(\eta_\mathcal{F}) : sc(\mathcal{F}) \to \mathbb{H}_X(\mathcal{F})$ with $\lambda_\mathcal{F} : \mathcal{F} \to sc(\mathcal{F})$, given by the descent structure on $(\text{Sh}(\mathcal{X},\mathcal{D}),\mathcal{S})$.

Thomason extensively studied those presheaves of fibrant spectra for which the morphism $\rho_\mathcal{F}$ is a global equivalence. Such an $\mathcal{F}$ was said to satisfy descent with respect to the site.

Definition 4.2.2. A sheaf $\mathcal{F} \in \text{Sh}(\mathcal{X},\mathcal{D})$ is said to satisfy Thomason’s descent if the natural morphism $\rho_\mathcal{F} : \mathcal{F} \to \mathbb{H}_X(\mathcal{F})$ is in $\mathcal{S}$. 
4.2.2. To study the properties of the hypercohomology sheaf $H_X(\mathcal{F})$, we need to understand the behaviour of the descent structure $(s, \mu, \lambda)$ on $(\mathcal{D}, E)$ with respect to the Godement pair $(p^\ast, p_\ast, \eta, \varepsilon)$. On the one hand, the comparison of $s$ with $p_\ast$ presents no difficulty: since $p_\ast$ only involves products and $s$ commutes with limits, $p_\ast s = s p_\ast$.

On the other hand, although $p^\ast$ does not commute in general with simple functors, we still do have a natural comparison morphism

$$\theta_{\mathcal{F}} : p^\ast s(\mathcal{F}^\ast) \rightarrow s p^\ast(\mathcal{F}^\ast)$$

(4.2.1)
defined as the adjoint through $(p^\ast, p_\ast)$ of $s(\eta_{\mathcal{F}}) : s(\mathcal{F}^\ast) \rightarrow sT(\mathcal{F}^\ast) = s p^\ast(\mathcal{F}^\ast) = p_\ast s p^\ast(\mathcal{F}^\ast)$ for each cosimplicial sheaf $\mathcal{F}^\ast$. Applying $p_\ast$ to (4.2.1), we obtain the natural transformation

$$\theta_{\mathcal{F}} : Ts(\mathcal{F}^\ast) \rightarrow sT(\mathcal{F}^\ast)$$

adjoint to the composition $p_\ast p^\ast s(\mathcal{F}^\ast) \xrightarrow{\varepsilon_{p_\ast s(\mathcal{F}^\ast)}} p^\ast s(\mathcal{F}^\ast) \xrightarrow{\theta_{\mathcal{F}}} s p^\ast(\mathcal{F}^\ast)$.

The next technical lemma summarizes the compatibility relations between the descent structure on $(\mathcal{D}, E)$ and the Godement pair which will be needed later on.

**Lemma 4.2.3.** The diagrams

\[
\begin{array}{ccc}
\text{s(}\mathcal{F}^\ast\text{)} & \xrightarrow{s(\eta_{\mathcal{F}})} & \text{sT(}\mathcal{F}^\ast\text{)} \\
\downarrow{\eta_{s(}\mathcal{F}^\ast\text{)}} & & \downarrow{\theta_{\mathcal{F}}} \\
\text{T}s(\mathcal{F}^\ast) & & \text{T}sT(\mathcal{F}^\ast)
\end{array}
\]

\[
\begin{array}{ccc}
\text{T}s(\mathcal{F}^\ast) & \xrightarrow{T(\theta_{\mathcal{F}})} & \text{T}sT(\mathcal{F}^\ast) \\
\downarrow{\nu_{s(}\mathcal{F}^\ast\text{)}} & & \downarrow{\theta_{\mathcal{F}}} \\
\text{T}s(\mathcal{F}^\ast) & & \text{s}(\nu_{\mathcal{F}})
\end{array}
\]

are commutative in $\text{Sh}(\mathcal{X}, \mathcal{D})$. In addition, the diagrams below are commutative, respectively, in $\mathcal{D}^X[E^{-1}]$ and $\text{Sh}(\mathcal{X}, \mathcal{D})[S^{-1}]$

\[
\begin{array}{ccc}
p^\ast(\mathcal{F}) & \xrightarrow{\lambda^\ast(p)} & \text{scp}(\mathcal{F}) = \text{sp}(\mathcal{F}) \\
\downarrow{p^\ast(\lambda_{\mathcal{F}})} & & \downarrow{\theta_{\mathcal{F}}} \\
p^\ast(\mathcal{F}) & & \text{p}(\mathcal{F})
\end{array}
\]

\[
\begin{array}{ccc}
\text{T}(\mathcal{F}) & \xrightarrow{T(\lambda_{\mathcal{F}})} & \text{scT}(\mathcal{F}) = \text{t}(\mathcal{F}) \\
\downarrow{T(\lambda_{\mathcal{F}})} & & \downarrow{\theta_{\mathcal{F}}} \\
\text{T}(\mathcal{F}) & & \text{T}(\mathcal{F})
\end{array}
\]

**Proof.** The commutativity of diagrams (4.2.2) may be checked by an easy adjunction argument. On the other hand, note that since $p_\ast(E) \subset \mathcal{S}$ and $p^\ast(\mathcal{S}) \subset E$,

$$\text{Sh}(\mathcal{X}, \mathcal{D})[S^{-1}] \xrightarrow{p^\ast} \mathcal{D}^X[E^{-1}]$$

is again an adjoint pair after localizing. The image of $\theta_{\mathcal{F}} : p^\ast s(\mathcal{F}^\ast) \rightarrow s p^\ast(\mathcal{F}^\ast)$ in $\mathcal{D}^X[E^{-1}]$ is then also the canonical morphism induced by the adjunction $(p^\ast, p_\ast)$ at the localized level. But $s : \Delta \mathcal{D}^X[E^{-1}] \rightarrow \mathcal{D}^X[E^{-1}]$ is right adjoint to the constant functor, and it follows formally that $\theta_{\mathcal{F}} : p^\ast s(\mathcal{F}^\ast) \rightarrow s p^\ast(\mathcal{F}^\ast)$ coincides with the adjoint morphism through $(\mathcal{C}, s)$ of the natural map $p^\ast(\varepsilon_{\mathcal{F}}) : p^\ast s(\mathcal{F}^\ast) = c p^\ast s(\mathcal{F}^\ast) \rightarrow p^\ast(\mathcal{F}^\ast)$, where $\varepsilon_{\mathcal{F}}$ is the unit of $(\mathcal{C}, s)$. This in turn implies that the two remaining diagrams are commutative as claimed.

$\square$
4.2.3. The properties of the natural comparison transformation (4.2.1) imply the following lemma, which will be of use for the next results.

**Lemma 4.2.4.** Assume that $\mathcal{X}$ and $(\mathcal{D}, E)$ satisfy hypotheses (4.1.1). If so, for any sheaf $\mathcal{F} \in \mathbf{Sh}(\mathcal{X}, \mathcal{D})$ the following conditions hold:

1. $T(\rho_F) : T(\mathcal{F}) \to T\mathbb{H}_\mathcal{X}(\mathcal{F})$ has a natural section in $\mathbf{Sh}(\mathcal{X}, \mathcal{D})[S^{-1}]$.
2. $\rho_{\mathbb{H}_\mathcal{X}}(\mathcal{F}) : \mathbb{H}_\mathcal{X}(\mathcal{F}) \to \mathbb{H}_\mathcal{X}(\mathcal{F})^2 = \mathbb{H}_\mathcal{X}\mathbb{H}_\mathcal{X}(\mathcal{F})$ has a natural section in $\mathbf{Sh}(\mathcal{X}, \mathcal{D})[S^{-1}]$.

**Remark 4.2.5.** In the proof of the above lemma and other results of this section, we are going to work with bicosimplicial objects. A bicosimplicial object will be denoted, respectively, by $\sigma$, or by $\mathbb{Z} \circ$, to distinguish both cosimplicial indexes. To such a bicosimplicial object we may apply the simple functor with respect to $\sigma$ or with respect to $\circ$. The resulting cosimplicial objects will be denoted, respectively, by $s_\circ \mathbb{Z}^\circ$ and $s_\sigma \mathbb{Z}^\circ$.

**Proof.** Given a sheaf $\mathcal{F} \in \mathbf{Sh}(\mathcal{X}, \mathcal{D})$, let us exhibit a natural morphism $\sigma_F : T\mathbb{H}_\mathcal{X}(\mathcal{F}) \to T(\mathcal{F})$ in $\mathbf{Sh}(\mathcal{X}, \mathcal{D})[S^{-1}]$ such that $\sigma_F T(\rho_F) = id$.

Recall that the coaugmentation $\eta_F : c(\mathcal{F}) \to G^\bullet(\mathcal{F})$ associated with the triple $T = (T, \eta, \nu)$ is such that $T(\eta_F) : cT(\mathcal{F}) \to TG^\bullet(\mathcal{F})$ has an extra degeneracy given by the maps $\{\nu_T(\mathcal{F}) : T^{n+2}(\mathcal{F}) \to T^{n+1}(\mathcal{F})\}_{n \geq 0}$. In particular,

$$\nu' = \{\nu_F \nu_T(\mathcal{F}) \cdots \nu_T(n, \mathcal{F})\}_{n \geq 0} : TG^\bullet(\mathcal{F}) \to cT(\mathcal{F})$$

is a natural cosimplicial morphism such that $\nu' T(\eta_F) = id$. Then $s(\nu') sT(\eta_F) = id$. Define $\sigma_F$ as the composition

$$TsG^\bullet(\mathcal{F}) \xrightarrow{\theta^\bullet c(\mathcal{F})} sT^\bullet(\mathcal{F}) \xrightarrow{s(\nu')} sT(\mathcal{F}) \xrightarrow{\lambda^{-1}(\mathcal{F})} T(\mathcal{F}) \ .$$

Since $\theta^\bullet$ is a natural transformation, then $\theta^\bullet c(\mathcal{F}) T(\eta_F) = sT(\eta_F) \theta^\bullet c(\mathcal{F})$, so $s(\nu') \theta^\bullet c(\mathcal{F}) T(\eta_F) = s(\nu') sT(\eta_F) \theta^\bullet c(\mathcal{F}) = \theta^\bullet c(\mathcal{F})$, and

$$\sigma_F T(\rho_F) = (\lambda^{-1}_1(\mathcal{F}) s(\nu') \theta^\bullet c(\mathcal{F})) (Ts(\eta_F) T(\lambda_F)) = \lambda^{-1}(\mathcal{F}) \theta^\bullet c(\mathcal{F}) T(\lambda_F) = \lambda^{-1}(\mathcal{F}) \lambda T(\mathcal{F}) = id \ .$$

To see (2), note that the canonical morphism $\theta^\bullet$ may be iterated to $\theta^\bullet_{T^2} : T^{n+1} sL(\mathcal{F}^\bullet) \to sT^{n+1}(\mathcal{F}^\bullet)$, with $(\theta^\bullet_{T^2})_{n+1} T^{n+1}(\lambda_F) = \lambda T^{n+1}(\mathcal{F})$. Explicitly, $(\theta^\bullet_{T^2})_{n+1}$ is given by the composition

$$sT^{n+1}(\mathcal{F}^\bullet) \xrightarrow{T(\theta^\bullet_{T^2})_{n+1}} T^{n+1} sL(\mathcal{F}^\bullet) \xrightarrow{\theta^\bullet_{T^2}} sT^{n+1}(\mathcal{F}^\bullet) \ .$$

Since the diagrams (4.2.2) are commutative, it follows that $(\theta^\bullet)^\circ : G^\circ s_{\bullet}(\mathcal{F}^\bullet) \to s_{\bullet} G^\circ(\mathcal{F}^\bullet)$ is a cosimplicial morphism. In addition, using the fact that $\theta^\bullet_{T^2} \eta_{sL(\mathcal{F}^\bullet)} = s_{\bullet}(\eta_{T^2}(\mathcal{F}^\bullet))$ it may be proved by induction that $\theta^\bullet_{T^2} \eta_{sL(\mathcal{F}^\bullet)} = s_{\bullet}(\eta_{T^2}(\mathcal{F}^\bullet))$. In particular, for $\mathcal{F}^\bullet = G^\circ(\mathcal{F})$ we obtain the commutative diagram of cosimplicial sheaves

$$c^\circ s_{\bullet} G^\circ(\mathcal{F}) \xrightarrow{s_{\bullet}(\eta_{c^\circ}(\mathcal{F}))} s_{\bullet} G^\circ G^\circ(\mathcal{F}) \ .$$
Hence, applying the simple functor we deduce that $s_0(\theta_0^G(\mathcal{F})) s_0(n_{\mathcal{G}^*}(\mathcal{F})) = s_0 s_0(n_{\mathcal{G}^*}(\mathcal{F}))$. Assume it proved that $s_0 s_0(n_{\mathcal{G}^*}(\mathcal{F})) \in \mathcal{S}$. In this case,

$$\phi = s_0 s_0(n_{\mathcal{G}^*}(\mathcal{F})) \lambda_{s_0 \mathcal{G}^*} = s_0(\theta_0^G(\mathcal{F})) s_0(n_{\mathcal{G}^*}(\mathcal{F})) \lambda_{s_0 \mathcal{G}^*} = s_0(\theta_0^G(\mathcal{F})) \rho_{\mathcal{H}X}(\mathcal{F})$$

is an isomorphism of $\text{Sh}(\mathcal{X}, \mathcal{D})[\mathcal{S}^{-1}]$, so $\sigma_{\mathcal{F}} = \phi^{-1} s_0(\theta_0^G(\mathcal{F}))$ is a section of $\rho_{\mathcal{H}X}(\mathcal{F})$. To finish, it remains to be shown that $s_0 s_0(n_{\mathcal{G}^*}(\mathcal{F})) \in \mathcal{S}$. By Lemma 2.1.5 this happens if and only if $s_0 s_0(n_{\mathcal{G}^*}(\mathcal{F})) \in \mathcal{S}$. For a fixed $n \geq 0$, the coaugmentation $\eta_{Tn(\mathcal{F})} = \eta_{Tn+1(\mathcal{F})} : \epsilon_0 T^{n+1}(\mathcal{F}) \to G^* T^{n+1}(\mathcal{F})$ has an extra degeneracy. Hence, by Lemma 2.1.4 we infer that $s_0(n_{\mathcal{G}^*}(\mathcal{F}))$ is in $\mathcal{S}$ for each $n \geq 0$. But then it follows from axiom (S4) that $s(n \to s_0(n_{\mathcal{G}^*}(\mathcal{F}))) = s_0 s_0(n_{\mathcal{G}^*}(\mathcal{F})) \in \mathcal{S}$ as required. \[ \square \]

4.2.4. The class $\mathcal{W}$ of local equivalences is by definition equal to $(p^*)^{-1} \mathcal{E}$. Below we prove that $\mathcal{W} = T^{-1} \mathcal{S} = \mathbb{H}^{-1} \mathcal{S}$ as well.

**Proposition 4.2.6.** Assume that $\mathcal{X}$ and $(\mathcal{D}, \mathcal{E})$ satisfy the hypotheses (4.1.1). Then, for a morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves, the following conditions are equivalent:

1. $f$ is a local equivalence.
2. $T(f) : T(\mathcal{F}) \to T(\mathcal{G})$ is a global equivalence.
3. $H_\mathcal{X}(f) : H_\mathcal{X}(\mathcal{F}) \to H_\mathcal{X}(\mathcal{G})$ is a global equivalence.

**Proof.** (1) implies (2) since $T(\mathcal{W}) \subset \mathcal{S}$. Conversely, if $T(f)$ is a global equivalence, it is in particular a local one, so $p^* T(f) \in \mathcal{E}$. On the other hand, it follows from the triangle identities of the adjoint pair $(p^*, p_*)$ that $p^*(f)$ is a retract of $p^* T(f) = p^* p_* p^*(f)$. But $\mathcal{E}$ being saturated, it is closed under retracts, and we deduce that $p^*(f) \in \mathcal{E}$ as well. But this is the same as saying that $f \in \mathcal{S}$. Therefore, (1) and (2) are equivalent.

Let us see that (2) implies (3). Assume that $T(f) \in \mathcal{S}$. Since $T(\mathcal{S}) \subset T(\mathcal{W}) \subset \mathcal{S}$, then $G^n(f) = T^{n+1}(\mathcal{F}) \in \mathcal{S}$ for all $n \geq 0$, and it follows from (S4) that $H_\mathcal{X}(f) = s G^*(f) \in \mathcal{S}$ as required. Finally, if $H_\mathcal{X}(f) \in \mathcal{S}$ then also $T H_\mathcal{X}(f) \in \mathcal{S}$. By Lemma 4.2.4 $T(f)$ is a retract of $T H_\mathcal{X}(f)$, so $T(f) \in \mathcal{S}$ and (2) and (3) are equivalent as well. \[ \square \]

4.2.5. As announced, we deduce that the hypercohomology sheaf is always CE-fibrant.

**Proposition 4.2.7.** Assume that $\mathcal{X}$ and $(\mathcal{D}, \mathcal{E})$ satisfy hypotheses (4.1.1). Then, for any sheaf $\mathcal{F}$, $H_\mathcal{X}(\mathcal{F})$ is a CE-fibrant sheaf.

**Proof.** Hypothesis (4.1.1) guarantee that $T(\mathcal{S}) \subset \mathcal{S}$, hence $H_\mathcal{X}(\mathcal{S}) \subset \mathcal{S}$. By Lemma 4.2.4 it is equipped with natural transformations $\rho : \text{id} \to H_\mathcal{X}$ and $\sigma : H_\mathcal{X}^2 \to H_\mathcal{X}$ such that $\sigma \rho = \text{id}$.

As a first consequence, a morphism $g : \mathcal{G} \to H_\mathcal{X}(\mathcal{F})$ of $\text{Sh}(\mathcal{X}, \mathcal{D})[\mathcal{S}^{-1}]$ is uniquely determined by $H_\mathcal{X}(g)$. Indeed, from the commutative diagram

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{g} & H_\mathcal{X}(\mathcal{F}) \\
\rho G & \searrow & \sigma T \downarrow \\
H_\mathcal{X}(\mathcal{G}) & \xrightarrow{H_\mathcal{X}(g)} & H_\mathcal{X}(\mathcal{F})
\end{array}
$$
we deduce that \( g = \sigma_F \mathbb{H}_X(g) \rho g \) as claimed. Consider now a lifting problem

\[
\begin{array}{ccc}
G & \xrightarrow{w} & G' \\
\downarrow f & & \downarrow \ \\
\mathbb{H}_X(F) & & \\
\end{array}
\]

where \( f \) is a morphism of \( \text{Sh}(\mathcal{X}, \mathcal{D})|S^{-1}| \) and \( w \) is a morphism of \( \text{Sh}(\mathcal{X}, \mathcal{D}) \) that is a local equivalence. Since \( \mathbb{H}_X(W) \subset S \), given two solutions \( g, g' : G' \to \mathbb{H}_X(F) \) of this lifting problem, we would have \( \mathbb{H}_X(g) = \mathbb{H}_X(f)(\mathbb{H}_X(w))^{-1} = \mathbb{H}_X(g') \). Hence \( g = g' \), and we need only see that there is at least one lifting for the above diagram. But \( g = \sigma_F \mathbb{H}_X(f)(\mathbb{H}_X(w))^{-1} \rho_{\mathbb{H}_X(F)} \) is easily seen to satisfy \( g w = f \), so we are done.

4.3. Characterization. In view of the last proposition, we conclude that if for any sheaf \( \eta_F : F \to \mathbb{H}_X(F) \) were in \( W \), then \( (\text{Sh}(\mathcal{X}, \mathcal{D}), S, W) \) would be a Cartan-Eilenberg category with \( (\mathbb{H}_X, \rho) \) as a resolvent functor. Below we show that this fact is indeed equivalent to two other conditions: one of them is Thomason’s descent property for hypercohomology sheaves, while the other one consists of a weak commutation between the simple functor and stalks.

4.3.1. Let us state precisely what we mean by the latter condition.

**Definition 4.3.1.** Let \( \mathcal{X} \) be a Grothendieck site and \( (\mathcal{D}, E) \) a descent category. We say that the simple functor commutes weakly with stalks if for each sheaf \( F \) the map \( \theta_{G*F} : p^*\mathbb{H}_X(F) = p^*sG^*(F) \to sp^*G^*(F) \) in (4.2.1) belongs to \( E \).

Equivalently, \( s \) commutes weakly with stalks if for each point \( x \in X \) the canonical map \( \theta_{G*F}(x) : sG^*(F)_x \to sG^*(F)_x \) is a weak equivalence.

4.3.2. We can now state and prove our first main result.

**Theorem 4.3.2.** Let \( \mathcal{X} \) be a Grothendieck site and \( (\mathcal{D}, E) \) a descent category satisfying the hypotheses (4.1.1). Then, the following statements are equivalent:

1. \((\text{Sh}(\mathcal{X}, \mathcal{D}), S, W)\) is a right Cartan-Eilenberg category and for every sheaf \( F \in \text{Sh}(\mathcal{X}, \mathcal{D}) \), \( \rho_F : F \to \mathbb{H}_X(F) \) is a CE-fibrant model.
2. For every sheaf \( F \in \text{Sh}(\mathcal{X}, \mathcal{D}) \), \( \rho_F : F \to \mathbb{H}_X(F) \) is in \( W \).
3. The simple functor commutes weakly with stalks.
4. For every sheaf \( F \in \text{Sh}(\mathcal{X}, \mathcal{D}) \), \( \mathbb{H}_X(F) \) satisfies Thomason’s descent; that is, \( \rho_{\mathbb{H}_X(F)} : \mathbb{H}_X(F) \to \mathbb{H}_X^2(F) \) is in \( S \).

**Definition 4.3.3.** We say that a descent category \( (\mathcal{D}, E) \) is compatible with the site \( \mathcal{X} \) if the equivalent conditions of this theorem are satisfied.

**Remark 4.3.4.** As we will see in the examples, this is not necessarily the case for general \( \mathcal{X} \) and \( (\mathcal{D}, E) \). Furthermore, it may happen that \((\text{Sh}(\mathcal{X}, \mathcal{D}), S, W)\) is indeed a Cartan-Eilenberg category, but the CE-fibrant model of a sheaf \( F \) does not agree with \( \mathbb{H}_X(F) \) in general. However, this does not pose much of a problem, and these drawbacks only occur when \( \mathcal{X} \) is “cohomologically big”: a suitable finite cohomological dimension hypothesis on \( \mathcal{X} \) ensures that the hypercohomology sheaf \( \mathbb{H}_X(F) \) is always a (CE-fibrant) model for \( F \).
Proof of Theorem 4.3.2. By Proposition 4.2.7 we know that $\mathbb{H}_X(F)$ is CE-fibrant for any sheaf $F$. Hence, the equivalence between (1) and (2) is clear. Let us see that (2) and (3) are equivalent. On the one hand, by definition, (2) holds if and only if $p^*(\rho_F)$ is in $E^X$ for any sheaf $F$. On the other hand, the cosimplicial Godement resolution is such that the coaugmentation $p^*\eta_F : c\rho^*(F) \rightarrow p^*G^*(F)$ has an extra degeneracy. It then follows from Proposition 3.2.3 and Lemma 2.4.4 that $sp^*(\eta_F)$ belongs to $E$. Since $\lambda_G : G \rightarrow sc(G)$ is also in $E$ for any sheaf $G$, we have the following commutative diagram in which the arrows decorated with $\sim$ are in $E$:

$$
\begin{array}{ccc}
\rho^*(F) & \xrightarrow{p^*(\lambda_F)} & p^*sc(F) \\
\downarrow{\lambda_p^*(F)} & & \downarrow{\theta^*(F)} \\
sp^*\xi(F) & \xrightarrow{sp^*(\eta_F)} & sp^*G^*(F) \\
\end{array}
$$

Note that the composition of the morphisms in the top row is precisely $p^*(\rho_F) : p^*(F) \rightarrow p^*\mathbb{H}_X(F)$. By the 2-out-of-3 property, we conclude that $p^*(\rho_F)$ is in $E$ if and only if $\theta^*(F)$ is in $E$. In other words, (2) and (3) are equivalent.

To finish with, we now show that (4) and (2) are equivalent. Because of Proposition 4.2.6 $W = \mathbb{H}_X^{-1}S$. Hence, $\rho_F : F \rightarrow \mathbb{H}_X(F)$ is in $W$ if and only if $\mathbb{H}_X(\rho_F)$ is in $S$. It is then enough to check that $\rho_{\mathbb{H}_X(F)}$ is in $S$ if and only if $\mathbb{H}_X(\rho_F)$ is. As in the proof of Lemma 4.2.4 the iteration of $\theta^*$ gives a canonical morphism of cosimplicial objects $\theta^*F : G^o\ast(F^*) \rightarrow sG^o(F^*)$ that makes the following diagrams commute

$$
\begin{array}{ccc}
s_oG^o(F) & \xrightarrow{s_oG^o(\lambda_F)} & s_oG^o(F) \\
\downarrow{s_o(\lambda_{G^o(F)})} & & \downarrow{s_o(\theta^*F)} \\
\mathbb{H}_X(\rho_F) & \xrightarrow{s_oG^o(\eta_F)} & s_oG^o(F) \\
\end{array}
$$

Note that all the arrows decorated with $\sim$ are global equivalences: for those arrows involving $\lambda$ this is clear (in particular this is so for $s_o(\theta^*(F))$). We already proved that $s_oG^o(\eta_{G^o(F)}) \in S$, and again using an extra degeneracy argument it readily follows that $s_oG^o(\eta_{G^o(F)}) \in S$. Consequently, $\rho_{\mathbb{H}_X(F)} \in S$ if and only if $\mathbb{H}_X(\rho_F) \in S$.

4.3.3. The first consequence of our main theorem is the following characterization of Thomason’s descent property for sheaves of spectra.

Corollary 4.3.5. If $(D, E)$ is compatible with the site $X$, then a sheaf $F \in \text{Sh}(X, D)$ satisfies Thomason’s descent if and only if it is a CE-fibrant sheaf.

4.3.4. The existence of an associated sheaf functor, or sheafification, $(-)^a : \text{PrSh}(X, D) \rightarrow \text{Sh}(X, D)$ guarantees that the homotopy theory of presheaves is the same as the homotopy theory of sheaves, because the adjoint pair $(-)^a : \text{PrSh}(X, D) \rightleftarrows \text{Sh}(X, D) : i$ induces an
equivalence of categories $\text{PrSh}(\mathcal{X}, \mathcal{D})[W^{-1}] \simeq \text{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}]$. Although an associated sheaf functor may not exist for $\mathcal{D}$, when $(\mathcal{D}, E)$ is compatible with the site $\mathcal{X}$ the hypercohomology sheaf may be thought of as a ‘homotopical’ sheafification functor. More precisely, the adjoint pair $(p^*, p_*)$ is also an adjoint pair

$$\text{PrSh}(\mathcal{X}, \mathcal{D}) \xrightarrow{p^*} \mathcal{D}^X \xleftarrow{p_*} \text{Sh}(\mathcal{X}, \mathcal{D})$$

and the induced triple on $\text{PrSh}(\mathcal{X}, \mathcal{D})$ allows an analogous definition $H^X_F = sG^*(F)$ for a presheaf $F$, which enjoys the same properties as in the sheaf case. In addition, $T(F) = p_*p^*(F)$ is a sheaf, and so is $H^X(F)$.

**Corollary 4.3.6.** Let $(\mathcal{D}, E)$ be a descent category compatible with the site $\mathcal{X}$. Then

$$\text{PrSh}(\mathcal{X}, \mathcal{D})[W^{-1}] \xrightarrow{H^X} \text{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}]$$

are inverse equivalences of categories.

**Proof.** By hypothesis, $\rho_F : F \to H^X_i(F)$ is in $W$, so it is an isomorphism of $\text{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}]$ for any sheaf $F$. It remains to be shown that if $F$ is now a presheaf then $\rho_F : F \to iH^X(F)$ is in $W$. Since $H^X(F)$ is a sheaf, $\rho_{H^X}(F) \in W$. But $\rho_{H^X}(F)$ is a morphism between CE-fibrant sheaves and hence belongs to $S$. By the same proof as in Theorem 4.3.2, we infer that $H^X(\rho_F) \in S$ as well. Again, this means that $\rho_F$ is a local equivalence as required.

4.3.5. We have seen that a descent structure on $(\mathcal{D}, E)$ always induces one on $(\text{Sh}(\mathcal{X}, \mathcal{D}), S)$ defined objectwise. We have another descent structure, though.

**Proposition 4.3.7.** Assume that a descent category $(\mathcal{D}, E)$ is compatible with the site $\mathcal{X}$ and that filtered colimits commute with finite products in $\mathcal{D}$. Then, $(\text{Sh}(\mathcal{X}, \mathcal{D}), W)$ is a descent category with simple functor

$$s' = sH^X : \Delta \text{Sh}(\mathcal{X}, \mathcal{D}) \to \text{Sh}(\mathcal{X}, \mathcal{D}).$$

**Proof.** The commutation of finite products with filtered colimits guarantees that $\mathcal{W} \prod W \subset W$. The fact that $s'$ is a simple functor for $(\text{Sh}(\mathcal{X}, \mathcal{D}), W)$ may be proved using that $H^X(W) \subset S$ and that $s$ is a simple functor for $(\text{Sh}(\mathcal{X}, \mathcal{D}), S)$.

It follows from the results in [Rod1] that path and loop functors may be constructed for $(\text{Sh}(\mathcal{X}, \mathcal{D}), W)$ in a natural way. They give rise to well behaved fiber sequences, satisfying the usual properties in $\text{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}]$. In particular, $\text{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}]$ is a triangulated category provided that the loop functor is an equivalence of categories.

4.4. Derived functors for sheaves.
4.4.1. The second consequence of our characterization of CE-fibrant sheaves, the existence of the right derived direct image functor, follows immediately (cf. [De] th.6]).

**Corollary 4.4.1.** Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a continuous functor of Grothendieck sites and \((\mathcal{D}, E)\) a descent category compatible with the site \( \mathcal{X} \). Then, \( f_* : \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \text{Sh}(\mathcal{Y}, \mathcal{D}) \) admits a right derived functor \( Rf_* : \text{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}] \rightarrow \text{Sh}(\mathcal{Y}, \mathcal{D})[W^{-1}] \) given by

\[
Rf_*(\mathcal{F}) = f_* \mathbb{H}_x(\mathcal{F}).
\]

**Proof.** In view of Theorem 4.3.2 and Proposition 2.2.6 we only need to show that \( f_* \) sends global equivalences to local equivalences. But this is obvious: if \( \varphi : \mathcal{F} \rightarrow \mathcal{G} \in \mathcal{S} \), then, for every object \( V \in \mathcal{Y} \), we have \( f_*(\varphi)(V) = \varphi(f^{-1}(V)) : \mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{G}(f^{-1}(V)) \in \mathcal{E} \). So \( f_*(\varphi) \) is also a global equivalence and hence, a fortiori, a local one. \( \square \)

If \( U \) is an object of \( X \), the same proof works for the \( U \)-sections functor \( \Gamma(U, -) : \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{D} \) because, by definition, \( \Gamma(U, \mathcal{F}) = \mathcal{F}(U) \) sends global equivalences in \( \text{Sh}(\mathcal{X}, \mathcal{D}) \) to equivalences in \( \mathcal{D} \). Hence,

**Corollary 4.4.2.** Let \((\mathcal{D}, E)\) be a descent category compatible with the site \( \mathcal{X} \). Then \( \Gamma(U, -) : \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{D} \) admits a right derived functor \( R\Gamma(U, -) : \text{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}] \rightarrow \mathcal{D}[E^{-1}] \) given by

\[
R\Gamma(U, \mathcal{F}) = \Gamma(U, \mathbb{H}_x(\mathcal{F})).
\]

4.4.2. When \( \mathcal{X} \) has a terminal object \( X \), e.g. in case \( \mathcal{X} \) is the site associated with a topological space \( X \), sheaf cohomology is by definition the right derived functor of the global sections functor \( \Gamma(X, -) : \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{D} \). So, under the above assumptions, sheaf cohomology is well defined and agrees with \( \Gamma(X, \mathbb{H}_x(\mathcal{F})) \).

Following [Jä3], if the coefficient category \( \mathcal{D} \) has limits, the notion of global sections functor \( \Gamma(\mathcal{X}, -) : \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{D} \) generalizes to a general site \( \mathcal{X} \), possibly without a terminal object, as:

\[
\Gamma(\mathcal{X}, \mathcal{F}) = \varinjlim_{U \in \mathcal{X}} \mathcal{F}(U).
\]

Note that in this case \( \Gamma(\mathcal{X}, -) \) does not necessarily send a global equivalence to a weak equivalence of \( \mathcal{D} \). But, being \((\mathcal{D}, E)\) a descent category in which arbitrary products are \( E \)-exact, the right derived functor of \( \varinjlim_{\mathcal{X}} : \mathcal{D}^\mathcal{X} \rightarrow \mathcal{D} \) exists, and is given by the composition of the simple functor with the cosimplicial replacement \( \mathcal{D}^\mathcal{X} \rightarrow \Delta \mathcal{D} \) (see [Rod2]). The resulting functor \( \text{holim}_{\mathcal{X}} : \text{Sh}(\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{D} \) sends global equivalences to weak ones; hence, it admits a right derived functor \( \text{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}] \rightarrow \mathcal{D}[E^{-1}] \) that may be seen to agree with the right derived functor of \( \Gamma(\mathcal{X}, -) \). That is, \( R\Gamma(\mathcal{X}, -) : \text{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}] \rightarrow \mathcal{D}[E^{-1}] \) exists and is given by

\[
R\Gamma(\mathcal{X}, \mathcal{F}) = \text{holim}_{U \in \mathcal{X}} \mathbb{H}_x(\mathcal{F})(U).
\]

4.4.3. Recall that when there is a sheafification functor then \( \text{Sh}(\mathcal{X}, \mathcal{D}) \) is complete (resp. cocomplete) when \( \mathcal{D} \) is. A homotopical version of this fact is that when \( \mathbb{H}_x \) is a 'homotopical' sheafification functor (that is, when \( (\mathcal{D}, E) \) is compatible with the site \( \mathcal{X} \)) then \( (\text{Sh}(\mathcal{X}, \mathcal{D}), \mathcal{W}) \) is homotopically complete, and homotopically cocomplete provided \( (\mathcal{D}, E) \) is.
The key points to seeing this are that the resolvent functor \((H^X, \rho)\) is also a resolvent functor for presheaves, and that it may be lifted to diagram categories: for each small category \(I\), \((H^X, \rho)\) induces objectwise a resolvent functor on \(\mathsf{PrSh}(\mathcal{X}, \mathcal{D})^I = \mathsf{PrSh}(\mathcal{X}, \mathcal{D}^I), S, W)\). This in turn implies that there is an adjunction natural in \(I\)

\[
\begin{align*}
\mathsf{PrSh}(\mathcal{X}, \mathcal{D})^I[S^{-1}] & \xrightarrow{\text{id}} \mathsf{PrSh}(\mathcal{X}, \mathcal{D})^I[W^{-1}] \\
\mathsf{Sh}(\mathcal{X}, \mathcal{D})^I[S^{-1}] & \xrightarrow{\mathbb{H}^X} \mathsf{Sh}(\mathcal{X}, \mathcal{D})^I[W^{-1}] \\
\end{align*}
\]

where the right adjoint \(\mathbb{H}^X\) is fully faithful. This natural adjunction then transfers homotopy limits and colimits existing for \((\mathsf{PrSh}(\mathcal{X}, \mathcal{D}^I), S) = (\mathcal{D}^X, \mathcal{E}^X)\) to \(\mathsf{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}]\). In particular \((\mathsf{Sh}(\mathcal{X}, \mathcal{D}), W)\) is homotopically complete and

\[
\begin{align*}
\text{holim}_I \mathbb{H}^{(\mathcal{X}, \mathcal{D}), \mathcal{S}} & = \text{holim}_I \mathbb{H}^{(\mathcal{X}, \mathcal{D}), \mathcal{S}} \\
\end{align*}
\]

5. Examples

In this section we show how the above results apply to classic and not so classic examples of categories of sheaves. More concretely, we will prove that a finite cohomological dimension assumption on the site \(\mathcal{X}\) guarantees its compatibility with the natural descent structures seen on categories of coefficients \(\mathcal{D}\) such as complexes, simplicial sets and spectra. Consequently, from the results of the previous section we conclude that for such \(\mathcal{X}\) and \(\mathcal{D}\) we have:

- For every sheaf \(\mathcal{F}\), the natural arrow \(\rho_F : \mathcal{F} \to \mathbb{H}_X(\mathcal{F})\) is a fibrant model of \(\mathcal{F}\). Or, what amounts to the same, \((\mathsf{Sh}(\mathcal{X}, \mathcal{D}), S, W)\) is a CE-category with resolvent functor \((H^X, \rho)\).
- The localized category \(\mathsf{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}]\) is naturally equivalent to \(\mathsf{Sh}(\mathcal{X}, \mathcal{D})_{\text{fib}}[S^{-1}]\).
- The CE-fibrant objects of \(\mathsf{Sh}(\mathcal{X}, \mathcal{D})\) are precisely those sheaves satisfying Thomason’s descent.
- Derived sections \(\mathbb{R}\Gamma(U, -)\) and derived direct image functor \(\mathbb{R}f_*\) may be computed by precomposing with \(\mathbb{H}_X\).
- The hypercohomology sheaf \(\mathbb{H}_X\) is a ‘homotopical’ sheafification functor that gives an equivalence \(\mathsf{Sh}(\mathcal{X}, \mathcal{D})[W^{-1}] \simeq \mathsf{PrSh}(\mathcal{X}, \mathcal{D})[W^{-1}]\).

5.1. Bounded complexes of sheaves.

5.1.1. Consider the descent category structure on the category of uniformly bounded cochain complexes \(\mathcal{C}^b(\mathcal{A})\) described in example 2.1.9.

In this case the simple functor is \(s = \text{Tot}^\Pi = \text{Tot}^\oplus : \Delta \mathcal{C}^b(\mathcal{A}) \to \mathcal{C}^b(\mathcal{A})\) by the boundedness assumption. The category of sheaves of uniformly bounded cochain complexes \(\mathsf{Sh}(\mathcal{X}, \mathcal{C}^b(\mathcal{A}))\) is a descent category where the weak equivalences are the global equivalences and the simple functor is the total-sum functor applied objectwise: \((\text{Tot}\mathcal{F})(U) = \text{Tot}(\mathcal{F}(U))\).

It follows that \(s\) commutes in this case with all colimits, since it is defined degree-wise through a finite direct sum. Hence, \(s\) commutes trivially with stalks. Therefore, we deduce from Theorem 4.3.2...
Theorem 5.1.1. Assume that $\mathcal{A}$ is an abelian category satisfying $(AB4)^*$ and $(AB5)$ (that is, arbitrary products and filtered colimits exist and are exact). Then, the descent category $C^{\geq b}(\mathcal{A})$ is compatible with any site $\mathcal{X}$. In particular, properties $\S$ hold for $(\mathcal{Sh}(\mathcal{X}, C^{\geq b}(\mathcal{A})), \mathcal{S}, \mathcal{W})$.

In this case a local equivalence $f \in \mathcal{W}$ is just a quasi-isomorphism of $\mathcal{Sh}(\mathcal{X}, C^{\geq b}(\mathcal{A})) = C^{\geq b}(\mathcal{Sh}(\mathcal{X}, \mathcal{A}))$. On the other hand, a global equivalence $f \in \mathcal{S}$ is a morphism $f : \mathcal{F} \to \mathcal{G}$ of complexes of sheaves such that $f(U)$ is a quasi-isomorphism of $C^{\geq b}(\mathcal{A})$ for each object $U \in \mathcal{X}$.

Consequently, a functor $F : \mathcal{Sh}(\mathcal{X}, C^{\geq b}(\mathcal{A})) \to C$ sending global equivalences to isomorphisms admits a right derived functor $RF : D^{\geq b}(\mathcal{Sh}(\mathcal{X}, \mathcal{A})) = \mathcal{Sh}(\mathcal{X}, C^{\geq b}(\mathcal{A}))[W^{-1}] \to C$ given by $RF(\mathcal{F}) = F(\mathbb{H}_X(\mathcal{F}))$. Note that this derivability criterion does not assume the existence of enough injectives in $\mathcal{A}$. Particularly, for the case $\mathcal{A} = \mathcal{R}$--modules, we recover the classic construction of abelian sheaf hypercohomology and derived direct image of sheaves constructed through canonical Godement resolutions by flasque sheaves.

5.2. Unbounded complexes of sheaves.

5.2.1. When the boundedness assumption on complexes of sheaves is dropped, Theorem 5.1.1 is not longer true for a general site $\mathcal{X}$, even in the case $\mathcal{A} = \mathcal{R}$--modules.

Consider the category $C^\ast(R)$ of unbounded cochain complexes of $R$-modules with the descent structure of example 2.1.10. In this case, the simple functor $s = \text{Tot}^\Pi : \Delta C^\ast(R) \to C^\ast(R)$ is an infinite product degree-wise, and consequently it does not commute (even weakly) with filtered colimits. This in turn means that the hypercohomology sheaf $\mathbb{H}_X(\mathcal{F})$ associated with an unbounded complex $\mathcal{F}$ of sheaves of $R$-modules does not necessarily produce a CE-fibrant model for $\mathcal{F}$ in $(\mathcal{Sh}(\mathcal{X}, C^\ast(R)), \mathcal{S}, \mathcal{W})$, for a general site $\mathcal{X}$.

Example 5.2.1. To illustrate this fact, consider a family $\{\mathcal{F}^k\}_k$ of abelian sheaves for which $(\prod_{k > 0} \mathbb{H}^b(\mathcal{F}^{-k}))_x \neq 0$ (for instance those described in [We, A.5] or [MV, 1.30]). Then construct the complex of sheaves $\mathcal{F}$ with zero differential that is 0 in positive degrees and equal to $\mathcal{F}^{-k}$ in negative degrees. It is not hard to verify that $p_F : \mathcal{F} \to \mathbb{H}_X(\mathcal{F})$ is not a quasi-isomorphism in this case, so it does not provide a CE-fibrant model for $\mathcal{F}$.

We remark however that $(\mathcal{Sh}(\mathcal{X}, C^\ast(R)), \mathcal{S}, \mathcal{W})$ is still a Cartan-Eilenberg category for any site $\mathcal{X}$: $K$-injective complexes of sheaves are easily seen to be CE-fibrant, and by $\text{Sp}$ each complex of sheaves is locally equivalent to some $K$-injective one (see also [We, appendix]). Hence the CE-fibrant model of an unbounded complex $\mathcal{F}$ of sheaves does not agree in general with its hypercohomology sheaf $\mathbb{H}_X(\mathcal{F})$, unless some extra assumption is imposed on site $\mathcal{X}$.

5.2.2. We are going to show that finite cohomological dimension is a sufficient condition for the site $\mathcal{X}$ in order that the hypercohomology sheaf $\mathbb{H}_X$ produces a resolvent functor for the Cartan-Eilenberg category $(\mathcal{Sh}(\mathcal{X}, C^\ast(R)), \mathcal{S}, \mathcal{W})$.

Recall that a system of neighbourhoods for a point $x \in \mathcal{X}$ is, by definition, a full cofinal subcategory of the category of neighbourhoods $\text{Nbh}(x)$ of $x$ in $\mathcal{X}$ ([SGA4] 6.8.2).

Definition 5.2.2 ([GS]). A site $\mathcal{X}$ is said to have finite cohomological dimension if for any point $x \in \mathcal{X}$ there exists $d \geq 0$ and a system $\Lambda$ of neighbourhoods of $x$ such that for any sheaf...
of abelian groups $\mathcal{F} \in \text{Sh}(\mathcal{X}, \text{Ab})$ and any neighbourhood $U \in \Lambda$ it holds that $H^n(U; \mathcal{F}) = 0$ whenever $n > d$.

For instance, the following sites have finite cohomological dimension:

1. The small Zariski site of a noetherian topological space of finite Krull dimension; e.g., the Zariski site of a noetherian scheme of finite Krull dimension. This follows from Grothendieck’s vanishing Theorem (\cite{Har} III, Theorem 2.7).
2. The big Zariski site of a noetherian scheme $X$ of finite Krull dimension consisting of all schemes of finite type over $X$, or all noetherian schemes of bounded Krull dimension (\cite{GS}, page 6).
3. The small site of a topological manifold of finite dimension. This follows from the vanishing Theorem of \cite{KS1}.

Theorem 5.2.3. The descent category $\mathcal{C}^*(R)$ is compatible with any finite cohomological dimension site $\mathcal{X}$. In this case, properties 5 hold for $(\text{Sh}(\mathcal{X}, \mathcal{C}^*(R)), \mathcal{S}, \mathcal{W})$.

The proof is based on a spectral sequence argument, the Colimit Lemma, for which we need some preliminaries. The same spectral sequence argument will also be used in the examples of simplicial sets and spectra.

5.2.3. Let $\mathcal{C}$ be a category with filtered colimits and $I$ a filtered indexing set. For us “spectral sequence” means a functorial right half-plane cohomological spectral sequence $E_\ast$ of abelian groups, commuting with filtered colimits: $E_\ast(\text{colim}_i X_i) = \text{colim}_i E_\ast(X_i)$.

For an object $X \in \mathcal{C}$, we say that the spectral sequence $E_\ast(X)$ is bounded on the right if there exists $d$ such that $E_2^{ps}(X) = 0$ for $p > d$. Note that, for conditionally convergent spectral sequences, this implies strong convergence (\cite{Boa}, Theorem 7.4). Given a filtered system $\{X_i\}_{i \in I}$ of objects of $\mathcal{C}$, we say that the family of spectral sequences $\{E_\ast(X_i)\}_{i \in I}$ is uniformly bounded on the right if there is a fixed $d$ that works for all $i \in I$.

Proposition 5.2.4 (Colimit Lemma). Assume as given the following data:

1. An object $X \in \mathcal{C}$ and a filtered system $X_\ast = \{X_i\}_{i \in I}$ of objects $X_i \in \mathcal{C}$.
2. A cone $\{f_i : X_i \rightarrow X\}_{i \in I}$ from the base $X_\ast$ to the vertex $X$ and, hence, an induced map $f : \text{colim}_i X_i \rightarrow X$.

Moreover, assume also that:

1. The spectral sequences $\{E_\ast X_i\}_{i \in I}$ and $E_\ast X$ converge conditionally to $\{H_i\}_{i \in I}$ and $H$, respectively.
2. The spectral sequences $\{E_\ast X_i\}_{i \in I}$ are uniformly bounded on the right.
3. The map $E_r(f) : \text{colim}_i E_r(X_i) \rightarrow E_r(X)$ is an isomorphism for some $r \geq 0$.

Then the map $H(f) : \text{colim}_i H_i \rightarrow H$ is an isomorphism too.

Proof. See \cite{Mit}, Proposition 3.3. \qed
Proof of Theorem 5.2.3. The first filtration of a double complex \( K \in C^{**}(R) \), \( F^p(\text{Tot}^\Pi K)^n = \prod_{s \geq p} K^{s,n-s} \) gives us a conditionally convergent spectral sequence

\[ E_2^{pq}(\text{Tot}^\Pi K) = H_p^p H_q^R(K) \implies H^{p+q}(\text{Tot}^\Pi K), \quad p \geq 0. \]

By Theorem 4.3.2, to prove that for any sheaf \( \mathcal{F} \in \text{Sh}(\mathcal{X}, C^*(R)) \) it holds that \( \rho_{\mathcal{F}} : \mathcal{F} \to \mathbb{H}_{\mathcal{X}}(\mathcal{F}) \) is a CE-fibrant model we may equivalently show that the canonical morphism

\[ \theta_{\mathcal{F}}(x) : \underbrace{\lim_{(U,u) \in \text{Nbh}(x)}}_{\to} \text{Tot}^\Pi (G^* \mathcal{F})(U) \to \text{Tot}^\Pi \lim_{(U,u) \in \text{Nbh}(x)} (G^* \mathcal{F}) \]

is a quis of \( C^*(R) \) for any sheaf \( \mathcal{F} \) and any point \( x \) in the set of enough points \( X \). These colimits may be computed using the neighbourhoods \( (U, u) \) in the system of neighbourhoods \( \Lambda \) that exists by assumption.

Therefore, we have an object \( \text{Tot}^\Pi x^*(G^* \mathcal{F}) \in C^*(R) \), a filtered system \( \{ \text{Tot}^\Pi (G^* \mathcal{F})(U) \}_{(U,u)} \), where \( (U, u) \) runs over all neighbourhoods of \( x \) in \( \Lambda \) and the induced map \( \theta_{\mathcal{F}}(x) \).

Let us verify the hypotheses of the Colimit Lemma: the spectral sequences

\[ E_2^{pq}(U) = H_p^p H_q^R((G^* \mathcal{F})(U)) \implies H^{p+q}(\text{Tot}^\Pi (G^* \mathcal{F})(U)), \quad p \geq 0 \]

and

\[ E_2^{pq}(x) = H_p^p H_q^R((G^* \mathcal{F})_x) \implies H^{p+q}(\text{Tot}^\Pi ((G^* \mathcal{F})_x)), \quad p \geq 0 \]

converge conditionally.

To compute \( E_2^{pq}(U) \) we use that \( T : \text{Sh}(\mathcal{X}, C^*(R)) \to \text{Sh}(\mathcal{X}, C^*(R)) \) commutes with cohomology in \( \text{Sh}(\mathcal{X}, C^*(R)) \). At the presheaf level, clearly \( H^*(T(F)) = T(H^*(F)) \) for any presheaf \( F \), because cohomology in \( C^*(R) \) commutes with products and filtered colimits. Since the stalks of a presheaf \( G \) are isomorphic to the ones of its associated sheaf \( G^a \), then \( T(G) = T(G^a) \). Hence, if \( \mathcal{F} \) is a sheaf

\[ T(\mathcal{H}^* \mathcal{F}) = T(\mathcal{H}^*(\mathcal{F})) = T(\mathcal{H}^* \mathcal{F}) = \mathcal{H}^*(T \mathcal{F}) \]

In particular \( H^*(T \mathcal{F}) = T(\mathcal{H}^* \mathcal{F}) \) is a sheaf, so it agrees with its associated sheaf. Therefore \( \mathcal{H}^*(T \mathcal{F}) = H^*(T \mathcal{F}) = T(\mathcal{H}^* \mathcal{F}) \), and \( \mathcal{H}^*(G^* \mathcal{F}) = G^*(\mathcal{H}^* \mathcal{F}) \). We then have, for all \( p > d \),

\[ E_2^{pq}(U) = H_p^p H_q^R(G^* \mathcal{F})(U) = H^p(\Gamma(U, G^a \mathcal{H}^d \mathcal{F})) = H^p(\Gamma(U, G^a \mathcal{H}^d \mathcal{F})) = H^p(U, \mathcal{H}^d \mathcal{F}) = 0 \]

because of the finite cohomological dimension assumption. Finally, already for \( r = 0 \), we have an isomorphism

\[ \colim E_0^{pq}(U) = \colim G^p \mathcal{F}^q(U) = (G^p \mathcal{F}^q)_x = E_0^{pq}(x). \]

Hence the Colimit Lemma tells us that

\[ H^n(\text{Tot}^\Pi (G^* \mathcal{F}))_x \to H^n \text{Tot}^\Pi ((G^* \mathcal{F})_x) \]

is an isomorphism for all \( n \).

\[ \square \]

5.3. Sheaves of fibrant simplicial sets.
5.3.1. Let $\mathcal{D} = s\mathbf{S}_f$ with the descent structure of [2.111]. As in the case of unbounded complexes, the simple functor may not commute weakly with stalks. Again, for this to hold we must either restrict to simplicial sets with vanishing higher homotopy groups, or impose some finiteness assumption on the site $\mathcal{X}$. Here we study the second alternative, showing that $\rho_{\mathcal{F}}: \mathcal{F} \to \mathbb{H}_{\mathcal{X}}(\mathcal{F})$ is a CE-fibrant model for each $\mathcal{F}$ in $(\mathbf{Sh}(\mathcal{X}, s\mathbf{S}_f), \mathcal{W}, \mathcal{S})$ if and only if $\mathcal{X}$ is a site of finite type in the sense of [MV].

5.3.2. By a theorem of Joyal, the category $\mathbf{Sh}(\mathcal{X}, s\mathbf{S})$ possesses a simplicial model category structure in which all objects are cofibrant and the weak equivalences are the local equivalences [Jal]. The fibrant objects in this model structure are then defined through a lifting property, and they are objectwise fibrant simplicial sets. Therefore, there is a fibrant replacement functor $Ex$ that takes a simplicial sheaf to a fibrant one, in particular $Ex(\mathcal{F}) \in \mathbf{Sh}(\mathcal{X}, s\mathbf{S}_f)$.

Given a simplicial sheaf $\mathcal{F} \in \mathbf{Sh}(\mathcal{X}, s\mathbf{S})$ and $n \geq 0$, let $\tilde{P}^{(n)} \mathcal{F}$ be the simplicial sheaf associated to the presheaf $U \mapsto P^{(n)} \mathcal{F}(U) = \text{Im}\{\mathcal{F}(U) \to \cosk_n \mathcal{F}(U)\}$. It is equipped with natural maps $\mathcal{F} \to \tilde{P}^{(n)} \mathcal{F}$ and $\tilde{P}^{(n+1)} \mathcal{F} \to \tilde{P}^{(n)} \mathcal{F}$.

If the stalks of $\mathcal{F}$ are fibrant simplicial sets, the tower $\{x^*\tilde{P}^{(n)} \mathcal{F} = P^{(n)} x^* \mathcal{F}\}$ is precisely the Moore-Postnikov tower of $x^* \mathcal{F}$. In this case the natural map $x^* \mathcal{F} \simeq \lim_{n \geq 0} x^* \tilde{P}^{(n)} \mathcal{F} \to \text{holim}_{n \geq 0} x^* \tilde{P}^{(n)} \mathcal{F}$ is a weak equivalence.

**Definition 5.3.1.** [MV] A site $\mathcal{X}$ is of finite type if for each simplicial sheaf $\mathcal{F} \in \mathbf{Sh}(\mathcal{X}, s\mathbf{S})$, the natural morphism $\mathcal{F} \to \text{holim}_{n \geq 0} Ex(\tilde{P}^{(n)} \mathcal{F})$ is a local equivalence.

**Theorem 5.3.2.** The descent category $s\mathbf{S}_f$ is compatible with site $\mathcal{X}$ if and only if $\mathcal{X}$ is of finite type. In this case, properties [5] hold for $(\mathbf{Sh}(\mathcal{X}, s\mathbf{S}_f), \mathcal{S}, \mathcal{W})$.

That is, for the site $\mathcal{X}$, either both of the natural morphisms

$$\text{holim}_{n \geq 0} Ex(\tilde{P}^{(n)} \mathcal{F}) \leftarrow \mathcal{F} \leftarrow \text{holim}_{p \geq 0} G^p \mathcal{F}$$

are local equivalences for all sheaves $\mathcal{F}$ simultaneously, or neither one is.

**Proof.** Assume that $\mathcal{X}$ is a site of finite type. If $\mathcal{F} \in \mathbf{Sh}(\mathcal{X}, s\mathbf{S}_f)$, since fibrant objects are preserved by filtered colimits, it holds that $\mathcal{F}$ is locally fibrant. Then, [MV 1.65] ensures that $\rho_{\mathcal{F}}: \mathcal{F} \to \mathbb{H}_{\mathcal{X}}(\mathcal{F})$ is a local equivalence, so $\rho_{\mathcal{F}}: \mathcal{F} \to \mathbb{H}_{\mathcal{X}}(\mathcal{F})$ is a CE-fibrant model of $\mathcal{F}$.

Conversely, assume that $\rho_{\mathcal{F}}: \mathcal{F} \to \mathbb{H}_{\mathcal{X}}(\mathcal{F})$ is a local equivalence for each $\mathcal{F} \in \mathbf{Sh}(\mathcal{X}, s\mathbf{S}_f)$. We must prove that $\mathcal{F} \to \text{holim}_{n \geq 0} Ex(\tilde{P}^{(n)} \mathcal{F})$ is a weak equivalence for each simplicial sheaf $\mathcal{F}$. As in the proof of [MV 1.37], taking a suitable replacement of the tower $\{\tilde{P}^{(n)} \mathcal{F}\}$ we can assume that $\mathcal{F}$ is a fibrant sheaf, and in particular that $\mathcal{F} \in \mathbf{Sh}(\mathcal{X}, s\mathbf{S}_f)$. 


Secondly, to compute $\lim_{n \geq 0} Ex(P^{(n)}F)$ we may take any choice of fibrant replacement $Ex$. It follows from [MV, 1.59] that $\mathbb{H}_X(P^{(n)}F)$ is a fibrant simplicial sheaf for all $n$, so we may equivalently show that $F \to \lim_{n \geq 0} \mathbb{H}_X(P^{(n)}F)$ is a weak equivalence.

Again, if $F$ is a simplicial presheaf then $T(F)$, defined in the same way, is a sheaf and agrees with $T(F^n)$, where $F^n$ is the sheaf associated to the presheaf $F$. Then $\mathbb{H}_X(F)$ is isomorphic to $\mathbb{H}_X(P^{(n)}F)$. In particular, $\mathbb{H}_X(P^{(n)}F) = \mathbb{H}_X(P^{(n)}F)$.

Recall that $P^{(n)}F(U) = (\text{Im} \phi_n)(U)$ where $\phi_n : F \to \text{cosk}_n F$, and that $\text{cosk}_n$ is given degreewise by a finite limit. Then, since the Godement resolution and the simple functor commute with finite limits and images, we have $\mathbb{H}_X(P^{(n)}F) = \mathbb{H}_X(P^{(n)}F) = P^{(n)}\mathbb{H}_X(F)$. In particular, $P^{(n)}\mathbb{H}_X(F)$ is already a sheaf, so

$$\mathbb{H}_X(P^{(n)}F) = P^{(n)}\mathbb{H}_X(F) = \tilde{P}^{(n)}\mathbb{H}_X(F).$$

As $F(U)$ is a fibrant simplicial set, so is $\mathbb{H}_X(F)(U)$. Consequently $\{P^{(n)}\mathbb{H}_X(F)(U)\}_n$ is the Moore-Postnikov tower of $\mathbb{H}_X(F)(U)$ and $\mathbb{H}_X(F)(U) \to \lim_{n \geq 0} P^{(n)}\mathbb{H}_X(F)(U)$ is a weak equivalence of simplicial sets. But then

$$\mathbb{H}_X(F) \to \lim_{n \geq 0} \tilde{P}^{(n)}\mathbb{H}_X(F) = \lim_{n \geq 0} \mathbb{H}_X(\tilde{P}^{(n)}F)$$

is a global equivalence. Composing with the local equivalence $F \to \mathbb{H}_X(F)$ we conclude that $F \to \lim_{n \geq 0} \mathbb{H}_X(\tilde{P}^{(n)}F)$ is a local equivalence as required.

Finally, let us remark that finite cohomological dimension implies finite type:

**Proposition 5.3.3.** Every finite cohomological dimension site is of finite type.

**Proof.** This may be seen as in [MV], Theorem 1.37, or using a spectral sequence argument such as the one given below for spectra, and previously for unbounded complexes. □

### 5.4. Sheaves of fibrant spectra.

#### 5.4.1. Let $D = Sp_f$ with the descent structure of [2.11]. Then Theorem 4.3.2 applies and gives a CE structure to $Sh(\mathcal{X}, Sp_f)$, if $\mathcal{X}$ is of finite cohomological dimension.

**Theorem 5.4.1.** The descent category $Sp_f$ is compatible with any finite cohomological dimension site $\mathcal{X}$. In this case, properties 5 hold for $(Sh(\mathcal{X}, Sp_f), S, W)$.

**Proof.** Let $X^* \in \Delta Sp_f$ be a cosimplicial object of $Sp_f$. According to [Bous], 2.9 (see also [Hir], remark 18.1.11), we have a conditionally convergent spectral sequence (Boa)

$$E_2^{pq}(X^*) = \pi^p\pi_q(X^*) \implies \pi_{q-p}(\lim_{n \geq 0} X^*), \quad p \geq q \geq 0.$$

As in the case of unbounded complexes, we want to show that the canonical morphism

$$\theta_F(x) : \lim_{U \in \text{Nbh}(x)} \lim_{n \geq 0} (G^p F)(U) \to (G^p \lim_{n \geq 0} F)(U)$$

...
is a weak equivalence of $\mathbf{Sp}_f$ for any sheaf $\mathcal{F}$ and any point $x \in X$ and the colimit may be computed using the neighbourhoods $(U, u)$ in the system $\Lambda$ which exists by hypothesis. Let us verify the hypotheses of the Colimit Lemma: the spectral sequences $E_2^{pq}(U) = \pi^p\pi_q((\mathcal{G}^\bullet\mathcal{F})(U)) \implies \pi_{q-p}(\text{holim} (\mathcal{G}^\bullet\mathcal{F})(U)), \ p \geq q \geq 0$ and $E_2^{pq}(x) = \pi^p\pi_q((\mathcal{G}^\bullet\mathcal{F})_x) \implies \pi_{q-p}(\text{holim} (\mathcal{G}^\bullet\mathcal{F})_x), \ p \geq q \geq 0$ converge conditionally. Moreover, the Godement cosimplicial resolution commutes with homotopy groups (argue as in the case of cohomology, or see [Th], page 452, formula (1.26)). Hence, for all $p > d$,

$$E_2^{pq}(U) = \pi^p\pi_q((\mathcal{G}^\bullet\mathcal{F})(U)) = H^p(\Gamma(U, \pi_q(\mathcal{G}^\bullet\mathcal{F}))) = H^p(\Gamma(U, \mathcal{G}^\bullet\pi_q(\mathcal{F}))) = H^p(U, \pi_q(\mathcal{F})) = 0.$$ 

Finally, for $r = 2$, we have an isomorphism

$$\text{colim} E_2^{pq}(U) = \text{colim} \pi^p\pi_q((\mathcal{G}^\bullet\mathcal{F})(U)) = H^p(\text{colim} \Gamma(U, \pi_q(\mathcal{G}^\bullet\mathcal{F})))$$

$$= \pi^p(\pi_q(\mathcal{G}^\bullet\mathcal{F})_x) = \pi^p\pi_q((\mathcal{G}^\bullet\mathcal{F})_x) = E_2^{pq}(x).$$

Hence, the Colimit Lemma tells us that

$$\pi_{q-p}(\text{holim} (\mathcal{G}^\bullet\mathcal{F})_x) \longrightarrow \pi_{q-p}(\text{holim} (\mathcal{G}^\bullet\mathcal{F})_x)$$

is an isomorphism for all $p \geq q \geq 0$.

5.5. Sheaves of filtered complexes.

5.5.1. As happens in the case of bounded complexes, the simple functors described in example 2.1.12 clearly commutes with the formation of stalks. Hence

**Theorem 5.5.1.** Assume that $\mathcal{A}$ is an abelian category satisfying $(AB4)^*$ and $(AB5)$. Then, the descent categories $(\mathcal{F}\mathcal{C}_{\geq b}(\mathcal{A}), E_r)$ are compatible with any site $\mathcal{X}$. In particular, properties 3 hold for $(\mathbf{Sh}(\mathcal{X}, \mathcal{F}\mathcal{C}_{\geq b}(\mathcal{A})), \mathcal{S}_r, \mathcal{W}_r)$.

Note that $\mathbf{Sh}(\mathcal{X}, \mathcal{F}\mathcal{C}_{\geq b}(\mathcal{A}))$ agrees with the category of filtered complexes of sheaves on $\mathcal{A}$. For $r = 0$, $\mathbf{Sh}(\mathcal{X}, \mathcal{F}\mathcal{C}_{\geq b}(\mathcal{A}))[[E_0^{-1}]]$ is then $\mathcal{F}\mathcal{D}_{\geq b}(\mathbf{Sh}(\mathcal{X}, \mathcal{A}))$, the filtered derived category of sheaves on $\mathcal{A}$. That is, for $\mathcal{A} = R$-modules, our sheaf cohomology agrees with classic filtered sheaf cohomology.

**References**

[Ba] A. Banerjee, *Tensor structures on smooth motives*, Journal of K-theory, 9 (2012), 57–101.

[Boa] J.M. Boardman, *Conditionally convergent spectral sequences*, Cont.Math. 239 (1999), 49–84.

[Bour] Bourbaki, *Algebra I. Chapters 1-3*, Springer (1989).

[Bous] A.K. Bousfield, *Cosimplicial resolutions and homotopy spectral sequences in model categories*, Geometry & Topology, 7 (2003), 1001–1053.

[BK] A.K. Bousfield, D.M. Kan *Homotopy limits, completions and localizations* Lecture Notes in Math. 304, (1972).

[B] Brown, K.S., *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Trans. Amer. Math. Soc, 186 (1973), 419–458.

[C1] J. Ciri, *Cofibrant models of diagrams: mixed Hodge structures in rational homotopy*, Available at [arXiv:1307.4968](http://arXiv.org/)

[C2] J. Ciri, *Homotopy Theory of Mixed Hodge Complexes*, Available at [arXiv:1304.6236](http://arXiv.org/)

[CE] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, (1956).
[DP] A. Dold, D. Puppe, Homologie Nicht-Additiver Funktoren. Anwendungen, Ann. Inst. Fourier, 11 (1961), 201-312.
[GL] T. Geisser, M. Levine The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky, Journal fr die reine und angewandte Mathematik (Crelles Journal), 530 (2001), 55-103.
[GS] H. Gillet, C. Soulé, Filtrations on Higher Algebraic K-theory, in Algebraic K-Theory, Proc. of Sym. in Pure Math., 67, AMS (1999), 41-88.
[Go] R. Godement, Théorie des faisceaux, Actualités sci. et ind. 1252, Hermann (1973).
[GN] F. Guillén, V. Navarro Aznar, Un critère d'extension des foncteurs définis sur les schémas lisses, Publ. Math. IHES, 95 (2002), 1-91.
[GNPR1] F. Guillén Santos, V. Navarro, P. Pascual, Agustí Roig, A Cartan-Eilenberg approach to homotopical algebra, J. Pure and Appl. Algebra 214 (2010), 140-164.
[GNPR2] F. Guillén, V. Navarro, P. Pascual, Agustí Roig, The differentiable chain functor is not homotopy equivalent to the continuous chain functor, Topol. App. 156 (2009), 65-680.
[Har] R. Hartshorne, Algebraic geometry, Springer GTM 52 (1977).
[Hir] P.S. Hirschhorn, Model Categories and Their Localizations, Math. Surveys and Monographs, 99, Amer. Math. Soc., Providence (2002).
[Ja1] J.F. Jardine, Simplicial objects in a Grothendieck topos, Contemp. Math. 551 (1986), 193-239.
[Ja3] J.F. Jardine, Generalised sheaf cohomology theories, in Axiomatic, Enriched and Motivic Homotopy Theory, NATO Science Series II 131 (2004), 29-68.
[KS1] M. Kashiwara, P. Schapira, Sheaves on manifolds, Grundlehren der Mathematischen Wissenschaften 292 Springer (1990).
[McL] S. Mac Lane, Categories for the working mathematician (second edition), Springer GTM 5, (1998).
[McLM] S. Mac Lane, I. Moerdijk Sheaves in Geometry and Logic: A First Introduction to Topos Theory, Springer (1992)
[Mit] S.A. Mitchell, Hypercohomology spectra and Thomason's descent theorem, in Algebraic K-Theory, Fields Institute Communications, (1997), 221-278.
[MV] F. Morel and V. Voevodsky, $\Lambda^1$-homotopy theory of schemes, Pub. Math. I.H.E.S. 90 (1999), 45–143.
[N] V. Navarro Aznar Sur la théorie de Hodge-Deligne, Inv. Math. 90 (1987), 11–76.
[P] P. Pascual, Some remarks on Cartan-Eilenberg categories, Collect.Math. 63 (2012), 203-216.
[Q] D. Quillen, Homotopical algebra, Springer LNM 43, (1967).
[Rod1] B. Rodríguez González, Simplicial descent categories, J. Pure and Appl. Algebra 216 no. 4 (2012), 775-788.
[Rod2] B. Rodríguez González, Realizable homotopy colimits. Available at arXiv:1104.0640
[SdS] F. Sancho de Salas, P. Sancho de Salas, A direct proof of the theorem on formal functions, Proc. Amer. Math. Soc. 137 (2009), 4083–4088.
[SGA4] Séminaire de Géometrie Algébrique SGA4 Théorie des topos et cohomologie étale des schémas, Springer LNM 269 (1972)
[Sp] N. Spaltenstein, Resolutions of unbounded complexes, Compositio Math. 65, (1988), p. 121–154.
[Th] R. W. Thomason Algebric K-theory and etale cohomology, Ann. Sci. ENS 18 (1985), 437–552.
[We] C. A. Weibel, Cyclic Homology of Schemes, Proc. AMS 124 (1996), 1655–1662.

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