Supersymmetric 3D gravity with torsion: asymptotic symmetries and black hole stability

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Abstract. We show that $N = 1 + 1$ supersymmetric extension of 3D gravity with torsion, with suitable boundary conditions, has the asymptotic superconformal symmetry. The existence of exact supersymmetries implies the stability conditions for the black hole solutions.

1. Introduction

Since the basic understanding of quantum gravity is still missing, the researchers are naturally oriented toward models that are technically simpler but share the same conceptual features with general relativity (GR). In this sense, the study of three-dimensional (3D) gravity within the realm of Riemannian geometry of spacetime was particularly fruitful [1, 2, 3, 4, 5]. However, 3D gravity can also be used to explore a more general, gauge-theoretic conception of gravity, based on Riemann-Cartan geometry of spacetime—the geometry that is characterized by both the curvature and the torsion [6]. Nowadays, fifteen years after a general topological model for 3D gravity with torsion was proposed by Mielke and Baekler [7], it is clear that this approach has led to a number of respectable results [8, 9, 10, 11, 12, 13, 14].

It was shown recently that $N = 1 + 1$ supersymmetric extension of 3D gravity with torsion, with a suitable generalization of the anti-de Sitter (AdS) asymptotic conditions, possesses superconformal symmetry at spatial infinities [13]. In this paper, we use this result to explore exact supersymmetries and the stability properties of the black hole with torsion. Our results represent a natural generalization of those obtained for the BTZ black hole in [15].

The paper is organized in the following way. In section 2, we review some basic aspects of the topological 3D gravity with torsion. Section 3 is devoted to exploring the supersymmetric extension of the Mielke-Baekler model with $N = 1 + 1$ gravitini and its asymptotic structure. In section 4, we explore exact supersymmetries of the AdS and black hole configurations and combine them with the asymptotic superconformal algebra, with the conclusion that the zero-energy black hole can be interpreted as the ground state of the Ramond sector, while the AdS solution is the ground state of the Neveu-Schwartz sector, with periodic and antiperiodic boundary conditions, respectively. Finally, section 5 is devoted to concluding remarks.

We use the same conventions as in [13]: the Latin indices $(i, j, k, \ldots)$ refer to the local orthonormal frame, the Greek indices $(\mu, \nu, \rho, \ldots)$ refer to the coordinate frame, and both run over $0, 1, 2$; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric object $\varepsilon^{ijk}$ is normalized by $\varepsilon^{012} = +1$; gamma matrices are pure imaginary and Majorana spinors are real.
2. 3D gravity with torsion

Theory of gravity with torsion can be naturally described as Poincaré gauge theory (PGT), with an underlying spacetime structure corresponding to Riemann-Cartan geometry [6]. Basic gravitational variables in PGT are the triad field $b^i$ and the Lorentz connection $A^{ij} = -A^{ji}$ (1-forms), and the corresponding fields strengths are the torsion $T^i$ and the curvature $R^{ij}$ (2-forms). In 3D, we use the simplified notation $A^{ij} =: -\varepsilon^{ij} k \omega^k$, $R^{ij} := -\varepsilon^{ij} k R^k$, so that

$$T^i = db^i + \varepsilon^{ijk} \omega^j \wedge b^k, \quad R^i = d\omega^i + \frac{1}{2} \varepsilon^{ijk} \omega^j \wedge \omega^k.$$ (2.1)

Gauge symmetries of the theory are local translations and local Lorentz rotations. The covariant derivative $\nabla \equiv \nabla(\omega)$ acts on a general tangent-frame spinor/tensor in accordance with its spinorial/tensorial structure; when $X$ is a form, $\nabla X := \nabla \wedge X$.

The metric structure of PGT is defined by $g = \eta_{ij} b^i \otimes b^j$. Metric and connection are related to each other by the metricity condition, $\nabla g = 0$, which defines Riemann-Cartan geometry of spacetime. In PGT, we have a useful identity

$$\omega^i \equiv \tilde{\omega}^i + K^i,$$ (2.2)

where $\tilde{\omega}^i$ is the Levi-Civita connection, and $K^i$ is the contortion defined by $T^i = \varepsilon_{mn}^{\ i} K^m \wedge b^n$.

Generalized dynamics. The topological Mielke-Baekler model for 3D gravity [7], represents a natural generalization of general relativity (GR) with a cosmological constant:

$$I_0 = 2a \int b^i R_i - \frac{\Lambda}{3} \int \varepsilon_{ijk} b^i b^j b^k + \alpha_3 \int \left( \omega^i d\omega_i + \frac{1}{3} \varepsilon^{ijk} \omega^j \omega^k \right) + \alpha_4 \int b^i T_i,$$ (2.3)

where the wedge product sign $\wedge$ is omitted for simplicity. The first term with $a = 1/16\pi G$ is the usual Einstein-Cartan action, the second term is a cosmological term, the third term is the Chern-Simons action for the Lorentz connection, and the last term is a torsion counterpart of the first one.

The vacuum field equations, in the sector $\alpha_3 \alpha_4 - a^2 \neq 0$, take the simple form

$$2T^i = p \varepsilon^{ijk} b^j b^k, \quad 2R^i = q \varepsilon^{ijk} b^j b^k,$$ (2.4)

where

$$p := \frac{\alpha_3 \Lambda + \alpha_4 a}{\alpha_3 \alpha_4 - a^2}, \quad q := \frac{(\alpha_4)^2 + a \Lambda}{\alpha_3 \alpha_4 - a^2}.$$ (2.5)

Thus, the vacuum solutions are characterized by constant torsion and constant curvature.

In Riemann-Cartan spacetime, one can use the identity (2.2) to express the curvature in terms of its Riemannian piece $\tilde{R}^{ij} = R^{ij}(\tilde{\omega})$ and the contortion. The resulting identity, combined with the equations of motion (2.4), leads to the on-shell relation

$$\tilde{R}^{ij} = -\Lambda_{\text{eff}} b^i \wedge b^j, \quad \Lambda_{\text{eff}} := q - \frac{1}{4} p^2,$$ (2.5)

where $\Lambda_{\text{eff}}$ is the effective cosmological constant. The form of $\tilde{R}^{ij}$ implies that our spacetime has maximally symmetric metric.

The black hole with torsion. For negative $\Lambda_{\text{eff}}$ (AdS sector), the Mielke-Baekler model has an exact vacuum solution, the black hole with torsion [8, 9], which is a natural generalization

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of the well-known BTZ black hole [4]. In static coordinates $x^\mu = (t, r, \varphi)$ (with $0 \leq \varphi < 2\pi$), the BTZ metric is given as

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 (d\varphi + N_\varphi dt)^2,$$

$$N^2 = \left( -8Gm + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2} \right), \quad N_\varphi = \frac{4GJ}{r^2}. \quad (2.6)$$

For the black hole with torsion, the triad field is taken as

$$b^0 = N dt, \quad b^1 = N^{-1} dr, \quad b^2 = r (d\varphi + N_\varphi dt), \quad (2.7a)$$

while the connection, in accordance with (2.2), has the form

$$\omega^i = \bar{\omega}^i + \frac{p}{2} b^i, \quad (2.7b)$$

where the Riemannian connection $\bar{\omega}^i$ reads:

$$\bar{\omega}^0 = -N d\varphi, \quad \bar{\omega}^1 = N^{-1} N_\varphi dr, \quad \bar{\omega}^2 = -\frac{r dt}{\ell} - r N_\varphi d\varphi. \quad (2.7c)$$

Energy and angular momentum of the black hole are given by the expressions

$$E = m + \frac{\alpha_3}{a} \left( \frac{pm}{2} - \frac{J}{\ell^2} \right), \quad M = J + \frac{\alpha_3}{a} \left( \frac{pJ}{2} - m \right). \quad (2.8)$$

Locally isometric but globally distinct AdS solution (AdS$_3$) can be formally obtained from the black hole by taking $J = 0, 8Gm = -1$.

3. Supersymmetric extension

There exists a simple locally supersymmetric extension of 3D gravity with torsion, based on the action (2.3), which includes two gravitini fields and is usually referred to as $N = 1 + 1$ AdS supergravity [2, 12, 13]. Consider the action:

$$I = I_0 - g \int (\bar{\psi} \nabla \psi - i \mu \bar{\psi} b^i \gamma_i \psi) - g' \int (\bar{\psi'} \nabla' \psi' - i \mu' \bar{\psi'} b^i \gamma_i \psi'), \quad (3.1)$$

where $\psi^I = (\psi, \psi')$ are the gravitini fields (1-forms) and $\nabla \psi^I = (d - \frac{i}{2} \omega^m \gamma_m) \psi^I$ are their covariant derivatives. The action is invariant under the following locally supersymmetry transformations with spinorial parameters $\varepsilon^I = (\varepsilon, \varepsilon')$:

$$\delta_S b^i = i \bar{\varepsilon} \gamma^i \psi + i \bar{\varepsilon'} \gamma^i \psi',$$

$$\delta_S \omega^i = -2i \mu \bar{\varepsilon} \gamma^i \psi - 2i \mu' \bar{\varepsilon'} \gamma^i \psi',$$

$$\delta_S \psi^I = 4a \left( \bar{\varepsilon} \varepsilon^I - i \mu \bar{\varepsilon} b^k \gamma_k \varepsilon^I \right), \quad (3.2)$$

provided the coupling constants $g, g'$ and $\mu^I = (\mu, \mu')$ satisfy the relations

$$2ag = a - 2\mu' \alpha_3, \quad 2ag' = a - 2\mu \alpha_3,$$

$$2\mu + \frac{p}{2} = \frac{1}{\ell}, \quad 2\mu' + \frac{p}{2} = -\frac{1}{\ell}.$$

Here, $\ell$ is the AdS radius, and for $\mu$ and $\mu'$ real and different from each other, the effective cosmological constant is negative: $\Lambda_{\text{eff}} = -(\mu - \mu')^2 = -1/\ell^2 < 0$. 


The variation of the action with respect to $b^i, \omega^i, \bar{\psi}$ and $\bar{\psi}'$ yields the supersymmetric field equations. As one can easily verify, the black hole and AdS$_3$ are exact solutions of the supersymmetric field equations with zero gravitini, $\psi = \psi' = 0$.

**Asymptotic symmetries.** For $\Lambda_{\text{eff}} < 0$, the asymptotic structure of 3D gravity with torsion is well understood [10]. It can be generalized to the supersymmetric theory (3.1) by a suitable completion of the asymptotic conditions in the fermionic sector. The asymptotic symmetry is characterized by four chiral parameters: two of them are bosonic, $T^\mp(x^\mp)$, while the other two are fermionic, $\epsilon^\mp(x^\mp)$. The asymptotic Poisson bracket algebra of the canonical generators $\hat{G}(T^\mp, \epsilon^\mp)$ is given (in the quantum-mechanical notation) by two independent copies of the super-Virasoro algebra [13]:

\[
[L_n^+, L_m^+] = (n-m)L_{n+m}^+ + \frac{c^+}{12}n^3\delta_{m+n,0},
\]

\[
[L_n^+, Q_m^+] = \left(\frac{1}{2}n-m\right)Q_{m+n}^+,
\]

\[
\{Q_n^+, Q_m^+\} = 2L_{n+m}^+ + \frac{c^+}{3}n^2\delta_{m+n,0},
\]

where $L_n^+$ and $Q_n^+$ are Fourier modes of $\hat{G}(T^\mp, \epsilon^\mp)$, and $c^+$ are classical central charges: $c^- = 12 \cdot 2\pi\ell ag, c^+ = 12 \cdot 2\pi\ell ag'$.

In the sectors with periodic (Ramond) or anti-periodic (Neveau-Schwartz) boundary conditions for fermions, index of $Q_n^+$ takes on integer or half-integer values, respectively. The reality properties of the modes are: $(L_n^+)_n = L_{-n}^+, (Q_n^+)_n = Q_{-n}$.

The eigenvalues of the bosonic operators $L_0^+$ can be expressed in terms of the energy $E$ and angular momentum $M$ of the system. For the black hole/AdS$_3$ solution, we have:

\[
L_0^+ = \frac{\ell E \pm M}{2} = (\ell m \pm J)\frac{c^+}{48\pi\ell a},
\]

4. The black hole stability

**Killing spinors.** Since the black hole and AdS$_3$ are maximally symmetric, their symmetries are locally identical, but the difference in their global structures implies completely different symmetries in the large. In the supersymmetric theory (3.1), the supersymmetry transformations that leave the black hole or AdS$_3$ configuration with $\psi = \psi' = 0$ invariant, are called exact supersymmetries. Since $\delta_S b^i = \delta_S \omega^i = 0$ for $\psi = \psi' = 0$, the spinor parameters $(\varepsilon, \varepsilon')$ of exact supersymmetries are defined solely by the requirements $\delta_S \psi = \delta_S \psi' = 0$. Expressing $\omega^i$ in terms of the Levi-Civita connection $\omega^i$ as in (2.7b), these requirements take the form

\[
\nabla \varepsilon = \frac{i}{2\ell} b^k \gamma_k \varepsilon, \quad \nabla \varepsilon' = -\frac{i}{2\ell} b^k \gamma_k \varepsilon',
\]

where $\nabla$ is Riemannian covariant derivative. Equations (4.1) are called the Killing spinor equations; they define the spinor parameters $(\varepsilon, \varepsilon')$ of the exact supersymmetries. Among their solutions, only those that are in agreement with the global properties of the black hole/AdS$_3$ are acceptable.

One can show that AdS$_3$ possesses four Killing spinors, two for $\varepsilon$ and two for $\varepsilon'$:

\[
\varepsilon = \left(\sqrt{\frac{N_+ + 1}{2}} \sigma^3 - \sqrt{\frac{N_+ - 1}{2}} \right) \left(\frac{\cos \frac{x^-}{2} + i\sigma^2 \sin \frac{x^-}{2}}{2}\right) \zeta,
\]

\[
\varepsilon' = \left(\sqrt{\frac{N_+ + 1}{2}} \sigma^3 + \sqrt{\frac{N_+ - 1}{2}} \right) \left(\frac{\cos \frac{x^+}{2} + i\sigma^2 \sin \frac{x^+}{2}}{2}\right) \zeta'.
\]
Here, \( N_* = \sqrt{1 + r^2 / \ell^2} \) is the value of \( N \) at AdS, \( x^\pm = t/\ell \pm \varphi \) and \( \zeta, \zeta' \) are constant spinors.

The black hole possesses locally the same number of Killing spinors as AdS. However, since the black hole is obtained from AdS by a process of identifications, spinors on the black hole manifold, compatible with these identifications, can be either periodic or anti-periodic [4, 15]. The form of this restriction depends on the values of the parameters \( m \) and \( J \).

1. For a generic black hole with \( |J| < m \ell \), there are no periodic/anti-periodic solutions of (4.1), and no Killing spinors.

2. Consider now the extreme black hole with \( |J| = m \ell \) and \( m \neq 0 \).
   (a) For \( J = m \ell \), the periodic/anti-periodic solution for \( \varepsilon' \) does not exist, while the solution for \( \varepsilon \), compatible with the periodicity in \( \varphi \), has the form \((u = r/\ell - 4Gm\ell/r)\):
   \[
   \varepsilon = \left( \left| u \right|^{1/2} + \left| u \right|^{-1/2} \right) - \text{sgn} \left( \left| u \right|^{1/2} - \left| u \right|^{-1/2} \right) 3 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) .
   \] (4.3a)
   (b) Similarly, in the case \( J = -m \ell \), there are no periodic/anti-periodic solutions for \( \varepsilon \), while globally acceptable solution for \( \varepsilon' \) takes the form
   \[
   \varepsilon' = \left( \left| u \right|^{1/2} + \left| u \right|^{-1/2} \right) + \text{sgn} \left( \left| u \right|^{1/2} - \left| u \right|^{-1/2} \right) 3 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) .
   \] (4.3b)

3. The case \( m = J = 0 \) corresponds to the zero-energy black hole, for which both \( E \) and \( M \) vanish. The related Killing spinors can be obtained from the extreme black hole solutions (4.3) in the limit \( m \to 0 \). The periodicity requirement implies the existence of two Killing spinors:
   \[
   \varepsilon \sim \sqrt{\frac{\ell}{\pi}} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) , \quad \varepsilon' \sim \sqrt{\frac{\ell}{\pi}} \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
   \] (4.4)

Thus, a generic black hole has no Killing spinors, the extreme black hole has only one, while the zero-energy black hole has two.

The existence of Killing spinors, combined with the asymptotic supersymmetry algebra, has serious implications on the stability of the black hole/AdS3.

**Ramond sector.** In the Ramond sector (with periodic boundary conditions), the subalgebra of (3.1) with vanishing central charge, generated by \((L_0^+, Q_0^-)\), yields:
   \[
   L_0^+ = Q_0^- Q_0^- = Q_0^- (Q_0^-)^\dagger \geq 0 \.
   \] (4.5)

These relations are equivalent to \( |M| \leq E \), which implies \( E \geq 0 \).

The zero-energy black hole has two Killing spinors, which are the parameters of two exact supersymmetries, generated by the action of \( Q_0^- \). Moreover, this configuration saturates the bounds (4.5), which means that it has the lowest values of the conserved charges \( L_0^+ \), compared to any other black hole configuration. Consequently, the zero-energy black hole can be naturally interpreted as the black hole ground state (the ground state of the Ramond sector).

One should note that for an extremal black hole, only one of the bounds in (4.5) is saturated, while for a generic black hole, both \( L_0^- \) and \( L_0^+ \) are strictly positive.

**Neveu-Schwarz sector.** The AdS solution belongs to the Neveu-Schwarz sector (anti-periodic boundary conditions), with (the value of \( L_0^+ \)) = \(-c^\mp /24\). If one makes the shift \( L_0^+ \to L_0^+ - c^\mp /24 \), the value of the new \( L_0^+ \) on AdS3 vanish. The osp(1|2) \( \oplus \) osp(1|2) subalgebra with vanishing charge is generated by \((L_{\pm 1}, L_0, Q_{\pm 1/2})^\mp\), and it implies
   \[
   2L_0^+ = Q_{1/2}^+ Q_{-1/2}^+ + Q_{-1/2}^+ Q_{1/2}^+ = Q_{1/2}^+ (Q_{1/2}^+)^\dagger + Q_{-1/2}^+ (Q_{-1/2}^+)^\dagger \geq 0 \.
   \] (4.6)

The AdS solution has four Killing spinors, which are the parameters of four exact supersymmetries, generated by the action of \((Q_{1/2}^+, Q_{-1/2}^+)\), and it saturates the bounds (4.6). Thus, we are led to interpret AdS3 as the ground state of the Neveu-Schwarz sector.
5. Concluding remarks
In conclusion, our results can be summarized as follows.

- The asymptotic symmetry of $N = 1+1$ supersymmetric extension of 3D gravity with torsion is described by two independent super-Virasoro algebras with different central charges.
- The zero-energy black hole and AdS$_3$ can be naturally interpreted as the ground states of the sectors with periodic and anti-periodic boundary conditions, respectively.

These results are a natural generalization of those found earlier in Riemannian GR [15].

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