Kosterlitz-Thouless phase transition
in the two dimensional linear $\sigma$-model

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**Abstract:**
We investigate the $O(N)$ symmetric linear $\sigma$-model in two dimensions by means of an exact nonperturbative evolution equation. The perturbative infrared divergences are absent in this formulation. We use a simple approximative solution of the flow equation which corresponds to a derivative expansion for the effective action. For $N = 2$ this gives a good picture of the Kosterlitz-Thouless phase transition.
In dimensions smaller than four the \(N\)-component linear \(\sigma\)-model is not directly accessible to perturbation theory. The obstacle arises from infrared divergences due to massless modes - either the Goldstone modes in the regime with spontaneous symmetry breaking for \(N \geq 2\) or the massless mode at the phase transition. One usually resorts to an extrapolation of the \(4 - \epsilon\) expansion around the four dimensional model, or to the \(2 + \epsilon\) expansion around the corresponding two dimensional nonlinear \(\sigma\)-model. We present here an approach working in arbitrary dimension \(d\) which can cope with the infrared problems. It is based on the concept of the average action \(\Gamma_k\) which is the continuum analogue of the block spin action on the lattice \(\mathbb{Z}^d\). The computation of \(\Gamma_k\) includes all fluctuations with momenta \(q^2\) larger than the infrared cutoff \(k^2\). For \(k \to 0\) one recovers the effective action, i.e. the generating functional for the one particle irreducible Green functions or the free energy. The dependence of \(\Gamma_k\) on the scale \(k\) is determined by an exact evolution equation \(\frac{\partial}{\partial t} \Gamma_k = \frac{1}{2} Tr \left\{ (\Gamma_k^{(2)} + R_k)^{-1} \frac{\partial}{\partial t} R_k \right\}\) (1).

(In a Fourier basis the trace reads \(Tr = \sum_{a=1}^{N} \int \frac{d^d q}{(2\pi)^d}\) and \(t = \ln(k/\Lambda)\).) In distinction to earlier versions of exact renormalization group equations \(\frac{\partial}{\partial t} \Gamma_k\) is characterized by its close resemblance to perturbation theory: The main differences as compared to a one loop expression are the appearance of the exact inverse propagator \(\Gamma_k^{(2)}\) (the second functional variation of \(\Gamma_k\) with respect to the \(N\)-component scalar fields \(\phi_a\)) and the infrared cutoff \(R_k\).

With \(R_k(q^2) = Z_k q^2 \exp(-q^2/k^2) / (1 - \exp(-q^2/k^2))\) (2) (where \(Z_k\) is an appropriate wave function renormalization constant specified below) we observe that the momentum integral in (1) is both infrared and ultraviolet finite. The short distance physics has to be specified by the “initial value” \(\Gamma_\Lambda\) (the “classical action”) at some high momentum scale \(\Lambda\). A solution of the flow equation (1) allows to extrapolate from the classical action \(\Gamma_\Lambda\) to the effective action \(\Gamma_0\) and therefore constitutes a solution of the model.

Our nonperturbative evolution equation remains a complicated nonlinear functional differential equation which cannot be solved exactly. One needs approximate nonperturbative solutions which correspond to a truncation of the general form of \(\Gamma_k\) and therefore reduce the problem to a manageable number of degrees of freedom. We consider here the \(O(N)\) symmetric linear \(\sigma\)-model with a truncation corresponding to the lowest terms in a derivative expansion of \(\Gamma_k\), i.e.

\[
\Gamma_k = \int d^d x \left\{ U_k(\rho) + \frac{1}{2} Z_k \partial_\mu \phi_a \partial_\mu \phi_a \right\},
\]

where \(\rho = \frac{1}{2} \phi_a \phi^a\). The truncation (3) neglects all terms with more than two derivatives. The most general two derivative expression would contain for \(N > 1\) a second term \(Y_k \partial_\mu \rho \partial_\mu \rho\) and \(Z_k\) and \(Y_k\) could depend on \(\rho\). For simplicity we consider first only a constant \(Z_k\) and \(Y_k = 0\).

With the Ansatz (3) we obtain from (1) an evolution equation for the effective average potential:

\[
\frac{\partial}{\partial t} U_k(\rho) = v_d \int_0^\infty dx x^{d/2} s_k(x) \left( \frac{N-1}{M_0} + \frac{1}{M_1} \right)
\]

where \(x = q^2\) with

\[
M_0 = x(1 + r_k(x)) + Z_k^{-1} U_k'(\rho)
\]

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\[ M_1 = x(1 + r_k(x)) + Z_k^{-1} U_k' + 2Z_k^{-1} \rho U_k''(\rho) \]
\[ v_d^{-1} = 2^{d+1} \pi^{d/2} \Gamma(d/2). \]

Here primes denote derivatives with respect to \( \rho \) and we observe the appearance of \( \rho \) dependent mass terms \( \sim U_k' \) in the “Goldstone” and \( \sim U_k' + 2\rho U_k'' \) in the “radial” directions. The dimensionless functions \( s_k \) and \( r_k \) depend on the ratio \( x/k^2 \) and are given by
\[
\begin{align*}
r_k(x) &= \frac{R_k(x)}{Z_k x}, \\
s_k(x) &= Z_k^{-1} \frac{\partial}{\partial t} \left( \frac{R_k(x)}{x} \right) = -2x \frac{\partial}{\partial x} r_k(x) - \eta r_k(x). \quad (6)
\end{align*}
\]

The second term in \( s_k(x) \) is proportional to the anomalous dimension
\[
\eta = -\frac{\partial}{\partial t} \ln Z_k \quad (7)
\]
and arises from the factor \( Z_k \) in (3). We will neglect it in the following.

It is convenient to work with dimensionless and renormalized quantities:
\[
\rho = k^{2-d} Z_k \rho \quad ; \quad u_k(\rho) = k^{-d} U_k(\rho(\rho)) \quad ; \quad y = \frac{x}{k^2}. \quad (8)
\]

To evaluate the equation for the potential we make a further approximation and expand around the minimum of \( u_k \) for non zero \( \rho = \kappa \) up to the quadratic order in \( \rho \):
\[
u_k(\rho) = u_k(\kappa) + \frac{1}{2} \lambda (\rho - \kappa)^2 \quad (9)
\]

The condition \( \partial u/\partial \rho \big|_{\rho=\kappa} = 0 \) holds independent of \( t \) and gives us the evolution equation for the location of the minimum of the potential parametrized by \( \kappa \). A similar evolution equation can be derived (3) for the symmetric regime where \( \kappa = 0 \) and an appropriate variable is \( \partial u/\partial \rho \big|_{\rho=0} \).

The flow equations for \( \kappa \) and \( \lambda \) follow from (4) by variable substitution and appropriate differentiations with respect to \( \rho \):
\[
\begin{align*}
\beta_\kappa &\equiv \frac{d\kappa}{dt} = -(d - 2 + \eta)\kappa + 2v_d(N - 1)t_1^d(0) + 6v_d t_1^d(2\lambda \kappa) \\
\beta_\lambda &\equiv \frac{d\lambda}{dt} = (d - 4 + 2\eta)\lambda + 2v_d(N - 1)\lambda^2 t_2^d(0) + 18v_d \lambda^2 t_2^d(2\lambda \kappa). \quad (10)
\end{align*}
\]

We notice that the “threshold functions”
\[
l_n^d(\omega) = \frac{n}{2} \int_0^\infty dy y^n s(y)[y(1 + r(y)) + \omega]^{-(n+1)} \quad (11)
\]
vanish for \( \omega \to \infty \) and describe appropriately the decoupling of a heavy radial mode for \( 2\lambda \kappa \gg 1 \).

The integrals are normalized such that \( t_1^d(0) = 1 \), \( t_2^d(0) = 1 \). Eq. (7) describes how the average potential \( U_k \) changes its shape, with “initial condition” given at some short distance scale \( k = \Lambda \) by \( \kappa(\Lambda) \) and \( \lambda(\Lambda) \). The solution for \( k = 0 \) determines the 1PI two and four point functions at zero momentum. In particular, the theory is in the symmetric phase if \( \kappa(0) = 0 \) - this happens if \( \kappa \) reaches zero for some nonvanishing \( k_s > 0 \). On the other hand, the phase with spontaneous symmetry breaking corresponds to \( \rho_0(0) > 0 \) where \( \rho_0(k) = k^{d-2} Z_k^{-1} \kappa(k) \). A second order phase transition is characterized by a scaling solution corresponding to fixpoints for \( \kappa \) and \( \lambda \). For small deviations from the fixpoint there is typically one infrared unstable direction which is related to
the relevant mass parameter. The phase transition can be studied as a function of \( \kappa(\Lambda) \) with a critical value \( \kappa(\Lambda) = \kappa_c \). The difference \( \kappa(\Lambda) - \kappa_c \) can be assumed to be proportional to \( T_c - T \), with \( T_c \) the critical temperature. This allows to define and compute critical exponents in a standard way. We should mention a particular possibility for \( d = 2 \), namely that \( \kappa(0) \) remains strictly positive whereas \( \rho_0(0) \) vanishes due to \( \lim_{k \to 0} Z_k \to \infty \). This is a somewhat special form of spontaneous symmetry breaking, where the renormalized expectation value, which determines the renormalized mass, is different from zero whereas the expectation value of the unrenormalized field vanishes. We will see that this scenario is indeed realized for \( d = 2, N = 2 \). The phase with this special form of spontaneous symmetry breaking exhibits a massive radial and a massless Goldstone boson - and remains nevertheless consistent with the Mermin-Wagner theorem \( \text{[6]} \) that the expectation value of the (unrenormalized) field \( \phi_a \) must vanish for \( N \geq 2 \). The Kosterlitz-Thouless phase transition \( \text{[7]} \) describes the transition from this phase to the standard symmetric phase of the linear \( \sigma \)-model, i.e. the phase where \( \kappa(0) = 0 \) with a spectrum of two degenerate massive modes.

In order to solve the flow equation \( \text{(10)} \) we further need the anomalous dimension \( \eta \) \( \text{[8]} \) which is related to the scale dependence of the wave function renormalization. We identify the wave function renormalization in the infrared cutoff \( \text{[2]} \) with the coefficient multiplying the kinetic term in the average action \( \text{[3]} \), evaluated for zero momentum and \( \bar{\rho} = \kappa \). The flow of \( Z_k \) can then be computed from the momentum dependence of \( \frac{\partial}{\partial k} \Gamma^{(2)}_k \) \( \text{[4]} \) and one obtains with the truncation \( \text{[3]} \)

\[
\eta = \frac{16\nu d}{d} \lambda^2 \kappa m^d_{2,2}(0, 2\lambda \kappa).
\]

The integral \( m^d_{2,2} \) defines again an appropriate threshold function

\[
m^d_{2,2}(0, \omega) = -\frac{1}{2} \int_0^\infty dy y^{d/2} \frac{\partial}{\partial t} \left\{ \left( 1 + r(y) + y \frac{\partial}{\partial y} r(y) \right)^2 [y(1 + r(y))]^{-2} [y(1 + r(y)) + \omega]^{-2} \right\}. \tag{13}
\]

Here \( \frac{\partial}{\partial t} \) stands symbolically for a derivative which acts only on the infrared cutoff \( R_k \) in \( r \), with \( \frac{\partial}{\partial t} r(y) \equiv s(y) \), and we omit again the second term \( \sim \eta \) in \( s \) \( \text{[6]} \). We have now a system of three coupled nonlinear differential equations for \( \kappa, \lambda \) and \( \eta \) which can be solved numerically for arbitrary \( N \) and \( d \).

We next specialize to the two dimensional linear \( \sigma \)-model \( (d = 2) \). We observe that the flow equations can be solved analytically in the limiting case of a large mass \( \omega = 2\lambda \kappa \) of the radial mode. The threshold functions vanish with powers of \( \omega^{-1} \) and for \( N > 1 \) the leading contributions to the \( \beta \)-functions are those from the Goldstone modes. Therefore this limit is called the Goldstone regime. In this approximation the \( \beta \)-functions can be expanded in powers of \( \omega^{-1} \). In particular, the leading order of the anomalous dimension can be extracted immediately from \( \text{[12]} \):

\[
\eta = \frac{1}{4\pi \kappa} + O(\kappa^{-2}). \tag{14}
\]

Inserting this result in \( \text{(10)} \) we have

\[
\beta_\kappa = \left( \frac{N - 2}{4\pi} \right) + O(\kappa^{-1}) \tag{15}
\]

and the leading order of \( \beta_\lambda \) is

\[
\beta_\lambda = -2\lambda + \left( \frac{N - 1}{2\pi} \right) \ln 2 \lambda^2 + O(\kappa^{-1}). \tag{16}
\]
Eq. (16) has a fixpoint solution $\lambda_* = \frac{4\pi}{(N-1) \ln 2} \approx 18.13/(N-1)$.

For $N > 2$ there exists a simple relation between the linear and the nonlinear $\sigma$-model: The effective coupling between the Goldstone bosons of the nonabelian nonlinear $\sigma$-model can be extracted directly from (3) and reads in an appropriate normalization [8]

$$g^2 = \frac{1}{2\kappa}. \quad (17)$$

The lowest order contribution to $\beta_\kappa$ (3) coincides with the one loop expression for the running of $g^2$ as computed in the nonlinear $\sigma$-model. We emphasize in this context the importance of the anomalous dimension $\eta$ which changes the factor $(N - 1)$ appearing in (10) into the appropriate factor $(N - 2)$ in (15). In correspondence with the universality of the two loop $\beta$-function for $g^2$ in the nonlinear $\sigma$-model we expect the next to leading term $\sim \kappa^{-1}$ in $\beta_\kappa$ (3) to be also proportional to $(N - 2)$. In order to verify this one has to go beyond the truncation (3) and systematically keep all terms contributing in the appropriate order of $\kappa^{-1}$. (This calculation is similar to the extraction of the two loop $\beta$-function of the linear $\sigma$-model in four dimensions by means of an “improved one loop calculation” using the flow equation (1).) We have calculated the expansion of $\beta_\kappa$ up to the order $O(\kappa^{-1})$ for the most general two derivative action, i.e. neglecting only the momentum dependence of $Z_k$ and $Y_k$. The result agrees with the two loop term of the nonlinear $\sigma$-model within a few per cent, and the discrepancy should be attributed to the neglected momentum dependence of the wave function renormalization. The issue of the contribution to $\beta_\kappa$ in order $\kappa^{-2}$ is less clear: Of course, the direct contribution of the Goldstone bosons (combined with their contribution to $\eta$) should always vanish for $N = 2$ since no nonabelian coupling exists in this case. The radial mode however, could generate a contribution which is not proportional to $(N - 2)$. This contribution is possibly nonanalytic in $\kappa^{-1}$ and would correspond to a nonperturbative contribution in the language of the nonlinear $\sigma$-model.

Let us now turn to the two dimensional abelian model $(d = 2, N = 2)$ for which we want to describe the Kosterlitz-Thouless phase transition. In the limit of vanishing $\beta_\kappa$ for large enough $\kappa$ the location of the minimum of $u_k(\hat{\rho})$ (3) is independent of the scale $k$. Therefore the parameter $\kappa$, or, alternatively, the temperature difference $T_\kappa - T$, can be viewed as a free parameter. If we go beyond the lowest order estimate (16) the fixpoint for $\lambda$ remains, but $\lambda_*$ becomes dependent on $\kappa$. This implies that the system has a line of fixpoints which is parametrized by $\kappa$ as suggested by results obtained from calculations with the nonlinear $\sigma$-model (10). In particular, the anomalous dimension $\eta$ depends on the temperature $T_\kappa - T$ (14). Even if this picture is not fully accurate for nonvanishing $\beta_\kappa$, it is a very good approximation for large $\kappa$: The possible running of $\kappa$ is extremely slow, especially if $\beta_\kappa$ vanishes in order $\kappa^{-1}$. We associate the low temperature or large $\kappa$ phase with the phase of vortex condensation in the nonlinear $\sigma$-model. The correlation length is always infinite due to the Goldstone boson. Since $\eta > 0$ we expect the inverse propagator of this Goldstone degree of freedom $\sim (q^2)^{1-\eta/2}$, thus avoiding Coleman’s no go theorem (1) for free massless particles in two dimensions. On the other hand, for small values of $\lambda_\kappa$ the threshold functions (11) (13) can be expanded in powers of $\lambda \kappa$. The anomalous dimension is small and $\kappa$ is driven to zero for $k_\kappa > 0$. This corresponds to the symmetric phase of the linear $\sigma$-model with a massive complex scalar field. We associate this high temperature phase with the phase of vortex disorder in the picture of the nonlinear $\sigma$-model. The transition between the behaviour for large and small $\kappa$ is described by the Kosterlitz-Thouless transition. In the language of the linear $\sigma$-model it is the transition from a special type of spontaneous symmetry breaking to symmetry restoration.

Finally we give a summary of the results obtained from the numerical integration of the
evolution equations (10) and (12) for the special case $d = 2, N = 2$. We use a Runge-Kutta method starting at $t = 0$ with arbitrary initial values for $\kappa$ and $\lambda$ and solve the flow equations for large negative values of $t$. For the integrals defined by (11) and (13) we use numerical fits. Results are shown in figs.1-4 where we plot typical trajectories. The distance between points corresponds to equal steps in $t$ such that very dense points or lines indicate the very slow running in the vicinity of fixpoints.

The understanding of the trajectories needs a few comments: The work of Kosterlitz and Thouless suggests that the correlation length is divergent for all temperatures below a critical temperature $T_c$ and that the critical exponent $\eta$ depends on temperature. The consequence for our model is that above a critical value for $\kappa$ all $\beta$-functions should vanish for a line of fixpoints parametrized by $\kappa$. From the results in the Goldstone regime and from earlier calculations we conclude that $\beta_\kappa$ should vanish faster than $\kappa^{-1}$ for large $\kappa$. Our truncation (3), however, yields a function $\beta_\kappa$ which vanishes only like $\kappa^{-1}$. The consequence is that even if the system reaches the supposed line of fixpoints the parameter $\kappa$ decreases very slowly until the transition to the symmetric regime is reached. The anomalous dimension first grows with decreasing $\kappa$ until the critical value is reached. Then the system runs into the symmetric regime and $\eta$ vanishes. So we expect that $\eta$ reaches a maximum near the phase transition. We use this as a criterion for the critical value $\kappa_c$. In summary, the truncation (3) smoothes the phase transition and this prevents a very accurate determination of the critical value $\kappa_c$ and the corresponding anomalous dimension $\eta_c$. From the numerical point of view the absence of a true phase transition in the truncation (3) makes life easier: One particular trajectory can show both the features of the low and the high temperature phase since it crosses from one to the other.

The numerical results fulfill our expectations. Fig. shows the evolution of the anomalous dimension with decreasing $t$ for several different initial values $\kappa(\Lambda), \lambda(\Lambda)$. The maximum is reached with $\eta_c = 0.24$ which has to be compared with the result of Kosterlitz and Thouless $\eta_c = 0.25$. The approximate “line of fixpoints” for $\kappa > \kappa_c$ ($\eta < \eta_c$) is demonstrated by the selfsimilarity of the curves for large $-t$. Trajectories with different initial conditions hit the line of fixpoints at different $\kappa$. Subsequently they follow the line of fixpoints. Except for the value of $\kappa(k)$ all “memory” of the initial conditions is lost for $t \leq -3$. Another manifestation of the line of fixpoints in the $(\kappa, \lambda)$ plane is demonstrated in fig.2. After some fast “initial running” (dotted parts of the trajectories) all trajectories with large enough $\kappa(\Lambda)$ follow this line independent of the initial $\lambda(\Lambda)$. We emphasize that the nonlinear $\sigma$-model corresponds to $\lambda(\Lambda) \to \infty$. Our investigation shows that the linear $\sigma$-model is in the same universality class, even for very small $\lambda(\Lambda)$. In fig.2 we plot $\eta(\kappa)$. Along the line of fixpoints we find perfect agreement with the analytical estimate (14) for large $\kappa$. For $\kappa = 0.3$ the deviation from the lowest order result is 34% in the present truncation. Finally we show in fig.3 the value of $\beta_\lambda$ for different trajectories. The dense parts of the “ingoing curves” show the fixpoint behaviour at $\lambda_\ast(\kappa_1)$ where $\kappa_1$ denotes the value of $\kappa$ where the line of fixpoints is hit. (For small $\kappa(\Lambda)$ there is a substantial difference between $\kappa_1$ and $\kappa(\Lambda)$ which depends also on $\Lambda(\Lambda)$. This can be seen from the curves with $\kappa(\Lambda) = 1$.) After hitting the line of fixpoints the trajectories stay for a large $t$-interval at $\beta_\lambda$ very close to zero. Subsequently, the “outgoing curve” indicates the transition to the symmetric phase.

In conclusion, both the analytical and the numerical investigations demonstrate all important characteristics of the Kosterlitz-Thouless phase transition for the linear $\sigma$-model. This belongs to the same universality class as the nonlinear $\sigma$-model and we have demonstrated a close correspondence between the linear and the nonlinear $\sigma$-model with abelian symmetry. In

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1This work has also been done for $d = 2, N = 1$. There we find a fixpoint which corresponds to the second order phase transition in the Ising model.
particular, the phase with vortex disorder in the nonlinear $\sigma$-model corresponds simply to the symmetric phase of the linear $\sigma$-model. We emphasize that we have never needed the explicit investigation of vortex configurations. The exact nonperturbative flow equation includes automatically all configurations. Its ability to cope with the infrared problems of perturbation theory is confirmed by the present work.

Despite the simple and clear qualitative picture arising from the truncation (3) this letter only constitutes a first step for a quantitative investigation. It is not excluded that the coincidence of our critical $\eta_c \approx 0.24$ with $1/4$ is somewhat accidental. In order to answer this question one needs to go beyond the truncation (3). In view of the relatively large value of $\eta_c$ we expect in particular that the momentum dependence of the wave function renormalization $Z_k$ (or the deviation of the inverse propagator from $q^2$) could play an important role at the phase transition. This effect should be included in a more detailed quantitative investigation.

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Figure Captions

- fig 1: The scale dependent anomalous dimension $\eta$
- fig 2: Critical line in the $\kappa - \lambda$ plane
- fig 3: The $\kappa$ dependence of the critical exponent $\eta$
- fig 4: The $\beta$-function for $\lambda$
Figure 1: $\eta(t)$

Figure 2: $\lambda(\kappa)$
Figure 3: $\eta(\kappa)$

Figure 4: $\beta_{\lambda}$