Time-Varying Multi-Objective Optimization: Tradeoff Regret Bounds

Allahkaram Shafiei, Vyacheslav Kungurtsev, Jakub Marecek *

November 18, 2022

Abstract

Multi-objective optimization studies the process of seeking multiple competing desiderata in some operation. Solution techniques highlight marginal tradeoffs associated with weighing one objective over others. In this paper, we consider time-varying multi-objective optimization, in which the objectives are parametrized by a continuously varying parameter and a prescribed computational budget is available at each time instant to algorithmically adjust the decision variables to accommodate for the changes. We prove regret bounds indicating the relative guarantees on performance for the competing objectives.

1 Introduction

During the last decades, substantial research efforts have been devoted to learning and decision making in environments with functionally relevant streaming data with potentially changing statistical properties. In many engineering design problems with social impact, including optimal power flow and sensor networks [4, 1], mobile robots [17], and non-linear distributed flow equations [2], there are potentially multiple criteria to consider in characterizing the best learning or decision-making performance. Formally, such multi-criteria optimization problems are classed as multi-objective optimization.

The setting of a dynamically changing and uncertain environment lends itself to what is classed as online optimization, where the cost function changes over time and an adaptive decision pertaining only to the past information has to be made at each stage. The standard convergence criteria in online optimization is the level of regret, a quantity capturing the difference between the accumulated cost incurred up to some arbitrary time and the cost obtained from the best fixed point chosen in hindsight.

In machine-learning applications of multi-objective optimization, the time-varying aspects could capture, e.g., time-varying group structure or some form of a concept drift. In game theory, the time-varying aspects could capture time-varying pay-offs (or time-varying price elasticity of the demand) in extensive forms of Stackelberg-like games, or time-varying demands in congestion games.

Our contributions include:

- introduction of regret tradeoffs as the appropriate metric for grading solvers for online multi-objective optimization

*All authors are at the Czech Technical University.
• an on-line proximal-gradient algorithm for handling multiple time-varying convex objectives,
• theoretical guarantees for the algorithm.

2 Related work

Proximal Gradient Descent is a natural approach for minimizing both single and multiple objectives. One of the most widely studied methods for multiobjective optimization problems is steepest descent, e.g., [5, 6].

Subsequently, a proximal point method [3], that can be applied to non-smooth problems, was considered. However, this method is just a conceptual scheme and does not necessarily generate subproblems that are easy to solve. For non-smooth problems, a subgradient method was also developed in [10]. A very engrossing recent paper [18] has presented the regret bounds for classical for n online convex optimization OCO algorithms for relative Lipschitz, but possibly non-differentiable, that has been proven to converge at a rate of $O(\frac{1}{\sqrt{K}})$ in K iteration. More closely to our work, H.Tanabe [15] proposed proximal gradient methods for unconstrained multiobjective optimization problems with, and without line searches in which every component of objective functions are defined by $F_i(x) = f_i(x) + g_i(x)$.

Next we describe the literature on online time varying convex single objective optimization. As the first innovative paper in in this space, Zinkevich [19] proposed a gradient descent algorithm with a regret bound of $O(\sqrt{K})$. In the case that cost functions are strongly convex, the regret bound of the online gradient descent algorithm was further reduced to be $O(\log(K))$ with suitably chosen step size by several online algorithms presented in [7].

3 Problem Formulation

We begin with describing the problem of Time-Varying Multi-Objective Optimization. Suppose we have a sequence of convex cost functions $\phi_{i,t}(x) := f_{i,t}(x) + g_{i,t}(x)$ where $f_{i,t}$ are smooth and $g_{i,t}$ non-smooth. The index $t$ corresponds to the time step, and $i$ indexes the objective function among the set of interest. At each time step $t \in [T] := \{1, 2, ..., T\}$, a decision maker chooses an action $x^t \in \mathbb{R}$ and after committing to this decision, a convex cost functions $\phi_{i,t}(x)$ is disclosed and the decision maker is faced with a loss of $\phi_{i,t}(x^t)$. In this scenario, due to lack of access to the cost functions before the decision is made, the decision does not necessarily correspond to the minimizers, and the decision maker faces a so-called regret. Regret is defined as the difference between the accumulated cost over time and the cost incurred by the best fixed decision, when all functions are known in advance, see [7, 8, 19]. Let us consider

$$F(x, t) = \begin{pmatrix}
    f_{1,t}(x) + g_{1,t}(x) \\
    f_{2,t}(x) + g_{2,t}(x) \\
    \vdots \\
    f_{N,t}(x) + g_{N,t}(x)
\end{pmatrix} = \begin{pmatrix}
    \phi_{1,t}(x) \\
    \phi_{2,t}(x) \\
    \vdots \\
    \phi_{N,t}(x)
\end{pmatrix}.$$ 

At time $t$, we consider the following time-varying vector optimization

$$\min_{x \in \mathbb{R}^n} (F(x, t) := F_t(x))$$ (1)
where $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and, each $f_{i,t}$ is $L_{f_{i,t}}$ Lipschitz continuously differentiable and $g_i$ is convex with a simple prox evaluation. Throughout the entire paper, our discussion on (1) and proposed algorithm are motivated by the following: As can be surmised from the definition, there is rarely a singleton that is a Pareto optimal point. Usually, there is a continuum of solutions. As such one can consider a Pareto front which indicates the set of Pareto optimal points. The front represents the objective values reached by the components of the range of $F(x)$ and it is usually a surface of $m-1$ dimensions. One can consider it as representing the tradeoffs associated with the optimization problem, to lower $i$’s value, i.e. $f_i(x)$, how much would you have to raise $f_j(x)$ or, if not that, $f_l(x)$ (with $i \neq j \neq k$)? Because of the fundamental generality of the concept of a solution to a vector optimization problem, finding a solution can be defined as (see, e.g., for a survey [11] and for a text [10]):

1. Visualizing the entire Pareto front, or some portion of it
2. Finding any point on the Pareto front
3. Finding some point that satisfies an additional criteria, effectively making this a bilevel optimization problem.

In regards to the second option, one can notice that this can be done, in the convex case, by solving the so called “scalarized” problem:

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{N} \omega_i f_i(x), \sum \omega_i = 1, 0 \leq \omega_i \leq 1$$

for any valid choice of $\{\omega_i\}$. This reduces the problem to simple unconstrained optimization.

This leaves the choice of said constants, however, arbitrary, and thus not all that informative. Although the parameters are weights balancing the relative importance of the objective functions, poor relative scaling across $f_i(x)$ can make any informed choice of $\{\omega_i\}$ insurmountable. For example if $f_1(x) = 1000x^2$ and $f_2(x) = 0.001(x - 2)^2$, taking $\omega_1 = \omega_2 = 0.5$ clearly pushes the solution of the scalarized problem to be the solution of minimizing $f_1(x)$.

As an additional problem aspect, we consider the time varying case, i.e., each $f_i(x)$ changes over time, e.g., due to data streaming with concept drift. With a finite processing capacity at each time instant, we seek an Algorithm that appropriately balances the objectives at each time instant. This problem is considered in [15] and regret is quantified in terms of distance to some Pareto optimal point.

In this paper, we introduce scalarization at the algorithmic level for time varying multi-objective optimization. In particular, at each iteration we consider computing a set of steps each of which intend to move an iterate towards the solution of the problem of minimizing $f_i(x)$ exclusively. We then form a convex combination of these steps with chosen coefficients. We then derive tradeoff regret bounds indicating how the choice of said coefficients results in guarantees in regards to suboptimality for every objective. We assert that this would be the most transparently informative theoretical guarantees, in terms of exactly mapping algorithmic choices to comparative performance for every objective function, and as such a natural and important contribution to the consideration of time varying multi-objective optimization.

Now we shall present our formal assumptions in regards to the problem, in particular the functional properties of $F$ as well as the Algorithm we are proposing and studying the properties
of in this paper. With these definitions in place, On-Line Multi-Objective Proximal Gradient Decent Algorithm is presented above, where the parameter $\alpha_i$ is such as $\sum_{i=1}^{N} \alpha_i = 1$ for $0 \leq \alpha_i \leq 1$.

**Assumption 3.1 (Problem Structure)**  
(i) For all $i, t$ functions $f_{i,t}(\cdot): \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable such that the gradient is Lipschitz with constant $L_{f_{i,t}}$:

$$\|\nabla f_{i,t}(x) - \nabla f_{i,t}(y)\| \leq L_{f_{i,t}} \|x - y\|, \quad \forall x, y \in \mathbb{R}^n$$

(ii) For all $t$, the function $g_{i,t}(\cdot): \mathbb{R}^n \to (-\infty, \infty]$ is proper, lower semi-continuous, and convex, but not necessarily differentiable. Also, assume that $\text{dom}(g_{i,t}(\cdot)) = \{x \in \mathbb{R}^n : g_{i,t}(x) < \infty\}$ is non-empty and closed.

(iii) Corresponding to each objectives $\phi_{i,t}$ we consider $T_{i,t}(x) = \text{prox}_{C_i g_{i,t}}(x - C_i \nabla f_{i,t}(x))$

This following assumptions are typical in the analysis of online algorithms and make real-time algorithmic path-following of solutions feasible. In particular, we consider the online streaming setting with a finite sampling rate, which we assume permits for $K$ iterations of the proximal-gradient steps between two updates of the inputs.

**Assumption 3.2 (Sufficient Processing Power)** At all times $t \in [T]$, the algorithm executes at least $K$ iterations before receiving the new input.

### 4 Preliminaries and Algorithm

We also assume a bound on the magnitude of change between two inputs at times $t, t + 1$:

**Assumption 4.1 (Slow Changes)** The observations as compared to estimates of the function values from the previous time step are bounded, i.e.,

$$\sup_{t \geq 1} \max_{i \in [N]} \left\{ \left| f_{i,t}(x_{t+1}^i) - f_{i,t}(x_t^i)\right|, \left| g_{i,t}(x_{t+1}^i) - g_{i,t}(x_t^i)\right| \right\} \leq \epsilon$$

where $x^i_t$ is generated by Algorithm 1.

We consider two measures of quality of the solution trajectory:

(A): The dynamic regret bound (see, e.g., [19], and reference therein) defined as:

$$\text{Reg}_i = \sum_{t=1}^{T} \phi_{i,t}(x_t^i) - \sum_{t=1}^{T} \phi_{i,t}(x_{\text{opt},t,i}^i), \quad i \in [N]$$

where $\phi_{i,t} = f_{i,t} + g_{i,t}$ and $x_{\text{opt},t,i} \in \arg\min_x \phi_{i,t}(x)$.

In the case of static regret [9], $x_{\text{opt},t,i}^i$ is replaced by $x_{\text{opt},i} \in \arg\min_{x \in \mathbb{X}} \sum_{t=1}^{T} \phi_{i,t}(x)$, i.e,

$$S - \text{Reg}_i = \sum_{t=1}^{T} \phi_{i,t}(x_t^i) - \min_{x \in \mathbb{X}} \sum_{t=1}^{T} \phi_{i,t}(x)$$
Algorithm 1: On-Line Multi-Objective Proximal Gradient Decent

1 **Input:** Initial iterate $x^1$ solving the problem with data $f_{i,1}(x), g_{i,1}(x)$ parameters 

$C_i \in (0, \frac{1}{L_{f_i}}]$, $\alpha_i > 0$, and let $x^{1.0} \leftarrow x^1$

2 for $t = 1, 2, ..., T$ do

3 $x^{t,0} \leftarrow x^t$;

4 Receive data $f_{i,t}(x^t), g_{i,t}(x^t)$;

5 for $k = 0, 1, 2, ..., K$ do

6 $y^{t,k+1,i} \leftarrow \text{prox}_{C_i g_{i,t}}(x^{t,k} - C_i \nabla f_{i,t}(x^{t,k})) \forall i$;

7 $x^{t,k+1} \leftarrow \sum_{i=1}^{N} \alpha_i y^{t,k+1,i}$;

8 $k \leftarrow k + 1$

9 end for

10 $x^{t+1,0} \leftarrow x^{t,K}$ and $x^{t+1,0} \leftarrow x^{t,K+1}$;

11 end for

In addition, we will consider the following quantities:

$$v_t = \sum_{i=1}^{t} \|x^{i+1} - x^i\|$$  \hspace{1cm} (2)

$$w_t = \sum_{j=1}^{t} \max_{i \in [N]} \|x^{\text{opt},j+1,i} - x^{\text{opt},j,i}\|$$  \hspace{1cm} (3)

$$\sigma_t = \sum_{j=1}^{t} \sum_{i=1}^{N} d(0, \partial \phi_{i,j}(x^j))$$  \hspace{1cm} (4)

$$\phi_t(x) = \min_{i \in [N]} \phi_{i,t}(x), \quad x^{\text{opt},t} = \sum_{i=1}^{N} \alpha_i x^{\text{opt},t,i}$$  \hspace{1cm} (5)

The following lemma is a key result throughout the paper.

**Lemma 4.1** Let $f$ be convex and smooth, and $g$ be non-smooth and $\phi = f + g$ then

$$\phi(T(x)) - \phi(y) \leq \frac{1}{2C} \|x - y\|^2 - \|T(x) - y\|^2$$  \hspace{1cm} (6)

and

$$\phi(T(x)) \leq \phi(x).$$  \hspace{1cm} (7)

where $T(x) = \text{prox}_{C g}(x - C \nabla f(x))$, $C \in (0, \frac{1}{L_f}]$ and $L_f$ is Lipschitz constant for $\nabla f$.

**Proof.** Take $G(x) := \frac{1}{C}(x - T(x))$. Owing to the convexity of $f$ one can obviously see that

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L_f}{2} \|x - y\|^2 \quad \forall x, y \in \mathbb{R}^n.$$  \hspace{1cm} (8)
Plugging $y = x - CG(x)$ in (8) one obtains that

$$f(x - CG(x)) \leq f(x) + \nabla f(x)^T((x - CG(x)) - x) + \frac{L_f C^2}{2} ||G(x)||^2$$

$$\leq f(x) - C\nabla f(x)^T(G(x)) + \frac{C}{2} ||G(x)||^2.$$  \hfill (9)

Now, from $x - CG(x) = \text{prox}_{CG}(x - C\nabla f(x))$ it follows that

$$G(x) - \nabla f(x) \in \partial g(x - CG(x)).$$

Therefore

$$g(x - CG(x)) \leq g(y) - (G(x) - \nabla f(x))^T(y - x - CG(x)).$$  \hfill (10)

Therefore, by simplifying and taking into account (11), one has

$$\phi(x - CG(x)) = f(x - CG(x)) + g(x - CG(x))$$

$$\leq f(x) - C\nabla f(x)^T(G(x)) + \frac{C}{2} ||G(x)||^2 + g(x - CG(x))$$

$$\leq f(y) - \nabla f(x)^T(y - x) - C\nabla f(x)^T(G(x)) + \frac{C}{2} ||G(x)||^2 + g(x - CG(x))$$

$$\leq f(y) - \nabla f(x)^T(y - x) - C\nabla f(x)^T(G(x))$$

$$+ \frac{C}{2} ||G(x)||^2 + g(y) - (G(x) - \nabla f(x))^T(y - x + CG(x))$$

$$= \phi(y) - \nabla f(x)^T(y - x) - C\nabla f(x)^T(G(x)) + \frac{C}{2} ||G(x)||^2 - G(x)^T(y - x)$$

$$- C||G(x)||\nabla f(x)^T(y - x) + C\nabla f(x)^T(G(x))$$

$$\leq \phi(y) + \frac{1}{2C} ||x - y||^2 - ||(x - y) - CG(x)||^2],$$

and since $x - CG(x) = T(x)$ the desired result follows by the previous.

$\blacksquare$

## 5 Main Results

Our main result provides a bound on the expected dynamic regret of the online multi-objective proximal gradient descent (Algorithm 1). Depending on the coefficients $\alpha_i$, there are two cases, i) if there is $i \in [N]$ such that $\alpha_i \neq 1$ and ii) if there is $i \in [N]$ such that $\alpha_i = 1$ and for all $j \neq i$, $\alpha_j = 0$. For case i), we have

**Theorem 5.1** Let $x^t$, $(t=1,...,T)$ be a sequence generated by running Algorithm 1 over $T$ time steps. Then, we have

$$\text{Reg}_i = \sum_{t=1}^{T} \phi_i(t) - \sum_{t=1}^{T} \phi_i(t^\text{opt},t) \leq (v_T + w_T + K\sigma_T)^2 \left(\sum_{i=1}^{N} \frac{1}{2\alpha C_i} ||x^1 - x^\text{opt,1,1}||^2\right)^2$$

In order to prove the result, we need a technical lemma:
Lemma 5.1 The following holds.

(a) For all $t$ one has

$$
\|x^{t+1} - x^t\| \leq K\sigma_t
$$

(b) For all $i \in [N]$ and $t$ one has

$$
\|x^{t+1} - x^{\opt,t,i}\| \leq 2v_t + w_t + \|x^1 - x^{\opt,1,i}\|
$$

(c) For all $i \in [N]$ and $t$ one has

$$
\|x^{t,K} - x^{\opt,t,i}\| \leq 2v_t + w_t + K\sigma_t + \|x^1 - x^{\opt,1,i}\|
$$

Proof.

(a) We can easily see by the Algorithm that

$$
\|x^{t,j+1} - x^{t,j}\| \leq \|x^{t,j} - x^{t,j-1}\| \leq \ldots \leq \|x^{t,2} - x^{t,1}\| \tag{12}
$$
on the other hand since

$$
\|x^{t,2} - x^{t,1}\| = \left\| \sum_{i=1}^{N} \alpha_i(T_{i,t}(x^{t,1}) - x^{t,1}) \right\| \leq \sum_{i=1}^{N} \left\| T_{i,t}(x^{t,1}) - x^{t,1} \right\| \leq \sum_{i=1}^{N} d(0, \partial \phi_{i,t}(x^t)) \tag{13}
$$

where in the last inequality we used [14, Proposition 15]. Thanks to (12), one obtains that

$$
\|x^{t+1} - x^t\| = \|x^{t,K+1} - x^{t,1}\| = \|(x^{t,K+1} - x^{t,K}) + (x^{t,K} - x^{t,K-1}) + \ldots + (x^{t,2} - x^{t,1})\|
$$

$$
\leq \sum_{k=1}^{K} \|x^{t,j+1} - x^{t,j}\| \leq \sum_{k=1}^{K} \left( \sum_{i=1}^{N} d(0, \partial \phi_{i,k}(x^k)) \right) = K \sum_{i=1}^{N} d(0, \partial \phi_{i,t}(x^t)) = K\sigma_t
$$

(b) One can write

$$
\|x^{t+1} - x^{\opt,t,i}\| = \|x^{t+1} - x^t + x^t - x^{t-1,i,\opt} + x^{\opt,t-1,i} - x^{\opt,t,i}\| \leq \|x^{t+1} - x^t\| + \|x^{\opt,t-1,i} - x^{\opt,t,i}\| + \|x^t - x^{\opt,t-1,i}\|
$$

and then one gets

$$
\|x^{t+1} - x^{\opt,t,i}\| - \|x^t - x^{\opt,t-1,i}\| \leq \|x^{t+1} - x^t\| + \|x^{\opt,t-1,i} - x^{\opt,t,i}\|
$$

summing up parties the above result from $t = 2$ to $t = T$ leads to

$$
\|x^{t+1} - x^{\opt,t,i}\| - \|x^2 - x^{1,i,\opt}\| \leq \sum_{t=1}^{T} \|x^{t+1} - x^t\| + \sum_{t=2}^{T} \max_{i \in [N]} \|x^{\opt,t-1,i} - x^{\opt,t,i}\|
$$

therefore,

$$
\|x^{t+1} - x^{\opt,t,i}\| \leq v_t + w_t + \|x^2 - x^{1,i,\opt}\|
$$

which follows that

$$
\|x^{t+1} - x^{\opt,t,i}\| \leq v_t + w_t + \|x^2 - x^{1,i,\opt}\| + \|x^{1,i,\opt} - x^{\opt,t,i}\| \leq 2v_t + w_t + \|x^1 - x^{\opt,1,i}\|
$$
and finally the desired result can be obtained. ■

(c) Using the last part (b) and combining (12) and (13) derives that

$$\|x^t - x_{opt}^t\| \leq \|x^{t+1} - x^t\| + \|x^{t+1} - x_{opt}^t\| = \|x^{t+1} - x_{opt}^t\| \leq 2v_t + w_t + K\sigma_t + \|x^1 - x_{opt}^1\|$$

Returning to the proof of the main result,

**Proof.** Take $\alpha = \min\{\alpha_i : \alpha_i \neq 0\}$. Using the convexity of $\phi_{i,t}$ we follow that

$$\phi_{i,t}(x^{t+1}) - \phi_{i,t}(x_{opt}^{t,i}) = \phi_{i,t}(x^{t+1}) - \phi_{i,t}(x_{opt}^{t,i}) \leq \left( \sum_{i=1}^{N} \alpha_i \phi_{i,t}(x^{t+1,i}) - \phi_{i,t}(x_{opt}^{t,i}) \right)$$

$$= \sum_{i=1}^{N} \alpha_i [\phi_{i,t}(y^{i,t+1}) - \phi_{i,t}(x_{opt}^{t,i})] + \left( \sum_{i=1}^{N} \alpha_i \phi_{i,t}(x_{opt}^{t,i}) \right) - \phi_{i,t}(x_{opt}^{t,i})$$

$$= \sum_{i=1}^{N} \alpha_i [\phi_{i,t}(T_{i,t}(x^{t,i})) - \phi_{i,t}(x_{opt}^{t,i})] + \left( \sum_{i=1}^{N} \alpha_i \phi_{i,t}(x_{opt}^{t,i}) \right) - \phi_{i,t}(x_{opt}^{t,i})$$

$$\leq \sum_{i=1}^{N} \alpha_i \frac{1}{2\alpha_i} \|x^{t+1} - x_{opt}^{t,i}\|^2 + \left( \sum_{i=1}^{N} \alpha_i \phi_{i,t}(x_{opt}^{t,i}) \right) - \phi_{i,t}(x_{opt}^{t,i})$$

Multiplying last inequality at $\alpha_i$ and summing up the result over $i$ and taking into account part (c) and applying Lemma (5.1) conclude that

$$\sum_{i=1}^{N} \alpha_i \left( \phi_{i,t}(x^{t+1}) - \phi_{i,t}(x_{opt}^{t,i}) \right) \leq \sum_{i=1}^{N} \frac{\alpha_i}{2\alpha_i} \left( 2v_t + w_t + K\sigma_t + \|x^1 - x_{opt}^1\| \right)^2$$

and since $x_{opt}^{t,i}$ is optimal point this implies that

$$\alpha_i \left( \phi_{i,t}(x^{t+1}) - \phi_{i,t}(x_{opt}^{t,i}) \right) \leq \sum_{i=1}^{N} \frac{1}{2\alpha_i} \left( 2v_t + w_t + K\sigma_t + \|x^1 - x_{opt}^1\| \right)^2$$

and subsequently,

$$\phi_{i,t}(x^{t+1}) - \phi_{i,t}(x_{opt}^{t,i}) \leq \sum_{i=1}^{N} \frac{1}{2\alpha_i} \left( 2v_t + w_t + K\sigma_t + \|x^1 - x_{opt}^1\| \right)^2$$

Using the previous and Assumption (4.1) one can illustrate that

$$\sum_{t=1}^{T} \phi_{i,t}(x^t) - \phi_{i,t}(x_{opt}^{t,i}) = \sum_{t=1}^{T} \phi_{i,t}(x^t) - \phi_{i,t}(x^{t+1}) + \sum_{t=1}^{T} \phi_{i,t}(x^{t+1}) - \phi_{i,t}(x_{opt}^{t,i})$$

$$\leq \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{1}{2\alpha_i} \left( 2v_t + w_t + K\sigma_t + \|x^1 - x_{opt}^1\| \right)^2$$

$$\leq \left( \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{1}{2\alpha_i} \left( 2v_t + w_t + K\sigma_t + \|x^1 - x_{opt}^1\| \right)^2 \right)^{\frac{1}{2}}$$

$$= (v_T + w_T + K\sigma_T)^2 \left( \sum_{i=1}^{N} \frac{1}{2\alpha_i} \left( \|x^1 - x_{opt}^1\| \right)^2 \right)$$

where in the last inequality, one uses Assumption (4.1). ■

8
Corollary 5.1 In the case that there exists \( i, \alpha_i = 1 \) and for all \( j \neq i \) we have \( \alpha_j = 0 \) the problem reduces to time-varying single objective optimization, i.e.,

\[
x^{t,k+1} = \text{prox}_{C_i \phi_{i,t}} (x^{t,k} - C \nabla f_{i,t}(x^{t,k})).
\]

Then

\[
\sum_{t=1}^{T} \phi_{i,t}(x^{t+1}) - \phi_{i,t}(x^{\text{opt},t,i}) \leq \frac{C}{K+1} \|x^{1} - x^{\text{opt},1,i}\|^2,
\]

and moreover

\[
\sum_{t=1}^{T} \phi_{i,t}(x^{t}) - \phi_{i,t}(x^{\text{opt},t,i}) \leq Te + \frac{C}{K+1} \|x^{1} - x^{\text{opt},1,i}\|^2.
\]

Proof. As can be seen from Lemma 4.1 we have

\[
\phi_{i,t}(x^{t,k+1}) - \phi_{i,t}(x^{\text{opt},t,i}) \leq \frac{1}{C_i} \left[ \|x^{t,k} - x^{\text{opt},t,i}\|^2 - \|x^{t,k+1} - x^{\text{opt},t,i}\|^2 \right].
\]

Summing the result over \( k \) from 1 to \( K \) we conclude that

\[
\sum_{k=1}^{K} \left[ \phi_{i,t}(x^{t,k+1}) - \phi_{i,t}(x^{\text{opt},t,i}) \right] \leq C \left[ \|x^{t,1} - x^{\text{opt},t,i}\|^2 - \|x^{t,K+1} - x^{\text{opt},t,i}\|^2 \right],
\]

where \( C = \max_{i \in [N]} \frac{1}{C_i} \), therefore

\[
\phi_{i,t}(x^{t,K+1}) - \phi_{i,t}(x^{\text{opt},t,i}) \leq \frac{C}{K+1} \left[ \|x^{t} - x^{\text{opt},t,i}\|^2 - \|x^{t+1} - x^{\text{opt},t,i}\|^2 \right],
\]

and subsequently, by summing the previous inequality over \( t \) from 1 to \( T \) one gives the required results.

It is important to note that due to being decreasing the sequence \( \{ \phi_{i,t}(x^{t}) \} \), i.e., \( \phi_{i,t}(x^{t+1}) \leq \phi_{i,t}(x^{t}) \) we can have

\[
\left\{ \begin{array}{l}
\phi_{i,t}(x^{t+1}) - \phi_{i,t}(x^{\text{opt},t,i}) \leq \frac{C}{T(K+1)} \|x^{t} - x^{\text{opt},t,i}\| \\
\phi_{i,T}(x^{T}) - \phi_{i,t}(x^{\text{opt},K,i}) \leq e + \frac{C}{T(K+1)} \|x^{T} - x^{\text{opt},1,i}\|
\end{array} \right.
\]

From definitions, one gets the following Lemma.

Lemma 5.2 one has

\[
\min_{x_1, \ldots, x_N \in \mathbb{R}^n} \sum_{i=1}^{N} \alpha_i \phi_{i,t}(x_i) = \phi_{t}(x^{\text{opt},t})
\]

Proposition 5.1 Let \( \{x^t\} \) be the iterates generated by Algorithm 1 then the following holds For every \( x \in \mathbb{R}^n \) and \( t \) one has

\[
\|\nabla f_{i,t}(x^t) - \nabla f_{i,t}(x^{\text{opt},t,i})\|^2 \leq 2L(w_T + w_T + K\sigma_T)^2 \left( \sum_{i=1}^{N} \frac{1}{2\alpha C_i} \left( \|x^1 - x^{\text{opt},1,i}\| \right) \right)^2
\]

where \( L = \max_{i \in [N], t \in [T]} L_{f_{i,t}} \).
Proof. Since $f_{i,t}$ is Lipschitz continuous with constant $L_f$, using [13, Theorem 2.1.5], one obtain that

$$
\|\nabla f_{i,t}(x) - \nabla f_{i,t}(x^{\text{opt},t,i})\|^2 \leq f_{i,t}(x) - f_{i,t}(x^{\text{opt},t,i}) - \nabla f_{i,t}(x^{\text{opt},t,i})^T(x - x^{\text{opt},t,i}).
$$

For the optimality of $x^{\text{opt},t,i}$, we see that there exists $\xi^{\text{opt},t,i} \in \partial g_{i,t}(x^{\text{opt},t,i})$ such that $\xi^{\text{opt},t,i} + \nabla f_{i,t}(x^{\text{opt},t,i}) = 0$. Hence for all $x$ one has

$$
\|\nabla f_{i,t}(x) - \nabla f_{i,t}(x^{\text{opt},t,i})\|^2 \leq f_{i,t}(x) - f_{i,t}(x^{\text{opt},t,i}) - \nabla f_{i,t}(x^{\text{opt},t,i})^T(x - x^{\text{opt},t,i})
$$

$$
= f_{i,t}(x) - f_{i,t}(x^{\text{opt},t,i}) + \xi^{\text{opt},t,i} + \nabla f_{i,t}(x^{\text{opt},t,i})^T(x - x^{\text{opt},t,i})
$$

$$
\leq f_{i,t}(x) - f_{i,t}(x^{\text{opt},t,i}) + g_{i,t}(x) - g_{i,t}(x^{\text{opt},t,i})
$$

$$
\phi_{i,t}(x) - \phi_{i,t}(x^{\text{opt},t,i})
$$

Invoking Theorem 5.1 follows the desired result. □

6 Conclusions

We have studied a time-varying multi-objective optimization problem in a setting, which has not been considered previously. We have shown properties of a natural, online proximal-gradient algorithm, when the processing power between iterations is bounded. Going forwards, one could clearly consider alternative uses of the same algorithm (e.g., how many operations one requires per update to achieve a certain bound in terms of the dynamic regret), variant algorithms, or completely novel settings. In parallel with our work, Tarzanagh and Balzano [16] studied online bilevel optimization under assumptions of strong convexity throughout, which could be seen as one such novel setting.

Acknowledgements  Allahkaram Shafiei acknowledges support of the OP RDE funded project CZ.02.1.01/0.0/0.0/16_019/0000765 “Research Center for Informatics”. This work has received funding from the European Union’s Horizon Europe research and innovation programme under grant agreement No. GA 101070568.

References

[1] Mesut E Baran and Felix F Wu. Optimal capacitor placement on radial distribution systems. IEEE Transactions on power Delivery, 4(1):725–734, 1989.

[2] Jonathan F Bard. Convex two-level optimization. Mathematical programming, 40(1-3):15–27, 1988.

[3] Henri Bonnel, Alfredo Noel Iusem, and Benar Fux Svaiter. Proximal methods in vector optimization. SIAM Journal on Optimization, 15(4):953–970, 2005.

[4] Jason C Derenick and John R Spletzer. Convex optimization strategies for coordinating large-scale robot formations. IEEE Transactions on Robotics, 23(6):1252–1259, 2007.

[5] Arthur M Geoffrion. Proper efficiency and the theory of vector maximization. Journal of mathematical analysis and applications, 22(3):618–630, 1968.
[6] MLN Gonçalves, FS Lima, and LF Prudente. Globally convergent newton-type methods for multiobjective optimization, 2021.

[7] Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2):169–192, 2007.

[8] Elad Hazan and Satyen Kale. Extracting certainty from uncertainty: regret bounded by variation in costs. *Machine Learning*, 2010.

[9] Elad Hazan, Karan Singh, and Cyril Zhang. Online learning of linear dynamical systems. In *Advances in Neural Information Processing Systems*, pages 6686–6696, 2017.

[10] Johannes Jahn et al. *Vector optimization*. Springer, 2009.

[11] R Timothy Marler and Jasbir S Arora. Survey of multi-objective optimization methods for engineering. *Structural and multidisciplinary optimization*, 26(6):369–395, 2004.

[12] Kaisa Miettinen. *Nonlinear multiobjective optimization*, volume 12. Springer Science & Business Media, 2012.

[13] Yurii Nesterov et al. *Lectures on convex optimization*, volume 137. Springer, 2018.

[14] Allahkaram Shafiei, Vyacheslav Kungurtsev, and Jakub Marecek. Trilevel and multilevel optimization using monotone operator theory. *arXiv preprint arXiv:2105.09407*, 2021.

[15] Hiroki Tanabe, Ellen H Fukuda, and Nobuo Yamashita. Proximal gradient methods for multiobjective optimization and their applications. *Computational Optimization and Applications*, 72(2):339–361, 2019.

[16] Davoud Ataee Tarzanagh and Laura Balzano. Online bilevel optimization: Regret analysis of online alternating gradient methods, 2022.

[17] Jianan Wang and Ming Xin. Optimal consensus algorithm integrated with obstacle avoidance. *International Journal of Systems Science*, 44(1):166–177, 2013.

[18] Yihan Zhou, Victor Sanches Portella, Mark Schmidt, and Nicholas Harvey. Regret bounds without lipschitz continuity: online learning with relative-lipschitz losses. *Advances in Neural Information Processing Systems*, 33:15823–15833, 2020.

[19] Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the 20th international conference on machine learning (icml-03)*, pages 928–936, 2003.
A Time-Varying MOP as a Problem in Graph Theory

In this section, vectors will be considered as a column vectors and use \( \text{col}\{x_1, x_2, ..., x_N\} \) to indicate a column vector formed by stacking \( x_1, x_2, ..., x_N \) on top of each other. Set \( V = \{1, 2, ..., N\} \) and \( S = \{1, 2, ..., \mathcal{X}\} \). The adjacent matrix of a graph is defined as \( A^s = [a^s_{ij}] \) where \( a^s_{ij} = 1 \) if the edge pair \((j, i) \in E^s\), and zero otherwise. The degree matrix is defined as \( D^s = \text{diag}(d^s_1, d^s_2, ..., d^s_N) \), with \( d^s_i = \sum_{j=1}^{N} a^s_{ij} \). Then, the Laplacian matrix is defined as \( L^s = D^s - A^s \).

The neighbouring set of agent \( i \) is defined as \( N^s_i = \{j \in V : (j, i) \in E^s\} \) where \( E^s = \{(i, j) : i, j \in V\} \).

**Definition A.1 (Time varying Pareto optimal solution)[12]** A decision vector \( z^t, \text{opt} \in \mathbb{R}^n \) is called time-varying Pareto solution to the problem (1) if for any time \( k \) there does not exist another decision vector \( z^t \in \mathbb{R}^n \) such that \( \phi_i(t)(z^t) \leq \phi_i(t)(z^t, \text{opt}) \) for all \( i \in [N] \) and \( \phi_j(t)(z^t) < \phi_j(t)(z^t, \text{opt}) \) for at least one \( j \in [N] \). Equivalently, \( z^t, \text{opt} \) is time-varying Pareto optimal solution for \( F \) if there is no \( z^t \in \mathbb{R}^n \) such that \( F_i(z^t) \leq F_k(z^t, \text{opt}) \) and \( F_i(z^t) \neq F_i(z^t, \text{opt}) \).

Due to the local objective \( f_i(x, t) \) is privately known to the agent \( i \) only, distributed algorithms should be implemented to solve the problem. Introducing a local decision variable \( x_i \in \mathbb{R}^n \) for agent \( i \), problem (1) can be reformulated as

\[
\min_{x_1, ..., x_N \in \mathbb{R}^n} \{f_1(x_1, t), ..., f_N(x_N, t)\} \\
\text{subject to } \mathcal{T}^\sigma \bar{x} = 0
\]

where \( \mathcal{T}^\sigma = \mathcal{T}^\sigma \otimes I_n \) and \( \bar{x} = \text{col}(x_1, ..., x_N) \).