Timelike surfaces into 4–dimensional Minkowski space via spinors

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Abstract
We prove that an isometric immersion of a timelike surface in four-dimensional Minkowski space is equivalent to a normalized spinor field which is a solution of a Dirac equation on the surface. Using the quaternions and the complex numbers, we obtain a spinor representation formula that relates the spinor field and the isometric immersion. Applying the representation formula, we deduce a new spinor representation of a timelike surface in three-dimensional De Sitter space; we give a formula for the Laplacian of the Gauss map of a minimal timelike surface in four-dimensional Minkowski space in terms of the curvatures of the surface; we obtain a local description of a flat timelike surface with flat normal bundle and regular Gauss map in four-dimensional Minkowski space, and we also give a conformal description of a flat timelike surface in three-dimensional De Sitter space.

Keywords: Timelike surfaces; spinors; immersions; Weierstrass representation

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1 Introduction
We consider $\mathbb{R}^{3,1}$ the four-dimensional Minkowski space defined by $\mathbb{R}^4$ endowed with indefinite metric of signature $(3, 1)$ given by

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

A surface $M \subset \mathbb{R}^{3,1}$ is said to be timelike if the metric $\langle \cdot, \cdot \rangle$ induces on $M$ a metric of signature $(1, 1)$. In this paper, we are interested in the spinorial description of a timelike surface in $\mathbb{R}^{3,1}$, with given normal bundle and given mean curvature vector, and in its applications to the geometry of timelike surfaces in $\mathbb{R}^{3,1}$. With this, we pretend to complete the spinorial description of semi-Riemannian surfaces in four-dimensional semi-Riemannian Euclidean spaces [4, 5, 6].
Below we will state the main result of this paper. Let $M$ be an abstract simply connected timelike surface, $E \to M$ be a bundle of rank 2 with a Riemannian metric and a compatible connection. We assume moreover that spin structures are given on $TM$ and on $E$, and we define $\Sigma := \Sigma M \otimes \Sigma E$, the tensor product of the corresponding bundles of spinors. Let $\mathbb{H}^C$ be the space of quaternions with coefficients in $C$ defined by

$$\mathbb{H}^C := \{ q_1 I + q_2 J + q_3 K \mid q_1, q_2, q_3 \in \mathbb{C} \},$$

where $I, J$ and $K$ are such that $I^2 = J^2 = K^2 = -1$ and $IJ = -JI = K$.

We will see (Section 2) that two natural bilinear maps $H : \Sigma \times \Sigma \to \mathbb{C}$ and $\langle \cdot, \cdot \rangle : \Sigma \times \Sigma \to \mathbb{H}^C$ are defined on $\Sigma$. We have the following:

**Theorem 1.1.** Let $\vec{H}$ be a section of $E$. The following three statements are equivalent.

1. There exists a spinor field $\varphi \in \Gamma(\Sigma)$ with $H(\varphi, \varphi) = 1$ solution of the Dirac equation

$$D \varphi = \vec{H} \cdot \varphi.$$

2. There exists a spinor field $\varphi \in \Gamma(\Sigma)$ with $H(\varphi, \varphi) = 1$ solution of

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1,2} \epsilon_j e_j \cdot B(X, e_j) \cdot \varphi,$$

where $B : TM \times TM \to E$ is a bilinear and symmetric map with $\frac{1}{2} \text{tr} B = \vec{H}$, and where $(e_1, e_2)$ is an orthonormal frame of $TM$ and $\epsilon_j = \langle e_j, e_j \rangle$.

3. There exists an isometric immersion $F : M \to \mathbb{R}^{3,1}$ with normal bundle $E$, second fundamental form $B$ and mean curvature vector $\vec{H}$.

Moreover, the isometric immersion is given by the spinor representation formula

$$F = \int \xi : M \to \mathbb{R}^{3,1} \quad \text{with} \quad \xi(X) := \langle (X \cdot \varphi, \varphi) \rangle,$$

for all $X \in TM$, where $\xi$ is a closed 1-form on $M$ with values in $\mathbb{R}^{3,1}$.

The definitions of the Clifford product "\cdot" on the spinor bundle $\Sigma$, of the Dirac operator $D$ acting on $\Gamma(\Sigma)$ and of the immersion of $\mathbb{R}^{3,1}$ into $\mathbb{H}^C$ are given in Section 2. The proof of this theorem will be given in Section 3.

Using the representation formula, we give various applications concerning to the geometry of timelike surfaces in $\mathbb{R}^{3,1}$: we start with a spinorial proof of
the fundamental theorem of submanifolds (Remark 2.1, Corollary 3.4); see in [4, 5, 6] a similar application in other contexts. We also give a classical formula for the Laplacian of the isometric immersion (Corollary 3.5).

As a second application of Theorem 1.1, we give (using only one intrinsic spinor) a new representation of a timelike surface in three-dimensional De Sitter space (Remark 4.4); our representation is different to that given in [9] where two spinors are needed. We also recover the representation of a timelike surface in three-dimensional Minkowski space given in [6].

The third application of Theorem 1.1 is a formula for the Laplacian of the Gauss map of a minimal timelike surface in $\mathbb{R}^{3,1}$ in terms of the Gauss and normal curvatures of the surface (Corollary 5.5); this formula generalizes a classical formula for minimal surfaces in Euclidean space.

The fourth application of Theorem 1.1 is the local description of a flat timelike surface with flat normal bundle and regular Gauss map in $\mathbb{R}^{3,1}$ (Corollary 6.8 and 6.9). Using the extrinsic geometry of the immersion, we give to the surface a Riemann surface structure with respect to which its Gauss map is holomorphic, and we prove that these surfaces are described by two holomorphic functions and two smooth functions satisfying a condition of compatibility; this is the main result of [1], that we prove here using spinors.

The last application obtained from Theorem 1.1 is a conformal description of a flat timelike surface in three-dimensional De Sitter space (Corollary 7.1); our representation coincides with the description given by Aledo, Gálvez and Mira in [1, Corollary 5.1].

We quote the following related papers: the spinor representation of surfaces in $\mathbb{R}^{3}$ was studied by many authors, especially by Friedrich [7], who interpreted a spinor field representing a surface in $\mathbb{R}^{3}$ as a constant spinor field of $\mathbb{R}^{3}$ restricted to the surface; following this approach, surfaces in $\mathbb{S}^{3}$ and $\mathbb{H}^{3}$ was studied by Morel [10] and surfaces in three-dimensional semi-Riemannian space forms was studied by Lawn and Roth [8, 9]. The last two authors together with Bayard studied in [5] surfaces in four-dimensional space forms; spacelike surfaces in $\mathbb{R}^{3,1}$ was studied by Bayard [4]. The author together with Bayard studied in [6] Lorentzian surfaces in $\mathbb{R}^{2,2}$ and, different applications of this representation were given by the author in [13].

The paper is organized as follows. In Section 2 we describe the preliminaries concerning the spinors of $\mathbb{R}^{3,1}$ and the spin geometry of a timelike surface in $\mathbb{R}^{3,1}$. In Section 3 we prove the spinor representation theorem and we also give the spinor representation formula of the immersion by the spinor field. In Section 4 we study the isometric immersion of a timelike surface in three-dimensional De Sitter space. Section 5 is devoted to compute the Laplacian of the Gauss map of a timelike surface in $\mathbb{R}^{3,1}$. We obtain the local description of a flat timelike surface with flat normal bundle and regular Gauss map in $\mathbb{R}^{3,1}$ in Section 6. Finally, in Section 7 we deduce a conformal description of a flat timelike surface in three-dimensional De Sitter space.
2 Preliminaries

2.1 Spinors of \( \mathbb{R}^{3,1} \)

In this section we describe the Clifford algebra of \( \mathbb{R}^{3,1} \), the spinorial group and their representations (see [4, Section 1]).

Using the Clifford map

\[
\mathbb{R}^{3,1} \rightarrow \mathbb{H}^C(2)
\]

\[
(x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} 0 & i x_1 I + x_2 I + x_3 J + x_4 K \\ -i x_1 I + x_2 I + x_3 J + x_4 K & 0 \end{pmatrix}
\]

where \( \mathbb{H}^C(2) \) stands for the set of \( 2 \times 2 \) matrices with entries belonging to \( \mathbb{H}^C \), we get the Clifford algebra of \( \mathbb{R}^{3,1} \)

\[
Cl(3,1) = \left\{ \left( \begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix} \in \mathbb{H}^C(2) \mid p, q \in \mathbb{H}^C \right) \right\}
\]

where \( \overline{q} := q_1 I + q_2 J + q_3 I + q_4 J \) (\( \overline{q}_j \) means the usual conjugation in \( \mathbb{C} \) of \( q_j \)) for all \( q = q_1 I + q_2 I + q_3 J + q_4 K \in \mathbb{H}^C \). The Clifford sub-algebra of elements of even degree is

\[
Cl_{0}(3,1) = \left\{ \begin{pmatrix} p & 0 \\ 0 & \overline{p} \end{pmatrix} \in \mathbb{H}^C(2) \mid p \in \mathbb{H}^C \right\} \simeq \mathbb{H}^C
\]

and the subspace of elements of odd degree is

\[
Cl_{1}(3,1) = \left\{ \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \in \mathbb{H}^C(2) \mid q \in \mathbb{H}^C \right\} \simeq \mathbb{H}^C.
\]

We consider the map \( H : \mathbb{H}^C \times \mathbb{H}^C \rightarrow \mathbb{C} \) defined by

\[
H(p, p') = p_1 p'_1 + p_2 p'_2 + p_3 p'_3 + p_4 p'_4
\]

where \( p = p_1 I + p_2 I + p_3 J + p_4 K \) and \( p' = p'_1 I + p'_2 I + p'_3 J + p'_4 K \). It is \( \mathbb{C} \)-bilinear and symmetric. Its real part, denoted by \( \langle \cdot, \cdot \rangle \), is a real scalar product of signature (4, 4) on \( \mathbb{H}^C \). The spinorial group is given by

\[
Spin(3,1) := \left\{ p \in \mathbb{H}^C \mid H(p, p) = 1 \right\} \subset Cl_{0}(3,1).
\]

Now, if we consider the identification

\[
\mathbb{R}^{3,1} \simeq \{ i x_1 I + x_3 I + x_3 J + x_4 K \in \mathbb{H}^C \mid x_j \in \mathbb{R} \} \simeq \{ q \in \mathbb{H}^C \mid q = -\overline{q} \},
\]

where, if \( q = q_1 I + q_2 I + q_3 J + q_4 K \in \mathbb{H}^C \), \( \overline{q} := q_1 I - q_2 I - q_3 J - q_4 K \) is the usual conjugation in \( \mathbb{H}^C \), we get the double cover

\[
\Phi : \ Spin(3,1) \rightarrow SO(3,1)
\]

\[
p \mapsto (q \in \mathbb{R}^{3,1} \mapsto p q \overline{q}^{-1} \in \mathbb{R}^{3,1}).
\]
Here and below $SO(3,1)$ stands for the component of the identity of the semiorthogonal group $O(3,1)$ (see [12]).

Let us denote by $\rho : Cl(3,1) \rightarrow End_{\mathbb{C}}(\mathbb{H}_{\mathbb{C}})$ the complex representation of $Cl(3,1)$ on $\mathbb{H}_{\mathbb{C}}$ given by

$$\rho \left( \begin{pmatrix} p & q \\ q & p \end{pmatrix} \right) : \xi \mapsto \begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \simeq p\xi + q\xi,$$

where the complex structure on $\mathbb{H}_{\mathbb{C}}$ is given by the multiplication by $K$ on the right. The spinorial representation of $Spin(3,1)$ is the restriction to $Spin(3,1)$ of the representation $\rho$ and reads

$$\rho|_{Spin(3,1)} : Spin(3,1) \rightarrow End_{\mathbb{C}}(\mathbb{H}_{\mathbb{C}}) \quad p \mapsto (\xi \in \mathbb{H}_{\mathbb{C}} \mapsto p\xi \in \mathbb{H}_{\mathbb{C}}).$$

This representation splits into $\mathbb{H}_{\mathbb{C}} = S^+ \oplus S^-$, where $S^+ = \{ \xi \in \mathbb{H}_{\mathbb{C}} \mid \xi K = i\xi \}$ and $S^- = \{ \xi \in \mathbb{H}_{\mathbb{C}} \mid \xi K = -i\xi \}$; explicitly we have

$$S^+ = (\mathbb{C} \oplus \mathbb{C}J)(1 - iK) \quad \text{and} \quad S^- = (\mathbb{C} \oplus \mathbb{C}J)(1 + iK).$$

Note that, if $(e_1, e_2, e_3, e_4)$ stands for the canonical basis of $\mathbb{R}^{3,1}$, the complexified volume element $i e_1 \cdot e_2 \cdot e_3 \cdot e_4$ acts as $+Id$ on $S^+$ and as $-Id$ on $S^-$.  

Spinors under the splitting $\mathbb{R}^{3,1} = \mathbb{R}^{1,1} \times \mathbb{R}^2$. We consider the splitting $\mathbb{R}^{3,1} = \mathbb{R}^{1,1} \times \mathbb{R}^2$ and the corresponding inclusion $SO(1,1) \times SO(2) \subset SO(3,1)$. Using the definition (3) of $\Phi$, we get

$$\Phi^{-1}(SO(1,1) \times SO(2)) = \{ \cos z + \sin z I \mid z \in \mathbb{C} \} =: S^1_{\mathbb{C}} \subset Spin(3,1);$$

more precisely, setting $z = r + is, r, s \in \mathbb{R}$, we have in $\mathbb{H}_{\mathbb{C}}$,

$$\cos z + \sin z I = (\cosh s + i \sinh s I)(\cos r + i \sin r I),$$

and $\Phi(\cos z + \sin z I)$ is the Lorentz transformation of $\mathbb{R}^{3,1}$ which consists of a Lorentz transformation of angle $2s$ in $\mathbb{R}^{1,1}$ and a rotation of angle $2r$ in $\mathbb{R}^2$. Thus, defining

$$Spin(1,1) := \{ \pm(\cosh s + i \sinh s I) \mid s \in \mathbb{R} \} \subset Spin(3,1)$$

and

$$Spin(2) := \{ \cos r + i \sin r I \mid r \in \mathbb{R} \} \subset Spin(3,1),$$

we have

$$S^1_{\mathbb{C}} = Spin(1,1).Spin(2) \simeq Spin(1,1) \times Spin(2)/\mathbb{Z}_2$$

and the double cover $\Phi : S^1_{\mathbb{C}} \rightarrow SO(1,1) \times SO(2)$.

Finally, the representation

$$Spin(1,1) \times Spin(2) \rightarrow End_{\mathbb{C}}(\mathbb{H}_{\mathbb{C}}) \quad (g_1, g_2) \mapsto \rho(g) : \xi \mapsto g\xi,$$

where $g = g_1g_2 \in S^1_{\mathbb{C}} = Spin(1,1).Spin(2)$, is equivalent to the representation $\rho_1 \otimes \rho_2$ of $Spin(1,1) \times Spin(2)$, where $\rho_1$ and $\rho_2$ are the spinorial representations of $Spin(1,1)$ and $Spin(2)$; see [4, Remark 1.1].
2.2 Spin geometry of a timelike surface in $\mathbb{R}^{3,1}$

**Fundamental equations.** Let $M$ be an oriented timelike surface in $\mathbb{R}^{3,1}$ with normal bundle $E$ and second fundamental form $B : TM \times TM \to E$ defined by

$$B(X,Y) = \nabla_X Y - \nabla_Y X,$$

where $\nabla$ and $\nabla$ are the Levi-Civita connections of $M$ and $\mathbb{R}^{3,1}$ respectively. The second fundamental form satisfies the following fundamental equations ([12]):

1. $K = |B(e_1,e_2)|^2 - \langle B(e_1,e_1), B(e_2,e_2) \rangle$ (Gauss equation),
2. $K_N = \langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2 \rangle$ (Ricci equation),
3. $(\tilde{\nabla}_X B)(Y, Z) - (\tilde{\nabla}_Y B)(X, Z) = 0$ (Codazzi equation),

where $K$ and $K_N$ are the curvatures of $M$ and $E$, $(e_1,e_2)$ and $(e_3,e_4)$ are orthonormal basis of $TM$ and $E$ respectively, and where $\nabla$ is the natural connection induced on $T^* M \otimes^2 E$. As usual, if $\nu \in E$, $S_\nu$ stands for the symmetric operator on $TM$ such that, for all $X,Y \in TM$,

$$\langle S_\nu(X), Y \rangle = \langle B(X,Y), \nu \rangle.$$

**Remark 2.1.** Let $M$ be an abstract timelike surface, $E \to M$ be a bundle of rank 2, equipped with a Riemannian metric and a compatible connection. We assume that $B : TM \times TM \to E$ is a bilinear map satisfying the equations 1-, 2- and 3- above; the fundamental theorem of submanifolds says that there exists locally a unique isometric immersion of $M$ in $\mathbb{R}^{3,1}$ with normal bundle $E$ and second fundamental form $B$. We will prove this theorem in Corollary 3.4.

**Spinorial Gauss formula.** There exists an identification between the spinor bundle of $\mathbb{R}^{3,1}$ over $M$, $\Sigma \mathbb{R}^{3,1}_M$, and the spinor bundle of $M$ twisted by the spinorial normal bundle, $\Sigma := \Sigma M \otimes \Sigma E$ (see [3] and the end of Section 2.1). Moreover, as in the Riemannian case we obtain a spinorial Gauss formula: for all $\varphi \in \Gamma(\Sigma)$ and all $X \in TM$,

$$\nabla_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1,2} \epsilon_j e_j \cdot B(X, e_j) \cdot \varphi,$$

where $\epsilon_j = \langle e_j, e_j \rangle$, $\nabla$ is the spinorial connection of $\Sigma \mathbb{R}^{3,1}$, $\nabla$ is the spinorial connection of $\Sigma$ defined by $\nabla = \nabla^M \otimes \nabla^E$ the tensor product of the spinor connections on $\Sigma M$ and on $\Sigma E$, and the dot “.” is the Clifford action of $\mathbb{R}^{3,1}$. Thus, if we take $\varphi \in \Sigma \mathbb{R}^{3,1}_M$ parallel, its restriction to $M$, $\varphi := \varphi |_M$ satisfies

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1,2} \epsilon_j e_j \cdot B(X, e_j) \cdot \varphi,$$

for all $X \in TM$. Taking the trace, we have the following Dirac equation

$$D\varphi = \bar{H} \cdot \varphi,$$

where $D\varphi := -e_1 \nabla_{e_1} \varphi + e_2 \nabla_{e_2} \varphi$ and where $\bar{H} = \frac{1}{2} tr(\overline{\cdot}) B$ is the mean curvature vector of $M$ in $\mathbb{R}^{3,1}$.
2.3 Twisted spinor bundle

Let $M$ be an abstract oriented timelike surface, $E \to M$ be a bundle of rank 2 equipped with a Riemannian metric and a compatible connection, with given spin structures. We consider
\[ \Sigma := \Sigma M \otimes \Sigma E, \]
the tensor product of spinor bundles constructed from $TM$ and $E$. We endow $\Sigma$ with the spinorial connection
\[ \nabla := \nabla M \otimes \nabla E, \]
the tensor product of the spinor connections on $\Sigma M$ and on $\Sigma E$, and with the natural action of the Clifford bundle
\[ Cl(TM \oplus E) \simeq Cl(TM) \hat{\otimes} Cl(E), \]
see [4, 5, 6]. This permits to define the Dirac operator $D$ on $\Gamma(\Sigma)$ by
\[ D \varphi = - e_1 \cdot \nabla e_1 \varphi + e_2 \cdot \nabla e_2 \varphi, \]
where $(e_1, e_2)$ is an orthonormal frame of $TM$. If we denote by $Q_1$ and $Q_2$ the $SO(1, 1)$ and $SO(2)$ principal bundles of the oriented and orthonormal frames of $TM$ and $E$, and by $Q_1 \to Q_1$ and $Q_2 \to Q_2$ the given spin structures on $TM$ and $E$, then $\Sigma$ is the vector bundle associated to the $\text{Spin}(1, 1) \times \text{Spin}(2)$ principal bundle $\tilde{Q} := \tilde{Q}_1 \times_M \tilde{Q}_2$, and to the representation $\rho_1 \otimes \rho_2 \simeq \rho$ of the structure group $\text{Spin}(1, 1) \times \text{Spin}(2)$, that is
\[ \Sigma = \tilde{Q} \times \mathbb{H}^c / \rho. \]
Since the group $S^1_C = \text{Spin}(1, 1).\text{Spin}(2)$ belongs to $\text{Spin}(3, 1)$, which preserves the complex bilinear map $H$ defined on $\mathbb{H}^c$, the spinor bundle $\Sigma$ is also equipped with a complex bilinear map $H$ and with a real scalar product $\langle \cdot, \cdot \rangle := \Re H(\cdot, \cdot)$ of signature $(4, 4)$. We note that $H$ vanishes on the bundles $\Sigma^+$ and $\Sigma^-$ since $H$ vanishes on $S^+$ and $S^-$. We also define a $\mathbb{H}^c$-valued scalar product on $\Sigma$ by
\[ \langle \langle \psi, \psi' \rangle \rangle := \xi^\prime \xi, \]
where $\xi$ and $\xi'$ are the components of $\psi$ and $\psi'$ in some local section of $\tilde{Q}$; this scalar product satisfies the following properties:
\[ \langle \langle \psi, \psi' \rangle \rangle = \overline{\langle \langle \psi', \psi \rangle \rangle} \quad \text{and} \quad \langle \langle X \cdot \psi, \psi' \rangle \rangle = - \langle \langle \psi, X \cdot \psi' \rangle \rangle \]
for all $\psi, \psi' \in \Sigma$ and for all $X \in TM \oplus E$. Note that, by definition, $H(\psi, \psi')$ is the coefficient of 1 in the decomposition of $\langle \langle \psi, \psi' \rangle \rangle$ in the basis $I, J, K, \bar{K}$ of $\mathbb{H}^c$, and that (5) yields
\[ H(\psi, \psi') = H(\psi', \psi) \quad \text{and} \quad H(X \cdot \psi, \psi') = - \overline{H(\psi, X \cdot \psi')} \]
Notation. We will use the next notation: if \( \tilde{s} \in \tilde{Q} \) is a given spinorial frame, the brackets \([\cdot]\) will denote the coordinates in \( \mathbb{H}^C \) of the spinor fields in the frame \( \tilde{s} \), that is, for all \( \phi \in \Sigma \),

\[
\phi \simeq [\tilde{s}, [\phi]] \in \Sigma \simeq \tilde{Q} \times \mathbb{H}^C / \rho.
\]

We will also use the brackets to denote the coordinates in \( \tilde{s} \) of the elements of the Clifford algebra \( Cl(TM \oplus E) \): for \( X \in Cl_0(TM \oplus E) \) and \( Y \in Cl_1(TM \oplus E) \) will be respectively represented by \([X], [Y] \in \mathbb{H}^C\) such that, in \( \tilde{s} \),

\[
X \simeq \begin{pmatrix} [X] & 0 \\ 0 & [\tilde{X}] \end{pmatrix} \quad \text{and} \quad Y \simeq \begin{pmatrix} 0 & [Y] \\ [Y] & 0 \end{pmatrix}.
\]

Note that

\[
[X \cdot \phi] = [X][\phi] \quad \text{and} \quad [Y \cdot \phi] = [Y][\tilde{\phi}]
\]

and that, in a spinorial frame \( \tilde{s} \in \tilde{Q} \) such that \( \pi(\tilde{s}) = (e_1, e_2, e_3, e_4) \), where \( \pi: \tilde{Q} \rightarrow Q_1 \times M Q_2 \) is the natural projection onto the bundle of the orthonormal frames of \( TM \oplus E \) adapted to the splitting, \( e_1, e_2, e_3 \) and \( e_4 \in Cl_1(TM \oplus E) \) are respectively represented by \( i1, I, J \) and \( K \in \mathbb{H}^C \).

3 Spinor representation of timelike surfaces

In this section we will prove the spinor representation theorem of a timelike surface in \( \mathbb{R}^{3,1} \). This result is the generalization of the principal theorems of \([4, 5, 6]\) and completes the spinorial description of semi-Riemannian surfaces in four dimensional semi-Riemannian Euclidean space.

3.1 The proof of Theorem 1.1

The proof of affirmations \( 3 \Rightarrow 2 \Rightarrow 1 \) are given by the spinorial Gauss formula (see Section 2.2). As in [7] (and after in [8, 9, 10, 11] and in [4, 5, 6]) the proof of \( 1 \Rightarrow 3 \) relies on the fact that such spinor field necessarily solves a Killing type equation:

**Proposition 3.1.** If \( \phi \) is a solution of \( D\phi = \tilde{H} \cdot \phi \), with \( H(\phi, \phi) = 1 \), then \( \phi \) satisfies

\[
\nabla_X \phi = -\frac{1}{2} \sum_{j=1,2} e_j e_j \cdot B(X, e_j) \cdot \phi,
\]

for all \( X \in \Gamma(TM) \), where \( B : TM \times TM \rightarrow E \) is the bilinear and symmetric map defined by

\[
\langle B(X, Y), \nu \rangle = 2 \langle X \cdot \nabla_Y \phi, \nu \cdot \phi \rangle
\]

for all \( X, Y \in \Gamma(TM) \) and all \( \nu \in \Gamma(E) \).

Moreover, the map \( B \) satisfies the Gauss, Ricci and Codazzi equations and is such that \( \tilde{H} = \frac{1}{2} \text{tr}_\nu B \).
Note that, in the proposition we use the same notation \( \langle \cdot, \cdot \rangle \) to denote the scalar products on \( TM \), on \( E \) and on \( \Sigma \).

**Proof.** We consider the complex structure \( i := -e_1 \cdot e_2 \cdot e_3 \cdot e_4 \), defined on the Clifford bundle \( Cl(TM \oplus E) \) by the multiplication on the left, and on the spinor bundle \( \Sigma \) by the Clifford action. The map \( H : \Sigma \times \Sigma \to \mathbb{C} \) is \( \mathbb{C} \)-bilinear with respect to this complex structure, whereas the Clifford action satisfies
\[
i(X \cdot \varphi) = (iX) \cdot \varphi = -X \cdot (i\varphi),
\]
for all \( \varphi \in \Sigma \) and \( X \in TM \oplus E \). Now, we consider the following spinors
\[
\{ \varphi, e_1 \cdot e_2 \cdot \varphi, e_2 \cdot e_3 \cdot \varphi, e_3 \cdot e_1 \cdot \varphi \}.
\]
Using the identities in (6), we can show that these spinors form an \( H \)-orthonormal set of \( \Sigma \); in particular, for all \( X \in TM \) we have
\[
\begin{align*}
\nabla_X \varphi &= H(\nabla_X \varphi, \varphi) - H(\nabla_X \varphi, e_1 \cdot e_2 \cdot \varphi) e_1 \cdot e_2 \cdot \varphi \\
&\quad + H(\nabla_X \varphi, e_2 \cdot e_3 \cdot \varphi) e_2 \cdot e_3 \cdot \varphi - H(\nabla_X \varphi, e_3 \cdot e_1 \cdot \varphi) e_3 \cdot e_1 \cdot \varphi.
\end{align*}
\]
Using \( H(\varphi, \varphi) = 1 \), we get \( H(\nabla_X \varphi, \varphi) = 0 \) for all \( X \in TM \); on the other hand, using the Dirac equation \( D\varphi = \bar{H} \cdot \varphi \), we obtain \( H(\nabla_X \varphi, e_1 \cdot e_2 \cdot \varphi) = 0 \), for all \( X \in TM \): if \( X = e_1 \) (the case when \( X = e_2 \) is analogous) we have
\[
H(\nabla_{e_1} \varphi, e_1 \cdot e_2 \cdot \varphi) = -H(e_1 \cdot \nabla_{e_1} \varphi, e_2 \cdot \varphi) = H(-e_2 \cdot \nabla_{e_2} \varphi + \bar{H} \cdot \varphi, e_2 \cdot \varphi)
\]
\[
= -H(\nabla_{e_2} \varphi, \varphi) + H(\bar{H} \cdot \varphi, e_2 \cdot \varphi) = 0
\]
since \( H(\nabla_{e_2} \varphi, \varphi) = 0 \) and
\[
H(\bar{H} \cdot \varphi, e_2 \cdot \varphi) = -H(\varphi, \bar{H} \cdot e_2 \cdot \varphi) = -H(e_2 \cdot \varphi, \bar{H} \cdot \varphi) = -H(\bar{H} \cdot \varphi, e_2 \cdot \varphi).
\]
Thus, we can write \( \nabla_X \varphi = \eta(X) \cdot \varphi \), where
\[
\eta(X) := H(\nabla_X \varphi, e_2 \cdot e_3 \cdot \varphi) e_2 \cdot e_3 - H(\nabla_X \varphi, e_3 \cdot e_1) e_3 \cdot e_1.
\]
Using the relations \( i e_2 \cdot e_3 = e_1 \cdot e_4 \) and \( i e_3 \cdot e_1 = e_4 \cdot e_2 \), we can see that \( \eta(X) \) has the form
\[
\eta(X) = e_1 \cdot \nu_1 + e_2 \cdot \nu_2,
\]for some \( \nu_1, \nu_2 \in E \).Now, for each \( \nu \in E \) and \( j = 1, 2 \) we have
\[
\langle B(e_j, X), \nu \rangle = 2\langle e_j \cdot \nabla_X \varphi, \nu \cdot \varphi \rangle = -2\langle \nabla_X \varphi, e_j \cdot \nu \cdot \varphi \rangle = -2\langle \eta(X) \cdot \varphi, e_j \cdot \nu \cdot \varphi \rangle,
\]
using the expression (8) of \( \eta(X) \), we get
\[
\langle B(e_j, X), \nu \rangle = -2\langle e_1 \cdot \nu_1 \cdot \varphi, e_j \cdot \nu \cdot \varphi \rangle - 2\langle e_2 \cdot \nu_2 \cdot \varphi, e_j \cdot \nu \cdot \varphi \rangle.
\]We note that for all \( \nu, \nu' \in E \) we have
\[
\langle e_1 \cdot e_2 \cdot \varphi, \nu \cdot \nu' \cdot \varphi \rangle = 0
\]
(the proof is analogous as Lemma 3.1 of [4]). Thus, the identity (9) gives
\[\langle B(e_1, X), \nu \rangle = 2\langle \nu_1 \cdot \varphi, \nu \cdot \varphi \rangle = 2\langle \nu_1, \nu \rangle,\]
\[\langle B(e_2, X), \nu \rangle = -2\langle \nu_2 \cdot \varphi, \nu \cdot \varphi \rangle = -2\langle \nu_2, \nu \rangle,\]
therefore \(\nu_1 = \frac{1}{2}B(e_1, X)\) and \(\nu_2 = -\frac{1}{2}B(e_2, X)\), and thus, by (8), we obtain
\[\eta(X) = -\frac{1}{2} \sum_{j=1,2} \epsilon_j e_j \cdot B(X, e_j).\]
Finally, the Gauss, Ricci and Codazzi equations appear to be the integrability condition of (7). The proof is analogous to that given in [4, Theorem 2] and will therefore be omitted.

Using the fundamental theorem of submanifolds (see Remark 2.1) we obtain the proof of the first part of Theorem 1.1; the proof of the spinor representation formula is given in the next section.

### 3.2 The spinor representation formula

With the hypothesis of Theorem 1.1, assume that we have a spinor field \(\varphi \in \Gamma(\Sigma)\) such that
\[D\varphi = \vec{H} \cdot \varphi \quad (10)\]
with \(H(\varphi, \varphi) = 1\). We define the 1-form \(\xi : TM \oplus E \to \mathbb{H}^C\) by
\[\xi(X) = \langle \langle X \cdot \varphi, \varphi \rangle \rangle \in \mathbb{H}^C\]
where the pairing \(\langle \langle ., . \rangle \rangle : \Sigma \times \Sigma \to \mathbb{H}^C\) is defined in (4).

**Proposition 3.2.** The 1–form \(\xi\) satisfies the following fundamental properties:

1. \(\xi = -\vec{\xi}\), thus \(\xi\) takes its values in \(\mathbb{R}^{3,1} \subset \mathbb{H}^C\), and
2. \(\xi : TM \to \mathbb{R}^{3,1}\) is closed, i.e. \(d\xi = 0\).

**Proof.** The proof of the first affirmation is a consequence of the identities in (5) of the scalar product \(\langle \langle ., . \rangle \rangle\) (we recall the identification (2) of \(\mathbb{R}^{3,1}\) as a subset \(\mathbb{H}^C\)). The second property is a consequence of the Dirac equation (10); see [4, Proposition 4.1] and [6, Lemma 2.3] for similar properties and detailed proofs.

If we moreover assume that \(M\) is simply connected, since \(\xi : TM \to \mathbb{R}^{3,1}\) is a closed 1–form there exists a differentiable map \(F : M \to \mathbb{R}^{3,1}\) such that \(dF = \xi\), that is
\[F = \int \xi : M \to \mathbb{R}^{3,1} \quad \text{where} \quad \xi(X) = \langle \langle X \cdot \varphi, \varphi \rangle \rangle,\]
for all \(X \in TM\). The next theorem is fundamental:
**Theorem 3.3.** 1- The map $F : M \to \mathbb{R}^{3,1}$ is an isometric immersion.

2- The map $\Phi : E \to M \times \mathbb{R}^{3,1}$ given by
\[ X \in E_m \mapsto (F(m), \xi(X))^* \]
is an isometry between $E$ and the normal bundle $N(F(M))$ of $F(M)$ in $\mathbb{R}^{3,1}$, preserving connections and second fundamental form.

**Proof.** The proof is consequence of the properties of the Clifford action and is analogous to that given in [4, Theorem 3] and will therefore be omitted. \(\square\)

As in [4, 5, 6], Theorem 3.3 gives a spinorial proof of the fundamental theorem of submanifolds (see Remark 2.1).

**Corollary 3.4.** We may integrate the Gauss, Ricci and Codazzi equations in two steps:

1- first solving
\[ \nabla_X \varphi = \eta(X) \cdot \varphi \]
where
\[ \eta(X) = -\frac{1}{2} \sum_{j=1,2} \epsilon_j e_j \cdot B(X, e_j), \]
there exists a solution $\varphi$ in $\Gamma(\Sigma)$ such that $H(\varphi, \varphi) = 1$, unique up to the natural right-action of $\text{Spin}(3,1)$ on $\Gamma(\Sigma)$,

2- then solving
\[ dF = \xi \]
where $\xi(X) = \langle \langle X \cdot \varphi, \varphi \rangle \rangle$, the solution is unique, up to translations of $\mathbb{R}^{3,1} \subset \mathbb{H}^C$.

Note that the multiplication on the right by a constant belonging to $\text{Spin}(3,1)$ in the first step, and the addition of a constant belonging to $\mathbb{R}^{3,1}$ in the second step, correspond to a rigid motion in $\mathbb{R}^{3,1}$. Another consequence of Theorem 3.3 is the following classical formula:

**Corollary 3.5.** The Laplacian of the isometric immersion $F : M \to \mathbb{R}^{3,1}$ is given by
\[ \Delta F = 2\tilde{H} \]
where $\tilde{H}$ is the mean curvature vector of the immersion.

**Proof.** Using the properties in (5) we get
\[ \nabla dF(e_i, e_i) = \nabla \xi(e_i, e_i) = \langle \langle e_i \cdot \nabla e_i \varphi, \varphi \rangle \rangle - \langle \langle \tilde{e}_i \cdot \nabla \tilde{e}_i \varphi, \varphi \rangle \rangle, \]
thus, from Dirac equation (10) we obtain
\[ \Delta F = -\nabla dF(e_1, e_1) + \nabla dF(e_2, e_2) = \langle \langle D \varphi, \varphi \rangle \rangle - \langle \langle \tilde{D} \varphi, \varphi \rangle \rangle = 2\langle \langle \tilde{H} \cdot \varphi, \varphi \rangle \rangle \]
where $\xi(\tilde{H}) = \langle \langle \tilde{H} \cdot \varphi, \varphi \rangle \rangle$ is the mean curvature vector of the immersion. \(\square\)
Applications of the spinor representation formula in Sections 4, 5, 6 and 7 will rely on the following simple observation: assume that \( F_0 : M \to \mathbb{R}^{3,1} \) is an isometric immersion and consider \( \varphi = \pm |I|_M \) the restriction to \( M \) of the constant spinor field \( +I \) or \(-I \in \mathbb{H}^C \) of \( \mathbb{R}^{3,1} \); if
\[
F = \int \xi, \quad \xi(X) = \langle (X \cdot \varphi, \varphi) \rangle
\]
is the immersion given in the theorem, then \( F \simeq F_0 \). This is in fact trivial since
\[
\xi(X) = \langle (X \cdot \varphi, \varphi) \rangle = [\varphi][X][\varphi] = [X],
\]
in a spinorial frame \( \hat{s} \) of \( \mathbb{R}^{3,1} \) which is above the canonical basis (in such a frame \([\varphi] = \pm I\)). The representation formula (11), when written in moving frames adapted to the immersion, will give nontrivial formulas.

## 4 Timelike surfaces in the De Sitter space

In this section we deduce spinor characterizations of timelike surfaces in three-dimensional Minkowski space \( \mathbb{R}^{2,1} \) and De Sitter space: in the first case, we recover the characterization given in [6]; in the second case, we obtain a new characterization which is different to the given in [9].

We suppose that \( E = \mathbb{R}e_3 \oplus \mathbb{R}e_4 \) where \( e_3 \) and \( e_4 \) are unit, orthogonal and parallel sections of \( E \) and such that \((e_3, e_4)\) is positively oriented. We consider the isometric embedding of \( \mathbb{R}^{2,1} \) and the De Sitter space in \( \mathbb{R}^{3,1} \subset H^C \) given by
\[
\mathbb{R}^{2,1} := (K)^{-1} \quad \text{and} \quad S^{2,1} := \{ x \in \mathbb{R}^{3,1} \mid \langle x, x \rangle = 1 \},
\]
where \( K \) is the fourth vector of the canonical basis of \( \mathbb{R}^{3,1} \subset H^C \). Let \( \vec{H} \) be a section of \( E \) and \( \varphi \in \Gamma(\Sigma) \) be a solution of
\[
D\varphi = \vec{H} \cdot \varphi, \quad H(\varphi, \varphi) = 1.
\]
According to Theorem 1.1, the spinor field \( \varphi \) defines an isometric immersion \( M \to \mathbb{R}^{3,1} \) (unique, up to translations), with normal bundle \( E \) and mean curvature vector \( \vec{H} \). We give a characterization of the isometric immersion in \( \mathbb{R}^{2,1} \) and \( S^{2,1} \) (up to translations) in terms of \( \varphi \):

**Proposition 4.1.** 1- Assume that
\[
\vec{H} = He_3 \quad \text{and} \quad e_4 \cdot \varphi = \pm i\varphi.
\]
Then the isometric immersion \( M \to \mathbb{R}^{3,1} \) belongs to \( \mathbb{R}^{2,1} \).
2- Consider the function \( F = \langle (e_4 \cdot \varphi, \varphi) \rangle \) and assume that
\[
\vec{H} = He_3 - e_4 \quad \text{and} \quad dF(X) = \langle (X \cdot \varphi, \varphi) \rangle.
\]
Then the isometric immersion $M \to \mathbb{R}^{3,1}$ belongs to $\mathbb{S}^{2,1}$.

Reciprocally, if $M \to \mathbb{R}^{3,1}$ belongs to $\mathbb{R}^{2,1}$ (resp. to $\mathbb{S}^{2,1}$), then (14) (resp. (15)) holds for some unit, orthogonal and parallel sections $(e_3, e_4)$ of $E$.

Proof. 1- We suppose that (14) holds, and we compute

$$\xi(e_4) = \langle \langle e_4 \cdot \varphi, \varphi \rangle \rangle = \pm \langle \langle i \varphi, \varphi \rangle \rangle = \pm |\varphi|(|\varphi|K) = \pm K.$$ 

The constant vector $K$ is thus normal to the immersion (by Theorem 3.3, since this is $\xi(e_4)$), and the result follows.

2- Analogously, assuming that (15) holds, the function $F = \langle \langle e_4 \cdot \varphi, \varphi \rangle \rangle$ is a primitive of the 1-form $\xi(X) = \langle \langle X \cdot \varphi, \varphi \rangle \rangle$, and is thus the isometric immersion defined by $\varphi$ (uniquely defined, up to translations); since the Minkowski norm of $\langle \langle e_4 \cdot \varphi, \varphi \rangle \rangle \in \mathbb{R}^{3,1} \subset \mathbb{H}^C$ coincides with the norm of $e_4$, and is thus constant equal to 1, the immersion belongs to $\mathbb{S}^{2,1}$.

For the converse statements, we choose $(e_3, e_4)$ such that $\langle \langle e_4 \cdot \varphi, \varphi \rangle \rangle = \pm K$ in the first case and such that $\langle \langle e_4 \cdot \varphi, \varphi \rangle \rangle$ is the normal vector to $\mathbb{S}^{2,1}$ in $\mathbb{R}^{3,1}$ in the second case. Writing these identities in some frame $\tilde{s}$, we easily deduce (14) and (15).

We now assume that $M \subset H \subset \mathbb{R}^{3,1}$, where $H$ is $\mathbb{R}^{2,1}$ or $\mathbb{S}^{2,1}$, and consider $e_3$ and $e_4$ unit vector fields such that $\mathbb{R}^{3,1} = TH \oplus_{\perp} \mathbb{R} e_4$ and $TH = TM \oplus_{\perp} \mathbb{R} e_3$.

The intrinsic spinors of $M$ indentify with the spinors of $H$ restricted to $M$, which in turn identify with the positive spinors of $\mathbb{R}^{3,1}$ restricted to $M$ : this is the content of Proposition 4.2 below, which, together with the previous result, will give the representation of timelike surfaces in $\mathbb{R}^{2,1}$ and $\mathbb{S}^{2,1}$ by means of spinors of $\Sigma M$ only.

We define the scalar product on $\mathbb{C}^2$ by setting

$$\langle \langle a + ib, c + id \rangle \rangle = \frac{ad' + a'd - bc' - b'c}{2},$$ 

of signature $(2, 2)$. This scalar product is $Spin(1, 1)$-invariant, thus induces a scalar product $\langle \cdot, \cdot \rangle$ on the spinor bundle $\Sigma M$. It satisfies the following properties:

$$\langle \psi, \psi' \rangle = \langle \psi', \psi \rangle \quad \text{and} \quad \langle X \cdot_M \psi, \psi' \rangle = -\langle \psi, X \cdot_M \psi' \rangle,$$

for all $\psi, \psi' \in \Sigma M$ and all $X \in TM$. This is the scalar product on $\Sigma M$ that we use in this section (and in this section only). We moreover define $|\psi|^2 := \langle \psi, \psi \rangle$ and, we denote by $i$ the natural complex structure of $\Sigma M$, which is such that the Clifford action is $\mathbb{C}$–linear. The following proposition is analogous to the given in [4, 5, 6] (see also [10, Proposition 2.1], and the references therein).
Proposition 4.2. There exists an identification

$$\Sigma M \xrightarrow{\sim} \Sigma^+_M$$

$$\psi \mapsto \psi^*$$

such that, for all $X \in TM$ and all $\psi \in \Sigma M$, $(\nabla_X \psi)^* = \nabla_X \psi^*$, the Clifford actions are linked by

$$(X \cdot_M \psi)^* = X \cdot e_3 \cdot \psi^*$$

and the following two properties holds:

$$H(\psi^*, ie_4 \cdot \psi^*) = -\frac{1}{2} |\psi|^2,$$  \hspace{1cm} (16)

and

$$d\langle\langle e_4 \cdot \psi^*, \psi^* \rangle\rangle(X) = \langle\langle X \cdot \psi^*, \psi^* \rangle\rangle \iff d\left( |\psi|^2 \right) (X) = \langle i(X \cdot_M \overline{\psi}), \psi \rangle. \hspace{1cm} (17)$$

Using this identification, the intrinsic Dirac operator on $M$, defined by

$$D_M \psi := -e_1 \cdot_M \nabla e_1 \psi + e_2 \cdot_M \nabla e_2 \psi,$$

where $(e_1, e_2)$ is an orthogonal basis tangent to $M$ such that $|e_1|^2 = -1$ and $|e_2|^2 = 1$, is linked to $D$ by

$$(D_M \psi)^* = -e_3 \cdot D \psi^*$$  \hspace{1cm} (18)

We suppose that $\varphi \in \Gamma(\Sigma)$ is a solution of equation (13), we may consider $\psi \in \Sigma M$ such that $\psi^* = \varphi^+$; it satisfies

$$(D_M \psi)^* = -e_3 \cdot D \psi^* = -e_3 \cdot \vec{H} \cdot \psi^*.$$

(19)

Note that $\psi \neq 0$, since

$$H(\varphi, \varphi) = 2H(\varphi^+, \varphi^-) = 1,$$  \hspace{1cm} (20)

where the decomposition $\varphi = \varphi^+ + \varphi^-$ is the decomposition in $\Sigma = \Sigma^+ \oplus \Sigma^-$, and recall that $H$ vanishes on $\Sigma^+$ and $\Sigma^-$; see Section 2.3.

We consider the case of a timelike surface in $\mathbb{R}^{2,1}$, i.e. $\mathcal{H} = \mathbb{R}^{2,1}$. Then, $\vec{H}$ is of the form $He_3$ and (19) reads

$$D_M \psi = H \psi;$$

moreover, (20), (14) and (16) imply that $|\psi|^2 = \pm 1$. This is the spinorial characterization of an isometric immersion in $\mathbb{R}^{2,1}$ given in [6].

Now, we examine the case of a timelike surface in $\mathbb{S}^{2,1}$. If $\mathcal{H} = \mathbb{S}^{2,1}$, then $\vec{H}$ is of the form $He_3 - e_4$, and using (19) we get

$$(D_M \psi)^* = -e_3 \cdot \vec{H} \cdot \psi^* = -e_3 \cdot (He_3 - e_4) \cdot \psi^* = H \psi^* + e_3 \cdot e_4 \cdot \psi^* = H \psi^* - (i\psi)^*,$$
where $\psi = \psi^+ - \psi^-$ denotes the usual conjugation in $\Sigma M$. Moreover, it is not difficult to prove that (15) implies that (17) holds. We thus get

$$D_M \psi = H \psi - i\bar{\psi} \quad \text{and} \quad d\left(|\psi|^2\right)(X) = \langle i(X \cdot_M \bar{\psi}), \psi \rangle.$$ (21)

Reciprocally, let $M$ be a timelike surface and $H : M \to \mathbb{R}$ a given differentiable function, and suppose that $\psi \in \Gamma(\Sigma M)$ satisfies (21). We define $\varphi^+ := \psi^* \in \Sigma^+$ and $\bar{H} := H e_3 - e_4$, where $e_3$ and $e_4$ are unit, orthogonal and parallel sections of $E$, and such that $(e_3, e_4)$ is positively oriented. Using (21), (18) and (17) we obtain

$$D \varphi^+ = \bar{H} \cdot \varphi^+ \quad \text{and} \quad d\langle (e_4 \cdot \psi^*, \psi^*) \rangle(X) = \langle (X \cdot \psi^*, \psi^*) \rangle.$$ (21)

**Proposition 4.3.** Let $\psi \in \Gamma(\Sigma M)$ be a solution of (21). There exists a spinor field $\varphi \in \Gamma(\Sigma)$ solution of

$$D \varphi = \bar{H} \cdot \varphi \quad \text{and} \quad H(\varphi, \varphi) = 1,$$

with $\varphi^+ = \psi^*$ and such that the immersion defined by $\varphi$ is given by $F = \langle (e_4 \cdot \varphi, \varphi) \rangle$. In particular $F(M)$ belongs to $S^{2,1}$.

**Proof.** We need to find $\varphi^-$ solution of the system

$$F_1 = \langle (e_4 \cdot \varphi^-, \varphi^-) \rangle$$
$$dF_1(X) = \langle (X \cdot \varphi^-, \varphi^-) \rangle$$

with $[\varphi^+]|\varphi^-| = 1/2$; this system is equivalent to

$$\varphi^- = -2e_4 \cdot (\varphi^+ \cdot F_1),$$

with $2[\varphi^+ \cdot F_1]|\varphi^+] = K$, where $F_1 : M \to \mathbb{H}^C$ solves the equation

$$\varphi^+ \cdot dF_1(X) = -\omega(X) \cdot (\varphi^+ \cdot F_1),$$ (22)

where $\omega(X) = X \cdot e_4$, for all $X \in TM$. Above, $\cdot$ means the natural action of $\mathbb{H}^C$ on $\Sigma$ on the right given in coordinates by $[\varphi \cdot q] = [\varphi]q$. The compatibility equation of (22) is given by

$$[\omega(X), \omega(Y)] = [\omega(X), \eta(Y)] - [\omega(Y), \eta(X)],$$ (23)

where $\eta$ is such that $\nabla_X \varphi^+ = \eta(X) \cdot \varphi^+$, and where $[p, p'] = pp' - p'p$, for all $p, p' \in Cl_0(3,1)$; by a direct computation (23) is satisfied, and thus (22) is solvable.

**Remark 4.4.** A solution of (21) is thus equivalent to an isometric immersion in three-dimensional De Sitter space $S^{2,1}$. We thus obtain a spinorial characterization of an isometric immersion of a timelike surface in $S^{2,1}$, which is simpler than the characterization given in [9], where two spinor fields are needed.
5 The Laplacian of the Gauss map of a timelike surface in $\mathbb{R}^{3,1}$

The main goal of this section is to compute the Laplacian of the Gauss map of a timelike surface in $\mathbb{R}^{3,1}$.

5.1 The Grassmannian of the timelike planes in $\mathbb{R}^{3,1}$

The Grassmannian of the oriented timelike planes in $\mathbb{R}^{3,1}$ identifies to

$$Q = \{u_1 \cdot u_2 \mid u_1, u_2 \in \mathbb{R}^{3,1}, |u_1|^2 = -|u_2|^2 = -1\} \subset Cl_0(3, 1).$$

Setting $\Im\mathbb{H} := C_iI \oplus C J \oplus C iK$ and since $e_1 \cdot e_2 \simeq iI, e_2 \cdot e_4 \simeq -J$ and $e_4 \cdot e_1 \simeq -iK$ in the identification $Cl_0(3, 1) \simeq \mathbb{H}$ given in (1), we easily get

$$Q = \{\xi \in \Im\mathbb{H} \mid H(\xi, \xi) = -1\}.$$

We define the cross product of two vectors $\xi, \xi' \in \Im\mathbb{H}$ by

$$\xi \times \xi' := \frac{1}{2}(\xi \xi' - \xi' \xi) \in \Im\mathbb{H}.$$

We also define the mixed product of three vectors $\xi, \xi', \xi'' \in \Im\mathbb{H}$ by

$$[\xi, \xi', \xi''] := H(\xi \times \xi', \xi'') \in \mathbb{C};$$

it is easily seen to be, up to sign, the determinant of the vectors $\xi, \xi', \xi'' \in \Im\mathbb{H}$ in the basis $(iI, J, iK)$ of $\Im\mathbb{H}$ (considered as a complex space). The mixed product is a complex volume form on $\Im\mathbb{H}$, and induces a natural complex area form $\omega_Q$ on $Q$ by

$$\omega_Q(p)(\xi, \xi') := [\xi, \xi', p],$$

for all $p \in Q$ and all $\xi, \xi' \in T_p Q$. Note that $\omega_Q(p)(\xi, \xi') = 0$ if and only if $\xi$ and $\xi'$ are dependent over $\mathbb{C}$.

5.2 The Gauss map of a timelike surface in $\mathbb{R}^{3,1}$

Let $M$ be an oriented timelike surface in $\mathbb{R}^{3,1}$. We consider its Gauss map

$$G : M \rightarrow Q \quad \text{ where, at } x \in M, (u_1, u_2) \text{ is a positively oriented orthogonal basis of } T_x M \text{ such that } |u_1|^2 = -|u_2|^2 = -1.$$
Proposition 5.1. We have
\[ G^* \omega_Q = (K + iK_N) \omega_M, \]
where \( \omega_M \) is the area form, \( K \) is the Gauss curvature and \( K_N \) is the normal curvature of \( M \). In particular, assuming moreover that
\[ dG_x : T_x M \rightarrow T_{G(x)} Q \]
is one-to-one at some point \( x \in M \), then \( K = K_N = 0 \) at \( x \) if and only if the linear space \( dG_x(T_x M) \) is a complex line in \( T_{G(x)} Q \), i.e.
\[ dG_x(T_x M) = \{ z U | z \in \mathbb{C} \} \]
where \( U \) is some vector belonging to \( T_{G(x)} Q \subset \mathbb{H}^C \).

As a consequence of this proposition, if \( K = K_N = 0 \) and \( G : M \rightarrow Q \) is a regular map (i.e. if \( dG_x \) is injective at every point \( x \) of \( M \)), there is a unique complex structure \( J \) on \( M \) such that
\[ dG_x(Ju) = i dG_x(u) \]
for all \( x \in M \) and all \( u \in T_x M \). Indeed, (24) implies that \( dG_x(T_x M) \) is stable by multiplication by \( i \), and we may define
\[ Ju := dG_x^{-1}(i dG_x(u)). \]
This complex structure coincides with the complex structure considered in [1], and we will use this in Section 6.

5.3 The Laplacian of the Gauss map

We suppose that the immersion of \( M \) in \( \mathbb{R}^{3,1} \) is given by some spinor field \( \varphi \in \Gamma(\Sigma) \) solution of the Dirac equation \( D\varphi = \bar{H} \cdot \varphi \) with \( H(\varphi, \varphi) = 1 \). We first express the Gauss map of the immersion in terms of \( \varphi \):

Lemma 5.2. The Gauss map is given by
\[ G : M \rightarrow Q \]
\[ x \mapsto \langle e_1 \cdot e_2 \cdot \varphi, \varphi \rangle \]
where, for all \( x \in M \), \( (e_1, e_2) \) is a positively oriented and orthonormal basis of \( T_x M \).

Proof. Setting \( u_1 = \langle e_1 \cdot \varphi, \varphi \rangle \) and \( u_2 = \langle e_2 \cdot \varphi, \varphi \rangle \in \mathbb{R}^{3,1} \subset \mathbb{H}^C \), the basis \( (u_1, u_2) \) is an orthonormal basis of the immersion (Theorem 3.3), and
\[ u_1 \cdot u_2 \simeq u_1 \widehat{u}_2 = \langle \overline{e_1} \cdot \overline{e_2} \cdot \overline{\varphi}, \varphi \rangle = \left( \overline{[\varphi]} e_1[\varphi] \right) \left( \overline{[\varphi]} e_1[\varphi] \right) \]
\[ = \overline{[\varphi]} e_1[\varphi] = \langle \overline{e_1} \cdot \overline{e_2} \cdot \varphi, \varphi \rangle, \]
where \( [e_1], [e_2] \) and \( [\varphi] \in \mathbb{H}^C \) represent \( e_1, e_2 \) and \( \varphi \) in some frame \( \tilde{s} \) of \( \tilde{Q} \). \( \square \)
According to Theorem 1.1, the spinor field $\varphi$ also satisfies $\nabla_X \varphi = \eta(X) \cdot \varphi$, for all $X \in TM$, where

$$\eta(X) = -\frac{1}{2} \sum_{j=1,2} \epsilon_j e_j \cdot B(X, e_j);$$

(26)

the second fundamental form $B$ was defined in Proposition 3.1. The differential of the Gauss map is linked to the second fundamental form $B$ as follows:

**Lemma 5.3.** The 1-form $\tilde{\eta} := \langle \eta \cdot \varphi, \varphi \rangle$ satisfies

$$dG = 2G \tilde{\eta}.$$ 

**Proof.** We suppose that $(e_1, e_2)$ is a moving frame on $M$ such that $\nabla e_{ijp} = 0$ and compute

$$dG(X) = \langle (e_1 \cdot e_2 \cdot \nabla_X \varphi), \varphi \rangle + \langle (e_1 \cdot e_2 \cdot \varphi, \nabla_X \varphi) \rangle$$

$$= \langle (e_1 \cdot e_2 \cdot \eta(X) \cdot \varphi), \varphi \rangle + \langle (e_1 \cdot e_2 \cdot \varphi, \eta(X) \cdot \varphi) \rangle$$

$$= 2 \langle (e_1 \cdot e_2 \cdot \eta(X) \cdot \varphi), \varphi \rangle.$$ 

But

$$\langle (e_1 \cdot e_2 \cdot \eta(X) \cdot \varphi, \varphi) \rangle = [\varphi][e_1 \cdot e_2][\eta(X)][\varphi] = \langle (e_1 \cdot e_2 \cdot \varphi, \varphi) \rangle \langle \eta(X) \cdot \varphi, \varphi \rangle$$

where $[\varphi], [e_1 \cdot e_2]$ and $[\eta(X)] \in \mathbb{H}^C$ represent $\varphi, e_1 \cdot e_2$ and $\eta(X)$ respectively in some local frame $\tilde{s}$ of $Q$. 

Using the lemma above, in the same moving frame, the Laplacian of the Gauss map $G$ seen as a map from $M$ to $\mathbb{H}^C$ (i.e. we will use the connection of the ambient space instead of the connection of the Grassmannian) is given by

$$\Delta G = -\nabla dG(e_1, e_1) + \nabla dG(e_2, e_2)$$

$$= -2 \{e_1(G\tilde{\eta}(e_1)) - G\tilde{\eta}(\nabla e_1 e_1)\} + 2 \{e_2(G\tilde{\eta}(e_2)) - G\tilde{\eta}(\nabla e_2 e_2)\}$$

$$= 2G(-2\tilde{\eta}(e_1)\tilde{\eta}(e_1) + 2\tilde{\eta}(e_2)\tilde{\eta}(e_2) - e_1(\tilde{\eta}(e_1)) + e_2(\tilde{\eta}(e_2))).$$

(27)

Now, we note that

$$-\tilde{\eta}(e_1)\tilde{\eta}(e_1) + \tilde{\eta}(e_2)\tilde{\eta}(e_2) = -[\varphi][\eta(e_1)][\varphi][\eta(e_1)][\varphi] + [\varphi][\eta(e_2)][\varphi][\eta(e_2)][\varphi]$$

$$= -[\varphi][\eta(e_1) \cdot \eta(e_1)][\varphi] + [\varphi][\eta(e_2) \cdot \eta(e_2)][\varphi]$$

$$= [\varphi][-\eta(e_1) \cdot \eta(e_1) + \eta(e_2) \cdot \eta(e_2)][\varphi]$$

$$= \langle (\eta(e_1) \cdot \eta(e_1) + \eta(e_2) \cdot \eta(e_2)) \cdot \varphi, \varphi \rangle;$$

and

$$-e_1(\tilde{\eta}(e_1)) + e_2(\tilde{\eta}(e_2)) = -\langle (\nabla e_1 \eta(e_1) \cdot \varphi, \varphi) \rangle - \langle (\eta(e_1) \cdot \varphi, \nabla e_1 \varphi) \rangle$$

$$+ \langle (\nabla e_2 \eta(e_2) \cdot \varphi, \varphi) \rangle + \langle (\eta(e_2) \cdot \varphi, \nabla e_2 \varphi) \rangle$$

$$= \langle (\eta(e_1) \cdot \varphi, \eta(e_1) \cdot \varphi) \rangle + \langle (\eta(e_2) \cdot \varphi, \eta(e_2) \cdot \varphi) \rangle,$$
By a direct computation we get
\[
\nabla \eta = \eta (e_i) \cdot \varphi \quad \text{and} \quad \nabla \varphi = 0.
\]
Using the expression of $\eta$ given in (26), we have
\[
\eta (e_i) = \frac{1}{2} (e_1 \cdot B_{i1} - e_2 \cdot B_{i2}) \quad \text{where} \quad B_{ij} = B(e_i, e_j).
\]
1- By a direct computation we get
\[
-\eta (e_1) \eta (e_1) = \frac{1}{4} (B_{11}^2 - B_{12}^2 + e_1 \cdot e_2 \cdot (B_{12} \cdot B_{11} - B_{11} \cdot B_{12}))
\]
and
\[
\eta (e_2) \eta (e_2) = \frac{1}{4} (B_{22}^2 - B_{12}^2 + e_1 \cdot e_2 \cdot (B_{12} \cdot B_{22} - B_{22} \cdot B_{12})).
\]
Using the Guass and Ricci equations, we easily get
\[
B_{11}^2 - 2B_{12}^2 + B_{22}^2 = -4|\bar{H}|^2 + 2K;
\]
and
\[
B_{12} \cdot B_{11} - B_{11} \cdot B_{12} + B_{12} \cdot B_{22} - B_{22} \cdot B_{12} = -2K e_3 \cdot e_4.
\]
2- We have
\[
-\nabla_{e_1} \eta (e_1) + \nabla_{e_2} \eta (e_2) = \frac{1}{2} (-e_1 \cdot \nabla_{e_1} B_{11} + e_2 \cdot \nabla_{e_1} B_{12} + e_1 \cdot \nabla_{e_2} B_{12} - e_2 \cdot \nabla_{e_2} B_{22});
\]
using the Codazzi equation (recall that $(e_1, e_2)$ is a moving frame on $TM$ such that $\nabla e_{ij} = 0$) we obtain
\[
\nabla_{e_1} B_{12} = \nabla_{e_2} B_{11} \quad \text{and} \quad \nabla_{e_2} B_{12} = \nabla_{e_1} B_{22},
\]
and since $\bar{H} = \frac{1}{2} (B_{11} + B_{22})$ we obtain the result. \qed
Therefore, using the identities of the lemma above, we get
\[
\langle\langle -\eta(e_1) \cdot \eta(e_1) + \eta(e_2) \cdot \eta(e_2) \rangle \cdot \varphi, \varphi \rangle = \left(-|\vec{H}|^2 + \frac{K}{2}\right) \langle\langle \varphi, \varphi \rangle \rangle + \frac{K_N}{2} \langle\langle -e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot \varphi, \varphi \rangle \rangle
\]

since \( \langle\langle \varphi, \varphi \rangle \rangle = H(\varphi, \varphi) = 1 \), and since the scalar product \( \langle\langle \cdot, \cdot \rangle \rangle \) is \( \mathbb{C} \)-bilinear if \( \Sigma \) is endowed with the complex structure given by the Clifford action of \( -e_1 \cdot e_2 \cdot e_3 \cdot e_4 \) (which corresponds to the multiplication by \( i \) on \( \mathbb{H}^\mathbb{C} \)), we finally get
\[
\langle\langle -\eta(e_1) \cdot \eta(e_1) + \eta(e_2) \cdot \eta(e_2) \rangle \cdot \varphi, \varphi \rangle = \left(-|\vec{H}|^2 + \frac{K}{2}\right) + \frac{K_N}{2}.
\]

On the other hand, using the Dirac equation \( D\varphi = \vec{H} \cdot \varphi \), we have
\[
D(\vec{H} \cdot \varphi) = -e_1 \cdot \nabla_{e_1}(\vec{H} \cdot \varphi) + e_2 \cdot \nabla_{e_2}(\vec{H} \cdot \varphi)
\]
\[= -\left(1 \cdot \nabla_{e_1} \vec{H} - e_2 \cdot \nabla_{e_2} \vec{H}\right) \cdot \varphi - \vec{H} \cdot D\varphi
\]
\[= -\left(1 \cdot \nabla_{e_1} \vec{H} - e_2 \cdot \nabla_{e_2} \vec{H}\right) \cdot \varphi + |\vec{H}|^2 \varphi,
\]

thus
\[
\langle\langle -(\nabla_{e_1} \eta(e_1) + \nabla_{e_2} \eta(e_2)) \cdot \varphi, \varphi \rangle \rangle = |\vec{H}|^2 \langle\langle \varphi, \varphi \rangle \rangle - \langle\langle D(\vec{H} \cdot \varphi), \varphi \rangle \rangle
\]
\[= |\vec{H}|^2 - \langle\langle D(\vec{H} \cdot \varphi), \varphi \rangle \rangle.
\]

Finally, replacing this expressions in (28) we obtain the formula for the Laplacian of the Gauss map
\[
\Delta G = G(K + iK_N) - 2G\langle\langle D(\vec{H} \cdot \varphi), \varphi \rangle \rangle.
\]

As a consequence of this, if the immersion \( M \subset \mathbb{R}^{3,1} \) have parallel mean curvature vector, using (29) we get
\[
\Delta G = (-2|\vec{H}|^2 + K + iK_N)G.
\]

This formula generalizes a classical result for surfaces in Euclidean space with constant mean curvature whose Gauss map is seen as a map from the surface in \( \mathbb{R}^3 \); see [2]. As a particular case of (30), we obtain the following result concerning the Laplacian of the Gauss map of a minimal timelike surface in \( \mathbb{R}^{3,1} \).

**Corollary 5.5.** Assume that \( M \) is a minimal timelike surface in \( \mathbb{R}^{3,1} \). Then the Laplacian of its Gauss map is given by the following formula
\[
\Delta G = (K + iK_N)G
\]

where \( K \) and \( K_N \) are the Gauss and normal curvatures of the surface.
6 Flat timelike surfaces with flat normal bundle and regular Gauss map in $\mathbb{R}^{3,1}$

We suppose that $M$ is simply connected and that the bundles $TM$ and $E$ are flat ($K = K_N = 0$). Recall that the spinor bundle $\Sigma := \Sigma M \otimes \Sigma E$ is associated to the principal bundle $\tilde{Q}$ and to the representation $\rho$ of the structure group $Spin(1, 1) \times Spin(2)$ in $\mathbb{H}^C$ (Section 2.3). Since the curvatures $K$ and $K_N$ are zero, the spinorial connection on the bundle $\tilde{Q}$ is flat, and $\tilde{Q}$ admits a parallel local section $\tilde{s}$; since $M$ is simply connected, the section $\tilde{s}$ is in fact globally defined. We consider $\varphi \in \Gamma(\Sigma)$ a solution of the Dirac equation

$$D\varphi = \bar{H} \cdot \varphi$$

such that $H(\varphi, \varphi) = 1$, and define $g := [\varphi] : M \to Spin(3, 1) \subset \mathbb{H}^C$ such that

$$\varphi = [\tilde{s}, g] \in \Sigma = \tilde{Q} \times \mathbb{H}^C/\rho,$$

that is, $g$ in $\mathbb{H}^C$ represents $\varphi$ in the parallel section $\tilde{s}$. Recall that, by Theorem 1.1, $\varphi$ also satisfies

$$\nabla_X \varphi = \eta(X) \cdot \varphi$$

for all $X \in TM$, where

$$\eta(X) = -\frac{1}{2} \sum_{j=1,2} \epsilon_j e_j \cdot B(X, e_j)$$

for some bilinear map $B : TM \times TM \to E$.

In the following, we will denote by $(e_1, e_2)$ and $(e_3, e_4)$ the parallel, orthonormal and positively oriented frames, respectively tangent, and normal to $M$, corresponding to $\tilde{s}$, i.e. such that $\pi(\tilde{s}) = (e_1, e_2, e_3, e_4)$ where $\pi : \tilde{Q} \to Q_1 \times Q_2$ is the natural projection. We moreover assume that the Gauss map $G$ of the immersion defined by $\varphi$ is regular, and consider the complex structure $J$ induced on $M$ by $G$, defined by (25).

Below we will prove that $g : M \to Spin(3, 1) \subset \mathbb{H}^C$ is a holomorphic map and that the immersion defined by $\varphi$ depends on two holomorphic maps and two smooth functions. We need the following lemmas; see in [4, 6] similar results.

**Lemma 6.1.** The Gauss map of the immersion defined by $\varphi$ is given by

$$G : M \longrightarrow \mathbb{Q} \subset \mathfrak{Im} \mathbb{H}^C$$

where $g : [\varphi] : M \to Spin(3, 1) \subset \mathbb{H}^C$ represents $\varphi$ in some local section of $\tilde{Q}$.

**Proof.** This is the identity given in Lemma 5.2

$$G = \{(e_1 \cdot e_2) \cdot \varphi, \varphi)\}$$

written in a section of $\tilde{Q}$ above $(e_1, e_2)$. □
Lemma 6.2. Denoting by $[\eta] \in \Omega^1(M, \mathbb{H}^C)$, the $1$–form which represents $\eta$ in $\tilde{s}$, we have
\[
[\eta] = dg \; g^{-1} = \theta_1 J + \theta_2 iK,
\] (34)
where $\theta_1$ and $\theta_2$ are two complex $1$–forms on $M$.

Proof. This is (31) in the parallel frame $\tilde{s}$, taking into account the special form (32) of $\eta$ for the last equality. $\square$

Lemma 6.3. The $1$–form $\tilde{\eta} := \langle \langle \eta \cdot \varphi, \varphi \rangle \rangle$ satisfies
\[
\tilde{\eta} = \frac{1}{2} G^{-1} dG = g^{-1} dg.
\]

Proof. Writing $\tilde{\eta}$ in $\tilde{s}$ together with (34) imply that $\tilde{\eta} = g^{-1} dg$. The other identity was given in Lemma 5.3. $\square$

The properties (33) and (34) may be rewritten as follows (see similar results in [4, 6]):

Lemma 6.4. Consider the projection
\[
p: \text{Spin}(3,1) \subset \mathbb{H}^C \longrightarrow \mathcal{Q} \subset \Im \mathbb{H}^C
g \mapsto ig^{-1}1g
\]
as a $S^1_C$–principal bundle, where the action of $S^1_C$ on $\text{Spin}(3,1)$ is given by the multiplication on the left. It is equipped with the horizontal distribution given at every $g \in \text{Spin}(3,1)$ by
\[
\mathcal{H}_g := d(R_{g^{-1}})^{-1} (CJ \oplus CiK) \subset T_g \text{Spin}(3,1),
\]
where $R_{g^{-1}}$ stands for the right multiplication by $g^{-1}$ on $\text{Spin}(3,1)$. The distribution $(\mathcal{H}_g)_{g \in \text{Spin}(3,1)}$ is $H$–orthogonal to the fibers of $p$, and, for all $g \in \text{Spin}(3,1)$, $dp_g : \mathcal{H}_g \rightarrow T_{p(g)}\mathcal{Q}$ is an isomorphism which preserves $i$ and such that
\[
H(dp_g(u), dp_g(u)) = -4H(u, u),
\]
for all $u \in \mathcal{H}_g$. With these notations, we have
\[
G = p \circ g, \quad (35)
\]
and the map $g : M \rightarrow \text{Spin}(3,1)$ appears to be a horizontal lift to $\text{Spin}(3,1)$ of the Gauss map $G : M \rightarrow \mathcal{Q}$.

Thus, from (35), we get
\[
dG = dp \circ dg.
\]
Since $dp$ and $dG$ commute to the complex structures $i$ defined on $\text{Spin}(3,1)$, $\mathcal{Q}$ and $M$, so does $dg$, and thus $g : M \rightarrow \text{Spin}(3,1) \subset \mathbb{H}^C$ is a holomorphic map.
Using the identity (34), the complex 1–forms \( \theta_1 \) and \( \theta_2 \) are holomorphic, therefore there exists two holomorphic functions \( f_1 \) and \( f_2 \) such that
\[
\theta_1 = f_1 dz \quad \text{and} \quad \theta_2 = f_2 dz,
\]
where \( z \) is a conformal parameter of \((M, J)\). We note that, \( f_1 \) and \( f_2 \) do not vanish simultaneously since \( dG \) is assumed to be injective at every point.

The aim now is to show that the immersion \( F : M \to \mathbb{R}^{3,1} \) induced by \( \varphi \) is determined by the holomorphic functions \( f_1 \) and \( f_2 \), and by the two smooth functions \( h_1, h_2 : M \to \mathbb{R} \) such that
\[
\vec{H} := h_1 e_3 + h_2 e_4,
\]
the components of the mean curvature vector in the parallel frame \((e_3, e_4)\) of \( E \).

We first observe that the immersion is determined by \( g : M \to \text{Spin}(3,1) \subset \mathbb{H}^C \) and by the orthonormal and parallel frame \((e_1, e_2)\) of \( TM \).

**Proposition 6.5.** The immersion \( F : M \to \mathbb{R}^{3,1} \) is such that
\[
dF(X) = g^{-1}(\omega_1(X) iI + \omega_2(X) I) \hat{g},
\]
for all \( X \in TM \), where \( \omega_1, \omega_2 : TM \to \mathbb{R} \) are the dual forms of \( e_1 \) and \( e_2 \).

**Proof.** We have
\[
dF(X) = \langle (X \cdot \varphi, \varphi) \rangle = g^{-1} [X] \hat{g},
\]
where \([X] \in \mathbb{H}^C\) stands for the coordinates of \( X \in \text{Cl}(TM \oplus E) \) in \( \tilde{s} \). Recalling that \([e_1] = iI \) and \([e_2] = I \) in \( \tilde{s} \), we have \([X] = X_1 iI + X_2 I\), where \( X_1, X_2 \) are the coordinates of \( X \in TM \) in \((e_1, e_2)\).

In the following proposition, we precise how to recover the map \( g \) and the frame \((e_1, e_2)\) from the holomorphic functions \( f_1 \) and \( f_2 \) and from the smooth functions \( h_1 \) and \( h_2 \):

**Proposition 6.6.**
1. \( g \) is determined by \( f_1 \) and \( f_2 \), up to the multiplication on the right by a constant belonging to \( \text{Spin}(3,1) \).
2. Define \( \alpha_1, \alpha_2 : M \to \mathbb{C} \) such that
\[
e_1 = \alpha_1 \quad \text{and} \quad e_2 = \alpha_2
\]
in the parameter \( z \). The functions \( \alpha_1, \alpha_2, f_1, f_2, h_1 \) and \( h_2 \) are linked by
\[
(\alpha_1 iI + \alpha_2 I)(f_1 J + f_2 iK) = (h_1 J + h_2 K).
\]
In particular, if \( f_1^2 - f_2^2 \neq 0 \), we get
\[
\alpha_1 iI + \alpha_2 I = -(h_1 J + h_2 K)\frac{f_1 J + f_2 iK}{f_1^2 - f_2^2},
\]
that is, the frame \((e_1, e_2)\) in the coordinates \( z \) is determined by \( f_1, f_2, h_3 \) and \( h_4 \).
Proof. 1- The solution $g$ of the equation $dg \ g^{-1} = [\eta]$ is unique, up to multiplication on the right by a constant belonging to $Spin(3,1)$.

2- In $\check{s}$, the Dirac equation $D\varphi = \check{H} \cdot \varphi$ is given by

$$- [e_1] [\nabla_{e_1} \varphi] + [e_2] [\nabla_{e_2} \varphi] = [\check{H}] [\varphi];$$

since $dg(X) = [\nabla_X \varphi] = [\eta(X)] g$, we get

$$i \frac{1}{2} \; dg(e_1) g^{-1} + I \; dg(e_2) g^{-1} = h_1 J + h_2 K,$$

using (34) and (36) we have $dg(e_1) g^{-1} = \alpha_1 (f_1 J + f_2 i K)$ and $dg(e_2) g^{-1} = \alpha_2 (f_1 J + f_2 i K)$ that implies (37). Equation (38) is a consequence of (37), together with the following observation: $\xi \in \mathbb{H}^C$ is invertible if and only if $H(\xi, \xi) = \overline{\xi} \xi \neq 0$; its inverse is then $\xi^{-1} = \frac{\xi}{m(\xi, \xi)}$.

Remark 6.7. The complex numbers $\alpha_1$ and $\alpha_2$, considered as real vector fields on $M$, are independent and satisfy $[\alpha_1, \alpha_2] = 0$: since the metric on $M$ is flat, there is a local diffeomorphism $\psi : \mathbb{R}^2 \to M$ such that $e_1 = \frac{\partial \psi}{\partial x}$ and $e_2 = \frac{\partial \psi}{\partial y}$.

The interpretation of the condition $f_1^2 - f_2^2 = 0$ is the following: using (34) and the identities given in Lemma 6.4 we get

$$G^* H = H(dG, dG) = -4H(dg, dg) = -4(f_1^2 - f_2^2) dz^2; \quad (39)$$

thus, if $f_1^2 - f_2^2 = 0$ in $x \in M$, $dG_x(T_x M)$ belongs to the union of two complex lines through $G(x)$ in the Grassmannian $Q$; in particular, the osculator space in $x$ is degenerate (i.e. the first normal space in $x$ is $1$–dimensional); a similar and more detailed description for spacelike surfaces is given in [4, Section 6.2.3].

We will gather the previous results to construct flat timelike immersions with flat normal bundle from initial data.

Corollary 6.8. Let $(U, z)$ be a simply connected domain in $\mathbb{C}$, and consider

$$\theta_1 = f_1 dz \quad \text{and} \quad \theta_2 = f_2 dz$$

where $f_1, f_2 : U \to \mathbb{C}$ are two holomorphic functions such that $f_1^2 - f_2^2 \neq 0$. Suppose that $h_1, h_2 : U \to \mathbb{R}$ are smooth functions such that

$$\alpha_1 := -\frac{h_1 f_1 + h_2 f_2}{f_1^2 - f_2^2} \quad \text{and} \quad \alpha_2 := \frac{h_2 f_1 - h_1 f_2}{f_1^2 - f_2^2} \quad (40)$$

considered as real vector fields on $U$, are independent at every point and satisfy $[\alpha_1, \alpha_2] = 0$. Then, if $g : U \to Spin(3,1) \subset \mathbb{H}^C$ is a map solving

$$dg \ g^{-1} = \theta_1 J + \theta_2 \ i K, \quad (41)$$

and if we set

$$\xi := g^{-1} (\omega_1 \ i + \omega_2 \ I) \ \hat{g} \quad (42)$$
where $\omega_1, \omega_2 : TU \to \mathbb{R}$ are the dual 1-forms of $\alpha_1, \alpha_2 \in \Gamma(TU)$, the function $F = \int \xi : U \to \mathbb{R}^{3,1}$ defines a timelike isometric immersion with $K = K_N = 0$.

Reciprocally, the isometric immersion of a timelike surface $M$ in $\mathbb{R}^{3,1}$ such that $K = K_N = 0$, with regular Gauss map and whose osculating spaces are everywhere not degenerate, are locally of this form.

**Proof.** We consider $E = U \times \mathbb{R}^2$ the trivial vector bundle on $U$ and we denote by $(e_3, e_4)$ the canonical basis of $\mathbb{R}^2$. Let us define $s = (e_1, e_2, e_3, e_4)$ where $e_1 = \alpha_1$ and $e_2 = \alpha_2$ in $\mathbb{R}^2 \simeq \mathbb{C}$, and let us consider the metric on $U$ such that $(e_1, e_2)$ is a orthonormal frame; this metric is flat and the frame $(e_1, e_2)$ is parallel since $[e_1, e_2] = 0$ by hypothesis. Let $\tilde{s}$ be a section of the trivial bundle $Q \to U$ such that $\pi(\tilde{s}) = s$, where $\pi : Q = S^1_C \times U \to (SO(1, 1) \times SO(2)) \times U$ is the natural projection. We consider $g : U \to Spin(3, 1) \subset \mathbb{H}^C$ the unique solution, up to the natural right action of $Spin(3, 1)$, of the equation (41): this equation is solvable since $\eta' := \theta_1 J + \theta_2 iK$ satisfies the structure equation $d\eta'(X, Y) - [\eta'(X), \eta'(Y)] = 0$, for all $X, Y \in \Gamma(TU)$. The definition (40) of $\alpha_1$ and $\alpha_2$ is equivalent to (38), which traduces that $\varphi := [\tilde{s}, g] \in \Sigma = Q \times \mathbb{H}^C/\rho$ is a solution of the Dirac equation $D\varphi = \bar{H} \cdot \varphi$ where $\bar{H} := h_1 e_3 + h_2 e_4$ (see the proof of Proposition 6.6). Moreover, setting $\omega_1, \omega_2$ for the dual 1-forms of $e_1, e_2 \in \Gamma(TU)$, the 1-form $\xi$, given in (42), is such that $\xi(X) = \langle X \cdot \varphi, \varphi \rangle$; thus $\xi$ is closed, and a primitive of $\xi$ defines a timelike isometric immersion in $\mathbb{R}^{3,1} \subset \mathbb{H}^C$ with induced metric $-\omega_1^2 + \omega_2^2$. Since the Gauss map of the immersion is $G = i g^{-1} I g$ (Lemma 6.1) and since $g$ is a holomorphic map (by (41)), we get that $G$ is a holomorphic map, and thus that $K = K_N = 0$ (Proposition 5.1). □

**Remark 6.9.** A flat timelike immersion with flat normal bundle and regular Gauss map, and whose osculating spaces are everywhere not degenerate (i.e. such that $G^* H \neq 0$ at every point), is determined by two holomorphic functions $f_1, f_2 : U \to \mathbb{C}$ such that $f_1^2 - f_2^2 \neq 0$ on $U$ and by two smooth functions $h_1, h_2 : U \to \mathbb{R}$ such that the two complex numbers $\alpha_1$ and $\alpha_2$ defined by (40), considered as real vector fields, are independent at every point and such that $[\alpha_1, \alpha_2] = 0$ on $U$.

Considering further a holomorphic map $h : U \to \mathbb{C}$ such that $h^2 = f_1^2 - f_2^2$, and setting $z'$ for the parameter such that $dz' = h(z)dz$, we have

$$g^* H = H(dg, dg) = H(dgg^{-1}, dgg^{-1}) = (f_1^2 - f_2^2)dz^2 = dz'^2,$$

and thus, in $z'$,

$$g' g^{-1} = \cosh \psi J + \sinh \psi iK$$

(43)

for some holomorphic function $\psi : U' \to \mathbb{C}$. The parameter $z'$ may be interpreted as the complex arc length of the holomorphic curve $g : U \to Spin(3, 1)$, and the holomorphic function $\psi$ as the complex angle of $g'$ in the trivialization $T Spin(3, 1) = Spin(3, 1) \times T_1 Spin(3, 1)$. Observe that, from the definition (43) of $\psi$, the derivative $\psi'$ may be interpreted as the complex geodesic curvature of the holomorphic curve $g : U \to Spin(3, 1)$. The immersion thus only depends
on the single holomorphic function $\psi$, instead of the two holomorphic functions $f_1$ and $f_2$. Moreover, the two relations in (40) then simplify to

$$\alpha_1 = -i(h_1 \cosh \psi + h_2 \sinh \psi) \quad \text{and} \quad \alpha_2 = h_2 \cosh \psi - h_1 \sinh \psi.$$  (44)

Note that the new parameter $z'$ may be only locally defined, since the map $z \to z'$ may be not one-to-one in general.

**Corollary 6.10.** Let $U \subset \mathbb{C}$ be a simply connected domain, and let $\psi : U \to \mathbb{C}$ be a holomorphic function. Suppose that $h_1, h_2 : U \to \mathbb{R}$ are smooth functions such that $\alpha_1$ and $\alpha_2$, real vector fields defined by (44), are independent at every point and satisfy $[\alpha_1, \alpha_2] = 0$ on $U$. Then, if $g : U \to \text{Spin}(3,1) \subset \mathbb{H}^C$ is a holomorphic map solving

$$g'g^{-1} = \cosh \psi J + \sinh \psi iK,$$

and if we set

$$\xi := g^{-1} (\omega_1 i + \omega_2 I) \hat{g},$$

where $\omega_1, \omega_2 : TU \to \mathbb{R}$ are the dual 1-forms of $\alpha_1, \alpha_2 \in \Gamma(TU)$, the function $F = \int \xi : U \to \mathbb{R}_{3,1}$ defines a timelike isometric immersion with $K = K_N = 0$. Reciprocally, the isometric immersion of a timelike surface $M$ in $\mathbb{R}_{3,1}$ such that $K = K_N = 0$, with regular Gauss map and whose osculating spaces are everywhere not degenerate, are locally of this form.

### 7 Flat timelike surfaces in the De Sitter space

In this section, using spinors we deduce a result of Aledo, Gálvez and Mira given in [1, Corollary 5.1] concerning the conformal representation of a flat timelike surface in three-dimensional De Sitter space.

Keeping the notation of Section 2, we consider the isomorphism of algebras

$$A : \mathbb{H}^C \to M_2(\mathbb{C})$$

$$q = q_1 i + q_2 I + q_3 J + q_4 K \quad \mapsto \quad A(q) = \begin{pmatrix} q_1 + iq_2 & q_3 + iq_4 \\ -q_3 + iq_4 & q_1 - iq_2 \end{pmatrix}.$$

We note the following properties:

$$A(\overline{q}) = A(q)^* \quad \text{and} \quad H(q, q) = \det(A(q))$$  (45)

for all $q \in \mathbb{H}^C$, where $A(q)^*$ is the conjugate transpose of $A(q)$. Using (45), we get an identification

$$\mathbb{R}_{3,1} = \{ \xi \in \mathbb{H}^C \mid \overline{\xi} = -\xi \} \simeq i\text{Herm}(2),$$
where the metric $⟨·,·⟩$ of $\mathbb{R}^{3,1}$ identifies with $\det$ defined on $i\text{Herm}(2)$; moreover, the De Sitter space $S^{2,1} \subset \mathbb{R}^{3,1}$ (defined in (12)) is described as

$$S^{2,1} = \left\{ B \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} B^* \mid B \in Sl_2(\mathbb{C}) \right\} \subset i\text{Herm}(2).$$

**Corollary 7.1.** Let $M$ be a Riemann surface, $B : M \to Sl_2(\mathbb{C})$ be a holomorphic map such that there exists $\theta, \omega$ nowhere vanishing holomorphic 1–forms that satisfy

$$B^{-1}dB = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}.$$

Assume moreover that $\Im(m(\frac{\omega}{\theta})) \neq 0$. Then

$$F := B \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} B^* : M \longrightarrow S^{2,1}$$

defines, with the induced metric, a flat timelike isometric immersion.

Conversely, an isometric immersion of a simply connected flat timelike surface $M$ in the De Sitter space may be described as above.

**Proof.** The proof of the direct statement is obtained by a direct computation; see [1]. We thus only prove the converse statement. We suppose that there exists a flat isometric immersion $F : M \to S^{2,1}$ of a simply connected timelike surface $M$. Using the natural isometric embedding $S^{2,1} \hookrightarrow \mathbb{R}^{3,1}$, we get a flat timelike immersion $M \hookrightarrow \mathbb{R}^{3,1}$ with flat normal bundle and regular Gauss map, and we can consider the complex structure $J$ on $M$ such that its Gauss map is holomorphic (see Section 5.1). We denote by $E$ its normal bundle, $\vec{H} \in \Gamma(E)$ its mean curvature vector field and $\Sigma := M \times \mathbb{H}^C$ the spinor bundle of $\mathbb{R}^{3,1}$ restricted to $M$. The immersion $F$ is given by

$$F = \int \xi \quad \text{where} \quad \xi(X) = \langle (X \cdot \varphi, \varphi) \rangle,$$

for some spinor field $\varphi \in \Gamma(\Sigma)$ solution of $D\varphi = \vec{H} \cdot \varphi$ and such that $H(\varphi, \varphi) = 1$ (the spinor field $\varphi$ is the restriction to $M$ of the constant spinor field $+1$ or $-1 \in \mathbb{H}^C$ of $\mathbb{R}^{3,1}$). Using Proposition 4.1, we have

$$F = \langle \langle e_4 \cdot \varphi, \varphi \rangle \rangle \quad (46)$$

where $e_4 \in \Gamma(E)$ is normal to $S^{2,1}$ in $\mathbb{R}^{3,1}$. We choose a parallel frame $\tilde{s} \in \Gamma(\tilde{Q})$ adapted to $e_4$, i.e. such that $e_4$ is the fourth vector of $\pi(\tilde{s}) \in \Gamma(Q_1 \times_M Q_2)$: in $\tilde{s}$, equation (46) reads

$$F = \overline{[\varphi]}K[\varphi] \simeq A(\overline{[\varphi]}K[\varphi]) = A(\overline{[\varphi]})A(K)A(\overline{[\varphi]}) \quad (47)$$

where $[\varphi] \in \mathbb{H}^C$ represents $\varphi$ in $\tilde{s}$. Thus, setting $B := A(\overline{[\varphi]})$ and using (45) we have that $B$ belongs to $Sl_2(\mathbb{C})$ (since $H(\varphi, \varphi) = 1$) and $B^* = A(\overline{[\varphi]})$. From (47) we thus get

$$F = B \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} B^*.$$
With respect to the complex structure induced on $M$ (by the Gauss map), $B : M \to Sl_2(\mathbb{C})$ is a holomorphic map (since $[\varphi] : M \to Spin(3,1) \subset \mathbb{H}^C$ is a holomorphic map and $A$ is $\mathbb{C}$–linear). Note that $dB = A(d[\varphi])$, using (34) we obtain

$$B^{-1}dB = A([\varphi] d[\varphi]) = -A(d[\varphi] [\varphi])$$

$$= -A(\theta_1 J + \theta_2 iK) = \begin{pmatrix} 0 & -\theta_1 + \theta_2 \\ \theta_1 + \theta_2 & 0 \end{pmatrix},$$

where $\theta_1 + \theta_2 =: \omega$ and $-\theta_1 + \theta_2 =: \theta$ are holomorphic 1–forms (formula (34)). We also note that $\omega$ and $\theta$ nowhere vanish: if we suppose that $\omega = 0$ or $\theta = 0$ in $x \in M$, using (36) we get $0 = \omega \theta = -\theta_1^2 + \theta_2^2 = -(f_1^2 - f_2^2)dz^2$, thus, from (39) we obtain $G^*H = 0$ in $x$, in particular, the first normal space in $x$ is 1–dimensional which is not possible since $G$ is regular; see [1, Lemma 2.2]. Finally, it is not difficult to verify that $dF$ injective reads $\Im m(\frac{\omega}{\theta}) \neq 0$.\hfill\qed

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