A VERTEX OPERATOR REFORMULATION OF THE
KANADE-RUSSELL CONJECTURE MODULO 9

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Abstract. We reformulate the Kanade-Russell conjecture modulo 9 via the
vertex operators for the level 3 standard modules of type $D_4^{(3)}$. Along the
same line, we arrive at three partition theorems which may be regarded as an
$A_4^{(2)}$ analog of the conjecture. Andrews-van Ekeren-Heluani has proven one of
them, and we point out that the others are easily proven from their results.

1. Introduction

1.1. The Rogers-Ramanujan identities. The Rogers-Ramanujan (RR, for short)
identities

$$\sum_{n \geq 0} q^{n^2} = \frac{1}{(q, q^4; q^9)_{\infty}}, \quad \sum_{n \geq 0} q^{n^2+n} = \frac{1}{(q^2, q^3; q^9)_{\infty}}$$

are undoubtedly one of the most famous $q$-series identities in mathematics (see [2]
[19] [48]). Here, the Pochhammer symbols are defined by

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad (a_1, \ldots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n.$$ 

As noted by Schur and MacMahon, the RR identities are equivalent to the fol-
lowing statement, often called the RR partition theorem.

Let $i = 1$ or $2$. For each non-negative integer $n$, the number
of partitions of $n$ whose successive differences are at least two and
whose minimum parts are at least $i$ equals the number of partitions
of $n$ whose parts are congruent to $i$ or $5 - i$ modulo 5.

Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of a non-negative integer $n$ is a weakly
decreasing sequence of positive integers (called parts) whose sum $|\lambda|$ is $n$. We denote
the length $\ell$ by $\ell(\lambda)$ and put the multiplicities as $m_j(\lambda) = |\{1 \leq i \leq \ell \mid \lambda_i = j\}|$.

Let $\text{Par}(n)$ (resp. $\text{Par}$) denote the set of partitions of $n$ (resp. partitions). We say
that two subsets $C$ and $D$ of $\text{Par}$ are partition theoretically equivalent (abbreviated
to $C \underset{PT}{\sim} D$) if we have $|C \cap \text{Par}(n)| = |D \cap \text{Par}(n)|$ for $n \geq 0$. For example, the RR
partition theorem is briefly written as $\text{RR}_1 \underset{PT}{\sim} T_{1,5}^{(5)}$ for $i = 1, 2$, where

$$\text{RR}_1 = \{ \lambda \in \text{Par} \mid \lambda_i - \lambda_{i+1} \geq 2 \text{ for } 1 \leq j < \ell(\lambda) \},$$

$$\text{RR}_2 = \text{RR}_1 \cap \{ \lambda \in \text{Par} \mid m_1(\lambda) = 0 \},$$

$$T_{a_1, \ldots, a_k}^{(N)} = \{ \lambda \in \text{Par} \mid \lambda_j \equiv a_1, \ldots, a_k \pmod{N} \text{ for } 1 \leq j \leq \ell(\lambda) \}.$$
1.2. The Kanade-Russell conjecture modulo 9. About a decade ago, Kanade-Russell found a celebrated conjectural partition theorem below (see [27, §4]), which is now well-known in the community under the name of the Kanade-Russell conjecture (e.g., see [14] [34] [37]).

Conjecture 1.1 ([27]). For $1 \leq a \leq 3$, we have $\text{KR}_a \cong T_{9}^{\text{PT}}(2a-1,3,6,9-2a-1)$, where

$$\text{KR}_1 = \{ \lambda \in \text{Par} \mid \lambda_j - \lambda_{j+2} \geq 3 \text{ for } 1 \leq j \leq \ell(\lambda) - 2 \text{ and } P(\lambda, 0) \text{ holds} \},$$

$$\text{KR}_2 = \text{KR}_1 \cap \{ \lambda \in \text{Par} \mid m_1(\lambda) = 0 \},$$

$$\text{KR}_3 = \text{KR}_1 \cap \{ \lambda \in \text{Par} \mid m_1(\lambda) = m_2(\lambda) = 0 \}.$$

Moreover, the condition $P(\lambda, k)$ for a partition $\lambda$ and an integer $k$ stands for

$$\lambda_j - \lambda_{j+1} \leq 1 \text{ implies } \lambda_j + \lambda_{j+1} \equiv k \pmod{3} \text{ for } 1 \leq j < \ell(\lambda).$$

Like the RR partition theorem, the Kanade-Russell conjecture has an equivalent $q$-series identity reformulation [34, §3]. Because the double sums due to Kursungoz are well-known, we give a reformulation regarding triple sums. Recall that Kanade-Russell also conjectured $\text{KR}_4 \cong T_{2,3,5,8}^{\text{PT}}$ and $\text{KR}_5 \cong T_{1,4,6,7}^{\text{PT}}$ [27, 37], where

$$\text{KR}_4 = \{ \lambda \in \text{Par} \mid \lambda_j - \lambda_{j+2} \geq 3 \text{ for } 1 \leq j \leq \ell(\lambda) - 2, \ P(\lambda, 2) \text{ holds and } m_1(\lambda) = 0 \},$$

$$\text{KR}_5 = \{ \lambda \in \text{Par} \mid \lambda_j - \lambda_{j+2} \geq 3 \text{ for } 1 \leq j \leq \ell(\lambda) - 2, \ P(\lambda, 1) \text{ holds and } m_2(\lambda) \leq 1 \}.$$

Proposition 1.2. For $1 \leq a \leq 5$, we have

$$\sum_{\lambda \in \text{KR}_a} \ell(\lambda) q^{\ell(\lambda)} = \sum_{i,j,k \geq 0} q^{3(5)_{i}+8(3)_{j}+6(2)_{k}+4ij+3ik+6j+3n(1)}q^{n+1}q^{i+2j+2k},$$

where $A_1 = (1, 4, 3)$, $A_2 = (2, 6, 6)$, $A_3 = (3, 8, 6)$, $A_4 = (2, 6, 5)$ and $A_5 = (1, 6, 4)$.

Because it is routine to derive a $q$-difference equation for the right (resp. left) hand side by a $q$-analog of Sister Celine’s technique [35] (resp. by Andrews’ linked partition ideals [12] or the regularly linked sets by Takigiku and the author [53]), we omit an automatic proof of Proposition 1.2. See also [51] [52, §7.1] and [14, §4].

1.3. The main result. Recall that for a standard module (a.k.a., integrable highest weight module) $V$ of an affine Kac-Moody Lie algebra $\mathfrak{g}$, the character $\chi(\Omega_V)$ of the vacuum space $\Omega_V$ with respect to the principal Heisenberg subalgebra (see also [27]) is obtained by the Lepowsky numerator formula [39, Proposition 8.4].

Let $i = 1$ or 2 and consider the level 3 standard module $V = V((i+1)\Lambda_0 + (2 - i)\Lambda_1)$ of the affine algebra $\mathfrak{g}(A^{(1)}_1)$ (see Figure 1). Lepowsky-Milne observed that

$$\chi(\Omega_V) = \frac{1}{(q^4, q^{5-i}; q^5)_\infty}.$$
is the infinite product in the RR identities \([37]\). In a seminal paper \([39]\) (see also \([38]\)), Lepowsky-Wilson showed (see \([39]\) Theorem 10.4))

\[
\chi((\Omega_V)^{[n]}/(\Omega_V)^{[n-1]}) = \frac{q^{n^2+n(i-1)}}{(q;q)_n},
\]

for \(n \geq 0\), where \((\Omega_V)^{[n]}\) is the associated \(Z\)-filtration, which coincides with the \(s\)-filtration \((\Omega_V)^{[n]}\) \([39]\ §5\) (and \((\Omega_V)^{[-1]} = (\Omega_V)^{[-1]} = \{0\}\)). For that purpose, Lepowsky-Wilson proved that the set of \(Z\)-monomials parameterized by \(\RR_i\)

\[\{Z_{-\lambda_1} \cdots Z_{-\lambda_{\ell}} v_0 \mid (\lambda_1, \ldots, \lambda_{\ell}) \in \RR_i\}\]

forms a basis of \(\Omega_V\), where \(v_0\) is a highest weight vector of \(V\) and \(Z_i\) is the \(Z\)-operator (see \([39]\) (3.13)) and a review in \([32]\) associated with the root \(\beta_1\) of \(A_1\).

Our main result gives a weaker vertex operator interpretation of the Kanade-Russell conjecture in the following way.

**Theorem 1.3.** For \(1 \leq a \leq 3\), the set of \(Z\)-monomials parameterized by \(\KR_a\)

\[\{Z_{-\lambda_1} \cdots Z_{-\lambda_{\ell}} v_0 \mid (\lambda_1, \ldots, \lambda_{\ell}) \in \KR_a\}\]

spans \(\Omega_{V(\Lambda(a))}\), where \(Z_i\) is the \(Z\)-operator associated with the root \(\beta_1\) of \(D_4\) (see Figure \([7]\), \(w_0^{(a)}\) is a highest weight vector of the standard module \(V(\Lambda(a))\) of type \(D_4^{(3)}\) whose highest weight is given by \(\Lambda(1) = \Lambda_0 + \Lambda_1, \Lambda(2) = 3\Lambda_0\) and \(\Lambda(3) = \Lambda_2\).

An immediate consequence of Theorem \([1.3]\) and the infinite product expression of \(\chi(\Omega_{V(\Lambda(a))})\) is that the Kanade-Russell Conjecture (Conjecture \([1.1]\)) is equivalent to the claim that the \(Z\)-monomials in Theorem \([1.3]\) are linearly independent.

**Corollary 1.4.** We have \(|\KR_a \cap \Par(n)| \geq |T^{(9)}_{2s-1,3,6,9-2s-1} \cap \Par(n)|\) for \(n \geq 0\) and \(a = 1, 2, 3\).

Our proof is a variant of \([39]\ §6\) and thus standard in vertex operator theory (see also similar calculations in \([10, 11, 23, 25, 39, 40, 41, 43, 50, 51, 52, 55]\) once a suitable set of relations between the \(Z\)-operators is available. In our case, four kinds of “generalized commutation relations" (see Theorem \([3.2]\)) which are sometimes called with adjectives “anti” and “partial” (see \([22]\ §3\) and \([35]\ §3.4\) \([33]\ §3.7\) suffice. It would be interesting to find a higher structure (e.g., vertex operator algebra structures as in \([21]\ [22]\)) in our calculation as well as to find a proof of linear independence (e.g., as in \([11, 39, 41, 55]\)). We remark that the arguments in \([4.7, 4.8, 4.9]\) are related to overlap ambiguities (resp. critical pairs) in non-commutative Gröbner basis theory \([6]\) (resp. theory of term rewriting \([9, 14]\)).

1.4. An \(A_4^{(2)}\) analog. The work of Lepowsky-Wilson \([39]\) initiated intensive research on explicit realizations of the generalized RR identity

\[\chi(\Omega_V) = \sum_{n \geq 0} \chi((\Omega_V)^{[n]}/(\Omega_V)^{[n-1]})\]

for a standard module \(V\) of type \(X_N^{(r)}(a_{ij})_{i,j \in I}\) (see also \([39]\) Theorem 7.5]). We give a brief review for types \(A_1^{(1)}, A_2^{(2)}\) and \(A_1^{(1)}\) with levels greater than 2.

For \(A_1^{(1)}\) arbitrary level, Lepowsky-Wilson showed that the \(Z\)-monomials parameterized by partitions in the Andrews-Gordon-Bressoud partition theorem (see \([48]\ §3.2\) and the references therein) spans \(\Omega_V\) \([40]\ Corollary 13.2, Lemma 14.3\), which were proven to be linearly independent by Mureman-Primc \([41]\ §9, Appendix\).
For $A_2(2)$ level 3 (resp. 4), Capparelli [10] (resp. Nandi [33]) obtained a set of $X$-monomials that spans $V$ as a module over the principal Heisenberg algebra. Subsequently, they were proven to be linearly independent by [35, 55] (resp. [33]). On corresponding $q$-series identities, see [8, 35, 54] and the references therein.

For $A_2(1)$ level 3, the author showed that a set of $X$-monomials parameterized by certain 2-color partitions (or bipartitions) forms a basis of $V$ as a module over the principal Heisenberg algebra [52, Theorem 1.3].

Although there is much literature devoted to higher levels, such as [12, 20, 29, 42, 54] (resp. [4, 13, 15, 17, 18, 26, 30, 52, 56, 58, 59, 60]) for $A_2(2)$ (resp. $A_2(1)$), relations to representation theory remain unknown except the aforementioned levels. As in [39] (6.8)], it is natural to expect that there exist certain $n(X_N^r)$-color partitions whose corresponding $Z$-monomials form a basis of $\Omega_V$, where (see [10, §8, §9])

$$n(X_N^r) = \frac{\text{the number of roots of type } X_N}{\text{the } r\text{-twisted Coxeter number of } X_N} = |I| - 1.$$ For example, $n(A_2(1)) = n(A_2(2)) = 1$ and $n(A_4(1)) = n(A_4(2)) = n(D_4(3)) = 2$. Thus, the above results can be regarded as instances of that expectation. On the other hand, for low levels such as level 2, sometimes $\Omega_V$ has a basis of $Z$-monomials parameterized by certain partitions as in [23, 35, 52, 54]. While conjectural, Theorem 1.5 may be thought of as a similar example.

It would be interesting to find more examples similar to Theorem 1.3 for other types $A$ with $n(A) > 1$ (or other levels). We note that, as shown in [23], “the Kanade-Russell conjecture modulo 12” [28], which is now a theorem [46], can be understood as a kind of these examples for type $A_2(2)$ level 2.

**Theorem 1.5.** Let $F$ be the set consisting of the following 13 partitions

$$\begin{align*}
(1, 1, 1), & \quad (2, 1, 1), \quad (2, 2, 1), \quad (3, 2, 1), \quad (3, 3, 1), \quad (5, 3, 3), \quad (4, 4, 1, 1), \\
(5, 4, 1, 1), & \quad (5, 4, 2, 1), \quad (5, 5, 2, 1), \quad (6, 5, 3, 1, 1), \quad (6, 6, 3, 1, 1), \quad (7, 6, 4, 2, 1)
\end{align*}$$

and define three sets $I_1, I_2, I_3$ as follows.

- $I_1 = \{(1), (5, 4, 2, 2), (9, 8, 6, 4, 2, 2)\}$
- $I_2 = \{(1, 1), (2, 2), (4, 3, 1)\}$
- $I_3 = \{(1, 1), (2, 1), (2, 2), (3, 2), (3, 3), (4, 3, 1), (4, 4, 1), (5, 4, 2), (6, 5, 3, 1)\}$

For $1 \le a \le 3$, let $L_a$ be the set of partitions that does not begin with $c$ for $c \in I_a$ and does not match $(b_1 + k, \ldots , b_p + k)$ for $(b_1, \ldots , b_p) \in F$ and $k \ge 0$ (see [27]). We have $L_1 \sim T_2^{(16)}$, $L_2 \sim T_1^{(2)}$, and $L_3 \sim T_1^{(16)}$.

We arrived at the statement of Theorem 1.5 by checking

$$\{Z_{-\lambda_1} \cdots Z_{-\lambda_\ell} u_0 \mid (\lambda_1, \ldots , \lambda_\ell) \in L_a \cap \text{Par}(n)\}$$

is a set of linearly independent vectors for $1 \le a \le 3$ and $0 \le n \le 17 - 2^{a-1}$. Here, $Z_i$ is the $Z$-operator associated with the root $\beta_1$ of $A_i$ (see Figure 1), and $u_0$ is a highest weight vector of the standard module $V(\Upsilon^{(a)})$ of type $A_i^{(2)}$ whose highest weight is given by $\Upsilon^{(1)} = 3A_0$, $\Upsilon^{(2)} = \Lambda_0 + A_1$ and $\Upsilon^{(3)} = \Lambda_0 + A_2$.

After submission to arXiv of the first version of this paper, where Theorem 1.5 was stated as a conjecture, we learned from Matthew Russell that Theorem 1.5 for $a = 1$ is a theorem due to Andrews-van Ekeren-Heluani [5, Theorem 3]. In §5.4 we point out that the case $a = 2, 3$ is also easily proven based on their work [5, Proposition 4.4]. Interestingly, they seemed to find Theorem 1.5 for $a = 1$ via the
simple vertex algebra associated with the $(3, 4)$ Virasoro minimal model of central charge $c = 1/2$, unlike via the level 3 standard modules of type $A_4^{(2)}$.

**Organization of the paper.** We review the $Z$-operators in \[\text{§}2\] establish the four relations among them in \[\text{§}3\] and prove Theorem 1.3 in \[\text{§}4\]. In \[\text{§}5\] we advertise the regularly linked sets \[\text{§}3\] by demonstrating an automatic derivation of a $q$-difference equation for the set of roots $\Phi = \{\beta \in L \mid (\beta, \beta) = 2\}$. The twisted Coxeter automorphism \[\nu : L \to L\] is defined by $\nu = \sigma_{i_1} \cdots \sigma_{i_j} \sigma'$, where

1. $\sigma_i : L \to L, x \mapsto x - \langle x, \beta_i \rangle \beta_i$ is a reflection,
2. $\sigma' : L \to L, \beta_i \mapsto \beta_{\sigma(i)}$ is the Dynkin diagram automorphism of order $r$
3. $\{i_1, \ldots, i_j\}$ is a set of complete representatives for the orbits of the action of the cyclic group $\{\sigma^s \mid 0 \leq s < r\}$ on the set of indices $\{1, \ldots, N\}$.

In the following, we fix an integer $m$ to the order of $\nu$ and fix a $m$-th primitive root of unity $\omega$. As in \[\text{[16]}\, (5.1)\], we define a map $\varepsilon : L \times L \to \mathbb{C}^\times$ by

$$\varepsilon(\beta, \beta') = \prod_{p=1}^{m-1} (1 - \omega^{-p})^{\nu^p(\beta), \beta'}.$$ 

Let $a = \mathbb{C} \otimes \mathbb{Z} L$ be a complexification of $L$ and define a complex vector space $\mathfrak{g} = a \oplus \bigoplus_{\beta \in \Phi} \mathbb{C} x_\beta$, where $x_\beta$ is a formal symbol attached to a root $\beta$. It is known that $\mathfrak{g}$ with brackets determined by

$$[\beta_i, x_\beta] = \langle \beta_i, \beta \rangle x_\beta = -[x_\beta, \beta_i], \quad [x_\beta, x_{\beta'}] = \begin{cases} \varepsilon(\beta, \beta') \beta & \text{(if } \langle \beta, \beta' \rangle = -2) \\ \varepsilon(\beta, \beta') x_{\beta + \beta'} & \text{(if } \langle \beta, \beta' \rangle = -1) \\ 0 & \text{(if } \langle \beta, \beta' \rangle \geq 0), \end{cases}$$

where $\beta, \beta' \in \Phi$ and $1 \leq i \leq N$, is isomorphic to the Kac-Moody Lie algebra $\mathfrak{g}(X_N)$ of type $X_N$. The form $\langle \cdot, \cdot \rangle$ is extended to $\mathfrak{g}$ by $\langle \beta_i, x_\beta \rangle = 0 = \langle x_\beta, \beta_i \rangle$ and $\langle x_\beta, x_{\beta'} \rangle = \varepsilon(\beta, \beta') \delta_{\beta + \beta', 0}$. The map $\nu$ is also extended to $\mathfrak{g}$ by $\nu(x_\beta) = x_{\nu(\beta)}$. Then, $\nu$ is a principal automorphism of $\mathfrak{g}$, and $\langle \cdot, \cdot \rangle$ is a non-degenerate invariant form which is $\nu$-invariant (see \[\text{[16]}\, \S 6, \S 9\]).

**2.2. The Lepowsky-Wilson $Z$-algebras.** The $\nu$-twisted affinization $\tilde{\mathfrak{g}}$ given by

$$\tilde{\mathfrak{g}} = \bigoplus_{i \in \mathbb{Z}} (\mathfrak{g}_{(i)} \otimes t^i) \oplus \mathbb{C} c \oplus \mathbb{C} d,$$

where $\mathfrak{g}_{(i)} = \{x \in \mathfrak{g} \mid \nu^i(x) = \omega^i x\}$, with brackets determined by

$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j} + \frac{i \delta_{i+j, 0}}{m}(x, y)c, \quad [d, x \otimes t^i] = tx \otimes t^i,$$
and by the condition that $c$ is central, is isomorphic to $g(X_N^{(r)})$ (see [24] Chapter 8 and [16] Proposition 9.4).

For $k \in \mathbb{C}^*$, as in [39] §3, let $C_k$ be the full subcategory of the category of $\hat{\mathfrak{g}}$-modules whose objects $V$ satisfy the following three conditions.

1. $c$ acts as a scalar multiplication by $k$, i.e., $V$ has level $k$.
2. $V$ has a simultaneous eigenspace decomposition $V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$ with respect to the Cartan subalgebra $\mathfrak{t} = (\mathfrak{g}_{(0)} \otimes 1) \oplus \mathbb{C}c \oplus \mathbb{C}d$ (see [10] §9.3).
3. For $z \in \mathbb{C}$, there exists $i_0 \in \mathbb{Z}$ such that $V_{z+i} = \{0\}$ for $i > i_0$.

For an object $V$ in $C_k$, the Lepowsky-Wilson $Z$-algebra $Z_V$ (in the principal picture) is defined to be the subalgebra of $\text{End} V$ generated by $c$, $d$ and $Z_i(\beta)$ for $i \in \mathbb{Z}$ and $\beta \in \Phi$ [39] p.222, where $\text{pr}_i : \mathfrak{g} \to \mathfrak{g}_{(i)}$ for $i \in \mathbb{Z}$ is the projection and

\[ Z(\beta, \zeta) = \sum_{i \in \mathbb{Z}} Z_i(\beta)\zeta^i = E^-(\beta, \zeta, k)X(\beta, \zeta)E^+(\beta, \zeta, k), \]

\[ X(\beta, \zeta) = \sum_{i \in \mathbb{Z}} (\text{pr}_i(x_\beta) \otimes t^i)\zeta^i, \]

\[ E^\pm(\beta, \zeta, r) = \sum_{\pm i \geq 0} E_i^\pm(\beta)\zeta^i = \exp \left( m \sum_{\pm j > 0} \frac{\text{pr}_j(\beta) \otimes t^j}{r_j} \zeta^j \right). \]

As in [39] Proposition 4.7, the vacuum space $\Omega_V$ is defined as

\[ \Omega_V = \{v \in V \mid av = 0 \text{ for } a \in \hat{\mathfrak{a}}_+\}, \]

where $\hat{\mathfrak{a}} = [\mathfrak{a}, \mathfrak{a}] = \hat{\mathfrak{a}}_+ \oplus \hat{\mathfrak{a}}_- \oplus \mathbb{C}c$ is the principal Heisenberg subalgebra defined by

\[ \hat{\mathfrak{a}} = \hat{\mathfrak{a}}_+ \oplus \hat{\mathfrak{a}}_- \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \hat{\mathfrak{a}}_\pm = \bigoplus_{\pm i > 0} (\mathfrak{g}_{(i)} \cap \mathfrak{a}) \otimes t^i. \]

For a complex vector space $U$, let $U\{\zeta_1, \ldots, \zeta_r\}$ be the complex vector space of formal Laurent series in the variables $\zeta_1, \ldots, \zeta_r$ with coefficients in $U$, i.e.,

\[ U\{\zeta_1, \ldots, \zeta_r\} = \{\sum_{n_1, \ldots, n_r, n \in \mathbb{Z}} u_{n_1, \ldots, n_r} \zeta_1^{n_1} \cdots \zeta_r^{n_r} \mid u_{n_1, \ldots, n_r} \in U\}. \]

The equalities in the following citations are those in $\text{End}(V)\{\zeta_1, \zeta_2\}$ or $\text{End}(V)\{\zeta\}$, where $V$ is an object in $C_k$, $\beta, \beta' \in \Phi$ and $r, s \geq 1, p \in \mathbb{Z}$. As usual, the formal delta is defined by $\delta(\zeta) = \sum_{n \in \mathbb{Z}} \zeta^n$ and let $D = \zeta \cdot d/d\zeta$ so that $D\delta(\zeta) = \sum_{n \in \mathbb{Z}} n\zeta^n$.

**Theorem 2.1** ([39] Theorem 3.10), [16] Theorem 7.3).

\[ F_{\beta, \beta'}(\zeta_1/\zeta_2)Z(\beta, \zeta_1)Z(\beta', \zeta_2) - F_{\beta', \beta}(\zeta_2/\zeta_1)Z(\beta', \zeta_2)Z(\beta, \zeta_1) \]

\[ = \frac{1}{m} \sum_{p \in C_{-1}} \varepsilon(\nu^p(\beta), \beta')Z(\nu^p(\beta) + \beta', \zeta_2)\delta(\omega^{-p}\zeta_1/\zeta_2) + \frac{k(x_\beta, x_-)}{m^2} \sum_{p \in C_{-2}} D\delta(\omega^{-p}\zeta_1/\zeta_2) \]

where $C_{\ell} = \{0 \leq p < m \mid \varepsilon(\nu^p(\beta), \beta') = \ell\}$ and $F_{\beta, \beta'}(x) = \prod_{p=0}^{m-1} (1 - \omega^{-p})^{(\nu^p(\beta), \beta')}/k$.

**Proposition 2.2** ([39] Proposition 3.4).

\[ E^+(\beta, \zeta, r)E^-(\beta', \zeta_2, s) = E^-(\beta', \zeta_2, s)E^+(\beta, \zeta, r)F_{\beta, \beta'}(\zeta_1/\zeta_2)^{1/\nu rs}. \]

**Proposition 2.3** ([39] Theorem 3.3), [16] Theorem 7.2).

\[ Z(\nu^p(\beta), \zeta) = Z(\beta, \omega^p\zeta). \]

**Proposition 2.4** ([39] Proposition 3.2.(3)).

\[ DE^\pm(\beta, \zeta, r) = \mp \beta^\pm(\zeta)E^\pm(\beta, \zeta, r), \]

where $\beta^\pm(\zeta) = \sum_{\pm j > 0} (\text{pr}_j(\beta) \otimes t^j)\zeta^j$. \]
2.3. The basic representations. As shown in [16] Theorem 9.7, giving (an isomorphism class of a basic representation of \(\hat{g}\) is the same as giving a \(\nu\)-invariant (i.e., \(\rho(\nu(\beta)) = \rho(\beta)\) for \(\beta \in L\) group homomorphism \(\rho : L \to \mathbb{C}^\times\). The corresponding basic representation \(V^{{\phi}}\) is given by the underlying \(\hat{a}\)-module \(V = \text{Ind}_{\mathfrak{a}_+ \oplus \mathbb{C} \mathfrak{c}}^{\hat{g}} \mathbb{C}(\equiv \overline{U(\mathfrak{a}_+)}\), where \(\mathfrak{a}_+ \oplus \mathbb{C} \mathfrak{c}\) acts trivially (resp. as the identity) on \(\mathbb{C}\) and the action by \(X(\beta, \zeta) = \frac{\rho(\beta)}{m} E^- (-\beta, \zeta, 1) E^+ (-\beta, \zeta, 1)\).

3. The (partial) generalized (anti)commutation relations

3.1. The affine type \(D_4^{(3)}\). We apply the general theory reviewed in \([2]\) to \(D_4^{(3)}\) (see Figure 1). Our convention of the Dynkin diagram automorphism \(\sigma\) is given by \(\sigma(1) = 3, \sigma(3) = 4, \sigma(4) = 1, \sigma(2) = 2\), and that of the twisted Coxeter automorphism is given by \(\nu = \sigma_1 \sigma_2 \sigma'\).

Then, the order \(m\) of \(\nu\) is 12, and the set of roots \(\{\beta_j^{(i)} | i = 1, 2\text{ and } 0 \leq j < m\}\), where \(\beta_j^{(i)} = \rho(\beta_j)\) is given by \(\nu^0 = -\text{id}\) and

- \(\beta_1^{(1)} = \beta_1 + \beta_2 + \beta_3, \quad \beta_2^{(1)} = \beta_1 + \beta_2 + \beta_3 + \beta_4, \quad \beta_3^{(1)} = \beta_1 + 2\beta_2 + \beta_3 + \beta_4, \quad \beta_4^{(1)} = \beta_1 + \beta_2 + \beta_3 + \beta_4,\)
- \(\beta_1^{(2)} = -\beta_1 - \beta_2, \quad \beta_2^{(2)} = -\beta_3, \quad \beta_3^{(2)} = -\beta_1 - \beta_2 - \beta_4, \quad \beta_4^{(2)} = -\beta_2 - \beta_3,\)

3.2. Summary. Recall \(m = 12\). As in \([2]\) \(\omega\) is a fixed primitive \(m\)-th root of unity, a solution of the \(m\)-th cyclotomic polynomial \(\Phi_m(x) = x^4 - x^2 + 1\). Let

- \(S = 4 + 4\omega - 2\omega^3, \quad T = -24 - 28\omega + 14\omega^3, \quad U = 42 + 48\omega - 24\omega^3,\)
- \(M = -6 - 8\omega + 4\omega^3, \quad N = 14 + 16\omega - 8\omega^3, \quad P = 4 - 8\omega^2 - 6\omega^3, \quad Q = 2 - 4\omega^2.\)

Definition 3.1. For integers \(a_0, \ldots, a_5\), we define a formal power series

\[H_{a_0, \ldots, a_5}(x) = \prod_{p=0}^{5} (1 - \omega^{-p} x)^{a_p/3}(1 + \omega^{-p} x)^{-a_p/3}.\]

We also define \(G_i(x) = \sum_{p \geq 0} c_p^{(i)} x^p\) for \(1 \leq i \leq 6\) by

- \(G_1 = H_2,1,0,1,0,1,1, \quad G_2 = H_{-1,1,0,1,0,1,1}, \quad G_3 = H_{-1,1,-2,0,2,-1},\)
- \(G_4 = H_{2,2,-2,0,2,2}, \quad G_5 = H_{2,2,1,0,1,1,0,1,2}, \quad G_6 = G_4 - \frac{P}{Q} G_5.\)

The four relations in Theorem 3.2 will be proven in this order in \([3.4\] \[3.5\] \[3.6\] \[3.7\] after preparation in \([3.3\).

Theorem 3.2. Let \(W = (V^{{\phi}})^{\otimes 3}\) be the triple tensor product of the basic representation \(V^{{\phi}}\) of type \(D_4^{(3)}\), where \(\rho : \mathbb{Z}^4 \to \mathbb{C}^\times\) is the trivial group homomorphism. Put \(Z_i = Z_i(\beta_1)\) and \(Z'_i = Z_i(\beta_2)\). In the \(Z\)-algebra \(Z_W\), for \(A, B \in \mathbb{Z}\), we have

\[\sum_{p \geq 0} c_p^{(1)}(Z_{A-p} Z_{B+p} - Z_{B-p} Z_{A+p})\]

\[= \sum_{p \geq 0} c_p^{(2)}(Z_{A-p} Z_{B+p} + Z_{B-p} Z_{A+p})\]

\[= \sum_{p \geq 0} c_p^{(3)}(Z_{A-p} Z_{B+p} - Z_{B-p} Z_{A+p})\]

\[= \sum_{p \geq 0} c_p^{(4)}(Z_{A-p} Z_{B+p} + Z_{B-p} Z_{A+p})\]

\[= \sum_{p \geq 0} c_p^{(5)}(Z_{A-p} Z_{B+p} - Z_{B-p} Z_{A+p})\]

\[= \sum_{p \geq 0} c_p^{(6)}(Z_{A-p} Z_{B+p} + Z_{B-p} Z_{A+p})\]
If $A + B$ is not divisible by 3, then we have
\[
\sum_{p \geq 0} c_p^{(3)} (Z_{A-p} Z_{B+p} + Z_{B-p} Z_{A+p}) = \frac{M}{m} (\omega^{4A+9B} + \omega^{9A+4B}) Z_{A+B} + \frac{3N}{m^2} \delta_{A+B,0} (-1)^A + \frac{12}{m} (-1)^{A+B} Z_{A+B},
\]
\[
\sum_{p \geq 0} c_p^{(6)} (Z_{A-p} Z_{B+p} - Z_{B-p} Z_{A+p}) = \frac{P}{m} (\omega^{-2A+2B} - \omega^{2A-2B}) Z_{A+B} + \frac{12}{m^2} (1 - \frac{3P}{Q}) \delta_{A+B,0} (-1)^A A.
\]

3.3. Auxiliary Fourier expansions. Let $\gamma : \mathbb{C}\{x\} \to \mathbb{C}\{x\}$ be a $\mathbb{C}$-linear map defined by $\sum_{p \in \mathbb{Z}} c_p x^p = \sum_{p \in \mathbb{Z}} c_p x^{-p}$, which is also a $\mathbb{C}$-algebra homomorphism when restricted to $\mathbb{C}\{x\}$ (or $\mathbb{C}\{x^{-1}\}$). The following results are elementary complex analysis exercises, and we omit the detail (see also [55] Lemma 3.2, Lemma 3.5).

**Proposition 3.3.**
\[
G_1^G G_2 + G_2^G G_1 = S(\delta(\omega^4 x) + \delta(\omega^{-4} x)) + T(\delta(\omega^5 x) + \delta(\omega^{-5} x)) + U \delta(-x),
\]
\[
G_1^{-1} G_2 + G_2^{-1} G_1 = 2 \delta(x),
\]
\[
G_3^G G_3 + G_3^{-1} G_3 = M(\delta(\omega^5 x) + \delta(\omega^{-5} x)) + N \delta(-x),
\]
\[
G_1^{-1} G_3 + G_1^G G_3 = 6 \delta(x) - 2(\delta(\omega^2 x) + \delta(\omega^{-2} x)),
\]
\[
G_4^G G_4 - G_4^{-1} G_4 = 4 \delta(-x),
\]
\[
G_1^{-1} G_4 - G_1^G G_4 = P(\delta(\omega x) - \frac{1}{3} \delta(\omega^2 x) + \frac{1}{3} \delta(\omega^{-2} x) - \delta(-1)),
\]
\[
G_5^G G_5 - G_5^{-1} G_5 = 12 \delta(-x) + Q(\delta(\omega^4 x) - \delta(\omega^{-4} x)),
\]
\[
G_1^{-1} G_5 - G_1^G G_5 = Q(\delta(\omega x) - \delta(-1)).
\]

3.4. A proof of the generalized commutation relation. Apply Theorem 2.1 for $k = 3$ and $\beta = \beta' = \beta_1$ by using $F_{\beta, \beta'} = G_1$ and $C_{-1} = \{4, 5, 7, 8\}$ with
\[
\nu^{\pm 1}(\beta_1) + \beta_1 = \nu^{\pm 2}(\beta_1), \quad \nu^5(\beta_1) + \beta_1 = \nu^9(\beta_2), \quad \nu^7(\beta_1) + \beta_1 = \nu^4(\beta_2),
\]
\[
\varepsilon(\nu^4(\beta_1), \beta_1) = P = -\varepsilon(\nu^{-4}(\beta_1), \beta_1), \quad \varepsilon(\nu^5(\beta_1), \beta_1) = -52 + 104 \omega^2 + 90 \omega^3 = -\varepsilon(\nu^7(\beta_1), \beta_1).
\]

We get the result by comparing the coefficients of $\zeta_1^A \zeta_2^B$ of the identity in Theorem 2.1 using Proposition 2.3.

3.5. A proof of the generalized anticommutation relation. As in [55] Lemma 9.1, for $\gamma \in \Phi$, we have $Z(\gamma, \zeta) = Z^{(1)}(\gamma, \zeta) + Z^{(2)}(\gamma, \zeta) + Z^{(3)}(\gamma, \zeta)$, where, for $1 \leq j \leq 3$, $m Z^{(j)}(\gamma, \zeta)$ is the tensor product of three factors, the $i(\neq j)$-th (resp. $j$-th) tensorand being (see also [55] (3.6), (3.7), (3.8)) and [22], [23],
\[
E^- (\gamma, \zeta, 3) E^+(\gamma, \zeta, 3) \quad \text{(resp. } E^- (\gamma, \zeta, 3)^{-2} E^+(\gamma, \zeta, 3)^{-2}).
\]

Recall $F_{\beta_1, \beta_3} = G_1$ as in [3.4] where $F_{\beta, \beta'}$ is defined as in Theorem 2.1 for $k = 3$. By Proposition 2.2, together with
\[
\begin{align*}
\frac{1}{3} : \frac{1}{3} : \frac{2}{3} & = \frac{2}{3}, & \frac{1}{3} : \frac{2}{3} : \frac{1}{3} & = \frac{1}{3}, & \frac{1}{3} : \frac{1}{3} : \frac{2}{3} & = \frac{2}{3}.
\end{align*}
\]
we see the following two results.

First, as in [55] (3.26), we have $Z^{(j)} (\beta_1, \zeta_1) Z^{(j)} (\beta_1, \zeta_2) = \frac{1}{m} F_{\beta_1, \beta_1} (\zeta_1 / \zeta_2)^2 I^{(j)}$, where $I^{(i)}$ is the tensor product of three factors, the $i$-th tensorand being
\[
E^- (\beta_1, \zeta_1, 3)^{1-3\delta_{ij}} E^- (\beta_1, \zeta_2, 3)^{1-3\delta_{ij}} E^+ (\beta_1, \zeta_1, 3)^{1-3\delta_{ij}} E^+ (\beta_1, \zeta_2, 3)^{1-3\delta_{ij}}.
\]
Here, \(1 \leq i \leq 3\) and \(\delta_{ij}\) is the Kronecker delta.

Second, for \(1 \leq s \neq t \leq 3\), take a unique \(1 \leq u \leq 3\) such that \(\{s, t, u\} = \{1, 2, 3\}\). Then, as in [55, (3.34)], we have \(Z^{(s)}(\beta_1, \zeta_1)Z^{(t)}(\beta_1, \zeta_2) = \frac{1}{m^2} F_{\beta_1, \beta_1}(\zeta_1/\zeta_2)^{-1} J^{(s,t)}\), where \(J^{(s,t)}\) is the tensor product of three factors, the \(s\)-th tensorand being

\[
E^{-}(\beta_1, \zeta_1, 1)^{-2} E^{-}(\beta_1, \zeta_2, 3) E^{+}(\beta_1, \zeta_1, 3)^{-2} E^{+}(\beta_1, \zeta_2, 3),
\]

the \(t\)-th tensorand being

\[
E^{-}(\beta_1, \zeta_1, 3) E^{-}(\beta_1, \zeta_2, 2)^{-2} E^{+}(\beta_1, \zeta_1, 3) E^{+}(\beta_1, \zeta_2, 3)^{-2},
\]

and the \(u\)-th tensorand being

\[
E^{-}(\beta_1, \zeta_1, 3) E^{-}(\beta_1, \zeta_2, 3)^{-2} E^{+}(\beta_1, \zeta_1, 3) E^{+}(\beta_1, \zeta_2, 3). \]

Let \(l = l^{(1)} + l^{(2)} + l^{(3)}\) and \(J = \sum_{1 \leq s \neq t \leq 3} J^{(s,t)}\). By Proposition 3.3 and in virtue of “the residue theorem” [55 Proposition 3.9.(2)], we see as in [55, (3.26), (3.34)]

\[
G_2(\zeta_1/\zeta_2)Z(\beta_1, \zeta_1)Z(\beta_1, \zeta_2) + G_2(\zeta_2/\zeta_1)Z(\beta_1, \zeta_1)Z(\beta_1, \zeta_2) = \frac{S}{m^2} \delta(\omega^4\zeta_1/\zeta_2) I(\omega^4\zeta_2, \zeta_1) + \frac{U}{m^2} \delta(-\zeta_1/\zeta_2) I(-\zeta_2, \zeta_2) + \frac{T}{m^2} \delta(\omega^5\zeta_1/\zeta_2) I(\omega^5\zeta_2, \zeta_1) + \frac{V}{m^2} \delta(\omega^5\zeta_2/\zeta_1) I(\omega^5\zeta_1, \zeta_2) + \frac{W}{m^2} \delta(\omega^5\zeta_1/\zeta_2) I(\omega^5\zeta_2, \zeta_1) + \frac{X}{m^2} \delta(\omega^5\zeta_2/\zeta_1) I(\omega^5\zeta_1, \zeta_2),
\]

It is easy to see that \(l^{(3)}(-\zeta_2, \zeta_2) = 1\) (see also [55 (3.29)]) and \(J^{(u,t)}(\zeta_2, \zeta_2) = \frac{mZ^{(u)}(-\beta_1, \zeta_2)}{Z^{(u)}(-\beta_1, \zeta_2)}\) (see also [55 (3.36)]). It is also easy to see that

\[
l^{(3)}(\omega^{-4}\zeta_2, \zeta_2) = mZ^{(3)}(\nu^{-2}(\beta_1), \zeta_2), \quad l^{(3)}(\omega^{4}\zeta_2, \zeta_2) = mZ^{(3)}(\nu^{2}(\beta_1), \zeta_2), \quad l^{(3)}(\omega^{-5}\zeta_2, \zeta_2) = mZ^{(3)}(\nu^{5}(\beta_1), \zeta_2), \quad l^{(3)}(\omega^{5}\zeta_2, \zeta_2) = mZ^{(3)}(\nu^{5}(\beta_2), \zeta_2),
\]

by Proposition 3.3 and \(\nu^{+4}(\beta_1) = 1\) \(\nu^{-5}(\beta_1) + \beta_1 = \nu^{4}(\beta_2)\). Thus, we have

\[
G_2(\zeta_1/\zeta_2)Z(\beta_1, \zeta_1)Z(\beta_1, \zeta_2) + G_2(\zeta_2/\zeta_1)Z(\beta_1, \zeta_1)Z(\beta_1, \zeta_2) = \frac{S}{m^2} \delta(\omega^4\zeta_1/\zeta_2) Z(\nu^2(\beta_1), \zeta_2) + \frac{S}{m^2} \delta(\omega^4\zeta_2/\zeta_1) Z(\nu^2(\beta_1), \zeta_2) + \frac{3U}{m^2} \delta(-\zeta_1/\zeta_2) Z(-\zeta_2, \zeta_2) + \frac{T}{m^2} \delta(\omega^5\zeta_1/\zeta_2) Z(\nu^5(\beta_1), \zeta_2) + \frac{T}{m^2} \delta(\omega^5\zeta_2/\zeta_1) Z(\nu^5(\beta_2), \zeta_2) + \frac{4}{m^2} \delta(\zeta_1/\zeta_2) Z(-\zeta_1, \zeta_2).
\]

We get the result by comparing the coefficients of \(\zeta_1^{\alpha} \zeta_2^{\beta}\) as in 3.4.

3.6. A proof of the partial generalized anticommutation relation. As in 3.5, \(G_3(\zeta_1/\zeta_2)Z(\beta_1, \zeta_1)Z(\beta_1, \zeta_2)\) is expanded as

\[
\frac{3N}{m^2} \delta(-\zeta_1/\zeta_2) + \frac{M}{m^2} \delta(\omega^5\zeta_1/\zeta_2) Z(\nu^5(\beta_2), \zeta_2) + \frac{2}{m^2} \delta(\omega^5\zeta_1/\zeta_2) J(\omega^2\zeta_2, \zeta_2) + \delta(\omega^5\zeta_1/\zeta_2) J(\omega^2\zeta_2, \zeta_2).
\]

Let \(J(\omega^2\zeta_2, \zeta_2) = \sum_{i \in \mathbb{Z}} K_i^\pm \zeta_2^i\). In order to complete the proof, it is enough to show \(K_i^\pm = 0\) when \(i \notin 3\mathbb{Z}\). The argument is the same as [55 Corollary 3.9]. We duplicate it below for completeness.
We see that $K^{(s,t)}(\zeta_2) = \sum_{i \in \mathbb{Z}} K^{(s,t)}_i \zeta_2^i$ defined as $J^{(s,t)}(\omega^{-2}\zeta_2, \zeta_2)$ is the tensor product of three factors, the $v$-th tensorand being $E^- (\gamma_v, \zeta_2, 3)E^+ (\gamma_v, \zeta_2, 3)$, where
\[
\begin{align*}
\gamma_s &= -2\nu^{-2}(\beta_1) + \beta_1 = \beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4, \\
\gamma_t &= \nu^{-2}(\beta_1) - 2\beta_1 = -2\beta_1 - \beta_2 - \beta_3 - \beta_4, \\
\gamma_u &= \nu^{-2}(\beta_1) + \beta_1 = \beta_1 - \beta_2 - \beta_3 - \beta_4
\end{align*}
\]
for $v = s, t, u$. Note that $\gamma_t = \nu^4(\gamma_s)$ and $\gamma_u = \nu^{-4}(\gamma_s)$. Thus, we have
\[
\begin{align*}
K^{(3,1)}(\zeta_2) &= K^{(1,2)}(\omega^4 \zeta_2), \\
K^{(2,3)}(\zeta_2) &= K^{(2,3)}(\omega^4 \zeta_2), \\
K^{(3,2)}(\zeta_2) &= K^{(2,1)}(\omega^4 \zeta_2),
\end{align*}
\]
and $J(\omega^{-2}\zeta_2, \zeta_2) = \sum_{i \in \mathbb{Z}} (1 + \omega^4 + \omega^{-4})K^{(3,1)}(\zeta_2)$, which shows $K_i^- = 0$ when $i \notin 3\mathbb{Z}$. The proof of $K_i^+ = 0$ when $i \notin 3\mathbb{Z}$ is the same.

3.7. A proof of the partial generalized commutation relation. We note $D\delta(-\zeta_1/\zeta_2)^{(j)} = D\delta(-\zeta_1/\zeta_2)$ by the same argument as [55] Proposition 3.10 in virtue of “the residue theorem” [39] Proposition 3.9,(3), Proposition 2.4 and $rac{-2}{3} + \frac{1}{v} + \frac{1}{w} = 0$ (see also [55] Lemma 3.6). Thus, as in [3.6] we have
\[
G_6(\zeta_1/\zeta_2)Z(\beta_1, \zeta_1)Z(\beta_1, \zeta_2) - G_6(\zeta_2/\zeta_1)Z(\beta_1, \zeta_1)Z(\beta_1, \zeta_2) = \frac{12}{m}\left(1 - \frac{3P}{Q}\right)D\delta(-\zeta_1/\zeta_2) + \frac{P}{m}\left(\delta(\omega^{-4}\zeta_1/\zeta_2)Z(\nu^2(\beta_1), \zeta_2) - \delta(\omega^4 \zeta_1/\zeta_2)Z(\nu^{-2}(\beta_1), \zeta_2)\right)
\]
\[
+ \frac{P}{3m^2}\left(\delta(\omega^{-2}\zeta_1/\zeta_2)J(\omega^2 \zeta_2, \zeta_2) - \delta(\omega^2 \zeta_1/\zeta_2)J(\omega^{-2} \zeta_2, \zeta_2)\right),
\]
because by Proposition 3.3 we know
\[
G_6^2 G_6 - G_6^2 G_6 = 4\left(1 - \frac{3P}{Q}\right)D\delta(-x) - P(\delta(\omega^4 x) - \delta(\omega^{-4} x)),
\]
\[
G_6^{-1} G_6 - G_6^{-1} G_6 = \frac{P}{3}(\delta(\omega^{-2} x) - \delta(\omega^2 x)).
\]
We get the result by comparing the coefficients of $\zeta_1^A \zeta_2^B$ as in [3.6]

4. A proof of Theorem 1.3

Recall $\Lambda^{(a)}$ and $w_0^{(a)}$ for $1 \leq a \leq 3$ in Theorem 1.3 and let $V = V(\Lambda^{(a)})$. By $\chi(\Omega V(\Lambda^{(a)})) = 1/(q^{2a-1}, q^3, q^6, q^{9-2a-1}; q^9)_\infty$, we have $Z_{-1} w_0^{(2)} = 0$, $Z_{-1} w_0^{(3)} = 0$ and $Z_{-2} w_0^{(3)} = 0$.

4.1. An elimination of $Z_i'$. In virtue of [39] Theorem 7.1, we have $\Omega V = Z V w_0^{(a)}$. By substituting $A = B$ or $A = B + 1$ of the second equality in Theorem 4.2 (i.e., the generalized anticommutation relation proven in [35]), we see that $\Omega V = ZV^{(1)} w_0^{(a)}$ holds, where $ZV^{(1)}$ is a subalgebra of $ZV$ generated by $Z_i$ for $i \in \mathbb{Z}$.

4.2. A Synopsis.

Definition 4.1 ([39] §6, [55] §4). For $i = (i_1, \ldots, i_\ell) \in \mathbb{Z}^\ell$ and $j = (j_1, \ldots, j_\ell) \in \mathbb{Z}^\ell$, where $\ell$ and $\ell'$ are non-negative integers, we say that $i$ is higher than $j$ (denoted by $i > j$) if $\ell < \ell'$ or
\[
\ell = \ell', i \neq j \text{ and } i_p + \cdots + i_\ell \geq j_p + \cdots + j_\ell \text{ for } 1 \leq p \leq \ell.
\]
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For \( j = (j_1, \ldots, j_r) \in \mathbb{Z}^r \), we define \( T(j) \) (resp. \( T^=(j) \)), which is also denoted by \( T(j_1, \ldots, j_r) \) (resp. \( T^=(j_1, \ldots, j_r) \)), to be a vector space spanned by all \( Z_{i_1} \cdots Z_{i_r} \) with \((i_1, \ldots, i_r) > j \) (resp. \((i_1, \ldots, i_r) \geq j \)). In order to prove Theorem 4.3, it is enough to show the following seven statements.

(i) \( V(A_0 + \Lambda_1 - \delta) \) and \( V(A_2 - 2\delta) \) are \( \mathfrak{g} \)-submodules of \( W \).

Here, \( R_k \) is the set of expansions of the form (see also [39 (6.19)])

\[
\xi = \sum_{\nu' \geq 0} \sum_{p=(p_1,\ldots,p_{\nu'}) \in \mathbb{Z}^{\nu'}} c_p Z_{p_1} \cdots Z_{p_{\nu'}},
\]

which may be infinite formal sums, where \( \text{Supp}_i(\xi) \) is finite for any \( i \in \mathbb{Z} \) and \( \text{Supp}_k(\xi) = \emptyset \). Recall (see [39 (4.8)]) that \( \text{Supp}_i(\xi) \) is defined to be the set

\[
\{ p = (p_1, \ldots, p_{\nu'}) \in \mathbb{Z}^{\nu'} | c_p \neq 0 \text{ and } p_{\nu'} + \cdots + p_{\nu'} \leq i \text{ for } 1 \leq \ell' \leq \ell \}.
\]

The statement (F1) (resp. (I)) is proven in §4.a+3 (resp. §4.10) for \( 1 \leq a \leq 6 \).

4.3. Convention. In the following, for \( 1 \leq a \leq 4 \), we denote by \( \theta_{A,B}^{(a)} \) the left-hand side minus the right-hand side of the \( a \)-th equation in Theorem 4.3. Note that \( \theta_{A,B}^{(3)} \) and \( \theta_{A,B}^{(4)} \) are defined for \( A, B \in \mathbb{Z} \) such that \( A + B \notin 3\mathbb{Z} \).

In the calculation, we use the following explicit values and \( c_0^{(p)} = 1 \) for \( 1 \leq p \leq 5 \).

\[
T = -24 - 28\omega + 14\omega^3, \quad M = -6 - 8\omega + 4\omega^3, \quad P = 4 - 8\omega - 6\omega^3, \quad Q = 2 - 4\omega^2, \quad \varepsilon(\nu^5(\beta_1), \beta_1) = -52 + 104\omega^2 + 90\omega^3, \quad c_0^{(6)} = -1 - 2\omega - \omega^3,
\]

\[
c_1^{(1)} = \frac{-6 - 4\omega + 2\omega^3}{3}, \quad c_1^{(2)} = \frac{-4\omega + 2\omega^3}{3}, \quad c_1^{(3)} = \frac{6 - 4\omega + 2\omega^3}{3},
\]

\[
c_1^{(4)} = \frac{8\omega - 4\omega^3}{3}, \quad c_1^{(5)} = \frac{-6 + 8\omega - 4\omega^3}{3}, \quad c_1^{(6)} = \frac{4\omega - 2\omega^3}{3}.
\]

4.4. A proof of (F1). For \( A > B \), we define

\[
\Delta_{A,B} = \theta_{A,B}^{(1)} T(\omega^{A+9B} + \omega^{9A+4B}) - \theta_{A,B}^{(2)} \varepsilon(\nu^5(\beta_1), \beta_1)(\omega^{A+9B} - \omega^{9A+4B}).
\]

The coefficient \( d_{A,B} \) of \( Z_A Z_B \) in \( \Delta_{A,B} \) is given by

\[
d_{A,B} = c_0^{(1)} T(\omega^{A+9B} + \omega^{9A+4B}) - c_0^{(2)} \varepsilon(\nu^5(\beta_1), \beta_1)(\omega^{A+9B} - \omega^{9A+4B}).
\]

One can check by case-by-case substitution that it is non-zero if \( A + 5 \equiv B \) (mod 12).

In the case \( A + 5 \equiv B \) (mod 12), we consider \( -\theta_{A+1,B-1}^{(1)} + \theta_{A+1,B-1}^{(2)} \). Then, the coefficient of \( Z_A Z_B \) (resp. \( Z_{A+1} Z_{B-1} \)) is \( c_1^{(2)} - c_1^{(1)} = 2(\neq 0) \) (resp. \( c_0^{(2)} - c_0^{(1)} = 0 \)), and that of \( Z_{A+1}Z_{B+1} \) is 0 because the following two values are equal.

\[
\varepsilon(\nu^5(\beta_1), \beta_1)(\omega^{A+9B} - \omega^{9A+4B}) = \omega^{A+8}(-52 + 104\omega^2 + 90\omega^3)(\omega - 1),
\]

\[
T(\omega^{A+9B} + \omega^{9A+4B}) = \omega^{A+8}(-24 - 28\omega + 14\omega^3)(\omega + 1).
\]
4.5. A proof of (F2). Consider \( \theta_{A,A}^{(2)} - \frac{1}{M} \theta_{A,A}^{(3)} \). The coefficient of \( Z_A Z_A \) is \( 2(-1 - 2\omega + \omega^3) \), which is non-zero.

4.6. A proof of (F3). Consider \( c_6(\theta_{A,A+1}^{(2)} - \frac{1}{M} \theta_{A,A+1}^{(3)}) + (1 - \frac{1}{M}) \theta_{A,A+1}^{(4)} \). The coefficient of \( Z_A Z_{A+1} \) is non-zero because of

\[
\frac{c_6((c_1^{(2)} + c_1^{(2)}) - \frac{1}{M}(c_0^{(3)} + c_1^{(3)}) + (1 - \frac{1}{M})(c_6 - c_1^{(6)}) = 8(2 + 2\omega - \omega^3).
\]

4.7. A proof of (F4). Recall \( \S 4.4 \) and \( \S 4.6 \) We define

\[
\Delta'_{A,B} = \Delta_{A,B}/d_{A,B} = \sum_{p \geq 0} c_p(A,B) Z_{A-p} Z_{B+p} + d(A,B) Z_{A+B} + \varepsilon(A,B),
\]

\[
\Delta'_{C,C+1} = \left( \sum_{p \geq 0} c_p(C,C+1) Z_{C-p} Z_{C+p} + d(C,C+1) Z_{C+1} + \varepsilon(C,C+1) \right) / (8(2 + 2\omega - \omega^3))
\]

for \( A, B, C \in \mathbb{Z} \) such that \( A > B \) and \( A + 5 \not\equiv B \pmod{12}, 2C + 1 \not\equiv 3Z \). Note that \( c_0(A,B) = c_0(C,C+1) = 1 \) and \( c_p(C,C+1) \) does not depend on \( C \).

Consider \( \Delta'_{3i,3i+1} Z_{3i-1} - Z_{3i} \Delta'_{3i+1,3i-1} \). It is zero, and has the expansion

\[
\sum_{1 \leq p \leq k-6i} c_p^1(3i,3i+1) Z_{3i-p} Z_{3i+1+p} Z_{3i-1} - (c_1^{(3i+1,3i-1)} Z_{3i} Z_{3i} + \sum_{2 \leq r \leq k+1-3i} c_r^{(3i+1,3i-1)} Z_{3i} Z_{3i+1-r} Z_{3i-1+r}).
\]

modulo \( R_k \). For \( p \geq 1 \), we have \( Z_{3i+1+p} Z_{3i-1} \in T^p(3i + p, 3i) + R_k \) by \( \S 4.4 \). This, together with \( Z_{3i+1-r} Z_{3i-1+r} \in T(3i, 3i) \) for \( r \geq 2 \), implies

\[
Z_{3i-p} Z_{3i+1+p} Z_{3i-1} Z_{3i} Z_{3i+1-r} Z_{3i-1+r} \in T(3i, 3i, 3i) + R_k.
\]

Thus, we have \( -c_1^{(3i+1,3i-1)} Z_{3i} Z_{3i} \in T(3i, 3i, 3i) + R_k \) and

\[
c_1^{(3i+1,3i-1)} = \frac{c_1^{(1)}(1 + \omega^{10}) - c_1^{(2)}(1 - \omega^{10})}{T(1 + \omega^{10}) - \varepsilon(1 + \omega^{10})} = \frac{\omega(2 - \omega^2)}{3}.
\]

4.8. A proof of (F5). Recall \( \S 4.5 \) and define

\[
\Delta'_{A,A} = \left( \frac{\theta_{A,A}^{(2)} - \frac{1}{M} \theta_{A,A}^{(3)}}{(2(-1 - 2\omega + \omega^3))} \right) \sum_{p \geq 0} c_p(A,A) Z_{A-p} Z_{A+p} + d(A,A) Z_{2A} + \varepsilon(A,A)
\]

for \( A \in \mathbb{Z} \) with \( 2A \not\equiv 3\mathbb{Z} \). Note that \( c_0^{(A,A)} = 1 \) and \( c_p^{(A,A)} \) does not depend on \( A \).

By a similar argument as in \( \S 4.7 \) applied to \( \Delta'_{3i,3i+1} Z_{3i+1} - Z_{3i} \Delta'_{3i+1,3i+1} \), we have

\[
-c_1^{(3i+1,3i+1)} Z_{3i} Z_{3i+2} \in T(3i, 3i, 3i + 2) + R_k \text{ and }
\]

\[
c_1^{(3i+1,3i+1)} = \frac{2(c_1^{(2)} - \frac{1}{M} c_1^{(3)})}{3} = \frac{3 + 2\omega - \omega^3}{3}.
\]

4.9. A proof of (F6). Apply a similar argument as in \( \S 4.8 \) to \( \Delta'_{3i-1,3i+1} Z_{3i} - Z_{3i-1} \Delta'_{3i-1,3i+1} \), we have \( c_1^{(3i-1,3i-1)} Z_{3i-2} Z_{3i} Z_{3i} \in T(3i - 2, 3i, 3i) + R_k \).
4.10. **A proof of (I).** We assume that readers are familiar with Kashiwara crystals \([32]\). Note that \(3\delta_0 - \alpha_0 = \delta_0 + \delta_1 - \delta = 3\delta_0 - 2\alpha_0 - \alpha_1 = \delta_2 - 2\delta\) (see \([24\text{ §}6.2, \text{ §}12.4]\)). In order to prove (I), it is enough to show that \(\emptyset \otimes \emptyset \otimes \widehat{f}_0 \emptyset\) and \(\emptyset \otimes \widehat{f}_0 \emptyset \otimes \widehat{f}_3 \widehat{f}_0 \emptyset\) are maximal (i.e., are sent to \(\emptyset\) by \(\overline{e}_0, \overline{e}_1, \overline{e}_2\)), where \(\emptyset\) is the highest weight element in the highest weight crystal \(B(\Lambda_0)\). This follows from \(\widehat{f}_3 \widehat{f}_0 \emptyset \neq \emptyset\), which is easily checked by explicit realizations such as Kyoto path models \([31]\), Littelmann path models \([36]\), etc.

5. **Automatic derivations via the regularly linked sets**

Recall Theorem \([1, 2]\). The purpose of this section is to give an automatic derivation of the generating function \(f_{\lambda}(x, q)\), where we write \(f_{\lambda}(x, q) = \sum_{\rho \in F} x^{\rho} q^{\ell(\rho)}\) for \(\lambda \subseteq \text{Par}\), by the regularly linked sets \([53]\), which generalize Andrews’ linked partition ideals \([1, 2]\) by finite automata in formal language theory \([33]\).

5.1. **A survey of \([53]\) for modulus 1.** As in \([53\text{ §}3]\), for a non-empty set \(\Sigma\), let \(\Sigma^*\) be the set of words \(w_1 \cdots w_\ell\) of finite length of \(\Sigma\). By the word concatenation \(\cdot\) and the empty word \(\varepsilon\), the set \(\Sigma^*\) is regarded as a free monoid generated by \(\Sigma\).

For \(A, B \subseteq \Sigma^*\), we define the sum, the product, and the Kleene star by

\[
A + B = A \cup B, \quad AB = \{ww' \mid w \in A, w' \in B\}, \quad A^* = \{\varepsilon\} + A + A^2 + \cdots .
\]

**Definition 5.1.** For a positive integer \(m\) and subsets \(F, 1 \subseteq \text{Par} \setminus \{\emptyset\}\), let \(C\) be a subset of \(\text{Par}\) which consists of partitions \(\lambda\) such that

1. \(m_j(\lambda) < m\) for \(j \geq 1\),
2. \(\lambda\) does not begin with \(c\) for \(c \in F\), and
3. \(\lambda\) does not match \((b_1 + k, \ldots, b_p + k)\) for \((b_1, \ldots, b_p) \in F\) and \(k \geq 0\).

Here, \(\emptyset\) is the empty partition (i.e., the partition of 0). Recall that as in \([53\text{ Definition 1.1}]\), we say that a partition \(\lambda = (\lambda_1, \ldots, \lambda_\ell)\) begins with (resp. matches) \((d_1, \ldots, d_r)\) if \(\ell \geq r\) and \((\lambda_{r+1}, \ldots, \lambda_\ell) = (d_1, \ldots, d_r)\) (resp. \((\lambda_{r+1}, \ldots, \lambda_\ell) = (d_1, \ldots, d_r)\)) for some \(0 \leq i \leq \ell - r\).

**Definition 5.2.** For a non-empty partition \(\lambda\) and \(m \geq 1\), let \(\text{sat}_m(\lambda)\) (resp. \(\text{sat}'_m(\lambda)\)) be a finite subset of \(\text{Par}\) which consists of partitions \(\mu\) such that \(m_1(\mu) < m\) for \(j \geq 1\), and \(\mu\) matches \(\lambda\) (resp. \(\mu_1 = \lambda_1, m_\lambda(\mu) < m, \) and \(\mu\) begins with \(\lambda\)).

**Example 5.3.** For \(\lambda = (5, 3, 2)\), we have \(\text{sat}_3((5, 3, 2)) = \{(5, 3, 2, 2), (5, 3, 2, 2), (5, 3, 2, 2), (5, 3, 2, 2), (5, 3, 2, 2, 2, 1, 1), (5, 3, 2, 2, 1), (5, 3, 2, 1, 1), (5, 3, 2, 1, 1), (5, 3, 2, 1, 1), (5, 3, 2, 1, 1), (5, 3, 2, 1, 1), (5, 3, 2, 1, 1), (5, 3, 2, 1, 1), (5, 3, 2, 1, 1), (5, 3, 2, 1, 1)\}.

Let \(K = \{0, \ldots, m - 1\}\) and define a map \(\pi : K \to \text{Par}\) by \(\pi(i) = (1, \ldots, 1)\) for \(i \in K\). As in \([53\text{ Definition 2.5}]\), we have an injection

\[
\pi^* : \text{Seq}(K, \pi) \hookrightarrow \text{Par},
\]

where \(\text{Seq}(K, \pi)\) is the set of infinite sequences \(i = (i_1, i_2, \ldots)\) in \(K\) (i.e., \(i_j \in K\) for \(j \geq 1\)) such that \(i_j \neq 0\) holds only for finitely many \(j\), and the partition \(\mu = \pi^*(i)\) is characterized by \(m_j(\mu) = i_j\) for \(j \geq 1\).

By construction, \(\pi^*\) gives a bijection to \(C\) when restricted to the set

\[
\{(i_1, i_2, \ldots) \in \text{Seq}(K, \pi) \mid (i_1, \ldots, i_j) \notin K^* \cdot \iota(\text{sat}_m(F)) \cdot K^* + \iota(\text{sat}'_m(F)) \cdot K^* \text{ for } j \geq 1\},
\]
where \( \iota(\nu) = m_1(\nu) \cdots m_{\nu_i}(\nu) \in K^{\nu_i} (\subseteq K^*) \) (i.e., \( \iota(\nu) = (m_1(\nu), \ldots, m_{\nu_i}(\nu)) \in K^{\nu_i} \)) for a non-empty partition \( \nu \), and \( \text{sat}_m(F) = \bigcup_{\lambda \in F} \text{sat}_m(\lambda) \), \( \text{sat}_m^t(l) = \bigcup_{\lambda \in l} \text{sat}^t_m(\lambda) \).

**Remark 5.4.** By the notation in [53, Definition 3.8], the set above is written as avoid(\( \text{Seq}(K, \pi), \iota(\text{sat}_m(F)), \iota(\text{sat}_m^t(l)) \)). Note also that \( F = \emptyset \) (resp. \( 1 = \emptyset \)) is allowed while we exclude the case that the empty partition \( \emptyset \) belongs to \( F \) (resp. \( 1 \)) to satisfy the condition “\( X, X' \subseteq \Sigma^+ \)” in [53, Definition 3.7], where \( \Sigma^+ = \Sigma \setminus \{\varepsilon\} \).

From now on, we assume \( \iota(\text{sat}_m(F)), \iota(\text{sat}_m^t(l)) \subseteq K^* \) are regular [53, Definition 3.3] so that the right-hand side of (1) below is regular [53, Proposition 3.5]. For example, the assumption is satisfied if \( F \) and \( l \) are finite (as in [52] and [53]).

Let \( M = (Q, K, \delta, s, F) \) be a deterministic finite automaton (DFA, for short) such that (see [53, Definition 3.1], [53, Definition 3.2] and [53, Appendix A])

\[
L(M) = K^* \cdot \iota(\text{sat}_m(F)) \cdot K^* + \iota(\text{sat}_m^t(l)) \cdot K^*.
\]

Then, [53, Theorem 3.14] gives a simultaneous \( q \)-difference equation

\[
f_{C^x}(x, q) = \sum_{u \in Q \setminus F} \left( \sum_{u \in K} x^{\iota(\pi(a))} q^{\pi(a)} \right) f_{C^u}(xq, q),
\]

where (see [53, Definition 3.11] and [53, (3.7)]) \( M_v = (Q, K, \delta, v, F) \) and

\[
C_v \equiv \pi^* \{(i_1, i_2, \ldots) \in \text{Seq}(K, \pi) \mid (i_1, \ldots, i_j) \notin L(M_q) \text{ for } j \geq 1\}.
\]

Because of \( C_v \equiv C \), it gives a \( q \)-difference equation for \( f_C(x, q) \) in virtue of the Murray-Miller algorithm (see a review in [53, Appendix B]).

### 5.2. An application to \( L_3 \)

Apply \( m = 3 \), \( F = F \) (or \( F = F \setminus \{(1, 1, 1)\} \)) and \( l = I_3 \) (see Theorem 1.3), we get the minimum DFA (unique up to isomorphism) \( M = (Q, K, \delta, q_0, \{q_3\}) \) where \( Q = \{q_0, \ldots, q_{10}\}, K = \{0, 1, 2\} \) and the values \( \delta(v, a) \) of the transition function \( \delta : Q \times K \to Q \) are displayed as follows.

| \( a \) | \( v \) | \( q_0 \) | \( q_1 \) | \( q_2 \) | \( q_3 \) | \( q_4 \) | \( q_5 \) | \( q_6 \) | \( q_7 \) | \( q_8 \) | \( q_9 \) | \( q_{10} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | \( q_1 \) | \( q_4 \) | \( q_3 \) | \( q_6 \) | \( q_8 \) | \( q_6 \) | \( q_4 \) | \( q_5 \) | \( q_4 \) | \( q_4 \) | \( q_4 \) |
| 1 | \( q_2 \) | \( q_5 \) | \( q_3 \) | \( q_7 \) | \( q_3 \) | \( q_7 \) | \( q_2 \) | \( q_{10} \) | \( q_3 \) | \( q_3 \) | \( q_3 \) |
| 2 | \( q_3 \) | \( q_3 \) | \( q_3 \) | \( q_3 \) | \( q_9 \) | \( q_3 \) | \( q_3 \) | \( q_3 \) | \( q_3 \) | \( q_3 \) | \( q_3 \) |

Thus, we have a simultaneous \( q \)-difference equation

\[
\begin{pmatrix}
  f_{C^{(q_0)}}(x, q) \\
  f_{C^{(q_1)}}(x, q) \\
  f_{C^{(q_2)}}(x, q) \\
  f_{C^{(q_3)}}(x, q) \\
  f_{C^{(q_4)}}(x, q) \\
  f_{C^{(q_5)}}(x, q) \\
  f_{C^{(q_6)}}(x, q) \\
  f_{C^{(q_7)}}(x, q) \\
  f_{C^{(q_8)}}(x, q) \\
  f_{C^{(q_9)}}(x, q) \\
  f_{C^{(q_{10})}}(x, q)
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 & xq & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & xq & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & xq & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & xq & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & xq & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  f_{C^{(q_0)}}(xq, q) \\
  f_{C^{(q_1)}}(xq, q) \\
  f_{C^{(q_2)}}(xq, q) \\
  f_{C^{(q_3)}}(xq, q) \\
  f_{C^{(q_4)}}(xq, q) \\
  f_{C^{(q_5)}}(xq, q) \\
  f_{C^{(q_6)}}(xq, q) \\
  f_{C^{(q_7)}}(xq, q) \\
  f_{C^{(q_8)}}(xq, q) \\
  f_{C^{(q_9)}}(xq, q) \\
  f_{C^{(q_{10})}}(xq, q)
\end{pmatrix}
\]

The pseudo-code in [53, Algorithm 1] stops at the 9-th iteration and gives a \( q \)-difference equation for \( f_{C^{(q_0)}}(x, q) \), which is written in Theorem 5.2 below.
**Theorem 5.5.** For $1 \leq a \leq 3$, we have 
\[
\sum_{r=0}^{8} p_r^{(a)}(x, q) f_{r+1}(x q^a, q) = 0,
\]
where
\[
p_0^{(1)} = (1 - x q^5)(1 - x^2 q^9), \quad p_1^{(1)} = -(1 - x q^5)(1 + x q + x q^2 - x q^3 - x q^4 + x^2 q^5 - x^2 q^6 - x^2 q^8 - x^2 q^9),
\]
\[
p_2^{(1)} = x q(1 - q^2 - q^3 + x q^2 + x q^3 - 2 x q^5 - x q^6 + x q^7 + x q^8 - x^2 q^9 + x^2 q^{11} + x^2 q^{12} - x^3 q^{13}),
\]
\[
p_3^{(1)} = -x^2 q^8(1 - q^2)(1 + q - q^3 + x q^3 + x q^4 + x q^5 - x q^7 - x^2 q^{10} - x^2 q^{11}),
\]
\[
p_4^{(1)} = -x^2 q^5(1 + q)(1 + x q^4 - x q^5 - x q^6 + x^2 q^8),
\]
\[
p_5^{(1)} = x^3 q^9(1 - q^2)(1 + q - x q - x q^2 - x q^3 + x q^4 + x q^5 - x^2 q^6 - x^2 q^7 + x^2 q^9),
\]
\[
p_6^{(1)} = -x^3 q^{10}(1 + x q^4 - x q^6 - x q^7 - x^2 q^5 - x^2 q^6 + 2 x^2 q^8 + x^2 q^9 - x^2 q^{10} - x^2 q^{11} - x^3 q^{11} + x^3 q^{13} + x^3 q^{14}),
\]
\[
p_7^{(1)} = x^4 q^{15}(1 - x q^3)(1 - q^2 - q^3 + x q^4 + x q^5 - x q^6 - x^2 q^{10} + x^2 q^{11}),
\]
\[
p_8^{(1)} = x^4 q^{18}(1 - x q^3)(1 - x^2 q^7),
\]
\[
p_0^{(2)} = -(1 - x q^6)(1 + x q - x q^3 - x q^4 - x^2 q^5 - x^2 q^6 - x^2 q^7 - x^3 q^{10} + x^3 q^{11} + x^3 q^{12}),
\]
\[
p_1^{(2)} = -x q^6(1 - x q^5)(1 - x q^7)(q - x + x^2 q^3 - x^2 q^4 - x^2 q^8 + x^3 q^{11}),
\]
\[
p_2^{(2)} = -x q^6(1 - x q^5)(1 - x q^7)(q - x + x^2 q^3 - x^2 q^4 - x^2 q^8 + x^3 q^{11} + x^3 q^{12} + x^3 q^{13} + x^3 q^{14}),
\]
\[
p_3^{(2)} = -x^2 q^{2}(1 - q^2)(1 - x q^3)(1 + q - x q^2 - x^2 q^4 - x^2 q^5 - x^2 q^6 - x^2 q^7 + x^2 q^9),
\]
\[
p_4^{(2)} = -x^2 q^{2}(1 - q^2)(1 - x q^3)(1 + q - x q^2 - x^2 q^4 - x^2 q^5 - x^2 q^6 - x^2 q^7 + x^2 q^9),
\]
\[
p_5^{(2)} = x^3 q^8(1 - q^2)(1 - x q^2)(1 + q^2 + q^3 - x q^3 - 2 x q^6 - x^2 q^8 - x^2 q^{10} + x^3 q^{13} + x^3 q^{15}),
\]
\[
p_6^{(2)} = -x^3 q^{11}(1 - x q^2)(1 - x q^3)(1 + x q - x q^4 - x^2 q^7 - x^2 q^8 + x^3 q^{16}),
\]
\[
p_7^{(2)} = x^4 q^{17}(1 - x q^2)(1 - x q^3)(1 - q^2 + x q^2 - x^2 q^4 - x^2 q^7 - x^2 q^9 + x^2 q^{10} - x^3 q^{14}),
\]
\[
p_8^{(2)} = x^4 q^{19}(1 - x q^2)(1 - x q^3)(1 - x q^5)(1 - x^2 q^7),
\]
\[
p_0^{(3)} = (1 - x q^2)(1 - x^2 q^{10}), \quad p_1^{(3)} = -(1 - x q^2)(1 + x q^2 - x q^4 - x^2 q^5 - x^2 q^9),
\]
\[
p_2^{(3)} = x q^2(1 - q^2 - q^3 + x q^3 + x q^4 + 2 x q^5 - x q^6 + x q^8 + x^2 q^9 - x^2 q^{10} + x^2 q^{11} + x^3 q^{13} + x^3 q^{15} + x^3 q^{16} + x^3 q^{18}),
\]
\[
p_3^{(3)} = -x^2 q^5(1 + q)(1 - q^3)(1 - q + x q + x q^3 - x q^5 + x^2 q^8 - x^2 q^9 - x^3 q^{13} + x^3 q^{14} + x^3 q^{16}),
\]
\[
p_4^{(3)} = -x^2 q^7(1 + q)(1 - x q^2 + x q^3 + x q^5 - x q^6 - x^2 q^3 - x^2 q^5 + x^2 q^6 + 2 x^2 q^8 - x^2 q^9 - x^2 q^{10} - x^2 q^{11} + x^3 q^{10} + x^3 q^{11} + x^3 q^{13} + x^3 q^{14} + x^4 q^{16}),
\]
\[
p_5^{(3)} = x^3 q^{11}(1 + q)(1 - q^3)(1 - x q^3 + x q^4 - x^2 q^4 + x^2 q^6 + x^2 q^8 - x^3 q^{11} + x^3 q^{12}),
\]
\[
p_6^{(3)} = -x^3 q^{14}(1 + x q^2 + x q^3 - x q^5 + x q^6 - x^2 q^5 - x q^8 - x^2 q^{10} - x^2 q^{11} - x^3 q^7 - x^3 q^{10} - x^3 q^{13} - x^3 q^{14} + x^4 q^{14} + x^4 q^{16} + x^4 q^{17}),
\]
\[
p_7^{(3)} = -x^4 q^{22}(1 - x q^2)(1 - x q + x q^3 + x q^4 - x^2 q^5), \quad p_8^{(3)} = x^4 q^{23}(1 - x^2 q^6)(1 - x^2 q^7).
\]

**Remark 5.6.** The \(q\)-difference equations in Theorem 5.5 can be obtained by Andrews’ linked partition ideals (see [33] Appendix E) because \(F\) and \(I\) are finite. Yet, an approach via the regularly linked sets has an advantage: the minimum forbidden patterns and forbidden prefixes are automatically detected [33] Appendix D. In other words, one has a chance to get smaller simultaneous \(q\)-difference equations.
5.3. A proof of Theorem 5.5 For $L_1$ (resp. $L_2$), the calculations are similar. One only needs to perform the automatic calculation (e.g., by computer algebra as in [53, Remark 4.3]) for the minimum DFA $M$ such that

$$\text{for } m = 3, F = F' \text{ (or } F = F \setminus \{(1, 1, 1)\}) \text{ and } l = I_1 \text{ (resp. } l = I_2\).$$

We omit the details.

5.4. A proof of Theorem 1.5 Note that the case $a = 1$ was shown in [5] by

$$f_{k_1}(x, q) = \sum_{m, k \geq 0} \frac{q^{m^2 + 3km + 4k^2}}{(q; q)_k (q; q)_m} (1 - q^k + q^{k+m}) x^{2k+m}$$

(see [5] Theorem 1.7). We follow a more or less similar line.

We perform an automatic calculation of a $q$-difference equation for each of

$$g_2(x, q) = \sum_{m, k \geq 0} \frac{q^{m^2 + 3km + 4k^2 + k+m}}{(q; q)_k (q; q)_m} (1 + xq^{3k+1}) x^{2k+m},$$

$$g_3(x, q) = \sum_{m, k \geq 0} \frac{q^{m^2 + 3km + 4k^2 + 2k}}{(q; q)_k (q; q)_m} (1 - x^2 q^{4m+8k+6}) x^{2k+m},$$

via a $q$-analog of Sister Celine’s technique (see [52 §7.1]). We see that it is the same as that of $f_{k_1}(x, q)$ in Theorem 5.5 for $a = 2, 3$, which implies that we have $f_{k_1}(x, q) = g_a(x, q)$. By taking the limit $n \to \infty$ in [6] Proposition 4.4, we have

$$f_{L_1}(1, q) = \sum_{k \geq 0} \frac{q^{2k^2 + k}}{(q; q)^{2k+1}}, \quad f_{L_2}(1, q) = \sum_{k \geq 0} \frac{q^{2k^2 + 2k}}{(q; q)^{2k+1}}.$$
K. Bringmann, C. Jennings-Shaffer and K. Mahlburg, Proofs and reductions of various conjectured partition identities of Kanade and Russell, J.Reine Angew.Math. 766 (2020), 109–135.

K. Baker, S. Kanade, M. Russell and C. Sadowski, Principal subspaces of basic modules for twisted affine Lie algebras, q-series multisums, and Nandi’s identities, to appear in Algebr. Comb.

F. Baader and T. Nipkow, Term rewriting and all that, Cambridge University Press, Cambridge, 1998.

S. Capparelli, On some representations of twisted affine Lie algebras and combinatorial identities, J.Algebra 154 (1993), 335–355.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some representations of twisted affine Lie algebras and combinatorial identities, J.Algebra 154 (1993), 335–355.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.

S. Capparelli, A construction of the level 3 modules for the affine algebra $A_{2}^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type, Trans.Amer.Math.Soc. 348 (1996), 481–501.

S. Capparelli, On some theorems of Hirschhorn, Comm.Algebra 32 (2004), 629–635.
[35] K. Kurşungöz, Andrews-Gordon Type Series for Capparelli’s and Göllnitz-Gordon Identities, J. Combin. Theory Ser. A 165 (2019), 117–138.
[36] P. Littelmann, Paths and root operators in representation theory, Ann. Math. 142 (1995), 499–525.
[37] J. Lepowsky and S. Milne, Lie algebraic approaches to classical partition identities, Adv. Math. 29 (1978), 15–59.
[38] J. Lepowsky and R. Wilson, A Lie theoretic interpretation and proof of the Rogers-Ramanujan identities, Adv. Math. 45 (1982), 21–72.
[39] J. Lepowsky and R. Wilson, The structure of standard modules. I. Universal algebras and the Rogers-Ramanujan identities, Invent. Math. 77 (1984), 199–290.
[40] J. Lepowsky and R. Wilson, The structure of standard modules. II. The case A1(1), principal gradation, Invent. Math. 79 (1985), 417–442.
[41] A. Meurman and M. Primc, Annihilating ideals of standard modules of sl(2, C)∼ and combinatorial identities, Adv. Math. 64 (1987), 177–240.
[42] J. McLaughlin and A.V. Sills, Ramanujan-Slater type identities related to the moduli 18 and 24, J. Math. Anal. Appl. 344 (2008), 765–777.
[43] D. Nandi, Partition identities arising from the standard A2(2)-modules of level 4, Ph.D. Thesis (Rutgers University), 2014.
[44] M.H.A. Newman, On theories with a combinatorial definition of ‘equivalence’, Ann. of Math. (2) 43 (1942), 223–243.
[45] A. Riese, qMultiSum – a package for proving q-hypergeometric multiple summation identities, J. Symbolic Comput. 35 (2003), 349–376.
[46] H. Rosengren, Proofs of some partition identities conjectured by Kanade and Russell, Ramanujan J. 61 (2023), 295–317.
[47] M. Russell, Using experimental mathematics to conjecture and prove theorems in the theory of partitions and commutative and non-commutative recurrences, PhD thesis (Rutgers University, New Brunswick, NJ, 2016).
[48] A.V. Sills, An invitation to the Rogers-Ramanujan identities. With a foreword by George E. Andrews, CRC Press, (2018).
[49] L.J. Slater, Further identities of the Rogers-Ramanujan type, Proc. London Math. Soc. (2) 54 (1952), 147–167.
[50] M. Tamba, Level two standard D4(1)-modules, J. Algebra 166 (1994) 651–666.
[51] M. Tamba, Structure of the level two standard modules for the affine Lie algebra D4(3), Comm. Algebra 21 (1993) 1037–1041.
[52] S. Tsuchioka, An example of A2 Rogers-Ramanujan bipartition identities of level 3, [arXiv:2205.04811]
[53] M. Takigiku and S. Tsuchioka, A proof of conjectured partition identities of Nandi, to appear in the American Journal of Mathematics, arXiv:1910.12461
[54] M. Takigiku and S. Tsuchioka, Andrews-Gordon type series for the level 5 and 7 standard modules of the affine Lie algebra A2(2), Proc. Amer. Math. Soc. 149 (2021) 2763–2776.
[55] M. Tamba and C. Xie, Level three standard modules for A2(2) and combinatorial identities, J. Pure Appl. Algebra 105 (1995), 53–92.
[56] A. Uncu, Proofs of Modulo 11 and 13 Cylindric Kanade-Russell Conjectures for A2 Rogers-Ramanujan Type Identities, [arXiv:2301.01359]
[57] A. Uncu and W. Zudilin, Reflecting (on) the modulo 9 Kanade–Russell (conjectural) identities, [arXiv:2106.02959]
[58] O. Warnaar, Hall-Littlewood functions and the A2 Rogers-Ramanujan identities, J. Pure Appl. Algebra 200 (2006) 403–434.
[59] O. Warnaar, The A2 Andrews-Gordon identities and cylindric partitions, to appear in Transactions of the AMS, Series B, arXiv:2111.07550
[60] O. Warnaar, An A2 Bailey tree and A2(1) Rogers-Ramanujan-type identities, arXiv:2303.09069.