ON AUTOMORPHISMS OF SOME FINITE $p$-GROUPS

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Abstract. We give a sufficient condition on a finite $p$-group $G$ of nilpotency class 2 so that $\text{Aut}_c(G) = \text{Inn}(G)$, where $\text{Aut}_c(G)$ and $\text{Inn}(G)$ denote the group of all class preserving automorphisms and inner automorphisms of $G$ respectively. Next we prove that if $G$ and $H$ are two isoclinic finite groups (in the sense of P. Hall), then $\text{Aut}_c(G) \cong \text{Aut}_c(H)$. Finally we study class preserving automorphisms of groups of order $p^5$, $p$ an odd prime and prove that $\text{Aut}_c(G) = \text{Inn}(G)$ for all the groups $G$ of order $p^5$ except two isoclinism families.

Key Words. Finite $p$-group, Isoclinism, Central automorphism, Class preserving automorphism.

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1. Introduction

Let $G$ be a finite $p$-group and $|G| = p^n$, where $p$ is a prime and $n$ is a positive integer. For $x \in G$, $x^G$ denotes the conjugacy class of $x$ in $G$. By $\text{Aut}(G)$ we denote the group of all automorphisms of $G$. An automorphism $\alpha$ of $G$ is called class preserving if $\alpha(x) \in x^G$ for all $x \in G$. The set of all class preserving automorphisms of $G$, denoted by $\text{Aut}_c(G)$, is a normal subgroup of $\text{Aut}(G)$. Notice that $\text{Inn}(G)$, the group of all inner automorphisms of $G$, is a normal subgroup of $\text{Aut}_c(G)$. Let $\text{Out}_c(G)$ denote the group $\text{Aut}_c(G)/\text{Inn}(G)$.

In 1911, W. Burnside [2, pg. 463] posed the following question: Does there exist any finite group $G$ such that $G$ has a non-inner class preserving automorphism? In 1913, Burnside [3] himself gave an affirmative answer to this question. He constructed a group $G$ of order $p^6$, $p$ an odd prime, such that $\text{Out}_c(G) \neq 1$. In [7], [8, pg. 102-103], [11], [14], [15] and [16] more groups were constructed such that $\text{Out}_c(G) \neq 1$. But the order of all these groups is $\geq p^6$. It follows from [12] that $\text{Out}_c(G) = 1$ for all the groups $G$ of order $p^4$. That there exist groups $G$ of order $2^5$ such that $\text{Out}_c(G) \neq 1$ follows from [13] or [18]. In this paper we study the class preserving automorphisms of groups of order $p^5$ for odd primes and prove the following theorem:

Theorem (Theorem 5.5). Let $G$ be a finite $p$-group of order $p^5$, where $p$ is an odd prime. Then $\text{Out}_c(G) \neq 1$ if and only if $G$ is isoclinic to one of the groups $G_7$, $G_{10}$ and $H$, defined in (5.2), (5.3) and (5.4) respectively.

Thus for any prime $p$, $n = 4$ is the largest number such that $\text{Out}_c(G) = 1$ for all the groups $G$ of order $\leq p^n$.

In section 4 we prove that if $G$ and $H$ are two isoclinic finite groups (see Section 4 below), then $\text{Aut}_c(G) \cong \text{Aut}_c(H)$. This result allows us to study $\text{Aut}_c(G)$ for...
a group $G$ only upto isoclinism. A list of the groups of order $p^5$ for odd primes $p$, ordered into ten isoclinism families, is available from James’ work [10]. Our method of proof is to take one group $G$ from each isoclinism family and compute the order of $\text{Aut}_c(G)$ by using the upper and lower bounds derived in Section 2. In section 3 we prove some results regarding class preserving automorphisms of finite $p$-group of class 2.

Our notation for objects associated with a finite multiplicative group $G$ is mostly standard. We use 1 to denote both the identity element of $G$ and the trivial subgroup $\{1\}$ of $G$. The abelian group of all homomorphisms from an abelian group $H$ to an abelian group $K$ is denoted by $\text{Hom}(H, K)$. We write $(x)$ for the cyclic subgroup of $G$ generated by a given element $x \in G$. To say that some $H$ is a subset or a subgroup of $G$ we write $H \subseteq G$ or $H \leq G$ respectively. To indicate, in addition, that $H$ is properly contained in $G$, we write $H < G$ respectively. If $x, y \in G$, then $x^y$ denotes the conjugate element $y^{-1} xy \in G$ and $[x, y] = [x, y]_G$ denotes the commutator $x^{-1} y^{-1} xy = x^{-1} x^y \in G$. If $x \in G$, then $x^G$ denotes the $G$-conjugacy class of all $x^w$, for $w \in G$, and $[x, G]$ denotes the set of all $[x, w]$, for $w \in G$. For $x \in G$, $C_H(x)$ denotes the centralizer of $x$ in $H$, where $H \leq G$. The center of $G$ will be denoted by $Z(G)$. The Frattini subgroup of $G$ is denoted by $\Phi(G)$.

We write the subgroups in the lower central series of $G$ as $\gamma_n(G)$, where $n$ runs over all strictly positive integers. And we write the subgroups in the upper central series of $G$ as $Z_n(G)$, where $n$ runs over all non-negative integers. We will be using the commutator identities

$$[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$$

and

$$[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$$

without any reference.

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### 2. Some Useful Lemmas

An automorphism $\phi$ of a group $G$ is called central if $g^{-1} \phi(g) \in Z(G)$ for all $g \in G$. The set of all central automorphisms of $G$, denoted by $\text{Aut}_c(G)$, is a normal subgroup of $\text{Aut}(G)$. It follows from [15] Proposition 1.7 that $\text{Aut}_c(G) = C_{\text{Aut}(G)}(\text{Inn}(G))$. Following [1], we shall say that a finite group $G$ is purely non-abelian if it does not have a non-trivial abelian direct factor.

The following lemma follows from [12].

**Lemma 2.1.** Let $G$ be a purely non-abelian finite $p$-group. Then $|\text{Aut}_c(G)| = |	ext{Hom}(G/\gamma_2(G), Z(G))|.$

**Lemma 2.2.** Let $G$ be a finite $p$-group such that $Z(G) \subseteq [x, G]$ for all $x \in G - \gamma_2(G)$. Then $|\text{Aut}_c(G)| \geq |\text{Aut}_c(G)||G/\gamma_2(G)|.$

**Proof.** Let $f \in \text{Aut}_c(G)$. Then $f$ fixes $\gamma_2(G)$ element wise. Therefore $f(x) = x \in x^G$ for all $x \in \gamma_2(G)$. So let $x \in G - \gamma_2(G)$. Since $f \in \text{Aut}_c(G)$, $x^{-1} f(x) \in Z(G) \subseteq [x, G]$. Thus $x^{-1} f(x) = [x, g]$ for some $g \in G$. This implies that $f(x) =
Such a \( g^{-1}xg \in xG \). Thus \( f(x) \in xG \) for all \( x \in G \), which proves that \( f \in \text{Aut}_c(G) \). Thus \( \text{Autcent}(G) \leq \text{Aut}_c(G) \). Since \( \text{Inn}(G) \leq \text{Aut}_c(G) \), it follows that

\[
|\text{Aut}_c(G)| \geq |\text{Autcent}(G)||\text{Inn}(G)|/|\text{Autcent}(G) \cap \text{Inn}(G)|.
\]

Set \( A = \text{Autcent}(G) \cap \text{Inn}(G) \). Then \( A \cong \text{Z} (\text{Inn}(G)) \). And \( \text{Z} (\text{Inn}(G)) \cong \text{Z} (G/Z(G)) = Z_2(G)/Z(G) \). Thus \( A \cong Z_2(G)/Z(G) \) and therefore \( |A| = |Z_2(G)/Z(G)| = |Z_2(G)|/[Z(G)] \). Hence

\[
|\text{Aut}_c(G)| \geq |\text{Autcent}(G)||\text{Inn}(G)|/|\text{Autcent}(G) \cap \text{Inn}(G)| = (|\text{Aut}(G)||G/[Z(G)])/(|Z_2(G)||Z(G)) = |\text{Aut}_c(G)||G/[Z_2(G)].
\]

This completes the proof of the lemma. \( \square \)

**Lemma 2.3.** Let \( G \) be a finite group. Let \( \tau \) be an endomorphism of \( G \) such that \( \tau(x) \in xG \) for all \( x \in G \). Then \( \tau \in \text{Aut}_c(G) \).

**Proof.** Such a \( \tau \) obviously has trivial kernel. \( \square \)

The following lemma is \([3, \text{Proposition 14.4}]\).

**Lemma 2.4.** Let \( G \) be a finite group and \( H \) be an abelian normal subgroup of \( G \) such that \( G/H \) is cyclic. Then \( \text{Out}_c(G) = 1 \).

**Lemma 2.5.** Let \( G \) be a finite \( p \)-group of order \( p^n \) and \( G \) have a cyclic subgroup of order \( p^{n-2} \), where \( p \) is an odd prime. Then \( \text{Out}_c(G) = 1 \).

**Proof.** If \( G \) has a cyclic subgroup of order \( p^{n-1} \), then the lemma follows from Lemma 2.4. Otherwise it follows from \([5]\) and \([13]\). \( \square \)

**Lemma 2.6** (\([11]\)). Let \( G \) be a finite group. Let \( \{x_1, x_2, \ldots, x_d\} \) be a minimal generating set for \( G \). Then \( |\text{Aut}_c(G)| \leq \prod_{i=1}^d |x_i^G| \).

**Proof.** There are no more than \( \prod_{i=1}^d |x_i^G| \) choices for the images of the given generators to define a class preserving automorphism of \( G \). \( \square \)

### 3. Groups of class 2

Let \( G \) be a finite nilpotent group of class 2. Let \( \phi \in \text{Aut}_c(G) \). Then the map 

\[
g \mapsto g^{-1}\phi(g)
\]

is a homomorphism of \( G \) into \( \gamma_2(G) \). This homomorphism sends \( Z(G) \) to 1. So it induces a homomorphism \( f_\phi: G/Z(G) \to \gamma_2(G) \), sending \( gZ(G) \) to \( g^{-1}\phi(g) \), for any \( g \in G \). It is easily seen that the map \( \phi \mapsto f_\phi \) is a monomorphism of the group \( \text{Aut}_c(G) \) into \( \text{Hom}(G/Z(G), \gamma_2(G)) \).

Any \( \phi \in \text{Aut}_c(G) \) sends any \( g \in G \) to some \( \phi(g) \in G \). Then \( f_\phi(gZ(G)) = g^{-1}\phi(g) \) lies in \( g^{-1}g^G = [g, G] \). Denote

\[
\{ f \in \text{Hom}(G/Z(G), \gamma_2(G)) \mid f(gZ(G)) \in [g, G], \text{ for all } g \in G \}
\]

by \( \text{Hom}_c(G/Z(G), \gamma_2(G)) \). Then \( f_\phi \in \text{Hom}_c(G/Z(G), \gamma_2(G)) \) for all \( \phi \in \text{Aut}_c(G) \).

On the other hand, if \( f \in \text{Hom}_c(G/Z(G), \gamma_2(G)) \), then the map sending any \( g \in G \) to \( g f(gZ(G)) \) is an automorphism \( \phi \in \text{Aut}_c(G) \) such that \( f_\phi = f \). Thus we have

**Proposition 3.1.** Let \( G \) be a finite nilpotent group of class 2. Then the above map \( \phi \mapsto f_\phi \) is an isomorphism of the group \( \text{Aut}_c(G) \) onto \( \text{Hom}_c(G/Z(G), \gamma_2(G)) \).
The following lemmas are well known.

**Lemma 3.2.** Let $A$, $B$ and $C$ be finite abelian groups. Then
(i) $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$;
(ii) $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$.

**Lemma 3.3.** Let $C_n$ and $C_m$ be two cyclic groups of order $n$ and $m$ respectively.
Then $\text{Hom}(C_n, C_m) \cong C_d$, where $d$ is the greatest common divisor of $n$ and $m$, and $C_d$ is the cyclic group of order $d$.

Let $G$ be a finite $p$-group of class 2. Notice that $[x, G]$ is a non-trivial proper normal subgroup of $G$ for all $x \in G - Z(G)$. Let $\{x_1, x_2, \ldots, x_d\}$ be a minimal generating set for $G$. Then $G/Z(G) = \times_{i=1}^d \langle x_i \rangle$, where $\bar{x}_i = x_i Z(G)$ and some of the factors may possibly be trivial (this may happen in the case when $Z(G) \not\subseteq \Phi(G)$). Let $f \in \text{Hom}(\bar{G}/Z(G), \gamma_2(G))$. So $f(gZ(G)) \in [g, G]$ for all $g \in G$. In particular, $f(x_i Z(G)) \in [x_i, G]$, $1 \leq i \leq d$. Thus it follows that $|\text{Hom}(\bar{G}/Z(G), \gamma_2(G))| \leq \Pi_{i=1}^d |\text{Hom}([\bar{x}_i], [x_i, G])|$. Since there is an isomorphism from $\text{Aut}(\bar{G})$ onto $\text{Hom}(\bar{G}/Z(G), \gamma_2(G))$, we have the following.

**Proposition 3.4.** Let $G$ be a finite $p$-group of class 2 and $\{x_1, x_2, \ldots, x_d\}$ be a minimal generating set for $G$. Then $|\text{Aut}(\bar{G})| \leq \Pi_{i=1}^d |\text{Hom}([\bar{x}_i], [x_i, G])|$. 

**Theorem 3.5.** Let $G$ be a finite $p$-group of class 2. Let $\{x_1, x_2, \ldots, x_d\}$ be a minimal generating set for $G$ such that $[x_i, G]$ is cyclic, $1 \leq i \leq d$. Then $\text{Out}_c(G) = 1$.

**Proof.** Since $|\text{Hom}([\bar{x}_i], [x_i, G])| \leq |\langle x_i \rangle|$, it follows from Proposition 3.3 that $|\text{Aut}_c(G)| \leq \Pi_{i=1}^d |\langle x_i \rangle|$. Now $|\text{Inn}(G)| \leq |\text{Aut}_c(G)| \leq \Pi_{i=1}^d |\langle x_i \rangle| = |G/\gamma_2(G)| = |\text{Inn}(G)|$.

Thus $|\text{Aut}_c(G)| = |\text{Inn}(G)|$ and therefore $\text{Out}_c(G) = 1$. 

**Corollary 3.6.** Let $G$ be a finite $p$-group of class 2 such that $\gamma_2(G)$ is cyclic. Then $\text{Out}_c(G) = 1$.

### 4. Isoclinic Groups

The following concept was introduced by P. Hall [2] (also see [3] pg. 39-40 for details).

Let $X$ be a finite group and $\bar{X} = X/Z(X)$. Then commutation in $X$ gives a well defined map $\alpha_X : \bar{X} \times \bar{X} \rightarrow \gamma_2(X)$ such that $\alpha_X(xZ(X), yZ(X)) = [x, y]$ for $(x, y) \in X \times X$. Two finite groups $G$ and $H$ are called isoclinic if there exists an isomorphism $\phi$ of the factor group $\bar{G} = G/Z(G)$ onto $\bar{H} = H/Z(H)$, and an isomorphism $\theta$ of the subgroup $\gamma_2(G)$ onto $\gamma_2(H)$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
G \times \bar{G} & \overset{\alpha_G}{\longrightarrow} & \gamma_2(G) \\
\downarrow_{\phi \times \phi} & & \downarrow_{\theta} \\
\bar{H} \times \bar{G} & \overset{\alpha_H}{\longrightarrow} & \gamma_2(H)
\end{array}
$$

The resulting pair $(\phi, \theta)$ is called an isoclinism of $G$ onto $H$. Notice that isoclinism is an equivalence relation among finite groups.

**Theorem 4.1.** Let $G$ and $H$ be two finite non-abelian isoclinic groups. Then $\text{Aut}_c(G) \cong \text{Aut}_c(H)$. 
Proof. Since $G$ and $H$ are isoclinic, there exist isomorphisms $\phi : G/\mathbb{Z}(G) \to H/\mathbb{Z}(H)$ and $\theta : \gamma_2(G) \to \gamma_2(H)$ such that $\theta([x, y]) = [x'Z(H), y'Z(H)] = [x', y']$, where $x'Z(H) = \phi(xZ(G))$ and $y'Z(H) = \phi(yZ(G))$.

Let $\tau \in \text{Aut}_c(G)$. Let $x' \in H - \mathbb{Z}(H)$. Then $x'Z(H) \in H/\mathbb{Z}(H)$ and $xZ(G) = \phi^{-1}(x'Z(H)) \in G/\mathbb{Z}(G)$. So $x \in G$ and therefore there exists an element $a_x \in G$ such that $\tau(x) = a_x^{-1}xa_x$.

Let $a_x'$ be a coset representative of $\phi(a_xZ(G))$. Now define a map $\sigma_\tau : H \to H$ by

\[
\sigma_\tau(x') = \begin{cases} 
(a_x')^{-1}x'a_x', & \text{if } x' \in H - \mathbb{Z}(H); \\
x', & \text{if } x' \in \mathbb{Z}(H).
\end{cases}
\]

To make the proof more readable, we prove it in several steps.

**Step 1.** $\sigma_\tau$ is well defined.

**Proof.** We prove that for each $x' \in H$, $\sigma_\tau(x')$ is unique. Let $w$ and $x$ be two coset representatives of $\phi^{-1}(x'Z(H))$ such that $w \neq x$. Then $w = xz$, where $z \in Z(G)$.

Now

\[
\tau(w) = \tau(wxz) = \tau(x)\tau(z) = a_x^{-1}xa_xz = a_x^{-1}xza_x = a_x^{-1}wa_x.
\]

Thus it follows that $a_x$ in $\tau(x)$ is independent of the choice of coset representative of the coset $xZ(G)$.

Now suppose that there exist two elements $a_x$ and $b_x$ in $G$ such that $\tau(x) = a_x^{-1}xa_x = b_x^{-1}xb_x$. Then, for $\sigma_\tau(x')$ there are two choices $(a_x')^{-1}x'a_x'$ and $(b_x')^{-1}x'b_x'$, where $a_x'$ and $b_x'$ are coset representatives of $\phi(a_xZ(G))$ and $\phi(b_xZ(G))$ respectively.

We claim that $(a_x')^{-1}x'a_x' = (b_x')^{-1}x'b_x'$. Since $a_x^{-1}xa_x = b_x^{-1}xb_x$, it follows that

\[
[x, b_xa_x^{-1}] = 1.
\]

Now applying $\theta$ on it we get

\[
1 = \theta([x, b_xa_x^{-1}]) = [\phi(xZ(G)), \phi(b_xa_x^{-1}Z(G))]
\]

\[
= [\phi(xZ(G)), \phi(b_xZ(G))\phi(a_x^{-1}Z(G))] = [x'Z(H), b_x'Z(H)(a_x')^{-1}Z(H)]
\]

Thus it follows that $(a_x')^{-1}x'a_x' = (b_x')^{-1}x'b_x'$ and therefore our claim is true.

Finally suppose that $w' \in a_x'Z(H)$. Then $w' = a_x'z'$ for some $z' \in \mathbb{Z}(H)$. Now

\[
(w')^{-1}x'w' = (a_x'z')^{-1}x'a_x'z' = (a_x')^{-1}x'a_x'.
\]

Thus $\sigma_\tau(x')$ is independent of the choice of coset representative of $a_x'Z(H)$. This proves that $\sigma_\tau$ is well defined.

**Step 2.** $\sigma_\tau \in \text{Aut}_c(H)$.

**Proof.** Let $x', y' \in H$. If both $x', y' \in Z(H)$, then $\sigma_\tau(x'y') = x'y' = \sigma_\tau(x')\sigma_\tau(y')$.

Now let $x' \in H - \mathbb{Z}(H)$ and $y' \in Z(H)$. Then $x'y' \in H - \mathbb{Z}(H)$ and $x'y'Z(H) = x'Z(H)$. Thus $\sigma_\tau(x'y') = (a_x')^{-1}x'y'a_x' = (a_x')^{-1}x'a_x'y = \sigma_\tau(x')\sigma_\tau(y')$, since $y' \in Z(H)$. So assume that $x', y' \in H - \mathbb{Z}(H)$.

Now

\[
\phi^{-1}(x'y'Z(H)) = \phi^{-1}(x'Z(H))\phi^{-1}(y'Z(H)) = xZ(G)yZ(G) = xyZ(G),
\]

where $x, y \in G$. Let $\tau(xy) = a_x^{-1}xya_{xy}$, $\tau(x) = a_x^{-1}xa_x$ and $\tau(y) = a_y^{-1}ya_y$. Since $\tau(xy) = \tau(x)\tau(y)$, we get

\[
a_x^{-1}xya_{xy} = a_x^{-1}xa_xa_y^{-1}ya_y
\]

or

\[
(xy)^{-1}a_x^{-1}ya_{xy} = y^{-1}x^{-1}a_x^{-1}xa_x(yy^{-1})a_y^{-1}ya_y
\]
or

\[ [xy, ax_y] = [x^y, (ax_x)^y][y, a_y], \]

where \((ax_x)^y = y^{-1}a_x y\). Now applying \(\theta\) on both the sides, we get

\[ [x'y', a_{x'y}^y] = [(x'y')^y, (a_{x'y})^y][y', a_y'] \]

or

\[ (a_{x'y})^{-1}x'y'a_{x'y}^y = (a_{x'y})^{-1}x'\phi_y^{-1}(a_{x'y})^{-1}y'a_y'. \]

Thus from the definition (4.1) it follows that \(\sigma_x(x'y') = \sigma_x(x')\sigma_x(y')\). Hence \(\sigma_x\) is an endomorphism of \(H\). That \(\sigma_x \in \text{Aut}_c(H)\) follows from Lemma 2.3. This completes the proof of Step 2.

**Step 3.** The map \(\tau \mapsto \sigma_\tau\) is a homomorphism from \(\text{Aut}_c(G)\) to \(\text{Aut}_c(H)\).

**Proof.** Let \(\tau_1, \tau_2 \in \text{Aut}_c(G)\). Let \(x' \in H\). Let \(x\) be a coset representative of \(\phi^{-1}(x'Z(H))\). Since \(\tau_1 \tau_2 \in \text{Aut}_c(G)\), there exists \(a_x \in G\) such that \(\tau_1 \tau_2(x) = a_x^{-1}xa_x\). Also there exists \(b_x, c_x \in G\) such that \(\tau_2(x) = b_x^{-1}xb_x\) and \(\tau_1(\tau_2(x)) = c_x^{-1}(b_x^{-1}xb_x)c_x\). Since \(\tau_1 \tau_2(x) = \tau_1(\tau_2(x))\), we get

\[ a_x^{-1}xa_x = c_x^{-1}b_x^{-1}xb_xc_x \]

or

\[ x^{-1}a_x^{-1}xa_x = x^{-1}(b_xc_x)^{-1}xb_xc_x \]

or

\[ [x, a_x] = [x, b_xc_x]. \]

Now applying \(\theta\) on both the sides, we get

\[ [x', a_x'] = [x', b_x'c_x'], \]

since \(\phi(b_xc_x Z(G)) = \phi(b_x Z(G))\phi(c_x Z(G)) = b_x' Z(H)c_x' Z(H) = b_x'c_x' Z(H)\), where \(a_x', b_x'\) and \(c_x'\) are the coset representatives of \(\phi(a_x Z(G))\), \(\phi(b_x Z(G))\) and \(\phi(c_x Z(G))\) respectively. Thus from the last equality we have

\[ (a_x')^{-1}x'a_x' = (c_x')^{-1}(b_x')^{-1}x'b_x'c_x'. \]

Now it follows from the definitions of \(\sigma_{\tau_1 \tau_2}, \sigma_{\tau_1}\) and \(\sigma_{\tau_2}\) that \(\sigma_{\tau_1 \tau_2}(x') = (a_x')^{-1}x'a_x'\) and \(\sigma_{\tau_1}(\sigma_{\tau_2}(x')) = (c_x')^{-1}(b_x')^{-1}x'b_x'c_x'\), for all \(x' \in H\). Hence \(\sigma_{\tau_1 \tau_2} = \sigma_{\tau_1}\sigma_{\tau_2}\). This proves Step 3.

Similarly for each \(\sigma \in \text{Aut}_c(H)\) we can define \(\tau_\sigma\in \text{Aut}_c(G)\) and the map sending \(\sigma\) to \(\tau_\sigma\) is a homomorphism from \(\text{Aut}_c(H)\) to \(\text{Aut}_c(G)\). It is not difficult to prove that for each \(\tau \in \text{Aut}_c(G)\) and \(\sigma \in \text{Aut}_c(H)\), \(\tau_\sigma = \tau\) and \(\sigma_{\tau_\sigma} = \sigma\). Thus it follows that the homomorphism from \(\text{Aut}_c(G)\) to \(\text{Aut}_c(H)\), defined above, becomes an isomorphism. This completes the proof of the theorem. \(\square\)

5. **Groups of order \(p^5\)**

We’ll use the classification of groups of order \(p^5\) by R. James [10, Section 4.5]. Throughout this section \(p\) always denotes an odd prime.

**Lemma 5.1.** Let \(G\) be the group \(\phi_7(1^5)\) in the isoclinism family (7) of [10, Section 4.5]. Then \(|\text{Aut}_c(G)| = p^5\).
Proof. Let $G$ be the group $\phi_7(1^5)$. Then $G$ is a nilpotent group of class 3 such that $Z(G) \leq \gamma_2(G) = \Phi(G)$, $|\gamma_2(G)| = p^2$ and $|Z(G)| = p$. Now it follows from \cite{17} Theorem 4.7 that $Z(G) \leq [x, G]$ for all $x \in G - Z(G)$. Since $Z(G) \leq \gamma_2(G)$, therefore $Z(G) \leq [x, G]$ for all $x \in G - \gamma_2(G)$. Thus it follows from Lemma 2.2 that

\begin{equation}
|\text{Aut}_c(G)| \geq |\text{Autcent}(G)|/Z_2(G)| = |\text{Autcent}(G)|/p^5 |Z_2(G)|.
\end{equation}

From Lemma 2.1 we have $|\text{Autcent}(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))| = p^3$, since $G/\gamma_2(G)$ is an elementary abelian group of order $p^3$. It follows that $|Z_2(G)| = p^3$. Thus from (5.1) we get $|\text{Aut}_c(G)| \geq p^5$. It follows from \cite{10} Section 4.1 that in $G$ there are $p^3 - p$ conjugacy classes of length $p^2$. Thus all these conjugacy classes covers $p^3 - p$ elements out of $p^5 - p^2$ elements in $G - \gamma_2(G)$. So there must exists an element $y \in G - \gamma_2(G)$ such that $|y^G| \leq p$. Since $Z(G) \leq \gamma_2(G)$, $y$ is not a central element and therefore $|y^G| = p$. We can extend the set $\{y\}$ to get a minimal generating set (say) $\{y, x_1, x_2\}$ of $G$, since $\gamma_2(G) = \Phi(G)$. Then from Lemma 2.6 we get $|\text{Aut}_c(G)| \leq p^5$, since $|x_i^G| \leq p^2$, $i = 1, 2$. Hence $|\text{Aut}_c(G)| = p^5$. \hfill $\square$

Lemma 5.2. Let $G$ be the group $\phi_{10}(1^5)$ in the isoclinism family (10) of $[10]$ Section 4.5. Then $|\text{Aut}_c(G)| = p^5$.

Proof. The group $G$ is a $p$-group of maximal class. $G$ is generated by $\alpha$ and $\alpha_1$ such that the elements $\alpha_{i+1} := [\alpha_i, \alpha]$, $1 \leq i \leq 3$, generate $\gamma_2(G)$ and $[\alpha_1, \alpha_2] = \alpha_4$. Here $\alpha_2$, $\alpha_3$ and $\alpha_4$ commutes with one another. Thus $\gamma_2(G)$ is abelian. Also $[\alpha_1, \alpha_3] = 1$. It is easy to prove that every element $g \in G$ can be written as

$$
g = \beta \alpha_1^k \alpha^k,$$

where $0 \leq k_1, k \leq p - 1$ and $\beta \in \gamma_2(G)$. Let $k \neq 0$. Then

$$[g, \alpha_3] = \beta \alpha_1^k \alpha^k, [\alpha_3] = [\alpha^k, \alpha_3] = \alpha_4^{-k} \in Z(G),$$

since $[\beta \alpha_1^k, \alpha_3] = 1$. Now let $k = 0$ and $k_1 \neq 0$. Then

$$[g, \alpha_2] = \beta \alpha_1^k, \alpha_2 = [\alpha_1^k, \alpha_2] = \alpha_4^{k_1} \in Z(G).$$

Thus it follows that $Z(G) \subseteq [g, G]$ for $g \in G - \gamma_2(G)$, since $|Z(G)| = p$. Now from Lemma 2.2 we have

$$|\text{Aut}_c(G)| \geq |\text{Autcent}(G)|/p^5 = p^5,$$

since $|Z_2(G)| = p^2$ and $|\text{Autcent}(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))| = p^2$. Since there are only $p^3 - p$ conjugacy classes of length $p^3$, there must exists an element $y \in G - \gamma_2(G)$ such that $|y^G| \leq p^2$. We can always extend $\{y\}$ to some minimal generating set (say) $\{y, x\}$ of $G$, therefore we have from Lemma 2.6 that $|\text{Aut}_c(G)| \leq |y^G||x^G| \leq p^5$. Hence $|\text{Aut}_c(G)| = p^5$. \hfill $\square$

Lemma 5.3. Let $G$ be the group $\phi_6(1^5)$ in the isoclinism family (6) of $[10]$ Section 4.5. Then $\text{Out}_c(G) = 1$.

Proof. $G$ is a group of class 3 and $|Z(G)| = p^2$. Suppose that $\text{Out}_c(G) \neq 1$. Since $|\text{Inn}(G)| = |G/Z(G)| = p^3$, we must have $|\text{Aut}_c(G)| \geq p^4$. It follows from $[10]$ Section 4.1 that $|x^G| = p^2$ for all $x \in G - Z(G)$. Also $|\gamma_2(G)| = |\Phi(G)| = p^3$. Then $G$ can be generated by two elements (say) $x, y$ of $G$. Now it follows from Lemma 2.4 that $|\text{Aut}_c(G)| \leq |x^G||y^G| \leq p^4$. Since $|\text{Inn}(G)| = p^3$ and $\text{Out}_c(G) \neq 1$, $|\text{Aut}_c(G)|$
must be $p^4$. Thus it follows that for any two elements $u \in x^G$ and $v \in y^G$, there must exist an automorphism $\tau \in \text{Aut}_c(G)$ such that $\tau(x) = u$ and $\tau(y) = v$.

The group $G$ is generated by $\alpha_1, \alpha_2$ such that $\gamma_2(G)$ is generated by $\beta := [\alpha_1, \alpha_2]$, $\beta_1 := [\beta, \alpha_1]$ and $\beta_2 := [\beta, \alpha_2]$, and $Z(G)$ is generated by $\beta_1$ and $\beta_2$. Since $[\alpha_1, \alpha_2] \neq 1$, $\alpha_1 \neq \alpha_2^{-1} \alpha_1 \alpha_2 \in G$, and $\alpha_2 \neq \alpha_1^{-1} \alpha_1 \alpha_2 \in G$. Let $\tau \in \text{Aut}_c(G)$ such that $\tau(\alpha_1) = \alpha_2^{-1} \alpha_1 \alpha_2$ and $\tau(\alpha_2) = \alpha_1^{-1} \alpha_2 \alpha_1$. Since $\tau \in \text{Aut}_c(G)$, $(\alpha_1 \alpha_2)^{-1} \tau(\alpha_1 \alpha_2) \in [\alpha_1 \alpha_2, G]$. Now

$$(\alpha_1 \alpha_2)^{-1} \tau(\alpha_1 \alpha_2) = \alpha_2^{-1} \alpha_1^{-1} \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2 \alpha_1 = \beta_2.$$ 

Thus $\beta_2 \in [\alpha_1 \alpha_2, G]$. So there is an element $y \in G$ such that $\beta_2 = [\alpha_1 \alpha_2, y]$. We claim that $[\alpha_1 \alpha_2, \beta] 
eq 1$. For, if $[\alpha_1 \alpha_2, \beta] = 1$, then $\gamma_2(G) < C_G(\alpha_1 \alpha_2)$, since $\alpha_1 \alpha_2 \in G - \gamma_2(G)$. Thus $|C_G(\alpha_1 \alpha_2)| \geq p^4$, which is a contradiction to the fact that $|[\alpha_1 \alpha_2, G]| = p^2$. Hence our claim is true. Now $[\alpha_1 \alpha_2, \beta] = [\alpha_1, \beta][\alpha_2, \beta] = \beta_1^{-1} \beta_2^{-1}$, since $[\alpha_i, \beta] \in Z(G), i = 1, 2$. Thus $\beta_1^{-1} \beta_2^{-1} \in [\alpha_1 \alpha_2, G]$. Now

$$\beta_1^{-1} \beta_2^{-1} \beta_2 = [\alpha_1 \alpha_2, \beta][\alpha_1 \alpha_2, y] = [\alpha_1 \alpha_2, \beta].$$

Thus it follows that $\beta_1, \beta_2 \in [\alpha_1 \alpha_2, G]$ and therefore $Z(G) \subseteq [\alpha_1 \alpha_2, G]$. Since $|[\alpha_1 \alpha_2, G]| = p^2$, we get $Z(G) = [\alpha_1 \alpha_2, G]$. This gives a contradiction, since $[\alpha_1 \alpha_2, \alpha_1] = [\alpha_2, \alpha_1] = \beta^{-1} \in [\alpha_1 \alpha_2, G]$, but $\beta^{-1} \notin Z(G)$. This completes the proof of the lemma.

\[\square\]

Define a set of relations $\mathcal{R}$ by

$$\mathcal{R} = \{a^p = b^p = x^p = y^p = z^p = 1\} \cup \{[x, y] = [y, z] = [x, z] = 1\}$$

$$\cup \{x^b = xz, y^b = y, z^b = z\} \cup \{x^a = xy, y^a = yz, z^a = z\}.$$ 

Now set

$$G_7 = \langle a, b, x, y, z \mid \mathcal{R}, [b, a] = 1 \rangle \quad \text{for} \quad p \geq 3$$

and

$$G_{10} = \langle a, b, x, y, z \mid \mathcal{R}, [b, a] = x \rangle \quad \text{for} \quad p \geq 5.$$ 

The groups $G_7$ and $G_{10}$ are of the form $\langle a \rangle \ltimes \langle (b) \ltimes ((x) \times (y) \times (z)) \rangle$ and have order $p^5$. For $p = 3$ define a group $H$ of order $3^5$ by

$$H = \langle a, b, c \mid a^3 = b^3 = c^9 = 1, [b, c] = c^3, [a, c] = b^3, [b, a] = c \rangle = \langle a \rangle \ltimes (\langle b \rangle \ltimes \langle c \rangle).$$

Remark 5.4. The group $\phi_7(1^5)$ in the isoclinism family (7) of [10, Section 4.5] is isomorphic to $G_7$. The group $\phi_9(1^5)$ in the isoclinism family (10) of [10, Section 4.5] is isomorphic to $G_{10}$ for $p \geq 5$ and is isomorphic to $H$ for $p = 3$.

Now we prove our main theorem.

Theorem 5.5. Let $G$ be a finite $p$-group of order $p^5$, where $p$ is an odd prime. Then $\text{Out}_c(G) \neq 1$ if and only if $G$ is isoclinic to one of the groups $G_7$, $G_{10}$ and $H$.

Proof. In view of Theorem 4.1, it is sufficient to study $\text{Out}_c(G)$ only for one group $G$ from each isoclinism family of groups of order $p^5$. If $G$ is abelian, then obviously $\text{Out}_c(G) = 1$. Let $G$ be either $\phi_2(311)a$, $\phi_3(311)a$ or $\phi_8(32)$. Then there exists an element $x \in G$ such that order of $x$ is $p^5$. So it follows from Lemma 2.9 that $\text{Out}_c(G) = 1$. Now let $G$ be the group $\phi_4(1^5)$. Then $G$ has a maximal abelian subgroup $H$ such that $G/H$ is cyclic. Thus it follows from Lemma 2.3 that
Out$_c(G) = 1$. If $G$ is some group from the isoclinism family (5), then the class of $G$ is 2 and $\gamma_2(G)$ is cyclic. Thus from Corollary [3.1], we have that Out$_c(G) = 1$. Next consider any group $G$ from the isoclinism family (9). Then it follows from [10] Section 4.1 that the nilpotency class of length 1. Thus $|G| = |\Phi(G)| = p^3$. Any conjugacy class of length $p^3$ must be contained in $G - \gamma_2(G)$, since $|\gamma_2(G)| = p^3$. Let $C = \{x \in G|x^G| = p^3\}$. Then $|C| = p^3(p^2 - p) = p^5 - p^3$ and $C \subset G - \gamma_2(G)$. Since $|G - \gamma_2(G)| = p^5 - p^3$, there must exist an element $y \in G - \gamma_2(G)$ such that $|y^G| < p^3$. Thus $|y^G| = p$. Since $\gamma_2(G) = \Phi(G)$, $y \in G - \Phi(G)$. Therefore the set $\{y\}$ can be extended to a minimal generating set (say) $\{y, x\}$ of $G$. Then $|y^G||x^G| \leq p^4$. Now it follows from Lemma 2.6 that Aut$_c(G) \leq |p^G||x^G|$. Thus

\[
\text{Inn}(G) \leq \text{Aut}_c(G) \leq |y^G||x^G| \leq p^4 = |G/Z(G)| = |\text{Inn}(G)|.
\]

Hence Inn$(G) = \text{Aut}_c(G)$, which gives Out$_c(G) = 1$. Now it follows, along with Lemma 5.3 that if Out$_c(G) \neq 1$, then $G$ can not lie in the isoclinism families (1) - (6), (8) and (9). Thus $G$ must lie either in the isoclinism family (7) or (10). Hence $G$ is isoclinic to $\phi_7(1^5)$ or $\phi_{10}(1^5)$. Thus it follows from Remark 5.3 that $G$ is isoclinic to $G_7$, $G_{10}$ or $H$.

Conversely suppose that $G$ is isoclinic to $G_7$, $G_{10}$ or $H$. Thus $G$ is isoclinic to $\phi_7(1^5)$ or $\phi_{10}(1^5)$. Then it follows from Lemma 5.4 and Lemma 5.2 that Out$_c(G) \neq 1$.

\[\square\]

6. Some alternative proofs

The following alternative proofs of some of our lemmas were provided by the referee.

\textit{Alternative proof of Lemma 5.1 and Lemma 5.2.} Let $G$ be one of the groups $G_7$ and $G_{10}$. Then the center of $G$ is generated by $z$ and has order $p$. Let $\tau \in \text{Aut}_c(G)$. Since Aut$_c(G) = \text{Inn}(G)$ for all groups $G$ of order $p^4$ (12), $\tau$ induces an inner automorphism on $G/Z(G)$ given by conjugation with (say) $aZ(G)$. Thus $\tau(x)Z(G) = x^aZ(G)$ for all $x \in G$. So for each $x \in G$, there exists some element $z_x \in Z(G)$ such that $\tau(x) = x^a z_x = (xz_x)^a$. Let us define a map $\tau'$ from $G$ to $G$ such that $\tau'(x) = \tau(x) a^{-1}$ for all $x \in G$. Then it is fairly easy to prove that $\tau' \in \text{Aut}_c(G)$ and $\tau'(x) = x z_x$, $x \in G$. Now it follows that $\tau'$ is a central automorphism of $G$ and $\tau = i_a \tau'$, where $i_a$ denotes the inner automorphism given by conjugation with $a$. Thus we have Aut$_c(G) \leq \text{Autcent}(G) \text{Inn}(G)$. It follows (as in [17] Section 3) that $Z(G) \leq [g, G]$ for all $g \in G - Z(G)$. Thus we have Aut$_c(G) = \text{Autcent}(G) \text{Inn}(G)$. Now using Lemma 2.1 one can prove that $|\text{Autcent}(G) \text{Inn}(G)| = p^5$. Hence $|\text{Aut}_c(G)| = p^5$. It is not difficult to prove the result for the group $H$. \[\square\]

\textit{Alternative proof of Lemma 5.3.} The nilpotency class of $G$ is 3 and $G$ has an elementary abelian center of order $p^2$. Let $\tau \in \text{Aut}_c(G)$. Let $N_1, \ldots , N_{p+1}$ be the central subgroups of order $p$ in $G$. Since Aut$_c(G) = \text{Inn}(G)$ for all groups $G$ of order $p^4$ (12), $\tau$ induces an inner automorphism on $G/N_i$, $1 \leq i \leq p + 1$. Let $x_1, \ldots , x_{p+1} \in G$ such that $\tau$ induces on $G/N_i$ the inner automorphism given by conjugation with $x_iN_i$. These inner automorphisms agree on $G/Z(G)$, so the $p + 1$ elements $x_1^{-1} x_i$ all lie in $Z_2(G)$. Since $|Z_2(G)/Z(G)| = p$, we have
$x_1^{-1}x_s Z(G) = x_1^{-1}x_t Z(G)$ for distinct indices $s$ and $t$. But then $x_s Z(G) = x_t Z(G)$ and $\tau$ is the inner automorphism given by conjugation with $x_s$. This proves that $\text{Out}_c(G) = 1$. □

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