Abstract—The direct-current Superconducting Quantum Interference Device (dc-SQUID) is a hybrid circuit with Josephson junctions and normal circuit elements, considering the parasitic capacitors and inductors introduced by tightly coupled input coils. Its equivalent circuit is becoming more and more complicated with the development of practical SQUID integrated circuits. It is difficult to derive the analytical expression of current-voltage characteristics directly from the nonlinear circuit equations. This article introduces a frequency domain analysis method to derive the analytical current-voltage expression using four network impedances instead of the nonlinear circuit equations. A frequency-phase-locking model is derived to depict the working principle inside the dc-SQUID. In this model, two Josephson junctions and the linear network form two cross-linked lock-in amplifiers, where the Josephson junction functions as both the voltage-controlled oscillator (VCO) and the mixer; the current-voltage characteristics of the dc-SQUID are the projections of the network impedances. The application of the model is demonstrated in the analysis of a typical dc SQUID magnetometer; the current-voltage characteristics predicted by the analytical expression agree well with the results through the numerical simulation. This frequency-phase-locking model improves our understanding of the working mechanism inside the dc-SQUID circuit.

Index Terms—Direct-current superconducting quantum interference device (dc-SQUID), frequency-locking, phase-locking, current-voltage characteristic.

I. INTRODUCTION

The direct-current Superconducting Quantum Interference Device (dc-SQUID) [1] is widely applied as the sensitive flux-to-voltage converter in the very-weak magnetic field measurements, such as biomagnetism [2] and geophysical applications [3], due to its unique flux-modulated current-voltage characteristics. The current-voltage characteristic of a dc-SQUID is usually expressed as a function in the form of \( f(V_s, I_b, \Phi_e) = 0 \), where \( I_b \) is the bias current source; \( \Phi_e \) is the applied external flux; \( V_s \) is the average Josephson junction voltage; we measure \( V_s \) at the two terminals of dc-SQUID with the readout circuit. In order to design a proper dc-SQUID for practical applications, we need to study the current-voltage characteristics under different circuit parameters for design optimization.

There are two approaches to calculate the current-voltage characteristics for a given dc-SQUID. The first method is the analytical derivation; it finds the analytical expression of the function \( f(V_s, I_b, \Phi_e) \) directly from the circuit equations [4-7]. The numerical simulation [8-10] is the second method approach. It extracts the average voltage \( (V_s) \) by solving the real-time Josephson junction voltages from the nonlinear circuit equations. It shows that both methods depend on the circuit equations derived from the equivalent circuit.

Fig. 1. A brief schematic of the typical niobium dc-SQUID magnetometer with the washer tightly coupled with an input coil; the input coil connects to a pick-up coil to pick the external flux input \( \Phi_e \); \( J_1 \) and \( J_2 \) represent two Josephson junctions usually with the Nb/AlOx/Nb structure; \( R_1 \) and \( R_2 \) are two shunt resistors for the Josephson junctions.

However, the equivalent circuit is becoming more and more complicated with the development of the practical SQUID magnetometers or gradiometers [11], if we take mutually coupled inductors and the parasitic capacitors into consideration. For instance, the dc-SQUID in the practical magnetic field sensors is tightly coupled with the input coil, as shown in Fig. 1. Its equivalent circuit involves the inductively...
coupled superconducting loops and the normal components such as the shunt resistors and the parasitic capacitors, as shown in Fig. 2.

Different equivalent circuits of SQUID sensors result in different circuit equations. Therefore, it is a challenge to find the general analytical solution for different circuit equations. Similarly, the numerical simulation method finds the numerical solutions of circuit equations; it can be regarded as the virtual experiment of the SQUID equivalent circuit; the real-time numerical solutions are the discrete experimental data, which only exhibits the effect of circuit parameters on the current-voltage characteristics rather than the analytical explanation of working principles inside dc-SQUID.

In this article, we introduce a general frequency domain analysis method to derive the analytical current-voltage expression for any dc-SQUID circuits. This method is independent of the circuit equations and derives the current-voltage expression using only four network impedance parameters of SQUID. It depicts the working mechanism inside the dc-SQUID with a general frequency-phase-locking model. The application of this method is demonstrated in the analysis of a practical dc-SQUID magnetometer. The current-voltage characteristics calculated with the analytical expression are compared with the results by numerical simulation. It shows that the current-voltage characteristic of dc-SQUID circuits is the exhibition of the network impedances between two Josephson junctions.

![Fig. 2. The equivalent circuit of the dc-SQUID magnetometer in Fig. 1. It takes all the possible parasitic capacitors and inductances into considerations. In the Josephson junctions $J_1$ and $J_2$, the $i_{j1} = I_0 \sin \theta_1$ and $i_{j2} = I_0 \sin \theta_2$ are two pure superconducting Josephson currents; the $\theta_1$ and $\theta_2$ are Josephson phases; $I_{01}$ and $I_{02}$ are the critical currents; $R_1$ and $R_2$ are the shunt resistors; $C_1$ and $C_2$ are the parasitic capacitors; $I_{a1}$ and $I_{a2}$ are two current noises. Two Josephson junctions are biased with the dc-current source $I_j$; the bias current is applied to the terminals led from the SQUID washer $L_i$ are the inductance of the SQUID washer; $L_j$ is tightly coupled to the SQUID washer with the mutual inductance $M_{ij}$; $L_p$ is the pick-up coil inductance. The external flux $\Phi_s$ is applied to the pick-up coil by the current source $I_j$ through mutual inductance $M_{ip}$; $\Phi_s$ is the final applied flux by $I_j$ to the SQUID washer. Moreover, $C_1$ is the parasitic capacitor between two junctions; $R_s$ is the extra shunt resistor for damping the LC-resonance in the SQUID loop; $C_2$ is the parasitic capacitor of the input coil; $R_s$ is the shunt resistor for damping the LC-resonance inside the input coil.]

II. THEORY

A. Principles of Josephson Current

The Josephson current exhibits two working principles in the dc-SQUID with a steady current-voltage characteristic. The first principle states that the Josephson current works as a voltage-controlled oscillator (VCO) in the voltage state. According to the Josephson equations [12], if there is a steady time average voltage $V_i$ at two terminals of the Josephson junction, the real-time Josephson voltage $u(t)$, Josephson phase $\theta(t)$, and the pure superconducting Josephson current $i_j(t)$ can be rewritten as

$$u(t) = \frac{\Phi_0}{2\pi} \frac{d\theta(t)}{dt} = V_i + u_{ac}(t)$$

$$\theta(t) = \theta(t_0) + \frac{2\pi}{\Phi_0} \int_0^t u(t) dt = \omega_0 t + \delta$$

$$i_j(t) = I_0 \sin \theta(t) = I_0 \sin (\omega_0 t + \delta)$$

(1)

$$V_i = \frac{1}{T_0} \int_0^T u(t) dt; \quad T_0 = \frac{2\pi}{\omega_0}; \quad \omega_0 = \frac{2\pi V_i}{\Phi_0}$$

$$\delta = \theta(t) - \omega_0 t_0 + \frac{2\pi}{\Phi_0} \int_0^t u_{ac}(t) dt$$

Here, $\Phi_0 = \hbar/2e = 2.07 \times 10^{-15}$ Wb is flux quantum; $I_0$ is the critical current of the Josephson junction; $\omega_0$ is the so-called Josephson frequency defined with $V_i$; $T_0 = 2\pi/\omega_0$ is the period corresponding to $\omega_0$. The Josephson voltage $u(t)$ is expressed with $V_i$ and the alternating current (ac) component $u_{ac}(t)$; the average of $u_{ac}(t)$ is zero since the average of $u(t)$ is $V_i$. In the Josephson phase $\theta(t)$, $\delta$ is the phase fluctuation from the integration of $u_{ac}(t)$. The rewritten Josephson current expression shows that $i_j(t)$ is a quasi-periodical signal oscillating at the frequency $\omega_0$ with a fluctuating phase $\delta$.

The second principle is that the Josephson current does not consume the active power. The average active power flowing through the Josephson current, namely $P_s$, is derived as

$$P_s = \frac{1}{T} \int_0^T i_j(t) \cdot u(t) dt \bigg|_{(T=2\pi)}$$

$$= \frac{I_0 \Phi_0}{2\pi} \cos \theta(0) - \cos \theta(T) \approx 0$$

(2)

It shows that the average power dissipated by the Josephson current is zero during a $T$ which is much larger than $T_0$.

B. Sinusoidal Steady-State Analysis

According to the first principle of Josephson currents, if the dc-SQUID has a steady average voltage output, two Josephson currents are the quasi-periodical signals with $\omega_0$ as the base frequency. It indicates that we can apply the Fourier analysis method to decompose the Josephson currents into dc and ac components.

First, by using the Fourier integral transformation on $i_{j1}$ and $i_{j2}$, we define two complex variables, namely $\tilde{I}_{ac1}$ and $\tilde{I}_{ac2}$, as...
\[ t_{\text{dc}} = \frac{2}{T_0} \int_0^T i_{j1}(t) \cdot e^{-j\omega t} \, dt \]
\[ = 2 \cdot e^{j\varphi_{1k}} \]  
\[ t_{\text{ac}} = \frac{2}{T_0} \int_0^T i_{j2}(t) \cdot e^{-j\omega t} \, dt \]
\[ = 2 \cdot e^{j\varphi_{2k}} \]  
\[ i_{j1}(t) = i_{k1} + \sum_{k=1}^{\infty} t_{\text{dc}} \cos(k \omega t + \varphi_{1k}) \]
\[ = i_{k1} + \sum_{k=1}^{\infty} t_{\text{dc}} \cdot e^{j\omega t} \]  
\[ i_{j2}(t) = i_{k2} + \sum_{k=1}^{\infty} t_{\text{ac}} \cos(k \omega t + \varphi_{2k}) \]
\[ = i_{k2} + \sum_{k=1}^{\infty} t_{\text{ac}} \cdot e^{j\omega t} \]  

Here, \( k \) is an integer that denotes the order of the Fourier series \((k = 1, 2, \ldots)\). The complex components are the phasors for the electric circuit sinusoidal steady-state analysis method [13]; the \( t_{\text{dc}} \) and \( t_{\text{ac}} \) are the phasor amplitudes; the \( \varphi_{1k} \) and \( \varphi_{2k} \) are the phasor angles.

Based on the phasors \( t_{\text{dc}} \) and \( t_{\text{ac}} \), the Josephson currents \( i_{j1} \) and \( i_{j2} \) are expressed in the form of Fourier series as

\[ u_1(t) = V_s + \frac{2\pi}{\Phi_0} \int_{0}^{\pi} \frac{1}{\pi} \int_{0}^{\pi} i_{j1}(t) \, dt \]
\[ u_2(t) = V_s + \frac{2\pi}{\Phi_0} \int_{0}^{\pi} \frac{1}{\pi} \int_{0}^{\pi} i_{j2}(t) \, dt \]

For dc-SQUIDs, two Josephson voltages achieve the same average voltage \( V_s \).

Accordingly, the Josephson phases \( \theta_1(t) \) and \( \theta_2(t) \) derived from the \( u_1(t) \) and \( u_2(t) \) are rewritten as

\[ \theta_1(t) = \theta_1(t_0) + \frac{2\pi}{\Phi_0} \int_0^t u_1(t) \, dt \]
\[ \theta_2(t) = \theta_2(t_0) + \frac{2\pi}{\Phi_0} \int_0^t u_2(t) \, dt \]

The Fourier transform shifts the dc-SQUID circuit analysis from the time domain to the frequency domain. The frequency-domain method analyzes the dc and ac signals individually according to the harmonic order. Thus, driven by the ac Josephson currents \( t_{\text{dc}} \) and \( t_{\text{ac}} \), the ac voltage responses, namely \( V_{\text{dc}} \) and \( V_{\text{ac}} \), at two ports of the RLC network are derived as

\[ V_{\text{dc}} = \begin{bmatrix} V_{\text{dc}}^{k1} \\ V_{\text{dc}}^{k2} \end{bmatrix} = \begin{bmatrix} Z_{11}(\omega) & Z_{12}(\omega) \\ Z_{21}(\omega) & Z_{22}(\omega) \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \]

Here, the trans function of the two-port network is an impedance matrix \([Z]\) composed of four frequency-dependent impedances [13], namely \( Z_{11}(\omega) \), \( Z_{12}(\omega) \), \( Z_{21}(\omega) \) and \( Z_{22}(\omega) \).

The Josephson voltage \( u_1(t) \) and \( u_2(t) \) are expressed in the form of Fourier series as

\[ u_1(t) = V_s + \frac{2\pi}{\Phi_0} \int_{0}^{\pi} \frac{1}{\pi} \int_{0}^{\pi} i_{j1}(t) \, dt \]
\[ u_2(t) = V_s + \frac{2\pi}{\Phi_0} \int_{0}^{\pi} \frac{1}{\pi} \int_{0}^{\pi} i_{j2}(t) \, dt \]

Each Josephson phase is decomposed into three parts. The \( \theta_{k1} \) and \( \theta_{k2} \) are the time-invariant dc components; The \( \omega_0 t \) is the linear time-varying item. The \( \theta_{k1} \) and \( \theta_{k2} \) are the integrals of the \( V_{\text{dc}}^{k1} \) and \( V_{\text{dc}}^{k2} \); they are also decided by the \( t_{\text{dc}} \) and \( t_{\text{ac}} \), according to (5).

C. De Response Analysis

First, according to Kirchhoff’s current law, the \( i_{k1} \) and \( i_{k2} \) generate the average voltage \( V_s \) on the \( Z_{11}(\omega) \), namely the impedance \( Z_{11} \) with \( \omega = 0 \),

\[ V_s = \left( I_b - i_{dc1} - i_{dc2} \right) \cdot Z_{11}(0) \]

Second, according to the flux-quantization law [12], the total flux applied in the SQUID washer causes the difference between \( \theta_{k1} \) and \( \theta_{k2} \), i.e.,
\[ \Delta \theta_{df} = (\Phi_{d1} + \Phi_{d2} + \Phi_b + \Phi_r) \cdot \frac{2\pi}{\Phi_0} \]

\[ \Delta \theta_{dc} = \theta_{dc2} - \theta_{dc1} \]  \hspace{1cm} (10)

\[ \Phi_{d1} = M_{d1} \cdot i_{dc1}; \quad \Phi_{d2} = M_{d2} \cdot i_{dc2} \]

\[ \Phi_b = M_b \cdot I_b; \quad \Phi_r = M_r \cdot I_r \]

The total flux generated by four dc currents, i.e., \( i_{dc1}, i_{dc2}, I_b \) and \( I_r \), is coupled to the SQUID washer through inductances \( M_{d1}, M_{d2}, M_b \) and \( M_r \), respectively.

It shows that we can derive the final current-voltage expressions from (9) and (10) if we find the solutions of the \( i_{dc1} \) and \( i_{dc2} \).

D. Mixing Function of Josephson Current

Based on the orthogonality of the Fourier series [13], the total active power of Josephson currents, namely \( P_{J1}, P_{J2} \), is expressed as

\[ P_{J1} = V_s \cdot i_{dc1} + \frac{1}{2} \text{Re} \left( \sum_{k=1}^{\infty} I_{ac1}^k \cdot (I_{ac1})^* \right) \]

\[ P_{J2} = V_s \cdot i_{dc2} + \frac{1}{2} \text{Re} \left( \sum_{k=1}^{\infty} I_{ac2}^k \cdot (I_{ac2})^* \right). \]  \hspace{1cm} (11)

Here, * indicates the conjugated format of the variable.

According to the second principle that the total active power of the Josephson current is zero, we can derive the \( i_{dc1} \) and \( i_{dc2} \) with the current and voltage phasors as

\[ i_{dc1} = -\frac{1}{2V_s} \text{Re} \left( \sum_{k=1}^{\infty} I_{ac1}^k \cdot (I_{ac1})^* \right) \]  \hspace{1cm} (12)

\[ i_{dc2} = -\frac{1}{2V_s} \text{Re} \left( \sum_{k=1}^{\infty} I_{ac2}^k \cdot (I_{ac2})^* \right) \]

It shows that the \( i_{dc1} \) and \( i_{dc2} \) are the output by mixing the ac voltages with the ac currents. The Josephson current functions as the mixer inside the dc-SQUID.

E. Ac Component Extraction

Referring to the composition of \( \theta_1(t) \) and \( \theta_2(t) \) defined in (7), we rewrite the expression of the Josephson currents as

\[ i_{J1}(t) = I_{01} \sin(\theta_{ac1} + \omega_{ac1} t + \theta_{ac1}) \]

\[ = I_{01} \sin(\theta_{dc1} + \omega_{dc1} t) \cos \theta_{ac1} + \cos(\theta_{ac1}) \]

\[ + I_{01} \cos(\theta_{dc1} + \omega_{dc1} t) \sin \theta_{ac1} \]

\[ = \text{Re}(I_{01} e^{j(\omega_{dc1} - 0.5\omega_{ac1})} e^{j\theta_{ac1}}) + \alpha(\theta_{ac1}) \]  \hspace{1cm} (13)

\[ i_{J2}(t) = I_{02} \sin(\theta_{ac2} + \omega_{dc1} t + \theta_{ac2}) \]

\[ = I_{02} \sin(\theta_{dc2} + \omega_{dc2} t + \theta_{ac2}) \cos(\theta_{dc2} + \omega_{dc2} t + \theta_{ac2}) \]

\[ + I_{02} \cos(\theta_{dc2} + \omega_{dc2} t + \theta_{ac2}) \sin \theta_{ac2} \]

\[ = \text{Re}(I_{02} e^{j(\omega_{dc2} - 0.5\omega_{ac2})} e^{j\theta_{ac2}}) + \alpha(\theta_{ac2}) \]

Where \( I_{01} \) and \( I_{02} \) are the critical currents of two Josephson junctions; The Josephson current is decomposed into a fundamental ac component and a minor component. Functions of two minor components, namely \( \alpha(\theta_{ac1}) \) and \( \alpha(\theta_{ac2}) \), are

\[ \alpha(\theta_{ac1}) = I_{01} \sin(\theta_{dc1} + \omega_{dc1} t) \cdot (\cos \theta_{ac1} - 1) \]

\[ + I_{01} \cos(\theta_{dc1} + \omega_{dc1} t) \cdot \sin \theta_{ac1} \]  \hspace{1cm} (14)

\[ \alpha(\theta_{ac2}) = I_{02} \sin(\theta_{dc2} + \omega_{dc2} t) \cdot (\cos \theta_{ac2} - 1) \]

\[ + I_{02} \cos(\theta_{dc2} + \omega_{dc2} t) \cdot \sin \theta_{ac2} \]

They are decreasing with the increase of \( V_s \), according to (8).

The minor component will be small to ignore with the increase of \( V_s \); the \( P_{J1} \) and \( P_{J2} \) can be approximated as

\[ I_{01} e^{j(\theta_{dc1} - 0.5\omega_{ac1})} \]

\[ I_{02} e^{j(\theta_{dc2} - 0.5\omega_{ac2})} \]

With those specific current phasors, we can find the \( i_{dc1} \) and \( i_{dc2} \) with (12) as

\[ i_{dc1} = \frac{I_{01}}{2V_s} \text{Re} \left( Z_{11}(\omega_{ac1}) + \frac{I_{02}}{I_{01}} Z_{12}(\omega_{ac1}) e^{j\theta_{ac1}} \right) \]

\[ i_{dc2} = \frac{I_{02}}{2V_s} \text{Re} \left( Z_{22}(\omega_{ac2}) + \frac{I_{01}}{I_{02}} Z_{21}(\omega_{ac2}) e^{j\theta_{ac2}} \right). \]  \hspace{1cm} (16)

Finally, based on this solution, we can derive the expression in the form of \( f(V_s, I_b, \Phi_0) = 0 \) from (9) and (10).

F. Frequency-Phase-Locking Model

The whole derivation of the current-voltage expression is depicted with a frequency-phase-locking model, as shown in Fig. 4. In this model, two Josephson currents implement two

\[ V_s = (I_b - i_{dc1} - i_{dc2}) \cdot Z_{11} \]

\[ I_c = \Delta \theta_{dc} = (\Phi_{d1} + \Phi_{d2} + \Phi_b + \Phi_r) \cdot \frac{2\pi}{\Phi_0} \]

\[ \Delta \theta_{dc} = (\Phi_{d1} + \Phi_{d2} + \Phi_b + \Phi_r) \cdot \frac{2\pi}{\Phi_0} \]

The general frequency-phase-locking model of dc-SQUID, in which two Josephson junctions function as two VCOs and mixers; the linear RLC network implements the interference between two junctions with both the dc and ac network impedances.

It is a feedback system; two VCOs and mixers drive the RLC network and generate the dc currents flowing through the junctions; the dc currents alter the average voltage output \( V_s \) and the phase-difference \( \Delta \theta_{dc} \) to adjust the frequency and phases of two VCOs.
lock-in amplifiers are cross-linked through the RLC network; $V_s$ and $\Delta \theta_{dc}$ set their frequency and phase-difference; the $i_{dc1}$ and $i_{dc2}$ are the output correspondingly.

In the self-mixing mode, the first lock-in amplifier driven by the $I_{dc1}$ reads out the real part of the impedance $Z_{11}$; the second lock-in amplifier driven by the $I'_{dc2}$ reads out the real part of the impedance $Z_{22}$. Meanwhile, the $Z_{12}$ and $Z_{21}$ realize the interaction between two lock-in amplifiers. The inter-mixing between two VCOs extracts the real part of $Z_{12}$ to $i_{dc1}$ and the real part of $Z_{21}$ to $i_{dc2}$. Thus, $i_{dc1}$ and $i_{dc2}$ include the information of four impedances and the phase-difference between two VCOs, as expressed in (16).

The whole system works as a frequency-phase-locking loop when the output of the lock-in amplifiers adjusts the frequency and phase of two VCOs with $V_s$ and $\Delta \theta_{dc}$. From this model, we find that the overall current-voltage characteristics of the dc-SQUID are the projections of the impedances of the two-port network driven by the Josephson currents; the cross-mixing principle between two lock-in amplifiers exhibits the interference mechanism inside the dc-SQUID. The flux input modulates the dc output through the cross-mixing, in which $Z_{12}$ and $Z_{21}$ determine the modulation depth.

III. APPLICATION EXAMPLE

We choose the dc-SQUID circuit shown in Fig. 2 as the example to demonstrate the application of the frequency-phase-locking model. First, we redraw the RLC network with the circuit impedances in complex format, as shown in Fig. 5.

\[
Z_{11}(\omega) = Z_{22}(\omega) = \frac{Z_0(Z_0 + Z_1 \cdot Z_0/Z_2 + Z_2)}{2Z_0 + Z_1 / Z_2 + Z_2 / Z_0}.
\]

Meanwhile, the equivalent circuit for the dc analysis is shown in Fig. 6, in which the bias current $I_b$ is assigned equally into two junctions.

From Fig. 6, we extract the mutual inductances as
\[
\begin{align*}
    l_i &= L_s - M_{il} = \frac{l_s}{2} \\
    M_{il1} &= -M_{il2} = \frac{l_s}{2} \\
    M_{ib} &= 0; M_{ib} = M_{il} = \frac{l_s}{2}
\end{align*}
\]

Here, $l_s$ is the equivalent inductance of the SQUID, including the effects of the pick-up and input coils to the washer.

With the specific parameters in (17) and (18), we obtain the final current-voltage expression as
\[
I_b = \frac{2V_s + l_b^2}{R_0} \text{Re} \left( Z_{11}(\omega) + Z_{12}(\omega) \cdot \cos \Delta \theta_{dc} \right)
\]

\[
\Phi = \frac{\Phi_0}{2\pi} \cdot \Delta \theta_{dc} + \frac{l_b^2}{2V_s} \text{Im} \left( Z_{12}(\omega) \cdot \sin \Delta \theta_{dc} \right).
\]

Here, Im( ) is the operation of getting the imaginary part of the complex.

The first analytical expression defines the $I_b$ as the function of $V_s$. It exhibits the current-voltage characteristic measured with the so-called voltage-bias readout scheme [14], which reads out the bias current $I_b$ by locking the SQUID voltage output $V_s$ at the given value with the feedback circuit.

In the first expression, the bias current $I_b$ includes both the mixing current through $Z_{11}$ and the inter-mixing current modified by the $\Delta \theta_{dc}$.

In the second analytical expression, the $\Delta \theta_{dc}$ corresponds to $\Phi$, point-by-point. Roughly, $\Delta \theta_{dc} \Phi_0/2\pi \approx \Phi$, with a deviation $\Phi_{err} = \Phi - \Delta \theta_{dc} \Phi_0/2\pi$ is proportional to $\sin \Delta \theta_{dc}$.

Thus, when $\Delta \theta_{dc} = \pi n (n = 0, 1, \ldots)$, $\Phi_{err} = 0$ and $\Phi = n \Phi_0/2$. Accordingly, $I_b$ reaches the maximum value $I_{b1}$ when $\Phi = 0$; $I_b$ reaches the minimum value $I_{b2}$ when $\Phi = 0.5 \Phi_0$. Meanwhile, when $\Delta \theta_{dc} = 0.5 \pi$, $I_b = (I_{b1} + I_{b2})/2$ and $\Phi = 0$ with the maximum positive $\Phi_{err}$. When $\Delta \theta_{dc} = 1.5 \pi$, $I_b = (I_{b1} + I_{b2})/2$ and...
the maximum negative $\Phi_{err}$. Those four values of $I_b$ and $\Phi_i$ correspond to four points in the current-voltage characteristics expressed as

$$I_{b1} = I_{s}(V_s)\rceil_{\Phi_i = 0}; \Phi_i = 0$$
$$I_{b2} = I_{s}(V_s)\rceil_{\Phi_i = 0.5\Phi_0}$$
$$I_{b3} = (I_{b1} + I_{b2})/2; \Phi_i = \Phi_i(V_s)\rceil_{\Phi_i = 0.5\Phi_0}$$
$$I_{b4} = (I_{b1} + I_{b2})/2; \Phi_i = \Phi_i(V_s)\rceil_{\Phi_i = 1.5\Phi_0}$$

According to (19), we can characterize the $Z_{11}$ and $Z_{12}$ from four points as

$$\text{Re}(Z_{11}(\omega)) = \frac{V}{I_0} \left( \frac{I_{b1} + I_{b2}}{2} - \frac{2V}{R_0} \right)$$
$$\text{Re}(Z_{12}(\omega)) = \frac{V}{I_0} \left( \frac{I_{b1} - I_{b2}}{2} \right)$$
$$\text{Im}(Z_{12}(\omega)) = \frac{V}{I_0} \left( \frac{\Phi_{i1} - \Phi_{i2}}{L_p} - 0.5\Phi_0 \right)$$

Therefore, from the measured current-voltage characteristics of a practical dc-SQUID, we can extract four curves according to four points defined in (20) and characterize the amplitude-frequency characteristics of the impedance $Z_{12}(\omega)$ and $Z_{11}(\omega)$ according to (21). In this way, the dc-SQUID can be utilized as the impedance analyzer to characterize the RLC network inside dc SQUIDs.

IV. RESULTS AND DISCUSSION

A. Results

The current-voltage characteristics of the example were calculated using both the analytical expression and the numerical simulation to make the comparisons. The numerical simulation is implemented with the circuit equations derived from the equivalent circuit [15].

The equivalent circuit shown in Fig. 2 depicts both the topologies and the parasitic elements of the practical dc-SQUID. Based on this equivalent circuit, the numerical simulation can simulate all the effects of the circuit elements on the overall current-voltage characteristics.

In order to verify the analytical expression in (19), we choose five test cases to compare the current-voltage characteristics provided by the analytical calculation and the numerical simulation; each test case is configured with different circuit parameters.

| TABLE I. ELEMENT PARAMETERS OF THE TEST CIRCUIT |
|-----------------------------|-----------------|---------------------|
| Parameter | Symbol | Value(Unit) |
| Inductance of washer | $\beta_L$ | 2.0 |
| Shunt resistor $R_3$ | $\gamma_3$ | 1000 |
| Capacitor $C_3$ | $\beta_C$ | 0.01 |
| Inductance $L_1$ | $\beta_L$ | 100 |
| Inductance $L_2$ | $\beta_L$ | 500 |

The typical circuit parameters of the test cases are shown in Table I; the other parameters are noted in the results. For simplicity, all the circuit parameters are normalized. For example, $\beta_L$ is the normalized parameter of the inductance $L$, $\beta_C = 2\pi I_0 R_0^2 C / \Phi_0$; $\beta_C$ is the normalized parameter of the capacitance $C$. $\beta_L = 2\pi I_0 R_0 L / \Phi_0$; $\gamma_3$ is the normalized parameter of the resistance $R_3$.

According to (19), we can characterize the $Z_{11}$ and $Z_{12}$ from four points as

$$\text{Re}(Z_{11}(\omega)) = \frac{V}{I_0} \left( \frac{I_{b1} + I_{b2}}{2} - \frac{2V}{R_0} \right)$$
$$\text{Re}(Z_{12}(\omega)) = \frac{V}{I_0} \left( \frac{I_{b1} - I_{b2}}{2} \right)$$
$$\text{Im}(Z_{12}(\omega)) = \frac{V}{I_0} \left( \frac{\Phi_{i1} - \Phi_{i2}}{L_p} - 0.5\Phi_0 \right)$$

Therefore, from the measured current-voltage characteristics of a practical dc-SQUID, we can extract four curves according to four points defined in (20) and characterize the amplitude-frequency characteristics of the impedance $Z_{12}(\omega)$ and $Z_{11}(\omega)$ according to (21). In this way, the dc-SQUID can be utilized as the impedance analyzer to characterize the RLC network inside dc SQUIDs.

IV. RESULTS AND DISCUSSION

A. Results

The current-voltage characteristics of the example were calculated using both the analytical expression and the numerical simulation to make the comparisons. The numerical simulation is implemented with the circuit equations derived from the equivalent circuit [15].

The equivalent circuit shown in Fig. 2 depicts both the topologies and the parasitic elements of the practical dc-SQUID. Based on this equivalent circuit, the numerical simulation can simulate all the effects of the circuit elements on the overall current-voltage characteristics.

In order to verify the analytical expression in (19), we choose five test cases to compare the current-voltage characteristics provided by the analytical calculation and the numerical simulation; each test case is configured with different circuit parameters.

TABLE I. ELEMENT PARAMETERS OF THE TEST CIRCUIT

| Parameter | Symbol | Value(Unit) |
|-----------------------------|-----------------|---------------------|
| Inductance of washer | $\beta_L$ | 2.0 |
| Shunt resistor $R_3$ | $\gamma_3$ | 1000 |
| Capacitor $C_3$ | $\beta_C$ | 0.01 |
| Inductance $L_1$ | $\beta_L$ | 100 |
| Inductance $L_2$ | $\beta_L$ | 500 |

The typical circuit parameters of the test cases are shown in Table I; the other parameters are noted in the results. For simplicity, all the circuit parameters are normalized. For example, $\beta_L$ is the normalized parameter of the inductance $L$, $\beta_C = 2\pi I_0 R_0^2 C / \Phi_0$; $\beta_C$ is the normalized parameter of the capacitance $C$. $\beta_L = 2\pi I_0 R_0 L / \Phi_0$; $\gamma_3$ is the normalized parameter of the resistance $R_3$.
those results, Fig. 8-10 also simulate the characteristics of the three SQUIDs reported in the reference [7].

![Graph](image1)

**Fig. 9.** The current-voltage characteristics of Case 3: (a) The curves by the analytical calculation, (b) the curves by the numerical simulation.

![Graph](image2)

**Fig. 10.** The current-voltage characteristics of Case 4: (a) The curves by the analytical calculation, (b) the curves by the numerical simulation.

![Graph](image3)

**Fig. 11.** The current-voltage characteristics of Case 5: (a) The curves by the analytical calculation, (b) the curves by the numerical simulation.

B. Discussion

From the current-voltage characteristics calculated in five test cases, we find that the current-voltage curves calculated with the analytical expression agree well with the results by the numerical simulation, except the results at the region with the small voltage. The reason is that the Josephson current is non-sinusoidal when $V_s$ is small, and its harmonics are not negligible; thus, the approximation in (15) cannot be well satisfied, and the analytical expression will lose accuracy.

On the other hand, the analytical expressions in (19) is valid when the current-voltage curves by analytical calculation agree with results by the numerical simulation. This analytical expression exhibits the frequency-phase-locking mechanism inside dc-SQUID, and develops a network impedance characterization method for the practical dc-SQUID from the measured current-voltage characteristics.

The frequency-domain analysis method depicts the working principles inside the dc-SQUID when two Josephson junctions are oscillating at the Josephson frequency. The corresponding current-voltage expression depends on only the two-port $RLC$ network impedances instead of the nonlinear differential equations; it is reliable since it is based on two solid principles directly derived from the Josephson equations.

V. Conclusion

This article presents a new frequency-domain analysis method for dc-SQUIDs; it derives the analytical current-voltage expression using only the two-port network impedances rather than the nonlinear differential equations. This method clarifies the working principle inside the dc-SQUID with a frequency-phase-locking model, in which two Josephson currents achieve both the VCO and mixer
functions and implement two crossed-linked lock-in amplifiers with the network impedances. The final analytical expressions show that the current-voltage curves are the projection of the network impedances inside the dc-SQUID. One can use the analytical expressions to either predict the current-voltage characteristics with the given network parameters or characterize the network impedances from the measured characteristics for the practical dc-SQUID design.

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