Disorder Effects in Two-Dimensional d-wave Superconductors

A. A. Nersesyan
Institute of Theoretical Physics, Chalmers University of Technology, S-41296 Göteborg, Sweden
and Institute of Physics, Georgian Academy of Sciences, Tamarashvili 6, 380077, Tbilisi, Georgia

A. M. Tsvelik
Department of Physics, University of Oxford, 1 Keble Road, Oxford, OX1 3NP, UK

F. Wenger
Institute of Theoretical Physics, Chalmers University of Technology, S-41296 Göteborg, Sweden
(September 26, 2018)

Abstract

Influence of weak nonmagnetic impurities on the single-particle density of states \( \rho(\omega) \) of two-dimensional electron systems with a conical spectrum is studied. We use a nonperturbative approach, based on replica trick with subsequent mapping of the effective action onto a one-dimensional model of interacting fermions, the latter being treated by Abelian and non-Abelian bosonization methods. It is shown that, in a d-wave superconductor, the density of states, averaged over randomness, follows a nontrivial power-law behavior near the Fermi energy: \( \rho(\omega) \sim |\omega|^\alpha \). The exponent \( \alpha > 0 \) is calculated for several types of disorder. We demonstrate that the property \( \rho(0) = 0 \) is a direct consequence of a continuous symmetry of the effective fermionic model, whose breakdown is forbidden in two dimensions. As a counter example, we consider another model with a conical spectrum - a two-dimensional orbital antiferromagnet, where static disorder leads to a finite \( \rho(0) \) due to breakdown of a discrete (particle-hole) symmetry.

74.20.Fg, 11.10.Gh, 71.10.+x
I. INTRODUCTION

The problem of weak static disorder in electronic systems with extended Fermi surface has been extensively studied.\(^1\) It is well understood that randomness has a profound influence on the transport properties and weakly affects the density of states. The latter is no longer true, if the density of states (DOS) of a pure system vanishes linearly at zero energy: \(\rho(\omega) \sim |\omega|\). In this case the standard procedure of averaging over impurities is complicated by the appearance of logarithmic singularities in the perturbative expansion of the single-electron self energy:\(^2\)–\(^5\)

\[
\Sigma(\omega_n) = i\omega_n (-g \ln(\Lambda/|\omega_n|) + \ldots).
\]

Such a situation can be realized in two-dimensional systems with a Dirac-like, conical spectrum \(E^2(p) = v_x^2p_x^2 + v_y^2p_y^2\), describing low-energy states near a degeneracy point (node), as well as in three-dimensional polar superconductors, where the order parameter has a nodal line. In both cases the pure systems exhibit a \(T^2\) low temperature specific heat. Since the exponents of thermodynamic quantities are used as an experimental criterium for selection of possible order parameters, it is important to study their stability with respect to disorder. For 3D polar superconductors, the problem is simplified by the fact that, due to the finite size of the Fermi surface, the diagrams with crossing impurity lines are not divergent, and one can easily sum the remaining diagrammatic series. This was done by Gor’kov and Kalugin,\(^2\) who showed that weak nonmagnetic disorder gives rise to a special energy scale \(Q_0 \sim \Lambda \exp[-\text{const}/c]\), \(c\) being the impurity concentration, below which the DOS becomes finite \(\rho(0) \sim Q_0\).

The renewed interest in quenched disorder in 2D systems\(^2,3\) has been stimulated by a number of experimental indications that the pairing state in the copper-oxide superconductors may be of \(d_{x^2-y^2}\) symmetry. In a recent publication, Lee\(^6\) applied the results of Ref.\(^2\) to analyze the role of nonmagnetic impurities in a 2D d-wave superconductor. He argued that, since a system with a nonzero DOS formally resembles a metal with a finite Fermi surface, the standard scaling arguments\(^1\) would indicate localization of all low-energy states.

In this paper representing an extended version of our short report,\(^7\) we show that, when averaging over impurities in 2D systems with a finite number of isolated nodes, the logarithmic singularities appear in all self-energy diagrams, including the crossing ones. No specific subset of diagrams can be selected from the whole self-energy series expansion. What one finds here is a typical ”logarithmic” situation when logarithmic singularities appear in all self-energy diagrams, including the crossing ones. Approximations based on partial summation of the (”rainbow”) diagrams\(^2,4\) do not apply. We overcome this difficulty by means of the field-theoretical approach based on replica trick. The static nature of the disorder allows to represent the effective, disorder-free action in terms of a one-dimensional model of interacting fermions. Using then bosonization technique, we demonstrate that, instead of creating a finite DOS at \(\omega = 0\), in the 2D d-wave superconductor the disorder changes the exponent of the DOS: \(\rho(\omega) \sim |\omega|^\alpha\), \((0 < \alpha < 1)\). The magnitude of \(\alpha\) depends on the type of disorder. For a slowly varying (quasiclassical) random potential, when only forward scattering is present, \(1 - \alpha \sim c\). The internode backscattering processes strongly reduce \(\alpha\) at energies \(\sim Q_0\), making it concentration independent: \(\alpha = 1/7\). The picture indeed becomes very close to the one with a finite DOS, but still not the same. In this situation, the
renormalization group for the conductivity can lead to an intermediate fixed point, which suggests a finite conductivity.

Superconductors are not the only systems where a conical spectrum and, as a consequence, linear DOS can appear. Such a spectrum occurs in zero-gap (degenerate) semiconductors, and in heterojunctions where the contact is made between semiconductors with inverted symmetry of bands. It can be realized in 2D graphite sheets, as well as for lattice electrons in a strong magnetic field, the most well known example being a tight-binding model of fermions with 1/2 of a magnetic flux quantum per plaquette (flux phase). The conical spectrum is also a property of hypothetical orbital antiferromagnet and spin nematic states.

Despite the common feature of all these systems, the existence of the degeneracy points (nodes) in their spectrum, the role of disorder can be quite different in each particular case, depending on the symmetry of the pure system. The result is determined by the structure of random terms which appear in the effective massless Dirac Hamiltonian, describing low-energy states of the pure system. In the 2D d-wave superconductor, nonmagnetic impurities add to the Dirac model random gauge fields, abelian or non-abelian, depending on the type of included scattering processes. The random gauge fields do not break particle-hole symmetry in the vicinity of the nodes and lead to a critical, power-law behavior of \( \rho(\omega) \) at \( \omega \to 0 \). As we demonstrate below, this is a direct consequence of the fact that at \( \omega \to 0 \) the equivalent 1D fermionic model possesses continuous replica and chiral \( \gamma^5 \) symmetries, remaining unbroken in two dimensions.

On the other hand, in 2D systems with a particle-hole condensate, such as the flux phase or orbital antiferromagnet, the situation is different. The nonmagnetic disorder gives rise to a random chemical potential and random charge-density wave, the latter being equivalent to a random relativistic mass of the Dirac quasiparticles. We argue that the random Dirac mass alone would also lead to a critical behavior with \( \rho(0) = 0 \). However, the random chemical potential, which is always present even for a weak quasiclassical disorder, breaks the particle-hole symmetry and drives the system away from criticality. The special role of the random chemical potential is that it lowers the continuous chiral symmetry of the corresponding 1D Fermi model down to a discrete one. The spontaneous breakdown of the latter is signalled by a finite \( \rho(0) \), as previously pointed out by Fisher and Fradkin. It is worth noticing that, for a d-wave superconductor, the appearance of a random chemical potential in the effective Dirac model could be only caused by random magnetic fields.

The paper is organized as follows. In Section II we formulate a model of two-dimensional d-wave superconductor with non-magnetic impurities. In Section III we give a perturbative analysis of the disorder in terms of the diagram expansion for the single electron Green’s function. The expansion demonstrates that the number of singular diagrams in the given \( n \)-th order grows with \( n \), and the problem is non-perturbative. In Section IV we average over the randomness using the replica trick and reformulate the replicated model as a two-dimensional Euclidean relativistic field theory. In Section V we consider a simplified version of this theory, namely, we consider only scattering within each node. This simplified theory is solved by the ordinary bosonization. In Section VI we consider the scattering between nodes; the corresponding model is solved by the non-Abelian bosonization. In Section VII we describe another model with a conical spectrum - a 2D orbital antiferromagnet. We claim that this model has a finite \( \rho(0) \). The paper has an Appendix where we discuss some
properties of the Bethe-ansatz solution of the replicated model of impure d-wave superconductors.

II. A MODEL OF IMPURE D-WAVE SUPERCONDUCTOR

We start from the following model describing low energy electronic degrees of freedom in a 2D d-wave superconductor:

\[ H = \sum_{\mathbf{k},\sigma} \epsilon(\mathbf{k}) c_{\mathbf{k}\sigma}^+ c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} [\Delta(\mathbf{k}) c_{\mathbf{k}1}^+ c_{\mathbf{k}1} + H.c.]. \]  

(1)

Here \( \epsilon(\mathbf{k}) \) is a tight-binding spectrum in the normal state which respects full point symmetry of the underlying (square for the copper oxides) lattice, and \( \Delta(\mathbf{k}) \) is a pairing amplitude, odd under \( \pi/2 \)-rotations in the \( \mathbf{k} \)-space (the \( d_{x^2-y^2} \) symmetry). Without loss of generality we can choose \( \epsilon(\mathbf{k}) = -2t(cos k_x + cos k_y) - \mu \) and \( \Delta(\mathbf{k}) = \Delta_0(cos k_x - cos k_y) \). It will be assumed that \( \Delta_0 \ll t \). The quasiparticle spectrum \( E(\mathbf{k}) = \pm \sqrt{\epsilon^2(\mathbf{k}) + \Delta^2(\mathbf{k})} \) has four nodes at the Fermi level which we denote \( 1, \bar{1}, 2, \bar{2} \) (see Fig. [1]). Their positions are \( k_1 = -\bar{k}_1 = (k_0, k_0) \); \( k_2 = -\bar{k}_2 = (-k_0, k_0) \), where \( k_0 = \arccos(\mu/4t) \). Associated with these nodes are gapless excitations with formally relativistic (conical) spectrum at \( |\omega| \ll \Delta_0 \). The single-electron density of states at these energies goes linearly with \( \omega \): \( \rho(\omega) \sim |\omega| \), giving rise to extra \( T \)-factors in thermodynamic quantities of pure d-wave superconductors.

Let us introduce the Nambu spinor

\[ \Phi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}1}^+ \\ c_{-\mathbf{k}1}^- \end{pmatrix}. \]

The Hamiltonian (1) in this notations is given by

\[ H = \sum_{\mathbf{k}} \Phi_{\mathbf{k}}^+[\epsilon_{\mathbf{k}} - \mu] \tau_3 + \Delta(\mathbf{k}) \tau_2] \Phi_{\mathbf{k}}, \]

(2)

where \( \tau_i \) are the Pauli matrices. We linearize the spectrum close to the nodes and pass to a continuum description in terms of four Fermi fields \( \psi_j(x), j = 1, \bar{1}, 2, \bar{2} \). We choose a new coordinate system rotated by angle \( \pi/4 \) with respect to the original one. It is also convenient to make a \( \pi/2 \) rotation in the Nambu space about the \( \tau_2 \)-axis. The resulting continuum model is then given by the sum of four Dirac-like Hamiltonians:

\[ H = \sum_{j=1,\bar{1},2,\bar{2}} \int d^2x \psi_j^+(x) \tilde{H}_j(x) \psi_j(x), \]

(3)

\[ \tilde{H}_1(x) = -\tilde{H}_{\bar{1}}(x) = -iv_1 \partial_1 \tau_1 - iv_2 \partial_2 \tau_2, \]

(4)

\[ \tilde{H}_2(x) = -\tilde{H}_{\bar{2}}(x) = -iv_1 \partial_2 \tau_1 - iv_2 \partial_1 \tau_2, \]

(5)

with \( v_1 = 2\sqrt{2}ta \sin k_0, v_2 = \sqrt{2}\Delta_0 \sin k_0 \).

Processes of scattering on nonmagnetic impurities are described by

\[ H_{\text{imp}} = \sum_{jj'} \int d^2x V_{jj'}(x) \psi_j^+(x) \tau_1 \psi_{j'}(x), \]

(6)
where $V_{jj'}(x) = V_{j'}^*(x)$ are random fields with Gaussian distributions. Since $V_{jj'}(x)$ are wave packets with average wave vectors $k_j - k_{j'}$, not all of them are independent. For the square geometry presented on Fig. 1 we have the following independent components:

(i) the real field $V_{jj}(x) \equiv V_0(x)$ representing a slowly varying (quasiclassical) component of the impurity potential;

(ii) the complex fields $V_{11}(x) \equiv V_1(x)$ and $V_{22}(x) \equiv V_2(x)$ representing a back scattering with the momentum transfer $2k_0$;

(iii) the complex fields $V_{12}(x) = V_{21}(x) \equiv W_1(x)$ and $V_{12}(x) = V_{21}(x) \equiv W_2(x)$ corresponding to transitions between nearest nodes.

In what follows we assume that all these fields are distributed according to the Gauss’ law:

$$<V_0(x)V_0(x')> = \lambda_0 \delta^{(2)}(x - x'),$$
$$<V_j(x)V_j^*(x')> = \lambda_1 \delta^{(2)}(x - x'), \quad (j = 1, 2),$$
$$<W_j(x)W_j^*(x')> = \lambda_2 \delta^{(2)}(x - x'), \quad (j = 1, 2).$$

(7)

III. SELF ENERGY DIAGRAMS

For a given realization of static random potential, the one-particle Green’s function tensor written in the mixed $x - \omega_n$ representation ($\omega_n = (2n + 1)\pi/\beta$ being the Matsubara frequency) satisfies the following integral equation

$$\hat{G}_{jj'}(x, x'; \omega_n) = \delta_{jj'} \hat{G}_0^j(x - x'; \omega_n)$$
$$+ \sum_{j''} \int d^2 x'' \hat{G}_j^0(x - x''; \omega_n)V_{jj''}(x'') \hat{T}_1 \hat{G}_{j''j'}(x'', x'; \omega_n),$$

(8)

where

$$\hat{G}_j^0(x - x'; \omega_n) = [i\omega_n - \hat{H}_j(x)]^{-1}\delta(x - x')$$

(9)

is the Green’s function for the pure system.

For weak electron-impurity scattering, the standard procedure (see, e.g. §3) consists in developing a perturbation theory expansion in Eq.(8), with subsequent averaging of products of the Gaussian random fields $V_{jj'}(x)$, appearing in each term of this series. The averaged Green’s function

$$<\hat{G}_{jj'}(x, x'; \omega_n)> = \delta_{jj'} \hat{G}_j(x - x'; \omega_n),$$

(10)

written in momentum representation, satisfies the Dyson equation

$$\hat{G}_j^{-1}(k, \omega_n) = i\omega_n - \hat{H}_j(k) - \hat{\Sigma}_j(k, \omega_n).$$

(11)

Here

$$\hat{H}_1(k) = -\hat{H}_1(k) = k_1v_1\hat{T}_1 + k_2v_2\hat{T}_2,$$
\( \hat{H}_2(k) = -\hat{H}_2(k) = k_2 v_1 \hat{\tau}_1 + k_1 v_2 \hat{\tau}_2. \) \hfill (13)

\( \hat{\Sigma}_j(k, \omega_n) \) is the self-energy operator incorporating all effects of the electron-impurity scattering. In what follows, we shall analyze lowest-order diagrams for \( \hat{\Sigma}_1 \). In calculations, a cutoff prescription, defined by conditions \( |H_j(k)| \leq \Delta_0 \), will be used.

In the lowest (Born) approximation \( \hat{\Sigma}_1 \) is given by the first diagram shown in Fig. 2. It is momentum independent and contains a logarithmic singularity, as previously pointed out in a number of papers:

\[
\hat{\Sigma}_1^{(1)}(k, \omega_n) = \hat{\Sigma}_1^{(1)}(\omega_n) = \int \frac{d^2 k'}{(2\pi)^2} \hat{\tau}_1 [\lambda_0 \hat{G}_j^0(k', \omega_n) + \lambda_1 \hat{G}_j^0(k, \omega_n)] \\
+ \lambda_2 (\hat{G}_j^0(k', \omega_n) + \hat{G}_j^0(k, \omega_n))] \hat{\tau}_1 \\
= -i \omega_n (g_3 + g_1 + 2g_2) \int_{|p|<\Delta_0} \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + \omega_n^2} \\
= -i \omega_n \frac{g_3 + g_1 + 2g_2}{2\pi} \ln \frac{\Delta_0}{|\omega_n|}. \quad (14)
\]

Here

\[
g_{1,2} = \frac{\lambda_{1,2}}{v_1 v_2}, \quad g_3 = \frac{\lambda_0}{v_1 v_2} \quad (15)
\]

are small dimensionless coupling constants, proportional to the impurity concentration \( c \).

In fact, logarithmic singularities appear in all orders of perturbation theory. Consider second-order diagrams shown in Fig. 2. The "rainbow" diagram is obtained from the Born one by first-order renormalization of the internal electron line. The node indices \( j_1 \) and \( j_2 \) take arbitrary values from the set \((1,1,2,2)\). The result is easily found to be

\[
\hat{\Sigma}^{(2a)}(\omega_n) = -i \omega_n \frac{(g_3 + g_1 + 2g_2)^2}{(2\pi)^2} [\ln^2 \frac{\Delta_0}{|\omega_n|} + O(1)]. \quad (16)
\]

The diagram with crossing impurity lines, shown in Fig. 2, corresponds to a vertex renormalization in the Born self-energy diagram. The momentum conservation restricts the allowed distribution of the node indices \((j_1, j_2, j_3)\) by the following nine possibilities: \((111), (111), (11\bar{1}), (221), (122), (221), (32)\). Using power counting and symmetry arguments, one can show that, in each diagram, the maximum power of \( \log |\omega_n| \) appears with the prefactor \( \sim i \omega_n \). This allows to simplify calculations by setting the external momentum \( k = 0 \), in which case

\[
\hat{\Sigma}^{(2b)}(k = 0, \omega_n) \big|_{(j_1, j_2, j_3)} = \int \frac{d^2 k}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \\
\times \hat{\tau}_1 \hat{G}_{j_1}^0(k, \omega_n) \hat{\tau}_1 \hat{G}_{j_2}^0(q, \omega_n) \hat{\tau}_1 \hat{G}_{j_3}^0(q - k, \omega_n) \hat{\tau}_1. \quad (17)
\]

Consider, for example, the impurity scattering in the vicinity of a single node, corresponding to the case \((111)\). Here we are able to rescale all the momenta to get a circular cutoff in each integral:
\[ \Sigma_{(111)}^{(2b)}(\omega_n) = -i\omega_n g_3^2 \int_{|k|<\Delta_0} \frac{d^2 k}{(2\pi)^2} \int_{|q|<\Delta_0} \frac{d^2 q}{(2\pi)^2} \frac{k \cdot (q - k)}{\omega_n^2 + k^2}. \] 

Integrating over the angle between \( k \) and \( q \) yields \(-2\pi\Theta(\omega_n)(k^2 - q^2)\), where \( \Theta(\omega) \) is a smeared step function, with the width of the order of \( |\omega| \) at \( x = 0 \). The remaining integration over the region \(|k| > |q|\) gives a \( \log^2 |\omega_n| \) contribution:

\[ \Sigma_{(111)}^{(2b)}(\omega_n) \simeq i\omega_n \frac{g_3^2}{(2\pi)^2} \int_{0}^{\Delta_0} \frac{q \ dq}{\omega_n^2 + q^2} \int_{0}^{\Delta_0} \frac{k \ dk}{\omega_n^2 + k^2}. \]

Quite similarly

\[ \Sigma_{(111)}^{(2b)}(\omega_n) + \Sigma_{(111)}^{(2b)}(\omega_n) \simeq -i\omega_n \frac{g_3 g_1}{(2\pi)^2} \ln^2 \frac{\Delta_0}{|\omega_n|}. \] 

In cases when intermediate states belong to neighboring nodes, as it takes place for diagrams (221), (122), (221)(122),(212),(212) the situation is somewhat more complicated due to the difference between the two velocities, \( v_1 \gg v_2 \). The two ellipses, representing the surfaces of constant energy for states 1 (\( \bar{1} \)) and 2 (\( \bar{2} \)), are strongly elongated in perpendicular directions and, therefore, cannot be simultaneously transformed to circles by a global rescaling of the momenta. The available phase space is then reduced by the amount determined by the anisotropy parameter

\[ \gamma = v_2/v_1 \sim \Delta_0/t \ll 1. \] 

As a result, an additional energy scale \( \gamma \Delta_0 \) enters the problem, and the corresponding diagrams become dependent on the ratio \( |\omega_n|/\gamma \Delta_0 \). For example,

\[ \Sigma_{(212)}^{(2b)}(\omega_n) + \Sigma_{(212)}^{(2b)}(\omega_n) = 2i\omega_n g_2^2 \int_{|k|<\Delta_0} \frac{d^2 k}{(2\pi)^2} \int_{|q|<\gamma\Delta_0,|q_2|<\Delta_0} \frac{d^2 q}{(2\pi)^2} \frac{k \cdot (q - k)}{(\omega_n^2 + k^2)(\omega_n^2 + \gamma^{-2}q_1^2 + \gamma^{-2}q_2^2)[\omega_n^2 + (q - k)^2]}. \]

Integrating first over \( k \), we get

\[ \Sigma_{(212)}^{(2b)}(\omega_n) + \Sigma_{(212)}^{(2b)}(\omega_n) \simeq -4 \frac{2i\omega_n}{(2\pi)^3} g_2^2 \int_{0}^{\Delta_0} dq_1 \int_{0}^{\Delta_0} dq_2 \ln \frac{\Delta_0/\max(|\omega_n|,|q|)}{\omega_n^2 + \gamma^{-2}q_1^2 + \gamma^{-2}q_2^2}. \] 

At energies \( \gamma\Delta_0 < |\omega_n| \ll \Delta_0 \), the integral in \( \bar{23} \) is mostly contributed by regions \( q_1 \ll \gamma\Delta_0 \), \( \gamma\Delta_0 \ll q_2 \ll \Delta_0 \), and is nearly independent of \( \omega_n \):

\[ \Sigma_{(212)}^{(2b)}(\omega_n) + \Sigma_{(212)}^{(2b)}(\omega_n) \simeq -\frac{2i g_2^2}{(2\pi)^2} \gamma\Delta_0 [1 + 0(\gamma \ln \gamma)] \text{sign}(\omega_n). \]
The log^2-term only appears at lower energies, |ω_n| ≪ γΔ_0, in which case integration over |q| < γΔ_0 yields

\[ \Sigma^{(2b)}_{(212)}(ω_n) + \Sigma^{(2b)}_{(212)}(ω_n) \simeq -iω_n \frac{g_2^2}{(2π)^2} \ln^2 \frac{γΔ_0}{|ω_n|}. \]  (25)

For the remaining four diagrams ((221), (122), (221), (122)) the effect of the anisotropy γ is even stronger. It can be shown that at |ω_n| ≪ γΔ_0 these diagrams contain only first power of the logarithm: Σ ∼ iω_n g_3 g_2 ln(γΔ_0/|ω_n|).

The above analysis shows that, in 2D electron systems with a conical spectrum, impurity scattering drastically differs from that in a normal metallic state with a finite Fermi surface. The structure of the perturbation theory expansion for the self energy reveals a ”logarithmic situation” resembling mass renormalization in models of interacting fermions in one space dimension. (This analogy, being realized in earlier papers\[11,8\], will become more transparent in subsequent sections, where the effective field theory in (1 + 1) dimensions is discussed.) Notice that, after analytic continuation, iω_n → ω + iδ, the leading logarithmic singularities occur in the real part of Σ(ω). This means that, in systems with a conical spectrum, disorder mostly affects renormalization of the single-particle spectrum, as opposed to the usual picture in systems with a finite DOS, where life-time effects dominate.

Actually, the logarithmic singularities appear in all orders of perturbation theory, including crossing diagrams. In two- and higher-dimensional metallic systems with a finite Fermi surface, such diagrams are known to be relatively small\[13\]. In the case of weak disorder, k_F l ≫ 1, l being the mean-free path, strong restrictions over the relative angles between the momenta of intermediate one-electron states effectively reduce the available phase space, making diagrams with crossing impurity lines proportional to powers of the factor (1/k_F l). However, in 1D systems, where the Fermi surface is represented by two points, ±k_F, no such reduction is possible, and all diagrams are equally important, as is known from the theory of one-dimensional localization\[14,15\]. In this respect, the electron-impurity scattering in 2D systems with conical spectra resembles the 1D case: the crossing diagrams also contain logarithmic singularities and therefore should be treated on the same footing as the rainbow ones. For a finite number of degeneracy points (nodes), no selection of diagrams is possible. An approximation, in which one confines consideration to the class of rainbow diagrams and then solves self-consistently the resulting Dyson equation for Σ(ω)\[6\], can be justified only in the limit of (infinitely) large number of nodes\[8\]; or when one extends consideration to a higher dimensional case, e.g. 3D polar superconductors with a line of nodes\[2\]. This approximation leads to a finite DOS at ω = 0.

In the next sections, all singular self-energy diagrams will be taken into account, using replica trick. Treating the effective, disorder-free field-theoretical model by abelian and non-abelian bosonization, we shall show that, for a weakly disordered 2D d-wave superconductor, the low-energy DOS follows a power-law behavior. The corresponding critical exponent depends on the type of electron-impurity processes included into consideration.
IV. REPLICA TRICK AND EFFECTIVE TWO-DIMENSIONAL FIELD THEORY

In what follows, we shall be mostly concerned with the forward and backward scattering processes, putting \( \lambda_2 = 0 \). In this case the nodes \((1, \bar{1})\) and \((2, \bar{2})\) can be considered independently. Without loss of generality we can choose the \((1, \bar{1})\) pair. Rescaling the coordinates and the fields

\[
x \rightarrow v_1 x_1, \quad y \rightarrow v_2 x_2, \\
\psi_1(x) \rightarrow \frac{1}{\sqrt{v_1 v_2}} \psi_1(x), \quad \psi_1(x) \rightarrow \frac{1}{\sqrt{v_1 v_2}} \psi_1(x)
\]

we arrive at following generating functional for the (unaveraged) Green’s function with a fixed Matsubara frequency \( \omega_n \):

\[
Z[A] = \int D\bar{\eta} D\eta \exp(-S[A]), \quad (26)
\]

\[
S = \int d^2 x \bar{\eta}(x) [(-i\partial_1 + A_3(x)\sigma_3)\tau_1 \\
+ (-i\partial_2 + A_1(x)\sigma_1 + A_2(x)\sigma_2)\tau_2 - i\omega_n] \eta(x).
\]

Action (27) describes two-dimensional massless fermions interacting with a random non-Abelian gauge potential. The Pauli matrices \( \sigma_i \) act on the isotopical indices \((1, \bar{1})\), and the probability distribution of the gauge field is

\[
P[A] = \int DA \exp[\int d^2 x \frac{A_i(x)^2}{2g_i}], \quad (28)
\]

The matrices \( \tau_2 = \gamma_0 \) and \( \tau_1 = \gamma_1 \) form a representation of the two dimensional Clifford algebra, i.e they are the Dirac matrices of our problem. The new fermionic fields are related to the initial ones as follows:

\[
\bar{\eta} = -i\psi^+, \quad \eta = \psi.
\]

In terms of the new fermionic variables, the DOS is given by

\[
\rho(\omega) = -\frac{1}{\pi v_1 v_2} Re[Tr < \bar{\eta}(x)\eta(x)>]|_{i\omega_n \rightarrow \omega + i\delta}.
\]

We average over the disorder using the standard replica trick. For this purpose we replicate the action (27) \( r \)-times \((r \) is the number of replicas) and integrate the fermionic partition function with the replicated action over the gauge fields. The result is

\[
Z_r = \int D\bar{\eta} D\eta \exp(-S_r), \quad (31)
\]

\[
S_r = \int d^2 x \{ \bar{\eta}_{a,\alpha} (\gamma_\mu \partial_\mu + \omega_n) \eta_{a,\alpha} + \frac{g_3}{2} (\bar{\eta}_{a,\alpha} \gamma_0 \sigma_3^{a,\alpha} \eta_{a,\alpha})^2 \\
+ \frac{g_1}{2} (\bar{\eta}_{a,\alpha} \gamma_1 \sigma_2^{a,\alpha} \eta_{a,\alpha})^2 + (\bar{\eta}_{a,\alpha} \gamma_1 \sigma_2^{a,\alpha})^2 \}, \quad (32)
\]
where the Greek indices are isotopic and the Latin ones are reserved for replicas \((a = 1, \ldots r)\). It will be convenient to rewrite the problem in the Hamiltonian formalism treating \(x_1\) as imaginary time and \(x_2\) as space coordinate. The quantization rules are \(\bar{\eta} \gamma_0 = \eta^+\), where the spinors \((\eta_R, \eta_L)\) and their Hermitian conjugate satisfy the standard anticommutation relations. The Hamiltonian is given by

\[
H = \int dx \left\{ \eta_{a,\alpha}^+ (-i \partial_x \tau_3 + \omega_n \tau_2) \eta_{a,\alpha} + \frac{g_3}{2} \left[ \left\{ \eta_{a,\alpha}^+ \sigma_3^a \eta_{a,\beta} \right\}^2 + \frac{9}{2} \left\{ \left\{ \eta_{a,\alpha}^+ \sigma_1^a \eta_{a,\beta} \right\}^2 + \frac{9}{2} \left\{ \left\{ \eta_{a,\alpha}^+ \sigma_2^a \eta_{a,\beta} \right\}^2 \right\} \right\} \right\},
\]

(33)

where the Matsubara frequency \(\omega_n\) plays the role of relativistic mass.

Notice that the Hamiltonian (33) possesses continuous \(SU(r)\) replica symmetry. At \(\omega_n = 0\) it is also invariant under chiral rotations of Fermi fields: (the continuous \(\gamma^5\) symmetry)

\[
\eta \rightarrow \exp(i \tau_3 \varphi) \eta.
\]

As follows from (30), a finite \(\rho(0)\) would mean the existence of a nonzero order parameter \(\langle \bar{\eta} \eta \rangle\), indicating a spontaneous breakdown of these symmetries. However, continuous symmetries cannot be broken in two dimensions; therefore \(\rho(\omega)\) should vanish at \(\omega \rightarrow 0\). This is confirmed by direct calculations presented below.

V. MODEL WITHOUT BACKSCATTERING. BOSONIZATION

In this Section we shall focus on the slowly varying (quasiclassical) part of the random potential, i.e. we set \(g_1 = g_2 = 0\). In this case the problem is reduced to the single node one. The isotopic indices can be suppressed, and the resulting model can be treated by the standard abelian bosonization methods.

The bosonization rules are well known:

\[
J_{R,a} + J_{L,a} = \frac{1}{\sqrt{\pi}} \partial_x \phi_a, \quad J_{R,a} - J_{L,a} = -\frac{1}{\sqrt{\pi}} \Pi_a,
\]

\[
: \eta_{R,a}^+ \eta_{L,a} + \eta_{L,a}^+ \eta_{R,a} : = -\Lambda \cos(\sqrt{4\pi} \phi_a).
\]

(34)

Here

\[
J_{R(L),a} = : \eta_{R(L),a}^+ \eta_{R(L),a} : \quad (35)
\]

are the current operators for right(left)-moving particles with the replica index \(a\); \(\phi_a(x)\) and \(\Pi_a(x)\) are scalar fields and their conjugate momenta, respectively, satisfying the canonical commutation relations: \(\left[ \phi_a(x), \Pi_b(y) \right] = i \delta_{ab} \delta(x - y); \quad \Lambda \sim \Delta_0\) is the ultraviolet cut-off. The Bose model equivalent to (33) reads:

\[
H = \int dx \left\{ \frac{1}{2} \sum_a [\Pi_a^2 + (\partial_x \phi_a)^2] - \frac{9}{2\pi} \sum_{ab} \Pi_a \Pi_b - \omega_n \Lambda \sum_a \cos(\sqrt{4\pi} \phi_a) \right\}.
\]

(36)

In order to proceed further we need to diagonalize the quadratic form \(\sum_{ab} \Pi_a \Pi_b \equiv \Pi^T M \Pi\), where \(M_{ab} = 1\) for all \(a\) and \(b\). We do it with an orthogonal transformation \(U\):
\[ \Pi = UP, \quad U^T U = I; \]
\[ U^T MU = M_D, \quad (M_D)_{ab} = \nu \delta_{ar} \delta_{br}; \]
\[ U_{ar} = \frac{1}{\sqrt{r}} \quad (a = 1, \ldots r). \quad (37) \]

and obtain
\[
H = \int dx \left\{ \frac{1}{2} \sum_{a=1}^{r-1} \left[ P_a^2 + (\partial_x \phi_a)^2 \right] + \frac{1}{2} \left[ (1 - \frac{rg_3}{\pi}) P_r^2 + (\partial_x \phi_r)^2 \right] 
- \omega_n \Lambda \sum_a \cos(\sqrt{4\pi V_{ab}} \phi_b) \right\}. \quad (38)
\]

The last step consists in rescaling \( \phi_r \) and \( P_r \):
\[
\phi_r \rightarrow \gamma \phi_r, \quad P_r \rightarrow \frac{1}{\gamma} P_r, \\
\gamma^2 = (1 - \frac{g_3 r}{\pi})^{1/2}. \quad (39)
\]

The resulting theory is described by the Hamiltonian
\[
H = \int dx \left\{ \sum_{a=1}^{r} \frac{u_a}{2} \left[ P_a^2 + (\partial_x \phi_a)^2 \right] - \omega_n \Lambda \sum_a \cos(\sqrt{4\pi V_{ab}} \phi_b) \right\}, \quad (40)
\]
where
\[
u_a = 1 \quad (a < r), \quad u_r = \gamma, \\
V_{ab} = U_{ab} \quad (a, b < r), \quad V_{ar} = \frac{\gamma}{r}.
\]

The critical dimension of the cosine term in Eq.(40) is given by
\[
\Delta_a = \sum_{b=1}^{r} V_{ab}^2 = 1 + \frac{\gamma^2 - 1}{r}. \quad (41)
\]

We see that it does not depend on \( a \) and remains well defined in the replica limit:
\[
\Delta \equiv \Delta_a (r \rightarrow 0) = 1 - \frac{g_3}{2\pi}. \quad (42)
\]

Simple scaling arguments allow us to estimate the DOS at finite \( \omega \). First we note that in Eq.(40) the cosine term is a relevant perturbation generating a new energy scale \( \tilde{\omega} \) (or, equivalently, correlation length \( \xi_c \) ), given by
\[
\tilde{\omega} \sim \xi_c^{-1} \sim \omega \left( \frac{\Lambda}{|\omega|} \right)^{\frac{1}{2-\Delta}}. \quad (43)
\]
This formula correctly reproduces perturbation theory expansion for the self-energy operator in the single-node approximation \( (g_1 = g_2 = 0) \):
\[
\frac{\Sigma(i\omega_n)}{i\omega_n} \bigg|_{i\omega_n\to\omega+i\delta} = 1 - \frac{\tilde{\omega}}{\omega} = 1 - \left( \frac{\Lambda}{|\omega|} \right)^{\frac{\Delta}{2}} \\
= -\frac{g_3}{2\pi} \ln \frac{\Lambda}{|\omega|} - \frac{1}{2} \left( \frac{g_3}{2\pi} \right)^2 \ln^2 \frac{\Lambda}{|\omega|} + \ldots, \quad (44)
\]
in agreement with the estimation of first- and second-order diagrams (see Eqs. (14), (16) and (19)). Applying then bosonization rules (34) to Eq. (30) we establish that \( \rho(\omega) \sim |\omega|^\alpha \) (45) with a nonuniversal exponent

\[
\alpha = \frac{\Delta}{2 - \Delta} = \frac{1 - g_3/2\pi}{1 + g_3/2\pi} < 1, \quad (46)
\]
depending on the impurity concentration.

**VI. IMPURITY SCATTERING IN MODEL WITH SEVERAL FERMI POINTS**

In this Section we consider the model where scattering processes transfer electrons between the opposite Fermi points. The corresponding replicated model was described in Section IV, Eq. (33); here we rewrite it explicitly in terms of chiral fermions:

\[
H = H_0 + H_1 + H_2, \\
H_0 = \int dx \eta^+_{a,\alpha} (-i\partial_x \tau_3 + \omega_n \tau_2) \eta_{a,\alpha}, \\
H_1 = \int dx \{ -2g_3 (J^3_R J^3_R : + J^3_L J^3_L :) + 2g_1 \sum_{i=1,2} (J^i_R J^i_R : + J^i_L J^i_L :) \}, \\
H_2 = 4 \int dx [g_3 (J^3_R J^3_L : + g_1 \sum_{i=1,2} J^i_R J^i_L :)], \quad (47) \quad (48) \quad (49) \quad (50)
\]

where

\[
J^i_R = \eta^+_{R,\alpha} \frac{\sigma^i_{\alpha\beta}}{2} \eta_{R,\alpha\beta}, \\
J^i_L = \eta^+_{L,\alpha} \frac{\sigma^i_{\alpha\beta}}{2} \eta_{L,\alpha\beta} \quad (51)
\]

are chiral components of the \( SU(2) \) currents.

We shall apply to the model (47) the procedure of non-Abelian bosonization developed by Witten. The approach is based on the following key moments. The first one is that the Hamiltonian of free fermions (48) can be expressed in terms of current operators (the Sugawara construction). The Sugawara form of the Hamiltonian of free massless fermions with a general \( U(1) \times SU(N) \times SU(r) \)-symmetry (in Eq. (48) \( N = 2 \)) is
\[ H_0 = H_{U(1)}^0 + H_{SU(N)}^0 + H_{SU(r)}^0 \]
\[ = \int dx \left[ \frac{\pi}{N_r} (\cdot \cdot J_R(x) J_R(x) : + : J_L(x) J_L(x) :) \right. \]
\[ + \frac{2\pi}{N + r} \sum_{i=1}^{G_N} (\cdot \cdot J_R^i(x) J_R^i(x) : + : J_L^i(x) J_L^i(x) :) \right. \]
\[ + \frac{2\pi}{N + r} \sum_{a=1}^{G_r} (\cdot \cdot J_R^a(x) J_R^a(x) : + : J_L^a(x) J_L^a(x) :) \right], \]

where \( J, J^i \) and \( J^a \) are the \( U(1) \), the \( SU(N) \) and the \( SU(r) \) currents defined as in Eq.(51), but with generators of the corresponding algebras instead of the Pauli matrices. \( G_N \) and \( G_r \) are the total number of generators of the \( su(N) \) and the \( su(r) \) Lie algebras. The second key moment is that the currents from different algebras commute. Therefore \( H_0 \) is a sum of three mutually commuting operators.

Now notice that the interaction terms (49), (50) contain only spin currents and therefore do not affect the spectra of \( SU(2) \)-singlets. Since this interaction is attractive, the spectrum in the \( SU(2) \) channel becomes gapful and, as we show later, this gap persists in the replica limit. Therefore, if \( \omega_n = 0 \), the spectrum well below the gap \( Q_0 \) is described by the rest of the Hamiltonian (57), in other words by \( H_{\text{eff}} = H_{U(1)}^0 + H_{SU(r)}^0 \). The Hamiltonian \( H_{SU(r)}^0 \) is the Hamiltonian of the Wess-Zumino-Witten model. Its spectrum is the subsector of the free fermionic spectrum generated by the \( SU(r) \) current operators. The model is conformally invariant and exactly solvable. Its primary fields transform according to representations of the \( SU(r) \) group. In particular, the field from the fundamental representation, which we denote \( g_{ab} \), has the following scaling dimension:

\[ \Delta_g = \frac{r - 1/r}{N + r}. \]

We see that \( \Delta_g \) is singular at \( r \to 0 \), which indicates that the \( g \)-field cannot appear alone in the physical sector. In fact, as we shall see, a local bilinear form of two Fermi operators, \( \bar{\eta} \eta \), which appears in the "mass" (i.e. \( \omega_n \)-proportional) term of the Hamiltonian, as well as in the definition (30) of DOS, is represented by a combination of \( g \) and a phase exponential of the scalar (\( U(1) \)-symmetric) field \( \phi \)

\[ \bar{\eta} \eta \equiv Q \sim Q_0 \Re [g \exp(i\sqrt{4\pi/Nr} \phi)]. \]

The scaling dimension of the exponential is equal to \( 1/Nr \). So, the two singularities cancel, and the resulting scaling dimension of the operator \( Q \) is finite in the replica limit:

\[ \Delta_Q = \lim_{r \to 0} \left[ \frac{1}{N_r} + \frac{r - 1/r}{N + r} \right] = \frac{1}{N^2}. \]

It is instructive to repeat the above derivation in a more traditional form introducing the replica tensor \( Q_{ab} \). For the sake of simplicity we assume \( g_1 = g_3 \). Let us rewrite the action (52) in the form suitable for the Hubbard-Stratonovich transformation:
\[ S = \int d^2 x \{ \bar{\eta}_{a,\alpha} (i \gamma_\mu \partial_\mu + i \omega_n) \eta_{a,\alpha} \\
- g \{ (\bar{\eta}_{a,\alpha} \eta_{b,\alpha}) (\bar{\eta}_{b,\beta} \eta_{a,\beta}) - (\bar{\eta}_{a,\alpha} \gamma_5 \eta_{b,\alpha}) (\bar{\eta}_{b,\beta} \gamma_5 \eta_{a,\beta}) \} + S_{irr}, \] (56)

where \( \gamma_5 = i \tau_1 \) and \( S_{irr} \) contains terms with interactions near the same Fermi point. Such terms renormalize velocities and are not important for the current discussion. Now we introduce the auxiliary field \( Q_{\alpha\beta} \) and decouple the interaction term by the Hubbard-Stratonovich transformation. Integrating formally over the fermions, we obtain the partition function for the replica tensor field:

\[ Z = \int DQ^+ DQ \exp(-\int d^2 x L), \]

\[ L = \frac{1}{2} Tr(Q^+ - \omega I)(Q - \omega I) \\
- N Tr \ln[i \gamma_\mu \partial_\mu + (1 + i \gamma_5) Q/2 + (1 - i \gamma_5) Q^+/2]. \] (57)

Following the conventional wisdom, we shall look for a saddle point of the exponent and then derive an effective action for fluctuations around this saddle point. We assume that the saddle point configuration of \( Q \) is coordinate independent and as such can be chosen as a diagonal real matrix: \( Q(x) = diag(\lambda_1,...\lambda_r) \). The density of effective action on this configuration is equal to

\[ S_{eff} = \sum_a \left[ \frac{(\lambda_a - \omega)^2}{2g} + \frac{N \lambda_a^2}{2\pi} \ln \frac{\lambda_a}{\Lambda} \right]. \] (58)

The saddle point value of \( Q \), being a point of minimum of this function, satisfies the equation:

\[ \frac{\lambda_a - \omega}{g} + \frac{N \lambda_a}{\pi} \ln \frac{\lambda_a}{\Lambda} = 0. \] (59)

At \( \omega = 0 \) we have the following solution:

\[ \lambda_a = \Lambda \exp[-\pi/Ng] \equiv Q_0. \] (60)

The existence of the saddle point does not mean, however, that the system has a non-zero order parameter \( \langle Q \rangle \). Such order parameter in the present case is the density of states at the Fermi energy:

\[ \rho(0) = \lim_{r \to 0} r^{-1} \frac{\partial Z}{\partial \omega} \bigg|_{\omega=0} = \lim_{r \to 0} \frac{1}{gr} Tr(Q + Q^+/2). \] (61)

The appearance of the average \( \langle Q \rangle \) breaks the continuous symmetry \( U(r) \), and it is well known that such symmetry breaking cannot occur in two dimensional systems due to strong transverse fluctuations of \( Q \). A finite \( Re \langle Q \rangle \) arises only at finite \( \omega \) when the transverse fluctuations acquire a finite correlation length \( \xi_c = \omega^{-1/(2 - \Delta)} \). \( \Delta \) is the scaling dimension of the operator \( ReQ \). In this case we have:

\[ Re \langle Q \rangle \sim Q_0 \omega^{\frac{\Delta}{2 - \Delta}}. \] (62)
Now we are going to demonstrate that the described qualitative picture survives the replica limit and maintains a contact with the previous treatment. In order to show that the radial fluctuations of $Q$ remain gapful even at $r = 0$, we use the exact solution of the model (56). This exact solution was found by one of the authors [19]. It was shown that the model (56) at $\omega = 0$ is a relativistic limit of the model directly solvable by the Bethe ansatz:

$$H = \int dx [\psi_{a,\alpha}^+ ( - \frac{1}{2} \partial^2_x - \frac{1}{2} k_F^2 ) \psi_{a,\alpha} - g (\psi_{a,\alpha}^+ \psi_{b,\alpha})(\psi_{b,\beta}^+ \psi_{a,\beta})].$$  \hfill (63)

The basic features of the exact solution coincide with those derived from the non-Abelian bosonization. In particular, it was established in Ref. [19] that, in the relativistic limit, the spectrum of model (63) consists of three sectors. Excitations of one sector are $U(r)$-singlets and have a spectral gap given by Eq. (60). The low energy sector contains gapless excitations which are $SU(N)$-singlets. One branch of gapless excitations consists of the U(1) scalar field described by the effective action

$$S_1 = \frac{1}{2} \int d^2x (\partial_\mu \phi)^2.$$  \hfill (64)

while the other is described by the Wess-Zumino-Witten model at the critical point, i.e. by $H_{SU(r)}^0$ in Eq. (52). The latter can be also written in the Lagrangian form with the following action:

$$S_2 = \int d^2x \{ \frac{N}{8\pi} Tr(\partial_\mu g^+ \partial_\mu g) + \frac{N}{24\pi^3} \int_0^1 d\xi \epsilon_{abc} Tr(g^+ \partial_\mu gg^+ \partial_\mu gg^+ \partial_\mu g) \},$$ \hfill (65)

where $g$ is a matrix from the $SU(r)$ group ($g^+ g = I, \det g = 1$). The second term in the r.h.s. of Eq. (65) is the so-called Wess-Zumino term. It has a topological origin. Despite of the fact that it is written as an integral including the additional dimension, its actual value (modulus $2\pi i N$) depends on the boundary values of $g(x, \xi = 0) = g(x) (g(x, \xi = 1) = 0$). This property follows from the fact that the Wess-Zumino term is proportional to the integral of the Jacobian of transformation from the three dimensional euclidean coordinates to the group coordinates, thus being a total derivative in disguise. Excitations of the Wess-Zumino-Witten model are gapless and the correlation functions are well studied (see Ref. [20]). The presence of the Wess-Zumino term is absolutely crucial for the criticality - the model without it undergoes a strong renormalization.

Let us establish a connection between the exact solution and the semiclassical approach. The massive sector obviously corresponds to radial fluctuations of the order parameter field $Q$. The rest of the action (except of the Wess-Zumino term) represents the first term in the gradient expansion of the $Tr \ln$-term in Eq. (57). Indeed, at small momenta $|p| \ll Q_0$ this term is equal to

$$\frac{N}{8\pi Q_0^2} \int d^2x Tr(\partial_\mu Q^+ \partial_\mu Q).$$ \hfill (66)

Now let us write down the $Q$-field in the form (54). Substituting this expression into Eq. (66) and taking into account that $Tr g^+ \partial_\mu g = 0$ as a trace of an element of the algebra, we reproduce Eq. (54) and the first term of Eq. (65). It comes not entirely unexpected that the
naive gradient expansion misses the important Wess-Zumino term: the latter is a Berry phase and requires special care. In order to avoid possible calculational difficulties, we have resorted to the exact solution. In this way we obtain the result having an independent value (see also Ref. [21]):

\[-Tr \ln[i\gamma_\mu \partial_\mu + (1 + i\gamma_5)gQ_0/2 + (1 - i\gamma_5)g^+Q_0/2] =
\int d^2x \left\{ \frac{1}{8\pi} Tr(\partial_\mu g^+\partial_\mu g) + \frac{1}{24\pi^2} \int_0^1 d\xi \epsilon_{abc} Tr(g^+\partial_\mu gg^+\partial_\mu gg^+\partial_\mu g) \right\} + O(Q_0^{-2}), \quad (67)\]

where \(g\) belongs to \(U(r)\) group (\(r\) is now an arbitrary number not related to the number of replicas).

The fact that radial and transverse fluctuations are decoupled at the level of the Bethe ansatz equations allows one to consider the corresponding replica limits separately. The proof of the fact that the replica limit is well defined for the gapful sector is given in the Appendix. Here we consider only the replica limit for the correlation functions of \(Q\). For the pair correlation function we have

\[< Q(x)Q^+(y) > = Q_0^2 < \exp[i\sqrt{4\pi/Nr}\phi(x)] \exp[-i\sqrt{4\pi/Nr}\phi(y)] > < g(x)g^+(y) > \sim |x - y|^{-2(1/Nr + \Delta_\phi)} = |x - y|^{-(2/N^2)}, \quad (68)\]

in agreement with the above estimation of the scaling dimension of the operator \(Q\) (see Eq. (55)). According to Eq. (62) the dimension \(1/N^2\) gives the following estimate for the density of states at \(|\omega| \ll Q_0:\]

\[\rho(\omega) \sim \omega^{1/(2N^2 - 1)}. \quad (69)\]

In fact, even in the case \(N = 2\) the corresponding exponent is extremely small: \(1/7\) and the specific heat at low temperatures goes almost linearly: \(c_v \sim T^{8/7}\).

So far the impurity scattering between the nearest-neighbor nodes has been neglected. When these processes are taken into account, the effective replicated action can be easily shown to have the form:

\[S_{eff} = S_1 + S_2 + S_{12}, \quad (70)\]

\[S_1 = \int d^2x \{ \bar{\eta}_{\lambda aa} (v_1 \partial_1 \tau_1 + v_2 \partial_2 \tau_2 + \omega_n) \eta_{1aa} + \frac{1}{2} g_3 (\bar{\eta}_{\lambda 1a} \tau_1 \sigma_{a\beta}^1 \eta_{1\beta a})^2 + \frac{1}{2} g_1 [(\bar{\eta}_{\lambda 1a} \tau_2 \sigma_{a\beta}^1 \eta_{1\beta a})^2 + (\bar{\eta}_{\lambda 1a} \tau_2 \sigma_{a\beta}^2 \eta_{1\beta a})^2] \}, \quad (71)\]

\[S_2 = \int d^2x \{ \bar{\eta}_{\lambda aa} (v_1 \partial_1 \tau_1 + v_2 \partial_2 \tau_2 + \omega_n) \eta_{1aa} + \frac{1}{2} g_3 (\bar{\eta}_{\lambda aa} \tau_1 \sigma_{a\beta}^1 \eta_{2\beta a})^2 + \frac{1}{2} g_1 [(\bar{\eta}_{\lambda aa} \tau_2 \sigma_{a\beta}^1 \eta_{2\beta a})^2 + (\bar{\eta}_{\lambda aa} \tau_2 \sigma_{a\beta}^2 \eta_{2\beta a})^2] \}, \quad (72)\]

\[S_{12} = \frac{1}{2} g_2 \sum_{i \neq j} \int d^2x \{ (\bar{\eta}_{i a a} \tau_1 \sigma_{a\beta}^1 \eta_{j a a})^2 - (\bar{\eta}_{i a a} \tau_1 \bar{\eta}_{j a a})^2 \]

\[+ (\bar{\eta}_{i a a} \tau_2 \sigma_{a\beta}^2 \eta_{j a a})^2 + (\bar{\eta}_{i a a} \tau_2 \sigma_{a\beta}^1 \eta_{j a a})^2 \}, \quad (73)\]
where fermionic fields $\eta_1$ and $\eta_2$ describe two pairs of opposite nodes, $(1, \bar{1})$ and $(2, \bar{2})$, respectively. The action (70) is still symmetric under continuous $\gamma^5$-transformations

$$\eta_{ja} \rightarrow e^{i\alpha \tau_3} \eta_{ja}, \quad \bar{\eta}_{ja} \rightarrow \bar{\eta}_{ja} e^{i\alpha \tau_3}$$

(74)

Therefore the DOS vanishes as $\omega \rightarrow 0$, even when the nearest-neighbor internode scattering is included into consideration. This conclusion does not depend on the ratio $v_1/v_2$. However, the crossover from the two-node critical regime to the four-node one will be shifted towards lower energies on increasing $v_1/v_2$.

VII. NONMAGNETIC IMPURI TIES IN 2D ORBITAL ANTIFERROMAGNET

In this section, we consider effects of nonmagnetic disorder in another 2D electron system with conical energy spectrum - the orbital antiferromagnet state (OAF). This state was earlier suggested as one of possible phases of weakly interacting fermions on a square lattice with a half-filled energy band. The OAF phase can be viewed as resulting from anomalous particle-hole pairing $< c_{k,\alpha}^+ c_{k+Q,\alpha} >$ with momentum transfer equal to the nesting vector $Q = (\pi, \pi)$. Such a pairing breaks translational, point and time reversal symmetries, but preserves spin-rotational invariance. At the microscopical level, the OAF is characterized by nonzero local charge currents circulating around plaquettes in an alternating way. It actually represents a weak-coupling analog of the flux phase studied by Affleck and Marston in the large-$U$ Hubbard model.

Since the OAF is a state with a spin-singlet particle-hole pairing, and since we wish to discuss effects of nonmagnetic impurities, spin degrees of freedom are irrelevant and will be suppressed in what follows. Then the mean-field Hamiltonian describing a 2D OAF state can be written as

$$H_0 = \sum_{\mathbf{k}} [\epsilon(\mathbf{k}) c_{\mathbf{k}}^+ c_{\mathbf{k}} - i\Delta(\mathbf{k}) c_{\mathbf{k}}^+ c_{\mathbf{k}+\mathbf{Q}}],$$

(75)

where the tight-binding spectrum $\epsilon(\mathbf{k})$ and the gap function $\Delta(\mathbf{k})$ have the same two-cosine structures as those for the d-wave superconductor (see (1)) with $\mu = 0$. The quasiparticle spectrum $E(\mathbf{k}) = \pm \sqrt{\epsilon^2(\mathbf{k}) + \Delta^2(\mathbf{k})}$ has zeros at points $\mathbf{k}_1 = -\mathbf{k}_1 = (\pi/2, \pi/2)$, $\mathbf{k}_2 = -\mathbf{k}_2 = (-\pi/2, \pi/2)$. However, due to unit cell doubling taking place in the OAF state, the reduced Brillouin zone corresponding to the new translational symmetry, coincides with the area enclosed by the original square Fermi surface $ABCD$, shown in Fig. 3. Therefore, there are actually two inequivalent nodes 1 = $\bar{1}$ and 2 = $\bar{2}$ in the OAF phase, with two associated branches of low-energy excitations with conical spectra.

Assuming that $\mathbf{q}$ lies inside $ABCD$, we introduce a 2-spinor

$$\Phi_\mathbf{q} = \left( \begin{array}{c} c_\mathbf{q} \\ c_{\mathbf{q}+\mathbf{Q}} \end{array} \right)$$

and rewrite Hamiltonian (75) in a matrix form

$$\hat{H}_0 = \sum_\mathbf{q} \Phi_\mathbf{q}^+ [\epsilon(\mathbf{q})\hat{\tau}_3 + \Delta(\mathbf{q})\hat{\tau}_2] \Phi_\mathbf{q},$$

(76)
Scattering on nonmagnetic impurities is described by

\[ H_{\text{imp}} = \sum_{\mathbf{k}, \mathbf{k}'} V(\mathbf{k} - \mathbf{k}') c^+_\mathbf{k} c_{\mathbf{k}'} \]

\[ = \sum_{\mathbf{q}, \mathbf{q}'} [V_0(\mathbf{q} - \mathbf{q}') \Phi^+_{\mathbf{q}} \Phi_{\mathbf{q}'} + V_1(\mathbf{q} - \mathbf{q}') \Phi^+_{\mathbf{q}} \tau_1 \Phi_{\mathbf{q}'}] \]  

(77)

where \( V(\mathbf{k} - \mathbf{k}') \) is the Fourier transform of a Gaussian random potential defined on the original lattice. In the OAF state, this potential gives rise to two random fields, \( V_0(\mathbf{q}) = V(\mathbf{q}) \) and \( V_1(\mathbf{q}) = V(\mathbf{q} + \mathbf{Q}) \), whose small-momentum parts describe the chemical potential and charge-density-wave (CDW) fluctuations, respectively.

The physical difference between \( V_0 \) and \( V_1 \) allows to expect qualitatively different effects of these two types of disorder on the low-energy DOS in the OAF phase. Consider a local, slowly varying fluctuation of the chemical potential, such that \( V_0(x) \) is nearly constant in a macroscopically large region. The finite value of \( V_0 \) will then (locally) break the particle-hole symmetry of the OAF state and open a finite Fermi surface, thus resulting in a finite DOS in the given region: \( N_{\text{loc}} \sim |V_0| \). The latter conclusion does not depend on the magnitude and sign of \( V_0(x) \). Therefore, the DOS averaged over realizations of the field \( V_0(x) \) is expected to be finite at zero energy. On the other hand, local CDW fluctuations tend to open a gap in the excitation spectrum, so that, for \( V_2(x) \approx \text{const} \), the local DOS will vanish in the whole energy range \( |E| < |V_2| \). Therefore, we expect in this case the averaged DOS to vanish at \( E \to 0 \) even faster than in the pure system. These expectations will be confirmed by the analysis presented below.

Following the same strategy as in Sec.4, we pass to the continuum limit based on separating states near the nodes. For simplicity, we shall ignore scattering between nodes 1 and 2. We shall argue later that the internode scattering does not alter main conclusions obtained in this section. Specializing to the node 1, we arrive at the Euclidean action

\[ S[V_i] = \int d^2x \bar{\eta}(x)[-i\partial_1 \tau_1 - i\partial_2 \tau_2 + \mu(x) - m(x) \tau_3 - i\omega_n] \eta(x) \]

(78)

which describes two-dimensional fermions with a random mass \( m(x) = V_1(x) \) and chemical potential \( \mu(x) = V_0(x) \). The Gaussian probability distributions for random fields \( V_i(x), (i = 1, 2) \) are given by

\[ P[V_i] = \int D\theta_i \exp \left[-\frac{1}{2g_i} \int \left( V^2_i(x) + \mathcal{D}^2 \right) \right], \quad (i = 0, 1). \]

(79)

Comparing Eqs.(78) and (27), we observe the important difference between the d-wave superconductor and OAF. In both cases we have a continuum description in terms of 2D “relativistic” fermions. However, nonmagnetic impurities introduce different types of randomness to the corresponding Dirac Hamiltonians: random gauge field in the former case, and random mass and chemical potential in the latter case. This is related to different symmetries of the condensates in the two cases.

Averaging over disorder results in the following quantum Hamiltonian of 1D interacting fermions:

\[ H = \int dx [\eta^+_a (-i\partial_2 \tau_3 + \omega_n \tau_2) \eta_a + \frac{1}{2} g_0 (\eta^+_a \tau_2 \eta_a )^2 - \frac{1}{2} g_1 (\eta^+_a \tau_1 \eta_a )^2]. \]

(80)
As before, $a = 1, 2, ..., r$ is the replica index.

At $\omega_n = 0$, the case $g_0 = -g_1$ would correspond to the massless chiral Gross-Neveu model, whose symmetry group is $U(r) = U(1) \times SU(r)$. However, since both coupling constants are positive by definition, this case is never realized; so even at $\omega_n = 0$ the model (80) has the $O(2r) = Z_2 \times SO(2r)$ symmetry. The important difference between these two groups is that the latter does not contain the continuous subgroup $U(1)$ which survives in the replica limit. Therefore it is possible that at $r = 0$ the model (80) has no gapless excitations. Thus the symmetry can be spontaneously broken, and contrary to the case of the d-wave superconductor, the DOS at $\omega_n = 0$, being the corresponding order parameter, $\langle \bar{\eta}_a \eta_a \rangle$, can be nonzero.

The breakdown of the discrete symmetry must be signalled by the development of strong-coupling regime in the zero-replica limit. To identify the effective couplings that are subject to renormalization, we rewrite the interaction terms in (80) as follows:

$$H_{int} = -\frac{1}{4} \int dx (\bar{\psi}_{R,a} \psi_{R,b} \tilde{\Gamma}_{aba'b'} \psi_{L,b'} \psi_{L,a'} + H.c.) - \frac{1}{4} \int dx (\bar{\psi}_{R,a} \psi_{L,b} \Gamma_{aba'b'} \psi_{L,b'} \psi_{R,a'} + H.c.).$$

(81)

Here

$$\tilde{\Gamma}_{aba'b'} = \tilde{\Gamma}(\delta_{aa'} \delta_{bb'} - \delta_{ab'} \delta_{ba'}),$$
$$\Gamma_{aba'b'} = \Gamma_2 \delta_{aa'} \delta_{bb'} - \Gamma_1 \delta_{ab'} \delta_{ba'}$$

(82)

are amplitudes of the Umklapp and momentum (chirality) conserving scattering processes, respectively. The bare values of the parameters in Eqs.(82) are

$$\tilde{\Gamma}(0) = -(g_0 + g_1), \quad \Gamma_2(0) = -(g_0 - g_1), \quad \Gamma_1(0) = 0.$$

(83)

In the leading logarithmic approximation (one-loop accuracy), the renormalization group equations are

$$\frac{\partial \tilde{\Gamma}}{\partial l} = -[(r - 1) \Gamma_1 - 2 \Gamma_2] \tilde{\Gamma},$$
$$\frac{\partial \Gamma_1}{\partial l} = -\frac{1}{2} [r \Gamma_1^2 + (r - 2) \tilde{\Gamma}^2],$$
$$\frac{\partial \Gamma_2}{\partial l} = \frac{1}{2} (\tilde{\Gamma}^2 - \Gamma_1^2),$$

(84)

where $l = \ln(\Lambda/|\epsilon|)$ is a logarithmic variable.

It is instructive to consider first a somewhat artificial case, when only backward impurity scattering, corresponding to a random Dirac mass $m(x)$ in Eq.(78), is present, $g_0 = 0$, $g_1 \neq 0$. The solution of Eqs.(84)

$$\Gamma_2 = 0, \quad \Gamma_1(l) = \tilde{\Gamma}(l) = -\frac{g_1}{1 - (r - 1)g_1 l}$$

(85)

indicates that a strong-coupling regime, taking place at all $r > 1$, is changed by a weak-coupling (“zero-charge”) infrared behavior in the replica limit.
\[
\Gamma_1(l; r \to 0) \simeq -\frac{1}{\ln (\Lambda/|\epsilon|)}.
\]

Therefore, in this case the perturbation caused by impurities is marginally irrelevant, the discrete $\gamma^5$-symmetry remains unbroken, and the DOS will show a power-law energy dependence.

On the other hand, when only forward scattering is considered (random chemical potential $\mu(x)$), $g_1 = 0$, $g_0 \neq 0$, the solution of Eqs. (84) is

\[
\Gamma_2 = 0, \quad \Gamma_1(l) = -\tilde{\Gamma}(l) = \frac{g_0}{1 + (r - 1)g_0l}
\]

and the situation is inverted. The weak-coupling regime, taking place at $r > 1$, is changed by a strong-coupling one at $r \to 0$, signalling breakdown of the discrete $\gamma^5$-symmetry. The position of the pole of the amplitude $\Gamma_1(l; r \to 0)$ determines the correlation length of the system in the zero-energy limit, $\xi_{c}^{-1} \sim \Lambda \exp(-1/g_0)$. As a result, the DOS at $\omega = 0$ is finite.

It can be shown that, when both scattering processes are present, the solution of renormalization group equations (84) is always singular. This clarifies the special role of random chemical potential which drives the system towards strong coupling regime with a finite DOS at arbitrarily small $g_0$. Therefore, as long as $g_0 \neq 0$, taking into account the scattering between the nodes 1 and 2 would not qualitatively change this conclusion.

**VIII. CONCLUSIONS**

In this paper we have presented a detailed analysis of several two-dimensional models with quenched randomness. The obtained results are essentially non-perturbative and, apart from their practical significance, give us a new understanding of the theoretical side of the problem.

We have shown that, for weak nonmagnetic impurities in a 2D d-wave superconductor, the whole diagrammatic series for the single-electron self energy has to be summed up. By explicit calculations and symmetry arguments, we have demonstrated that the DOS $\rho(\omega)$, averaged over disorder, preserves the property of being zero at zero energy. A new power-law behavior $\rho(\omega) \sim |\omega|^\alpha$ was found, with critical exponent $\alpha$ depending on the type of included scattering processes.

The feature most essential for our solution is that our models have a relativistic spectrum. This fact allows us to use a powerful machinery of Abelian and non-Abelian bosonization and the Bethe ansatz. Some of the models turned out to be exactly solvable and the main technical problem was to derive the replica limit of this exact solution. On the basis of our experience we can formulate a principle that the right procedure is to deal not with the energy eigenvalues, but with correlation functions. Energy eigenvalues in different symmetry sectors are singular at $r \to 0$, but these singularities cancel in correlation functions.
ACKNOWLEDGMENTS

We acknowledge the generous hospitality of the International Centre for Theoretical Physics in Trieste (Italy), where this collaboration was started. A.N. and F.W. acknowledge the financial support from Chalmers University of Technology and would like to thank H. Johannesson, S. Östlund, B.-S. Skagerstam and T. Einarsson for helpful discussions. A.N. is grateful to A. Luther for his generous hospitality at Nordita and interesting conversations. A.T. is grateful to J. Chalker and B. Altshuler for their interest to the work.

APPENDIX:

Let us consider the thermodynamic Bethe ansatz equations derived for the massive sector of the model (63) in [19]. For the sake of simplicity we consider only the case $N = 2$. The free energy is described by the following system of nonlinear integral equations:

\[
\frac{F_r}{L} = -\frac{Q_0}{4} \int dx \cosh(\pi x/2) \ln\{1 + \exp[\beta \epsilon_r(x)]\}, \quad (A1)
\]

\[
\beta \epsilon_n = s \ln\{1 + \exp[\beta \epsilon_{n-1}(x)]\}\{1 + \exp[\beta \epsilon_{n+1}(x)]\} - \beta Q_0 \cosh(\pi x/2) \delta_{n,r}, \quad (A2)
\]

where $n = 1, 2, ...$ and

\[
s \ast f(x) = \int_{-\infty}^{\infty} dy \frac{f(y)}{4 \cosh[\pi (x - y)/2]}.
\]

In the replica limit we need to show that the following quantity has a finite value:

\[
F = \lim_{r \to 0} \frac{F_r}{r}.
\]

(A3)

It what follows we shall consider two limiting cases: $\beta Q_0 \ll 1$ and $\beta Q_0 \gg 1$. In the first case, the main contribution to the free energy comes from large $|x|$ where one can neglect the spectral gap and make the following approximation:

\[
Q_0 \cosh(\pi x/2) \approx \exp(-\pi |x|/2 + \ln Q_0).
\]

(A4)

The free energy at such $\beta$ is a free energy of a theory with linear spectrum. It has been shown that this theory is the Wess-Zumino-Witten model on the $SU(2)$ group. The free energy was calculated in Refs. 23 and 24:

\[
\frac{F_r}{L} = -\frac{\pi}{6} - 3r + 2 \beta^{-2}. \quad (A5)
\]

The replica limit exists:

\[
\frac{F}{L} = -\frac{\pi}{4} \beta^{-2}. \quad (A6)
\]
In the second case, we shall expand $F_r$ order by order in $\exp(-\beta Q_0)$ and calculate the replica limit of this expansion. The existence of the replica limit at $\beta Q_0 \ll 1$ guarantees a convergence of this expansion. In the first approximation at $\beta Q_0 \gg 1$ one can substitute $\epsilon_r = -\infty$ in Eqs.(A2). The solution for $n \neq r$ has the following form:

$$1 + \exp[\beta \epsilon_n(x)] = \begin{cases} 
  (n - r + 1)^2 & (n > r) \\
  \left( \frac{\sin \pi(n+1)}{\sin \pi r} \right)^2 & (n < r).
\end{cases} \quad (A7)$$

Substituting $\epsilon_{r\pm1}$ into the equation for $\epsilon_r$ we get the second iteration:

$$\beta \epsilon_r^{(2)} = -\beta Q_0 \cosh(\pi x/2) + \ln \left[ 2 \frac{\sin(\pi r/r + 2)}{\sin(\pi/r + 2)} \right]. \quad (A8)$$

Substituting it into Eq.(41) and taking the replica limit, we get the first iteration for the free energy:

$$\frac{F^{(1)}}{L} = -\beta^{-1} \frac{\pi Q_0}{4} \int dx \cosh(\pi x/2) \exp[-Q_0 \beta \cosh(\pi x/2)]$$

$$= -\frac{1}{2} \int_{-\infty}^{\infty} dp \exp[-\beta \sqrt{p^2 + Q_0^2}]. \quad (A9)$$

The first iteration is well defined and contains the scale $Q_0$. It is possible to calculate the next term in our expansion. To do this we solve Eqs.(A2) linearizing them in $d_n(x) = \ln\{1 + \exp[\beta \epsilon_n(x)]\} - \ln\{1 + \exp[\beta \epsilon_n^{(0)}]\}$.

A similar procedure was applied in Ref.25 for the O(3) nonlinear sigma model. Taking the corresponding solutions from Appendix of Ref.25 and substituting $r = 0$ we get the third iteration for $\epsilon_r$ at $r \to 0$:

$$\beta \epsilon_r^{(3)}(x) = -\beta Q_0 \cosh(\pi x/2) + \ln(\pi r)$$

$$-\frac{1}{2} \int dy \frac{1}{\cosh \left[ \pi(x-y)/4 \right]} \exp[-\beta Q_0 \cosh(\pi y/2)]. \quad (A10)$$

From here we get the following expression for the second iteration of the free energy:

$$\frac{F^{(2)}}{L} = \beta^{-1} \frac{\pi Q_0}{8} \int dx dy \exp[-Q_0 \cosh(\pi x/2) - Q_0 \cosh(\pi y/2)] \frac{\cosh \pi(x+y)/4}{\cosh \pi(x-y)/4}. \quad (A11)$$

Thus we see that the procedure does work and the $r = 0$ limit exists.
REFERENCES

* e-mail: ners@fy.chalmers.se

1 P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985).
2 L. P. Gor’kov and P. A. Kalugin, Pis’ma v ZhETF 41, 208 (1985) [JETP Lett. 41, 253 (1985)].
3 K. Ueda and T. M. Rice, in “Theory of Heavy Fermions and Valence Fluctuation”, ed. by T. Kazuya, Springer (1985).
4 F. Gross, B. S. Chandrasekhar, D. Einzel, K. Andres, P. J. Hirschfeld, H. R. Ott, J. Beuers, Z. Fisk and J. L. Smith, Z. Phys. B 64, 175 (1986).
5 F. V. Kusmartsev and A. M. Tsvelik, JETP Lett. 42, 178 (1985).
6 P. A. Lee, Phys. Rev. Lett. 71, 1887 (1993).
7 A. A. Nersesyan, A. M. Tsvelik and F. Wenger, to be published.
8 E. Fradkin, Phys. Rev. B 33, 3257 (1986).
9 B. A. Volkov and O. A. Pankratov, JETP Lett. 42, 145 (1985).
10 G. W. Semenoff, Phys. Rev. Lett. 53, 2449 (1984).
11 M. P. A. Fisher and E. Fradkin, Nucl. Phys. B 251, 457 (1985).
12 A. A. Nersesyan and A. Luther, Nordita report (1988); H. J. Schulz, Phys. Rev. B 39, 2940 (1989); A. A. Nersesyan and G. E. Vanchadze, J. Low Temp. Phys. 77, 293 (1989); A. A. Nersesyan, G. I. Japaridze and I. G. Kimeridze, J. Phys.: Condens. Matter. 3, 3353 (1991).
13 A. A. Abrikosov, L. P. Gor’kov and I. E. Dzyaloshinskii, “Methods of Quantum Field Theory in Statistical Physics”, Prentice-Hall, Englewood Cliffs, NJ 1963.
14 V. L. Berezinsky, Zh. Exp. Teor. Fiz. 22, 7 (1970).
15 A. A. Abrikosov and I. A. Ryzhkin, Adv. Phys. 27, 147 (1978).
16 E. Witten, Comm. Math. Phys. 92, 455 (1984).
17 See, for instance, in “Statistical Field Theory” by C. Itzykson and J.-M. Drouffe, v. 2., Chapter 9. Cambridge University Press, 1989.
18 See, for instance, I. Affleck, Nucl. Phys. B 265 [FS 15], 409 (1986).
19 A. M. Tsvelik, Zh. Eksp. Teor. Fiz. 93, 1329 (1987) [Sov. Phys. JETP 66, 754 (1987)].
20 V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B 247, 83 (1984).
21 A. M. Tsvelik, Phys. Rev. Lett. (1994).
22 I. Affleck and J. B. Marston, Phys. Rev. B 37, 3774 (1988).
23 A. M. Tsvelik, J. Phys. C 18, 159 (1985).
24 I. Affleck, Phys. Rev. Lett. 56, 746 (1986).
25 A. M. Tsvelik, Zh. Eksp. Teor. Fiz. 93, 385 (1987) [Sov. Phys. JETP 66, 221 (1987)].
FIGURES

FIG. 1. Brillouin zone with the four nodes in the quasiparticle spectrum of the 2D $d$-wave superconductor labeled by $1, \bar{1}, 2, \bar{2}$. Typical lines of constant quasiparticle energy are indicated.

FIG. 2. The standard diagrammatic expansion of the self-energy $\Sigma(k, \omega_n)$ for weak static non-magnetic impurity scattering. The full line stands for the propagation of a quasiparticle with a certain momentum and each broken semi-circle corresponds to the Born scattering at the same impurity center. The frequency $\omega_n$ is conserved and therefore omitted.

FIG. 3. The two nodal points for the 2D orbital antiferromagnet (OAF). The Fermi surface of the underlying half-filled tight-binding band is the square $ABCD$ which is also the true Brillouin zone for the OAF. Therefore there are only two independent nodes $1, 2$ in the OAF.