THE EVANS-KRYLOV THEOREM FOR NONLOCAL PARABOLIC FULLY NONLINEAR EQUATIONS

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Abstract. In this paper, we prove the Evans-Krylov theorem for nonlocal parabolic fully nonlinear equations.

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1. Introduction

L. Evans and N. Krylov proved independently an interior regularity for elliptic partial differential equations which states that any solution \( u \in C^2(B_1) \) of a uniformly elliptic and fully nonlinear concave equation
\[
F(D^2 u) = 0 \text{ in the unit ball } B_1 \subset \mathbb{R}^n
\]
satisfies an interior estimate
\[
\| u \|_{C^{2,\alpha}(B_1/2)} \leq C \| u \|_{C^{1,1}(B_1)}
\]
with some universal constants \( C > 0 \) and \( \alpha \in (0,1) \), so-called the Evans-Krylov theorem (see [Ev], [Kr] and [CS2]). Recently, L. Caffarelli and L. Silvestre [CS1] proved a nonlocal version of the Evans-Krylov theorem which describes that any viscosity solution \( u \in L^\infty(\mathbb{R}^n) \) of concave homogeneous equation on \( B_1 \subset \mathbb{R}^n \) formulated by elliptic integro-differential operators of order \( \sigma \in (0,2) \) satisfies an estimate
\[
\| u \|_{C^{\sigma+\alpha}(B_1/2)} \leq C \| u \|_{L^\infty(\mathbb{R}^n)}
\]
with some universal constants \( C > 0 \) and \( \alpha \in (0,1) \). This nonlocal result makes it possible to recover the Evans-Krylov theorem as \( \sigma \to 2^- \). In this paper, we prove a parabolic version of the nonlocal elliptic result of Caffarelli and Silvestre.

We consider the linear parabolic integro-differential operators given by
\[
Lu(x,t) - \partial_t u(x,t) = \text{p.v.} \int_{\mathbb{R}^n} \mu_t(u,x,y)K(y) \, dy - \partial_t u(x,t)
\]
for \( \mu_t(u,x,y) = u(x+y,t) + u(x-y,t) - 2u(x,t) \). Here we write \( \mu(u,x,y) = u(x+y) + u(x-y) - 2u(x) \) if \( u \) is independent of \( t \). We refer the detailed definitions of notations to [CS1, KL1, KL2, KL3]. Then we see that \( Lu(x,t) \) is well-defined.
provided that \( u \in C^{1,1}_x(x, t) \cap B(\mathbb{R}^2_+) \) where \( B(\mathbb{R}^2_+) \) denotes the family of all real-valued bounded functions defined on \( \mathbb{R}^2_+ := \mathbb{R}^n \times (-T, 0] \) and \( C^{1,1}_x(x, t) \) means \( C^{1,1} \) function in \( x \)-variable at a given point \( (x, t) \). Moreover, \( L u(x, t) \) is well-defined even for \( u \in C^{1,1}_x(x, t) \cap L^\infty(L^2_+) \) (see [KL4]).

We say that the operator \( L \) belongs to \( \mathcal{L}_0 = \mathcal{L}_0(\sigma) \) if its corresponding kernel \( K \in \mathcal{K}_0 = \mathcal{K}_0(\sigma) \) satisfies the uniform ellipticity assumption:

\[
(2 - \sigma) \frac{\Lambda}{|y|^{n+\sigma}} \leq K(y) \leq (2 - \sigma) \frac{\Lambda}{|y|^{n+\sigma}}, \quad 0 < \sigma < 2.
\]

Also we say the operator \( L \in \mathcal{L}_0 \) belongs to \( \mathcal{L}_1 = \mathcal{L}_1(\sigma) \) if its corresponding kernel \( K \in \mathcal{K}_1 = \mathcal{K}_1(\sigma) \) satisfies \( K \in C^1 \) away from the origin and satisfies

\[
|\nabla K(y)| \leq \frac{C}{|y|^{n+1+\sigma}}.
\]

Finally we say that the operator \( L \in \mathcal{L}_1 \) belongs to \( \mathcal{L}_2 = \mathcal{L}_2(\sigma) \) if its corresponding kernel \( K \in \mathcal{K}_2 = \mathcal{K}_2(\sigma) \) satisfies \( K \in C^2 \) away from the origin and satisfies

\[
|D^2 K(y)| \leq \frac{C}{|y|^{n+2+\sigma}}.
\]

The maximal operators are defined by

\[
M^+_1 u(x, t) = \sup_{L \in \mathcal{L}_0} L u(x, t) = (2 - \sigma) \int_{\mathbb{R}^n} \Lambda \mu^+_t(u, x, y) - \lambda \mu^-_t(u, x, y) \frac{dy}{|y|^{n+\sigma}},
\]

\[
M^+_1 u(x, t) = \sup_{L \in \mathcal{L}_1} L u(x, t) \quad \text{and} \quad M^+_2 u(x, t) = \sup_{L \in \mathcal{L}_2} L u(x, t).
\]

We shall consider nonlinear integro-differential operators, which originates from stochastic control theory with jump processes related with

\[
\text{InfL}(x, t) = \inf_{\beta \in \mathcal{B}} L^\beta u(x, t),
\]

where \( L^\beta u(x, t) = \text{p.v.} \int_{\mathbb{R}} \mu_t(u, x, y) K(y) dy \) (see [AK, CS1, KL1, KL2, MP, MR] for the elliptic case and [KL3, KL4] for the parabolic case). In this paper, we are mainly interested in the nonlocal parabolic concave equations

\[
\text{InfL}(x, t) - \partial_t u(x, t) = 0 \quad \text{in} \ Q_1.
\]

[Notations and definition] (1) We denote by \( Q_r = B_r \times (-r^\sigma, 0] \) for \( r > 0 \).

(2) The parabolic distance \( d \) for \( P_1 = (x, t) \) and \( P_2 = (y, s) \) is defined to be

\[
d(P_1, P_2) = \begin{cases} 
|x-y|^{\sigma} + |t-s|^{1/\sigma}, & t \leq s, \\
\infty, & t > s.
\end{cases}
\]

For \( (x_0, t_0) \in \mathbb{R}^2_+ \), we set \( B^2(x_0, t_0) = \{(x, t) \in \mathbb{R}^2_+ : d((x, t), (x_0, t_0)) < r\} \).

(3) Let \( f : \mathbb{R}^n \times I \rightarrow \mathbb{R} \) be a continuous function and let \( J := (a, b) \subset I := (-T, 0) \).

Then a function \( u : \mathbb{R}^n \times I \rightarrow \mathbb{R} \) which is upper (lower) semicontinuous on \( \overline{\Omega} \times J \) is said to be a viscosity subsolution (res. viscosity supersolution) of an equation

\[
J u - \partial_t u = f \quad \text{on} \ \Omega \times J
\]

and we write \( J u - \partial_t u \geq f \) (res. \( J u - \partial_t u \leq f \)) on \( \Omega \times J \) in the viscosity sense, if for any \( (x, t) \in \Omega \times J \) there is a neighborhood \( Q_r(x, t) \subset \Omega \times J \) of \( (x, t) \) such that \( J u(x, t) - \partial_t \varphi(x, t) \) is well-defined and \( J u(x, t) - \partial_t \varphi(x, t) \geq f(x, t) \) (res. \( J u(x, t) - \partial_t \varphi(x, t) \leq f(x, t) \)) for \( v = \varphi_{1_{Q_r(x, t)} + u} \equiv 1_{Q_r(x, t)} \) whenever \( \varphi \in C^2(Q_r(x, t)) \) with \( \varphi(x, t) = u(x, t) \) and \( \varphi > u \) (\( \varphi < u \)) on \( Q_r(x, t) \setminus \{(x, t)\} \) exists.

Here, we denote such a function \( \varphi \) by \( \varphi \in C^2_{\Omega \times J}(u; x, t)^+ \) (res. \( \varphi \in C^2_{\Omega \times J}(u; x, t)^- \)).
Also a function \( u \) is called as a \textit{viscosity solution} if it is both a viscosity subsolution and a viscosity supersolution to \( f u - \partial_t u = f \) on \( \Omega \times J \) (see [KL3, KL4]).

(4) We denote by \( \omega_\sigma(y) = 1/(1 + |y|^{n+\sigma}) \) for \( \sigma \in (0,2) \) and we write \( \omega := \omega_\sigma \) for some \( \sigma_0 \in (1,2) \) very close to 1. Let \( \mathcal{F} \) denote the family of all real-valued measurable functions defined on \( \mathbb{R}^n_+ := \mathbb{R}^n \times (-T,0] \). For \( u \in \mathcal{F} \), we define the mixed norm \( \|u\|_{L_T^\infty(L^1_\omega)} \) by

\[
\|u\|_{L_T^\infty(L^1_\omega)} := \sup_{t \in (-T,0]} \int_{\mathbb{R}^n} |u(x,t)| \omega(x) \, dx.
\]

Also we denote by \( L_T^\infty(L^1_\omega) \) the family of all \( \mathcal{F} \)-measurable functions defined on \( \mathbb{R}^n_+ \) such that

\[
\|u\|_{L_T^\infty(L^1_\omega)} < \infty.
\]

We shall now state the main theorem. The following \( C^{\sigma+\alpha} \)-estimate for nonlocal parabolic concave equation for \( \sigma + \alpha \geq 2 \) and \( \sigma \in (1,2) \) makes it possible to recover the well-known Evans-Krylov estimate as \( \sigma \to 2^- \). If \( \sigma + \alpha < 2 \), then \( C^{\sigma+\alpha} \)-estimate is covered by \( C^{1,\beta} \)-estimate in [KL3]. Our proof of the main theorem is based on the nonlocal elliptic results of Silvestre and Caffarelli [CS1] and the regularity results on nonlocal parabolic equations [KL3, KL4].

\textbf{Theorem 1.1.} Let \( u \in C(\mathbb{R}^n_T) \cap L_T^\infty(L^1_\omega) \) be a viscosity solution of the concave equation

\[
\mathbf{I} u - \partial_t u = 0 \quad \text{in } Q_1,
\]

where \( \mathbf{I} \) is defined on \( \mathcal{L}_2(\sigma) \) for \( \sigma \in (0,2) \) as in (1.5). Then there exist a universal constant \( c > 0 \) and some \( \alpha \in (0,1) \) such that

\[
\|u\|_{C^{\sigma+\alpha}(Q_{1/2})} \leq c \|u\|_{L_T^\infty(L^1_\omega)}.
\]

\textit{Remark.} As mentioned above, given any \( \sigma_0 \in (1,2) \) very close to 1, it suffices to prove this theorem only for \( \sigma + \alpha \geq 2 \) and \( \sigma \in [\sigma_0,2) \).

\section{Approximation of solutions and average of subsolutions}

In the first part of this section, we show that any viscosity solution of (1.5) can be approximated by \( C^{2,\alpha} \)-functions solving an approximate equation with the same shape as (1.5), by using a standard regularization argument. This useful result makes it possible to extend an estimate on \( C^{2,\alpha} \)-solutions to the estimate on viscosity solutions by passing to the limit process.

We say that a function \( u : \mathbb{R}^n_T \to \mathbb{R} \) is in \( C^{1,1}(Q(y,s)) \) for \( (y,s) \in \mathbb{R}^n_T \), if there is a constant \( C > 0 \) (independent of \( (x,t) \)) such that

\[
|u(x,t) - u(y,t) - (x-y) \cdot \nabla_y u(y,t)| \leq C |x-y|^2
\]

for all \((x,t) \in Q(y,s)\). We denote by the norm \( \|u\|_{C^{1,1}(Q(y,s))} \) the smallest \( C > 0 \) satisfying (2.1).

The following definitions are the parabolic version corresponding to the elliptic case in [CS1] (see also [KL4]).

\textbf{Definition 2.1.} For a nonlocal parabolic operator \( \mathbf{I} \) and \( \tau \in (0,T] \), we define \( |||\| \) in \( \Omega_\tau \) with respect to some weight \( \omega \) as

\[
|||\| = \sup_{(y,s) \in \Omega_\tau} \sup_{u \in \mathcal{F}^M_{y,s}} \frac{|\mathbf{I} u(y,s)|}{1 + \|u\|_{L_T^\infty(L^1_\omega)} + \|u\|_{C^{1,1}(Q_1(y,s))}}
\]

where \( \mathcal{F}^M_{y,s} = \{ u \in \mathcal{F} \cap C^2(y,s) : \|u\|_{L_T^\infty(L^1_\omega)} \vee \|u\|_{C^{1,1}(Q_1(y,s))} \leq M \} \) for some \( M > 0 \).
Under the parabolic topology, it is natural to consider the partial derivative by \( \alpha \) are some \( \phi \) where 

\[ Q \]

weakly to \( I \) in 

\[ (2.2) \]

the idea of the proof can be applied also to the equations of type 

with respect to the past time defined by 

\[ B \]

Moreover, we have that 

\[ (a) \] Note that the condition \( \lim \]

Remark.

Proof. We observe that if \( L \)

We let 

Throughout this paper, let \( \eta \in (0, 1) \) be a small number and we set \( \epsilon = \eta/2 \).

Lemma 2.2. Let \( u \in C(\mathbb{R}^n) \cap L^\infty_T(L^1) \) be a viscosity solution satisfying the nonlocal parabolic concave equation 

\[ \mathbf{I} u - \partial_t u = 0 \quad \text{in} \quad Q_{1+\eta}, \]

where every \( L_\beta \) belong to the class \( \mathcal{L}_m(\sigma) \) (\( m = 0, 1, 2 \)) for \( \sigma \in (0, 2) \). Then there are some \( \alpha \in (0, 1) \) and a sequence \( \{u^\epsilon\} \subset C^{2,\alpha}(Q_{1+\epsilon}) \) such that 

\[ \lim_{\epsilon \to 0} \sup_{Q_{1+\eta}} |u^\epsilon - u| = 0, \quad \lim_{\epsilon \to 0} \partial_t u^\epsilon = \partial_t u \quad \text{on} \quad B_{1+\epsilon} \times (-1+\epsilon)^\sigma, 0), \]

\[ \lim_{\epsilon \to 0} \partial_\nu u^\epsilon(x, 0) = \partial_\nu u(x, 0) \quad \text{for any} \quad x \in B_{1+\epsilon} \quad \text{and} \]

\[ (2.2) \]

\[ \left\{ \begin{array}{ll}
\mathbf{I}^\epsilon u^\epsilon - \partial_t u^\epsilon = 0 & \text{in} \quad Q_{1+\eta}, \\
u^\epsilon = u & \text{in} \quad \mathbb{R}^n_T \setminus Q_{1+\eta}.
\end{array} \right. \]

Moreover, we have that \( \lim_{\epsilon \to 0} \| \mathbf{I}^\epsilon - I \| = 0 \).

Remark. (a) Note that the condition \( \lim_{\epsilon \to 0} \| \mathbf{I}^\epsilon - I \| = 0 \) implies that \( \mathbf{I}^\epsilon \) converges weakly to \( I \) in \( Q_{1+\eta} \) as in [KL4].

(b) The concavity of the equation is never used in the following proof. In fact, the idea of the proof can be applied also to the equations of type 

\[ \inf_{\alpha} \sup_{\beta} L_{\alpha \beta} u - \partial_\nu u = 0 \quad \text{or} \quad \sup_{\alpha} \inf_{\beta} L_{\alpha \beta} u - \partial_\nu u = 0 \quad \text{in} \quad Q_{1+\eta}. \]

Proof. We observe that if \( L_\beta \in \mathcal{L}_m \) for \( m = 0, 1, 2 \), then \( L_\beta \in \mathcal{L}_m \). For any \( \epsilon \in (0, 1) \), let \( u^\epsilon \) be the viscosity solution of (2.2). Then it follows from Theorem 6.6 [KL4] that \( u^\epsilon \in C^{2,\alpha}(Q_{1+\epsilon}) \) for some \( \alpha \in (0, 1) \).

If \( v \in \mathcal{F}_{M,\alpha}^T \) for \( M > 0 \) and \( (y, s) \in Q_{1+\epsilon} \), then \( \| v \|_{L^\infty_T(L^1)} \vee \| v \|_{C^{1,1}(Q_{1+\epsilon})} \leq M \)

\[ v \in \mathcal{F} \cap C^2(\mathbb{R}^n, Q_{1+\epsilon}), \]

and so we have that 

\[ |v(x, t) - v(y, t) - (x - y) \cdot \nabla_x v(y, t)| \leq \| v \|_{C^{1,1}(Q_{1+\epsilon})} |x - y|^2 \]

for all \( (x, t) \in Q_{1+\epsilon} \). Thus by simple computation we obtain that 

\[ |v(x, t) - v(\mathbf{I} v(x, t)| \leq C \epsilon^{2-\sigma}, \]

so that \( \| \mathbf{I}^\epsilon - I \| \leq C \epsilon^{2-\sigma} \to 0 \) as \( \epsilon \to 0 \) because \( \sigma \in (0, 2) \). Thus by Lemma 4.7 [KL4] we conclude that \( u^\epsilon \) converges to \( u \) uniformly in \( Q_{1+\eta} \) as \( \epsilon \to 0 \).
For $\varepsilon \in (0, 1)$, $h \in (-1, 1)$ and $(x, t) \in Q_{1+\varepsilon}$, we set
\[
ge_{\varepsilon, h}(x, t) = \frac{u^\varepsilon(x, t + h) - u^\varepsilon(x, t)}{h} \quad \text{and} \quad g_h(x, t) = \frac{u(x, t + h) - u(x, t)}{h}.
\]
For every fixed $h \in (0, 1)$, it is easy to check that $g_{\varepsilon, h}$ converges uniformly to $g_h$ on $Q_{1+\varepsilon}$ as $\varepsilon \to 0$, and moreover $g_{\varepsilon, h}$ has a pointwise limit $\partial_t u^\varepsilon$ on $Q_{1+\varepsilon}$ as $h \to 0$. Thus, by commutative property of double limits, $g_h$ has a pointwise limit on $Q_{1+\varepsilon}$ as $h \to 0$, and moreover
\[
\partial_t u(x, t) = \lim_{h \to 0} g_h(x, t) = \lim_{\varepsilon \to 0} \lim_{h \to 0} g_{\varepsilon, h}(x, t) = \lim_{\varepsilon \to 0} \partial_t u^\varepsilon(x, t)
\]
for any $(x, t) \in B_{1+\varepsilon} \times ((-1 + \varepsilon)^n, 0)$ and $\lim_{\varepsilon \to 0} \partial_t u^\varepsilon(x, 0) = \partial_t^- u(x, 0)$ for any $x \in B_{1+\varepsilon}$. Hence we are done. $\square$

From Lemma 2.2 and Theorem 2.2 [KL3], we can easily derive the following corollary which shall be useful in the final step of the proof of the main theorem.

**Corollary 2.3.** If $u \in C(\mathbb{R}^n_T) \cap L^\infty_T(L^1_V)$ be a viscosity solution of the nonlocal parabolic concave equation
\[
I u - \partial_t u = 0 \quad \text{in} \quad Q_{1+\eta},
\]
where every $L_\beta$ belong to $\Sigma_m(\sigma)$ $(m = 0, 1, 2)$ for $\sigma \in (0, 2)$, then $I u - \partial_t u$ is well-defined on $Q_{1+\eta}$ in the classical sense and
\[
I u(x, t) - \partial_t u(x, t) = 0 \quad \text{for any} \quad (x, t) \in Q_{1+\eta}.
\]

In the second part, we shall show that an averages of viscosity subsolutions to the nonlocal parabolic concave equation is a viscosity subsolution to the same equation. This implies that the convolution of the viscosity subsolution with a mollifier with compact support is also a viscosity subsolution, which shall be very useful in obtaining local uniform boundedness of linear operators in Section 5.

**Proposition 2.4.** If $u, v \in C(\mathbb{R}^n_T) \cap L^\infty_T(L^1_V)$ be viscosity subsolutions satisfying the concave equations $I u - \partial_t u = 0$ and $I v - \partial_t v = 0$ in $\Omega \times I$, then we have that
\[
I \left(\frac{u + v}{2}\right) - \partial_t \left(\frac{u + v}{2}\right) \geq 0 \quad \text{in} \quad \Omega \times I
\]
in the viscosity sense. In particular, if $u \in C(\mathbb{R}^n_T) \cap L^\infty_T(L^1_V)$ is a viscosity solution of the concave equation $I u - \partial_t u = 0$ in $Q_1$ and $\varphi \in C^\infty(\mathbb{R}^n)$ is a mollifier supported in a small ball $B_\delta$ such that $\varphi \geq 0$ and $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$, then $I(\varphi \ast u) - \partial_t(\varphi \ast u) \geq 0$ in $Q_1$ in the viscosity sense.

**Remark.** Note that the convolution $\varphi \ast u$ of $\varphi$ and $u$ means
\[
\varphi \ast u(x, t) = \int_{\mathbb{R}^n} \varphi(x - y) u(y, t) \, dy = \int_{\mathbb{R}^n} u(x - y, t) \varphi(y) \, dy, \quad x \in \mathbb{R}^n, t \in (-T, 0].
\]

**Proof.** We consider approximate equations $I^\varepsilon u^\varepsilon - \partial_t u^\varepsilon = 0$ and $I^\varepsilon v^\varepsilon - \partial_t v^\varepsilon = 0$ in $Q_{1+\varepsilon}$ with boundary values as in (2.2). By Lemma 2.2, we see that $u^\varepsilon, v^\varepsilon \in C^2(Q_{1+\varepsilon})$ and $u^\varepsilon, v^\varepsilon$ converges uniformly to $u, v$ in $Q_{1+\varepsilon}$, respectively. Thus the operators $L_\beta^\varepsilon u^\varepsilon, L_\beta^\varepsilon v^\varepsilon$ are well-defined and continuous on $Q_{1+\varepsilon}$. Now it follows from simple computation that
\[
I^\varepsilon \left(\frac{u^\varepsilon + v^\varepsilon}{2}\right) - \partial_t \left(\frac{u^\varepsilon + v^\varepsilon}{2}\right) \geq \inf_{\beta \in \Lambda} L_\beta^\varepsilon u^\varepsilon + \inf_{\beta \in \Lambda} L_\beta^\varepsilon v^\varepsilon - \partial_t \left(\frac{u^\varepsilon + v^\varepsilon}{2}\right)
\]
\[
= \left( I^\varepsilon u^\varepsilon - \partial_t u^\varepsilon \right) + \left( I^\varepsilon v^\varepsilon - \partial_t v^\varepsilon \right) \geq 0 \quad \text{in} \quad \Omega \times I
\]
in the viscosity sense. Since it is obvious that \( \lim_{\varepsilon \to 0} \| u^\varepsilon - u \|_{L^\infty_q(L^1_\sigma)} = 0 \) and \( \lim_{\varepsilon \to 0} \| u^\varepsilon - v \|_{L^\infty_q(L^1_\sigma)} = 0 \), by Lemma 4.3 \[KL4\] and Lemma 2.2 we obtain the first required result. Finally, the second part is a natural by-product of the first part we obtained just before in the above. \( \square \)

3. Linear parabolic integro-differential equations

In this section, we shall obtain regularity results for linear parabolic integro-differential equations much better than those for the nonlinear equations.

\textbf{Theorem 3.1.} Let \( L \) be a linear integro-differential operator in the class \( \mathcal{L}_1(\sigma) \) for \( \sigma \in (\sigma_0, 2) \) with \( \sigma_0 \in (1, 2) \). If \( u \in L^\infty_q(L^1_\sigma) \) is a viscosity solution of

\[ Lu - \partial_t u = 0 \quad \text{in} \quad Q_{1+\eta}, \]

then \( u \in C^{2,\alpha}(Q_{1+\epsilon}) \) and moreover there is a constant \( C > 0 \) and \( \alpha \in (0, 1) \) (depending only on \( n, \lambda, \sigma, \eta \) and \( \sigma_0 \) but not on \( \sigma \)) such that

\[ \| u \|_{C^{2,\alpha}(Q_{1+\epsilon})} \leq C \left( \sup_{Q_{1+\eta}} |u| + \| u \|_{L^\infty_q(L^1_\sigma)} \right). \]

\textbf{Proof.} Applying Theorem 2.6 in \[KL4\], we see that there is a constant \( C > 0 \) and \( \alpha \in (0, 1) \) (depending only on \( n, \lambda, \sigma, \eta \) and \( \sigma_0 \) but not on \( \sigma \)) such that \( u \in C^{1,\alpha}(Q_{1+\epsilon}) \) and

\[ \| u \|_{C^{1,\alpha}(Q_{1+\epsilon})} \leq C \left( \sup_{Q_{1+\eta}} |u| + \| u \|_{L^\infty_q(L^1_\sigma)} \right). \]

We note that \( Lu(x, t) - \partial_t u(x, t) = 0 \) for \( (x, t) \in Q_{1+\epsilon} \) where \( u \) means the weak derivative of \( u \) in the direction \( e \). Also, by (3.1), we note that \( u \) coincides with the strong type directional derivative of \( u \) in the direction \( e \) on \( Q_{1+\epsilon} \).

Next we show that \( u \in L^\infty_q(L^1_\sigma) \). For \( (x, t) \in Q_{1+\epsilon} \), we consider a function \( w \in C^1_0(\mathbb{R}^n) \) such that \( w(y) = 1 \) for \( |y| < 1/2 \), \( |u(y)| \leq 1 \) and \( w(y) \geq 1 \) for \( 1/2 \leq |y| < 1 \), and \( w(y) = K(y) \) for \( |y| \geq 1 \). Take any \( (x, t) \in Q_{1+\epsilon} \). Then by integration by parts and (1.3), we have that

\[ \left| \int_{\mathbb{R}^n} u_e(y, t)w(y) \, dy \right| = \int_{\mathbb{R}^n} u(y, t)w(y) \, dy \]

\[ \leq \int_{|y| < 1/2} |u(y, t)| \, dy + \int_{|y| \geq 1} |u(y, t)| \, |e, \nabla K(y)| \, dy \]

\[ \leq C \left( \sup_{Q_{1+\eta}} |u| + \| u \|_{L^\infty_q(L^1_\sigma)} \right). \]

This implies that \( (u_e)^+, (u_e)^- \in L^1(w \, dy) \). Thus we see that \( |u_e| \in L^1(w \, dy) \). Moreover we conclude that \( u_e \in L^\infty_q(L^1_\sigma) \).

Then it follows from Theorem 2.6 \[KL4\] that \( u_e \in C^{1,\alpha}_x(Q_{1+\epsilon}) \). Thus we obtain that \( u \in C^{2,\alpha}_x(Q_{1+\epsilon}) \). Here we note that we could choose some \( \alpha > 0 \) so that \( \alpha < \sigma_0 - 1 \) in Theorem 2.6 \[KL4\]. Since \( (2 + \alpha)/\sigma > 1 \) for such \( \alpha > 0 \), we see that

\[ \frac{2 + \alpha - \sigma}{\sigma} + 1 = \frac{2 + \alpha}{\sigma} \]

and \( 0 < \alpha < 2 + \alpha - \sigma < 1 \). Since \( 0 < 2 + \alpha - \sigma < 1 < 1 + \alpha \), by (3.1) we can obtain that \( u \) is \( \frac{2 + \alpha - \sigma}{\sigma} \)-Hölder continuous in the \( t \)-direction. We consider the difference quotients in the \( t \)-direction

\[ w^r(x, t) = \frac{u(x, t + \tau) - u(x, t)}{\tau}. \]
By applying the idea of the proof of Theorem 6.6 [KL4], the $C^{1, \frac{2+\alpha}{2}}_t$-regularity of $u$ can be achieved. This implies that $u \in C^{2,\alpha}_t(Q_{1+\epsilon})$. Therefore we conclude that $u \in C^{2,\alpha}(Q_{1+\epsilon})$. \hfill \Box

Let $\mathcal{F}$ denote the family of all real-valued measurable functions defined on $\mathbb{R}^n_t$. Then we introduce a function space $L^\infty_t(L^2_x)$ consisting of all $f \in \mathcal{F}$ satisfying

$$\sup_{t \in (-T,0]} \left( \int_{\mathbb{R}^n} |f(x,t)|^2 \, dx \right)^{\frac{1}{2}} < \infty.$$ 

**Theorem 3.2.** If $\|L_0 u\|_{L^\infty_t(L^2_x)} < \infty$ for some $L_0 \in \mathcal{L}_0(\sigma)$ with $\sigma \in (0,2)$ and $u \in \mathcal{F}$, then we have that

$$\sup_{L \in \mathcal{L}_0(\sigma)} \|L u\|_{L^\infty_t(L^2_x)} \leq C \inf_{L \in \mathcal{L}_0(\sigma)} \|L u\|_{L^\infty_t(L^2_x)}$$

for a universal constant $C > 0$ possibly depending on $\lambda, \Lambda$ and the dimension $n$.

**Proof.** If we denote the Fourier transform $\widehat{u}$ of $u \in \mathcal{F}$ in terms of $x$-variable given by $\widehat{u}(\xi,t) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x,t) \, dx$, then it easily follows from Plancherel’s Theorem that

$$\widehat{u}(\xi,t) = \left( -\int_{\mathbb{R}^n} 2(1-\cos(y \cdot \xi)) K(y) \, dy \right) \widehat{u}(\xi,t) := -m(\xi) \widehat{u}(\xi,t)$$

for any $L \in \mathcal{L}_0(\sigma)$. By simple computation as in [CSI], we have that

$$\frac{1}{c_0} |\xi|^\sigma \leq m(\xi) \leq c_0 |\xi|^\sigma$$

for a universal constant $c_0 > 0$ possibly depending on $\lambda, \Lambda$ and the dimension $n$, but not depending on $t$. Applying standard harmonic analysis, there is a universal constant $C > 0$ possibly depending on $\lambda, \Lambda$ and the dimension $n$, but not depending on $t$ such that

$$\sup_{t \in (-T,0]} \sup_{v(\cdot,t) \parallel L^2(\mathbb{R}^n)} \frac{\|L_1 \circ L_2^{-1} v(\cdot,t)\|_{L^2(\mathbb{R}^n)}}{\|v(\cdot,t)\|_{L^2(\mathbb{R}^n)}} = \|m_1 m_2^{-1}\|_{L^\infty(\mathbb{R}^n)} < C < \infty$$

for any $L_1, L_2 \in \mathcal{L}_0(\sigma)$, where $m_1$ and $m_2^{-1}$ denote the symbols of $L_1$ and the inverse $L_2^{-1}$ of the operator $L_2$, respectively. Hence this implies the required result. \hfill \Box

Let $s$ be a real number. Then the homogeneous mixed Sobolev space $L^\infty_t(\mathcal{H}^s_x)$ is defined as the function space of all $f \in \mathcal{F}$ satisfying

$$\|f\|_{L^\infty_t(\mathcal{H}^s_x)} := \sup_{t \in (-T,0]} \left( \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi,t)|^2 \right)^{\frac{1}{2}} < \infty.$$ 

For $p_\sigma := 2n/(n-2\sigma)$ with $\sigma \in (0,2)$, we define a function space $L^\infty_t(L^p_x)$ consisting of all $f \in \mathcal{F}$ satisfying

$$\|f\|_{L^\infty_t(L^p_x)} := \sup_{t \in (-T,0]} \left( \int_{\mathbb{R}^n} |f(x,t)|^{p_\sigma} \right)^{\frac{1}{p_\sigma}} < \infty.$$ 

For $r > 0$, we consider the function space of all measurable functions $f$ on $Q_r$ such that

$$\|f\|_{L^\infty_t(L^r_x)_{Q_r}} := \sup_{t \in (-T,0]} \left( \int_{B_r} |f(x,t)|^r \right)^{\frac{1}{r}} < \infty.$$
Theorem 3.3. Suppose that a function \( u \in C(\mathbb{R}^n) \cap L^\infty_T(L^1_n) \) is a viscosity solution of the equation

\[
L_0 u - \partial_t u = h \quad \text{in } Q_{1+\eta}
\]

for some \( h \in L^\infty_T(L^2_n) \), where \( L_0 \in \mathcal{L}_0(\sigma) \) for \( \sigma \in [\sigma_0, 2) \) with \( \sigma_0 \in (1, 2) \). Then there exist a solution \( v \in L^\infty_T(H^2_f) \) of the equation \( L_0 v - \partial_t v = h \mathbb{I}_{Q_{1+\eta}} \) in \( \mathbb{R}^n_T \) and a constant \( C > 0 \) depending on \( n, \lambda, \Lambda, \eta \) and \( \sigma_0 \), but not depending on \( u \) such that

\[
\sup_{L \in \mathcal{L}_0(\sigma)} \| Lu \|_{L^\infty_T(L^2_\infty(Q_{1+\eta}))} \leq C \left( \sup_{Q_{1+\eta}} |u - v| + \| u - v \|_{L^\infty_T(L^2_\infty)} + \| h \|_{L^\infty_T(L^2_\infty)} + \| v \|_{L^\infty_T(H^2_f)} \right).
\]

Proof. Take any \( L \in \mathcal{L}_0(\sigma) \) for \( \sigma \in [\sigma_0, 2) \) with \( \sigma_0 \in (1, 2) \). Let \( v \in L^\infty_T(H^2_f) \) be a solution of the equation \( L_0 v - \partial_t v = h \mathbb{I}_{Q_{1+\eta}} \) in \( \mathbb{R}^n_T \). By the Sobolev embedding theorem, we see that

\[
(3.3) \quad v \in L^\infty_T(H^2_f) \subset L^\infty_T(L^p_n).
\]

Since \( v \in L^\infty_T(H^2_f) \) is equivalent to \((-\Delta)^{\sigma/2} v \in L^\infty_T(L^2_n)\), it follows from Lemma 3.2 that \( L_0 v \in L^\infty_T(L^2_n) \), and so \( \partial_t v \in L^\infty_T(L^2_n) \). By Hölder’s inequality and (3.3), we have that \( v \in L^\infty_T(L^1_n) \). From Theorem 3.1, we obtain that

\[
(3.4) \quad \| u - v \|_{C^{0,\alpha}(Q_{1+\eta})} \leq C \left( \sup_{Q_{1+\eta}} |u - v| + \| u - v \|_{L^\infty_T(L^2_\infty)} \right).
\]

Since \( \mu_t(u - v, x, y) = \int_0^1 (D^2(u - v)((x + \tau y) - 2s\tau y, t) y, y) \, ds \, d\tau \) by the mean value theorem, we have that

\[
|\mu_t(u - v, x, y)| \leq C \left( \sup_{Q_{1+\eta}} |u - v| + \| u - v \|_{L^\infty_T(L^2_\infty)} \right) |y|^2
\]

for any \((x, t) \in Q_{1+\eta}\) and \( y \in B_{\frac{1}{4} + \epsilon}. \) So we get that

\[
(3.5) \quad |L(u - v)(x, t)| \leq C \left( \sup_{Q_{1+\eta}} |u - v| + \| u - v \|_{L^\infty_T(L^2_\infty)} \right) \int_{|y| < \frac{1}{4} + \epsilon} |y|^2 K(y) \, dy
\]

\[
+ |u - v| * K_\epsilon(y) \leq C \left( \sup_{Q_{1+\eta}} |u - v| + \| u - v \|_{L^\infty_T(L^2_\infty)} \right)
\]

for any \((x, t) \in Q_{1+\eta}\), where \( K_\epsilon(y) = \mathbb{1}_{\mathbb{R}^n \setminus B_{\frac{1}{4} + \epsilon}}(y) K(y). \) This implies that

\[
\| L(u - v) \|_{L^\infty_T(L^2_n(Q_{1+\eta}))} \leq C \left( \sup_{Q_{1+\eta}} |u - v| + \| u - v \|_{L^\infty_T(L^2_\infty)} \right).
\]

Hence we conclude that

\[
\sup_{L \in \mathcal{L}_0(\sigma)} \| Lu \|_{L^\infty_T(L^2_\infty(Q_{1+\eta}))} \leq C \left( \sup_{Q_{1+\eta}} |u - v| + \| u - v \|_{L^\infty_T(L^2_\infty)} + \| h \|_{L^\infty_T(L^2_\infty)} + \| v \|_{L^\infty_T(H^2_f)} \right). \quad \Box
\]

4. Local uniform upper boundedness of viscosity subsolutions

In this section, local uniform upper boundedness of viscosity subsolutions in \( L^\infty_T(L^1_n) \) will be achieved by using almost the same idea of the proof of the Harnack inequality in [KL3].

Theorem 4.1. Let \( \sigma \in (1, 2) \). If \( u \in L^\infty_T(L^1_n) \cap C(\overline{Q}_2) \) satisfies the equation

\[
M^+ u - \partial_t u \geq -\| u \|_{L^\infty_T(L^2_\infty)} \quad \text{in } Q_2
\]

in the viscosity sense, then there is a universal constant \( C > 0 \) such that

\[
\sup_{Q_{1/2}} u \leq C \| u \|_{L^\infty_T(L^2_\infty)}.
\]
Proof. Without loss of generality, we may assume that $u \in B(\mathbb{R}^n_T)$. Indeed, if we set $u_1 = u|_{Q_2}$ and $u_2 = u|_{\mathbb{R}^n_T \setminus Q_2}$, then it easily follows that

$$M_0^+ u_1 - \partial_t u_1 \geq -\|u\|_{L^\infty(L^n_1)} \text{ in } Q_2.$$  

Since $u$ is continuous on $\overline{Q}_2$, $u_1$ is bounded on $\mathbb{R}^n_T$. So we could use $u_1$ instead of $u$. Also we may assume that $\|u\|_{L^\infty(L^n_1)} = 1$ by dividing $u$ by the norm $\|u\|_{L^\infty(L^n_1)}$. Thus it suffices to show that $\sup_{Q_{1/2}} u \leq C$. If $u$ is non-positive on $Q_{1/2}$, then there is nothing to prove it. Thus we may now suppose that $u$ is non-negative on $Q_{1/2}$. We set $s_0 = \inf\{s > 0 : u(x,t) \leq s d((x,t), \partial_pQ_1)^{-n-\sigma}, \forall (x,t) \in Q_1\}$. Then we see that $s_0 > 0$ and there is some $(\tilde{x}, \tilde{t}) \in Q_1$ such that

$$u(\tilde{x}, \tilde{t}) = s_0 d((\tilde{x}, \tilde{t}), \partial_pQ_1)^{-n-\sigma} = s_0 d_0^{-n-\sigma}$$

where $d_0 = d((\tilde{x}, \tilde{t}), \partial_pQ_1) \leq 2^{1/\sigma} < 2$ for $\sigma \in (1, 2)$. We note that

$$B^d_r(x_0, t_0) \subset Q_r(x_0, t_0) \subset B^d_{2r}(x_0, t_0)$$

for any $r > 0$ and $(x_0, t_0) \in \mathbb{R}^n_T$.

To finish the proof, we have only to show that $s_0$ cannot be too large because $u(x,t) \leq C_1 d((x,t), \partial_pQ_1)^{-n-\sigma} \leq C$ for any $(x,t) \in Q_{1/2} \subset Q_1$ if $C_1 > 0$ is some constant with $s_0 \leq C_1$. Assume that $s_0$ is very large. Then by Chebyshev’s inequality we have that

$$\left|\left\{u \geq u(\tilde{x}, \tilde{t})/2\right\} \cap Q_1\right| \leq \frac{2}{u(\tilde{x}, \tilde{t})} \|u\|_{L^\infty(L^n_1)} \leq Cs_0^{-1} d_0^{n+\sigma}.$$  

Since $B^d_r(\tilde{x}, \tilde{t}) \subset Q_1$ and $|B^d_r| = C d_0^{n+\sigma}$ for $r = d_0/2 \leq 2^{-(1-1/\sigma)} < 1$ for $\sigma \in (1, 2)$, we easily obtain that

$$\left|\left\{u \geq u(\tilde{x}, \tilde{t})/2\right\} \cap B^d_r(\tilde{x}, \tilde{t})\right| \leq Cs_0^{-1} |B^d_r|.$$

In order to get a contradiction, we estimate $|\{u \leq u(\tilde{x}, \tilde{t})/2\} \cap B^d_{\delta/2}(\tilde{x}, \tilde{t})|$ for some very small $\delta > 0$ (to be determined later). For any $(x,t) \in B^d_{\delta/2}(\tilde{x}, \tilde{t})$, we have that $u(x,t) \leq s_0(d_0 - \delta d_0)^{-n-\sigma} = u(\tilde{x}, \tilde{t})(1 - \delta)^{-n-\sigma}$ for $\delta > 0$ so that $(1 - \delta)^{-n-\sigma}$ is close to 1. We consider the function

$$v(x,t) = \frac{u(x,t)}{(1 - \delta)^{n+\sigma}} - u(x,t).$$

Then we see that $v \geq 0$ on $B^d_{2\delta r}(\tilde{x}, \tilde{t})$, and also $M_0^- v - \partial_t v \leq 1$ on $Q_{3\delta r}(\tilde{x}, \tilde{t})$ because $M_0^+ u - \partial_t u \geq -1$ on $Q_{3\delta r}(\tilde{x}, \tilde{t})$. In order to apply Theorem 4.12 [KL3] to $v$, we consider $w = v^+$ instead of $v$. Since $w = v + v^-$, we have that

$$M_0^- w - \partial_t w \leq M_0^- v - \partial_t v + M_0^+ v^+ - \partial_t v^- \leq 1 + M_0^+ v^- - \partial_t v^-$$

on $Q_{3\delta r}(\tilde{x}, \tilde{t})$. Since $v^- \equiv 0$ on $B^d_{3\delta r}(\tilde{x}, \tilde{t})$, if $(x,t) \in Q_{3\delta r}(\tilde{x}, \tilde{t})$ then we have that $\mu_1 (v^-, x, y) = v^- (x+y,t) + v^- (x-y,t)$ for $y \in \mathbb{R}^n$. 


Thus by (4.3), we obtain that
\[ \mathbf{M}_0^+ v^-(x, t) - \partial_t \varphi(x, t) = (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda u^+(v^-, x, y) - \lambda \mu^-(v^-, x, y)}{|y|^{n+\sigma}} \, dy \]
\[ \leq 2(2 - \sigma) \Lambda \int_{y \in \mathbb{R}^n: v(x+y,t) < 0} \frac{-v(x+y,t)}{|y|^{n+\sigma}} \, dy \]
\[ \leq 2(2 - \sigma) \Lambda \int_{B^r_{\delta r}} \frac{(u(x+y,t) - (1 - \delta)^{-n-\sigma} u(\bar{x}, \bar{t}))^+}{|y|^{n+\sigma}} \, dy \]
\[ \leq C(2 - \sigma) \Lambda ((\delta r)^{-n-\sigma} + 1) \int_{\mathbb{R}^n} \frac{|u(y, t)|}{1 + |y|^{n+\sigma}} \, dy. \]

This implies that
\[ \mathbf{M}_0^+ v^- - \partial_t v^- \leq C\|u\|_{L^\infty(L^\infty)} ((\delta r)^{-n-\sigma} \leq C(\delta r)^{-n-\sigma} \text{ on } Q_{\delta r}(\bar{x}, \bar{t}). \]

Thus by (4.3), we obtain that \( w \) satisfies
\[ \mathbf{M}_0^- w(x, t) - \partial_t w \leq C(\delta r)^{-n-\sigma} \text{ on } Q_{\delta r}(\bar{x}, \bar{t}) \]
in viscosity sense. Since \( u(\bar{x}, \bar{t}) = s_0q_0^{-\beta} = 2 - \beta s_0r^{-\beta}, \) by Theorem 4.12 [KL3] there is some \( \varepsilon_* > 0 \) such that
\[ \{u \leq u(\bar{x}, \bar{t})/2\} \cap B^d_{\delta r/2}(\bar{x}, \bar{t}) \leq \{(u \leq u(\bar{x}, \bar{t})/2\} \cap Q_{\delta r/2}(\bar{x}, \bar{t}) \]
\[ = \{|w \geq u(\bar{x}, \bar{t})((1 - \delta)^{-\beta} - 1/2)) \cap Q_{\delta r/2}(\bar{x}, \bar{t})| \leq C(\delta r)^{n+\sigma} \left\{ (1 - \delta)^{-\beta} - 1/2 \right\} u(\bar{x}, \bar{t}) + C(\delta r)^{-\sigma} (\delta r)^{n+\sigma} \] \[ \times \left\{ u(\bar{x}, \bar{t})((1 - \delta)^{-\beta} - 1/2) \right\} \varepsilon_* \]
\[ \leq C(\delta r)^{n+\sigma} \left\{ \left( \frac{(1 - \delta)^{-\beta} - 1}{(1 - \delta)^{-\beta} - 1/2} \right)^{\varepsilon_*} + s_0^{-\varepsilon_*} \varepsilon_*^{n+\sigma} \right\} \]
\[ \leq C(\delta r)^{n+\sigma} \left\{ ((1 - \delta)^{-\beta} - 1)^{\varepsilon_*} + s_0^{-\varepsilon_*} \varepsilon_*^{n+\sigma} \right\}. \]

We now choose \( \delta > 0 \) so small enough that \( C(\delta r)^{n+\sigma} ((1 - \delta)^{-\beta} - 1)^{\varepsilon_*} \leq |B^d_{\delta r/2}|/4. \)

Since \( \delta \) was chosen independently of \( s_0, \) so \( s_0 \) is large enough for such fixed \( \delta \) then we get that \( C(\delta r)^{n+\sigma} s_0^{-\varepsilon_*} \varepsilon_*^{n+\sigma} \leq |B^d_{\delta r/2}|/4. \) Therefore we obtain that
\[ \{u \leq u(\bar{x}, \bar{t})/2\} \cap B^d_{\delta r/2}(\bar{x}, \bar{t}) \leq |B^d_{\delta r/2}|/2. \]

Thus we conclude that
\[ \{u \geq u(\bar{x}, \bar{t})/2\} \cap B^d_{\delta r}(\bar{x}, \bar{t}) \geq \{u \geq u(\bar{x}, \bar{t})/2\} \cap B^d_{\delta r/2}(\bar{x}, \bar{t}) \]
\[ \geq \{u > u(\bar{x}, \bar{t})/2\} \cap B^d_{\delta r/2}(\bar{x}, \bar{t}) \]
\[ \geq |B^d_{\delta r/2}(\bar{x}, \bar{t})| - |B^d_{\delta r/2}|/2 \]
\[ = |B^d_{\delta r/2}|/2 = C|B^d_{\delta r}|, \]

which contradicts (4.2) if \( s_0 \) is large enough. Hence we complete the proof. \( \square \)

5. LOCAL UNIFORM BOUNDEDNESS OF LINEAR OPERATORS

The main theme of this section is to establish local uniform boundedness of linear operators from the result obtained in Section 4, which facilitates obtaining local uniform boundedness of extremal operators to be given in the next section.
Lemma 5.1. Let $u \in C(\mathbb{R}^n_+) \cap L_+^{\infty}(L^1_>)$ be a viscosity solution satisfying the equation $\text{I}u - \partial_t u = 0$ in $Q_1$. If $K$ is a symmetric kernel satisfying $K(y) \leq (2 - \sigma)\Lambda|y|^{-n-\sigma}$, then for any radial cut-off function $\varphi \in C^\infty_c(\mathbb{R}^n)$ supported in $B_1$ and with $0 \leq \varphi \leq 1$ in $\mathbb{R}^n$, we have that

$$M^+_2 u \varphi - \partial_t u \varphi \geq 0 \text{ in } Q_1$$

in the viscosity sense, where

$$u \varphi(x,t) = \int_{\mathbb{R}^n} \mu_t(u,x,y)K(y)\varphi(y)\,dy.$$ 

Proof. By Lemma 4.3 [KL4] and Lemma 2.2, without loss of generality we may assume that $u \in C^{2,\alpha}(Q_1)$ for some $\alpha \in (0,1)$ as in the above. So we see that $u \in C^{2,\alpha}_c(Q_1) \cap C^1_c(Q_1)$, and moreover integro-differential type operators like $u_{\varphi}$ are well-defined and continuous in $Q_1$. For $\ell \in \mathbb{N}$, we set $\varphi_\ell(y) = 1_{B_1/\ell}(y)K(y)\varphi(y)$. Then we see that $\varphi_\ell \in L^1(\mathbb{R}^n)$ for all $\ell \in \mathbb{N}$. By Lebesgue’s dominated convergence theorem, we have that

$$u \varphi = \lim_{\ell \to \infty} \int_{\mathbb{R}^n} \mu_t(u,\cdot,y)\varphi_\ell(y)\,dy = 2 \lim_{\ell \to \infty} u \ast \varphi_\ell - u \|\varphi_\ell\|_{L^1}.$$ 

Now it follows from Lemma 2.3 that

$$\text{I} \left( u \ast \frac{\varphi_\ell}{\|\varphi_\ell\|_{L^1}} \right) - \partial_t \left( u \ast \frac{\varphi_\ell}{\|\varphi_\ell\|_{L^1}} \right) \geq 0 \text{ in } Q_1.$$ 

Also we have that $\text{I}u - \partial_t u = 0$ in $Q_1$. Thus by applying Theorem 2.0.4 [KL3], we easily obtain that

$$M^+_2 \left( u \ast \varphi_\ell - u \|\varphi_\ell\|_{L^1} \right) - \partial_t \left( u \ast \varphi_\ell - u \|\varphi_\ell\|_{L^1} \right) = \|\varphi_\ell\|_{L^1} \left[ M^+_2 \left( u \ast \frac{\varphi_\ell}{\|\varphi_\ell\|_{L^1}} - u \right) - \partial_t \left( u \ast \frac{\varphi_\ell}{\|\varphi_\ell\|_{L^1}} - u \right) \right] \geq 0 \text{ in } Q_1$$

for any $\ell \in \mathbb{N}$. Hence we can obtain the required result by taking limit $\ell \to \infty$. \hfill $\square$

Lemma 5.2. Let $u \in C(\mathbb{R}^n_+) \cap L_+^{\infty}(L^1_>)$ be any viscosity solution satisfying the equation $\text{I}u - \partial_t u = 0$ in $Q_2$. Then there is a universal constant $C > 0$ such that

$$M^+_2 (Lu) - \partial_t (Lu) \geq -C \|u\|_{L_+^{\infty}(L^1_>)} \text{ in } Q_1$$

for any $L \in \mathcal{L}_2$.

Proof. By Lemma 4.3 [KL4] and Lemma 2.2, without loss of generality we may assume that $u \in C^{2,\alpha}(Q_1)$ for some $\alpha \in (0,1)$ as in the above. So we see that $u \in C^{2,\alpha}_c(Q_1) \cap C^1_c(Q_1)$. For $\ell \in \mathbb{N}$, let $\eta_\ell(y) = 1_{B_{3/2\ell}(y)}K(y)$. Take any $L \in \mathcal{L}_2$. Then as in Lemma 5.1 we have that

$$Lu = \lim_{\ell \to \infty} \int_{\mathbb{R}^n} \mu_t(u,\cdot,y)\eta_\ell(y)\,dy = 2 \lim_{\ell \to \infty} (u \ast \eta_\ell - u \|\eta_\ell\|_{L^1}).$$ 

Let $\varphi \in C^\infty_c(\mathbb{R}^n)$ be any radial cut-off function supported in $B_2$ such that $\varphi \equiv 1$ in $B_{3/2}$ and $0 \leq \varphi \leq 1$ in $\mathbb{R}^n$. We set $\phi_\ell(y) = \eta_\ell(y)\varphi(y)$ and $\psi_\ell(y) = \eta_\ell(y)(1 - \varphi(y))$. By Lemma 5.1, we have that

$$M^+_2 \left( u \ast \phi_\ell - u \|\phi_\ell\|_{L^1} \right) - \partial_t \left( u \ast \phi_\ell - u \|\phi_\ell\|_{L^1} \right) \geq 0 \text{ in } Q_1.$$
Also we now estimate $I(u * \psi_t) - \partial_t (u * \psi_t)$ in $Q_1$. Take any point $(x, t) \in Q_1$. We note that

$$I(u * \psi_t) - \partial_t (u * \psi_t) = \inf_{\beta} L_\beta (u * \psi_t) - \partial_t (u * \psi_t)$$

$$= \inf_{\beta} u * (L_\beta \psi_t) - \partial_t (u * \psi_t)$$

and

$$u * L_\beta (\psi_t)(x, t) = \int_{\mathbb{R}^n} u(x - y, t) \int_{|z| \leq 2^{-1}} \mu(\psi_t, y, z) K(z) \, dz \, dy$$

$$+ \int_{|z| > 2^{-1}} u(x - y, t) \int_{|z| < 2^{-1}} \mu(\psi_t, y, z) K(z) \, dz \, dy$$

$$:= I(x, t) + II(x, t)$$

by the definition of $\psi_t$. Then it is easy to check that

$$I = 2(u * \psi_t * \eta_2) - 2c (u * \psi_t)$$

for a universal constant $c > 0$. By the mean value theorem and triangle inequality, we see that for any $y \in \mathbb{R}^n \setminus B_1$ and $z \in B_{1/2}$,

$$\mu(\psi_t, y, z) = \int_0^1 \int_0^1 |D^2 \psi_t((y + \tau z) - 2s\tau z), z| \, d\tau, ds$$

$$|y + \tau z - 2s\tau z| = |y + \tau(1 - 2s)z| \geq |y| - |z| \geq |y|/2.$$  

Since $D^2 \psi_t = (D^2 \eta_\beta)(1 - \varphi) - 2(D\eta_\beta)(D\varphi) - \eta_\beta(D^2 \varphi)$, by (1.2) and (1.4) we have that

$$|D^2 \psi_t((y + \tau z) - 2s\tau z, t)| \leq \frac{C}{|y|^{n+\sigma}} \mathbb{1}_{\mathbb{R}^n \setminus B_1}(y) := k(y)$$

for any $y \in \mathbb{R}^n \setminus B_1$, $z \in B_{1/2}$ and $s, \tau \in [0, 1]$. Thus we obtain that

$$|II(x, t)| \leq |u| * k(x, t) \int_{|z| < 2^{-1}} |z|^2 K(z) \, dz \leq C |u| * k(x, t) \leq C \|u\|_{L^\infty(L^1)}$$

for a universal constant $C > 0$. Hence it easily follows from (5.2), (5.3) and Young's inequality that

$$|u * L_\beta (\psi_t)(x, t)| \leq C \|u\|_{L^\infty(L^1)}$$

for any $\beta$, and thus we have that

$$I(u * \psi_t)(x, t) \geq -C \|u\|_{L^\infty(L^1)}.$$  

Since $u \in C_1^1(Q_1)$, as in the above estimate we can obtain that

$$\partial_t (u * \psi_t)(x, t) = (\partial_t u) * \psi_t(x, t) = (Iu) * \psi_t(x, t)$$

$$\leq (I_\beta u) * \psi_t(x, t) = u * (I_\beta \psi_t)(x, t)$$

$$\leq C \|u\|_{L^\infty(L^1)}.$$  

Hence by (5.1), (5.5) and (5.6), we conclude that

$$M_2^+ (L u) - \partial_t (Lu) \geq -C \|u\|_{L^\infty(L^1)} \text{ in } Q_1.$$  

Therefore we complete the proof. \qed
Lemma 5.3. If \( u \in C(\mathbb{R}^n) \cap L^\infty_c(L^1_\nu) \) is a viscosity solution satisfying the equation \( Lu - \partial_t u = 0 \) in \( Q_{2+\eta} \), then there is a universal constant \( C > 0 \) such that
\[
Lu \leq C \|u\|_{L^\infty_c(L^1_\nu)} \quad \text{in } Q_{1/4}
\]
for any \( L \in \mathcal{L}_2 \).

Proof. By Lemma 4.3 [KL4] and Lemma 2.2, without loss of generality we may assume that \( u \in C^{2,\alpha}(Q_2) \) for some \( \alpha \in (0,1) \). So we see that \( u \in C^2_c(Q_2) \cap C^1_c(Q_2) \). By Lemma 5.2, we see that there is a universal constant \( C > 0 \) such that
\[
\tag{5.9} M^+_0(Lu) - \partial_t(Lu) \geq M^+_0(Lu) - \partial_t(Lu) \geq -C \|u\|_{L^\infty_c(L^1_\nu)} \quad \text{in } Q_1
\]
for any \( L \in \mathcal{L}_2 \). Since it is easy to check that \( L \) is a nonlocal parabolic operator, we see that \( Lu \in C(Q_2) \) (see [KL4]).

Let \( \varphi \in C_c^\infty(\mathbb{R}^n) \) be a function such that \( 0 \leq \varphi \leq 1 \), \( \varphi = 1 \) in \( B_2 \), \( \varphi = 0 \) in \( \mathbb{R}^n \setminus B_2^{1+\varepsilon} \) and \( |D^2 \varphi| \leq N_0 \) in \( B_2^{1+\varepsilon} \) for some \( N_0 > 0 \). Then by the change of variables we have that
\[
\tag{5.8} \int_{\mathbb{R}^n} Lu(x,t) \varphi(x) \, dx = \int_{\mathbb{R}^n} u(x,t) L\varphi(x) \, dx.
\]
We note that \( |(x + \tau y) - 2s\tau| = |x + \tau(1 - 2s)y| \leq |x| + |y| \) for \( s, \tau \in [0,1] \) and
\[
\mu(\varphi, x, y) = \int_0^1 \int_0^1 (D^2 \varphi((x + \tau y) - 2s\tau)y, y) \, ds \, d\tau
\]
for any \( x \in B_1 \) and \( y \in B_1 \). We now have that
\[
L\varphi(x) = \int_{B_1} \mu(\varphi, x, y) K(y) \, dy + \int_{\mathbb{R}^n \setminus B_1} \mu(\varphi, x, y) K(y) \, dy := b(x) + c(x)
\]
and \( c(x) = 2 \varphi * \eta_1(x) - 2c_0 \varphi(x) \) where \( c_0 = \int_{\mathbb{R}^n \setminus B_1} K(y) \, dy < \infty \) and \( \eta_\tau(y) = \mathbb{1}_{\mathbb{R}^n \setminus B_1}(y)K(y) \). Then it is easy to check that \( |b(x)| \leq N_0 \int_{B_1} |y|^2K(y) \, dy \leq C < \infty \) for \( |x| < 5 \) and \( |b(x)| = 0 \) for \( |x| \geq 5 \), and \( |c(x)| \leq C \) for \( |x| < 5 \) and \( |c(x)| \leq C/|x|^{n+\sigma} \) for \( |x| \geq 5 \), where \( C > 0 \) is a universal constant. So we see that \( |L\varphi(x)| \leq C \omega(x) \) for some universal constant \( C > 0 \). Thus by (5.8), we obtain that
\[
\tag{5.9} \left| \int_{\mathbb{R}^n} Lu(x,t) \varphi(x) \, dx \right| \leq C \|u\|_{L^\infty_c(L^1_\nu)}
\]
for a universal constant \( C > 0 \). We set \( \phi(x) = 1 - \varphi(x) \) and \( w(x,t) = \varphi(x)Lu(x,t) \), and we denote by \( f^\varphi(y) = f(x + y) \). Then (5.9) implies that \( w \in L^\infty_c(L^1_\nu) \cap C(Q_2) \).

We now estimate \( M^+_0 w(x,t) \) for \( x \in B_1 \) and \( t \in (-T,0] \). For this, as in (5.4) we have that
\[
\tag{5.10} \sup_{(x,t) \in Q} |u * L(\phi^\varphi K)(x,t)| \leq C \|u\|_{L^\infty_c(L^1_\nu)},
\]
because \( \phi^\varphi K \) is a smooth function with nice decay such that \( \phi^\varphi K = 0 \) on \( B(x; 1) \) for each \( x \in B_1 \). If \( (x,t) \in Q_1 \), then by the change of variables and (5.10), we have
the estimate
\[ L_\beta w(x, t) = \int_{\mathbb{R}^n} \mu_t(Lu, x, y)K(y)\,dy - \int_{\mathbb{R}^n} \mu_t((Lu)\phi, x, y)K(y)\,dy \]
\[ = \int_{\mathbb{R}^n} \mu_t(Lu, x, y)K(y)\,dy - 2\int_{\mathbb{R}^n} Lu(x + y, t)\phi^\sigma(y)K(y)\,dy \]
\[ = \int_{\mathbb{R}^n} \mu_t(Lu, x, y)K(y)\,dy - 2u * L(\phi^\sigma K)(x, t) \]
\[ \geq \int_{\mathbb{R}^n} \mu_t(Lu, x, y)K(y)\,dy - C\|u\|_{L_\infty^p(L_\beta)} \]
(5.11)

for any $L_\beta \in \mathcal{L}_2$. Hence by (5.7) and (5.11) we conclude that
\[ M^+_\beta w - \partial_t w \geq M^+_2(Lu) - \partial_t(Lu) - C\|u\|_{L_\infty^p(L_\beta)} \geq -C\|u\|_{L_\infty^p(L_\beta)} \] on $Q_1$.

Therefore the required result can be achieved by applying Theorem 4.1. \(\square\)

### 6. Local uniform boundedness of extremal operators

In this section, we show that if $u \in C(\mathbb{R}^n) \cap L_\infty^p(L_\beta)$ is a viscosity solution of the nonlocal parabolic concave equation $Lu - \partial_t u = 0$ in $Q_2$, then $M^+_\beta u$ and $M^+_0 u$ are bounded uniformly on $Q_{1/2}$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. This plays an important role as a cornerstone in proving the main theorem in the final section.

**Lemma 6.1.** Let $u \in C(\mathbb{R}^n) \cap L_\infty^p(L_\beta)$ be a viscosity solution satisfying the equation $Lu - \partial_t u = 0$ in $Q_2$. If $K$ is a symmetric kernel with $K(y) \leq (2 - \sigma)\Lambda|y|^{-n-\sigma}$, then for any function $\gamma \in C^\infty_c([-T, T])$ with $\gamma = 1$ in $(-2^{1-\sigma}, 0]$ and supp($\gamma$) $\subset (-1, \varepsilon]$, and any radial cut-off function $\psi \in C^\infty_c(\mathbb{R}^n)$ supported in $B_2$ such that $\psi = 1$ in $B_{8/5}$, $\psi = 0$ in $\mathbb{R}^n \setminus B_2$ and $0 \leq \psi \leq 1$ in $\mathbb{R}^n$, we have that
\[ M^+_2(\psi \gamma u_\varphi) - \partial_t(\psi \gamma u_\varphi) \geq -C\|u\|_{L_\infty^p(L_\beta)} \]
in the viscosity sense, where
\[ u_\varphi(x, t) = \int_{\mathbb{R}^n} \mu_t(u, x, y)K(y)\varphi(y)\,dy \]
for a radial cut-off function $\varphi \in C^\infty_c(\mathbb{R}^n)$ supported in $B_{1/4}$ with $0 \leq \varphi \leq 1$ in $\mathbb{R}^n$.

**Proof.** By Lemma 5.1, we see that $M^+_2 u_\varphi - \partial_t u_\varphi \geq 0$ in $Q_1$ in the viscosity sense. Set $\phi = 1 - \psi \gamma$ in $\mathbb{R}^n$. Take any $L_\beta \in \mathcal{L}_2$ and $(x, t) \in Q_{1/2}$. Then we have that
\[ L_\beta(\psi \gamma u_\varphi)(x, t) = \int_{\mathbb{R}^n} \mu_t(u_\varphi, x, y)K_\beta(y)\,dy - E(x, t) \]
(6.1)
where $E(x, t) = \int_{\mathbb{R}^n} \mu_t(\phi u_\varphi, x, y)K_\beta(y)\,dy$. By the mean value theorem and triangle inequality, we see that
\[ \mu(\phi^\sigma K_\beta(y, z) \geq \int_{0}^{1} \int_{0}^{1} \langle D^2(\phi^\sigma K_\beta)((y + \tau z) - 2s\tau z), z \rangle\,ds\,d\tau \]
(6.2)
and $|x + (y + \tau z) - 2s\tau z| = |x + y + \tau(1 - 2s)z| \geq |y| - \frac{15}{16}|y| \geq \frac{1}{16}|y|$ for any $y \in \mathbb{R}^n \setminus B_{4/5}$, $z \in B_{1/4}$ and $-2^{1-\sigma} < t \leq 0$. Also we note that $\mu(\phi^\sigma K_\beta(y, z) = 0$
for any \( y \in B_{4/5}, \ z \in B_{1/4} \) and \(-2^{1-\sigma} < t \leq 0\). Thus by (6.2) we obtain that
\[
E(x, t) = 2 \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \mu_t(u, x + y, z) K(z) \varphi(z) \, dz \right) \phi(x + y) K_\beta(y) \, dy
\]
\[
= 2 \sup_{|y| \geq \frac{1}{5}} \left| u(x + y, t) \right| \frac{1}{|y|^{n+2+\sigma}} \int_{|z| < \frac{1}{5}} |z|^2 K(z) \, dz \leq C \| u \|_{L_2^\infty(L_{2\beta})}
\]
for any \( |x| < 1/2 \) and \(-2^{1-\sigma} < t \leq 0\). Hence by (6.1) we conclude that
\[
\mathbf{M}_2^\infty (\psi \gamma u_\varphi)(x, t) - \partial_t (\psi \gamma u_\varphi)(x, t)
\geq \mathbf{M}_2^\infty u_\varphi(x, t) - \partial_t u_\varphi(x, t) - E(x, t) \geq -C \| u \|_{L_2^\infty(\omega)}
\]
for any \((x, t) \in Q_{1/2}\). Therefore we complete the proof. \(\square\)

**Lemma 6.2.** Let \( u \in C(\mathbb{R}_0^+) \cap L_{2\beta}^\infty(L_{2\beta}^\infty) \) be any viscosity solution satisfying the equation \( Lu - \partial_t u = 0 \) in \( Q_2 \), where \( L \) is defined on \( \Sigma_2(\sigma) \) for \( \sigma \in (\sigma_0, 2) \) with \( \sigma_0 \in (1, 2) \). Then for any operator \( L \) with a symmetric kernel \( K \) satisfying \( K(y) \leq (2-\sigma)\Lambda |y|^{-n-\sigma} \), there is a constant \( C > 0 \) (depending only on \( n, \lambda, \Lambda, \eta \) and \( \sigma_0 \) but not on \( \sigma \)) such that
\[
\sup_{Q_{1/2}} |Lu| \leq C \| u \|_{L_2^\infty(L_{2\beta}^\infty)}.
\]

**Proof.** Take any \( \sigma \in (\sigma_0, 2) \) with \( \sigma_0 \in (1, 2) \). As in Lemma 5.3, without loss of generality, we may assume that \( u \in C^{2, \alpha}(Q_1) \) for some \( \alpha \in (0, 1) \), so that \( u \in C^2_2(Q_1) \cap C_1(Q_1) \). For convenience, we normalize \( \| u \|_{L_2^\infty(L_{2\beta}^\infty)} = 1 \).

Take any \( L_\beta \in \Sigma_2 \). Then by Lemma 5.3, we see that \( |L_\beta u| \) is bounded in \( Q_{1+\epsilon} \) because \(-u \) is another viscosity solution of our equation. Thus it follows from that
\[
|L_\beta u - \partial_t u|_{L_2^\infty(Q_{1+\epsilon})} \leq |L_\beta u|_{L_2^\infty(Q_{1+\epsilon})} + \| \partial_t u \|_{L_2^\infty(Q_{1+\epsilon})} < \infty.
\]
Combining Theorem 3.3 with this yields that
\[
\sup_{L \in \mathcal{L}_0(\sigma)} \| Lu \|_{L_2^\infty(Q_{1+\epsilon})} < \infty.
\]

Take any operator \( L \) with a symmetric kernel \( K \) satisfying \( K(y) \leq (2-\sigma)\Lambda |y|^{-n-\sigma} \). Then we split \( Lu \) into two integrals
\[
Lu(x, t) = \int_{\mathbb{R}^n} \mu_t(u, x, y) K(y) \varphi(y) \, dy + \int_{\mathbb{R}^n} \mu_t(u, x, y) K(y)(1 - \varphi(y)) \, dy
\]
\[
= u_\varphi(x, t) + u_{1-\varphi}(x, t),
\]
where \( \varphi \in C^\infty_c(\mathbb{R}^n) \) is a radial cut-off function supported in \( B_1 \) such that \( \varphi = 1 \) in \( B_{1/2} \) and \( 0 \leq \varphi \leq 1 \) in \( \mathbb{R}^n \). Since \( K \in L_1(\mathbb{R}^n \setminus B_{1/2}) \), it is easy to check that \( \sup_{Q_{1+\epsilon}} |u_{1-\varphi}| < \infty \), and thus we have that
\[
\| u_{1-\varphi} \|_{L_2^\infty(Q_{1+\epsilon})} < \infty.
\]
Thus by (6.3), we obtain that
\[(6.4) \quad \|u_\varphi\|_{L^\infty_T(L^1_\varphi(Q)))} < \infty.\]

From Lemma 5.1, we have that
\[(6.5) \quad M^T_2 u_\varphi - \partial_t u_\varphi \geq 0 \quad \text{in} \quad Q_1.\]

Let \(\psi \in C^\infty_c(\mathbb{R}^n)\) be a function such that \(\psi = 1\) in \(B_{1/2}\) and \(\text{supp}(\psi) \subset B_{1/4}\), and let \(\gamma \in C_c([-T,T])\) be a function such that \(\gamma = 1\) in \((-\frac{1+\varepsilon}{2},0]\) and \(\text{supp}(\gamma) \subset (-\frac{1}{2}\varepsilon)^\gamma - \varepsilon, \varepsilon]\). Set \(v_\varphi(x,t) = \psi(x)\gamma(t) u_\varphi(x,t)\). Then by (6.4) it is easy to check that \(v_\varphi \in L^\infty_T(L^1_\varphi) \cap C(\mathbb{R}^n_T)\). So it follows from Lemma 6.1 that
\[M^T_2 v_\varphi - \partial_t v_\varphi \geq -C \quad \text{in} \quad Q_{1/2}.\]

Applying Theorem 4.1, we obtain that \(v_\varphi \leq C\) in \(Q_{1/8}\). Thus the required upper bound for \(L u\) on \(Q_{1/2}\) follows from a standard covering and scaling argument.

For the lower bound for \(L u\) on \(Q_{1/2}\), we take an operator \(L_\beta \in \mathcal{L}_2(\sigma)\) with kernel \(K_\beta\) and consider an operator \(L_\ast\) with kernel \(K_\ast = \frac{2}{\lambda}K_\beta - \frac{1}{\lambda}K\). Then it is easy to check that
\[2 - \sigma < K_\ast(y) \leq \frac{2 - \sigma(2\lambda - 4)}{|y|^{n+\sigma}}.\]

As in the first half, we obtain that \(L_\ast u \leq C\) in \(Q_{1/2}\). This implies that \(L u \geq -C\) in \(Q_{1/2}\). Therefore the required result can be achieved. \(\square\)

From the above result, it is natural to obtain the following corollaries.

**Corollary 6.3.** Let \(u \in C(\mathbb{R}^n_T) \cap L^\infty_T(L^1_\ast)\) be any viscosity solution satisfying the equation \(I u - \partial_t u = 0\) in \(Q_2\), where \(I\) is defined on \(\mathcal{L}_2(\sigma)\) for \(\sigma \in (\sigma_0, 2)\) with \(\sigma_0 \in (1, 2)\). Then \(M^T_0 u, M^T_0 u\) and \(\partial_t u\) are uniformly bounded in \(Q_{1/2}\).

**Corollary 6.4.** Let \(u \in C(\mathbb{R}^n_T) \cap L^\infty_T(L^1_\ast)\) be any viscosity solution satisfying the equation \(I u - \partial_t u = 0\) in \(Q_2\), where \(I\) is defined on \(\mathcal{L}_2(\sigma)\) for \(\sigma \in (\sigma_0, 2)\) with \(\sigma_0 \in (1, 2)\). Then we have that
\[\sup_{Q_{1/2}} \int_{\mathbb{R}^n} \mu(u, \cdot, y) \frac{2 - \sigma}{|y|^{n+\sigma}} dy \leq \|u\|_{L^\infty_T(L^1_\ast)}.\]

**7. Proof of the Main Theorem**

Let \(u \in C(\mathbb{R}^n_T) \cap L^\infty_T(L^1_\ast)\) be any viscosity solution satisfying the equation
\[(7.1) \quad I u - \partial_t u = 0 \quad \text{in} \quad Q_2,\]

where \(I\) is defined on \(\mathcal{L}_2(\sigma)\) for \(\sigma \in (\sigma_0, 2)\) with \(\sigma_0 \in (1, 2)\). From Corollary 6.4, we see that there is a universal constant \(c_0 > 0\) such that
\[(7.2) \quad \sup_{Q_{1/2}} \int_{\mathbb{R}^n} \mu(u, \cdot, y) \frac{2 - \sigma}{|y|^{n+\sigma}} \varphi(y) dy \leq c_0 \|u\|_{L^\infty_T(L^1_\ast)},\]

where \(\varphi \in C^\infty_c(\mathbb{R}^n)\) is a function such that \(\varphi = 1\) in \(B_1\), \(\varphi = 0\) in \(\mathbb{R}^n \setminus B_{3/2}\) and \(0 \leq \varphi \leq 1\) in \(\mathbb{R}^n\).

In order to prove Theorem 1.1, our main goal is to obtain that there are some universal constants \(c > 0\) and \(\alpha \in (0, 1)\) such that
\[(7.3) \quad \int_{\mathbb{R}^n} |\mu_t(u, x, y) - \mu_0(u, 0, y)| \frac{2 - \sigma}{|y|^{n+\sigma}} \varphi(y) dy \leq c(|x| + |t|)^{1-\alpha} \|u\|_{L^\infty_T(L^1_\ast)}.\]
for any \((x, t) \in Q_{1/2}\). This implies that the fractional Laplacian \((-\Delta)^{\sigma/2}\) admits the Hölder continuity, and moreover the viscosity solutions of the nonlocal parabolic equation in Theorem 1.1 enjoy the \(C^{\sigma+\alpha}\)-regularity.

Let \(\psi \in C_c^\infty(\mathbb{R}^n)\) be a function such that \(\psi = 1\) in \(B_{1/2}\) and \(\text{supp}(\psi) \subset B_{2\epsilon/3}\), and let \(\gamma \in C_c(-T, T)\) be a function such that \(\gamma = 1\) in \((-\frac{1+\epsilon}{2\epsilon}, 0]\) and \(\text{supp}(\gamma) \subset (-\frac{1+\epsilon}{2\epsilon})^{\sigma} - \epsilon, \epsilon]\). Set \(w_\varphi(x, t) = \psi(x)\gamma(t)\varphi(x, t)\), where

\[
v_\varphi(x, t) = \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \frac{2 - \sigma}{|y|^{n+\sigma}} \varphi(y) \, dy
\]

for a radial cut-off function \(\varphi \in C_c^\infty(\mathbb{R}^n)\) supported in \(B_{1/4}\) with \(0 \leq \varphi \leq 1\) in \(\mathbb{R}^n\). Then, as in Lemma 6.2, it is easy to check that \(w_\varphi \in C(\mathbb{R}^n_T) \cap L_2^\infty(Q_{1/2})\) and it follows from Lemma 6.1 that

\[
M_2 w_\varphi - \partial_t w_\varphi \geq -C \|u\|_{L_2^\infty(L_2^1)} \text{ in } Q_{1/2},
\]

We set

\[
w_\varphi^\pm(x, t) = \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \frac{2 - \sigma}{|y|^{n+\sigma}} \varphi(y) \, dy
\]

and set

\[
w_{\varphi}^S(x, t) = \psi(x)\gamma(t) \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \frac{2 - \sigma}{|y|^{n+\sigma}} \varphi(y) \|_S(y) \, dy
\]

for a symmetric set \(S \subset \mathbb{R}^n\) (i.e. \(S = -S\)). Also we consider the positive part \(P_\varphi u\) and negative part \(N_\varphi u\) of \(w_\varphi\) defined by \(P_\varphi u(x, t) = \psi(x)\gamma(t) v_\varphi^+(x, t)\) and \(N_\varphi u(x, t) = \psi(x)\gamma(t) v_\varphi^-(x, t)\). Then we see that \(P_\varphi u = \sup_S w_{\varphi}^S\) and \(N_\varphi u = -\inf_S w_{\varphi}^S\), and moreover \(P_\varphi u = w_{\varphi}^{S_0}\) and \(N_\varphi u = -w_{\varphi}^{S_0}\) where \(S_0\) is the symmetric set given by \(S_0 = \{ y \in \mathbb{R}^n : \mu_t(u, x, y) > \mu_0(u, 0, y) \}\).

**Lemma 7.1.** Let \(u \in C(\mathbb{R}^n_T) \cap L_2^\infty(L_2^1)\) be a viscosity solution satisfying the equation

\[
I u - \partial_t u = 0 \text{ in } Q_2,
\]

where \(I\) is defined on \(\mathcal{L}_2(\sigma)\) for \(\sigma \in (\sigma_0, 2)\) with \(\sigma_0 \in (1, 2)\). Then there exist a universal constant \(c_1 > 0\) and some \(\alpha \in (0, 1)\) such that

\[
P_\varphi u(x, t) \leq c_1 (|x|^\sigma + |t|)^{\frac{\alpha}{2}} \|u\|_{L_2^\infty(L_2^1)}
\]

for any \((x, t) \in Q_{1/8}\).

**Proof.** We may assume that \(\|u\|_{L_2^\infty(L_2^1)} = 1\) by dividing the equation by \(\|u\|_{L_2^\infty(L_2^1)}\). Take any \((x, t) \in Q_{1/8}\) and \(L \in \mathcal{L}_2(\sigma)\). We let \(u^{x,t}(z, s) = u(x + z, t + s)\). Then we have that

\[
L (u^{x,t} - u)(0, 0) = \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \varphi(y) K(y) \, dy
\]

\[
+ \int_{\mathbb{R}^n} [\mu_t(u, x, y) - \mu_0(u, 0, y)] \phi(y) K(y) \, dy
\]

\[
:= L_\varphi u(x, t) + L_\phi u(x, t),
\]

where \(\phi = 1 - \varphi\). Then we see that

\[
w_{\varphi}^-(x, t) \leq L_\varphi(x, t) \leq w_{\varphi}^+(x, t)
\]
where \( w^-_\sigma(x,t) = \lambda Pu(x,t) - \Lambda Nu(x,t) \) and \( w^+_\sigma(x,t) = \Lambda Pu(x,t) - \Lambda Nu(x,t) \). By easy calculation, the second term in the right hand side of (7.4) becomes

\[
L_\phi u(x,t) = 2 \int_{\mathbb{R}^n} u(y,t) [K(y-x)\psi(y-x) - K(y)\psi(y)] \, dy \\
+ 2 \int_{\mathbb{R}^n} [u(y,t) - u(y,0)]K(y)\psi(y) \, dy \\
+ 2[u(0,0) - u(x,t)] \int_{\mathbb{R}^n} K(y)\psi(y) \, dy.
\]

Thus it follows from (1.3) and Theorem 2.5 [KL4] that

\[
\text{for some universal constants } c > 0 \text{ and } \beta > 0. \text{ Here we note that } \beta \text{ could be chosen freely in the open interval } (0, 1) \text{ (see } [KL3]). \text{ Since } u \text{ and } u^{x,t} \text{ solve (7.1) in a parabolic neighborhood of } (0,0), \text{ by (7.5) and (7.6) we see that}
\]

\[
A(x,t) \leq I(u^{x,t} - u)(0,0) = \partial_t(u^{x,t} - u)(0,0) \leq B(x,t)
\]

where \( A(x,t) = w^-_\sigma(x,t) - c(|x|^\sigma + |t|)^{\beta/\sigma} \) and \( B(x,t) = w^+_\sigma(x,t) + c(|x|^\sigma + |t|)^{\beta/\sigma} \).

Then we have only three possible cases; either (a) \( A(x,t) \leq 0 \) and \( B(x,t) \geq 0 \), or (b) \( A(x,t) \geq 0 \) and \( B(x,t) \geq 0 \), or (c) \( A(x,t) \leq 0 \) and \( B(x,t) \leq 0 \).

**Case I:** (a) \( A(x,t) \leq 0 \) and \( B(x,t) \geq 0 \) (a) implies that

\[
\frac{\lambda}{\Lambda} Nu(x,t) - c_1 (|x|^\sigma + |t|)^{\beta} \leq Pu(x,t) \leq \frac{\lambda}{\Lambda} Nu(x,t) + c_1 (|x|^\sigma + |t|)^{\beta}
\]

for any \((x,t) \in Q_{1/8}, \) where \( c_1 = c/\Lambda. \)

**Case II:** (b) \( A(x,t) \geq 0 \) and \( B(x,t) \geq 0 \) (b) implies that

\[
Nu(x,t) \leq Pu(x,t).
\]

**Case III:** (c) \( A(x,t) \leq 0 \) and \( B(x,t) \leq 0 \) (c) implies that

\[
Nu(x,t) \geq Pu(x,t).
\]

We note that \(-u\) is another viscosity solution of (7.1). Using \(-u\) instead of \(u\), we see that \(N(-u)(x,t) = Pu(x,t)\) and \(P(-u)(x,t) = Nu(x,t)\). In this case, the proof can be achieved exactly in the same way as Case II. Thus we have only to consider Case I and Case II.

Our main goal is to show that there is a universal constant \( c > 0 \) such that \( \sup_{Q_r} Pu \leq cr^\alpha \) for any small enough \( r > 0 \). Since \( B^d_r \subset Q_r \subset B^d_{2r} \), it suffices to show that \( \sup_{B^d_{2r}} Pu \leq cr^\alpha \) for any small enough \( r > 0 \). If we take a rescaled function \( w^S_\phi(x,t) = \frac{1}{c_0} w^\phi(rx, r^\sigma t) \) where \( c_0 \) is the constant in (7.2), then we may assume that

(i) \( |w^S_\phi| \leq 1 \) in \( \mathbb{R}^n \) and \( M^S_\phi w^S_\phi - \partial_t w^S_\phi \geq -r^\sigma/c_0 \) in \( B^d_1 \), for all symmetric sets \( S \subset \mathbb{R}^n \),

(ii) for any \((x,t) \in B^d_1\), we have that either

\[
\frac{\lambda}{\Lambda} Nu(x,t) - c_1 r^\sigma (|x|^\sigma + |t|)^{\beta} \leq Pu(x,t) \leq \frac{\lambda}{\Lambda} Nu(x,t) + c_1 r^\sigma (|x|^\sigma + |t|)^{\beta}
\]

or (7.9) holds, for any small enough \( r > 0 \), where \( c_1 \) is the constant in (7.8). From Lemma 2.2, we can also assume that \( u \) is \( C^{2,\alpha_0} \) for some \( \alpha_0 \in (0,1) \), and so \( w^S_\phi, Pu \) and \( Nu \) are continuous.
For our aim, we need only to prove that there are some \( r \in (0,1) \) and \( \varrho \in (0,1) \) such that
\[
(7.11) \quad \sup_{B_1^d} |Pu| \leq (1 - \varrho)^k = r^\alpha k \text{ for } \alpha = \frac{\ln(1 - \varrho)}{\ln r}.
\]

We are going to proceed this proof by using mathematical induction. If \( k = 0 \), then it is trivial by (i). Assume that (7.11) holds in the \( k \)-th step. Then we shall show that (7.11) holds also for the \((k + 1)\)-th step. By (7.11) and geometric observation, we have that
\[
(7.12) \quad -1 \leq w^{\varphi}(x,t) \leq Pu(x,t) \leq \frac{1}{1 - \varrho} (|x|^\alpha + |t|) \quad \quad \text{for any } (x,t) \text{ with } (|x|^\alpha + |t|)^{1/\alpha} > r^k.
\]

We consider the following rescaled functions
\[
\begin{align*}
\tilde{w}^{\varphi}(x,t) &:= (1 - \varrho)^{-k} w^{\varphi}(r^k x, r^k t), \\
\tilde{Pu}(x,t) &:= (1 - \varrho)^{-k} Pu(r^k x, r^k t) = \sup_S \tilde{w}^{\varphi}(x,t), \\
\tilde{Nu}(x,t) &:= (1 - \varrho)^{-k} Nu(r^k x, r^k t) = -\inf_S \tilde{w}^{\varphi}(x,t).
\end{align*}
\]

Then the function \( \tilde{Pu} \) satisfies that
\[
\tilde{Pu}(x,t) \leq \begin{cases} 
1/\varrho \ (|x|^\alpha + |t|)^{\beta} & \text{in } B_1^d, \\
1/(1 - \varrho) (|x|^\alpha + |t|)^{\beta} & \text{outside } B_1^d.
\end{cases}
\]

Choosing \( \beta = \alpha \) in (7.10), by (7.9) and (7.10) we have that
\[
(7.13) \quad \frac{\Lambda}{\lambda} \tilde{Nu}(x,t) - c_1 r^\alpha \leq \tilde{Pu}(x,t) \leq \frac{\Lambda}{\lambda} \tilde{Nu}(x,t) + c_1 r^\alpha \quad \text{in } B_1^d
\]
and
\[
(7.14) \quad \tilde{Nu}(x,t) \leq \tilde{Pu}(x,t) \quad \text{in } B_1^d.
\]

Next, we shall show that if \( \varphi \) and \( r \) are chosen so small enough that \( 1 - \varrho = r^\alpha \) for some \( \alpha \in (0,1) \), then \( \tilde{Pu} \leq 1 - \varrho \) in \( B_1^d \). This makes it possible to complete the induction process. For this proof, we assume that there are some small enough \( r \) and \( \varrho \) such that \( \tilde{Pu} \not\leq 1 - \varrho \) in \( B_1^d \), i.e. \( \tilde{Pu}(x_0, t_0) > 1 - \varrho \) for some \( (x_0, t_0) \in B_1^d \). Without loss of generality, we may suppose that \( (x_0, t_0) \) be the point at which the maximum value of \( \tilde{Pu} \) is attained in \( B_1^d \). Then we see that
\[
(7.15) \quad \tilde{Pu}(x_0, t_0) = \tilde{w}^{\varphi}(x_0, t_0) > 1 - \varrho
\]
and
\[
(7.16) \quad \tilde{Pu}(x,t) = \tilde{w}^{\varphi}(x,t) \leq \begin{cases} 
1/(1 - \varrho) (|x|^\alpha + |t|)^{\beta} & \text{in } B_1^d, \\
1/\varrho (|x|^\alpha + |t|)^{\beta} & \text{outside } B_1^d,
\end{cases}
\]
where \( S_0 \) is the symmetric set given by \( S_0 = \{ y \in \mathbb{R}^n : \mu_1(u, x, y) > \mu_0(u, 0, y) \} \). Then we note that
\[
(7.17) \quad M_2^k \tilde{w}^{\varphi} - \partial_i \tilde{w}^{\varphi} \geq -\frac{\rho^\sigma}{c_0} \left( \frac{r^\sigma}{1 - \varrho} \right)^k \geq -\frac{\rho^\sigma}{c_0} \quad \text{in } B_1^{d/2},
\]
because \( \alpha < \sigma_0 < \sigma < 2 \). Since it is easy to check that
\[
(1 - \tilde{w}_\varphi^{S_0})_- \leq \left( \frac{1}{1 - \varrho} (|x|^\sigma + |t|)^{\tilde{\nu}} - 1 \right)_+ := h(x,t) \quad \text{in } \mathbb{R}_T^n
\]
by (7.16), we derive that
\[
(7.18) \quad M_2^+ (1 - \tilde{w}_\varphi^{S_0})_- \leq M_2^+ h \leq c < \infty \quad \text{in } B_{1/2}^d
\]
for some universal constant \( c > 0 \). We also observe that
\[
\partial_t (1 - \tilde{w}_\varphi^{S_0})_- = 0 \quad \text{in } B_{1/2}^d,
\]
because \( B_{1/2}^d \subset \{(1 - \tilde{w}_\varphi^{S_0})_- = 0\} \) by (7.16). Let \( v_\varphi^{S_0} = (1 - \tilde{w}_\varphi^{S_0})_+ \). Then we have that \( v_\varphi^{S_0}(x_0,t_0) = \inf_{B_1^d} v_\varphi^{S_0} \leq \varrho \) by (7.15), and moreover by (7.17) and (7.18) we conclude that
\[
M_2^- v_\varphi^{S_0} - \partial_t v_\varphi^{S_0} \leq M_2^- (1 - \tilde{w}_\varphi^{S_0}) - \partial_t (1 - \tilde{w}_\varphi^{S_0})
\]
\[
+ M_2^+ (1 - \tilde{w}_\varphi^{S_0})_+ - \partial_t (1 - \tilde{w}_\varphi^{S_0})_-
\]
\[
\leq (M_2^+ \tilde{w}_\varphi^{S_0} - \partial_t \tilde{w}_\varphi^{S_0})
\]
\[
+ M_2^+ (1 - \tilde{w}_\varphi^{S_0})_+ - \partial_t (1 - \tilde{w}_\varphi^{S_0})_+ \leq c \quad \text{in } B_{1/2}^d.
\]
By Theorem 4.11 [KL3], there are some universal constants \( c > 0 \) and \( \mu > 0 \) such that
\[
(7.19) \quad \left| \{v_\varphi^{S_0} > \lambda \varrho \} \cap Q_r(x_0,t_0) \right| \leq c r^{n+\sigma} (v_\varphi^{S_0}(x_0,t_0) + cr^\sigma)^\mu (\lambda \varrho)^{-\mu}
\]
for any \( \lambda > 0 \) and \( r \in (0,1/4) \). If we choose \( r \) so that \( cr^\sigma < \varrho \), then (7.19) becomes
\[
(7.20) \quad \left| \{v_\varphi^{S_0} > \lambda \varrho \} \cap Q_r(x_0,t_0) \right| \leq c r^{n+\sigma} \lambda^{-\mu} = c \lambda^{-\mu} |Q_r|
\]
for any \( \lambda > 0 \). Set \( D = \{v_\varphi^{S_0} \leq \lambda \varrho \} \cap Q_r(x_0,t_0) \). By (7.20), we have that
\[
(7.21) \quad |D| \geq (1 - c \lambda^{-\mu}) |Q_r|
\]
for all large enough \( \lambda > 0 \). Since \( v_\varphi^{S_0} > \lambda \varrho \Leftrightarrow \tilde{w}_\varphi^{S_0} < 1 - \lambda \varrho \), we see that
\[
D = \{\tilde{w}_\varphi^{S_0} \geq 1 - \lambda \varrho \} \cap Q_r(x_0,t_0). \quad \text{Since } D \subset B_{1/2}^d \text{ and } \tilde{P}u \leq 1 \text{ in } D \text{ by (7.16), we also see that } \tilde{P}u - \tilde{w}_\varphi^{S_0} \leq \lambda \varrho \text{ in } D. \quad \text{So we have the estimate}
\]
\[
\tilde{P}u - \tilde{w}_\varphi^{S_0} \leq \lambda \varrho \quad \text{in } D,
\]
because \( \tilde{w}_\varphi^{S_0} + \tilde{w}_\varphi^{S_0} = \tilde{P}u - \tilde{\eta}u \). For (Case I), it follows from (7.13) and (7.22) that
\[
(7.23) \quad \tilde{w}_\varphi^{S_0} \leq -\frac{\lambda}{\Lambda} (1 - \lambda \varrho) + \lambda \varrho + c_1 r^\sigma \leq -\frac{\lambda}{2 \Lambda} \quad \text{in } D,
\]
provided that \( r \) and \( \varrho \) are chosen small enough. For (Case II), by (7.14) and (7.22) we have that
\[
(7.24) \quad \tilde{w}_\varphi^{S_0} \leq -(1 - \lambda \varrho) + \lambda \varrho \leq -\frac{\lambda}{2 \Lambda} \quad \text{in } D,
\]
if \( r \) and \( \varrho \) are chosen small enough. From (7.23), (7.24) and (7.21), we obtain that
\[
(7.25) \quad \left| \{\tilde{w}_\varphi^{S_0} \leq -\frac{\lambda}{2 \Lambda} \} \cap Q_r(x_0,t_0) \right| \geq (1 - c \lambda^{-\mu}) |Q_r|
\]
for any \( \lambda > 0 \) and \( r \in (0,1/4) \).
For any small $\eta > 0$, let $g(x, t) = (\tilde{w}_r^S(r\eta(x - x_0), (r\eta)\sigma(t - t_0)) + \frac{\lambda}{2\eta})_+$. Then it follows from (7.25) that

\begin{equation}
(7.26) \quad \{g > 0\} \cap Q_{\eta^{-1}} \leq c\lambda^{-\mu}|Q_{\eta^{-1}}|.
\end{equation}

When $r$ is small enough, by (i) it is also easy to check that

\begin{equation}
(7.27) \quad M^+_0 g - \partial_t g \geq -\|u\|_{L^\infty(\mathbb{R}^n)} \text{ in } Q_2.
\end{equation}

Applying Theorem 4.1 to $g$ with small enough $r \in (0, 1/4)$, by (7.12), (7.16) and (7.26) we obtain that

\[
g(x_0, t_0) \leq C \sup_{s \in (-T, 0)} \int_{\mathbb{R}^n} \frac{g(y, s)}{1 + |y|^{n+\sigma}} dy,
\]

\[
\leq C \sup_{s \in (-T, 0)} \int_{B_{\eta^{-1}}} \frac{g(y, s)}{1 + |y|^{n+\sigma}} dy + C \sup_{s \in (-T, 0)} \int_{\mathbb{R}^n \setminus B_{\eta^{-1}}} \frac{g(y, s)}{1 + |y|^{n+\sigma}} dy
\]

\[
\leq C \eta^{-n-\sigma}\lambda^{-\mu} + C \eta^\alpha \sup_{s \in (-T, 0)} \int_{\mathbb{R}^n \setminus B_{\eta^{-1}}} \frac{|y|^\alpha + |s|^{n/\sigma}}{1 + |y|^{n+\sigma}} dy
\]

\[
\leq C \eta^{-n-\sigma}\lambda^{-\mu} + C \frac{\eta^\sigma}{\sigma - \alpha} + C \frac{\eta^{\sigma + \alpha}}{\sigma}.
\]

In this estimate, choose $\eta$ so small that $C \frac{\eta^\sigma}{\sigma - \alpha} + C \frac{\eta^{\sigma + \alpha}}{\sigma} < \frac{\lambda}{8\Lambda}$, and then select $\mu$ so large that $C \eta^{-n-\sigma}\lambda^{-\mu} < \frac{\lambda}{8\Lambda}$. Then we have that

\[
g(x_0, t_0) \leq \frac{\lambda}{4\Lambda}.
\]

This implies that $\tilde{w}_r^S(0, 0) \leq -\frac{\lambda}{4\Lambda}$, which contradicts to the fact that $\tilde{w}_r^S(0, 0) = 0$. Hence we conclude that $\partial u \leq 1 - \theta$ in $B_{4\Lambda}^d$, that is to say, $\partial u \leq (1 - \theta)^{k+1}$ in $B_{4\Lambda}^{d+1}$. Therefore we complete the proof.

We can also obtain the following corollary in the same manner as Lemma 7.1.

**Corollary 7.2.** Let $u \in C(\mathbb{R}^n_+ \cap L^\infty(\mathbb{R}^n_1))$ be a viscosity solution satisfying the equation

\[
\mathbf{I} u - \partial_t u = 0 \quad \text{in } Q_2,
\]

where $\mathbf{I}$ is defined on $\mathbf{L}_2(\sigma)$ for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. Then there exist a universal constant $c_1 > 0$ and some $\alpha \in (0, 1)$ such that

\[
\mathbf{N} u(x, t) \leq c_1 (|x|^{\sigma} + |t|)^{\frac{\sigma}{2}} \|u\|_{L^\infty(\mathbb{R}^n_1)}
\]

for any $(x, t) \in Q_{1/8}$.

**Proof of Theorem 1.2.** As mentioned above, the case $\sigma \in (0, 1]$ could be treated in [KLI]. Thus we have only to prove our main theorem only for the case $\sigma \in (1, 2)$.

We note that the fractional Laplacian of order $\sigma \in (0, 2)$ is given by

\[
-(-\Delta)^{\sigma/2} u(x, t) = c_\sigma \int_{\mathbb{R}^n} \mu_t(u, x, y) \frac{2 - \sigma}{|y|^{n+\sigma}} dy,
\]

where $c_\sigma$ is the normalization constant comparable to $\sigma$ defined by

\[
c_\sigma = \frac{1}{2(2 - \sigma)} \int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+\sigma}} d\xi.
\]
As in (7.4), if \((x, t) \in Q_{1/8}\), then we have that
\[
-(-\Delta^{\sigma/2})u(x, t) + (-\Delta^{\sigma/2})u(0, 0) = c_\sigma \int_{\mathbb{R}^n} \left[ \mu_\ell(u, x, y) - \mu_0(u, 0, y) \right] \varphi(y) \frac{2 - \sigma}{|y|^{n+\sigma}} dy + c_\sigma \int_{\mathbb{R}^n} \left[ \mu_\ell(u, x, y) - \mu_0(u, 0, y) \right] \psi(y) \frac{2 - \sigma}{|y|^{n+\sigma}} dy
\]
\[
= c_\sigma \left( Pu(x, t) - Nu(x, t) + \int_{\mathbb{R}^n} \left[ \mu_\ell(u, x, y) - \mu_0(u, 0, y) \right] \psi(y) \frac{2 - \sigma}{|y|^{n+\sigma}} dy \right),
\]
where \(\varphi\) is the radial cut-off function in (7.4) and \(\phi = 1 - \varphi\). Thus it follows from Lemma 7.1, Corollary 7.2 and (7.6) that there is a universal constant \(c > 0\) such that
\[
\left| (-\Delta^{\sigma/2})u(x, t) - (-\Delta^{\sigma/2})u(0, 0) \right| \leq c \left( |x|^\sigma + |t| \right)^{\frac{\sigma}{\theta}} \| u \|_{L^\infty_{x,t}(L^\infty)}
\]
for any \((x, t) \in Q_{1/8}\). Now, by Corollary 2.3, it is easy to check that
\[
M^2_\sigma (u^{x,t} - u)(0,0) \leq \partial_t u(x,t) - \partial_t u(0,0)
\]
\[
= I_u(x,t) - I_u(0,0) \leq M^2_\sigma (u^{x,t} - u)(0,0).
\]
Thus, by (7.5), Lemma 7.1 and Corollary 7.2, we have the estimate
\[
\left| \partial_t u(x,t) - \partial_t u(0,0) \right| \leq \left| M^2_\sigma (u^{x,t} - u)(0,0) \right| \leq \Lambda \left( Pu(x,t) + Nu(x,t) \right)
\]
\[
\leq c \left( |x|^\sigma + |t| \right)^{\frac{\sigma}{\theta}} \| u \|_{L^\infty_{x,t}(L^\infty)}
\]
for any \((x, t) \in Q_{1/8}\). Hence by a standard translation argument of (7.28) and (7.29), we conclude that
\[
\| u \|_{C^{\alpha,n}_{x,t}(Q_{1/2})} \leq c \| u \|_{L^\infty_{x,t}(L^\infty)}
\]
for a universal constant \(c > 0\). Therefore we complete the proof. \(\square\)

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