Solvability of a Regular Polynomial Vector Optimization Problem without Convexity

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Abstract In this paper we consider the solvability of a non-convex regular polynomial vector optimization problem on a nonempty closed set. We introduce regularity conditions for the polynomial vector optimization problem and study properties and characterizations of the regularity conditions. Under the regularity conditions, we study nonemptiness and boundedness of the solution sets of the problem. As a consequence, we establish two Frank-Wolfe type theorems for the non-convex polynomial vector optimization problem. Finally, we investigate the solution stability of the non-convex regular polynomial vector optimization problem.

Keywords Polynomial vector optimization problem · Regularity condition · Frank-Wolfe type theorem · Non-convexity · Stability

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Throughout, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space with the norm $\| \cdot \|$ and the inner product $\langle \cdot, \cdot \rangle$, and $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \ldots, n\}$. In this paper we are interested in the following polynomial vector optimization problem on $K$:

$$\text{PVOP}(K, f) : \mathbb{R}^n_+ - \min_{x \in K} f(x),$$

where $f = (f_1, \ldots, f_s) : \mathbb{R}^n \rightarrow \mathbb{R}^s$ is a vector polynomial such that each component function $f_i$ is a polynomial with its degree $\deg f_i = d_i$, and $K \subseteq \mathbb{R}^n$ is a nonempty closed set (not necessarily convex set or semi-algebraic set [1, 2]).

Recall that a point $x^* \in K$ is a Pareto efficient solution of $\text{PVOP}(K, f)$ if

$$f(x) - f(x^*) \notin \mathbb{R}^s_+ \setminus \{0\}, \quad \forall x \in K$$

and $x^* \in K$ is a weak Pareto efficient solution of $\text{PVOP}(K, f)$ if

$$f(x) - f(x^*) \notin \text{int} \mathbb{R}^s_+, \quad \forall x \in K.$$

The Pareto efficient solution set and the weak Pareto efficient solution set of $\text{PVOP}(K, f)$ are denoted by $\text{SOL}^s(K, f)$ and $\text{SOL}^w(K, f)$ respectively. Clearly, $\text{SOL}^s(K, f) \subseteq \text{SOL}^w(K, f)$. When $s = 1$, $\text{PVOP}(K, f)$ collapses to the polynomial scalar optimization problem:

$$\text{PSOP}(K, f) : \min_{x \in K} f(x),$$

whose solution set is denoted by $\text{SOL}(K, f)$.

For the polynomial scalar optimization problem, regularity condition has been used in [3] to investigate the nonemptiness of the solution set and the continuity of the solution mapping for a quadratic programming problem. Hieu [4] established a Frank-Wolfe type theorem for a polynomial scalar optimization problem on a nonempty closed set by proving the nonemptiness and boundedness of its solution set when the objective function is bounded from below on the constraint set and the regularity condition holds. Hieu et al. [5] proved that the solution set of an optimization problem corresponding to a polynomial complementarity problem is nonempty and compact by using the regularity condition of the polynomial complementarity
problem. Kim et al. [6] proved the existence of Pareto efficient solutions of an unconstrained polynomial vector optimization problem when the Palais-Smale-type condition holds and the image of the objective function has a bounded section. When \( K \) is a convex semi-algebraic set and \( f \) is convex, Jiao et al. [7] proved that \( \text{PVOP}(K, f) \) has a Pareto efficient solution if and only if the image \( f(K) \) of \( f \) has a nonempty bounded section. Inspired by the above works, in this paper, we study the nonemptiness and boundedness of the solution sets of \( \text{PVOP}(K, f) \) without assuming any convexity of the objective function.

The rest of this paper is organized as follows: In Section 2, we present some notations and preliminary results. In Section 3, we give some sufficient and necessary conditions for regularity conditions. In Section 4, we discuss local properties of the regularity conditions. Section 5 is devoted to the study of solvability of \( \text{PVOP}(K, f) \) under regularity conditions. In Section 6, we discuss the solution stability under the regularity conditions. Finally, we makes a concluding remark in Section 7.

2 Preliminaries

In this section, we recall some concepts and results that will be used in this paper. A nonempty subset \( C \) of \( \mathbb{R}^n \) is called a cone if \( tx \in C \) for any \( x \in C \) and any \( t > 0 \). Given a nonempty closed set \( K \subset \mathbb{R}^n \), the asymptotic cone \( K_\infty \) of \( K \) is defined by

\[
K_\infty = \{ v \in \mathbb{R}^n : \text{there exist } t_k \to +\infty \text{ and } x_k \in K \text{ such that } \lim_{k \to +\infty} \frac{x_k}{t_k} = v \}.
\]

As known, \( K_\infty \) is a closed cone and \( (K_\infty)_\infty = K_\infty \), and \( K \) is bounded if and only if \( K_\infty = \{0\} \). If \( K \) is a convex set, then \( K_\infty \) is also a convex cone and \( K_\infty = \text{Rec} \ K \), where \( \text{Rec} \ K \) is the recession cone of \( K \) defined by

\[
\text{Rec} \ K = \{ v \in \mathbb{R}^n : x + tv \in K, \forall x \in K, \forall t \geq 0 \}.
\]

It is known that \( K + K_\infty = K \). The above results can be found in [8, 9].

Definition 2.1 Let \( p = (p_1, \ldots, p_s) : \mathbb{R}^n \to \mathbb{R}^s \) be a vector polynomial with \( \deg p_i = d_i, i = 1, \ldots, s \). We say that \( p^\infty \) is the vector recession polynomial (or the vector leading term) of \( p \), where

\[
p^\infty(x) = (p_1^\infty(x), p_2^\infty(x), \ldots, p_s^\infty(x)) \quad \text{and} \quad p_i^\infty(x) = \lim_{\lambda \to +\infty} \frac{p_i(\lambda x)}{\lambda^{d_i}}, \quad \forall x \in \mathbb{R}^n.
\]
Remark 2.1 When \( s = 1 \), \( p^\infty \) is a recession polynomial of \( p \) (see [4]).

**Definition 2.2** (See e.g. [6, 10, 11]) A function \( h : \mathbb{R}^n \mapsto \mathbb{R}^s \) is said to be bounded from below on \( K \) if there exists \( r \in \mathbb{R}^s \) such that \( h(x) \in r + \mathbb{R}_+^s \) for all \( x \in K \). Clearly, \( h \) is bounded from below on \( K \) if and only if its component \( h_i \) is bounded from below on \( K \) for all \( i \).

**Definition 2.3** We say that \( \text{PVOP}(K, f) \) is weakly regular (resp. strongly regular) if \( \text{SOL}^w(K_\infty, f^\infty) \) (resp. \( \text{SOL}^w(K_\infty, f^\infty) \)) is bounded.

**Remark 2.2** Clearly, strong regularity implies weak regularity. When \( s = 1 \), both weak regularity and strong regularity coincide with the regularity in [4, Definition 2.1].

The following result plays an important role in establishing the existence of the Pareto efficient solutions of \( \text{PVOP}(K, f) \).

**Lemma 2.1** [12, Proposition 13] Given \( \lambda \in \text{int} \mathbb{R}_+^s \) and \( x_0 \in K \), define \( g(x) = \sum_{j=1}^s \lambda_j f_j(x) \) and \( G_{x_0} = \{ x \in K : f_i(x) \leq f_i(x_0), i = 1, 2, \ldots, s \} \). If \( x^* \in \text{SOL}(G_{x_0}, g) \), then \( x^* \in \text{SOL}^w(K, f) \).

In what follows we always assume the each component polynomial \( f_i \) of the objective function \( f \) has a degree \( d_i \geq 1 \).

**3 Regularity of \( \text{PVOP}(K, f) \)**

In this section, we shall discuss properties and characterizations of regularity of \( \text{PVOP}(K, f) \).

### 3.1 Conditions for regularity

In this subsection we shall show that the regularity of \( \text{PVOP}(K, f) \) is closely related to the regularity of \( \text{PSOP}(K, f_i) \). To do so, we first give a characterization of \( \text{SOL}^w(K_\infty, f^\infty) = \emptyset \).

**Proposition 3.1** \( \text{SOL}^w(K_\infty, f^\infty) = \emptyset \) if and only if \( 0 \notin \text{SOL}^w(K_\infty, f^\infty) \).

**Proof** We only need to prove the sufficiency. Suppose that \( 0 \notin \text{SOL}^w(K_\infty, f^\infty) \). Then there exists \( v_0 \in K_\infty \setminus \{0\} \) such that

\[
 f^\infty(v_0) - f^\infty(0) = f^\infty(v_0) \in - \text{int} \mathbb{R}_+^s ,
\]
which yields

\[ f_i^\infty(v_0) < 0, \quad i = 1, \ldots, s. \]

Let \( v \in K_\infty \). It follows that

\[ f_i^\infty(tv_0) - f_i^\infty(v) = t^{d_i}f_i^\infty(v_0) - f_i^\infty(v) < 0, \quad i = 1, \ldots, s \]

for all sufficiently large \( t > 0 \). Since \( v \in K_\infty \) is arbitrary, \( SOL^w(K_\infty, f^\infty) = \emptyset \). \( \square \)

**Example 3.1** Consider the vector polynomial \( f = (f_1, f_2) \) with

\[ f_1(x_1, x_2) = x_1^3 - x_1^2x_2 - 3x_1 + 2x_2 + 1, \quad f_2(x_1, x_2) = -x_2^2 - x_1x_2 + x_1 - 1 \]

and

\[ K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 - x_1 \geq 0\}. \]

It is easy to verify that \( K = K_\infty \), \( f_1^\infty(x_1, x_2) = x_1^3 - x_1^2x_2 \), and \( f_2^\infty(x_1, x_2) = -x_2^2 - x_1x_2 \). Then \( 0 \notin SOL^w(K_\infty, f^\infty) \) since \( f_1^\infty(1, 2) = -1 < 0 = f_1^\infty(0, 0) \) and \( f_2^\infty(1, 2) = -6 < 0 = f_2^\infty(0, 0) \). By Proposition 3.1, \( SOL^w(K_\infty, f^\infty) = \emptyset \).

**Proposition 3.2** If \( SOL^w(K_\infty, f^\infty) \neq \emptyset \), then \( SOL^w(K_\infty, f^\infty) \) is a nonempty cone.

**Proof** Since \( SOL^w(K_\infty, f^\infty) \neq \emptyset \), by Proposition 3.1, \( 0 \in SOL^w(K_\infty, f^\infty) \). Suppose on the contrary that there exist \( v_0 \in SOL^w(K_\infty, f^\infty) \) and \( t_0 > 0 \) such that \( t_0v_0 \notin SOL^w(K_\infty, f^\infty) \). Then there exists \( v_1 \in K_\infty \) such that

\[ f_i^\infty(v_1) - f_i^\infty(t_0v_0) < 0, \quad i = 1, \ldots, s. \tag{1} \]

Dividing the both sides of the inequality (1) by \( t_0^{d_i} \), we obtain \( f_i^\infty(v_1)/t_0^{d_i} - f_i^\infty(v_0) < 0 \) for all \( i \in \{1, \ldots, s\} \).

This reaches a contradiction to \( v_0 \in SOL^w(K_\infty, f^\infty) \) since \( v_1/t_0^{d_i} \in K_\infty \). Thus, \( SOL^w(K_\infty, f^\infty) \) is a nonempty cone. \( \square \)

**Remark 3.1** By Proposition 3.2, PVOP\((K, f)\) is strongly regular if and only if \( SOL^w(K_\infty, f^\infty) \) is empty or \( SOL^w(K_\infty, f^\infty) = \{0\} \).

**Proposition 3.3** If \( SOL^w(K_\infty, f^\infty) = \emptyset \), then \( f_i \) is unbounded from below on \( K \) for all \( i \in \{1, \ldots, s\} \).
Proof Suppose on the contrary that there exists \( i_0 \in \{1, 2, \ldots, s\} \) such that \( f_{i_0} \) is bounded from below on \( K \). Then \( SOL(K_\infty, f_{i_0}^\infty) = \emptyset \) since \( SOL(K_\infty, f_i^\infty) \subseteq SOL^w(K_\infty, f^\infty) \). By Lemma 3.1, there exists \( v_0 \in K_\infty \setminus \{0\} \) such that \( f_{i_0}^\infty(v_0) < f_{i_0}^\infty(0) = 0 \). Since \( v_0 \in K_\infty \setminus \{0\} \), there exist \( t_k > 0 \) with \( t_k \to +\infty \) and \( x_k \in K \) such that \( t_k^{-1}x_k \to v_0 \) as \( k \to +\infty \). Since \( f_{i_0} \) is bounded from below on \( K \), there exists a constant \( l \) such that

\[
\frac{f_{i_0}(x_k)}{t_k^{d_{i_0}}} \geq \frac{l}{t_k^{d_{i_0}}}.
\]

Letting \( k \to +\infty \), we obtain \( f_{i_0}^\infty(v_0) \geq 0 \) which is a contradiction to \( f_{i_0}^\infty(v_0) < 0 \). \( \square \)

When \( f_i \) is bounded from below on \( K \) for some \( i \in \{1, \cdots, s\} \), by Proposition 3.3 and Proposition 3.1, we know that \( 0 \in SOL^w(K_\infty, f^\infty) \). In the following proposition, we further show \( 0 \in SOL^s(K_\infty, f^\infty) \) when each component \( f_i \) is bounded from below on \( K \).

Proposition 3.4 If \( f \) is bounded from below on \( K \), then \( 0 \in SOL^s(K_\infty, f^\infty) \).

Proof Suppose that \( f \) is bounded from below on \( K \). Then \( f_i \) is bounded from below on \( K \) for all \( i \). By Proposition 3.3 and Proposition 3.1, \( f_i^\infty(x) \geq f_i^\infty(0) = 0 \) for all \( x \in K_\infty \) and for all \( i \in \{1, \ldots, s\} \). This means \( 0 \in SOL^s(K_\infty, f^\infty) \). \( \square \)

The following result shows that the weak (strong) regularity of PVOP \((K, f)\) is closely related to the regularity of PVOP \((K, f_i)\).

Theorem 3.1

(i) \( SOL^w(K_\infty, f^\infty) = \{0\} \) if and only if \( SOL(K_\infty, f_i^\infty) = \{0\} \) for all \( i \in \{1, \ldots, s\} \).

(ii) If \( SOL^s(K_\infty, f^\infty) = \{0\} \), then \( SOL(K_\infty, f_i^\infty) \neq \emptyset \) for all \( i \in \{1, \ldots, s\} \).

Proof (i) Assume that \( SOL^w(K_\infty, f^\infty) = \{0\} \). Since

\[
SOL(K_\infty, f_i^\infty) \subseteq SOL^w(K_\infty, f^\infty),
\]

it suffices to show \( SOL(K_\infty, f_i^\infty) \neq \emptyset \) for all \( i \in \{1, \ldots, s\} \). Suppose on the contrary that there exists \( i_0 \in \{1, \ldots, s\} \) such that \( SOL(K_\infty, f_{i_0}^\infty) = \emptyset \) which is equivalent to \( 0 \notin SOL(K_\infty, f_{i_0}^\infty) \) (by Lemma 3.1). Then there exists \( v_0 \in K_\infty \setminus \{0\} \) such that \( f_{i_0}^\infty(v_0) < 0 \). Let \( \lambda \in \text{int} \mathbb{R}_+^s \). Consider the function

\[
F_\lambda(x) = \sum_{i=1}^s \lambda_i f_i^\infty(x) \quad \text{and} \quad S_{v_0} = \{ x \in K_\infty : f^\infty(x) \leq f^\infty(v_0) \}. \tag{2}
\]
Then $S_{v_0}$ is nonempty and closed. Next we shall show that $S_{v_0}$ is bounded. If not, then there exists the sequence $\{x_k\} \subseteq S_{v_0}$ such that $\|x_k\| \to +\infty$ as $k \to +\infty$. Without loss of generality, we assume that $\|x_k\| \neq 0$ and $\frac{x_k}{\|x_k\|} \to x^* \in K_\infty \setminus \{0\}$. Since $x_k \in S_{v_0}$, we have

$$f_i^\infty(x_k) \leq f_i^\infty(v_0), \quad \forall i = 1, \ldots, s.$$  

Dividing the both sides of the above inequality by $\|x_k\|^d$ and letting $k \to +\infty$, we get

$$f_i^\infty(x^*) \leq 0 = f_i^\infty(0), \quad \forall i = 1, \ldots, s,$$

which together with $0 \in SOL^w(K_\infty, f^\infty)$ yields $x^* \in SOL^w(K_\infty, f^\infty)$, a contradiction. Thus, $S_{v_0}$ is bounded. By the known Weierstrass’ theorem, we have $SOL(S_{v_0}, F_\lambda) \neq \emptyset$. It follows from Lemma 2.1 that

$$\emptyset \neq SOL(S_{v_0}, F_\lambda) \subseteq SOL^s(K_\infty, f^\infty) \subseteq SOL^w(K_\infty, f^\infty) = \{0\},$$

which implies $SOL(S_{v_0}, F_\lambda) = \{0\}$, and so $0 \in S_{v_0}$. By the definition of $S_{v_0}$, we get $f_i^\infty(v_0) \geq 0$, a contradiction to $f_i^\infty(v_0) < 0$. Therefore, $SOL(K_\infty, f^\infty) \neq \emptyset$ for all $i \in \{1, \ldots, s\}$.

For the converse, assume that $SOL(K_\infty, f_i^\infty) = \{0\}$ for all $i \in \{1, \ldots, s\}$ and there exists $v_0 \in SOL^w(K_\infty, f^\infty) \setminus \{0\}$. Then

$$f^\infty(0) - f^\infty(v_0) \notin \text{int } R^+_s,$$

which implies that $f_i^\infty(v_0) \leq f_i^\infty(0)$ for some $i_0 \in \{1, \ldots, s\}$. This reaches a contradiction to $SOL(K_\infty, f_i^\infty) = \{0\}$.

(ii) Suppose on the contrary that there exists $i_0 \in \{1, \ldots, s\}$ such that $SOL(K_\infty, f_i^\infty) = \emptyset$. By Lemma 3.1, $0 \notin SOL(K_\infty, f_i^\infty)$. Then there exists $v_0 \in K_\infty \setminus \{0\}$ such that $f_i^\infty(v_0) < 0$. Let $\lambda \in \text{ int } R^+_s$. By similar arguments as in the proof of (i), we have

$$\emptyset \neq SOL(S_{v_0}, F_\lambda) \subseteq SOL^s(K_\infty, f^\infty) = \{0\},$$

where $S_{v_0}$ and $F_\lambda$ are defined as in (2). The rest is same as the one of (i), and so we omit it. \qed

Remark 3.2 The following example shows that $SOL^s(K_\infty, f^\infty) = \{0\}$ does not imply $SOL(K_\infty, f_i^\infty) = \{0\}$ for each $i \in \{1, \ldots, s\}$. 

Example 3.2 Consider the vector polynomial \( f = (f_1, f_2) \) with

\[
f_1(x_1, x_2) = x_1^3 - x_2 + 1, \quad f_2(x_1, x_2) = x_2^3 - x_1 - 1
\]

and the constraint set

\[
K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}.
\]

It is easy to verify that \( K = K_\infty, f_1^\infty(x_1, x_2) = x_1^3 \) and \( f_2^\infty(x_1, x_2) = x_2^3 \). Clearly, \( SOL(K_\infty, f_1^\infty) = \{0\} \). However, \( SOL(K_\infty, f_1^\infty) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \geq 0\} \neq \{0\} \) and \( SOL(K_\infty, f_2^\infty) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\} \neq \{0\} \).

Proposition 3.5 PVOP\((K, f)\) is strongly regular and \( f \) is bounded from below on \( K \) if and only if \( SOL(K_\infty, f_i^\infty) = \{0\} \) for all \( i \in \{1, \ldots, s\} \).

Proof The sufficiency follows immediately from Proposition 3.4 and Theorem 3.1 (i).

Next we prove the sufficiency. By Theorem 3.1 (i), PVOP \((K, f)\) is strongly regular. Suppose on the contrary that there exists \( i_0 \in \{1, \ldots, s\} \) such that \( f_{i_0} \) is not bounded from below on \( K \). Let \( x \in K \). Then there exists \( \{x_k\}_{k=1}^\infty \subseteq K \) such that

\[
f_{i_0}(x_k) \leq -k \leq f_{i_0}(x)
\]

for all sufficiently large \( k \). We claim that \( \{x_k\}_{k=1}^\infty \subseteq K \) is unbounded. Indeed, if not, then we may assume that \( \|x_k\| \to +\infty \) and \( \frac{x_k}{\|x_k\|} \to v \in K_\infty \setminus \{0\} \) as \( k \to +\infty \). Dividing the both sides of (3) by \( \|x_k\|^{d_i} \) and letting \( k \to +\infty \), we get \( f_i^\infty(v) \leq 0 \), a contradiction to \( SOL(K_\infty, f_i^\infty) = \{0\} \). Hence, \( \{x_k\}_{k=1}^\infty \) is bounded. Without loss of generality, we may assume that \( x_k \to x^* \in K \) as \( k \to +\infty \). It follows from (3) that

\[
f_{i_0}(x^*) = \lim_{k \to +\infty} f_{i_0}(x_k) \leq \lim_{k \to +\infty} -k = -\infty,
\]

a contradiction. Thus, \( f \) is bounded from below on \( K \). \( \square \)

3.2 Regularity and \( R_0 \)-property

It has been shown in [13–17] that \( R_0 \)-property plays an important role in studying the compactness of the solution sets of complementarity problems as well as the upper continuity of the solution mappings. In
this subsection, we shall show that the weak (strong) regularity of PVOP $(K, f)$ is closely related to the $R_0$-property of the following weak vector complementarity problem \cite{18, 19}

$$WVCP(K_{\infty}, \nabla f^{\infty}): \text{Find } x \in K_{\infty} \text{ such that } \nabla f^{\infty}(x) \in (K_{\infty})^{\infty+}_{R^s_+} \text{ and } \langle \nabla f^{\infty}(x), x \rangle \notin \text{ int } R_+^s,$$

where $\nabla f^{\infty} = (\nabla f^{\infty}_1, \nabla f^{\infty}_2, \ldots, \nabla f^{\infty}_s)$ is the gradient of $f^{\infty}$,

$$(K_{\infty})^{\infty+}_{R^s_+} = \{ v \in L(R^n, R^s) : \langle v, x \rangle \notin \text{ int } R_+^s, \forall x \in K_{\infty} \}$$

is the weak dual cone of $K_{\infty}$ with respect to $R_+^s$, and $L(R^n, R^s)$ is the space of all bounded linear operators from $R^n$ to $R^s$. We denote the solution set of $WVCP(K_{\infty}, \nabla f^{\infty})$ by $SOL_{WVCP}(K_{\infty}, \nabla f^{\infty})$. Recall that $WVCP(K_{\infty}, \nabla f^{\infty})$ is of type $R_0$ \cite{13} if $SOL_{WVCP}(K_{\infty}, \nabla f^{\infty}) = \{0\}$.

**Theorem 3.2** Assume that $K$ is convex. If $SOL^w(K_{\infty}, f^{\infty}) = \{0\}$, then $WVCP(K_{\infty}, \nabla f^{\infty})$ is of type $R_0$. Conversely, if $WVCP(K_{\infty}, \nabla f^{\infty})$ is of type $R_0$, then $PVOP(K, f)$ is strongly regular.

**Proof** It is clear that $0 \in SOL_{WVCP}(K_{\infty}, \nabla f^{\infty})$.

Suppose that $SOL^w(K_{\infty}, f^{\infty}) = \{0\}$. By Theorem 3.1(i), we have

$SOL(K_{\infty}, f^{\infty}_i) = \{0\}, i = 1, \ldots, s.$

This implies $f^{\infty}_i(v) > 0$ for all $v \in K_{\infty} \setminus \{0\}, i = 1, \ldots, s$. This together with the Euler’s Homogeneous Function Theorem yields

$$\langle \nabla f^{\infty}(v), v \rangle = (\langle \nabla f^{\infty}_1(v), v \rangle, \ldots, \langle \nabla f^{\infty}_s(v), v \rangle) = (d_1 f^{\infty}_i(v), \ldots, d_s f^{\infty}_s(v)) \in \text{ int } R^s_+$$

for all $v \in K_{\infty} \setminus \{0\}$, where $d_i$ is the degree of $f_i, i = 1, \ldots, s$. As a consequence, $SOL_{WVCP}(K_{\infty}, \nabla f^{\infty}) = \{0\}$ and so $WVCP(K_{\infty}, \nabla f^{\infty})$ is of type $R_0$.

For the converse, suppose $WVCP(K_{\infty}, \nabla f^{\infty})$ is of type $R_0$. To complete the proof, it suffices to show $SOL^w(K_{\infty}, f^{\infty}) = \{0\}$ (by Remark 3.1). Suppose on the contrary that there exists $v^* \in SOL^w(K_{\infty}, f^{\infty}) \setminus \{0\}$. By [20, Theorem 4], we get

$$\langle \nabla f^{\infty}(v^*), x - v^* \rangle \notin \text{ int } R^s_+, \forall x \in K_{\infty}.$$

This implies $v \in SOL_{WVCP}(K_{\infty}, \nabla f^{\infty})$ since $K_{\infty}$ is a closed convex cone. This reaches a contradiction. \qed

The following example illustrates the conclusion of Theorem 3.2.
Consider the vector polynomial $f = (f_1, f_2)$ with $f_1(x_1, x_2) = x_1^2 + x_2^2$, $f_2(x_1, x_2) = x_2^2$ and $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0, x_1 \geq 0\}$.

Then $K = K_\infty$, $\nabla f_1^\infty(x_1, x_2) = (2x_1, 2x_2)$, and $\nabla f_2^\infty(x_1, x_2) = (0, 2x_2)$. It is easy to verify that $0 \in SOL^w(K_\infty, f^\infty)$. Let $x = (x_1, x_2) \in K_\infty$ be such that $\langle \nabla f^\infty(x), x \rangle \notin \text{int} \mathbb{R}^2_+$. It follows that $(2x_1^2 + 2x_2^2, 2x_2^2) \notin \text{int} \mathbb{R}^2_+$. This implies $x = (0, 0)$. As a consequence, WVCP($K_\infty, \nabla f^\infty$) is of type $R_0$. By Theorem 3.2, PVOP($K, f$) is strongly regular.

**Remark 3.3** Assume that $K$ is convex and $f$ is bounded from below on $K$. Then by Theorem 3.2 and Proposition 3.4, $SOL_{WVCP}(\nabla f^\infty, K_\infty) = \{0\} \Rightarrow SOL^*(K_\infty, f^\infty) = \{0\}$. The following example shows that the converse is not true in general.

Consider the vector polynomial $f = (f_1, f_2)$ with $f_1(x_1, x_2) = x_1^2 + x_2^2$, $f_2(x_1, x_2) = x_2^2$ and $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0, x_1 \geq 0\}$.

Clearly, $K$ is convex, $K = K_\infty$, and $f$ is bounded from below on $K$. It is easy to verify that $f_1(x_1, x_2) = f_1^\infty(x_1, x_2)$, $f_2(x_1, x_2) = f_2^\infty(x_1, x_2)$, $\nabla f_1^\infty(x_1, x_2) = (2x_1, 0)$, $\nabla f_2^\infty(x_1, x_2) = (0, 2x_2)$, and $SOL^*(K_\infty, f^\infty) = \{0\}$. Let $x_0 = (0, 1)$. Then $\langle \nabla f^\infty(x_0), x_0 \rangle = (0, 2) \notin \text{int} \mathbb{R}^2_+$ and $\langle \nabla f^\infty(x_0), w \rangle = (0, 2w_2) \notin \text{int} \mathbb{R}^2_+$ for all $w = (w_1, w_2) \in K_\infty$. This means $x_0 \in SOL_{WVCP}(K_\infty, \nabla f^\infty) \setminus \{0\}$, and so WVCP($K_\infty, \nabla f^\infty$) is not of type $R_0$.

4 Local Properties of Regularity Conditions

In this section, we investigate local properties of (strong) weak regularity of PVOP $(K, f)$. Given an integer $d$, in what follows, we always let $P_d$ denote the family of all polynomials of degree at most $d$, and

$$X_d^p(x) := (1, x_1, \ldots, x_n, x_1^2, \ldots, x_1^d, \ldots, x_n^2, \ldots, x_n^d, x_1^{d-1}x_2, x_1^{d-1}x_3, \ldots, x_1^{d-1}x_d, x_2^{d-1}x_3, \ldots, x_n^{d-1}x_d),$$

whose components are listed by the lexicographic ordering. The dimension of $P_d$ is denoted by $\kappa_d$. Then, for each polynomial $p \in P_d$, there exists a unique $\alpha \in \mathbb{R}^{\kappa_d}$ such that $p(x) = \langle \alpha, X_d^p(x) \rangle$. $P_d$ can be endowed with a norm $\|p\| := \|\alpha\| = \sqrt{\alpha_1^2 + \cdots + \alpha_{\kappa_d}^2}$. Let $p^k \in P_d$ with $p^k \to p \in P_d$ and $x^k \in \mathbb{R}^n$ with $x^k \to x \in \mathbb{R}^n$. It is easy to verify that $(p^k)^\infty \to p^\infty$ and $p^k(x^k) \to p(x)$ as $k \to +\infty$. 


Given \( \mathbf{d} = (d_1, \ldots, d_s) \in \mathbb{R}^s \) with \( d_i \) being an integer, \( i = 1, \ldots, s \), let \( \mathbf{P}_d = \mathbf{P}_{d_1} \times \cdots \times \mathbf{P}_{d_s} \). Denoted by \( \mathbf{GR}_w^d \) (resp. \( \mathbf{GR}_s^d \)) the family of all vector polynomials \( p \) with \( \deg p_i = d_i, i = 1, \ldots, s \), such that \( \text{PVOP}(K, p) \) is strongly (resp. weakly) regular. Then we have the following results.

**Proposition 4.1** \( \mathbf{GR}_s^d \) and \( \mathbf{GR}_w^d \) are nonempty.

**Proof** We only need to prove that \( \mathbf{GR}_s^d \) is nonempty since \( \mathbf{GR}_s^d \subseteq \mathbf{GR}_w^d \). If \( K \) is bounded, then \( K_\infty = \{0\} \).

In this case \( \text{SOL}^w(K_\infty, f^\infty) = \{0\} \), and so \( \text{PVOP}(K, f) \) is strongly regular for every vector polynomial \( f \).

Suppose that \( K \) is unbounded. Then there exists \( x^* = (x_1^*, \ldots, x_s^*) \in K_\infty \setminus \{0\} \). Without loss of generality, we suppose that \( x_{i_0}^* \neq 0 \). Consider the vector polynomial \( f = (f_1, \ldots, f_s) : \mathbb{R}^s \to \mathbb{R}^s \) with \( f_i(x) = -(x_{i_0}^*)^{d_i} x_{i_0}^{d_i}, i = 1, \ldots, s \). Then \( f_i(x) \) is a-polynomials \( f \) of degree \( d_i \) and \( f_i(tx^*) = -(x_{i_0}^*)^{2d_i} t^{d_i} \to -\infty \) as \( t \to +\infty \). As a consequence, \( \text{SOL}^w(K_\infty, f^\infty) = \emptyset \), and so \( f \in \mathbf{GR}_s^d \).

**Proposition 4.2** \( \mathbf{GR}_s^d \) is open in \( \mathbf{P}_d \).

**Proof** We shall prove that \( \mathbf{P}_d \setminus \mathbf{GR}_s^d \) is closed in \( \mathbf{P}_d \). Let \( \{f^k\} \subseteq \mathbf{P}_d \setminus \mathbf{GR}_s^d \) with \( f^k = (f_1^k, \ldots, f_s^k) \) such that \( f^k \to f = (f_1, \ldots, f_s) \) as \( k \to +\infty \). We can suppose that \( \deg f_i = d_i \) for all \( i \in \{1, 2, \ldots, s\} \) since \( f \notin \mathbf{GR}_s^d \) when \( \deg f_{i_0} < d_{i_0} \) for some \( i_0 \in \{1, \ldots, s\} \), where \( d_i \) is the \( i \)-th component of \( \mathbf{d} \). Since \( \text{SOL}^w(K_\infty, (f_k)^\infty) \) is unbounded for all \( k \), there exists \( x_k \in \text{SOL}^w(K_\infty, (f_k)^\infty) \) such that \( \|x_k\| \to +\infty \).

Without loss of generality, we assume that \( \frac{x_k}{\|x_k\|} \to x^* \in K_\infty \setminus \{0\} \). We claim that \( x^* \in \text{SOL}^w(K_\infty, f^\infty) \).

Indeed, if not, then there exists \( v \in K_\infty \) such that

\[
 f_i^\infty(v) < f_i^\infty(x^*), \quad i = 1, \ldots, s. \tag{4}
\]

Since \( x_k \in \text{SOL}^w(K_\infty, (f_k)^\infty) \), we have

\[
 (f_k)^\infty(\|x_k\|v) - (f_k)^\infty(x_k) \notin \text{int} \mathbb{R}^s_+.
\]

Then for each \( k \), there exists \( i_{k,v} \in \{1, \ldots, s\} \) such that

\[
 (f_k^{i_{k,v}})^\infty(\|x_k\|v) - (f_k^{i_{k,v}})^\infty(x_k) \geq 0.
\]

Since the set \( \{1, 2, \ldots, s\} \) is finite, without loss of generality, we suppose that there exists \( i_{0,v} \in \{1, \ldots, s\} \) such that

\[
 (f_k^{i_{0,v}})^\infty(\|x_k\|v) - (f_k^{i_{0,v}})^\infty(x_k) \geq 0, \quad \forall k.
\]
Since $(f_k^i)^\infty \to f_i^\infty$ as $k \to +\infty$, dividing the both sides of the above inequality by $\|x_k\|^{d_{0,v}}$ and letting $k \to +\infty$, we get

$$f_{i_{0,v}}^\infty (v) \geq f_{i_{0,v}}^\infty (x^*).$$

This reaches a contradiction to (4), and so $x^* \in SOL^w(K_\infty, f^\infty)\{0\}$. By Proposition 3.2, $SOL^w(K_\infty, f^\infty)$ is unbounded. Hence, $P_d \setminus GR^d_s$ is closed. \hfill \Box

Remark 4.1 When $s = 1$, Proposition 4.2 reduces to [4, Lemma 4.1].

In the following result, we shall show that strong regularity of a polynomial vector optimization problem remains stable under a small perturbation.

**Theorem 4.1** The following conclusions hold:

(i) If $SOL^w(K_\infty, f^\infty) = \{0\}$, then there exists $\epsilon > 0$ such that $SOL^w(K_\infty, g^\infty) = \{0\}$ for all $g \in P_d$ satisfying $\|g - f\| < \epsilon$;

(ii) If $SOL^w(K_\infty, f^\infty) = \emptyset$, then there exists $\epsilon > 0$ such that $SOL^w(K_\infty, g^\infty) = \emptyset$ for all $g \in P_d$ satisfying $\|g - f\| < \epsilon$.

**Proof** Since $GR^d_s$ is open in $P_d$ (by Proposition 4.2) and $f \in GR^d_s$, there exists an open ball $B(f, \delta) \subseteq GR^d_s$ such that either $SOL(K_\infty, g^\infty) = \{0\}$ or $SOL(K_\infty, g^\infty) = \emptyset$ for all $g \in B(f, \delta)$.

(i) It suffices to show that there exists $\epsilon \in (0, \delta)$ such that $SOL(K_\infty, g^\infty) = \{0\}$ for all $g \in B(f, \epsilon)$ when $SOL^w(K_\infty, f^\infty) = \{0\}$. Suppose on the contrary that for any $\epsilon \in (0, \delta)$, there exists $g^\epsilon \in P_d$ with $\|g^\epsilon - f\| < \epsilon$ such that $SOL^w(K_\infty, (g^\epsilon)^\infty) = \emptyset$. By Lemma 3.1, there exists $x_\epsilon \in K_\infty \setminus \{0\}$ such that

$$ (g_i^\epsilon)^\infty (x_\epsilon) < (g_i^\epsilon)^\infty (0) = 0, \quad i = 1, \cdots, s. $$

(5)

Since $g^\epsilon \to f$ as $\epsilon \to 0$, we have $(g^\epsilon)^\infty \to f^\infty$ as $\epsilon \to 0$. Without loss of generality, we assume that $\|x_\epsilon\| \to x^* \in K_\infty \setminus \{0\}$ as $\epsilon \to 0$. Since $g^\epsilon \in B(f, \epsilon) \subseteq GR^d_s$, we get $\deg (g_i^\epsilon)^\infty = d_i$. Dividing the both sides of (5) by $\|x_\epsilon\|^{d_i}$ and letting $\epsilon \to 0$, we get

$$ f_i^\infty (x^*) \leq 0, \quad i = 1, \cdots, s, $$

which reaches a contradiction to $SOL(K_\infty, f^\infty) = \{0\}$. 

!!!
(ii) It suffices to show that there exists \( \epsilon \in (0, \delta) \) such that \( SOL(K_{\infty}, g^\infty) = \emptyset \) for all \( g \in B(f, \epsilon) \) when \( SOL^w(K_{\infty}, f^\infty) = \emptyset \). Suppose on the contrary that for any \( \epsilon \in (0, \delta) \), there exists \( g^\epsilon \in P_d \) with \( \| g^\epsilon - f \| < \epsilon \) such that \( SOL^w(K_{\infty}, (g^\epsilon)^\infty) = \{0\} \). By (i) of Theorem 3.1, we have

\[
0 = (g_i^\epsilon)^\infty(0) \leq (g_i^\epsilon)^\infty(x), \quad i = 1, \ldots, s
\]

for any \( x \in K_{\infty} \). Since \( g^\epsilon \to f \) as \( \epsilon \to 0 \), we have \( (g_i^\epsilon)^\infty \to f_i^\infty \) as \( \epsilon \to 0 \). Letting \( \epsilon \to 0 \) in the above inequality, we get

\[
f_i^\infty(0) \leq f_i^\infty(x), \quad i = 1, \ldots, s.
\]

Since \( x \in K_{\infty} \) is arbitrary, again from (i) of Theorem 3.1 we get \( 0 \in SOL^w(K_{\infty}, f^\infty) \), a contradiction. \( \Box \)

Observe that \( f^\infty = (f + g)^\infty \) for all \( g = (g_1, \ldots, g_s) \in P_d \) with \( \text{deg} \, g_i < \text{deg} \, f_i, \quad i = 1, \ldots, s \). As a consequence, we have the following result.

**Proposition 4.3** Let \( f = (f_1, \ldots, f_s) : \mathbb{R}^n \to \mathbb{R}^s \) be a vector polynomial. Then for any vector polynomial \( g = (g_1, \ldots, g_s) \) with \( \text{deg} \, g_i < \text{deg} \, f_i, \quad i = 1, \ldots, s \), the following conclusions hold:

(i) If \( SOL^w(K_{\infty}, f^\infty) = \{0\} \), then \( SOL^w(K_{\infty}, (f + g)^\infty) = \{0\} \).

(ii) If \( SOL^w(K_{\infty}, f^\infty) = \emptyset \), then \( SOL^w(K_{\infty}, (f + g)^\infty) = \emptyset \).

(iii) If \( SOL^w(K_{\infty}, f^\infty) = \{0\} \), then \( SOL^w(K_{\infty}, (f + g)^\infty) = \{0\} \).

(iv) If \( SOL^w(K_{\infty}, f^\infty) = \emptyset \), then \( SOL^w(K_{\infty}, (f + g)^\infty) = \emptyset \).

The following result is a direct consequence of Theorem 4.1 and Proposition 4.3.

**Corollary 4.1** Let \( f = (f_1, \cdots, f_s) : \mathbb{R}^n \to \mathbb{R}^s \) be a vector polynomial. Then for any vector polynomial \( g = (g_1, \ldots, g_s) \) with \( \text{deg} \, g_i < \text{deg} \, f_i, \quad i = 1, \ldots, s \), the following conclusions hold:

(i) If \( PVOP(K, f) \) is weakly regular, then \( PVOP(K, f + g) \) is weakly regular.

(ii) If \( PVOP(K, f) \) is strongly regular, then \( PVOP(K, f + g) \) is strongly regular.

5 Existence Results for PVOP(K, f) with Regularity

In this section we shall study emptiness and boundedness of the solution sets of PVOP(K, f) under the regularity condition.
5.1 The weak regularity case

First, we give a necessary condition for the existence of Pareto efficient solutions of PVOP($K, f$).

**Theorem 5.1** Assume that one of the following conditions hold:

(i) $\text{SOL}^s(K, f^\infty) = \{0\}$.
(ii) there exists $i_0 \in \{1, \ldots, s\}$ such that $\text{SOL}(K, f^\infty_{i_0}) = \{0\}$.

Then $\text{SOL}^s(K, f)$ is nonempty.

**Proof** Let $\lambda \in \text{int } \mathbb{R}^s_+$ and $x_0 \in K$. Define $g_{\lambda}(x) = \sum_{i=1}^s \lambda_i f_i(x)$ and $G_{x_0} = \{x \in K : f(x) \leq f(x_0)\}$.

Clearly, $G_{x_0}$ is nonempty and closed. We assert that $G_{x_0}$ is bounded. If not, then there exists $\{x_k\} \subset G_{x_0}$ such that $\|x_k\| \to +\infty$ and $\frac{x_k}{\|x_k\|} \to \bar{x} \in K^\infty \setminus \{0\}$ as $k \to +\infty$. It follows from $x_k \in G_{x_0}$ that $f_i(x_k) \leq f_i(x_0), \ i = 1, \ldots, s$.

Dividing the both sides of the above inequality by $\|x_k\|^d_i$ and letting $k \to +\infty$, we get $f^\infty_i(\bar{x}) \leq f^\infty_i(0), \ i = 1, \ldots, s$, a contradiction to $\text{SOL}^s(K^\infty, f^\infty) = \{0\}$ as well as $\text{SOL}(K^\infty, f^\infty_{i_0}) = \{0\}$. Hence, $G_{x_0}$ is compact. By the Weierstrass Theorem, $\text{SOL}(G_{x_0}, g_{\lambda}) \neq \emptyset$. Since $\text{SOL}(G_{x_0}, g_{\lambda}) \subseteq \text{SOL}^s(K, f)$ (by Lemma 2.1), we have $\text{SOL}^s(K, f) \neq \emptyset$.

The following example shows that $\text{SOL}^s(K, f)$ may be unbounded when $\text{SOL}^s(K^\infty, f^\infty) = \{0\}$.

**Example 5.1** Consider the vector polynomial $f = (f_1, f_2)$ with $f_1(x_1, x_2) = x_1^2 - x_2 - 1, f_2(x_1, x_2) = x_2^3 + 1$ and

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq x_1 \geq 0\}.$$ 

It is easy to see that $K = K^\infty, f^\infty_1(x_1, x_2) = x_1^2, \text{ and } f^\infty_2(x_1, x_2) = x_2^3$. Clearly, $\text{SOL}^s(K^\infty, f^\infty) = \{0\}$. On the other hand, $\text{SOL}^s(K, f)$ is unbounded since

$$\{(x_1, x_2) \in K : x_1 = 0, x_2 \geq 0\} \subseteq \text{SOL}^s(K, f).$$

The following result gives a Frank-Wolf type theorem for textPVOP($K, f$) under the weak regularity condition.
Corollary 5.1 If PVOP(K, f) is weakly regular and f : \( \mathbb{R}^n \mapsto \mathbb{R}^s \) is bounded from below on K, then SOL*(K, f) is nonempty.

**Proof** It follows directly from Proposition 3.4, Remark 3.1 and Theorem 5.1.

The following result shows that the existence of the Pareto efficient solutions of a polynomial vector optimization problem is preserved when the vector objective function is perturbed by a lower degree polynomial.

**Theorem 5.2** Assume that one of the following conditions hold:

(i) SOL*(K, f∞) = {0}.

(ii) there exists \( i_0 \in \{1, \ldots, s\} \) such that SOL(K∞, fi0) = {0}.

Then SOL*(K, f + g) is nonempty for all \( g = (g_1, \ldots, g_s) \in \mathbb{P}_d \) with \( \deg g_i < \deg f_i, \ i = 1, \ldots, s. \)

**Proof** It follows directly from Proposition 4.3(i) and Theorem 5.1.

5.2 The strong regularity case

In this subsection, we study nonemptiness and boundedness of the solution sets of PVOP (K, f) under the strong regularity condition. Next, we give a necessary condition for the existence of the weak Pareto efficient solutions of PVOP (K, f).

**Theorem 5.3** If SOLw(K∞, f∞) = \( \emptyset \), then SOLw(K, f) = \( \emptyset \).

**Proof** Suppose that there exists \( x_0 \in SOL^w(K, f) \). By Proposition 3.1, \( 0 \notin SOL^w(K_\infty, f_\infty) \). Then there exists \( v_1 \in K_\infty \setminus \{0\} \) such that

\[
f^\infty(v_1) - f^\infty(0) = f^\infty(v_1) \in - \text{int} \mathbb{R}_+^s.
\]

This means that

\[
f_i^\infty(v_1) < f_i^\infty(0), \quad i = 1, \ldots, s.
\]

(6)

Since \( v_1 \in K_\infty \), there exist \( t_k > 0 \) with \( t_k \to +\infty \) and \( x_k \in K \) such that \( t_k^{-1} x_k \to v_1 \) as \( k \to +\infty \). Since \( x_0 \in SOL^w(K, f) \) and \( x_k \in K \), there exists \( i_{x_k} \in \{1, \ldots, s\} \) such that

\[
f_{i_{x_k}}(x_k) - f_{i_{x_k}}(x_0) \geq 0, \quad \forall k.
\]
Without loss of generality, we can assume that there exists $i_0 \in \{1, \ldots, s\}$ such that

$$f_{i_0}(x_k) - f_{i_0}(x_0) \geq 0.$$  

Dividing the both sides of the above inequality by $t_k^{d_{i_0}}$ and letting $k \to +\infty$, we have

$$f_{i_0}^{\infty}(r_1) \geq 0,$$

a contradiction to (6). \[\Box\]

**Remark 5.1** By Theorem 5.3, $SOL^w(K, f) \neq \emptyset$ implies that $SOL^w(K, f_{\infty}) \neq \emptyset$. The following example shows that the converse does not hold in general.

**Example 5.2** Consider the vector polynomial $f = (f_1, f_2)$ with

$$f_1(x_1, x_2) = (x_1^4 + x_2^4 - 1)^2 + x_1^4, f_2(x_1, x_2) = (x_1^2 + x_2^4 - 1)^2 + x_1^2$$

and $K = \mathbb{R}^n$. It is easy to verify that $0 \in SOL^w(K_{\infty}, f_{\infty})$. On the other hand, $f_1 > 0$ and $f_2 > 0$, but $f(\frac{1}{n}, n) = (\frac{1}{n^2}, \frac{1}{n^2}) \to (0, 0)$ as $n \to +\infty$. This implies $SOL^w(K, f) = \emptyset$.

As a direct consequence of Proposition 3.1 and Theorem 5.3, we have the following result.

**Theorem 5.4** If $0 \notin SOL^w(K_{\infty}, f_{\infty})$, then $SOL^w(K, f) = \emptyset$.

**Remark 5.2** By Theorem 5.3 (or Theorem 5.4), $SOL^w(K, f) \neq \emptyset$ implies $SOL^w(K, f_{\infty}) \neq \emptyset$. The following example shows that $SOL^*(K_{\infty}, f_{\infty})$ may be empty when $SOL^*(K, f) \neq \emptyset$.

**Example 5.3** Consider the vector polynomial $f = (f_1, f_2)$ with

$$f_1(x_1, x_2) = -x_2^2 - 1, f_2(x_1, x_2) = -x_1^3 + x_2 + 1$$

and

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, x_2 \geq 2\}.$$  

Then $K_{\infty} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \geq 0\}$ and $f_{\infty}(x_1, x_2) = (f_{1,\infty}(x_1, x_2), f_{2,\infty}(x_1, x_2)) = (-x_2^2, -x_1^3)$. It is easy to verify that $SOL^*(K_{\infty}, f_{\infty}) = \emptyset$ and $(1, 2) \in SOL^*(K, f)$.

The following result shows that $SOL^w(K_{\infty}, f_{\infty}) = \{0\}$ is sufficient for the existence of the Pareto efficient solutions as well as boundedness of the weak Pareto efficient solution set.
Theorem 5.5 If $\text{SOL}^w(K, f^\infty) = \{0\}$, then $\text{SOL}^s(K, f)$ is nonempty and $\text{SOL}^w(K, f)$ is compact.

Proof $\text{SOL}^s(K, f) \neq \emptyset$ follows from Theorem 3.1(i) and Theorem 5.1(ii). The closedness of $\text{SOL}^w(K, f)$ is clear. Next we prove that $\text{SOL}^w(K, f)$ is bounded. If not, then there exists $y_k \in \text{SOL}^w(K, f)$ such that $\|y_k\| \to +\infty$ as $k \to +\infty$. Without loss of generality, assume that $\|y_k\| \neq 0$ and $\frac{y_k}{\|y_k\|} \to v \in K_\infty \setminus \{0\}$. Let $y \in K$. Since $y_k \in \text{SOL}^w(K, f)$, there exists $i_k \in \{1, \ldots, s\}$ such that $f_{i_k}(y_k) - f_{i_k}(y) \geq 0$.

Since the set $\{1, \ldots, s\}$ is finite, without loss of generality, we can assume that there exists $i_0 \in \{1, \ldots, s\}$ such that $f_{i_0}(y) - f_{i_0}(y_k) \geq 0$, $\forall k$.

Dividing the both sides of the above inequality by $\|y_k\|^d_{i_0}$ and letting $k \to +\infty$, we have $f_{i_0}^\infty(v) \leq f_{i_0}^\infty(0) = 0$. This implies $v \in \text{SOL}(K_\infty, f_{i_0}^\infty) \setminus \{0\}$, a contradiction. $\square$

As a consequence of Theorem 5.5 and Theorem 3.1(i), we have the following result.

Corollary 5.2 If $\text{SOL}(K_\infty, f^\infty_i) = \{0\}$ for each $i \in \{1, 2, \ldots, s\}$, then $\text{SOL}^s(K, f)$ is nonempty and $\text{SOL}^w(K, f)$ is bounded.

Now we give an example to illustrate the conclusion of Corollary 5.2.

Example 5.4 Consider the vector polynomial $f = (f_1, f_2)$ with

$$f_1(x_1, x_2) = x_2^3 - x_1^2 - x_1 x_2 + 1, f_2(x_1, x_2) = x_2^2 - x_1 - 1$$

and

$$K = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 - 1 \geq 0, x_2 - x_1 + 1 \geq 0, e^{x_1} - 1 - x_2 \geq 0 \}.$$ 

It is easy to verify that $f_1^\infty(x_1, x_2) = x_2^3, f_2^\infty(x_1, x_2) = x_2^2$.

$$K_\infty = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 - x_1 \geq 0 \},$$

$\text{SOL}(K_\infty, f_1^\infty) = \{(0, 0)\}$, and $\text{SOL}(K_\infty, f_2^\infty) = \{(0, 0)\}$. By Corollary 5.2, $\text{SOL}^s(K, f)$ is nonempty and $\text{SOL}^w(K, f)$ is compact. It is worth mentioning that [6, Theorem 4.1] and [7, Theorem 3.1] cannot be applied in this example since $f$ is non-convex on $\mathbb{R}^2$ and $K$ is neither convex nor semi-algebraic set.
Remark 5.3 By Theorem 5.1(ii), SOL*(K, f) is nonempty when there exists some \( i_0 \in \{1, \cdots, s\} \) such that \( SOL(K_{\infty}, f_{i_0}^{\infty}) = \{0\} \). If \( SOL(K_{\infty}, f_i^{\infty}) = \{0\} \) for all \( i \in \{1, \cdots, s\} \), then SOL*(K, f) is nonempty and bounded (by Corollary 5.2). The following example shows that SOL*(K, f) may be unbounded when there exists \( \{i_0, j_0\} \subset \{1, \cdots, s\} \) such that \( SOL(K_{\infty}, f_{i_0}^{\infty}) = \{0\} \) and \( SOL(K_{\infty}, f_{j_0}^{\infty}) \neq \{0\} \).

Example 5.5 Consider the vector polynomial \( f = (f_1, f_2) \) with

\[
    f_1(x_1, x_2) = x_1^3, \quad f_2(x_1, x_2) = -x_1^2 + x_2
\]

and

\[
    K = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0, x_1 - x_2 \geq 0\}.
\]

It is easy to verify that \( K = K_{\infty}, f_1^{\infty}(x_1, x_2) = x_1^3, f_2^{\infty}(x_1, x_2) = -x_1^2, SOL(K_{\infty}, f_1^{\infty}) = \{0\}, SOL(K_{\infty}, f_2^{\infty}) = \emptyset \). On the other hand, SOL*(K, f) is unbounded since

\[
    \{(x_1, x_2) \in K : x_1 \geq 0, x_2 = 0\} \subseteq SOL^*(K, f).
\]

It has been shown in Corollary 5.1 that PVOP (K, f) with weak regularity admits a weak Pareto efficient solution provided that f is bounded from below on K. In the following Frank-Wolf type theorem, we further prove the existence of Pareto efficient solutions and compactness of the weak Pareto efficient solution set if we strengthen weak regularity by strong regularity.

Corollary 5.3 If PVOP(K, f) is strongly regular and \( f : \mathbb{R}^n \mapsto \mathbb{R}^s \) is bounded from below on K, then SOL*(K, f) is nonempty and SOL*(K, f) is compact.

Proof It follows directly from Proposition 3.5 and Theorem 5.5. \( \square \)

Remark 5.4 Corollary 5.3 extends [4, Theorem 3.1] to the vector case.

The following example shows that the converse of Theorem 5.5, Corollary 5.2 and Corollary 5.3 does not hold in general.

Example 5.6 Consider the polynomial \( f = (f_1, f_2) \) with

\[
    f_1(x_1, x_2) = x_1x_2 + 1, \quad f_2(x_1, x_2) = x_1x_2 + x_1 - 1
\]
and

\[ K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1\}. \]

Then \( f_1^\infty(x_1, x_2) = f_2^\infty(x_1, x_2) = x_1 x_2 \) and \( K_\infty = \mathbb{R}_+^2 \). It is easy to verify that \( SOL^*(K, f) = SOL^w(K, f) = \{(1, 1)\} \). However, \( SOL(K_\infty, f_1^\infty) = SOL(K_\infty, f_2^\infty) = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 x_2 = 0\} \) are unbounded.

The following result shows that some properties of the solution sets of a strongly regular polynomial vector optimization problem are preserved when its objective polynomial is perturbed by a lower degree polynomial.

**Theorem 5.6** Let \( g = (g_1, \cdots, g_s) : \mathbb{R}^n \mapsto \mathbb{R}^s \) be a vector polynomial with \( \deg g_i < \deg f_i, i = 1, \cdots, s \).

Then the following conclusions hold:

(i) If \( SOL^w(K_\infty, f^\infty) = \emptyset \), then \( SOL^w(K, f + g) = \emptyset \).

(ii) If \( SOL^w(K_\infty, f^\infty) = \{0\} \), then \( SOL^*(K, f + g) \) is nonempty and \( SOL^w(K, f + g) \) is compact.

**Proof** (i) follows from Proposition 4.3(iv) and Theorem 5.3, and (ii) follows from Proposition 4.3(ii) and Theorem 5.5.

\[ \square \]

6 Stability Analysis

In this section we investigate the solution stability of a regular polynomial vector optimization problem. First, we shows that some properties of the solution sets of a strongly regular polynomial vector optimization problem are stable under a small perturbation.

**Theorem 6.1** The following conclusions hold:

(i) If \( SOL^w(K_\infty, f^\infty) = \emptyset \), then there exists \( \epsilon > 0 \) such that \( SOL^w(K, f + g) = \emptyset \) for all \( g \in P_d \) satisfying \( \|g\| < \epsilon \).

(ii) If \( SOL^w(K_\infty, f^\infty) = \{0\} \), then there exists \( \epsilon > 0 \) such that \( SOL^*(K, f + g) \) is nonempty and \( SOL^w(K, f + g) \) is compact for all \( g \in P_d \) satisfying \( \|g\| < \epsilon \).

**Proof** (i) follows from Theorem 4.1 and Theorem 5.3 and (ii) follows from Theorem 4.1 and Theorem 5.5.

\[ \square \]
In the sequel we shall investigate the local boundedness and upper semicontinuity of the weak Pareto efficient solution mapping. Recall that a set-valued mapping $T : \mathbb{R}^n \Rightarrow 2^{\mathbb{R}^n}$ is said to be upper semicontinuous at $x$ if for any neighborhood $U$ of $T(x)$, there exists a neighborhood $V$ of $x$ such that $T(y) \subseteq U$ for any $y \in V$. A set-valued mapping $T : \mathbb{R}^n \Rightarrow 2^{\mathbb{R}^n}$ is locally bounded at $x$ if there exists an open neighborhood $V$ of $x$ such that $\cup_{y \in V} T(y)$ is bounded.

**Lemma 6.1** [8, Theorem 5.7 and Theorem 5.19] If $T : \mathbb{R}^n \Rightarrow 2^{\mathbb{R}^n}$ is locally bounded at $x$ and $T$ has a closed graph $\text{Gph}(T)$, where $\text{Gph}(T) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in T(x)\}$, then $T$ is upper semi-continuous at $x$.

**Theorem 6.2** Assume that $K$ is convex and $\text{PVOP}(K, f)$ is strongly regular. Then the following conclusions hold:

(i) $\text{SOL}^w(K, \cdot)$ is locally bounded at $f$, i.e., there exists $\delta > 0$ such that

$$Q_\delta = \cup_{h \in \mathbb{B}(0, \delta)} \text{SOL}^w(K, f + h)$$

is bounded, where the $\mathbb{B}(0, \delta)$ is an open ball in $\mathbb{P}_d$ with center at 0 and radius $\delta > 0$.

(ii) $\text{SOL}^w(K, \cdot)$ is upper semi-continuous on $\text{GR}_d$.

**Proof (i)** Since $\text{GR}_d$ is open in $\mathbb{P}_d$ (by Proposition 4.2) and $f \in \text{GR}_d$, there exists $\delta > 0$ such that $f + \bar{\mathbb{B}}(0, \delta) \subseteq \text{GR}_d$, where $\bar{\mathbb{B}}(0, \delta)$ denotes the closure of $\mathbb{B}(0, \delta)$. Suppose on the contrary that $Q_\delta = \text{SOL}^w(K, f + \mathbb{B}(0, \delta))$ is unbounded. Then there exist $\{h^k\} \subseteq \mathbb{B}(0, \delta)$ and $x_k \in \text{SOL}^w(K, f + h^k)$ such that $\|x_k\| \to +\infty$ as $k \to +\infty$. Without loss of generality, we may assume that $\frac{x_k}{\|x_k\|} \to x^* \in K_\infty \setminus \{0\}$ and $h^k \to h \in \mathbb{B}(0, \delta)$. Let $x \in K$ and $v \in K_\infty$. Then $x + \|x_k\|v \in K$ for all $k$. Since $x_k \in \text{SOL}^w(K, f + h^k)$, we have

$$(f + h^k)(x + \|x_k\|v) - (f + h^k)(x_k) \notin \text{int } \mathbb{R}_+^n.$$ 

Then for each $k$, there exists $i_{x, v} \in \{1, \ldots, s\}$ such that

$$(f_{i_{x, v}} + h^k_{i_{x, v}})(x + \|x_k\|v) - (f_{i_{x, v}} + h^k_{i_{x, v}})(x_k) \geq 0.$$ 

Since $\{1, \ldots, s\}$ is finite, we can assume that there exists $i_{0, v} \in \{1, \ldots, s\}$ such that

$$(f_{i_{0, v}} + h^k_{i_{0, v}})(x + \|x_k\|v) - (f_{i_{0, v}} + h^k_{i_{0, v}})(x_k) \geq 0, \quad \forall k.$$
Dividing the both sides of the above inequality by \( \|x_k\|^{d_{\alpha,v}} \) and letting \( k \to +\infty \), we get

\[
(f_{\alpha,v} + h_{\alpha,v})^\infty(v) \geq (f_{\alpha,v} + h_{\alpha,v})^\infty(x^*).
\]

Since \( v \in K_\infty \) is arbitrary, we have \( x^* \in SOL^w(K_\infty,(f + h)^\infty)\setminus\{0\} \). On the other hand, since \( f + h \in f + B(0,\delta) \subseteq \text{GR}^d \), by Proposition 3.2 we have \( x^* \not\in SOL^w(K_\infty,(f + h)^\infty) \), a contradiction. Thus, \( SOL^w(K,\cdot) \) is locally bounded at \( f \).

(ii) Let \( f \in \text{GR}^d \). By (i) and Lemma 6.1, we only need to prove that the graph \( \text{Gph}(SOL^w(K,\cdot)) \) of \( SOL^w(K,\cdot) \) is closed in \( P_d \times R^n \). Let \( (f_k, y_k) \to (f, y) \) as \( k \to +\infty \) with \( y_k \in SOL^w(K, f^k) \). Then for any \( z \in K \), we have

\[
f^k(z) - f^k(y_k) \not\in - \text{int} R^*_+.
\]

Letting \( k \to +\infty \), we have

\[
f(z) - f(y) \not\in - \text{int} R^*_+, \quad \forall z \in K,
\]

which means \( y \in SOL^w(K, f) \). Thus, \( \text{Gph}(SOL^w(K,\cdot)) \) is closed in \( P_d \times R^n \). \( \square \)

7 Conclusion

In this paper we extend the concept of regularity due to Hieu [4] to the polynomial vector optimization problem. Under regularity conditions, we investigate nonemptiness and boundedness of the solution sets of a non-convex polynomial vector optimization problem on a nonempty closed set (not necessarily semi-algebraic set). As a consequence, we derive two Frank-Wolfe type theorems for a non-convex polynomial vector optimization problem. We also discuss the solution stability. Our results extend and improve the corresponding results of [4, 6, 7, 10, 11].

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