Point singularities of 3D stationary Navier-Stokes flows

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Abstract

This article characterizes the singularities of very weak solutions of 3D stationary Navier-Stokes equations in a punctured ball which are sufficiently small in weak $L^3$.

Keywords: stationary Navier-Stokes equations, point singularity, very weak solution, Landau solution.

1 Introduction

We consider point singularities of very weak solutions of the 3D stationary Navier-Stokes equations in a finite region $\Omega$ in $\mathbb{R}^3$. The Navier-Stokes equations for the velocity $u : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ with external force $f : \Omega \to \mathbb{R}^3$ are

$$- \Delta u + (u \cdot \nabla) u + \nabla p = f, \quad \text{div } u = 0, \quad (x \in \Omega). \tag{1.1}$$

A very weak solution is a vector function $u$ in $L^2_{loc}(\Omega)$ which satisfies (1.1) in distribution sense:

$$\int -u \cdot \Delta \varphi + u_j u_i \partial_j \varphi_i = \langle f, \varphi \rangle, \quad \forall \varphi \in C^\infty_{c,\sigma}(\Omega), \tag{1.2}$$

and $\int u \cdot \nabla h = 0$ for any $h \in C^\infty_c(\Omega)$. Here the force $f$ is allowed to be a distribution and

$$C^\infty_{c,\sigma}(\Omega) = \{ \varphi \in C^\infty_c(\Omega, \mathbb{R}^3) : \text{div } \varphi = 0 \}. \tag{1.3}$$

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In this definition the pressure is not needed. Denote \( B_R = \{ x \in \mathbb{R}^3 : |x| < R \} \) and \( B_R^c = \mathbb{R}^3 \setminus B_R \) for \( R > 0 \).

We are concerned with the behavior of very weak solutions which solve \( (1.1) \) in the punctured ball \( B_2^c \setminus \{0\} \) with zero force, i.e., \( f = 0 \). There are a lot of studies on this problem \([5, 14, 15, 4, 10]\). A typical result is to show that, under some conditions, the solution is a very weak solution across the origin without singular forcing supported at the origin (removable singularity), and is regular, i.e., locally bounded, under possibly more assumptions (regularity). Dyer-Edmunds \([5]\) proved removable singularity and regularity assuming both \( u, p \in L^{3+\varepsilon}(B_2) \) for some \( \varepsilon > 0 \). Shapiro \([14, 15]\) proved removable singularity and regularity assuming \( u \in L^{3+\varepsilon}(B_2) \) for some \( \varepsilon > 0 \) and \( u(x) = o(|x|^{-1}) \) as \( x \to 0 \), without assumption on \( p \). Choe and Kim \([4]\) proved removable singularity assuming \( u \in L^3(B_2) \) or \( u(x) = o(|x|^{-1}) \) as \( x \to 0 \), and regularity assuming \( u \in L^{3+\varepsilon}(B_2) \) for some \( \varepsilon > 0 \). Kim and Kozono \([10]\) recently proved removable singularity under the same assumptions as \([4]\), and regularity assuming \( u \in L^3(B_2) \) or \( u \) is small in weak \( L^3 \). As mentioned in \([10]\), their result is optimal in the sense that if their assumption is replaced by

\[
|u(x)| \leq C_* |x|^{-1} \tag{1.4}
\]

for \( 0 < |x| < 2 \), then the singularity is not removable in general, due to the existence of Landau solutions, which is the family of explicit singular solutions calculated by L. D. Landau in 1944 \([8]\), and can be found in standard textbooks, see e.g., \([9\text{, p. } 82]\) or \([1\text{, p. } 206]\).

The purpose of this article is to characterize the singularity and to identify the leading order behavior of very weak solutions satisfying the threshold assumption \( (1.4) \) when the constant \( C_* \) is sufficiently small. We show that it is given by Landau solutions.

We now recall Landau solutions in order to state our main theorems. Landau solutions can be parametrized by vectors \( b \in \mathbb{R}^3 \) in the following way: For each \( b \in \mathbb{R}^3 \) there exists a unique \((-1)\)-homogeneous solution \( U_b^\bullet \) of \( (1.1) \) together with an associated pressure \( P_b^\bullet \) which is \((-2)\)-homogeneous, such that \( U_b^\bullet, P_b^\bullet \) are smooth in \( \mathbb{R}^3 \setminus \{0\} \) and they solve

\[
-\Delta u + (u \cdot \nabla)u + \nabla p = b \delta, \quad \text{div} \ u = 0, \tag{1.5}
\]

in \( \mathbb{R}^3 \) in the sense of distributions, where \( \delta \) denotes the Dirac \( \delta \) function. When \( b = (0, 0, \beta) \) with \( \beta \geq 0 \), they have the following explicit formulas in spherical coordinates \( r, \theta, \phi \) with \( x = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \):

\[
U = \frac{2}{r} \left( \frac{A^2 - 1}{(A - \cos \theta)^2} - 1 \right) e_r - \frac{2 \sin \theta}{r(A - \cos \theta)} e_\theta, \quad P = \frac{-4(A \cos \theta - 1)}{r^2(A - \cos \theta)^2} \tag{1.6}
\]
where \( e_r = \frac{x}{r} \) and \( e_\theta = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \). The parameters \( \beta \geq 0 \) and \( A \in (1, \infty] \) are related by the formula
\[
\beta = 16\pi \left( A + \frac{1}{2} A^2 \log \frac{A - 1}{A + 1} + \frac{4A}{3(A^2 - 1)} \right).
\] (1.7)

The formulas for general \( b \) can be obtained from rotation. One checks directly that \( \|rU^b\|_{L^\infty} \) is monotone in \(|b|\) and \( \|rU^b\|_{L^\infty} \to 0 \) (or \( \infty \)) as \(|b| \to 0 \) (or \( \infty \)). Recently Sverak [16] proved that Landau solutions are the only solutions of (1.1) in \( \mathbb{R}^3 \backslash \{0\} \) which are smooth and \((-1)-homogeneous\) in \( \mathbb{R}^3 \backslash \{0\} \), without assuming axisymmetry. See also [18, 2, 11] for related results.

If \( u, p \) is a solution of (1.1), we will denote by
\[
T_{ij}(u, p) = p\delta_{ij} + u_i u_j - \partial_i u_j - \partial_j u_i
\] (1.8)
the momentum flux density tensor in the fluid, which plays an important role to determine the equation for \((u, p)\) at 0. Our main result is the following.

**Theorem 1.1** For any \( q \in (1, 3) \), there is a small \( C_* = C_*(q) > 0 \) such that, if \( u \) is a very weak solution of (1.1) with zero force in \( B_2 \backslash \{0\} \) satisfying (1.4) in \( B_2 \backslash \{0\} \), then there is a scalar function \( p \) satisfying \(|p(x)| \leq C|x|^{-2} \) unique up to a constant, so that \((u, p)\) satisfies (1.5) in \( B_2 \) with \( b_i = \int_{|x|=1} T_{ij}(u, p) n_j(x) \), and
\[
\|u - U^b\|_{W^{1,q}(B_1)} + \sup_{x \in B_1} |x|^{3/q-1}|u - U^b(x)| \leq C C_*,
\] (1.9)
where the constant \( C \) is independent of \( q \) and \( u \).

The exponent \( q \) can be regarded as the degree of the approximation of \( u \) by \( U^b \). The closer \( q \) gets to 3, the less singular \( u - U^b \) is. But in our theorem, \( C_*(q) \) shrinks to zero as \( q \to 3^- \). Ideally, one would like to prove that \( u - U^b \in L^\infty \). However, it seems quite subtle in view of the following model equation for a scalar function,
\[
- \Delta v + cv = 0, \quad c = \Delta v/v.
\] (1.10)
If we choose \( v = \log |x| \), then \( c(x) \in L^{3/2} \) and \( \lim_{|x| \to 0} |x|^2 |c(x)| = 0 \), but \( v \not\in L^\infty \). In equation (3.2) for the difference \( w = u - U^b \), there is a term \((w \cdot \nabla)U^b \) which has similar behavior as \( cv \) above.

In fact, we have the following stronger result. Denote by \( L_{wk}^r \) the weak \( L^r \) spaces. We claim the same conclusion as in Theorem 1.1 assuming only a small \( L^3_{wk} \) bound of \( u \) but not the pointwise bound (1.4).
Theorem 1.2 There is a small $\varepsilon_\ast > 0$ such that, if $u$ is a very weak solution of (1.1) with zero force in $\Omega = B_{2,1}\{0\}$ satisfying $\|u\|_{L^3_{\text{wk}}(\Omega)} =: \varepsilon \leq \varepsilon_\ast$, then $u$ satisfies $|u(x)| \leq C_1 \varepsilon |x|^{-1}$ in $B_2 \setminus \{0\}$ for some $C_1$. Thus the conclusion of Theorem 1.1 holds if $C_1 \varepsilon \leq C_\ast(q)$.

Our results are closely related to the regularity problem of very weak solutions, which could be considered when $u$ is only assumed to be in $L^2_{\text{loc}}$. In fact, the problem with the assumption $u$ being large in $L^3_{\text{wk}}$ already exhibits a great difficulty. Recall the scaling property of (1.1): If $(u, p)$ is a solution of (1.1), then so is $(u_\lambda, p_\lambda)(x) = (\lambda u(\lambda x), \lambda^2 p(\lambda x))$, $(\lambda > 0)$. (1.11)

The known methods are primarily perturbation arguments. Since $L^3_{\text{wk}}$-quasi-norm is invariant under the above scaling and does not become smaller when restricted to smaller regions, one would need to exploit the structure of the Navier-Stokes equations in order to get a positive answer. Compare the recent result \[3\] on axisymmetric solutions of nonstationary Navier-Stokes equations, which also considers a borderline case under the natural scaling.

This work is inspired by Korolev-Sverak \[11\] in which they study the asymptotic as $|x| \to \infty$ of solutions of (1.1) satisfying (1.4) in $\mathbb{R}^3 \setminus B_1$. They show that the leading behavior is also given by Landau solutions if $C_\ast$ is sufficiently small. Our theorem can be considered as a dual version of their result. However, their proof is based on the unique existence in $\mathbb{R}^3$ of the equation for $v = \varphi(u - U^b) + \zeta$ where $\varphi$ is a cut-off function supported near infinity and $\zeta$ is a suitable function chosen to make $\text{div } v = 0$. If one tries the same approach for our problem, since one needs to remove the origin as well as the region $|x| \geq 2$ while extending $u - U^b$, one needs to choose a sequence $\varphi_k$ with the supports of $1 - \varphi_k$ shrinking to the origin, which produce very singular force terms near the origin.

Instead, we first prove Lemma 2.3 which gives the equation for $(u, p)$ near the origin. Since the equation for $u$ is same as $U^b$ near the origin for $b = b(u)$, the $\delta$-functions at the origin cancel in the equation for their difference. We then apply the approach of Kim-Kozono \[10\] to the difference equation, and prove its unique existence in $W^{1, r}_{0}(B_2)$ for $3/2 \leq r < 3$ and uniqueness in $W^{1, r}_0 \cap L^3_{\text{wk}}(B_2)$ for $1 < r < 3/2$, which improves the regularity of the original difference. Above $W^{1, r}_0(B_2)$ is the closure of $C^\infty_c(B_2)$ in the $W^{1, r}(B_2)$-norm.

As an application, we give the following corollary. Recall $u_\lambda$ for $\lambda > 0$ is defined in (1.11). A solution $u$ on $B_2 \setminus \{0\}$ is called discretely self-similar if there is a $\lambda_1 \in (0, 1)$ so that $u_\lambda = u$. Such a solution is completely determined by its values in the annulus $B_1 \setminus B_{\lambda_1}$, since $u(\lambda_1 x) = \lambda_1^{-k} u(x)$. They contain minus-one homogeneous solutions as a special subclass.
Corollary 1.3 If $u$ satisfies the assumptions of Theorem 1.1 and furthermore $u$ is discretely self-similar in $B_2\{0\}$, then $u \equiv U^b$.

This corollary also follows from [11] (with domain $\mathbb{R}^3 \setminus B_1$ and $\lambda_1 > 1$). In the case of small $C_*$, this corollary extends the result of Sverak [16] on minus-one homogeneous solutions. The classification of discretely self-similar solutions with large $C_*$ is unknown.

As another application, we consider a conjecture by Sverak [16, §5]:

Conjecture 1.4 If $u$ is a solution of the stationary Navier-Stokes equations (1.1) with zero force in $\mathbb{R}^3\{0\}$ satisfying (1.4) with some $C_* > 0$. Then $u$ is a Landau solution.

We give a partial answer for this problem.

Corollary 1.5 Conjecture 1.4 is true, provided the constant $C_*$ is sufficiently small.

The above corollary can be also shown to be true by either our main theorem or the result of Korolev-Sverak [11], see section 3.4. The corresponding conjecture for large $C_*$ is related to the regularity problem of evolutionary Navier-Stokes equations via the usual blow-up procedures.

2 Preliminaries

In this section we collect some lemmas for the proof of Theorem 1.1. The first lemma recalls Hölder and Sobolev type inequalities in Lorentz spaces. We denote the Lorentz spaces by $L^{p,q}$ ($1 < p < \infty$, $1 \leq q \leq \infty$). Note $L^{3,\infty} = \text{weak } L^3$.

Lemma 2.1 Let $B = B_2 \subset \mathbb{R}^n$, $n \geq 2$.

i) Let $1 < p_1, p_2 < \infty$ with $1/p := 1/p_1 + 1/p_2 < 1$ and let $1 \leq r_1, r_2 \leq \infty$. For $f \in L^{p_1,r_1}$ and $g \in L^{p_2,r_2}$, we have

$$
\|fg\|_{L^{p,r}(B)} \leq C\|f\|_{L^{p_1,r_1}(B)}\|g\|_{L^{p_2,r_2}(B)} \quad \text{for } r := \min\{r_1, r_2\},
$$

(2.1)

where $C = C(p_1, r_1, p_2, r_2)$.

ii) Let $1 < r < n$. For $f \in W^{1,r}(B)$, we have

$$
\|f\|_{\text{weak } L^{n,r}(B)} \leq C\|f\|_{W^{1,r}(B)},
$$

(2.2)

where $C = C(n, r)$. 

Part (i) of Lemma 2.1 was proved in [13]. Part (ii) was proved in [13] for $\mathbb{R}^n$ and in [12, 10] for bounded domains.

By this lemma, when $n = 3$ and $1 < r < 3$, we have
\[
\|fg\|_{L^r(B)} \leq C \|f\|_{L^{3/w_k}(B)} \|g\|_{L^{\frac{3}{w_r},r}(B)} \leq C_r \|f\|_{L^3(B)} \|g\|_{W^{1,r}(B)}.
\] (2.3)

This estimate first appeared in [10] and plays an important role for our application.

The next lemma is on interior estimates for Stokes system with no assumption on the pressure.

**Lemma 2.2** Assume $v \in L^1$ is a distribution solution of the Stokes system
\[
-\Delta v_i + \partial_j f_{ij} = 0 \quad \text{div} v = 0 \quad \text{in } B_{2R}
\] (2.4)
and $f \in L^r$ for some $r \in (1, \infty)$. Then $v \in W^{1,r}_{loc}$ and, for some constant $C_r$ independent of $v$ and $R$,
\[
\|\nabla v\|_{L^r(B_{R})} \leq C_r \|f\|_{L^r(B_{2R})} + C_r R^{-4+3/r} \|v\|_{L^1(B_{2R})}.
\] (2.5)

This lemma is [17], Theorem 2.2. Although the statement in [17] assumes $v \in W^{1,r}_{loc}$, its proof only requires $v \in L^1$. This lemma can be also considered as [3, Lemma A.2] restricted to time-independent functions.

The following lemma shows the first part of Theorem 1.1, except (1.9). In particular, it shows that $(u, p)$ solves (1.5).

**Lemma 2.3** If $u$ is a very weak solution of (1.1) with zero force in $B_2 \setminus \{0\}$ satisfying (1.4) in $B_2 \setminus \{0\}$ (with $C_*$ allowed to be large), there is a scalar function $p$ satisfying $|p(x)| \leq C|x|^{-2}$, unique up to a constant, such that $(u, p)$ satisfies (1.5) in $B_2$ with
\[
b_i = \int_{|x|=1} T_{ij}(u, p)n_j(x).
\]
Moreover, $u, p$ are smooth in $B_{3/2}\setminus\{0\}$.

**Proof.** For each $R \in (0, 1/2]$, $u$ is a very weak solution in $B_2 - \bar{B}_R$ in $L^\infty$. Lemma 2.2 shows $u$ is a weak solution in $W^{1,2}_{loc}$. The usual theory shows that $u$ is smooth and there is a scalar function $p_R$, unique up to a constant, so that $(u, p_R)$ solves (1.1) in $B_2 - \bar{B}_R$, see e.g. [7]. By the scaling argument in Sverak-Tsai [17] using Lemma 2.2, we have for $x \in B_{3/2} - B_{2R}$,
\[
|\nabla^k u(x)| \leq \frac{C_k C_*}{|x|^{k+1}} \quad \text{for } k = 1, 2, \ldots,
\] (2.6)
where $C_k = C_k(C_*)$ are independent of $R \in (0, 1/2]$ and its dependence on $C_*$ can be dropped if $C_* \in (0, 1)$. Varying $R$, (2.6) is valid for $x \in B_{3/2}\setminus\{0\}$. For $0 < R < R'$, by uniqueness of $p_{R'}$, the difference $p_{R'|B_2 - B_{R'}} - p_{R'}$ is a constant. Thus we can fix
the constant by requiring $p_R = p_{1/2}$ in $B_2 \setminus \bar{B}_{1/2}$, and define $p(x) = p_R(x)$ for any $x \in B_2 \setminus \{0\}$ with $R = |x|/2$. By the equation, $|\nabla p(x)| \leq CC_* |x|^{-3}$. Integrating from $|x| = 1$ we get $|p(x)| \leq CC_* |x|^{-2}$. In particular

$$|T_{ij}(u, p)(x)| \leq CC_* |x|^{-2} \quad \text{for } x \in B_{3/2} \setminus \{0\}. \quad (2.7)$$

Denote $NS(u) = -\Delta u + (u \cdot \nabla)u + \nabla p$. We have $NS(u)_i = \partial_j T_{ij}(u)$ in the sense of distributions. Thus, by divergence theorem and $NS(u) = 0$ in $B_2 \setminus \{0\}$,

$$b_i = \int_{|x|=1} T_{ij}(u, p)n_j(x) = \int_{|x|=R} T_{ij}(u, p)n_j(x) \quad (2.8)$$

for any $R \in (0, 2)$. Let $\phi$ be any test function in $C^\infty_c(B_1)$. For small $\varepsilon > 0$,

$$\langle NS(u)_i, \phi \rangle = -\int T_{ij}(u) \partial_j \phi$$

$$= -\int_{B_1 \setminus B_\varepsilon} T_{ij}(u) \partial_j \phi - \int_{B_\varepsilon} T_{ij}(u) \partial_j \phi$$

$$= \int_{B_1 \setminus B_\varepsilon} \partial_j T_{ij}(u) \phi + \int_{\partial B_\varepsilon} T_{ij}(u) \phi n_j - \int_{\partial B_1} T_{ij}(u) \phi n_j - \int_{B_\varepsilon} T_{ij}(u) \partial_j \phi.$$

In the last line, the first integral is zero since $NS(u) = 0$ and the third integral is zero since $\phi = 0$. By the pointwise estimate (2.7), the last integral is bounded by $C\varepsilon^{3-2}$. On the other hand, by (2.8),

$$\int_{\partial B_\varepsilon} T_{ij}(u) \phi n_j \to b_i \phi(0) \quad \text{as } \varepsilon \to 0. \quad (2.9)$$

Thus $(u, p)$ solves (1.5) and we have proved the lemma. \hfill \Box

It follows from the proof that $|b| \leq CC_*$ for $C_* < 1$. With this lemma, we have completely proved Theorem 1.1 in the case $q < 3/2$. In the case $3/2 \leq q < 3$, it remains to prove (1.9).

### 3 Proof of main theorem

In this section, we present the proof of Theorem 1.1. We first prove that solutions belong to $W^{1, q}$. We next apply this result to obtain the pointwise estimate. For what follows, denote

$$w = u - U, \quad U = U^b, \quad (3.1)$$
where $U^b$ is the Landau solution with $b$ given by (2.8). By Lemma 2.3 there is a function $\tilde{p}$ such that $(w, \tilde{p})$ satisfies in $B_2$ that

$$
-\Delta w + U \cdot \nabla w + w \cdot \nabla (U + w) + \nabla \tilde{p} = 0, \quad \text{div } w = 0,
$$

$$
|w(x)| \leq \frac{CC_*}{|x|}, \quad |\tilde{p}(x)| \leq \frac{CC_*}{|x|^2}.
$$

(3.2)

Note that the $\delta$-functions at the origin cancel.

### 3.1 $W^{1,q}$ regularity

In this subsection we will show $w \in W^{1,q}(B_1)$. Fix a cut off function $\varphi$ with $\varphi = 1$ in $B_{9/8}$ and $\varphi = 0$ in $B_{11/8}^c$. We localize $w$ by introducing

$$
v = \varphi w + \zeta
$$

(3.3)

where $\zeta$ is a solution of the problem $\text{div } \zeta = -\nabla \varphi \cdot w$. By Galdi [6, Ch.3] Theorem 3.1, there exists such a $\zeta$ satisfying

$$
supp \zeta \subset B_{3/2} \setminus B_1, \quad \|\nabla \zeta\|_{L^{100}} \leq C \|\nabla \varphi \cdot w\|_{L^{100}} \leq CC_*.
$$

(3.4)

The vector $v$ is supported in $\overline{B}_{3/2} \setminus B_1$, satisfies $v \in W^{1,r}_w \cap L^3$ for $r < 3/2$ by (1.4), (2.6) and (3.4), and

$$
-\Delta v + U \cdot \nabla v + v \cdot \nabla (U + v) + \nabla \pi = f, \quad \text{div } v = 0,
$$

(3.5)

where $\pi = \varphi \tilde{p}$, and

$$
f = -2(\nabla \varphi \cdot \nabla)w - (\Delta \varphi)w + (U \cdot \nabla \varphi)w + (\varphi^2 - \varphi)w \cdot \nabla w + (w \cdot \nabla \varphi)w + \tilde{p} \nabla \varphi - \Delta \zeta + (U \cdot \nabla)\zeta + \zeta \cdot \nabla (U + \varphi w + \zeta) + \varphi w \cdot \nabla \zeta
$$

(3.6)

is supported in the annulus $B_{3/2} \setminus B_1$. One verifies directly that, for some $C_1$,

$$
\sup_{1 \leq r \leq 100} \|f\|_{W^{-1,r}_0(B_2)} \leq C_1 C_*.
$$

(3.7)

Our proof is based on the following lemmas.

**Lemma 3.1 (Unique existence)** For any $3/2 \leq r < 3$, for sufficiently small $C_* = C_*(r) > 0$, there is a unique solution $v$ of (3.5) and (3.7) in the set

$$
V = \{v \in W^{1,r}_0(B_2), \quad \|v\|_V := \|v\|_{W^{1,r}_0(B_2)} \leq C_2 C_*\}
$$

(3.8)

for some $C_2 > 0$ independent of $r \in [3/2, 3)$.
Lemma 3.2 (Uniqueness) Let $1 < r < 3/2$. If both $v_1$ and $v_2$ are solutions of (3.5) and (3.7) in $W^{1,r}_0 \cap L^3_{uw}$ and $C_s + \|v_1\|_{L^3_{uw}} + \|v_2\|_{L^3_{uw}}$ is sufficiently small, then $v_1 = v_2$.

Assuming the above lemmas, we get $W^{1,q}$ regularity as follows. First we have a solution $\tilde{v}$ of (3.5) in $W^{1,q}_0(B_2)$ by Lemma 3.1. On the other hand, both $v = \varphi w + \zeta$ and $\tilde{v}$ are small solutions of (3.5) in $W^{1,r}_0 \cap L^3_{uw}(B_2)$ for $r = 5/4$, and thus $v = \tilde{v}$ by Lemma 3.2. Thus $v \in W^{1,q}_0(B_2)$ and $w \in W^{1,q}(B_1)$.

Proof of Lemma 3.1. Consider the following mapping $\Phi$: For each $v \in V$, let $\tilde{v} = \Phi v$ be the unique solution in $W^{1,r}_0(B_2)$ of the Stokes system

$$
- \Delta \pi + \nabla \pi = f - \nabla \cdot (U \otimes v + v \otimes (U + v)), \quad \operatorname{div} \pi = 0. \tag{3.9}
$$

By estimates for the Stokes system, see Galdi [6, Ch.4] Theorem 6.1, in particular (6.9), for $1 < r < \infty$, we have

$$
\|\tilde{v}\|_{W^{1,r}_0(B_2)} \leq N_r \|f\|_{W^{-1,r}_0} + N_r \|\nabla \cdot (U \otimes v + v \otimes (U + v))\|_{W^{-1,r}} \tag{3.10}
$$

for some constant $N_r > 0$ which is uniformly bounded for $r$ in any compact regions of $(1, \infty)$. By (3.7) and Lemma 2.1 in particular (2.3), for $1 < r < 3$,

$$
\|\tilde{v}\|_{W^{1,r}_0(B_2)} \leq N_r C_1 C_s + N_r \|U \otimes v + v \otimes (U + v)\|_{L^r} \leq N_r C_1 C_s + N_r C_r (\|U\|_{L^3_{uw}} + \|v\|_{L^3_{uw}}) \|v\|_V. \tag{3.11}
$$

We now choose $C_2 = 2(C_1 + 1) \sup_{3/2 \leq r < 3} N_r$. Since $V \subset L^3_{uw}$ if $r \geq 3/2$, we get $\tilde{v} = \Phi v \in V$ if $C_s$ is sufficiently small.

We next consider the difference estimate. Let $v_1, v_2 \in V$, $\tilde{v}_1 = \Phi v_1$, and $\tilde{v}_2 = \Phi v_2$. Then

$$
\|\Phi v_1 - \Phi v_2\|_{W^{1,r}} \leq C C_r (\|U\|_{L^3_{uw}} + \|v_1\|_{L^3_{uw}} + \|v_2\|_{L^3_{uw}}) \|v_1 - v_2\|_{W^{1,r}}. \tag{3.12}
$$

Taking $C_s$ sufficiently small for $3/2 \leq r < 3$, we get $\|\Phi v_1 - \Phi v_2\|_V \leq \frac{1}{2} \|v_1 - v_2\|_V$, which shows that $\Phi$ is a contraction mapping in $V$ and thus has a unique fixed point. We have proved the unique existence of the solution for (3.5)-(3.7) in $V$.

Remark. Since the constant $C_r$ from Lemma 2.1 (ii) blows up as $r \to 3$, our $C_s$ shrinks to zero as $r \to 3$.

Proof of Lemma 3.2. By the difference estimate (3.12), we have

$$
\|v_1 - v_2\|_{W^{1,r}} \leq C (\|U\|_{L^3_{uw}} + \|v_1\|_{L^3_{uw}} + \|v_2\|_{L^3_{uw}}) \|v_1 - v_2\|_{W^{1,r}}. \tag{3.13}
$$

Thus, if $C (\|U\|_{L^3_{uw}} + \|v_1\|_{L^3_{uw}} + \|v_2\|_{L^3_{uw}}) < 1$, we conclude $v_1 = v_2$.

9
3.2 Pointwise bound

In this subsection, we will prove pointwise bound of $w$ using $\|w\|_{W^{1,q}} \lesssim C_s$.

For any fixed $x_0 \in B_{1/2} \setminus \{0\}$, let $R = |x_0|/4$ and $E_k = B(x_0, kR)$, $k = 1, 2$.

Note $q^* \in (3, \infty)$. Let $s$ be the dual exponent of $q^*$, $1/s + 1/q^* = 1$. We have

$$\|w\|_{L^1(E_2)} \lesssim \|w\|_{L^{q^*}(E_2)} \|1\|_{L^{s}(E_2)} \lesssim C_s R^{4-3/q}.$$ \hfill (3.14)

By the interior estimate Lemma 2.2

$$\|\nabla w\|_{L^{q^*}(E_1)} \lesssim \|f\|_{L^{q^*}(E_2)} + R^{-4+3/q^*} \|w\|_{L^{1}(E_2)}$$ \hfill (3.15)

where $f = U \otimes w + w \otimes (U + w)$. Since $|U| + |w| \lesssim C_s |x|^{-1} \lesssim C_s R^{-1}$ in $E_2$,

$$\|f\|_{L^{q^*}(E_2)} \lesssim C_s R^{-1} \|w\|_{L^{q^*}(E_2)} \lesssim C_s^2 R^{-1}.$$ \hfill (3.16)

We also have $R^{-4+3/q^*} \|w\|_{L^{1}(E_2)} \lesssim R^{-4+3/q^*} C_s R^{4-3/q} = C_s R^{-1}$. Thus

$$\|\nabla w\|_{L^{q^*}(E_1)} \lesssim C_s R^{-1}.$$ \hfill (3.17)

By Gagliardo-Nirenberg inequality in $E_1$,

$$\|w\|_{L^{\infty}(E_1)} \lesssim \|w\|_{L^{q^*}(E_1)}^{1-\theta} \|\nabla w\|_{L^{q^*}(E_1)}^\theta + R^{-3} \|w\|_{L^{1}(E_1)},$$ \hfill (3.18)

where $1/\infty = (1 - \theta)/q^* + \theta(1/q_s - 1/3)$ and thus $\theta = 3/q - 1$. We conclude $\|w\|_{L^{\infty}(E_1)} \lesssim C_s R^{-\theta}$. Since $x_0$ is arbitrary, we have proved the pointwise bound, and completed the proof of Theorem 1.1.

Remark. Equivalently, one can define $v(x) = u(x_0 + Rx)$, find the equation of $v$, estimate $v$ in $L^{\infty}(B_1)$, and then derive the bound for $w(x_0)$.

3.3 Proof of Theorem 1.2

In this subsection we prove Theorem 1.2. For any $x_0 \in B_2 \setminus \{0\}$, let $v(x) = \lambda u(\lambda x + x_0)$ with $\lambda = \min(0.1, |x_0|)/2$. By our choice of $\lambda$, $v$ is a very weak solution in $B_2$ and $\|v\|_{L^{\infty}_{W}B_2} \leq \varepsilon = \|u\|_{L^{\infty}_{W}B_2 \setminus \{0\}}$. By [10], we have $\|v\|_{L^{\infty}(B_1)} \leq C_2 \varepsilon$ for some constant $C_2$ if $\varepsilon$ is sufficiently small. Thus $|u(x_0)| \leq C_2 \varepsilon \lambda^{-1} \leq 40 C_2 \varepsilon |x_0|^{-1}$.

3.4 Proof of Corollary 1.5

In this subsection we prove Corollary 1.5. Suppose $u$ satisfies (1.4) with $C_* = C_s(q)$, $q = 2$, given in Theorem 1.1. Let $b$ be given by (2.8), $U = U^b$ and $w = u - U$. 

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Let \( u_\lambda = \lambda u(\lambda x) \) be the rescaled solution and \( w_\lambda(x) = \lambda w(\lambda x) \). Note \( U \) is scaling-invariant. Then \( u_\lambda = U + w_\lambda \) also satisfies (1.4) with same \( C_* \). By Theorem 1.1 with \( q = 2 \), we have the bound

\[
|w_\lambda(x)| \leq C C_* |x|^{-1/2}, \quad |x| < 1,
\]

which is uniform in \( \lambda \). In terms of \( w \) and \( y = \lambda x \), we get

\[
|w(y)| \leq C C_* \lambda^{-1} |\lambda^{-1} y|^{-1/2}, \quad |y| \leq \lambda.
\]

Now fix \( y \) and let \( \lambda \to \infty \). We conclude \( w \equiv 0 \).

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