Efficient drift parameter estimation for ergodic solutions of backward SDEs

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Abstract. We derive consistency and asymptotic normality results for quasi-maximum likelihood methods for drift parameters of ergodic stochastic processes observed in discrete time in an underlying continuous-time setting. The special feature of our analysis is that the stochastic integral part is unobserved and non-parametric. Additionally, the drift may depend on the (unknown and unobserved) stochastic integrand. Our results hold for ergodic semi-parametric diffusions and backward SDEs. Simulation studies confirm that the methods proposed yield good convergence results.

Keywords. asymptotic normality; backward SDEs; consistency; ergodic diffusion processes; maximum-likelihood-type estimation; unobserved volatility processes

1 Introduction

The paper analyzes statistical inference for Markovian ergodic forward backward stochastic differential equations (BSDEs). Ergodic solutions of backward SDEs may be seen as a generalization of an ergodic Markovian diffusion process with unknown but ergodic diffusion part. Specifically, consider a probability space \((\Omega, \mathcal{F} = (\mathcal{F}_t), P)\) with filtration \(\mathcal{F}\) being generated by a \(d\)-dimensional Brownian motion \(W\). Let \(Y\) be a \(d\)-dimensional Markov diffusion process depending on an unknown parameter \(\theta \in \mathbb{R}^m\). \(Y\) will in the sequel be also referred to as a data generating process. In the classical statistical inference problem for stochastic processes \(Y\) satisfies a stochastic differential equation of the form

\[dY_t = \psi(t, Y_t, \theta) + \sigma(t, Y_t, \theta) dW_t\] (1)

where \(\psi\) and \(\sigma\) are known functions and \(Y_0\) is assumed to be known as well. A classical example for \(Y\) is given by a Brownian motion with drift or, rather popular in finance, a geometric Brownian motion. Statistical inference results for (1) are analyzed through quasi-maximum likelihood methods in Yoshida (1992, 2011), Kessler (1997) and Uchida and Yoshida (2012). They have been extended to jump–diffusion processes by Shimizu and Yoshida (2006) and Ogihara and Yoshida (2011).

Now assume that the diffusion function \(\sigma\) in (1) is unknown and that we only know that the integrand of the diffusion part is given by a positive definite \(\mathbb{R}^{d \times d}\)-valued ergodic predictable process, say \(V_t V_t^T\) bounded away from zero. This leads to the stochastic differential equation

\[dY_t = \psi(t, Y_t, \theta) + V_t dW_t\]

where \(V\) may be identified with a triagonal ergodic stochastic process. Next, suppose that we additionally allow the integrand of the drift, \(\psi\), to possibly also depend on \(V_t V_t^T\) and furthermore on an observed additional Markov process \(X\). Then we have that \(Y\) satisfies

\[dY_t = \psi(t, X_t, Y_t, V_t V_t^T, \theta) dt + V_t dW_t.\] (2)

This equation is also called a backward stochastic differential equation with solution \((Y, V)\) and driver function \(\psi\). The goal of this paper to give consistency and asymptotic normality results to estimate \(\theta\) in (2) with data generating processes \((Y, X)\) and discrete time observations.
BSDEs have been introduced by Peng and Pardoux (1991) and have since been extended in many directions regarding assumptions on the driver function, connections to PDEs and Hamilton-Jacobi-Bellman equations, applications to stochastic optimal control theory, smoothness of \( (Y, V) \), robustness, numerical approximations and invariance principles. Although originally developed for a finite maturity, in many situations the terminal time is either random or there is no natural terminal time at all and the decision maker faces instead an infinite time horizon. Usually in the theory of BSDEs existence and uniqueness of a solution can be guaranteed by Lipschitz conditions on the driver. Now for an infinite time horizon the BSDE may be ill posed which has been addressed by Briand and Hu (1998) by imposing a monotonic assumption on the driver. However, for our statistical analysis we will simply assume that the data generating process satisfies an equation of the form \( (2) \) and is ergodic. In this case we refer to \( (2) \) as also an ergodic BSDE.

Ergodic backward SDEs for finite or infinite dimensional Brownian motion have for instance been considered in Buckdahn and Peng (1999), Fuhrmann, Hu and Tessitore (2009), Richou (2009), Debusche, Hu and Tessitore (2011), Hu and Wang (2018), Madec (2015), Hu et al. (2015), Liang and Zariphopoulou (2017), Chong et al. (2019), Hu and Leonnier (2019), Hu, Liang and Tang (2020) and Guatto and Tessitore (2020).

For statistical inference on BSDEs there is in general not much literature available. For nonparametric estimation of linear drivers see Su and Lin (2009), Chen and Lin (2010) and Zhang (2013). Zhang and Lin (2019), Hu and Lemonnier (2019), Hu, Liang and Zariphopoulou (2017), Chong et al. (2019), Hu and Lemonier (2019), Hu, Liang and Tang (2020) and Guatto and Tessitore (2020).

The paper is structured as follows: In Section 2 we describe the setting our assumptions and give the main results. Section 3 gives a number of applications and examples. Section 4 contains numerical studies in the one- and multidimensional case. The proofs can be found in Section 5.

2 Main results

Given a probability space \( (\Omega, \mathcal{F}, P) \) with a right-continuous filtration \( \mathbf{F} = (\mathcal{F}_t)_{t \geq 0} \), let \( Y = (Y_t)_{t \geq 0} \) be a \( d_Y \)-dimensional \( \mathbf{F} \)-adapted process satisfying

\[
Y_t = Y_T - \int_t^T \psi(X_s, Y_s, V_s^T, \theta_0)ds - \int_t^T V_s dW_s, \quad 0 \leq t \leq T < \infty,
\]

where \( W = (W_t)_{t \geq 0} \) is a \( d_W \)-dimensional standard \( \mathbf{F} \)-Wiener process (\( dW \geq dy \)), \( \theta_0 \in \Theta \) is an unknown parameter, \( \Theta \) is a bounded open subset in \( \mathbb{R} \), \( \psi \) is an \( \mathbb{R}^{d_Y} \)-valued function, \( X = (X_t)_{t \geq 0} \) is a \( d_X \)-dimensional continuous \( \mathbf{F} \)-adapted process, \( V = (V_t)_{t \geq 0} \) is a \( d_Y \times d_W \) matrix-valued continuous \( \mathbf{F} \)-adapted process. The dimension \( d_X \) of \( X_t \) is possibly zero. In that case, we ignore \( X_t \). We observe \( \{(X_{k h_n}, Y_{k h_n})\}_{k=0}^\infty \), and consider asymptotics: \( h_n \to 0, n h_n \to \infty \) and \( n h_n^2 \to 0 \) as \( n \to \infty \).

We construct a maximum-likelihood-type estimator for the parameter \( \theta_0 \). For this purpose, we construct a quasi-likelihood function \( H_n(\theta) \). Let \( \Delta_l U = U_{t_l} - U_{t_{l-1}} \) for a stochastic process \( (U_t)_{t \geq 0} \). Let \((c_n)_{n \in \mathbb{N}} \) be a sequence of positive integers such that

\[
c_n n^{-\epsilon} \to \infty \quad \text{and} \quad c_n h_n n^\epsilon \to 0, \quad (3)
\]

for some \( \epsilon > 0 \). Let \( L_n = [n/c_n], t_m = (m + c_n l)h_n \), and let

\[
\hat{Z}_l = \frac{1}{c_n h_n} \sum_{m=1}^{c_n} (Y_{t_m} - Y_{t_{m-1}})(Y_{t_m} - Y_{t_{m-1}})^T \quad (0 \leq l \leq L_n - 1),
\]

where \( \tau \) denotes transpose. We define a quasi-log-likelihood function by

\[
H_n(\theta) = -\frac{1}{2} \sum_{l=1}^{L_n-1} \left\{ \left( \Delta_l Y - c_n h_n \hat{\psi}_l(\theta) \right) \frac{\hat{Z}_{l-1}}{c_n h_n} (\Delta_l Y - c_n h_n \hat{\psi}_l(\theta)) \right\} 1_{\{\hat{Z}_{l-1} > 0\}},
\]

(4)
where $\Theta$ is the closure of $\Theta$ and $\hat{\psi}(\theta) = \psi(X_{t_0}, Y_{t_0}, \hat{Z}_{t-1}, \theta)$. Let $\Delta_t U = U_{t+1} - U_t$ for a stochastic process $(U_t)_{t \geq 0}$.

Then we can construct a maximum-likelihood-type estimator $\hat{\theta}_n$ as a random variable which maximizes $H_n$; $\hat{\theta}_n \in \arg\max_{\theta \in \Theta} H_n(\theta)$.

Let $\mathcal{P}$ be the space of $d_Y \times d_Y$ symmetric, positive definite matrices. For a vector $v = (v_i)_{1 \leq i \leq k}$ and a matrix $m = (m_{ij})_{1 \leq j \leq k_2}$, we denote

$$
\delta_v^k = \left( \frac{\partial}{\partial v_{i_1}} \cdots \frac{\partial}{\partial v_{i_l}} \right)_{i_1, \ldots, i_l = 1} \quad \text{and} \quad \delta_m^k = \left( \frac{\partial}{\partial m_{i_{j_1}, \ldots, i_{j_l}}} \right)_{1 \leq j_1, \ldots, j_l \leq k_1}.
$$

We assume that $\Theta$ admits Sobolev’s inequality, that is, for any $p > d$, there exists a positive constant $C_p$ depending only $p$ and $\Theta$ such that

$$
\sup_{x \in \Theta} |u(x)| \leq C \sum_{k=0,1} \left( \int_{\Theta} \left| \frac{\partial^k u(x)}{\partial x} \right|^p dx \right)^{1/p} \quad (5)
$$

for any $u \in C^1(\Theta)$. Sobolev’s inequality is satisfied if $\Theta$ has a Lipschitz boundary (see Adams and Fournier (2003)).

Let $\mathcal{P}$ be the closure of $\mathcal{P}$ in $\mathbb{R}^{d_Y} \otimes \mathbb{R}^{d_Y}$, and $\mathcal{P}_{\delta} = \{ z \in \mathcal{P} | z - \delta I \in \mathcal{P} \}$ for any $\delta > 0$, where $I$ is the unit matrix. For $p \geq 1$ and $r \geq 1$, we consider the following assumptions.

**Assumption (A1-p).** $\sup_{t \geq 0} ||(V_t V_t^T)^{-1}|| < \infty$ almost surely and there exists a positive constant $C$ such that

$$
E[|V_t - V_s|^2]^{1/2} + E[|X_t - X_s|^2]^{1/2} \leq C|t - s|^{1/2},
$$

$$
E \left[ \frac{E[|V_t - V_s|^2]^{2p}}{(t - s)} \right] \leq C,
$$

$$
E[|X_s|^2] \leq C.
$$

for $0 \leq s < t$.

**Assumption (A2-r).** $\delta_y^l \psi(x, y, z, \theta)$ exists and is continuous on $\mathbb{R}^{d_X} \times \mathbb{R}^{d_Y} \times \mathcal{P} \times \bar{\Theta}$, for $l = 0, 1, 2$, and there exists a constant $C$ such that

$$
|\delta_y^l \psi(x, y, z, \theta)| \leq C(1 + |x| + |y| + |z|)^r.
$$

Moreover, for any $\delta > 0$, there exists a constant $C_\delta$ such that

$$
|\delta_y^l \psi(x_1, y_1, z_1, \theta) - \delta_y^l \psi(x_2, y_2, z_2, \theta)| \leq C_\delta(1 + |x_1| + |y_1| + |z_1|)^r(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)
$$

for $l = 0, 1, 2$, $x_1, x_2 \in \mathbb{R}^{d_X}$, $y_1, y_2 \in \mathbb{R}^{d_Y}$, $z \in \mathcal{P}$, $z_1, z_2 \in \mathcal{P}$, and $\theta \in \Theta$.

**Assumption (A3-p).** At least one of the following two conditions holds true.

1. The function $\psi(x, y, z, \theta)$ does not depend on $y$ and $(X_t, V_t V_t^T)$ is ergodic, that is, there exists an invariant distribution $\pi$ such that for any measurable function $f$,

$$
\frac{1}{T} \int_0^T f(X_t, V_t V_t^T) dt \overset{P}{\to} \int f(x, z) \pi(dx dz),
$$

as $T \to \infty$. Moreover,

$$
\int \left( \frac{1 + |x| + |z|}{(\det z) \wedge 1} \right)^p \pi(dx dz) < \infty.
$$
2. \((X_t, Y_t, V_t V_t^\top)\) is ergodic, that is, there exists an invariant distribution \(\pi\) such that for any measurable function \(f\),
\[
\frac{1}{T} \int_0^T f(X_t, Y_t, V_t V_t^\top) dt \xrightarrow{P} \int f(x, y, z) \pi(dx dy dz),
\]
as \(T \to \infty\). Moreover,
\[
\int \left( \frac{1 + |x| + |y| + |z|}{(\det z) \wedge 1} \right)^p \pi(dx dy dz) < \infty.
\]

**Assumption (A4).** (Identifiability condition) For \(\theta_1, \theta_2 \in \bar{\Theta}, \psi(x, y, z, \theta) = \psi(x, y, z, \theta_2)\) for all \((x, y, z)\) on \(\text{supp}(\pi)\) implies \(\theta_1 = \theta_2\).

Most of the above assumptions are standard for asymptotic theory of maximum-likelihood-type estimation to ergodic diffusion processes, and similar (or stronger) assumptions are required in Kessler (1997) and Uchida and Yoshida (2012). A similar statement applies to Condition \((A2')\) appearing later. Here, the upper bound \(C_\delta\) of \(\partial_\theta^r \psi\) in \((A2-r)\) depends on \(\delta\). While this assumption is not a typical one, by doing so, \((A2-r)\) is satisfied even the case that \(\psi\) is not smooth at \(z = 0\) (for example, \(\psi(x, y, z, \theta) = \theta \sqrt{1 + \frac{1}{2} |z|}\) with \(a_Y = 1\)). For sufficient conditions of ergodicity for \((X_t, Y_t, V_t V_t^\top)\), we refer readers to Remark 1 of Uchida and Yoshida (2012).

Fix \(\delta > 0\) satisfying (4). Under the assumptions above, we obtain consistency of our estimator.

**Theorem 2.1** (consistency). Let \(p, r \geq 1\) such that
\[
\frac{p}{4r} > d \vee \frac{2}{r} \vee 4. \tag{6}
\]
Assume \((A1-p), (A2-r), (A3-p),\) and \((A4)\). Then \(\hat{\theta}_n \xrightarrow{P} \theta_0\) as \(n \to \infty\).

Under \((A2-r)\) and \((A3-p)\), we define
\[
\Gamma = \int \partial_\theta \psi(x, z, \theta_0)^T z^{-1} \partial_\theta \psi(x, z, \theta_0) \pi(dx dz)
\]
if the function \(\psi(x, y, z, \theta)\) does not depend on \(y\), and otherwise we define
\[
\Gamma = \int \partial_\theta \psi(x, y, z, \theta_0)^T z^{-1} \partial_\theta \psi(x, y, z, \theta_0) \pi(dx dy dz).
\]

To deduce asymptotic normality of our estimator, we need a further condition. Let \(\mathcal{O}\) be an open set in \(\mathbb{R}^{d_Y} \otimes \mathbb{R}^{d_V}\) such that \(\mathbb{P} \subset \mathcal{O}\).

**Assumption \((A2'-r)\).** \((A2-r)\) is satisfied. \(\partial_\theta^r \partial_\theta^s \partial_\theta^t \partial_\theta^u \psi(x, y, z, \theta)\) exists and is continuous on \(\mathbb{R}^{dx} \times \mathbb{R}^{dy} \times \mathcal{O} \times \bar{\Theta}\) for \(l \in \{0, 1, 2, 3\}\) and \(i, j, k \in \{0, 1, 2\}\) with \(i + j + k \leq 2\), and for any \(\delta > 0\), there exists a constant \(C'_\delta\) such that
\[
|\partial_\theta^r \partial_\theta^s \partial_\theta^t \partial_\theta^u \psi(x, y, z, \theta)| \leq C'_\delta (1 + |x| + |y| + |z|)^r
\]
for \(x \in \mathbb{R}^{dx}, y \in \mathbb{R}^{dy}, z \in \mathbb{P}_s, l \in \{0, 1, 2, 3\}\) and \(i, j, k \in \{0, 1, 2\}\) with \(i + j + k \leq 2\).

Moreover, there exist a Wiener process \((W'_t)_{t \geq 0}\) independent of \((W_t)_{t \geq 0}\) and \(\mathbb{F}\)-progressively measurable processes \((a'_t)_{t \geq 0}\) for \(j \in \{1, 2, 3\}\) such that
\[
X_t = X_0 + \int_0^t a'_t^1 ds + \int_0^t a'_t^2 dW_s + \int_0^t a'_t^3 dW'_s,
\]
and \(\sup_{t \geq 0} E[|a'_t|^p] < \infty\) for any \(p > 0\) and \(j \in \{1, 2, 3\}\).

Suppose that \(n^3 h_n^5 \to 0\). Then we can choose \(c_n\) in the definition of \(H_n\) satisfying
\[
n h_n^2 c_n \to 0 \quad \text{and} \quad n h_n / c_n \to 0. \tag{7}
\]
For such \(c_n\), fix \(\epsilon > 0\) satisfying (3).
Theorem 2.2 (Asymptotic normality). Let $p, r \geq 1$ such that (+) is satisfied. Assume (A1-p), (A2-r), (A3-p), (A4), and that $n^3 h_n^{5,5} \to 0$ as $n \to \infty$. Assume further that $\Gamma$ is positive definite and $c_n$ satisfies $\delta$. Then
\[
\sqrt{nh_n} (\hat{\theta}_n - \theta_0) \overset{d}{\to} N(0, \Gamma^{-1}).
\]

The condition $n^3 h_n^{5,5} \to 0$ is stronger than the ones in previous works (for instance $nh_n^2 \to 0$ in Yoshida (2011), and $nh_n^2 \to 0$ for $p \geq 2$ in Uchida and Yoshida (2012) and Kessler (1997)). Unlike previous studies, we need to construct an estimator $\hat{Z}_t$ of $Z_t$ whose structure is not specified. For this purpose, (7) and consequently $n^3 h_n^{5,5} \to 0$ is required.

Remark 2.1. If $V_t$ is a diffusion process with SDE-coefficients not depending on $\theta$, $\hat{\theta}_n$ is asymptotically efficient under the assumptions of Gobet (2002) because $\Gamma^{-1}$ corresponds the efficient asymptotic variance in Gobet (2002).

3 Examples

1. The first example to which our results apply is a data generating process of the form
\[
X_0 = x_0,
\]
\[
dX_t = \psi(t, X_t, \theta)dt + V_t dW_t,
\]
where $V$ is an unknown predictable ergodic process. We remark that previous literature only treated the case $dX_t = \psi(t, X_t, \theta)dt + \sigma(t, X_t, \theta)dW_t$ with $\psi$ and $\sigma$ known.

2. As a further example consider
\[
dP_s := \left( \frac{dP_s^1}{dP_s^2} \right) = \left( \frac{\mu P_s^1 + \sqrt{Z_s^1} \sqrt{\nu \theta^1} ds + \sqrt{Z_s^1} dW_s^1}{\mu P_s^2 + \sqrt{Z_s^2} \sqrt{\nu \theta^2} ds + \sqrt{Z_s^2} dW_s^2} \right). \tag{8}
\]
with $\mu \leq 0$. This backward SDE is motivated by extending the evolution of a price process in the Heston model to a random and possibly arbitrary large time horizon.

3. Ergodic BSDEs appear naturally in forward performance processes which are utility functionals which do not depend on the specific time horizon, see for instance Hu, Liang and Tang (2020). In Liang and Zariphopoulou (2017) for instance a forward performance process is described which has the factor form $U(x, t) = e^{X_t - \lambda t}$ with $Y$ being the ergodic solution of an BSDE with quadratic driver function.

4 Simulation studies

In the sequel, we will consider different possibilities for our sequences converging to zero or to infinity. In particular, consider $c_n = n^{-0.05k}, k = 1, 2, \ldots, l - 1$.

$h_n = n^{-0.05l}, l = 11, \ldots, 19$. Then we must have

a) $n h_n^2 c_n = n^{1 + 0.05k - 0.4l} \to 0$
\[\Rightarrow 0.05k < 0.1l - 1 \Rightarrow k < 2l - 20\]

b) $\frac{n h_n^{2,5}}{c_n} = n^{0.5 - 0.25l - 0.05k} \to 0$
\[\Rightarrow 0.5 - 0.25l < 0.05k \Rightarrow 10 - \frac{5l}{2} < k\]

c) $n^3 h_n^5 \to 0$
\[\Rightarrow n^3 c_n^{-0.25l} \to 0 \Rightarrow 3 - 0.25l < 0 \Rightarrow 12 < l\]

Combining three cases yields $13 \leq l \leq 19$, $\max(10 - \frac{5l}{2}) \leq k \leq \min(19, 2l - 20) = 2l - 20$. We will below try every one of these combinations.
4.1 Simulation Results for the Vasicek model

Suppose that \( X_t \) evolves according to the Vasicek model, that is, 
\[
\frac{d X_t}{X_t} = a(b - X_t)dt + \sigma dW_t,
\]
where \( W_t \) is the standard Brownian motion, with parameters \( a = 2, b = 0.3 \) and \( \sigma = 0.025 \). The initial value \( X_0 \) is set as 0.3.

Let us estimate \( \theta \) in the equation
\[
dY_t = \theta \sqrt{|X_t|} + 0.1 \, dt + \sqrt{|X_t|} + 0.1 \, dW_t,
\]
where \( Y_0 = 1 \).

In the following, \( h_n \) is set to be \( n^{-0.05l} \) and \( c_n \) to be \( n^{0.05k} \). We consider integers \( l \) and \( k \) where to satisfy the conditions of Theorem 2.1 and Theorem 2.2: \( 13 \leq l \leq 19 \) and \( \max(1, 10 - \frac{4}{l}) \leq k \leq 2l - 20 \). To look for the pair of \( (l, k) \) which best estimates \( \theta \), we run simulations for each combination of \( (l, k) \) and calculate the average of the errors as the sum of differences between \( \hat{\theta}_n \) and \( \theta \) in percentage for the \( n \)'s simulated, which means
\[
\text{Error} = \frac{\sum_{n \in A} |\hat{\theta}_n - \theta|/\theta}{|A|},
\]
where \( A \) denotes the set of \( n \)'s simulated. Two sets of \( n \)'s are considered: \( A_1 = \{1 \times 10^5, 2 \times 10^5, \ldots, 1 \times 10^6\} \) and \( A_2 = \{1 \times 10^6, 2 \times 10^6, \ldots, 1 \times 10^7\} \). We let \( \theta = 1 \).

The results are summarized in the following tables.

| \( l \) | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|
| 1 | 4.79986 | 4.39994 |
| 2 | 0.55168 | 0.59204 | 0.64261 | 0.63193 |
| 3 | 0.13564 | 0.19179 | 0.17408 | 0.43068 | 0.45217 | 0.82545 |
| 4 | 0.065 | 0.16896 | 0.0839 | 0.21815 | 0.36891 | 0.46106 | 0.86921 |
| 5 | 0.11211 | 0.14296 | 0.24044 | 0.29471 | 0.30672 | 0.36704 | 0.72157 |
| 6 | 0.07487 | 0.10097 | 0.21671 | 0.19126 | 0.2234 | 0.44338 | 0.57126 |
| 7 | 0.10343 | 0.16694 | 0.20898 | 0.19727 | 0.48259 | 0.55946 |
| 8 | 0.1056 | 0.22114 | 0.24371 | 0.25512 | 0.63417 | 0.7991 |
| 9 | 0.11754 | 0.19612 | 0.29589 | 0.32613 | 0.51654 |
| 10 | 0.14666 | 0.17857 | 0.24282 | 0.18316 | 0.56393 |
| 11 | 0.31039 | 0.22011 | 0.63986 | 0.71099 |
| 12 | 0.23643 | 0.22018 | 0.31369 | 0.51456 |
| 13 | 0.40641 | 0.50407 | 0.43586 |
| 14 | 0.27931 | 0.50327 | 0.29167 |
| 15 | 0.43433 | 0.38009 |
| 16 | 0.52718 | 0.41497 |
| 17 | 0.65534 |
| 18 | 0.52093 |

Table 1: Errors from different combinations of \( (l, k) \) simulated for \( A_1 \).
Table 2: Errors from different combinations of \((l,k)\) simulated for \(n \in A_2\).

From the tables it can be seen that the choices for \(l\) and \(k\) strongly matter. The pairs with \(l = 13\) gives the smallest error and estimates \(\theta\) most accurately under both sets of \(n\)'s. When simulations are repeated, any of the three pairs could result in the smallest error. Overall, for the same \(k\), the smaller \(l\) is, the better the estimation for \(\theta\) is.

Below, Figure 1 shows an analysis for the Vasicek model where \(k\) and \(l\) are chosen to be 6 and 13 respectively, with \(\theta_0 = 10\). The number of simulation times \(n\) is set as \(\{1000, 2000, \ldots, 10000, 20000, \ldots, 100000, 200000, \ldots, 500000\}\).

For each \(n\), we repeat the process by 500 times and calculate the Mean Error of the estimators \(\hat{\theta}'s\).

4.2 The Heston model

Next, the two-dimensional case is simulated. The process \(\nu_t\) evolves according to the Heston model, that is, \(\nu_t = L(\beta - \nu_t)dt + \sigma \sqrt{\nu_t}dW_t\), with parameters \(L = 1, \beta = 1.5\) and \(\sigma = 0.5\), and the initial value is \(\nu_0 = 1.5\). We want to estimate \(\theta^1\) and \(\theta^2\) in equation (8), where \(\sqrt{Z^1_t} = \sqrt{Z^2_t} = \sqrt{Z^3_t} = 0.4\). \(k\) and \(l\) remains to be 6 and 13 respectively, and \(\theta^1_0 = \theta^2_0 = 5\).

The number of simulation times \(n\) is set as \(\{10000, 30000, \ldots, 90000, 100000, 300000, \ldots, 900000, 1000000, 2000000, \ldots, 5000000\}\).

For each \(n\), we repeat the process by 500 times and calculate the Mean-Absolute-Error (MAE) of the estimators \(\hat{\theta}'s\). Figure 2 shows the result.

5 Proofs

In this section, we prove the results in Section 2. In Section 5.1 we introduce two functions \(\tilde{H}_{n,\delta}\) and \(\hat{H}_{n,\delta}\) which are approximation of the quasi-log-likelihood \(H_n\). The function \(\tilde{H}_{n,\delta}\) is introduced to control the event that either \(Z_t\) or \(\hat{Z}_t\) is close to degenerate for some \(t\) or \(l\), and is equal to \(H_n\) except on that event. The function \(\hat{H}_{n,\delta}\) is obtained by replacing the estimator \(\hat{Z}_{t-1}\) in \(\tilde{H}_{n,\delta}\) with \(Z_{t-1}\). In Section 5.2, we will show that the difference of \(\partial^1_{\theta} \tilde{H}_{n,\delta}\) and \(\partial^1_{\theta} \hat{H}_{n,\delta}\) can be asymptotically ignored, and we consequently obtain consistency of \(\hat{\theta}_n\).
To show Theorem 2.2, we need an accurate estimate for the difference of $\partial_{\theta} \tilde{H}_{n,\delta}(\theta_0)$ and $\partial_{\theta} \hat{H}_{n,\delta}(\theta_0)$, which is given in Proposition 5.1 of Section 5.3. Together with asymptotic estimate Lemma 5.3 of $\partial_{\theta} \hat{H}_{n,\delta}$, we obtain then the desired results.

5.1 Approximation of $H_n$

For a vector $v$ and a matrix $A$, $[v]_i$ and $[A]_{ij}$ denote $(i, j)$ element of a matrix $A$ and $i$-th element of $v$, respectively. For $q > 0$ and a sequence $p_n$ of positive numbers, let us denote by $\{\tilde{R}_{n,q}(p_n)\}_{n \in \mathbb{N}}$ and $\{\tilde{R}_{n,q}(p_n)\}_{n \in \mathbb{N}}$ sequences of random variables (which may also depend on $l$ and $\theta$) satisfying

$$\sup_{\theta,i} E[|p_n^{-1}\tilde{R}_{n,q}(p_n)[\vartheta]^1/q] < \infty \quad \text{and} \quad \sup_{\theta,i} E[|p_n^{-1}\tilde{R}_{n,q}(p_n)[\vartheta]^1/q] \to 0. \quad (10)$$

Then (A1-p) and (A2-r) imply

$$\Delta_t Y = \int_{t_0}^{t_{n-1}} \psi(X_t, Y_t, V_t V_t^T) dt + \int_{t_0}^{t_{n-1}} V_t dW_t = \tilde{R}_{n,p/r}(\sqrt{c_n h_n}). \quad (11)$$

Let $Z_t = V_t V_t^T$. We first introduce a family of stopping times controlling the degeneracy of $Z_t$ and $\hat{Z}_t$. For any $\delta > 0$, let

$$T_{n,\delta} = \inf\{t_{n+1}; 0 \leq l \leq L_n - 1, \hat{Z}_t \not\in \mathcal{F}_s \text{ or } Z_t \not\in \mathcal{F}_s \text{ for some } t \in [0, \bar{t}_{n+1}],\}$$

where $\inf \emptyset = \infty$. Under (A1-p), $t_{n+1} - \hat{Z}_t \geq \delta^{d_Y}$ and $\hat{Z}_t \geq \delta^{d_Y}$ for $t \in [0, \bar{t}_{n+1}]$ because $V$ has a continuous path.

Let $\tilde{\psi}(\theta) = \psi(X_{t_0}, Y_{t_0}, Z_{t_0}, \theta)$, and let

$$\tilde{H}_{n,\delta}(\theta) = -\frac{1}{2} \sum_{l=1}^{L_n-1} (\Delta_l Y - c_n h_n \tilde{\psi}(\theta))^T \frac{\bar{Z}_{l-1}}{c_n h_n} (\Delta_l Y - c_n h_n \tilde{\psi}(\theta)) 1_{\{l < T_{n,\delta}\}},$$

and

$$\hat{H}_{n,\delta}(\theta) = -\frac{1}{2} \sum_{l=1}^{L_n-1} (\Delta_l Y - c_n h_n \tilde{\psi}(\theta))^T \frac{\hat{Z}_{l-1}}{c_n h_n} (\Delta_l Y - c_n h_n \tilde{\psi}(\theta)) 1_{\{l < T_{n,\delta}\}}.$$

When $\delta$ is sufficiently small and $n$ sufficiently large, $\tilde{H}_n$ corresponds to $H_n$ with high probability (see 10). $\tilde{H}_n$ is an approximation of $H_n$ which is useful when we deduce the asymptotic behavior.

The Burkholder-Davis-Gundy inequality and Jensen’s inequality yield

$$E\left[\left|\int_{t_{n-1}}^{t_n} V_t dW_t\right|^{2p}\right] \leq C_p E\left[\left(\int_{t_{n-1}}^{t_n} |V|^2 dt\right)^p\right] \leq C_p h_n^{p-1} E\left[\int_{t_{n-1}}^{t_n} |V_t|^2 dt\right] \leq C_p h_n^{p-1} \sup_t E[|V_t|^2],$$

which implies that $\Psi_{1,l,m} := \int_{t_{n-1}}^{t_n} V_t dW_t = \tilde{R}_{n,2p}(\sqrt{c_n h_n})$ by (A1-p). Similarly, (A1-p) and (A2-r) yield $\Psi_{2,l,m} := \int_{t_{n-1}}^{t_n} \psi(X_t, Y_t, Z_t, \theta_0) dt = \tilde{R}_{n,p/r}(h_n)$. Then by Itô’s formula and the Cauchy-Schwarz inequality, (A1-p), and (A2-r) yield

$$\hat{Z}_t = \frac{1}{c_n h_n} \sum_{m=1}^{c_n} (Y_{t_m} - Y_{t_{m-1}})(Y_{t_m} - Y_{t_{m-1}})^T$$

$$= \frac{1}{c_n h_n} \sum_{m=1}^{c_n} \left\{ \int_{t_{m-1}}^{t_m} Z_t dt + 2 \Psi_{1,l,m} + \Psi_{2,l,m} Y_{1,l,m}^T + (\Psi_{1,l,m} + \Psi_{2,l,m}) \Psi_{2,l,m}^T \right\}$$

$$= \frac{1}{c_n h_n} \sum_{m=1}^{c_n} \left\{ \int_{t_{m-1}}^{t_m} Z_t dt + 2 \Psi_{1,l,m} + \tilde{R}_{n,\vartheta}(h_n^{1/2}) \right\}$$

$$= Z_t + \frac{2}{c_n h_n} \sum_{m=1}^{c_n} \Psi_{1,l,m} + \tilde{R}_{n,\vartheta}(\sqrt{c_n h_n})$$

$$= Z_t + \tilde{R}_{n,\vartheta}(c_n^{-1/2} + \sqrt{c_n h_n}), \quad (12)$$

$$= Z_t + \tilde{R}_{n,\vartheta}(c_n^{-1/2} + \sqrt{c_n h_n}), \quad (13)$$
where
\[ [A_{l,m}]_{ij} = \frac{1}{2} \sum_{k} \int_{t_{l-1}^{n}}^{t_{l}^{n}} \left( [Y_{t} - Y_{t-1}]_{i}[V_{t}]_{jk} + [Y_{t} - Y_{t-1}]_{j}[V_{t}]_{ik} \right) dt_{l}^{n} k. \]

Therefore, for any \( \delta > 0 \) and \( q = p/(2r) \), we obtain
\[ P(\max_{r} |\tilde{Z}_{l} - Z_{t}^{q}| > \delta) \leq \delta^{-q} \sum_{l} E[|\tilde{Z}_{l} - Z_{t}^{q}|^q] = O(L_n(c_n^{-1/2} + \sqrt{c_n h_n})^q) \to 0, \]
(14) as \( n \to \infty \) if \( q > 1/\epsilon \).

Then (A1-p) yields
\[ \lim_{\delta \to 0} \lim_{n \to \infty} \inf_{\hat{H}_{n,\delta} = \hat{H}_{n}} P(T_{n,\delta} = +\infty) = 1, \]
(15) and therefore, we have
\[ \lim_{\delta \to 0} \lim_{n \to \infty} \inf_{\hat{H}_{n,\delta} = \hat{H}_{n}} P(\hat{H}_{n,\delta}(\theta) = H_n(\theta)) \text{ for any } \theta = 1. \]
(16)

Equation (10) implies that the asymptotic behavior of \( H_n \) is essentially the same as more tractable \( \hat{H}_{n,\delta} \) for sufficiently small \( \delta > 0 \). We further show that \( \hat{H}_{n,\delta} \) is asymptotically equivalent to \( \hat{H}_{n,\delta} \) in Lemma 5.1 of the following section.

5.2 Proof of consistency

**Lemma 5.1.** Let \( p, r \geq 1 \) such that (B) is satisfied. Assume (A1-p) and (A2-r). Then
\[ (nh_n)^{-1} \sup_{\theta} |\hat{H}_{n,\delta}(\theta) - \hat{H}_{n,\delta}(\theta_0) - \hat{H}_{n,\delta}(\theta_0) + \hat{H}_{n,\delta}(\theta_0)| \to 0, \]
(17) as \( n \to \infty \) for any \( \delta > 0 \).

**Proof.** By the definitions of \( \hat{H}_{n} \) and \( \hat{H}_{n} \), we can decompose the difference as
\[ \hat{H}_{n,\delta}(\theta) - \hat{H}_{n,\delta}(\theta_0) - \hat{H}_{n,\delta}(\theta_0) + \hat{H}_{n,\delta}(\theta_0) \]
\[ = -\frac{c_n h_n}{2} \sum_{l=1}^{L_{l-1}} \left( \hat{\psi}_l(\theta)^T (\tilde{Z}_{l-1} - Z_{t_0}^{q-1}) \hat{\psi}_l(\theta) - \hat{\psi}_l(\theta_0)^T (\tilde{Z}_{l-1} - Z_{t_0}^{q-1}) \hat{\psi}_l(\theta_0) \right) 1_{\{t_0 < T_{n,\delta}\}} \\
+ \sum_{l=1}^{L_{l-1}} \Delta l Y^T (\tilde{Z}_{l-1} - Z_{t_0}^{q-1}) (\hat{\psi}_l(\theta) - \hat{\psi}_l(\theta_0)) 1_{\{t_0 < T_{n,\delta}\}} \\
- \frac{c_n h_n}{2} \sum_{l=1}^{L_{l-1}} \left( \hat{\psi}_l(\theta)^T \tilde{Z}_{l-1} \hat{\psi}_l(\theta) - \hat{\psi}_l(\theta_0)^T \tilde{Z}_{l-1} \hat{\psi}_l(\theta_0) \right) \\
- \hat{\psi}_l(\theta)^T \tilde{Z}_{l-1} \hat{\psi}_l(\theta_0) + \hat{\psi}_l(\theta_0)^T \tilde{Z}_{l-1} \hat{\psi}_l(\theta_0) 1_{\{t_0 < T_{n,\delta}\}} \\
+ \sum_{l=1}^{L_{l-1}} \Delta l Y^T (\tilde{Z}_{l-1} (\hat{\psi}_l(\theta) - \hat{\psi}_l(\theta_0)) - \hat{\psi}_l(\theta) + \hat{\psi}_l(\theta_0)) 1_{\{t_0 < T_{n,\delta}\}} \\
=: \Lambda_1(\theta) + \Lambda_2(\theta) + \Lambda_3(\theta) + \Lambda_4(\theta). \]

Then it is sufficient to show that \( \sup_{\theta} |\Lambda_j(\theta)| = R_{n,p,r'/2}(nh_n) \) for \( 1 \leq j \leq 4 \), where \( r' = 1/r \). (A2-r) yields
\[ \sup_{\theta} |\hat{\psi}_l(\theta) \hat{\psi}_l(\theta)^T - \hat{\psi}_l(\theta) \hat{\psi}_l(\theta)^T| = \sup_{\theta} |\hat{\psi}_l(\theta) (\hat{\psi}_l(\theta) - \hat{\psi}_l(\theta))^T + (\hat{\psi}_l(\theta) - \hat{\psi}_l(\theta)) \hat{\psi}_l(\theta)^T| \]
\[ \leq CC(1 + |X_{t_0}^{q-1} + |Y_{t_0}^{q-1} + |\tilde{Z}_{l-1} - Z_{t_0}^{q-1}|^2r |\tilde{Z}_{l-1} - Z_{t_0}^{q-1}|, \]
on \{t_0 < T_{n,\delta}\} for any \( \delta > 0 \). Then (A1-p), (B), and the Cauchy-Schwartz inequality yield \( \sup_{\theta} |\hat{\psi}_l(\theta) \hat{\psi}_l(\theta)^T - \hat{\psi}_l(\theta) \hat{\psi}_l(\theta)^T| 1_{\{t_0 < T_{n,\delta}\}} = R_{n,p,r'/2}(1) \). We also have
\[ (\tilde{Z}_{l-1} - Z_{t_0}^{q-1}) 1_{\{t_0 < T_{n,\delta}\}} = \tilde{Z}_{l-1} - Z_{t_0}^{q-1} Z_{t_0}^{q-1} 1_{\{t_0 < T_{n,\delta}\}} = R_{n,pr'/2}(1), \]

and hence we obtain
\[
\sup_{\theta} |\Lambda_j(\theta)| = R_{n,pr'/4}(c_n h_n, L_n) = R_{n,pr'/4}(n h_n),
\]
for \( j \in \{1, 3\} \).
Moreover, since
\[
\Delta_t Y = \int_{t_0}^{t+1} V_s dW_s + \hat{\psi}_l(\theta_0)(t_{L_n}^{n+1} - t_0) + R_{n,pr'/3}(c_n h_n)^{3/2}
\]
for \( l \in \{0, 1\} \). The Burkholder-Davis-Gundy inequality and the triangle inequality yield
\[
E \left[ \left| \sum_{i=1}^{L_n-1} \Delta_t W^T V_t^T (\tilde{Z}_{i-1}^{-1} - Z_{i-1}^{-1}) \partial_t^l (\tilde{\psi}_l(\theta) - \hat{\psi}_l(\theta_0)) 1_{\{t^i < T_{n, \delta}\}} \right|^q \right] 
\leq C_q \left( \sum_{i=1}^{L_n-1} E \left[ \left| \Delta_t W^T V_t^T (\tilde{Z}_{i-1}^{-1} - Z_{i-1}^{-1}) \partial_t^l (\tilde{\psi}_l(\theta) - \hat{\psi}_l(\theta_0)) 1_{\{t^i < T_{n, \delta}\}} \right|^q \right] \right)^{q/2},
\]
for \( q \geq 2 \). Then we obtain
\[
\partial_t^l \Lambda_2 = \tilde{R}_{n,p/(r+1)}(\sqrt{L_n c_n h_n}) + R_{n,pr'/4}(n h_n) = R_{n,pr'/4}(n h_n),
\]
for \( l \in \{0, 1\} \), and similarly we have
\[
\partial_t^l \Lambda_4 = \tilde{R}_{n,p/(r+1)}(\sqrt{L_n c_n h_n}) + R_{n,pr'/4}(n h_n) = R_{n,pr'/4}(n h_n),
\]
for \( l \in \{0, 1\} \).
Sobolev’s inequality \([5]\) yields \( \sup_{\theta} |\Lambda_j(\theta)| = R_{n,pr'/4}(n h_n) \) for \( j \in \{2, 4\} \), which completes the proof.

Proof of Theorem 2.1.
We first deduce the limit of \( (nh_n)^{-1}(H_n(\theta) - H_{n,0}(\theta)) \). \([13]\) yields
\[
\tilde{H}_{n, \delta}(\theta) - H_{n, \delta}(\theta_0) = \frac{1}{2} \sum_{l=1}^{L_n-1} \left( c_n h_n (\tilde{\psi}_l(\theta)^T Z_{l-1}^{-1} \tilde{\psi}_l(\theta) - \tilde{\psi}_l(\theta_0)^T Z_{l-1}^{-1} \tilde{\psi}_l(\theta_0)) - 2 \Delta_t Y^T Z_{l-1}^{-1} (\tilde{\psi}_l(\theta) - \tilde{\psi}_l(\theta_0)) \right) 1_{\{t^i < T_{n, \delta}\}}
\]
\[
= \frac{1}{2} \sum_{l=1}^{L_n-1} \left( \tilde{\psi}_l(\theta)^T Z_{l-1}^{-1} \tilde{\psi}_l(\theta) - \tilde{\psi}_l(\theta_0)^T Z_{l-1}^{-1} \tilde{\psi}_l(\theta_0) - 2 \tilde{\psi}_l(\theta)^T Z_{l-1}^{-1} (\tilde{\psi}_l(\theta) - \tilde{\psi}_l(\theta_0)) \right) 1_{\{t^i < T_{n, \delta}\}}
\]
\[
+ R_{n,pr'/4}(n h_n \sqrt{c_n h_n} + \sqrt{n h_n})
\]
\[
= J_n(\theta) + R_{n,pr'/4}(n h_n),
\]
on \( \{t^i < T_{n, \delta}\} \), where
\[
J_n(\theta) = - \int_{t_0}^{nh_n} (\psi(X_t, Y_t, Z_t, \theta) - \psi(X_t, Y_t, Z_t, \theta_0))^T Z_t^{-1} (\psi(X_t, Y_t, Z_t, \theta) - \psi(X_t, Y_t, Z_t, \theta_0)) dt.
\]
Similarly, we have \( \partial_\theta \hat{H}_{n, \delta}(\theta) = \partial_\theta \mathcal{Y}_n(\theta) + \frac{\mathbb{R}_{n, pr'/4}(nh_n)}{t_{0, n}^{L_n-1}} \) on \( \{ t_{0, n}^{L_n-1} < T_{n, \delta} \} \), and hence Sobolev’s inequality yields

\[
\sup_{\theta} | \hat{H}_{n, \delta}(\theta) - \hat{H}_{n, \delta}(\theta_0) - \mathcal{Y}_n(\theta) | = \frac{\mathbb{R}_{n, pr'/4}(nh_n)}{t_{0, n}^{L_n-1}}.
\]

(23)

Let

\[
\mathcal{Y}(\theta) = \begin{cases} 
- \int (\psi(x, z, \theta) - \psi(x, z, \theta_0)) \tau z^{-1} \psi(x, z, \theta) \pi(dz) & \text{if Point 1 of (A3-p) is satisfied} \\
- \int (\psi(x, y, z, \theta) - \psi(x, y, z, \theta_0)) \tau z^{-1} \psi(x, y, z, \theta) \pi(dx dy dz) & \text{if Point 2 of (A3-p) is satisfied}
\end{cases}
\]

for \( \theta \in \Theta \), sufficiently large \( n \), and sufficiently small \( \delta \). Then we have

\[
(nh_n)^{-1} (H_n(\theta) - H_n(\theta_0)) \xrightarrow{P} \mathcal{Y}(\theta),
\]

(24)
as \( n \to \infty \) for any \( \theta \in \Theta \).

Next, we show that consistency of \( \hat{\theta}_n \) is obtained if \( \mathcal{Y}(\theta) \) holds uniformly in \( \theta \).

(A2-\( \tau \)), (A3-p), and (A4) imply that \( \mathcal{Y}(\theta) \) is continuous on \( \theta \) and

\[
\mathcal{Y}(\theta) = 0 \quad \implies \quad \theta = \theta_0.
\]

Then for any \( \epsilon, \delta > 0 \), there exists \( \eta > 0 \) such that

\[
P \left( \inf_{|\theta - \theta_0| \geq \delta} (-\partial_\theta \mathcal{Y}(\theta)) < \eta \right) < \frac{\epsilon}{2}.
\]

(25)

Because \( H_n(\hat{\theta}_n) - H_n(\theta_0) \geq 0 \) by the definition of \( \hat{\theta}_n \), together with \( \mathcal{Y}(\theta) \), we have

\[
P(|\hat{\theta}_n - \theta_0| \geq \delta) < P \left( \sup_{\theta} \left| \frac{1}{nh_n} (H_n(\theta) - H_n(\theta_0)) - \mathcal{Y}(\theta) \right| > \eta \right) + \frac{\epsilon}{2},
\]

for any \( n \).

Then it is sufficient to show

\[
\sup_{\theta \in \Theta} \left| \frac{1}{nh_n} (H_n(\theta) - H_n(\theta_0)) - \mathcal{Y}(\theta) \right| \xrightarrow{P} 0,
\]

(26)
as \( n \to \infty \). By \( \mathcal{Y}(\theta) \), it is sufficient to show C-tightness of \( (nh_n)^{-1} (H_n(\cdot) - H_n(\theta_0)) \).

Finally, we show C-tightness. Similarly to the proof of Lemma \( 5.1 \), we obtain \( \sup_{\theta} |\partial_\theta \mathcal{Y}_n(\theta)| = \frac{\mathbb{R}_{n, pr'/4}(nh_n)}{t_{0, n}^{L_n-1}} \) for \( 1 \leq j \leq 4 \) and \( l \in \{1, 2\} \). Together with a similar argument to \( \mathcal{Y}(\theta) \), for \( q = pr'/4 \) and \( l \in \{1, 2\} \), there exists \( N' \in \mathbb{N} \) such that

\[
\sup_{n \geq N', \theta} E \left[ |(nh_n)^{-1} \partial_\theta \hat{H}_{n, \delta}(\theta)|^q 1_{\{ t_{0, n}^{L_n-1} < T_{n, \delta} \}} \right] \
\leq \sup_{n \geq N', \theta} E \left[ |(nh_n)^{-1} \partial_\theta \hat{H}_{n, \delta}(\theta)|^q 1_{\{ t_{0, n}^{L_n-1} < T_{n, \delta} \}} \right] + 1 \
\leq \sup_{n \geq N', \theta} E \left[ |(nh_n)^{-1} \partial_\theta \mathcal{Y}_n(\theta)|^q 1_{\{ t_{0, n}^{L_n-1} < T_{n, \delta} \}} \right] + 2 < \infty.
\]

Then Sobolev’s inequality yields

\[
\lim_{n \to \infty} \sup_{\theta} \left| (nh_n)^{-1} \partial_\theta H_{n, \delta}(\theta) \right|^q 1_{\{ t_{0, n}^{L_n-1} < T_{n, \delta} \}} < \infty.
\]
Together with (15) and (16), for any $\epsilon > 0$, there exists $K > 0$ and $N'' \in \mathbb{N}$ such that

$$\sup_{n \geq N''} P(\sup_{\theta} |(nh_n)^{-1}\partial_{\theta} H_n(\theta)| > K) < \epsilon.$$ 

Then C-tightness condition (Theorem 7.3) in Billingsley (1999) yields the desired result.

5.3 The proof of asymptotic normality

We show asymptotic normality of $\hat{\theta}_n$ in this section. For this purpose, we show a stronger estimate of $\partial_{\theta} \tilde{H}_{n,\delta}(\theta_0) - \partial_{\theta} \hat{H}_{n,\delta}(\theta_0)$ in Proposition 5.1. We first prepare a fundamental result which is repeatedly used in the following.

**Lemma 5.2.** Let $(F_l)_{l=1}^{L_n-1}$ be random variables satisfying that $F_l$ is $\mathcal{F}_{t_l}$-measurable and that $E[F_l|\mathcal{F}_{t_l-1}]=0$ for $1 \leq l \leq L_n-1$. Then

$$E\left[\left|\sum_{l=1}^{L_n-1} F_l 1_{\{t_l < T_{n,\delta}\}}\right|^2\right] \leq 4 \sum_{l=1}^{L_n-1} E[|F_l|^2 1_{\{t_l < T_{n,\delta}\}}].$$

**Proof.** Because $E[F_l 1_{\{t_l < T_{n,\delta}\}}|\mathcal{F}_{t_l-1}]=0$ and $\sum_{l=1}^{L_n-1} 1_{\{t_l < T_{n,\delta}\}} \leq 1$, the Cauchy-Schwarz inequality yields

$$E\left[\left|\sum_{l=1}^{L_n-1} F_l 1_{\{t_l < T_{n,\delta}\}}\right|^2\right] \leq 2 \sum_{l=1}^{L_n-1} E[|F_l|^2 1_{\{t_l < T_{n,\delta}\}}] \leq 4 \sum_{l=1}^{L_n-1} E[|F_l|^2 1_{\{t_l < T_{n,\delta}\}}].$$

**Proposition 5.1.** Let $p, r \geq 1$ such that (A2) is satisfied. Assume (A1-p), (A2-r) and that $n^3 h_n^5 \to 0$. Then

$$\frac{1}{\sqrt{nh_n}} \partial_{\theta} \tilde{H}_{n,\delta}(\theta_0) - \frac{1}{\sqrt{nh_n}} \partial_{\theta} H_{n,\delta}(\theta_0) \xrightarrow{P} 0,$$

as $n \to \infty$ for any $\delta > 0$.

**Proof.** For a positive sequence $(c_n)_{n \in \mathbb{N}}$ and random variables $(U_n)_{n \in \mathbb{N}}$, we denote $U_n = O_P(c_n)$ if $(c_n^{-1}U_n)_{n \in \mathbb{N}}$ is tight, and we denote $U_n = o_P(c_n)$ if $c_n^{-1}U_n \xrightarrow{P} 0$. First, (12) and (13) imply

$$\partial_{\theta} \tilde{\psi}^T (\theta_0) - \partial_{\theta} \hat{\psi}^T (\theta_0) = \sum_{i,j} \int_0^1 \partial_{z_{ij}} (\partial_{\theta} \tilde{\psi}^T)(X_{t_l}, Y_{t_l}, u \tilde{Z}_{t_l} - (1-u)Z_{t_l-1}, \theta_0) du \cdot [\hat{Z}_{t_l-1} - Z_{t_l-1}]_{ij}$$

$$= 2 \sum_{i,j} \frac{\partial_{z_{ij}} (\partial_{\theta} \tilde{\psi}^T)(\theta_0)}{c_n h_n} \sum_{m=1}^{c_n} [\hat{Z}_{t_l-1,m}]_{ij} + O_P(c_n^{-1} + \sqrt{c_n h_n})$$

$$= O_P(c_n^{-1/2} + \sqrt{c_n h_n}), \quad (27)$$
and similarly
\[
\partial_\theta \bar{\psi}_t(\theta_0) - \partial_\theta \tilde{\psi}_t(\theta_0) = 2 \sum_{i,j} \frac{\partial_{ij}}{c_n h_n} \sum_{m=1}^{c_n} [\mathfrak{M}_{i-1,m}]_{ij} + O_p(c_n^{-1} + \sqrt{c_n h_n}) \tag{28}
\]

\[
= O_p(c_n^{-1/2} + \sqrt{c_n h_n}),
\]

if \( t_0^p < T_{n,\delta} \), where \( \psi_t(\theta) = \psi(X_{t_0}^{-}, Y_{t_0}^{-}, Z_{t_0}^{-1}, \theta) \).

Moreover, (12), (13) and (14) yield
\[
\hat{Z}_{t_0}^{-1} - Z_{t_0}^{-1} = Z_{t_0}^{-1}(Z_{t_0}^{-1} - \hat{Z}_{t_0}^{-1})Z_{t_0}^{-1} + Z_{t_0}^{-1}(Z_{t_0}^{-1} - \hat{Z}_{t_0}^{-1})Z_{t_0}^{-1}
\]

\[
= -\frac{2}{c_n h_n} \sum_{m=1}^{c_n} \hat{Z}_{t_0}^{-1} Z_{t_0}^{-1} + R_{n,pe/4}((nh_n)^{-1/2}) \tag{29}
\]
\[
= R_{n,pe/4}(c_n^{-1/2} + (nh_n)^{-1/2}), \tag{30}
\]
on \( \{ t_0^p < T_{n,\delta} \} \).

Then Lemma 5.2 yields
\[
\frac{1}{\sqrt{nh_n}} \partial_\theta \Lambda_1(\theta_0)
\]
\[
= -\frac{c_n h_n}{\sqrt{nh_n}} \sum_{l=1}^{L_n-1} \partial_\theta \tilde{\psi}_l(\theta_0)^T (\hat{Z}_{l-1}^{-1} - Z_{t_0}^{-1}) \tilde{\psi}_l(\theta_0) 1_{\{ t_0^p < T_{n,\delta} \}}
\]
\[
= \frac{2}{\sqrt{nh_n}} \sum_{l=1}^{L_n-1} \sum_{m=1}^{c_n} \partial_\theta \tilde{\psi}_l(\theta_0)^T Z_{t_0}^{-1} \mathfrak{M}_{l,m} Z_{t_0}^{-1} \tilde{\psi}_l(\theta_0) 1_{\{ t_0^p < T_{n,\delta} \}} + o_P(1)
\]
\[
= \frac{2}{\sqrt{nh_n}} \sum_{l=1}^{L_n-1} \sum_{m=1}^{c_n} \partial_\theta \tilde{\psi}_l(\theta_0)^T Z_{t_0}^{-1} \mathfrak{M}_{l,m} Z_{t_0}^{-1} \tilde{\psi}_l(\theta_0) 1_{\{ t_0^p < T_{n,\delta} \}} + o_P(1)
\]
\[
= O_P\left( \frac{1}{\sqrt{nh_n}} \sqrt{L_n \sqrt{c_n h_n}} \right) + o_P(1) \xrightarrow{p} 0, \tag{31}
\]

and (14), (27), and (29) yield
\[
\frac{1}{\sqrt{nh_n}} \partial_\theta \Lambda_3(\theta_0)
\]
\[
= -\frac{c_n h_n}{\sqrt{nh_n}} \sum_{l=1}^{L_n-1} \left( \partial_\theta \tilde{\psi}_l(\theta_0)^T \hat{Z}_{l-1}^{-1} \tilde{\psi}_l(\theta_0) - \partial_\theta \tilde{\psi}_l(\theta_0)^T Z_{l-1}^{-1} \tilde{\psi}_l(\theta_0) \right) 1_{\{ t_0^p < T_{n,\delta} \}}
\]
\[
= \frac{1}{\sqrt{nh_n}} \sum_{m=1}^{c_n} \sum_{l=1}^{L_n-1} \text{tr}(\mathfrak{B}_{l,m} \hat{Z}_{l-1}^{-1}) 1_{\{ t_0^p < T_{n,\delta} \}} + o_P(1)
\]
\[
= \frac{1}{\sqrt{nh_n}} \sum_{m=1}^{c_n} \sum_{l=1}^{L_n-1} \text{tr}(\mathfrak{B}_{l,m} Z_{t_0}^{-1}) 1_{\{ t_0^p < T_{n,\delta} \}} + o_P(1)
\]
\[
+ \frac{2}{c_n h_n \sqrt{nh_n}} \sum_{l=1}^{L_n-1} \sum_{m,m'=1}^{c_n} \text{tr}(\mathfrak{B}_{l,m} Z_{t_0}^{-1} \mathfrak{M}_{l-1,m'} Z_{t_0}^{-1}) 1_{\{ t_0^p < T_{n,\delta} \}} + o_P(1),
\]

where \( \mathfrak{B}_{l,m} = 2 \sum_{i,j} \partial_{ij} (\partial_\theta \tilde{\psi}_l(\theta_0)^T) [\mathfrak{M}_{l-1,m}]_{ij} \).

The first term in the right-hand side of the above equation is equal to
\[
O_P((nh_n)^{-1/2} \sqrt{L_n c_n h_n}) \xrightarrow{p} 0,
\]

\[
13
\]
by Lemma 5.2. The second term is equal to

\[ O_P \left( \frac{L_n c_n h_n^2}{c_n h_n \sqrt{nh_n}} \right) + O_P \left( \frac{\sqrt{L_n c_n h_n^2}}{c_n h_n \sqrt{nh_n}} \right) \xrightarrow{P} 0, \]

by Lemma 5.2, \(1\), and \(E[\sum_{m \neq m'} \text{tr} (\mathcal{B}_{l,m} Z_{t_0}^{-1} \mathcal{A}_{l-1,m'} Z_{t_0}^{-1}) | \mathcal{F}_{t_0}] = 0\). Therefore, we have

\[ (nh_n)^{-1/2} \partial_\theta \Lambda_{\delta}(\theta_0) \xrightarrow{P} 0, \quad (32) \]

as \(n \to \infty\).

Moreover, \(34\), \(13\), \(26\) Lemma 5.2 and the parallelogram law yield

\[
E \left[ \left( \frac{1}{\sqrt{nh_n}} \partial_\theta \Lambda_{\delta}(\theta_0) \right)^2 \right] \\
\leq \frac{C}{nh_n} E \left[ \sum_{i=1}^{L_{n-1}} \left( t_{i+1}^0 - t_i^0 \right) \psi_l(\theta_0)^T (\hat{Z}_{l-1} - Z_{t_0}^{-1} \mathcal{A}_{l-1,m} Z_{t_0}^{-1}) \partial_\theta \psi_l(\theta_0) 1_{\{t_i^0 < T_{n,i}\}} \right]^2 \\
+ C \frac{L_{n-1}}{nh_n} \left( \sum_{n=1}^L \psi_l(\theta_0)^T \mathcal{A}_{l-1,n} Z_{t_0}^{-1} \partial_\theta \psi_l(\theta_0) \right)^2 1_{\{t_{i+1}^0 < T_{n,i}\}} \\
+ O \left( \left( \frac{L_n c_n h_n}{nh_n} \right)^{3/2} \right) \\
\leq \frac{C}{nh_n} \sum_{i=1}^{L_{n-1}} \left[ \sum_{n=1}^L \psi_l(\theta_0)^T \mathcal{A}_{l-1,n} Z_{t_0}^{-1} \partial_\theta \psi_l(\theta_0) \right]^2 1_{\{t_{i+1}^0 < T_{n,i}\}} \\
+ O \left( \frac{1}{nh_n} c_n h_n \cdot (nh_n)^{-1} \right) + O(nh_n^2 c_n) \\
= O \left( \frac{1}{nh_n} (c_n h_n)^2 \right) + o((nh_n)^{-1}) + o(1) \to 0. \quad (33) \]

Similarly Lemma \(5.2\) and \(28\) yield

\[
E \left[ \left( \frac{1}{\sqrt{nh_n}} \partial_\theta \Lambda_{\delta}(\theta_0) \right)^2 \right] \\
\leq \frac{C}{nh_n} E \left[ \sum_{i=1}^{L_{n-1}} \left( \int_{t_i^0}^{t_{i+1}^0} V_i \text{d}W_i \right)^T (\hat{Z}_{l-1} - 1_{\{t_i^0 < T_{n,i}\}} \partial_\theta \psi_l(\theta_0) 1_{\{t_i^0 < T_{n,i}\}} \right)^2 + o(1) \\
\leq O \left( \frac{1}{nh_n} L_n c_n h_n (c_n^{-1/2} + \sqrt{c_n h_n})^2 \right) + o(1) \to 0. \quad (34) \]

\(31\) - \(34\) complete the proof. \(\square\)

Let

\[ \tilde{H}_n(\theta) = -\frac{1}{2} \sum_{i=1}^{L_{n-1}} (\Delta_i Y - c_n h_n \hat{\psi}_l(\theta))^T Z_{t_0}^{-1}_{t_0} (\Delta_i Y - c_n h_n \hat{\psi}_l(\theta)). \]

The following lemma gives the asymptotic behavior of \(\partial_\theta \tilde{H}_n(\theta)\), which consequently give the asymptotic behavior of \(\partial_\theta^2 \tilde{H}_n(\theta)\).
Lemma 5.3. Let $p, r \geq 1$ such that (A2) is satisfied. Assume (A1-p), (A2-r), and (A3-p) and that $n^3 h_n^2 \to 0$ as $n \to \infty$. Then for any positive numbers $\epsilon_n$ tends to zero,

$$\sup_{|\theta - \theta_0| \leq \epsilon_n} \left| \frac{1}{nh_n} \partial_\theta^2 \tilde{H}_n(\theta) + \Gamma \right| \xrightarrow{P} 0, \quad \text{and} \quad \frac{1}{\sqrt{nh_n}} \partial_\theta \tilde{H}_n(\theta_0) \xrightarrow{d} N(0, \Gamma),$$

as $n \to \infty$.

**Proof.** Let $\partial_\theta^2 \tilde{\psi}_{1,0} = \partial_\theta^2 \tilde{\psi}_1(\theta_0)$ for $l \in \{0, 1, 2, 3\}$. Since (A2-r) implies

$$\sup_{|\theta - \theta_0| \leq \epsilon_n} |\partial_\theta \tilde{\psi}_1(\theta) - \partial_\theta \tilde{\psi}_1(\theta_0)| \leq \epsilon_n \sup_\theta |\partial_\theta \tilde{\psi}_1(\theta)| \leq C\epsilon_n(1 + |X_{t_0-1}| + |Y_{t_0-1}| + |Z_{t_0-1}|)^r,$$

then (A3-p) and Sobolev’s inequality yield

$$\frac{1}{nh_n} \partial_\theta^2 \tilde{H}_n(\theta)$$

$$= -\frac{1}{nh_n} \sum_{l=1}^{L_n-1} \left\{ c_n h_n \partial_\theta \tilde{\psi}_1^{(l)}(\theta) Z_{t_0-1} \partial_\theta \tilde{\psi}_1(\theta) - \partial_\theta \tilde{\psi}_1^{(l)}(\theta) Z_{t_0-1} (\Delta Y - c_n h_n \tilde{\psi}_1(\theta)) \right\}$$

$$= -\frac{c_n}{n} \sum_{l=1}^{L_n-1} \partial_\theta \tilde{\psi}_1^{(l)}(\theta) Z_{t_0-1} \partial_\theta \tilde{\psi}_1(\theta) + \frac{c_n}{n} \sum_{l=1}^{L_n-1} \partial_\theta \tilde{\psi}_1^{(l)}(\theta) Z_{t_0-1} (\tilde{\psi}_{1,0} - \tilde{\psi}_1(\theta))$$

$$+ \frac{1}{nh_n} \sum_{l=1}^{L_n-1} \partial_\theta \tilde{\psi}_1^{(l)}(\theta) Z_{t_0-1} \int_{t_0}^{t_{l+1}} V_s dW_s + O_P \left( \frac{L_n(c_n h_n)^{3/2}}{nh_n} \right)$$

$$= -\frac{c_n}{n} \sum_{l=1}^{L_n-1} \partial_\theta \tilde{\psi}_1^{(l)}(\theta_0) Z_{t_0-1} \partial_\theta \tilde{\psi}_1(\theta_0) + O_P \left( \frac{L_n c_n h_n}{nh_n} \epsilon_n \right) + O_P \left( \frac{\sqrt{L_n c_n h_n}}{nh_n} \right) + o_P(1)$$

$$\xrightarrow{P} -\Gamma,$$

and

$$\partial_\theta \tilde{H}_n(\theta_0) = \sum_{l=1}^{L_n-1} \partial_\theta \tilde{\psi}_1^{(l)}(\theta_0) Z_{t_0-1} \int_{t_0}^{t_{l+1}} V_s dW_s + O_P(h_n \sqrt{c_n h_n}).$$

We have $O_P(nh_n \sqrt{c_n h_n}) = O_P(\sqrt{nh_n} \cdot \sqrt{nh_n} c_n) = o_P(\sqrt{nh_n})$,

$$\sum_{l=1}^{L_n-1} E \left[ \left( \frac{1}{\sqrt{nh_n}} \partial_\theta \tilde{\psi}_1^{(l)}(\theta_0) Z_{t_0-1} \int_{t_0}^{t_{l+1}} V_s dW_s \right)^2 \right] \xrightarrow{P} 0,$$

and

$$\sum_{l=1}^{L_n-1} E \left[ \left( \frac{1}{\sqrt{nh_n}} \partial_\theta \tilde{\psi}_1^{(l)}(\theta_0) Z_{t_0-1} \int_{t_0}^{t_{l+1}} V_s dW_s \right)^4 \right] = O_P \left( \frac{L_n(c_n h_n)^2}{n^2 h_n^2} \right) = O_P(L_n^{-1}) \xrightarrow{P} 0.$$

Then, Lemma 9 in Genon-Catalot and Jacob (1993) and the martingale central limit theorem (Corollary 3.1 and the remark after that in Hall and Heyde (1980)) imply

$$\frac{1}{\sqrt{nh_n}} \partial_\theta \tilde{H}_n(\theta_0) \xrightarrow{d} N(0, \Gamma).$$
Proof of Theorem 2.2.

Similarly to (16), we obtain
\[ \lim_{\delta \to 0} \liminf_{n \to \infty} P(\tilde{H}_{n, \delta}(\theta) = \tilde{H}_n(\theta) \text{ for any } \theta) = 1. \] (36)

Since \( \partial_\theta H_n(\hat{\theta}_n) = 0 \) by definition, Taylor’s formula yields
\[-\partial_\theta H_n(\theta_0) = \partial_\theta H_n(\hat{\theta}_n) - \partial_\theta H_n(\theta_0) = \int_0^1 \partial^2_\theta H_n(\theta_u) du (\hat{\theta}_n - \theta_0),\]
if \( (\theta_u)_{u \in [0,1]} \subset \Theta \), where \( \theta_u = u\hat{\theta}_n - (1-u)\theta_0 \) for \( 0 \leq u \leq 1. \)

Similarly to Lemma 5.1 we have
\[ (nh_n)^{-1} \sup_\theta |\partial_\theta \Lambda_j(\theta)| P \to 0, \]
for \( 1 \leq j \leq 4. \) Then discussions in Section 5.2, Lemma 5.3, and (36) yield
\[ \frac{1}{nh_n} \int_0^1 \partial^2_\theta H_{n, \delta}(\theta_u) du = \frac{1}{nh_n} \int_0^1 \left\{ \partial^2_\theta \tilde{H}_{n, \delta}(\theta_u) + 4 \sum_{j=1}^4 \partial^2_\theta \Lambda_j(\theta_u) \right\} du \overset{P}{\to} \Gamma, \]
on \{\liminf_{n \to \infty} T_{n, \delta} = \infty\} \text{ for any } \delta > 0, \text{ and together with (16) we obtain}
\[ \frac{1}{nh_n} \int_0^1 \partial^2_\theta H_n(\theta_u) du \overset{P}{\to} \Gamma. \] (37)

Furthermore, Proposition 5.1, Lemma 5.5 and (36) yield
\[ \frac{1}{\sqrt{nh_n}} \partial_\theta \tilde{H}_{n, \delta}(\theta_0) = \frac{1}{\sqrt{nh_n}} \partial_\theta \tilde{H}_n(\theta_0) + o_P(1) \overset{d}{\to} N(0, \Gamma), \] (38)
on \{\liminf_{n \to \infty} T_{n, \delta} = \infty\}. Then together with (37) and (16), we obtain
\[ \sqrt{n h_n} (\hat{\theta}_n - \theta_0) = \Gamma^{-1} \frac{1}{\sqrt{nh_n}} \partial_\theta \tilde{H}_n(\theta_0) + o_P(1) \overset{d}{\to} N(0, \Gamma^{-1}). \]

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References

[1] Adams, R. A. and Fournier, J. J. F. (2003) Sobolev spaces, Elsevier/Academic Press, Amsterdam.
[2] Billingsley, P. (1999) Convergence of probability measures (Second edition), John Wiley & Sons, Inc., New York.
[3] Briand, P. and Hu, Y. (1998). Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDEs. Journal of Functional Analysis 155(2), 455-494
[4] Buckdahn, R. and Peng, S. (1999) Ergodic backward SDE and associated PDE. In Seminar on Stochastic Analysis, Random Fields and Applications, pp. 73-85. Birkhauser, Basel.
[5] Chen, X. and Lin, L. (2010) Nonparametric estimation for FBSDEs models with applications in finance. Communications in Statistics-Theory and Methods 39(14), pp. 2492-2514.
[6] Chong, W.F., Hu, Y., Liang, G. and Zariphopoulou, T. (2019) An ergodic BSDE approach to forward entropic risk measures: representation and large-maturity behavior. Finance and Stochastics 23(1), pp. 239-273.
[7] Debussche, A., Hu, Y. and Tessitore, G. (2011) Ergodic BSDEs under weak dissipative assumptions. *Stochastic Processes and their Applications* 121(3), pp. 407-426.

[8] Fuhrman, M., Hu, Y. and Tessitore, G. (2009) Ergodic BSDEs and optimal ergodic control in Banach spaces. *SIAM Journal on Control and Optimization* 48(3), pp. 1542-1566.

[9] Gobet, E. (2002) LAN property for ergodic diffusions with discrete observations. *Annales de l’Institut Henri Poincaré. Probabilités et Statistiques* 38(5), pp. 711-737.

[10] Genon-Catalot, V. and Jacob, J. (1993) On the estimation of the diffusion coefficient for multi-dimensional diffusion processes, *Annales de l’Institut Henri Poincaré. Probabilités et Statistiques* 29(1), pp. 119-151.

[11] Guatteri, G. and Masiero, F. (2009) Infinite horizon and ergodic optimal quadratic control for an affine equation with stochastic coefficients. *SIAM Journal on Control and Optimization* 48(3), pp. 1600-1631.

[12] Guatteri, G. and Tessitore, G. (2020) Ergodic BSDEs with Multiplicative and Degenerate Noise. *SIAM Journal on Control and Optimization* 58(4), pp. 2050-2077.

[13] Hall, P. and Heyde, C. C. (1980) Martingale limit theory and its application, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London.

[14] Hu, M. and Wang, F. (2018) Ergodic BSDEs driven by G-Brownian motion and applications. *Stochastics and Dynamics* 18(06), 1850050.

[15] Hu, Y. and Lemonnier, F. (2019) Ergodic BSDE with unbounded and multiplicative underlying diffusion and application to large time behaviour of viscosity solution of HJB equation. *Stochastic Processes and their Applications* 129(10), pp. 4009-4050.

[16] Hu, M., Li, H., Wang, F. and Zheng, G. (2015) Invariant and ergodic nonlinear expectations for $G$-diffusion processes. *Electronic Communications in Probability* 20.

[17] Hu, Y., Liang, G. and Tang, S. (2020) Systems of Ergodic BSDEs arising in regime switching forward performance processes. *SIAM Journal on Control and Optimization* 58(4), pp. 2503-2534.

[18] Kessler, M. (1997) Estimation of an ergodic diffusion from discrete observations. *Scandinavian Journal of Statistics* 24(2), pp. 211-229.

[19] Liang, G. and Zariphopoulou, T. (2017) Representation of homothetic forward performance processes in stochastic factor models via ergodic and infinite horizon BSDE. *SIAM Journal on Financial Mathematics* 8(1), pp. 344-372.

[20] Madec, P.Y. (2015) Ergodic BSDEs and related PDEs with Neumann boundary conditions under weak dissipative assumptions. *Stochastic Processes and their Applications* 125(5), pp. 1821-1860.

[21] Ogihara, T. and Yoshida, N. (2011) Quasi-likelihood analysis for the stochastic differential equation with jumps. *Statistical Inference for Stochastic Processes* 14(3), pp. 189.

[22] Richou, A. (2009) Ergodic BSDEs and related PDEs with Neumann boundary conditions. *Stochastic Processes and their Applications* 119(9), pp. 2945-2969.

[23] Shimizu, Y. and Yoshida, N. (2006) Estimation of parameters for diffusion processes with jumps from discrete observations. *Statistical Inference for Stochastic Processes* 9(3), pp. 227-277.

[24] Song, Y. (2014) Terminal-dependent statistical inference for the FBSDEs models. *Mathematical Problems in Engineering*, 2014.

[25] Su, Y. and Lin, L. (2009) Semi-parametric estimation for forward–backward stochastic differential equations. *Communications in Statistics-Theory and Methods* 38(11), pp. 1759-1775.

[26] Uchida, M. and Yoshida, N. (2012) Adaptive estimation of an ergodic diffusion process based on sampled data. *Stochastic Processes and their Applications* 122(8), pp. 2885-2924.

[27] Yoshida, N. (1992) Estimation for diffusion processes from discrete observation. *Journal of Multivariate Analysis* 41, pp. 220-242.
[28] YOSHIDA, N. (2011) Polynomial type large deviation inequalities and quasi-likelihood analysis for stochastic differential equations. *Annals of the Institute of Statistical Mathematics* 63(3), pp. 431-479.

[29] ZHANG, Q. (2013) Terminal-dependent statistical inference for the integral form of FBSDE. *Discrete Dynamics in Nature and Society*, 2013.

[30] ZHANG, Q. AND LIN, L. (2014) Terminal-dependent statistical inferences for FBSDE. *Stochastic Analysis and Applications* 32(1), pp. 128-151.
Figure 1: Simulation result under a one-dimensional Vasicek model.
Figure 2: Mean Absolute Error under a two-dimensional Heston model.