EQUIVARIANT COBORDISM OF SCHEMES

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Abstract. Let $k$ be a field of characteristic zero. For a linear algebraic group $G$ over $k$ acting on a $k$-scheme $X$, we define the equivariant algebraic cobordism of $X$, which extends the definition of equivariant cobordism of smooth schemes in [7] to all schemes. Using a refined version of the Levine-Morel localization, we establish the localization sequence for the equivariant cobordism for all $G$-schemes. We explicitly describe the relation of equivariant cobordism with equivariant Chow groups, $K$-groups and complex cobordism.

We show that the rational equivariant cobordism of a $G$-variety can be expressed as the Weyl group invariants of the equivariant cobordism for the action of a maximal torus of $G$. As applications, we show that the rational algebraic cobordism of the classifying space of a complex algebraic group is isomorphic to its complex cobordism.

1. Introduction

Let $k$ be a field of characteristic zero. Based on the construction of the motivic algebraic cobordism spectrum $MGL$ by Voevodsky, Levine and Morel [25] gave a geometric construction of the algebraic cobordism and showed that this is a universal oriented Borel-Moore homology theory in the category of varieties over the field $k$. Their definition was extended by Deshpande [7] in the equivariant set-up that led to the notion of the equivariant cobordism of smooth varieties acted upon by linear algebraic groups. This in particular allowed one to define the algebraic cobordism of the classifying spaces analogous to their complex cobordism.

Apart from its many applications in the equivariant set up which are parallel to the ones in the non-equivariant world, an equivariant cohomology theory often leads to the description of the corresponding non-equivariant cohomology by mixing the geometry of the variety with the representation theory of the underlying groups.

Our aim in this first part of a series of papers is to develop the theory of equivariant cobordism in the category of all $k$-schemes with action of a linear algebraic group. We establish the fundamental properties of this theory and give applications. In the second part [18] of this series, we shall give other important applications of the results of this paper. Some further applications of the results of this paper to the computation of the non-equivariant cobordism rings appear in [19] and [20]. We now describe some of the main results in of this paper.

Let $G$ be a linear algebraic group over $k$. In this paper, a scheme will mean a quasi-projective $k$-scheme and all $G$-actions will be assumed to be linear. If $X$ is a smooth scheme with a $G$-action, Deshpande defined the equivariant cobordism $\Omega^G_*(X)$ using the coniveau filtration on the Levine-Morel cobordism of certain smooth mixed spaces. This was based on the construction of the Chow groups of classifying spaces in [33] and the equivariant Chow groups in [8].

Using a niveau filtration on the algebraic cobordism, which is based on the analogous filtration on any Borel-Moore homology theory as described in [2, Section 3],

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we define the equivariant algebraic cobordism of any $k$-scheme with $G$-action in Section 4. This is defined by taking a projective limit over the quotients of the Levine-Morel cobordism of certain mixed spaces by various levels of the niveau filtration. In order to make sense of this construction, one needs to prove various properties of the above niveau filtration which is done in Section 3. These equivariant cobordism groups coincide with the one in [7] for smooth schemes. We also show in Section 5 how one can recover the formula for the cobordism group of certain classifying spaces directly from the above definition, by choosing suitable models for the underlying mixed spaces.

The first main result on these equivariant cobordism groups is to show that they satisfy the localization exact sequence, a fundamental property of any cohomology theory. For smooth schemes, a version of the localization sequence was proven in [7, Theorem 7.3]. However, the proof given there is not complete (cf. Remark 5.2). The problem stems from the fact that the projective limit is not in general a right exact functor. Hence to prove the equivariant version of the localization sequence, one needs a stronger version of the Levine-Morel localization sequence involving the above niveau filtration. We prove this refined localization sequence in Theorem 3.4 which is then used to give the complete proof of the equivariant localization sequence in Theorem 5.1. We prove the other expected properties such as functoriality, homotopy invariance, exterior product, projection formula and existence of Chern classes for equivariant vector bundles in Theorem 5.4.

In Section 7, we show how the equivariant cobordism is related to other equivariant cohomology theories such as equivariant Chow groups, equivariant $K$-groups and equivariant complex cobordism. Using some properties of the niveau filtration and known relation between the non-equivariant cobordism and Chow groups, we deduce an explicit formula (cf. Proposition 7.1) which relates the equivariant cobordism and the equivariant Chow groups of $k$-varieties. Using this and the main results of [15], we give a formula in Theorem 7.3 which relates the equivariant cobordism with the equivariant $K$-theory of smooth schemes. We also construct a natural transformation from the algebraic to the equivariant version the complex cobordism for varieties over the field of complex numbers.

Our next main result of this paper is Theorem 8.7 where we show that for a connected linear algebraic group $G$ acting on a scheme $X$, there is a canonical isomorphism $\Omega^G_\ast(X) \cong (\Omega^T_\ast(X))^W$ with rational coefficients, where $T$ is a split maximal torus of a Levi subgroup of $G$ with Weyl group $W$. This is mainly achieved by the Morita isomorphism of Proposition 5.5 and a detour to the motivic cobordism $MGL$ and its extension $MGL'$ to singular schemes by Levine [22]. This motivic cobordism $MGL'$ comes into play due to the advantage that it is a bigraded Borel-Moore homology which has long exact localization sequences. What helps us in using this theory in our context is the recent comparison result of Levine [23] which shows that the Levine-Morel cobordism theory is a piece of the more general $MGL'$-theory.

As an easy consequence of Proposition 7.1, we recover Totaro’s cycle class map (cf. [33])

$$CH^\ast(BG) \to MU^\ast(BG) \otimes_{L} \mathbb{Z} \to H^\ast(BG)$$

for a complex linear algebraic group $G$. It is conjectured that this map is an isomorphism of rings. This conjecture has been shown to be true by Totaro for some classical groups such as $BGL_n$, $O_n$, $Sp_{2n}$ and $SO_{2n+1}$. Although, we can not say anything about this conjecture here, we do show as a consequence of Theorem 8.7 that the map $CH^\ast(BG) \to MU^\ast(BG) \otimes_{L} \mathbb{Z}$ is indeed an isomorphism of rings with the rational coefficients (see Theorem 8.10 for the full statement). We do this by first showing that there is a natural ring homomorphism $\Omega^\ast(BG) \to MU^\ast(BG)$
(with integer coefficients) which lifts Totaro’s map. We then show that this map is in fact an isomorphism with rational coefficients using Theorem 8.7.

As some more applications of Theorem 8.7, we aim to compute the cobordism ring of certain spherical varieties in [13].

2. Recollection of algebraic cobordism

In this section, we briefly recall the definition of algebraic cobordism of Levine-Morel. We also recall the other definition of this object as given by Levine-Pandharipande. Since we shall be concerned with the study of schemes with group actions and the associated quotient schemes, and since such quotients often require the original scheme to be quasi-projective, we shall assume throughout this paper that all schemes over \( k \) are quasi-projective.

Notations. We shall denote the category of quasi-projective \( k \)-schemes by \( \mathcal{V}_k \). By a scheme, we shall mean an object of \( \mathcal{V}_k \). The category of smooth quasi-projective schemes will be denoted by \( \mathcal{V}_k^s \). If \( G \) is a linear algebraic group over \( k \), we shall denote the category of quasi-projective \( k \)-schemes with a \( G \)-action and \( G \)-equivariant maps by \( \mathcal{V}_G \). The associated category of smooth \( G \)-schemes will be denoted by \( \mathcal{V}_G^s \). All \( G \)-actions in this paper will be assumed to be linear. Recall that this means that all \( G \)-schemes are assumed to admit \( G \)-equivariant ample line bundles. This assumption is always satisfied for normal schemes (cf. [30, Theorem 2.5], [31, 5.7]).

2.1. Algebraic cobordism. Before we define the algebraic cobordism, we recall the Lazard ring \( \mathbb{L} \). It is a polynomial ring over \( \mathbb{Z} \) on infinite but countably many variables and is given by the quotient of the polynomial ring \( \mathbb{Z}[A_{ij} | (i, j) \in \mathbb{N}^2] \) by the relations, which uniquely define the universal formal group law \( F_\mathbb{L} \) of rank one on \( \mathbb{L} \). This formal group law is given by the power series

\[
F_\mathbb{L}(u, v) = u + v + \sum_{i,j \geq 1} a_{ij} u^i v^j,
\]

where \( a_{ij} \) is the equivalence class of \( A_{ij} \) in the ring \( \mathbb{L} \). The Lazard ring is graded by setting the degree of \( a_{ij} \) to be \( 1 - i - j \). In particular, one has \( \mathbb{L}_0 = \mathbb{Z} \), \( \mathbb{L}_{-1} = \mathbb{Z} a_{11} \) and \( \mathbb{L}_i = 0 \) for \( i \geq 1 \), that is, \( \mathbb{L} \) is non-positively graded. We shall write \( \mathbb{L}_s \) for the graded ring such that \( \mathbb{L}_{s+i} = \mathbb{L}_{-i} \) for \( i \in \mathbb{Z} \). We now define the algebraic cobordism of Levine and Morel [25].

Let \( X \) be an equi-dimensional \( k \)-scheme. A cobordism cycle over \( X \) is a family \( \alpha = [Y \xrightarrow{f} X, L_1, \ldots, L_r] \), where \( Y \) is a smooth scheme, the map \( f \) is projective, and \( L_i \)'s are line bundles on \( Y \). Here, one allows the set of line bundles to be empty. The degree of such a cobordism cycle is defined to be \( \text{deg}(\alpha) = \dim_k(Y) - r \) and its codimension is defined to be \( \dim(X) - \text{deg}(\alpha) \). Let \( \mathcal{Z}^*(X) \) be the free abelian group generated by the cobordism cycles of the above type. Note that this group is graded by the codimension of the cycles. In particular, for \( j \in \mathbb{Z} \), \( \mathcal{Z}_j(X) \) is the free abelian group on cobordism cycles \( \alpha = [Y \xrightarrow{f} X, L_1, \ldots, L_r] \), where \( Y \) is smooth and irreducible and codimension of \( \alpha \) is \( j \).

We impose several relations on \( \mathcal{Z}^*(X) \) in order to define the algebraic cobordism group. The first among these is the so called dimension axiom: let \( \mathcal{R}_{\dim}^*(X) \) be the graded subgroup of \( \mathcal{Z}^*(X) \) generated by the cobordism cycles \( \alpha = [Y \xrightarrow{f} X, L_1, \ldots, L_r] \) such that \( \dim_k Y < r \). Let

\[
\mathcal{Z}^*_\dim(X) = \frac{\mathcal{Z}^*(X)}{\mathcal{R}^*_\dim(X)}.
\]
For a line bundle $L$ on $X$ and cobordism cycle $\alpha$ as above, we define the Chern class operator on $Z^*_{\text{dim}}(X)$ by letting $c_1(L)(\alpha) = [Y \xrightarrow{f} X, L_1, \cdots, L_r, f^*(L)]$.

Next, we impose the so-called section axiom. Let $\mathcal{R}^*_{\text{sec}}(X)$ be the graded subgroup of $Z^*_{\text{dim}}(X)$ generated by cobordism cycles of the form $[Y \to X, L] - [Z \to X]$, where $Y \xrightarrow{s} L$ is a section of the line bundle $L$ on $Y$ which is transverse to the zero-section, and $Z \hookrightarrow Y$ is the closed subvariety of $Y$ defined by the zeros of $s$. The transversality of $s$ ensures that $Z$ is a smooth variety. In particular, $[Z \to X]$ is a well-defined cobordism cycle on $X$. Define

$$\Omega^*(X) = \frac{Z^*_{\text{dim}}(X)}{\mathcal{R}^*_{\text{sec}}(X)}$$

The assignment $X \mapsto \Omega^*(X)$ is called the pre-cobordism theory.

Finally, we impose the formal group law on the cobordism using the following relation. For $X$ as above, let $\mathcal{R}^*_{\text{FGL}}(X) \subset \mathbb{L} \otimes_{\mathbb{Z}} \Omega^*(X)$ be the graded $\mathbb{L}$-submodule generated by elements of the form

$$\{F_\mathbb{L}(c_1(L), c_1(M))(x) - c_1(L \otimes M)(x) | x \in \Omega^*(X), L, M \in \text{Pic}(X)\}.$$

We define the algebraic cobordism group of $X$ by

$$\Omega^*(X) = \frac{\mathbb{L} \otimes \Omega^*(X)}{\mathcal{R}^*_{\text{FGL}}(X)}.$$  \hspace{1cm} (2.1)

If $X$ is not necessarily equi-dimensional, we define $Z_*(X)$ to be same as $Z^*(X)$ except that $Z_*(X)$ is now graded by the degree of the cobordism cycles. In particular, $Z_i(X)$ is the free abelian group on cobordism cycles $[Y \xrightarrow{f} X, L_1, \cdots, L_r]$ such that $f$ is projective and $Y$ is smooth and irreducible such that $\dim(Y) - r = i$. One then defines $\Omega_*(X)$ to be the quotient of $\mathbb{L}_* \otimes_{\mathbb{Z}} \Omega_*(X)$ in the same way as above. Note that for $X$ equi-dimensional of dimension $d$ and $i \in \mathbb{Z}$, one has $\Omega^*(X) \cong \Omega_{d-i}(X)$.

Observe that $\Omega^*(X)$ is a graded $\mathbb{L}$-module such that $\Omega^j(X) = 0$ for $j > \dim(X)$ and $\Omega^j(X)$ can be non-zero for any given $-\infty < j \leq \dim(X)$. Similarly, $\Omega_*(X)$ is a graded $\mathbb{L}_*$-module which has no component in the negative degrees and it can be non-zero in arbitrarily large positive degree.

The following is the main result of Levine and Morel from which most of their other results on algebraic cobordism are deduced. We refer to loc. cit. for more properties.

**Theorem 2.1.** The functor $X \mapsto \Omega_*(X)$ is the universal Borel-Moore homology on the category $\mathcal{V}_k$. In other words, it is universal among the homology theories on $\mathcal{V}_k$ which have functorial push-forward for projective morphism, pull-back for smooth morphism (any morphism of smooth schemes), Chern classes for line bundles, and which satisfy Projective bundle formula, homotopy invariance, the above dimension, section and formal group law axioms. Moreover, for a $k$-scheme $X$ and closed subscheme $Z$ of $X$ with open complement $U$, there is a localization exact sequence

$$\Omega_*(Z) \to \Omega_*(X) \to \Omega_*(U) \to 0.$$  

It was also shown in loc. cit. that the natural composite map

$$\Phi : \mathbb{L} \to \mathbb{L} \otimes_{\mathbb{Z}} \Omega^*(k) \to \Omega^*(k)$$

$$a \mapsto [a]$$

is an isomorphism of commutative graded rings.
As an immediate corollary of Theorem 2.1, we see that for a smooth $k$-scheme $X$ and an embedding $\sigma : k \to \mathbb{C}$, there is a natural morphism of graded rings
\[ \Phi_{X}^{\operatorname{top}} : \Omega^{*}(X) \to \text{MU}^{2*}(X_{\sigma}(\mathbb{C})), \]
where $\text{MU}^{*}(X_{\sigma}(\mathbb{C}))$ is the complex cobordism ring of the complex manifold $X_{\sigma}(\mathbb{C})$ given by the complex points of $X \times_{k} \mathbb{C}$. This map is an isomorphism for $X = \text{Spec}(k)$. In particular, there are isomorphisms of graded rings
\[ \mathbb{L} \cong \Omega^{*}(k) \cong \text{MU}^{2*} \cong \text{MU}^{*}, \]
where $\text{MU}^{*}$ is the complex cobordism ring of a point. As a corollary, we see that for any field extension $k \hookrightarrow K$, the natural map $\Omega^{*}(k) \to \Omega^{*}(K)$ is an isomorphism.

2.2. Cobordism via double point degeneration. To enforce the formal group law on the algebraic cobordism in order to make it an oriented cohomology theory on the category of smooth varieties, Levine and Morel artificially imposed this condition by tensoring their pre-cobordism theory with the Lazard ring. Although they were able to show that the resulting map $\bar{Z}_{*}(X) \to \Omega_{*}(X)$ is still surjective, they were unable to describe the explicit geometric relations in $Z_{*}(X)$ that define $\Omega_{*}(X)$. This was subsequently accomplished by Levine-Pandharipande [26]. We conclude our introduction to the algebraic cobordism by briefly discussing the construction of Levine-Pandharipande. For $n \geq 1$, let $\Box^{n}$ denote the space $(\mathbb{P}^{1}_{k} - \{1\})^{n}$.

**Definition 2.2.** A morphism $Y \xrightarrow{\pi} \Box^{1}$ is called a double point degeneration, if $Y$ is a smooth scheme and $\pi^{-1}(0)$ is scheme-theoretically given as the union $A \cup B$, where $A$ and $B$ are smooth divisors on $Y$ which intersect transversely. The intersection $D = A \cap B$ is called the double point locus of $\pi$. Here, $A$, $B$ and $D$ are allowed to be disconnected or, even empty.

For a double point degeneration as above, notice that the scheme $D$ is also smooth and $\mathcal{O}_{D}(A + B)$ is trivial. In particular, one sees that $N_{A/D} \otimes_{D} N_{B/D} \cong \mathcal{O}_{D}$. This turn implies that the projective bundles $\mathbb{P}(\mathcal{O}_{D} \oplus N_{A/D}) \to D$ and $\mathbb{P}(\mathcal{O}_{D} \oplus N_{B/D}) \to D$ are isomorphic, where $N_{A/D}$ and $N_{B/D}$ are the normal bundles of $D$ in $A$ and $B$ respectively. Let $\mathbb{P}(\pi) \to D$ denote any of these two projective bundles.

Let $X$ be a $k$-scheme and let $Y \xrightarrow{f} X \times \Box^{1}$ be a projective morphism from a smooth scheme $Y$. Assume that the composite map $\pi : Y \to X \times \Box^{1} \to \Box^{1}$ is a double point degeneration such that $Y_{\infty} = \pi^{-1}(\infty)$ is smooth. We define the cobordism cycle on $X$ associated to the morphism $f$ to be the cycle
\[ C(f) = [Y_{\infty} \to X] - [A \to X] - [B \to X] + \mathbb{P}(\pi) \to X]. \]

Let $\mathcal{M}_{*}(X)$ be the free abelian group on the isomorphism classes of the morphisms $[Y \xrightarrow{f} X]$, where $Y$ is smooth and irreducible and $f$ is projective. Then $\mathcal{M}_{*}(X)$ is a graded abelian group, where the grading is by the dimension of $Y$. Let $\mathcal{R}_{*}(X)$ be the subgroup of $\mathcal{M}_{*}(X)$ generated by all cobordism cycles $C(f)$, where $C(f)$ is as in (2.1). Note that $\mathcal{R}_{*}(X)$ is a graded subgroup of $\mathcal{M}_{*}(X)$. Define
\[ \omega_{*}(X) = \frac{\mathcal{M}_{*}(X)}{\mathcal{R}_{*}(X)}. \]

**Theorem 2.3** ([26]). There is a canonical isomorphism
\[ \omega_{*}(X) \xrightarrow{\cong} \Omega_{*}(X) \]
of oriented Borel-Moore homology theories on $X$. 


3. Niveau filtration on algebraic cobordism

In this section, we introduce the niveau filtration on the algebraic cobordism which plays an important role in the definition of the equivariant algebraic cobordism. Our main result here is a refined localization sequence for the cobordism which preserves the niveau filtration. This new localization sequence will have interesting consequences in the study of the equivariant cobordism.

Let $X$ be a $k$-scheme of dimension $d$. For $j \in \mathbb{Z}$, let $Z_j$ be the set of all closed subschemes $Z \subset X$ such that $\dim_t(Z) \leq j$ (we assume $\dim(\emptyset) = -\infty$). The set $Z_j$ is then ordered by the inclusion. For $i \geq 0$, we define

$$\Omega_i(Z_j) = \lim_{Z \in Z_j} \Omega_i(Z)$$

and put

$$\Omega_*(Z_j) = \bigoplus_{i \geq 0} \Omega_i(Z_j).$$

It is immediate that $\Omega_*(Z_j)$ is a graded $\mathbb{L}_*$-module and there is a graded $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$.

Following [2, Section 3], we let $Z_j/Z_{j-1}$ denote the ordered set of pairs $(Z, Z') \in Z_j \times Z_{j-1}$ such that $Z' \subset Z$ with the ordering

$$(Z, Z') \geq (Z_1, Z'_1) \text{ if } Z_1 \subseteq Z \text{ and } Z'_1 \subseteq Z'.$$

We let

$$\Omega_*(Z_j/Z_{j-1}(X)) := \lim_{(Z, Z') \in Z_j/Z_{j-1}} \Omega_i(Z - Z').$$

**Lemma 3.1.** For $f : X' \to X$ projective, the push-forward map $\Omega_*(X') \xrightarrow{f_*} \Omega_*(X)$ induces a push-forward map $\Omega_*(Z_j/Z_{j-1}(X')) \to \Omega_*(Z_j/Z_{j-1}(X))$.

**Proof.** Let $(Z, Z') \in Z_j/Z_{j-1}(X')$. Then $(W, W') = (\text{Im}(Z), \text{Im}(Z')) \in Z_j/Z_{j-1}(X)$. It suffices now to show that $f_*$ induces a natural map $\Omega_*(Z - Z') \to \Omega_*(W - W')$. However, this follows directly from the localization exact sequences

$$\Omega_*(Z') \to \Omega_*(Z) \to \Omega_*(Z - Z') \to 0$$

$$f_* \downarrow \quad \quad \downarrow f_*$$

$$\Omega_*(W') \to \Omega_*(W) \to \Omega_*(W - W') \to 0$$

and the fact that the square on the left is commutative. \qed

For $x \in Z_j$, let

$$\Omega_*(\hat{k}(x)) = \lim_{U \subseteq \{x\}} \Omega_*(U),$$

where the limit is taken over all non-empty open subsets of $\{x\}$. Taking the limit over the localization sequences

$$\Omega_*(Z) \to \Omega_*(Z) \to \Omega_*(Z - Z') \to 0$$

for $(Z, Z') \in Z_j/Z_{j-1}$, one now gets an exact sequence

$$\Omega_*(Z_{j-1}) \to \Omega_*(Z_j) \to \bigoplus_{x \in (Z_j/Z_{j-1})} \Omega_*(\hat{k}(x)) \to 0.$$
Definition 3.2. We define $F_j\Omega_*(X)$ to be the image of the natural $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$. In other words, $F_j\Omega_*(X)$ is the image of all $\Omega_*(W) \to \Omega_*(X)$, where $W \to X$ is a projective map such that $\dim(\text{Image}(W)) \leq j$.

One checks at once that there is a canonical niveau filtration
\begin{equation}
0 = F_{-1}\Omega_*(X) \subseteq F_0\Omega_*(X) \subseteq \cdots \subseteq F_{d-1}\Omega_*(X) \subseteq F_d\Omega_*(X) = \Omega_*(X).
\end{equation}

Lemma 3.3. If $f : X' \to X$ is a projective morphism, then $f_* (F_j\Omega_*(X')) \subseteq F_j\Omega_*(X)$. If $g : X' \to X$ is a smooth morphism of relative dimension $r$, then $g^*(F_j\Omega_*(X)) \subseteq F_{j+S}\Omega_*(X')$.

Proof. The first assertion is obvious from the definition. In fact, the push-forward map preserves the niveau filtration at the level of the free abelian groups of cobordism cycles. The second assertion also follows immediately using the fact that for a cobordism cycle $[Y \to X]$, one has $g^*([Y \to X]) = [Y \times_X X' \to X']$. This in turn implies that $g^* \circ f_* = f'_* \circ g_*^*$ for a Cartesian square
\[
\begin{array}{ccc}
W' & \xrightarrow{f'} & X' \\
\downarrow{g'} & & \downarrow{g} \\
W & \xrightarrow{f} & X
\end{array}
\]
such that $f$ is projective and $g$ is smooth. \qed

Theorem 3.4. (Refined localization sequence) Let $X$ be a $k$-scheme and let $Z$ be a closed subscheme of $X$ with the complement $U$. Then for every $j \in \mathbb{Z}$, there is an exact sequence

\[
F_j\Omega_*(Z) \to F_j\Omega_*(X) \to F_j\Omega_*(U) \to 0.
\]

Proof. All these groups are zero if $j < 0$, so we assume $j \geq 0$. Let $F_j\mathcal{Z}_*(X)$ be the free abelian group on cobordism cycles $[Y \xrightarrow{f} X]$ such that $Y$ is irreducible and $\dim(f(Y)) \leq j$. Note that $f(Y)$ is a closed and irreducible subscheme of $X$ since $Y$ is irreducible and $f$ is projective. It is then clear that $F_j\mathcal{Z}_*(X) \subseteq \mathcal{Z}_*(X)$ and $F_j\mathcal{Z}_*(X) \to F_j\Omega_*(X)$. We first show the following.

Claim 3.5.

\[
\text{Ker} (F_j\mathcal{Z}_*(X) \to F_j\Omega_*(X)) \to \text{Ker} (F_j\mathcal{Z}_*(U) \to F_j\Omega_*(U))
\]
is surjective.

Proof of the claim: In one of the steps in the proof of the localization sequence in [25], it is shown that the map
\begin{equation}
(3.4) \quad \text{Ker} (\mathcal{Z}_*(X) \to \Omega_*(X)) \to \text{Ker} (\mathcal{Z}_*(U) \to \Omega_*(U))
\end{equation}
is surjective. Let $\alpha = \sum_{l=1}^n a_l[W_l \xrightarrow{f_l} U]$ be an element in the kernel of the map $F_j\mathcal{Z}_*(U) \to F_j\Omega_*(U)$. By (3.4), there is an element $\beta = \sum_{s=1}^m b_s[Y_s \xrightarrow{g_s} X] \in \mathcal{Z}_*(X)$ which restricts to $\alpha$. We can assume that any two summands of this sum are not isomorphic. For a map $Y \to X$, let $Y_U \to U$ be its restriction to $U$. Then we get
\[
\sum_{l=1}^n a_l[W_l \xrightarrow{f_l} U] = \sum_{s=1}^m b_s[(Y_s)_U \xrightarrow{g_s} U].
\]
Since $\mathcal{Z}_s(U)$ is a free abelian group, we see that $m = n$ and for each $1 \leq s \leq n$, $[(Y_s)_U \xrightarrow{g_s} U] \cong [W_l \xrightarrow{f_l} U]$ for some $l$. In particular, $\dim (g_s|_U((Y_s)_U)) \leq j$. Since $Y_s$ is irreducible and $g_s$ is projective, one easily checks that $\dim(g_s(Y_s)) \leq j$. That is $\beta \in F_j\mathcal{Z}_s(X)$. This proves the claim.

Since $F_j\mathcal{Z}_s(X)$ is a free abelian group, we see using Theorem 2.3 Claim 3.5 and the surjection $F_j\mathcal{Z}_s(X) \twoheadrightarrow F_j\Omega_s(X)$ that $\ker (F_j\Omega_s(X) \to F_j\Omega_s(U))$ is generated by the cobordism cycles $\alpha = [Y \xrightarrow{f} X] - [Y' \xrightarrow{f'} X]$, where $Y$ and $Y'$ are smooth and irreducible and $Y_U \cong Y'_U$ such that the dimension of their images are at most $j$.

Put $S = f(Y), S' = f'(Y'), T = S \cap U$ and $T' = S' \cap U$. Then the irreducibility of the sources and the projectivity of the maps imply that all these are irreducible closed subschemes of dimension at most $j$. Put $W = S \cup S'$ and $V = T \cup T'$. Then it is easy to see that $[Y \to X]$ and $[Y' \to X]$ are the images of the cycles $[Y \to W]$ and $[Y' \to W]$ in $\mathcal{Z}_s(W)$ under the push-forward via the closed immersion $W \hookrightarrow X$. Moreover, $Y_V \cong Y'_V$. Hence we see that we have a cycle $\beta = [Y \to W] - [Y' \to W] \in \mathcal{Z}_s(W)$ such that $[Y_V \to V] \cong [Y'_V \to V]$. Now, it follows from the proof of the localization sequence in loc. cit. that $\beta \in \image (\mathcal{Z}_s(W \cap Z) \to \mathcal{Z}_s(W))$. In particular, $\alpha = \image(\beta) \in \image (F_j\mathcal{Z}_s(Z) \to F_j\mathcal{Z}_s(X))$. Hence we have shown that $\ker (F_j\Omega_s(X) \to F_j\Omega_s(U))$ comes from $F_j\Omega_s(Z)$. We now show that the map $F_j\Omega_s(X) \to F_j\Omega_s(U)$ is surjective to complete the proof of the theorem.

We can assume that $X$ is irreducible and $j < \dim(X)$. By the generalized degree formula (cf. [25, Theorem 4.4.7]), any $\alpha \in F_j\Omega_s(U)$ can be written as

$$\alpha = \sum_{i=1}^{s} u_i[\tilde{U}_i \to U],$$

where $\tilde{U}_i \to U_i$ is a resolution of singularities of an irreducible closed subscheme $U_i \subset U$ and $u_i \in L_s$. Then we must have $[\tilde{U}_i \to U] \in F_j\Omega_s(U)$. Letting $X_i = \bar{U}_i \subset X$, we see that $\sum_{i=1}^{s} u_i[\tilde{X}_i \to X] \in F_j\Omega_s(X)$ and its image in $\Omega_s(U)$ is $\alpha$. This finishes the proof of the theorem.

The following is an immediate consequence of Theorem 3.4.

**Corollary 3.6.** Let $X$ be a $k$-scheme. Then for any $j \geq 0$ and any closed subscheme $Z \subset X$ of dimension at most $j$, the natural map $\Omega_s(X) \to \Omega_s(X - Z)$ induces an isomorphism

$$\frac{\Omega_s(X)}{F_j\Omega_s(X)} \cong \frac{\Omega_s(X - Z)}{F_j\Omega_s(X - Z)}.$$

**Lemma 3.7.** For a $k$-scheme $X$ and $i \geq 0$, the natural map $\Omega_i(X) \to CH_i(X)$ has the factorization

$$\Omega_i(X) \to \frac{\Omega_i(X)}{F_{i-1}\Omega_i(X)} \to CH_i(X).$$

**Proof.** By Theorem 2.3, $\Omega_s(X)$ is generated by the cobordism cycles $[Y \to X]$, where $Y$ is smooth and $f$ is projective. It follows from the definition of the niveau filtration that $F_j\Omega_s(X)$ is generated by the cobordism cycles of the form $i_*([Y \to Z])$, where $Z \xrightarrow{\phi} X$ is a closed subscheme of $X$ of dimension at most $j$. 

Since $\Omega_* \to CH_*$ is a natural transformation of oriented Borel-Moore homology theories, we get a commutative diagram

$$
\begin{array}{ccc}
\Omega_i(Z) & \longrightarrow & CH_i(Z) \\
\phi_* & & \phi_* \\
\Omega_i(X) & \longrightarrow & CH_i(X).
\end{array}
$$

The lemma now follows from the fact that $CH_i(Z) = 0$ if $j \leq i - 1$. \qed

**Lemma 3.8.** Let $E \xrightarrow{f} X$ be a vector bundle of rank $r$. Then the pull-back map $f^*: \Omega_*(X) \to \Omega_*(E)$ induces an isomorphism

$$F_j \Omega_*(X) \xrightarrow{\sim} F_{j+r} \Omega_*(E)$$

for all $j \in \mathbb{Z}$. In particular, $F_{<r} \Omega_*(E) = 0$.

**Remark 3.9.** The reader should be warned that the map $f^*$ shifts the degree of the grading by $r$.

**Proof.** Using the generalized degree formula, this can be proved in the same way as [7, Lemma 3.3], where a similar result is proven for smooth varieties and coniveau filtration. We sketch the proof in the singular case.

By the homotopy invariance of the algebraic cobordism, the natural map $\Omega_*(X) \xrightarrow{f^*} \Omega_*(E)$ is an isomorphism. So we only need to show that this map is surjective at each level of the niveau filtration. So let $e \in F_j \Omega_*(E)$. Assume that the dimension of $X$ is $d$. Since $F_{d+r} \Omega_*(E) = \Omega_*(E)$, the homotopy invariance implies that we can assume that $j < d + r$. The generalized degree formula now implies that there are irreducible closed subschemes $\{E_1, \ldots, E_s\}$ of $E$ such that in the $L_*$-module $\Omega_*(E)$, one has

$$e = \sum_{l=1}^{s} w_l [\tilde{E}_l \to E],$$

where $\tilde{E}_l \to E_l$ is a resolution of singularities of $E_l$ and $w_l \in \mathbb{L}_*$. In particular, we see that for each $l$, $[\tilde{E}_l \to E] \in \Omega_{p_l}(E)$ for some $p_l \leq j$.

On the other hand, the isomorphism of $f^*$ and the generalized degree formula for $\Omega_*(X)$ imply that for each $l$,

$$[\tilde{E}_l \to E] = \sum_{n=1}^{n_l} w_{l,n} \left[ \left( \tilde{X}_{l,n} \times_{X_l} E \right) \to E \right],$$

where $\tilde{X}_{l,n} \to X_{l,n}$ is a resolution of singularities of a closed subscheme $X_{l,n}$ of $X$. Since each component of this sum is a homogeneous element, we must have that

$$\left[ \left( \tilde{X}_{l,n} \times_{X_l} E \right) \to E \right] \in \Omega_{p_l}(E),$$

that is, $[\tilde{X}_{l,n} \to X] \in \Omega_{p_l-r}(X)$. This in turn implies that $x_l = \sum_{n=1}^{n_l} w_{l,n} [\tilde{X}_{l,n} \to X] \in \Omega_{p_l-r}(X)$. Hence we get

$$x = \sum_{l=1}^{s} w_l x_l \in F_{j-r} \Omega_*(X) \text{ and } e = f^*(x).$$

\qed
4. EQUIVARIANT ALGEBRAIC COBORDISM

In this text, $G$ will denote a linear algebraic group of dimension $g$ over $k$. All representations of $G$ will be finite dimensional. The definition of equivariant cobordism needs one to consider certain kind of mixed spaces which in general may not be a scheme even if the original space is a scheme. The following well known (cf. [8, Proposition 23]) lemma shows that this problem does not occur in our context and all the mixed spaces in this paper are schemes with ample line bundles.

**Lemma 4.1.** Let $H$ be a linear algebraic group acting freely and linearly on a $k$-scheme $U$ such that the quotient $U/H$ exists as a quasi-projective variety. Let $X$ be a $k$-scheme with a linear action of $H$. Then the mixed quotient $X^H/U$ exists for the diagonal action of $H$ on $X \times U$ and is quasi-projective. Moreover, this quotient is smooth if both $U$ and $X$ are so. In particular, if $H$ is a closed subgroup of a linear algebraic group $G$ and $X$ is a $k$-scheme with a linear action of $H$, then the quotient $G \times X$ is a quasi-projective scheme.

**Proof.** It is already shown in [8, Proposition 23] using [10, Proposition 7.1] that the quotient $X^H/U$ is a scheme. Moreover, as $U/H$ is quasi-projective, [10, Proposition 7.1] in fact shows that $X^H/U$ is also quasi-projective. The similar conclusion about $G^H/X$ follows from the first case by taking $U = G$ and by observing that $G/H$ is a smooth quasi-projective scheme (cf. [8, Theorem 6.8]).

The assertion about the smoothness is clear since $X \times U \to X^H/U$ is a principal $H$-bundle. \hfill \Box

For any integer $j \geq 0$, let $V_j$ be an $l$-dimensional representation of $G$ and let $U_j$ be a $G$-invariant open subset of $V_j$ such that the codimension of the complement $(V_j - U_j)$ in $V_j$ is at least $j$ and $G$ acts freely on $U_j$ such that the quotient $U_j/G$ is a quasi-projective scheme. Such a pair $(V_j, U_j)$ will be called a good pair for the $G$-action corresponding to $j$ (cf. [15, Section 2]). It is easy to see that a good pair always exists (cf. [8, Lemma 9]). Let $X_G$ denote the mixed quotient $X^G/U_j$ of the product $X \times U_j$ by the diagonal action of $G$, which is free.

Let $X$ be a $k$-scheme of dimension $d$ with a $G$-action. Fix $j \geq 0$ and let $(V_j, U_j)$ be an $l$-dimensional good pair corresponding to $j$. For $i \in \mathbb{Z}$, set

$$\Omega_i^G(X)_j = \frac{\Omega_{i+l-g}(X^G/U_j)}{F_{d+l-g-j}\Omega_{i+l-g}(X^G/U_j)}.$$  \tag{4.1}

**Lemma 4.2.** For a fixed $j \geq 0$, the group $\Omega_i^G(X)_j$ is independent of the choice of the good pair $(V_j, U_j)$.

**Proof.** Let $(V'_j, U'_j)$ be another good pair of dimension $l'$ corresponding to $j$. Following the proof of a similar result for the equivariant Chow groups in [8, Proposition 1], we let $V = V_j \oplus V'_j$ and $U = (U_j \oplus V'_j) \cup (V_j \oplus U'_j)$. Let $G$ act diagonally on $V$. Then it is easy to see that the complement of the open subset $X^G/(U_j \oplus V'_j)$ in $X \times U$ has dimension at most $d + l + l' - g - j$. Hence by Corollary 3.6, the
In the same way, we also get an isomorphism.

Moreover, there is a natural surjection

\[ \Omega_{i+l'+g}(X \times U) \to \Omega_{i+l'+g}(X \times (U_j \oplus V'_{j'})) \]

is an isomorphism. On the other hand, the map \( X \times (U_j \oplus V'_{j'}) \to X \times U_j \) is a vector bundle of rank \( l' \) and hence by Lemma 3.8, the map

\[ \Omega_{i+l-g}(X \times U_j) \to \Omega_{i+l'+g}(X \times (U_j \oplus V'_{j'})) \]

is also an isomorphism. Combining the above two isomorphisms, we get the isomorphism

\[ \Omega_{i+l'+g}(X \times U) \cong \Omega_{i+l-g}(X \times U_j) \]

In the same way, we also get an isomorphism

\[ \Omega_{i+l'+g}(X \times U) \cong \Omega_{i+l-g}(X \times U_j) \]

which proves the result. \( \square \)

**Lemma 4.3.** For \( j' \geq j \geq 0 \), there is a natural surjective map \( \Omega_i^G(X)_{j'} \to \Omega_i^G(X)_j \).

**Proof.** Choose a good pair \((V_{j'}, U_{j'})\) for \( j' \). Then it is clearly a good pair for \( j \) too. Moreover, there is a natural surjection

\[ \Omega_{i+l-g}(X \times U_{j'}) \to \Omega_{i+l-g}(X \times U_{j'}) \]

On the other hand, the left and the right terms are \( \Omega_i^G(X)_{j'} \) and \( \Omega_i^G(X)_j \) respectively by Lemma 3.8. \( \square \)

**Definition 4.4.** Let \( X \) be a \( k \)-scheme of dimension \( d \) with a \( G \)-action. For any \( i \in \mathbb{Z} \), we define the **equivariant algebraic cobordism** of \( X \) to be

\[ \Omega_i^G(X) = \lim_{\to j} \Omega_i^G(X)_j. \]
The reader should note from the above definition that unlike the ordinary cobordism, the equivariant algebraic cobordism $\Omega^G_i(X)$ can be non-zero for any $i \in \mathbb{Z}$. We set

$$\Omega^G_i(X) = \bigoplus_{i \in \mathbb{Z}} \Omega^G_i(X).$$

If $X$ is an equi-dimensional $k$-scheme with $G$-action, we let $\Omega^G_i(X) = \Omega^G_{d-i}(X)$ and $\Omega^*_G(X) = \bigoplus_{i \in \mathbb{Z}} \Omega^G_i(X)$. We shall denote the equivariant cobordism $\Omega^G_G(k)$ of the ground field by $\Omega^*(BG)$. It is also called the algebraic cobordism of the classifying space of $G$.

Remark 4.5. If $G$ is the trivial group, we can take the good pair $(V, V_j)$ for every $j$ where $V_j$ is any $l$-dimensional $k$-vector space. In that case, we get $\Omega^G_{i+l}(X \times V_j) \cong \Omega^G_i(X \times V_j)$ which is isomorphic to $\Omega^G_i(X)$ by the homotopy invariance of the non-equivariant cobordism. Moreover, $F^d_{d-j} \Omega^G_i(X \times V_j)$ is isomorphic to $F^d_{d-j} \Omega^G_i(X)$ by Lemma [3.8] and this last term is zero for all large $j$. In particular, we see from (4.1) and the definition of the equivariant cobordism that there is a canonical isomorphism $\Omega^G_i(X) \cong \Omega^j_i(X)$.

Remark 4.6. It is easy to check from the above definition of the niveau filtration that if $X$ is a smooth and irreducible $k$-scheme of dimension $d$, then $F^d_{j} \Omega^G_i(X) = F^d_{j} \Omega^G_{d-i}(X)$, where $F^* \Omega^*(X)$ is the coniveau filtration used in [7]. Furthermore, one also checks in this case that if $G$ acts on $X$, then

$$\Omega^G_i(X) = \lim_{\rightarrow j} \frac{\Omega^j_i \left( X \times U_j \right)}{F^j \Omega^j_i \left( X \times U_j \right)},$$

where $(V, U_j)$ is a good pair corresponding to any $j \geq 0$. Thus the above definition [4.4] of the equivariant cobordism coincides with that of [7] for smooth schemes.

Remark 4.7. As is evident from the above definition (see Example 6.4), the equivariant cobordism $\Omega^G_i(X)$ can not in general be computed in terms of the algebraic cobordism of one single mixed space. This makes these groups more complicated to compute than the equivariant Chow groups, which can be computed in terms of a single mixed space. This also motivates one to ask if the equivariant cobordism can be defined in such a way that they can be calculated using one single mixed space in a given degree. It follows however from Lemmas 4.2 and 4.3 that for a given $i$ and $j$, each component of the projective system $\{ \Omega^G_i(X) \}_{j \geq 0}$ can be computed using a single mixed space.

4.1. Change of groups. If $H \subset G$ is a closed subgroup of dimension $h$, then any $l$-dimensional good pair $(V, U_j)$ for $G$-action is also a good pair for the induced $H$-action. Moreover, for any $X \in V_G$ of dimension $d$, $X \times U_j \rightarrow X \times U_j$ is an étale locally trivial $G/H$-fibration and hence a smooth map (cf. [3, Theorem 6.8]) of relative dimension $g - h$. This induces the inverse system of pull-back maps

$$\Omega^G_i(X)_j = \frac{\Omega^G_{i+l-g} \left( X \times U_j \right)}{F^d_{d+l-g} \Omega^G_{i+l-g} (X \times U_j)} \rightarrow \frac{\Omega^H_{i+l-h} \left( X \times U_j \right)}{F^d_{d+l-h} \Omega^H_{i+l-h} (X \times U_j)} = \Omega^H_i(X)_j.$$
and hence a natural restriction map

\[(4.3) \quad r^G_{Y,H} : \Omega^G_*(X) \to \Omega^H_*(X).\]

Taking \(H = \{1\}\) and using Remark 4.3, we get the forgetful map

\[(4.4) \quad r^G_X : \Omega^G_*(X) \to \Omega_*(X)\]

from the equivariant to the non-equivariant cobordism. Since \(r^G_{Y,H}\) is obtained as a pull-back under the smooth map, it commutes with any projective push-forward and smooth pull-back (cf. Theorem 5.4). We remark here that although the definition of \(r^G_{Y,H}\) uses a good pair, it is easy to see as in Lemma 4.2 that it is independent of the choice of such good pairs.

4.2. Fundamental class of cobordism cycles. Let \(X \in \mathcal{V}_G\) and let \(Y \xrightarrow{f} X\) be a morphism in \(\mathcal{V}_G\) such that \(Y\) is smooth of dimension \(d\) and \(f\) is projective. For any \(j \geq 0\) and any \(l\)-dimensional good pair \((V_j, U_j)\), \([Y, V_j] \xrightarrow{f} X_G]\) is an ordinary cobordism cycle of dimension \(d + l - g\) by Lemma 5.3 and hence defines an element \(\alpha_j \in \Omega^G_{d,l}(X)\). Moreover, it is evident that the image of \(\alpha_{j'}\) is \(\alpha_j\) for \(j' \geq j\). Hence we get a unique element \(\alpha \in \Omega^G_l(X)\), called the \(G\)-equivariant fundamental class of the cobordism cycle \([Y, f] \to X\). We also see from this more generally that if \([Y, f] \to X, L_1, \ldots, L_l\) is as above with each \(L_i\) a \(G\)-equivariant line bundle on \(Y\), then this defines a unique class in \(\Omega^G_{d,l}(X)\). It is interesting question to ask under what conditions on the group \(G\), the equivariant cobordism group \(\Omega^G_l(X)\) is generated by the fundamental classes of \(G\)-equivariant cobordism cycles on \(X\). It turns out that this question indeed has a positive answer if \(G\) is a split torus by [18 Theorem 4.11].

5. Localization sequence and other properties

In this section, we establish the localization sequence for the equivariant cobordism using Theorem 3.4. We also prove other basic properties of the equivariant algebraic cobordism which are analogous to the non-equivariant case.

Theorem 5.1. Let \(X\) be a \(G\)-scheme of dimension \(d\) and let \(Z \subset X\) be a \(G\)-invariant closed subscheme with the complement \(U\). Then there is an exact sequence

\[\Omega^G_*(Z) \to \Omega^G_*(X) \to \Omega^G_*(U) \to 0.\]

Proof. We fix integers \(i \in \mathbb{Z}\) and \(j \geq 0\) and first show that the sequence

\[(5.1) \quad \Omega^G_*(Z)_j \to \Omega^G_*(X)_j \to \Omega^G_*(U)_j \to 0\]

is exact. Choose a good pair \((V_j, U_j)\) of dimension \(l\) for \(j\). Then we see that \(Z_G \subset X_G\) is a closed subscheme with the complement \(U_G\). Hence by applying Theorem 3.4 at the appropriate levels of the niveau filtration and taking the quotients, we get an exact sequence

\[\frac{\Omega_{i+l-g}(Z_G)}{F_{d+l-g-j} \Omega_{i+l-g}(Z_G)} \to \frac{\Omega_{i+l-g}(X_G)}{F_{d+l-g-j} \Omega_{i+l-g}(X_G)} \to \frac{\Omega_{i+l-g}(U_G)}{F_{d+l-g-j} \Omega_{i+l-g}(U_G)} \to 0.\]

If \(d' = \dim(Z)\), then \(F_{d'+l-g-j} \Omega_{i+l-g}(Z_G) \subset F_{d+l-g-j} \Omega_{i+l-g}(Z_G)\) and hence by Lemma 4.2 we get an exact sequence

\[(5.2) \quad \Omega^G_*(Z)_j \to \Omega^G_*(X)_j \to \Omega^G_*(U)_j \to 0.\]
Finally, the exact sequences by Lemma 4.2 to get an exact sequence 
\[ (5.4) \ 0 \to \frac{\Omega_{i+1-g}(Z)}{F_j \Omega_{i+1-g}(Z)} \to \frac{\Omega_{i+1-g}(X)}{F_j \Omega_{i+1-g}(X)} \to \frac{\Omega_{i+1-g}(U)}{F_j \Omega_{i+1-g}(U)} \to 0. \]

Since \( \{E/E'\} \) and \( \left\{ \frac{\Omega_{i+1-g}(Z)}{F_j \Omega_{i+1-g}(Z)} \right\} \) are the inverse systems of surjective maps and hence satisfy the Mittag-Leffler (ML) condition, we identify the various terms in the exact sequences by Lemma 4.2 to get an exact sequence 
\[ (5.5) \ \lim_j \frac{\Omega_{i+1-g}(Z)}{F_{d+1-g-j} \Omega_{i+1-g}(Z)} \to \lim_j \Omega_i^G(X) \to \lim_j \Omega_i^G(U) \to 0. \]

Finally, the exact sequences
\[ 0 \to \frac{F_{d+1-g-j} \Omega_{i+1-g}(Z)}{F_{d+1-g-j} \Omega_{i+1-g}(Z)} \to \frac{\Omega_{i+1-g}(Z)}{F_{d+1-g-j} \Omega_{i+1-g}(Z)} \to \frac{\Omega_{i+1-g}(Z)}{F_{d+1-g-j} \Omega_{i+1-g}(Z)} \to 0 \]
and the Mittag-Leffler property of the first terms imply that
\[ \lim_j \Omega_i^G(Z) \to \lim_j \frac{\Omega_{i+1-g}(Z)}{F_{d+1-g-j} \Omega_{i+1-g}(Z)}. \]

Combining this with (5.5), we get the desired exact sequence
\[ \lim_j \Omega_i^G(Z) \to \lim_j \Omega_i^G(X) \to \lim_j \Omega_i^G(U) \to 0, \]
which proves the theorem. \( \square \)

**Remark 5.2.** If \( X \) is a smooth scheme, a proof of the above localization sequence is given in [7, Theorem 7.3]. However, the proof given there is not complete, since it only proves the exact sequence (5.2) for a fixed \( j \) by a different method. As the reader will observe, this exact sequence does not imply the localization theorem. The crucial ingredient in the above proof (including the direct proof of (5.2)) is the refined localization sequence of Theorem 3.4.

Before we prove the other properties of the equivariant cobordism, we mention the following elementary result.

**Lemma 5.3.** Let \( f : X \to Y \) be a projective \( G \)-equivariant map in \( \mathcal{V}_G \) such that \( G \) acts freely on \( X \) and \( Y \). Then the induced map \( \overline{f} : X/G \to Y/G \) of quotients is also projective.

**Proof.** It follows from our assumption and Lemma 4.1 that \( \overline{f} : X' = X/G \to Y/G = Y' \) is a morphism in \( \mathcal{V}_k \). Furthermore, the square
\[ \begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \overline{f} \\ Y & \to & Y' \end{array} \]
is Cartesian. Since both the horizontal maps are the principal $G$-bundles, they are smooth and surjective. Since proper maps have smooth descent (in fact fpqc descent), we see that $\overline{f} : X \to Y'$ is proper. Since these schemes are quasi-projective, we leave it as an exercise to show that $\overline{f}$ is also quasi-projective and hence must be projective.

\textbf{Theorem 5.4.} The equivariant algebraic cobordism satisfies the following properties.

(i) \textbf{Functoriality :} The assignment $X \mapsto \Omega_*(X)$ is covariant for projective maps and contravariant for smooth maps in $V_G$. It is also contravariant for l.c.i. morphisms in $V_G$. Moreover, for a fiber diagram

$$
\begin{array}{ccc}
X' & \overset{g'}{\to} & X \\
\downarrow f' & & \downarrow f \\
Y' & \overset{g}{\to} & Y
\end{array}
$$

in $V_G$ with $f$ projective and $g$ smooth, one has $g^* \circ f_* = f'_* \circ g'^*$ : $\Omega^G_*(X) \to \Omega^G_*(Y')$.

(ii) \textbf{Homotopy :} If $f : E \to X$ is a $G$-equivariant vector bundle, then $f^* : \Omega^G_*(X) \xrightarrow{\sim} \Omega^G_*(E)$.

(iii) \textbf{Chern classes :} For any $G$-equivariant vector bundle $E \xrightarrow{\phi} X$ of rank $r$, there are equivariant Chern class operators $c_m^G(\phi) : \Omega^G_*(X) \to \Omega^G_{r-m}(X)$ for $0 \leq m \leq r$ with $c_0^G(\phi) = 1$. These Chern classes have same functoriality properties as in the non-equivariant case. Moreover, they satisfy the Whitney sum formula.

(iv) \textbf{Free action :} If $G$ acts freely on $X$ with quotient $Y$, then $\Omega^G_*(X) \xrightarrow{\sim} \Omega_*(Y)$.

(v) \textbf{Exterior Product :} There is a natural product map

$$
\Omega^G_*(X) \otimes \Omega^G_*(X') \to \Omega^G_{i + j}(X \times X').
$$

In particular, $\Omega^G_*(k)$ is a graded algebra and $\Omega^G_*(X)$ is a graded $\Omega^G_*(k)$-module for every $X \in V_G$.

(vi) \textbf{Projection formula :} For a projective map $f : X' \to X$ in $V^S_G$, one has for $x \in \Omega^G_*(X)$ and $x' \in \Omega^G_*(X')$, the formula $f_*(x' \cdot f^*(x)) = f_*(x') \cdot x$.

\textbf{Proof.} Assume that the dimensions of $X$ and $Y$ are $m$ and $n$ respectively and let $d = m - n$ be the relative dimension of a projective $G$-equivariant morphism $f : X \to Y$. For a fixed $j \geq 0$, let $(V_j, U_j)$ be an $l$-dimensional good pair for $f$. Since $f$ is projective, Lemma $2.3$ implies that $\overline{f} : X_G \to Y_G$ is projective and hence by Theorem $2.1$ and Lemma $3.3$, there is a push-forward map

$$
\frac{\Omega_{i + l - g}(X_G)}{F_{m + l - g - j} \Omega_{i + l - g}(X_G)} \to \frac{\Omega_{i + l - g}(Y_G)}{F_{m + l - g - j} \Omega_{i + l - g}(Y_G)} = \frac{\Omega_{i + l - g}(Y_G)}{F_{m + l - g - (j - d)} \Omega_{i + l - g}(Y_G)}.
$$

In particular, we get a compatible system of maps

$$
\Omega^G_*(X)_{j + d} \to \Omega^G_*(Y)_j.
$$

Taking the inverse limits, one gets the desired push-forward map $\Omega^G_*(X) \xrightarrow{f^*} \Omega^G_*(Y)$.

If $f$ is smooth of relative dimension $d$, then $\overline{f} : X_G \to Y_G$ is also smooth of same relative dimension. Hence, we get a compatible system of pull-back maps $\Omega^G_*(Y)_j \xrightarrow{\overline{f}^*} \Omega^G_*(X)_j$. Taking the inverse limit, we get the desired pull-back map of
the equivariant algebraic cobordism groups. If $f$ is a l.c.i. morphism of $G$-schemes, the same proof applies using the existence of similar map in the non-equivariant case. The required commutativity of the pull-back and push-forward maps follows exactly in the same way from the corresponding result for the non-equivariant cobordism groups.

To prove the homotopy property, let $E \xrightarrow{f} X$ be a $G$-equivariant vector bundle of rank $r$. For any $j \geq 0$, let $(V_j, U_j)$ be a good pair for $j$. Then the map of mixed quotients $E \rightarrow X_G$ is a vector bundle of rank $r$ (cf. [28 Lemma 1]). Hence by Lemma 3.8 the pull-back map $\Omega_i^G(X) \rightarrow \Omega_i^{G^r}(E)$ is an isomorphism. If $j' \geq j$, then we can choose a common good pair for both $j$ and $j'$. Hence, we a pull-back map of the inverse systems $\{\Omega_i^G(X)\}_{j} \rightarrow \{\Omega_i^{G^r}(E)\}_{j}$ which is an isomorphism at each level. Hence $f^* : \Omega_i^G(X) \rightarrow \Omega_i^{G^r}(E)$ is an isomorphism.

To define the Chern classes of an equivariant vector bundle $E$ of rank $r$, we choose an $l$-dimensional good pair $(V_j, U_j)$ and consider the vector bundle $E \rightarrow X_G$ as above and let $c_{m,j}^G : \Omega_i \times_{l-g}(X_G) \rightarrow \Omega_i \times_{l-g-m}(X_G)$ be the non-equivariant Chern class as in [25 4.1.7]. For a closed subscheme $Z \hookrightarrow X_G$, the projection formula for the non-equivariant cohomology

$$(5.6) \quad c_{m,j}^G(E_G) \circ \tau_* = \tau_* \circ (c_{m,j}^G(E_G)).$$

implies that $c_{m,j}^G(E_G)$ descends to maps $c_{m,j}^G : \Omega_i \times_{l} (X_G) \rightarrow \Omega_i \times_{l-m} (X_G)$.

One shows as in Lemma 4.2 that this is independent of the choice of the good pairs. Furthermore, choosing a common good pair for $j' \geq j$, we see that $c_{m,j}^G$ actually defines a map of the inverse systems. Taking the inverse limit, we get the Chern classes $c_m^G(E) : \Omega_i^G(X) \rightarrow \Omega_i^{G^r}(X)$ for $0 \leq m \leq r$ with $c_0^G(E) = 1$.

The functoriality and the Whitney sum formula for the equivariant Chern classes are easily proved along the above lines using the analogous properties of the non-equivariant Chern classes.

The statement about the free action follows from [7 Lemma 7.2] and Remark 1.5.

We now show the existence of the exterior product of the equivariant cobordism which requires some work. Let $d$ and $d'$ be the dimensions of $X$ and $X'$ respectively. We first define maps

$$(5.7) \quad \Omega_i^G(X) \times \Omega_i^{G'}(X') \rightarrow \Omega_i^{G+G'}(X \times X')$$

for $j \geq 0$.

Let $(V_j, U_j)$ be an $l$-dimensional good pair for $j$ and let $\alpha = [Y \xrightarrow{f} X]$ and $\alpha' = [Y' \xrightarrow{f'} X']$ be the cobordism cycles on $X_G$ and $X'_G$ respectively. Using the fact that $X \times U_j \rightarrow X_G$ and $X' \times U_j \rightarrow X'_G$ are principal $G$-bundles, we get the unique cobordism cycles $[\tilde{Y} \rightarrow X \times U_j]$ and $[\tilde{Y}' \rightarrow X' \times U_j]$ whose $G$-quotients are the above chosen cycles. We define $\alpha \ast \alpha' = [\tilde{Y} \times \tilde{Y}' \rightarrow (X \times X')_G]$.

Note that $(V_j \times V_j, U_j \times U_j)$ is a good pair for $j$ of dimension $2l$ and $(X \times X')_G$ is the quotient of $X \times X' \times U_j \times U_j$ for the free diagonal action of $G$ and $\alpha \ast \alpha'$ is a well defined cobordism cycle by Lemma 5.3.

Suppose now that $W \xrightarrow{p} X_G \times \Box^1$ is a projective morphism from a smooth scheme $W$ such that the composite map $\pi : W \rightarrow X_G \times \Box^1 \rightarrow \Box^1$ is a double point degeneration with $W_{\infty} = \pi^{-1}(\infty)$ smooth. Letting $G$ act trivially on $\Box^1$, this gives a unique $G$-equivariant double point degeneration $\tilde{W} \xrightarrow{\tilde{p}} X \times U_j \times \Box^1$ of $G$-schemes. This implies in particular that $\tilde{W} \times \tilde{Y} \xrightarrow{\tilde{p} \times \tilde{Y}} X \times X' \times U_j \times U_j \times \Box^1$
is also a $G$-equivariant double point degeneration whose quotient for the free $G$-action gives a double point degeneration $\tilde{W} \times \tilde{Y} \xrightarrow{q} (X \times X')_G \times \square^1$. Moreover, it is easy to see from this that $C(p) \star \alpha' = C(g)$ (cf. (2.4)). Reversing the roles of $X$ and $X'$ and using (2.4) and Theorem 2.3, we get the maps

$$\Omega_{i+t-g}(X_G) \otimes \Omega'_{i+t-g}(X'_G) \to \Omega_{i+i'-2t-g}((X \times X')_G).$$

It is also clear from the definition of $\alpha \star \alpha'$ and the niveau filtration that

$$\{ F_{d+i-t-g}(X_G) \otimes \Omega'_{i+t-g}(X'_G) \} + \{ \Omega_{i+t-g}(X_G) \otimes F_{d+i-t-g}(X'_G) \} \to F_{d+d'+2t-g}(X \times X')_G.$$

This defines the maps as in (5.7). One can now show as in Lemma 4.2 that these maps are independent of the choice of the good pairs. We get the desired exterior product as the composite map

$$\Omega^G_i(X) \otimes \Omega^G_{i'}(X') = \lim_{j} \Omega^G_i(X)_j \otimes \Omega^G_{i'}(X')_j \to \lim_{j} \left( \Omega^G_i(X)_j \otimes \Omega^G_{i'}(X')_j \right) \to \lim_{j} \Omega^G_{i+i'}(X \times X')_j = \Omega^G_{i+i'}(X \times X').$$

Finally for $X$ smooth, we get the product structure on $\Omega^*_G(X)$ via the composite

$$\Omega^*_G(X) \otimes \Omega^*_G(X) \to \Omega^*_G(X \times X) \xrightarrow{\Delta^*} \Omega^*_G(X).$$

The projection formula can now be proven by using the non-equivariant version of such a formula (cf. [25, 5.1.4]) at each level of the projective system $\{ \Omega^n_G(X)_j \}$ and then taking the inverse limit.

**Proposition 5.5 (Morita Isomorphism).** Let $H \subset G$ be a closed subgroup and let $X \in V_H$. Then there is a canonical isomorphism

$$\Omega^*_G \left( H \times X \right) \cong \Omega^*_G(X).$$

**Proof.** Define an action of $H \times G$ on $G \times X$ by

$$(h,g) \cdot (g',x) = (gg'h^{-1},hx),$$

and an action of $H \times G$ on $X$ by $(h,g) \cdot x = hx$. Then the projection map $G \times X \xrightarrow{p_1} (H \times X)$ is $G$-equivariant which is a $G$-torsor. Hence by [25, Lemma 7.2], the natural map $\Omega^*_G(X) \xrightarrow{p^*_G} \Omega^*_{H \times G}(G \times X)$ is an isomorphism. On the other hand, the projection map $G \times X \to G \times X$ is $(H \times G)$-equivariant which is an $H$-torsor. Hence we get an isomorphism $\Omega^*_G \left( G \times X \right) \xrightarrow{\sim} \Omega^*_H \left( G \times X \right)$. The proposition follows by combining these two isomorphisms. \qed

6. **Computations**

Let $X$ be a $k$-scheme of dimension $d$ with a $G$-action. We have seen above that unlike the situation of Chow groups, the cobordism group $\Omega_{i+l-g}(X \times U)$ is not independent of the choice of the $l$-dimensional good pair $(V_j, U_j)$. This anomaly is rectified by considering the quotients of the cobordism groups of the good pairs by the niveau filtration. Our main result in this section is to show that if we suitably choose a sequence of good pairs $\{(V_j, U_j)\}_{j \geq 0}$, then the above equivariant cobordism group can be computed without taking quotients by the
niveau filtration. This reduction is often very helpful in computing the equivariant cobordism groups.

**Theorem 6.1.** Let \( \{(V_j, U_j)\}_{j \geq 0} \) be a sequence of \( l_j \)-dimensional good pairs such that

1. \( V_{j+1} = V_j \oplus W_j \) as representations of \( G \) with \( \dim(W_j) > 0 \) and
2. \( U_j \oplus W_j \subset U_{j+1} \) as \( G \)-invariant open subsets.

Then for any scheme \( X \) as above and any \( i \in \mathbb{Z} \),

\[
\Omega_i^G(X) \cong \lim_{j} \Omega_{i+l_j-g}(X \times U_j).
\]

Moreover, such a sequence \( \{(V_j, U_j)\}_{j \geq 0} \) of good pairs always exists.

**Proof.** Let \( \{(V_j, U_j)\}_{j \geq 0} \) be a sequence of good pairs as in the theorem. We have natural maps

\[
\Omega_{i+l_j+1-g}(X \times U_{j+1}) \to \Omega_{i+l_j+1-g}(X \times (U_j \oplus W_j)) \cong \Omega_{i+l_j-g}(X \times U_j),
\]

where the first map is the restriction to an open subset and the second is the pull-back via a vector bundle. Taking the quotients by the niveau filtrations, we get natural maps (cf. proof of Lemma 4.2)

\[
\begin{array}{ccc}
\Omega_{i+l_j+1-g}(X \times U_{j+1}) & \xrightarrow{F_d} & \Omega_{i+l_j+1-g}(X \times (U_j \oplus W_j)) \\
\Omega_{i+l_j-g}(X \times U_j) & \xrightarrow{g} & \Omega_{i+l_j-g}(X \times U_j)
\end{array}
\]

where the right vertical arrow is an isomorphism by Lemma 3.8. Setting \( X_j = X \times U_j \), we get natural maps

\[
\begin{array}{ccc}
\Omega_{i+l_j+1-g}(X_{j+1}) & \xrightarrow{\nu_{j+1}} & \Omega_{i+l_j-g}(X_j) \\
\Omega_{i+l_j+1-g}(X_{j+1}) & \xrightarrow{F_d} & \Omega_{i+l_j-g}(X_j)
\end{array}
\]

Since \( (V_j, U_j) \) is a good pair for each \( j \), we see that \( \Omega_{i+l_j-g}(X_j) \cong \Omega_i^G(X)_j \).

Hence, we only have to show that the map

\[
\lim_j \Omega_{i+l_j-g}(X_j) \to \lim_{j} \Omega_{i+l_j-g}(X_j)
\]

is an isomorphism in order to prove the theorem.
To prove (6.3), we only need to show that for any given \( j \geq 0 \), the map
\[
\Omega_{i+l_j-g}(X_j') \xrightarrow{\nu_j'} \Omega_{i+l_j-g}(X_j)
\]
factors through
\[
\frac{\Omega_{i+l_j-g}(X_j')}{F_{d+l_j-g-j} \Omega_{i+l_j-g}(X_j')} \to \Omega_{i+l_j-g}(X_j) \quad \text{for all } j' \gg j.
\]

However, it follows from (6.1) that \( \nu_j' \) induces the map
\[
\frac{\Omega_{i+l_j-g}(X_j')}{F_{d+l_j-g-j} \Omega_{i+l_j-g}(X_j')} \to \frac{\Omega_{i+l_j-g}(X_j)}{F_{d+l_j-g-j} \Omega_{i+l_j-g}(X_j)}.
\]

On the other hand \( F_{d+l_j-g-j} \Omega_{i+l_j-g}(X_j) \) vanishes for \( j' \gg j \). This proves (6.4) and hence (6.3).

Finally, it follows easily from the proof of Lemma 4.2 (see also [33, Remark 1.4]) that a sequence of good pairs as in Theorem 6.1 always exists. \( \square \)

One consequence of Theorem 6.1 is that for a linear algebraic group \( G \) acting on a scheme \( X \) of dimension \( d \), the forgetful map \( r^G_X : \Omega^G_\ast(X) \to \Omega_\ast(X) \) (cf. (4.4)) can be easily shown to be analogous to the one used in [8, Subsection 2.2] for the Chow groups. This interpretation of the forgetful map will have some interesting applications in the computation of the non-equivariant cobordism using the equivariant techniques (cf. [18], [19]).

So let \( \{ (V_j, U_j) \}_{j \geq 0} \) be a sequence of good pairs as in Theorem 6.1. We choose a \( k \)-rational point \( x \in U_0 \) and let \( x_j \) be its image in \( U_j/G \) under the natural map \( U_0 \to U_0/G \to U_j/G \). Setting \( X_j = X \times U_j \), this yields a commutative diagram
\[
\begin{array}{ccc}
X \times U_j & \xrightarrow{\psi_j} & X_j \\
\downarrow \pi_j & & \downarrow \psi_j \\
U_j & \xrightarrow{\phi_j} & U_j/G \to U_{j+1}/G
\end{array}
\]

such that the left square is Cartesian and
\[
X \cong \pi^{-1}(x) \xrightarrow{\phi_0^{-1}(x)} \phi_j^{-1}(x_j) \xrightarrow{\phi_0^{-1}(x)} \phi_{j+1}^{-1}(x_{j+1}).
\]
Let \( \nu_j : \phi_j^{-1}(x_j) \to X_j \) be the closed embedding. Notice that since \( U_j/G \) is smooth and \( \psi_j \) is flat, it follows that \( \nu_j \) is a regular embedding. Using the identification in (6.6), we get maps \( \nu_j^G : \Omega_\ast(X_j) \to \Omega_\ast(X) \) such that \( \nu_j^G \circ \phi_j^* = \nu_{j+1}^G \). Taking the limit over \( j \geq 0 \), this yields for any \( i \in \mathbb{Z} \), a restriction map \( \tilde{r}^G_X : \Omega^G_i(X) = \lim \Omega^G_i(X)_j \to \Omega_i(X) \). Since any two rational points in \( U_j/G \) define the same class in \( \Omega_0(U_j/G) \) by [25, Lemma 4.5.10], we see that \( \tilde{r}^G_X \) is well-defined.

**Corollary 6.2.** The maps \( r^G_X, \tilde{r}^G_X : \Omega_\ast^G(X) \to \Omega_\ast(X) \) coincide.

**Proof.** Using the construction of the map \( r^G_X \) in (4.4) and the diagram (6.5), it suffices to show that the natural maps
\[
\frac{\Omega_\ast(X)}{F_{i-j} \Omega_\ast(X)} \xrightarrow{F_{d+l_j-j} \Omega_\ast(X \times V_j)} \frac{\Omega_\ast(X \times U_j)}{F_{d+l_j-j} \Omega_\ast(X \times U_j)}
\]

are equal.
are isomorphisms for all \( j \gg 0 \), where the two maps are the restrictions to the zero-section and the open subset. But this follows immediately from Corollary \( \ref{cor:3.6} \) and Lemma \( \ref{lem:3.8} \).

### 6.1. Formal group law in equivariant cobordism

Let \( G \) be a linear algebraic group over \( k \) acting on a scheme \( X \) of dimension \( d \). Let \( \{(V_j, U_j)\}_{j \geq 0} \) be a sequence of \( \Omega \)-dimensional good pairs as in Theorem \( \ref{thm:6.1} \). Letting \( X_j = X \times U_j \), we see that for every \( j \geq 0 \), \( \Omega_\ast(X_j) = \bigoplus_{i \in \mathbb{Z}} \Omega_{i+j-\delta}(X_j) \) is an \( \mathbb{L} \)-module and for \( j' \geq j \), there is a natural surjection \( \Omega_\ast(X_{j'}) \to \Omega_\ast(X_j) \) of \( \mathbb{L} \)-modules.

Given \( G \)-equivariant line bundles \( L, M \) on \( X \), where \( L_j = L \times X_j \) for \( j \geq 0 \). The formal group law of the non-equivariant cobordism yields

\[
c_1((L \otimes M)_j) = c_1(L_j \otimes M_j) = c_1(L_j) + c_1(M_j) + \sum_{i,i' \geq 1} a_{i,i'} (c_1(L_j))^{i} \circ (c_1(M_j))^{i'}.
\]

Note that if \( (x_j) \in \Omega^G_i(X) \), then the evaluation of the operator \( c_1^\mathbb{T}(L)(x_j) \) at any level \( j \geq 0 \) is a finite sum above.

Taking the limit over \( j \geq 0 \) and noting that the sum (and the product) in the equivariant cobordism groups are obtained by taking the limit of the sums (and the products) at each level of the inverse system, we get the same formal group law for the equivariant Chern classes:

\[
(6.7) \quad c_1^G(L \otimes M) = c_1^\mathbb{T}(L) + c_1^\mathbb{T}(M) + \sum_{i,i' \geq 1} a_{i,i'} (c_1^\mathbb{T}(L))^{i} \circ (c_1^\mathbb{T}(M))^{i'}.
\]

One should also observe that unlike the case of ordinary cobordism, the evaluation of the above sum on any given equivariant cobordism cycle may no longer be finite. In other words, the equivariant Chern classes are not in general locally nilpotent.

### 6.2. Cobordism ring of classifying spaces

Let \( R \) be a Noetherian ring and let \( A = \bigoplus_{j \in \mathbb{Z}} A_j \) be a \( \mathbb{Z} \)-graded \( R \)-algebra with \( R \subset A_0 \). Recall that the graded power series ring \( S^{(n)} = \bigoplus_{i \in \mathbb{Z}} S_i \) is a graded ring such that \( S_i \) is the set of formal power series of the form \( f(t) = \sum_{m(t) \in \mathbb{C}} a_{m(t)}m(t) \) such that \( a_{m(t)} \) is a homogeneous element of \( A \) of degree \( |a_{m(t)}| \) and \( |a_{m(t)}| + |m(t)| = i \). Here, \( \mathbb{C} \) is the set of all monomials in \( t = (t_1, \ldots, t_n) \) and \( |m(t)| = i_1 + \cdots + i_n \) if \( m(t) = t_1^{i_1} \cdots t_n^{i_n} \). We call \( |m(t)| \) to be the degree of the monomial \( m(t) \).

We shall often write the above graded power series ring as \( A[[t]]_{\text{gr}} \) to distinguish it from the usual formal power series ring. Notice that if \( A \) is only non-negatively graded, then \( S^{(n)} \) is nothing but the standard polynomial ring \( A[t_1, \cdots, t_n] \) over \( A \). It is also easy to see that \( S^{(n)} \) is indeed a graded ring which is a subring of the formal power series ring \( A[[t_1, \cdots, t_n]] \). The following result summarizes some basic properties of these rings. The proof is straightforward and is left as an exercise.

**Lemma 6.3.** (i) There are inclusions of rings \( A[t_1, \cdots, t_n] \subset S^{(n)} \subset A[[t_1, \cdots, t_n]] \), where the first is an inclusion of graded rings.

(ii) These inclusions are analytic isomorphisms with respect to the \( t \)-adic topology. In particular, the induced maps of the associated graded rings

\[
A[t_1, \cdots, t_n] \to \text{Gr}_t S^n \to \text{Gr}_t A[[t_1, \cdots, t_n]]
\]
are isomorphisms.

(iii) \( S^{(n-1)}[[t]]_{gr} \cong S^{(n)}. \)

(iv) \( \frac{S^{(n)}}{(t_1, \ldots, t_r)} \cong S^{(n-r)} \) for any \( n \geq r \geq 1, \) where \( S^{(0)} = A. \)

(v) The sequence \( \{t_1, \ldots, t_r\} \) is a regular sequence in \( S^{(n)}. \)

(vi) If \( A = R[x_1, x_2, \ldots] \) is a polynomial ring with \( |x_i| < 0 \) and \( \lim_{i \to \infty} |x_i| = -\infty, \)
then \( S^{(n)} \cong \varprojlim_i R[x_1, \ldots, x_i][[t]]_{gr}. \)

Since we shall mostly be dealing with the graded power series ring in this text, we make the convention of writing \( A[[t]]_{gr} \) as \( A[[t]], \) while the standard formal power series ring will be written as \( \widehat{A[[t]]}. \)

**Examples 6.4.** In the following examples, we compute \( \Omega^*(BG) = \Omega^*_G(k) \) for some classical groups \( G \) over \( k. \) These computations follow directly from the definition of equivariant cobordism and suitable choices of good pairs. We first consider the case when \( G = \mathbb{G}_m \) is the multiplicative group. For any \( j \geq 1, \) we choose the good pair \( (V_j, U_j) \), where \( V_j \) is the \( j \)-dimensional representation of \( \mathbb{G}_m \) with all weights \(-1\) and \( U_j \) is the complement of the origin. We see that \( U_j/\mathbb{G}_m \cong \mathbb{P}^{j-1}_k. \) Let \( \zeta \) be the class of \( c_1(O(-1))(1) \in \Omega^1(\mathbb{P}^{j-1}_k). \) The projective bundle formula for the ordinary algebraic cobordism implies that \( \Omega^i_{G_m} = \bigoplus_{0 \leq p \leq j-1} \mathbb{L}^{i-p}\zeta^p. \) Taking the inverse limit over \( j \geq 1, \) we find from this that for \( i \in \mathbb{Z}, \)

\[
\Omega^i_{\mathbb{G}_m}(k) = \prod_{p \geq 0} \mathbb{L}^{-p}\zeta^p.
\]

It particular, it is easy to see that there is a natural map

\[
(6.8) \quad \Omega^*(B\mathbb{G}_m) \to \mathbb{L}[[t]]
\]

\[
(x_{i_1}, \ldots, x_{i_n} = \prod a_{i_p}^{i_p}\zeta^p, \ldots, x_{i_n} = \prod a_{i_p}^{i_n}\zeta^p) \mapsto \sum_{p \geq 0} \left( \sum_{1 \leq j \leq n} a_{i_j}^{i_j} \right) t^p,
\]

which is an isomorphism of graded \( \mathbb{L} \)-algebras.

For a general split torus \( T \) of rank \( n, \) we choose a basis \( \{\chi_1, \ldots, \chi_n\} \) of the character group \( \hat{T}. \) This is equivalent to a decomposition \( T = T_1 \times \cdots \times T_n \) with each \( T_i \) isomorphic to \( \mathbb{G}_m \) and \( \chi_i \) is a generator of \( \hat{T}_i. \) Let \( L_{\chi} \) be the one-dimensional representation of \( T, \) where \( T \) acts via \( \chi. \) For any \( j \geq 1, \) we take the good pair \( (V_j, U_j) \) such that \( V_j = \prod_{i=1}^n L_{\chi_i}^{a_{i_j}}, U_j = \prod_{i=1}^n \left( L_{\chi_i}^{a_{i_j}} \setminus \{0\} \right) \) and \( T \) acts on \( V_j \) by \( (t_1, \ldots, t_n)(x_1, \ldots, x_n) = (\chi_1(t_1)(x_1), \ldots, \chi_n(t_n)(x_n)). \) It is then easy to see that \( U_j/T \cong X_1 \times \cdots \times X_n \) with each \( X_i \) isomorphic to \( \mathbb{P}^{j-1}_k. \) Moreover, the \( T \)-line bundle \( L_{\chi_i} \) gives the line bundle \( L_{\chi_i} T_i \times \left( L_{\chi_i}^{a_{i_j}} \setminus \{0\} \right) \to X_i \) which is \( O(\pm 1). \) Letting \( \zeta_i \) be the first Chern class of this line bundle, the projective bundle formula for the non-equivariant cobordism shows that

\[
\Omega^i_T(k) = \prod_{p_1, \ldots, p_n \geq 0} \mathbb{L}^{-\left( \sum_{i=1}^n p_i \right)}\zeta_1^{p_1} \cdots \zeta_n^{p_n},
\]
which is isomorphic to the set of formal power series in \( \{ \zeta_1, \ldots, \zeta_n \} \) of degree \( i \) with coefficients in \( \mathbb{L} \). In particular, one concludes as in the rank one case above that

**Proposition 6.5.** Let \( \{ \chi_1, \ldots, \chi_n \} \) be a chosen basis of the character group of a split torus \( T \) of rank \( n \). The assignment \( t_i \mapsto c^\gamma_i(L\chi)_i \) yields a graded \( \mathbb{L} \)-algebra isomorphism

\[
\mathbb{L}[t_1, \ldots, t_n] \rightarrow \Omega^*(BT).
\]

For \( G = GL_n \), we can take a good pair for \( j \) to be \( (V_j, \mathcal{U}_j) \), where \( V_j \) is the vector space of \( n \times p \) matrices with \( p > n \) with \( GL_n \) acting by left multiplication, and \( \mathcal{U}_j \) is the open subset of matrices of maximal rank. Then the mixed quotient is the Grassmannian \( Gr(n,p) \). We can now calculate the cobordism ring of \( Gr(n,p) \) using the projective bundle formula (by standard stratification technique) and then we can use the similar calculations as above to get a natural isomorphism

\[
\Omega^*(BGL_n) \rightarrow \mathbb{L}[\gamma_1, \ldots, \gamma_n]
\]

of graded \( \mathbb{L} \)-algebras, where \( \gamma_i \)'s are the elementary symmetric polynomials in \( t_1, \ldots, t_n \) that occur in \([6,8]\). Furthermore, using the fact that \( \mathcal{U}_j/SL_n \) fibers over \( Gr(n,p) \) as the complement of the zero-section of the determinant bundle of the tautological rank \( n \) bundle on \( Gr(n,p) \) and using the localization sequence of Theorem [2.1] we see that \( \Omega^*(BSL_n) \xrightarrow{\cong} \mathbb{L}[\gamma_2, \ldots, \gamma_n] \).

**Remark 6.6.** The cobordism rings of \( BGL_n \) and \( BSL_n \) have also been calculated in [7, Section 4] by an indirect method which involves the comparison of these groups with the known complex cobordism rings of the corresponding topological classifying spaces. However, the comparison result in loc. cit. assumes the existence of a natural ring homomorphism from the algebraic to the complex cobordism of classifying spaces, which is not immediately obvious.

### 7. Comparison with other cohomology theories

In this paper, we fix the following notation for the tensor product while dealing with inverse systems of modules over a commutative ring. Let \( A \) be a commutative ring with unit and let \( \{ L_n \} \) and \( \{ M_n \} \) be two inverse systems of \( A \)-modules with inverse limits \( L \) and \( M \) respectively. Following [32], one defines the **topological tensor product** of \( L \) and \( M \) by

\[
L \hat{\otimes}_A M := \lim_n (L_n \otimes_A M_n).
\]

In particular, if \( D \) is an integral domain with quotient field \( F \) and if \( \{ A_n \} \) is an inverse system of \( D \)-modules with inverse limit \( A \), one has \( A \hat{\otimes}_D F = \lim_n (A_n \otimes_D F) \).

The examples \( \hat{\mathbb{Z}}(p) = \lim_n \mathbb{Z}/p^n \) and \( \mathbb{Z}[[x]] \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \lim_n \mathbb{Z}[[x]]/(x^n) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[[x]] \) show that the map \( A \hat{\otimes}_D F \rightarrow A \hat{\otimes}_D F \) is in general neither injective nor surjective.

If \( R \) is a \( \mathbb{Z} \)-graded ring and if \( M \) and \( N \) are two \( R \)-graded modules, then recall that \( M \otimes_R N \) is also a graded \( R \)-module given by the quotient of \( M \otimes_{R_0} N \) by the graded sub module generated by the homogeneous elements of the type \( ax \otimes y - x \otimes ay \) where \( a, x \), and \( y \) are the homogeneous elements of \( R, M \), and \( N \) respectively. If all the graded pieces \( M_i \) and \( N_i \) are the limits of inverse systems \( \{ M_i^\lambda \} \) and \( \{ N_i^\lambda \} \) of \( R_0 \)-modules, we define the graded topological tensor product as \( M \hat{\otimes}_R N = \)}
\[ \bigoplus_{i \in \mathbb{Z}} (M \otimes R \mathcal{N})_i, \] where

\[ (M \otimes_R \mathcal{N})_i = \lim_{\lambda} \left( \bigoplus_{j, j' = i} \frac{M^\lambda_j \otimes_{R_0} N^\lambda_{j'}}{a x \otimes y - x \otimes ay} \right). \]

(7.2)

Notice that this reduces to the ordinary tensor product of graded \( R \)-modules if the underlying inverse systems are trivial.

### 7.1. Comparison with equivariant Chow groups

Let \( X \) be a \( k \)-scheme of dimension \( d \) with a \( G \)-action. It was shown by Levine and Morel [25] that there is a natural map \( \Omega_*(X) \to CH_*^X(X) \) of graded abelian groups which is a ring homomorphism if \( X \) is smooth. Moreover, this map induces a graded isomorphism

\[ \Omega_*(X) \otimes_{\mathbb{Z}} \mathcal{Z} \cong CH_*^X(X). \]

(7.3)

Recall from [33] and [8] that the equivariant Chow groups of \( X \) are defined as \( CH^G_i(X) = CH_{i+l-g}^G(X \times U) \), where \((V, U)\) is an \( l \)-dimensional good pair corresponding to \( d - i \). It is known that \( CH^G_i(X) \) is well-defined and can be non-zero for any \(-\infty < i \leq d\). We set \( CH^G_*^X(X) = \bigoplus_i CH^G_i(X) \). If \( X \) is equi-dimensional, we let \( CH^G_i(X) = CH^G_{d-i}(X) \) and set \( CH^G_*^X(X) = \bigoplus_{i \geq 0} CH^G_i(X) \). Notice that in this case, \( CH^G_i(X) \) is same as \( CH^i \left( X \times U \right) \), where \((V, U)\) is an \( l \)-dimensional good pair corresponding to \( i \).

If we fix \( i \in \mathbb{Z} \) and choose an \( l \)-dimensional good pair \((V_j, U_j)\) corresponding to \( d - j \geq \max(0, d - i + 1) \), the universality of the algebraic cobordism gives a unique map \( \Omega_{i+l-g}^X(X \times U_j) \to CH_{i+l-g}^G(X \times U_j) \). By Lemma 3.7, this map factors through

\[ \Omega_{i+l-g}^X(X \times U_j) \to F_{i+l-g-1} \Omega_{i+l-g}^X(X \times U_j) \to CH_{i+l-g}^G(X \times U_j). \]

(7.4)

Since \( j \leq i - 1 \) by the choice, we have \( d + l - g - (d - j) \leq i + l - g - 1 \) and hence we get the map

\[ \Omega^G_{i+l-g}(X)_{d-j} = \frac{\Omega_{i+l-g}^X(X \times U_j)}{F_{d+l-g-(d-j)} \Omega_{i+l-g}^X(X \times U_j)} \to CH_{i+l-g}^G(X \times U_j). \]

(7.5)

It is easily shown using the proof of Lemma 4.2 that this map is independent of the choice of the good pair \((V_j, U_j)\). Taking the inverse limit over \( d - j \geq 0 \), we get a natural map \( \Omega^G_i(X) \to CH^G_i(X) \) and hence a map of graded abelian groups

\[ \Phi_X : \Omega_*^G(X) \to CH_*^G(X) \]

(7.6)
which is in fact a map of graded \( \mathbb{L} \)-modules. Notice that the right side of (7.5) does not depend on \( j \) as long as \( d - j \gg 0 \). If \( X \) is equi-dimensional, we write the above map cohomologically as \( \Omega^*_G(X) \to CH^*_G(X) \).

**Proposition 7.1.** The map \( \Phi_X \) induces an isomorphism of graded \( \mathbb{L} \)-modules

\[
\Phi_X : \Omega^*_G(X) \otimes_{\mathbb{L}} \mathbb{Z} \xrightarrow{\cong} CH^*_G(X).
\]

**Proof.** We fix \( i \in \mathbb{Z} \) and choose an \( l \)-dimensional good pair \((V_j, U_j)\) corresponding to \( d - j \geq \max(0, d - i + 1) \) and set \( X_j = X \times U_j \). This gives the natural map as in (7.5) which in turn yields an exact sequence

\[
0 \to \left( \mathbb{L}^{<0} \frac{\Omega^*_s(X_j)}{F_{i+j-g} \Omega^*_s(X_j)} \cap \frac{\Omega^*_{i+t-g}(X_j)}{F_{l+j-g} \Omega^*_{i+t-g}(X_j)} \right) \to \frac{\Omega^*_{i+t-g}(X_j)}{F_{l+j-g} \Omega^*_{i+t-g}(X_j)} \to CH^*_{i+t-g}(X_j) \to 0
\]

by (7.3). Since \( \frac{\Omega^*_{\Omega^*_s(X_j)}}{F_{i+j-g} \Omega^*_s(X_j)} \to \frac{\Omega^*_s(X_j)}{F_{i+j-g} \Omega^*_{i+t-g}(X_j)} \) is a surjective map of graded \( \mathbb{L} \)-modules (as can be easily seen by choosing a good pair corresponding to \( d - j + 1 \)), we see that the left and the middle terms of the above exact sequence form inverse systems of surjective maps. Taking the limit, we get an exact sequence

\[
0 \to \lim_{d-j \geq 0} \left( \mathbb{L}^{<0} \frac{\Omega^*_s(X_j)}{F_{i+j-g} \Omega^*_s(X_j)} \cap \frac{\Omega^*_{i+t-g}(X_j)}{F_{l+j-g} \Omega^*_{i+t-g}(X_j)} \right) \to \Omega^*_G(X) \to CH^*_G(X) \to 0.
\]

Taking the direct sum over \( i \in \mathbb{Z} \) and using (7.2), we get

\[
\Omega^*_G(X) \otimes_{\mathbb{L}} \mathbb{Z} \xrightarrow{\cong} CH^*_G(X).
\]

Let \( C(G) = CH^*_G(k) \) denote the equivariant Chow ring of the field \( k \).

**Corollary 7.2.** For a \( k \)-scheme \( X \) with a \( G \)-action, the natural map

\[
\Omega^*_G(X) \otimes_{S(G)} C(G) \to CH^*_G(X)
\]

is an isomorphism of \( C(G) \)-modules. This is a ring isomorphism if \( X \) is smooth.

**Proof.** It is clear that the above map is a ring homomorphism if \( X \) is smooth. So we only need to prove the first assertion. But this follows directly from the isomorphisms \( \Omega^*_G(X) \otimes_{S(G)} C(G) \cong \Omega^*_G(X) \otimes_{S(G)} (S(G) \otimes_{\mathbb{L}} \mathbb{Z}) \cong \Omega^*_G(X) \otimes_{\mathbb{L}} \mathbb{Z} \), using Proposition 7.1.

### 7.2. Comparison with equivariant \( K \)-theory.

It was shown by Levine and Morel in [24, Corollary 11.11] that the universal property of the algebraic cobordism implies that there is a canonical isomorphism of oriented cohomology theories

\[
\Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\cong} K_0(X)[\beta, \beta^{-1}]
\]

in the category of smooth \( k \)-schemes. This was later generalized to a complete algebraic analogue of the Conner-Floyd isomorphism

\[
MGL^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\cong} K_*(X)[\beta, \beta^{-1}]
\]

between the motivic cobordism and algebraic \( K \)-theory by Panin, Pimenov and Röndigs [27]. Since the equivariant cobordism is a Borel style cohomology theory, one can not expect an equivariant version of the isomorphism (7.9) even with the rational coefficients. However, we show here that the equivariant Conner-Floyd isomorphism holds after we base change the above by the completion of
the representation ring of $G$ with respect to the ideal of virtual representations of rank zero. In fact, it can be shown easily that such a base change is the minimal requirement. In Theorem 7.3, all cohomology groups are considered with rational coefficients (cf. Section 3).

For a linear algebraic group $G$, let $R(G)$ denote the representation ring of $G$. Let $I$ denote the ideal of of virtual representations of rank zero in $R(G)$ and let $\widehat{R(G)}$ denote the associated completion of $R(G)$. Let $\widehat{C(G)}$ denote the completion of $C(G)$ with respect to the augmentation ideal of algebraic cycles of positive codimensions. For a scheme $X$ with $G$-action, let $K_0^G(X)$ denote the Grothendieck group of $G$-equivariant vector bundles on $X$. By [3] Theorem 4.1, there is a natural ring isomorphism $\widehat{R(G)} \cong \widehat{C(G)}$ given by the equivariant Chern character. We identify these two rings via this isomorphism. In particular, the maps $S(G) \to C(G) \to \widehat{C(G)}$ yield a ring homomorphism $S(G) \to \widehat{R(G)}$. 

**Theorem 7.3.** For a smooth scheme $X$ with a $G$-action, there is a natural isomorphism of rings

$$\Psi_X : \Omega^*_G(X) \otimes_{S(G)} \widehat{R(G)} \cong K_0^G(X) \otimes_{R(G)} \widehat{R(G)}.$$ 

**Proof.** By [15] Theorem 1.2, there is a Chern character isomorphism $K_0^G(X) \otimes_{R(G)} \widehat{R(G)} \cong CH^*(X) \otimes_{C(G)} \widehat{C(G)}$ of cohomology rings. Thus, we only need to show that the map $\Omega^*_G(X) \otimes_{S(G)} \widehat{C(G)} \to CH^*(X) \otimes_{C(G)} \widehat{C(G)}$ is an isomorphism. However, we have

$$\Omega^*_G(X) \otimes_{S(G)} \widehat{C(G)} \cong (\Omega^*_G(X) \otimes_{S(G)} C(G)) \otimes_{C(G)} \widehat{C(G)} \cong CH^*_G(X) \otimes_{C(G)} \widehat{C(G)},$$

where the last isomorphism follows from Corollary 7.2. This finishes the proof. \[\square\]

### 7.3. Comparison with complex cobordism

Let $G$ be a complex Lie group acting on a finite CW-complex $X$. We define the **equivariant complex cobordism** ring of $X$ as

$$(7.10) \quad MU^*_G(X) := MU^* \left( X \times EG \right)$$

where $EG \to BG$ is universal principal $G$-bundle over the classifying space $BG$ of $G$. If $E'G \to B'G$ is another such bundle, then the projection $(X \times EG \times E'G)/G \to X \times EG$ is a fibration with contractible fiber. In particular, $MU^*_G(X)$ is well-defined. Moreover, if $G$ acts freely on $X$ with quotient $X/G$, then the map $X \times EG \to X/G$ is a fibration with contractible fiber $EG$ and hence we get $MU^*_G(X) \cong MU^*(X/G)$.

For a linear algebraic group $G$ over $\mathbb{C}$ acting on a $\mathbb{C}$-scheme $X$, let $H^*_G(X, A)$ denote the (equivariant) cohomology of the complex analytic space $X(\mathbb{C})$ with coefficients in the ring $A$.

**Proposition 7.4.** Assume that $X \in V^*_G$ is such that $H^*_G(X, \mathbb{Z})$ is torsion-free. Then there is a natural homomorphism of graded rings

$$\rho^*_X : \Omega^*_G(X) \to MU^*_G(X).$$
Proof. If \( \{(V_j, U_j)\} \) is a sequence of good pairs as in Theorem \[6.1\] then the universality of the Levine-Morel cobordism gives a natural \( \mathbb{L} \)-algebra map of inverse systems

\[
\Omega_i^G \left( X \times U_j \right) \to MU^{2i} \left( X \times U_j \right)
\]

which after taking limits yields the map

\[
\Omega_G^i(X) = \lim_{j \geq 0} \Omega_i^G \left( X \times U_j \right) \to \lim_{j \geq 0} MU^{2i} \left( X \times U_j \right) \cong \lim_{j \geq 0} MU^{2i} \left( X \times (BG)_j \right).
\]

Thus, we only need to show that the map

\[
\Omega_G^i(X) = \lim_{j \geq 0} \Omega_i^G \left( X \times U_j \right) \to \lim_{j \geq 0} MU^{i} \left( X \times (BG)_j \right)
\]

is an isomorphism. But this follows from our assumption, \[21, Corollary 1\] and the Milnor exact sequence

\[
0 \to \lim_{j \geq 0} MU^{i} \left( X \times (BG)_j \right) \to MU^{i} \left( X \times EG \right) \to \lim_{j \geq 0} MU^{i} \left( X \times (BG)_j \right) \to 0.
\]

\[\square\]

The following is an immediate consequence of Propositions \[7.1\] and \[7.4\].

**Corollary 7.5.** For any \( X \in \mathcal{V}_G^S \), there is a natural map of graded \( \mathbb{L}_Q \)-algebras

\[
\rho^G_X : \Omega^*_G(X)_Q \to MU^{2*}_G(X)_Q.
\]

In particular, there is a natural ring homomorphism

\[
\rho^G_X : CH^*_G(X)_Q \to MU^{2*}_G(X)_Q \otimes_{\mathbb{L}_Q} Q
\]

which factors the cycle class map \( CH^*_G(X) \to H^{2*}_G(X, Q) \).

**Corollary 7.6.** There is a natural morphism \( \Omega^*(BG) \to MU^{2*}(BG) \) of graded \( \mathbb{L} \)-algebras. In particular, there is a natural ring homomorphism \( CH^*(BG) \to MU^{2*}(BG) \otimes_{\mathbb{L}} \mathbb{Z} \) which factors the cycle class map \( CH^*(BG) \to H^{2*}(BG, \mathbb{Z}) \).

**Proof.** The first assertion follows immediately from \[7.11\], \[7.12\] and \[21, Theorem 1\] using the fact that \( BG \) is homotopy equivalent to the classifying space of its maximal compact subgroup. The second assertion follows from the first and Proposition \[7.4\] using the identification \( \mathbb{L} \cong MU^* \). \[\square\]

**Remark 7.7.** The map \( CH^*(BG) \to MU^{2*}(BG) \otimes_{\mathbb{L}} \mathbb{Z} \) has also been constructed by Totaro \[33\] by a more direct method.

We shall study the above realization maps in more detail in the next section.

### 8. Reduction of arbitrary groups to tori

The main result of this section is to show that with the rational coefficients, the equivariant cobordism of schemes with an action of a connected linear algebraic group can be written in terms of the Weyl group invariants of the equivariant cobordism for the action of the maximal torus. This reduces the problems about the equivariant cobordism to the case where the underlying group is a torus. We draw some consequences of this for the cycle class map from the rational Chow
groups to the complex cobordism groups of classifying spaces. We first prove some reduction results about the equivariant cobordism which reflect the relations between the $G$-equivariant cobordism and the equivariant cobordism for actions of subgroups of $G$. The results of this section are used in \[17\] and \[19\] to compute the non-equivariant cobordism ring of flag varieties and flag bundles.

**Notation:** All results in this section will be proven with the rational coefficients. In order to simplify our notations, an abelian group $A$ from now on will actually mean the $\mathbb{Q}$-vector space $A \otimes \mathbb{Z} \mathbb{Q}$, and an inverse limit of abelian groups will mean the limit of the associated $\mathbb{Q}$-vector spaces. In particular, all cohomology groups will be considered with the rational coefficients and $\Omega_i^G(X)$ will mean $\Omega_i^G(X) := \lim_{\leftarrow j} \left( \Omega_i^G(X)_j \otimes \mathbb{Z} \mathbb{Q} \right)$.

Notice that this is same as $\Omega_i^G(X) \otimes \mathbb{Z} \mathbb{Q}$ in our earlier notation.

**Proposition 8.1.** Let $G$ be a connected and reductive group over $k$. Let $B$ be a Borel subgroup of $G$ containing a maximal torus $T$ over $k$. Then for any $X \in V_G$, the restriction map

$$\Omega^B_*(X) \xrightarrow{r^B_{H,T}} \Omega^T_*(X),$$

is an isomorphism.

**Proof.** By Proposition \[5.5\] we only need to show that

$$\Omega^B_*(B/T \times X) \cong \Omega^B_*(B/T \times X).$$

By \[6, XXII, 5.9.5\], there exists a characteristic filtration $B^u = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = \{1\}$ of the unipotent radical $B^u$ of $B$ such that $U_{i-1}/U_i$ is a vector group, for example, $U_i$ normal in $B$ and $TU_i = T \times U_i$. Moreover, this filtration also implies that for each $i$, the natural map $B/BU_i \rightarrow B/TU_{i-1}$ is a torsor under the vector bundle $U_{i-1}/U_i \times B/TU_{i-1}$ on $B/TU_{i-1}$. Hence, the homotopy invariance (cf. Theorem \[5.4\]) gives an isomorphism

$$\Omega^B_*(B/TU_{i-1} \times X) \xrightarrow{\cong} \Omega^B_*(B/TU_i \times X).$$

Composing these isomorphisms successively for $i = 1, \cdots, n$, we get

$$\Omega^B_*(B/T \times X) \xrightarrow{\cong} \Omega^B_*(B/T \times X).$$

The canonical isomorphism of $B$-varieties $B/T \times X \cong B/T \times X$ (cf. Proposition \[5.5\]) now proves \[8.2\] and hence \[8.1\].

**Proposition 8.2.** Let $H$ be a possibly non-reductive group over $k$. Let $H = L \times H^u$ be the Levi decomposition of $H$ (which exists since $k$ is of characteristic zero). Then the restriction map

$$\Omega^H_*(X) \xrightarrow{r^H_{L,X}} \Omega^L_*(X)$$

is an isomorphism.

**Proof.** Since the ground field is of characteristic zero, the unipotent radical $H^u$ of $H$ is split over $k$. Now the proof is exactly same as the proof of Proposition \[8.1\] where we just have to replace $B$ and $T$ by $H$ and $L$ respectively. \qed
8.1. **The motivic cobordism theory.** Before we prove our main results of this section, we recall the theory of motivic algebraic cobordism $MGL_{*,*}$ introduced by Voevodsky in \[25\]. This is a bi-graded ring cohomology theory in the category of smooth schemes over $k$. Levine has recently shown in \[22\] that $MGL_{*,*}$ extends uniquely to a bi-graded oriented Borel-Moore homology theory $M_{2*,*}$ on the category of all schemes over $k$. This homology theory has exterior products, homotopy invariance, localization exact sequence and Mayer-Vietoris among other properties (cf. \[loc. cit., Section 3\]). Moreover, the universality of Levine-Morel cobordism theory implies that there is a unique map
\[
\vartheta : \Omega_* \to MGL'_{2*,*}
\]
of oriented Borel-Moore homology theories. Our motivation for studying the motivic cobordism theory in this text comes from the following result of Levine.

**Theorem 8.3 (\[23\]).** For any $X \in \mathcal{V}_k$, the map $\vartheta_X$ is an isomorphism.

We draw some simple consequences of working with the rational coefficients. We recall from \[22\] that for a smooth $k$-variety $X$, there is a Hopkins-Morel spectral sequence
\[
E_2^{p,q}(n) = CH^n(X,2n - p - q) \otimes \mathbb{L}^q \Rightarrow MGL^{p+q,n}(X)
\]
which is the algebraic analogue of the Atiyah-Hirzebruch spectral sequence in complex cobordism.

If $X$ is possibly singular, we embed it as a closed subscheme of a smooth scheme $M$. Then, the functoriality of the above spectral sequence with respect to an open immersion yields a spectral sequence
\[
E_2^{p,q}(n) = CH^n(M,2n - p - q) \otimes \mathbb{L}^q \Rightarrow MGL^{p+q,n}(M)
\]
of cohomology with support. Since the higher Chow groups and the motivic cobordism groups of $M$ with support in $X$ are canonically isomorphic to the higher Chow groups and the Borel-Moore motivic cobordism groups of $X$ (cf. \[1, 23\] Section 3), the above spectral sequence is identified with
\[
E_2^{p,q}(n) = CH^n(X,p) \otimes \mathbb{L}^q \Rightarrow MGL'_{2n+p+q,n+q}(X).
\]

Now, suppose that a finite group $G$ acts on $X$. By embedding $X$ equivariantly in a smooth $G$-scheme $M$, the formula
\[
MGL'_{p,q}(X) := \text{Hom}_{\text{SH}(k)}(\Sigma^q_T M/(M - X), \Sigma^{p',d'} MGL)
\]
(where $p' = 2\dim(M) - p$, $q' = \dim(M) - q$) shows that $G$ acts naturally on $MGL'_{p,q}(X)$. It also acts on the higher Chow groups $CH_p(X,q)$ likewise.

We also observe that $G$ in fact acts on the cycle complex $Z_j(X, \cdot)$ by acting trivially on $\Delta^\bullet$. This action is given by
\[
(g, \sigma) \mapsto \mu^*_g(\sigma)
\]
where $\mu_g$ is the automorphism of $X \times \Delta^i$ associated with $g \in G$ and $\sigma$ is an irreducible admissible cycle on $X \times \Delta^i$. This action extends linearly to all $\sigma \in Z_j(X,i)$. Since $G$ acts trivially on $\Delta^\bullet$, it preserves the boundary map of the complex $Z_j(X, \cdot)$. In particular, $G$ acts on the cycle complex of $X$. If we let $Z_j(X, \cdot)^G$ denote the subcomplex of the invariant cycles, then the exactness of the functor “$G$-invariants” on the category of $\mathbb{Q}[G]$-modules implies that
\[
H_i \left( Z_j(X, \cdot)^G \right) \cong CH_j(X, i)^G.
\]
Moreover, as the $E^1$-terms of the spectral sequence (8.3) are given by $E^1_{p,q} = Z_n(X,p) \otimes \mathbb{L}^q$ with differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$, we see that the $E^1$ is just the cycle complex of $X$ and hence it is a complex with $G$-action as described above. In other words, the $E^1$ terms are complexes of $\mathbb{Q}[G]$-modules. We conclude that the spectral sequence (8.5) is equipped with a natural $G$-action. Since we are dealing with cohomology with rational coefficients, the exactness of the functor of taking $G$-invariants on the category of $\mathbb{Q}[G]$-modules implies that the above spectral sequence descends to the spectral sequence of ‘$G$-invariants’

\[(8.7)\quad E'^2_{p,q}(n) = (CH_n(X,p))^G \otimes \mathbb{L}^q \Rightarrow (MGL^p_{2n+p+n,q}(X))^G.\]

As a consequence, we get the following.

**Lemma 8.4.** Let $G$ be a finite group acting freely on a $k$-scheme $X$ with quotient $Y$. There is an isomorphism

\[MGL^p_{p,q}(Y) \xrightarrow{\cong} (MGL^p_{p,q}(X))^G.\]

**Proof.** This follows immediately from the spectral sequences (8.5) and (8.7), combined with [15, Corollary 8.3].

Recall that a connected and reductive group $G$ over $k$ is said to be *split*, if it contains a split maximal torus $T$ over $k$ such that $G$ is given by a root datum relative to $T$. One knows that every connected and reductive group containing a split maximal torus is split (cf. [8] Chapter XXII, Proposition 2.1). In such a case, the normalizer $N$ of $T$ in $G$ and all its connected components are defined over $k$ and the quotient $N/T$ is the Weyl group $W$ of the corresponding root datum.

**Lemma 8.5.** Let $G$ be a connected reductive group and let $T$ be a split maximal torus in $G$. Put $H = G/N$, where $N$ is the normalizer of $T$ in $G$. Then, any étale locally trivial $H$-fibration $f : X \to Y$ over a smooth variety $Y$ induces an isomorphism $f^* : MGL^{*,*}(Y) \xrightarrow{\cong} MGL^{*,*}(X)$.

**Proof.** This follows immediately from the similar result for the higher Chow groups in [15, Lemma 3.7] and the spectral sequence (8.4).

**Proposition 8.6.** Let the algebraic group $G$ be as in Lemma 8.5 and let $(V,U)$ be a good pair for the $G$-action corresponding to $j \geq 0$. Then for any $X \in V_G$, the pull-back map

\[(8.8)\quad \Omega_*(X) \stackrel{f_X^*}{\to} \Omega_*(X/U)\]

is an isomorphism.

**Proof.** Since $G$ acts on $X$ linearly, we can find a $G$-equivariant closed embedding $X \hookrightarrow M$, where $M \in V_G$. Let $W$ be the complement of $X$ in $M$. Set $X_G = X \times_G U$. We have similar meaning for $X_N$. Then $X_G$ is a closed subscheme of $M_G$ with complement $W_G$. Moreover, $M_N \xrightarrow{f_M} M_G$ is an étale locally trivial $G/N$-fibration. This yields a commutative diagram of long exact sequences

\[(8.9)\quad MGL^{*,*}(M_G) \to MGL^{*,*}(W_G) \to MGL^{*,*}_{X_G}(M_G) \to MGL^{*,*}(M_G) \to MGL^{*,*}(W_G)\]

\[\xrightarrow{f_M^*} \xrightarrow{f_W^*} \xrightarrow{f_X^*} \xrightarrow{f_M^*} \xrightarrow{f_W^*}\]

\[MGL^{*,*}(M_N) \to MGL^{*,*}(W_N) \to MGL^{*,*}_{X_N}(M_N) \to MGL^{*,*}(M_N) \to MGL^{*,*}(W_N).\]
Since $M_G$ and $W_G$ are smooth, the pull-back maps $f_M^*$ and $f_W^*$ are isomorphisms by Lemma 8.3. This implies in particular that $f_N^*$ is also an isomorphism. Since

$$
\Omega_*(X_G) \cong MGL'_{*,*}(X_G) := MGL_{X_G}^{2\dim(M_G) - 2\dim(M_G) - *}(M_G)
$$

by Theorem 8.3 and since the pull-back $f_N^*$ : $\Omega_*(X_G) \rightarrow \Omega_*(X_N)$ is independent of the choice of the embedding, we conclude that the map in (8.8) is an isomorphism.

Theorem 8.7. Let $G$ be a connected linear algebraic group and let $L$ be a Levi subgroup of $G$ with a split maximal torus $T$. Let $W$ denote the Weyl group of $L$ with respect to $T$. Then for any $X \in \mathcal{V}_G$, the natural map

$$
(8.10) \quad \Omega_*(X) \rightarrow (\Omega_*^T(X))^W
$$

is an isomorphism.

Proof. By Proposition 8.2, we can assume that $G = L$ and hence $G$ is a connected reductive group with split maximal torus $T$.

Let $N$ denote the normalizer of $T$ in $G$ so that $W = N/T$ and $H = G/N$. Fix $i \in \mathbb{Z}$. We choose a sequence of $l_j$-dimensional good pairs $\{(V_j, U_j)\}$ as in Theorem 6.1 for the $G$-action. Then, this is also a sequence of good pairs for the action of $N$ and $T$. Setting $X^j_G = X^G \times U_j$, we see that there is an étale locally trivial $H$-fibration $X^j_N \rightarrow X^j_G$. Hence by Proposition 8.4 the smooth pull-back $\Omega_{i+l_j-n}(X^j_G) \rightarrow \Omega_{i+l_j-n}(X^j_N)$ is an isomorphism, where $n$ is the dimension of $N$. Taking the inverse limit and using Theorem 6.1, we get an isomorphism

$$
(8.11) \quad \Omega_i^G(X) \cong \Omega_i^N(X).
$$

On the other hand, as $W$ acts freely on $X_T$ with quotient $X_N$, it follows from Theorem 8.3 and Lemma 8.4 that the natural map

$$
(8.12) \quad \Omega_{i+l_j-n}(X_N) \rightarrow (\Omega_{i+l_j-n}(X_T))^W
$$

is an isomorphism. Since the action of $W$ on the inverse system $\{\Omega_{i+l_j-n}(X_T)\}_j$ induces the similar action on the inverse limit and since the inverse limit commutes with taking the $W$-invariants, we get

$$
(8.13) \quad \lim_{\leftarrow j} \Omega_{i+l_j-n}(X_N) \cong \left( \lim_{\leftarrow j} \Omega_{i+l_j-n}(X_N) \right)^W.
$$

Since the left and the right terms are same as $\Omega_i^N(X)$ and $\Omega_i^T(X)$ respectively by choice of our good pairs and Theorem 6.1, we conclude that $\Omega_i^N(X) \cong (\Omega_i^T(X))^W$.

We complete the proof of the theorem by combining this with (8.11).

Corollary 8.8. Let $X \in \mathcal{V}_G$ be as in Theorem 8.7. Then the natural map

$$
(8.14) \quad \Omega_*(X) \xrightarrow{\rho_{G,X}^T} \Omega_*^T(X)
$$

is a split monomorphism which is natural for the morphisms in $\mathcal{V}_G$. In particular, if $H$ is any closed subgroup of $G$, then there is a split injective map

$$
(8.15) \quad \Omega_*(H) \xrightarrow{\rho_{H,X}^T} \Omega_*^T \left( G \times X \right).
$$
Proof. The first statement follows directly from Theorem 8.7, where the splitting is given by the trace map. The second statement follows from the first and Proposition 5.5. □

Before we draw some more consequences, we have the following topological analogue of Theorem 8.7, which is much simpler to prove. Recall from (7.10) that if $G$ is a complex Lie group and $X$ is a finite CW-complex with a $G$-action, then its equivariant complex cobordism is defined as

\begin{equation}
MU^*_G(X) := MU^* \left( X \times G \right).
\end{equation}

**Theorem 8.9.** Let $G$ be a complex Lie group with a maximal torus $T$ and Weyl group $W$. Then for any $X$ as above, the natural map

\begin{equation}
MU^*_G(X) \rightarrow (MU^*_T(X))^W
\end{equation}

is an isomorphism.

Proof. As in the proof of Theorem 8.7, we can reduce to the case when $G$ is reductive. It follows from the above definition of the equivariant complex cobordism and the similar definition of the equivariant singular cohomology of $X$, plus the Atiyah-Hirzebruch spectral sequence in topology that there is a spectral sequence

\begin{equation}
E_2^{p,q} = H^p_G(X, Q) \otimes Q MU^q \Rightarrow MU^p_G(X).
\end{equation}

Since the Atiyah-Hirzebruch spectral sequence degenerates rationally, we see that the above spectral sequence degenerates too. Since one knows that $H^*_G(X) \cong (H^*_T(X))^W$ (cf. [4, Proposition 1]), the corresponding result for the cobordism follows. □

**Theorem 8.10.** For a connected linear algebraic group $G$ over $\mathbb{C}$, the map $\rho^G : \Omega^*(BG) \rightarrow MU^*(BG)$ (cf. Corollary 7.6) is an isomorphism of $\mathbb{L}$-algebras. In particular, the natural map

\[ CH^*(BG) \rightarrow MU^*(BG) \otimes L \mathbb{Q} \]

is an isomorphism of $\mathbb{Q}$-algebras.

Proof. To prove the first isomorphism, we can use Theorems 8.7 and 8.9 to reduce to the case of a torus. But this is already known even with the integer coefficients (cf. (6.8) and [33]). The second isomorphism follows from the first and Proposition 7.1. □

**Remark 8.11.** The map $\pi^G : CH^*(BG) \rightarrow MU^*(BG) \otimes L \mathbb{Z}$ was discovered by Totaro in [33] even before Levine and Morel discovered their algebraic cobordism. It is conjectured that the map $\pi^G$ is an isomorphism with the integer coefficients for a connected complex algebraic group $G$. Totaro modified this conjecture to an expectation that $\pi^G$ should be an isomorphism after localization at a prime $p$ such that $MU^*(BG)_{(p)}$ is concentrated in even degree. The above theorem proves the isomorphism in general with the rational coefficients. We also remark that the map $MU^*(BG) \otimes L \mathbb{Q} \rightarrow H^*(BG)$ is an isomorphism (cf. [32]). The above result then shows that the cycle class map for the classifying space is an isomorphism with the rational coefficients. One wonders if the techniques of this paper could be applied to the algebraic version of the Brown-Peterson cobordism theory to prove the Totaro’s modified conjecture.
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