Tidal and Tidal-Resonant Effects in Coalescing Binaries

Kostas D. Kokkotas\textsuperscript{1,2} and Gerhard Schäfer\textsuperscript{1}

\textsuperscript{1}Max-Planck-Gesellschaft, Arbeitsgruppe Gravitationstheorie an der Friedrich-Schiller-Universität Jena, 07743 Jena, Germany

\textsuperscript{2}Department of Physics, Aristotle University of Thessaloniki, 540 06 Thessaloniki, Macedonia, Greece.

24 March 2022

ABSTRACT

Tidal and tidal-resonant effects in coalescing compact binary systems are investigated by direct numerical integration of the equations of motion. For the stars polytropic models are used. The tidal effects are found to be dominated by the (non-resonant) $f$-modes. The effect of the $g$-mode-tidal resonances is obtained. The tidal interaction is shown to be of interest especially for low-mass binaries. There exists a characteristic final plunge orbit beyond which the system cannot remain stable even if radiation reaction is not taken into account; in agreement with results obtained by Lai et al. (1993). The importance of the investigated effects for the observation of gravitational waves on Earth is discussed.

Key words: Radiation mechanisms : gravitational - Stars : binaries:close - neutron

1 INTRODUCTION

Close neutron star binaries are among the primary sources of gravitational waves that will be detected by the future gravitational-wave observatories on Earth (\cite{Thorne1987}). Using the matched filtering technique proposed by Thorne (\cite{Thorne1987}), close binaries should be detectable out to cosmological distances in the Gpc regime. The observed gravitational waves will be produced during the final inspiral and coalescence stages of the motion of the both compact objects (neutron stars and/or black holes). The detection process will last 2 – 3 secs as the gravitational waves are expected to be measured when the binaries will be in the gravitational-wave-frequency window of 100 – 1000 Hz. Future improved detector
technology is expected to widen this window down to 10 Hz and up to 10 kHz, which means longer observation time and higher resolution of the signal, leading in this way to a more accurate determination of the system parameters like the masses, spins, and radii of the bodies. Nevertheless, already the 100 – 1000 Hz window can give us important information about the above parameters of the system if we know the orbital motion and the interaction mechanism of the two extended bodies with reasonable accuracy (during the final inspiral the bodies cannot be considered as point-like). The better our theoretical knowledge will be the more information will be extractable from the detected signal while any effect that will be omitted will reduce the signal to noise ratio of the detector. For example, the omission of post-Newtonian orbital terms in the construction of the template waveform will not change significantly its amplitude but it will produce destructive effects on the phase of the wave leading to a decrease in the correlation of the incoming wave and the applied template \cite{Schutz1993}. Recent numerical simulations for testing the post-Newtonian approximation \cite{Kokkotas1994a} have shown that the correlation falls down by \(\approx 40\%\) for nearly equal-mass systems while the phase effects become more dramatic for systems with unequal masses. Moreover, this phase mismatch will produce analogous effects in the accuracy of the calculation of the system parameters (e.g. masses) as well as of the observational distance and, as a consequence, of the rate of the detected events \cite{Kokkotas1994a}. For these reasons a lot of attention has been paid recently to the study of possible fluid characteristics that may corrupt the signal \cite{Kochanek1992, Jaranowski1992, Bildsten1992, Spyrou1994}.

What is expected from theoreticians in the remaining 5 to 6 years before operating gravitational–wave observatories might be available is the qualitative and quantitative study of every aspect of the compact binary–star coalescence in order to construct the best possible theoretical models that our theoretical approximations allow \cite{Cutler1992, Schutz1993}. Tidal effects have been proposed \cite{Fabian1975} as the reason for the formation of close binary systems. Tidal capture has been extensively studied by various authors in the past two decades leading to a quite well understanding of the influence of the tides not only on the formation but also on the evolution of close binary systems \cite{Press1977, Lee1986, Alexander1987}.

In this line of research we are investigating tidal and tidal-resonant effects in the final stages of the inspiral of two neutron stars. Beyond this we investigate the information that can be gained about these effects from the detected gravitational waveforms. We also calcu-
late the evolution of the binaries in the 1st-post-Newtonian (1PN) approximation to general relativity, including 3.5PN gravitational radiation damping, and compare it with the tidal evolution.

In the next section we present equations that govern the evolution of close binary systems under the influence of Newtonian tidal and 2.5PN gravitational (quadrupole-)radiation reaction forces. In the third section we describe numerical procedures and discuss the obtained results.

2 EQUATIONS OF MOTION

Let us consider a close binary system consisting of two bulk-non-rotating neutron stars with masses $m_1$ and $m_2$. Both of the stars are more or less tidally distorted by their respective companions. The Hamiltonian governing the motion of the two extended bodies, in our case, consists of four parts: the contribution from the stationary orbital motion, a part describing the oscillations of the stars, a term arising from the gravitational interaction of the tidally deformed star 2 with the undeformed star 1, and vice-versa, and, finally, a part which describes the gravitational radiation reaction or radiation damping.

The orbital Hamiltonian, at Newtonian order, in polar coordinates reads

$$H_{orb} = \frac{1}{2\mu} \left( P_R^2 + P_\Phi^2 \right) - \frac{G\mu M}{R},$$

where $R$ denotes the separation of the two stars and $\Phi$ the orbital rotation angle. $P_R$ and $P_\Phi$ are the corresponding canonical momenta.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{and} \quad M = m_1 + m_2$$

are the reduced mass and the total mass of the system, respectively.

The Hamiltonian describing the oscillations of the stars corresponds to an infinite set of harmonic oscillators. It has the form (Alexander 1987):

$$H_{osc}^{(a)} = \frac{1}{2} \sum_n \left( p_n^2 + \omega_n^2 q_n^2 \right),$$

where $q_n$ are generalized coordinates and $p_n$ their conjugate momenta; $\omega_n$ are real frequency eigenvalues. The index $a$ on $H_{osc}$ denotes the stars; it takes the values 1 and 2. Like in eq. (3) we often drop indices for simplicity.

The tidal gravitational potential at a point $(r, \theta, \phi)$ inside, let us say, the star 2 (the centre of mass of this star is put into the origin of the coordinate system) due to the star 1, which is located at $(R, \pi/2, \Phi)$ and treated as point-like, reads in spherical coordinates:
\[ \phi_T = -\frac{4\pi G m_1}{R} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell + 1} Y_{\ell m}(\pi/2, \Phi) Y_{\ell m}^*(\theta, \phi)(r/R)^{\ell}, \]  

where \( Y_{\ell m} \) are normalized spherical harmonics. Using this expression, the gravitational potential energy \( U_T \) of the deformed star 2 (with the corresponding mass change \( \delta\rho \)) in the tidal field of star 1 is given by

\[ U_T^{(1,2)} = \int \phi_T \delta\rho \, d^3r. \]

By the use of the continuity equation (\( \rho_0 \) is the unperturbed mass density)

\[ \delta\rho = -\text{div}\rho_0 \xi \]

and integration by parts, the expression (5) becomes

\[ U_T^{(1,2)} = \int \rho_0 \xi \cdot \nabla \phi_T \, d^3r, \]

where \( \xi \) is a vector describing the Lagrangian displacement of the fluid. The vector can be written as a linear superposition of normalized eigenvectors \( \xi_n \),

\[ \xi(r, t) = \sum_n q_n(t) \xi_n(r), \]

which satisfy a self-adjoint eigenvalue equation of the type

\[ L \xi_n = \rho_0 \omega_n^2 \xi_n. \]

The normal modes \( \xi_n \) are characterized by their spherical harmonic indices \( \ell \) and \( m \), and the radial index \( n \). They can be decomposed into radial and poloidal spherical harmonics (Chandrasekhar 1964):

\[ \xi_n \equiv \xi_{n\ell m}(r, \theta, \phi) = [\xi_{n\ell}^R(r)e_r + \xi_{n\ell}^S(r)r\nabla] Y_{\ell m}(\theta, \phi). \]

By the aid of the unitary transformation

\[ \xi^{(e)}_{n\ell m} = \frac{1}{\sqrt{2}} [\xi_{n\ell m} + \xi_{n\ell - m}] \quad (0 \leq m \leq \ell), \]

\[ \xi^{(o)}_{n\ell m} = \frac{1}{i\sqrt{2}} [\xi_{n\ell m} - \xi_{n\ell - m}] \quad (1 \leq m \leq \ell) \]

we can create an orthonormal real basis in which the generalized coordinates \( q_n^{(\sigma)} \) (\( \sigma = e, o \)) are real. Herewith \( U_T \) can be written as

\[ U_T^{(1,2)} = -\frac{G m_1 m_2}{R} \sum_n \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{\sigma=e, o} \frac{W_{\ell m} q_n^{(\sigma)}(r)}{r_2} \left( \frac{r_2}{R} \right)^{\ell} Q_{n\ell} 2^{1/2} \text{trig}_{m}^{(\sigma)}(\Phi), \]

where \( \text{trig}_{m}^{(e)}(\Phi) = \cos(m\Phi), \text{trig}_{m}^{(o)}(\Phi) = \sin(m\Phi). \) \( \Phi \) decomposes into \( \Phi = v + g \), where \( v \) is the true orbital anomaly and \( g \) = constant. The prime on the fourth sum means exclusion of the case \( m = 0, \sigma = o \), and \( r_2 \) is the radius of the star 2. \( Q_{n\ell} \) denotes the overlap integral.
\[ Q_{nt} = \ell \int_0^1 \rho \tilde{r}^\ell+1 \left[ \xi_{nt}^R(\tilde{r}) + (\ell + 1) \xi_{nt}^S(\tilde{r}) \right] d\tilde{r}, \]  
\[ W_{\ell m} = (-)^{\ell m} \frac{2\pi}{2\ell + 1} \left[ \frac{4\pi}{2\ell + 1} (\ell - m)! (\ell + m)! \right] \frac{1}{2} \left[ 2\ell \left( \frac{\ell + m}{2} \right)! \left( \frac{\ell - m}{2} \right)! \right], \] 

where \( \tilde{r} = r/r_2 \) and \( \rho = \rho_0 r_2^3/m_2 \) is a dimensionless density and \( W_{\ell m} \) is given by

where \( (-)^k \) is zero if \( k \) is not an integer. The functions \( q_{n\ell m}^{(\sigma)} \) of the motions of the harmonic oscillator, defined by the Hamiltonian (3), in terms of action–angle variables, \( J_{n\ell m}^{(\sigma)} \) and \( \theta_{n\ell m}^{(\sigma)} \), respectively, can be represented as

\[ q_{n\ell m}^{(\sigma)} = (2J_{n\ell m}^{(\sigma)}/\omega_{nt})^{1/2} \cos \theta_{n\ell m}^{(\sigma)}, \]

where \( \theta_{n\ell m}^{(\sigma)} = \omega_{nt} t + \phi_{n\ell m}^{(\sigma)} \), \( \phi_{n\ell m}^{(\sigma)} = \text{constant} \). Herewith, finally, \( U_T^{(1,2)} \) takes the form

\[ U_T^{(1,2)} = - \frac{Gm_1 m_2}{R} \sum_{n} \sum_{\ell=2}^\infty \sum_{\sigma=e,o} \sum_{m=0}^\ell \rho W_{\ell m} \frac{1}{r_2} \left( \frac{r_2}{R} \right)^\ell Q_{n\ell} \omega_{nt}^{-1/2} \]

\[ \times \left\{ \frac{(J_{n\ell m}^{(e)})^{1/2} \left( \cos(\theta_{n\ell m}^{(e)} + m\Phi) + \cos(-\theta_{n\ell m}^{(e)} + m\Phi) \right)}{(J_{n\ell m}^{(o)})^{1/2} \left( \sin(\theta_{n\ell m}^{(o)} + m\Phi) + \sin(-\theta_{n\ell m}^{(o)} + m\Phi) \right)} \right\}. \]

Circular orbits play an important role in coalescing binaries. There, \( \Phi \) is given by the single Keplerian orbital angular frequency \( \omega_K \). The orbital motion induces perturbations on the star modes which rotate with the phase velocities \( m\omega_K \), see eq. (13). For an adiabatic inspiral situation \( \omega_K \) is a slowly varying function with time. Whenever \( \omega_{nt}/\omega_K \) approaches integer values, resonances can occur; see eq. (17) and, for \( W_{\ell m} \), remember the condition \( (\ell + m)/2 = \text{integer} \).

The Hamiltonian of the entire system – orbital motion, star oscillations, tidal coupling of the two stars, radiation reaction – takes the form:

\[ H(t) = H_{\text{orb}} + H_{\text{osc}}^{(1)} + H_{\text{osc}}^{(2)} + H_{\text{tid}}^{(1,2)} + H_{\text{tid}}^{(2,1)} + H_{\text{reac}}(t), \]

where \( H_{\text{tid}}^{(1,2)} = U_T^{(1,2)} \), etc.

The (explicitly) time-dependent reaction Hamiltonian \( H_{\text{reac}} \) is given by, cf. \( \text{Schäfer} [1990] \):

\[ H_{\text{reac}}(t) = \frac{2G}{5c^5} \frac{d^3 Q_{ij}(t)}{dt^3} \left( \frac{P_i P_j}{\mu} - \frac{GM\mu R^i R^j}{R^3} \right), \]

where \( Q_{ij} = \mu \left( R^i R^j - \frac{1}{3} \delta_{ij} R^2 \right) \) denotes the mass-quadrupole tensor of the two-body system.

A spin-orbit-interaction part \( H_{SO} \) could have been also included into the Hamiltonian, e.g. see \( \text{Damour & Schäfer 1988} \). But it has been found that in our case of bodies without
bulk rotation it does not contribute significantly to the evolution of the system, and in what follows we shall not discuss it any further.

The system evolves in time as dictated by the Hamiltonian equations of motion:

\[
\frac{d\mathbf{P}}{dt} = -\frac{\partial H(t)}{\partial \mathbf{R}}, \quad \frac{d\mathbf{R}}{dt} = +\frac{\partial H(t)}{\partial \mathbf{P}},
\]

\[
\frac{dp_n}{dt} = -\frac{\partial H(t)}{\partial q_n}, \quad \frac{dq_n}{dt} = +\frac{\partial H(t)}{\partial p_n}.
\]

The respective reaction parts, \(F_P\) and \(F_R\), in the above equations, in polar coordinates, take the form, cf. (Schäfer 1985):

\[
F_{PR} = + \frac{8}{3} \frac{G^2 P_R}{c^5 R^4} \left( \frac{GM^3 \nu}{5} - \frac{P^2_\Phi}{\nu R} \right),
\]

\[
F_{P\Phi} = - \frac{8}{5} \frac{G^2 P_\Phi}{c^5 R^3 \nu} \left( \frac{2GM^3 \nu^2}{R} + 2 \frac{P^2_\Phi}{R^2} - P_R^2 \right),
\]

\[
F_R = - \frac{8}{15} \frac{G^2}{c^5 R^2 \nu} \left( 2P^2_R + 6 \frac{P^2_\Phi}{R^2} \right),
\]

\[
F_\Phi = - \frac{8}{3} \frac{G^2 P_R P_\Phi}{c^5 \nu R^4}.
\]

The gravitational waves are emitted primarily by the orbital motion. The amplitudes of the leading (quadrupole) waves are proportional to \(1/R\) and their phases vary with time through \(2\Phi\).

3 RESULTS AND DISCUSSION

The Hamiltonian equations of motion (20) – (21) have been integrated for various binary systems consisting of neutron stars from the less relativistic to the extreme relativistic regime. The equation of state was taken to be of polytropic type,

\[ p = K \rho^\Gamma, \quad \Gamma = 1 + \frac{1}{N} \]

with polytropic index \(N = 0.5, 1, 2\) while the central density was \(\approx 10^{14} - 10^{15} \text{gr/cm}^3\). It is known that the best approximation for constructing realistic polytropic neutron stars is to take polytropes with indices 0.5 up to 2 (Finn 1987). The adiabatic index \(\Gamma_1\) we have chosen in such a way that \(g\)-modes exist. This is possible as far as the adiabatic index \(\Gamma_1\) is larger than \(\Gamma\); in the limit \(\Gamma_1 = \Gamma\) there are no \(g\)-modes. There are two other important aspects regarding the selection of the appropriate adiabatic index: As \(\Gamma_1 \to \Gamma\) the \(g\)-mode frequencies are slowing down significantly and the \(g\)-modes become degenerate at zero frequency. This property of the \(g\)-modes in principle increases the influence of the
interaction part of the Hamiltonian (since $U_T \approx \omega^{-1/2}$) but there are also the values of the overlap integrals which decrease much faster as it can be seen from Tables 2a and 2b. The result is that the interaction Hamiltonian becomes less important as $\Gamma_1 \rightarrow \Gamma$. For a detailed discussion of the g-modes we refer to Finn (1987), McDermott et al. (1985), and Reisenegger and Goldreich (1992) and references therein. For the f-modes it does practically not matter which values of $\Gamma_1$ we use, the frequency and the overlap integral are not very sensitive to the choice of $\Gamma_1$. In what follows we shall discuss polytropic stars which show typical external characteristics, i.e. masses and radii of accepted relativistic neutron stars models. More general equations of state will surely alter the results, but the main outcome of the paper, in particular those effects which are dominated by the f-modes, should have wider applicability. We have decided to study the binary evolution for nine different star models, A, B, C with polytropic indices 0.5, 1, and 2 (the value of the constant $K$ has been chosen appropriately in each case). These models span the whole range of the neutron star family, from the very condensed state with high surface redshift to the less compact state with small surface redshift, see Table 1. In all cases the equations of structure are Newtonian-like, i.e. $\rho$ in eq. (26) is the Newtonian mass density. Thus, with the models we have constructed, the less relativistic neutron stars are the better approximated ones. For these models normal-mode eigenfrequencies and eigenfunctions have been found and overlap integrals have been calculated. For $\ell = 2$, the part of the spectrum around the f-mode together with the first few gravity and pressure modes for the $N = 0.5, 1, 2$ polytropes are shown in Tables 2a and 2b where additionally the corresponding values of the overlap integrals are listed. Although for our numerical calculations we have mainly used the values of Table 2a we present also Table 2b in order to show the dependence of the g-mode frequencies and the overlap integrals on the values of the adiabatic index. The frequencies agree with those given by Cox (1980). For an easier use of Tables 2a and 2b, in the last column of Table 1, the fundamental frequencies are given in units of Hz. When we checked the procedure for the calculation of the overlap integrals we have calculated also the normal modes and the overlap integrals for $N = 3$ polytropes ($\Gamma_1 = 5/3$). The agreement with the results of Lee & Ostriker (1986) was very good, usually the difference was less than 0.1% even for the lowest g-modes for which special care must be taken in order to avoid numerical errors.

The procedure followed for the calculation of the normal-mode eigenfrequencies is similar to Dziembowski (1971) with the exception that the final frequencies and eigenfunctions
calculated by the aid of this procedure have been used as trial expressions for a variational calculation of the eigenfrequencies. Practically, this final step together with an extra check for the orthogonality of the eigenfunctions was just a verification for the accuracy of our method.

The Hamiltonian equations of motion (20) – (21), for each value of \( n, \ell, m \) (\( m \) positive), and \( \sigma \), form a system of 6 first-order ODEs. As can be seen from the form of \( U_T \) (see eq. (17)), because of the existence of the term \( (r_2/R)^\ell \), respectively \( (r_1/R)^\ell \), only the harmonics with \( \ell = 2 \) and \( 3 \) are important (\( m = 0, \ldots, \ell \)). Furthermore, from the Tables 2a and 2b follows that the \( f \)-mode together with the first \( g \)- and \( p \)-modes maximize the overlap integral, but since \( U_T \) is inverse proportional to the square root of \( \omega_{n\ell} \) the \( p \)-modes do not contribute significantly and thus their incorporation in the following calculations is not needed.

The initial conditions put two non-oscillating stars into circular orbits with orbital frequencies of 50 Hz (correspondingly, the frequencies of the emitted gravitational waves are 100 Hz). Notice that at the late stages of the evolution of a binary system orbits are expected to be circular because of the tendency of the gravitational radiation damping to circularize the orbits. Then the system is left to evolve according to the equations (20) – (21). The evolution of the system depends on the radiation damping and on the tidal interaction of the two stars.

Since the important aspect of this work is the comparison of the rate of evolution of binary systems with and without tidal interaction we have to discuss also a refinement of the binary motion where the components are treated as point-like. For this we consider the 1PN orbital dynamics. In case of circular motion the (secular) radiation-reaction change of the 1PN orbital angular frequency reads, e.g see (Cutler & Flanagan 1994)

\[
\frac{d\omega}{dt} = \frac{96G^{5/3}\mu M^{2/3}}{5c^5} \mu M^{2/3} \omega^{11/3} \left[ 1 - \frac{(743 + 924\mu/M)}{336c^2}(GM\omega)^{2/3} \right].
\] (27)

In this relation the gravitational radiation damping has been taken into account up to the 3.5PN level (for non-circular orbits, see (Junker & Schäfer 1992)) using balance equations. A local expression for the 3.5PN reaction force is still not available.

As seen from Figure 1, the evolution of the system as calculated by the damped 1PN approximation (equation 27) is slower than the evolution which assumes damped Newtonian approximation. If we additionally turn on the tidal part of the Hamiltonian, \( H_{\text{tid}} \), then the evolution becomes faster since the orbital motion will excite oscillations of the initially non-oscillating stars. The amount of the energy transferred from the orbital dynamics to the
stellar dynamics depends on the stellar equation of state and the excitation efficiency of the tidal coupling. As mentioned earlier, the overlap integral $Q_{n2}$ is maximum for the $f$-mode, and it has been shown numerically that all other modes together cannot absorb more orbital energy than the $f$-mode alone (see also Reisenegger 1994). If radiation reaction is turned off, the tidal interaction makes the orbits oscillating around the initially chosen circular orbit. Without radiation damping the system is a near-integrable system since the part of the Hamiltonian which describes the orbital motion and the oscillations of the star is integrable while the other part which describes the tidal interaction is the “perturbation Hamiltonian”. Therefore, techniques for the treatment of evolution and resonances of near-integrable systems can be applied (Alexander 1987; Goldstein 1980; Lichtenberg & Lieberman 1983). For an extensive analysis of the resonances in Newtonian binary systems the reader is referred to a paper by Alexander (1987).

For the cases we are studying, i.e. coalescing binary systems consisting of two neutron stars, strong $f$-mode resonances can occur only when the two stars are near their final plunge stage. Since the orbit there shrinks very fast, the passage through the tidal resonances is instantaneous and thus it does not result in a significant energy exchange between the quasi-periodic orbital part and the periodic internal part of the system. But if we allow for the lower frequency $g$-modes then resonances can occur as it is shown in Figures 2 and 3. Obviously, for $\omega_{n2} = 2\omega_K$, i.e. $\ell = 2$ and $m = 2$, the tidal-interaction Hamiltonian leads to strongest resonance, see eq. (17), and the $g$-mode frequencies are low enough to fulfil this condition, see Table 2. However, the Figures 2 and 3 show that the $f$-mode (non-resonant) effects are much more important than $g$-modes resonant ones. The effect of the resonances depend on the adiabatic index. We find that the $g$-mode resonances are unimportant for the polytropes with $N = 0.5$ but have some relevant contribution as $\Gamma_1$ becomes larger which is mainly due to the small values of the overlapping integrals. The inverse is true for the $f$-modes. The $f$-mode frequency for the case $N = 0.5$ is smaller and it results that the deviation from the point-particle evolution is larger, see Figures 1 to 4.

In Figures 2 and 3 one can see the differences between the distances of the two stars as calculated with radiation reaction + tidal interaction on the one side and with radiation reaction alone on the other. The differences start oscillating as the orbital motion of the two stars approaches frequencies of one half of the oscillation frequencies. The figures show the effects of the $f$-, $g_1$-, and $g_2$-modes for the binary systems $A_{0.5} - C_{0.5}$ and $A_1 - C_1$. Both figures clearly show that the $f$-modes dominate the $g$-modes. The effects of the tidal
effects mildly corrupt the nice sinusoidal form that the standard matched filtering technique is supposing. This leads to a smaller correlation and results in an increase of the errors in the calculation of the mass parameters of binaries from measured waveforms. Evidently, near the high–frequency end of the gravitational-wave-observation window tidal effects should be incorporated into the template waveform. This is supported also by the recent works by Reisenegger and Goldreich [1994] and Shibata [1994].

For the total phase, see Figure 4, we have obtained a phase difference between the (orbital) motion with radiation reaction and the motion with radiation reaction + tidal interaction of the order $15\pi$ for a binary consisting of two stars of type $A_1$-$C_1$ (see Table 1) and $21\pi$ for a binary consisting of one type $A_{0.5}$ and one type $C_{0.5}$ star, while in the case of a binary with stars of types $A_i$ and $A_i$ the phase difference is quite small and unimportant. In both cases these phases have been accumulated during the 100 – 1000 Hz emission stage of gravitational radiation while the stars have completed about 300 revolutions around each other.

While the tidal resonances contribute seemingly only during the late stages of the binary-systems evolution, the tidal effects accumulate energy on the stellar oscillations during the whole period of the coalescence. The transfer of energy becomes significant if the system has the following characteristics: (i) long coalescing time, which implies small stellar masses since the coalescing time is proportional to $\approx M^{-3}$ for nearly equal-mass binaries, and (ii) small ratios $m_i/r_i$, because in this case the tidal effects are more significant (it results in a larger energy transfer from the orbital motion to the oscillations). Even though, the contribution of the tidal effects outside the region where resonances get important is in general small.

For the system $A_1$-$C_1$ we have mentioned earlier that during the whole evolution of the system, from the beginning of the observation window, i.e. 100Hz which correspond to a distance of $44.8M$, the phase difference accumulates to $14.2\pi$, while the evolution of the same system from a distance of $20M$ (frequency of the gravitational waves 335 Hz) to $9.7M$ (end of the observational window, where also the stars touch each other) accumulates a phase difference of $12.9\pi$. The corresponding numbers for the system $A_{0.5}$-$C_{0.5}$ are $21\pi$ and $18.3\pi$. This means that 90% of the total phase difference is created during the final inspiral. This late-stages corruption of the signal due to the tidal effects is also apparent in the work of Kochanek [1992] where irrotational Roche-Riemann ellipsoids have been used as models for the neutron stars.
An interesting feature is the fact that if the two stars are close enough the tidal effects can
drive them into coalescence even in the case that no radiation reaction is taken into account,
see Figure 5. The *tidally induced plunge* region strongly depends on the masses of the two
stars and on their degree of compactness. The more compact the stars are the smaller their
separation is for the tidally induced plunge. For the three models that we referred to earlier
this distance takes the values: $11.6M$ (8.9$M$) for the system $A_2-C_2$, $16M$ (8.9$M$) for the
system $A_1-C_1$, $17.4M$ (8.9$M$) for the system $A_{0.5}-C_{0.5}$ and $19M$ (15.9$M$) for the system $C_1-
C_1$ (in parentheses are shown the distances for which the stars touch each other). We should
notice in addition that this tidally induced plunge, in reality, should be more complicated
since it happens for distances where the two stars have passed already their Roche limit
(Bildsten & Cutler 1992). Recently Lai et al. (1993), (1994) came to similar conclusions for
binary systems with also polytropic star components, but they have not used the approach
which is applied in the present work. The results support the idea of introducing the notion
of extended bodies in the construction of the waveform templates for the future detection of
gravitational waves. Nevertheless, we should keep in mind that we have used linear analysis
and that the “plunge orbit” occurs at small separations at which the linear theory is not
strictly valid. Thus, although the existence of “plunge orbits” seems to be established the
numbers that we give might change if non-linear theory is applied.

Concluding this work we will discuss the effects of our models on the waveform of the
gravitational waves emitted during the binary coalescence stage. As it can be seen from the
Figure 4 (remember the end of Sect. 2) the waves accumulate very quickly a phase difference
which seriously corrupts the shape of the Newtonian waveform leading to difficulties in
the detection procedure if not taken into account in the template waveform (for a numerical
simulation of the detection procedure and the exact influence on the detection procedure the
reader is referred to (Kokkotas et al. 1994a)). Although the 1PN corrections are weakening
the high phase differences (Kokkotas et al. 1994b) the effects on the phase are very large
and cannot be overcome. For the evolution of the system $A_i-C_i$, from $20M$ to coalescence,
the phase difference produced by the 1PN theory is $32\pi$, while from the tidal effects, as
mentioned earlier, $21\pi$ for the system $A_{0.5}-C_{0.5}$ and $14\pi$ for the system $A_1-C_1$. The large
tidal contributions depend on the choice of the binary system. If we would have chosen
a system of the type $A_i-A_i$ then the post-Newtonian effects would have been of the same
order while the tidal ones would have turned out much smaller. This can be considered as a
The system of equations that have been used in this work would be more precise if the radiation loss due to non-radial oscillations of the stars would have been included. Nevertheless, to leading order, the above analysis is still valid since the energy stored in the stellar pulsations is a fraction of the total energy of the system (notice that the reaction Hamiltonian, equation (19), is proportional to the non-spherical-symmetric part of the sum of two times the kinetic energy tensor plus the potential energy tensor and, implicitly, to the first time derivative of that part). Furthermore, the damping time of the stellar oscillations which for identical stars is of about the same order as the coalescence time (coalescence understood as the stage where the interbody-body separation amounts to some few single-body diameters only) becomes longer as the stars become less relativistic; this can be proven easily from the relations given in Balbinski and Schutz (1982) eq. (1) and/or Reisenegger and Goldreich (1994) eq. (11). In our work we have taken the most compact star as a point-like mass while the other star which is less relativistic was the one that has undergone oscillations, oscillations with indeed much longer damping time than the coalescence time of the binary system. But if the evolution of a binary system is studied from the regime of 10Hz, since the coalescence time is long, the damping of the stellar oscillations should be taken into account.

Acknowledgements

KDK acknowledges the kind hospitality and the financial support provided by the Max-Planck-Research-Group Gravitationstheorie at the University of Jena. Helpful suggestions and discussions with Bernard Schutz are also gratefully acknowledged by KDK. We are grateful to Andrzej Krolak for his verification for some of our numerical results. Finally, we express our gratitude to an anonymous referee for his suggestions which improved the final form of the paper.

REFERENCES

M. E. Alexander, Mon. Not. R. astr. Soc. 227, 843 (1987).
E. Balbinski and B. F. Schutz, Mon. Not. R. astr. Soc. 200, 43P (1982).
L. Bildsten and C. Cutler, Astrophys. J. 400, 175 (1992).
S. Chandrasekhar, Astrophys. J. 139, 664 (1964).
J. P. Cox, *Theory of Stellar Pulsation*, Princeton University Press, Princeton (1980).

C. Cutler, T. A. Apostolatos, L. Bildsten, L. S. Finn, E. E. Flanagan, D. Kennefick, D. M. Markovic, A. Ori, E. Poisson, G. J. Sussman, and K. S. Thorne, *Phys. Rev. Lett.* **70**, 2984 (1993).

C. Cutler and E. E. Flanagan, *Phys. Rev. D* **49**, 2658 (1994).

T. Damour and G. Schäfer, *Nouvo Cimento* **101B**, 127 (1988).

W. Dziembowski, *Acta Astr.* **21**, 289 (1971).

A. C. Fabian, J. E. Pringle, and M. J. Rees, *Mon. Not. R. astr. Soc.* **175**, 15P (1975).

L. S. Finn *Mon. Not. R. astr. Soc.* **227**, 265 (1987).

H. Goldstein, *Classical Mechanics*, 2nd edn, Addison-Wesley, Massachusetts (1980).

P. Jaranowski and A. Krolak, *Astrophys. J.* **394**, 586 (1992).

W. Junker and G. Schäfer, *Mon. Not. R. astr. Soc.* **254**, 146 (1992).

C. S. Kochanek, *Astrophys. J.* **398**, 234 (1992).

K. D. Kokkotas, A. Krolak, and G. Tsegas, *Class. Quantum Grav.* **11**, 1901 (1994).

K. D. Kokkotas, A. Krolak, and G. Schäfer, *Preprint*, (1994).

D. Lai, F. A. Rasio and S. L. Shapiro *Astrophys. J.* **406**, L63 (1993).

D. Lai, F. A. Rasio and S. L. Shapiro *Astrophys. J.* **420**, 811 (1994).

H. M. Lee and J. P. Ostriker, *Astrophys. J.* **310**, 176 (1986).

A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, Springer-Verlag, New York (1992).

P. N. McDermott, P. N. Hansen, C. J. Van Horn and R. Bouland *Astrophys. J.* **297**, L37 (1985).

W. H. Press and S. A. Teukolsky, *Astrophys. J.* **213**, 183 (1977).

A. Reisenegger and P. Goldreich, *Astrophys. J.* **395**, 240 (1992).

A. Reisenegger and P. Goldreich, *Astrophys. J.* **426**, 688 (1994).

A. Reisenegger, *Astrophys. J.* **432**, 296 (1994).

G. Schäfer, *Ann. Phys. (N. Y.)* **161**, 81 (1985).

G. Schäfer, *Astron. Nachrichten* **311**, 213 (1990).

B. F. Schutz, *Class. Quantum Grav.* **10**, S135 (1993).

M. Shibata, *Prog. of Theor. Physics* **91**, 871 (1994).

N. K. Spyrou and K. D. Kokkotas, *Astrophys. J.* **431**, 254 (1994).

K. S. Thorne, in: *300 Years of Gravitation*, eds S. W. Hawking and W. Israel, Cambridge Univ. Press, Cambridge (1987), p. 330.
TABLES

Table 1: For \( \ell = 2 \), nine polytropic neutron stars models are listed for three polytropic indices \( N = 0.5, N = 1 \) and \( N = 2 \). The index on the letters A, B, C shows the polytropic index. For these models, which span a wide range of possible neutron stars radii, masses, and densities, the \( f \)-mode frequency is given in units of Hz.

Table 2: For \( \ell = 2 \), normalized frequencies for three polytropic-adiabatic index pairs are listed together with the corresponding values for the overlap integral \( Q_{n2} \).
**Table 1**

| Model | \( r \) (Km) | \( m \) (Km) | \( \frac{m}{M_\odot} \) | \( \rho_c \) (gr/cm\(^3\)) | \( f \)-mode (Hz) |
|-------|--------------|--------------|-----------------|-----------------|-----------------|
| \( A_{0.5} \) | 10.105 | 2.036 | 1.38 | \( 1.17 \times 10^{15} \) | 2212.8 |
| \( B_{0.5} \) | 12.533 | 1.561 | 1.06 | \( 4.69 \times 10^{14} \) | 1402.4 |
| \( C_{0.5} \) | 18.162 | 1.144 | 0.77 | \( 1.13 \times 10^{14} \) | 688.4 |
| \( A_1 \) | 10.105 | 2.036 | 1.38 | \( 2.09 \times 10^{15} \) | 2607.6 |
| \( B_1 \) | 12.533 | 1.561 | 1.06 | \( 8.40 \times 10^{14} \) | 1324.8 |
| \( C_1 \) | 18.162 | 1.144 | 0.77 | \( 2.02 \times 10^{14} \) | 809.9 |
| \( A_2 \) | 10.105 | 2.036 | 1.38 | \( 7.25 \times 10^{15} \) | 3742.6 |
| \( B_2 \) | 12.533 | 1.561 | 1.06 | \( 2.91 \times 10^{15} \) | 2372.0 |
| \( C_2 \) | 18.162 | 1.144 | 0.77 | \( 7.02 \times 10^{14} \) | 1164.3 |
Table 2a

Eigenfrequencies and Overlap Integrals

| Mode | $\omega_n^2 [Gm/r^3]$ | $|Q_{n2}|$ | $\omega_n^2 [Gm/r^3]$ | $|Q_{n2}|$ |
|------|---------------------|----------|---------------------|----------|
| $p_2$ | 48.2425             | .1155×10^{-3} | 30.1934             | .2541×10^{-2} |
| $p_1$ | 18.8887             | .7475×10^{-2}  | 12.3929             | .2588×10^{-1}  |
| $f$   | 1.0883              | .6238×10^{+0}  | 1.5064              | .5580×10^{+0}  |
| $g_1$ | 0.0088              | .3193×10^{-3}  | 0.0340              | .1764×10^{-2}  |
| $g_2$ | 0.0038              | .5054×10^{-4}  | 0.0161              | .4250×10^{-3}  |
| $g_3$ | 0.0022              | .3106×10^{-4}  | 0.0095              | .1256×10^{-3}  |

Table 2b

Eigenfrequencies and Overlap Integrals

| Mode | $\omega_n^2 [Gm/r^3]$ | $|Q_{n2}|$ | $\omega_n^2 [Gm/r^3]$ | $|Q_{n2}|$ |
|------|---------------------|----------|---------------------|----------|
| $N = 0.5, \Gamma_1 = 3.05$ |                       |          | $N = 1, \Gamma_1 = 2.05$ |          |
| $p_2$ | 61.1329             | .3095×10^{-2} | 40.6315             | .8070×10^{-2} |
| $p_1$ | 35.1837             | .2204×10^{-2} | 24.0741             | .2287×10^{-1} |
| $f$   | 14.7537             | .2121×10^{-1} | 11.5549             | .7574×10^{-1} |
| $g_1$ | 1.5112              | .5580×10^{+0}  | 3.1133              | .4219×10^{+0}  |
| $g_2$ | 0.1902              | .1110×10^{-1}  | 0.5633              | .2105×10^{-1}  |
| $g_3$ | 0.0921              | .2607×10^{-2}  | 0.2967              | .8596×10^{-2}  |
|      | 0.0549              | .7634×10^{-3}  | 0.1839              | .3844×10^{-2}  |
|      | 0.1254              | .1821×10^{-2}  | 0.0911              | .8946×10^{-3}  |
|      | 0.0693              | .4494×10^{-3}  |                      |            |
FIGURES

Figure 1: The evolution of the separation of a binary system is shown. The system is left to evolve from a quite small separation of 24M or 125Hz orbital frequency. The dash-dotted and dashed lines represent the evolution as determined from the Newtonian and 1PN approximation, respectively, each time including the appropriate radiation reaction terms, i.e. the 2.5PN approximation for the Newtonian motion and 3.5PN approximation for the 1PN motion. The continuous line corresponds to the evolution of the system as dictated only from the Newtonian dynamics together with radiation reaction and tidal interaction. Slightly inconsistently, 2.5PN (or 3.5PN) means only the radiation reaction part without inclusion of the lower ‘integer’ approximations.

Figure 2: The difference of the separation of the stars is shown as function of time considering, additionally to the Keplerian motion, radiation reaction on the one side and radiation reaction + tidal interaction on the other side. The figure presents the effects of the f-, g1- and g2-mode of star 1 (model C0.5) with its corresponding frequencies of about 688.4, 61.8 and 40.9 Hz. The contributions of the analogous modes of star 2 (model A0.5) are negligible. Time is plotted in terms of $R = R_{\text{rad}}$.

Figure 3: The difference of the separation of the stars is shown as function of time considering, additionally to the Keplerian motion, radiation reaction on the one side and radiation reaction + tidal interaction on the other side. The figure presents the effects of the f-, g1- , and g2-mode of star 1 (model C1) with its corresponding frequencies of 809.9, 121.7 and 83.8 Hz, respectively. The contributions from the analogous modes of star 2 (model A1) are negligible. The $(m = 2)$-resonances for the g1 and g2 occur near $R/M = 39.3$ and 50.5, respectively. The resonance of the g1-mode can be clearly seen in this Figure. Time is plotted in terms of $R = R_{\text{rad}}$.

Figure 4: For $N = 0.5$ and $N = 1$, the phase differences between the evolution predicted by Newtonian and 1PN approximation are shown. The tidal effects slowly accumulate a phase difference which becomes large only at the final stage of the coalescence leading there to a phase difference of a few complete cycles.

Figure 5: The evolution of the separation of the two stars due to tidal effects only (radiation reaction is not included). For each system there is a characteristic distance beyond which the system is driven into coalescence without the need of radiation reaction.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502034v1
Figure 1
Evolution of the system A - C

N=1
N=0.5
2.5PN+N
2.5PN+N+Tid.
3.5PN+1PN

Orbital Frequency (Hz)

Time (secs)
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502034v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502034v1
This figure "fig1-4.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502034v1
This figure "fig1-5.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502034v1
Figure 5
Tidally Induced Plunge (System A-C)

N=1
N=0.5

Time (in secs)