Conditional expanding bounds for two-variable functions over arbitrary fields

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Abstract

In this short note, we use Rudnev’s point-plane incidence bound to improve some results on conditional expanding bounds for two-variable functions over arbitrary fields due to Hegyvári and Hennecart [4].

1 Introduction

Throughout this chapter, by $\mathbb{F}$ we refer to any arbitrary field, while by $\mathbb{F}_p$, we only refer to the fields of prime order $p$. We denote the set of non-zero elements by $\mathbb{F}^*$ and $\mathbb{F}_p^*$, respectively. Furthermore, we use the following convention: if the characteristic of $\mathbb{F}$ is positive, then we denote its characteristic by $p$; if the characteristic of $\mathbb{F}$ is zero, then we set $p = \infty$. So a term like $N < p^{5/8}$ is restrictive in positive characteristic, but vacuous for zero one.

For $A \subset \mathbb{F}$, the sum and the product sets are defined as follows:

$$A + A = \{a + a' : a, a' \in A\}, \quad A \cdot A = \{a \cdot a' : a, a' \in A\}.$$ 

For $A \subset \mathbb{F}_p$, Bourgain, Katz and Tao ([2]) proved that if $p^\delta < |A| < p^{1-\delta}$ for some $\delta > 0$, then we have

$$\max \{|A + A|, |A \cdot A|\} \gg |A|^{1+\epsilon},$$

for some $\epsilon = \epsilon(\delta) > 0$. Here, and throughout, by $X \ll Y$ we mean that there exists the constant $C > 0$ such that $X \leq CY$.

In a breakthrough paper [8], Roche-Newton, Rudnev, and Shkredov improved and generalized this result to arbitrary fields. More precisely, they showed that for $A \subset \mathbb{F}$, the sum set and the product set satisfy

$$\max \{|A + A|, |A \cdot A|\} \gg |A|^{6/5}, \quad \max \{|A + A|, |A \cdot A|\} \gg |A|^{6/5}.$$ 

We note that the same bound also holds for $|A(1+A)|$ [11], and $|A+A^2|$, max $\{|A + A|, |A^2 + A^2|\}$ [7]. We refer the reader to [1, 3, 8, 6] and references therein for recent results on the sum-product topic.

Let $G$ be a subgroup of $\mathbb{F}^*$, and $g : G \to \mathbb{F}^*$ be an arbitrary function. We define

$$\mu(g) = \max_{t \in \mathbb{F}^*} |\{x \in G : g(x) = t\}|.$$
For $A, B \subseteq \mathbb{F}_p$ and two-variable functions $f(x, y)$ and $g(x, y)$ in $\mathbb{F}_p[x, y]$, Hegvári and Hennecart [4], using graph theoretic techniques, proved that if $|A| = |B| = p^\alpha$, then

$$\max \{|f(A, B)|, |g(A, B)|\} \gg |A|^{1+\Delta(\alpha)},$$

for some $\Delta(\alpha) > 0$. More precisely, they established the following results.

**Theorem 1.1 (Hegvári and Hennecart, [4]).** Let $G$ be a subgroup of $\mathbb{F}_p^*$. Consider the function $f(x, y) = g(x)(h(x) + y)$ on $G \times \mathbb{F}_p^*$, where $g, h: G \to \mathbb{F}_p^*$ are arbitrary functions. Define $m = \mu(g \cdot h)$. For any subsets $A \subseteq G$ and $B, C \subseteq \mathbb{F}_p^*$, we have

$$|f(A, B)| |B \cdot C| \gg \left\{ \frac{|A||B|^2|C|}{pm^2}, \frac{p|B|}{m} \right\}.$$  

**Theorem 1.2 (Hegvári and Hennecart, [4]).** Let $G$ be a subgroup of $\mathbb{F}_p^*$. Consider the function $f(x, y) = g(x)(h(x) + y)$ on $G \times \mathbb{F}_p^*$, where $g, h: G \to \mathbb{F}_p^*$ are arbitrary functions. Define $m = \mu(g)$. For any subsets $A \subseteq G$, $B, C \subseteq \mathbb{F}_p^*$, we have

$$|f(A, B)||B + C| \gg \left\{ \frac{|A||B|^2|C|}{pm^2}, \frac{p|B|}{m} \right\}.$$  

It is worth noting that Theorem 6 established by Bukh and Tsimerman [3] does not cover such a function defined in Theorem 1.2. The reader can also find the generalizations of Theorems 1.1 and 1.2 in the setting of finite valuation rings in [5].

Suppose $f(x, y) = g(x)(h(x) + y)$ with $\mu(g), \mu(h) = O(1)$ and $A = B = C$. Then, it follows from Theorems 1.1 and 1.2 that

1. If $|A| \gg p^{2/3}$, then we have
   $$|f(A, A)||A \cdot A|, |f(A, A)||A + A| \gg p|A|.$$

2. If $|A| \ll p^{2/3}$, then we have
   $$|f(A, A)||A \cdot A|, |f(A, A)||A + A| \gg |A|^4/p. \quad (1)$$

The main goal of this paper is to improve and generalize Theorems 1.1 and 1.2 to arbitrary fields for small sets. Our first result is an improvement of Theorem 1.1.

**Theorem 1.3.** Let $f(x, y) = g(x)(h(x) + y)$ be a function defined on $\mathbb{F}^* \times \mathbb{F}^*$, where $g, h: \mathbb{F}^* \to \mathbb{F}^*$ are arbitrary functions. Define $m = \mu(g \cdot h)$. For any subsets $A, B, C \subseteq \mathbb{F}^*$ with $|A|, |B|, |C| \leq p^{5/8}$, we have

$$\max \{|f(A, B)|, |B \cdot C|\} \gg \left\{ \frac{|A|^{\frac{2}{5}}|B|^\frac{2}{5}|C|^{\frac{2}{5}}}{m^{\frac{2}{5}}}, \frac{|B||C|^{\frac{1}{5}}|A|^{\frac{2}{5}}}{m}, \frac{|B|^\frac{2}{5}|C|^{\frac{2}{5}}|A|^{\frac{1}{5}}}{m^{\frac{2}{5}}} \right\}.$$  

The following are consequences of Theorem 1.3.

**Corollary 1.4.** Let $f(x, y) = g(x)(h(x) + y)$ be a function defined on $\mathbb{F}^* \times \mathbb{F}^*$, where $g, h: \mathbb{F}^* \to \mathbb{F}^*$ are arbitrary functions with $\mu(g \cdot h) = O(1)$. For any subset $A \subseteq \mathbb{F}$ with $|A| \leq p^{5/8}$, we have

$$\max \{|f(A, A)|, |A \cdot A|\} \gg |A|^{\frac{5}{6}}.$$  

**Corollary 1.5.** Consider the subsets $A \subseteq \mathbb{F}$, and $B, C \subseteq \mathbb{F}$ with $|A|, |B|, |C| \leq p^{5/8}$. 

1. By fixing $g(x) = 1$ and $h(x) = x^{-1}$, we get

$$\max \{ |A^{-1} + B|, |B \cdot C| \} \gg \min \left\{ \frac{|A|^{\frac{5}{2}} |B|^{\frac{3}{2}} |C|^{\frac{1}{2}}}{m^{\frac{1}{2}}}, \frac{|B||C|^{\frac{1}{2}}}{m}, \frac{|B||A|^{\frac{1}{2}}}{m}, \frac{|B|^{\frac{3}{2}} |C|^{\frac{1}{2}} |A|^{\frac{1}{2}}}{m^{\frac{1}{2}}} \right\}.$$  

2. By fixing $g(x) = x$ and $h(x) = 1$, we have

$$\max \{ |A(B + 1)|, |B \cdot C| \} \gg \min \left\{ \frac{|A|^{\frac{5}{2}} |B|^{\frac{3}{2}} |C|^{\frac{1}{2}}}{m^{\frac{1}{2}}}, \frac{|B||C|^{\frac{1}{2}}}{m}, \frac{|B||A|^{\frac{1}{2}}}{m}, \frac{|B|^{\frac{3}{2}} |C|^{\frac{1}{2}} |A|^{\frac{1}{2}}}{m^{\frac{1}{2}}} \right\}.$$  

It follows from Corollary 15(2) that if $B = A$ and $C = A + 1$ then we have $|A(A + 1)| \gg |A|^{6/5}$, which recovers the result of Stevens and de Zeeuw [11].

Our next result is the additive version of Theorem 13, which improves Theorem 12.

**Theorem 1.6.** Let $f(x, y) = g(x)(h(x) + y)$ be a function defined on $\mathbb{F}^* \times \mathbb{F}^*$, where $g : \mathbb{F}^* \to \mathbb{F}^*$ are arbitrary functions. Define $m = \mu(g)$. For any subsets $A, B, C \subset \mathbb{F}^*$ with $|A|, |B|, |C| \leq p^{5/8}$, we have

$$\max \{ |f(A, B)|, |A + A| \} \gg |A|^{\frac{5}{8}}.$$  

Let $g(x) = x$ and $h(x) = 1$, we have the following corollary.

**Corollary 1.7.** For $A, B, C \subset \mathbb{F}$ with $|A|, |B|, |C| \leq p^{5/8}$, we have

$$\max \{ |A(B + 1)|, |B + C| \} \gg \min \left\{ \frac{|A|^{\frac{5}{2}} |B|^{\frac{3}{2}} |C|^{\frac{1}{2}}}{m^{\frac{1}{2}}}, \frac{|B||C|^{\frac{1}{2}}}{m}, \frac{|B||A|^{\frac{1}{2}}}{m}, \frac{|B|^{\frac{3}{2}} |C|^{\frac{1}{2}} |A|^{\frac{1}{2}}}{m^{\frac{1}{2}}} \right\}.$$  

By fixing $g(x) = x$ and $h(x) = 0$, we have the following result.

**Corollary 1.8.** For $A, B, C \subset \mathbb{F}$ with $|A|, |B|, |C| \leq p^{5/8}$, we have

$$\max \{ |A(B + 1)|, |B + C| \} \gg \min \left\{ \frac{|A|^{\frac{5}{2}} |B|^{\frac{3}{2}} |C|^{\frac{1}{2}}}{m^{\frac{1}{2}}}, \frac{|B||C|^{\frac{1}{2}}}{m}, \frac{|B||A|^{\frac{1}{2}}}{m}, \frac{|B|^{\frac{3}{2}} |C|^{\frac{1}{2}} |A|^{\frac{1}{2}}}{m^{\frac{1}{2}}} \right\}.$$  

In the case $A = B = C$, we recover the following result due to Roche-Newton, Rudnev, and Shkredov [8], which says that max $\{ |A + A|, |A \cdot A| \} \gg |A|^{6/5}$.

It has been shown in [11] that if $f(x, y) = x(x + y)$, then $|f(A, A)| \gg |A|^{5/4}$ under the condition $|A| \leq p^{2/3}$. In the following theorem, we show that if either $|A + A|$ or $|A \cdot A|$ is sufficiently small, the exponent $5/4$ can be improved from the polynomials to a larger family of functions on $\mathbb{F}^* \times \mathbb{F}^*.$

**Theorem 1.10.** Let $f(x, y) = g(x)(h(x) + y)$ be a function defined on $\mathbb{F}^* \times \mathbb{F}^*$, where $g, h : \mathbb{F}^* \to \mathbb{F}^*$ are arbitrary functions with $\mu(f), \mu(g) = O(1)$. Consider the subset $A \subset \mathbb{F}^*$ with $|A| \leq p^{5/8}$, satisfying

$$\min \{ |A + A|, |A \cdot A| \} \leq |A|^{\frac{5}{4} - \epsilon}$$

for some $\epsilon > 0$. Then, we have

$$|f(A, A)| \gg |A|^{\frac{5}{4} + \frac{\epsilon}{2}}.$$
2 Proofs of Theorems 1.3, 1.6, and 1.10

Let \( \mathcal{R} \) be a set of points in \( \mathbb{F}^3 \) and \( \mathcal{S} \) be a set of planes in \( \mathbb{F}^3 \). We write \( \mathcal{I}(\mathcal{R}, \mathcal{S}) = |\{(r, s) \in \mathcal{R} \times \mathcal{S} : r \in s\}| \) for the number of incidences between \( \mathcal{R} \) and \( \mathcal{S} \). To prove Theorems 1.3 and 1.6, we make use of the following point-plane incidence bound due to Rudnev [10]. A short proof can be found in [12].

**Theorem 2.1 (Rudnev, [10]).** Let \( \mathcal{R} \) be a set of points in \( \mathbb{F}^3 \) and let \( \mathcal{S} \) be a set of planes in \( \mathbb{F}^3 \), with \( |\mathcal{R}| \ll |\mathcal{S}| \) and \( |\mathcal{R}| \ll p^2 \). Assume that there is no line containing \( k \) points of \( \mathcal{R} \). Then

\[
\mathcal{I}(\mathcal{R}, \mathcal{S}) \ll |\mathcal{R}|^{1/2} |\mathcal{S}| + k|\mathcal{S}|.
\]

**Proof of Theorem 1.3.** Define \( f(A, B) = \{f(a, b) : a \in A, b \in B\}, g(A) = \{g(a) : a \in A\}, h(A) = \{h(a) : a \in A\} \). For \( \lambda \in B \cdot C \), let

\[
E_\lambda = \left| \{(f(a, b), c \cdot g(a)^{-1}, c \cdot h(a)) : (a, b, c) \in A \times B \times C, f(a, b) \cdot c \cdot g(a)^{-1} - c \cdot h(a) = \lambda \} \right|,
\]

where by \( g(a)^{-1} \) we mean the multiplicative inverse of \( g(a) \) in \( \mathbb{F}^* \). For a given triple \( (x, y, z) \in (\mathbb{F}^*)^3 \), we count the number of solutions \( (a, b, c) \in A \times B \times C \) to the following system

\[
g(a)(h(a) + b) = x, \ c \cdot g(a)^{-1} = y, \ c \cdot h(a) = z.
\]

This implies that

\[
g(a) h(a) = z y^{-1}.
\]

Since \( \mu(g \cdot h) = m \), there are at most \( m \) different values of \( a \) satisfying the equation \( g(a) h(a) = z y^{-1} \), and \( b, c \) are uniquely determined in term of \( a \) by the first and second equations of the system. This implies that

\[
|A| |B| |C| /m \leq \sum_{\lambda \in B \cdot C} E_\lambda.
\]

By the Cauchy-Schwarz inequality, we get

\[
(|A||B||C|/m)^2 \leq \left( \sum_{\lambda \in B \cdot C} E_\lambda \right)^2 \leq E \cdot |B \cdot C|,
\]

(2)

where \( E = \sum_{\lambda \in B \cdot C} E_\lambda^2 \).

Define the point set \( \mathcal{R} \) as

\[
\mathcal{R} = \{(c \cdot g(a)^{-1}, c \cdot h(a), g(a')(h(a') + b')) : a, a' \in A, b' \in B, c \in C\}
\]

and the set of planes \( \mathcal{S} \) as

\[
\mathcal{S} = \{g(a)(h(a) + b)X - Y - c' g(a')^{-1}Z = -c' \cdot h(a') : a, a' \in A, b \in B, c' \in C\}.
\]

We have \( E \leq I(\mathcal{R}, \mathcal{S}), \) and \( |\mathcal{R}| = |\mathcal{S}| \leq |f(A, B)||A||C| \). To apply Theorem 2.1, we need to find an upper bound on \( k \) which is the maximum number of collinear points in \( \mathcal{R} \). The projection of \( \mathcal{R} \) into the first two coordinates is the set \( \mathcal{T} = \{(c \cdot g(a)^{-1}, c \cdot h(a)) : a \in A, c \in C\} \). The set \( \mathcal{T} \) can be covered by the lines of the form \( y = g(a) h(a) x \) with \( a \in A \). This implies that \( \mathcal{T} \) can be covered by at most \( |A| \) lines passing through the origin, with each line containing \( |C| \) points of \( \mathcal{T} \). Therefore, a line in \( \mathbb{F}^3 \) contains at most \( \max\{|A|, |C|\} \) points of
unless it is vertical, in which case it contains at most \(|f(A, B)|\) points. In other words, we get

\[ k \leq \max\{|A|, |C|, |f(A, B)|\}. \]

If \(|\mathcal{R}| \gg p^2\), then we get \(|f(A, B)||A||C| \gg p^2\). Since \(|A|, |C| \leq p^{5/8}\), we have \(|f(A, B)| \gg p^{3/4} \gg |A|^3/4 |C|^3/4\), and we are done in this case. Thus, we can assume that \(|\mathcal{R}| \ll p^2\).

Applying Theorem 2.1, we obtain

\[ I(\mathcal{R}, \mathcal{S}) \leq |f(A, B)|^{3/2}|A|^{3/2}|C|^{3/2} + k|f(A, B)||A||C|. \] (3)

Putting (2) and (3) together gives us

\[
\max \{|f(A, B)|, |B \cdot C|\} \gg \min \left\{ \frac{|A|^{3/4}|B|^{3/4}|C|^{3/4}}{m^{3/4}}, \frac{|B||C|^{1/2}}{m}, \frac{|B|^3|C|^{1/2}|A|^{1/2}}{m^{3/2}} \right\}.
\]

This completes the proof of the theorem. \(\Box\)

**Proof of Theorem 1.6**  The proof goes in the same direction as Theorem 1.3 but for the sake of completeness, we include the detailed proof. For \(\lambda \in B + C\), let

\[ E_{\lambda} = \left| \left\{ (f(a, b), g(a)^{-1}, c - h(a)) : (a, b, c) \in A \times B \times C, \; f(a, b) \cdot g(a)^{-1} + (c - h(a)) = \lambda \right\} \right|. \]

For a given triple \((x, y, z) \in (\mathbb{F}^*)^3\), we count the number of solutions \((a, b, c) \in A \times B \times C\) to the following system

\[ g(a)(h(a) + b) = x, \; g(a)^{-1} = y, \; c - h(a) = z. \]

Since \(\mu(g) = m\), there are at most \(m\) different values of \(a\) satisfying the equation \(g(a) = y^{-1}\), and \(b, c\) are uniquely determined in term of \(a\) by the first and third equations of the system. This implies that

\[ |A||B||C|/m \leq \sum_{\lambda \in B+C} E_{\lambda}. \]

By the Cauchy-Schwarz inequality, we have

\[ (|A||B||C|/m)^2 \leq \left( \sum_{\lambda \in B+C} E_{\lambda} \right)^2 \leq E \cdot |B + C|, \] (4)

where \(E = \sum_{\lambda \in B+C} E_{\lambda}^2\). Define the point set \(\mathcal{R}\) as

\[ \mathcal{R} = \left\{ (g(a)^{-1}, c - h(a), g(a')(h(a') + b)) : a, a' \in A, b' \in B, c \in C \right\}, \]

and the collection of planes \(\mathcal{S}\) as

\[ \mathcal{S} = \left\{ g(a)(h(a) + b)X + Y - g(a')^{-1}Z = c' - h(a') : a, a' \in A, b \in B, c' \in C \right\}. \]

It is clear that \(|\mathcal{R}| = |\mathcal{S}| \leq |f(A, B)||A||C|\), and \(E \leq I(\mathcal{R}, \mathcal{S})\). To apply Theorem 2.1, we need to find an upper bound on \(k\) which is the maximum number of collinear points in \(\mathcal{R}\). The projection of \(\mathcal{R}\) into the first two coordinates is the set \(\mathcal{T} = \{(g(a)^{-1}, c - h(a)) : a \in A, c \in C\}\). The set \(\mathcal{T}\) can be covered by at most \(|A|\) lines of the form \(x = g(a)^{-1}\) with \(a \in A\), where each line contains \(|C|\) points of \(\mathcal{T}\). Therefore, a line in \(\mathbb{F}^3\) contains at most
max{|A|, |C|} points of $\mathcal{R}$, unless it is vertical, in which case it contains at most |$f(A, B)$| points. So we get
\[
k \leq \max\{|A|, |C|, |f(A, B)|\}.
\]
If $|\mathcal{R}| \gg p^2$, this implies that $|f(A, B)||A||C| \gg p^2$. Since $|A|, |C| \leq p^{5/8}$, we have $|f(A, B)| \gg p^{3/4} \gg |A|^{3/4}|B|^{3/4}|C|^{3/4}$, and we are done. Thus, we can assume that $|\mathcal{R}| \ll p^2$. Applying Theorem 2.1, we obtain
\[
I(\mathcal{R}, \mathcal{S}) \leq |f(A, B)|^{3/2}|A|^{3/2}|C|^{3/2} + k|f(A, B)||A||C|.
\]
(5)
Putting (4) and (5) together gives us
\[
\max \{|f(A, B)|, |B + C|\} \gg \min \left\{ \frac{|A|^{3/4}|B|^{3/4}|C|^{3/4}}{m^{4/5}}, \frac{|B||C|^{1/2}}{m}, \frac{|B||A|^{1/2}}{m}, \frac{|B|^{3/4}|C|^{3/4}|A|^{1/2}}{m^{4/5}} \right\}.
\]
This completes the proof. □

**Proof of Theorem 1.10.** One can assume that $|f(A, A)| \leq |A|^2$, since otherwise we are done. Now by the proofs of Theorems 1.3 and 1.6 for $A \subset \mathbb{F}^*$ with $|A| \leq p^{5/8}$, we have
\[
|f(A, A)|^{3/2}|A \cdot A| \gg |A|^3, \quad |f(A, A)|^{3/2}|A + A| \gg |A|^3.
\]
Since $\min \{|A + A|, |A \cdot A|\} \leq |A|^{3/2} + \varepsilon$, we get $|f(A, A)|^{3/2} \gg |A|^{3 - 3/2 + \varepsilon}$, which concludes the proof of the theorem. □

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