Dusty turbulence

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The feedback forces exerted by particles suspended in a turbulent flow is shown to lead to a new scaling law for velocity fluctuations associated to a power-spectra $\propto k^{-2}$. The mechanism at play relies on a direct transfer of kinetic energy to small scales through Kelvin–Helmholtz instabilities occurring in regions of high particle density contrast. This finding is confirmed by two-dimensional direct numerical simulations.

It is common to face environmental, industrial or astrophysical situations where impurities such as dust, droplets, sediments, and other kinds of colloids are transported by a turbulent fluid. When the suspended particles have finite sizes and masses, they detach from the flow by inertia and form uneven distributions where intricate interactions and collisions take place. The physical processes at play are rather well established, leading to quantitative predictions on the rates at which cloud droplets coalesce, dust accrete to form planets, or heavy sediments settle in a turbulent environment.

Still, basic and important questions remain largely open as to the backward influence of particles on the carrier flow structure and geometry. Some situations involve particle mass loadings so large that the fluid turbulent microscales are altered and, in turn, several macroscopic processes are drastically impacted. These include spray combustion in engines, aerosol salination in dust storms, biomixing by microorganisms in the oceans, and formation of planetesimals by streaming instabilities in circumstellar disks. Currently such systems are unsatisfactorily handled by empirical approaches or specific treatments. A better modelling requires identifying and understanding the universal physical mechanisms at play in turbulence modulation by dispersed particles. In this spirit, we focus here on the alteration of small scales by tiny heavy spherical particles. We show that the fluid velocity is unstable in regions with a high particle density contrast, leading to energy transfers shortcutting the classical turbulent cascade. This effect leads to a novel scaling regime of the turbulent velocity field associated to a power-law spectrum $\propto k^{-2}$.

The fluid velocity field $\mathbf{u}$ solves the incompressible Navier–Stokes equations: $\nabla \cdot \mathbf{u} = 0$ with

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho_t} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}_{\text{ext}} + \mathbf{f}_{p\rightarrow f}.$$  

(1)

$\rho_t$ is here the fluid mass density and $\nu$ its kinematic viscosity. A homogeneous isotropic turbulence is maintained in a statistical steady state by an external forcing $\mathbf{f}_{\text{ext}}$. The fluid flow is perturbed by a monodisperse population of small solid particles whose effects are entailed in the force $\mathbf{f}_{p\rightarrow f}$. These particles are assumed sufficiently small, dilute and heavy for approximating their distribution and dynamics in terms of fields, namely a mass density $\rho_p$ and a particle velocity field $\mathbf{v}_p$ satisfying

$$\partial_t \rho_p + \nabla \cdot (\rho_p \mathbf{v}_p) = 0$$  

(2)

$$\partial_t \mathbf{v}_p + (\mathbf{v}_p \cdot \nabla) \mathbf{v}_p = -\frac{1}{\tau_p} (\mathbf{v}_p - \mathbf{u}),$$  

(3)

where $\tau_p = 2\rho_p a^2/(9\rho_t \nu)$ is the particles response time, $a$ being their radius and $\rho_p$ the mass density of the material constituting the particles. The hydrodynamical system (2)-(3) has proven to be a valid approximation for relatively small Stokes numbers $St = \tau_p/\tau_c$, that is when the particle response time is smaller than the smallest active timescale $\tau_c$ of the fluid flow. In this limit, fold caustics appear with an exponentially small probability, preventing the development of multivalued branches in the particle velocity profile and thus ensuring the validity of a hydrodynamical description.

The force exerted by the particles on the fluid reads

$$\mathbf{f}_{p\rightarrow f} = \frac{1}{\tau_p} \frac{\rho_p}{\rho_t} (\mathbf{v}_p - \mathbf{u}).$$  

(4)

It is proportional to the mass density of the dispersed phase and thus combines the heaviness of the particles with their number density. The strength of feedback is measured by the comprehensive non-dimensional parameter $\Phi = \langle \rho_p \rangle / \rho_t$. It involves the particle density spatial average $\langle \rho_p \rangle = N_p m_p / V$, where $N_p$ is the total number of particles, $m_p$ their individual mass, and $V$ the volume of the domain. All these quantities being conserved by the dynamics, so is the coupling parameter $\Phi$.

We first draw some straightforward comments pertaining to the limit of small Stokes numbers. There, particles almost follow the flow with a tiny compressible correction, namely $\mathbf{v}_p \approx \mathbf{u} - \tau_p \mathbf{a}$, where $\mathbf{a} = \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}$ denotes the fluid flow acceleration field. The feedback force exerted on the fluid is hence, to leading order,

$$\mathbf{f}_{p\rightarrow f}(x,t) \approx -\frac{1}{\rho_t} \rho_p(x,t) \mathbf{a}(x,t).$$  

(5)

The effect of particles can thus be seen as an added mass, which does not depend upon their response time and is responsible for an increase of the fluid inertia. The fluid is accelerated as if it has an added density equal to that of the particles. Such considerations predict that the presence of particles decreases the effective kinematic viscosity of the fluid and thus increases its level of turbulence.
This vision is however too naive as it overlooks the spatial fluctuations of the particle density. It is indeed known that an even infinitesimal inertia of the particles creates extremely violent gradients of their density through the mechanism of preferential concentration. As we will now see, these variations are responsible for instabilities that shortcut the turbulent energy cascade by directly transferring kinetic energy to the smallest turbulent scales.

A key attribute of turbulence is the vigorous local spinning of the fluid flow, weighed by the vorticity \( \omega = \nabla \times \mathbf{u} \). The effect of particles on the vorticity dynamics is entailed in the curl of the feedback force (1), reading

\[
\nabla \times \mathbf{f}_{p\rightarrow t} = \frac{1}{\rho_p \tau_p} \left[ \rho_p \left( \omega_p - \omega \right) + \nabla \rho_p \times (\mathbf{v}_p - \mathbf{u}) \right],
\]

where \( \omega_p = \nabla \times \mathbf{v}_p \) is the vorticity of the dispersed phase. The action of particles is thus twofold. The first term accounts for a friction of the fluid vorticity with that of the particles, which amounts at small Stokes numbers to the above-mentioned added-mass effect. The second term gives a source of vorticity proportional to the gradients of particle density. The combined effects of preferential concentration and turbulent mixing is responsible for very sharp spatial variations of \( \rho_p \). Centrifugal forces indeed eject heavy particles from coherent vortical structures and Lagrangian transport stretches particles patches in stirring regions. This leads to the development of substantial fluctuations of \( \nabla \rho_p \) as illustrated in two dimensions on the left panels of Fig. 1. This mechanism creates regions with very strong shear in the fluid flow, which, in turn, develop small-scale vortical structures through Kelvin–Helmholtz instability. It is indeed well known that flows presenting a quasi-discontinuity of velocity are linearly unstable and develop wavy vortical streaks at the interface of the two motions (see, e.g., [13]). Such phenomenological arguments thus suggest that the feedback of particles lead to the formation of small-scale eddies, as can be seen in the right panels of Fig. 1. Particles thus actively participate in the transfer of kinetic energy toward the smallest turbulent scales.

This effect and the resulting modification of the fluid flow scaling properties can be quantified by examining the scale-by-scale kinetic energy budget given by Kármán–Howarth–Monin relation (see, e.g., [16]). Denoting the velocity increment over a separation \( \mathbf{r} \) by \( \delta, \mathbf{u} = \mathbf{u}' - \mathbf{u} \) with \( \mathbf{u}' = \mathbf{u}(\mathbf{x} + \mathbf{r}, t) \) and \( \mathbf{u} = \mathbf{u}(\mathbf{x}, t) \), one can easily check that statistically homogeneous solutions to the Navier–Stokes equation [1] satisfy

\[
\frac{1}{2} \partial_t \langle \mathbf{u} \cdot \mathbf{u}' \rangle = \frac{1}{4} \nabla_r \cdot \left[ \langle \delta, \mathbf{u} \rangle^2 \delta, \mathbf{u} \rangle + \nu \nabla_r^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle \right] + \langle \mathbf{u} \cdot \mathbf{f}_{\text{ext}} \rangle + \langle \mathbf{f}_{p\rightarrow t} \cdot \mathbf{u} \rangle,
\]

where the overbar denotes the average over the two points located at \( \mathbf{x} \pm \mathbf{r} \), that is \( \mathbf{f} = [\mathbf{f}(\mathbf{x} + \mathbf{r}, t) + \mathbf{f}(\mathbf{x} - \mathbf{r}, t)]/2 \). In classical stationary turbulence, the above relation suggests a balance between the non-linear transfer term (first term on the right-hand side) and viscous dissipation (second term), leading for isotropic flows to the celebrated Kolmogorov 4/5 law. In the presence of coupling with particles, this equilibrium is broken by the feedback force. In the asymptotics \( St \ll 1 \) of low inertia, this force is approximated by (5), so that its contribution to (7) reads

\[
\langle \mathbf{f}_{p\rightarrow t} \cdot \mathbf{u} \rangle \approx \frac{1}{\rho_t} \langle \rho_p \mathbf{a} \cdot \mathbf{u} \rangle - \frac{1}{\rho_t} \langle \rho_p \mathbf{a} \cdot \delta, \mathbf{u} \rangle.
\]

The first term on the right-hand side involves the correlation between the particle density field and the instantaneous power acting on fluid elements. To leading order when \( St \to 0 \), we have \( \langle \rho_p \mathbf{a} \cdot \mathbf{u} \rangle \approx \langle \rho_p \rangle \langle \mathbf{a} \cdot \mathbf{u} \rangle = 0 \). Non-vanishing corrections at small but finite Stokes numbers might arise from a combined effect of the small compressibility of the particle velocity together with the biased sampling due to preferential concentration, as already seen for the radial distribution function [17, 18]. However such correlations are in the best case of the order of \( St^2 \). The second term on the right-hand side of (8) does not vanish in the limit \( St \to 0 \) and thus gives the dominant contribution.

Such arguments lead to predict that the scale-by-scale
energy balance \[7\] reduces in the inertial range to
\[
\frac{1}{4} \nabla_r \cdot \left( |\delta_r u|^2 \delta_r u \right) \simeq \frac{1}{\rho_f} \left( \rho_p a \cdot \delta_r u \right). \tag{9}
\]

Now, assuming that the fluid velocity field obeys some scaling property \( \delta_r u \sim r^h \), one deduces from the above balance that \( 3h - 1 = h \), and thus \( h = 1/2 \). Such a scaling behavior is associated to an angle-averaged kinetic energy power spectrum \( E(k) \propto k^{-2} \).

In order to test such prediction, we perform two-dimensional simulations of the fluid-particle system defined by \([1],[2],[3],[4]\) and \([5]\) in a periodic domain. We make use of a Fourier-spectral solver with 1024\(^2\) collocation points for estimating spatial derivatives and of a second-order Runge-Kutta scheme for time marching. We focus on the direct enstrophy cascade, so that the external forcing \( f_{\text{ext}} \) is the sum of an Ekman friction with timescale \( 1/\alpha \) and of a random Gaussian field \( \eta \) white noise in time and concentrated at wavenumbers \( |k| \leq 2 \). We make use of hyper-viscosity and hyper-diffusivity (fourth power of the Laplacian) in order to maximize the extent of the inertial range and prevent Eqs. \([2]\) and \([3]\) from blowing up. The particle response time is fixed in such a way that \( St = \tau_p \langle \omega^2 \rangle^{1/2} \approx 10^{-2} \) in the uncoupled case and various values of the coupling parameter \( \Phi = 0, 0.1, 0.2, \) and 0.4 are simulated.

Figure 1 shows snapshots of the fluid vorticity field together with the particle density field, without and with coupling between the two phases. In the absence of feedback from the particles (left panels), the flow develops the traditional picture of two-dimensional direct cascade consisting of large-scale vortices separated by a bath of filamentary structures where enstrophy is dissipated. The particles density field is characterized by large voids in the vortical structures separated by a filamentary distribution that is symptomatic of turbulent mixing. These qualitative pictures are strongly altered when the particle feedback is turned on. In the presence of coupling (right panels), the fluid flow still shows large-scale structures but which are this time surrounded by a bath of small-scale vortices. These eddies form wavy structures along the lines associated to quasi-discontinuities of the particle density field. This is a clear signature that Kelvin–Helmholtz instability is at play.

Figure 2 shows the angle-averaged power spectra of the fluid kinetic energy obtained when varying the coupling parameter. In the case of no feedback (\( \Phi = 0 \)), the specific choices of the Ekman coefficient \( \alpha \) and of the energy injection amplitude yield a kinetic energy spectrum \( E(k) \propto k^{-\delta} \) with \( \delta \approx 3.3 \). For any non-vanishing value of the coupling parameter \( \Phi \), one observes remarkable changes in the spectral behavior of the fluid velocity. The first effect is a clear decrease of the total kinetic energy. Similarly to what is obtained in the asymptotic of large Stokes numbers \([19]\), this is due to a net dissipative effect of the coupling with the particle phase. However this im-

\[\text{FIG. 2. (color online) Angle-averaged kinetic energy power spectra of the fluid velocity represented for various values of the coupling parameter \( \Phi \), as labelled.}\]

\[\text{FIG. 3. (color online) Angle-averaged Fourier amplitudes of the various terms contributing to the kinetic energy budget shown here for \( \Phi = 0.4 \). Coupling stands for the contribution of the forces exerted by the particles on the fluid, transfer for the nonlinear advection terms, dissipation for viscous forces and friction for Ekman damping.}\]
Further insight is given by measuring the amplitude of the various terms entering in the energy budget \([7]\). Figure 3 shows the angle-averaged amplitude of their Fourier transforms with respect to the separation \(r\). One observes that the non-linear transfer term gives a positive contribution at small wavenumbers. This is a strong signature of two-dimensional turbulence for which, conversely to three dimensions, the nonlinear terms are not transferring kinetic energy toward small scales participating to its accumulation at largest lengthscales of the flow. This term is exactly compensated by the linear Ekman friction and the coupling with the particle phase which are both negative and of the same order. Coupling is thus pumping energy at large scales but restitutes it at larger wavenumbers as it is positive for \(k \geq 4\). In the inertial range for \(10 \lesssim k \lesssim 100\) where both the contribution of Ekman friction and viscous dissipation are negligible, it is exactly compensated by a negative value of the nonlinear transfer term. Both curves decrease as \(k^{-1}\), in agreement with the scaling observed earlier. At the smallest scales, coupling becomes negligible, nonlinear transfer changes sign and is compensated by viscous dissipation. The whole two-dimensional picture thus confirms the prediction made above.

We have thus evidenced from this work a new regime of turbulent flow where the feedback of suspended particles onto the fluid flow dominates inertial-range energy transfers. This regime is evidenced by numerical simulations in two dimensions but such strong effects should also be present in three dimensions, at least at sufficiently small scales. A remarkable feature of this turbulent enhancement due to dust-like particles is the creation of small-scale eddies whose spectral signature is a \(k^{-2}\) power-law range for the fluid velocity. These vortices profoundly affect particle concentration. On the one hand, their spatial distribution tends to weaken large-scale inhomogeneities, to reduce potential barriers to transport and enhance mixing. On the other hand, the dispersion in the flow and the interactions between these long-living structures trigger density fluctuations that are much more intense than in the absence of coupling between the two phases. Such effects clearly need being investigated in a more systematic manner: They might indeed strongly modify at both qualitative and quantitative levels the rate at which particles interact together.

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