RIESZ-TYPE REPRESENTATION FORMULAS FOR SUBHARMONIC FUNCTIONS IN SUB-RIEMANNIAN SETTINGS

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Abstract. In this paper we use a potential-theoretic approach to establish various representation theorems and Poisson-Jensen-type formulas for subharmonic functions in sub-Riemannian settings. We also characterize the Radon measures in \( \mathbb{R}^N \) which are the Riesz-measures of bounded-above subharmonic functions in the whole space \( \mathbb{R}^N \).

1. Introduction. Throughout this note, we shall be concerned with second-order partial differential operators (PDOs, in short) of the following divergence form

\[
\mathcal{L} = \text{div}(A(x) \nabla) = \sum_{i=1}^{N} \left( \sum_{j=1}^{N} a_{ij}(x) \partial x_j \right) \quad \text{for } x \in \mathbb{R}^N,
\]

where \( A(x) := (a_{ij})_{i,j=1,...,N} \) is a symmetric and positive semi-definite matrix whose entries are \( C^\infty \) functions on \( \mathbb{R}^N \). Under suitable structural assumptions on \( \mathcal{L} \) (see, precisely, Section 2), we shall exploit a potential-theoretic approach in order to establish several representation theorems and Poisson-Jensen-type formulas for general \( \mathcal{H}_L \)-subharmonic functions.

While we refer to the Appendix for the relevant definitions and notation of abstract Potential Theory, here we review the following basic notions:

(a) a function \( u \in C^\infty(\Omega) \) is \( \mathcal{H}_L \)-harmonic in \( \Omega \) if \( \mathcal{L}u = 0 \) pointwise in \( \Omega \);

(b) an u.s.c. function \( u : \Omega \to [-\infty, \infty) \) is \( \mathcal{H}_L \)-subharmonic in \( \Omega \) if, for any open set \( V \subseteq \Omega \) and any \( h \in C(\overline{V}) \) such that \( h|_V \) is \( \mathcal{H}_L \)-harmonic, one has

\[
u \leq h \text{ on } \partial V \implies u \leq h \text{ throughout } V.
\]

We denote by \( \mathcal{H}_L(\Omega) \) (resp. \( \mathcal{S}(\Omega) \)) the set of the \( \mathcal{H}_L \)-harmonic (resp. \( \mathcal{H}_L \)-subharmonic) functions in \( \Omega \); moreover, we define

\[
\mathcal{S}(\Omega) := -\mathcal{S}(\Omega).
\]
The Riesz decomposition theorem has its roots in the classical Potential Theory for the Laplace operator; in particular, it relies on some selected properties of the Green function \(G\), and on the general theory of the Green potentials.

Assuming that \(L\) is \(C^\infty\)-hypoelliptic and that it possesses a ‘well-behaved’ global fundamental solution for \(L\) (see Section 2 for the relevant definitions), we shall show that it is possible to develop a satisfactory Potential Theory for \(L\), and to construct a ‘sufficiently rich’ theory of the \(L\)-Green potentials; thanks to these facts, we can prove the following theorem, which is one of the main results of the paper.

\[ \mathbf{Theorem 1.1} \] (The Riesz representation). Let \(\Omega \subseteq \mathbb{R}^N\) be an arbitrary open set, and let \(u \in \mathcal{S}(\Omega)\). Then, the following conditions are equivalent:

(i) \(G_\Omega \ast \mu_u \in \mathcal{S}(\Omega)\) and there exists \(h \in \mathcal{H}_L(\Omega)\) such that

\[
    u = h - G_\Omega \ast \mu_u \quad \text{pointwise in} \; \Omega, \tag{1.1}
\]

(ii) there exists \(v \in \mathcal{S}(\Omega)\) such that \(v \geq u\) in \(\Omega\),

(iii) every connected component of \(\Omega\) contains a point \(x_0\) such that

\[
    (G_\Omega \ast \mu)(x_0) < \infty.
\]

Furthermore, if one of the above (equivalent) conditions is satisfied, the function \(h\) appearing in (1.1) is the least harmonic majorant of \(u\) in \(\Omega\), i.e.,

\[
    h = \inf \left\{ f \in \mathcal{H}_L(\Omega) : f \geq u \text{ in } \Omega \right\}.
\]

With reference to the statement of Theorem 1.1, \(G_\Omega\) is the \(L\)-Green function related to \(\Omega\) and \(\mu_u\) is the \(L\)-Riesz measure of \(u\), that is, \(\mu_u\) is the unique Radon measure on \(\Omega\) such that \(\Delta u = \mu_u\) in \(\mathcal{D}'(\Omega)\) (see Section 3).

The Riesz-type Theorem 1.1 has some important consequences: first of all, it allows to derive a global representation theorem for bounded-above \(\mathcal{H}_L\)-subharmonic functions in the whole space \(\mathbb{R}^N\), see Corollary 4.7; moreover, it is the key ingredient to establish the following Poisson-Jensen-type formula for \(L\) (see the Appendix for the notion of \(\mathcal{H}_L\)-regular measure).

\[ \mathbf{Theorem 1.2}. \] Let \(U \subseteq \mathbb{R}^N\) be an arbitrary open set, and let \(\Omega \subseteq U\) be \(\mathcal{H}_L\)-regular. Moreover, let \(u \in \mathcal{S}(U)\). Then, we have the representation formula

\[
    u(x) = \int_{\partial \Omega} u \, d\mu^\Omega_z - (G_\Omega \ast \mu_u)(x) \quad \text{pointwise in} \; \Omega,
\]

where \(G_\Omega\) is the \(L\)-Green function of \(\Omega\) and \(\mu^\Omega_z\) is the \(\mathcal{H}_L\)-harmonic measure related with \((\text{the \(\mathcal{H}_L\)-regular open set}) \; \Omega\) and \(x\).

By making use of the Poisson-Jensen-type formula in Theorem 1.2, we can provide a full characterization of those Radon measures \(\mu \in \mathbb{R}^N\) for which there exists
$u \in S(\mathbb{R}^N)$ such that $\mu_u = \mu$. This characterization involves the super-level sets of the global fundamental solution $\Gamma$ for $\mathcal{L}$, i.e.,

$$\Omega_r(x) := \{y \in \mathbb{R}^N \setminus \{x\} : \Gamma(x, y) > 1/r\} \cup \{x\}.$$ 

**Theorem 1.3 (L-Riesz measure of a bounded-above $u \in S(\mathbb{R}^N)$).** Let $\mu$ be a Radon measure in $\mathbb{R}^N$, and let $x_0 \in \mathbb{R}^N$. Then, the following conditions are equivalent:

1. there exists a function $u \in S(\mathbb{R}^N)$ such that
   (i) $u(x_0) > -\infty$;
   (ii) $u$ is bounded-above in $\mathbb{R}^N$, i.e., $U := \sup_{\mathbb{R}^N} u < \infty$;
   (iii) $\mu_u = \mu$ in $\mathbb{R}^N$;
2. $\mu$ satisfies the integrability property

$$\int_0^\infty \frac{\mu(\Omega_{\rho}(x_0))}{\rho^2} d\rho < \infty. \tag{1.2}$$

It is worth mentioning that Theorem 1.3 is exploited in [13] in order to prove the validity of a Weak Maximum Principle on unbounded domains for $\mathcal{L}$. Moreover, Theorems 1.1-1.2 extend and generalize some results contained in [2].

As already pointed out, Theorems 1.1-to-1.3 are obtained after having developed a good Potential Theory for $\mathcal{L}$, which is based on two global assumptions:

1. $\mathcal{L}$ is $C^\infty$-hypoelliptic in $\mathbb{R}^N$;
2. there exists a ‘well-behaved’ global fundamental solution $\Gamma$ for $\mathcal{L}$.

Hence, the approach we follow in this paper is somehow axiomatic: whenever $\mathcal{L}$ is a divergence-form operator satisfying (1)-(2), we can construct a Potential Theory for $\mathcal{L}$, and Theorems 1.1-to-1.3 hold true. While, in general, the validity of assumptions (1)-(2) for a general PDO could be a serious issue, there are some meaningful classes of operators which satisfy these ‘structural assumptions’:

(i) the sub-Laplace operators on (real) Carnot groups;
(ii) the homogeneous Hörmander sums of squares.

The above classes of PDOs are presented in detail in Example 2.3; here, however, we spend a few words about the validity of (1)-(2) for these operators.

As regards the sub-Laplace operators on Carnot groups (which are extensively studied in the monograph [19]), the validity of assumption (1) is a consequence of the celebrated Hörmander’s hypoellipticity theorem [33], while the validity of assumption (2) is proved in the seminal paper by Folland [31]. Starting from Folland’s result, a very rich global theory has been developed for this kind of operators; in particular, in [19, Chap.9] it is constructed a good Potential Theory, and the analog of Theorems 1.1-to-1.3 are established. We also mention the papers [15, 16] for other representation formulas in the context of Carnot groups.

As regards the homogeneous Hörmander sums of squares, instead, the validity of assumption (1) follows again from Hörmander’s theorem, while the validity of assumption (2) is proved in [6, 10]. Starting from the results in [6], a global theory for this kind of homogeneous operators, has been developed in [5, 8, 9, 11, 12].

An ‘axiomatic’ approach to representation theorems has been exploited also in the recent paper [25]: in this paper, the Authors assume the existence of ‘well-behaved’ global fundamental solution for $\mathcal{L}$ and prove necessary and sufficient conditions for a $C^2$-solution of $-\mathcal{L}u \geq 0$ in the whole space $\mathbb{R}^N$ to satisfy various representation formulas (different from ours).

As pointed out in the introduction of [25], representation formulas for $\mathcal{H}_\mathcal{L}$-subharmonic functions can be a useful tool to investigate Liouville-type theorems and
positive-preserving properties for semilinear/quasilinear equations driven by $\mathcal{L}$; in this direction, we highlight the papers [23, 26, 27, 28, 29, 34]; see also [14].

Plan of the paper. A short plan of the paper is now in order.

- In Section 2 we properly introduce the main assumptions and notation concerning our operators $\mathcal{L}$; we also review some potential-theoretic results which will be fundamental in our approach.
- In Section 3 we use the results in Section 2 to study the $\mathcal{L}$-Green function associated with an arbitrary open set, and we develop a general theory of the $\mathcal{L}$-Green potentials of Radon measures.
- In Section 4 we prove Theorems 1.1-1.2, together with some other (local and global) representation formulas for $\mathcal{L}$.
- In Section 5 we prove Theorem 1.3.
- Finally, to make the paper as self-contained as possible, we collect in Appendix A all the definitions and the results of abstract Potential Theory which have been used throughout Sections 2-to-5.

2. Assumptions, notation and preliminary results. The aim of this section is to collect all the assumptions, notation and preliminary results which will be exploited in the rest of the paper.

2.1. The operator $\mathcal{L}$. Throughout this paper, we shall be concerned with general second-order differential operators $\mathcal{L}$ taking the divergence form

$$
\mathcal{L} = \sum_{i=1}^{N} \partial_{x_i} \left( \sum_{j=1}^{N} a_{ij}(x) \partial_{x_j} \right) \quad \text{in } \mathbb{R}^N,
$$

(2.1)

and satisfying the following structural assumptions:

(S): $\mathcal{L}$ has smooth coefficients, that is, $a_{ij} \in C^\infty(\mathbb{R}^N)$ for every $i, j \in \{1, \ldots, N\}$;

(DE): $\mathcal{L}$ is degenerate-elliptic, that is, $A(x) = (a_{ij}(x))_{i,j} \geq 0$ for every $x \in \mathbb{R}^N$;

(NTD): $\mathcal{L}$ is non-totally degenerate, that is, $\exists i \in \{1, \ldots, N\}$ such that $\inf_{\mathbb{R}^N} a_{ii} > 0$;

(HY): there exists some $\eta > 0$ such that both $\mathcal{L}$ and $\mathcal{L}_\eta := \mathcal{L} - \eta$ are $C^\infty$-hypelliptic differential operators in $\mathbb{R}^N$;

(FS): $\mathcal{L}$ is endowed with a well-behaved global fundamental solution: this means, precisely, that there exists a function

$$
\Gamma : \mathcal{O} := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\} \to \mathbb{R}
$$

satisfying the properties listed below:

1. $\Gamma \in L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N)$ and, for every fixed $x \in \mathbb{R}^N$, one has

$$
\int_{\mathbb{R}^N} \Gamma(x, y) \mathcal{L}\varphi(y) \, dy = -\varphi(x) \quad \forall \varphi \in C^\infty_c(\mathbb{R}^N);
$$

(2.2)

2. $\Gamma \in C^2(\mathcal{O})$ and $\Gamma > 0$ on $\mathcal{O}$;

3. $\Gamma(x, y) = \Gamma(y, x)$ for every $x, y \in \mathcal{O}$;
4. for every fixed $x \in \mathbb{R}^N$, one has
\[
\lim_{|y| \to \infty} \Gamma(x, y) = 0 \quad \text{and} \quad \lim_{y \to x} \Gamma(x, y) = \infty.
\] (2.3)
Following, e.g., Treves [37], we remind that a linear PDO $P$ on $\mathbb{R}^N$ is $C^\infty$-hypoelliptic on an open set $\Omega \subseteq \mathbb{R}^N$ if, for every distribution $u \in \mathcal{D}'(\Omega)$, every open set $U \subseteq \Omega$ and every $f \in C^\infty(U)$, the equation
\[
Pu = f \quad \text{in} \quad \mathcal{D}'(U)
\]
implies that $u$ is a function-type distribution associated with a smooth function on $U$. The problem of establishing, in general, whether the $C^\infty$-hypoellipticity of $\mathcal{L}$ implies that of $\mathcal{L} - \eta$ (for a certain $\eta > 0$) seems non-trivial; for example, in the complex-coefficient case, the presence of a zero-order term can drastically alter hypoellipticity. We refer to [35] for an investigation on this topic.

**Remark 2.1.** We explicitly highlight that the $C^\infty$-hypoellipticity of $\mathcal{L}_\eta = \mathcal{L} - \eta$ is crucially exploited in [3] in order to establish a homogeneous non-invariant Harnack inequality for $\mathcal{L}$. As we will see, this result ensures that $\mathcal{L}$ endows $\mathbb{R}^N$ with a structure of harmonic space, in which the Harnack Axiom holds.

In the particular case when $\mathcal{L} = \sum_{i=1}^m X_i^2$ is a Hörmander sums of squares, the hypoellipticity of $\mathcal{L}_\eta$ is ensured by the hypoellipticity of $\mathcal{L}$, via the celebrated Hörmander Hypoellipticity Theorem [33].

**Remark 2.2.** In this remark we list, for a future reference, some standard consequences of assumptions (S)-to-(FS) which will be exploited in the sequel.

(a) $\mathcal{L}$ satisfies the Weak Maximum Principle (WMP, in short) on every bounded open set $\Omega \subseteq \mathbb{R}^N$ (see, e.g., [7, Prop. 8.22]): more precisely, we have
\[
\begin{cases}
  u \in C^2(\Omega) \\
  \mathcal{L}u \geq 0 \quad \text{in} \quad \Omega \\
  \limsup_{x \to \xi} u(x) \leq 0 \quad \text{for every} \quad \xi \in \partial \Omega
\end{cases}
\implies u \leq 0 \text{ in } \Omega.
\]

(b) As a consequence of (a), the operator $\mathcal{L}$ also satisfies the following WMP on the whole space $\mathbb{R}^N$ (see, e.g., [7, Prop. 8.18]):
\[
\begin{cases}
  u \in C^2(\mathbb{R}^N) \\
  \mathcal{L}u \geq 0 \quad \text{in} \quad \mathbb{R}^N \\
  \limsup_{|x| \to \infty} u(x) \leq 0
\end{cases}
\implies u \leq 0 \text{ in } \mathbb{R}^N.
\]

(c) If $\Gamma$ is a well-behaved global fundamental solution for $\mathcal{L}$ and $x \in \mathbb{R}^N$ is arbitrarily fixed, it follows from (2.2) that
\[
\mathcal{L}\Gamma(x, \cdot) = -\text{Dir}_x \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N),
\]
where $\text{Dir}_x$ is the Dirac measure at $\{x\}$. In particular, assumption (HY) and the regularity of $\Gamma$ on $\mathcal{O}$ measure at $\{x\}$ and the regularity of $\Gamma$ on $\mathcal{O}$ measure at $\{x\}$ imply that $\Gamma(x, \cdot) \in C^\infty(\mathbb{R}^N \setminus \{x\})$, and
\[
\mathcal{L}\Gamma(x, \cdot) = 0 \text{ pointwise in } \mathbb{R}^N \setminus \{x\}.
\] (2.4)

(d) The fundamental solution for $\mathcal{L}$ is unique. In fact, let $\Gamma, \Gamma' : \mathcal{O} \to \mathbb{R}$ be two functions on $\mathcal{O}$ satisfying (1)-to-(4) in assumption (FS), and let $x \in \mathbb{R}^N$ be fixed. On account of (c), it is readily seen that the map
\[
h_x := \Gamma(x, \cdot) - \Gamma'(x, \cdot)
\]
satisfies $\mathcal{L}h_x = 0$ in $\mathcal{D}'(\mathbb{R}^N)$; thus, using assumption (HY) and the properties of $\Gamma$ and $\Gamma'$, we get the existence of $\hat{h}_x \in C^\infty(\mathbb{R}^N)$ such that

(i) $\mathcal{L}\hat{h}_x = 0$ in $\mathbb{R}^N$ and $\hat{h}_x(y) \to 0$ as $|y| \to \infty$;

(ii) $\hat{h}_x \equiv h_x$ pointwise in $\mathbb{R}^N \setminus \{x\}$.

Now, by (i) and the WMP on unbounded domains we deduce that $\hat{h}_x \equiv 0$ on the whole of $\mathbb{R}^N$; as a consequence, from (ii) we conclude that

$$\Gamma(x, \cdot) \equiv \Gamma'(x, \cdot) \text{ on } \mathbb{R}^N \setminus \{x\}.$$  

Due to their relevance in the subsequent sections, we introduce a notation for the super-level sets of $\Gamma$: for every fixed $x \in \mathbb{R}^N$ and every $r > 0$, we define

$$\Omega_r(x) := \left\{ y \in \mathbb{R}^N \setminus \{x\} : \Gamma(x, y) > \frac{1}{r} \right\} \cup \{x\}.$$  \hspace{1cm} (2.5)

Using (2.3) and the fact that $\Gamma$ is continuous out of the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$, it is easy to recognize that $\Omega_r(x)$ is a bounded open neighborhood of $x$; moreover,

$$\bigcup_{r > 0} \Omega_r(x) = \mathbb{R}^N \quad \text{and} \quad \bigcap_{r > 0} \Omega_r(x) = \{x\}. \hspace{1cm} (2.6)$$

Finally, since $\Gamma(x, \cdot) \in C^\infty(\mathbb{R}^N \setminus \{x\})$, the Sard Lemma ensures that

$$\partial \Omega_r(x) = \left\{ y \in \mathbb{R}^N \setminus \{x\} : \Gamma(x, y) = \frac{1}{r} \right\}$$

is a smooth $(N - 1)$-dimensional manifold for a.e. $r > 0$.

We now provide concrete examples of operators $\mathcal{L}$ satisfying (S)-to-(FS).

Example 2.3. (1) The classical Laplace operator $\mathcal{L} = \Delta$ in $\mathbb{R}^N$ obviously satisfies all the assumptions (S)-to-(HY); moreover, if $N \geq 3$, $\mathcal{L}$ fulfills also assumption (FS), and the global fundamental solution $\Gamma$ for $\mathcal{L}$ is given by

$$\Gamma(x, y) = \frac{1}{N(N - 2)\omega_N} |x - y|^{2-N},$$

where $\omega_N > 0$ is the volume of the unit ball $B(0, 1) \subseteq \mathbb{R}^N$.

(2) Let $G = (\mathbb{R}^N, \ast, D_\lambda)$ be a homogeneous Carnot group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of the (smooth) left-invariant vector fields on $G$ (see, e.g., [19, Chap. 1] for the relevant definitions). It is well-known that $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \bigoplus_{k=1}^r \mathfrak{g}_k, \quad \text{where} \quad \mathfrak{g}_k = [\mathfrak{g}_1, \mathfrak{g}_{k-1}] \quad \text{for} \quad k = 2, \ldots, r,$$

$$[\mathfrak{g}_1, \mathfrak{g}_r] = \{0\}.$$  

Then, if $Z = \{Z_1, \ldots, Z_m\}$ is a linear basis of $\mathfrak{g}_1$ (the first layer of $\mathfrak{g}$), the second-order differential operator (usually referred to as a sub-Laplacian on $G$)

$$\Delta_Z = Z_1^2 + \cdots + Z_m^2$$

satisfies assumptions (S)-to-(FS). In fact, the validity of assumptions (S)-to-(HY) is proved in [19, Sec. 1.5]; moreover, a notable result by Folland [31] shows that $\Delta_Z$ possesses a well-behaved global fundamental solution (see also [19, Chap. 5]).

(3) More generally, let $X = \{X_1, \ldots, X_m\}$ be a family of linearly independent smooth vector fields in $\mathbb{R}^N$ satisfying the following two properties:

(H1) $X_1, \ldots, X_m$ satisfy Hörmander’s rank condition at $x = 0$, i.e.,

$$\dim\{Y(0) : Y \in \text{Lie}\{X_1, \ldots, X_m\}\} = N;$$
(H2) $X_1, \ldots, X_m$ are homogeneous of degree 1 with respect to a family of non-isotropic dilations $\{\delta_\lambda\}_{\lambda > 0}$ in $\mathbb{R}^N$ of the form

$$\delta_\lambda(x) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N),$$

where $\sigma_1, \ldots, \sigma_N \in \mathbb{N}$ and $1 = \sigma_1 \leq \ldots \leq \sigma_N$.

Then, the second-order differential operator

$$L_X := X_1^2 + \cdots + X_m^2$$

satisfies assumptions (S)-to-(FS). In fact, the validity of assumptions (S)-to-(HY) is proved in [10, Sec. 7] as a direct consequence of (H1)-(H2); as for assumption (FS), it follows from the results contained in [6, 10].

2.2. The $\mathcal{L}$-harmonic space. By using assumptions (S)-to-(FS), we now turn to develop a satisfactory Potential Theory for $L$. Throughout what follows, we tacitly exploit all the definitions and notations collected in the Appendix.

To begin with, if $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ is a fixed open set, we define

$$\mathcal{H}_L(\Omega) := \{ u \in C^\infty(\Omega) : Lu = 0 \text{ in } \Omega \}.$$

We then observe that, since $L$ is linear, the map $\Omega \mapsto \mathcal{H}_L(\Omega)$ is a sheaf of functions on $\mathbb{R}^N$; moreover, from [3, Lem. 1.7 and Thm. 1.10] it easily follows that

(i) there exists a basis $\mathcal{B}_L$ for the Euclidean topology of $\mathbb{R}^N$ made of $\mathcal{H}_L$-regular open sets, and thus $(\mathbb{R}^N, \mathcal{H}_L)$ satisfies the Regularity Axiom;

(ii) $(\mathbb{R}^N, \mathcal{H}_L)$ satisfies the Harnack axiom.

As a consequence, also the Brelot Convergence Axiom is satisfied, and thus

$$(\mathbb{R}^N, \mathcal{H}_L)$$

is an abstract harmonic space, which will be referred to as the $\mathcal{L}$-harmonic space. Following the notation in the Appendix, we denote by $\mathcal{S}(\Omega)$ (resp. $\mathcal{S}(\Omega)$) the cone of the $\mathcal{H}_L$-superharmonic (resp. $\mathcal{H}_L$-subharmonic) functions in $\Omega$.

We remind that a l.s.c. function $u : \Omega \to (-\infty, \infty]$ is called $\mathcal{H}_L$-superharmonic in $\Omega$ (and we write $u \in \mathcal{S}(\Omega)$) if it satisfies the following two properties:

1. the set $\{ x \in \Omega : u(x) < \infty \}$ is dense in $\Omega$;

2. for every $\mathcal{H}_L$-regular open set $V \subseteq \overline{V} \subseteq \Omega$ and for every function $\varphi \in C(\partial V)$ such that $u \geq \varphi$ on $\partial V$, one has $u \geq H^V_\varphi$ throughout $V$.

An u.s.c. function $u : \Omega \to [-\infty, \infty)$ is $\mathcal{H}_L$-subharmonic in $\Omega$ if $-u \in \mathcal{S}(\Omega)$.

Let now $x \in \mathbb{R}^N$ be arbitrarily fixed, and let

$$\Gamma_x : \mathbb{R}^N \to (-\infty, \infty], \quad \Gamma_x(y) := \begin{cases} \Gamma(x, y), & \text{if } y \neq x, \\ \infty, & \text{if } y = x. \end{cases}$$

(2.7)

On account of (2.3), it is not difficult to prove that $\Gamma_x \in \mathcal{S}(\mathbb{R}^N)$; moreover, from the WMP in Remark 2.2-(c) it easily follows that

$\Gamma_x$ is a strictly positive $\mathcal{H}_L$-potential in $\mathbb{R}^N$.

(2.8)

As a consequence, it is possible to characterize the $\mathcal{H}_L$-regular open sets in $\mathbb{R}^N$ by using the Bouligand theorem (see Theorem A.7 in the Appendix). For a future reference, we highlight the following application of this fact.

**Example 2.4.** Let $x \in \mathbb{R}^N$ be fixed, and let $r > 0$. We claim that

$$\Omega_r(x)$$

is $\mathcal{H}_L$-regular.

(2.9)
Indeed, owing to Bouligand’s theorem, to prove (2.9) it suffices to show that there exists a $H^r$-barrier for $\Omega_r(x)$ at every point $\xi \in \partial \Omega_r(x)$; on the other hand, taking into account the very definition of $\Omega_r(x)$, it is readily seen that

$$ w := \Gamma_x - \frac{1}{r}, $$

does the job (independently of the point $\xi \in \partial \Omega$).

In addition to (2.8), assumption (FS) has another consequence of great importance for our study: in fact, the existence of a ‘well-behaved’ fundamental solution $\Gamma$ allows to define suitable mean-value operators, which can be used to characterize the $H^r$-harmonic/subharmonic functions (see, precisely, [17, Thm. 4.2]).

In order to properly introduce these operators, we first fix some notation which will be used in next sections. To begin with, for every $x \in \mathbb{R}^N$ we set

$$ K^x : \mathbb{R}^N \setminus \{x\} \to [0, \infty), \quad K^x(y) := \frac{\langle A(y) \nabla \Gamma_x(y), \nabla \Gamma_x(y) \rangle}{|\nabla \Gamma_x(y)|^2 + \alpha}; $$

and moreover, if $\alpha > 0$ is arbitrarily chosen, we define

$$ K^{\alpha}_x : \mathbb{R}^N \setminus \{x\} \to [0, \infty), \quad K^{\alpha}_x(y) := \frac{\langle A(y) \nabla \Gamma_x(y), \nabla \Gamma_x(y) \rangle}{\Gamma_x(y)^{2+\alpha}}. $$

Let now $\Omega \subset \mathbb{R}^N$ be an open set, and let $u : \Omega \to [-\infty, \infty)$ be a u.s.c. function on $\Omega$. For every $x \in \Omega$ and every $r > 0$ such that $\Omega_r(x) \subseteq \Omega$, we define

$$ m_r(u)(x) := \int_{\partial \Omega_r(x)} u K_x \, d\sigma, \quad M^\alpha_r(u)(x) := \frac{\alpha + 1}{r^{\alpha+1}} \int_{\Omega_r(x)} u K^{\alpha}_x \, dy, $$

(2.10)

where $d\sigma$ denotes the standard $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.

We say that $m_r$ is the surface mean-integral operator related to $\mathcal{L}$ and $M^\alpha_r$ is the solid mean-integral operator (related to $\mathcal{L}$).

**Remark 2.5.** A couple of remarks are in order.

(a) Using the Federer Coarea Formula, it is not difficult to recognize that the operator $M^\alpha_r(u)$ has the following alternative representation:

$$ M^\alpha_r(u)(x) = \frac{\alpha + 1}{r^{\alpha+1}} \int_0^r \rho^{\alpha} m_\rho(u)(x) \, d\rho. $$

(b) By exploiting the Mean Value formulas for $\mathcal{L}$ in [17, Thm. 3.3], together with the fact that $u \equiv 1$ is $H^r$-harmonic in $\mathbb{R}^N$, we readily obtain

$$ m_r(1) = M^\alpha_r(1) = 1 \quad \text{for every } x \in \mathbb{R}^N \text{ and every } r > 0. $$

(2.11)

In order to simplify the notation, we fix once and for all a real $\alpha > 0$ and we use the simpler notation $M_r$ and $K_r$ (instead of $M^\alpha_r$ and $K^{\alpha}_r$).

**Remark 2.6.** Let $x \in \mathbb{R}^N$ be fixed, and let $r > 0$. We have already recognized in Example 2.4 that $\Omega_r(x)$ is $H^r$-regular; then, by exploiting the surface mean-value formula for $\mathcal{L}$ in [17, Thm. 3.3], we can easily prove that

$$ \mu_{\Omega^r}(x) = K_x \, d\sigma, $$

(2.12)

where $d\sigma$ is the standard surface measure in $\mathbb{R}^N$ (see, e.g., [17, Lem. 5.8]). We explicitly notice that (2.12) does not provide an expression for

$$ \mu_{\Omega^r}(x) \quad \text{when } z \neq x. $$
We conclude this section by reviewing the notion of $L$-Riesz measure associated with an $H\mathcal{L}$-subharmonic/superharmonic function. Let then $\Omega \subseteq \mathbb{R}^N$ be open, and let $u \in S(\Omega)$. On account of [17, Thm. 4.2], we know that $u \in L^1_{\text{loc}}(\Omega)$ and $Lu \geq 0$ in the sense of distributions on $\Omega$; as a consequence, there exists a unique Radon measure $\mu_u$ on $\Omega$ such that $Lu = \mu_u$ in $D'(\Omega)$, i.e.,

$$\int_{\Omega} u \mathcal{L} \varphi \, dx = \int_{\Omega} u \, d\mu_u \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

The measure $\mu_u$ will be called the $L$-Riesz measure of $u$.

If $v := -u \in S(\Omega)$, we define the $L$-Riesz measure of $u$ as the $L$-Riesz measure of $v$. We denote this measure again by $\mu_u$; moreover, we explicitly notice that, by definition, $Lu = -\mu_u$ in $D'(\Omega)$, i.e.,

$$\int_{\Omega} u \mathcal{L} \varphi \, dx = -\int_{\Omega} u \, d\mu_u \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

**Example 2.7.** Let $x \in \mathbb{R}^N$ be fixed, and let $\Gamma_x$ be as in (2.7). We know that $\Gamma_x$ is $H\mathcal{L}$-superharmonic in $\mathbb{R}^N$ and, by Remark 2.2-(c), we have $-L\Gamma_x = \text{Dir}_x$ in $D'(\mathbb{R}^N)$; hence, the $L$-Riesz measure of $\Gamma_x$ is the Dirac measure at $\{x\}$.

3. **$L$-Green functions and potentials.** In this section we extend to our context some results of the classical Potential Theory for the Laplacian: we define the $L$-Green function of an arbitrary open set, and we introduce the notion of $L$-Green potential of a Radon measure. In what follows, we tacitly inherit all the assumptions and notation introduced so far.

We explicitly point out that, in the particular case when $L$ is a sub-Laplacian on a homogeneous Carnot group $G$, the results we are going to present are established in [19, Sec.s 9.1-9.3]; however, since most of the arguments in [19] only rely on the fact that any sub-Laplacian satisfies assumptions (S)-to-(FS) (see Example 2.3), the very same arguments also work in our more general context. For this reason, in the forthcoming proofs we shall often skip all the details, and we shall refer directly to the corresponding statements in [19].

**$L$-Green functions.** Let $\Omega$ be a $H\mathcal{L}$-regular open set. We call $L$-Green function of $\Omega$ with pole at $x \in \Omega$, the function $G_\Omega(x, \cdot) : \Omega \to (-\infty, \infty]$ defined as

$$G_\Omega(x, y) := \Gamma_x(y) - H_{\Gamma_x}^\Omega(y) = \begin{cases} 
\Gamma(x, y) - H_{\Gamma_x}^\Omega(y), & \text{if } y \neq x, \\
\infty, & \text{if } y = x,
\end{cases}$$

(3.1)

where $H_{\Gamma_x}^\Omega \in C^\infty(\Omega)$ is the solution to the $H\mathcal{L}$-Dirichlet problem

$$\begin{cases}
\mathcal{L}h = 0 & \text{in } \Omega \\
h(z) = \Gamma(x, z) & \text{for every } z \in \partial\Omega,
\end{cases}$$

(3.2)

We remind that (3.2) has a (unique) solution since $\Omega$ is $H\mathcal{L}$-regular.

**Remark 3.1.** Taking into account (2.3) and (2.4), we see that

1. $G_\Omega(x, \cdot)$ is $H\mathcal{L}$-harmonic in $\Omega \setminus \{x\}$;
2. $G_\Omega(x, y) \to \infty$ as $y \to x$;
3. for every fixed $z \in \partial\Omega$, we have $G_\Omega(x, y) \to 0$ as $y \to z$. 
Moreover, we can represent $G_\Omega$ as follows:

$$G_\Omega(x, y) = \Gamma_x(y) - \int_{\partial \Omega} \Gamma(x, \zeta) d\mu^\Omega_\zeta(\zeta), \quad (3.3)$$

where $\mu^\Omega_\zeta$ is the $\mathcal{H}_\zeta$-harmonic measure related to $\Omega$ and $y$.

The next theorem contains the main properties of $G_\Omega$.

**Theorem 3.2.** For every $x, y \in \Omega$ with $x \neq y$, we have:

1. $G_\Omega(x, y) \geq 0$,
2. $G_\Omega(x, y) > 0$ if and only if $x$ and $y$ are in the same component of $\Omega$,
3. $G_\Omega(x, y) = G_\Omega(y, x)$.

**Proof.** One can argue exactly as in the proof of [19, Thm. 9.1.2], by using the Strong Maximum Principle for $\mathcal{H}_\zeta$-harmonic functions established in [3].

**Remark 3.3.** Let $x_0 \in \mathbb{R}^N$ be fixed, and let $r > 0$. We know from Example 2.4 that $\Omega_r(x_0)$ is $\mathcal{H}_\zeta$-regular; moreover, since $\Gamma_{x_0} \equiv \frac{1}{r}$ on $\partial \Omega_r(x_0)$, we have

$$H^{\Omega}_{\Gamma_{x_0}} \equiv \frac{1}{r}. \quad (3.4)$$

It should be noticed that (3.4) provides an expression of $G_\Omega$ only when the pole is precisely the ‘centre’ $x_0$: in fact, if $x \neq x_0$, we have no information on $H^{\Omega}_{\Gamma_{x}}$.

We now proceed by extending the notion of $\mathcal{L}$-Green function to general open sets. Let then $\Omega \subseteq \mathbb{R}^N$ be open, and let $x \in \Omega$. On account of (2.8), the function $\Gamma_x$ is $\mathcal{H}_\zeta$-superharmonic and non-negative in $\Omega$; hence, it has a greatest harmonic minorant in $\Omega$, say $\gamma_x \in \mathcal{H}_\zeta(\Omega)$. The function

$$\Omega \times \Omega \ni (x, y) \mapsto G_\Omega(x, y) := \Gamma_x(y) - \gamma_x(y) \in [0, \infty] \quad (3.5)$$

is the $\mathcal{L}$-Green function for $\Omega$.

**Remark 3.4.** A couple of remarks are in order.

1. Since $\gamma_x \in \mathcal{H}_\zeta(\Omega)$ and $\Gamma_x \in \overline{\mathcal{S}}(\Omega)$, the function $G_\Omega(x, \cdot)$ is $\mathcal{H}_\zeta$-superharmonic in $\Omega$ and $\mathcal{H}_\zeta$-harmonic in $\Omega \setminus \{x\}$; moreover, we have

$$\gamma_x = \sup \{v \in \mathcal{S}(\Omega) : v \leq \Gamma_x \text{ in } \Omega\}, \quad (3.6)$$

so that $0 \leq \gamma_x \leq \Gamma_x$ and $G_\Omega \geq 0$.

2. In the particular case when $\Omega$ is $\mathcal{H}$-regular, by Lemma A.6 we have

$$\gamma_x = \int_{\partial \Omega} \Gamma_x d\mu^\Omega_x = H^{\Omega}_{\Gamma_x}$$

(recall that $\Gamma_x$ is continuous on $\mathbb{R}^N \setminus \{x\} \supseteq \partial \Omega$); as a consequence, the definition of $G_\Omega$ given in (3.5) coincides with that in (3.1).

The following theorem contains some relevant properties of $G_\Omega$ which are simple consequences of its very definition and of the Harnack Axiom (see [19, Sec. 9.2]).

**Theorem 3.5.** The following assertions hold.

1. Let $x \in \Omega$ be fixed. Then $G_\Omega(x, \cdot)$ is a $\mathcal{H}_\zeta$-potential in $\Omega$. 
2. Let \( x \in \Omega \), and let \( u \in \overline{S}(\Omega) \) be such that \( u \geq 0 \) in \( \Omega \) and \( u = \Gamma_x + v \), for a suitable function \( v \in \overline{S}(\Omega) \). Then, we have

\[
u \geq G_\Omega(x, \cdot).
\]

3. Let \( \Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^N \) be arbitrary open sets. Then, we have

\[
G_{\Omega_1}(x, y) \leq G_{\Omega_2}(x, y) \quad \text{for every } x, y \in \Omega_1.
\]

4. Let \( \{ \Omega_n \}_n \) be a monotone increasing sequence of open sets in \( \mathbb{R}^N \), that is, \( \Omega_n \subseteq \Omega_{n+1} \) for every \( n \in \mathbb{N} \). Then, setting \( \Omega := \bigcup_{n \in \mathbb{N}} \Omega_n \), we have

\[
\lim_{n \to \infty} G_{\Omega_{p+n}}(x, y) = G_\Omega(x, y) \quad \forall p \in \mathbb{N} \text{ and } x, y \in \Omega_p.
\]

\[(3.7)\]

Remark 3.6. The \( \mathcal{L} \)-Green function for \( \mathbb{R}^N \) is

\[
G_{\mathbb{R}^N}(x, y) = \Gamma_x(y), \quad x, y \in \mathbb{R}^N.
\]

In fact, by (2.8) we have \( \gamma_x \equiv 0 \) in \( \mathbb{R}^N \) (for every fixed \( x \)).

In comparison with the case of \( \mathcal{H}_{\mathcal{L}} \)-regular open sets, there is a property of \( G_\Omega \) which is not contained in Theorem 3.5: its symmetry w.r.t. \( x, y \). To establish this property, we first need the following result of independent interest.

Lemma 3.7. Let \( \Omega \subseteq \mathbb{R}^N \) be an arbitrary open set. Then, there exists a monotone increasing sequence \( \{ \Omega_n \}_n \) of \( \mathcal{H}_{\mathcal{L}} \)-regular open sets in \( \mathbb{R}^N \) such that

\[
\bigcup_{n \in \mathbb{N}} \Omega_n = \Omega.
\]

Proof. We first suppose that \( \Omega \) is bounded. In this case, for every \( n \in \mathbb{N} \) we can cover the compact set \( \partial \Omega \) with a finite family of super-level sets of \( \Gamma \), say

\[
\mathcal{F}_n = \{ \Omega_{r^n_j}(x^n_j) \}_{j=1}^{p_n} \quad \text{(for some } p_n \geq 1),
\]

such that \( 0 < r^n_j < 1/n \) and

\[
\Omega_{r^n_j+1}(x^n_j) \subseteq \bigcup_{i=1}^{p_n} \Omega_{r^n_i}(x^n_i).
\]

Then, defining

\[
\Omega_n := \Omega \setminus \bigcup_{i=1}^{p_n} \Omega_{r^n_i}(x^n_i),
\]

it is immediate to recognize that \( \{ \Omega_n \}_n \) is increasing and \( \bigcup_n \Omega_n = \Omega \). Most importantly, we claim that any \( \Omega_n \) is \( \mathcal{H}_{\mathcal{L}} \)-regular. In fact, let \( z_0 \in \partial \Omega_n \) be fixed and let \( j \in \{ 1, \ldots, p_n \} \) be such that \( z_0 \in \partial \Omega_{r^n_j}(x^n_j) \). By definition, the function

\[
w = \frac{1}{r^n_j} - \Gamma(x^n_j, \cdot)
\]

is \( \mathcal{H}_{\mathcal{L}} \)-harmonic and strictly positive in \( \mathbb{R}^N \setminus \overline{\Omega_{r^n_j}(x^n_j)} \supseteq \Omega_n \). Moreover,

\[
w(z) \to 0 \text{ as } z \to z_0.
\]

As a consequence, \( w \) provides a \( \mathcal{H}_{\mathcal{L}} \)-barrier for \( \Omega_n \) at \( z_0 \), and the Bouligand theorem ensures that \( z_0 \) is \( \mathcal{H}_{\mathcal{L}} \)-regular for \( \Omega_n \), as desired.

Let us now suppose that \( \Omega \) is unbounded, and let

\[
O_n := \Omega \cap \Omega_n(0), \quad n \in \mathbb{N}.
\]
Since $O_n$ is (open and) bounded, we can find an increasing sequence of $\mathcal{L}$-regular open sets, say $\{O_n^k\}_{k \in \mathbb{N}}$, such that $\bigcup_k O_n^k = O_n$. Moreover, using the first part of the proof, we can choose $O_n^k$ in such a way that

$$O_n^k \subseteq O_n^{k+1} \text{ for every } n, k \in \mathbb{N}.$$  

As a consequence, we have $O_n^k \subseteq O_m^m$ if $k, n \leq m$. Let us now set

$$\Omega_n = O_n^1 \cup O_n^2 \cup \cdots \cup O_n^n \subseteq \Omega.$$  

By definition, $\Omega_n$ is $\mathcal{H}_\mathcal{L}$-regular and $\Omega_n \subseteq \Omega_{n+1}$. Furthermore,

$$\Omega = \bigcup_n \Omega_n = \bigcup_n \left( \bigcup_k O_n^k \right) \subseteq \bigcup_n O_n \subseteq \Omega.$$  

Hence, $\Omega = \bigcup_n \Omega_n$, and the proof is complete. \hfill \Box

By combining Lemma 3.7, Theorem 3.2 and Theorem 3.2-(3), we immediately obtain the following proposition on the symmetry of $G_{\Omega}$.

**Proposition 3.8.** Let $\Omega \subseteq \mathbb{R}^N$ be an arbitrary open set. Then

$$G_{\Omega}(x, y) = G_{\Omega}(y, x) \quad \text{for every } x, y \in \Omega.$$  

In particular, for every fixed $y \in \Omega$, the function

$$x \mapsto G_{\Omega}(x, y)$$  

is $\mathcal{L}$-harmonic in $\Omega \setminus \{y\}$.

Taking into account all the previous results, we can now proceed by introducing the $\mathcal{L}$-potential of a Radon measure. Before doing this, for a future reference we summarize in the following theorem all the results stated so far.

**Theorem 3.9.** Let $\Omega \subseteq \mathbb{R}^N$ be an arbitrary open set, and let $x \in \Omega$ be fixed. Moreover, let $h_x$ denote the greatest harmonic minorant of $\Gamma_x \geq 0$ in $\Omega$, i.e.,

$$\gamma_x = \sup \{ u \in \mathcal{L}(\Omega) : u \leq \Gamma_x \text{ on } \Omega \}.$$  

Then, the following facts hold.

1. The $\mathcal{L}$-Green function of $\Omega$ with pole at $x$ is defined as

$$G_{\Omega}(x, y) = \Gamma_x(y) - \gamma_x(y).$$  

Moreover, $G_{\Omega}$ enjoys several properties:

(i) $G_{\Omega} \geq 0$ on $\Omega \times \Omega$;

(ii) $G_{\Omega}$ is a symmetric, i.e., $G_{\Omega}(x, y) = G_{\Omega}(y, x)$ for all $x, y \in \Omega$.

(iii) $G_{\Omega}(x, \cdot)$ is $\mathcal{H}_\mathcal{L}$-harmonic in $\Omega \setminus \{x\}$ and $\mathcal{H}_\mathcal{L}$-subharmonic on $\Omega$;

(iv) the greatest $\mathcal{L}$-harmonic minorant of $G_{\Omega}(x, \cdot)$ in $\Omega$ is the zero function.

2. If $\Omega$ is $\mathcal{H}_\mathcal{L}$-regular, then $\gamma_x$ is the unique ‘classical’ solution of

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega \\ \lim_{y \to \xi} u(y) = \Gamma_x(\xi) \text{ for every } \xi \in \partial \Omega. \end{cases}$$

Moreover, $G_{\Omega}(x, \cdot) > 0$ in the connected component of $x$ in $\Omega$, and

$$\lim_{y \to \xi} G_{\Omega}(x, y) = 0 \quad \text{for every } \xi \in \partial \Omega.$$
**Potentials of Radon Measures.** Let $\Omega \subseteq \mathbb{R}^N$ be an arbitrary open set, and let $G_{\Omega}$ be the associated $\mathcal{L}$-Green function. If $\mu$ is a Radon measure in $\Omega$, the map

$$G_{\Omega} \ast \mu : \Omega \to [0, \infty], \quad (G_{\Omega} \ast \mu)(x) := \int_{\Omega} G_{\Omega}(x, \cdot) \, d\mu$$

is called the $G_{\Omega}$-potential of $\mu$ in $\Omega$.

**Remark 3.10.** Since $G_{\Omega}(x, \cdot)$ is non-negative and $\mathcal{H}_\mathcal{L}$-superharmonic in $\Omega$, an application of Fatou’s lemma shows that $G_{\Omega} \ast \mu$ is well-defined and l.s.c.

For the sake of simplicity, in all the subsequent results we shall deal with connected open sets; this is not restrictive, since one can obtain analogous results for an arbitrary open set by considering every connected component. To begin with, we prove the following theorem.

**Theorem 3.11.** Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set, and let $\mu$ be a Radon measure on $\Omega$. Then, the following conditions are equivalent:

(i) $G_{\Omega} \ast \mu \in \mathcal{S}(\Omega)$;

(ii) there exists $x_0 \in \Omega$ such that $(G_{\Omega} \ast \mu)(x_0) < \infty$.

**Proof.** (i) $\Rightarrow$ (ii). Since $G_{\Omega} \ast \mu \in \mathcal{S}(\Omega)$, by definition we actually have

$$G_{\Omega} \ast \mu < \infty \text{ in a dense subset of } \Omega.$$

(ii) $\Rightarrow$ (i). In view of assumption (ii), and since $G_{\Omega} \ast \mu$ is l.s.c. on $\Omega$, we can use the characterization of $\mathcal{H}_\mathcal{L}$-superharmonicity provided in [17, Thm. 4.2]: given any $x \in \Omega$ and any $r > 0$ such that $\Omega_r(x) \subseteq \Omega$, we then prove that

$$G_{\Omega} \ast \mu(x) \geq M_r(G_{\Omega} \ast \mu)(x), \quad \text{(3.8)}$$

where $M_r$ is the solid mean-value operator defined in (2.10). First of all, since we know that $G_{\Omega}(\zeta, \cdot) \in \mathcal{S}(\Omega)$ for every $\zeta \in \Omega$, again by [17, Thm. 4.2] we have

$$M_r(G_{\Omega}(\zeta, \cdot))(x) \leq G_{\Omega}(\zeta, x);$$

as a consequence, reminding that $G_{\Omega}$ is symmetric, we obtain

$$M_r(G_{\Omega} \ast \mu)(x)$$

$$=\frac{\alpha + 1}{r^{\alpha+1}} \int_{\Omega_r(x)} K_x(y)(G_{\Omega} \ast \mu)(y) \, dy$$

$$=\frac{\alpha + 1}{r^{\alpha+1}} \int_{\Omega_r(x)} K_x(y) \left( \int_{\Omega} G_{\Omega}(y, \zeta) \, d\mu(\zeta) \right) \, dy = \int_{\Omega} M_r(G_{\Omega}(\zeta, \cdot))(x) \, d\mu(\zeta)$$

$$\leq \int_{\Omega} G_{\Omega}(\zeta, x) \, d\mu(\zeta) = (G_{\Omega} \ast \mu)(x).$$

This is exactly the desired (3.8), and the proof is complete.

By combining Theorem 3.11 with the Harnack inequality for $\mathcal{L}$ proved in [3], we easily obtain the following important result.

**Theorem 3.12.** Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set, and let $\mu$ be a Radon measure in $\Omega$ such that $K := \text{supp}(\mu) \subseteq \Omega$ is compact. Moreover, let $y_0 \in K$ be fixed, and let $U \subseteq \Omega$ be a bounded and connected open set such that

$$K \subseteq U \subseteq \overline{U} \subseteq \Omega.$$

Then, there exists a constant $C = C(U, K) \geq 1$ such that

$$(G_{\Omega} \ast \mu)(x) \leq C G_{\Omega}(x, y_0) \quad \text{for every } x \in \Omega \setminus U.$$  \quad (3.9)
In particular, $G_\Omega * \mu \in \mathcal{S}(\Omega)$.

**Proof.** For every $x \in \Omega \setminus U$, the function $G_\Omega(x, \cdot)$ is $\mathcal{H}_L$-harmonic and non-negative in $U$; then, by the Harnack inequality for $L$ in [3, Thm. 1.10], it is possible to find a constant $C = C(U, K) \geq 1$ such that, for every $y \in K$, one has

$$G_\Omega(x, y) \leq \sup_K (G_\Omega(x, \cdot)) \leq C \inf_K (G_\Omega(x, \cdot)) \leq C G_\Omega(x, y_0).$$

As a consequence, since $\mu(\Omega \setminus K) = 0$, for every $x \in \Omega \setminus U$ we obtain

$$(G_\Omega * \mu)(x) = \int_K G_\Omega(x, y) d\mu(y) \leq C \mu(K) G_\Omega(x, y_0),$$

which is precisely the desired (3.9). Finally, since (3.9) shows that $G_\Omega * \mu$ is finite on the whole of $\Omega \setminus U$, from Theorem 3.11 we get $G_\Omega * \mu \in \mathcal{S}(\Omega)$. $\square$

The next theorem contains a fundamental property of $G_\Omega * \mu$.

**Theorem 3.13.** Let $\Omega \subseteq \mathbb{R}^N$ be open and connected, and let $\mu$ be a Radon measure on $\Omega$. We assume that $G_\Omega * \mu$ is finite in at least one point $x_0 \in \Omega$. Then,

$$L(G_\Omega * \mu) = -\mu \quad \text{in} \quad \mathcal{D}'(\Omega).$$

(3.10)

In particular, $G_\Omega * \mu$ is $\mathcal{H}_L$-harmonic in $\Omega \setminus \text{supp}(\mu)$.

**Proof.** Since $(G_\Omega * \mu)(x_0) < \infty$, by Theorem 3.11 we have $G_\Omega * \mu \in \mathcal{S}(\Omega)$; in particular, we derive from [17, Thm. 4.2] that $G_\Omega * \mu \in L^1_{\text{loc}}(\Omega)$. Using this fact, and taking into account that $L$ is $C^\infty$-hypoelliptic (in view of assumptions (FS)), one can then proceed as in the proof of [19, Thm. 9.3.5]. $\square$

Using Theorems 3.12 and 3.13, we obtain the useful result.

**Corollary 3.14.** Let $\Omega \subseteq \mathbb{R}^N$ be open and connected, and let $\mu$ be a Radon measure in $\Omega$ with compact support $K$. Then, the following facts hold:

1. $G_\Omega * \mu \in \mathcal{S}(\Omega)$.
2. $G_\Omega * \mu$ is $\mathcal{H}_L$-harmonic in $\Omega \setminus K$;
3. if $v \in \mathcal{S}(\Omega)$ and $v \leq G_\Omega * \mu$ on $\Omega$, then $v \leq 0$ on $\Omega$.

As a consequence, $G_\Omega * \mu$ is a $\mathcal{H}_L$-potential in $\Omega$.

**Proof.** One can argue as in the proof of [19, Cor. 9.3.6], by using the Minimum Principle in Theorem A.2-(2) (notice that $\theta \equiv 1$ is $\mathcal{H}_L$-harmonic in $\mathbb{R}^N$). $\square$

We end the section by extending the result in Corollary 3.14-(3) to the $G_\Omega$-potential of an arbitrary Radon measure, not necessarily compactly supported.

**Theorem 3.15.** Let $\Omega \subseteq \mathbb{R}^N$ be open and connected, and let $\mu$ be a Radon measure on $\Omega$. We assume that $G_\Omega * \mu$ is finite in at least one point $x_0 \in \Omega$. Then,

$$(h \in \mathcal{H}_L(\Omega), h \leq G_\Omega * \mu \quad \text{in} \quad \Omega) \implies h \leq 0 \quad \text{in} \quad \Omega.$$  

(3.11)

As a consequence, $G_\Omega * \mu$ is a $\mathcal{H}_L$-potential in $\Omega$.

**Proof.** One can argue as in the proof of [19, Thm. 9.3.7], by using Theorem 3.11, Corollary 3.14 and Proposition A.5. $\square$
A comparison between two notions of $L$-Green function. We conclude the section by briefly comparing the notion of $L$-Green function introduced here with the one introduced by Bony [20] in the context of Hörmander operators (and later extended in [3] to general divergence-form operators).

To begin with, let $B_L$ be the topological basis of $\mathbb{R}^N$ appearing in the statement of [3, Lem. 1.7] (remind that the elements of $B_L$ are $H_L$-regular open sets, see Section 2), and let $\Omega \in B_L$. For every fixed $f \in C(\Omega)$, it is proved in [3] that there exists a unique function $u = G(f) \in C(\Omega)$ such that

$$
\begin{cases}
Lu_f = -f & \text{in } D'(\Omega), \\
u_f = 0 & \text{pointwise in } \Omega.
\end{cases}
$$

Moreover, by [3, Thm. 1.9] we know that the operator

$$
C(\Omega) \ni f \mapsto G(f) \in C(\Omega)
$$

can be actually represented as an integral operator associated with a smooth kernel. To be more precise, there exists a function $g: \Omega \times \Omega \to \mathbb{R}$ such that

$$
G(f) = \int_{\Omega} g(x,y)f(y) \, dy \quad \text{pointwise in } \Omega. 
$$

(3.12)

Moreover, $g$ satisfies the following properties:

1. $g$ is smooth and strictly positive out of the diagonal of $\Omega \times \Omega$;
2. $g$ is symmetric, that is, $g(x,y) = g(y,x)$ for every $x \neq y \in \Omega$;
3. for every fixed $x \in \Omega$, one has $g(x,\cdot) \in L^1_{\text{loc}}(\Omega)$ and

$$
\lim_{y \to \xi} g(x,y) = 0 \quad \forall \xi \in \partial \Omega.
$$

This function $g$ is called in [3] the Green kernel related to $L$.

We now aim at proving that

$$
g(x,y) = G_{\Omega}(x,y) \quad \text{for every } x \neq y \in \Omega.
$$

(3.13)

To this end we first observe that, on account of (3.12), we have

$$
\int_{\Omega} k(x,\cdot) L\varphi \, dy = -\varphi(x) \quad \forall \varphi \in C^\infty_0(\Omega) \text{ and } x \in \Omega;
$$

in particular, if $x \in \Omega$ is fixed, we deduce that $Lg(x,\cdot) = -\text{Dir}_x$ in $D'(\Omega)$. On the other hand, since $G_{\Omega}(x,\cdot)$ is the sum of $\Gamma_x$ plus a $H_L$-harmonic function in $\Omega$, from Remark 2.2-(c) we derive that $L G_{\Omega}(x,\cdot) = -\text{Dir}_x$ in $D'(\Omega)$. Hence, we obtain

$$
L(g(x,\cdot) - G_{\Omega}(x,\cdot)) = 0 \quad \text{in } D'(\Omega). 
$$

(3.14)

We now exploit in a crucial way the $C^\infty$-hypoellipticity of $L$: in view of (3.14), it is possible to find a unique function $h = h_x \in H_L(\Omega)$ such that

$$
h_x \equiv g(x,\cdot) - G_{\Omega}(x,\cdot) \quad \text{a.e. in } \Omega \text{ and pointwise in } \Omega \setminus \{x\}.
$$

Moreover, taking into account that both $g(x,y)$ and $G_{\Omega}(x,y)$ vanishes when $y \to \partial \Omega$ (see Remark 3.1 and remind that $\Omega \in B_L$ is $H_L$-regular), we obtain

$$
\lim_{y \to \xi} h_x(y) = 0 \quad \text{for every } \xi \in \partial \Omega.
$$

As a consequence, from the WMP in Remark 2.2-(a) we infer that $h_x \equiv 0$ pointwise in $\Omega$, and thus $g(x,\cdot) \equiv G_{\Omega}(x,\cdot)$ pointwise in $\Omega \setminus \{x\}$, as desired.
4. Riesz representation theorems for $\mathcal{H}_L$-subharmonic functions and Poisson-Jensen-type formulas. In this section we use the theory of the $G\Omega$-potentials to establish some representation theorems for $\mathcal{H}_L$-subharmonic functions. In particular, we extend the classical Poisson-Jensen formula for the Laplacian.

Riesz representations theorems. We remind that, if $\Omega \subseteq \mathbb{R}^N$ is an arbitrary open set and $u \in \mathcal{S}(\Omega)$, the $L$-Riesz measure $\mu_u$ of $u$ is the unique Radon measure in $\Omega$ such that
\[
\int_{\Omega} u \ L \varphi \, dx = \int_{\Omega} u \, d\mu_u \quad \forall \varphi \in C_0^\infty(\Omega).
\]
When $\mu_u$ is compactly supported, we readily obtain the following theorem.

**Theorem 4.1** (Riesz representation - I). Let $\Omega \subseteq \mathbb{R}^N$ be open, and let $u \in \mathcal{S}(\Omega)$. We assume that the $L$-Riesz measure $\mu_u$ has compact support $K \subseteq \Omega$.

Then, there exists a function $h \in \mathcal{H}_L(\Omega)$ such that
\[
u = h - G\Omega * \mu_u \quad \text{pointwise in } \Omega.
\]

**Proof.** Since $\mu_u$ has compact support in $\Omega$, from Theorems 3.12 and 3.13 we derive that $v := G\Omega * \mu_u$ is $\mathcal{H}_L$-superharmonic in $\Omega$ and $\mathcal{L}v = -\mu_u$ in $\mathcal{D}'(\Omega)$; this, together with the very definition of $\mu_u$, implies that $\mathcal{L}(u + v) = 0$ in $\mathcal{D}'(\Omega)$.

Now, since the operator $\mathcal{L}$ is $C^\infty$-hypoelliptic, from (4.2) we infer the existence of a unique $\mathcal{H}_L$-harmonic function in $\Omega$ such that
\[
u = h - v = h - G\Omega * \mu_u \quad \text{a.e. in } \Omega;
\]
thus, to complete the proof we need to show that (4.3) actually holds pointwise in $\Omega$. To this end, we arbitrarily fix $x \in \Omega$ and we choose $r > 0$ such that $\Omega_r(x) \subseteq \Omega$.

On account of (4.3), for every $0 < \rho < r$ we have
\[
M_{\rho}(u)(x) = M_{\rho}(h - G\Omega * \mu_u)(x),
\]
where $M_{\rho}$ is the solid mean-integral operator defined in (2.10). As a consequence, since $u, h - G\Omega * \mu_u \in \mathcal{S}(\Omega)$, from [17, Thm. 4.2] we get
\[
u(x) = \lim_{\rho \to 0^+} M_{\rho}(u)(x) = \lim_{\rho \to 0^+} M_{\rho}(h - G\Omega * \mu_u)(x) = (G\Omega * \mu_u)(x).
\]
Due to the arbitrariness of $x \in \Omega$, this proves (4.1). □

If $\mu_u$ is not compactly supported, we have the following ‘local’ result.

**Theorem 4.2** (Riesz representation - II). Let $\Omega \subseteq \mathbb{R}^N$ be open, and let $u \in \mathcal{S}(\Omega)$. Moreover let $U \subseteq \mathbb{R}^N$ be an arbitrary open set and let $\Omega_1 \subseteq U \cap \Omega$.

Then, there exists a function $h \in \mathcal{H}_L(\Omega_1)$ such that
\[
u = h - G_U * (\mu_u|_{\Omega_1}) \quad \text{pointwise in } \Omega_1.
\]

In particular, if $U = \mathbb{R}^N$ and $\Omega_1 \Subset \Omega$, we have
\[
u = h - \int_{\Omega_1} \Gamma_x \, d\mu_u \quad \text{pointwise in } \Omega,
\]
where $h$ is a suitable $\mathcal{H}_L$-harmonic function in $\Omega_1$. 
Proof. Since the measure $\nu := \mu_u|_{\Omega_1}$ has compact support $\overline{\Omega}_1 \subseteq U$, we can extend it to a compactly supported Radon measure on $U$; thus, Theorems 3.12 and 3.13 imply that $v := Gv * \nu \in S(U)$ and $Lv = -\nu$ in $D'(U)$. In particular,

$$Lv = -\mu_u \text{ in } D'(\Omega_1).$$

On the other hand, since $Lu = -\mu_u$ in $D'(\Omega)$, this last identity gives

$$L(u + v) = 0 \text{ in } D'(\Omega_1).$$

Then, starting from (4.6) and arguing exactly as in the proof of Theorem 4.1, we obtain (4.4) with a suitable $h \in \mathcal{H}_L(\Omega_1)$. As for (4.5), it follows from (4.4) and from the fact that, on account of Remark 3.6, we have

$$G_{\mathbb{R}^N}(x, y) = \Gamma_x(y) \text{ for every } x, y \in \mathbb{R}^N,$$

This ends the proof. \qed

The next lemma contains a somehow ‘global’ version of Theorem 4.2.

**Lemma 4.3.** Let $\Omega \subseteq \mathbb{R}^N$ be open, and let $u \in S(\Omega)$. Moreover, let $K \subseteq \Omega$ be a compact set. Then, there exists a function $w \in S(\Omega)$ such that

$$w = u - G_{\Omega} * (\mu_u|_K) \text{ pointwise in } \Omega. \quad (4.7)$$

**Proof.** Let $\Omega_1 \subseteq \mathbb{R}^N$ be a bounded open set satisfying $K \subseteq \Omega_1$ and $H := \overline{\Omega}_1 \subseteq \Omega$. Owing to Theorem 4.2, we can find $h \in H_L(\Omega_1)$ such that

$$u = h - G_{\Omega} * (\mu_u|_H) \text{ pointwise in } \Omega_1; \quad (4.8)$$

then, we consider the function $w : \Omega \rightarrow [-\infty, \infty]$ defined as follows:

$$w = \begin{cases} u + G_{\Omega} * (\mu_u|_K) & \text{in } \Omega \setminus \Omega_1 \\ h - G_{\Omega} * (\mu_u|_{\Omega \setminus K}) & \text{in } \Omega_1. \end{cases}$$

We first observe that this definition is well-posed: indeed, since $\mu_u|_K$ is compactly supported in $K$, by Corollary 3.14 we have $G_{\Omega} * (\mu_u|_K) \in S(\Omega) \cap H_L(\Omega \setminus K)$; as a consequence, $G_{\Omega} * (\mu_u|_K)$ is real-valued on $\Omega \setminus K$, and thus

$$w_1 := u + G_{\Omega} * (\mu_u|_K)$$

is well-defined in $\Omega \setminus K \supseteq \Omega \setminus \Omega_1$. Furthermore, we claim that $w \in S(\Omega)$. In fact, since $u \in S(\Omega)$ and $G_{\Omega} * (\mu_u|_K) \in H_L(\Omega \setminus K)$, one has

$$w_1 \in S(\Omega \setminus K); \quad (4.9)$$

on the other hand, since the measure $\mu_u|_{\Omega \setminus K}$ has compact support in $\Omega$ and $h$ is $\mathcal{H}_L$-harmonic in $\Omega_1$, again from Corollary 3.14 we derive that

$$w_2 := h - G_{\Omega} * (\mu_u|_{\Omega \setminus K}) \in S(\Omega_1). \quad (4.10)$$

Owing to (4.9)-(4.10), and taking into account the ‘local’ criterion for $\mathcal{H}_L$-subharmonicity in Theorem A.3, we can conclude that $w \in S(\Omega)$ if we prove that

$$w_1 \equiv w_2 \text{ on } (\Omega \setminus K) \cap \Omega_1 = \Omega_1 \setminus K.$$

In its turn, this identity easily follows from (4.8): indeed, one has

$$w_1 = u + (G_{\Omega} * (\mu_u|_K)) = (h - G_{\Omega} * (\mu_u|_H)) + G_{\Omega} * (\mu_u|_K)
= h - G_{\Omega} * (\mu_u|_{\Omega \setminus K}) = w_2 \text{ pointwise on } \Omega_1 \setminus K. \quad (4.11)$$

Finally, using (4.11) and taking into account the very definition $w$, it is readily seen that identity (4.7) is satisfied. This ends the proof. \qed
Thanks to Lemma 4.3, we can establish the following result.

**Theorem 4.4.** Let \( \Omega \subseteq \mathbb{R}^N \) be an arbitrary open set, and let \( u \in \mathcal{S}(\Omega) \). Then, the following conditions are equivalent:

(i) \( G_\Omega \ast \mu_u \in \mathcal{S}(\Omega) \);

(ii) there exists \( v \in \mathcal{S}(\Omega) \) such that \( v \geq u \).

**Proof.** (i) \( \Rightarrow \) (ii). Since, by assumption, \( w := G_\Omega \ast \mu_u \in \mathcal{S}(\Omega) \), from Theorem 3.13 we derive that \( Lw = -\mu_u \) in \( \mathcal{D}'(\Omega) \); thus, by definition of \( \mu_u \) we have

\[
L(u + w) = 0 \text{ in } \mathcal{D}'(\Omega). \tag{4.12}
\]

Starting from (4.12), and arguing exactly as in the proof of Theorem 4.1, we get the existence of a function \( h \in H_{\mathcal{L}}(\Omega) \) such that \( u = h - G_\Omega \ast \mu_u \) pointwise in \( \Omega \).

Since \( G_\Omega \ast \mu_u \geq 0 \), we then have \( u \leq h \) in \( \Omega \), and (ii) holds with \( v := h \).

(ii) \( \Rightarrow \) (i). Let \( \{K_n\}_n \) be an increasing sequence of compact sets exhausting \( \Omega \), that is, \( \Omega = \bigcup_n K_n \). Moreover, for every fixed \( n \in \mathbb{N} \), let \( \mu_n := \mu_u|_{K_n} \). On account of Lemma 4.3, it is possible to find \( w_n \in \mathcal{S}(\Omega) \) such that \( u = w_n - G_\Omega \ast \mu_n \) pointwise in \( \Omega \); as a consequence, if \( v \in \mathcal{S}(\Omega) \) satisfies \( v \geq u \) in \( \Omega \), we get

\[
G_\Omega \ast \mu_n \geq w_n - v \text{ in } \Omega.
\]

Now, since \( w_n - v \in \mathcal{S}(\Omega) \), from Corollary 3.14 we get \( w_n - v \leq 0 \) in \( \Omega \); hence,

\[
v - u = (v - w_n) + G_\Omega \ast \mu_n \geq G_\Omega \ast \mu_n \quad \forall n \in \mathbb{N}.
\]

Letting \( n \to \infty \) and using Beppo Levi's theorem, we then obtain

\[
v - u \geq G_\Omega \ast \mu_u \text{ pointwise in } \Omega. \tag{4.13}
\]

Since \( v - u \in \mathcal{S}(\Omega) \), from (4.13) we derive that \( G_\Omega \ast \mu_u \) is finite in a dense subset of \( \Omega \), and thus Theorem 3.11 ensures that \( G_\Omega \ast \mu_u \) is \( H_{\mathcal{L}} \)-superharmonic in \( \Omega \). \( \square \)

After all these results, we are finally ready to prove Theorem 4.5. For the sake of readability, we rewrite the statement of this theorem here.

**Theorem 4.5 (The Riesz representation).** Let \( \Omega \subseteq \mathbb{R}^N \) be an arbitrary open set, and let \( u \in \mathcal{S}(\Omega) \). Then, the following conditions are equivalent:

(i) \( G_\Omega \ast \mu_u \in \mathcal{S}(\Omega) \) and there exists \( h \in H_{\mathcal{L}}(\Omega) \) such that

\[
u = h - G_\Omega \ast \mu_u \text{ pointwise in } \Omega, \tag{4.14}
\]

(ii) there exists \( v \in \mathcal{S}(\Omega) \) such that \( v \geq u \) in \( \Omega \);

(iii) every connected component of \( \Omega \) contains a point \( x_0 \) such that

\[
(G_\Omega \ast \mu)(x_0) < \infty.
\]

Furthermore, if one of the above (equivalent) conditions is satisfied, the function \( h \) appearing in (4.14) is the least harmonic majorant of \( u \) in \( \Omega \), i.e.,

\[
h = \inf \{ f \in H_{\mathcal{L}}(\Omega) : f \geq u \text{ in } \Omega \}.
\]
Proof. (i) ⇒ (ii). Since \( G_\Omega * \mu_u \) is \( \mathcal{H}_L \)-superharmonic and non-negative in \( \Omega \), from identity (4.14) we get \( u \leq h \) in \( \Omega \); hence, (ii) holds with \( v := h \in \mathcal{H}_L(\Omega) \).

(ii) ⇔ (iii). This follows by combining Theorems 4.4 and 3.11.

(iii) ⇒ (i). Without of loss of generality, we can assume that \( \Omega \) is connected. Then, the validity of assumption (iii) and Theorem 3.11 imply that

\[
G_\Omega * \mu_u \in \mathcal{S}(\Omega). \tag{4.15}
\]

Let now \( \{ \Omega_n \}_n \) be a sequence of bounded open sets in \( \mathbb{R}^N \) such that

\[
\Omega_n \subseteq \Omega_{n+1} \subseteq \Omega \quad \text{and} \quad \Omega = \bigcup_n \Omega_n.
\]

Moreover, for every fixed \( n \in \mathbb{N} \), we set \( \mu_n := \mu_u|_{\overline{\Omega_n}} \). On account of Theorem 4.2 (applied here with \( \Omega_1 = \Omega_n \) and \( U = \Omega \)), there exists \( h_n \in \mathcal{H}_L(\Omega_n) \) such that

\[
u(x) = h_n(x) - (G_\Omega * \mu_n)(x) \quad \forall x \in \Omega_n \quad \text{and} \quad n \in \mathbb{N}; \tag{4.16}
\]

on the other hand, since \( G_\Omega * \mu \not\uparrow \Omega * \mu \) pointwise in \( \Omega \) as \( n \to \infty \), from (4.15) we deduce the existence of a dense set \( D \subseteq \Omega \) such that

\[
(G_\Omega * \mu_n)(x) \leq (G_\Omega * \mu)(x) < \infty \quad \forall x \in \Omega \cap D \quad \text{and} \quad n \in \mathbb{N}.
\]

This, together with (4.16), implies that

\[
\nu_n(x) \leq \nu_{n+k}(x) \leq \nu(x) + (G_\Omega * \mu_n)(x) \quad \forall x \in \Omega_n \cap D \quad \text{and} \quad n, k \in \mathbb{N}. \tag{4.17}
\]

Now, since \( h_n \in C(\Omega_n) \) and \( D \) is dense, from (4.17) we have

\[
h_n(x) \leq \nu_{n+k}(x) \quad \forall x \in \Omega_n \quad \text{and} \quad n, k \in \mathbb{N};
\]

moreover, since \( u < \infty \) in \( \Omega \) (as \( u \in \mathcal{S}(\Omega) \)), we also have

\[
\sup_{\Omega_n \cap D} \nu_{n+k} < \infty.
\]

As a consequence, the Brelot Convergence Axiom ensures the existence of a (unique) function \( h : \Omega \to \mathbb{R} \), \( \mathcal{H}_L \)-harmonic in \( \Omega \), such that

\[
h(x) = \lim_{k \to \infty} h_{n+k}(x) \quad \forall x \in \Omega_n \quad \text{and} \quad k \in \mathbb{N}. \tag{4.18}
\]

By combining (4.18) and (4.16), we then obtain (4.14).

Finally, let us assume that one of the (equivalent) conditions (i)-to-(iii) is satisfied, and let \( h \in \mathcal{H}_L(\Omega) \) be such that (4.14) holds. Since \( G_\Omega \geq 0 \), we have

\[
u = h - G_\Omega * \mu \leq h \quad \text{pointwise in} \ \Omega,
\]

and thus \( h \) is a harmonic majorant of \( u \). On the other hand, if \( f \in \mathcal{H}_L(\Omega) \) is such that \( f \geq u \) in \( \Omega \), it follows from (4.14) that

\[
h - f \leq G_\Omega * \mu \quad \text{pointwise in} \ \Omega;
\]

hence, by Theorem 3.15 we get \( h - f \leq 0 \), and thus \( h \leq f \). Due the arbitrariness of \( f \), this proves that \( h \) is the least harmonic majorant of \( u \) in \( \Omega \).

As an application of this theorem, we get the following results.

**Corollary 4.6** (Riesz representation on compact sets). Let \( U \subseteq \mathbb{R}^N \) be an arbitrary open set, and let \( u \in \mathcal{S}(U) \). Moreover, let \( \Omega \subseteq U \). Then,

\[
u(x) = h(x) - (G_\Omega * \mu_u)(x) \quad \text{pointwise in} \ \Omega, \tag{4.19}
\]

where \( h \in \mathcal{H}_L(\Omega) \) is the least harmonic majorant of \( u \) in \( \Omega \).
Proof. First of all, since \( u \) is u.s.c. in \( U \) and \( \overline{\Omega} \subseteq U \) is compact, there exists \( M \in \mathbb{R} \) such that \( M \geq u \) on \( \Omega \); as a consequence, since the constant function \( h := M \) is \( \mathcal{H}_L \)-harmonic, the desired (4.19) follows immediately from Theorem 4.5.

**Corollary 4.7** (Riesz representation - III). Let \( \Omega \subseteq \mathbb{R}^N \) be open, and let \( u \in \mathcal{S}(\Omega) \) be such that \( \mathcal{L}u = 0 \) outside a compact set \( K \subseteq \Omega \). Then,

\[
u = h - G_{\Omega} \ast \mu \quad \text{pointwise in} \quad \Omega,
\]
where \( h \in \mathcal{H}(\Omega) \) is the least harmonic majorant of \( u \) in \( \Omega \).

**Proof.** Since \( \mathcal{L}u = 0 \) on \( \Omega \setminus K \), the measure \( \mu_u \) has compact support \( K \); we then infer from Corollary 3.14 that \( G_{\Omega} \ast \mu_u \in \mathcal{S}(\Omega) \), and thus \( G_{\Omega} \ast \mu_u \) is finite in a dense subset of \( \Omega \).

This, together with Theorem 4.5, immediately implies (4.20).

**Corollary 4.8** (Riesz representation in space). Let \( u \in \mathcal{S}(\mathbb{R}^N) \) be such that

\[
U := \sup_{\mathbb{R}^N} u < \infty.
\]

Then, there exists \( h \in \mathcal{H}(\mathbb{R}^N) \), \( h \leq 0 \), such that

\[
u = h + U - \int_{\mathbb{R}^N} \Gamma x \, d\mu_u \quad \text{pointwise in} \quad \mathbb{R}^N.
\]

**Proof.** On account of (4.21), the constant function \( U \) is \( \mathcal{H}_L \)-harmonic in \( \mathbb{R}^N \) and satisfies \( U \geq u \); thus, by Theorem 4.5 and Remark 3.6 we have

\[
u = h_0 - G_{\mathbb{R}^N} \ast \mu_u = h_0 - \int_{\mathbb{R}^N} \Gamma x \, d\mu_u \quad \text{pointwise in} \quad \mathbb{R}^N,
\]
where \( h_0 \in \mathcal{H}_L(\mathbb{R}^N) \) is the least harmonic majorant of \( u \) in \( \mathbb{R}^N \). On the other hand, since the constant \( U \) is a harmonic majorant of \( u \) we get

\[
h_0 \leq U \quad \text{pointwise in} \quad \mathbb{R}^N,
\]
and (4.22) follows from (4.23) by choosing \( h := h_0 - U \leq 0 \).

In the particular case when the operator \( \mathcal{L} \) is known to satisfy a Liouville-type theorem, Corollary 4.8 boils down to the following representation theorem.

**Theorem 4.9.** Let \( u \in \mathcal{S}(\mathbb{R}^N) \) be such that \( U := \sup_{\mathbb{R}^N} u < \infty \). Moreover, let us assume that \( \mathcal{L} \) satisfies the following Liouville-type theorem:

**\( \textbf{(L)} \):** if \( h \in \mathcal{H}_L(\mathbb{R}^N) \) is bounded from above, then \( h \) is constant in \( \mathbb{R}^N \).

Then, we have the global representation formula

\[
u = U - \int_{\mathbb{R}^N} \Gamma x \, d\mu_u \quad \text{pointwise in} \quad \mathbb{R}^N.
\]

**Proof.** On account of Corollary 4.8, there exists a function \( h : \mathbb{R}^N \to \mathbb{R} \), \( \mathcal{H}_L \)-harmonic and non-positive on the whole of \( \mathbb{R}^N \), such that,

\[
u = h + U - \int_{\mathbb{R}^N} \Gamma x \, d\mu_u \quad \text{pointwise in} \quad \mathbb{R}^N.
\]

On the other hand, since \( h \leq 0 \) in \( \mathbb{R}^N \), from the Liouville-type Theorem for \( \mathcal{L} \) we derive that \( h \) is constant throughout \( \mathbb{R}^N \), that is,

\[
h \equiv h_0 \in \mathbb{R}^N \quad \text{for some} \quad h_0 \in \mathbb{R}, \quad h_0 \leq 0.
\]
This, together with (4.25), implies that
\[ U = \sup_{\mathbb{R}^N} u \leq U + h_0 \leq U, \]
and thus \( h_0 = 0 \). The desired (4.24) now follows immediately from (4.25).

**Remark 4.10.** The validity of (one-side) Liouville-type theorems for a general second-order operator \( L \) is far from being obvious, and it usually requires some global assumptions on \( L \). In this direction, we highlight the following examples:

1. if \( L = \Delta_G \) is a sub-Laplace operator on some Carnot group \( G = (\mathbb{R}^N, *, D_\lambda) \),
   then \( L \) satisfies the one-side Liouville-type theorem \((L)\);
2. if \( L = \sum_{j=1}^m X_j^2 \) is a sum of squares of homogeneous Hörmander vector fields,
   then \( L \) satisfies the one-side Liouville-type theorem \((L)\).

For a proof of these facts see, respectively, [19, Thm. 5.8.1] and [13, Prop. 5.6].

**The Poisson-Jensen formula.** By exploiting the representation theorems established so far, we can now prove Theorem 1.2. For the sake of readability, we rewrite the statement of this Poisson-Jensen-type theorem here.

**Theorem 4.11.** Let \( U \subseteq \mathbb{R}^N \) be an arbitrary open set, and let \( \Omega \Subset U \) be \( \mathcal{H}_L \)-regular. Moreover, let \( u \in \mathcal{S}(U) \). Then, we have the representation formula
\[
    u(x) = \int_{\partial \Omega} u \, d\mu_x^\Omega - (G_\Omega \ast \mu_u)(x) \quad \text{pointwise in } \Omega, \tag{4.26}
\]
where \( G_\Omega \) is the \( L \)-Green function of \( \Omega \) and \( \mu_x^\Omega \) is the \( \mathcal{H}_L \)-harmonic measure related with (the \( \mathcal{H}_L \)-regular open set) \( \Omega \) and \( x \).

**Proof.** We begin with some preliminary observations. First of all, since \( u \in \mathcal{S}(U) \) and the open set \( \Omega \) is compactly contained in \( U \), by Corollary 4.6 we can write
\[
    u(x) = f(x) - (G_\Omega \ast \mu_u)(x) \quad \text{pointwise in } \Omega,
\]
where \( f \in \mathcal{H}_L(\Omega) \) is the least harmonic majorant of \( u \) in \( \Omega \). In particular, \( G_\Omega \ast \mu_u \) is \( \mathcal{H}_L \)-superharmonic in \( \Omega \). Moreover, by Theorem A.3 we have
\[
    \Omega \ni x \mapsto \int_{\partial \Omega} u \, d\mu_x^\Omega \in \mathcal{H}_L(\Omega).
\]
Let now \( O \Subset \mathbb{R}^N \) be a fixed open set satisfying \( \Omega \Subset O \) and \( O \Subset U \). On account of Theorem 4.2, there exists a function \( h \in \mathcal{H}_L(O) \) such that
\[
    u(x) = h(x) - \int_{\partial \Omega} \Gamma_x \, d\mu_u =: h(x) + v(x) \quad \text{pointwise in } O; \tag{4.27}
\]
in particular, \( v = -G_{\mathbb{R}^N} \ast (\mu_u|_{\overline{O}}) \in \mathcal{S}(\mathbb{R}^N) \) and
\[
    \mu_v = \mu_u|_{\overline{O}} = \mu_u \quad \text{on } O \ni \Omega.
\]
We then turn to prove that (4.26) holds with \( u \) replaced by \( v \), that is,
\[
    v(x) = \int_{\partial \Omega} v \, d\mu_x^\Omega - (G_\Omega \ast \mu_u)(x) \quad \text{pointwise in } \Omega. \tag{4.28}
\]
To this end, we arbitrarily fix \( x_0 \in \Omega \) and we distinguish two cases. If \( v(x_0) = -\infty \), we apply Theorem 4.5: since \( v \leq 0 \) in \( \Omega \) (and \( w := 0 \) is \( \mathcal{H}_L \)-harmonic), we have
\[
    -\infty = v(x_0) = g(x_0) - (G_\Omega \ast \mu_u)(x_0) = g(x_0) - (G_\Omega \ast \mu_u)(x_0), \tag{4.29}
\]
where \( g \in \mathcal{H}_L(\Omega) \) is the least harmonic majorant of \( v \) in \( \Omega \). On the other hand, since \( v \in \mathcal{S}(\mathbb{R}^N) \) and \( \Omega \) is \( \mathcal{H}_L \)-regular, by Theorem A.3-(3) we know that

\[
\Omega \ni x \mapsto \int_{\partial\Omega} v \, d\mu^\Omega_x
\]

is \( \mathcal{H}_L \)-harmonic in \( \Omega \), and hence real-valued. This, together with (4.29), readily implies that (4.28) is satisfied at \( x_0 \). We then suppose that

\[
v(x_0) = -\int_{\Omega} \Gamma_{x_0} d\mu_u > -\infty, \tag{4.30}
\]

and we observe that, by Fubini’s theorem and the symmetry of \( \Gamma \), we have

\[
\int_{\partial\Omega} v \, d\mu^\Omega_{x_0} = -\int_{\Omega} \left( \int_{\partial\Omega} \Gamma_z \, d\mu^\Omega_{x_0} \right) d\mu_u(z).
\]

Then, we turn to establish the following key identity:

\[
\int_{\partial\Omega} \Gamma_z \, d\mu^\Omega_{x_0} = \begin{cases} 
\Gamma_z(x_0) & \text{if } z \in \overline{\Omega} \setminus \Omega, \\
\Gamma_z(x_0) - G_{\Omega}(z, x_0) & \text{if } z \in \Omega \setminus \{x_0\},
\end{cases} \tag{4.31}
\]

If \( z \in \overline{\Omega} \setminus \Omega \), we know from (2.4) that \( \Gamma_z \) is \( \mathcal{H}_L \)-harmonic in \( \mathbb{R}^N \setminus \{z\} \supset \Omega \); hence, reminding that \( \Omega \) is \( \mathcal{H}_L \)-regular, we obtain (see also Lemma A.1)

\[
\int_{\partial\Omega} \Gamma_z \, d\mu^\Omega_{x_0} = \Gamma_z(x_0).
\]

If, instead, \( z \in \Omega \setminus \{x_0\} \), we exploit the representation of \( G_{\Omega} \) given in (3.3): reminding that \( \Omega \) is \( \mathcal{H}_L \)-regular (and \( z \neq x_0 \)), we immediately derive

\[
h_z(x_0) = \int_{\partial\Omega} \Gamma_z \, d\mu^\Omega_{x_0} = \Gamma(z, x_0) - G(z, x_0).
\]

Finally, we prove (4.31) when \( z \in \partial\Omega \). To this end, we consider the map

\[
w(t) := \Gamma_t(x_0) - \int_{\partial\Omega} \Gamma_t \, d\mu^\Omega_{x_0} = \Gamma_{x_0}(t) - (G_{\mathbb{R}^N} * \mu^\Omega_{x_0})(t),
\]

defined for \( t \in O \setminus \{x_0\} \), and we show that \( w \equiv 0 \) on \( \partial\Omega \).

First of all, since \( \Gamma_{x_0} \) is \( \mathcal{H}_L \)-harmonic in \( \mathbb{R}^N \setminus \{x_0\} \) and \( \mu^\Omega_{x_0} \) can be thought of as a Radon measure in \( \mathbb{R}^N \) with compact support \( \partial\Omega \), by Corollary 3.14 we have that \( w \in \mathcal{S}(O) \); moreover, since identity (4.31) holds in \( O \setminus \partial\Omega \) and

\[
G(t, x_0) = G(x_0, t) \to 0 \text{ as } t \to \xi \in \partial\Omega
\]

(see Remark 3.1 and remind that \( \Omega \) is \( \mathcal{H}_L \)-regular), one has

\[
\lim_{t \to \xi \atop t \in \partial\Omega} w(t) = 0 \quad \text{for every } \xi \in \partial\Omega. \tag{4.32}
\]

In particular, since \( w \) is u.s.c. on \( O \) (as \( w \in \mathcal{S}(O) \)), from (4.32) we readily infer that \( w \geq 0 \) on \( \partial\Omega \). We then suppose, by contradiction, that

\[
M := \sup_{\partial\Omega} w = \max_{\partial\Omega} w > 0,
\]

and we choose \( z_0 \in \partial\Omega \) such that \( w(z_0) = M \). On account of (4.32), it is possible to find a small \( r > 0 \) such that \( B := B(z_0, r) \subseteq O, \; x_0 \notin B \) and

\[
\sup_B w = \max_{\partial\Omega} w = w(z_0) > 0.
\]
On the other hand, since \( w \in \mathcal{S}(B) \) (and the constant functions are \( \mathcal{H}_L \)-harmonic), from the Minimum Principle in Theorem A.2, we deduce that
\[
w \equiv w(z_0) > 0 \quad \text{throughout} \quad B = B(z_0, r).
\]
As a consequence, we obtain
\[
\lim_{t \to z_0} w(t) = w(z_0) > 0,
\]
but this is contradiction with (4.32). Hence, \( w \equiv 0 \) on \( \partial \Omega \), as desired.

Now we have established (4.31), we are ready to prove (4.28). In fact, on account of (4.30) we have \( \mu_u(\{ x_0 \}) = 0 \); as a consequence, from (4.31) we obtain
\[
\int_{\partial \Omega} v \, d\mu^\Omega_{x_0} = - \int_{\Omega \setminus \{ x_0 \}} \Gamma_x(x_0) \, d\mu_u(z) + \int_{\Omega \setminus \{ x_0 \}} G_{\Omega}(z, x_0) \, d\mu_u(z)
\]
\[
= - \int_{\Omega} \Gamma_x(x_0) \, d\mu_u(z) + \int_{\Omega} G_{\Omega}(z, x_0) \, d\mu_u(z)
\]
(by the symmetry of \( \Gamma \) and \( G_{\Omega} \))
\[
= v(x_0) + (G_{\Omega} \ast \mu_u)(x_0).
\]
Reminding that \( x \mapsto \int_{\partial \Omega} v \, d\mu^\Omega_{x} \in \mathcal{H}_L(\Omega) \), this gives (4.28). Finally, using (4.27), (4.28) and taking into account that \( h \in \mathcal{H}_L(O) \), we get
\[
u(x) = v(x) + h(x) = \int_{\partial \Omega} v \, d\mu^\Omega_{x} - (G_{\Omega} \ast \mu_u)(x) + h(x)
\]
(since \( \Omega \subseteq O \) is \( \mathcal{H}_L \)-regular)
\[
= \int_{\partial \Omega} v \, d\mu^\Omega_{x} - (G_{\Omega} \ast \mu_u)(x) + \int_{\partial \Omega} h \, d\mu^\Omega_{x} = \int_{\partial \Omega} u \, d\mu^\Omega_{x} - (G_{\Omega} \ast \mu_u)(x).
\]
This ends the proof. \( \square \)

If, in the previous theorem, we take \( \Omega = \Omega_r(x) \), we obtain the following extension of the \( \mathcal{L} \)-representation formulas contained in [17, Thm. 3.3].

**Theorem 4.12.** Let \( \Omega \subseteq \mathbb{R}^N \) be an arbitrary open set, and let \( u \in \mathcal{S}(\Omega) \). Moreover, let \( x_0 \in \Omega \) and \( r > 0 \) be such that \( \Omega_r(x_0) \subseteq \Omega \). Then, we have
\[
u(x_0) = m_r(u)(x_0) - \int_{\Omega_r(x_0)} \left( \Gamma_{x_0} - \frac{1}{r} \right) \, d\mu_u,
\]
(4.33)
\[
u(x_0) = M_r(u)(x_0) - \frac{\alpha + 1}{\alpha + 1} \int_0^r \rho^{\alpha} \left( \int_{\Omega_r(x)} \left( \Gamma_{x_0} - \frac{1}{\rho} \right) \, d\mu_u \right) \, d\rho,
\]
(4.34)
where \( m_r, M_r \) are the mean-integral operators introduced in (2.10).

**Proof.** We first remind that, on account of Example 2.4, the open set \( \Omega_r(x_0) \subseteq \Omega \) is \( \mathcal{H}_L \)-regular; hence, we are entitled to apply Theorem 4.11, obtaining
\[
u(x_0) = \int_{\partial \Omega_r(x_0)} u \, d\mu^\Omega_{x_0}(x_0) - \int_{\Omega_r(x_0)} G_{\Omega_r(x_0)}(x_0, \cdot) \, d\mu_u.
\]
(4.35)
We now claim that formula (4.33) is just a restatement of (4.35). Indeed, on the one hand, we have already recognized in Remark 3.3 that
\[
G_{\Omega_r(x_0)}(x_0, \cdot) = \Gamma_{x_0} - \frac{1}{r};
\]
on the other hand, using the expression of $\mu_{x_0}^{Ω_r(x_0)}$ given in Remark 2.6, we get
\[
\int_{\partial Ω_r(x_0)} u \, d\mu_{x_0}^{Ω_r(x_0)} = \int_{\partial Ω_r(x_0)} u \mathcal{K}_{x_0} \, d\sigma = m_r(u)(x_0).
\]
By inserting these information into (4.35), we immediately obtain (4.33). As for identity (4.34), it readily follows from (4.33), bearing in mind that
\[
M_r(u)(x_0) = \frac{\alpha + 1}{r^{\alpha + 1}} \int_0^r \rho^\alpha m_\rho(u)(x_0) \, d\rho
\]
(see Remark 2.5-(4)). This ends the proof. □

Remark 4.13. We end this section with a couple of final remarks.

(1) The representation formulas (4.33)-(4.34) have also been proved, with a different approach, in [18, Cor. 1.3]. In that paper, however, the Poisson-Jensen formula is established only for $Ω = Ω_r(x)$.

(2) Taking into account the results presented at the end of Section 3, we deduce that all the representation theorems established in [2] when $Ω \in \mathcal{B}_L$ are particular cases of the ones proved in this section.

5. Bounded-above $\mathcal{L}$-subharmonic functions in $\mathbb{R}^N$. In this section we exploit the Poisson-Jensen formula in Theorem 4.11 in order to prove Theorem 1.3. Also in this case, for the sake of readability we rewrite the statement of this theorem here.

Theorem 5.1 ($\mathcal{L}$-Riesz measure of a bounded-above $u \in S(\mathbb{R}^N)$). Let $\mu$ be a Radon measure in $\mathbb{R}^N$, and let $x_0 \in \mathbb{R}^N$. Then, the following conditions are equivalent:

1. there exists a function $u \in S(\mathbb{R}^N)$ such that
   (i) $u(x_0) > -\infty$;
   (ii) $u$ is bounded-above in $\mathbb{R}^N$, i.e., $U := \sup_{\mathbb{R}^N} u < \infty$;
   (iii) $\mu_u = \mu$ in $\mathbb{R}^N$;
2. $\mu$ satisfies the integrability property
   \[
   \int_0^\infty \frac{\mu(Ω_\rho(x_0))}{\rho^2} \, d\rho < \infty. \tag{5.1}
   \]

Proof. (1) $\Rightarrow$ (2). We can suppose, without loss of generality, that $x_0 = 0$; moreover, to simplify the notation, we define $n(\rho) := \mu(Ω_\rho(0))$.

Let now $u \in S(\mathbb{R}^N)$ satisfy (i)-to-(iii) (with $x_0 = 0$). Using [17, Thm. 4.2] and the fact that $m_r(1)(0) = 1$ for every $r > 0$ (see identity (2.11)), we get
\[
-\infty < u(0) \leq m_r(u)(0) \leq U \quad \text{for every } r > 0;
\]
this, together with the Poisson-Jensen formula (4.33), implies that
\[
\int_{Ω_r(0)} \left( \Gamma_0 - \frac{1}{r} \right) \, d\mu = m_r(u)(0) - u(0) \leq U - u(0) \quad \forall r > 0. \tag{5.2}
\]
On the other hand, bearing in mind the definition $Ω_r(0)$, we see that
\[
\Gamma_0(y) - \frac{1}{r} \geq \frac{1}{2} \Gamma_0(y) \quad \forall y \in Ω_\rho(0) \text{ and } 0 < \rho < r/2;
\]
as a consequence, from (5.2) we deduce that
\[
\frac{n(\rho)}{\rho} = \frac{\mu(Ω_\rho(0))}{\rho} \leq \int_{Ω_\rho(0)} \Gamma_0 \, d\mu \leq 2 \int_{Ω_\rho(0)} \left( \Gamma_0 - \frac{1}{r} \right) \, d\mu
\]
\[ \leq 2 \int_{\Omega_r(0)} \left( \Gamma_0 - \frac{1}{r} \right) d\mu \leq U - u(0) \quad \forall \ 0 < \rho < r/2. \]

In particular, \( n(t) \) is bounded in \((0, r/2)\), and
\[ \mu(\{0\}) = \lim_{\rho \to 0^+} n(\rho) = 0. \]

Gathering together all these facts, we then obtain
\[ U - u(0) \geq \int_{\Omega_r(0)\setminus\{0\}} \left( \Gamma_0(y) - \frac{1}{r} \right) d\mu(y) = \int_{\Omega_r(0)\setminus\{0\}} \left( \int_{1/\Gamma_0(y)}^{r} \frac{1}{\rho^2} \, d\rho \right) d\mu(y) \]
\[ = \int_0^r \frac{\mu(\Omega_\rho(0) \setminus \{0\})}{\rho^2} \, d\rho = \int_0^r \frac{n(\rho)}{\rho^2} \, d\rho, \]

and this estimate holds for every \( r > 0 \). Finally, letting \( r \to \infty \), we get
\[ \int_0^\infty \frac{n(\rho)}{\rho^2} \, d\rho \leq U - u(0) < \infty, \]

and this proves that \( \mu = \mu_u \) fulfills the integrability property (5.1).

(2) \( \Rightarrow \) (1). We assume that \( \mu \) satisfies (5.1), and we define
\[ u(x) := -\int_{\mathbb{R}^N} \Gamma_x \, d\mu = -(G_{\mathbb{R}^N} \ast \mu)(x) \quad (x \in \mathbb{R}^N). \]

Then, we claim that \( u \) enjoys the following property:
\[ u(0) > -\infty. \] (5.3)

Taking this claim for granted for a moment, we can easily complete the proof of the theorem. Indeed, on account of (5.3), by Theorems 3.11 and 3.13 we have
\[ u \in \mathcal{S}(\mathbb{R}^N) \quad \text{and} \quad \mu_u = \mu \in \mathbb{R}^N; \]

as a consequence, since one obviously has \( u \leq 0 \) in \( \mathbb{R}^N \), we conclude that \( u \) satisfies all the properties (i)-to-(iii) in the statement (with \( x_0 = 0 \)).

Hence, we turn to establish (5.3). To this end we first observe that, since \( \mu \) satisfies (5.1) and \( n(\rho) = \mu(\Omega_\rho(0)) \) is increasing on \((0, \infty)\), we necessarily have
\[ \mu(\{0\}) = \lim_{\rho \to 0^+} n(\rho) = 0. \]

As a consequence, we obtain the following computation:
\[ u(0) \geq -\int_{\mathbb{R}^N \setminus \{0\}} \Gamma_0(y) \, d\mu(y) = -\lim_{\lambda \to 0} \int_{\mathbb{R}^N \setminus \Omega_\lambda(0)} \Gamma_0(y) \, d\mu(y) \]
\[ = -\lim_{\lambda \to 0} \left\{ \frac{1}{\lambda} \int_{\mathbb{R}^N \setminus \Omega_\lambda(0)} \left( \int_{1/\Gamma_0(y)}^{\infty} \frac{1}{\rho^2} \, d\rho \right) \, d\mu(y) \right\} \]
\[ = -\lim_{\lambda \to 0} \int_{\lambda}^{\infty} \frac{\mu(\Omega_{\rho}(0) \setminus \{0\})}{\rho^2} \, d\rho = -\lim_{\lambda \to 0} \int_{\lambda}^{\infty} \frac{n(\rho)}{\rho^2} \, d\rho = -\int_{0}^{\infty} \frac{n(\rho)}{\rho^2} \, d\rho. \]

This, together with (5.1), implies that \( u(0) > -\infty \), and the proof is complete. \( \Box \)

As a matter of fact, it is possible to remove the explicit mention of the point \( x_0 \) from the statement of Theorem 5.1: indeed, the following result holds.

**Theorem 5.2** \( \mathcal{L} \)-Riesz measure of a bounded-above \( u \in \mathcal{S}(\mathbb{R}^N) \) - II. Let \( \mu \) be a Radon measure in \( \mathbb{R}^N \). Then, the following assertions are equivalent:
1. there exists a function \( u \in S(\mathbb{R}^N) \) such that
\[
U := \sup_{\mathbb{R}^N} u < \infty \quad \text{and} \quad \mu_u = \mu \text{ in } \mathbb{R}^N; \quad (5.4)
\]

2. \( \mu \) satisfies the integrability property
\[
\int_{1}^{\infty} \frac{\mu(\Omega_\rho(0))}{\rho^2} \, d\rho < \infty. \quad (5.5)
\]

Proof. (1) \( \Rightarrow \) (2). Let \( u \in S(\mathbb{R}^N) \) satisfy (5.4), and let \( V := \Omega_{1/2}(0) \). Reminding that \( V \) is \( \mathcal{H}_L \)-regular, we consider the Perron regularization of \( u \) on \( V \), i.e.,
\[
\hat{u} := \begin{cases} u(x), & \text{if } x \notin V; \\
\int_{\partial V} u \, d\mu_{V}, & \text{if } x \in V.
\end{cases}
\]

Since \( u \in S(\mathbb{R}^N) \), we know from Theorem A.4 that \( \hat{u} \in S(\mathbb{R}^N) \) and \( \hat{u}|_V \in H_L(V) \); in particular, \( \hat{u} \) is real-valued on \( V \), and \( \hat{u}(0) > -\infty \). Moreover, we have
\[
\hat{u} \leq u \leq \sup_{\mathbb{R}^N} u = U < \infty.
\]

We are then entitled to apply Theorem 5.1 to \( \hat{v} \), obtaining
\[
\int_{0}^{\infty} \frac{\mu_\rho(\Omega_\rho(0))}{\rho^2} \, d\rho < \infty. \quad (5.6)
\]

We now observe that, since \( u \equiv \hat{u} \) on \( \mathbb{R}^N \setminus V \) and (5.4) holds, one has
\[
\mu_v = \mu \text{ on } \mathbb{R}^N \setminus V;
\]
as a consequence, for every \( \rho > 1 \) we get
\[
\mu(\Omega_\rho(0)) = \mu(\Omega_1(0)) + \mu(\Omega_\rho(0) \setminus \Omega_1(0)) = \mu(\Omega_\rho(0)) + \mu_v(\Omega_\rho(0) \setminus \Omega_1(0)),
\]

By using this identity in (5.6), we immediately obtain (5.5).

(2) \( \Rightarrow \) (1). We assume that \( \mu \) satisfies (5.5), and we define
\[
\mu_1 := |_{\Omega_{1/2}(0)}, \quad \mu_2 := |_{\mathbb{R}^N \setminus \Omega_{1/2}(0)}.
\]

Since, by definition, \( \mu_2(\Omega_\rho(0)) = 0 \) for every \( 0 < \rho < 2 \), from (5.5) we get
\[
\int_{0}^{\infty} \frac{\mu_2(\Omega_\rho(0))}{\rho^2} \, d\rho = \int_{1}^{\infty} \frac{\mu_2(\Omega_\rho(0))}{\rho^2} \, d\rho \leq \int_{1}^{\infty} \frac{\mu(\Omega_\rho(0))}{\rho^2} \, d\rho < \infty;
\]
as a consequence, we derive from Theorem 5.1 that there exists \( w \in S(\mathbb{R}^N) \) such that \( W := \sup_{\mathbb{R}^N} w < \infty \) and \( \mu_w = \mu_2 \) in \( \mathbb{R}^N \). On the other hand, since \( \mu_1 \) has compact support, by Corollary 3.14 and Theorem 3.13 we have
\[
v := -(G_{\mathbb{R}^N} * \mu_1) \in S(\mathbb{R}^N) \quad \text{and} \quad \mu_v = \mu_1 \text{ in } \mathbb{R}^N.
\]

Gathering together all these facts, and setting \( u := v + w \), we then get
(a) \( u \in S(\mathbb{R}^N) \);
(b) \( u \leq w \leq W \) on \( \mathbb{R}^N \);
(c) \( \mu_u = \mu_v + \mu_w = \mu \) in \( \mathbb{R}^N \).

Hence, \( u \) satisfies (5.4), and the proof is complete. \( \square \)

By combining Theorem 5.2 with Corollary 4.8, we easily obtain the following complete characterization of bounded-above \( \mathcal{H}_L \)-subharmonic functions.
Theorem 5.3. Let $\mu$ be a Radon measure in $\mathbb{R}^N$ such that
$$\int_1^{\infty} \frac{\mu(\Omega_\rho(0))}{\rho^2} \, d\rho < \infty. \quad (5.7)$$
Moreover, let $U \in \mathbb{R}$ be fixed. Then, the function
$$u(x) := U - (G_{\mathbb{R}^N} \ast \mu)(x) = U - \int_{\mathbb{R}^N} \Gamma_x \, d\mu, \quad (5.8)$$
is $H_\mathcal{L}$-subharmonic in $\mathbb{R}^N$ and satisfies
$$\sup_{\mathbb{R}^N} u = U, \quad \mu_u = \mu \text{ in } \mathbb{R}^N. \quad (5.9)$$
Moreover, if $\hat{u} \in \mathcal{S}(\mathbb{R}^N)$ is another function satisfying (5.9), then
$$\hat{u} = u + h, \quad \text{for some } h \in H_{\mathcal{L}}(\mathbb{R}^N), \ h \leq 0 \text{ in } \mathbb{R}^N.$$ 

Proof. First of all, since $\mu$ satisfies (5.7), from Theorem 5.2 we derive the existence of a function $w \in \mathcal{S}(\mathbb{R}^N)$ such that $W := \sup_{\mathbb{R}^N} w < \infty$ and $\mu_w = \mu$ in $\mathbb{R}^N$; as a consequence, by Corollary 4.8 there exists $h_0 \in H_{\mathcal{L}}(\mathbb{R}^N)$, $h_0 \leq 0$, such that
$$w(x) = W + h_0 - (G_{\mathbb{R}^N} \ast \mu)(x) = W + h_0 - \int_{\mathbb{R}^N} \Gamma_x \, d\mu \quad \text{pointwise in } \mathbb{R}^N.$$ 
Reminding that the constant functions are $H_\mathcal{L}$-harmonic, we then get
$$u = w + (U - W - h_0) \in \mathcal{S}(\mathbb{R}^N) \quad \text{and} \quad \mu_u = \mu_w = \mu \text{ in } \mathbb{R}^N.$$ 
In particular, there exists a dense set $D \subseteq \mathbb{R}^N$ such that
$$u(x) > -\infty \iff (G_{\mathbb{R}^N} \ast \mu)(x) < \infty \quad \text{for every } x \in D.$$ 
We now turn to prove that $\sup_{\mathbb{R}^N} u = U$. To this end we notice that, since $\Gamma_x > 0$, we have $M := \sup_{\mathbb{R}^N} u \leq U$; hence, again by Corollary 4.8, we can write
$$U - \int_{\mathbb{R}^N} \Gamma_x \, d\mu = u(x) = M + f - \int_{\mathbb{R}^N} \Gamma_x \, d\mu \quad \text{pointwise in } \mathbb{R}^N,$$
where $f : \mathbb{R}^N \to \mathbb{R}$ is a non-positive $H_{\mathcal{L}}$-harmonic function in $\mathbb{R}^N$. On the other hand, choosing a point $x_0 \in \mathbb{R}^N$ such that $(G_{\mathbb{R}^N} \ast \mu)(x_0) < \infty$, we obtain
$$U = M + f(x_0) \leq M,$$
and thus $U = M$, as desired. Summing up, we have proved that the function $u$ defined in (5.8) is $H_{\mathcal{L}}$-subharmonic in $\mathbb{R}^N$ and satisfies (5.9). Finally, if $\hat{u} \in \mathcal{S}(\mathbb{R}^N)$ is another function satisfying (5.9), by Corollary 4.8 we get
$$\hat{u}(x) = U + h - \int_{\mathbb{R}^N} \Gamma_x \, d\mu = u(x) + h \quad \text{pointwise in } \mathbb{R}^N,$$
for a suitable $h \in H_{\mathcal{L}}(\mathbb{R}^N)$, $h \leq 0$, and the proof is complete. \hfill \Box

Corollary 5.4. Suppose that $\mathcal{L}$ satisfies the following Liouville-type theorem:

(L): if $h \in H_{\mathcal{L}}(\mathbb{R}^N)$ is bounded from above, then $h$ is constant in $\mathbb{R}^N$.
Moreover, let $\mu$ be a Radon measure in $\mathbb{R}^N$ such that
$$\int_1^{\infty} \frac{\mu(\Omega_\rho(0))}{\rho^2} \, d\rho < \infty.$$ 
Then, for every $U \in \mathbb{R}$ there exists a unique $u \in \mathcal{S}(\mathbb{R}^N)$ such that
$$\sup_{\mathbb{R}^N} u = U \quad \text{and} \quad \mu_u = \mu \text{ in } \mathbb{R}^N. \quad (5.10)$$
More precisely, this function \( u \) is given by
\[
  u(x) = U - \int_{\mathbb{R}^N} \Gamma_x \, d\mu.
\] (5.11)

**Proof.** On account of Theorem 5.3, the function \( u \) defined in (5.11) is \( \mathcal{H}_{\mathcal{L}} \)-subharmonic in \( \mathbb{R}^N \) and satisfies (5.10). On the other hand, if \( \hat{u} \in \mathcal{S}(\mathbb{R}^N) \) is another function satisfying (5.10), it follows from Theorem 4.9 that
\[
  \hat{u}(x) = U - \int_{\mathbb{R}^N} \Gamma_x \, d\mu = u(x) \quad \text{pointwise in } \mathbb{R}^N,
\]
and this proves the uniqueness of \( u \).

**Appendix A. Basic results of abstract Potential Theory.** In this Appendix we collect the basic notions and results of abstract Potential Theory which have been used in the paper. Our main references are the survey notes [21, 22, 32] and the appendix of [30] (see also [4, 24] and [19, Chap. 6]).

**Harmonic spaces.** Let \((X, \tau)\) be a connected, locally connected, locally compact and non-compact Hausdorff space satisfying the Second Countability Axiom. For every fixed set \( \emptyset \neq A \subseteq X \), we indicate by \( C(A) \) the linear space of the real-valued functions which are continuous on \( A \). We say that a map \( \tau \ni V \mapsto H(V) \subseteq C(V) \) is a harmonic sheaf on \( X \) if the following properties are satisfied:
1. for every \( V \in \tau \), \( H(V) \) is a linear subspace of \( C(V) \);
2. if \( V_1, V_2 \in \tau \) and \( V_1 \subseteq V_2 \), then \( H(V_1) \subseteq H(V_2) \);
3. if \( \{V_i\}_{i \in I} \subseteq \tau \) and \( u : V := \bigcup_{i \in I} V_i \rightarrow \mathbb{R} \), then
   \[
   (u|_{V_i} \in H(V_i) \text{ for every } i \in I) \implies u \in H(V).
   \]
If \( V \in \tau \) and \( u \in H(V) \), we say that \( u \) is \( H \)-harmonic in \( V \).

Let now \( H \) be a fixed harmonic sheaf of the topological space \( X \). We say that an open set \( V \subseteq X \) is \( H \)-regular if it satisfies the properties listed below:

(i) \( \overline{V} \) is compact and \( \partial V \neq \emptyset \);
(ii) for every \( \varphi \in C(\partial V) \) there exists a unique \( H^V_\varphi \in H(V) \) such that
   (ii)_1 \( \lim_{x \rightarrow \xi} H^V_\varphi(x) = \varphi(\xi) \) for every \( \xi \in \partial V \);
   (ii)_2 \( H^V_\varphi \geq 0 \) on \( V \) if \( \varphi \geq 0 \) on \( \partial V \).

If \( V \in \tau \) is \( H \)-regular and \( x \in V \) is fixed, it follows from (ii) that the map
\[
  T : C(\partial V) \rightarrow \mathbb{R}, \quad T(\varphi) := H^V_\varphi(x)
\]
is linear and positive; as a consequence, the Riesz Representation Theorem (see, e.g., [36]) ensures the existence of a unique Radon measure \( \mu^V_x \) on \( V \) such that
\[
  H^V_\varphi(x) = T(\varphi) = \int_{\partial V} \varphi \, d\mu^V_x.
\]
This measure \( \mu^V_x \) is called the \( H \)-harmonic measure related with \( V \) and \( x \).

Thanks to these preliminaries, we can introduce the notion of abstract harmonic space: if \( H \) is a harmonic sheaf on the topological space \( (X, \tau) \), we say that the pair \( (X, H) \) is an abstract harmonic space if the following axioms hold:

(A1) there exists a basis for \( \tau \) consisting of \( H \)-regular open sets;
(A2) if $\Omega \subseteq X$ is a connected open set and \( \{u_n\}_n \) is an increasing sequence of $H$-harmonic functions in $\Omega$, then either
\[
\sup_{n \in \mathbb{N}} u_n \equiv \infty \text{ in } \Omega \quad \text{or} \quad u := \sup_{n \in \mathbb{N}} u_n \in \mathcal{H}(\Omega).
\]
Axiom (A1) is usually referred to as the Regularity Axiom, while axiom (A2) is referred to as the Brelot Convergence Axiom.

Quite surprisingly, a deep result by Mokobodzki (and contained in the survey notes [21]) shows that, if axiom (A1) holds, the ‘qualitative’ axiom (A2) is indeed equivalent to the following ‘qualitative’ Harnack axiom:

(AH) if $\Omega \subseteq X$ is a connected open set and $K \subseteq \Omega$ is compact, it is possible to find a constant $C = C(\Omega, K) \geq 1$ such that,
\[
\sup_K u \leq C \inf_K u
\]
for every function $u \in \mathcal{H}(\Omega)$ satisfying $u \geq 0$ in $\Omega$.

Throughout what follows it will be tacitly understood that $(X, H)$ is an abstract harmonic space, so that axioms (A1) and (A2)/(AH) are satisfied.

Super/sub-harmonic functions. Let $\emptyset \neq \Omega \subseteq X$ be an open set. A l.s.c. function $u : \Omega \rightarrow (-\infty, \infty]$ is called $H$-superharmonic in $\Omega$ if
\[
1. \ \{x \in \Omega : u(x) < \infty\} \text{ is dense in } U;
2. \ \text{for every } H\text{-regular open set } V \subseteq V \subseteq \Omega \text{ and for every function } \varphi \in C(\partial V) \text{ such that } u \geq \varphi \text{ on } \partial V, \text{ one has } u \geq H^V \varphi \text{ throughout } V.
\]
An u.s.c. function $s : \Omega \rightarrow [-\infty, \infty)$ is called $H$-subharmonic in $\Omega$ if $u := -s$ is $H$-superharmonic in $\Omega$.

We denote by $\mathfrak{S}(\Omega)$ and $\mathfrak{S}(U)$, respectively, the set of the $H$-superharmonic functions and the set of the $H$-subharmonic functions in $\Omega$.

By crucially exploiting the Regularity Axiom (and the ‘local’ nature of the notion of $H$-harmonicity), it is not difficult to recognize that
\[
\mathfrak{S}(\Omega) \cap \mathfrak{S}(\Omega) = \mathcal{H}(\Omega).
\]
In particular, we obtain the following simple (yet important) result.

Lemma A.1. A function $u \in C(\Omega)$ is $H$-harmonic in $U$ if and only if
\[
u(x) = \int_{\partial V} u d\mu^\Omega_x
\]
for every $H$-regular open set $V \subseteq V \subseteq \Omega$ and every $x \in V$.

A remarkable property of the $H$-superharmonic functions is that they satisfy the following Strong and Weak Minimum Principles.

Theorem A.2. Let $\Omega \subseteq X$ be open, and let $u \in \mathfrak{S}(\Omega)$. The following facts hold:
\[
1. \ \text{if } \Omega \text{ is connected and } u \geq 0 \text{ in } \Omega, \text{ then either } u \equiv 0 \text{ or } u > 0 \text{ in } \Omega;
2. \ \text{if } \Omega \text{ is relatively compact and there is } h \in \mathcal{H}(\Omega) \text{ s.t. } \inf_{\Omega} h > 0, \text{ then}
\[
\liminf_{x \to \xi} u(x) \geq 0 \text{ for every } \xi \in \partial \Omega \implies u \geq 0 \text{ in } \Omega.
\]
Using Theorem A.2, it is possible to establish a useful characterization of $H$-superharmonicity. More precisely, we have the following result.

**Theorem A.3.** Let $\Omega \subseteq X$ be open, and let $u : \Omega \to (-\infty, \infty]$ be a l.s.c. on $\Omega$ such that the set $D := \{x \in \Omega : u(x) < \infty\}$ is dense in $\Omega$.

Then, the following conditions are equivalent:

1. $u \in \mathcal{S}(\Omega)$;
2. for every $x_0 \in \Omega$ there exists a neighborhood basis $B_u(x_0)$ for $x_0$, possibly depending on $u$ and consisting of $H$-regular open sets, such that
   $$u(x_0) \geq \int_{\partial V} u d\mu_x^V$$
   for every $V \in B_u(x_0)$ satisfying $V \subseteq \overline{V} \subseteq \Omega$;
3. for every $H$-regular open set $V \subseteq \overline{V} \subseteq \Omega$, the map
   $$V \ni x \mapsto \int_{\partial V} u d\mu_x^V$$
is $H$-harmonic in $V$.

Using Theorem A.3, it is not difficult to prove the following result.

**Theorem A.4.** Let $\Omega \subseteq X$ be an arbitrary open set, and let $u \in \mathcal{S}(\Omega)$. Given a $H$-regular open set $V \subseteq \Omega$, we consider the function

$$u_V : \Omega \to (-\infty, \infty], \quad u_V(x) := \begin{cases} u(x), & \text{if } x \notin V; \\ \int_{\partial V} u d\mu_x^V, & \text{if } x \in V. \end{cases}$$

Then, the following properties hold:

(i) $u_V \leq u$ on $\Omega$;
(ii) $u_V \in \mathcal{S}(\Omega)$ and $(u_V)|_V \in \mathcal{H}(V)$.

The function $u_V$ in Theorem A.4 is called the Perron regularization of $u$ in $V$.

**Harmonic minorants and potentials.** Let $\Omega \subseteq X$ be a fixed open set, and let $u \in \mathcal{S}(\Omega)$. If there exists $s_0 \in \mathcal{S}(\Omega)$ such that $s_0 \leq u$ in $\Omega$, the map

$$u^\infty(x) := \sup \{s(x) : s \in \mathcal{S}(\Omega) \text{ and } s \leq u \text{ in } \Omega\}$$

is well-defined; moreover, by crucially exploiting the Brelot Convergence Axiom, it can be proved that $u^\infty$ is actually $H$-harmonic in $\Omega$. For this reason, $u^\infty$ is referred to as the greatest harmonic minorant of $u$ (in $\Omega$). Analogously, if $s \in \mathcal{S}(\Omega)$ and if there exists some $u_0 \in \mathcal{S}(\Omega)$ such that $s \leq u_0$ in $\Omega$, the map

$$s^\infty(x) := \inf \{u(x) : u \in \mathcal{S}(\Omega) \text{ and } s \leq u \text{ in } \Omega\}$$

is well-defined and $H$-harmonic in $\Omega$; for this reason, $s^\infty$ is referred to as the least harmonic majorant of $s$ (in $\Omega$).

The following proposition holds.

**Proposition A.5.** Let $\Omega \subseteq X$ be open, and let $u_1, u_2 \in \mathcal{S}(\Omega)$. If there exists the greatest harmonic minorant of $u_1$ and $u_2$ in $\Omega$, then

$$(u_1 + u_2)^\infty = u_1^\infty + u_2^\infty.$$
Lemma A.6. Let $U \subseteq X$ be open, and let $u \in \mathcal{F}(U)$. Moreover, let $\Omega \subseteq \overline{U} \subseteq U$ be $\mathcal{H}$-regular. If $u$ is continuous on an open neighborhood of $\partial \Omega$, we have

$$u^\infty(x) = \int_{\partial \Omega} u \, d\mu^\Omega_x = H^\Omega_u(x) \quad \text{for every } x \in \Omega.$$

Let now $\Omega \subseteq X$ be a fixed open set (not necessarily $\mathcal{H}$-regular), and let $p \in \mathcal{F}(\Omega)$ be non-negative. Since $s_0 = 0 \in \mathcal{H}(\Omega)$, the greatest harmonic minorant of $p$ in $\Omega$ is well-defined and non-negative; we say that $p$ is a $\mathcal{H}$-potential in $\Omega$ if

$$p^\infty_\Omega \equiv 0,$$

and we denote by $\mathcal{P}(\Omega)$ the set of the $\mathcal{H}$-potentials in $\Omega$.

If there exists a strictly positive potential $p_0$ in the whole space $X$, it is possible to completely characterize the $\mathcal{H}$-regular open sets via the notion of $\mathcal{H}$-barrier.

Given an relatively compact open set $\Omega \subseteq X$ and a point $\xi_0 \in \partial \Omega$, a $\mathcal{H}$-barrier (or a Bouligand function) for $\Omega$ at $\xi_0$ is a function $w : \Omega \cap U \rightarrow (-\infty, \infty]$, where $U$ is a suitable open neighborhood of $\xi_0$, satisfying the following properties:

(i) $w \in \mathcal{F}(\Omega \cap U)$;
(ii) $w > 0$ in $\Omega \cap U$;
(iii) $w(x) \rightarrow 0$ as $x \rightarrow \xi_0$.

Then, we have the following remarkable theorem due to Bouligand.

Theorem A.7 (Bouligand’s regularity theorem). Assume that there exists a strictly positive potential $p_0$ in the whole space $X$. Moreover, let $\Omega \subseteq X$ be a relatively compact open set with $\partial \Omega \neq \emptyset$. Then, the following conditions are equivalent:

1. $\Omega$ is $\mathcal{H}$-regular;
2. for every point $\xi_0 \in \partial \Omega$ there exists a $\mathcal{H}$-barrier for $\Omega$ at $\xi_0$.

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