ZERO-SURGERY CHARACTERIZES INFINITELY MANY KNOTS

JOHN A. BALDWIN AND STEVEN SIVEK

Abstract. We prove that 0 is a characterizing slope for infinitely many knots, namely the genus-1 knots whose knot Floer homology is 2-dimensional in the top Alexander grading, which we classified in recent work and which include all \((-3,3,2n+1)\) pretzel knots. This was previously only known for 5_2 and its mirror, as a corollary of that classification, and for the unknot, trefoils, and the figure eight by work of Gabai from 1987.

1. Introduction

A rational number \(r \in \mathbb{Q}\) is said to be a characterizing slope for a knot \(K \subset S^3\) if the orientation-preserving homeomorphism type of the manifold obtained via Dehn surgery on \(K\) of slope \(r\) uniquely determines \(K\); that is,

\[
\text{if } S^3_r(J) \cong S^3_r(K) \text{ then } J = K.
\]

It seems very hard to prove for most knots that any given integral slope is characterizing. This is especially true for slope 0: in his celebrated 1987 work [Gab87], Gabai proved that \(S^3_0(K)\) detects the genus of \(K\) and whether or not \(K\) is fibered, which immediately implies that 0-surgery characterizes the unknot (resolving the Property R Conjecture), trefoils, and figure eight. To our knowledge, the only other knots known to be characterized by their 0-surgeries are 5_2 and its mirror, which we proved in our recent work [BS22a]. The main result of this paper is that infinitely many knots are characterized by their 0-surgeries:

**Theorem 1.1.** Let \(K\) be any of the knots

\[15n_{43522}, \text{ Wh}^-(T_{2,3}, 2), \text{ Wh}^+(T_{2,3}, 2), P(-3,3,2n+1) \ (n \in \mathbb{Z}),\]

or their mirrors. Then 0 is a characterizing slope for \(K\).

Here, \(\text{Wh}^\pm(T_{2,3}, 2)\) is the 2-twisted Whitehead double of the right-handed trefoil, with a positive or a negative clasp, respectively, and the \(P(-3,3,2n+1)\) are pretzel knots. See Figure 1.

By contrast, there are many knots that are not characterized by their 0-surgeries. Brakes [Bra80] gave the first pairs of examples, and later Osoinach [Oso06] used annulus twisting to construct infinite families of examples. In fact, there can be infinitely many knots \(K_n\) with pairwise diffeomorphic 0-traces \(X_0(K_n)\), the result of attaching a 0-framed 2-handle to \(B^4\) along \(K_n\) [AJOT13]. Knots

![Figure 1](image-url)

**Figure 1.** The knots that Theorem 1.1 says are characterized by their 0-surgeries.
which are not smoothly concordant, or which have different slice genera, can nonetheless have diffeomorphic 0-surgeries [Yas15] or even 0-traces [MP18, Pic19]. Indeed, Piccirillo [Pic20] famously proved that the Conway knot is not slice by exhibiting a non-slice knot with the same 0-trace. Recently, Manolescu and Piccirillo [MP21] have given a systematic construction of pairs of knots with the same 0-surgeries, and used it as a source of potentially exotic 4-spheres.

In general, a major difficulty in Floer-theoretic approaches to proving that some integral slope characterizes a knot $K$ is that one must first identify all knots with the same knot Floer homology as $K$, and this was out of reach until recently for all but a handful of knots. However, Theorem 1.1 is made possible by our recent classification [BS22b] of all genus-1 nearly fibered knots:

**Theorem 1.2** ([BS22b, Theorem 1.2]). Let $K \subset S^3$ be a genus-1 knot with $\dim_{\mathbb{Q}} \hat{HFK}(K, 1) = 2$. Then up to mirroring $K$ must be one of

\begin{equation}
5_2, \ 15n_{43522}, \ \text{Wh}^- (T_{2,3}, 2)
\end{equation}

or

\begin{equation}
\text{Wh}^+(T_{2,3}, 2), \ P(-3, 3, 2n + 1) \ (n \in \mathbb{Z}),
\end{equation}

where the knots in (1.1) have Alexander polynomial $\Delta_K(t) = 2t - 3 + 2t^{-1}$ and determinant $|\Delta_K(-1)| = 7$, and those in (1.2) have Alexander polynomial $\Delta_K(t) = -2t + 5 - 2t^{-1}$ and determinant $|\Delta_K(-1)| = 9$.

For example, we were able to use this classification to prove in [BS22a] that all rational slopes besides the positive integers (i.e., not just 0) are characterizing for $5_2$:

**Theorem 1.3** ([BS22a, Theorem 1.1]). Every $r \in \mathbb{Q} \setminus \mathbb{Z}_{>0}$ is a characterizing slope for $5_2$.

We do not expect anything as strong as Theorem 1.3 to hold for the knots in Theorem 1.1. Indeed, Baker and Motegi [BM18, Example 4.1] proved that $P(-3, 3, 5)$ is not characterized by any non-zero integer surgeries. On the other hand, Theorem 1.1 gives an affirmative answer to [BM18, Question 4.4], which asked whether 0 might be a characterizing slope for $P(-3, 3, 5)$.

In this paper we assume some background in Heegaard Floer homology, but the Floer-theoretic techniques we use were all present in [BS22a]; the casual reader may be relieved to know that unlike in [BS22a], we make no use of the “mapping cone” formula for the Heegaard Floer homology of surgeries on a knot. On the other hand, Floer theoretic invariants cannot distinguish the 0-surgeries on any of the pretzel knots $P(-3, 3, 2n + 1)$, so we will eventually need to introduce some perturbative invariants defined by Ohtsuki [Oht10] which can tell them apart.

**Organization.** Theorem 1.1 is proved in several steps. In Section 2 we prove some general facts about 0-surgery on knots of genus one, and then we use these in Section 3 to prove Theorem 3.1 stating that 0-surgery characterizes $15n_{43522}$ and $\text{Wh}^-(T_{2,3}, 2)$ as well as their mirrors. In Section 4 we use JSJ decompositions to deal with $\text{Wh}^+(T_{2,3}, 2)$ and its mirror in Theorem 1.3. Then in Section 5 we use Ohtsuki’s invariants to prove in Theorem 5.4 that 0 is a characterizing slope for each of the pretzel knots $P(-3, 3, 2n + 1)$. We prove as a bonus in Proposition 5.5 that $r$-surgery distinguishes these pretzel knots for any $r \in \mathbb{Q}$.

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2. Zero-surgery on genus-one knots

We begin by introducing some general results that will let us reduce Theorem \[1.1\] to the case where \( J \) is one of the knots listed in Theorem \[1.2\].

**Proposition 2.1.** Let \( K \subset S^3 \) be a knot with Seifert genus 1, and suppose for some other knot \( J \subset S^3 \) that there is an orientation-preserving homeomorphism

\[
S^3_0(K) \cong S^3_0(J).
\]

Then \( J \) has genus 1 and the same Alexander polynomial as \( K \), and moreover

\[
dim \widehat{HFK}(K, 1) = \dim \widehat{HFK}(J, 1)
\]

over any field \( \mathbb{F} \).

**Proof.** The manifold \( S^3_0(J) \) determines the Alexander polynomial of \( J \), because the infinite cyclic covers of both \( S^3_0(J) \) and the knot exterior \( S^3 \setminus N(J) \) have the same first homology as \( \mathbb{Z}[t^\pm 1] \)-modules, so \( \Delta_K(t) = \Delta_J(t) \). Gabai [Gab87] proved that it also determines the Seifert genus \( g(J) \), so \( g(J) = g(K) = 1 \).

We now apply the surgery exact triangle of various surgeries on \( K \), which for the remainder of this proof we will always take with coefficients in a fixed field \( \mathbb{F} \). We recall that there is a smooth concordance invariant \( V_0(K) \in \mathbb{Z} \), defined by Rasmussen [Ras03], which can be extracted from the knot Floer complex \( CFK^\infty(K) \). Its precise definition does not matter here, except to note that it appears in computing the Heegaard Floer correction terms of surgeries on \( K \), by a formula of Ni and Wu [NW15, Proposition 1.6] which implies

\[
d(S^3_1(K)) = -2V_0(K)
\]

as a special case.

The correction terms of the zero-surgery on \( K \) satisfy

\[
d_{1/2}(S^3_0(K)) = \frac{1}{2} - 2V_0(K)
\]

\[
d_{-1/2}(S^3_0(K)) = -\frac{1}{2} + 2V_0(K),
\]

by [OS03, Proposition 4.12] and (2.1). The same is true for \( J \), and these correction terms for \( S^3_0(K) \) and \( S^3_0(J) \) must agree since \( S^3_0(K) \cong S^3_0(J) \), so we have

\[
V_0(K) = V_0(J).
\]

Now since \( g(K) = 1 \) we can apply [BS22a, Lemma 2.8] to see that \( HF^+_{\text{red}}(S^3_1(K)) \) is an \( \mathbb{F}[U] \)-module with trivial \( U \)-action, and that

\[
\dim HF^+_{\text{red}}(S^3_1(K)) = \dim \widehat{HFK}(K, 1) - V_0(K).
\]

This means that

\[
HF^+(S^3_1(K)) = \mathbb{F}[U, U^{-1}] \frac{1}{U \cdot \mathbb{F}[U]} \oplus \mathbb{F}^{\dim \widehat{HFK}(K, 1) - V_0(K)}
\]

as ungraded \( \mathbb{F}[U] \)-modules, so from the exact triangle

\[
\cdots \to \widehat{HF}(S^3_1(K)) \to HF^+(S^3_1(K)) \to HF^+(S^3_0(K)) \to \cdots
\]

we deduce that

\[
\dim \widehat{HF}(S^3_1(K)) = 2 \left( \dim \widehat{HFK}(K, 1) - V_0(K) \right) + 1.
\]

Now we apply the surgery exact triangle

\[
\cdots \to \widehat{HF}(S^3) \to \widehat{HF}(S^3_0(K)) \to \widehat{HF}(S^3_1(K)) \to \cdots
\]
to see that
\[ \dim \widehat{HF}(S^3_0(K)) = 0. \tag{2.3} \]
The same is true for $J$ since $g(J) = 1$ as well, namely
\[ \dim \widehat{HF}(S^3_0(J)) = 2 \left( \dim \widehat{HF}(J, 1) - V_0(J) \right) + 1 \pm 1. \tag{2.4} \]

But $\widehat{HF}(S^3_0(K)) \cong \widehat{HF}(S^3_0(J))$ since the two manifolds are the same, so we combine (2.3) and (2.4) together with (2.2) to get
\[ 2 \left( \dim \widehat{HF}(K, 1) - \dim \widehat{HF}(J, 1) \right) \in \{-2, 0, 2\}. \tag{2.5} \]

Now we recall that $\widehat{HF}(K)$ carries a $\mathbb{Z}$-valued Maslov grading, and that each $\widehat{HF}(K, i)$ has Euler characteristic equal to the $t^i$-coefficient of $\Delta_K(t)$. Since $\Delta_K(t) = \Delta_J(t)$, this means that
\[ \chi(\widehat{HF}(K, 1)) = \chi(\widehat{HF}(J, 1)), \]
and in particular this implies that
\[ \dim \widehat{HF}(K, 1) \equiv \dim \widehat{HF}(J, 1) \pmod{2}. \]

But then the left side of (2.5) is a multiple of 4, so it must be zero, and thus $\dim \widehat{HF}(K, 1) = \dim \widehat{HF}(J, 1)$ as claimed. \hfill \Box

**Remark 2.2.** The analogue of the $\widehat{HF}$ claim in Proposition 2.1 for $g \geq 2$ is that if $S^3_0(K) \cong S^3_0(J)$ then $\widehat{HF}(K, g) \cong \widehat{HF}(J, g)$. This has long been known because in that case [OS04, Corollary 4.5] identifies $\widehat{HF}(K, g)$ with $HF^+(S^3_0(K), s_{g-1})$ for a certain Spin$^c$ structure $s_{g-1}$.

### 3. The determinant-7 case

Proposition 2.1 allows us to take care of the knots in Theorem 1.2 with Alexander polynomial $2t - 3 + 2t^{-1}$, using only classical invariants from now on.

**Theorem 3.1.** Let $K$ be one of $15n_{43522}$, $Wh^-(T_{2,3}, 2)$, or their mirrors. If $S^3_0(K) \cong S^3_0(J)$ for some knot $J$, then $J$ is isotopic to $K$.

**Proof.** In each case we have $\Delta_K(t) = 2t - 3 + 2t^{-1}$ and $\dim_Q \widehat{HF}(K, 1) = 2$. Thus Proposition 2.1 says that the same is true of $J$, and then by Theorem 1.2 we know that $J$ must be one of the knots listed in (1.1) up to mirroring. In fact, it cannot be $5_2$ or its mirror, because we know from Theorem 1.3 that 0 is a characterizing slope for each of these.

Next, we claim that $J$ cannot be isotopic to the mirror $\overline{K}$. Indeed, if this is the case then
\[ S^3_0(K) \cong S^3_0(\overline{K}) \cong -S^3_0(K), \]
so if $\chi : H_1(S^3_0(K)) \cong \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the unique surjection then the Casson–Gordon invariant $\sigma_1(S^3_0(K), \chi)$ (see [CG78] Lemma 3.1) must be zero. This invariant is equal to minus the signature of $K$ [CG78], so it follows that $\sigma(K) = 0$. However, this is impossible because $\Delta_K(t)$ has a conjugate pair of simple roots on the unit circle, at
\[ t = \pm \frac{1}{4} (3 \pm i\sqrt{7}), \]
and these are its only roots. Thus the Tristram–Levine signature $\sigma_K(-1) = \sigma(K)$ must be $\pm 2$, giving a contradiction.

It now remains to be shown that if $K$ is $15n_{43522}$ or its mirror, then $J$ cannot be $Wh^-(T_{2,3}, 2)$ or its mirror, and vice versa. In other words, we need to show that
\[ \pm S^3_0(15n_{43522}) \neq \pm S^3_0(Wh^-(T_{2,3}, 2)), \]
and we do this by checking that they have different fundamental groups. This can be done in SnapPy \cite{CDGW} by counting 6-fold covers of each:

\texttt{In[1]: M = Manifold('15n43522(0,1)')}
\texttt{In[2]: N = Manifold('16n696530(0,1)')}
\texttt{In[3]: len(M.covers(6))}
\texttt{Out[3]: 3}
\texttt{In[4]: len(N.covers(6))}
\texttt{Out[4]: 21}

In particular, the fundamental groups of each have different numbers of index-6 subgroups, so they cannot be homeomorphic. \hfill \Box

\textbf{Remark 3.2.} Even with Proposition 2.1, we will need more than just classical invariants to address the knots in Theorem 1.2 with Alexander polynomial $-2t + 5 - 2t^{-1}$. For example, if $P$ is one of the pretzel knots $P(-3,3,2n+1)$, then $P$ is slice and so $\sigma(P) = 0$, meaning that the arguments used in Theorem 3.1 cannot even distinguish the 0-surgery on $P$ from the 0-surgery on its mirror.

4. The determinant-9 case, part 1

We now turn to the knots in Theorem 1.2 with Alexander polynomial $-2t + 5 - 2t^{-1}$. In order to do this, we will first discuss the JSJ decompositions of their 0-surgeries.

\textbf{Lemma 4.1.} Let $Y$ be the result of 0-surgery on $P(-3,3,2n+1)$ for some $n \in \mathbb{Z}$. Then $Y$ is a graph manifold: it has a single, non-separating JSJ torus, whose complement is Seifert fibered over the annulus.

\textbf{Proof.} We know that $Y$ is toroidal, because if $\Sigma$ is a genus-1 Seifert surface for $P = P(-3,3,2n+1)$ then it extends to a non-separating torus $\hat{\Sigma}$ after performing 0-surgery on $P$, and $\hat{\Sigma}$ is incompressible by \cite[Corollary 8.2]{Gab87}. Since $P$ is a Montesinos knot other than a trefoil, Ichihara and Jong \cite{JJ10} proved that $S^3_0(P)$ cannot be toroidal and Seifert fibered, so $Y$ is not Seifert fibered. On the other hand, if we cut $Y$ open along the torus $\hat{\Sigma}$ then Cantwell and Conlon \cite[Theorem 1.5]{CC93} proved that the resulting manifold is the complement of the $(2,4)$-torus link $T_{2,4} \subset S^3$, which is Seifert fibered over the annulus. \hfill \Box

\textbf{Lemma 4.2.} Let $Y$ be the result of 0-surgery on $W^{+}(T_{2,3},2)$. Then $Y$ is a graph manifold, and its JSJ decomposition consists of two pieces: one piece is the exterior of $T_{2,3}$, and the other is Seifert fibered over a pair of pants.

\textbf{Proof.} Let $W = W^{+}(T_{2,3},2)$. We observe that $W$ is a satellite, with companion $C = T_{2,3}$; its pattern $P$ has winding number 0, hence is not a 0- or 1-bridge braid in the solid torus $V = S^1 \times D^2$. This means that 0-surgery on the pattern $P \subset V$ produces a manifold with incompressible torus boundary, by \cite[Theorem 1.1]{Gab89}. Thus the companion torus $T = \partial N(C)$ in the exterior of $W$ remains incompressible in $Y = S^3_0(W)$. In particular $T$ is one of the JSJ tori of $S^3_0(W)$, and moreover it separates $S^3_0(W)$ into the union of $S^3 \setminus N(T_{2,3})$ (which is Seifert fibered) and $V_0(P)$.

We claim that $V_0(P)$ is not Seifert fibered. Indeed, if it were then all but at most one Dehn filling of its boundary would also be Seifert fibered. But for any $n$ we can realize one of these Dehn fillings by doing $(0,\frac{1}{n})$-surgery on the Whitehead link, and these are homeomorphic to 0-surgeries on infinitely many different twist knots. The only twist knots with a toroidal, Seifert fibered surgery are the trefoils \cite{JJ10}, however, so $V_0(P)$ cannot be Seifert fibered after all.

On the other hand, that the pattern $P$ has a genus-1 Seifert surface $\Sigma$ which lies entirely inside $V$, and which extends to a non-separating, incompressible torus $\hat{\Sigma}$ in $V_0(P) \subset S^3_0(W)$. According
Theorem 7.1], if we cut $S_0^3(W)$ open along $\hat{\Sigma}$ then we are left with the complement of the $(2,4)$-cable of $T_{2,3}$, where the companion torus is the same torus $T$ discussed above. It follows that cutting $V_0(P)$ along $\hat{\Sigma}$ produces the complement of a $(2,4)$-torus link in the solid torus, and this is Seifert fibered over a pair of pants. We conclude that $T$ and $\hat{\Sigma}$ are the JSJ tori of $S_0^3(W)$, and that $S_0^3(W)$ has the claimed JSJ decomposition.

Lemma 4.1 and 4.2 make it easy to distinguish 0-surgery on $Wh^+(T_{2,3}, 2)$ from the 0-surgeries on the $P(-3, 3, 2n + 1)$ pretzel knots.

**Theorem 4.3.** Let $K$ be either $Wh^+(T_{2,3}, 2)$ or its mirror. If $S_0^3(J) \cong S_0^3(K)$ for some knot $J \subset S^3$, then $J$ is isotopic to $K$.

**Proof.** By Proposition 2.1 we see that $J$ has genus 1 and top knot Floer homology

$$\overline{HFK}(J, 1; \mathbb{Q}) \cong \overline{HFK}(K, 1; \mathbb{Q}) \cong \mathbb{Q}^2,$$

and its Alexander polynomial is $-2t + 5 - 2t^{-1}$. According to Theorem 1.2, we therefore know that $J$ is either $K$, its mirror $\overline{K}$, or some pretzel knot $P(-3, 3, 2n + 1)$. (We note here that the mirror of $P(-3, 3, 2n + 1)$ is $P(-3, 3, -2n - 1)$.)

In order to show that $J$ cannot be $\overline{K}$, we consider the JSJ decompositions of $S_0^3(K)$ and $S_0^3(\overline{K}) \cong -S_0^3(K)$. One of these two manifolds is $S_0^3(Wh^+(T_{2,3}, 2))$, and by Lemma 4.2 its JSJ decomposition consists of two pieces, one of which is the exterior of $T_{2,3}$ and the other of which is not a knot complement. But then the other manifold decomposes into the exterior of $T_{-2,3}$ and another piece, which is again not a knot complement. By the uniqueness of the JSJ decomposition, any orientation-preserving homeomorphism $S_0^3(K) \xrightarrow{\phi} -S_0^3(\overline{K})$ would have to restrict to an orientation-preserving homeomorphism

$$S^3 \setminus N(T_{2,3}) \cong S^3 \setminus N(T_{-2,3}),$$

and this is impossible.

Now if $J = P(-3, 3, 2n + 1)$ then Lemma 4.1 says that the JSJ decomposition of $S_0^3(J)$ consists of a single Seifert fibered piece. This does not match the decomposition of $S_0^3(K)$, so again we must have $S_0^3(K) \not\cong S_0^3(J)$. We have now shown that $J$ cannot be either $\overline{K}$ or any of the pretzel knots $P(-3, 3, 2n + 1)$, so $J$ must be isotopic to $K$ after all. \hfill $\square$

5. The determinant-9 case, part 2

In this section we prove that 0 is a characterizing slope for each pretzel knot $P(-3, 3, 2n + 1)$. We begin with the following.

**Lemma 5.1.** If $S_0^3(J) \cong S_0^3(P(-3, 3, 2n + 1))$ for some $n \in \mathbb{Z}$, then $J$ is isotopic to the pretzel knot $P(-3, 3, 2m + 1)$ for some $m \in \mathbb{Z}$.

**Proof.** Just as in the proof of Theorem 4.3, we apply Proposition 2.1 and Theorem 1.2 to see that if we write $W = Wh^+(T_{2,3}, 2)$ then $J$ must be one of $W$, $\overline{W}$, or $P(-3, 3, 2m + 1)$ ($m \in \mathbb{Z}$).

On the other hand, Theorem 4.3 tells us that $S_0^3(W) \not\cong S_0^3(P(-3, 3, 2n + 1))$ and $S_0^3(\overline{W}) \not\cong S_0^3(P(-3, 3, 2n + 1))$, so $J$ cannot be $W$ or $\overline{W}$, hence it must be some $P(-3, 3, 2m + 1)$. \hfill $\square$
In order to distinguish the 3-manifolds $S_0^3(P(-3,3,2n+1))$ for different values of $n$, we use Ohtsuki’s perturbative invariants of 3-manifolds $M$ with $b_1(M) = 1$ [Oht10], which take the form of a power series

$$\tau(M; c) = \sum_{\ell=0}^{\infty} \lambda_{\ell}(M; c)(q - 1)^\ell \in \mathbb{C}[[q - 1]]$$

that can be evaluated at $c = 0$ or at any root $c$ of the Alexander polynomial $\Delta_M(t)$. Each $\lambda_{\ell}(M; c)$ is itself an invariant of $M$, and $\lambda_0(M; c)$ is determined by the Alexander polynomial of $M$ [Oht10, Proposition 5.3], so we will compute $\lambda_1(S_0^3(P(-3,3,2n+1)), 0)$.

According to the discussion in [Oht10, §1], we have

$$\lambda_{\ell}(S_0^3(K); c) = -\frac{1}{2} \cdot \frac{1 + c}{1 - c} \left( \text{Res}_{t=c} \frac{(1 - t^{-1})^2 P_t(t)}{\Delta_K(t)^{2\ell+1}} \right),$$

where the Laurent polynomials $P_t(t)$ are the coefficients of the loop expansion

$$J_n(K; q) = \sum_{\ell=0}^{\infty} \frac{P_t(q^n)}{\Delta_K(t)^{2\ell+1}} (q - 1)^\ell$$

of the colored Jones polynomial. We have $P_0(t) = 1$ regardless of $K$, and then Ohtsuki [Oht04, Proposition 6.1] computed that

$$P_1(t) = -(t^{1/2} - t^{-1/2})^2 \cdot \hat{\Theta}_K(t),$$

where the last factor

$$\hat{\Theta}_K(t) = \frac{\Theta_K(t, 1)}{(t^{1/2} - t^{-1/2})^2} \in \mathbb{Q}[t, t^{-1}]$$

is a specialization of a polynomial called the “2-loop polynomial” $\Theta_K(t_1, t_2)$ arising from the Kontsevich integral of $K$. (We note that the polynomial $J_n(K; q)$ in [Oht10] is the same as the one denoted $V_n(K; q)$ in [Oht04] – both are normalized to take the value 1 when $K$ is the unknot – and also that (5.1) may differ from the value in [Oht10] by a sign, but this only changes the invariants $\lambda_1(S_0^3(K); c)$ that we will compute by an overall sign.)

The calculation of these polynomials was described in part by Ohtsuki [Oht07], including a computation of both $\Theta_K(t_1, t_2)$ and $\hat{\Theta}_K(t)$ when $K$ is a 3-stranded pretzel knot:

**Lemma 5.2 ([Oht07, Example 3.6]).** For the pretzel knot $K = P(p, q, r)$, if we let

$$d = \frac{pq + qr + rp + 1}{4}$$

then the reduced 2-loop polynomial of $K$ is given by

$$\hat{\Theta}_K(t) = \frac{1}{16} \left( (p + q + r)(4d + 1) + pqr \right) \left( -2 - \frac{2d + 1}{3}(t - 2 + t^{-1}) \right).$$

Applying Lemma 5.2 when $(p, q, r) = (-3, 3, 2n + 1)$, we have $d = -2$ and then

$$\hat{\Theta}_{P(-3,3,2n+1)}(t) = -(2n + 1)(t - 4 + t^{-1}),$$

whence for $K = P(-3,3,2n+1)$ we have $\Delta_K(t) = -2t + 5 - 2t^{-1}$ and

$$P_1(t) = -(t - 2 + t^{-1}) \cdot \hat{\Theta}_K(t)$$

$$= (2n + 1)(t - 2 + t^{-1})(t - 4 + t^{-1})$$

$$= (2n + 1)(t^2 - 6t + 10 - 6t^{-1} + t^{-2})$$

$$= (2n + 1) \left( \frac{1}{2} \Delta_K(t)^2 + \frac{1}{2} \Delta_K(t) - \frac{3}{4} \right).$$

The reason for writing it this way is that we can compute $\lambda_1(S_0^3(K), 0)$ via the following lemma.
Lemma 5.3 ([Oht10 Proposition 1.7(2)]). Suppose that the Alexander polynomial of $K$ has degree 1, and write

$$\Delta_K(t) = b_0 - b_1(t - 2 + t^{-1}),$$

$$P_1(t) = f(t)\Delta_K(t)^3 + a_2\Delta_K(t)^2 + a_1\Delta_K(t) + a_0$$

for some constants $b_0, b_1, a_0, a_1, a_2 \in \mathbb{Q}$ and Laurent polynomial $f(t)$. Then

$$\lambda_1(S^3_0(K); 0) = -\frac{d}{2} + \frac{a_2}{2b_1}$$

where $d$ is the constant term of $(t - 2 + t^{-1})f(t)$.

Theorem 5.4. Fix an integer $n \in \mathbb{Z}$. If $S^3_0(P(-3, 3, 2n + 1)) \cong S^3_0(K)$ for some knot $K \in S^3$, then $K$ is isotopic to $P(-3, 3, 2n + 1)$.

Proof. Lemma 5.1 guarantees that $K$ is $P(-3, 3, 2m + 1)$ for some $m \in \mathbb{Z}$. We use Lemma 5.3 for $P(-3, 3, 2n + 1)$: we have $(b_0, b_1) = (1, 2)$, and (5.3) tells us that

$$(f(t), a_2, a_1, a_0) = \left(0, \frac{2n + 1}{4}, \frac{2n + 1}{2}, -\frac{3(2n + 1)}{4}\right).$$

The constant term of $(t - 2 + t^{-1})f(t) = 0$ is $d = 0$, so we end up with

$$\lambda_1(S^3_0(P(-3, 3, 2n + 1)); 0) = \frac{a_2}{2b_1} = \frac{2n + 1}{16}.$$

But then an identical calculation says that

$$\lambda_1(S^3_0(P(-3, 3, 2m + 1)); 0) = \frac{2m + 1}{16},$$

and since these two invariants agree, we must have $m = n$. □

In fact, we can distinguish surgeries of any slope on these pretzel knots.

Proposition 5.5. If $r \in \mathbb{Q}$ is non-zero and $m$ and $n$ are distinct integers, then

$$S^3_r(P(-3, 3, 2m + 1)) \not\cong S^3_r(P(-3, 3, 2n + 1)).$$

Proof. This uses an LMO invariant obstruction due to Ito [Ito20], just as in [BS22a §7]: both knots have the same Conway polynomial $\nabla_K(z) = 1 - 2z^2$, with the same $z^4$-coefficient

$$a_4(P(-3, 3, 2m + 1)) = a_4(P(-3, 3, 2n + 1)) = 0.$$

Thus if their $r$-surgeries are homeomorphic, then by [Ito20 Corollary 1.3(iv)] these knots must have the same finite type invariants

$$v_3(P(-3, 3, 2m + 1)) = v_3(P(-3, 3, 2n + 1)).$$

But Ohtsuki [Oht07 Proposition 1.1] proved that $v_3(K) = \frac{1}{2}\hat{\Theta}_K(1)$, and so (5.2) says that

$$v_3(P(-3, 3, 2n + 1)) = 2n + 1,$$

hence these pretzel knots have different $v_3$ invariants unless $m = n$. (We note that Ohtsuki’s normalization of $v_3$ differs from Ito’s by a scalar, but this does not affect the argument.) □

We remark that Ito’s obstruction cannot be used to prove Theorem 5.4 however, because it only applies to non-zero surgeries. Moreover, Proposition 5.5 does not prove that non-zero slopes are characterizing for these pretzel knots, because for example the Heegaard Floer homology of $S^3_r(K) \cong S^3_r(P(-3, 3, 2n + 1))$ may not suffice to determine $\text{HF}\hat{K}(K)$ when $r \neq 0$. 

\[\text{\[Oht07\] Proposition 1.1\]}\]
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Department of Mathematics, Boston College

Email address: john.baldwin@bc.edu

Department of Mathematics, Imperial College London

Email address: s.sivek@imperial.ac.uk