Large intersection classes on fractals

David Färn and Tomas Persson

Institute of Mathematics, Polish Academy of Sciences, ulica Śniadeckich 8, PO Box 21, 00-956 Warszawa, Poland

E-mail: david@maths.lth.se and tomasp@maths.lth.se

Received 17 August 2010, in final form 17 February 2011
Published 11 March 2011
Online at stacks.iop.org/Non/24/1291

Recommended by C P Dettmann

Abstract

We consider limit sets of conformal iterated function systems, and introduce classes of subsets of these limit sets, with the property that the classes are closed under countable intersections and that all sets in the classes have a large Hausdorff dimension. Using these classes we determine the Hausdorff dimension and large intersection properties of some sets occurring in ergodic theory, Diophantine approximation and complex dynamics.

Mathematics Subject Classification: 28A78, 28A80, 37C45, 37E05, 11J83

1. Introduction

A classical result in the Diophantine approximation due to Dirichlet is that for any number \( x \in \mathbb{R} \), there are infinitely many \( p \in \mathbb{Z}, q \in \mathbb{N} \) with \( \gcd(p, q) = 1 \) such that \( |x - \frac{p}{q}| \leq \frac{1}{q^2} \). It means that all real numbers can be approximated at least at a certain rate by rationals. Many numbers can be approximated even faster. One way to classify the speed of approximation is to consider the sets

\[
K(\alpha) = \left\{ x \in \mathbb{R} : |x - \frac{p}{q}| < q^{-\alpha} \text{ for infinitely many } p \in \mathbb{Z}, q \in \mathbb{N} \right\}
\]

for \( \alpha > 2 \). These sets all have zero Lebesgue measure but Jarník [18] showed that the Hausdorff dimension of \( K(\alpha) \) is \( \frac{2}{\alpha} \).

In [13], Falconer introduced classes \( \mathcal{G}^t \) of subsets of \( \mathbb{R}^n \) with the property that any set in \( \mathcal{G}^t \) has a Hausdorff dimension of at least \( t \), and countable intersections as well as bi-Lipschitz images of sets from \( \mathcal{G}^t \) are also in \( \mathcal{G}^t \). Such properties are not what one expects of sets in \( \mathbb{R}^n \) with Hausdorff dimension less than \( n \). Indeed, the intersection of two sets \( E_1 \) and \( E_2 \) in \( \mathbb{R}^n \) is typically expected to be \( \dim_H(E_1) + \dim_H(E_2) - n \), see [14]. As an application of his classes, Falconer proved that \( K(\alpha) \) belongs to \( \mathcal{G}^{2/\alpha} \). This implies that not only does \( K(\alpha) \) have
a Hausdorff dimension at least $2/\alpha$, but so does its intersection with any sequence of sets from $G^\alpha$, or any image of $K(\alpha)$ under bi-Lipschitz maps.

Falconer [13] proved that the classes $G^\alpha$ can be defined in a number of equivalent ways, expressing different aspects of the sets involved. One of these ways to define $G^\alpha$, which resembles the classes found in Falconer’s earlier paper [12] and his book [14], can be written as follows. The definition we give here uses a variation due to Bugeaud [5]. Recall that $G_\alpha$-sets are those sets that can be expressed as countable intersections of open sets.

**Definition 1.** For any set $E \subset \mathbb{R}^n$ and $0 < t \leq n$, let

$$M^*_\infty(E) = \inf \left\{ \sum_i |D_i|^t : E \subset \bigcup_i D_i \right\},$$

where $|\cdot|$ denotes diameter, and each $D_i$ is a hypercube with dyadic side length, i.e. a set of the form $[k_j, k_j+1) \times \cdots \times [k_n, k_n+1)$, where $j, k_1 \ldots k_n \in \mathbb{Z}$. Let $G'$ be the class of all $G_\alpha$ sets $E \subset \mathbb{R}^n$ such that for all $s < t$ there is a $c > 0$ such that

$$M^*_\infty(E \cap D) \geq c|D|^t$$

for each set $D$ of the form $[k_j, k_j+1) \times \cdots \times [k_n, k_n+1)$, where $j, k_1 \ldots k_n \in \mathbb{Z}$.

The above definition is rather technical, but Falconer [13] provided a useful sufficient condition for sets to belong to the classes $G'$. It can be stated as follows. If $(E_i)_{i=1}^\infty$ is a sequence of open sets such that for each $s < t$ there exists a $c > 0$ such that

$$\lim_{i \to \infty} M^*_\infty(E_i \cap D) \geq c|D|^t$$

for each set $D$ of the form $[k_j, k_j+1) \times \cdots \times [k_n, k_n+1)$, where $j, k_1 \ldots k_n \in \mathbb{Z}$, then $\bigcap_{i=1}^\infty \bigcup_{j=1}^\infty E_i \in G'$. Such limsup sets occur naturally in the Diophantine approximation. For example, the sets $K(\alpha)$ can be written as lim sup $i \to \infty E_i$ where each $E_i$ is a finite union of open intervals. See [13] for more examples.

Diophantine approximation is not the only area where sets from the classes $G'$ appear. Given an integer $b > 1$, any number $x \in [0, 1]$ can be expanded in base $b$ as $x = \sum_{i=1}^\infty \frac{p_i}{b^i}$, where each $p_i$ is an integer $0, 1, \ldots, b-1$. Let

$$\tau(x,k,n) = \# \{ i \in \mathbb{N} : i \leq n \text{ and } x_i = k \}.$$ 

For each vector $p = (p_0, \ldots, p_{b-1})$ such that $0 \leq p_i \leq 1$ and $\sum p_i = 1$, let

$$G(p) = \{ x \in [0, 1] : \tau(x,k,n) \to p_k \text{, } k = 0, \ldots, b-1 \}.$$ 

Eggleston [11] proved that the Hausdorff dimension of $G(p)$ is $\sum_{i=0}^{b-1} \frac{p_i \log p_i}{\log b}$. However, there are plenty of points $x$ for which $\tau(x,k,n)$ does not converge. For $\epsilon > 0$, let

$$G(k,p_k,n,\epsilon) = \{ x \in [0, 1] : p_k - \epsilon < \tau(x,k,n) < p_k + \epsilon \}.$$ 

Sets of the form $\limsup_{n \to \infty} G(k,p_k,n,\epsilon)$ were studied in [15] and were shown to belong to a natural restriction of $G'$ from $\mathbb{R}$ to $[0,1]$, for any

$$t < \sup \{ \dim_h(G(q)) : p_k - \epsilon < q_k < p_k + \epsilon \}.$$ 

Using the large intersection properties of $G'$ it was shown in [15] that the set of points for which $\tau(x,k,n)$ does not converge for any $k$ for any base $b$, has Hausdorff dimension 1.

Similarly to Hausdorff measures, Falconer’s classes $G^\alpha$ can be generalized to classes $G^\beta$, where $x \mapsto x'$ from (1) is replaced by more general gauge functions $g : \mathbb{R}^+ \to \mathbb{R}^+$. This was done by Bugeaud [5], and even more general by Durand [6], in order to find more applications and finer characterizations of sets occurring in the Diophantine approximation. Durand wrote...
a series of papers [7–10], applying his classes from [6] to different settings, including Lévy processes and random wavelet series. See also the papers by Amou and Bugeaud [1] and by Barral and Seuret [2] for more applications.

In this paper we study classes of subsets of limit sets Λ of conformal graph directed Markov systems. Simple examples of such sets Λ are the interval, the middle third Cantor set and Julia sets of polynomials of degree larger than 1. We are interested in such classes which have the intersection and dimension properties of $G'$. As is easily seen, the complement of a set in $G'$ (as well as in $G^g$ from [5] or [6]) cannot contain an open non-empty set. Since the complement of Λ may contain open non-empty sets, the classes $G'$ or $G^g$ cannot be directly used in our case. We therefore introduce a modification of Falconer’s classes $G'$, consisting of subsets of Λ with the desired properties.

The main purpose of these new classes is to apply them to sets that are similar to the examples discussed above, but are subsets of attractors Λ. In particular, we study subsets of the middle third Cantor set, with Diophantine properties similar to those of $K(α)$, and show that they belong to our classes, concluding that they thereby have large intersection properties. Another application is the set of points in an attractor that they belong to our classes, concluding that they thereby have large intersection properties. We apply this result to piecewise expanding interval maps and Julia sets of polynomials.

This paper is organized as follows. First we recall some basic facts about graph directed Markov systems and state our main assumptions in section 2. In section 3 we introduce our new classes and their main properties. The applications can be found in section 4 and the proofs in sections 5, 6, 7 and 8.

2. Graph directed Markov systems

Let $Σ_A$ be a transitive one-sided subshift of finite type over the alphabet $[1,...,q]$ governed by a transition matrix $A$. We endow $Σ_A$ with the product topology. Elements in $Σ_A$ will be denoted by $i = i_1i_2...$ and any finite combination of symbols from the alphabet $[1,...,q]$, such as $i_1i_2...i_k$, will be called a word. The topology of $Σ_A$ is generated by the metric $d_Σ$ defined by $d_Σ(i,j) = 2^{-n}$, where $n$ is the smallest number such that $i_n ≠ j_n$. We define a cylinder to be a set of the form

$$C_{i_1,...,i_k} = \{j ∈ Σ_A : j_1...j_k = i_1...i_k\}$$

and we refer to $k$ as the generation of the cylinder $C_{i_1,...,i_k}$. On $Σ_A$ we have the shift map $σ$ defined by $σ : i_1i_2... → i_2i_3...$

We now use $Σ_A$ to define a graph directed Markov system. This can be done in a much more general way, see for instance [21]. However, in this paper we only consider the following simplification. Let $X$ be a compact and non-empty subset of $ℝ^n$, endowed with the usual metric. Assume that there are contractions on $X$ denoted by $(f_i)_{i=1}^q$ and that for each word $ij$ from $Σ_A$, there is a contraction on $X$ denoted by $f_{i,j}$. Such contractions can be composed to form $f_{i_1} ◦ f_{i_1i_2} ◦ ... ◦ f_{i_{n-1}i_n}$ for each element $i_1i_2... ∈ Σ_A$. We call such a system of contractions together with $Σ_A$, a graph directed Markov system.

For any element $i_1i_2... ∈ Σ_A$, there is a point $x ∈ X$ such that

$$\{x\} = \bigcap_{n=1}^{∞} f_{i_1} ◦ f_{i_1i_2} ◦ ... ◦ f_{i_{n-1}i_n}(X).$$

This defines a mapping $π : Σ_A → X$. Since all $f_i$ and $f_{i,j}$ are contractions, $π$ is continuous. The limit set or attractor of the graph directed Markov system is the set $π(Σ_A)$ and will be
denoted by \( \Lambda_A \). It is compact since \( \Sigma_A \) is compact and \( \pi \) is continuous. We endow \( \Lambda_A \) with the subset topology that comes from the topology of \( X \).

It is time to state the assumptions on the graph directed Markov system, which will be used in this paper. First of all, each contraction of the graph directed Markov system is assumed to be a conformal map. Such a system is called a conformal graph directed Markov system. The main assumptions are

The compact set \( X \) is the closure of its interior, and the images of the interior of \( X \) under the conformal contractions are disjoint

\( (2) \)

and

There are numbers \( 0 < \lambda_1 < \lambda_2 < 1 \) such that we have the inequalities \( \lambda_1 < \|d_x f_{i,j}\| < \lambda_2 \) and \( \lambda_1 < \|d_x f_i\| < \lambda_2 \) for all \( i, j \) and \( x \in X \).

\( (3) \)

In addition, we need the following bounded distortion property.

There is a constant \( \kappa \) such that for any sequence \( i \in \Sigma_A \), any integer \( n > 0 \) and any \( x, y \in X \) it holds that

\[
\kappa^{-1} \leq \frac{d_x(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_{n-1} \circ f_i})}{d_y(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_{n-1} \circ f_i})} \leq \kappa.
\]

\( (4) \)

Assumption \( (4) \) is satisfied if for example there is an \( \alpha > 0 \) such that

\[
\|d_x f_{i,j}\| - \|d_y f_{i,j}\| \leq \sup_{x \in X} \|d_z f_{i,j}\|^{-1} \|x - y\|^\alpha
\]

holds for all \( i, j \) and \( x, y \), and similarly for the maps \( f_i \), see [21]. It is shown in [21] that, under assumptions \( (2) \) and \( (3) \), such an \( \alpha \) exists as long as the dimension \( n \) of the space is larger than one. So, assumption \( (4) \) is in fact only needed when \( n = 1 \).

3. A class with large intersection properties

Let \( d \) be the semi-metric on \( \Sigma_A \) given by \( d(i, j) = |\pi(i) - \pi(j)| \). For sets \( E \subset \Sigma_A \), let \( d(E) \) denote the diameter of \( E \). Then

\[
d(C_{i_1 \ldots i_n}) = |\pi(C_{i_1 \ldots i_n})|
\]

for each cylinder \( C_{i_1 \ldots i_n} \). Note, however, that \( d \) does not necessarily generate the topology of \( \Sigma_A \). In fact, it need not be a metric, since there can be different \( i \) and \( j \) such that \( \pi(i) = \pi(j) \).

Let \( M^\infty_t \) be the outer measure on \( \Sigma_A \) defined by

\[
M^\infty_t(E) = \inf \left\{ \sum_i d(C_i)^t : E \subset \bigcup_i C_i \right\},
\]

where each \( C_i \) is a cylinder. We will work mainly with \( M^\infty_t \), but for some future use we also introduce the outer measures \( M^t \) and \( M^t_\cdot \) on \( \Sigma_A \), defined by

\[
M^t_t(E) = \inf \left\{ \sum_i d(C_i)^t : E \subset \bigcup_i C_i, \ d(C_i) \leq \delta \right\},
\]

where each \( C_i \) is a cylinder and \( M^t(E) = \lim_{\delta \to 0} M^t_\delta(E) \). The \( t \)-dimensional Hausdorff measure is

\[
H^t(E) = \lim_{\delta \to 0} \inf \left\{ \sum_i d(U_i)^t : E \subset \bigcup_i U_i, \ d(U_i) \leq \delta \right\}.
\]
Note that $M'(E) \geq M'_\infty(E)$ for all sets $E \subset \Sigma_A$, and that $M'$ is equivalent to the $t$-dimensional Hausdorff measure $H^t$ on $\Sigma_A$.

The outer measures $M'_\infty, M'_t, M'$ and $H^t$ are defined on subsets of $\Sigma_A$. Using the projection $\pi: \Sigma_A \to X$, we can project these outer measures to form outer measures on $X$ denoted by $M'_\infty, M'_t, M'$ and $H^t$, respectively. Equivalently we may define

$$M'_\infty(A) = \inf \left\{ \sum_i |\pi(C_i)|^t : A \subset \bigcup_i \pi(C_i) \right\},$$

where each $C_i$ is a cylinder, and similarly for the outer measures $M'_t, M'$ and $H^t$. Now that we have the Hausdorff measures $H^t$ and $H^t$ we can define the Hausdorff dimension of a set $E \subset \Lambda_A$ to be the unique number $s$ such that $H^t(E) = 0$ if $t > s$ and $H^t(E) = \infty$ if $t < s$.

The following classes of sets are modifications of Falconer’s intersection classes from [13], but constructed to fit the structure of the attractor $\Lambda_A$.

**Definition 2.** Let $G^t(\Lambda_A), 0 < t \leq \dim_H(\Lambda_A)$ be the class of $G_\delta$-sets $F \subset \Lambda_A$ such that

$$M'_\infty(F \cap \pi(C)) = M'_\infty(\pi(C))$$

holds for all cylinders $C$.

Although we are mainly interested in the classes $G^t(\Lambda_A)$ we prefer working with subsets of $\Sigma_A$ when developing the theory. Therefore we also introduce the following classes.

**Definition 3.** Let $G^t(\Sigma_A), 0 < t \leq \dim_H(\Lambda_A)$ be the class of $G_\delta$-sets $F \subset \Sigma_A$ such that

$$M'_\infty(F \cap C) = M'_\infty(C)$$

holds for all cylinders $C$.

The key properties of these classes are contained in the following theorem.

**Theorem 1.** The classes $G^t(\Lambda_A)$ and $G^t(\Sigma_A)$ are closed under countable intersections and the Hausdorff dimension of any set in one of these classes is at least $t$.

The proof can be found in section 5.

### 4. Applications

#### 4.1. Non-typical points in ergodic theory

Consider the behaviour of the ergodic averages

$$\frac{1}{n} \sum_{k=0}^{n-1} g(\sigma^k x),$$

as $n \to \infty$, where $g: \Sigma_A \to \mathbb{R}$. For $x \in \Lambda_A$ such that $\pi(i) = x$, let $A_x(x)$ be the set of accumulation points for the expression (5). Using the large intersection properties of our classes we will prove the following.

**Theorem 2.** Consider a conformal graph directed Markov system, satisfying assumptions (2), (3) and (4). For any sequence $(g_i)_{i=1}^\infty$ of real valued continuous functions on $\Sigma_A$, and each sequence $(x_i)_{i=1}^\infty$ of points in $\mathbb{R}$, it holds that

$$\dim_H(\bigcap_{i=1}^\infty \{ y \in \Lambda_A : x_i \in A_{g_i}(y) \}) = \inf_i \dim_H(\{ y \in \Lambda_A : x_i \in A_{g_i}(y) \}).$$
The statement above does not say very much unless we are able to determine the dimensions \( \dim_H (\{ y \in \Lambda_A : x_1 \in A_{g_i}(y) \}). One way to ensure that they can be determined is to assume a bit more, namely that the functions \( g_i \) are Hölder continuous. Recall that for a space \( \Sigma \) equipped with the metric \( d \), a function \( g : \Sigma \to \mathbb{R} \) is Hölder continuous if there are constants \( K \) and \( \alpha > 0 \) such that \( |g(i) - g(j)| < K d(i, j)^\alpha \) holds for all \( i, j \in \Sigma \).

Assuming that the functions \( g_i \) are Hölder continuous we can use the results of Barreira and Saussol in [3], and combined with theorem 2 one obtains the following corollary.

**Corollary 1.** Consider a conformal graph directed Markov system, satisfying assumptions (2), (3) and (4). Let \( (g_i)_{i=1}^\infty \) be a sequence of Hölder continuous maps on \( \Sigma_A \) such that no \( g_i \) is cohomologous to a constant. Then the set of points for which the ergodic averages do not converge for any of the maps \( g_i \) has the same Hausdorff dimension as \( \Lambda_A \).

Theorem 2 and corollary 1 are generalizations of results in [15], where the case \( \Lambda_A = \{0, 1\} \) is considered, i.e. where there are no holes. Moreover, in [15], each function \( g_i \) is the characteristic functions of a cylinder \( C_{j_1 \ldots j_k} \), so that the ergodic average (5) is the frequency of occurrence of the word \( j_1 \ldots j_k \) in a sequence \( i \in \Sigma_A \). Note that to prove theorem 2 it is sufficient to prove that each of the sets \( \{ y : x_1 \in A_{g_i}(y) \} \) is in \( \mathcal{G}'(\Lambda_A) \) for any \( t \) less than \( \dim_H (\{ y : x_1 \in A_{g_i}(y) \}) \). The proofs of theorem 2 and corollary 1 are in section 6.

**Corollary 2.** Let \( 0 = a_0 < \cdots < a_q = 1 \), and let \( f : [0, 1] \to [0, 1] \) be monotone and \( C^2 \) on each of the intervals \( (a_k, a_{k+1}] \), with \( 1 < \lambda_1 \leq |f'| \leq \lambda_2 < \infty \). The system \( (f, [0, 1]) \) generates a subshift \( \Sigma_f \) on an alphabet of \( q \) symbols. Assume \( \Sigma_f \) is transitive and that \( (0, 1) \setminus \bigcup_{k=0}^{q-1} f((a_k, a_{k+1}]) \) does not contain any isolated points. Let \( (g_i)_{i=1}^\infty \) be a family of Hölder continuous maps on \( [0, 1] \) such that no \( g_i \) is cohomologous to a constant. Then the set of points in \( [0, 1] \), such that no ergodic average of any \( g_i \) converges, has Hausdorff dimension 1.

The proof of corollary 2 is in section 7. In [4] another method is used to prove theorems similar to corollary 2. Corollary 3 in [16] considers a special case of corollary 2, namely maps of the form \( f_\beta : x \mapsto \beta x \) modulo 1, and with \( g_i \) being characteristic functions. For this special case, corollary 3 in [16] is stronger by stating that the set of points for which the ergodic averages do not converge contains a set from a class similar to \( \mathcal{G}'([0, 1]) \), which is independent of the parameter \( \beta \). This makes it possible to handle countably many maps \( g_i \) simultaneously and to take intersections between sets defined using different maps \( f_\beta \).

### 4.2. Diophantine approximation and Cantor sets

Let \( K \) be the middle third Cantor set. With \( n = 3, \Sigma_A = \{0, 2\}^\mathbb{N} \), and an appropriate definition of the iterated function system, we may regard \( K \) as an attractor \( \Lambda_A \). Let \( \alpha > 1 \) and consider the sets

\[
W(\alpha) = \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| < q^{-\alpha} \text{ for infinitely many } p \in \mathbb{Z}, q = 3^k \right\}.
\]

It is not difficult to show that \( \dim_H (W(\alpha)) = 1/\alpha \), i.e. half that of \( K(\alpha) \) from section 1 where the denominator \( q \) is allowed to take on any value in \( \mathbb{N} \). The intersection of \( W(\alpha) \) with \( K \) was studied in the paper [19] by Levesley, Salp and Velani. They proved that the Hausdorff dimension of \( W(\alpha) \cap K \) is \( \frac{1}{\alpha} \log_3 1/2 \). We will strengthen this result by proving the following theorem.

**Theorem 3.** \( W(\alpha) \cap K \) contains sets from \( \mathcal{G}'(K) \) for all \( t < \frac{1}{\alpha} \log_3 1/2 \).
In [2], Barral and Seuret indicate a way to extend theorem 1.6 from [2] in order to obtain a similar result. The proof of theorem 3 is in section 8. As a corollary of theorem 3 and the proof of theorem 2 we obtain the following.

**Corollary 3.** The set of points in $W(\alpha) \cap K$ for which, when expanded to base 3, the frequency of any finite word from $\{0, 2\}^N$ is undefined, has Hausdorff dimension $\frac{1}{\alpha} \log \frac{2}{\log 3}$.

Following [19], define the exact order of a number $x \in [0, 1]$ as

$$\tau(x) = \sup \{\alpha : x \in W(\alpha)\}$$

and consider the set

$$E(\alpha) = \{x \in [0, 1] : \tau(x) = \alpha\}.$$

It was proven in [19] that the Hausdorff dimension of $E(\alpha) \cap K$ is $\frac{1}{\alpha} \log \frac{2}{\log 3}$. Since the sets $E(\alpha) \cap K$ are disjoint for different $\alpha$, it is clear that they cannot contain sets from $G'(K)$. However, using theorem 3 and its proof we can deduce the following corollary. The proof can be found in section 8.

**Corollary 4.** The set of points in $E(\alpha) \cap K$ for which, when expanded to base 3, the frequency of any finite word from $\{0, 2\}^N$ is undefined, has Hausdorff dimension $\frac{1}{\alpha} \log \frac{2}{\log 3}$.

### 4.3. Julia sets

Corollary 1 can be applied to non-typical points of the Julia set $J(f)$ of a polynomial $f(z)$, yielding the following corollary, which can be regarded as a two-dimensional version of corollary 2.

**Corollary 5.** Let $f(z)$ be a polynomial of degree larger than 1, such that $|f'| > 1$ on the Julia set $J(f)$. Consider a sequence $(g_k)_{k=1}^\infty$ of Hölder continuous functions $g_k : J(f) \to \mathbb{R}$, such that no $g_k$ is cohomologous to a constant with respect to $(f, J(f))$. Then the set of points in $J(f)$ such that the ergodic averages of $g_k$ do not converge for any $g_k$, has the same Hausdorff dimension as $J(f)$.

**Proof.** Since $f$ is of degree $d > 1$, the Julia set is compact and non-empty. There are only finitely many critical points since $f$ is a polynomial, and by assumption none of them are in $J(f)$. Since $J(f)$ is compact and invariant, there exists an $R > 0$ such that the distance from $J(f)$ to any image of a critical point is at least $R$, and $|f'| > 1$ on any ball $B_R(x)$, $x \in J(f)$. Moreover, the compactness of $J(f)$ implies that there exists a constant $c$ such that $|f'(x)| > c > 1$ for all $x \in J(f)$. We can even choose $c > 1$ and $0 < r < R$ such that if $x \in J(f)$, then the pre-image of the ball $B_r(x)$ of radius $r$ around $x$ satisfies

$$f^{-1}(B_r(x)) \subset \bigcup_{y \in f^{-1}(x)} B_{r/c}(y),$$

and $|f'| > c$ on $B_r(x)$.

Let $U$ be an open neighbourhood around $J(f)$ defined by

$$U = \bigcup_{x \in J(f)} B_r(x),$$

and put $X = \overline{U}$. Now $X$ contains no image of a critical point, so we can define inverse branches $(f_j)_{j=1}^\infty$ of $f$ on $X$, and put $f_i,j = f_j$. It is clear that $f_j$ are conformal contractions on $X$. Moreover $f_j : X \to X$. Indeed we have

$$f^{-1}(X) = f^{-1}(U) \subset \bigcup_{x \in J(f)} B_{r/c}(x) \subset \overline{U} = X,$$

so, $f_j(X) \subset X$. 


Note that some $f_j$ may be discontinuous at the curves $f(\partial f_j(X))$. However $f(\partial f_j(X))$ are piecewise smooth curves and if necessary, we can cut up $X$ along these curves, so that $f_j$ satisfies assumptions (2), (3) and (4). Now, corollary 1 completes the proof. □

5. Proof of theorem 1

Let us first comment on the relation between the classes $G^t(\Lambda_A)$ and $G^t(\Sigma_A)$. If $U \subset \Lambda_A$ is an open set, then $\pi^{-1}(U) \subset \Sigma_A$ is an open set since $\pi$ is continuous. Hence, if $F \subset \Lambda_A$ is a $G_\delta$-set, then so is $\pi^{-1}(F) \subset \Sigma_A$. We are going to show that the class $G^t(\Sigma_A)$ is closed under countable intersections. Since countable intersections of $G_\delta$-sets are $G_\delta$-sets, this implies that the class $G^t(\Lambda_A)$ is also closed under countable intersections. Hence, to prove theorem 1, we only need to prove the intersection statement for the class $G^t(\Sigma_A)$. This is also true for the statement on the Hausdorff dimension in theorem 1.

Now we will briefly use the concept of pressure of a potential. For a definition and investigation of pressure in our setting, see [20, 21]. Define the potential $\phi: \Sigma_A \rightarrow \mathbb{R}$ by
$$
\phi_i = (i_k)_{k=1}^\infty \mapsto \log \|d\pi_1 f_{i_1}^1, f_{i_2}^2\|
$$
and let
$$
S_n \phi(i) = \log \|d\pi_1 f_{i_1}^1\| + \sum_{k=1}^{n-1} \phi(\sigma^k(i)).
$$
Then
$$
e^{\inf_{C_{j_1,..,j_n}} S_n \phi(i)} \leq d(C_{j_1,..,j_n}) \leq e^{\sup_{C_{j_1,..,j_n}} S_n \phi(i)}.
$$
Since $\phi$ is a Hölder continuous potential, we can write the pressure of $s\phi$ as
$$
P(s\phi) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{C_{j_1,..,j_n}} e^{\inf_{C_{j_1,..,j_n}} S_n \phi(i)} \right)
$$
and
$$
P(s\phi) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{C_{j_1,..,j_n} \subset C} e^{\sup_{C_{j_1,..,j_n}} S_n \phi(i)} \right),
$$
where the sums run over all cylinders $C_{j_1,..,j_n}$ of generation $n$. It is known, see for instance [21], that $P(s\phi) = 0$ is equivalent to $\dim_H(\Lambda_A) = s$. Note that for each non-empty cylinder $C$ we have
$$
P(s\phi) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{C_{j_1,..,j_n} \subset C} e^{\inf_{C_{j_1,..,j_n}} S_n \phi(i)} \right)
$$
and
$$
P(s\phi) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{C_{j_1,..,j_n} \subset C} e^{\sup_{C_{j_1,..,j_n}} S_n \phi(i)} \right),
$$
so given $s < \dim_H(\Lambda_A)$, we have $P(s\phi) > 0$ and there are constants $m = m(s)$ and $c > 0$ such that
$$
\sum_{C_{j_1,..,j_n} \subset C} e^{\inf_{C_{j_1,..,j_n}} sS_n \phi(i)} > e^{cn} > 1,
$$
for all generation 1 cylinders $C_a$ and all $n \geq m$.

Observe that if $C$ is a cylinder, then it need not be true that $M'_\infty(C) = d(C)'$, since $C$ might not be the optimal cover of $C$. However, because of (6), we only need to consider covers of $C$ with cylinders that are at most of $m = m(t)$ generations higher than $C$. Since there
are only finitely many such covers, we conclude that there is a largest constant $0 < c_1 \leq 1$ such that
\[ M_\infty'(C) \geq c_1 d(C) \] holds for all cylinders $C$. A similar statement is of course true for the outer measure $M_\infty$. Note also that it always holds that $M_\infty'(C) \leq d(C)'.

The proof of theorem 1 will be divided into a couple of propositions and lemmata. The proofs of these are in most cases quite similar to the corresponding ones in Falconer’s paper [13].

The following lemma lets us extend the property that defines $G_{t/\Lambda_1 A}$ to any open set.

**Lemma 1.** If $c > 0$ and $F$ is a set such that
\[ M_\infty'(F \cap C) \geq c M_\infty'(C) \]
holds for all cylinders $C$, then
\[ M_\infty'(F \cap U) \geq c M_\infty'(U) \]
holds for all open sets $U$.

**Proof.** Let $U$ be open. Then we can write $U$ as a disjoint union $U = \bigcup_{j=1}^\infty C_j$, where each $C_j$ is a cylinder. Let $(D_j)$ be cylinders covering $F \cap U$. We can assume that this cover is disjoint.

By the net property of cylinders, for any $i$, either there are no $D_j \subset C_i$ and instead $C_i \subset D_j$ for some $j$, or there are $D_j \subset C_i$ and
\[ \bigcup_{D_j \subset C_i} D_j \cap C_i \neq \emptyset \]
In case there are $D_j \subset C_i$ we have that
\[ \sum_{D_j \subset C_i} d(D_j)^t \geq M_\infty'(F \cap C_i) \geq c M_\infty'(C_i). \] (8)

If we take $(D_j)$ and for each $i$ such that there are $D_j \subset C_i$, we replace all these $D_j$ by one copy of $C_i$, then we obtain a disjoint cover $(E_r)$ of $U$, where all $E_r$ are cylinders. Indeed, any $C_i$ from the union $U = \bigcup_{j=1}^\infty C_j$ is either a subset of some element $E_r = D_j$, or it is itself an element $E_r$. Using (8) and the fact that $c \leq 1$, we obtain
\[ \sum_{D_j \subset C_i} d(D_j)^t \geq c \sum_{E_r \cap C_i} M_\infty'(E_r) \geq c M_\infty'(U). \]
Since $(D_j)$ is arbitrary, this proves the lemma. $\square$

We will also need the following lemma.

**Lemma 2.** Suppose that $F$ is a set with the property that there is a constant $c > 0$ such that
\[ M_\infty'(F \cap C) \geq c M_\infty'(C) \]
holds for all cylinders $C$. Then
\[ M_\infty'(F \cap C) \geq M_\infty'(C) \]
holds for all cylinders $C$, and $0 < t \leq t_0$.

**Proof.** We start by proving that $M_\infty'(F \cap C) \geq M_\infty'(C)$ holds for $t = t_0$. This result can then be used to obtain $M_\infty'(F \cap C) \geq M_\infty'(C)$ when $t < t_0$.

Let $(C_i)$ be a collection of cylinders covering $F \cap C$. We may assume that the sets $C_i$ are pairwise disjoint. Since $M_\infty'(F \cap C)$ is finite, we may assume that $\sum d(C_i)^{t_0}$ is finite. Let $I(m) = \{ i : C_i \text{ is of generation } m \}$. Then, for any $\varepsilon > 0$, there is an $m_0$ such that
\[ \sum_{m \geq m_0} \sum_{i \in I(m)} d(C_i)^{t_0} = \sum_{i} d(C_i)^{t_0} - \sum_{m < m_0} \sum_{i \in I(m)} d(C_i)^{t_0} < \varepsilon. \] (9)
Let \((D_j)\) denote a finite cover of the cylinder \(C\) by cylinders, such that for any \(j\) holds either

(i) \(D_j = C_i\) for some \(i\) and \(D_j\) is of generation less than \(m_0\),

or

(ii) \(D_j\) is of generation \(m_0\), and those \(C_i\) that intersect \(D_j \cap F\) are contained in \(D_j\).

Let \(Q(j) = \{i : C_i \subset D_j\}\). If \(j\) satisfies (i) then

\[
\sum_{i \in Q(j)} d(C_i)^b = d(D_j)^b. \tag{10}
\]

For those \(j\) that satisfies (ii), we have that

\[
\sum_{i \in Q(j)} d(C_i)^b \geq c M_{\infty}^b (D_j) \geq c c_n d(D_j)^b, \tag{11}
\]

where \(c_n\) is defined by (7). Using (9), (10) and (11), we conclude that

\[
\sum_{i} d(C_i)^b = \sum_{m < m_0} \sum_{i \in I(m)} d(C_i)^b + \sum_{m \geq m_0} \sum_{i \in I(m)} d(C_i)^b
\]

\[
= \sum_{m < m_0} \sum_{i \in I(m)} d(C_i)^b + c^{-1} c_n c_0 \sum_{m \geq m_0} \sum_{i \in I(m)} d(C_i)^b
\]

\[
\geq \sum_j d(D_j)^b + (1 - c^{-1} c_n c_0 ) \varepsilon \geq M_{\infty}^b (C) + (1 - c^{-1} c_n c_0 ) \varepsilon. \tag{12}
\]

Hence, \(M_{\infty}^b (C \cap F) \geq M_{\infty}^b (C) + (1 - c^{-1} c_n c_0 ) \varepsilon\). Since \(\varepsilon\) is arbitrary, we conclude that

\(M_{\infty}^b (C \cap F) \geq M_{\infty}^b (C)\).

We now turn to the case \(0 < t < t_0\). Using what we proved above, we note that it is sufficient to prove that there is a constant \(c'\) such that \(M_{\infty}^b (C \cap F) \geq c'M_{\infty}^b (C)\) holds for all cylinders \(C\). Let \(C\) be an \(n\)-cylinder. Take \(n_0 \geq n\) such that \(\lambda_{n+1}^e = \lambda_{n}^e\), or equivalently

\[
\lambda_{2}^{(n+1)(t-t_0)} = \lambda_{1}^{(n+1)(t-t_0)}. \tag{12}
\]

Let \((C_i)\) be a collection of cylinders covering \(F \cap C\). We may assume that the sets \(C_i\) are pairwise disjoint. We let \((D_j)\) denote a finite cover of the cylinder \(C\) by cylinders, such that for any \(j\) holds either

(i) \(D_j = C_i\) for some \(i\) and \(D_j\) is of generation less than \(n_0\),

or

(ii) \(D_j\) is of generation \(n_0\), and those \(C_i\) that intersect \(D_j \cap F\) are contained in \(D_j\).

Let \(Q(j) = \{i : C_i \subset D_j\}\). If \(j\) satisfies (i) then

\[
\sum_{i \in Q(j)} d(C_i)^b = d(D_j)^b = d(D_j)^{t-t_0}d(D_j)^b \geq d(C)^{t-t_0}d(C)^b. \tag{10}
\]

If \(j\) satisfies (ii) then, for \(i \in Q(j)\), holds

\[
d(C_i)^b = d(C_i)^{t-t_0}d(C_i)^b \geq (\lambda_{2}^{n_0+1})^{t-t_0}d(C_i)^b
\]

\[
\geq (\lambda_{1}^{n+1})^{t-t_0}d(C_i)^b \geq d(C)^{t-t_0}d(C)^b, \tag{11}
\]

\[
\sum_{i} d(C_i)^b \geq \sum_j d(D_j)^b + (1 - c^{-1} c_n c_0 ) \varepsilon \geq M_{\infty}^b (C) + (1 - c^{-1} c_n c_0 ) \varepsilon. \tag{12}
\]

Hence, \(M_{\infty}^b (C \cap F) \geq M_{\infty}^b (C) + (1 - c^{-1} c_n c_0 ) \varepsilon\). Since \(\varepsilon\) is arbitrary, we conclude that

\(M_{\infty}^b (C \cap F) \geq M_{\infty}^b (C)\).
Large intersection classes on fractals

by (12). Hence, if \( j \) satisfies (ii), then

\[
\sum_{i \in Q(j)} d(C_i) t^j \geq d(C) t^j - t^j_0 \sum_{i \in Q(j)} d(D_i) t^j_0 \geq d(C) t^j - t^j_0 d(D_j) t^j_0.
\]

It follows that

\[
\sum_i d(C_i) t^j \geq c_0 d(C) t^j - t^j_0 \sum_i d(D_i) t^j_0 \geq c_0 d(C) t^j - t^j_0 M_{\infty}^i(D_j) \geq c t^j_0 d(C) t^j - t^j_0 M_{\infty}^i(C) \geq c^2 t^j_0 M_{\infty}^i(C).
\]

This proves that

\[
M_{\infty}^i(F \cap C) \geq c^2 t^j_0 M_{\infty}^i(C)
\]

holds for all cylinders \( C \), and \( 0 < t < t_0 \). □

The proof of theorem 1 is completed, if we prove the following two propositions.

**Proposition 1.** If \( F \in G^t(\Sigma_A) \), then \( \dim_H(F) \geq t \), and thereby \( \dim_H(\pi(F)) \geq t \) with respect to Euclidean metric on \( \Lambda_A \).

**Proposition 2.** If \( F_i \in G^t(\Sigma_A) \) for all \( i \in \mathbb{N} \), then

\[
M_{\infty}^i\left( \bigcap_{i=1}^{\infty} F_i \cap U \right) = M_{\infty}^i(U),
\]

for all open \( U \), and thereby \( \bigcap_{i=1}^{\infty} F_i \in G^t(\Sigma_A) \).

**Proof of proposition 1.** Clearly \( M_{\infty}^i(F) > 0 \). Hence \( M^i(F) > 0 \), and since \( M^i \) is equivalent to the \( t \)-dimensional Hausdorff measure \( H^t \), we conclude that \( H^t(F) > 0 \) and \( \dim_H(F) \geq t \). □

**Proof of proposition 2.** The proof of this proposition is very similar to the corresponding proof in [13]. It is based on an increasing set lemma from Roger’s book [22].

Let \( E \subset \Sigma_A \) and \( \delta > 0 \). We will denote by \( E_{(-\delta)} \) the set

\[
E_{(-\delta)} = \left\{ x \in E : \inf_{y \neq x} d(x, y) > \delta \right\},
\]

and we note that \( E_{(-\delta)} \) is an open set. Take \( \varepsilon > 0 \). Let first \( \{F_i\}_{i=1}^{\infty} \) be a decreasing sequence of open sets, with the property that

\[
M_{\infty}^i(F_i \cap U) \geq M_{\infty}^i(U)
\]

holds for any open set \( U \). We will choose a sequence of numbers \( (\delta_k) \) and open sets \( (U_k) \), such that

\[
U_0 = U,
\]

\[
U_k = (F_k \cap U_{k-1})_{(-\delta_k)},
\]

and \( M_{\infty}^i(U_k) > M_{\infty}^i(U) - \varepsilon \).

We will choose \( \delta_k \) and \( U_k \) inductively. Assume that they have been chosen for \( k = 1, \ldots, n \). Since \( F_n \in G^t(\Lambda_A) \), lemma 1 implies that

\[
M_{\infty}^i(F_n \cap U_{n-1}) \geq M_{\infty}^i(U_{n-1}) > M_{\infty}^i(U) - \varepsilon.
\]
The set \((F_n \cap U_{n-1})(-\delta_n)\) is open and increases to \(F_n \cap U_{n-1}\) as \(\delta_n\) vanishes. Using theorem 52 from [22], there is a \(\delta_n\) such that
\[ M'_\infty(U_n) = M'_\infty((F_n \cap U_{n-1})(-\delta_n)) > M'_\infty(U) - \varepsilon. \]
We have that \(\overline{U}_k \subseteq F_k\). Let \((C_i)\) be a cover by cylinders of \(\cap U_k\). Since \((\overline{U}_k)\) is a nested sequence of compact sets and \(C_i\) is open, there is an \(m\) such that \(\overline{U}_m \subseteq \cup C_i\). Hence
\[ \sum_i d(C_i) \geq M'_\infty(\overline{U}_m) \geq M'_\infty(U) - \varepsilon. \]
As \(\varepsilon\) is arbitrary we conclude that
\[ M'_\infty\left( \bigcap_i F_i \cap U \right) \geq M'_\infty(U). \]

Now note that any finite intersection of open sets in \(G'(\Lambda_A)\) is in \(G'(\Lambda_A)\) and that any countable intersection of \(G_A\) sets can be expressed as the intersection of a countable decreasing sequence of open sets. Together with the result proved above, this completes the proof of the proposition.

6. Proofs of theorem 2 and corollary 1

Let \(G_g(p)\) be the set of points in \(\Sigma_A\) for which \(p\) is an accumulation point of the ergodic averages of \(g\), see (5), and let \(G_g(p, n, \varepsilon)\) be defined as
\[ G_g(p, n, \varepsilon) = \left\{ i \in \Sigma_A : p - \varepsilon < \frac{1}{n} \sum_{k=0}^{n-1} g(\sigma^k i) < p + \varepsilon \right\}. \]
We also introduce the notation \(\hat{G}_g(p)\) for the set of points for which the ergodic averages of \(g\) converge to \(p\). Hence \(\hat{G}_g(p)\) is a subset of \(G_g(p)\) and so \(\dim_H G_g(p) \geq \dim_H \hat{G}_g(p)\).

The key step in proving theorem 2 and corollary 1 is the following proposition.

Proposition 3. For any real valued continuous function \(g\) on \(\Sigma_A\), it holds that \(G_g(p) \in G'(\Sigma_A)\) for all \(t < \dim_H(\hat{G}_g(p))\).

Proof of theorem 2. According to proposition 3, the set \(G_g(x_i) = \{ y \in \Lambda_A : x_i \in A_g(y) \}\) is in \(G'(\Sigma_A)\) for \(t < \dim_H G_g(x_i)\). Now theorem 2 follows from theorem 1.

Proof of corollary 1. Let \(\varepsilon > 0\). We now use theorem 9 of [3]. This theorem implies that there is a non-empty open interval \(I_{\hat{g}}\) such that \(p \mapsto \dim_H \hat{G}_g(p)\) is a real analytic function. Since there are only finitely many maps \(f_i\) and \(f_{i,j}\) the conformal graph directed markov system is regular and there exists an ergodic measure of full dimension, see [21]. By the Birkhoff ergodic theorem, it is therefore clear that \(p \mapsto \dim_H \hat{G}_g(p)\) attains the value \(\dim_H \Lambda_A\) somewhere in the interval \(I_{\hat{g}}\). Hence, for each \(g_i\) there are \(x_{1,i} \neq x_{2,i}\) such that \(\dim_H \hat{G}_{g_i}(x_{1,i}) > \dim_H \Lambda_A - \varepsilon\) and \(\dim_H \hat{G}_{g_i}(x_{2,i}) > \dim_H \Lambda_A - \varepsilon\). We then have \(\dim_H G_{g_i}(x_{1,i}) > \dim_H \Lambda_A - \varepsilon\) and \(\dim_H G_{g_i}(x_{2,i}) > \dim_H \Lambda_A - \varepsilon\).

Proposition 3 implies that \(G_{g_i}(x_{1,i})\) and \(G_{g_i}(x_{2,i})\) are in \(G^{\dim_H \Lambda_A - \varepsilon}(\Sigma_A)\). Thereby \(G_{g_i}(x_{1,i})\) and \(G_{g_i}(x_{2,i})\) are in \(G^{\dim_H \Lambda_A - \varepsilon}(\Sigma_A)\). This set consists of points for which both \(x_{1,i}\) and \(x_{2,i}\) are accumulation points of the ergodic averages of \(g_i\). Hence the set of points where the ergodic averages do not converge for any \(g_i\) is in \(G^{\dim_H \Lambda_A - \varepsilon}(\Sigma_A)\). Let \(\varepsilon \to 0\).
The proof of proposition 3 is broken down to a series of lemmata. The following lemma will be used to show that \( \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} G_{\delta}(p, k, \varepsilon) \in \mathcal{G}'(\Sigma_{\Lambda}) \). For this approach to work, we need the sets \( G_{\delta}(p, k, \varepsilon) \) to be open, since then \( \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} G_{\delta}(p, k, \varepsilon) \) is \( G_{\delta} \). That the sets \( G_{\delta}(p, k, \varepsilon) \) are open is guaranteed if \( g \) is continuous. So, from now on we assume, without mentioning, that \( g \) is a continuous function on \( \Sigma_{\Lambda} \).

**Lemma 3.** Let \( (F_{k})_{k=1}^{\infty} \) be a sequence of open sets in \( \Sigma_{\Lambda} \). If there is a \( c > 0 \) such that
\[
\limsup_{i \to \infty} M_{i}^{i}(F_{k} \cap C) \geq c M_{i}^{i}(C)
\]
for all cylinders \( C \), then
\[
\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} F_{k} \in \mathcal{G}'(\Sigma_{\Lambda}), \forall t' \leq t.
\]

**Proof.** First note that \( \bigcup_{k=1}^{\infty} F_{k} \) is open, so it is clearly \( G_{\delta} \). It follows from lemma 2 that \( \bigcup_{k=1}^{\infty} F_{k} \in \mathcal{G}'(\Sigma_{\Lambda}) \) for all \( n \). The lemma then follows from theorem 1.

Before continuing we fix an \( s < \dim_{H} \Lambda_{\Lambda} \), and an \( m = m(s) \) according to (6). We define new outer measures \( N_{t}^{m,i}, 0 < t < s, \) of the same type as \( M_{t}' \), but instead of considering all covers by cylinders, we only consider covers by cylinders of generation \( km \), where \( k \in \mathbb{N} \).

Note that (6) ensures that \( N_{n,i}^{m}(C) = d(C) \) for all cylinders \( C \) of generation \( km \), where \( k \in \mathbb{N} \). This makes it easier to work with \( N_{n,i}^{m} \), than with \( M_{t}' \). Note also that the outer measures \( M_{t}' \) and \( N_{t}^{m,i} \) are equivalent. Hence there is some constant \( c_{i} \), depending on \( m \) and \( t \), such that
\[
M_{t}(U) \leq N_{t}^{m,i}(U) \leq c_{i} M_{t}(U)
\]
holds for any open set \( U \). Therefore, by lemma 2, to prove that \( \bigcup_{n=1}^{\infty} \cup_{k=1}^{\infty} G_{\delta}(p, k, \varepsilon) \in \mathcal{G}'(\Sigma_{\Lambda}) \), it is sufficient to consider the outer measures \( N_{t}^{m,i} \) instead of \( M_{t}' \).

**Lemma 4.** Assume that there is a \( c < 1 \) and an integer \( M \) such that
\[
N_{t}^{m,i}(C_{i_{1} \ldots i_{m}} \cap G_{\delta}(p, M, \varepsilon)) < c N_{\infty}^{m,i}(C_{i_{1} \ldots i_{m}}),
\]
holds for the cylinder \( C_{i_{1} \ldots i_{m}} \). Then for each cylinder \( C = C_{c_{1} \ldots c_{jm}} \) such that \( j \in \mathbb{N} \) and \( c_{(j-1)m+1} \ldots c_{jm} = i_{1} \ldots i_{m} \), there is a constant \( K_{C} \), depending only on \( C \) and \( g \), such that
\[
N_{t}^{m,i}(C \cap G_{\delta}(p, M, \varepsilon/2)) < \kappa c N_{\infty}^{m,i}(C)
\]
holds if \( M \geq K_{C} \). (The constant \( \kappa \) is defined in (4).)

**Proof.** Let \( C \) and \( C_{i_{1} \ldots i_{m}} \) be as in the statement of the lemma. Take \( A \) such that \( A > |g| \).

By assumption, there is a cover of \( C_{i_{1} \ldots i_{m}} \cap G_{\delta}(p, M, \varepsilon) \) by cylinders \( (U_{i})_{i=1}^{\infty} \), of generation \( km \), with value less than \( c N_{t}^{m,i}(C_{i_{1} \ldots i_{m}}) \). For each \( U_{i} \) there is a corresponding set \( \tilde{U}_{i} \), created by appending \( c_{1} \ldots c_{(i-j)m} \) to the beginning of the coding of each element in \( U_{i} \).

Note that
\[
d(\tilde{U}_{i})' = \kappa d(C') = \frac{d(C')}{N_{t}^{m,i}(C_{i_{1} \ldots i_{m}})},
\]
by (4). Moreover, \( (\tilde{U}_{i})_{i=1}^{\infty} \) is a cover of
\[
C \cap G_{\delta}(p, M, \varepsilon) = 2A(j-1)m/M.
\]

It now follows that
\[
N_{t}^{m,i}(C \cap G_{\delta}(p, M, \varepsilon - 2A(j-1)m/M)) \leq \sum_{i} d(\tilde{U}_{i})'
\]

\[
\leq \frac{\kappa d(C')}{N_{t}^{m,i}(C_{i_{1} \ldots i_{m}})} \sum_{i} d(\tilde{U}_{i})' < \kappa c d(C') = \kappa c N_{\infty}^{m,i}(C).
\]

The lemma now follows if we choose \( K_{C} \) so large that \( 2A(j-1)m/K_{C} < \varepsilon/2 \). □
Lemma 5. Assume that there is a $c > 0$ and an integer $M$ such that

$$N_{m,j}^{m,j}(C_{i_1,\ldots,i_n} \cap G_g(p, M, \delta)) > c N_{\infty}^{m,j}(C_{i_1,\ldots,i_n}),$$

holds for the cylinder $C_{i_1,\ldots,i_n}$. Then for each cylinder $C = C_{c_1,\ldots,c_m}$ such that $j \in \mathbb{N}$ and $c_{(j-1)m+1} \ldots c_{jm} = i_1 \ldots i_m$, there is a constant $K_C$, depending only on $C$ and $g$, such that

$$N_{\infty}^{m,j}(C \cap G_g(p, M, 2\delta)) > \kappa^{-1} N_{\infty}^{m,j}(C),$$

holds if $M \geqslant K_C$.

Proof. Let $C$ and $C_{i_1,\ldots,i_n}$ be as in the statement of the theorem. Let $A > |g|$ and consider the set $C_{i_1,\ldots,i_n} \cap G_g(p, M, \delta)$. We can write this set as a union of cylinders. If we append $c_1 \ldots c_{(j-1)m}$ to the beginning of the coding of each of these cylinders, we obtain a collection of cylinders in

$$C \cap G_g(p, M, \delta + 2A(j - 1)m/M).$$

Let $(\tilde{U}_i)$ be a cover of $C \cap G_g(p, M, \delta + 2A(j - 1)m/M)$. For each $\tilde{U}_i$ there is a corresponding set $U_i$, created by removing the first $(j - 1)m$ symbols in the coding. Note that $d(\tilde{U}_i)^j \geqslant \kappa^{-1} d(C\tilde{\tilde{U}}_i)^j N_{\infty}^{m,j}(C_{i_1,\ldots,i_n})$, by (4), and that $(\tilde{U}_i)_{i=1}^\infty$ is a cover of $C_{i_1,\ldots,i_n} \cap G_g(p, M, \delta)$. It now follows that

$$\sum_i d(\tilde{U}_i)^j \geqslant \kappa^{-1} (C\tilde{\tilde{U}}_i)^j N_{\infty}^{m,j}(C_{i_1,\ldots,i_n}) \sum_i d(U_i)^j > \kappa^{-1} c d(C)^j = \kappa^{-1} c N_{\infty}^{m,j}(C).$$

Since the cover $(\tilde{U}_i)$ was arbitrary, we obtain

$$N_{\infty}^{m,j}(C \cap G_g(p, M, \delta + 2A(j - 1)m/M)) > \kappa^{-1} c N_{\infty}^{m,j}(C).$$

The lemma now follows if we choose $K_C$ so large that $2A(j - 1)m/K_C < \delta$. \qed

Lemma 6. There is a constant $L > 0$ such that if there is a $c > 0$ and an $M$ such that

$$N_{\infty}^{m,j}(C_{i_1,\ldots,i_n} \cap G_g(p, M, \delta)) > c N_{\infty}^{m,j}(C_{i_1,\ldots,i_n}),$$

for one cylinder $C_{i_1,\ldots,i_n}$ of generation $m$, then for each cylinder $C$, of generation $m$, there is a number $K_C$, depending only on $C$ and $g$, such that

$$N_{\infty}^{m,j}(C \cap G_g(p, M, 2\delta)) > L c N_{\infty}^{m,j}(C),$$

if $M \geqslant K_C$.

Proof. Since $\Sigma_A$ is transitive, there is a number $N \in \mathbb{N}$ such that any cylinder $C = C_{c_1,\ldots,c_m}$ contains a cylinder $C_{c_{(N-1)m+1} \ldots c_{Nm}}$ of generation $Nm$ for which $c_{(N-1)m+1} \ldots c_{Nm} = i_1 \ldots i_m$. Since $\Sigma_A$ is of finite type and the maps $f = f_i$ or $f = f_{i,j}$ all satisfy $\lambda_1 \leqslant |f'| \leqslant \lambda_2$, it is clear that there is a uniform constant $L' > 0$ such that

$$\frac{d(C_{c_{(N-1)m+1} \ldots c_{Nm}})^j}{d(C_{c_{1,\ldots,c_m}})^j} > L',$$

regardless of which cylinders $C_{i_1,\ldots,i_n}$ and $C_{c_{(N-1)m+1} \ldots c_{Nm}}$ we started with. (For instance, $L' = \lambda_1^{(N-1)m}$ will do.) Using Lemma 5 we can find $K_C$ such that

$$N_{\infty}^{m,j}(C_{c_{(N-1)m+1} \ldots c_{Nm}} \cap G_g(p, M, 2\delta)) \geqslant \kappa^{-1} c N_{\infty}^{m,j}(C_{c_{1,\ldots,c_m}}) > L' \kappa^{-1} c N_{\infty}^{m,j}(C_{c_{1,\ldots,c_m}})$$

if $M \geqslant K_C$. Let $L = L' \kappa^{-1}$. \qed
Lemma 7. For each $t < \dim_H(G_g(p))$ there is a constant $c > 0$ such that for any cylinder $C_{i_1,...,i_m}$ of generation $m$ and any $\epsilon > 0$, it holds that
\[ \liminf_{M \to \infty} N_{\infty}^{m,t}(C_{i_1,...,i_m} \cap G_g(p,M,\epsilon)) \geq c N_{\infty}^{m,t}(C_{i_1,...,i_m}), \]
for any $\epsilon > 0$.

Proof. Let $L$ be given by lemma 6. Assume in contrast that there exist a cylinder $C_{i_1,...,i_m}$ and a strictly increasing sequence $(M_k)_{k=1}^{\infty}$ such that
\[ N_{\infty}^{m,t}(C_{i_1,...,i_m} \cap G_g(p,M_k,\epsilon)) < \kappa - \frac{1}{2} N_{\infty}^{m,t}(C_{i_1,...,i_m}), \forall k. \] (13)
By lemma 6 we obtain
\[ N_{\infty}^{m,t}(C_{i_1,...,i_m} \cap G_g(p,M_k,\epsilon)) < \frac{1}{2} N_{\infty}^{m,t}(C_{i_1,...,i_m}), \forall k \geq K_{C_{i_1,...,i_m}} \]
for each $C_{i_1,...,i_m}$. Indeed, if would have
\[ N_{\infty}^{m,t}(C_{i_1,...,i_m} \cap G_g(p,M_k,\epsilon)) > \frac{1}{2} N_{\infty}^{m,t}(C_{i_1,...,i_m}), \]
for some $C_{i_1,...,i_m}$ and $M_k \geq K_{C_{i_1,...,i_m}}$, then lemma 6 gives us a contradiction to (13). By lemma 4 we obtain that
\[ N_{\infty}^{m,t}(C \cap G_g(p,M_k,\epsilon/2)) \leq \frac{1}{2} N_{\infty}^{m,t}(C), \forall k \geq \max\{K_C, K_{C_{i_1,...,i_m}}\} \]
for each cylinder $C$ of any generation $jm$, $j \in \mathbb{N}$. Thus, there is a number $k_1$ and finite cover $(C_{i_1})_i$ of $C_{i_1,...,i_m}$ such that
\[ \sum_i d(C_i)^t \leq 2^{1/2} N_{\infty}^{m,t}(C_{i_1,...,i_m}). \]
Consider each $C_i$ and its intersection with $G_g(p,M_k,\epsilon/2)$ for $k > k_1$. There is now a $k_2 > k_1$ and a finite cover $(C_{i,j})_j$ such that $(C_{i,j})_j$ covers $C_i \cap G_g(p,M_k,\epsilon/2)$ and
\[ \sum_j d(C_{i,j})^t \leq \frac{2}{3} N_{\infty}^{m,t}(C_i) = 2^{-2/3} d(C_i)^t. \]
We obtain
\[ \sum_{i,j} d(C_{i,j})^t \leq \left(\frac{2}{3}\right)^2 N_{\infty}^{m,t}(C_{i_1,...,i_m}). \]
Continuing like this we obtain $\dim_H(G_g(p)) < t$ which contradicts the assumptions. \qed

Lemma 8. For each $t < \dim_H(G_g(p))$ there is a constant $c > 0$ such that for any cylinder $C$ and any $\epsilon > 0$, it holds that
\[ \liminf_{M \to \infty} N_{\infty}^{m,t}(C \cap G_g(p,M,\epsilon)) \geq c N_{\infty}^{m,t}(C). \]

Proof. For cylinders of generation $jm$, where $j \in \mathbb{N}$, this follows from lemmas 7 and 5. Since any other cylinder contains at least one cylinder of some generation $jm$, and since there is a uniform bound on the relative size of these two sets, the lemma follows. \qed

We are now ready to prove proposition 3.

Proof of proposition 3. We first note that
\[ G_g(p) = \bigcap_{e>0} \bigcup_{n=1}^{\infty} G_g(p,M,e) \]
and we recall that the outer measures $M_{\infty}$ and $N_{\infty}^{m,t}$ are equivalent. Recall that $G_g(p,M,\epsilon)$ is open since $g$ is continuous. By applying first lemma 8, then lemma 3 and finally theorem 1, we obtain that $G_g(p)$ is in $G_t(\Sigma_A)$ for any $t < \dim_H(G_g(p))$. \qed
7. Proof of corollary 2

We will approximate $\Sigma_f$ by subshifts of finite type denoted by $\Sigma_A$. Let $\pi_A$ be the projection associated with $\Sigma_A$, and let $d_A$ be the semi-metric on $\Sigma_A$ defined in section 3. We also define the projection $\pi_f: \Sigma_f \rightarrow [0, 1]$.

Given $\epsilon > 0$, by proposition 1 in [17], we can approximate $\Sigma_f$ from the inside by a subshift of finite type $\Sigma_A$, such that the attractor $\Lambda_A = \pi_A(\Sigma_A) \subset [0, 1]$ has Hausdorff dimension at least $1 - \epsilon$. The subshift $\Sigma_A$ is created from $\Sigma_f$ by forbidding some words of length $n(\epsilon)$, corresponding to cylinders close to the endpoints of the intervals $(a_k, a_{k+1})$.

To be able to use corollary 1 we first need to show that $\Sigma_A$ can be chosen such that no map $g_i|_{\Lambda_A} \circ \pi_A: \Sigma_A \rightarrow R$ is cohomologous to a constant. To do this we use the fact that the functions $g_i$, and thereby their restrictions $g_i|_{\Lambda_A}$, are Hölder continuous. Denote by $C^K_u(\Sigma_A)$ the family
\[ C^K_u(\Sigma_A) = \{ h: \Sigma_A \rightarrow \mathbb{R} : |h(i) - h(j)| < K d_A(i, j)^u, \forall i, j \in \Sigma_A \} , \]
and by $C^K_u(\Lambda_A)$ the family
\[ C^K_u(\Lambda_A) = \{ h: \Lambda_A \rightarrow \mathbb{R} : |h(x) - h(y)| < K |x - y|^u, \forall x, y \in \Lambda_A \} . \]
Since we have finitely many Hölder continuous functions $g_i|_{\Lambda_A}$, they all belong to $C^K_u(\Lambda_A)$ for some $K_u$ and $u$ which only depend on $g_1, \ldots, g_m$. But $d_A(i, j) = |\pi_A(i), \pi_A(j)|$, so $g_i|_{\Lambda_A} \circ \pi_A$ is in $C^K_u(\Sigma_A)$ for all $i$.

Lemma 9. For each function $g_i$ there is a subshift $\Sigma_A \subset \Sigma_f$ such that $g_i|_{\Lambda_A} \circ \pi_A$ is not cohomologous to a constant on $\Sigma_A$.

Proof. To obtain a contradiction, assume $g_i$ is cohomologous to a constant on each $\Sigma_A$. It means that there is a constant $C_A$ and a continuous function $\chi_A: \Sigma_A \rightarrow \mathbb{R}$ such that
\[ g \circ \pi = C_A = \chi_A \circ \sigma - \chi_A \] on $\Sigma_A$. We claim that the function $\chi_A: \Sigma_A \rightarrow \mathbb{R}$ is in $C^K_u(\Sigma_A)$, where $K$ only depends on $f$ and $K_u$.

The assumption that $(0, 1) \setminus \bigcup_{i=0}^{n-1} f((a_k, a_{k+1}))$ does not contain isolated points implies that any small enough interval in $\bigcup_{i=0}^{n-1} f((a_k, a_{k+1}))$ is contained in some $f((a_k, a_{k+1}))$. Since $f$ is uniformly expanding and there are only $q$ inverse branches of $f$, it is clear that for some $n_0 \in \mathbb{N}$, each cylinder $C_{x_1 \ldots x_n}$ in $\Sigma_f$, where $n \geq n_0$, satisfies $\pi(C_{x_1 \ldots x_n}) = f(\pi(C_{a_1 \ldots a_n}))$ for some $a \in [0, 1, \ldots, q - 1]$. Hence, if $x$ and $y$ are such that $x_1 \ldots x_n = y_1 \ldots y_n$, then if $n \geq n_0$, there is always a digit $a$ such that $ax \in \Sigma_f$ and $ay \in \Sigma_f$. Since $\Sigma_A$ is created from $\Sigma_f$ by removing words corresponding to points close to the endpoints of the intervals $(a_k, a_{k+1})$, we can make sure that we always have $ax \in \Sigma_A$ and $ay \in \Sigma_A$ by choosing $\Sigma_A$ and $n_0$ large enough.

Define
\[ A_n = \sup_{x,y \in \Sigma_A, x = y_{n-1} \ldots y_0} |\chi_A(x) - \chi_A(y)|. \]
Let $x$ and $y$ be elements of $\Sigma_A$, with $x_1 \ldots x_n = y_1 \ldots y_n$, such that there is a digit $a$ with $ax, ay \in \Sigma_A$. Then by (14), we have
\[ |\chi_A(x) - \chi_A(y)| \leq |\chi_A(ax) - \chi_A(ay)| + |g(ax) - g(ay)| \]
\[ \leq |\chi_A(ax) - \chi_A(ay)| + K_u 2^{-u(n+1)}. \]
Taking supremum, we obtain that
\[ |\chi_A(x) - \chi_A(y)| \leq A_{n+1} + K_u 2^{-u(n+1)} \]
Figure 1. Picture of $F_n$ and $E_n$ for $n = 2$.

holds for all $x$ and $y$ with $x_1 \ldots x_n = y_1 \ldots y_n$, such that there is a digit $a$ with $ax, ay \in \Sigma_A$. If $n \geq n_0$ there is always such a digit $a$ if $x_1 \ldots x_n = y_1 \ldots y_n$. Hence we obtain

$$A_n \leq A_{n+1} + K_0 2^{-\alpha(n+1)},$$

for large enough $n$. Since $\chi_A$ is continuous on the compact set $\Sigma_A$, we know that $A_n \to 0$ as $n \to 0$. This implies that

$$A_n \leq K 2^{-\alpha n}, \quad \forall n,$$

for some constant $K$ that does not depend on $\Sigma_A$. This proves the claim that $\chi_A$ is in $C^K_A(\Sigma_A)$.

It is clear that we can use the same constant $C = C_A$ on each subshift $\Sigma_A \subset \Sigma_f$. Since the functions $\chi_A$ are uniformly Hölder continuous, they can be extended to a function $\chi$ in $C^K_A(\Sigma_f)$. We obtain that

$$g \circ \pi_f - C = \chi \circ \sigma - \chi$$

on $\Sigma_f$. Hence $g \circ \pi_f$ is cohomologous to a constant. This implies that $g$ is cohomologous to a constant, a contradiction. □

By lemma 9 we can choose $\Sigma_A$ such that no $g_i|_{\Lambda_A} \circ \pi_A$ is cohomologous to a constant on $\Sigma_A$. Now corollary 2 follows from corollary 1 by letting $\varepsilon \to 0$.

8. Proofs of theorem 3 and corollary 4

First we will prove theorem 3, that $W(\alpha) \cap K$ contains sets from $G^t(K)$ for all $t < \frac{1}{\alpha \log 3}$. We emphasize that $W(\alpha) \cap K$ is not in $G^t(K)$ for any $t > \frac{1}{\alpha \log 3}$, since $\dim_1 (W(\alpha) \cap K) \leq \frac{1}{\alpha \log 3}$, as is easily shown by standard arguments.

Proof of theorem 3. Let

$$E_n = \{x \in [0, 1) : |x - p/3^n| < 3^{-\alpha n}, \text{ for some } p \in \mathbb{Z}\},$$

and let $F_n$ be the $n$th step in the construction of the middle third Cantor set. There is a picture showing $F_n$ and $E_n$ for $n = 2$ in figure 1.

Let $n$ be so large that $2/3^{\alpha n} < 1/3^n$. Then $E_n$ consists of $3^n + 1$ intervals of length $2/3^{\alpha n}$ (apart from two, which are of length $1/3^{\alpha n}$) and $F_n$ consists of $2^n$ intervals of length $1/3^n$. For any interval from $E_n$ that intersects $K$, the midpoint of the interval is an endpoint of an interval in $F_n$. Moreover, any endpoint of an interval in $F_n$ is also a midpoint of an interval in $E_n$. It follows that exactly $2^{\alpha n}$ intervals from $E_n$ intersect $K$ and that all these intersections are congruent, that is, they are translations and reflections of each other.

Let $C$ be a cylinder in $K$ of generation $m$. Then $C$ is an interval of length $1/3^m$. (Rather, the convex hull of $C$ is an interval of length $1/3^m$.) We need to estimate $\mathcal{M}_G(C \cap E_n \cap K)$ from below, for large $n$. For this purpose we consider a cover of $C \cap E_n \cap K$ by cylinders $\{U_i\}$ in $K$. (This means that each $U_i$ is a projection by $\pi$ from a cylinder in the symbolic space $\Sigma_A$.) We may assume that this cover consists of sets with pairwise disjoint interior, all contained in $C$. Obviously $\mathcal{M}_G(C \cap E_n \cap K) \leq |C'|$, so to obtain a lower bound we can assume that the generations of the cylinders in the cover are strictly larger than $m$. 

Consider one of the cylinders $U_i$ and let $n_i$ be its generation. We have two cases. Either $n_i \leq n$ or $n_i > n$. In both cases we have $m$ smaller than $n$ and $n_i$.

If $n_i \leq n$ then $U_i$ intersects $2^{n-n_i+1}$ intervals from $E_n$.

If $n_i > n$, then $U_i$ intersects at most one interval from $E_n$. We may then assume that $U_i$ intersects exactly one interval and that $|U_i| \leq 3/3^n$.

Let $\mu$ be a uniform mass distribution of mass 1 on $C \cap K \cap E_n$. This means that $\mu$ is the normalized restriction of the Hausdorff measure $\mathcal{H}$, for $s = \frac{\log 2}{\log 3}$, on $C \cap K \cap E_n$. (The measure $\mathcal{H}$ is finite for this $s$.)

In the first case when $n_i \leq n$ we have

$$\mu(U_i) = \frac{2^{n-n_i+1}}{2^{n-m+1}} = \frac{2^{-n_i}}{2^{-m}} = \left(\frac{|U_i|}{|C|}\right)^{\frac{\log 2}{\log 3}} \leq \left(\frac{|U_i|^t}{|C|^t}\right)^{\frac{\log 2}{\log 3}}$$

if $t \leq \frac{\log 2}{\log 3}$.

In case $n_i > n$ we first observe that if $I$ is the interval from $E_n$ that intersects $U_i$, then $\mu(U_i) \mu(I) \leq 6(|U_i||I|)^{\frac{\log 2}{\log 3}}$. (The constant 6 is not optimal.) We have that $|I| = 1/3^n$ since the intervals of $E_n$ are of length $2/3^n$, and we loose half of each interval when intersecting with $K$. Moreover, $\mu(I) = 1/2^{m-m+1}$, since there are $2^{m-m+1}$ such intervals of equal measure. Hence

$$\mu(U_i) \leq 6 \left(\frac{|U_i|}{3^{-2n}}\right)^{\frac{\log 2}{\log 3}} = \frac{3}{2^{m-3n}} \left(\frac{|U_i|}{3^{-2n}}\right)^{\frac{\log 2}{\log 3}}$$

$$= \frac{3}{2^n} \left(\frac{|U_i|}{3^{-2n}|C|}\right)^{\frac{\log 2}{\log 3}} = \frac{3}{2^n} \left|U_i\right|^t \left(\frac{|U_i|}{3^{-2n}|C|}\right)^{\frac{\log 2}{\log 3}}.$$

If $t < \frac{1}{2} \frac{\log 2}{\log 3}$ and $n$ is sufficiently large, we have

$$\frac{3}{2^n} \left|U_i\right|^t \left(\frac{|U_i|}{3^{-2n}|C|}\right)^{\frac{\log 2}{\log 3}} \left|C\right|^t \leq \frac{3}{2^n} \left(\frac{3}{|C|}\right)^{\frac{\log 2}{\log 3}} \left|C\right|^t \leq 2,$$

so

$$\mu(U_i) \leq 2 \left|U_i\right|^t \left|C\right|^t.$$

Summing over $i$ we obtain

$$1 = \sum_i \mu(U_i) \leq \sum_i 2 \left|U_i\right|^t \left|C\right|^t,$$

and $\sum_i \left|U_i\right|^t \geq \frac{1}{2}\left|C\right|^t$. Since $U_i$ is an arbitrary cover we obtain that $\mathcal{M}_\infty(C \cap E_n \cap K) \geq \frac{1}{2}\left|C\right|^t$, provided that $t < \frac{1}{2} \frac{\log 2}{\log 3}$ and $n$ is large. The set $\limsup(E_n \cap K)$ is $G_\delta$ since $E_n$ are open sets. Now lemma 3 implies that $\limsup(E_n \cap K)$ is in $G^t(K)$ for $t < \frac{1}{2} \frac{\log 2}{\log 3}$. This proves theorem 3.

We complete the section by giving a brief proof of corollary 4.

**Proof of corollary 4.** Let $s = \frac{1}{\alpha} \frac{\log 2}{\log 3}$. We will consider two different sets.

First, consider the set

$$W(\alpha) = \left\{ x \in [0, 1] : |x - \frac{p}{3^k}| \leq \frac{k}{3^{\alpha k}} \text{ for infinitely many } p \in \mathbb{Z}, k \in \mathbb{N} \right\}.$$  

Replacing $W(\alpha)$ by $W'(\alpha)$ in the proof of theorem 3 we obtain that $W'(\alpha) \cap K$ contains sets from $G^t(K)$.  

□
Next, consider the set of points in $K$ for which, when expanded to base 3, the frequency of any finite word from $\{0, 2\}$ is undefined. It follows from proposition 3 that it contains sets from the class $G^\prime(K)$.

By the properties of the class $G^\prime(K)$, it follows that the intersection of these two sets has positive $s$-dimensional Hausdorff measure. Since

$$W'(\alpha) \setminus \bigcup_{n=1}^{\infty} W(\alpha + 1/n) \subset E(\alpha)$$

and the $s$-dimensional Hausdorff measure of $W(\alpha + 1/n)$ is zero for all positive $n$, the statement of the corollary follows. □

Acknowledgments

Parts of this paper was written when David Färn held a position at the Centre for Mathematical Sciences at Lund University, Sweden. David Färn was supported by EC FP6 Marie Curie ToK programme CODY, Tomas Persson was supported by EC FP6 Marie Curie ToK programmes SPADE2, and CODY. This paper was completed when the authors were visiting Institut Mittag-Leffler in Djursholm, Sweden. The authors are grateful for the hospitality of the institute.

References

[1] Amou M and Bugeaud Y 2010 Exponents of Diophantine approximation and expansions in integer bases J. Lond. Math. Soc. 81 297–316
[2] Barral J and Seuret S 2008 Ubiquity and large intersections properties under digit frequencies constraints Math. Proc. Camb. Phil. Soc. 145 527–48
[3] Barreira L and Saussol B 2001 Variational principles and mixed multifractal spectra Trans. Am. Math. Soc. 353 3919–44
[4] Barreira L and Schmelting J 2000 Sets of ‘non-typical’ points have full topological entropy and full Hausdorff dimension Israel J. Math. 116 29–70
[5] Bugeaud Y 2004 Intersective sets and Diophantine approximation Michigan Math. J. 52 667–82
[6] Durand A 2008 Sets with large intersection and ubiquity Math. Proc. Camb. Phil. Soc. 144 119–44
[7] Durand A 2008 Ubiquitous systems and metric number theory Adv. Math. 218 368–94
[8] Durand A 2008 Random wavelet series based on a tree-indexed Markov chain Commun. Math. Phys. 283 451–77
[9] Durand A 2009 Large intersection properties in Diophantine approximation and dynamical systems J. Lond. Math. Soc. 79 377–98
[10] Durand A 2009 Singularity sets of Lévy processes Probab. Theory Relat. Fields 143 517–44
[11] Eggleston H 1949 The fractional dimension of a set defined by decimal properties Q. J. Math. Oxford. Second Series 20 31–6
[12] Falconer K 1985 Classes of sets with large intersections Mathematika 32 191–205
[13] Falconer K 1994 Sets with large intersection properties J. Lond. Math. Soc. 49 267–80
[14] Falconer K 2003 Fractal Geometry: Mathematical Foundations and Applications 2nd edn (Chichester: Wiley)
[15] Färn D 2011 Simultaneously non-convergent frequencies of words in different expansions Monatsh. Math. accepted (doi:10.1007/s00605-009-0183-2)
[16] Färn D, Persson T and Schmelting J 2010 Dimension of countable intersections of some sets arising in expansions in non-integer bases Fundam. Math. 209 157–76
[17] Hofbauer F, Raith P and Simon K 2007 Hausdorff dimension for some hyperbolic attractors with overlaps and without finite Markov partition Ergod. Theory Dyn. Syst. 27 1143–65
[18] Jarník V 1929 Diophantischen Approximationen und Hausdorffsches Mass Mat. Sbor. 36 371–82 http://mi.mathnet.ru/eng/msb/v36/i3/p371
[19] Levesley J, Salp C and Velani S 2007 On a problem of K. Mahler: Diophantine approximation and Cantor sets Math. Ann. 338 97–118
[20] Mauldin D and Urbański M 2001 Gibbs states on the symbolic space over an infinite alphabet Israel J. Math. 125 93–136
[21] Mauldin D and Urbański M 2003 Graph directed Markov systems. Geometry and Dynamics of Limit Sets (Cambridge Tracts in Mathematics vol 148) (Cambridge: Cambridge University Press)
[22] Rogers C A 1970 Hausdorff Measures (Cambridge: Cambridge University Press)