A VERSION OF THE THEOREM OF JOHNSON, PALMER AND SELL FOR QUASICOMPACT COCYCLES

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Abstract. The well-known theorem of Johnson, Palmer and Sell asserts that the endpoints of the Sacker–Sell spectrum of a given cocycle of invertible matrices over a topological dynamical system \((M, f)\) are realized as Lyapunov exponents with respect to some ergodic invariant probability measure for \(f\). In this note we establish the version of this result for quasicompact cocycles of operators acting on an arbitrary Banach space.

1. Introduction

In their landmark paper [16], Sacker and Sell introduced the notion of (what is now called) the Sacker–Sell spectrum for cocycles over topological dynamical systems and they described all possible structures of the spectrum. Furthermore, they have obtained several results which indicated that there exists a strong connection between their spectral theory and the theory of Lyapunov exponents which plays a central role in the modern dynamical systems theory. Those developments culminated with the remarkable paper by Johnson, Palmer and Sell [11], where the authors proved that the endpoints of the Sacker-Sell spectrum of a given cocycle are realized as Lyapunov exponents of the cocycle with respect to some ergodic probability measure which is invariant for the base space on which the cocycle acts.

We emphasize that the results in [11] and [16] deal with invertible cocycles on a finite-dimensional space. More recently, Magalhães [12] developed the Sacker-Sell spectral theory for compact cocycles on an arbitrary Banach space and Schreiber [17] established the version of the theorem of Johnson, Palmer and Sell in this setting under the additional assumption that the cocycle is injective.

In the present paper, we go step further and we develop the Sacker-Sell spectral theory and establish the version of the theorem of Johnson, Palmer and Sell for not not necessarily injective quasicompact cocycles acting on an arbitrary Banach space. The principal motivation for this work are the most recent versions [18, 8, 9, 2] (that build on the earlier work of Ruelle [15] and Mañé [13]) of the classical Oseledets multiplicative ergodic theorem [14] that work under the assumption that the cocycle is quasicompact. These developments culminated in a recent remarkable paper by Blumenthal and

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Young \[3\] in which the authors have extended a large part of the modern smooth ergodic theory to the case of maps \(f\) acting on arbitrary Banach spaces. The results in \[3\] are valid precisely under the assumption that the derivative cocycle \(Df\) associated to \(f\) is quasicompact. It is believed that those results will have applications in the study of partial and delay differential equations. In another direction, above described advancements in the multiplicative ergodic theory have also inspired new directions in the study of statistical properties of random dynamical systems \[7\].

The present paper is organized as follows. In Section 2 we introduce notation and recall several concepts and useful results that will be used throughout the paper. In Section 3 we introduce the notion of a Sacker-Sell spectrum for quasicompact cocycles and describe all possible structures of the spectrum. We note that our results and proofs in this section are inspired by the results in \[1\], where the authors have developed the version of the Sacker-Sell theory with respect to the notion of nonuniform hyperbolicity. Finally, in Section 4 we establish the main result of this paper, i.e. the version of the theorem of Johnson, Palmer and Sell for quasicompact cocycles. Here our strategy follows the one we outlined in \[6\] (in the simple case of finite-dimensional dynamics) and consists of using deep results of Cao \[5\] on the growth of subadditive functions over semiflows.

2. Preliminaries

2.1. Uniformly hyperbolic cocycles. Throughout this paper \(M\) will be a compact topological space and \(f: M \to M\) will be a homeomorphism. Furthermore, assume that \(X\) is a Banach space and denote by \(B(X)\) the space of all bounded operators on \(X\). Finally, let \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\). A map \(A: M \times \mathbb{N}_0 \to B(X)\) is said to be a cocycle over \(f\) if:
   1. \(A(q, 0) = \text{Id}\) for each \(q \in M\);
   2. \(A(q, n + m) = A(f^n(q), m)A(q, n)\) for every \(q \in M\) and \(n, m \in \mathbb{N}_0\);
   3. the map \(A: M \to B(X)\) defined by
      \[
      A(q) = A(q, 1), \quad q \in M
      \]
      is continuous.

We recall that the map \(A\) given by (1) is called the generator of the cocycle \(A\). We say that a cocycle \(A\) over \(f\) is uniformly hyperbolic if:
   1. there exists a family of projections \(P(q), q \in M\) satisfying
      \[
      A(q)P(q) = P(f(q))A(q), \quad q \in M
      \]
      such that each map \(A(q)|\text{Ker} P(q): \text{Ker} P(q) \to \text{Ker} P(f(q))\) is invertible;
   2. \(q \mapsto P(q)\) is a continuous map from \(M\) to \(B(X)\);
   3. there exist \(D, \lambda > 0\) such that for each \(q \in M\) and \(n \geq 0\)
      \[
      \|A(q, n)P(q)\| \leq De^{-\lambda n}
      \]
      and
      \[
      \|A(q, -n)(\text{Id} - P(q))\| \leq De^{-\lambda n},
      \]
      where \(A(q, -n) = (A(f^{-n}(q), n)|\text{Ker} P(f^{-n}(q)))^{-1}\).
Let $B_X$ denote the closed unit ball in $X$ centered at 0. For an operator $A \in B(X)$, we define $\|A\|_{ic(X)}$ to be an infimum over all $r > 0$ with the property that $A(B_X)$ can be covered by finitely many balls of radius $r$. It is easy to verify that $\|A\|_{ic(X)} \leq \|A\|$. The following result will play an important role in our arguments.

**Lemma 1.** Assume that a cocycle $A$ is uniformly hyperbolic and that

$$\limsup_{n \to \infty} \frac{1}{n} \log \|A(q, n)\|_{ic(X)} \leq 0,$$  \hspace{1cm} (5)

for some $q \in M$. Then, $\dim \ker P(q) < \infty$.

**Proof.** Suppose that $\dim \ker P(q) = \infty$. By Riesz lemma, we can find a sequence $(e_n)_{n \in \mathbb{N}} \subset B_X \cap \ker P(q)$ such that $\|e_n - e_m\| \geq 1/2$ for $n \neq m$. It follows from (4) that

$$\|A(q, n)(e_k - e_l)\| \geq \frac{1}{D} e^{\lambda n} \|e_k - e_l\| \geq \frac{1}{2D} e^{\lambda n},$$

for $n \in \mathbb{N}$ and $k, l \in \mathbb{N}$ such that $k \neq l$. Hence, $A(q, n)(B_X)$ cannot be covered by finitely many balls of radius $\frac{1}{4D} e^{\lambda n}$. Thus,

$$\|A(q, n)\|_{ic(X)} \geq \frac{1}{4D} e^{\lambda n},$$

which implies that

$$\limsup_{n \to \infty} \frac{1}{n} \log \|A(q, n)\|_{ic(X)} \geq \lambda > 0.$$  \hspace{1cm} □

This yields a contradiction with (5).

2.2. **Quasicompactness.** We first note that since $A$ is continuous and $M$ is compact, we have that

$$\sup_{q \in M} \|A(q)\| < \infty.$$  

Hence, it follows from the subadditive ergodic theorem that for each ergodic $f$-invariant measure $\mu$, there exist $\kappa(\mu), \lambda(\mu) \in [-\infty, \infty)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \|A(q, n)x\| = \lambda(\mu) \text{ and } \lim_{n \to \infty} \frac{1}{n} \log \|A(q, n)x\|_{ic(X)} = \kappa(\mu),$$

for $\mu$-a.e. $q \in M$. We say that $A$ is *quasicompact* with respect to $\mu$ if

$$\kappa(\mu) < \lambda(\mu).$$  \hspace{1cm} (6)

To the best of our knowledge, the notion of quasicompactness in the context of cocycles over some measure-preserving dynamical system originated in the work of Thieullen [18].

The following result which is proved in [9, Lemma C.5.] (building on the work Buzzi [4]) provides sufficient conditions under which the cocycle is quasicompact.

**Proposition 1.** Let $(X', \|\cdot\|)$ be a Banach space such that $X \subset X'$ and that the inclusion $(X, \|\cdot\|) \hookrightarrow (X', \|\cdot\|)$ is compact. Furthermore, suppose that $A(q)$ can be extended to a bounded operator on $(X', \|\cdot\|)$ for each $q \in M$. Finally, let $\mu$ be an ergodic, $f$-invariant Borel probability measure such that
there exist Borel-measurable functions $\alpha, \beta, \gamma : M \to (0, \infty)$ with $\gamma$ being log-integrable such that for $\mu$-a.e. $q \in M$ and $x \in X$,

$$\|A(q)x\| \leq \alpha(q)\|x\| + \beta(q)|x|$$

and

$$\|A(q)\| \leq \gamma(q).$$

Finally, let us assume that

$$\int \log \alpha(q) \, d\mu(q) < \lambda(\mu).$$

Then,

$$\kappa(\mu) \leq \int_M \log \alpha(q) \, d\mu(q).$$

In particular, $A$ is quasicompact with respect to $\mu$.

In the context of cocycles of the so-called transfer operators which play a major role in modern dynamical systems theory, conditions (7) and (8) are called the strong and weak Lasota-Yorke inequality respectively. We refer to [4, 7, 8, 9] for details.

2.3. Multiplicative ergodic theorem. Let us state the following version of the multiplicative ergodic theorem.

**Theorem 2.** Assume that $\mu$ is ergodic, $f$-invariant Borel probability measure. Furthermore, suppose that the cocycle $A$ is quasicompact with respect to $\mu$. Then, we have the following:

1. there exists $l \in [1, \infty]$ and a sequence of numbers $(\lambda_i)_{i=1}^l$ such that

   $$\lambda(\mu) = \lambda_1 > \lambda_2 > \ldots > \lambda_i > \ldots > \kappa(\mu).$$

   Furthermore, if $l = \infty$ we have that $\lim_{i \to \infty} \lambda_i = \kappa(\mu)$;

2. for $\mu$-a.e. $q \in M$ there exists a unique and measurable decomposition

   $$B = \bigoplus_{i=1}^l E_i(q) \oplus F(q),$$

   where $E_i$ are closed, finite-dimensional subspaces of $X$ and $A(q)E_i(q) = E_i(f(q))$. Furthermore, $F(q)$ are closed subspaces of $X$ and $A(q)F(q) \subset F(f(q));$

3. for each $1 \leq i \leq l$ and $v \in E_i(q) \setminus \{0\}$, we have

   $$\lim_{n \to \infty} \frac{1}{n} \log \|A(q,n)v\| = \lambda_i.$$

   In addition, for every $v \in F(x)$,

   $$\limsup_{n \to \infty} \frac{1}{n} \log \|A(q,n)v\| \leq \kappa(\mu).$$

The numbers $\lambda_i$ are called Lyapunov exponents of the cocycle $A$ with respect to $\mu$. 
A VERSION OF THE THEOREM OF JOHNSON

Theorem 2 was established by Froyland, Lloyd and Quas [8], building on the earlier work of Thieullen [18] who has obtain analogous result but under additional assumption that the cocycle $A$ is injective. We also note that the version of Theorem 2 established in [8] works under milder assumptions both for the base space $(M, f)$ and for the cocycle $A$. In particular, for the conclusion of Theorem 2 to hold it is not necessary to assume that $A$ is a continuous map. Subsequent generalizations of the main result in [8] were obtained by González-Tokman and Quas [9, 10] and Blumenthal [2].

2.4. Growth of subadditive functions over semiflows. In this subsection we recall the very useful result of Cao [5]. A sequence of continuous functions $(F_n)_{n \in \mathbb{N}}, F_n: M \to \mathbb{R}$ is said to be subadditive if

$$F_{n+m}(q) \leq F_m(f^n(q)) + F_n(q), \quad q \in M \text{ and } n, m \in \mathbb{N}. $$

Let $\mathcal{E}(f)$ denote the set of all ergodic $f$-invariant measures. The following result is due to Cao [5, Theorem 4.2].

**Theorem 3.** There exists $\nu \in \mathcal{E}(f)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \max_{q \in M} F_n(q) = \Lambda(\nu) = \max_{\mu \in \mathcal{E}(f)} \Lambda(\mu),$$

where $\Lambda(\mu) = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_M F_n d\mu$.

3. Sacker-Sell spectrum for quasicompact cocycles

In this section we introduce the notion of a Sacker-Sell spectrum for quasicompact cocycles and describe all possible structures of this spectrum. As we have already mentioned, our arguments follow closely the approach developed in [1].

¿From now on, we consider the cocycle $A$ over $f$ such that (6) holds for each $\mu \in \mathcal{E}(f)$. The following simple consequence of Proposition 1 provides sufficient conditions under which this is satisfied.

**Proposition 4.** Let $(X, \| \|)$ and $(X', | |)$ be as in the statement of Proposition 1. Furthermore, assume that there exist Borel measurable functions $\alpha, \beta, \gamma: M \to (0, \infty)$ with $\gamma$ being log-integrable such that:

1. (7) and (8) hold for each $q \in M$ and $x \in X$;
2. (9) holds for each $\mu \in \mathcal{E}(f)$.

Then, (6) holds for each $\mu \in \mathcal{E}(f)$.

**Proof.** By applying Proposition 1 for an arbitrary $\mu \in \mathcal{E}(f)$, we obtain that

$$\kappa(\mu) \leq \int_M \log \alpha(q) \, d\mu(q) < \lambda(\mu),$$

and thus (6) holds. \qed

**Proposition 5.** We have that $\max_{\mu \in \mathcal{E}(f)} \kappa(\mu)$ exists.

**Proof.** This is a direct consequence of Theorem 1 applied to a subadditive sequence

$$F_n(q) = \|A(q, n)\|_{ic(X)}, \quad q \in M, \ n \in \mathbb{N}. \qed$$
Let
\[ \kappa := \max_{\mu \in E} \kappa(\mu) \in [\infty, \infty) \]
Furthermore, for any \( a \in \mathbb{R} \) we can define a cocycle \( A_a \) over \( f \) by
\[ A_a(q, n) = e^{-an}A(q, n), \quad q \in M, \ n \geq 0. \]
Finally, we define the Sacker-Sell spectrum \( \Sigma = \Sigma(A) \) of \( A \) by
\[ \Sigma = \left \{ a \in \mathbb{R} : a > \kappa \text{ and } A_a \text{ is not uniformly hyperbolic} \right \} \subset (\kappa, \infty). \]
The main aim of this section is to describe all possible structures of \( \Sigma \). We first introduce some useful notation. For \( a \in \mathbb{R} \) and \( q \in M \), set
\[ S_a(q) = \left \{ x \in X : \sup_{n \geq 0} e^{-an}\|A(q, n)x\| < \infty \right \}. \]
Moreover, let \( U_a(q) \) denote the set of all \( x \in X \) for which there exists a sequence \( (x_n)_{n \leq 0} \subset X \) such that \( \sup_{n \leq 0} (e^{-an}\|x_n\|) < \infty \) and
\[ x_n = A(f^{n-1}(q))x_{n-1} \text{ for } n \leq 0. \]
It is easy to verify that if \( A_a \) is uniformly hyperbolic with projections \( P(q) \), \( q \in M \) then
\[ \text{Im } P(q) = S_a(q) \quad \text{and} \quad \text{Ker } P(q) = U(q), \]
for every \( q \in M \). Moreover, for \( a_1 < a_2 \) we have
\[ S_{a_1}(q) \subset S_{a_2}(q) \quad \text{and} \quad U_{a_2}(q) \subset U_{a_1}(q), \]
for each \( q \in M \). We now collect basic properties of the Sacker-Sell spectra which can be found in [16]. Although the original work of Sacker and Sell considers only the case of finite-dimensional and invertible dynamics, all the auxiliary results we list below can be easily proved by repeating the arguments in [16] (see also [1]).

Lemma 2. \( \Sigma \) is a closed set. More precisely, for \( a > \kappa \) such that \( a \notin \Sigma \), there exists \( \varepsilon > 0 \) such that for each \( b \in (a - \varepsilon, a + \varepsilon) \) we have that:
1. \( b \notin \Sigma; \)
2. for every \( q \in M \),
\[ S_b(q) = S_a(q) \quad \text{and} \quad U_b(q) = U_a(q). \]
Note that for \( a > \kappa \) such that \( a \notin \Sigma \), \( \dim U_a(q) \) doesn’t depend on \( q \) and thus we can simply write \( \dim U_a \). Moreover, it follows from Lemma 1 that \( \dim U_a < \infty \) for each such \( a \).

Lemma 3. Take \( a_2 > a_1 > \kappa \) such that \( a_1, a_2 \notin \Sigma \). Then, the following properties are equivalent:
1. \( [a_1, a_2] \subset (\kappa, \infty) \ \setminus \Sigma; \)
2. \( \dim U_{a_1} = \dim U_{a_2}. \)

Lemma 4. We have that \( \sup \Sigma < \infty \).
Proof. Since $A$ is continuous and $M$ is compact, there exists $C > 0$ such that $\|A(q)\| \leq C$ for $q \in M$. Hence,
\[
\|A(q, n)\| \leq C^n \quad \text{for } q \in M \text{ and } n \in \mathbb{N}. \tag{10}
\]
It follows from (10) that for each $a > \log C$, the cocycle $A_n$ is uniformly hyperbolic with projections $P(q) = \text{Id}$, $q \in M$. This readily implies the desired conclusion.

The following lemma is crucial for our purposes.

**Lemma 5.** Let $c > \kappa$ such that $c \notin \Sigma$. Then, $\Sigma \cap [c, \infty)$ is the union of finitely many closed intervals.

**Proof.** Let $d = \dim U_C$ and assume that $\Sigma \cap [c, +\infty)$ contains at least $d + 2$ disjoint closed intervals $I_i = [\alpha_i, \beta_i]$, for $i = 1, \ldots, d + 2$, where
\[
\alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \cdots < \alpha_{d+2} \leq \beta_{d+2} < +\infty.
\]
For $i \in \{1, \ldots, d + 1\}$, take $c_i \in (\beta_i, \alpha_{i+1})$. By Lemma 3,
\[
d > \dim U_{c_1} > \dim U_{c_2} > \cdots > \dim U_{c_{d+1}},
\]
which is clearly impossible.

The following is the main result of this section.

**Theorem 6.** One of the following alternatives holds:

1. $\Sigma = \emptyset$;
2. $\Sigma = \bigcup_{n=1}^k [a_n, b_n]$, for some numbers $b_1 \geq a_1 > b_2 \geq a_2 > \cdots > b_k \geq a_k > \kappa$; \tag{11}
3. $\Sigma = \bigcup_{n=1}^{k-1} [a_n, b_n] \cup (\kappa, b_k]$, for some numbers $a_n$ and $b_n$ as in (11);
4. $\Sigma = \bigcup_{n=1}^{\infty} [a_n, b_n]$, for some numbers $b_1 \geq a_1 > b_2 \geq a_2 > \cdots$ \tag{12}
with $\lim_{n \to +\infty} a_n = \kappa$;
5. $\Sigma = \bigcup_{n=1}^{\infty} [a_n, b_n] \cup (\kappa, b_\infty]$, for some numbers $a_n$ and $b_n$ as in (12)
with $b_\infty := \lim_{n \to +\infty} a_n > \kappa$.

**Proof.** Since $\Sigma$ is closed and $\Sigma \neq (\kappa, \infty)$ (see Lemma 4), if $\Sigma$ is nonempty and has finitely many connected components, then it has one of the forms in alternatives 2 and 3.

Now we consider the case when $\Sigma$ has infinitely many connected components. Namely, assume that $\Sigma$ is not given by one of the first three alternatives in the theorem and take $c_1 \notin \Sigma$. By Lemma 5 the set $\Sigma \cap (c_1, +\infty)$ consists of finitely many disjoint closed intervals $I_1, \ldots, I_k$. We note that $\Sigma \cap (\kappa, c_1) \neq \emptyset$, since otherwise we would have $\Sigma = I_1 \cup \cdots \cup I_k$, which contradicts to our assumption. Now we observe that there exists $c_2 < c_1$ such that $c_2 \notin \Sigma$ and $(c_2, c_1) \cap \Sigma \neq \emptyset$. Indeed, otherwise we would have $(\kappa, c_1) \cap \Sigma = (\kappa, a]$ for some $a < c_1$ and thus,
\[
\Sigma = (\kappa, a] \cup I_1 \cdots \cup I_k,
\]
which again contradicts to our assumption. Proceeding inductively, we obtain a decreasing sequence $(c_n)_{n \in \mathbb{N}}$ such that $c_n \notin \Sigma$ and $(c_{n+1}, c_n) \cap \Sigma \neq \emptyset$.
for each \( n \in \mathbb{N} \). Now there are two possibilities: either \( \lim_{n \to +\infty} c_n = \kappa \) or \( \lim_{n \to +\infty} c_n = b_\infty \) for some \( b_\infty > \kappa \). In the first case, it follows from Lemma 5 that \( \Sigma \) is given by alternative 5 in the theorem. In the second case, it follows from Lemma 5 that

\[
(b_\infty, +\infty) \cap \Sigma = I_1 \cup \bigcup_{n=2}^{\infty} [a_n, b_n],
\]

where \( I_1 = [a_1, b_1] \) or \( I_1 = [a_1, +\infty) \), for some sequences \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) as in \([12]\) with \( b_\infty = \lim_{n \to +\infty} a_n \). Again by Lemma 5 we have \((\kappa, b_\infty) \subset \Sigma\) and so \( \Sigma \) is given by the last alternative in the theorem. \( \square \)

We note that the structure of the Sacker-Sell spectrum for compact operators on an arbitrary Banach space under the additional assumption that \( f \) has a fixed or a periodic orbit was discussed in \([12]\). There, a result similar to Theorem 6 was established. In particular, it was proved that the last alternative in the statement of Theorem 6 never occurs in that setting.

4. A VERSION OF THE THEOREM OF JOHNSON, PALMER AND SELL

We begin this section by noting that our assumption that \( (6) \) holds for any \( \mu \in \mathcal{E}(f) \), enable us to apply Theorem 2 for each \( \mu \in \mathcal{E}(f) \). Hence, we obtain the set of Lyapunov exponents of our cocycle \( A \) with respect to some ergodic \( f \)-invariant measure. The following version of the theorem of Johnson, Palmer and Sell \([11]\) for quasicompact cocycles is the main result of our paper.

**Theorem 7.** Assume that \( \Sigma \neq \emptyset \). The endpoints \( a_n \) and \( b_n \) for \( n \in \mathbb{N} \) of spectral intervals contained in \( \Sigma \) are Lyapunov exponents of the cocycle \( A \) with respect to some ergodic \( f \)-invariant measure.

**Proof.** Take a spectral interval \([a_m, b_m] \subset \Sigma \) and let us prove that \( b_m \) is a Lyapunov exponent of \( A \) with respect to some ergodic \( f \)-invariant measure. The argument for \( a_m \) is completely analogous. Assume that \( b_m \) is not a Lyapunov exponent of \( A \) and take \( c = b_m + \varepsilon \), where \( \varepsilon > 0 \). For a sufficiently small \( \varepsilon > 0 \), \( c \notin \Sigma \) and therefore the cocycle \( A_c \) is uniformly hyperbolic with respect to projections \( P(q) \), \( q \in M \) that satisfy

\[
\text{Im} \, P(q) = S_c(q) \quad \text{and} \quad \text{Ker} \, P(q) = U_c(q).
\]

In addition, by Lemma 8 the subspaces \( S_c(q) \) and \( U_c(q) \) don’t depend on the choice of \( c \). In particular, there exist \( D, \lambda > 0 \) such that

\[
\|e^{-cn}A(q,n)P(q)\| \leq De^{-\lambda n}, \quad \text{for } n \geq 0,
\]

which implies that for \( v \in S_c(q) \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A(q,n)v\| \leq c - \lambda < c.
\]

Letting \( \varepsilon \to 0 \), we have that \( c \to b_m \) and therefore

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A(q,n)v\| \leq b_m.
\]

Set

\[
F_n(q) = \log \|A(q,n)P(q)\| \quad \text{for } q \in M \text{ and } n \in \mathbb{N}.
\]
It follows from the continuity of $A$ and the map $q \mapsto P(q)$ that $F_n \colon M \to \mathbb{R}$ is continuous map for each $n \in \mathbb{N}$.

**Lemma 6.** The sequence $(F_n)_{n \in \mathbb{N}}$ is subadditive.

**Proof of the lemma.** It follows from (2) that
\[
\|A(q, n + m)P(q)\| = \|A(q, n + m)P(q)^2\|
\leq \|A(f^n(q), m)A(q, n)P(q)P(q)\|
\leq \|A(f^n(q), m)P(f^n(q))\| \cdot \|A(q, n)P(q)\|,
\]
for $q \in M$ and $n, m \in \mathbb{N}$ which immediately implies the desired conclusion. \qed

For $\mu \in \mathcal{E}(f)$, set
\[
\Lambda(\mu) = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_M F_n \, d\mu.
\]

It follows from Kingman’s subadditive ergodic theorem that
\[
\Lambda(\mu) = \lim_{n \to \infty} \frac{1}{n} F_n(q) \quad \text{for } \mu\text{-a.e. } q \in M.
\]

**Lemma 7.** $\Lambda(\mu)$ is a Lyapunov exponent of $A$ with respect to $\mu$.

**Proof of the lemma.** Using the arguments similar to those we used to establish (13), one can easily prove that for each nonzero $v \in S_c(q) \cap U_{a_m - \epsilon}(q)$ (we note that such $v$ exists by Lemma 2), we have
\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A(q, n)v\| \geq a_m,
\]
which immediately implies that $\Lambda(\mu) \geq a_m$. Hence,
\[
\Lambda(\mu) \geq a_m > \kappa(\mu).
\]

Let $\lambda_1 > \lambda_2 > \ldots$ denote (distinct) Lyapunov exponent of $A$ with respect to $\mu$ and assume that $\Lambda(\mu) \neq \lambda_j$ for each $j$. Choose $i$ such that $\Lambda(\mu) \in (\lambda_{i+1}, \lambda_i)$. Then, for $\mu$-a.e. $q \in M$ and each $v \in \text{Im} P(q)$, we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \|A(q, n)v\| \leq \lim_{n \to \infty} \frac{1}{n} \log \|A(q, n)P(q)\| < \lambda_i,
\]
and thus it follows from Theorem 2 that
\[
\lim_{n \to \infty} \frac{1}{n} \log \|A(q, n)v\| \leq \lambda_{i+1}
\]
By [8, Proposition 14.], we have that
\[
\lim_{n \to \infty} \frac{1}{n} \log \|A(q, n)\| \text{ Im } P(q)\| \leq \lambda_{i+1},
\]
which together with the continuity of the map $q \mapsto P(q)$ implies that $\Lambda(\mu) \leq \lambda_{i+1}$. Hence, we have obtained an contradiction. \qed
We now note that (13), Lemma 7 and our assumption on $b_m$ imply that $\Lambda(\mu) < b_m$ for each $\mu \in \mathcal{E}(f)$. By Theorem 3 we have that
\[ \lim_{n \to \infty} \frac{1}{n} \max_{q \in \mathcal{M}} F_n(q) < b_m, \]
which readily implies that there exist $D, \lambda > 0$ such that
\[ \|e^{-b_m n} A(q, n) P(q)\| \leq De^{-\lambda n} \quad \text{for } q \in \mathcal{M} \text{ and } n \geq 0. \] (14)
Similarly, one can show that there exist $D', \lambda' > 0$ such that
\[ \|e^{b_m n} A(q, -n) P(q)\| \leq D'e^{-\lambda'n} \quad \text{for } q \in \mathcal{M} \text{ and } n \geq 0. \] (15)
It follows from (14) and (15) that $A_{b_m}$ is uniformly hyperbolic. Therefore, $b_m \not\in \Sigma$ which yields a contradiction.

We emphasize that Theorem 7 says nothing about $b_\infty$. In the case when $f$ is uniquely ergodic (which means that $\mathcal{E}(f)$ consists of only one measure), $b_\infty$ cannot be realized as a Lyapunov exponent since Lyapunov exponents with respect to a given measure $\mu$ can only accumulate in $\kappa(\mu)$ (see Theorem 2). As for what happens in general case, it is unclear to me.

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A VERSION OF THE THEOREM OF JOHNSON

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