NEWELL-LITTLEWOOD NUMBERS III: EIGENCONES AND GIT-SEMIGROUPS

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ABSTRACT. The Newell-Littlewood numbers are tensor product multiplicities of Weyl modules for the classical groups in the stable range. Littlewood-Richardson coefficients form a special case. Klyachko connected eigenvalues of sums of Hermitian matrices to the saturated LR-cone and established defining linear inequalities. We prove analogues for the saturated NL-cone:

• an eigenvalue interpretation;
• a minimal list of defining linear inequalities;
• a description by Extended Horn inequalities, as conjectured in part II of this series; and
• a factorization of NL-numbers, on the boundary.

1. INTRODUCTION

This is the third installment in a series [9, 10] about the Newell-Littlewood numbers [21, 19]

\[ N_{\lambda,\mu,\nu} = \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta}^\lambda c_{\beta,\gamma}^\mu c_{\gamma,\alpha}^\nu; \]

the indices are partitions in \(\text{Par}_n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}\). In [1], \(c_{\alpha,\beta}^\lambda\) is the Littlewood-Richardson coefficient. The Littlewood-Richardson coefficients are themselves Newell-Littlewood numbers: if \(|\nu| = |\lambda| + |\mu|\) then \(N_{\lambda,\mu,\nu} = c_{\lambda,\mu}^\nu\). The goal of this series is to establish analogues of results known for Littlewood-Richardson coefficients. This paper proves NL-generalizations of breakthrough results of Klyachko [14].

Fix \(n \in \mathbb{N} = \{1, 2, 3, \ldots\}\). The paper [9] investigated

\[ \text{NL-semigroup}(n) = \{(\lambda, \mu, \nu) \in (\text{Par}_n)^3 : N_{\lambda,\mu,\nu} > 0\}. \]

Indeed, NL-semigroup is a finitely generated semigroup [9, Section 5.2]. A good approximation of it is the saturated semigroup:

\[ \text{NL-sat}(n) = \{(\lambda, \mu, \nu) \in (\text{Par}_n^Q)^3 : \exists t > 0 \text{ s.t. } N_{t\lambda,t\mu,t\nu} \neq 0\}, \]

where \(\text{Par}_n^Q = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Q}^n : \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}\). Our main results give descriptions of NL-sat\((n)\), including with a minimal list of defining linear inequalities.

Fix \(m \in \mathbb{N}\) and consider the symplectic Lie algebra \(\text{sp}(2m, \mathbb{C})\). The irreducible \(\text{sp}(2m, \mathbb{C})\)-representations \(V(\lambda)\) are parametrized by their highest weight \(\lambda \in \text{Par}_m\) (see Section 3.1 for details). The tensor product multiplicities \(\text{mult}_{\lambda,\mu,\nu}^m\) are defined by

\[ V(\lambda) \otimes V(\mu) = \sum_{\nu \in \text{Par}_m} V(\nu)^{\otimes \text{mult}_{\lambda,\mu,\nu}^m}. \]

Since \(\text{sp}(2m, \mathbb{C})\)-representations are self-dual, \(\text{mult}_{\lambda,\mu,\nu}^m\) is symmetric in its inputs. The supports of these multiplicities (and more generally when \(\text{sp}(2m, \mathbb{C})\) is replaced by any

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Lemma 2.2(II), the sl\(_n\)-eigencone is a facet of the sp\(_{2n}\)-eigencone.

The finitely generated semigroup

\[
\text{sp-semigroup}(m) = \{ (\lambda, \mu, \nu) \in (\text{Par}_m)^3 : \text{mult}_{\lambda,\mu,\nu}^m > 0 \},
\]

and the cone generated by it

\[
\text{sp-sat}(m) = \{ (\lambda, \mu, \nu) \in (\text{Par}_m^Q)^3 : \exists t > 0 \quad \text{mult}_{\lambda,\mu,\nu}^m t > 0 \}.
\]

For \( m \geq n \), by postpending 0's, Par\(_n\) embeds into Par\(_m\). Newell-Littlewood numbers are tensor product multiplicities for sp\((2m, \mathbb{C})\) in the stable range [16] Corollary 2.5.3:

\[
\forall (\lambda, \mu, \nu) \in (\text{Par}_n)^3 \quad \text{if } m \geq 2n \text{ then } \text{mult}_{\lambda,\mu,\nu}^m = N_{\lambda,\mu,\nu}.
\]

Now, (2) immediately implies

\[
(3) \quad \text{NL-sat}(n) = \text{sp-sat}(m) \cap (\text{Par}_n^Q)^3, \text{ for any } m \geq 2n.
\]

Our first result says the relationship of NL-sat to sp-sat is independent of the stable range.

**Theorem 1.1.** For any \( m \geq n \geq 1 \),

\[
\text{NL-sat}(n) = \text{sp-sat}(m) \cap (\text{Par}_n^Q)^3.
\]

Theorem 1.1 has a number of consequences. Define

\[
\text{LR-sat}(n) = \{ (\lambda, \mu, \nu) \in (\text{Par}_n^Q)^3 : \exists t > 0 \quad c_{t\lambda,t\mu}^\nu > 0 \}.
\]

Klyachko [14] showed that LR-sat(n) describes the possible eigenvalues \( \lambda, \mu, \nu \) of three \( n \times n \) Hermitian matrices \( A, B, C \) (respectively) such that \( A + B = C \). Similarly, Theorem 1.1 shows that NL-sat(n) describes solutions to a more general eigenvalue problem; see Section 2.6 and Proposition 3.1.

Another major accomplishment of [14] was the first proved description of LR-sat(n) via linear inequalities. We have three such descriptions of NL-sat(n). We now state the first of these. Set \([n] = \{1, \ldots, n\}\). For \( A \subset [n] \) and \( \lambda \in \text{Par}_n \), let \( \lambda_A \) be the partition using the only parts indexed by \( A \); namely, if \( A = \{i_1 < \cdots < i_r\} \) then \( \lambda_A = (\lambda_{i_1}, \ldots, \lambda_{i_r}) \). In particular, \( |\lambda_A| = \sum_{i \in A} \lambda_i \). Using the known descriptions of sp-sat(n) [11, 23, 25] we deduce from Theorem 1.1 a minimal list of inequalities defining NL-sat(n):

**Theorem 1.2.** Let \( (\lambda, \mu, \nu) \in (\text{Par}_n)^3 \). Then \( (\lambda, \mu, \nu) \in \text{NL-sat}(n) \) if and only if

\[
0 \leq |\lambda_A| - |\lambda_A'| + |\mu_B| - |\mu_B'| + |\nu_C| - |\nu_C'| \quad \text{for any subsets } A, A', B, B', C, C' \subset [n] \text{ such that}
\]

1. \( A \cap A' = B \cap B' = C \cap C' = \emptyset \);
2. \( |A| + |A'| = |B| + |B'| = |C| + |C'| = |A'| + |B'| + |C'| = r \);
3. \( t_{(A,A')}^{(2n-2r)^+} t_{(B,B')}^{(2n-2r)^+} t_{(C,C')}^{(2n-2r)^+} = t_{(A',A')}^{(r^+)} t_{(B',B')}^{(r^+)} t_{(C',C')}^{(r^+)} = 1 \).

Moreover, this list of inequalities is irredundant.

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1We remark that since \( (\lambda, \mu, \nu) \in \text{LR-sat}(n) \) is also in \( \text{NL-sat}(n) \) if and only if \( |\lambda| + |\mu| = |\nu| \) (see [9] Lemma 2.2(II)), the sl\(_n\)-eigencone is a facet of the sp\(_{2n}\)-eigencone.
The definition of the partitions occurring in condition (3) is in Section 3.2.

The proofs of Theorems 1.1 and 1.2 use ideas of P. Belkale-S. Kumar [1] on their deformation of the cup product on flag manifolds, as well as the third author’s work on GIT-semigroups/cones [23][25]. We interpret NL-sat(n) from the latter perspective in Section 5 (see Proposition 5.2) by study of the truncated tensor cone. Our argument requires us to generalize [1, Theorem 28] [23, Theorem B] (recapitulated here as Theorem 2.3); see Theorem 5.1. As an application, we obtain Theorem 1.3 below, which is a factorization of the cup product on flag manifolds, as well as the third author’s work on NL-Saturation [9, Conjecture 5.5].

Conjecture 1.4

\[ \lambda_{A,A'} = (\lambda_{i_1}, \ldots, \lambda_{i_t}, -\lambda_{i_t}, \ldots, -\lambda_{i_1}) \text{ and } \lambda^{A,A'} = \lambda_{[n]-(\bigcup A')}, \text{ etc.} \]

**Theorem 1.3.** Let \( A, A', B, B', C, C' \subset [n] \) such that

1. \( A \cap A' = B \cap B' = C \cap C' = \emptyset; \)
2. \( |A| + |A'| = |B| + |B'| = |C| + |C'| = |A'| + |B'| + |C'| =: r; \)
3. \( \tau^{\lambda_{(A,A')}}(C,C')^{\lambda_{(B,B')}}(C,C')^{\lambda_{(C,C')}} = (\tau^{\lambda_{(A,A')}})^{\lambda_{(B,B')}}(\tau^{\lambda_{(C,C')}})^{\lambda_{(C,C')}} = 1, \)

as in Theorem 1.2. For \( (\lambda, \mu, \nu) \in (\mathbb{P}ar_n)^3 \) such that

\[ 0 = |\lambda_A| - |\lambda_A| + |\mu_B| - |\mu_B| + |\nu_C| - |\nu_C| , \]

\[ N_{\lambda,\mu,\nu} = \nu_{C,C'}^{\lambda_{(A,A')}} N_{\lambda_{A,A'},\mu_{B,B'},\nu_{C,C'}}. \]

Theorem 1.3 is analogous to [5] Theorem 7.4 and [13] Theorem 1.4 for \( e_{\alpha,\beta}^{\gamma} \).

Knutson-Tao’s celebrated Saturation Theorem [15] proves, inter alia, that LR-sat(n) is described by Horn’s inequalities (see, e.g., Fulton’s survey [7]). This posits a generalization:

**Conjecture 1.4 (NL-Saturation [9 Conjecture 5.5]).** Let \( (\lambda, \mu, \nu) \in (\mathbb{P}ar_n)^3 \). Then \( N_{\lambda,\mu,\nu} \neq 0 \) if and only if \( |\lambda| + |\mu| + |\nu| \) is even and there exists \( t > 0 \) such that \( N_{t,\lambda,\mu,\nu} \neq 0 \).

Theorem 1.2 permits us to prove Conjecture 1.4 for \( n \leq 5 \), by computer-aided calculation of Hilbert bases; see Section 6. This is the strongest evidence of the conjecture to date; previously, [9] Corollary 5.16 proved the \( n = 2 \) case, by combinatorial reasoning.

The conditions parametrizing the inequalities occurring in the Horn’s inequalities for LR-sat(n) are inductive, as they depend on LR-sat(n') for \( n' < n \). Theorem 1.2 is not inductive. In [10], a conjectural description of NL-sat(n), somewhat closer to the Horn conjecture for LR-sat(n), was given. More precisely, the formulation [10] Conjecture 1.4 subsumes both Conjecture 1.4 and a description of NL-sat using extended Horn inequalities [10] Definition 1.2). Our last result is a proof the latter part of the conjecture.

Let \( \lambda_1, \ldots, \lambda_s \in Par_n \) for \( s \geq 3 \). Treat the indices \( 1, \ldots, s \) as elements of \( \mathbb{Z}/s\mathbb{Z} \). We introduce the multiple Newell-Littlewood number as

\[ N_{\lambda_1, \ldots, \lambda_s} = \sum_{(\alpha_1, \ldots, \alpha_s) \in (\text{Par}_n)^s} \prod_{i \in \mathbb{Z}/s\mathbb{Z}} c_{\alpha_1, \alpha_{i+1}}^{\lambda_i}. \]

When \( s = 3 \), we recover Newell-Littlewood numbers [2]. To \( A = \{i_1 < \cdots < i_r\} \subset [n] \), associate the partition

\[ \tau(A) = (i_r - r \geq \cdots \geq i_1 - 1). \]

\[ ^2N_{\lambda_1, \ldots, \lambda_s} \text{ also has a representation-theoretic interpretation. Discussion may appear elsewhere.} \]
Theorem 1.5. Let \((\lambda, \mu, \nu) \in (\text{Par}_n^Q)^3\). Then \((\lambda, \mu, \nu) \in \text{NL-sat}(n)\) if and only if
\[
0 \leq |\lambda_\alpha| - |\lambda_{\alpha}'| + |\mu_B| - |\mu_{B'}| + |\nu_C| - |\nu_{C'}|
\]
for any subsets \(A, A', B, B', C, C' \subset [n]\) such that

1. \(A \cap A' = B \cap B' = C \cap C' = \emptyset\);
2. \(|A| + |A'| = |B| + |B'| = |C| + |C'| = |A'| + |B'| + |C'|\);
3. \(N_{\tau(A), \tau(B), \tau(C'), \tau(A), \tau(B'), \tau(C)} \neq 0\).

As explained in Section 10.1, the inequalities of Theorem 1.5 are easily equivalent to the extended Horn inequalities. However, the reformulation of Theorem 1.5 is more compact. The main part of the proof is to show that the inequalities of Theorem 1.2 are all instances of extended Horn inequalities. That argument, which occupies Sections 7–10, requires heavy-lifting in tableau combinatorics. Along the way, we establish the relationship between the demotion algorithm [9] and the RSK correspondence (Theorem 8.2), as well as a nonvanishing result about multiple Newell-Littlewood numbers (Theorem 9.1).

2. Generalities on the tensor cones

2.1. Finitely generated semigroups. \(\Gamma \subseteq \mathbb{Z}^n\) is a semigroup if \(\vec{0} \in \Gamma\) and is closed under addition. A finitely generated \(\Gamma\) generates a closed convex polyhedral cone \(\Gamma_\mathbb{Q} \subseteq \mathbb{Q}^n\):
\[
\Gamma_\mathbb{Q} = \{x \in \mathbb{Q}^n : \exists t \in \mathbb{Z}_{>0} \quad tx \in \Gamma\}.
\]
The subgroup of \(\mathbb{Z}^n\) generated by \(\Gamma\) is
\[
\Gamma_\mathbb{Z} = \{x - y : x, y \in \Gamma\}.
\]
The semigroup \(\Gamma\) is saturated if \(\Gamma = \Gamma_\mathbb{Z} \cap \Gamma_\mathbb{Q}\).

2.2. GIT-semigroups. We recall the GIT-perspective of [23]. Let \(G\) be a complex reductive group acting on an irreducible projective variety \(X\). Let \(\text{Pic}^G(X)\) be the group of \(G\)-linearized line bundles. Given \(\mathcal{L} \in \text{Pic}^G(X)\), let \(H^0(X, \mathcal{L})\) be the space of sections of \(\mathcal{L}\); it is a \(G\)-module. Let \(H^0(X, \mathcal{L})^G\) be the subspace of invariant sections. Define
\[
\text{GIT-semigroup}(G, X) = \{\mathcal{L} \in \text{Pic}^G(X) : H^0(X, \mathcal{L})^G \neq \{0\}\}.
\]
This is a semigroup since \(X\) being irreducible says the product of two nonzero \(G\)-invariant sections is a nonzero \(G\)-invariant section. The saturated version of it is
\[
\text{GIT-sat}(G, X) = \{\mathcal{L} \in \text{Pic}^G(X) \otimes \mathbb{Q} : \exists t > 0 \quad H^0(X, \mathcal{L}^\otimes t)^G \neq \{0\}\}.
\]

2.3. The tensor semigroup. Let \(\mathfrak{g}\) be a semisimple complex Lie algebra, with fixed Borel subalgebra \(\mathfrak{b}\) and Cartan subalgebra \(\mathfrak{t} \subset \mathfrak{b}\). Denote by \(\Lambda^+(\mathfrak{g}) \subset \mathfrak{t}'\) the semigroup of the dominant weights. It is contained in the weight lattice \(\Lambda(\mathfrak{g}) \simeq \mathbb{Z}^r\), where \(r\) is the rank of \(\mathfrak{g}\). Given \(\lambda \in \Lambda^+(\mathfrak{g})\), denote by \(V_\mathfrak{g}(\lambda)\) (or simply \(V(\lambda)\)) the irreducible representation of \(\mathfrak{g}\) with highest weight \(\lambda\). Let \(V(\lambda)^*\) be the dual representation. Consider the semigroup
\[
\mathfrak{g}\text{-semigroup} = \{(\lambda, \mu, \nu) \in (\Lambda^+(\mathfrak{g}))^3 : V(\nu)^* \subset V(\lambda) \otimes V(\mu)\},
\]
and the generated cone \(\mathfrak{g}\text{-sat} \in (\Lambda(\mathfrak{g}) \otimes \mathbb{Q})^3\). When \(\mathfrak{g} = \mathfrak{sp}(2m, \mathbb{C})\) we have \(V(\nu)^* \simeq V(\nu)\) and \(\mathfrak{g}\text{-semigroup}\) is what we denoted by \(\mathfrak{sp}\text{-semigroup}(m)\) in the introduction. The set
\( g \)-semigroup spans the rational vector space \((\Lambda(g) \otimes \mathbb{Q})^3\), or equivalently, the cone \( g \)-sat has nonempty interior. The group \((g \text{-semigroup})_Z\) is well-known (see, e.g., [22], Theorem 1.4):

\[
(g \text{-semigroup})_Z = \{(\lambda, \mu, \nu) \in (\Lambda(g))^3 : \lambda + \mu + \nu \in \Lambda_R(g)\},
\]

where \(\Lambda_R(g)\) is the root lattice of \(g\).

We now interpret \(g\)-semigroup in terms of Section 2.2. Consider the semisimple, simply-connected algebraic group \(G\) with Lie algebra \(g\). Denote by \(B\) and \(T\) the connected subgroups of \(G\) with Lie algebras \(b\) and \(t\) respectively. The character groups \(X(B) = X(T) = \Lambda(g)\) of \(B\) and \(T\) coincide. For \(\lambda \in X(T)\), \(\mathcal{L}_\lambda\) is the unique \(G\)-linearized line bundle on the flag variety \(G/B\) such that \(B\) acts on the fiber over \(B/B\) with weight \(-\lambda\).

Assume \(X = (G/B)^3\). Then \(\text{Pic}^G(X)\) identifies with \(X(T)^3\). For \((\lambda, \mu, \nu) \in X(T)^3\), define \(\mathcal{L}_{(\lambda, \mu, \nu)}\). By the Borel-Weil Theorem,

\[
H^0(X, \mathcal{L}_{(\lambda, \mu, \nu)}) = V(\lambda)^* \otimes V(\mu)^* \otimes V(\nu)^*.
\]

In particular, GIT-semigroup \((G, X) \simeq g\)-semigroup.

Given three parabolic subgroups \(P, Q, R\) containing \(B\), we consider more generally \(X = G/P \times G/Q \times G/R\). Then \(\text{Pic}^G(X)\) identifies with \(X(P) \times X(Q) \times X(R)\) which is a subgroup of \(X(T)^3\). Moreover,

\[\text{GIT-semigroup}(G, X) = \text{GIT-semigroup}(G, (G/B)^3) \cap (X(P) \times X(Q) \times X(R)).\]

2.4. Schubert calculus. We need notation for the cohomology ring \(H^*(G/P, \mathbb{Z})\); \(P \supset B\) being a parabolic subgroup. Let \(W\) (resp. \(W_P\)) be the Weyl group of \(G\) (resp. \(P\)). Let \(\ell : W \rightarrow \mathbb{N}\) be the Coxeter length, defined with respect to the simple reflections determined by the choice of \(B\). Let \(W^P\) be the minimal length representatives of the cosets in \(W/W_P\).

For a closed irreducible subvariety \(Z \subset G/P\), let \([Z]\) be its class in \(H^*(G/P, \mathbb{Z})\), of degree \(2(\dim(G/P) - \dim(Z))\). For \(v \in W^P\), set \(\tau_v = [BV\mathcal{P}/P]\) (dim(BvP/P) = \(\ell(v))\). Let \(w_0\) be the longest element of \(W\) and \(w_{0, P}\) be the longest element of \(W_P\). Set \(v^\vee = w_0v_0, P\) and \(\tau_v = \tau^v\); \(\tau^v\) and \(\tau_v\) are Poincaré dual.

Let \(\rho\) be the half sum of the positive roots of \(G\). To any one-parameter subgroup \(\tau : \mathbb{C}^* \rightarrow T\), associate the parabolic subgroup (see [20])

\[P(\tau) = \{g \in G : \lim_{t \to 0} \tau(t)g\tau(t^{-1}) \text{ exists}\} .\]

Fix such a \(\tau\) such that \(P = P(\tau)\).

For \(v \in W^P\), define the BK-degree of \(\tau^v \in H^*(G/P, \mathbb{Z})\) to be \(\text{BK-deg}(\tau^v) := \langle v^{-1}(\rho) - \rho, \tau\rangle\). Let \(v_1, v_2\) and \(v_3\) in \(W^P\). By [1], Proposition 17, if \(\tau^{v_3}\) appears in the product \(\tau^{v_1} \cdot \tau^{v_2}\) then

\[
\text{BK-deg}(\tau^{v_3}) \leq \text{BK-deg}(\tau^{v_1}) + \text{BK-deg}(\tau^{v_2}).
\]

In other words, the BK-degree filters the cohomology ring. Let \(\otimes_0\) denote the associated graded product on \(H^*(G/P, \mathbb{Z})\).

2.5. Well-covering pairs. Let \(X\) be a projective variety. In [23], \(\text{GIT-sat}(G, X)\) is described in terms of well-covering pairs. When \(X = (G/B)^3\), it recovers the description made by Belkale-Kumar [1]. We now discuss the case \(X = G/P \times G/Q \times G/R\) is the product of three partial flag varieties of \(G\). Let \(\tau\) be a dominant one-parameter subgroup of \(T\). The centralizer \(G^\tau\) of the image of \(\tau\) in \(G\) is a Levi subgroup. Moreover, \(P(\tau)\) is the parabolic subgroup generated by \(B\) and
$G^\tau$. Let $C$ be an irreducible component of the fixed set $X^\tau$ of $\tau$ in $X$. It is well-known that $C$ is the $(G^\tau)^3$-orbit of some $T$-fixed point:

$C = G^\tau u^{-1} P/P \times G^\tau v^{-1} Q/Q \times G^\tau w^{-1} R/R,$

with $u \in W_P \backslash W/W_P(\tau)$ and similarly for $v$ and $w$. Set

$C^+ = P(\tau)u^{-1} P/P \times P(\tau)v^{-1} Q/Q \times P(\tau)w^{-1} R/R.$

Then the closure of $C^+$ is a Schubert variety (for $G^3$) in $X$. By [23 Proposition 11], the pair $(C, \tau)$ is well-covering if and only if

$$[PuP(\tau)/P(\tau)] \circ_0 [QvP(\tau)/P(\tau)] \circ_0 [RwP(\tau)/P(\tau)] = [pt] \in H^*(G/P(\tau), \mathbb{Z}).$$

It is said to be dominant if

$$[PuP(\tau)/P(\tau)] \cdot [QvP(\tau)/P(\tau)] \cdot [RwP(\tau)/P(\tau)] \neq 0 \in H^*(G/P(\tau), \mathbb{Z}).$$

In this paper, the reader can take these characterizations as definitions of well-covering and dominant pairs. They are used in [23] to produce inequalities for the GIT-cones:

**Proposition 2.1.** Let $(\lambda, \mu, \nu) \in X(P) \times X(Q) \times X(R)$ be dominant such that $L_{(\lambda, \mu, \nu)} \in \text{GIT-sat}(G, X)$. Let $(C, \tau)$ be a dominant pair. Then,

$$\langle u\tau, \lambda \rangle + \langle v\tau, \mu \rangle + \langle w\tau, \nu \rangle \leq 0.$$

Here $(u, v, w)$ are determined by $C$ and equation (12).

In Proposition 2.1 and below, $\langle \cdot, \cdot \rangle$ denotes the pairing between one parameter subgroups and characters of $T$. Among the inequalities given by Proposition 2.1, those associated to well-covering pairs are sufficient to define the GIT-cone (see [23 Proposition 4]):

**Proposition 2.2.** Let $(\lambda, \mu, \nu) \in X(P) \times X(Q) \times X(R)$ be dominant. Then, $L_{(\lambda, \mu, \nu)} \in \text{GIT-sat}(G, X)$ if and only if for any dominant one-parameter subgroup $\tau$ of $T$ and any well-covering pair $(C, \tau)$,

$$\langle u\tau, \lambda \rangle + \langle v\tau, \mu \rangle + \langle w\tau, \nu \rangle \leq 0.$$

Here $(u, v, w)$ are determined by $C$ and equation (12). \(^3\)

In the case $P = Q = R = B$, there is a more precise statement. The fact that the inequalities define the cone is Belkale-Kumar [11 Theorem 28]. The irredundancy is [23 Theorem B]. Let $\alpha$ be a simple root of $G$. Denote by $P^\alpha$ the associated maximal parabolic subgroup of $G$ containing $B$. Denote by $w_\alpha^\vee$ the associated fundamental one-parameter subgroup of $T$ characterized by $\langle w_\alpha^\vee, \beta \rangle = \delta_\alpha^\beta$ for any simple root $\beta$.

**Theorem 2.3 ([11 Theorem 28],[23 Theorem B]).** Here $X = (G/B)^3$. Let $(\lambda, \mu, \nu) \in X(T)^3$ be dominant. Then, $L_{(\lambda, \mu, \nu)} \in \text{GIT-sat}(G, X)$ if and only if for any simple root $\alpha$, for any $u, v, w$ in $W^{P^\alpha}$ such that

$$[BuP^\alpha/P^\alpha] \circ_0 [BvP^\alpha/P^\alpha] \circ_0 [BwP^\alpha/P^\alpha] = [pt] \in H^*(G/P^\alpha, \mathbb{Z}),$$

$$\langle uw_\alpha^\vee, \lambda \rangle + \langle vw_\alpha^\vee, \mu \rangle + \langle w_\alpha^\vee, \nu \rangle \leq 0.$$

Moreover, this list of inequalities is irredundant.

\(^3\)Proposition 2.2 implies Proposition 2.1 is also a characterization of GIT-sat$(G, X)$, but we have stated the weaker form of the latter to emphasize it is an easier result.
Theorem 2.3 can be obtained from Proposition 2.2 by showing that it is sufficient to consider the one-parameter subgroups $\tau$ equal to $\omega_\alpha^\vee$ for some simple root $\alpha$. See the proof of Theorem 5.1 below for a similar argument.

2.6. The eigencone. A relationship between $g$-sat and projections of coadjoint orbits was discovered by Heckman [12]. Theorem 2.4 below interprets $g$-sat in terms of eigenvalues.

Fix a maximal compact subgroup $U$ of $G$ such that $T \cap U$ is a Cartan subgroup of $U$. Let $u$ and $t$ denote the Lie algebras of $U$ and $T$ respectively. Let $t^+$ be the Weyl chamber of $t$ corresponding to $B$. Let $\sqrt{-1}$ denote the usual complex number. It is well-known that $\sqrt{-1}t^+$ is contained in $u$ and that the map
\[
t^+ \mapsto u/U
\]
\[
\xi \mapsto U.(\sqrt{-1}\xi)
\]
is a homeomorphism. Here $U$ acts on $u$ by the adjoint action. Consider the set
\[
\Gamma(U) := \{(\xi, \zeta, \eta) \in (t^+)^3 : U.(\sqrt{-1}\xi) + U.(\sqrt{-1}\zeta) + U.(\sqrt{-1}\eta) \ni 0\}.
\]

Let $u^*$ (resp. $t^*$) denote the dual (resp. complex dual) of $u$ (resp. $t$). Let $t^{**}$ denote the dominant chamber of $t^*$ corresponding to $B$. By taking the tangent map at the identity, one can embed $X(T)^+$ in $t^{**}$. Note that this embedding induces a rational structure on the complex vector space $t^*$. Moreover it allows to embed the tensor cone $g$-sat in $(t^{**})^3$.

The Cartan-Killing form allows to identify $t^+$ and $t^{**}$. In particular $\Gamma(U)$ also embeds in $(t^{**})^3$; the so obtained subset of $(t^{**})^3$ is denoted by $\tilde{\Gamma}(U)$ to avoid any confusion. The following result is well-known; see e.g., [17, Theorem 5] and the references therein.

**Theorem 2.4.** The set $\tilde{\Gamma}(U)$ is a closed convex polyhedral cone. Moreover, $g$-sat is the set of the rational points in $\tilde{\Gamma}(U)$.

3. THE CASE OF THE SYMPLECTIC GROUP

3.1. The root system of type $C$. Let $V = \mathbb{C}^{2n}$ with the standard basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{2n}\}$. Let $J_n$ be the $n \times n$ “anti-diagonal” identity matrix and define a skew-symmetric bilinear form $\omega(\bullet, \bullet) : V \times V \to \mathbb{C}$ using the block matrix $\Omega := \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}$. By definition, the *symplectic group* $G = \text{Sp}(2n, \mathbb{C})$ is the group of automorphisms of $V$ that preserve this bilinear form.

Given a $n \times n$ matrix $A = (A_{ij})_{1 \leq i,j \leq n}$ define $^TA$ by $(^TA) = A_{n+1-j,n+1-i}$, obtained from $A$ by reflection across the antidiagonal. The Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$ is the set of matrices $M \in \text{Mat}_{2n \times 2n}(\mathbb{C})$ such that $^tM\Omega + \Omega M = 0$; namely,
\[
\mathfrak{sp}(2n, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & ^tA \end{pmatrix} : A, B, C \text{ of size } n \times n, \quad ^tB = B \text{ and } ^tC = C \right\}
\]
which has complex dimension $2n^2 + n$. The Lie algebra $\mathfrak{u}(2n, \mathbb{C})$ of the unitary group $U(2n, \mathbb{C})$ is the set of anti-Hermitian matrices. Thus, (20) gives
\[
\mathfrak{sp}(2n, \mathbb{C}) \cap \mathfrak{u}(2n, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ -^t\bar{B} & ^tA \end{pmatrix} : ^tA = A \text{ and } ^tB = B \right\}
\]
which has real dimension $2n^2 + n$. As a consequence, $U(2n) \cap \text{Sp}(2n, \mathbb{C})$ is a maximal compact subgroup of $\text{Sp}(2n, \mathbb{C})$. 

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Let $B$ be the Borel subgroup of upper triangular matrices in $G$. Let

$$T = \{ \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) : t_i \in \mathbb{C} \}$$

be the maximal torus contained in $B$. For $i \in [1, n]$, let $\varepsilon_i$ denote the character of $T$ that maps $\text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$ to $t_i$; then $X(T) = \oplus_{i=1}^n \mathbb{Z} \varepsilon_i$. Here

$$\phi^+ = \{ \varepsilon_i \pm \varepsilon_j : 1 \leq i \leq j \leq n \} \cup \{ 2\varepsilon_i : 1 \leq i \leq n \},$$

$$\Delta = \{ \alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_n - \varepsilon_1, \alpha_n = 2 \varepsilon_n \},$$

and

$$X(T)^+ = \{ \sum_{i=1}^n \lambda_i \varepsilon_i : \lambda_i \geq \ldots \geq \lambda_n \geq 0 \} = \text{Par}_n.$$

For $i \in [1, 2n]$, set $\bar{i} = 2n + 1 - i$. The Weyl group $W$ of $G$ may be identified with a subgroup of the Weyl group $S_{2n}$ of $SL(V)$. More precisely,

$$W = \{ w \in S_{2n} : w(\bar{i}) = \bar{w(i)} \ \forall i \in [1, 2n] \}.$$

Observe that $T \cap U(2n, \mathbb{C})$ has real dimension $n$ and is a maximal torus of $U(2n) \cap \text{Sp}(2n, \mathbb{C})$. The bijection (19) implies that any matrix $M_1 = \left( \begin{array}{cc} A & B \\ -A^{-1}B & T \end{array} \right)$ in $\text{sp}(2n, \mathbb{C}) \cap u(2n, \mathbb{C})$ (see (21)) is diagonalizable with eigenvalues in $\sqrt{-1}\mathbb{R}$. Moreover, ordering the eigenvalues by nonincreasing order, we get

$$\lambda(\sqrt{-1}M_i) \in \{ (\lambda_1 \geq \ldots \geq \lambda_n \geq -\lambda_n \geq \ldots \geq -\lambda_1) : \lambda_i \in \mathbb{R} \}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \text{Par}_n$, set $\hat{\lambda} = (\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)$. Now, Theorems 1.1 with $m = n$ and Theorem 2.4 give an interpretation of $\text{NL-sat}(n)$ in terms of eigenvalues:

**Proposition 3.1.** Let $\lambda, \mu, \nu \in \text{Par}_n$. Then $(\lambda, \mu, \nu) \in \text{NL-sat}(n)$ if and only if there exist three matrices $M_1, M_2, M_3 \in \text{sp}(2n, \mathbb{C}) \cap u(2n, \mathbb{C})$ such that $M_1 + M_2 + M_3 = 0$ and

$$(\hat{\lambda}, \hat{\mu}, \hat{\nu}) = (\lambda(\sqrt{-1}M_1), \lambda(\sqrt{-1}M_2), \lambda(\sqrt{-1}M_3)).$$

**3.2. Isotropic Grassmannians and Schubert classes.** Our reference for this subsection is [25] Section 5]. For $r = 1, \ldots, n$, the one-parameter subgroup $\omega_{\alpha' \gamma}$ is given by

$$\omega_{\alpha' \gamma}(t) = \text{diag}(t, \ldots, t, 1, \ldots, 1, t^{-1}, \ldots, t^{-1}),$$

where $t$ and $t^{-1}$ occur $r$ times.

A subspace $W \subseteq V$ is isotropic if for all $\vec{v}, \vec{v}' \in W$, $\omega(\vec{v}, \vec{v}') = 0$. Given an $r$-subset $I \subset [2n]$, we set $F_I = \text{Span}(\vec{e}_i : i \in I)$. Clearly, $F_I$ is isotropic if and only if $I \cap \bar{I} = \emptyset$, where $I = \{ i : i \in I \}$. Now, $P^\alpha\nu$ is the stabilizer of the isotropic subspace $F_{\{1, \ldots, r\}}$. Thus, $G/P^\alpha\nu = \text{Gr}_\omega(r, 2n)$ is the Grassmannian of isotropic $r$-dimensional vector subspaces of $V$.

Let $S(r, 2n)$ denote the set of subsets of $[1, \ldots, 2n]$ with $r$ elements. Set

$$\text{Schub}(\text{Gr}_\omega(r, 2n)) := \{ I \in S(r, 2n) : I \cap \bar{I} = \emptyset \}.$$

If $I = \{ i_1 < \cdots < i_r \} \in \text{Schub}(\text{Gr}_\omega(r, 2n))$, let $i_k^r := i_k$ for $k \in [r]$, and $\{ i_{r+1} < \cdots < i_{2n} \} = [2n] - (I \cup \bar{I})$. Therefore $w_I = (i_1, \ldots, i_{2n}) \in S_{2n}$ is the element of $W^P^\alpha\nu$ corresponding to $F_I$; that is, $F_I = w_IP^\alpha\nu/P^\alpha\nu$.

Set

$$\text{Schub}'(\text{Gr}_\omega(r, 2n)) := \left\{ (A, A') : A \in S(a, n), A' \in S(a', n) \text{ for some } a \text{ and } a' \text{ s.t. } a + a' = r \text{ and } A \cap A' = \emptyset \right\}.$$
This map is a bijection:

\[
\text{Schub}(\text{Gr}_\omega(r, 2n)) \xrightarrow{I \mapsto} \text{Schub}'(\text{Gr}_\omega(r, 2n)) \quad (\bar{I} \cap [n], I \cap [n]).
\]

Recall from the introduction the definition of \(\tau(I)\) and hence \(\tau(A)\) and \(\tau(A')\). The relationship between these three partitions is depicted in Figure 1.

**Definition 3.2.** For subsets \(X \subseteq Y \subseteq [2n]\), define

\[
\chi_{(X,Y)} : [2n] \to \{0, 1, 2\}
\]

\[
i \mapsto \begin{cases} 
1 & \text{if } i \in X \\
2 & \text{if } i \in Y \setminus X \\
0 & \text{if } i \notin Y
\end{cases}
\]

Treat \(\chi_{(X,Y)}\) as a word of length \(2n\) with letters in \(\{0, 1, 2\}\). Removing all the 2’s in the word, we obtain the characteristic function \(\chi^0_{(X,Y)}\) of some subset \(I^0\) of \([[2n] - |Y| + |X|]\) with \(|X|\) elements. Similarly, removing all 0’s in the word, and then, replacing all the 2’s with 0’s, gives the characteristic function \(\chi^2_{(X,Y)}\) of some subset \(I^2\) of \([[|Y|]\) with \(|X|\) elements.

Given \(I \in \text{Schub}(\text{Gr}_\omega(r, 2n))\) for some \(1 \leq r \leq n\), we apply this construction to \(I \subset ([2n] - \bar{I})\). One obtains \(I^2 \in S(r, 2r)\) and \(I^0 \in S(r, 2n - r)\).

**Definition 3.3.** For a partition \(\lambda = (\lambda_1, \ldots, \lambda_k) \subseteq (a^b)\), i.e., the rectangle with \(a\) columns and \(b\) rows. Define \(\lambda^Y\) with respect to \((a^b)\) to be the partition \((a - \lambda, a - \lambda - 1, \ldots, a - \lambda_1)\) where we set \(\lambda_i = 0\) for \(i > k\). We will denote this by \(\lambda^Y[a^b]\).

Now,

\[
\text{codim}(\overline{BF_I}) = |\tau(I^0)^{\lambda^Y[(2n-2r)^r]}| + 1/2(|\tau(I^2)^{\lambda^{r^r}}| + |I \cap [n]|).
\]

Moreover,

\[
\dim(\text{Gr}_\omega(r, 2n)) = r(2n - 2r) + \frac{r(r + 1)}{2}.
\]

Let \(I \in \text{Schub}(\text{Gr}_\omega(r, 2n))\) and \(A = \bar{I} \cap [n], A' = I \cap [n]\) be the corresponding pair in \(\text{Schub}'(\text{Gr}_\omega(r, 2n))\). Set

\[
\tau^0(A, A') = \tau(I^0), \quad \tau^2(A, A') = \tau(I^2).
\]
While the above discussion defines \( \tau^0(A, A'), \tau^2(A, A') \) through the bijection (22), we emphasize that these partitions from Theorem 1.2 can be defined explicitly:

**Definition-Lemma 3.4.** Set \( a = |A| \) and \( a' = |A'| \). Write \( A = \{\alpha_1 < \cdots < \alpha_a\} \) and \( A' = \{\alpha'_1 < \cdots < \alpha'_{a'}\} \). Then

\[
\begin{align*}
\tau^2(A, A')_k &= a + |A' \cap [\alpha_k; n]|, & \forall k = 1, \ldots, a; \\
\tau^2(A, A')_{l+a} &= |A \cap [\alpha_{a'+1-l}]; n|, & \forall l = 1, \ldots, a'; \\
\tau^0(A, A')_k &= n - a - a' + |[\alpha_k; n] - (A \cup A')|, & \forall k = 1, \ldots, a; \\
\tau^0(A, A')_{l+a} &= |[\alpha_{a'+1-l}] - (A \cup A')|, & \forall l = 1, \ldots, a'.
\end{align*}
\]

**Proof.** Write \( I = A' \cup \bar{A} = \{i_1 < \cdots < i_r\} \) with \( r = a + a' \). By definition,

\[
\tau^2(A, A')_k := \tau(I^2)_k = |\bar{I} \cap [i_{a+a'+1-k}]|, \quad \text{for } 1 \leq k \leq a + a'.
\]

If \( k \leq a, i_{a+a'+1-k} \in \bar{A} \subset [n + 1; 2n], i_{a+a'+1-k} = \bar{\alpha}_k \) and \( \bar{I} \cap [i_{a+a'+1-k}] = A \cup A' \cap [\alpha_k; n] \).

The first assertion follows.

If \( k = a + l \) for some positive \( l, \bar{i}_{a+a'+1-k} \in A' \subset [n], \bar{i}_{a+a'+1-k} = \alpha'_{a'+1-l} \) and \( \bar{I} \cap [\bar{i}_{a+a'+1-k}] = A \cap [\alpha'_{a'+1-l}] \).

Similarly,

\[
\tau^0(A, A')_k := \tau(I^0)_k = |[\bar{i}_{a+a'+1-k}] \cap ([2n] - (I \cup \bar{I}))|, \quad \text{for } 1 \leq k \leq a + a'.
\]

If \( k \leq a, [\bar{i}_{a+a'+1-k}] \cap ([2n] - (I \cup \bar{I})) = ([n] - (A \cup A')) \cup [\alpha_k; n] - (A \cup A') \) (a disjoint union).

This proves the third claim.

If \( k = a + l \) with some positive \( l, \alpha'_{a'+1-l} = \bar{i}_{a+a'+1-k} \in [n] \); last assertion follows. \( \square \)

### 3.3. The parabolic subgroup \( P_0 \)

Fix \( m \geq n \). Let \( P_0 \) be the subgroup of \( \text{Sp}(2m, \mathbb{C}) \) of matrices

\[
\begin{pmatrix}
T_1 & * & * \\
0 & A & * \\
0 & 0 & \bar{T}_2
\end{pmatrix},
\]

where \( T_1 \) and \( T_2 \) are \( n \times n \) upper triangular matrices and \( A \) is a matrix in \( \text{Sp}(2m - 2n, \mathbb{C}) \).

\( P_0 \) is the standard parabolic subgroup of \( \text{Sp}(2m, \mathbb{C}) \) corresponding to the simple roots \( \{\alpha_{n+1}, \ldots, \alpha_m\} \). A character \( \lambda = \sum_{i=1}^m \lambda_i \varepsilon_i \in X(T) \) extends to \( P_0 \) if and only if \( \lambda_{n+1} = \cdots = \lambda_m = 0 \). Thus the set of dominant characters of \( X(P_0) \) identifies with \( \text{Par}_n \). Hence

\[
\text{sp-sat}(m) \cap (\text{Par}_m^3)^3 = \text{GIT-sat}(\text{Sp}(2m, \mathbb{C}), (\text{Sp}(2m, \mathbb{C})/P_0)^3).
\]

Let \( \text{Schub}^{P_0}(\text{Gr}_\omega(r, 2m)) \) be the set of \( I \in \text{Schub}(\text{Gr}_\omega(r, 2m)) \) such that the Schubert variety \( BF_I \) is \( P_0 \)-stable. One checks (details omitted) that

\[
I \in \text{Schub}^{P_0}(\text{Gr}_\omega(r, 2m)) \iff I \cap [n + 1; 2m - n] = [k; 2m - n] \quad \text{for some } k \geq m + 1.
\]

### 4. Proof of Theorems 1.1 and 1.2

**Proposition 4.1.** The inequalities (4) in Theorem 1.2 characterize \( \text{sp-sat}(n) \).

**Proof.** Since \( \text{sp-sat}(n) = \text{GIT-sat}(\text{Sp}(2n), (\text{Sp}(2n)/B)^3) \) (see Section 2.3), we may apply Theorem 2.3. Let \( (\lambda, \mu, \nu) \in (\text{Par}_n)^3 \). Write \( \lambda = \sum_i \lambda_i \varepsilon_i \) and similarly for \( \mu \) and \( \nu \).
Fix $1 \leq r \leq n$ and $\alpha = \alpha_r \in \Delta$. Given $I \in \text{Schub}(\text{Gr}_\omega(r, 2n))$, from the description of $w_{\alpha \gamma}$ and $w_I$ it is easy to check that

\begin{equation}
\langle w_I w_{\alpha \gamma}, \lambda \rangle = \sum_{i \in I \cap [n]} \lambda_i - \sum_{i \in I \cap [n]} \lambda_i
\end{equation}

Then (4) is obtained from (18) associated to the triple of Schubert classes $(I, J, K) \in \text{Schub}(\text{Gr}_\omega(r, 2n))^3$ by setting

$$
A = \bar{I} \cap [n], A' = I \cap [n];
B = \bar{J} \cap [n], B' = J \cap [n];
C = \bar{K} \cap [n], C' = K \cap [n].
$$

Since the map (22) is bijective, it suffices to show (17) from Theorem 2.3 is equivalent to

1. $|A'| + |B'| + |C'| = r$, and
2. $c_{\tau^2(\omega', \omega)}(I, J, K) = c_{\tau^2(\omega', \omega)}(I', J', K') = 1$.

By [25, Theorem 19], condition (17) is equivalent to

1. $\text{codim}(BF_I) + \text{codim}(BF_J) + \text{codim}(BF_K) = \text{dim}(\text{Gr}_\omega(r, 2n))$, and
2. $c_{\tau^2(\omega', \omega)}(I, J, K) = c_{\tau^2(\omega', \omega)}(I', J', K') = 1$.

By definition, the two conditions involving Littlewood-Richardson are the same. Assuming these two Littlewood-Richardson coefficients equal to one, it remains to prove that $\text{codim}(BF_I) + \text{codim}(BF_J) + \text{codim}(BF_K) = \text{dim}(\text{Gr}_\omega(r, 2n))$ if and only if $|A'| + |B'| + |C'| = r$. This directly follows from (23) and (24).

**Proof of Theorem 1.1** By Theorem 2.4, the inclusion $\text{sp-sat}(n) \subset \text{sp-sat}(m)$ is equivalent to the inclusion $\Gamma(\text{Sp}(2n, \mathbb{C}) \cap U(2n, \mathbb{C})) \subset \Gamma(\text{Sp}(2m, \mathbb{C}) \cap U(2m, \mathbb{C}))$. Here we use the symplectic form defined in Section 3.1 to embed $\text{Sp}(2n, \mathbb{C})$ in $\text{GL}(2n, \mathbb{C})$.

Clearly, the following map is well-defined

$$
\text{Lie}(\text{Sp}(2n, \mathbb{C}) \cap U(2n, \mathbb{C})) \longrightarrow \text{Lie}(\text{Sp}(2m, \mathbb{C}) \cap U(2m, \mathbb{C}))
$$

$$
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \tilde{M} = \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ C & 0 & D \end{pmatrix}
$$

where $A, B, C$ and $D$ are square matrices of size $n$, and the matrices of these Lie algebras are described by (21).

Let $(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3) \in \Gamma(\text{Sp}(2n, \mathbb{C}) \cap U(2n, \mathbb{C}))$. Let

$$(M_1, M_2, M_3) \in (\text{Sp}(2n, \mathbb{C}) \cap U(2n, \mathbb{C}))^3, (\sqrt{-1} \tilde{h}_1, \sqrt{-1} \tilde{h}_2, \sqrt{-1} \tilde{h}_3)$$

such that $M_1 + M_2 + M_3 = 0$.

The fact that $\tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3 = 0$ implies that $(\hat{h}_1, \hat{h}_2, \hat{h}_3) \in \Gamma(\text{Sp}(2m, \mathbb{C}) \cap U(2m, \mathbb{C}))$, where $h_1, h_2, h_3$ are viewed as elements of $\text{Par}_m$ is the spectrum of $\tilde{M}_1$.

To obtain the converse inclusion

$$
\text{GIT-sat}(\text{Sp}(2m, \mathbb{C}), (\text{Sp}(2m, \mathbb{C})/P_0)^3) = \text{sp-sat}(m) \cap (\text{Par}_m)^3 \subset \text{sp-sat}(n),
$$

we have to prove that any inequality (4) from Proposition 4.1 is satisfied by the points of $\text{sp-sat}(m) \cap (\text{Par}_m)^3$; here we have used (27). Fix such an inequality $(A, A', B, B', C, C')$. 

Set \( I = A' \cup \bar{A} \subset [2n] \), \( J = B' \cup \bar{B} \subset [2n] \) and \( K = C' \cup \bar{C} \subset [2n] \). Similarly for \( m \), set \( \bar{I} = A' \cup \{2m+1-i : i \in A\} \), \( \bar{J} = B' \cup \{2m+1-i : i \in B\} \) and \( \bar{K} = C' \cup \{2m+1-i : i \in C\} \); these are subsets of \([2m]\). Set also \( a' = |A'|, b' = |B'| \) and \( c = |C| \).

Notice that \((\bar{I}^2)^0, (\bar{J}^2)^0, (\bar{K}^2)^0 \subset 0^r = \emptyset\). Thus, trivially,

\[
\tau(\bar{K}^0) = [2(m-n)]^c + \tau(K^0),
\]

\[
\tau(\bar{I}^0)^\vee([2m-2r]^*) = [2(m-n)]^{a'} + \tau(I^0)^\vee([2n-2r]^*),
\]

\[
\tau(\bar{J}^0)^\vee([2m-2r]^*) = [2(m-n)]^{b'} + \tau(J^0)^\vee([2n-2r]^*).
\]

The assumption \( a' + b' = c \) and the semigroup property of LR-semigroup implies that

\[
\tau(\bar{I}^0)^\vee([2m-2r]^*) \neq 0.
\]

Next we apply [25, Proposition 8.1] to the \( \bar{I}, \bar{J}, \bar{K} \) and the space \( Gr_\omega(r, 2m) \); equations (32) and (33) mean that condition (iii) holds. Hence by (i) of [ibid. and (28)],

\[
[B\bar{F}]_2 \cdot [B\bar{F}]_2 \cdot [B\bar{F}]_2 = [pt] \in H^*(Gr_\omega(r, 2r), \mathbb{Z}).
\]

One can easily check that

\[
\tau(\bar{K}^0) = [2(m-n)]^c + \tau(K^0),
\]

\[
\tau(\bar{I}^0)^\vee([2m-2r]^*) = [2(m-n)]^{a'} + \tau(I^0)^\vee([2n-2r]^*),
\]

\[
\tau(\bar{J}^0)^\vee([2m-2r]^*) = [2(m-n)]^{b'} + \tau(J^0)^\vee([2n-2r]^*).
\]

The assumption \( d' + b' = c \) and the semigroup property of LR-semigroup implies that

Next we apply [25, Proposition 8.1] to the \( \bar{I}, \bar{J}, \bar{K} \) and the space \( Gr_\omega(r, 2m) \); equations (32) and (33) mean that condition (iii) holds. Hence by (i) of [ibid. and (28)],

\[
[\bar{P}_0 F]_2 \otimes_0 [\bar{P}_0 F]_2 \otimes_0 [\bar{P}_0 F]_2 = d[pt] \in H^*(Gr_\omega(r, 2m), \mathbb{Z}),
\]

for some nonzero \( d \). Now use Proposition 2.1 which shows that (4) is a case of (15) which holds on GIT-sat(Sp(2m, \mathbb{C}), (Sp(2m, \mathbb{C})/\bar{P}_0)^3) = sp-sat(m) \cap (\text{Par}_n^2)^3, \text{ as desired.} \]

**Proof of Theorem 1.2.** This follows from Theorem 1.1 and Proposition 4.1.

**Example 4.2.** Let \( n = 4, r = 3 \). Let

\[
A = B' = C' = \emptyset, \; A' = \{2, 3, 4\}, \; B = \{1, 2, 4\}, \; C = \{1, 3, 4\},
\]

giving a triple \((A, A'), (B, B'), (C, C')\) in \((\text{Schub}'(Gr_\omega(3, 8)))^3\) satisfying conditions (1) and (2) from Theorem 1.2. The corresponding triple in \(\text{Schub}(Gr_\omega(3, 8))\) is

\[
\bar{I} = \{2, 3, 4\}, \; \bar{J} = \{5, 7, 8\}, \; \bar{K} = \{5, 6, 8\} \subseteq [8].
\]

Thus

\[
\tau(\bar{I}) = (1, 1, 1), \; \tau(\bar{J}) = (5, 5, 4), \; \tau(\bar{K}) = (5, 4, 4) \subseteq (5^3);
\]

The three associated characteristic functions \(\chi_{I \subset [2n] \setminus \bar{I}}; \chi_{J \subset [2n] \setminus \bar{J}}; \chi_{K \subset [2n] \setminus \bar{K}}\) respectively are

\[
\begin{align*}
\chi_{I^2} &= 111000, \quad \chi_{J^2} = 000111, \quad \chi_{K^2} = 000111; \\
\chi_{I^0} &= 01110, \quad \chi_{J^0} = 01011, \quad \chi_{K^0} = 01101.
\end{align*}
\]
Now,
\[ I^2 = \{1, 2, 3\}, \quad \tau(I^2) = 000 \]
\[ J^2 = K^2 = \{4, 5, 6\}, \quad \tau(J^2) = \tau(K^2) = 333 \]
\[ I^0 = \{2, 3, 4\}, \quad \tau(I^0) = 111 \]
\[ J^0 = \{2, 4, 5\}, \quad \tau(J^0) = 221 \]
\[ K^0 = \{2, 3, 5\}, \quad \tau(K^0) = 211 \]

The reader can check that
\[ c^{\tau(C,C')}_{\tau_0(A,A') \cap (2n-2r)^2} \cdot c^{\tau(B,B') \cap (2n-2r)^2} = c^{\tau(K_0)}_{\tau_0(I_0) \cap (2n-2r)^2} \]
\[ c^{\tau_2(C,C')}_{\tau_2(A,A') \cap (2n-2r)^2} = c^{\tau(K_2)}_{\tau_2(I_2) \cap (2n-2r)^2} = c^{(3,3,3)}_{(3,3,3), (0,0,0)} = 1. \]

Hence, by Theorem 1.2, \(-\lambda_2 - \lambda_3 - \lambda_4 + \mu_1 + \mu_2 + \mu_4 + \nu_1 + \nu_3 + \nu_4 \geq 0\) is one of the inequalities defining \(\text{sp-sat}(4)\). \(\square\)

### 5. The truncated tensor cone

In this section, we characterize the truncated tensor cone of \(\text{sp-sat}(m)\), that is, \(\text{sp-sat}(m) \cap (\text{Par}_n^Q)^3\) where \(m > n\). By (3), this implies another set of inequalities for \(\text{NL-sat}(n)\).

We first need the following result, a generalization of Theorem 2.3:

**Theorem 5.1.** Here \(X = G/P \times G/Q \times G/R\). Let \((\lambda, \mu, \nu) \in X(P) \times X(Q) \times X(R)\) be dominant. Then \(\mathcal{L}_{(\lambda,\mu,\nu)} \in \text{GIT-sat}(G, X)\) if and only if for any simple root \(\alpha\), for any

\[(u, v, w) \in W_P \setminus W/W_{P^\alpha} \times W_Q \setminus W/W_{P^\alpha} \times W_R \setminus W/W_{P^\alpha}\]

such that

\[ [PuP^\alpha/P^\alpha] \circ [QuP^\alpha/P^\alpha] \circ [RwP^\alpha/P^\alpha] = [pt] \in H^*(G/P^\alpha, \mathbb{Z}), \]

\[ \langle u\varpi_\alpha, \lambda \rangle + \langle v\varpi_\alpha, \mu \rangle + \langle w\varpi_\alpha, \nu \rangle \leq 0. \]

**Proof.** GIT-sat\((G, X)\) is characterized by Proposition 2.2: let \((C, \tau)\) and a choice of

\[(u', v', w') \in W_P \setminus W/W_{P(\tau)} \times W_Q \setminus W/W_{P(\tau)} \times W_R \setminus W/W_{P(\tau)}\]

be as in that proposition. Since every inequality (36) appears in Proposition 2.2 with \(\tau = \varpi_\alpha\), it suffices to show that (16) is implied by the inequalities in Theorem 5.1.

Write

\[ \tau = \sum_{\alpha \in \Delta} n_\alpha \varpi_\alpha, \]

where \(\Delta\) is the set of simple roots. Since \(\tau\) is dominant the \(n_\alpha\)’s are nonnegative. Set

\[ \text{Supp}(\tau) := \{\alpha \in \Delta : n_\alpha \neq 0\}. \]

Fix any \(\alpha \in \text{Supp}(\tau)\), \(P^\alpha := P(\varpi_\alpha)\) contains \(P(\tau)\). Let

\[ \pi : G/P(\tau) \longrightarrow G/P^\alpha \]

denote the associated projection. By [27, Theorem 1.1 and Section 1.1] (see also [24]), condition (13) implies there are

\[(u, v, w) \in W_P \setminus W/W_{P^\alpha} \times W_Q \setminus W/W_{P^\alpha} \times W_R \setminus W/W_{P^\alpha}, \]

such that condition (35) holds and such that \((u, v, w)\) and \((u', v', w')\) define the same cosets in \(W_P \setminus W/W_{P^\alpha} \times W_Q \setminus W/W_{P^\alpha} \times W_R \setminus W/W_{P^\alpha}\). Therefore, inequality (36) is the same as

\[ \langle u'\varpi_\alpha, \lambda \rangle + \langle v'\varpi_\alpha, \mu \rangle + \langle w'\varpi_\alpha, \nu \rangle \leq 0. \]
Therefore each of the inequalities (16) can be written as a linear combination of (36). Hence the inequalities of the theorem imply and are implied by the inequalities of Proposition 2.2, so the result follows.

We now deduce from Theorem 5.1 the following statement.

Proposition 5.2. Let \((\lambda, \mu, \nu)\) in \(\text{Par}_n\) and \(m \geq n\). Then \((\lambda, \mu, \nu) \in \text{sp-sat}(m)\) if and only if
\[
|\lambda \cap [n]| - |\lambda \cap [n]| + |\mu \cap [n]| - |\mu \cap [n]| + |\nu \cap [n]| - |\nu \cap [n]| \leq 0,
\]
for any \(1 \leq r \leq m\) and \((I, J, K) \in \text{Schub}^P_0(\text{Gr}_\omega(r, 2m))^3\) such that
\[
\begin{align*}
(1) & \ |I \cap [m]| + |J \cap [m]| + |K \cap [m]| = r, \text{ and} \\
(2) & \ c_{\tau(J_0)}^{r(J_0)^{\nu([2m-2r])}} = c_{\tau(J_2)^{r(J_2)^{\nu([2m-2r])}}} = 1.
\end{align*}
\]

Proof. We already observed (29) that inequality (37) is inequality (16) in our context. Regarding Theorem 5.1, the only thing to prove is that condition (13) associated to \((I, J, K)\) is equivalent to the two conditions of the proposition. This is [25, Theorem 8.2].

A priori, Proposition 5.2 could contain redundant inequalities. In view of Theorem 1.2, an affirmative answer to this question would imply irredundancy:

Question 1. Does any \((I, J, K) \in \text{Schub}^P_0(\text{Gr}_\omega(r, 2m))^3\) occurring in Proposition 5.2 satisfy
\[
\begin{align*}
(1) & \ |I \cap [n+1, 2m-n]| = |J \cap [n+1, 2m-n]| = K \cap [n+1, 2m-n] = \emptyset; \\
(2) & \ c_{\tau(J_0)^{\nu([2m-2r])}}^{r(J_0)^{\nu([2m-2r])}} = 1,
\end{align*}
\]
where \(\hat{I} = I \cap [n] \cup \{i-2(m-n) : i \in I \cap [m+1, 2m]\}\), and \(\hat{J}\) and \(\hat{K}\) are defined similarly.

Proof of Theorem 5.3. Fix an inequality \((A, A', B, B', C, C')\) from (4). It is minimal for the full-dimensional cone \(\text{NL-sat}(n) = \text{sp-sat}(2n) \cap (\text{Par}_n)^3 \subset \mathbb{R}^{3n}\). Thus, it has to appear in Proposition 5.2 for \(m = 2n\). Let \((\hat{I}, \hat{J}, \hat{K}) \in \text{Schub}^P_0(\text{Gr}_\omega(\tilde{r}, 4n))^3\) be the associated Schubert triple. Set \(\tilde{A}' = \hat{I} \cap [2n], \tilde{A} = \hat{I} \cap [2n], \) etc. Since \((\hat{I}, \hat{J}, \hat{K}) \in \text{Schub}^P_0(\text{Gr}_\omega(\tilde{r}, 4n))^3\), \(\tilde{A}', \tilde{B}', \tilde{C}' \subset [n]\) (by (28)). Thus, comparing (4) and (37), we have
\[
\begin{align*}
A &= \tilde{A} \cap [n] \\
B &= \tilde{B} \cap [n] \\
C &= \tilde{C} \cap [n] \\
A' &= \tilde{A}' \cap [n] \\
B' &= \tilde{B}' \cap [n] \\
C' &= \tilde{C}' \cap [n] = \tilde{C}'.
\end{align*}
\]

Now, Proposition 5.2(1) and Theorem 1.2(2) imply that \(r = \tilde{r}\). In particular, \(|\tilde{A}| + |\tilde{A}'| = |A| + |A'| = r\) and \(A = \tilde{A}\). Similarly, \(B = \tilde{B}\) and \(C = \tilde{C}\).

Let \(\alpha\) be the simple root of \(\text{Sp}(4n, \mathbb{C})\) associated to \(r\). Observe that the Levi subgroup of \(P^n\) has type \(A_{r-1} \times C_{2n-r}\). Let \(u, v, w \in W^{P^n}\) corresponding to \((A', \tilde{A}), (\tilde{B}', \tilde{B})\) and \((\tilde{C}', \tilde{C})\), respectively. Proposition 5.2 and its proof show that (35) holds with \(P = Q = R = P_0\). In particular, one can apply the reduction rule proved in [28, Theorem 3.1] or [26, Theorem 1]: \(\text{mult}_{\lambda, \alpha, \mu, \nu}^{A_{r-1} \times C_{2n-r}}\) is a tensor multiplicity for the Levi subgroup of \(P^n\) of type \(A_{r-1} \times C_{2n-r}\). The factor \(c_{\lambda, \alpha, A', \mu, B, B'}^{C, C'}\) in the theorem corresponds to the factor of type \(A_{r-1}\). Adding zeros, consider \(\lambda\) as an element of \(\text{Par}_{2n}\). Then the dominant weights for the factor \(C_{2n-r}\) are \(\lambda_{[2n]} - (\tilde{A}, \tilde{A})\), \(\mu_{[2n]} - (\tilde{B}, \tilde{B}')\), \(\nu_{[2n]} - (\tilde{C}, \tilde{C}')\). Since these partitions have length at most \(n-r\), the tensor multiplicity for the factor of \(C_{2n-r}\) is a Newell-Littlewood coefficient. The theorem follows.
6. Application to Conjecture 1.4

**Corollary 6.1 (of Theorem 1.2).** Conjecture 1.4 holds for \( n \leq 5 \).

The proof is computational and uses the software Normaliz [4].

Fix \( n \geq 2 \) and consider the cone \( \text{sp}-\text{sat}(n) \). Consider the two lattices \( \Lambda = \mathbb{Z}^{3n} \) and \( \Lambda_2 = \{ (\lambda, \mu, \nu) \in (\mathbb{Z}^n)^3 : |\lambda| + |\mu| + |\nu| \text{ is even} \} \).

Then NL-semigroup(\( n \)) \( \subset \Lambda_2 \cap \text{sp}-\text{sat}(n) \). Conjecture 1.4 asserts that the converse inclusion holds. The set \( \Lambda_2 \cap \text{sp}-\text{sat}(n) \) is a semigroup of \( \Lambda_2 \) defined by a family of linear inequalities (explicitly given by Theorem 1.2). Using Normaliz [4] one can compute (for small \( n \)) the minimal set of generators, i.e., the Hilbert basis, for this semigroup. Hence, to prove Corollary 6.1 one can proceed as follows:

1. Compute the list of inequalities given by Theorem 1.2.
2. Compute the Hilbert basis of \( \Lambda_2 \cap \text{sp}-\text{sat}(n) \) using Normaliz.
3. Check \( N_{\lambda,\mu,\nu} > 0 \) for any \( (\lambda, \mu, \nu) \) in the Hilbert basis.

The table below summarizes our computations; see [8].

| \( n \) | \# facets | \# EHI | \# rays | \# Hilb \( \Lambda_2 \cap \text{sp}-\text{sat} \) | \# Hilb \( \Lambda \cap \text{sp}-\text{sat} \) |
|------|---------|-------|--------|-----------------|-----------------|
| 2    | 6+18    | 18    | 12     | 13              | 20              |
| 3    | 9+93    | 100   | 51     | 58              | 93              |
| 4    | 12+474  | 662   | 237    | 302             | 451             |
| 5    | 15+2421 | 5731  | 1122   | 1598            | 2171            |

In the column “\# facets” there are the number of partition inequalities (like \( \lambda_1 \geq \lambda_2 \)) plus the number of inequalities [4] given by Theorem 1.2. The next column counts the inequalities [9] given by applying Theorem 1.5. The number of extremal rays of the cone \( \text{sp}-\text{sat}(n) \) is also given. The two last column are the cardinalities of the Hilbert bases of the two semigroups \( \Lambda_2 \cap \text{sp}-\text{sat}(n) \) and \( \Lambda \cap \text{sp}-\text{sat}(n) \).

7. Preliminaries on Littlewood-Richardson coefficients

We recall standard material [6] on Littlewood-Richardson coefficients.

**7.1. Representations of** \( \text{GL}(n, \mathbb{C}) \). The irreducible representations \( V(\lambda) \) of \( \text{GL}(n, \mathbb{C}) \) are indexed by their highest weight

\[
\lambda \in \Lambda^+_{n} = \{ (\lambda_1 \geq \cdots \geq \lambda_n) : \lambda_i \in \mathbb{Z} \} \supset \text{Par}_n.
\]

In this setting, the Littlewood-Richardson coefficients appear because

\[
V(\lambda) \otimes V(\nu) = \bigoplus_{\mu} V(\nu)^{\oplus c_{\lambda,\mu}^{\nu}}.
\]

The dual representation \( V(\lambda)^* \) has highest weight \( \lambda^* = (-\lambda_n \geq \cdots \geq -\lambda_1) \in \Lambda^+_{n} \). Moreover, for any \( a \in \mathbb{Z} \),

\[
V(\lambda + a^n) = (\text{det})^a \otimes V(\lambda).
\]

Consequently, for any \( \lambda, \mu, \nu \in \Lambda^+_{n} \),

\[
c_{\lambda,\mu}^{\nu} = c_{\lambda+a^n,\mu+b^n}^{\nu+a^n+b^n} = c_{\lambda^*,a^n,\mu^*,b^n}^{\nu^*,(a+b)^n};
\]
this is [3] Theorem 4. Let \( \nu' \) be the transpose of \( \nu \). Since \( c_{\lambda,\mu}^\nu = c_{\lambda,\mu}^{\nu'} \), by (38),

\[
(39) \quad c_{\lambda,\mu}^\nu = c_{\lambda,\mu}^{\nu'} = c_{(\nu')^T,\mu}^{(\nu')^T} = c_{(\nu')^T,(\nu')^T,\mu}^{(\nu')^T,(\nu')^T,\mu} = c_{\lambda,(\nu')^T,\mu}^{\nu,(\nu')^T,\mu} = c_{\lambda,\mu}^{\nu,(\nu')^T,\mu} = c_{\lambda,\mu}^{\nu,(\nu')^T,\mu},
\]

for any \( m \geq \ell(\mu) \).

7.2. Littlewood-Richardson tableaux. A tableau filling \( T \) of shape \( \nu/\lambda \) by \( \mathbb{N} \) is semistandard if it is weakly increasing along rows and strictly increasing along columns; \( T \) is furthermore standard if the filling bijects the boxes of \( \nu/\lambda \) with \( \{1, 2, \ldots, |\nu/\lambda|\} \). The content of \( T \) is \( \text{cont}(T) = (i_1, i_2, \ldots) \) where \( i_k \) is the number of \( k \)'s that appear in \( T \). Let \( \text{SYT}(\nu/\lambda) \) be the set of semistandard tableaux of shape \( \nu/\lambda \). Let \( \text{SSYT}(\nu/\lambda, \mu) \) be the subset consisting of those tableaux of content \( \mu \). Finally, \( \text{SYT}(\nu/\lambda) \) is the subset of standard fillings.

Define \( \text{rowword}(T) = (w_1, w_2, \ldots, w_{|\nu/\lambda|}) \) to be the reading word obtained by reading right to left along rows and from top to bottom. Define \( \text{revrowword}(T) \) to be the reverse of \( \text{rowword}(T) \). Either \( \text{rowword}(T) \) or \( \text{revrowword}(T) \) are ballot if, for every \( i \geq 2, k \geq 1 \), the number of \( i - 1 \) that appear among the initial subword \( w_1, w_2, \ldots, w_k \) is weakly greater than the number of \( i \) that appear in that subword. We say \( T \) is ballot if \( \text{rowword}(T) \) is ballot.

We will make use of a number of (re)formulations of the Littlewood-Richardson rule:

**Theorem 7.1** (Littlewood-Richardson rule, version 1).

\[
c_{\lambda,\mu}^\nu = \#\{T \in \text{SSYT}(\nu/\lambda, \mu) : T \text{ is ballot}\}.
\]

If \( \lambda \) and \( \mu \) are straight-shape, let \( \lambda \star \mu \) be the skew shape obtained by placing \( \lambda \) and \( \mu \) corner to corner with the former southwest relative to the latter.

**Theorem 7.2** (Littlewood-Richardson rule, version 2).

\[
c_{\lambda,\mu}^\nu = \#\{T \in \text{SSYT}(\lambda \star \mu, \nu) : T \text{ is ballot}\}.
\]

We review basics of the theory of jeu de taquin. An inner corner \( x \) of \( \nu/\lambda \) is a maximally southeast box of \( \lambda \). For \( T \in \text{SSYT}(\nu/\lambda) \), a jeu de taquin slide \( \text{jdt}_x(T) \) is obtained as follows. Place \( \bullet \) in \( x \), and apply one of the following slides, depending on how \( T \) looks near \( x \):

1. \( \bullet [a] \mapsto [b] \bullet \) (if \( b \leq a \), or \( a \) does not exist)
2. \( [a] \mapsto \bullet [b] \) (if \( a < b \), or \( b \) does not exist)

Repeat use of (1) or (2) on the new box \( x' \) where \( \bullet \) lands. Terminate when \( \bullet \) arrives at a box \( y \) of \( \lambda \) that has no labels south or east of it. Then \( \text{jdt}_x(T) \) is the result after erasing \( \bullet \).

A rectification of \( T \in \text{SSYT}(\nu/\lambda) \) is defined iteratively. Pick an inner corner \( x_0 \) of \( \nu/\lambda \) and determine \( T_1 := \text{jdt}_{x_0}(T) \in \text{SSYT}(\nu^{(1)}/\lambda^{(1)}) \). Let \( x_1 \) be an inner corner of \( \nu^{(1)}/\lambda^{(1)} \) and compute \( T_2 := \text{jdt}_{x_1}(T_1) \in \text{SSYT}(\nu^{(2)}/\lambda^{(2)}) \). Repeat \(|\lambda| \) times, arriving at \( \text{Rect}_{\{x_i\}}(T) \).

**Theorem 7.3** (First fundamental theorem of jeu de taquin). For \( T \in \text{SSYT}(\nu/\lambda) \), \( \text{Rect}_{\{x_i\}}(T) \) is independent on the choice of inner corners \( \{x_i\} \).

Thus we can unambiguously refer to the rectification \( \text{Rect}(T) \).

Knuth equivalence on words is the equivalence relation \( \equiv_K \) generated by the relations:

\( \ldots, b, c, a, \ldots \) \( \equiv_K \) \( \ldots, b, a, c, \ldots \) if \( a < b \leq c \),
\[(\ldots,a,c,b,\ldots) \equiv_K (\ldots,c,a,b,\ldots) \text{ if } a \leq b < c.\]

The following fact is well-known, and in any case, easy to see from the local relations:

**Theorem 7.4.** Suppose \(w,u\) are two words such that \(w \equiv_K u\). Then \(w\) is reverse-ballot if and only if \(u\) is reverse-ballot.

_Jeu de taquin_ preserves Knuth word and reverse-ballotness:

**Theorem 7.5.** Let \(T \in \text{SSYT}(\nu/\lambda)\) and \(T' = \text{jdt}_x(T)\).

(I) \(\text{revrowword}(T) \equiv_K \text{revrowword}(T')\)

(II) \(T\) is ballot if and only if \(T'\) is ballot.

Let \(Y_\mu\) be the super-semistandard tableau of shape \(\mu\), the tableau using only \(i\)'s in row \(i\).

**Theorem 7.6** (Littlewood-Richardson rule, version 3). Fix \(U \in \text{SSYT}(\mu)\). Then

\[
(40) \quad c^\nu_{\lambda,\mu} = \{T \in \text{SSYT}(\nu/\lambda) : \text{Rect}(T) = U\}.
\]

Moreover, if \(U = Y_\mu\), then \(T\) is in the set given in (40) if and only if \(T\) is ballot.

The final claim follows from Theorem 7.5(II) and the fact that \(Y_\mu\) is ballot.

### 7.3. RSK correspondence

A biword of length \(m\) is

\[
a = \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q_1 & q_2 & \cdots & q_m \\ p_1 & p_2 & \cdots & p_m \end{pmatrix},
\]

such that \(q_1 \leq q_2 \leq \cdots \leq q_m\) and if \(q_i = q_j\) with \(i \leq j\) then \(p_i \leq p_j\).

The Robinson-Schensted-Knuth (RSK) correspondence bijectively maps biwords to pairs of semistandard tableaux \((P(a), Q(a))\) of the same shape. We refer to _ibid._ for details of this correspondence. Here, we just state that we use row insertion, so that the contents of the insertion tableau \(P(a)\) is \(p_1, \ldots, p_m\), and the contents of the recording tableau \(Q(a)\) is \(q_1, \ldots, q_m\). Whereas \(\text{RSK} : a \mapsto (P(a), Q(a))\), \(\text{RSK}^{-1} : (P(a), Q(a)) \mapsto a\) is defined by reverse row insertion.

If \(p = (p_1, \ldots, p_m)\) is a word, \(\text{RSK}_2(p) = Q(a)\) where \(a\) is the biword whose bottom row is \(p\) and top row is \(1\) through \(m\). \(\text{RSK}_1(p) = P(a)\).

**Proposition 7.7.**

(1) \(\text{revrowword}(P(a)) \equiv_K p\).

(II) if \(a \equiv_K a'\) then \(P(a) = P(a')\).

(III) In each Knuth equivalence class \(C\), there is a unique straight-shape semistandard tableau \(T\) such that \(\text{revrowword}(T) \in C\).

(IV) Suppose \(a = \begin{pmatrix} q \\ p \end{pmatrix}\) and \(a' = \begin{pmatrix} q' \\ p \end{pmatrix}\) are two biwords. Assume \(\min(q') > \max(q)\). Then \(Q(aa')^{\max(q)}\) is a skew semistandard tableau such that \(\text{Rect}(Q(aa')^{\max(q)}) = Q(q')\).

**Proof.** (I), (II) and (III) are well-known. For (IV), see [6, Section 5.1, Proposition 1].

We recall a construction in [6, Chapter 5]. Fix \(V_0 \in \text{SSYT}(\nu)\) and \(U_0 \in \text{SSYT}(\mu)\). Define

\[
\mathcal{T}(\lambda, \mu, V_0) = \{T \ast U \in \text{SSYT}(\lambda \ast \mu) : \text{Rect}(T \ast U) = V_0\}
\]

and

\[
\mathcal{S}(\nu/\lambda, U_0) = \{S \in \text{SSYT}(\nu/\lambda) : \text{Rect}(S) = U_0\}.
\]
Fix \( T_0 \in \text{SSYT}(\lambda) \) where we use entries in \( \mathbb{Z}_{\leq 0} \). Define
\[
\xi : S(\nu/\lambda, U_0) \to T(\lambda, \mu, V_0)
\]
as follows. For any \( S \in S(\nu/\lambda, U_0) \), define \( T_0 \cup S \) to be the tableau of shape \( \nu \) obtained by placing \( T_0 \) in \( \lambda \) and \( S \) in \( \nu/\lambda \). Consider the biword
\[
\text{RSK}^{-1}(V_0, T_0 \cup S) = \begin{pmatrix} t_1 & \ldots & t_n & v_1 & \ldots & v_m \\ x_1 & \ldots & x_n & w_1 & \ldots & w_m \end{pmatrix},
\]
where \( n = |\lambda|, m = |\mu| \). There exist \( T \in \text{SSYT}(\lambda) \) and \( U \in \text{SSYT}(\mu) \) such that
\[
(T, T_0) = \text{RSK} \left( \begin{pmatrix} t_1 & \ldots & t_n \\ x_1 & \ldots & x_n \end{pmatrix} \right), (U, U_0) = \text{RSK} \left( \begin{pmatrix} v_1 & \ldots & v_m \\ w_1 & \ldots & w_m \end{pmatrix} \right).
\]
Let \( \xi(S) := T \ast U \).

**Proposition 7.8 ([6 Chapter 5]).** The map \( \xi : S(\nu/\lambda, U_0) \to T(\lambda, \mu, V_0) \) is a bijection for any \( U_0 \in \text{SSYT}(\mu) \) and \( V_0 \in \text{SSYT}(\nu) \). The cardinality of either set is \( c_{\lambda,\mu}^\nu \).

### 7.4 Some consequences.

Given \( \nu \in \text{Par}_n, a \geq \nu_1 \) and \( \ell \in \mathbb{Z}_{>0} \), we set
\[
a^\ell \cup \nu = (a, \ldots, a, \nu_1, \ldots, \nu_n).
\]

**Lemma 7.9.** Let \( \lambda, \mu, \nu \in \text{Par}_n \). Let \( a, b, k \in \mathbb{Z}_{\geq 0} \) such that \( a + b \geq \nu_1 - k, a \geq \mu_1 - k \) and \( b \geq \lambda_1 - k \). Then
\[
c_{\lambda+a^n,\mu+b^n}^{(a+b+k)^n} = c_{\lambda,\mu}^{\nu+k^n}.
\]

**Proof.** By (38),
\[
c_{\lambda,\mu}^{\nu+k^n} = c_{\lambda^*,(b+k)^n,\mu^*+(a+k)^n}^{(\nu+k^n)^*+(a+b+2k)^n} = c_{\lambda^*,(b+k)^n}^{(\nu+k^n)^*} = c_{\lambda^*,(b+k)^n,\mu^*+(a+k)^n}^{\nu+[a+b+k]^n} = c_{\lambda^*,(b+k)^n}^{\nu+[a+b+k]^n,\mu^*+(a+k)^n}.
\]
Combining with (39), we obtain
\[
c_{\lambda,\mu}^{\nu+k^n} = c_{\lambda^*,(b+k)^n}^{\nu+[a+b+k]^n,\mu^*+(a+k)^n} = c_{\lambda^*,(b+k)^n}^{\nu+[a+b+k]^n,\mu^*+(a+k)^n} = c_{\lambda+a^n,\mu+b^n}^{a+b+k+n}.
\]

**Lemma 7.10.** If \( T \in \text{SSYT}(\nu/\lambda) \) and \( \text{Rect}_{\{x_i\}}(T) = S \in \text{SSYT}(\mu) \) (for some choice of inner corners \( \{x_i\} \)) then \( c_{\lambda,\mu}^\nu > 0 \).

**Proof.** Immediate from Theorem 7.6

For \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \text{Par}_m \) and any \( k \in [m] \), write \( \lambda_{\leq k} = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), and \( \lambda_{>k} = (\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_m) \). If \( T \) is a tableau of shape \( \lambda \), let \( T_{\leq k} \) (respectively, \( T_{>k} \)) be the subtableau of \( T \) of shape \( \lambda_{\leq k} \) (respectively, \( \lambda_{>k} \)).

**Proposition 7.11.** Let \( \lambda, \mu, \nu \in \text{Par}_n \) such that \( c_{\lambda,\mu}^\nu > 0 \). For any \( k \in [n-1] \), there exist partitions \( \alpha, \beta \) such that
\[
c_{\nu,\alpha}^{\nu,\alpha} > 0, c_{\lambda,\beta}^{\nu,\beta} > 0 \text{ and } c_{\alpha,\beta}^{\nu,\nu} > 0.
\]

**Proof.** If \( \nu_{\leq k}/\lambda_{\leq k} = \nu/\lambda \), we are done by using \( \alpha = \mu \) and \( \beta = 0 \). We can then assume that the skew partition \( \nu/\lambda \) is non-empty below row \( k \). By Theorem 7.1, there exists a ballot tableau \( T \in \text{SSYT}(\nu/\lambda, \mu) \). Since \( T \) is ballot and semistandard, clearly \( T_{\leq k} \) is ballot and semistandard. As a result, the content \( \alpha \) of \( T_{\leq k} \) is a partition. Therefore, \( c_{\nu,\alpha}^{\nu,\nu} > 0 \).
Since $T$ is semistandard, $T_{>k}$ is also semistandard. Set $T' \in \text{SSYT}(\beta)$ to be the jeu de taquin rectification of $T_{>k} \in \text{SSYT}(\nu_{>k}/\lambda_{>k})$. Hence, by Lemma 7.10, $c_{\lambda_{>k},\beta} > 0$.

Let $\tilde{T}$ be the filling of the skew shape $\beta \ast \alpha$ where we use a super-semistandard filling $Y_\alpha$ in the $\alpha$ part and place $T' \in \text{SSYT}(\beta)$ in the $\beta$ part; clearly $\tilde{T} \in \text{SSYT}(\beta \ast \alpha)$.

By Theorem 7.5(I),
\[
\text{revrowword}(Y_\alpha) \equiv_K \text{revrowword}(T_{\leq k}) \\
\text{revrowword}(T') \equiv_K \text{revrowword}(T_{>k}).
\]
Hence
\[
\text{revrowword}(\tilde{T}) = \text{revrowword}(T') \cdot \text{revrowword}(Y_\alpha) \\
\equiv_K \text{revrowword}(T_{>k}) \cdot \text{revrowword}(T_{\leq k}) \\
= \text{revrowword}(T).
\]

Since revrowword$(T)$ is reverse-ballot, by Theorem 7.4, revrowword$(\tilde{T})$ is reverse-ballot. Thus $\tilde{T}$ is ballot. Hence by Theorem 7.2, $\tilde{T} \in \text{SSYT}(\beta \ast \alpha, \mu)$ witnesses $c_{\beta, \alpha, \mu} = c_{\alpha, \beta} > 0$. □

Lemma 7.12. ([11, Theorem 3.1]) Let $\lambda, \mu, \nu$ be partitions with $m, n, k \in \mathbb{Z}_{\geq 0}$ and $m \geq n$. Then
\[
\frac{c_{\mu}}{c_{\lambda, \mu}} \leq \frac{c_{\nu+(k^m)}}{c_{\lambda+(k^n)+\mu+(k^m-n)}}.
\]

8. Demotion

8.1. Definition. We relate the demotion algorithm ([9, Claim 3.5]) to the Robinson-Schensted-Knuth (RSK) correspondence. This is used to prove Theorem 8.15 (needed in Theorem 9.1).

Fix a ballot tableau $S \in \text{SSYT}(\nu/\lambda, \mu)$ and a corner box $c_0$ of $\lambda$. Let $\lambda^c \subseteq \lambda$ be obtained by removing $c_0$. Demotion is defined as follows. Place a 1 in $c_0$. Find the first 1 after $c_0$ (if it exists) in the column reading order. If it does not exist, terminate the process. Otherwise let $c_1$ be the box containing this 1 and turn that into a 2. Similarly, find the first 2 (if it exists, say in $c_2$) in the column reading word order after $c_1$ and change that to a 3. Terminate and output $S^\dagger$ when, after replacing the $k - 1$ in $c_{k-1}$ with $k$, there is no later $k$ in the column reading order.

Proposition 8.1 ([9, Claim 3.5]). $S^\dagger \in \text{SSYT}(\nu/\lambda^c, \mu^\dagger)$ for some $\mu \subseteq \mu^\dagger$. $S^\dagger$ is ballot.

For a corner $c$ of $\lambda$, let demote$(c) \in \mathbb{N}$ be the row index of $\mu^\dagger/\mu$ resulting from using demotion starting with $c = \lambda/\lambda^c$.

8.2. Relationship between demotion and RSK. We fixed a ballot $S \in \text{SSYT}(\nu/\lambda, \mu)$. Also fix $T_0 \in \text{SSYT}(\lambda)$ where we use entries in $\mathbb{Z}_{<0}$ as in Section 7.3.

Theorem 8.2 (Demotion-RSK relationship). Let $c_0$ be the box in $\lambda$ that is occupied by the largest entry in $T_0$ (or the rightmost of such entries, if there are repeats), then demote$(c_0) = x_n$ where demotion is applied to (a ballot tableau) $S$, and $x_n$ is defined in ([41]) where $V_0 = Y_\nu$.

If $U \in \text{SSYT}(\gamma/\delta)$, let shape$(U) = \gamma/\delta$. Let $\alpha \subseteq \gamma$ be two partitions and let $S_1, S_2 \in \text{SSYT}(\gamma/\alpha)$. Define $S_1$ and $S_2$ to L-R correspond by $V_0$ if
\[
\text{shape}($\text{Rect}(S_1)$) = \text{shape}($\text{Rect}(S_2)$) = \beta
\]
for some partition $\beta$ and

$$\xi_1(S_1) = \xi_2(S_2)$$

where $\xi_1$ and $\xi_2$ are $\xi$ defined by $U_0 = \text{Rect}(S_1)$ and $U_0 = \text{Rect}(S_2)$ respectively:

$$S(\gamma/\alpha, \text{Rect}(S_1)) \xrightarrow{\xi_1} T(\alpha, \beta, V_0) \xleftarrow{\xi_2} S(\gamma/\alpha, \text{Rect}(S_2)).$$

(There is another “$T_0$” used in the definition of $\xi_1$ and $\xi_2$ that we fix, but will play no role in our argument.)

Define $S_1$ and $S_2$ to be $Q$-equivalent if

$$(44) \quad \text{RSK}_2(\text{revrowword}(S_1)) = \text{RSK}_2(\text{revrowword}(S_2)).$$

**Proposition 8.3** ([6, Chapter A.3]). $S_1$ and $S_2$ L-R correspond by some $V_0 \in \text{SSYT}(\beta)$ if and only if $S_1$ and $S_2$ are $Q$-equivalent.

**Lemma 8.4** (Pieri property of RSK). Let $W = (w_1, \ldots, w_n)$ be a word, and $x, y \in \mathbb{N}$. Let $\text{col}(x), \text{col}(y)$ be the column indices of the added boxes when inserting $x, y$ during computation of $\text{RSK}_1(Wxy)$. Then $\text{col}(x) < \text{col}(y)$ if and only if $x \leq y$.

**Proof.** ($\Rightarrow$) We prove the contrapositive. Suppose $x > y$. Let

$$\kappa = \begin{pmatrix} -n & -n + 1 & \ldots & -1 & 1 & 2 \\ w_1 & w_2 & \ldots & w_n & x & y \end{pmatrix}.$$

Let $S = Q(\kappa) > 0$. By Proposition 7.7(IV),

$$\text{Rect}(S) = Q \left( \begin{pmatrix} 1 & 2 \\ x & y \end{pmatrix} \right) = 12.$$

Therefore, by Theorem 7.5, $S$ is a ballot; this means $\text{col}(x) \geq \text{col}(y)$.

($\Leftarrow$) If $x \leq y$, then by the same reasoning as in the “$\Rightarrow$” argument:

$$\text{Rect}(S) = Q \left( \begin{pmatrix} 1 & 2 \\ x & y \end{pmatrix} \right) = 12,$$

which means $S$ is not a ballot, and thus $\text{col}(x) < \text{col}(y)$. $\square$

Define

$$(45) \quad \omega := \text{RSK}^{-1}(Y_\nu, T_0 \cup S) = \begin{pmatrix} t_1 & \ldots & t_n & v_1 & \ldots & v_m \\ x_1 & \ldots & x_n & w_1 & \ldots & w_m \end{pmatrix}.$$

**Lemma 8.5.** The $P$-tableau of any final segment of $\omega$ is super-semistandard.

**Proof.** By Proposition 7.7(I),

$$x_1 \ldots x_n w_1 \ldots w_m \equiv_K \text{revrowword}(Y_\nu).$$

Hence, by Theorem 7.4, $x_1 \ldots x_n w_1 \ldots w_m$ is a reverse-ballot (that is $w_m w_{m-1} \ldots x_1$ is ballot). As a result, any final segment $f$ of it is also reverse-ballot. By Proposition 7.7(I), $\text{revrowword}(\text{RSK}_1(f)) \equiv_K f$; moreover by Theorem 7.4 $\text{revrowword}(\text{RSK}_1(f))$ is reverse-ballot. This immediately implies that $\text{RSK}_1(f)$ super-semistandard, as desired. $\square$

**Lemma 8.6.** Let $Y = (y_1, \ldots, y_p)$ and $x, y, z \in \mathbb{Z}$ such that $x < y \leq z$, then

$$(46) \quad \text{RSK}_1(Yyxz) = \text{RSK}_1(Yyxx).$$

Also, $x$ is inserted into the same box in both the computation of $\text{RSK}_1(Yyxz)$ and $\text{RSK}_1(Yyxx)$. 20
Proof. Since $x < y \leq z$, $Y y z z \equiv_K Y y z x$. Therefore by Proposition 7.7(II), (46) holds.

For the remaining assertion, let
\[
\pi_1 = \begin{pmatrix}
-p & -(p-1) & \ldots & -1 & 1 & 2 & 2 \\
y_1 & y_2 & \ldots & y_p & y & x & z
\end{pmatrix}
\quad \text{and} \quad
\pi_2 = \begin{pmatrix}
-p & -(p-1) & \ldots & -1 & 1 & 1 & 2 \\
y_1 & y_2 & \ldots & y_p & y & z & x
\end{pmatrix}.
\]

By Lemma 8.4, col($x) < \text{col}(z) \text{ in } RSK_1(Y y z z) = P(\pi_1) = P(\pi_2) = RSK_1(Y y z x)$. Thus, it is enough to show that col($x) < \text{col}(z) \text{ in } RSK(\pi_2)$. Notice that
\[
RSK\left(\begin{array}{ccc}
1 & 1 & 2 \\
y & z & x
\end{array}\right) = \begin{pmatrix}
x \\
y \\
z \\
x
\end{pmatrix}.
\]

Thus, by Proposition 7.7(IV) and Theorem 7.5(II), $Q(\pi_2) > 0$ is a ballot tableau with content $(2, 1)$. By Lemma 8.4, col($z) > \text{col}(y) \text{ in } P(\pi_2)$. We then conclude, by ballotness, that col($x) \leq \text{col}(z) \text{ in } P(\pi_2)$. Since $x$ and $z$ are inserted to the same set of two boxes in $\pi_1$ and $\pi_2$, col($x) < \text{col}(z) \text{ in } P(\pi_2)$. \hfill \Box

Lemma 8.7. Let $Y = y_1 \ldots y_p$ and suppose $x \leq y < z \in \mathbb{Z}$. Let col($x), \text{col}(y), \text{col}(z),$ respectively, be the column indices of the added box when inserting $x, y, z$ during the computation of $RSK_1(Y y z z)$. Similarly, let col'($x), \text{col}'(y), \text{col}'(z)$ be the indices when computing $RSK_1(Y y z x)$.

1. If col($x) < \text{col}(z) < \text{col}(y)$ then col'($y) < \text{col}'(x) < \text{col}'(z).
2. If col($x) < \text{col}(y) < \text{col}(z)$ then col'($x) < \text{col}'(y) < \text{col}'(z).

Proof. Consider the biwords
\[
\eta_1 = \begin{pmatrix}
-p & -(p-1) & \ldots & -1 & 1 & 2 & 3 \\
y_1 & y_2 & \ldots & y_p & z & x & y
\end{pmatrix}
\quad \text{and} \quad
\eta_2 = \begin{pmatrix}
-p & -(p-1) & \ldots & -1 & 1 & 2 & 3 \\
y_1 & y_2 & \ldots & y_p & x & z & y
\end{pmatrix}.
\]

Let
\[
\alpha = \text{shape}(RSK_1(Y)), \beta = \text{shape}(RSK_1(z x y)), \gamma = \text{shape}(RSK_1(y z x)).
\]

Also, set
\[
S_1 = Q(\eta_1) > 0, S_2 = Q(\eta_2) > 0 \in \text{SSYT}(\gamma/\alpha).
\]

Since $Y y z z \equiv_K Y z x y$, $P(\eta_1) = P(\eta_2)$ (by Proposition 7.7(II)); let $V_\gamma$ be this tableau. Set $T_0 = Q(\eta_1) < 0 = Q(\eta_2) < 0 \in \text{SSYT}(\alpha)$ and let
\[
\xi_1 : \mathcal{S}(\gamma/\alpha, \text{Rect}(S_1)) \longrightarrow T(\alpha, \beta, V_\gamma)
\]
and
\[
\xi_2 : \mathcal{S}(\gamma/\alpha, \text{Rect}(S_2)) \longrightarrow T(\alpha, \beta, V_\gamma)
\]
be the maps as defined in Proposition 7.8. Since
\[
RSK^{-1}(V_\gamma, Q(\eta_1) < 0 \cup S_1) = \eta_1 \quad \text{and} \quad RSK^{-1}(V_\gamma, Q(\eta_2) < 0 \cup S_2) = \eta_2,
\]
we obtain
\[
\xi_1(S_1) = RSK_1(Y) \ast RSK_1(z x y) \quad \text{and} \quad \xi_2(S_2) = RSK_1(Y) \ast RSK_1(x y z).
\]

Since $x y z z \equiv_K z x y$, we have $RSK_1(x y z) = RSK_1(z x y)$ and thus $\xi_1(S_1) = \xi_2(S_2)$. Notice further that we have shape(\text{Rect}(S_1)) = \beta = \text{shape}(\text{Rect}(S_2)). Therefore $S_1$ and $S_2$ L-R correspond by $V_\gamma$. By Proposition 8.3, $S_1$ and $S_2$ are also $Q$-equivalent.

In (1), since col($x) < \text{col}(z) < \text{col}(y)$, we know that \text{revrowword}(S_1) = 213, and thus
\[
RSK_2(\text{revrowword}(S_1)) = \begin{pmatrix} 1 & 3 \\ 2 \end{pmatrix}.
Since \( x, y < z \), by Lemma 8.4, we have \( \text{col}'(x), \text{col}'(y) < \text{col}'(z) \). Hence, \( \text{revrowword}(S_2) = 132 \) if \( \text{col}'(x) < \text{col}'(y) \) and \( \text{revrowword}(S_2) = 312 \) if \( \text{col}'(y) < \text{col}'(x) \). Now,

\[
\text{RSK}_2(132) = \begin{pmatrix} 1 & 2 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \text{RSK}_2(312) = \begin{pmatrix} 1 & 3 \\ 2 \\ 2 \end{pmatrix}
\]

Since \( S_1 \) and \( S_2 \) are \( Q \)-equivalent, (44) holds; this implies we are in case \( \text{col}'(y) < \text{col}'(x) \). Thus \( \text{col}'(y) < \text{col}'(x) < \text{col}'(z) \), as desired.

In (2), since \( \text{col}(x) < \text{col}(y) < \text{col}(z) \), \( \text{revrowword}(S_1) = 231 \). Hence

\[
\text{RSK}_2(\text{revrowword}(S_1)) = \begin{pmatrix} 1 & 2 \\ 3 \\ 2 \end{pmatrix}
\]

Since \( x, y < z \), by Lemma 8.4, \( \text{col}'(x), \text{col}'(y) < \text{col}'(z) \). By the same reasoning as in (1)

\[
\text{RSK}_2(132) = \begin{pmatrix} 1 & 2 \\ 3 \\ 2 \end{pmatrix}
\]

we see that \( \text{col}'(x) < \text{col}'(y) < \text{col}'(z) \).

For \( 1 \leq j \leq \ell(\mu) \), let \( i_j \) be the largest index such that \( v_{i_j} = j \) and declare \( i_0 = 0 \).

**Lemma 8.8.** Each \( i_j \) (for \( 1 \leq j \leq \ell(\mu) \)) is finite, and

\[
w_{i_{j-1}} \ldots w_{i_j} = 1, \ldots, 1, 2, \ldots, 2, \ldots, \ell(\mu) - j + 1, \ldots, \ell(\mu) - j + 1.
\]

**Proof.** Let \( U_0 = \text{Rect}(S) \). Since \( S \) is assumed to be ballot, it follows from Theorem 7.5(II) that \( U_0 = Y_\mu \) for some \( \mu \). Comparing (45) and (41) we conclude from (42) that

\[
(U, Y_\mu) = \text{RSK}\left(\begin{pmatrix} v_1 & \ldots & v_m \\ w_1 & \ldots & w_m \end{pmatrix}\right).
\]

The first claim follows, since the multiset of labels of \( v \) is the same as than in \( Y_\mu \). Also, by Lemma 8.5, we also have \( U = Y_\mu \), which implies the second claim, for the same reason.

**Lemma 8.9.** For any \( 1 \leq j \leq x_n - 1 \), there exists a largest \( k_j \) such that \( \binom{v_{k_j}}{w_{k_j}} = \binom{j}{x_n-j} \).

**Proof.** By Lemma 8.5,

\[
P\left(\begin{pmatrix} t_n & v_1 & \ldots & v_m \\ x_n & w_1 & \ldots & w_m \end{pmatrix}\right) \quad \text{and} \quad P\left(\begin{pmatrix} v_1 & \ldots & v_m \\ w_1 & \ldots & w_m \end{pmatrix}\right)
\]

are super-semistandard. Thus, it follows that \( \mu_{x_n-1} > \mu_{x_n} \). By Lemma 8.8, in \( w_{i_{j-1}+1} \ldots w_{i_j} \), there are \( \mu_{x_n-1} - \mu_{x_n} > 0 \) many \( x_n - j \)'s. Let \( i_{j-1} + 1 \leq k_j \leq i_j \) be the index of any such occurrence. Since \( v_{i_{j-1}+1}, v_{i_{j-1}+2}, \ldots, v_{i_j} = j \), we are done.

Let \( S^\dagger \in \text{SSYT}(V/\lambda^\dagger, \mu^\dagger) \) be the ballot tableau from demoting \( S \) at \( c_0 \). Define

\[
\omega_j = \begin{pmatrix} t_1 & \ldots & t_n & v_1 & \ldots & v_{i_j} \\ x_1 & \ldots & x_n & w_1 & \ldots & w_{i_j} \end{pmatrix}, \quad \text{for} \ j \in [x_n - 1].
\]

Define

\[
\omega'_j = \begin{pmatrix} t_1 & \ldots & t_{n-1} & v_1 & \ldots & v_{k_{j-1}} & 1 & v_{k_{j-1}+1} & \ldots & v_{k_{j-2}} & 2 & v_{k_{j-1}+1} & \ldots & v_{i_j} & j + 1 \\ x_1 & \ldots & x_{n-1} & w_1 & \ldots & w_{k_{j-1}} & x_n & w_{k_{j-1}} & \ldots & w_{k_{j-2}} & w_{k_{j-1}} & \ldots & w_{i_j} & w_{k_j} \end{pmatrix},
\]

this is the biword obtained from \( \omega_j \) by replacing \( \binom{v_{k_{j-1}}}{w_{k_{j-1}}} \) with \( \binom{i}{w_{k_{j-1}}-1} \) for \( 0 < i < j \) (where \( w_{k_0} = x_n \)), removing \( \binom{t_n}{x_n} \) and appending \( \binom{j+1}{w_{k_j}} \) to the right-end.
Lemma 8.10. $\omega_j'$ is a biword.

Proof. Since $v_{k_j+1} > i \geq v_{k_i-1}$ for $0 < i \leq j$ and $v_{ij} = j < j + 1$, the first row of $\omega_j'$ is weakly increasing. For the second row, first recall that by definition $i = v_{k_i} \geq v_{k_i-1}$. If the inequality is strict there is nothing to show, so we may assume $i = v_{k_i} = v_{k_i-1}$. Since $\omega$ is a biword, $w_{ki} \geq w_{ki-1}$. Since $x_n = w_{k_0} > w_{k_1} > w_{k_2} > \cdots > w_{kj}$, we have $w_{k_i-1} > w_{ki} \geq w_{ki-1}$ for all $0 < i \leq j$.

Therefore $\omega_j'$ is a biword. □

By definition, $RSK(\omega_j) = \langle P(\omega_j), T_0 \cup S^{\leq j} \rangle$.

For each such $j$, define $(S^{\leq j})^\uparrow \in SSYT(\nu/\lambda^\downarrow, \mu^{\uparrow}_{\leq j})$ to be the ballot tableau obtained by applying demotion to $S^{\leq j}$ at the $t_n$ entry in $T_0$, and $T_0^\downarrow \in SYT(\lambda^\downarrow)$ the tableau obtained from $T_0$ by removing the $t_n$ entry.

Proposition 8.11. For each $j \in [x_n - 1]$, $RSK(\omega_j') = \langle P(\omega_j), T_0^\downarrow \cup (S^{\leq j})^\uparrow \rangle$.

Proof of Proposition 8.11: We proceed by induction on $j$. For the base case $j = 1$, set

$$\omega_1 = \begin{pmatrix} t_1 & t_2 & \ldots & t_{n-1} & t_n & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_{n-1} & x_n & w_1 & w_2 & \ldots & w_{i_1} \end{pmatrix}.$$ 

Since $\omega$ is a biword, by definition, $w_1 \leq w_2 \leq \ldots \leq w_{i_1}$. Now,

$$\omega_1' = \begin{pmatrix} t_1 & t_2 & \ldots & t_{n-1} & 1 & \ldots & 1 & 1 & \ldots & 1 & 2 \\ x_1 & x_2 & \ldots & x_{n-1} & w_1 & \ldots & w_{k_{i_1}-1} & x_n & w_{k_{i_1}+1} & \ldots & w_{i_1} & w_{k_1} \end{pmatrix}.$$ 

Claim 8.12. Suppose $t_n$ appears in box $c_0$ of $T_0 \cup S^{\leq 1}$ and all $1$’s of $T_0 \cup S^{\leq 1}$ occur weakly west of $c_0$. Then $x_n > w_{i_1}$.

Proof. Let $d_1, d_2, \ldots, d_{i_1}$ be boxes of the 1’s in $S^{\leq 1}$, as listed from right to left (they form a horizontal strip). Let $P^{(1)}$ be the tableau obtained by reverse insertion of $P^{(0)} := P(\omega_1)$ at $d_1$. $P^{(2)}$ is obtained by reverse inserting $P^{(1)}$ at $d_2$. Likewise define $P^{(3)}$, $P^{(4)}, \ldots, P^{(i_1)}$. By definition of RSK, the output of the reverse inserting $P^{(k)}$ at $d_{k+1}$ is $w_{i_{k-1} - k}$, for $0 \leq k \leq i_1 - 1$.

The box $c_0$ in $\lambda \subset \nu = \text{shape}(T_0 \cup S^{\leq 1}) = \text{shape}(P(\omega_1))$ also appears in each of $\text{shape}(P^{(k)})$ for $0 \leq k \leq i_1$. Thus, define $r_k$ to be the result of reverse inserting $P^{(k)}$ at $c_0$, for $1 \leq k \leq i_1$ ($r_0$ does not make sense if $d_1$ is directly below $c_0$).

Now, by Lemma 8.4

$$r_1 > w_{i_1}. \quad (47)$$

We claim:

$$r_{i_1} \geq r_{i_1 - 1} \geq r_{i_1 - 2} \geq \cdots \geq r_1 \quad (48)$$

Combining this with (47) gives us $x_n = r_{i_1} \geq r_1 > w_{i_1}$, as desired.

Hence it remains to prove (48). Let $1 = y_1 = P^{(1)}(d_2)$ (i.e., the entry of $P^{(1)}$ at box $d_2$). When reverse inserting $y_1$ into the previous row, suppose $y_2$ is reverse bumped etc. Suppose that $y_n$ is reverse inserted into the row of $c_0$ and $y_{n+1}$ is reverse bumped. By the Lemma’s hypothesis, $c_0$ is the rightmost box in its row. Hence there are two cases:
Case 1: \((y_{u+1} < P^{(1)}(c_0))\) By definition of RSK reverse row insertion, \(y_{u+i} < P^{(1)}(c_0)\) for \(i \geq 2\). If \(b\) is any box of \(P^{(2)}\) in a row strictly above that of \(c_0\) then
\[
P^{(1)}(b) \leq P^{(2)}(b)
\]
and
\[
P^{(1)}(b) \neq P^{(2)}(b) \implies P^{(2)}(b) < P^{(1)}(c_0).
\]
By (49) and (50) combined, the boxes used in reverse inserting \(P^{(1)}\) starting at \(c_0\) are the same when reverse inserting \(P^{(2)}\) at \(c_0\). Therefore \(r_2 \geq r_1\).

Case 2: \((y_{u+1} = P(c_0))\) Consider the RSK reverse insertion path \(P\) starting at \(d_2\) that outputs \(w_{i_1-1}\); these are the boxes of \(P^{(1)}\) that have been changed. Our assumption implies \(P\) includes \(c_0\). Notice that when computing \(r_2\) (by reverse inserting \(P^{(2)}\) at \(c_0\)) we use a path \(P'\) that is weakly right (in each row) of the subpath of \(P\) starting at \(c_0\). Since each of the boxes in \(P\) boxes contain a larger label in \(P^{(2)}\) than in \(P^{(1)}\), and \(P^{(2)}\) is semistandard, it again follows that \(r_2 \geq r_1\).

Having established \(r_2 \geq r_1\), we consider the reverse insertion of \(P^{(2)}\) at \(d_3\). The analysis above applies, mutatis mutandis, to show \(r_3 \geq r_2\). Repeating this, we obtain the claim. \(\square\)

Claim 8.13. The base case \(j = 1\) follows from the special case where \(k_1 = i_1\).

Proof. For \(i \in [k_1, i_1]\), set
\[
\theta'_i := \begin{pmatrix} t_1 & \cdots & t_{n-1} & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 2 \\ x_1 & \cdots & x_{n-1} & w_1 & \cdots & w_{k_1-1} & x_n & w_{k_1+1} & \cdots & w_i & w_{k_1} \end{pmatrix}
\]
and let
\[
S^\diamond := Q \left( \begin{pmatrix} t_1 & \cdots & t_n & 1 & \cdots & 1 \\ x_1 & \cdots & x_n & w_1 & \cdots & w_{k_1} \end{pmatrix} \right)^{>0}.
\]

Suppose the base case is correct when \(k_1 = i_1\). This implies that the unique 2 in \(Q(\theta'_{k_1})\) is in the same box as the rightmost 1 of \(S^\diamond\) that is weakly west of \(c_0\). By the contrapositive statement of Claim 8.12 shape(\(Q(\theta'_{k_1})\)) contains all the boxes occupied by 1 in \(S\) that are weakly west of \(c_0\). Hence the unique 2 in \(Q(\theta'_{k_1})\) will be in the same box as the rightmost 1 weakly left of the \(t_n\) in \(T_0 \cup S^{\leq 1}\). Consider
\[
\tilde{\theta}'_i := \begin{pmatrix} t_1 & \cdots & t_{n-1} & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 2 & 3 \\ x_1 & \cdots & x_{n-1} & w_1 & \cdots & w_{k_1-1} & x_n & w_{k_1+1} & \cdots & w_i & w_{k_1} & w_{i+1} \end{pmatrix}.
\]
It is clear that the unique 2 in \(Q(\theta'_{i+1})\) is in the same box as the unique 2 in \(Q(\tilde{\theta}'_i)\) for any \(i \in [k_1, i_1 - 1]\). Since \(w_{k_1} < w_i \leq w_{i+1}\), by Lemma 8.6 we know that the unique 2 in \(Q(\tilde{\theta}'_i)\) is also in the same box as the unique 2 in \(Q(\theta'_{i+1})\). As a result, the 2 in \(Q(\theta'_{k_1})\) is in the same box as the 2 in \(Q(\theta'_{i+1}) = T_0^\uparrow \cup (S^{\leq 1})^\uparrow = Q(\omega'_i)\) which proves the base case. \(\square\)

Now by Claim 8.13 it is enough to prove the base case with the assumption that \(k_1 = i_1\). Since the second rows of \(\omega'_i\) and \(\omega'_1\) are Knuth equivalent, we get \(P(\omega'_i) = P(\omega'_1)\).

Now we show \(Q(\omega'_1) = T_0^\uparrow \cup (S^{\leq 1})^\uparrow\). If some 1 appears immediately below the \(t_n\) in \(T_0 \cup S^{\leq 1}\), then we are done since \(T_0^\uparrow \cup (S^{\leq 1})^\uparrow\) is semistandard. We can thus assume there is at most one box in each column of \(T_0^\uparrow \cup (S^{\leq 1})^\uparrow\) that is filled by 1 or 2. For \(i \in [k_1]\), consider
\[
\theta_i = \begin{pmatrix} t_1 & \cdots & t_{n-1} & 1 & \cdots & 1 & 2 & \cdots & 2 \\ x_1 & \cdots & x_{n-1} & w_1 & \cdots & w_{i-1} & x_n & w_i & \cdots & w_{k_1} \end{pmatrix}.
\]
Let $\text{col}_i(w_k)$ and $\text{col}_i(x_n)$ be the column index where $w_k$ and $x_n$ is inserted in the computation of $Q(\theta_t)$ respectively. Notice that $\text{col}_i(w_k) = \text{col}_i(w_k)$ for all $i > k$. Also since $x_1 \cdots x_n w_1 \cdots w_{k-1} \equiv_K x_1 \cdots x_{n-1} w_1 \cdots w_{i-1} x_n w_i \cdots w_{k-1}$ it follows that

\begin{equation}
\text{col}_i(w_k) = \text{col}_i(w_k) \quad \text{for all } i < k.
\end{equation}

By Lemma \ref{lem:8.7}(2), if $\text{col}_i(w_i) < \text{col}_i(w_{i+1}) < \text{col}_i(x_n)$, then

\begin{equation}
\text{col}_i(w_i) = \text{col}_{i+1}(w_i) < \text{col}_i(w_{i+1}) = \text{col}_{i+1}(w_{i+1}) < \text{col}_i(x_n) = \text{col}_{i+1}(x_n).
\end{equation}

Let $\ell$ be the index such that $w_{\ell}$ is inserted to the rightmost 1 that is weakly west of $c_0$ in the computation of $Q(\omega_1) = T_0 \cup S^{\le 1}$. By \ref{eq:52} and a simple induction argument,

$\text{col}_1(w_{\ell}) = \text{col}_\ell(w_{\ell}) < \text{col}_1(x_n) = \text{col}_\ell(x_n) < \text{col}_\ell(w_{\ell+1}).$

Hence, by Lemma \ref{lem:8.7}(1), we have

\begin{equation}
\text{col}_1(w_{\ell}) = \text{col}_{\ell+1}(w_{\ell+1}) < \text{col}_{\ell+1}(w_{\ell}) < \text{col}_{\ell+1}(x_n) = \text{col}_{\ell}(w_{\ell+1}).
\end{equation}

By \ref{eq:51} and Lemma \ref{lem:8.4}

\begin{equation}
\text{col}_{\ell+1}(w_{\ell+2}) = \text{col}_1(w_{\ell+2}) > \text{col}_1(w_{\ell+1}) = \text{col}_{\ell}(w_{\ell+1}).
\end{equation}

Thus by \ref{eq:53} and \ref{eq:54},

$\text{col}_{\ell+1}(w_{\ell+1}) < \text{col}_{\ell+1}(x_n) < \text{col}_{\ell+1}(w_{\ell+2}).$

Hence, by Lemma \ref{lem:8.7}(1) and induction, for any $k \in [\ell, k_1 - 1]$, we have

$\text{col}_k(w_k) = \text{col}_\ell(w_k) < \text{col}_1(x_n) < \text{col}_k(w_{k+1}).$

Thus by Lemma \ref{lem:8.7}(1), we get

$\text{col}_1(w_{\ell}) = \text{col}_{k+1}(w_{k+1}) < \text{col}_{k+1}(w_k) < \text{col}_{k+1}(x_n).$

As a result, $\text{col}_1(w_{\ell}) = \text{col}_{k_1}(w_{k_1})$ and the unique 2 in $Q(\theta_{k_1}) = T_0^j \cup (S^{\le 1})^\dagger$ is in the same box as the rightmost 1 in the columns weakly left of the $t_n$ in $T_0 \cup S^{\le 1}$ as we desired. This completes the proof of the base case $j = 1$ of Proposition \ref{prop:8.11}.

We assumed nothing about the inner semistandard tableau $T_0$ in the proof of the base case. In addition, notice that the largest entry in $T_0^j \cup (S^{<j})^\dagger$ is the final $j$ entry that results from demotion of $T_0 \cup S^{<j}$ at $c_0$. Therefore the same argument we used in the base case holds for any $j \in [n-1]$ by replacing $T_0$ with $T_0^j \cup (S^{<j})^\dagger$. As a result we have $\text{RSK}(\omega') = (P(\omega_j), T_0^j \cup (S^{<j})^\dagger)$ as we wanted. This completes the proof of Proposition \ref{prop:8.11}.

**Conclusion of the proof of Theorem \ref{thm:8.2}**

By Proposition \ref{prop:8.11}, $\text{RSK}(\omega'_{x_n-1}) = (P(\omega_j), T_0^j \cup (S^{<x_n})^\dagger)$ and so demote($c_0$) $\geq x_n$. The $x_n$-entries in $S$ occupy shape($P(\omega_{x_n})/P(\omega_{x_n-1})$).

In addition, the leftmost entry in $P(\omega_{x_n})/P(\omega_{x_n-1})$ is added by inserting $(x_n \ w_{x_n-1+1})$ to $P(\omega_{x_n-1})$ by RSK. Since $w_{x_n-1+1} = 1 \leq w_{x_n+1+1}$ and $P(\omega_{x_n-1}) = P(\omega_{x_n-1})$, by Lemma \ref{lem:8.4}, the unique $x_n$ entry in $T_0^j \cup (S^{<x_n})^\dagger$ is strictly left of all $x_n$ entries in $S$. By definition, demotion terminates and we have demote($c_0$) $= x_n$.

Let $\omega'$ be the biword obtained from $\omega'_{x_n-1}$ by appending

\begin{equation}
\begin{pmatrix}
v_{i_{x_n-1}+1} & v_{i_{x_n-1}+2} & \cdots & v_m \\
w_{i_{x_n-1}+1} & w_{i_{x_n-1}+2} & \cdots & w_m
\end{pmatrix}
\end{equation}
at the end. Since \( v_{i_{x_n-1}+1} = x_n \), the first row of \( \omega' \) is weakly increasing. Since \( w_{k_{x_n-1}} = 1, w_{k_{x_n-1}+1} \leq w_{i_{x_n-1}+1} \) and thus \( \omega' \) is a biword. Let \( S^\dagger \) be the ballot tableau obtained by applying demotion to \( S \) at the \( t_n \) entry in \( T_0 \).

**Corollary 8.14.** \( \text{RSK}(\omega') = (Y_\nu, T_0^\dagger \cup S^\dagger) \)

**Proof.** By Proposition 8.11, \( \omega'_{x_n-1} \equiv K \omega_{x_n-1} \) and thus \( \omega' \equiv K \omega \). Therefore \( P(\omega') = Y_\nu \). By Theorem 8.2, \( \text{demote}(e_0) = x_n \) and thus \( S^\dagger = (S \leq x_n -1) \cup S \geq x_n \). Since \( P(\omega'_{x_n-1}) = P(\omega_{x_n-1}) \),

\[
Q(\omega') = Q(\omega'_{x_n-1}) \cup S \geq x_n = T_0^\dagger \cup (S \leq x_n -1) \cup S \geq x_n = T_0^\dagger \cup S^\dagger. \]

The important consequence of Corollary 8.14 for Section 9 is:

**Theorem 8.15.** Let \( \lambda, \mu, \nu \) be three partitions, \( a' \) and \( \ell \) two positive integers. Assume that \( \ell(\lambda) \leq \ell \) and there exists a ballot tableau \( T \in \text{SSYT}(\nu/\mu, \lambda) \) with all entries in rows below \( \ell \) at most \( a' \).

Then, there exists a partition \( \rho \) such that

\[
\ell^\nu_{\rho, \lambda \leq a'} > 0, \quad \ell^\nu_{\mu \geq a'} > 0, \quad \mu > a' = \rho > a'.
\]

8.3. **Proof of Theorem 8.15** We first recall **tableau infusion** (also known as **tableau switching**) following [2]. For \( T \in \text{SSYT}(\nu/\lambda) \) and \( S \in \text{SSYT}(\lambda) \), let \( x_0 \) be the corner of the \( \lambda \) occupied by the largest entry in \( \text{std}_{\text{cont}(S)}(S) \). Define \( T^{(1)} \in \text{SSYT}(\nu^{(1)}/\lambda^{(1)}) = \text{jdt}_{\nu}(T) \) and \( S^{(1)} \in \text{SSYT}(\lambda^{(1)}) \) the subtableau of \( S \) obtained by removing the entry in \( x_0 \). Fill in the box \( (\nu^{(1)}/\lambda^{(1)}/S(x_0), \nu^{(1)}) \) by \( S(x_0) \), the entry of \( x_0 \) in \( S \), and denote this filling \( S^{(1)} \).

For the triple \( T^{(i)} \in \text{SSYT}(\nu^{(i)}/\lambda^{(i)}), \emptyset \neq S_i \in \text{SSYT}(\lambda^{(i)}) \) and \( S^{(i)} \), let \( x_i \) be the corner of \( \lambda^{(i)} \) occupied by the largest entry in \( \text{std}_{\text{cont}(S_i)}(S_i) \). Set \( T^{(i+1)} = \text{jdt}_{x_i}(T^{(i)}) \in \text{SSYT}(\nu^{(i+1)}/\lambda^{(i+1)}) \) and \( S_{i+1} \in \text{SSYT}(\lambda^{(i)}) \) the subtableau of \( S_i \) obtained by removing the entry in \( x_i \). Let us obtain \( S^{(i+1)} \) from \( S^{(i)} \) by attaching the cell \( \nu^{(i)}/\nu^{(i+1)} \) and fill in \( S_i(x_i) \).

For \( T \) and \( S \) as above, define

\[
\text{Infusion}(S, T) = (\text{Infusion}_1(S, T), \text{Infusion}_2(S, T)) := (T^{(n)}, S^{(n)})
\]

where \( n = |\lambda| \). Observe that \( T^{(n)} = \text{Rect}_{x_1}(T) \).

**Theorem 8.16** ([2] Theorem 2.2 and Theorem 3.1). \( S^{(n)} \in \text{SSYT}(\nu/\nu^{(n)}) \) and

\[
\text{Infusion}(T^{(n)}, S^{(n)}) = (S, T).
\]

For the ballot \( T \in \text{SSYT}(\nu/\mu, \lambda) \) as in the statement of Theorem 8.15 define

\[
\text{Inf}(T) := \text{Infusion}_2(Y_\mu, T).
\]

**Claim 8.17.** The number of entries in the first column of \( \text{Inf}(T)_{\leq \ell - a'} \) is at most \( \ell - \ell(\lambda) \).

**Proof.** Our proof is by contradiction. Suppose there are more than \( \ell - \ell(\lambda) \) entries in the first column of \( \text{Inf}(T)_{\leq \ell - a'} \). Let \( k \) be the label in matrix coordinate \( (\ell + 1, 1) \) of \( \text{Inf}(T) \in \text{SSYT}(\nu/\lambda) \). Then

\[
\ell - \ell(\lambda) < k \leq \ell - a',
\]

where the strict inequality is the by column strictness \( \text{Inf}(T) \).

By Theorem 8.16 since \( \text{Infusion}_1(Y_\mu, T) = Y_\lambda \)

\[
\text{Infusion}(Y_\lambda, \text{Inf}(T)) = (Y_\mu, T).
\]
Let $x_i$ be the box in $\lambda$ occupied by the $i$-th largest entry in $\text{std}_\lambda(Y_\lambda)$ and set
\begin{equation}
\inf(T)^{(i)} = \text{jdt}_{x_i} \text{jdt}_{x_{i-1}} \cdots \text{jdt}_{x_1} (\inf(T)) \in \text{SSYT}(\nu^{(i)}/\lambda^{(i)}).
\end{equation}

Let $i_k$ be the index such that $x_{i_k}$ is the leftmost box in row $\ell + 1 - k$ of $\lambda$. By the upper bound in (55), $\ell + 1 - k > a'$. Since, by hypothesis, all entries strictly south of the $\ell$-th row of $T$ are $\leq a'$, by (56), $\inf(T)$ and $\inf(T)^{(1)}, \inf(T)^{(2)}, \ldots, \inf(T)^{(i_k-1)}$ coincide below row $\ell$. In particular,
\begin{equation}
\inf(T)^{(i_k-1)}(\ell + 1, 1) = k.
\end{equation}

By our choice of (partial) rectification order (57), in column 1 of $\nu$, the rows $\ell + 2 - k$ through $\ell + 1$ are all in $\inf(T)^{(i_k-1)}$. Thus, since $\inf(T)^{(i_k-1)}$ is semistandard,
\begin{equation}
\inf(T)^{(i_k-1)}(\ell + 1 - j, 1) = k - j, \text{ for } 0 \leq j < k.
\end{equation}

Thus, in the computation $\inf(T)^{(i_k)} = \text{jdt}_{x_{i_k}} (\inf(T)^{(i_k-1)})$, (58) implies jeu de taquin uses only (J1) as its first $k$ slides. This combined with (55) says that $\ell + 1 - k > a'$ appears in a row $> \ell$ in $T$. This is a contradiction of our hypothesis about $T$. \qed

Claim 8.18. Let $c_0$ be the southmost corner of $\lambda$. If demotion is applied to $S \in \text{SSYT}(\nu/\lambda, \mu)$ with respect to the corner box $c_0$, then demote($c_0$) = $m$ where $m \in \mathbb{Z}_{>0}$ is the smallest number that is not in the first column of $S$.

Proof. Since $c_0$ is a corner of the bottom row of $\lambda$, the first 1 after $c_0$ in the column reading word, if exists, must be the rightmost 1 in row $\ell(\lambda) + 1$ of $S$. Therefore we have
\begin{equation}
S(\ell(\lambda) + 1, 1) = 1 \text{ and } c_1 \text{ is in row } \ell(\lambda_1) + 1.
\end{equation}

Similarly, using that $S$ is semistandard, and an easy induction, if $c_k$ exists, it is in row $\ell(\lambda) + k$ and $S(\ell(\lambda) + k, 1) = k$. So demote($c_0$) $\leq m$. On the other hand, since $m - 1$ appears in that first column, it follows $c_{m-1}$ exists and so demote($c_0$) $\geq m$, and we are done. \qed

Label the boxes of the southmost row of $\lambda$ by $d_1$ through $d_j$, from right to left. Fix $U_1 = U$ to be a ballot semistandard tableau. Let demote($d_j$) be computed by applying demotion to $U_j$ with respect to to $d_j$, where $U_j$ is the result of $U_{j-1}$ bumped at $d_{j-1}$ (if $j > 1$).

Claim 8.19. demote($d_1$) $\geq$ demote($d_2$) $\geq \cdots \geq$ demote($d_i$).

Proof. Let $m_i \in \mathbb{Z}_{>0}$ be the smallest label not appearing in the first column of $U_i$. By definition of demotion, $m_1 \geq m_2 \geq \cdots \geq m_i$. Now apply Claim 8.18. \qed

Let $R := \ell(\lambda) - a > 0$. If $R = 0$ then $\ell - a' = 0$ and Theorem 8.15 is trivial since we can take $\rho = \mu$. Therefore we may assume $R > 0$. Now, iteratively apply demotion, beginning with $\inf(T)$, using the boxes $\{q_i\}_{i=1}^R$ in $\lambda$ taken from right to left, bottom to top. Call the result $\inf(T)' \in \text{SSYT}(\nu/\lambda_{\leq a'}, \rho)$. This completes the proof of Theorem 8.15

Claim 8.20. The partition $\rho$ satisfies:

(i) $\ell(\lambda_{\leq a'}, \rho) > 0$;
(ii) $\ell(\lambda_{> a'}, \rho) > 0$; and
(iii) $\rho_{>\ell-a'} = \mu_{>\ell-a'}$. 

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Proof of Claim 8.20: (i): Since demotion converts one ballot tableau into another, \( \inf(T') \in \text{SSYT}(\nu/\lambda, \rho) \) is ballot. Hence \( c_{\lambda \triangleleft a'}^\rho = c_{\rho, \lambda \triangleleft a'}^\rho > 0. \)

(ii): Let \( T_0 \in \text{SYT}(\lambda) \) be the superstandard tableau using negative entries. Let \( S = \inf(T) \in \text{SSYT}(\nu/\lambda, \mu). \) Let

\[
\omega := \text{RSK}^{-1}(Y_\nu, T_0 \cup S) = \begin{pmatrix} t_1 & \cdots & t_n & v_1 & \cdots & v_m \\ x_1 & \cdots & x_n & w_1 & \cdots & w_m \end{pmatrix}.
\]

Let \( \ell \in [n] \) be the index such that

\[
(T', (T_0)_{\leq a'}) = \text{RSK} \left( \begin{pmatrix} t_1 & \cdots & t_\ell \\ x_1 & \cdots & x_\ell \end{pmatrix} \right)
\]

for some \( T' \). By Corollary 8.14, we can find \( v'_1, \ldots, v'_{m+n-\ell} \) and \( w'_1, \ldots, w'_{m+n-\ell} \) such that

\[
(Y_\nu, (T_0)_{\leq a'} \cup \inf(T')), (Y_\rho, Y_\rho) = \text{RSK} \left( \begin{pmatrix} t_1 & \cdots & \ell & v'_1 & \cdots & v'_{m+n-\ell} \\ x_1 & \cdots & \ell & w'_1 & \cdots & w'_{m+n-\ell} \end{pmatrix} \right).
\]

Write

\[
(\tilde{T}, \tilde{T}_0) = \text{RSK} \left( \begin{pmatrix} t_{\ell+1} & \cdots & t_n \\ x_{\ell+1} & \cdots & x_n \end{pmatrix} \right)
\]

and let \( \pi \) be the shape of \( \tilde{T} \). Let \( \tilde{T} := \text{RSK}_1(x_1 x_2 \cdots x_n) \in \text{SSYT}(\lambda) \). Observe that \( T' \star \tilde{T} \in \mathcal{T}(\lambda_{\triangleleft a'}, \pi, \tilde{T}) \), by Proposition 7.7(III) and Theorem 7.5(I), since \( \text{revrowword}(T' \star \tilde{T}) = x_1 x_2 \cdots x_n \). Since \( c_{\lambda_{\triangleleft a'}, \pi}^\rho = |\mathcal{T}(\lambda_{\triangleleft a'}, \pi, T)| > 0 \), we get \( \pi = \lambda_{\triangleright a'} \).

Notice by Lemma 8.5 and Corollary 8.14, we have \( x_{\ell+1} \cdots x_n w_1 \cdots w_m \equiv_K \text{revrowword}(Y_\rho) \). Therefore \( \tilde{T} \star Y_\mu \in \mathcal{T}(\lambda_{\triangleright a'}, \mu, Y_\rho) \), again by Proposition 7.7(III) and Theorem 7.5(I). Hence, \( c_{\lambda_{\triangleright a'}, \mu}^\rho = |\mathcal{T}(\lambda_{\triangleright a'}, \mu, Y_\rho)| > 0. \)

(iii): For \( 1 \leq t \leq L+1 \), let \( \text{Inf}(T)[t] \) be the (ballot) tableau obtained by iteratively applying demotion at \( q_1, \ldots, q_{t-1} \). Suppose \( \rho[t] = \text{cont}(\text{Inf}(T)[t]) \); hence

\[
\mu = \rho[1] \subsetneq \rho[2] \subsetneq \rho[3] \subsetneq \cdots \subsetneq \rho[L+1] = \rho.
\]

Call \( \rho[t] \) good if \( (\rho[t])_i = \mu_i \) for \( i > b \); we will show all \( \rho[t] \) are good.

Let \( r_1, \ldots, r_R \) be the indices such that \( q_{r_j} \) is the rightmost box of the \( j \)-th southmost row of \( \lambda \); thus \( q_{r_1} = q_1 \). Suppose \( 1 \leq u \leq R \). By Claim 8.17 and the construction of demotion, \( (\text{Inf}(T)[r_u])_{\leq \ell-a'} \) has at most \( \ell - \ell(\lambda) + (u-1) = \ell - a' - R + (u-1) < \ell - a' \) elements in the first column. Hence \( \rho[r_u+1] \) is good by Claim 8.18. Claim 8.19 either says that \( \rho[L+1] = \rho \) is good if \( u = R \) (and we are done), or \( \rho[r_u+1] \) is good if \( u < R \).

9. A nonvanishing result about multiple Newell-Littlewood numbers

To prove Theorem 1.5 we need:
Theorem 9.1. Let $1 \leq r < n$ and $\lambda, \mu, \nu \subset (n-r)^r$ be such that $c_{\lambda, \mu}^{\nu} > 0$. For $a', b', c' \in \mathbb{Z}_{\geq 0}$ such that $a' + b' + c' = r$, set
\[
\begin{align*}
\lambda^- &= (\lambda_{\leq a'})^{\nu[(n-r)^r]} \quad \lambda^+ = \lambda_{> a'} \\
\mu^- &= (\mu_{\leq b'})^{\nu[(n-r)^r]} \quad \mu^+ = \mu_{> b'} \\
\nu^- &= (\nu_{\leq c'})^{\nu[(n-r)^r]} \quad \nu^+ = \nu_{> c'}
\end{align*}
\]
Then $N_{\lambda^-, \mu^-, \nu^-, \lambda^+, \mu^+, \nu^+} > 0$.

9.1. Proof of Theorem 9.1. We must show that there exist $\alpha_1, \ldots, \alpha_6 \in \text{Par}_r$ such that the Littlewood-Richardson coefficients corresponding to each of the six triangles in Figure 2 are positive. Set $a = b' + c'$, $b = a' + c'$, $c = a' + b'$. We begin with a special case:

9.1.1. The case when $\lambda^+ = \emptyset$. Since $\lambda^+ = \emptyset$, we need $\alpha_2 = \alpha_3 := \emptyset$ to get $c_{\alpha_2, \alpha_3}^{\lambda^+} > 0$. Also, $\alpha_1 := \mu^-$ and $\alpha_4 := \nu^-$ for $c_{\alpha_1, \alpha_2}^{\mu^+}, c_{\alpha_3, \alpha_4}^{\nu^+} > 0$. It remains to find $\alpha_5, \alpha_6$ such that
\begin{equation}
(61) \quad c_{\alpha_5, \alpha_6}^{\mu^+}, c_{\nu^-, \alpha_5}^{\nu^+}, c_{\alpha_5, \alpha_6}^{\lambda^-} > 0.
\end{equation}

Consider Figure 3. We numbered the regions by ① through ⑨ (some regions could be empty). We declare that $\mu$ is region ①④⑦. Also $\nu$ is region ③⑥⑨ rotated 180 degrees, denoted as rotate(③⑥⑨). Thus $\nu'/\mu$ is region ②⑤⑧.
For any partitions $\alpha_5, \alpha_6 \subseteq (n - r)^{a'}$, by (64), we have

\begin{align*}
(62) \quad c_{\mu^+, \alpha_6}^\alpha &= c_{\nu^+, \alpha_6}^\alpha \\
(63) \quad c_{\mu^-, \alpha_5}^\alpha &= c_{\nu^-, \alpha_5}^\alpha \\
(64) \quad c_{\lambda^-, \alpha_5, \alpha_6} &= c_{\lambda^-, \alpha_5, \alpha_6}^{-} \\
&= c_{\lambda^-, \alpha_5, \alpha_6}^\alpha \\
&= c_{\lambda^-, \alpha_5, \alpha_6}^{(n-r)^a \cup \lambda} \\
&= c_{\lambda^-, \alpha_5, \alpha_6}^{(n-r)^a \cup \lambda}
\end{align*}

Since $\lambda^+ = \emptyset$, 

\[(\lambda^-)^{(n-r)^a} = \lambda.\]

Combining this with (64) gives

\[(65) \quad c_{\lambda^-, \alpha_5, \alpha_6}^{(n-r)^a \cup \lambda} = c_{\lambda^-, \alpha_5, \alpha_6}^{\lambda'.}
\]

Observe that 

\begin{align*}
\mu^+ &\text{ is } \{4, 7\} \\
\nu^+ &\text{ is rotate}(3, 6) \\
\mu^- &\text{ is rotate}(2, 3) \\
\nu^- &\text{ is } \{7, 8\}.
\end{align*}

Thus $\nu^+_{(n-r)^c} / \nu^-_{(n-r)^c'}$ is $\{2, 4, 5\}$, and $(\mu^+)^{(n-r)^h} / (\nu^-)^{(n-r)^y'}$ is rotate$(5, 6, 8)$. By (62), (63) and (65), if there exists

1. a ballot $U \in \text{SSYT}(\{2, 4, 5\}, \alpha)$ with $\alpha \in \text{Par}_{a'}$; 
2. a ballot $V \in \text{SSYT}(\{5, 6, 8\}, \beta)$ with $\beta \in \text{Par}_{a'}$ such that
3. $c_{\alpha_5, \alpha_6}^{(n-r)^a \cup \lambda} > 0$,

then $\alpha_6 = \alpha^{\nu^{(n-r)^a}}$ and $\alpha_5 = \beta^{\nu^{(n-r)^a}}$ will satisfy (61). (In (ii) we have used the LR-symmetry $c_{\alpha, \beta}^{\nu} = c_{\beta, \alpha}^{\nu}$.)

Two cases occur:

**Case 1 (region 5 contains a column of length $a'$):** Since $\lambda^+ = \emptyset$, therefore the longest column of $\{2, 5, 8\}$ is also of length $a'$. Now, choose any such column; say its index is $\text{col}_1$. Notice that such a column divides region 5 into two parts: namely, $5' =$ the part of region 5 weakly left of $\text{col}_1$ and $5'' =$ the remainder. Now by Proposition 7.11 (and its proof), there exist partitions $\rho, \pi$ where $\rho$ is the content of a ballot tableau $U'$ whose shape is region $\{2, 5, 8\}$ and $\pi$ is the content of a ballot tableau $V'$ whose shape is region $\{5', 8\}$ such that

\[(66) \quad c_{\lambda^+, \rho, \pi}^\lambda > 0.
\]

In particular, $\rho, \pi \in \text{Par}_{a'}$.

Since $(\nu^+)^{(n-r)^c} / (\nu^-)^{(n-r)^y'}$ (region $\{2, 4, 5\}$) is obtained by adding a rectangle $(\text{col}_1)^{a'}$ to the left end of region $\{2, 5, 8\}$, we can obtain a ballot tableau $U \in \text{SSYT}(\{2, 4, 5\}, \rho + (\text{col}_1)^{a'})$ from $U'$ by filling in each column of the rectangle $(\text{col}_1)^{a'}$ with $[a']$. This is (i).
Similarly, since region \((5\,6\,8)\) is obtained by adding a rectangle \((n-r-\text{col}_1)a'\) to the right end of region \((5'\,8)\), we can construct a ballot tableau
\[
V \in \text{SSYT}(5\,6\,8, \pi + (n-r-\text{col}_1)a')
\]
from \(V'\) by filling in each column of \((n-r-\text{col}_1)a'\) by \([a']\). This is (ii).

By (66) and Lemma 7.9 where we set \(k = 0\),
\[
c_{\rho + (\pi'') \cup \text{col}_1, \pi + (\rho'')} > 0.
\]
Now since \(\pi_1 \leq \text{col}_1\) and \(\rho_1 \leq n-r-\text{col}_1\), it follows that by setting \(\alpha = \rho + (\text{col}_1 a')\) and \(\beta = \pi + (n-r-\text{col}_1)a'\),
\[
c_{\alpha, \beta} > 0,
\]
as required by (iii).

**Case 2 (no column in region (5) has length \(a'\)):** Let \(\text{col}_1 = (\nu'')_{\mu+a'}\) and \(\text{col}_2 = \mu_{\nu+1}\). In Figure 4(A), \(\text{col}_1\) is the index of the right-most column that does not intersect region \((6)\). If no such column exists, then \(\text{col}_1 = 0\). Similarly, \(\text{col}_2\) is the index of the right-most column that intersect region \((4)\). If no such column exists, then \(\text{col}_2 = 0\).

Set \(k = \text{col}_2 - \text{col}_1\). Since \(c_{\lambda, \mu} > 0\), by Lemma 7.12
\[
(67)
c_{\lambda, \mu} > 0.
\]
The skew shape \((\nu'' + (k(a'+b')))/(\mu + (k''))\) is depicted in Figure 4(B). Set \((5')\) to be the region in the new region \(5\) whose column is weakly less than \(\text{col}_2' = \text{col}_2 = \text{col}_1 + k\) and \((5'')\) to be the remaining of the new region \(5\). By (67) there is a ballot \(V \in \text{SSYT}(2\,5'(5'')\,8, \lambda + (k''))\). As in Case 1, by Proposition 7.11 we can find two partitions \(\rho, \pi\) where \(\rho\) is the content of a ballot tableau \(U'\) whose shape is region \(2\,5''\) and \(\pi\) is the content of a ballot tableau \(V'\) whose shape is region \(5'(8)\) such that
\[
(68)
c_{\rho, \pi} > 0.
\]
Since \((\nu')_+(n-r-c')/((\mu')_+(n-r))\) (region \(2\,4\,5\)) is obtained by attaching a rectangle \((\text{col}_1 a')\) to the left end of region \(2\,5''\), we can construct a ballot tableau
\[
U \in \text{SSYT}(2\,4\,5, \rho + (\text{col}_1 a'))
\]
from \(U'\) by filling in each column of the rectangle \((\text{col}_1 a')\) by \([a']\). This is (i).

Similarly, since region \((5\,6\,8)\) is obtained by adding a rectangle \((n-r-\text{col}_2)a'\) to the right end of region \((5'\,8)\), we obtain a ballot tableau
\[
V \in \text{SSYT}(5\,6\,8, \pi + (n-r-\text{col}_2)a')
\]
from \(V'\) by filling in each column of \((n-r-\text{col}_2)a'\) by \([a']\). This is (ii).

Notice that since \(\rho_1 \leq n-r-\text{col}_1\) and \(\pi_1 \leq \text{col}_2\), we have
\[
\rho_1 - k = \rho_1 - \text{col}_2 + \text{col}_1 \leq n-r-\text{col}_2\) and \(\pi_1 - k = \pi_1 - \text{col}_2 + \text{col}_1 \leq \text{col}_1\).

Furthermore, since
\[
(68)
\lambda_1 \leq n-r-(n-r-\text{col}_2) + \text{col}_1 + k;
\]
we can apply Lemma 7.9 to (68) to see
\[ c^{(n-r)a' \cup \lambda}_{\rho+(\text{col}_{a'} \pi+(n-r-\text{col}_2)a')} > 0. \]

Thus (iii) holds by setting \( \alpha = \rho + (\text{col}_{a'}) \) and \( \beta = \pi + (n-r-\text{col}_2)a' \).

9.1.2. The general case. Since \( c_{\lambda,\mu}^{\nu} > 0 \), by Theorem 7.1, there is a ballot \( T \in \text{SSYT}(\nu^{\vee}/\mu, \lambda) \).

Let \( \tilde{T} = T^{\leq a'} \cup T_{\leq a'+b'} \) be the subtableau of \( T \) that keeps only the first \( c = a'+b' \) rows, together with the boxes of \( T \) whose entries are at most \( a' \). Notice that \( \tilde{T} \in \text{SSYT}(\nu^{(1)}/\mu, \lambda^{(1)}) \) is ballot, where \( \nu^{(1)}, \lambda^{(1)} \) are some partitions with \( \lambda^{(1)} \in \Par_{a'+b'}, \nu^{(1)} \subset \nu^{\vee} \) and \( \lambda^{(1)} \subset \lambda \).

By Theorem 8.15, we can find a partition \( \rho \) such that
\[ (69) \quad c^{\nu^{(1)}}_{\rho, (\lambda^{(1)}) \leq a'}, \quad c^{\rho}_{\mu, (\lambda^{(1)}) > a'}, \quad \rho > b' = \mu > b'. \]

Set
\[ (70) \quad \alpha_1 = (\rho \leq b')^{\nu^{\vee}[n-r]b'} \text{ and } \alpha_2 = (\lambda^{(1)}) > a'. \]

Since \( \rho/\mu = \rho_{\leq b'}/\mu_{\leq b'} \), by the second non-vanishing in (69),
\[ (71) \quad c_{\alpha_1, \alpha_2}^{\mu} > 0. \]

Consider \( T^{>a'} \subset T \), the tableau obtained from \( T \) by only keeping the entries that are greater than \( a' \). Since \( T \) is ballot, for any \( k > a' \) and any initial segment of the column (or row) reading word of \( T \), \( k \) appears at least as much as \( k+1 \). Thus \( k \) appears at least as much as \( k+1 \) in any initial segment of the column (or row) reading word of \( T^{>a'} \). Hence, \( \overline{T^{>a'}} = \text{the tableau obtained from } T^{>a'} \text{ by subtracting } a' \text{ from every entry}, \) is ballot.

Let \( U_1 \in \text{SSYT}(\rho/\mu, (\lambda^{(1)})_{>a'}) \) be a ballot tableau that witnesses the second non-vanishing in (69). Let \( U_2 \) be the tableau of shape \( \nu^{\vee}/\nu^{(1)} \) where for each box \( (i, j) \in \nu^{\vee}/\nu^{(1)} \), we set
\[ U_2(i, j) = T(i, j) - a'. \]

Since \( T(i, j) \geq a' \) for all \( (i, j) \in \nu^{\vee}/\nu^{(1)} \),
\[ U_2(i, j) = \overline{T^{>a'}}(i, j). \]
Let \((\overline{T}^{a'})_{b'+a'}^a\) and \((\overline{T}^{a'})_{b'+a'}^a\) be the first \(b'+a'\) and bottom \(c'\) rows of \(\overline{T}^{a'}\) respectively. Notice in particular that \((\overline{T}^{a'})_{b'+a'}^a = U_2\), and thus

\[
\text{revrowword}(\overline{T}^{a'})_{b'+a'}^a = \text{revrowword}(U_2).
\]

Since \(\overline{T}^{a'}\) is ballot, so is \((\overline{T}^{a'})_{b'+a'}^a\). Moreover, by the definition of \(\lambda^{(1)}\), we have

\[
\text{cont}((\overline{T}^{a'})_{b'+a'}^a) = (\lambda^{(1)})_{b'+a'} = \text{cont}(U_1).
\]

Since \(U_1\) is also ballot,

\[
\text{revrowword}(\overline{T}^{a'})_{b'+a'}^a \equiv_K \text{revrowword}(U_1).
\]

By ballotness of \(\overline{T}^{a'}\), we have

\[
\text{revrowword}(Y_{\lambda > a'}) \equiv_K \text{revrowword}(\overline{T}^{a'}) = \text{revrowword}(\overline{T}^{a'}) \cdot \text{revrowword}(\overline{T}^{a'}) \cdot \text{revrowword}(U_1).
\]

Combining with (72) and (73), we reach

\[
\text{revrowword}(Y_{\lambda > a'}) \equiv_K \text{revrowword}(U_2) \cdot \text{revrowword}(U_1).
\]

As a result, by Theorem 7.5, \(U_2 \ast U_1\) is ballot and \(\text{cont}(U_2 \ast U_1) = \lambda_{>a'} = \lambda^+\).

Then, by the construction in Proposition 7.11, there exists \(\beta\) such that

\[
\nu^y_{\nu^{(1)}, \beta} > 0 \quad \text{and} \quad (\lambda^{(1)})_{>a'} > 0.
\]

Setting \(\alpha_3 = \beta\), one gets

\[
\nu^{(1)}_{\lambda^{(1)}, \alpha_3} > 0
\]

Set now

\[
\alpha_4 = (\nu^{(1)})_{>c}.
\]

Since \((\nu^{(1)})_{\leq c} = (\nu^y)_{\leq c}\), we have \(\nu^-/\alpha_4 = \nu^y/\nu^{(1)}\) and

\[
\nu^{y}_{\nu^{(1)}, \alpha_4} > 0.
\]

We just constructed \(\alpha_1, \ldots, \alpha_4\) such that the three LR-coefficients associated to \(\lambda^+, \mu^-\) and \(\nu^-\) do not vanish. Consider now three alternate partitions

\[
\bar{\lambda} = \lambda_{\leq a'}, \bar{\mu} = \mu, \text{ and } \bar{\nu}^{y([n-r])} = \nu^{(1)}.
\]

and define \(\bar{\lambda}^+, \bar{\lambda}^-, \bar{\mu}^+, \bar{\mu}^-, \bar{\nu}^+, \bar{\nu}^-\) as in Theorem 9.1 accordingly. In fact, we have

\[
\begin{align*}
\bar{\lambda}^- &= \lambda^- & \bar{\lambda}^+ &= \emptyset \\
\bar{\mu}^- &= \alpha_4 & \bar{\mu}^+ &= \mu^+ \\
\bar{\nu}^- &= \alpha_4 & \bar{\nu}^+ &= \nu^+.
\end{align*}
\]

By (69),

\[
\nu^{y}_{\bar{\lambda}, \bar{\mu}} > 0.
\]

We can now apply Section 9.1.1 to get

\[
N_{\bar{\lambda}^-, \bar{\mu}^+, \bar{\lambda}^+, \bar{\mu}^-, \bar{\nu}^+, \bar{\nu}^-} > 0.
\]

In particular, we have

\[
\bar{\alpha}_1 = \alpha_1, \bar{\alpha}_2 = \emptyset, \bar{\alpha}_3 = \emptyset, \bar{\alpha}_4 = \alpha_4.
\]
Therefore,
\[ c_{\alpha_1, \alpha_6}^+ = c_{\alpha_1, \alpha_6}^- > 0, \]
\[ c_{\alpha_4, \alpha_5}^+ = c_{\alpha_4, \alpha_5}^- > 0, \]
\[ c_{\alpha_5, \alpha_6}^+ = c_{\alpha_5, \alpha_6}^- > 0. \]
Combined with (71), (74) and (76), we showed that the sextuple \((\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)\) makes all six Littlewood-Richardson coefficients in Figure 2 non-zero. □

10. Extended Horn inequalities and the proof of Theorem 1.5

10.1. Extended Horn inequalities. We recall the following notion from [10]:

**Definition 10.1.** An extended Horn inequality on \(\text{Par}_n^3\) is
\[ 0 \leq |\lambda_A| - |\lambda_{A'}| + |\mu_B| - |\mu_{B'}| + |\nu_C| - |\nu_{C'}| \]
where \(A, A', B, B', C, C' \subseteq [n]\) satisfy
(1) \(A \cap A' = B \cap B' = C \cap C' = \emptyset\)
(2) \(|A| = |B'| + |C'|, |A'| = |B| + |C|\)
(3) There exists \(A_1, A_2, B_1, B_2, C_1, C_2 \subseteq [n]\) such that:
   (i) \(|A_1| = |A_2|, |B_1| = |B_2|, |C_1| = |C_2|\)
   (ii) \(c_{\tau(A_1), \tau(A_2)}^{\tau(A')}, c_{\tau(B_1), \tau(B_2)}^{\tau(B')}, c_{\tau(C_1), \tau(C_2)}^{\tau(C')} > 0\)
   (iii) \(c_{\tau(B_1), \tau(A_2)}^{\tau(B')}, c_{\tau(C_1), \tau(A_2)}^{\tau(C)} > 0\).

**Definition 10.2.** The extended Horn cone is:
\[ \text{EH}(n) := \{(\lambda, \mu, \nu) \in (\text{Par}_n^3)^3 : \text{inequalities (77) are satisfied}\}. \]

Let
\[ \overline{\text{EH}}(n) = \text{EH}(n) \cap \{(\lambda, \mu, \nu) \in (\text{Par}_n^3)^3 : |\lambda| + |\mu| + |\nu| \text{ is even}\}. \]

**Conjecture 10.3 ([10], Conjecture 1.4).** If \((\lambda, \mu, \nu) \in \overline{\text{EH}}(n)\) then \(N_{\lambda, \mu, \nu} > 0\).

We will prove a weakened version of Conjecture 10.3.

**Theorem 10.4** (cf. [10], Conjecture 1.4). \(\text{EH}(n) = \text{NL-sat}(n)\).

Consequently, we are able to answer an issue raised in [10] Section 1:

**Corollary 10.5.** Conjecture 1.4 implies Conjecture 10.3.

Corollary 10.5 is analogous to the situation in Zelevinsky’s [29], before [15].

The following shows that Theorem 1.5 is equivalent to Theorem 10.4.

**Lemma 10.6.** A sextuple \((A, A', B, B', C, C')\) of subsets of \([n]\) parametrizes an extended Horn inequality if and only if it appears in Theorem 1.5

**Proof.** Definition 10.1 implies
\[ c_{\tau(A_1), \tau(A_2)}^{\tau(A')}, c_{\tau(B_1), \tau(B_2)}^{\tau(B')}, c_{\tau(C_1), \tau(C_2)}^{\tau(C')} > 0. \]
Since \(\tau(A_1), \tau(A_2)\ldots\) have length at most \(n\), this implies \(N_{\tau(A'), \tau(B'), \tau(C'), \tau(A), \tau(B'), \tau(C)} \neq 0.\)
Conversely, if \( N_{\tau(A'),\tau(B),\tau(C'),\tau(A),\tau(B'),\tau(C)} \neq 0 \), there exists \( \alpha_1, \alpha_2, \ldots, \alpha_6 \in \text{Par}_n \) such that

\[
\begin{array}{l}
c_{\alpha_1,\alpha_2} c_{\alpha_2,\alpha_3} c_{\alpha_3,\alpha_4} c_{\alpha_4,\alpha_5} c_{\alpha_5,\alpha_6} c_{\alpha_6,\alpha_1} > 0.
\end{array}
\]

Set \( a' = |A'| \). Then, the Young diagram of \( \tau(A') \) is contained in the rectangle \( a' \times (n - a') \). But the nonvanishing of \( c_{\alpha_1,\alpha_2}' \) implies that \( \alpha_1 \subset \tau(A') \). Hence there exists \( A_1 \subseteq [n] \) such that \( \tau(A_1) = \alpha_1 \). Similarly, we can pick \( A_2, B_1, B_2, C_1, C_2 \subseteq [n] \) such that \( \tau(A_2) = \alpha_2, \tau(C_1) = \alpha_3 \) etc. that satisfy Definition 10.1.

10.2. Proof of Theorem 1.5. (\( \Rightarrow \)) By Lemma 10.6, \( \text{EH}(n) \) is the cone defined by the inequalities in Theorem 1.5. Now, \( \text{NL-sat}(n) \subseteq \text{EH}(n) \) is immediate from [10, Theorem 1].

(\( \Leftarrow \)) Fix an inequality (4) associated to \( (A, A', B, B', C, C') \) appearing in Theorem 1.2. We prove that this inequality appears in Theorem 1.5, that is

\[
N_{\tau(A'),\tau(B),\tau(C'),\tau(A),\tau(B'),\tau(C)} \neq 0.
\]

Set \( r = |A| + |A'| \). Let \( I \in \text{Schub}(\text{Gr},(r, 2n)) \) be associated to \( (A, A') \in \text{Schub}(\text{Gr},(r, 2n)) \), under (22). Similarly define \( J \) and \( K \). Now apply Theorem 9.1 with \( \lambda = \tau(I), \mu = \tau(J) \) and \( \nu = \tau(K) \), and \( a' = |A'|, b' = |B'| \) and \( c' = |C'| \) (refer to Figure 1) to see, \( \tau(A') = \lambda^-, \tau(B) = \mu^+ \) etc. The assumption \( a' + b' + c' = r \) comes from condition (2) in Theorem 1.5. The assumption \( c'_{\lambda^-}\mu > 0 \) is implied by condition (3) in Theorem 1.2 combined with the semigroup property of LR coefficients. Specifically, we are claiming, from the definitions that \( \tau^0(C, C') + \tau^2(C, C') = \tau(K) = \nu^\vee \) etc. By Theorem 9.1, (79) holds, since, \( \lambda^- = \tau(A'), \lambda^+ = \tau(A), \) etc., as desired.

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