Reconstruction of Randomly Sampled Sparse Signals Using an Adaptive Gradient Algorithm

Ljubiša Stanković, Miloš Daković

Abstract

Sparse signals can be recovered from a reduced set of samples by using compressive sensing algorithms. The case when available samples are a random subset of a uniformly or nonuniformly sampled signal is considered in this paper. A recalculation procedure is used to reconstruct the nonuniformly sampled signal. Signal recovery is done using an adaptive gradient-based algorithm in the time domain. A new criterion for the parameter adaptation in this algorithm, based on the gradient direction angles, is proposed. It improves the algorithm computational efficiency. The methods are illustrated on statistical examples.

1. Introduction

A discrete-time signal can be transformed into other domains in various ways. Some signals that cover whole considered interval in one domain could be sparse in a transformation domain. Compressive sensing theory, in general, deals with a lower dimensional set of linear observations of a sparse signal, in order to recover its all values [1]-[26]. This area intensively develops in the last decade. A common form of observations are the signal samples. A reduced set of samples is considered in the compressive sensing in order to represent a signal with the lowest possible number of samples and to achieve compressed sensing. This theory may also be applied when the signal samples are missing due to their physical constraints or unavailability of the measurements. In applications it could happen that some arbitrarily positioned samples of a signal are so heavily corrupted by disturbances that it is better to omit them and consider as missing in the analysis [6], [27]. Several approaches to reconstruct sparse signals from their lower dimensional set of linear observations are introduced [8]-[19]. The most common are the reconstruction algorithms based either on the gradient formulations [8] or the orthogonal matching pursuit approaches [18].

In this paper, sparse signals with available samples being a random subset of uniformly or nonuniformly sampled signal, are considered [28]-[32]. A possibility to recalculate the signal sample values at the sampling theorem positions is exploited in the nonuniform case before a reconstruction algorithm is applied [33]. For the signal reconstruction a simple adaptive gradient-based algorithm in the time domain is used [26]. This algorithm uses the sparsity measure function in a direct way to reconstruct the missing samples through adaptive iterations in the time domain. A new criterion for the parameter adaptation of this simple algorithm, based on the gradient directions, is proposed. Computational time to achieve the target reconstruction accuracy is significantly reduced. The Fourier transform domain is used as a study case, although the algorithm application is not restricted to this transform. The algorithm efficiency is statistically checked.

The paper is organized as follows. After the definitions in the next section, the adaptive gradient algorithm, with a new criterion for the algorithm parameter adaptation, is presented in Section 3. Reconstruction of nonuniformly sampled sparse signals is considered in Section 4.

2. Definitions

Consider a discrete-time signal \( x(n) \) obtained by sampling a continuous-time signal \( x(t) \). Since the DFT will be used in the analysis, we can assume that the continuous-time signal is periodically extended with a period \( T \). According to the sampling theorem, the period \( T \) is related to the number of samples \( N \), the
sampling interval $\Delta t$, and the maximal frequency $\Omega_m$ as $\Omega_m = \pi/\Delta t = \pi N/T$. The continuous-time signal can be written as an inverse Fourier series

$$x(t) = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kt/T},$$

with the Fourier series coefficients being related to the DFT as $X_k N = X(k) = \text{DFT}[x(n)]$ and $x(n) = x(n\Delta t)$. The discrete-time index $n$ corresponds to the continuous-time instant $t = n\Delta t$. Discrete-frequency indices are $k \in \{- (N - 1)/2, \ldots, -1, 0, 1, \ldots, (N - 1)/2\}$. Any signal value can be reconstructed from the samples taken according to the sampling theorem,

$$x(t) = \sum_{n=0}^{N-1} x(n\Delta t) \frac{\sin[(n - \frac{1}{2N})\pi]}{N \sin[(n - \frac{1}{2N})\pi/N]}.$$  

This relation holds for an odd $N$. Slightly corrected relation holds for an even $N$, [33], [7].

A signal $x(t)$ is sparse in the transformation domain if the number of nonzero transform coefficients $s$ is much lower than the number of the original signal samples $N$ within $T$, $s \ll N$, i.e., $X_k = 0$ for $k \notin \{k_1, k_2, \ldots, k_s\}$. A signal

$$x(t) = \sum_{k \in \{k_1, k_2, \ldots, k_s\}} X_k e^{j2\pi kt/T}.$$  

of sparsity $s$ can be reconstructed from $M$ samples, where $M < N$, if the recovery conditions are met.

3. Gradient-based Reconstruction

Assume that a set of $M < N$ signal samples in the time domain is available at the instants corresponding to the discrete-time positions

$$n_i \in \mathbb{N}_A = \{n_1, n_2, \ldots, n_M\} \subset \mathbb{N} = \{0, 1, 2, \ldots, N - 1\}.$$  

In general, the signal recovery within the compressive sensing framework consists in reconstructing the signal (calculation of missing/unavailable/discarded samples) so that the number of nonzero transform coefficients $X(k)$ is minimal, subject to the available sample values. Counting of the nonzero transform coefficients is achieved by a simple mathematical form $\sum_{k=0}^{N-1} |X(k)|^0$, sometimes referred to as the “$\ell_0$-norm”, [1], [2]. Thus, the problem statement is

$$\min_{X(k)} \sum_{k=0}^{N-1} |X(k)|^0 \text{ subject to } y = AX,$$

where $X = [X(1) \ X(2) \ \ldots \ X(N)]^T$ is the vector of unknown transform coefficients, $y = [x(n_1) \ x(n_2) \ \ldots \ x(n_M)]^T$ is the vector of the available signal samples, and $A$ is the inverse transform matrix with omitted rows corresponding to the unavailable signal samples. By definition $|X(k)|^0 = 0$ for $X(k) = 0$. The $\ell_0$-norm based formulation is an NP-hard combinatorial optimization problem. Its calculation complexity is of order $(N_s)^N$. In theory, the NP-hard problems can be solved by an exhaustive search. However, as the problem parameters $N$ and $s$ increase the running time increases and the problem becomes unsolvable. Moreover, in practical signal processing applications, the $\ell_0$-norm cannot be used even for small values of parameters $N$ and $s$, when the combinatorial approach could be used within a reasonable computational time. In all real-life signals at least a very small, but nonzero, A/D quantization noise exists. Even if all values of an originally sparse signal in the DFT domain are known, the $\ell_0$-norm of $X(k)$ for a signal with a finite precision samples will be equal to $N$ (probability that a transform coefficient with a random error is equal to 0 is a zero probability event).
These are the reasons why the $\ell_1$-norm of the signal transform is commonly used as a sparsity measure function. The minimization problem is

$$\min_{N-1} \sum_{k=0}^{N-1} |X(k)| \quad \text{subject to} \quad y = AX.$$  \hfill (6)

This minimization problem, under the conditions defined within the restricted isometry property (RIP), produces the same result as (5). Note that other norms $\ell_p$ between the $\ell_0$-norm and the $\ell_1$-norm, with values $0 < p < 1$, are also used in the minimization in attempts to combine good properties of these two norms [1, 26, 34].

### 3.1. Algorithm

A simple gradient-based algorithm, to iteratively calculate the missing signal samples according to (6), is presented next. The basic idea for the algorithm comes from the gradient-based minimization approaches. The missing samples are considered as variables. The influence of their change on the sparsity measure is checked by appropriate variations of the missing sample values [26]. By performing an iterative procedure, the missing samples are changed toward the sparsity measure lower values, in order to approach the minimum of the $\ell_1$-norm based sparsity measure (6). If the recovery conditions are met then the $\ell_1$-norm minimum will be at the same position as the $\ell_0$-norm minimum, representing the true values of the missing samples.

The initial signal $y^{(0)}(n)$ is defined for $n \in \mathbb{N} = \{0, 1, \ldots, N-1\}$ as:

$$y^{(0)}(n) = \begin{cases} 0 & \text{for missing samples}, \ n \in \mathbb{N}_A \\ x(n) & \text{for available samples}, \ n \in \mathbb{N}_A \end{cases},$$  \hfill (7)

where $\mathbb{N}_A$ is the complement of $\mathbb{N}_A$ with respect to $\mathbb{N}$ [4].

The missing signal samples are then corrected in an iterative procedure as

$$g^{(m)}(n) = y^{(m-1)}(n) - y^{(m)}(n),$$  \hfill (8)

where $g^{(m)}(n)$ is an estimate of the sparsity measure gradient vector coordinate along the variable $y(n)$ direction, in the $m$th iteration. At the positions of the available signal samples, $n \in \mathbb{N}_A$, $g^{(m)}(n) = 0$.

At the positions of missing samples, $n_i \in \mathbb{N}_A$, its values are calculated by changing the signal values and forming new signals $y_1(n)$ and $y_2(n)$ as

$$y_1(n) = y^{(m)}(n) + \Delta \delta(n - n_i)$$

$$y_2(n) = y^{(m)}(n) - \Delta \delta(n - n_i).$$  \hfill (9)

The algorithm step is denoted by $\Delta$. The values of $g^{(m)}(n_i)$ at $n_i \in \mathbb{N}_A$ are

$$g(n_i) = \frac{\sum_{k=0}^{N-1} |Y_1(k)| - \sum_{k=0}^{N-1} |Y_2(k)|}{N},$$  \hfill (10)

where $Y_1(k) = \text{DFT}[y_1(n)]$ and $Y_2(k) = \text{DFT}[y_2(n)]$.

The initial value for the algorithm adaptation step $\Delta$ is estimated as

$$\Delta = \max_{n \in \mathbb{N}_A} |x(n)|.$$

(11)

The gradient algorithm will approach the minimum point of the $\ell_1$-norm based sparsity measure with a precision related to the algorithm step $\Delta$. Rate of the algorithm convergence for different steps is considered in [26]. The algorithm performance is significantly improved by using adaptive step $\Delta$. A criterion that efficiently detects the event that the algorithm has reached the vicinity of the sparsity measure minimum is proposed in this paper. It is based on the direction change of the gradient vector. When the vicinity of the optimal point is reached, the gradient estimate in the $\ell_1$-norm based sparsity measure function changes...
direction for almost 180 degrees. For each two successive gradient estimations \( g^{(m-1)}(n) \) and \( g^{(m)}(n) \), the angle \( \beta_m \) between gradient vectors is calculated as

\[
\beta_m = \arccos \frac{\sum_{n=0}^{N-1} g^{(m-1)}(n)g^{(m)}(n)}{\sqrt{\sum_{n=0}^{N-1} (g^{(m-1)}(n))^2} \sqrt{\sum_{n=0}^{N-1} (g^{(m)}(n))^2}}
\]

If the angle \( \beta_m \) is above 170° it means that the values reached oscillatory nature around the minimal measure value position. When this kind of the angle change is detected the step \( \Delta \) is reduced, for example, \( \Delta/\sqrt{10} \to \Delta \), and the same calculation procedure is continued from the reached reconstructed signal values.

When the optimal point is reached with a sufficiently small \( \Delta \), then this value of \( \Delta \) is also an indicator of measure value position. When this kind of angle change is detected the step \( \Delta \) is reduced, for example, \( \Delta/\sqrt{10} \to \Delta \), and the same calculation procedure is continued from the reached reconstructed signal values.

A common way to estimate the precision of the result in iterative algorithms is based on the change of the result in the last iteration. An average of changes in last iteration in a large number of missing samples is a good estimate of the achieved precision. Thus, the value of

\[
T_r = 10 \log_{10} \frac{\sum_{n \in \mathbb{N}_A} |y_p(n) - y^{(m)}(n)|^2}{\sum_{n \in \mathbb{N}_A} |y^{(m)}(n)|^2}
\]

can be used as a rough estimate of the reconstruction error to signal ratio. Here \( y_p(n) \) is the reconstructed signal prior to execution of the algorithm inner loop (lines 7-20 in Algorithm 1) and \( y^{(m)}(n) \) is the reconstructed signal after the inner loop execution. This value can also be used as a criterion to stop the algorithm. If \( T_r \) is above the required precision threshold \( T_{\max} \) (for example, if \( T_r > -100dB \)), the calculation procedure should be repeated with smaller values \( \Delta \).

A pseudo code of this algorithm is presented in Algorithm 1.

**Comments on the algorithm:**
- The inputs to the algorithm are the signal length \( N \), the set of available samples \( \mathbb{N}_A \), the available signal values \( x(n_i), n_i \in \mathbb{N}_A \), and the required precision \( T_{\max} \).
- Instead of calculating signals \( \bar{y} \) and their DFTs for each \( n_i \in \mathbb{N}_A \) we can calculate

\[
|Y_1(k)| = |Y^{(m)}(k) + \Delta D_{n_i}(k)|
\]

\[
|Y_2(k)| = |Y^{(m)}(k) - \Delta D_{n_i}(k)|
\]

with \( Y^{(m)}(k) = \text{DFT}[y^{(m)}(n)] \) and \( D_{n_i}(k) = \text{DFT}[\delta(n - n_i)] = \exp(-j2\pi n_i k/N) \), for each \( n_i \in \mathbb{N}_A \). Since \( D_{n_i}(k) \) are independent of the iteration number \( m \) they can be calculated independently from the DFT of the signal.

- In a gradient-based algorithm, a possible divergence is related to the algorithm behavior for a large step \( \Delta \). Small steps influence the rate of the algorithm approach to the solution only, with the assumption that it exists. Influence of small steps to the calculation complexity is considered in [26]. Here, we will examine the algorithm behavior for a large value of step \( \Delta \). We can write

\[
|Y_1(k)| - |Y_2(k)| = \left| Y^{(m)}(k) + \Delta D_{n_i}(k) \right| - \left| Y^{(m)}(k) - \Delta D_{n_i}(k) \right|
\]

\[
= \Delta |D_{n_i}(k)| \left( 1 + \frac{Y^{(m)}(k)}{\Delta D_{n_i}(k)} \right) \left( \frac{1}{1 + \frac{Y^{(m)}(k)}{\Delta D_{n_i}(k)}} - \frac{1}{1 - \frac{Y^{(m)}(k)}{\Delta D_{n_i}(k)}} \right).
\]

Considering the complex number \( a = Y^{(m)}(k)/(\Delta D_{n_i}(k)) \), with \( |a| \ll 1 \) for a large \( \Delta \), from the problem geometry it is easy to show that the following bounds hold

\[
0 \leq |1 + a| - |1 - a| \leq 2|a|
\]

Exact value of this expression depends on the phase of \( a \). Therefore,

\[
0 \leq |Y_1(k)| - |Y_2(k)| \leq 2|Y^{(m)}(k)|.
\]

Lower limit 0 is obtained if \( a \) is imaginary-valued, while the upper limit \( 2|Y^{(m)}(k)| \) follows if \( a \) is real-valued.
Algorithm 1 Reconstruction

Require:
• Set of missing/omitted sample positions $N_x$
• Available samples $x(n)$, $n \notin N_x$

1: Set $y^{(0)}(n) ← x(n)$ \textcolor{gray}{\triangleright for $n \notin N_x$}
2: Set $y^{(0)}(n) ← 0$ \textcolor{gray}{\triangleright for $n \in N_x$}
3: Set $m ← 0$
4: Set $Δ ← \max \{y^{(0)}(n)\}$
5: repeat
6: Set $y_p(n) = y^{(m)}(n)$ \textcolor{gray}{\triangleright for each $n$}
7: repeat
8: $m ← m + 1$
9: for $n_i ← 0$ to $N - 1$ do
10: \textcolor{gray}{if $n_i \in N_x$ then}
11: \hspace{1em} $Y_1(k) ← \text{DFT}\{y^{(m)}(n) + Δδ(n - n_i)\}$
12: \hspace{1em} $Y_2(k) ← \text{DFT}\{y^{(m)}(n) - Δδ(n - n_i)\}$
13: \hspace{1em} $g^{(m)}(n_i) ← \frac{1}{N} \sum_{k=0}^{N-1} |Y_1(k)| - |Y_2(k)|$
14: \textcolor{gray}{else}
15: \hspace{1em} $g^{(m)}(n_i) ← 0$
16: \textcolor{gray}{end if}
17: \hspace{1em} $y^{(m+1)}(n_i) ← y^{(m)}(n_i) - g^{(m)}(n_i)$
18: \textcolor{gray}{end for}
19: $β_m = \arccos\left(\frac{\sum_{n=0}^{N-1} g^{(m-1)}(n)g^{(m)}(n)}{\sqrt{\sum_{n=0}^{N-1} (g^{(m-1)}(n))^2} \sqrt{\sum_{n=0}^{N-1} (g^{(m)}(n))^2}}\right)$
20: until $β_m < 170°$
21: $Δ ← Δ/\sqrt{10}$
22: $T_r = 10 \log_{10} \frac{\sum_{n\in N_x} |y_p(n) - y^{(m)}(n)|^2}{\sum_{n\in N_x} |y^{(m)}(n)|^2}$.
23: until $T_r < T_{max}$
24: return $y^{(m)}(n)$

Output:
• Reconstructed signal $y^{(m)}(n)$
Figure 1: Signal-to-reconstruction-error (SRR) obtained by using Algorithm 1, averaged over 100 realizations for various sparsity $s$ and number of available samples $M$: (a) The total number of samples is $N = 128$. (b) The total number of samples is $N = 64$. (c) With a Gaussian noise in the input signal, $SNR = 20 \,[dB]$ and $N = 64$.

It means that the value of the finite difference $|Y_1(k) - Y_2(k)|$, that is used to correct the missing signal samples, does not depend on the value of the step $\Delta$, if $\Delta$ is large. The missing signal values will be adapted for a value independent on $\Delta$ in that case. The values of missing samples will oscillate within the range of the original signal values of order $|Y^{(m)}(k)|/N$, until $\Delta$ is reduced in the iterations below the signal magnitude. Then the missing samples will start approaching to the position of the sparsity measure minimum. The initial values will be arbitrary changed within the signal amplitude order as far as $\Delta$ is too large. It will not influence further convergence of the algorithm, when the step $\Delta$ assumes appropriate values.

- Since two successive gradient vectors are required to calculate the gradient angle $\beta_m$, it is calculated starting from the second iteration for each $\Delta$.
- The algorithm output is the reconstructed signal $y(n)$, $n = 0, 1, \ldots, N - 1$.
- Other signal transforms can be used instead of the DFT. The only requirement is that signal is sparse in that transform domain.

**Example 1:** Consider a signal

$$x(t) = \sum_{i=1}^{K} A_i \cos(2\pi k_i t / T + \phi_i),$$

with $t = n\Delta t$, $\Delta t = 1$, and the total number of samples $N = T/\Delta t$. The sparsity parameter $s = 2K$ is changed from $s = 2$ to $s = N/2$. The amplitudes $A_i$, frequencies $k_i$, and phases $\phi_i$ are taken randomly. Amplitude values are modeled as Gaussian random variables with variance 1, the frequency indices assume random numbers within $1 \leq k_i \leq N - 1$, and the phases assume uniform random values within $0 \leq \phi_i \leq 2\pi$, in each realization. The reconstruction is performed by using 100 realizations for each $s$ with random sets of missing $N - M$ samples in each realization. The reconstructed signals $x_R(n)$ are obtained. The results are presented in Fig.1 in a form of the signal-to-reconstruction-error ratio (SRR) in [dB]

$$SRR = 10 \log \frac{\sum_{n=0}^{N-1} |x(n)|^2}{\sum_{n=0}^{N-1} |x(n) - x_R(n)|^2}. \quad (13)$$

Red colors indicate the region where the algorithm had fully recovered missing samples in all realizations, while blue colors indicate the region where the algorithm could not recover missing samples in any realization. In the transition region for $M$ slightly greater than $2s$ we have cases when the signal recovery is not achieved and the cases of full signal recovery. The simulations are done for $N = 128$ and for $N = 64$, Fig.1(a),(b). A stopping criterion for the accuracy of $120 \,[dB]$ is used. It corresponds to a precision in the recovered signal of an input samples precision if they are acquired by a 20-bit A/D converter. The case with $N = 64$ is
repeated with an additive input Gaussian noise such that the input signal-to-noise ratio is 20 [dB] in each realization Fig.1(c). The reconstruction error in this case is limited by the input signal-to-noise value.

The average reconstruction error in the noise-free cases is related to the number of the full recovery events. For $N = 64$ the number of the full recovery events is checked and presented in Fig.2 (a),(b). The average number of the algorithm iterations to produce the required precision, as a function of the number of missing samples and signal sparsity $s$, is presented as well, Fig.2(c), along with the corresponding average computation time (in seconds) for the Windows PC with Intel Dual Core processor, Fig.2(d). The average computation time is proportional to the average number of iterations multiplied by the number of missing samples (variables) $N - M$.

![Figure 2: (a)-(b) The percentage of the full recovery events as a function of the number of available samples $M$ and the sparsity $s$ in the case of $N = 64$. (c) The average number of iterations as a function of the number of missing samples and sparsity. (d) The average computation time.](image)

Calculation of the algorithm is very efficient. Finite difference method and the adaptation procedure presented in this paper overcome the problem of the derivative existence in the case of the $\ell_1$-norm near the optimal point. Efficiency of the presented algorithm is compared with the standard routines where the $\ell_1$-norm problem is solved using the linear programming, although the main contribution of the presented algorithm is in a different way of the solving the problem, by a direct adaptation of the signal values. Direct adaptation can be used in various applications (including recovery of nonuniformly sampled signals, detection of corrupt sample positions, or the solution of problems where a linear relation cannot be established. The performance of the proposed algorithm are compared with the algorithm that recasts the recovery problem (6) into a linear program and uses the primal-dual interior point method (L1-magic code in MATLAB). Both algorithms are run with the default parameters using 100 sparse signals with random parameters. The results are presented in the Table 1. Columns notation in the table is: $s$ for sparsity, $N - M$ for the number of missing samples, MAE stands for the mean absolute error, LP-DP denotes the values obtained by running the linear program dual-prime (MATLAB L1-Magic code), and AS is for the presented adaptive algorithm with variable step. Calculation time using MATLAB is presented in both cases.

An illustration of the algorithm performance regarding to the SRR and the gradient angle $\beta_m$ in one
Table 1: MAE and elapsed time for L1-magic (LP-DP) and proposed algorithm (AS)

| s   | N - M | MAE LP-DP        | MAE AS        | time LP-DP | time AS |
|-----|-------|------------------|---------------|------------|---------|
| 6   | 16    | 1.719 × 10^{-4}  | 3.959 × 10^{-7} | 0.043390  | 0.013433 |
| 10  | 16    | 1.124 × 10^{-4}  | 3.730 × 10^{-7} | 0.041121  | 0.013393 |
| 16  | 16    | 2.575 × 10^{-4}  | 5.943 × 10^{-7} | 0.041569  | 0.014003 |
| 6   | 32    | 3.238 × 10^{-4}  | 8.000 × 10^{-7} | 0.038492  | 0.025733 |
| 10  | 32    | 3.454 × 10^{-4}  | 1.133 × 10^{-6} | 0.038578  | 0.027270 |
| 16  | 32    | 1.068 × 10^{-3}  | 1.818 × 10^{-6} | 0.046595  | 0.029636 |
| 6   | 45    | 1.000 × 10^{-3}  | 1.295 × 10^{-6} | 0.041317  | 0.036442 |
| 10  | 45    | 4.731 × 10^{-3}  | 1.878 × 10^{-6} | 0.039162  | 0.041843 |
| 16  | 45    | 2.415 × 10^{-3}  | 2.751 × 10^{-6} | 0.042910  | 0.054350 |

Figure 3: Angle between successive gradient estimations $\beta_m$ and the signal-to-reconstruction-error ratio (SRR) as a function of the number of iterations in the algorithm for one signal realization with 6 nonzero DFT coefficients and $M = 64$.

3.2. Random Subset of Nonuniformly Sampled Values

Consider now a random set of possible sampling instants $\{t_1, t_2, ..., t_N\}$,

$$t_i = i\Delta t + \nu_i,$$

where $\nu_i$ is a uniform random variable $-\Delta t/2 \leq \nu_i \leq \Delta t/2$. Here $t_i$ denotes a time instant, while in the uniform sampling the discrete-time index $n_i$ has been used to indicate instant corresponding to $n_i\Delta t$. Assume that a random set of $M$ signal samples is available at

$$t_i \in T_A = \{t_1, t_2, ..., t_M\}.$$

Since the signal is available at randomly positioned instants the Fourier transform coefficients estimated as

$$X_k = \sum_{t_i \in T_A} x(t_i) \exp(-j2\pi k t_i/T).$$
will not be sparse even if a large number $M$ of samples is available. To improve the results, the problem can be reformulated to produce a sparse signal during the recovery process. If the signal values were available at $t_i \in T_A$ for $M = N$ the signal values at the sampling theorem positions could be recovered. The transformation matrix relating samples taken at $t_i$ with the signal values at the sampling theorem positions, according to (2), is

$$\begin{bmatrix}
x(t_1) \\
x(t_2) \\
\vdots \\
x(t_N)
\end{bmatrix} = \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1N} \\
b_{21} & b_{22} & \cdots & b_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
b_{N1} & b_{N2} & \cdots & b_{NN}
\end{bmatrix} \begin{bmatrix} x(1) \\
x(2) \\
\vdots \\
x(N)
\end{bmatrix}$$

$$\hat{x} = Bx$$

with

$$b_{ij} = \frac{\sin[(j - t_i/\Delta t)\pi]}{N\sin[(j - t_i/\Delta t)\pi/N]}$$

A problem here is that we know just $M < N$ of signal samples. The values at unavailable positions $t_i \notin T_A$ are assumed to be zero. Their positions are assumed at the sampling theorem instants, $t_i = i\Delta t$ for $t_i \notin T_A$, since they are not known anyway. The sampling interval signal value $s$, defined by vector $s$, are recalculated to

$$y_i = \hat{y}_1 = B^{-1}\hat{y}_1$$

and

$$y_2 = B^{-1}\hat{y}_2.$$
property guarantees the full recovery, although we can not check it since it is an NP-hard problem. In this region, there is also a probability that some samples are not independent and that the recovery condition is not met for a given signal and set of samples. Therefore, in the detection of a sparse signal by using a sparsity measure close to the $\ell_0$-norm, as a detection criterion, we can have two possible scenarios: 1) A particular realization with $M$ samples satisfies the recovery condition and the result of the reconstruction is a sparse signal. This result according to the RIP is unique and will be properly detected since its sparsity measure, close to the $\ell_0$-norm, will be small. 2) A particular realization contains $M$ samples not satisfying the recovery condition. The presented reconstruction process is based on the $\ell_1$-norm minimization, which by definition does not look for solutions with many transform zero values but for a solution with the smallest possible sum of their absolute values. The event that this kind of minimization, without the conditions for a unique sparse solution being met, will result in a signal with a large number of zero coefficients in the transformation domain is a zero probability event. As a result of the reconstruction process an arbitrary signal with many nonzero transform coefficients is obtained, with a high probability. This case is properly detected as well, since its sparsity measure, based on a norm close to the $\ell_0$, will be close to $N$. In all scenarios it has been assumed that the reconstruction algorithm behavior is proper.

4. Conclusion

Analysis of nonuniformly sampled sparse signals is performed. A gradient-based algorithm with adaptive step is used for the reconstruction. A new criterion for the parameter adaptation in the algorithm, based on the gradient directions analysis, is proposed. It significantly improves the calculation efficiency of the algorithm. The random nonuniformly positioned available samples are recalculated based on the sampling theorem reconstruction formula. Based on the new set of samples the recovery is performed. The methods are checked and illustrated on numerical examples.

References

[1] D. L. Donoho, “Compressed sensing,” IEEE Trans. on Information Theory, vol. 52, no. 4, pp. 1289–1306, 2006.
