An alternative theorem for gradient systems

BIAGIO RICCERI

Dedicated to the memory of Professor Felix E. Browder

Abstract: Here is one of the result obtained in this paper: Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain and let \( F, G : \mathbb{R} \to \mathbb{R} \) be two \( C^1 \) functions satisfying the following conditions:

(i) for some \( p > 0 \), one has
\[
\limsup_{|\xi| \to +\infty} \frac{|F'(\xi)| + |G'(\xi)|}{|\xi|^p} < +\infty;
\]

(ii) \( F \) is non-negative, non-decreasing, \( \lim_{\xi \to +\infty} F(\xi) \xi^2 = 0 \), \( \lim_{\xi \to 0^+} F(\xi) \xi^2 = +\infty \) and the function \( \xi \to \frac{F'(\xi)}{\xi} \) is strictly decreasing in \( (0, +\infty] \);

(iii) \( G \) is positive and convex.

Then, for every positive function \( \alpha \in L^\infty(\Omega) \), the problem
\[
\begin{array}{ll}
-\Delta u = \alpha(x)G(v(x))F'(u) & \text{in } \Omega \\
-\Delta v = -\alpha(x)F(u(x))G'(v) & \text{in } \Omega \\
u = v = 0 & \text{on } \partial\Omega
\end{array}
\]
has a non-zero weak solution belonging to \( L^\infty(\Omega) \times L^\infty(\Omega) \).

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The present paper lies in the extensive program of studying consequences and applications of certain general minimax theorems ([9], [10], [12]-[15], [17]-[25]) which cannot be directly deduced by the classical Fan-Sion theorem ([5], [26]).

Here, we are interested in gradient systems. Precisely, given two Banach spaces \( X, Y \) and a \( C^1 \) functional \( \Phi : X \times Y \to \mathbb{R} \), we are interested in the existence of critical points for \( \Phi \), that is in the solvability of the system
\[
\begin{array}{ll}
\Phi'_x(x, y) = 0 \\
\Phi'_y(x, y) = 0,
\end{array}
\]
where \( \Phi'_x \) (resp. \( \Phi'_y \)) is the derivative of \( \Phi \) with respect to \( x \) (resp. \( y \)).

Let \( I : X \to \mathbb{R} \). As usual, \( I \) is said to be coercive if \( \lim_{\|x\| \to +\infty} I(x) = +\infty \). \( I \) is said to be quasi-concave (resp. quasi-convex) if the set \( I^{-1}([r, +\infty[) \) (resp. \( I^{-1}(-\infty, r]) \)) is convex for all \( r \in \mathbb{R} \). When \( I \) is \( C^1 \), it is said to satisfy the Palais-Smale condition if each sequence \( \{x_n\} \) in \( X \) such that \( \sup_{n \in \mathbb{N}} \|I'(x_n)\| \to 0 \) admits a strongly convergent subsequence.

Here is our main abstract theorem:
THEOREM 1. - Let $X, Y$ be two real reflexive Banach spaces and let $\Phi : X \times Y \to \mathbb{R}$ be a $C^1$ functional satisfying the following conditions:

(a) the functional $\Phi(x, \cdot)$ is quasi-concave for all $x \in X$ and the functional $-\Phi(x_0, \cdot)$ is coercive for some $x_0 \in X$;

(b) there exists a convex set $S \subseteq Y$ dense in $Y$, such that, for each $y \in S$, the functional $\Phi(\cdot, y)$ is weakly lower semicontinuous, coercive and satisfies the Palais-Smale condition.

Then, either the system

$$
\begin{cases}
\Phi'_x(x, y) = 0 \\
\Phi'_y(x, y) = 0
\end{cases}
$$

has a solution $(x^*, y^*)$ such that

$$
\Phi(x^*, y^*) = \inf_{x \in X} \Phi(x, y^*) = \sup_{y \in Y} \Phi(x^*, y) ,
$$

or, for every convex set $T \subseteq S$ dense in $Y$, there exists $\tilde{y} \in T$ such that equation

$$
\Phi'_x(x, \tilde{y}) = 0
$$

has at least three solutions, two of which are global minima in $X$ of the functional $\Phi(\cdot, \tilde{y})$.

PROOF. Assume that there is no solution $(x^*, y^*)$ of the system

$$
\begin{cases}
\Phi'_x(x, y) = 0 \\
\Phi'_y(x, y) = 0
\end{cases}
$$

such that

$$
\Phi(x^*, y^*) = \inf_{x \in X} \Phi(x, y^*) = \sup_{y \in Y} \Phi(x^*, y) .
$$

We consider both $X, Y$ endowed with the weak topology. Notice that, by (a), $\Phi(x, \cdot)$ is weakly upper semicontinuous in $Y$ for all $x \in X$ and weakly sup-compact for $x = x_0$. As a consequence, the functional $y \to \inf_{x \in X} \Phi(x, y)$ is weakly sup-compact and so it attains its supremum. Likewise, by (b), $\Phi(\cdot, y)$ is weakly inf-compact for all $y \in S$. By continuity and density, we have

$$
\sup_{y \in Y} \Phi(x, y) = \sup_{y \in S} \Phi(x, y) \quad (1)
$$

for all $x \in X$. As a consequence, the functional $x \to \sup_{y \in Y} \Phi(x, y)$ is weakly inf-compact and so it attains its infimum. Therefore, the occurrence of the equality

$$
\sup_{y \in Y} \inf_{x \in X} \Phi = \inf_{x \in X} \sup_{y \in Y} \Phi
$$

is equivalent to the existence of a point $(\hat{x}, \hat{y}) \in X \times Y$ such that

$$
\sup_{y \in Y} \Phi(\hat{x}, y) = \Phi(\hat{x}, \hat{y}) = \inf_{x \in X} \Phi(x, \hat{y}) .
$$

But, for what we are assuming, no such a point can exist and hence we have

$$
\sup_{y \in Y} \inf_{x \in X} \Phi < \inf_{x \in X} \sup_{y \in Y} \Phi \quad (2)
$$

So, in view of (1) and (2), we also have

$$
\sup_{S \subseteq Y} \inf_{x \in X} \Phi < \inf_{x \in X} \sup_{S \subseteq Y} \Phi .
$$
At this point, we are allowed to apply Theorem 1.1 of [20]. Therefore, there exists \( \tilde{y} \in S \) such that the functional \( \Phi(\cdot, \tilde{y}) \) has at least two global minima in \( X \) and so, thanks to Corollary 1 of [8], the same functional has at least three critical points.

The next result is a consequence of Theorem 1.

**THEOREM 2.** Let \( X, Y \) be two real Hilbert spaces and let \( J : X \times Y \to \mathbb{R} \) be a \( C^1 \) functional satisfying the following conditions:
\( (a_1) \) the functional \( y \to \frac{1}{2} \| y \|_Y^2 + J(x, y) \) is quasi-convex for all \( x \in X \) and coercive for some \( x \in X \);
\( (b_1) \) there exists a convex set \( S \subseteq Y \) dense in \( Y \) such that, for each \( y \in S \), the operator \( J'_x(\cdot, y) \) is compact and
\[
\limsup_{\|x\| \to +\infty} \frac{J(x, y)}{\| x \|_X^2} < \frac{1}{2};
\] (3)

Then, either the system
\[
\begin{cases}
  x = J'_x(x, y) \\
y = -J'_y(x, y)
\end{cases}
\] has a solution \((x^*, y^*)\) such that
\[
\frac{1}{2}(\| x^* \|_X^2 - \| y^* \|_Y^2) - J(x^*, y^*) = \inf_{x \in X} \left( \frac{1}{2}(\| x \|_X^2 - \| y^* \|_Y^2) - J(x, y^*) \right) = \sup_{y \in Y} \left( \frac{1}{2}(\| x^* \|_X^2 - \| y \|_Y^2) - J(x^*, y) \right),
\]
or, for every convex set \( T \subseteq S \) dense in \( Y \), there exists \( \tilde{y} \in T \) such that the equation
\[
x = J'_x(x, \tilde{y})
\] has at least three solutions, two of which are global minima in \( X \) of the functional \( x \to \frac{1}{2} \| x \|_X^2 - J(x, \tilde{y}) \).

**PROOF.** Consider the function \( \Phi : X \times Y \to \mathbb{R} \) defined by
\[
\Phi(x, y) = \frac{1}{2}(\| x \|_X^2 - \| y \|_Y^2) - J(x, y)
\] for all \((x, y) \in X \times Y\). Clearly, \( \Phi \) is \( C^1 \) and one has
\[
\Phi'_x(x, y) = x - J'_x(x, y),
\]
\[
\Phi'_y(x, y) = -y - J'_y(x, y)
\] for all \((x, y) \in X \times Y\). We want to apply Theorem 1 such a \( \Phi \). Of course, \( \Phi \) satisfies \((a_1)\) in view of \((a_1)\). Concerning \((b)\), notice that, for each \( y \in S \), the functional \( J(\cdot, y) \) is sequentially weakly continuous since \( J'_y(\cdot, y) \) is compact ([27], Corollary 41.9). Moreover, from (3) it immediately follows that \( \Phi(\cdot, y) \) is coercive and so, by the Eberlein-Šmulian theorem, it is weakly lower semicontinuous. Finally, \( \Phi(\cdot, y) \) satisfies the Palais-Smale condition in view of Example 38.25 of [27]. Now, the conclusion follows directly from Theorem 1. \( \triangle \)

We now present an application of Theorem 2 to non-cooperative elliptic systems.

In what follows, \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) is a bounded smooth domain. We consider \( H^1_0(\Omega) \) equipped with the scalar product
\[
\langle u, v \rangle = \int_\Omega \nabla u(x) \nabla v(x) dx.
\]

We denote by \( A \) the class of all functions \( H : \Omega \times \mathbb{R}^2 \to \mathbb{R} \), with \( H(x, 0, 0) = 0 \) for all \( x \in \Omega \), which are measurable in \( \Omega \), \( C^1 \) in \( \mathbb{R}^2 \) and satisfy
\[
\sup_{(x, u, v) \in \Omega \times \mathbb{R}^2} \frac{|H_u(x, u, v)| + |H_v(x, u, v)|}{1 + |u|^p + |v|^q} < +\infty
\]
where $p, q > 0$, with $p < \frac{n+2}{n-2}$ and $q \leq \frac{n+2}{n-2}$ when $n > 2$.

Given $H \in \mathcal{A}$, we are interested in the problem

$$
\begin{cases}
-\Delta u = H_u(x,u,v) & \text{in } \Omega \\
-\Delta v = -H_v(x,u,v) & \text{in } \Omega \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}
$$

($P_H$)

$H_u$ (resp. $H_v$) denoting the derivative of $H$ with respect to $u$ (resp. $v$).

As usual, a weak solution of ($P_H$) is any $(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$ such that

$$
\int_\Omega \nabla u(x) \nabla \varphi(x) dx = \int_\Omega H_u(x,u(x),v(x)) \varphi(x) dx,
$$

$$
\int_\Omega \nabla v(x) \nabla \psi(x) dx = -\int_\Omega H_v(x,u(x),v(x)) \psi(x) dx
$$

for all $\varphi, \psi \in H^1_0(\Omega)$.

Define the functional $I_H : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ by

$$
I_H(u,v) = \frac{1}{2} \left( \int_\Omega |\nabla u(x)|^2 dx - \int_\Omega |\nabla v(x)|^2 dx - \int_\Omega H(x,u(x),v(x)) dx \right)
$$

for all $(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$.

Since $H \in \mathcal{A}$, the functional $I_H$ is $C^1$ in $H^1_0(\Omega) \times H^1_0(\Omega)$ and its critical points are precisely the weak solutions of ($P_H$).

Our result on ($P_H$) is as follows:

THEOREM 3. - Let $H \in \mathcal{A}$ be such that

$$
\limsup_{|u| \to +\infty} \sup_{v \leq r} \frac{H(x,u,v)}{u^2} \leq 0
$$

(4)

for all $r > 0$, and

$$
\text{meas} \left( \left\{ x \in \Omega : \sup_{u \in \mathbb{R}} H(x,u,0) > 0 \right\} \right) > 0.
$$

(5)

Moreover, assume that either $H(x,u,\cdot)$ is convex for all $(x,u) \in \Omega \times \mathbb{R}$, or

$$
L := \sup_{(v,\omega) \in \mathbb{R}^2, v \neq \omega} \frac{\max_{(x,u) \in \Omega \times \mathbb{R}} |H_v(x,u,v) - H_v(x,u,\omega)|}{|v - \omega|} < +\infty.
$$

(6)

Set

$$
\lambda^* = \frac{1}{2} \inf \left\{ \frac{\int_\Omega |\nabla w(x)|^2 dx}{\int_\Omega H(x,w(x),0) dx} : w \in H^1_0(\Omega), \int_\Omega H(x,w(x),0) dx > 0 \right\}
$$

and assume that $\lambda^* < \frac{1}{4}$ when (6) holds.

Then, for each $\lambda > \lambda^*$, with $\lambda < \frac{1}{4}$ when (6) holds, either the problem

$$
\begin{cases}
-\Delta u = \lambda H_u(x,u,v) & \text{in } \Omega \\
-\Delta v = -\lambda H_v(x,u,v) & \text{in } \Omega \\
u = v = 0 & \text{on } \partial \Omega
\end{cases}
$$

has a solution $(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$.
has a non-zero weak solution belonging to $L^\infty(\Omega) \times L^\infty(\Omega)$, or, for each convex set $S \subseteq H^1_0(\Omega) \cap L^\infty(\Omega)$ dense in $H^1_0(\Omega)$, there exists $\tilde{v} \in S$ such that the problem

$$
\begin{cases}
  -\Delta u = \lambda H_u(x, u, \tilde{v}(x)) & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega
\end{cases}
$$

has at least three weak solutions, two of which are global minima in $H^1_0(\Omega)$. Concerning $(a)$, $J$ convex and coercive. This is clear when $L$ has a non-zero weak solution belonging to $L^\infty(\Omega)$. Fix $\lambda > \lambda^*$ for all $(u, v)$ such that $\lambda J(u, v)$ is uniformly monotone and then the claim follows from a classical result ([27], pp. 247-249). Concerning $(b_1)$, notice that, for each $x, \xi \in A H^1(\Omega)$, notice that the operator $J'_u(\cdot, v)$ is compact due to restriction on $p$ (recall that $H \in A$). Moreover, in view of (4), for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$
H(x, t, s) \leq \epsilon t^2
$$

for all $(x, t, s) \in [-\|v\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}] \times \mathbb{R}$, $t \in \mathbb{R}$, and $t \in \mathbb{R} \setminus [-\delta, \delta]$. But $H$ is bounded on each bounded subset of $\Omega \times \mathbb{R}^2$, and so, for a suitable constant $c > 0$, we have

$$
H(x, t, s) \leq \epsilon t^2 + c
$$

for all $(x, t, s) \in A \times \mathbb{R} \times [-\|v\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}]$. Of course, from (7) it follows that

$$
\limsup_{\|u\| \to +\infty} \frac{J(u, v)}{\|u\|^2} \leq \epsilon
$$

and so

$$
\limsup_{\|u\| \to +\infty} \frac{J(u, v)}{\|u\|^2} \leq 0
$$

since $\epsilon > 0$ is arbitrary. Hence, $\lambda J$ satisfies (3). Now suppose that there exists a convex set $S \subseteq H^1_0(\Omega) \cap L^\infty(\Omega)$ dense in $H^1_0(\Omega)$ such that, for each $v \in S$, the problem

$$
\begin{cases}
  -\Delta u = \lambda H_u(x, u, \tilde{v}(x)) & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega
\end{cases}
$$

has at most two weak solutions. Then, Theorem 2 ensures the existence of a weak solution $(u^*, v^*)$ of the problem

$$
\begin{cases}
  -\Delta u = \lambda H_u(x, u, v) & \text{in } \Omega \\
  -\Delta v = -\lambda H_v(x, u, v) & \text{in } \Omega \\
  u = v = 0 & \text{on } \partial \Omega
\end{cases}
$$

such that

$$
I_{\lambda H}(u^*, v^*) = \inf_{u \in H^1_0(\Omega)} I_{\lambda H}(u, v^*) = \sup_{v \in H^1_0(\Omega)} I_{\lambda H}(u^*, v). \tag{8}
$$
From (8), in view of Theorem 1 of [3] (see Remark 5, p. 1631), it follows that \( u^*, v^* \in L^\infty(\Omega) \). We show that \((u^*, v^*) \neq (0, 0)\). If \( v^* \neq 0 \), we are done. So, assume \( v^* = 0 \). Since \( \lambda > \lambda^* \), we have

\[
\inf_{u \in H^1_0(\Omega)} \left( \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx - \lambda \int_\Omega H(x, u(x), 0) \, dx \right) < 0 .
\]

But then, since \( \int_\Omega H(x, 0, 0) \, dx = 0 \), from (9) and the first equality in (8), it follows that \( u^* \neq 0 \), and the proof is complete.

For previous results on problem \((P_H)\) (markedly different from Theorem 3) we refer to [1], [4], [6], [7]. A joint application of Theorem 3 with the main result in [2] gives the following:

**THEOREM 4.** Let \( H \in A \) satisfy the assumptions of Theorem 3. Moreover, suppose that \( \inf_{x \in \Omega} H_u \geq 0 \) and that, for each \((x, v) \in \Omega \times \mathbb{R} \), the function \( u \mapsto \frac{H_u(x, u, v)}{u} \) is strictly decreasing in \([0, +\infty[\).

Then, for every \( \lambda > \lambda^* \), with \( \lambda < \frac{1}{\lambda^*} \) when (6) holds, the problem

\[
\begin{align*}
-\Delta u &= \lambda H_u(x, u, v) \quad \text{in } \Omega \\
-\Delta v &= -\lambda H_v(x, u, v) \quad \text{in } \Omega \\
u &= v = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

has a non-zero weak solution belonging to \( L^\infty(\Omega) \times L^\infty(\Omega) \).

**PROOF.** Fix \( \lambda > \lambda^* \), with \( \lambda < \frac{1}{\lambda^*} \) when (6) holds. Fix also \( v \in C^\infty_0(\Omega) \). Since \( \inf_{x \in \Omega} H_u \geq 0 \), the bounded weak solutions of the problem

\[
\begin{align*}
-\Delta u &= \lambda H_u(x, u, v(x)) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

are continuous and non-negative in \( \overline{\Omega} \). As a consequence, in view of Theorem 1 of [2], the problem

\[
\begin{align*}
-\Delta u &= \lambda H_u(x, u, v(x)) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

has at most one non-zero bounded weak solution. Now, the conclusion follows directly from Theorem 3. \( \triangle \)

Finally, notice the following corollary of Theorem 4:

**THEOREM 5.** Let \( F, G : \mathbb{R} \to \mathbb{R} \) be two \( C^1 \) functions, with \( FG - F(0)G(0) \in A \), satisfying the following conditions:

(a2) \( F \) is non-negative, non-decreasing, \( \lim_{u \to +\infty} \frac{F(u)}{u^2} = 0 \) and the function \( u \mapsto \frac{F'(u)}{u} \) is strictly decreasing in \([0, +\infty[,\);  

(b2) \( G \) is positive and convex.

Finally, let \( \alpha \in L^\infty(\Omega) \), with \( \alpha > 0 \). Set

\[
\lambda^*_\alpha = \frac{1}{2G(0)} \inf \left\{ \int_\Omega |\nabla w(x)|^2 \, dx : \int_\Omega \alpha(x)F(w(x)) \, dx > 0 \right\} .
\]

Then, for every \( \lambda > \lambda^*_\alpha \), the problem

\[
\begin{align*}
-\Delta u &= \lambda \alpha(x)G(v(x))F'(u) \quad \text{in } \Omega \\
-\Delta v &= -\lambda \alpha(x)F(u(x))G'(v) \quad \text{in } \Omega \\
u &= v = 0 \quad \text{on } \partial \Omega
\end{align*}
\]
has a non-zero weak solution belonging to $L^\infty(\Omega) \times L^\infty(\Omega)$.

**Proof.** Apply Theorem 4 to the function $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$H(x, u, v) = \alpha(x)(F(u)G(v) - F(0)G(0))$$

for all $(x, u, v) \in \Omega \times \mathbb{R}^2$. Checking that $H$ satisfies the assumptions of Theorem 4 is an easy task. △

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Department of Mathematics and Informatics
University of Catania
Viale A. Doria 6
95125 Catania, Italy

 e-mail address: ricceri@dmi.unict.it