Reheating-volume measure for random-walk inflation

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The recently proposed “reheating-volume” (RV) measure promises to solve the long-standing problem of extracting probabilistic predictions from cosmological “multiverse” scenarios involving eternal inflation. I give a detailed description of the new measure and its applications to generic models of eternal inflation of random-walk type. For those models I derive a general formula for RV-regulated probability distributions that is suitable for numerical computations. I show that the results of the RV cutoff in random-walk type models are always gauge-invariant and independent of the initial conditions at the beginning of inflation. In a toy model where equal-time cutoffs lead to the “youngness paradox,” the RV cutoff yields unbiased results that are distinct from previously proposed measures.

I. INTRODUCTION AND MOTIVATION

It was realized in recent years that in many cosmological scenarios the fundamental theory does not predict with certainty the values of observable cosmological parameters, such as the effective cosmological constant and the masses of elementary particles. This is the case for the “landscape of string theory” [1, 2, 3] (see also the “recycling universe” [4]) and for models of inflation driven by a scalar field (see e.g. [5, 6] for early work). A common feature of these cosmological models is the presence of eternal inflation, i.e. the absence of a global end to inflation in the entire spacetime (see Refs. [7, 8, 9] for reviews). Eternal inflation gives rise to infinitely many causally disconnected regions of the spacetime where the cosmological observables may have significantly different values. Hence the program outlined in the early works [10, 11, 12] was to obtain the probability distribution of the cosmological parameters as measured by an observer randomly located in the spacetime. The main difficulty in obtaining such probability distributions is due to the infinite volume of regions where an observer may be located.

Since the spacetime during inflationary evolution is cold and empty, observers may appear only after reheating. The standard cosmology after reheating is tightly constrained by current experimental knowledge. Hence, the average number of observers produced in any freshly-reheated spatial domain is a function of cosmological parameters in that domain. Calculating that function is, in principle, a well-defined astrophysical problem that does not involve any infinities. Therefore we focus on the problem of obtaining the probability distribution of cosmological observables at reheating.

The set of all spacetime points where reheating takes place is a spacelike three-dimensional hypersurface [12, 13, 14] called the “reheating surface.” The hallmark feature of eternal inflation is that a finite, initially inflating spatial 3-volume typically gives rise to a reheating surface having an infinite 3-volume (see Fig. 1). The geometry and topology of the reheating surface is quite complicated. For instance, the reheating surface contains infinitely many future-directed spikes around never-thermalizing comoving worldlines called “eternally inflating geodesics” [15, 16, 17]. It is known that the set of “spikes” has a well-defined fractal dimension that can be computed in the stochastic approach [15]. Since the reheating surface is a highly inhomogeneous, noncompact 3-manifold without any symmetries, a “random location” on such a surface is mathematically ill-defined. This feature of eternal inflation is at the root of the technical and conceptual difficulties known collectively as the “measure problem” (see Refs. [8, 4, 18, 19, 20, 21] for reviews).

To visualize the measure problem, it is convenient to consider an initial inflating spacelike region $S$ of horizon size (an “H-region”) and the portion $R \equiv R(S)$ of the reheating surface that corresponds to the comoving future of $S$. If the 3-volume of $R$ were finite, the volume-weighted average of any observable quantity $Q$ at reheating would be defined simply by averaging $Q$ over $R$,

$$
\langle Q \rangle = \frac{\int_R Q \sqrt{\gamma} d^3 x}{\int_R \sqrt{\gamma} d^3 x},
$$

Figure 1: A 1+1-dimensional slice of the spacetime in an eternally inflating universe (numerical simulation in Ref. [22]). Shades of different color represent different regions where reheating took place. The reheating surface is the line separating the white (inflating) domain and the shaded domains.
where \( \gamma \) is the induced metric on the 3-surface \( R \). This would have been the natural prescription for the observer-based average of \( Q \); all higher moments of the distribution of \( Q \), such as \( \langle Q^2 \rangle \), \( \langle Q^3 \rangle \), etc., would have been well-defined as well. However, in the presence of eternal inflation\(^1\) the 3-volume of \( R \) is infinite with a nonzero probability \( X(\phi_0) \), where \( \phi = \phi_0 \) is the initial value of the inflaton field at \( S \). The function \( X(\phi_0) \) has been computed in slow-roll inflationary models\(^12\) where typically \( X(\phi_0) \approx 1 \) for \( \phi_0 \) not too close to reheating. In other words, the volume of \( R \) is infinite with a probability close to 1. In that case, the straightforward average of a fluctuating quantity \( Q(x) \) over \( R \) is mathematically undefined since \( \int_R \sqrt{\gamma} d^3x = \infty \) and \( \int_R Q \sqrt{\gamma} d^3x = \infty \).

The average \( \langle Q \rangle \) can be computed only after imposing a volume cutoff on the reheating surface, making its volume finite in a controlled way. What has become known in cosmology as the “measure problem” is the difficulty of coming up with a physically motivated cutoff prescription (informally called a “measure”) that makes volume averages \( \langle Q \rangle \) well-defined.

Volume cutoffs are usually implemented by restricting the infinite reheating domain \( R \) to a large but finite subdomain having a volume \( V \). Then one defines the “regularized” distribution \( p(Q|V) \) of an observable \( Q \) by gathering statistics about the values of \( Q \) over the finite volume \( V \). More precisely, \( p(Q|V) \) is defined as the 3-volume of regions (within the finite domain \( V \)) where the observable \( Q \) has values in the interval \( [Q, Q + dQ] \). The final probability distribution \( p(Q) \) is then defined as

\[
p(Q) \equiv \lim_{V \to \infty} p(Q|V),
\]

provided that the limit exists.

Several cutoffs have been proposed in the literature, differing in the choice of the compact subset \( V \) and in the way \( V \) approaches infinity. It has been found early on (e.g.\(^13\)\(^12\)) that probability distributions, such as \( p(Q) \), depend sensitively on the choice of the cutoff. This is the root of the measure problem. Since a “natural” mathematically consistent definition of the measure is absent, one judges a cutoff prescription viable if its predictions are not obviously pathological. Possible pathologies include the dependence on choice of spacetime coordinates\(^13\)(\(^24\)\(^25\)), the “youngness paradox”\(^26\)(\(^27\)), and the “Boltzmann brain” problem\(^28\)(\(^29\)(\(^30\)(\(^31\)(\(^32\)(\(^33\)(\(^34\)(\(^35\)))).

The presently viable cutoff proposals fall into two rough classes that may be designated as “worldline-based” and “volume-based” measures (a more fine-grained classification of measure proposals can be found in Refs.\(^17\)(\(^18\)). The “worldline” or the “holographic” measure\(^36\)(\(^37\)) avoids considering the infinite total 3-volume of the reheating surface in the entire spacetime. Instead it focuses only on the reheated 3-volume of one \( H \)-region surrounding a single randomly chosen comoving worldline. This measure, by construction, is sensitive to the initial conditions at the location where the worldline starts and is essentially equivalent to performing calculations with the comoving-volume probability distribution. Proponents of the “holographic” measure have argued that the infinite reheating surface cannot be considered because the spacetime beyond one \( H \)-region is not adequately described by semiclassical gravity\(^37\).

However, the semiclassical approximation was recently shown to be valid in a large class of inflationary models\(^38\). In my view, an attempt to count the total volume of the reheating surface corresponds more closely to the goal of obtaining the probability distribution of observables in the entire universe, as measured by a “typical” observer (see Refs.\(^34\)(\(^39\)(\(^10\)(\(^11\)(\(^12\))) for recent discussions of “typicality” and accompanying issues). The sensitive dependence of “holographic” proposals on the conditions at the beginning of inflation also appears to be undesirable. Volume-based proposals are insensitive to the initial conditions because the 3-volume of the universe is, in a certain well-defined sense, dominated by regions that spent a long time in the inflationary regime.\(^2\)

Existing volume-based proposals include the equal-time cutoff\(^3\)(\(^6\)(\(^10\)), the “spherical cutoff”\(^27\), the “comoving horizon cutoff”\(^44\)(\(^45\)(\(^46\)), the “stationary measure”\(^21\)(\(^47\)), the “no-boundary” measure with volume weighting\(^48\)(\(^49\)(\(^50\)(\(^51\)), the “pseudo-comoving” measure\(^31\)(\(^52\)), and the most recently proposed “reheating-volume” (RV) measure\(^53\).

The focus of this article is a more detailed study of the RV measure in the context of random-walk eternal inflation. As a typical generic model I choose a scenario where inflation is driven by the potential \( V(\phi) \) of a minimally coupled scalar field \( \phi \). In this model, there exists a range of \( \phi \) where large quantum fluctuations dominate over the deterministic slow-roll evolution, which gives rise to eternal self-reproduction of inflationary domains. I extensively use the stochastic approach to inflation, which is based on the Fokker-Planck or “diffusion” equations (see Ref.\(^9\) for a pedagogical review). The results can be straightforwardly generalized to multiple-field or non-slow-roll models are straightforward since the Fokker-Planck formalism is already developed in those contexts\(^38\)(\(^54\)). Applications of the RV measure to “landscape” scenarios will be considered elsewhere.

An attractive feature of the RV measure is that its construction lacks extraneous geometric elements that could

\(^1\) Various equivalent conditions for the presence of eternal inflation were examined in more detail in Refs.\(^13\)(\(^24\)(\(^23\)) and \(^14\). Here I adopt the condition that \( X(\phi) \) is nonzero for all \( \phi \) in the inflating range.

\(^2\) It has been noted that 3-volume is a coordinate-dependent quantity, and hence statements involving 3-volume need to be formulated with care\(^43\). Indeed there exist time foliations where the 3-volume of inflationary space does not grow with time. The issues of coordinate dependence were analyzed in Ref.\(^24\).
introduce a bias. An example of a biased measure is the equal-time cutoff where one considers the subdomain of the reheating surface to the past of a hypersurface of fixed proper time $t = t_c$, subsequently letting $t_c \to \infty$. It is well known that the volume-weighted distribution of observables within a hypersurface of equal proper time is strongly dominated by regions where inflation ended very recently. A time delay $\delta t$ in the onset of reheating due to a rare quantum fluctuation is overwhelmingly rewarded by an additional volume expansion factor $\propto \exp[3H_{\text{max}} \delta t]$, where $H_{\text{max}}$ is roughly the highest Hubble rate accessible to the inflaton. This is the essence of the so-called youngness paradox that seems unavoidable in an equal-time cutoff (see Refs. [27] and [34] for recent discussions).

Moreover, the results of the equal-time cutoff are sensitive to the choice of the time coordinate (“time gauge”). For instance, the proper time can be replaced by the family of time gauges labeled by a constant $\alpha$, \[ t_{(\alpha)} \equiv \int^t H^\alpha dt, \] which interpolate between the proper time ($\alpha = 0$, $t_{(0)} \equiv t$) and the $e$-folding time ($\alpha = 1$, $t_{(1)} = \ln a$). It has been shown that the results of the equal-time cutoff depend sensitively on the value of $\alpha$, and that no “correct” value of $\alpha$ could be specified so as to remove the bias \[23].\] Since the time coordinate is an arbitrary label in the spacetime, we may impose the requirement that a viable measure prescription be invariant with respect to choosing even more general time gauges, such as

\[ \tau \equiv \int^t T(\phi) dt, \] where $T(\phi) > 0$ is an arbitrary function of the inflaton field (and possibly of other fields), and the integration is performed along comoving worldlines $x^{1,2,3} = \text{const}$.

The “spherical cutoff” \[27\] and the “stationary measure” \[24\] prescriptions were motivated by the need to remove the bias inherent in the equal-time cutoff. In particular, the spherical cutoff selects as a compact subset $\mathcal{V}$ the interior of a large sphere drawn within the reheating surface $R$ around a randomly chosen center. The spherical cutoff is manifestly gauge-invariant since its construction uses only the intrinsically defined 3-volume of the reheating surface rather than the spacetime coordinates $(t, x)$. Some results were obtained in the spherical cutoff using numerical simulations \[22\]. A disadvantage of the spherical cutoff is that its direct implementation requires one to perform costly numerical simulations of random-walk inflation on a spacetime grid, for instance, using the techniques of Refs. \[1, 22, 56\]. Instead, one would prefer to obtain a generally valid analytic formula for the probability distribution of cosmological observables. For instance, one could ask whether the results of the spherical cutoff depend in an essential way on the spherical shape of the region, on the position of the center of the sphere, and on the initial conditions. Satisfactory answers to these questions (in the negative) were obtained in Refs. \[22, 27\] in some tractable cases where results could be obtained analytically. However, it is difficult to analyze these questions in full generality since one lacks a general analytic formula for the probability distribution in the spherical cutoff.

The RV measure is similar in spirit to the spherical cutoff because the RV cutoff uses only the intrinsic geometrical information defined by the reheating surface. It can be argued that the RV cutoff is “more natural” than other cutoffs in that it selects a finite portion $\mathcal{V}$ of the reheating surface without using artificial constant-time hypersurfaces, spheres, worldlines, or any other extraneous geometrical data. Instead, the selection of $\mathcal{V}$ in the RV cutoff is performed using a certain well-defined selection of subensemble in the probability space, which is determined by the stochastic evolution itself.

The central concept in the RV cutoff is the “finitely produced volume.” The basic idea is that there is always a nonzero probability that a given initial $H$-region $S$ does not give rise to an infinite reheating surface in its comoving future. For instance, it is possible that by a rare coincidence the inflaton field $\phi$ rolls towards reheating at approximately the same time everywhere in $S$. Moreover, there is a nonzero (if small) probability $\rho(\mathcal{V})d\mathcal{V}$ that the total volume $\text{Vol}(R)$ of the reheating surface $R$ to the future of $S$ belongs to a given interval $[\mathcal{V}, \mathcal{V} + d\mathcal{V}]$.

\[ \rho(\mathcal{V}) \equiv \lim_{d\mathcal{V} \to 0} \frac{\text{Prob}\{\text{Vol}(R) \in [\mathcal{V}, \mathcal{V} + d\mathcal{V}]\}}{d\mathcal{V}}. \] I call $\rho(\mathcal{V})$ the “finitely produced volume distribution.” This distribution is nontrivial because the probability of the event $\text{Vol}(R) < \infty$ is nonzero, if small, for any given (non-reheated) initial region $S$. The distribution $\rho(\mathcal{V})$ is, by construction, normalized to that probability:

\[ \int_0^\infty \rho(\mathcal{V}) d\mathcal{V} = \text{Prob}\{\text{Vol}(R) < \infty\} < 1. \]

The RV cutoff consists of a selection of a certain ensemble $E_\mathcal{V}$ of the histories that produce a total reheated volume equal to a given value $\mathcal{V}$ starting from an initial $H$-region. In the limit of large $\mathcal{V}$, the ensemble $E_\mathcal{V}$ consists of $H$-regions that evolve “almost” to the regime of eternal inflation. Thus, heuristically one can expect that the ensemble $E_\mathcal{V}$ provides a representative sample of the infinite reheating surface.\[3\] Given the ensemble $E_\mathcal{V}$, one can determine the volume-weighted probability distribution $\rho(Q|E_\mathcal{V})$ of a cosmological parameter $Q$ by ordinary sampling of the values of $Q$ throughout the finite volume $\mathcal{V}$. Finally, the probability distribution $\rho(Q)$ is defined as the limit of $\rho(Q|E_\mathcal{V})$ at $\mathcal{V} \to \infty$, provided that the limit exists.

\[3\] Of course, this heuristic statement cannot be made rigorous since there exists no natural measure on the infinite reheating surface. We use this statement merely as an additional motivation for considering the RV measure.
To clarify the construction of the ensemble $E_V$, it is helpful to begin by considering the distribution $\rho(V)$ in a model that does not permit eternal inflation. In that case, the volume of the reheating surface is finite with probability 1, so the distribution $\rho(V)$ is an ordinary probability distribution normalized to unity. In that context, the distribution $\rho(V)$ was introduced in the recent work [14] where the authors considered a family of inflationary models parameterized by a number $\Omega$, such that eternal inflation is impossible in models where $\Omega > 1$. It was then found by a direct calculation that all the moments of the distribution $\rho(V)$ diverge at the value $\Omega = 1$ where the possibility of eternal inflation is first switched on. One can show that the \textit{finitely produced} distribution $\rho(V)$ for $\Omega < 1$ is again well-behaved and has finite moments (see Sec. II D). This FPRV distribution $\rho(V)$ is the formal foundation of the RV cutoff. It is worth emphasizing that the RV cutoff does not regulate the volume of the reheating surface by modifying the dynamics of a given inflationary model and making eternal inflation impossible. Rather, finite volumes $V$ are generated by rare chance (i.e., within the ensemble $E_V$) through the unmodified dynamics of the model, directly in the regime of eternal inflation.

Below I compute the distribution $\rho(V)$ asymptotically for very large $V$ in models of slow-roll inflation (Sec. II D). Specifically, I will compute the distribution $\rho(V; \phi_0)$, where $\phi_0$ is the (homogeneous) value of the inflaton field in the initial region $S$. To implement the RV cutoff explicitly for predicting the distribution of a cosmological parameter $Q$, it is necessary to consider the joint finitely produced distribution $\rho(V, V_{Q_n}; \phi_0, Q_0)$ for the reheating volume $V(R)$ and the portion $V_{Q_n}$ of the reheating volume in which $Q = Q_n$. (As before, $\phi_0$ and $Q_0$ are the values in the initial $H$-region.) If the distribution $\rho(V, V_{Q_n}; \phi_0, Q_0)$ is found, one can determine the mean volume $\langle V_{Q_n} | V \rangle$ while the total reheating volume $V$ is held fixed,

$$\langle V_{Q_n} | V \rangle = \frac{\int \rho(V, V_{Q_n}; \phi_0, Q_0) V_{Q_n} dV_{Q_n}}{\rho(V; \phi_0, Q_0)}.$$  

Then one computes the probability of finding the value of $Q$ in the interval $[Q_R, Q_R + dQ]$ at a random point in the volume $V$,

$$p(Q = Q_R; V) = \frac{\langle V_{Q_n} | V \rangle}{V}. \tag{8}$$

The RV cutoff defines the probability distribution $p(Q)$ for an observable $Q$ as the limit of the distribution $p(Q; V)$ at large $V$,

$$p(Q) = \lim_{V \to \infty} \frac{\langle V_{Q_n} | V \rangle}{V}. \tag{9}$$

One expects that this limit is independent of the initial values $\phi_0, Q_0$ because the large volume $V$ is generated by regions that spent a very long time in the self-reproduction regime and forgot the initial conditions.

In Ref. [53] I derived equations from which the distributions $\rho(V, V_{Q_n}; \phi_0, Q_0)$ and $\rho(V; \phi_0, Q_0)$ can be in principle determined. However, a direct computation of the limit $V \to \infty$ (for instance, by a numerical method) will be cumbersome since the relevant probabilities are exponentially small in that limit. One of the main results of the present article is an analytic evaluation of the limit $V \to \infty$ and a derivation of a more explicit formula, Eq. (87), for the distribution $p(Q)$. The formula shows that the distribution $p(Q)$ can be computed as a ground-state eigenfunction of a certain modified Fokker-Planck equation. The explicit representation also proves that the limit (9) exists, is gauge-invariant, and is independent of the initial conditions $\phi_0$ and $Q_0$.

It was argued qualitatively in Ref. [53] that the RV measure does not suffer from the youngness paradox. In this article I demonstrate the absence of the youngness paradox in the RV measure by an explicit calculation. To this end, I will consider a toy model where every $H$-region starts in the fluctuation-dominated (or “self-reproduction”) regime with a constant expansion rate $H_0$ and proceeds to reheating via two possible channels. The first channel consists of a short period $\delta t_1$ of deterministic slow-roll inflation, yielding $N_1$ e-folds until reheating; the second channel has a different period $\delta t_2 \neq \delta t_1$ of deterministic inflation, yielding $N_2$ e-folds. (For simplicity, in this model one neglects fluctuations that may return the field from the slow-roll regime to the self-reproduction regime, and thus the time periods $\delta t_1$ and $\delta t_2$ are sharply defined.) Thus there are two types of reheated regions corresponding to the two possible slow-roll channels. The task is to compute the relative volume-weighted probability $P(2)/P(1)$ of regions of these types within the reheating surface. (Essentially the same model was considered, e.g., in Refs. [12, 21, 23, 27]. See Fig. 2 for a sketch of the potential $V(\phi)$ in this model.)

This toy model serves as a litmus test of measure prescriptions. The “holographic” or “worldline” prescription yields $P(2)/P(1)$ equal to the probability ratio of exiting through the two channels for a single comoving worldline. This probability ratio depends on the initial conditions. Thus, the worldline measure is (by design) blind to the volume growth during the slow-roll periods. On the other hand, the volume-weighted prescriptions of Refs. [21, 27] both yield

$$\frac{P(2)}{P(1)} = \frac{\exp(3N_2)}{\exp(3N_1)}.$$  \tag{10}$$

rewarding the reheated $H$-regions that went through channel $j$ by the additional volume factor $\exp(3N_j)$. This ratio is now independent of the initial conditions. For comparison, an equal-time cutoff gives

$$\frac{P(2)}{P(1)} = \frac{\exp[3N_2 - (3H_{\max} - \Gamma_1 - \Gamma_2)\delta t_2]}{\exp[3N_1 - (3H_{\max} - \Gamma_1)\delta t_1]}.$$  \tag{11}$$

The overwhelming exponential dependence on $\delta t_1$ and $\delta t_2$ manifests the youngness paradox: Even a small difference
\( \delta t_2 - \delta t_1 \) in the duration of the slow-roll inflationary epoch leads to the exponential bias towards the “younger” universes. The bias persists regardless of the choice of the time gauge \( \text{23} \), essentially because the presence of \( \delta t_1 \) and \( \delta t_2 \) in the ratio \( P(2)/P(1) \) cannot be eliminated by using a different time coordinate.\(^4\) One expects that the RV measure will be free from this bias because the RV prescription does not involve the time coordinate \( R \) and \( V \). The RV measure will be free from this bias because the \( R \) and \( V \) are independent of the slow-roll durations \( \delta t_{1,2} \). The RV-regulated result [shown in Eq. \( \text{33} \) below] depends only on the gauge-invariant quantities such as \( N_1 \) and \( N_2 \) and is, in general, different from Eq. \( \text{10} \). A calculation for an analogous landscape model was performed in Ref. \( \text{23} \), yielding a result qualitatively similar to Eq. \( \text{33} \).

These calculations confirm that the RV measure has the desirable properties expected of a volume-based measure: coordinate invariance, independence of initial conditions, and the absence of the youngness paradox. Thus the RV measure is a promising solution to the long-standing problem of obtaining probabilities in models of eternal inflation. Ultimately, the viability of the RV measure proposal will depend on its performance in various example cases. In the calculations available so far, it is found that RV measure yields results that do not identically coincide with the results of any other measure proposal. Hence, the RV measure is not equivalent to earlier proposals and needs to be studied in detail.

As formulated here and in Ref. \( \text{53} \), the RV measure prescription is directly applicable only to comparisons of reheating volumes, or in general of terminal states in the landscape (such as the anti-de Sitter bubbles). The RV proposal needs to be extended to predicting distributions of properties not directly related to terminal states, such as the relative number of observations performed in different nonterminal bubbles. Then it will be possible to investigate whether the RV measure suffers from the “Boltzmann brain” problem or from other difficulties encountered by some previous measure proposals.

An extension of the RV measure to landscape scenarios can be achieved in several ways. For instance, one can consider the set of all possible future evolutions of a single nonterminal bubble and define the ensemble \( E_N \) of evolutions yielding a finite total number \( N \) of daughter bubbles (of all types). One can also consider the ensemble \( E'_N \) of evolutions yielding a finite total number \( N \) of observers in bubbles of all types. After computing the distribution of some desired quantity by counting the observations made within the finite set of \( N \) bubbles (or observers), the cutoff parameter \( N \) can be increased to infinity. It remains to be seen whether the limit distributions are different for differently defined ensembles, such as \( E_N \) and \( E'_N \), and if so, which definition is more suitable. Future work will show whether some extension of the RV measure can provide a satisfactory answer to the problem of predictions in eternal inflation.

II. OVERVIEW OF THE RESULTS

In this section I describe the central results of this paper; in particular, I develop simplified mathematical procedures for practical calculations in the RV measure. For convenience of the reader, the results are stated here without proof, while the somewhat lengthy derivations are given in Sec. III.

A. Preliminaries

I consider a model of slow-roll inflation driven by an inflaton \( \phi \) with the action

\[
\int \left[ \frac{R}{16\pi G} + \frac{1}{2} (\partial_{\mu} \phi)^2 - V(\phi) \right] \sqrt{-gd^4x}. \tag{12}
\]

In the semiclassical stochastic approach to inflation,\(^5\) the semiclassical dynamics of the field \( \phi \) averaged over an \( H \)-region is regarded as a superposition of a deterministic slow roll,

\[
\dot{\phi} = v(\phi) \equiv -\frac{V_{,\phi}(\phi)}{3H(\phi)} = -\frac{H_{,\phi}}{4\pi G}, \tag{13}
\]

and a random walk with root-mean-squared step size

\[
\sqrt{\langle \delta \phi \rangle^2} = \frac{H(\phi)}{2\pi} \equiv \sqrt{\frac{2D(\phi)}{H(\phi)}}, \quad D \equiv \frac{H^3}{8\pi^2}, \tag{14}
\]

during time intervals \( \delta t = H^{-1} \), where \( H(\phi) \) is the function defined by

\[
H(\phi) \equiv \sqrt{\frac{8\pi G}{3}} V(\phi). \tag{15}
\]

A useful effective description of the evolution of the field at time scales \( \delta t \lesssim H^{-1} \) can be given as

\[
\phi(t + \delta t) = \phi(t) + v(\phi)\delta t + \xi(t) \sqrt{2D(\phi)\delta t}, \tag{16}
\]

where \( \xi(t) \) is a normalized “white noise” function,

\[
\langle \xi \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \delta(t - t'), \tag{17}
\]

\(^4\) It should be noted that the youngness bias becomes very small, possibly even negligible, if one uses the number \( N \) of inflationary e-foldings as the time variable rather than the proper time \( t \). I am grateful to A. Linde and A. Vilenkin for bringing this to my attention.

\(^5\) See Refs. \( \text{57, 58, 59} \) for early works on the stochastic approach and Refs. \( \text{7, 8} \) for pedagogical reviews.
which is approximately statistically independent between different $H$-regions. This stochastic process describes the evolution $\phi(t)$ and the accompanying cosmological expansion of space along a single comoving worldline. For simplicity, we assume that inflation ends in a given horizon-size region when $\phi = \phi_*$, where $\phi_*$ is a fixed value such that the relative change of $H$ during one Hubble time $\delta t = H^{-1}$ becomes of order 1, i.e.

$$\left| \frac{H_\phi v H^{-1}}{H} \right|_{\phi = \phi_*} = \left| \frac{H^2}{4\pi G H'^2} \right|_{\phi = \phi_*} \sim 1. \quad (18)$$

From the point of view of the stochastic approach, an inflationary model is fully specified by the kinetic coefficients $D, D_\phi$. These coefficients are found from Eqs. (13)–(15) in models of canonical slow-roll inflation and by suitable analogues in other models.

Dynamics of any fluctuating cosmological parameter $Q$ is described in a similar way. One assumes that the value of $Q$ is homogeneous in $H$-regions. The evolution of $Q$ is described by an effective Langevin equation,

$$Q(t + \delta t) = Q(t) + v_Q(\phi, Q)\delta t + \xi_Q(t)\sqrt{2D_Q(\phi, Q)\delta t}, \quad (19)$$

where the kinetic coefficients $D_Q$ and $v_Q$ can be computed, similarly to $D$ and $v$, from first principles. For simplicity we assume that the “noise variable” $\xi_Q$ is independent of the “noise” $\xi$ used in Eq. (10). A correlated set of noise variables can be considered as well (see e.g. Ref. [38]).

B. Probability of finite inflation

Let us consider an initial $H$-region $S$ where the inflaton field $\phi$ as well as the parameter $Q$ are homogeneous and have values $\phi = \phi_0$ and $Q = Q_0$. For convenience we assume that reheating starts when $\phi = \phi_*$, and the Planck energy scales are reached at $\phi = \phi_{\text{PH}}$ independently of the value of $Q$. (If necessary, the field variables $\phi, Q$ can be redefined to achieve this.)

Although eternal inflation to the future of $S$ is almost always the case, it is possible that reheating is reached at a finite time everywhere to the future of $S$, due to a rare fluctuation. In that event, the total reheating volume $V$ to the future of $S$ is finite. The (small) probability of that event, denoted by

$$\text{Prob}(V < \infty | \phi_0, Q_0) \equiv \tilde{X}(\phi_0, Q_0), \quad (20)$$

can be found as the solution of the following nonlinear equation,

$$\frac{D}{H} \tilde{X}_\phi + \frac{D_Q}{H} \tilde{X}_{QQ} + \frac{v}{H} \tilde{X}_\phi + \frac{v_Q}{H} \tilde{X}_Q + 3\tilde{X} \ln \tilde{X} = 0, \quad (21)$$

where for brevity we dropped the subscript 0 in $\phi_0$ and $Q_0$. This basic equation, first derived in Ref. [15], is of reaction-diffusion type and can be viewed as a nonlinear modification of the Fokker-Planck equations used previously in the literature on the stochastic approach to inflation.

While $\tilde{X}(\phi, Q) \equiv 1$ is always a solution of Eq. (21), it is not the correct one for the case of eternal inflation. A nontrivial solution, $\tilde{X}(\phi, Q) \neq 1$, exists and has small values $\tilde{X}(\phi, Q) \ll 1$ for $\phi, Q$ away from the thermalization boundary. If the coefficients $D/H$ and $v/H$ happen to be $Q$-independent, the solution of Eq. (21) will be also independent of $Q$, i.e. $\tilde{X}(\phi, Q) = \tilde{X}(\phi)$, and thus determined by a simpler equation obtained from Eq. (21) by omitting derivatives with respect to $Q$,

$$\frac{D}{H} \tilde{X}_\phi + \frac{v}{H} \tilde{X}_\phi + 3\tilde{X} \ln \tilde{X} = 0. \quad (23)$$

It is easy to see that Eqs. (21) and (23) are manifestly gauge-invariant. Indeed, a change of time variable according to Eq. (4) results in dividing the coefficients $D, D_Q, v, v_Q, H$ by the function $T(\phi)$ [24], which leaves Eqs. (21) and (23) unmodified.

Some approximate solutions of Eq. (23) were given in Ref. [15], where it was shown that $\tilde{X}(\phi)$ is typically exponentially small for $\phi$ in the inflationary regime. While small, $\tilde{X}(\phi)$ is never zero; hence, there is a well-defined statistical ensemble of initial $H$-regions that have a finite total reheating volume in the future. The construction of the RV measure relies on this fact.

C. Finitely produced volume

In a scenario where eternal inflation is possible, we now consider the probability density $\rho(V; \phi_0)$ of having a finite total reheating volume $V$ to the comoving future of an initial $H$-region with homogeneous value $\phi = \phi_0$ (focusing attention at first on the case of inflation driven by a single scalar field). The distribution $\rho(V; \phi_0)$ is normalized to the overall probability $\bar{X}(\phi_0)$ of having a finite total reheating volume,

$$\int_0^\infty \rho(V; \phi_0)dV = \bar{X}(\phi_0). \quad (24)$$

The distribution $\rho(V; \phi_0)$ can be calculated by first determining the generating function $g(z; \phi_0)$, which is defined by

$$g(z; \phi_0) \equiv \langle e^{-zV} \rangle_{V < \infty} \equiv \int_0^\infty e^{-zV} \rho(V; \phi_0)dV. \quad (25)$$

This generating function is a solution of the nonlinear Fokker-Planck equation,

$$\hat{L}g + 3g \ln g = 0, \quad (26)$$

where the differential operator $\hat{L}$ is defined by

$$\hat{L} \equiv \frac{D}{H} \partial_\phi \partial_\phi + \frac{v}{H} \partial_\phi. \quad (27)$$
In the case of several fields, say \( \phi \) and \( Q \), one needs to use the corresponding Fokker-Planck operator such as

\[
\hat{L} = \frac{D_{\phi\phi}}{H}\partial_\phi \partial_\phi + \frac{D_{QQ}}{H}\partial_Q \partial_Q + \frac{v_\phi}{H}\partial_\phi + \frac{v_Q}{H}\partial_Q. \tag{28}
\]

The boundary conditions for Eq. (26) are

\[
g(z; \phi, Q) = 1 \quad \text{for} \quad \{ \phi, Q \} \in \text{Planck boundary}, \tag{29}
\]

\[
g(z; \phi, Q) = e^{-zH^{-3}(\phi, Q)} \quad \text{for} \quad \{ \phi, Q \} \in \text{reheating boundary}. \tag{30}
\]

Note that the parameter \( z \) enters the boundary conditions but is not explicitly involved in Eq. (26). Also, the operator \( \hat{L} \) and Eq. (26) are manifestly gauge-invariant with respect to redefinitions of the form \( \{ \phi, Q \} \rightarrow \{ \phi', Q' \} \).

The generating function \( g \) plays a central role in the calculations of the RV cutoff. It will be shown below that the solution \( g(z; \phi, Q) \) of Eq. (26) needs to be obtained only at an appropriately determined negative value of \( z \). This solution can be obtained by a numerical method or through an analytic approximation if available.

### D. Asymptotics of \( \rho(V; \phi_0) \)

The finitely produced distribution \( \rho(V; \phi) \) can be found through the inverse Laplace transform of the function \( g(z; \phi) \),

\[
\rho(V; \phi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dz e^{zV} g(z; \phi), \tag{31}
\]

where the integration contour in the complex \( z \) plane can be chosen along the imaginary axis. The asymptotic behavior of \( \rho(V; \phi) \) at large \( V \) is determined by the rightmost singularity of \( g(z; \phi) \) in the complex \( z \) plane. It turns out that the function \( g(z; \phi) \) always has a singularity at a real, nonpositive \( z = z_* \) of the type

\[
g(z; \phi) = g(z_*; \phi) + \sigma(\phi) \sqrt{z - z_*} + O(z - z_*), \tag{32}
\]

where \( z_* \) and \( \sigma(\phi) \) are determined as follows. One considers the \((z\text{-dependent})\) linear operator

\[
\hat{\tilde{L}} = \hat{L} + 3(\ln g(z_*; \phi) + 1), \tag{33}
\]

where \( \hat{L} \) is the Fokker-Planck operator described above. For \( z > 0 \) this operator is invertible in the space of functions \( f(\phi) \) satisfying zero boundary conditions. The value of \( z_* \) turns out to be the algebraically largest real number (in any case, \( z_* \leq 0 \)) such that there exists an eigenfunction \( \sigma(\phi) \) of \( \hat{\tilde{L}} \) with zero eigenvalue and zero boundary conditions,

\[
\hat{\tilde{L}} \sigma(\phi) = 0, \quad \sigma(\phi_*) = \sigma(\phi_{n1}) = 0. \tag{34}
\]

The specific normalization of the eigenfunction \( \sigma(\phi) \) can be derived analytically but is unimportant for the present calculations.

The singularity type shown in Eq. (32) determines the leading asymptotic of \( \rho(V; \phi) \) at \( V \to \infty \):

\[
\rho(V; \phi) \approx \frac{1}{2\sqrt{\pi}} \sigma(\phi) V^{-3/2} e^{z_*V}. \tag{35}
\]

The explicit form (35) allows one to investigate the moments of the distribution \( \rho(V; \phi) \). It is clear that all the moments are finite as long as \( z_* < 0 \). However, if \( z_* = 0 \) all the moments diverge, namely for \( n \geq 1 \) we have

\[
\langle V^n \rangle = \int_0^\infty \rho(V; \phi) V^n dV \propto \int_0^\infty V^{n-3/2} dV = \infty. \tag{36}
\]

The case \( z_* = 0 \) corresponds to the “transition point” analyzed in Ref. [14], corresponding to \( \Omega = 1 \) in their notation. This is the borderline case between the presence and the absence of eternal inflation. The fact that \( z_* = 0 \) in the borderline case can be seen directly by noting that the Fokker-Planck operator \( \hat{L} + 3 \) has in that case a zero eigenvalue, meaning that the 3-volume of equal-time surfaces does not expand with time (reheating of some regions is perfectly compensated by inflationary expansion of other regions). In that case, the only solution \( g(z = 0; \phi) = X(\phi) \) of Eq. (26) is \( X = 0 \) because there are no eternally inflating comoving geodesics. Hence \( \ln g(z = 0; \phi) = 0 \), and so the operator \( \hat{L} \) is simply \( \hat{L} = \hat{L} + 3 \). It follows that the operator \( \hat{L} \) also has a zero eigenvalue at \( z = 0 \), and thus \( z = z_* = 0 \) is the dominant singularity of \( g(z; \phi) \). This argument reproduces and generalizes the results obtained in Ref. [14] where direct calculations of various moments of \( \rho(V; \phi) \) were performed for the case of the absence of eternal inflation.

We note that the only necessary ingredients in the computation of \( \sigma(\phi) \) is the knowledge of the singularity point \( z_* \) and the corresponding function \( g(z_*; \phi) \), which is a solution of the nonlinear reaction-diffusion equation (26). Determining \( z_* \) and \( g(z_*; \phi) \) in a given inflationary model does not require extensive numerical simulations.

### E. Distribution of a fluctuating field

Above we denoted by \( Q \) a cosmological parameter that fluctuates during inflation but is in principle observable after reheating. One of the main questions to be answered using a multiverse measure is to derive the probability distribution \( p(Q) \) for the values of \( Q \) observed in a “typical” place in the multiverse. I will now present a formula for the distribution \( p(Q) \) in the RV cutoff. This formula is significantly more explicit and lends itself more easily to practical calculations than the expressions first shown in Ref. [53].

As in the previous section, we assume that the dynamics of the inflaton field \( \phi \) and the parameter \( Q \) is described by a suitable Fokker-Planck operator \( \hat{L} \), e.g. of the form (25), and that reheating occurs at \( \phi = \phi_* \) independently of the value of \( Q \). We then consider
Eq. (20) for the function \( g(z; \phi, Q) \) and the operator \( \hat{L} \equiv \hat{L} + 3(\ln g + 1) \); we need to determine the value \( z^* \) at which \( g(z; \phi, Q) \) has a singularity. The operator \( \hat{L} \) has an eigenfunction with zero eigenvalue for this value of \( z \). This eigenfunction \( f_0(z; \phi, Q) \) needs to be determined with zero boundary conditions (at reheating and Planck boundaries). Then the RV-regulated distribution of \( Q \) at reheating is

\[
p(Q_R) = \text{const} \left[ \frac{\partial f_0(z; \phi, Q)}{\partial \phi} D_{\phi\phi} e^{-z_0 H^{-3}} \right]_{\phi = \phi_*, Q = Q_R} H^4,
\]

where the normalization constant needs to be chosen such that \( \int p(Q_R) dQ_R = 1 \). The derivation of this result occupies Sec. III D.

We note that \( f_0 \) is the eigenfunction \( f_0 \) of a gauge-invariant operator, and that the result in Eq. (37) depends on the kinetic coefficients only through the gauge-independent ratio \( D/H \) times the volume factor \( H^{-3} \). The distribution \( p(Q_R) \) is independent of the initial conditions, which is due to a specific asymptotic behavior of the finitely produced volume distributions, as shown in Sec. III D.

### F. Toy model of inflation

We now apply the RV cutoff to the toy model described at the end of Sec. I. We consider a model of inflation driven by a scalar field with a potential shown in Fig. 2. For the purposes of the present argument, we may assume that there is exactly zero “diffusion” in the deterministic regimes \( \phi_0^{(1)} < \phi < \phi_1 \) and \( \phi_2 < \phi < \phi_0^{(2)} \), while the range \( \phi_1 < \phi < \phi_2 \) is sufficiently wide to allow for eternal self-reproduction. Thus there are two slow-roll channels that produce respectively \( N_1 \) and \( N_2 \) e-folds of slow-roll inflation after exiting the self-reproduction regime. Since the self-reproduction range generates arbitrarily large volumes of space that enter both the slow-roll channels, the total reheating volume going through each channel is infinite. We apply the RV cutoff to the problem of computing the regularized ratio of the reheating volumes in regions of types 1 and 2.

In this toy model it is possible to obtain the results of the RV cutoff using analytic approximations. The required calculations are somewhat lengthy and can be found in Sec. III D. The result for a generic case where one of the slow-roll channels has many more e-folds than the other (say, \( N_2 \gg N_1 \)) can be written as

\[
\frac{P(2)}{P(1)} \approx O(1) \frac{H^{-3}(\phi_0^{(2)})^2 \exp[3N_2]}{H^{-3}(\phi_0^{(1)})^2 \exp[3N_1]} \exp[3N_{12}],
\]

where we have defined

\[
N_{12} = \frac{\pi^2}{\sqrt{2H_0^3}} (\phi_2 - \phi_1)^2, \quad H_0^2 = \frac{8\pi G}{3} V_0.
\]

The pre-exponential factor \( O(1) \) can be computed numerically, as outlined in Sec. III D.

We note that the ratio (38) is gauge-invariant and does not involve any spacetime coordinates. This result can be interpreted as the ratio of volumes \( e^{3N_1} \) and \( e^{3N_2} \) gained during the slow-roll regime in the two channels multiplied by a correction factor \( e^{3N_{12}} \). The dimensionless number \( N_{12} \) can be suggestively interpreted (up to the factor \( \sqrt{2} \)) as the mean number of “steps” of size \( \delta \phi \sim \frac{1}{2} H_0 \) required for a random walk to reach the boundary of the flat region \([\phi_1, \phi_2] \) starting from the middle point \( \phi_0 \equiv \frac{1}{2} (\phi_1 + \phi_2) \). Since each of the “steps” of the random walk takes a Hubble time \( H_0^{-1} \) and corresponds to one e-folding of inflation, the volume factor gained during such a traversal will be \( e^{3N_{12}} \). Note that the correction factor increases the probability of channel 2 that was already the dominant one due to the larger volume factor \( e^{3N_{12}} \gg e^{3N_1} \). Depending on the model, this factor may be a significant modification of the ratio (10) obtained in previously used volume-based measures.

### III. DERIVATIONS

#### A. Positive solutions of nonlinear equations

It is not easy to demonstrate directly the existence of nontrivial solutions of reaction-diffusion equations such as Eq. (21). However, there is a connection between solutions of such nonlinear equations and solutions of the linearized equations. Rigorous results are available in the mathematical literature on nonlinear functional analysis and bifurcation theory.

Heuristically, consider a solution of Eq. (28) that is approximately \( \bar{X}(\phi) \approx 1 \). The equation can be linearized in the neighborhood of \( \bar{X} \approx 1 \) as \( \bar{X} = 1 - \chi(\phi) \) and yields the Fokker-Planck (FP) equation

\[
\left[ \hat{L} + 3 \right] \chi = 0, \quad \hat{L} \equiv \frac{D}{H} \partial_{\phi\phi} + \frac{v}{H} \partial_{\phi}.
\]

The FP operator \( \hat{L} + 3 \) is adjoint to the operator

\[
\left[ \hat{L} + 3 \right] P = \partial_{\phi\phi} \left( \frac{D}{H} P \right) - \partial_{\phi} (\frac{v}{H} P) + 3P.
\]

Figure 2: A model potential with a flat self-reproduction regime \( \phi_1 < \phi < \phi_2 \) and deterministic slow-roll regimes \( \phi_0^{(1)} < \phi < \phi_1 \) and \( \phi_2 < \phi < \phi_0^{(2)} \) producing \( N_1 \) and \( N_2 \) inflationary e-folds respectively. In the interval \( \phi_1 < \phi < \phi_2 \) the potential \( V(\phi) \) is assumed to be constant, \( V(\phi) = V_0 \).
which enters the FP equation for the 3-volume distribution $P(\phi, t)$ in the $e$-folding time parameterization. If eternal inflation is allowed in a given model, the operator $\hat{L} + 3$ has a positive eigenvalue. The largest eigenvalue of that operator is zero in the borderline case when eternal inflation is just about to set in. The spectrum of the operator $\hat{L} + 3$ is the same as that of the adjoint operator $\hat{L}^\dagger + 3$. Hence, in the borderline case the largest eigenvalue of the operator $\hat{L} + 3$ will be zero, and there will exist a nontrivial, everywhere nonnegative solution $\chi$ of Eq. (10). Thus, heuristically one can expect that a nontrivial solution $\hat{X}(\phi) \not\equiv 1$ will exist away from the borderline case, i.e. when the operator $\hat{L} + 3$ has a positive eigenvalue.

Following the approach of Ref. [14], one can imagine a family of inflationary models parameterized by a label $\Omega$, such that eternal inflation is allowed when $\Omega < 1$. Then Eq. (23) will have only the trivial solution, $\hat{X}(\phi) \equiv 1$, for $\Omega = 1$. The case $\Omega = 1$ where eternal inflation is on the borderline of existence is the bifurcation point for the solutions of Eq. (23). At the bifurcation point, a nontrivial solution $\hat{X}(\phi) \not\equiv 1$ appears, branching off from the trivial solution. A rigorous theory of bifurcation can be developed using methods of nonlinear functional analysis (see e.g. chapter 9 of the book [60]). In particular, it can be shown that a nontrivial solution of a nonlinear equation, such as Eq. (23), exists if and only if the dominant eigenvalue of the linearized operator $\hat{L} + 3$ with zero boundary conditions is positive.

There remains a technical difference between the eigenvalue problem for the operator $\hat{L} + 3$ with zero boundary conditions and with the “no-diffusion” boundary conditions normally used in the stochastic approach, \[
\frac{\partial}{\partial \phi} [D(\phi)P(\phi)] = 0. \tag{42}
\]
It was demonstrated in Ref. [15] that the eigenvalue of $\hat{L} + 3$ with the boundary conditions (24) is positive if a nontrivial solution of Eq. (23) exists. In principle, the eigenvalue of $\hat{L} + 3$ with zero boundary conditions is not the same as the eigenvalue of the same operator with the boundary conditions (24). One can have a borderline case when one of these two eigenvalues is positive while the other is negative. In this case, the two criteria for the presence of eternal inflation (based on the positivity of the two different eigenvalues) will disagree. However, the alternative boundary conditions are imposed at reheating, i.e. in the regime of very small fluctuations where the value of the eigenfunction $P(\phi)$ is exponentially small compared with its values in the fluctuation-dominated range of $\phi$. Hence, the difference between the two eigenvalues is always exponentially small (it is suppressed at least by the factor $e^{-3N}$, where $N$ is the number of $e$-folds in the deterministic slow-roll regime before reheating). Therefore, we may interpret the discrepancy as a limitation inherent in the stochastic approach to inflation. In other words, one cannot use the stochastic approach to establish the presence of eternal inflation more precisely than with the accuracy $e^{-3N}$. Barring an extremely fine-tuned borderline case, this accuracy is perfectly adequate for establishing the presence or absence of eternal inflation.

The main nonlinear equation in the calculations of the RV cutoff is Eq. (24) for the generating function $g(z; \phi)$. That equation differs from Eq. (24) mainly by the presence of the parameter $z$ in the boundary conditions. Therefore, solutions of Eq. (24) may exist for some values of $z$ but not for other values. Note that $g(z = 0; \phi) = X(\phi)$; hence, nontrivial solutions $g(z; \phi)$ exist for $z = 0$ under the same conditions as nontrivial solutions $\hat{X}(\phi) \not\equiv 1$ of Eq. (23). While it is certain that solutions $g(z; \phi)$ exist for $z \geq 0$, there may be values $z < 0$ for which no real-valued solutions $g(z; \phi)$ exist at all. However, the calculations in the RV cutoff require only to compute $g(z; \phi)$ for a certain value $z_* > 0$, which is the algebraically largest value $z$ where $g(z; \phi)$ has a singularity in the $z$ plane. The structure of that singularity will be investigated in detail below, and it will be shown that $g(z_*; \phi)$ is finite while $\partial g/\partial z \sim (z - z_*)^{-1/2}$ diverges at $z = z_*$. Hence, the solution $g(z; \phi)$ remains well-defined at least for all real $z$ in the interval $[z_*, +\infty]$. It follows that $g(z; \phi)$ may be obtained e.g. by a numerical solution of a well-conditioned problem with $z = z_* + \varepsilon$, where $\varepsilon > 0$ is a small real constant.

### B. Nonlinear Fokker-Planck equations

In this section I derive Eq. (24), closely following the derivation of Eq. (21) in Ref. [14].

We begin by considering the case when inflation is driven by a single scalar field $\phi$, such that reheating is reached at $\phi = \phi_*$. Let $\rho(\mathcal{V}; \phi_0)$ be the probability density of obtaining the finite reheated volume $\mathcal{V}$. We will derive an equation for a generating function of the distribution of volume, rather than an equation directly for $\rho(\mathcal{V}; \phi_0)$. Since the volume $\mathcal{V}$ is by definition non-negative, it is convenient to define a generating function $g(z; \phi_0)$ through the expectation value of the expression $\exp(-z \mathcal{V})$, where $z > 0$ is the formal parameter of the generating function,

\[
g(z; \phi_0) \equiv \langle e^{-z \mathcal{V}} \rangle_{\mathcal{V} < \infty} \equiv \int_0^{\infty} e^{-z \mathcal{V}} \rho(\mathcal{V}; \phi_0) d\mathcal{V}. \tag{43}
\]

Note that for any $z$ such that $\text{Re } z \geq 0$ the integral in Eq. (43) converges, and the events with $\mathcal{V} = +\infty$ are automatically excluded from consideration. However, we use the subscript “$\mathcal{V} < \infty$” to indicate explicitly that the statistical average is performed over a subset of all events. The distribution $\rho(\mathcal{V}; \phi_0)$ is not normalized to unity; instead, the normalization is given by Eq. (6).

The parameter $z$ has the dimension of inverse 3-volume. Physically, this is the 3-volume measured along the reheating surface and hence is defined in a gauge-invariant manner. If desired for technical reasons, the
variable $z$ can be made dimensionless by a constant rescaling.

The generating function $g(z; \phi)$ has the following multiplicative property: For two statistically independent regions that have initial values $\phi = \phi_1$ and $\phi = \phi_2$ respectively, the sum of the (finitely produced) reheating volumes $V_1 + V_2$ is distributed with the generating function
\[ \langle e^{-z(V_1 + V_2)} \rangle = \langle e^{-zV_1} \rangle \langle e^{-zV_2} \rangle = g(z; \phi_1)g(z; \phi_2). \] (44)

We now consider an $H$-region at some time $t$, having an arbitrary value $\phi(t)$ not yet in the reheating regime. Suppose that the finitely produced volume distribution for this $H$-region has the generating function $g(z; \phi)$. After time $\delta t$ the initial $H$-region grows to $N \equiv e^{3H\delta t}$ statistically independent, “daughter” $H$-regions. The value of $\phi$ in the $k$-th daughter region ($k = 1, \ldots, N$) is found from Eq. (16),
\[ \phi_k = \phi + v(\phi)\delta t + \xi_k \sqrt{2D(\phi)\delta t}, \] (45)
where the “noise” variables $\xi_k$ ($k = 1, \ldots, N$) are statistically independent because they describe the fluctuations of $\phi$ in causally disconnected $H$-regions. The finitely produced volume distribution for the $k$-th daughter region has the generating function $g(z; \phi_k)$. The combined reheating volume of the $N$ daughter regions must be distributed with the same generating function as reheating volume of the original $H$-region. Hence, by the multiplicative property we obtain
\[ g(z; \phi) = \prod_{k=1}^{N} g(z; \phi_k). \] (46)

We can average both sides of this equation over the noise variables $\xi_k$ to get
\[ g(z; \phi) = \left\langle \prod_{k=1}^{N} g(z; \phi_k) \right\rangle_{\xi_1, \ldots, \xi_N}. \] (47)
Since all the $\xi_k$ are independent, the average splits into a product of $N$ identical factors,
\[ g(z; \phi) = \left[ \left\langle g(z; \phi + v(\phi)\delta t + \sqrt{2D(\phi)\xi}) \right\rangle_{\xi} \right]^N. \] (48)

The derivation now proceeds as in Ref. [13]. We first compute, to first order in $\delta t$,
\[ \left\langle g(z; \phi + v(\phi)\delta t + \sqrt{2D(\phi)\xi}) \right\rangle_{\xi} = g + (vg,\phi + Dg,\phi)\delta t. \] (49)
Substituting $N = e^{3H\delta t}$ and taking the logarithmic derivative of both sides of Eq. (48) with respect to $\delta t$ at $\delta t = 0$, we then obtain
\[ 0 = \frac{\partial}{\partial \delta t} \ln g(z; \phi) = 3H \ln g + \frac{vg,\phi + Dg,\phi}{g}. \] (50)

The equation for $g(z; \phi)$ follows,
\[ Dg,\phi + vg,\phi + 3H \ln g = 0. \] (51)

This is formally the same as Eq. (21). However, the boundary conditions for Eq. (51) are different. The condition at the end-of-inflation boundary $\phi = \phi_*$ is
\[ g(z; \phi_*) = e^{-zH^{-3}(\phi_*)} \] (52)
because an $H$-region starting with $\phi = \phi_*$ immediately reheats and produces the reheating volume $H^{-3}(\phi_*)$. The condition at Planck boundary $\phi_{Pl}$ (if present), or other boundary where the effective field theory breaks down, is “absorbing,” i.e. regions that reach $\phi = \phi_{Pl}$ do not generate any reheating volume:
\[ g(z; \phi_{Pl}) = 1. \] (53)

The variable $z$ enters Eq. (51) as a parameter and only through the boundary conditions. At $z = 0$ the solution is $g(0; \phi) = X(\phi)$.

A fully analogous derivation can be given for the generating function $g(z; \phi_Q, Q)$ in the case when additional fluctuating fields, denoted by $Q$, are present. The generating function $g(z; \phi_Q, Q_0)$ is defined by
\[ g(z; \phi_Q, Q_0) = \int_0^\infty e^{-zV} \rho(V; \phi_Q, Q_0) dV, \] (54)
where $\rho(V; \phi_Q, Q_0)$ is the probability density for achieving a total reheating volume $V$ in the future of an $H$-region with initial values $\phi_0, Q_0$ of the fields. In the general case, the fluctuations of the fields $\phi, Q$ can be described by the Langevin equations
\[ \phi(t + \delta t) = \phi(t) + v_\phi \delta t + \xi_{\phi} \sqrt{2D_{\phi\phi} \delta t} + \xi_Q \sqrt{2D_{\phi Q} \delta t}, \] (55)
\[ Q(t + \delta t) = Q(t) + v_Q \delta t + \xi_{\phi} \sqrt{2D_{\phi Q} \delta t} + \xi_Q \sqrt{2D_{QQ} \delta t}, \] (56)
where the “diffusion” coefficients $D_{\phi\phi}$, $D_{\phi Q}$, and $D_{QQ}$ have been introduced, as well as the “slow roll” velocities $v_\phi$ and $v_Q$ and the “noise” variables $\xi_{\phi}$ and $\xi_Q$. The resulting equation for $g(z; \phi_0, Q_0)$ is (dropping the subscript $0$)
\[ \dot{L}g + 3H \ln g = 0, \] (57)
where the differential operator $\dot{L}$ is defined by
\[ \dot{L} = \frac{D_{\phi\phi}}{H} \partial_\phi \partial_\phi + \frac{2D_{\phi Q}}{H} \partial_\phi \partial_Q + \frac{D_{QQ}}{H} \partial_Q \partial_Q + \frac{v_\phi \partial_\phi + v_Q \partial_Q}{H}. \] (58)
The ratios $D_{\phi\phi}/H$, etc., are manifestly gauge-invariant with respect to time parameter changes of the form [14].

Performing a redefinition of the fields if needed, one may assume that reheating is reached when $\phi = \phi_*$ independently of the value of $Q$. Then the boundary conditions for Eq. (57) at the reheating boundary can be written as
\[ g(z; \phi_*, Q) = e^{-zH^{-3}(\phi_*, Q)}. \] (59)

The Planck boundary still has the boundary condition $g(z; \phi_{Pl}) = 1$. 
C. Singularities of \( g(z) \)

For simplicity we now focus attention on the case of single-field inflation; the generating function \( g(z; \phi) \) then depends on the initial value of the inflaton field \( \phi \). The corresponding analysis for multiple fields is carried out as a straightforward generalization.

By definition, \( g(z; \phi) \) is an integral of a probability distribution \( \rho(V; \phi) \) times \( e^{-zV} \). It follows that \( g(z; \phi) \) is analytic in \( z \) and has no singularities for Re \( z \) > 0. Then the probability distribution \( \rho(V; \phi) \) can be recovered from the generating function \( g(z; \phi) \) through the inverse Laplace transform,

\[
\rho(V; \phi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dz e^{zV} g(z; \phi),
\]

(60)

where the integration contour in the complex \( z \) plane can be chosen along the imaginary axis because all the singularities of \( g(z; \phi) \) are to the left of that axis. The RV cutoff procedure depends on the limit of \( \rho(V; \phi) \) and related distributions at \( V \to \infty \). The asymptotic behavior of \( g(z; \phi) \) is determined by the type and the location of the right-most singularity of \( g(z; \phi) \) in the half-plane Re \( z \) < 0. For instance, if \( z = z_* \) is such a singularity, the asymptotic is \( \rho(V; \phi) \propto e^{zV} \). The prefactor in this expression needs to be determined; for this, a detailed analysis of the singularities of \( g(z; \phi) \) will be carried out.

It is important to verify that the singularities of \( g(z; \phi) \) are \( \phi \)-independent. We first show that solutions of Eq. (57) cannot diverge at finite values of \( \phi \). If that were the case and say \( g(z; \phi) \to \infty \) as \( \phi \to \phi_1 \), the function \( \ln \ln g \) as well as derivatives \( g, \phi \) and \( g,\phi \) would diverge as well. Then

\[
\lim_{\phi \to \phi_1} \partial_\phi \ln \ln g = \lim_{\phi \to \phi_1} \partial_\phi g \partial_\phi \ln g = \infty.
\]

(61)

It follows that the term \( g(\ln g) \) is negligible near \( \phi = \phi_1 \) in Eq. (57) compared with the term \( \partial_\phi g \partial_\phi \ln g \) and hence also with the term \( \partial_\phi g \partial_\phi g \). In a very small neighborhood of \( \phi = \phi_1 \), the operator \( \hat{L} \) can be approximated by a linear operator \( \hat{L}_1 \) with constant coefficients, such as

\[
\hat{L} \approx \hat{L}_1 \equiv A_1 \partial_\phi \partial_\phi + B_1 \partial_\phi.
\]

(62)

Since at least one of the coefficients \( A_1, B_1 \) is nonzero at \( \phi = \phi_1 \), it follows that \( g(z; \phi) \) is approximately a solution of the linear equation \( \hat{L}_1 g = 0 \) near \( \phi = \phi_1 \). However, solutions of linear equations cannot diverge at finite values of the argument. Hence, the function \( g(z; \phi) \) cannot diverge at a finite value of \( \phi \).

The only remaining possibility is that the function \( g(z; \phi) \) has singular points \( z = z_* \) such that \( g(z_*; \phi) \) remains finite while \( \partial g/\partial z \), or a higher-order derivative, diverges at \( z = z_* \). We will now investigate such divergences and show that \( g(z; \phi) \) has a leading singularity of the form

\[
g(z; \phi) = g(z_*; \phi) + \sigma(\phi)\sqrt{z - z_*} + O(z - z_*),
\]

(63)

where \( z_* \) is a \( \phi \)-independent location of the singularity such that \( z_* \leq 0 \), while the function \( \sigma(\phi) \) is yet to be determined.

Denoting temporarily \( g_1(z; \phi) \equiv \partial g/\partial z \), we find a linear equation for \( g_1 \),

\[
\hat{L} g_1 + 3(\ln g + 1) g_1 = 0,
\]

(64)

with inhomogeneous boundary conditions

\[
g_1(\phi_*) = -H^{-3}(\phi_*) e^{-zH^{-3}(\phi_*)}, \quad g_1(\phi_{p1}) = 0.
\]

(65)

The solution \( g_1(z; \phi) \) of this linear problem can be found using a standard method involving the Green’s function. The problem with inhomogeneous boundary conditions is equivalent to the problem with zero boundary conditions but with an inhomogeneous equation. To be definite, let us consider the operator \( \hat{L} \) of the form used in Eq. (51),

\[
\hat{L} = \frac{D(\phi)}{H(\phi)} \partial_\phi \partial_\phi + \frac{v(\phi)}{H(\phi)} \partial_\phi.
\]

(66)

Then Eqs. (64)–(65) are equivalent to the inhomogeneous problem with zero boundary conditions,

\[
\hat{L} g_1 + 3(\ln g + 1) g_1 = D H^{-4} e^{-zH^{-3}} \delta'(\phi - \phi_*),
\]

(67)

\[
g_1(\phi_*) = g_1(\phi_{p1}) = 0.
\]

(68)

The solution of this inhomogeneous equation exists as long as the linear operator \( \hat{L} + 3(\ln g + 1) \) does not have a zero eigenfunction with zero boundary conditions.

Note that the operator \( \hat{L} + 3(\ln g + 1) \) is explicitly \( z \)-dependent through the coefficient \( g(z; \phi) \). Note also that \( g(z; \phi) \neq 0 \) by definition (43) for values of \( z \) such that the integral in Eq. (43) converges; hence \( \ln g \) is finite for those \( z \). Let us denote by \( G(z; \phi, \phi') \) the Green’s function of that operator with zero boundary conditions,

\[
\hat{L} G + 3(\ln g(z, \phi) + 1) G = \delta(\phi - \phi'),
\]

(69)

\[
G(z; \phi_*, \phi') = G(z; \phi_{p1}, \phi') = 0.
\]

(70)

This Green’s function is well-defined for values of \( z \) such that \( \hat{L} + 3(\ln g(z, \phi) + 1) \) is invertible. For these \( z \) we may express the solution \( g_1(z; \phi) \) of Eqs. (64)–(65) explicitly through the Green’s function as

\[
g_1(z; \phi) = -\frac{D}{H^2} e^{-zH^{-3}} \left| \frac{\partial G(z; \phi, \phi')}{\partial \phi'} \right| \bigg|_{\phi' = \phi_*}.
\]

(71)

Hence, for these \( z \) the function \( g_1(z; \phi) \equiv \partial g/\partial z \) remains finite at every value of \( \phi \). A similar argument shows that all higher-order derivatives \( \partial^n g/\partial z^n \) remain finite at every \( \phi \) for these \( z \). Therefore, the singularities of \( g(z; \phi) \) can occur only at certain \( \phi \)-independent points \( z = z_*, z = z'_* \), etc.

Since the generating function \( g(z; \phi) \) is nonsingular for all complex \( z \) with Re \( z \) > 0, it is assured that \( g_1(z; \phi) \) and \( G(z; \phi, \phi') \) exist for such \( z \). However, there will be values of \( z \) for which the operator \( \hat{L} + 3(\ln g + 1) \) has a
zero eigenfunction with zero boundary conditions, so the Green’s function $G$ is undefined. Denote by $z_*$ such a value with the algebraically largest real part; we already know that $\text{Re } z_* \leq 0$ in any case. Let us now show that the function $g_1(z; \phi)$ actually diverges when $z \to z_*$. In other words, $\lim_{z \to z_*} g_1(z; \phi) = \infty$ for every value of $\phi$.

To show this, we need to use the decomposition of the Green’s function in the eigenfunctions of the operator $\hat{L} + 3(\ln g + 1)$,

$$G(z; \phi, \phi') = \sum_{n=0}^{\infty} \frac{1}{\lambda_n(z)} f_n(\phi) f_n^*(\phi'),$$

(72)

where $f_n(z; \phi)$ are the (appropriately normalized) eigenfunctions with eigenvalues $\lambda_n(z)$ and zero boundary conditions,

$$\left[\hat{L} + 3(\ln g(z) + 1)\right] f_n(z; \phi) = \lambda_n(z) f_n(z; \phi),$$

(73)

$$f_n(z; \phi) = 0 \text{ for } \phi = \phi_*, \phi = \phi_{\phi_1}.$$  

(74)

The decomposition (72) is possible as long as the operator $\hat{L}$ is self-adjoint with an appropriate choice of the scalar product in the space of functions $f(\phi)$. The scalar product can be chosen in the following way,

$$\langle f_1, f_2 \rangle = \int f_1(\phi) f_2^*(\phi) M(\phi) d\phi,$$  

(75)

where $M(\phi)$ is a weighting function. One can attempt to determine $M(\phi)$ such that the operator $\hat{L}$ is self-adjoint,

$$\langle f_1, \hat{L} f_2 \rangle = \langle \hat{L} f_1, f_2 \rangle.$$  

(76)

In single-field models of inflation where the operator $\hat{L}$ has the form (60), it is always possible to choose $M(\phi)$ appropriately [24]. However, in multi-field models this is not necessarily possible. One can show that in standard slow-roll models with $K$ fields $\phi_1, \ldots, \phi_K$ and kinetic coefficients

$$D_{ij} = \frac{H^3}{8\pi^2} \delta_{ij}, \quad v_i = -\frac{1}{4\pi G} \frac{\partial H}{\partial \phi_i}, \quad H = H(\phi_1, \ldots, \phi_K),$$  

(77)

there exists a suitable choice of $M(\phi)$, namely

$$M(\phi_1, \ldots, \phi_K) = \frac{\pi G}{H^2} \exp\left[\frac{\pi G}{H^2}\right],$$  

(78)

such that the operator

$$\hat{L} = H^{-1} \sum_{i,j} D_{ij} \frac{\partial^2}{\partial \phi_i \partial \phi_j} + H^{-1} \sum_i v_i \frac{\partial}{\partial \phi_i}.$$  

(79)

is self-adjoint in the space of functions $f(\phi)$ with zero boundary conditions and the scalar product (76). However, the operator $\hat{L}$ may be non-self-adjoint in more general inflationary models where the kinetic coefficients are given by different expressions. We omit the formulation of precise conditions for self-adjointness of $\hat{L}$ because this property is not central to the present investigation. In non-self-adjoint cases a decomposition similar to Eq. (72) needs to be performed using the left and the right eigenfunctions of the non-self-adjoint operator $\hat{L} + 3(\ln g + 1)$. One expects that such a decomposition will still be possible because (heuristically) the nondiagonalizable operators are a set of measure zero among all operators. The requisite left and right eigenfunctions can be obtained numerically. We leave the detailed investigation of those cases for future work. Presently, let us focus on the case when the decomposition of the form (72) holds, with appropriately chosen scalar product and the normalized eigenfunctions

$$\langle f_m, f_n \rangle = \delta_{mn}.$$  

(80)

The eigenfunctions $f_n(z; \phi)$ can be obtained e.g. numerically by solving the boundary value problem (73)–(74).

In the limit $z \to z_*$, one of the eigenvalues $\lambda_n$ approaches zero. Since linear operators such as $\hat{L}$ always have a spectrum bounded from above [24], we may renumber the eigenvalues such that $\lambda_0(z_*) = 0$. By construction, for all $z$ with $\text{Re } z > \text{Re } z_*$ all the eigenvalues $\lambda_n$ are negative. Note that the (algebraically) largest eigenvalue $\lambda_0(z)$ is always nondegenerate, and the corresponding eigenfunction $f_0(z; \phi)$ can be chosen real and positive for all $\phi$, except at the boundaries $\phi = \phi_*$ and $\phi = \phi_{\phi_1}$ where $f$ satisfies the zero boundary conditions.

For $z$ near $z_*$, only the nondegenerate eigenvalue $\lambda_0$ will be near zero, so the decomposition (72) of the Green’s function will be dominated by the term $1/\lambda_0$. Hence, we can use Eqs. (71) and (72) to determine the function $g_1(z; \phi)$ approximately as

$$g_1(z; \phi) \approx -\frac{f_0(z; \phi)}{\lambda_0(z)} \left. \frac{\partial f_0(z; \phi_*)}{\partial \phi} \right|_{\phi_*} \frac{D}{H^2 e^{-zH^{-3}}}.$$  

(81)

It follows that indeed $g_1(z; \phi) \to \infty$ as $z \to z_*$ because $\lambda_0(0) \to 0$.

This detailed investigation allows us now to determine the behavior of $g(z; \phi)$ at the leading singularity $z = z_*$. We will consider the function $g(z; \phi)$ for $z$ near $z_*$ and show that the singularity indeed has the structure (83).

We have already shown that the function $g$ itself does not diverge at $z = z_*$ but its derivative $g_1 \equiv \partial g/\partial z$ does. Hence, the function $g(z_*; \phi)$ is continuous, and the difference $g(z_*; \phi) - g(z; \phi)$ is small for $z \approx z_*$, so that we have the expansion

$$\delta g(z; \phi) \equiv g(z; \phi) - g(z_*; \phi)$$  

(82)

$$\approx g_1(z; \phi) (z - z_*) + O((z_* - z)^2).$$  

(83)

\footnote{I am grateful to D. Podolsky for pointing this out to me. The hermiticity of operators of diffusion type in the context of eternal inflation was briefly discussed in Ref. [23].}
(Note that we are using the finite value $g_1(z; \phi)$ rather than the divergent value $g_1(z_*; \phi)$ in the above equation.) On the other hand, we have the explicit representation (31). Let us examine the values of $\lambda_0(z)$ for $z \approx z_*$. At $z = z_*$, we have $\lambda_0(z_*) = 0$, so the (small) value $\lambda_0(z)$ for $z \approx z_*$ can be found using standard perturbation theory for linear operators. If we denote the change in the operator $L$ by

$$\delta L \equiv 3(\ln g(z; \phi) - \ln g(z_*; \phi)) \approx \frac{3\delta g(z; \phi)}{g(z_*; \phi)}, \quad (84)$$

we can write, to first order,

$$\lambda_0(z) \approx \left\langle f_0, \delta L f_0 \right\rangle = \int |f_0(\phi)|^2 \frac{3\delta g(z; \phi)}{g(z_*; \phi)} d\phi. \quad (85)$$

Now, Eqs. (31) and (32) yield

$$\frac{\delta g(z; \phi)}{z - z_*} \approx -\frac{f_0(z; \phi)}{\lambda_0(z)} \frac{1}{\lambda_0(z)} \left[ \frac{\partial f_0}{\partial \phi} \frac{D}{\partial \phi} H^4 e^{-z H^{-3}} \right] \phi. \quad (86)$$

Integrating the above equation in $\phi$ with the prefactor

$$|f_0(\phi)|^2 \frac{3}{g(z_*; \phi)} d\phi \quad (87)$$

and using Eq. (35), we obtain a closed equation for $\lambda_0(z)$ in which terms of order $(z - z_*)^2$ have been omitted.

$$\lambda_0(z) \approx -\frac{1}{\lambda_0(z)} \left[ \frac{\partial f_0}{\partial \phi} \frac{D}{\partial \phi} H^4 e^{-z H^{-3}} \right] \phi. \quad (88)$$

It follows that $\lambda_0(z) \propto \sqrt{z - z_*}$ and $g_1(z; \phi) \propto (z - z_*)^{-1/2}$, confirming the leading asymptotic of the form (63).

Let us also obtain a more explicit form of the singularity structure of $g(z; \phi)$. We can rewrite Eq. (36) as

$$\lambda_0(z) \approx \sigma_0 \sqrt{z - z_*} + O(z - z_*), \quad (89)$$

where $\sigma_0$ is a constant that may be obtained explicitly. Then Eq. (31) yields

$$g_1(z; \phi) \approx \frac{f_0(z; \phi)}{\sqrt{z - z_*}} \sigma_1, \quad (90)$$

with a different constant $\sigma_1$. Finally, we can integrate this in $z$ and obtain

$$g(z; \phi) = g(z_*; \phi) + 2\sigma_1 f_0(z; \phi) \sqrt{z - z_*} + O(z - z_*). \quad (91)$$

We may rewrite this by substituting $z = z_*$ into $f_0(z; \phi)$,

$$g(z; \phi) = g(z_*; \phi) + \sigma(\phi) \sqrt{z - z_*} + O(z - z_*), \quad (92)$$

$$\sigma(\phi) \equiv 2\sigma_1 f_0(z_*; \phi). \quad (93)$$

The result is now explicitly of the form (63). It will turn out that the normalization constant $2\sigma_1$ cancels in the final results. So in a practical calculation the eigenfunction $f_0(z_*; \phi)$ may be determined with an arbitrary normalization.

As a side note, let us remark that the argument given above will apply also to other singular points $z'_* \neq z_*$ as long as the eigenvalue $\lambda_k(z)$ of the operator $L + \delta g(\phi)$ is nondegenerate when it vanishes at $z = z_*$. If the relevant eigenvalue becomes degenerate, the singularity structure will not be of the form $\sqrt{z - z_*}$ but rather $(z - z'_*)^{1/2}$ with some other power $0 < s < 1$.

Now we are ready to obtain the asymptotic form of the distribution $\rho(V; \phi)$ for $V \to \infty$. We deform the integration contour in the inverse Laplace transform (60) such that it passes near the real axis around $z = z_*$. Then we use Eq. (92) for $g(z; \phi)$ and obtain the leading asymptotic

$$\rho(V; \phi) \approx \frac{1}{2\pi i} \sigma(\phi) \int_{-\infty}^{\infty} \frac{e^{-z V}}{\sqrt{z - z_*} e^{z V} dz} \quad (94)$$

The subdominant terms come from the higher-order terms in the expansion in Eq. (92) and are of the order $V^{-1}$ times the leading term shown in Eq. (94).

Finally, we show that $z_*$ must be real-valued and that there are no other singularities $z'_*$ with $\text{Re} z'_* \neq \text{Re} z_*$. This is so because the integral

$$g_1(z_*; \phi) = -\int_0^\infty \rho(V; \phi) e^{-z V} dV = \infty \quad (95)$$

will definitely diverge for purely real $z_*$ if it diverges for a nonreal value $z'_* = z_* + iA$. If, on the other hand, the integral (95) diverges for a real $z_*$, it will converge for any nonreal $z'_* = z_* + iA$ with real $A \neq 0$ because the function $\rho(V; \phi)$ has the large-$V$ asymptotic of the form (91) and the oscillations of $\exp(iA V)$ will make the integral (95) convergent.

### D. FPRV distribution of a field $Q$

In this section we follow the notation of Ref. (52). Consider a fluctuating field $Q$ such that the Fokker-Planck operator $L$ is of the form (63). We are interested in the portion $\nu_{QR}$ of the total reheated volume $\nu$ where the field $Q$ has a value within a given interval $[Q_R, Q_R + dQ]$. We denote by $\rho(V, \nu_{QR}; \phi_0, Q_0)$ the joint distribution function of the volumes $V$ and $\nu_{QR}$ for initial $H$-regions with initial values $\phi = \phi_0$ and $Q = Q_0$. The generating function $\tilde{g}(z, q; \phi, Q)$ corresponding to that distribution is defined by

$$\tilde{g}(z, q; \phi, Q) = \int e^{-z V - q V_\nu} \rho(V, \nu; \phi, Q) dV d\nu. \quad (96)$$

Since this generating function satisfies the same multiplicative property (14) as the generating function
\(g(z; ϕ, Q)\), we may repeat the derivation of Eq. (57) without modifications for the function \(\hat{g}(z; q; ϕ, Q)\). Hence, \(g(z; q; ϕ, Q)\) is the solution of the same equation as \(g(z; ϕ, Q)\). The only difference is the boundary conditions at reheating, which are given not by Eq. (59) but by

\[
\hat{g}(z, q; ϕ, Q) = \exp \left[ - (z + q δ_{QQR}) H^{-3}(ϕ, Q) \right],
\]

where (with a slight abuse of notation) \(δ_{QQR}\) is the indicator function of the interval \([QR, Q + dQ]\), i.e.

\[
δ_{QQR} = \theta(Q - QR)θ(Q + dQ - Q).
\]

We employ this “finite” version of the \(δ\)-function only because we cannot use a standard Dirac \(δ\)-function under the exponential. This slight technical inconvenience will disappear shortly.

The solution for the function \(\hat{g}(z; q; ϕ, Q)\) may be obtained in principle and will provide complete information about the distribution of possible values of the volume \(V_Q\) together with the total reheating volume \(V\) to the future of an initial \(H\)-region. In the context of the RV cutoff, one is interested in the event when \(V\) is finite and very large. Then one expects that \(V_Q\) also becomes typically very large while the ratio \(V_Q/V\) remains roughly constant. In other words, one expects that the distribution of \(V_Q\) is sharply peaked around a mean value \(⟨V_Q⟩\), and that the limit \(⟨V_Q⟩/V\) is well-defined at \(V \to ∞\). The value of that limit is the only information we need for calculations in the RV cutoff. Therefore, we do not need to compute the entire distribution \(ρ(V, V_Q; ϕ, Q)\) but only the mean value \(⟨V_Q⟩_V\) at fixed \(V\).

Let us therefore define the generating function of the mean value \(⟨V_Q⟩_V\) as follows,

\[
h(ϕ, Q) \equiv ⟨V_Q e^{-zV}⟩_{V < ∞} = \frac{∂}{∂q}\hat{g}(ϕ, Q).
\]

(The dependence on the fixed value of \(QR\) is kept implicit in the function \(h(ϕ, Q)\) in order to make the notation less cumbersome.) The differential equation and the boundary conditions for \(h(ϕ, Q)\) follow straightforwardly by taking the derivative \(\phi_q\) at \(q = 0\) of Eqs. (57) and (97). It is clear from the definition of \(\hat{g}\) that \(\hat{g}(ϕ, Q) = g(z, Q)\). Hence we obtain

\[
\hat{L}h + 3 (ln g(z, Q) + 1) h = 0,
\]

\[
h(z; ϕ, Q) = \frac{e^{-zH^{-3}(ϕ, Q)}}{H^{3}(ϕ, Q)} δ_{QQR},
\]

\[
h(z; ϕ, Q) = 0.
\]

Note that it is the generating function \(g\), not \(\hat{g}\), that appears as a coefficient in Eq. (100).

Since the “finite” \(δ\)-function \(δ_{QQR}\) now enters only linearly rather than under an exponential, we may replace \(δ_{QQR}\) by the ordinary Dirac \(δ\)-function \(δ(Q - QR)\). To maintain consistency, we need to divide \(h\) by \(dQ\), which corresponds to computing the probability density of the reheated volume with \(Q = QR\). This probability density is precisely the goal of the present calculation.

The RV-regularized probability density for values of \(Q\) is defined as the limit

\[
p(Q) = \lim_{V \to ∞} \frac{⟨V_Q⟩_{V < ∞}}{V ρ(V; ϕ, Q)} = \frac{1}{V} \int_{−∞}^{∞} e^{-V h(z; ϕ, Q)} dz.
\]

To compute this limit, we need to consider the asymptotic behavior of \(⟨V_Q⟩_{V < ∞}\) at \(V \to ∞\). This behavior is determined by the leading singularity of the function \(h(z; ϕ, Q)\) in the complex \(z\) plane. The arguments of Sec. III C apply also to \(h(z; ϕ, Q)\) and show that \(h\) cannot have a \(ϕ\) - or \(Q\)-dependent singularity in the \(z\) plane.

Moreover, \(h(z; ϕ, Q)\) has precisely the same singular points, in particular \(z = z_s\), as the basic generating function \(g(z; ϕ, Q)\) of the reheating volume. Indeed, the function \(h(z; ϕ, Q)\) can be expressed through the Green’s function \(G(z; ϕ, Q, φ', Q')\) of the operator \(L + 3(ln g + 1)\), similarly to the function \(g_{1}(z; ϕ)\) considered in Sec. III C.

For \(z \neq z_s\), this operator is invertible on the space of functions \(f(ϕ, Q)\) satisfying zero boundary conditions. Hence, \(h(z; ϕ, Q)\) is nonsingular at \(z \neq z_s\) and becomes singular precisely at \(z = z_s\).

Let us now obtain an explicit form of \(h(z; ϕ, Q)\) near the singular point \(z = z_s\). We assume again the eigenfunction decomposition of the Green’s function (with the same caveats as in Sec. III C),

\[
G(z; ϕ, Q, φ', Q') = \sum_{n=0}^{∞} \frac{1}{λ_n(z)} f_n(z; ϕ, Q) f_n^*(z; φ', Q'),
\]

where \(f_n\) are appropriately normalized eigenfunctions of the \(z\)-dependent operator \(L + 3(ln g + 1)\) with eigenvalues \(λ_n(z)\). The eigenfunctions \(f_n\) must satisfy zero boundary conditions at reheating and Planck boundaries. Similarly, to the way we derived Eq. (81), we obtain the explicit solution

\[
h(z; ϕ, Q) = \sum_{n=0}^{∞} \frac{f_n(z; ϕ, Q)}{λ_n(z)} \left[ \frac{∂f_n}{∂φ} D_{ϕφ} e^{−zH^{−3}} \right]_{ϕ=Q_R}.
\]

The value of \(h(z; ϕ, Q)\) for \(z ≈ z_s\) is dominated by the contribution of the large factor \(1/λ_0(z) \propto (z - z_s)^{−1/2}\), so the leading term is

\[
h(z; ϕ, Q) ≈ \frac{f_0(z_s; ϕ, Q)}{λ_0(z)} \left[ \frac{∂f_0}{∂φ} D_{ϕφ} e^{−zH^{−3}} \right]_{ϕ=Q_R}.
\]
the large-$V$ asymptotic of $\langle V_{Q_0} \rangle_{V<\infty}$ as follows,
\[
\langle V_{Q_0} \rangle_{V} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{zV} h(z; \phi, Q) dz \\
\approx \frac{f_0(z; \phi, Q)}{\sqrt{\pi \sigma_0 V}} e^{zV} \left[ \frac{\partial f_0}{\partial \phi} D_{\phi \phi} e^{-zH^{-3}} \right]_{\phi = \phi_0, V = V_0},
\]
(107)
where $\sigma_0$ is the constant defined by Eq. (89). We now complete the analytic evaluation of the limit (103). Since the denominator of Eq. (103) has the large-$V$ asymptotics of the form
\[
\mathcal{V} \int_0^\infty g(z; \phi, Q)e^{zV}dV \propto f_0(z; \phi, Q)V^{-1/2}e^{zV},
\]
(108)
where $f_0$ is the same eigenfunction, the dependence on $\phi$ and $Q$ identical cancels in the limit (103). Hence, that limit is independent of the initial values $\phi$ and $Q$ but is a function only of $Q$, on which $h(z; \phi, Q)$ implicitly depends. Using this fact, we can significantly simplify the rest of the calculation. It is not necessary to compute the denominator of Eq. (103) explicitly. The distribution of the values of $Q$ at $\phi = \phi_0$ is simply proportional to the $Q_{N}$-dependent part of Eq. (107); the denominator of Eq. (103) serves merely to normalize that distribution. Hence, the RV cutoff yields
\[
p(Q_R) = \text{const} \left[ \frac{\partial f_0(z; \phi, Q)}{\partial \phi} D_{\phi \phi} e^{-zH^{-3}} \right]_{\phi = \phi_0, Q = V_0}.
\]
(109)
where the normalization constant needs to be chosen such that $\int p(Q_R)dQ_R = 1$. This is the final analytic formula for the RV cutoff applied to the distribution of $Q$ at reheating. The value $z_*$ and the corresponding solution $g(z_*; \phi, Q)$ of Eq. (26), and the eigenfunction $f_0(z_*; \phi, Q)$ need to be obtained numerically unless an analytic solution is possible.

Let us comment on the presence of the factor $D_{\phi \phi}$ in the formula (109). The “diffusion” coefficient $D_{\phi \phi}$ is evaluated at the reheating boundary and is thus small since the fluctuation amplitude at reheating is (in slow-roll inflationary models)
\[
\frac{\delta \phi}{\phi} \sim H^2 \phi = \frac{\sqrt{8\pi^2 D_{\phi \phi} \phi}}{v_\phi} \sim 10^{-5}.
\]
(110)
Nevertheless, it is not possible to set $D_{\phi \phi} = 0$ directly in Eq. (109). This is so because the existence of the Green’s function of the Fokker-Planck operator such as $\tilde{L}$ depends on the fact that $\tilde{L}$ is a second-order differential operator of elliptic type. If one sets $D_{\phi \phi} = 0$ near the reheating boundary, the operator $\tilde{L}$ becomes first-order in $\phi$ at that boundary. Then one needs to use a different formula than Eq. (67) for reducing an equation with inhomogeneous boundary conditions to an inhomogeneous equation with zero boundary conditions. Accordingly, one cannot use formulas such as Eq. (71) for the solutions. Alternative ways of solving the relevant equations in that case will be used in Sec. 111.

E. Calculations for an inflationary model

In this section we perform explicit calculations of RV cutoff for a model of slow-roll inflation driven by a scalar field with a potential shown in Fig. 2. The kinetic coefficients $D(\phi)$ and $v(\phi)$ are such that $D(\phi) = D_0$, $v(\phi) = 0$, and $H(\phi) = H_0$ in the flat region $\phi_1 < \phi < \phi_2$, where the constants $D_0$ and $H_0$ are
\[
H_0 = \frac{\sqrt{8\pi G}}{3} V_0, \quad D_0 = \frac{H_0^3}{8\pi^2}.
\]
(111)
In the slow-roll regions $\phi_1 < \phi < \phi_1^{(1)}$ and $\phi_2 < \phi < \phi_2^{(2)}$, the coefficient $D(\phi)$ is set equal to zero, while $v(\phi) \neq 0$ and $H(\phi)$ is not constant any more. The number of e-folds in the two slow-roll “shoulders” can be computed by the standard formula,
\[
N_j = \int_0^{\phi_j} \frac{H}{v} d\phi = -4\pi G \int_0^{\phi_j} \frac{H}{H} d\phi, \quad j = 1, 2.
\]
(112)
The first step of the calculation is to determine the singular point $z = z_*$ of solutions $g(z; \phi)$ of Eq. (51). We expect $z_*$ to be real and negative. The boundary conditions for $g(z; \phi)$ are
\[
\int_0^{\phi_1} H_d\phi = -4\pi G \int_0^{\phi_1} \frac{H}{H} d\phi.
\]
(113)
In each of the two deterministic regions, $\phi_1 < \phi < \phi_1^{(1)}$ and $\phi_2 < \phi < \phi_2^{(2)}$, Eq. (51) becomes
\[
\frac{v}{H} \partial_\phi g + 3g \ln g = 0,
\]
(114)
with the general solution
\[
g(z; \phi) = \exp \left[ C \exp \left( -3 \int_0^\phi \frac{H}{v} d\phi \right) \right],
\]
(115)
where $C$ is an integration constant. Since the equation is first-order within the deterministic regions, the solutions are fixed by the boundary condition (113) in the respective region,
\[
g(z; \phi) = \exp \left[ -zH^{-3}(\phi_1^{(1)}) \exp \left( -3 \int_0^\phi \frac{H}{H} d\phi \right) \right],
\]
(116)
We may therefore compute the values of $g(z; \phi)$ at the boundaries $\phi_{1,2}$ of the self-reproduction region as
\[
g(z; \phi_{1,2}) = \exp \left[ -zH^{-3}(\phi_{1,2}^{(1,2)}) \exp (3N_{1,2}) \right] .
\]
(117)
Now we need to solve Eq. (118) with these boundary conditions in the region $\phi_1 < \phi < \phi_2$. The equation has then the form
\[
\frac{D_0}{H_0} \frac{\partial g}{\partial z} + 3g \ln g = 0. \tag{118}
\]
Exact solutions of Eq. (118) were studied in Ref. [15], to which the reader is referred for more details. It is easy to show that Eq. (118) is formally equivalent to a one-dimensional motion of a particle with coordinate $g(\phi)$ in a potential $U(g)$,
\[
U(g) = \frac{6\pi^2}{H_0^2} g^2 (2 \ln g - 1), \tag{119}
\]
while $\phi$ plays the role of time. A solution $g(z; \phi)$ with boundary conditions (117) corresponds to a trajectory that starts at the given value $g(z; \phi_1)$ with the initial velocity chosen such that the motion takes precisely the specified time interval $\phi_2 - \phi_1$ and reaches $g(z; \phi_2)$. For $z < 0$ the boundary conditions specify $g(z; \phi_1, 2) > 1$, i.e. the trajectory begins and ends to the right of the minimum of the potential (see Fig. 3). Since the system is conservative, there is a constant of motion $E$ such that $E = U(g_0)$ at the highest point of the trajectory $g_0$ where the “kinetic energy” vanishes. The solution $g(z; \phi)$ can be written implicitly as one of the two alternative formulas,
\[
\pm \int_{g(z; \phi)}^{g(z; \phi_1, 2)} \frac{dg}{\sqrt{2E(z) - 2U(g)}} = \phi - \phi_1, \tag{120}
\]
valid in appropriate intervals $\phi_1 < \phi < \phi_0$ and $\phi_0 < \phi < \phi_2$ respectively, where $\phi_0$ is the value of $\phi$ corresponding to the turning point $g_0 = g(z; \phi_0)$. The value $E = E(z)$ in Eq. (120) must be chosen such that the total “time” is $\phi_2 - \phi_1$,
\[
\left[ \int_{g_0}^{g(z; \phi_1, 2)} + \int_{g_0}^{g(z; \phi_2)} \frac{dg}{\sqrt{2E(z) - 2U(g)}} \right] = \phi_2 - \phi_1. \tag{121}
\]
This condition together with $E = U(g_0)$ implicitly determine the values $E = E(z)$ and $g_0 = g_0(z)$.

The singularity $z = z_*$ of the solution $g(z; \phi)$ is found by using the condition $\partial g / \partial z \to \infty$. Differentiating Eq. (120) with respect to $z$ and substituting Eq. (117) for $g(z; \phi_1, 2)$, we obtain the condition
\[
- \frac{\partial g}{\partial z} \frac{1}{\sqrt{2E(z) - 2U(g)}} + \frac{e^{3N_{1,2}} H^{-3}(\phi^{(1,2)}_*)}{\sqrt{2E(z) - 2U(g(z; \phi_1, 2))}} - E'(z) \int_{g(z; \phi)}^{g(z; \phi_1, 2)} \frac{dg}{[2E(z) - 2U(g)]^{3/2}} = 0. \tag{122}
\]
It follows that $\partial g / \partial z \to \infty$ when
\[
E(z) = U(g(z; \phi_1, 2)). \tag{123}
\]
This condition is interpreted in the language of Fig. 3 as follows. As the value of $z$ becomes more negative, the initial and the final values of $g$ given by Eq. (117) both grow. The last available trajectory starts from rest at $\phi = \phi_2$ and at the value of $g$ such that $U(g) = E$.

To obtain a specific result, let us assume that $\phi_2 - \phi_1$ is sufficiently large to provide self-reproduction ($\phi_2 - \phi_1 \gg H_0$) and that the number of $e$-folds in channel 1 is smaller than that in channel 2,
\[
H^{-3}(\phi^{(1)}_*) \exp (3N_1) < H^{-3}(\phi^{(2)}_*) \exp (3N_2). \tag{124}
\]
Then the value $g(z; \phi_2)$ will grow faster than $g(z; \phi_1)$ as $z$ becomes more negative. It follows that $g(z; \phi_2)$ will reach the singular point first. Since the “time” $\phi_2 - \phi_1$ is large, the constant $E$ will be close to 0 so that the trajectory spends a long “time” near $g = 0$. Then the value $g(z_*, \phi_2)$ will be close to $e^{1/2}$. Hence the value of $z_*$ is approximately
\[
z_* \approx -\frac{1}{2} H^3(\phi^{(2)}_*) \exp (-3N_2). \tag{125}
\]
For this value of $z_*$, the starting point of the trajectory will be
\[
g(z_*, \phi_1) \approx \exp \left[ \frac{1}{2} \exp (3N_1 - 3N_2) \right] \approx 1. \tag{126}
\]
Hence, the solution $g(z; \phi)$ at the singular point $z = z_*$ can be visualized as the thin line in Fig. 3 starting approximately at $g(z; \phi_1) = 1$ and finishing at $g(z; \phi_2) \approx e^{1/2}$.

An approximate expression for $g(z; \phi)$ can be obtained by setting $E \approx 0$ in Eq. (120); then the integral can be evaluated analytically. In the range $\phi_0 < \phi < \phi_2$ we obtain

$$\phi_2 - \phi \approx \int_{g(z; \phi)}^{g(z; \phi_2)} \frac{dg}{\sqrt{-2U(g)}} \approx \frac{H_0}{\sqrt{12\pi^2}} \sqrt{1 - 2\ln g(z; \phi_2)}^\phi,$$

so the solution is

$$g(z; \phi) \approx \exp \left[ \frac{1}{2} - \frac{6\pi^2}{H_0^2} (\phi_2 - \phi)^2 \right], \quad \phi_0 < \phi < \phi_2. \quad (127)$$

In the range $\phi_1 < \phi < \phi_0$ we obtain within the same approximation

$$g(z; \phi) \approx \exp \left[ \frac{1}{2} - \frac{6\pi^2}{H_0^2} \left( \phi - \phi_1 + \frac{H_0}{\sqrt{12\pi^2}} \right)^2 \right]. \quad (128)$$

These approximations are valid for $\phi$ within the indicated ranges and away from the turning point $\phi_0$. The value of $\phi_0$ can be estimated by requiring that the value of $g(z; \phi_0)$ obtained from Eq. (128) be equal to that obtained from Eq. (129). This yields

$$\phi_0 \approx \frac{1}{2} \left[ \phi_2 + \phi_1 - \frac{H_0}{\sqrt{12\pi^2}} \right] \approx \frac{\phi_2 + \phi_1}{2}. \quad (130)$$

We note that the value $g(z; \phi_0)$ can be obtained somewhat more precisely by approximating the solution $g(z; \phi)$ in a narrow interval near $\phi = \phi_0$ by a function of the form $[A + B(\phi - \phi_0)^2]$ and matching both the values and the derivatives of $g(z; \phi)$ to the approximations (128) and (129) at some intermediate points straddling $\phi = \phi_0$. In this way, a uniform analytic approximation for $g(z; \phi)$ can be obtained. However, the accuracy of the approximations (128) and (129) is sufficient for the present purposes.

Having obtained adequate analytic approximations for $g$ and $g(z; \phi)$, we can now proceed to the calculation of the mean volumes $\langle V_{1,2}\rangle$ of regions reheated through channels 1 and 2 respectively, conditioned on the event that the total volume of all reheated regions is $V$. We use the formalism developed in Sec. III D where the variable $Q$ now takes only the discrete values 1 and 2, so instead let us denote that value by $j$. The relevant generating function $h_j(z; \phi)$ is defined by

$$h_j(z; \phi) = \langle V_j e^{-zV}\rangle_{V<\infty}, \quad j = 1, 2, \quad (131)$$

and is found as the solution of Eq. (103), which now takes the form

$$\left[ \frac{D(\phi)}{H(\phi)} \partial_\phi \partial_\phi + \frac{v(\phi)}{H(\phi)} \partial_\phi + 3(\ln g(z; \phi) + 1) \right] h_j(z; \phi) = 0, \quad (132)$$

with boundary conditions imposed at the reheating boundaries,

$$h_1(z; \phi_1) = H^{-3}(\phi_1) e^{-zH^{-3}(\phi_1)}, \quad h_1(z; \phi_2) = 0; \quad (133)$$

$$h_2(z; \phi_1) = 0, \quad h_2(z; \phi_2) = H^{-3}(\phi_2) e^{-zH^{-3}(\phi_2)}. \quad (134)$$

In the present toy model the diffusion coefficient is set to zero at reheating, so the formalism developed in Sec. III D needs to be modified. We will first solve Eq. (132) analytically in the no-diffusion intervals of $\phi$ and obtain the boundary conditions for $h$ at the boundaries of the self-reproduction regime $[\phi_1, \phi_2]$ where $D(\phi) \neq 0$. Then the methods of Sec. III D can be applied to the boundary value problem for the interval $[\phi_1, \phi_2]$.

Implementing this idea in the first no-diffusion region $\phi_1 < \phi < \phi_0$, we use the solution (116) for $g(z; \phi)$ and reduce Eq. (132) to

$$\partial_\phi h_j + \frac{3H}{v} \left[ 1 - zH^{-3}(\phi_1) \exp \left(-3\int_{\phi_0}^{\phi_1} \frac{H}{v} d\phi \right) \right] h_j = 0. \quad (135)$$

This equation is easily integrated together with the boundary conditions (133)-(134) and yields the values of $h_j$ at $\phi_1$.

$$h_j(z; \phi_1) = \delta_{j1} H^{-3}(\phi_1) \exp \left[ 3N_1 - zH^{-3}(\phi_1) e^{3N_1} \right]. \quad (136)$$

Similarly we can determine the values $h_j(z; \phi_2)$. Since the value $z = z_*$ is important for the present calculation, we now find the values of $h_j$ at $z = z_*$ using the assumption $N_2 \gg N_1$ and the estimate (125),

$$h_j(z_*; \phi_i) \approx \delta_{ij} \frac{\exp \left[ \frac{3N_1 + \frac{1}{2} \delta_{j2}}{H^3(\phi_1)} \right]}{H^3(\phi_1)}, \quad i, j = 1, 2. \quad (137)$$

We have thus reduced the problem of determining $h_j(z; \phi)$ to the boundary-value problem for the interval $[\phi_1, \phi_2]$ where the methods of Sec. III D apply but the boundary conditions are given by Eq. (137).

The next step, according to Sec. III D, is to compute the eigenfunction $f_0(z; \phi)$ of the operator

$$\hat{L} \equiv \frac{D_0}{H_0} \partial_\phi \partial_\phi + 3(\ln g(z; \phi) + 1) \quad (138)$$

such that

$$\hat{L} f_0 = 0; \quad f_0(z_*, \phi_1, 2) = 0. \quad (139)$$

As we have shown, this eigenfunction with eigenvalue 0 exists precisely at $z = z_*$. Once this eigenfunction is computed, the ratio of the RV-regulated mean volumes in channels 1 and 2 will be expressed through the derivatives of $f_0$ at the endpoints and through the modified boundary conditions (137) as follows,

$$\frac{P(2)}{P(1)} = \frac{h_2(z_*; \phi_2) \partial_\phi f_0(z_*; \phi_2)}{h_1(z_*; \phi_1) \partial_\phi f_0(z_*; \phi_1)}. \quad (140)$$
The absolute value is taken to compensate for the negative sign of the derivative \( \partial_\phi f_0 \) at the right boundary point (assuming that \( f_0 \geq 0 \) everywhere). Since \( h_{1,2}(z; \phi_{1,2}) \) are already known, it remains to derive an estimate for \( f_0(z; \phi) \).

The eigenvalue equation \( \hat{L} f_0 = 0 \) formally resembles a one-dimensional Schrödinger equation with the coordinate \( \phi \) and the “potential”

\[
\sqrt{V}(\phi) \equiv -\frac{12\pi^2}{H_0^2} (\ln g(z; \phi) + 1). \tag{141}
\]

The eigenfunction \( f_0(z; \phi) \) is then interpreted as the “wavefunction” of a stationary state with zero energy and zero boundary conditions at \( \phi = \phi_{1,2} \). According to Eqs. (128) and (129), the function \( \sqrt{V}(\phi) \) has a maximum at \( \phi \approx \phi_0 \) (see Fig. 4), while its values at the endpoints are

\[
\sqrt{V}(\phi_1) \approx -\frac{12\pi^2}{H_0^2}, \quad \sqrt{V}(\phi_2) \approx -\frac{18\pi^2}{H_0^2}. \tag{142}
\]

Using the terminology of quantum mechanics, there is a potential barrier separating two classically allowed regions near \( \phi = \phi_1 \) and \( \phi = \phi_2 \). Since the “potential well” at \( \phi = \phi_2 \) is deeper, the ground state is approximately the ground state of that one well, with an exponentially small amplitude of being near \( \phi = \phi_1 \). The shape of the eigenfunction is sketched in Fig. 4. The exponential suppression of the amplitude near \( \phi = \phi_1 \) can be found using the WKB approximation, which yields

\[
\frac{\partial_\phi f_0(z; \phi)}{\partial_\phi f_0(z; \phi)} = A_{21} \exp \left[ \int_{\phi}^{\phi_2} \sqrt{V}(\phi)d\phi \right], \tag{143}
\]

where \( \phi_{1,2} \) are the turning points such that \( V(\phi_{1,2}) = 0 \). The pre-exponential factor \( A_{21} \) is of order 1 and can, in principle, be obtained from a more detailed matching of the WKB-approximated solution across the barrier to the solutions in the “classically allowed” regions, or by determining the solution \( f_0(z; \phi) \) numerically. However, we will omit this calculation since the main result will consist of an exponentially small factor. That factor can be estimated using Eqs. (128), (129), and (141) as

\[
\int_{\phi_1}^{\phi_2} \sqrt{V}(\phi)d\phi \approx 2 \int_{\phi_0}^{\phi_1} \sqrt{\frac{12\pi^2}{H_0^2} \ln g d\phi} \approx \frac{3\pi^2}{\sqrt{2}H_0^2} (\phi_2 - \phi_1)^2. \tag{144}
\]

Hence, the ratio \( P(2) \) is simplified to

\[
\frac{P(2)}{P(1)} = A_{21} \frac{H^{-3}(\phi_{1}^2)}{H^{-3}(\phi_{1}^1)} e^{3N_2 + \frac{1}{2}} e^{3N_1} \exp \left[ \frac{3\pi^2}{\sqrt{2}H_0^2} (\phi_2 - \phi_1)^2 \right]. \tag{145}
\]

This is the main result quoted above in Eq. (38).

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