A conjectured bound on the spanning tree number of bipartite graphs

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Abstract
The Ferrers bound conjecture is a natural graph-theoretic extension of the enumeration of spanning trees for Ferrers graphs. We document the current status of the conjecture and provide a further conjecture which implies it.

1 Introduction

Ferrers graphs are a bipartite analogue of threshold graphs. They were introduced by Hammer, Peled, and Srinivasan [7], where they were called difference graphs. Ferrers graphs and threshold graphs are both realizations of Ferrers digraphs, which were introduced by Riguet [12]. Threshold graphs and Ferrers graphs obey many analogies. For example, in the polytope of degree sequences, the extreme points correspond to threshold graphs, while in the polytope of bipartite degree sequences, the extreme points correspond to Ferrers graphs [11].

Ehrenborg and van Willigenburg [4] studied Ferrers graphs and proved that the spanning tree number of a Ferrers graph depends only on its degree sequence and the size of each color class. This defines a Ferrers invariant for any bipartite graph. In 2006, Ehrenborg conjectured that the Ferrers invariant is an upper bound for the spanning tree number of any bipartite graph. The purpose of this note is to document this conjecture and provide some evidence which suggests the conjecture is reasonable.

First, let us establish some definitions and notation. The equivalency of the following conditions is demonstrated in [7].

Definition 1 ([7]). Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \sqcup Y$. Then $G$ is a Ferrers graph if and only if any of the following equivalent conditions hold:

1.

2.

3.

4.

5.
There exist a vertex weighting $w$ and a real number $T$ such that for any $u \neq v \in V$, the vertices $u$ and $v$ are adjacent if and only if $|w(u) - w(v)| \geq T$.

The graph $G'$ constructed from $G$ by adding all possible edges between vertices in $X$ is a threshold graph.

The graph $G$ contains no induced $2K_2$.

The neighborhoods of vertices in $X$ are linearly ordered by inclusion.

The degree sequences for vertices in $X$ and vertices in $Y$ are conjugate.

A sample Ferrers graph is depicted in Figure 1.

Figure 1: The Ferrers graph corresponding to the conjugate degree sequences $(3, 3, 2, 1)$ and $(4, 3, 2)$.

Let $T(G)$ denote the spanning tree number of the graph $G$.

**Definition 2.** Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \sqcup Y$. The Ferrers invariant of $G$ is the quantity

$$F(G) = \frac{1}{|X||Y|} \prod_{v \in V} \deg(v).$$

Ehrenborg and van Willigenburg proved [4, Theorem 2.1] that for Ferrers graphs, $T(G) = F(G)$.

**Conjecture 1** (Ferrers bound conjecture). Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \sqcup Y$. Then

$$T(G) \leq \frac{1}{|X||Y|} \prod_{v \in V} \deg(v),$$

that is, $T(G) \leq F(G)$. 

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Let us call a bipartite graph $G$ Ferrers-good (respectively Ferrers-bad) if $T(G) \leq F(G)$ (respectively $T(G) > F(G)$). Thus Conjecture $\text{I}$ may be expressed more briefly as the claim that all bipartite graphs are Ferrers-good.

For definitions and notation for majorization-related terms, including reordering of terms, we follow Marshall, Olkin, and Arnold $[9]$.

2 Status of the conjecture

Conjecture $\text{I}$ is trivially true for disconnected graphs and all Ferrers graphs.

In 2009, Jack Schmidt (personal communication) computationally verified by an exhaustive search that all bipartite graphs on at most 13 vertices are Ferrers-good.

In 2013, Praveen Venkataramana proved an inequality weaker than Conjecture $\text{I}$ valid for all bipartite graphs.

Proposition 1 (Venkataramana, unpublished). Let $G$ be a bipartite graph with red vertices having degrees $d_1, \ldots, d_p$ and blue vertices having degrees $e_1, \ldots, e_q$. Then

$$T(G) \leq \prod_{i=2}^{p} \left( d_i + \frac{1}{2} \right) \prod_{j=2}^{q} \left( e_j + \frac{1}{2} \right) \sqrt{e_1}$$

In 2014, Garrett and Klee $[6]$ proved that Conjecture $\text{I}$ is equivalent to an inequality on a particular homogeneous polynomial. They used this to verify the conjecture for trees and all bipartite graphs on at most 11 vertices.

In his 2016 senior thesis, Koo $[8]$ summarized what was then known about Conjecture $\text{I}$. He proved that even cycles are Ferrers-good and that the operation of connecting two graphs by a new edge preserves Ferrers-goodness. Moreover, he showed that Conjecture $\text{I}$ holds for a sufficiently edge-dense graph with a cutvertex of degree 2.

Proposition 2 (Theorem 5.12 in $[8]$). Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \sqcup Y$. Suppose that $G$ has a cutvertex of degree 2 and furthermore that

$$\frac{|E|}{|X||Y|} \geq 0.544.$$ 

Then $G$ is Ferrers-good.

Multiple authors have noted that Ferrers-goodness is preserved under the operation of adding pendant vertices. The following proposition is somewhat stronger and requires little additional work.
Proposition 3. Let $G_1$ and $G_2$ be disjoint Ferrers-good graphs, and let $G$ be the graph formed by gluing $G_1$ and $G_2$ along an arbitrary vertex. Then $G$ is also Ferrers-good.

Proof. For each $i$, let $G_i = (V_i, E_i)$ have bipartition $V_i = X_i \sqcup Y_i$, and let $x_i \in X_i$ be arbitrary. Let $G = (V, E)$ be the graph formed from $G_1 \cup G_2$ by identifying $x_1$ with $x_2$, calling the identified vertices $x$. Thus $G$ has bipartition $V = X \sqcup Y$, where $X = X_1 \cup X_2 \cup \{x\} \setminus \{x_1, x_2\}$ and $Y = Y_1 \cup Y_2$.

By construction, $T(G) = T(G_1) \cdot T(G_2)$. Since each $G_i$ is Ferrers-good, it follows that

$$T(G) \leq F(G_1) \cdot F(G_2) = \frac{\deg_{G_1}(x_1) \deg_{G_2}(x_2) \prod_{v \in V \setminus \{x\}} \deg_G(v)}{|X_1||X_2||Y_1||Y_2|}.$$ 

Since $0 < \deg_{G_i}(x_i) \leq |Y_i|$ and $\deg_G(x) = \deg_{G_1}(x_1) + \deg_{G_2}(x_2)$, it follows that

$$\frac{\deg_{G_i}(x_1) \deg_{G_2}(x_2)}{\deg_G(x)} \leq \frac{|Y_1||Y_2|}{|Y|} \leq \frac{|X_1||X_2|}{|X|} \cdot \frac{|Y_1||Y_2|}{|Y|},$$

that is,

$$\frac{\deg_{G_1}(x_1)}{|X_1||Y_1|} + \frac{\deg_{G_2}(x_2)}{|X_2||Y_2|} \leq \frac{\deg_G(x)}{|X||Y|}.$$

Since $\deg_{G_i}(v) = \deg_G(v)$ for any $v \neq x_i$,

$$F(G_1) \cdot F(G_2) \leq \frac{1}{|X||Y|} \prod_{v \in V} \deg_G(v) = F(G).$$

Hence $G$ is also Ferrers-good. \hfill \Box

The following corollary may be useful.

Corollary 1. A minimal Ferrers-bad graph is 2-connected.

Proof. Let $G$ be a minimal Ferrers-bad graph. If $G$ had a cutvertex $x$, we could decompose it into subgraphs $G_1$ and $G_2$ glued along the vertex $x$. By minimality, $G_1$ and $G_2$ are Ferrers-good. Now apply Proposition 3. \hfill \Box

For a Ferrers-good graph $G$ the following inequality holds:

$$\frac{1}{n} \prod_{i=1}^{n-1} \lambda_i \leq \frac{1}{pq} \prod_{i=1}^{n} d_i,$$

where $d$ is the degree sequence of $G$ and $\lambda$ is the spectrum of the Laplacian. These possible values of $d$ and $\lambda$ are known to be restricted by the Gale–Ryser theorem and the Grone–Merris conjecture, proved by Bai [1].

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Theorem 1 (Gale [5], Ryser [13]). Let \( a \) and \( b \) be partitions of an integer. There is a bipartite graph whose blue degree sequence is \( a \) and whose red degree sequence is \( b \) if and only if \( a \preceq b^* \).

Theorem 2 (Grone–Merris conjecture, proved in [1]). The Laplacian spectrum of a graph is majorized by the conjugate of its degree sequence.

One might hope that the constraints provided by these theorems would restrict \( \lambda \) enough for Inequality 1 to hold. We can state this as the following (incorrect) conjecture.

Conjecture 2. Let \( n = p + q > 1 \) be an integer. Let \( a \vdash p \) and \( b \vdash q \) be integer partitions, and let \( d = d_1 \geq d_2 \geq \cdots \geq d_n \) be their union. Let \( \lambda = \lambda_1 \geq \cdots \geq \lambda_{n-1} \) be a weakly decreasing sequence of positive real numbers.

If \( a \preceq b^* \) and \( d \preceq \lambda \preceq d^* \) then

\[
\frac{1}{n} \prod_{i=1}^{n-1} \lambda_i \leq \frac{1}{pq} \prod_{i=1}^{n} d_i
\]

However, the following simple example, provided by Evan Chen (personal communication), shows that Conjecture 2 is false. Let \( d = (2, 2, 2, 1, 1) = (2, 2) \oplus (2, 1, 1) \) (so \( a = (2, 2) \) and \( b = (2, 1, 1) \)) and \( \lambda = (2, 2, 2, 2) \). One can verify that the majorization inequalities of Conjecture 2 hold. However, the conclusion is false:

\[
\frac{1}{5} \prod_{i=1}^{4} \lambda_i = \frac{16}{5} \not\leq \frac{4}{3} = \frac{1}{2 \cdot 3} \prod_{i=1}^{5} d_i.
\]

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