Ricci tensor on smooth metric measure space with boundary

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Abstract

The aim of this note is to study the measure-valued Ricci tensor on smooth metric measure space with boundary, which is a generalization of Bakry-Émery’s modified Ricci tensor on weighted Riemannian manifold. As an application, we offer a new approach to study curvature-dimension condition of smooth metric measure space with boundary.

Keywords: metric measure space, curvature-dimension condition, boundary, Bakry-Émery theory.

1 Introduction

Let $M = (X, g, e^{-V} \text{Vol}_g)$ be a $n$-dimensional weighted Riemannian manifold (or smooth metric measure space) equipped with a metric tensor $g : [TM]^2 \rightarrow C^\infty(M)$. The well-known Bakry-Émery’s Bochner type formula

$$\Gamma_2(f) = \text{Ricci}(\nabla f, \nabla f) + H_V(\nabla f, \nabla f) + |H_f|^2_{\text{HS}},$$

valid for any smooth function $f$, where $H_V = \nabla^2 V$ is the Hessian of $V$ and $|H_f|_{\text{HS}}$ is the Hilbert-Schmidt norm of the Hessian $H_f$. The operator $\Gamma_2$ is defined by

$$\Gamma_2(f) := \frac{1}{2}L\Gamma(f, f) - \Gamma(f, Lf), \quad \Gamma(f, f) := \frac{1}{2}L(f^2) - fLf$$

where $\Gamma(\cdot, \cdot) = g(\nabla \cdot, \nabla \cdot)$, and $L = \Delta - \nabla V$ is the Witten-Laplacian on $M$. It is known that $\Gamma_2 \geq K$ could characterize many important geometric and analysis properties of $M$.

The aim of this paper is to study the Bakry-Émery’s $\Gamma_2$-calculus on smooth metric measure space with boundary. It can be seen that smooth metric measure space with boundary is actually a non-smooth space, since the geodesics are not even $C^2$ in general (see e.g. [1]). Therefore, it will not be more difficult to study this

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problem in an abstract framework. In this paper, we will use the theory of (non-smooth) metric measure space with lower Ricci curvature bound, which was founded by Lott-Sturm-Villani, and systematically studied using different techniques which originally come from differential geometry, metric geometry, probability theory, etc.

We will see that the non-smooth Bochner inequality and the measure-valued Ricci tensor $\text{Ricci}_\Omega$, which are introduced in [14] and [10] have precise representations on weighted Riemannian manifold $(\Omega, d_g, e^{-V}\text{Vol}_g)$ with boundary, where $d_g$ is the intrinsic distance on $\Omega \subset X$ induced by the Riemannian metric $g$:

$$\text{Ricci}_\Omega = \text{Ricci}_V e^{-V}d\text{Vol}_g + II e^{-V}d\mathcal{H}^{n-1}|_{\partial\Omega} \tag{1.2}$$

where $\text{Ricci}_V = \text{Ricci} + H_V$ is the Bakry-Émery Ricci tensor and $II$ is the second fundamental form.

From [4,5] and [10] we know that $(\Omega, d_g, \text{Vol}_g)$ is a RCD($K, \infty$) space, or in other words, the Boltzman entropy is $K$-displacement convex, if and only if $\text{Ricci}_\Omega \geq K$. By (1.2) we know $\text{Ricci}_\Omega \geq K$ if and only if $\text{Ricci} \geq K$ and $II \geq 0$. Then we immediately know $(\Omega, d_g)$ is locally convex if it is RCD($K, \infty$). Even though this result could also be proved by combining the result of Ambrosio-Gigli-Saváré ([3,4]) and Wang (see e.g. Chapter 3, [17]). Our approach here is the first one totally ‘inside’ the framework of metric measure space.

In this paper, we will review the construction of measure-valued Ricci tensor and give a quick proof to our main formula. Then we end this note with some simple applications. More applications and generalizations will be studied in the future.

## 2 Measure valued Ricci tensor and application

Let $M := (X, d, m)$ be a complete, separable geodesic space. We define the local Lipschitz constant $\text{lip}(f) : X \to [0, \infty]$ of a function $f$ by

$$\text{lip}(f)(x) := \begin{cases} \lim_{y \to x} \frac{|f(y) - f(x)|}{d(x,y)}, & x \text{ is not isolated} \\ 0, & \text{otherwise.} \end{cases}$$

We say that $f \in L^2(X, m)$ is a Sobolev function in $W^{1,2}(M)$ if there exists a sequence of Lipschitz functions functions $\{f_n\} \subset L^2$, such that $f_n \to f$ and $\text{lip}(f_n) \to G$ in $L^2$ for some $G \in L^2(X, m)$. It is known that there exists a minimal function $G$ in $m$-a.e. sense. We call this minimal $G$ the minimal weak upper gradient (or weak gradient for simplicity) of the function $f$, and denote it by $|Df|$. It is known that the locality holds for $|Df|$, i.e. $|Df| = |Dg|$ $m$-a.e. on the set $\{x \in X : f(x) = g(x)\}$. If $M$ is a Riemannian manifold, it is known that $|Df|_M = |\nabla f| = \text{lip}(f)$ for any $f \in C^\infty$. Furthermore, let $\Omega \subset M$ be a domain such that $\partial\Omega$ is $(n-1)$-dimensional. Then we know $|Df|_\Omega = |\nabla f|$ $m$-a.e. (see Theorem 6.1, [8]). It can also be seen that the weighted measure $e^{-V}m$ does not change the value of weak gradients.

We equip $W^{1,2}(X, d, m)$ with the norm

$$\|f\|_{W^{1,2}(X, d, m)}^2 := \|f\|_{L^2(X, m)}^2 + \|\text{D}f\|_{L^2(X, m)}^2.$$
It is known that $W^{1,2}(X)$ is a Banach space, but not necessarily a Hilbert space. We say that $(X, d, m)$ is an infinitesimally Hilbertian space if $W^{1,2}$ is a Hilbert space. Obviously, Riemannian manifolds (with or without boundary) are infinitesimally Hilbertian spaces.

On an infinitesimally Hilbertian space $M$, we have a natural pointwise bilinear map defined by

$$[W^{1,2}(M)]^2 \ni (f, g) \mapsto \langle \nabla f, \nabla g \rangle := \frac{1}{4} \left( |D(f + g)|^2 - |D(f - g)|^2 \right).$$

Then we can define the Laplacian by duality.

**Definition 2.1** (Measure valued Laplacian, [10, 11]). The space $D(\Delta) \subset W^{1,2}(M)$ is the space of $f \in W^{1,2}(M)$ such that there is a measure $\mu$ satisfying

$$\int h \, d\mu = -\int \langle \nabla h, \nabla f \rangle \, dm, \forall h : M \to \mathbb{R}, \text{ Lipschitz with bounded support}.$$

In this case the measure $\mu$ is unique and we shall denote it by $\Delta f$. If $\Delta f \ll m$, we denote its density by $\Delta f$.

We have the following proposition characterizing the curvature-dimensions conditions $\text{RCD}(K, \infty)$ and $\text{RCD}^*(K, N)$ through non-smooth Bakry-Émery theory. We say that a space is $\text{RCD}(K, \infty)/\text{RCD}^*(K, N)$ if it is a $\text{CD}(K, \infty)/\text{CD}^*(K, N)$ space which are defined by Lott-Sturm-Villani in [13, 15, 16] and Bacher-Sturm in [6], equipped with an infinitesimally Hilbertian Sobolev space. For more details, see [4] and [2].

We define $\text{TestF}(M) \subset W^{1,2}(M)$, the set of test functions by

$$\text{TestF}(M) := \left\{ f \in D(\Delta) \cap L^\infty : |Df| \in L^\infty \text{ and } \Delta f \in W^{1,2}(M) \cap L^\infty(M) \right\}.$$ 

It is known that $\text{TestF}(M)$ is dense in $W^{1,2}(M)$ when $M$ is $\text{RCD}(K, \infty)$.

Let $f, g \in \text{TestF}(M)$. We know (see [14]) that the measure $\Gamma_2(f, g)$ is well-defined by

$$\Gamma_2(f, g) = \frac{1}{2} \Delta \langle \nabla f, \nabla g \rangle - \frac{1}{2} \left( \langle \nabla f, \nabla \Delta g \rangle + \langle \nabla g, \nabla \Delta f \rangle \right) m,$$

and we put $\Gamma_2(f) := \Gamma_2(f, f)$. Then we have the following Bochner inequality on metric measure space, which can be regarded as variant definitions of $\text{RCD}(K, \infty)$ and $\text{RCD}^*(K, N)$ conditions.

**Proposition 2.2** (Bakry-Émery condition, [4, 5], [9]). Let $M = (X, d, m)$ be an infinitesimally Hilbertian space satisfying Sobolev-to-Lipschitz property (see [5] or [12] for the definition). Then it is a $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty]$ if and only if

$$\Gamma_2(f) \geq \left( K |Df|^2 + \frac{1}{N} (\Delta f)^2 \right) m$$

for any $f \in \text{TestF}(M)$. 

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Let \( f \in \text{TestF}(M) \). We define the Hessian \( H_f : \{ \nabla g : g \in \text{TestF}(M) \}^2 \to L^0(M) \) by
\[
2H_f(\nabla g, \nabla h) = \langle \nabla g, \nabla (\nabla f, \nabla h) \rangle + \langle \nabla h, \nabla (\nabla f, \nabla g) \rangle - \langle \nabla f, \nabla (\nabla g, \nabla h) \rangle
\]
for any \( g, h \in \text{TestF}(M) \). Using the estimate obtained in [14], it can be seen that \( H_f \) can be extended to a symmetric \( L^\infty(M) \)-bilinear map on \( L^2(TM) \) (see [10] for the definition) and continuous with values in \( L^0(M) \), see Theorem 3.3.8 in [10] for a proof. On Riemannian manifolds (with boundary), it can be seen that \( H_f \) coincides with the usual Hessian \( \nabla^2 f \), \( m \)-a.e., and the Hilbert-Schmidt norms are also identified.

Furthermore, we have the following proposition.

**Proposition 2.3** (See [10]). Let \( M \) be an infinitesimally Hilbertian space satisfying Sobolev-to-Lipschitz property. Then \( M \) is RCD\((K, \infty)\) if and only if
\[
\text{Ricci}(\nabla f, \nabla f) \geq K|Df|^2 m
\]
for any \( f \in \text{TestF}(M) \), where \( \text{Ricci}(\nabla f, \nabla f) := \Gamma_2(f) - |H_f|_{\text{HS}}^2 \).

Now we introduce our main theorem.

**Theorem 2.4** (Measure-valued Ricci tensor). Let \( M = (X, g, e^{-V} \text{Vol}_g) \) be a \( n \)-dimensional weighted Riemannian manifold and \( \Omega \subset M \) be a submanifold with \((n - 1)\)-dimensional smooth orientable boundary. Then the measure valued Ricci tensor on \((\Omega, d_\Omega, e^{-V} \text{Vol}_g)\) can be computed as
\[
\text{Ricci}_\Omega(\nabla g, \nabla g) = \text{Ricci}_V(\nabla g, \nabla g) e^{-V} d\text{Vol}_g + II(\nabla g, \nabla g) e^{-V} d\mathcal{H}^{n-1}_{|\partial \Omega} \quad (2.1)
\]
for any \( g \in C^\infty_c \) with \( g(N, \nabla g) = 0 \), where \( N \) is the outwards normal vector field on \( \partial \Omega \), and \( \text{Ricci}_V \) is the usual Bakry-Émery Ricci tensor on \( M \).

**Proof.** By integration by part formula (or Green’s formula) on Riemannian manifold, we know
\[
\int g(\nabla f, \nabla g) e^{-V} d\text{Vol}_g = -\int f \Delta_V g e^{-V} d\text{Vol}_g + \int_{\partial \Omega} f g(N, \nabla g) e^{-V} d\mathcal{H}^{n-1}_{|\partial \Omega}
\]
for any \( f, g \in C^\infty_c \), where \( \Delta_V := (\Delta - \nabla V) \) and \( N \) is the outwards normal vector field, \( \mathcal{H}^{n-1}_{|\partial \Omega} \) is the \((n - 1)\)-dimensional Hausdorff measure on \( \partial \Omega \). From the discussions before we know
\[
\int g(\nabla f, \nabla g) e^{-V} d\text{Vol}_g = -\int f \Delta_V g e^{-V} d\text{Vol}_g + \int_{\partial \Omega} f g(N, \nabla g) e^{-V} d\mathcal{H}^{n-1}_{|\partial \Omega}.
\]

Therefore we know \( g \in D(\Delta_\Omega) \) and we obtain the following formula concerning the measure-valued Laplacian
\[
\Delta_\Omega g = \Delta_V g e^{-V} d\text{Vol}_g - g(N, \nabla g) e^{-V} d\mathcal{H}^{n-1}_{|\partial \Omega}.
\]
Therefore for any $g \in C^\infty_c$ with $g(N, \nabla g) = 0$ on $\partial \Omega$, we know $g \in \text{TestF}(\Omega)$.

Now we can compute the measure-valued Bakry–Émery tensor. Let $g \in C^\infty_c$ with $g(N, \nabla g) = 0$ on $\partial \Omega$. We have

$$\text{Ricci}_\Omega(\nabla g, \nabla g) = \frac{1}{2} \Delta_\Omega [Dg]_{\Omega}^2 - \langle \nabla g, \nabla \Delta_\Omega g \rangle_\Omega e^{-V} d\text{Vol}_g - \|\text{Hess}_g\|_{\text{HS}}^2 e^{-V} d\text{Vol}_g$$

$$= \frac{1}{2} \Delta_V |\nabla g|^2 e^{-V} d\text{Vol}_g - g(\nabla g, \nabla \Delta_V g) e^{-V} d\text{Vol}_g - \|\text{Hess}_g\|_{\text{HS}}^2 e^{-V} d\text{Vol}_g$$

$$= \text{Ricci}(\nabla g, \nabla g) e^{-V} d\text{Vol}_g + H_V(\nabla g, \nabla g) e^{-V} d\text{Vol}_g$$

$$= \text{Ricci}_V(\nabla g, \nabla g) e^{-V} d\text{Vol}_g - \frac{1}{2} g(N, \nabla |\nabla g|^2) e^{-V} d\mathcal{H}^{n-1}|_{\partial \Omega},$$

where we use Bochner formula at the third equality and $\text{Ricci}_V = \text{Ricci} + H_V$ is the Bakry–Émery Ricci tensor on weighted Riemannian manifold w.r.t the weight $e^{-V}$.

By definition of second fundamental form, we have

$$II(\nabla g, \nabla g) = g(\nabla_{\nabla g} N, \nabla g) = g(\nabla g(N, \nabla g), \nabla g) - \frac{1}{2} g(N, \nabla |\nabla g|^2).$$

However, we assume that $g(N, \nabla g) = 0$ on $\partial \Omega$. Hence $g(\nabla_{\nabla g} N, \nabla g) = -\frac{1}{2} g(N, \nabla |\nabla g|^2)$.

Finally, we obtain

$$\text{Ricci}_\Omega(\nabla g, \nabla g) = \text{Ricci}_V(\nabla g, \nabla g) d\text{Vol}_g + II(\nabla g, \nabla g) e^{-V} d\mathcal{H}^{n-1}|_{\partial \Omega} \quad (2.2)$$

for any $g \in C^\infty_c$ with $g(N, \nabla g) = 0$.

In the next corollary we will see that the space $\{ g : g \in C^\infty_c, g(N, \nabla g) = 0 \} \subset \text{TestF}(\Omega)$ is big enough to characterize the Ricci curvature and the mean curvature.

**Corollary 2.5** (Rigidity: convexity of the boundary). Let $(\Omega, d_\Omega, e^{-V} \text{Vol}_g)$ be a space as in Theorem 2.4. Then it is RCD($K, \infty$) if and only if $\partial \Omega$ is convex and $\text{Ricci}_V \geq K$ on $\Omega$.

**Proof.** If $\Omega$ is RCD($K, \infty$), then from Proposition 2.3 we know $\text{Ricci}_\Omega(\nabla g, \nabla g) \geq K|\nabla g|^2 \text{Vol}_g$ for any $g \in \text{TestF}(\Omega)$. By Theorem 2.4 we know $\text{Ricci}_V(\nabla g, \nabla g) \geq K|\nabla g|^2$ and $II(\nabla g, \nabla g) \geq 0$ for any $g \in C^\infty_c$ with $g(N, \nabla g) = 0$.

On one hand, for any $g \in C^\infty_c(\Omega)$ with support inside $\Omega$, we know $g \in \text{TestF}$. Applying Theorem 2.4 with any of these $g$, we know $\text{Ricci}_V(\nabla g, \nabla g) \geq K|\nabla g|^2$, hence $\text{Ricci}_V \geq K$. On the other hand, for any $g \in C^\infty_c(\partial \Omega)$. By Cauchy–Kovalevskaya theorem we know the Cauchy problem:

1) $f = g$ on $\partial \Omega$,

2) $g(\nabla f, N) = 0$ on $\partial \Omega$.
has a local analytical solution $\tilde{g}$. Furthermore, by multiplying an appropriate smooth cut-off function we can assume further that $\tilde{g} \in C^\infty_c(\Omega)$ and $\tilde{g} \in \text{TestF}(\Omega)$. Applying Theorem 2.3 with $\tilde{g}$, we know $II(\nabla g, \nabla g) \geq 0$. Since $g$ is arbitrary, we know $II \geq 0$.

Conversely, if $\partial \Omega$ is convex we know $\Omega$ is locally convex in the ambient space $X$ (see e.g. [7]). Combining with $\text{Ricci}_V \geq 0$ we know $\Omega$ is locally RCD($K, \infty$). By local to global property of RCD($K, \infty$) condition (see e.g. [15]), we prove the result.

**Remark 2.6.** In this corollary, we only study the manifolds with boundary which can be regarded as a submanifold with orientable boundary. Since the problem we are considering is local, it is not more restrictive than general case.

**Remark 2.7.** In [3] Ambrosio-Gigli-Savare identify the gradient flow of Boltzman entropy with the (Neumann) heat flow. In [4] they prove the exponential contraction of heat flows in Wasserstein distance. Combining the result of Wang (see Theorem 3.3.2 in [17]) we can also prove this result.

**Corollary 2.8.** A $N$-dimensional Riemannian manifold with boundary is RCD($K, \infty$) if and only if it is RCD$^\ast$($K, N$).

The next corollary characterize the Ricci-flat space as a metric measure space.

**Corollary 2.9.** Let $M$ and $\Omega$ be as above. Then $\Omega$ is a Ricci flat space, i.e. $\text{Ricci}_\Omega = 0$, if and only if it is a minimal hypersurface with zero Ricci curvature inside.

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