COMPLETE INTERPOLATING SEQUENCES FOR THE GAUSSIAN
SHIFT-ININVARIANT SPACE

ANTON BARANOV, YURI BELOV, AND KARLHEINZ GRÖCHENIG

Abstract. We give a full description of complete interpolating sequences for the shift-invariant space generated by the Gaussian. As a consequence, we rederive the known density conditions for sampling and interpolation.

1. Main results

Consider the shift-invariant space of functions on $\mathbb{R}$ with Gaussian generator $g(z) = e^{-az^2}$ for $a > 0$ defined as

$$V^2 = \left\{ f(z) = \sum_{n \in \mathbb{Z}} c_n e^{-a(z-n)^2} : (c_n) \in \ell^2(\mathbb{Z}) \right\}.$$ 

We consider the space $V^2$ as a subspace of $L^2(\mathbb{R})$ with the usual $L^2$-norm.

The space $V^2$ belongs to the general family of shift-invariant spaces. Given a generator $g \in L^2(\mathbb{R})$, such a space is defined as

$$V^2(g) = \left\{ f(z) = \sum_{n \in \mathbb{Z}} c_n g(z-n) : (c_n) \in \ell^2(\mathbb{Z}) \right\} \subseteq L^2(\mathbb{R}).$$

The primary example is the classical Paley–Wiener space $PW = \{ f \in L^2(\mathbb{R}) : \text{supp} \hat{f} \subseteq [-1/2,1/2] \}$, which, by the sampling theorem of Shannon–Whittaker–Kotelnikov, can be identified with the shift-invariant space $V^2(\sin \pi x / \pi x)$. In signal processing shift-invariant spaces are often taken as a substitute for the Paley–Wiener space. A unifying feature of both $V^2$ (Gaussian generator) and $PW$ (sinc-generator) is the fact that both spaces can be viewed as spaces of entire functions by interpreting the variable $z$ to be in $\mathbb{C}$.

It is easy to see that the norm equivalence $\|f\|_{L^2(\mathbb{R})} \asymp \|(c_n)\|_{\ell^2(\mathbb{Z})}$ holds on $V^2$ with some absolute constants. In what follows it will be convenient for us to work with the second quantity and so we put $\|f\|_{V^2} := \|(c_n)\|_{\ell^2(\mathbb{Z})}$, and we often identify $V^2$ with $\ell^2(\mathbb{Z})$ via the mapping $(c_n) \mapsto f(z) = \sum_{n \in \mathbb{Z}} c_n e^{-(z-n)^2}$.

A sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{R}$ is said to be sampling for $V^2$, if

$$\sum_{n \in \mathbb{Z}} |f(\lambda_n)|^2 \asymp \|f\|^2_{V^2}, \quad f \in V^2,$$

2000 Mathematics Subject Classification. Primary 30H20; Secondary 30D10, 30E05, 42C15, 94A20.

Key words and phrases. Sampling, interpolation, Riesz bases, small Fock spaces, shift-invariant space, Avdonin-type condition.

A.B. and Yu.B. were supported by the Ministry of Science and Higher Education of the Russian Federation, agreement No 075-15-2021-602, and by the Russian Foundation for Basic Research grant 20-51-14001-ANF-a. K.G. was supported in part by the project P31887-N32 of the Austrian Science Fund (FWF).

1
and we say that \( \Lambda \) is interpolating for \( V^2 \) if for every \((a_n) \in \ell^2(\mathbb{Z})\) there exists \( f \in V^2 \) such that \( f(\lambda_n) = a_n \). If \( \Lambda \) is both sampling and interpolating (or, equivalently, the solution of the interpolation problem is unique), we say that \( \Lambda \) is a complete interpolating sequence.

Sampling and interpolation in the space \( V^2 \) and in more general shift-invariant spaces were studied in \([8, 9]\). The main result can be formulated as follows. Recall that \( \Lambda \) is said to be separated if \( \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0 \), and denote by \( D^+(\Lambda) \) and \( D^-(\Lambda) \) the usual upper and lower Beurling densities

\[
D^+(\Lambda) = \lim_{r \to \infty} \sup_{x \in \mathbb{R}} \frac{\operatorname{card}(\Lambda \cap [x, x + r])}{r}, \quad D^-(\Lambda) = \lim_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\operatorname{card}(\Lambda \cap [x, x + r])}{r}.
\]

**Theorem 1.1.**

(i) Sufficiency: Every separated sequence \( \Lambda \subset \mathbb{R} \) with \( D^-(\Lambda) > 1 \) is sampling for \( V^2 \). Moreover, it contains a complete interpolating sequence. 

(ii) Necessity: If \( \Lambda \) is sampling for \( V^2 \), then \( \Lambda \) is a finite union of separated subsequences and \( \Lambda \) contains a separated sequence \( \tilde{\Lambda} \) such that \( D^-(\tilde{\Lambda}) \geq 1 \).

**Theorem 1.2.**

(i) Every separated sequence \( \Lambda \) with \( D^+(\Lambda) < 1 \) is a set of interpolation for \( V^2 \). Moreover, it can be enlarged to a complete interpolating sequence. 

(ii) If \( \Lambda \) is a set of interpolation for \( V^2 \), then \( \Lambda \) is separated and \( D^+(\Lambda) \leq 1 \).

One of the main insights obtained in \([8, 9]\) is the similarity of \( V^2 \) with the Paley–Wiener space \( PW \) with respect to sampling and interpolation. In particular, for both \( V^2 \) and \( PW \) the same density conditions hold. For \( PW \) these go back to Beurling, Kahane, and Landau \([6, 14, 15, 18]\).

Theorems 1.1 and 1.2 provide an almost characterization of sampling sets and of interpolating sets, but they leave open the case of critical density. If a set is simultaneously sampling and interpolating, then \( D(\Lambda) = 1 \). The case of the critical density is much more subtle, because anything may happen. For \( PW \) the problem of complete interpolating sequences was solved in \([12]\) and \([16]\), for \( V^2 \) it was open so far.

Our main result is a complete and explicit description of complete interpolating sequences for \( V^2 \). Moreover, this characterization can be extended to the larger class of complex Gaussians

\[
g_c(z) = e^{-cz^2}, \quad c = a + ib, \quad a > 0, \quad b \in \mathbb{R}.
\]

We denote the space \( V^2(g_c) \) by \( V^2_c \).

It is easy to see that for \( f(z) = \sum_{n \in \mathbb{Z}} c_n e^{-c(z-n)^2} \) we still have \( \|f\|_{L^2(\mathbb{R})} \approx \|(c_n)\|_{\ell^2(\mathbb{Z})} \) and so we can define the norm in \( V^2_c \) by \( \|f\|_{V^2_c} := \|(c_n)\|_{\ell^2(\mathbb{Z})} \).

**Theorem 1.3.** Given \( c \in \mathbb{C} \) with \( a = \Re c > 0 \), an increasing sequence \( \Lambda \subset \mathbb{R} \) is a complete interpolating sequence for \( V^2_c \), if and only if \( \Lambda \) is separated and there exists an enumeration \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}, \lambda_n = n + \delta_n, n \in \mathbb{Z} \), such that

(a) \( \{\delta_n\}_{n \in \mathbb{Z}} \in \ell^\infty \); 
(b) there exists \( N \geq 1 \) and \( \delta > 0 \) such that

\[
\sup_{n \in \mathbb{Z}} \frac{1}{N} \left| \sum_{k=n+1}^{n+N} \delta_k \right| \leq \delta < \frac{1}{2}.
\]

Theorem 1.3 is stronger than the results in \([8]\). Indeed, as a corollary, we will deduce the density results for sampling or interpolation obtained previously in \([8]\) and extend them to the spaces generated by complex Gaussians.
At least to us, Theorem 1.3 is quite surprising, as it shows a marked difference between $V_c^2$ and $PW$. The characterizing conditions go back to Avdonin who showed that conditions (a) and (b) are sufficient for complete interpolating sequences in $PW$. However, they are far from necessary in $PW$. By contrast, in the shift-invariant spaces $V_c^2$ these conditions provide a complete characterization.

To prove Theorem 1.3, we reduce the problem to the study of similar problems in some Fock-type space of entire functions. After this reduction is done, one can apply the description of complete interpolating sequences in this space obtained in [4].

2. Unitary equivalence with a Fock-type space

For a parameter $a > 0$, we consider the Fock-type space of entire functions, sometimes referred to as a small Fock space, defined by

$$\mathcal{F} = \mathcal{F}_a = \left\{ F \text{ entire} : \|F\|^2_{\mathcal{F}_a} = \frac{1}{2\sqrt{2\pi a}} \int_{\mathbb{C}} |F(w)|^2 e^{-\frac{1}{2a} (\log |w|)^2} dm_2(w) < \infty \right\},$$

where $m_2$ is the Lebesgue measure $\frac{1}{\pi} dx dy$. A simple computation shows that for a function $F(w) = \sum_{n \geq 0} b_n w^n$ one has

$$\|F\|^2_{\mathcal{F}_a} = \sum_{n \geq 0} |b_n|^2 e^{2a(n+1)^2}.$$

In what follows we assume $c \in \mathbb{C}$ to be fixed. To see the connection between $\mathcal{F}_a$ and $V_c^2$, we note that

$$\sum_{n \in \mathbb{Z}} c_n e^{-c(z-n)^2} = e^{-cz} \sum_{n \in \mathbb{Z}} c_n e^{-cn^2} e^{2cnz}.$$ 

Introducing a new variable $w = e^{2cz}$, $V_c^2$ becomes unitarily equivalent to the space of functions representable as

$$\left\{ g(w) = \sum_{n \in \mathbb{Z}} d_n w^n : \|g\|^2 := \sum_{n \in \mathbb{Z}} |d_n|^2 e^{2am^2} < \infty \right\}.$$

This space consists of functions analytic in $\mathbb{C} \setminus \{0\}$. It is more convenient, however, to split this space into three parts, corresponding to positive and negative powers of $w$, and the constant term. Namely, for $f \in V_c^2$ we write

$$f(z) = \sum_{n \in \mathbb{Z}} c_n e^{-c(z-n)^2} = \sum_{n < 0} c_n e^{-c(z-n)^2} + c_0 e^{-cz^2} + \sum_{n > 0} c_n e^{-c(z-n)^2}$$

(1)

$$= f_-(z) + c_0 e^{-cz^2} + f_+(z).$$

With respect to this decomposition $V_c^2$ splits into a direct sum $V_c^2 = V_+^2 \oplus e^{-cz^2} \mathbb{C} \oplus V_-^2$, which is, in terms of the coefficients, simply $\ell^2(\mathbb{Z}) = \ell^2(\mathbb{N}_-) \oplus \mathbb{C} \oplus \ell^2(\mathbb{N})$.

Next, for $w = e^{2cz}$, set

$$F_+(w) = \sum_{n=1}^{\infty} c_n e^{-cn^2} w^{n-1}, \quad F_-(w) = \sum_{n=1}^{\infty} c_{-n} e^{-cn^2} w^{n-1},$$

then in view of (1) we can write

$$f(z) = e^{-cz^2} (w^{-1} F_-(w^{-1}) + c_0 + w F_+(w)).$$
It follows from the discussion above that $F_+, F_- \in \mathcal{F}_a$ and

$$\|F_+\|_{\mathcal{F}_a} = \left( \sum_{n>0} |c_n e^{-cn^2}|^2 e^{2an^2} \right)^{1/2} = \|c\|_2 = \|f_+\|_{\mathcal{V}_c^2},$$

and $\|F_-\|_{\mathcal{F}_a} = \|f_-\|_{\mathcal{V}_c^2}$. In addition every function $F_+, F_- \in \mathcal{F}_a$ can appear in this representation and

$$wF_+(w) = e^{cz^2} f_+(z)|_{e^{2czw}=w}. $$

This representation of elements in $\mathcal{V}_c^2$ makes it possible to reduce the study of complete interpolating sequences in $\mathcal{V}_c^2$ to the same problem in the Fock-type space $\mathcal{F}_a$.

Let us introduce the notions of sampling and interpolation for the space $\mathcal{F}_a$. Let $\varphi(z) = \frac{1}{4a} \log^2 |z|$. In the following we drop the reference to the parameter $a$, when it is not needed. We say that the sequence $\{w_n\}$ is sampling for $\mathcal{F}$, if

$$\sum_n (1 + |w_n|^2) e^{2\varphi(w_n)} |F(w_n)|^2 \leq \|F\|_{\mathcal{F}}^2, \quad F \in \mathcal{F}. \tag{2}$$

Analogously, $\{w_n\}$ is interpolating for $\mathcal{F}$ if for every sequence $\{a_n\}$ satisfying $\sum_n (1 + |w_n|^2) e^{-2\varphi(w_n)} |a_n|^2 < \infty$ there exists $F \in \mathcal{F}$ with $F(w_n) = a_n$. Recall from [5, Lemma 2.7] that the reproducing kernel $k_w$ of $\mathcal{F}$ has the norm

$$\|k_w\|_{\mathcal{F}}^2 \lesssim \frac{e^{2\varphi(w)}}{1 + |w|^2}. \tag{3}$$

Thus the weights in the sampling inequality are nothing but $\|k_{w_n}\|_{\mathcal{F}}^2$. So the norm equivalence (2) reads as $\sum_n |\langle f, k_{w_n}\rangle_{\mathcal{F}}|^2 \|k_{w_n}\|_{\mathcal{F}}^2 \lesssim \|f\|_{\mathcal{F}}^2$. We may thus formulate the questions about sampling and interpolation in the equivalent language of frames and Riesz sequences of normalized reproducing kernels $\tilde{k}_{w_n} = k_{w_n}/\|k_{w_n}\|_{\mathcal{F}}$ in $\mathcal{F}$. Namely, $\{w_n\}$ is sampling for $\mathcal{F}$ if and only if $\{\tilde{k}_{w_n}\}$ is a frame in $\mathcal{F}$, while $\{w_n\}$ is interpolating if and only if $\{\tilde{k}_{w_n}\}$ is a Riesz sequence. Finally, $\{w_n\}$ is a complete interpolating sequence if and only if $\{\tilde{k}_{w_n}\}$ is a Riesz basis in $\mathcal{F}$.

We now formulate the description of complete interpolating sequences for $\mathcal{F}_a$. First it was shown in [3] that the sequence $\{e^{2an}\}_{n \geq 1}$ is a complete interpolating sequence for $\mathcal{F}_a$. Then a full characterization of complete interpolating sequences was obtained in [4]. Remarkably, this characterization is formulated in terms of perturbations of the complete interpolating sequence $\{e^{2an}\}_{n \geq 1}$, somewhat in the spirit of the theorems of Kadets and Avdonin for complex exponentials (although this condition is only sufficient in $PW$). Let $\{w_n\}_{n \geq 1}$ be a sequence such that $0 < |w_n| \leq |w_{n+1}|$ and let

$$w_n = e^{2an} e^{\delta_n} e^{i\theta_n}$$

with $\delta_n, \theta_n \in \mathbb{R}$. In [4] the following result was proved.

**Theorem 2.1.** [4] A sequence $\{w_n\}_{n \geq 1}$ is a complete interpolating sequence for $\mathcal{F}_a$ if and only if

(i) $\{w_n\}$ is $\mathcal{F}_a$-separated, i.e., there exists $\gamma > 0$ such that $|w_m - w_n| \geq \gamma |w_n|$, $m \neq n$;

(ii) $\{\delta_n\}_{n \geq 1} \in \ell^\infty$; \footnote{Note that in [4] the normalization for the small Fock space is different; to formulate the results from [4] one needs to set $\alpha = 1/(4a)$.}

1
(iii) there exists $N \geq 1$ and $\delta > 0$ such that
\[
\sup_{n \geq 1} \frac{1}{N} \left| \sum_{k=n+1}^{n+N} \delta_k \right| \leq \delta < a.
\]

Note that the result does not depend on the arguments of $w_n$, but only on their moduli. For the description of complete interpolating sequences for a more general class of Hilbert spaces of entire functions see [3].

3. Proof of Theorem 1.3: Sufficiency

Assume that $\Lambda \subset \mathbb{R}$ is separated and that there exists an enumeration $\Lambda = \{\lambda_m\}_{m \in \mathbb{Z}}$, $\lambda_m = m + \delta_m$, $m \in \mathbb{Z}$, such that conditions (a) and (b) of Theorem 1.3 are satisfied. We need to show that the mapping
\[
J : (c_n)_{n \in \mathbb{Z}} \mapsto (f(\lambda_m))_{m \in \mathbb{Z}}
\]
is an isomorphism of $\ell^2(\mathbb{Z})$ onto itself. Again we identify the sequence $(c_n)$ and the corresponding function $f \in V_e^2$.

Recall that $\varphi(w) = \frac{1}{4a} (\log |w|)^2$, and put
\[
w_m = e^{2c\lambda_m} = e^{2am} e^{2b(m+\delta_m)}.
\]
The Avdonin-type condition (iii) of Theorem 2.1 reads as \(\frac{1}{N} \left| \sum_{k=n+1}^{n+N} (2a\delta_k) \right| \leq \delta < a\), which amounts precisely to the assumption (b) of Theorem 1.3. Thus each of the sequences \(\{w_m\}_{m \geq 1}\) and \(\{w_{-1,m}\}_{m \geq 1}\) is a complete interpolating sequence for $F_a$.

Claim 1. The mapping $U_+ : (c_n)_{n \geq 1} \mapsto (f_+(\lambda_m))_{m \geq 1}$ is an isomorphism from $\ell^2(\mathbb{N})$ onto itself. Indeed, since $\varphi(w) = (\log |w|)^2/(4a) = \lambda^2$ whenever $w = e^{2c\lambda}$, we have
\[
f_+(\lambda_m) = e^{-c\lambda^2_m} w_m F_+(w_m) = e^{-\varphi(w_m)} e^{-ib\lambda^2_m} w_m F_+(w_m),
\]
and we conclude that $(f_+(\lambda_m))_{m \geq 1} \in \ell^2(\mathbb{N})$, because $\{w_m\}$ is sampling, and that every sequence in $\ell^2(\mathbb{N})$ can be obtained in this way from some function in $F_+$, because $\{w_m\}$ is interpolating. Thus $U_+$ is one-to-one and onto. Similarly the mapping $U_- : (c_n)_{n \leq -1} \mapsto (f_-^{-1}(\lambda_m))_{m \leq -1}$ is an isomorphism on $\ell^2(\mathbb{N}_-)$.

Claim 2. The mapping $K_+ : (c_n)_{n \geq 1} \mapsto (f_+(\lambda_m))_{m \leq -1}$ is compact from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N}_-)$. Since
\[
f_+(\lambda_m) = \sum_{n \geq 1} c_n e^{-c(m+\delta_m-n)^2},
\]
the matrix of $K_+$ has the entries $e^{-c(m+\delta_m-n)^2}$, $m < 0$, $n > 0$. The boundedness of $\delta_m$ yields that $\sum_{m < 0, n > 0} e^{-a(m+n-\delta_m)^2} < \infty$, thus $K_+$ is a Hilbert–Schmidt operator. Similarly, the operator $K_- : (c_n)_{n \leq -1} \mapsto (f_-(\lambda_m))_{m \geq 1}$ is compact from $\ell^2(\mathbb{N}_-)$ to $\ell^2(\mathbb{N})$.

Now we define the operator $U$ on $\ell^2(\mathbb{Z}) = \ell^2(\mathbb{N}_+) \oplus \mathbb{C} \oplus \ell^2(\mathbb{N})$ by the formula
\[
U : (\{c_n\}_{n \leq -1}; c_0; (c_n)_{n \geq 1}) \mapsto ((f_-(\lambda_m))_{m \leq -1}; f(\lambda_0); (f_+(\lambda_m))_{m \geq 1})
\]
By Claim 1 $U$ is an isomorphism from $\ell^2(\mathbb{Z})$ onto itself. At the same time, by Claim 2, the map $J : (c_n)_{n \in \mathbb{Z}} \mapsto (f(\lambda_m))_{m \in \mathbb{Z}}$ differs from $U$ by a compact operator, since $f(\lambda_m) = f_+(\lambda_m) + c_0 e^{-c\lambda^2_m} + f_-(\lambda_m)$. Thus, to prove that $J$ is invertible it is sufficient to show that $J$ is one-to-one, which is usually easier. In our context it means that $\Lambda$ is a uniqueness set for the space $V_e^2$. 

Claim 3. \( \Lambda \) is a uniqueness set for the space \( V_c^2 \). Assume that \( f \in V_c^2 \) and \( f(\lambda_m) = 0 \) for all \( m \). Then the function

\[
F(w) = w^{-1}F_-(w^{-1}) + c_0 + wF_+(w)
\]

vanishes on the set \( w_m = e^{2\lambda_m} \).

Since \( W_+ = \{w_m\}_{m \geq 1} \) is a complete interpolating sequence for \( F \), there exists a so-called generating function \( G_+ \) for this sequence. This is a function with simple zeros exactly at \( w_m \), \( m \geq 1 \), and no other zeros, such that \( G_+ \notin F \), but \( \frac{G_+}{w-w_m} \in F \) for all \( m \). In fact, the system \( \{\frac{G_+(w_m)(w-w_m)}{w-w_m}\}_{m \geq 1} \) is the biorthogonal system in \( F \) to the system of reproducing kernels \( \{k_{w_m}\}_{m \geq 1} \).

We will need several estimates for the generating functions. First, let \( G_0 \) be the generating function for the sequence \( W_0 = \{e^{2am}\}_{m \geq 1} \). It was shown in [5, Lemma 2.6] that

\[
|G_0(w)| \approx e^{\varphi(w)} \frac{\text{dist}(w, W_0)}{1 + |w|^{3/2}}, \quad w \in \mathbb{C}.
\]

Using standard estimates of lacunary canonical products it is easy to show (see [4, Section 3, p. 1373]) that for the generating function \( G_+ \) of the perturbed sequence \( W_+ = \{w_m\}_{m \geq 1} \) with \( |w_m| = e^{2am + 2a\delta_m} \) one has

\[
\frac{|G_+(w)|}{|G_0(w)|} \approx \exp \left( -2a \sum_{k=1}^{m} \delta_k \right) \frac{\text{dist}(w, W_+)}{\text{dist}(w, W_0)}
\]

whenever \( e^{a(2m-1)} \leq |w| \leq e^{a(2m+1)} \). By condition (b) of Theorem 1.3 for sufficiently large \( m \) we have \( \exp(-2a \sum_{k=1}^{m} \delta_k) \geq \exp(-2a\delta m) \approx |w|^{-\delta} \), where \( \delta < \frac{1}{2} \). Hence,

\[
(7) \quad |G_+(w)| \gtrsim \frac{\text{dist}(w, W_+)}{(1 + |w|)^{\frac{1}{2} + \delta}} e^{\varphi(w)}.
\]

Analogously, we define the function \( G_- \), the generating function of the sequence \( W_- = \{w_n^{-1}\}_{n \leq -1} \). Consequently, the zero set of the function \( (w-w_0)G_+(w)G_-(w^{-1}) \) is precisely \( \Lambda = \{w_n : n \in \mathbb{Z}\} \). Since by assumption \( F \) vanishes also on \( \Lambda \), \( F \) possesses the factorization

\[
F(w) = (w-w_0)G_+(w)G_-(w^{-1})H(w)
\]

for some function \( H \) analytic in \( \mathbb{C} \setminus \{0\} \). We will show that \( H \) is an entire function which tends to 0 at infinity, and thus must be identically zero. Clearly, \( w^{-1}F_-(w^{-1}) \to 0 \), as \( |w| \to \infty \) and

\[
|F_+(w)| = |\langle F_+, k_w \rangle| \leq \|F_+\|_{\mathcal{F}} \|k_w\|_{\mathcal{F}} \lesssim \frac{e^{\varphi(w)}}{1 + |w|}.
\]

by (3). Consequently,

\[
(8) \quad |F(w)| \lesssim 1 + |wF_+(w)| \lesssim |w| \frac{e^{\varphi(w)}}{1 + |w|} \lesssim e^{\varphi(w)}, \quad |w| \to \infty.
\]

Thus, using (7), (8), and the fact that \( G_-(w^{-1}) \to G_-(0) \neq 0 \), \( |w| \to \infty \), we get

\[
|H(w)| \geq \frac{|F(w)|}{|(w-w_0)G_+(w)G_-(w)|} \lesssim e^{\varphi(w)} \cdot \frac{(1 + |w|)^{\frac{1}{2} + \delta}}{|w| \text{dist}(w, W_+)} e^{-\varphi(w)} \lesssim (1 + |w|)^{\frac{1}{2} + \delta} \text{dist}(w, W_+), \quad |w| \to \infty.
\]
It follows from the $\mathcal{F}$-separation of $W_+$ and the inclusion $W_+ \subset \mathbb{R}$ that there exists a sequence of radii $r_k \to \infty$ such that $\text{dist}(w,W_+) \geq \gamma|w|$ when $|w| = r_k$, with $\gamma > 0$ independent on $k$. Since $\delta < \frac{1}{2}$, we conclude that $\text{max}_{|w|=r_k}|H(w)| \to 0$, as $k \to \infty$. The maximum principle implies that also $H(w) \to 0$, as $|w| \to \infty$.

Analogously, replacing $w$ by $w^{-1}$ and using the estimate (7) for $G_-$, we conclude that $H(w) \to 0$ as $w \to 0$. Consequently the singularity of $H$ at 0 is removable, and $H$ is thus entire and bounded. Thus $H \equiv 0$ and so $F \equiv 0$, as was to be shown.

To conclude we formulate a simple, sufficient condition for complete interpolating sets in the style of Kadets, see e.g. [19].

**Corollary 3.1.** Assume that $|\delta_n| \leq \delta < \frac{1}{2}$ for all $n \in \mathbb{Z}$. Then $\{n+\delta_n\}$ is a complete interpolating sequence for $V_c^2$.

Note that in the Paley–Wiener space $PW$ the corresponding result holds with $\delta < 1/4$. The constant $1/2$ is sharp, as the set $\{n+1/2: n \in \mathbb{Z}\}$ is not a sampling sequence for $V_a^2$ with $a > 0$ [13].

### 4. Proof of Theorem 1.3 necessity

For the proof of necessity we need to reverse the argument of the sufficiency part. Assume that $\Lambda$ is a complete interpolating sequence for $V_c^2$. This means that the operator $J$ given by (4) is an isomorphism of $\ell^2(\mathbb{Z})$ onto itself.

Since a complete interpolating sequence for $V_c^2$ is always separated, the operators $K_+$ and $K_-$ defined in Claim 2 are compact, whence $U$ is a compact perturbation of the isomorphism $J$.

It follows that the kernel of $U$ has finite dimension and that the range of $U$ has finite codimension:

\[
\text{dim Ker } U < \infty \quad \text{and} \quad \text{codim Range } U < \infty.
\]

In the language of frame theory [10, Sec. 8.7] one speaks of the deficit and the excess of the set of reproducing kernels. The condition $\text{Ker } U \neq \{0\}$ means that the reproducing kernels associated to the sampling points $\lambda_n, n \in \mathbb{Z}$, are not complete, but span a proper subspace (the deficit). The finite codimension of the Range of $U$ means that there are linear dependencies of the reproducing kernels, in other words, too many functions for a Riesz basis (the excess). The finiteness condition of (9) implies that we can construct a Riesz basis of reproducing kernels by adding and/or removing finitely many points.

In our context we will use an important property of the Fock space $\mathcal{F}$. Assume that $F \in \mathcal{F}$ satisfies $F(w_1) = 0$ and $w_2 \in \mathbb{C}, w_2 \neq w_1$. Then the function $\tilde{F}(z) = \frac{z-w_2}{z-w_1}F(z)$ is again in $\mathcal{F}$. This property is called the **division property** of $\mathcal{F}$.

**Step 1.** To start with the proof of necessity, assume that (9) holds. Recall that $V_c^2 = \{f \in V_c^2: f = f_+\} \cong \ell^2(\mathbb{N})$, and consider the restriction $U_+ = U|_{V_c^2}$ of $U$ to the subspace $V_c^2$, given by $U_+(e_n)_{n \geq 1} = (f_+(\lambda_m))_{m \geq 1}$. From the properties of $U$ we conclude that the kernel $\mathcal{N} = \text{Ker } U_+$ in $V_c^2$ also is finite-dimensional and that Range $U_+$ is a closed subspace of $\ell^2(\mathbb{N})$ of finite codimension. Put $w_m = e^{2i\lambda_m}$, as before. If $U_+$ is an isomorphism between $V_c^2$ and $\ell^2(\mathbb{N})$, then, by (5), $\{w_m\}_{m \geq 1}$ is a complete interpolating sequence for $\mathcal{F}$. If not, we distinguish two cases.
Case 1. Assume that $U_+$ has a nontrivial kernel $\mathcal{N}$. Then $U_+|_{V_+^2 \ominus \mathcal{N}}$ is an isomorphism on its image. Thus,
\[
\sum_{m \geq 1} |f_+(\lambda_m)|^2 \asymp \|f_+\|_{V_+^2}^2, \quad f_+ \in V_+^2 \ominus \mathcal{N}.
\]
Note that $f \in \mathcal{N}$ (i.e. $f_+(\lambda_m) = 0$, $m \geq 1$) if and only if $F_+(w_m) = 0$, $m \geq 1$. By (5), we have
\[
\sum_{m \geq 1} (1 + |w_m|^2)e^{-2\varphi(w_m)}|F_+(w_m)|^2 \asymp \|F_+\|_{\mathcal{F}}^2, \quad F_+ \in \mathcal{F} \ominus \mathcal{M},
\]
where $\mathcal{M}$ is the finite-dimensional subspace in $\mathcal{F}$ which consists of all functions vanishing on $\{w_m\}_{m \geq 1}$.

We will need the following simple lemma.

Lemma 4.1. Let $\mathcal{M} = \{F \in \mathcal{F} : F(w_m) = 0, m \geq 1\}$ and $M = \dim \mathcal{M} < \infty$. Then there exists a set of points $\{\mu_k\}_{1 \leq k \leq M}$ which is a uniqueness set for $\mathcal{M}$, while $\{\mu_k\}_{1 \leq k \leq M-1}$ is a non-uniqueness set for $\mathcal{M}$.

Proof. We argue by induction on $M$. The base $M = 1$ (i.e. $\mathcal{M} = \text{Lin} \{F_1\}$) is trivial since one can take as $\mu_1$ any point such that $F_1(\mu_1) \neq 0$. Assume that we can prove the statement for $M-1$. Take any point $\mu_1 \in \mathbb{C}$ such that there is a function $F_0 \in \mathcal{M}$ with $F_0(\mu_1) \neq 0$ and consider $\mathcal{M}_1 = \{F \in \mathcal{M} : F(\mu_1) = 0\}$. It is obvious that $\dim \mathcal{M}_1 = M-1$.

By the induction hypothesis there exist points $\mu_2, \ldots, \mu_M$ which form a uniqueness set for $\mathcal{M}_1$, while $\{\mu_2, \ldots, \mu_{M-1}\}$ is a non-uniqueness set for $\mathcal{M}_1$. Then the points $\{\mu_k\}_{1 \leq k \leq M}$ have the required property.

Let $\{\mu_k\}_{1 \leq k \leq M}$ be the set constructed in the lemma. Since $\mathcal{M}$ is finite-dimensional, it follows that
\[
\sum_{k=1}^{M} (1 + |\mu_k|^2)e^{-2\varphi(\mu_k)}|F_+(\mu_k)|^2 \asymp \|F_+\|_{\mathcal{F}}^2, \quad F_+ \in \mathcal{M}.
\]
Let $\tilde{W} = \{w_m\}_{m \geq 1} \cup \{\mu_k\}_{1 \leq k \leq M}$ the enlarged set. By construction, the sampling operator $F_+ \mapsto (F_+(\tilde{w}_k))_{w_k \in \tilde{W}}$ is one-to-one and it has closed range of finite codimension in $\ell^2(\tilde{W})$.

Consequently, by [10] Thm. 8.29, (c)], $\tilde{W}$ is a sampling set for $\mathcal{F}$.

By Lemma 4.1 the set $\tilde{W} \setminus \{\mu_M\}$ is a non-uniqueness set for $\mathcal{F}$ and therefore there exists a function $H \in \mathcal{F}$, such that $H(w_m) = H(\mu_k) = 0$ for $m \in \mathbb{N}$ and $k = 1, \ldots, M-1$, but $H(\mu_M) \neq 0$. We now use the division property of $\mathcal{F}$ and show that every set $\tilde{W} \setminus \{v\}$, where $v \in \tilde{W} \setminus \{\mu_M\}$, is also a non-uniqueness set in $\mathcal{F}$. Consider $\tilde{H}(w) = \frac{w - \mu_M}{w - v}H(w)$. Then $\tilde{H}$ is non-zero in $\mathcal{F}$, and $\tilde{H}$ vanishes on $\tilde{W} \setminus \{v\}$, but $\tilde{H}$ cannot vanish at $v$, because $\tilde{W}$ is sampling on $\mathcal{F}_+$.

It follows that the frame of reproducing kernels $\{k_w\}_{w \in \tilde{W}}$ fails to be complete after removal of any of its members. We conclude that the normalized reproducing kernels corresponding to the sampling set $\tilde{W}$ form an exact frame, i.e., a frame which fails to remain a frame after removal of any of its vectors. By well-known results (see, e.g., [10, 19]) the normalized reproducing kernels at $\tilde{W}$ form a Riesz basis in $\mathcal{F}$, and, equivalently, $\tilde{W}$ is a complete interpolating sequence for $\mathcal{F}$.

Case 2. Assume now that $\text{Ker} U_+ = 0$, whence $U_+$ is an isomorphism onto its image and so $\{w_m\}_{m \geq 1}$ is already sampling for $\mathcal{F}$. Let $\text{codim} \text{Range} U_+ = M$. This means that
In order to show that any separated sequence \( \Lambda \) with \( D^{\leq 1} \) satisfies conditions (a) and (b).

Analogously, any separated \( \Lambda \) with \( \delta_m \) satisfying Avdonin-type conditions (a) and (b) by Theorem 2.1. We apply the same procedure to the sequence \( \{\lambda_m\}_{m=1}^{\infty} \) or \( \{\lambda_m\}_{m=0}^{\infty} \) or \( \{\lambda_m\}_{m<0} \) and \( \{\lambda_m\}_{m<0} \) satisfies conditions (a) and (b).

By taking the union of both sequences and returning to \( \lambda_m, \mu_j, \nu_j \) (instead of \( w_m \)), we see that there exist finite sets \( F_1 \) and \( F_2 \), satisfying \( F_1 \subseteq \Lambda \) and \( F_2 \cap \Lambda = \emptyset \), such that \( \Lambda' = (\Lambda \setminus F_1) \cup F_2 \) can be enumerated to satisfy (a) and (b). It is possible that \( F_1 = \emptyset \) (we add points to both \( \{w_m\}_{m=0}^{\infty} \) and to \( \{w_m\}_{m<0} \) or \( F_2 = \emptyset \) (we remove points from both sets).

Hence, by sufficiency part of Theorem 1.3, \( \Lambda' \) is a complete interpolating sequence for \( V_c^2 \). Note that \( \Lambda = (\Lambda' \setminus F_2) \cup F_1 \). Let card \( F_1 = \text{card} \ F_2 + \ell \). If \( \ell > 0 \), we have moved card \( F_2 \) points and added \( \ell \) additional points. Then \( \Lambda \) fails to be a complete interpolating sequence for \( V_c^2 \) in contradiction to the assumption. If \( \ell < 0 \), we have moved card \( F_1 \) points and removed \( \ell \) additional points. Again \( \Lambda \) fails to be a complete interpolating sequence.

We conclude that card \( F_1 = \text{card} \ F_2 \) and \( \Lambda \) is obtained from \( \Lambda' \) by moving finitely many points. As observed above, since \( \Lambda' \) satisfies conditions (a) and (b), so must the original sequence \( \Lambda \), as was to be proved.

\[ \square \]

5. Proof of the Density Results and Remarks

Finally we indicate how Theorems 1.1 and 1.2 follow from the characterization of complete interpolating sequences in \( V_c^2 \). We use a simple “combinatorial” argument from [17] in order to show that any separated sequence \( \Lambda \) with \( D^{-}(\Lambda) > 1 \) contains a subsequence \( \Lambda' = \{m+\delta_m\}_{m \in \mathbb{Z}} \) with \( \delta_m \) satisfying Avdonin-type conditions (a) and (b) of Theorem 1.3. Analogously, any separated \( \Lambda \) with \( D^{+}(\Lambda) < 1 \) can be enhanced to such a set \( \Lambda' \). We refer to [4] for the precise details, as they are almost identical.
The necessary density conditions for sampling and interpolation in a shift-invariant space are well-known, see, e.g., [1]. Let us show that they can also be deduced from the results for $\mathcal{F}$ in [4].

Let $\Lambda$ be an interpolating set for $V^2_c$. Then the mapping $f \mapsto \{f(\lambda_m)\}_{m \geq 1}$ acts from $V^2_c$ onto $\ell^2(\mathbb{N})$. By Claim 2 of the proof of sufficiency part in Theorem 1.3 the map $f \mapsto \{f_-(\lambda_m)\}_{m \geq 1}$ is a compact operator from $V^2_c$ to $\ell^2(\mathbb{N})$, whence the operator $f \mapsto \{f_+(\lambda_m)\}_{m \geq 1}$ has the closed range of finite codimension. Then it is easy to see that for some $m_0 \geq 1$ the map $f \mapsto \{f_+(\lambda_m)\}_{m \geq m_0}$ will be onto. Hence, the set $\{\lambda_m\}_{m \geq m_0}$ will be an interpolating set for $V^2_c$ and so $\{w_m\}_{m \geq m_0}$ will be interpolating for $\mathcal{F}$. We conclude that $D^+(\Lambda) \leq 1$.

Now assume that $\Lambda$ is a sampling set for $V^2_c$ and, in particular, for $V^2_2$. Since $\Lambda$ is a finite union of separated sets, the sequence $\{f_+(\lambda_m)\}_{m \leq -1}$ decays very fast and it is easy to show that the set $\{\lambda_m\}_{m \geq m_0}$ will be sampling for $V^2_2$.

Further remarks. 1. Let $V^p_c = \{f(z) = \sum_{n \in \mathbb{Z}} c_n e^{-c(z-n)^2} : (c_n) \in \ell^p(\mathbb{Z})\}$ be the subspace generated by shifts of the Gaussian with $\ell^p$ coefficients. This is a closed subspace of $L^p(\mathbb{R})$. Again, a sequence $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$ is called a sampling set for $V^p_c$, if

$$\sum_{n \in \mathbb{Z}} |f(\lambda_n)|^p \asymp \|f\|_{V^p_c}^p \asymp \|c\|_{\ell^p}^p; \quad f \in V^p_c,$$

and interpolating for $V^p_c$ if any $(a_n) \in \ell^p(\mathbb{Z})$ can be interpolated by a function $f \in V^p_c$.

Based on the general theory of sampling in shift-invariant spaces with a nice generator, we can assert that a set $\Lambda \subseteq \mathbb{R}$ is sampling for $V^2_2$, if and only if $\Lambda$ is sampling for some $V^p_c$, if and only if $\Lambda$ is sampling for $V^p_c$ for all $p, 1 \leq p \leq \infty$ [3, Thm. 3.1]. A similar statement holds for the interpolation property. Theorem 1.3 therefore provides a characterization for complete interpolating sequences for all spaces $V^p_c$.

2. Sign retrieval. The sign retrieval problem asks whether a real-valued function $f$ in some function space is uniquely determined by its unsigned values $|f(\lambda)|$ on some set $\Lambda$. For $V^2_a$ with a real-valued Gaussian generator, Theorem 1.3 implies the following result on sign retrieval.

**Corollary 5.1.** Let $\Lambda = \{n/2+\delta_n : n \in \mathbb{Z}\}$ be uniformly separated such that $\{\delta_n\}$ satisfies the Avdonin-type conditions (a) and (b) with $\delta < 1/4$. Then every $f \in V^2_a$ is uniquely determined by its unsigned values $\{|f(n/2+\delta_n)| : n \in \mathbb{Z}\}$.

The proof is the same as in [7], we simply replace sampling sets of density $D^-(\Lambda) > 2$ by a complete interpolating sequence for a dilated version of $V^2_a$. Thus sign retrieval is possible even at the critical density.

3. It would be interesting to obtain similar results for complete interpolating sequences in shift-invariant spaces with other generators. Even a sharp version of Corollary 5.1 would have important consequences for Gabor frames.

**References**

[1] A. Aldroubi, K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.* **43** (2001), 4, 585–620.

[2] S.A. Avdonin, On the question of Riesz bases of exponential functions in $L^2$, *Vestnik Leningrad. Univ.* No. **13** Mat. Meh. Astronom. Vyp. 3 (1974), 5–12 (in Russian).
[3] Yu. Belov, T. Mengestie, K. Seip, Discrete Hilbert transforms on sparse sequences, *Proc. London Math. Soc.* **103** (2011), 73–105.

[4] A. Baranov, A. Dumont, A. Hartmann, K. Kelbay, Sampling, interpolation and Riesz bases in small Fock spaces, *J. Math. Pures Appl.* **103** (2015), 6, 1358–1389.

[5] A. Borichev, Yu. Lyubarskii, Riesz bases of reproducing kernels in Fock type spaces, *J. Inst. Math. Jussieu* **9** (2010), 449–461.

[6] A. Beurling, *The Collected Works of Arne Beurling. Vol. 2*. Contemporary Mathematicians, Birkhäuser Boston Inc., Boston, MA, 1989. Harmonic analysis, Edited by L. Carleson, P. Malliavin, J. Neuberger and J. Wermer.

[7] K. Gröchenig, Phase-retrieval in shift-invariant spaces with Gaussian generator. *J. Fourier Anal. Appl.* **26** (2020), 3, Paper No. 52.

[8] K. Gröchenig, J.L. Romero, J. Stöckler, Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions, *Invent. Math.* **211** (2018), 3, 1119–1148.

[9] K. Gröchenig, J.L. Romero, J. Stöckler, Sharp results on sampling with derivatives in shift-invariant spaces and multi-window Gabor frames, *Constr. Approx.* **51** (2020), 1, 1–25.

[10] C. Heil, *A Basis Theory Primer, Expanded Edition*. Birkhäuser, Boston, 2011.

[11] J. R. Holub, Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert spaces. *Proc. Amer. Math. Soc.*, **122** (1994), 1, 779–785.

[12] S. Hruscev, N. Nikolski, B. Pavlov, Unconditional bases of exponentials and of reproducing kernels. Complex Analysis and Spectral Theory, 214–335, Lecture Notes in Math. **864**, Springer-Verlag, Berlin, 1981.

[13] A. Janssen, The Zak transform and sampling theorems for wavelet subspaces. *IEEE Trans. Signal Process.* **41** (1993) 3360–3365.

[14] J.-P. Kahane, Pseudo-périodicité et séries de Fourier lacunaires. *Ann. Sci. École Norm. Sup.* **79** (1962), 93–150.

[15] H. J. Landau, Necessary density conditions for sampling and interpolation of certain entire functions. *Acta Math.* **117** (1967), 37–52.

[16] J. Ortega-Cerdà, K. Seip. Fourier frames. *Ann. of Math. (2) 155* (2002), 3, 789–806.

[17] K. Seip, A simple construction of exponential bases in $L^2$ of the union of several intervals, *Proc. Edinburgh Math. Soc.* **38** (1995), 1, 171–177.

[18] K. Seip. *Interpolation and Sampling in Spaces of Analytic Functions*, University Lecture Series, Vol. 33, American Mathematical Society, Providence, RI, 2004.

[19] R.M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.

---

**DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA**  
*Email address: anton.d.baranov@gmail.com*

**DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA**  
*Email address: juri.belov@mail.ru*

**FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, A-1090 VIENNA, AUSTRIA**  
*Email address: karlheinz.groechenig@univie.ac.at*