Analysis of parametric finite-difference schemes for the system of linear advection equations

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Abstract. Explicit finite-difference schemes for the linear advection equations are considered. The schemes may be used in the advective step of the splitting method for the kinetic equations. The time derivative is approximated with first order by special asymmetric approximation. Schemes from first to fourth orders of approximation on space are constructed. Stability analysis is realized by von Neumann method. Stability criteria in the form of inequalities on Courant parameter are obtained. As it is demonstrated, obtained schemes demonstrate better stability properties in cases of high order asymmetric approximations in comparison with well-known explicit schemes.

1. Introduction
Mathematical models, based on Boltzmann or Bhatnaghar — Gross — Krook (BGK) kinetic equations are considered for the description of fluid and gas flows at systems with different space scales. These models are considered in many practical problems, e. g. in the construction of Knudsen pump [1] or in porous systems, where the size of pores varies from nanometer to millimeter, such as coal [2]. In last decades lattice Boltzmann method which is based on the finite-difference (FD) schemes for BGK equations has established itself as a powerful tool for numerical modelling of fluid and gas flows on multiprocessor and multicore systems [3, 4].

Many of numerical algorithms for kinetic equations are based on the splitting on physical processes. At every time step this method is realized by two steps named advection and collision. During the advection step linear advection equation is solved. During the collision step equation with nonlinear collision term is solved. The main advantage of the splitting method is based on the opportunities for the stability improvement by the application of various FD schemes on its steps. As it is demonstrated by S. V. Bogomolov [5] and T. Ohwada [6], the splitting procedure is realized only with first order on time step.

In the presented paper FD schemes for the advective step of splitting procedure are investigated. In section 2 modified splitting method for the kinetic equations with discrete velocities is considered. In section 3 FD schemes for linear advection equations are constructed. In section 3 stability analysis of parametric schemes is performed and necessary stability conditions are formulated. Some concluding remarks are made in section 4.
2. Splitting method

Let us consider the kinetic equation without body force term:

\[ \frac{\partial f}{\partial t} + \mathbf{v} \nabla f = I(f), \]  

(1)

where \( f = f(t, \mathbf{r}, \mathbf{v}) \) is a particle distribution function, \( t \) is a time, \( \mathbf{r} \) is a vector of space variables, \( \mathbf{v} \) is a velocity, \( I(f) \) is a collision term. After the discretization of (1) by discrete velocity method [7] on velocity grid \( \mathbf{v}_i, i = 1, n \), the following system of equations for \( \hat{f}_i(t, \mathbf{r}) = f(t, \mathbf{r}, \mathbf{v}_i) \) is obtained:

\[ \frac{\partial \hat{f}_i}{\partial t} + \mathbf{v}_i \nabla \hat{f}_i = I_i(\mathbf{f}), \]  

(2)

where \( \mathbf{f} = (f_1, \ldots, f_n) \).

The splitting method for solution of (2) is realized on time interval \([t_j, t_{j+1}]\), when \( \Delta t = t_{j+1} - t_j < \delta t \), where \( \delta t \) is a mean free time. The method consists of two steps:

1. Advection. The following system of linear advection equations is solved:

\[ \frac{\partial \hat{f}_i}{\partial t} + \mathbf{v}_i \nabla \hat{f}_i = 0, \]  

(3)

with the initial condition: \( \hat{f}_i(t_j, \mathbf{r}) = f_i(t_j, \mathbf{r}) \).

2. Collision. The system without space derivatives is considered:

\[ \frac{\partial \hat{f}_i}{\partial t} = I_i(\hat{\mathbf{f}}), \]  

(4)

with the initial condition: \( \hat{f}_i(t_j, \mathbf{r}) = \hat{f}_i(t_{j+1}, \mathbf{r}) \). As it is demonstrated in [5, 6], this computational procedure provide to obtain results only of first order on time step \( \Delta t \). Results on space steps may be obtained with higher order by means of special approximations constructed by finite-difference, finite-element or finite-volume techniques [8, 9, 10, 11]. Advantage of splitting method consist with the opportunities for improvement of its stability by application of various discretization techniques.

Solution of the problem for (4) may be realized by implicit Euler method, which is A-stable:

\[ \hat{f}_i(t_{j+1}, \mathbf{r}) = \hat{f}_i(t_j, \mathbf{r}) + \Delta t I_i(\hat{\mathbf{f}}(t_{j+1}, \mathbf{r})). \]  

(5)

This scheme discretise (4) only with first order on time step. But the accuracy order of splitting method is equal to unity, so the application of schemes of higher order for (4) does not make sense.

Nonlinear system (5) may be solved by following iterative method:

\[ \hat{f}_i^{(s+1)}(t_{j+1}, \mathbf{r}) = \hat{f}_i(t_j, \mathbf{r}) + \Delta t I_i(\hat{\mathbf{f}}^{(s)}(t_{j+1}, \mathbf{r})), \]  

(6)

where \( s = 0, 1, 2, \ldots \) is the iteration number.

One of the main difficulties (except the convergence) of the realization of (6) is a choice of initial approximation \( \hat{f}_i^{(0)}(t_{j+1}, \mathbf{r}) \) at every \( j \). To avoid this problem, the following modification of the splitting method is proposed — at every time step instead of the solution of the Cauchy problem for (4) the following sequence of the problems is solved:

\[ \frac{\partial \hat{f}_i^{(q)}}{\partial t} = I_i(\hat{\mathbf{f}}^{(q)}), \quad t \in (t_j, t_{j+1}], \quad q = 1, \ldots, \quad \hat{f}_i^{(q+1)}(t_j, \mathbf{r}) = \hat{f}_i^{(q)}(t_{j+1}, \mathbf{r}), \quad q = 0, \ldots, \]  

(7)
where \( \tilde{f}_i^{(0)}(t_{j+1}, \mathbf{r}) = \tilde{f}_i(t_{j+1}, \mathbf{r}) \). The solution of these problems is realized by implicit Euler method (5)–(6), but with only one iteration step with initial approximation \( \tilde{f}_i^{(q,0)}(t_{j+1}, \mathbf{r}) = \tilde{f}_i^{(q+1)}(t_j, \mathbf{r}) = \tilde{f}_i^{(q)}(t_{j+1}, \mathbf{r}) \):

\[
\tilde{f}_i^{(q+1)}(t_{j+1}, \mathbf{r}) = \tilde{f}_i^{(q,0)}(t_{j+1}, \mathbf{r}) + \Delta t I_i(\tilde{f}_i^{(q,0)}(t_{j+1}, \mathbf{r})) = \tilde{f}_i^{(q)}(t_{j+1}, \mathbf{r}) + \Delta t I_i(\tilde{f}_i^{(q)}(t_{j+1}, \mathbf{r})).
\] (8)

The cycle on \( q \) for (7)–(8) may be stopped, when the values of the density \( \rho \) at moment \( t_{j+1} \) stop changed. These values are computed by following formulae [12]:

\[
\rho^{(q)}(t_{j+1}, \mathbf{r}) = \int f(t_{j+1}, \mathbf{r}, \mathbf{v}) d\mathbf{v} \approx \sum_{i=1}^{n} f_i^{(q)}(t_{j+1}, \mathbf{r}).
\]

So the convergence condition of the procedure (7)–(9) can be formulated by following inequality:

\[
\frac{||\rho^{(q+1)}(t_{j+1}, \mathbf{r}) - \rho^{(q)}(t_{j+1}, \mathbf{r})||}{||\rho^{(q)}(t_{j+1}, \mathbf{r})||} \leq Tol,
\]

where value of \( Tol \) is set in advance. It must be noted, that other approaches for the completion of the iteration procedure may be considered.

3. Finite-difference schemes for the linear advection equation

As it can be seen, equations of the system (3) are independent of each other. So difference schemes for scalar linear advection equation can be considered. This equation in the one-dimensional case is written as:

\[
\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,
\] (9)

where \( u = u(t, x) \), \( c = \text{const} \) is a velocity of the advection along the characteristic. In all further formulas the case of \( c > 0 \) is considered.

The following approximation of time derivative is proposed:

\[
\frac{\partial u}{\partial t}(t_j, x_n) \approx \frac{u_{j+1}^n - \frac{1}{2} \left(u_n^j + u_{n}^{j-2}\right)}{2\Delta t},
\] (10)

where \( u_{0}^j \approx u(t_j, x_n) \), \( t_j \) is a node of time grid constructed with step \( \Delta t \), \( x_n \) is a node of grid on space variable constructed with step \( h \). Scheme with (10) approximate eq. (9) only with the first order on time step. As it is demonstrated in [13, 14] stability properties of schemes for nonlinear BGK equations with approximation (10) are better, than for the schemes with second order central approximation.

Let us consider FD schemes for (9) obtained after approximation on space variable \( x \) in following form:

\[
u_{j+1}^n = \frac{1}{2} \left(u_n^j + u_n^{j-2}\right) - 2\Delta t \mathcal{L}_h(u_{0}^j),
\] (11)

where \( \mathcal{L}_h \) is a discrete operator, which approximate term \( \partial u/\partial x \) at node \( (t_j, x_n) \). The following representations of this term are considered:

1) First order backward difference:

\[
\mathcal{L}_h(u_{0}^j) = \frac{c}{h} \left(u_n^j - u_n^{j-1}\right).
\] (12.1)
2) Second order differences:

\[ \mathcal{L}_h(u^j_n) = \frac{c}{2h} \left( u^j_{n+1} - u^j_{n-1} \right), \quad (12.2) \]

\[ \mathcal{L}_h(u^j_n) = \frac{c}{2h} \left( 3u^j_n - 4u^j_{n-1} + u^j_{n-2} \right). \quad (12.3) \]

3) Third order difference:

\[ \mathcal{L}_h(u^j_n) = \frac{c}{6h} \left( u^j_{n-2} - 6u^j_{n-1} + 3u^j_n + 2u^j_{n+1} \right). \quad (12.4) \]

4) Fourth order differences:

\[ \mathcal{L}_h(u^j_n) = \frac{c}{12h} \left( u^j_{n-2} - 8u^j_{n-1} + 8u^j_{n+1} - u^j_{n+2} \right), \quad (12.5) \]

\[ \mathcal{L}_h(u^j_n) = \frac{c}{12h} \left( 3u^j_{n+1} + 10u^j_n - 18u^j_{n-1} + 6u^j_{n-2} - u^j_{n-3} \right). \quad (12.6) \]

According to the approach proposed in [15], let us approximate term \( c \partial u / \partial x \) in following form:

\[ c \partial u / \partial x (t_j, x_n) \approx \mathcal{L}_h(u^j_n) = \varepsilon F_1 + (1 - \varepsilon) F_2, \]

where \( F_1 \) and \( F_2 \) are the FD approximations of this term and \( \varepsilon \in [0, 1] \) is a dimensionless parameter. So schemes with parametric coefficients represented by following formulae are constructed:

\[ u^{j+1}_n = \frac{1}{2} \left( u^j_n + u^{j-2}_n \right) - 2\Delta t \left( (\varepsilon F_1 + (1 - \varepsilon) F_2) \right), \quad (13) \]

where \( F_1 \) and \( F_2 \) are presented by formulas (12.1)–(12.6). The existence of the parameter \( \varepsilon \) in the right part of (13) provide possibilities to regulate different properties of the scheme, such as stability, numerical dispersion and numerical dissipation [16].

4. Stability analysis

Stability analysis is realized by application of von Neumann method, which is proposing to obtain necessary stability conditions of multistep schemes represented by (11) and (13). The method is based on the following representation of the solution:

\[ u^j_n = \lambda^j(\varphi) \exp(\imath n\varphi), \quad (14) \]

where \( \imath^2 = -1, \varphi \in [0, 2\pi] \) and \( \lambda \) is a spectral function. Stability condition is represented by following inequality:

\[ |\lambda(\varphi)| \leq 1, \quad \forall \varphi \in [0, 2\pi]. \]

After the substitution of (14) into (11) or (13), the following third-order equation on \( \lambda \) is obtained:

\[ \lambda^3 + a\lambda - \frac{1}{2} = 0, \quad (15) \]

where \( \gamma = c\Delta t / h \) is a Courant number, \( a = a(\gamma, \varphi) \) for schemes from (11) and \( a = a(\gamma, \varphi, \varepsilon) \) for schemes represented by (13). Roots of eq. (15) are obtained by Cardano’s formulas.
4.1. Stability conditions
The following stability conditions are obtained for the schemes represented by eq. (11): $\gamma \leq 0.25$ for the case of approximation (12.1), $\gamma \leq 0.5$ for (12.2), $\gamma \leq 0.12$ for (12.3), $\gamma \leq 0.25$ for (12.4), $\gamma \leq 0.36$ for (12.5) and $\gamma \leq 0.15$ for (12.6).

Stability criteria for the schemes presented by (13) are obtained in following form: $\gamma \leq \tilde{\gamma}(\varepsilon)$, where $\tilde{\gamma}(\varepsilon)$ represent the function, which approximate the boundary of the stability domain in the space of parameters $\varepsilon$ and $\gamma$. The expressions for $\tilde{\gamma}(\varepsilon)$ are obtained as a polynomials of low degree by the application of Matlab’s procedure polyfit() and are written as:

$$\tilde{\gamma}(\varepsilon) = -0.11\varepsilon^3 + 0.32\varepsilon^2 - 0.46\varepsilon + 0.49,$$

for the case of the scheme, where $F_1$ is approximated by (12.1), $F_2$ — by (12.2),

$$\tilde{\gamma}(\varepsilon) = -0.47\varepsilon^3 + 1.15\varepsilon^2 - 1.05\varepsilon + 0.49,$$

for the case, where $F_1$ is approximated by (12.3) and $F_2$ by (12.2),

$$\tilde{\gamma}(\varepsilon) = -0.11\varepsilon + 0.36,$$

for the case, where $F_1$ is approximated by (12.4) and $F_2$ by (12.5),

$$\tilde{\gamma}(\varepsilon) = 0.1\varepsilon^2 - 0.31\varepsilon + 0.35,$$

for the case, where $F_1$ is approximated by (12.5) and $F_2$ by (12.6).

4.2. Comparison with other schemes
Despite the fact that an accuracy order of the schemes (11) and (13) on time step is equal to unity, these schemes provide more opportunities to practical computations. It corresponds to their stability for the cases of high order asymmetric approximations on space variable. Let us demonstrate this fact in comparison with well known schemes.

For the explicit schemes of the first order on time step with following representation of time derivative:

$$\frac{\partial u}{\partial t}(t_j, x_n) \approx \frac{u^{j+1}_n - u^j_n}{\Delta t},$$

only the scheme with approximation (12.1) is stable with the stability condition $\gamma \leq 1$. Schemes with higher orders of approximation on spatial steps are unstable [17], while the proposed schemes are conditionally stable.

In the cases of the schemes with the second order of approximation on time step, when:

$$\frac{\partial u}{\partial t}(t_j, x_n) \approx \frac{u^{j+1}_n - u^{j-1}_n}{2\Delta t},$$

only the schemes with symmetric approximations (12.2) and (12.5) (with stability conditions $\gamma \leq 1$ and $\gamma \leq 0.75$ respectively) are stable. Schemes with other approximations are unstable. So, the presented schemes (11) and (13) with approximations (12.1)–(12.6) provided opportunities to realize calculations with high orders on spatial steps in case of asymmetric spatial approximations. It may be useful in the realization of splitting method for kinetic equations.
5. Conclusion
Explicit finite-difference schemes for the linear advection equations are considered. The schemes may be used in the advective step of the splitting method for the kinetic equations. The time derivative is approximated with first order by special asymmetric approximation. Schemes from first to fourth orders of approximation on space are constructed. Stability analysis is realized by von Neumann method. Stability criteria in the form of inequalities on Courant parameter are obtained. As it is demonstrated, obtained schemes demonstrate better stability properties in cases of high order asymmetric approximations in comparison with well-known explicit schemes.

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