On the application of ergodic condition to averaging principle for multiscale stochastic systems*

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Abstract

This work concerns the asymptotic behavior for fully coupled multiscale stochastic systems. We focus on studying the impact of the ergodicity of the fast process on the limit process and the averaging principle. The key point is to investigate the continuity of the invariant probability measures relative to parameters in various distances over the Wasserstein space. An illustrative example is constructed to show the complexity of the fully coupled multiscale system compared with the uncoupled multiscale system, which shows that the averaged coefficients may become discontinuous even they are originally Lipschitz continuous and the fast process is exponentially ergodic.

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1 Introduction

In this work we are concerned with the following two time-scale stochastic systems:

\[
\begin{align*}
\text{d}X_t^\varepsilon &= b(X_t^\varepsilon, Y_t^\varepsilon)\text{d}t + \sigma(X_t^\varepsilon, Y_t^\varepsilon)\text{d}W_t, \\
\text{d}Y_t^\varepsilon &= \frac{1}{\varepsilon}f(X_t^\varepsilon, Y_t^\varepsilon)\text{d}t + \frac{1}{\sqrt{\varepsilon}}g(X_t^\varepsilon, Y_t^\varepsilon)\text{d}B_t,
\end{align*}
\]

where \((W_t)\) and \((B_t)\) are \(d\)-dimensional mutually independent Wiener processes, \(b(x, y) \in \mathbb{R}^d\) and \(f(x, y) \in \mathbb{R}^{d \times d}\) are drifts, \(\sigma(x, y) \in \mathbb{R}^{d \times d}\) and \(g(x, y) \in \mathbb{R}^{d \times d}\) are diffusion coefficients. The parameter \(\varepsilon > 0\) represents the ratio between the time scale of processes \((X_t^\varepsilon)\) and \((Y_t^\varepsilon)\). We are interested in the case \(\varepsilon \ll 1\), in which \((X_t^\varepsilon)\) is called the slow component, and \((Y_t^\varepsilon)\) is called the fast component. Our purpose is to study the ergodicity condition of the fast component so that the limit system is wellposed and the averaging principle holds. In particular, we investigate the continuous dependence of the invariant probability measure \(\pi_x\) of the fast component \((Y_t^\varepsilon)\) on the fixed state \(x\) of the slow component \((X_t^\varepsilon)\).

In applications many real systems can be viewed as a combination of slow and fast motions such as in modeling climate-weather interaction \([6, 19]\), in biology \([16]\), in mathematical finance \([7, 8]\) and references therein. The averaging principle has been established for various multiscale systems which says that the slow component \((X_t^\varepsilon)\) will converge to some limit process \((\bar{X}_t)\) as \(\varepsilon \to 0\) in suitable sense. Also, there are many works devoted to the study of central limit theorems and large deviations of multiscale stochastic models. For a two-time scale system where both slow and fast components are continuous processes given as solutions of SDEs, these problems have been extensively studied in e.g. \([17, 18, 22, 21, 27, 31, 32]\), in \([13]\) for SDEs driven by fractional Brownian motions. For a two-time scale system where the slow component is a continuous process but the fast process is a jump process over a discrete space, these problems have been studied in e.g. \([3, 9, 24]\) and references therein.

When the fast component does not depend on the slow component, we arrive at the classical uncoupled setup, and the averaging principle usually holds in quite general conditions. However, when the fast component depends on the slow component, the situation becomes more complicated. Anosov \([1]\) established the first relatively general result on fully coupled averaging principle. See the monograph of Kifer \([20]\) for a detailed discussion on averaging of fully coupled dynamical systems. The averaging principle for fully coupled stochastic processes have been investigated in e.g. Freidlin and Wentzell \([10]\) and Veretennikov \([30]\), the corresponding large deviation principle has been studied by
Veretennikov [31, 32] and recently by Puhalskii [27]. This work focuses on the averaging principle for fully coupled two time-scale stochastic processes.

To be precise, for each fixed slow component \( x \), let \( \pi^x \) denote the invariant probability measure of the fast component. As usual, put \( \bar{b}(x) = \int_{\mathbb{R}^d} b(x, y)\pi^x(dy) \) and \( \bar{\sigma}(x) = \int_{\mathbb{R}^d} \sigma(x, y)\pi^x(dy) \), and consider the SDE

\[
\text{d}\bar{X}_t = \bar{b}(\bar{X}_t)\text{d}t + \bar{\sigma}(\bar{X}_t)\text{d}W_t, \quad \bar{X}_0 = x_0. \tag{1.2}
\]

The averaging principle suggests that often \((X^\varepsilon_t)\) converges to a process \((\bar{X}_t)\). To show this, a premise is the wellposedness of the SDE (1.2). In the uncoupled setup, Lipschitz continuity of \( b \) and \( \sigma \) implies already that \( \bar{b} \) and \( \bar{\sigma} \) are also Lipschitz continuous in \( x \), and so there exists a unique solution \((\bar{X}_t)\) to SDE (1.2). However, in the fully coupled case even when \( b \) and \( \sigma \) are Lipschitz continuous, we construct an example (see Example 2.1 in Section 2) to show that the averaged coefficients \( \bar{b}(x) \) may even not be continuous in \( x \), let alone Lipschitz. So, SDE (1.2) may not have solution at all.

The wellposedness of SDE (1.2) depends not only on the regularity of coefficients \( b \) and \( \sigma \) but also on the ergodicity of the fast component at each fixed slow component \( x \) via the invariant probability measure \( \pi^x \). So, to establish the averaging principle we first study the continuity of \( \pi^x \) in \( x \) under the two kinds of strongly ergodic conditions for the fast component w.r.t. total variation distance, bounded Lipschitz distance and \( L^1 \)-Wasserstein distance respectively. As the space of probability measures is an infinite dimensional space, the previous mentioned distances are not equivalent, and so the continuity of \( \pi^x \) w.r.t. different distances can be applied to deal with the regularity of \( \bar{b}, \bar{\sigma} \) for different kinds of \( b \) and \( \sigma \). Furthermore, under the wellposedness of the limit process \((\bar{X}_t)\) the averaging principle is established for fully coupled two time-scale stochastic processes under certain conditions.

In the study of fully coupled stochastic systems, there are limited works on the ergodicity of the fast component to ensure the averaging principle holds, especially when the fast component locates in a noncompact space. Usually, the focus is on how to weaken the conditions on the slow component. Among these limited study, the results in Veretennikov [30] are representative, where the key condition on the fast component is the following ergodicity condition.

- There exist functions \( \bar{b}, \bar{\sigma} \) and \( K(T) \) such that \( \lim_{T \to \infty} K(T) = 0 \), \( \bar{\sigma}(x) \) is nonde-
generate, continuous in $x$, and for all $t \geq 0, T > 0, x, y \in \mathbb{R}^d$,
\[
\left| \frac{1}{T} \mathbb{E} \int_t^{t+T} b(x_s, Y^{x,y}_s) ds - \bar{b}(x) \right| \leq K(T)(1 + |x|^2 + |y|^2),
\]
\[
\left| \frac{1}{T} \mathbb{E} \int_t^{t+T} \sigma(x_s, Y^{x,y}_s) ds - \bar{\sigma}(x) \right| \leq K(T)(1 + |x|^2 + |y|^2),
\]
where $(Y^{x,y}_t)$ is the solution to
\[
dY^{x,y}_t = f(x, Y^{x,y}_t)dt + g(x, Y^{x,y}_t)dB_t, \quad Y^{x,y}_0 = y. \tag{1.3}
\]
In application, the verification of the previous ergodicity and continuity condition on $\bar{\sigma}$ is not easy. In particular, as explained by Example 2.1 below, $\bar{\sigma}(x)$ may be discontinuous even $\sigma$ is Lipschitz continuous and $(Y^{x,y}_t)$ is exponentially ergodic. So, a widely used condition is the monotonicity condition on the fast component (cf. e.g. [12, 22]), that is, there exists a constant $\beta > 0$ such that
\[
2(y_1 - y_2) \cdot (f(x, y_1) - f(x, y_2)) + \|g(x, y_1) - g(x, y_2)\|^2 \leq -\beta |y_1 - y_2|^2, \quad x, y_1, y_2 \in \mathbb{R}^d. \tag{1.4}
\]
Besides the monotonicity condition to characterize the ergodicity of the fast component, certain strict coercivity condition is also needed to establish the averaging principle (cf. [22, Remark 2.1]).

Based on our constructed example, we shall impose strong ergodicity condition on the fast component using respectively the total variation distance and the $L^1$-Wasserstein distance to measure the convergence of the semigroup to its invariant probability measure. The strong ergodicity of diffusion processes is well studied topic, and there are many criteria in the existing literature. The monotonicity condition is a sufficient condition to guarantee the strong ergodicity in the $L^1$-Wasserstein distance. Accordingly, the continuity of invariant probability measure $\pi^x$ in $x$ is proved, which helps us to show the continuity of $\bar{b}$ and $\bar{\sigma}$ and further the wellposedness of the SDE for limit process $(\bar{X}_t)$. At last, we can use the time discretization method and the coupling method to establish the averaging principle.

This work is organized as follows. In Section 2, we first construct an example in Subsection 2.1 to show the essential impact of $\pi^x$ on the regularity of $\bar{b}(x), \bar{\sigma}(x)$. Then, we investigate the continuity of $\pi^x$ in $x$ w.r.t. different distances. In Section 3, the averaging principle for the fully coupled system (1.1) is established under different conditions.
2 Continuous dependence on parameters of invariant probability measures

2.1 An illustrative example

In this part we aim to present the complexity of fully coupled stochastic systems via an explicit example.

Example 2.1 Let \((X_\varepsilon^x)\) and \((Y_\varepsilon^y)\) be stochastic processes respectively on \([0, 1]\) and on \([0, \infty)\) with reflection boundary satisfying

\[
\begin{align*}
\frac{dX_\varepsilon^x}{dt} &= Y_\varepsilon^x dt + Y_\varepsilon^x dW_t, \quad X_0^x = x_0 \in (0, 1), \\
\frac{dY_\varepsilon^y}{dt} &= \frac{1}{\varepsilon} \tilde{f}(X_\varepsilon^x, Y_\varepsilon^y) dt + \frac{1}{\sqrt{\varepsilon}} dB_t, \quad Y_0^y = y_0 \in (0, \infty),
\end{align*}
\]

(2.1)

where

\[
\tilde{f}(x, y) = -\frac{x^3 e^{-xy} - (1 - x)e^{-y}}{x^2 e^{-xy} + (1 - x)e^{-y}}, \quad x \in [0, 1], \ y \in [0, \infty).
\]

As \(\tilde{f}(x, y)\) is continuous on \([0, 1] \times [0, \infty)\), the solution to (2.1) exists and is unique in distribution according to [29]. In this example, \(b(x, y) = \sigma(x, y) = y\) and \(g(x, y) = 1\) are both Lipschitz continuous.

For each \(x \in [0, 1]\), the invariant probability measure \(\pi^x\) associated with the SDE

\[
\frac{dY_{x,y}^x}{dt} = \tilde{f}(x, Y_{x,y}^x) dt + dB_t, \quad Y_0^{x,y} = y \in (0, \infty),
\]

(2.2)

is given by

\[
\pi^x(dy) = \left(x^2 e^{-xy} + (1 - x)e^{-y}\right)dy.
\]

Then

\[
\bar{b}(x) := \int_0^\infty b(x, y)\pi^x(dy) = \int_0^\infty y\left(x^2 e^{-xy} + (1 - x)e^{-y}\right)dy
\]

\[
= \begin{cases} 
2 - x, & \text{if } x \in (0, 1], \\
1, & \text{if } x = 0.
\end{cases}
\]

It is clear that \(\bar{b}(x)\) is not continuous at \(x = 0\). Hence, this example is our desired example to show the complexity of the limit behavior of the fully coupled system \((X_\varepsilon^x, Y_\varepsilon^y)\) as \(\varepsilon \to 0\).
Next, we investigate the ergodic property of the process \((Y_t^{x,y})\) given by (2.2). To this end, introduce the notation

\[
C_x(y) = \ln \left( x^2 e^{-xy} + (1-x)e^{-y} \right), \quad \mu^x(dy) = e^{C_x(y)}dy.
\]

So, \(\mu^x([0, \infty)) = 1\) and \(\pi^x(dy) = \mu^x(dy)/\mu^x([0, \infty)) = \mu^x(dy)\).

1) The process \((Y_t^{x,y})\) is unique and ergodic. To this end, one needs to check

\[
\int_0^\infty \mu^x([y, \infty)) e^{-C_x(y)}dy = \infty \quad \text{and} \quad \mu^x([0, \infty)) < \infty.
\]

Indeed, \(\mu^x([0, \infty)) = 1\),

\[
\int_0^\infty \left( \int_0^y \left( x^2 e^{-xz} + (1-x)e^{-z} \right)dz \right) \frac{dy}{x^2 e^{-xy} + (1-x)e^{-y}} \geq \int_1^\infty \frac{x(1-e^{-x}) + (1-x)(1-e^{-1})}{x^2 e^{-x} + (1-x)e^{-1}}dy = \infty.
\]

This implies that for each \(x \in [0, 1]\) the process \((Y_t^{x,y})\) is ergodic on \([0, \infty)\) by the criterion of ergodicity for one-dimensional diffusion processes; see, for instance, [5, Chapter 5].

2) According to [5, Chapter 5], to study the exponential ergodicity of \((Y_t^{x,y})\), we need to study

\[
\mu^x([z, \infty)) \int_z^\infty e^{-C_x(y)}dy = \left( x e^{-xz} + (1-x)e^{-z} \right) \int_0^z \frac{dy}{x^2 e^{-xy} + (1-x)e^{-y}}.
\]

Using L'Hôpital's rule,

\[
\lim_{z \to \infty} \left( x e^{-xz} + (1-x)e^{-z} \right) \int_0^z \frac{dy}{x^2 e^{-xy} + (1-x)e^{-y}} = \lim_{z \to \infty} \left( \frac{x e^{(1-x)z} + 1-x}{x^2 e^{(1-x)z} + 1-x} \right)^2 < \infty.
\]

Hence, we have \(\sup_{z>0} \left\{ \mu^x([z, \infty)) \int_0^z e^{-C_x(y)}dy \right\} < \infty\), and then the process \((Y_t^{x,y})\) is exponentially ergodic.

3) According to [23, Theorem 2.1], \((Y_t^{x,y})\) is strongly ergodic (also called uniformly ergodic) if and only if

\[
\int_0^\infty \mu^x([y, \infty)) e^{-C_x(y)}dy < \infty.
\]
However, for this example,
\[
\int_0^\infty \mu^x([y, \infty)) e^{-C_x(y)} dy = \int_0^\infty \frac{xe^{-xy} + (1-x)e^{-y}}{x^2e^{-xy} + (1-x)e^{-y}} dy = \infty,
\]
so \((Y_t^{x,y})\) is not strongly ergodic.

2.2 Continuity of \(\pi^x\) in the Wasserstein space

The space of probability measures over \(\mathbb{R}^d\), denoted by \(\mathcal{P}(\mathbb{R}^d)\), is an infinite dimensional space, on which various distances have been defined. These distances are not mutually equivalent. To deal with different coefficients, one needs to consider the continuity of \(\pi^x\) in \(x\) in different distances. We shall consider three kinds of distances on \(\mathcal{P}(\mathbb{R}^d)\) including the total variation distance \(\| \cdot \|_{\text{var}}\), \(L_1\)-Wasserstein distance \(W_1\), bounded Lipschitz distance \(W_{bL}\) (also called Fortet-Mourier distance). These distances are defined as follows: for two probability measures \(\mu, \nu\) on \(\mathbb{R}^d\),

\[
\|\mu - \nu\|_{\text{var}} := 2 \sup_{A \in \mathcal{B}(\mathbb{R}^d)} |\mu(A) - \nu(A)| = \sup_{|h| \leq 1} |\mu(h) - \nu(h)|,
\]

\[
W_p(\mu, \nu) := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \Gamma(dx, dy); \Gamma \in \mathcal{C}(\mu, \nu) \right\}^{1/p}, \quad p \geq 1,
\]

\[
W_{bL}(\mu, \nu) := \sup \left\{ \mu(h) - \nu(h); |h(x)| \leq 1, |h(x) - h(y)| \leq |x - y|, x, y \in \mathbb{R}^d \right\}.
\]

By Kantorovich’s dual representation theorem for the Wasserstein distance,

\[
W_1(\mu, \nu) = \sup \left\{ \mu(h) - \nu(h); |h(x) - h(y)| \leq |x - y|, x, y \in \mathbb{R}^d \right\}.
\]

We refer the readers to [33, Chapter 6] for more discussion on the relationship of various distances on \(\mathcal{P}(\mathbb{R}^d)\).

In this work we shall use two kinds of ergodicity condition on \((Y_t^{x,y})\) with respect to the total variation distance and the \(L_1\)-Wasserstein distance respectively. Throughout this section, \(P_t^x(y, \cdot)\) denotes the semigroup associated with the Markov process \((Y_t^{x,y})\) given by (1.3), i.e. \(P_t^x h(y) := \mathbb{E}[h(Y_t^{x,y})]\) for \(h \in \mathcal{B}_b(\mathbb{R}^d)\), \(\pi^x\) denotes its invariant probability measure and we always assume its existence.

(E1) There exist constants \(\kappa_1, \lambda_1 > 0\) such that

\[
\sup_{y \in \mathbb{R}^d} \|P_t^x(y, \cdot) - \pi^x\|_{\text{var}} \leq \kappa_1 e^{-\lambda_1 t}, \quad t > 0, x \in \mathbb{R}^d.
\]
(E2) There exist constants $\kappa_2, \lambda_2 > 0$ such that
\[
\sup_{y \in \mathbb{R}^d} \mathcal{W}_1(P_t^x(y, \cdot), \pi^x) \leq \kappa_2 e^{-\lambda_2 t}, \quad t > 0, x \in \mathbb{R}^d.
\]

Ergodicity for diffusion processes is an extensively studied topic (cf. [4, 25]). We shall present some criteria on the coefficients $f, g$ to verify (E1) and (E2) in Section 3; see, Lemma 3.1 and Lemma 3.2.

Now we collect the conditions on the coefficients of $(X_t^\varepsilon, Y_t^\varepsilon)$ used in this work.

**Assumptions on slow component:**

(A1) There exists a constant $K_1 > 0$ such that for any $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$,
\[
|b(x_1, y_1) - b(x_2, y_2)|^2 + \|\sigma(x_1, y_1) - \sigma(x_2, y_2)\|^2 \leq K_1 (|x_1 - x_2|^2 + |y_1 - y_2|^2).
\]

(A2) There exists a constant $K_2 > 0$ such that
\[
|b(x, y)| + \|\sigma(x, y)\| \leq K_2 (1 + |x|), \quad x, y \in \mathbb{R}^d.
\]

(A3) Assume that
\[
\inf_{x, y \in \mathbb{R}^d} \inf_{\xi \in \mathbb{R}^d, |\xi| = 1} \xi^* (\sigma \sigma^*)(x, y) \xi > 0.
\]

**Assumptions on fast component:**

(B1) There exists $K_3 > 0$ such that
\[
(f(x_1, y) - f(x_2, y)) \cdot z \leq K_3 |x_1 - x_2||z|, \quad x_1, x_2, y, z \in \mathbb{R}^d,
\]
\[
(f(x_1, y_1) - f(x_2, y_2)) \cdot (y_1 - y_2) + \|g(x_1, y_1) - g(x_2, y_2)\|^2 \leq K_3 (|x_1 - x_2|^2 + |y_1 - y_2|^2)
\]
for $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$.

(B2) There exists a constant $K_4 > 0$ such that $|f(x, y)| + |g(x, y)| \leq K_4$, $x, y \in \mathbb{R}^d$.

(B3) There exists a constant $\lambda_3 > 0$ such that and
\[
\eta^* (gg^*) (x, y) \eta \geq \lambda_3 |\eta|^2, \quad x, y, \eta \in \mathbb{R}^d.
\]
Consider the following parabolic equation:

$$\partial_t u(t, y) = \mathcal{L}^x u(t, y), \quad t > 0, \ y \in \mathbb{R}^d,$$

(2.3)

with initial condition \( u(0, y) = h(y), \ y \in \mathbb{R}^d \) and \( h \in C_b(\mathbb{R}^d) \), where \( \mathcal{L}^x \) is the infinitesimal generator of the process \( (Y_{t,x}^y) \) given by

$$\mathcal{L}^x v(y) = \sum_{k=1}^d f_k(x, y) \frac{\partial v(y)}{\partial y_k} + \frac{1}{2} \sum_{k,l=1}^d G_{kl}(x, y) \frac{\partial^2 v(y)}{\partial y_k \partial y_l}, \quad v \in C^2(\mathbb{R}^d),$$

(2.4)

and \( (G_{kl}(x, y)) = (gg^*)(x, y) \). Let \( \Theta^x(t, z; s, y) \), \( 0 \leq s < t \), denotes the fundamental solution to PDE (2.3); see, e.g. [11] for the definition of fundamental solutions to parabolic equations. We shall use the Gaussian type estimates on the derivatives of \( \Theta^x(t, z; s, y) \): there exist constants \( c_1, c_2 > 0 \) such that for each \( x \in \mathbb{R}^d \)

$$\frac{1}{c_1(t-s)^{d/2}} e^{-\frac{|y-z|^2}{2c_2(t-s)}} \leq \Theta^x(t, z; s, y) \leq \frac{c_1}{(t-s)^{d/2}} e^{-\frac{c_2|y-z|^2}{t-s}},$$

(2.5)

$$|\nabla_z \Theta^x(t, z; s, y)| \leq \frac{c_1}{(t-s)^{(d+1)/2}} e^{-\frac{c_2|y-z|^2}{t-s}}$$

(2.6)

for \( 0 \leq s \leq t, y, z \in \mathbb{R}^d \). According to [11, Chapter 9, Theorem 2], under conditions (B1)-(B3), the estimates (2.5) and (2.6) hold. Moreover, there are many works to study the Gaussian estimates for the fundamental solutions when the coefficients are singular such as in some Kato’s class. See, e.g. [35, 36] and references therein.

**Proposition 2.1** Assume (E1), (B1) and the estimate (2.6) hold. In addition, assume \( g(x, y) \) depends only on \( y \), i.e. \( g(x, y) = g(y) \). Then there exists a constant \( C > 0 \) such that

$$\|\pi^{x_1} - \pi^{x_2}\|_{\text{var}} \leq C|x_1 - x_2|^{2/3}, \quad x_1, x_2 \in \mathbb{R}^d.$$ 

(2.7)
Proof. For any $h \in C(\mathbb{R}^d)$ with $|h|_\infty := \sup_{x \in \mathbb{R}^d} |h(x)| \leq 1$, due to (E1),

$$|\pi^{x_1}(h) - \pi^{x_2}(h)| \leq |\pi^{x_1}(h) - \frac{1}{t} \int_0^t P_{s}^{x_1} h(y_0) ds| + |\pi^{x_2}(h) - \frac{1}{t} \int_0^t P_{s}^{x_2} h(y_0) ds|$$

$$+ \left| \frac{1}{t} \int_0^t |P_{s}^{x_1} h(y_0) - P_{s}^{x_2} h(y_0)| ds \right|$$

$$\leq \frac{1}{t} \int_0^t \left( \|P_{s}^{x_1}(y_0, \cdot) - \pi^{x_1}\|_{\text{var}} + \|P_{s}^{x_2}(y_0, \cdot) - \pi^{x_2}\|_{\text{var}} \right) ds$$

$$+ \frac{1}{t} \int_0^t |P_{s}^{x_1} h(y_0) - P_{s}^{x_2} h(y_0)| ds$$

$$\leq \frac{2K_1}{t} \int_0^t e^{-\lambda s} ds + \frac{1}{t} \int_0^t |P_{s}^{x_1} h(y_0) - P_{s}^{x_2} h(y_0)| ds. \tag{2.8}$$

Consider the function $\phi(r) := P_{r}^{x_1}(P_{s-r}^{x_2} h)(y_0)$ for $r \in [0, s]$. It is easy to see

$$P_{s}^{x_1} h(y_0) - P_{s}^{x_2} h(y_0) = \int_0^s \phi'(r) dr = \int_0^s P_{r}^{x_1}(\mathcal{L}^{x_1} - \mathcal{L}^{x_2}) P_{s-r}^{x_2} h(y_0) dr. \tag{2.9}$$

Notice that $u(s, y) := P_{s}^{x_2} h(y)$ is the unique solution to (1.4) with $x = x_2$, which can be represented by the fundamental solution $\Theta^{x_2}(s, y; t, z)$ via

$$u(s, y) = \int_{\mathbb{R}^d} \Theta^{x_2}(s, y; 0, z) h(z) dz$$

by virtue of [11, Chapter 1, Theorem 12]. Then, by (B1) and (2.6), it follows from (2.9) that for $s \in [0, t]$

$$|P_{s}^{x_1} h(y_0) - P_{s}^{x_2} h(y_0)| \leq \int_0^s \sup_{y \in \mathbb{R}^d} |(f(x_1, y) - f(x_2, y)) \cdot \nabla P_{s-r}^{x_2} h(y)| dr$$

$$\leq K_3 |x_1 - x_2| \int_0^s \left( \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} r^{-\frac{d+1}{2}} e^{-c_1 |x-y|^2 \frac{r}{s}} h(z) dz \right) dr$$

$$\leq K_3 |x_1 - x_2| \int_0^s r^{-\frac{1}{2}} dr \int_{\mathbb{R}^d} e^{-c_1 |z|^2} dz$$

$$\leq C \sqrt{s} |x_1 - x_2|.$$

Inserting this estimate into (2.8), we get

$$|\pi^{x_1}(h) - \pi^{x_2}(h)| \leq \frac{2K_1}{t} (1 - e^{-\lambda t}) + C \sqrt{t} |x_1 - x_2|.\tag{2.10}$$
Furthermore, by taking $t = |x_1 - x_2|^{-2/3}$, it holds for some $C > 0$ that
\[ |\pi^{x_1}(h) - \pi^{x_2}(h)| \leq C|x_1 - x_2|^{2/3}. \tag{2.10} \]
By the arbitrariness of $h \in C(\mathbb{R}^d)$ with $|h|_\infty \leq 1$, we get that
\[ \|\pi^{x_1} - \pi^{x_2}\|_V := \sup\{|\pi^{x_1}(h) - \pi^{x_2}(h)|; h \in C(\mathbb{R}^d), |h|_\infty \leq 1\} \leq C|x_1 - x_2|^{2/3}. \]

To show the desired conclusion (2.7), we only need to show
\[ \|\pi^{x_1} - \pi^{x_2}\| = \|\pi^{x_1} - \pi^{x_2}\|_{\text{var}}. \tag{2.11} \]
Indeed, by Lusin’s theorem (cf. [28, Theorem 2.23]), for every $h \in \mathcal{B}(\mathbb{R}^d)$ with $|h|_\infty \leq 1$, for any $\varepsilon > 0$, there exists a function $h_\varepsilon \in C_c(\mathbb{R}^d)$ such that
\[ (\pi^{x_1} + \pi^{x_2})(\{h \neq h_\varepsilon\}) < \varepsilon, \text{ and } |h_\varepsilon|_\infty \leq |h|_\infty. \]
Hence,
\[ |\pi^{x_1}(h) - \pi^{x_2}(h)| \leq |\pi^{x_1}(h_\varepsilon) - \pi^{x_2}(h_\varepsilon)| + |\pi^{x_1}(h - h_\varepsilon)| + |\pi^{x_2}(h - h_\varepsilon)| \leq |\pi^{x_1}(h_\varepsilon) - \pi^{x_2}(h_\varepsilon)| + 2\varepsilon \leq \|\pi^{x_1} - \pi^{x_2}\|_V + 2\varepsilon. \]
From this, it is easy to see (2.11) holds. Thus, the proof is completed. \hfill \square

**Corollary 2.2** Let the conditions of Proposition 2.1 be valid. In addition, suppose (A1) and (A2) hold. Let
\[ \bar{b}(x) = \int_{\mathbb{R}^d} b(x, y)\pi^x(dy), \quad \bar{\sigma}(x) = \int_{\mathbb{R}^d} \sigma(x, y)\pi^x(dy). \tag{2.12} \]
Then $\bar{b}$ and $\bar{\sigma}$ are locally H"older continuous of exponent 2/3.

**Proof.** By (2.7), (A1), and (A2), we obtain that
\[ |\bar{b}(x_1) - \bar{b}(x_2)| \leq \int_{\mathbb{R}^d} b(x_1, y)|\pi^{x_1}(dy) - \pi^{x_2}(dy)| + \int_{\mathbb{R}^d} |b(x_1, y) - b(x_2, y)|\pi^{x_2}(dy) \leq \sup_{y \in \mathbb{R}^d} |b(x_1, y)|\|\pi^{x_1} - \pi^{x_2}\|_{\text{var}} + \sqrt{K_1}|x_1 - x_2| \leq K_2(1 + |x_1|)|x_1 - x_2|^{2/3} + \sqrt{K_1}|x_1 - x_2|, \]
which implies that $\bar{b}$ is locally H"older continuous of exponent 2/3. Similar deduction yields the result for $\bar{\sigma}$. Hence, this corollary is proved. \hfill \square
Proposition 2.3  (i) Assume (E1), (B1) hold. Then for $x_1, x_2 \in \mathbb{R}^d$

$$\mathbb{W}_{bl}(\pi^{x_1}, \pi^{x_2}) \leq |x_1 - x_2| \mathbb{1}_{\{|x_1-x_2| \geq 2\kappa_1\}} + |x_1 - x_2|^{\frac{\lambda_1}{\lambda_1 + \kappa_3}} \mathbb{1}_{\{|x_1-x_2| < 2\kappa_1\}}.$$  \hspace{1cm} (2.13)

(ii) Assume (E2), (B1) hold. Then for $x_1, x_2 \in \mathbb{R}^d$

$$\mathbb{W}_1(\pi^{x_1}, \pi^{x_2}) \leq |x_1 - x_2| \mathbb{1}_{\{|x_1-x_2| \geq 2\kappa_2\}} + |x_1 - x_2|^{\frac{\lambda_2}{\lambda_2 + \kappa_2}} \mathbb{1}_{\{|x_1-x_2| < 2\kappa_2\}}.$$  \hspace{1cm} (2.14)

Proof. (i) For $x_1, x_2, y \in \mathbb{R}^d$, consider

$$
dY_t^{x_1, y} = f(x_1, Y_t^{x_1, y})dt + g(x_1, Y_t^{x_1, y})dB_t, \quad Y_0^{x_1, y} = y,
$$
$$
dY_t^{x_2, y} = f(x_2, Y_t^{x_2, y})dt + g(x_2, Y_t^{x_2, y})dB_t, \quad Y_0^{x_2, y} = y.
$$

Then, by Itô’s formula and (B1),

$$
\mathbb{E}|Y_t^{x_1, y} - Y_t^{x_2, y}|^2 \leq \mathbb{E} \int_0^t K_3(|x_1 - x_2|^2 + |Y_s^{x_1, y} - Y_s^{x_2, y}|^2)ds.
$$

This yields

$$
\mathbb{E}|Y_t^{x_1, y} - Y_t^{x_2, y}|^2 \leq (e^{K_3t} - 1)|x_1 - x_2|^2. \hspace{1cm} (2.15)
$$

For any $h \in \mathscr{B} (\mathbb{R}^d)$ with $|h|_{\infty} \leq 1$ and $|h|_{\text{Lip}} := \sup_{z_1 \neq z_2} \frac{|h(z_1) - h(z_2)|}{|z_1 - z_2|} \leq 1$, by virtue of (2.15) and (E1),

$$
|\pi^{x_1}(h) - \pi^{x_2}(h)| \leq |\pi^{x_1}(h) - P_t^{x_1}h(y)| + |\pi^{x_2}(h) - P_t^{x_2}h(y)| + |P_t^{x_1}h(y) - P_t^{x_2}h(y)|
$$
$$
\leq \|P_t^{x_1}(y, \cdot) - \pi^{x_1}\|_{\text{var}} + \|P_t^{x_2}(y, \cdot) - \pi^{x_2}\|_{\text{var}} + \mathbb{E}|h(Y_t^{x_1, y}) - h(Y_t^{x_2, y})|
$$
$$
\leq 2\kappa_1 e^{-\lambda_1 t} + |h|_{\text{Lip}} \mathbb{E}|Y_t^{x_1, y} - Y_t^{x_2, y}|^2)^{1/2}
$$
$$
\leq 2\kappa_1 e^{-\lambda_1 t} + (e^{K_3t} - 1)^{1/2}|x_1 - x_2| =: (I).
$$

To estimate (I), when $|x_1 - x_2| \geq 2\kappa_1$, it holds

$$(I) \leq |x_1 - x_2| + (e^{K_3t} - 1)^{1/2}|x_1 - x_2|,$$

which yields $|x_1 - x_2|$ by letting $t \to 0$. When $|x_1 - x_2| < 2\kappa_1$, take $t = -\frac{1}{\lambda_1} \ln \frac{|x_1-x_2|^p}{2\kappa_1}$ for some $p > 0$ to be determined later. Then $e^{-\lambda_1 t} = \frac{|x_1-x_2|^p}{2\kappa_1}$, and

$$(I) \leq |x_1 - x_2|^p + e^{K_3t}|x_1 - x_2|.$$
Take $p = \frac{\lambda_1}{\lambda_1 + K_3}$, then $p = 1 - \frac{K_3}{\lambda_1}$. Finally, (I) \leq \left(1 + \frac{2\kappa_1}{\lambda_1 + K_3}\right)|x_1 - x_2|^{1 - \frac{K_3}{\lambda_1}}.\]

By the arbitrariness of $h$ with $|h|_{\infty} \leq 1$ and $|h|_{\text{Lip}} \leq 1$, we get (2.13) from the definition of $W_{bL}(\pi^{x_1}, \pi^{x_2})$.

(ii) For any $h \in C(\mathbb{R}^d)$ with $|h|_{\text{Lip}} \leq 1$, for $y \in \mathbb{R}^d$,

$$|\pi^{x_1}(h) - \pi^{x_2}(h)| \leq |\pi^{x_1}(h) - P_t^{x_1}h(y)| + |\pi^{x_2}(h) - P_t^{x_2}h(y)| + |P_t^{x_1}h(y) - P_t^{x_2}h(y)|$$

$$\leq W_1(P_t^{x_1}(y, \cdot), \pi^{x_1}) + W_1(P_t^{x_2}(y, \cdot), \pi^{x_2}) + |h|_{\text{Lip}}(\mathbb{E}|Y_t^{x_1,y} - Y_t^{x_2,y}|^2)^{\frac{1}{2}}$$

$$\leq 2\kappa_2 e^{-\lambda_3 t} + (e^{K_3 t} - 1)\frac{1}{2}|x_1 - x_2|,$$

due to (2.15). Then, completely similar to assertion (i), by choosing suitable $t > 0$, we have

$$|\pi^{x_1}(h) - \pi^{x_2}(h)| \leq |x_1 - x_2|1_{\{|x_1 - x_2| \geq 2\kappa_2\}} + |x_1 - x_2|^{\frac{\lambda_3}{\lambda_2 + K_3}}1_{\{|x_1 - x_2| < 2\kappa_3\}},$$

which yields (2.14) by taking supremum over $h \in C(\mathbb{R}^d)$ with $|h|_{\text{Lip}} \leq 1$. \hfill \Box

**Corollary 2.4** (i) Assume (E1), (B1) and (A1), (A2) hold. Then $\bar{b}, \bar{\sigma}$ defined in (2.12) are locally H"older continuous of exponent $\frac{\lambda_1}{\lambda_1 + K_3}$.

(ii) Assume (E2), (B1) and (A1) hold. Then $\bar{b}, \bar{\sigma}$ are H"older continuous of exponent $\frac{\lambda_2}{\lambda_2 + K_3}$.

**Proof.** (i) We only prove the assertion for $\bar{b}$, and the assertion for $\bar{\sigma}$ can be proved in the same way. By (A1) and (A2), for $x_1, x_2 \in \mathbb{R}^d$,

$$|\bar{b}(x_1) - \bar{b}(x_2)| \leq \int_{\mathbb{R}^d} b(x_1, y)(\pi^{x_1}(dy) - \pi^{x_2}(dy)) + \int_{\mathbb{R}^d} b(x_1, y) - b(x_2, y)\pi^{x_2}(dy)$$

$$\leq K_2(1 + |x_1|)||\pi^{x_1} - \pi^{x_2}||_{\text{var}} + \sqrt{K_1}|x_1 - x_2|.$$
Thus, $\bar{b}$ is locally Hölder continuous of exponent $\frac{\lambda_1}{\lambda_1 + K_3}$.

(ii) By (A1), it holds $|b(x_1, y) - b(x_1, y')| \leq \sqrt{K_1}|y - y'|$ for $y, y' \in \mathbb{R}^d$. Using (2.14),

$$|\bar{b}(x_1) - \bar{b}(x_2)| \leq \sqrt{K_1} |\pi_{x_1}(dy) - \pi_{x_2}(dy)| + \sqrt{K_1}|x_1 - x_2|$$

This yields immediately that $\bar{b}$ is Hölder continuous of exponent $\frac{\lambda_2}{\lambda_2 + K_3}$. This corollary is proved.

3 Averaging principle for fully coupled systems

Before establishing the averaging principle using the ergodicity condition (E1) or (E2), we would like introduce some sufficient conditions to verify (E1) and (E2) based on the coefficients $f, g$ of the fast component. We refer the readers to the monograph [5] for more discussion on (E1), especially its connection of functional inequalities, that is, Nash inequality can yield (E1) (cf. [5, Chapter 1]). Recall that the diffusion process $(Y_{t}^{x,y})$ is defined in (1.3) for each fixed slow component $x$ and its semigroup is denoted by $P_t^{x}(y, \cdot)$.

Lemma 3.1 ([23]) (1) When $(Y_{t}^{x,y})$ is a diffusion process on $[0, \infty)$ with reflection boundary at 0. Assume $g(x, y) > 0$ for $x, y \in [0, \infty)$. Let $C_x(y) = \int_1^y \frac{f(x,z)}{g(x,z)}dz$. Then $(Y_{t}^{x,y})$ is strongly ergodic in the sense of (E1) if and only if

$$\int_0^\infty e^{-C_x(y)}dy \int_0^y g(x, z)^{-z} e^{C_x(z)}dz = \infty,$$

$$\int_0^\infty e^{-C_x(y)}dy \int_y^\infty g^{-2}(x, z)e^{C_x(z)}dz < \infty.$$

(2) Generally, assume $(Y_{t}^{x,y})$ is a reversible process in $\mathbb{R}^d$, i.e. for each $x \in \mathbb{R}^d$ there exists a function $V^x \in C^2(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} e^{V^x(y)}dy < \infty$,

$$g(x, y) = \sum_{j=1}^d G_{ij}(x, y) \frac{\partial}{\partial y_j} V^x(y) + \sum_{j=1}^d \frac{\partial}{\partial y_j} G_{ij}(x, y),$$
where \( G(x, y) = (G_{ij}(x, y)) = (g g^*)(x, y) \). Suppose that \( G(x, y) = \text{diag}(G_{ii}(x, y)) \) and \( G_{ii}(x, y) = G_{ii}(x, y_i), i = 1, \ldots, d \), for \( x \in \mathbb{R}^d \), \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d \). Let

\[
\rho_x(y, y') = \left( \sum_{i=1}^d \left( \int_{y_i}^{y_i'} G_{ii}(x, z)^{-\frac{1}{2}} dz \right) \right)^{\frac{1}{2}},
\]

\[
h_i^x = \sqrt{G_{ii}(x, y)} \frac{\partial}{\partial y_i} V^x(y) + \left( 2 \sqrt{G_{ii}(x, y)} \right)^{-1} \frac{\partial}{\partial y_i} G_{ii}(x, y),
\]

\[
\gamma_x(r) = \sup_{\rho_x(y, y') = r} \sum_{i=1}^d \left( h_i^x(y) - h_i^x(y') \right) \int_{y_i}^{y_i'} G_{ii}(x, z)^{-\frac{1}{2}} dz,
\]

\[
C_x(r) = \exp \left[ \int_0^r \frac{\gamma_x(s)}{4s} ds \right].
\]

If \( \int_0^\infty \left( C_x(s)^{-1} \int_s^\infty \frac{C_x(t)}{4} dt \right) ds < \infty \), then \( (Y_t^x) \) is strongly ergodic in the sense of (E1).

The exponential decay of the semigroup \( (P^x_t) \) in the Wasserstein distance is heavily related to geometry of the underlying space as illustrated in [34]. The best convergence rate in the inequality is called the Wasserstein curvature in [15] or the coarse Ricci curvature in [26]. Here, we shall provide a criterion for (E2) based on the coupling method.

Recall the generator \( \mathcal{L}^x \) of \( (Y_t^x) \) given by

\[
\mathcal{L}^x = \frac{1}{2} \sum_{k,l=1}^d G_{kl}(x, y) \frac{\partial^2}{\partial y_k \partial y_l} + \sum_{k=1}^d f_k(x, y) \frac{\partial}{\partial y_k}.
\]

For simplicity, we write \( \mathcal{L}^x \sim (G(x, y), f(x, y)) \). An operator \( \widetilde{\mathcal{L}} \) on \( \mathbb{R}^d \times \mathbb{R}^d \) is called a coupling operator of \( \mathcal{L}^x \) and itself if

\[
\widetilde{\mathcal{L}} h(y, y') = \mathcal{L}^x h(y) \quad \text{if} \ h \in C_0^2(\mathbb{R}^d) \ \text{and independent of} \ y,
\]

\[
\widetilde{\mathcal{L}} h(y, y') = \mathcal{L}^x h(y') \quad \text{if} \ h \in C_0^2(\mathbb{R}^d) \ \text{and independent of} \ y'.
\]

The coefficients of any coupling operator must be of the form \( \widetilde{\mathcal{L}} \sim (a_x(y, y'), \tilde{f}_x(y, y')) \) with

\[
a_x(y, y') = \begin{pmatrix} G(x, y) & c_x(y, y') \\ c_x(y, y') & G(x, y') \end{pmatrix}, \quad \tilde{f}_x(y, y') = \begin{pmatrix} f(x, y) \\ f(x, y') \end{pmatrix},
\]

where \( c_x(y, y') \) is a matrix such that \( a_x(y, y') \) is nonnegative definite. When taking \( c_x(y, y') = 0 \), \( \widetilde{\mathcal{L}} \) is called an independent coupling of \( \mathcal{L}^x \) and itself, denoted by \( \mathcal{L}_{ind} \).
The monotonicity condition (1.4) means that
\[ \tilde{L}_{\text{ind}}(|y - y'|^2) = 2(y-y') \cdot (f(x, y) - f(x, y')) + \|g(x, y) - g(x, y')\|^2 \leq -\beta |y - y'|^2. \] (3.1)
Together with the following coercivity condition, i.e.
\[ L^x(|y|^2) = -y \cdot f(x, y) + \|g(x, y)\|^2 \leq c_1 - \beta_1 |y|^2 \]
for some \( c_1, \beta_1 > 0 \), similar to Lemma 3.2 below, we obtain from the monotonicity condition (3.1) that
\[ W_1(P^x_t(y, \cdot), \pi^x) \leq W_2(P^x_t(y, \cdot), \pi^x) \leq \frac{c_1}{\beta_1} e^{-\beta t}, \quad t > 0, \]
and hence (E2) holds.

**Lemma 3.2** Assume (B3) holds. Let \( \rho : [0, \infty) \to [0, \infty) \) be in \( C^2([0, \infty)) \) satisfying \( \rho(0) = 0, \rho' > 0, \rho'' < 0, \) and \( \rho(x) \to \infty \) as \( x \to \infty \). There is a constant \( c_2 > 0 \) such that \( \rho(x) > c_2 x \) for \( x \geq 0 \). If there exist constants \( c_3, \beta_2, \beta_3 > 0 \), a coupling \( \tilde{L}^x \) of \( L^x \) and itself such that
\[ \tilde{L}^x \rho(|y - y'|) \leq -\beta_2 \rho(|y - y'|), \quad x, y, y' \in \mathbb{R}^d, \] (3.2)
and
\[ L^x \rho(|y|) \leq c_3 - \beta_3 \rho(|y|), \quad x, y \in \mathbb{R}^d. \] (3.3)
Then for any \( t > 0 \)
\[ W_1(P^x_t(y, \cdot), \pi^x) \leq \frac{c_3}{\beta_3 c_2} e^{-\beta_3 t}, \quad x, y \in \mathbb{R}^d. \]

**Proof.** Associated with the coupling operator \( \tilde{L}^x \), there is a coupling process \((Y^x_t, \tilde{Y}^x_t)\) with initial value \( Y^x_0 = y \) and \( \tilde{Y}^x_0 = \xi \), where \( \xi \) is a random variable with distribution \( \pi^x \). By virtue of (3.3), it holds
\[ \mathbb{E}[\rho(|Y^x_{t}|)] \leq \mathbb{E}[\rho(|Y^x_{0}|)] + \int_0^t (c_3 - \beta_3 \mathbb{E}[\rho(|Y^x_{s}|)]) ds. \]
Gronwall’s inequality yields that
\[ \mathbb{E}[\rho(|Y^x_{t}|)] \leq \frac{c_3}{\beta_3} (1 - e^{-\beta_3 t}) + \rho(|y|) e^{-\beta_3 t}. \] (3.4)
This implies the existence of invariant probability measure $\pi^x$ for the process $(Y^{x,y}_t)$. Letting $t \to \infty$ in (3.4), Fatou’s lemma implies that

$$\int_{\mathbb{R}^d} \rho(|y'|) \pi^x(dy') \leq \frac{c_3}{\beta_3}. \quad (3.5)$$

Note that the uniform nondegenerate condition (B3) yields that $\pi^x$ admits a density w.r.t. the Lebesgue measure.

By Itô’s formula and (3.2), for $0 \leq s \leq t$,

$$\mathbb{E}[\rho(|Y^x_t - \tilde{Y}^x_t|)] \leq \mathbb{E}[\rho(|Y^x_s - \tilde{Y}^x_s|)] - \beta_2 \int_s^t \mathbb{E}[\rho(|Y^x_r - \tilde{Y}^x_r|)] dr.$$  

Using Gronwall’s inequality, we get

$$\mathbb{E}[\rho(|Y^x_t - \tilde{Y}^x_t|)] \leq \mathbb{E}[\rho(|y - \xi|)] e^{-\beta_2 t}.$$  

Hence, by (3.5),

$$\mathbb{W}_1(P^x_t(y, \cdot), \pi^x) \leq c_2^{-1} \int_{\mathbb{R}^d} \rho(|y - y'|) \pi^x(dy') e^{-\beta_2 t} \leq \frac{c_3}{\beta_3 c_2} e^{-\beta_2 t}.$$  

The proof is completed. \hfill \Box

According to Stroock and Varadhan [29] (see, also, [14, Chapter IV]), when $\bar{b}, \bar{\sigma}$ are continuous satisfying the linear growth condition, and $\bar{\sigma}$ is nondegenerate, SDE

$$d\bar{X}_t = \bar{b}(\bar{X}_t)dt + \bar{\sigma}(\bar{X}_t)dB_t, \quad \bar{X}_0 = x_0, \quad (3.6)$$

admits a weak solution whose distribution is unique. Of course, the wellposedness of SDE (3.6) is a precondition of the averaging principle. Corollaries 2.2 and 2.4 provide us different sufficient conditions for the wellposedness of SDE (3.6).

Denote

$$\mathcal{L} h(x) = \bar{b}(x) \cdot \nabla h(x) + \frac{1}{2} \text{tr}(\bar{a}(x) \nabla^2 h(x))$$

$$= \sum_{k=1}^d \bar{b}_k(x) \frac{\partial h(x)}{\partial x_k} + \frac{1}{2} \sum_{k,l=1}^d \bar{a}_{kl}(x) \frac{\partial^2 h(x)}{\partial x_k \partial x_l}, \quad h \in C^2(\mathbb{R}^d), \quad (3.7)$$
where \( \bar{a}(x) = (\hat{\sigma} \hat{\sigma}^\ast)(x) \).

Let us introduce some notations. For \( T > 0 \), \( C([0, T]; \mathbb{R}^d) \) denotes the set of continuous functions from \([0, T]\) to \( \mathbb{R}^d \) endowed with uniform norm, i.e. \( \|x - y\|_{\infty} = \sup_{t \in [0, T]} |x_t - y_t| \) for \( x, y \in C([0, T]; \mathbb{R}^d) \). Denote by \( \mathcal{L}_X \) and \( \mathcal{L}_Y \) the law of the stochastic processes \( (X_t^\varepsilon) \) and \( (\tilde{X}_t) \) respectively. Let

\[
\mathcal{L}_\varepsilon h(x, y) = b(x, y) \cdot \nabla h(x) + \frac{1}{2} \text{tr} ((\sigma \sigma^\ast)(x, y) \nabla^2 h(x)), \quad h \in C^2(\mathbb{R}^d), x, y \in \mathbb{R}^d. \tag{3.8}
\]

**Theorem 3.3** Let \((X_t^\varepsilon, Y_t^\varepsilon)\) be the solution to (1.1) and \((\tilde{X}_t)\) the solution to (3.6). Assume (E1), (A1)-(A3), and (B1) hold. Then for any \( T > 0 \) the process \((X_t^\varepsilon)_{t \in [0,T]}\) converges weakly in \( C([0, T]; \mathbb{R}^d) \) as \( \varepsilon \to 0 \) to the process \((\tilde{X}_t)_{t \in [0,T]}\).

**Proof.** By virtue of Corollary 2.4, the limit process \((\tilde{X}_t)\) exists and is unique in distribution. Due to the linear growth condition (A2), it is standard to show

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^\varepsilon|^p \right] \leq C(T, x_0, p), \quad \text{for } p \geq 1.
\]

By Itô’s formula,

\[
\mathbb{E} |X_t^\varepsilon - X_s^\varepsilon|^4 \leq 8 \mathbb{E} \left| \int_s^t b(X_r^\varepsilon, Y_r^\varepsilon)dr \right|^4 + 8 \mathbb{E} \left| \int_s^t \sigma(X_r^\varepsilon, Y_r^\varepsilon)dW_r \right|^4
\]

\[
\leq 8(t-s)^3 \mathbb{E} \int_s^t |b(X_r^\varepsilon, Y_r^\varepsilon)|^4dr + 288(t-s)^2 \mathbb{E} \int_s^t |\sigma(X_r^\varepsilon, Y_r^\varepsilon)|^4dr
\]

\[
\leq C(t-s)^2
\]

for some constant \( C > 0 \). Combining this with the fact \( X_0^\varepsilon = x_0 \), the collection of the laws of \( \{(X_t^\varepsilon)_{t \in [0,T]}; \varepsilon > 0\} \) over \( C([0, T]; \mathbb{R}^d) \) is tight by [2, Theorem 12.3]. As a consequence, there is a subsequence \( \{\mathcal{L}_{X_t^\varepsilon}; \varepsilon' > 0\} \) and a limit law \( \mathcal{L}_{\tilde{X}} \) over \( C([0, T]; \mathbb{R}^d) \) such that \( \mathcal{L}_{X_t^{\varepsilon'}} \) converges weakly to \( \mathcal{L}_{\tilde{X}} \) as \( \varepsilon' \to 0 \). According to Skorokhod’s representation theorem, we may assume that \((X_t^{\varepsilon'})_{t \in [0,T]}\) converges almost surely to some random process \((\tilde{X}_t)_{t \in [0,T]}\) in \( C([0, T]; \mathbb{R}^d) \) as \( \varepsilon' \to 0 \).

We proceed to characterize the process \((\tilde{X}_t)_{t \in [0,T]}\). To this end, we shall prove that for any \( h \in C^2(\mathbb{R}^d) \), the space of functions with compact support and continuous second order derivatives,

\[
h(\tilde{X}_t) - h(x_0) - \int_0^t \mathcal{L} h(\tilde{X}_s)ds \quad \text{is a martingale}, \tag{3.9}
\]

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Hence, for measurable function $\Phi$, the almost sure convergence of $(\tilde{X}_t)$ equals to $X_t$ where $\bar{\cal L}$ is defined in (3.7). This means that $(\tilde{X}_t)$ is a solution to SDE (3.6), and hence $\bar{\cal L}$ equals to $\cal L$ due to the uniqueness of distribution for solutions to SDE (3.6).

To prove (3.9), it is suffices to show that for $0 \leq s < t \leq T$, for any bounded $\mathfrak{F}_s$ measurable function $\Phi$,

$$
\mathbb{E}
\left[
(h(\tilde{X}_t) - h(\tilde{X}_s) - \int_s^t \bar{\cal L} h(\tilde{X}_r) dr) \Phi
\right] = 0, \quad \forall h \in C_c^2(\mathbb{R}^d).
$$

As a solution to SDE (1.1), $(X_t')$ satisfies that

$$
\mathbb{E}
\left[
(h(X_t') - h(X_t') - \int_s^t \cal L h(X_r', Y_r') dr) \Phi
\right] = 0.
$$

The almost sure convergence of $(X_t')_{t \in [0, T]}$ to $(\tilde{X}_t)_{t \in [0, T]}$ in $\mathcal{C}([0, T]; \mathbb{R}^d)$ yields that

$$
\lim_{\varepsilon' \to 0} \mathbb{E}
\left[
(h(X_t') - h(X_t')) \Phi
\right] = \mathbb{E}
\left[
(h(\tilde{X}_t) - h(\tilde{X}_s)) \Phi
\right].
$$

Hence, for (3.10) we only need to show

$$
\lim_{\varepsilon' \to 0} \mathbb{E}
\left[
\int_s^t (\cal L h(X_r', Y_r') - \bar{\cal L} h(\tilde{X}_r)) dr \bigg| \mathfrak{F}_s
\right] = 0.
$$

In view of the expression of $\cal L$ and $\bar{\cal L}$, we need to show

$$
\lim_{\varepsilon' \to 0} \mathbb{E}
\left[
\int_s^t \left(b(X_r', Y_r') \cdot \nabla h(X_r') - \bar{b}(\tilde{X}_r) \cdot \nabla h(\tilde{X}_r) \right) dr \bigg| \mathfrak{F}_s
\right] = 0, \quad (3.11)
$$

$$
\lim_{\varepsilon' \to 0} \mathbb{E}
\left[
\int_s^t \left(\text{tr}(\sigma \sigma^*) (X_r', Y_r') \nabla^2 h(X_r') - \text{tr}(\bar{\sigma} \bar{X}_r) \nabla^2 h(\tilde{X}_r) \right) dr \bigg| \mathfrak{F}_s
\right] = 0. \quad (3.12)
$$

We shall use the time discretization method and the coupling method to show (3.11) and (3.12). Since the method is similar, we only present the proof of (3.11).

For $\delta \in (0, 1)$, let $r(\delta) = s + \left\lfloor \frac{t-s}{\delta} \right\rfloor \delta$ for $r \in [s, t]$, where $\left\lfloor \frac{t-s}{\delta} \right\rfloor$ denotes the integer part of $\frac{t-s}{\delta}$.

$$
\mathbb{E}
\left[
\int_s^t (b(X_r', Y_r') \cdot \nabla h(X_r') - \bar{b}(\tilde{X}_r) \cdot h(\tilde{X}_r)) dr \bigg| \mathfrak{F}_s
\right] = \mathbb{E}
\left[
\int_s^t \left\{b(X_r', Y_r') \cdot (\nabla h(X_r') - \nabla h(\tilde{X}_r)) + (b(X_r', Y_r') - b(X_{r(\delta)}', Y_{r(\delta)}')) \cdot \nabla h(\tilde{X}_r) \right\} dr \bigg| \mathfrak{F}_s
\right]
$$

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\[ + \left( b(X_{r(\delta)}^\varepsilon, Y_{r}^\varepsilon) - \bar{b}(X_{r(\delta)}^\varepsilon) \right) \cdot \nabla h(\bar{X}_r) + \left( \bar{b}(X_{r(\delta)}^\varepsilon) - \bar{b}(X_r) \right) \cdot \nabla h(\bar{X}_r) \right\} dr \bigg| \mathcal{F}_s \]

By the continuity of \( b \) and \( \bar{b} \) due to (A1) and Corollary 2.2, it follows from the almost sure convergence of \((X^\varepsilon_r)\) to \((\bar{X}_r)\) in \(C([0,T]; \mathbb{R}^d)\) that

\[
\lim_{\varepsilon', \delta \to 0} \mathbb{E} \left[ \int_s^t \left\{ b(X_{r(\delta)}^\varepsilon, Y_{r}^\varepsilon) \cdot (\nabla h(X_{r(\delta)}^\varepsilon) - \nabla h(\bar{X}_r)) + \left( b(X_{r(\delta)}^\varepsilon, Y_{r}^\varepsilon) - b(X_{r(\delta)}^\varepsilon, Y_{r}^\varepsilon) \right) \cdot \nabla h(\bar{X}_r) \right. \\
+ \left( \bar{b}(X_{r(\delta)}^\varepsilon) - \bar{b}(X_r) \right) \cdot \nabla h(\bar{X}_r) \right\} dr \bigg| \mathcal{F}_s \right] = 0.
\]

Therefore, to prove (3.11) we only to show that for suitable choice of \( \delta \),

\[
\mathbb{E} \left[ \int_s^t (b(X_{r(\delta)}^\varepsilon, Y_{r}^\varepsilon) - \bar{b}(X_{r(\delta)}^\varepsilon)) \cdot \nabla h(\bar{X}_r) dr \bigg| \mathcal{F}_s \right] \to 0, \quad \text{as } \varepsilon' \to 0. \quad (3.13)
\]

Let \( N_t = [(t-s)/\delta], s_k + s + k\delta \) for \( 0 \leq k \leq N_t \) and \( s_{N_t} = t \). Then,

\[
\mathbb{E} \left[ \int_s^t (b(X_{r(\delta)}^\varepsilon, Y_{r}^\varepsilon) - \bar{b}(X_{r(\delta)}^\varepsilon)) \cdot \nabla h(\bar{X}_r) dr \bigg| \mathcal{F}_s \right] = \sum_{k=0}^{N_t} \mathbb{E} \left[ \int_{s_k}^{s_{k+1}} (b(X_{s_k}^\varepsilon, Y_{s_k}^\varepsilon) - \bar{b}(X_{s_k}^\varepsilon)) \cdot \nabla h(\bar{X}_r) dr \bigg| \mathcal{F}_{s_k} \right]. \quad (3.14)
\]

On each time interval \([s_k, s_{k+1})\), \( 0 \leq k \leq N_t \), we introduce a coupling process \((Y_{r}^\varepsilon, \tilde{Y}_{r}^{\varepsilon,k})\) by the following SDEs

\[
\begin{align*}
\frac{dY_{r}^\varepsilon}{\varepsilon'} &= \frac{1}{\varepsilon'} f(X_{r}^\varepsilon, Y_{r}^\varepsilon) dr + \frac{1}{\sqrt{\varepsilon'}} g(Y_{r}^\varepsilon) dB_r, \\
\frac{d\tilde{Y}_{r}^{\varepsilon,k}}{\varepsilon'} &= \frac{1}{\varepsilon'} f(X_{s_k}^\varepsilon, \tilde{Y}_{s_k}^{\varepsilon,k}) dr + \frac{1}{\sqrt{\varepsilon'}} g(\tilde{Y}_{s_k}^{\varepsilon,k}) dB_r, \\
\tilde{Y}_{s_k}^{\varepsilon,k} &= Y_{s_k}^\varepsilon. \quad (3.15)
\end{align*}
\]

Notice that under the conditional expectation \( \mathbb{E}[\cdot | \mathcal{F}_{s_k}] \), the distribution of \( \tilde{Y}_{r}^{\varepsilon,k} \) equals to \( P_{(r-s_k)/\varepsilon'}^{X_{s_k}^\varepsilon}(Y_{s_k}^\varepsilon, \cdot) \) due to the homogeneity and the uniqueness of solution to (3.15). It follows from (A2), (B1) and Itô’s formula that for \( r \in [s_k, s_{k+1}) \),

\[
\mathbb{E}[|Y_{r}^\varepsilon - \tilde{Y}_{r}^{\varepsilon,k}|^2 | \mathcal{F}_{s_k}] \leq \frac{C}{\varepsilon'} \int_{s_k}^{r} \mathbb{E}[|X_u^\varepsilon - X_{s_k}^\varepsilon|^2 + |Y_u^\varepsilon - \tilde{Y}_{u}^{\varepsilon,k}|^2 | \mathcal{F}_{s_k}] du \\
\leq \frac{C}{\varepsilon'} \int_{s_k}^{r} (u-s_k) + \mathbb{E}[|Y_u^\varepsilon - \tilde{Y}_{u}^{\varepsilon,k}|^2 | \mathcal{F}_{s_k}] du \\
\leq \frac{C\delta^2}{\varepsilon'} + \frac{C}{\varepsilon'} \int_{s_k}^{r} \mathbb{E}[|Y_u^\varepsilon - \tilde{Y}_{u}^{\varepsilon,k}|^2 | \mathcal{F}_{s_k}] du.
\]

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By Gronwall’s inequality,
\[
\mathbb{E}[|Y_r^{e'} - \tilde{Y}_r^{e', k}|^2 | \mathcal{F}_{s_k}] \leq \frac{C\delta^2}{\varepsilon'} e^{C(r-s_k)/\varepsilon'}.
\]
(3.16)

Therefore, using (A1), (A2), (E1), and (3.16),
\[
\begin{aligned}
&|\mathbb{E}\left[ \int_{s_k}^{s_h+1} (b(X_{s_k}^{e'}, Y_{s_k}^{e'}) - \tilde{b}(X_{s_k}^{e'})) \cdot \nabla h(\tilde{X}_r) \, dr | \mathcal{F}_{s_k} \right] | \\
&\leq |\nabla h|_{\infty} \mathbb{E}\left[ \int_{s_k}^{s_h+1} |b(X_{s_k}^{e'}, Y_{s_k}^{e'}) - b(X_{s_k}^{e'}, \tilde{Y}_{s_k}^{e', k})| + |b(X_{s_k}^{e'}, \tilde{Y}_{s_k}^{e', k}) - \tilde{b}(X_{s_k}^{e'})| \, dr | \mathcal{F}_{s_k} \right] \\
&\leq |\nabla h|_{\infty} \int_{s_k}^{s_h+1} \left\{ \mathbb{E}[K_1|Y_r^{e'} - \tilde{Y}_r^{e', k}|^2 | \mathcal{F}_{s_k}] \right\}^\frac{1}{2} \\
&\quad + K_2(1 + |X_{s_k}^{e'}|) \| P_{s_k}^{X_{s_k}^{e'}} (Y_{s_k}^{e'}, \cdot) - \pi_{X_{s_k}^{e'}} \|_{\text{var}} \right\} \, dr \\
&\leq |\nabla h|_{\infty} \int_{s_k}^{s_h+1} \left\{ \frac{C\delta^2}{\varepsilon'} e^{C(r-s_k)/2\varepsilon'} + K_2C_0 e^{-\kappa r - \varepsilon' s_k} (1 + |X_{s_k}^{e'}|) \right\} \, dr \\
&\leq |\nabla h|_{\infty} \frac{C\delta^2}{\varepsilon'} e^{\frac{C\delta}{2\varepsilon'}} + |\nabla h|_{\infty} K_2C_0 (1 + |X_{s_k}^{e'}|) \int_0^\delta e^{-\kappa \tilde{r}} \, dr.
\end{aligned}
\]
(3.17)

Inserting this estimate into (3.14), we obtain that
\[
\begin{aligned}
|\mathbb{E}\left[ \int_s^t (b(X_{s}^{e', \theta_{s}}), Y_r^{e'}) - \tilde{b}(X_{s}^{e', \theta_{s}})) \cdot \nabla h(\tilde{X}_r) \, dr | \mathcal{F}_s \right] | & \\
&\leq |\nabla h|_{\infty} \sum_{k=0}^{N_t} \left( \frac{C\delta^2}{\varepsilon'} e^{\frac{C\delta}{\varepsilon'}} + C_0K_2 \mathbb{E}[1 + |X_{s_k}^{e'}| | \mathcal{F}_s] \int_0^\delta e^{-\kappa \tilde{r}} \, dr \right) \\
&\leq C|\nabla h|_{\infty} \frac{t\delta}{\varepsilon'} e^{\frac{C\delta}{2\varepsilon'}} + C|\nabla h|_{\infty} \frac{t\varepsilon'}{\kappa\delta} (1 - e^{-\kappa \tilde{r}}).
\end{aligned}
\]
(3.18)

Take \( \delta = \varepsilon' \ln \left( \frac{1}{\varepsilon'} \right) \), then
\[
\lim_{\varepsilon' \to 0} \frac{\delta}{\varepsilon'} = \infty, \quad \lim_{\varepsilon' \to 0} \frac{\delta}{\varepsilon'} e^{\frac{C\delta}{\varepsilon'}} = \lim_{\varepsilon' \to 0} \sqrt{\varepsilon'} \left( \ln \ln \left( \frac{1}{\varepsilon'} \right) \right) \left( \ln \left( \frac{1}{\varepsilon'} \right) \right)^{\frac{C}{2}} = 0.
\]

Hence, using this choice of \( \delta \), we get from (3.18) that
\[
\lim_{\varepsilon' \to 0} \mathbb{E}\left[ \int_s^t (b(X_{s}^{e', \theta_{s}}), Y_r^{e'}) - \tilde{b}(X_{s}^{e', \theta_{s}})) \cdot \nabla h(\tilde{X}_r) \, dr | \mathcal{F}_s \right] = 0.
\]

This is the desired (3.13), and further (3.11) holds.
Consequently, we have shown that \((X'_t)_{t \in [0,T]} \) converges weakly to \((\bar{X}_t)_{t \in [0,T]} \). The arbitrariness of weakly convergent subsequence of \((X'_t)_{t \in [0,T]} \) and the uniqueness of \((\bar{X}_t)_{t \in [0,T]} \) imply that the whole sequence \((X'_t)_{t \in [0,T]} \) converges weakly to \((\bar{X}_t)_{t \in [0,T]} \). We have completed the proof. \(\square\)

**Theorem 3.4** Assume \((E2), (B1), (A1), (A3)\) hold. In addition, suppose that

\(\text{(A2')}\) there exists \(K'_2 > 0\) such that

\[
|b(x, y)| + \|\sigma(x, y)\| \leq K'_2(1 + |x| + |y|), \quad x, y \in \mathbb{R}^d.
\]

Then for any \(T > 0\) \((X'_t)_{t \in [0,T]} \) converges weakly in \(C([0,T]; \mathbb{R}^d)\) to \((\bar{X}_t)_{t \in [0,T]} \) as \(\varepsilon \to 0\). 

**Proof.** This theorem can be proved along the line of the argument of Theorem 3.3. We only point out the different point caused by using the \(L^1\)-Wasserstein distance instead of the total variation distance in ergodicity condition. Namely, instead of (3.17), we now have

\[
\left| \mathbb{E} \left[ \int_{s_k}^{s_{k+1}} \left( b(X'_{s_k}, Y'_{r}) - b(X'_{s_k}) \right) \cdot \nabla h(\bar{X}_r) dr \right] \right|
\]

\[
\leq |\nabla h|_\infty \mathbb{E} \left[ \int_{s_k}^{s_{k+1}} |b(X'_{s_k}, Y'_{r}) - b(X'_{s_k}, \bar{Y}'_{r,k})| + |b(X'_{s_k}, \bar{Y}'_{r,k}) - b(X'_{s_k})| dr \right] \mathcal{F}_{s_k}
\]

\[
\leq |\nabla h|_\infty \int_{s_k}^{s_{k+1}} \left\{ \mathbb{E} \left[ K_1|Y'_{r} - \bar{Y}'_{r,k}|^2 \right] \mathcal{F}_{s_k} \right\}^{\frac{1}{2}} + \sqrt{K_1} \mathcal{W}_1 \left( P^{X'_{s_k}}(Y'_{s_k}, \cdot), \pi^{X'_{s_k}} \right) dr
\]

\[
\leq |\nabla h|_\infty \int_{s_k}^{s_{k+1}} \left\{ C_\delta \sqrt{K_1} e^{\frac{C(\tau-s_k)}{\varepsilon^2}} + \sqrt{K_1} \kappa_2 e^{-\lambda_2 \frac{r-s_k}{\varepsilon^2}} \right\} dr
\]

\[
\leq |\nabla h|_\infty \frac{C_\delta^2}{\sqrt{\varepsilon^2}} + |\nabla h|_\infty \kappa_2 \sqrt{K_1} \int_0^\delta e^{-\lambda_2 \tau} dr.
\]

Other details are omitted to save space. Then this theorem can be proved. \(\square\)

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