Abstract

We associate to an algebraic quantum group a C*-algebraic quantum group and show that this C*-algebraic quantum group essentially satisfies an upcoming definition of Masuda, Nakagami & Woronowicz.

Introduction.

In 1979, Woronowicz proposed the use of the C*-algebra language in the field of quantum groups [27]. Since then, a lot of work has been done in this area but there is still no satisfactory definition of a quantum group in the C*-algebra framework.

However, the following subjects are better understood:

- compact & discrete quantum groups and their duality theory
- some examples: quantum SU(2), quantum Heisenberg, quantum E(2), quantum Lorentz, duals of locally compact groups,...

A good definition of a quantum group should satisfy the following requirements:

- It should incorporate all the known examples and well understood parts of the theory.
- There will have to be a good balance between the theory which can be extracted from the definition and the non complexity of the definition.
- The definition has to allow a consistent duality theory.

The most difficult part of finding a satisfactory axiom scheme for quantum groups seems to be the ability to prove the existence and uniqueness of a left Haar weight from the proposed definition.

At the moment, Masuda, Nakagami and Woronowicz are working on a quasi-final definition of a quantum group in the C*-algebra framework (see also [10]). It is quasi-final in the sense that there is some hope to find simpler axioms which imply their current definition. Later, we will give a preview of this definition.

In [22], A. Van Daele introduced the notion of Multiplier Hopf-algebras. They are natural generalizations of Hopf algebras to the case of non-unital algebras. Recently, A. Van Daele has looked into the case where such a Multiplier Hopf algebra possesses a non-zero left invariant functional (Haar functional) and found some very interesting properties [18]. It should be noted that everything in this theory is of an algebraic nature.

This category of Multiplier Hopf algebras with a Haar functional behaves very well in different ways:

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• This category includes the compact & discrete quantum groups.
• It is possible to construct the dual within this category.
• The category is closed under the double construction of Drinfeld (see [5]).

The last and the first property imply that this category contains the Lorentz quantum group [12]. However, this new category will not exhaust all quantum groups. It is not so difficult to find many classical groups which are not Multiplier Hopf algebras. Also, quantum E(2) will not fit in this scheme. Nevertheless, we still have a nice class of algebraic quantum groups.

The main purpose of this paper is the construction of a C∗-algebraic quantum group (in the sense of Masuda, Nakagami & Woronowicz) out of a Multiplier Hopf∗-algebra which possesses a positive left invariant functional. In a first section, we wil give an overview of the results of A. Van Daele about such Multiplier Hopf∗-algebras. In a second section, we introduce the C∗-algebra together with the comultiplication. From there on, we gradually prove that this C∗-algebra fits almost in the scheme of Masuda, Nakagami & Woronowicz.

First, we introduce some notations and conventions. We will always use the minimal tensor product between C∗-algebras and use the symbol ⊗ for this complete tensor product. For any C∗-algebra A, we denote the Multiplier C∗-algebra by M(A). The flip map between two C∗-algebras will be denoted by χ.

For the algebraic tensorproducts of vectorspaces and linear mappings, we use the symbol ⊙. The algebraic dual of a vectorspace V will be denoted by V′.

Let H be a Hilbert space. Then B(H) will denote the C∗-algebra of bounded operators on H, whereas B₀(H) will denote the C∗-algebra of compact operators on H. Consider vectors v, w ∈ H, then ωᵥ,w is the element in B₀(H)* such that ωᵥ,w(x) = ⟨xv, w⟩ for all x ∈ B₀(H).

The domain of an unbounded operator T on H is denoted by D(T).

The domain of an element α which is affiliated with some C∗-algebra, will be denoted by D(α) (we will use the same notation for closed mappings in a C∗-algebra which arise from one-parameter groups).

Whenever we say that an unbounded operator is positive, it is included that this operator is also self-adjoint. The same rules apply to elements affiliated with a C∗-algebra. Let α be an element affiliated with a C∗-algebra A, then α is called strictly positive if α is positive and has dense range (it is then automatically injective).

A one-parameter group σ on a C∗-algebra A is called norm-continuous if and only if for every a ∈ A, the mapping ℜ→ A : t ↦→ σₜ(a) is norm-continuous.

We refer to the appendix for some notations and results about weights.

As promised, we give now a preview of the definition of a C∗-algebraic quantum group according to Masuda, Nakagami & Woronowicz:

Let B be a C∗-algebra and Δ a non-degenerate ∗-homomorphism from B into M(B ⊗ B) such that

1. Δ is coassociative, i.e. (Δ ⊗ ε)Δ = (ε ⊗ Δ)Δ.
2. Δ satisfies the following density conditions: Δ(B)(B ⊗ 1) and Δ(B)(1 ⊗ B) are dense subsets of B ⊗ B.

Furthermore, we assume the existence of the following objects:

1. a KMS-weight ϕ on B with modular group σ,
2. a norm continuous one parameter group τ on B,
3. an involutive ∗-anti-automorphism R on B,
which satisfy the following properties:

1. For every $a \in M_\varphi$, we have that $\Delta(a)$ belongs to $\overline{M_\varphi \otimes \varphi}$ and $(i \otimes \varphi)\Delta(a) = \varphi(a)1$.

2. Consider $a, b \in N_\varphi$. Let $\omega \in B^*$ such that $\omega R\tau - i$ is bounded and call $\theta$ the unique element in $B^*$ which extends $\omega R\tau - i$. Then

$$\varphi((b^* (\omega \otimes i)\Delta(a))) = \varphi((\theta \otimes i)(\Delta(b^*))a).$$

3. • $\varphi$ is invariant under $\tau$.
   • $\varphi$ commutes with $\varphi R$.

4. • For every $t \in \mathbb{R}$, we have that $R\tau_t = \tau_t R$.
   • We have that $\Delta\tau_t = (\tau_t \otimes \tau_t)\Delta$ for all $t \in \mathbb{R}$.
   • $\Delta R = \chi(R \otimes R)\Delta$

Then, we call $(B, \Delta, \varphi, \tau, R)$ a $C^*$-algebraic quantum group.

We call $\varphi$ the left Haar weight, $R$ the anti-unitary antipode and $\tau$ the scaling group of our quantum group. We put $\kappa = R\tau - i$, then $\kappa$ plays the role of the antipode of our quantum group.

This definition seems to be a $C^*$-version of a definition of a quantum group in the von Neumann algebra setting (see [10]), which in turn was a generalization of the (too restrictive) definition of a Kac-algebra (see [8]). We are not sure that this will be the ultimate definition of a $C^*$-algebraic quantum group proposed by Masuda, Nakagami & Woronowicz, but we expect that this one gives a fairly good idea of it.

A possible drawback of this definition is the complexity of the axioms. However, we will show that the $C^*$-algebraic versions of Van Daele’s algebraic objects fit almost in this scheme. The only difference lies in the fact that we only can prove that $\varphi$ is relatively invariant with respect to $\tau$ in stead of invariant.

It is not clear at the moment whether this definition of Masuda, Nakagami & Woronowicz should be modified in this respect.

1 Algebraic quantum groups.

In this first section, we will introduce the notion of an algebraic quantum group as can be found in [18]. Moreover, we will give an overview of the properties of this algebraic quantum group. The proofs of these results can be found in the same paper [18]. After this section, we will construct a $C^*$-algebraic quantum group out of this algebraic one, thereby heavily depending on the material gathered in this section. We will first introduce some terminology.

We call a $*-\text{algebra}$ $A$ non-degenerate if and only if we have for every $a \in A$ that:

$$(\forall b \in A : ab = 0) \Rightarrow a = 0 \quad \text{and} \quad (\forall b \in A : ba = 0) \Rightarrow a = 0.$$

For a non-degenerate $*-\text{algebra}$ $A$, you can define the multiplier algebra $M(A)$, this is a unital $*-\text{algebra}$ in which $A$ sits as a selfadjoint ideal (the definition of this multiplier algebra is the same as in the case of $C^*$-algebras).

If you have two non-degenerate $*-\text{algebras}$ $A, B$ and a multiplicative linear mapping $\pi$ from $A$ to $M(B)$, we call $\pi$ non-degenerate if and only if the vectorspaces $\pi(A)B$ and $B\pi(A)$ are equal to $B$. Such a non-degenerate multiplicative linear map has a unique multiplicative linear extension to $M(A)$, this extension will be denoted by the same symbol as the original mapping. Of course, we have similar definitions and
results for antimultiplicative mappings. If we work in an algebraic setting, we will always use this form of non degeneracy as opposed to the non degeneracy of \(^*-\)homomorphisms between \(C^*\)-algebras!

For a linear functional \(\omega\) on a non-degenerate \(^*-\)algebra \(A\) and any \(a \in M(A)\) we define the linear functionals \(\omega a\) and \(a\omega\) on \(A\) such that \((a\omega)(x) = \omega(ax)\) and \((\omega a)(x) = \omega(ax)\) for every \(x \in A\).

You can find some more information about non-degenerate algebras in the appendix of [22].

Now, let \(\omega\) be a linear functional on a \(^*-\)algebra \(A\), then:

1. \(\omega\) is called positive if and only if \(\omega(a^*a)\) is positive for every \(a \in A\).
2. We say that \(\omega\) is faithful if and only if for every \(a \in A\), we have that 
   \[
   (\forall b \in A : \omega(ab) = 0) \Rightarrow a = 0 \quad \text{and} \quad (\forall b \in A : \omega(ba) = 0) \Rightarrow a = 0.
   \]
3. If \(\omega\) is positive, then \(\omega\) is faithful if and only if for every \(a \in A\), we have that
   \[
   \omega(a^*a) = 0 \Rightarrow a = 0.
   \]

Let \(\omega\) be a positive linear functional on a \(^*-\)algebra \(A\). A GNS-pair for \(\omega\) is by definition a pair \((K, \Gamma)\), where \(K\) is a Hilbert space and \(\Gamma\) is a linear map from \(A\) into \(K\) such that \(\Gamma(A)\) is dense in \(K\) and 
\[
\langle \Gamma(a), \Gamma(b) \rangle = \omega(b^*a)
\]
for every \(a, b \in A\). It is clear that such a pair exist and that it is unique up to a unitary transformation.

We have now gathered the necessary information to understand the following definition

**Definition 1.1** Consider a non-degenerate \(^*-\)algebra \(A\) and a non-degenerate \(^*-\) homomorphism \(\Delta\) from \(A\) into \(M(A \odot A)\) such that 

1. \((\Delta \odot \epsilon)\Delta = (\epsilon \odot \Delta)\Delta\).
2. The linear mappings \(T_1, T_2\) from \(A \odot A\) into \(M(A \odot A)\) such that
   \[
   T_1(a \otimes b) = \Delta(a)(b \otimes 1) \quad \text{and} \quad T_2(a \otimes b) = \Delta(a)(1 \otimes b)
   \]
   for all \(a, b \in A\), are bijections from \(A \odot A\) to \(A \odot A\).

Then we call \((A, \Delta)\) a Multiplier Hopf\(^*-\)-algebra.

In [22], A. Van Daele proves the existence of a unique non-zero \(^*-\) homomorphism \(\epsilon\) from \(A\) to \(\mathbb{C}\) such that 
\[
\langle \epsilon \odot \iota \rangle \Delta = (\iota \odot \epsilon)\Delta = \iota.
\]
Furthermore, he proves the existence of a unique anti-automorphism \(S\) on \(A\) such that
\[
m(S \odot \iota)((\Delta(a)(b \otimes 1))) = \epsilon(a)b \quad \text{and} \quad m(\iota \odot S)((b \otimes 1)\Delta(a)) = \epsilon(a)b
\]
for every \(a, b \in A\) (here, \(m\) denotes the multiplication map from \(A \odot A\) to \(A\)). As usual, \(\epsilon\) is called the counit and \(S\) the antipode of \((A, \Delta)\). Moreover, 
\[
S(S(a^*)) = a \quad \text{for all} \quad a \in A.
\]
Also, \(\chi(S \odot S)\Delta = \Delta S\).

Let \(\omega\) be a linear functional on \(A\). We call \(\omega\) left invariant (with respect to \((A, \Delta)\)) if and only if 
\[
(\iota \odot \omega)(\Delta(a)(b \otimes 1)) = \omega(ab)
\]
for every \(a, b \in A\). Right invariance is defined in a similar way.

**Definition 1.2** Consider a Multiplier Hopf\(^*-\)-algebra \((A, \Delta)\) such that there exists a non-zero positive linear functional \(\varphi\) on \(A\) which is left invariant. Then we call \((A, \Delta)\) an algebraic quantum group.

For the rest of this paper, we will fix an algebraic quantum group \((A, \Delta)\) together with a non-zero left invariant positive linear functional \(\varphi\) on it.

An important feature of such an algebraic quantum group is the faithfulness and unicity of left invariant functionals:
1. Consider a left invariant linear functional $\omega$ on $A$, then there exists a unique element $c \in \mathbb{C}$ such that $\omega = c\varphi$.

2. Consider a non-zero left invariant linear functional $\omega$ on $A$, then $\omega$ is faithful.

In particular, $\varphi$ is faithful.

A first application of this unicity result concerns the antipode: Because $\varphi S^2$ is left invariant, there exists a unique complex number $\mu$ such that $\varphi S^2 = \mu \varphi$ (in [18], our $\mu$ is denoted by $\tau$!). It is not so difficult to prove in an algebraic way that $|\mu| = 1$. The question remains open if there exists an example of an algebraic quantum group (in our sense) with $\mu \neq 1$.

It is clear that $\varphi S$ is a non-zero right invariant linear functional on $A$. However, in general, $\varphi S$ will not be positive. Later, we will use our C*-algebra approach to prove the existence of a non-zero positive right invariant linear functional on $A$.

Of course, we have similar faithfulness and uniqueness results about right invariant linear functionals.

Another non-trivial property about $\varphi$ is the existence of a unique automorphism $\rho$ on $A$ such that $\varphi(ab) = \varphi(b\rho(a))$ for every $a, b \in A$. We call this the weak KMS-property of $\varphi$ (In [18], our mapping $\rho$ is denoted by $\sigma$!). This weak KMS-property will be crucial to extend $\varphi$ to a weight on the C*-algebra level.

Moreover, we have that $\rho(\rho(a^*)^*) = a$ for every $a \in A$.

As usual there exists a similar object $\rho'$ for the right invariant functional $\varphi S$, i.e. $\rho'$ is an automorphism on $A$ such that $(\varphi S)(ab) = (\varphi S)(b\rho'(a))$ for every $a, b \in A$.

Using the antipode, we can connect $\rho$ and $\rho'$ via the formula $S\rho' = \rho S$. Furthermore, we have that $S^2$ commutes with $\rho$ and $\rho'$. The interplay between $\rho, \rho'$ and $\Delta$ is given by the following formulas:

$$\Delta \rho = (S^2 \circ \rho) \Delta \quad \text{and} \quad \Delta \rho' = (\rho' \circ S^{-2}) \Delta.$$

It is also possible to introduce the modular function of our algebraic quantum group. This is an invertible element $\delta$ in $M(A)$ such that $(\varphi \otimes \iota)(\Delta(a)(1 \otimes b)) = \varphi(a) \delta b$ for every $a, b \in A$.

Concerning the right invariant functional, we have that $(\iota \otimes \varphi S)(\Delta(a)(b \otimes 1)) = (\varphi S)(a) \delta^{-1} b$ for every $a, b \in A$.

This modular function is, like in the classical group case, a one dimensional (generally unbounded) corepresentation of our algebraic quantum group:

$$\Delta(\delta) = \delta \otimes \delta \quad \varepsilon(\delta) = 1 \quad S(\delta) = \delta^{-1}.$$

As in the classical case, we can relate the left invariant functional to our right invariant functional via the modular function: for every $a \in A$, we have that

$$\varphi(S(a)) = \varphi(a\delta) = \mu \varphi(\delta a).$$

Not surprisingly, we have also that $\rho(\delta) = \rho'(\delta) = \mu^{-1}\delta$.

Another connection between $\rho$ and $\rho'$ is given by the equality $\rho'(a) = \delta \rho(a)\delta^{-1}$ for all $a \in A$.

We have also a property which says, loosely speaking, that every element of $A$ has compact support:

Consider $a_1, \ldots, a_n \in A$. Then there exists an element $c$ in $A$ such that $ca_i = a_i c = a_i$ for every $i \in \{1, \ldots, n\}$.

In a last part, we are going to say something about duality.

We define the subspace $\hat{A}$ of $A'$ as follows:

$$\hat{A} = \{\varphi a \mid a \in A\} = \{a\varphi \mid a \in A\}.$$

Like in the theory of Hopf*-algebras, we turn $\hat{A}$ into a non-degenerate *-algebra:
1. For every \( \omega_1, \omega_2 \in \hat{A} \) and \( a \in A \), we have that \( (\omega_1 \omega_2)(a) = (\omega_1 \odot \omega_2)(\Delta(a)) \).

2. For every \( \omega \in \hat{A} \) and \( a \in A \), we have that \( \omega^*(a) = \overline{\omega(S(a)^*)} \).

We should remark that a little bit of care has to be taken by defining the product and the \(*\)-operation in this way.

Also, a comultiplication \( \Delta \) can be defined on \( \hat{A} \) such that \( \Delta(\omega)(x \otimes y) = \omega(xy) \) for every \( \omega \in \hat{A} \) and \( x, y \in A \).

Again, this has to be made more precise. This can be done by embedding \( M(A \odot A) \) into \( (A \odot A)' \) in the right way but we will not go into this subject. A definition of the comultiplication \( \hat{\Delta} \) without the use of such an embedding can be found in definition 4.4 of [18].

In this way, \( (\hat{A}, \hat{\Delta}) \) becomes a Multiplier Hopf*-algebra. The counit \( \hat{\varepsilon} \) and the antipode \( \hat{S} \) are such that

1. For every \( \omega \in \hat{A} \), we have that \( \hat{\varepsilon}(\omega) = \omega(1) \).

2. For every \( \omega \in \hat{A} \) and every \( a \in A \), we have that \( \hat{S}(\omega)(a) = \omega(S(a)) \).

For any \( a \in A \), we define \( \hat{a} = a\varphi \in \hat{A} \). The mapping \( A \to \hat{A} : a \mapsto \hat{a} \) is a bijection, which is in fact nothing else but the Fourier transform.

Next, we define the linear functional \( \hat{\psi} \) on \( \hat{A} \) such that \( \hat{\psi}(\hat{a}) = \varepsilon(a) \) for every \( a \in A \). It is possible to prove that \( \hat{\psi} \) is right invariant.

Also, we have that \( \hat{\psi}(a^*\hat{a}) = \varphi(a^*a) \) for every \( a \in A \). This implies that \( \hat{\psi} \) is a non-zero positive right invariant linear functional on \( \hat{A} \).

Later, we will also prove that \( A \) possesses a non-zero positive right invariant linear functional. In a similar way, this functional will give rise to a non-zero positive left invariant linear functional on \( \hat{A} \). This will imply that \( (\hat{A}, \hat{\Delta}) \) is again an algebraic quantum group.

## 2 The reduced bi-C*-algebra.

In the rest of this paper we are going to construct gradually a C*-algebraic quantum group in the sense of Masuda, Nakagami & Woronowicz out of our algebraic quantum group \( (A, \Delta) \). In this second section, we are first going to construct the C*-algebra together with the comultiplication. In fact, we will always be working with the reduced C*-algebraic quantum group, because all our constructions are on the GNS-space of our left invariant functional \( \varphi \) on \( A \). It should be noted however that it is also possible to construct a universal C*-algebraic quantum group out of our algebraic one. This section revolves around a multiplicative unitary like in the case of Kac-algebras and the work of Baaj & Skandalis [18]. Therefore, the definitions and results appearing in this section will resemble the ones in those works.

For the rest of the paper, we fix a GNS-pair \((H, \Lambda)\) for \( \varphi \). Note that \( \Lambda \) is injective because \( \varphi \) is faithful.

**Proposition 2.1** There exists a unique unitary operator \( W \) on \( H \otimes H \) such that \( W(\Lambda(a)\Lambda(b)(a \otimes 1)) = \Lambda(a) \otimes \Lambda(b) \) for all \( a, b \in A \).

The proof of this proposition is straightforward. The left invariance of \( \varphi \) implies that

\[
\langle (\Lambda \odot \Lambda)(\Delta(b)(a \otimes 1)), (\Lambda \odot \Lambda)(\Delta(d)(c \otimes 1)) \rangle = \langle \Lambda(a) \otimes \Lambda(b), \Lambda(c) \otimes \Lambda(d) \rangle
\]

for all \( a, b, c, d \in A \). Remembering that \( \Delta(A)(A \otimes 1) = A \otimes A \), the result follows easily.

There is yet another way of expressing \( W \):

**Proposition 2.2** For all \( a, b \in A \), we have that \( W(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \odot \Lambda)((S^{-1} \odot \iota)(\Delta(b))(a \otimes 1)) \).
We define the mapping \( \pi \) by \[ \pi = \sum_{i=1}^{n} S(p_i) \otimes q_i = (S(a) \otimes 1) \Delta(b). \]

We would like to represent \( A \) by left multiplication as operators on \( H \). In general, this would result in unbounded operators. However, in the case of our left invariant functional \( \varphi \), we get bounded operators. In order to prove this, we will need the following short lemma.

\textbf{Lemma 2.3} If every \( a, b, c \in A \), we have that 
\[ \lambda = (\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(W) = \Lambda(\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a))c. \]

\textbf{Proof:} For all \( d \in A \), we have that 
\[ \lambda = (\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(W) = \Lambda(\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a))c. \]

By the previous lemma, this lemma is true for every element of the form \( x = (\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a)) \) with \( a, b \in A \). Because every element in \( A \) is a sum of such elements, our lemma is true for all \( x \in A \). This allows us to introduce the GNS-representation \( \varphi \) of \( \Lambda \):

\textbf{Definition 2.5} We define the mapping \( \pi \) from \( A \) into \( B(H) \) such that \( \pi(a) \Lambda(b) = \Lambda(ab) \) for all \( a, b \in A \). Then \( \pi \) is an injective \( \star \)-homomorphism such that \( \pi(A)H \) is dense in \( H \).

The last statement of this definition follows easily from the fact that \( A^2 = A \). Furthermore, using lemma \[ \textbf{2.3} \] we have for all \( a, b \in A \):

\[ \pi((\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a))) = (\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(W) \]

Now, we are in a position to define our reduced C*-algebra \( A_r \) associated to \( (A, \Delta) \).
Definition 2.6 We define $A_r$ as the closure of $\pi(A)$ in $B(H)$. So, $A_r$ is a non-degenerate sub-$C^*$-algebra of $B(H)$.

Equation 2 implies that

$$A_r = \text{closure of } \{(\iota \otimes \omega)(W) \mid \omega \in B_0(H)^*\} \text{ in } B(H),$$

which is something familiar to us.

We will define our comultiplication on the $C^*$-algebra level. It will be denoted by the same symbol as the comultiplication on the $*$-algebra level but it will always be clear from the context which comultiplication is under consideration.

Definition 2.7 We define the mapping $\Delta$ from $A_r$ into $B(H \otimes H)$ such that $\Delta(x) = W^*(1 \otimes x)W$ for all $x \in A_r$. Then $\Delta$ is an injective $^*$-homomorphism.

The next lemma guarantees that the comultiplication on $A_r$ is an extension of the comultiplication on $A$.

Lemma 2.8 For any $a \in A$ and $x \in A \otimes A$, we have that $(\pi \otimes \pi)(x) \Delta(\pi(a)) = (\pi \otimes \pi)(x \Delta(a))$ and $\Delta(\pi(a)) (\pi \otimes \pi)(x) = (\pi \otimes \pi)(\Delta(a)x)$.

Proof: Choose $b, c \in A$, then

$$(\pi \otimes \pi)(x) W^* (1 \otimes \pi(a)) (\Lambda(b) \otimes \Lambda(c)) = (\pi \otimes \pi)(x) W^* (\Lambda(b) \otimes \Lambda(ac))$$
$$= (\pi \otimes \pi)(x) (\Lambda \otimes \Lambda)(\Delta(ac)(b \otimes 1)) = (\Lambda \otimes \Lambda)(x \Delta(ac)(b \otimes 1))$$
$$= (\Lambda \otimes \Lambda)(x \Delta(a)) \Delta(c)(b \otimes 1) = (\pi \otimes \pi)(x \Delta(a)) (\Lambda \otimes \Lambda)(\Delta(c)(b \otimes 1))$$
$$= (\pi \otimes \pi)(x \Delta(a)) W^* (\Lambda(b) \otimes \Lambda(c)).$$

Therefore,

$$(\pi \otimes \pi)(x) W^* (1 \otimes \pi(a)) = (\pi \otimes \pi)(x \Delta(a)) W^*,$$

so

$$(\pi \otimes \pi)(x) \Delta(\pi(a)) = (\pi \otimes \pi)(x) W^* (1 \otimes \pi(a)) W = (\pi \otimes \pi)(x \Delta(a)).$$

The other equality can be proven in a similar way.

Using this lemma, it is easy to infer formulas like:

$$\Delta(\pi(a))(1 \otimes \pi(b)) = (\pi \otimes \pi)(\Delta(a)(1 \otimes b))$$

for all $a, b \in A$. Hence, we can conclude the following.

Lemma 2.9 We have that $\Delta$ is a non-degenerate $^*$-homomorphism from $A_r$ into $M(A_r \otimes A_r)$ such that $\Delta(A_r)(A_r \otimes 1)$ and $\Delta(A_r)(1 \otimes A_r)$ are dense subsets of $A_r \otimes A_r$.

The coassociativity on the $*$-algebra level is transferred to the coassociativity on the $C^*$-algebra level.

Lemma 2.10 We have that $\Delta$ is coassociative on the $C^*$-algebra level.
Proof: Choose \( a \in A \). For any \( b, c \in A \), we have that

\[
(\Delta \otimes \iota)\Delta (\pi(a)) (\pi(b) \otimes 1 \otimes \pi(c)) = (\Delta \otimes \iota)(\Delta (\pi(a))(1 \otimes \pi(c))) (\pi(b) \otimes 1 \otimes 1)
\]

\[
= (\Delta \otimes \iota)((\pi \otimes \pi)(\Delta(a)(1 \otimes c))) (\pi(b) \otimes 1 \otimes 1)
\]

\[
= (\pi \otimes \pi \otimes \pi)((\Delta \otimes \iota)(\Delta(a)(1 \otimes c))(b \otimes 1 \otimes 1))
\]

\[
= (\pi \otimes \pi \otimes \pi)((i \otimes \Delta)(\Delta(a)(b \otimes 1))(1 \otimes 1 \otimes c))
\]

\[
= (i \otimes \Delta)(\Delta(\pi(a))(\pi(b) \otimes 1)) (1 \otimes 1 \otimes \pi(c))
\]

\[
= (i \otimes \Delta)\Delta(\pi(a)) (\pi(b) \otimes 1 \otimes \pi(c)).
\]

The nondegeneracy of \( A_r \) implies that \( (\Delta \otimes \iota)\Delta(\pi(a)) = (i \otimes \Delta)\Delta(\pi(a)) \). Because \( \pi(A) \) is dense in \( A_r \), we must have that \( (\Delta \otimes \iota)\Delta = (i \otimes \Delta)\Delta \).

In the next theorem, we gather all these results.

**Theorem 2.11** We have that \( A_r \) is a non-degenerate sub-C*-algebra of \( B(H) \) and \( \Delta \) is a non-degenerate injective *-homomorphism from \( A_r \) to \( M(A_r \otimes A_r) \) such that:

1. \( (\Delta \otimes \iota)\Delta = (i \otimes \Delta)\Delta \)

2. The vectorspaces \( \Delta(A_r)(A_r \otimes 1) \) and \( \Delta(A_r)(1 \otimes A_r) \) are dense subsets of \( A_r \otimes A_r \).

In fact, this theorem says that \( (A_r, \Delta) \) satisfies the first part of the definition of Masuda, Nakagami & Woronowicz.

The coassociativity on the *-algebra level implies also easily the next proposition.

**Proposition 2.12** We have that \( W \) is multiplicative: \( W_{12}W_{13}W_{23} = W_{23}W_{12} \).

In a second part of this section, we will also represent the dual Multiplier Hopf*-algebra \( \hat{A} \) on \( H \). This will be done in a similar way as before and therefore the proofs will be left out.

Remember from section 1 that we have a non-zero positive right invariant linear functional \( \hat{\psi} \) on \( \hat{A} \). Moreover, \( \hat{\psi}(b^*a) = \phi(b^*a) \) for all \( a, b \in A \) (a).

Next, we define the linear map \( \hat{\Lambda} \) from \( \hat{A} \) into \( H \) such that \( \Lambda(\hat{a}) = \Lambda(a) \) for every \( a \in A \). By using equality (a), we see that \( (H, \hat{\Lambda}) \) is a GNS-pair for \( \hat{\psi} \).

In a first lemma, we want to prove an expression for \( W \) in terms of our new GNS-pair.

In the proof of this lemma, we will need the following formula:

\[
(i \otimes \phi)((1 \otimes a)\Delta(b)) = S((i \otimes \phi)(\Delta(a)(1 \otimes b)))
\]

(3)

for all \( a, b \in A \). A proof of this result can be found in proposition 3.11 of [8]. It is in fact nothing else but an algebraic form of the strong left invariance in the definition of Masuda, Nakagami & Woronowicz.

**Lemma 2.13** For every \( \omega_1, \omega_2 \in \hat{A} \), we have that \( W(\Lambda(\omega_1) \otimes \hat{\Lambda}(\omega_2)) = (\hat{\Lambda} \otimes \Lambda)(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)) \)

Proof: There exist \( a, b \in A \) such that \( \omega_1 = \hat{a} \) and \( \omega_2 = \hat{b} \).

Choose \( x, y \in A \), then we have that

\[
((S^{-1} \otimes \iota)(\Delta(b))(a \otimes 1))(x \otimes y) = (\phi \otimes \phi)((x \otimes y)(S^{-1} \otimes \iota)(\Delta(b))(a \otimes 1))
\]

\[
\phi(x (i \otimes \phi)((1 \otimes y)(S^{-1} \otimes \iota)(\Delta(b))(a) = \phi(x S^{-1}(i \otimes \phi)((1 \otimes y)(\Delta(b))(a),
\]
Theorem 2.18 We have that
\[(\hat{\mathcal{L}}(\omega_1)(1\otimes\omega_2))(x\otimes y)\]
which, by definition 4.4 of [18], equals
\[((\hat{\mathcal{L}}(\omega_1)(1\otimes\omega_2))(x\otimes y)\]
It follows that\[((S^{-1}\circ\iota)(\Delta(b))(a\otimes1))^\sim = \hat{\mathcal{L}}(\omega_1)(1\otimes\omega_2)\]
Using proposition 2.2, we see that
\[\hat{\mathcal{L}}(\omega_1)(1\otimes\omega_2)\]
It is also interesting to prove another formula about \(\hat{\mathcal{L}}\)
Using proposition 2.2, we see that
\[\hat{\mathcal{L}}(\omega_1)(1\otimes\omega_2)\]
With this expression, we can do the same things for the dual \(\hat{\mathcal{L}}\) as we did for \(\mathcal{L}\) itself.

**Definition 2.14** We define the mapping \(\hat{\pi}\) from \(\hat{\mathcal{L}}\) into \(B(H)\) such that \(\hat{\pi}(\omega)\hat{\mathcal{L}}(\theta) = \hat{\mathcal{L}}(\omega\theta)\) for all \(\omega,\theta \in \hat{\mathcal{L}}\). Then \(\hat{\pi}\) is an injective \(*\)-homomorphism such that \(\hat{\pi}(\hat{\mathcal{L}})H\) is dense in \(H\).

Moreover, for every \(\theta, \eta \in \hat{\mathcal{L}}\), we have that
\[(\omega_{\hat{\mathcal{L}}(\theta),\hat{\mathcal{L}}(\eta)} \otimes \iota)(W) = \hat{\pi}(\hat{\psi} \circ \iota)((\eta^* \otimes 1)\hat{\mathcal{L}}(\theta)). \quad (4)\]

**Definition 2.15** We define \(\hat{\mathcal{L}}_r\) as the closure of \(\hat{\pi}(\hat{\mathcal{L}})\) in \(B(H)\). So, \(\hat{\mathcal{L}}_r\) is a non-degenerate sub-C*-algebra of \(B(H)\).

Equation 4 implies that
\[\hat{\mathcal{L}}_r = \text{ closure of } \{\omega \otimes \iota(W) \mid \omega \in B_0(H)^*\} \text{ in } B(H),\]
which is again something familiar.

**Definition 2.16** We define \(\hat{\mathcal{L}}_r\) as the closure of \(\hat{\pi}(\hat{\mathcal{L}})\) in \(B(H)\). So, \(\hat{\mathcal{L}}_r\) is a non-degenerate sub-C*-algebra of \(B(H)\).

**Lemma 2.17** For any \(\omega \in \hat{\mathcal{L}}\) and \(\theta \in \hat{\mathcal{L}} \circ \hat{\mathcal{L}}\), we have that \((\hat{\pi} \circ \hat{\pi})(\theta)\hat{\mathcal{L}}(\omega) = (\hat{\pi} \circ \hat{\pi})(\theta\hat{\mathcal{L}}(\omega))\) and \(\hat{\mathcal{L}}(\hat{\pi}(\omega))(\hat{\pi} \circ \hat{\pi})(\theta) = (\hat{\pi} \circ \hat{\pi})(\hat{\mathcal{L}}(\omega)\theta)\).

**Theorem 2.18** We have that \(\hat{\mathcal{L}}_r\) is a non-degenerate sub-C*-algebra of \(B(H)\) and \(\hat{\mathcal{L}}_r\) is a non-degenerate injective \(*\)-homomorphism from \(\hat{\mathcal{L}}_r\) to \(M(\hat{\mathcal{L}}_r \circ \hat{\mathcal{L}}_r)\) such that:
1. \((\hat{\mathcal{L}}_r \circ \iota)\hat{\mathcal{L}}_r = (\iota \circ \hat{\mathcal{L}}_r)\hat{\mathcal{L}}_r\)
2. The vector spaces \(\hat{\mathcal{L}}(\hat{\mathcal{L}}_r)(\hat{\mathcal{L}}_r \otimes 1)\) and \(\hat{\mathcal{L}}(\hat{\mathcal{L}}_r)(1 \otimes \hat{\mathcal{L}}_r)\) are dense subsets of \(\hat{\mathcal{L}}_r \circ \hat{\mathcal{L}}_r\).

It is also interesting to prove another formula about \(\hat{\pi}\).

**Lemma 2.19** For every \(\omega \in \hat{\mathcal{L}}\), we have that \(\hat{\pi}(\omega)\Lambda(x) = \Lambda((\hat{S}^{-1}(\omega) \circ \iota)\Delta(x))\)
Proof: Choose \( a \in A \). Then
\[
(\omega \hat{x})(a) = (\omega(x \varphi))(a) = (\omega \odot \varphi)((\Delta(a)(1 \otimes x)) = \omega((\iota \odot \varphi)((\Delta(a)(1 \otimes x)))
\]

By using equation 3, we see that
\[
(\omega \hat{x})(a) = \omega(S^{-1}((\iota \odot \varphi)((1 \otimes a)\Delta(x)))) = \hat{S}^{-1}(\omega)((\iota \odot \varphi)((1 \otimes a)\Delta(x)))
\]
\[
= \varphi(\hat{S}^{-1}(\omega) \odot \iota)((1 \otimes a)\Delta(x)) = \varphi(a(\hat{S}^{-1}(\omega) \odot \iota)\Delta(x))
\]
\[
= \varphi((\hat{S}^{-1}(\omega) \odot \iota)\Delta(x))(a).
\]

So, \( \omega \hat{x} = (\hat{S}^{-1}(\omega) \odot \iota)\Delta(x) \). Hence,
\[
\hat{\pi}(\omega)\Lambda(x) = \hat{\pi}(\omega)(\hat{x}) = \Lambda(\omega \hat{x}) = \Lambda((\hat{S}^{-1}(\omega) \odot \iota)\Delta(x))
\]


\[\blacksquare\]

Lemma 2.20 For every \( a, b \in A \) we have that \( (\omega_{\Lambda(a),\Lambda(b)} \odot \iota)(W) = \hat{\pi}(a \varphi b^*) \).

Proof: Choose \( y \in A \), then
\[
\langle (\omega_{\Lambda(a),\Lambda(b)} \odot \iota)(W) \rangle \Lambda(\hat{x}), \Lambda(y) \rangle = \langle (\omega_{\Lambda(a),\Lambda(b)} \odot \iota)(W) \rangle \Lambda(x), \Lambda(y) \rangle
\]
\[
= \langle W(\Lambda(a) \odot \Lambda(x)), \Lambda(b) \odot \Lambda(y) \rangle
\]

Therefore, proposition 2.2 implies that
\[
\langle (\omega_{\Lambda(a),\Lambda(b)} \odot \iota)(W) \rangle \Lambda(\hat{x}), \Lambda(y) \rangle = \langle (\Lambda \odot \Lambda)((S^{-1}(\iota)(\Delta(x))(a \otimes 1)), \Lambda(b) \odot \Lambda(y) \rangle.
\]

By using this equality, we see that
\[
\langle (\omega_{\Lambda(a),\Lambda(b)} \odot \iota)(W) \rangle \Lambda(\hat{x}), \Lambda(y) \rangle = \varphi \odot \varphi((b^* \otimes y^*)(S^{-1}(\iota)(\Delta(x))(a \otimes 1))
\]
\[
= (a \varphi b^* \odot \varphi)((S^{-1}(\iota)(1 \otimes y^*)\Delta(x))) = (\hat{S}^{-1}(a \varphi b^*) \odot \varphi)((1 \otimes y^*)\Delta(x))
\]
\[
= \varphi((\hat{S}^{-1}(a \varphi b^*) \odot \iota)((1 \otimes y^*)\Delta(x))) = \varphi(y^*(\hat{S}^{-1}(a \varphi b^*) \odot \iota)\Delta(x))
\]
\[
= \langle \Lambda((\hat{S}^{-1}(a \varphi b^*) \odot \iota)\Delta(x)), \Lambda(y) \rangle = \langle \Lambda((a \varphi b^*) \hat{x}), \Lambda(y) \rangle,
\]

where we used the previous lemma in the last step.

\[\blacksquare\]

In the last part of this section, we want to prove that \( W \) belongs to \( M(A_r \odot \hat{A}_r) \).

Remembering that \( A \) is a subset of the \( A' \), we can identify \( A \odot \hat{A} \) with a subspace of the vector space of linear mappings from \( A \) into \( A \) in the natural way.

Lemma 2.21 Let \( \omega \in \hat{A} \) and \( a \in A \). Define the linear mapping \( F \) from \( A \) into \( A \) such that
\[
F(x) = (\iota \odot \omega)(\Delta(x)(a \otimes 1)).
\]
Then \( F \) belongs to \( A \odot \hat{A} \) and \( W(\pi \odot \hat{\pi})(a \odot \omega) = (\pi \odot \hat{\pi})(F) \).

Proof: There exist \( b \in A \) such that \( \omega = b \varphi \). Moreover, there exist \( p_1, \ldots, p_n \), \( q_1, \ldots, q_n \in A \) such that
\[
a \otimes b = \sum_{i=1}^n \Delta(p_i)(q_i \otimes 1).
\]

Now, we have for every \( x \in A \) that
\[
F(x) = (\iota \odot \varphi)(\Delta(x)(a \otimes b)) = \sum_{i=1}^n (\iota \odot \varphi)(\Delta(xp_i)(q_i \otimes 1)) = \sum_{i=1}^n q_i \varphi(xp_i).
\]
Therefore, $F = \sum_{i=1}^{n} q_i \otimes p_i \varphi$, which clearly belongs to $A \otimes \hat{A}$.
Because $a \otimes b = \sum_{i=1}^{n} \Delta(p_i)(q_i \otimes 1)$ and $\chi(S \otimes S) \Delta = \Delta S$, we have that
\[
S(b) \otimes S(a) = \sum_{i=1}^{n} (1 \otimes S(q_i)) \Delta(S(p_i)). \quad (a)
\]
Choose $c, d \in A$. Using first proposition 2.19 and then proposition 2.2, we see that
\[
W(\pi \circ \hat{\pi})(a \otimes \omega)(\Lambda(c) \otimes \Lambda(d)) = W(\Lambda(ac) \otimes \Lambda((S^{-1}(\omega) \circ i) \Delta(d))) \\
= (\Lambda \circ \Lambda)((S^{-1} \circ i)((S^{-1}(\omega) \circ i) \Delta(d))) (ac \otimes 1) \\
= (\Lambda \circ \Lambda)((S^{-1} \circ i)((S(ac) \otimes 1)(\Delta \circ i) \Delta(d))) . \quad (b)
\]
Furthermore,
\[
(S(ac) \otimes 1) \Delta((S^{-1}(\omega) \circ i) \Delta(d)) = (S(ac) \otimes 1) \Delta((\omega S^{-1} \circ i) \Delta(d)) \\
= (S(ac) \otimes 1) \Delta((\varphi S^{-1} \circ i)((S(b) \otimes 1) \Delta(d))) \\
= (\varphi S^{-1} \circ i)(1 \otimes S(ac) \otimes 1)(\Delta \circ i)((S(b) \otimes 1) \Delta(d)) \\
= (\varphi S^{-1} \circ i)(S(b) \otimes S(c) S(a) \otimes 1)(\Delta \circ i) \Delta(d) \\
= (\varphi S^{-1} \circ i)(S(b) \otimes S(c) S(a) \otimes 1)(\Delta \circ i) \Delta(d). \quad (c)
\]
Refering to (a), we see that
\[
(S(ac) \otimes 1) \Delta((S^{-1}(\omega) \circ i) \Delta(d)) \\
= \sum_{i=1}^{n} (\varphi S^{-1} \circ i)(1 \otimes S(c) S(q_i) \otimes 1)(\Delta(S(p_i)) \otimes 1)(\Delta \circ i) \Delta(d) \\
= \sum_{i=1}^{n} (\varphi S^{-1} \circ i)(1 \otimes S(q_i c) \otimes 1)(\Delta \circ i)((S(p_i) \otimes 1) \Delta(d)) \\
= \sum_{i=1}^{n} S(q_i c) \otimes (\varphi S^{-1} \circ i)((S(p_i) \otimes 1) \Delta(d)) \\
= \sum_{i=1}^{n} S(q_i c) \otimes (S^{-1}(p_i \varphi) \circ i)(\Delta(d)),
\]
where in step (c), we used the right invariance of $\varphi S^{-1}$.
Combining this result with equation (b), we get that
\[
W(\pi \circ \hat{\pi})(a \otimes \omega)(\Lambda(c) \otimes \Lambda(d)) = \sum_{i=1}^{n} \Lambda(q_i c) \otimes \Lambda((S^{-1}(p_i \varphi) \circ i) \Delta(d)) \\
= \sum_{i=1}^{n} \pi(q_i) \Lambda(c) \otimes \hat{\pi}(p_i \varphi) \Lambda(d) = (\pi \circ \hat{\pi})(F)(\Lambda(c) \otimes \Lambda(d)).
\]
Hence, $W(\pi \circ \hat{\pi})(a \otimes \omega) = (\pi \circ \hat{\pi})(F).$

**Proposition 2.22** We have that $W(\pi(A) \circ \hat{\pi}(\hat{A})) = \pi(A) \circ \hat{\pi}(\hat{A})$.

**Proof:** From the previous lemma, it follows that $W(\pi(A) \circ \hat{\pi}(\hat{A})) \subseteq \pi(A) \circ \hat{\pi}(\hat{A})$.
Now, we prove the other inclusion. Choose $a, b \in A$. Then there exist $p_1, \ldots, p_n, q_1, \ldots, q_n \in A$ such that
\[
\sum_{i=1}^{n} p_i \otimes q_i = \Delta(b)(a \otimes 1).
\]
For every $x \in A$, we have that
\[
\sum_{i=1}^{n} (\iota \circ q_{i}\varphi)(\Delta(x)(p_{i} \circ 1)) = \sum_{i=1}^{n} (\iota \circ \varphi)(\Delta(x)(p_{i} \circ q_{i})) \\
= \sum_{i=1}^{n} (\iota \circ \varphi)(\Delta(xb)(a \circ 1)) = a \varphi(xb) = (a \otimes b\varphi)(x).
\]

By using 2.21, we see that
\[
W(\pi \circ \hat{\pi})(\sum_{i=1}^{n} p_{i} \otimes q_{i}\varphi) = \pi(a) \otimes \hat{\pi}(b\varphi).
\]

Proposition 2.23 We have that $W$ belongs to $M(A_{r} \otimes \hat{A}_{r})$.

Proof: From the previous lemma, we can conclude that $W(A_{r} \otimes \hat{A}_{r}) = A_{r} \otimes \hat{A}_{r}$. If we multiply with $W^{*}$ to the left, we get $A_{r} \otimes \hat{A}_{r} = W^{*}(A_{r} \otimes \hat{A}_{r})$. Taking the adjoint of this equation, gives $A_{r} \otimes \hat{A}_{r} = (A_{r} \otimes \hat{A}_{r})W$. Hence, $W$ belongs to $M(A_{r} \otimes \hat{A}_{r})$.

3 The modular group of the left Haar weight.

In this section, we want to introduce a norm continuous one-parameter group $\sigma$ on $A_{r}$ which, in a later stage, will play the role of the modular group of the left Haar weight. We will define this one-parameter group with the use of the theory of left Hilbert algebras. However, the one-parameter group which we get in this way has initially two shortcomings:

- $\sigma$ leaves only the von Neumann algebra $A_{r}''$ invariant, where we want $\sigma$ to leave the $C^{*}$-algebra $A_{r}$ invariant.

- $\sigma$ is strongly continuous, we would like to have that $\sigma$ is norm continuous.

The main part of this section consists of showing that $\sigma$ does not have this shortcomings. For this purpose, we will have to depend on the quantum group structure. In a last part of this section, we prove that $\pi(A)$ consists of elements which are analytic with respect to $\sigma$ and we show that $\rho$ is the restriction of $\sigma_{-1}$ to $A$.

First, we introduce a natural left Hilbert algebra $U$.

Definition 3.1 We define $U = \Lambda(A)$, so $U$ is a dense subspace of $H$. We will make $U$ into a $^{*}$-algebra in the following way.

1. For all $a, b \in A$, we have that $\Lambda(a)\Lambda(b) = \Lambda(ab)$.

2. For all $a \in A$, we have that $\Lambda(a)^{*} = \Lambda(a^{*})$.

Proposition 3.2 We have that $U$ is a left Hilbert algebra on $H$.

Proof:

1. Because $A^{2} = A$, $U^{2} = U$. So, $U^{2}$ is dense in $H$.

2. Choose $a \in A$. It follows immediately that $\Lambda(a)v = \pi(a)v$ for all $v \in U$. Hence, the mapping $U \rightarrow U : v \mapsto \Lambda(a)v$ is bounded.
We also define $\nabla$ and that $U$. Then $Tv$.

Later in this section, we will define our one-parameter group $\sigma$.

Lemma 3.3

Let $a \in A$.

1. $\varphi(a^*da)$ is positive

2. $\varphi(a^*da) = 0$ if and only if $a = 0$.

Proof:

1. Choose $a \in A$. There exist $b \in A$ such that $\varphi(b^*b) = 1$, then $\delta = (\varphi \circ \iota)\Delta(b^*b)$. So,

   $$\varphi(a^*da) = \varphi(a^*(\varphi \circ \iota)(\Delta(b^*b)) a) = (\varphi \circ \varphi)((1 \otimes a^*)\Delta(b^*b)(1 \otimes a))$$

   $$= ((\Lambda \otimes \Lambda)(\Delta(b)(1 \otimes a)), (\Lambda \otimes \Lambda)(\Delta(b)(1 \otimes a)))$$

   Hence, $\varphi(a^*da)$ is positive.

2. Assume that $a \in A$ and that $\varphi(a^*da) = 0$. By using the Schwarz equality for positive sesquilinear forms, it follows for every $b \in A$ that

   $$\varphi(b^*da) \leq \varphi(b^*db) \varphi(a^*da),$$

   therefore $\varphi(b^*da) = 0$. Because $\varphi$ is faithful, $\delta a = 0$. The invertibility of $\delta$ implies that $a = 0$.

This lemma allows us to define a Hilbert space $H_\delta$ together with an injective linear map $\Lambda_\delta$ from $A$ to $H_\delta$ such that
1. $\Lambda_{\delta}$ has dense range in $H_{\delta}$
2. $\langle \Lambda_{\delta}(a), \Lambda_{\delta}(b) \rangle = \varphi(b^*\delta a)$ for all $a, b \in A$.

We will use this new Hilbert space to construct a useful operator.

**Lemma 3.4** Let $a, b \in A$, then $\langle \Lambda_{\delta}(S(a^*)), \Lambda_{\delta}(b) \rangle = \langle \Lambda(\delta^{-1}S(b)^*\delta), \Lambda(a) \rangle$.

**Proof:** We have that

$$\langle \Lambda_{\delta}(S(a^*)), \Lambda_{\delta}(b) \rangle = \varphi(b^*\delta S(a^*)) = \varphi(S(a^*\delta^{-1}S(b)^*))$$

From section 1, we know that $\varphi S = \delta \varphi$, hence

$$\langle \Lambda_{\delta}(S(a^*)), \Lambda_{\delta}(b) \rangle = \varphi(a^*\delta^{-1}S(b)^*\delta) = \langle \Lambda(\delta^{-1}S(b)^*\delta), \Lambda(a) \rangle.$$  

Because $\Lambda_{\delta}(A)$ is dense in $H_{\delta}$, this lemma justifies the following definition.

**Definition 3.5** We define the closed antilinear mapping $E$ from within $H$ into $H_{\delta}$ such that $\Lambda(A)$ is a core for $E$ and $E\Lambda(a) = \Lambda_{\delta}(S(a^*))$ for all $a \in A$.

It is clear that $E$ is a densely defined injective operator which has a dense range.

The previous lemma implies easily that $\langle Ev, \Lambda_{\delta}(a) \rangle = \langle \Lambda(\delta^{-1}S(a^*)\delta), v \rangle$ for all $a \in A$ and $v \in D(E)$. In fact, you can use this equality to prove that $E$ is injective. Also, we can infer from this lemma that $\Lambda_{\delta}(A)$ is a subset of $D(E^*)$ and that

$$E^*\Lambda_{\delta}(a) = \Lambda(\delta^{-1}S(a)^*\delta) \quad (5)$$

for all $a \in A$.

The following operator plays a crucial role in the rest of this section.

**Definition 3.6** We define $P = E^*E$. Then $P$ is an injective positive operator in $H$.

From definition 3.5 and equation 3, it follows that $\Lambda(A)$ is a subset of $D(P)$ and that $PA(a) = \Lambda(\delta^{-1}S^{-2}(a)\delta)$ for every $a \in A$.

Now, we want to prove some commutation relations involving $W$, $P$ and $\nabla$. The technique that we will use, will be helpful to us in several situations. First, we prove the following frequently used lemma.

**Lemma 3.7** Consider Hilbert spaces $K, L, H_1, H_2$, a unitary operator $U$ from $K$ to $H_2$ and a unitary operator $V$ from $H_1$ to $L$, $F$ a closed linear operator from within $K$ into $H_1$, $G$ a closed linear operator from within $H_2$ into $L$.

Suppose there exists a core $C$ for $F$ such that $U(C)$ is a core for $G$ and such that $V(F(v)) = G(U(v))$ for every $v \in C$. Then we have that $VF = GU$.

**Proof:**

- Choose $v \in D(F)$. Then there exists a sequence $(v_n)_{n=1}^{\infty}$ in $C$ such that $(v_n)_{n=1}^{\infty} \to v$ and $(F(v_n))_{n=1}^{\infty} \to F(v)$.

For every $n \in \mathbb{N}$, we have that $U(v_n)$ belongs to $D(G)$ and $G(U(v_n)) = V(F(v_n))$. So, $(U(v_n))_{n=1}^{\infty} \to U(v)$ and $(G(U(v_n)))_{n=1}^{\infty} \to V(F(v))$.

Because $G$ is closed, $U(v)$ must belong to the domain of $G$ and $G(U(v)) = V(F(v))$. 

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Choose \( v \in K \) such that \( U(v) \) belongs to \( D(G) \). Then there exists a sequence \( (v_n)_{n=1}^{\infty} \) in \( C \) such that \( (U(v_n))_{n=1}^{\infty} \to U(v) \) and \( (G(U(v_n)))_{n=1}^{\infty} \to G(U(v)) \).

For every \( n \in \mathbb{N} \), we have that \( V(F(v_n)) = G(U(v_n)) \). So, \( (v_n)_{n=1}^{\infty} \to v \) and \( (F(v_n))_{n=1}^{\infty} \to V^*(G(U(v))) \). Because \( F \) is closed, we must have that \( v \) belongs to the domain of \( F \) and \( F(v) = V^*(G(U(v))) \), hence \( V(F(v)) = G(U(v)) \).

\[ \]

**Lemma 3.8** Let \( a, b, c, d \in A \), then

\[
\langle (\Lambda_\delta \odot \Lambda_\delta)(\chi(\Delta(b))(a \otimes 1) ), (\Lambda_\delta \odot \Lambda_\delta)(\chi(\Delta(d))(c \otimes 1) ) \rangle \\
= \langle (\Lambda_\delta(a) \otimes \Lambda_\delta(b), \Lambda_\delta(c) \otimes \Lambda_\delta(d) \rangle.
\]

**Proof:** We have that

\[
\langle (\Lambda_\delta \odot \Lambda_\delta)(\chi(\Delta(b))(a \otimes 1) ), (\Lambda_\delta \odot \Lambda_\delta)(\chi(\Delta(d))(c \otimes 1) ) \rangle \\
= (\varphi \circ \varphi)((c^* \otimes 1)\chi(\Delta(d^*))\delta \otimes \delta)(\chi(\Delta(b))(a \otimes 1) ) \\
= (\varphi \circ \varphi)((1 \otimes c^*)\Delta(d^*)(\delta \otimes \delta)\Delta(b)(1 \otimes a)).
\]

If we use first that \( \Delta(\delta) = \delta \otimes \delta \) and then the fact that \( (\varphi \circ \iota)\Delta(x) = \varphi(x)\delta \) for all \( x \in A \), we get that

\[
\langle (\Lambda_\delta \odot \Lambda_\delta)(\chi(\Delta(b))(a \otimes 1) ), (\Lambda_\delta \odot \Lambda_\delta)(\chi(\Delta(d))(c \otimes 1) ) \rangle \\
= (\varphi \circ \varphi)((1 \otimes c^*)\Delta(d^*)\delta(b)(1 \otimes a)) = \varphi(c^*\delta a) \varphi(d^*\delta b) \\
= (\Lambda_\delta(a) \otimes \Lambda_\delta(b), \Lambda_\delta(c) \otimes \Lambda_\delta(d) \rangle.
\]

\[ \]

Now, we are able to prove a first commutation relation.

**Proposition 3.9** We have that \( (P \otimes P) W = W (P \otimes P) \).

**Proof:** From the previous lemma, it follows that there exists a unique unitary operator \( V \) on \( H_\delta \otimes H_\delta \) such that

\[
V (\Lambda_\delta(a) \otimes \Lambda_\delta(b)) = (\Lambda_\delta \odot \Lambda_\delta)(\chi(\Delta(b))(a \otimes 1) ).
\]

We know that \( W^*(\Lambda(A) \odot \Lambda(A)) = \Lambda(A) \odot \Lambda(A) \). Hence, \( \Lambda(A) \odot \Lambda(A) \) and \( W^*(\Lambda(A) \odot \Lambda(A)) \) are cores for \( E \otimes E \).

Choose \( a, b \in A \). Then

\[
(E \otimes E)W^* (\Lambda(a) \otimes \Lambda(b)) = (E \otimes E)(\Lambda(a) \otimes \Lambda(b))(a \otimes 1) \\
= (\Lambda_\delta \odot \Lambda_\delta)((S \otimes S)(\Delta(b)(a \otimes 1))^*) \\
= (\Lambda_\delta \odot \Lambda_\delta)((S \otimes S)(\Delta(b^*))(S(a^*) \otimes 1)) \\
= (\Lambda_\delta \odot \Lambda_\delta)((\chi(\Delta(b^*)))(S(a^*) \otimes 1)) \\
= V (\Lambda_\delta \odot \Lambda_\delta)(S(a^*) \otimes S(b^*)) = V(E \otimes E)(\Lambda(a) \otimes \Lambda(b)).
\]

Using lemma 3.7, we get that \( (E \otimes E)W^* = V(E \otimes E) \).
By taking the adjoint of this equation, we infer that \( W(E^* \otimes E^*) = (E^* \otimes E^*)V^* \). By using this two equalities, we conclude that
\[
W(P \otimes P) = W(E^* \otimes E^*)(E \otimes E) = (E^* \otimes E^*)V^*(E \otimes E)
\]
\[
= (E^* \otimes E^*)(E \otimes E)W = (P \otimes P)W.
\]

In order to prove another commutation relation, we will first need a little result in the algebraic case.

**Lemma 3.10** We have that \( (\rho^{-1} \circ \rho')\Delta = \Delta S^{-2} \).

**Proof:** Choose \( a \in A \). From section 1, we know that \( \rho'(b) = \delta \rho(b) \delta^{-1} \) for all \( b \in A \). It follows easily, using the fact that \( \rho(\delta) = \frac{1}{\mu} \delta \), that \( \rho^{-1}(b) = \delta \rho'(b) \delta^{-1} \) for all \( b \in A \).

We use these two equations to conclude that
\[
(\rho^{-1} \circ \rho')\Delta(a) = (\delta \otimes \delta)(\rho'(a))(\Delta(a))\delta \otimes \delta. \quad (b)
\]

From section 1, we know that
\[
(\iota \circ \rho)\Delta = (S^{-2} \circ \iota)\Delta \rho \quad \text{and} \quad (\rho^{-1} \circ \iota)\Delta = (\iota \circ S^{-2})\Delta \rho^{-1}.
\]

We also have that \( S^{-2} \) commutes with \( \rho \) and \( \rho' \). Using all this information, equation (b) implies that
\[
(\rho^{-1} \circ \rho')\Delta(a) = (\delta \otimes \delta)(S^{-2} \circ S^{-2})(\Delta(\rho(\rho^{-1}(a))))(\delta \otimes \delta).
\]

Equation (a) implies that \( \rho(\rho^{-1}(a)) = \delta^{-1}a \delta \). So, remembering that \( \Delta(\delta) = \delta \otimes \delta \) and \( S^{-2}(\delta) = \delta \), we see that
\[
(\rho^{-1} \circ \rho')\Delta(a) = (S^{-2} \circ S^{-2})\Delta(a),
\]

which equals \( \Delta(S^{-2}(a)) \).

**Lemma 3.11** Consider \( a, b, c, d \in A \), then
\[
\langle (A \otimes A)((\iota \circ S)((a \otimes 1)\Delta(b))) \rangle, (A \otimes A)((\iota \circ S)((c \otimes 1)\Delta(d))) \rangle = (A(a) \otimes A(b), A(c) \otimes A(d)).
\]

**Proof:** We have that
\[
\langle (A \otimes A)((\iota \circ S)((a \otimes 1)\Delta(b))) \rangle, (A \otimes A)((\iota \circ S)((c \otimes 1)\Delta(d))) \rangle
\]
\[
= (\varphi \circ \varphi)((\iota \circ S)((c \otimes 1)\Delta(d)))(1 \otimes \delta)(\iota \circ S)((a \otimes 1)\Delta(b)))
\]
\[
= (\varphi \circ \varphi)((\iota \circ S^{-1})(\Delta(d^*))((c^* \otimes 1))(1 \otimes \delta)(\iota \circ S)((a \otimes 1)\Delta(b)))
\]
\[
= (\varphi \circ \varphi)((\iota \circ S^{-1})(\Delta(d^*))((c^* a \otimes \delta)(\iota \circ S)(\Delta(b)))
\]

From this equation, we get by the definition of \( \rho \) that
\[
\langle (A \otimes A)((\iota \circ S)((a \otimes 1)\Delta(b))) \rangle, (A \otimes A)((\iota \circ S)((c \otimes 1)\Delta(d))) \rangle
\]
\[
= (\varphi \circ \varphi)((\rho^{-1} \circ \rho^{-1})((\iota \circ S)\Delta(b))(\iota \circ S^{-1})(\Delta(d^*))((c^* a \otimes \delta))
\]

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From section 1, we know that $\rho^{-1}S = S\rho'$. Using this fact and the previous lemma, we get that

$$
\rho
\begin{aligned}
\langle (\Lambda \otimes \Lambda_b)((\iota \circ S)((a \circ 1)\Delta(b))) , (\Lambda \otimes \Lambda_b)((\iota \circ S)((c \circ 1)\Delta(d))) \rangle
&= (\varphi \circ \varphi)((\iota \circ S)((\rho^{-1} \circ \rho')\Delta(b))(\iota \circ S^{-1})(\Delta(d^*))((\rho^* a \circ \delta)) \\
&= (\varphi \circ \varphi)((S^{-2} \circ S^{-1})(\Delta(b))(\iota \circ S^{-1})(\Delta(d^*))((\rho^* a \circ \delta^{-1} \circ 1)(\delta \circ \delta)).
\end{aligned}
$$

So, remembering that $\varphi S = \delta \varphi$, this equation implies that

$$
\rho
\begin{aligned}
\langle (\Lambda \otimes \Lambda_b)((\iota \circ S)((a \circ 1)\Delta(b))) , (\Lambda \otimes \Lambda_b)((\iota \circ S)((c \circ 1)\Delta(d))) \rangle
&= (\varphi \circ \varphi)((S(c^* a \delta^{-1}) \circ 1)(S \circ c)(\Delta(d^*))(S^{-1} \circ c)(\Delta(b))) \\
&= (\varphi \circ \varphi)((\delta S(a)S^{-1}(c)^* \circ 1)(S^{-1} \circ c)(\Delta(d))^*(S^{-1} \circ c)(\Delta(b))) \\
&= (\varphi \circ \varphi)((S^{-1}(c)^* \circ 1)(S^{-1} \circ c)(\Delta(d))^*(S^{-1} \circ c)(\Delta(b))(\rho(\delta S(a)) \otimes 1)) \\
&= (\varphi \circ \varphi)((S^{-1}(c)^* \circ 1)(\Delta(d))(S^{-1}(c \circ 1))^*[S^{-1} \circ c)(\Delta(b))(\rho(\delta S(a)) \otimes 1])
\end{aligned}
$$

Because $W$ is unitary and because of proposition 3.12, this last term equals

$$
(\varphi \circ \varphi)((S^{-1}(c) \circ d)^*(\rho(\delta S(a)) \otimes b))
$$

Hence,

$$
\rho
\begin{aligned}
\langle (\Lambda \otimes \Lambda_b)((\iota \circ S)((a \circ 1)\Delta(b))) , (\Lambda \otimes \Lambda_b)((\iota \circ S)((c \circ 1)\Delta(d))) \rangle
&= \varphi(S^{-1}(c)^* \rho(\delta S(a)))(\varphi(d^* b) = \varphi(\delta S(a)S(c^*)) \varphi(d^* b) \\
&= \varphi(S(c^* a \delta^{-1})) \varphi(d^* b).
\end{aligned}
$$

Because $\varphi S = \delta \varphi$, we have that

$$
\rho
\begin{aligned}
\langle (\Lambda \otimes \Lambda_b)((\iota \circ S)((a \circ 1)\Delta(b))) , (\Lambda \otimes \Lambda_b)((\iota \circ S)((c \circ 1)\Delta(d))) \rangle
&= \varphi(c^* a) \varphi(d^* b) = \langle \Lambda(a) \otimes \Lambda(b) , \Lambda(c) \otimes \Lambda(d) \rangle.
\end{aligned}
$$

We are now able to prove the second commutation relation.

**Proposition 3.12** We have that $(\nabla \otimes \nabla) W = W(\nabla \otimes P)$.

**Proof:** Because of the previous lemma, there exists a unique unitary operator from $H \otimes H$ to $H \otimes H_5$ such that

$$
V(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda_b)((\iota \circ S)((a \circ 1)\Delta(b))).
$$

Again, $\Lambda(A) \otimes \Lambda(A)$ is a core for $T \otimes T$ and $W^*(\Lambda(A) \otimes \Lambda(A))$ is a core for $T \otimes E$.

Choose $a, b \in A$. Then

$$
(T \otimes E) W^*(\Lambda(a) \otimes \Lambda(b)) = (T \otimes E)(\Lambda \otimes \Lambda)((\Delta(b))(a \circ 1))
$$

$$
= (\Lambda \otimes \Lambda_b)((\iota \circ S)([\Delta(b)(a \circ 1)]^*)) = (\Lambda \otimes \Lambda_b)((\iota \circ S)((a^* \circ 1)\Delta(b^*)))
$$

$$
= V(\Lambda(\Lambda_b)(a^* \circ b^*) = V(T \otimes T)(\Lambda(a) \otimes \Lambda(b)).
$$

Using once again lemma 3.13, we see that $(T \otimes E) W^* = V(T \otimes T)$. Just like in proposition 3.13, we deduce that $(\nabla \otimes \nabla) W = W(\nabla \otimes P)$. $
$
Consider $\omega \in B_0(H)^*$ and $s \in \mathbb{R}$. We use the preceding proposition to infer that
\[(\nabla^{is} \otimes 1)W(\nabla^{-is} \otimes 1) = (1 \otimes \nabla^{-is})W(1 \otimes P^{is}).\]
Applying $t \otimes \omega$ to this equality gives us
\[\nabla^{is}(t \otimes \omega)(W)\nabla^{-is} = (t \otimes P^{is} \omega \nabla^{-is})(W).\]
Hence,

1. For all $t \in \mathbb{R}$, we have that $\nabla^{it}(t \otimes \omega)(W)\nabla^{-it}$ belongs to $A_r$.
2. The function $\mathbb{R} \to A_r : t \mapsto \nabla^{it}(t \otimes \omega)(W)\nabla^{-it}$ is norm continuous.

Using these results and formula 2 of section 2, we are able to conclude for any $x \in A_r$,

1. For all $t \in \mathbb{R}$, we have that $\nabla^{it}x\nabla^{-it}$ belongs to $A_r$.
2. The function $\mathbb{R} \to A_r : t \mapsto \nabla^{it}x\nabla^{-it}$ is norm continuous.

This discussion justifies the following definition.

**Definition 3.13** We define the norm-continuous one-parameter group $\sigma$ on $A_r$ such that $\sigma_t(x) = \nabla^{it}x\nabla^{-it}$ for all $t \in \mathbb{R}$ and $x \in A_r$.

Remember from the preceding discussion that
\[\sigma_t((t \otimes \omega)(W)) = (t \otimes P^{it} \omega \nabla^{-it})(W)\]
for every $t \in \mathbb{R}$ and $\omega \in B_0(H)^*$.

By proposition 3.9, we can justify in a similar way as before the following definition.

**Definition 3.14** We define the norm-continuous one-parameter group $K$ on $A_r$ such that $K_t(x) = P^{it}xP^{-it}$ for all $t \in \mathbb{R}$ and $x \in A_r$.

Just as before, we have that
\[K_t((t \otimes \omega)(W)) = (t \otimes P^{it} \omega P^{-it})(W)\]
for every $t \in \mathbb{R}$ and $\omega \in B_0(H)^*$.

The proof of the following proposition is now rather straightforward.

**Proposition 3.15** For every $t \in \mathbb{R}$, we have that $(\sigma_t \otimes K_t)\Delta = \Delta \sigma_t$.

**Proof**: Choose $x \in A_r$. Then
\[
\Delta(\sigma_t(x)) = W^*(1 \otimes \sigma_t(x))W = W^*(1 \otimes \nabla^{it}x\nabla^{-it})W = W^*(\nabla^{it} \otimes \nabla^{-it})(1 \otimes x)(\nabla^{-it} \otimes \nabla^{-it})W = (\nabla^{it} \otimes P^{it})W^*(1 \otimes x)W(\nabla^{-it} \otimes P^{-it}) = (\nabla^{it} \otimes P^{it})\Delta(x)(\nabla^{-it} \otimes P^{-it}) = (\sigma_t \otimes K_t)\Delta(x).
\]
In a next step, we want to prove that for every element $a \in A$, we have that $\pi(a)$ belongs to the domain of $D(\sigma_{-1})$ and $\sigma_{-1}(\pi(a)) = \pi(\rho(a))$. Because $\sigma$ is implemented by $\nabla$, we have to prove that $\pi(a)\nabla^{-1} \subseteq \nabla^{-1}\pi(\rho(a))$.

We know that

$$\pi(a)\nabla^{-1}\Lambda(b) = \pi(a)\Lambda(\rho^{-1}(b)) = \Lambda(a\rho^{-1}(b)) = \Lambda(\rho^{-1}(a\rho(b))) = \nabla^{-1}\Lambda(\rho(a)b) = \nabla^{-1}\pi(\rho(a))\Lambda(b).$$

This gives an indication that the above claim is true, but we cannot conclude from this that $\pi(a)\nabla^{-1} \subseteq \nabla^{-1}\pi(\rho(a))$ because $\Lambda(A)$ does not have to be a core for $\nabla$. Therefore, we have to prove it in another way. First, we will prove a small result in left Hilbert algebra theory.

**Lemma 3.16** Consider $v, w \in \mathcal{U}' \cap D(\nabla)$ such that $\nabla(v), \nabla(w)$ belong to $\mathcal{U}'$. Then $vw$ belongs to $D(\nabla)$ and $\nabla(vw) = \nabla(v)\nabla(w)$.

**Proof:** Because $\nabla = T^*T$, $v, w$ belong to $D(T)$, $T(v), T(w)$ belong to $D(T^*)$ and $T^*(T(w)) = \nabla(v)$ and $T^*(T(w)) = \nabla(w)$.

The fact that $T(w)$ belongs to $D(T^*)$ and $T^*(T(w)) = \nabla(w)$ implies that $T(w)$, $\nabla(w)$ are right closable and $R_{\nabla(w)} \subseteq R_{T(w)}$. Remark also that $\mathcal{U}'$ is a subset of $D(R_{\nabla(w)}), D(R_{T(w)})$.

Choose $u \in \mathcal{U}'$. Then

$$\langle T(u), T(vw) \rangle = \langle u^*, (vw)^* \rangle = \langle u^*, w^*v^* \rangle$$

$$= \langle uw^*, v^* \rangle = \langle T(uw^*), T(v) \rangle$$

$$= \langle T^*(T(v)), uw^* \rangle = \langle \nabla(v), uw^* \rangle = \langle \nabla(v), R_{T(w)}(u) \rangle$$

$$= \langle R_{T(w)}(\nabla(v)), u \rangle = \langle R_{\nabla(w)}(\nabla(v)), u \rangle = \langle \nabla(v)\nabla(w), u \rangle.$$

Because $\mathcal{U}'$ is a core for $T$, we have for all $u \in D(T)$ that $\langle T(u), T(vw) \rangle = \langle \nabla(v)\nabla(w), u \rangle$. Hence, $vw$ belongs to $D(\nabla)$ and $\nabla(vw) = \nabla(v)\nabla(w)$. So $vw$ belongs to $D(\nabla)$ and $\nabla(vw) = \nabla(v)\nabla(w)$.

We can even have weaker assumptions:

**Lemma 3.17** Consider $v \in \mathcal{U}' \cap D(\nabla)$ such that $\nabla(v)$ belongs to $\mathcal{U}'$ and consider $w \in D(\nabla)$. Then $vw$ belongs to $D(\nabla)$ and $\nabla(vw) = \nabla(v)\nabla(w)$.

**Proof:** Put $C = \{u \in \mathcal{U}' \cap D(\nabla) \mid \nabla(u) \in \mathcal{U}'\}$, because the Tomita algebra is a core for $\nabla$, $C$ must be a core for $\nabla$. Hence, there exists a sequence $(u_n)_{n=1}^\infty$ in $C$ such that $(u_n)_{n=1}^\infty \to w$ and $(\nabla(u_n))_{n=1}^\infty$. We know by the previous lemma that $vu_n \in D(\nabla)$ and $\nabla(vu_n) = \nabla(v)\nabla(u_n)$ for all $n \in \mathbb{N}$. Because $v$ and $\nabla(v)$ are left bounded, we get that $(vu_n)_{n=1}^\infty \to vu$ and $(\nabla(vu_n))_{n=1}^\infty \to \nabla(v)\nabla(u)$. The closedness of $\nabla$ implies that $vu$ belongs to $D(\nabla)$ and $\nabla(vu) = \nabla(v)\nabla(u)$.

The next lemma is an easy consequence of the previous one.

**Lemma 3.18** Consider $v \in \mathcal{U}' \cap D(\nabla^{-1})$ such that $\nabla^{-1}(v)$ belongs to $\mathcal{U}'$ and consider $w \in D(\nabla^{-1})$. Then $vw$ belongs to $D(\nabla^{-1})$ and $\nabla^{-1}(vw) = \nabla^{-1}(v)\nabla^{-1}(w)$.

We formulate these two lemmas in another way which is more useful to us.

**Lemma 3.19** 1. Consider $v \in \mathcal{U}' \cap D(\nabla)$ such that $\nabla(v) \in \mathcal{U}'$.

Then $L_{\nabla(v)} \nabla \subseteq \nabla L_v$. 20
2. Consider \( v \in \mathcal{U}'' \cap D(\nabla^{-1}) \) such that \( \nabla^{-1}(v) \in \mathcal{U}'' \).
Then \( L_{\nabla^{-1}(v)} \nabla^{-1} \subseteq \nabla^{-1} L_v \).

Now, we are in a position to use these results in our case.

**Proposition 3.20** Consider \( a \in A \). Then \( \pi(a) \nabla \subseteq \nabla \pi(\rho^{-1}(a)) \) and \( \pi(a) \nabla^{-1} \subseteq \nabla^{-1} \pi(\rho(a)) \).

**Proof:** We know that \( \Lambda(\rho^{-1}(a)) \) belongs to \( \mathcal{U}'' \cap D(\nabla) \) and \( \nabla \Lambda(\rho^{-1}(a)) = \Lambda(a) \), which also belongs to \( \mathcal{U}'' \). Hence, by the previous proposition,
\[
\pi(a) \nabla = L_{\Lambda(a)} \nabla = L_{\nabla \Lambda(\rho^{-1}(a))} \nabla \subseteq \nabla L_{\nabla \Lambda(\rho^{-1}(a))} = \nabla \pi(\rho^{-1}(a)).
\]
The other equality is proven in a similar way. \( \blacksquare \)

**Corollary 3.21** Consider \( a \in A \) and \( n \in \mathbb{Z} \). Then \( \pi(a) \nabla^n \subseteq \nabla^n \pi(\rho^{-n}(a)) \).

Remembering that \( \sigma \) is implemented by \( \nabla \) and using the previous corollary, we get the following

**Proposition 3.22** Consider \( a \in A \). Then \( \pi(a) \) is an analytic element for \( \sigma \) and \( \sigma_n(\pi(a)) = \pi(\rho^{-n}(a)) \) for all \( n \in \mathbb{Z} \).

4 A construction procedure connected with the polar decomposition of the antipode.

In this section, we will introduce a fairly general construction procedure which in later sections provides a tool for making a polar decomposition of the antipode. We will use this procedure in different cases to get different implementations of the scaling group and anti-unitary antipode. The techniques used in this section resemble the ones we used for the construction of the modular group \( \sigma \).

We consider a positive linear functional \( \eta \) on \( A \) such that there exist an invertible element \( x \in M(A) \) such that \( \eta(a) = \varphi(S(a)x) \) for all \( a \in A \). Furthermore, we take an invertible element \( y \in M(A) \). Let \((K, \Gamma)\) be a GNS-pair of \( \eta \).

**Lemma 4.1** Consider \( a, b \in A \). Then \( \langle \Gamma(S(a)^* y), \Gamma(b) \rangle = \langle \Lambda(S(b)^* x \rho(S(y))), \Lambda(a) \rangle \).

**Proof:** We have that
\[
\langle \Gamma(S(a)^* y), \Gamma(b) \rangle = \eta(b^* S(a)^* y) = \eta(b^* S^{-1}(a^*) y) = \eta(S^{-1}(S(y)a^* S(b^*)) = \varphi(S(y)a^* S(b^*) x) = \varphi(a^* S(b^*) x \rho(S(y))) = \langle \Lambda(S(b)^* x \rho(S(y))), \Lambda(a) \rangle.
\]

Because, by definition, \( \Gamma(A) \) is dense in \( K \), the following definition is justified.

**Definition 4.2** We define \( G \) as the closed antilinear map from within \( H \) into \( K \) such that \( \Lambda(A) \) is a core for \( G \) and such that \( G \Lambda(a) = \Gamma(S(a)^* y) \) for every \( a \in A \).

It is clear that \( G \) is a densely defined injective operator which has dense range.
Again, it follows easily that
\[ \langle Gv, \Gamma(b) \rangle = \langle \Lambda(S(b^*) x \rho(S(y))), v \rangle \]
for every $v \in D(G)$. So $\Gamma(A)$ is a subset of $D(G^*)$ and $G^* \Gamma(b) = \Lambda(S(b^*) x \rho(S(y)))$ for all $b \in A$.

**Corollary 4.3** We have that $\Gamma(A)$ is a core for $G^{-1}$ and that $G^{-1} \Gamma(a) = \Lambda(S(a)^* S(y^{-1})^*)$ for all $a \in A$.

**Proof:** Because $\Lambda(A)$ is a core for $G$, we have that $GA(A)$ is a core for $G^{-1}$. It is clear that $GA(A) = \Gamma(A)$, so $\Gamma(A)$ is a core for $G^{-1}$. The formula for $G^{-1}$ is easy to prove. \qed

Now, we want to make a polar decomposition of $G$. We define $\mathcal{J}$ as the anti-unitary operator from $H$ to $K$ arising from the polar decomposition of $G$. Furthermore, we define $\mathcal{P} = G^*G$, so $\mathcal{P}$ is a positive injective operator in $H$ such that $G = \mathcal{J} \mathcal{P}^{\frac{1}{2}}$.

Again, we want to prove some commutation relations. We will use the same kind of techniques as in the previous section.

The left invariance of $\varphi$ allows us to define a unitary operator on $K \otimes H$ such that $V(\Gamma \circ \Lambda)(\Delta(b)(a \otimes 1)) = \Gamma(a) \otimes \Lambda(b)$ for all $a, b \in A$.

**Proposition 4.4** We have that $(\mathcal{J} \otimes J)W = V^*(\mathcal{J} \otimes J)$ and $(\mathcal{P} \otimes \nabla)W = W(\mathcal{P} \otimes \nabla)$.

**Proof:** For all $a, b \in A$, we have that
\[
V^*(G \otimes T) (\Lambda(a) \otimes \Lambda(b)) = V^*(\Gamma(S(a)^* y) \otimes \Lambda(b^*))
\]
\[
= (\Gamma \circ \Lambda)(\Delta(b^*)(S(a)^* y \otimes 1)) = (\Gamma \circ \Lambda)((S(a) \otimes 1)\Delta(b)^* (y \otimes 1))
\]
\[
= (\Gamma \circ \Lambda)((S(a) \otimes 1)(\Delta(b)) (a \otimes 1))^* (y \otimes 1))
\]
\[
= (G \otimes T)(\Lambda \circ \Lambda)((S(a) \otimes 1)(\Delta(b)) (a \otimes 1)) = (G \otimes T)W (\Lambda(a) \otimes \Lambda(b)),
\]
where, in the last equality, we used proposition 2.4. Using lemma 3.7 once again, we see that $V^*(G \otimes T) = (G \otimes T)W$. As in the proof of proposition 3.9, we get that $(\mathcal{P} \otimes \nabla)W = W(\mathcal{P} \otimes \nabla)$

Hence, $W^*(\mathcal{P}^{\frac{1}{2}} \otimes \nabla^{\frac{1}{2}})W = \mathcal{P}^{\frac{1}{2}} \otimes \nabla^{\frac{1}{2}}$. Therefore, we see that
\[
V(\mathcal{J} \otimes J) W (\mathcal{P}^{\frac{1}{2}} \otimes \nabla^{\frac{1}{2}}) = V(\mathcal{J} \otimes J) (\mathcal{P}^{\frac{1}{2}} \otimes \nabla^{\frac{1}{2}}) W
\]
\[
= V(G \otimes T) W = G \otimes T = (\mathcal{J} \otimes J) (\mathcal{P}^{\frac{1}{2}} \otimes \nabla^{\frac{1}{2}}).
\]

Because $\mathcal{P}^{\frac{1}{2}} \otimes \nabla^{\frac{1}{2}}$ has dense range, it follows that $V(\mathcal{J} \otimes J)W = \mathcal{J} \otimes J$. \qed

The proof of the following lemma is similar to the proof of lemma 2.3.

**Lemma 4.5** Consider $a, b, c \in A$. Then $(\iota \otimes \omega_{\Lambda(a), \Lambda(b)})(V) \Gamma(c) = \Gamma((\iota \circ \varphi)(\Delta(b^*)(1 \otimes a)) c)$.

Then, in the same way as in the first section, we get for every $a$ in $A$ the existence of a unique bounded operator $T$ on $K$ such that $T \Gamma(c) = \Gamma(ac)$ for all $c \in A$.

**Definition 4.6** We define the mapping $\theta$ from $A$ into $B(K)$ such that $\theta(a) \Gamma(c) = \Gamma(ac)$ for all $a, c \in A$.

Then $\theta$ is a $^*$-homomorphism from $A$ to $B(K)$ such that $\theta(A)K$ is dense in $K$.

We will need the following easy consequence of lemma 1.3.
\[
(\iota \otimes \omega_{\Lambda(a), \Lambda(b)})(V^*) = \theta((\iota \circ \varphi)((1 \otimes b^*)\Delta(a)))
\]
for every $a, b \in A$.

Now, it is time for some new commutation relations.
Proposition 4.7 Consider $a \in A$. Then

1. $\theta(a) G \subseteq G \pi(S(a^*))$
2. $\pi(a) G^* \subseteq G^* \theta(S(a^*))$
3. $\pi(a) G^{-1} \subseteq G^{-1} \theta(S(a^*))$
4. $\theta(a) (G^{-1})^* \subseteq (G^{-1})^* \pi(S(a^*))$

Proof:

1. Consider $b \in A$. Then we have for all $c \in A$ that

\[
\theta(b) G \Lambda(c) = \theta(b) \Gamma(S(c)^* y) = \Gamma(b S(c)^* y) = \Gamma(S(b)^* c)^* y = G \Lambda(S(b)^* c) = G \pi(S(b^*)) \Lambda(c).
\]

Because $\theta(b)$ and $\pi(S(b^*))$ are bounded and $\Lambda(A)$ is a core for $G$, it follows easily that $\theta(b) G \subseteq G \pi(S(b^*))$.

2. Consider $b \in A$. Apply 1) with $a$ equal to $S^{-1}(b)$, then $\theta(S^{-1}(b)) G \subseteq G \pi(b^*)$. Taking the adjoint of this equation gives

\[
(G \pi(b^*))^* \subseteq (\theta(S^{-1}(b)))^* G^*.
\] (a)

We know that

\[
(G \pi(b^*))^* \supseteq \pi(b^*)^* G^* = \pi(b) G^*.
\]

The boundedness of $\theta(S^{-1}(b))$ implies that

\[
(\theta(S^{-1}(b)))^* G^* = G^* \theta(S^{-1}(b))^* = G^* \theta(S(b^*)�).
\]

Combining this two relations with relation (a) gives us $\pi(b) G^* \subseteq G^* \theta(S(b^*))$.

3. This is proven in a similar way as 1), using corollary 1.3.

4. This is proven in a similar way as 2), using 3).

Proposition 4.8 Consider $a \in A$. Then

1. $\pi(a) \mathcal{P} \subseteq \mathcal{P} \pi(S^{-2}(a))$
2. $\pi(a) \mathcal{P}^{-1} \subseteq \mathcal{P}^{-1} \pi(S^2(a))$.

Proof:

1. Using the previous proposition, we see that

\[
\pi(a) \mathcal{P} = \pi(a) G^* G \subseteq G^* \theta(S(a^*)) G \subseteq G^* G \pi(S(a^*))^* = \mathcal{P} \pi(S^{-2}(a)).
\]

2. This is proven in a similar way, using the fact that $\mathcal{P}^{-1} = (G^{-1})(G^{-1})^*$. 

\[
\]
Corollary 4.9 Consider $a \in A$ and $n \in \mathbb{Z}$. Then $\pi(a)P^n \subseteq P^n \pi(S^{-2n}(a))$.

The following results must be seen in the light of the manageability of our multiplicative unitary $W$, as defined by Woronowicz in [20].

Proposition 4.10 Consider $u_1 \in D(P^\frac{1}{2})$, $u_2 \in D(P^{-\frac{1}{2}})$ and $v_1, v_2 \in H$. Then

$$\langle W^*(P^\frac{1}{2}u_1 \otimes Jv_1), P^{-\frac{1}{2}}u_2 \otimes Jv_2 \rangle = \langle W(u_1 \otimes v_2), u_2 \otimes v_1 \rangle.$$

Proof: Choose $a_1, a_2 \in A$.

Because $G = JF\frac{1}{2}$ and $(G^{-1})^* = F^{-\frac{1}{2}}$, we have that $u_1 \in D(G)$, $u_2 \in D((G^{-1})^*)$ and $P^\frac{1}{2}u_1 = J^*(Gu_1)$, $P^{-\frac{1}{2}}u_2 = J^*((G^{-1})^*)u_2$.

Using equation [1] and proposition [4.7], we see that

$$\langle (i \otimes \omega_{\Lambda(a_2)\Lambda(a_1)})(V^*) (G^{-1})^* = \theta(i \otimes \varphi)((1 \otimes a_1)\Delta(a_2)) (G^{-1})^*$$

$$\subseteq (G^{-1})^* \pi(S(i \otimes \varphi)((1 \otimes a_1)\Delta(a_2))^*))$$

$$= (G^{-1})^* \pi(S(i \otimes \varphi)(\Delta(a_2^*)(1 \otimes a_1)^*))$$

$$= (G^{-1})^* \pi(i \otimes \varphi)((1 \otimes a_2^*)\Delta(a_1)).$$

where, in the last equality, we used equation [3] of section 2.

So, using equation [1], we get that

$$\langle (i \otimes \omega_{\Lambda(a_2)\Lambda(a_1)})(V^*) (G^{-1})^* \subseteq (G^{-1})^* (i \otimes \omega_{\Lambda(a_2)\Lambda(a_1)})(W^*).$$

Hence, $(i \otimes \omega_{\Lambda(a_1)\Lambda(a_2)})(W^*) u_2$ belongs to $D((G^{-1})^*)$ and

$$(G^{-1})^* (i \otimes \omega_{\Lambda(a_1)\Lambda(a_2)})(W^*) u_2 = (i \otimes \omega_{\Lambda(a_2)\Lambda(a_1)})(W^*) (G^{-1})^* u_2. \quad (a)$$

Furthermore,

$$\langle W^*(P^\frac{1}{2}u_1 \otimes J\Lambda(a_1)), P^{-\frac{1}{2}}u_2 \otimes J\Lambda(a_2) \rangle$$

$$= \langle W^*(J^*Gu_1 \otimes J\Lambda(a_1)), J^*(G^{-1})^*u_2 \otimes J\Lambda(a_2) \rangle$$

$$= \langle W^*(J^* \otimes J)(Gu_1 \otimes \Lambda(a_1)), (J^* \otimes J)((G^{-1})^*u_2 \otimes \Lambda(a_2)) \rangle$$

$$= \langle (J^* \otimes J)V(Gu_1 \otimes \Lambda(a_1)), (J^* \otimes J)((G^{-1})^*u_2 \otimes \Lambda(a_2)) \rangle$$

$$= \langle (G^{-1})^*u_2 \otimes \Lambda(a_2), V(Gu_1 \otimes \Lambda(a_1)) \rangle$$

$$= \langle V^*((G^{-1})^*u_2 \otimes \Lambda(a_2)), Gu_1 \otimes \Lambda(a_1) \rangle$$

$$= \langle (i \otimes \omega_{\Lambda(a_2)\Lambda(a_1)})(V^*) (G^{-1})^* u_2, Gu_1 \rangle.$$  

where, in step (b), we used proposition [4.4].

Combining this result with equation (a), we get that

$$\langle W^*(P^\frac{1}{2}u_1 \otimes J\Lambda(a_1)), P^{-\frac{1}{2}}u_2 \otimes J\Lambda(a_2) \rangle$$

$$= \langle ((G^{-1})^* (i \otimes \omega_{\Lambda(a_1)\Lambda(a_2)})(W^*) u_2), Gu_1 \rangle$$

$$= \langle u_1, (i \otimes \omega_{\Lambda(a_1)\Lambda(a_2)})(W^*) u_2 \rangle$$

$$= \langle (i \otimes \omega_{\Lambda(a_2)\Lambda(a_1)})(W) u_1, u_2 \rangle$$

$$= \langle W(u_1 \otimes \Lambda(a_2)), u_2 \otimes \Lambda(a_1) \rangle.$$  

Now, the result follows easily. \hfill \blacksquare
Lemma 4.11 Suppose that $S(y^*) x \rho(S(y)) \in \mathcal{C}1$. Then we have that

$$
((\Gamma \circ \Gamma)(\chi(\Delta(b)(1 \otimes a))(y \otimes y)), (\Gamma \circ \Gamma)(\chi(\Delta(d)(1 \otimes c))(y \otimes y))
= ((\Gamma \circ \Gamma)((a \otimes b)(y \otimes y)), (\Gamma \circ \Gamma)((c \otimes d)(y \otimes y))).
$$

Proof: By supposition, there exists a complex number $r$ such that $S(y^*) x \rho(S(y)) = r1$. We have that

$$
\langle (\Gamma \circ \Gamma)(\chi(\Delta(b)(1 \otimes a))(y \otimes y)), (\Gamma \circ \Gamma)(\chi(\Delta(d)(1 \otimes c))(y \otimes y)) \rangle
= \langle (\Gamma \circ \Gamma)((a \otimes b)(y \otimes y)), (\Gamma \circ \Gamma)((c \otimes d)(y \otimes y)) \rangle.
$$

Using our connection between $\varphi$ and $\eta$, we get that

$$
\langle (\Gamma \circ \Gamma)(\chi(\Delta(b)(1 \otimes a))(y \otimes y)), (\Gamma \circ \Gamma)(\chi(\Delta(d)(1 \otimes c))(y \otimes y)) \rangle
= \langle (\varphi \circ \eta)((S \circ \iota)((y^* \otimes (cy)^*)\Delta(d^* b)(y \otimes ay))(x \otimes 1)) \rangle
= \langle (\varphi \circ \eta)((S(y^*) \otimes 1)(S \circ \iota)((1 \otimes (cy)^*)\Delta(d^* b)(1 \otimes ay))(S(y^*) \otimes 1)(x \otimes 1)) \rangle
= \langle (\varphi \circ \eta)((S \circ \iota)((1 \otimes (cy)^*)\Delta(d^* b)(1 \otimes ay))(S(y^*) x \rho(S(y)) \otimes 1)) \rangle
= r \langle \varphi \circ \eta((S \circ \iota)((1 \otimes (cy)^*)\Delta(d^* b)(1 \otimes ay))) \rangle
= r \langle \varphi S \circ \eta((1 \otimes (cy)^*)\Delta(d^* b)(1 \otimes ay)) \rangle.
$$

Hence, the right invariance of $\varphi S$ implies that

$$
\langle (\Gamma \circ \Gamma)(\chi(\Delta(b)(1 \otimes a))(y \otimes y)), (\Gamma \circ \Gamma)(\chi(\Delta(d)(1 \otimes c))(y \otimes y)) \rangle
= r \varphi(S(d^* b)) \eta((cy)^*(ay)) = \varphi(S(d^* b)S(y^*) x \rho(S(y)) \otimes 1) \eta((cy)^*(ay))
= \varphi(S(y)S(d^* b)S(y^*) x) \eta((cy)^*(ay)) = \varphi(S(y) d^* by) x) \eta((cy)^*(ay))
= \eta((dy)^*(by)) \eta((cy)^*(ay))
= \langle (\Gamma \circ \Gamma)((a \otimes b)(y \otimes y)), (\Gamma \circ \Gamma)((c \otimes d)(y \otimes y)) \rangle.
$$

Proposition 4.12 If $S(y^*) x \rho(S(y)) \in \mathcal{C}1$, then we have that $W(P \otimes P) = (P \otimes P)W$.

Proof: The previous lemma implies the existence of a unitary operator $U$ on $K \otimes K$, such that

$$
U(\Gamma \circ \Gamma)(\chi(\Delta(b)(1 \otimes a))(y \otimes y)) = (\Gamma \circ \Gamma)((a \otimes b)(y \otimes y))
$$

for all $a, b \in A$.

Choose $c, d \in A$, then

$$
(G \otimes G)W(\Lambda \otimes \Lambda)(\Delta(d)(c \otimes 1)) = (G \otimes G)(\Lambda \otimes \Lambda)(c \otimes d)
= (\Gamma \circ \Gamma)((S(c)^* \otimes S(d)^*)(y \otimes y))
= U(\Gamma \circ \Gamma)(\chi(\Delta(S(d)^*)(1 \otimes S(c)^*))(y \otimes y))
= U(\Gamma \circ \Gamma)(((S(c) \otimes 1)\chi(\Delta(S(d))))^*(y \otimes y))
= U(\Gamma \circ \Gamma)((S(c) \otimes 1)(S \otimes S)(\Delta(d))^*(y \otimes y))
= U(\Gamma \circ \Gamma)((S \otimes S)(\Delta(d)(c \otimes 1))^*(y \otimes y))
= U(G \otimes G)(\Lambda \otimes \Lambda)(\Delta(d)(c \otimes 1)).
$$

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Refering to lemma 3.7 once again, we see that \((G \otimes G)W = U(G \otimes G)\). As before, we can conclude that \((\mathcal{P} \otimes \mathcal{P})W = W(\mathcal{P} \otimes \mathcal{P})\).

In a later stage, we will apply our construction procedure in a case where the assumption of this proposition is satisfied. Then we will have by definition that \(W\) is amenable.

5 The polar decomposition of the antipode.

In this section, we are going to make a polar decomposition of the antipode \(S\). We will use the construction procedure of section 4 for this. In this case, we take \(\eta = \varphi\) and \(y = 1\). For all \(a \in A\), we have that
\[
\frac{1}{\mu} \varphi(S(a)\delta) = \varphi(a). \quad \text{Hence, } x \text{ is equal to } \frac{1}{\mu} \delta.
\]
Furthermore, we take in this case \(K = H\) and \(\Gamma = \Lambda\). So, \(\theta = \pi\). We put \(I = J\) and \(M = \mathcal{P}\). First, we give a summary of the results of section 4 in this particular case.

We have that \(G\) is the closed antilinear map from within \(H\) into \(H\) such that \(\Lambda(A)\) is a core for \(G\) and \(GA(a) = \Lambda(S(a)^*)\) for every \(a \in A\). It is clear that \(G\) is involutive in this case.

Therefore, \(I\) is an involutive anti-unitary transformation on \(H\), \(M = G^*G\) is an injective positive operator in \(H\) such that \(G = IM^{\frac{1}{2}} = M^{-\frac{1}{2}}I\). For \(t \in \mathbb{R}\) we have that \(M^t = IM^tI\) and \(M^t = IM^{-t}I\).

We have the following commutations:
\[
\pi(a) G \subseteq G \pi(S(a)^*) \quad \text{and} \quad \pi(a) G^* \subseteq G^* \pi(S(a^*))
\]
for all \(a \in A\). Moreover, we have for all \(a \in A\) and \(n \in \mathbb{Z}\) that
\[
\pi(a) M^n \subseteq M^n \pi(S^{-2n}(a)). \quad (7)
\]

In this case, \(V\) is equal to \(W\), so
\[
(M \otimes \nabla)W = W(M \otimes \nabla) \quad \text{and} \quad (I \otimes J)W = W^* (I \otimes J).
\]

Using these two commutation relations, the following definitions are justified in the same way as definition 3.13.

**Definition 5.1** We define the norm-continuous one-parameter group \(\tau\) on \(A_r\) such that \(\tau_t(x) = M^{it}xM^{-it}\) for every \(x \in A_r\) and \(t \in \mathbb{R}\).

Again, it follows that
\[
\tau_t( (\iota \otimes \omega)(W) ) = (\iota \otimes \nabla^{it}\omega\nabla^{-it})(W)
\]
for all \(t \in \mathbb{R}\) and \(\omega \in B_0(H)^*\).

**Definition 5.2** We define the involutive \(^*\)-anti automorphism \(R\) on \(A_r\) such that \(R(x) =Ix^*I\) for every \(x \in A_r\).

Similarly, we have that
\[
R( (\iota \otimes \omega)(W) ) = (\iota \otimes \omega(J^*J))(W)
\]
for every \(\omega \in B_0(H)^*\).

**Proposition 5.3** For every \(t \in \mathbb{R}\), we have that \(\tau_t R = R\tau_t\).
Proof: Choose $x \in A_r$. We know that $M^t$ and $I$ commute. Hence,

$$\tau_t(R(x)) = M^t I x^* IM^{-it} = IM^t x^* M^{-it} I$$

$$= I(M^t x M^{-it})^* I = R(\tau_t(x)).$$

\[\square\]

Corollary 5.4 For every $z \in \mathfrak{g}$, we have that $\tau_z R = R \tau_z$.

Using relation 7 and remembering that $\tau$ is implemented by $M$, we get the following

Proposition 5.5 Consider $a \in A$. Then $\pi(a)$ is an analytic element of $\tau$ and $\tau_{ni}(\pi(a)) = \pi(S^{-2n}(a))$ for every $n \in \mathbb{Z}$.

Finally, we are able to describe the polar decomposition of the antipode.

Theorem 5.6 Consider $a \in A$, then $\pi(S(a)) = R(\tau_{-\frac{i}{2}}(\pi(a)))$.

Proof: We know from the beginning of this section that $\pi(a) G \subseteq G \pi(S(a)^*)$, so $\pi(a) M^{-\frac{i}{2}} I \subseteq M^{-\frac{i}{2}} I \pi(S(a)^*)$. Therefore,

$$\pi(a) M^{-\frac{i}{2}} \subseteq M^{-\frac{i}{2}} I \pi(S(a))^* I = M^{-\frac{i}{2}} R(\pi(S(a))).$$

Remembering that $\tau$ is implemented by $M$, we get that $\tau_{-\frac{i}{2}}(\pi(a)) = R(\pi(S(a)))$. The result follows because $R$ is involutive. \[\square\]

The rest of this section will devoted to proving some interesting commutation relations between $\tau, R$ and $\Delta$. They are always consequences of commutation relations between $W, M, \nabla, I$ and $J$.

Proposition 5.7 For every $t \in \mathbb{R}$, we have that $\Delta \sigma_t = (\tau_t \otimes \sigma_t) \Delta$.

Proof: Choose $x \in A_r$. Then

$$\Delta(\sigma_t(a)) = W^*(1 \otimes \sigma_t(x))W = W^*(1 \otimes \nabla^it x \nabla^{-it})W$$

$$= W^*(M^{it} \otimes \nabla^it)(1 \otimes x)(M^{-it} \otimes \nabla^{-it})W.$$ We know that $W$ commutes with $M \otimes \nabla$. Hence,

$$\Delta(\sigma_t(a)) = (M^{it} \otimes \nabla^it)W^*(1 \otimes x)W(M^{-it} \otimes \nabla^{-it})$$

$$= (M^{it} \otimes \nabla^it)\Delta(x)(M^{-it} \otimes \nabla^{-it}) = (\tau_t \otimes \sigma_t)\Delta(x).$$ \[\square\]

Using this result, it is not difficult to prove the following one.

Proposition 5.8 For every $t \in \mathbb{R}$, we have that $\Delta \tau_t = (\tau_t \otimes \tau_t)\Delta$.  

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**Proof:** We have that

\[
((\tau_1 \otimes \tau_1)(\Delta \otimes \sigma_t))\Delta = (\tau_1 \otimes \tau_1 \otimes \sigma_t)((\Delta \otimes \iota)\Delta)
\]
\[
= (\tau_1 \otimes \tau_1 \otimes \sigma_t)(\iota \otimes \Delta)\Delta = (\tau_1 \otimes \Delta \sigma_t)\Delta
\]
\[
= (\iota \otimes \Delta)\Delta \sigma_t = (\Delta \otimes \iota)\Delta \sigma_t = (\Delta \tau_1 \otimes \sigma_t)\Delta.
\]

Therefore,

\[
((\tau_1 \otimes \tau_1)(\Delta \otimes \iota)\Delta = (\Delta \tau_1 \otimes \iota)\Delta.
\]

From this and the fact that \(A_r = \{(\iota \otimes \omega)\Delta(x) \mid \omega \in A_r^*, x \in A_r^*\}\), it follows easily that \((\tau_1 \otimes \tau_1)(\Delta = \Delta \tau_1\).

\(\blacksquare\)

A last commutation property concerns \(R\) and \(\Delta\).

**Proposition 5.9** We have that \(\chi(R \otimes R)\Delta = \Delta R\).

**Proof:** Choose \(\omega \in B_0(H)^*\). Then

\[
(\chi(R \otimes R)\Delta)((\iota \otimes \omega)(W))
\]
\[
= \chi((I \otimes I)\Delta((\iota \otimes \omega)(W))^{*}(I \otimes I))
\]
\[
= \chi((I \otimes I)[W^*\iota(\iota \otimes \omega)(W)](I \otimes I))
\]
\[
= \chi((I \otimes I)[W^*(\iota \otimes \omega)(W)](I \otimes I))
\]
\[
= \chi((I \otimes I)(\iota \otimes \omega)(W_{12}^*W_{23}^*W_{12})(I \otimes I))
\]
\[
= \chi((I \otimes I)(\iota \otimes \omega)((W_{13}W_{23})^*)(I \otimes I)).
\]

where, in the last equality, we used the pentagon equation (proposition 2.12). Using this equality, it is not so difficult to check that

\[
(\chi(R \otimes R)\Delta)((\iota \otimes \omega)(W))
\]
\[
= \chi((\iota \otimes \iota \otimes \omega)((I^* \iota \otimes I^* \iota)(W_{13}W_{23})))
\]
\[
= \chi((\iota \otimes \iota \otimes \omega)((I^* \iota \otimes I^* \iota)(W_{13}W_{23})))
\]

Because \((I \otimes J)W = W^*(I \otimes J)\), we have that

\[
((I^* \iota \otimes \iota)(W) = (\iota \otimes (J^* \iota))(W).
\]

Hence,

\[
(\chi(R \otimes R)\Delta)((\iota \otimes \omega)(W))
\]
\[
= \chi((\iota \otimes \iota \otimes \omega)((\iota \otimes (J^* \iota))(W_{13}W_{12})))
\]
\[
= \chi((\iota \otimes \iota \otimes \omega)((\iota \otimes (J^* \iota))(W_{13}W_{12})))
\]
\[
= W^*(\iota \otimes \iota \otimes \omega)((\iota \otimes (J^* \iota))(W_{13}W_{12}))W
\]
\[
= W^*(1 \otimes (\iota \otimes \omega)(J^* \iota))(W)W
\]
\[
= W^*(1 \otimes R((\iota \otimes \omega)(W)))W = \Delta(R((\iota \otimes \omega)(W))).
\]

The proposition follows by equation 3 of section 2. \(\blacksquare\)

In a later section, we will look at some other implementations of \(\tau\) and \(R\).
6 The left Haar weight.

In this section, we want to extend our positive Haar functional on the *-algebra \( A \) to a weight on the C*-algebra \( A_r \). We will show that this weight satisfies the KMS-condition and that it is left invariant in some sense.

First, we are going to define the left Haar weight on our C*-algebra \( A_r \). In order to make our notation not too heavy, we will use for this weight the same symbol \( \varphi \) as for our left Haar functional on our *-algebra \( A \).

The context will make clear which meaning \( \varphi \) has. The term ‘linear functional \( \varphi \)’ always means the left invariant functional \( \varphi \) on the *-algebra \( A \) whereas the term ‘weight \( \varphi \)’ always means the left invariant weight \( \varphi \) on the C*-algebra \( A_r \). Also, when \( \varphi \) works on elements of \( A \), \( \varphi \) will always be the linear functional \( \varphi \). When \( \varphi \) works on elements of \( A_r \), \( \varphi \) will always be the weight \( \varphi \).

Later in this paper, we also have other symbols with two meanings. In every case, it handles about a symbol which denotes an object connected to the *-algebra \( A \) and an object connected to the C*-algebra \( A_r \) which are each other equivalent. In these cases, we will use the same conventions as above.

Definition 6.1 Denote \( \hat{\varphi} \) as the semifinite faithful normal weight on the von Neumann algebra \( A_r^\prime \) associated to the left Hilbert algebra \( \cal{U} \). Then we define \( \varphi \) as the restriction of \( \hat{\varphi} \) to \( A_r^+ \), hence \( \varphi \) is a faithful lower semi-continuous weight on \( A_r \).

The next proposition guarantees that our \( \varphi \) on the C*-algebra level is an extension of our \( \varphi \) on the *-algebra level.

Proposition 6.2 We have that \( \pi(A) \) is a subset of \( \cal{M}_\varphi \) and \( \varphi(\pi(a)) = \varphi(a) \) for all \( a \in A \).

Proof: In this proof, we will use the theory of left Hilbert algebras. Choose \( b \in A \). Then \( L_{\Lambda(b)} = \pi(b) \). According to the theory of left Hilbert algebras, \( L_{\Lambda(b)}^* L_{\Lambda(b)} \) belongs to \( \cal{M}_\varphi^+ \) and \( \varphi(L_{\Lambda(b)}^* L_{\Lambda(b)}) = \langle \Lambda(b), \Lambda(b) \rangle = \varphi(b^* b) \). Because \( L_{\Lambda(b)}^* L_{\Lambda(b)} = \pi(b)^* \pi(b) = \pi(b^* b) \), and \( \varphi \) is the restriction of \( \hat{\varphi} \) to \( A_r^+ \), it follows that \( \pi(b^* b) \) belongs to \( \cal{M}_\varphi \) and \( \varphi(\pi(b^* b)) = \varphi(b^* b) \).

The proposition follows from polarization and the fact that \( A = A^* A \).

From this proposition, we can conclude dat \( \varphi \) is densely defined.

From now on, we will use the following GNS-triple of \( \varphi \):

1. \( H_\varphi = H \)
2. We define the linear map \( \Lambda_\varphi \) from \( \cal{N}_\varphi \) into \( H \) as follows. Let \( a \in \cal{N}_\varphi \), then there exists a unique \( v \in H \) such that \( v \) is left bounded with respect to \( \cal{U} \) and we define \( \Lambda_\varphi(a) = v \).
3. \( \pi_\varphi = \) the identity map on \( A_r \)

Remark that \( \Lambda_\varphi(\pi(a)) = \Lambda(a) \) for all \( a \in A \), this guarantees in fact that \( \Lambda_\varphi \) has dense range in \( H \).

The following result follows immediately from the theory of left Hilbert algebras.

Proposition 6.3 We have that \( \varphi \) is invariant under \( \sigma \). Moreover, \( \Lambda_\varphi(\sigma_t(x)) = \nabla^{it}\Lambda_\varphi(x) \) for all \( t \in \mathbb{R} \).

Now, we are going to prove that the weight \( \varphi \) is determined by its values on \( \pi(A) \). More precisely, we will prove that \( \Lambda(A) \) is a core for \( \Lambda_\varphi \).

We define \( A_0 \) as the closure of the mapping \( \pi(A) \rightarrow H : x \mapsto \Lambda_\varphi(x) \) and \( A_0 \) will be the domain of \( \Lambda_\varphi \). It is clear that \( A_0 \) is a restriction of \( \Lambda_\varphi \). Furthermore, it is easy to check that \( A_0 \) is a left ideal of \( A_r \).
**Lemma 6.4** Consider \( x, y \in \pi(A) \) and \( \omega \in A_r^+ \). Then \((y\omega \otimes \iota)\Delta(x) \) belongs to \( \pi(A) \) and \( \varphi((y\omega \otimes \iota)\Delta(x)) = \omega(y) \varphi(x) \).

**Proof:** Choose \( a, b \in A \). Then
\[
(\pi(b)\omega \otimes \iota)\Delta(\pi(a)) = (\omega \otimes \iota)((\Delta(\pi(a))(\pi(b) \otimes 1)) = (\omega \otimes \iota)((\Delta(\pi(a))(\pi(b) \otimes 1)) = \pi((\omega \otimes \iota)(\Delta(\pi(a))(\pi(b) \otimes 1))).
\]
So, \((\pi(b)\omega \otimes \iota)\Delta(\pi(a))\) belongs to \( \pi(A) \) and the left invariance of \( \varphi \) on the \( \ast \)-algebra level implies that
\[
\varphi((\pi(b)\omega \otimes \iota)\Delta(\pi(a))) = \varphi(a) (\omega \otimes \iota)(b) = \varphi(\pi(a)) \omega(\pi(b)).
\]

\[\blacksquare\]

**Lemma 6.5** Consider \( x \in A_0 \) and \( \omega \in (A_r)_{+}^\ast \). Then \((\omega \otimes \iota)\Delta(x^\ast x) \) belongs to \( \mathcal{M}_\varphi^+ \) and \( \varphi((\omega \otimes \iota)\Delta(x^\ast x)) \leq \|\omega\| \varphi(x^\ast x) \).

**Proof:** It is a well known result that there exist \( \theta \in (A_r)_{+}^\ast \) and \( y \in A_r \) such that \( \omega = y\theta y^\ast \) (see \[10\]). Now, there exist sequences \((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \) in \( \pi(A) \) such that \((y_n)_{n=1}^\infty \rightarrow y, (x_n)_{n=1}^\infty \rightarrow x \) and \((\Lambda_{\varphi}(x_n))_{n=1}^\infty \rightarrow \Lambda_{\varphi}(x)\). It is clear that \(( (y_n\theta y_n^\ast \otimes \iota)\Delta(x_n^\ast x_n))_{n=1}^\infty \rightarrow (\omega \otimes \iota)\Delta(x^\ast x)\). The lower semicontinuity of \( \varphi \) implies that
\[
\varphi((\omega \otimes \iota)\Delta(x^\ast x)) \leq \liminf((\varphi((y_n\theta y_n^\ast \otimes \iota)\Delta(x_n^\ast x_n)))_{n=1}^\infty.
\]
By the previous lemma, the right hand side of this equality equals
\[
\liminf((\varphi((y_n\theta y_n^\ast \otimes \iota)\Delta(x_n^\ast x_n)))_{n=1}^\infty = \liminf(\|y_n\theta y_n^\ast\| (\Lambda_{\varphi}(x_n), \Lambda_{\varphi}(x_n)))_{n=1}^\infty
= \|\omega\| (\Lambda_{\varphi}(x), \Lambda_{\varphi}(x)) = \|\omega\| \varphi(x^\ast x).
\]
The lemma follows. \[\blacksquare\]

**Lemma 6.6** Consider \( x \in A_0 \) and \( \omega \in A_r^+ \). Then \((\omega \otimes \iota)\Delta(x) \) belongs to \( \mathcal{N}_\varphi \) and \( \|\Lambda_{\varphi}((\omega \otimes \iota)\Delta(x))\| \leq \|\omega\| \|\Lambda_{\varphi}(x)\|\).

**Proof:** We know that there exist an element \( \theta \in (A_r)_{+}^\ast \) with \( \|\theta\| = \|\omega\| \) such that
\[
[(\omega \otimes \iota)(\Delta(y))]^\ast [(\omega \otimes \iota)(\Delta(y))] \leq \|\omega\| ((\omega \otimes \iota)(\Delta(y^\ast y))
\]
for all \( y \in A_r \) (see proposition 4.6 of \[17\]). Hence, using the previous lemma, we see that
\[
\varphi([(\omega \otimes \iota)(\Delta(x))]^\ast [(\omega \otimes \iota)(\Delta(x))]) \leq \|\omega\| \varphi((\theta \otimes \iota)(\Delta(x^\ast x))
\leq \|\omega\| \|\theta\| \varphi(x^\ast x) = \|\omega\|^2 \varphi(x^\ast x).
\]
The lemma follows. \[\blacksquare\]

Finally, we can prove the lemma which will be essential to us:

**Lemma 6.7** Consider \( \omega \in A_r^+ \) and \( x \in A_0 \). Then \((\omega \otimes \iota)\Delta(x) \) belongs to \( A_0 \).
Proof: There exist $\theta \in A^*_r$ and $y \in A_r$ such that $\omega = y\theta$. Furthermore, there exist sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in $\pi(A)$ such that $(y_n)_{n=1}^\infty \to y$, $(x_n)_{n=1}^\infty \to x$ and $(\Lambda_\varphi(x_n))_{n=1}^\infty \to \Lambda_\varphi(x)$.

We know already that $(y_n \otimes \iota)(\Delta(x_n))$ belongs to $\pi(A)$ for all $n \in \mathbb{N}$. It is also clear that $(\omega \otimes \iota)(\Delta(x))$. There exists a positive number $M$ such that $\|y_n\| \leq M$ and $\|\Lambda_\varphi(x_n)\| \leq M$ for all $n \in \mathbb{N}$.

So, for all $m, n \in \mathbb{N}$, we have that

$$
\|\Lambda_\varphi((y_n \otimes \iota)(\Delta(x_n))) - \Lambda_\varphi((y_m \otimes \iota)(\Delta(x_m)))\|
\leq \|\Lambda_\varphi((y_n \otimes \iota)(\Delta(x_n))) - \Lambda_\varphi((y_n \otimes \iota)(\Delta(x_n)))\|
+ \||\Lambda_\varphi((y_n \otimes \iota)(\Delta(x_n))) - \Lambda_\varphi((y_m \otimes \iota)(\Delta(x_m)))\|
\leq M\|\Lambda_\varphi(x_n) - \Lambda_\varphi(x_m)\| + \|y_n - y_m\| \cdot M,
$$

where in inequality (a), we used the previous lemma.

From this chain of inequalities, we infer that $(\Lambda_\varphi((y_n \otimes \iota)(\Delta(x_n))))_{n=1}^\infty$ is Cauchy and hence convergent in $H$. Because $\Lambda_0$ is closed, $(\omega \otimes \iota)(\Delta(x))$ belongs to $A_0$.

Now, we will prove a result which is essential in the rest of the paper. First, we need an easy lemma.

Lemma 6.8 Consider elements $a, b, c \in A$. Then

1. $(\omega_{\Lambda(b), \Lambda(c)} \otimes \iota)(\Delta(\pi(a))) = \pi((\varphi \otimes \iota)((c^* \otimes 1)(\Delta(a))(b \otimes 1)))$

2. $(\iota \otimes \omega_{\Lambda(b), \Lambda(c)})(\Delta(\pi(a))) = \pi((\iota \otimes \varphi)((1 \otimes c^*)(\Delta(a))(1 \otimes b)))$.

Proof: We know there exist an element $d \in A$ such that $b = db$. Hence,

$$
\pi((\varphi \otimes \iota)((c^* \otimes 1)(\Delta(a))(b \otimes 1))) = \pi((\varphi \otimes \iota)((c^* \otimes 1)(\Delta(d)(d \otimes 1))(b \otimes 1)))
= (\omega_{\Lambda(b), \Lambda(c)} \otimes \iota)((\tau \circ \pi)(\Delta(a))(d \otimes 1))
= (\omega_{\Lambda(b), \Lambda(c)} \otimes \iota)(\Delta(\pi(a))(d \otimes 1))
= (\omega_{\Lambda(d), \Lambda(c)} \otimes \iota)((\Delta(\pi(a))(d \otimes 1))
= (\omega_{\Lambda(d), \Lambda(c)} \otimes \iota)(\Delta(\pi(a)))
= (\omega_{\Lambda(d), \Lambda(c)} \otimes \iota)(\Delta(\pi(a)))
= (\omega_{\Lambda(d), \Lambda(c)} \otimes \iota)(\Delta(\pi(a)) = (\omega_{\Lambda(b), \Lambda(c)} \otimes \iota)(\Delta(\pi(a))).
$$

The other equality is proven in a similar way.

Proposition 6.9 Consider a dense left ideal $N$ in $A_r$ such that for all $a \in N$ and all $\omega \in A^*_r$, we have that $(\iota \otimes \omega)(\Delta(a)) \in N$. Then $\pi(A)$ is a subset of $N$.

Proof: Choose $b \in A$. Because $\varphi \neq 0$, there exist $c, d \in A$ such that $c \varphi d^* \neq 0$. We have that $\omega_{\Lambda(b), \Lambda(c)} \circ \pi = c \varphi d^*$, therefore $\omega_{\Lambda(b), \Lambda(c)} \neq 0$. The fact that $N$ is dense in $A_r$ implies the existence of an element $x \in N$ such that $\omega_{\Lambda(b), \Lambda(c)}(x) = 1$.

There exist $p_1, \ldots, p_n, q_1, \ldots, q_n, r_1, \ldots, r_n \in A$ such that

$$(b \otimes 1)(\Delta(\rho^{-1}(c)d^*)) = \sum_{i=1}^n (1 \otimes p_i q_i)(\Delta(r_i)). \quad (*)$$
For every \( y \in A \), we have that

\[
\omega_{\Lambda(c),\Lambda(d)}(\pi(y)) \pi(b) = \varphi(d^*yc) \pi(b) = \pi(\varphi(\rho^{-1}(c)d^*y)b) \\
= \pi( (\iota \circ \varphi)((b \otimes 1)\Delta(\rho^{-1}(c)d^*)y) ) \\
= \sum_{i=1}^{n} \pi((\iota \circ \varphi)((1 \otimes p_i)q_i)\Delta(r_iy)) \\
= \sum_{i=1}^{n} \pi((\iota \circ \varphi)((1 \otimes q_i)\Delta(r_iy)(1 \otimes \rho(p_i))) ) \\
= \sum_{i=1}^{n}(\iota \otimes \omega_{\Lambda(\rho(p_i)),\Lambda(q_i^*)})\Delta(\pi(r_iy)) \\
= \sum_{i=1}^{n}(\iota \otimes \omega_{\Lambda(\rho(p_i)),\Lambda(q_i^*)})\Delta(\pi(r_i)\pi(y)) .
\]

Because \( \pi(A) \) is dense in \( A_r \), we must have that

\[
\omega_{\Lambda(c),\Lambda(d)}(z) \pi(b) = \sum_{i=1}^{n}(\iota \otimes \omega_{\Lambda(\rho(p_i)),\Lambda(q_i^*)})\Delta(\pi(r_i)z).
\]

for all \( z \in A_r \). In particular, we have that

\[
\pi(b) = \omega_{\Lambda(c),\Lambda(d)}(x) \pi(b) = \sum_{i=1}^{n}(\iota \otimes \omega_{\Lambda(\rho(p_i)),\Lambda(q_i^*)})\Delta(\pi(r_i)x).
\]

Because \( x \) belongs to \( N \) and \( N \) is a left ideal in \( A_r \), \( \pi(r_i)x \) will belong to \( N \) for all \( i \in \{1, \ldots, n\} \). By the above equality, we see that \( \pi(b) \) belongs to \( N \).

Of course, also the following proposition is true.

**Proposition 6.10** Consider a dense left ideal \( N \) in \( A_r \) such that for all \( a \in N \) and all \( \omega \in A_r^* \), we have that \( (\omega \otimes \iota)\Delta(a) \in N \). Then \( \pi(A) \) is a subset of \( N \).

**Sketch of proof:** The proof starts in the same way as the previous one but step (*) is replaced by:

There exist \( p_1, \ldots, p_n, q_1, \ldots, q_n, r_1, \ldots, r_n \in A \) such that

\[
(1 \otimes b\delta^{-1})\Delta(\rho^{-1}(c)d^*) = \sum_{i=1}^{n}(p_iq_i \otimes 1)\Delta(r_i). 
\]

Then you prove in a similar way as in the previous proof that

\[
\omega_{\Lambda(c),\Lambda(d)}(\pi(y)) \pi(b) = \sum_{i=1}^{n}(\omega_{\Lambda(\rho(p_i)),\Lambda(q_i^*)} \otimes \iota)\Delta(\pi(r_i)\pi(y))
\]

for every \( y \in A \) but in stead of the left invariance of \( \varphi \), you use the fact that \( (\varphi \otimes \iota)\Delta(a) = \varphi(a)\delta \) for any \( a \in A \). From there one, the proof runs along the same lines as the preceding one.

Now, we will give a first application of this result.

**Lemma 6.11** Consider \( t \in \mathbb{R} \), then \( \sigma_t(A_0) \) is a subset of \( A_0 \).

**Proof:** First, we proof that \( \sigma_t(\pi(A)) \) is a subset of \( A_0 \).
Because $A_0$ is a dense left ideal in $A_r$, $\sigma_{-t}(A_0)$ is a dense left ideal in $A_r$.

Choose $a \in A_0$ and $\omega \in A_r^*$. Using proposition $[\mathbb{J}]$, we see that

$$(\omega \otimes t)\Delta(\sigma_{-t}(a)) = (\omega \otimes t)(\tau_{-t} \otimes \sigma_{-t})\Delta(a) = \sigma_{-t}( (\omega\tau_{-t} \otimes t)\Delta(a)),$$

Lemma $[\mathbb{L}]$ implies that $(\omega\tau_{-t} \otimes t)\Delta(a)$ belongs to $A_0$, so $(\omega \otimes t)\Delta(\sigma_{-t}(a))$ belongs to $\sigma_{-t}(A_0)$.

By proposition $[\mathbb{M}]$, we have that $\pi(A) \subseteq \sigma_{-t}(A_0)$, so $\sigma_t(\pi(A)) \subseteq A_0$.

Choose $x \in A_0$. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ in $\pi(A)$ such that $(x_n)_{n=1}^{\infty} \to x$ and $(\Lambda_{\varphi}(x_n))_{n=1}^{\infty} \to \Lambda_{\varphi}(x)$.

From the first part of the proof, we know that $\sigma_t(x_n)$ belongs to $A_0$ for all $n \in \mathbb{N}$. Furthermore, it is clear that $(\sigma_t(x_n))_{n=1}^{\infty} \to \sigma_t(x)$. Because $\sigma$ is invariant under $\varphi$, we know that

$$\|\Lambda_{\varphi}(\sigma_t(x_m)) - \Lambda_{\varphi}(\sigma_t(x_n))\| = \|\Lambda_{\varphi}(x_m) - \Lambda_{\varphi}(x_n)\|$$

for every $m, n \in \mathbb{N}$. This implies that $(\Lambda_{\varphi}(\sigma_t(x_n)))_{n=1}^{\infty}$ is Cauchy and hence convergent in $H$. The closedness of $A_0$ implies that $\sigma_t(x)$ belongs to $A_0$.

We are now able to prove a major result of this paper which says that the left invariant weight is completely determined by its values on $\pi(A)$.

**Theorem 6.12** We have that $\pi(A)$ is a core for $\Lambda_{\varphi}$.

**Proof**: Because $\pi(A)$ is dense in $A_r$, there exists a bounded net $(e_j)_{j \in J}$ in $\pi(A)$ such that $(e_j)_{j \in J}$ converges strictly to 1.

For every $j \in J$, we define

$$u_j = \frac{1}{\sqrt{2\pi}} \int \exp(-t^2) \sigma_t(e_j) \, dt \in A_r,$$

it is clear that $u_j$ belongs to $D(\sigma_{\frac{1}{2}})$ and

$$\sigma_{\frac{1}{2}}(u_j) = \frac{1}{\sqrt{2\pi}} \int \exp(-(t - \frac{i}{2})^2) \sigma_t(e_j) \, dt.$$

Furthermore, $(u_j)_{j \in J}$ and $(\sigma_{\frac{1}{2}}(u_j))_{j \in J}$ are both bounded nets in $A_r$ which converge strictly to 1, because $(e_j)_{j \in J}$ converges strictly to 1.

Next, we show that $u_j$ belongs to $A_0$ for all $j \in J$.

From lemma $[\mathbb{L}]$, we know that $\sigma_t(e_j)$ belongs to $A_0$ for every $t \in \mathbb{R}$. Also, $\Lambda_0(\sigma_t(e_j)) = \nabla^2 \Lambda_0(e_j)$ for every $t \in \mathbb{R}$. Therefore, the function $\mathbb{R} \to H : t \mapsto \exp(-t^2) \Lambda_0(\sigma_t(e_j))$ is integrable. The closedness of $A_0$ implies that $u_j$ belongs to $A_0$.

Choose $x \in \mathcal{N}_{\varphi}$. Then $(xu_j)_{j \in J} \to x$.

For all $j \in J$, we have that $xu_j$ belongs to $A_0$ (because $A_0$ is a left ideal) and

$$\Lambda_0(xu_j) = \Lambda_{\varphi}(xu_j) = J\sigma_{\frac{1}{2}}(u_j)^*J\Lambda_{\varphi}(x)$$

(here, we used lemma $[\mathbb{L}]$). Therefore, $(\Lambda_0(xu_j))_{j \in J} \to \Lambda_{\varphi}(x)$.

Because $\Lambda_0$ is closed, we see that $x \in A_0$ and $\Lambda_0(x) = \Lambda_{\varphi}(x)$.

At the end, we see that $A_0 = \mathcal{N}_{\varphi}$, so $\pi(A)$ is a core for $\Lambda_{\varphi}$.

This result allows us to prove the left invariance of $\varphi$ on the $C^*$-algebra level.
Theorem 6.13 Consider $x \in \mathcal{M}_\varphi$, then $\Delta(x)$ belongs to $\overline{\mathcal{M}}_{x \otimes \varphi}$ and $(\iota \otimes \varphi)\Delta(x) = \varphi(x)1$.

Proof: Choose $y \in \mathcal{N}_\varphi.

Take $z \in A_r$. Then there exist sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ in $A$ such that $(\pi(a_n))_{n=1}^\infty \rightarrow y$, $(\varphi(a_n))_{n=1}^\infty \rightarrow z$ and $(\varphi(a_n))_{n=1}^\infty \rightarrow \Lambda\varphi(y)$. Choose $c_1, c_2, d_1, d_2 \in A$. Then

$$
(\iota \otimes \varphi)((\Delta(c_1))(\pi(d_1) \otimes 1)), (\iota \otimes \varphi)((\Delta(c_2))(\pi(d_2) \otimes 1))
= (\iota \otimes \varphi)((\Delta(c_1))(\pi(d_1) \otimes 1)), (\iota \otimes \varphi)((\Delta(c_2))(\pi(d_2) \otimes 1))
= (\pi \otimes \varphi)((d_2^* \otimes 1)\Delta(c_2^*(c_1))(d_1 \otimes 1)) = \varphi(c_2^*c_1)\pi(d_2^*d_1)
$$

Therefore,

$$
(\iota \otimes \Lambda\varphi)((\Delta(c_1))(\pi(d_1) \otimes 1)), (\iota \otimes \Lambda\varphi)((\Delta(c_2))(\pi(d_2) \otimes 1))
= (\Lambda\varphi(c_1)), (\Lambda\varphi(c_2)) \pi(d_2^*\pi(d_1)). \quad (a)
$$

It is clear that $(\Delta(\pi(a_n))(\pi(b_n) \otimes 1))_{n=1}^\infty \rightarrow \Delta(y)(z \otimes 1)$. Using equation (a), we get for any $m, n \in \mathbb{N}$ that

$$
\|(\iota \otimes \Lambda\varphi)((\Delta(\pi(a_m))(\pi(b_m) \otimes 1)) - (\iota \otimes \Lambda\varphi)((\Delta(\pi(a_n))(\pi(b_n) \otimes 1))\|^2
= \|\Lambda\varphi(\pi(a_m)), \Lambda\varphi(\pi(a_n))\|^2 \pi(b_m)^*\pi(b_m) - \Lambda\varphi(\pi(a_m)), \Lambda\varphi(\pi(a_n))\|^2 \pi(b_n)^*\pi(b_n)
= \Lambda\varphi(\pi(a_m)), \Lambda\varphi(\pi(a_n))\|^2 \pi(b_n)^*\pi(b_n)
$$

This equation implies that $(\iota \otimes \Lambda\varphi((\Delta(\pi(a_n))(\pi(b_n) \otimes 1)))_{n=1}^\infty$ is a Cauchy sequence and hence convergent in $H \otimes A$. The closedness of $\iota \otimes \Lambda\varphi$ implies that $\Delta(y)(z \otimes 1)$ belongs to $\mathcal{N}_{x \otimes \varphi}$ and that $(\Delta(\pi(a_n))(\pi(b_n) \otimes 1))_{n=1}^\infty$ converges to $(\iota \otimes \Lambda\varphi)((\Delta(y)(z \otimes 1))$.

By equation (a), we have that

$$
(\iota \otimes \Lambda\varphi)((\Delta(\pi(a_n))(\pi(b_n) \otimes 1)), (\iota \otimes \Lambda\varphi)((\Delta(\pi(a_n))(\pi(b_n) \otimes 1))
= (\Lambda\varphi(\pi(a_n)), \Lambda\varphi(\pi(a_n))\|^2 \pi(b_n)^*\pi(b_n)
$$

for every $n \in \mathbb{N}$. So,

$$
(\iota \otimes \Lambda\varphi((\Delta(\pi(a_n))(\pi(b_n) \otimes 1)), (\iota \otimes \Lambda\varphi((\Delta(\pi(a_n))(\pi(b_n) \otimes 1)))_{n=1}^\infty
$$

converges to $(\Lambda\varphi(a), \Lambda\varphi(a))^\ast b$. Hence,

$$
(\iota \otimes \Lambda\varphi)((\Delta(y)(z \otimes 1)), (\iota \otimes \Lambda\varphi)((\Delta(y)(z \otimes 1))
= (\Lambda\varphi(y), \Lambda\varphi(y))^\ast z^\ast z.
$$

This implies that

$$
(\iota \otimes \varphi)((z^\ast \otimes 1)\Delta(y^\ast y)(z \otimes 1)) = \varphi(y^\ast y) z^\ast z.
$$

For every $\omega \in \mathcal{G}_\varphi$, we have that

$$
z^\ast (\iota \otimes \omega)(\Delta(y^\ast y)) z = (\iota \otimes \omega)((z^\ast \otimes 1)\Delta(y^\ast y)(1 \otimes z)). \quad (a)
$$

By the previous discussion, we know that

$$
((\iota \otimes \omega)((z^\ast \otimes 1)\Delta(y^\ast y)(1 \otimes z)))_{\omega \in \mathcal{G}_\varphi} \rightarrow \varphi(y^\ast y) z^\ast z.
$$

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The left invariance of \( \Delta(y^* y) \) belongs to \( \overline{\mathcal{M}}_{\otimes \varphi} \) and \((i \otimes \varphi)\Delta(y^* y) = \varphi(y^* y)1.\) Consequently, we get that \( \Delta(y^* y) \) belongs to \( \overline{\mathcal{M}}_{\otimes \varphi} \) and \((i \otimes \varphi)\Delta(y^* y) = \varphi(y^* y)1.\) The proposition follows by polarization.  

**Corollary 6.14** Consider \( x \in \mathcal{M}_\varphi \) and \( \omega \in A_+^\ast. \) Then \((\omega \otimes i)\Delta(x) \) belongs to \( \mathcal{M}_\varphi \) and \( \varphi((\omega \otimes i)\Delta(x)) = \varphi(x)\omega(1).\)

It is also possible to prove the strong left invariance proposed in the definition of Masuda, Nakagami & Woronowicz. This is in fact a direct consequence of formula 3.

**Proposition 6.15** Consider \( a, b \in \mathcal{N}_\varphi \) and \( \omega \in A_+^\ast \) such that \( \omega R_{-\frac{1}{2}} \) is bounded and call \( \theta \) the unique element in \( A_+^\ast \) which extends \( \omega R_{-\frac{1}{2}}. \) Then \( b^* (\omega \otimes i)\Delta(a) \) and \((\theta \otimes i)(\Delta(b^*)) a \) belong to \( \mathcal{M}_\varphi \) and

\[
\varphi(b^* (\omega \otimes i)\Delta(a)) = \varphi((\theta \otimes i)(\Delta(b^*)) a).
\]

**Proof:** Because of the left invariance of \( \varphi, \) it follows that \((\omega \otimes i)\Delta(a) \) belongs to \( \mathcal{N}_\varphi \) and \((\theta \otimes i)(\Delta(b^*)) \) belongs to \( \mathcal{N}_\varphi. \) This implies that \( b^* (\omega \otimes i)\Delta(a) \) and \((\theta \otimes i)(\Delta(b^*)) a \) belong to \( \mathcal{M}_\varphi. \)

Because \( \pi(A) \) is a core for \( \Lambda_\varphi, \) there exist sequences \((a_n)_{n=1}^\infty \) and \((b_n)_{n=1}^\infty \) in \( A \) such that \( (\pi(a_n))_{n=1}^\infty \to a, \) \( (\pi(b_n))_{n=1}^\infty \to b, \) \((\Lambda_\varphi(\pi(a_n)))_{n=1}^\infty \to \Lambda_\varphi(a) \) and \((\Lambda_\varphi(\pi(b_n)))_{n=1}^\infty \to \Lambda_\varphi(b).\)

The left invariance of \( \varphi \) implies that \((\Lambda_\varphi((\omega \otimes i)\Delta(a)))_{n=1}^\infty \to \Lambda_\varphi((\omega \otimes i)\Delta(a)) \) and \((\Lambda_\varphi((\theta \otimes i)(\Delta(b^n))))_{n=1}^\infty \to \Lambda_\varphi((\theta \otimes i)(\Delta(b^n))).\)

This implies that

\[
(\Lambda_\varphi((\omega \otimes i)\Delta(a))), \Lambda_\varphi(\pi(b^n)))_{n=1}^\infty
\]

converges to \( \varphi(b^*(\omega \otimes i)\Delta(a)) \) \( \text{(a)} \) and

\[
(\Lambda_\varphi(\pi(a^n)), \Lambda_\varphi((\theta \otimes i)(\Delta(b^n))))_{n=1}^\infty
\]

converges to \( \varphi((\theta \otimes i)(\Delta(b^n)) a) \) \( \text{(b)} \).

Using equation 3, we have for every \( n \in \mathbb{N} \) that

\[
(\Lambda_\varphi((\omega \otimes i)\Delta(a^n)), \Lambda_\varphi(\pi(b^n))) = \varphi(\pi(b^n)((\omega \otimes i)\Delta(a^n)))
\]

\[
= \varphi((\omega \otimes i)(1 \otimes \pi(b^n))\Delta(a^n))) \quad = \varphi((\omega \otimes i)((1 \otimes \pi)(\Delta(b^n)))(\Delta(a^n)))
\]

\[
= \omega(\pi((\omega \otimes i)((1 \otimes \pi(b^n))\Delta(a^n)))) \quad = \omega(\pi(S((\omega \otimes i)((\Delta(b^n)))(\Delta(a^n)))])
\]

\[
= \omega(R(\tau_{-\frac{1}{2}}\pi((\omega \otimes i)((\Delta(b^n))(1 \otimes a^n)))))) \quad = \theta(\pi((\omega \otimes i)((\Delta(b^n))(1 \otimes a^n))))
\]

\[
= \varphi((\theta \otimes i)((\Delta(b^n))(1 \otimes a^n))) \quad = \varphi((\theta \otimes i))(\Delta(\pi(b^n))(1 \otimes \pi(a^n)))
\]

\[
= \varphi((\theta \otimes i)(\Delta(\pi(b^n)))\pi(a^n)) \quad = (\Lambda_\varphi(\pi(a^n)), \Lambda_\varphi((\theta \otimes i)(\Delta(b^n))))
\]

Together with (a) and (b), this gives us that

\[
\varphi(b^*(\omega \otimes i)\Delta(a)) = \varphi((\theta \otimes i)(\Delta(b^*)) a).
\]
7 Invariance properties of bi-C*-isomorphisms and group-like elements.

In this section, we will consider group-like elements and *-isomorphisms which commute with the comultiplication. In particular, their behaviour with respect to the one-parameter groups $\sigma$ and $\tau$, as well as their behaviour with respect to the anti-unitary antipode $R$, is investigated. The propositions involved will mostly handle about some form of relative invariance. We will use a lot of these results for the first time in the next section, where we introduce the modular function of our C*-algebraic quantum group $(A_r, \Delta)$.

**Proposition 7.1** Consider a *-automorphism $\alpha$ on $A_r$ such that there exists a *-automorphism $\beta$ on $A_r$ such that $\Delta\alpha = (\beta \otimes \alpha)\Delta$. Then there exists a unique strictly positive number $r$ such that $\phi \alpha = r \phi$.

**Proof:** Because $N_\phi$ is a dense left ideal of $A_r$, $\alpha^{-1}(N_\phi)$ is a dense left ideal of $A_r$. Choose $\omega \in A_r^*$ and $a \in \alpha^{-1}(N_\phi)$. Then
\[
\alpha((\omega \otimes i)\Delta(a)) = (\omega\beta^{-1} \otimes i)(\beta \otimes \alpha)\Delta(a),
\]

Because $\alpha(a)$ belongs to $N_\phi$ and because of the left invariance of $\phi$, we have that $(\omega\beta^{-1} \otimes i)\Delta(\alpha(a))$ belongs to $N_\phi$. Therefore, $(\omega \otimes i)\Delta(a)$ belongs to $\alpha^{-1}(N_\phi)$. Proposition 7.10 implies that $\pi(A) \subseteq \alpha^{-1}(N_\phi)$, hence, $\alpha(\pi(A)) \subseteq N_\phi$. Now, using the fact that $A^*A = A$, it is not difficult to see that $\alpha(\pi(A)) \subseteq M_\phi$.

This allows us to define the positive linear functional $\phi$ on $A$ such that $\phi(a) = \phi(\alpha(\pi(a)))$ for every $a \in A$.

Choose $a \in A$. For any $b, c \in A$, we have that
\[
\varphi(c^* (i \otimes \varphi)(\Delta(a)) b) = \varphi( (\varphi \otimes i)((c^* \otimes 1)\Delta(a)(b \otimes 1)) ) \\
= \varphi( (\varphi \otimes i)((c^* \otimes 1)\Delta(a)(b \otimes 1)) ) = \varphi( (\omega_{\Lambda(b),\Lambda(c)} \otimes 1)(\Delta(\pi(a))) ) \\
= \varphi( (\omega_{\Lambda(b),\Lambda(c)}\beta^{-1} \otimes 1)((\beta \otimes \alpha)\Delta(\pi(a)))) = \varphi( (\omega_{\Lambda(b),\Lambda(c)}\beta^{-1} \otimes 1)\Delta(\alpha(\pi(a))) ) \\
= (\omega_{\Lambda(b),\Lambda(c)}\beta^{-1})(1) \varphi(\alpha(\pi(a))) = \omega_{\Lambda(b),\Lambda(c)}(1) \phi(a) = \varphi(c^* b) \phi(a)
\]

where we used the left invariance of the weight $\varphi$ in the third last equality. The faithfulness of $\varphi$ implies that $(i \otimes \varphi)\Delta(a) = \phi(a)1$.

By unicity of left invariant functionals on the *-algebra level, we get the existence of a strictly positive number $r$ such that $\phi = r \varphi$. So, $\varphi(\alpha(x)) = r \varphi(x)$ for every $x \in \pi(A)$.

So, we have for any $y \in \pi(A)$, that $\alpha(y)$ belongs to $N_\varphi$ and $\|\Lambda_\varphi(\alpha(y))\| = \sqrt{r} \|\Lambda_\varphi(y)\|$. Remembering that $\pi(A)$ is a core for $\Lambda_\varphi$, we get that $\alpha(y)$ belongs to $N_\varphi$ and $\|\Lambda_\varphi(\alpha(y))\| = \sqrt{r} \|\Lambda_\varphi(y)\|$ for every $y \in N_\varphi$.

Using the fact that $(\beta^{-1} \otimes \alpha^{-1})\Delta = \Delta \alpha^{-1}$, we infer completely analogously that $\alpha^{-1}(N_\varphi) \subseteq N_\varphi$, so $N_\varphi \subseteq \alpha(N_\varphi)$.

Combining these two results, we conclude that $\alpha(N_\varphi) = N_\varphi$ and $\|\Lambda_\varphi(\alpha(y))\| = \sqrt{r} \|\Lambda_\varphi(y)\|$ for all $y \in N_\varphi$.

The result follows.

**Proposition 7.2** Consider a *-automorphism $\alpha$ on $A_r$ such that there exists a *-automorphism $\beta$ on $A_r$ such that $\Delta\alpha = (\beta \otimes \alpha)\Delta$. Then $\sigma_t \alpha = \alpha \sigma_t$ and $\tau_t \beta = \beta \tau_t$ for every $t \in \mathbb{R}$. 

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Proof: By the previous proposition, we know that there exists a strictly positive number \( r \) such that \( \varphi \alpha = r \varphi \). Because \( \varphi \) is KMS with respect to \( \sigma \), we have that \( \sigma_t \alpha = \alpha \sigma_t \) for every \( t \in \mathbb{R} \).

Choose \( s \in \mathbb{R} \). Then

\[
(\beta \tau_s \otimes \sigma_s) \Delta = (\beta \otimes \iota) \Delta \sigma_s = (\iota \otimes \alpha^{-1}) \Delta \alpha \sigma_s
= (\iota \otimes \alpha^{-1}) \Delta \alpha \tau_s = (\iota \otimes \alpha^{-1})(\tau_s \otimes \sigma_s) \Delta \alpha
= (\tau_s \otimes \sigma_s)(\iota \otimes \alpha^{-1})(\beta \otimes \alpha) \Delta = (\tau_s \beta \otimes \sigma_s) \Delta.
\]

Therefore, \( (\beta \tau_s \otimes \iota) \Delta = (\tau_s \beta \otimes \iota) \Delta \), which, as before, implies that \( \beta \tau_s = \tau_s \beta \).

**Corollary 7.3** There exist a unique strictly positive number \( \nu \) such that \( \varphi \tau_t = \nu^t \varphi \) for all \( t \in \mathbb{R} \). For all \( s, t \in \mathbb{R} \), we have that \( \tau_s \sigma_t = \sigma_t \tau_s \).

We see that \( \varphi \) is relatively invariant with respect to the one-parameter group \( \tau \). In the definition of Masuda, Nakagami & Woronowicz, it is assumed that \( \tau \) is invariant with respect to \( \tau \). Until now, we did not find any examples where we have a real relative invariance, nor could we prove that \( \varphi \) is invariant with respect to \( \tau \). Therefore, the question remains open whether the definition of Masuda, Nakagami & Woronowicz should be adapted in this respect.

**Proposition 7.4** Consider a \( * \)-automorphism \( \alpha \) on \( A_r \) such that there exists a \( * \)-automorphism \( \beta \) on \( A_r \) such that \( \Delta \alpha = (\beta \otimes \alpha) \Delta \). Then \( R \beta = \beta R \).

Proof: We know from the previous results that there exists a strictly positive number \( r \) such that \( \varphi \alpha = r \varphi \). Moreover, we have for every \( t \in \mathbb{R} \) that \( \beta \tau_t = \tau_t \beta \), which implies that \( \tau_{-t} \beta = \beta \tau_{-t} \).

Choose \( a \in N_\varphi \) and \( \omega \in A_r^* \). Take \( \theta \in A_r^* \). For every \( n \in \mathbb{N} \), we define \( \theta_n \in A_r^* \) such that

\[
\theta_n(x) = \frac{n}{\sqrt{n}} \int \exp(-n^2 t^2) \theta(\tau_t(x)) \, dt
\]

for every \( x \in A_r \). It is clear that \( (\theta_n)_{n=1}^\infty \) converges weakly* to \( \theta \).

Fix \( m \in \mathbb{N} \) and define \( \eta \in A_r^* \) such that

\[
\eta(x) = \frac{m}{\sqrt{\pi}} \int \exp(-m^2(t - \frac{i}{2})) \theta(\tau_t(x)) \, dt
\]

for every \( x \in A_r \). It is not so difficult to see that \( \eta(\tau_{-\frac{1}{2}}(x)) = \theta_m(x) \) for every \( x \in D(\tau_{-\frac{1}{2}}) \). This implies that \( \eta \tau_{-\frac{1}{2}} \) is bounded and \( \eta \tau_{-\frac{1}{2}} = \theta_m(\alpha) \).

Choose \( b \in N_\varphi \). Then we see that

\[
\varphi( (\theta_m R \beta \otimes \iota)(\Delta(\alpha^*)) b) = \varphi(\alpha^{-1}( (\theta_m R \otimes \iota)(\Delta(\alpha^*) ) b)
= \varphi(\alpha^{-1}( (\theta_m R \otimes \iota)(\Delta(\alpha^*) ) \alpha(b) ) = r^{-1} \varphi( (\theta_m R \otimes \iota)(\Delta(\alpha^*) ) \alpha(b) ).
\]

We have that \( \eta R \tau_{-\frac{1}{2}} = \eta \tau_{-\frac{1}{2}} R \). Together with (a), this implies that \( \eta R \tau_{-\frac{1}{2}} \) is bounded and \( \eta R \tau_{-\frac{1}{2}} = \theta_m R \). Hence, using the foregoing chain of equalities and proposition 6.15 we get that

\[
\varphi( (\theta_m R \beta \otimes \iota)(\Delta(\alpha^*)) b) = r^{-1} \varphi(\alpha^*(\eta \otimes \iota)\Delta(\alpha(b) ) = r^{-1} \varphi(\alpha^*(\eta \beta \otimes \iota)\Delta(b) ) = r^{-1} \varphi(\alpha^*(\eta \beta \otimes \iota)\Delta(b) )
= \varphi(\alpha^*(\eta \beta \otimes \iota)\Delta(b) ).
\]
Furthermore, $\eta \beta R \tau_+ \beta R = \eta \tau_+ \beta R$. So, by using (a) once more, we see that $\eta \beta R \tau_+ \beta R$ is bounded and $\eta \beta R \tau_+ \beta R = \theta_m \beta R$. Referring to proposition 6.11 once again, we get that

$$\varphi(a^*(\eta \beta \otimes \iota)\Delta(b)) = \varphi((\theta_m \beta R \otimes \iota)(\Delta(a^*))b),$$

which implies that

$$\varphi((\theta_m \beta R \otimes \iota)(\Delta(a^*))b) = \varphi((\theta_m \beta R \otimes \iota)(\Delta(a^*))b).$$

Because $\varphi$ is faithful, we can conclude from this that

$$(\theta_m \beta R \otimes \iota)(\Delta(a^*)) = (\theta_m \beta R \otimes \iota)(\Delta(a^*)).$$

So, applying $\omega$ to this equation, gives

$$\theta_m((\beta R)((\iota \otimes \omega)\Delta(a^*))) = \theta_m((R\beta)((\iota \otimes \omega)\Delta(a^*)).$$

Because $(\theta_n)_{n=1}^{\infty}$ converges weakly* to $\theta$, we get that

$$\theta((\beta R)((\iota \otimes \omega)\Delta(a^*))) = \theta((R\beta)((\iota \otimes \omega)\Delta(a^*)).$$

Consequently, $(\beta R)((\iota \otimes \omega)\Delta(a^*)) = (R\beta)((\iota \otimes \omega)\Delta(a^*)).$ The density conditions imply that $\beta R = R\beta$.

We would like to mention the following application of these results:

Consider a locally compact group $G$ together with a norm continuous one parameter group $\alpha$ on $A_r$ such that $(\alpha_t \otimes \alpha_t)\Delta = \Delta \alpha_t$ for every $t \in \mathbb{R}$. Then we get from the preceding propositions that the weight $\varphi$ is relatively invariant under $\alpha$ and that $\alpha$ commutes with $\sigma, \tau$ and $R$. These kind of properties allow us to make a left Haar weight with corresponding modular group, a scaling group and a anti-unitary antipode on the level of the crossed product $G \times_{\alpha} A_r$, turning it into a $C^*$-algebraic quantum group according to Masuda, Nakagami & Woronowicz.

Now, we turn our attention to group-like elements.

**Proposition 7.5** Consider a unitary element $u \in M(A_r)$ such that there exists an element $v \in M(A_r)$ such that $\Delta(u) = v \otimes u$. Then there exist a unique strictly positive number $\lambda$ such that $\sigma_t(u) = \lambda^t u$ for every $t \in \mathbb{R}$. Moreover, $\tau_t(v) = v$ for all $t \in \mathbb{R}$.

**Proof:** We first proof the assertion about $\sigma$.

1. Define the *-automorphisms $\alpha, \beta$ on $A_r$ such that $\alpha(a) = u^*au$ and $\beta(a) = v^*av$ for all $a \in A$. The formula $\Delta(u) = v \otimes u$ implies easily that $(\beta \otimes \alpha)\Delta = \Delta \alpha$. By proposition 7.1 there exists a strictly positive number $\mu$ such that $\varphi \alpha = \mu \varphi$. Consequently, we have for every $x \in M_\varphi$ that $u^*xu$ belongs to $M_\varphi$ and $\varphi(u^*xu) = \mu \varphi(x)$.

   Hence, we have for all $y \in N_\varphi$ that $(yu)^*(yu) = u^*y^*yu \in M_\varphi^+$ so that $yu$ belongs to $N_\varphi$. Thus, $N_\varphi u \subseteq N_\varphi$.

2. Analogously, working with $u^*$ in stead of $u$, one proves that $N_\varphi u^* \subseteq N_\varphi$, so $uN_\varphi^* \subseteq N_\varphi^*$.

From 1) and 2), it follows immediately that $uM_\varphi \subseteq M_\varphi$ and $M_\varphi u \subseteq M_\varphi$.

Choose $x, y \in N_\varphi$. From 2), it follows that $uy^*$ belongs to $N_\varphi^*$, so $uy^*x$ belongs to $M_\varphi$. Using 1), we get that

$$\varphi(y^*xu) = \varphi(u^*(uy^*)u) = \mu \varphi(u^*x).$$
Using lemma [10.3] we conclude that \( \sigma_t(u) = \mu^{-it} u \) for all \( t \in \mathbb{R} \). Putting \( \lambda = \frac{1}{\mu} \), we see that \( \sigma_t(u) = \lambda^{it} u \) for all \( t \in \mathbb{R} \).

Now, we will prove the assertion about \( \tau \). Choose \( t \in \mathbb{R} \). Then

\[
\tau_t(v) \otimes \sigma_t(u) = (\tau_t \otimes \sigma_t) \Delta(u) = \Delta(\sigma_t(u)) = \lambda^{it} \Delta(u) = \lambda^{it} v \otimes u = v \otimes \sigma_t(u).
\]

Therefore, \( \tau_t(v) = v \).

In the following proposition, we prove that every one dimensional unitary corepresentation of \( A_r \) is of an algebraic nature. We first prove a little lemma.

**Lemma 7.6** Consider \( a, b, c \in A \) and \( x \in M(A_r) \), then \( (\iota \otimes \omega_{\Lambda(b),\Lambda(c)})(\Delta(x)) \pi(a) \) belongs to \( \pi(A) \).

**Proof:** We know that there exist \( p_1, \ldots, p_n, q_1, \ldots, q_n, r_1, \ldots, r_n \in A \) such that

\[
a \otimes b \rho(c^*) = \sum_{i=1}^{n} \Delta(p_i \rho(q_i^*))(r_i \otimes 1).
\]

For every \( y \in A \), we have that

\[
(i \otimes \omega_{\Lambda(b),\Lambda(c)})(\Delta(\pi(y))(\pi(a) \otimes 1)) = (i \otimes \omega_{\Lambda(b),\Lambda(c)})(\pi \otimes \pi)(\Delta(y)(a \otimes 1))
\]

\[= \pi(i \otimes \varphi)((1 \otimes c^*)\Delta(y)(a \otimes b)) = \pi((i \otimes \varphi)(\Delta(y)(a \otimes b \rho(c^*)))
\]

\[= \sum_{i=1}^{n} \pi(i \otimes \varphi)(\Delta(yp_i \rho(q_i^*))r_i \otimes 1)) = \sum_{i=1}^{n} \pi(\varphi(yp_i \rho(q_i^*))r_i)
\]

\[= \sum_{i=1}^{n} \varphi(q_i^* yp_i) \pi(r_i) = \sum_{i=1}^{n} \omega_{\Lambda(p_i),\Lambda(q_i)}(\pi(y))\pi(r_i).
\]

Because \( \pi(A) \) is dense in \( A_r \) and because of strict continuity arguments we can replace \( \pi(y) \) in this equation by any element in \( M(A_r) \). Hence,

\[
(i \otimes \omega_{\Lambda(b),\Lambda(c)})(\Delta(x)(\pi(a) \otimes 1)) = \sum_{i=1}^{n} \omega_{\Lambda(p_i),\Lambda(q_i)}(x)\pi(r_i).
\]

Consequently, \( (i \otimes \omega_{\Lambda(b),\Lambda(c)})(\Delta(x)) \pi(a) \) belongs to \( \pi(A) \).

**Proposition 7.7** Consider a unitary \( u \) in \( M(A_r) \) such that \( \Delta(u) = u \otimes u \). Then there exists a unique \( x \in M(A) \) such that \( u \pi(a) = \pi(xa) \) and \( \pi(a) u = \pi(ax) \) for every \( a \in A \). Moreover, \( \Delta(x) = x \otimes x \).

**Proof:** Choose \( a \in A \). Now, there exist \( b, c \in A \) such that \( \langle u \Lambda(b), \Lambda(c) \rangle = 1 \). Therefore,

\[
u \pi(a) = u \pi(a) \omega_{\Lambda(b),\Lambda(c)}(u) = (i \otimes \omega_{\Lambda(b),\Lambda(c)})(u \otimes u) \pi(a)
\]

\[= (i \otimes \omega_{\Lambda(b),\Lambda(c)})(\Delta(u)) \pi(a).
\]

which, by the previous lemma, implies that \( u \pi(a) \) belongs to \( \pi(A) \).

Using the fact that \( \Delta(u^*) = u^* \otimes u^* \), we conclude also that \( u^* \pi(A) \subseteq \pi(A) \), hence \( \pi(A) u \subseteq \pi(A) \).

Because \( u \pi(A) \) and \( \pi(A) u \) are subsets of \( \pi(A) \), it follows easily that there exist \( x \in M(A) \) such that \( u \pi(a) = \pi(xa) \) and \( \pi(a) u = \pi(ax) \) for every \( a \in A \). It is also not so difficult to prove that \( x \) is unitary and that \( \Delta(x) = x \otimes x \).

\[\blacksquare\]
Proposition 7.8 Consider a unitary element \( u \in M(A_r) \) such that \( \Delta(u) = u \otimes u \). Then \( R(u) = u^* \).

**Proof:** The previous proposition ensures the existence of a unique unitary element \( x \in M(A) \) such that \( \pi(a)u = \pi(ax) \) and \( u\pi(a) = \pi(xa) \) for all \( a \in A \). Moreover, \( \Delta(x) = x \otimes x \). This last equation implies that \( S(x) = x^* \).

Choose \( a \in A \). For any \( t \in \mathbb{R} \), we have that

\[
\tau_t(\pi(S(ax^*)) = \tau_t(\pi(S(a))^* = \tau_t(\pi(S(a)))^* u^*,
\]

where we used proposition 7.5. This implies that

\[
\tau_t(\pi(S(a))^*) = \tau_t(\pi(S(a)))^* = R(\pi(a))^* u^*.
\]

Furthermore,

\[
R(\pi(a)) R(u) = R(u\pi(a)) = R(\pi(xa)) = \tau_t(\pi(S(xa)))
= \tau_t(\pi(S(a)S(x))) = \tau_t(\pi(S(a)x^*)) = R(\pi(a))^* u^*.
\]

So, we can conclude that \( R(u) = u^* \).

Now, we want to use these results about unitary elements to prove results about strictly positive elements affiliated with the \( C^* \)-algebra \( A_r \). They will apply to the modular function of our quantum group.

Proposition 7.9 Consider a strictly positive element \( \alpha \in A_r \) such that there exists a strictly positive element \( \beta \in A_r \) such that \( \Delta(\alpha) = \beta \otimes \alpha \). Then there exists a unique strictly positive number \( \lambda \) such that \( \sigma_t(\alpha) = \lambda^t \alpha \) for every \( t \in \mathbb{R} \). Moreover, \( \tau_t(\beta) = \beta \) for every \( t \in \mathbb{R} \).

**Proof:** First, we will prove the assertion about \( \sigma \).

Choose \( s \in \mathbb{R} \). Then \( \Delta(\alpha^s) = \beta^s \otimes \alpha^s \). By proposition 7.5, there exists a unique strictly positive number \( \lambda_s \) such that \( \sigma_t(\alpha^s) = (\lambda_s)^t \alpha^s \) for every \( t \in \mathbb{R} \).

Choose \( s_1, s_2 \in \mathbb{R} \).

For every \( t \in \mathbb{R} \), we have that

\[
(\lambda_{s_1+s_2})^t \alpha^{(s_1+s_2)i} = \sigma_t(\alpha^{(s_1+s_2)i}) = \sigma_t(\alpha^{s_1i})\sigma_t(\alpha^{s_2i})
= (\lambda_{s_1})^t \alpha^{s_1i}(\lambda_{s_2})^t \alpha^{s_2i} = (\lambda_{s_1})^t \alpha^{(s_1+s_2)i},
\]

hence, \( (\lambda_{s_1+s_2})^t = (\lambda_{s_1})^t (\lambda_{s_2})^t \).

This implies that \( \lambda_{s_1+s_2} = \lambda_{s_1} \lambda_{s_2} \).

We put \( \lambda = \lambda_1 \in \mathbb{R}_0^+ \).

From the previous result, it follows easily that \( \lambda_q = \lambda^q \) for every \( q \in \mathbb{Q} \).

Fix \( t \in \mathbb{R} \).

For any \( s \in \mathbb{R} \), we have that \( (\lambda_s)^t 1 = \sigma_t(\alpha^s) \alpha^{-s} \). Therefore, because the mapping \( \mathbb{R} \to M(A_r) : s \mapsto \sigma_t(\alpha^s) \alpha^{-s} \) is strictly continuous, the mapping \( \mathbb{R} \to \mathbb{C} : s \mapsto (\lambda_s)^t \) is continuous.

We know also that \( (\lambda_q)^t = \lambda^q t \) for \( q \in \mathbb{Q} \), thus \( (\lambda_s)^t = \lambda^{ist} \) for every \( s \in \mathbb{R} \).

Hence,

\[
\sigma_t(\alpha) = \sigma_t(\alpha^s) = (\lambda_s)^t \alpha^s = \lambda^{ist} \alpha^s = (\lambda^t \alpha)^s
\]

for all \( s \in \mathbb{R} \). Consequently, \( \sigma_t(\alpha) = \lambda^t \alpha \).

Next, we quickly prove the assertion concerning \( \tau \).

Choose \( t \in \mathbb{R} \).

We have for any \( s \in \mathbb{R} \) that \( \Delta(\alpha^s) = \beta^s \otimes \alpha^s \), so proposition 7.5 implies that \( \tau_t(\beta^s) = \beta^s \), hence \( \tau_t(\beta^s) = \beta^s \). Therefore, we see that \( \tau_t(\beta) = \beta \).
Proposition 7.10 Consider a strictly positive element $\alpha \eta A_r$ such that $\Delta(\alpha) = \alpha \otimes \alpha$. Then $R(\alpha) = \alpha^{-1}$.

Proof: Choose $t \in R$. Then $\Delta(\alpha^{it}) = \alpha^{it} \otimes \alpha^{it}$. Proposition 7.10 implies that $R(\alpha^{it}) = \alpha^{-it}$, hence $R(\alpha)^{it} = (\alpha^{-1})^{it}$. This implies that $R(\alpha) = \alpha^{-1}$.

Lemma 7.11 Consider $x \in M(A_r)$ such that $\Delta(x) = x \otimes 1$. Then $x$ belongs to $\mathcal{C}1$.

Proof: By proposition 3.15, we have that
\[ \Delta(\sigma_t(x)) = (\sigma_t \otimes K_t)\Delta(x) = (\sigma_t \otimes K_t)(x \otimes 1) = \sigma_t(x) \otimes 1 \quad (a) \]
for every $t \in \mathbb{R}$. Fix $n \in \mathbb{N}$ and define $x_n \in M(A_r)$ such that
\[ x_n a = \frac{n}{\sqrt{n}} \int \exp(-nt^2) \sigma_t(x) a \, dt \]
for all $a \in A_r$. Then, equation (a) implies that $\Delta(x_n) = x_n \otimes 1$.

Define the function $F$ from $\mathcal{C}$ to $M(A_r)$ such that
\[ F(z) a = \frac{n}{\sqrt{n}} \int \exp(-nt^2(t-z)^2) \sigma_t(x) a \, dt \]
for all $z \in \mathcal{C}$ and $a \in A_r$. It is clear that $F$ is strictly analytic and hence analytic on $\mathcal{C}$. Moreover, we have that $F(u) = \sigma_u(x_n)$ for every $u \in \mathbb{R}$.

Using a standard technique, it is not difficult to prove, using lemma 6.10, that $\mathcal{N}_\varphi x_n \subseteq \mathcal{N}_\varphi$ and that $\Lambda_\varphi(ax_n) = JF(\frac{1}{t})^*J\Lambda_\varphi(a)$ for every $a \in \mathcal{N}_\varphi$. So, we get that $\mathcal{M}_\varphi x_n \subseteq \mathcal{M}_\varphi$.

Now, we are in a position to use a technique which we borrowed from Woronowicz.

There exists an element $d \in \mathcal{M}_\varphi$ such that $\varphi(d) = 1$. By the previous result, we know that $dx_n$ belongs also to $\mathcal{M}_\varphi$. Moreover, $\Delta(d)(x_n \otimes 1) = \Delta(dx_n)$.

Choose $\omega \in A_r^*$ and apply $\omega \otimes 1$ to the previous equation, this gives $(x_n \omega \otimes 1)\Delta(d) = (\omega \otimes 1)\Delta(dx_n)$ \quad (b).

The left invariance of $\varphi$ implies that $(x_n \omega \otimes 1)\Delta(d)$ belongs to $\mathcal{M}_\varphi$ and
\[ \varphi((x_n \omega \otimes 1)\Delta(d)) = (x_n \omega)(1) \varphi(d) = \omega(x_n) \]

At the same time, we get that $(\omega \otimes 1)\Delta(dx_n)$ belongs to $\mathcal{M}_\varphi$ and
\[ \varphi((\omega \otimes 1)\Delta(dx_n)) = \omega(1) \varphi(dx_n) \]

Combining these two equalities, using equation (b), gives us that $\omega(x_n) = \omega(\varphi(dx_n)1)$.

Therefore, we have that $x_n = \varphi(dx_n)1$.

It is clear that $(x_n)_{n=1}^\infty$ converges strictly to $x$. Using the Cauchy criterium, this implies easily that there exists a complex number $c$ such that $(\varphi(dx_n))_{n=1}^\infty \rightarrow c$. Then, $x$ must be equal to $c1$.

Using the fact that $\chi(R \otimes R) \Delta = \Delta R$, it is straightforward to arrive at the following conclusion.

Lemma 7.12 Consider $x \in M(A_r)$ such that $\Delta(x) = 1 \otimes x$. Then $x$ belongs to $\mathcal{C}1$.

It is clear that all the things we prove about $A_r$, we can also prove about $\hat{A}_r$. For instance, we can extend the right Haar functional $\hat{\psi}$ on $\hat{A}$ to a weight on $\hat{A}_r$ in the same way as we did for $\varphi$. We will go a little bit further into this because this gives a nice implementation of the polar decomposition of the antipode.

As can be expected, we will introduce again a left Hilbert algebra:
**Definition 7.13** We define $\mathcal{U} = \hat{\Lambda}(A)$, so $\mathcal{U}$ is a dense subspace of $H$. We will make $\mathcal{U}$ into a $^*$-algebra in the following way.

1. For all $a, b \in A$, we have that $\hat{\Lambda}(a)\hat{\Lambda}(b) = \hat{\Lambda}(ab)$.
2. For all $a \in A$, we have that $\hat{\Lambda}(a)^\circ = \hat{\Lambda}(a^*)$.

Just as before, we have that $\mathcal{U}$ is a left Hilbert algebra on $H$.

Again, we will introduce some terminology in connection with this left Hilbert algebra. We will have that $\hat{\Lambda}(a) = \hat{\Lambda}(\hat{a})$ for all $a \in A$.

Define the mapping $\hat{\Lambda}$ as the closed antilinear map from within $H$ into $H$ such that $\hat{\Lambda}(a)$ is a core for $\hat{T}$.

Furthermore, we define $\hat{\nabla} = T^* \hat{T}$. We also define $\hat{J}$ as the anti-unitary operator arising from the polar decomposition of $\hat{T}$. Hence, $\hat{T} = \hat{J}\hat{\nabla}^{\frac{1}{2}}$.

We are going to interpret these maps $\hat{T}$ in another way which will have some implications on the level of the antipode. Remember that, by definition, $\Lambda(a) = \hat{\Lambda}(\hat{a})$ for every $a \in A$. So, $\mathcal{U}$ equals $\Lambda(A)$ as a subspace of $H$. Therefore $\Lambda(A)$ is a core for $T$.

**Lemma 7.14** We have that $\hat{a}^* = (S(a)^* \delta)^\dagger$ for every $a \in A$.

**Proof:** Choose $x \in A$. Then

$$\hat{a}^*(x) = \overline{a(S(x)^*)} = \varphi(S(x)^*a) = \varphi(a^*S(x)) = \varphi(S(xS(a)^*)) .$$

Because $\varphi S = \delta \varphi$, we get that

$$\hat{a}^*(x) = \varphi(xS(a)^* \delta) = (S(a)^* \delta)^\dagger(x) .$$

This lemma implies easily that $\hat{T}\Lambda(a) = \Lambda(S(a)^* \delta)$ for every $a \in A$. A short investigation learns that we are in a situation where the construction of section 4 applies. Consider the construction procedure of section 4 with $\eta = \varphi$, $x = \frac{1}{\tau} \delta$ and $y = \delta$. Moreover, take $K = H$ and $\Gamma = \Lambda$.

If we go through this construction procedure, we see that $G = T$. This implies that $\mathcal{P} = \hat{\nabla}$ and $\mathcal{J} = \hat{J}$.

In this case $V = W$, so proposition 4.4 of section 4 implies the following result.

**Proposition 7.15** We have that $(\hat{J} \otimes J)W = W^*(\hat{J} \otimes J)$ and $(\hat{\nabla} \otimes \nabla)W = W(\hat{\nabla} \otimes \nabla)$.

A similar result can be found in [10]. Using this proposition, we can prove easily the following one.

**Proposition 7.16**

1. For every $t \in \mathbb{R}$ and $x \in A_r$, we have that $\tau_t(x) = \hat{\nabla}^{it} x \hat{\nabla}^{-it}$.

2. For every $x \in A_r$, we have that $R(x) = \hat{J} x^* \hat{J}$.

**Proof:** Fix $t \in \mathbb{R}$. Choose $\omega \in B_0(H)^*$. We know by the above commutation relation that

$$(\hat{\nabla}^{it} \otimes 1) W (\hat{\nabla}^{-it} \otimes 1) = (1 \otimes \nabla^{-it}) W (1 \otimes \nabla^{it}) .$$

Applying $t \otimes \omega$ to this equation, gives us that

$$\hat{\nabla}^{it} (t \otimes \omega)(W) \hat{\nabla}^{-it} = (t \otimes \nabla^{it} \omega \nabla^{-it})(W) .$$
Therefore, equation 8 implies that
\[ \hat{\nabla}^{it}(\iota \otimes \omega)(W) \hat{\nabla}^{-it} = \tau_t(\iota \otimes \omega)(W) \]
Hence, it follows that \( \tau_t(x) = \hat{\nabla}^{it}x \hat{\nabla}^{-it} \) for every \( x \in A_r \).
The assertion about \( R \) is proven in a similar way.

This implementation of \( R \), together with lemma 7.11 allows us to prove another result which can be found in [10].

**Proposition 7.17** We have that \( M(A_r) \cap M(\hat{A}_r) \equiv \mathbb{C}1 \).

**Proof:** The following argument is due to [10], proposition 3.11.
Choose \( x \in M(A_r) \cap M(\hat{A}_r) \). Because \( x \) belongs to \( M(\hat{A}_r) \), we have that \( x \) belongs to \( L(\hat{U}) \). By the theory of left Hilbert algebras, we know that \( Jx^*J \) belongs to \( L(\hat{U})' \). Therefore \( R(x) \) belongs to \( L(\hat{U})' \),
Remembering that \( W \) belongs to \( M(A_r \otimes \hat{A}_r) \), this implies that \( W^*(1 \otimes R(x))W = 1 \otimes R(x) \).
We know also that \( R(x) \) belongs to \( M(A_r) \) and
\[ \Delta(R(x)) = W^*(1 \otimes R(x))W = 1 \otimes R(x) . \]
Lemma 7.11 implies that \( R(x) \) belongs to \( \mathbb{C}1 \), so \( x \) belongs to \( \mathbb{C}1 \).

8 The modular function.

We already have a modular function \( \delta \) on the \( * \)-algebra level. In this section, we are going to introduce the modular function of our \( C^* \)-algebraic quantum group. For the notations used in this part, we refer to section 3.

It is easy to see that \( \langle \Lambda_\delta(a), \Lambda_\delta(b) \rangle = \langle \Lambda(a), \Lambda(\delta b) \rangle \) for all \( a, b \in A \). As before, this result justifies the following definition

**Definition 8.1** We define the closed linear operator \( L \) from within \( H \) into \( H_\delta \) such that \( \Lambda(A) \) is a core for \( L \) and \( L\Lambda(a) = \Lambda_\delta(a) \) for every \( a \in A \).
Then \( L \) is a densely defined injective operator with dense range.

As before, it is easy to check that
\[ \langle Lv, \Lambda_\delta(a) \rangle = \langle v, \Lambda(\delta a) \rangle \]
for every \( a \in A \). It follows that \( \Lambda_\delta(A) \) is a subset of \( D(L^*) \) and \( L^*\Lambda_\delta(a) = \Lambda(\delta a) \) for all \( a \in A \).

Next, we will give the definition of the modular function of the our quantum group \( A_r \). Again, we will use the same symbol \( \delta \) as for the modular function on the \( * \)-algebra level. By looking on which elements \( \delta \) acts, it should be clear which \( \delta \) is meant (elements in \( A \) vs. elements in \( A_r \)).

**Definition 8.2** We define \( \delta = L^*L \), so \( \delta \) is an injective positive operator in \( H \).

It is clear that \( \Lambda(A) \) is a subset of \( D(\delta) \) and that \( \delta \Lambda(a) = \Lambda(\delta a) \). Later, we will prove that \( \Lambda(A) \) is a core for \( \delta \). In the next part, we want to prove that \( \delta \) is affiliated with \( A_r \).
Lemma 8.3 Consider $a, b \in A$, then
\[
\langle (\Lambda_{\delta} \circ \Lambda_{\delta})(\Delta(b)(a \otimes 1)), (\Lambda_{\delta} \circ \Lambda_{\delta})(\Delta(d)(c \otimes 1)) \rangle \\
= \langle \Lambda(a) \otimes \Lambda_{\delta}(b), \Lambda(c) \otimes \Lambda_{\delta}(d) \rangle.
\]

Proof: We have that
\[
\langle (\Lambda_{\delta} \circ \Lambda_{\delta})(\Delta(b)(a \otimes 1)), (\Lambda_{\delta} \circ \Lambda_{\delta})(\Delta(d)(c \otimes 1)) \rangle \\
= (\varphi \otimes \varphi)((c^* \otimes 1)\Delta(d')(d \otimes \delta)\Delta(b)(a \otimes 1)) \\
= (\varphi \otimes \varphi)((c^* \otimes 1)\Delta(d')\Delta(h)(a \otimes 1)) \\
= (\varphi \otimes \varphi)((c^* \otimes 1)\Delta(d' \delta b)(a \otimes 1)).
\]
So, by the left invariance of $\varphi$, we find that
\[
\langle (\Lambda_{\delta} \circ \Lambda_{\delta})(\Delta(b)(a \otimes 1)), (\Lambda_{\delta} \circ \Lambda_{\delta})(\Delta(d)(c \otimes 1)) \rangle \\
= \varphi(c^* a) \varphi(d' \delta b) = \langle \Lambda(a) \otimes \Lambda_{\delta}(b), \Lambda(c) \otimes \Lambda_{\delta}(d) \rangle.
\]

Lemma 8.4 We have that $(1 \otimes \delta)W = W(\delta \otimes \delta)$.

Proof: By the previous lemma, there exists a unitary operator $U$ from $H \otimes H_{\delta}$ to $H_{\delta} \otimes H_{\delta}$ such that
\[
U(\Lambda(a) \otimes \Lambda_{\delta}(b)) = (\Lambda_{\delta} \circ \Lambda_{\delta})(\Delta(b)(a \otimes 1))
\]
for every $a, b \in A$. We easily infer that
\[
(L \otimes L)W^* (\Lambda(a) \otimes \Lambda(b)) = (L \otimes L)(\Lambda \circ \Lambda)(\Delta(b)(a \otimes 1)) \\
= (\Lambda_{\delta} \circ \Lambda_{\delta})(\Delta(b)(a \otimes 1)) = U(\Lambda(a) \otimes \Lambda_{\delta}(b)) \\
= U(1 \otimes L)(\Lambda(a) \otimes \Lambda(b))
\]
for every $a, b \in A$. Using, lemma once more, we get that $(L \otimes L)W = W(1 \otimes L)$. As before, this implies that $W(\delta \otimes \delta) = (1 \otimes \delta)W$.

We are now in a position to use a technique from Woronowicz.

Proposition 8.5 We have that $\delta$ is a strictly positive element affiliated with $A_r$ (in the $C^*$-algebra sense).

Proof: Choose $t \in \mathbb{R}$ and $\omega \in B_0(H)$. We know that $(1 \otimes \delta^i t)W(1 \otimes \delta^{-i} t) = W(\delta^i t \otimes 1)$. Applying $\iota \otimes \omega$ to this equation gives that $(\iota \otimes \omega)(W) \delta^i t = (\iota \otimes \delta^{-i} t \omega \delta^i t)(W)$.

Consequently, we have for every $\omega \in B_0(H)$ that

1. For every $t \in \mathbb{R}$, $(\iota \otimes \omega)(W) \delta^i t$ belongs to $A_r$.

2. The mapping $\mathbb{R} \to A_r : t \mapsto (\iota \otimes \omega)(W) \delta^i t$ is continuous.

It follows easily that every $a \in A_r$ satisfies:

1. For every $t \in \mathbb{R}$, a $\delta^i t$ belongs to $A_r$. 

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2. The mapping $\mathbb{R} \to A_r : t \mapsto a \delta^t$ is continuous.

The conclusion follows from these two results.

sometimes, we will look at $\delta$ as an operator in $H$. In other cases, we will look at $\delta$ as an element affiliated with $A_r$. It will be clear from the context which viewpoint is under consideration.

**Proposition 8.6** We have that $\Delta(\delta) = \delta \otimes \delta$.

**Proof:** Choose $t \in \mathbb{R}$. We know that $(1 \otimes \delta^t) W = W(\delta^t \otimes \delta^t)$. Hence,

$$\Delta(\delta)^t = \Delta(\delta^t) = W^*(1 \otimes \delta^t) W \delta^t \otimes \delta^t = (\delta \otimes \delta)^t.$$ 

Therefore, $\Delta(\delta) = \delta \otimes \delta$.

Because of this equation and the results of the previous section, we have already some nice results about $\delta$.

1. For every $t \in \mathbb{R}$, we have that $\delta^t \pi(A) \subseteq \pi(A)$ and $\pi(A) \delta^t \subseteq \pi(A)$, so $\delta^t$ belongs to the algebraic multiplier algebra of $\pi(A)$.

2. There exist a unique strictly positive number $\gamma$ such that $\sigma_t(\delta) = \gamma \delta$. Later, we will prove that $\gamma = \nu^{-1}$.

3. For every $t \in \mathbb{R}$, we have that $\tau_t(\delta) = \delta$.

4. We have that $R(\delta) = \delta^{-1}$.

**Lemma 8.7** Consider $a \in A$ and $n \in \mathbb{Z}$. Then $\Lambda(a)$ belongs to $D(\delta^n)$ and $\delta^n \Lambda(a) = \Lambda(\delta^n a)$.

This proposition is trivially true for $n = 0$. We know already that it is true for $n = 1$. This implies easily the result for $n = -1$. Induction will give us the general result.

**Lemma 8.8** For every $a \in A$ and $z \in \mathbb{C}$, we have that $\Lambda(a)$ belongs to $D(\delta^z)$.

In the next propositions, we will look at $\delta$ as an element affiliated with $A_r$.

**Proposition 8.9** Consider $a \in A$ and $n \in \mathbb{Z}$. Then $\pi(a)$ belongs to $D(\delta^n)$ and $\delta^n \pi(a) = \pi(\delta^n a)$.

**Proof:** Choose $x \in A$. Then $\pi(a) \Lambda(x) = \Lambda(ax) \in D(\delta^n)$ and

$$\delta^n(\pi(a) \Lambda(x)) = \Lambda(\delta^n ax) = \pi(\delta^n a) \Lambda(x).$$

Using the closedness of $\delta^n$, it is not so difficult to prove that $\delta^n \pi(a) = \pi(\delta^n a)$ as operators on $H$.

Therefore, $\pi(a)$ belongs to $D(\delta^n)$ and $\delta^n \pi(a) = \pi(\delta^n a)$.

**Corollary 8.10** For every $z \in \mathbb{C}$ and $a \in A$, we have that $\pi(a)$ belongs to $D(\delta^z)$.

**Lemma 8.11** Consider an element $\alpha \eta A_r$ and elements $a \in D(\alpha)$, $b \in A_r$. Then $\Delta(a)(b \otimes 1)$ belongs to $D(\Delta(\alpha))$ and $\Delta(\alpha)(\Delta(a)(b \otimes 1)) = \Delta(\alpha(a))(b \otimes 1)$.
Proof: Take an approximate unit \((e_k)_{k \in K}\) for \(A_r\). We know by the general theory of affiliated elements that we have for any \(k \in K\) that \(\Delta(a) (b \otimes e_k)\) belongs to \(\mathcal{D}(\Delta(a))\) and \(\Delta(a)(\Delta(a)(b \otimes e_k)) = \Delta(\alpha(a))(b \otimes e_k)\).

Because \(\Delta(a)(b \otimes 1)\) and \(\Delta(\alpha(a))(b \otimes 1)\) belong to \(A_r \otimes A_r\), we see that

\[
(\Delta(a)(b \otimes e_k))_{k \in K} \to \Delta(a)(b \otimes 1)
\]

and

\[
(\Delta(\alpha(a))(b \otimes e_k))_{k \in K} \to \Delta(\alpha(a))(b \otimes 1).
\]

The closedness of \(\Delta(a)\) implies that \(\Delta(a)(b \otimes 1)\) belongs to \(\mathcal{D}(\Delta(a))\) and \(\Delta(a)(\Delta(a)(b \otimes 1)) = \Delta(\alpha(a))(b \otimes 1)\).

We will now show that \(\delta^z\) is of an algebraic nature. The proof is a slight adaptation of the proof of proposition \(\ref{prop:delta-z-algebraic}\).

**Proposition 8.12** Consider \(z \in \mathcal{C}\), then \(\delta^z \pi(A) \subseteq \pi(A)\).

**Proof:** Choose \(a \in A\). Take also a non zero element \(b \in A\), then \(\pi(b)\) belongs to \(\mathcal{D}(\delta^z)\) and \(\delta^z \pi(b) \neq 0\) (because \(\delta^z\) is injective). Hence, there exist \(c, d \in A\) such that \((\delta^z \pi(b)) \Lambda(c, \Lambda(d)) = 1\). Moreover, there exist \(p_1, \ldots, p_n, q_1, \ldots, q_n \in A\) such that

\[
a \otimes b = \sum_{i=1}^{n} \Delta(p_i)(q_i \otimes 1).
\]

We know that \(\Delta(\delta^z) = \delta^z \otimes \delta^z\). So, by the previous lemma,

\[
\delta^z \pi(a) \otimes \delta^z \pi(b) = \Delta(\delta^z)(\pi(a) \otimes \pi(b))
= \sum_{i=1}^{n} \Delta(\delta^z)(\Delta(\pi(p_i))(\pi(q_i) \otimes 1))
= \sum_{i=1}^{n} \Delta(\delta^z \pi(p_i))(\pi(q_i) \otimes 1).
\]

Therefore, we get that

\[
\delta^z \pi(a) = (\delta^z \pi(a)) \omega_{\Lambda(c), \Lambda(d)}(\delta^z \pi(b))
= (\iota \otimes \omega_{\Lambda(c), \Lambda(d)})(\delta^z \pi(a) \otimes \delta^z \pi(b))
= \sum_{i=1}^{n} (\iota \otimes \omega_{\Lambda(c), \Lambda(d)})(\Delta(\delta^z \pi(p_i))(\pi(q_i) \otimes 1)).
\]

By lemma \(\ref{lem:delta-z-algebraic}\), we have that \(\delta^z \pi(a)\) belongs to \(\pi(A)\).

This proposition allows us to define any power of \(\delta\) on the algebraic level. Proposition \(\ref{prop:delta-z-algebraic}\) guarantees that this definition is consistent with the usual definition of integer powers of \(\delta\).

**Definition 8.13** Consider \(z \in \mathcal{C}\), then there exist a unique \(\delta^z\) in \(M(A)\) such that \(\pi(\delta^z a) = \delta^z \pi(a)\) for every \(a \in A\).

It is not difficult to prove the usual rules for exponentiation on the \(*\)-algebra level:

1. For any \(z \in \mathcal{C}\), we have that \((\delta^z)^* = \delta^{\bar{z}}\).
2. For any \( y, z \in \mathbb{C} \), we have that \( \delta^y \delta^z = \delta^{y+z} \).

3. For any \( t \in \mathbb{R} \), we have that \( \delta^t \) is a unitary in \( M(A) \).

4. For any \( t \in \mathbb{R} \), we have that \( \delta^t \) is positive in the sense that \( \delta^t = \delta^* \delta \), where \( \delta^* \) is self-adjoint.

In the next proposition, we show that \( \delta \) on the \( C^* \)-algebra level is determined by its values on \( \pi(A) \).

**Proposition 8.14** Consider \( z \in \mathbb{C} \), then \( \pi(A) \) is a core for \( \delta^z \).

**Proof:** Define \( \delta_z \) as the closure of the mapping \( \pi(A) \to A_r : x \mapsto \delta^z x \). Then \( \delta^z \) an extension of \( \delta_z \).

Choose \( n \in \mathbb{N} \) and \( x \in \pi(A) \). We define

\[
x_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \delta^t x \, dt.
\]

For every \( t \in \mathbb{R} \), we have that \( \exp(-n^2t^2) \delta^t x \) belongs to \( \pi(A) \) (which is a subset of \( \delta_z \)) and

\[
\delta_z(\exp(-n^2t^2) \delta^t x) = \delta^z(\exp(-n^2t^2) \delta^t x) = \exp(-n^2t^2) \delta^t(\delta^z x).
\]

Consequently, the function \( \mathbb{R} \to A_r : t \mapsto \delta_z(\exp(-n^2t^2) \delta^t x) \) is integrable. The closedness of \( \delta_z \) implies that \( x_n \) belongs to the domain of \( \delta_z \).

Because \( \pi(A) \) is dense in \( A_r \), we have that \( (x_n | x \in \pi(A) \text{ and } n \in \mathbb{N}) \) is a core for \( \delta^* \). It follows that \( \delta_z = \delta^z \), so the proposition is proven. \( \blacksquare \)

We know already that there exists a strictly positive number \( \gamma \) such that \( \sigma_t(\delta) = \gamma^t \delta \). In proposition 7.3, we introduced a strictly positive number \( \nu \) such that \( \varphi_t = \nu^t \varphi \) for every \( t \in \mathbb{R} \). We will prove in the next part that \( \gamma = \nu^{-1} \).

**Lemma 8.15** Consider \( a \in \mathcal{N}_\varphi \) and \( b \in \mathcal{D}(\delta^* \delta) \). Then \( \Delta(a)(1 \otimes b) \) belongs to \( \mathcal{N}_\varphi \otimes_\lambda \) and

\[
(\varphi \otimes \iota)((1 \otimes b^*)\Delta(a^*a)(1 \otimes b)) = \varphi(a^*a)(\delta^* \delta)(b^* b).
\]

**Proof:** There exist sequences \( \{a_n\}_{n=1}^\infty \) and \( \{b_n\}_{n=1}^\infty \) in \( A \) such that \( \pi(a_n) \to a \), \( \pi(b_n) \to b \), \( \Lambda_\varphi(\pi(a_n)) \to \Lambda_\varphi(a) \) and \( \Lambda_\varphi(\pi(b_n)) \to \delta^* \delta b \).

Choose \( c_1, c_2, d_1, d_2 \in A \). Then

\[
\langle (\Lambda_\varphi \otimes \iota)(\Delta(\pi(c_1)))(1 \otimes \pi(d_1)), (\Lambda_\varphi \otimes \iota)(\Delta(\pi(c_2)))(1 \otimes \pi(d_2)) \rangle
\]

\[
= \langle (\Lambda_\varphi \otimes \iota)((\pi \otimes \pi)(\Delta(c_1))(1 \otimes d_1)), (\Lambda_\varphi \otimes \iota)((\pi \otimes \pi)(\Delta(c_2))(1 \otimes d_2)) \rangle
\]

\[
= \langle (\varphi \otimes \pi)(1 \otimes d_2^*)\Delta(c_2^* c_1)(1 \otimes d_1) \rangle
\]

\[
= \langle \Lambda_\varphi(\pi(c_1)), \Lambda_\varphi(\pi(c_2)) \rangle \pi(d_2^* d_1)^* \pi(d_2^* d_1).
\]

Therefore,

\[
\langle (\Lambda_\varphi \otimes \iota)(\Delta(\pi(c_1)))(1 \otimes \pi(d_1)), (\Lambda_\varphi \otimes \iota)(\Delta(\pi(c_2)))(1 \otimes \pi(d_2)) \rangle
\]

\[
= \langle \Lambda_\varphi(\pi(c_1)), \Lambda_\varphi(\pi(c_2)) \rangle (\delta^* \delta(\pi(d_1)))^* (\delta^* \delta(\pi(d_1))).
\]

It is clear that \( (\Delta(\pi(a_n))(1 \otimes \pi(b_n)))_{n=1}^\infty \to \Delta(a)(1 \otimes b) \). Using equation (a), we get for any \( m, n \in \mathbb{N} \) that

\[
||\langle \Lambda_\varphi \otimes \iota)(\Delta(\pi(a_m))(1 \otimes \pi(b_n)) \rangle - (\Lambda_\varphi \otimes \iota)(\Delta(\pi(a_n))(1 \otimes \pi(b_n)))||^2
\]

\[
= ||\langle \Lambda_\varphi(\pi(a_m)), \Lambda_\varphi(\pi(a_n)) \rangle (\delta^* \delta(\pi(b_n)))^* (\delta^* \delta(\pi(b_n))) - (\Lambda_\varphi(\pi(a_m)), \Lambda_\varphi(\pi(a_n)) \rangle ||^2
\]

\[
= \langle \Lambda_\varphi(\pi(a_m)), \Lambda_\varphi(\pi(a_n)) \rangle (\delta^* \delta(\pi(b_n)))^* (\delta^* \delta(\pi(b_n))) + (\Lambda_\varphi(\pi(a_m)), \Lambda_\varphi(\pi(a_n)) \rangle (\delta^* \delta(\pi(b_n)))^* (\delta^* \delta(\pi(b_n)) ||.
\]
This equation implies that \( (\Lambda_\varphi \otimes \iota)(\Delta(\pi(a_n))(1 \otimes \pi(b_n))) \) converges to \( (\Lambda_\varphi \otimes \iota)(\Delta(\pi(a))(1 \otimes \pi(b))) \) is a Cauchy sequence and hence convergent in \( H \otimes A \). The closedness of \( \Lambda_\varphi \otimes \iota \) implies that \( \Delta(a)(1 \otimes b) \) belongs to \( \mathcal{N}_{\varphi \otimes \iota} \) and \( \left( (\Lambda_\varphi \otimes \iota)(\Delta(\pi(a_n))(1 \otimes \\
\pi(b_n))) \right)_{n=1}^\infty \) converges to \( (\Lambda_\varphi \otimes \iota)(\Delta(a)(1 \otimes b)) \).

By equation (a), we have that

\[
\langle (\Lambda_\varphi \otimes \iota)(\Delta(\pi(a_n))(1 \otimes \pi(b_n))), (\Lambda_\varphi \otimes \iota)(\Delta(\pi(a_n))(1 \otimes \pi(b_n))) \rangle = \langle \Lambda_\varphi(\pi(a_n)), \Lambda_\varphi(\pi(a_n)) \rangle (\delta_\varphi^* \pi(b_n))^*(\delta_\varphi^* \pi(b_n))
\]

for every \( n \in \mathbb{N} \). So,

\[
\left( \langle (\Lambda_\varphi \otimes \iota)(\Delta(\pi(a_n))(1 \otimes \pi(b_n))), (\Lambda_\varphi \otimes \iota)(\Delta(\pi(a_n))(1 \otimes \pi(b_n))) \rangle \right)_{n=1}^\infty
\]

converges to \( \langle \Lambda_\varphi(a), \Lambda_\varphi(a) \rangle (\delta_\varphi^* b)^*(\delta_\varphi^* b) \). Hence,

\[
\langle (\Lambda_\varphi \otimes \iota)(\Delta(a)(1 \otimes b)), (\Lambda_\varphi \otimes \iota)(\Delta(a)(1 \otimes b)) \rangle = \langle \Lambda_\varphi(a), \Lambda_\varphi(a) \rangle (\delta_\varphi^* b)^*(\delta_\varphi^* b).
\]

This implies that

\[
(\varphi \otimes \iota)((1 \otimes b^*)\Delta(a^*a)(1 \otimes b)) = \varphi(a^*a)(\delta_\varphi^* b)^*(\delta_\varphi^* b).
\]

The following lemma is an easy consequence of the previous one.

**Lemma 8.16** Consider \( a \in \mathcal{N}_\varphi \) and \( b \in \mathcal{D}(\delta) \). Then

\[
(\varphi \otimes \iota)((1 \otimes b^*)\Delta(a^*a)(1 \otimes b)) = \varphi(a^*a)(\delta_\varphi b)^* b = \varphi(a^*a) b^*(\delta_\varphi^* b).
\]

**Proposition 8.17** We have that \( \sigma_t(\delta) = \nu^{-t}\delta \) for every \( t \in \mathbb{R} \).

**Proof:** Remember from the beginning of this section the existence of a strictly positive number \( \gamma \) such that \( \sigma_t(\delta) = \gamma^t \delta \) for every \( t \in \mathbb{R} \).

We take an element \( a \in \mathcal{N}_\varphi \) such that \( \varphi(a^*a) = 1 \). Moreover, there exists a non-zero element \( b \in \mathcal{D}(\delta) \).

The strict positivity of \( \delta \) implies that \( b^*(\delta_\varphi^* b) \neq 0 \).

1. Because \( \sigma_t(\delta) = \gamma^t \delta \), we have that \( \sigma_{-t}(b) \) belongs to \( \mathcal{D}(\delta) \) and \( \sigma_t(\delta \sigma_{-t}(b)) = \gamma^t \delta b \). The previous lemma implies that

\[
(1 \otimes \sigma_{-t}(b)^*)\Delta(a^*a)(1 \otimes \sigma_{-t}(b))
\]

belongs to \( \mathcal{M}_{\varphi \otimes \iota} \) and

\[
(\varphi \otimes \iota)((1 \otimes \sigma_{-t}(b)^*)\Delta(a^*a)(1 \otimes \sigma_{-t}(b))) = \varphi(a^*a) \sigma_{-t}(b)^*(\delta_\varphi^* \sigma_{-t}(b)) = \sigma_{-t}(b)^*(\delta_\varphi^* \sigma_{-t}(b)).
\]

The fact that \( \varphi \tau_t = \nu^t \varphi \) implies that

\[
(\tau_t \otimes \sigma_t)((1 \otimes \sigma_{-t}(b)^*)\Delta(a^*a)(1 \otimes \sigma_{-t}(b)))
\]

belongs to \( \mathcal{M}_{\varphi \otimes \iota} \) and

\[
(\varphi \otimes \iota)((\tau_t \otimes \sigma_t)((1 \otimes \sigma_{-t}(b)^*)\Delta(a^*a)(1 \otimes \sigma_{-t}(b))))) = \nu^t \sigma_t((\varphi \otimes \iota)((1 \otimes \sigma_{-t}(b)^*)\Delta(a^*a)(1 \otimes \sigma_{-t}(b))))
\]

\[
= \nu^t \sigma_t(\sigma_{-t}(b)^*(\delta_\varphi^* \sigma_{-t}(b))) = \nu^t \gamma^t b^*(\delta_\varphi^* b).
\]
Using the fact that \((\tau_t \otimes \sigma_t)\Delta = \Delta \sigma_t\), we see that the element \((a)\) equals \((1 \otimes b^*)\Delta(\sigma_t(a)^* \sigma_t(a))(1 \otimes b)\). Consequently, we get that \((1 \otimes b^*)\Delta(\sigma_t(a)^* \sigma_t(a))(1 \otimes b)\) belongs to \(\mathcal{M}_{\varphi \otimes \iota}\) and
\[
(\varphi \otimes \iota)((1 \otimes b^*)\Delta(\sigma_t(a)^* \sigma_t(a))(1 \otimes b)) = \nu^t \gamma^t b^*(\delta b).
\]

2. We have also that \(\sigma_t(a)\) belongs to \(\mathcal{N}_{\varphi}\), so the previous lemma gives us that \((1 \otimes b^*)\Delta(\sigma_t(a)^* \sigma_t(a))(1 \otimes b)\) belongs to \(\mathcal{M}_{\varphi \otimes \iota}\). Consequently, we get that \((1 \otimes b^*)\Delta(\sigma_t(a)^* \sigma_t(a))(1 \otimes b)\) belongs to \(\mathcal{M}_{\varphi \otimes \iota}\) and
\[
(\varphi \otimes \iota)((1 \otimes b^*)\Delta(\sigma_t(a)^* \sigma_t(a))(1 \otimes b)) = b^*(\delta b).
\]

Combining these two results, we see that \(b^*(\delta b) = \nu^t \gamma^t b^*(\delta b)\), so \(\nu^t \gamma^t\) must be equal to 1. Hence \(\gamma^t = \nu^{-t}\).

This result implies the following result on the *-algebra level.

**Proposition 8.18** For every \(z \in \mathbb{C}\), we have that \(\rho(\delta^z) = \nu^{iz} \delta^z\).

**Proof:** Choose \(a \in A\). Fix \(n \in \mathbb{N}\) and define
\[
x_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(s + iz)^2) \delta^{iz} \pi(a) \ ds.
\]
Take \(s \in \mathbb{R}\). By the previous proposition, we have for all \(t \in \mathbb{R}\) that
\[
\sigma_t(\delta^{iz} \pi(a)) = \nu^{-ist} \delta^{is} \sigma_t(\pi(a)).
\]
Because \(\pi(a)\) is analytic with respect to \(\sigma\), this implies that \(\delta^{is} \pi(a)\) belongs to \(\mathcal{D}(\sigma_{-i})\) and
\[
\sigma_{-i}(\delta^{is} \pi(a)) = \nu^{-s \delta^{is} \sigma_{-i}(\pi(a)))} = \nu^{-s \delta^{is} \pi(a)}.
\]
This implies that the function
\[
\mathbb{R} \to A_r : s \mapsto \exp(-n^2(s + iz)^2) \sigma_{-i}(\delta^{is} \pi(a))
\]
is integrable. Therefore, the closedness of \(\sigma_{-i}\) implies that \(x_n\) belongs to \(\mathcal{D}(\sigma_{-i})\) and
\[
\sigma_{-i}(x_n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(s + iz)^2) \sigma_{-i}(\delta^{is} \pi(a)) \ ds
\]
\[
= \frac{n}{\sqrt{\pi}} \int \exp(-n^2(s + iz)^2) \nu^{-s \delta^{is} \pi(a)} \ ds.
\]
Using the fact that all elements of \(\pi(A)\) are analytic with respect to \(\delta\), we get the following expressions for \(x_n\) and \(\sigma_{-i}(x_n)\).
\[
x_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 s^2) \delta^{is} (\delta^z \pi(a)) \ ds
\]
and
\[
\sigma_{-i}(x_n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 s^2) \nu^{-s+i z \delta^{is} (\delta^z \pi(a))) \ ds.
\]
These two equalities imply that \((x_n)_{n=1}^{\infty} \to \delta^z \pi(a)\) and \((\sigma_{-i}(x_n))_{n=1}^{\infty} \to \nu^{iz} \delta^z \pi(a))\). Because \(\sigma_{-i}\) is closed, this implies that \(\delta^z \pi(a)\) belongs to \(\mathcal{D}(\sigma_{-i})\) and
\[
\sigma_{-i}(\delta^z \pi(a)) = \nu^{iz} \delta^z \pi(a).
\]
We can restate this equality in the following form
\[ \pi(\rho(\delta^za)) = \nu iz \pi(\delta^z \rho(a)), \]
which implies that \( \rho(\delta^za) = \nu iz \delta^z \rho(a) \).
From this, we conclude that \( \rho(\delta^z) = \nu iz \delta^z \).

Remember from section 1 that \( \mu \) is by definition the unique complex number such that \( \varphi S^2 = \mu \varphi \). Now, we can make a connection with our constant \( \nu \).

**Corollary 8.19** We have that \( \mu = \nu^{-i} \).

**Proof:** From section 1, we know that \( \rho(\delta) = \frac{1}{\mu} \delta \), whereas from the previous proposition, we infer that \( \rho(\delta) = \nu i \delta \). Comparing these two equations, we get our equality.

---

**9 The right Haar weight.**

In this section, we are going to introduce the right Haar weight \( \psi \). First, we will define it using the left Haar weight and the anti-unitary antipode. In a later stage we will show that \( \psi \) is in some way absolutely continuous with respect to \( \varphi \) and that the Radon Nikodym derivative is equal to \( \delta \). It will also be shown that there exists a non-zero positive right invariant functional on the \( \ast \)-algebra level.

Let us start of with the definition of our right Haar weight \( \psi \).

**Definition 9.1** We define the weight \( \psi = \varphi R \) on \( A_r \), then \( \psi \) is a faithful densely defined lower semi-continuous weight on \( A_r \).

We can also define the modular group of \( \psi \):

**Definition 9.2** We define the norm-continuous one-parameter group \( \sigma' \) on \( A_r \) such that \( \sigma'_t = R \sigma_{-t} R \) for every \( t \in \mathbb{R} \). Then \( \psi \) is a KMS-weight with modular group \( \sigma' \).

The formula \( \Delta R = \chi(R \otimes R) \Delta \) implies immediately the right invariance of \( \psi \):

**Theorem 9.3** Consider \( x \in \mathcal{M}_\psi \), then \( \Delta(x) \) belongs to \( \overline{\mathcal{M}_\psi} \otimes_{\ast} 1 \) and \( (\psi \otimes \iota) \Delta(x) = \psi(x) 1 \).

**Corollary 9.4** Consider \( x \in \mathcal{M}_\psi \) and \( \omega \in A^*_r \). Then \( (\iota \otimes \omega) \Delta(x) \) belongs to \( \mathcal{M}_\psi \) and \( \psi((\iota \otimes \omega) \Delta(x)) = \psi(x) \omega(1) \).

We can also prove easily the following result:

**Proposition 9.5** For every \( t \in \mathbb{R} \), we have that \( \Delta \sigma'_t = (\sigma'_t \otimes \tau_{-t}) \Delta \).

**Proof:** We have that
\[
\Delta \sigma'_t = \Delta R \sigma_{-t} R = \chi(R \otimes R) \Delta \sigma_{-t} R = \chi(R \tau_{-t} \otimes R \sigma_{-t}) \Delta R = (R \sigma_{-t} \otimes R \tau_{-t}) \chi R = (R \sigma_{-t} \otimes R \tau_{-t}) \Delta = (\sigma'_t \otimes \tau_{-t}) \Delta.
\]

We want to use this right Haar weight \( \psi \) to define a non-zero positive right invariant linear functional on our \( \ast \)-algebra \( A \) by the formula \( \psi \pi \). We need for this that \( \pi(A) \) is a subset of \( \mathcal{M}_\psi \), which is the content of the following lemma.
Lemma 9.6 We have that $\pi(A)$ is a subset of $\mathcal{M}_\psi$.

Proof: Because $\mathcal{N}_\varphi$ is a dense left ideal in $A_r$, $R(\mathcal{N}_\varphi)^*$ is a dense left ideal in $A_r$. Choose $x \in \mathcal{N}_\varphi$ and $\omega \in A_r^\ast$. Using the fact that $(R \otimes R)\Delta = \chi \Delta R$, we see that

$$(\iota \otimes \omega)\Delta(R(x)^*) = ((\iota \otimes \chi)(R \otimes R)\Delta(x))^* = R((\overline{\chi} \otimes \iota)\Delta(x))^*.$$

By the left invariance of $\varphi$, we have that $(\overline{\chi} \otimes \iota)\Delta(x)$ belongs to $\mathcal{N}_\varphi$, so $(\iota \otimes \omega)\Delta(R(x)^*)$ belongs to $R(\mathcal{N}_\varphi)^*$. By proposition 6.10, we see that $R(\pi(A)) \subseteq \mathcal{N}_\varphi^\ast$. This implies that

$$R(\pi(A)) = R(\pi(A^\ast A)) = R(\pi(A)) R(\pi(A))^\ast \subseteq \mathcal{N}_\varphi^\ast \mathcal{N}_\varphi = \mathcal{M}_\varphi.$$

Therefore, $\pi(A)$ is a subset of $\mathcal{M}_\psi$. 

We will now define a faithful positive right invariant linear functional on the $^\ast$-algebra $A$. For this linear functional on $A$, we will use the same symbol $\psi$ as for our right Haar weight on our $C^\ast$-algebra $A_r$. The context will make clear which viewpoint is under consideration. For instance, it is clear that in the following definition the weight $\psi$ only turns up in the expression ‘$\psi(\pi(a))$’.

Definition 9.7 We define the linear functional $\psi$ on $A$ such that $\psi(a) = \psi(\pi(a))$ for every $a \in A$, then $\psi$ is a faithful positive linear functional on $A$.

We could prove the right invariance of the linear functional $\psi$ by using the right invariance of the weight $\psi$. However, we will prove first another interesting formula which connects $\psi$ with $\varphi S$ and this formula implies also immediately the right invariance of the linear functional $\psi$.

Proposition 9.8 We have that $\psi(a) = \nu^\ast \varphi(S(a))$ for every $a \in A$.

Proof: Choose $a \in A$. By the polar decomposition of the antipode, we have that $R(\pi(a)) = \tau_\psi(\pi(S(a)))$ (a). We know already that $\pi(S(a))$ belongs to $\mathcal{M}_\varphi$. By the previous lemma and equality (a), we see that $\tau_\psi(\pi(S(a)))$ also belongs to $\mathcal{M}_\varphi$. Because $\varphi \tau_t = \nu^t \varphi$ for every $t \in \mathbb{R}$, these two facts imply that

$$\varphi(\tau_\psi(\pi(S(a)))) = \nu^\ast \varphi(\pi(S(a))).$$

Therefore,

$$\psi(a) = \psi(\pi(a)) = \varphi(R(\pi(a))) = \varphi(\tau_\psi(\pi(S(a)))) = \nu^\ast \varphi(\pi(S(a))) = \nu^\ast \varphi(S(a)).$$

Theorem 9.9 We have that $\psi$ is a faithful positive right invariant functional on $A$.

So we have proved a result in the algebraic case, using our $C^\ast$-algebraic framework.

Another nice formula for $\psi$ on the algebraic level is given in the following proposition (Remember that we can also define $\delta^\ast$ on the algebraic level).

Proposition 9.10 We have that $\psi(a) = \varphi(\delta^\ast a \delta^\ast)$ for all $a \in A$. 

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Proof: We already know that \( \psi(a) = \nu^a \varphi(S(a)) \). So, by section 1, we get that \( \psi(a) = \nu^a \varphi(a \delta) \).

By proposition 8.18 we have that \( \rho(\delta^a) = \nu^a \delta^a \). Hence,

\[
\psi(a) = \nu^a \varphi(a \delta^a \delta^a) = \varphi(a \delta^a \rho(\delta^a)) = \varphi(\delta^a a \delta^a).
\]

Just like in the case of our left Haar weight \( \varphi \), we want to show that our right Haar weight \( \psi \) is completely determined by its values on \( \pi(A) \). It won’t be a great surprise that both proofs run along the same lines.

Let us take a GNS-triple \((H_\psi, \Lambda_\psi, \pi_\psi)\) for the weight \(\psi\).

Define \( \Gamma_0 \) as the closure of the mapping \( \pi(A) \to H_\psi : x \mapsto \Lambda_\psi(x) \) and denote the domain of \( \Gamma_0 \) by \( B_0 \).

It is not difficult to see that \( B_0 \) is a dense left ideal in \( A_r \) which is a subset of \( N_\psi \). Furthermore, \( \Gamma_0 \) is a closed linear map from \( B_0 \) into \( H_\psi \) which is a restriction of \( \Lambda_\psi \).

The following lemma is the equivalent of lemma 6.6, but we can immediately use the right invariance of \( \psi \) without proving the preceding lemma’s (as in section 6).

Lemma 9.11 Consider \( x \in N_\psi \) and \( \omega \in A^*_r \). Then \( (\iota \otimes \omega)\Delta(x) \) belongs to \( N_\psi \) and \( \| \Lambda_\psi((\iota \otimes \omega)\Delta(x)) \| \leq \| \omega \| \| \Lambda_\psi(x) \| \).

Proof: We know that there exist an element \( \theta \in (A_r)^*_+ \) with \( \| \theta \| = \| \omega \| \) such that

\[
[(\iota \otimes \omega)\Delta(y)]^*[\iota \otimes \omega)\Delta(y)] \leq \| \omega \| (\iota \otimes \theta)\Delta(y^*y)
\]

for all \( y \in A_r \). Hence, using the right invariance of \( \psi \), we see that

\[
\psi([(\iota \otimes \omega)\Delta(x)]^*[\iota \otimes \omega)\Delta(x)]) \leq \| \omega \| \psi((\iota \otimes \theta)\Delta(x^*x)) \leq \| \omega \| \| \theta \| \psi(x^*x) = \| \omega \|^2 \psi(x^*x).
\]

The lemma follows.

The proof of the following lemma is completely analogous to the proof of lemma 6.7 and will therefore be left out.

Lemma 9.12 Consider \( \omega \in A^*_r \) and \( x \in B_0 \). Then \( (\iota \otimes \omega)\Delta(x) \) belongs to \( B_0 \).

Similar to lemma 6.11, we have the following lemma.

Lemma 9.13 Consider \( t \in \mathbb{R} \), then \( \sigma'_r(B_0) \) is a subset of \( B_0 \).

The proof is again completely similar to the proof of lemma 6.11 with the following slight modifications.

1. We use proposition 6.9 in stead of 6.10.

2. Proposition 9.5 is in this situation more useful than proposition 5.7.

At last, we get our desired result:

Theorem 9.14 We have that \( \pi(A) \) is a core for \( \Lambda_\psi \).
The proof is analogous to the proof of theorem 6.12, but we have to use the modular group \( \sigma' \) in stead of \( \sigma \).

In the next part, we want to prove that \( \psi \) is absolutely continuous with respect to \( \varphi \) and that the Radon Nikodym derivative is equal to the modular function \( \delta \).

We know that \( \sigma_t(\delta) = \nu^{-t} \delta \) for every \( t \in \mathbb{R} \). Like in the paper of Takesaki & Pedersen (11), it is possible to define a weight \( \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}}) \). However, we cannot apply their theory because \( \delta \) is not invariant under \( \sigma \). Our construction procedure of this weight \( \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}}) \) is different from the one used by Takesaki & Pedersen. For further details, we refer to (7). Now, we give the basic properties of this weight.

We denote \( \Upsilon = \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}}) \), then \( \Upsilon \) is a faithful densely defined lower semicontinuous weight on \( A_r \). Next, we will describe a GNS-construction for \( \Upsilon \). In fact, in (7), the weight \( \Upsilon \) is defined via this GNS-construction.

As the GNS-space of \( \Upsilon \), we take our Hilbert space \( H \). We define a closed linear map \( \Lambda_\Upsilon \) from within \( A_r \) into \( H \) in the following way. Put

\[
C_\Upsilon = \{ a \in A_r | a \delta^{\frac{1}{2}} \text{ is bounded and } a \delta^{\frac{1}{2}} \text{ belongs to } \mathcal{N}_\varphi \}.
\]

Then, \( C_\Upsilon \) is a core for \( \Lambda_\Upsilon \) and \( \Lambda_\Upsilon(a) = \Lambda_\varphi(a \delta^{\frac{1}{2}}) \) for every \( a \in C \). In this setting, the identity mapping is the GNS-representation of \( \Upsilon \).

We also define the norm-continuous one-parameter group \( \Sigma \) on \( A_r \) such that \( \Sigma_t(a) = \delta^{it} \sigma_t(a) \delta^{-it} \) for every \( a \in A_r \) and \( t \in \mathbb{R} \). We have that \( \Upsilon \) is KMS with respect to \( \Sigma \).

If we denote the modular group of \( \Upsilon \) by \( \Psi \), we have that \( \Psi^{it} = J \delta^{it} J \delta^{it} \nabla^{it} \) for every \( t \in \mathbb{R} \).

If we denote the modular conjugation of \( \Upsilon \) by \( J' \), we have that \( J' = \lambda_\Upsilon^* J \).

Now, we are going to prove some results about \( \Upsilon \) in this specific case.

**Lemma 9.15** Let \( a \in A \). Then \( \pi(a) \) belongs to \( C_\Upsilon \subseteq \mathcal{N}_\Upsilon \) and \( \Lambda_\Upsilon(\pi(a)) = \Lambda(\pi(a) \delta^{\frac{1}{2}}) \).

**Proof:** We know that \( \pi(a^*) \) belongs to \( D(\delta^{\frac{1}{2}}) \) and \( \delta^{\frac{1}{2}} \pi(a^*) = \pi(\delta^{\frac{1}{2}} a^*) \). This implies that \( \pi(a) \delta^{\frac{1}{2}} \) is bounded and

\[
\pi(a) \overline{\delta^{\frac{1}{2}}} = (\delta^{\frac{1}{2}} \pi(a^*))^* = \pi(a \overline{\delta^{\frac{1}{2}}}),
\]

which belongs to \( \mathcal{N}_\varphi \). By definition, we have that \( \pi(a) \) belongs to \( C_\Upsilon \) and

\[
\Lambda_\Upsilon(\pi(a)) = \Lambda_\varphi(\pi(a \delta^{\frac{1}{2}})) = \Lambda(\pi(a) \delta^{\frac{1}{2}}).
\]

**Lemma 9.16** Consider \( x \in \pi(A) \), then \( x \) belongs to \( \mathcal{M}_\Upsilon \) and \( \Upsilon(x) = \psi(x) \).

**Proof:** Choose \( b, c \in A \). We know from the previous lemma that \( \pi(b), \pi(c) \) belong to \( \mathcal{N}_\Upsilon \). This implies that \( \pi(c^* b) \) belongs to \( \mathcal{M}_\Upsilon \) and

\[
\Upsilon(\pi(c^* b)) = \langle \Lambda_\Upsilon(\pi(b)), \Lambda_\Upsilon(\pi(c)) \rangle = \langle \Lambda(b \delta^{\frac{1}{2}}), \Lambda(c \delta^{\frac{1}{2}}) \rangle = \varphi(\delta^{\frac{1}{2}} c^* b \delta^{\frac{1}{2}}).
\]

From proposition 9.10, we know that this last term equals \( \psi(\pi(c^* b)) \).

The lemma follows because \( A^* A = A \).

The next lemma is the last step towards a real equality between \( \Upsilon \) and \( \psi \).
Lemma 9.17 We have that \( \pi(A) \) is a core for \( \Lambda_T \).

**Proof:** Define \( \Gamma \) as the closure of the map \( \pi(A) \rightarrow H : x \mapsto \Lambda_T(x) \) and call \( B \) the domain of \( \Gamma \). It is clear that \( \Lambda_T \) is an extension of \( \Gamma \).

Choose \( x \in \mathcal{N}_\varphi \cap \mathcal{N}_T \).

Take \( n \in \mathbb{N} \) and put

\[
x_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) x \delta^{it} dt.
\]

Because \( \Sigma_n(\delta) = \nu^{-s} \delta \) for all \( s \in \mathbb{R} \), we get that \( x_n \) belongs to \( \mathcal{N}_T \) and

\[
\Lambda_T(x_n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \nu^{\frac{j}{2}} J\delta^{-it} J\Lambda_T(x) dt \quad \text{(a)}
\]

(cf. lemma [10.2]).

Furthermore, we have that \( x_n \delta^{\frac{i}{2}} \) is bounded and

\[
\overline{x_n \delta^{\frac{i}{2}}} = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(t + \frac{i}{2})^2) x \delta^{it} dt.
\]

Because \( \sigma_s(\delta) = \nu^{-s} \delta \) for all \( s \in \mathbb{R} \), this implies that \( x_n \delta^{\frac{i}{2}} \) belongs to \( \mathcal{N}_\varphi \) and

\[
\Lambda_\varphi(x_n \delta^{\frac{i}{2}}) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(t + \frac{i}{2})^2) \nu^{\frac{j}{2}} J\delta^{-it} J\Lambda_\varphi(x) dt.
\]

By definition, we get that

\[
\Lambda_T(x_n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(t + \frac{i}{2})^2) \nu^{\frac{j}{2}} J\delta^{-it} J\Lambda_\varphi(x) dt \quad \text{(b)}
\]

Choose \( \varepsilon \in \mathbb{R}_0^+ \) and \( y \in \mathcal{N}_\varphi \cap \mathcal{N}_T \). From (a), we infer the existence of a natural number \( m \) such that \( \|y - y_m\| \leq \frac{\varepsilon}{2} \) and \( \|\Lambda_T(y) - \Lambda_T(y_m)\| \leq \frac{\varepsilon}{2} \). Because \( \pi(A) \) is a core for \( \Lambda_\varphi \), (b) implies the existence of an element \( z \in \pi(A) \) such that \( \|z_m - y_m\| \leq \frac{\varepsilon}{2} \) and \( \|\Lambda_T(z_m) - \Lambda_T(y_m)\| \leq \frac{\varepsilon}{2} \). This implies that \( \|y - z_m\| \leq \varepsilon \) and \( \|\Lambda_T(y) - \Lambda_T(z_m)\| \leq \varepsilon \). Next, we show that \( z_m \) belongs to \( B \).

Remembering proposition [7.3], we get for any \( t \in \mathbb{R} \) that \( z\delta^{it} \) belongs to \( \pi(A) \) and

\[
\Gamma(z\delta^{it}) = \Lambda_T(z\delta^{it}) = \nu^{\frac{j}{2}} J\delta^{-it} J\Lambda_T(z).
\]

Consequently, the function \( \mathbb{R} \rightarrow H : t \mapsto \exp(-m^2t^2) \Gamma(z\delta^{it}) \) is integrable. The closedness of \( \Gamma \) implies that \( z_m \) belongs to \( B \).

It is not so difficult to prove, in general, that \( \mathcal{N}_\varphi \cap \mathcal{N}_T \) is a core for \( \Lambda_T \) (see [7]). This implies with the foregoing results that \( \Lambda_T = \Gamma \). So, \( \pi(A) \) is a core for \( \Lambda_T \)

So, we arrive at the following nice conclusion:

**Theorem 9.18** We have that \( \psi \) equals \( \Upsilon \).

The unicity of the modular group implies that \( \sigma' \) must be equal to \( \Sigma \). So, we get the following nice proposition.

**Proposition 9.19** For every \( t \in \mathbb{R} \) and \( a \in A_r \), we have that \( \sigma'_t(a) = \delta^{it} \sigma_t(a) \delta^{-it} \).
This formula for $\sigma'$ allows us to prove a continuous version of lemma 3.10.

**Proposition 9.20** For every $t \in \mathbb{R}$, we have that $\Delta \tau_t = (\sigma_t \otimes \sigma'_t) \Delta$.

**Proof:** Choose $x \in A_r$. Then

$$(\sigma_t \otimes \sigma'_t) \Delta(x) = (\delta^{-it} \otimes \delta^{-it})(\sigma'_t \otimes \sigma_{-t})(\Delta(x))(\delta^{it} \otimes \delta^{it})$$

$$= (\delta^{-it} \otimes \delta^{-it})(i \otimes \sigma_{-t} \tau_t)(\Delta(\sigma'_t(x)))(\delta^{it} \otimes \delta^{it})$$

$$= (\delta^{-it} \otimes \delta^{-it})(i \otimes \tau_t \sigma_{-t})(\Delta(\sigma'_t(x)))(\delta^{it} \otimes \delta^{it})$$

$$= (\delta^{-it} \otimes \delta^{-it})(\tau_t \otimes \tau_t)(\Delta(\sigma_{-t}(\sigma'_t(x))))(\delta^{it} \otimes \delta^{it})$$

$$= (\tau_t \otimes \tau_t)((\delta^{it} \otimes \delta^{it})\Delta(\delta^{it}x\delta^{-it})(\delta^{it} \otimes \delta^{it}))$$

$$= (\tau_t \otimes \tau_t)(\Delta(\delta^{-it})\Delta(\delta^{it}x\delta^{-it})\Delta(\delta^{it}))$$

$$= (\tau_t \otimes \tau_t)\Delta(x) = \Delta(\tau_t(x)).$$

\[ \square \]

**Corollary 9.21** We have that $K_t(a) = \delta^{-it} \tau_{-t}(a) \delta^{it}$ for every $t \in \mathbb{R}$ and $a \in A_r$.

**Proof:** Choose $x \in A_r$ and $\omega \in A^*_r$. By the previous proposition, we have that $\Delta(\sigma'_t(x)) = (\sigma'_t \otimes \tau_{-t})\Delta(x)$. Remembering that $\sigma'_t(y) = \delta^{it} \sigma_t(y) \delta^{-it}$ for all $y \in A_r$, we infer from this that

$$(\delta^{-it} \otimes \delta^{it})\Delta(\sigma_t(x))(\delta^{-it} \otimes \delta^{it}) = (\delta^{it} \otimes 1)(\sigma_t \otimes \tau_{-t})(\Delta(x))(\delta^{it} \otimes 1),$$

so, $\Delta(\sigma_t(x)) = (1 \otimes \delta^{-it})(\sigma_t \otimes \tau_{-t})(\Delta(x))(1 \otimes \delta^{it})$. From proposition 3.15, we know already that $\Delta(\sigma_t(x)) = (\sigma_t \otimes K_t)\Delta(x)$, therefore,

$$(1 \otimes \delta^{-it})(i \otimes \tau_{-t})(\Delta(x))(1 \otimes \delta^{it}) = (i \otimes K_t)\Delta(x).$$

Applying $\omega \otimes i$ on this equation, results in the equality $\delta^{-it}\tau_{-t}((\omega \otimes i)\Delta(x))\delta^{it} = K_t((\omega \otimes i)\Delta(x))$. The proposition follows now from the density conditions. \[ \square \]

The existence of a non-zero positive right invariant linear functional $\psi$ on the algebraic quantum group $(A, \Delta)$ allows us to find other implementations of the polar decomposition of the antipode. Also, we can prove the manageability of $W$ (see [24]). For this, we use the construction procedure of section 4 with $\eta$ equal to $\psi$. For every $a \in A$, we have that $\psi(a) = \nu^{\frac{1}{2}} \varphi(S(a))$. Therefore, we have in this case that $x = \nu^{\frac{1}{2}}1$. We take $y = 1$. Because $\psi(a) = \varphi(\delta^a \varphi \delta^a)$ for all $a \in A$, we can choose our GNS-pair $(K, \Gamma)$ such that $K = H$ and $\Gamma(a) = \Lambda(a \delta^a)$ for all $a \in A$. We put $Q = \mathcal{P}$ and $D = \mathcal{J}$. Again, we give a summary of some of the results of section 4 in this case.

We have that $G$ is the closed antilinear map from within $H$ into $H$ such that $\Lambda(a)$ is a core for $G$ and $GA(a) = \Gamma(S(a)^*) = \Lambda(S(a)^* \delta^a)$ for every $a \in A$. It is clear that $G$ is involutive in this case.

Therefore, $D$ is an involutive anti-unitary transformation on $H$, $Q = G^*G$ is an injective positive operator in $H$ such that $G = DQ^{\frac{1}{2}} = Q^{-\frac{1}{2}}D$. For $t \in \mathbb{R}$ we have that $Q^{it} = IQ^{it}I$ and $Q^t = IQ^{-t}I$.

For any $a, b \in A$, we have by definition of $V$ and $W$ that

$$V(\Lambda \otimes \Lambda)(\Delta(b)(a \delta^b \otimes 1)) = V(\Gamma \otimes \Lambda)(\Delta(b)(a \otimes 1)) = (\Gamma \otimes \Lambda)(a \otimes b)$$

$$(\Lambda \otimes \Lambda)(a \delta^b \otimes b) = W(\Lambda \otimes \Lambda)(\Delta(b)(a \delta^b \otimes 1)).$$
So, we get in this case that $W = V$. This implies that

$$(Q \otimes \nabla)W = W(Q \otimes \nabla) \quad \text{and} \quad (D \otimes J)W = W^*(D \otimes J).$$

These two commutation relations allow us to prove the next proposition in an analogous way as proposition 7.16.

**Proposition 9.22**

1. For every $t \in \mathbb{R}$ and $x \in A_r$, we have that $\tau_t(x) = Q^it \cdot Q^{-it}$.

2. For every $x \in A_r$, we have that $R(x) = D^*D$.

It is clear that in this case $S(y^*)x\rho(S(y)) \in \mathfrak{C}1$. So the conclusion of proposition 4.12 holds. We also have the result of proposition 4.10:

- Consider $u_1 \in D(Q^\frac{1}{2})$, $u_2 \in D(Q^{-\frac{1}{2}})$ and $v_1, v_2 \in H$. Then
  $$\langle W^*(Q^\frac{1}{2}u_1 \otimes Jv_1), Q^{-\frac{1}{2}}u_2 \otimes Jv_2 \rangle = \langle W(u_1 \otimes v_2), u_2 \otimes v_1 \rangle.$$

- We have that $W(Q \otimes Q) = (Q \otimes Q)W$.

By definition 1.2 of [26], we arrive at the following conclusion.

**Theorem 9.23** We have that $W$ is manageable.

## 10 Appendix: some information about weights

In this section, we will collect some necessary information and conventions about weights.

Consider a $C^*$-algebra $A$ and a densely defined lower semi-continuous weight $\varphi$ on $A$. We will use the following notations:

- $M^+_\varphi = \{ a \in A^+ \mid \varphi(a) < \infty \}$
- $N_\varphi = \{ a \in A \mid \varphi(a^*a) < \infty \}$
- $M_\varphi = \text{span } M^+_\varphi \cap N_\varphi$

A GNS-construction of $\varphi$ is by definition a triple $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ such that

- $H_\varphi$ is a Hilbert space
- $\Lambda_\varphi$ is a linear map from $N_\varphi$ into $H_\varphi$ such that
  1. $\Lambda_\varphi(N_\varphi)$ is dense in $H_\varphi$
  2. For every $a, b \in N_\varphi$, we have that $\langle \Lambda_\varphi(a), \Lambda_\varphi(b) \rangle = \varphi(b^*a)$

Because $\varphi$ is lower semi-continuous, $\Lambda_\varphi$ is closed.

- $\pi_\varphi$ is a non-degenerate representation of $A$ on $H_\varphi$ such that $\pi_\varphi(a)\Lambda_\varphi(b) = \Lambda_\varphi(ab)$ for every $a \in M(A)$ and $b \in N_\varphi$. (The non-degeneracy of $\pi_\varphi$ is a consequence of the lower semi-continuity of $\varphi$.)
The following concepts play a central role in the theory of lower semi-continuous weights. We define \( \mathcal{F}_\varphi = \{ \omega \in A^*_+ \mid \omega \leq \varphi \} \). Because \( \varphi \) is lower semi-continuous, we have that
\[
\varphi(x) = \sup_{\omega \in \mathcal{F}_\varphi} \omega(x)
\]
for all \( x \in A^+ \).

Also, we put \( \mathcal{G}_\varphi = \{ \alpha \omega \mid \omega \in \mathcal{F}_\varphi, \alpha \in [0,1] \} \). Then \( \mathcal{G}_\varphi \) is a directed subset of \( \mathcal{F}_\varphi \) such that
\[
\varphi(x) = \lim_{\omega \in \mathcal{G}_\varphi} \omega(x)
\]
for all \( x \in A^+ \).

It follows easily that
\[
\varphi(x) = \lim_{\omega \in \mathcal{G}_\varphi} \omega(x)
\]
for every \( x \in \mathcal{M}_\varphi \).

Any lower semi-continuous weight \( \varphi \) has a natural extension to a weight \( \overline{\varphi} \) on \( M(A)^+ \) by putting \( \overline{\varphi}(x) = \sup\{ \omega(x) \mid \omega \in \mathcal{F}_\varphi \} \) for every \( x \in M(A)^+ \).

We define \( \overline{\mathcal{M}}_\varphi = \mathcal{M}_\varphi \) and \( \overline{\mathcal{N}}_\varphi = \mathcal{N}_\varphi \). For any \( x \in \overline{\mathcal{M}}_\varphi \), we put \( \varphi(x) = \overline{\varphi}(x) \).

Then, we have that
\[
\varphi(x) = \lim_{\omega \in \mathcal{G}_\varphi} \omega(x)
\]
for every \( x \in \overline{\mathcal{M}}_\varphi \).

Now, we will introduce the class of KMS-weights. All weights used in this paper, will belong to this class.

**Definition 10.1** Consider a \( C^* \)-algebra \( A \) and a weight \( \varphi \) on \( A \), we say that \( \varphi \) is a KMS-weight on \( A \) if and only if \( \varphi \) is a faithful densely defined lower semi-continuous weight on \( A \) such that there exists a norm-continuous one-parameter group \( \sigma \) on \( A \) satisfying the following properties:

1. \( \varphi \) is invariant under \( \sigma \): \( \varphi \sigma_t = \varphi \) for every \( t \in \mathbb{R} \).
2. For every \( a \in \mathcal{D}(\sigma_\frac{t}{2}) \), we have that \( \varphi(a^*a) = \varphi(\sigma_\frac{t}{2}(a)\sigma_\frac{t}{2}(a)^*) \).

If a weight is KMS in the sense of the previous definition, then the one- parameter group \( \sigma \) is unique and is called the modular group of \( \sigma \).

It is possible to replace condition 2) with a weaker condition like:

There exist a core \( K \) for \( \Lambda_\varphi \) such that

- \( \sigma_t(K) \subseteq K \) for every \( t \in \mathbb{R} \).
- We have that \( K \subseteq \mathcal{D}(\sigma_\frac{t}{2}) \), \( \sigma_\frac{t}{2}(K)^* \subseteq \mathcal{N}_\varphi \) and
  \[
  \|\Lambda_\varphi(x)\| = \|\Lambda_\varphi(\sigma_\frac{t}{2}(x)^*)\|
  \]
  for every \( x \in K \).

We are going to give some properties about KMS-weights.

**Lemma 10.2** Consider a \( C^* \)-algebra \( A \), and a KMS-weight \( \varphi \) on \( A \) with modular group \( \sigma \). Then:

1. There exists a unique anti-unitary operator \( J \) on \( H_\varphi \) such that \( J\Lambda_\varphi(x) = \Lambda_\varphi(\sigma_\frac{t}{2}(x)^*) \) for every \( x \in \mathcal{N}_\varphi \cap \mathcal{D}(\overline{\varphi}) \).
We want to mention that in the definition of a KMS-weight the condition 2) can be replaced by the existence of an anti-unitary $J$ on $H_\varphi$ such that condition 2) of the previous lemma is satisfied (we can even allow this equality to hold for fewer (but enough) elements).

Define $\mathcal{U} = \Lambda_\varphi(N_\varphi \cap N_\varphi^*)$, then $\mathcal{U}$ is turned into a $^*$- algebra in the usual way:

- For every $a, b \in N_\varphi \cap N_\varphi^*$, we have that $\Lambda_\varphi(a)\Lambda_\varphi(b) = \Lambda_\varphi(ab)$.
- For every $a \in N_\varphi \cap N_\varphi^*$, we have that $\Lambda_\varphi(a)^* = \Lambda_\varphi(a^*)$.

It is not so difficult to prove that $\mathcal{U}$ is a left Hilbert algebra on $H_\varphi$.

For every element $a \in A$, we have that $\Lambda_\varphi(a)$ is left bounded with respect to $\mathcal{U}$ and $L_{\Lambda_\varphi(a)} = \pi_\varphi(a)$. Define $T$ as the closed antilinear mapping from within $H_\varphi$ into $H_\varphi$ such that $\mathcal{U}$ is a core for $T$ and $Tv = v^*$ for every $v \in \mathcal{U}$. Denote the modular operator by $\nabla = T^*T$. Then $J, \nabla^{1/2}$ is the polar decomposition of $T$. We call $\nabla$ the modular operator of $\varphi$ and $J$ the modular conjugation of $\varphi$.

We also have that $\nabla^{it}\Lambda_\varphi(a) = \Lambda_\varphi(\sigma_t(a))$ for every $t \in \mathbb{R}$ and $a \in N_\varphi$. As a consequence, $\pi_\varphi(\sigma_t(a)) = \nabla^{it}\pi(a)\nabla^{-it}$ for every $t \in \mathbb{R}$ and $a \in A$.

As a last remark, we want to say that $\varphi$ satisfies the so-called KMS-condition with respect to $\sigma$.

We will need the following generalization of theorem 3.6 of [11].

**Lemma 10.3** Consider a $C^*$-algebra $A$ and a KMS-weight $\varphi$ on $A$ with modular group $\sigma$. Let $a$ be an element in $M(A)$ such that $aM_\varphi \subset M_\varphi$ and $M_\varphi a \subset M_\varphi$ and such that there exists a strictly positive number $\lambda$ such that $\varphi(ax) = \lambda\varphi(xa)$ for every $x \in M_\varphi$. Then we have that $\sigma_t(a) = \lambda^t a$ for every $t \in \mathbb{R}$.

In a last part we say something about slicing with weights. Therefore, we fix a $C^*$-algebra $A$ and a KMS-weight $\varphi$ on $A$.

**Definition 10.4** We define the map $\iota \otimes \varphi$ from within $(A \otimes A)^+$ into $A^+$ as follows:

- We define the set $\mathcal{M}_{i\otimes \varphi}^+ = \{a \in (A \otimes A)^+ \mid \text{the net } (\iota \otimes \omega)(a)_{\omega \in \mathcal{G}_\varphi} \text{ is norm convergent in } A\}$.
- The mapping $\iota \otimes \varphi$ will have as domain the set $\mathcal{M}_{i\otimes \varphi}^+$, and for any $a \in \mathcal{M}_{i\otimes \varphi}^+$, we have by definition that the net $(\iota \otimes \omega)(a)_{\omega \in \mathcal{G}_\varphi}$ converges to $(\iota \otimes \varphi)(a)$.

It is clear that $\mathcal{M}_{i\otimes \varphi}^+$ is a dense hereditary cone in $(A \otimes A)^+$. Furthermore we define the following sets:

1. We define $\mathcal{N}_{i\otimes \varphi} = \{a \in A \otimes A \mid a^*a \text{ belongs to } \mathcal{M}_{i\otimes \varphi}^+\}$.
2. Also, we define $\mathcal{M}_{i\otimes \varphi} = \text{span } \mathcal{M}_{i\otimes \varphi}^+ = \mathcal{M}_{i\otimes \varphi}^*N_{i\otimes \varphi}$.

Of course, there exist a unique linear map $\psi$ from $\mathcal{M}_{i\otimes \varphi}$ to $A$ which extend $\iota \otimes \varphi$. For any $a \in \mathcal{M}_{i\otimes \varphi}$, we put $(\iota \otimes \varphi)(a) = \psi(a)$. It is then clear that $(\iota \otimes \varphi)(a)_{\omega \in \mathcal{G}_\varphi}$ converges to $(\iota \otimes \varphi)(a)$ for every $a \in \mathcal{M}_{i\otimes \varphi}$.

We also have for any $a \in \mathcal{M}_{i\otimes \varphi}$ and $\theta \in A^*$ that $(\theta \otimes \iota)(a)$ belongs to $M_\varphi$ and $\varphi((\theta \otimes \iota)(a)) = \theta(\iota \otimes \varphi)(a)$.

We are now going to describe a GNS-construction for $\iota \otimes \varphi$. It is possible to prove that the mapping $\iota \otimes \Lambda_\varphi$ from $A \otimes N_\varphi$ into $A \otimes H$ is closable (as a mapping from the $C^*$-algebra $A \otimes A$ into the Hilbert-$C^*$-module $A \otimes H$). We define $\iota \otimes \Lambda_\varphi$ to be the closure of this mapping $\iota \otimes \Lambda_\varphi$. 

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It is also possible to prove that \( D(\iota \otimes \Lambda_\varphi) \) is a subset of \( \mathcal{N}_\otimes \varphi \) and that
\[
(\iota \otimes \varphi)(b^*a) = \langle (\iota \otimes \Lambda_\varphi)(a), (\iota \otimes \Lambda_\varphi)(b) \rangle
\]
for every \( a, b \in D(\iota \otimes \Lambda_\varphi) \).

All that is said about \( \iota \otimes \varphi \) until now goes through for lower semi-continuous weights (and can be found in \([13]\) and \([14]\)). Because we assumed that \( \varphi \) is also KMS, we have also the rather non-trivial result that \( D(\iota \otimes \Lambda_\varphi) = \mathcal{N}_\otimes \varphi \). This last result is also true if \( \varphi \) would obey a weaker condition called regularity (see \([3]\)).

We would like to have an extension of \( \iota \otimes \varphi \) to \( M(A \otimes A) \). This is done in the following way:

**Definition 10.5** We define the map \( \overline{\iota \otimes \varphi} \) from within \( M(A \otimes A)^+ \) into \( M(A)^+ \) as follows:

- We define the set \( \overline{\mathcal{M}}_{\iota \otimes \varphi}^+ = \{ a \in M(A \otimes A)^+ \mid \text{ the net } ( (\iota \otimes \omega)(a) )_{\omega \in \mathcal{G}_\varphi} \text{ is strictly convergent in } M(A) \} \).

- The mapping \( \overline{\iota \otimes \varphi} \) will have as domain the set \( \overline{\mathcal{M}}_{\iota \otimes \varphi}^+ \) and for any \( a \in \overline{\mathcal{M}}_{\iota \otimes \varphi}^+ \), we have by definition that the net \( ( (\iota \otimes \omega)(a) )_{\omega \in \mathcal{G}_\varphi} \) converges strictly to \( (\overline{\iota \otimes \varphi})(a) \).

This definition is in fact not entirely correct because it depends on \( \varphi \) and not on \( \iota \otimes \varphi \). It is possible to give a definition in terms of the mapping \( \iota \otimes \varphi \) and our definition would then be a proposition.

The next proposition reveals a nice feature about \( \overline{\iota \otimes \varphi} \).

**Proposition 10.6** Consider \( a \in M(A \otimes A)^+ \). Then \( a \) belongs to \( \overline{\mathcal{M}}_{\iota \otimes \varphi}^+ \) if and only if the net \( ( b^*(\iota \otimes \omega)(a)b )_{\omega \in \mathcal{G}_\varphi} \) is norm convergent for every \( b \in A \).

It is clear that \( \overline{\mathcal{M}}_{\iota \otimes \varphi}^+ \) is a hereditary cone in \( M(A \otimes A)^+ \). Furthermore we define the following sets:

1. We define \( \overline{\mathcal{N}}_{\iota \otimes \varphi} = \{ a \in A \otimes A \mid a^*a \text{ belongs to } \overline{\mathcal{M}}_{\iota \otimes \varphi}^+ \} \).

2. Also, we define \( \overline{\mathcal{M}}_{\iota \otimes \varphi} = \text{span } \overline{\mathcal{M}}_{\iota \otimes \varphi}^+ = \overline{\mathcal{N}}_{\iota \otimes \varphi} \). \( \overline{\mathcal{N}}_{\iota \otimes \varphi} \).

Of course, there exist a unique linear map \( \overline{\psi} \) from \( \overline{\mathcal{M}}_{\iota \otimes \varphi} \) to \( M(A) \) which extend \( \overline{\iota \otimes \varphi} \). For any \( a \in \overline{\mathcal{M}}_{\iota \otimes \varphi} \), we put \( (\overline{\iota \otimes \varphi})(a) = \overline{\psi}(a) \).

It is then clear that \( ( (\iota \otimes \omega)(a) )_{\omega \in \mathcal{G}_\varphi} \) converges strictly to \( (\overline{\iota \otimes \varphi})(a) \) for every \( a \in \overline{\mathcal{M}}_{\iota \otimes \varphi} \).

We also have for any \( a \in \overline{\mathcal{M}}_{\iota \otimes \varphi} \) and \( \theta \in A^* \) that \( (\theta \otimes \iota)(a) \) belongs to \( \overline{\mathcal{M}}_\varphi \) and \( \varphi((\theta \otimes \iota)(a)) = \theta((\iota \otimes \varphi)(a)) \).

It is clear that we can do the same things for the slice-mapping \( \varphi \otimes \iota \).

**References**

[1] E. Abe, Hopf Algebras. *Cambridge University Press* (1977).

[2] S. Baaj & G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres. *Ann. scient. Éc. Norm. Sup.*, 4e série, t. 26 (1993), 425–488.

[3] E.G. Effros & Z.-J. Ruan, Discrete Quantum Groups I. The Haar Measure. *Int. J. of Math.* (1994).

[4] M. Enock & J.-M. Schwartz, Kac Algebras and Duality of Locally Compact Groups. *Springer-Verlag, Berlin* (1992).
[5] B. DRABANT & A. VAN DAEL, Pairing and Quantum Double of Multiplier Hopf Algebras. Preprint K.U. Leuven (1996).

[6] E.C. GOOTMAN & A.J. LAZAR, Quantum Groups and Duality. Reviews in Math. Physics 5 No. 2 (1993), 417–451.

[7] J. KUSTERMANS, A construction procedure for KMS-weights on C*-algebras. In preparation.

[8] J. KUSTERMANS, Regular C*-valued weights on C*-algebras. In preparation.

[9] C. LANCE, Hilbert C*-modules, a toolkit for operator algebrists. Leed. (1993).

[10] T. MASUDA & Y. NAKAGAMI, A von Neumann Algebra Framework for the Duality of Quantum Groups. Publications of the RIMS Kyoto University 30 (1994), 799–850.

[11] G.K. PEDERSEN & M. TAKESAKI, The Radon-Nikodym theorem for von Neumann algebras. Acta Math. 130 (1973), 53–87.

[12] P. PODLEŚ & S.L. WORONOWICZ, Quantum Deformation of the Lorentz Group. Commun. Math. Phys. 130 (1990), 381–431.

[13] J. QUAEGERBEUR & J. VERDING, A construction for weights on C*-algebras. Dual weights for C*-crossed products. Preprint K.U. Leuven (1994).

[14] J. QUAEGERBEUR & J. VERDING, Left invariant weights and the left regular corepresentation for locally compact quantum semi-groups. Preprint K.U. Leuven (1994).

[15] S. STRATILA & L. ZSIDO, Lectures on von Neumann algebras. Abacus Press, Tunbridge Wells, England (1979).

[16] D.C. TAYLOR, The Strict Topology for Double Centralizer Algebras. Trans. Am. Math. Soc 150 (1970), 633–643.

[17] M. TAKESAKI, Theory of Operator Algebras I. Springer-Verlag, New York (1979).

[18] A. VAN DAEL, An Algebraic Framework for Group Duality. Preprint K.U. Leuven (1996).

[19] A. VAN DAEL, Dual Pairs of Hopf *-algebras. Bull. London Math. Soc. 25 (1993), 209–230.

[20] A. VAN DAEL, Discrete Quantum Groups. Journal of Algebra 180 (1996), 431–444.

[21] A. VAN DAEL, The Haar Measure on a Compact Quantum Group. Proc. Amer. Math. Soc. 123 (1995), 3125–3128.

[22] A. VAN DAEL, Multiplier Hopf Algebras. Trans. Am. Math. Soc. 342 (1994), 917–932.

[23] J. VERDING, Weights on C*-algebras. Phd-thesis. K.U. Leuven (1995).

[24] S.L. WORONOWICZ, Compact matrix pseudogroups. Commun. Math. Phys. 111 (1987), 613–665.

[25] S.L. WORONOWICZ, Compact quantum groups. Preprint Warszawa (1993).

[26] S.L. WORONOWICZ, From multiplicative unitaries to quantum groups. Preprint Warszawa (1995).

[27] S.L. WORONOWICZ, Pseudospaces, pseudogroups and Pontriagin duality. Proceedings of the International Conference on Mathematical Physics, Lausanne (1979), 407–412.

[28] S.L. WORONOWICZ, Unbounded elements affiliated with C*-algebras and non-compact quantum groups. Commun. Math. Phys. 136 (1991), 399–432.