On the dynamic pull-in instability in a mass-spring model of electrostatically actuated MEMS devices

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Abstract

In this work we study the mass-spring system

\[ \ddot{x} + \alpha \dot{x} + x = -\frac{\lambda}{(1 + x)^2}, \]

which is a simplified model for an electrostatically actuated MEMS device. The static pull-in value is \( \lambda^* = \frac{4}{27} \), which corresponds to the largest value of \( \lambda \) for which there exists at least one stationary solution. For \( \lambda > \lambda^* \) there are no stationary solutions and \( x(t) \) achieves the value \(-1\) in finite time: touchdown occurs. We establish the existence of a dynamic pull-in value \( \lambda^*_d(\alpha) \in (0, \lambda^*) \), defined for \( \alpha \in [0, \infty) \), which is a threshold in the sense that \( x(t) \) approaches a stable stationary solution as \( t \to \infty \) for \( 0 < \lambda < \lambda^*_d(\alpha) \), while touchdown occurs for \( \lambda > \lambda^*_d(\alpha) \). This dynamic pull-in value is a continuous, strictly increasing function of \( \alpha \) and \( \lim_{\alpha \to \infty} \lambda^*_d(\alpha) = \lambda^* \).

Key words: Dynamic pull-in value, quenching, MEMS, mass-spring system.
1 Introduction

The operation of many micro electromechanical systems (MEMS) relies upon the action of electrostatic forces. Many such devices, including pumps, switches or valves, can be modeled by electrostatically deflected elastic membranes. In a typical situation, a MEMS device consists of an elastic membrane held at a constant voltage and suspended above a rigid ground plate placed in series with a fixed voltage source. The voltage difference causes a deflection of the membrane. For a more detailed description we refer to the book by Pelesko and Bernstein [7].

Taking inertial and viscous forces into account, assuming that the membrane is thin and using a linear approximation for the elastic energy, which is the analogue of a linear Hooke’s law, the motion of the membrane is described by a wave equation with damping and a singular forcing. Rescaling time yields, in the viscous dominated regime, the equation

\[ \gamma^2 u_{tt} + u_t - \Delta u = -\frac{\lambda}{(1 + u)^2} \text{ in } \Omega. \] (2)

In the regime dominated by inertia the equation is:

\[ u_{tt} + \alpha u_t - \Delta u = -\frac{\lambda}{(1 + u)^2} \text{ in } \Omega. \] (3)

where \( \gamma \) is the “quality factor” and \( \alpha = \frac{1}{\gamma} \).

An important nonlinear phenomenon in electrostatically deflected membranes is the so-called “pull-in” instability. For moderate values of the voltage the system is in the stable operation regime in which the membrane approaches a stable steady state and remains separate from the ground plate; when the voltage is increased beyond a critical value, the device is in the touchdown regime: the membrane collapses onto the ground plate. This phenomenon is known as “touchdown” or “pull-in”.

The critical value of the voltage required for touchdown to occur is termed the pull-in voltage. The determination of the pull-in voltage is important for the design and manufacture of MEMS devices. In most cases it is desirable to achieve the stable operation regime, except for some devices such as microvalves, for which touchdown is a desirable property.

Nathanson et al. [6] introduced the first model for an electrostatically actuated device, a millimeter-sized resonant gate transistor was modeled by a mass-spring system. In this model, the moving structure is a plate attached to a spring. The elastic properties of the moving plate are described by the restoring force of the spring,
which is assumed to be given by Hooke’s law in the linear regime. The voltage applied to the moving plate results in an electrostatic force acting on the system by setting it in motion. The governing equation for the displacement of the moving mass is

\[ m\ddot{x} + b\dot{x} + kx = -\frac{\lambda}{(1+x)^2}, \] (4)

in which the relevant parameter \( \lambda \) is proportional to the square of the applied voltage. For systems in which damping dominates, we introduce a dimensionless time with scaling factor \( k/b \), which yields

\[ \gamma^2\ddot{x} + \dot{x} + x = -\frac{\lambda}{(1+x)^2}, \] (5)

where \( \gamma^2 = mk/b^2 \). In this formulation, it is easy to see that if the inertia is not taken into account, that is, \( \gamma = 0 \), then the static and dynamic critical values coincide. Indeed, (5) reduces to the first order equation

\[ \dot{x} + x + \frac{\lambda}{(1+x)^2} = 0. \] (6)

With \( f(x, \lambda) := x + \frac{\lambda}{(1+x)^2} \) and \( g(x) := x(1+x)^2 \), the stationary solutions of (6) are the solutions of \( f(x, \lambda) = 0 \), which correspond to solutions of \( g(x) = -\lambda \). The cubic \( g \) has a local minimum at \( x = -\frac{1}{3} \). The number of stationary solutions in the region of interest \( -1 < x < 0 \) is determined by \( \lambda^* := -g(-\frac{1}{3}) = \frac{4}{27} \). There are two solutions \( x_1(\lambda) < x_2(\lambda) \) for \( 0 < \lambda < \lambda^* \), one \( (x_1(\lambda) = x_2(\lambda) = -\frac{1}{3}) \) at \( \lambda = \lambda^* \) and none for \( \lambda > \lambda^* \).

Moreover, for \( 0 < \lambda < \lambda^* \) and for any initial condition in \( (x_1(\lambda), \infty) \), the corresponding solution converges to \( x_2(\lambda) \) as \( t \to \infty \).

At \( \lambda = \lambda^* \) we have \( x(t) \to -\frac{1}{3} \) as \( t \to \infty \) provided \( x(0) > -\frac{1}{3} \), in particular if \( x(0) = 0 \). For \( \lambda > \lambda^* \) and \( x(0) > -1 \), we have \( x(t) \to -1 \) and the value \( -1 \) is achieved in finite time. This is known as quenching in the mathematical literature.

The coincidence of the static and dynamic pull-in values has also been established for the parabolic equation obtained by setting \( \gamma = 0 \) in (2). See for instance, Flores et al. [4].

Based on numerical evidence, several authors have reported that the dynamic pull-in value is smaller than the static pull-in value, both for the wave equation and for the mass-spring system. This means that when inertia is taken into account, the
moving structure may collapse onto the substrate even if there is a stable stationary solution. Chang and Levine [1] observed this behavior for the conservative wave equation, which corresponds to $\alpha = 0$ in (3), Kavallaris et al. [5] in a nonlocal version of the conservative wave equation and Flores [3] in the damped wave equation (2).

For the mass-spring system, Zhang et al. [9] described the dynamic pull-in as the collapse of the moving structure towards the substrate, due to the combined action of kinetic and potential energies. They also stated that, in general, dynamic pull-in requires a lower voltage to be triggered compared to the static pull-in threshold.

The main result in the present work establishes the existence of a dynamic pull-in value for the mass-spring system.

The equation for systems dominated by inertia is obtained from (1) by introducing a dimensionless time scaled by the natural frequency of the system $\omega = \sqrt{k/m}$, which yields (1), with $\alpha = 1/\gamma$. See Pelesko and Bernstein [7]. Our results are formulated for this regime. The existence of the dynamic pull-in value is obtained for all positive values of $\alpha$, so that the viscous dominated regime also has this property.

We assume that the motion starts from rest: $x(0) = 0 = \dot{x}(0)$.

We begin by writing (1) as the first order system

$$\dot{x} = y, \quad \dot{y} = -[\alpha y + f(x, \lambda)] \tag{7}$$

The stationary solutions of (7) are given by $y = 0, f(x, \lambda) = 0$. The structure of the steady states is obtained from the first order equation (6). There are two solutions $(x_1(\lambda), 0)$ and $(x_2(\lambda), 0)$ for $0 < \lambda < \lambda^*$, one (with $x_1(\lambda) = x_2(\lambda) = -\frac{1}{3}$) at $\lambda = \lambda^*$ and none for $\lambda > \lambda^*$.

We now describe the main steps in the proof of the existence of a dynamic pull-in value. In Section 2 we determine explicitly the dynamic pull-in value for $\alpha = 0$, namely $\lambda^*_d(0) = \frac{1}{8}$. In Section 3 we prove that for each $\lambda \in (\frac{1}{8}, \lambda^*)$ there exists a unique value $\alpha^*(\lambda)$ of $\alpha$ such that (7) is in the touchdown regime for $\alpha \leq \alpha^*$ and it is in the stable operation regime for $\alpha > \alpha^*$. This threshold is a continuous, strictly increasing function of $\lambda$, and $\lim_{\lambda \to \lambda^*} \alpha^*(\lambda) = \infty$. The dynamic pull-in value $\lambda^*_d(\alpha)$ is then the inverse function of $\alpha^*(\lambda)$. In view of the asymptotic behavior of $\alpha^*(\lambda)$ as $\lambda \to \lambda^*$, the dynamic pull-in value is defined for all $\alpha > 0$, $\lambda^*_d(\alpha) \in (\frac{1}{8}, \lambda^*)$ and $\lim_{\alpha \to \infty} \lambda^*_d(\alpha) = \lambda^*$. Therefore, our results remain valid in the viscous dominated regime which corresponds to $\alpha$ large. In the limiting case $\alpha = \infty$ we obtain the coincidence of the static and
dynamic pull-in values for the first order equation mentioned above. A key property in
the analysis is the monotonicity of the stable manifold of \((x_1(\lambda), 0)\), which determines
the domain of attraction of \((x_2(\lambda), 0)\).

We conclude this introduction by mentioning that one of the findings in Rocha et al. [8] is the fact that for an overdamped device, the dynamics in the touchdown regime has three distinguished regions characterized by different time scales: in the first region the structure moves fast until it gets near the static pull-in distance, then there is a metastable region of very slow motion and finally a third region in which collapse takes place on a fast time scale. We shall see that these regions correspond to the approach to the unstable stationary solution, which occurs on a fast (order 1) time scale, followed by a slow motion close to the stable manifold of the unstable steady state until the solution gets away from the stationary point and enters the region of collapse where the dynamics occurs on a fast time scale again.

2 The conservative case: \(\alpha = 0\).

In this case it is possible to determine explicitly the dynamic pull-in value which
separates the stable operation regime from the touchdown regime. In the present
situation, the stable operation regime means that the solution is periodic.

When \(\alpha = 0\), (7) becomes a conservative system. The integral curves are determined explicitly as graphs of functions by means of

\[ y = \pm \sqrt{2\sqrt{E_0} - F(x, \lambda)} \]  

where \(E_0\) is the total energy of an initial condition and \(F(x, \lambda) = \frac{x^2}{2} - \frac{\lambda}{1 + x}\) is a primitive of \(f(x, \lambda)\).

For each \(\lambda \in (0, \lambda^*)\), \((x_1(\lambda), 0)\) is a saddle, \((x_2(\lambda), 0)\) is a center surrounded by periodic orbits and a homoclinic orbit at \((x_1(\lambda), 0)\).

It is clear from the picture in the phase plane that the solution starting at \((0, 0)\)
is periodic if and only if this initial condition is enclosed by the homoclinic. It is also clear that this happens if and only if \(F((x_1(\lambda), \lambda) > F(0, 0) = -\lambda\). Using these observations, we prove

**Proposition 1.** There exists \(\lambda_d(0) \in (0, \lambda^*)\) such that the solution starting at \((0, 0)\)
is periodic if and only if \(\lambda < \lambda_d(0)\), and it approaches \((-1, -\infty)\) for \(\lambda > \lambda_d(0)\). In fact, \(\lambda_d(0) = \frac{1}{8}\).
Proof. Indeed, \( \phi(\lambda) := F(x_1(\lambda), \lambda) - F(0, \lambda) = \frac{x_1^2(\lambda)}{2} + \lambda \frac{x_1(\lambda)}{1 + x_1(\lambda)} \) satisfies \( \frac{d\phi}{d\lambda} = f(x_1(\lambda), \lambda) \frac{dx_1}{d\lambda} + \frac{\partial F}{\partial \lambda}(x_1(\lambda), \lambda) - \frac{\partial F}{\partial \lambda}(0, \lambda) = -x_1(\lambda) \left( \frac{1}{1 + x_1(\lambda)} \right) < 0 \). Thus, \( \phi \) is a strictly decreasing function. Moreover, \( \phi\left(\frac{1}{27}\right) = -\frac{1}{54} \) and \( \lim_{\lambda \to 0^+} \phi(\lambda) = 1/2 \) since \( \frac{x_1(\lambda)}{1 + x_1(\lambda)} = -x_1^2(\lambda)[1 + x_1(\lambda)] \) and \( x_1(\lambda) \to -1 \) as \( \lambda \to 0^+ \). The continuity and monotonicity of \( \phi \) guarantee the existence of a unique value \( \lambda_d(0) \in (0, 4/27) \) such that \( \phi(\lambda_d(0)) = 0 \). This is the dynamic pull-in value. The root is determined explicitly using the previous identity: \( x_1(\lambda_d(0)) = -\frac{1}{2} \) and \( \lambda_d(0) = \frac{1}{8} \). In terms of the phase portrait, this means that the homoclinic orbit at \( (x_1(\lambda), 0) \) crosses the \( x \)-axis at a point \( (\bar{x}(0), 0) \) with \( \bar{x}(0) > 0 \) for \( \lambda \in \left(0, \frac{1}{8}\right) \), while \( \bar{x}(0) < 0 \) for \( \lambda \in \left(\frac{1}{8}, \frac{4}{27}\right) \). The required properties of the solution starting at the origin follow from this. The proof is finished.

3 The dissipative case: \( \alpha > 0 \)

We begin with the local stability analysis of the stationary solutions.

It is clear that \( (-1)^j \frac{\partial f}{\partial x_j}(x_j(\lambda), \lambda) > 0 \) for \( 0 < \lambda < \lambda^* \). The jacobian matrix of the vector field at the stationary solution \( (x_j(\lambda), 0) \), which we denote by \( A_j(\lambda, \alpha) \) is given by

\[
A_j(\lambda, \alpha) = \begin{pmatrix}
0 & 1 \\
-\frac{\partial f}{\partial x_j}(x_j(\lambda), \lambda) & -\alpha
\end{pmatrix}.
\]

Its characteristic polynomial is \( p(\mu) = \mu^2 + \alpha \mu + \frac{\partial f}{\partial x_j}(x_j(\lambda), \lambda) \), with roots

\[
\mu_\pm = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \frac{\partial f}{\partial x_j}(x_j(\lambda), \lambda)} \tag{9}
\]

It follows that for \( \lambda \in \left(0, \frac{4}{27}\right) \), \( (x_1(\lambda), 0) \) is a saddle, while \( (x_2(\lambda), 0) \) is a stable node if \( \alpha > 2 \sqrt{\frac{\partial f}{\partial x_j}(x_2(\lambda), \lambda)} \), and it is a stable focus for values of \( \alpha \) such that the reversed inequality holds. At \( \lambda = \frac{4}{27} \) we have a degenerate stationary solution: \( (x_1(\lambda), 0) = (x_2(\lambda), 0) \) with eigenvalues \( \mu_+ = 0 \) and \( \mu_- = -\alpha \).

The stable operation regime corresponds to the values of \( \lambda \) for which \( x(t; \alpha, \lambda) \to x_2(\lambda) \) as \( t \to \infty \). In dynamical terms, this means that the initial condition \( (0, 0) \)
belongs to the domain of attraction of \((x_2(\lambda), 0)\).

By means of a phase plane analysis, we establish the existence of a dynamic pull-in value \(\lambda^*_d < \lambda^*\) such that the stable operation regime is the interval \((0, \lambda^*_d)\), while the touchdown regime corresponds to \((\lambda^*_d, \infty)\).

Indeed, (7) is a dissipative system with energy

\[ E(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{\lambda}{1+x} \]  

such that along integral curves, \(\frac{dE}{dt} = -\alpha y\).

It follows that the system does not have periodic or homoclinic orbits, and every solution which is bounded for \(t \geq 0\) converges to a stationary solution.

We denote by \(\gamma(t; \alpha, \lambda)\) the solution of (7) with \(\gamma(0; \alpha, \lambda) = (0,0)\), and consider the relevant region of parameters: \(\Omega = \{(\lambda, \alpha) : \lambda > 0, \alpha \geq 0\}\), which is divided into \(\Omega_1 = \{(\lambda, \alpha) \in \Omega : \gamma(t; \alpha, \lambda) \rightarrow (x_2(\lambda), 0) \text{ as } t \rightarrow \infty\}\), \(\Omega_2 = \{(\lambda, \alpha) \in \Omega : \gamma(t; \alpha, \lambda) \rightarrow (x_1(\lambda), 0) \text{ as } t \rightarrow \infty\}\), and \(\Omega_3 = \{(\lambda, \alpha) \in \Omega : \gamma(t; \alpha, \lambda) \rightarrow (-1, -\infty) \text{ in finite time}\}\). The set \(\Omega_1\) corresponds to the stable operation regime, \(\Omega_3\) corresponds to the touchdown regime and \(\Omega_2\) corresponds to the critical behavior.

We shall prove that \(\Omega = \bigcup_{j=1}^3 \Omega_j\).

Our first result concerning (7) is the existence of a stable operation regime for each \(\alpha > 0\). Indeed, we show that \(\Omega_1\) contains a vertical strip in \(\Omega\). We also give an explicit description of part of the domain of attraction of the stable steady state. The energy \(E\) defined in (10) and the euclidean distance \(D(x, y) = x^2 + y^2\) are useful tools in the analysis.

**Proposition 2.** For fixed \(\alpha > 0\) and \(\lambda < \frac{1}{32}\), the set \(U = \{(x_0, y_0) : D(x_0, y_0) < \frac{1}{16}, \text{ and } E(x_0, y_0) \leq -\lambda\}\) is positively invariant. Integral curves of (7) corresponding to initial conditions in \(U\) satisfy \(x(t) \rightarrow x_2(\lambda)\) as \(t \rightarrow \infty\).

**Proof.** For a given \((x_0, y_0) \in U\), take \(T > 0\) such that \(D(x(t), y(t)) < \frac{1}{4}\) for \(0 \leq t \leq T\). Since \(x(t) \geq -\frac{1}{2}\) for \(0 \leq t \leq T\), it follows that \(D(x(t), y(t)) = 2E(x(t), y(t)) + \frac{2\lambda}{1+x(t)} \leq -2\lambda + 4\lambda = 2\lambda < \frac{1}{16}\). It follows that \((x(t), y(t)) \in U\) for \(0 \leq t \leq T\). Since \(T\) depends on the Lipschitz constant of the vector field on a fixed domain, we conclude that \((x(t), y(t)) \in U\) for all \(t > 0\). This establishes the positive invariance of \(U\). Moreover, \((0,0) \in U\). Therefore, the corresponding integral curve converges to a stationary solution as \(t \rightarrow \infty\). In Proposition 1 we established that \(x_1(\lambda) < -\frac{1}{2}\) for \(\lambda < \frac{1}{8}\). In particular, the same is true for the values of \(\lambda\) under consideration. Hence,
\( x(t) \to x_2(\lambda) \) as \( t \to \infty \). The proof is finished.

The above result is similar to Theorem 2 of \([3]\) in which the existence of the stable operation regime is established for the damped wave equation model. Since \((0, 0) \in U\), it follows that \((0, \frac{1}{32}) \times (0, \infty) \subset \Omega_1\).

The stable operation regime and the touchdown regime do persist under small perturbations of the parameters and initial conditions. This implies that \(\Omega_1\) and \(\Omega_3\) are open subsets in \(\Omega\). This is the content of the following result.

**Proposition 3.** Denote by \(\gamma(t; \alpha, \lambda)\) the integral curve of (7) with \(\gamma(0; \alpha, \lambda) = (0, 0)\).

Assume that for fixed \(\alpha_0 > 0\) and \(\lambda_0 \in (0, \frac{1}{27})\) the corresponding integral curve satisfies either of the following two conditions:

a) \(\gamma(t; \alpha_0, \lambda_0) \to (x_2(\lambda_0), 0)\) as \(t \to \infty\)

b) \(\gamma(t; \alpha_0, \lambda_0) \to (-1, -\infty)\) in finite time

Then the same is true for all nearby values of \(\alpha\) and \(\lambda\).

**Proof**

a) Since \((x_2(\lambda_0), 0)\) is a hyperbolic sink, Taylor’s theorem guarantees the existence of \(\delta(\lambda_0) > 0\) such that on the circle centered at this fixed point and radius \(\delta\), the vector field defined by (7) points inside the corresponding disk. By continuous dependence on parameters, the same is true for all values of \(\alpha\) and \(\lambda\) sufficiently close to \(\alpha_0\) and \(\lambda_0\) respectively. For such values of \(\alpha\) and \(\lambda\), the corresponding integral curve \(\gamma(t; \alpha, \lambda)\) enters the invariant disk at some positive time and it remains there for all later times. It follows that the integral curve must approach \((x_2(\lambda), 0)\), provided we restrict \(\delta\) further if necessary, to make sure that the saddles \((x_1(\lambda), 0)\) lie outside the invariant disk.

b) Since for any positive values of \(\alpha\) and \(\lambda\) we have \(\ddot{x}(0; \alpha, \lambda) = \dot{y}(0; \alpha, \lambda) = -\lambda\), the integral curve starting at the origin enters the third quadrant immediately and \(\dot{y}(t) < 0\) for sufficiently small positive values of \(t\). At some \(t_1 > 0\) we must have \(\dot{y}(t_1) = 0\) and the integral curve enters the region \(\dot{y}(t) > 0\) immediately. By the hypothesis, the integral curve cannot remain in this region for all \(t > t_1\). Therefore, there exists \(t_2 > t_1\) such that \(\gamma(t_2; \alpha_0, \lambda_0)\) is in the invariant region defined by \(-1 < x < x_1(\lambda_0), y < 0\) and \(\dot{y} = -[\alpha_0 y + f(x, \lambda_0)] < 0\). By the continuity of solutions with respect to parameters, the same will be true for all nearby values of \(\alpha\) and \(\lambda\). On each of the invariant regions corresponding to such values of \(\alpha\) and \(\lambda\) we have \(x(t) \to -1\) and \(y(t) \to -\infty\) in finite time. The proof is finished.

The argument in (b) above allows us to show that \(\gamma(t; \alpha, \lambda)\) either converges to a
stationary solution or else it approaches $(-1, -\infty)$.

**Proposition 4.** For any $(\alpha, \lambda) \in \Omega$, either $\gamma(t; \alpha, \lambda) \to (x_j(\lambda), 0)$ as $t \to \infty$ for $j = 1$ or 2, or else $\gamma(t; \alpha, \lambda) \to (-1, \infty)$ in finite time. In other words, $\Omega = \bigcup_{j=1}^{3} \Omega_j$.

**Proof.** If $\gamma(t; \alpha, \lambda)$ is bounded, then it is defined for all $t > 0$ and has a non-empty $\omega$-limit set, which consists of stationary solutions. Since this set is finite, $\gamma(t; \alpha, \lambda)$ converges to a stationary solution as $t \to \infty$.

To analyze the other case, assume that $\gamma(t; \alpha, \lambda)$ is unbounded. The first observation is that the integral curve does not cross the line $x = 0$ for positive times, since $E(0, y) = y^2/2 - \lambda > -\lambda = E(0, 0)$ for $y \neq 0$. Therefore, $-1 < x(t; \alpha, \lambda) < 0$ for $t > 0$ as long as the solution is defined. It follows that the $y$ component is unbounded. Since it is bounded above by the maximum of $-f(x, \lambda)$, it follows that the $y$ component must approach $-\infty$ through a sequence of times $t_n \to \infty$. This is possible only in the invariant region $-1 < x < 0$, $y < 0$ and $\dot{y} < 0$, where, as we have seen in part (b) of the previous result, $x(t) \to -1$ and $y(t) \to -\infty$ in finite time.

**Remark** A consequence of Proposition 4 is that $(\alpha, \lambda) \in \Omega_3$ for any $\alpha \geq 0$ and all $\lambda > \lambda^*$, since in this case there are no stationary solutions.

Our next result is part of the description of the phase portrait of (7), in which the invariant manifolds of the stationary solutions play a fundamental role. We prove the existence of a heteroclinic orbit from $(x_1(\lambda), 0)$ to $(x_2(\lambda), 0)$ for $\alpha > 2\sqrt{s(\lambda)}$, where $s(\lambda) := \sup_x \frac{f(x, \lambda)}{x - x_2(\lambda)}$. Since $\frac{\partial f(x, \lambda)}{\partial x} = 1 - \frac{2\lambda}{(1 + x)^3}$ is an increasing function of $x$, it follows that $s(\lambda) = \frac{\partial f(x_2(\lambda), \lambda)}{\partial x}$. Hence, there is a heteroclinic orbit for every value of $\alpha$ for which $(x_2(\lambda), 0)$ is a stable node. The nonlinearity in the equation is of the type of Fisher’s equation, which explains the existence of the saddle-node connections for the stated values of $\alpha$.

**Proposition 5.** For each $\alpha > 2\sqrt{s(\lambda)}$, system (7) has a heteroclinic connection from $(x_1(\lambda), 0)$ to $(x_2(\lambda), 0)$.

**Proof.** We consider the triangular region defined by the line $x = x_1(\lambda)$, with $y > 0$, the segment $[x_1(\lambda), x_2(\lambda)]$, with $y = 0$ and a line segment $y = m(x - x_2(\lambda))$ with $x_1(\lambda) \leq x \leq x_2(\lambda)$, and $m < 0$. It is clear that on the horizontal and vertical sides of the triangle, the vector field defined by (7) points inward the triangular region. We shall determine negative values of $m$ for which the vector field also points inwards on the third side. Choosing $N = (-m, 1)$ as a normal vector for the slanted side of the
triangle, and denoting by $\mathbf{V}$ the vector field defined by (7), the condition on $m$ so that the vector field points inwards is $\langle \mathbf{N}, \mathbf{V} \rangle < 0$. Since $\langle \mathbf{N}, \mathbf{V} \rangle = -\frac{x - x_2(\lambda)}{m^2 + \alpha m + s(\lambda)} \{m^2 + \alpha m + f(x, \lambda)\}$,

$$< \mathbf{N}, \mathbf{V}> = -\frac{x - x_2(\lambda)}{m^2 + \alpha m + f(x, \lambda)} \{m^2 + \alpha m + s(\lambda)\}, \quad (11)$$

it follows that $< \mathbf{N}, \mathbf{V}> \leq -\frac{x - x_2(\lambda)}{m^2 + \alpha m + s(\lambda)}$. The quadratic polynomial in $m$ has roots $m_{\pm} = -\frac{\alpha}{2} \pm \frac{1}{2} \sqrt{\alpha^2 - 4s(\lambda)}$. For $\alpha > 2 \sqrt{s(\lambda)}$, the root $m_-$ is negative. Therefore, there are negative values of $m$ for which the quadratic takes negative values. This completes the construction of an invariant region. The branch of the unstable manifold of $(x_1(\lambda), 0)$ that points into the region $y > 0$, $x > x_1(\lambda)$ enters this invariant region and never leaves it. Therefore, it converges to $(x_2(\lambda), 0)$ as $t \to \infty$. The proof is finished.

The next result establishes the monotonicity of the integral curve starting at $(0, 0)$ as a function of $\lambda$, as well as a criterion for touchdown.

**Proposition 6.** Fix $\alpha > 0$, let $\gamma(t; \alpha, \lambda) = (x(t; \alpha, \lambda), y(t; \alpha, \lambda))$, then $y(t; \alpha, \lambda)$ is a decreasing function of $\lambda$ as long as $y(t; \alpha, \lambda) < 0$. Moreover, $(\alpha, \lambda) \in \Omega_3$ if there exists $\lambda_0 < \lambda$ such that $(\alpha, \lambda_0) \in \Omega_2 \cup \Omega_3$.

Proof. Take $0 < \lambda_1 < \lambda_2$, and let $y_j(t; \alpha) := y(t; \alpha, \lambda_j)$. Then $y_j(0; \alpha) = 0$ and $\dot{y}_j(0; \alpha) = -\lambda_j$. It follows that $y_2(t; \alpha) < y_1(t; \alpha)$ for $t > 0$ and small. Note that the second component of the vector field in (7) is monotonic in $\lambda$ because $\frac{\partial f(x, \lambda)}{\partial \lambda} = \frac{1}{1 + x^2} > 0$. This implies that the inequality above is valid as long as each $y_j$ is negative. This means that the integral curves do not cross as long as they remain in the third quadrant.

For the second part of the statement, the conclusion follows immediately from the monotonicity if $(\alpha, \lambda_0) \in \Omega_3$. In the other case, $(\alpha, \lambda_0) \in \Omega_2$, the integral curve starting at $(0, 0)$ approaches $(x_1(\lambda_0), 0)$ as $t \to \infty$. Since $x_1(\lambda)$ is increasing in $\lambda$ and $x_2(\lambda)$ is decreasing, the monotonicity of the integral curves guarantee that $\gamma(t; \alpha, \lambda)$ cannot approach either of the critical points. Hence it has to approach $(-1, -\infty)$ and the integral curve has a finite time of existence since the $y$ component is eventually decreasing. The proof is finished.

The stable manifold of the saddle $(x_1(\lambda), 0)$ plays a crucial role in the determination of the dynamic pull-in value. The domain of attraction of $(x_2(\lambda), 0)$ is determined by the connected component of the stable manifold of $(x_1(\lambda), 0)$ that approaches this saddle from the third quadrant. It is more convenient to analyze the behavior of the stable manifold by fixing $\lambda$ and varying $\alpha$. 

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We prove that for each $\lambda \in (0, \frac{1}{4})$, the connected component of the stable manifold described above is a strictly monotonic function of $\alpha$. The point of intersection with the horizontal axis is a monotonic and continuous function of $\alpha$. We shall consider $\lambda \in (\frac{1}{8}, \frac{1}{4})$. We shall prove that for small values of $\alpha$, the stable manifold crosses the negative $x$-axis, which corresponds to touchdown because the solution starting at $(0,0)$ cannot approach $(x_2(\lambda),0)$. For large values of $\alpha$, the stable manifold crosses the positive $x$-axis. In this case, the solution $(x(t), y(t))$ is bounded for $t \geq 0$ and it converges to $(x_2(\lambda),0)$ as $t \to \infty$.

It follows that there is a unique value $\alpha^*(\lambda)$ of $\alpha$ such that the stable manifold crosses the $x$-axis at $x = 0$. We also prove that $\alpha^*(\lambda)$ is a continuous and strictly increasing function of $\lambda$. The dynamic pull-in value $\lambda^*_d(\alpha)$ is the inverse function of $\alpha^*(\lambda)$. The dynamic pull-in value is defined for all? positive values of $\alpha$.

For convenience, we change $t$ by $-t$, $y$ by $-y$, and rewrite (7) in terms of $u = x - x_1(\lambda)$ and $v = y$, obtaining

$$\dot{u} = v, \quad \dot{v} = \alpha v - f(u + x_1(\lambda), \lambda)$$

(12)

This system has a saddle point at $(0,0)$, with eigenvalues given by:

$$\mu_{\pm} = \frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \frac{\partial f}{\partial x}(x_1(\lambda), \lambda)}$$

(13)

The branch of the local unstable manifold that points into the first quadrant is the graph of a continuous function $v = \Phi(u; \lambda, \alpha)$ and it can be continued as a graph as long as $v > 0$. Moreover, $\Phi(0; \alpha, \lambda) = 0$ and $\frac{d\Phi}{du}(0; \alpha, \lambda) = \mu_+$. By the Chain Rule,

$$\frac{d\Phi}{du} = \alpha - \frac{f(u + x_1(\lambda), \lambda)}{\Phi(u; \alpha, \lambda)}$$

(14)

Our next result is the monotonicity of $\Phi$ with respect to $\alpha$.

**Proposition 7.** For fixed $\lambda > 0$, $\Phi(u; \alpha, \lambda)$ is an strictly increasing function of $\alpha$.

**Proof.** Fix $\lambda > 0$, take $0 < \alpha_1 < \alpha_2$ and denote $\Phi(u; \alpha_j, \lambda)$ by $\Phi_j(u)$ for $j = 1, 2$. Since $\mu_+$ is an strictly increasing function of $\alpha$, it follows that $\Phi_1(u) < \Phi_2(u)$ for small positive values of $u$. It is clear from (14) that the graph of $\Phi_1$ cannot intersect the graph of $\Phi_2$ as long as they are defined. The proof is finished.

For fixed $\lambda \in (\frac{1}{6}, \frac{4}{27})$, let

$$I(\lambda) := \{ \alpha \geq 0 : \text{there exists } \bar{u}(\alpha) > 0 \text{ with } \Phi(\bar{u}(\alpha); \alpha, \lambda) = 0 \}$$
and \( J(\lambda) := \{ \bar{u}(\alpha) : \alpha \in I(\lambda) \} \).

A crucial step in the proof of the existence of the dynamic pull-in value is the determination of the set \( J(\lambda) \). To do this, it is convenient to analyze the intersection of the unstable manifolds with the vertical line \( L \) in the phase plane given by \( u = x_2(\lambda) - x_1(\lambda) \). By the monotonicity of the unstable manifolds and the transversality of \( L \) with respect to the vector field in (12), the set

\[
K(\lambda) := \{ (x_2(\lambda) - x_1(\lambda), \Phi(x_2(\lambda) - x_1(\lambda); \alpha, \lambda) : \alpha \geq 0 \}
\]
defines an interval on the line \( L \), since the points of intersection define a continuous function of \( \alpha \). See Conley [2].

Let \( v_0 := \Phi(x_2(\lambda) - x_1(\lambda); 0, \lambda) \) denote the height of the homoclinic orbit corresponding to \( \alpha = 0 \) at \( u = x_2(\lambda) - x_1(\lambda) \). Our next result determines the set \( K(\lambda) \).

**Lemma 1.** \( K(\lambda) = \{ x_2(\lambda) - x_1(\lambda) \} \times [v_0, \infty) \).

**Proof.** Since \( f(u + x_1(\lambda), \lambda) < 0 \), it follows from (14) that \( \frac{d \Phi}{du} \geq \alpha \) for every \( \alpha > 0 \) and \( u \in [0, x_2(\lambda) - x_1(\lambda)] \). On this interval we have \( \Phi(u; \alpha, \lambda) \geq \alpha u \). In particular \( \Phi(x_2(\lambda) - x_1(\lambda); \alpha, \lambda) \geq \alpha[x_2(\lambda) - x_1(\lambda)] \). The proof is finished.

**Remark.** The above estimate for \( \Phi \) suggests that the critical value of \( \alpha \) tends to \( \infty \) as \( \lambda \) approaches \( \lambda^* \).

**Lemma 2.** \( I(\lambda) \) and \( J(\lambda) \) are non-empty intervals. Moreover, \( I(\lambda) = [0, \bar{\alpha}(\lambda)) \) with \( \bar{\alpha}(\lambda) := \sup I(\lambda) \), and \( \bar{u}(\lambda) := \sup J(\lambda) = \infty \), so that \( J(\lambda) = [\bar{u}(0), \infty) \).

**Proof.** In Section 2 we verified that \( 0 \in I(\lambda) \). The monotonicity and the continuity of \( \Phi \) with respect to \( \alpha \) guarantee that \( I(\lambda) \) and \( J(\lambda) \) are intervals.

We claim that \( I(\lambda) = [0, \bar{\alpha}(\lambda)) \). This is clear if \( \bar{\alpha}(\lambda) = \infty \). Now assume that the supremum is finite and that it belongs to the set \( I(\lambda) \), then \( \bar{u}(\lambda) < \infty \) and \( \Phi(\bar{u}(\lambda); \bar{\alpha}(\lambda), \lambda) = 0 \). Take \( \varepsilon > 0 \), then, by the continuity and monotonicity of \( \Phi \) with respect to \( \alpha \), the set

\[
\{(x_2(\lambda) - x_1(\lambda), \Phi(x_2(\lambda) - x_1(\lambda); \alpha, \lambda) : \alpha \in [0, \bar{\alpha}(\lambda) + \varepsilon) \}
\]
describes a closed interval on the line \( L \) in the phase plane. Moreover, the integral curve starting at \((\bar{u}(\bar{\alpha}(\lambda)), 0)\) immediately enters the region \( v < 0 \). By continuity with respect to initial conditions, the same is true for values of \( \alpha \in (\bar{\alpha}(\lambda), \bar{\alpha}(\lambda) + \varepsilon) \) for \( \varepsilon \) sufficiently small. This contradicts the definition of \( \bar{u}(\lambda) \) as the supremum of \( J(\lambda) \) and establishes the claim.
The next task is to verify that $\bar{u}(\lambda) = \infty$. We distinguish two cases, according to whether $\bar{\alpha}(\lambda)$ is finite or infinity. In the first case, $\bar{\alpha}(\lambda) < \infty$, assume that $\bar{u}(\lambda) < \infty$. From (14) we see that $\Phi$ has a finite derivative on every finite interval on which it is defined. Using this fact and the continuous dependence on $\alpha$ we conclude that $\Phi(u; \bar{\alpha}(\lambda), \lambda) > 0$ for all $u > 0$. By continuous dependence on initial conditions we get unstable manifolds for $\bar{\alpha}(\lambda) - \varepsilon < \alpha < \bar{\alpha}(\lambda)$ with $\varepsilon$ small enough, for which $\Phi > 0$ for $u > \bar{u}(\lambda)$. This is a contradiction since $\Phi(\bar{u}(\alpha); \alpha, \lambda) = 0$ for $0 \leq \alpha < \bar{\alpha}(\lambda)$. This contradiction proves that $\bar{u}(\lambda) = \infty$ if $\bar{\alpha}(\lambda) < \infty$.

In the case $\bar{\alpha}(\lambda) = \infty$ we have that $\bar{u}(\alpha)$ is defined for all $\alpha > 0$ and $\Phi(\bar{u}(\alpha); \alpha, \lambda) = 0$. The point of intersection of the graph of $\Phi$ with the line $L$ lies in the region where $\dot{v} > 0$. In the case under consideration, the graph of $\Phi$ must leave this region at a point with first component $u$ satisfying $u > h(\alpha[x_2(\lambda) - x_1(\lambda)], \lambda)$, where $h$ is the inverse function of $f$ on the interval $(x_2(\lambda) - x_1(\lambda), \infty)$. Since $f$ is increasing and tends to $\infty$ as $u$ approaches $\infty$, we get $\bar{u}(\alpha) > h(\alpha[x_2(\lambda) - x_1(\lambda)], \lambda) \to \infty$ as $\alpha \to \infty$.

The proof is finished.

For the values of $\lambda$ under consideration, the next result establishes the existence of a critical value $\alpha^*(\lambda)$ such that the touchdown regime corresponds to $(0, \alpha^*(\lambda))$, while the stable operation corresponds to $(\alpha^*(\lambda), \infty)$. The critical value occurs when $\bar{u}(\alpha) = -x_1(\lambda)$.

**Theorem.** For each $\lambda \in \left(\frac{1}{5}, \frac{4}{27}\right)$ there exists $\alpha^*(\lambda) > 0$ such that the touchdown regime corresponds to $(0, \alpha^*(\lambda))$, while the stable operation corresponds to $(\alpha^*(\lambda), \infty)$. Moreover, $\alpha^*(\lambda)$ is an strictly increasing, continuous function of $\lambda$, and $\alpha^*(\lambda) \to \infty$ as $\lambda \to \frac{4}{27}$.

**Proof.** The critical value $\alpha^*(\lambda)$ is the value of $\alpha$ for which $\bar{u}(\alpha) = -x_1(\lambda)$. It is well defined by Lemma 2. The monotonicity is a consequence of Proposition 7. To verify the stated properties, we return to the original equation (17). In this setting, the critical value satisfies $\bar{x}(\alpha^*(\lambda)) = 0$.

The content of Lemma 2 is that for $\alpha < \bar{\alpha}(\lambda)$, the branch of the stable manifold of the saddle $(x_1(\lambda), 0)$ that enters from the third quadrant intersects the horizontal axis at $(\bar{x}(\alpha), 0)$ and the points of intersection comprise the interval $[\bar{u}(0), \infty)$, or equivalently, the interval $[\bar{x}(0) - x_1(\lambda), \infty)$ where $\bar{u}(0) < -x_1(\lambda)$, or equivalently $\bar{x}(0) < 0$. The left end-point of the interval is thus determined by the homoclinic orbit in the conservative case $\alpha = 0$. By the monotonicity and continuity of the points of intersection, there exists a unique value $\alpha^*$ of $\alpha$ such that the point of intersection satisfies $\bar{x}(\alpha^*) = 0$. 

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For $\alpha < \alpha^*$, the point of intersection satisfies $\bar{x}(\alpha) < 0$. In this case, it is clear that $(0, 0)$ is not in the domain of attraction of $(x_2(\lambda), 0)$. It follows that the integral curve starting at $(0, 0)$, which enters the third quadrant immediately, in fact, it enters the region where $y < 0$ and $\dot{y} < 0$. Since integral curves in this region cannot approach the point $(x_2(\lambda), 0)$, it follows that there exists $T_1 > 0$ such that $\dot{y}(T_1) = 0$ and the integral curve enters immediately the region where $\dot{y} > 0$. But, it cannot remain there for all $t \geq T_1$ since it cannot cross the stable manifold. It follows that there exists $T_2 > T_1$ such that $\dot{y}(T_2) = 0$. Now the integral curves enter the region where $-1 < x < x_1(\lambda)$, $y < 0$ and $\dot{y} < 0$. As it was established in Proposition 4, solutions in this positively invariant region satisfy $x(t) = -1$ is achieved in finite time.

In the case $\alpha > \alpha^*$, the point of intersection satisfies $\bar{x}(\alpha) > 0$. The stable manifold provides a lower bound on the second component of the integral curve starting at $(0, 0)$. The graph $v = -\frac{1}{2} f(x, \lambda)$ on the interval $[x_1(\lambda), x_2(\lambda)]$ provides the upper bound. Hence, the integral curve converges to $(x_2(\lambda), 0)$ as $t \to \infty$.

For the continuity, take $\lambda_0$ in the interval under consideration, and $0 < \varepsilon < \alpha^*(\lambda_0)$, then $\gamma(t; \alpha^*(\lambda_0) - \varepsilon, \lambda_0) \to (-1, -\infty)$ and $\gamma(t; \alpha^*(\lambda_0) + \varepsilon, \lambda_0) \to \gamma(\alpha^*(\lambda_0), 0)$. By Proposition 3, there exists $\delta > 0$ such that the above properties are maintained if $|\lambda - \lambda_0| < \delta$. Now we use the continuity and monotonicity of the unstable manifolds to conclude that for such values of $\lambda$ we have $\alpha^*(\lambda_0) - \varepsilon < \alpha^*(\lambda) < \alpha^*(\lambda_0) + \varepsilon$, which establishes the continuity of the critical value $\alpha^*(\lambda)$.

The last step is the asymptotic behavior of $\alpha^*(\lambda)$ as $\lambda \to \lambda^*$. The proof is by contradiction. Assume that for $\lambda = \lambda^* = \frac{4}{27}$, there exists a positive real number $\alpha_0$ such that if $\gamma(t; \alpha)$ is the solution of (12) for $\lambda = \lambda^*$ with $\gamma(0; \alpha) = \gamma(0; 0)$, except that we keep the variables $(x, y)$, then $\gamma(t; \alpha_0) \to (-\frac{1}{3}, 0)$ as $t \to \infty$. In this case, $(\alpha, \lambda)$ is in the stable operation regime for all $\alpha > \alpha_0$ and $0 < \lambda < \lambda_*$. Since $(\alpha, \lambda)$ is in the touchdown regime for all $\alpha > 0$ and $\lambda > \lambda_*$, it follows that $\gamma(t; \alpha) \to (-\frac{1}{3}, 0)$ as $t \to \infty$ for all $\alpha > \alpha_0$. Now we have a one-parameter family of unstable manifolds of $(-\frac{1}{3}, 0)$ with a branch that points into the second quadrant and such that their first crossing with the line $y = 0$ occurs at $x = 0$. Each of these branches of the unstable manifolds is the graph of a function $y = \Phi(x; \alpha)$ defined for $-\frac{1}{3} \leq x \leq 0$. Moreover, $\frac{d\Phi}{dx}(x; \alpha) = \alpha - \frac{f(x; \lambda^*)}{\Phi(x; \alpha)}$. In particular, $\frac{d\Phi}{dx}(-\frac{1}{3}, \alpha) = \mu_+ = \alpha$. An argument similar to the one used in Proposition 7 for $0 < \lambda < \lambda_*$ shows that $\Phi$ is an increasing function of $\alpha$ for $-\frac{1}{3} \leq x \leq 0$. Now let $(x_0, y_0)$ be the point of intersection of the branch of the unstable manifold with the graph of $y = f(x, \lambda_*)$, then $y_0 = \Phi(x_0, \alpha_0)$. Since $\Phi$ is an increasing function of $x \in (-\frac{1}{3}, x_0)$ for $\alpha \geq \alpha_0$, we take $\delta > 0$ such that
\( \Phi(x_0 - \delta, \alpha_0) = \frac{y_0}{2} \), and consider values of \( \alpha > \frac{4\lambda_*}{y_0} \). Then, for \( x \in [x_0 - \delta, x_0] \) we have

\[
\frac{d\Phi}{dx}(x; \alpha) = \alpha - \frac{f(x; \lambda^*)}{\Phi(x; \alpha)} \geq \alpha - \frac{2\lambda^*}{y_0} > \frac{\alpha}{2}.
\]

It follows from the Mean Value Theorem that

\[
\Phi(x; \alpha) = \Phi(x_0 - \delta; \alpha) + \Phi(x; \alpha) - \Phi(x_0 - \delta; \alpha) \geq \frac{y_0}{2} + \frac{\alpha}{2}(x - (x_0 - \delta))
\]

and at \( x = x_0 \) we get \( \Phi(x_0; \alpha) > \frac{y_0}{2} + \frac{\alpha}{2}\delta \). Now we take \( \alpha \) such that \( \frac{y_0}{2} + \frac{\alpha}{2}\delta > \lambda^* = f(0, \lambda^*) \), then for such values of \( \alpha \) the following inequality holds: \( \Phi(0; \alpha) > \lambda^* > 0 \). This contradiction shows that the assumption is not valid. The proof is finished.

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