FINDING TWO-DIMENSIONAL PEAKS

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Abstract. Two-dimensional generalization of the original peak finding algorithm suggested earlier is given. The ideology of the algorithm emerged from the well known quantum mechanical tunneling property which enables small bodies to penetrate through narrow potential barriers. We further merge this “quantum” ideology with the philosophy of Particle Swarm Optimization to get the global optimization algorithm which can be called Quantum Swarm Optimization. The functionality of the newborn algorithm is tested on some benchmark optimization problems.

Key words. Numerical optimization, Global optimum, Quantum Swarm Optimization

1. Introduction. Some time ago we suggested a new algorithm for automatic photopeak location in gamma-ray spectra from semiconductor and scintillator detectors [1]. The algorithm was inspired by quantum mechanical property of small balls to penetrate through narrow barriers and find their way down to the potential wall bottom even in the case of irregular potential shape.

In one dimensional case the idea was realized by means of finite Markov chain and its invariant distribution [1]. States of this Markov chain correspond to channels of the original histogram. The only nonzero transition probabilities are those which connect a given state to its closest left and right neighbor states. Therefore the transition probability matrix for our Markov chain has the form

\[ P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ P_{21} & 0 & P_{23} & 0 & 0 & \cdots \\ 0 & P_{32} & 0 & P_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \]

As for the transition probabilities, the following expressions were used

\[ P_{i,i\pm1} = \frac{Q_{i,i\pm1}}{Q_{i,i-1} + Q_{i,i+1}} \]

with

\[ Q_{i,i\pm1} = \sum_{k=1}^{m} \exp \left( \frac{N_{i \pm k} - N_i}{\sqrt{N_{i \pm k} + N_i}} \right) \]

The number \( m \) is a parameter of the model which mimics the (inverse) mass of the quantum ball and therefore allows to govern its penetrating ability.

The invariant distribution for the above described Markov chain can be given by a simple analytic formula [2]

\[ u_2 = \frac{P_{12}}{P_{21}} u_1 , \quad u_3 = \frac{P_{12}P_{23}}{P_{32}P_{21}} u_1 , \quad \cdots , \quad u_n = \frac{P_{12}P_{23} \cdots P_{n-1,n}}{P_{n,n-1}P_{n-1,n-2} \cdots P_{21}} u_1 , \]

where \( u_1 \) is defined from the normalization condition

\[ \sum_{i=1}^{n} u_i = 1 \]
Local maximums in the original spectrum are translated into the very sharp peaks in the invariant distribution and therefore their location is facilitated.

The algorithm proved helpful in uniformity studies of NaJ(Tl) crystals for the SND detector \[3\]. Another application of this “peak amplifier”, to refine the amplitude fit method in ATLAS B$_s$-mixing studies, was described in \[4\]. In this paper we will try to extend the method also in the two-dimensional case.

2. Two-dimensional generalization. The following two-dimensional generalization seems straightforward. For two-dimensional $n \times n$ histograms the corresponding Markov chain states will also form a two-dimensional array $(i, j)$. Let $u_{ij}^{(k)}$ be a probability for the state $(i, j)$ to be occupied after $k$-steps of the Markov process. Then

$$u_{lm}^{(k+1)} = \sum_{i,j=1}^{n} P_{ij,lm} u_{ij}^{(k)},$$

where $P_{ij,lm}$ is a transition probability from the state $(i, j)$ to the state $(l, m)$. We will assume that the only nonzero transition probabilities are those which connect a given state to its closest left, right, up or down neighbor states. Then the generalization of equations (1.1) and (1.2) is almost obvious. Namely, for the transition probabilities we will take

$$P_{ij,i\pm1,l,j} = \frac{Q_{ij,i\pm1,l,j} + Q_{ij,i,l\pm1,j} + Q_{ij,i,l,j}}{Q_{ij,i,l,j} + Q_{ij,i,l\pm1,j}} \cdot (2.1)$$

with

$$Q_{ij,i\pm1,l,j} = \sum_{k=1}^{m} \sum_{l=-k}^{k} \exp \left[ \frac{N_{i\pm1,l,j} - N_{ij}}{\sqrt{N_{i\pm1,l,j} + N_{ij}}} \right],$$

$$Q_{ij,i,l\pm1,j} = \sum_{k=1}^{m} \sum_{l=-k}^{k} \exp \left[ \frac{N_{i,l\pm1,j} - N_{ij}}{\sqrt{N_{i,l\pm1,j} + N_{ij}}} \right]. \cdot (2.2)$$

We are interested in invariant distribution $u_{ij}$ for this Markov chain, such that

$$\sum_{i,j=1}^{n} P_{ij,lm} u_{ij} = u_{lm}.$$

Unfortunately, unlike to the one-dimensional case, this invariant distribution can not be given by a simple analytic formula. But there is a way out: having at hand the transition probabilities $P_{ij,lm}$, we can simulate the corresponding Markov process starting with some initial distribution $u_{ij}^{(0)}$. Then after a sufficiently large number of iterations we will end with almost invariant distribution irrespective to the initial choice of $u_{ij}^{(0)}$. For example, in the role of $u_{ij}^{(0)}$ one can take the uniform distribution:

$$u_{ij}^{(0)} = \frac{1}{n^2}.$$

For practical realization of the algorithm, it is desirable to have precise meaning of words “sufficiently large number of iterations”. In our first tests the following
Fig. 2.1. The upper histograms represent the initial data in the contour and lego formats respectively. The middle histograms show the corresponding probability distribution after 258 iterations for the penetrating ability $m = 3$. The lower histograms represent the invariant distribution for the penetrating ability $m = 30$. 
stopping criterion was used. One stops at \( k \)-th iteration if the averaged relative difference between \( u_{ij}^{(k)} \) and \( u_{ij}^{(k-1)} \) probability distributions is less than the desired accuracy \( \epsilon \):

\[
\sum_{u_{ij}^{(k)} \neq 0} 2 \frac{|u_{ij}^{(k)} - u_{ij}^{(k-1)}|}{u_{ij}^{(k)} + u_{ij}^{(k-1)}} u_{ij}^{(k)} < \epsilon.
\]  

(2.3)

The performance of the algorithm is illustrated by Fig. 2.1 for a 100x100 histogram representing three overlapping Gaussians with different widths. As expected, it works much like to its one-dimensional cousin: the invariant probability distribution shows sharp peaks at locations where the initial data has broad enough local maximums. Note that in this concrete example the one iteration variability \( \epsilon = 10^{-3} \) was reached after 258 iterations for \( m = 3 \) and after 113 iterations for \( m = 30 \).

Convergence to the invariant distribution can be slow. In the example given above by Fig. 2.1 the convergence is indeed slow for small penetrating abilities. If we continue iterations for \( m = 3 \) further, the side peaks will slowly decay in favor of the main peak corresponding to the global maximum. In the case of \( m = 3 \) it takes too much iterations to reach the invariant distribution. However, as Fig. 2.1 indicates, the remarkable property to develop sharp peaks at locations of local maxima of the initial histogram is already revealed by \( u_{ij}^{(k)} \) when number of iterations \( k \) is of the order of 300.

One can make the algorithm to emphasize minimums, not maximums, by just reversing signs in the exponents:

\[
\exp \left[ \frac{N_{im} - N_{ij}}{\sqrt{N_{im} + N_{ij}}} \right] \rightarrow \exp \left[ \frac{-N_{im} - N_{ij}}{\sqrt{N_{im} + N_{ij}}} \right].
\]

This is illustrated by Fig. 2.2. Here the initial histogram is generated by using a variant of the Griewank function \[5\]

\[
F(x, y) = \frac{(x - 100)^2 + (y - 100)^2}{4000} - \cos (x - 100) \cos \frac{y - 100}{\sqrt{2}} + 1.
\]  

(2.4)

This function has the global minimum at a point \( x = 100, y = 100 \) and in the histogramed interval \( 50 \leq x \leq 150, 50 \leq y \leq 150 \) exhibits nearly thousand local minima. Many of them are still visible in the probability distribution for penetrating ability \( m = 3 \). But for \( m = 30 \) only one peak, corresponding to the global minimum, remains.

3. Quantum swarm optimization. The discussion above was focused on two-dimensional histograms, while in practice more common problem is finding global optimums of nonlinear functions. The algorithm in the form discussed so far is not suitable for this latter problem. However it is possible to merge its ideology with the one of particle swarm optimization \[6, 7, 8\] to get a workable tool.

The particle swarm optimization was inspired by intriguing ability of bird flocks to find spots with food, even though birds in the flock had no previous knowledge of their location and appearance. “This algorithm belongs ideologically to that philosophical school that allows wisdom to emerge rather than trying to impose it, that emulates nature rather than trying to control it, and that seeks to make things simpler rather than more complex” \[6\]. This charming philosophy is indeed very attractive. So we
attempted to develop quantum swarm optimization - when each particle in the swarm mimics the quantum behavior.

The algorithm that emerged goes as follows:

- initialize a swarm of \( n_p \) particles at random positions in the search space \( x_{\min} \leq x \leq x_{\max}, \ y_{\min} \leq y \leq y_{\max} \).
- find a particle \( i_b \) in the swarm with the best position \((x_b, y_b)\), such that the function \( F(x, y) \) under investigation has the most optimal value for the swarm at \((x_b, y_b)\).
- for each particle in the swarm find the distance from the best position \( d = \sqrt{(x - x_b)^2 + (y - y_b)^2} \). For the best particle instead take the maximal value of these distances from the previous iteration (or for the first iteration take \( d = \sqrt{(x_{\max} - x_{\min})^2 + (y_{\max} - y_{\min})^2} \)).
- generate a random number \( r \) uniformly distributed in the interval \( 0 \leq r \leq 1 \) and find the random step \( h = rd \).
- check for a better optimum the closest left, right, up and down neighbor states with the step \( h \). If the result is positive, change \( i_b \) and \((x_b, y_b)\) respectively.
Otherwise

- move the particle to left, right, up or down by the step \( h \) according to the corresponding probabilities of such jumps:

\[
\begin{align*}
q_L &= \frac{q}{q_L + q_R + q_U + q_D}, \quad q_R = \frac{q}{q_L + q_R + q_U + q_D}, \\
q_U &= \frac{q}{q_L + q_R + q_U + q_D}, \quad q_D = \frac{q}{q_L + q_R + q_U + q_D},
\end{align*}
\]

(3.1)

where

\[
\begin{align*}
q_L &= \sum_{y' = y_u, y_d} \exp \left( I_s \frac{F(x_d, y') - F(x, y)}{h} \right), \\
q_R &= \sum_{y' = y_u, y_d} \exp \left( I_s \frac{F(x_u, y') - F(x, y)}{h} \right), \\
q_D &= \sum_{x' = x_u, x_d} \exp \left( I_s \frac{F(x, y_d) - F(x, y)}{h} \right), \\
q_U &= \sum_{x' = x_u, x_d} \exp \left( I_s \frac{F(x', y_u) - F(x, y)}{h} \right),
\end{align*}
\]

(3.2)

and

\[
\begin{align*}
x_u &= \min(x + h, x_{\text{max}}), \quad x_d = \max(x - h, x_{\text{min}}), \\
y_u &= \min(y + h, y_{\text{max}}), \quad y_d = \max(y - h, y_{\text{min}}).
\end{align*}
\]

(3.3)

At last \( I_s = 1 \), if optimization means to find the global maximum, and \( I_s = -1 \), if the global minimum is searched.

- do not stick at walls. If the particle is at the boundary of the search space, it jumps away from the wall with the probability equaled one (that is the probabilities of other three jumps are set to zero).
- check whether the new position of the particle leads to the better optimum.
  - If yes, change \( i_b \) and \((x_b, y_b)\) accordingly.
- do not move the best particle if not profitable.
- when all particles from the swarm make their jumps, the iteration is finished.
  - Repeat it at a prescribed times or until some other stopping criteria are met.

To test the algorithm performance, we tried it on some benchmark optimization test functions. For each test function and for each number of iterations one thousand independent numerical experiments were performed and the success rate of the algorithm was calculated. The criterion of success was the following

\[
\begin{align*}
|x_f - x_m| &\leq \begin{cases} 
10^{-3} |x_m|, & \text{if } |x_m| > 10^{-3} \\
10^{-3}, & \text{if } |x_m| \leq 10^{-3}
\end{cases}, \\
|y_f - y_m| &\leq \begin{cases} 
10^{-3} |y_m|, & \text{if } |y_m| > 10^{-3} \\
10^{-3}, & \text{if } |y_m| \leq 10^{-3}
\end{cases},
\end{align*}
\]

(3.4)

where \((x_m, y_m)\) is the true position of the global optimum and \((x_f, y_f)\) is the position found by the algorithm. The results are given in the table 3.1. The test functions itself are defined in the appendix. Here we give only some comments about the algorithm performance.
Table 3.1
Success rate of the algorithm in percentages for various test functions and for various numbers of iterations. Swarm size \( n_p = 20 \).

| Function Name          | Iterations |
|------------------------|------------|
|                        | 50 | 100 | 200 | 300 | 400 | 500 | 600 | 700 |
| Chichinadze            | 35.5 | 97 | 100 | 100 | 100 | 100 | 100 | 100 |
| Schwefel               | 99.4 | 99.5 | 99.8 | 99.3 | 99.2 | 99.8 | 100 | 99.6 |
| Ackley                 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Matyas                 | 88.9 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Booth                  | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Easom                  | 93.6 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Levy5                  | 98.4 | 99.5 | 99.4 | 99.3 | 99 | 99 | 99.1 | 99.5 |
| Goldstein-Price        | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Griewank               | 76.3 | 99.7 | 100 | 100 | 100 | 100 | 100 | 100 |
| Rastrigin              | 100 | 100 | 99.8 | 99.9 | 100 | 99.9 | 99.9 | 100 |
| Rosenbrock             | 43.6 | 90.4 | 99.8 | 100 | 100 | 100 | 100 | 100 |
| Leon                   | 13.8 | 52.1 | 82 | 91.6 | 97.6 | 99.1 | 99.6 | 99.8 |
| Giunta                 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Beale                  | 99.7 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Bukin2                 | 61.8 | 84.4 | 93.8 | 97.8 | 98.6 | 99.3 | 99.7 | 99.8 |
| Bukin4                 | 99.6 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Bukin6                 | 0.2 | 0.1 | 0 | 0.2 | 0 | 0.1 | 0.2 | 0.1 |
| Styblinski-Tang        | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Zettl                  | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Three Hump Camel       | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Schaffer               | 8.2 | 34.7 | 60.7 | 71.2 | 77.8 | 78.9 | 80.4 | 83.9 |
| Levy13                 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| McCormic               | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |

For some test problems, such as Chichinadze, Ackley, Matyas, Booth, Easom, Goldstein-Price, Griewank, Giunta, Beale, Bukin4, Styblinski-Tang, Zettl, Levy13, McCormic and Three Hump Camel Back, the algorithm is triumphant.

Matyas problem seems easy, because the function is only quadratic. However it is very flat near the line \( x = y \) and this leads to problems for many global optimization algorithms.

Easom function is a unimodal test function which is expected to be hard for any stochastic algorithms, because vicinity of its global minimum has a small area compared to the search space. Surprisingly our algorithm performs quite well for this function and one needs only about 100 iterations to find the needle of the global minimum in a haystack of the search space.

Schwefel function is deceptive enough to cause search algorithms to converge in the wrong direction. This happens because the global minimum is geometrically distant from the next best local minima. In some small fraction of events our algorithm is also prone to converge in the wrong direction and in these cases the performance seems not to improve by further increasing the number of iterations. But the success rate is quite high. Therefore in this case it is more sensible to have two or more independent tries of the algorithm with rather small number of iterations each.

Rastrigin function is a multimodal test function which have plenty of hills and
valleys. Our algorithm performs even better for this function, but the success is not universal either.

Rosenbrock function is on contrary unimodal. Its minimum is situated in a banana shaped valley with a flat bottom and is not easy to find. The algorithm needs more than 200 iterations to be successful in this case. Leon function is of the similar nature, with even more flat bottom and the convergence in this case is correspondingly more slow.

Griewank, Levy5 and Levy13 are multimodal test functions. They are considered to be difficult for local optimizers because of the very rugged landscapes and very large number of local optima. For example, Levy5 has 760 local minima in the search domain but only one global minimum and Levy13 has 900 local minima. Test results reveal a small probability that our algorithm becomes stuck in one of the local minima for the Levy5 function.

Giunta function simulates the effects of numerical noise by means of a high frequency, low amplitude sine wave, added to the main part of the function. The algorithm is successful for this function.

Convergence of the algorithm is rather slow for Bukin2 function, and especially for the Schaffer function. This latter problem is hard because of the highly variable data surface features many circular local optima, and our algorithm becomes, unfortunately, often stuck in the optima nearest to the global one.

At last, the algorithm fails completely for the Bukin6 function. This function has a long narrow valley which is readily identified by the algorithm. But the function values differ very small along the valley. Besides the surface is non-smooth in the valley with numerous pitfalls. This problem seems hopeless for any stochastic algorithm based heavily on random walks, because one has to chance upon a very vicinity of the global optimum to be successful. The non-stochastic component of our algorithm (calculation of jump probabilities to mimic the quantum tunneling) turns out to be of little use for this particular problem.

4. Concluding remarks. The Quantum Swarm Optimization algorithm presented above emerged while trying to generalize in the two-dimensional case a “quantum mechanical” algorithm for automatic location of photopeaks in the one dimensional histograms.11

“Everything has been said before, but since nobody listens we have to keep going back and beginning all over again”22. After this investigation was almost finished, we discovered the paper by Xie, Zhang and Yang with the similar idea to use the simulation of particle-wave duality in optimization problems. However their realization of the idea is quite different.

Even earlier, Levy and Montalvo used the tunneling method for global optimization, but without referring to quantum behavior. Their method consisted in a transformation of the objective function, once a local minimum has been reached, which destroys this local minimum and allows to tunnel classically to another valley.

We found also that the similar ideology to mimic Nature’s quantum behavior in optimization problems emerged in quantum chemistry and led to such algorithms as quantum annealing and Quantum Path Minimization.

Nevertheless, the Quantum Swarm Optimization is conceptually rather different from these developments. We hope it is simple and effective enough to find an ecological niche in a variety of global optimization algorithms.

Appendix. Here we collect the test functions definitions, locations of their optima and the boundaries of the search space. The majority of them was taken from
Chichinadze function \[13, 16\]

\[F(x, y) = x^2 - 12x + 11 + 10 \cos \frac{\pi}{2} x + 8 \sin (5\pi x) - \frac{1}{\sqrt{5}} \exp \left( -\frac{(y - 0.5)^2}{2} \right),\]

\[-30 \leq x, y \leq 30, \quad F_{\text{min}}(x, y) = F(5.90133, 0.5) = -43.3159.\]

Schweffel function \[18\]

\[F(x, y) = -x \sin \sqrt{|x|} - y \sin \sqrt{|y|},\]

\[-500 \leq x, y \leq 500, \quad F_{\text{min}}(x, y) = F(420.9687, 420.9687) = -837.9658.\]

Ackley function \[19\]

\[F(x, y) = 20[1 - e^{-0.2\sqrt{0.5(x^2 + y^2)}}] - e^{0.5[\cos (2\pi x) + \cos (2\pi y)]} + \epsilon,\]

\[-35 \leq x, y \leq 35, \quad F_{\text{min}}(x, y) = F(0, 0) = 0.\]

Matyas function \[15\]

\[F(x, y) = 0.26(x^2 + y^2) - 0.48xy,\]

\[-10 \leq x, y \leq 10, \quad F_{\text{min}}(x, y) = F(0, 0) = 0.\]

Booth function \[16\]

\[F(x, y) = (x + 2y - 7)^2 + (2x + y - 5)^2\]

\[-10 \leq x, y \leq 10, \quad F_{\text{min}}(x, y) = F(1, 3) = 0.\]

Easom function \[20\]

\[F(x, y) = -\cos x \cos y \exp \left[-(x - \pi)^2 - (y - \pi)^2\right],\]

\[-100 \leq x, y \leq 100, \quad F_{\text{min}}(x, y) = F(\pi, \pi) = -1.\]

Levy5 function \[15\]

\[F(x, y) = \sum_{i=1}^{5} i \cos [(i - 1)x + i] \sum_{j=1}^{5} j \cos [(j + 1)y + j] +

\(+(x + 1.42513)^2 + (y + 0.80032)^2,\)

\[-100 \leq x, y \leq 100, \quad F_{\text{min}}(x, y) = F(-1.30685, -1.424845) = -176.1375.\]
Goldstein-Price function \[15\]

\[F(x, y) = \left[1 + (x + y + 1)^2 (19 - 14x + 3x^2 - 14y + 6xy + 3y^2)\right] \times \left[30 + (2x - 3y)^2 (18 - 32x + 12x^2 + 48y - 36xy + 27y^2)\right],\]

\[-2 \leq x, y \leq 2, \quad F_{\text{min}}(x, y) = F(0, -1) = 3.\]

Griewank function \[5, 15\]

\[F(x, y) = \frac{x^2 + y^2}{200} - \cos x \cos \frac{y}{\sqrt{2}} + 1,\]

\[-100 \leq x, y \leq 100, \quad F_{\text{min}}(x, y) = F(0, 0) = 0.\]

Rastrigin function \[21\]

\[F(x, y) = x^2 + y^2 - 10 \cos (2\pi x) - 10 \cos (2\pi y) + 20,\]

\[-5.12 \leq x, y \leq 5.12, \quad F_{\text{min}}(x, y) = F(0, 0) = 0.\]

Rosenbrock function \[15\]

\[F(x, y) = 100(y - x^2)^2 + (1 - x)^2,\]

\[-1.2 \leq x, y \leq 1.2, \quad F_{\text{min}}(x, y) = F(1, 1) = 0.\]

Leon function \[22\]

\[F(x, y) = 100(y - x^3)^2 + (1 - x)^2,\]

\[-1.2 \leq x, y \leq 1.2, \quad F_{\text{min}}(x, y) = F(1, 1) = 0.\]

Giunta function \[23\]

\[F(x, y) = \sin \left(\frac{16}{15} x - 1\right) + \sin^2 \left(\frac{16}{15} x - 1\right) + \frac{1}{50} \sin \left[4 \left(\frac{16}{15} x - 1\right)\right] + \sin \left(\frac{16}{15} y - 1\right) + \sin^2 \left(\frac{16}{15} y - 1\right) + \frac{1}{50} \sin \left[4 \left(\frac{16}{15} y - 1\right)\right] + 0.6,\]

\[-1 \leq x, y \leq 1, \quad F_{\text{min}}(x, y) = F(0.45834282, 0.45834282) = 0.0602472184\]

Beale function \[15\]

\[F(x, y) = (1.5 - x + xy)^2 + (2.25 - x + xy^2)^2 + (2.625 - x + xy^3)^2,\]

\[-4.5 \leq x, y \leq 4.5, \quad F_{\text{min}}(x, y) = F(3, 0) = 0.\]
Bukin2 function \[24\]
\[F(x, y) = 100(y - 0.01x^2 + 1) + 0.01(x + 10)^2,\]
\[-15 \leq x \leq -5, \ -3 \leq y \leq 3, \ F_{\text{min}}(x, y) = F(-10, 0) = 0.\]

Bukin4 function \[24\]
\[F(x, y) = 100y^2 + 0.01|x + 10|,\]
\[-15 \leq x \leq -5, \ -3 \leq y \leq 3, \ F_{\text{min}}(x, y) = F(-10, 0) = 0.\]

Bukin6 function \[24\]
\[F(x, y) = 100\sqrt{|y - 0.01x^2|} + 0.01|x + 10|,\]
\[-15 \leq x \leq -5, \ -3 \leq y \leq 3, \ F_{\text{min}}(x, y) = F(-10, 1) = 0.\]

Styblinski-Tang function \[25\]
\[F(x, y) = \frac{1}{2} \left[ x^4 - 16x^2 + 5x + y^4 - 16y^2 + 5y \right],\]
\[-5 \leq x, y \leq 15, \ F_{\text{min}}(x, y) = F(-2.903534, -2.903534) = -78.332.\]

Zettl function \[22\]
\[F(x, y) = (x^2 + y^2 - 2x)^2 + 0.25x,\]
\[-5 \leq x, y \leq 5, \ F_{\text{min}}(x, y) = F(-0.0299, 0) = -0.003791.\]

Three Hump Camel back function \[14\]
\[F(x, y) = 2x^2 - 1.05x^4 + \frac{x^6}{6} + xy + y^2,\]
\[-5 \leq x, y \leq 5, \ F_{\text{min}}(x, y) = F(0, 0) = 0.\]

Schaffer function \[26\]
\[F(x, y) = 0.5 + \frac{\sin \sqrt{x^2 + y^2} - 0.5}{[1 + 0.001(x^2 + y^2)]^2},\]
\[-100 \leq x, y \leq 100, \ F_{\text{min}}(x, y) = F(0, 0) = 0.\]

Levy13 function \[13\]
\[F(x, y) = \sin^2(3\pi x) + (x - 1)^2 \left[ 1 + \sin^2(3\pi y) \right] + (y - 1)^2 \left[ 1 + \sin^2(2\pi y) \right],\]
\[-10 \leq x, y \leq 10, \ F_{\text{min}}(x, y) = F(1, 1) = 0.\]

McCormic function \[27\]
\[F(x, y) = \sin(x + y) + (x - y)^2 - 1.5x + 2.5y + 1,\]
\[-1.5 \leq x \leq 4, \ -3 \leq y \leq 4, \ F_{\text{min}}(x, y) = F(-0.54719, -1.54719) = -1.9133.\]
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