THETA-CHARACTERISTICS ON TROPICAL CURVES

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Abstract. We give an explicit description of theta-characteristics on tropical curves and characterize the effective ones. We construct the moduli space, $T_g^{\text{trop}}$, for tropical theta-characteristics of genus $g$ as a generalized cone complex. We describe the fibers of the specialization map from the moduli space $S_g$ of theta-characteristics on algebraic curves, to $T_g^{\text{trop}}$.

MSC (2020): 14H10, 14H40, 14T20.

Keywords: Moduli space, theta-characteristic, tropical curve, algebraic curve.

1. Introduction

Let $X$ be a smooth algebraic curve of genus $g$ over an algebraically closed field of characteristic different from 2. A theta-characteristic on $X$ is a line bundle of degree $g - 1$ on $X$ such that $L^{\otimes 2} \cong K_X$, where $K_X$ is the canonical line bundle of $X$. Theta-characteristics on algebraic curves have been extensively studied over the years because of their remarkable geometry. In particular, there is a notion of parity on theta-characteristics (given by the parity of the dimension of its space of global sections) which is deformation invariant (see [Mum71]). Each algebraic curve admits $2^{2g}$ theta-characteristics, $(2^g + 1)2^{g-1}$ of which are even and $(2^g - 1)2^{g-1}$ of which are odd. The moduli space $S_g$ of theta-characteristics on curves of genus $g$ splits in two connected components, according to the parity.

Let now $\Gamma$ be a tropical curve of genus $g$ and let $K_\Gamma$ be the canonical divisor on $\Gamma$. A tropical theta-characteristic on $\Gamma$ is a divisor class $[T] \in \text{Pic}^{g-1}(\Gamma)$ such that $[2T] = [K_\Gamma]$. Tropical theta-characteristics on pure tropical curves have been studied by Zharkov in [Zha10] and shown to be associated to orientations given by the choice of a cyclic subcurve of $\Gamma$. There are therefore $2^g$ theta-characteristics on a pure tropical curve of genus $g$, and among these exactly one is non-effective.

In this paper, we study tropical theta-characteristics on any tropical curve (not necessarily pure). In Proposition 3.6 we generalize Zharkov’s results by exhibiting explicit representatives for tropical theta-characteristics obtained.

The first and second authors were supported by funds from MIUR via the Excellence Department Project awarded to the Department of Mathematics and Physics of Roma Tre and by the project PRIN2017SSNZAW: Advances in Moduli Theory and Birational Classification. The second author is a member of the Centre for Mathematics of the University of Coimbra – UIDB/00324/2020), funded by the Portuguese Government through FCT/MCTES. The third author was supported by CNPq-PQ, 301671/2019-2.
from orientations associated to suitable flows on the curve and classify the ones that are effective (see Theorem 3.7). Our choice of representatives is different from Zharkov’s, and it is inspired by the geometry of the moduli space of stable spin curves $\mathcal{S}_g$ defined by Cornalba in [C89]. Our representatives are well behaved with respect to specialization, and we are able to construct, in Theorem 4.1, a moduli space, $T^\text{trop}_g$, for tropical theta-characteristics as a generalized cone complexes.

Unlike the situation for classical algebraic curves, there is no natural notion of parity on tropical theta-characteristics. In our paper [CMP20], we introduced the notion of tropical spin curve, which is a tropical curve together with a spin structure, i.e., a theta-characteristic together with a suitable sign function, which encodes the notion of parity. Theta-characteristics on tropical curves are obtained from spin structures simply by forgetting the sign function. In loc. cit, we constructed a moduli space $S^\text{trop}_g$ for tropical spin curves of given genus and we showed that this space is related with Cornalba’s moduli space via Berkovich analytification. There is a natural morphism of generalized cone complexes, $S^\text{trop}_g \to M^\text{trop}_g$, to the moduli space of tropical curves, $M^\text{trop}_g$. From the results of Section 4 we obtain a natural factorization

$$S^\text{trop}_g \to T^\text{trop}_g \to M^\text{trop}_g$$

via morphisms of generalized cone complexes.

Let now $X_K$ be a smooth algebraic curve of genus $g$ over a non-Archimedean algebraically closed field $K$ of characteristic different from 2, and let $\Gamma$ be the genus $g$ tropical curve which is the skeleton of $X_K$. There is a specialization map $\tau: \text{Pic}(X_K) \to \text{Pic}(\Gamma)$, which is well-known to be a homomorphism. Let $S_{X_K}$ be the set of theta-characteristics on $X_K$ and let $T^\text{trop}_\Gamma$ be the set of (tropical) theta-characteristics on $\Gamma$. Since $\tau$ is a homomorphism, it restricts to a map $S_{X_K} \to T^\text{trop}_\Gamma$. It is quite interesting to understand the fibers of the specialization map over a given tropical theta-characteristic of $\Gamma$. This problem has been considered by various authors: in the case of tropical plane quartics by Baker, Len, Morrison, Pflueger and Ren in [BLMPR16], Chan, Jiradilok in [CJ] and Len, Markwig in [LM20], for hyperelliptic curves by Panizzut in [Pan16] and by Jensen and Len in [JL18] in the case of pure tropical curves. Jensen and Len showed that the specialization map $S_{X_K} \to T^\text{trop}_\Gamma$ is surjective. Moreover, there are $2^g$ even theta-characteristics specializing to the non-effective tropical theta-characteristic, and each effective theta-characteristic is the image of $2^{g-1}$ even theta-characteristics and $2^{g-1}$ odd theta-characteristics. Jensen and Len’s result follows by carefully analyzing the Weil pairing on the set of 2-torsion points in the Jacobian of smooth curves.

In the second part of this paper we propose a different approach to study the specialization map for theta-characteristics, based on the description of tropical theta-characteristics given in the first part. Our results rely on our approach to tropical theta-characteristics in [CMP20], which allows us
to give an interpretation to the specialization map using the combinatorial
description of the boundary of Cornalba’s moduli space $\overline{\mathcal{M}}_g$. In Theorem 5.4 we use this approach to obtain a generalization of Jensen and Len’s results for any tropical curve (not necessarily pure). We point out that by using the results in [CMP20] the proof of Theorem 5.4 is quite simple compared to the previous approaches.

2. Preliminaries

2.1. Graphs. In this paper we denote by $G = (V, E)$ a connected graph. Given a vertex $v \in V$, we write $\deg_G(v)$, sometimes $\deg(v)$, for the number of edges of $G$ incident to $v$, with loops counting twice. Sometimes we will use the notation $E(G)$ and $V(G)$ for the set of edges and vertices of $G$.

Given a subset $R \subset E$, we will frequently abuse the notation by writing $R$ to denote the subgraph of $G$ generated by $R$. For $R \subset E$, the $R$-subdivision (or simply subdivision) of $G$, written $\hat{G}_R$, is the graph obtained from $G$ by inserting exactly one vertex in the interior of each edge $e \in R$.

An orientation $O$ on $G$ is a pair $O = (\sigma, \tau)$ where $\sigma, \tau : E \to V$, called source and target map, are such that $\{\sigma(e), \tau(e)\} = \{u_e, v_e\}$, where we write $u_e, v_e$ for the ends of $e$. A sub-orientation on $G$ is an orientation on some subdivision of $G$.

Given an orientation $O$ on $G$, if $\hat{G}$ is the $R$-subdivision of $G$ for $R \subset E$, we have a natural orientation induced on $\hat{G}$ (i.e. a sub-orientation on $G$), which we call, again, $O$. More precisely, let $e_0, e_1$ be the edges of $\hat{G}$ obtained after inserting one vertex in the interior of each edge $e \in R$. If $e_0$ is incident to $\sigma(e)$ and $e_1$ to $\tau(e)$, then we set $\sigma(e_0) = \sigma(e)$ and $\tau(e_1) = \tau(e)$, as in the picture.

A (vertex weighted) graph $(G, w)$ is a (connected) graph $G$ endowed with a function $w : V(G) \to \mathbb{Z}_{\geq 0}$. The genus of $(G, w)$ is

$$g = \sum_{v \in V} w(v) + b_1(G) = \sum_{v \in V} w(v) + |E| - |V| + 1.$$ 

We say that $(G, w)$ is stable if $2w(v) - 2 + \deg_G(v) > 0$, for every $v \in V$.

We denote by $\mathcal{E}_G$, respectively $\mathcal{V}_G$, the vector space over $\mathbb{F}_2$ spanned by $E$, respectively by $V$. We consider the linear map $\partial : \mathcal{E}_G \to \mathcal{V}_G$ such that for every subset $S \subset E$,

$$\partial \left( \sum_{e \in S} e \right) = \sum_{e \in S} (u_e + v_e).$$
We denote by $C_G$ the kernel of $\partial$, and we call an element of $C_G$ cyclic or a cyclic subgraph of $G$. It is well known that $C_G$ is generated by the cycles of $G$, (i.e. by the connected subgraphs all of whose vertices have degree 2.)

2.2. Tropical curves. A (connected) tropical curve is a triple $\Gamma = (G,w,\ell)$, where $(G,w)$ is a stable graph and $\ell$ a length function, $\ell : E \to \mathbb{R}_{>0}$. We call $(G,w)$ the combinatorial type of $\Gamma$. The genus of $\Gamma = (G,w,\ell)$ is the genus of $(G,w)$, usually denoted by $g$.

We view $\Gamma$ as a metric space as follows. Every edge $e$ is identified with a closed interval of length $\ell(e)$ if $e$ is not a loop, and with a circle of length $\ell(e)$ if $e$ is a loop. Then every path in $\Gamma$ has a well defined length. Now given two points $p,q \in \Gamma$ we define the distance between them, $d(p,q)$, as the shortest length of a path in $\Gamma$ from $p$ to $q$.

For any edge $e$ of $G$, we let $p_e \in \Gamma$ be the mid-point of $e$.

We extend the weight function to all points of $\Gamma$ as follows

$$w_\Gamma(p) = \begin{cases} w(p) & \text{if } p \in V \\ 0 & \text{otherwise.}\end{cases}$$

Similarly, we set

$$\deg_\Gamma(p) = \begin{cases} \deg_V(p) & \text{if } p \in V \\ 2 & \text{otherwise.}\end{cases}$$

We write $w = w_\Gamma$ and $\deg = \deg_\Gamma$ when no confusion is likely.

A divisor on $\Gamma$ is a formal sum $D = \sum_{p \in \Gamma} D(p)p$, with $D(p) \in \mathbb{Z}$, where $D(p)$ is nonzero only for a finite set of points of $\Gamma$. We write $D \geq 0$ if $D(p) \geq 0$ for all $p \in \Gamma$, and we say $D$ is effective. The degree of $D$ is the integer $\sum_{p \in \Gamma} D(p)$. We let $\text{Div}(\Gamma)$ be the group of divisors on $\Gamma$ and $\text{Div}^d(\Gamma) \subset \text{Div}(\Gamma)$ the subset of divisors of degree $d$.

A rational function $f$ on $\Gamma$ is a continuous piecewise linear function $f : \Gamma \to \mathbb{R}$ with integral slopes and finitely many pieces. The points of $\Gamma$ where $f$ is not linear will be called critical points of $f$. Given a rational function $f$ on $\Gamma$, for every $p \in \Gamma$, we let $\text{ord}_p(f)$ be the sum of the slopes of $f$ outgoing $p$ (there are $\deg_\Gamma(p)$ of them). If $f$ is linear at $p$ then $\text{ord}_p(f) = 0$, therefore the following

$$\text{div}(f) := \sum_{p \in \Gamma} \text{ord}_p(f)p$$

is a finite sum, hence a divisor, and it is easy to see that $\text{div}(f) \in \text{Div}^0(\Gamma)$. The divisors of the form $\text{div}(f)$ are called principal divisors. The set of all principal divisors is a subgroup of $\text{Div}(\Gamma)$ denoted by $\text{Prin}(\Gamma)$. The Picard group of $\Gamma$ is

$$\text{Pic}(\Gamma) := \text{Div}(\Gamma)/\text{Prin}(\Gamma),$$

and $\text{Pic}^d(\Gamma) \subset \text{Pic}(\Gamma)$ is the set of divisor classes of degree $d$. For $D \in \text{Div}(\Gamma)$ we denote by $[D] \in \text{Pic}(\Gamma)$ its class and we write $D \sim D'$ if $[D] = [D']$. 

Example 2.1. Fix a non-loop edge $e$ of $\Gamma$ and let $u_e, v_e$ be its ends. Let us identify $e$ with the interval $[-\ell(e)/2, \ell(e)/2]$ so that its mid-point, $p_e$, gets identified with 0. Now consider the rational function $f_e$ on $\Gamma$ defined as follows.

$$f_e(x) = \begin{cases} \ell(e)/2 + x & \text{if } x \in [-\ell(e)/2, 0] \\ \ell(e)/2 - x & \text{if } x \in [0, \ell(e)/2] \\ 0 & \text{otherwise, i.e. if } x \not\in e. \end{cases}$$

Then

$$\text{div}(f_e) = u_e + v_e - 2p_e.$$ 
If $e$ is a loop based at a vertex $v_e \in V$, we identify $e$ with the circle obtained from the interval $[-\ell(e)/2, \ell(e)/2]$ by identifying its ends. Then $f_e$ defines again a rational function, and we have $\text{div}(f_e) = 2v_e - 2p_e$.

From the previous example we highlight the following basic relation:

$$u_e + v_e \sim 2p_e$$

for any edge $e$ of $\Gamma$ with (possibly equal) ends $u_e, v_e$.

Let $O = (\sigma, \tau)$ be a sub-orientation on $G$ supported on the subdivision $\hat{G}$. For $p \in \Gamma$ we denote by $\deg^-_O(p)$ the number of edges directed towards $p$, i.e.

$$\deg^-_O(p) = \begin{cases} |\{e \in E(\hat{G}) : \tau(e) = p\}|, & \text{if } p \in V(\hat{G}); \\ 1, & \text{otherwise.} \end{cases}$$

We define the following divisor on $\Gamma$:

$$D^-_O := \sum_{p \in \Gamma} (\deg^-_O(p) - 1 + w(p))p.$$ 

The divisor $D^-_O$ has degree $g - 1$ and is supported on some subset of $V(\hat{G})$.

3. Theta-characteristics on tropical curves

3.1. Tropical square roots and theta-characteristics. The canonical divisor of a tropical curve $\Gamma = (G, w, \ell)$ is

$$K_\Gamma := \sum_{p \in \Gamma} (2w(p) - 2 + \deg(p))p = \sum_{v \in V} (2w(v) - 2 + \deg(v))v.$$ 

The divisor $K_\Gamma$ has degree $2g - 2$, with $g$ the genus of $\Gamma$.

Definition 3.1. A (tropical) square root (of zero) of $\Gamma$ is a divisor class $[D] \in \text{Pic}(\Gamma)$ such that $[2D] = 0$. The set of square roots of $\Gamma$ is denoted by $R_{\Gamma}^{\text{trop}}$.

A (tropical) theta-characteristic of $\Gamma$ is a divisor class $[D] \in \text{Pic}(\Gamma)$ such that $[2D] = [K_\Gamma]$. The set of theta-characteristics of $\Gamma$ is denoted by $T_{\Gamma}^{\text{trop}}$. 
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Obviously, $R_{\Gamma}^{trop} \subset \text{Pic}^0(\Gamma)$ and $T_{\Gamma}^{trop} \subset \text{Pic}^{g-1}(\Gamma)$.

Our next goal is to give an explicit description of all square roots and all theta-characteristics.

Let $P \in C_G$, then $\deg_P(v)$ is even for every $v \in V(G)$. We can thus define the following divisor on $\Gamma$

$$F_P := \sum_{v \in V} \frac{\deg_P(v)}{2} v - \sum_{e \in P} p_e.$$

We have, using Example 2.1

$$2F_P = \sum_{v \in V} \deg_P(v)v - 2\sum_{e \in P} p_e = \sum_{e \in P} (v_e + u_e - 2p_e) = \sum_{e \in P} \text{div}(f_e).$$

Therefore $2F_P$ is a principal divisor, in other words, $[F_P]$ is a square root (of zero). Let us show that, as $P$ varies in $C_G$, the classes of the divisors $F_P$ are distinct and give all the square roots of $\Gamma$.

**Proposition 3.2.** Let $\Gamma = (G,w,\ell)$ be a tropical curve. Then the following is a bijection

$$C_G \longrightarrow R_{\Gamma}^{trop}; \quad P \mapsto [F_P]$$

**Proof.** Recall from [MZ08] and [BMV11] that there is a natural group isomorphism,

$$\text{Pic}^0(\Gamma) \cong H_1(G,\mathbb{R})/H_1(G,\mathbb{Z}),$$

and the set of 2-torsion points of $\text{Pic}^0(\Gamma)$ (the square roots of $\Gamma$) is identified with $H_1(G,\mathbb{Z}/2\mathbb{Z}) = C_G$. Hence $R_{\Gamma}^{trop}$ has the same cardinality as $C_G$. There remains to prove that the $[F_P]$ are all distinct. It is easy to check that for every $P, P' \in C_G$ we have

$$F_P + F_{P'} - F_{P+P'} = \sum_{e \in P \cap P'} \text{div}(f_e).$$

Hence

$$F_P - F_{P'} \sim F_P - F_{P'} + 2F_{P} = F_P + F_{P'} \sim F_{P+P'}.$$ 

Therefore it suffices to show that $[F_P] \neq 0$ for every nonzero $P \in C_G$. By (2) we have $F_P(p) = 0$ unless $p$ is either a vertex of $P$ (in which case $F_P(p) \geq 1$) or the mid-point $p_e$ of an edge of $P$ (in which case $F_P(p_e) = -1$). By contradiction, assume that for some (nonzero) cycle $P$ we have $\text{div}(f) = F_P$, with $f$ a rational function on $\Gamma$. Let $\Gamma_{\text{max}}$ be the locus of $\Gamma$ where $f$ attains its maximum. Since $\Gamma$ is compact, $\Gamma_{\text{max}}$ is a non-empty closed subcurve, moreover $\Gamma_{\text{max}} \neq \Gamma$ (as otherwise $\text{div}(f) = 0$).

We know that the slopes of $f$ outgoing $\Gamma_{\text{max}}$ are all negative, while the slopes of $f$ internal to $\Gamma_{\text{max}}$ are zero ($f$ is constant on $\Gamma_{\text{max}}$). Therefore $\text{div}(f)(p) < 0$ for every $p \in \Gamma_{\text{max}} \cap (\Gamma \setminus \Gamma_{\text{max}})$. Since the only points of $\Gamma$ where $F_P$ is negative are the mid-points $p_e$ of the edges in $P$, we deduce

$$\Gamma_{\text{max}} \cap (\Gamma \setminus \Gamma_{\text{max}}) \subset \{p_e, e \in P\}.$$
Let us show that no vertex of $P$ can be contained in $\Gamma_{\max}$. If $v \in V(P)$ were in $\Gamma_{\max}$, it would have to lie in its interior, by (3), and hence $\text{div}(f)(v) = 0$ (as $f$ is constant on $\Gamma_{\max}$), which contradicts $F_P(v) \geq 1$.

We then conclude that $\Gamma_{\max}$ is a union of mid-points $p_e$, for some edges $e$ of $P$. Since the slopes outgoing $\Gamma_{\max}$ are negative we obtain $\text{div}(f)(p_e) \leq -2$ for every such $p_e$, in contradiction with $F_P(p_e) = -1$.

We now turn to theta-characteristics; again, let $P \in \mathcal{C}_G$. Consider the following divisor on $\Gamma$,

$$T_P := \sum_{v \in V} \left( \frac{\deg_P(v)}{2} - 1 + w(v) \right) v + \sum_{e \in E \setminus P} p_e.$$  

Let us show that as $P$ varies the $[T_P]$ give all the theta-characteristics of $\Gamma$.

**Proposition 3.3.** Let $\Gamma = (G, w, \ell)$ be a tropical curve and $P \in \mathcal{C}_G$. Then $[T_P]$ is a theta-characteristic and the following is a bijection $\mathcal{C}_G \beta \rightarrow T_{\text{trop}}^\Gamma; \quad P \mapsto [T_P]$.

**Proof.** Recalling the definition of $F_P$ we have

$$T_P = \sum_{v \in V} (w(v) - 1) v + F_P + \sum_{e \in E} p_e.$$  

As $2F_P \sim 0$ we get

$$2T_P \sim \sum_{v \in V} (2w(v) - 2) v + 2 \sum_{e \in E} p_e \sim \sum_{v \in V} (2w(v) - 2) v + \sum_{e \in E} (u_e + v_e)$$

using (1). Since $\sum_{e \in E}(u_e + v_e) = \sum_{v \in V} \deg_G(v)v$ we have $2T_P \sim K_\Gamma$, i.e. $[T_P]$ is a theta-characteristic.

It remains to show that if $P \neq P'$ then $[T_P] \neq [T_{P'}]$. Using (5) we have

$$T_P - T_{P'} = F_P - F_{P'}$$

and $F_P - F_{P'} \not\sim 0$, by Proposition [5.2]. The proof is complete.

Notice that the following

$$K_{\Gamma/2} = \sum_{v \in V}(2w(v) - 2 + \deg(v))v$$

is a divisor on $\Gamma$ if and only if $G$ has only vertices of even degree, i.e. if and only if $G$ is cyclic. Now, for any $P \in \mathcal{C}_G$ we denote $\overline{P} = P \cup V$ and

$$\Gamma_{\overline{P}} = (\overline{P}, w|_{\overline{P}}, \ell|_{\overline{P}}) \subset \Gamma,$$

in other words $\Gamma_{\overline{P}}$ is the subcurve of $\Gamma$ whose underlying graph is $\overline{P}$, with the same weight function as $\Gamma$. Then $K_{\Gamma_{\overline{P}}}/2$ is a divisor (on both $\Gamma$ and $\Gamma_{\overline{P}}$) and we have

$$T_P = K_{\Gamma_{\overline{P}}}/2 + \sum_{e \in E \setminus P} p_e.$$
3.2. **Effective theta-characteristics via flows.** We say that a theta-characteristic \([TP]\) on a tropical curve \(\Gamma\) is **effective** if its rank is nonnegative, i.e. if \(r_\Gamma(TP) \neq -1\). Equivalently, \([TP]\) is effective if there exists an effective divisor \(E\) on \(\Gamma\) such that \(E \sim TP\). Our next goal is to characterize effective theta-characteristics, and describe the effective representatives.

Recall the definition \(TP = \sum_{v \in V} \left( \frac{\deg_P(v)}{2} - 1 + w(v) \right) v + \sum_{e \in E \setminus P} p_e\). We write \(TP = TP^+ - TP^-\) with \(TP^+\) and \(TP^-\) effective. We have \(TP^- = \sum_{v \in V(V(P)) \setminus V(P)} (\deg_{\Gamma}(v) - 1 + w_\Gamma(v)) v\).

In other words, \(TP \geq 0\) if and only if every vertex of \(G\) having weight zero is contained in \(P\).

**Definition 3.4.** Let \(P \in C_G\) and \(W \subset V\). Assume that \(P \cup W\) is not empty. The **cyclic subcurve** of \(\Gamma\) associated to \(P\) and \(W\) is the tropical subcurve \(\Gamma_{P,W} \subset \Gamma\) supported on the graph \(P \cup W\) (we consider \(W\) to be a subgraph of \(G\) with vertex set \(W\) and no edges).

Consider a cyclic subcurve \(\Gamma_{P,W}\) of \(\Gamma\). Let us define a sub-orientation \(O_{P,W}\) on \(G\) and a divisor \(DP_{P,W}\) on \(\Gamma\). Informally, \(O_{P,W}\) will be the flow away from \(\Gamma_{P,W}\). More precisely, we choose on \(P\) a cyclic orientation; there are \(2^{h_1(P)}\) ways to do it, the choice of which will be irrelevant. Next, we orient the edges in the complement of \(P\) away from \(\Gamma_{P,W}\), as follows. Consider the distance function \(d_{P,W}: \Gamma \to \mathbb{R}_{\geq 0}\)

\[
d_P^W(p, \Gamma_{P,W}) = \sum_{v \in V(G_{P,W})} \left( \deg_{O_{P,W}}(v) - 1 + w_\Gamma(v) \right) v.
\]

**Remark 3.5.** If \(W = V\) every vertex \(v\) not lying on \(P\) is a source (i.e. \(\deg_{O_{P,V}}(v) = 0\)) and \(p_e\) is a sink (hence \(\deg_{O_{P,V}}(p_e) = 2\)) for every edge \(e\) not in \(P\). Therefore by \(4\) we have \(DP_{P,V} = TP\).

**Proposition 3.6.** Let \(\Gamma_{P,W}\) and \(\Gamma_{P',W'}\) be two cyclic subcurves of a tropical curve \(\Gamma\). Then

1. \(DP_{P,W} \sim DP_{P',W'}\) if and only if \(P = P'\).
(2) $[D_{P,W}]$ is a theta-characteristic on $\Gamma$, that is, $2D_{P,W} \sim K_\Gamma$.

Proof. We first prove $D_{P,W} \sim D_{P,W \cup \{v\}}$ for any vertex $v \in \Gamma$. Consider the following function $f$ on $\Gamma$

$$f := \frac{d_{P,W \cup \{v\}} - d_{P,W}}{2}.$$ 

Of course, $f$ is continuous and piecewise linear with finitely many pieces. We will show that $f$ has integer slopes, i.e. is a rational function, and that $D_{P,W} - D_{P,W \cup \{v\}} = \text{div}(f)$.

Set $O_1 := O_{P,W}$ and $O_2 := O_{P,W \cup \{v\}}$. Assume without loss of generality that $O_1$ and $O_2$ coincide over $P$. For $p \in \Gamma$ we have

$$D_{P,W}(p) - D_{P,W \cup \{v\}}(p) = \deg_{O_1}(p) - \deg_{O_2}(p).$$

Let $B(p, \epsilon) \subset \Gamma$ be the closed ball with center $p$ and radius $\epsilon \in \mathbb{R}_{>0}$. If $\epsilon$ is small enough $B(p, \epsilon)$ is the union of segments $h_i$ of length $\epsilon$ and incident to $p$, for $i = 1, \ldots, \deg(\Gamma)$, over which both $d_{P,W}$ and $d_{P,W \cup \{v\}}$ are linear. Fix one such $h_i$.

If $O_1$ and $O_2$ coincide along $h_i$, then $d_{P,W}$ and $d_{P,W \cup \{v\}}$ have the same slope on it. Therefore, their difference is a constant function on $h_i$, and the slope of $f$ at $p$ along $h_i$ is 0.

If $O_1$ and $O_2$ do not coincide along $h_i$, then $h_i$ is not contained in $P$, the slopes of $d_{P,W}$ and $d_{P,W \cup \{v\}}$ have absolute value 1 and opposite signs. Therefore their sum is even and $f$ has slope $\pm 1$ along $h_i$. So $f$ is a rational function. Suppose that $p$ is the target of $h_i$ according to $O_1$, and the source according to $O_2$. Then the contribution to $\deg_{O_1}(p)$ along $h_i$ is 1, and the contribution to $\deg_{O_2}(p)$ is zero, so their difference is 1. Moreover, the slope of $d_{P,W}$ on $h_i$ is equal to $-1$ whereas the slope of $d_{P,W \cup \{v\}}$ is $+1$, hence the contribution to $\text{div}(f)(p)$ along $h_i$ is $(1 - (-1))/2 = 1$.

Repeating this for all segments $h_i$ we get $\text{div}(f)(p) = \deg_{O_1}(p) - \deg_{O_2}(p)$, hence by (4) we have $\text{div}(f)(p) = D_{P,W}(p) - D_{P,W \cup \{v\}}(p)$, as wanted.

Therefore $D_{P,W} \sim D_{P,W \cup \{v\}}$, hence $D_{P,W} \sim D_{P,W'}$ for any $W, W'$.

In particular $D_{P,W} \sim D_{P,V}$, hence, as $D_{P,V} = T_V$, by Proposition 3.3 $D_{P,W}$ is a theta-characteristic. Part (2) is proved.

It remains to prove that $D_{P,W}$ and $D_{P',W'}$ are not equivalent if $P \neq P'$. Now it is enough to show that $D_{P,V}$ and $D_{P',V}$ are not equivalent, i.e. that $T_P$ and $T_{P'}$ are not equivalent, which follows again by Proposition 3.3.

We write $V_+(\Gamma) = \{v \in V: w(v) > 0\}$, sometime just $V_+ = V_+(\Gamma)$. We say that $\Gamma$ is a pure tropical curve if $V_+(\Gamma) = \emptyset$, equivalently, if $w = 0$.

We now prove that theta-characteristics are always effective with the only exception of the theta-characteristic $[T_0]$ on a pure tropical curve.

Theorem 3.7. Let $\Gamma = (G, w, \ell)$ be a tropical curve and $P \in C_G$. Then $r_T(T_P) = -1$ if and only if $P = 0$ and $\Gamma$ is pure.
The case of a pure tropical curve is known; see \[\text{Zha10}\]. We include the full proof for completeness and better clarity. First we note the following simple fact.

**Lemma 3.8.** Let \(P \in C_G\) and \(W \subset V\) with \(P \cup W\) non empty. Then

1. \(D_{P,W} \geq 0\) if and only if \(w(v) > 0\) for every \(v \in W \setminus V(P)\).
2. \(D_{0,W} \geq 0\) if and only if \(W \subset V_+\).
3. \(D_{P,\emptyset} \geq 0\).

**Proof.** Notice that (2) and (3) follow trivially from (1). Let us prove (1); we have

\[
\deg_{O_{P,W}}(v) \begin{cases} 0 & \text{if } v \in W \setminus V(P) \\ \geq 1 & \text{otherwise.} \end{cases}
\]

Therefore, writing \(D_{P,W} = D_{P,W}^+ - D_{P,W}^-\) as the difference of two effective divisors, we have

\[
D_{P,W}^- = \sum_{v \in W \setminus V(P)} v \quad \text{where } w(v) = 0
\]

which is zero if and only if \(w(v) > 0\) for every \(v \in W \setminus V(P)\).

Now we prove Theorem 3.7

**Proof.** By Lemma 3.8, if \(P \neq 0\) then \(D_{P,\emptyset} \geq 0\). By Proposition 3.6 we have \(D_{P,\emptyset} \sim D_{P,V}\) and since \(D_{P,V} = T_P\) we conclude that \([T_P]\) is effective.

We are left with the case \(P = 0\).

If \(\Gamma\) is not pure then \(V_+ \neq \emptyset\), hence we can consider \(D_{0,V_+}\), which is a representative for \([T_0]\). By Lemma 3.8 we have \(D_{0,V_+} \geq 0\), hence \([T_0]\) is effective. This concludes the proof in case \(\Gamma\) is not pure.

Now suppose \(\Gamma\) is pure. To finish the proof we need to show

\[
r_{T}(T_0) = -1
\]

It suffices to prove that \(r_{T}(D_{0,v}) = -1\) for any vertex \(v\) of \(\Gamma\). By definition, \(D_{0,v}\) is associated to the sub-orientation \(O_{0,v}\), which is clearly acyclic (having \(v\) as a source). Therefore, \(D_{0,v}\) is a reduced divisor with respect to \(v\) in the sense of [BN07], so the fact that \(D_{0,v}(v) < 0\) implies that it has rank equal \(-1\) (see e.g., Lemma 3.8(a) in [CLM15]). In fact, \(D_{0,v}\) is a moderator in the sense of [MZ08, Def. 7.8], so the fact that it has rank \(-1\) follows also from [MZ08, Lm. 7.10].

**Remark 3.9.** If \(\Gamma\) is a pure tropical curve, then the theta-characteristics described by Zharkov in \[\text{Zha10}\] are the classes of the divisors \(D_{P,W}\) where \(W = \emptyset\) for \(P \neq 0\), and \(|W| = 1\) for \(P = 0\).
4. THE MODULI SPACE OF TROPICAL THETA-CHARACTERISTICS

The moduli space, $M_g^{\text{trop}}$, of tropical curves of genus $g$, first constructed in [BMV11], has the structure of a generalized cone complex (see [ACP15]). The points in $M_g^{\text{trop}}$ are in bijective correspondence with (equivalence classes of) tropical curves of genus $g$.

Similarly, the moduli space $S_g^{\text{trop}}$, of spin tropical curves of genus $g$, constructed in [CMP20, sec. 2.5] is a generalized cone complex, and there is a natural morphism of generalized cone complexes

$$\pi_{\text{trop}} : S_g^{\text{trop}} \longrightarrow M_g^{\text{trop}}.$$

This morphism has an explicit connection with the analogous moduli spaces of Deligne-Mumford stable algebraic curves, $\overline{M}_g$, and stable spin curves, $\overline{S}_g$, for which we refer to loc. cit. Theorem C.

We shall now construct the moduli space of tropical theta-characteristics on tropical curves of genus $g$ as a generalized cone complex, and relate it to $M_g^{\text{trop}}$ and $S_g^{\text{trop}}$. Since the procedure is very similar to the one used in [ACP15] and [CMP20] we will skip many details.

By Proposition 3.3, for any tropical curve $\Gamma = (G, w, \ell)$ of combinatorial type $(G, w)$, we have the isomorphism $C_G \cong T_{\Gamma}^{\text{trop}}$ associating to each cyclic subgraph $P$ of $G$ the theta-characteristic $[T_P]$. Therefore, for each tropical curve of combinatorial type $(G, w)$, there is exactly one theta-characteristic associated to the choice of each cyclic subgraph $P$ of $G$.

Notice that a non-trivial automorphism of $\Gamma$ may fix a cyclic subgraph of $G$, that is, $\text{Aut}(\Gamma)$ may act non-trivially on $T_{\Gamma}^{\text{trop}}$.

For any $P \in C_G$ we denote by $T_{(G, P)}^{\text{trop}}$ the set of isomorphism classes of all theta-characteristics of type $[T_P]$ on all tropical curves $\Gamma = (G, w, \ell)$.

We consider the poset of cyclic graphs of genus $g$, i.e.:

$$\mathcal{C}_g := \bigsqcup_{G \in \mathcal{G}_g} C_G$$

where $\mathcal{G}_g$ is the poset of stable graphs of genus $g$, partially ordered by edge contraction. Elements in $\mathcal{C}_g$ are written as pairs $(G, P)$ with $G$ a stable graph and $P \in C_G$. The poset structure on $\mathcal{C}_g$ is given by edge contraction, as follows. For $(G, P)$ and $(G', P')$ in $\mathcal{C}_g$ we say that $(G, P) \geq (G', P')$ if there exists a contraction $\gamma : G \to G'$ such that $\gamma_*P = P'$ (in this case we have $G \geq G'$ by definition). By [CMP20, Prop. 2.3.1] the poset $\mathcal{C}_g$ is connected and the forgetful map $\mathcal{C}_g \to \mathcal{G}_g$ is a quotient of posets. We also consider the poset $[\mathcal{C}_g] := \sqcup_{G \in \mathcal{G}_g} C_G / \text{Aut}(G)$.

We introduce the category, $\text{cyc}_g$, whose objects are isomorphism classes of pairs $(G, P)$ with $G \in \mathcal{G}_g$ and $P \in C_G$, and whose arrows are generated by contractions and automorphisms of pairs (that is, the elements of the subgroup $\text{Aut}(G, P)$ of $\text{Aut}(G)$ fixing $P$).

We now define a contravariant functor from the category $\text{cyc}_g$ to the category of rational polyhedral cones. To the isomorphism class of $(G, P)$
we associate the cone
\[ \sigma_{(G, P)} = \mathbb{R}^{E(G)}_{\geq 0}, \]
with the integral structure determined by the sub-lattice parametrizing tropical curves having integral edge-lengths. As usual, we write \( \sigma^o_{(G, P)} = \mathbb{R}^{E(G)}_{> 0} \).

To a contraction \( \gamma: (G, P) \to (G', P') \) we associate the injection of cones \( \iota_\gamma: \sigma_{(G', P')} \hookrightarrow \sigma_{(G, P)} \) whose image is the face of \( \sigma_{(G, P)} \) where the coordinates corresponding to \( E(G) \setminus E(G') \) vanish. If \( \gamma \in \text{Aut}(G, P) \) then \( \iota_\gamma \) is the corresponding automorphism of \( \mathbb{R}^{E(G)}_{\geq 0} \). By the results in [CMP20, sect. 2], this is indeed a contravariant functor. We can therefore consider the colimit of the diagram of cones \( \sigma_{(G, P)} \) using the inclusions \( \iota_\gamma \), for all arrows, \( \gamma \), in \( \text{cyc}_{C_g} \), which is the following generalized cone complex
\[ T_{\text{trop}} g := \lim_{\longrightarrow} \left( \sigma_{(G, P)}, \iota_\gamma \right). \]

**Theorem 4.1.** The following properties hold.

1. The space \( T_{g, \text{trop}} \) is the moduli space of tropical theta-characteristics and we have a stratification
   \[ T_{g, \text{trop}} = \bigsqcup_{[G, P] \in C_g} T_{(G, P)} \]

2. We have \( T_{(G, P)} \cong \sigma^o_{(G, P)}/\text{Aut}(G, P) \).

3. \( T_{g, \text{trop}} \) is connected and has pure dimension \( 3g - 3 \).

4. \( T_{(G', P')} \subset T_{(G, P)} \) if and only if \( (G, P) \geq (G', P') \).

**Proof.** A point \( l = (l_e, e \in E) \in \sigma_{(G, P)} \) corresponds to a pair \((\Gamma_l, P)\) where \( \Gamma_l = (G, w, \ell) \) is the tropical curve whose length function is \( \ell(e) = l_e \) for all \( e \in E \). Hence \( l \) corresponds to the theta-characteristic \( T_P \in T_{\Gamma_l} \). By extending this reasoning to all tropical curves of genus \( g \) we have a surjection
\[ \bigcup_{\Gamma \in M_{g, \text{trop}}} T_{\Gamma} \longrightarrow T_{g, \text{trop}} \]
which identifies \((\Gamma, T_P)\) with \((\Gamma', T_P')\) if and only if \( \Gamma = \Gamma' \) and there is an automorphism of \( \Gamma \) mapping \( T_P \) to \( T_{P'} \). From this \( \blacksquare \) follows.

The rest is similar to the proofs of [CMP20] Props 2.5.1 and 2.5.2. \( \blacklozenge \)

We have a canonical morphism of generalized cone complexes
\[ \psi_{\text{trop}}: T_{g, \text{trop}} \longrightarrow M_{g, \text{trop}} \]
sending a tropical theta-characteristic \([T_P]\) in \( \Gamma = (G, w, \ell) \) (associated to the cyclic subgraph \( P \) of \( G \)) to the tropical curve \((G, w, \psi_{\text{trop}}(\ell))\), where \( \psi_{\text{trop}}(\ell)(e) = 2\ell(e) \) if \( e \in E \setminus P \), and \( \psi_{\text{trop}}(\ell)(e) = \ell(e) \) if \( e \in P \).

By the same argument as in [CMP20] Prop. 2.5.3] we have, for every tropical curve \([\Gamma] \in M_{g, \text{trop}} \),
\[ (\psi_{\text{trop}})^{-1}([\Gamma]) \cong T_{\Gamma, \text{trop}} / \text{Aut}(\Gamma). \]
Remark 4.2. The canonical map $\pi^{\text{trop}} : S^{\text{trop}}_g \to M'^{\text{trop}}_g$ factors through $\psi^{\text{trop}}$.

5. Lifting theta-characteristics on tropical curves

From now on, we fix an algebraically closed field $k$ of characteristic different from 2.

Let $X$ be a stable curve over $k$. We let $(G,w)$ be the (stable) dual graph of $X$, with $G = (V,E)$. Recall that $V$ corresponds to the set of irreducible components of $X$ and $E$ to the set of its nodes. We often identify edges with nodes. The weight of a vertex is the genus of the desingularization of the corresponding component, therefore $(G,w)$ is a stable graph and has the same genus as $X$.

Recall that a spin curve over $X$ is a pair $(\hat{X},\hat{L})$, where $\hat{X} = X^\nu_R \cup Z$ is a quasistable curve, with $X^\nu_R$ the normalization of $X$ at the subset of nodes corresponding to some $R \subset E$ and $Z$ the disjoint union of smooth connected rational components meeting $X^\nu_R$ at 2 points (called exceptional components), and $\hat{L}$ is a line bundle on $\hat{X}$ such that

(1) the restriction, $L^R$, of $\hat{L}$ to $X^\nu_R$ satisfies $L^2 \cong \omega_{X^\nu_R}$;

(2) the restriction of $\hat{L}$ to each exceptional component $E$ has degree 1.

We denote by $S_X$ the scheme of spin curves over $X$. We will use the terminology theta-characteristic for spin curves such that $\hat{X} = X$ (in this case, $\hat{L}^2 \cong \omega_X$). The parity of a spin curve refers to the parity of $h^0(\hat{X}, \hat{L})$.

The dual spin graph of a spin curve $(\hat{X}, \hat{L})$ on $X$ is the tern $(G,P,s)$, where:

1. $G$ is the dual graph of $X$;
2. $P = E \setminus R$ (it is a cyclic subgraph of $G$);
3. $s : V(G/P) \to \mathbb{Z}/2\mathbb{Z}$ is a function taking $v \in V(G/P)$ to $s(v)$, the parity of $h^0(Z_v, \hat{L}|_{Y_v})$, where $Y_v$ is the connected component of $X^\nu_R$ corresponding to $v$ (the set of connected components of $X^\nu_R$ is in bijection with the set of vertices $V(G/P)$).

Recall that $(G,P,s)$ is a spin graph in the sense of [CMP20, Def. 2.1.1] and its parity is the parity of $\sum_{v \in V(G/P)} s(v)$.

In what follows, we fix an algebraically closed non-Archimedean field $K$ whose valuation ring is $R$ and residue field is $k$.

We let $X_K$ be a genus-$g$ smooth curve over $K$, we assume that $X_K$ extends to a stable curve $\mathcal{X}$ over $\text{Spec} R$, whose special fiber we denote by $X$.

We let $\Gamma = (G,w,\ell)$ be the tropical curve given as the skeleton of the Berkovich analitification $X^\text{an}_K$. Recall that the length $\ell$ is defined by setting $\ell(e) = \text{val}_K(f_e)$ where $xy = f_e$ is an étale local equation of $\mathcal{X}$ at the node corresponding to the edge $e$. 
The natural retraction map \( \tau : X^\an_K \to \Gamma \) induces by linearity a specialization homomorphism:

\[
\tau_{\Div} : \Div(X_K) \longrightarrow \Div(\Gamma)
\]

where \( \Div(X_K) \) is the group of divisors on \( X_K \).

Let us describe \( \tau_{\Div} \) explicitly. Let \( D_K \) be a divisor on \( X_K \). Then there is a semistable curve \( \hat{X} \) over \( \Spec R \) with \( \mathcal{X} \) as stable model, such that \( D_K \) extends to a Cartier divisor \( D \) over \( \hat{X} \); we let \( \hat{X} \) be the special fiber of \( \hat{X} \). The dual graph \( \hat{G} \) of \( \hat{X} \) is obtained by inserting \( n_e \) vertices of weight zero in the interior of any edge \( e \) of \( G \), where \( n_e \) is the number of rational components of \( \hat{X} \) lying over the node of \( X \) corresponding to \( e \). Hence we can view \( V(\hat{G}) \) as a subset of \( \Gamma \) (see also Remark 5.1). The multidegree, \( \deg|_{\hat{X}} \), of the restriction of \( D \) to \( \hat{X} \) is thus a divisor of \( \Gamma \) supported at \( V(\hat{G}) \), and we have

\[
(9) \quad \tau_{\Div}(D_K) = \deg|_{\hat{X}}.
\]

**Remark 5.1.** In passing from \( \mathcal{X} \) to \( \hat{X} \), the tropical equivalence class of the skeleton \( \Gamma \) does not change. Indeed, this amounts to inserting weight zero vertices in the interior of the edges of \( \Gamma \), getting a tropical curve, \( \hat{\Gamma} = (\hat{G}, \hat{w}, \hat{\ell}) \), having the same metric structure, and hence equivalence class, of \( \Gamma \). More precisely (following [ACP15] Section 8) let \( v \) be the vertex of \( \hat{G} \) corresponding to the exceptional component, \( C_v \), of \( \hat{X} \). Let \( e_0 \) and \( e_1 \) be the edges of \( \hat{\Gamma} \) corresponding to the two nodes in \( C_v \cap \hat{X} \setminus C_v \), and, for \( i = 0, 1 \), let \( xy = f_{e_i} \) be an étale local equation of \( \hat{X} \) at \( e_i \). Then an étale local equation of \( \mathcal{X} \) at the node \( e \) to which \( C_v \) gets contracted is \( xy = f_{e_0}f_{e_1} \), therefore

\[
\ell(e) = \val_K(f_{e_0}f_{e_1}) = \val_K(f_{e_0}) + \val_K(f_{e_1}) = \hat{\ell}(e_0) + \hat{\ell}(e_1).
\]

The map \( \tau_{\Div} \) takes principal divisors on \( X_K \) to principal divisors on \( \Gamma \) (see [BR15]), hence we get a homomorphism

\[
\tau_{\Pic} : \Pic(X_K) \longrightarrow \Pic(\Gamma).
\]

Let \( S_{X_K} \subset \Pic(X_K) \) and \( T^\trop_\Gamma \subset \Pic(\Gamma) \) be, respectively, the sets of theta-characteristics of \( X_K \) and of \( \Gamma \). Since \( \tau_{\Pic} \) is a homomorphism taking class of the canonical divisor of \( X_K \) to the class of the canonical divisor of \( \Gamma \), it restricts to a map:

\[
\tau_S : S_{X_K} \longrightarrow T^\trop_\Gamma.
\]

In Proposition 3.3 we introduced the bijection \( \beta : C_G \longrightarrow T^\trop_\Gamma \) mapping \( P \) to \( [T_P] \). We now use it to construct a useful factorization of the map \( \tau_S \).

First of all, we define a map

\[
\alpha : S_{X_K} \longrightarrow C_G
\]

as follows. Given a theta-characteristic \( L_K \) on \( X_K \), there is a unique pair \( (\hat{X}, \hat{\mathcal{L}}) \) with the following properties. \( \hat{X} \) is a semistable curve over \( \Spec R \) having \( \mathcal{X} \) as stable model, \( \hat{\mathcal{L}} \) is a line bundle on \( \hat{X} \) extending \( L_K \); we let \( \hat{\mathcal{X}} \)
be the special fiber of \( \hat{X} \) and \( \hat{L} = \hat{L}|_{\hat{X}} \). Finally (which ensures uniqueness) \((\hat{X}, \hat{L})\) is a spin curve on \( X \). We denote by \((G, P, s)\) the dual spin graph of \((\hat{X}, \hat{L})\). By definition, \( P \in C_G \), and we define

\[
\alpha(L_K) = P.
\]

**Lemma 5.2.** With the above notation, we have the following factorization

\[
\tau_S : S_{X_K} \xrightarrow{\alpha} C_G \xrightarrow{\beta} T^\text{trop}_{\Gamma}.
\]

**Proof.** We continue to use the notation before the statement. We need to show that \( \tau_S(L_K) \) is equal to \([TP]\). By \([1]\)

\[
\tau_S(L_K) = \deg \hat{L}|_{\hat{X}} = \deg \hat{L}.
\]

To compute the multidegree of \( \hat{L} \), observe that the dual graph, \( \hat{G} \), of \( \hat{X} \) is the \( R \)-subdivision of \( G \), where \( R = E \setminus P \). For a vertex \( v \in V(\hat{G}) \), we let \( \hat{C}_v \) be the irreducible component of \( \hat{X} \) corresponding to \( v \). Recall that \( \hat{L} \) restricts to a theta-characteristic on the complement of the exceptional components of \( \hat{X} \), i.e. on the subcurve of \( \hat{X} \) whose dual graph is \((V(G), P)\).

Therefore \( \forall v \in V(G) \subset V(\hat{G}) \) we have

\[
\deg_{\hat{C}_v} \hat{L} = w(v) - 1 + \frac{\deg_P(v)}{2}.
\]

Comparing this with \([4]\) we conclude that the coefficient of \( v \) in \( \deg \hat{L} \) and \( T_P \) coincide, as wanted.

Consider now \( v \in V(\hat{G}) \setminus V(G) \), so that \( \hat{C}_v \) is an exceptional component lying over a node of \( X \) corresponding to an edge \( e \in E \setminus P \). We know that \( \deg_{\hat{C}_v} \hat{L} = 1 \). To conclude the proof, in view of \([4]\), we need to show that in the tropical curve \( \Gamma \) the vertex \( v \) coincides with \( p_e \), the mid-point of \( e \).

We use Remark \([5]\). Let \( e_0 \) and \( e_1 \) be the nodes of \( \hat{X} \) lying on \( \hat{C}_v \). Recall that there are étale neighborhoods of \( \hat{X} \) around \( e_0 \) and \( e_1 \) in which the local equations of \( \hat{X} \) are, respectively, \( x_0y_0 = h_e \) and \( x_1y_1 = h_e \), for the same \( h_e \in R \) (see \([CCC07, Eq. (4), Section 3.2])\). This implies that the lengths of \( e_0 \) and \( e_1 \) are equal, and therefore \( v \) is the mid point of \( e \). \( \blacksquare \)

We let \( S^+_X \) and \( S^-_X \), respectively, be the loci in \( S_{X_K} \) corresponding to even and odd theta-characteristics of \( X_K \). Thus \( \tau_S \) restricts to maps

\[
\tau^+_S : S^+_X \rightarrow T^\text{trop}_{\Gamma} \quad \text{and} \quad \tau^-_S : S^-_X \rightarrow T^\text{trop}_{\Gamma}.
\]

Similarly, we restrict \( \alpha \)

\[
\alpha^+ : S^+_X \rightarrow C_G \quad \text{and} \quad \alpha^- : S^-_X \rightarrow C_G.
\]

Next, let \( S_X \) be the scheme of spin curves over \( X \). For a spin structure \((P, s)\) on \( G \) we define

\[
S_{(X, P, s)} = \{(\hat{X}, \hat{L}) \in S_X : \text{the dual spin graph of } (\hat{X}, \hat{L}) \text{ is } (G, P, s)\}.
\]
Denote by $\mathcal{S}_X^+$ and $\mathcal{S}_X^-$, respectively, the subschemes of $\mathcal{S}_X$ corresponding to even and odd spin curves.

Recall that $SP_G^+$ and $SP_G^-$ denote, respectively, the set of pairs $(P, s)$ such that $(G, P, s)$ is an even and odd spin graph. For every $P \in C_G$, we define
$$S_{(X, P)}^+ = \bigcup_{(P, s) \in SP_G^+} S_{(X, P), s}^+ \quad \text{and} \quad S_{(X, P)}^- = \bigcup_{(P, s) \in SP_G^-} S_{(X, P), s}^-.$$ 

**Lemma 5.3.** With the above notation, for every $P \in C_G$ we have
$$|(\alpha^+)^{-1}(P)| = 2^{b_1(G) - b_1(P)} |S_{(X, P)}^+|.$$ 

**Proof.** As we saw above, for every theta-characteristic $L_K$ on $X_K$ there is a unique pair $(\hat{X}, \hat{L})$ over Spec $R$ extending $(X_K, L_K)$. By construction, the special fiber of $(\hat{X}, \hat{L})$ is contained in $\mathcal{S}_X$. For every $(\hat{X}, \hat{L}) \in \mathcal{S}_X$, we let $N(\hat{X}, \hat{L})$ be the number of theta-characteristics $L_K$ on $X_K$ whose extension $(\hat{X}, \hat{L})$ has $(\hat{X}, \hat{L})$ as a special fiber. Clearly, we have
$$|(\alpha^+)^{-1}(P)| = \sum_{(\hat{X}, \hat{L}) \in S_{(X, P)}^+} N(\hat{X}, \hat{L}).$$

Let $\pi: X_K \to \text{Spec } R$ be the moduli space of spin curves of the family $X \to \text{Spec } R$. The map $\pi$ is finite and flat and the multiplicity of the special fiber of $\pi$ at a point $(\hat{X}, \hat{L}) \in S_{(X, P)}^+$ is $2^{b_1(G) - b_1(P)}$ (see \[C89\] and \[CC03\]). Thus $\hat{L}$ lifts to $\hat{L}$ theta-characteristics on $X_K$, that is, $N(\hat{X}, \hat{L}) = 2^{b_1(G) - b_1(P)}$, and we are done. $\clubsuit$

Given an integer $m$, we set
$$N_m^+ := 2^{m-1}(2^m + 1) \quad \text{and} \quad N_m^- := 2^{m-1}(2^m - 1).$$

**Theorem 5.4.** For every $P \in C_G$, the following properties hold.

(1) If $P \neq 0$, then
$$|(\tau^-)^{-1}(T_P)| = |(\tau^+)^{-1}(T_P)| = 2^{2g - b_1(G) - 1}.$$ 

(2) If $P = 0$, then
$$|(\tau^-)^{-1}(T_0)| = 2^{b_1(G)} \sum_{U \subset V, |U| \equiv 0 \mod (2)} \left( \prod_{v \in U} N_{w(v)}^- \prod_{v \in V \setminus U} N_{w(v)}^+ \right),$$
and
$$|(\tau^+)^{-1}(T_0)| = 2^{b_1(G)} \sum_{U \subset V, |U| \equiv 1 \mod (2)} \left( \prod_{v \in U} N_{w(v)}^- \prod_{v \in V \setminus U} N_{w(v)}^+ \right).$$

**Proof.** Consider $(\hat{X}, \hat{L}) \in S_{(X, P)}$. The dual graph of $\hat{X}$ is the $R$-subdivision of $G$, where $R = E \setminus P$. We let $Z \subset \hat{X}$ be the union of the exceptional components, and $X_1, \ldots, X_n$ the connected components of $\hat{X} \setminus Z$. It is well
known (as the restriction of $\widehat{L}$ to every exceptional component is $O(1)$) that we have the following identity

$$h^0(\widehat{X}, \widehat{L}) = \sum_{1 \leq i \leq n} h^0(X_i, \widehat{L}|_{X_i}). \tag{10}$$

Assume $P \neq 0$. We can assume that the dual graph $P_1$ of $X_1$ has $b_1(P_1) \neq 0$. By [Har82 Cor. 2.13], we have the same number, $M$, of odd and even theta-characteristics on $X_1$. We argue that $|S^+_{(X,P)}| = |S^-_{(X,P)}|$. By (10), this is clear if $n = 1$. Otherwise, let $Y$ be the disjoint union of $X_2, \ldots, X_n$, and denote by $A$ and $B$ the number of even and odd theta-characteristics on $Y$, respectively. We have $h^0(\widehat{X}, \widehat{L}) = h^0(X_1, \widehat{L}|_{X_1}) + h^0(Y, \widehat{L}|_Y)$, therefore

$$|S^+_{(X,P)}| = |S^-_{(X,P)}| = AM + BM.$$

Therefore, by [CC03 Sect. 1.3],

$$|S^+_{(X,P)}| = |S^-_{(X,P)}| = 2^2 \sum_{v \in V} w(v) 2^{h_1(P) - 1} = 2^{2g - 2b_1(G)} 2^{h_1(P) - 1}.$$

Then by Lemmas 5.2 and 5.3 we have

$$|(\tau^+_S)^{-1}(T_P)| = 2^{b_1(G) - b_1(P)} |S^+_{(X,P)}| = 2^{2g - b_1(G) - 1}.$$

This concludes the proof of (11).

Assume now $P = 0$. A spin curve in $S_{(X,0)}$ is given by the datum of a theta-characteristic on each component of the normalization of $X$. This spin curve is even if and only if we have an even number of components for which the theta-characteristic is odd. Hence

$$|S^+_{(X,0)}| = \sum_{|U| \equiv 0 \mod(2)} \left( \prod_{v \in U} N^-_{w(v)} \prod_{v \in V \setminus U} N^+_{w(v)} \right),$$

which, by Lemmas 5.2 and 5.3, gives the stated number for $|(\tau^+_S)^{-1}(T_0)|$. A similar reasoning gives the stated number for $|(\tau^-_S)^{-1}(T_0)|$.

**Remark 5.5.** Notice that Theorem 5.4 recovers [JL18 Theorem 1.1, (2) and (3)]. Indeed, assume that $\Gamma$ is pure, so that $b_1(\Gamma) = g$. For every $P \neq 0$, by Theorem 5.4 we get

$$|(\tau^+_S)^{-1}(T_P)| = |(\tau^-_S)^{-1}(T_P)| = 2^{g - 1}.$$  

Moreover, for $\epsilon \in \{0, 1\}$, using that $N^+_0 = 1$ and $N^-_0 = 0$, we have

$$\sum_{|U| \equiv \epsilon \mod(2)} \left( \prod_{v \in U} N^-_0 \prod_{v \in V \setminus U} N^+_0 \right) = \begin{cases} 1, & \text{if } \epsilon = 0, \\ 0, & \text{if } \epsilon = 1, \end{cases}$$

(the unique non-zero summand in the left hand side is the one corresponding to $\epsilon = 0$ and $U = \emptyset$). By Theorem 5.4

$$|(\tau^+_S)^{-1}(T_0)| = 2^g$$  and  $$|(\tau^-_S)^{-1}(T_0)| = 0.$$
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