Computational Complexity of Spatial Reasoning with Directional Relationship

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1 Introduction

The directional information is a kind of important spatial information in GIS, and plays important roles in the problems dealing with visualization of spatial information, pattern recognition and spatial information inquiring etc. There are successful researches on the computational complexity about temporal reasoning and topologic reasoning, except for the computational complexity about directional reasoning. As one of important aspects of spatial qualitative reasoning, computational complexity of directional reasoning should be studied, and it will be meaningful for the development of spatial representation and reasoning systems in GIS and other systems.

In this paper we focus our researches on the NP-completeness (if a problem can be determinate in polynomial time, then the problem is an NP problem) property of directional reasoning problem with four directional relationship in two-dimensional space, and the computation model for directional relationship between spatial objects is a cone-shaped directional model.

2 Directional computation of spatial objects in two dimensional space with the cone-shaped model

Directional relationship is a binary relationship, and the computation of directional relationship between two spatial objects should be based on reference frames, and different reference frames have different principles for the determination of spatial objects’ directional relationship.

The cone-shaped directional model[1] is one of the famous models used for the directional computation between objects. In
this model the plane is divided into several half-planes according to the number of directions, and object’s directional relation to the reference object can be got by determining which half-plane the object is located in, and directional relationship is reflexive. For instance, if a set of directional relations between spatial objects in two dimensional space is declared, which include four directional relation \{East (E), North (N), West (W), South (S)\}, then the cone-shaped directional model for the four directional relationships can be got, which is shown in Fig. 1 (the intersection point represents the reference object).

![Fig. 1 The cone-shaped directional model](image)

On the basis of the directional model above, a composition table for the four basic directional relations can be obtained, which is shown in Table 1. From this composition table, it can be known that directional relations between objects are uncertain in the process of spatial reasoning, which is similar to the process of spatial reasoning with topologic relationship.

|       | E   | N   | W   | S   |
|-------|-----|-----|-----|-----|
| E     | E   | E,N | NoInfo | E,S |
| N     | N,E | N   | N   | W   | NoInfo |
| W     | NoInfo | W   | W   | S   |
| S     | E,S | NoInfo | S   | W   | S   |

Table 1 Composition table for the four basic directional relations

According to the property of uncertainty, the problem of topologic spatial reasoning contains another property of NP-completeness\(^2\) (NP-complete problems are difficult ones in NP problems). Because of the similarity of the two problems above, the problem of directional spatial reasoning can be considered as an NP-complete problem.

3 Directional constraint satisfactory problem

the directional reasoning problem can be translated into a constraint satisfactory problem, and with this translation, variables and domain of values in the constraint satisfactory problem can be transformed to spatial objects and constraints of the directional reasoning problems. On the basis of four-direction model shown in Fig. 1, the directional reasoning problem can be translated into the following directional constraint satisfactory problem.

- a finite set of spatial objects, \(O\),
- a finite set of directional relationship, \(D = \{E, N, W, S\}\),
- a finite set of constraints of directional relationship between objects, \(C\).

And whether the directional reasoning problem has a solution equals to whether the directional constraints are consistent.

4 Proof of NP-complete property of the directional constraint satisfactory problem

4.1 NOT-ALL-EQUAL-3SAT problem

NOT-ALL-EQUAL-3SAT (it is abbreviated as 3SAT in the following text) problem, which is a famous NP-complete problem, can be described as follows:

- Input: a set of clauses \((C) = c_1, c_2, \ldots, c_m\), each of which just includes three of the variables \((V) = v_1, v_2, \ldots, v_n\), \((m, n > 0)\).
- Output: if there is a truth assignment to each clause, for which at least one literal is assigned true and one literal assigned false, then the answer is "TRUE"; otherwise, the answer is "FALSE".

4.2 Transformation from 3SAT problem to directional constraint satisfactory problem

Every variable \(v\) of the 3SAT instances can be transformed into two directional constraints \(O_x|E, W| O_y\) and \(O_y|E, W| O_x\), where \(O_x, O_y, O_{x,y}\) and \(O_{y,x}\) are spatial objects introduced by \(v\), and \(v\) is assigned true if and only if the constraint \(O_x|E| O_y\) holds, at the same time, the constraint \(O_{y,x}|W| O_{x,y}\) holds; \(v\) is assigned false if and only if constraint \(O_y|E| O_x\) holds, at the same time constraint \(O_{x,y}|W| O_{y,x}\) holds. And there are two
important properties for this transformation.

Property 1: Let $O_x, O_y, O_z$ be spatial objects. If there is a path in the constraint graph starting from and ending at $O_x$, passing $O_y, O_z$, and only directional constraints $\{E\}$ or $\{W\}$ hold between all of them, then the directional constraints are inconsistent, which is shown in Fig. 2.

Property 2: Let $O_x, O_y, O_z$ be spatial objects, and $O_x|N, S|O_y, O_z|E, W|O_x, O_z|E, W|O_x$ hold, then directional constraints between objects are consistent if and only if directional constraints between $O_x, O_y$ and $O_y, O_z$ are the same, which is shown in Fig. 3.

Depending on the above two properties, the problem of 3SAT can be transformed to directional constraint satisfactory problem as follows:

1) For each variable $v_i \in V$, four spatial objects $O_x, O_y, O_z, O_0$ will be introduced by adding directional constraints $O_x|E, W|O_y$ and $O_z|E, W|O_0$ to directional constraint set $\varphi$. At the same time, the following directional constraints $O_x|N, S|O_0, O_x|E, W|O_0$ and $O_x|N, S|O_0, O_x|E, W|O_0$ are also added to $\varphi$ according to property 2, which is shown in Fig. 4:

2) According to directional constraints between objects introduced by variable of 3SAT, directional constraints introduced by literal of 3SAT can be described as follows:

For each literal $l_0$ in 3SAT, spatial objects $O_x$ and $O_y$ are introduced by adding directional constraint $O_x|E, W|O_y$ to $\varphi$. Whether different polarity constraints have to be added to $\varphi$ depends on whether the literal occurrence is positive or negative.

(a) If $l_0$ is a positive literal, the spatial variables constraint $O_x|E, W|O_y$ to $\varphi$, and according to property 2, other directional constraints should also be added to $\varphi$ (Fig. 5 (a)).

(b) If $l_0$ is a negative literal, the spatial variables constraint $O_x|E, W|O_y$ to $\varphi$, and according to property 2, other directional constraints should also be added to $\varphi$ (Fig. 5 (b)).
3) Depending on the transformation from variable and literal of 3SAT to directional constraints, the transformation of clauses should be as follows:

For each clause $c_i = \{ l_{i1}, l_{i2}, l_{i3} \}$, according to the properties of 3SAT problem, the following directional constraints $O_{a1} \{E, W\} O_{a2}$, $O_{a2} \{E, W\} O_{a3}$, $O_{a3} \{E, W\} O_{a1}$ are added to $\varphi$. Because the following constraints $O_{a1} \{E, W\} O_{a1}, O_{a2} \{E, W\} O_{a2}, O_{a3} \{E, W\} O_{a3}$ exist, according to property 2, these constraints, $O_{a1} \{N, S\}$, $O_{a2} \{N, S\}$, $O_{a3} \{N, S\}$, $O_{a4}$, should be added to $\varphi$. And according to the definition of 3SAT problem, at least one of these constraints $O_{a1} \{N, S\}$, $O_{a3}$, $O_{a2} \{N, S\}$, $O_{a1}$, $O_{a3} \{N, S\}$, $O_{a2}$ must be false, and one must be true, which is shown in Fig. 6 (at least one of the constraint graphs is true, and one is false). And similar constraints hold between spatial objects $O_{1, x1}$, $O_{2, x2}$, $O_{3, y3}$, $O_{1, y1}$, $O_{2, y2}$, $O_{3, x3}$.

![Diagram](image)

**Fig. 6 Directional constraint of clause**

It is obvious that all the transformation above can be transformed in polynomial time.

4.3 Proof of NP-complete property for the directional constraint satisfactory problem

1) The directional constraint satisfactory problem is an NP problem. To solve the directional constraint problem, we can suppose a kind of constraint assignments and verify the assignments’ consistence, which can be achieved in polynomial time obviously.

2) If the directional constraint is consistent, then the 3SAT problem is satisfied. Supposing that the set of directional constraint $\varphi$ obtained by transformation from a given instance of 3SAT is consistent, then an assignment that satisfies the instance can be obtained in the following way: for each variable $v_i \in V$, if $X_i \{E\} Y_i$ holds, then the assignment is true; otherwise, the assignment is false. According to this assignment, the 3SAT problem will be satisfied if and only if the directional constraint is consistent.

3) If the 3SAT problem satisfied, then the directional constraint is consistent. This can be proved by constructing a spatial configuration that satisfies all directional constraints in $\varphi$.

Considering the rules for directional constraints in the literal: if the literal is a negative variable, then $O_x, O_y$ can be replaced with $O_{-x}, O_{-y}$; if the literal is a positive variable, then $O_x, O_y$ can be replaced with $O_x, O_y$. Therefore, directional constraints shown in Fig. 5 are the only possible constraints between the four spatial objects introduced by the literal.

According to the directional constraints introduced by clauses, there are at most 12 spatial objects corresponding to one clause. The directional constraints between the spatial objects introduced by the literal are shown in Fig. 6. For example, in a clause $c_1 = v_1 \lor v_2 \lor \neg v_3$, $v_1$ introduces spatial objects $x_1, y_1, x_1, y_1$; $v_2$ introduces spatial objects $x_2, y_2, x_2, y_2$; $v_3$ introduces spatial objects $x_3, y_3, x_3, y_3$. Then according to the constraint rules for clauses, relations $x_1 \{E, W\} x_2$, $x_1 \{E, W\} y_1$, $x_2 \{E, W\} x_3$, $x_2 \{E, W\} y_2$, $x_3 \{E, W\} y_3$ will hold, and at least one of these relations $x_2 \{N, S\} y_1$, $x_3 \{N, S\} y_2$, $x_1 \{N, S\} y_3$ can hold, and one can not hold. At the same time, relations $x_1 \{E, W\} x_2$, $x_1 \{E, W\} y_1$, $x_2 \{E, W\} x_3$, $x_2 \{E, W\} y_2$, $x_3 \{E, W\} y_3$ will hold, and at least one of these relations $x_2 \{N, S\} y_1$, $x_3 \{N, S\} y_2$, $x_1 \{N, S\} y_3$ can hold, and one can not hold. According to these constraints, we can construct a constraint configuration diagram (Fig. 7), in which relations $x_1 \{N, S\} y_3$ and $x_1 \{N, S\} y_1$ are false.
and all others are true. If there are more spatial objects introduced by other clauses, then all of them can be inserted in the diagram according to the same rules. It is obvious that these constraints in the diagram are consistent.

Because the transformation is polynomial and directional constraints are consistent, the directional constraint satisfiable problem is NP-complete, and the directional reasoning problem is NP-complete as well.

5 Conclusion and Perspectives

The directional reasoning problem can be translated to a directional constraint satisfiable problem, and depending on the transformation from the 3SAT problem to the directional constraint satisfiable problem in polynomial time, an important property of NP-completeness of directional reasoning problem can be obtained. This conclusion is meaningful to the development of directional reasoning systems and other spatial and temporal reasoning systems in GIS. Jochen Renz has provided a maximal tractable fragment of RCC-8 used in topologic reasoning systems[2]. Whether there are tractable fragments in directional relation and whether the computational complexity of spatial and temporal reasoning problem can be simplified depend on an integrating method between directional, topological and other geometrical reasoning methods, which are the future objectives of research.

References

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