Ovoids of generalized quadrangles of order $(q, q^2 - q)$ and Delsarte cocliques in related strongly regular graphs

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**Abstract**

We investigate strongly regular graphs for which Hoffman's ratio bound and Cvetković's inertia bound are equal. This means that $ve^{-} = m^{-}(e^{-} - k)$, where $v$ is the number of vertices, $k$ is the regularity, $e^{-}$ is the smallest eigenvalue, and $m^{-}$ is the multiplicity of $e^{-}$. We show that Delsarte cocliques do not exist for all Taylor’s 2-graphs and for point graphs of generalized quadrangles of order $(q, q^2 - q)$ for infinitely many $q$. For cases where equality may hold, we show that for nearly all parameter sets, there are at most two Delsarte cocliques.

**KEYWORDS**  
Delsarte coclique, generalized quadrangle, ovoid, quasisymmetric design, strongly regular graph

**MSC CODES:**  
51E12, 05B05, 05C69, 05E30
1 | INTRODUCTION

Hoffman’s ratio bound and Cvetković’s inertia bound are two of the best known bounds for cocliques in regular graphs. We investigate the case where the unweighted versions of both bounds are equal for strongly regular graphs. We refer to [3,4] for a general discussion of strongly regular graphs. In this paper, a strongly regular graph $\Gamma$ has parameters $v, k, \lambda, \mu$, where $v$ denotes the number of vertices of $\Gamma$, $k$ denotes the degree of each vertex of $\Gamma$, $\lambda$ denotes the size of the common neighborhood of two adjacent vertices, and $\mu$ denotes the size of the common neighborhood of two nonadjacent vertices. For two vertices $u, v$ in a graph $\Gamma$, we use $u \sim v$ to denote that $u$ is adjacent to $v$. The adjacency matrix $A$ of a strongly regular graph has three eigenvalues, $k, e^+, e^-$, where $k \geq e^+ \geq e^-$. A strongly regular graph $\Gamma$ is primitive if both $\Gamma$ and $\overline{\Gamma}$ are connected; this implies that $e^+ > 0$. In this paper, we only consider primitive strongly regular graphs. We denote the eigenspaces that correspond to $k, e^+$, and $e^-$ by $\langle j \rangle$, $V^+$, and $V^-$, respectively (where $j$ denotes the all ones vector). We denote the multiplicity of $e^+$ by $m^+$, and the multiplicity of $e^-$ by $m^-$. Throughout this paper every graph that is called $\Gamma$ is strongly regular and its parameters are named as above.

Hoffman’s ratio bound and Cvetković’s inertia bound state (respectively) that for a primitive strongly regular graph $\Gamma$, a coclique $Y$ in $\Gamma$ satisfies

$$|Y| \leq \frac{ve^-}{e^- - k}, \quad |Y| \leq m^-.$$

A coclique of size $\frac{ve^-}{e^- - k}$ is called a Delsarte coclique. We refer to Section 2 for further definitions.

It has long been known that the case where both bounds are tight is special; for example Haemers investigated this in his Ph.D. thesis in 1979 [9, Th. 2.1.7]. Our work extends an investigation by Haemers and Higman [10] for strongly regular graphs in general, and results by Makhnev and Makhnev for generalized quadrangles of order $(q, q^2 - q)$ [13]. For our purposes, we phrase some of these results in a different way than the existing literature. For example, we provide a short translation of [4, Th. 9.4.1] as Theorem 2.11.

Our main results are based on designs derived from strongly regular graphs. A 2-$(\bar{v}, \bar{k}, \bar{\lambda})$ design with intersection numbers $s_1$ and $s_2$ is a set of $\bar{k}$-sets $B$ such that

(a) all $b \in B$ satisfy $b \subseteq \{1, \ldots, \bar{v}\}$,
(b) each pair $\{i, j\} \subseteq \{1, \ldots, \bar{v}\}$ lies in exactly $\bar{\lambda}$ elements of $B$, and
(c) $|b \cap b'| \in \{s_1, s_2, \bar{k}\}$ for all $b, b' \in B$.

If $s_1 \neq s_2$, then $B$ is called a quasisymmetric design. If $s_1 = s_2$, then $B$ is called a symmetric design. The replication number $\bar{r}$ denotes the number of blocks that contain one given element and satisfies

$$\bar{r}(\bar{k} - 1) = (\bar{v} - 1)\bar{\lambda}.$$

Notice that we have $\bar{r} = \bar{k}$ if and only if the design is symmetric.

Our next result describes how to construct designs from the strongly regular graphs that we consider in this paper. We will see that this result is a reformulation of Theorem 9.4.1 from [4].

Theorem 1.1. Let $\Gamma$ be a primitive strongly regular graph, $Y$ a Delsarte coclique of $\Gamma$ and $ve^- = m^-(e^- - k)$. Let $\overline{Z}$ denote the set of vertices of $\Gamma$ that are not in $Y$. For $z \in \overline{Z}$, let $b_z$ denote $\{y \in Y : y \sim z\}$. Then $\overline{B} := \{b_z : z \in \overline{Z}\}$ is a quasisymmetric 2-$(m^-, e^-, \mu)$ design with replication number $k$. Two adjacent vertices in $\overline{Z}$ correspond to two blocks with intersection size $-(e^+)^2 - e^+ - e^-$, while two nonadjacent vertices in $\overline{Z}$ correspond to two blocks with intersection size $-(e^+)^2 - e^-$. 
We combine this result with existence results for quasisymmetric designs due to Blokhuis and Calderbank [2] that rule out equality in the Hoffman bound for many feasible parameter sets for strongly regular graphs. We first consider strongly regular graphs constructed from generalized quadrangles.

A generalized quadrangle of order \((s, t)\) consists of a set \(P\) of points, a set \(L\) of lines, and an incidence structure \(I \subseteq P \times L\). We say that a line \(l \in L\) contains a point \(p\) if \((p, l) \in I\). The incidence structure must satisfy the following:

(a) for each \(p \in P\), there are exactly \(t + 1\) lines in \(L\) that contain \(p\),
(b) each \(l \in L\) contains exactly \(s + 1\) points in \(P\),
(c) if a point \(p\) is not on a line \(l\), then there is a unique point \(p'\) and line \(l'\) such that \(l'\) contains \(p\) and \(p'\), and \(l\) contains \(p'\).

A generalized quadrangle induces a strongly regular graph \(\Gamma\), called the point graph, by setting the vertices of \(\Gamma\) to be the points \(P\), and setting \(u \sim v\) if and only if there is a line containing both \(u\) and \(v\). A Delsarte coclique of the point graph of a generalized quadrangle is traditionally called an ovoid.

We refer to [14, Chapter 1] for details. Whether certain generalized quadrangles possess an ovoid is a long-standing open question that has attracted various researchers, see [14, Sections 1.8 and 3.4]. In particular, we show that for infinitely many choices of \(q\), generalized quadrangles of order \((q, q^2 - q)\) do not possess ovoids.

Besides the nonexistence proofs for some Delsarte cocliques, our main result is the following.

**Theorem 1.2.** Let \(\Gamma\) be a primitive strongly regular graph, let \(Y, Z\) be different Delsarte cocliques of \(\Gamma\) and \(v e^- = m^- (e^- - k)\). For a vertex \(z\) in \(Z \setminus Y\) let \(b_z\) denote \(\{y \in Y : y \sim z\}\). Then the following hold.

(a) The set \(B := \{b_z : z \in Z \setminus Y\}\) is a symmetric \(2-(\frac{(e^+)^2(e^-)}{(e^+)^2 + (e^-)}), -e^-, -(e^+)^2 - e^-\) design.
(b) \(|Y \cap Z| = m^- - \frac{(e^+)^2(e^-)}{(e^+)^2 + e^-} = \frac{(e^+ + 1)e^+}{(e^+)^2 + e^-} - e^-\).
(c) Every block of the quasisymmetric design of Theorem 1.1 corresponding to a vertex that is not in \(Z\) contains exactly \(e^+\) elements of \(Y \cap Z\).
(d) The graph \(\Gamma\) contains at most \(m^- + 1\) Delsarte cocliques. Equality in this bound implies that a symmetric \(2-(\frac{(e^+)^2+e^+e^-+e^-}{(e^+)^2+e^-}, \frac{(e^+)^2+e^+}{(e^+)^2+e^-}, \frac{(e^+)^2+e^+}{(e^+)^2+e^-})\) design exists and that every vertex of \(\Gamma\) lies in exactly \(1 + \frac{(e^+ + 1)e^+}{(e^+)^2 + e^-}\) Delsarte cocliques.
(e) The common intersection of three Delsarte cocliques in \(\Gamma\) has size \(-\frac{(e^+)^2 + e^+}{(e^+)^2 + e^-}\).
(f) If \(e^+\) and \(e^-\) are coprime and \(e^+ > 1\), then \(\Gamma\) contains at most two Delsarte cocliques.

The main motivation for Theorem 1.2 is to bound the number of Delsarte cocliques in a strongly regular graph. For most feasible parameter sets, the intersection numbers above are not integers. As an example, for a strongly regular graph with parameters \(v = 287, k = 126, \lambda = 45, \mu = 63\), the intersection of three cocliques given by Part (e) is not an integer, therefore such a graph can have at most two Delsarte cocliques. In cases like this where the number of Delsarte cocliques is bounded, if a graph exists and possesses a single Delsarte coclique then it must be very asymmetric.

Notice that a strongly regular graph that has exactly one or two Delsarte cocliques is not unusual. It can be easily verified that some strongly regular graphs with parameters \((45, 32, 22, 24)\) have this property.

* See http://www.maths.gla.ac.uk/~es/srgraphs.php for a list of these strongly regular graphs.
Theorem 1.2 also gives a straight-forward route of constructing such graphs by extending a symmetric design with parameters as in Theorem 1.2 to a quasisymmetric design as in Theorem 1.1.

This paper is organized as follows. After proving Theorem 1.2, we apply it with Theorem 1.1 to various infinite families of feasible parameter sets. We conclude by investigating all feasible parameter sets for graphs with up to 1300 vertices by going through Brouwer’s database of strongly regular graphs.*

2 | PRELIMINARIES

2.1 | Block designs

For the following results, let $B$ be a quasisymmetric $2-(\tilde{v}, \tilde{k}, \tilde{\lambda})$ design with intersection numbers $s_1$ and $s_2$.

**Theorem 2.1** (Calderbank [5, Theorem A]). If $s_1 \equiv s_2 \pmod{2}$ and $\tilde{r} \not\equiv \tilde{\lambda} \pmod{4}$, then $\tilde{v} \equiv \pm 1 \pmod{8}$.

**Theorem 2.2** (Blokhuis and Calderbank [2, Theorem 4.3]). If $s_1 \equiv s_2 \equiv s \pmod{p}$ for some odd prime number $p$ and $\tilde{r} \not\equiv \tilde{\lambda} \pmod{p^2}$, then one of the following occurs:

(a) $\tilde{v}$ is even, $\tilde{v} \equiv s \equiv 0 \pmod{p}$ and $(-1)^{\tilde{v}/2}$ is a square modulo $p$,

(b) $\tilde{v}$ is even, $\tilde{v} \not\equiv s \equiv 0 \pmod{p}$, $\tilde{\lambda} \equiv 0 \pmod{p}$ and $(-1)^{\frac{\tilde{v}+2}{2}} k(\tilde{v} - \tilde{k})$ is a square modulo $p$,

(c) $\tilde{v}$ is odd, $\tilde{v} \not\equiv s \equiv 0 \pmod{p}$, $\tilde{\lambda} \equiv 0 \pmod{p}$, and $-\tilde{v}(-1)^{\frac{\tilde{v}+1}{2}}$ is a square modulo $p$, or

(d) $\tilde{v}$ is odd, $\tilde{v} \equiv s \equiv 0 \pmod{p}$, and $-s(-1)^{\frac{\tilde{v}+1}{2}}$ is a square modulo $p$.

**Theorem 2.3** (Blokhuis and Calderbank [2, Theorem 5.1]). Let $p$ be an odd prime number and let $e$ be an odd positive integer. For an integer $z$ define $\psi(z) = \max\{\epsilon : p^\epsilon$ divides $z, \epsilon \leq e\}$. If $s_1 \equiv s_2 \equiv s \pmod{p^e}$, $\tilde{r} \not\equiv \tilde{\lambda} \pmod{p^{e+1}}$ and $\tilde{v}$ is odd, then one of the following occurs:

(a) $\psi(s)$ is odd and $(-1)^{\frac{\tilde{v}-1}{2}} \tau$ is a square modulo $p$, where $\tilde{v} - s = p^{\psi(v-s)} \tau$, or

(b) $\psi(s)$ is even and $(-1)^{\frac{\tilde{v}-1}{2}} \sigma$ is a square modulo $p$, where $s = p^{\psi(s)} \sigma$.

The following lemma is surely known, but we do not know a reference. We include a short proof for the sake of completeness.

**Lemma 2.4.** Let $B$ be a set of $\tilde{k}$-sets of $\{1, \ldots, \tilde{v}\}$ with $|B| = \tilde{v}$ and suppose that there is a constant $s$ such that $|b \cap b'| \in \{s, \tilde{k}\}$ for all $b, b' \in B$. Then $B$ is a symmetric $2-(\tilde{v}, \tilde{k}, s)$ design if and only if $\tilde{k}(\tilde{k} - 1) = s(\tilde{v} - 1)$.

**Proof.** We verify that (i) each element of $M := \{1, \ldots, \tilde{v}\}$ lies on $\tilde{k}$ elements of $B$ and that (ii) each pair of $M$ lies on $s$ elements of $B$.

For (i) let $\ell_i$ denote the number of elements in $M$ that lies in exactly $i$ elements of $B$. Standard counting arguments show that

$$\sum \ell_i = \tilde{v}, \quad \sum i \ell_i = \tilde{v} \tilde{k}, \quad \sum i(i-1) \ell_i = \tilde{v}(\tilde{v} - 1)s.$$  

* https://www.win.tue.nl/~aeb/graphs/srg/srgtab.html.
Thus,
\[
0 \leq \sum (\bar{k} - i)^2 c_i = \bar{v}((\bar{v} - 1)s + \bar{k}) - \bar{v}\bar{k}^2. \tag{1}
\]
Hence, each element of \( M \) lies in exactly \( \bar{k} \) elements of \( B \) if and only if we have equality in (1), which occurs if and only if \( \bar{k}(\bar{k} - 1) = (\bar{v} - 1)s \).

For (ii) let \( r_i \) denote the number of ordered pairs of elements in \( M \) that lie in \( i \) elements of \( B \). Notice that we can assume \( \bar{k}(\bar{k} - 1) = (\bar{v} - 1)s \). As before we obtain
\[
\sum r_i = \bar{v}(\bar{v} - 1)s,
\]
\[
\sum ir_i = \bar{v}\bar{k}(\bar{k} - 1) = \bar{v}(\bar{v} - 1)s,
\]
\[
\sum i(i - 1)r_i = \bar{v}(\bar{v} - 1)s(s - 1).
\]
From this we obtain \( \sum (s - i)^2 r_i = 0 \). Hence, each pair of \( M \) lies in exactly \( s \) elements of \( B \). \( \square \)

2.2 Strongly regular graphs

For a strongly regular graph \( \Gamma \), if \( \mu > 0 \), then the following equations are well-known [3, Theorem 1.3.1]:
\[
e^+e^- = \mu - k, \quad v = \frac{(k - e^+)(k - e^-)}{\mu}, \quad e^+e^- = \lambda - \mu, \tag{2}
\]
\[
m^+ = \frac{(e^- + 1)k(k - e^-)}{\mu(e^- - e^+)}, \quad m^- = v - 1 - m^+. \tag{3}
\]
We are only considering graphs with \( ve^- = m^-(e^- - k) \). Together with (2) and (3), we have
\[
k = \frac{(e^-)^2 - e^-e^+}{e^+ + 1}, \quad \mu = \frac{e^-e^+e^- + (e^-)^2}{e^+ + 1}, \quad m^- = \frac{(e^+)^2 + e^-e^+ + e^+ - (e^-)^2}{(e^+)^2 + e^-}, \tag{4}
\]
\[
v = \frac{(2e^+ - e^- + 1)((e^+)^2 + e^-e^+ + e^+ - (e^-)^2)}{(e^+ + 1)((e^+)^2 + e^-)}. \tag{5}
\]

**Theorem 2.10** ([3, Proposition 1.3.2]). Let \( \Gamma \) be a strongly regular graph. If \( Y \) is a coclique of \( \Gamma \), then the following statements are equivalent.

(a) \( |Y|e^- - k = ve^- \).

(b) Every vertex not in \( Y \) has exactly \( -e^- \) neighbors in \( Y \).

(c) The characteristic vector \( \chi \) of \( Y \) lies in \( (j) + V^- \).

Note that the coclique \( Y \) in Theorem 2.10 is a Delsarte coclique.

**Theorem 2.11** (Theorem 9.4.1 [4]). Let \( \Gamma \) be a primitive strongly regular graph with \( ve^- = m^-(e^- - k) \) and let \( Y \) be a Delsarte coclique. Then the subgraph \( \Gamma' \) of \( \Gamma \) induced by the vertices not in \( Y \) is strongly regular with degree \( k' = k + e^- \), positive eigenvalues \( k' \) and \( e^+ \), and negative eigenvalue \( e^+ + e^- \).
Theorem 1.1 is a restatement of this result. To see this, let \( \overline{B} \) be the quasisymmetric 2-(\( v, k, \lambda, \mu \)) design defined in Theorem 1.1 constructed from a strongly regular graph \( \Gamma \) with parameters (\( v, k, \lambda, \mu \)). By the definition of \( \overline{B} \), \( \overline{v} = |Y| = m^2 \). By Theorem 2.10, \( \overline{k} = -e^- \). By the definition of \( \Gamma \), \( \overline{\lambda} = \mu \) and the replication number is \( k \).

Let \( \Gamma' \) be the subgraph of \( \Gamma \) induced by the vertices not in a given Delsarte coclique \( Y \). Clearly the degree of \( \Gamma' \) is \( k' = k + e^- \). Applying equation (2) with these eigenvalues, we obtain that the size \( \mu' \) of the common neighborhood of two nonadjacent vertices in \( \Gamma' \) satisfies

\[
\mu' = e^+(e^- + e^+) + k' = e^+(e^- + e^+) + k + e^-.
\]

Hence, by (2), one of the intersection sizes for the blocks in the quasisymmetric design is

\[
\mu - \mu' = (e^+e^- + k) - (e^+(e^- + e^+) + k + e^-) = -(e^+)^2 - e^-.
\]

Similarly, by (2), the size \( \lambda' \) of the common neighborhood of two adjacent vertices in \( \Gamma' \) satisfies

\[
\lambda' = e^+(e^+ + e^-) + \mu' = 2e^+ + e^- + \mu'.
\]

Hence, by (2), the other intersection size is

\[
\lambda - \lambda' = e^+ + e^- - (2e^+ + e^-) + (\mu - \mu') = -(e^+)^2 - e^+ - e^-.
\]

3 | PROOF OF THEOREM 1.2

Parts (a) and (b) of Theorem 1.2 are surely known, but we have only found them for a special case in literature due to Makhnev et al. [13, Lemma 2], which limits itself to strongly regular graphs with the same parameters as the point graph of a generalized quadrangle of order \( (q, q^2 - q) \). Part (c) may be known, but it was only observed for the special case mentioned before [13, Lemma 2]. Part (d) is new. Part (e) and Part (f) are straight-forward generalizations of [13, Proposition 2].

Proof of Theorem 1.2. Recall that \( B := \{ b_z : z \in Z \setminus Y \} \) where \( b_z = \{ y \in Y : y \sim z \} \). Since \( Y \) and \( Z \) are both Delsarte cocliques, we can switch the roles of \( Y \) and \( Z \) in the definition of \( \overline{B} \) in Theorem 1.1. By Theorem 1.1, each set in \( B \) has size \( -e^- \) and any two distinct sets have intersection size \( -(e^+)^2 - e^- \).

Apply standard double-counting arguments to triples \( (b, p, p') \), where \( p, p' \in Y \setminus Z \) and \( b \in Z \setminus Y \), with both \( p \) and \( p' \) adjacent to \( b \) in \( \Gamma \), to obtain

\[
|B| = \frac{(e^+)^2 - (e^-)^2}{(e^+)^2 + e^-}.
\]

Then by Lemma 2.4 \( B \) is a symmetric 2-(\( e^+e^- \), \( e^- \), \( -(e^+)^2 - e^- \)).

Thus Part (a) holds. Part (b) is implied by (a) and equation (4).

Part (c) follows from Theorem 2.10 if we can show that there is equality in Hoffman’s bound for the coclique \( Z \setminus Y \) in the graph \( \Gamma' \) induced by the vertices of \( \Gamma \) not in \( Y \). Using the parameters from Theorem 2.11, we need to show that

\[
\frac{(v - m^-)(e^+ + e^-)}{(e^+ + e^-) - (k + e^-)} = \frac{(e^+)^2 - (e^-)^2}{(e^+)^2 + e^-}.
\]
holds. Using \( ve^- = m^- (e^- - k) \) and (2), this is equivalent to
\[
\mu(e^+ - e^-) = -k((e^+)^2 + e^-).
\]
By (4), this is true. Thus, we can apply Part (b) of Theorem 2.10 and the eigenvalues given by Theorem 2.11 to show that each vertex in \( \Gamma' \) that is not in \( Z \) is adjacent to exactly \(-e^+ - e^-\) vertices in \( Z \setminus Y \). Since the vertices in \( \Gamma' \) that are not in \( Z \) are adjacent to \(-e^-\) vertices in \( Z \), they must be adjacent to \( e^+\) vertices in \( Z \cap Y \).

For the bound in (d), suppose that we have \( c \) Delsarte cocliques \( Y_1, \ldots, Y_c \). Consider the \( c \times v \) matrix \( M \) whose rows are the characteristic vectors of the cocliques \( Y_i \). By Theorem 2.10, the row span of \( M \) lies in \( (j) + V^- \), so \( \text{rank}(M) \leq 1 + m^- \). By Part (b),
\[
((e^+)^2 + e^-)MM^T = ((e^+)^2 - (e^-)^2)I + (e^- e^+ + e^+)J,
\]
where \( I \) is the identity matrix and \( J \) is the all-ones matrix. Since \((e^+)^2 - (e^-)^2 > e^- e^+ + e^+\) the eigenvalues of this matrix are strictly positive. Hence, \( \text{rank}(MM^T) = c \). As \( \text{rank}(MM^T) \leq \text{rank}(M) \leq 1 + m^- \), we obtain the desired bound.

For Part (e) and Part (f), let \( Y_1, Y_2, \) and \( Y_3 \) be three different Delsarte cocliques. Let \( S \) denote \( Y_1 \cap Y_2 \cap Y_3 \). Let \( \beta = |S| \). Double counting the number of edges between \( (Y_1 \cap Y_2) \setminus S \) and \( Y_3 \setminus (Y_1 \cup Y_2) \), we obtain
\[
\left(\frac{(e^- + 1)e^+}{(e^+)^2 + e^-} - \beta \right) (-e^-) = \left( m^- - 2 \frac{(e^- + 1)e^+}{(e^+)^2 + e^-} + \beta \right) (e^+).
\]
From equation (4), we obtain
\[
((e^+)^2 + e^-)(e^- - e^+)\beta = e^+ ((e^+)^2 - e^- e^+ + e^- - e^+).
\]
This implies (e). If \( e^- \) and \( e^+ \) are coprime, then this implies \( \beta \equiv 0 \pmod{e^+} \). Hence, \( \beta = 0 \) or \( \beta = \gamma e^+ \) for some positive integer \( \gamma \). If \( \beta = 0 \), then equation (6) gives
\[
0 = (e^+)^2 - e^- e^+ + e^- - e^+ = (e^+ - e^-)(e^+ - 1).
\]
As the first factor is always positive, we obtain \( e^+ = 1 \). If \( \beta = \gamma e^+ \), then (6) gives
\[
0 = \gamma(e^+)^2 + \gamma e^- + e^+ - 1 < \gamma((e^+)^2 + e^- + e^+).
\]
The number \(-(e^+)^2 - e^- - e^+\) is one of our intersection numbers, so it is nonnegative. Hence, the right hand side of the above equation is nonpositive and therefore \( \gamma(e^+)^2 - e^- - e^+ \) is negative. Thus, the case \( \beta = \gamma e^+ \) cannot occur and therefore Part (f) holds.

For the second and third part of (d), consider \( Y_1 \cap Y_2, Y_1 \cap Y_3, Y_1 \cap Y_4, \ldots \) as a family of \( k \)-subsets of \( Y_1 \). By (b) and (e), we can apply Lemma 2.4 with
\[
\tilde{v} = m^- , \quad \tilde{k} = \frac{(e^- + 1)e^+}{(e^+)^2 + e^-} , \quad s = \frac{-(e^+)^2 + e^+}{(e^+)^2 + e^-}.
\]
Here, using equation (4), the identity \( \tilde{k} (\tilde{k} - 1) = (\tilde{v} - 1) s \) is easily verified. Due to this construction we know that every vertex of \( \Gamma \) lies in \( 0 \) or \( 1 + \frac{(e^- + 1)e^+}{(e^+)^2 + e^-} \) Delsarte cocliques. A Delsarte coclique contains \( m^- \) elements. Using equations (4) and (5), we see that \( v(1 + \frac{(e^- + 1)e^+}{(e^+)^2 + e^-}) = m^-(m^- + 1) \). This concludes the proof of (d).
4 | KNOWN EXAMPLES WITH DELSARTE COCLIQUES

We start by applying Theorems 1.1 and 1.2 to the only known examples for such graphs, namely the complements of triangular graphs and the $M_{22}$ graph on 77 vertices.

**Example 4.1.** The complements of the triangular graphs $T(n)$ can be defined in the following way: the 2-subsets of $\{1, \ldots, n\}$ are the vertices of the graph and two vertices are adjacent if their intersection is empty. It is well-known that this is a graph with parameters

$$v = \binom{n}{2}, \quad m^- = n - 1,$$

$$e^+ = 1, \quad e^- = -n + 3.$$

For $n > 4$, from Theorem 1.1 it is easy to verify that the largest independent sets in this graph correspond to the set of all 2-sets that contain a fixed element.*

By Theorem 1.2, these independent sets pairwise intersect in exactly 1 element. Notice that this is no longer the case when $n = 4$. For example, $\{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ and $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ are independent sets that share two elements. It is easy to check that the common intersection of three Delsarte cocliques is 0 as claimed by Theorem 1.2.

Notice that for $n = 5$ the corresponding graph is the Petersen graph; for $n = 6$ the corresponding graph is the point graph of the unique generalized quadrangle of order $(2, 2)$; for $n = 7$ the corresponding graph is Taylor’s 2-graph with $e^+ = 1$.

**Example 4.2.** The $M_{22}$ graph has parameters

$$v = 77, \quad m^- = 21,$$

$$e^+ = 2, \quad e^- = -6.$$

It is well-known and can be easily checked that this graph possesses 22 cocliques of size 21 that pairwise intersect in five elements, while the common intersection of three Delsarte cocliques is 1. This is also implied by Theorem 1.2. As we have equality in Theorem 1.2 (d), we can identify the intersections $Y_1 \cap Y_2, \ldots, Y_{21} \cap Y_{21}$ of the Delsarte cocliques with a 2-(20, 5, 1) design. That is the unique projective plane of order 4. The design from Theorem 1.2 (a) is a 2-(21, 6, 2) design, so a biplane of order 4. It is well-known that there are three such biplanes, but from $M_{22}$ we only obtain the unique biplane with an automorphism group of order 11520.

5 | GENERALIZED QUADRANGLES OF ORDER $(q, q^2 - q)$

Although the results in this section are valid for all strongly regular graphs having the parameters listed below, we state the results in terms of generalized quadrangles, as there has been great interest in the existence of Delsarte cocliques in the point graphs of generalized quadrangles. Recall that for a generalized quadrangle, a Delsarte coclique is called an ovoid. It is known that generalized quadrangles of order $(q, q')$ with $q' > q^2 - q$ do not possess ovoids, while it is an open question whether generalized quadrangles of order $(q, q^2 - q)$ possess ovoids [14, Section 1.8]. We will rule out the existence of ovoids for various parameters $q$. Except for $q \in \{2, 3\}$, the existence of a generalized

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* This is a very special case of the famous Erdős-Ko-Rado theorem.
quadrangle of order \((q, q^2 - q)\) is open, so our results may only apply to an empty set. See [14, Chapter 6] for the unique existing generalized quadrangle of order \((2, 2)\) and the nonexistence of generalized quadrangles of order \((3, 6)\). As shown in [3, Section 1.15], the parameters of the point graph of a generalized quadrangle of order \((q, q^2 - q)\) are as follows, where \(q\) is an integer larger than 1.

\[
\begin{align*}
v &= (q + 1)(q^3 - q^2 + 1), \\
m^+ &= q^3 - q^2 + 1, \\
e^+ &= q - 1, \\
e^- &= -q^2 + q - 1.
\end{align*}
\]

**Theorem 5.1.** A generalized quadrangle of order \((q, q^2 - q)\) does not possess an ovoid if one of the following cases occurs:

(i) \(q \equiv 3 \pmod{8}\), or

(ii) \(q = \ell p^e + 1\), where \(p\) is a prime with \(p \equiv 3 \pmod{4}\), \(e\) is odd, and \(\ell \equiv 2 \pmod{4}\).

**Proof.** By Theorem 1.1, the existence of an ovoid is equivalent to the existence of a quasisymmetric \(2\)-\((q^3 - q^2 + 1, q^2 - q + 1, q^2 - q + 1)\) design with intersection numbers \(\{1, q\}\) and replication number \(q(q^2 - q + 1)\).

If \(q \equiv 3 \pmod{8}\), then

\[
q \equiv 1 \pmod{2}, \\
r - \bar{\lambda} \equiv (q - 1)(q^2 - q + 1) \equiv 2 \cdot 3 \not\equiv 0 \pmod{4}, \\
\bar{v} \equiv q^3 - q^2 + 1 \equiv 3 \not\equiv \pm 1 \pmod{8}.
\]

Hence, Theorem 2.1 implies nonexistence of an ovoid for Case (i).

If \(q = \ell p^e + 1\), then

\[
\begin{align*}
q &\equiv 1 \pmod{p^e}, \\
r - \bar{\lambda} &\equiv (q - 1)(q^2 - q + 1) \equiv \ell p^e \pmod{p^{e+1}}, \\
\bar{v} &\equiv q^2(q - 1) + 1 \equiv 1 \cdot 2 + 1 \equiv 3 \pmod{4}, \\
\psi(s) &\equiv 0 \pmod{2}.
\end{align*}
\]

The condition that \(p \equiv 3 \pmod{4}\) is equivalent to the statement that \(-1\) is not a square modulo \(p\). It then follows that \(-\sigma(-1)^{-\frac{1}{2}}\) is not a square modulo \(p\). Hence, Theorem 2.3 implies nonexistence for Case (ii).

Theorem 5.1 rules out the existence of an ovoid for \(q = 7\), but not for \(q \in \{4, 5, 6\}\). If several ovoids exist, then, by Theorem 1.2, they pairwise intersect in \((q - 1)^2\) points. This is well-known for the unique generalized quadrangle of order \((2, 2)\). This quadrangle belongs to a family of generalized quadrangles of order \((q, q)\) for which strong intersection conditions between ovoids are known [1].

The next open case is the generalized quadrangle of order \((4, 12)\). By Theorem 1.2, we obtain a symmetric \(2\)-\((40, 13, 4)\) design. Many such designs are known [6,15] (for example, we can take the 1-dimensional subspaces of \(\mathbb{F}_3^4\) as elements and the 3-dimensional subspaces of \(\mathbb{F}_3^4\) as blocks) but it is an open problem to use such a design to construct a generalized quadrangle:

**Problem 1.** Construct a generalized quadrangle of order \((4, 12)\), starting with a \(2\)-\((40, 13, 4)\) design.
We doubt that this is possible due to the following lemma:

**Lemma 5.2.** A point-transitive generalized quadrangle of order \((q, q^2 - q)\), where \(q > 2\), does not possess an ovoid.

**Proof.** Recall that an ovoid has size \(m^+\). Suppose that the generalized quadrangle contains at least one ovoid. Then we have at least \(v/m^- = q + 1\) ovoids due to transitivity. By [13] (or Theorem 1.2 (f)) a generalized quadrangle of order \((q, q^2 - q)\), where \(q > 2\), can have at most two ovoids. This is a contradiction, so the quadrangle possesses no ovoids. \(\Box\)

By Lemma 5.2, a generalized quadrangle of order \((4, 12)\) would be very asymmetric, as it could not be point-transitive.

### 6 | TAYLOR'S 2-GRAph

For the case that \(q\) is an odd prime power, we refer to [16] for a definition of Taylor's 2-graph for \(U(3, q)\). The Taylor 2-graph for \(U(3, q)\) has parameters

\[
\begin{align*}
v &= (q + 1)(q^2 - q + 1), \\
m^- &= q^2 - q + 1, \\
e^+ &= \frac{q - 1}{2}, \\
e^- &= -\frac{q^2 + 1}{2}.
\end{align*}
\]

Again, our results hold for all graphs with the same parameters.

**Theorem 6.1.** A strongly regular graph \(\Gamma\) with the parameters \(v = (q + 1)(q^2 - q + 1)\), \(e^+ = \frac{q - 1}{2}\), and \(e^- = -\frac{q^2 + 1}{2}\), where \(q > 1\) is odd, does not possess a Delsarte coclique if one of the following occurs:

- (a) \(q \equiv 5 \pmod{8}\), or
- (b) \(q = 2\ell p^e + 1\), where \(p\) is a prime, \(p \equiv 3 \pmod{4}\), \(e\) is odd, \(\ell\) is odd, and \(\text{gcd}(\ell, p) = 1\).

Furthermore, \(\Gamma\) possesses at most two Delsarte cocliques if \(q > 3\).

**Proof.** By Theorem 1.1, the existence of an ovoid is equivalent to the existence of a quasisymmetric 2-(\(q^2 - q + 1\), \(\frac{q^2 + 1}{2}\), \(\frac{q^3 + q^2 + q + 1}{4}\)) design with intersection numbers \(\{\frac{q^2 + 3}{4}, \frac{(q + 1)^2}{4}\}\) and replication number \(\frac{q^3 + q}{2}\).

If \(q \equiv 5 \pmod{8}\), then

\[
q \equiv 1 \pmod{2},
\]

\[
\lambda - r \equiv \frac{(q^2 + 1)(1 - q)}{4} \equiv 2 \not\equiv 0 \pmod{4},
\]

\[
v \equiv q^2 - q + 1 \equiv 5 \not\equiv \pm 1 \pmod{8}.
\]

Hence, Theorem 2.1 implies nonexistence of a Delsarte coclique for Case (a).

If \(q = 2\ell p^e + 1\), then

\[
s := 1 \equiv \frac{q^2 + 3}{4} \equiv \frac{(q + 1)^2}{4} \pmod{p^{e+1}},
\]

\[
q \equiv 1 \pmod{p^e},
\]
\[ \lambda - r \equiv \frac{-q^3 + q^2 - q + 1}{4} \equiv \ell p^e \not\equiv 0 \pmod{p^{e+1}}, \]
\[ \tilde{v} \equiv q^2 - q + 1 \equiv 1 - 2 \cdot \ell' \cdot 3 - 1 + 1 \equiv 3 \pmod{4}, \]
\[ \psi(s) \equiv 0 \pmod{2}, \]
\[ \tilde{v} - 1 \equiv \frac{q^2 - q}{2} \equiv 1 \pmod{2}. \]

Again, having \( p \equiv 3 \pmod{4} \) means that \( \sigma(-1) \frac{\ell'-1}{2} = -1 \) is not a square modulo \( p \). Therefore Theorem 2.3 implies nonexistence of a Delsarte coclique for Case (b).

We now show that \( \Gamma \) has at most two Delsarte cocliques. Let \( q = 2\ell' + 1 \) for some positive integer \( \ell' \). Then
\[ e^+ = \frac{q - 1}{2} = \ell', \quad -e^- = \frac{q^2 + 1}{2} = 2\ell'^2 + 2\ell' + 1. \]
Hence, \( e^+ \) and \( e^- \) are always coprime and we can apply Theorem 1.2.

Corollary 6.2. Taylor's 2-graph does not possess a Delsarte coclique.

Proof. Suppose that Taylor's 2-graph does possess at least one Delsarte coclique. By [16, Corollary 2], Taylor's graph has a transitive automorphism group. Hence, it has at least \( \frac{v - m^-}{m^-} = q + 1 > 2 \) Delsarte cocliques. This contradicts Theorem 6.1.

7 | GENERALIZED \( M_{22} \) GRAPHS

The parameters of the \( M_{22} \) graph are part of the following infinite family, where \( q \) is a positive integer.
\[ v = (q^2 + 2q - 1)(q^2 + 3q + 1), \quad m^- = (q + 1)(q^2 + 2q - 1), \]
\[ e^+ = q, \quad e^- = -q^2 - q. \]

One noteworthy property of these graphs is that \( \lambda = 0 \). No such graphs seem to be known for \( q > 2 \). For \( q = 1 \) these are the parameters of the Petersen graph.

The smallest open case is \( q = 3 \). Here the symmetric design of Theorem 1.2 (a) has parameters 2-(45, 12, 3) and many such designs are known. In particular all of the designs with a nontrivial automorphism group are classified [7]. Using an MIP solver, we verified that none of these designs can be extended to a quasisymmetric 2-(56, 12, 9) design. Hence, we conjecture that no graph of the above family with \( q = 3 \) contains a Delsarte coclique. Therefore, the most promising open case for a construction is \( q = 4 \). Here one would take a symmetric 2-(96, 20, 4) design and try to extend it to a quasisymmetric 2-(115, 20, 16) design with intersection numbers 0 and 4.

Following an idea by Alexander L. Gavrilyuk,\(*\) we have the following lemma about the number of cocliques.

Lemma 7.1. If a strongly regular graph with \( v = (q^2 + 2q - 1)(q^2 + 3q + 1) \), \( e^+ = q \) and \( e^- = -q^2 - q \) has \( m^- + 1 \) Delsarte cocliques, then there exists a strongly regular graph with parameters \( v = q^2(q + 3)^2 \), \( e^+ = q \) and \( e^- = -q^2 - 2q \).

\(*\) Private communication for the \( q = 3 \) case.
Our graph $\Gamma$ has parameters $(v, k, \lambda, \mu) = (q^2 + 2q - 1, q^2 + 3q + 1, q^2(q + 2), 0, q^2)$. Suppose that we have equality in Theorem 1.2 (d) that implies that $q^3 + 3q^2 + q$ cocliques of size $q^3 + 3q^2 + q - 1$ form a symmetric 2-$(q^3 + 3q^2 + q - 1, q^2 + q - 1, q - 1)$ design. Now we can construct a strongly regular graph $\Gamma'$ with $(v, k, \lambda, \mu) = (q^2(q^3 + 3q^2 + q, 0, q^2 + q)$. Suppose that we have equality in Theorem 1.2 (d) that implies that $q^3 + 3q^2 + q$ cocliques of size $q^3 + 3q^2 + q$ form a symmetric 2-$(q^3 + 3q^2 + q, 0, q^2 + q)$ design. Now we can construct a strongly regular graph $\Gamma'$ with $(v, k, \lambda, \mu) = (q^2(q^3 + 3q^2 + q, 0, q^2 + q)$ as follows: The vertices of $\Gamma'$ consist of the vertices of $\Gamma$, the $q^3 + 3q^2 + q$ new vertices representing the $q^3 + 3q^2 + q$ Delsarte cocliques, and a new vertex $z^*$ representing the set of all $q^3 + 3q^2 + q$ Delsarte cocliques. Adjacency is defined as follows:

- Two vertices of $\Gamma'$ are adjacent if they are adjacent in $\Gamma$.
- A vertex of $x \in \Gamma$ and a vertex $z$ representing a Delsarte coclique $Z$ are adjacent if $x \in Z$.
- The neighborhood of $z^*$ is exactly the set of all vertices representing the $q^3 + 3q^2 + q$ Delsarte cocliques.

Using Theorem 1.2 (d) it is easy to verify that $\Gamma'$ is a strongly regular graph. □

Strongly regular graphs with parameters $(v, k, \lambda, \mu) = (324, 57, 0, 12)$ do not exist (see Gavrilyuk and Makhnev [8]). Hence, we obtain the following.

**Corollary 7.2.** A strongly regular graph with $v = 266, e^+ = 3$, and $e^- = -12$ has at most $m^-$ Delsarte cocliques.

Besides the triangular graphs, this is the only family of parameters for which we could not rule out the existence of $m^- + 1$ Delsarte cocliques in general. Among all graphs up to 1300 vertices, there is one more examples that can have more than two Delsarte cocliques.

**Lemma 7.3.** A strongly regular graph with $v = 1036, e^+ = 5$, and $e^- = -45$ has at most $m^-$ Delsarte cocliques.

**Proof.** Suppose that we have equality in Theorem 1.2 (d). Then a symmetric 2-$(111, 11, 1)$ design exists. This is equivalent to the existence of a projective plane of order 10 for which nonexistence is known [12]. □

If the set of $m^- + 1$ Delsarte cocliques forms a projective plane, then $-(e^+)^2 + e^+ = (e^+)^2 + e^-$. By equation (5), the number of vertices is an integer if and only if $e^+ \in \{1, 2, 5\}$. Hence, we cannot find any other strongly regular graphs where equality in Theorem 1.2 (d) induces a projective plane. Indeed we can say the following.

**Theorem 7.4.** Let $\Gamma$ be a primitive strongly regular graph with $ve^- = m^-(e^- - k)$. If $\Gamma$ has $m^- + 1$ Delsarte cocliques, then a 2-$(v, \bar{k}, c)$ design exists for some constant $c$ and $e^+ + 1$ divides $2c + 4$.

**Proof.** By Theorem 1.2, we have

$$e^- = \frac{e^+ - (c + 1)(e^+)^2}{c}.$$  

By equation (5),

$$v = \frac{(e^+ + 1)((c + 1)e^+ + c - 2)((c + 1)^2((e^+)^2 + 1) + (2c - 1)(c + 1)e^+ + 1)) + 2c + 4}{c^2(e^+ + 1)}.$$
TABLE I All strongly regular graphs with \(v\epsilon^- = m^-(\epsilon^- - k)\) and \(\epsilon^+ > 1\) for \(v \in [200, 1300]\)

| \(v\) | \(k\) | \(\lambda\) | \(\mu\) | \(\epsilon^+\) | \(\epsilon^-\) | \(m^-\) | \(s_1\) | \(s_2\) | \# | Reference | Remark |
|-------|------|----------|------|-----------|-----------|-------|------|------|----|--------|-------|
| 245   | 52   | 3        | 13   | 3         | -13       | 49    | 1    | 4    | 2  | Sec. 5, \(q = 4\) | GQ(4, 12) |
| 246   | 85   | 20       | 34   | 3         | -17       | 41    | 5    | 8    | 0  | Th. 2.2, \(p = 3\)  |
| 261   | 176  | 112      | 132  | 2         | -22       | 29    | 16   | 18   | 0  | [10,17]  |
| 266   | 45   | 0        | 9    | 3         | -12       | 56    | 0    | 3    | 56 | Cor. 7.2  | Gen. \(M_{22}\) |
| 287   | 126  | 45       | 63   | 3         | -21       | 41    | 9    | 12   | 2  | Th. 1.2 (e) |
| 344   | 175  | 78       | 100  | 3         | -25       | 43    | 13   | 16   | 0  | Sec. 6, \(q = 7\)  | Taylor's  |
| 490   | 297  | 168      | 198  | 3         | -33       | 49    | 21   | 24   | 2  | Th. 1.2 (e) |
| 532   | 156  | 30       | 52   | 4         | -26       | 76    | 6    | 10   | 2  | Th. 1.2 (e) |
| 568   | 217  | 66       | 93   | 4         | -31       | 71    | 11   | 15   | 2  | Th. 1.2 (e) |
| 606   | 105  | 4        | 21   | 4         | -21       | 101   | 1    | 5    | 2  | Sec. 5, \(q = 5\)  | GQ(5, 20) |
| 639   | 288  | 112      | 144  | 4         | -36       | 71    | 16   | 20   | 2  | Th. 1.2 (e) |
| 667   | 96   | 0        | 16   | 4         | -20       | 115   | 0    | 4    | 116| Sec. 7  | Gen. \(M_{22}\) |
| 672   | 451  | 290      | 328  | 3         | -41       | 56    | 29   | 32   | 0  | Th. 2.2, \(p = 3\)  |
| 730   | 369  | 168      | 205  | 4         | -41       | 73    | 21   | 25   | 0  | Sec. 6, \(q = 9\)  | Taylor's  |
| 836   | 460  | 234      | 276  | 4         | -46       | 76    | 26   | 30   | 2  | Th. 1.2 (e) |
| 1003  | 300  | 65       | 100  | 5         | -40       | 118   | 10   | 15   | 2  | Th. 1.2 (e) |
| 1016  | 259  | 42       | 74   | 5         | -37       | 127   | 7    | 12   | 0  | Th. 2.2, \(p = 5\)  |
| 1017  | 344  | 91       | 129  | 5         | -43       | 113   | 13   | 18   | 0  | Th. 2.2, \(p = 5\)  |
| 1036  | 375  | 110      | 150  | 5         | -45       | 111   | 15   | 20   | 111| Lem. 7.3 |
| 1080  | 221  | 22       | 51   | 5         | -34       | 144   | 4    | 9    | 0  | Th. 2.2, \(p = 5\)  |
| 1090  | 441  | 152      | 196  | 5         | -49       | 109   | 19   | 24   | 2  | Th. 1.2 (e) |
| 1122  | 209  | 16       | 44   | 5         | -33       | 153   | 3    | 8    | 0  | Th. 2.2, \(p = 5\)  |
| 1136  | 855  | 630      | 684  | 3         | -57       | 71    | 45   | 48   | 1  | Th. 1.2 (b) |
| 1199  | 550  | 225      | 275  | 5         | -55       | 109   | 25   | 30   | 2  | Th. 1.2 (e) |
| 1267  | 186  | 5        | 31   | 5         | -31       | 181   | 1    | 6    | 2  | Sec. 5, \(q = 6\)  | GQ(6, 30) |

Dividing the numerator by \(\epsilon^+ + 1\) leaves a remainder of \(2c + 4\), and thus \(\epsilon^+ + 1\) must divide \(2c + 4\).

For \(c = 2\), the \(2-(\tilde{v}, \tilde{k}, c)\) design is a biplane. The only nontrivial choice of parameters is \(\epsilon^+ = 3\) and \(\epsilon^- = -12\), so the design in Theorem 1.2 (d) has parameters \(2-(56, 11, 2)\). Biplanes with these parameters were classified by Kaski and Östergård [11]. Hence, the smallest case for which we might be able to use the result to construct new symmetric designs are triplanes, so \(c = 3\). Here \((\epsilon^+, \epsilon^-) = (4, -20)\) and \((\epsilon^+, \epsilon^-) = (9, -105)\) are the two interesting parameter sets.

8 | OTHER GRAPHS WITH UP TO 1300 VERTICES

In Table I we list strongly regular graphs with up to 1300 vertices. We do not include the complements of triangular graphs or graphs with less than 200 vertices. The parameters \(s_1\) and \(s_2\) are the intersection numbers of the quasi-symmetric 2-design in Theorem 1.1. The entry \# gives the maximal number of Delsarte cocliques. Notice that except for Taylor’s 2-graphs it is not known if strongly regular graphs with the given parameters exist.
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