Top–down holographic $G$-structure glueball spectroscopy at (N)LO in $N$ and finite coupling

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Abstract The top–down type IIB holographic dual of large-$N$ thermal QCD as constructed in Mia et al. (Nucl Phys B 839:187, 2010) involving a fluxed resolved warped deformed conifold, its delocalized type IIA Strominger–Yau–Zaslow-mirror (SYZ-mirror) as well as its M-theory uplift constructed in Dhuria and Misra (JHEP 1311:001, 2013) – both in the finite coupling ($g_s \sim 1$)’MQGP’ limit of Dhuria and Misra (JHEP 1311:001, 2013) – were shown explicitly to possess a local $SU(3)/G_2$-structure in Sil and Misra (Nucl Phys B 910:754, 2016). Glueballs spectra in the finite-gauge-coupling limit (and not just large ’t Hooft coupling limit) – a limit expected to be directly relevant to strongly coupled systems at finite temperature such as QGP (Natsuume in String theory and quark–gluon plasma, 2007) – has thus far been missing in the literature. In this paper, we fill this gap by calculating the masses of the $0^{++}, 0^{--}, 0^{-+}, 1^{++}, 2^{++}$ (‘glueball’) states (which correspond to fluctuations in the dilaton or complexified two-forms or appropriate metric components) in the aforementioned backgrounds of $G$-structure in the ‘MQGP’ limit of Dhuria and Misra (JHEP 1311:001, 2013). We use WKB quantization conditions on one hand and impose Neumann/Dirichlet boundary conditions at an IR cut-off ($r_0$)/horizon radius ($r_h$) on the solutions to the equations of motion on the other hand. We find that the former technique produces results closer to the lattice results. We also discuss the $r_h = 0$ limits of all calculations. In this context we also calculate the $0^{++}, 0^{--}, 1^{++}, 2^{++}$ glueball masses up to Next to Leading Order (NLO) in $N$ and find a $g_s N_f^2/(N^2)$-suppression similar to and further validating semi-universality of NLO corrections to transport coefficients, observed in Sil and Misra (Eur Phys J C 76(11):618, 2016).

1 Introduction

The AdS/CFT correspondence [1] remarkably establishes an equivalence between the partition functions of a five-dimensional gravitational theory (bulk theory) and a four-dimensional supersymmetric and scale invariant gauge theory (boundary theory). A generalization of the AdS/CFT correspondence is necessary to explore more realistic gauge theories (less supersymmetric and non-conformal) such as QCD with $SU(3)$ gauge group. The top–down model that we have considered in this work is motivated by the desire to capture QCD-like gauge theories from a suitable gravitational background. QCD is a strongly coupled theory at low energies. The low energy dynamics of QCD involves the color-neutral bound states of gluons, known as glueballs. Hence, the non-perturbative aspects of QCD can be largely understood from the glueball sector of the theory. Moreover, the plasma phase of QCD (QGP) occurs at high temperatures $T > T_c$. In a QGP medium the quarks and the gluons stay in a deconfined state due to Debye screening. However, the recent RHIC experiments indicate strongly that non-perturbative effects of QCD are present in the plasma phase. In fact the lattice results of [2] conclude that QGP must be non-perturbative in the temperature regime $T_c \leq T \leq 5T_c$. This is precisely the reason why we concentrate on the glueball spectra in the finite-gauge-coupling limit (and not just large ’t Hooft coupling limit) – a limit expected to be directly relevant to strongly coupled systems at finite temperature such as QGP [3].

QCD is a non-abelian gauge theory, in which gauge fields play the role of dynamical degrees of freedom. Non-abelian nature of QCD allows the gauge bosons to form color-neutral bound states of gluons known as glueballs (gg, ggg, etc.). Therefore, the study of glueballs and their spectra enables us to gain a better understanding of the non-perturbative regime of QCD. The glueball state is represented by quantum numbers $j^{PC}$, where $J$, $P$ and $C$ correspond to total angular momentum, parity and charge conjugation, respectively.
Different generalized versions of the AdS/CFT correspondence has thus far been proposed to study non-supersymmetric field theories with a running gauge coupling constant. The original proposal was given by Witten to obtain a gravity dual for non-supersymmetric field theories. As per Witten’s formalism, non-supersymmetric Yang–Mills theory can be obtained by compactifying one of the spatial direction on a circle and imposing antiperiodic boundary conditions on the fermions around this circle. This makes the fermions and scalars massive and they get decoupled leaving only gauge fields as degrees of freedom. The gravity dual of this compactified theory was asymptotically AdS. In a particular case of $\mathcal{N} = 4$ SU(N) super-Yang–Mills theory dual to type IIB string theory on $AdS_5 \times S^5$, the procedure described above gives an effective model of three-dimensional Yang–Mills theory, i.e., $QCD_3$.

The gravity dual of non-supersymmetric theories in the low energy limit is typically given by supergravity backgrounds involving $AdS_p \times M_q$ where $AdS$ is the anti de Sitter space with dimension $p$ and $M_q$ is the internal manifold with dimension $q$. In supergravity theory the Kaluza–Klein modes on $M_q$ can be classified according to the spherical harmonics of $M_q$, which forms representations of the isometry group of $M_q$. The states carrying the non-trivial isometry group quantum numbers are heavier and do not couple to the pure gluonic operators on the boundary. Thus the glueballs are identified with singlet states of the isometry group.

In the past decade, glueballs have been studied extensively to gain new insight into the non-perturbative regime of QCD. Various holographic setups such as soft-wall model, hard-wall model, modified soft wall model, etc. have been used to obtain the glueball’s spectra. In [4, 5] a soft wall holographic model was used to study the glueball spectrum. In [4] glueballs and scalar mesons were studied at finite temperature. It was found that the masses of the hadronic states decreases and the widths become broader as $T$ increases. But for a temperature range of the order of 40–60 MeV , states disappear from the glueball and meson spectral function. Both hard-wall and soft-wall holographic models were considered [6, 7] to obtain the glueball correlation functions to study the dynamics of QCD. Decay rates were obtained for glueballs in both models. Dynamical content of the correlators was investigated [7] by obtaining their spectral density and relating it with various other quantities to obtain the estimates for three lowest-dimensional gluon condensates. In [8] a two-flavor quenched dynamical holographic QCD model was considered with two different forms for the dilaton field given as $\Phi = \mu^2 G z^2$ and $\Phi = \mu^2 G z^2 \tanh(\mu^2 G z^2 / \mu^2 G)$, ($z$ being a radial coordinate). In [9] an AdS$_5$ mass renormalization was implemented in a modified holographic soft-wall model to obtain the spectrum of scalar and higher even glueball spin states with $P = C = +1$. In [10, 11] a bottom–up approach was used to obtain the mass spectra of the scalar and vector glueballs. In this case, the vector glueball masses were found to be heavier than that of the scalar glueballs while higher values for both were reported in other approaches. In [12] a holographic description was used for supersymmetric and non-supersymmetric, non-commutative dipole gauge theory in 4D. The WKB approximation was used to obtain the mass by solving the dilaton and antisymmetric tensor field equations in the bulk. For the supersymmetric theory, dipole length plays the role of an intrinsic scale while for non-supersymmetric theory the same role is played by the temperature. Two different phases for baryons were found, a big baryon dual to the static string and a small baryon dual to a moving string. In [13] spectrum for scalar, vector and tensor two-gluon and triluon glueballs were obtained in 5-D holographic QCD model with a metric structure deformed by the dilaton field. The spectrum was compared with the results obtained from both soft-wall and hard-wall holographic QCD models. Here, the spectra of the two-gluon glueball was found to be in agreement with the lattice data. For triluon glueballs, the masses for $1^{++}$, $2^{++}$ were matched while masses for $0^{--}$, $0^{++}$ and $2^{--}$ were lighter than lattice data which indicates that the latter glueballs are dominated by the three-gluon condensate. In [14] a holographic glueball spectrum was obtained in the singlet sector of $\mathcal{N} = 1$ supersymmetric Klebanov–Strassler model. States containing the bifundamental $A_1$ and $B_1$ fields were not considered. Comparison with the lattice data showed a nice agreement for $1^{++}$ and $1^{--}$ states, while $0^{++}$ results were different because of its fermionic component.

Glueballs appear in the meson spectra of QCD and the difficulty in their identification in the meson spectra is largely due to lack of information as regards their coupling with mesons in strongly coupled QCD. Lattice QCD gives an estimate for the masses but it does not give any information as regards the glueball couplings and their decay widths both of which are required for identification of glueballs. Holographic approach gives a better understanding of glueball decay rates than lattice QCD. Various holographic models such as Witten–Sakai–Sugimoto model, Soft wall model and supersymmetric Klebanov–Strassler model, etc. have been used to obtain the coupling between mesons and glueballs to obtain expressions for the glueball decay widths.

In [15–20] top–down Witten–Sakai–Sugimoto model was used as a holographic setup for low energy QCD to obtain the coupling of scalar glueballs to mesons and subsequently obtain their decay widths. Results obtained were compared with the experimental data available for lattice counterparts $f_0(1500)$ and $f_0(1710)$ of scalar glueballs. In [15] results obtained for decay widths and branching ratios for scalar glueball decays were found to be consistent with experimental data for $f_0(1500)$ state in [15] while in [17] results favored $f_0(1710)$ as scalar glueball candidate instead of $f_0(1500)$. Decay patterns were obtained for scalar glueball candidate
$f_0(1710)$ in top–down holographic Witten–Sakai–Sugimoto model for low energy QCD in [16]. It was shown that there exists a narrow pseudoscalar glueball heavier than the scalar glueball whose decay pattern involves $\eta$ and $\eta'$ mesons. In [18–20] Witten–Sakai–Sugimoto model was used to study the phenomenology of scalar glueball states. A dilaton and an exotic mode were obtained as two sets of scalar glueball states in [18,19]. Calculation of mass spectra showed that out of two modes, dilaton mass is quite close to both $f_0(1710)$ and $f_0(1500)$ scalar glueball candidates while calculation of decay width showed that $f_0(1710)$ is the favored glueball candidate corresponding to dilaton mode. In [21] the holographic top–down Witten–Sakai–Sugimoto model was used to study the tensor $2^{++}$ glueball mass spectrum and decay width. Decay width was found to be above 1 GeV for glueball mass $M_f = 2400$ MeV while for $M_T = 2000$ MeV it was reduced to 640 MeV. In [22] modified holographic soft-wall model was used to calculate the mass spectrum and Regge trajectories of lightest scalar glueball and higher spin glueball states. Results were obtained for both even and odd spins glueball states.

In this paper, we use a large-$N$ top–down holographic dual of QCD to obtain the spin $2^{++}, 1^{++}, 0^{++}, 0^{--}, 0^{-+}$ glueball spectrum explicitly for QCD3 from type IIB, type IIA and M-theory perspectives. Now for the computation of the glueball masses, we need to introduce a scale in our theory. In other words, the conformal invariance has to be broken. This can be done in two different ways. The first approach, after Witten [23], corresponds to the compactification of the time direction on a circle of finite radius, forming a black hole in the background. In this case the masses are determined in units of the horizon radius $r_h$ of the black hole. The other approach is to consider a cut-off at $r = r_0$ in the gravitational background ($r$ being the non-compact radial direction) [24]. This forbids the arbitrary low energy excitations of the boundary field theory and hence breaks the conformal invariance. So, in this case the required scale to address strong interaction is introduced by the IR cut-off $r_0$. From a top–down perspective this IR cut-off will in fact be proportional to two-third power of the Ouyang embedding parameter obtained from the minimum radial distance (corresponding to the lightest quarks) requiring one to be at the South Poles in the $\theta_{1,2}$ coordinates, in the holomorphic Ouyang embedding of flavor $D7$-branes. In the spirit of [23], the time direction for both cases will be compact with fermions obeying antiperiodic boundary conditions along this compact direction, and hence we will be evaluating three-dimensional glueball masses.

Glueball masses can be obtained by evaluating the correlation functions of gauge invariant local operator. The first step to obtain the glueball spectrum in QCD3 is to identify the operators in the gauge theory that have quantum numbers corresponding to the glueballs of interest. According to the gauge/gravity duality each supergravity mode corresponds to a gauge-theory operator. This operator couples to the supergravity mode at the boundary of the AdS space, for example, the lowest dimension operator with quantum numbers $J^{PC} = 0^{++}$ is $TrF^{2} = TrF^{\mu\nu}F^{\mu\nu}$ and this operator couples to the dilaton mode on the boundary. To calculate $0^{++}$ glueball mass we need to evaluate the correlator $\langle TrF^{2}(x)TrF^{2}(y) \rangle = \Sigma_i \epsilon_i e^{-m_i|x-y|}$, where $m_i$ give the value for glueball mass. However, the masses can also be obtained by solving the wave equations for supergravity modes which couples to the gauge-theory operators on the boundary. The latter approach is used in this paper.

The 11D metric obtained as the uplift of the delocalized Strominger–Yau–Zaslow (SYZ) type IIA metric, up to LO in $N$, can be interpreted as a black $M3$-brane wrapping a two-cycle, i.e. a black $M3$-brane [25,26]. Taking this as the starting point, compactifying again along the M-theory circle, we land up at the type IIA metric and then compactifying again along the periodic temporal circle (with the radius given by the reciprocal of the temperature), one obtains QCD3 corresponding to the three non-compact directions of the black $M3$-brane world volume. The Type IIB background of [27], in principle, involves $M_4 \times$ RWDC(μ=Resolved Warped Deformed Conifold); asymptotically the same becomes $AdS_5 \times T^{1,1}$. To determine the gauge-theory fields that would couple to appropriate supergravity fields à la gauge-gravity duality, ideally one should work the same out for the $M_4 \times$ RWDC background (which would also involve solving for the Laplace equation for the internal RWDC). We do not attempt to do the same here. Motivated, however, by, e.g.,

(a) asymptotically the type IIB background of [27] and its delocalized type IIA mirror of [28] consist of $AdS_5$ and

(b) terms of the type $Tr(F^{2}(AB)^{K})$, $(F^{4}(AB)^{K})$ where $F^{2} = F^{\mu\nu}F^{\mu\nu}$, $F^{4} = \frac{1}{2} (F^{\mu\nu}F^{\rho\sigma})^{2}$, $A, B$ being the bifundamental fields that appear in the gauge-theory superpotential corresponding to $AdS_5 \times T^{1,1}$ in [29], form part of the gauge-theory operators corresponding to the solution to the Laplace equation on $T^{1,1}$ [30] (the operator $TrF^{2}$, which shares the quantum numbers of the $0^{++}$ glueball, couples to the dilaton and $TrF^{4}$ which also shares the quantum numbers of the $0^{++}$ glueball couples to trace of metric fluctuations and the four-form potential, both in the internal angular directions).

we calculate in this paper:

• type IIB dilaton fluctuations, which we refer to as $0^{++}$ glueball

• type IIB complexified two-form fluctuations that couple to $d^{abc} Tr(F^{a}_{\mu\nu}F^{b}_{\rho\sigma}F^{c}_{\mu\nu})$, which we refer to as $0^{--}$ glueball
• type IIA one-form fluctuations that couple to \( \text{Tr}(F \wedge F) \), which we refer to as \( 0^{-} \) glueball
• M-theory metric’s scalar fluctuations which we refer to as another (lighter) \( 0^{++} \) glueball
• M-theory metric’s vector fluctuations which we refer to as \( 1^{++} \) glueball,
  and
• M-theory metric’s tensor fluctuations which we refer to as \( 2^{++} \) glueball.

All holographic glueball spectra calculations done thus far, have only considered a large ‘t Hooft coupling limit: 
\[ s_{YM}^2 N \gg 1, \quad N \gg 1. \]
However, holographic duals of thermal QCD laboratories like sQGP also require a finite gauge coupling [3]. This was addressed as part of the ‘MQGP limit’ in [28]. It is in this regard that results of this paper – which discusses supergravity glueball spectra at finite string coupling – are particularly significant. Also, the recent observation – see, e.g., [31] – that the non-perturbative properties of quark–gluon plasma can be related to the change of properties of scalar and pseudoscalar glueballs, makes the study of glueballs quite important.

The rest of the paper is organized as follows. In Sect. 2, via five subsections, we summarize the top–down type IIB holographic dual of large-\( N \) thermal QCD of [27], its delocalized SYZ type IIA mirror and its M-theory uplift of [25, 28]. In Sect. 3, we discuss a supergravity calculation of the spectrum of \( 0^{++} \) glueball at finite horizon radius \( r_h \) (Sect. 3.1) and setting \( r_h = 0 \) (Sect. 3.2). The \( r_h \neq 0 \) computations are given in Sect. 3.1, corresponding to use of WKB quantization conditions using coordinate/field redefinitions of [32]. The \( r_h = 0 \) calculations are subdivided into Sect. 3.2.1 corresponding to solving the \( 0^{++} \) equation of motion up to LO in \( N \) and imposing the Neumann/Dirichlet boundary condition at the horizon, and Sect. 3.2.2 corresponding to WKB quantization conditions inclusive of non-conformal/NLO-in-\( N \) corrections using the redefinitions of [32]. Section 4 has to do with the \( 0^{-} \) glueball spectrum. Further therein, Sects. 4.1.1 and 4.1.2, respectively, are on obtaining the \( r_h \neq 0 \) spectrum and its \( r_h = 0 \) limit using Neumann/Dirichlet boundary conditions on the solutions up to LO in \( N \), respectively, at the horizon and the IR cut-off. Then Sects. 4.1.3 and 4.1.4, respectively, are on WKB quantization at finite and zero \( r_h \) up to LO in \( N \), using the redefinitions of [32]. Section 5 is on \( 0^{-} \) glueball spectrum. Therein, Sects. 5.1 and 5.2 are on getting the spectrum by imposing Neumann/Dirichlet boundary condition on the solutions up to LO in \( N \) to the EOM, respectively, at the horizon and the IR cut-off. Section 5.3 has to do with obtaining the spectrum using WKB quantization up to LO in \( N \) at \( r_h \neq 0 \) using the redefinitions of [32]; Sect. 5.4 has to do with a similar calculation in the \( r_h = 0 \) limit at LO in \( N \) in Sect. 5.4.1 and up to NLO in \( N \) in Sect. 5.4.2. Section 6 has to do with M-theory calculations of \( 0^{++}, 1^{++}, 2^{++} \) glueballs arising from appropriate metric fluctuations. Section 6.1 is on such a \( 0^{++} \) glueball spectrum, whereas in Sect. 6.1.1 and Sect. 6.1.2 we obtain the same with a finite horizon radius \( r_h \) and an IR cut-off \( r_0 \), respectively, both by imposing Neumann/Dirichlet boundary conditions (at the horizon/IR cut-off) and using WKB quantization conditions and the redefinitions of [32]. Section 6.2 is on such a \( 2^{++} \) glueball spectrum with a finite horizon radius \( r_h \) in Sect. 6.2.1 and an IR cut-off \( r_0 \) in Sect. 6.2.2. The results are obtained by imposing Neumann/Dirichlet boundary conditions at the horizon/IR cut-off and also via WKB quantization conditions using redefinitions of [32] at LO in \( N \) in Sect. 6.2.1 (\( r_h \neq 0 \)) and both at LO as well as up to NLO in \( N \) in Sect. 6.2.1 (\( r_h = 0 \) limit). Section 6.3 is on such a \( 1^{++} \) glueball spectrum, where similar to the previous sections, we have imposed Neumann/Dirichlet boundary conditions (at the horizon/IR cut-off) and used WKB quantization conditions and the redefinitions of [32]. The spectrum corresponding to the finite horizon radius \( r_h \) at LO in \( N \) is discussed in Sect. 6.3.1. On the other hand, the results corresponding to that with an IR cut-off \( r_0 \) both at LO and up to NLO in \( N \) is given in Sect. 6.3.2. From the point of view of comparing the string theory and the M-theory glueball spectrum calculations, we obtain the \( 2^{++} \) glueball spectrum arising from tensor mode of metric fluctuations in the type IIB background of [27] in Sect. 7. Section 7.1 has to do with a supergravity calculation (via Neumann/Dirichlet boundary conditions at the horizon in Sect. 7.1.1 and WKB quantization condition using redefinitions of [32] in Sect. 7.1.2), and Sect. 7.2 has to do with zero-horizon radius limit calculation (WKB quantization condition using redefinitions of [32] at LO in \( N \) in Sect. 7.2.1 and up to NLO in \( N \) in Sect. 7.2.2). Section 8 contains a summary and discussion of the results obtained in this paper. There a Appendix A on the square of different fluxes that appear in EOM relevant to spin-two perturbations of the type IIB metric.

2 Background: a top–down type IIB holographic large-\( N \) thermal QCD and its M-theory uplift in the ‘MQGP’ limit

In this section, via five subsections we will:
• provide a short review of the type IIB background of [27] which is supposed to provide a UV complete holographic dual of large-\( N \) thermal QCD, as well as their precursors in Sect. 2.1,
• discuss the ‘MQGP’ limit of [28] and the motivation for considering the same in Sect. 2.2,
• briefly review issues as discussed in [28] pertaining to construction of delocalized SYZ type IIA mirror and approximate supersymmetry, in Sect. 2.3,
briefly review the new results of [25,33] pertaining to construction of explicit $SU(3)$ and $G_2$ structures, respectively, of type IIB/IIA and M-theory uplift,

briefly discuss the new Physics-related results of [25,33], in Sect. 2.4.

2.1 Type IIB dual of large-$N$ thermal QCD

In this subsection, we will discuss a UV complete holographic dual of large-$N$ thermal QCD as given in Dasgupta–Mia et al. [27]. As mentioned in Sect. 1, this was inspired by the zero-temperature Klebanov–Witten model [29], the non-conformal Klebanov–Tseytlin model [34], its IR completion as given in the Klebanov–Strassler model [35] and Ouyang’s inclusion [36] of flavor in the same,1 as well as the non-zero temperature/non-extremal version of [37] (the solution, however, was not regular as the non-extremality/black-hole function and the ten-dimensional warp factor vanished simultaneously at the horizon radius) [38,39] (valid only at high temperatures) of the Klebanov–Tseytlin model and [40] (addressing the IR region), in the absence of flavors.

(a) Brane construction

In order to include fundamental quarks at non-zero temperature in the context of type IIB string theory, to the best of our knowledge, the following model proposed in [27] is the closest to a UV complete holographic dual of large-$N$ thermal QCD. The KS model (after a duality cascade) and QCD have similar IR behavior: $SU(M)$ gauge group and IR confinement. However, they differ drastically in the UV as the former yields a logarithmically divergent gauge coupling (in the UV) – Landau pole. This necessitates modification of the UV sector of the KS model apart from inclusion of non-extremality factors. With this in mind and building up on all of the above, the type IIB holographic dual of [27] was constructed. The setup of [27] is summarized below.

- From a gauge-theory perspective, the authors of [27] considered $N$ black D3-branes placed at the tip of six-dimensional conifold, $M$ D5-branes wrapping the vanishing two-cycle and $M$ $D5$-branes distributed along the resolved two-cycle and placed at the outer boundary of the IR–UV interpolating region/inner boundary of the UV region.

- More specifically, the $M$ $D5$ are distributed around the antipodal point relative to the location of $M$ D5 branes on the blown-up $S^2$. If the $D5/D5$ separation is given by $R_{D5/D5}$, then this provides the boundary common to the outer UV–IR interpolating region and the inner UV region. The region $r > R_{D5/D5}$ is the UV. In other words, the radial space, in [27] is divided into the IR region, the IR–UV interpolating region and the UV. To summarize the above:

$$- r_0/r_h < r < |\mu_{Ouyang}|^{\frac{1}{2}}(r_h = 0)/R_{D5/D5}(r_h \neq 0): \text{the IR/IR–UV interpolating regions with } r \sim \Lambda: \text{deep IR where the } SU(M) \text{ gauge theory confines}$$

$$- r > |\mu_{Ouyang}|^{\frac{1}{2}}(r_h = 0)/R_{D5/D5}(r_h \neq 0): \text{the UV region.}$$

- $N_f$ D7-branes, via Ouyang embedding, are holomorphically embedded in the UV (asymptotically $AdS_5 \times T^{1,1}$), the IR–UV interpolating region and dipping into the (confining) IR (up to a certain minimum value of $r$ corresponding to the lightest quark) and $N_f$ $D7$-branes present in the UV and the UV–IR interpolating (not the confining IR). This is to ensure turning off of three-form fluxes, constancy of the axion–dilaton modulus and hence conformality and absence of Landau poles in the UV region.

- The resultant ten-dimensional geometry hence involves a resolved warped deformed conifold. Back-reactions are included, e.g., in the ten-dimensional warp factor. Of course, the gravity dual, as in the Klebanov–Strassler construct, at the end of the Seiberg-duality cascade will have no D3-branes and the D5-branes are smeared/dissolved over the blown-up $S^3$ and thus replaced by fluxes in the IR region.

The delocalized S(trongimer) Y(au) Z(asilow) type IIA mirror of the aforementioned type IIB background of [27] and its M-theory uplift had been obtained in [25,28,33].

(b) Seiberg-duality cascade, IR confining $SU(M)$ gauge theory at finite temperature and $N_c = N_{\text{eff}}(r) + M_{\text{eff}}(r)$

1. IR Confinement after Seiberg Duality Cascade: Footnote numbered 3 shows that one effectively adds on to the number of $D3$-branes in the UV and hence, one has $SU(N + M) \times SU(N + M)$ color gauge group (implying an asymptotic $AdS_5$) and $SU(N_f) \times SU(N_f)$ flavor gauge group, in the UV: $r \geq R_{D5/D5}$. It is expected that there will be a partial Higgsing of $SU(N + M) \times SU(N + M)$ to $SU(N + M) \times SU(N)$ at $r = R_{D5/D5}$ [42]. The two gauge couplings, $g_{SU(N + M)}$ and $g_{SU(N)}$ flow logarithmically and oppositely in the IR region:

$$4\pi^2 \left( \frac{1}{g_{SU(N + M)}^2(r)} + \frac{1}{g_{SU(N)}^2(r)} \right) e^\phi \sim \pi$$

$$4\pi^2 \left( \frac{1}{g_{SU(N + M)}^2(r)} - \frac{1}{g_{SU(N)}^2(r)} \right) e^\phi \sim \frac{1}{2\pi a'} \int_{S^2} B_2.$$

1 See [41] for earlier attempts at studying back-reacted $D3/D7$ geometry at zero temperature; we thank Zayas for bringing [40,41] to our attention.
Obtaining $N_c = 3$, and color-flavor enhancement of length scale in the IR region: So, in the IR region, at the end of the duality cascade, what gets identified with the number of colors $N_c$ is $M$, which in the ‘MQGP limit’ to be discussed below, can be tuned to equal 3. One can identify $N_c$ with $N_{\text{eff}}(r) + M_{\text{eff}}(r)$, where $N_{\text{eff}}(r) = \int_{S^3} \tilde{F}_3$ (the $S^3$ being dual to $e_{\phi} \wedge (\sin \theta_1 d\theta_1 \wedge d\phi_1 - B_1 \sin \theta_2 \wedge d\phi_2)$, wherein $B_1$ is an asymmetry factor defined in [27], and $e_{\phi} \equiv d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2$ where $\tilde{F}_3 \equiv \tilde{F}_3 - \tau H_3 \propto M(r) \equiv 1 - \frac{e^{-\mu r}}{1 + e^{-\mu r}}$, $\mu \equiv \frac{1}{\Lambda^2}$ [43]. The effective number $N_{\text{eff}}$ of D3-branes varies between $N \gg 1$ in the UV and 0 in the deep IR, and the effective number $M_{\text{eff}}$ of D5-branes varies between 0 in the UV and $M$ in the deep IR (i.e., at the end of the duality cascade in the IR region). Hence, the number of colors $N_c$ varies between $M$ in the deep IR and a large value [even in the MQGP limit of (11) (for a large value of $N$)] in the UV region. Hence, at very low energies, the number of colors $N_c$ can be approximated by $M$, which in the MQGP limit is taken to be finite and can hence be taken to be equal to three. However, in this discussion, the low energy or the IR region is relative to the string scale. But these energies which are much less than the string scale, can still be much larger than $T_c$. Therefore, for all practical purposes, as regard the energy scales relevant to QCD, the number of colors can be tuned to three.

In the IR region in the MQGP limit, with the inclusion of terms higher order in $g_s N_f$ in the RR and NS–NS three-form fluxes and the NLO terms in the angular part of the metric, there occurs an IR color-flavor enhancement of the length scale as compared to a Planckian length scale in KS for $O(1)$ $M$, thereby showing that quantum corrections will be suppressed. Using [27]:

$$N_{\text{eff}}(r) = N \left(1 + \frac{3g_s M_{\text{eff}}}{2\pi N} \right) \log r + \left(\frac{3g_s N_{\text{eff}}}{2\pi} \right) (\log r)^2,$$

$$M_{\text{eff}}(r) = M + \frac{3g_s N_f M}{2\pi} \log r + \sum_{m \geq 1} \sum_{n \geq 1} N_f^m M^n f_{mn}(r),$$

$$N_{\text{eff}}^\text{IR}(r) = N_f + \sum_{m \geq 1} \sum_{n \geq 1} N_f^m M^n g_{mn}(r).$$

It was argued in [25] that the length scale of the OKS-BH metric in the IR region will be given by

$$L_{\text{OKS-BH}} \sim \sqrt{M N_f^3} \left(\sum_{m \geq 0} \sum_{n \geq 0} N_f^m M^n f_{mn}(\Lambda)\right)^{\frac{1}{4}}\sqrt{\alpha^2} \frac{1}{g_s^L \sqrt{\alpha}}$$

$$\equiv N_f^3 \left(\sum_{m \geq 0} \sum_{n \geq 0} N_f^m M^n f_{mn}(\Lambda)\right)^{\frac{1}{4}} L_{\text{KS}} \left|\Lambda; \log \Lambda < \frac{2\pi}{g_s \sqrt{\Lambda}}\right|,$$

which implies that in the IR region, relative to KS, there is a color-flavor enhancement of the length scale in the OKS-BH metric. Hence, in the IR region, even for $N_c^{\text{IR}} = M = 3$ and $N_f = 2$ (light flavors) upon inclusion of $n, m > 1$ terms in $M_{\text{eff}}$ and $N_{\text{eff}}$ in (2), $L_{\text{OKS-BH}} \gg L_{\text{KS}}$ ($\sim L_{\text{Planck}}$) in the MQGP limit involving $g_s < 1$, implying that the stringy corrections are suppressed and one can trust supergravity calculations.

A reminder: one will generate higher powers of $M$ and $N_f$ in the double summation in $M_{\text{eff}}$ in (2), e.g., from the terms higher order in $g_s N_f$ in the RR and NS–NS three-form fluxes that become relevant for the aforementioned values of $g_s, N_f$.

3. Further, the global flavor group in the UV–IR interpolating and UV regions, due to presence of $N_f D7$ and $N_d D7$-branes, is $SU(N_f) \times SU(N_f)$, which is
broken in the IR region to $SU(N_f)$ as the IR case has only $N_f$ D7-branes.

Hence, the following features of the type IIB model of [27] make it an ideal holographic dual of thermal QCD:

- the theory having quarks transforming in the fundamental representation, is UV conformal and IR confining with the required chiral symmetry breaking in the IR region and restoration at high temperatures;
- the theory is UV complete with the gauge coupling remaining finite in the UV (absence of Landau poles), the theory is not just defined for high temperatures but for low and high temperatures;
- with the inclusion of a finite baryon chemical potential (as will become evident in Sect. 3), the theory provides a lattice-compatible QCD confinement–deconfinement transition with the M-theory perspective with the M-theory uplift also being thermodynamically stable.

(d) Supergravity solution on resolved warped deformed conifold

The working metric is given by

$$g \quad \text{assumed to be} \quad \text{Klebanov–Strassler)-BH (Black-Hole) background and are}$$

$$\theta \quad \text{Eur. Phys. J. C (2017) 77 :381 Page 7 of 34}$$

The $g_i$ (metric in (4) is given as

$\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\\,
In [28], we had considered the following two limits:

2.2 The ‘MQGP limit’

In [28], we had considered the following two limits:

(i) weak ($g_s$) coupling—large ’t Hooft coupling limit:

\[
g_s \ll 1, \quad g_s M^2 / N \ll 1, \quad g_s M \gg 1, \quad g_s N \gg 1
\]

effected by: \( g_s \sim e^d, \quad M \sim (O(1)e)^{- \frac{3d}{2}}, \quad N \sim (O(1)e)^{-19d}, \quad \epsilon \ll 1, \quad d > 0 \)

(ii) MQGP limit: \( g_s M^2 / N \ll 1, \quad g_s N \gg 1 \), finite \( g_s, \quad M \)

effected by: \( g_s \sim e^d, \quad M \sim (O(1)e)^{- \frac{3d}{2}}, \quad N \sim (O(1)e)^{-39d}, \quad \epsilon \lesssim 1, \quad d > 0 \). (11)

Let us enumerate the motivations for considering the MQGP limit which was discussed in detail in [25]. There are principally two.

1. Unlike the AdS/CFT limit wherein \( g_{YM} \to 0, \quad N \to \infty \) such that \( g_{YM}^2 N \) is large, for strongly coupled thermal systems like sQGP, what is relevant is \( g_{YM} \sim O(1) \) and \( N_c = 3 \). From the discussion in the previous paragraphs, specially the one in point (c) of Sect. 2.1, one sees that in the IR region after the Seiberg-duality cascade, effectively \( N_c = M \), which in the MQGP limit of (11) can be tuned to 3. Further, in the same limit, the string coupling \( g_s \approx 1 \). The finiteness of the string coupling necessitates addressing the same from an M-theory perspective. This is the reason for coining the name: ‘MQGP limit’. In fact this is the reason why one is required to first construct a type IIA mirror, which was done in [28] à la delocalized Strominger–Yau–Zaslow mirror symmetry, and then take its M-theory uplift.

2. From the perspective of calculational simplification in supergravity, the following are examples of the same and constitute therefore the second set of reasons for looking at the MQGP limit of (11):

- In the UV–IR interpolating region and the UV, \( (M_{\text{eff}}, N_{\text{eff}}, N_{\text{f}}) \approx (M, N, N_f) \).
- Asymmetry factors \( A_i, \quad B_j \) (in three-form fluxes) \( M_{\text{QGP}} \to 1 \) in the UV–IR interpolating region and the UV.
- Simplification of ten-dimensional warp factor and non-extremality function in the MQGP limit.

With \( R_{D5/\overline{D5}} \) denoting the boundary common to the UV–IR interpolating region and the UV region, \( \tilde{F}_{l_{mn}}, \quad H_{l_{mn}} = 0 \) for \( r \geq R_{D5/\overline{D5}} \) is required to ensure conformality in the UV region. Near the \( \theta_1 = \theta_2 = 0 \)-branch, assuming \( \theta_1, \theta_2 \to 0 \) as \( e^{\gamma_0} \to 0 \) and \( r \to R_{UV} \to \infty \) as \( e^{-\psi} \to 0 \), \lim_{r \to \infty} \tilde{F}_{l_{mn}} = 0 \) and \lim_{r \to \infty} H_{l_{mn}} = 0 \) for all components except \( H_{\theta_1 \theta_2 \phi_{1,2}} \); in the MQGP limit and near the \( \theta_1, \theta_2 = \pi/0\)-branch, \( H_{\theta_1 \theta_2 \phi_{1,2}} = 0 / (O(1)g_s)^{3(1 + 2N_f)} \), \( N_f = 2, \quad g_s = 0.6, M = (O(1)g_s)^{-1} / 2 \). So, the UV nature too is captured near \( \theta_1, \theta_2 = 0 \)-branch in the MQGP limit. This mimics addition of \( D5 \)-branes in [27] to ensure cancellation of \( \tilde{F}_3 \).

2.3 Approximate supersymmetry, construction of the delocalized SYZ IIA mirror and its M-theory uplift in the MQGP limit

A central issue to [26,28] has been the implementation of delocalized mirror symmetry via the Strominger–Yau–Zaslow prescription according to which the mirror of a Calabi–Yau can be constructed via three T-dualities along a special Lagrangian \( T^3 \) fibered over a large base in the Calabi–Yau. This subsection is a quick review of precisely this.
To implement the quantum mirror symmetry à la Strominger–Yau–Zaslow [44], one needs a special Lagrangian (sLag) $T^3$ fibered over a large base (to nullify contributions from open-string disc instantons with boundaries as non-contractible one-cycles in the sLag). Defining delocalized T-duality coordinates, $(\phi_1, \phi_2, \psi) \rightarrow (x, y, z)$ valued in $T^3(x, y, z)$ [28]:

$$
\begin{align*}
    x &= \sqrt{h_3} \sin(\theta_1)(r) \phi_1, \\
    y &= \sqrt{h_3} \sin(\theta_2)(r) \phi_2, \\
    z &= \sqrt{h_3} \tan(\psi),
\end{align*}
$$

(12)

using the results of [45] it was shown in [26,33] that the following conditions are satisfied:

$$
\begin{align*}
    i^* f_{\text{RC/DC}} & \approx 0, \\
    s\text{t}(i^* \Omega)_{\text{RC/DC}} & \approx 0, \\
    \text{R}\text{e}(i^* \Omega)_{\text{RC/DC}} & \text{ volume form (}$T^3(x, y, z)$),
\end{align*}
$$

(13)

for the $T^2$-invariant sLag of [45] for a deformed conifold. Hence, if the resolved warped deformed conifold is predominantly either resolved or deformed, the local $T^3$ of (12) is the required sLag to effect SYZ-mirror construction.

Interestingly, in the ‘delocalized limit’ [46] $\psi \rightarrow \langle \psi \rangle$, under the coordinate transformation:

$$
\begin{align*}
    \left( \sin \theta_2 \phi_2 \right) \rightarrow \left( \cos \langle \psi \rangle \sin \langle \psi \rangle \right),
\end{align*}
$$

(14)

and $\psi \rightarrow \psi - \cos(\tilde{\theta}_2)\phi_2 + \cos(\theta_2)\phi_2 - \tan(\psi) \ln \tilde{\theta}_2$, the $h_5$ term becomes $h_5[d\theta_1d\theta_2 - \sin \tilde{\theta}_1 \sin \theta_2d\phi_1d\phi_2]$, $e \psi \rightarrow e \psi$, i.e., one introduces an isometry along $\psi$ in addition to the isometries along $\phi_1$. This clearly is not valid globally – the deformed conifold does not possess a third global isometry.

To enable use of SYZ-mirror duality via three T-dualities, one also needs to ensure a large base (implying large complex structures of the aforementioned two two-tori) of the $T^3(x, y, z)$ fibration. This is effected via [47]:

$$
\begin{align*}
    d\psi &= d\psi + f_1(\theta_1) \cos \theta_1 d\theta_1 + f_2(\theta_2) \cos \theta_2 d\theta_2, \\
    df_1,2 &\rightarrow df_1,2 - f_1,2(\theta_1,2) d\theta_1,2,
\end{align*}
$$

(15)

for appropriately chosen large values of $f_{1,2}(\theta_1,2)$. The three-form fluxes remain invariant. The fact that one can choose such large values of $f_{1,2}(\theta_1,2)$, was justified in [28]. The guiding principle is that one requires the metric obtained after SYZ-mirror transformation applied to the non-Kähler resolved warped deformed conifold is like a non-Kähler warped resolved conifold at least locally. Then $G_{\text{IIA}}^{\text{Kähler}}$ needs to vanish [28]. This was explicitly shown in [25].

As in the Klebanov–Strassler construction, a single T-duality along a direction orthogonal to the D3-brane world volume, e.g., $z$ of (12), yields D4 branes straddling a pair of NS5-branes consisting of world-volume coordinates $(\theta_1, x)$ and $(\theta_2, y)$. Further, T-dualizing along $x$ and then $y$ would yield a Taub–NUT space from each of the two $NS5$-branes [48]. The $D7$-branes yield $D6$-branes which get uplifted to Kaluza–Klein monopoles in M-theory [49], which too involve Taub–NUT spaces. Globally, probably the 11-dimensional uplift would involve a seven-fold one of the $G2$-structure, analogous to the uplift of $D5$-branes wrapping a two-cycle in a resolved warped conifold [50].

2.4 G-structures

The mirror type IIA metric after performing three T-dualities, first along $x$, then along $y$ and finally along $z$, utilizing the results of [46] was worked out in [28]. The type IIA metric components were worked out in [28].

Now, any metric-compatible connection can be written in terms of the Levi-Civita connection and the contorsion tensor $\kappa$. Metric compatibility requires $\kappa \in \Lambda^1 \otimes \Lambda^2, \Lambda^6$ being the space of $n$-forms. Alternatively, in $d$ complex dimensions, since $\Lambda^2 \cong so(d)$, $\kappa$ also can be thought of as $\Lambda^1 \otimes so(d)$. Given the existence of a $G$-structure, we can decompose $so(d)$ into a part in the Lie algebra $g$ of $G \subset SO(d)$ and its orthogonal complement $g^0 = so(d)/g$. The contorsion $\kappa$ splits accordingly into $\kappa = \kappa^0 + \kappa^g$, where $\kappa^0 - \text{the intrinsic torsion - is part in } \Lambda^1 \otimes g^0$. One can decompose $\kappa^0$ into irreducible $G$ representations providing a classification of $G$-structures in terms of which representations appear in the decompostion. Let us consider the decomposition of $T^0$ in the case of $SU(3)$-structure. The relevant representations are $\Lambda^1 \sim 3 \oplus \bar{3}, g \sim 8, g^0 \sim 1 \oplus 3 \oplus \bar{3}$. Thus the intrinsic torsion, an element of $\Lambda^1 \otimes su(3)^\perp$, can be decomposed into the following $SU(3)$ representations:

$$
\begin{align*}
    \Lambda^1 \otimes su(3)^\perp &= (3 \oplus \bar{3}) \oplus ((1 \oplus 3) \oplus (3 \oplus \bar{3})) \\
    &= (1 \oplus 1) \oplus (8 \oplus 8) \oplus (6 \oplus \bar{6}) \oplus (3 \oplus \bar{3}) \oplus (3 \oplus \bar{3})' \\
    &\equiv W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5.
\end{align*}
$$

(16)

The $SU(3)$ structure torsion classes [51] can be defined in terms of $J, \Omega, dJ$ and $\Omega$ and the contraction operator $\mathcal{J} : \Lambda^2 T^* \otimes \Lambda^2 T^* \rightarrow \Lambda^{n-k} T^*$, $J$ being given by $J = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6$, and the $(3, 0)$-form $\Omega$ being given by $\Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)$.

The torsion classes are defined in the following way:

- $W_1 \leftrightarrow [dJ]^{(3,0)}$, given by real numbers $W_1 = W_1^+ + W_1^-$ with $d\Omega^+ \wedge J = \Omega^+ \wedge dJ = W_1^+ J \wedge J$ and $d\Omega^- \wedge J = \Omega^- \wedge dJ = W_1^- J \wedge J$;
- $W_2 \leftrightarrow [d\Omega]^{(2,2)}$; $(d\Omega^+)^{(2,2)} = W_2^+ J + W_2^- J$ and $(d\Omega^-)^{(2,2)} = W_2^- J + W_2^+ J$;
- $W_3 \leftrightarrow [dJ]^{(2,1)}$, defined as $W_3 = dJ^{(2,1)} - [J \wedge W_4]^{(2,1)}$. 

\[ \text{Springer} \]
be calibrated SYZ type IIA mirror metric were shown in [25].

In [26], we saw that the five $SU(3)$ structure torsion classes, in the UV–IR interpolating region/UV, implying a Klebanov–Strassler-like supersymmetry [52]. Locally around

\[ \sum_{1}^{2} N_{1}^{2} + g_{s} N_{2}^{3} + g_{t} \sqrt{N_{1} N_{2}} e^{-3r} + \frac{2}{3} + \frac{1}{2} \]

(17)

\( r \sim e^{\tau} \), such that

\[ \frac{2}{3} W_{1}^{3} = W_{2}^{3} \]

(18)

in the UV–IR interpolating region/UV, implying a Klebanov–Strassler-like supersymmetry [52]. Locally around \( \theta_{1} \sim \frac{1}{N_{2}^{2}}, \theta_{2} \sim \frac{1}{N^{12}} \), the type IIA torsion classes of the delocalized SYZ type IIA mirror metric were shown in [25] to be

\[ T_{S^2(3)}^{IIA} \subset W_{2}^{3} + W_{3}^{3} + W_{4}^{3} + W_{5}^{3} \]

\[ \sim \gamma_{2} g_{s} N_{2}^{3} + g_{t} N_{2}^{3} + N_{1}^{2} + g_{s} N_{2}^{3} + g_{t} N_{2}^{3} \]

\[ \approx \gamma W_{2}^{3} + W_{4}^{3} + W_{5}^{3} \]

(19)

indicative of supersymmetry after constructing the delocalized SYZ-mirror.

Apart from quantifying the departure from $SU(3)$ holonomy due to intrinsic contorsion supplied by the NS–NS three-form $H$, via the evaluation of the $SU(3)$ structure torsion classes, to the best of our knowledge for the first time in the context of holographic thermal QCD at finite gauge coupling in [25]:

(i) the existence of approximate supersymmetry of the type IIB holographic dual of [27] in the MQGP limit near the coordinate branch $\theta_{1} = \theta_{2} = 0$ was demonstrated, which apart from the existence of a special Lagrangian three-cycle (as shown in [25,26]) is essential for construction of the local SYZ type IIA mirror;

(ii) it was demonstrated that the large-$N$ suppression of the deviation of the type IIB resolved warped deformed conifold from being a complex manifold, is lost on being duality-chased to type IIA – it was also shown that one further fine tuning $\gamma_{2} = 0$ in $W_{1}^{IIA}$ can ensure that the local type IIA mirror is complex;

(iii) for the local type IIA $SU(3)$ mirror, the possibility of a surviving approximate supersymmetry was demonstrated, which is essential from the point of view of the end result of the application of the SYZ-mirror prescription.

We can get a one-form type IIA potential from the triple $T$-dual (along $x, y, z$) of the type IIB $F_{1,3,5}$ in [28], using which the following $D = 11$ metric was obtained in [28] ($u \equiv \frac{g_{*}}{\sqrt{\tau}}$):

\[ ds_{IIA}^{2} = e^{-2\rho_{IIA}} [g_{tt} dt^{2} + g_{IIA}(dx^{2} + dy^{2} + dz^{2}) + g_{uu} du^{2} + d\phi_{IIA}(\theta_{1,2}, \phi_{1,2}, \psi)] + e^{-\tau_{IIA}} (dx_{11} + A_{1} + A_{3} + A_{5})^{2} \]

\[ = \text{Black } M3 - \text{Brane} + O \left( \left( \frac{g_{*} M^{2} \log N}{N} \right) (g_{*} M) N_{f} \right) . \]

(21)

If $V$ is a seven-dimensional real vector space, then a three-form $\varphi$ is said to be positive if it lies in the $GL(7, \mathbb{R})$ orbit of $\varphi_{0}$, where $\varphi_{0}$ is a three-form on $\mathbb{R}^{7}$ which is preserved by the $G_{2}$-subgroup of $GL(7, \mathbb{R})$. The pair $(\varphi, g)$ for a positive three-form $\varphi$ and the corresponding metric $g$ constitute a $G_{2}$-structure. The space of $p$-forms decomposes as the following irreps of $G_{2}$ [53]:

\[ \Lambda^{1} = \Lambda_{1}, \]
\[ \Lambda^{2} = \Lambda_{2}^{2} \oplus \Lambda_{14}^{2}, \]
\[ \Lambda^{3} = \Lambda_{3}^{3} \oplus \Lambda_{27}^{3}, \]
\[ \Lambda^{4} = \Lambda_{4}^{4} \oplus \Lambda_{27}^{4}, \]
\[ \Lambda^{5} = \Lambda_{5}^{5} \oplus \Lambda_{14}^{5}, \]
\[ \Lambda^{6} = \Lambda_{6}^{6} . \]

(22)

The subscripts denote the dimension of the representation, and components of the same representation/dimensionality are isomorph. Let $M$ be a 7-manifold with a $G_{2}$-structure $(\varphi, g)$. Then the components of spaces of two-, three-, four-, and five-forms are given in [53,54]. The metric $g$ defines a reduction of the frame bundle $F$ to a principal $SO(7)$-sub-bundle $Q$, that is, a sub-bundle of oriented orthonormal frames. Now, $g$ also defines a Levi-Civita connection $\nabla$ on the tangent bundle $TM$, and hence on $F$. However, the $G_{2}$-invariant three-form $\varphi$ reduces the orthonormal bundle further to a principal $G_{2}$-sub-bundle $Q$. The Levi-Civita connection can be pulled back to $Q$. On $Q, \nabla$ can be uniquely decomposed as

\[ \nabla = \tilde{\nabla} + T \]

(23)

where $\tilde{\nabla}$ is a $G_{2}$-compatible canonical connection on $P$, taking values in the sub-algebra $g_{2} \subset \mathfrak{so}(7)$, while $T$ is a one-form taking values in $g_{2}^{+} \subset \mathfrak{so}(7)$; $T$ is known as the intrinsic torsion of the $G_{2}$-structure – the obstruction to the Levi-Civita connection being $G_{2}$-compatible. Now $\mathfrak{so}(7)$ splits under $G_{2}$ as
\[ \text{so}(7) \cong \Lambda_2^2 V \cong \Lambda_7^3 \oplus \Lambda_{14}^2. \]  
(24)

But \( \Lambda_{14}^3 \cong g_2 \), so the orthogonal complement \( g_1^2 \cong \Lambda_7^2 \cong V \). Hence \( T \) can be represented by a tensor \( T_{ab} \) which lies in \( W \cong V \otimes V \). Now, since \( \varphi \) is \( G_2 \)-invariant, it is \( \nabla \)-parallel. So, the torsion is determined by \( \nabla \varphi \) from Lemma 2.24 of [55]:

\[ \nabla \varphi \in \Lambda_1^1 \otimes \Lambda_7^3 \cong W. \]  
(25)

Due to the isomorphism between the \( \Lambda^a = 1, \ldots, 5 \), \( \nabla \varphi \) lies in the same space as \( T_{AB} \) and thus completely determines it. Equation (25) is equivalent to

\[ \nabla_A \varphi_{BCD} = T_A \varepsilon_{EBCD} \]  
(26)

where \( T_{AB} \) is the full torsion tensor. Equation (26) can be inverted to yield

\[ T_A^M = \frac{1}{24} \left( \nabla_{A} {\varphi}_{BCD} \right) \psi^{MBCD}. \]  
(27)

The tensor \( T_A^M \), like the space \( W \), possesses 49 components and hence fully defines \( V \). In general \( T_{AB} \) can be split into torsion components as

\[ T = T_1 g + T_2 \varphi + T_4 + T_27 \]  
(28)

where \( T_1 \) is a function and gives the 1 component of \( T \). We also have \( T_2 \), which is a one-form and hence gives the 7 component, and \( T_4 + T_{27} \) is traceless symmetric and gives the 27 component. Writing \( T_i \) as \( W_i \), we can split \( W \) as

\[ W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}. \]  
(29)

From [56], we see that a \( G_2 \) structure can be defined as

\[ \varphi_0 = \frac{1}{3!} f_{ABC} e^{ABC} = e^{-\varphi} c_{abc} e^{abc} + e^{-2\varphi} n \bigwedge \, \bigwedge \, x_{10}, \]  
(30)

where \( A, B, C = 1, \ldots, 6, 10; a, b, c = 1, \ldots, 6 \) and \( f_{ABC} \) are the structure constants of the imaginary octonions. Using the same, the \( G_2 \)-structure torsion classes were worked out around \( \theta_1 \approx 1/N, \theta_2 \approx N^{-1} \) in [25] to be found to be

\[ T_{G_2} \in W_{14} \oplus W_{27} \sim \frac{1}{(g_1 N)^{\frac{1}{2}}} + \frac{1}{(g_1 N)^{\frac{1}{2}}}. \]  
(31)

Hence, the approach of the seven-fold, locally, to having a \( G_2 \) holonomy \( (W_1^{G_2} \otimes W_2^{G_2} = W_3^{G_2} = W_4^{G_2} = 0) \) is accelerated in the MQGP limit.

As stated earlier, the global uplift to M-theory of the type IIB background of [27] is expected to involve a seven-fold of the \( G_2 \) structure (not a \( G_2 \)-holonomy due to the non-zero \( G_4 \)). It is hence extremely important to be able to see this, at least locally. It is in this sense that the results of [28] are of great significance; as one explicitly sees, for the first time, in the context of holographic thermal QCD at finite gauge coupling, though locally, the aforementioned \( G_2 \) structure has been worked out in terms of the non-trivial \( G_2 \)-structure torsion classes.

3 0++(+++...) Glueball spectrum from type IIB supergravity background

In this section we discuss the 0++ glueball spectrum by solving the dilaton wave equation in the type IIB background discussed in Sect. 2. The type IIB metric as given in Eq. (4) with the warp factor \( h \) given in (7) can be simplified by working around a particular value of \( \theta_1 \) and \( \theta_2 \): \( \theta_1 = \frac{1}{N}, \theta_2 = \frac{1}{N} \), then keeping terms up to NLO in \( N \) in the large \( N \) limit. The simplifed type IIB metric is given as

\[ \text{ds}^2 = g_{tt} \text{d}t^2 + g_{s1s1} (\text{d}x_1^2 + \text{d}x_1^2 + \text{d}x_2^2) \]
\[ + g_{rr} \text{d}r^2 + \sqrt{r^2} dM_5^2, \]  
(32)

with the components \( g_{tt}, g_{s1s1} (= g_{s2s2} = g_{s3s3}), g_{rr} \) as
given now:

\[ g_{tt} = \frac{2}{\sqrt{\pi} \sqrt{N} \sqrt{g_s}}, \]
\[ g_{s1s1} = \frac{r^2 (1-B(r))}{2 \sqrt{\pi} \sqrt{N} \sqrt{g_s}}, \]
\[ g_{rr} = \frac{2 \sqrt{\pi} \sqrt{N} (r^2 - 3a^2) (B(r) + 1) \sqrt{g_s}}{r^4 \left( 1 - \frac{r_h^2}{r^2} \right)}, \]  
(33)

where

\[ B(r) = \frac{\log(r) (12N_f g_s \log(r) + 6N_f g_s)}{32 \pi^2 N}, \]

and \( a \) being the resolution parameter is proportional to the horizon radius \( r_h \). Hence while computing the spectrum with a cut-off in the radial direction and no horizon, we must put both \( r_h \) and \( a \) to zero in the above equation.

Moreover, the dilaton profiles with/without the black-hole are given below as

\[ r_h \neq 0 : e^{-\Phi} = \frac{1}{g_s} - \frac{N_f}{8 \pi} \log(r^6 + a^2 r^4) \]
\[ - \frac{N_f}{2 \pi} \log \left( \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right), \quad r < R_{DS}/D_S, \]
\[ e^{-\Phi} = \frac{1}{g_s}, \quad r > R_{DS}/D_S; \]

\[ r_h = 0 : e^{-\Phi} = \frac{1}{g_s} - \frac{3 N_f}{4 \pi} \log r \]
\[ - \frac{N_f}{2 \pi} \log \left( \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right), \quad r < |\mu_{Ouyang}| \frac{2}{3}, \]
\[ e^{-\Phi} = \frac{1}{g_s}, \quad r > |\mu_{Ouyang}| \frac{2}{3}. \]  
(34)
Again working around the particular choices of $\theta_1$ and $\theta_2$ the above profile can be simplified up to NLO in $N$. The dilaton equation that has to be solved is given as

$$\partial_\mu (e^{-2\phi} \sqrt{g} g^{\mu\nu} \partial_\nu \phi) = 0.$$  \hfill (35)

To solve the above dilaton equation we assume $\phi$ in (35) to be of the form $\phi = e^{ik_+ \tilde{\phi}(r)}$. Now, with this, we adopt the WKB method to get to the final result. The first step towards the WKB method is to convert the glueball equation of motion into a Schrödinger-like equation. Then the WKB quantization condition can be applied on the potential term obtained from the Schrödinger-like equation. For the $(0^{++})$ glueball spectrum with no horizon ($r_h = 0$), one of the solutions was obtained by imposing the Neumann boundary condition at the cut-off.

3.1 $r_h \neq 0$ using WKB quantization method

In the following we discuss the spectrum of the $0^{++}$ glueball in the type IIB background with a black hole, implying a horizon of radius $r_h$ in the geometry. The results corresponding to the coordinate and field redefinitions of [32] are discussed below.

Using the redefinitions of [32] with $r = \sqrt{y}$, $r_h = \sqrt{y_h}$ and finally $y = y_h(1 + e^z)$, the $0^{++}$ EOM (35), with $k^2 = -m^2$, can be written as

$$\partial_z (E_z \partial_z \tilde{\phi}) + y_h^2 F_z m^2 \tilde{\phi} = 0,$$  \hfill (36)

where at leading order in $N$, $E_z$ and $F_z$ are given with $L = (4\pi g_s N)^{1/4}$ as

$$E_z = \frac{1}{128 \pi^2 L^5 g_s^2} (e^z + 2) y_h^2 (8 \pi + g_s N_f \log 256 + 2 N_f g_s \log N - 3 g_s N_f \log \left[(e^z + 1) y_h\right]) \times \left[2 (3 a^2 [4 \pi + g_s N_f \log 16 - 6]) + (e^z + 1) y_h (8 \pi + g_s N_f \log 256) + 2 g_s N_f \log N \{3 a^2 + 2 (e^z + 1) y_h\} - 3 g_s N_f \{3 a^2 + 2 (e^z + 1) y_h\} \log [e^z + 1) y_h]\right].$$  \hfill (37)

$$F_z = \frac{1}{128 \pi^2 g_s^2 L y_h (e^z + 1)^4} e^z (4 \pi + g_s N_f \log 16 + g_s N_f \log N - \frac{3}{2} g_s N_f \log [e^z + 1) y_h]) \times \left[(e^z + 1) y_h (8 \pi + g_s N_f \log N + g_s N_f \log 256 - 3 g_s N_f \log [e^z + 1) y_h]) - 3 a^2 (4 \pi + g_s N_f + g_s N_f \log N + g_s N_f \log 16 - \frac{3}{2} g_s N_f \log (e^z + 1) y_h)\right].$$  \hfill (38)

Now, redefining the wave function $\tilde{\phi}$ as $\psi(z) = \sqrt{E_z} \tilde{\phi}(z)$ Eq. (36) reduces to a Schrödinger-like equation

$$\left(\frac{d^2}{dy^2} + V(z)\right) \psi(z) = 0$$  \hfill (39)

where the potential $V(z)$ has a rather cumbersome expression, which we will not explicitly write out. The WKB quantization condition becomes $\int_{z_1}^{z_2} \sqrt{V(z)} = (n + \frac{1}{2})\pi$ where $z_{1,2}$ are the turning points of $V(z)$. We will work below with a dimensionless glueball mass $\tilde{m}$ assumed to be large and defined via $m = \tilde{m} \frac{L}{2 L}$. To determine the turning points of the potential $V(z)$, we consider two limits of the same $- r \in [r_h, \sqrt{3} a (r_h) \approx \sqrt{3} b r_h] \cup [\sqrt{3} b r_h, \infty] \equiv$ (IR, IR/UV interpolating + UV). In the IR region, we have to take the limit $z \rightarrow - \infty$. Now in the large-$\tilde{m}$ and large-log $N$ limit this potential at small $z$ can be shown to be given as:

$$V(z \ll 0) = \frac{1}{8} (1 - 3 b^2 e^z \tilde{m}^2) + \mathcal{O}\left(e^{2z}, \left(\frac{1}{\tilde{m}}\right)^2, \frac{1}{\log N}\right) < 0$$  \hfill (40)

Hence, there are no turning points in the IR region.

Now, in the UV region, apart from taking the large-$z$ limit we also have to take $N_f = M = 0$, to get

$$V(z \gg 1) = - \frac{3 (b^2 + 3) (\lambda_0 \tilde{m}^2 + 3)}{4 \lambda_0 e^z} \frac{3 b^2 + \lambda_0 \tilde{m}^2 + 6}{4 \lambda_0 e^z} - 1 + \mathcal{O}(e^{-3z}).$$  \hfill (41)

The turning points of (41) are $z_1 = \frac{1}{2} (3 b^2 - \sqrt{9 b^4 - 6 b^2 (7 y_h \tilde{m}^2 + 18) + y_h \tilde{m}^4 - 36 y_h \tilde{m}^2 - 108 + y_h \tilde{m}^2 + 6)}$, $z_2 = \frac{1}{2} (3 b^2 + \sqrt{9 b^4 - 6 b^2 (7 y_h \tilde{m}^2 + 18) + y_h \tilde{m}^4 - 36 y_h \tilde{m}^2 - 108 + y_h \tilde{m}^2 + 6})$, which in the large-$\tilde{m}$ limit is given as $[z_1 = (3 + 3 b^2) + \mathcal{O}(\frac{1}{\tilde{m}^2})]$. $z_2 = \frac{z_2^2}{4} = \frac{3 (2 + 2 \lambda_0)}{4} + \mathcal{O}(\frac{1}{\tilde{m}^2})$.

To obtain a real spectrum, one first notes

$$\sqrt{V(z \gg 1, N_f = M = 0; b = 0.6)} = \sqrt{0.25 e^{-z} - 1.02 e^{-2z} \tilde{m} + \mathcal{O}\left(e^{-3z}, \frac{1}{\tilde{m}^2}\right)}$$  \hfill (42)

and
∫ \sqrt{V(z)} dz = \frac{\sqrt{e^{-2z}(0.25e^z - 1.0023)}}{\sqrt{0.25e^z - 1.0023}} \left[ 0.124856e^z \tan^{-1} \left( \frac{0.124856e^z - 1.00115}{\sqrt{0.25e^z - 1.0023}} \right) - 1.\sqrt{0.25e^z - 1.0023} \right] \cdot \frac{(0.123768e^z \tan^{-1} \left( \frac{0.123768e^z - 1.00099}{\sqrt{0.25e^z - 1.02}} \right) - 1.\sqrt{0.25e^z - 1.02})}{\sqrt{0.25e^z - 1.02}}.

Therefore

\int_{0.508}^{1} \sqrt{V(z)} = 0.39\tilde{m} - 2 = \left( n + \frac{1}{2} \right) \pi.

yielding

n_{n_{n^+}} = 9.18 + 8.08n.

3.2 Glueball mass for \( r_h = 0 \) and an IR cut-off \( r_0 \)

In this background the type IIB metric and the dilaton profile has to be modified by the limit \( r_h \to 0 \) and hence with \( a \to 0 \). This time also we have provided two solutions to the dilaton equation. The first solution was obtained by first following the redefinition of the variables in [64] and then imposing a Neumann boundary condition at the radial cut-off. For the other solution we again consider the WKB method after a redefinition of variables as given in [32].

3.2.1 Neumann boundary condition at \( r_0 \)

Following [64], we redefine the radial coordinate as \( z = \frac{1}{r} \).

With this change of variable, the radial cut-off now maps to \( z = z_0 \), with \( z_0 = \frac{1}{r_0} \). The dilaton equation using the metric and the dilaton background in the limit \( (r_h, a) \to 0 \) is given as

\[ e^{2U} \partial_z (e^{-2U} \partial_z \tilde{\phi}) + \left( \frac{4g_sN\pi}{(1 - B(z))} \right) (m^2) \tilde{\phi} = 0, \]

where up to NLO in \( N \) we have

\[ e^U = \frac{8 \times 21^{1/4}g_s^{13/8}N^{13/8}N^5/8e^{3/2}}{4(\pi \log(4)Nf g_s + \pi) + 2\pi N_f g_s \log(N) + 3N_f g_s \log(z)} - 15M^2z^{3/2}g^{21/8} \log(\log(z))(-12Nf g_s \log(z) + 6NF g_s + N_f g_s (-\log(N)) - \log(16)N_f g_s + 8\pi)}{8 \times 3^{3/4}N^{3/8}N^3/8(3N_f g_s \log(z) + 2\pi N_f g_s \log(N) + 4(\pi \log(4)N_f g_s + \pi)).

Now to convert the above equation in a one-dimensional Schrödinger-like form we introduce a new field variable \( \psi(z) = e^{-U} \tilde{\phi}(z) \).

With this one can write the equation in the following Schrödinger-like form:

\[ \frac{d^2 \psi(z)}{dz^2} = V(z) \psi(z). \]

The potential \( V(z) \), in the large-\( N \) large-log \( N \) limit is given as

\[ V(z) = 4\pi g_s m^2 N + \frac{6}{\pi z^2 \log(N)} - \frac{15}{4z^2} + O \left( \frac{1}{(\log N)^{2.5}} \right). \]

Hence, the Schrödinger equation becomes

\[ \psi''(z) + \psi(z) \left( 4\pi g_s m^2 N + \frac{6}{\pi z^2 \log(N)} - \frac{15}{4z^2} \right) = 0, \]

whose solution is given as follows:

\[ \psi \left( z \right) = c_1 \sqrt{z} J_{Nf g_s \log(z)} (2\sqrt{g_s m \sqrt{N} \sqrt{\pi} z}) + c_2 \sqrt{z} Y_{Nf g_s \log(z)} (2\sqrt{g_s m \sqrt{N} \sqrt{\pi} z}), \]

Finite\( n(z) \) at \( z = 0 \) requires setting \( c_2 = 0 \). Then imposing the Neumann boundary condition on \( \phi(z) \) at \( z = z_0 \) implies

\[ \phi(z_0) = \left( -2\pi g_s N_f \log(z_0) + 3N_f \log(z_0) - N_f \log(\log(z_0)) + 4(\pi g_s \log(4) + \pi) \right)^2 \times \frac{4 \sqrt{2} \pi^{13/8}g_s^{13/8}N^{3/8}z_0^{3/2}}{ \log(z_0) - 2\pi g_s \log(\log(z_0)) + 4\pi g_s \log(4) + 4\pi \log(z_0) + N_f \log(4\pi g_s \log(4) + \pi)) \times \frac{J_{N_f g_s \log(z_0)}}{ \sqrt{\pi \log(z_0)}} \left( 2\sqrt{m \sqrt{N} \sqrt{\pi} z_0} \right) \]
\[ \frac{1}{2} x_0 J \sqrt{4 - \frac{\pi}{N \log(N)}} - 1(\chi_0) - \frac{1}{2} x_0 J \sqrt{4 - \frac{\pi}{N \log(N)}} + 1(\chi_0) + 2 J \sqrt{4 - \frac{\pi}{N \log(N)}}(\chi_0) = 0, \]  

(53)

where \( x_0 = 2\sqrt{g_s N \pi m z_0} \). The graphical solution points out that the ground state has a zero mass and the lightest (first excited state) glueball mass is approximately given by \( 3.71 \frac{m}{N} \) (Fig. 1).

### 3.2.2 WKB method: including the non-conformal/NLO (in \( N \)) corrections

Again following the redefinition of variables as given in [32]: \( r = \sqrt{N}, y = y_0(1 + e^z) \), where \( r_0 = \sqrt{y_0} \) is the radial cut-off, and using the type IIB metric as well as the dilaton profile in the limit \( (r h, a) \to 0 \), the dilaton equation (35) can be written as

\[
V(\varphi) = -\frac{3}{4} e^{2z} g_s^2 2\log N M^2 \tilde{m}^2 N_f \log(y_0) \]

\[
+ \frac{e^{2z}(2g_s N_f(\log N(\tilde{m}^2 - 3) + \tilde{m}^2 \log(16) + 12 - 6\log(4) + 4\pi(\tilde{m}^2 - 3)) - 3g_s(\tilde{m}^2 - 3)N_f \log(y_0))}{4(2g_s N_f(\log N + \log(16)) - 3g_s N_f \log(y_0) + 8\pi)}
\]

\[ + O\left(\frac{1}{N^2}\right). \]

(54)

\[ \partial_z(C_z \partial_z \tilde{\phi}) + y_0^2 D_z \tilde{m}^2 \tilde{\phi} = 0, \]

where \( C_z \) and \( D_z \) are given up to NLO in \( N \) by

\[
C_z = \frac{1}{32768\sqrt{2} \pi^{21/4} N^{9/4} g_s^{13/4}} e^{-z}(e^z + 1) \]

\[
\times \frac{y_0^3(8\pi + 2g_s N_f \log N + 4g_s N_f \log 4 - 3g_s N_f \log(y_0)(e^z + 1))^{2}(128\pi^2 N^2)}{18g_s M^2(-8\pi + g_s N_f(16 - 6))}
\]

The domain of integration over which \( V(\varphi) > 0 \) can be shown to be \( \{ \log \left( \frac{3g_s^2 M^2 N_f \log N \log y_0}{64\pi^2 N^2} + \frac{1}{m} \right), \log(\delta^2 - 1) \} \), where \( m = \delta \sqrt{N^2} \). Note that the IR cut-off \( r_0 \) or \( y_0 \) is not put in by hand but is proportional to the Ouyang embedding parameter (corresponding to the embedding of the flavor D7-branes) raised to the two-third power. The proportionality constant \( \delta \) could be determined, as discussed in point number 5 in Sect. 8, by matching with lattice calculations and turns out to be \( O(1) \).

Expanding \( V \) first in \( N \) and then in \( \tilde{m} \) and then integrating over the above mentioned domain, one gets the following quantization condition:

\[
\int \log(\delta^2 - 1) m \left( \frac{3g_s^2 M^2 N_f \log N \log y_0}{64\pi^2 N^2} + \frac{1}{m} \right) \sqrt{V(\varphi)}
\]

\[
= \frac{1}{2} (\delta^2 - 1) \tilde{m} \left( \frac{3g_s^2 M^2 N_f \log N \log y_0}{64\pi^2 N^2} + 1 \right) - 0.75
\]

\[ + O\left(\frac{1}{N}, \frac{1}{\tilde{m}^2}\right) \]

(59)

yielding

\[
m_{0++}^N = \left( \frac{6.28319 \pi + 4.64159 (1 - \frac{0.00474943 g_s^2 M^2 N_f \log(y_0)}{\delta^2 - 1})}{\delta^2} \right). \]

(60)
4 Scalar glueball (0−+(*…*)) masses

As $\text{Tr}(F\tilde{F})$ has $P=-, C=+$ and it couples to $A_1$ in the Wess–Zumino term for the type IIA D4–brane: $\int_{\Sigma_{4,1}} A_1 \wedge F \wedge F$, one considers $A_1$’s EOM:

$$\partial_\nu \left( \sqrt{g} g^{\mu\nu} s_{\text{IIA}} A_{\mu} \right) = 0,$$

(61)

$$U \left( 1 - \frac{32/3 \pi g_s m^2 N}{5r_h^2} \right)^2 \frac{5r_h}{r_h - 5} = - \frac{- r_h^3}{\sqrt{g} g^{\mu\nu} s_{\text{IIA}} A_{\mu}}$$

$$\begin{align*}
\psi^{(0)} \left( 1 - \frac{32/3 g_s m^2 N}{5r_h^2} \right) + \log \left( \frac{5r_h}{r_h - 5} \right) + 2\gamma - 1
\end{align*}$$

$$\begin{align*}
\left( r - r_h \right) \left( 5r_h^2 - \frac{32/3 \pi g_s m^2 N}{5r_h^2} \right) \left( 2\psi^{(0)} \left( 1 - \frac{32/3 g_s m^2 N}{5r_h^2} \right) + 2\log \left( \frac{5r_h}{r_h - 5} \right) + 4\gamma - 5 \right)
\end{align*}$$

$$+ \mathcal{O}(r - r_h^2),$$

(66)

where $\mu, \nu, \ldots = a(\equiv 0, 1, 2, 3), r, \alpha(\equiv 5, \ldots, 9)$. Like [57], assume $A_\mu = \delta_\mu^a a_\mu(r) e^{ik \cdot x}, k^2 = -m^2$ as the fluctuation about the type IIA $A_1$ that was worked out in [28]. The $0^{−+}$ EOM then reduces to

$$\begin{align*}
&\sqrt{g} g^{\mu\nu} s_{\text{IIA}} A_{\mu} \left( e^{\nu^a} \right) + \partial_\nu \left( \sqrt{g} g^{\mu\nu} s_{\text{IIA}} A_{\mu} \right) a_\nu(r) \\
&+ \sqrt{m^2} \sqrt{g} g^{\mu\nu} s_{\text{IIA}} A_{\mu} a_\nu(r) = 0.
\end{align*}$$

(62)

4.1 Neumann/Dirichlet boundary conditions

4.1.1 $r_h \neq 0$

Then taking the large-N limit followed by a small-$\theta_1$, small-$\theta_2$ limit one can show that the equation of motion (62) yields

$$\begin{align*}
8\pi (r^4 - r_h^4) a_\nu''(r) + \frac{32\pi (r^4 - 2r_h^4) a_\nu'(r)}{\sqrt{3}r} \\
- 32 \sqrt{3} \pi^2 g_s m^2 N a_\nu(r) = 0.
\end{align*}$$

(63)

Working near $r = r_h$, approximating (64) by

$$\begin{align*}
&\frac{32\pi r_h^3 (r - r_h) a_\nu''(r)}{\sqrt{3}} \\
&+ \left( \frac{160\pi r_h^2 (r - r_h)}{\sqrt{3}} - \frac{32\pi r_h^3}{\sqrt{3}} \right) a_\nu'(r) \\
&- 32 \sqrt{3} \pi^2 g_s m^2 N a_\nu(r) = 0,
\end{align*}$$

(64)

whose solution is given as follows:

$$a_\nu(r) = c_1 U \left( \frac{32/3 g_s m^2 N}{5r_h^2}, 1, \frac{5r_h}{r_h - 5} \right)$$

$$+ c_2 L \frac{32/3 g_s m^2 N}{5r_h^2} \left( \frac{5r_h}{r_h - 5} \right).$$

(65)

Imposing the Neumann boundary condition, $a_\nu'(r = r_h) = 0$, utilizing

$$r_h = T \sqrt{4\pi g_s N}$$

up to LO in $N$ – requires $c_2 = 0$ and

$$\frac{32/3 m^2}{20T^2} = n,$$

(67)

implying

$$m_n^{0−+} = \frac{2\sqrt{3} \sqrt{\pi T}}{\sqrt{3}}.$$ (68)

One can show that imposing the Dirichlet boundary condition $a_\nu(r = r_h) = 0$ yields the same spectrum as (68). If the temperature $T$ gets identified with $a$ of [57], then the ground state, unlike [57], is massless; the excited states for lower $n$ are closer to $a = 0$ and the higher excited states are closer to $a \to \infty$ in [57].

4.1.2 $r_h = 0$ Limit of (63)

The $r_h = 0$ limit of (63) gives

$$\sqrt{3} r^4 a_\nu''(r) - 4\sqrt{3} r^3 a_\nu'(r) + 3 \sqrt{3} m^2 r^2 a_\nu(r) = 0,$$

(69)

which near $r = r_0$ yields

$$a_\nu(r) = (4r - 3m)^{5/4} \left( c_1 U \left( \frac{5}{4}, \frac{m^2}{4\sqrt{3}}, \frac{9}{4}, \frac{3r}{m} - \frac{9}{4} \right) \\
+ c_2 L \frac{5}{4} (32/3 m^2 - 15) \left( \frac{3r}{m} - \frac{9}{4} \right) \right).$$

(70)

Imposing the Neumann boundary condition on (70) yields

$$m_n^{0−+}(r_h = 0) = 0,$$

$$m_n^{0−+}(r_h = 0) \approx 3.4 \frac{r_0}{L^2},$$

$$m_n^{0−+}(r_h = 0) \approx 4.35 \frac{r_0}{L^2}.$$ (71)
4.1.3 WKB Quantization for \( r_h \neq 0 \)

The potential corresponding to the Schrödinger-like equation à la [32], substituting \( m = \tilde{m} \frac{\sqrt{N}}{L} \), is given by

\[
V = e^{-z}(4 \frac{3^{2/3} \tilde{m}^2}{2} (3e^z + e^{2z} + 2) - 64e^z - 108e^{2z} - 25e^{3z} + 96) \frac{1}{16(e^z + 1)^2(e^z + 2)^2}.
\]  

Therefore, in the IR region:

\[
V(z \ll 0) = \left( -\frac{3}{16} 3^{2/3} \tilde{m}^2 - \frac{11}{2} \right) e^{2z} + \left( \frac{1}{8} 3^{2/3} \tilde{m}^2 + \frac{3}{2} \right) e^z + O(e^{-3z}),
\]

the turning points being given by \(-\infty\) and \( \log(\frac{3^{2/3} \tilde{m}^2 + 32}{3^{2/3} \tilde{m}^2 + 88}) \approx -0.405 \). But only \( z \in (-\infty, -2.526) \) corresponds to the IR region in our calculations. So,

\[
\int_{-\infty}^{-2.526} \sqrt{V} = 0.283 \tilde{m} = \left( n + \frac{1}{2} \right) \pi,
\]

from which one obtains

\[
n_n^{0+}(T, \text{IR}) = 5.56(1 + 2n) \frac{r_h}{L^2}.
\]  

Similarly, in the UV region

\[
V(UV, T) = \left( \frac{1}{4} \frac{3^{2/3} \tilde{m}^2}{2} + \frac{21}{8} \right) e^{-z} + \left( \frac{9}{16} - \frac{3}{4} \frac{3^{2/3} \tilde{m}^2}{2} \right) e^{-2z} - \frac{25}{16} + O(e^{-3z}),
\]

whose turning points are \( \log(\frac{1}{\sqrt{3}} (2 \frac{3^{2/3} \tilde{m}^2}{2} \pm \sqrt{6\sqrt{2} \frac{3^{2/3} \tilde{m}^2}{2} + 116 + 1} - 116 + 111 + 21)) = \log(3 + O(1 / \tilde{m}^2)) \), \( \log(0.35 \tilde{m}^2 - 1.32 + O(1 / \tilde{m}^2)) \). Now \( \sqrt{V(UV, T)} = \frac{1}{2} \frac{3^{2/3} \tilde{m}}{2} e^{-z} / \sqrt{e^{-z} - \frac{3}{2} + O(\frac{1}{\tilde{m}^2})} \). Therefore,

\[
\int_{\log(0.35 \tilde{m}^2 - 1.32)}^{\log 3} \sqrt{V(UV, T)} = 0.654 \tilde{m} - 2.5 = \left( n + \frac{1}{2} \right) \pi,
\]

from which one obtains

\[
m_n^{0+}(UV, T) = (6.225 + 4.804n) \pi T.
\]

4.1.4 WKB quantization at \( r_h = 0 \)

The ‘potential’ is given by

\[
V(0^{-+}, r_h = 0) = \frac{4^{3/2} \tilde{m}^2 e^{2z} + e^{(z+1)(3e^z+12)^2}}{e^{e^z} + 1} - 2(e^z + 1)(14e^z + 25e^{2z} + 4) + O\left( \frac{8 \tilde{m}^2}{N} \right).
\]

Therefore, in the IR region

\[
V(\text{IR}, r_h = 0) = -\frac{1}{2} e^{\frac{3}{4} e^z} - \left( \frac{9}{8} + \frac{3}{4} \tilde{m}^2 \right) e^{2z} + O(e^{3z}),
\]

and in the IR region, the domain of integration becomes \( \log(\frac{2}{3} \tilde{m}^2), \log(\delta^2 - 1) \): \( \int_{\log(\delta^2 - 1)}^{\log(2)} \sqrt{V(\text{IR}, r_h = 0)} = \frac{\sqrt{3}}{2} (\delta^2 - 1)^{1/2} - 1.126 \), yielding

\[
m_n^{0+}(\text{IR}, r_h = 0) = \left( 3.72 + 4.36n \right) \frac{r_0}{L^2}.
\]

Also, in the UV region

\[
V(UV, r_h = 0) = \left( -\frac{3}{4} \frac{3^{2/3} \tilde{m}^2}{2} - \frac{103}{16} \right) e^{-2z} + \left( \frac{1}{4} \frac{3^{2/3} \tilde{m}^2}{2} + \frac{21}{8} \right) e^{-z} - \frac{25}{16},
\]

whose turning points are \( \frac{1}{\sqrt{3}} (2 \frac{3^{2/3} \tilde{m}^2}{2} \pm \sqrt{12 \sqrt{3} \tilde{m}^2 - 21632 \frac{3^{2/3} \tilde{m}^2}{2} - 2134 + 21}) = (3 + O(1 / \tilde{m}^2), \frac{4}{3} \frac{3^{2/3}}{\sqrt{3} \tilde{m}} \tilde{m}^2 - \frac{33}{\sqrt{3}} \tilde{m}) \), yielding \( \frac{3}{2} \frac{3^{2/3} \tilde{m}^2}{2} - \frac{33}{\sqrt{3}} \tilde{m} \). \( \sqrt{V(UV, r_h = 0)} = \frac{\pi}{43^4} = (n + \frac{1}{2}) \pi, \) from which one obtains

\[
m_n^{0+}(UV, r_h = 0) = 4.804 \left( n + \frac{1}{2} \right) \frac{r_0}{L^2}.
\]

5 Glueball (0-(*...*)) masses

5.1 \( r_h \neq 0 \) and Neumann/Dirichlet boundary conditions at \( r = r_h \)

Given the Wess–Zumino term \( A^{\mu} d^{\alpha \beta \gamma \delta} \text{Tr}(F_{\mu \nu}^a F_{\nu \gamma}^b F_{\gamma \delta}^c F_{\delta \mu}^d) \) and the two-form potential \( A_{\mu \nu} \) being dual to a pseudoscalar, for \( r_h \neq 0 \), corresponding to QCD3, one writes down the EOM for the fluctuation \( \delta A \). The \( B_{MN}, CMN \) EOMs are

\[
D^M H_{MN P} = \frac{2}{3} F_{NP QR S} F_{QR S},
\]

\[
D^M F_{MNP} = -\frac{2}{3} F_{NP QR S} H_{QR S},
\]

\[ \text{Springer} \]
or defining $A_{MN} = B_{MN} + iC_{MN}$, (85) can be rewritten as

$$D^M \delta_{[M A_N P]} = -\frac{2i}{3} F_{NPQR} \delta^{[Q A_{RS}]}.$$

(86)

Now, $A_{MN} \rightarrow A^{(0)}_{MN} + \delta A_{MN}$ with $\delta A^{MN} = \delta^M_2 \delta^N_3 \delta A_{23}$, the EOM satisfied by $\delta A_{23}(\omega, 0, 1, 2, 3, r) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} g_{22} G(r)$ reduces to

$$\partial_y (\sqrt{-g} g^{22} \delta^{\mu\nu} \partial_y \delta A_{23}) = 0.$$  

(87)

Assuming $a = (\alpha + \beta g_M^2 \frac{m^2}{2} + \gamma g_N^2 \log r_N) r_N h [33, 42]$ for $(\alpha, \beta, \gamma) = (0.6, 4, 4)$ [33] and $k^M = (\omega, k_1, 0, 0) : k^2 = -m^2$, and defining $G(r) \equiv g_{22} G(r)$ the EOM for $G(r)$ is

implying

$$\frac{G'(r)}{g_{22}} = \frac{1}{\Gamma \left( \frac{-2a_2 + b_1 + \sqrt{b_1^2 - 4b_2}}{2\sqrt{b_1^2 - 4b_2}} \right)} \times \sum_{n=-1}^{\infty} a_n (N, M, N_f, g_s, r_h) (r - r_h)^n.$$  

(92)

Assuming $c_2 = 0$, the Neumann boundary condition at $r = r_h$ can be satisfied by setting the argument of the gamma function to a negative integer $n$. It runs out set-

$$G''(r) + G'(r) \left( \frac{3g_M^2 M^2 (g_N^2 \log N - 6 + \log(16)) - 24g_N^2 \log(r) - 8\pi}{64\pi^2 N r} \right)$$

$$- \frac{75r_N^2 (4g_M^2 \log(r_N) + 4g_N^2 M^2 + 0.6N^2) + 5r^4 - r_h^4}{N^2 r^3} + \frac{5r^4 - r_h^4}{N^2 r^3}$$

$$- \frac{g_M^2 G(r)}{4\pi r^2 (r^4 - r_h^4)} \left( \frac{3r_N^2 (4g_M^2 \log(r_N) + 4g_N^2 M^2 + 0.6N^2)}{N^2} - r^2 \right)$$

$$\times 36g_M^2 M^2 N_f \log(2r) - 3g_N^2 M^2 \log(r)$$

$$\times (g_N \log N - 6 + \log(16)) - 8\pi + 16\pi^2 N = 0.$$  

(88)

The EOM (88) near $r = r_h$ can be approximated by

$$G''(r) + \left( b_1 + \frac{1}{r - r_h} \right) G'(r) + G(r) \left( \frac{a_2}{r - r_h} + b_2 \right) = 0,$$  

(89)

where

$$b_1 = \frac{g_M^2 (9.005 \log N - 0.015) N_f + (-0.114g_N M - 360.1) \log(r_N) - 360.119}{r_N h} - 24.5,$$

$$a_2 = \frac{g_M^2 m^2 (\log(r_N) (g_M (0.24 \log N - 0.775) N_f - 2279.99) - 2.88g_N \log^2(r_N) - 2.88g_N \log(r_N) - 2273.96)}{r_N h^3} - 0.251g_M^2 N_f,$$

$$b_2 = \frac{2.04g_M^2 m^2 (\log(r_N) (g_M (8.158 - 3.42 \log N) + 4065.38) + g_N (0.12 \log N - 0.387) N_f + 41.04g_N M \log^2(r_N) + 3976.41)}{r_N h^4}$$

$$+ \frac{7.163g_M^2 m^2}{r_N h^4}.$$  

(90)

The solution to (89) is given by

$$G(r) = e^{\frac{1}{2} \left( -\sqrt{b_1^2 - 4b_2} - b_1 \right)} \left[ c_1 U \left( \frac{2a_2 - b_1 - \sqrt{b_1^2 - 4b_2}}{2\sqrt{b_1^2 - 4b_2}}, \frac{b_1^2 - 4b_2}{2\sqrt{b_1^2 - 4b_2}} \right) \right.$$

$$\times 1, \sqrt{b_1^2 - 4b_2} r - \sqrt{b_1^2 - 4b_2} r_h \right]$$

$$+ c_2 L \left( \frac{2a_2 - b_1 - \sqrt{b_1^2 - 4b_2}}{2\sqrt{b_1^2 - 4b_2}}, \frac{b_1^2 - 4b_2}{2\sqrt{b_1^2 - 4b_2}} \right) \left( r \sqrt{b_1^2 - 4b_2} - r_h \sqrt{b_1^2 - 4b_2} \right),$$

$$T^2 \left( \frac{1.5 \sqrt{187219 \times 10^{12} T^2^2 - 166774 \times 10^9 m^2}}{T \sqrt{187219 \times 10^{12} T^2} - 166774 \times 10^9 m^2} - 675867 \right) + 265.153 m^2$$

$$= -n \in \mathbb{Z}^{-} \cup \{0\},$$  

(93)

the solutions to which are given now:

$$m_{0-} = 32.461 T,$$

$$m_{0+} = 32.887 T,$$

$$m_{0-} = 32.989 T,$$
\[ m_{0-} = 33.0333T, \]
\[ m_{0-} = 33.055T. \]  
\[ \text{(94)} \]

One can show that one obtains the same spectrum as in (94) after imposing the Dirichlet boundary condition \( G(r = r_h) = 0. \)

5.2 \( r_h = 0 \) Limit of (88)

We have

\[ G''(r) + \frac{G'(r)(3g_s M^2 (g_s N_f (\log N - 6 + \log(16)) - 24g_s N_f \log(r - 8\pi)) + 320)}{64r} \]
\[ + \frac{g_s m_0^2 (36g_s M^2 N_f \log^2(r) - 3g_s M^2 \log(r)(g_s N_f (\log N - 6 + \log(16)) - 8\pi) + 16\pi^2 N)}{4\pi^2} G(r) = 0. \]
\[ \text{(95)} \]

The EOM (95) near \( r = r_0 - \text{IR cut-off at } r_h = 0 \) reduces to

\[ G''(r) + (\alpha_1 + \beta_1 (r - r_0))G'(r) \]
\[ + (\alpha_2 + \beta_2 (r - r_0))G(r) = 0 \]  
\[ \text{(96)} \]

where

\[ \alpha_1 = \frac{3g_s M^2 (g_s N_f (\log N - 6 + \log(16)) - 24g_s N_f \log(r - 8\pi)) + 320}{64r}, \]
\[ \beta_1 = \frac{3g_s M^2 N_f}{64r^2} - \frac{5}{r_0^2}, \]
\[ \alpha_2 = \frac{4\pi g_s M^2 N_f}{r_0^4} - \frac{3g_s M^2 N_f \log(r_0)}{4\pi r_0^4}, \]
\[ \beta_2 = \frac{3g_s M^2 N_f \log N (4 \log(r_0) - 1)}{5\pi r_0^5} - \frac{16g_s m_0^2 N}{r_0^5}. \]
\[ \text{(97)} \]

The solution to (96) is given by

\[ G(r) = e^{-\alpha_1 r + \frac{\beta_1 r}{r_0}} \left[ \right. \]
\[ \left. \begin{array}{c}
  c_1 F_1 \left( b + \frac{2b^2}{2b^3} \right) \\
  + c_1 H_{\alpha_1 + \beta_1 + \frac{2b^2}{2b^3}} - \frac{1}{2} (r - r_0) \beta_1^2 (r - r_0) - \frac{2b^2}{2b^3} \\
  + \left. \begin{array}{c}
    c_1 \left( \frac{\alpha_1 + \beta_1^2 (r - r_0) - \frac{2b^2}{2b^3}}{\sqrt{2b^2}} \right) \\
    \left( \frac{\alpha_1 + \beta_1^2 (r - r_0) - \frac{2b^2}{2b^3}}{\sqrt{2b^2}} \right) \right. \\
  \end{array} \right] \right] \]  
\[ \text{(98)} \]

One can then work out \( G'(r = r_0) = \left( \frac{G(r)}{r_0^2} \right)' \bigg|_{r=r_0} \). Now, setting \( c_2 = 0 \), defining \( \tilde{m} \) via \( m = \tilde{m} \tilde{t}^2 \), and using the large-\( \tilde{m} \) limit of Hermite functions:

\[ H_{\alpha_1 + \beta_1^2 (r - r_0) - \frac{2b^2}{2b^3}} \left( \frac{\alpha_1 + \beta_1^2 (r - r_0) - \frac{2b^2}{2b^3}}{\sqrt{2b^2}} \right) \]
\[ \rightarrow H_{\frac{16m^4}{\ell^2}} \left( \frac{2 \tilde{m}^2}{\sqrt{5} \tilde{m}^2} \right) \]  
\[ \text{(99)} \]

and

\[ H_n(x) \rightarrow \frac{2^{n+1} e^{n/2} (2n-1/2 \pi)^{1/2}}{\sqrt{\left( \frac{x}{\pi} - x \left( \frac{2n-1}{2} \right) + 1 \right)^{1/2}}} \]
\[ \text{(100)} \]

one can show that the Neumann / Dirichlet: \( G(r = r_0) = 0 \) boundary condition at \( r = r_0 \) is equivalent to the condition

\[ \frac{8}{375} \left( \sqrt{6\tilde{m}^2} \sqrt{375 - 64\tilde{m}^4} - 6i\tilde{m}^4 \right) = i\pi (2n + 1), \]
\[ \text{(101)} \]

yielding

\[ m_{0-} (r_h = 0) \]
\[ = \frac{1}{2} \left( \frac{3}{8} \right)^{3/4} \left( -2(\sqrt{6} \sqrt{375 - 64\tilde{m}^4} - 6i\tilde{m}^4) \right) \]
\[ \approx \frac{1}{2} \left( \frac{3}{8} \right)^{3/4} \left( -2(\sqrt{6} \sqrt{375 - 64\tilde{m}^4} - 6i\tilde{m}^4) \right) \]
\[ \times \left( \frac{3\pi^2 - 32}{3\pi^2 - 32} \right) \]
\[ \text{(102)} \]

5.3 \( 0^- \) Glueball spectrum from WKB method for \( r_h \neq 0 \)

Using the variables of [32], the potential in the IR region is given as

\[ V(\text{IR}, T) = (6 - 0.01\tilde{m}^2)e^{\tilde{r}} + (0.15\tilde{m}^2 - 16.1875)e^{2\tilde{r}} \]
\[ + O(e^{3\tilde{r}}), \]
\[ \text{(103)} \]

where in the ‘large’ \( \tilde{m} \) limit, \( V(\text{IR}, T) > 0 \) for (i) \( e^{\tilde{r}} > \frac{3.1979 \times 10^{13} - 8.63632 \times 10^{20} \tilde{m}^2}{6.25374 \times 10^{31} - 5.79497 \times 10^{30} \tilde{m}^2} = 0.067 + O\left( \frac{1}{\tilde{m}^2} \right) \) if \( \tilde{m} > 24.495 \) and (ii) \( e^{\tilde{r}} < \frac{3.1979 \times 10^{13} - 8.63632 \times 10^{20} \tilde{m}^2}{6.25374 \times 10^{31} - 5.79497 \times 10^{30} \tilde{m}^2} = 0.067 + O\left( \frac{1}{\tilde{m}^2} \right) \) if \( 10.388 < \tilde{m} < 24.495 \). One can show that

\[ \int_{\log(0.067)}^{\log(-0.067)} \sqrt{V(\text{IR}, T)} \approx 0, \]

implies there is no contribution to the WKB quantization condition in the IR region.
Now, consider:

\[
V(UV, T) = (-1.02 \tilde{m}^2 - 22.5)e^{-2z} + (0.25 \tilde{m}^2 + 8.25)e^{-z} - 1 + O(e^{-3z}).
\]

(104)

For \( \tilde{m} > 4.29 \) the turning points of \( V(UV, T) \) are 0.125\( \tilde{m}^2 - 0.025\sqrt{25.\tilde{m}^4 + 18.\tilde{m}^2 - 8775} \). Hence, \( e^z < 0.125\tilde{m}^2 + 0.025\sqrt{25.\tilde{m}^4 + 18.\tilde{m}^2 - 8775} + 4.125 = 0.25\tilde{m}^2 + 4.17 + O(\frac{1}{\tilde{m}^2}) \).

With turning points:

\[
\int_{\log(0.25\tilde{m}^2+4.17)}^{\log(0.25\tilde{m}^2+4.17)} \sqrt{V(UV, T)}
\]

(105)

\[
= \int_{\log(0.25\tilde{m}^2+4.17)}^{\log(0.25\tilde{m}^2+4.17)} e^{-z}\sqrt{0.25e^z - 1.02} + O\left(\frac{1}{\tilde{m}}\right)
\]

\[
= 0.389\tilde{m} - 2 + O\left(\frac{1}{\tilde{m}}\right) = \left(n + \frac{1}{2}\right)\pi.
\]

Hence one obtains isospectrality with \( 0^{++} \); for large \( n \), there is also isospectrality with \( 2^{++} \).

5.4 WKB method at \( r_h = 0 \)

In this section we will discuss obtaining the spectrum at \( r_h = 0 \) using WKB quantization condition at LO in \( N \) in Sect. 5.4.1 and up to NLO in \( N \) in Sect. 5.4.2.

5.4.1 LO in \( N \)

In the IR region, the WKB ‘potential’ can be shown to be given by

\[
V(IR, r_h = 0) = -\frac{1}{4} + \frac{1}{4}(-3 + \tilde{m}^2)e^{2z} + O(e^{3z}).
\]

(106)

with turning points: \( (\log(\frac{1}{m_0}) + O(\frac{1}{m_0^2})) \approx -\log(m_0), (\log(\delta^2 - 1)) \). Further dropping \( O(\frac{1}{m_0^3}) \) terms, \( \int_{\log(m_0)}^{\sqrt{V(IR, r_h = 0)}} \)

\[
= \frac{(\delta^2 - 1)}{2}\tilde{m} - 0.785 = \left(n + \frac{1}{2}\right)\pi \text{ yielding}
\]

\[
n^{0--}_{n}(IR, r_h = 0) = \left(\frac{3 + 4n\pi}{2(\delta^2 - 1)}\right) \frac{r_0}{L^2} = \left(\frac{4.71 + 6.28n}{\delta^2 - 1}\right) \frac{r_0}{L^2}.
\]

(107)

In the UV region,

\[
V(UV, r_h = 0) = -\frac{3}{2}(\tilde{m}^2 + 3)e^{-2z} + \frac{1}{4}(\tilde{m}^2 + 6)e^{-z} - 1,
\]

(108)

with turning points: \( (\log(3 + O(\frac{1}{m^2}))), (\log\left(\frac{\tilde{m}^2}{4} - \frac{3}{2} + O(\frac{1}{m^2})\right)), \) and \( \sqrt{V(UV, r_h = 0)} = \frac{e^{-z}}{2}\sqrt{e^z - 3\tilde{m} + O(\frac{1}{m^2})}: \)

\[
\int_{\log\frac{3}{2}}^{\log\left(\frac{\tilde{m}^2}{2} - \frac{1}{2}\right)} e^{-z} = \frac{\sqrt{e^z - 3\tilde{m}}}{2} = \left(n + \frac{1}{2}\right)\pi.
\]

(109)

implying:

\[
m^{0--}_{n}(UV, r_h = 0) = (7.87 + 6.93n)\frac{r_0}{L^2}.
\]

(110)

5.4.2 NLO (in \( N \) )/non-conformal corrections

Up to NLO in \( N \), in the IR region, the potential ‘\( V(IR, r_h = 0) \)’ is given by:

\[
V(IR, r_h = 0) = \frac{1}{256\pi^2 N} (e^{2z}(-g_s^2 M^2 N_f (6\log N - 72 + \log(16777216)) + 36g_s^2 M^2 \tilde{m}^2 N_f \log^2(y_0)) + g_s M^2 \log(y_0) \times (g_s N_f (72 - \tilde{m}^2 (6\log N - 36 + \log(16777216))) + 48\pi \tilde{m}^2) + 48\pi g_s M^2 + 64\pi^2 (m^2 - 3N)) - \frac{1}{4} + O(e^{-3z}).
\]

(111)

The turning points of (111) up NLO in \( N \) are given by

\[
\left[ \log\left(\frac{1}{m_0}\right) - \frac{1}{2} \frac{M^2 \log(y_0) - g_s N_f (6\log N - 36 + \log(16777216)) + 36g_s N_f \log(y_0) + 48\pi r_0}{128\pi^2 N} \right]
\]

(112)

\[
\log(\delta^2 - 1) \right].
\]

After evaluation of the integral of \( \sqrt{V(IR, r_h = 0)} \) between the aforementioned turning points, in the large-\( \tilde{m} \) limit, one obtains the following quantization condition:

\[
\left(\frac{\delta^2 - 1}{2}\right) - 3(\delta^2 - 1)g_s M^2 (g_s N_f \log(r_0) - \log(r_0)) = \left(n + \frac{1}{2}\right)\pi,
\]

which yields

\[
n^{0--}_{n}(r_h = 0) = \frac{6.28319n + 4.71239}{\delta^2 - 1} \times \left(1 + \frac{0.01g_s^2 N M^2 N_f \log(r_0)}{N}\right).
\]

(113)

6 Glueball masses from M-theory

The glueball spectrum for spin \( 0^{++}, 1^{++} \) and \( 2^{++} \) is calculated in this section from the M-theory perspective both by considering \( r_h \neq 0 \) and an IR radial cut-off (for \( r_h = 0 \)) in the background. The 11-dimensional M-theory action is given as
\[ S_M = \frac{1}{16\pi} \int_M d^{11}x \sqrt{G} R \]
\[ - \frac{1}{4} \sqrt{G} \int_M d^{11}x \Lambda_4 \ast \ast_1 \Lambda_4, \tag{114} \]

where \( G_4 = dC_3 + A_1 \wedge dA_2 + dx_10 \wedge dA_2 \), and \( C_{\mu_1 \mu_2 \mu_3} = C^{D}\rangle_{\mu_1 \mu_2 \mu_3} \). Now, as shown in [28], no \( F^{IA} \) (to be obtained via a triple T-dual of type IIB \( F_{1,3,5} \)) where \( F_1 \sim F_{x/y^2}, F_3 \sim F_{xy/|y|}, F_{x/\bar{y}/|y|}, F_{y/\bar{z}/|y|} \) and \( F_3 \sim F_{xy/\bar{y}2} \) where \( \beta_i = r/\theta_i \) can be generated. Thus, the four-term flux \( G_4 = d(C_{\mu_1 \mu_2}dx^\mu \wedge dx^\nu \wedge dx_10) + (A_1^F + A_1^F)^2 \) \( H_3 = H_1 \wedge dx_10 + A \wedge H_3 \), where \( C_{\mu_1 \mu_2 \mu_3} = B_{\mu_1 \mu_2 \mu_3} \).

\[ \int G_4 \ast \ast_1 \Lambda_4 = \int (H_3 \wedge dx_10 + A \wedge H_3) \wedge \ast_1 (H_3 \wedge dx_10 + A \wedge H_3). \tag{115} \]

Now, \( H_3 \wedge dx_10 \ast \ast_1 (H_3 \wedge A) = 0 \) as neither \( H_3 \) nor \( A \) has support along \( x_10 \). Hence,

\[ H_3 \wedge dx_10 \ast \ast_1 (H_3 \wedge dx_10) \]
\[ = \sqrt{G} H_{\mu_1 \mu_2 \mu_3} (G^\mu_1 G^\nu_1 G^\rho_1 G^{10 \lambda_1} H_{\mu_1 \nu_1 \rho_1 \lambda_1}) \wedge \cdots \]
\[ = \sqrt{G} H_{\mu_1 \mu_2 \mu_3} (G^\mu_1 G^\nu_1 G^\rho_1 G^{10 \lambda_1} H_{\mu_1 \nu_1 \rho_1 \lambda_1}) \wedge \cdots \]
\[ \frac{1}{4} \sqrt{G} \int d^{11}x \Lambda_4 \ast \ast_1 \Lambda_4, \]

where \( H_{\mu_1 \mu_2 \mu_3} A_{\mu_4} = H_{\mu_1 \mu_2 \mu_3 \mu_4} - (H_{\mu_1 \mu_2 \mu_3} A_{\mu_4} - H_{\mu_1 \mu_2 \mu_3 \mu_4} A_{\mu_4}) \).

It was shown in [28] that in the MQGP limit (11), in Eq. (116), contribution of \( H_3 \wedge dx_10 \ast \ast_1 (H_3 \wedge dx_10) \) is always dominated by \( \sqrt{G} H_{\mu_1 \mu_2 \mu_3} G^{\mu_1} G^{\nu_1} G^{\rho_1} G^{10 \lambda_1} H_{\mu_1 \nu_1 \rho_1 \lambda_1} \).

\[ \begin{align*}
&(H \wedge A) \ast \ast_1 (H \wedge A) \\
&= \sqrt{G} H_{\mu_1 \mu_2 \mu_3} A_{\mu_4} = H_{\mu_1 \mu_2 \mu_3 \mu_4} - (H_{\mu_1 \mu_2 \mu_3} A_{\mu_4} - H_{\mu_1 \mu_2 \mu_3 \mu_4} A_{\mu_4}) \tag{117}
\end{align*} \]

where \( H_{\mu_1 \mu_2 \mu_3} A_{\mu_4} = H_{\mu_1 \mu_2 \mu_3 \mu_4} - (H_{\mu_1 \mu_2 \mu_3} A_{\mu_4} - H_{\mu_1 \mu_2 \mu_3 \mu_4} A_{\mu_4}) \).

Eqs. (116) and (117), as shown in [28], are given by the following analytical expressions:

\[ H_3 \wedge dx_10 \ast \ast_1 (H_3 \wedge dx_10) \bigg|_{\theta_1 = \frac{a_1}{N}, \theta_2 = \frac{a_2}{N}} = \frac{\sqrt{3}(-4\alpha_1^2 + 27\alpha_1^6) r^N}{2\alpha_1^6 (2r^4 - r_4^4) N^\frac{3}{2}} \]
\[ = \frac{2\alpha_1^6 (2r^4 - r_4^4) N^\frac{3}{2}}{2\sqrt{3} \alpha_1^6 |s_3| N^\frac{3}{2}}, \tag{118} \]

and

\[ (H \wedge A) \ast \ast_1 (H \wedge A) \bigg|_{\theta_1 = \frac{a_1}{N}, \theta_2 = \frac{a_2}{N}} = \mathcal{F}(a_1, a_2; a, g_s, M, N_f) N^\frac{3}{2}, \tag{119} \]

where \( \mathcal{F}(a_1, a_2; a, g_s, M, N_f) \) is a well-defined function of the parameters indicated. One hence notes that \( \lim_{r_3 \rightarrow \infty} \int_0^{r_3} (H \wedge A) \ast \ast_1 (H \wedge A) = \frac{\pi^2}{12} N^\frac{3}{2} \) is UV-divergent. Also, this yields a large cosmological constant in the IR because: \( G_4 \wedge G_4 \sim \frac{|H_{10}|^2}{\sqrt{G}} \sim \frac{N^2}{r^6 N^\frac{2}{3}} = \frac{1}{r^6} \). To take care of both these issues, from the discussion of holographic renormalizability of the \( D = 11 \) supergravity action in [28], one sees that this term can be cancelled by a boundary counter term: \( \int_{\partial \mathcal{M}} \sqrt{G} |G_{14}|^2 \).

Now, using:

\[ \sqrt{G} \bigg|_{\theta_1 = \frac{a_1}{N}, \theta_2 = \frac{a_2}{N}} = \frac{2 N^{17/20} \lambda^3}{27 \frac{35}{6} \sqrt{\pi} \alpha_1^6 \alpha_0^3 35/12}, \tag{120} \]

and (118), one sees that one obtains a large-\( N \) suppressed cosmological constant from the second term in \( \frac{|H_{10}|^2}{\sqrt{G}} \), which remains small \( \lambda \gg r \). To ensure one does not generate an \( N \)-enhanced cosmological constant from the first term in (118), one imposes the condition: \(-4\alpha_1^2 + 27\alpha_1^6 = 0 \), i.e., \( a_0 = \frac{3}{4} \alpha_1^6 \).

One hence obtains the following flux-generated cosmological constant (with a slight abuse of notation):

\[ G_4 \wedge \ast G_4 \bigg|_{\theta_1 = \frac{a_1}{N}, \theta_2 = \frac{a_2}{N}} = \frac{3 (27 \alpha_1^6 |s_3| (82r^4 - r_4^4))^3}{N^2 \sqrt{2\pi} r^4}, \tag{121} \]

**Metric fluctuations:** The background metric \( g_{\mu\nu}^{(0)} \) is linearly perturbed as \( g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} \). With this perturbation the equation of motion follows from the action (114) as

\[ R_{\mu\nu} = -\frac{1}{12} G_{\mu\nu}, \tag{122} \]

Now we assume the perturbation to have the following form:

\[ h_{\mu\nu} = \epsilon_{\mu\nu}(r)e^{i k x_1}. \]
metry in the \(x_2-x_3\) plane which allow us to classify different perturbations into three categories, namely tensor, vector and scalar type of metric perturbations.

The mass spectrum was obtained by (i) solving equation (122) and applying Neumann/Dirichlet boundary condition near \(r_h/r_0\), (ii) following the redefinition of variables in [32] and then considering the WKB quantization condition.

6.1 0\(+\) Glueball spectrum

The 0\(+\) glueball in M-theory corresponds to scalar metric perturbations [59]:

\[ h_{tt} = g_{tt} e^{i q x_1} q_1(r), \]
\[ h_{tx} = h_{x1} = i q \ g_{x1x} e^{i q x_1} q_3(r), \]
\[ h_{rr} = g_{rr} e^{i q x_1} q_2(r), \]

where \( g_{tt}, g_{x1x}, \) and \( g_{rr} \) are the metric components of the M-theory and is given in Eq. (21).

6.1.1 M-theory background with \( r_h \neq 0 \)

Considering these components at leading order in \( N \), and taking into account the above perturbation, we get the following differential equation for \( q_3(r) \) with \( q^2 = -m^2 \) from (122):

\[
q_3''(r) + \frac{q_3'(r)}{12 \pi g_s N r^3 (r^4 - r_h^4)} [r^2 (16 \pi^2 g_s^2 m^2 N^2 r^2 + 12 \pi g_s N (9 r^4 - r_h^4) - 3 r (r^4 - r_h^4)^2) - 3 q_3'(r)] + \frac{q_3(r)}{12 \pi g_s N r^4 (r^4 - r_h^4)} \times \left\{ r^2 (32 \pi^2 g_s^2 m^2 N^2 r^2 + 12 \pi g_s N (15 r^4 + r_h^4) - 3 (5 r^8 - 6 r^4 (r^4 + r_h^4) - 36 a^2 (4 \pi^2 g_s^2 m^2 N^2 r^2 + \pi g_s N (11 r^4 + r_h^4) + r^9 - 5 r^4)^2)) \right\} = 0. \tag{124}
\]

(a) Spectrum from Neumann/Dirichlet Boundary Condition: Equation (124), for \( a = 0.6 r_h \) near \( r = r_h \) (writing \( m = \sqrt{\frac{\tilde{m}}{r}} \)) simplifies to

\[
q_3''(r) + \frac{q_3'(r)}{(r - r_h) (r - r_h)} + \frac{(0.76 - 0.103333 \tilde{m}^2) q_3(r)}{r_h (r - r_h)} = 0. \tag{125}
\]

Let's write the above equation of the following form:

\[
h''(r) + \frac{p}{r - r_h} h'(r) + \frac{s}{r - r_h} h(r) = 0, \tag{126}
\]

where we have, \( p = (2 - 0.0066667 \tilde{m}^2), \)
\( s = (0.76 - 0.103333 \tilde{m}^2). \)

The solution to (126) is given by

\[
h(r) = c_1 (2 r - 2 r_h)^{p/2} (r - r_h)^{-p/2} \times (-s (r - r_h) )^{1/2} \left( \frac{1}{1 - \frac{1}{2} s} \right) \right)^{1/2} + (-1)^{1-p} c_2 (2 r - 2 r_h)^{p/2} (r - r_h)^{-p/2} \times (-s (r - r_h) )^{1/2} K_{p-1}(2 \sqrt{s} (r - r_h)). \tag{127}
\]

Setting \( c_2 = 0, \) one can verify that one satisfy the Neumann boundary condition: \( h'(r = r_h) = 0 \) provided:

\[
p = -n \in \mathbb{Z}^{-} \cup \{0\}, \tag{128}
\]

implying:

\[
\tilde{m} = 12.25 \sqrt{2 + n}. \tag{129}
\]

One can similarly show that by imposing the Dirichlet boundary condition, \( h(r = r_h) = 0, \)

\[
\tilde{m} = 12.25 \sqrt{1 + n}. \tag{130}
\]

(b) Spectrum using the WKB method: Following the redefinition of [32], Eq. (124) can be rewritten in a Schrödinger-like form, where for \( a = 0.6 r_h \) and setting \( g_s = 0.9, N \sim (g_s)^{-39} \sim 100 \) in the MQGP limit of [28] – the ‘potential’ in the IR region can be shown to be given by

\[
V_{IR}(z) = e^z (-0.00576389 \tilde{m}^4 - 0.0708333 \tilde{m}^2 + 0.)
+ 0.00201389 \tilde{m}^4 + 0.0033333 \tilde{m}^2
- 0.25 + \mathcal{O}(e^{2z}). \tag{131}
\]

The potential (131) is positive for \( z \in (-\infty, \log(0.349)] \), but to remain within the IR region, one truncates this domain to \( z \in (-\infty, -2.526] \) and the same yields

\[
\int_{-\infty}^{-2.526} \sqrt{V_{IR}(z)} = 0.0898 \tilde{m}^2 \log(\tilde{m}) - 0.0439 \tilde{m}^2
= \pi \left( n + \frac{1}{2} \right) = \left( n + \frac{1}{2} \right) \pi, \tag{132}
\]

or

\[
m_n^{0++} = \frac{\sqrt{70.0055n + 35.0027}}{\sqrt{p} \zeta(26.3065n + 13.1533)}. \tag{133}
\]

In the UV region, one can show that

\[
V_{UV}(z) = -0.00694444 \tilde{m}^4 + 0.295 \tilde{m}^2 + 1.62
\frac{e^{2z}}{e^z - 0.0833333 \tilde{m}^2 - 0.54} - 0.25 < 0, \tag{134}
\]

implying no turning points in the UV region.
6.1.2 M-theory background with IR cut-off $r_0$

In this case, Eq. (124) is modified only by the limit $(r_h, a) \to 0$. The equation in this limit is given as

$$q^3_{2}(r) + q^3_{2}(r) \left( \frac{4\pi g_s m^2 N}{3 r^3} - \frac{r^4}{4 \pi g_s N} + \frac{9}{r} \right)$$

$$+ q^3_{2}(r) \left( \frac{2 \pi g_s m^2 N}{3 r^4} - \frac{5 \pi}{4 \pi g_s N} + \frac{15}{r^2} \right) = 0. \quad (135)$$

(a) Spectrum from Neumann/Dirichlet boundary condition:
Near the cut-off at $r = r_0$, Eq. (135) is given by

$$q^3_{2}(r) + \left( \frac{4 m^2 + 108}{12 r_0} - \frac{(m^2 + 9)(r - r_0)}{r_0^2} \right) q^3_{2}(r)$$

$$+ q^3_{2}(r) \left( \frac{8 m^2 + 180}{12 r_0^2} - \frac{(32 m^2 + 360)(r - r_0)}{12 r_0^3} \right) = 0,$$

where solution is given by

$$q^3_{2}(r) = e^{-\frac{2(4m^2 + 45)z}{3(m^2 + 9)r_0}} \left[ c_1 H_{2(m^2 + 4)} \left( \frac{2m^6 + 71m^4 + 828m^2 + 2835}{18(m^2 + 9)^2} \right) \right. \frac{2m^6 + 71m^4 + 828m^2 + 2835}{18(m^2 + 9)^2} + \left. \frac{1}{2} \left( \frac{m^6 + 20m^2 + 63}{18(m^2 + 9)^2} \right) \frac{2m^6 + 71m^4 + 828m^2 + 2835}{18(m^2 + 9)^2} \right]. \quad (136)$$

Thus,

$$q^3_{2}(r = r_0) = \frac{1}{27 r_0^2} e^{-\frac{2(4m^2 + 45)}{3(m^2 + 9)}} \left[ (2m^6 + 71m^4 + 828m^2 + 2835) \left( \frac{2m^6 + 71m^4 + 828m^2 + 2835}{18(m^2 + 9)^2} \right) + \frac{1}{2} \left( \frac{m^6 + 20m^2 + 63}{18(m^2 + 9)^2} \right) \frac{2m^6 + 71m^4 + 828m^2 + 2835}{18(m^2 + 9)^2} \right].$$

Numerically/graphically we see that, for $c_1 = -0.509c_2$, one gets $q_3(r = r_0, \tilde{m} \approx 4.1) = 0$. We hence estimate the ground state of $0^{++}$ from metric fluctuations in M-theory to be $4.1 \sqrt{L^2}$.

(b) Spectrum using the WKB method: Defining $m = m_0 \sqrt{N}$ and following [32], the potential term in the Schrödinger-like equation from (135) is given as

$$V(z) = \frac{1}{2304\pi^2 g_s^2 N^2} (e^{2z} - 1)^4 \left( -16 \pi^2 g_s^2 N^2 \right)$$

$$\times (12(\tilde{m}^2 + 12)e^{3z} + (\tilde{m}^2 + 12m^2 + 216)e^{2z} + 144e^z + 36e^z + 36) + 24\pi g_s N y_0 e^{2z}$$

$$\times (e^z + 1)^2 (\tilde{m}^2 + 9e^z + 9 - 9y_0^2 e^{2z}(e^z + 1)^2). \quad (139)$$

We hence see that in the large-\(N\) limit, $V(z) < 0$, and hence it has no turning points. The WKB method à la [32] does not work in this case.

6.2 2++ Glueball spectrum

To study the spectrum of a spin 2++ glueball, we consider the tensor type of metric perturbations where the non-zero perturbations are given as

$$h_{x_2 x_3} = h_{x_3 x_2} = g_{x_1 x_1} H(r) e^{ikx_1},$$

$$h_{x_2 x_2} = h_{x_3 x_3} = g_{x_1 x_1} H(r) e^{ikx_1}, \quad (140)$$

where $g_{x_1 x_1}$ is given in (21).
6.2.1 M-theory background with \( r_h \neq 0 \)

Considering the tensor modes of metric perturbations and the M-theory metric components corrected up to NLO in \( N \), given in (21), we obtain a second order differential equation in \( H(r) \) from (122),

\[
H''(r) + H'(r) \left( \frac{-3a^2}{r^3} + \frac{15g_s M^2 (g_s N_f \log(N) - 24g_s N_f \log(r) - 6g_s N_f + g_s N_f \log(16) - 8\pi)}{64\pi^2 N r} + \frac{5r^4 - rh^4}{r^3 - rh^4} \right)
+ \left( \frac{1}{4\pi r^4 (r^4 - rh^4)^2} \right) [8\pi (3a^2 + 2\pi g_s N m^2 r^2 - r^4 + rh^4 + 2\pi g_s N m^2 r^4 + 4r^6) - 3g_s^2 M^2 m^2 r^2 (r^2 - 3a^2)]
\times \log(r) [g_s N_f \log(N) + g_s N_f (\log(16) - 6) - 8\pi] + 36g_s^3 M^2 N_f m^2 r^2 (r^2 - 3a^2) \log^2(r) \right) H(r) = 0,
\]

where we assume \( k^2 = -m^2 \) with \( m \) being the mass of the corresponding glueball.

(a) Spectrum from Neumann/Dirichlet boundary condition: Near \( r = r_h \), the solution to the above equation will be given along the same lines as Sect. 5.1 for \( 0^- \) glueballs, and the analog of (93) is

\[
\tau^2 \left( \frac{1.5\sqrt{0.05366987^2 - 0.00186263 m^2}}{0.00186196 m^2} - 0.115834 \right) + 0.00018196 m^2
= -n,
\]

the solutions to which are given as

\[
m_{2+}^2 = 5.086T,
\]

\[
m_{2-}^2 = 5.269T,
\]

\[
m_{3+}^2 = 5.318T,
\]

\[
m_{3-}^2 = 5.338T,
\]

\[
m_{4+}^2 = 5.348T.
\]

One can impose Dirichlet boundary condition: \( H(r = r_h) = 0 \), and one can show that

\[
m_{n}^{2+} \text{(Neumann)} = m_{n+1}^{2+} \text{(Dirichlet)}, \quad n = 0, 1, 2, \ldots
\]

(b) Spectrum from WKB method: Using the variables of [32], the potential term in the Schrödinger-like equation for \( 2^{++} \) glueball can be obtained from (141) and in the IR region,

\[
\text{written in terms of } m = \bar{m} \frac{r_h}{L^2} = \bar{m} \frac{\sqrt{N}}{L^2}, \text{ can be shown to be given by}
\]

\[
V_{IR}(z) = e^{z} (0.52 - 0.01 \bar{m}^2) + (0.15 \bar{m}^2 - 1.0275) e^{2z} + \mathcal{O} \left( \frac{g_s M^2}{N}, e^{3z} \right).
\]

Now, negative \( z \) implies being closer to the IR region and the boundary would correspond to \( r \leq \sqrt{3} a = 0.6 \sqrt{3} r_h \), corresponding to \( z = -2.53 \). It can be shown that for \( \bar{m} > 7.211 \),

\[
V(z) \in [-2.71, -2.53]) > 0. \text{ Now } \int_{z=-2.53}^{z=0} V_{IR}(z) \approx 0. \text{ Hence, the IR region does not contribute to the } 2^{++} \text{ glueball spectrum.}
\]

In the UV region we must consider the limit (\( z \to \infty \)). Moreover, in the UV region \( N_f = M = 0 \). We have

\[
V_{UV}(z) = e^{-2z} (6.56 - 1.02 \bar{m}^2) + (0.25 \bar{m}^2 - 2.77) e^{-z} + 1 + \mathcal{O} \left( \frac{1}{\bar{m}}, e^{-3z} \right),
\]

whose turning points are \( \{ \log (4.08 + \mathcal{O}(\frac{1}{\bar{m}^2})) \}, \infty \), giving the WKB quantization as

\[
\int_{\log 4.08}^{\infty} \sqrt{V_{UV}(z)} = 0.39 \bar{m} = \left( n + \frac{1}{2} \right) \pi,
\]

implying

\[
m_{n}^{2+} (T) = 8.08 \left( n + \frac{1}{2} \right) \frac{r_h}{L^2}.
\]

6.2.2 M-theory background with an IR cut-off \( r_0 \)

Considering the limit (\( r_h, a \to 0 \)), Eq. (141) is given by

\[
H''(r) + \left( \frac{5(3M^2 g_s (-24 N_f g_s \log(r) - 6N_f g_s + g_s N_f \log(N) + \log(16) N_f g_s - 8\pi) + 64\pi^2 N)}{64\pi^2 N r} \right) H'(r)
+ \frac{1}{4\pi r^4} [36m^2 M^2 N_f g_s^3 \log^2(r) - 3m^2 M^2 g_s^2 \log(r) N_f g_s \log(N)]
+ (\log(16) - 6) N_f g_s - 8\pi) + 16\pi (\pi m^2 N_f g_s + 2r^2)) H(r) = 0.
\]

\[\square \text{ Springer} \]
(a) Neumann/Dirichlet boundary condition at \( r = r_0 \): Up to LO in \( N \) near \( r = r_0 \), the above equation is given by

\[
H''(r) + \left( \frac{5}{r_0} - \frac{5(r - r_0)}{r_0^2} \right) H'(r) + H(r) \left( \frac{\tilde{m}^2 + 8}{r_0^2} - \frac{4(\tilde{m}^2 + 4)(r - r_0)}{r_0^3} \right) = 0. \tag{149}
\]

The solution of (149) is given by

\[
H(r) = e^{\frac{4(\tilde{m}^2 + 4)v}{r_0}} \left( c_1 H \left( \frac{1}{153} (16\tilde{m}^4 + 53\tilde{m}^2 + 56) \right) \right.
\]

\[
\times \left( \frac{2(4\tilde{m}^2 - 9)v_0 + 25r}{5\sqrt{10}r_0} \right)^{\frac{1}{2}} \left( \frac{1}{250}(-16\tilde{m}^4 - 53\tilde{m}^2 - 56); \frac{1}{2}; \right.
\]

\[
\left. \left( 25r + 2(4\tilde{m}^2 - 9)v_0^2 \right) \right) \). \tag{150}
\]

The Neumann boundary condition \( H'(r = r_0) = 0 \), numerically shows that, for \( c_1 = -0.509c_2 \), the lightest \( 2^{++} \) glueball has a mass 1.137 \( \tilde{m} \). Similarly, by imposing the Dirichlet boundary condition: \( H(r = r_h) = 0 \), for \( c_1 = -0.509c_2 \), the lightest \( 2^{++} \) glueball has a mass 0.665 \( \tilde{m} \).

(b) Spectrum using the WKB method: Following [32], the \('potential\' term, in the IR region, up to leading order in \( N \) with \( m = \tilde{m} \), \( m = \frac{\tilde{m}}{2} \), \( \tilde{m} \), is given as

\[
V_{\text{IR}}(z) = \frac{1}{4}(m^2 - 5)e^{-z} - \frac{1}{4} + O(e^{3z}), \tag{151}
\]

and \( V_{\text{IR}}(z) \geq 0 \) for \( z \in [\log(\frac{1}{m} + \frac{1}{m})], \log(\delta^2 - 1)] \approx [-\log \tilde{m}, \log(\delta^2 - 1)] \). Hence the WKB quantization condition gives

\[
\int_{-\log \tilde{m}}^{-2.526} \left( \frac{(\delta^2 - 1)m}{2} - 0.785 + O\left( \frac{1}{m^2} \right) \right) = \left( n + \frac{1}{2} \right) \pi, \tag{152}
\]

implying

\[
m_n^{2++}(\text{IR}, r_h = 0) = m_n^{0--}(\text{IR}, r_h = 0). \tag{153}
\]

In the UV region, we have

\[
V_{\text{UV}}(z) = \frac{1}{4}(m^2 - 10)e^{-z} - \frac{3}{4}(m^2 - 5)e^{-2z} + 1 + O(e^{3z})
\]

\[
= e^{-z} \sqrt{e^{-3\tilde{m}}} + O\left( \frac{e^{-3z}}{m} \right) \tag{154}
\]

and \( V_{\text{UV}}(z) > 0 \) for \( z \geq \log \left( \frac{1}{5}(m^2 + \sqrt{m^4 + 28m^2 - 140} + 10) \right) \approx \log (3 + O\left( \frac{1}{m^2} \right)) \). Further, \( \int_{-\infty}^{\log 3} \sqrt{V_{\text{UV}}(z)} = \frac{\pi \tilde{m}}{8\sqrt{3}} \), implying

\[
m_n^{2++}(\text{UV}) = (3.46 + 6.93)\frac{r_0}{L^2}. \tag{155}
\]

(c) NLO-in-\( N \)/non-conformal corrections using the WKB method: The \('potential\' inclusive of NLO-in-N terms, in the IR region in the \( r_h = 0 \) limit, is given by

\[
V(\text{IR}, r_h = 0) = \frac{1}{512\pi^2 N}\left( e^{2z}(-60g_s^2M^2N_f^2)
\times (\log N - 12 + \log(16)) + 72g_s^2M^2m_0^2N_f\log^2(y_0)
+ g_sM^2\log(y_0) \times (g_sN_f(m_0^2 - 12\log N + 72)
+ \log(4096) - 15\log(16) + 720 + 96\pi m_0^2)
+ 480\pi g_sM^2 + 128\pi^2(m_0^2 + 5N)
\right) - \frac{1}{4} + O(e^{3z}). \tag{156}
\]

whose turning points are given by

\[
\left\{ \begin{array}{l}
\log \left( 1 - \frac{e^{2z}}{512\pi^2 N}\left( e^{2z}(-60g_s^2M^2N_f^2)
\times (\log N - 12 + \log(16)) + 72g_s^2M^2m_0^2N_f\log^2(y_0)
+ g_sM^2\log(y_0) \times (g_sN_f(m_0^2 - 12\log N + 72)
+ \log(4096) - 15\log(16) + 720 + 96\pi m_0^2)
+ 480\pi g_sM^2 + 128\pi^2(m_0^2 + 5N)\right) \right) \\
\log \left( \delta^2 - 1 \right) \end{array} \right. \right) \tag{156}
\]

The integral of \( \sqrt{V(\text{IR}, r_h = 0)} \) between these turning points, in the large-\( \tilde{m} \) limit, yields the same spectrum as \( \tilde{m} \) up to NLO in \( N \).

6.3 Spin-1++ glueball spectrum

Here we need to consider the vector type of metric perturbation with the non-zero components given as \( h_{11} = h_{11} = g_{x_1x_1}G^{(r)}(r)e^{i\kappa r}, l = x_2, x_3 \).

6.3.1 M-theory background with \( r_h \neq 0 \)

Substituting the above ansatz for the perturbation in (122), the differential equation in \( G(r) \) is given with \( k^2 = -m^2 \) as
(a) Neumann/Diriclet boundary condition at \( r = r_h \): Near
\( r = r_h \), Eq. (157) up to LO in \( N \), is given by
\[
G''(r) + \left( \frac{3.92}{r_h} \right) G'(r)
+ \left( \frac{2 - 0.02\tilde{m}^2}{r_h(r - r_h)} + \frac{-1.16 + 0.57\tilde{m}^2}{r_h^2} \right) G'(r) = 0,
\]
whose solution is given by
\[
G(r) = \exp \left( \frac{0.5r(2.52 - 0.57\tilde{m}^2 - 3.921 + r_h \log(r - r_h))}{r_h} \right)
\times \left[ c_1 \left( -\frac{0.01\tilde{m}^2 - 5.0016 - 0.57\tilde{m}^2 + 1}{\sqrt{5.0016 - 0.57\tilde{m}^2}} \right) + \frac{2.\sqrt{5.0016 - 0.57\tilde{m}^2} - r_h}{r_h} \right.
\]
\[ \left. -2.\sqrt{5.0016 - 0.57\tilde{m}^2} + c_2 L^{1}\left( \frac{1}{\sqrt{5.0016 - 0.57\tilde{m}^2}} \right) + \frac{2r\sqrt{5.0016 - 0.57\tilde{m}^2}}{r_h} - 2.\sqrt{5.0016 - 0.57\tilde{m}^2} \right] \right] \].

(158)

Imposing the Neumann boundary condition at \( r = r_h \) yields
\[
\begin{align*}
-1.8765 \times 10^{14} \tilde{m}^2 &+ 6.022232598554301 \times 10^{-7} \sqrt{9.70917 \times 10^{40} \tilde{m}^4 + 1.26219 \times 10^{44} \tilde{m}^2 - 5.04877 \times 10^{44} + 1.8765 \times 10^{16}} \\
&= 5.70456 \times 10^{10} - 5.6295 \times 10^{15} \tilde{m}^2
\end{align*}
\]

\[
= -\frac{25}{\tilde{m}^2} + O\left( \frac{1}{\tilde{m}^2} \right),
\]

\[
\lim_{z \to 0} \frac{1}{r_h} \left\{ e^{-\sqrt{5.0016 - 0.57\tilde{m}^2} \frac{0.140858 c_1 r_h U}{\sqrt{5.0016 - 0.57\tilde{m}^2} \frac{1}{\sqrt{5.0016 - 0.57\tilde{m}^2} + 1, 2, z}} \right. \\
+ \frac{0.140858 c_2 r_h L^{1}}{\sqrt{5.0016 - 0.57\tilde{m}^2} \frac{1}{\sqrt{5.0016 - 0.57\tilde{m}^2} + 1, 2, z}} \left( \begin{array}{c} 1 \end{array} \right) \right\}.
\]

(160)

Considering \( p = \frac{0.01\tilde{m}^2}{\sqrt{5.0016 - 0.57\tilde{m}^2}} - \frac{1}{\sqrt{5.0016 - 0.57\tilde{m}^2}} + 1 \) and
setting \( c_2 = 0 \) in (160), and then using \( \lim_{z \to 0} U(p, 2, z) \sim 0 \) \( \sim e^{1} F_1(p - 1; 0; z) = \lim_{b \to 0} \lim_{z \to 0} F_1(p - 1; b; z) \)
i.e. first set \( z \) to 0 and then \( b \), \( p = -n \in \mathbb{Z}^- \). Hence,
\[
m_{1+n}(T) = 2.6956\pi T,
\]
\[
m_{1+n}(T) = 2.8995\pi T,
\]
\[
m_{1+n}(T) = 2.9346\pi T,
\]
\[
m_{1+n}(T) = 2.9467\pi T.
\]

(161)

One can show that one obtains the same spectrum as (161) even upon imposing the Dirichlet boundary condition: \( G(r = r_h) = 0 \).

(b) Spectrum using the WKB method: Following [32], the ‘potential’ \( V \) in the Schrödinger-like equation working with the dimensionless mass variable \( \tilde{m} \) defined via \( m = \tilde{m} \frac{\rho L}{\hbar} = \tilde{m} \sqrt{\frac{\rho L}{\hbar}} \), in the IR region, can be shown to be given by
\[
V_{IR}(z) = e^{2z}(0.15\tilde{m}^2 - 1.52) + e^z(1 - 0.01\tilde{m}^2)
- \frac{1}{4} + O\left(e^{z}; \frac{1}{N} \right).
\]

(162)

The zeros of the potential, as a function of \( e^z \), in (162) are given by
\[
V_{UV}(z) = \frac{e^{3z}(-3b^2 + \tilde{m}^2 + 10) + e^{2z}(-3b^2(\tilde{m}^2 + 2) + 2\tilde{m}^2 + 9) + e^z((1 - 3b^2)\tilde{m}^2 + 1) + 4e^{4z} - 2}{4(e^z + 1)^3(e^z + 2)}.
\]

(164)
by which for \( b = 0.6 \) one obtains
\[
V(N_f = M = 0, \text{UV}) = e^{-2z}(6.56 - 1.02m^2) + (0.25m^2 - 2.77)e^{-z} + 1 + O(e^{-3z}).
\]
(165)

The zeros of the potential in the UV region, as a function of \( e^z \), in (165) are given by \(-0.125m^2 \pm 0.005 \sqrt{625m^4 + 20960m^2 - 185671} + 1.385 = (-0.25m^2 - 1.31, 4.08 + O(\frac{1}{m}))\); the former not being permissible. Hence, the allowed domain of integration, over which the potential is positive, is \((\log(0.08), \infty)\). Performing a large-\( \hat{m} \) expansion, one obtains
\[
\int_{\log(0.08)}^{\infty} e^{-2z}(6.56 - 1.02m^2) + (0.25m^2 - 2.77)e^{-z} + 1 = \int_{z = \log(0.08)}^{\infty} e^{-z} \sqrt{0.25e^{z} - 1}\ 0.389\hat{m} = \left( n + \frac{1}{2} \right)\pi,
\]
(166)
yielding
\[
m^{1++}(T) = 4.04(1 + 2n)\ r_h \ L^2.
\]
(167)

6.3.2 M-theory background with an IR cut-off \( r_0 \)

(a) Neumann/Dirichlet boundary condition at \( r = r_0 \); Considering the limit of \( (r_3, a) \to 0 \) in Eq. (157) up to LO in \( N \) and imposing the Neumann boundary condition at the IR cut-off \( r = r_0 \), yields isospectrality with the \( 2^{++} \) glueball spectrum at \( r_h = 0 \).

(b) Spectrum using the WKB method: Using the redefinition of [32], the ‘potential’ up to leading order in \( N \) is given by
\[
V(z) = \frac{(\hat{m}^2 + 2)e^{2z} - 3e^z + 4e^3 - 1}{4(e^z + 1)^3} + O\left( \frac{g_s M^2}{N} \right).
\]
(168)

\[
\int_{0}^{\log(\delta^2 - 1)} \sqrt{V_{IR}(z)} = \frac{\delta^2 - 1}{2}\hat{m} - \frac{\pi}{4} = \left( n + \frac{1}{2} \right)\pi.
\]
(170)

Therefore,
\[
m^{1++}_{n}(\text{IR}, r_h = 0) = m^{2+}_{n}(\text{IR}, r_h = 0) = m^{0-}_{n}(\text{IR}, r_h = 0).
\]
(171)

Further, in the UV region:
\[
V_{UV}(z) = \frac{1}{4}(\hat{m}^2 - 10)e^{-z} - \frac{3}{4}(\hat{m}^2 - 5)e^{-2z} + 1 + O(e^{-3z}),
\]
(172)

implying that \( V_{UV}(z) > 0 \) for \( z \in [\log(\frac{1}{8}(\hat{m}^2 + \sqrt{\hat{m}^4 + 28\hat{m}^2 - 140} + 10)) = \log(3 + O(\frac{1}{\hat{m}^7})), \infty) \). This yields the following WKB quantization condition:
\[
\int_{0}^{\infty} \sqrt{V_{UV}(z)} = \frac{\pi\hat{m}}{4\sqrt{3}} = \left( n + \frac{1}{2} \right)\pi,
\]
(173)

from which one obtains
\[
m^{1++}_{n}(\text{UV}, r_h = 0) = (3.46 + 6.93n)\ r_0 \ L^2.
\]
(174)

(c) NLO-in-N/non-conformal corrections using WKB method: In the IR region, the ‘potential’ including NLO-in-N corrections in the \( r_h = 0 \) limit, is given by
\[
V(\text{IR}, r_h = 0) = \frac{1}{4}e^{2z}(\hat{m}^2 + 5)
\]
\[
+ \frac{1}{512\pi^2 N}\{g_s M^2 e^{2z}(\log(\gamma_0)(g_s N_f (\hat{m}^2(-12\log N + 72
\]
\[
+ \log(4096) - 15\log(16)) + 720) + 96\pi\hat{m}^2)
\]
\[
+ 60(8\pi - g_s N_f (\log N - 12 + \log(16)))
\]
\[
+ 72g_s \hat{m}^2 N_f \log^2(\gamma_0)) - \frac{1}{4},
\]
(175)

whose turning points are
\[
\log \left( \frac{1}{m} \left[ 1 - g_s M^2 \log(\gamma_0)(g_s N_f (-12\log N + 72 + \log(4096) - 15\log(16)) + 72g_s N_f \log(\gamma_0) + 96\pi) \right] \frac{256\pi^2 N}{256\pi^2 N} \right), \log(\delta^2 - 1) \right] ,
\]

In the IR region we get the potential as
\[
V_{IR}(z) = -\frac{1}{4} + \frac{1}{4}(5 + \hat{m}^2)e^{2z} + O(e^{3z}),
\]
(169)

giving the turning points \( z \in [\log(\frac{1}{m} + O(\frac{1}{m^7})) \approx -\log \hat{m}, \log(\delta^2 - 1)] \) and the WKB quantization condition becomes

and the integral of \( \sqrt{V(\text{IR}, r_h = 0)} \) between these turning points yields isospectrality with the \( 0^{--} \) and \( 2^{++} \) NLO-in-N spectrum.
7 2++ Glueball masses from type IIB

7.1 $r_h \neq 0$

The 10-dimensional type IIB supergravity action in the low energy limit is given by

$$\frac{1}{2k_{10}^2} \left\{ \int d^{10}x \, e^{-2\phi} \sqrt{-G} \left( R - \frac{1}{2} H_3^2 \right) - \frac{1}{2} \int d^{10}x \sqrt{-G} \left( F_3^2 + \tilde{F}_3^2 + \frac{1}{2} \tilde{F}_5^2 \right) \right\}, \quad (176)$$

where $\phi$ is the dilaton, $G_{MN}$ is the 10-d metric and $F_1$, $H_3$, $\tilde{F}_3$, $\tilde{F}_5$ are different fluxes.

The five-form flux $F_5$ and the three-form flux $\tilde{F}_3$ are defined by

$$\tilde{F}_3 = F_3 + \frac{1}{2} B_2 \wedge F_3, \quad \tilde{F}_3 = F_3 - C_0 \wedge H_3, \quad (177)$$

where $F_3$ and $F_5$ are sourced by the $D_3$ and $D_5$ branes, respectively. $B_2$ is the NS–NS two-form and $C_0$ is the axion. For the three-form fluxes $\tilde{F}_3$, $H_3$, the two-form $B_2$ and the axion $C_0$ [27], see (9). Now varying the action in (176) with respect to the metric $g_{\mu\nu}$, one gets the following equation of motion:

$$R_{\mu\nu} = \frac{5}{4} e^{2\phi} \tilde{F}_{\mu p_2 p_3 p_4 p_5} \tilde{F}_{\nu p_2 p_3 p_4 p_5} - \left( \frac{g_{\mu\nu}}{8} \right) e^{2\phi} \tilde{F}_{5}^2 + \frac{3}{2} H_{\mu\alpha \gamma \alpha \gamma} H_{\nu \alpha \gamma \alpha \gamma} - \left( \frac{g_{\mu\nu}}{8} \right) H_3^2 + \frac{3}{2} e^{2\phi} \tilde{F}_{\mu \alpha \alpha \gamma a \gamma} \tilde{F}_{\nu \alpha \alpha \gamma a \gamma} - \left( \frac{g_{\mu\nu}}{8} \right) e^{2\phi} \tilde{F}_{3}^2 + \frac{1}{2} e^{2\phi} F_{\mu \nu} F_{\nu \mu}, \quad (178)$$

and we consider the following linear perturbation of the metric:

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}, \quad (179)$$

where as before $\mu$, $\nu = \{ r, x_1, x_2, x_3, \theta_1, \theta_2, \phi_1, \phi_2, \psi \}$. Here the only non-zero component according to the tensor mode of metric fluctuation is $h_{x_2 x_3}$. Since the non-zero components of $F_1$, $\tilde{F}_3$ and $H_3$ have no indices as $\{ x_2, x_3 \}$, the final equation of motion gets simplified and can be shown to be given as

$$R_{x_2 x_3}^{(1)} = \frac{5}{4} e^{2\phi} (4 \tilde{F}_{x_2 x_3 p_1 p_2 p_3} \tilde{F}_{x_2 x_3 p_4 p_5 p_6} P_{P_1 P_2 P_3} P_{P_4 P_5 P_6} h_{x_2 x_3}) - \left( \frac{\delta h_{x_2 x_3}}{8} \right) e^{2\phi} \tilde{F}_{5}^2 - \left( \frac{\delta h_{x_2 x_3}}{8} \right) H_3^2 - \left( \frac{\delta h_{x_2 x_3}}{8} \right) e^{2\phi} \tilde{F}_{3}^2. \quad (180)$$

Working at a particular value of $\theta_1$ and $\theta_2$ given as $\theta_1 = N^{-1/5}$ and $\theta_2 = N^{-3/10}$, the square of different fluxes figuring in (180) at the lowest-order in $N$, are given in (A1). Writing the perturbation $h_{x_2 x_3}$ as $h_{x_2 x_3} = \frac{r^2}{2r^2 + N M^2} H(r) e^{ikx_1}$.

Eq. (180) reduces to the following second order differential equation in $H(r)$:

$$H''(r) + \left( \frac{5r^4 - r_h^4}{r^6 - r_h^6} \right) \frac{9a^2}{r^3} + \left( \frac{3}{256\pi^2 N^2 M^2} \right) \times \left\{ \begin{array}{c}
-5a^2 \xi^2 M^2 N_f - 72a^2 \xi^2 M^2 - 768a^2 x^2 \\
+ 12a^2 \xi^2 M^2 N_f r^2 + 9a^2 \xi^2 M^2 N_f \log(16) \\
- 2a^2 \xi^2 M^2 r^2 \log(16) + 16a^2 \xi^2 M^2 x^2 \\
+ g_0^2 \xi^2 M^2 N_f (9a^2 - 2x^2) \log(N) - 2g_0^2 \xi^2 M^2 N_f \\
\times (9a^2 - 2x^2) \log(r) \end{array} \right\} H'(r) + \left( \frac{4\pi e^r (r^6 - r_h^6)}{r^3} \right) \times (8\pi a^2 (6\pi g_0 N_f x^2 - 9a^2 + 9a^2 x^2) \\
- 2\pi g_0 N_f x^2 r^2 + 4\pi a^2) + 3g_0^2 \xi^2 M^2 r^2 (r^2 - 3a^2) \\
\times \log(r) (\pi \xi^2 M^2 \log(16) - 6\pi \xi^2 N_f - 8r) \\
- 36\pi g_0^2 M^2 N_f x^2 (r^2 - 3a^2) \log(r) \right\} \times H(r) = 0. \quad (181)$$

7.1.1 Mass spectrum from Neumann boundary condition at $r = r_h$

To get a sensible answer, one has to perform a small-$T$ expansion when one rewrites and solves (181) around $r = r_h$. This time the analog of (93) becomes

$$0.5 - \frac{0.17401 m^2}{m^2} \left( 1 - \frac{3 (g_0, M^2 (2 \log(N) + 1) + 11.9063 + 4.90^2 M^2 \log(T + 0.6 N)^2)}{m^2} \right) \left( 4 - 7 \frac{3 (g_0, M^2 (2 \log(N) + 11.9063) + 4.90^2 M^2 \log(T + 0.6 N)^2)}{m^2} \right) \right) = -n, \quad (182)$$

which for $g_s = 0.8$, $N = g_s^{3+9} \sim 6000$, $M = 3$ yields

$$m_{2++} = 4.975 T, \quad \left( 183 \right)$$

Thus, one obtains an approximate match between the ground state $2^{++}$ mass from M-theory and type IIB string theory.
Using the variables of [32], the ‘potential’, defining \( m = \tilde{m} \frac{\sqrt{\pi}}{\sqrt{z}} \), yields

\[
V(2^{++}, \text{IIB}, r_h \neq 0) = \frac{1}{4(e^z + 1)^3(e^z + 2)^2} \times [e^z(3b^2(e^z + 2) - (m^2 - 6)e^z - m^2 + 3e^{2z} + 6) + (e^z - 1)((25 - 3m^2)e^z - (m^2 - 18)e^{2z} - 2(m^2 - 6) + 4e^{3z})] + \mathcal{O}\left(\frac{g_s M^2}{N}\right).
\]  

(184)

In the IR region, the potential is given by

\[
V(\text{IR}, T) = e^z(0.15\tilde{m}^2 - 1.3375)e^z - 0.01\tilde{m}^2 + 0.06 + \mathcal{O}(e^{3z}),
\]

and in the IR region, for \( \tilde{m} > 2.986, V(\text{IR} < T) > 0 \) for \( z \in [\log(0.067 + \mathcal{O}(\frac{1}{m})) \approx -2.708, -2.526] \) and \( \int_{-2.526}^{-2.708} \sqrt{V(\text{IR}, T)} \approx 0 \) – hence the IR region provides no contribution to the WKB quantization.

In the UV region,

\[
V(\text{UV}, T) = \frac{1}{4}(\tilde{m}^2 - 9.24)e^{-z} - \frac{3}{4} \tilde{m}^2 + 0.36(\tilde{m}^2 + 9) + 3
\]

\[ + \mathcal{O}(e^{-3z}), e^{-2z} - 1, \]

(185)

the turning points for \( \tilde{m} > 7.141 \) are \( (\log(4.08 + \mathcal{O}(\frac{1}{\tilde{m}}), 0.25\tilde{m}^2 - 1.77)) \). Hence,

\[
\int_{\log(0.25\tilde{m}^2 - 1.77)}^{\log(2.5\tilde{m}^2 - 1.77)} \sqrt{V(\text{UV}, T)}
\]

\[ = \int_{\log(0.25\tilde{m}^2 - 1.77)}^{\log(2.5\tilde{m}^2 - 1.77)} \frac{e^{-z}}{2} \sqrt{e^z - 4.08\tilde{m}} + \mathcal{O}\left(\frac{1}{\tilde{m}}\right)
\]

\[ = 0.389\tilde{m} - 2 = \left(n + \frac{1}{2}\right)\pi, \]

(186)

from which one obtains

\[
m^{2++}_n (T) = (9.18 + 8.08n) \frac{r_h}{LT}.
\]

(187)

Hence, the string theory \( 2^{++} \) glueball is isospectral with \( 0^{++} \); in the large-\( n \) limit of the spectrum, M-theory and type IIB spectra coincide.

\[
7.2 \ r_h = 0
\]

\[
7.2.1 \ WKB \ method
\]

Once again defining \( r = \sqrt{\tilde{r}}, r_h = \sqrt{\tilde{r}_0}, y = y(1 + e^z) \) analogous to their \( r_h \neq 0 \) redefinitions of [32], the ‘potential’ in the IR region, is given by

\[
V(r_h = 0) = -\frac{1}{4} + \frac{1}{4}(1 + \tilde{m}^2)e^{2z} + \mathcal{O}(e^{-3z}).
\]

(189)

The domain in the IR region over which \( V(y) = 0 \) is \([\log(5 + \tilde{m}^2), \log(\delta^2 - 1)] \) and

\[
\int_{-2.526}^{-2.708} \sqrt{V(\text{IR}, T)} \approx 0
\]

yields

\[
m^{2++}_n (\text{IR, IIB}, r_h = 0) = m^{2++}_n (\text{IR, M theory}, r_h = 0).
\]

(190)

In the UV region,

\[
V(UV, r_h = 0) = \frac{1}{4}(\tilde{m}^2 - 10)e^{-z} - 3\frac{1}{4} \tilde{m}^2 - 5)e^{-2z}
\]

\[ + 1 + \mathcal{O}(e^{-3z}), \]

(191)

the zeros of which, as functions of \( e^z \), are at \( \left(\frac{1}{2}(-\tilde{m}^2 \pm \sqrt{\tilde{m}^4 + 28\tilde{m}^2 - 140}) + 10\right) = (-\tilde{m}^2 - \frac{1}{2} \tilde{m}^2 + \mathcal{O}(1)) \). Hence the domain of integration over which \( V(UV, r_h = 0) > 0 \) is \([\log 3, \infty] \). Therefore,

\[
\int_{\log 3}^{\infty} \sqrt{V(UV, r_h = 0)} = \frac{1}{2} \int_{\log 3}^{\infty} e^{-z} \sqrt{e^z - 3\tilde{m}} + \mathcal{O}\left(\frac{1}{\tilde{m}}\right)
\]

\[ = \tilde{m}\pi - \frac{3}{4}\sqrt{3} = \left(n + \frac{1}{2}\right)\pi, \]

(192)

yielding

\[
m^{2++}(r_h = 0) = (3.464 + 6.928n) \frac{r_0}{L^2}.
\]

(193)

\[
7.2.2 \ NLO-in-N/non-conformal \ corrections \ in \ the \ IR \ region \ in \ the \ r_h = 0 \ limit
\]

The ‘potential’ inclusive of the NLO-in-N corrections in the IR region in the \( r_h = 0 \) limit, reads

\[
V(\text{IR}, r_h = 0) = e^{2z} \left(\frac{g_s M^2}{N}\left(\frac{1}{7}\right) 2^{1/5} (g_s N_f (6 \log N - 72 + \log(16777216)) - 72 g_s N_f \log(y_0) - 48\pi) + \frac{1}{4}(\tilde{m}^2 - 3)\right)
\]

\[ - \frac{1}{4} + \mathcal{O}(e^{-3z}), \]

(194)
whose turning points, in the large-$\tilde{m}$ limit, are $\log(\frac{1}{\tilde{m}} + O(\frac{1}{\tilde{m}^2})) \equiv \log(\delta^2 - 1) \approx \log(\delta^2 - 1)$. Now,

$$
\int_{-\log \tilde{m}}^{\log(\delta^2-1)} \sqrt{V(\text{IR}, r_h = 0)} = \frac{(\delta^2 - 1) \tilde{m} - \pi}{2} + O(\frac{1}{\tilde{m}}) = (n + \frac{1}{2}),
$$

which yields the same $LO$ spectrum as $0^{--}, 1^{++}$ and the $2^{++}$ spectrum obtained from M-theory. Hence, the type IIB at $r_h = 0$ is unable to capture the non-conformal NLO-in-$N$ corrections in the $2^{++}$ corrections, as the same either precisely cancel out or are $\frac{1}{\tilde{m}}$-suppressed in the large-$\tilde{m}$ limit, in a IIB computation.

### 8 Summary and discussion

Supergravity calculations of glueball spectra in top–down holographic duals of large-$N$ thermal QCD at finite string/gauge coupling and not just $g_sN \gg 1$, have thus far been missing in the literature. Such a limit is particularly relevant to sQGP [3]. This work fills in this gap by working out the spectra of $0^{++}, 0^{--}, 1^{++}, 2^{++}$ glueballs in a type IIB/delocalized SYZ IIA mirror/(its) M-theory (uplift) model corresponding to the top–down holographic dual of [27] in the MQGP limit introduced in [28]. As discussed in Sect. 2.1, towards the end of a Seiberg-duality cascade in the IR region, despite setting $M(= \bar{N}_c$ in the IR region) to three in the MQGP limit, due a flavor–color enhancement of the length scale as compared to the KS model, one can trust supergravity calculations without worrying about stringy corrections. Further, in the MQGP limit, all physical quantities,
Table 2: Comparison of [62]'s $N \rightarrow \infty$ lattice results for $0^{++}$ glueball with our supergravity results obtained using WKB quantization condition and redefinitions of [32] for M-theory scalar metric fluctuations

| State | $N \rightarrow \infty$ entry in Table 34 of [62] in units of square root of string tension | M-theory scalar metric perturbations (Sect. 6.1.2 – in units of $\frac{1}{L^2}$) | Type IIB dilaton fluctuations of [60] in units of reciprocal of temporal circle’s diameter |
|-------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| $0^{++}$ | 4.065 ± 0.055 | 4.267 | 4.07 (normalized to match lattice) |
| $0^{+++}$ | 6.18 ± 0.13 | 6.251 | 7.02 |
| $0^{++++}$ | 7.99 ± 0.22 | 7.555 | 9.92 |
| $0^{+++++}$ | – | 8.588 | 12.80 |
| $0^{++++++}$ | – | 9.464 | 15.67 |

as seen in all calculations in [33], receive non-conformal corrections that appear at the NLO in $N$ and display a universal $\frac{g_s M^2(M,N_f)}{N}$ suppression.

It should be noted that a numerical computation like the ‘shooting computation’ used in a lot of holographic glueball spectrum computations will not be feasible to use for the following reason. In the ‘shooting method’, like [60], one can first solve the EOMs in the UV using the infinite series/Frobenius method and then numerically (via Euler’s method, etc.) obtain the solution at the horizon where one imposes a Neumann boundary condition. By matching the value obtained by numerically ‘shooting’ from the UV to the horizon in the IR region and matching the radial derivative of the solution so obtained to zero, one can obtain quantized values of the glueball masses. The caveat is that one should have at hand the exact radial profile of the effective number of fractional $D3$-branes ($D5$-branes wrapping the small two-cycle) and the number of flavor branes which would correctly interpolate between $(M, N_f) = (0,0)$ in the UV and $(M = 3, N_f = 2)$ in the IR region. But we do not have this information – we know the values in the IR and the UV region but not for the interpolating region. Hence, numerical methods such as the ‘shooting method’ could at best be used, to obtain only the LO-in-$N$ results, not the NLO-in-$N$ results, which is one of the main objectives of our computations.

The summary of all calculations is given in Table 1 (and Fig. 2) and Table 3 – the former table/graph having to do with a WKB quantization calculation using the coordinate/field redefinitions of [32] and the latter table having to do with obtaining the mass spectrum by imposing Neumann/Dirichlet boundary condition at $r_h/IR$ cut-off $r_0$. Some of the salient features of the results are given as separate bullets.

It should be noted that the last two columns in Tables 1 and 3 have been prepared in the same spirit as the last columns in Table 2 of [61].
Table 3 Summary of glueball spectra from Type IIB, IIA and M Theory for $r_h \neq 0/r_h = 0$ using Neumann/Dirichlet boundary conditions at the horizon $r_h/\text{IR}$ cut-off $r_0$

| S. no. | Glueball | Spectrum using N(eumann)/D(irichlet) b.c., $r = r_h$ (units of $\pi T$) | Spectrum using N(eumann)/D(irichlet) b.c., $r = r_0$ (units of $\pi T^2$) |
|--------|----------|-------------------------------------------------|-------------------------------------------------|
| 1      | $0^{++}$ | (M-theory) (N) $12.25\sqrt{2+n}$ | (M-theory) (N) $4.1$ |
| 2      | $0^{--}$ | (Type IIA) (N/D) $\frac{1.1}{T}\sqrt{n}$ | (Type IIA) (N) $m_{n=0}^{0-} = 0, m_{n=1}^{0-} \approx 3.4, m_{n=2}^{0-} \approx 4.35$ |
| 3      | $0^{--}$ | (Type IIB) (N/D) $m_{n=0}^{0--}(T) = 0, m_{n=1}^{0--}(T) = \frac{32.46}{n}$, $m_{n=2}^{0--}(T) = \frac{32.88}{n}$ | (Type IIB) (N/D) $m_{n=0}^{0--}(r_h = 0)$ $\approx 1.137$ |
| 4      | $1^{++}$ | (M-theory) (N/D) $m_{n=0}^{1++}(T) = 2.6956, m_{n=1}^{1++}(T) = 2.8995$ | (M-theory) (N) $m_{n=0}^{1++}(r_h = 0)$ $\approx 0.665$ |
| 5      | $2^{++}$ | (M-theory) (N) $m_{n=0}^{2++}(T) = \frac{5.086}{n}, m_{n=1}^{2++}(T) = \frac{5.269}{n}$ | $m_{n=0}^{2++}(D, T) = 0, m_{n=1}^{2++}(D, T) = 0.518$ |

The $r_h \neq 0$ glueball spectrum is plotted in Fig. 2. Some of the salient features of Table 1 and Fig. 2 are presented below:

1. Interestingly, via a WKB quantization condition using coordinate/field redefinitions of [32], the lightest $0^{++}$ glueball spectrum for $r_h \neq 0$ coming from scalar metric fluctuations in M-theory compares rather well with the $N \to \infty$ lattice results of [62] – refer to Table 2. Also, similar to [63], the $0^{++}$ coming from the scalar fluctuations of the M-theory metric is lighter than the $0^{++}$ coming from type IIB dilaton fluctuations. Furthermore, interestingly, one can show that by using the coordinate and field redefinitions of [64] when applied to the EOM for the dilaton fluctuation to yield a WKB quantization condition, for $a = 0.6r_h$ – as in [33] – one obtains a match with the UV limit of the $0^{++}$ glueball spectrum as obtained in [32]. For our purpose, the method based on coordinate/field redefinitions of [64] is no good for obtaining the $0^{++}$ glueball ground state and was not used for any other glueball later on in subsequent calculations in this paper.

2. Also, from Table 1/Fig. 2, $m_{n>0}^{2++} > m_{n>0}^{0++}$ (scalar metric perturbations), similar to [63].

3. The higher excited states of the type IIA $0^{--}$ glueball, for both $r_h \neq 0$ and $r_h = 0$, are isospectral. This is desirable because large $n$ corresponds to the UV and that takes one away from the BH geometry, i.e., towards $r_h = 0$.

4. The non-conformal corrections up to NLO in $N$, have a semi-universal behavior of $\frac{(g_s M^2)(g_s N_f) \log(r_h/r_0)}{N}$ and turn out to be multiplied by a numerical pre-factor of $O(10^{-2})$; we could disregard the same in the MQGP limit.

5. As per a more recent lattice calculation [65], the $0^{--}$-glueball has a mass $4.16 \pm 0.11 \pm 0.04$ (in units of the reciprocal of the ‘hadronic scale parameter’ of [66]), which compares rather well with $m_{n=0}^{0--} = 4.267$ (in units of $r_h T$) of Table 2 coming from scalar fluctuations of the M-theory metric. Similarly, the $0^{++}$-glueball in [65] has a mass $6.25 \pm 0.06 \pm 0.06$ and from Table 1, which matches rather nicely with $m_{n=0}^{0++}(\delta = 1.26) = 6.25$ (in units of $r_h T$) of Table 1 coming from type IIA one-form fluctuation.

6. The ground state and the $n \gg 1$ excited states of $1^{++}$ and $0^{--}$ glueballs are isospectral.

7. The higher excited $r_h \neq 0$ $2^{++}$ glueball states corresponding to metric fluctuations of the M-theory metric

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3 We thank P. Majumdar for bringing this reference to our attention.
The following is the comparison of ratios of $0^{++}$ glueball masses obtained in this work from Neumann/Dirichlet boundary conditions at the horizon, with [60]:

Hence, for higher excited states, the ratio of masses of successive excited states approaches unity faster as per our results as compared to [60].

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### Appendix A: $\tilde{F}_2^2$, $\tilde{F}_3^2$, $H_3^2$

The expressions of squares of various fluxes that figure in the EOM (180) are given below for ready reference:

$$\tilde{F}_2^2 = -\frac{8}{\sqrt{\pi N \sqrt{g_s}}} - \frac{1}{254803968\pi^{13/2}N^{7/10}} \left[ M^4 r^6 N_f g_s^{11/2}(r^4 - r_h^4)(\phi_1 + \phi_2 - \psi)^2(N_f g_s, \log(N)(2(r + 1) \log(r) + 1) + 2(-9(r + 1)N_f g_s \log^2(r) - 2(r + 1) \log(r)2\pi - \log(4)N_f g_s) + \log(4)N_f g_s)^2 \right]$$

$$\tilde{F}_3^2 = \frac{729M^2 N_f \sqrt{g_s}(r^4 - r_h^4)(72a^2 N^{2/5} \log(r) + a^2(-3N^{2/5} + 4) + 2N^{2/5}r^2)}{128\pi^{7/2}N^{11/10}r_6};$$

$$H_3^2 = \frac{243M^2 N_f^2 g_s^{5/2}(r^4 - r_h^4)(144a^2(\sqrt{N} + 3)r \log(r) + a^2(9 - 15\sqrt{N}) + 2(\sqrt{N} + 1)r^2)}{256\pi^{7/2}N^6};$$

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\[
\tilde{F}_{132} = \tilde{F}_{132}^{P_4 P_4} = \frac{60r^4}{\pi N^3 g_s^6} - \frac{1}{169869312\pi^{13/2} N^{7/10}} \\
\times [5M^4 r^{10} N_f g_s^4 (r^4 - r^4_f) (\phi_1 + \phi_2 - \psi)^2 (N_f g_s \log(N)(2 + 1) \log(r) + 1) \\
+ 2(-9(r + 1) N_f g_s \log^2(r) - 2(r + 1) \log(r) (2\pi - 2g_s (N_f g_s) + 4(N_f g_s)))^2] \\
+ 2(-9(r + 1) N_f g_s \log^2(r) - 2(r + 1) \log(r) (2\pi - 2g_s (N_f g_s) + 4(N_f g_s)))^2] \\
\times (2\pi - 2g_s (N_f g_s) + 4(N_f g_s))^2) \\
\times (2\pi - 2g_s (N_f g_s) + 4(N_f g_s))^2.
\]

(A1)

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