Abstract

The main result of this text is a generalization of Perrin-Riou’s $p$-adic Gross-Zagier formula to the case of Shimura curves over totally real fields. Let $F$ be a totally real field. Let $f$ be a Hilbert modular form over $F$ of parallel weight 2, which is a new form and is ordinary at $p$. Let $E$ be a totally imaginary quadratic extension of $F$ of discriminant prime to $p$ and to the conductor of $f$. We may construct a $p$-adic $L$ function that interpolates special values of the complex $L$ functions associated to $f$, $E$ and finite order Hecke characters of $E$. The $p$-adic Gross-Zagier formula relates the central derivative of this $p$-adic $L$ function to the $p$-adic height of a Heegner divisor on a certain Shimura curve.

The strategy of the proof is close to that of the original work of Perrin-Riou. In the analytic part, we construct the analytic kernel via adelic computations; in the geometric part, we decompose the geometric kernel into two parts: places outside $p$ and places dividing $p$. For places outside $p$, the $p$-adic heights are essentially intersection numbers and are computed in works of S. Zhang, and it turns out that this part is closely related to the analytic kernel. For places dividing $p$, we use the method in the work of J. Nekovář to show that the contribution of this part is zero.

Résumé

Le résultat principal de ce texte est une généralisation de la formule de Gross-Zagier $p$-adique de Perrin-Riou au cas de courbes de Shimura sur les corps totalement réels. Soit $F$ un corps totalement réel. Soit $f$ une forme modulaire de Hilbert sur $F$ de poids parallel 2, qui est une forme nouvelle et est ordinaire en $p$. Soit $E$ une extension quadratique totalement imaginaire de $F$ de discriminant premier à $p$ et au
conducteur de $f$. On peut construire une fonction $L$ $p$-adique qui interpole valeurs spéciales de la fonction $L$ complexe associée à $f$, $E$ et caractères de Hecke d’ordre fini de $E$. La formule $p$-adique de Gross-Zagier relie la dérivée centrale de cette fonction $L$ $p$-adique à la hauteur d’un diviseur de Heegner sur une certaine courbe de Shimura.

La stratégie de la preuve est proche de celle du travail original de Perrin-Riou. Dans la partie analytique, on construit le noyau analytique par calculs adéliques; dans la partie géométrique, on décompose le noyau géométrique en deux parties: places hors de $p$ et places divisant $p$. Pour les places hors de $p$, les hauteurs $p$-adiques sont essentiellement des nombres d’intersection et sont calculées dans les travaux de S. Zhang, et il s’avère que cette partie est bien liée au noyau analytique. Pour les places divisant $p$, on utilise la méthode dans le travail de J. Nekovár pour montrer que la contribution de cette partie est nulle.

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1 Introduction

The original Gross-Zagier formula, proved in [3], expresses the central derivative of the $L$ function associated to a new form of weight 2 over $\mathbb{Q}$ and an imaginary quadratic field as the (real-valued) height of a Heegner divisor on the modular curve.

Later in [11], Perrin-Riou attached a $p$-adic $L$ function to a $p$-ordinary new form of weight 2 and an imaginary quadratic field, which interpolates special values of the complex $L$ functions. She then proved that the central derivative of this $p$-adic $L$ function can be expressed as the $p$-adic height of a Heegner divisor. This was generalized by Nekovár [8] to the case of higher weight modular forms over $\mathbb{Q}$.

On the other hand, the original (complex, or real) Gross-Zagier formula has been generalized by Zhang [17] (and subsequent works) to Hilbert modular forms over any totally real field. In this paper we would like to prove a $p$-adic version of Zhang’s formula, which is a generalization of Perrin-Riou’s work to the case of any totally real field.

Let $p$ be a fixed prime number. Fix injections $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ and $\mathbb{Q} \hookrightarrow \mathbb{C}$. Let $F$ be a totally real number field, $\mathcal{O} = \mathcal{O}_F$ its integer ring and $N$ an ideal of $\mathcal{O}$ prime to $p$. Let $f$ be a new form of level $N$, parallel weight 2 and trivial central character. The form $f$ admits a $q$-expansion:

$$f = \sum_{0 \neq I \subseteq \mathcal{O}} a(I, f) q^I.$$ 

We assume that the form $f$ is ordinary at $p$, i.e. the coefficient $a(p\mathcal{O}, f)$ has $p$-adic absolute value 1.

Let $E$ be a totally imaginary quadratic extension of $F$ with relative discriminant $D = D_{E/F}$, such that all prime ideals dividing $2NP$ split in $E$. Then for any finite order character $\chi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$, the complex $L$ function is given by:

$$L(s, f, \chi) = L(2s - 1, \epsilon_{\chi_F}) D(s, f, \chi),$$

where $\epsilon : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ is the quadratic character associated to the extension $E/F$, $\chi_F$ is the restriction of $\chi$ to $F$, and $D(s, f, \chi)$ is:

$$\sum_{I \subseteq \mathcal{O}_E} a(N_{E/F}(I), f) \chi(I) |I|_\infty^{-s}$$

(We use $|I|_\infty$ to denote the absolute norm of $I$).
We are only interested in characters $\chi$ that factor through the quotient $G = \mathbb{A}_E^\times / E^\times \prod_{v \mid p} O_E^\times$, which is the Galois group $\text{Gal}(E^{(p)}/E)$ with $E^{(p)}$ the maximal unramified outside $p$ abelian extension of $E$. Our first result is that there exists a $p$-adic pseudo-measure $K_f$ on the group $G$, with possible poles only on characters $\chi$ such that $\chi_F$ is the $p$-cyclotomic character, and satisfies the interpolation property: for any finite order character $\chi : G \to \mathbb{C}^\times$, the integral $\int_G \chi K_f$ is, up to some simple factor, equal to $L(1, f, \chi^{-1})$.

Let $X$ be the rigid analytic space $\text{Hom}_{\text{cont}}(G, \mathbb{C}_p^\times)$, then the pseudo-measure $K_f$ induces a meromorphic function $L_p$ on the space $X$ via: $L_p(x) = \int x K_f$, which is analytic on the anti-cyclotomic line $\{ x \in X : x = \tau \}$. Thus we may take the derivative of the function $L_p$ on the trivial character and in the direction of the cyclotomic character $\xi_E$, $L'_p, \xi_E(1) = \frac{d}{ds} L_p(\xi_E^s) \big|_{s=0}$.

We would like to relate this derivative to the $p$-adic height of a Heegner divisor on a certain Shimura curve. We make the assumption that $\epsilon(\mathcal{N}) = (-1)^{|F:Q|-1}$. Fix an archimedean place $\tau$ of $F$. The assumption $\epsilon(\mathcal{N}) = (-1)^{|F:Q|-1}$ ensures that there is a quaternion algebra $B$ over $F$, which ramifies exactly at all infinite places different from $\tau$ and all finite places $v$ such that $\epsilon_v(\mathcal{N}) = -1$. Since every prime ideal dividing $\mathcal{N}$ splits in the field $E$, we may embed $E$ into $B$ and thus view $E$ as a sub-algebra of $B$.

Let $R$ be an order of $B$ of type $(\mathcal{N}, E)$, i.e. an order that contains $O_E$ and has discriminant $\mathcal{N}$. The existence of such an order is proved in [17] section 1.5. We then have a Shimura curve $X$, whose complex points are given by:

$$X(\mathbb{C}) = B^\times \backslash \mathcal{H}^\pm \times \hat{B}^\times / \hat{\mathcal{F}}^\times \hat{R}^\times.$$ 

Here we let $B^\times$ act on the Poincaré double half plane $\mathcal{H}^\pm$ via the fixed place $\tau$ and a fixed isomorphism $B_{\tau} \simeq M_2(\mathbb{R})$.

Unlike the classical case, there is no "cusp" for the curve $X$ if the field $F$ is not equal to $\mathbb{Q}$, which we always assume. But Zhang [17] defined a canonical divisor class $\xi \in \text{Pic}(X) \otimes \mathbb{Q}$, the "Hodge class", which has degree 1 on every connected component of $X$. This replaces the cusp $\infty$ in the classical case, and allows us to define a map $\phi : X \to \text{Jac}(X) \otimes \mathbb{Q}$ by sending a point $y$ to the class of $y - \xi$.

Under the complex description, there are CM points on the curve $X$, which are represented by pairs $(z_0, b) \in \mathcal{H}^\pm \times \hat{B}^\times$, where $z_0$ is the unique fixed point of $E^\times$ in $\mathcal{H}^\pm$. Shimura’s theory shows that these points are algebraic and defined over the
maximal abelian extension of $E$.

A CM point that is defined over the Hilbert class field $H$ of $E$ is called a "Heegner point". The group $\text{Gal}(H/E)$ acts simply transitively on the set of Heegner points. Let $z$ be the divisor class $\frac{1}{\#(\mathcal{O}_E^*/\mathcal{O}_E^r)} \sum_x \phi(x) \in \text{Jac}(X) \otimes \mathbb{Q}$, where the sum runs over all Heegner points. This is the trace (or the average) of any Heegner point. Let $z_f$ be the $f$-isotypic part of $z$ via the Jacquet-Langlands correspondence.

The general theory of $p$-adic height pairings of Zarhin and Nekovář ([16], [7]) constructs a $p$-adic height pairing $\langle \cdot, \cdot \rangle$ on $\text{Jac}(X)(E) \times \text{Jac}(X)(E)$. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_l$ be all prime ideals of $\mathcal{O}$ above $p$, and for each $i$, let $\alpha_i$ be the unique root of the polynomial $X^2 - a(\mathfrak{P}_i, f)X + |\mathfrak{P}_i|_{\infty}^f$ that is a $p$-adic unit. Our main result can be stated as:

**Theorem 1.1.** We have the following identity:

$$L'_p,\xi_E(1) = \left( \prod_{i} \frac{(\alpha_i - 1)^3}{(\alpha_i + 1)(\alpha_i^2 - |\mathfrak{P}_i|_{\infty})} \right) \langle z_f, z_f \rangle.$$

**Remark** In [2], D. Disegni obtained independently the same result. It turns out that the analytic part of his work is almost the same as ours (with slight difference in the construction of Eisenstein series), but the geometric methods are different.

Let us mention one application of this theorem. As in the classical case, let $A$ be a modular elliptic curve over $F$ with conductor $N$, parameterized by the curve $X$, i.e. we have a surjective morphism $X \to A$, and hence a morphism $\text{Jac}(X) \to \text{Jac}(A) = A$. We assume that $A$ has ordinary good reduction at places above $p$, thus corresponds to a $p$-ordinary new form $f$ of parallel weight 2 and level $N$. Let $z_A$ be the projection of the divisor $z$ on $A$, which is an $E$-rational point on the curve $A$. We then have a $p$-adic $L$ function $L_p(A_E, \cdot)$ associated to the curve $A$ and to the quadratic extension $E$, which interpolating special values of the complex $L$ functions $L(A_E, \chi, 1)$, and our formula becomes:

$$L'_p,\xi_E(A_E, 1) = C(z_A, z_A),$$

where $C$ is a non-zero factor.

**Corollary 1.2.** If the order of vanishing of the $p$-adic $L$ function $L_p(A_E, \cdot)$ at the trivial character is exactly 1, then the group $\mathbb{Z} \cdot z_A$ has finite index in $A(E)$, and the Tate-Safarevič group $\text{III}(A/E)$ is finite.
Proof  This follows from a generalization of Kolyvagin’s method in the classical case, c.f. for example [9] Theorem 3.2.

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Notations and conventions We fix the following default settings throughout the text:

- Fix injections $\mathbb{Q} \hookrightarrow \mathbb{C}$ and $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$;
- $p$: an odd rational prime number;
- $F$: a totally real field, with integer ring $\mathcal{O}$;
- $N$: an ideal of $F$ as in the introduction;
- $\mathbb{A} = \mathbb{A}_F$: the ad`ele ring of $F$;
- $d_F$: the absolute different of $F$;
- $P_1, \ldots, P_l$: prime ideals of $\mathcal{O}$ above $p$;
- $P$: the product of all $P_i$;
- $\text{val}_1, \ldots, \text{val}_l$: corresponding valuations;
- $E$: a totally imaginary quadratic extension of $F$ as in the introduction, with relative discriminant $D = D_{E/F}$;
- $d_{E/F}$: the relative different of $E/F$;
- $\xi: \mathbb{A}_F^\times \rightarrow \hat{\mathbb{Z}}^\times$: the cyclotomic character;
- $\xi_p: \mathbb{A}_F^\times \rightarrow \mathbb{Z}_p^\times$: the $p$-adic cyclotomic character;
- $\psi: \mathbb{A}_F \rightarrow \mathbb{C}^\times$: the additive character defined by $\psi = \psi_Q \circ \text{tr}_{F/Q}$, with $\psi_Q$ the usual additive character on $\mathbb{A}_Q^\times$ such that $\psi_Q,\infty(x) = e^{2\pi ix}$.

If $N$ is an ideal of $\mathcal{O}$, a function $\phi: \mathbb{A}_F^\times \rightarrow \mathbb{C}$ is called a function mod $N$ if it factors through the quotient space $\mathbb{A}_F^\times(1 + \hat{N})^\times F_{\infty,>0}^\times$. A Hecke character $\chi: \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ is called a character mod $N$ if it is a function mod $N$. If $\chi$ is a character mod $N$, it is called primitive if it is not a character mod $D$ for any proper divisor $D$ of $N$. In this case, the ideal $N$ is called the conductor of the character $\chi$.

If $\phi$ is a function mod $N$, we may define a function on the integral ideals of $\mathcal{O}$, denoted by $\phi[N]$, such that:

$$
\phi[N](I) = \begin{cases} 
\phi(i), & \text{if } I \text{ is prime to } N \text{ and } i \in \mathbb{A}_F^\times \text{ satisfies } i\mathcal{O} = I \text{ and } i_v = 1 \text{ for all } v | N; \\
0, & \text{if } I \text{ is not prime to } N.
\end{cases}
$$

The Haar measure on $\mathbb{A}_F$ is normalized to be self-dual with respect to the character $\psi$. 

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All Frobenius maps are by default the "geometric Frobenius", and the reciprocity map of class field theory is normalized in the geometric way.
2 Hilbert Modular Forms

2.1 Definition and $q$-expansion

Let $N$ be an ideal of $O$. Define the following open compact subgroups of $GL_2(\hat{O})$:

\[
K_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{O}) : c \equiv 0 \mod N \right\},
\]
\[
K_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) : d \equiv 1 \mod N \right\}.
\]

Also define $K_\infty := SO_2(F_\infty)$, the standard maximal compact subgroup of $GL_2(F_\infty)$.

Let $\kappa$ be a positive integer. A modular form of (parallel) weight $\kappa$ for $K_1(N)$ is a measurable function $\phi : GL_2(A) \to \mathbb{C}$ such that:

1. $\phi(\gamma g) = \phi(g)$ for any $\gamma \in GL_2(F)$;
2. $\phi(gk r(\theta) a) = \phi(g) e^{i \kappa \theta}$ for any $k \in K_1(N)$, any $\theta \in F_\infty$ and any $a$ in $F_\infty, >0$;
   where for any $\theta \in F_\infty$, we write $r(\theta)$ for the element $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ in $K_\infty$,
   and we write $e^{i \kappa \theta}$ for the product $\prod_{v|\infty} e^{i \kappa \theta_v}$;
3. $\phi$ is slowly increasing, i.e. for every real number $c > 0$ and any compact subset $\Omega$ of $GL_2(A)$, there exist constants $C$ and $M$ such that $\phi \left( \begin{pmatrix} a \\ 1 \end{pmatrix} g \right) \leq C |a|^M$
   for any $g \in \Omega$ and any $a \in \mathbb{A}^\times$ with $|a| \geq c$.

Let $\chi : \mathbb{A}^\times \to \mathbb{C}^\times$ be a character mod $N$, such that $\chi_v(-1) = (-1)^\kappa$ for every infinite place $v$. A modular form of weight $\kappa$ for $K_0(N)$ with central character $\chi$ is a modular form $\phi$ of weight $\kappa$ for $K_1(N)$ such that $\phi(\alpha g) = \chi(\alpha) \phi(g)$ for any $\alpha \in Z(\mathbb{A}^\times)$ (scalar matrices).

If $\phi$ is a modular form, then it has a Fourier expansion:

\[
\phi(g) = W_0(g) + \sum_{\alpha \in F_\infty} W \left( \begin{pmatrix} \alpha \\ 1 \end{pmatrix} g \right),
\]
where:

\[
W_0(g) = \int_{\mathbb{A}/F} \phi \left( \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} g \right) dx,
\]

\[
W(g) = \int_{\mathbb{A}/F} \phi \left( \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} g \right) \psi(-x) dx.
\]

The functions \( W_0 \) and \( W \) satisfy the same condition 2 as the modular form \( \phi \), and have the following property:

\[
W_0 \left( \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} g \right) = W_0(g),
\]

\[
W \left( \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} g \right) = W(g) \psi(x).
\]

If we have \( W_0(g) = 0 \) for almost all \( g \), then the modular form \( \phi \) is called \textbf{cuspidal}; if for any fixed elements \( x_f \in \mathbb{A}_f \) and \( y_f \in \mathbb{A}_f^\times \), the function

\[
\mathcal{H}^n \to \mathbb{C}
\]

\[
x_\infty + iy_\infty \mapsto y_\infty^{-\kappa/2} \phi \left( \begin{pmatrix} y_f & x_f \\ 1 & 1 \end{pmatrix} \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) \right)
\]

is holomorphic on \( \mathcal{H}^n \), then the modular form \( \phi \) is called \textbf{holomorphic}. Here \( y_\infty^{-\kappa/2} \) stands for \( \prod_{v | \infty} y_v^{-\kappa/2} \).

**Proposition 2.1.** Let \( \phi \) be a holomorphic modular form for \( K_1(N) \). Then there exists a unique function on the set of non-zero ideals of \( \mathcal{O} \), \( I \mapsto a(I) \), and a function on the narrow ideal class group \( \mathbb{A}^\times/F^\times F^\times_{\infty,>0} \hat{\mathcal{O}}^\times \), \( y \mapsto a_0(y) \), such that for any \( x \in \mathbb{A} \) and \( y \in \mathbb{A}^\times \) with \( y_\infty > 0 \), we have:

\[
\phi \left( \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} \right) = |y|^{\kappa/2} \left( a_0(y_f d_F) + \sum_{\alpha > 0} a(\alpha y_f d_F) \psi(i\alpha y_\infty) \psi(\alpha x) \right),
\]

where the sum ranges through totally positive elements \( \alpha \) in \( F \), and \( a(I) \) is understood to be zero if \( I \) is not an integral ideal. Furthermore, the modular form \( \phi \) is determined by these functions.

If \( \phi \) is a holomorphic form, we may formally write

\[
\phi = a_0 + \sum_{0 \neq I \subseteq \mathcal{O}} a(I) q^I.
\]

This is called the \textbf{q-expansion} of \( \phi \). The form \( \phi \) is cuspidal if and only if the function \( a_0 \) is zero. We sometimes write \( a(I, \phi) \) to emphasize the dependence on \( \phi \).
The proof is the same as [17] Proposition 3.1.2, but note that the constant term depends on the narrow ideal class, which seems to be an error of loc. cit.

The space of holomorphic modular forms (resp. cusp forms) of weight $\kappa$, level $N$ and central character $\chi$ will be denoted $\mathcal{M}_\kappa(N, \chi)$ (resp. $\mathcal{S}_\kappa(N, \chi)$). When the central character $\chi$ is trivial, we denote also by $\mathcal{M}_\kappa(N)$ and $\mathcal{S}_\kappa(N)$ the corresponding spaces.

### 2.2 Hecke operators

Let $M$ be a non-zero ideal of $\mathcal{O}$. The set

$$U(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\hat{\mathcal{O}}) : c \equiv 0 \pmod{N}, (ad - bc)\mathcal{O} = M \right\}$$

is left and right invariant by $K_0(N)$, and thus defines a Hecke operator $T(M)$ on the space $\mathcal{M}_\kappa(N, \chi)$:

$$(T(M)\phi)(g) := N(M)^{\kappa/2-1} \int_{U(M)} \phi(gh) dh,$$

where the Haar measure on $\text{GL}_2(\mathbb{A}_f)$ is normalized such that the set $K_0(N)$ has volume 1. Equivalently, if we write $U(M)$ as a disjoint union $\bigsqcup_j h_j K_0(N)$, then we have:

$$(T(M)\phi)(g) := N(M)^{\kappa/2-1} \sum_j \phi(gh_j).$$

From the definition it is clear that the operators $T_M$ are multiplicative, i.e. if $M$ and $M'$ are coprime, then we have $T(MM') = T(M)T(M')$.

**Proposition 2.2.** Let $\phi$ be a form in the space $\mathcal{M}_\kappa(N, \chi)$ and $M$ be a non-zero ideal of $\mathcal{O}$. If the $q$-expansions of the forms $\phi$ and $T(M)\phi$ are $a_0 + \sum a(L)q^L$ and $b_0 + \sum b(L)q^L$, respectively, then we have:

$$b(L) = \sum_{J|(L,M)} \chi_{[N]}(J)N(J)^{\kappa-1}a(LM/J^2),$$

$$b_0(L) = \sum_{J|M} \chi_{[N]}(J)N(J)^{\kappa-1}a_0(LM/J^2).$$
Proof} This is the classical calculation of $q$-expansion of Hecke operators. A proof may be found in [17] Proposition 3.1.4.

**Corollary 2.3.** Let $P$ be a prime ideal of $\mathcal{O}$. Then on the space $\mathcal{M}_\kappa(N, \chi)$ we have, for any integer $m \geq 2$:

$$T(P^m) = T(P^{m-1})T(P) - \chi_{[N]}(P)N(P)^{\kappa-1}T(P^{m-2}),$$

Thus all the Hecke operators $T_M$ commute with each other, and satisfy the following formal identity:

$$\sum_M T(M)N(M)^{-s} = \prod_P (1 - T(P)N(P)^{-s} + \chi_{[N]}(P)N(P)^{-2s+\kappa-1})^{-1}$$

**Remark** This can also be deduced directly from the definition of the Hecke operators (interpreted as operators on the Bruhat-Tits tree).

We would also like to define the $V$ operator. Let $N$ and $D$ be ideals of $\mathcal{O}$. Let $d \in \mathbb{A}_\mathbb{F}$ be a generator of $D$. We then define the operator $V_D : \mathcal{M}_\kappa(N, \chi) \rightarrow \mathcal{M}_\kappa(ND, \chi)$ by: $V_D(\phi)(g) := N(D)^{-\kappa/2}\phi\left(g \begin{pmatrix} d^{-1} & 1 \\ 1 & 0 \end{pmatrix}\right)$. An easy calculation shows that if the form $\phi$ has $q$-expansion $a_0(I) + \sum_{0 \neq I \subseteq \mathcal{O}} a(I)q^I$, then the form $V_D(\phi)$ has $q$-expansion $a_0(D^{-1}I) + \sum_{0 \neq I \subseteq \mathcal{O}} a(I)q^{DI}$.

**2.3 Peterson inner product**

Let $\phi_1$ and $\phi_2$ be two modular forms of same weight and same central character. Then we may define their Peterson product as:

$$(\phi_1, \phi_2) := \int_{\text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A})/Z(\mathbb{A})} \overline{\phi_1(g)}\phi_2(g)dg,$$

where $dg$ is the right $\text{GL}_2(\mathbb{A})$-invariant Borel measure on the space $\text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A})/Z(\mathbb{A})$ (cf. [13] Theorem 3.17 for the existence and uniqueness up to a positive scalar). The measure is induced by the measure on $\text{GL}_2(\mathbb{A})/Z(\mathbb{A})$ normalized as follows:

$$\int_{\text{GL}_2(\mathbb{A})/Z(\mathbb{A})} f(g)dg := \int_{A^\times} \int_{\mathbb{A}} \int_{K_0(1)} f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}k\right) |y|^{-1}dkdxd^\times y.$$

Now fix a level $N$, a weight $\kappa$ and a character $\chi$. The Peterson inner product is then a positive definite Hermitian form on the space $\mathcal{S}_\kappa(K_0(N), \chi)$, making it a Hilbert space.
2.4 Fricke and Atkin-Lehner involutions and adjoints of Hecke operators

Still fix $N$, $\kappa$ and $\chi$. We define an application:

$$w_N : \mathcal{M}_\kappa(N, \chi) \rightarrow \mathcal{M}_\kappa(N, \chi^{-1})$$

$$\phi \mapsto \left( g \mapsto \phi \left( g \left( n_f \right)^{-1} \right) \right) \chi^{-1}(\det g),$$

where $n_f \in \mathcal{A}_f^\times$ is any element such that $n_f \mathcal{O} = N$. It is easy to see that this operator is well-defined. It is called the Fricke involution. The word "involution" comes from the fact that $w_N^2$ is the identity map.

More generally, suppose that the ideal $N$ factors as a product $N = MD$ with $(M, D) = 1$, and suppose that the character $\chi$ also factors as $\chi = \chi_M \chi_D$, with $\chi_M$ (resp. $\chi_D$) a character mod $M$ (resp. mod $D$). Then we may similarly define the (partial) Atkin-Lehner involutions $w_M$ by:

$$(w_M \phi)(g) := \phi \left( g \left( m_f \right)^{-1} \right) \chi_M^{-1}(\det g),$$

with $m_f \in \mathcal{A}_f^\times$ a generator of $M$. This operator takes values in the space $\mathcal{M}_\kappa(N, \chi_M^{-1} \chi_D)$.

In particular, in the case that $\chi$ is trivial, we see that for any factor $M$ of $N$ such that $(M, N/M) = 1$, the operator $w_M : \mathcal{M}_\kappa(N) \rightarrow \mathcal{M}_\kappa(N)$ is defined.

The Atkin-Lehner involutions can be used to describe adjoints of Hecke operators. On the Hilbert space $\mathcal{S}_\kappa(N)$, let $T(M)^*$ denote the adjoint of the operator $T(M)$. We then have:

**Proposition 2.4.**

1. $T(M)^* = w_N^{-1}T(M)w_N$;

2. If $M$ is prime to $N$, then we have: $T(M)^* = T(M)$.

**Proof** Since Hecke operators are commutative and generated by $T_P$ for $P$ prime ideals of $\mathcal{O}$, it suffices to prove the results for $T(P)$. This can be done by direct computation, remarking that if $K_0(N) \left( \begin{array}{c} p \\ 1 \end{array} \right) K_0(N)$ has decomposition $\bigsqcup \beta_j K_0(N)$, then $K_0(N) \left( \begin{array}{c} p^{-1} \\ 1 \end{array} \right) K_0(N)$ has decomposition $\bigsqcup p^{-1}u_N^{-1} \beta_j u_N K_1(N)$, where $u_N = \left( n_f \right)^{-1}$ with $n_f$ a generator of $N$. Here $p \in \mathcal{A}_f^\times$ is a fixed generator of $P$. □
2.5 Old forms and new forms

Let $N$ be an ideal of $\mathcal{O}$. Let $M$ be a proper divisor of $N$ and write $D$ for the quotient $N/M$. If $\phi$ is a modular form in $S_\kappa(M)$, then the form $V_D(\phi)$ is in $S_\kappa(N)$.

The subspace of $S_\kappa(N)$ generated by all the forms $V_D(\phi)$ (for $M$ running over all the proper divisors of $N$ and $\phi$ running over all the forms in $S_\kappa(M)$) is called the space of old forms, denoted by $S_\kappa(N)^{\text{old}}$. Any form in this subspace will be called an old form.

The subspace of new forms is the orthogonal complement of the space of old forms with respect to the Peterson inner product, denoted by $S_\kappa(N)^{\text{new}}$.

**Proposition 2.5.** The subspaces $S_\kappa(K_0(N))^{\text{old}}$ and $S_\kappa(K_0(N))^{\text{new}}$ are stable under Hecke operators.

**Proof** It suffices to prove that the old forms are stable under the Hecke operators and their adjoints. By Proposition 2.4 it suffices to prove that the old forms are stable under the operators $T(M)$ and $w_N$. These are all easy to verify. \qed

3 $p$-adic modular forms and Hida’s theory

3.1 $p$-adic modular forms and Hecke operators

We will define $p$-adic modular forms à la Serre.

Let $R$ be any ring. The ring of formal $q$-expansions with coefficients in $R$, denoted by $R[[q]]$, is the set of all formal series of the form $a_0 + \sum_{I \subseteq \mathcal{O}} a(I)q^I$, where $a_0$ is a function on the narrow ideal class group of $F$ and $a$ is a function on the set of non-zero ideals of $\mathcal{O}$, both taking values in $R$. The addition of two series in $R[[q]]$ is the pointwise addition, and multiplication is defined as if they were $q$-expansions of modular forms: if $f = a_0 + \sum a(I)q^I$ and $g = b_0 + \sum b(I)q^I$ are two series in $R[[q]]$, their product $fg$ will be the formal series $c_0 + \sum c(I)q^I$ such that:

\[
\begin{align*}
c_0(I) &= a_0(I)b_0(I), \\
c(I) &= \sum_{\substack{\alpha, \beta I \subseteq I^{-1} \leq \mathcal{O} \\alpha \geq 0, \beta \geq 0 \\alpha + \beta = 1}} a(\alpha I)b(\beta I),
\end{align*}
\]
where in the last equation, the value \( a(\alpha I) \) for \( \alpha = 0 \) is understood to be \( a_0(I) \).

If \( M \) is an ideal of \( \mathcal{O} \), \( f \) is a holomorphic form for \( K_0(M) \) and \( R \) is a sub-ring of \( \mathbb{C} \) that contains all coefficients of \( f \), then we may identify \( f \) with its \( q \)-expansion, considered as an element of the ring \( R[[q]] \).

If \( R = L \) is a fixed \( p \)-adic field, the modular forms lie in the smaller ring of **bounded formal \( q \)-expansions**, \( L\langle q \rangle \), which is by definition the sub-ring of \( L[[q]] \) consisting of those series whose coefficients \( a \) and \( a_0 \) are bounded functions. The ring \( L\langle q \rangle \) is equipped with the sup norm on its coefficients, which makes it an \( L \)-Banach algebra.

Let \( \kappa \) be a positive integer. For \( M \) a non-zero ideal of \( \mathcal{O} \), let \( \mathcal{M}_\kappa(M, \mathbb{Q}) \) denote the intersection of \( \mathcal{M}_\kappa(M) \) with \( \mathbb{Q}[[q]] \). The complex vector space \( \mathcal{M}_\kappa(M) \) is finite dimensional and has a basis consisting of forms having rational \( q \)-expansions, i.e. we have: \( \mathcal{M}_\kappa(M, \mathbb{Q}) \otimes \mathbb{C} = \mathcal{M}_\kappa(M) \).

Define \( \mathcal{M}_\kappa(M, L) \) to be the image of \( \mathcal{M}_\kappa(M, \mathbb{Q}) \otimes L \) in the space \( L[[q]] \). Thus we know that this image lies in the subspace \( L\langle q \rangle \).

Denote by \( \mathcal{M}_\kappa(Mp^\infty, L) \) the closure (with respect to the topology of \( L\langle q \rangle \) induced by its norm) of the union of all the spaces \( \mathcal{M}_\kappa(Mp^r, L) \) for all \( r > 0 \). This is then an \( L \)-Banach space.

If \( \chi \) is any finite order character of conductor dividing \( Mp^\infty \), we define in the same way the space \( \mathcal{M}_\kappa(Mp^\infty, \chi, L) \), just by enlarging \( \mathbb{Q} \) to a number field that contains all values of \( \chi \).

Finally, we define \( S_\kappa(Mp^\infty, L) \) or \( S_\kappa(Mp^\infty, \chi, L) \) to be those \( p \)-adic modular forms with zero constant terms.

It turns out that Hecke operators on the space of (complex) modular forms induce operators on the space of \( p \)-adic modular forms.

In fact, for all \( r > 0 \), the ideals \( Mp^r \) always have the same prime divisors, so the Hecke operators always have the same effects on the coefficients of the \( q \) expansion, and the effects are obviously continuous with respect to the sup norm. Thus the \( p \)-adic Hecke operators can be defined via these actions on \( q \)-expansions.
3.2 Hida’s theory

Let \( N \) be an ideal of \( \mathcal{O} \) prime to \( p \).

A \( p \)-adic modular form in \( \mathcal{M}_\kappa(Np^\infty, L) \) is called “ordinary” if the coefficients \( a(\mathfrak{P}_i) \) are \( p \)-adic units for each \( \mathfrak{P}_i \) dividing \( p \). Hida’s theory ([4]) shows that the ordinary forms in \( \mathcal{M}_\kappa(Np^\infty, L) \) are essentially in \( \mathcal{M}_\kappa(Np, L) \).

More precisely, the operator \( e_{\text{ord}} = \lim_{k \to \infty} T(p)^k \) acts on the space \( \mathcal{M}_\kappa(Np^\infty, L) \), and takes values in \( \mathcal{M}_\kappa(N\mathfrak{P}, L) \), where \( \mathfrak{P} = \prod \mathfrak{P}_i | p \mathfrak{P}_i \). If \( f \) is a Hecke eigenform, then we have:

\[
e_{\text{ord}}(f) = \begin{cases} f, & \text{if } f \text{ is ordinary;} \\ 0, & \text{else.} \end{cases}
\]

Now let \( f \) be the modular form in the introduction, i.e. a cuspidal new form of weight 2, of level \( N \) prime to \( p \) and of trivial central character, which is a Hecke eigenform and is ordinary at \( p \). When we regard \( f \) as a form in \( \mathcal{M}_2(Np^\infty, L) \), however, it is no longer an eigenvector for the Hecke operators dividing \( p \), because they are different from the corresponding level \( N \) operators. Thus we define its \( p \)-stabilization as follows. Let \( \alpha_i \) (resp. \( \beta_i \)) be the unique root of the polynomial \( X^2 - a(\mathfrak{P}_i, f) X + |\mathfrak{P}_i|_\infty \) that is (resp. is not) a \( p \)-adic unit. Then the form \( f_0 \) defined by:

\[
f_0 = \sum_{J \subseteq \{1, \ldots, l\}} (-1)^{\#J} \left( \prod_{j \in J} \beta_j \right) V_{\prod \mathfrak{P}_j}(f)
\]

is the unique ordinary form in \( \mathcal{S}_2(Np^\infty, L) \) which is an eigenvector for all Hecke operators, and has the same eigenvalues as \( f \) for Hecke operators with index prime to \( p \). It actually is a form in \( \mathcal{S}_2(N_0) \), where \( N_0 \) is the ideal \( N\mathfrak{P} \).

By results of Hida, the line \( L \cdot f_0 \) is a direct summand of the ordinary part, regarded as modules over the Hecke algebra generated by all Hecke operators. Let \( e_{f_0} \) be the corresponding projector. We then define a linear form \( l_f \) on \( \mathcal{M}_2(Np^\infty, L) \) by:

\[
l_f(g) := a(f_0, e_{\text{ord}}(g)).
\]

This linear form then “interpolates” the Peterson inner product with \( f_0 \) in the following sense. If \( h \) is a holomorphic modular form, let \( h^* \) be the form such that \( h^*(g) := h \left( g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \), which is another holomorphic modular form, with Fourier coefficients \( a(I, h^*) = a(I, h) \) for any ideal \( I \). We then have:
Proposition 3.1. The linear form \( l_f \) on \( M_2(K_0(NP^\infty), L) \) has the following interpolation property: if \( P \) is a divisor of \( p^\infty \), and \( g \) is a form in \( M_2(K_0(N_0P), L) \) with algebraic Fourier coefficients, then we have:

\[
l_f(g) = |P|_\infty \left( \prod \alpha_i^{-\text{val}(P)} \right) \frac{w_{NP}(f_0^*), g}{w_{N_0}(f_0^*), f_0}\]

Proof This is similar to Hida’s original proof. If \( \gamma = (\gamma_1, \cdots) \) is any array in \( \prod \{\alpha_i, \beta_i\} \), let \( f_\gamma \) be the form \( \prod_i (1 - \gamma_i V_{p_i}) f \). Thus if \( \gamma \) is the array such that \( \gamma_i = \beta_i \) for all \( i \), then \( f_\gamma \) is just the form \( f_0 \). Without confusion, we denote this special array by 0.

In the case \( P = 1 \), the only thing to verify is that for any array \( \gamma \neq 0 \), the product \( S_\gamma := (w_{N_0}(f_0^*), f_\gamma) \) is zero. Suppose without loss of generality that \( \gamma_1 = \alpha_1 \), then this is shown by the following calculations:

\[
(w_{N_0}(f_0^*), T(\mathcal{P}_1)(f_\gamma)) = (w_{N_0}(f_0^*), \beta_1(f_\gamma)) = \beta_1 S_\gamma;
\]

\[
= (T(\mathcal{P}_1)^* w_{N_0}(f_0^*), f_\gamma) = (w_{N_0} T(\mathcal{P}_1)(f_0^*), f_\gamma) = (\alpha_1 w_{N_0}(f_0^*), f_\gamma) = \alpha_1 S_\gamma.
\]

For general \( P \), the operator \( T(P) \) takes a form \( g \) in \( M_2(K_0(N_0P), L) \) to \( M_2(K_0(N_0), L) \), and we may compute:

\[
l_f(T(P)g) = a(1, \epsilon_{f_0} e_{\text{ord}} T(P)g) = a(1, T(P) e_{f_0} e_{\text{ord}} g)
\]

\[
= a \left( 1, \left( \prod \alpha_i^{-\text{val}(P)} \right) e_{f_0} e_{\text{ord}} g \right) = \left( \prod \alpha_i^{-\text{val}(P)} \right) l_f(g).
\]

Applying the case \( P = 1 \) to the form \( T(P)g \), we get:

\[
\left( \prod \alpha_i^{-\text{val}(P)} \right) l_f(g) = l_f(T(P)g)
\]

\[
= \frac{(w_{N_0}(f_0^*), T(P)(g))}{(w_{N_0}(f_0^*), f_0)} = \frac{(T_{NP}(P)^* w_{N_0}(f_0^*), g)}{(w_{N_0}(f_0^*), f_0)}.
\]

Here \( T_{NP}(P)^* \) is the dual operator on the space \( M_2(N_0P) \), thus is equal to \( w_{NP} T(P) w_{NP} \). Moreover, it is easy to see (from the corresponding matrix identity) that \( w_{N_0P} w_{N_0} = |P|_\infty V_P \), and that \( T(P) V_P \) is identity, so the form \( T(P)^* w_{N_0}(f_0^*) \) simplifies to: \( |P|_\infty w_{NP} f_0^* \), and gives the desired result.

From the definition and the fact that the operator \( T(\mathcal{P}_1) \) commutes with \( \epsilon_{f_0} \) and \( e_{\text{ord}} \), it is clear that we have \( l_f(T(\mathcal{P}_1)(g)) = \alpha_i l_f(g) \) for every \( i \). As in the classical case, we may compare "Peterson inner product with \( f_0 \)" with "Peterson inner product with \( f \)".

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Lemma 3.2. If $g$ is a modular form in the space $\mathcal{M}_2(N)$ with algebraic coefficients, then we have:

$$\prod_i \left(1 - \frac{\|\mathfrak{p}_i\|_\infty}{\alpha_i^2}\right) l_f(g) = \frac{(f, g)}{(f, f)}.$$ 

Proof  This is true for all modular forms $g$ with complex coefficients, if we replace the left hand side by the Peterson inner product expression in Proposition 3.1. The proof is the same as in the classical case of elliptic modular forms \cite{11} Lemme 2.2 (the detailed calculations are in \cite{12} lemme 27). \qed

4 Complex Theta functions

In this section, let $N \neq 1$ be an ideal of $\mathcal{O}_E$. Let $\chi : A_E^\times / E^\times (1 + \widehat{N})^\times E^\times_\infty \to \mathbb{C}$ be a character mod $N$. Using the method of \cite{15}, we are going to attach to the character $\chi$ a holomorphic modular form in the space $M_1(D_{E/F}N_{E/F}(N), \chi)$, the theta function.

4.1 Weil representations

For simplicity, here we only recall the Weil representation over $F$ attached to the quadratic space $(E, N)$. Also, instead of introducing the action of the whole group $GO(A)$ (cf. \cite{15} section 2.1), we only introduce the action of $A_E^\times$.

Let $v$ be a place of $F$. We define Schwartz functions on the space $E_v \times F_v^\times$ as follows: if $v$ is a finite place, a Schwartz function is a locally constant and compactly supported function; if $v$ is infinite, a Schwartz function is of the form:

$$(t, u) \mapsto (P_1(uN(t)) + \text{sgn}(u)P_2(uN(t))) e^{-2\pi |u|N(t)},$$

with $P_1, P_2$ polynomials over $F_v$, and sgn is the sign of $u$.

For any place $v$, we let $E_v^\times$ act on the space $E_v \times F_v^\times$ via: $e \cdot (t, u) := (e^{-1}t, N(e)u)$.

Proposition 4.1. There is a unique representation of $\text{GL}_2(F_v) \times E_v^\times$ on the space of Schwartz functions on $E_v \times F_v^\times$, such that for any Schwartz function $\phi$, we have:

- $(e\phi)(t, u) = \phi(e^{-1}(t, u))$, for any $e \in E_v^\times$;

- $\left(\begin{array}{cc} y & 0 \\ 1 & 0 \end{array} \right) \phi(t, u) = \phi(yt, u) |y|_v \epsilon_v(y)$, for any $y \in F_v^\times$;
\[
\left( \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \right) \phi(t, u) = \phi(t, d^{-1}u) |d|_v^{-1/2}, \text{ for any } d \in F_v^\times;
\]
\[
\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \phi(t, u) = \phi(t, u)\psi(xuN(t)), \text{ for any } x \in F_v;
\]
\[
\left( \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right) \phi(t, u) = \gamma_v \epsilon_v(u) |u|_v \int_{E_v} \phi(s, u)\psi(uN(t, s))ds,
\]
where \( \gamma_v \) is a fourth root of unity (the Weil index) depending only on \( E \) and \( F \), and \( N(t, s) \) stands for \( N(t + s) - N(t) - N(s) \).

We also have: \( \prod_v \gamma_v = 1 \) (cf. [14] Proposition 5).

**Remark** From the above formulae, it is easy to deduce the following:

\[
\left( \begin{pmatrix} y \\ 1 \end{pmatrix} \right) \phi(t, u) = \phi(yt, y^{-1}u) |y|_v^{1/2} \epsilon_v(y), \text{ for any } y \in F_v^\times;
\]
\[
\left( \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \right) \phi(t, u) = (z\phi)(t, u)\epsilon_v(z), \text{ for any } z \in F_v^\times;
\]

In the global case, define Schwartz functions on the space \( A_E \times A^\times_c \) as finite linear combinations of products \( \prod_v \phi_v \) of Schwartz functions \( \phi_v \) on each place \( v \), where for almost all \( v \), the function \( \phi_v \) is spherical, i.e. is the characteristic function of \( O_{E,v} \times O_v^\times \). Then the local Weil representations induce a global representation of the group \( GL_2(A) \times A_E^\times \) on the space of Schwartz functions on \( A_E \times A^\times_c \).

### 4.2 Theta functions

Let \( \phi \) be a Schwartz function on \( A_E \times A^\times_c \). Then there is a subgroup \( U \) of finite index of \( O_E^\times \), such that \( \phi \) is invariant under the action of \( U \) (cf. [15] section 3.1). Since the action of \( GL_2(A) \) commutes with the action of \( A_E^\times \), we see that \( g\phi \) is invariant under \( U \) for any \( g \in GL_2(A) \).

**Proposition 4.2.** Let \( \phi \) be a Schwartz function on \( A_E \times A^\times_c \) and \( U \) be a subgroup of finite index of \( O_E^\times \) such that \( \phi \) is invariant under \( U \). For any \( g \in GL_2(A) \), define:

\[
\theta_\phi(g) := \sum_{(t, u) \in U \setminus (E \times F^\times)} (g\phi)(t, u),
\]
then the sum converges uniformly for \( g \) in any compact subset of \( \text{GL}_2(\mathbb{A}) \), and defines an automorphic form on \( \text{GL}_2(\mathbb{A}) \).

**Proof**  The only problem is the convergence, which is proved in [15] section 3.1.  

**Remark**  If the group \( U \) in the proposition is replaced by some smaller group \( U' \), then the resulting function \( \theta_\phi \) will be \( [U : U'] \) times bigger. Thus the notation \( \theta_\phi \) is well-defined up to a constant factor.

Now write \( x \mapsto \pi \) for the action of the non-trivial element of \( \text{Gal}(E/F) \). Let \( N \) be an ideal of \( \mathcal{O}_E \). Define a Schwartz function \( \rho \) as follows:

\[
\rho_f(t, u) := \mathbf{1}_{t \in (1 + \hat{N})} \mathbf{1}_{u \in \mathcal{O} \times (1 + \hat{N})} e^{u \epsilon(t)};
\rho_\infty(t, u) := \mathbf{1}_{u > 0} e^{-2\pi uN(0)}.
\]

The function \( \rho \) (and thus all the functions \( \sigma \rho \) for \( \sigma \in \mathcal{A}_E \times E \times F \)) is invariant under action of the group \( U := \mathcal{O}_E \times (1 + \hat{N}) \times E \times F \), so we may define, for every \( \sigma \in \mathcal{A}_E \times E \times F \), the associated theta function:

\[
\theta_\sigma(g) := \sum_{U \setminus (E \times F)} (g \rho)(t, u).
\]

This is the \( \text{(mod } N) \) partial theta function associated to \( \sigma \in \mathcal{A}_E^\times \).

**Lemma 4.3.** The function \( \rho \) has the following properties:

- \( \left( \begin{array} {cc} 1 & x \\ 0 & 1 \end{array} \right) \rho_f(t, u) = \rho_f(t, u), \text{ for any } x \in \hat{O}; \)
- \( \left( \begin{array} {cc} y & 0 \\ 0 & 1 \end{array} \right) \rho_f(t, u) = \rho_f(t, u), \text{ for any } y \in \hat{O}^\times; \)
- \( \left( \begin{array} {cc} 1 & a \\ 0 & 1 \end{array} \right) \rho_f(t, u) = \rho_f(t, u), \text{ for any } a \in D_{E/F} \hat{N}(N); \)
- \( \left( \begin{array} {cc} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{array} \right) \rho_\infty(t, u) = \rho_\infty(t, u)e^{i\alpha}, \text{ for any } \alpha \in F_\infty; \)
- \( (\tau \rho)(t, u) = \rho(t, u), \text{ for any } \tau \in (1 + \hat{N}) \times E_\infty^\times; \)
Proof  Only the third assertion needs some explanation. Write the matrix \( \begin{pmatrix} 1 & -a \\ 1 & 1 \end{pmatrix} \) as \( W^{-1} \begin{pmatrix} 1 & -a \\ 1 & 1 \end{pmatrix} W \), where \( W \) denotes the matrix \( \begin{pmatrix} -1 & 1 \end{pmatrix} \). The function \( W \rho_f \) can be evaluated as follows:

\[
W \rho_f(t, u) = \gamma_f \prod_{u \in d_F \hat{O}_v} |d_F|^{1/2} \int_{u^{-1}(1+\hat{N})} \psi(uN(t, s)) ds \\
= A(t, u) \int_{\hat{N}} \psi_E(\tilde{\tau}) ds,
\]

where \( A \) is a function. It follows that the above function is supported on the compact set \( \{(t, u) : u \in d_F \hat{O}_v^\times, t \in (d_E \hat{N})^{-1} \} \) (notice that \( d_E = d_E \)). Since for any \( a \in D_{E/F \hat{N}((N))}, u \in d_F \hat{O}_v^\times \) and \( t \in (d_E \hat{N})^{-1} \) the number \( auN(t) \) belongs to \( d_F^{-1} \hat{O}_v \), we have \( \begin{pmatrix} 1 & -a \\ 1 & 1 \end{pmatrix} W \rho_f = W \rho_f \) for any \( a \in D_{E/F \hat{N}((N))}. \) So finally we get:

\[
\begin{pmatrix} 1 & -a \\ 1 & 1 \end{pmatrix} \rho_f = W^{-1} W \rho_f = \rho_f,
\]

for any \( a \in D_{E/F \hat{N}((N))}. \)

\[\square\]

Corollary 4.4. For any \( \sigma \in \mathbb{A}_{E,F}^\times \), the function \( \theta_\sigma \) is a holomorphic modular form of weight 1 for \( K_1(D_{E/F \hat{N}((N))} \).

We also have: \( \theta_{e \tau} = \theta_\sigma \) for any \( e \in E^\times \) and any \( \tau \in (1 + \hat{N})^\times E_\infty^\times \), so we may define \( \theta_\sigma \) for \( \sigma \in \mathbb{A}_{E/F}^\times (E^\times (1 + \hat{N})^\times E_\infty^\times \).

Proof  The holomorphicity is easily checked from the form of \( \rho_\infty \). The other things are clear from the above lemma. For example, to prove that \( \theta_\sigma \) is right invariant under \( K_1(D_{E/F \hat{N}((N))} \), take any \( d \in (1 + D_{E/F \hat{N}((N))})^\times \) and compute:

\[
\begin{pmatrix} 1 & -a \\ 1 & 1 \end{pmatrix} \rho_f = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \rho_f = (d \rho_f) e(d) = \rho_f,
\]

then remark that for any ideal \( M \), the group \( K_1(M) \) is generated by elements of the form \( \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} y & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix} \), with \( x \in \hat{O}, y \in \hat{O}^\times, d \in (1 + \hat{M})^\times, a \in \hat{M}. \)

\[\square\]
Now let $\chi : \mathbb{A}_E^\times (1+N)^\times \to \mathbb{C}^\times$ be a character mod $N$. The **theta function** associated to the character $\chi$ is defined by:

$$\Theta(\chi) := \sum_{\sigma \in \mathbb{A}_E^\times (1+N)^\times \mathbb{E}_\infty^\times} \chi(\sigma^{-1})\theta_\sigma.$$ 

**Proposition 4.5.** The theta function associated to $\chi$ is a holomorphic modular form in the space $\mathcal{M}_1(D_E/FN(N), \epsilon \chi_F)$, where $\chi_F$ denotes the restriction of $\chi$ to $\mathbb{A}^\times$.

**Proof**  The only thing left to verify is the central character. In fact, for any $z \in \mathbb{A}^\times$, we have:

$$\theta_\sigma\left(\begin{pmatrix} z \\ z \end{pmatrix} \right) g = \sum_{(t,u)} (g \sigma z \rho)(t,u) \epsilon(z) = \theta_{\sigma z}(g) \epsilon(z),$$

so that:

$$\Theta\left(\begin{pmatrix} z \\ z \end{pmatrix} \right) g = \sum_{\sigma} \chi(\sigma^{-1})\theta_{\sigma z}(g) \epsilon(z) = \Theta(g)(\chi \epsilon)(z).$$

\[\square\]

### 4.3 $q$-expansion of the theta function

Here we compute the $q$-expansions of the partial theta functions.

For any element $\sigma \in \mathbb{A}_E^\times / \mathbb{E}_\infty^\times (1+N)^\times \mathbb{E}_\infty^\times$, write $1_\sigma : \mathbb{A}_E^\times / \mathbb{E}_\infty^\times (1+N)^\times \mathbb{E}_\infty^\times \to \mathbb{C}$ for the characteristic function of $\sigma$.

**Proposition 4.6.** Let $\sigma$ be an element of $\mathbb{A}_E^\times / \mathbb{E}_\infty^\times (1+N)^\times \mathbb{E}_\infty^\times$. Let $a$ and $a_0$ be the coefficients of the $q$-expansion of $\theta_\sigma$. Let $I$ be any non-zero ideal of $\mathcal{O}_E$. Then we have:

$$a(I) = \sum_{J \subseteq \mathcal{O}_E \substack{N(J) = I}} 1_{\sigma^{-1}}(J),$$

$$a_0(I) = 0.$$ 

**Proof**  Let $I$ be any non-zero ideal of $\mathcal{O}_E$, and take any $y \in \mathbb{A}_f^\times$ such that $yd_F = I$. We have:

$$a(I) = |y|^{-1/2} \psi(-iy_{\infty}) \int_{\mathbb{A}_F} \theta_\sigma\left(\begin{pmatrix} y \\ x \end{pmatrix} \right) \psi(-x) dx$$

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\[ y^{-1/2} \psi(-iy_{\infty}) \sum_{U \backslash (E \times F^\times)} \left( \begin{pmatrix} y & 0 \\ 1 & 1 \end{pmatrix} \sigma \rho \right) (t, u) \int_{A/F} \psi((uN(t) - 1)x) dx \]

\[ = \sum_{U \backslash (E \times F^\times)} \rho_f(y \sigma t, y^{-1}N(\sigma)^{-1}u) \epsilon(y) \]

\[ = \sum_{t \in E^\times/U} 1_{(\sigma)^{-1} \in (1 + \hat{N})} 1_{N(\sigma)^{-1} \in \hat{N}^\times} \]

\[ = \sum_{J \subseteq O_E} 1_{\sigma^{-1}(J)}. \]

The same calculation for the constant term gives:

\[ a_0(I) = \sum_{u \in F^\times/U} 1_{u > 0} 1_{0 \in (1 + \hat{N})} 1_{uN(\sigma)^{-1} \in \hat{N}^\times}. \]

This vanishes under the hypothesis that \( N \) is not equal to \( O_E \).

\[ \Box \]

**Corollary 4.7.** Let \( \chi : \mathbb{A}_E^\times / E^\times (1 + \hat{N})^\times E_{\infty}^\times \to \mathbb{C} \) be a character mod \( N \). Then the \( q \)-expansion of the modular form \( \Theta_{\chi} \) is given by:

\[ \theta_\phi = \sum_{0 \neq J \subseteq O_E} \chi_{[N]}(J) q^{N(J)}. \]

## 5 Complex Eisenstein series

In this section, let \( \kappa \) be a positive integer, \( N \) be an ideal of \( O \) and \( \chi : \mathbb{A}^\times / F^\times \to \mathbb{C}^\times \) be a Hecke character mod \( N \) and having the same parity as \( \kappa \), i.e., such that \( \chi_v(-1) = (-1)^\kappa \) for any infinite place \( v \). We are going to recall the construction of an Eisenstein series in the space \( \mathcal{M}_\kappa(N, \chi) \).

### 5.1 The case of primitive characters

We first consider the case when the character \( \chi \) is of conductor \( N \). For any finite place \( v \) of \( F \), define a function \( H_v \) on \( \text{GL}_2(O_v) \) as follows: if \( v \) does not divide \( N \), put \( H_v = 1 \); otherwise, for \( k = \begin{pmatrix} u & v \\ w & t \end{pmatrix} \in \text{GL}_2(O_v) \), put \( H_v(k) = 1_{K_0(N)(k)} \chi_v(t) \). Also, for \( v = \infty \) define \( H_\infty \) as: \( H_\infty(r(\theta)) = e^{in\theta} \), where \( r(\theta) \) is the matrix \( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \).
The product of $H_v$ for all places $v$ then defines a function $H$ on the group $K_0(1)K_{\infty}$, i.e. $H(g) = \prod_v H_v(g_v)$.

Let $s$ be a complex number such that $\text{Re}(s) > 1$. The function $H$ then extends to a function $H_s : \text{GL}_2(\mathbb{A}) \to \mathbb{C}$ via the Iwasawa decomposition: for any $g \in \text{GL}_2(\mathbb{A})$, write $g = \begin{pmatrix} a & b \\ d & c \end{pmatrix} k$ with $k \in K_0(1)K_{\infty}$. We then define:

$$H_s(g) := \left| \frac{a}{d} \right|^s \chi(d) H(k).$$

It is easy to see that the function $H_s$ is well defined (because of the hypothesis on the parity of $\chi$), and it also decomposes as a product: $H_s(g) = \prod_v H_{s,v}(g_v)$.

Define the Eisenstein series $G_s : \text{GL}_2(\mathbb{A}) \to \mathbb{C}$ and its normalization $E_s$ as follows:

$$G_s(g) := \sum_{\gamma \in B(F) \backslash \text{GL}_2(F)} H_s(\gamma g),$$

$$E_s := C_s G_s,$$

where we write $C_s$ for the number $2^{-n}L_N(1-2s, \chi)$. The two Eisenstein series are absolutely convergent if $\text{Re}(s) > 1$. An easy calculation shows that in this case the function $G_s$ (and hence the function $E_s$) is a modular form for $K_0(N)$ of weight $\kappa$ and of character $\chi$.

For any ideal $I$ of $\mathcal{O}$ and any integer $m$, write $\sigma_{\chi[N],m}(I) := \sum_J \chi[N](J)N(J)^m$.

**Proposition 5.1.** Let $\chi$ be as above.

- The function $E_s$, originally defined only for those $s$ with $\text{Re}(s) > 1$, can be meromorphically continued (with respect to $s$) to all $s \in \mathbb{C}$, and is holomorphic at the point $s = \kappa/2$.

- When $(F, \kappa, \chi) \neq (\mathbb{Q}, 2, 1)$, the form $E_{\kappa/2}$ is a holomorphic modular form. The coefficients of its $q$-expansion are given by:

$$a(I) = \sigma_{\chi[N],\kappa-1}(I), \text{ for any } I \neq 0.$$

The constant term is given by:

$$a_0(I) = 2^{-n}L_N(1-\kappa, \chi), \text{ for any } I \neq 0,$$

except for the case that $\kappa = 1$ and $\chi$ is unramified on all finite places. In that case, the constant term is:

$$a_0(I) = 2^{-n}(L_1(0, \chi) + (-1)^n \chi[1](I)L_1(0, \chi^{-1})).$$
The proof is the same as [17] Proposition 3.5.2. Here we briefly recall the main steps. Let $W_s$ and $W_{0,s}$ be the Whittaker functions of the form $E_s$. Then for any $x \in \mathbb{A}$ and any $y \in \mathbb{A}^\times$ with $y_\infty > 0$ we have:

$$E_s\left(\begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}\right) = W_{0,s}\left(\begin{pmatrix} y & 1 \\ 1 & 1 \end{pmatrix}\right) + \sum_{\alpha \in F^\times} W_s\left(\begin{pmatrix} \alpha y & 1 \\ 1 & 1 \end{pmatrix}\right) \psi(\alpha x).$$

Note that a modular form is determined by its values on these matrices. Thus it suffices to prove analytic continuation of the Whittaker functions.

From the Bruhat decomposition:

$$\text{GL}_2(F) = B(F) \bigsqcup \left( \bigsqcup_{u \in F} B(F) \left( \begin{pmatrix} 1 & 1 \\ u & 1 \end{pmatrix} \right) \right)$$

we get:

$$W_s\left(\begin{pmatrix} y & 1 \\ 1 & 1 \end{pmatrix}\right) = C_s \int_{\mathbb{A}/F} H_s\left(\begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}\right) + \sum_{u \in F} H_s\left(\begin{pmatrix} y & 1 \\ u & 1 \end{pmatrix}\right) \psi(-x) dx$$

$$= C_s \int_{\mathbb{A}} H_s\left(\begin{pmatrix} 1 & 1 \\ x & 1 \end{pmatrix}\right) \psi(-x) dx$$

$$= C_s |y|^{-s} \chi(y) \int_{\mathbb{A}} H_s\left(\begin{pmatrix} 1 & 1 \\ xy^{-1} & 1 \end{pmatrix}\right) \psi(-x) dx$$

$$= C_s |y|^{1-s} \chi(y) \prod_v V_{s,v}(y_v),$$

where for each place $v$, the functions $V_{s,v} : F_v \to \mathbb{C}$ is defined as:

$$V_{s,v}(y) := \int_{F_v} H_{s,v}\left(\begin{pmatrix} 1 & 1 \\ x & 1 \end{pmatrix}\right) \psi_v(-yx) dx.$$

The same calculation for the function $W_{0,s}$ yields:

$$W_{0,s}\left(\begin{pmatrix} y & 1 \\ 1 & 1 \end{pmatrix}\right) = C_s(|y|^s + |y|^{1-s} \chi(y) \prod_v V_{s,v}(0)).$$

It then suffices to compute the functions $V_{s,v}$ for each place $v$. We refer to [17] for the details.

We write $E_{\kappa/2}(\chi)$ for the holomorphic form $E_{\kappa/2}$ defined above.
5.2 The case of non-primitive characters

Now assume that the character $\chi$ is of conductor $D$, a divisor of $N$. We would also like to construct a modular form $E = E_{\kappa/2}(\chi)$ in the space $\mathcal{M}_\kappa(N, \chi)$ with Fourier coefficients $a(I) = \sigma_{\chi[N],\kappa-1}(I)$ and $a_0(I) = 2^{-n}L_N(1 - \kappa, \chi)$.

There are several possible constructions. Let $E_D$ be the modular form constructed in the above subsection, i.e. the Fourier coefficients $b(I)$ of $E_D$ is equal to $\sigma_{\chi[D],\kappa-1}(I)$. For any prime ideal $Q$ dividing $N$, denote by $F(Q)$ the operator $T(Q) - \chi[D](Q)N(Q)^{\kappa-1}$ on the space $\mathcal{M}_\kappa(N, \chi)$. We then define the form $E$ as $\prod_{Q|N} E_D$. It is then easy to verify that this form $E$ has the desired Fourier coefficients.

Equivalently, we may define a function $H_s = \prod_v H_{s,v}$ on the group $K_0(1)K_\infty$ as follows: for places $v$ dividing $N$ but not dividing $D$ and for any $k = \begin{pmatrix} u & v \\ w & t \end{pmatrix} \in \text{GL}_2(\mathcal{O}_v)$, put:

$$H_{s,v}(k) = \begin{cases} 1 - |\pi_v|, & \text{if } |w|_v = 1; \\ 1 - \chi(\pi_v)|\pi_v|^{-1 - 2s}, & \text{if } |w|_v < 1, \end{cases}$$

and for other places $v$, define $H_{s,v}(k)$ the same way as the above subsection. The function $H_s$ then extends to a function on $\text{GL}_2(\mathbb{A})$ via the formula:

$$H_s \left( \begin{pmatrix} a & b \\ d \end{pmatrix} k \right) = \left| \frac{a}{d} \right| \chi(d)H_s(k).$$

The Eisenstein series $G_s(g) := \sum_{\gamma \in B(F) \setminus \text{GL}_2(F)} H_s(\gamma g)$ and its normalization $E_s := 2^{-n}L_D(1 - 2s, \chi)G_s$ then admit holomorphic extensions to all $s$, and the form $E_{\kappa/2}$ is holomorphic. The same calculation as the subsection above shows that this form has the desired $q$-expansion. We will use this construction in the calculations below.

**Remark** There is another construction of this Eisenstein series in [2], which is $E := w_N E_D$, where $E_D$ is the form constructed in the above subsection. This defines the same Eisenstein series as ours, because an easy calculation shows that their Fourier coefficients are the same. 

Remark

There is another construction of this Eisenstein series in [2], which is $E := w_N E_D$, where $E_D$ is the form constructed in the above subsection. This defines the same Eisenstein series as ours, because an easy calculation shows that their Fourier coefficients are the same. 

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6 Measures on the Galois group

From now on, let \( E \) be a totally imaginary quadratic extension of \( F \), with relative discriminant \( D = D_{E/F} \) and relative difference \( d_{E/F} \).

We would like to construct (pseudo-)measures on the Galois group \( G = \text{Gal}(E^{(p)}/E) \), where \( E^{(p)} \) is the maximal abelian extension of \( E \) unramified outside places dividing \( p \). By class field theory, we may rewrite the group \( G \) as:

\[
G = A \times E / E \times \prod_{v \mid p} O_{E,v} \times E,\]

here the bar means taking closure.

The natural morphism \( \prod_{v \mid p} O_{E,v} \rightarrow G \) induces an exact sequence:

\[
1 \rightarrow \overline{O}_E^\times \rightarrow \prod_{v \mid p} O_{E,v}^\times \rightarrow G \rightarrow \text{Cl}(E) \rightarrow 1.
\]

Thus the group \( G \) is isomorphic to a direct product of a finite group and some copies of \( \mathbb{Z}_p \).

### 6.1 Theta measure

To define the Theta measure, we first define a measure on the group \( G \) with values in \( L\langle q \rangle \), the space of bounded formal \( q \)-expansions, as follows: for any continuous function \( \phi : G \rightarrow L \), put

\[
\Theta(\phi) = \sum_{0 \neq J \subseteq O_E} \phi[J] q^{N(J)}.
\]

It is obvious that this is a measure on the group \( G \).

By results of section 4.2, we have \( \Theta(\chi) = \Theta(\chi) \) for any finite order character \( \chi : G \rightarrow \mathbb{C}^\times \), so that \( \Theta(\chi) \) is actually in the subspace \( S_1(Dp_\infty, \epsilon_X F, L) \).

### 6.2 Eisenstein pseudo-measure

We first define a pseudo-measure on the group \( G_F := \mathbb{A}^\times / F^\times F_{\infty, > 0}^\times \prod_{v \mid p} (1 + DO_v)^\times \) with values in \( L\langle q \rangle \).
For an element \( s \in F_\infty^\times \), define its sign as \( \text{sgn}(s) = \prod_{v|\infty} \text{sgn}(s_v) \). For any function \( \phi : \mathbb{A}_F^\times / F_\infty^\times F_{\infty,>0}^\times \to \mathbb{C} \), define:

\[
\phi^\#(x) := 2^{-n} \sum_{s \in F_\infty^\times / F_{\infty,>0}^\times} \text{sgn}(s) \phi(sx).
\]

By result of Deligne-Ribet, the map sending a locally constant function \( \phi : G_F \to L \) to the value \( L_{pD_E/F}(0, \phi^\#) \) extends to a pseudo-measure on \( G_F \), which we still denote by \( L_{pD_E/F}(0, \phi^\#) \). More precisely, this pseudo-measure \( \mu \) is an element in the total quotient ring of \( \Lambda_{G_F} \) (the Iwasawa algebra of \( G_F \)), such that for any element \( C \) of the group \( G_F \), the element \( \mu^C := (1 - \xi_p(C)^{-1}C)\mu \) lies in the sub-algebra of \((\mathbb{Q}_p^-)\)measures over \( G_F \).

We then define the Eisenstein pseudo-measure on \( G_F \) by:

\[
E_F(\phi) := 2^{-n} L_{pD}(0, \phi^\#) + \sum_I \sigma_{\phi^\#_{pD}}(I) q^I,
\]

where for any function \( \varphi \) on the set of integral ideals of \( \mathcal{O} \), we define: \( \sigma_\varphi(I) := \sum_{J|I} \varphi(J) \).

By results in section 5 for any finite odd character \( \chi : G_F \to \mathbb{C}_\times \), we have: \( E_F(\chi) = E_{1/2}(\chi) \), so that \( E_F(\chi) \) is actually in the subspace \( M_1(Dp^\infty, L) \). Also it is obvious that \( E_F(\chi) = 0 \) if the character \( \chi \) is not odd.

Now define a pseudo-measure \( \mathbf{E} \) on the group \( G \) as follows:

\[
\int_G \phi \mathbf{E} := \int_{G_F} \epsilon(g) \phi_F(g^{-1}) E_F(g).
\]

For an element \( C \) of \( G_F \), we also write \( \mathbf{E}^C \) for the measure \( \mathbf{E}(1 - \xi_p(C)^{-1}C) \). If \( \phi = \chi : G \to \mathbb{C}_\times \) is a finite order character, then (since the character \( \epsilon\chi_F \) is automatically odd) \( \mathbf{E}(\phi) \) is a modular form of central character \( \epsilon\chi_F^{-1} \).

6.3 The analytic kernel

Define the convoluted pseudo-measure \( \tilde{K} := \Theta(V_N \mathbf{E}) \). (And \( \tilde{K}^C \) means the measure \( \Theta(V_N \mathbf{E}^C) \).)

If \( \phi = \chi \) is a finite order character, then we have:

\[
\int_G \chi \tilde{K} = \int_{G\times G} \chi(xy) \Theta(x(V_N \mathbf{E})y = \Theta(\chi)V_N \mathbf{E}(\chi),
\]

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which is a modular form in the space $\mathcal{M}_2(NDp^r)$ for sufficiently large $r$. Since any locally constant function is a linear combination of finite order characters, we see that the pseudo-measure $\tilde{K}$ takes values in the $p$-adic Banach space $\mathcal{M}_2(NDp^\infty, L)$.

Now let $f$ be the new form of level $N$ in the introduction. Recall the linear form $l_f$ defined on the space $\mathcal{M}_2(Np^\infty, L)$ that interpolates "Peterson inner product with $f$".

We would like to take a trace to pass from forms of level $NDp^\infty$ to forms of level $Np^\infty$.

For any integer $r > 0$, the trace map $tr_D : \mathcal{M}_2(NDp^r) \to \mathcal{M}_2(Np^r)$ is defined in the usual way, i.e. for any form $\phi \in \mathcal{M}_2(NDp^r)$ we have $tr_D(\phi)(g) = \sum_{h \in K_0(NDp^r)} \phi(gh)$. But under our hypothesis that $N$, $D$, $p$ are prime to each other, the set $K_0(NDp^r) \setminus K_0(Np^r)$ can be identified with $K_0(D) \setminus K_0(1)$, which does not depend on $r$, hence the trace maps $tr_D$ on different levels are coherent, and induce a map, which we still denote by $tr_D$, from the space $\mathcal{M}_2(NDp^\infty, L)$ to the space $\mathcal{M}_2(Np^\infty, L)$.

Denote by $K$ the pseudo-measure $tr_D(\tilde{K})$ and by $K_f$ the pseudo-measure $l_f(K)$. That is to say,

$$K(\phi) := tr_D(\tilde{K}(\phi)),$$

$$K_f(\phi) := l_f(K(\phi)).$$

The pseudo-measure $K$ is called the analytic kernel.

We have the following interpolation property:

**Proposition 6.1.** For any finite order character $\chi$ on $G$ of conductor $C \mid p^\infty$, we have:

$$\int_G \chi K_f = r(\chi)D_F^2 |DN(C)|^{1/2} \chi^{-1}(d_{E/F}) \left( \prod_{\psi_i : p} \frac{\alpha_i^{2-\text{val}_{\psi_i}(C)}}{(\alpha_i^2 - 1)(\alpha_i^2 - |\psi_i|_\infty)} \right) \cdot L(1, f, \chi^{-1}).$$

This proposition is proved by Disegni in [2] section 4.4, using his construction of Eisenstein series. But since the two definitions lead to the same form, and the interpolation formula is not needed in the following work, we won’t reprove it here.

Now that we have the pseudo-measure $K_f$, we may define the $p$-adic $L$ function associated to $f$ and $E$ as the Mellin transform of $K_f$. More precisely, let $\mathcal{X}$ be the rigid
analytic space \( \text{Hom}_{\text{cont}}(G, \mathbb{C}_p^\times) \), then the pseudo-measure \( K_f \) induces a meromorphic function \( L_p \) on \( \mathfrak{X} \):

\[
L_p(x) := \int_G xK_f.
\]

The only possible poles of \( L_p \) are on the line \( \{ x \in \mathfrak{X} : x_F = \xi_p \} \), which come from the pole of the Deligne-Ribet \( L \) function. In particular, the \( L \) function is analytic on the anti-cyclotomic line \( \{ x \in \mathfrak{X} : x = \overline{x} \} \). Here the bar means action of the non-trivial element of \( \text{Gal}(E/F) \).

Suppose that \( x_0 \in \mathfrak{X} \) is not a pole of \( L_p \), and let \( x \) be an element in the "neutral component" of \( \mathfrak{X} \) (i.e. \( x \) is trivial on the torsion subgroup of \( G \)). The derivative of \( L_p \) on the point \( x_0 \) along the direction \( x \) is by definition:

\[
L'_{p,x}(x_0) := \frac{d}{ds} L_p(x_0 x^s)|_{s=0} = \lim_{s \to 0} \frac{L_p(x_0 x^s) - L_p(x_0)}{s},
\]

here the character \( x^s \) is understood as \( \exp(s \log x) \).

It is then easy to calculate that:

\[
L'_{p,x}(x_0) = \frac{d}{ds} \int_G x_0 x^s K_f|_{s=0} = \int_G x_0 x^s \log x K_f|_{s=0} = \int_G x_0 \log x K_f.
\]

### 7. \( q \)-expansion of the analytic kernel

Since the pseudo-measure \( K \) takes values in the space of modular forms, for each ideal \( I \) of \( \mathcal{O} \) there are pseudo-measures \( a_0(I, K) \) and \( a(I, K) \) taking values in \( L \), which correspond to the Fourier coefficients of \( K \). In this section we are going to give explicit formulas for these coefficients.

More precisely, let \( \chi \) be a character mod \( p^r \), we are going to calculate the Fourier coefficients of the form \( K(\chi) \). By definition, this form is: \( \text{tr}_D \left[ \Theta(\chi) V_N E_{1/2}(\epsilon \chi p^{-1}) \right] \).
Note that a complete set of representatives of $K(1)/K(D)$ is given by:

$$K(1)/K(D) = \bigsqcup_{D|D} \left\{ \gamma_{D_1,a} : a \in \hat{O}/D_1 \right\},$$

where $\gamma_{D_1,a}$ is the matrix whose $v$-th component is $\begin{pmatrix} a & 1 \\ 1 & 1 \end{pmatrix}$ if $v$ divides $D_1$ and is the identity matrix if $v$ does not divide $D_1$.

Fix $d_1 \in \mathbb{A}_f^\times$ a generator of $D_1$. Denote by $\gamma_{D_1}$ the matrix whose $v$-th component is $\begin{pmatrix} d_1 & 1 \\ 1 & 1 \end{pmatrix}$ if $v$ divides $D_1$ and is the identity matrix if $v$ does not divide $D_1$. Note the identity: $d_1 \gamma_{D_1,a} = \begin{pmatrix} d_1 & a \\ 1 & 1 \end{pmatrix} \gamma_{D_1}$. Write $X$ for the form $\Theta(\chi)V_N E_{1/2}(\epsilon \chi_F^{-1})$. Since the form $X$ has trivial central character, we have:

$$K(\chi)(g) = \sum_{D_1|D} \sum_{a \in \hat{O}/D_1} X(g \gamma_{D_1,a})$$

$$= \sum_{D_1|D} \sum_{a \in \hat{O}/D_1} X\left(g \begin{pmatrix} d_1 & a \\ 1 & 1 \end{pmatrix} \gamma_{D_1} \right)$$

$$= \sum_{D_1|D} \sum_{a \in \hat{O}/D_1} X_{D_1} \left(g \begin{pmatrix} d_1 & a \\ 1 & 1 \end{pmatrix} \right)$$

$$= \sum_{D_1|D} Y_{D_1}(g),$$

here we have defined the forms $X_{D_1}$ and $Y_{D_1}$ by:

$$X_{D_1}(g) := X(g \gamma_{D_1}),$$

$$Y_{D_1}(g) := \sum_{a \in \hat{O}/D_1} X_{D_1} \left(g \begin{pmatrix} d_1 & a \\ 1 & 1 \end{pmatrix} \right).$$

To calculate the Fourier coefficients of the form $K(\chi)$ (i.e. the Whittaker functions $W\left(\begin{pmatrix} y \\ 1 \end{pmatrix}, K(\chi)\right)$), it then suffices to calculate the Whittaker functions of each $Y_{D_1}$. But it is easy to see that:

$$W\left(\begin{pmatrix} y \\ 1 \end{pmatrix}, Y_{D_1}\right) = \begin{cases} |D_1|_\infty W\left(\begin{pmatrix} d_1y \\ 1 \end{pmatrix}, X_{D_1}\right), & \text{if } yd_F \text{ is integral;} \\ 0, & \text{else.} \end{cases}$$
Thus it suffices to calculate the Whittaker functions of $X_{D_1}$.

Since $X_{D_1}$ is the product of the two forms:

$$\Theta_{D_1}(g) := \Theta(\chi)(g\gamma_{D_1}),$$
$$E_{D_1}(g) := V_{N}E_{1/2}(\epsilon\chi_F^{-1})(g\gamma_{D_1}),$$

we see that the Whittaker function of $X_{D_1}$ can be obtained from the Whittaker functions of $\Theta_{D_1}$ and $E_{D_1}$. We now do the calculations one by one.

### 7.1 The Whittaker function of $\Theta_{D_1}$

The theta functions are defined in section 4.2. By definition, we have:

$$\Theta_{D_1}(g) = \sum_{\sigma \in A \times E / E \times \infty} \chi(\sigma)^{-1}\theta_\sigma(g\gamma_{D_1}),$$

where:

$$\theta_\sigma(g\gamma_{D_1}) = \sum_{(t,u) \in U \setminus (E \times F \times)} g\gamma_{D_1}\sigma\rho(t,u) = \sum_{(t,u) \in U \setminus (E \times F \times)} g\sigma\gamma_{D_1}\rho(t,u).$$

Here we used the fact that the actions of $GL_2(A)$ and $A^\times$ commute with each other.

Writing $W_\sigma$ for the Whittaker function $W(\cdot, \theta_\sigma(g\gamma_{D_1}))$, we have:

$$W_\sigma \left( \begin{array}{c} y \\ 1 \end{array} \right) = \int_{A \setminus E \times F \times} \sum_{\sigma \in A \times E / E \times \infty} \left[ \begin{array}{c} x \\ 1 \end{array} \right] \left( \begin{array}{c} y \\ 1 \end{array} \right) \sigma\gamma_{D_1}\rho(t,u)\psi(-x)dx$$

$$= \sum_{\sigma \in A \times E / E \times} \left[ \begin{array}{c} y \\ 1 \end{array} \right] \sigma\gamma_{D_1}\rho \int_{A \setminus E \times F \times} \psi(x(uN(t) - 1)dx$$

$$= \sum_{\sigma \in A \times E / E \times} \left[ \begin{array}{c} y \\ 1 \end{array} \right] \sigma\gamma_{D_1}\rho \int_{A \setminus E \times F \times} \psi(x(uN(t) - 1)dx$$

The function $\left( \begin{array}{c} y \\ 1 \end{array} \right) \sigma\gamma_{D_1}\rho$ decomposes into product of local terms:

$$\left[ \begin{array}{c} y \\ 1 \end{array} \right] \sigma\gamma_{D_1}\rho \int_{A \setminus E \times F \times} \psi(x(uN(t) - 1)dx$$

Here the product ranges over all places $v$ of $F$. We now calculate each local term $B_v$. 

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Case 1: $v \mid \infty$ Here both $\sigma_v$ and $(\gamma_{D_1})_v$ act trivially, so we have:

$$B_v = \mathbf{1}_{y_v > 0} |y_v|^{1/2} e^{-2\pi y_v}.$$ 

Case 2: $v \nmid D_1 \infty$ Here again $(\gamma_{D_1})_v$ acts trivially, so we have:

$$B_v = \mathbf{1}_{N(\sigma)^{-1} \in \mathcal{O}_v} \mathbf{1}_{v \in \mathcal{D}_1} |y_v|^{1/2}.$$ 

Case 3: $v \mid D_1$ In this case we have $(\gamma_{D_1})_v = \begin{pmatrix} d_1 & 1 \\ -1 & 1 \end{pmatrix}$, so that (notice that $1 + p^r \mathcal{O}_{E,v} = \mathcal{O}_{E,v}$):

$$(\gamma_{D_1})_v \rho(t, u) = \gamma_v |u_v| |D_1|^{1/2} \mathbf{1}_{s \in \mathcal{D}_1} \mathbf{1}_{u \in \mathcal{D}_1} \psi(uN(t, s)) ds$$

$$= \gamma_v |u_v| |D_1|^{1/2} \mathbf{1}_{u \in \mathcal{D}_1} \int \psi(uN(t, s)) ds,$$

hence:

$$B_v = |y_v|^{1/2} \epsilon_v(y) \gamma_v \left( (yN(\sigma))^{-1} \right) |v| |D_1|^{1/2} \mathbf{1}_{yN(\sigma)}^{-1} \in \mathcal{D}_1 \mathbf{1}_{\mathcal{O}_v} \int \psi((yN(\sigma))^{-1}N(y\sigma, s)) ds.$$ 

Notice that:

$$(yN(\sigma))^{-1}N(y\sigma, s) = N(\sigma)^{-1}(\sigma t s + \sigma t s)$$

$$= \text{tr}_{E/F}((\sigma)^{-1} s),$$

and since $\psi(\text{tr}((\sigma)^{-1} s)) = \psi_E((\sigma)^{-1} s)$, we get:

$$B_v = \epsilon_v(y) \gamma_v |N(\sigma)|^{-1} |D_1|^{1/2} |y_v|^{-1/2} \mathbf{1}_{N(\sigma)}^{-1} \in \mathcal{D}_1 \mathbf{1}_{\mathcal{O}_v} \int \psi_E((\sigma)^{-1} s) ds$$

$$= \epsilon_v(y) \gamma_v \mathbf{1}_{|N(\sigma)|^{-1} \leq |yD_1|} \mathbf{1}_{yN(\sigma)} \mathbf{1}_{N(\sigma)} \mathbf{1}_{|yD_1| \leq d_{E,F,v}^{1/2}}$$

$$= \epsilon_v(y) \gamma_v \mathbf{1}_{|N(\sigma)|^{-1} \leq |yD_1|} \mathbf{1}_{N(\sigma)} \mathbf{1}_{|yD_1| \leq 1}.$$ 

Here we used the formula $d_{E,F,v} = d_{E/F,v}d_{F,v}$ and the fact that $d_{E/F,v}$ is the maximal ideal of $\mathcal{O}_{E,v}$ in this case.
Putting together Denote by \( \Delta_1 \) the ideal of \( \mathcal{O}_E \) such that \( \Delta_1^2 = D_1 \mathcal{O}_E \), and by \( \delta_1 \in \mathbb{A}_E^\times \) a fixed generator of \( \Delta_1 \). Combining the above results, we get:

\[
\left[ \begin{pmatrix} y \\ 1 \end{pmatrix} \sigma \gamma_{D_1} \right] (t, N(t)^{-1})
= \mathbf{1}_{y_\infty > 0} |y|^{1/2} e^{-2\pi y_\infty} \prod_{v \mid D_1} |y|^{1/2} \mathbf{1}_{(\sigma t)^{-1} \in (1 + p^r \mathcal{O}_E)^\times} \mathbf{1}_{N(\sigma t)^{-1} \in \gamma_{D_1} \mathcal{O}_E^\times}
\prod_{v \mid D_1} \epsilon_v(y) \gamma_v |y|^{1/2} \mathbf{1}_{y \in \mathcal{O}_E} \mathbf{1}_{N(\sigma t)^{-1} \in \gamma_{D_1} \mathcal{O}_E^\times}
= |y|^{1/2} e^{-2\pi y_\infty} \mathbf{1}_{y_\infty > 0} \mathbf{1}_{N(\sigma \delta_1)^{-1} \in \gamma_{D_1} \mathcal{O}_E^\times} \mathbf{1}_{(\sigma \delta_1)^{-1} \in (1 + p^r \mathcal{O}_E)^\times} \prod_{v \mid D_1} \epsilon_v(y) \gamma_v.
\]

Taking the sum over all \( t \):

\[
W_{\sigma} \left( \begin{pmatrix} y \\ 1 \end{pmatrix} \right) = \mathbf{1}_{y_\infty > 0} |y|^{1/2} e^{-2\pi y_\infty} \prod_{v \mid D_1} \epsilon_v(y) \gamma_v
= \mathbf{1}_{y_\infty > 0} |y|^{1/2} e^{-2\pi y_\infty} \prod_{v \mid D_1} \epsilon_v(y) \gamma_v \sum_{J \subseteq \mathcal{O}_E} \mathbf{1}_{N(J) = y \gamma_{D_1}} \mathbf{1}_{(\sigma \delta_1)^{-1}(J)} \sum_{\sigma} \mathbf{1}_{(\sigma \delta_1)^{-1}(J)} \chi(\sigma)^{-1}
= \mathbf{1}_{y_\infty > 0} |y|^{1/2} e^{-2\pi y_\infty} \sum_{J \subseteq \mathcal{O}_E} \mathbf{1}_{N(J) = y \gamma_{D_1}} \chi_{[p]}(\Delta_1) \prod_{v \mid D_1} \epsilon_v(y) \gamma_v \sum_{J \subseteq \mathcal{O}_E} \mathbf{1}_{\mathcal{O}_E} \chi_{[p]}(J).
\]

Finally, taking the sum over all \( \sigma \):

\[
W \left( \begin{pmatrix} y \\ 1 \end{pmatrix}, \Theta_{D_1} \right) = \sum_{\sigma} \chi(\sigma)^{-1} W_{\sigma} \left( \begin{pmatrix} y \\ 1 \end{pmatrix} \right)
= \mathbf{1}_{y_\infty > 0} |y|^{1/2} e^{-2\pi y_\infty} \prod_{v \mid D_1} \epsilon_v(y) \gamma_v \sum_{J \subseteq \mathcal{O}_E} \mathbf{1}_{N(J) = y \gamma_{D_1}} \sum_{\sigma} \mathbf{1}_{(\sigma \delta_1)^{-1}(J)} \chi(\sigma)^{-1}
= \mathbf{1}_{y_\infty > 0} |y|^{1/2} e^{-2\pi y_\infty} \chi_{[p]}(\Delta_1) \prod_{v \mid D_1} \epsilon_v(y) \gamma_v \sum_{J \subseteq \mathcal{O}_E} \mathbf{1}_{\mathcal{O}_E} \chi_{[p]}(J).
\]

It is easy to see that the constant term \( W_0 \left( \begin{pmatrix} y \\ 1 \end{pmatrix}, \Theta_{D_1} \right) \) vanishes, since a similar calculation shows that the constant term of each \( \theta_{\sigma} (g \gamma_{D_1}) \) vanishes.
7.2 The Whittaker function of $E_{D_1}$

Write $\eta$ for the character $\epsilon \chi^{-1}$ and $C$ for the conductor of $\chi$ (hence the conductor of $\eta$ is $CD$). We begin with a complex variable $s$ with $\text{Re}(s) > 1$ and write $\mathcal{E}_s$ for the form $V_N E_s(\eta)(g_{\gamma D_1})$ (thus $E_{D_1}$ is equal to $\mathcal{E}_{1/2}$). By definition, we have:

$$\mathcal{E}_s = 2^{-n} L_{CD}(1 - 2s, \eta) \sum_{\gamma \in B(F) \setminus \text{GL}_2(F)} Q(\gamma g),$$

where the function $Q = \prod_v Q_v$ satisfies $Q \left( \begin{pmatrix} a & b \\ d & c \end{pmatrix} k \right) = |\frac{a}{d}|^s \eta(d)Q(k)$ for any $k$ in the maximal compact subgroup, and each $Q_v$ is defined by:

- if $v | \infty$, then we have $Q_v(r(\theta)) = e^{i\theta}$;

- if $v \nmid pD\infty$, then for $k = \begin{pmatrix} u & v \\ w & t \end{pmatrix} \in \text{GL}_2(\mathcal{O}_v)$ we have $Q_v \left( k \begin{pmatrix} n_f \\ 1 \end{pmatrix} \right) = |N_v|^{1/2}$, here $n_f \in \mathbb{A}_f^\times$ denotes a generator of $N$;

- if $v | CD$ but $v \nmid D_1$, then for $k = \begin{pmatrix} u & v \\ w & t \end{pmatrix} \in \text{GL}_2(\mathcal{O}_v)$ we have

  $$Q_v(k) = \begin{cases} \eta_v(t), & \text{if } (CD)_v \mid w; \\ 0, & \text{else}; \end{cases}$$

- if $v | D_1$, then for $k = \begin{pmatrix} u & v \\ w & t \end{pmatrix} \in \text{GL}_2(\mathcal{O}_v)$ we have

  $$Q_v(k) = \begin{cases} 0, & \text{if } (D_1)_v \mid w; \\ |D_1|_v^{\sigma_v} \eta_v(w), & \text{else}; \end{cases}$$

- if $v | p$ but $v \nmid C$, then for $k = \begin{pmatrix} u & v \\ w & t \end{pmatrix} \in \text{GL}_2(\mathcal{O}_v)$ we have

  $$Q_v(k) = \begin{cases} 1 - |\pi_v|, & \text{if } |w|_v = 1; \\ 1 - \eta(\pi_v)|\pi_v|^{1-2s}, & \text{else}. \end{cases}$$

The following calculation of Fourier coefficients is standard.

$$W \left( \begin{pmatrix} y \\ 1 \end{pmatrix}, \mathcal{E}_s \right) = 2^{-n} L_{CD}(1 - 2s, \eta) \int_{\mathbb{A}_F} \left( Q \left( \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} \right) + \sum_{u \in F} Q \left( \begin{pmatrix} y & 1 \\ u + x & 1 \end{pmatrix} \right) \right) \psi(-x) dx$$

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\[
\begin{align*}
\ &= 2^{-n}L_{CD}(1 - 2s, \eta) \int_Q \left( y \begin{pmatrix} 1 \\ x \end{pmatrix} \right) \psi(-x)dx \\
\ &= 2^{-n}L_{CD}(1 - 2s, \eta) |y|^{1-s} \eta(y) \prod_v V_v(y_v),
\end{align*}
\]

where the functions \( V_v \) are defined as:

\[
V_v(y) := \int_{F_v} Q_v \left( 1 \begin{pmatrix} 1 \\ x \end{pmatrix} \right) \psi_v(-yx)dx.
\]

The constant term can be computed similarly:

\[
W_0 \left( \left( y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \mathcal{E}_s \right) = 2^{-n}L_{CD}(1 - 2s, \eta) \left( |y|^s Q \left( 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + |y|^{1-s} \eta(y) \prod_v V_v(0) \right).
\]

Note that the value of \( Q \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \) is easily seen to be \( 1 \).

It then remains to compute each local function \( V_v \).

**Case 1:** \( v \mid \infty \) In this case the standard calculation gives:

\[
V_v(y) = \int_{F_v} \frac{e^{-2\pi i yx}}{(x^2 + 1)^{s-1/2}(x + i)}dx.
\]

If \( y \) is non-zero, then this function can be analytically continued to all \( s \in \mathbb{C} \), and decreases exponentially with respect to \( |y| \). Furthermore, when \( s = 1/2 \), we have:

\[
V_v(y) = \begin{cases} (-2\pi i)e^{-2\pi y}, & \text{if } y > 0; \\
0, & \text{if } y < 0.
\end{cases}
\]

If \( y \) is zero, then we have:

\[
V_v(0) = -i\pi^{1/2} \Gamma(s) \Gamma(s + 1/2)^{-1}.
\]

**Case 2:** \( v \nmid pD\infty \) In this case we have:

\[
Q_v \left( \begin{pmatrix} 1 \\ 1 \\ x \end{pmatrix} \right) = \begin{cases} \eta_v(N)^{-1} |N|^{s+1/2}, & \text{if } |N| \leq 1; \\
\eta_v(x) |N|^{1/2-s} |x|^{-2s}, & \text{else.}
\end{cases}
\]

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It follows that:

\[
V_v(y) = \eta_v(N)^{-1} |N|^s \int_{|x| \leq |N|^{-1}} \psi_v(-yx) dx \\
+ |N|^{1/2-s} \sum_{i > \text{val}_v(N)} \int_{|x| = |\pi_v|^{-1}} \eta_v(x) |x|^{-2s} \psi_v(-yx) dx \\
= \eta_v(N)^{-1} |N|^s S_{\text{val}_v(N)} + |N|^{1/2-s} \sum_{i > \text{val}_v(N)} \eta_v(\pi_v)^{-i} |\pi_v|^{2is} (S_i - S_{i-1}),
\]

where \( S_i \) is defined as:

\[
S_i := \int_{|x| \leq |\pi_v|^{-1}} \psi_v(-yx) dx = \begin{cases} |\pi_v|^{-i} |d_F|^{1/2}, & \text{if } |yd_F|_v \leq |\pi_v|^{i}, \\ 0, & \text{else.} \end{cases}
\]

Hence the function \( V_v \) is non-zero only if \( |yd_F|_v \leq |N|_v \), and in this case we have (making the convention that \( S_i = 0 \) for \( i < \text{val}_v(N) \)):

\[
V_v(y) = |N|^{1/2-s} \sum_i \eta_v(\pi_v)^{-i} |\pi_v|^{2is} (S_i - S_{i-1}) \\
= |N|^{1/2-s} (1 - \eta_v(\pi_v)^{-1} |\pi_v|^{2s}) \sum_i \eta_v(\pi_v)^{-i} |\pi_v|^{2is} S_i \\
= |N|^{1/2-s} (1 - \eta_v(\pi_v)^{-1} |\pi_v|^{2s}) |d_F|^{1/2} \sum_{i = \text{val}_v(yd_F)} \eta_v(\pi_v)^{-1} |\pi_v|^{2s-1} i.
\]

Case 3: \( v \mid CD \text{ but } v \nmid D_1 \) In this case a similar calculation shows that the function \( V_v \) is reduced to a Gauss sum: it is zero if \( yd_F \) is not integral or if \( y = 0 \); otherwise we have:

\[
V_v(y) = |CD|^{2s-1/2} |yd_F|^{2s-1} |d_F|^{1/2} \eta_v(-y)^{-1} r_v(\eta, \psi),
\]

where \( r_v(\eta, \psi) \) is the root number:

\[
r_v(\eta, \psi) = |CDd_F|^{1/2} \int_{|x| = |CDd_F|^{-1}} \eta_v(x) \psi_v(x) dx.
\]

Note that for places \( v \mid D \) we have \( r_v(\eta, \psi) = r_v(\epsilon, \psi)\chi_{F,v}(Dd_F) \), and \( r_v(\epsilon, \psi) \) is equal to \( \gamma_v \), the Weil index.
Case 4: $v \mid D_1$  In this case we have:
\[
Q_v \left( 1 \begin{array}{c} 1 \\ x \end{array} \right) = \begin{cases} |D_1|^s_v, & \text{if } |x|_v \leq 1; \\ 0, & \text{else}, \end{cases}
\]
so that:
\[
V_v(y) = \int |D_1|^s_v \psi_v(-yx) dx = \begin{cases} |D_1|^s_v |d_F|_v^{1/2}, & \text{if } |yd_F|_v \leq 1; \\ 0, & \text{else}. \end{cases}
\]

Case 5: $v \mid p$ but $v \nmid C$  In this case we have:
\[
Q_v \left( 1 \begin{array}{c} 1 \\ x \end{array} \right) = \begin{cases} 1 - |\pi_v|_v, & \text{if } |x|_v \leq 1; \\ \eta_v(x) |x|^{-2s} (1 - \eta_v(\pi_v) |\pi_v|_v^{1-2s}), & \text{else}. \end{cases}
\]
From this we see that $V_v(y)$ is non-zero only if $yd_F$ is integral, in which case we have:
\[
V_v(y) = (1 - |\pi_v|_v) |d_F|_v^{1/2} + (1 - \eta_v(\pi_v) |\pi_v|_v^{1-2s}) \sum_{i>0} \eta_v(\pi_v)^{-i} |\pi_v|_v^{2s} (S_i - S_{i-1}),
\]
where:
\[
S_i := \int_{|x|_v \leq |\pi_v|_v^{-i}} \psi_v(-yx) = \begin{cases} |\pi_v|^{-i} |d_F|_v^{1/2}, & \text{if } i \leq \text{val}_v(yd_F); \\ 0, & \text{else}. \end{cases}
\]
After simplification, we get (when $yd_F$ is integral):
\[
V_v(y) = \begin{cases} (1 - \eta_v(\pi_v)^{-1} |\pi_v|_v^{2s}) |d_F|_v^{1/2} \eta_v(yd_F)^{-1} |yd_F|_v^{2s-1}, & \text{if } y \neq 0; \\ 0, & \text{if } y = 0. \end{cases}
\]

Putting together  Combining the above results, we may now compute the Whittaker function of $\mathcal{E}_s$:
\[
W \left( \left( \begin{array}{c} y \\ 1 \end{array} \right) ; \mathcal{E}_s \right) = 2^{-n} L_{CD}(1 - 2s, \eta) |y|^{1-s} \eta(y) \prod_{v \mid \infty} V_v(y_v)
\]
\[
\prod_{v \mid yd_F \mid \leq |N|_v} \mathbf{1} |N|_v^{1/2} (1 - \eta_v(\pi_v)^{-1} |\pi_v|_v^{2s}) |d_F|_v^{1/2} \sum_{i=\text{val}_v(N)}^{\text{val}_v(yd_F)} (\eta_v(\pi_v)^{-1} |\pi_v|_v^{2s-1})^i
\]

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Using the functional equation:

\[
\prod_{v \mid CD} 1_{|yd_F|_v \leq 1} |CD|_{v}^{2s-1/2} |yd_F|_{v}^{2s-1} |d_F|_{v}^{1/2} \eta_v(-y)^{-1} r_v(\eta, \psi)
\]

\[
\prod_{v \mid D_1} 1_{|yd_F|_v \leq 1} |D_1|_{v}^{s} |d_F|_{v}^{1/2} \prod_{v \mid |yd_F|_v \leq 1} (1 - \eta_v(\pi_v)^{-1} |\pi_v|_{v}^{2s}) |d_F|_{v}^{1/2} \eta_v(yd_F)^{-1} |yd_F|_{v}^{2s-1}
\]

\[
= 2^{-n} L_{CD}(1 - 2s, \eta) L_{CD}(2s, \eta^{-1})^{-1} |y|^{1-s} \eta(y) \prod_{v \mid |yd_F|_v \leq 1} |N|_{f}^{1/2-s} |D_1|_{f}^{1/2-s} |CD|_{f}^{2s-1/2} |d_F|_{f}^{1/2}
\]

\[
\prod_{v \mid \infty} V_v(y) \prod_{v \mid D_\infty} \sum_{i = \val_v(yd_F)} (\eta_v(\pi_v)^{-1} |\pi_v|_{v}^{2s-1})^i
\]

\[
\prod_{v \mid CD} |yd_F|_{v}^{2s-1} \eta_v(-y)^{-1} r_v(\eta, \psi) \prod_{v \mid C} \eta_v(yd_F)^{-1} |yd_F|_{v}^{2s-1}.
\]

Using the functional equation:

\[
L_{CD}(1 - 2s, \eta)/L_{CD}(2s, \eta^{-1})
\]

\[
= |CDd_F|_{\infty}^{2s-1/2} \left(-i \pi^{1/2-2s} \Gamma(1-s)^{-1} \Gamma(s+1/2)^{n} \prod_{v \mid |yd_F|_v \leq 1} \eta_v(d_F) \prod_{v \mid CD} r_v(\eta, \psi)^{-1},
\]

and then specialize to \( s = 1/2 \), we simplify the formula to:

\[
W(\left( \begin{array}{c} y \\ 1 \end{array} \right), E_{D_1})
\]

\[
= 1_{y_\infty > 0} |y|^{1/2} e^{-2\pi y_\infty} \sum_{J \subseteq O} \eta_{[D]}(J) \prod_{v \mid D_1} \eta_v(-y) r_v(\eta, \psi)^{-1}
\]

\[
= 1_{y_\infty > 0} |y|^{1/2} e^{-2\pi y_\infty} \sum_{J \subseteq O} \chi_{[J]}^{-1} (J \mathcal{O}_E \epsilon_{[D]}(J) \prod_{v \mid D_1} \epsilon_v(-y) r_v(\epsilon, \psi)^{-1} \chi_{F,s}(yd_F D_1).
\]

For the constant term, notice that for any place \( v \) dividing \( p \), we always have \( V_v(0) = 0 \). Hence:

\[
W_0(\left( \begin{array}{c} y \\ 1 \end{array} \right), E_s) = 2^{-n} L_{CD}(1 - 2s, \eta) |y|^s Q(\left( \begin{array}{c} 1 \\ 1 \end{array} \right))
\]

\[
= 1_{D_1 = 1} 2^{-n} |N|_{f}^{1/2-s} |y|^s L_{pD}(1 - 2s, \eta).
\]

When specialized to \( s = 1/2 \), this gives:

\[
W_0(\left( \begin{array}{c} y \\ 1 \end{array} \right), E_{D_1}) = 1_{D_1 = 1} |y|^{1/2} 2^{-n} L_{pD}(0, \chi_F^{-1} \epsilon).
\]

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7.3 $q$-expansion of the analytic kernel

We are ready to calculate the $q$-expansion of the analytic kernel. For each divisor $D_1$ of $D$ and each $y \in \mathbb{A}^x$ such that $y_\infty > 0$, we have:

$$W\left(\left(\begin{array}{c} y \\ 1 \end{array}\right), X_{D_1}\right) = \sum_{\alpha \geq 0, \beta \geq 0} W\left(\left(\begin{array}{c} \alpha y \\ 1 \end{array}\right), \Theta_{D_1}\right) W\left(\left(\begin{array}{c} \beta y \\ 1 \end{array}\right), E_{D_1}\right)$$

$$= |y| e^{-2\pi y_\infty} \sum_{\alpha \geq 0, \beta \geq 0} \sum_{J \subseteq O_E} \sum_{K \subseteq O} \chi_{[p]}(J/\Delta_1 K(\beta y d_F) D_1) e_{|D_j|}(K) \prod_{v | D_1} \epsilon_v(-\alpha \beta)$$

$$+ |y| e^{-2\pi y_\infty} \prod_{v | D_1} (\beta y d_F)_v \sum_{J \subseteq O_E} \sum_{N(J) = y d_F} \chi_{[p]}(J/N) (0, \chi_F^{-1} \epsilon) \chi_{[p]}(J),$$

here we write $(\beta y d_F)_{D_1}$ for the ideal $\prod_{v | D_1} (\beta y d_F)_v$ of $O$. The constant term vanishes, because that of $\Theta_{D_1}$ vanishes.

Finally, for any integral ideal $I$ of $O$, take $y \in \mathbb{A}^x$ such that $y_\infty > 0$ and $y d_F = I$, we may compute:

$$a(I, K(\chi)) = |y|^{-1} e^{2\pi y_\infty} W\left(\left(\begin{array}{c} y \\ 1 \end{array}\right), K(\chi)\right)$$

$$= |y|^{-1} e^{2\pi y_\infty} \sum_{D_1 | D} |D_1|_{\infty} W\left(\left(\begin{array}{c} d_1 y \\ 1 \end{array}\right), X_{D_1}\right)$$

$$= \sum_{D_1 | D} \sum_{\alpha \geq 0, \beta \geq 0} \sum_{J \subseteq O_E} \sum_{K \subseteq O} \chi_{[p]}(J/\Delta_1 K(\beta D_1 I) D_1) e_{|D_j|}(K) \prod_{v | D_1} \epsilon_v(-\alpha \beta)$$

$$+ 2^{-n} L_{pD}(0, \chi_F^{-1} \epsilon) \sum_{J \subseteq O_E} \chi_{[p]}(J/N) (0, \chi_F^{-1} \epsilon) \chi_{[p]}(J).$$

Since this formula is valid for all character $\chi$, by linearity it is also valid for all functions on the group $G$.

7.4 The central derivative

Here we specialize to the case of the central derivative of the $p$-adic $L$ function in the cyclotomic line. Writing $\xi_E$ for the character $\xi_p \circ N_{E/F}$ (the $p$-adic cyclotomic
character on $E$), we have:

$$L'_{p,ξ_E}(1) = \int_G (\log_p \circ ξ_E)K_f = l_f \left( \int_G (\log_p \circ ξ_E)K \right).$$

We denote by $Φ$ the $p$-adic modular form $\int (\log_p \circ ξ_E)K$. The calculations in previous subsections apply and gives:

**Proposition 7.1.** If $I$ is an ideal of $O$ divisible by $p$, then the coefficient $a(I, Φ)$ of the $q$-expansion of the form $Φ$ is given by $a(I, Φ) = \sum v a_v(I, Φ)$, where the sum ranges over finite places $v$, and each term $a_v(I, Φ)$ is given by:

1. if $v$ is inert in $E$ and $q_v$ is the associated prime ideal, then:
   $$a_v(I, Φ) = \sum_{\alpha+\beta=1, \alpha, \beta > 0 \atop \epsilon_v(-\alpha\beta)=1} \delta r(\alpha I)r(\beta I/Nq_v)\text{ord}_v(\beta Iq_v/N) \log_p(|q_v|_{\infty}),$$
   here $δ$ denotes $2^{\#\{v:Dv|\beta DI\}}$, and for an ideal $M$ of $O$, $r(M)$ denotes the number of ideals of $O_E$ with norm $M$;

2. if $v$ is ramified in $E$ and $q_v$ is the associated prime ideal, then:
   $$a_v(I, Φ) = \sum_{\alpha+\beta=1, \alpha, \beta > 0 \atop \epsilon_v(-\alpha\beta)=1, \forall w|D \atop N|\beta DI \atop (p,\beta DI) = 1} \delta r(\alpha I)r(\beta I/q_v)\log_p(|q_v|_{\infty});$$

3. if $v$ is split in $E$, then $a_v(I, Φ) = 0$.

**Proof** If we replace the character $χ$ in the formula in subsection 7.3 by the function $\log_p \circ ξ_E$, then the sum over all the $J$ becomes a multiplication by the factor $r(\alpha I)$, because every $J$ has the same norm to $F$, and also note that $r(\alpha D_1 I) = r(\alpha I)$ for any $D_1 | D$.

Since we only consider the case $p | I$, the contribution of the second term is zero. Thus we have:

$$a(I, Φ) = \sum_{\alpha+\beta=1} \sum_{D_1} \sum_K r(\alpha I) \log_p(ξ_p(\alpha I/K^2(\beta D_1 I)^2_{D_1}))\epsilon|D_1(K) \prod_{v|D_1} \epsilon_v(-\alpha\beta).$$

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Now for any ideal $M$ of $\mathcal{O}$, we have:

$$\log_p(\xi_p(M)) = \log_p \left( \prod_v \xi_p(q_v^{\text{ord}_v(M)}) \right) = \sum_v \text{ord}_v(M) \log_p(|q_v|_{\infty}),$$

thus the coefficient $a(I, \Phi)$ decomposes into a sum of local terms $a_v(I, \Phi)$, and each local term can be computed. $\square$
Part II. Geometric part

8 Shimura curves and Heegner points

In the following we assume that $\epsilon(N) = (-1)^{n-1}$. We are going to introduce a Shimura curve $X$ and construct Heegner points on the curve.

8.1 Shimura curves

Fix an infinite place $\tau$ of $F$. Under our hypothesis, there is a quaternion algebra $B$ over $F$ which ramifies exactly at all infinite places different from $\tau$ and all finite places $v$ such that $\epsilon_v(N) = -1$.

By fixing an isomorphism $B_\tau \simeq M_2(\mathbb{R})$, the group $B^\times$ acts on the left on the Poincaré double half plane $\mathcal{H}^\pm := \mathbb{C} - \mathbb{R}$ via the usual action of $GL_2(\mathbb{R})$. For any open compact subgroup $K$ of $\hat{B}^\times/\hat{F}^\times$, we then have a Shimura curve $X_K$, whose complex points are given by: $X_K(\mathbb{C}) = B^\times\backslash\mathcal{H}^\pm \times \hat{B}^\times/\hat{F}^\times K$.

Remark. When $F = \mathbb{Q}$ there may be a finite set of cusps, but we will exclude this case, which is the original work of Perrin-Riou.

8.2 Level structure and the Shimura curve $X$

By construction of the quaternion algebra $B$, the quadratic extension $E$ is non-split on every ramified place of $B$, so we can fix an embedding of $E$ into $B$ (which is unique up to conjugation by $B^\times$), and thus view $E$ as a sub-algebra of $B$.

In [17] section 1.5, an order $R$ of $B$ of type $(N, E)$ is constructed, that is, the order $R$ contains $O_E$ and has discriminant $N$. Using this order $R$, we define the level structure $K = \hat{F}^\times\hat{R}^\times$, and the Shimura curve $X := X_K$, whose complex points are given by: $X(\mathbb{C}) = B^\times\backslash\mathcal{H}^\pm \times \hat{B}^\times/\hat{F}^\times\hat{R}^\times$.

The moduli interpretation problem of the curve $X$ is discussed in [17] section 1. There is a finite map from $X(\mathbb{C})$ to another Shimura curve $X'(\mathbb{C})$ which parametrizes certain classes of abelian varieties. This interpretation gives an integral model of the curve $X$. 
8.3 CM points, Galois actions and Hecke operators

Let $z_0$ be the unique point in the Poincaré upper half plane which is fixed by the action of $E^\times$. A CM point on $X$ is a point in $X(\mathbb{C})$ which is represented by a pair $(z_0, b) \in \mathcal{H}^\pm \times \widehat{B}^\times$.

The Galois actions on CM points are described by Shimura’s reciprocity law, which states that all CM points are algebraic and defined over the maximal abelian extension $E^{ab}$ of $E$, and for any element $a \in \widehat{E}^\times$, we have:

$$\text{rec}(a)(z_0, b) = (z_0, ab),$$

where rec : $\widehat{E}^\times \to \text{Gal}(E^{ab}/E)$ is the Artin reciprocity map over $E$.

There are also Hecke operators defined on the group of divisors of $X$, in a way similar to Heck operators on modular forms. Fix isomorphisms $B_v \cong M_2(F_v)$ for split places $v$ of $B$. Let $M$ be a non-zero ideal of $\mathcal{O}$ prime to all rafimied places of $B$, and let $U(M)$ be the same set as in subsection 2.2. Again write $U(M)$ as a disjoint union $\bigsqcup_j h_j K_0(N)$, and the Hecke operator $T_N(M)$ is defined as:

$$T_N(M)(z, b) = \sum_j (z, bh_j) = \sum_{h \in U(M)/K_0(N)} (z, bh).$$

The Hecke operators are obviously multiplicative in $M$.

By Jacquet-Langlands theory, the Hecke algebra generated by these Hecke operators is a quotient of the Hecke algebra generated by the Hecke operators on modular forms of level $N$. Thus we may write $T_N(I)$ unambiguously to represent both of the operators.

8.4 Heegner divisors

A CM point on the curve $X$ is called a "Heegner point" if it is of conductor 1, i.e. it is defined over the conductor 1 ring class field $H$ of $E$ (i.e. the abelian extension of $E$ such that $\text{Gal}(H/E)$ is isomorphic to $\mathbb{A}_{E,f}/E^\times \widehat{\mathcal{O}}_E^\times \widehat{F}^\times$ via class field theory). The action of the group $\text{Gal}(H/E)$ acts transitively on the set of Heegner points.

More generally, if $C$ is a non-zero ideal of $\mathcal{O}$, let $H[C]$ be the ring class field of conductor $C$, i.e. the abelian extension of $E$ such that $\text{Gal}(H[C]/E) \simeq \mathbb{A}_{E,f}/E^\times (\mathcal{O} + C)^\times \widehat{F}^\times$. A CM point of conductor $C$ is a point that is defined over the field $H[C]$.  


In [17], a canonical divisor class of degree 1, $\xi \in \text{Pic}(X) \otimes \mathbb{Q}$, is constructed (see section 4.1.4 and Introduction of loc. cit.), and one may use it to define a map $\phi : X \to \text{Jac}(X) \otimes \mathbb{Q}$, which sends any point $y$ to the class of $y - \xi$. The choice of a divisor representing the Hodge class will be discussed later.

Now let $x$ be any Heegner point, and let $z$ be the divisor class:

$$\frac{1}{\#(\mathcal{O}_E^\times/\mathcal{O}_E^\times)} \sum_{\sigma \in \text{Gal}(H/E)} \phi(x^\sigma),$$

which is rational over $E$. This is the Heegner divisor in the introduction, and its $p$-adic height will be related to the derivative of the $p$-adic $L$ function.

## 9 The geometric kernel

### 9.1 $p$-adic height pairing

We are going to recall the basics of $p$-adic height pairings, following Zarhin [16] and Nekovár [7].

In this subsection, let $K$ be a number field and let $A/K$ be an Abelian variety over $K$, which has good reduction at every place $v$ of $K$ above $p$. For each integer $n \geq 1$, write $S(A/K,n)$ for the $n$-Selmer group and write $S_p(A/K)$ for the limit $\varprojlim_S(A/K,p^k)$. Also write $T_p(A) = \varprojlim_k A[p^k]$ for the $p$-adic Tate module, and put $V_p(A) = T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

The injection $A(K) \otimes \mathbb{Z}_p \hookrightarrow S_p(A/K)$ gives, after tensoring $\mathbb{Q}_p$, an injection of $A(K) \otimes \mathbb{Q}_p$ into $S_p(A/K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, which, by [1] 3.11, is equal to $H^1_{dR}(K,V_p(A))$, the Bloch-Kato Selmer group.

Let $K_\infty$ be the compositum of all $\mathbb{Z}_p$-extensions of $K$. Suppose that for every place $v \mid p$, we are given a $\mathbb{Q}_p$-linear splitting for the Hodge Filtration $F^0H^1_{dR}(A/K_v) \to H^1_{dR}(A/K_v)$, which we fix from now on. Then, by results of [16] and [7], a pairing

$$H^1_j(K,V_p(A)) \times H^1_j(K,V_p(\hat{A})) \to \text{Gal}(K_\infty/K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is defined. This is the abstract $p$-adic pairing. If we are also given a continuous morphism $\ell : A_{K_\infty}^\times/K^\times \to \mathbb{Q}_p$, it will induce a morphism from $\text{Gal}(K_\infty/K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to $\mathbb{Q}_p$, so the above pairing gives a pairing taking values in $\mathbb{Q}_p$ after composing with $\ell$. 

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Remark We may replace $V_p(\hat{A})$ by $V_p(A)^*(1)$ via the Weil pairing $V_p(A) \times V_p(\hat{A}) \to \mathbb{Q}_p(1)$.

Now we specialize to the case of Jacobians of curves and briefly recall the definition of the height pairing. Let $X$ be an irreducible smooth projective curve over $K$ with good reduction at all places above $p$, and let $A$ be the Jacobian of $X$, which is canonically isomorphic to its dual $\hat{A}$.

In this case, we have the $p$-adic Abel-Jacobi map: $A(K) \to H^1(K, V_p(A)) = \text{Ext}^1_{\text{Gal}(\overline{K}/K)}(\mathbb{Q}_p, V_p(A)).$ Thus for each divisor $D$ of $X$ of degree 0, we may represent the image of $D$ under the Abel-Jacobi map by an extension:

$0 \to V_p(A) \to \mathcal{E}_D \to \mathbb{Q}_p \to 0.$

Let $D_1, D_2$ be two divisors of $X$ of degree 0 which do not intersect. The height pairing $\langle \text{cl}(D_1), \text{cl}(D_2) \rangle$ is a sum of local pairings (here $\text{cl}$ means the class in $A(K)$). The decomposition depends on a mixed extension $\mathcal{E}$, i.e. a $\text{Gal}(\overline{K}/K)$-module $\mathcal{E}$ that fits into the commutative diagram:

Such a mixed extension arise naturally as subquotient of the relative cohomology $H^1_{cl}(X_{\overline{K}} - |D_1|, |D_2|; \mathbb{Q}_p)(1)$, c.f. [8] II 1.9.

If $v$ is a place not dividing $p$, then we have $H^1(K_v, V_p(A)) = H^1(K_v, V_p(A)^*(1)) = 0$, thus as a $\text{Gal}(\overline{K}_v/K_v)$-module, the module $\mathcal{E}$ splits into a direct sum $V_p(A) \oplus \mathcal{E}_v'$, where $\mathcal{E}_v'$ is an extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$. We denote by $C_v$ the class of $\mathcal{E}_v'$ in the group $\text{Ext}^1_{\text{Gal}(\overline{K}_v/K_v)}(\mathbb{Q}_p, \mathbb{Q}_p(1))$. 

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On the other hand, we have:

\[
\text{Ext}^1_{\text{Gal}(\mathbb{K}/\mathbb{K})}(\mathbb{Q}_p, \mathbb{Q}_p(1)) = H^1(\mathbb{K}, \mathbb{Q}_p(1)) = \left( \lim_{\to} K_v^\times/(K_v^\times)^p \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = K_v^\times \otimes \mathbb{Q}_p.
\]

Let \( \ell_v \) be the map \( K_v^\times \to A_\mathbb{K}^\times \otimes \mathbb{Q}_p \), where \( \ell \) is a fixed continuous morphism from \( A_\mathbb{K}^\times \) to \( \mathbb{Q}_p \), then the local height pairing is defined as \( \langle D_1, D_2 \rangle_v = -(\ell_v \otimes 1)(C_v) \).

If \( v \) is a place dividing \( p \), one constructs similarly a class \( C_v \) in the group \( H^1(\mathbb{K}, \mathbb{Q}_p(1)) \), and defines the local pairing by the same formula. For details, see [8] II 1.7 and [7].

In the following, we always take the morphism \( \ell \) to be the logarithm of the cyclotomic character: \( \ell = \log_p \circ \xi_p \).

### 9.2 Construction of the geometric kernel

We will define the geometric kernel \( \Psi \) to be a \( p \)-adic modular form in the space \( S_2(N, L) \) with \( q \)-expansion given by: \( a(I, \Psi) = \langle z, T_N(I)z \rangle \), where \( z \) is the Heegner divisor in subsection 8.4. By duality of \( p \)-adic modular forms (c.f. [5] section 2 for a setting close to ours), this \( q \)-expansion is really a \( p \)-adic modular form.

We also define:

\[
\Psi_p = \sum_I \sum_{w \mid p} \langle z, T_N(I)z \rangle_w q^I
\]

\[
\Psi_f = \sum_I \sum_{w \nmid p} \langle z, T_N(I)z \rangle_w q^I,
\]

which are also in the space \( S_2(N, L) \) and satisfy \( \Psi_p + \Psi_f = \Psi \).

### 10 \( p \)-adic heights at places outside \( p \)

In this section, let \( v \) be a finite place of \( F \) not dividing \( p \).

In this case the \( p \)-adic height pairing \( \langle \cdot, \cdot \rangle_w \) is unique, and takes values in the subset \( \mathbb{Q} \cdot \log_p(|q_v|_\infty) \). Since the corresponding real height pairings \( \langle \cdot, \cdot \rangle_{w,\infty} \) takes values in the subset \( \mathbb{Q} \cdot \log(|q_v|_\infty) \) of \( \mathbb{R} \) and satisfies the same conditions as the \( p \)-adic one, by uniqueness we have:

\[
\langle \cdot, \cdot \rangle_w/\log_p(|q_v|_\infty) = \langle \cdot, \cdot \rangle_{w,\infty}/\log(|q_v|_\infty).
\]
In [17], the real height pairings are computed, and we are going to use these results to show that the form $\Psi_f$ is closely related to the analytic kernel $\Phi$.

### 10.1 The quotient space

Following [17], we define $S$ to be the space of functions on the set of non-zero integral ideals of $\mathcal{O}$ with values in the $p$-adic field $L$, modulo the equivalence relation that two functions $a$ and $b$ are equivalent if there is a non-zero ideal $M$ such that $a(I) = b(I)$ for all $I$ prime to $M$. Identifying a $p$-adic modular form with its $q$-expansion, we may view the space $\mathcal{S}_2(N,L)$ as a subspace of $S$.

A function $f$ in $S$ is called "quasi-multiplicative" if there is a non-zero ideal $M$ such that $f(IJ) = f(I)f(J)$ for all $I, J$ such that $(I,J) = (M,IJ) = 1$. If $f$ is quasi-multiplicative function, a function $h$ is called an "$f$-derivative" if $h(IJ) = f(I)h(J) + h(I)f(J)$ for all $I, J$ as above.

The functions $I \mapsto \sigma_1(I)$ and $I \mapsto r(I)$ define functions $\sigma_1$ and $r$ in the space $S$. Let $D_N$ be the subspace of $S$ generated by $\sigma_1$, $r$, $\sigma_1$-derivatives, $r$-derivatives, and all forms that are "old at $N$", i.e. comes from a form of level $Mp\infty$ with $M \mid N$ and $M \neq N$.

**Proposition 10.1.** If the $q$-expansion of a form in $\mathcal{S}_2(N,L)$ lies in $D_N$, then it is old at $N$.

**Proof** This is the same as Proposition 4.5.1 of [17].

In view of this proposition, forms that differ by some function in $D_N$ will have the same value under the linear functional $l_f$.

### 10.2 Compare the two kernels

In [17], the real heights of the Heegner divisors are calculated, and we may deduce from it the corresponding $p$-adic heights, i.e. coefficients of the $q$-expansion of the geometric kernel $\Psi_f$.

**Proposition 10.2.** Modulo $D_N$, the function $I \mapsto a(I, \Psi_f)$ on the set of non-zero ideals of $\mathcal{O}$ is equal to a sum $\sum_v a_v(I, \Psi_f)$, where the sum ranges over all finite places of $F$, and each $a_v(I, \Psi_f)$ is given by:

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1. if \(v\) is inert in \(E\) and \(q_v\) is the associated prime ideal, then:

\[
a_v(I, \Psi_f) = \sum_{\alpha + \beta = 1, \alpha, \beta > 0} \delta r(\alpha I) r(\beta I/Nq_v) \text{ord}_v(\beta Iq_v/N) \log_p(|q_v|_\infty),
\]

here \(\delta\) and \(r\) has the same meaning as in Proposition 7.1.

2. if \(v\) is ramified in \(E\) and \(q_v\) is the associated prime ideal, then:

\[
a_v(I, \Psi_f) = \sum_{\alpha + \beta = 1, \alpha, \beta > 0} \delta r(\alpha I) r(\beta I/N) \text{ord}_v(\beta Iq_v) \log_p(|q_v|_\infty);
\]

3. if \(v\) is split in \(E\), then \(a_v(I, \Psi_f) = 0\).

**Proof** This is the combination of [17] Proposition 5.4.8, Proposition 6.4.5 and Proposition 7.1.1, but we changed the real logarithm to the \(p\)-adic logarithm everywhere.

The relation between the two forms \(\Phi\) and \(\Psi_f\) can then be stated as follows.

**Proposition 10.3.** The difference of the two forms \(\prod_{P_i \mid p} (T(P_i)^4 - T(P_i)^2)\) \(\Phi\) and \(\prod_{P_i \mid p} (T(P_i) - 1)^4\) \(\Psi_f\) is annihilated by the linear functional \(l_f\).

**Proof** This is essentially the same as [11] Proposition 3.20, but note that our \(\Psi\) takes the sum of the heights of all conjugates of \(x\), thus the translation by an element \(\sigma \in \text{Gal}(H/E)\) acts trivially.

If we can prove that the form \(\prod_{P_i \mid p} (T(P_i) - 1)^4\) \(\Psi_p\) is also annihilated by the linear form \(l_f\), then applying \(l_f\) will give:

\[
\left( \prod_i (\alpha_i^4 - \alpha_i^2) \right) L_{p,\xi_{\infty}}(1) = l_f \left( \prod_i (T(P_i)^4 - T(P_i)^2) \Phi \right) = l_f \left( \prod_i (T(P_i) - 1)^4 \Psi \right) = \left( \prod_i (\alpha_i - 1)^4 (1 - \frac{|P_i|_\infty}{\alpha_i^2})^{-1} \right) \frac{(f, \Psi)}{(f, f)},
\]

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where the last equality comes from Lemma 3.2. Since we have \( \frac{(f, \Psi)}{(f, f)} = \langle z_f, z \rangle = \langle z_f, z_f \rangle \), this gives our main result.

Thus it only remains to prove that the contribution of the form \( \Psi_p \) is zero. Actually, we will prove that for each place \( v \) of \( F \) over \( p \), the contribution of the form \( \Psi_v = \sum_{I} \sum_{w | v} \langle z, T(I)z \rangle_w q^I \) is zero.

11 \( p \)-adic heights at places above \( p \)

In this section we prove that the contribution of the local heights for places above \( p \) is zero. We are going to use a corrected version of the method in [8].

Thus recall that \( X/F \) is the Shimura curve constructed in subsection 8.2 and \( A \) is the Jacobian of \( X \).

11.1 General arguments

We first decompose the space of modular forms \( S_2(N) \) according to action of the spherical Hecke algebra. Let \( K \) be a number field, big enough to contain all Hecke eigenvalues of eigenforms in \( S_2(N) \). The spherical Hecke algebra \( \mathcal{T} = K[T_N(I), (I, \sqrt{N}p) = 1] \), a subalgebra of \( \text{End}_{L_0}(S_2(N, K)) \), then decomposes as a direct sum of \( \mathcal{T} \) modules:

\[
\mathcal{T} = \bigoplus_{\rho: \mathcal{T} \to K} K,
\]

where \( \rho \) runs through all morphisms from \( \mathcal{T} \) to \( K \). For each \( \rho \) as above, let \( e_\rho \) be the projector onto the \( \rho \)-component.

**Lemma 11.1.** There is a finite set \( S \) of ideals \( I \) of \( \mathcal{O} \) satisfying \( (I, \sqrt{N}p) = 1 \) and \( r(I) = 0 \), such that every projector \( e_\rho \) is a linear combination of the Hecke operators \( T_N(I) \).

**Proof** This follows from [8] Lemma II.5.7. \( \square \)

Let \( K_p \) be the closure of \( K \) in \( \mathbb{C}_p \), under the fixed embedding \( \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p \). We then have the corresponding decomposition of \( V_p(A) \) as representation of \( \text{Gal}(\overline{\mathbb{Q}}/F) \) over \( K_p \):

\[
V_p(A) \otimes_{\mathbb{Q}_p} K_p = \bigoplus_{\rho: \mathcal{T} \to K} V_\rho(1)^{m_\rho},
\]

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where $V_p$ is a $K_p$ representation of $\text{Gal}(\overline{\mathbb{Q}}/F)$ of dimension 2, characterized by the identity:
\[
\det(1 - X \cdot \text{Frob}(Q)|_{V_p}) = 1 - \rho(T_N(Q))X + |Q|_\infty X^2
\]
for any prime ideal $Q$ prime to $NP$, and $m_p \geq 1$ is the multiplicity.

Let $\rho_f : T \to K$ be the morphism corresponding to our fixed form $f$. Since $f$ is a new form, the multiplicity $m_{f|}$ is equal to 1, by [17], Theorem 3.2.1. Hence the space $V = V_p(A) \otimes_{\mathbb{Q}_p} K_p$ decomposes as: $V = V' \oplus V''$, with $V' = e_{f|}(V)$ a representation of dimension 2, the $f$-component, and $V'' = (1 - e_{f|})(V)$.

Now fix a prime ideal $\mathfrak{P}_i$ of $O$ above $p$. If we view $V'$ as representation of the Galois group $\text{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{P}_i})$, then the "reduction mod $\mathfrak{P}_i$" map on the variety $A$ gives a short exact sequence:
\[
0 \to V'_{+,i} \to V' \to V'_{-,i} \to 0,
\]
with $V'_{-,i} \simeq (V'_{+,i})^*(1)$ an unramified representation, on which the Frobenius acts as multiplication by a Weil number of weight $-1$.

Let $T'$ and $T''$ be fixed $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $O_{K_p}$-lattices of $V'$ and $V''$, respectively. Put $T'_{+,i} = T' \cap V'_{+,i}$ and $T'_{-,i} = T'/T'_{+,i}$.

**Proposition 11.2.** If $\Omega$ is an algebraic extension of $F_{\mathfrak{P}_i}$, such that the maximal unramified subextension $\Omega^{ur}$ is of finite degree over $F_{\mathfrak{P}_i}$, then we have $H^0(\Omega, V'_{-,i}) = 0$.

**Proof** The representation $V'_{-,i}$ being unramified, the non-triviality of $H^0(\Omega, V'_{-,i})$ would imply non-triviality of the cohomology $H^0(\Omega^{ur}, V'_{-,i})$, hence implies that some power of the Frobenius morphism acts trivially on the space $V'_{-,i}$. But none of the powers of a Weil number is equal to 1, a contradiction. \qed 

**Corollary 11.3.** The groups $H^0(\Omega, V'_{+,i}/T'_{-,i})$ and $H^0(\Omega, (V'_{+,i})^*(1)/(T'_{+,i})^*(1))$ are finite.

**Proof** Temporarily write $V$ for $V'_{+,i}$, $T$ for $T'_{-,i}$ and $G$ for $\text{Gal}(\overline{\mathbb{Q}}_p, \Omega)$. If the group $H^0(\Omega, V'_{+,i}/T'_{-,i})$ is infinite, then there is a sequence $(v_m)_m$ in $V$, all different modulo $T$, such that $v^\sigma_m - v_m$ is in $T$ for every $\sigma \in G$. Since the set $p^{-k}T/T$ is finite for every $k$, the sequence $(v_m)_m$ is unbounded, i.e. for every $k$, there is a term $v_{m_k}$ not belonging to $p^{-k}T$. Fix such $v_{m_k}$ for every $k$.

Let $a_k (\geq k)$ be the integer such that $v_{m_k} \in p^{-a_k - 1}T \setminus p^{-a_k}T$, and put $u_k = p^{a_k}v_{m_k}$, which lies in $p^{-1}T \setminus T$. We then have $u_k^\sigma - u_k \in p^{a_k}T \subseteq p^kT$ for any $\sigma \in G$. Since the
set $p^{-1}T \setminus T$ is compact, the sequence $(u_k)_k$ admits at least one accumulation point $u$, which must be a non-zero fixed point of $G$, contradicting the above proposition.

Changing the meanings of $V$ and $T$ proves that $H^0(\Omega, (V'_+,i)^*(1)/(T'_+,i)^*(1))$ is finite. \hfill \square

Now let $T$ be the lattice $T_p(A) \otimes \mathbb{Z}_p \mathcal{O}_K$ of $V = V_p(A) \otimes \mathbb{Q}_p K_p$. Take $T' = e_{\rho f} T$ and $T'' = (1 - e_{\rho f}) T$, which are lattices of $V'$ and $V''$, respectively. Then the sum $T' + T''$ is again a lattice of $V$, thus there is a constant $d_0$, such that: $T \subseteq T' + T'' \subseteq p^{-d_0}T$.

We will also fix a splitting for the Hodge filtration $F^0(\Omega_1 \otimes \mathbb{Q}_p K_p) \rightarrow H^1_{dR}(A/F_{\mathfrak{p}_i}) \otimes \mathbb{Q}_p K_p$. Under the decomposition

$$H^1_{dR}(A/F_{\mathfrak{p}_i}) \otimes \mathbb{Q}_p K_p = D_{\text{cris}}(V'_{\text{Gal}(\mathbb{Q}_p/F_{\mathfrak{p}_i})})\{-1\} \oplus D_{\text{cris}}(V''_{\text{Gal}(\mathbb{Q}_p/F_{\mathfrak{p}_i})})\{-1\},$$

the $F^0$ part decomposes as: $D_{\text{cris}}(V'_{\text{Gal}(\mathbb{Q}_p/F_{\mathfrak{p}_i})})\{-1\} \oplus (F^1 D_{\text{cris}}(V''_{\text{Gal}(\mathbb{Q}_p/F_{\mathfrak{p}_i})}))\{-1\}$. We fix the canonical splitting on the first factor, and any splitting on the second factor.

**Remark** If the curve have good ordinary reduction at $\mathfrak{p}_i$ (which is our case), the $p$-adic height pairing given by this splitting is the same as that defined by the universal norm, as in [7] Theorem 6.11. \hfill \square

We fix a proper smooth model $X_i$ of $X$ over the ring $\mathcal{O}_{\mathfrak{p}_i}$. Such a model exists under our hypothesis.

**Proposition 11.4.** Let $\Omega$ be a fixed extension of $F_{\mathfrak{p}_i}$ as in Proposition 11.2 and let $H$ be a finite subextension of $\Omega$. If $D_1$ and $D_2$ are two divisors of $X$ of degree zero over $H$, such that the Zariski closures of $D_1$ and $e_{\rho f}(D_2)$ in $X_i$ do not intersect, then there is a constant $C$, depending only on $\Omega$, such that:

$$\langle D_1, p^{d_0} e_{\rho f}(D_2) \rangle_{H} \leq p^{-C} \log p(\xi_H(\mathcal{O}_H^* \otimes \mathbb{Z}_p)).$$

**Proof** The proof is in [8] II Proposition 1.11, with the following modifications:

1. By definition of $d_0$, we see that $p^{d_0} e_{\rho f}(D_2)$ is in $T'$, so there is no $T''$ component, thus the constant $d_1$ in loc. cit. can be replaced by 0;

2. In this case the mixed extension $E$ is crystalline as a $\text{Gal}(\overline{\mathbb{Q}}_p/H)$-representation, thanks to the comparison theorem for relative cohomology ([10] Theorem 13.21), so
we can replace $H^* \otimes \mathbb{Z}_p$ by $O_H^* \otimes \mathbb{Z}_p$.

3. By [7] 6.9, the constant $p^{d_2}$ in loc. cit. divides the number

$$\#H^0(H, V'_{-/i} / T'_{-/i}) \#H^0(H, (V'_{+i})^* (1) / (T'_{+i})^* (1)),$$

which is bounded by the same number with the field $H$ replaced by $\Omega$. This last number is finite by Corollary 11.3, and only depends on the field $\Omega$.

11.2 Application to Heegner divisors

We now apply the above general arguments to Heegner divisors on the curve $X$ with complex points $X(\mathbb{C}) = B^ \times \times H^\times \times \tilde{B}^\times \times R^\times$. If $b$ is any element of $\tilde{B}^\times$, write $c(b)$ for the CM point represented by $(z_0, b)$, with $z_0$ the unique fixed point of $E^\times$ on the upper half plane.

Fix throughout this subsection a prime ideal $\mathfrak{P}_i$ of $O$ above $p$, and a place $v$ of the conductor 1 ring class field $H$ over $\mathfrak{P}_i$. Let $\mathfrak{P}_i = q_i \bar{q}_i$ be the decomposition of $\mathfrak{P}_i$ in $O_E$.

Let $x$ be a Heegner point as in subsection 8.4. If $I$ is an ideal of $O$ such that $r(I) = 0$, then the divisors $x$ and $T_N(I)(x)$ do not intersect. By Serre-Tate theory (c.f. [17] section 5.2.2, Case 1), their Zariski closures on the integral model $X_i$ are also disjoint.

Let $\xi$ be a divisor representing the Hodge class which does not intersect with any $T_N(I)(x)$. For each prime divisor $D$ occurring in $\xi$, let $Z_D$ be the set of ideals $I$ of $O$ such that $(I, N) = 1$ and the Zariski closure $\mathcal{D}$ of $D$ on $X_i$ intersects the Zariski closure of $T_N(I)(x)$.

**Lemma 11.5.** If $I$ is an ideal in $Z_D$ and $J$ is another ideal such that $r(J) = 0$, then the ideal $IJ$ is not in $Z_D$.

**Proof** The hypothesis $r(J) = 0$ implies that the divisors $T_N(I)(x)$ and $T_N(IJ)(x)$ do not intersect, thus their Zariski closures on $X_i$ do not intersect, again by [17] section 5.2.2, Case 1. Since the intersection of $\mathcal{D}$ and the special fiber is a prime divisor, this implies that the Zariski closure of $T_N(IJ)(x)$ does not intersect with $\mathcal{D}$ on the special fiber.
Let $M$ be the ideal $\prod_{Q \in S} Q$, where $S$ is the finite set in Lemma 11.1. For any prime divisor $D$ in the support of $\xi$ such that the set $Z_D$ is non-empty, choose an ideal $I_D$ of $Z_D$. Let $I$ be the product $\prod_D I_D$. Let $J$ be any ideal of $\mathcal{O}$ prime to $INM$, and let $\mathcal{J}$ be the ideal $IJ$. The above lemma then shows that the Zariski closures of $\xi$ and $T_N(I)(x)$ do not intersect if $I$ is an ideal prime to $N$ and divisible by $\mathcal{J}$.

Remark  The ideal $\mathcal{J}$ depends on $i$, but in the following we may replace it with the product for all $i$, thus the above property will be valid for every model $\mathcal{X}_i$.

We choose another representative $\xi'$ of the Hodge class, such that its Zariski closure on $\mathcal{X}_i$ does not intersect that of $x - \xi$. The moving lemma on the special fiber of $\mathcal{X}_i$ guarantees that such a divisor exists for every $i$, and then applying Chinese remainder theorem gives a divisor $\xi'$ that satisfies the above property for each $i$.

For every element $\sigma \in \text{Gal}(H/E)$, put:

$$\Psi_{v,\sigma} = \sum_I \langle x - \xi, T_N(I)(x^\sigma - \xi') \rangle_v q^I$$

$$= \sum_I \langle x - \xi, T_N(I)(x^\sigma) - \deg(T_N(I))\xi' \rangle_v q^I$$

as a formal $q$-expansion. We let an element $\tau \in \text{Gal}(H/E)$ act on functions $A$ over the group $\text{Gal}(H/E)$ by translation:

$$\tau A(\sigma) = A(\tau \sigma).$$

Write $\sigma_q$ for the Frobenius element of $\text{Gal}(H/E)$ corresponding to the ideal $q$, and put:

$$U_{v,\sigma} = \langle T(N) - \sigma_q, T(N) - 1 \rangle \Psi_{v,\sigma}$$

$$= \sum_I \langle x, (T_N(I\mathbb{P}_i) + T_N(I))(x^{\sigma q}) - T_N(I\mathbb{P}_i)(x^{\sigma q}) \rangle_v.$$

Lemma 11.6. ("norm relation", or "trace relation") Let $k \geq 0$ be an integer. There is a divisor $D_k$ defined over the ring class field $H[\mathbb{P}_i^{k+2}]$, such that the trace $\text{tr}_H[H^k/\mathbb{P}_i^{k+2}]$ is equal to the divisor $(T_N(I_k\mathbb{P}_i^2) + T_N(I_k))(x^{\sigma q}) - T_N(I_k\mathbb{P}_i)(x^{\sigma q})$, with $I_k := I\mathbb{P}_i^k$.

We will prove this lemma in subsection 11.3. Here we give the consequence. Let $d_0$ be the fixed constant as in Proposition 11.4.
Proposition 11.7. Let $I$ be an ideal of $\mathcal{O}$ prime to $NM$ and divisible by $J$ such that $r(I) = 0$. There is a constant $C'$, not depending on $k$, such that the height $\langle x, p^{d_0}e_{\rho_f}(T_N(I\mathfrak{P}_i^2) + T_N(I)) (x^{\sigma_0} - T_N(I\mathfrak{P}_i)) (x^\sigma + x^{\sigma_0}) \rangle_v$ has value in $p^{k-C'}\mathbb{Z}_p$.

Proof Let $w = w_k$ be a place of $H[\mathfrak{P}_i^{k+2}]$ above $v$. By Lemma 11.6, the above height is equal to: $\langle x - \xi, p^{d_0}e_{\rho_f}D_k \rangle_w$. Since the divisors do not intersect on the integral model, we may apply Proposition 11.4 with $\Omega$ equals to the union of all the extensions $H[\mathfrak{P}_i^k]_{w_k}$ for all $k$. This gives:

$$\langle x - \xi, p^{d_0}e_{\rho_f}D_k \rangle_w \in p^{-C} \log_p(\xi_p(N_{H[\mathfrak{P}_i^{k+2}]/H_v}(\mathcal{O}_{H[\mathfrak{P}_i^{k+2}]}^* \otimes \mathbb{Z}_p))).$$

The extension $H[\mathfrak{P}_i^{k+2}]/H_v$ is ramified with ramification index $N(I\mathfrak{P}_i)^{k+2}$, so by class field theory we know that the number $\log_p(\xi_p(N_{H[\mathfrak{P}_i^{k+2}]/H_v}(\mathcal{O}_{H[\mathfrak{P}_i^{k+2}]}^* \otimes \mathbb{Z}_p)))$ lives in $p^{k+2}\mathbb{Z}_p$. □

Now take the same of all $\sigma \in \text{Gal}(H/E)$ and all $x$ of conductor 1 in the above proposition, we get the similar result with $x$ and $x^\sigma$ replaced by the divisor $z$ in subsection 8.3. Note that, since the translation by any $\sigma \in \text{Gal}(H/E)$ permutes the group, the operator $(T(\mathfrak{P}_i) - \sigma q_i)(\sigma q_i T(\mathfrak{P}_i) - 1)$ is the same as $(T(\mathfrak{P}_i) - 1)^2$ when acting on the sum. Also note that the sum of $\Psi_v = \sum_I \langle z, T_N(I)z \rangle_v q^I$ for all places $v$ is equal to the form $\Psi_p$.

Corollary 11.8. We have $l_f \left( \prod I(T(\mathfrak{P}_i) - 1)^4 \right) \Psi_p = 0$.

Proof We have shown that there is a $p$-adic modular form $\Psi'$ in the space $S_2(N,L)$ which has the same coefficients as $\left( \prod I(T(\mathfrak{P}_i) - 1)^4 \right) \Psi_p$ on ideals $I$ divisible by $\mathfrak{P}J$ and such that $(I, NM) = 1$ and $r(I) = 0$. For these $I$, we have $a(I, e_{\rho_f}e_{\text{ord}}(\Psi')) = 0$ by letting $k$ tend to infinity in the above proposition. This implies that the form $e_{\rho_f}e_{\text{ord}}(\Psi')$ is equal to zero, by a variant of [8] Lemma 5.7. □

Remark The remark in [6] Remark 2.0.3 is not valid, because the form $e_{\rho_f}e_{\text{ord}}(\Psi')$ is a multiple of the form $f_0$, hence the counterexample in loc. cit. does not apply. □

11.3 The norm relation

In this subsection we prove Lemma 11.6. Without loss of generality, we may assume that the ideal $I$ has trivial prime-to-$\mathfrak{P}_i$ part, and that the Heegner point $x^\sigma$ is
the "basic Heegner point" represented by \((z_0, 1)\), with the complex description as in subsection \(8.3\).

Let \(CM(X)\) denote the set of CM points on \(X\). Since the prime ideal \(\mathfrak{P}_i\) splits in \(\mathcal{O}_E\) as \(\mathfrak{P}_i = q_i \mathfrak{q}_i\), the algebra \(E_{\mathfrak{P}_i} = E \otimes_F F_{\mathfrak{P}_i} = E_{\mathfrak{q}_i} \times E_{\mathfrak{q}_i}\) is canonically isomorphic to \(E_{\mathfrak{q}_i}^2\). We fix an isomorphism \(B \otimes_F F_{\mathfrak{P}_i} \simeq M_2(\mathcal{O}_{\mathfrak{P}_i})\), such that the restriction to the order \(R \otimes \mathcal{O}_{\mathfrak{P}_i}\) gives an isomorphism to \(M_2(\mathcal{O}_{\mathfrak{P}_i})\), and that the image of the element \((a, b) \in F_{\mathfrak{P}_i}^2 \simeq E_{\mathfrak{P}_i}\) is the diagonal matrix \(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\).

Once such an isomorphism is fixed, we get a map:
\[
c : GL_2(F_{\mathfrak{P}_i}) \simeq (B \otimes_F F_{\mathfrak{P}_i})^\times \to CM(X)
\]
\[b \mapsto \begin{bmatrix} z_0 \\ b \end{bmatrix},\]
here \([z_0, b]\) stands for the CM point represented by the pair \((z_0, b)\). This map obviously factors through the quotient: \(PGL_2(F_{\mathfrak{P}_i}) / PGL_2(\mathcal{O}_{\mathfrak{P}_i}) = \mathcal{T}\), the set of vertices of the Bruhat-Tits tree. We write \(d(\cdot, \cdot)\) for the distance function on this tree.

There is a special line \(\mathcal{L}\) on the tree \(\mathcal{T}\), given by the image of \(E_{\mathfrak{P}_i}^\times\), or equivalently, the image of matrices of the form \(\begin{pmatrix} F_{\mathfrak{P}_i}^* & 0 \\ 0 & F_{\mathfrak{P}_i}^* \end{pmatrix}\). If \(b\) is a vertex on the line \(\mathcal{L}\), then by Shimura’s reciprocity law, the corresponding CM point \(c(b)\) is defined over the conductor 1 ring class field \(H\), because \(b\) commutes with any element \(\sigma \in \hat{E}^\times\). Similarly, if \(b\) is a vertex of distance \(k\) to the line \(\mathcal{L}\), then the stabilizer of \(b\) in \(\hat{E}^\times\) contains the \(\mathfrak{P}_i^k\)-th order, hence the point \(c(b)\) is define over the ring class field \(H[\mathfrak{P}_i^k]\). The Galois group \(Gal(H[\mathfrak{P}_i^k]/H)\) acts as isometry on the tree and fixes the line \(\mathcal{L}\), and for a given point \(x\) on the line, the action permutes all the points of distance \(k\) to the point \(x\) that are not on the line.

Let \(\pi_i \in F_{\mathfrak{P}_i}\) be a uniformiser. The Hecke operators (or correspondences) \(T_N(\mathfrak{P}_i^k)\) act in the usual way on the tree \(\mathcal{T}\):
\[
T_N(\mathfrak{P}_i^k)(x) = \sum_{\substack{d(x, y) \leq k \\ d(x, y) \equiv k \mod 2}} y.
\]
It is then easy to see that, under the map \(c\), these actions coincide with the actions of the Hecke operators on the divisors of \(X\). (More precisely, this follows from the following decomposition:
\[
U(\mathfrak{P}_i^k) = K \left( \begin{pmatrix} \pi_i^k \\ 1 \end{pmatrix} \right) K \bigcup K \left( \begin{pmatrix} \pi_i^{k-1} \\ \pi_i \end{pmatrix} \right) K \bigcup \cdots \bigcup K \left( \begin{pmatrix} \pi_i^{[k/2]} \\ \pi_i^{[k/2]} \end{pmatrix} \right) K,
\]
57
where $U(\mathfrak{P}_i^k)$ is the set of matrices $g$ in $M_2(\mathcal{O}_{\mathfrak{P}_i})$ such that $(\det g)\mathcal{O}_{\mathfrak{P}_i} = \mathfrak{P}_i^k$, which was used to define the Hecke operators, and $K$ is the group $GL_2(\mathcal{O}_{\mathfrak{P}_i})$.

We also have the action of the Frobenius morphism $\sigma_i$, which sends the basic Heegner point $x$ to the point $c\left(\pi_i^\sigma, 1\right)$.

Now we can prove the lemma using the description of the tree. Put $x_A = x^\sigma$, $x_B = x^{\sigma\pi_i}$, $x_C = x^{\sigma\pi_i^2}$, these are three consecutive points on the line $\mathcal{L}$. We then have:

$$
(T_N(\mathfrak{P}_i^{k+2}) + T_N(\mathfrak{P}_i^{k}))(x_B) - T_N(\mathfrak{P}_i^{k+1})(x_A + x_C)
$$

$$
= \sum_{d(y,x_B) \equiv k + 2 \mod 2} y - \sum_{d(y,x_A) \equiv k + 1 \mod 2} y - \sum_{d(y,x_C) \equiv k + 1 \mod 2} y + \sum_{d(y,x_B) \equiv k \mod 2} y.
$$

Write $R_1$, $R_2$, $R_3$ and $R_4$ for the four sets of summation in the above formula. It is then clear that $R_2$, $R_3$ and $R_4$ are all subsets of $R_1$, and that the intersection of $R_2$ and $R_3$ is exactly $R_4$. The inclusion-exclusion principle then tells us that the above is equal to the sum over the set $R_1 \setminus (R_2 \cup R_3)$, which is the set of vertices $y$ such that $d(y, x_B) = k + 2$ and that the shortest path between $y$ and $x_B$ does not pass any other vertex on the line $\mathcal{L}$. Let $y$ be any one of these vertices, applying the map $c$ to the above formula shows that the divisor in Lemma 11.6 is equal to the trace $\text{tr}_{H(\mathfrak{P}_i^{k+2})/H(c(y))}$.

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