Supersymmetric Calogero and Calogero-Sutherland models from gauging

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Abstract. We describe how the new kinds of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric extensions of the rational and hyperbolic Calogero models can be derived by gauging $U(n)$ symmetry of the appropriate superfield matrix models. These systems feature non-standard numbers $\mathcal{N}_n$ of physical fermionic variables as compared with $\mathcal{N}n$ in the standard case. An essential ingredient of $\mathcal{N} = 4$ models is the necessary presence of semi-dynamical spin variables described by $d = 1$ Wess-Zumino terms. The bosonic cores of $\mathcal{N} = 4$ models are $U(2)$ spin Calogero and Calogero-Sutherland models. In the hyperbolic case two non-equivalent $\mathcal{N} = 4$ extensions exist, with and without the interacting center-of-mass coordinate in the bosonic sector. The talk is based on joint works with Sergey Fedoruk and Olaf Lechtenfeld.

1. Motivations

Calogero-type models (CM) [1] (see also review [2]) provide nice examples of integrable $d = 1$ systems. Simplest is the so-called rational $n$-particle Calogero model

$$S^c = \int dt \left[ \sum_a \dot{x}_a \dot{x}_a - \sum_{a \neq b} \frac{c^2}{4(x_a - x_b)^2} \right].$$

(1)

The Calogero-Moser system is obtained by adding the oscillator-type term $\sim \sum_{a \neq b} (x_a - x_b)^2$. The rational CM models are conformal: the action (1) is invariant under $d = 1$ conformal group $SO(1, 2)$

$$\delta t = \alpha, \quad \delta x_a = \frac{1}{2} \dot{\alpha} x_a, \quad \partial_j^3 \alpha = 0.$$  

Conformal CM models, being interesting on their own, can be closely related to superstrings and M-theory [3]. Besides conformal models, there are known other integrable CM-type models, e.g., the trigonometric and hyperbolic Calogero-Sutherland systems,

$$S^{cs} = \int dt \left[ \sum_a \dot{x}_a \dot{x}_a - \sum_{a \neq b} \frac{c^2}{4 \sinh^2 \frac{x_a - x_b}{2}} \right].$$  

(2)

The first example of $\mathcal{N} = 2$ superextended CM model was given in [4]: each bosonic coordinate $x_a$ was enlarged to the multiplet $(1, 2, 1)$, i.e. $n$ bosonic particles were completed by $2n$ fermionic $d = 1$ fields. The appropriate $\mathcal{N} = 2, d = 1$ superfield action can be easily constructed and shown to yield the rational CM model in the limit of vanishing fermions. Analogously, $\mathcal{N} = 2$ extension...
of the Calogero-Sutherland models can be constructed. An important role in gaining the correct pairwise potential terms is played by auxiliary fields of the supermultiplet.

There are some problems with higher $\mathcal{N}$ extensions [5], [6], [7]. For $\mathcal{N} = 4$ case, $x_a$ should be extended to the supermultiplets $(1, 4, 3)$, i.e. one deals with $n$ bosonic and $4n$ fermionic fields. It is very difficult to construct the appropriate superfield action yielding $n$-particle Calogero potential in the bosonic sector. There arise a few functions of $x_a$ related by the complicated WDVV [8], [9] equations the explicit solutions of which are known only for a few lowest values of $n$. This is in contrast to the standard one-particle $\mathcal{N} = 4$ supersymmetric mechanics where no similar difficulties appear.

Thus, no universal convenient method was invented so far for constructing the “standard” $\mathcal{N}n$- extended supersymmetric CM systems.

Fortunately, another type of supersymmetrization is possible, such that no the above-mentioned problems arise, though the models appearing in this way are “non-minimal”: they contain $\mathcal{N}n^2$ fermions for $n$ bosonic coordinates [10], [11], [12]. It proceeds from the gauge approach to bosonic CM models developed earlier in [13] [14], [15]. One starts from some simple free matrix model and gauge the appropriate linear isometries by non-propagating $d = 1$ gauge fields. After eliminating gauge fields by their algebraic equations of motion one or another CM model is reproduced.

Generalization to the supersymmetry case is straightforward: one starts from some free superfield matrix model action and gauge the isometry by a gauge superfield. After passing to components and gauge-fixing, the relevant supersymmetric CM model is recovered.

I will apply this approach to $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superfield matrix models and show that some new versions of the corresponding supersymmetric CM models can be discovered in this way.

The talk is based mainly on the papers [10], [11], [12].

2. Toy example: conformal mechanics by gauging
The renowned conformal mechanics model by [16] is described by the action:

$$S_0 = \int dt \left( \dot{x}^2 - \gamma^2 x^{-2} \right) \equiv \int dt \, \mathcal{L}_0.$$ 

Let us obtain it from another $d = 1$ action. As an input we take the Lagrangian of free complex field $z(t)$

$$L_z = \dot{z} \dot{\bar{z}} + i m (\dot{\bar{z}} \dot{z} - z \dot{\bar{z}}).$$

It is obviously invariant under rigid phase transformations

$$z' = e^{-i\lambda} z, \quad \bar{z}' = e^{i\lambda} \bar{z}.$$ 

Then we gauge this abelian symmetry, $\lambda \to \lambda(t)$, and introduce $d = 1$ gauge field $A(t)$, so that $\dot{z} \to \dot{z} + iA z$:

$$L_z \to L_z^g = (\dot{z} + iA z) \left( \dot{\bar{z}} - iA \bar{z} \right) + im \left( \dot{\bar{z}} \dot{z} - z \dot{\bar{z}} + 2iA \dot{z} \bar{z} \right) + 2\gamma A,$n

$$A' = A + \lambda.$$ 

Here, a “Fayet-Iliopoulos term” $\propto \gamma$ has been also added. It is gauge invariant (up to a total derivative) by itself.

The next step is to effect the appropriate gauge in $L^g$:

$$z = \bar{z} \equiv q(t).$$
We plug it back into $L$ and obtain:

$$L_g = (\dot{q} + iAq)(\dot{q} - iAq) + 2imA\dot{q}^2 + 2\gamma A = (\dot{q})^2 + A^2\dot{q}^2 - 2mAq^2 + 2\gamma A.$$ 

The field $A(t)$ can be eliminated by its algebraic equation of motion:

$$\delta A : \quad A = m - \gamma q^{-2}.$$ 

The final form of the gauge-fixed Lagrangian is

$$L_g \Rightarrow (\dot{q})^2 - (mq - \gamma q^{-1})^2 = (\dot{q})^2 - m^2q^2 - \gamma^2q^{-2} + 2\gamma m.$$ 

Up to an additive constant, this Lagrangian coincides with the mass-modified AFF. At $m = 0$, one recovers the standard conformal mechanics:

$$L_g^{(m=0)} = (\dot{q})^2 - \gamma^2 q^{-2}.$$ 

The initial action $S_z = \int dt L_z$ at $m = 0$ is invariant under the conformal $SO(1,2)$ transformations $\delta t = f(t), \delta z = \frac{1}{2}\dot{f}z, (\partial t)^3f = 0$. The conformal invariance is preserved by the gauging procedure, provided that the gauge field $A(t)$ transform as $\partial_t$, i.e. $\delta A(t) = -\dot{f}A(t)$.

3. Calogero and Calogero-Sutherland by gauging

The generalization of this setting to the CM case goes as follows.

We start from the $U(n)$ invariant free action of the $n \times n$ hermitian matrix field $X^a_b$ and complex $U(n)$-spinor field $Z^a(t), \bar{Z}^a = (\bar{Z}_a), a, b = 1, \ldots, n$. We gauge $U(n)$ symmetry by $n^2$ hermitian gauge fields $A^a_b$. The resulting gauge invariant action reads

$$S_C = \int dt \left[ \text{tr}(\nabla X \nabla X) + \frac{i}{2} (\bar{Z} \nabla Z - \nabla \bar{Z} Z) + c \text{tr} A \right],$$

$$\nabla X = \dot{X} + i[A, X], \quad \nabla Z = \dot{Z} + iAZ, \quad \nabla \bar{Z} = \dot{\bar{Z}} - i\bar{Z} A.$$ 

The last term (Fayet-Iliopoulos term) includes only $U(1)$ gauge field, $c$ is a real constant. As the next step we fix $U(n)$ gauge so as to kill all non-diagonal components of $X^a_b$

$$X^a_b = x_a \delta^b_a, \Rightarrow [X, A]_a^b = (x_a - x_b)A^a_b.$$ 

We have used just $n^2 - n$ gauge parameters, but there remains the residual abelian gauge subgroup $[U(1)]^n$, with local parameters $\varphi_a(t)$:

$$Z_a \rightarrow e^{i\varphi_a} Z_a, \bar{Z}^a \rightarrow e^{-i\varphi_a} \bar{Z}^a, A^a_a \rightarrow A^a_a - \varphi_a \quad (\text{no sum with respect to } a).$$

Then the next gauge-fixing can be effected:

$$\bar{Z}^a = Z_a,$$

which leads to the gauge-fixed action in the form

$$S_C = \int dt \sum_{a,b} \left[ x_a \dot{x}_a + (x_a - x_b)^2 A^b_a A^a_b - Z_a Z_b A^b_a + c A^a_a \right].$$
Varying it with respect to the non-propagating gauge fields, we obtain

\[ A_a^b = \frac{Z_a Z_b}{2(x_a - x_b)^2} \quad \text{for } a \neq b, \]

\[ Z_a Z_a = c \quad \forall a \quad \text{(no sum with respect to a)}. \]

The diagonal entries \( A_a^a \) drop out from the action and, after substituting the explicit expressions for the rest of \( A_a^{b'} \), we obtain

\[ S_C = \frac{1}{2} \int dt \left[ \sum_a \dot{x}_a x_a - \sum_{a \neq b} \frac{c^2}{(x_a - x_b)^2} \right], \]

that is just the rational Calogero model action. The action we started from is conformal, so the final Calogero action is conformal as well.

The Calogero-Sutherland model can be recovered by the same techniques, the only difference is that the initial action includes nonlinear sigma-model type kinetic term for the matrix \( X_a^{b'} \):

\[ S_{CS} = \int dt \left[ \text{tr} \left( X^{-1} \nabla X X^{-1} \nabla X \right) + \frac{i}{2} (\bar{Z} \nabla Z - \nabla \bar{Z} Z) + c \text{tr} A \right]. \] (4)

After passing through the same steps as in the rational case, we obtain the gauge-fixed action in the form

\[ S_C = \frac{1}{2} \int dt \left[ \sum_a \dot{x}_a x_a - \sum_{a \neq b} \frac{x_a x_b c^2}{(x_a - x_b)^2} \right]. \]

Introducing the new variables as \( x_a = e^{q_a} \) brings this action to the standard Calogero-Sutherland form

\[ S^{CS} = \int dt \left[ \sum_a \dot{q}_a q_a - \sum_{a \neq b} \frac{c^2}{4 \sinh^2 \frac{q_a - q_b}{2}} \right]. \]

This action is not conformal, since the initial action is not conformal.

4. \( N = 2 \) Calogero and Calogero-Sutherland

Once again, both cases follow the same strategy, are defined on the same set of \( d = 1 \) superfields and differ only in the choice of the initial \( N = 2 \) matrix model action.

The starting point in the first case is the free \( N = 2, d = 1 \) action of the \( n \times n \) matrix hermitian superfield \( X_a^{b'}(t, \theta, \bar{\theta}) \), \( a, b = 1, \ldots, n \), with each entry carrying \((1, 2, 1)\) multiplet, and of chiral \( U(n)\)-spinor superfield \( Z_a(t_R, \bar{\theta}), \bar{Z}^a(t_L, \theta), \) \( DZ_a = 0, \) \( D\bar{Z}^a = 0, \)

\[ S_{CS}^{(N=2)} = \int dt d\theta d\bar{\theta} \left[ \text{tr} \left( \bar{D}X D X \right) + \frac{1}{2} \bar{Z} \bar{Z} \right]. \]

This action is evidently invariant under rigid \( U(n) \) transformations acting as rotations of the fundamental and co-fundamental indices \( a, b \).

Next, in order to preserve the chiralities of \( Z_a, \bar{Z}^a \) we gauge this global symmetry by chiral and anti-chiral superfield parameters \( \lambda \) and \( \bar{\lambda} \),

\[ Z' = e^{i\lambda} Z, \quad \bar{Z}' = \bar{Z} e^{-i\bar{\lambda}}, \quad X' = e^{i\lambda} X e^{-i\bar{\lambda}}. \]
To construct the gauge invariant action, one introduces the hermitian gauge superfield $V$,

$$e^{2V'} = e^{i\lambda} e^{2V} e^{-i\lambda}.$$  

Then, the gauge-covariantized action reads

$$\tilde{S}^{(N=2)}_{sc} = \int dtd^2\theta \left[ \text{tr} \left( \overline{D}\chi e^{2V} D\chi e^{2V} \right) + \frac{1}{2} \overline{Z} e^{2V} Z - c \text{tr} V \right],$$

(5)

where

$$D\chi = D\chi + e^{-2V} (De^{2V}) \chi, \quad \overline{D}\chi = \overline{D}\chi - \chi e^{-2V} (\overline{D} e^{-2V}).$$

It can be shown that the original rigid matrix action, as well as the final gauge-covariantized one, are invariant under the $N=2$ superconformal symmetry $SU(1,1|1)$.

In the component expansion of the involved superfields

$$\chi = X + \theta \Psi - \bar{\theta} \bar{\Psi} + \theta \bar{\theta} Y, \quad Z = Z + 2i\theta \bar{Y} + i\theta \bar{\theta} Z, \quad \overline{Z} = \overline{Z} + 2i\bar{\theta} \bar{Y} - i\theta \bar{\theta} \overline{Z},$$

all the fields, besides $2n^2$ fermionic ones,

$$\Psi^a, \quad \bar{\Psi}^a,$$

are auxiliary and can be eliminated by their equations of motion.

Upon choosing the standard Wess-Zumino gauge $V = \theta \partial A(t)$, the component action acquires the form

$$S^{WZ}_{sc} = \int dt \left[ \text{tr} \nabla X \nabla X + \frac{1}{2} (\overline{Z} \nabla Z - \nabla \overline{Z} Z) - c \text{tr} A + i \text{tr} (\overline{\Psi} \nabla \Psi - \nabla \overline{\Psi} \Psi) \right],$$

where

$$\nabla \Psi = \dot{\Psi} + i[\Psi, A], \quad \nabla \bar{\Psi} = \dot{\bar{\Psi}} + i[\bar{\Psi}, A]$$

and $\nabla Z, \nabla X$ coincide with the purely bosonic expressions presented earlier. In the limit of zero fermions, the standard input gauge-invariant action of the rational Calogero is recovered.

So we have obtained a new $\mathcal{N}=2$ extension of the $n$-particle Calogero model with $n$ bosons and $2n^2$ fermions, as opposed to the standard $\mathcal{N}=2$ super Calogero system of Freedman and Mende with only $2n$ fermions.

In terms of the physical variables, the component action reads

$$S^{(N=2)}_{sc} = \int dt \left[ \sum_a \dot{x}_a \dot{x}_a + i \left( \Psi^a b \bar{\Psi}^a b - \Psi^a b \bar{\Psi}^a b - V \right) \right],$$

(6)

$$V = \sum_{a \neq b} \frac{4}{(x_a - x_b)^2} \left( Z_a \overline{Z}^a Z_b \overline{Z}^b + 2 \overline{Z}^a \{ \Psi, \overline{\Psi} \}_a Z_b + \{ \Psi, \Psi \}_a \overline{\Psi}^a b \{ \overline{\Psi}, \Psi \}^a b \right).$$

The constraint on $Z$ (after fixing the gauge $Z_a = \overline{Z}^a$) also essentially involves fermions:

$$(Z_a)^2 = c - R_a, \quad R_a \equiv \{ \Psi, \overline{\Psi} \}_a = \sum_b (\Psi^a b \bar{\Psi}^a b + \bar{\Psi}^a b \Psi^a b),$$

$$(R_a)^{2n-1} = 0 \text{ (nilpotency)}$$
To date, it is unclear how to treat this huge amount of fermionic fields, and whether this number could be somehow reduced by imposing some extra (perhaps, fermionic) gauge invariance. More detailed study of the fermionic sectors of such models was recently undertaken in [17], [18].

Like in the bosonic case, the $\mathcal{N} = 2$ CS system is obtained by starting from a sigma-model type gauged action

$$S_{cs}^{N=2} = \frac{1}{2} \int dt d^2 \theta \left[ \text{tr}(\mathcal{X}^{-1} D \mathcal{X} \mathcal{X}^{-1} D \mathcal{X}) - \bar{Z} e^{2V} Z + 2c \text{tr}V \right]$$

(7)

After passing through the same steps as before, we obtain the component action

$$S^{N=2} = \int dt L^{N=2},$$

$$L^{N=2} = \frac{1}{2} \text{tr}(X^{-1} \nabla X X^{-1} \nabla X) + \frac{i}{2} (\bar{Z} \nabla Z - \nabla \bar{Z} Z) + c \text{tr}A$$

$$+ \frac{i}{2} \text{tr}(X^{-1} \bar{\Psi} X^{-1} \nabla \Psi - X^{-1} \nabla \bar{\Psi} X^{-1} \Psi)$$

$$- \frac{1}{4} \text{tr}(X^{-1} \bar{\Psi} X^{-1} \Psi X^{-1} \bar{\Psi} X^{-1} \Psi)$$

In the bosonic limit the gauge-invariant CS action is recovered. A new interesting feature is the appearance of the quartic fermionic term as compared to $\mathcal{N} = 2$ rational Calogero.

5. Towards higher $\mathcal{N}$: $\mathcal{N} = 4$ Calogero

The $\mathcal{N} = 4$ case is of special interest because the same gauge procedure applied to it results in the $U(2)$ spin Calogero systems. The reason is that the additional multiplets $Z, \bar{Z}$ now cannot be entirely eliminated by using the gauge freedom and/or the constraints as earlier. Their remnants are just a sort of the target space $U(2)$ harmonics.

The universal superfield approach to $\mathcal{N} = 4$ mechanics, both one-particle and multi-particle, is provided by the $\mathcal{N} = 4, d = 1$ harmonic superspace (HSS) [19] which is the $d = 1$ version of the renowned $\mathcal{N} = 2, d = 4$ HSS [20].

The $d = 1$ HSS is an extension of the standard $\mathcal{N} = 4, d = 1$ superspace $(t, \theta, \bar{\theta}, u, \bar{u})$ by the harmonic coordinates $u^{\pm}$:

$$(t, \theta^{\pm}, \bar{\theta}^{\pm}, u^{\pm}_i), \quad \theta^{\pm} = \theta^i u^+_i, \quad \bar{\theta}^{\pm} = \bar{\theta}^i u^+_i, \quad u^{+i} u^{-i} = 1.$$

The commuting $SU(2)$ variables $u^{\pm}_i$ parametrize the 2-sphere $S^2 \sim SU(2)_R/U(1)_R$. The salient property of HSS is the presence of the harmonic analytic superspace in it, with half of the original Grassmann coordinates

$$(\zeta, u) = (t_A, \theta^{\pm}, \bar{\theta}^{\pm}, u^{\pm}_i), \quad t_A = t + i(\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+).$$

It is closed under $\mathcal{N} = 4$ supersymmetry transformations.

Most off-shell $\mathcal{N} = 4, d = 1$ multiplets are described by analytic superfields “living” on this subspace.

The direct analog of the $\mathcal{N} = 2$ multiplet $(1,2,1)$ is the multiplet $(1,4,3)$ represented by a general superfield $\mathcal{X}(t, \theta^{\pm}, \bar{\theta}^{\pm}, u)$ subjected to the constraints

$$D^{++} \mathcal{X} = 0, \quad D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} + 2i \theta^+ \bar{\theta}^+ \partial_{t_A},$$

$$D^+ D^- \mathcal{X} = 0, \quad (D^+ D^- + D^+ D^-) \mathcal{X} = 0.$$
These constraints can be solved for by the analytic prepotential $V$,

$$\mathcal{X}(t, \theta, \bar{\theta}) = \left. \int du \ V(t_A, \theta^+, \bar{\theta}^+, u) \right|_{\theta^\pm = \theta^{\pm}_u, \ \bar{\theta}^\pm = \bar{\theta}^{\pm}_u}.$$

The needed field content is ensured by the invariance under gauge analytic transformations

$$\delta V = D^{++}\lambda^{--}, \quad \lambda^{--} = \lambda^{--}(\zeta, u)$$

and is recovered in the appropriate WZ gauge for $V$.

The $N = 4$ analogs of the chiral $N = 2$ multiplet $Z_a, \bar{Z}^a$ are the complex analytic superfields $Z^+, \bar{Z}^+$, subjected to the additional constraint

$$D^{++} Z^+ = 0,$$

which ensures the off-shell content $(4, 4, 0)$ for these superfields:

$$Z^+ = z^i u^+_i + \theta^+ \phi + \bar{\theta}^+ \bar{\phi} - 2i \theta^+ \bar{\theta}^+ \partial_A z^i u^-_i.$$

Finally, the gauge field $A(t)$ is accommodated by the analytic unconstrained gauge prepotential $V^{++}$,

$$V^{++}' = e^{i\lambda} V^{++} e^{-i\lambda} - ie^{i\lambda}(D^{++} e^{-i\lambda}),$$

where $\lambda^a_b(\zeta, u^\pm) \in u(n)$ is the hermitian analytic matrix parameter. Using this gauge freedom we can choose the WZ gauge

$$V^{++} = 2i \theta^+ \bar{\theta}^+ A(t_A).$$

Now we have all ingredients required for constructing $N = 4$ Calogero and Calogero-Sutherland models. We will firstly discuss the first class of systems.

Our guiding principle will be invariance under the most general $N = 4, d = 1$ conformal supergroup $D(2, 1; \alpha)$. The appropriate matrix superfield action reads

$$S = -\frac{1}{4(1+\alpha)} \int \mu_H (\text{tr} \lambda^2)^{-1/2\alpha} - \frac{1}{2} \int \mu^{(-2)} A \bar{Z}^a Z^+_a + \frac{i}{2} c \int \mu^{(-2)} \text{tr} V^{++}$$

(8)

Here, all the superfields defined above are involved, with all derivatives properly covariantized with respect to local $U(n)$ group which acts as

$$\mathcal{X}' = e^{i\lambda} \mathcal{X} e^{-i\lambda}, \quad Z^+ = e^{i\lambda} Z^+, \quad \bar{Z}^+ = \bar{Z}^+ e^{-i\lambda}$$

E.g., $D^{++} Z^+ \to D^{++} \bar{Z}^+ = D^{++} Z^+ + iV^{++} Z^+$. The object $\mathcal{V}_0$ is a real analytic gauge prepotential for the $U(n)$ singlet $(1, 4, 3)$ superfield $X_0 \equiv \text{tr} (\mathcal{X})$. It is defined by the integral transform

$$X_0(t_A, \theta, \bar{\theta}) = \left. \int du \mathcal{V}_0 \left(t_A, \theta^+, \bar{\theta}^+, u^\pm\right) \right|_{\theta^\pm = \theta^{\pm}_u, \ \bar{\theta}^\pm = \bar{\theta}^{\pm}_u}.$$

In what follows, we will be interested in the choice $\alpha = -1/2$, which corresponds to the free superfield Lagrangian $\sim \text{tr} \mathcal{X}^2$ for the multiplet $(1, 4, 3)$, and at which $D(2, 1; \alpha) \sim \text{osp}(4|2)$. 

In WZ gauge, and with auxiliary fields eliminated, we end up with:

\[
S_4 = S_b + S_f,
\]

\[
S_b = \int dt \left[ \text{tr}(\nabla X \nabla X + c A) + \frac{n}{8} (\bar{Z}^{(i)k}) (\bar{Z}_i Z_k) + \frac{i}{2} X_0 (\bar{Z}_k \nabla Z^k - \nabla \bar{Z}_k Z^k) \right],
\]

\[
S_f = i \text{tr} \left( \int dt (\bar{\Psi}_k \nabla \Psi^k - \nabla \bar{\Psi}_k \Psi^k) - \int dt \frac{\Psi_0^i \Psi_0^k (\bar{Z}_i Z_k)}{X_0} \right),
\]

\[
X_0 := \text{tr}(X), \quad \Psi_0^i := \text{tr}(\Psi^i), \quad \bar{\Psi}_0^i := \text{tr}(\bar{\Psi}^i).
\]

After fixing gauges with respect to the residual gauge group, elimination of \( A^b_a, a \neq b \), and some redefinitions, the bosonic core of this action is found to be as follows

\[
S_b = \int dt \left\{ \sum_a \dot{x}^a \dot{x}_a + \frac{i}{2} \sum_a (\bar{Z}_a^k \dot{Z}_a^k - \dot{Z}_a^k Z_a^k) + \sum_{a \neq b} \frac{\text{tr}(S_b S_b)}{4(x_a - x_b)^2} - \frac{n \text{tr}(\dot{S} \dot{S})}{2(X_0)^2} \right\}.
\]

Here, the fields \( Z_a^k \) are subject to the constraints

\[
\bar{Z}_a^i Z_a^i = c \quad \forall \ a,
\]

and

\[
(S_a)^i_j := \bar{Z}_a^i Z_a^j, \quad (\dot{S})^i_j := \sum_a \left[ (S_a)^i_j - \frac{1}{2} \delta^i_j (S_a)^k_k \right].
\]

To reveal the meaning of these composite objects, we note that in the Hamiltonian approach, the kinetic WZ term for \( Z \) gives rise to the following Dirac brackets:

\[
[\bar{Z}_a^i, Z_b^j]_D = i \delta_a^b \delta_i^j,
\]

that implies

\[
[(S_a)^i_j, (S_b)^k_l]_D = i \delta_{ab} \left\{ \delta^i_k (S_a)^j_l - \delta^j_k (S_a)^i_l \right\},
\]

i.e., for each index \( a \) the quantities \( S_a \) form mutually commuting \( u(2) \) algebras. The object \( \dot{S} \) is just the conserved Noether \( SU(2) \) current for the total system. Thus, the new feature of the \( \mathcal{N} = 4 \) case is that not all out of the bosonic variables \( Z_a^k \) are eliminated by fixing gauges and solving the constraint; there survives a non-vanishing WZ term for them. After quantization these variables become purely internal (U(2)-spin) degrees of freedom. Since \( \text{tr} \dot{S} \dot{S} \) is a constant of motion, we have the conformal inverse-square potential even in the sector of the center-of-mass coordinate \( X_0 \). This is the essential difference of the \( \mathcal{N} = 4 \) case from the \( \mathcal{N} = 1, 2 \) cases where this coordinate decouples. Modulo this extra conformal potential, the bosonic limit of the \( \mathcal{N} = 4 \) system constructed is none other than the integrable U(2)-spin Calogero model as it was formulated in [22], [2].

6. \( \mathcal{N} = 4 \) Calogero-Sutherland

Like in the previous case, the input superfield action in the hyperbolic case is the sum of three parts

\[
S^{N=4} = S_X + S_{WZ} + S_{FI}.
\]

(10)
The basic distinguishing feature of this system is the choice of $\mathcal{X}$ action

$$S_{\mathcal{X}} = \frac{1}{2} \int \mu_H \text{tr} \left( \ln \mathcal{X} \right),$$  \hspace{1cm} (11)

while the structure of the remaining two pieces is the same, as well as the form of the superfield constraints.

The full structure of the component action is restored by passing through the same steps as in the rational Calogero case, i.e. imposing various gauges, elimination of the auxiliary fields, etc. It is rather involved, especially in the fermionic sector. In particular, it contains a few terms in the rational Calogero case, i.e. imposing various gauges, elimination of the auxiliary fields, constraints.

The bosonic sector is described by the action

$$S_{\text{bose}}^{N=4} = \frac{1}{2} \int dt \left[ \text{tr} \left( X^{-1} \nabla X X^{-1} \nabla X + 2c A \right) + i(\tilde{Z}_k \nabla Z^k - \nabla \tilde{Z}_k Z^k) \right] + \frac{(\bar{Z}(iZ^k))(\bar{Z}_a Z_b) \text{tr}(X^2)}{2(X_0)^2}. $$

After fixing $U(n)$ gauge, $X_a^b = 0, \ a \neq b$, eliminating fields $A^b_a$, using the constraint $\bar{Z}_i^a Z^i = c \forall a$, and passing to $x_a = e^{q_a}$, it becomes

$$S_{\text{bose}}^{N=4} = \frac{1}{2} \int dt \left\{ \sum_a [\dot{q}_a q_a + i(\dot{\tilde{Z}}_a Z^k_a - \dot{\tilde{Z}}_a Z^k_a)] - \sum_{a \neq b} \frac{(S_a)_i^k (S_b)_k^i}{\text{2 sinh}^2 \frac{q_a - q_b}{2}} \right\},$$

$$\text{Tr}(X^2) = \sum_c e^{2q_c}, \hfill X_0 = \sum_c e^{q_c}, \hfill (S_a)_i^k := \bar{Z}_i^a Z^k_a.$$  \hspace{1cm} (12)

The quantities $(S_a)_i^k$ generate $n$ copies of $U(2)$ algebra. As a result, the above action, up to the last term, describes the hyperbolic $U(2)$-spin Calogero-Sutherland system [14], [2].

The choice of the $\mathcal{Z}^+$ action in the $\mathcal{N} = 4$ rational Calogero model was mainly dictated by the requirement of superconformal invariance. In the hyperbolic case, no such a symmetry is present from the very beginning. In particular, the $\mathcal{X}$ part of the action already lacks such an invariance. So, there are no reasons to require it in other pieces.

Then it is natural to try the simplest action for the $(4, 4, 0)$ multiplets, just $-1/2 \int \mu_A^{-2} \tilde{Z}_a^+ Z_a^+$. In this case all steps are radically simplified, in particular, all fermionic auxiliary fields of $(4, 4, 0)$ multiplets are zero on shell. The bosonic sector of the component action reads

$$S_{\text{bose}}^{N=4} = \frac{1}{2} \int dt \left\{ \sum_a [\dot{q}_a q_a + i(\dot{Z}_a^k \dot{Z}^k_a - \dot{\tilde{Z}}_a Z^k_a)] - \sum_{a \neq b} \frac{(S_a)_i^k (S_b)_k^i}{4 \text{ sinh}^2 \frac{q_a - q_b}{2}} \right\}. $$

The previous bosonic action involved $\text{tr}(X^2)$ and $X_0$. The latter coordinate (the center-of-mass coordinate) decouples only for the trivial cases $n = 1, 2$. In contrast, the new action yields in the bosonic sector the pure hyperbolic $U(2)$-spin Calogero-Sutherland system for any $n$, without
any additional interaction. The center-of-mass coordinate is fully detached and described by the free action in this sector.

While for $n = 1$ in the system obtained the $X$ sector is fully decoupled from the $Z$ sector and describes a free dynamics, at $n = 2$ one is left with a non-trivial system. The relative coordinate $\phi := \frac{1}{2}(q_1 - q_2)$ involves non-trivial interaction with the spin variables $Z^k_1$ and $Z^k_2$

$$\sim \frac{\text{tr} (S^1 S^2)}{4 \sinh^2 \phi}.$$  \hfill (14)

So in the bosonic sector we obtain an extension of the standard hyperbolic Pöschl-Teller mechanics [23] by the spin variables. This new $\mathcal{N} = 4$ superextended Pöschl-Teller system certainly deserves a careful analysis. In the $\mathcal{N} = 1, 2$ cases analogous Pöschl-Teller potential without any additional variables is recovered (the known versions of supersymmetric Pöschl-Teller mechanics were given, e.g., in [24]).

7. **Summary and Outlook**

- We have outlined a universal method of constructing supersymmetric extensions of the Calogero-type models by the superfield gauging procedure directly generalizing the similar one for the $n$-particle bosonic Calogero systems. This method yields non-standard supersymmetrization, with $\mathcal{N} n^2$ physical fermionic fields, in contrast to $\mathcal{N} n$ such fields in the standard supersymmetrization.

- In this way, we have explicitly constructed new $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superfield systems containing the rational Calogero and hyperbolic Calogero-Sutherland models as the bosonic cores. In the hyperbolic models, new superxtensions of the Pöschl-Teller mechanics for the relative coordinate arise at $n = 2$, with the extra spin variables in the $\mathcal{N} = 4$ case.

- Recently, it was shown, in the on-shell Hamiltonian approach, that similar systems, at least in the rational case, can be formulated for arbitrary $\mathcal{N}$ [17], [25]. It is unclear whether such systems can be re-obtained within the superfield gauging since the off-shell superfield matrix models (the starting point of the gauging procedure) are known up to now only until $\mathcal{N} = 8$.

*Some further lines of development:*

(a) An intriguing question is as to whether the new super Calogero models preserve the remarkable classical and quantum integrability properties of the bosonic models.

(b) It would be interesting to inquire whether other $\mathcal{N} = 4, d = 1$ multiplets (e.g., the multiplets $(2, 4, 2)$ or $(3, 4, 1)$) can be used to represent spin variables in various $\mathcal{N} = 4$ Calogero systems.

(c) To generalize new $\mathcal{N}$ super Calogero to the case of “weak” $\mathcal{N} = 4$ $(SU(2|1))$ supersymmetry [26], [27], [28], as well as to analogous deformed versions of $\mathcal{N} = 8$ supersymmetry [29], [30]. Such generalizations involve an intrinsic mass-dimension parameter $m$ which would deform the Calogero models to a kind of Calogero-Moser systems, with extra oscillator-type terms.

(d) Quantizing all these models. In fact, the quantization of the deformed $SU(2|1)$ Calogero-Moser systems was recently done in [31]. The quantization of the $\mathcal{N} = 2$ models considered here was undertaken in a recent paper [32].

(e) At last, there are known other integrable Calogero-like multiparticle models [33], [34], e.g., the trigonometric Calogero-Sutherland models, the elliptic models, etc. All of them still wait their supersymmetrization.
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