CLASSIFICATION OF CONDITIONAL MEASURES ALONG CERTAIN INVARIANT ONE-DIMENSIONAL FOLIATIONS

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ABSTRACT. Let $(M, \mathcal{A}, \mu)$ be a probability space and $f : M \to M$ a homeomorphism preserving the Borel ergodic probability measure $\mu$. Given $\mathcal{F}$ a continuous one-dimensional $f$-invariant foliation of $M$ with $C^1$ leaves, we show that if $f$ preserves a continuous $\mathcal{F}$-arc length system, then we only have three possibilities for the conditional measures of $\mu$ along $\mathcal{F}$, namely:

- they are atomic for almost every leaf, or
- for almost every leaf their support is a Cantor subset of the leaf or
- for almost every leaf they are equivalent to the measure $\lambda_x$ induced by the invariant arc-length system over $\mathcal{F}$.

This trichotomy classifies, for example, the possible disintegrations of ergodic measures along foliations over which $f$ acts as an isometry, and also disintegrations of ergodic measures along the center foliation preserved by transitive partially hyperbolic diffeomorphisms with topological neutral center direction.

CONTENTS

1. Introduction
2. Basics on conditional measures
3. Invariant arc-length and invariant metric systems
4. Properties of non-atomic disintegrations over a continuous one-dimensional foliation
5. Proof of the main Theorem
6. Funding
References

1. INTRODUCTION

Given a topological space $X$, $\mathcal{B}$ its Borel sigma-algebra and $\mu$ a probability measure on $X$, it is well known that for any sub-sigma algebra $\mathcal{E} \subset \mathcal{B}$, $\mu$ may be disintegrated over the partition induced by $\mathcal{E}$ in the sense that we may find a system of probability measures $\{\mu_x\}_{x \in X}$, such that $\mu_x \left( \bigcap_{E \in \mathcal{E}} E \right) = 1$, $x \mapsto \mu_x$ is Borel measurable and $\mu(B) = \int \mu_x(B) d\mu$ for any $B \in \mathcal{B}$. This system is referred to as being the disintegration of $\mu$ along $\mathcal{E}$. This fact was later extended to the more general context of Lebesgue spaces, where V. Rokhlin (see [17]) proved that $\mu$ may be
disintegrated over any measurable partition, that is, over any partition that is countably generated by elements of the sigma-algebra of the Lebesgue space in consideration.

The mentioned theorem of Rokhlin is extensively used in dynamical systems and was further extended to more general contexts (see for example [19], [15]). In general, given a certain dynamics, in many contexts this dynamics admits certain invariant partitions which are dynamically defined and the study of the disintegration of the invariant measures along these partitions usually yields important properties of the dynamics. In some cases these partitions are naturally given by invariant foliations of the dynamics.

On smooth ergodic theory for example, Anosov and partially hyperbolic diffeomorphisms admit a pair of invariant foliations called stable and unstable foliations, which we will denote here by $F^s$ and $F^u$. In these contexts, measure disintegration techniques have been an essential tool to obtain ergodicity and rigidity of some properties such as regular conjugacy between certain $C^1$-close Anosov maps.

In his seminal work D. Anosov proved [2] that the disintegration of the volume measure along the unstable (resp. stable) foliation of a volume preserving Anosov diffeomorphism is absolutely continuous with respect to the leaf measure. This result was generalized to the stable and unstable foliation of partially hyperbolic diffeomorphisms (see [4] for example). This is clearly not the general case for an arbitrary foliation, an example for which absolute continuity does not occur was given by A. Katok [11] and there are now several examples of other natures in the literature. More specifically, Katok’s example shows a foliation by analytic curves of $(0,1) \times \mathbb{R}/\mathbb{Z}$ such that there exists a full Lebesgue measure set which intersects each leaf in exactly one point. In this case we say that the conditional measures along the leaves are atomic, or that the foliation is atomic with respect to the reference measure - which in the example of Katok is the standard two dimensional Lebesgue measure.

Atomicity and absolute continuity are two extremes among the possibilities that one could expect when studying the conditional measures along a foliation. The first one is equivalent to saying that the conditional measures are atomic measures and the last one implies that the conditional measures are absolutely continuous with respect to the Riemannian measure of the leaf, a property which is usually called leafwise absolute continuity or Lebesgue disintegration of measure. Although these are two extreme behaviors among, a priori, many possibilities for the disintegration of a measure, recent results have indicated that this dichotomy is more frequent than one would at first expect. In [18] D. Ruelle and A. Wilkinson proved that for certain skew product type of partially hyperbolic dynamics, if the fiberwise Lyapunov exponent is negative then the disintegration of the preserved measure along the fibers is atomic. Later A. Homburg [9] proved that some examples treated in [18] one can actually prove that the disintegration is composed by only one dirac measures. A. Avila, M. Viana and A. Wilkinson [3] proved that for $C^1$-volume preserving perturbations of the time-1 map of geodesic flows on negatively curved surfaces, the
disintegration of the volume measure along the center foliation is either atomic or absolutely continuous and that in the latter case the perturbation should be itself the time-1 map of an Anosov flow. Also inside the class of derived from Anosov diffeomorphisms, G. Ponce, A. Tahzibi and R. Varão [14] exhibited an open class of volume preserving diffeomorphisms which have (mono) atomic disintegration along the center foliation and, recently, A. Tahzibi and J. Zhang [20] proved that non-hyperbolic measures of derived from Anosov diffeomorphisms on $\mathbb{T}^3$ also must have atomic disintegration along the center foliation, answering a question from [13].

In this paper our main goal is to better understand the disintegration of an invariant measure along an invariant foliation for the dynamics without requiring hyperbolicity or partially hyperbolicity for $f$ but assuming that the invariant foliation has some type of metric rigidity with respect to $f$. In other words, we aim to investigate what are the possible characterizations of the conditional measures obtained when we disintegrate $\mu$ over a foliation $F$, assuming that the behavior of $f$ along $F$ is very far from being hyperbolic.

1.1. Setting and statement of results. A continuous $m$-dimensional foliation $F$ of a smooth manifold $M$ by $C^r$-submanifolds is a partition of $M$ into $C^r$-submanifolds which can be locally trivialized by local charts, that is, for each $x \in M$ one can find open sets $U \subset M$, $V \subset \mathbb{R}^m$, $W \subset \mathbb{R}^{n-m}$, $n = \dim M$, and a homeomorphism $\varphi : U \to W \times V$, such that for every $c \in W$ the set $\varphi^{-1}(\{c\} \times V)$, which is called a plaque of $F$, is a connected component of $L \cap U$ for a certain $L \in F$. Given a foliation $F$ of $M$, we denote by $F(x)$ the element of $F$ which contains $x$ and call such elements the leaves of $F$.

The following is the main result of this work.

**Theorem A.** Let $f : M \to M$ be a homeomorphism over a compact smooth manifold and $F$ be a $f$-invariant one dimensional continuous foliation of $M$ by $C^1$-submanifolds and $\{l_x\}$ a $F$-arc length system. If $f$ is ergodic with respect to a $f$-invariant measure $\mu$ then one of the following holds:

a) the disintegration of $\mu$ along $F$ is atomic.

b) for almost every $x \in M$, the conditional measure on $F(x)$ is equivalent to the measure $\lambda_x$ defined on simple arcs of $F(x)$ by:

$$\lambda_x(\gamma([0,1])) = l_x(\gamma), \text{ where } \gamma \text{ is a simple arc.}$$

c) for almost every $x \in M$, the conditional measure on $F(x)$ is supported in a Cantor subset of $F(x)$.

The existence of invariant systems of metrics was obtained in [6] for the context of transitive partially hyperbolic diffeomorphisms with topological neutral center, meaning that $f$ and $f^{-1}$ have Lyapunov stable center direction (see [16, section 7.3.1]), i.e., given any $\varepsilon > 0$ there exists $\delta > 0$ for which, given any $C^1$ path $\gamma$ tangent to the center direction, one has

$$\text{length}(\gamma) < \delta \Rightarrow \text{length}(f^n(\gamma)) < \varepsilon, \quad \forall n \in \mathbb{Z}.$$
For these diffeomorphisms, the center direction integrates to a continuous foliation $F^c$ of $M$ (Corollary 7.6). In particular an immediate consequence of Theorem A is the following.

**Theorem B.** Let $f : M \to M$ be a transitive $C^1$ partially hyperbolic diffeomorphism with one-dimensional topological neutral center direction. If $f$ is ergodic with respect to a $f$-invariant measure $\mu$ then one of the following holds:

- a) the disintegration of $\mu$ along $F^c$ is atomic.
- b) for almost every $x \in M$, the conditional measure on $F^c(x)$ is equivalent to the measure $\lambda_x$ defined on simple arcs of $F^c(x)$ by:
  \[ \lambda_x(\gamma([0,1])) = l_x(\gamma), \text{ where } \gamma \text{ is a simple arc}. \]
- c) for almost every $x \in M$, the conditional measure on $F^c(x)$ is supported in a Cantor subset of $F^c(x)$.

1.2. **Organization of the paper.** In Section 2 we give some preliminaries on measure theory and disintegration of measures along a foliation. In Section 3 we introduce the definition of $F$-arc length system with respect to a dynamics in $(M, \mathcal{A}, \mu)$ and the construction of the measures induced by this system. In Section 4 we prove several technical lemmas concerning the continuity/measurability of certain functions such as the evaluation of conditional measures on certain balls inside the leaves of $F$. Finally, in Section 5 we give the proof of Theorem A.

### 2. Basics on Conditional Measures

All along the paper $(M, \mathcal{A}, \mu)$ will be a probability space, where $M$ is a compact Riemannian manifold, with dimension at least two, $\mu$ is a non-atomic Borel measure and $\mathcal{A}$ is a completion of the Borel $\sigma$-algebra $\mathcal{B}$ of $M$ with respect to the measure $\mu$. In other words, $(M, \mathcal{A}, \mu)$ is measurably isomorphic to $(\mathbb{R}, \mathcal{A}_{\mathbb{R}}, \text{Leb}_{\mathbb{R}})$ where $\text{Leb}_{\mathbb{R}}$ is the standard Lebesgue measure on $[0,1]$ and $\mathcal{A}_{\mathbb{R}}$ is the $\sigma$-algebra of Lebesgue measurable sets of $[0,1]$. We will denote by $\mu(\cdot|U)$ the restriction of $\mu$ to a subset $U \subset M$, that is, it denotes the measure given by: $\mu(\cdot|U) = \mu(U)\cdot\mu(B \cap U)$.

Given a sub-$\sigma$-algebra $\mathcal{E} \subset \mathcal{B}$ generated by a countable family $\{E_n\}_{n \in \mathbb{N}}$, the atom of $x$ is the set given by

\[ [x] := \bigcap_{E \in \mathcal{E} : x \in E} E. \]

Since $\{E_n\}_{n \in \mathbb{N}}$ generates $\mathcal{E}$ we may also write

\[ [x] = \bigcap_{x \in E_n} E_n. \]

Consequently, $[x]$ is a Borel set for every $x \in M$ and $\{[x] : x \in M\}$ is a partition of $M$.

Given $\mathcal{E} \subset \mathcal{B}$ a countably generated sub-$\sigma$-algebra. A family of measures $\{\mu_x\}_{x \in M}$ is called a system of conditional measures of $\mu$ associated to $\mathcal{E}$ if
i) for every \( B \in \mathcal{B}, x \to \mu_x(B) \) is \( \mathcal{E} \)-measurable,

ii) for every \( x \in X, \mu_x([x]) = 1 \),

iii) \( \mu(B) = \int_{y \in B} \mu_y(B) d\mu(y) \).

As it is well known, every Borel measure \( \mu \) admits a system of conditional measures with respect to any countably generated sub-\( \sigma \) algebra \( \mathcal{E} \subset \mathcal{B} \) and such system is essentially unique, see for example [8, Theorem 5.14].

In our context it is usually convenient to consider systems of conditional measures along certain partitions by \( C^1 \) submanifolds, as we detail in the sequel.

We say that \( \mathcal{F} \) is a continuous foliation of dimension \( m \) by \( C^1 \)-manifold if \( \mathcal{F} = \{ \mathcal{F}(x) \}_{x \in \mathcal{M}} \) is a partition of \( \mathcal{M} \) into \( C^1 \) submanifolds of dimension \( m \), such that for every \( x \in \mathcal{M} \) there exist open sets \( U \subset \mathcal{M}, V \subset \mathbb{R}^m, W \subset \mathbb{R}^{n-m} \) and a homeomorphism \( \varphi : U \to V \times W \), called a local chart, such that for every \( c \in W \) the set \( \varphi^{-1}(V \times \{c\}) \), which is called a plaque of \( \mathcal{F} \) is a connected component of \( L \cap U \) for a certain \( L \in \mathcal{F} \). Given a foliation \( \mathcal{F} \) of \( \mathcal{M} \), we denote by \( \mathcal{F}(x) \) the element of \( \mathcal{F} \) which contains \( x \) and call such elements the leaves of \( \mathcal{F} \). Whenever \( U \) is a foliated chart, we denote by \( \mathcal{F}|U \) the continuous foliation of \( U \) given by plaques of \( \mathcal{F} \) restricted to \( U \).

Given a one-dimensional continuous foliation \( \mathcal{F} \) of \( \mathcal{M} \), consider a local chart of \( \mathcal{F} \)

\[
\varphi : U \to (0,1) \times B_1^{n-1}(0),
\]

we also call \( U \) a foliated box.

Let \( \{ \tilde{E}_{q,k} \} \) a countable collection of sets defined by

\[
\tilde{E}_{q,k} = (0,1) \times B(q,1/k) \subset \mathbb{R} \times \mathbb{R}^{n-1}, \quad q \in B_1^{n-1}(0) \cap \mathbb{Q}^{n-1}, \quad k \in \mathbb{N}.
\]

Let \( E_{q,k} = \varphi^{-1}(\tilde{E}_{q,k}) \) and \( \mathcal{E} \subset \mathcal{B} \) the sub-\( \sigma \)-algebra generated by the family of Borelian sets \( E_{q,k} \).

Notice that for every \( y \in \mathcal{M} \) the atom, \( [y] \) is the connected component of \( \mathcal{F}(y) \cap U \) that contains \( y \), in this case we also call the system of conditional measures of \( \mu \) associated to \( \mathcal{E} \), \( \{ \mu_y^U \}_{y \in \mathcal{U}} \), the disintegration of \( \mu \) along \( \mathcal{F} \) restricted to \( U \). We say that the disintegration of \( \mu \) along \( \mathcal{F} \) is atomic if for any local chart \( U \), for almost every \( y \in U \) there exists \( a(y) \) in the plaque \( \mathcal{F}(y) \cap U \) with \( \mu_y^U(a(y)) > 0 \).

As we may observe, the disintegration of \( \mu \) along \( \mathcal{F} \) is always done inside a local chart. However, the following well known result states that if two local charts intersect, the disintegration of both local charts are equal except by a multiplicative constant when restricted to the intersection.

**Proposition 2.1.** (see for example [7] Proposition 5.17) If \( U_1 \) and \( U_2 \) are domains of two local charts \( \varphi_1 \) and \( \varphi_2 \) of \( \mathcal{F} \), then for almost every \( x \) the conditional measures \( \mu_x^{U_1} \) and \( \mu_x^{U_2} \) coincide up to a constant on \( U_1 \cap U_2 \).

As observed in [3], this allows us to define a family of classes of measures \( \{ x \in \mathcal{M} : \Omega_x \} \), such that

- \( \omega_x(M \setminus \mathcal{F}(x)) = 0 \), for any representant \( \omega_x \) of \( \Omega_x \),
- any two represents of \( \Omega_x \) are equal modulo multiplication by a constant
• for any local chart $U$ of $M$ and $\{\mu^U_x\}_{x \in U}$ a disintegration of $\mu(\cdot|U)$ along $\mathcal{F}|U$, for almost every point $x \in U$ we have

$$\mu^U_x = \omega_x(\cdot|\mathcal{F}|U(x)),$$

where $\omega_x$ denotes a representant of $\Omega_x$.

The family $\{\Omega_x\}_{x \in M}$ will be called a disintegration of $\mu$ along $\mathcal{F}$.

3. INVARIANT ARC-LENGTH AND INVARIANT METRIC SYSTEMS

Given $f : M \to M$ and a foliation $\mathcal{F}$ of $M$, we say that $f$ preserves $\mathcal{F}$, or that $\mathcal{F}$ is $f$-invariant if for $x \in M$

$$\mathcal{F}(f(x)) = f(\mathcal{F}(x)).$$

From now on, $\mathcal{F}$ will denote a continuous and $f$-invariant one dimensional foliation.

By a simple arc $\gamma$ on a leaf $\mathcal{F}(x)$, we mean a $C^1$ curve $\gamma : [0, 1] \to \mathcal{F}(x)$ for which $\gamma(t) \neq \gamma(s)$ for all $t \neq s$ with $(t, s) \notin \{(0, 1), (1, 0)\}$. In the space of simple arcs we define an equivalence relation by saying that $\gamma \sim \sigma$ if $\sigma$ is a reparametrization of $\gamma$. We say that a sequence of simple arcs $\gamma_n$ converges to $\gamma$ (in the $C^0$-topology) if $\gamma_n$ converges pointwise to $\gamma$. By convention, by a degenerate arc we mean a point.

The following definition is inspired in the concept of center metric given in [6].

**Definition 3.1.** We will call $\{l_x\}$ a $\mathcal{F}$-arc length system, if for $x \in M$, $l_x$ is defined on the simple arcs on $\mathcal{F}(x)$, and $l_x$ satisfies the following properties:

1. strictly positive on the non-degenerate arcs, and vanishing on degenerate arcs,
2. $l_x(\gamma) = l_x(\sigma)$ if $\gamma \sim \sigma$,
3. let $\gamma : [0, 1] \to \mathcal{F}(x)$ be a simple arc and $a \in (0, 1)$ then

$$l_x(\gamma[0, a]) + l_x(\gamma[a, 1]) = l_x(\gamma[0, 1]),$$

4. let $\gamma : [0, 1] \to \mathcal{F}(x)$ a simple arc, then

$$l_x(\gamma[0, 1]) = l_{f(x)}(f(\gamma[0, 1])).$$

5. given a sequence of simple arcs $\gamma_n : [0, 1] \to \mathcal{F}(x_n)$, converging to a simple arc $\gamma : [0, 1] \to \mathcal{F}(x)$, then

$$l_{x_n}(\gamma_n) \to l_x(\gamma), \quad \text{as } n \to +\infty.$$

In general, it is easy to give examples of systems preserving some continuous foliation of dimension one $\mathcal{F}$ and admitting some $\mathcal{F}$-arc length system.

**Examples:**
a) Let \( M = \mathbb{T}^d, d \geq 2, \) and \( L : \mathbb{R}^d \rightarrow \mathbb{R}^d \) a linear map given by a matrix with integer entries and for which 1 is an eigenvalue. Let \( v \) be an eigenvector associated to 1 and take \( E = \mathbb{R} \cdot v \). The linear map \( L \) induces a linear function \( f_L : \mathbb{T}^d \rightarrow \mathbb{T}^d \) and \( E \) induced a foliation \( \mathcal{F} \) of \( \mathbb{T}^d \) which is one-dimensional and \( f_L \)-invariant. Clearly \( f_L \) is an isometry along \( \mathcal{F} \). In particular, the family of standard arc-lengths on the leaves of \( \mathcal{F} \) constitute a \( \mathcal{F} \)-arc length system. For this case we obtain that any ergodic measure invariant by \( f_L \) must have either atomic conditionals, conditionals supported on Cantor subsets of the leaves or they must be equivalent to the Lebesgue measure on the leaves.

**Question 1.** Does there exists an ergodic measure \( \mu \) preserved by some \( f_L \) whose conditional measures are supported on a Cantor subset of the leaves?

b) Let \( \varphi : \mathbb{R} \times M \rightarrow M \) be any \( C^1 \) flow. The foliation \( \mathcal{F} \) given by the orbits of \( \varphi \) is a \( \varphi_t \)-invariant \( C^1 \)-foliation of \( M \) for any fixed \( t \in \mathbb{R} \). There is a natural \( \mathcal{F} \)-arc length system in this case given by:

\[
l_x(\gamma) := l, \quad \text{with} \quad \varphi(l, \gamma(0)) = \gamma(1).
\]

Assume that almost every \( x \in M \) is not a periodic point of \( \varphi \). Given any \( \varphi_t \)-ergodic invariant measure \( \mu \), it follows from [10, Example 7.4] that the disintegration of \( \mu \) along \( \mathcal{F} \) is either Lebesgue or atomic.

c) Some skew-products also provide interesting examples. For example take \( f : \mathbb{T}^d \times S^1 \rightarrow \mathbb{T}^d \times S^1 \) given by

\[
f(x, y) = (g(x), R_\alpha(y)),
\]

where \( g : \mathbb{T}^d \rightarrow \mathbb{T}^d \) is any homeomorphism and \( R_\alpha : S^1 \rightarrow S^1 \) denotes a rotation of angle \( \alpha \). In this case, the foliation \( \mathcal{F} \) whose leaves are \( \{x\} \times S^1, x \in \mathbb{T}^d \), is \( f \)-invariant and by taking \( l_x \) on \( \{x\} \times S^1 \) to be given by the usual arc length on \( S^1 \), we conclude that \( \{l_x\} \) is a \( \mathcal{F} \)-arc length system. In this example it is easy to determine the measurable properties of \( \mathcal{F} \) in the sense that, given a Borel \( g \)-invariant measure \( \nu \), the measure \( \nu \times \lambda_{S^1} \) is \( f \)-invariant and, a direct application of the Fubini Theorem shows that, the disintegration of \( \mu \) along \( \mathcal{F} \) has the Lebesgue measures \( \lambda_{S^1} \) as its conditional measures.

d) Another, more interesting case, is provided by recent results of Bonatti-Zhang [6]. A \( C^1 \) diffeomorphism \( f : M \rightarrow M \), on a compact Riemannian manifold \( M \), is said to be partially hyperbolic if there is a nontrivial splitting

\[
TM = E^s \oplus E^c \oplus E^u
\]

such that

\[
Df(x)E^\tau(x) = E^\tau(f(x)), \quad \tau \in \{s, c, u\}
\]

Here we denote by \( \lambda_{S^1} \) the standard Lebesgue measure on \( S^1 \).
and a Riemannian metric for which there are continuous positive functions \( \mu, \hat{\nu}, \nu, \gamma, \hat{\gamma} \) with

\[
v(p), \hat{\nu}(p) < 1, \quad \text{and} \quad \mu(p) < \nu(p) < \gamma(p) < \hat{\gamma}(p)^{-1} < \hat{\nu}(p)^{-1},
\]

such that for any vector \( v \in T_pM, \)

\[
\mu(p)||v|| < ||Df(p) \cdot v|| < \nu(p)||v||, \quad \text{if} \quad v \in E^s(p)
\]

\[
\gamma(p)||v|| < ||Df(p) \cdot v|| < \hat{\gamma}(p)^{-1}||v||, \quad \text{if} \quad v \in E^c(p)
\]

\[
\hat{\nu}(p)^{-1}||v|| < ||Df(p) \cdot v|| < \hat{\mu}(p)^{-1}||v||, \quad \text{if} \quad v \in E^u(p).
\]

We say that \( f \) has topological neutral center if, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) for which: given any smooth curve \( \gamma : [0,1] \to M \) with \( \gamma'(t) \in E^c(\gamma(t)), 0 \leq t \leq 1 \), if length\( (\gamma) < \delta \) then length\( (f^n(\gamma)) < \varepsilon, \) for all \( n \in \mathbb{Z} \). For partially hyperbolic diffeomorphisms with neutral center, the center distribution \( E^c \) integrates to a \( f \)-invariant foliation \( \mathcal{F}^c \) (see [16 Corollary 7.6]) called center foliation of \( f \). In [6] the authors proved that if \( f : M \to M \) is a \( C^1 \) partially hyperbolic diffeomorphism with neutral center direction, then \( f \) admits a continuous \( \mathcal{F} \)-arc length system. To understand the measurable properties of the center foliation preserved by such maps, were one of the motivations of this work. As a consequence of our results, the disintegration of any \( f \)-invariant ergodic probability measure of such maps falls in three possible cases. When the conditional measures have full support, the second author proves in [12] the occurrence of an invariance principle. Further, if the measure is smooth, full support of the conditional measures imply the Bernoulli property for \( f \). If, moreover, \( f \) is locally accessible, then \( \mathcal{F}^c \) is as regular as the map \( f \) itself.

**Definition 3.2.** Given \( y, z, w \in \mathcal{F}(x) \) we say that \( y \) is between \( z \) and \( w \), if there exists a simple arc \( \gamma : [0,1] \to \mathcal{F}(x) \) such that \( \gamma(0) = z, \gamma(1) = w, \gamma(t) = y \) for some \( t \in (0,1) \), and

\[
l_x(\gamma) = \min\{l_x(\alpha) : \alpha : [0,1] \to \mathcal{F}(x), \alpha(0) = z, \alpha(1) = w\}.
\]

**Definition 3.3.** Let \( \{l_x\} \) be a \( \mathcal{F} \)-arc length system. For \( x \in M \) we define a metric \( d_x \) on \( \mathcal{F}(x) \) by

\[
d_x(y,z) := \min\{l_x(\gamma) : \gamma : [0,1] \to \mathcal{F}(x) \text{ is simple with } \gamma(0) = y, \gamma(1) = z\}.
\]

We call the family \( \{d_x\}_x \) the \( \mathcal{F} \)-metric system associated to the \( \mathcal{F} \)-arc length system \( \{l_x\}_x \).

In what follows, we prove that indeed \( d_x \) is a metric over \( \mathcal{F}(x) \) and, moreover, the metric system is \( f \)-invariant.

**Lemma 3.4.** Let \( \{d_x\} \) be a \( \mathcal{F} \)-metric system given by the Definition 3.3 then

i) \( d_x \) is an additive metric, that is, given \( y, z, w \in \mathcal{F}(x) \) such that \( y \) is between \( z \) and \( w \) then

\[
d_x(z,w) = d_x(z,y) + d_x(y,w);
\]
ii) \( d_x \) is invariant by \( f \), that is,
\[
d_f(x, f(y)) = d_x(z, y).
\]

\textbf{Proof.} Observe that the second item is trivial by the definition of \( \{d_x\}_x \), thus we only have to prove that \( d_x \) is indeed a metric on \( \mathcal{F}(x) \).

It is easy to see that \( d_x(x, y) = 0 \) if, and only if, \( x = y \) and that \( d_x(x, y) = d_x(y, x) \).

Since \( \mathcal{F}(x) \) is an one-dimensional submanifold of \( M \), \( \mathcal{F}(x) \) is homeomorphic either to \( \mathbb{R} \) or \( S^1 \). If \( \mathcal{F}(x) \) is homeomorphic to \( \mathbb{R} \) then, for \( z, w \in \mathcal{F}(x) \) there is only one simple connected path \( \gamma : [0, 1] \to \mathcal{F}(x) \), modulo reparametrization, with \( \gamma(0) = z \) and \( \gamma(1) = w \). Given \( y \) between \( z \) and \( w \) there exists \( a \in (0, 1) \) such that \( \gamma(a) = y \). In particular, \( \gamma|_{[0,a]} \) and \( \gamma|_{[a,1]} \) are the only simple connected paths, modulo reparametrization, from \( z \) to \( y \), and from \( y \) to \( w \) respectively. Thus by Definition 3.1 we have
\[
d_x(z, w) = l_x(\gamma) = l_x(\gamma|_{[0,a]}) + l_x(\gamma|_{[a,1]}) = d_x(z, y) + d_x(y, w).
\]

Now, if \( \mathcal{F}(x) \) is homeomorphic to \( S^1 \) then there are two paths \( \gamma_1, \gamma_2 : [0, 1] \to \mathcal{F}(x) \) that connect the points \( z \) and \( w \). Assume, without loss of generality, that \( \gamma_1 \) satisfies \( d_x(z, w) = l_x(\gamma_1) \) and \( \gamma_1(a) = y \) for some \( a \in (0, 1) \). Let us prove that \( d_x(z, y) = l_x(\gamma_1|_{[0,a]}) \).

If this is not the case, there is another path \( \gamma_3 : [0, 1] \to \mathcal{F}(x) \) for which \( \gamma_3(0) = z \), \( \gamma_3(1) = y \) and \( d_x(z, y) = l_x(\gamma_3) \). In this case we have \( l_x(\gamma_3) > l_x(\gamma_2) \), since \( \gamma_3 \) is the concatenation of \( \gamma_2 \) with \( -\gamma_1|_{[-1, -a]} \), where \( -\gamma_1 \) denotes the curve \( -\gamma_1 : [-1, 0] \to \mathcal{F}(x) \), \( -\gamma_1(t) := \gamma_1(-t) \). Thus,
\[
l_x(\gamma_1|_{[0,a]}) > d_x(z, y) = l_x(\gamma_3) \geq l_x(\gamma_2) \geq l_x(\gamma_1),
\]
which is a contradiction. Therefore, \( l_x(\gamma_1|_{[0,a]}) = d_x(z, y) \) and, analogously, \( l_x(\gamma_1|_{[a,1]}) = d_x(y, w) \).

By the second item of Definition 3.1 we have,
\[
d_x(z, w) = l_x(\gamma_1) = l_x(\gamma_1|_{[0,a]}) + l_x(\gamma_1|_{[a,1]}) = d_x(z, y) + d_x(y, w),
\]
concluding that \( d_x \) is an additive metric as we wanted to show.

It is not true that \( \{d_x\}_x \) is continuous in the sense that we may have sequences \( x_n \to x \), \( y_n \to y \), with \( y_n \in \mathcal{F}(x_n) \), \( y \in \mathcal{F}(x) \) but \( d_{x_n}(x_n, y_n) \to d_x(x, y) \). Indeed this happens, for example, for compact foliations where the leaves do not have uniformly bounded length. It is true, however, that this family of metrics are continuous when restricted to plaques inside local charts. We make this property more precise below.

\textbf{Definition 3.5.} Consider \( \mathcal{F} \) a continuous foliation of \( M \). A function \( F : \bigcup_{x \in M} \mathcal{F}(x) \times \mathcal{F}(x) \to [0, \infty) \) will be called plaque-continuous if given any \( p \in M \), there exists a local chart \( p \in U \) of \( \mathcal{F} \), such that for
any sequences \(x_n \to x, y_n \to y\) with \(y_n \in \mathcal{F}|U(x_n), x \in U\) and \(y \in \mathcal{F}|U(x)\), we have

\[
\lim_{n \to \infty} F(x_n, y_n) = F(x, y).
\]

Any such local chart \(U\) will be called a continuity-domain of \(F\).

**Definition 3.6.** We say that a family of metrics \(\{d_x : x \in M\}\), each \(d_x\) defined on \(\mathcal{F}(x)\), is plaque-continuous if \(F : \bigcup_{x \in M} \mathcal{F}(x) \times \mathcal{F}(x) \to [0, \infty)\) defined by

\[
F(x, y) := d_x(x, y),
\]

is plaque continuous. In this case if \(U\) is a continuity-domain of \(F\) we will also say that \(U\) is a continuity-domain of \(\{d_x\}\).

**Proposition 3.7.** The metric system given in Definition 3.3 is plaque-continuous.

**Proof.** Let \(\varphi : U \to (0, 1) \times V \subset \mathbb{R}^n\) be a local chart of \(\mathcal{F}\) where \(\varphi^{-1}((0, 1) \times \{c\}), c \in V\), are the plaques of \(\mathcal{F}\) in \(U\). For any \(p \in U\), consider \(\xi : W \subset U \to (0, 1) \times K \subset \mathbb{R}^n\) another local chart centered in \(p\) such that for any

\[
z \in W \Rightarrow d_z(\mathcal{F}|U(z)) > 3 \cdot d_z(\mathcal{F}|W(z)).
\]

This can be done by the continuity of \(\{l_z\}\). In particular, for any \(x \in W, y \in \mathcal{F}|W(x) = \xi^{-1}((0, 1), c')\), the simple curve \(\gamma(t) = \xi^{-1}((1 - t)\xi(x) + t\xi(y), c')\) minimizes the \(l_x\)-length connecting \(x\) and \(y\), that is, \(d_x(x, y) = l_x(\gamma)\).

On that account, consider \(x \in W, y \in \mathcal{F}|W(x) = \xi^{-1}((0, 1), c')\) and sequences \(x_n \in W, y_n \in \mathcal{F}|W(x_n) = \xi^{-1}((0, 1), c_n)\) with \(x_n \to x\) and \(y_n \to y\). Let \(\gamma\) be defined as in the previous paragraph and \(\gamma_n(t) := \xi^{-1}((1 - t)\xi(x_n) + t\xi(y_n), c_n)\). By the convergence of the sequences we have \(\gamma_n \to \gamma\). But by the previous discussion on the choice of the local chart \(W\) we have

\[
d_{x_n}(x_n, y_n) = l_{x_n}(\gamma_n) \quad \text{and} \quad d_x(x, y) = l_x(\gamma).
\]

We then conclude by the continuity of \(\{l_x\}\) that \(\lim_{n \to \infty} d_{x_n}(x_n, y_n) = \lim_{n \to \infty} l_{x_n}(\gamma_n) = l_x(\gamma) = d_x(x, y)\).

Therefore \(\{d_x\}\) is plaque-continuous as we wanted to show. \(\square\)

**Lemma 3.8.** Given any local open transversal \(T\) to \(\mathcal{F}\), for any \(r\) small enough, the set

\[
S := \bigcup_{x \in T} B_{d_x}(x, r)
\]

is open.

**Proof.** Assume that \(T\) is a local transversal associated to a certain local chart \((U, \varphi)\). Let \(r > 0\) small enough we have \(B_{d_x}(x, r) \subset U\) for every \(x \in T\). In particular \(U \setminus \overline{T}\) has two open connected components \(U_1\) and \(U_2\) with \(U_1 \cap U_2 = \overline{T}\).

Since \(U\) is a local chart, we may consider an orientation on the \(\mathcal{F}|U\)-plaques. Assume that \(S\) is not open. Then, there exists \(y \in S\) and a sequence \(y_k \notin S\), with \(y_k \rightarrow y\).
Consider $x \in T$ such that $y \in B_{d_x}(x, r)$ and denote by $\varphi$ the flow on the $\mathcal{F}|U$-plaques induced by the orientation fixed before and such that

$$d_p(\varphi_t(p), p) = |t|,$$

whenever $\varphi_t(p)$ is defined. Let $t_0$ be such that $x = \varphi_{t_0}(y)$. As $y \in B_{d_x}(x, r)$, there exists $\delta > 0$ for which

$$\varphi_t(y) \in S, \quad t \in [t_0 - \delta, t_0 + \delta].$$

Now, by the plaque continuity and the fact that $y_k \to y$, we have

$$\varphi_{t_0 - \delta}(y_k) \to \varphi_{t_0 - \delta}(y), \quad \varphi_{t_0 + \delta}(y_k) \to \varphi_{t_0 + \delta}(y).$$

Observe that $\varphi_{t_0 - \delta}(y)$ and $\varphi_{t_0 + \delta}(y)$ belong to different connected components, thus, for $k$ large enough the same happens for $\varphi_{t_0 - \delta}(y_k)$ and $\varphi_{t_0 + \delta}(y_k)$. Since $\gamma_k := \{ \varphi_t(y_k) : t \in [t_0 - \delta, t_0 + \delta] \}$ is an arc with points in the interior and in the exterior of $U_k$, it must intersect its boundary, namely $T$. This implies that $y_k \in S$ for large $k$, yielding a contradiction.

That is, $S$ is open as we wanted to show. 

\[\square\]

**Proposition 3.9.** Given a finite open cover $U$ of $M$ by local charts of $\mathcal{F}$. There exists $\varepsilon > 0$ such that for all $x \in M$, there is $U \in U$ with

$$B_{d_x}(x, \varepsilon) \subset U.$$

**Proof.** For each $x \in M$, take any $U_x \in U$ with $x \in U_x$. There exists $r_x > 0$ for which $B_{d_x}(x, r_x) \subset U_x$. By plaque continuity of $\{d_x\}$ there exists a neighborhood $x \in V_x \subset U_x$ for which

$$y \in V_x \Rightarrow B_{d_x}(y, r_x) \subset U_x.$$

Since $M$ is compact we may cover $M$ with a finite number of neighborhoods $V_x$, $1 \leq i \leq l$. Take $\varepsilon = \min\{r_x : 1 \leq i \leq l\}$. 

In the sequel we will prove a technical result which will be used along the proof of the main theorem. Namely, we prove that the continuous translation of a measurable set along the foliation $\mathcal{F}$ is also a measurable set.

**Lemma 3.10.** There exists $t_0 > 0$ such that for every $0 \leq t \leq t_0$ and every Borel subset $A \subset M$ the set

$$\Phi_t(A) := \{ x \in M : d_x(x, A) < t \},$$

is a measurable set.

**Proof.** Let $U$ be a finite cover of $M$ by local charts which are continuity-domains of $\{d_x\}$. Consider $\varepsilon$ the number given by Proposition 3.9. In particular the family $\{U_t/2 : U \in U\}$, defined by

$$U_{t/2} = \{ x \in U : d_x(x, \partial U) \geq r/2 \},$$
is still a cover of \( M \). Let \( A \subset M \) be a Borel subset. Observe that
\[
\Phi_t(A \cap U_{t/2}) \subset U, \quad U \in \mathcal{U}, \quad t < \tau/2.
\]

We will prove that for \( U \in \mathcal{U} \), the subset \( \Phi_t(A \cap U_{t/2}) \) is measurable.

Let \( \varphi_U : U \to B^{n-1}_1(0) \times (0, 1) \) be a local chart of \( \mathcal{F} \), and inside \( U \) consider the orientation in the plaques \( \mathcal{F}(x) \) induced by the orientation in the line segments of the form \( \{x\} \times (0, 1) \subset \mathbb{R}^{n-1} \times \mathbb{R} \). This orientation induces, at each plaque, an order relation which we will denote by \( \prec \) (the plaque being implicit in the context).

Now for \( s \in [-t, t] \), with \( 0 \leq t < \tau/2 \) fixed, we define \( \varphi_U^s : U_{t/2} \to U \) by:

- for \( s > 0 \), \( \varphi_U^s(x) \) is the only point of the plaque \( \mathcal{F}|U(x) \) such that \( d_x(x, \varphi_U^s(x)) = s \) and \( x \prec \varphi_U^s(x) \);
- for \( s < 0 \), \( \varphi_U^s(x) \) is the unique point of the plaque \( \mathcal{F}|U(x) \) such that \( d_x(x, \varphi_U^s(x)) = -s \) and \( \varphi_U^s(x) \prec x \).

Observe that \( \varphi_U^s \) is continuous for every \( |s| < t \) since \( U \) is a continuity-domain of \( \{d_x\} \) and, consequently, it is a homeomorphism. Thus \( \varphi_U^s(A \cap U_{t/2}) \) is a measurable subset of \( M \) for every \( s \in [-t, t] \).

Now, for each \( 1 \leq i \leq n \) take
\[
\Phi_i(U) := \bigcup_{q \in \mathbb{Q} \cap \mathcal{U}} \varphi_U^s(A \cap U_{t/2}), \quad 0 \leq t < \tau/2.
\]
Notice that \( \Phi_i(U) \) is a measurable set, since each set in the countable union is measurable as we have proved before. Consequently,
\[
\Phi_t(A) = \bigcup_{U \in \mathcal{U}} \Phi_t(U),
\]
is a measurable set, as we wanted to show. \( \square \)

**Definition 3.11.** Let \( \{l_x\} \) be a \( \mathcal{F} \)-arc length system. Then, we have a well defined homeomorphism
\[
h_x : \mathcal{F}(x) \to F,
\]
where \( F = \mathbb{R} \) or \( F = S^1 \), \( h_x(x) = 0 \) and such that, for any simple arc \( \gamma : [0, 1] \to \mathcal{F}(x) \) we have
\[
l_x(\gamma[0, 1]) = \lambda(h_x(\gamma[0, 1])),
\]
where \( \lambda \) denotes the Lebesgue measure on \( F \). In particular \( \lambda(h_x(\gamma[0, 1])) \) is the size of the interval \( h_x(\gamma[0, 1]) \). We now define the measure \( \lambda_x \) on \( \mathcal{F}(x) \) given by:
\[
\lambda_x = (h_x^{-1})_* \lambda.
\]

\(^2\)Here we are using the identification \( S^1 = [0, 1]/\sim \) where \( 0 \sim 1 \), thus the point 0 stands for the equivalence class of 0 in \( S^1 \).
Note that if $\gamma[0,1]$ is a simple arc in $F(x)$ then,

$$\lambda_x(\gamma[0,1]) = \lambda(h_x(\gamma[0,1])) = I_x(\gamma[0,1]).$$

Consequently, the measure $\lambda_x$ is a doubling measure $3$.  

4. Properties of non-atomic disintegrations over a continuous one-dimensional foliation

The proof of the main result of this paper follows from understanding the topological structure of $\text{supp} \mu_x^U$, for a disintegration $\{\mu_x^U\}_{x \in U}$ of $\mu(\cdot|U)$ of $\mu$ on a local chart $U$. To this end we first need to understand the behavior, in terms of measure theory, of the map

$$(x,r) \mapsto \mu_x^U(B_d(x,r)),$$

defined for a certain subset of $U \times \mathbb{R}$. This is the goal of this Section.

Along the rest of the paper we assume the following:

- $F$ is a $f$-invariant one dimensional continuous foliation,
- $U$ is a finite cover of $M$ by local charts $U$ of $F$ such that $\overline{U}$ is still inside a local chart of $F$,
- each $U \in U$ is a continuity-domain of $\{d_x\}$,
- for each $U \in U$, $\{\mu_x^U\}$ is a disintegration of $\mu(\cdot|U)$ along the plaques $F|U$,
- the disintegration of $\mu$ along $F$ is not atomic, in particular, for each $U \in U$ there exists a subset $A_U \subset U$ with $\mu(A_U) = 0$ and for which
  $$x \not\in A_U \Rightarrow \mu_x^U \text{ is not atomic},$$
- $r > 0$ is any constant given by Proposition 3.9,
- $\{\Omega_x\}_{x \in M}$ a disintegration of $\mu$ along $F$ and we denote $\omega_x$ any representative of $\Omega_x$, $x \in M$.

We also fix the following notation: for any subset $X \subset M$, we denote by $B_X$ the Borel sigma algebra of $X$ given by the topology induced by that of $M$. It is important to observe that, by definition, for any $U \in U$, the set $A_U$ is $F|U$-saturated in $U$.

**Lemma 4.1.** For each $0 < r < \tau$, $U \in U$ and $x \in U \setminus A_U$, the map

$$(y) \mapsto \mu_x^U(B_d(x,r)),$$

is continuous when restricted to the subset $F_{\tau} \subset F|U(x)$ given by

$$F_{\tau}(x) = \{y \in F|U(x) : B_d(x,y,r) \subset U\}.$$  

3Recall that given a metric space $(X,d)$, a measure $\nu$ on $X$ is said to be a **doubling measure** if there exists a constant $\Omega > 0$ such that for any $x \in X$ and any $r > 0$ we have

$$\nu(B(x,2r)) \leq \Omega \cdot \nu(B(x,r)).$$
Proof. Take \( x \in U, U \in \mathcal{U} \). Let \( y_n \to y, y_n, y \in F_r(x) \). We want to show that for \( r > 0 \)
\[
\lim_{n \to \infty} \mu^U_x(B_{d_y}(y_n, r)) = \mu^U_x(B_{d_y}(y, r)).
\]

Given any \( k \in \mathbb{N} \), since \( \mu^U_x \) is not atomic, we have that
\[
\mu^U_x(\partial B_d(y, r)) = 0 \quad \text{and} \quad \mu^U_x(\partial B_d(y_n, r)) = 0, \forall n \in \mathbb{N},
\]
where \( \partial B_d \) denotes the boundary of the set inside the leaf \( F(x) \). Now, let \( B_n := B_{d_y}(y_n, r) \Delta B_{d_y}(y, r) \)
where \( Y \Delta Z \) denotes the symmetric difference of the sets \( Y \) and \( Z \). From standard measure theory:
\[
\limsup_{n \to \infty} \mu^U_x(B_n) \leq \mu^U_x \left( \limsup_{n \to \infty} B_n \right).
\]

Thus
\[
\limsup_{n \to \infty} \mu^U_x(B_n) \leq \mu^U_x \left( \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} B_n \right) \leq \mu^U_x(\partial B_{d_y}(y, r)) = 0.
\]

Therefore
\[
\lim_{n \to \infty} \mu^U_x(B_n(y)) = \mu^U_x(B_{d_y}(y, r)) = 0
\]
and consequently
\[
\lim_{n \to \infty} \mu^U_x(B_{d_y}(y_n, r)) = \mu^U_x(B_{d_y}(y, r)),
\]
as we wanted to show. \( \square \)

**Proposition 4.2.** Let \( \mathcal{U} \in \mathcal{U} \) and \( 0 < r < \tau \). Consider \((V, \varphi)\) a local chart inside \( U \) such that
\[
x \in V \Rightarrow B_{d_y}(x, r) \subset U.
\]

Then, restricted to \( V \setminus A_U \) the map
\[
x \mapsto \mu^U_x(B_{d_y}(x, r))
\]
is \( B_{\mathcal{V} \setminus A_U} \)-measurable, and consequently \( B_{\mathcal{U} \setminus A_U} \)-measurable as \( V \subset U \).

**Proof.** If \( x \) and \( y \) belong to the same \( F \)-plaque in \( U \) then \( \mu^U_x = \mu^U_y \). We already know that for all
Borel subset \( W \subset U \)
\[
x \in V \mapsto \mu^U_x(W),
\]
is Borel measurable.

Consider the local chart be given by the homeomorphism \( \varphi : V \to (0, 1) \times G \), where \( G \subset \mathbb{R}^{n-1} \) is an open subset. Setting \( g_r : V \to [0, \infty) \)
\[
g_r(x) = \mu^U_x(B_{d_y}(x, r)),
\]
we have
\[
g_r \circ \varphi^{-1}(x_1, x_2) = \mu^U_{\varphi^{-1}(x_1, x_2)}(B_{d_{\varphi^{-1}(x_1, x_2)}}(\varphi^{-1}(x_1, x_2), r)).
\]
Let us prove that this function is continuous in $x_1$ and Borel measurable in $x_2$.

By Lemma 3.1 restricted to $V \setminus A_U$ we already have the continuity of $g_r \circ q^{-1}$ the first coordinate, since the second coordinate being fixed means we are evaluating the function on a single plaque where the conditional measure is non-atomic. Now, fix the first coordinate $x_1$ and consider the transversal $T = \{x_1\} \times B_1^{q^{-1}}(0)$. By Lemma 3.8 the set

\[ S := \bigcup_{x \in q^{-1}(T)} B_{d_1}(x, r), \]

is an open subset of $M$. Thus, $y \mapsto \mu_{\nu}^{U}(S)$ is a Borel measurable function in $V$, which implies that $g_r \circ q^{-1}(x_1, \cdot)$ is $B_T$-measurable. In particular, its restriction to $T \cap (V \setminus A_U) = T \setminus (\varphi(A_U))$ is $B_{T \setminus (\varphi(A_U))}$-measurable map. But observe that for $x \in T$ we have

\[ \mu_{\nu}^{U}(S) = \mu_{\nu}^{U}(B_{d_1}(x, r)). \]

Therefore, for $x_1$ fixed the map $x_2 \in G \setminus \tau_2(\varphi(A_U)) \mapsto \mu_{\nu}^{U}(x_1, x_2) (B_{d_{\varphi^{-1}(x_1, x_2)}(\varphi^{-1}(x_1, x_2), r))$ is $B_{G \setminus \tau_2(\varphi(A_U))}$-measurable, where $\tau_2 : (0, 1) \times G \mapsto G$ is the projection onto the second coordinate.

Consequently, $g_r \circ q^{-1}$ restricted to $((0, 1) \times G) \setminus \varphi(A_U)$ is a jointly measurable function with respect to the product sigma-algebra $B((0, 1) \times G) \setminus \varphi(A_U))$ (see for example [1] Lemma 4.51). As $\varphi$ is a homeomorphism, we conclude that $g_r$ is $B_{V \setminus A_U}$-measurable, as we wanted to show.

\[ \square \]

In the following Lemma, we prove that the subset of $M$ consisting of all points $x \in M$ for which there is a ball in $F(x)$ with null $\mu_{\nu}^{U}$ measure, is a relatively Borel set.

**Lemma 4.3.** For each $U \in \mathcal{U}$, the set

\[ Z_U = \bigcup_{x \in U \setminus A_U} F|U(x) \setminus \text{supp} \mu_{\nu}^{U}, \]

is $B_{U \setminus A_U}$-measurable set.

**Proof.** First let us give a better formulation for the definition of $Z_U$. Observe that

\[ Z_U = \{ x \in U \setminus A_U : \mu_{\nu}^{U}(I) = 0 \text{ for some open ball } x \in I \subset F(x) \}. \]

Consider $\{q_1, q_2, \ldots\}$ an enumeration of the $Q \cap [0, 1]$ and $\mathcal{U}$ be the given finite family of local charts covering $M$. For each $U \in \mathcal{U}$ and $i \in \mathbb{N}$, define $\phi_{\nu}^{U_i} : U_i \setminus A_U \rightarrow \mathbb{R}$ by

\[ \phi_{\nu}^{U_i}(x) = \mu_{\nu}^{U_i}(B_{d_i}(x, q_i)), \]

where $U_i = \{ x \in U : B_{d_i}(x, q_i) \subset U \}$. Observe that we may cover $U_i$ with a countable number of local charts $V_{ij} \subset U_i$, $j \in \mathbb{N}$ and, by Proposition 4.2, $\phi_{\nu}^{U_j}|V_{ij}$ is $B_{V_{ij} \setminus A_U}$-measurable for every $j$. In particular $\phi_{\nu}^{U_i}$ is $B_{U_i \setminus A_U}$-measurable for every $i$. On that account we have that $Z_{\nu}^{U_i} := (\phi_{\nu}^{U_i})^{-1}(\{0\}) \subset M$ is a $B_{U_i \setminus A_U}$-measurable subset, in particular, a $B_{U_i \setminus A_U}$-measurable subset. It is not difficult to
Proof. is continuous. Therefore (2)  

\[ Z_U = \bigcup_{i=1}^{\infty} Z_i^U. \]

Therefore \( Z_U \) is a \( B_{U,A_U} \)-measurable subset as we wanted to show. Moreover, \( \mu(Z_U) = 0 \). \( \square \)

Consider \( \mathcal{P} \) the set given by

\[ \mathcal{P} = \{ x : \exists U,V \in \mathcal{U}, x \in U \cap V, \mu^U_x (|U \cap V) \sim \mu^V_x (|U \cap V) \}, \]

that is, \( \mathcal{P} \) is the set of points \( x \) for which there exists two local charts \( U \) and \( V \) in \( \mathcal{U} \), both containing \( x \), where the respective conditional measures at the plaque of \( x \), \( \mu^U_x \) and \( \mu^V_x \), are not equivalent on the intersection \( \mathcal{F}|U(x) \cap \mathcal{F}|V(x) \). In particular this set has zero measure by Proposition 2.1. Set

\[ M_0 := \bigcap_{n \in \mathbb{Z}} f^n(\tilde{M}). \]

As \( \mu(\tilde{M}) = 1 \), we have \( \mu(M_0) = 1 \). For each \( x \in M_0 \), we will denote by \( \mu_x \) the measure on \( B_{d_s}(x,r) \) given by the conditional \( \mu^U_x \), for some \( U \in \mathcal{U} \) with \( x \in U \), normalized to give weight exactly one to \( B_{d_s}(x,r) \), that is, for a measurable \( F \subset \mathcal{F}|U(x) \)

\[ \mu_x (F) = \mu^U_x (F|B_{d_s}(x,r)). \]

Given any \( y \in B_{d_s}(x,r) \cap M_0 \), the measures \( \mu_y \) and \( \mu_x \) are proportional to each other at the intersection \( B_{d_s}(x,r) \cap B_{d_s}(y,r) \) by Proposition 2.1 that is, there exists a constant \( \beta \) for which \( \mu_y = \beta \cdot \mu_x \) restricted to \( B_{d_s}(x,r) \cap B_{d_s}(y,r) \).

In particular, evaluating both sides at \( B_{d_s}(x,r) \cap B_{d_s}(y,r) \) we see that

\[ \beta \cdot \mu_x (B_{d_s}(x,r) \cap B_{d_s}(y,r)) = \mu_y (B_{d_s}(x,r) \cap B_{d_s}(y,r)) \]

\[ \Rightarrow \beta = \frac{\mu_y (B_{d_s}(x,r) \cap B_{d_s}(y,r))}{\mu_x (B_{d_s}(x,r) \cap B_{d_s}(y,r))}. \]

Corollary 4.4. For each \( 0 < r < \tau \), \( x \in M_0 \)

\[ y \in B_{d_s}(x,r) \cap M_0 \mapsto \mu_y (B_{d_s}(y,r)), \]

is continuous.

Proof. For a certain \( 0 < r < \tau \) fixed, take any \( x \in M_0 \). Let \( y \in B_{d_s}(x,r) \cap M_0 \) and \( U \in \mathcal{U} \) with \( B_{d_s}(y,r) \subset U \), take \( y_n \in B_{d_s}(x,r) \cap M_0 \) with \( y_n \to y \) as \( n \to \infty \) and \( B_{d_s}(y_n,r) \subset U \).

By definition,

\[ \mu_y = \mu_y^U (|B_{d_s}(y,r)), \quad \mu_{y_n} = \mu_{y_n}^U (|B_{d_s}(y_n,r)). \]
Corollary 4.5. For each \( x \in M_0 \), the map
\[
(4) \quad \mu_{y_n}(B_{d_n}(y, r)) = \frac{\mu_{y_n}(B_{d_n}(y, r))}{\mu_y(B_{d_n}(y, r))} = \frac{\mu_y(B_{d_n}(y, r))}{\mu_y(B_{d_n}(y, r))}
\]
By Lemma 4.1 we have \( \mu_y(B_{d_n}(y, r)) \rightarrow \mu_y(B_{d}(y, r)) \) and \( \mu_y(B_{d_n}(y, r)) \rightarrow \mu_y(B_{d}(y, r)) \) as \( n \rightarrow \infty \). Therefore \( \mu_{y_n}(B_{d_n}(y, r)) \rightarrow \mu_y(B_{d}(y, r)) \) as we wanted to show.

\[\square\]

Corollary 4.5. For each \( x \in M_0 \), the map
\[
r \in [0, r] \mapsto \mu_x(B_{d_n}(x, r)),
\]
is continuous. Furthermore the map
\[
(5) \quad (x, r) \in M_0 \times [0, r] \mapsto \mu_x(B_{d_n}(x, r)),
\]
is jointly measurable.

Proof. Let \( x \in M_0 \), first, let us prove that \( r \in [0, r] \mapsto \mu_x(B_{d_n}(x, r)) \) is a continuous function. Let \( 0 = r < r \) and \( r_n \in [0, r] \) if \( r = r \) the argument is analogous, hence \( \mu_x(B_{d_n}(x, r_n)) = \mu_x(B_{d_n}(x, r)) + \mu_x(B_{d_n}(x, r_n) \setminus B_{d_n}(x, r)) \). As \( \mu_x \) is non-atomic we have
\[
\lim_{n \rightarrow \infty} \mu_x(B_{d_n}(x, r_n) \setminus B_{d_n}(x, r)) = 0.
\]
Then,
\[
\mu_x(B_{d_n}(x, r_n)) \rightarrow \mu_x(B_{d_n}(x, r)),
\]
showing the first part of the statement.

Let us show the second statement. For each \( x \in M_0 \), let \( x \in V_x \) a local chart with
\[
y \in V_x \Rightarrow B_{d_n}(y, r) \subset U_x, \text{ for some } U_x \in \mathcal{U}.
\]
As \( M \) is compact we may cover \( M \) with a finite number of such local charts, say \( V_1, V_2, \ldots, V_l \) and call \( U_1, U_2, \ldots, U_l \) the associated local charts in \( \mathcal{U} \). For any \( j \), consider
\[
y \in V_j \mapsto \mu_y(B_{d}(y, r)).
\]
Observe that
\[
\mu_y(B_{d}(y, r)) = \frac{\mu_y(B_{d}(y, r))}{\mu_y(B_{d}(y, r))}.
\]
Therefore, by Proposition 4.2, \( y \in V_j \cap M_0 \Rightarrow \mu_y(B_{d}(y, r)) \) is a \( B_{U_j \setminus A_{U_j}} \)-measurable map. As \( j \) is arbitrary, \( y \in M_0 \Rightarrow \mu_y(B_{d}(y, r)) \) is a \( B_{U \cap M_0} \)-measurable map.

Thus, the map given by (5) is jointly measurable as it is continuous in the first coordinate and \( B_{M_0} \)-measurable in the second.

\[\square\]
5. Proof of the Main Theorem

First of all, we can assume that \((M, \mu)\) is an atom-less probability space and that the disintegration of \(\mu\) along \(\mathcal{F}\) is not atomic, otherwise there would be nothing to do. We also consider the same objects fixed in the beginning of Section 4.

Now we define the distortion of the disintegration measure of \(\mu\) relative to the metric system in each leaf \(\mathcal{F}(x)\) by the following.

**Definition 5.1.** Let \((M, \mu)\) be a probability space and \(\mathcal{F}\) be a continuous one-dimensional foliation of \(M\). Let \(\{\mu_x\}\) to be the system of conditional measures along \(\mathcal{F}\) given by \((3)\), for \(x \in M_0\) and let \(d = \{d_x\}\) be the \(\mathcal{F}\)-metric system induced by the arc-length system \(\{l_x\}\) as in Definition 3.3. We define the \(\mu\)-distortion of the \(\mathcal{F}\)-metric system by

\[
\Delta(x) = \begin{cases} 
\limsup_{\varepsilon \to 0} \frac{\mu_x(B_{d_x}(x, \varepsilon))}{2\varepsilon} & \text{if } x \in M_0 \\
0 & \text{if } x \notin M_0.
\end{cases}
\]

Recall that \(B_{d_x}(x, \varepsilon)\) is the ball inside \(\mathcal{F}(x)\), centered in the point \(x\) and with radius \(\varepsilon\) with respect to the metric \(d_x\). Observe that, a priori, \(\Delta(x)\) is a measurable function but it is not immediately true that \(\Delta(x) < \infty\) for \(\mu\)-almost every \(x\). Also note that,

\[f_\ast \mu_x = \mu_{f(x)} \quad \text{and} \quad f(B_{d_x}(x, \varepsilon)) = B_{d_{f(x)}}(f(x), \varepsilon),\]

since

\[d_{f(x)}(f(x), f(y)) = d_x(x, y),\]

we conclude that \(\Delta(x)\) is \(f\)-invariant map. By ergodicity of \(f\) it follows that \(\Delta(x)\) is constant almost everywhere, let us call that constant by \(\Delta\), that is for almost every \(x\):

\[
\Delta(x) = \Delta.
\]

Let \(\hat{M} \subset M_0\) be a Borel \(f\)-invariant full measure set of points \(x\) for which \((6)\) occurs.

5.1. Technical Lemmas for the case \(\Delta = \infty\).

**Lemma 5.2.** If \(\Delta = \infty\), there exists a sequence \(\varepsilon_k \to 0\), as \(k \to +\infty\), and a full measure subset \(R^\infty \subset \hat{M}\) such that

i) \(R^\infty\) is \(f\)-invariant;
ii) for all \(x \in R^\infty\) we have

\[
\frac{\mu_x(B_{d_x}(x, \varepsilon_k))}{2\varepsilon_k} \geq k.
\]

\(^4\text{see Lemma 4.3 and recall that }\mu(M_0) = 1.\)
Proof. Let \( k \in \mathbb{N}^* \) arbitrary. Since \( \Delta(x) = \Delta \) for every \( x \in \hat{M} \), define
\[
\epsilon_k(x) := \sup \left\{ \epsilon \leq 1 : \frac{\mu_x(B_{\delta_k}(x, \epsilon))}{2\epsilon} \geq k \right\}, \quad x \in \hat{M}.
\]

Claim: The function \( \epsilon_k(x) \) is measurable for all \( k \in \mathbb{N} \).

Proof. Define
\[
w(x, \epsilon) = \frac{\mu_x(B_{\delta_k}(x, \epsilon))}{2\epsilon}.
\]

By Corollary 4.5, for any \( x \in M_0 \) the function \( w(x, \cdot) : (0, r) \rightarrow (0, \infty) \) is continuous and, for \( 0 < \epsilon < r \) fixed the function \( w(\cdot, \epsilon) : M_0 \rightarrow (0, \infty) \) is measurable function by Proposition 4.2. Given any \( k \in \mathbb{N}, \beta > 0 \), the continuity of \( w(x, \cdot) \) implies that
\[
\epsilon_k^{-1}((0, \beta)) = \{ x : \epsilon_k(x) \in (0, \beta) \}
\]
\[
= \bigcap_{\beta \leq r \leq 1} w(\cdot, r)^{-1}([0, k))
\]
\[
= \bigcap_{\beta \leq r \leq 1, r \in \mathbb{Q}} w(\cdot, r)^{-1}([0, k)).
\]

Therefore \( \epsilon_k^{-1}((0, \beta)) \) is measurable, as it is a countable intersection of measurable subsets of \( M_0 \), and consequently \( \epsilon_k \) is a measurable function for every \( k \).

Note that \( \epsilon_k(x) \) is \( f \)-invariant. Thus, by ergodicity, for every \( k \in \mathbb{N} \) the function \( \epsilon_k \) is constant almost everywhere, let \( R_\epsilon^\infty \) be a full measure set such that \( \epsilon_k(x) \) is constant equal to \( \epsilon_k \). It is easy to see that the sequence \( \epsilon_k \) goes to 0 as \( k \) goes to infinity. Take \( \hat{R}_{\epsilon}^\infty := \bigcap_{k=1}^{\infty} R_\epsilon^\infty \). Since each \( R_\epsilon^\infty \) has full measure, \( \hat{R}_{\epsilon}^\infty \) has full measure and clearly satisfies what we want for the sequence \( \{ \epsilon_k \} \). Finally, take \( R_{\epsilon}^\infty = \bigcap_{i \in \mathbb{Z}} f^i(\hat{R}_{\epsilon}^\infty) \). The set \( R_{\epsilon}^\infty \) is \( f \)-invariant, has full measure and satisfies (i) and (ii).

We now set, for each \( U \in \mathcal{U}, x \in U \setminus (Z_U \cup A_U) \),
\[
\Pi_{\epsilon_k}^{\infty}_{x, U} := \left\{ y \in \mathcal{F}|U(x) \setminus Z_U : \frac{\mu_y(B_{\delta_k}(y, \epsilon_k))}{\mu_y(B_{\delta_k}(y, r))} \geq k, \forall k \text{ with } B_{\delta_k}(y, \epsilon_k) \subset U \right\},
\]
and
\[
\Pi_{\epsilon_k}^{\infty} := \bigcup_{x \in U \setminus (Z_U \cup A_U)} \Pi_{\epsilon_k}^{\infty}_{x, U}.
\]

Observe that if \( x \in R_{\epsilon}^\infty \) then \( x \in \Pi_{\epsilon_k}^{\infty}_{x, U} \) therefore \( R_{\epsilon}^\infty \cap U \subset \Pi_{\epsilon_k}^{\infty} \). In particular \( U \setminus \Pi_{\epsilon_k}^{\infty} \subset U \setminus R_{\epsilon}^\infty \). Since \( \mu(R_{\epsilon}^\infty) = 1 \) then \( \Pi_{\epsilon_k}^{\infty} \) is measurable. Also we can clearly assume that \( \epsilon_k \) is strictly decreasing and \( \epsilon_1 < r \).

Lemma 5.3. For every \( x \in R_{\epsilon}^\infty \cap U \), consider \( \delta = \delta(x) > 0 \) for which
\[
B_{\delta_k}(x, 2 \cdot \delta + r) \subset U.
\]
The set \( \Pi_{\epsilon_k}^{\infty}_{x, U} \cap B_{\delta_k}(x, \delta) \) is a closed subset on the plaque \( \mathcal{F}|U(x) \).
Proof. Let \( y_n \to y, y_n \in \Pi_{x,U}^\infty \cap B_{d_e}(x, \delta), y \in F|U(x) \). In particular, \( B_{d_e}(y_n, \epsilon) \subset B_{d_e}(x, \delta + \epsilon) \subset U \) and by taking the limit over \( n \) we also have \( B_{d_e}(y, \epsilon) \subset B_{d_e}(x, \delta + \epsilon) \subset U \). Furthermore, it is clear that \( y \in B_{d_e}(x, \delta) \) since this is a closed set. By Lemma 4.1 for each \( k \in \mathbb{N} \) the map
\[
y \in B_{d_e}(x, \delta) \subset F_\epsilon(x) \mapsto \mu_y^U(B_{d_e}(y, \epsilon_k)),
\]
is continuous and the same holds for
\[
y \in B_{d_e}(x, \delta) \subset F_\epsilon(x) \mapsto \mu_y^U(B_{d_e}(y, \epsilon_k)).
\]
Thus,
\[
\lim_{n \to \infty} \frac{\mu_y^U(B_{d_e}(y_n, \epsilon_k))}{\mu_y^U(B_{d_e}(y, \epsilon_k))} = \frac{\mu_y^U(B_{d_e}(y, \epsilon_k))}{\mu_y^U(B_{d_e}(y, \epsilon_k))}, \quad k \geq 1,
\]
which implies that for all \( k \geq 1 \) we have
\[
\frac{\mu_y^U(B_{d_e}(y, \epsilon_k))}{\mu_y^U(B_{d_e}(y, \epsilon_k))} = \lim_{n \to \infty} \frac{\mu_y^U(B_{d_e}(y_n, \epsilon_k))}{\mu_y^U(B_{d_e}(y_n, \epsilon_k))} \geq k,
\]
that is, \( y \in \Pi_{x,U}^\infty \) as we wanted.

Now we consider the following sets
\[
D_U^\infty := F|U(\Pi_U^\infty) \setminus (F|U)(Z_U).
\]
We claim that \( D_U^\infty \) is a measurable subset. In fact, consider the natural projection \( \pi : U \to U/F \), as \( U \) is an open subset of a manifold (in particular it is a Polish space) we have that \( U/F \) is a Polish space with the quotient topology. Since \( Z_U = \chi_U \cap (U \setminus A_U) \), where \( \chi_U \) is a Borel subset and \( U \setminus A_U \) is \( F|U \)-saturated, then
\[
\pi(Z_U) = \pi(\chi_U) \cap \pi(U \setminus A_U),
\]
where \( \pi(\chi_U) \) is a Souslin set\(^5\) by [5] Corollary 1.10.9, therefore
\[
F|(Z_U) = \pi^{-1}(\pi(\chi_U) \cap \pi(U \setminus A_U)) = \pi^{-1}(\pi(\chi_U)) \cap (U \setminus A_U),
\]
is a measurable set.

Since \( R^\infty \cap U \subset F|U(\Pi_U^\infty) \) and \( \mu(R^\infty) = 1 \) we have that \( F|U(\Pi_U^\infty) \) is a measurable subset of \( U \), this implies that \( D_U^\infty = F|U(\Pi_U^\infty) \setminus F|U(Z_U) \) is a measurable set as we wanted to show.

Since \( D_U^\infty \) is a measurable, by ergodicity of \( f \) the \( f \)-invariant set:
\[
D^\infty := \bigcup_{n \in \mathbb{Z}, U \in U} f^n \left( \bigcup D_U^\infty \right),
\]
must satisfy either \( \mu(D^\infty) = 0 \) or \( \mu(D^\infty) = 1 \).

\(^5\) A subset of a Polish space \( Y \) is called a Souslin set, or an analytical set, if it is the image of a Polish space \( X \) by a continuous map from \( X \) to \( Y \).
5.2. **Technical Lemmas for the case** $\Delta < \infty$.

**Lemma 5.4.** If $\Delta < \infty$, there exists a sequence $\epsilon_k \to 0$, as $k \to +\infty$, and a full measure subset $R \subset \hat{M}$ such that

i) $R$ is $f$-invariant;

ii) for every $x \in R$, then

\[ | \frac{\mu_x(B_d(x, \epsilon_k))}{2\epsilon_k} - \Delta | \leq \frac{1}{k}. \]

**Proof.** The proof is very similar to the proof of Lemma 5.2. Let $k \in \mathbb{N}^*$ arbitrary. Since $\Delta(x) = \Delta$ for every $x \in \hat{M}$ define

$\epsilon_k(x) := \sup \left\{ \epsilon : \left| \frac{\mu_x(B_d(x, \epsilon))}{2\epsilon} - \Delta \right| \leq \frac{1}{k} \right\}.$

Observe that such $\epsilon_k$ exists because since the lim sup is $\Delta$ we can take a sequence $\epsilon_l \to 0$ such that the ratio given approaches $\Delta$.

**Claim:** The function $\epsilon_k(x)$ is measurable for all $k \in \mathbb{N}$.

**Proof.** Define

\[ w(x, \epsilon) = \frac{\mu_x(B_d(x, \epsilon))}{2\epsilon}. \]

As observed in the proof of Lemma 5.2 $r \mapsto w(x, r)$ is continuous and $x \mapsto w(x, r)$ is measurable. Given any $k \in \mathbb{N}$, $k > 0$, the continuity of $w(x, \cdot)$ implies that

\[ \epsilon_k^{-1}((0,\beta)) = \{ x : \epsilon_k(x) \in (0,\beta) \} \]

\[ = \bigcap_{\beta \leq r \leq 1} w(\cdot, r)^{-1}\left( [\Delta + \frac{1}{k}, \infty) \right) \cup w(\cdot, r)^{-1}\left( [0, \Delta - \frac{1}{k}) \right) \]

\[ = \bigcap_{\beta \leq r \leq 1, r \in \mathbb{Q}} w(\cdot, r)^{-1}\left( [\Delta + \frac{1}{k}, \infty) \right) \cup w(\cdot, r)^{-1}\left( [0, \Delta - \frac{1}{k}) \right). \]

Therefore $\epsilon_k^{-1}((0,\beta))$ is measurable, as it is a countable intersection of measurable sets, and consequently $\epsilon_k$ is a measurable function for every $k$. \qed

As $\epsilon_k(x)$ is $f$-invariant, by ergodicity we may take the full measure set $R_k$ where $\epsilon_k(x)$ is constant equal to $\epsilon_k$. The sequence $\epsilon_k$ goes to 0 as $k$ goes to infinity, so we set $\hat{R} := \bigcap_{k=1}^{\infty} R_k$. Since each $R_k$ has full measure, $\hat{R}$ has full measure and clearly satisfies what we want for the sequence $\{\epsilon_k\}_k$. The set $R = \bigcap_{l \in \mathbb{Z}} f^l(\hat{R})$ is $f$-invariant, has full measure and satisfies (i) and (ii) as we wanted. \qed

Similar to the definitions made in section 5.1 we set

\[ \Pi_U := \bigcup_{x \in U \setminus (\mathcal{Z} \cup \mathcal{A}_U)} \Pi_{x, U}. \]
where

\[
\Pi_{x,U} := \left\{ y \in \mathcal{F}|U(x) \setminus \mathcal{Z}_U \mid \frac{1}{2\varepsilon_k} \cdot \frac{\mu_y^U(B_d(y, \varepsilon_k))}{\mu_y^U(B_d(y, r))} - \Delta \leq \frac{1}{k}, \forall k \text{ with } B_{d_x}(y, r) \subset U \right\}.
\]

**Lemma 5.5.** For every \( x \in \mathcal{R} \cap U \), consider \( \delta(x) > 0 \) for which

\[
B_{d_x}[x, 2 \cdot \delta + r] \subset U.
\]

The set \( \Pi_{x,U} \cap B_{d_x}[x, \delta] \) is a closed subset on the plaque \( \mathcal{F}|U(x) \).

**Proof.** Identical to the proof of Lemma 5.3. \( \square \)

Similar to the definition made in section 5.1, we consider the set

\[
D_U := \mathcal{F}|U(\Pi_U) \setminus (\mathcal{F}|U)(\mathcal{Z}_U),
\]

and

\[
D := \bigcup_{n \in \mathbb{Z}, U \in \mathcal{U}} f^n \left( \bigcup D_U \right).
\]

Similarly \( D_U \) is measurable for all \( U \in \mathcal{U} \) and again by ergodicity we have \( \mu(D) = 0 \) or \( \mu(D) = 1 \).

After proving the auxiliary lemmas for \( \Delta^\infty \) (resp. \( \Delta \)) and obtaining the sets \( D \) (resp. \( D^\infty \)) we divide the next part of the proof into four cases.

5.3. **Case 1: \( \Delta < \infty \) and \( \mu(D) = 0 \).

In this case we will show that the support of the conditional measures is a Cantor set for almost every \( x \in \mathcal{M} \).

As fixed in the beginning, consider \( \{\omega_x\}_x \) the disintegration of \( \mu \) along \( \mathcal{F} \) and consider \( \mathcal{G} \) a full measure \( \mathcal{F} \)-saturated set of points where

\[
f_j^1 \omega_x = \omega_{f^j(x)}, \quad \forall j \in \mathbb{Z}.
\]

Let \( \mathcal{G}^U := \{ x \in U : \mu_x^U = \omega_x(\cdot|\mathcal{F}|U(x)) \} \cap \{ x : \mu_x^U \text{ is non-atomic} \} \).

Consider:

- \( \Phi_{1/n}^U (Z_U) := \{ x \in U : d_x(x, Z_U) < 1/n \} \),
- \( \mathcal{E}_n = \bigcup_j \mathcal{G} \cap f^j(\Phi_{1/n}^U(Z_U) \cap \mathcal{G}^U) \).

As \( Z_U = \chi_U \cap (U \setminus \mathcal{A}_U) \) we have

\[
\Phi_{1/n}^U (Z_U) = \{ x \in M : d_x(x, \mathcal{R}_U) < 1/n \} \cap (U \setminus \mathcal{A}_U),
\]

which is measurable for \( n \geq \hat{n} \), for some \( \hat{n} \in \mathbb{N} \) not depending on \( U \), by Lemma 3.10 since \( \chi_U \) is a Borel subset of \( U \). Therefore the set of the second item is a \( f \)-invariant measurable subset of \( M \), thus it either has full or null measure.

Now, again we separate two cases:
Case 3.1: Assume that for all \( n \geq \hat{n}, \mu(\mathcal{E}_n) = 1 \). Then,

\[
\mathcal{E}^U_n = \mathcal{G}^U \cap \left( \bigcup_j \mathcal{G} \cap f^j(\Phi_{1/n}^U(Z_U) \cap \mathcal{G}^U) \right),
\]

has full measure in \( U \) for every \( n \geq \hat{n} \). For \( z \in \mathcal{E}^U = \bigcap_{n \geq \hat{n}} \mathcal{E}^U_n \), let \( n_0 > 0 \) such that for \( n \geq n_0 \geq \hat{n} \) we have \( B_{d_1}(z, 1/n_0) \subset U \). For \( n \geq n_0 \), let \( j \) with

\[
f^{-j}(z) \in \Phi_{1/n}^U(Z_U) \cap \mathcal{G}^U,
\]

and \( p \in Z_U \) with \( d(z, f^{-j}(z)) < 1/n \). Then \( d_x(f^j(p), z) < 1/n \), which implies \( f^j(p) \in U \), and if \( \mu_p^U(B_{d_1}(p, \delta)) = 0 \), for \( \delta \) small, then since \( \mu_p^U \sim \omega_{f^{-j}(z)} \) (because \( f^{-j}(z) \in \mathcal{G}^U \) and \( z \in \mathcal{G} \)), it follows that \( \omega_z(f^{-j}(I_p)) = \omega_{f^j(z)}(I_p) = 0 \) and \( z \in \mathcal{G}^u \), thus \( \mu_p^U(f^{-j}(I_p)) = 0 \).

Therefore, \( f^j(p) \in Z_U \) and \( z \in \Phi_{1/n}(Z_U) \). That is, the set \( Z_U \cap \mathcal{F}|U(z) \) is dense in \( \mathcal{F}|U(x) \), for almost every \( z \in \mathcal{E}^U \).

Claim: For \( x \in \mathcal{E}^U \) then \( \mathcal{C}_x := \mathcal{F}|U(x) \cap Z_U \) is a Cantor set in \( \mathcal{F}|U(x) \).

Proof. To prove that \( \mathcal{C}_x \) is a Cantor set, we will show that this set is a nowhere dense and perfect set. Since \( Z_U \cap \mathcal{F}|U(x) \) is a dense and open set in \( \mathcal{F}|U(x) \) we have that \( \mathcal{C}_x \) is a nowhere dense and closed set. Now let us see that \( \mathcal{C}_x \) has no isolated points. Let \( y \in \mathcal{C}_x \), suppose that there exist \( r > 0 \) with \( B_{d_1}(y, r) \subset U \) and \( B_{d_1}(y, r) \cap \mathcal{C}_x = \{y\} \). Since \( y \in \mathcal{C}_x \),

\[
0 < \mu_p^U(B_{d_1}(y, r)) = \mu_p^U(B_{d_1}(y, r) \setminus \mathcal{C}_x) + \mu_p^U(\{y\}) = \mu_p^U(\{y\}),
\]

thus \( \mu_p^U(\{y\}) > 0 \), which is a contradiction since \( x \notin A_U \).

Therefore \( \mathcal{C}_x \) is indeed a Cantor subset. \( \square \)

Thus, for almost every \( x \in M \) and any local chart \( U \) the conditional measures \( \mu_p^U \) is supported in the Cantor set \( \mathcal{C}_x \). We remark that for the Claim we did not use the fact that \( \Delta = \infty \), thus the same argument will work when \( \Delta < \infty \).

Case 3.2: If, on the other hand, there exists \( N_0 \in \mathbb{N} \) with \( N_0 \geq \hat{n} \) such that \( \mu(\mathcal{E}_{N_0}) = 0 \), then \( \mu(\mathcal{E}_N) = 0 \) and moreover \( \mu(\mathcal{E}_N^U) = 0 \), for any \( N \geq N_0 \). Since

\[
\mathcal{G}^U \cap \mathcal{G} \cap \Phi_{1/N}(Z_U) \subset \mathcal{E}^U_{N_0},
\]

we have

\[
\mu(\Phi_{1/N}^U(Z_U)) = \mu(\mathcal{G}^U \cap \mathcal{G} \cap \Phi_{1/N}(Z_U)) \leq \mu(\mathcal{E}^U_{N_0}) = 0, \quad \forall N \geq N_0.
\]

In particular, for almost every \( x \in U \) we have

\[
\mu_x^U(\Phi_{1/N}^U(Z_U \cap \mathcal{F}|U(x))) = 0.
\]

(9)
As $\Phi_{1/N}^U(Z_U \cap F|U(x)))$ is an open subset of $F|U(x))$, by (9) we have, for almost every $x \in U$,
$$
\Phi_{1/N}^U(Z_U \cap F|U(x))) \subset Z_U \cap F|U(x)).
$$
But clearly the other continence holds, thus $Z_U \cap F|U(x) = \Phi_{1/N}^U(Z_U \cap F|U(x)))$ and this implies $Z_U \cap F|U(x))) = F|U(x))$. As this happens for almost every $x \in U$ we fall in contradiction with the fact that $\mu(Z_U \cap U) = 0$. Therefore this case does not happen.

5.4. Case 2: $\Delta = \infty$ and $\mu(D_\infty^o) = 0$.

In particular for every $U \in \mathcal{U}$ we must have $\mu(D_\infty^o) = 0$ which implies $\mu(F|U(Z_U))) = \mu(U)$.

In this case we will proceed very similarly to the previous Case. For $U \in \mathcal{U}$, consider the sets $\mathcal{E}_n$ and $\mathcal{E}_n^U$ defined in Case 1.

Again, if $\mu(\mathcal{E}_n) = 1$ for every $n \in \mathbb{Z}$, then there exists a full measure subset of $U$, namely $\mathcal{E}_n^U$, such that $z \in \mathcal{E}_n^U$ implies $Z_U \cap F|U(z)$ is dense is $F|U(z)$. Hence, as showed by the Claim in Case 1, it follows that the support of $\mu_x^U$ is a Cantor subset of the plaque $F|U(x)$ for almost every $x \in U$.

Otherwise, if $\mu(\mathcal{E}_{N_0}) = 0$ for some $N_0 \in \mathbb{N}$, then as in Case 1 we conclude that $Z_U \cap F|U(x) = F|U(x)$ contradicting the fact that $\mu(Z_U \cap U) = 0$, thus this case does not occur.

5.5. Case 3: $\Delta = \infty$ and $\mu(D_\infty) = 1$.

Let us prove that this case cannot occur. Since $\mu(D_\infty^o) = 1$, there exists $U \in \mathcal{U}$ with $\mu(D_\infty^o) > 0$.

Since $\mu(F|U(\Pi_\infty^o)) = \mu(U)$, for almost every point $\bar{x} \in D_\infty^o$ we have
$$
\mu_{\bar{x}}^U(\Pi_\infty^o \cap F|U(x))) = 1.
$$

Take any such typical $\bar{x}$ and consider $x \in F|U(\bar{x}) \cap \Pi_\infty^o$, in particular $B_{d_\bar{x}}(x, r) \subset U$. Also, $\Pi_{\bar{x}, U}^\infty \cap B_{d_\bar{x}}(x, \delta)$ is closed in $F|U(x))$ for some $\delta > 0$ small, if there exists $z \in B_{d_\bar{x}}(x, \delta) \cap \Pi_{\bar{x}, U}^\infty \cap B_{d_\bar{x}}(x, \delta)$ then for some $\delta_2 > 0$ we have $B_{d_\bar{x}}(z, \delta_2) \subset B_{d_\bar{x}}(x, \delta) \cap \Pi_{\bar{x}, U}^\infty \cap B_{d_\bar{x}}(x, \delta)$ and $\mu_{\bar{x}}^U(B_{d_\bar{x}}(z, \delta_2)) = 0$ by (10).

But this cannot happens since this would imply $z \in Z_U$ and, consequently, $F|U(Z_U) \cap D_\infty^o \neq \emptyset$, falling in contradiction with the definition of $D_\infty^o$. Therefore
$$
\Pi_{\bar{x}, U}^\infty \cap B_{d_\bar{x}}(x, \delta) = B_{d_\bar{x}}(x, \delta).
$$

Consider $0 < r_0 < \delta$ small enough so that $B_{d_\bar{x}}(x, r + 2 \cdot r_0) \subset U$. By hypothesis, for $k$ with $\varepsilon_k < r_0$, by Lemma 3.4 we can take $[r_0/\varepsilon_k]$ disjoint balls of radius $\varepsilon_k$ inside $B_{d_\bar{x}}(x, r_0)$, say with center $a_1, a_2, \ldots, a_{[r_0/\varepsilon_k]}$. Then
$$
\sum_{i=1}^{[r_0/\varepsilon_k]} \mu_{\bar{x}}^U(B_{d_\bar{x}}(a_i, \varepsilon_k)) \leq \mu_{\bar{x}}^U(B_{d_\bar{x}}(x, r_0))
$$
\[
\Rightarrow \sum_{i=1}^{[r_0/\varepsilon_k]} \mu_{\bar{x}}^U(B_{d_\bar{x}}(a_i, \varepsilon_k)) \leq \mu_{\bar{x}}^U(B_{d_\bar{x}}(x, r_0))
\]
for every $w$ is continuous, hence there exists $\eta > 0$ such that

$$\frac{\mu_U(B_d, (a_i, r))}{\mu_U(B_d, (x, r))} \leq \frac{\mu_U(B_d, (a_i, r_k))}{\mu_U(B_d, (x, r_k))},$$

By Lemma 4.1

$$w \in B_d, r_0 \subset F, \lim_{\epsilon \to 0} \mu(U, a_k) = \lim_{\epsilon \to 0} \mu(U, (a_k, r_0)).$$

is continuous, hence there exists $\eta > 0$ such that

$$\frac{\mu_U(B_d, (w, r))}{\mu_U(B_d, (x, r))} \geq \eta,$$

for every $w \in B_d, x, r_0$. Therefore, since $a_i \in B_d, x, \epsilon$ for all $i$, we have

$$\eta \cdot (2\epsilon_k \cdot k) \cdot \frac{|r_0|}{\epsilon_k} < \eta \cdot \frac{\sum_{i=1}^{\lfloor r^0/\epsilon_k \rfloor} \mu_U(B_d, (a_i, \epsilon_k))}{\mu_U(B_d, (x, r))} \leq \frac{\sum_{i=1}^{\lfloor r^0/\epsilon_k \rfloor} \mu_U(B_d, (a_i, \epsilon_k))}{\mu_U(B_d, (x, r))} \leq \frac{\mu_U(B_d, (x, r_0))}{\mu_U(B_d, (x, r))}.$$

Taking $k \to \infty$, the left side goes to infinity from where we conclude that $\mu_U(B_d, (x, r_0))$ falls in contradiction. Thus, this case does not occur.

5.6. **Case 4:** $\Delta < \infty$ and $\mu(D) = 1$.

We will prove that if this case occurs then for almost every $x \in U, U \in \mathcal{U}$, the conditional measure in $\mathcal{F}|U(x)$ is equivalent to the measure $\lambda_x$ given in definition 3.11

**Lemma 5.6.** The constant $\Delta$ is bounded away from zero and

$$\mu_X \ll \lambda_x,$$

for $\mu$-almost every $x \in M$.

**Proof.** Let $y \in D$. Then, for some $n_0 \in \mathbb{Z}$ and $U \in \mathcal{U}$ we have $f^{n_0}(y) \in D_U$. Call $x = f^{n_0}(y)$. As $x \in \mathcal{F}|U(P_n) \setminus \mathcal{F}|U(Z_\mathcal{U})$ we have $\mu_U(\mathcal{F}|U(P_n) \cap \mathcal{F}|U(x)) = 1$. Therefore we conclude (by the same argument used in Case 3 - Section 5.5) that

$$\Pi_{x, U} \cap B_d, x, \delta = B_d, x, \delta,$$

for some $\delta$ small. Consequently $\Pi_{x, U} \cap \mathcal{F}|U(x) \supset \{y \in \mathcal{F}|U(x) : d_x(y, \partial \mathcal{F}|U(x)) \geq r\}$.

For any given $k \in \mathbb{N}^+, U \in \mathcal{U}$ and $n \in \Pi_{x, U}$ we have

$$\frac{\mu_U(B_d, (x, \epsilon_k))}{\mu_U(B_d, (x, \epsilon_k))} - \Delta \leq \frac{1}{k}.$$

Given $\epsilon > 0$ take $k_0 \in \mathbb{N}$ such that $k_0^{-1} < \epsilon$. Again since $\{d_x\}$ is a $\mathcal{F}$-metric system, given a constant $r > 0$ we need at most $s(k) = \lfloor r/\epsilon_k \rfloor + 1$ points, say $a_1, a_2, ..., a_s(k)$, to cover the ball
Since limits can be taken to be arbitrarily small in the beginning, it follows that \( \varepsilon_i \). In particular, by taking the limit along the subsequence \( \beta \) exists. By Lemma 5.6 we know that

\[
\frac{\mu^U_x(B_d(x,r))}{\mu^U_x(B_d(x,\varepsilon_r))} \leq s(k) \frac{\mu^U_x(B_d(x,\varepsilon_k))}{\mu^U_x(B_d(x,\varepsilon_r))} = \sum_{i=1}^{s(k)} \frac{\mu^U_x(B_d(a_i,\varepsilon_k))}{\mu^U_x(B_d(a_i,\varepsilon_r))} \leq \beta \cdot \sum_{i=1}^{s(k)} \frac{\mu^U_x(B_d(a_i,\varepsilon_k))}{\mu^U_x(B_d(a_i,\varepsilon_r))} \leq \beta \sum_{i=1}^{s(k)} \frac{2\varepsilon_k}{k} + \Delta \varepsilon_k
\]

Thus \( \mu^U_x(B_d(x,\varepsilon_k)) \) goes to zero as \( k \to \infty \), we have

\[
\frac{\mu^U_x(B_d(x,\varepsilon_k))}{\mu^U_x(B_d(x,\varepsilon_r))} \leq 2\Delta \cdot \beta \cdot r.
\]

Therefore \( \mu^U_x << \lambda_x \) when restricted to \( \{ y \in \mathcal{F}|U(x) : d_x(y, \partial \mathcal{F}|U(x)) \geq r \} \) and \( \Delta > 0 \). As \( \tau \) can be taken to be arbitrarily small in the beginning, it follows that \( \mu^U_x << \lambda_x \) as we wanted to show.

Next we are able to conclude that \( \mu^U_x \) is equivalent to the measure \( \lambda_x \).

**Lemma 5.7.** For \( \mu \) almost every \( x \in M \)

\[
\mu^U_x \sim \lambda_x.
\]

**Proof.** By Lemma 5.6 we know that \( \mu^U_x << \lambda_x \). Since \( \lambda_x \) is a doubling measure we have that the Radon-Nikodym derivative \( d\mu^U_x/d\lambda_x \) exists and is given at \( \lambda_x \)-almost every point \( y \in \{ y \in \mathcal{F}|U(x) : d_x(y, \partial \mathcal{F}|U(x)) \geq r \} \) by

\[
\frac{d\mu^U_x}{d\lambda_x}(y) = \lim_{r \to 0} \frac{\mu^U_x(B_d(y,r))}{\lambda_x(B_d(y,r))}.
\]

In particular, by taking the limit along the subsequence \( \varepsilon_i, k \to \infty \), we conclude that

\[
\frac{d\mu^U_x}{d\lambda_x}(y) = \lim_{k \to \infty} \frac{\mu^U_x(B_d(y,\varepsilon_k))}{\lambda_x(B_d(y,\varepsilon_k))}, \lambda_x \text{-a.e. } y \in \{ y \in \mathcal{F}|U(x) : d_x(y, \partial \mathcal{F}|U(x)) \geq r \},
\]

which implies

\[
\frac{d\mu^U_x}{d\lambda_x}(y) = \beta(y) \cdot \Delta, \lambda_x \text{-a.e. } y \in \{ y \in \mathcal{F}|U(x) : d_x(y, \partial \mathcal{F}|U(x)) \geq r \},
\]

where \( \beta(y) = \mu^U_x(B_d(y,r)) \). Since \( \beta \) is a continuous function, given any compact \( I \subset \{ y \in \mathcal{F}|U(x) : d_x(y, \partial \mathcal{F}|U(x)) \geq r \} \) we have

\[
\beta_1 \Delta \leq \frac{d\mu^U_x}{d\lambda_x}(y) \leq \beta_2 \Delta, \lambda_x \text{ a.e } y \in I.
\]
Then,

\[ \beta_1 \Delta \cdot \lambda_x \leq \mu_x, \, \lambda_x \text{ a.e } y \in I. \]

In particular, if \( \mu^U_x(E) = 0 \) then, for any compact subset \( I \subset \{ y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geq \tau \} \) we have \( \lambda_x(E \cap I) = 0. \) Since \( \mathcal{F}(x) \) may be written as a countable union of increasing compact subsets, we conclude that \( \lambda_x(E \cap \{ y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geq \tau \}) = 0. \) Again, since \( \tau \) may be taken to be arbitrarily small we conclude that \( \lambda_x(E) = 0, \) thus \( \lambda_x \ll \mu^U_x \) as we wanted to show. \( \square \)

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