On Particles Creation and Renormalization in a Cosmological Model with a Big Rip

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Abstract. An exact solution is obtained for a massive scalar field conformally coupled with the curvature in a cosmological model with the scale factor $a(t) = a_0/|t|$, corresponding to background matter with the equation of state $p = -5\xi/3$. An expression for the number density of created particles is obtained, and its behavior is studied as the model approached the instant of a Big Rip. Renormalization of the energy-momentum tensor is considered, and it is shown that back reaction of the quantum effects of a conformally coupled scalar field on the space-time metric can be neglected if the field mass is much smaller than the Planck mass and if the time left to the Big Rip is greater than the Planck time.

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1. Introduction

Particle creation in curved space-time has been actively studied since the 70s of last century. Results were obtained which may have important applications in cosmology and astrophysics (see [1]). In particular, creation of particles with mass on the Grand Unification scale by the gravitational field of the early Universe may be used for explaining the observed density of visible and dark matter [2] and be related to observations of superhigh-energy cosmic rays [3].

In connection with the recent discovery of the accelerated expansion of the Universe, it is of interest to study quantum effects in the cosmological background determined by matter with negative pressure, in particular, phantom matter. In this paper, we consider particle creation in cosmological model that admits an exact solution of the scalar field equation. This exact solution makes it possible to analyze the behavior of expressions for the number density of created particles as the cosmological time tends to the instant of a Big Rip. We also consider renormalization of the energy-momentum tensor (EMT) and study the back reaction of quantum effects of the scalar field on the space-time metric.

We use the system of units in which $\hbar = c = 1$. The signs of the Riemann and Ricci tensors are chosen in such a way that $R^{ijkl} = \partial_i \Gamma^l_{jk} - \partial_k \Gamma^l_{ij} + \Gamma^l_{il} \Gamma^m_{jk} - \Gamma^l_{ik} \Gamma^m_{jl}$, where $\Gamma^m_{ij}$ are the Christoffel symbols.

2. Scalar field in a homogeneous isotropic space

Consider a complex scalar field $\varphi(x)$ of mass $m$ with the Lagrangian

$$L(x) = \sqrt{|g|} \left[ g^{ik} \partial_i \varphi^* \partial_k \varphi - (m^2 + \xi_c R) \varphi^* \varphi \right]$$

and the corresponding equation of motion

$$(\nabla^i \nabla_i + m^2 + \xi_c R) \varphi(x) = 0,$$

where $\nabla_i$ are covariant derivatives in $N$-dimensional space-time with the metric $g_{ik}$, $R$ is the scalar curvature, $\xi_c = (N - 2)/(4(N - 1))$ ($\xi_c = 1/6$ for $N = 4$), $g = \det(g_{ik})$. Eq. (2) is conformally invariant if $m = 0$.

For homogeneous isotropic space-time with the metric

$$ds^2 = dt^2 - a^2(t) dl^2 = a^2(\eta) (d\eta^2 - dl^2),$$

where $dl^2$ is the metric of $(N - 1)$-dimensional space-time with constant curvature $K = 0, \pm 1$, the full set of solutions to Eq. (2) can be found in the form

$$\varphi(x) = a^{-(N-2)/2}(\eta) g_{\lambda}(\eta) \Phi_J(x),$$

where

$$g''_{\lambda}(\eta) + \omega^2(\eta) g_{\lambda}(\eta) = 0,$$

$$\omega^2(\eta) = m^2 a^2(\eta) + \lambda^2,$$

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\[ \Delta_{N-1} \Phi_J(x) = -\left(\lambda^2 - \left(\frac{N-2}{2}\right)^2 K\right) \Phi_J(x), \]  
\[ J \text{ is the set of indices (quantum numbers) enumerating the eigenfunctions of the Laplace-Beltrami operator } \Delta_{N-1} \text{ in } (N - 1)-\text{dimensional space.} \]

In accordance with the Hamiltonian diagonalization method [1], the functions \( g_\lambda(\eta) \) should satisfy the following initial conditions [4]:
\[ g_\lambda'(\eta_0) = i\omega(\eta_0) g_\lambda(\eta_0), \quad |g_\lambda(\eta_0)| = \omega^{-1/2}(\eta_0). \]  

If the quantized scalar field is in a vacuum state for the time instant \( \eta_0 \), then the number density of particle pairs created by the instant \( \eta \), can be calculated (for the quasi-Euclidean metric with \( K = 0 \)) by the formula [11]
\[ n(\eta) = \frac{B_N}{2a^N} \int_0^\infty S_\lambda(\eta) \lambda^{N-2} d\lambda, \]  
where \( B_N = \left[2^{N-3}\pi^{(N-1)/2}\Gamma((N-1)/2)\right]^{-1} \), \( \Gamma(z) \) is the gamma function, and
\[ S_\lambda(\eta) = |g_\lambda'(\eta) - i\omega g_\lambda(\eta)|^2 / (4\omega). \]

As has been shown in [14], \( S_\lambda \sim \lambda^{-6} \), and the integral in [16] converges for \( N < 7 \).

3. A cosmological model with phantom matter

Consider a cosmological model with the background matter having the equation of state \( p = w \varepsilon \), where \( w < -1 \), from the Einstein equations
\[ R_{ik} - \frac{1}{2} g_{ik} R = -8\pi G T_{ik}, \]  
where \( T_{ik} = \text{diag}(\varepsilon, -p, \ldots, -p) \), it follows that, in the metric [3], the energy density of the background matter grows according to the law
\[ \varepsilon(\eta) \sim a^{-(1+w)(N-1)}(\eta). \]

For \( K = 0 \), from [11] we obtain
\[ a = a_0 / (-t)^q = a_1 / (-\eta)^\beta, \]  
where \( t \in (-\infty, 0), \ \eta \in (-\infty, 0), \)
\[ \beta = -\frac{2}{N-3 + w(N-1)}, \quad q = \frac{\beta}{1-\beta}. \]  

For \( w < -1, \ \beta \in (0,1) \), and there is a “Big rip” singularity [5] at \( t \rightarrow -0 \).

For the value \( w = -(N+1)/(N-1) \) (so that \( w = -5/3 \) in four-dimensional space-time), when \( q = 1, \ \beta = 1/2 \). Eq. [13] can be solved exactly in terms of degenerate hypergeometric functions (see (2.1.2.103) in Ref. [6]). The solution satisfying the condition [6] as \( \eta_0 \rightarrow -\infty \), has the form
\[ g_\lambda(\eta) = -2i\eta\sqrt{\lambda} \exp\left(\frac{\pi m^2 a_0^2}{4\lambda} + i(\lambda\eta + a_0)\right) \times \]
\[ \times \Psi\left(1 + \frac{im^2 a_0^2}{2\lambda}, 2;i2i\lambda\right), \]  
where \( \Psi(a, b; z) \) is Tricomi’s degenerate hypergeometric function, and \( a_0 \) is an arbitrary real constant. In what follows, we will suppose that quantized scalar field is in a vacuum state for a time instant \( \eta_0 \rightarrow -\infty \).

Let us introduce \( p_\lambda = \lambda/(ma) \), the physical momentum \( \lambda/a \) measured in the units of \( m \). Making the corresponding substitution in [16], we obtain for the density of particles created by the instant \( t \):
\[ n(t) = m^{N-1} \frac{B_N}{2} \int_0^\infty S_{p_\lambda}(t) p_\lambda^{N-2} dp_\lambda, \]  
where, in agreement with [111] and [115],
\[ S_{p_\lambda}(t) = \frac{p_\lambda e^{\pi mt/(4p_\lambda)}}{16\sqrt{1 + p_\lambda^2}} \left|\left(\sqrt{1 + p_\lambda^2} - p_\lambda\right) \times \right. \]
\[ \times i2mt - 4\right| \Psi\left(1 - \frac{imt}{4p_\lambda}, 2; -i\lambda mt\right) - \]
\[ - mt(mt + i4p_\lambda)\Psi\left(2 - \frac{imt}{4p_\lambda}, 3; -i\lambda mt\right)\right|^2. \]  

Let us find the asymptotic \( S_{p_\lambda}(t) \) for \( t \rightarrow -0 \). From Eq. 6.7.(13) from [7] we obtain, as \( z \rightarrow 0 \),
\[ \Psi(1 + \alpha z, 2; z) = \frac{1}{z}\Gamma(1 + \alpha z) + O(|z \ln z|), \]  
\[ \Psi(2 + \alpha z, 3; z) = \frac{1}{\Gamma(2 + \alpha z)}\left(\frac{1}{z^2} - \alpha\right) + O(|z \ln z|). \]  

Therefore,
\[ S_{p_\lambda}(t) \sim \frac{\left(\sqrt{1 + p_\lambda^2} - p_\lambda\right)^2}{4p_\lambda \sqrt{1 + p_\lambda^2}}, \quad t \rightarrow -0. \]

Let us note that this an asymptotic as \( t \rightarrow -0 \). The main term of the asymptotic \( S_{p_\lambda}(t) \) as \( p_\lambda \rightarrow \infty \) is proportional to \( p_\lambda^{-6} \) and has the form \( S_{p_\lambda}(t) \sim 1/(16p_\lambda^2 m^2 t^2) \).
be presented in the form

\[ \rho \sim \frac{1}{a^2(t)} \int_0^\infty \omega(\lambda) S_\lambda(t) \lambda^2 d\lambda, \]

where \( G^k_i \) is the Einstein tensor, the expressions for the tensors \((1)^H_k^k\) and \((3)^H_k^k\) are given, for example, in [1].

\[ \rho^0_0 \equiv \rho_0 = \frac{1}{\pi^2 \alpha^4(\eta)} \int_0^\infty \omega(\lambda) S_\lambda(\eta) \lambda^2 d\lambda, \]

and \( \rho^\alpha_\beta = -\delta^\alpha_\beta \rho_s \) can be found from [21]. \( \alpha, \beta = 1, 2, 3 \). In the \( N \)-dimensional case, renormalization of the EMT with the aid of the Zel'dovich-Starobinsky \( n \)-wave procedure [8] has been considered in [9].

4. Renormalization of the energy-momentum tensor

In obtaining the renormalized vacuum EMT, let us note that, in the homogeneous, isotropic space with \( K = 0 \), the vacuum EMT is diagonal, \((T^k_i)_{\text{ren}} = \text{diag}(\epsilon_\varphi, -p_\varphi, \ldots, -p_\varphi)\), and its components satisfy the covariant conservation condition

\[ (\epsilon_\varphi)' + \frac{a'(\eta)}{a(\eta)} (N-1)(\epsilon_\varphi + p_\varphi) = 0. \quad (21) \]

For a scalar field conformally coupled to the curvature, the vacuum EMT for \( N = 4 \) and \( K = 0 \) can be presented in the form

\[ (T^k_i)_{\text{ren}} = \tilde{T}^k_i = \frac{m^2G_i^k}{144\pi^2} + \frac{1}{1440\pi^2} \left( (3)^H_i^k - 1 \right) \left( 1 \right)^H_i^k, \]

where \( G^k_i = R^k_i - R g^k_i / 2 \) is the Einstein tensor, the expressions for the tensors \((1)^H_i^k\) and \((3)^H_i^k\) are given, for example, in [1].

\[ \frac{\Delta T^k_i(2)}{T^k_i} = \frac{-1}{18\pi} \left( \frac{m}{M_{Pl}} \right)^2 \]

which is negligibly small for \( m \ll M_{Pl} \). Note that the conclusion that it is possible to neglect the back reaction of created particles on the space-time metric in models with phantom matter was made in [10] in the case of a massless minimally coupled scalar field.

For the third term in the vacuum EMT [22], we obtain

\[ \frac{\Delta T^0_0(3)}{T^0_0} = \frac{1}{18\pi(t M_{Pl})^2}, \quad \frac{\Delta T^\alpha_\beta(3)}{T^\alpha_\beta} = \frac{7}{90\pi(t M_{Pl})^2}. \quad (28) \]

Therefore, the influence of the third polarization term in the vacuum EMT on the space-time metric becomes important only at times smaller than...
Planckian, $|t| < t_{Pl} = 1/M_{Pl}$, before the Big Rip. At such times, it would also be necessary to take into account not only the back reaction of a quantum field on the metric [11] but also effects of quantum gravity itself.

We conclude that the back reaction of quantum effects of a massive, conformally coupled scalar field on the space-time metric, in the cosmological model under consideration, with an exact solution of the field equations, can be neglected in the whole region where one can apply the approach of quantum field theory in curved space-time.

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