Rationality of the instability parabolic and related results

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Abstract

In this paper we study the extension of structure group of principal bundles with a reductive algebraic group as structure group on smooth projective varieties defined over algebraically closed field of positive characteristic. Our main result is to show that given a representation $\rho$ of a reductive algebraic group $G$, there exists an integer $t$ such that any semistable $G$-bundle whose first $t$ Frobenius pullbacks are semistable induces a semistable vector bundle on extension of structure group via $\rho$. Moreover we quantify the number of such Frobenius pullbacks required.

1 Introduction

Let $X$ be a smooth projective variety defined over an algebraically closed field $k$. Fix a very ample line bundle $H$. Let $G$ be a reductive algebraic group defined over $k$. All representations considered in this paper are rational finite-dimensional representations. Recall that a $G$-bundle is semistable with respect to the polarisation $H$ if for any reduction of structure group to a parabolic subgroup $P$ and any dominant character of $P$, the induced line bundle on $X$ has non-positive degree.

Now let $\rho : G \to \text{Gl}(V)$ be a rational representation of $G$ sending the connected component of the centre of $G$ to that of $\text{Gl}(V)$. If the characteristic $k$ is zero and $E$ is a semistable $G$-bundle on $X$ then the induced $\text{Gl}(V)$-bundle is also semistable. From this it follows easily that if the characteristic of the field is “sufficiently large”, then again a semistable $G$-bundle induces a semistable $\text{Gl}(V)$-bundle. This is quantified in [IMP] where in it is shown that if $\text{char } k > \text{ht}(\rho)$, then a semistable $G$-bundle induce a semistable $\text{Gl}(V)$-bundle on extension of structure group. In positive characteristic however, it is not in general true that a semistable $G$ bundle will induce a semistable $\text{Gl}(V)$-bundle. A principal $G$-bundle on $X$ is said to be strongly semistable if all its Frobenius pullbacks are semistable. In char 0, the Frobenius map is just identity and hence the notion of semistability and strong semistability coincide. In Ramanan-Ramanathan [RR], it is shown that a strongly semistable $G$-bundle induces a strongly semistable $\text{Gl}(V)$-bundle. This result is sharpened in the paper of Coiai-Holla [CH] where the authors show that given a representation $\rho$ as before, there exists a non-negative integer $t$ such that if $E$ is any $G$-bundle on $X$ which along with its first $t$ Frobenius pullbacks is semistable, then the induced $\text{Gl}(V)$-bundle is again semistable. This fact is crucial in their proof of boundedness of semistable $G$-bundles with fixed Chern classes. In this paper we give bounds for this $t$, in terms of certain numerical data attached to $G$ and $\rho$. The main ingredient of the proof is the use
of the instability parabolic (also known sometimes as the Kempf’s parabolic) associated to points of the representing space (see [Section 3] for definition). The basic idea is as follows: Let $E$ be a principal $G$-bundle on $X$. Let $k(X)$ denote the function field of $X$. Let $E(G)$ be the group scheme associated to $E$ (see section 2 for definition). Let $E_{Gl(V)}$ denote the induced $Gl(V)$ bundle. Let $E(G)_o$ denote the generic fiber of $E(G)$. It is a group scheme defined over the function field of $X$. Let $P$ be any maximal parabolic in $Gl(V)$. Let $E(Gl(V)/P)$ be the associated $Gl(V)/P$ fiber-space. Again let $E(Gl(V)/P)_o$ denote the generic fiber of $E(Gl(V)/P)$. Then $E(G)_o$ acts on $E(Gl(V)/P)_o$ which is linearized by a suitable very ample line bundle. If $E_{Gl(V)}$ admits a reduction of structure group to this maximal parabolic $P$, then we get a section (canonically) $\sigma$ of $E(Gl(V)/P)$. Restricting to the generic fiber gives a $k(X)$-valued point $\sigma_o$ of $E(Gl(V)/P)_o$. In [RR], it is shown that if either $\sigma_o$ is a semistable point for action described above or its instability parabolic (see [Section 3] for definition), which is in general defined over $k(\tilde{X})$, is actually defined over $k(X)$, then this section (or equivalently this reduction) does not contradict semistability. In char 0, using uniqueness of the instability parabolic and Galois descent, this proves the semistability of the induced bundles. In [CH] it is shown that there exists a non-negative integer $t$ such that for all possible reductions to all the maximal parabolics the instability parabolics of points corresponding to these reductions is actually defined over $k(X)^{\rho^t}$. This can be shown to imply that if $E$ is a semistable principal $G$-bundle with first $t$ frobenius pullbacks semistable, the induced $Gl(V)$-bundle is also semistable. The main aim of this paper is to give bounds for this $t$ in terms of certain numerical data attached to $G$ and $\rho$.

2 Basic definitions and preliminary notions

In this section we set up some notations and recall some of basic definitions and facts which will be used later.

$X$ will always denote a smooth projective variety over a field $k$. Let $G$ be a reductive algebraic group defined over $k$ and let $\mathfrak{g}$ denote its lie algebra. Fix a maximal torus $T \subset G$ and a Borel $B$ containing $T$. Let $X_\ast(T)$ denote the set of 1-parameter subgroups of $T$ and let $X^\ast(T)$ be the character group of $T$. There exists a nondegenerate pairing, denoted $(\cdot, \cdot) : X_\ast(T) \times X^\ast(T) \to \mathbb{Z}$. Let $\Phi \subset X^\ast(T)$ be the set of roots of $G$. Let $\Phi^+$ denote the set of positive roots corresponding to the choice of $B$ in $G$ and $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ a set of simple roots of $G$. Corresponding to this choice of simple roots, there exists a set of elements $\omega_i \in X_\ast(T) \otimes \mathbb{Q}$ known as the fundamental weights with the property that $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. For any root $\alpha$, there exists an isomorphism of $x_\alpha$ of $G_a$ with a closed subgroup $X_\alpha$ of $G$ with the property that $t \cdot x_\alpha(a) \cdot t^{-1} = x_\alpha(\alpha(t)a)$. $X_\alpha$ is known as the root group associated to $\alpha$. 

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By a parabolic in $G$, we mean a closed subgroup of $G$ containing $B$. There exists a natural bijection of the set of subsets of $\Delta$ with the set of parabolic subgroups of $G$ containing $B$ under which for a subset $I \subseteq \Delta$, we assign the parabolic $P_I$ to be the closed subgroup of $G$ generated by $B$ and $X_+ \setminus \{ -\alpha | \alpha \in \Delta \setminus I \}$. Let $W = N(T)/T$ be the Weyl group. Fix a $W$-invariant inner product $\langle \cdot, \cdot \rangle$ on $X(T) \otimes \mathbb{Q}$. Using this inner product we can define norm of any 1-PS $\lambda(t) \in T$ as $||\lambda(t)|| = \langle \lambda, \lambda \rangle$. For a arbitrary 1-PS in $G$ we can conjugate it into the fixed maximal torus and then define its norm. We begin by recalling the definitions of semistability of vector and principal bundles with respect to the fixed polarisation $H$.

**Definition 1.** For a vector bundle $E$ on $X$, define its slope to be the rational number:

$$\mu(E) = \text{deg}(E)/\text{rk}(E).$$

A vector bundle $E$ on $X$ is said to be $\mu$-semistable (w.r.t. the polarization $H$) if for any proper subbundle $F \subset E$, we have the inequality $\mu(F) \leq \mu(E)$, where $\mu$ denotes the slope of the bundles.

For any vector bundle $E$, there exists a canonical filtration of $E$ by $\mathcal{O}_X$-coherent subsheaves known as its Harder-Narasimhan filtration (denoted $\text{HN}(E)$).

$$0 \subset E_1 \subset \cdots \subset E_l = E$$

with the property that successive quotients $E_i/E_{i-1}$ are $\mu$-semistable and $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ for all $1 \leq i \leq l$.

Define $\mu_{\text{max}}(E) = \mu(E_1)$ and by $\mu_{\text{min}}(E) = \mu(E_l/E_{l-1})$

The quantity $\mu_{\text{max}}(E) - \mu_{\text{min}}(E)$ known as the instability degree is a measure of the instability of the vector bundle.

**Definition 2.** A principal $G$-bundle $E$ over $X$ is said to be semistable if for any reduction of structure group to a parabolic $P$ of $G$ and any dominant character on $P$, the induced line bundle has degree $\leq 0$.

Equivalently, a principal bundle $E$ on $X$ is said to be semistable is for any reduction of structure group to a parabolic $P$ of $G$, the pullback of the relative tangent bundle of $E(G/P)$ over $X$, via the section $\sigma : X \to E(G/P)$ corresponding to this reduction is a vector bundle on $X$ of degree $\geq 0$.

**Definition 3.** Let $E$ be a principal $G$ bundle on $X$. Let $E_P$ be a reduction of structure group of $E$ to a parabolic $P \subset G$. The reduction is said to be canonical (or the Behrend reduction) if the following conditions are satisfied:

1) $\text{deg } E_P(P) > 0$
2) For any parabolic subgroup scheme \( Q \subset E(G) \), \( \deg Q \leq \deg E_{P}(P) \).

3) For any subgroup scheme \( Q \supset E_{P}(P) \), \( \deg Q < \deg P \).

4) The unipotent radical bundle \( E_{P}(P)/R_{u}(P) \) is semistable.

With these conditions our definition of canonical reduction coincides with that of Behrend. \( P \) is known as the Behrend’s parabolic. The degree of \( E_{P}(P) \) is denoted by \( \deg_{HN}(E) \).

The canonical reduction can be shown to be equivalent to the following: For any nontrivial character on \( P \) which is a non-negative combination of simple roots with respect to the choice of \( B \), the induced line bundle on \( X \) obtained by extension of structure group has non-negative degree.

### 3 Frobenius morphism

Let \( X \) be a scheme over a algebraically closed field of char \( p > 0 \). The \( p \)-th power map \( \mathcal{O}_{X} \to \mathcal{O}_{X} \) given by \( f \to f^{p} \) gives rise to a morphism of schemes \( F_{X} : X \to X \) called the absolute frobenius. If \( k \) is a perfect field, this morphism is an isomorphism (although not a \( k \)-morphism in general). Let \( F^{m} \) denote the iterated frobenius map. If \( E \) is a \( G \)-bundle on \( X \) we an take its pullback \( F^{m'}(E) \) which will be a \( F^{m'}(G) \) bundle. We call this the \( m \)-th frobenius pullback. By twisting Spec \( k \) by the frobenius map (which will be an isomorphism), we can define a \( k \)-structure on \( F^{m'}(X) \), \( F^{m'}(G) \) as well as \( F^{m'}(E) \). The \( G \) bundle \( F^{m'}(E) \) on \( X \) is the same as the one obtained by extension of structure group under the homomorphism \( G \to G \) given by the \( m \)-th frobenius map.

Clearly if the frobenius pullback of a \( G \)-bundle is semistable with respect to the pulled back polarization, then so is the original bundle. A semistable \( G \) bundle may not however pullback to a semistable \( G \)-bundle. A \( G \)-bundle \( E \) is said to be strongly semistable if all its frobenius pullbacks are also semistable.

### 4 The instability parabolic

In this section we discuss the role of the instability parabolic which plays an important role in studying extension of structure groups in positive characteristic. We first begin by recalling some elementary notions and facts from Geometric Invariant Theory.

Let \( K \) be an algebraically closed field. Let \( G \) be a reductive algebraic group defined over \( K \). Let \( \rho : G \to Gl(V) \) be a representation of \( G \) defined over \( K \). A vector \( v \in V \) is said to be semistable for the \( G \)-action if \( 0 \notin \bar{G}v \). Equivalently there exists a \( G \)-invariant
\( \phi \in S^n(V) \) for some \( n > 0 \) such that \( \phi(v) \neq 0 \).

For a 1-PS \( \lambda(t) \) of \( G \) we get a decomposition of \( V = \oplus V_i \), where \( V_i = \{ v \in V \mid \lambda(t)(v) = t^i(v) \} \).

Define \( m(v, \lambda) = \min \{ i \mid v \text{ has a nonzero component in } V_i \} \).

Define slope of the 1-PS \( \lambda(t) \) by

\[
\nu(\lambda, v) = \frac{m(v, \lambda)}{\| \lambda \|}
\]

Note that for any vector \( v \in V(\bar{K}) \) and any 1-PS \( \lambda \), we have \( \nu(\lambda, v) = \nu(g\lambda g^{-1}, gv) \)

**Lemma 4.** (See [RR]) There exists a constant \( C \) such that for all \( v \in V \) and all 1-PS \( \lambda \), \( \nu(\lambda, v) \leq C \).

For a non-semistable vector \( v \in V \) define its **instability 1-PS** (denoted \( \lambda_v \)) to be one for which \( \nu(v, \lambda) \) attains the maximum value among all the 1-PS of \( G \). Intuitively, this is the 1-PS in \( G \) which takes the vector \( v \) to 0 fastest after proper scaling.

For a 1-PS \( \lambda \) define a parabolic \( P(\lambda) \) whose valued points consist of elements \( g \in G \) such that \( \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \) exists. This is known as the **instability parabolic** associated to \( \lambda \). If \( \lambda \) is an instability 1-PS of \( v \), then \( P(\lambda) \) will also be known as the instability parabolic of \( v \), denoted \( P(v) \).

Now if \( G \) acts on a projective variety \( M \) defined over \( K \) which is linearized by some very ample line bundle \( \mathcal{L} \), then we get a \( G \)-equivariant embedding \( i : M \hookrightarrow \mathbb{P}(H^0(M, \mathcal{L})) = \mathbb{P}(V) \). We then say that a point \( m \in M \) is semistable for the \( G \)-action if the corresponding point in \( V \) is semistable.

We recall some basic facts concerning instability 1-PS (See [RR] )

Suppose \( G \) acts on a projective variety \( M \) as above. Let \( m \in M \) be a nonsemistable point for the action of \( G \).

(a) The function which sends every 1-PS \( \lambda \) of \( G \) to \( \nu(\lambda, m) \) attains its maximum on the set of all 1-PS subgroups of \( G \). Following [RR], we denote this value by \( B \).

(b) There exists a parabolic subgroup \( P(m) \) of \( G \), called the instability parabolic associated to the point \( m \), such that for any instability 1-PS \( \lambda \) associated to \( m \), we have \( P(m) = P(\lambda) \).

(c) The instability parabolic \( P \) is generated by \( T \) together with the root groups \( U_\alpha \) corresponding to roots \( \alpha \) for which \( \alpha(\lambda) \geq 0 \).

(d) A maximal torus \( T \) in \( G \) contains a instability 1-PS \( \lambda \) for \( m \) if and only if \( T \subset P(\lambda) \). Such a 1-PS is necessarily unique.
(e) For a non-semistable \( m \in M \), if \( \lambda(t) \) is an instability 1-PS of \( m \), then \( g\lambda(t)g^{-1} \) is the instability 1-PS of \( gm \) and \( \nu(\lambda, m) = \nu(g\lambda g^{-1}, gm) \).

(f) For a 1-PS \( \lambda \) of \( G \) and any element \( g \in P(\lambda) \) we have \( \nu(m, \lambda) = \nu(gm, \lambda) \).

(g) For any \( g \in G \), we have \( P(gm) = gP(m)g^{-1} \).

(h) If \( m \in M \) is an unstable point for the action of \( G \) having an instability 1-PS defined over an extension field \( [L : K] \), then the instability parabolic \( P(m) \) is also defined over \( L \).

Now let \( K \) be an arbitrary field (not necessarily algebraically closed). Let \( K_s \) denote its separable closure. Let \( G \) be a reductive algebraic group defined over \( K \). Let \( T \) be a fixed maximal torus of \( G \) (which will always be split over \( K_s \), in fact over a finite extension of \( K \)). Let \( M \) be a projective variety defined over \( K \) on which \( G \) acts, linearized by a very ample line bundle \( L \) giving a \( G \)-equivariant embedding \( i : M \hookrightarrow \mathbb{P}(V) \). Fix an inner product on \( X, (T \otimes K_s) \) to be one which is invariant under the action of the Weyl group as well as the Galois group \( \text{Gal}(K_s | K) \) (See [Kempf]).

A point \( m \in M \) is said to be semistable if it semistable after base change to its algebraic closure, i.e. thought of as an element in \( V(\bar{K}) \).

Let \( m \in M \) be a \( K \)-rational point of \( M \). Let \( P(m) \) be the instability parabolic of \( m \) defined over \( \bar{K} \). By invariance of the inner product under the Galois action and uniqueness of \( P(m) \) we see that if \( P(m) \) is defined over \( K_s \), then it is already defined over \( K \). [See RR].

**Rationality of the instability parabolic and its consequences**

Let \( X, G \) and \( L \) be as before. Suppose \( \rho : G \to GL(V) \) be a representation of \( G \) which takes the connected component of the centre of \( G \) to the centre of \( GL(V) \). Let \( P \) be a maximal parabolic of \( GL(V) \). Choose the very ample generator \( L \) of \( GL(V)/P \). This is a linearized very ample line bundle giving an embedding of \( GL(V)/P \) inside a projective space \( \mathbb{P}(W) \).

Now let \( \pi : E \to X \) a principal \( G \)-bundle on \( X \). Let \( E(G) \) be the associated group scheme over \( X \). Let \( E(GL(V)/P) \) be the associated fiber space. Let \( T_\pi \) denote the relative tangent bundle on \( E(GL(V)/P) \). Let \( E(L) \) be the associated line bundle on \( E(GL(V)/P \) corresponding the line bundle \( L \) on \( SL(V)/P \). The group scheme \( E(G) \) acts on \( E(G/P) \). Let \( E(G)_\circ \) be the generic fiber of \( E(G) \). It is a group scheme defined over the function field of \( X \). \( E(G)_\circ \) acts on \( E(GL(V)/P)_\circ \) which is linearized by \( E(L)_\circ \). Let suppose \( \sigma \) be a reduction of the induced \( GL(V) \)-bundle to \( P \). Then corresponding to this reduction we get a section of (called \( \sigma \) again) of \( E(GL(V)/P \) over \( X \). Let \( \sigma_\circ \) be the associated \( k(X) \)-valued point of \( E(GL(V)/P_\circ) \). Suppose \( \sigma_\circ \) is a non-semistable point for the action of \( E(G)_\circ \) on
$E(Gl(V)/P)$. Let $P(\sigma_e)$ denote the instability parabolic associated to the point $\sigma_e$. We call $P(\sigma_e)$ the instability parabolic corresponding to this reduction. Let $T_{\sigma}$ denote the pullback of $T_\pi$ via the section $\sigma$.

**Proposition 5.** (See [RR, Proposition 3.10, (1)]) Let $\sigma_e$ be a semistable point for the action of $E(G)$, on $E(G/P)$ with respect to the polarization $E(L)$. Then $T_{\sigma}$ has degree $\geq 0$. In other words this reduction of structure group does not contradict semistability of $E(Gl(V))$.

**Proposition 6.** (See [RR]) Let $E$ be a semistable $G$-bundle. Suppose for every reduction to a parabolic $P$ in $Gl(V)$, the instability parabolic associated to this reduction is rational (defined over $k(X)$), then the induced $Gl(V)$ bundle is semistable.

**Proposition 7.** (See [HC], Proposition 4.5) Let $G$ be a reductive algebraic group defined over an arbitrary field $K$ (not necessary algebraically closed) acting on a projective variety $M$ defined over $K$. Then there exists an integer $t$ such that given any $K$-valued point $m \in M$ which is not semistable its instability parabolic $P(m)$ is defined over $K^{p^{-t}}$.

**The method of Holla-Coiai**

In this section we briefly explain the method of Holla-Coiai for proving the existence of the integer $t$ in proposition 7. We will be brief and sketchy in this exposition. Let $G$ and $M$ be as in above proposition 7. Let $\mathcal{L}$ be a linearized very ample line bundle on $M$ giving a $G$-equivariant embedding $i : M \hookrightarrow \mathbb{P}(H^0(M, \mathcal{L})) = \mathbb{P}(V)$.

For an affine algebra $A$ over $K$, we define its radical index to be the smallest integer $n$, such that $f^n = 0$ for all $f \in \text{Rad}(\bar{A})$ by def $\text{Rad} (A \otimes K \bar{K})$. Now let $m \in M$ be a $K$-rational point of $M$ which is not semistable for the $G$-action.

Recall that the action of $G$ is said to be strongly separable at a point $m \in M$ if the isotropy subgroup scheme at every $\bar{K}$-valued point in the closure of $O(m)$ is reduced, where $O(m)$ denotes the orbit of $m$. Let $P(m)$ be the instability parabolic of $m$. There exists $g \in G$ such that the parabolic $P = g P(m) g^{-1}$ is defined over $K_s$. By uniqueness of the instability parabolic and Galois descent, it is already defined over $K$. Let $x_m = gm$. Then the instability parabolic of $x_m$ is $P$. Since $P(x_m)$ is defined over $K$, it contains a maximal torus over $K$ (which is split over $K_s$). Hence there is a unique instability 1-PS of $x_m$ contained in this maximal torus which is defined over $K_s$ and hence by uniqueness defined over $K$.

Consider the decomposition of $V = \bigoplus V_i$ into simultaneous eigenspaces for the action of $\lambda$, where $V_i = \{ v \in V \mid \lambda(t)(v) = t^i(v) \}$. Let $j = m(x_m, \lambda)$ and $V_j = \bigoplus V_i$, $i \geq j$. Define the $K_s$-scheme $M(P) x_m$ to be the scheme theoretic intersection of the $K_s$-subscheme $\mathbb{P}(V_j)$ and $O(m)$ of $\mathbb{P}(V)$. The following proposition summarizes the basic properties of the scheme $M(P) x_m$.
**Proposition 8.** The $\bar{K}$-valued points of $M(P)_{x_m}$ are precisely those points in the $K$-scheme $O(m)$ for which the instability parabolic is $P(x_m)$. Also, when the action of $G$ on $m$ is strongly separable, then $M(P)_{x_m}$ is absolutely reduced.

Suppose one can find a $K_s$ rational point $m'$ in $M(P)_{x_m}$, then by proposition it's instability parabolic being $P(x_m)$ is hence defined over $K_s$. Since $m$ and $m'$ are both $K_s$ rational points, they are translates of each other by a $G(K_s)$-valued point $g$ and hence their instability parabolic are conjugates by $g$. This will prove that the instability parabolic for $m$ is defined over $K_s$ and hence by uniqueness and Galois descent it is defined over $K$.

Thus the problem of showing the existence of the integer $t$ in proposition boils down to finding a finite purely inseperable extension $L$ of $K_s$ (independent of the point $m$) over which the scheme $M(P)_{x_m}$ will have a $L$-valued point. This bound is obtained using the following lemma's:

**Lemma 9.** Let $f : Y \to X$ be a morphism of finite-type scheme over $\bar{K}$. Then there exists an integer $n$ such that the radical index of the schematic fiber of $x$ is less than or equal to $n$ for all closed points $x \in X$.

**Lemma 10.** Let $A$ be an affine $K_s$-algebra with radical index $\leq p^n$. Then $A$ admits a $K_s^p^{-n}$-rational point.

### 5 Bounds for the field of definition of the instability parabolic and its consequences

In this section we give explicit bounds for the field of definition of the instability parabolic associated to non-semistable points for the action of a reductive algebraic group $G$ acting on a vector space $V$ defined over an arbitrary field $K$. We do this by giving explicit bounds for the field of definition for the instability 1-PS associated to these points. We first do this $G = Sl(2)$, where we can get much better bounds than for a general $G$, then for the tensor power representation of $Sl(n)$, then for an arbitrary representation of $Sl(n)$ and then for an arbitrary representation of any arbitrary reductive algebraic group $G$.

We now begin with giving bounds for the field of definition of the instability parabolic for various $Sl(2)$-modules.

**Lemma 11.** Let $K$ be any field (not neccessarily algebraically closed) if char $p > 0$. Let $G = Sl(2, K)$. Let $\rho : Sl(2, K) \to S^N(V)$ be the standard symmetric power representation. Let $N = N_0 + N_1p + N_2p^2 + \cdots + N_t p^t$ be the $p$-adic expansion of $N$. Then for any non-semistable $K$-rational vector $v \in S^N(V)$, the instability parabolic $P(v)$ of $v$ is defined over $K^{1/p^t}$.
Proof  By uniqueness of instability parabolic and Galois descent explained before, we can assume that $K$ is seperably closed. Let $X, Y$ denote the basis for $V$ over $K$. Thus $S^N(V)$ can be identified with the vector space of all degree $N$ homogeneous polynomials in $X$ and $Y$. Let $f = \sum_{i+j=N} a_{ij}X^iY^j, a_{ij} \in K$ be an unstable vector in $S^n(V)$ for the action of $G$.

Claim 1: $f$ has a zero of multiplicity greater than $N/2$ on $\mathbb{P}_K^1$.
Proof of claim: Let $\lambda(t)$ be the instability 1-PS of $f$ defined over $\overline{K}$. Every 1-PS of $S^N(2)$ is conjugate over $\overline{K}$ to the 1-PS $\mu(t) = (t^0 0)$. Choose $g \in S^N(2, \overline{K})$ such that $g\lambda(t)g^{-1}$ is of the form $\mu(t)$. Then $\mu(t)$ is the instability 1-PS of $g \cdot f$. Let suppose $g \cdot f$ have the form:

$$g \cdot f = X^T g$$

for some nonnegative integer $T$ and some polynomial $g \in S^N(V)$ which is not divisible by $X$. Since $\mu(t)$ drives $g \cdot f$ to 0, $T$ necessarily satisfies $N/2 < T \leq N$. i.e. $f$ has a zero of multiplicity greater than $N/2$ on $\mathbb{P}_K^1$ and hence a unique such zero.

Now, by using the fact that $K$ is seperably closed, by a suitable change of basis made over $K$, we can assume that $f$ can be factorized in the form:

$$f = F_1 \cdot F_2 \cdots F_r$$

for some $0 \leq r \leq N$, with $\deg F_1 \geq \deg F_2 \geq \cdots \geq \deg F_r$ and each $F_i$ of the form $(X^{p^i} - \alpha_i Y^{p^i})$ for some non-negative integer $t_i$.
Note that $t_1 \geq t_2 \geq \cdots \geq t_r \geq 0$. Factorizing $f$ into product of linear polynomials over the field $K^{1/p^i}$, we get:

$$f = (X - \alpha_1^{1/p^i} Y)^{p^i} \cdots (X - \alpha_r^{1/p^i} Y)^{p^r}$$

Note that by Claim 1, $p^i = T$. By once again making a change of basis over the field $K^{1/p^i}$, sending

$$(X - \alpha_1^{1/p^i} Y) \rightarrow X'$$

$$Y \rightarrow Y'$$

and calling the resulting polynomial $f'$ (which is a translate of $f$ by an element in $Sl(2, K^{1/p^i})$), we see that $f'$ has the form

$$f' = X'^{p^i} (X' - \beta_1 Y')^{p^2} \cdots (X' - \beta_r Y')^{p^r}$$

with all the $\beta_i$’s distinct. Note that $\beta_1, \ldots, \beta_r$ belong to $K^{1/p^i}$. Since $f$ has a unique root of multiplicity $> N/2$, we see that the factor occuring in the above factorization with the
highest power is necessarily unique. i.e. $t_1$ is unique.

Claim 2: The 1-PS $\mu(t)$ is an instability 1-PS of $f'$.

Proof of claim 2: The proof of the claim is quite obvious. We only sketch it briefly. Note that $v(f', \mu) = t_1/(\| \mu \|)$. Suppose there exists another 1-PS $\mu'(t)$ such that $v(f', \mu') > v(f', \mu)$. Since all 1-PS's of $G$ are conjugates over $\bar{K}$, there exists an element $h \in G(\bar{K})$ which conjugates $\mu$ into $\mu'$. Then $\mu(t)$ will be the instability 1-PS of $hf'$. It is easy to see that the highest power of $X'$ occuring in $f'$ is greater than or equal to the highest power of $X'$ occuring in $hf'$. Hence we see that $m(f', \mu) \geq m(hf', \mu) = m(f', \mu')$. Since $\mu$ and $\mu'$ are conjugates over $\bar{K}$, we see that this implies that $v(f', \mu) \geq v(f', \mu')$. This proves that $\mu$ is an instability 1-PS of $f'$ and hence completes the proof of Claim 2.

Now since $f$ and $f'$ are translates of each other by an element in $K^{1/p^n}$ and an instability 1-PS of $f'$ is defined over $K$, we see that an instability 1-PS and hence the instability parabolic of $f$ is defined over $K^{1/p^n}$.

Corollary 12. Let $\rho : G \to S^N(V)$ be the representation as in lemma[17] If $N > p$, the instability parabolic of any non-semistable vector in $S^N(V)$ is rational.

Proof Obvious.

In general, for an arbitrary representation of $Sl(V)$, the method does not seem to work. This is because it is in general impossible to determine all the non-semistable points in the representing space. Hence we have adopt a more indirect way of bounding the field of definition of the instability 1-PS which does not use the knowledge of all the non-semistable vectors. We begin with a lemma which will be a brutal step in the bounding of the field of definition of the instability 1-PS:

Lemma 13. Let $K$ be an infinite field. Let $A = K[Y_1,...Y_n]/(f_1,...,f_i)$ be a finitely generated $K$-algebra. Let $g \in K[Y_1,...Y_n]$. Let suppose $\deg f_i = d_i$. Let $d = \prod d_i$. Let suppose $X = \text{Spec } A$ thought of as a closed subscheme of $\mathbb{A}^n_K$ has a $\bar{K}$-valued point at which $g$ is non-vanishing (thought of as a regular function on $X$). Then there exists an extension field $L$ of $K$ with $\deg [L : K] \leq d$ such that $X$ has a $L$-valued point at which $g$ is non-vanishing.

Proof Let $V(g) \subset X$ be the closed subscheme of $X$ defined by the intersection of the vanishing locus of $g$ with $X$. Let $X' = X\backslash V(g)$ be an open affine subscheme of $X$. Now by hypothesis $X$ has a $\bar{K}$-valued point. By restricting to a irreducible component of Spec $A$ containing the $\bar{K}$ valued point, we can assume that $X$ is irreducible. Let $\dim X = m$. By a linear change of coordinates, we can perform a Noether normalisation such there exists $m$ elements $t_1,...,t_m$ in $A$ such that $A$ is integral over $B = K[t_1,...,t_m]$ and the induced map $f : \text{Spec } A \to \text{Spec } B$ on affine schemes corresponding to the inclusion of $B$ in $A$ has degree atmost $d$. Let $p \in B$ be a $K$ valued point of $B$ which is not in the image of $V(g)$. This is
possible to choose since $f$ is a finite map. By going-up lemma, there exists a point $q \in X'$ lying over $p$. Let the residue field extension $[K(q) : K(p)]$ be $s$. Then $s \leq \deg f \leq d$. Taking $L$ to be $K(q)$, we get the lemma.

**Lemma 14.** Let $V$ be a vector space of dimension $n$ defined over a field $K$ of char $p > 0$. Let $G = SL(V)$. Let $K$ be an arbitrary field of char $p > 0$. Let $\rho : SL(V) \rightarrow SL(V^m)$ be the tensor power representation of $SL(n)$. Then for any non-semistable $K$-rational point $v \in V^m$, the instability parabolic $P(v)$ is defined over an extension field of $[L : K]$ of degree $\leq mn^m$. Equivalently if $t$ is such that $p^t > mn^m$, then the instability parabolic for unstable $K$-rational point is defined over $K^{1/p^t}$.

**Proof** Let $X_1, ..., X_n$ be a basis of $V$ over $K$. By uniqueness of instability parabolic and Galois descent, we may assume that all the objects are defined over the seperable closure $K_s$ of $K$. Hence without loss of generality we may assume $K = K_s$. Let $R = K\langle X_1, ..., X_n \rangle$ denote the non-commutative polynomial ring in the variables $X_1, ..., X_n$. Let $R^m$ denote the vector subspace of $R$ consisting of non-commutative monomials in $X_1, ..., X_n$ of degree $m$. Let $w_1, ..., w_M$ denote an ordered basis of $R^m$ consisting of non-commutative monomials of degree $m$ (words). Note that $M = n^m$. Then $V^m$ can be identified with $R^m$, the identification compatible with the action of $SL(V)$. For any extension field $[L : K]$ we will think of elements $g \in G(L)$ as $n \times n$ matrices $g_{ij}$ with coefficients in $L$.

Consider the commutative polynomial ring $B = K[G_{ij}]$. Any $g = g_{ij} \in G(L)$ can thus be thought of as a $L$-valued point of Spec $B$. Let $v = \sum a_j w_j$ be any element in $R^m$. We define the elementary polynomials associated to $v$ as follows:

Denote by $K[G_{ij}](X_1, ..., X_n)$ the noncommutative ring in the variables $X_i$, with coefficients in the commutative polynomial ring $K[G_{ij}]$. Consider the set mapping $\theta : K\langle X_1, ..., X_n \rangle \rightarrow K[G_{ij}](X_1, ..., X_n)$ defined as follows:

$\theta$ sends a variable $X_i$ in $K\langle X_1, ..., X_n \rangle$ to $\sum G_{ij} \cdot X_j$ and extends the action in the obvious way to $K\langle X_1, ..., X_n \rangle$. The ordered set of coefficients of the various noncommutative monomials in the $X_i$’s that occur in $\theta v$ (which are polynomials in the commutative ring $K[G_{ij}]$) will be called the elementary polynomials corresponding to $v$, denoted $EP_v$ (some of which may be the zero polynomial for a given $v$). More precisely, if $\theta(v) = \sum f_i w_i$, with $f_i \in K[G_{ij}]$, then the set ordered set $f_1, f_2, ..., f_M$ will be defined to be the elementary polynomials associated to $v$. Just for the sake of clarity we explain this definition (of elementary polynomials) by taking a simple example.

In the two-variable case, consider the action of $SL(2, K)$ on $V^m$ as above. If $\{X_1^2, X_1X_2, X_2X_1, X_2^2\}$ denote the ordered basis for $V^m$, then the elementary polynomials associated to the vector $v = X_1^2 + X_1 \cdot X_2$ will be computed as follows: Consider the image of $v$ under $\theta$:

$$X_1 \rightarrow G_{11}X_1 + G_{12}X_2$$

$$X_2 \rightarrow G_{21}X_1 + G_{22}X_2$$
Hence the image of $v = X_1^2 + X_1X_2$ will be:

$$(G_{11}X_1 + G_{12}X_2)^2 + (G_{11}X_1 + G_{12}X_2)(G_{21}X_1 + G_{22}X_2)$$

$$= (G_{11}^2X_1^2 + G_{11}G_{12}X_1X_2 + G_{12}G_{11}X_2X_1 + G_{12}^2X_2^2) + (G_{11}G_{21}X_1^2 + G_{11}G_{22}X_1X_2 + G_{12}G_{21}X_1X_2 + G_{12}G_{22}X_2^2)$$

$$= (G_{11}^2 + G_{11}G_{21})X_1^2 + (G_{11}G_{12} + G_{11}G_{22})X_1X_2 + (G_{12}G_{11} + G_{12}G_{22})X_2X_1 + (G_{12}^2 + G_{12}G_{22})X_2^2.$$ 

Thus the elementary polynomials corresponding to $X_1^2 + X_1X_2$ are:

$$f_v = (G_{11}^2 + G_{11}G_{21}), f_2 = (G_{11}G_{12} + G_{11}G_{22}), f_3 = (G_{12}G_{11} + G_{12}G_{21}), f_4 = (G_{12}^2 + G_{12}G_{22}).$$

Note that for any $v \in V^{\otimes m}$, the elementary polynomials $f_v$ all have degree $m$. If $f_v \in EP_v$ is an elementary polynomial and $g = g_{ij} \in G(\tilde{K})$ is any element, then by $f_v(g)$, we mean the element of $\tilde{K}$ obtained by substituting $G_{ij} = g_{ij}$ in $f_v$.

Let $v \in V^{\otimes m}$ (or equivalently in $R^m$) be a non-semistable vector for the action of $SL(V)$. Let $v = \sum a_i \cdot w_i$ be the expansion of $v$ in terms of the basis vectors. Let $\lambda(t) = \lambda_{ij}(t)$ be a 1-PS subgroup of $G(\tilde{K})$ which is an instability 1-PS for $v$. Then there exists an element $g = (g_{ij}) \in G(\tilde{K})$ such that $g \cdot \lambda(t) \cdot g^{-1}$ is of the form

$$\begin{pmatrix}
  t^{a_1} & 0 & \cdots & 0 \\
  0 & t^{a_2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & t^{a_n}
\end{pmatrix}$$

for some $a_1, ..., a_n$ such that $a_1 \geq a_2 \geq ... \geq a_n$.

Then $g \cdot \lambda(t) \cdot g^{-1} = \lambda'(t)$ is an instability 1-PS for $g \cdot v$ with $\nu(\lambda, v) = \nu(\lambda', gv)$. Let $g \cdot v = \sum b_i w_i$. Clearly $b_i = f_v(g_{ii})$. Let $f_{i_1}, ..., f_{i_n}$ (resp. $f_{i_{n+1}}, ..., f_{i_m}$) denote the set of elementary polynomials in $EP_v$ which vanish at $g$ (resp. are nonzero at $g$). By lemma[13] there exists an extension field $L$ of $K$ with $[L : K] \leq rm$ and an $L$-valued point $g' \in G(L)$ such that $f_{i_1}, ..., f_{i_n}$ all vanish at $g'$ and $f_{i_{n+1}}, ..., f_{i_m}$ are all non-vanishing at $g'$. Thus $g \cdot v$ and $g' \cdot v$ have the same set of monomials with non-zero coefficients. Note that since $\lambda'(t)$ is of the form

$$\begin{pmatrix}
  t^{a_1} & 0 & \cdots & 0 \\
  0 & t^{a_2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & t^{a_n}
\end{pmatrix}$$

an simple observation shows that $m(\lambda', gv) = m(\lambda', g'v)$ and hence $\nu(\lambda', gv) = \nu(\lambda', g'v)$. Also $\lambda'(t)$ is an instability 1-PS for $g' \cdot v$. This is seen as follows: $g' \cdot \lambda' g'^{-1}$ is an instability 1-PS of $g' \cdot v$ and $\nu(\lambda, v) = \nu(g' \cdot \lambda' g'^{-1}, g'v)$. But $\nu(\lambda, v) = \nu(\lambda', gv) = \nu(\lambda', g'v)$. Thus $\nu(g' \cdot \lambda' g'^{-1}, g'v) = \nu(\lambda', g'v)$ and hence $\lambda'(t)$ is also an instability 1-PS for $g'v$. This implies that $g'^{-1} \lambda' g'$ is an instability 1-PS of $v$. But $g'^{-1} \lambda' g'$ is defined over $L$. This shows that an instability 1-PS and hence the instability parabolic of $v$ is defined over $L$. Since $r \leq n^m$, we see that $\deg [L : k] \leq mn^m$. Since $K$ can be assumed to be seperably closed, the
only algebraic extensions possible are those obtained by taking \( p^l \)-th roots of generators of \( K \) for various non-negative integers \( l \). Since \( p^l > mn^m \), it is clear that the instability parabolic for \( v \) is defined over \( K^{1/p^l} \). This completes the proof of the lemma.

Notation: For any integers \( n \) and \( r \), with \( r < n \), set the symbol \( nC_r \) (\( n \) choose \( r \)) to be equal to \( n!/(r!(n-r)!)) \).

We use the above lemma to prove the following theorem:

**Theorem 15.** Let \( G = SL(n) \). Let \( X \) be a smooth projective variety over an algebraically closed field \( k \). Let \( K(X) \) denote its function field. Let \( V, m \) and \( \rho \) be as in lemma \( \text{[14]} \). Let \( E \) be any principal \( G \)-bundle on \( X \). Let \( N = \max \ n^m C_r \cdot (rm) \). Let \( t \) be any integer such that \( p^t > N \). Let suppose \( E \) together together with its first \( t \) Frobenius pullbacks is semistable. Then the induced \( SL(V) \) bundle is also semistable.

**Proof** Let \( W = V^{\otimes m} \). Let \( E_{SL(W)} \) denote the induced \( SL(W) \) bundle. We want to show that \( E_{SL(W)} \) is also semistable. By lemma \( \text{[6]} \), this is equivalent to showing that for any maximal parabolic \( P \) in \( SL(W) \) and any reduction of structure group to \( P \), the instability parabolic for the point \( \sigma_o \) in \( E(SL(W)/P) \) corresponding to this reduction is rational. Let \( E(G) \), be as before. \( E(G) \) acts on \( E(SL(W)/P) \) which is linearized by the very ample line bundle \( E(\mathcal{L}) \) explained before. Since \( E_o \) gets trivialized after a finite separable extension, we get isomorphisms \( E_o \otimes_{k(x)} k(x) \simeq G \otimes_{k(x)} k(x) \) and \( E(SL(W)/P)_o \otimes_{k(x)} k(x) \simeq (SL(W)/P) \otimes_{k(x)} k(x) \), the isomorphisms being compatible with the action. Since \( P \) is a maximal parabolic, \( SL(W)/P \) is isomorphic to the Grassmannian of \( r \) dimensional subspaces of \( W \) for some \( r < \dim W \). Using \( E(\mathcal{L})_o \otimes_{k(x)} k(x) \), we get an \( G(k(x)) \)-equivariant embedding of \( E(SL(W)/P)_o \otimes_{k(x)} k(x) \) inside \( \mathbb{P}(\wedge^r(W)) \). We need to show that for this action of \( G(k(x)) \) on \( \mathbb{P}(\wedge^r(W)) \), the instability parabolic for the point \( \sigma_o \) corresponding to this reduction is rational. By lifting this point to a point in \( \wedge^r(W) \) (call it \( \sigma_o \) again), it boils down to proving the same fact for the action of \( G(k(x)) \) on \( \wedge^r(W) \). This representation of \( G \) on \( \wedge^r(W) \) is the standard representation of \( G \) on \( \wedge^r(W) \), induced from the tensor power representation of \( G \) on \( V^{\otimes m} \).

Corresponding to the basic \( X_1, ..., X_n \) of \( V \), we get a standard basis of \( \wedge^r(W) \) consisting of vectors of the form \( w_{i_1} \wedge ... \wedge w_{i_r}, (i_1, ..., i_r) \in 1, ..., M \) with \( i_1 < ... < i_r \), where each \( w_i \) is a noncommutative monomial in the \( X_i \)'s of degree \( m \) as in lemma \( \text{[14]} \). Choose an ordering of this basis. Let \( \{W_1, ..., W_M\} \) denote the ordered basis. Note that \( S = n^m C_r \). For any non-semistable vector \( v \in \wedge^r(V^{\otimes m}) \), let \( v = \sum b_i W_i \) be its expansion in terms of the basis vector \( W_i \). Define the elementary polynomials \( EP_i \) similar to lemma \( \text{[14]} \) to be the polynomials in \( K(X)[G_{ij}] \) occuring in the coefficients of the image of \( W_i \) when acted upon by a \( n \times n \) matrix of indeterminates \( G_{ij} \). Note that the degree of the elementary polynomials in now \( mr \). Now by using the same argument as in lemma \( \text{[14]} \), by considering \( W_i \)'s instead of the noncommutative monomials.
In lemma [14] we see that the instability parabolic for the vector \( \sigma_0 \) is defined over an extension field \([L : k(X)]\), where \( \deg [L : k(X)] \leq \text{n}^m C_r (\text{rm}) < p' \). Hence by lemma [14] for any reduction of structure group to any maximal parabolic \( P \) in \( S_l(W) \) the instability 1-PS and hence the instability parabolic corresponding to \( \sigma_0 \) is defined over \( k(X)^{1/p'} \). Now consider the action of \( F^\pi (E(G))_0 \) on \( F^\pi (E(S_l(W)/P))_0 \). For this action \( F^\pi (\sigma_0) \) has its instability 1-PS and hence its instability parabolic defined over \( k(X) \) via the isomorphism in the commutative diagram shown below:

\[
\begin{array}{c}
\text{Spec } K \\
\downarrow \\
\text{Spec } K^{\text{ps}}
\end{array}
\]

Let \( F^\pi (\pi) : F^\pi (E(S_l(W)/P)) \to X \), denote the pullback of \( \pi \) under \( F^\prime \). Similarly let \( F^\pi (\pi_\sigma) \) denote the pullback of the relative tangent bundle of \( E(S_l(W)/P) \) under \( F^\prime \) which is the same as the relative tangent bundle of the pullback of \( E(S_l(W)/P) \) under \( F' \). Since \( F^\pi (E) \) is semistable and the instability 1-PS corresponding to every reduction to every maximal parabolic is rational, \( \deg F^\pi (\sigma)^* F^\pi (T_\sigma) > 0 \). But \( \deg F^\pi (\sigma)^* (F^\pi (\pi)) = \deg F^\pi (T_\sigma) = p'^{\deg T_\sigma} \). This follows from the fact that for any line bundle \( L \) on \( X \), \( F^\pi (L) \) is isomorphic to \( L^{p'} \). Hence \( \deg T_\sigma > 0 \) for every reduction of \( E_{S_l(W)} \) to every maximal parabolic \( P \) in \( S_l(W) \) and hence by lemma [6] \( E_{S_l(W)} \) is also semistable.

Now let \( \rho' \) be an arbitrary representation of \( S_l(V) \). We use the above lemma to get bounds for the number of semistable Frobenius pullbacks required for an \( S_l(V) \)-bundle \( E \), so that the induced bundle on extension of structure group via \( \rho' \) is again semistable.

Let \( \rho' : S_l(V) \to S_l(W) \) be an arbitrary representation of \( S_l(V) \). Let \( 0 = W_0 \subset W_1 \subset \cdots \subset W_i = W \) be the Jordan-Holder filtration of \( W \). Then \( W_i/W_{i-1} \) are simple \( S_l(V) \)-modules. Any simple \( S_l(V) \)-module \( L(\lambda) \) corresponding to a highest weight vector \( \lambda = \sum a_i \omega_i \) is an \( S_l(V) \)-submodule of \( V^{\otimes i} \), where \( 1 \cdot | \lambda | = \sum a_i \) is called the degree of \( \lambda \). Following Langer (see [L]), we call the maximum of the degrees of the dominant weights whose modules occur as the successive quotients in the Jordan-Holder filtration as the Jordan-Holder degree of \( W \), denoted \( \text{JH}(W) \).

**Lemma 16.** Let \( \rho' : S_l(V) \to S_l(W) \) be an arbitrary representation of \( W \). Let \( \text{JH}(W) = d \). Let \( E \) be a \( S_l(V) \)-bundle on \( X \) such that \( F^\pi (E) \) is semistable for some \( t \) such that \( p'^{t} > \max_{0<r<\text{ps}d-1} n^m C_r (\text{rd}) \). Then the induced \( S_l(W) \)-bundle is also semistable.

**Proof** Let \( 0 = W_0 \subset W_1 \subset \cdots \subset W_i = W \) be the Jordan-Holder filtration of \( W \) as before. Then each successive quotient is a \( S_l(V) \)-submodule of the \( S_l(V) \)-module \( V^{\otimes i} \), for some \( i \leq d \). Since \( F^\pi (E) \) is semistable, by lemma [15] each of the induced vector bundles obtained by extension of structure group of \( E \) using these tensor power representations
are also semistable and of degree zero. Since a degree zero subbundle of a semistable bundle of degree zero is also semistable, we see that the induced vector bundle $E_{S(l(W)}$ is filtered by semistable bundles of degree zero and hence $E_{S(l(W)}$ is also semistable of degree zero. This completes the proof of the lemma.

**Remark 17.** Let $V$ be a vector space defined over $K$. Let $0 \to W' \to W \to W'' \to 0$ be a short-exact sequence of $S(l(V))-modules defined over $K$. If the instability parabolic for each of the unstable $K$-rational points in $W$ is defined over $K^{1/p^t}$ then so is the case for all the unstable $K$-rational points in $W'$. However it does not seem easy to bound the field of definition of the instability parabolics for unstable $K$-rational points in $W$ in terms of similar bounds for $W'$ and $W''$. Similarly it does not seem possible to determine the field of definition of the instability parabolics for all the unstable $K$-rational points in $W''$ knowing the same for $W$. This is because unstable $K$-rational points in $W$ may not surject onto the unstable $K$-rational points in $W''$. However if an integer $t$ satisfies the property that any $S(l(V))-bundle with first $t$-frobenius pullbacks semistable induces semistable $S(l(W'))$ and $S(l(W''))-bundles on extensions of structure group, then clearly the induced $S(l(W))-bundle is also semistable. Similarly, if integer $s$ satisfies the property that any $S(l(V))-bundle with first $s$-frobenius pullbacks semistable induces a semistable $S(l(W)$ on extension of structure group, then clearly the induced $S(l(W''))-bundle is also semistable. This is because any degree zero quotient of a semistable bundle of degree zero is also semistable of degree zero. Hence for computing the number of semistable frobenius pullbacks required for a $S(l(V))-bundle to induce a semistable bundle on extension of structure group, it suffices to compute the same for the tensor power representation. Then using the fact that an arbitrary representation $W$ of $S(l(V)$ can be filtered by $S(l(V))-modules which are submodules of a suitable tensor-power representation, we get bounds for the number of semistable frobenius pullbacks required so that the induced $S(l(W))-bundle is semistable.

**Remark 18.** Note that one of the major differences between the methods for estimating the field of definition of the instability parabolic described here and the methods of [RR] and [CH] is that unlike their methods we do not use the orbit map $E(G)_{o} \times E(G/P)_{o} \to E(G/P)_{o}$ and try and bound its non-seperability. We directly estimate the field of definition of the instability parabolic which is probably weaker than trying to bound the non-reducedness of the stabilizers of the various unstable $K$-rational points which does not seen quantifiable. Indeed it is an open problem as to whether it is possible to have a representation of a semisimple group $G$ such that the stabilizers of some of the unstable $K$-rational points in the representing space are non-reduced but their instability parabolics are rational. We do not know the answer to this yet.
6 Case of an arbitrary reductive group

In this section we get bounds for the field of definition of the instability parabolic for an arbitrary representation of an arbitrary reductive algebraic group.

Let \( G \) be a reductive algebraic group defined over \( k \). Fix an embedding \( i : G \hookrightarrow Gl(V) \), where \( V \) is a \( n \)-dimensional vector space. Fix a maximal torus \( T \) in \( G \).

**Theorem 19.** Let \([ F : k ]\) be an extension of fields. Let \( \rho : G \to Gl(W) \) be a finite dimensional representation of \( G \) defined over \( F \). Then there exists an integer \( t \), such that for any unstable \( F \)-rational point in \( W \), its instability parabolic is defined over \( F^{1/p^t} \).

**Proof** The proof of the lemma is similar to the proof of lemma [14]. The main difference now is that we also have to consider the defining equations of \( G \) in \( Gl(V) \) along with the elementary polynomials of the unstable \( F \)-rational points. As before we may assume that \( F \) is seperably closed. Let \( \dim W = m \) and \( \dim V = n \). Fix a basis of \( V \) via which \( G(V) \) will be identified with \( Gl(n, F) \). \( Gl(n, F) \) will be thought of as an open subscheme of \( M_n(F) \) which will identified with \( \mathbb{A}^n_F \). Let the a \( F \)-coordinate ring of \( G \) for the embedding \( \bar{i} : G \hookrightarrow M_n(F) \) given by the composite of \( G \hookrightarrow Gl(V) \subset \mathbb{A}^n_F = \text{Spec} \ F[G_{ij}]_{1 \leq i,j \leq n} \) be isomorphic to \( F[G_{ij}, (\det G_{ij})^{-1}]/(h_1, \ldots, h_s)_{1 \leq i,j \leq n} \), for some \( h_1, \ldots, h_s \in F[G_{ij}, (\det G_{ij})^{-1}] \). The valued points of \( Gl(V) \) will be thought of as \( n \times n \) invertible matrices. The affine coordinate ring of \( Gl(V, F) \) is isomorphic to \( A = F[G_{11}, \ldots, G_{nn}, (\det G)^{-1}] \), where \( \det(G) \) is the determinant polynomials in the \( G_{ij} \)'s and a matrix element \( g = g_{ij} \in Gl(V, L) \), for any extension field \([ L : F ]\), will be thought of as an \( L \)-valued point of Spec \( A \) in the obvious way. Choose an ordered simultaneous eigen basis \( \{ w_1, \ldots, w_m \} \) of \( W \) for all the 1-PS of \( G \) which lie in \( T \). With respect to this basis, the matrix of \( \rho \) will be an \( m \times m \) matrix whose entries are regular functions on \( G \), which are by definition the restrictions of the regular functions on \( \mathbb{A}^n_F \) via the embedding \( \bar{i} \).

\[
\begin{pmatrix}
\tilde{f}_{11}(G_{ij})/(\det(G_{ij}))^{a_{11}} & \tilde{f}_{12}(G_{ij})/(\det(G_{ij}))^{a_{12}} & \cdots & \tilde{f}_{1n}(G_{ij})/(\det(G_{ij}))^{a_{1n}} \\
\tilde{f}_{21}(G_{ij})/(\det(G_{ij}))^{a_{21}} & \tilde{f}_{22}(G_{ij})/(\det(G_{ij}))^{a_{22}} & \cdots & \tilde{f}_{2n}(G_{ij})/(\det(G_{ij}))^{a_{2n}} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{f}_{n1}(G_{ij})/(\det(G_{ij}))^{a_{n1}} & \tilde{f}_{n2}(G_{ij})/(\det(G_{ij}))^{a_{n2}} & \cdots & \tilde{f}_{nn}(G_{ij})/(\det(G_{ij}))^{a_{nn}}
\end{pmatrix}
\]

where \( \tilde{f}_{ij} \) and \( \det(G_{ij}) \) are regular functions on \( G \) which are by definition the restrictions of the regular functions \( f_{ij}(G_{ij}) \) and \( \det(G_{ij}) \) resp. from \( M_n(F) \) to \( G \). By multiplying the numerator and denominator of each matrix entry by a suitable power of \( \det G \), we can assume that all the \( a_{ij} \)'s occurring in the matrix are all equal to some non-negative integer, say \( a \). Let \( w \in W \) be a non-semistable \( F \)-rational point of \( W \). Let \( \lambda(t) \) be an instability
1-PS of w. Then there exists an element \( g \in G \) such that \( gA(t)g^{-1} = \mu(t) \subset T \). Clearly \( \mu(t) \) is an instability 1-PS of \( gw \).

Now as in lemma [14] we will define the elementary polynomials associated to \( w \) to be certain “modified”coefficients that occur in the expansion of \( \rho(G) \cdot w \) in terms of the basis vectors \( \{ w_1, \ldots, w_m \} \). These will be certain polynomials in the \( F[Gl_i]_{1 \leq i \leq n} \).

More precisely, if \( w = \sum_{i=1}^{m} b_i w_i \), then

\[
\rho(G)(w) = \sum_{i=1}^{m} \frac{f_1(G_{ij})b_i}{(\det(G_{ij}))^a} w_1 + \sum_{i=1}^{m} \frac{f_2(G_{ij})b_i}{(\det(G_{ij}))^a} w_2 + \cdots + \sum_{i=1}^{m} \frac{f_m(G_{ij})b_i}{(\det(G_{ij}))^a} w_m.
\]

Define the elementary polynomials of associated to \( w \), denoted \( EP_w \), to be the polynomials \( \{ F_{1w} = \sum_{i=1}^{m} f_1(G_{ij})b_i; \ldots; F_{mw} = \sum_{i=1}^{m} f_m(G_{ij})b_i \} \).

Then clearly,

\[
gw = \sum_{i=1}^{m} \frac{f_1(g_{ij})b_i}{(\det(g_{ij}))^a} w_1 + \sum_{i=1}^{m} \frac{f_2(g_{ij})b_i}{(\det(g_{ij}))^a} w_2 + \cdots + \sum_{i=1}^{m} \frac{f_m(g_{ij})b_i}{(\det(g_{ij}))^a} w_m.
\]

Thus we see that the vanishing or non-vanishing of a particular coefficient of \( gw \) depends on whether or not the corresponding elementary polynomial vanishes at \( g \) or not.

Let \( F_{i_1w}, \ldots, F_{i_rw} \) be exactly the set of elementary polynomials which are vanishing at \( g \). Now as in lemma [14] we would like to find a quantifiable extension \([ L : F ]\) and a element \( g' \in G(L) \) such that \( g'w \) has the same set of coefficients as \( gw \) which are zero. Consider the affine \( F \)-algebra \( B = F[Gl_i]/(h_1, \ldots, h_s, F_{i_1w}, \ldots, F_{i_rw}) \). Let \( G_w \) be the product of all the elementary polynomials \( F_{iw} \) which are non-vanishing at \( g \). Note that \( g' = (g_{ij}) \) is a \( F \)-valued point of \( \text{Spec} A \) at which \( G_w \) in non-vanishing. Hence by lemma [13] there exists an extension field \([ L : F ]\) with \( \text{deg} [ L : F ] \leq \text{deg}( \prod_{j=1}^{r} F_{i_jw} \cdot \prod_{j=1}^{s} h_j ) = d \) (say) and an \( L \)-valued point \( g' \) of \( \text{Spec} A \) at which \( G_w \) is non-vanishing. Since the polynomials \( h_1, \ldots, h_s \) vanish at \( g' \), it follows that \( g' \in G(L) \). Now \( gw \) and \( g'w \) have the same set of coefficients of the \( w_i \)'s which are non-zero. Hence as in the proof of lemma [14] \( \mu(t) \) is also an instability 1-PS of \( g'w \) and hence \( g'^{-1} \mu(t)g' \) is an instability 1-PS of \( w \), which is clearly defined over \( L \). From this it follows easily that if \( t \) is any integer such that \( p' > d \), then for any unstable \( F \)-rational point \( w \in W \) it has an instability 1-PS and hence its instability parabolic defined over \( F^{1/p'} \).

**Definition 20.** Let \( G \) be a reductive algebraic group defined over \( K \). Let \( W \) be a vector space defined over \( K \). Let \( \rho : G \rightarrow GL(W) \) be a representation of \( G \) defined over \( K \). We say that the instability parabolic for subspaces is defined over \( L \) if for the induced action of
\( G \) on \( \wedge^i(W) \), the instability parabolic for any unstable \( K \)-rational point in \( \wedge^i(W) \) is defined over \( L \) for all \( i \) with \( 0 < i \leq m \). Similarly we will say that the instability parabolic for subspaces is rational if \( L \) can be choosen to be \( K \).

**Corollary 21.** Let \( G \) and \( \rho \) be as in the above definition. Then there exists an integer \( t' \) such that the instability parabolic for subspaces is defined over \( K^{1/p'} \). Consequently, if \( E \) is any principal \( G \)-bundle on \( X \) such that \( F^{t'}(E) \) is semistable then the induced vector bundle \( E_W \) is also semistable.

**Proof**  Proof follows immediately from theorem 19 and the proof of theorem 15.

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