ARENS REGULARITY OF PROJECTIVE TENSOR PRODUCTS

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Abstract. For completely contractive Banach algebras $A$ and $B$ (resp. operator algebras $A$ and $B$), necessary and sufficient conditions for the operator space projective tensor product $A\hat{\otimes}B$ (resp. the Haagerup tensor product $A\otimes^h B$) to be Arens regular are obtained. Using the non-commutative Grothendieck’s inequality, we show that, for $C^*$-algebras $A$ and $B$, the Arens regularity of Banach algebras $A\otimes^\alpha B$ and $A\otimes^\beta B$ is equivalent, where $\alpha$ and $\beta$ being any algebra cross-norms which lies between the Haagerup and the Banach space projective tensor norm.

1. Introduction

For a Hilbert space $H$, let $B(H)$ denote the space of all bounded operators on $H$. An operator space $X$ on $H$ is a closed subspace of $B(H)$. For operator spaces $X$ and $Y$, and $u$ an element in the algebraic tensor product $X\otimes Y$, the operator space projective tensor norm is defined to be

$$\|u\|_\wedge = \inf\{\|\alpha\|\|x\|\|y\|\|\beta\| : u = \alpha(x \otimes y)\beta\},$$

where $\alpha \in M_{1,pq}$, $\beta \in M_{pq,1}$, $x \in M_p(X)$ and $y \in M_q(Y)$, $p, q \in \mathbb{N}$, and $x \otimes y = (x_{ij} \otimes y_{kl})_{(i,j)(k,l)} \in M_{pq}(X \otimes Y)$. The normed space $(X \otimes Y, \| \cdot \|_\wedge)$ will be denoted by $X \otimes^\wedge Y$ and the completion of $X \otimes^\wedge Y$ is denoted by $X\hat{\otimes}Y$, known as the operator space projective tensor product of $X$ and $Y$. The Haagerup norm on the algebraic tensor product of two operator spaces $X$ and $Y$ is defined, for $u \in X \otimes Y$, by $\|u\|_h = \inf\{\|x\|\|y\|\}$, where infimum is taken over all the ways to write $u = x \odot y = \sum_{k=1}^r x_{1k} \otimes y_{k1}$, where $x \in M_{1,r}(X), y \in M_{r,1}(Y), r \in \mathbb{N}$. The Haagerup tensor product $X \otimes^h Y$ is defined to be the completion of $X \otimes Y$ in the norm $\| \cdot \|_h$.

It is well known that the Haagerup tensor norm is injective, associative, functorial, projective and may be used to linearize the complete bounded bilinear forms, that is, $CB(X \times Y, \mathbb{C}) \cong (X \otimes^h Y)^*$. $CB(X \times Y, \mathbb{C})$ denotes the collection of complete bounded bilinear forms on $X \times Y$, where a bilinear form $\varphi : X \times Y \to \mathbb{C}$ is said to be completely bounded if $\|\varphi\|_{cb} := \sup_n \|\varphi_n\| < \infty$.

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The $n^{th}$ amplification is a map $\varphi_n : M_n(X) \times M_n(Y) \to M_n(\mathbb{C})$ defined by specifying the $(i,j)$ entry of $\varphi_n((x_{ij}),(y_{ij}))$ to be $\sum_k \varphi(x_{ik},y_{kj})$.

The operator space projective tensor norm is projective, functorial, associative and symmetric. But it is not injective and may be used to linearize the jointly completely bounded bilinear forms, that is, $JCB(X \times Y, \mathbb{C}) \cong (X \hat{\otimes} Y)^*$. $JCB(X \times Y, \mathbb{C})$ denotes the Banach space of jointly completely bounded bilinear forms, which becomes an operator space by identifying $M_n(JCB(X \times Y, \mathbb{C}))$ with $JCB(X \times Y, M_n(\mathbb{C}))$, for each $n \in \mathbb{N}$, and a bilinear form $\varphi : X \times Y \to \mathbb{C}$ is said to be jointly completely bounded if the associated maps $\varphi_n : M_n(X) \times M_n(Y) \to M_n(\mathbb{C})$ given by

$$\varphi_n((x_{ij}),(y_{kl})) = (\varphi(x_{ij},y_{kl})),$$

are uniformly bounded, and in this case we denote $\|\varphi\|_{jcb} = \sup_n \|\varphi_n\|$, see [7], [18] for the development of tensor product of operator spaces.

In 1951, Arens showed that the second dual space $A^{**}$ of a Banach algebra $A$ admits two Banach algebra products known as the first and second Arens products. Each of these products extends the original multiplication on $A$ when $A$ is canonically imbedded in its second dual $A^{**}$. In this note, we wish to draw attention when the two Arens products agree on the second dual of $A \hat{\otimes} B$ (resp. $A \otimes^h B$) for completely contractive Banach algebras $A$ and $B$ (resp. for operator algebras $A$ and $B$). It is shown that for completely contractive Banach algebras $A$ and $B$ (resp. for operator algebras $A$ and $B$), $A \hat{\otimes} B$ (resp. $A \otimes^h B$) is Arens regular if and only if every jointly completely bounded bilinear form $m : A \times B \to \mathbb{C}$ (resp. every completely bounded bilinear form $m : A \times B \to \mathbb{C}$) is biregular. This is then used to show the astonishing fact that, for $C^*$-algebras $A$ and $B$, the Arens regularity of $A \hat{\otimes} B$ and $A \otimes^h B$ (resp. $A \otimes^\gamma B$) is equivalent, which further implies that the Arens regularity of Banach algebras $A \otimes^\alpha B$ and $A \otimes^\beta B$ is equivalent, where $\alpha$ and $\beta$ being any algebra cross-norms which lies between the Haagerup and the Banach space projective tensor norm. Furthermore, for exact operator algebras $V$ and $W$, the Arens regularity of $V \otimes^h W$ and $V \hat{\otimes} W$ is shown to be equivalent.

2. ARENS REGULARITY OF $A \hat{\otimes} B$

For operator space $X$, a closed subspace $\tilde{X}$ of $X$ is said to be completely complemented if there exists a completely bounded (cb) projection $P$ from $X$ onto $\tilde{X}$.

We begin by stating a lemma which follows easily using the functorial property of operator space projective tensor product.

**Lemma 2.1.** Let $\tilde{X}, \tilde{Y}$ be completely complemented subspaces of the operator spaces $X$ and $Y$ complemented by cb projection having cb norm equal to 1, respectively. Then $\tilde{X} \hat{\otimes} \tilde{Y}$ is a closed subspace of $X \hat{\otimes} Y$. 
For any normed space $X$, $X_1$ and $X_1^\alpha$ will denote the closed unit ball and the open unit ball of $X$, respectively. For normed space $X$ and $Y$, the normed linear space obtained by equipping $X \otimes Y$ with $\| \cdot \|_\alpha$ norm is denoted by $X \otimes_\alpha Y$, and the completion of $X \otimes_\alpha Y$ is denoted by $X \bar{\otimes}_\alpha Y$.

**Lemma 2.2.** For normed spaces $X$ and $Y$, the closed unit ball of $X \bar{\otimes}_\alpha Y$, $\alpha$ being any cross-norm which lies between the injective and the projective tensor norm, is the closed convex hull of the set $X_1 \otimes Y_1$.

**Proof.** It is easy to see that the closed unit ball of $X \otimes_\alpha Y$ is the closure of the closed unit ball of $X \otimes Y$. Therefore, it suffices to show that $(X \otimes_\alpha Y)_1 = \text{cl}(\text{co}(X_1 \otimes Y_1))$. For this, let $u$ be in the open unit ball of $X \otimes Y$, and so that $\|u\|_\lambda < 1$, where $\| \cdot \|_\lambda$ being the injective tensor norm. Since $u \in X \otimes Y$, so there is a smallest natural number $n$ such that $u = \sum_{i=1}^n a_i \otimes b_i$, with $\{a_i : i = 1, 2, \ldots, n\}$ and $\{b_i : i = 1, 2, \ldots, n\}$ linearly independent. Choose $f \in X^*$ and $g \in Y^*$ such that $f(a_i) = \|a_i\|$ and $g(b_i) = \|b_i\|$, for $i = 1, 2, \ldots, n$, with $\|f\| = \|g\| = 1$. So $\sum_{i=1}^n \|a_i\| \|b_i\| < 1$. Now let $w_i = \|a_i\|^{-1} a_i$, $z_i = \|b_i\|^{-1} b_i$ and $\lambda_i = \|a_i\| \|b_i\|$, for $i = 1, 2, \ldots, n$. Then, we have $u = \sum_{i=1}^n \lambda_i w_i \otimes z_i$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i < 1$. Hence $(X \otimes_\alpha Y)_1 \subseteq \text{cl}(\text{co}(X_1 \otimes Y_1))$. Using the fact that $\| \cdot \|_\alpha$ is a cross norm on $X \otimes Y$, we get $X_1 \otimes Y_1 \subseteq (X \otimes_\alpha Y)_1$. Since $\text{cl}(\text{co}(X_1 \otimes Y_1))$ is the smallest closed convex set containing $X_1 \otimes Y_1$, so $\text{cl}(\text{co}(X_1 \otimes Y_1)) \subseteq (X \otimes_\alpha Y)_1$, and hence the result. \[\square\]

Next we discuss the Arens regularity of $A \hat{\otimes} B$. The second dual $A^{**}$ of a Banach algebra $A$ possesses two natural Banach algebra products, say $\square$ and $\hat{\circ}$. We briefly recall the definition of these products. For $F \in A^{**}$, $G \in A^{**}$ and $f \in A^*$, the two multiplications on $A^{**}$ are given by

\[F \square G(f) = F(gf) \text{ and } F \hat{\circ} G(f) = G(fG).\]

Here $gf, fG$ are the elements of $A^*$ defined by $f_G(b) = F(bf)$ and $gf(b) = G(fb)$ for any $b \in A$, and $bf \in A^*$, $fb \in A^*$ given by $bf(a) = f(ab)$ and $fb(a) = f(ba)$ for any $a \in A$.

We say that $A$ is Arens regular if the two products coincide, i.e.

\[F \square G = F \hat{\circ} G \text{ for all } F, G \in A^{**}.\]

A functional $f \in A^*$ is said to be wap (weakly almost periodic) on $A$ if the set $\{af : a \in A_1\}$ is relatively weakly compact (rwc). This is equivalent to saying that the bounded operator $L_f : A \to A^*$, defined by $L_f(a) = fa$, is weakly compact or the following “Double Limit Criterion” \[11\] holds: for any two nets $(a_i)_i, (b_j)_j$ in $A_1$,

\[\lim_{i} \lim_{j} f(a_ib_j) = \lim_{j} \lim_{i} f(a_ib_j)\]
whenever both iterated limits exist. Let \( \text{wap}(A) \) be a set consisting of all weakly almost periodic functionals on \( A \). It is known that \( A \) is Arens regular if and only if \( \text{wap}(A) = A^* \). Every operator algebra, in particular, every \( C^* \)-algebra, is Arens regular.

A bilinear form \( m : A \times B \to \mathbb{C} \) is said to be biregular if for any two pairs of sequences \( (a_i, b_j) \in A_1 \) and \( (c_i, d_j) \in B_1 \), we have \( \lim_{i,j} m(a_i b_j, c_i d_j) = \lim_{j} \lim_{i} m(a_i b_j, c_i d_j) \), provided that these limits exist. It was shown in [21] that the Banach space projective tensor product \( A \otimes^\pi B \) is Arens regular if and only if every bounded bilinear form is biregular. We now prove its analogous result for the operator space projective and the Haagerup tensor product.

It has been observed that \( A \hat{\otimes} B \) and \( A \otimes^h B \) are Banach algebras for operator algebras \( A \) and \( B \) with respect to natural multiplication \( (a \otimes b)(c \otimes d) = ac \otimes bd \) for all \( a, c \in A \) and \( b, d \in B \). One can use the same technique given therein for completely contractive Banach algebras so as to obtain the following:

**Proposition 2.3.** For any completely contractive Banach algebras \( A \) and \( B \), \( A \hat{\otimes} B \) is a completely contractive Banach algebra, and it is a Banach \( * \)-algebra under the natural involution provided both \( A \) and \( B \) have isometric involution.

**Proposition 2.4.** For completely contractive Banach algebras \( A \) and \( B \) (for operator algebras \( A \) and \( B \)), \( A \hat{\otimes} B \) (resp. \( A \otimes^h B \)) is Arens regular if and only if every jointly completely bounded bilinear form \( m : A \times B \to \mathbb{C} \) (resp. every completely bounded bilinear form \( m : A \times B \to \mathbb{C} \)) is biregular.

**Proof.** Note that, by Proposition 2.3, \( A \hat{\otimes} B \) is a Banach algebra. Assume that \( A \hat{\otimes} B \) is Arens regular. Let \( m : A \times B \to \mathbb{C} \) be a jointly completely bounded bilinear form such that for any two pairs of sequences \( (a_i, b_j) \) in \( A_1 \) and \( (c_i, d_j) \) in \( B_1 \), the iterated limits \( \lim_{i,j} m(a_i b_j, c_i d_j) \) exist. Since \( (A \hat{\otimes} B)^* = JCB(A \times B, \mathbb{C}) \), so there exists \( f \in (A \hat{\otimes} B)^* \) such that \( f(a \otimes b) = m(a, b) \) for all \( a \in A \) and \( b \in B \). By Lemma 2.2, both the sequence \( (a_i \otimes c_i) \) and \( (b_j \otimes d_j) \) belong to the closed unit ball of \( A \hat{\otimes} B \). So \( \lim_{i,j} f(a_i b_j \otimes c_i d_j) = \lim_{i,j} f(a_i b_j \otimes c_i d_j) \) by the assumption and hence \( \lim_{i,j} m(a_i b_j, c_i d_j) = \lim_{i,j} m(a_i b_j, c_i d_j) \).

For the converse, let \( f \) be a linear functional on \( A \hat{\otimes} B \) and \( m : A \times B \to \mathbb{C} \) be the corresponding jointly completely bounded bilinear form. Consider \( L_f((A \hat{\otimes} B)_1) \subseteq \text{cl}(\text{co}L_f(A_1 \otimes B_1)) = \text{cl}(\text{co}H(f)) \), where \( H(f) = \{ fa \otimes b : a \in A_1 \text{ and } b \in B_1 \} \). By Krein-Smulian theorem, it suffices to show that \( H(f) \) is relatively weakly compact. By (21), Lemma 3.3), \( H(f) \) is relatively weakly compact if and only if for any two sequences \( (a_i \otimes b_i) \) and \( (\tilde{a}_j \otimes \tilde{b}_j) \) in \( A_1 \otimes B_1 \), we have \( \lim_{i,j} f\tilde{a}_j \otimes \tilde{b}_j(a_i \otimes b_i) = \lim_{i,j} f\tilde{a}_j \otimes \tilde{b}_j(a_i \otimes b_i) \), provided that
these limits exist; i.e. we have $\lim \lim f(a_i \tilde{a}_j \otimes b_i \tilde{b}_j) = \lim \lim f(a_i \tilde{a}_j \otimes b_i \tilde{b}_j)$, which is true as $m$ being biregular.

**Corollary 2.5.** Suppose that the algebras $A$ and $B$ are not trivial and that $A \hat{\otimes} B$ (resp. $A \otimes^h B$) is Arens regular, then $A$ and $B$ are Arens regular.

**Proof.** Suppose that $f \in A^*$ and $(a_i), (c_j)$ are sequences in $A_1$ such that $\lim \lim f(a_i c_j)$ and $\lim \lim f(a_i c_j)$ exist. Since $B$ is not trivial, there exist $b \neq 0$, $b' \neq 0$ such that $bb' \neq 0$. Using Hahn Banach theorem, select $g \in B^*$ such that $g(bb') = 1$. Now define, $f \otimes g : A \otimes B \to \mathbb{C}$ as $f \otimes g(a \otimes b) = f(a)g(b)$.

Then for $x = \sum_{i=1}^{n} a_i \otimes b_i \in A \otimes B$, $|f \otimes g(x)| \leq \|f\|\|g\|\|x\|_\lambda \leq \|f\|\|g\|\|x\|_\lambda$.

so that $f \otimes g$ can be extended to the continuous linear functional on $A \hat{\otimes} B$. Let the extension be denoted by $f \hat{\otimes} g$. Now, let $b_i = b$ and $d_j = b'$ for all $i$, $j$. Then $\lim \lim f \hat{\otimes} g(a_i c_j \otimes b d_j) = \lim \lim f \hat{\otimes} g(a_i c_j \otimes b d_j)$, by Theorem 2.4.

Hence $\lim \lim f(a_i c_j) = \lim \lim f(a_i c_j)$, so $A$ is Arens regular. Similarly, $B$ is Arens regular.

In particular, for an amenable locally compact Hausdorff groups $G$ and $H$.

by using the fact that $A(G) \hat{\otimes} A(H)$ is completely isometrically isomorphic to $A(G \times H)$ [6] and ([15], Proposition 3.3), $A(G) \hat{\otimes} A(H)$ is Arens regular if and only if $G$ and $H$ are finite. In general, for a locally compact groups $G$ and $H$, if $A(G) \hat{\otimes} A(H)$ ($A(G) \otimes^h A(H)$) is Arens regular then $G$ and $H$ are discrete ([8], Theorem 3.2). Furthermore, $A(G) \hat{\otimes} A(H)$ is not Arens regular if $G$ and $H$ contain an infinite abelian subgroup [9].

We are now prepared to present the main results.

**Theorem 2.6.** For completely contractive Banach algebras $A$ and $B$, if $A \otimes B$ is Arens regular then $A \otimes^h B$ is. Conversely, for $C^*$-algebras $A$ and $B$, if $A \otimes^h B$ is Arens regular then $A \hat{\otimes} B$ is.

**Proof:** First part follows directly by the fact that there is a contractive homomorphism map from $A \otimes B$ into $A \otimes^h B$ [7].

For the converse, we first claim that, for $C^*$-algebra $A$, if $A \otimes^h B$ is Arens regular then $A \hat{\otimes} A$ is. Assume that $f \in JCB(A \times A, \mathbb{C})$ be such that for sequences $(a_i)$, $(b_j)$ in $A_1$ and $(c_i)$, $(d_j)$ in $A_1$, the iterated limits $\lim \lim f(a_i b_j, c_i d_j)$ and $\lim \lim f(a_i b_j, c_i d_j)$ exist. By ([12], Lemma 3.1), $f$ can be decomposed as $f = f_1 + f_2$, where $f_1$ and $f_2$ are bounded bilinear forms with $\|f_1\|_cb \leq \|f\|_{jcb}$ and $\|f_2\|_cb \leq \|f\|_{jcb}$, where $f_2(a, b) = f_2(a, b)$ for all $a, b \in A$. Since $f_1$ is a bounded bilinear form, so there exist subsequences $(a_i)_{i}$, $(b_j)_{i}$, $(c_i)_{i}$ and $(d_j)_{i}$ such that the iterated limits $\lim \lim f_1(a_i b_{j_k}, c_{i_k} d_{j_k})$ and $\lim \lim f_1(a_i b_{j_k}, c_{i_k} d_{j_k})$ exist. Since $f_2 = f - f_1$, so we can assume that $\lim \lim f_2(a_i b_{j_k}, c_{i_k} d_{j_k})$ and $\lim \lim f_2(a_i b_{j_k}, c_{i_k} d_{j_k})$
also exist. Therefore, \( \lim_{i_k, j_k} f(a_{i_k}, b_{j_k}, c_{i_k}, d_{j_k}) = \lim_{i_k, j_k} f_1(a_{i_k}, b_{j_k}, c_{i_k}, d_{j_k}) + \lim_{i_k, j_k} f_2(a_{i_k}, b_{j_k}, c_{i_k}, d_{j_k}) \), which is further equal to \( \lim_{i_k, j_k} f_1(a_{i_k}, b_{j_k}, c_{i_k}, d_{j_k}) + \lim_{i_k, j_k} f_2(c_{i_k}, d_{j_k}, a_{i_k}, b_{j_k}) \). Now by the Arens regularity of \( A \otimes^h A \) we have 
\[
\lim_{i_k, j_k} f(a_{i_k}, b_{j_k}, c_{i_k}, d_{j_k}) = \lim_{i_k, j_k} f(a_{i_k}, b_{j_k}, c_{i_k}, d_{j_k}).
\]
Let \( \alpha = : \lim_{i} f(a_i b_j, c_i d_j) \) and \( \beta = : \lim_{j} f(a_i b_j, c_i d_j) \). For each \( i \) and \( j \), set \( \alpha_i = \lim f(a_i b_j, c_i d_j) \) and \( \beta_j = \lim f(a_i b_j, c_i d_j) \). Since \( f(a_{i_k}, b_{j_k}, c_{i_k}, d_{j_k}) \) is a sequence which converges to \( \alpha_{i_k} \), so every subsequence of it also converges to \( \alpha_{i_k} \). Therefore, we have 
\[
\lim_{i_k, j_k} \alpha_{i_k} = \lim_{i_k, j_k} \beta_{j_k} \text{ and hence } \alpha = \beta.
\]

Now assume that \( A \otimes^h B \) is Arens regular. So \( ((A \otimes^h B) \oplus (A \otimes^h B)) \), where \( \oplus_1 \) denotes the direct sum of the algebras with 
\[
\| (u, v) \|_1 = \| u \|_h + \| v \|_h.
\]
We now claim that \( (A \oplus B) \oplus^h (A \oplus B) \) is Arens regular, where \( \oplus_\infty \) being the direct product of algebras with \( \| (a, b) \|_\infty = \max \{ \| a \|, \| b \| \} \). Consider a map \( \theta \) define as 
\[
\theta \left( \sum_{i=1}^{\infty} a_i \otimes b_i, \sum_{i=1}^{\infty} c_i \otimes d_i \right) = \sum_{i=1}^{\infty} (a_i, d_i) \otimes (c_i, b_i),
\]
where \( \{ a_i \}_{i=1}^{\infty}, \{ b_i \}_{i=1}^{\infty}, \{ c_i \}_{i=1}^{\infty} \) and \( \{ d_i \}_{i=1}^{\infty} \) are strongly independent [1]. We claim that \( \theta \) is a bijective continuous algebra homomorphism. Using (13, Proposition 4.4) and the strongly independent of \( \{ (a_i, d_i) \} \), it follows easily that \( \theta \) is injective. For the continuity of \( \theta \), consider 
\[
\| \theta \left( \sum_{i=1}^{\infty} a_i \otimes b_i, \sum_{i=1}^{\infty} c_i \otimes d_i \right) \|_h = \| \sum_{i=1}^{\infty} (a_i, d_i) \otimes (c_i, b_i) \|_h
\]
\[
\leq \| \sum_{i=1}^{\infty} (a_i a_i^*, d_i d_i^*) \|_h \}
\[
\sum_{i=1}^{\infty} (c_i^* c_i, b_i b_i) \|_h \}
\]
\[
\leq \max \{ \| a_i a_i^* \|, \| d_i d_i^* \| \}
\]
\[
\leq \frac{1}{2} \max \{ \| a_i a_i^* \|, \| d_i d_i^* \| \}
\]
Thus \( \| \sum_{i=1}^{\infty} (a_i, d_i) \otimes (b_i, c_i) \|_h \leq \frac{1}{2} (\| \sum_{i=1}^{\infty} a_i a_i^* \| + \| \sum_{i=1}^{\infty} d_i d_i^* \| + \| \sum_{i=1}^{\infty} b_i b_i \| + \| \sum_{i=1}^{\infty} c_i^* c_i \|) \). We can rewrite it as 
\[
\| \sum_{i=1}^{\infty} (t^{1/2} a_i, t^{1/2} d_i) \otimes (t^{-1/2} b_i, t^{-1/2} c_i) \|_h \leq \frac{1}{2} (t \| \sum_{i=1}^{\infty} a_i a_i^* \| + t \| \sum_{i=1}^{\infty} d_i d_i^* \| + t^{-1} \| \sum_{i=1}^{\infty} b_i b_i \| + t^{-1} \| \sum_{i=1}^{\infty} c_i^* c_i \|) \text{ for any } t > 0.
\]
Take infimum over \( t > 0 \) and use the fact that 
\[
\inf_{t>0} \frac{ta + t^{-1}b}{2} = \sqrt{ab},
\]
we get
Thus $\|\sum_{i=1}^{\infty} a_i \otimes b_i, c_i \|_h \leq \| \sum_{i=1}^{\infty} a_i a_i^* \|^{1/2} \| \sum_{i=1}^{\infty} b_i^* b_i \|^{1/2} + \| \sum_{i=1}^{\infty} d_i d_i^* \|^{1/2} \| \sum_{i=1}^{\infty} c_i^* c_i \|^{1/2}$. Hence $\theta$ is continuous. One can easily verify that $\theta$ is an onto algebra homomorphism. Thus, by the bounded inverse theorem, $\theta^{-1}$ is continuous. This gives that $(A \oplus B)_{\infty} \hat{\otimes} (A \oplus B)_{\infty}$ is Arens regular. Now define $P : (A \oplus B)_{\infty} \to A$ as $(a, b) \to a$. Clearly, $P$ is a completely positive contraction, and $P(a_1(a, b)) = P((a_1, 0)(a, b)) = P(a_1a, 0) = a_1a = a_1P(a, b)$ for $a_1, a \in A$ and $b \in B$. Therefore, $P$ is a conditional expectation from $(A \oplus B)_{\infty}$ onto A. Thus $A \hat{\otimes} B$ is a closed subalgebra of $(A \oplus B)_{\infty} \hat{\otimes} (A \oplus B)_{\infty}$, by Lemma [2,1] and hence $A \hat{\otimes} B$ is Arens regular.

In particular for an amenable locally compact Hausdorff groups $G$ and $H$, $A(G) \hat{\otimes} h A(H)$ is Arens regular if and only if $G$ and $H$ are finite.

For any two vector spaces $V$ and $W$, since a sesquilinear form on $V \times W$ can be viewed as a complex bilinear form on $V \times \overline{W}$, $\overline{W}$ being the complex conjugate of a vector space $W$. So, for operator spaces $X$ and $Y$, we say that a sesquilinear form $\phi$ on $X \times Y$ is jointly completely bounded if $\|\phi\|_{jcb} := \sup\{||\phi(x_{ij}, y_{kl})|| : \|x_{ij}\|_{M_n(X)} \leq 1, \|y_{ij}\|_{M_n(Y)} \leq 1\} < \infty$, where $\overline{\gamma}$ is the conjugate of an operator space $Y$. It is known that, for a $C^*$-algebra $A$, the complex conjugate of the operator space $A$ and the opposite $C^*$-algebra $A^\theta$ are $C^*$-isomorphic. Therefore, for $C^*$-algebras $A$ and $B$, a sesquilinear form $\phi$ on $A \times B$ is jointly completely bounded if $\|\phi\|_{jcb} = \sup\{||\phi(x_{ij}, y_{kl})|| : \|x_{ij}\|_{M_n(A)} \leq 1, \|y_{ij}\|_{M_n(B^{op})} \leq 1\} < \infty$. For more details about the complex conjugate of an operator space and the opposite operator spaces, the reader may refer [18].

Now note that every jointly completely bounded bilinear form on $A \times B$ is biregular if and only if every sesquilinear jointly completely bounded form on $A \times B$ is biregular. Indeed, suppose that every jointly completely bounded bilinear form on $A \times B$ is biregular, and take $\phi$ to be a sesquilinear jointly completely bounded bilinear form on $A \times B$ such that for sequences $(a_i), (b_j)$ in $A_1$ and $(c_i), (d_j)$ in $B_1$, the iterated limits $\lim_{j} \lim_{i} \phi(a_i b_j, c_i d_j)$ and $\lim_{i} \lim_{j} \phi(a_i b_j, c_i d_j)$ exist. Now define $\psi : A \times B \to \mathbb{C}$ as $\psi(a, b) = \phi(a, b^*)$ for all $a \in A$ and $b \in B$. Clearly, $\psi$ is a sesquilinear form. Consider $\psi_n((a_{ij}), (b_{kl})) = (\psi(a_{ij}, b_{kl})) = (\phi(a_{ij}, b_{kl}^*)) = \phi_n((a_{ij}), (b_{kl}^*))$. Therefore,

\[
\|\psi_n((a_{ij}), (b_{kl}))\| = \|\phi_n((a_{ij}), (b_{kl}^*))\| \leq \|\phi\|_{jcb} \|\phi_n((a_{ij}), (b_{kl}^*))\|_{M_n(A)} \|b_{kl}^*\|_{M_n(B^{op})} \\
\leq \|\phi\|_{jcb} \|\phi_n((a_{ij}), (b_{kl}^*))\|_{M_n(A)} \|b_{kl}^*\|_{M_n(B^{op})} \\
= \|\phi\|_{jcb} \|\phi_n((a_{ij}), (b_{kl}^*))\|_{M_n(A)} \|b_{kl}^*\|_{M_n(B^{op})} \\
= \|\phi\|_{jcb} \|\phi_n((a_{ij}), (b_{kl}^*))\|_{M_n(A)} \|b_{kl}^*\|_{M_n(B^{op})}.
\]

Thus $\psi$ is a jointly completely bounded bilinear form. Further, note that $\psi(a_i b_j, c_i^* d_j) = \phi(a_i b_j, c_i^* d_j)$, and hence the claim. In fact, from Proposition
this observation leads to the fact that $A \hat{\otimes} B$ is Arens regular if and only if $A \hat{\otimes} B^{op}$ is.

**Theorem 2.7.** For completely contractive Banach algebras $A$ and $B$, if $A \otimes^\gamma B$ is Arens regular then $A \hat{\otimes} B$ is. Conversely, for $C^*$-algebras $A$ and $B$, if $A \hat{\otimes} B$ is Arens regular then $A \otimes^\gamma B$ is.

**Proof.** Using the fact that there is a canonical homomorphism $i : A \otimes^\gamma B \to A \hat{\otimes} B$ [7], the Arens regularity of $A \otimes^\gamma B$ implies that of $A \hat{\otimes} B$.

For the converse, let $m : A \times B \to \mathbb{C}$ be a bounded bilinear form such that for sequences $(a_i), (b_j)$ in $A_1$ and $(c_i), (d_j)$ in $A_1$, the iterated limits $\lim_{i \to j} \lim_{j \to i} m(a_ib_j, c_id_j)$ and $\lim_{j \to i} \lim_{i \to j} m(a_ib_j, c_id_j)$ exist. By the non-commutative version of Grothendieck’s inequality to the setting of bounded bilinear forms on $C^*$-algebras, $m$ can be decomposed as $m = m_1 + m_2$, where $m_1$ and $m_2$ are jointly completely bounded bilinear forms on $A \times B$ and $A \times B^{op}$ respectively [17]. Using the fact that $A \hat{\otimes} B$ is Arens regular if and only if $A \hat{\otimes} B^{op}$ is and a similar argument as in Theorem 2.6, we obtain the required result.

In particular, for a compact Hausdorff group $G$, $C(G) \hat{\otimes} C(G)$ and $C(G) \hat{\otimes}^h C(G)$ are not Arens regular Banach algebras as $C(G) \otimes^\gamma C(G)$ is not [23], $C(G)$ being the commutative $C^*$-algebra of continuous functions on $G$.

**Corollary 2.8.** For any two algebra cross-norms $\alpha$ and $\beta$ on $A \otimes B$ which lies between the Haagerup and the Banach space projective tensor norm, the Arens regularity of Banach algebras $A \otimes^\alpha B$ and $A \otimes^\beta B$ is equivalent.

In particular, the Arens regularity of all Banach algebras $A \otimes^h B$, the Schur tensor product of $C^*$-algebras $A$ and $B$ [20], $A \otimes^h B$, $A \otimes^\gamma B$ and $A \hat{\otimes} B$ are equivalent.

By [15], Theorem 7.6 and the above results, we have

**Corollary 2.9.** Let $A$ be a unital $C^*$-algebra such that the von Neumann algebra $A^{**}$ is not finite. Then the algebra $A \hat{\otimes} A$ (resp. $A \hat{\otimes}^h A$) is not Arens regular. In particular, $A \hat{\otimes} A^{**}$ (resp. $A \hat{\otimes}^h A^{**}$) and $A^{**} \hat{\otimes} A^{**}$ (resp. $A^{**} \hat{\otimes}^h A^{**}$) are not Arens regular.

In particular, for an infinite dimensional Hilbert space $H$, $B(H) \hat{\otimes} B(H)$, $B(H) \hat{\otimes} K(H)$ and $K(H) \hat{\otimes} B(H)$ are not Arens regular and so $B(H) \hat{\otimes} K(H)$ and $K(H) \hat{\otimes} B(H)$ is not Arens regular by using [5], Corollary 6.3. Also, by [16], Proposition 7.7 and Lemma 7.8), $K(H) \hat{\otimes} K(H)$ is not Arens regular.

Recall that an operator space $X$ is exact if and only if it is locally embeds into a nuclear $C^*$-algebra (say $A$), i.e. there is a constant $C$ such that for any finite dimensional $E \subseteq X$, there is a subspace $\tilde{E} \subseteq A$ and an isomorphism $u : E \to \tilde{E}$ with $\|u\|_{cb}^\gamma \|u^{-1}\|_{cb} \leq C$. Now using this definition of exact operator space and the fact that direct sum of two nuclear $C^*$-algebras is nuclear if and only if each one of them is, one can easily verify that if $V$ and $W$ are exact operator algebras then $V \oplus W$ with sup-norm is also exact.
Using the same idea as in Theorem 2.6 and appealing to the non-commutative version of Grothendieck’s inequality to the setting of jointly completely bounded bilinear forms on exact operator spaces ([17], Theorem 0.4), we have

**Proposition 2.10.** For exact operator algebras \( V \) and \( W \), the Arens regularity of \( V \hat{\otimes} W \) and \( V \otimes^h W \) is equivalent.

**Proposition 2.11.** Let \( A \) and \( B \) be any operator algebras such that every weakly compact operator from \( A \) to \( B^* \) is compact. Then \( B \otimes^h A \) is Arens regular. Conversely, assume that, for each \( a \in A \) and \( b \in B \), one of the left multiplication operators \( a^\tau \) or \( b^\tau \) is compact and the other is weakly compact. Then every weakly compact operator from \( A \) to \( B^* \) is compact.

**Proof.** Let \( m \) be a completely bounded bilinear form and \( \tilde{m} : A \to B^* \) be the corresponding operator given by \( \tilde{m}(a)(b) = m(a, b) \). By ([7], Lemma 13.3.1), \( \tilde{m} \) can be factored through a column Hilbert space so is weakly compact. Hence \( \tilde{m} \) is compact by hypothesis. Therefore, by ([21], Theorem 4.5), \( m \) is biregular and so \( B \otimes^h A \) is Arens regular. Converse part follows from ([22], Theorem 5.3).

In particular, for any operator algebra \( B \) for which \( B^* \) has Schur property, \( B \otimes^h A \) is Arens regular for any operator algebra \( A \). Thus, for a compact Hausdorff group \( G \), \( C^*(G) \otimes^h A \) is Arens regular for any operator algebra \( A \) by ([22], Theorem 4.5). Similarly, for a compact dispersed topological space \( K \), it follows from ([22], Theorem 6.4 and Corollary 6.5) that \( C(K) \otimes^h A \) is Arens regular for any operator algebra \( A \). Also, by ([10], Theorem 3.3), for a closed subalgebra \( B \) of \( K(H) \) having Dunford-Pettis property, \( B \otimes^h A \) is Arens regular for any operator algebra \( A \).

We now change the multiplication on one of the Banach algebra and look at the Arens regularity of the operator space projective, the Banach space projective and the Haagerup tensor product.

One can immediately verify that if the multiplication on one of the Banach algebras is trivial, then \( A \hat{\otimes} B \) (resp. \( A \otimes^h B \)) is always Arens regular. Now let \( B \) be any operator space. Define an algebra multiplication on \( B \) as

\[
b_1b_2 = \phi(b_1)b_2(b_1; b_2 \in B);
\]

where \( \phi \in B^* \) with \( \|\phi\| \leq 1 \). Note that such a \( \phi \) is automatically a multiplicative linear functional and multiplication will be non-trivial provided \( \phi \) is one-to-one. Also, one can easily verify that \( B \) is an associative Banach algebra under the above multiplication. Now by using Ruan’s axiom and
the fact that \(\|\phi\|_{cb} = \|\phi\|\), we have
\[
\|\phi(b_{ij})b'_{ij}\| = \|\sum_{k=1}^{n} \phi(b_{ik})b'_{kj}\| \\
= \|\phi(b_{ij})\|\|b'_{ij}\| \\
\leq \|\phi(b_{ij})\|\|b'_{ij}\| \\
\leq \|b_{ij}\|\|b'_{ij}\|.
\]
Thus \(B\) is an operator algebra by ([4], Theorem 1.3).

**Proposition 2.12.** Let \(A\) be an operator algebra, and let \(B\) be an operator space such that the multiplication on \(B\) is given by
\[
b_1b_2 = \phi(b_1)b_2(b_1; b_2 \in B);
\]
where \(\phi\) is as above. Then \(A \otimes^h B\) is Arens regular.

**Proof:** Consider the left slice map \(L_{\phi} : A \otimes^h B \to A (a \otimes b \to \phi(b)a)\). Note that \(L_{\phi}\) is an algebra homomorphism. We claim that \(L_{\phi}\) is a bijective map. Fix \(b_0 \in B\) such that \(\phi(b_0) = 1\). Thus for any \(a \in A\), \(L_{\phi}(a \otimes b_0) = a\).

Now let \(L_{\phi}\left(\sum_{i=1}^{\infty} a_i \otimes b_i\right) = 0\). As \(\phi\) is one-to-one. Therefore, \(\sum_{i=1}^{\infty} a_i \psi(b_i) = 0\) for any \(\psi \in B^*\) and hence \(\sum_{i=1}^{\infty} a_i \otimes b_i = 0\) by ([14], Proposition 4.4). Thus \(L_{\phi}\) is a bijective map. Now let \(m\) be a completely bounded bilinear form such that for any two pairs of sequences \((a_i), (b_j)\) in \(A_1\) and \((c_i), (d_j)\) in \(B_1\), the iterated limits \(\lim_{i,j} m(a_ib_j, c_id_j)\) and \(\lim_{i,j} m(a_ib_j, c_id_j)\) exist. Let \(f\) be the associated linear functional corresponding to \(m\). It is clear that \(L_{\phi}(a_i \otimes c_i)\) and \(L_{\phi}(b_j \otimes d_j)\) are in \(A_1\). Using the fact that the operator algebras are Arens regular, we have the following equality:
\[
\lim_{i,j} f \circ L_{\phi}^{-1}(L_{\phi}(a_i \otimes c_i)L_{\phi}(b_j \otimes d_j)) = \lim_{i,j} f \circ L_{\phi}^{-1}(L_{\phi}(a_i \otimes c_i)L_{\phi}(b_j \otimes d_j))
\]
Thus, by the algebra homomorphism of \(L_{\phi}\), we have
\[
\lim_{i,j} f(a_ib_j \otimes c_id_j) = \lim_{i,j} f(a_ib_j \otimes c_id_j)
\]
and hence the result follows from Theorem 2.4.

By Proposition 2.10 for exact operator algebras \(A\) and \(B\) for which multiplication on \(B\) is defined by \(b_1b_2 = \phi(b_1)b_2\), \(A \otimes B\) is Arens regular.

Our next aim is to present multiplications other than the natural multiplication with respect to which \(A \hat{\otimes} B\) or \(A \otimes^h B\) becomes an Arens regular. Let us consider the following multiplication defined on elementary tensor as:
\[
(a \otimes b)(c \otimes d) = m(a,b)(c \otimes d),
\]
for \(a, c \in A\) and \(b, d \in B\), where \(m\) is a bounded bilinear form on \(A \times B\) with \(\|m\| \leq 1\).
Proposition 2.13. For any two Banach algebras $A$ and $B$, $A \otimes_B^\gamma B$ with respect to the above multiplication is an Arens regular Banach algebra.

Proof. For $a, c \in A$ and $b, d \in B$, \( a \otimes bf(c \otimes d) = f((c \otimes d)(a \otimes b)) = f(m(c, d)(a \otimes b)) = m(c, d)f(a \otimes b) \) and \( fa \otimes b(c \otimes d) = f((a \otimes b)(c \otimes d)) = m(a, b)f(c \otimes d) \). Thus \( a \otimes bf = f(a \otimes b)m \) and \( fa \otimes b = m(a, b)f \), which further gives \( F(a \otimes b) = F(fa \otimes b) = F(m(a, b)f) = m(a, b)F(f) = \tilde{m}(a \otimes b)F(f) \) and \( F(a \otimes b) = F(a \otimes bf) = F(f(a \otimes b)\tilde{m}) = F(\tilde{m})f(a \otimes b) \), where \( \tilde{m} \) is a bounded linear functional on \( A \otimes_B^\gamma B \). So by linearity and continuity, it follows that \( F = F(f)\tilde{m} \) and \( Ff = F(\tilde{m})f \). Thus \( F\tilde{G}(f) = F(Gf) = F(G(\tilde{m})f) = G(\tilde{m})F(f) \) and \( F\tilde{G}(f) = G(fF) = F(f)F(\tilde{m}) \), and hence the result. \( \square \)

Note that if we consider the jointly completely bounded bilinear forms and the completely bounded bilinear forms instead of bounded bilinear forms in the above, then similar result holds for the Haagerup tensor product and the operator space projective tensor product.

Another multiplication is given as follows:

\[
(a \otimes b)(c \otimes d) = m(a, d)(b \otimes c),
\]

for $a, c \in A$ and $b, d \in B$, where $m$ is a bounded bilinear form on $A \times B$.

Proposition 2.14. For any two Banach algebras $A$ and $B$, $A \otimes_B^\gamma B$ with respect to the above multiplication is an Arens regular Banach algebra if and only if $m$ is biregular.

Proof. Assume that $m$ is biregular. Let $f \in (A \otimes_B^\gamma B)^*$ be such that for sequences \((a_i, b_j)\) in $A_1$ and \((c_i, d_j)\) in $B_1$ the iterated limits \(\lim \lim f(a_ib_j \otimes c_id_j)\) and \(\lim \lim f(a_i b_j \otimes c_id_j)\) exist. Since $m$ is a bounded bilinear form and $f$ is a bounded linear functional, so there exist subsequences \((a_{ik}, b_{jk})\), \((c_{ik}, d_{jk})\) such that \(\lim \lim m(a_{ik} d_{jk})\), \(\lim \lim m(a_{ik} d_{jk})\), \(\lim \lim f(c_{ik} \otimes b_{jk})\) and \(\lim \lim f(c_{ik} \otimes b_{jk})\) exist. Therefore, by the given hypothesis,

\[
\lim \lim f((a_{ik} \otimes c_{ik})(b_{jk} \otimes d_{jk})) = \lim \lim m(a_{ik} d_{jk})f(c_{ik} \otimes b_{jk})
\]

\[
= \lim \lim m(a_{ik} d_{jk}) \lim \lim f(c_{ik} \otimes b_{jk})
\]

\[
= \lim \lim f((a_{ik} \otimes c_{ik})(b_{jk} \otimes d_{jk})).
\]

Hence the result follows. Converse is trivial. \( \square \)

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