Geometry of the local equivalence of states

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Abstract

We present a description of locally equivalent states in terms of symplectic geometry. Using the moment map between local orbits in the space of states and coadjoint orbits of the local unitary group, we reduce the problem of local unitary equivalence to an easy part consisting of identifying the proper coadjoint orbit and a harder problem of the geometry of fibers of the moment map. We give a detailed analysis of the properties of orbits of ‘equally entangled states’. In particular, we show connections between certain symplectic properties of orbits such as their isotropy and coisotropy with effective criteria of local unitary equivalence.

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1. Introduction

In a recent paper [1], we presented a symplectic description of pure states of composite quantum systems in finite-dimensional Hilbert spaces. In particular, we showed that entanglement among subsystems of a multipartite quantum system can be quantified in terms of the degeneracy of the canonical symplectic form on the complex projective space restricted to orbits of local, i.e. entanglement preserving unitary groups. In this paper, we would like to continue this line of research by giving a precise geometric description of orbits in low-dimensional cases and, above all, by showing how the proposed geometric approach contributes to a solution of an important problem of local unitary equivalence of states.

The classification of states which are connected by local unitary transformations, i.e. operations on the whole system composed from unitary actions (purely quantum evolutions) each of which is restricted to a single subsystem, has recently become a topic of several studies [2, 3]. To appreciate the experimental importance of such a setting, let us recall that it is a basis for spectacular applications of quantum information technologies like teleportation or dense coding, where the fundamental parts of experiments consist of manipulations restricted to parts of the whole system in distant laboratories.
2. Symplectic geometry of entanglement

We start with a short outline of a symplectic description of quantum correlations in composite systems. For details consult [1]. Thorough expositions of the below employed constructions from symplectic geometry can be found in [4] and [5].

2.1. Space of quantum states as a symplectic manifold

The Hilbert space of a quantum system consisting of \( L \) identical \( N \)-level systems (qubits) is the tensor product

\[
\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_L,
\]

where each \( \mathcal{H}_k \) is isomorphic with the complex \( N \)-dimensional space \( \mathbb{C}^N \) equipped with the standard Hermitian scalar product \( \langle \cdot, \cdot \rangle \). (We will denote by the same symbol the standard scalar product in the whole \( \mathcal{H} \) as long as it does not lead to confusion.)

The set of pure states is the projective space \( \mathbb{P} (\mathcal{H}) \). We denote a canonical projection from \( \mathcal{H} \) to \( \mathbb{P} (\mathcal{H}) \) by \( \pi \) and use the notation \( [v] = \pi (v) \) for \( v \in \mathcal{H} \).

The projective space \( \mathbb{P} (\mathcal{H}) \) is equipped with a natural symplectic structure—the Fubini study form— inherited from the initial Hilbert space \( \mathcal{H} \), where the natural symplectic structure is defined in terms of the imaginary part of the scalar product. For further purposes, it is convenient to calculate the symplectic form on \( \mathbb{P} (\mathcal{H}) \) in the following way. First observe that the linear action of the unitary group \( U (\mathcal{H}) \) on \( \mathcal{H} \) projects in a natural way to \( \mathbb{P} (\mathcal{H}) \) as

\[
U [v] := [U v], \quad U \in U (\mathcal{H}), \quad v \in \mathcal{H}, \quad [v] = \pi (v).
\]

Let \( A \in u (\mathcal{H}) := \text{Lie} (U (\mathcal{H})) \) (the Lie algebra of \( U (\mathcal{H}) \) which also acts linearly on \( \mathcal{H} \)). Denote by \( T [v] \mathbb{P} (\mathcal{H}) \) the tangent space to \( \mathbb{P} (\mathcal{H}) \) at the point \([v] \), and by \( A [v] \) the vector in \( T [v] \mathbb{P} (\mathcal{H}) \) tangent to the curve \( t \mapsto \pi (\exp (i A t) v) \). When \( A \) runs through the whole Lie algebra \( u (\mathcal{H}) \), the corresponding \( A [v] \) spans \( T [v] \mathbb{P} (\mathcal{H}) \) and the symplectic form on \( \mathbb{P} (\mathcal{H}) \) at \([v] \) reads

\[
\omega ([v], B [v]) = \text{Im} \frac{\langle A v | B v \rangle - \langle A v | v \rangle \langle v | B v \rangle}{\langle v | v \rangle^2} = \frac{i}{2} \frac{\{A [v], B [v] \}}{\langle v | v \rangle}, \quad A, B \in u (\mathcal{H}),
\]

where \( \{ \cdot , \cdot \} \) is the Lie bracket (commutator) in \( u (\mathcal{H}) \). One checks that indeed \( \omega \) is nondegenerate and closed, \( d \omega = 0 \), on \( T \mathbb{P} (\mathcal{H}) \) and as such makes \( \mathbb{P} (\mathcal{H}) \) a symplectic manifold. Moreover, as is clear from the above construction, \( \omega \) is invariant with respect to the action (2) of \( U (\mathcal{H}) \). In other words, the action of \( U (\mathcal{H}) \) on \( \mathbb{P} (\mathcal{H}) \) is symplectic.

2.2. Symplectic group actions: moment map

Symplectic actions of semisimple groups lead to another important construction useful in our analysis—the moment map. Let a compact semisimple group \( G \) act on a symplectic manifold \( (M, \omega) \) via symplectomorphisms \( M \ni (g, x) \mapsto \Phi_g (x) \in M \), i.e. we demand that the pullback of the form \( \omega \) by \( \Phi_g \) is the form \( \omega \) itself, \( \Phi_g^* \omega = \omega \). For an arbitrary \( \xi \in g = \text{Lie} (G) \) (the Lie algebra of \( G \)), we define a vector field \( \xi \) (called in the following the fundamental vector field corresponding to \( \xi \))

\[
\xi (x) = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp t \xi} (x).
\]

Since \( G \) acts on \( M \) by symplectomorphisms, there exists a function \( \mu_\xi : M \to \mathbb{R} \) such that

\[
d \mu_\xi = t_\xi \omega := \omega (\xi , \cdot).
\]

A generalization to nonidentical subsystems, i.e. living in spaces of different dimensionality, is straightforward but more tedious.
It can be chosen to be linear in $\xi$, i.e. there exists $\mu(x)$ in the space of linear forms on $g$ (the dual space to $g$ denoted in the following by $g^*$) such that

$$\mu_\xi(x) = (\mu(x), \xi), \quad \mu(x) \in g^*, \quad (6)$$

where $(,)$ is the pairing between $g$ and $g^*$. In this way, we obtain a function $\mu : M \to g^*$ called the moment map.

The group $G$ acts on its Lie algebra $g$ via the adjoint action,

$$\text{Ad}_g \xi = \left. \frac{d}{dt} \right|_{t=0} g \exp t \xi g^{-1} =: g \xi g^{-1}, \quad g \in G, \quad \xi \in g,$$

which dualizes to the coadjoint action on $g^*$,

$$\langle \text{Ad}_g^* \alpha, \xi \rangle = \langle \alpha, \text{Ad}_g^{-1} \xi \rangle = \langle \alpha, g^{-1} \xi g \rangle,$$

for $g \in G, \xi \in g$ and $\alpha \in g^*$. Under our assumption of the semisimplicity of $G$, the momentum map can be chosen equivariant, i.e. for each $x \in M$ and $g \in G$,

$$\mu(\Phi_g(x)) = \text{Ad}_g^* \mu(x)$$

is fulfilled.

Coadjoint orbits, i.e. the orbits of a coadjoint action of $G$ on $g^*$ bear a canonical symplectic structure—the so-called Kirillov–Kostant–Souriau form. Let $\Omega_\alpha$ be the coadjoint orbit going through $\alpha \in g^*$:

$$\Omega_\alpha = \{ \text{Ad}_g^* \alpha : g \in G \}.$$  \hspace{1cm} (10)

For any $\xi \in g$, let $\dot{\xi}$ be a vector tangent at $\alpha$ to the curve $t \mapsto \text{Ad}_{\exp(t \xi)}^* \alpha$:

$$\dot{\xi} = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp(t \xi)}^* \alpha.$$  \hspace{1cm} (11)

When $\xi$ runs over the whole algebra $g$, such vectors span the tangent space to $\Omega_\alpha$ at the point $\alpha$. We define the desired symplectic form $\omega$ at the point $\alpha$ by its action on two vectors constructed via (11) from the $\xi$ and $\eta$ elements of $g$,

$$\omega_\alpha(\xi, \eta) = \langle \alpha, [\xi, \eta] \rangle.$$  \hspace{1cm} (12)

We can obviously repeat the construction at each point $\beta$ on $\Omega_\alpha$, obtaining thus a symplectic form on the whole orbit. It can be checked that $\omega$ constructed in this way is indeed closed and nondegenerate on $\Omega_\alpha$, as well as $G$-invariant, i.e. $(\text{Ad}_g^*)^* \omega = \omega$.

Due to the equivariance of the moment map (9), the orbit of the $G$-action on $M$ going through a point $x$,

$$\Omega_\xi := \{ \Phi_g(x), \ g \in G \},$$  \hspace{1cm} (13)

is mapped by $\mu$ onto a coadjoint orbit

$$\Omega_{\mu(x)} = \{ \text{Ad}_g^* \mu(x), \ g \in G \}. \hspace{1cm} (14)$$

Moreover, the map $\mu$ intertwines the symplectic structures on $M$ and coadjoint orbits; if we pull back $\omega$ from $\Omega_{\mu(x)}$ by $\mu$ to $M$, we recover the restriction of $\omega$ to $\Omega_\xi$,

$$\mu^* \omega = \omega|_{\Omega_\xi}. \hspace{1cm} (15)$$

In this way we obtain a map between two symplectic structures which can be used to investigate properties of $G$-orbits in $M$. First, natural questions which can be addressed with the help of the above constructions concern symplecticity of orbits. In general, the moment map does not map $\Omega_\xi$ onto $\Omega_{\mu(x)}$ diffeomorphically. If it were the case, then all $G$-orbits in $M$ would be symplectic, i.e. the restriction of $\omega$ to an orbit would be nondegenerate. To characterize
fully the situation when it is the case, let us consider two subgroups of $G$—the stabilizers of, respectively, $x$ and $\mu(x)$:

$$\text{Stab}(x) = \{g \in G : \Phi_g(x) = x\}, \quad (16)$$

$$\text{Stab}(\mu(x)) = \{g \in G : \text{Ad}_g^* \mu(x) = \mu(x)\}. \quad (17)$$

As a consequence of the equivariance of $\mu$, we have always $\text{Stab}(x) \subset \text{Stab}(\mu(x))$. The Kostant–Sternberg theorem [6] states that a $G$ orbit is symplectic if and only if both stabilizers are equal. As a corollary, we obtain that the degeneracy subspace at $x$ defined as

$$D_x = \{u \in T_xO_x : \omega|_{O_x}(u, v) = 0 \quad \forall v \in T_xO_x\} \quad (18)$$

has the dimension

$$D_x = \dim(D_x) = \dim(O_x) - \dim(\Omega_{\mu(x)}). \quad (19)$$

The dimension (20) is of course constant along the whole orbit.

### 2.3. Orbits in the space of states: entanglement

For the entanglement problem of $L$ identical $N$-level subsystems, the relevant group $G$ is the $L$-fold direct product of the special unitary group,

$$G = \text{SU}(N) \times \cdots \times \text{SU}(N), \quad (21)$$

acting in the natural way on the tensor product $\mathcal{H}$, i.e. $g \cdot v = U_1v_1 \otimes \cdots \otimes U_Lv_L$ for $g = (U_1, \ldots, U_L) \in G$, $v = v_1 \otimes \cdots \otimes v_L$, $v_k \in \mathcal{H}_k$. This action is projected to the symplectic manifold

$$M = \mathbb{P}(\mathcal{H}). \quad (22)$$

$G$ is a group of local unitary transformations where each $\text{SU}(N)$ represents unitary quantum operations exercised on a single subsystem placed in one laboratory. They preserve quantum correlations among subsystems, i.e. they leave the ‘amount of entanglement’ in the system intact.

The moment map for the action of the unitary group $U(\mathcal{H})$ (isomorphic to $U(N^L)$) on $\mathbb{P}(\mathcal{H})$ is easily calculated as

$$\langle \mu([v]), A \rangle = \frac{i}{2} \frac{\langle v|Av \rangle}{\langle v|v \rangle}, \quad A \in \mathfrak{u}(\mathcal{H}). \quad (23)$$

The group $G$ of local transformations (21) is a subgroup of $U(\mathcal{H})$. All relevant formulas for the symplectic forms and the moment map remain the same after appropriate restrictions to $G$ and its Lie algebra $\mathfrak{g} = \mathfrak{su}(N) \oplus \cdots \oplus \mathfrak{su}(N)$.

In [1], we showed that the only symplectic orbit in $\mathbb{P}(\mathcal{H})$ is the manifold of separable (nonentangled) states and the dimension $\dim$ of degeneracy space $D_{[v]}$ can be used to quantify entanglement of a state $[v]$ or, in other words, $D([v])$ is an entanglement measure.

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$^4$ Here, and in the following, by $\dim$ we always mean the real dimension of the corresponding linear spaces and manifolds, even if they bear complex structures.
3. Local unitary equivalence of states

For simplicity, in the following, as is customary, we will use the term ‘states’ also for vectors from $\mathcal{H}$ recalling that we have in mind their projections to $\mathbb{P}(\mathcal{H})$.

Two pure states $v, w \in \mathcal{H}$ are called locally unitary (LU) equivalent if and only if there exist $U_i \in SU(N)$ such that

$$[v] = U_1 \otimes \cdots \otimes U_L[w],$$

i.e. $[v]$ and $[w]$ belong to the same orbit of the action of $G$ on $\mathbb{P}(\mathcal{H})$.

We will show how the above-outlined symplectic description of entanglement can help in analyzing local unitary equivalence. As a first step, let us calculate the image of an arbitrary state $[v]$ under the moment map (23).

Choosing an orthonormal basis $\{e_k\}$, $k = 1, \ldots, n$, in $\mathcal{H}_i \cong \mathbb{C}^N$, we can write an arbitrary $v \in \mathcal{H}$ in the form

$$v = \sum_{k_1, \ldots, k_L} C_{k_1, \ldots, k_L} e_{k_1} \otimes \cdots \otimes e_{k_L}.$$  (25)

Without losing generality, we can assume that $v$ has unit length.

The Lie algebra $g = su(N) \oplus \cdots \oplus su(N)$ is spanned by the matrices $X_1 \otimes \cdots \otimes I, \ldots, I \otimes \cdots \otimes X_l \otimes \cdots \otimes I, \ldots, I \otimes \cdots \otimes X_l$ with anti-Hermitian $X_l$. The dual $g^*$ can be identified with $g$ via the invariant bilinear form $(X, Y) = -\text{tr}(XY)$, $X, Y \in g$, i.e. $g^* \ni \alpha \sim X \in g$ if $(\alpha, Y) = -\text{tr}(XY)$ for an arbitrary $Y \in g$. For convenience, we supplement this identification by the multiplication of $X$ by the imaginary unit making elements of $g^*$ Hermitian. This is irrelevant for the whole reasoning but allows us to treat elements of $g^*$ as physical observables. Let thus

$$X = X_1 \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes X_l \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes X_L$$  (26)

be an element of $g^*$. A straightforward calculation gives

$$\mu_X([v]) = \sum_{k}^{L} \sum_{n,m}^{N} C^{(k)}_{nm} (e_m | X_k e_n),$$  (27)

where

$$C^{(k)}_{nm} = \sum_{l_1, \ldots, l_{L-1}} \overline{C}_{l_1, \ldots, n, \ldots, l_{L-1}} C_{l_1, \ldots, m, \ldots, l_{L-1}},$$  (28)

where the overbar denotes the complex conjugation and the summation is over all corresponding pairs of indices except those on the $k$th places. The positive semidefinite matrices $C^{(k)}$ are in fact the reduced density matrices of the subsystems. In the following, we will occasionally use the notation $C^{(k)}([v])$ to exhibit explicitly the dependence of the reduced density matrices on the original state $[v]$.

It is known (see, e.g., [5]) that each coadjoint orbit of a compact group (such as our $G$) intersects the dual $t^*$ to the maximal commutative subalgebra $t$ of $g$. In our case, $t^*$ is spanned by $I \otimes \cdots \otimes Y_L \otimes \cdots \otimes I$ with diagonal Hermitian $Y_L$. In general, a coadjoint orbit intersects $t^*$ at several points connected by elements of the Weyl group. By restricting to a particular Weyl chamber, e.g., by demanding that the diagonal elements of the diagonal matrices $Y_L$ appear in the nonincreasing order, we get rid of this redundancy.

In [1], it was shown that for a state $[\alpha]$, its image under the moment map $\mu([\alpha])$ belongs to $t^*$ if and only if all matrices $C^{(k)}([\alpha])$ are diagonal, $C^{(k)}([\alpha]) = \text{diag}(p_{1k}, \ldots, p_{kl})$, and we have in this case

$$\mu([\alpha]) = Y_1 \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes Y_L,$$  (29)
where, up to an irrelevant multiplicative constant,
\[ Y_k = \text{diag} \left( -\frac{1}{N} + p^2_{1k}, \ldots, -\frac{1}{N} + p^2_{Nk} \right). \]  
(30)

Since the orbit \( O_{\nu} \) is mapped by the moment map onto the coadjoint orbit \( \Omega_{\mu(\nu)} \), each \( \nu \in \mathcal{H} \) can be transformed by some group element \( U_1 \otimes \cdots \otimes U_L \in G \) to \( \nu^{'} \in \mathcal{H} \) such that the moment map \( \mu(\nu^{'}) \) belongs to \( \Gamma^* \), i.e. the reduced density matrices \( C^{(k)}(\nu) \) are diagonal, \( C^{(k)}(\nu^{'}) = \text{diag}(p^2_{1k}, \ldots, p^2_{Nk}) \). The corresponding \( U_k \) are recovered from the matrices diagonalizing the reduced density matrices of the state \( \nu \). If \( U^*_1 C^{(k)}(\nu) U_k = C^{(k)}(\nu^{'}) \), then \( U_k = U^*_1 \) (where the superscript \( T \) denotes the transposition).

Such a state \( \nu^{'} = [\nu] \) is called in [2, 3] the ‘sorted trace decomposition’ of the state \( \nu = [v] \). Observe that in the case of two subsystems \( (L = 2) \), the transformation from \( x \) to \( x^{'} \) can be made unique using the Schmidt decomposition of a bipartite state.

Let us now return to the question of the local unitary equivalence of two states \( x = [u] \) and \( y = [v] \), \( x, y \in \mathbb{P}(\mathcal{H}) \). Obviously, the equalities \( C^{(k)}(x) = C^{(k)}(y) \), \( k = 1, \ldots, L \), give a necessary condition for the local unitary equivalence of \( x \) and \( y \). If they are fulfilled, we can use matrices \( \tilde{U}_1 \) and \( \tilde{V}_1 \) diagonalizing, respectively, \( C^{(k)}(x) \) and \( C^{(k)}(y) \) to transform \( x \) and \( y \) to their sorted trace decompositions \( x^{'} \) and \( y^{'} \). The equality \( x^{'} = y^{'} \) is then a sufficient condition for the local equivalence of \( x = [u] \) and \( y = [v] \) and, explicitly,
\[ [u] = U^*_1 V_1 \otimes \cdots \otimes U^*_L V_L [v]. \]  
(31)

It is also clear that if the spectra of reduced density matrices for \( x \) and \( y \) are equal but \( x^{'} \neq y^{'} \), the states \( x \) and \( y \) may still be locally unitary equivalent. Indeed, the equality of spectra of the reduced density matrices of \( x \) and \( y \) means that \( \mu(x^{'}) = \mu(y^{'}). \) If \( y^{'} = \Phi_g(x^{'}) \) for some \( g \in G \), which is equivalent to \( y = \Phi_g(x) \) for some \( g \in G \), then due to the equivariance of \( \mu \),
\[ \mu(x^{'}) = \mu(y^{'}) = \mu(\Phi_g(x^{'})) = \mu(\Phi_g(x)). \]  
(32)
i.e. \( g \in \text{Stab}(\mu(x^{'})) \). Summarizing, \( x = [u] \) and \( y = [v] \) with the same spectra of reduced density matrices are equivalent if and only if there exists \( g \in \text{Stab}(\mu(x^{'})) \) such that \( \Phi_g(x^{'}) = y^{'} \). Since \( \text{Stab}(\mu(x^{'})) \supset \text{Stab}(x^{'}), \) this can happen also for \( x^{'} \neq y^{'} \).

In a generic case when spectra of all reduced density matrices are nondegenerate (there are no multiple eigenvalues), the stabilizer \( \text{Stab}(\mu(x^{'})) \) consists of diagonal unitary matrices, as is clear from equation (30), giving explicitly the value of the moment map at an intersection with \( \Gamma^* \). In this case, the algorithm of deciding the local unitary equivalence can be effectively applied. In nongeneric cases \( \text{Stab}(\mu(x^{'})) \) is a subgroup of \( G = SU(N)^{\otimes L}. \) As long as it is a proper subgroup, the effort of checking local unitary equivalence can be considerably eased, but if for all \( k \) we have \( C^{(k)}(x^{'}) = \frac{1}{2N} I \) (maximally entangled states) we return to the full group \( G \) since in this case \( \text{Stab}(\mu(x^{'})) = SU(N) \otimes \cdots \otimes SU(N) \).

### 4. Fibers of the moment map

From the preceding section it is clear that to make progress in checking the local unitary equivalence, we have to investigate closely the fiber of the moment map at \( x \) (we will omit \( \Gamma \) in the following assuming that \( x \) is already reduced to its sorted trace form), i.e.
\[ F_x := \{ z \in M : \mu(z) = \mu(x) \} = \mu^{-1}(\mu(x)). \]  
(33)

Let \( \xi_k, k = 1, \ldots, d = \dim g \), be a basis in the Lie algebra \( g \). The corresponding vector fields \( \xi_k \) at \( x \) (cf equation (4)) span the tangent space \( T_x O_x \) to the orbit through \( x \) at \( x \). On the other hand, the fiber \( F_x \) is a common level set of the functions \( \mu_{\xi_k} \):
\[ F_x := \{ z \in M : \mu_{\xi_k}(z) = c_k \}, \quad c_k = \mu_{\xi_k}(x), \quad k = 1, \ldots, d. \]  
(34)
Let us define
\[ \text{Ker}_\omega(d\mu) := \{ a \in T_xM : d\mu_{\xi_k}(x)(a) = 0, \ k = 1, \ldots, d \}. \] (35)

From (5), we have
\[ d\mu_{\xi_k}(a) = \omega(\xi_k, a); \] (36)
hence \( a \in \text{Ker}_\omega(d\mu) \) if and only if \( a \) is \( \omega \)-orthogonal to all \( \xi_k \) and since the latter span \( T_xO_x \), we obtain
\[ \text{Ker}_\omega(d\mu) = (T_xO_x)^{1,\omega}. \] (37)

Since (cf 34) \( \mu_{\xi_k} \) are constant on \( \mathcal{F} \), we have \( d\mu_{\xi_k}(a) = 0 \) for \( a \in T_x\mathcal{F}_x \). Hence, \( T_x\mathcal{F}_x \subset \text{Ker}_\omega(d\mu) \) and finally \( T_x\mathcal{F}_x \subset (T_xO_x)^{1,\omega} \).

It is obvious that the above reasoning does not depend on the choice of a particular point in \( \mathcal{F}_x \), i.e.
\[ T_x\mathcal{F}_x \subset (T_xO_x)^{1,\omega}, \quad y \in \mathcal{F}_x. \] (38)

A submanifold \( P \) of a symplectic manifold \( M \) is called coisotropic if for arbitrary \( y \in P \), we have \( (T_yP)^{1,\omega} \subset T_yP \). We conclude thus, that if \( O_x \) is coisotropic then \( \mathcal{F}_x \subset O_x \). Indeed from (38) and the coisotropy of \( O_x \) at each \( y \in \mathcal{F}_x \), we have \( T_y\mathcal{F}_x \subset T_yO_x \). Hence, in this case examining whether some \( y \) belongs to \( O_x \) (and, consequently whether \( y \) and \( x \) are LU-equivalent) reduces to checking if their sorted trace forms \( x' \) and \( y' \) have the same image under the moment map.

The coisotropy of \( O_x \) is a sufficient but not necessary condition for \( \mathcal{F}_x \subset O_x \), since even for a non-coisotropic orbit the fiber can be fully contained in it.

Summarizing the reasonsings presented in the preceding sections, we may formulate the following observations. Let us assume that for two states \( x = [v] \) and \( y = [w] \), the necessary condition for LU equivalence is fulfilled, i.e. the spectra of the reduced density matrices \( C^k(x) \) and \( C^k(y) \) are equal for all \( k = 1, \ldots, L \) and let \( x' = [v'] \) and \( y' = [w'] \) be the sorted trace forms of \( x \) and \( y \). Therefore,

(i) if the spectra of \( C^k(x) \) for all \( k \) are non-degenerate, then establishing the LU equivalence of \( x \) and \( y \) consists of checking whether there exists a diagonal unitary \( U \) such that \( [v'] = [Uw'] \), which reduces to a straightforward calculation.

(ii) If some spectra of the reduced density matrices are degenerate, then the states are LU equivalent if the fiber of the moment map \( \mathcal{F}_x \) is contained in the orbit \( O_x \). A sufficient but not necessary condition for such an inclusion is the coisotropy of the orbit \( O_x \).

In the two-partite case \( (L = 2) \), the LU equivalence is easily checked by performing the Schmidt decomposition of both considered states. If the non-zero Schmidt coefficients, equal to the square roots of the reduced density matrices (in this case equal for both subsystems), are equal, the states are LU equivalent. This simple criterion is reflected in the geometry of orbits and fibers of the moment map, albeit not in the simplest possible way consisting of the coisotropy of orbits. In the following section, we give a detailed analysis of \( L = 2 \) case identifying coisotropic and non-coisotropic orbits and showing also that the latter contain the whole corresponding fibers of the moment map.

5. Two-partite states

As already mentioned, for \( L = 2 \) the reduction of a state \( x = [v] \) to its sorted trace form gives the Schmidt decomposition of \( v \). We will assume that this operation has already been
performed; hence, we assume that \( v \) reads as

\[
v = \sum_{k=1}^{N} p_k e_k \otimes f_k,
\]

where \( \{e_k\} \) and \( \{f_k\} \) are appropriate orthonormal bases in \( \mathbb{C}^N \). Let us denote by \( m_0 \) the number of vanishing \( p_k \) and by \( m_l \) the multiplicity of the consecutive nonzero coefficients \( p_k \), thus \( \sum_{l=0}^{r} m_l = N \), where \( r \) is the number of different nonvanishing coefficients in the Schmidt decomposition (39).

It was proved in [1] (see also [7]) that the dimensions of orbits \( \mathcal{O}_{\{e\}} \) and \( \Omega_{\mu(\{e\})} \) are given as

\[
dim(\mathcal{O}_{\{e\}}) = 2N^2 - 2m_0^2 - \sum_{n=1}^{r} m_n^2 - 1, \tag{40}
\]

\[
dim(\Omega_{\mu(\{e\})}) = 2N^2 - 2 - \dim(\mathcal{O}_{\{e\}}) = 2m_0^2 + \sum_{n=1}^{r} m_n^2 - 1. \tag{41}
\]

From the above, we can thus easily calculate the dimension of the space \( \omega \)-orthogonal to \( T_{\{e\}}^\ast \mathcal{O}_{\{e\}} \):

\[
dim((T_{\{e\}}^\ast \mathcal{O}_{\{e\}})^{\perp_\omega}) = \dim(\mathcal{P}(\mathcal{H})) - \dim(\mathcal{O}_{\{e\}})
= (2N^2 - 2) - \dim(\mathcal{O}_{\{e\}}) = 2m_0^2 + \sum_{n=1}^{r} m_n^2 - 1, \tag{42}
\]

and the dimension of the degeneracy space (see equation (20))

\[
D(\{e\}) = \dim(\mathcal{O}_{\{e\}}) - \dim(\Omega_{\mu(\{e\})}) = \sum_{n=1}^{r} m_n^2 - 1. \tag{43}
\]

Observe that the degeneracy space (18) consists of exactly those vectors from \( T_{\{e\}}^\ast \mathcal{O}_{\{e\}} \) which simultaneously belong to \( (T_{\{e\}}^\ast \mathcal{O}_{\{e\}})^{\perp_\omega} \); hence, it is the part of \( (T_{\{e\}}^\ast \mathcal{O}_{\{e\}})^{\perp_\omega} \) contained in \( T_{\{e\}}^\ast \mathcal{O}_{\{e\}} \).

Comparing (42) and (43), we infer that an orbit is coisotropic if and only if all coefficients in the Schmidt decomposition (39) differ from zero. In this case, as we showed above, fibers of the moment map are contained in the corresponding orbits. We will prove that this is the case also for non-coisotropic orbits.

First, observe that we have the following direct sum decomposition of subspaces:

\[
(T_{\{e\}}^\ast \mathcal{O}_{\{e\}})^{\perp_\omega} = D_{\{e\}} \oplus S, \tag{44}
\]

where \( D_{\{e\}} \) is the degeneracy space (18) and \( S \) is a symplectic subspace of dimension \( 2m_0^2 \) spanned by \( e_k \otimes f_l \) and \( i e_k \otimes f_l \), where \( k \) and \( l \) are such that the corresponding \( p_k \) and \( p_l \) in (39) vanish\(^5\). The symplecticity of \( S \) is obvious since \( \omega \) is nondegenerate on it. Checking that \( S \) is indeed spanned by the mentioned vectors is a matter of a short calculation. Let us note first that for any \( e_k \otimes f_l \) and \( i e_k \otimes f_l \) such that \( p_k = 0 = p_l \) in (39), we have

\[
\langle e_k \otimes f_l | v \rangle = 0 = \langle i e_k \otimes f_l | v \rangle, \quad \langle e_k \otimes f_l | e_k \otimes f_l \rangle = \langle i e_k \otimes f_l | i e_k \otimes f_l \rangle, \tag{45}
\]

which means that \( e_k \otimes f_l \) and \( i e_k \otimes f_l \) belong to \( T_{\{e\}}^\ast M \). On the other hand (see the remark above equation (3)), each element of \( T_{\{e\}}^\ast M \) has the form \( \xi_{\{e\}} \) with \( \xi = (A \otimes I + I \otimes B) \in \mathfrak{g} = \mathfrak{su}(N) \otimes \mathfrak{su}(N) \). Using formulas (3) and (45), we see that

\[
\omega_{\{e\}}(e_k \otimes f_l, (A \otimes I + I \otimes B)_{\{e\}}) = \text{Im} \langle e_k \otimes f_l | (A \otimes I + I \otimes B) v \rangle, \tag{46}
\]

\(^5\) Remember that we treat \( T_{\{e\}}^\ast M \) as a real vector space, hence \( e_k \otimes f_l \) and \( i e_k \otimes f_l \) are different vectors.
for any $A, B \in su(N)$. Direct calculations give
\[\langle e_k \otimes f_{i}(A \otimes I + I \otimes B)v = \sum_{i=1}^{N} p_i (\langle e_k | Ae_i \rangle \langle f_{i}|f_{i} \rangle + \langle e_k | e_i \rangle \langle f_{i}|B f_{i} \rangle).\]  
(47)
Note that $\langle e_k | Ae_i \rangle \langle f_{i}|f_{i} \rangle + \langle e_k | e_i \rangle \langle f_{i}|B f_{i} \rangle \neq 0$ if and only if $i = l$ or $i = k$, but then from our assumption $p_l = 0$ which means (47) and (46) vanish. Hence, $e_k \otimes f_i$ and $ie_k \otimes f_i$ are elements of $S \subset (T_{|v|}O_{|v|})^{1w}$. Comparing the dimensions of $\dim S = 2m_0^2$ with (42) and (43), we obtain (44).

It is now enough to show that fibers of the moment map are not tangent to $S$. Let us use again the notation $x = [v] = \pi(v)$ and assume the contrary, i.e. that there exists a curve $t \mapsto x(t) \in F$ with $x(0) = x$, such that the tangent $\dot{x}(0)$ to it at $x$ belongs to $S$. Since in the two-partite case the fiber is given as a level set of functions $\mu_{I \otimes A}$ and $\mu_{A \otimes I}$, where $A$ span $su(N)$, it follows that for an arbitrary $A \in su(N)$, we have $\mu_{I \otimes A}(x(t)) = const$, $\mu_{A \otimes I}(x(t)) = const$ and, consequently,
\[
\frac{d\mu_{I \otimes A}(x(t))}{dt} \Big|_{t=0} = \frac{d\mu_{A \otimes I}(x(t))}{dt} \Big|_{t=0} = 0.
\]
(48)
(49)
The first condition is always fulfilled due to the definition (5) of $\mu$,
\[
d\mu_{I \otimes A}(\dot{x}(0)) = \omega(I \otimes A, \dot{x}(0)) = 0,
\]
(50)
since $I \otimes A$ belongs to $T_{x}O_x$ and from the assumption, $\dot{x}(0) \in S \subset (T_{x}O_{x})^{1w}$. Condition (49) reads explicitly
\[
\dot{x}(0)^T \cdot [D^2 \mu_{I \otimes A}] \cdot \dot{x}(0) + D^1 \mu_{I \otimes A} \cdot \ddot{x}(0) = 0,
\]
(51)
where $D^1 \mu_{I \otimes A}$ is the first derivative vector and $[D^2 \mu_{I \otimes A}]$ is the second derivative matrix of the function $\mu_{I \otimes A}$ at $x$. Since $\dot{x}(0)$ is tangent to $M$ at $[v]$, we have $\dot{x}(0) = B_{[v]} = [Bv]$ for some $B \in su(N^2)$ (see again the remark above equation (3)), and by a direct calculation, we find
\[
\dot{x}(0)^T \cdot [D^2 \mu_{I \otimes A}] \cdot \dot{x}(0) = -i[[I \otimes A, B], B]v|v\rangle = -\frac{i[e_{I \otimes A}, B]v|v\rangle}{2\langle v|v \rangle}.
\]
(52)
Our aim now is to show that there exists such an $A \in su(N)$ that (51) is not fulfilled for any choice of $\dot{x}(0) \in S$. In appendix A, we show that the goal is achieved by taking $A \in su(N)$ as a diagonal traceless matrix with $A_{lk} = i$ if $p_k \neq 0$ and $A_{lk} = -i\frac{m_0}{m_0}$ if $p_k = 0$ in (39). Appendix B contains a complete description of orbits in the simplest non-trivial example of two qutrits ($N = 2, L = 3$).

6. Geometric structure of orbits through GHZ states
As remarked at the end of section 3, the method of checking the local unitary equivalence based on comparison of the moment map images gives no advantages when all reduced density matrices are proportional to the identity. In this section, we will show how it is reflected in the structure of orbits through the so-called Greenberger–Horn–Zeilinger (GHZ) states for $L \geq 3$ qubits. The Hilbert space will thus be $\mathcal{H} = (C^2)^{2^{L}}$ with the real dimension $\dim(\mathcal{H}) = 2^{L+1}$ so $\dim (\mathbb{P}(\mathcal{H})) = 2^{L+1} - 2$. We are interested in orbits of the action of $G = SU(2)^{2^L}$ on $\mathbb{P}(\mathcal{H})$. The Lie algebra $\mathfrak{g}$ of $G$ is spanned by
\[
\mathfrak{X}_k = iI \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes I
\]
(53)
where \( \sigma_x, \sigma_y, \sigma_z \) are the Pauli matrices and \( k = 1, \ldots, L \). The fibers of the moment map are given as level sets of the functions \( \mu_{\mathcal{X}_k}, \mu_{\mathcal{Y}_k}, \mu_{\mathcal{Z}_k}, k = 1, \ldots, L \).

Let us consider the \( L \)-partite GHZ state

\[
v_L = \frac{1}{\sqrt{2}} (|0\rangle^\otimes L + |1\rangle^\otimes L),
\]

where in order to make the formulas more readable we switched to the customary notation of the qubit states \( e_1 = |0\rangle, e_2 = |1\rangle \) together with \( |k\rangle \otimes |l\rangle = |kl\rangle \) and \( |kk\ldots k\rangle = |k\rangle^\otimes L \), etc.

The matrices \( C^k(|v_L\rangle) \) (28) are the same for all \( k \):

\[
C^k(|v_L\rangle) = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}.
\]

For the GHZ states, we have \( \mu(|v_L\rangle) = 0 \). Indeed,

\[
\langle \mu(|v_L\rangle), I \otimes I \otimes \cdots \otimes X_k \otimes \cdots \otimes I \rangle = \frac{i}{2} \langle v_L | I \otimes I \otimes \cdots \otimes X_k \otimes \cdots \otimes I v_L \rangle = \frac{i}{4} \text{ tr} X_k = 0
\]

for an arbitrary \( X_k \in su(2) \). Hence, for any two vectors \( A|v_L\rangle, B|v_L\rangle \in T|v_L\rangle \mathcal{O}_{|v_L\rangle} \), where \( A, B \in g \), we have

\[
\omega(A|v_L\rangle, B|v_L\rangle) = -\frac{i}{2} \langle [A, B]|v_L\rangle = \langle \mu(|v_L\rangle), [A, B] \rangle = 0,
\]

since \( \mu(|v_L\rangle) = 0 \).

Note that for any \( L \geq 3 \), (59) implies that \( T|v_L\rangle \mathcal{O}_{|v_L\rangle} \subseteq (T|v_L\rangle \mathcal{O}_{|v_L\rangle})^\perp_\omega \). We will show now that the orbit \( \mathcal{O}_{|v_L\rangle} \) through \( |v_L\rangle \) is Lagrangian\(^6\), i.e. \( T|v_L\rangle \mathcal{O}_{|v_L\rangle} = (T|v_L\rangle \mathcal{O}_{|v_L\rangle})^\perp_\omega \) whereas for \( L \geq 4 \) it is isotropic, i.e. \( T|v_L\rangle \mathcal{O}_{|v_L\rangle} \subset (T|v_L\rangle \mathcal{O}_{|v_L\rangle})^\perp_\omega \). In other words, for \( L = 3 \) the orbit \( \mathcal{O}_{|v_L\rangle} \) is (minimally) coisotropic and for \( L > 3 \) the orbit \( \mathcal{O}_{|v_L\rangle} \) is not coisotropic.

The space \( T|v_L\rangle \mathcal{O}_{|v_L\rangle} \) is spanned by the vectors

\[
\mathcal{X}_k v_L = \frac{i}{\sqrt{2}} \begin{pmatrix}
|0\ldots1\ldots0\rangle \wedge_k \\
|1\ldots0\ldots1\rangle \wedge_k
\end{pmatrix},
\]

\[
\mathcal{Y}_k v_L = \frac{1}{\sqrt{2}} \begin{pmatrix}
|0\ldots1\ldots0\rangle - |1\ldots0\ldots1\rangle \wedge_k
\end{pmatrix},
\]

\[
\mathcal{Z}_k v_L = \frac{i}{\sqrt{2}} \begin{pmatrix}
|0\rangle^\otimes L - |1\rangle^\otimes L
\end{pmatrix}
\]

with \( k = 1, \ldots, L \). The above \( 2L + 1 \) vectors are mutually orthogonal; hence,

\(^6\) The Lagrangian subspace \( U \) of symplectic space \((V, \omega)\) is the minimally coisotropic \((U \subseteq U^{\perp_\omega})\) and at the same time maximally isotropic \((U \supseteq U^{\perp_\omega})\) subspace of \( V \), i.e. for any coisotropic space \( W \subseteq U \), we have \( W = U \) and for any isotropic space \( W' \supseteq U \) we have \( W' = V \). These conditions imply that \( U \) is Lagrangian if and only if \( U = U^{\perp_\omega} \); hence, \( \omega|_U = 0 \) and \( \dim U = \frac{1}{2} \dim V \).
**Fact 1.** The orbit $O_{[v_L]}$ has dimension
\begin{equation}
\dim O_{[v_L]} = 2L + 1. \tag{63}
\end{equation}

Since $\frac{1}{2} \dim (\mathbb{P}(\mathcal{H})) = 2^L - 1$, we have as an immediate consequence,

**Fact 2.** The orbit $O_{[v_L]}$ is Lagrangian, and hence the fiber of the moment map is contained inside it. If $L \geq 4$, then $O_{[v_L]}$ is not Lagrangian.

Indeed, from (63) $\dim O_{[v_L]} = 7 = 2^3 - 1 = \frac{1}{2} \dim (\mathbb{P}(\mathcal{H}))$, whereas for $L \geq 4$ we have $2^L - 1 > 2L + 1$, hence the orbits have too small dimension to be Lagrangian.

The fact that $O_{[v_L]}$ is Lagrangian (so also coisotropic) implies that the necessary and sufficient condition for two states $|u\rangle$ and $|w\rangle$ of three qubits to belong to $O_{[v_L]}$ is $\mu(\{|u\rangle\}) = 0 = \mu(\{|w\rangle\})$.

For $L \geq 4$, the fiber of the moment map is not entirely contained in $O_{[v_L]}$. We will show that in fact $T_{[v_L]} F_{[v_L]} = (T_{[v_L]} O_{[v_L]})^{\perp}$.

Let $A_{[v_L]} \in T_{[v_L]} O_{[v_L]}$, i.e. $A_{[v_L]} = \{A v_L\}$, where $A$ is of the form (60)–(62),
\begin{equation}
A = i I \otimes \cdots \otimes I, \quad \beta \in \{x, y, z\}. \tag{64}
\end{equation}

The space $(T_{[v_L]} O_{[v_L]})^{\perp}$ is spanned by these $B_{[v_L]} \in T_{[v_L]} \mathbb{P}(\mathcal{H})$ for which $\alpha_{[v_L]}(A_{[v_L]}, B_{[v_L]}) = 0$. According to (3), such vectors $B_{[v_L]}$ have the form $[B v_L]$ with $B \in \mathfrak{u}(\mathcal{H})$ and
\begin{equation}
0 = \langle [A, B] v_L, v_L \rangle = -\langle v_L A v_L \rangle = \langle v_L A v_L \rangle. \tag{65}
\end{equation}

We can choose
\begin{equation}
B = i \sigma_k \otimes \cdots \otimes \sigma_k, \tag{66}
\end{equation}
with $\alpha_k \in \{x, y, z\}$ and $\alpha_0 = 1$ since such vectors span $\mathfrak{u}(\mathcal{H})$.

To prove that $T_{[v_L]} F_{[v_L]} = (T_{[v_L]} O_{[v_L]})^{\perp}$, we have to show that if $B_{[v_L]} = [B v_L]$ belongs to $(T_{[v_L]} O_{[v_L]})^{\perp}$ then the curve $t \mapsto [e^{iB} v_L]$ is contained in the fiber of the moment map, i.e.
\begin{equation}
\langle \mu(v_L), e^{-iB} A e^{iB} v_L \rangle = \langle v_L A v_L \rangle e^{-iB} A e^{iB} v_L = 0, \tag{67}
\end{equation}
fors arbitrary $A$ and $B$ of the forms, respectively, (64) and (66), fulfilling (65). To this end, we employ the Hadamard lemma,
\begin{equation}
e^{-iB} A e^{iB} = A + (-i)B \langle [B, A] \rangle + \frac{(-i)^2}{2!} \langle [B, [B, A]] \rangle + \frac{(-i)^3}{3!} \langle [B, [B, [B, A]]] \rangle + \cdots. \tag{68}
\end{equation}

Now, using $\sigma_k \otimes I$, we have from (64) and (66)
\begin{equation}
[B, [B, A]] = -i I \otimes \cdots \otimes [\sigma_k, [\sigma_k, [\sigma_k, [\sigma_k, I]]]] \otimes \cdots I. \tag{69}
\end{equation}

From the commutation relations for Pauli matrices
\begin{equation}
[\sigma_x, \sigma_y] = i \sigma_z, \quad [\sigma_y, \sigma_z] = i \sigma_x, \quad [\sigma_z, \sigma_x] = i \sigma_y, \tag{70}
\end{equation}
we infer that the double commutator $[B, [B, A]]$ equals $A$ (possibly up to the sign) or vanishes (if $\alpha_k = \beta$ or $\alpha_k = I$). Consequently, in expansion (68), we encounter only the terms proportional to $A$ and $[B, A]$. But $\langle v_L A v_L \rangle = \frac{1}{2} \text{tr} \sigma_\beta = 0$ and $\langle v_L [B, A] v_L \rangle = 0$ vanishes on the assumption (65). This concludes a proof of

**Fact 3.** The tangent space $T_{[v_L]} F_{[v_L]}$ to the fiber of the moment map over $\mu(|v_L\rangle)$ is exactly equal to $(T_{[v_L]} O_{[v_L]})^{\perp}$ and orbits $O_{[v_L]}$ are isotropic.
7. Multiqubit systems

In this section, using geometric properties of state \([v_1]\) described in the previous section, we present an easy method of checking whether two states \([u]\) and \([v]\) of three qubits are locally unitary equivalent. Note at the beginning that in case of two qubits states, the necessary and sufficient condition for this is given by equality of Schmidt decompositions. For three qubits we already know that states for which \(\mu([v]) = 0\) are locally equivalent and lie on the orbit \(O_{[v]}\) which is Lagrangian. For other states, the following reasoning is crucial.

Let us consider the action of \(G = U(H)\) on the complex projective space \(\mathbb{P}(H)\). Let \(x = [u]\) and \(y = [v]\) be two points from \(\mathbb{P}(H)\). Since \(G\)-action is transitive on \(\mathbb{P}(H)\) there is at least one unitary matrix \(U \in G\) joining \(x\) with \(y\), i.e.

\[ [Uu] = [v]. \]  

(71)

Let \(V \neq U\) have the property (71). Then,

\[ Ux = y = Vx \Rightarrow U^{-1}V \in \text{Stab}(x). \]  

(72)

Hence, there is \(W \in \text{Stab}(x)\) such that \(V = UW\). It means that all matrices joining \(x\) with \(y\) are of the form \(UW\) where \(W \in \text{Stab}(x)\). Let us consider now three vectors \(v_1\), \(v_2\) and \(v_3\) such that

\[ \langle v' | v' \rangle = 1, \quad \langle v_1 | v_2 \rangle = 0, \quad \langle v_1 | v_3 \rangle = 0, \]  

(73)

i.e. all \(v'\) are normalized to 1 and \(v_2\), \(v_3\) are orthogonal to \(v_1\). Note that \(v_2\) can be obtained from \(v_3\) by the action of the unitary matrix \(U_1 \in \text{Stab}(v_1)\). Hence, the general form of the unitary matrix joining \(v_2\) with \(v_3\) is

\[ U = U_1V, \quad U_1 \in \text{Stab}([v_1]), \quad V \in \text{Stab}([v_2]). \]  

(74)

In the case of three qubits, \(H = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2\) and the group of interest is \(G = SU(3)^3\). The direct consequence of property (74) is the following fact.

**Fact 4.** Two locally equivalent states \([x]\) and \([y]\) are orthogonal to some state \([z]\) if and only if there exist \(U \in \text{Stab}([z]) \cap G\) such that \([Ux] = [y]\), where \(G\) is the appropriate local unitary group.

Using this fact we will give a simple criterion to check the LU equivalence of two states \(x = [u]\) and \(y = [v]\). Let us assume at the beginning that \(x\) and \(y\) are already in the sorted trace form, i.e.

\[ \mu(x) = \mu(y) = X_1 \otimes I \otimes I + I \otimes X_2 \otimes I + I \otimes I \otimes X_3, \]  

(75)

where matrices \(X_i\) are diagonal and at least one of them, e.g., \(X_1\), has nondegenerate spectrum. Under this assumption, states \(x\) and \(y\) can be written in the form

\[ u = p_{11}|0\rangle \otimes |\Psi_1\rangle + p_{12}|1\rangle \otimes |\Psi_2\rangle, \]

\[ v = p_{11}|0\rangle \otimes |\Phi_1\rangle + p_{12}|1\rangle \otimes |\Phi_2\rangle, \]  

(76)

where \(\langle \Psi_i | \Psi_j \rangle = \delta_{ij}\) and \(\langle \Phi_i | \Phi_j \rangle = \delta_{ij}\) (\(|\Psi_i\rangle\) and \(|\Phi_i\rangle\) are two-qubit states). From (76) we see that the necessary condition for \(x\) and \(y\) to be locally equivalent is local equivalence of pairs \(|\Psi_1\rangle\), \(|\Phi_1\rangle\) and \(|\Psi_2\rangle\), \(|\Phi_2\rangle\), but this can be easily checked using Schmidt decomposition as these are two-qubit states. Assume that the necessary condition is fulfilled. Hence, there exists a matrix \(U_2 \otimes U_3\) joining state \(|\Psi_1\rangle\) with \(|\Phi_1\rangle\), i.e.

\[ v' = U_2 \otimes U_3 u = p_{11}|0\rangle \otimes |\Phi_1\rangle + p_{12}|1\rangle \otimes |\Psi'_2\rangle. \]  

(77)
where \( \langle \Phi_1 | \Psi_1' \rangle = 0 \) and \( |\Psi_2'\rangle = U_3 \otimes U_2 |\Psi_2\rangle \). Note that we can still act on \( \nu' \) with \( \text{Stab}(\langle |\Phi_1\rangle |) \cap (\text{SU}(2) \times \text{SU}(2)) \). But from fact 4, using the assumption that \( |\Phi_2\rangle \) is locally equivalent to \( |\Psi_2'\rangle \) and that both \( |\Phi_2\rangle \) and \( |\Psi_2'\rangle \) are orthogonal to \( |\Phi_1\rangle \), we obtain that \( x \) is locally equivalent to \( y \). Summing up, states of three qubits (76) are locally equivalent if and only if the corresponding pairs of states of two qubits \( |\Psi_1\rangle, |\Phi_1\rangle \) and \( |\Psi_2\rangle, |\Phi_2\rangle \) are locally equivalent. Note that this method can be used to investigate local equivalence of states of four qubits, but only if at least one of the matrices (28) \( C_k \) has a nondegenerate spectrum. The example of the state for which all four matrices \( C_k \) have a degenerate spectrum is \( |v_{4}\rangle \). In section 6, we proved that the orbit \( \mathcal{O}_{[v_4]} \) is not Lagrangian but isotropic and the fiber of the moment map over \( \mu([v_4]) \) is not entirely contained inside the orbit \( \mathcal{O}_{[v_4]} \). In fact, the dimension of the part which is not contained in \( \mathcal{O}_{[v_4]} \) is 12 and this makes the problem of local equivalence hard.

8. Summary

The presented symplectic approach to entanglement exhibited \textit{a priori} unexpected geometric richness of the space of pure states for multipartite, finite-dimensional quantum systems and sheds some light on the important problem of the local unitary equivalence of pure states, or in physical terms, possibility of transforming one state into another by means of quantum operations restricted to single parties.

Using a fundamental concept of symplectic geometry and symplectic group action theory, namely the moment map, the problem of the local equivalence of states is mapped from the space of states and corresponding orbits of local unitary groups onto geometry of (co)adjoint orbits in corresponding local Lie algebras and their duals. The procedure has an obvious advantage—checking whether two elements of the Lie algebra or its dual space belong to the same orbit (i.e. are ‘locally equivalent’) reduces to the comparison of spectra of (anti)symmetric matrices. On the other hand, since the moment map usually is not a diffeomorphism of an orbit in the space of states onto the corresponding coadjoint orbit, a detailed investigation of its fiber is needed for the ultimate check of the local equivalence of states. Such an analysis also clearly identifies situations in which a conclusive solution is hard to find.

The simplest situation occurs when an orbit of the local action in the space of states is coisotropic. In this case, the whole fiber is included in the orbit, and checking whether a state belongs to the orbit and hence is locally equivalent to all other states on it consists of checking if the spectra of all reduced density matrices are the same as for any other state on the orbit. However, such a situation typically occurs only in various ‘nondegenerate’ cases. On the other hand, fibers can be fully contained in the corresponding orbits also when the latter are not coisotropic. We have illustrated such phenomena by analyzing the bipartite case. For two particles, checking of the local unitary equivalence of states can be effectively and easily done by comparison of the Schmidt spectra. This fact should be reflected in a simple geometry of local orbits. Indeed, we have shown that only when no Schmidt coefficient vanishes the orbit is cosotropic; nevertheless, also non-coisotropic orbits contain the whole corresponding fibers.

In order to interpret geometrically the principal obstacles for effective checking of the local unitary equivalence, we analyzed the local orbits through multiqubit GHZ states. For such states all reduced density matrices are proportional to the identity (the ‘maximally mixed’ states). The geometry of orbits through the GHZ states depends on the number of parties. For three qubits, the orbit is Lagrangian, hence coisotropic. Consequently, the fiber of the moment map is contained in it which means that all states that have maximally mixed density matrices are locally unitary equivalent to the GHZ state. We also showed that if the number of qubits exceeds three, the orbits through the GHZ states are isotropic rather than coisotropic, and the
corresponding fibers are only partially included in them. This is the main obstacle for an easy effective checking of the local unitary equivalence.

We believe that our approach to quantum entanglement description, although involving relatively abstract concepts of symplectic geometry, has already proven to be fruitful. As we showed in [1], it gives rise to a purely geometric discrete entanglement measure. The measure is very robust but it yields an easy criterion to identify the separable (nonentangled) states for an arbitrary multipartite system. Moreover, for multipartite qubit states, it assigns the highest value to states which have maximally mixed one-qubit reduced density matrices, e.g., GHZ states. The states which are locally unitary equivalent are classified by our measure as equally entangled. It is thus a priori possible to use our approach to classify multipartite entanglement. The main difficulty we expect to occur during this process is the lack of effective description of the fibers of the moment map. This question is of course a problem investigated in symplectic geometry, but to our knowledge, little is known about particular situations interesting for entanglement theory presented in our paper. We have tried to give some flavor of the possible obstacles which can be encountered when dealing with such problems in section 6 where we analyzed multiqubit GHZ states. When fibers of the moment map are not completely contained inside orbits of the local unitary action, the same value of the geometric measure is assigned to different local orbits. One promising way of attacking the ensuing complication is to perform a reduction of the space of states by the action of the local group. Local orbits then become points in the resulting quotient space which to some extent inherits the geometric structure of the space before reduction. Since the action of the group is symplectic, the reduction procedure is closely connected to the symplectic reduction well-known from classical mechanics. This connection needs further investigations.

A satisfactory classification of multipartite entanglement with the help of symplectic geometry methods is still not complete. None the less, we believe that our approach to quantum entanglement shows at least the importance of the problem of effective description of the fibers of the moment map. This problem is interesting from both a geometric and quantum entanglement point of view. Our approach also establishes the curious link between these two seemingly unconnected concepts. The investigation and full understanding of this link is one of our future goals.

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Appendix A

We will fill in some details of the calculations showing that in the two-partite case, fibers of the moment map lie within the corresponding orbit. In particular, we will show that the fibers are not tangent to $S$ (see (44)).

Let us define the following operators:

\[ X_{ij} = i(E_{ij} - E_{ji}), \quad Y_{ij} = E_{ij} + E_{ji}, \quad H_{ij} = E_{ii} - E_{jj}, \quad i < j, \]

(A.1)

where $E_{ij}$ are matrices defined as

\[ (E_{ij})_{kl} = \begin{cases} 0 & \text{for } k \neq i, \quad l \neq j, \\ 1 & \text{for } k = i, \quad l = j. \end{cases} \]

(A.2)
Without losing generality, we assume that \( p_1 \neq 0 \) in (39). Note that vectors from \( S \) can be generated in the following way:

\[
i p_i^2 e_k \otimes f_i = [iY_{1k} \otimes Y_{1l}] v, \quad p_i^2 e_k \otimes f_i = [iY_{1k} \otimes X_{1l}] v. \quad (A.3)
\]

Let us choose \( A \in \mathfrak{su}(N) \) as a diagonal traceless matrix with \( A_{ik} = i \) if \( p_k \neq 0 \) and \( A_{ik} = -i \frac{N-1}{N} \) if \( p_k = 0 \) in (39). We have

\[
D^1 \mu \otimes (\hat{x}(0)) = \omega (I \otimes A, \hat{x}(0)) = \omega (0, \hat{x}(0)) = 0. \quad (A.4)
\]

We used \( I \otimes A = [I \otimes Av] \) which follows from the fact that \( I \otimes Av = iv \) and as such it corresponds to the zero vector in the tangent space \( T_\mu M \). What is left to be shown is thus

\[
\dot{x}(0)^T [D^2 \mu \otimes \hat{A}] \hat{x}(0) \neq 0 \quad (A.5)
\]

for any \( \hat{x}(0) \in S \). Let us thus write

\[
\dot{x}(0) = \sum_{k,l} (a_{lk} e_k \otimes f_l + b_{kl} i e_k \otimes f_l) = p_1^{-2} \sum_{k,l} i [a_{lk} Y_{1k} \otimes X_{1l} + b_{kl} Y_{1k} \otimes Y_{1l}] v, \quad (A.6)
\]

where the sum goes over such \( k, l \) that \( p_k = 0 = p_l \) in (39), and we used (A.3) to obtain the second equality.

To calculate explicitly the second derivative using (52), we need some commutators,

\[
[I \otimes A, iY_{1k} \otimes Y_{1l}] = iY_{1k} \otimes [A, Y_{1l}] = i\alpha Y_{1k} \otimes X_{1l}, \quad (A.7)
\]

\[
[I \otimes A, iY_{1k} \otimes X_{1l}] = iY_{1k} \otimes [A, X_{1l}] = -i\alpha Y_{1k} \otimes Y_{1l}, \quad (A.8)
\]

where \( \alpha = \frac{N}{N-1} \). Hence,

\[
\left[ I \otimes A, \sum_{k,l} i (a_{lk} Y_{1k} \otimes X_{1l} + b_{kl} Y_{1k} \otimes Y_{1l}) \right] = i\alpha \sum_{k,l} b_{kl} Y_{1k} \otimes X_{1l} - a_{kl} Y_{1k} \otimes Y_{1l}. \quad (A.9)
\]

And, finally,

\[
\dot{x}(0) [D^2 \mu \otimes \hat{A}] \hat{x}(0) = -i \left\{ \left[ I \otimes A, p_1^{-2} \sum_{k,l} i (a_{lk} Y_{1k} \otimes X_{1l} + b_{kl} Y_{1k} \otimes Y_{1l}) \right] v \right\} v
\]

\[
= -i \left\{ i\alpha p_1^{-2} \sum_{k,l} (b_{kl} Y_{1k} \otimes X_{1l} - a_{kl} Y_{1k} \otimes Y_{1l}) \right\} v \nu
\]

\[
= - i\alpha \cdot \omega \left( \sum_{kl} b_{kl} e_k \otimes f_l - a_{kl} i e_k \otimes f_l, \sum_{kl} a_{kl} e_k \otimes f_l + b_{kl} i e_k \otimes f_l \right)
\]

\[
= - 2i\alpha \sum_{kl} (a_{kl}^2 + b_{kl}^2). \quad (A.10)
\]

This clearly means that \( \dot{x}(0) [D^2 \mu \otimes \hat{A}] \hat{x}(0) \neq 0 \) for any \( \hat{x}(0) \in S \) and proves that in the bipartite case, fibers of the moment map are fully contained in the corresponding orbits.
Appendix B. Two qutrits

In the case of two qutrits ($N = 3, L = 2$), the Hilbert space is $\mathcal{H} = \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\dim(\mathcal{H}) = 18$, so $\dim(\mathbb{P}(\mathcal{H})) = 16$. The Lie algebra $\mathfrak{g} = su(3) \oplus su(3)$ of $G = SU(3) \times SU(3)$ is spanned by \{\(A_k \otimes I, I \otimes A_k\)\}, where \(A_k, k = 1, \ldots, 8\) is a basis in $su(3)$; hence $\dim(\mathfrak{g}) = 16$. The fibers of the moment map through $v$ are given as a common level set of 16 functions $\mu_{A_k \otimes f}, \mu_{I \otimes A_k}$. Without looseing generality, we assume that the bases \{\(e_k\)\} and \{\(f_k\)\} in both Hilbert spaces are equal. As previously, we switch to the customary notation $e_1 = |0\rangle = f_1, e_2 = |1\rangle = f_2, e_3 = |2\rangle = f_3$, together with \(|k\rangle = |k\rangle \otimes |l\rangle\).

The general form of a Schmidt-decomposed two-qutrit state is given by

$$ v = p_1 |00\rangle + p_2 |11\rangle + p_3 |22\rangle, $$

where $p_1^2 + p_2^2 + p_3^2 = 1$. There are six cases to consider.

1. $p_1 = 1, \ p_2 = p_3 = 0$ (a separable state). In this case $v = |00\rangle$. The orbit $\mathcal{O}_{\{v\}}$ through $[v]$ is symplectic ([11]); hence, the part of the fiber which is contained in $\mathcal{O}_{\{v\}}$ is zero dimensional. Orthogonal complement $(T_{\{v\}}\mathcal{O}_{\{v\}})^{\perp v}$ is spanned by

$$\{|22\rangle, \ i|22\rangle, \ |11\rangle, \ |12\rangle, \ |i12\rangle, \ |21\rangle, \ |i21\rangle\}, $$

and is a symplectic vector space $S$. The matrix $A \in su(3)$ used in the proof in appendix A has the form

$$ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 \\ 0 & 0 & \frac{i}{2} \end{bmatrix}. $$

There is no fiber and the orbit is not coisotropic.

2. $p_1 = p_2 = p_3 = \frac{1}{\sqrt{3}}$ (the maximally entangled state). In this case, the orbit $\mathcal{O}_{\{v\}}$ through $v = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ is coisotropic since all $p_k \neq 0$. In fact $\mathcal{O}_{\{v\}}$ is minimally coisotropic; hence, Lagrangian, i.e.

$$ (T_{\{v\}}\mathcal{O}_{\{v\}})^{\perp v} = T_{\{v\}}\mathcal{O}_{\{v\}}, $$

$$ \dim(T_{\{v\}}\mathcal{O}_{\{v\}}) = \frac{1}{2} \dim(\mathbb{P}(\mathcal{H})). $$

Using formula (43) it is easy to prove that in the case of two qutrits it is always true that orbit through

$$ v = \sum_{k=1}^{N} \frac{1}{\sqrt{N}} |kk\rangle, $$

is Lagrangian. Namely for (B.5), we have

$$ (T_{\{v\}}\mathcal{O}_{\{v\}})^{\perp v} = D([v]) = N^2 - 1 = \frac{1}{2} \dim(\mathbb{P}(\mathcal{H})); $$

hence, $\mathcal{O}_{\{v\}}$ is Lagrangian [8].

3. $p_1 \neq p_2 \neq p_3 \neq 0$ (a generic state). The orbit $\mathcal{O}_{\{v\}}$ through $v = p_1 |00\rangle + p_2 |11\rangle + p_3 |22\rangle$ is coisotropic since all $p_k \neq 0$. Formulas (40) and (41) give

$$ \dim(\mathcal{O}_{\{v\}}) = 14, \quad \dim(\mu(\mathcal{O}_{\{v\}})) = 12. $$

The whole fiber is contained in $\mathcal{O}_{\{v\}}$ and is two dimensional.

4. $p_1 = p_2 \neq 0, p_3 \neq 0$. The orbit $\mathcal{O}_{\{v\}}$ through $v = p_1 |00\rangle + |11\rangle + p_3 |22\rangle$ is coisotropic since all $p_k \neq 0$. Formulas (40) and (41) give

$$ \dim(\mathcal{O}_{\{v\}}) = 12, \quad \dim(\mu(\mathcal{O}_{\{v\}})) = 8. $$

The whole fiber is contained in $\mathcal{O}_{\{v\}}$ and is four dimensional.
(5) $p_1 = p_2 = \frac{1}{\sqrt{2}}$, $p_3 = 0$. The orbit $O_{[v]}$ through $v = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ is not coisotropic since $p_3 = 0$. Formulas (40) and (41) give

$$\dim(O_{[v]}) = 11, \quad \dim(\mu(O_{[v]})) = 8. \quad \text{(B.9)}$$

Hence, the part of the fiber contained in $O_{[v]}$ is three dimensional. The orthogonal complement $(T_{[v]}O_{[v]})^\perp$ is five dimensional and is spanned by three vectors contained in $T_{[v]}O_{[v]}$ and two other \{ $v_1 = |22\rangle$, $v_2 = i|22\rangle$ \}. The matrix $A \in \mathfrak{su}(3)$ used in the proof in appendix A has the form

$$A = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$ 

The whole fiber is contained inside the orbit, although the orbit is not coisotropic.

(6) $p_1 \neq p_2 \neq 0$, $p_3 = 0$. The orbit $O_{[v]}$ through $v = p_1|00\rangle + p_2|11\rangle$ is not coisotropic since $p_3 = 0$. Formulas (40) and (41) give

$$\dim(O_{[v]}) = 13, \quad \dim(\mu(O_{[v]})) = 12. \quad \text{(B.10)}$$

Hence, the part of the fiber contained in $O_{[v]}$ is one dimensional. The orthogonal complement $(T_{[v]}O_{[v]})^\perp$ is three dimensional and is spanned by one vector contained in $T_{[v]}O_{[v]}$ and two other \{ $v_1 = |22\rangle$, $v_2 = i|22\rangle$ \}. The matrix $A \in \mathfrak{su}(3)$ used in the proof in appendix A has the form

$$A = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$ 

Again the whole fiber is contained inside the orbit, although the orbit is not coisotropic.

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