NEGATIVE 4-PROBE CONDUCTANCES OF MESOSCOPIC, SUPERCONDUCTING WIRES.

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ABSTRACT

We analyze the longitudinal 4-probe conductance of mesoscopic normal and superconducting wires and predict that in the superconducting case, large negative values can arise for both the weakly disordered and localized regimes. This contrasts sharply with the behaviour of the longitudinal 4-probe conductance of normal wires, which in the localized limit is always exponentially small and positive.

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In contrast with the huge literature on quantum transport in normal, phase coherent, structures [1, 2], a detailed knowledge of mesoscopic superconducting structures is only now being developed. Such structures constitute new quantum objects, which during the past 3 years, have yielded many surprises [3, 4, 5]. Most theoretical work to date has focussed on two-probe transport coefficients, such as normal-superconducting (N-S) boundary [3] conductances and N-S-N 2-probe conductances [4, 5]. These are simpler to analyze than multi-probe coefficients and therefore without good reason for expecting new physics to emerge, there would seem to be little point in injecting unnecessary details into an already complex theory. The aim of this Letter is to demonstrate that multi-probe conductances contain new features which are absent from two-probe measurements. A key result of our work is that when a normal system with a positive 4-probe conductance is allowed to become superconducting, the sign of the conductance can change.

As an example, we study the 4-probe conductances $G_{jk,lm}$ of the 2 dimensional wire shown in figure 1a, where $G_{jk,lm} = I_{jk}/(V_l - V_m)$, with $V_l - V_m$ the potential difference between voltage probes $l, m$ and $I_{jk}$ the current flowing from probe $j$ to probe $k$. We consider the case for which the system size is smaller than the inelastic scattering length and focus attention on the longitudinal conductances $G_{13,24}$ and $G_{14,23}$. It has been shown [1] that for normal materials, both negative and positive multiprobe conductances can both occur depending on the geometry of the device. On the other hand, numerical simulations of electron waveguide couplers [10], have found that for clean materials, the analogues of $G_{13,24}$ and $G_{14,23}$ are always positive, in agreement with theoretical arguments valid for devices satisfying certain spatial symmetries [10, 11].

In this letter we first derive a general criterion which determines the sign of
the 4-probe conductance in any arrangement of normal conductors. In the case of localized normal wires, for which the quasi-particle transmission probability from one end of the sample to another is exponentially small, this shows that $G_{13,24}$ and $G_{14,23}$ are both positive and of order $\sim \exp -2L/\xi$, where $\xi$ is the localization length. In contrast, using a recently derived generalisation [12] of the multiprobe Büttiker formulae to the case in which Andreev scattering is permitted, we show that for a superconducting wire with exponentially small quasi-particle transmission, longitudinal conductances are finite and may be of either sign. We also present numerical results for the behaviour of 4-probe conductances in the presence of finite quasi-particle transmission along the wire.

Consider first the case of a normal wire, for which it is known [9] that

$$G_{jk,lm} = \frac{D}{(T_{jl}T_{km} - T_{jm}T_{ki})}$$

where $T_{ij} = \sum_{\alpha\beta} T_{ij}^{\alpha\beta}$ and $T_{ij}^{\alpha\beta}$ is the probability for a particle incident in channel $\beta$ of probe $j$ to be transmitted to channel $\alpha$ of probe $i$. $D$ is the determinant of the matrix obtained by crossing out any one row and column of the matrix of transport coefficients $A$, $A_{ij} = N\delta_{ij} - T_{ij}$ where $N$ is the number of channels in each probe (assumed equal). The value of this determinant is independent of which row and column are removed [13], although because of the relation $\sum_i A_{ij} = \sum_j A_{ij} = 0$ the expression may be written in many equivalent ways. In order to make the analysis clearer we will write those scattering coefficients involving transmission along the wire, for example $T_{13}, T_{24}$ (but not eg. $T_{12}$ or $T_{34}$) in the form $T_{ij} = t_{ij}\exp -2L/\xi$ where $t_{ij}$ has magnitude of order unity. We expand $D$ by removing row 4 and column 4 of $A$ and substituting $A_{ii} = -\sum_{j\neq i} A_{ij}$. This yields

$$D = \exp -2L/\xi (T_{12}t_{23}T_{34} + T_{12}t_{24}T_{34} + t_{13}T_{21}T_{34} + t_{14}T_{21}T_{34})$$

$$+ \exp -4L/\xi (T_{12}t_{24}T_{31} + T_{12}t_{24}T_{32} + T_{34}t_{13}T_{23} + T_{34}t_{13}t_{24})$$
\[+T_{21}t_{14}t_{31} + T_{21}t_{14}t_{32} + T_{34}t_{14}t_{23} + T_{34}t_{14}t_{24})
+ \exp^{-6L/\xi}(t_{13}t_{24}t_{32} + t_{14}t_{23}t_{31} + t_{14}t_{24}t_{31} + t_{14}t_{24}t_{32})\]

(2)

which demonstrates that for \(L \gg \xi\), \(D\) decays exponentially with \(L\). Although we have written \(D\) explicitly as a sum of powers of \(\exp^{-2L/\xi}\), expression (2) is exact and remains true when \(L \ll \xi\). The key observation here is that \(D\) is the sum of positive terms and hence is always positive. Hence the sign of \(G_{jk,lm}\) depends only on the relative magnitudes of \(T_{jl}T_{km}\) and \(T_{jm}T_{kl}\). This was noted separately by Avishai and Band [11] for a crossed wire arrangement and by Wang et al [10] for ballistically coupled wires. However both of these references rely on being able to apply certain symmetries to the system, which simplifies the form of \(D\). The above analysis shows that \(D\) is positive, independent of such symmetries.

Now consider the longitudinal conductances \(G_{13,24}\) and \(G_{14,23}\) in the limit \(L \gg \xi\). Since none of \(T_{12}, T_{34}\) and \(T_{43}\) are expected to decay with \(L\), the denominator of expression (1) for these conductances is positive, with magnitude of order unity. Hence in this limit

\[G_{13,24} \sim G_{14,23} \sim + \exp^{-2L/\xi}\]

(3)

Since the two-probe conductance \(G_2\) of a device with substantial localization is of order \(\sim \exp^{-2L/\xi}\), we see that for normal wires in the localised limit 4-probe conductance measurements will give results of the same sign and order of magnitude as 2-probe measurements.

Conductance formulae of the kind shown in equation (1), describe a normal scatterer connected to normal probes. The generalisation of this approach to the case where the scatterer incorporates superconductivity, but the probes remain normal, was first derived for 2 probes by [7, 8] and more recently for many probes by Lambert, Hui and Robinson [12]. This generalisation leads to the introduction
of a matrix of transport coefficients $A$, which at zero temperature, has elements $A_{ij} = N \delta_{ij} - T^O_{ij} + T^A_{ij}$, where superscripts $O$ and $A$ refer respectively to normal and Andreev scattering. It is shown in [12] that in general

$$G_{ij,kl} = d/\left(b_{ik} - b_{jk} - b_{il} + b_{kl}\right)$$

Here, $d = \det A$, and $b_{mn}$ is the cofactor of the matrix element $A_{mn}$. In order to highlight the behaviour of $G$ in the localized limit we work to zero’th order in $L/\xi$ so that $A$ becomes

$$A = \begin{pmatrix}
N - T^O_{11} + T^A_{11} & -T^O_{12} + T^A_{12} & 0 & 0 \\
-T^O_{21} + T^A_{21} & N - T^O_{22} + T^A_{22} & 0 & 0 \\
0 & 0 & N - T^O_{33} + T^A_{33} & -T^O_{34} + T^A_{34} \\
0 & 0 & -T^O_{43} + T^A_{43} & N - T^O_{44} + T^A_{44}
\end{pmatrix}$$

The respective determinants $d_{TL}$ and $d_{BR}$ of the top left and bottom right blocks of $A$ are both positive, because in general $N = \sum_{j=1}^{4} (T^O_{ij} + T^A_{ij})$ for any $i$, so that $A_{11} > A_{12}, A_{22} > A_{21}$ etc. Hence $d = d_{TL} d_{BR}$ is positive in the localized limit.

Now consider the longitudinal conductance $G_{13,24} = d/\left(b_{12} + b_{34} - b_{13} - b_{24}\right)$. To zeroth order in $L/\xi$ we have $b_{13} = b_{24} = 0$, $b_{12} = -a_{21} d_{BR}$, and $b_{34} = -a_{43} d_{TL}$. Hence

$$G_{13,24} = -\left(\frac{T^A_{21} - T^O_{21}}{d_{TL}} + \frac{T^A_{43} - T^O_{43}}{d_{BR}}\right)^{-1}$$

A similar expression can be derived for $G_{14,23}$. Since all the terms on the right hand side of (6) are of order unity, $G_{13,24}$ would also be expected to be of this order. However, in contrast with the behaviour of normal systems the conductance can be of arbitrary sign, depending on the relative magnitudes of the normal and Andreev scattering coefficients for transmission from a current to a voltage probe at the same end of the wire. In general stronger Andreev scattering favours negative longitudinal conductances, while stronger normal scattering favours positive conductances. It should also be noted that it is possible in principle for the two terms in (6) to
approximately cancel, so that the longitudinal conductance may become arbitrarily large, *even in the strongly localised limit* [14].

To test whether finite negative multi-probe conductances can occur in practice we have performed numerical simulations of both normal and superconducting multiprobe structures. Following Avishai and Pichard *et al* [13, 16], we model a 2d wire with 3 channels and 4 probes by a network of 1d wires connected at nodes, shown in figure 1b. In the figure $M$ denotes the number of slices between the pairs of probes. The figure shows the case $M = 10$, but simulations were performed over the range of lengths $M = 1$ to $M = 40$ for the normal wire and $M = 1$ to $M = 100$ for the superconducting wire. Disorder is introduced by specifying random scattering matrices at each node, while allowing perfect transmission along each 1d wire. For a normal system, the $N_w \times N_w$ scattering matrix of a node connecting $N_w$ wires is modelled by equating it to $\exp(iH)$, where $H$ is a $N_w \times N_w$ Hermitian matrix chosen as follows: each element along or above the main diagonal is a real number chosen at random and independently of the other elements of $H$ between $-\pi$ and $\pi$, and the elements below the diagonal are chosen to ensure $H = H^T$. For a superconducting sample the same procedure is adopted, except that the matrices are of size $2N_w \times 2N_w$ and the elements of $H$ are restricted to satisfy particle-hole symmetry. To obtain the scattering matrix for the whole network, we employ a numerical $S$-matrix reduction algorithm [17, 18], details of which are explained more fully elsewhere [21].

Figure 2 shows the logarithm of the conductance $G_{13,24}$ as a function of the number of slices $M$ in the wire. For each value of $M$, the conductances arising from 100 different realisations of the disorder are shown as dots in the figure. The inset shows corresponding results for the logarithm of the transmission coefficient $T_{13}$. Clearly the typical values of both $G_{13,24}$ and $T_{13}$ decay exponentially with
$M$ for large $M$. The different localization lengths for the two systems are to be expected, since the requirement that particle-hole symmetry must be satisfied for the superconductor will influence the level statistics of the random node scattering matrices. For a superconducting system, figure 3 shows corresponding results for $G_{13,24}$ (plotted on a linear scale) and $T_{13}$. This shows that for large $M$, whereas the transmission coefficient decays exponentially to zero, the conductance remains finite and can have arbitrary sign. This confirms our expectation in the strongly localized limit, based on equation (6) and additionally shows that negative conductances arise in the presence of quasi-particle transmission.

In the above simulations positive and negative conductances occur with roughly equal frequency, because our choice of random node scattering matrices favours neither normal nor Andreev scattering. Historically, the phenomenon of Andreev scattering, which yields charge transport in the absence of quasi-particle transmission, was used to explain the marked difference between thermal and electrical conductance across normal-superconducting interfaces. In the absence of quasi-particle transmission, the ends of the sample scatter quasi-particles independently and therefore apart from a dependence of the condensate potential on the applied potential difference $[7, 8, 12]$, the voltage probes become completely decoupled. In this limit, the voltage applied to probe 1 (3) serves to cancel the current due to quasi-particles from lead 2 (4) and therefore a 4-probe measurement can be viewed as two independent measurements of quasi-particle charge imbalance at the two ends of the system. In the absence of inelastic scattering there is no difference in principle between charge imbalance measurements $[22]$ and point contact spectroscopy of the kind described by Tsoi and Yakovlev $[23]$. Here the sign of the voltage due to quasi-particles was shown to be reversed by the application of a magnetic field. By analogy such experiments, one expects negative 4-probe conductances of the kind
predicted in this letter, to be particularly sensitive to the presence of applied or internal magnetic fields.

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Figure Captions

1. (a) A mesoscopic structure connected to four external probes numbered 1,2,3,4. (b) A network of one dimensional normal wires connected at nodes, with 3 wires in each external lead. The scatterer consists of $M$ slices of nodes. For a normal system, the scattering matrix of each node scatters particles into particles. For a superconducting system, the scattering matrix of each node also incorporates Andreev scattering.

2. The main figure is a scatter plot of the conductances $G_{13,24}$ of a normal system as a function of the number of slices $M$. The inset is a corresponding plot of the transmission coefficients $T_{13}$. All quantities are plotted on a logarithmic scale.

3. Results for the conductances and transmission coefficients of a superconducting system. In this case since the conductance no longer decays exponentially with $M$, only the transmission coefficients are plotted on a logarithmic scale.