THE SUPERSYMMETRIC TWO BOSON HIERARCHIES

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Abstract

We construct the most general supersymmetric two boson system that is integrable. We obtain the Lax operator and the nonstandard Lax representation for this system. We show that, under appropriate redefinition of variables, this reduces to the supersymmetric nonlinear Schrödinger equation without any arbitrary parameter which is known to be integrable. We show that this supersymmetric system has three local Hamiltonian structures just like the bosonic counterpart and we show how the supersymmetric KdV equation can be embedded into this system.
I. Introduction:

Integrable systems in 1+1 and 2+1 dimensions have been studied vigorously in the past [1-3]. These are nonlinear systems with a biHamiltonian structure and with infinitely many, independent, commuting conserved quantities and possess solitonic solutions. Furthermore, these equations have a standard Lax representation which makes them integrable through the method of inverse scattering. Such systems have also appeared in various studies in string theory [4].

More recently, a dispersive generalization of the long water wave equation [5-7] has received much attention [8-11]. It has the form

\[
\frac{\partial u}{\partial t} = (2h + u^2 - \alpha u')' \\
\frac{\partial h}{\partial t} = (2uh + \alpha h')'
\]

(1)

where \(u(x, t)\) and \(h(x, t)\) can be thought of as the horizontal velocity and the height, respectively, of the free surface (\(\alpha\) is an arbitrary parameter) and a prime denotes a derivative with respect to \(x\). The system of equations in (1) is integrable [6] and has a triHamiltonian structure [7]. It has a nonstandard Lax representation and reduces to various known integrable systems with appropriate identification [7]. Thus, for example, with the identification

\[
\alpha = 1 \\
u = -q'/q \\
h = q\overline{q}
\]  

(2)

equations (1) reduce to the nonlinear Schrödinger equation [8-11]. With \(\alpha = -1\), \(h = 0\), Eq. (1) gives the Burgers’ equation while for \(\alpha = 0\) we obtain Benney’s equation from (1). For the rest of our discussion, we will choose \(\alpha = 1\).

While the supersymmetric nonlinear Schrödinger equation has been studied in some detail in recent years [12-14] and while a fermionic extension of Eq. (1) has also been investigated [15], the supersymmetric form of Eq. (1) has not yet been obtained. The study of such a system, among other things, is expected to shed light on such properties as superturbulence in classical hydrodynamics. In this paper, we study the supersymmetric

2
generalization of the long water wave equation. In sec. II, we obtain the most general superfield equation consistent with dimensional analysis which can also be expressed as a superfield Lax equation. We give the component form of the equations and the supersymmetry transformations which leave the system invariant. In sec. III, we show how this equation reduces to the supersymmetric nonlinear Schrödinger equation which is integrable and which has been studied in some detail [13]. In sec. IV, we derive the Hamiltonian structures for this system and show that it is a triHamiltonian system much like the bosonic equation in (1). Finally, in sec. V we show how other supersymmetric equations, such as the susy KdV, can be embedded into this system and present a brief conclusion in sec. VI.

II. Supersymmetric Equation:

For the rest of our discussions, we will choose $\alpha = 1$ and for consistency with the two boson formulation of this system [8,16], we will make the identification $u = J_0$ and $h = J_1$. The equations in (1) then take the form

$$
\frac{\partial J_0}{\partial t} = (2J_1 + J_0^2 - J_0')'
$$

$$
\frac{\partial J_1}{\partial t} = (2J_0J_1 + J_1')'
$$

A simple dimensional analysis shows that we can assign the following canonical dimensions to the variables of the system.

$$
[x] = -1 \quad [t] = -2 \quad [J_0] = 1 \quad [J_1] = 2
$$

This system can be represented as a Lax equation [7,16]

$$
\frac{\partial L}{\partial t} = [L, (L^2)_{\geq 1}]
$$

where

$$
L = \partial - J_0 + \partial^{-1} J_1
$$

and $(\cdot)_{\geq 1}$ refers to the differential part of a pseudo-differential operator. (For details see ref. 16) We note that

$$
[L] = 1
$$
The simplest way to obtain the supersymmetric equation is to go to the superspace. Let $z = (x, \theta)$ define the superspace and

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$$

represent the supercovariant derivative. From the relation $D^2 = \partial$, it follows that

$$[\theta] = -\frac{1}{2}$$

Let us next introduce two fermionic superfields

$$\Phi_0 = \psi_0 + \theta J_0$$
$$\Phi_1 = \psi_1 + \theta J_1$$

The canonical dimensions of the new variables now follow to be

$$[\Phi_0] = [\psi_0] = \frac{1}{2}$$
$$[\Phi_1] = [\psi_1] = \frac{3}{2}$$

Given these, one can write the most general dynamical equations in superspace consistent with the canonical dimensions and which reduce to Eq. (3) in the bosonic limit as

$$\frac{\partial \Phi_0}{\partial t} = -(D^4 \Phi_0) + 2(D \Phi_0)(D^2 \Phi_0) + 2(D^2 \Phi_1) + a_1 D(\Phi_0(D^2 \Phi_0)) + a_2 D(\Phi_0 \Phi_1)$$

$$\frac{\partial \Phi_1}{\partial t} = (D^4 \Phi_1) + b_1 D((D^2 \Phi_1)\Phi_0) + 2(D^2 \Phi_1)(D \Phi_0) - b_2 D(\Phi_1(D^2 \Phi_0)) + 2(D \Phi_1)(D^2 \Phi_0) + b_3 \Phi_1 \Phi_0(D^2 \Phi_0) + b_4 D(\Phi_1 \Phi_0)(D \Phi_0) + b_5 D(\Phi_0(D^4 \Phi_0)) + b_6 D(\Phi_0(D^2 \Phi_0))(D \Phi_0)$$

Here $a_1, a_2, b_1, b_2, b_3, b_4, b_5$ and $b_6$ are arbitrary parameters and Eq. (12) represents the most general supersymmetric extension of Eq. (3). This system of equations, however, may not be integrable. To find an integrable, supersymmetric extension, we look for a Lax representation for the system of equations. We find that a consistent Lax representation can be obtained if we define

$$L = D^2 + \alpha(D \Phi_0) + \beta D^{-1} \Phi_1$$
where $\alpha, \beta$ are arbitrary parameters. (Note that for $\alpha = -1$ and $\beta = 1$ Eq. (13) reduces to Eq. (6) in the bosonic limit.) The nonstandard Lax equation

$$\frac{\partial L}{\partial t} = [L, (L^2)_{\leq 1}]$$  \hspace{1cm} (14)$$

in this case, gives

$$\frac{\partial \Phi_0}{\partial t} = -(D^4 \Phi_0) - 2\alpha(D\Phi_0)(D^2 \Phi_0) - \frac{2\beta}{\alpha} (D^2 \Phi_1)$$

$$\frac{\partial \Phi_1}{\partial t} = (D^4 \Phi_1) - 2\alpha D^2((D\Phi_0)\Phi_1)$$  \hspace{1cm} (15)$$

Comparing with Eq. (12), we conclude that $\alpha = -1$ and $\beta = 1$ so that

$$L = D^2 - (D\Phi_0) + D^{-1}\Phi_1$$  \hspace{1cm} (16)$$

and it would appear that the most general supersymmetric extension of Eq. (3) which is integrable is given by

$$\frac{\partial \Phi_0}{\partial t} = -(D^4 \Phi_0) + 2(D\Phi_0)(D^2 \Phi_0) + 2(D^2 \Phi_1)$$

$$\frac{\partial \Phi_1}{\partial t} = (D^4 \Phi_1) + 2D^2((D\Phi_0)\Phi_1)$$  \hspace{1cm} (17)$$

In components, the equations have the form

$$\frac{\partial J_0}{\partial t} = (2J_1 + J_0^2 - J_0')'$$

$$\frac{\partial \psi_0}{\partial t} = 2\psi_1' + 2\psi_0 J_0 - \psi_0''$$

$$\frac{\partial J_1}{\partial t} = (2J_0 J_1 + J_1' + 2\psi_0'\psi_1)'$$

$$\frac{\partial \psi_1}{\partial t} = (2\psi_1 J_0 + \psi_1')'$$  \hspace{1cm} (18)$$

This is a completely interacting system and is invariant under the supersymmetric transformations

$$\delta \psi_0 = \epsilon J_0$$

$$\delta J_0 = \epsilon \psi_0'$$

$$\delta \psi_1 = \epsilon J_1$$

$$\delta J_1 = \epsilon \psi_1'$$  \hspace{1cm} (19)$$
where $\epsilon$ is a constant Grassmann parameter of transformation.

**III. Reduction to SUSY NLS Equation:**

As we have noted in the introduction, the bosonic equation (3) reduces to the nonlinear Schrödinger equation with appropriate redefinition of variables (see Eq. (2)). It is, therefore, natural to expect that the supersymmetric equations (17) will reduce to the supersymmetric nonlinear Schrödinger equation [12-14] with appropriate redefinition of variables. This is indeed true and it is worth emphasizing that the only consistent set of field redefinitions that is possible, leads to the supersymmetric nonlinear Schrödinger equation without any arbitrary parameter. Without going into detail, we note that the field redefinitions

\[
\Phi_0 = -D \ln(DQ) + D^{-1}(\overline{Q}Q) \\
\Phi_1 = -\overline{Q}(DQ)
\]

where $Q = \psi + \theta q$ and $\overline{Q} = \overline{\psi} + \theta \overline{q}$ are fermionic superfields which are complex conjugates of each other, leads from Eq. (17) to (The derivation is slightly involved.)

\[
\frac{\partial Q}{\partial t} = -(D^4Q) + 2((D^2Q)\overline{Q} + (DQ)(D\overline{Q}))Q \\
\frac{\partial \overline{Q}}{\partial t} = (D^4\overline{Q}) - 2((D^2\overline{Q})Q + (DQ)(D\overline{Q}))\overline{Q}
\]

These are nothing other than the supersymmetric nonlinear Schrödinger equations without any free parameter. (See ref. 13 for details.) These equations are known to be integrable which also proves integrability of Eq. (17). We note that the field redefinitions in Eq. (20) reduce to Eq. (2) (with appropriate identification) in the bosonic limit.

**IV. Hamiltonian Structures:**

Most bosonic integrable systems are known to be biHamiltonian. The corresponding supersymmetric systems, on the other hand, have only a single Hamiltonian structure which is local (see [17-19] for the susy KdV system). As we have indicated in the introduction, the present bosonic integrable system in Eq. (3) is a triHamiltonian system [7]. It is, therefore, interesting to ask what will be the Hamiltonian structures of the supersymmetric equation in Eq. (17). We find that the present supersymmetric system is also triHamiltonian and this implies its integrability.
It is easy to verify with the usual Berezin rules for Grassmann variables that with 
\((z = (x, \theta))\)
\[
\{\Phi_0(z_1), \Phi_0(z_2)\}_1 = 0 = \{\Phi_1(z_1), \Phi_1(z_2)\}_1
\]
\[
\{\Phi_0(z_1), \Phi_1(z_2)\}_1 = D_{z_2} \partial(z_1 - z_2) = \{\Phi_1(z_1), \Phi_0(z_2)\}_1
\]  \hspace{1cm} (22)

and with
\[
H_1 = \int dz \left[ -\Phi_1(z)(D\Phi_1(z)) + (D^3\Phi_0(z))\Phi_1(z) \right.
\]
\[
- \Phi_0(z)(D\Phi_0(z))(D\Phi_1(z)) - (D^2\Phi_0(z))\Phi_1(z)\Phi_0(z) \bigg]  \hspace{1cm} (23)
\]

we can write Eq. (17) in the Hamiltonian form
\[
\frac{\partial \Phi_0(z)}{\partial t} = \{\Phi_0(z), H_1\}_1
\]
\[
\frac{\partial \Phi_1(z)}{\partial t} = \{\Phi_1(z), H_1\}_1
\] \hspace{1cm} (24)

On the other hand, we can choose
\[
\{\Phi_0(z_1), \Phi_0(z_2)\}_2 = 2D_{z_2} \partial(z_1 - z_2)
\]
\[
\{\Phi_0(z_1), \Phi_1(z_2)\}_2 = (D_{z_2} \Phi_0(z_2))D_{z_2} \partial(z_1 - z_2) + D^3_{z_2} \partial(z_1 - z_2)
\]
\[
\{\Phi_1(z_1), \Phi_0(z_2)\}_2 = (D_{z_1} \Phi_0(z_1))D_{z_2} \partial(z_1 - z_2) - D^3_{z_2} \partial(z_1 - z_2)
\] \hspace{1cm} (25)
\[
\{\Phi_1(z_1), \Phi_1(z_2)\}_2 = 2\Phi_1(z_1)D^2_{z_2} \partial(z_1 - z_2) - (D^2_{z_2} \Phi_1(z_2))\partial(z_1 - z_2)
\]

and
\[
H_2 = - \int dz \Phi_1(z)(D\Phi_0(z))
\] \hspace{1cm} (26)

to write Eq. (17) also in the Hamiltonian form
\[
\frac{\partial \Phi_0(z)}{\partial t} = \{\Phi_0(z), H_2\}_2
\]
\[
\frac{\partial \Phi_1(z)}{\partial t} = \{\Phi_1(z), H_2\}_2
\] \hspace{1cm} (27)
Finally, it is also easy to verify that the choice

$$\{\Phi_0(z_1), \Phi_0(z_2)\}_3 = 2\left[ (D_{z_1} \Phi_0(z_1)) + (D_{z_2} \Phi_0(z_2)) \right] D_{z_2} \delta(z_1 - z_2)$$

$$\{\Phi_0(z_1), \Phi_1(z_2)\}_3 = [D_{z_2}^2 + (D_{z_2} \Phi_0(z_2))]^2 D_{z_2} \delta(z_1 - z_2)$$

$$\{\Phi_1(z_1), \Phi_0(z_2)\}_3 = [D_{z_1}^2 + (D_{z_1} \Phi_0(z_1))] \left( \Phi_1(z_1) + \Phi_1(z_2) \right) D_{z_2}^2 \delta(z_1 - z_2)$$

$$\{\Phi_1(z_1), \Phi_1(z_2)\}_3 = [D_{z_1}^2 + (D_{z_1} \Phi_0(z_1))] \left( \Phi_1(z_1) + \Phi_1(z_2) \right) D_{z_2}^2 \delta(z_1 - z_2)$$

and the Hamiltonian

$$H_3 = -\int dz \Phi_1(z)$$

would also make Eq. (17) Hamiltonian. Thus, unlike other supersymmetric integrable systems, the present system allows generalization of all the three Hamiltonian structures of its bosonic counterpart. Let us note that if we denote the three Hamiltonian structures of Eqs. (22), (25) and (28) in the matrix form as $\omega^{-1}_1$, $\omega^{-1}_2$ and $\omega^{-1}_3$ respectively, then we can write

$$\omega^{-1}_2 = R \omega^{-1}_1$$

$$\omega^{-1}_3 = R \omega^{-1}_2 = R^2 \omega^{-1}_1$$

where

$$R = \omega^{-1}_2 \omega_1$$

This implies that the corresponding symplectic forms are also related through a matrix operator

$$S = R^{-1}$$

This immediately implies that the Nijenhuis tensor associated with $S$ must vanish [20] which is also a sufficient condition for the integrability of the system [21].

V. Embedding of SUSY KdV:

It is known that the KdV equation can be embedded into the long water wave equation and can also be expressed in the nonstandard Lax representation [7]. We show here that the
supersymmetric KdV equation [17] can, similarly, be embedded into the supersymmetric extension of the long water wave equation or the supersymmetric two boson hierarchy.

If we choose $\Phi_0 = 0$, then the Lax operator from Eq. (16) takes the form

$$L = D^2 + D^{-1}\Phi_1$$

(33)

We note that in this case,

$$(L^3)_{\geq 1} = D^6 + 3D\Phi_1D^2$$

(34)

It can now be checked in a straightforward manner that the nonstandard Lax equation

$$\frac{\partial L}{\partial t} = [L, (L^3)_{\geq 1}]$$

in the present case yields

$$\frac{\partial \Phi_1}{\partial t} = -(D^6\Phi_1) - 3D^2(\Phi_1(D\Phi_1))$$

(35)

which we recognize to be the supersymmetric KdV equation [17]. This is the embedding of the susy KdV equation in the nonstandard Lax representation. As is well known, while KdV is a biHamiltonian system, susy KdV has only one local Hamiltonian structure [17-19]. The embedding, therefore, is not compatible with all the Hamiltonian structures of the system.

VI. Conclusion:

We have constructed the most general supersymmetric extension of the two boson hierarchy (long water wave equation) which is integrable. We have shown that this system reduces to the susy nonlinear Schrödinger equation without any free parameter with appropriate redefinition of variables. We have shown that this supersymmetric system is triHamiltonian much like its bosonic counterpart. We have also shown how the supersymmetric KdV equation can be embedded into this system. The derivation of supersymmetric Burgers’ equation and the supersymmetric Benney equation can be carried out in a straightforward manner from our construction.

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