QCD meson spectrum in the large $N_C$ limit

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The low energy mass spectrum of QCD in the large $N_C$ limit is computed. The low-lying states in the meson spectrum are explicitly evaluated up to the fourth order in the strong coupling perturbative expansion. The 't Hooft limit is smooth and the meson masses are in very good agreement with the experimental values.

1. INTRODUCTION

Since the seminal work of 't Hooft the large $N_C$ limit has played an increasingly important role in studying gauge theories in the continuum, on the lattice (see for example [3] and references therein) and the duality between gauge and string theory [4]. However, there are still no estimates of physically observable quantities, where the large $N_C$ limit can be tested against experiments. Here we present the results of an explicit and successful calculation of the meson masses, in the large $N_C$ limit [5]. In [3] we extend to an arbitrary number of colors the procedure used by Banks et al. [6] in their, by now famous, evaluation of the low-lying states in the meson spectrum and then take the large $N_C$ limit. The 't Hooft limit is smooth [7] and all the divergent terms in the thermodynamic limit cancel against the ground state vacuum energy.

The theory is studied in the strong coupling limit using staggered fermions and the Hamiltonian approach to lattice gauge theory. We use $x = 1/g^2$ as the expansion parameter and the continuum limit is extrapolated using Padé approximants. The computation is performed by taking as the ground state that of the antiferromagnetic Ising model which is gauge invariant and breaks chiral symmetry. The gauge invariant eigenstates of the unperturbed Hamiltonian are used as a basis for the perturbative expansion. A similar approach has been successfully used to study the meson spectrum of some simpler toy models like the Schwinger models [8] and the two dimensional 't Hooft model [9].

2. HAMILTONIAN FORMULATION

We consider the Hamiltonian formulation of lattice gauge theory where the time is continuous and the space is discretized on a 3-dimensional cubic lattice with sites $x = (x_1, x_2, x_3)$, where $x_i$ are integers. The Hamiltonian reads

\[
H = \frac{1}{2a} \left( g^2 \sum_{[x,n]} E^a [x,n]^2 + \frac{1}{g^2} \sum_{[x,i,j]} Tr(UUUU) \right. \\
+ \sum_{[x,n]} \eta(\hat{n}) \Psi^\dagger_A (x + \hat{n}) U_{AB} [x,n] \Psi_B (x) + h.c.) \right) 
\]

(1)

\[(TrUUUU = Tr(\prod_{plaq[x,i,j]} U_{[x,i,j]}))\text{ where } \eta(\hat{x}) = (-1)^x \eta(\hat{y}) = (-1)^y \eta(\hat{z}) = (-1)^y\]

are the Dirac matrices for staggered fermions [10] and $a$ is the lattice spacing. The gauge field $U[x,n]$ is associated with the link $[x,n]$ and it is a group element in the fundamental representation of $SU(N_C)$. The electric field operator $E^a[x,n]$ is defined on the link and it obeys the algebra

\[ [E^a [x,n], E^b[y,j]] = i f^{abc} E^c[x,n] \delta([x,n] - [y,j]) \]

and $E[x,n] = E^a[x,n] T^a$, with $T^a$, $a = 1, \ldots, N_C^2 - 1$ the generators of the Lie algebra of $U(N_C)$. It generates the left-action of the Lie algebra on $U[x,n]$

\[ [E^a [x,n], U[y,j]] = -T^a U[x,n] \delta([x,n] - [y,j]) \]
The fermion fields $\Psi$ are defined on the lattice sites and obey the relation
$$\{\Psi_A(x), \Psi_B^\dagger(y)\} = \delta_{AB}\delta(x-y)$$

Each term in the Hamiltonian is gauge invariant under static gauge transformation. The Hamiltonian is also invariant under some discrete symmetries which are used to construct the operators generating the mesons from the vacuum. They are: lattice translation by a single link, shift along a face diagonal, parity, G-parity and Charge conjugation. In particular, the invariance under translation by a single link plays the role of a discrete chiral invariance of the theory. This symmetry is broken by an explicit mass term in the Hamiltonian.

### 2.1. The strong-coupling expansion

In the strong-coupling expansion $H_0$ is treated as unperturbed Hamiltonian, while $H_1$ and $H_2$ are the perturbations. $H_1$ is the interaction Hamiltonian between the quarks and the gauge fields and $H_2$ is the magnetic Hamiltonian:

$$H_0 = \frac{g^2}{2a} \sum_{[x,n]} E^a[x,n]^2$$

$$H_1 = \frac{1}{2a} \sum_{[x,n]} \eta(\hat{n}) \Psi_A^\dagger(\hat{n}) \Psi_B^\dagger(\hat{n}) U_{AB}[x,n] \Psi_B(x) + h.c.$$  

$$H_2 = \frac{1}{2g^2a} \sum_{[x,i,j]} Tr(UUUU) + h.c.$$  

The vacuum of the theory is a singlet of the electric field algebra. It is needed then to find the gauge invariant eigenstates of $H_0$ with the lowest energy $E_0$. The lowest eigenstates of $H_0$ with $E_0 = 0$ are singlets of the electric field algebra and since they are gauge invariant they must be color singlets. These states must also conserve charge, that means $\sum_x \rho(x)|0> = 0$ where $\rho$ is the fermion number operator $\rho(x) = \Psi^\dagger(x)\Psi(x) - N_C/2$. There are $M!(M/2)!$ degenerate ground states. The degeneracy can be resolved by diagonalizing the first nontrivial order in perturbation theory, that is the second

$$E_0^{(2)} = <0|H_2|0> + <0|H_1|0> \frac{\Pi_0}{E_0 - H_0} H_1|0> \quad (2)$$

where $<, > = \prod_{[x,n]} \int dU[x,n]|, >$ is the inner product in the full Hilbert space of the model. This energy has been computed by constructing an eigenstates of $H_0$ and using it to evaluate the functions present in the strong coupling expansion. If a state $|\Psi>$ is a singlet of the electric field algebra $E^a[x,n]|\Psi> = 0$ then $H_0|\Psi> = 0$ and

$$H_0U[x,n]|\Psi> = \frac{g^2}{2a} C_2(N_C) U[x,n]|\Psi> \quad (3)$$

where $C_2$ is the Casimir operator of $SU(N_C)$. After integration over the link variable $U$, the vacuum energy becomes

$$E_0^{(2)} = -K <0 \sum_{[x,n]} [\rho(x+\hat{n}) + \frac{N_C}{2}] - \rho(x) + \frac{N_C}{2}|$$

where $K = 1/g^2aC_2N_C$. The Hamiltonian has the two degenerate ground states of the antiferromagnetic Ising model; choosing one of them breaks the chiral symmetry. The vacuum energy has been computed up to the fourth order in the strong-coupling expansion [5].

### 3. THE MESON SPECTRUM

In the strong coupling expansion the lowest-lying states in the meson spectrum are those consisting of a quark and antiquark at opposite ends of a single link. For a given meson the wavefunction may be determined by taking the quark bilinear with the desired transformation properties, writing it in point-separated lattice form using the discrete symmetries of the theory and applying it to the vacuum. The mesons we have considered are $\pi_0$, $\rho$, $\omega$, $b_1$, $a_1$, $f_2$ and $f_0$. All the mesons are degenerate at the lowest order and their energy is $E_M^{(0)} = g^2 C_2/2a$. The meson energy has been computed up to the fourth order in the perturbative expansion using a procedure similar to the one used in the evaluation of the vacuum energy. If one computes the meson energy up to the fourth order, one gets

$$E_M = \frac{g^2 C_2}{2a} + E_M^{(2)} + E_M^{(4)}$$

(4)

where $a$ is the lattice spacing). If one rescales the coupling constant according to the 't Hooft
prescription $g^2 N_C \to g^2$ (large $N_C$ with $g^2 N_C$ fixed) one gets for the meson masses:

$$m_{\pi_0} = E_{\pi_0} - E_0 = \frac{1}{a\sqrt{y}} [\frac{1}{4} + 6y - 171y^2]$$

$$m_\omega = E_\omega - E_0 = \frac{1}{a\sqrt{y}} [\frac{1}{4} + 6y - 171y^2]$$

$$m_\rho = E_\rho - E_0 = \frac{1}{a\sqrt{y}} [\frac{1}{4} + 6y - 203y^2]$$

$$m_{b_1} = E_{b_1} - E_0 = \frac{1}{a\sqrt{y}} [\frac{1}{4} + 10y - 267y^2]$$

$$m_{a_1} = E_{a_1} - E_0 = \frac{1}{a\sqrt{y}} [\frac{1}{4} + 14y - 1435y^2]$$

$$m_{f_2} = E_{f_2} - E_0 = \frac{1}{a\sqrt{y}} [\frac{1}{4} + 14y - 875y^2]$$

$$m_{f_0} = E_{f_0} - E_0 = \frac{1}{a\sqrt{y}} [\frac{1}{4} + 18y - 1083y^2]$$

where $y = 1/g^4$.

### 3.1. Lattice versus continuum

The series are valid only for $g^2$ large. To compare the results of the strong-coupling expansion with the continuum theory we need to extrapolate these series to the region in which $y \gg 1$. This region corresponds to the continuum limit because $g^2 a^4 \to 0$ when $y \to \infty$. One may consider the mass ratios, expand them as power series in $y$ and then use [1, 1] Padé approximants by writing the mass ratios in the form $P_1^4 = (1 + ay)/(1 + by)$ where $a$ and $b$ are determined by expanding to order $y^2$ and equating coefficients. In the continuum limit this ratio yields $a/b$. The results obtained with this method are

$$\frac{m_{\pi_0}}{m_{b_1}} \to 0.75 \quad \frac{m_\omega}{m_{b_1}} \to 0.75 \quad \frac{m_\rho}{m_{b_1}} \to 0.71 \quad \frac{m_{b_1}}{m_{f_0}} \to 0.82 \quad \frac{m_{a_1}}{m_{f_2}} \to 0.92$$

For each mass ratio considered here, the [1, 1] Padé approximant exists with positive values for $a/b$. Therefore, an extrapolation from $y = 0$ to $y \to \infty$ is singularity free in this approximation. The results are in very good agreement with experiment except for the pion mass: this should not surprise due to the lack of a full chiral symmetry in the lattice theory.

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