Observation of the Eckhaus Instability in Whispering-Gallery Mode Resonators

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The Eckhaus instability is a secondary instability of nonlinear spatiotemporal patterns in which high-wavenumber periodic solutions become unstable against long-wavelength perturbations. We show in this letter that this instability can take place in Kerr combs generated with ultra-high Q whispering-gallery mode resonators. In our experiment, sub-critical Turing patterns undergo Eckhaus instabilities upon changes in the laser detuning leading to cracking patterns with long-lived transients. In the spectral domain, this results in a metastable Kerr comb dynamics with a timescale as large as few seconds. This ultra-slow timescale is at least six orders of magnitude larger than the intracavity photon lifetime, and is in sharp contrast will all the transient behaviors reported so far in dissipative nonlinear optics, that are typically only few photon lifetimes long (microseconds). We show that this phenomenology is well explained by the Lugiato-Lefever model, as the result of an Eckhaus instability. Our theoretical analysis is found to be in excellent agreement with the experimental measurements.

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Kerr optical frequency combs are obtained through pumping a high-Q whispering-gallery mode (WGM) cavity with a resonant laser [1]. In the last decade, the experimental and theoretical study of these combs has permitted major advances in photonics (see review articles [2–5]). From the applications standpoint, Kerr combs have been developed for time-frequency metrology, ultra-stable microwave generation, spectroscopy, and optical communications, just to name a few. From the fundamental perspective, Kerr combs have provided an ideal platform to investigate light-matter interactions in confined media. It has been shown that a wide variety of dissipative structures could be excited in the WGM resonators, being either stationary (azimuthal roll patterns, cavity solitons, platicons) or non-stationary (breather solitons, spatiotemporal chaos, rogue waves). The primary bifurcations leading to these various patterns have also been the focus of a detailed analysis in the literature [6–13]. However, only a limited attention has been also devoted to secondary bifurcations, which lead to the destabilization of the stationary patterns [13–16].

In this letter, we evidence experimentally one of these secondary bifurcations in 1D, namely the Eckhaus instability, which emerges when a roll (or stripes) pattern looses its stability against small-wavenumber perturbations. The Eckhaus instability has long been studied in fluid mechanics [17, 18], liquid crystals [19], nonlinear optics [13, 14, 20], or systems with delayed feedback [21]. Experimental observations are however much more limited since large aspect-ratio patterns are required, while being difficult to attain in most systems. Other secondary instabilities and parametric perturbations may also hinder Eckhaus instabilities [18, 19]. Counterintuitively, despite their relatively small size, WGM resonators can output large aspect-ratio roll patterns with tens or even hundreds of peaks [22], making the system...
The integer number of azimuthal rolls instability (MI) can be excited inside the cavity. They are characterized by an integer number of azimuthal rolls pattern and in the spectral domain, these roll patterns correspond to the so-called primary combs where the teeth have a $L \times$ FSR separation [6, 7, 23].

Fig. 2 shows two examples of primary combs corresponding to high-wavenumber roll patterns with $L = 47$ and $L = 87$. When the laser frequency is thermally driven across from the resonance, we observe the emergence of spurious peaks around the main primary comb, and the comb dynamics is characterized by a very slow timescale, that can be larger than a minute. This timescale appears a priori as inconsistent with the intrinsic Kerr comb dynamics, where the slowest timescale is generally the photon lifetime $\tau_{ph} = Q/\omega_0 \sim 1$ µs, with $Q \sim 10^9$ being the loaded quality factor of our resonator and $\omega_0$ is the angular frequency of the pumped mode [24–26]. However, as demonstrated hereafter, a detailed analysis unveils that the mechanism behind this ultra-slow timescale dynamics is an Eckhaus instability leading to cracking patterns.

The theoretical analysis of the Eckhaus instability starts with the Lugiato-Lefever equation [27], which is an accurate model to analyze the laser field dynamics in Kerr-nonlinear WGM resonators [28–30]. The slowly varying complex amplitude of the normalized intracavity field $\psi(\theta, \tau)$ obeys the equation

$$\frac{\partial \psi}{\partial \tau} = -(1 + i\alpha)\psi - \frac{\beta}{2} \frac{\partial^2 \psi}{\partial \theta^2} + i|\psi|^2 \psi + F,$$

(1)

where $\theta \in [-\pi, \pi]$ is the azimuthal coordinate along the ring of the resonator, and $\tau = t/2\tau_{ph}$ is the time scaled to the photon lifetime. The normalized parameters of this equation are the continuous-wave pump field $F$, the frequency detuning between laser and pumped resonance frequencies $\alpha$, and the group-velocity dispersion $\beta$ [29].

Equation (1) has homogeneous steady states $\psi_n$ implicitly given by $\rho_\alpha [1 + (\rho_\alpha - \alpha)^2] = F^2$ with $\rho_\alpha = |\psi_n|^2$. This equation is trivalued for $\alpha > \sqrt{3}$. The gray line $SN_\alpha$ in Fig. 3 corresponds to the saddle-node bifurcation where lower and middle branches meet. Correspondingly $SN_t$ for middle and upper branches. $SN_b$ and $SN_t$ unfold from a cusp at $\alpha = \sqrt{3}$ [6, 7]. For $\alpha < \sqrt{3}$ the solution is monovalued. In what follows we refer the lower homogeneous steady state as HSS.

In the anomalous regime ($\beta < 0$) and for $\alpha < 2$, $\rho_\alpha = 1$ is the MI threshold (yellow line in Fig. 3), above which the HSS is unstable to perturbations with wavenumber $L$ in a range around $L_n = \sqrt{\langle 2/\beta \rangle (\alpha - 2\rho_\alpha)}$. Roll patterns with different wavenumbers can emerge although typically the one with wavenumber $L_n$ dominates since it has the largest growth rate. This pattern is supercritical for $\alpha < 41/30$, and subcritical for $\alpha > 41/30$. Regarding the other possible roll patterns, it turns out that only those with wavenumber close to $L_n$ are stable, forming what is known as a Busse balloon [31, 32] while the others are unstable. Moreover, in the subcritical regime cavity solitons or localized states (LSs) coexist with the periodic patterns and the HSS. For $\alpha > 2$ the critical wavenumber is zero and the threshold $\rho_\alpha = 1$ is a Belyakov-Devaney (BD) transition of the HSS [7, 13] (yellow line in Fig. 3).

To study the secondary bifurcations that destabilize a roll pattern of wavenumber $L$ we perform a linear stability analysis. The stationary but $\theta$-dependent pattern can be expanded in Fourier series

$$\psi_L(\theta) = \sum_{n=-N}^{N-1} \psi_n e^{inL\theta},$$

(2)

with $L$ being the integer wavenumber (or order) of the pattern and $\psi_n$ the complex amplitudes of the Fourier modes. We take $N = 32$ and the amplitudes can be calculated numerically by solving the stationary problem using a Newton-Raphson algorithm. Linearizing Eq. (1) about the stationary pattern $\psi_L(\theta)$ yields the perturba-
waves

Eq. (3) can be written as the superposition of Bloch waves

\[ \partial_t \delta \psi = -(1 + i \alpha) \delta \psi - i(\beta/2) \partial^2 \psi \delta \psi + 2i|\psi|^2 \delta \psi + i \psi^* \delta \psi^* . \]  

(3)

Due to the periodicity of the pattern, the solution of Eq. (3) can be written as the superposition of Bloch waves

\[ \delta \psi(\theta, \tau) = e^{iq\theta} \delta a(\theta, \tau, q) + e^{-iq\theta} \delta a(\theta, \tau, -q) , \]  

(4)

where \( \delta a \) has the same periodicity of the pattern \( \psi(\theta) \), and can be written as

\[ \delta a(\theta, \tau, q) = \sum_{n=-N}^{N-1} \delta a_n(\tau, q) e^{inL\theta} \]  

(5)

with \( q \) being an integer number. Using Eq. (3), a set of linear equations for the amplitudes \( \delta a_n(\theta, q) \) can be derived [34], and in compact form they read as

\[ \partial_{\tau} \Upsilon(\tau, q) = M(\{\psi_n\}, q) \Upsilon(\tau, q) , \]  

(6)

where \( \Upsilon(\tau, q) = [\delta a_{-N}(\tau, q), \ldots, \delta a_{N-1}(\tau, q), \delta a_{0}^*(\tau, -q), \ldots, \delta a_{N-1}^*(\tau, -q)] \). The stability analysis of \( \psi(\theta) \) reduces to find the 2N eigenvalues \( \{\lambda_n(q)\} \) of the matrix \( M(\{\psi_n\}, q) \), and its corresponding eigenvectors, for each value of \( q \). The eigenvalues for a given integer \( q \) determine the stability of the pattern against perturbations containing any set of wavenumbers \( nL \pm q \). For this analysis it is sufficient to consider only the \( q \) values inside the first Brillouin zone \([0, L/2] \). We recall that \( q = 0 \) corresponds to the Goldstone mode associated to the translational invariance, and modes with \( q \gtrsim 0 \) form the branch of soft modes.

FIG. 3: Bifurcation lines of the HSS and the roll pattern with \( L = 55 \) in the parameter space \((\alpha, F)\). The HSS is stable below the MI line (in yellow) for \( \alpha < 2 \) and below the SN\(_b\) line (in grey) for \( \alpha > 2 \). The pattern is stable above the Eckhaus line (EC) (in red) and below the SN\(_{P2}\) (in black) or FWH (dot-dashed) line whichever comes first. The dashed line shows the ramp of parameters applied to the pattern, starting from \((\alpha, F) = (1,1.05)\) to \((3.8, 2)\) beyond the Eckhaus instability.

FIG. 4: Real part of the eigenvalues of the pattern with \( L = 55 \) for the branch of soft modes obtained from Eq. (6). Lines (a) to (d) correspond to the parameter values indicated by blue dots in Fig. 3. The curvature of the branch progressively changes from negative to positive signaling the Eckhaus instability.

Fig. 3 shows the bifurcation lines of the roll pattern created spontaneously with the most unstable wavenumber \( L_u \) for \( \alpha = 1 \) and \( F = 1.05 \) \((\rho_s = 1.095) \). For the value of \( \beta \) considered here \((\beta = -8 \times 10^{-4}) \), \( L_u = 55 \). Increasing the detuning, the pattern becomes subcritical for \( \alpha \simeq 41/30 \), and above this value it exists between the saddle-node lines SN\(_{P2}\) and SN\(_{P1}\) in black, although is unstable below the Eckhaus line (EC) (in red). Above a certain value of the detuning and the pump, we observe a finite-wavelength Hopf (FWH) instability (dot-dashed gray) leading to oscillatory patterns [13, 34]. We will not consider this regime here since we focus on the Eckhaus instability. Note that pattern and HSS are stable and coexist in the parameter region limited by MI, SN\(_b\), FWH and EC lines.

Fig. 4 shows the real part of the eigenvalues of the pattern as a function of the wavenumber \( q \) for the branch of soft modes [34]. The parameters correspond to those of the blue dots in Fig. 3, while crossing the Eckhaus instability. The change of convexity of the branch at \( q = 0 \) is what precisely signals the Eckhaus instability. After the instability, the pattern becomes unstable to small-wavenumber perturbations [Fig. 4(c)]. Well beyond the instability, the mode with maximum growth rate has a wavenumber close to the edge of the Brillouin zone, as shown in Fig. 4(d).

After encountering an Eckhaus instability a pattern with a wavenumber which is too large to be stable loses cells in such a way that the new wavenumber lies in the stability balloon [14]. For supercritical patterns this happens at a relatively fast time scale. For subcritical patterns, the HSS is stable and coexists with the pattern allowing the formation of LS. When a cell is lost the space is occupied by the HSS leading to a transient state formed by groups of LSs separated by the HSS, known as crack pattern [33]. If LSs have oscillatory tails they may lock at specific distances given by multiples of the oscilla-
tory tail wavelength, thus the cracking pattern pattern is stationary. On the contrary, if LS tails are monotonous, LS repel each other and the cracking pattern evolves towards a periodic solution with equally spaced peaks and a stable wavenumber. In practice, a similar behavior is observed if tails are oscillatory with a wavelength much larger than the typical separation between peaks. This transient behavior can be extremely slow as the interaction decays exponentially with the distance between peaks allowing for long-lived cracking patterns likely to be observed at second- and even minute-timescales in experiments.

The Eckhaus instability is triggered numerically by slowly ramping up the detuning and the pump parameter. This procedure is consistent with the experimental system where the detuning is thermally driven across the resonance [35]. The dotted line in Fig. 3 shows the ramp of parameter values used in the simulation shown in Fig. 5. The simulation starts at $t = 0$ with $\alpha = 1$ and $F = 1.05$, just above the MI, and a stable pattern with $L = 55$ emerges, corresponding to the wavenumber with maximum growth rate $L_u$. The parameters are ramped until $t = 0.05$ s with $\alpha = 3.8$ and $F = 2$, above the Eckhaus instability. The simulation then continues up to $t = 3$ s with clamped values for $\alpha$ and $F$. The original pattern, whose spatial profile and power spectrum is shown in Figs. 5 a) and b), remains stable through the ramp until it crosses the EC line. At this point the pattern becomes unstable and soft mode perturbations start to grow. As a consequence some pattern cells disappear as shown in Figs. 5 c) and d). Further development of the instability leads to a cracked pattern as shown in Figs. 5 e) and f) for time $t = 0.077$ s. For the parameters considered, LSs have oscillatory tails although the wavelength of the tail oscillations is much larger than the separation between consecutive peaks [36]. As a consequence, LSs do not get pinned rather they repel each other and the cracked pattern evolves at a very slow time scale towards a periodic solution. The power spectrum corresponds to the experimental combs displayed in Fig. 2. The asymptotic states after yield combs. We stopped our simulations at $t = 3$ s [Figs. 5 g) and h)], which corresponds to three million photon lifetimes in our resonator, but it can still take up to several minutes to converge asymptotically to another pattern, as observed experimentally in Fig. 2. In contrast for all nonlinear effects reported so far using the LLE, transient dynamics usually last only few $\tau_{ph}$. If the ramp is increased to much larger values of the detuning, one reaches the single-soliton regime described in [38] and eventually only a single peak survives.

In conclusion, we have experimentally evidenced the Eckhaus instability in a whispering-gallery mode resonator. The emerging timescale of the instability, dominated by the interaction between LSs, was shown to be six to eight orders of magnitude larger than the intracavity photon lifetime, which is the natural timescale for Kerr comb dynamics. These results permit to achieve a deeper understanding of secondary bifurcations in dissipative optical systems, and future work will investigate in detail the wide variety of spatiotemporal patterns that can be excited via these bifurcations.

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FIG. 5: Numerical evidence of Eckhaus instability. Nonlinear evolution of the $L = 55$ pattern after crossing the EC line as described in the main text. Left column shows the spatial profile of the pattern at different times while right column shows the corresponding power spectra. Time stamps given in real time $t = 2\tau_{ph} \times 10^{-6} \tau$. 

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The wavelength of the spatial oscillations in the tails of the LS calculated from the imaginary part of the spatial eigenvalues of the HSS [37] for the parameters used in the simulations is $\pi/6.59$, larger than the typical separation between peaks.

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