Closed 3-forms in five dimensions and embedding problems

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Abstract

We consider the question if a five-dimensional manifold can be embedded into a Calabi–Yau manifold of complex dimension 3 such that the real part of the holomorphic volume form induces a given closed 3-form on the 5-manifold. We define an open set of 3-forms in dimension five which we call strongly pseudoconvex, and show that for closed strongly pseudoconvex 3-forms, the perturbative version of this embedding problem can be solved if a finite-dimensional vector space of obstructions vanishes.

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In this paper, we begin the study of an embedding question for 3-forms on five-dimensional manifolds. Let $Z$ be a complex Calabi–Yau threefold, that is, a three-dimensional complex manifold with a nowhere-vanishing holomorphic 3-form, which we write as $\Psi + i\bar{\Psi}$ where $\Psi, \bar{\Psi}$ are real 3-forms. From another point of view, $Z$ is a manifold with a torsion-free $SL(3, \mathbb{C})$-structure. Let $M$ be a 5-manifold and $\psi$ a given closed 3-form on $M$. The question we consider is the existence of an embedding $F : M \rightarrow Z$ such that $F^*\Psi = \psi$. The main case we have in mind is when $Z$ is $\mathbb{C}^3$ with its standard holomorphic 3-form.

Our original motivation for considering this question comes from Hitchin’s approach to $SL(3, \mathbb{C})$-structures [10]. A special feature of the algebra of 3-forms in dimension 6 is that there is an open set of forms $\Psi$ which determine an almost-complex structure and the imaginary part $\bar{\Psi}$ algebraically. The structural equations are equivalent to the conditions that $\Psi$ and $\bar{\Psi}$ are closed 3-forms: they can be viewed as a system of partial differential equations for $\Psi$. On a closed 6-manifold, Hitchin gave a variational formulation in terms of the volume functional on forms $\Psi$.
in a given de Rham cohomology class. In this setting, there is a natural boundary value problem on a 6-manifold $Z_0$ with boundary $\partial Z_0 = M$ and with a given 3-form $\psi$ on $M$. One seeks a solution $\Psi$ to the partial differential equations which restrict to $\psi$ on the boundary. This is the analogue, in dimension 6, of the seven-dimensional theory for $G_2$-structures studied in [5]. We plan to develop this boundary value theory further in another article but, in the perturbative theory which we focus on in the body of the current paper, if $Z_0$ is, for example, a pseudoconvex domain with smooth boundary in $\mathbb{C}^3$, then by results of Hamilton [8], any deformation of $Z_0$ as an abstract complex manifold with boundary is realised as a deformation of the domain within $\mathbb{C}^3$. Then, the boundary value problem in this setting is essentially equivalent to the embedding problem for $M$ in $Z$.

An informal count tells us that a closed 3-form in five dimensions depends locally on six unconstrained functions. (That is, we can write such a form as $d \tau$ for a 2-form $\tau$, which gives 10 functions: we can change $\tau$ to $\tau + d \eta$, so we subtract 5, but if $\eta = df$, the change is ineffective, so we add 1, and our count is $10 - 5 + 1 = 6$.) Since a map from $M$ to a 6-manifold also depends on six functions, the count suggests that the embedding question is a reasonable one. This is special to the dimension 5: for $m > 3$, on a manifold of dimension $2m - 1$, the closed $m$-forms form a much ‘larger’ space than the maps to a $2m$-manifold. The situation is somewhat like that in Riemannian geometry where, at the level of function counting, in dimension $n = 2$, it is reasonable to seek an isometric embedding of an abstract Riemannian $n$-manifold as a hypersurface in $\mathbb{R}^{n+1}$, but not for $n > 2$. In that setting, the solution (by Nirenberg and Pogorelov) of the famous Weyl problem gives existence and uniqueness for the case when the metric on the surface has positive Gauss curvature: the image is then the boundary of a convex domain in $\mathbb{R}^3$. In a similar vein, in this article, we focus on data satisfying a pseudoconvexity condition. We will also see (in Subsection 1.4) that the Minkowski problem — another famous classical embedding problem — can be obtained as a dimensional reduction of our theory.

We begin Section 1 with an elementary study of the structure of closed 3-forms on 5-manifolds $M$ and define an open set of strongly pseudoconvex forms. If a solution to the embedding problem exists, these correspond to strongly pseudoconvex hypersurfaces in the ordinary sense of several complex variable theory. Such a 3-form defines a contact structure $H \subset TM$ with a contact form $\theta$ and an orthonormal pair of 2-forms $\omega, \alpha$ on $H$. A solution of the embedding problem gives a third 2-form $\beta$ on $H$ such that $\omega, \alpha, \beta$ make up an orthonormal triple, satisfying certain equations involving the exterior derivative $d$ and its restriction $d_H$ to $H$. $(\theta, \omega, \alpha, \beta)$ defines an SU(2)-structure, of a kind which we call contact hyperkähler. A special class of such structures is formed by the Sasaki–Einstein structures, which we define in Section 1.3. Contact hyperkähler SU(2)-structures are a special type of so-called nearly hypo SU(2)-structures which are induced on real hypersurfaces in six-dimensional manifolds with a torsion-free SU(3)-structure [4]. These in general come without a contact structure.

As Robert Bryant pointed out to us, the core of the embedding question can be formulated as a problem on the 5-manifold $M$. Starting with a strongly pseudoconvex 3-form $\psi$, and hence a pair $(\omega, \alpha)$, the problem is to extend this to a contact hyperkähler structure $(\omega, \alpha, \beta)$. This is a non-linear PDE for $\beta$ on the 5-manifold which can be viewed as a ‘contact version’ of the Calabi–Yau problem in four real dimensions (i.e. the existence of a hyperkähler structure). If we have a solution $\beta$, the pair $(\alpha, \beta)$ defines a CR structure. From then on, our embedding question becomes essentially the much-studied CR-embedding problem.

In the first part of Section 3, we analyse the perturbative version of our problem, for deformations around a given solution, and the associated linearised question. The SU(2)-structure defines a Euclidean metric on $H$ and a decomposition of the 2-forms on $H$ into self-dual and anti-self-dual
parts: $\Omega_H^2 = \Omega_H^+ \oplus \Omega_H^-$, where the self-dual subspace is spanned by $\omega, \alpha, \beta$. We find that the key operator in the linearised theory is

$$d_H^- : \Omega_H^1 \to \Omega_H^-.$$  

The relevant foundations from linear analysis are developed before, in Section 2. In particular, we show that the vector space

$$\mathcal{H} = \{ \sigma \in \Omega_H^- : d_H \sigma = 0 \},$$  

can be identified with the cokernel of $d_H^-$. This space $\mathcal{H}$ thus appears as the obstruction to solving the deformation problem (if $H^2(M, \mathbb{R})$ is non-zero, the obstruction space can be reduced to a possibly smaller space $\mathcal{H}_1$). We also show that the vanishing of $\mathcal{H}$ is an open condition. The main feature of the linear analysis is that, as in the CR theory, the relevant operators are subelliptic, not elliptic, and the inverses suffer a ‘loss of derivatives’ in Sobolev spaces. In the second part of Section 3, we apply the Nash–Moser inverse function theorem to obtain our deformation result, in the case when $\mathcal{H} = 0$. This requires a careful study of the dependence of the estimates for inverse operators on parameters.

Given $\psi$, an embedding $F : M \hookrightarrow Z$ with $F^* \Psi = \psi$ is in general not unique. If $F(M)$ is the boundary of a domain $U \subset Z$, and if $\Phi : U \to Z$ is a diffeomorphism to its image which is holomorphic and satisfies $\Phi^* \Psi = \Psi$, then $\Phi \circ F$ is another embedding which realises $\psi$. If, for example, the ambient space is $\mathbb{C}^3$, then the restriction of every element in $\text{SL}(3, \mathbb{C})$ is such a diffeomorphism.

We collect our results in the following theorem.

**Theorem 1.** Let $(\theta, \omega, \alpha, \beta)$ be a contact hyperkähler $\text{SU}(2)$-structure on $M$.

* The space $\mathcal{H}$ is finite dimensional.
* Suppose the $\text{SU}(2)$-structure is induced by an embedding $F : M \hookrightarrow Z$ and that $\mathcal{H} = 0$. Then, for every closed 3-form $\tilde{\psi}$ in the de Rham cohomology class of $\psi = F^* \Psi$ which is sufficiently close to $\psi$, there is an embedding $\tilde{F}$ close to $F$ such that $\tilde{F}^* \Psi = \tilde{\psi}$. If the ambient space is $Z = \mathbb{C}^3$, then $\tilde{F}$ in a neighbourhood of $F$ is unique up to holomorphic diffeomorphisms as above.
* If the $\text{SU}(2)$-structure is Sasaki–Einstein and $Z$ is Stein, then $\mathcal{H} = 0$. In particular, this is true for the standard embedding $S^5 \hookrightarrow \mathbb{C}^3$.

We prove the first item in the more general context of contact-metric 5-manifolds. For the proof of the last item in Theorem 1, we need some of the theory of the $\tilde{\partial}_b$ complex, which is reviewed in Appendix A.

We do not know any examples of contact hyperkähler manifolds which bound a strongly pseudoconvex region in a Stein manifold for which $\mathcal{H}$ is non-zero. It is possible that it is always zero, which would greatly extend the scope of our result. In Appendix B, we give an explicit neighbourhood of the standard structure on $S^5$ where $\mathcal{H}$ vanishes and in Appendix C, we obtain a curvature criterion for vanishing, via a Weitzenbock formula. This involves differential-geometric constructions which have independent interest.

The uniqueness statement in Theorem 1 only covers the case when $Z = \mathbb{C}^3$. We are confident that there is a similar statement in general but that seems to be more easily treated in the framework of the boundary value theory alluded to at the beginning of this introduction, so we do not go into it here.
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1 CLOSED 3-FORMS IN DIMENSION 5

1.1 The structure of closed 3-forms on a five-dimensional manifold

Let $M$ be an oriented 5-manifold and $\psi$ a closed 3-form on $M$. In this section, we describe the structure induced on $M$ by $\psi$ under a further convexity condition. The chosen terminology will become clear in the next section.

**Definition 1.1.** The 3-form $\psi$ is called *strongly pseudoconvex* if it satisfies the following three properties.

1. The skew-symmetric bilinear form on cotangent vectors with values in the real line of volume forms given by

   $$ (\lambda, \eta) \mapsto \lambda \wedge \eta \wedge \psi $$

   has maximal rank, that is, 4, at each point. This defines a rank 4 subbundle $H \subset TM$.

   To describe the next two conditions, let $\theta$ be a 1-form which at each point spans the one-dimensional space of cotangent vectors for which (1.2) is degenerate. Then, $\psi$ can be written as $\psi = \theta \wedge \alpha$ for some 2-form $\alpha$ and $H$ is the kernel of $\theta$.

2. $\theta \wedge \alpha^2$ does not vanish anywhere on $M$.

3. $\theta \wedge (d\theta)^2$ is a positive multiple of $\theta \wedge \alpha^2$. In particular, $d\theta$ is non-degenerate on $H$, which means that $H$ is a contact structure.

The one-dimensional space for which (1.2) is degenerate, which we will denote by $\ker(\psi)$, is characterised by $\ker(\psi) = \{ \theta : \theta \wedge \psi = 0 \}$. If $\Phi$ is a diffeomorphism of $M$, the calculation $\Phi^* \theta \wedge \Phi^* \psi = \Phi^* (\theta \wedge \psi) = 0$ shows $\ker(\Phi^* \psi) = \Phi^* \ker(\psi)$. This means that condition (1) is preserved by the action of the diffeomorphism group, and it is clear that conditions (2) and (3) are preserved as well. Thus, the diffeomorphism group acts on the set of strongly pseudoconvex 3-forms. Furthermore, if $\psi$ determines the contact distribution $H$, then $\Phi^* \psi$ determines the contact distribution $(\Phi^{-1})_*H$.

The decomposition $\psi = \theta \wedge \alpha$ is not unique as we can change $\theta$ to $f \theta$ and $\alpha$ to $f^{-1} \alpha$, where $f$ is a nowhere vanishing function on $M$, and add $\theta \wedge \chi$ to $\alpha$, where $\chi$ is any 1-form. Conditions (2) and (3) make sense independent of these choices. In the following, we choose preferred forms $\theta$ and $\alpha$.

First of all, fix the sign of $\theta$ such that $\theta \wedge \alpha^2$ is a positive form with respect to the orientation of $M$. Then scale as above by a positive function $f$ such that $\theta \wedge \alpha^2 = \theta \wedge (d\theta)^2$. This fixes the contact form $\theta$ and hence a Reeb vector field $v$ such that $v$ spans $\ker d\theta$ and $\theta(v) = 1$. Define

$$ \omega := d\theta. $$
We have a splitting
\[ TM = \mathbb{R}v \oplus H, \]  
which furthermore induces a splitting
\[ \Lambda^p TM^* = \theta \wedge \Lambda^{p-1} H^* \oplus \Lambda^p H^* \]
of the bundle of \( p \)-forms on \( M \) with the associated splitting
\[ \Omega^p = \theta \wedge \Omega^{p-1}_H \oplus \Omega^p_H, \]
of \( p \)-forms, where we write \( \Omega^p_H \) for sections of \( \Lambda^p H^* \). By the definition of \( v \), we have \( \omega \in \Omega^2_H \). We now fix \( \alpha \) such that \( \alpha \in \Omega^2_H \). This does not change the previous normalisation, that is, we have \( \omega^2 = \alpha^2 \). Write
\[ \text{Vol}_H := \omega^2 \in \Omega^4_H \]
for this volume form on \( H \).

Similar to the geometry of 4-manifolds, an important role will be played by the symmetric bilinear form on \( \Omega^2_H \) given by the wedge product. We write
\[ \sigma.\tau = \sigma \wedge \tau \text{Vol}_H, \quad \sigma, \tau \in \Omega^2_H. \]

Next, we describe the exterior derivative \( d \) under the splitting (1.5). For \( X \in \Gamma(H) \), we have \( \theta([v,X]) = -\omega(v,X) = 0 \), so that the Lie derivative \( \mathcal{L}_v \) along the Reeb vector field \( v \) preserves the splitting of \( TM \) and (1.5). The exterior derivative on \( \Omega^p_H \) is the sum of
\[ d_H : \Omega^p_H \to \Omega^{p+1}_H, \]
and
\[ \theta \wedge \mathcal{L}_v : \Omega^p_H \to \theta \wedge \Omega^p_H. \]
The equation \( d^2 = 0 \) induces the relations
\[ d_H^2 = -\omega \wedge \mathcal{L}_v, \quad \mathcal{L}_v d_H + d_H \mathcal{L}_v = 0. \]

Now we will use \( d\psi = 0 \) to derive relations for \( \omega \) and \( \alpha \). \( d\psi = 0 \) is equivalent to
\[ \omega \wedge \alpha = \theta \wedge d\alpha, \]
and since the left-hand side is a section of \( \Lambda^4 H^* \), we get \( \omega \wedge \alpha = 0 \) and \( \theta \wedge d\alpha = 0 \). To sum up, \( \psi \) gives orthonormal sections \( \alpha, \omega \) of \( \Lambda^2 H^* \) in the sense that
\[ \omega.\omega = 1, \quad \alpha.\alpha = 1, \quad \omega.\alpha = 0. \]
Furthermore, we have

\[ d\omega = 0, \quad d_H\alpha = 0. \]

The orthonormal pair \((\omega, \alpha)\) defines a complex structure \(K\) on \(H\) with complex volume form \(\omega + i\alpha\). At each point, the group of linear transformations of \(TM\) preserving the structure \((\theta, \omega, \alpha)\) is isomorphic to \(\text{SL}(2, \mathbb{C})\).

### 1.2 Closed 3-forms in dimension 5 realised by an embedding into a Calabi–Yau 3-fold

Let \(Z\) be a complex manifold of complex dimension 3 with a nowhere vanishing holomorphic form \(\Psi + i\hat{\Psi}\) of type (3,0). The local model is \(Z = \mathbb{C}^3\) with \(dz^1 \wedge dz^2 \wedge dz^3\). Let \(M \subset Z\) be a submanifold of real dimension 5. The pull-back of \(\Psi\) to \(M\) induces a closed 3-form \(\psi\) on \(M\). To understand the algebraic properties of \(\psi\), take \(\mathbb{C}^3\) with co-ordinates \(z_j = x_j + iy_j\). \(\mathbb{C}^3\) has the frame

\[ e_1 = \partial_{x_1}, \quad e_2 = \partial_{y_1}, \quad e_3 = \partial_{x_2}, \quad e_4 = \partial_{y_2}, \quad e_5 = \partial_{x_3}, \quad e_6 = \partial_{y_3}, \]

with dual frame \(\{e^1, \ldots, e^6\}\). In this frame,

\[ \Psi = e^{135} - e^{146} - e^{236} - e^{245}, \quad \hat{\Psi} = e^{136} + e^{145} + e^{235} - e^{246}, \]

where we write \(e^{ijk}\) for \(e^i \wedge e^j \wedge e^k\). Given a point \(p \in M\), we can always find a bi-holomorphism of \(\mathbb{C}^3\) which preserves the holomorphic volume form such that \(p = 0\) and \(M\) in a neighbourhood of \(p\) is given as the graph \(y_3 = f(x_1, y_1, x_2, y_2, x_3)\) of a function \(f\) with \(f(0) = 0\) and \(df(0) = 0\). Then, \(T_pM = \text{span}\{e_1, \ldots, e_5\}\) and

\[ \psi|_p = e^{135} - e^{245}. \]

At the point \(p\), the skew-symmetric form (1.2) is given by

\[ (e_{13} - e_{24}) \otimes e^{12345}. \]

We see that this form has rank 4, being degenerate on the span of \(\theta|_p = e^5\), and defines \(H_p = \text{span}\{e_1, \ldots, e_4\}\). Thus, we see that condition (1) in Definition 1.1 is necessary for a closed 3-form to be realisable as the restriction of \(\Psi\) by an embedding of \(M\) into \(Z\). We decompose \(\psi = \theta \wedge \alpha\) and \(\hat{\psi} = \theta \wedge \beta\), where

\[ \alpha|_p = e^{13} - e^{24}, \quad \beta|_p = e^{14} + e^{23}. \]

We have \(\theta \wedge \alpha^2|_p = 2e^{12345}\). Thus, condition (2) in Definition 1.1 is necessary, too, for \(\psi\) to be realised by the embedding into \((Z, \Psi + i\hat{\Psi})\).

We have the relations \(\alpha \wedge \beta = 0 \text{ and } \alpha^2 = \beta^2\). Thus, \(\alpha|_I + i\beta|_I = dz_1 \wedge dz_2\) induces an almost complex structure \(I\) on \(H\). \((H, I)\) is the real expression of the CR-structure induced by the complex structure of the ambient manifold \(Z\). The fundamental invariant of this CR-structure is its Levi
form
\[ L(X, Y) = d\theta(X, iY) - id\theta(X, Y), \quad X, Y \in \Gamma(H). \]

This form is definite if and only if \( \theta \wedge (d\theta)^2 \) is a positive multiple of \( \theta \wedge \alpha^2 \). Thus, condition (3) in Definition 1.1 means precisely that given conditions (1) and (2), any embedding \( \iota: M \hookrightarrow \mathbb{Z} \) such that \( \iota^*\Psi = \psi \) is a strongly pseudoconvex embedding. This motivates our chosen terminology. Strong pseudoconvexity is a helpful condition for embedding problems in CR-geometry. Boutet de Monvel [3] has shown that compact, strongly-pseudoconvex CR-manifolds of real dimension at least 5 can be embedded in some \( \mathbb{C}^n \) with possibly high codimension, and by the work of Kuranishi, Akahori and Webster (see [16] and references therein) local, strongly pseudoconvex CR-structures of real dimension at least 7 can be embedded as hypersurfaces.

We now proceed to describe the full structure induced on \( M \) in the case the embedding is strongly pseudoconvex. Let \( \omega, \alpha, \beta \in \Omega^2_H \) be the 2-forms on \( H \) obtained after the normalisation described in Section 1.1. Because \( \hat{\psi} = \theta \wedge \beta \) is closed as well, analogous to (1.10), we get the relations
\[ \omega \wedge \beta = 0, \quad \theta \wedge d\beta = 0. \]

The second condition is equivalent to \( d_H \beta = 0 \). The triple \( (\omega, \alpha, \beta) \) is orthonormal with respect to the wedge product pairing and thus defines an \( SU(2) \)-structure on \( M \). It can be thought of as a contact version of a hyperkähler structure in real dimension 4. This motivates us to make the following definition.

**Definition 1.12.** Let \( \theta \) be a contact 1-form on the 5-manifold \( M \) with contact distribution \( H \). Suppose \( \omega := d\theta \) and a pair of 2-forms \( \alpha, \beta \in \Omega^2_H \) satisfy
\[ \omega \wedge \omega = \alpha \wedge \omega = \beta \wedge \omega = 0. \]

The \( SU(2) \)-structure defined by \( (\theta, \omega, \alpha, \beta) \) is called contact hyperkähler if it satisfies
\[ d_H \alpha = 0, \quad d_H \beta = 0, \]

where \( d_H \) is the operator defined in (1.7).

\( (\omega, \alpha, \beta) \) spans a positive definite subspace \( \Lambda^+_H \). Together with the volume form \( \text{Vol}_H = \omega^2 \), this induces a metric \( g_H \) on \( H \), which is given by \( g_H(X, Y) = \omega(X, iY) \). We have a splitting
\[ \Lambda^2 H^* = \Lambda^+_H \oplus \Lambda^-_H \]
which is orthogonal with respect to \( g_H \) and write \( \Omega^+_H \) and \( \Omega^-_H \) for sections of \( \Lambda^+_H \) and \( \Lambda^-_H \), respectively. The forms \( \alpha + i\beta, \beta + i\omega, \omega + i\alpha \) induce the almost complex structures \( I, J, K \) on \( H \), which satisfy the quaternionic relations and are compatible with \( g_H \). Furthermore, we have
\[ \omega(X, Y) = g_H(IX, Y), \quad \alpha(X, Y) = g_H(JX, Y), \quad \beta(X, Y) = g_H(KX, Y). \]
We can think of the realisation problem for a strongly pseudoconvex 3-form $\psi$ as consisting of two parts.

**Problem 1.** Find $\beta \in \Omega^2_H$ such that $(\theta, \omega, \alpha, \beta)$ forms a contact hyperkähler SU(2)-structure.

**Problem 2.** Find an embedding for the strongly pseudoconvex CR-manifold $(M, H, \alpha + i\beta)$.

### 1.3 Sasaki–Einstein structures and invariants of closed 3-forms in dimension 5

Let $H$ be an oriented contact structure on a 5-manifold $M$ with contact 1-form $\theta$ and Reeb field $v$. Furthermore, let $\omega := d\theta, \alpha, \beta$ be an orthonormal triple on $\Lambda^2_H$. The SU(2)-structure $(\theta, \omega, \alpha, \beta)$ is called *Sasaki–Einstein* if $d\alpha = \theta \wedge \beta$ and $d\beta = -\theta \wedge \alpha$, or equivalently,

\[
\begin{align*}
    d_H \alpha &= 0, \\
    d_H \beta &= 0, \\
    \mathcal{L}_v \alpha &= \beta, \\
    \mathcal{L}_v \beta &= -\alpha.
\end{align*}
\]

This implies that on $(0, \infty) \times M$ the conical differential forms

\[
\begin{align*}
    \Lambda &= (3r^2 dr + ir^3 \theta) \wedge (\alpha - i\beta), \\
    \Omega &= 2r dr \wedge \theta + r^2 \omega,
\end{align*}
\]

satisfy

\[
\begin{align*}
    \Lambda \wedge \Omega &= 0, \\
    \Lambda \wedge \bar{\Lambda} &= 2i \Omega^3, \\
    d\Lambda &= 0, \\
    d\Omega &= 0.
\end{align*}
\]

This means that $(\Lambda, \Omega)$ defines a torsion-free SU(3)-structure on the cone $(0, \infty) \times M$. In particular, the induced Riemannian cone metric is Ricci-flat and the induced metric on $M$ is Einstein.

Now we return to the structure of a strongly pseudoconvex closed 3-form $\psi$ on the 5-manifold $M$, which induces the SL(2, $\mathbb{C}$)-structure $(\vartheta, \omega, \alpha)$ on $M$ as in Section 1.1. Write $\rho := \mathcal{L}_v \alpha$. Next we show that the condition that $(\vartheta, \omega, \alpha, \rho)$ is a Sasaki–Einstein SU(2)-structure can be expressed in terms of invariants of the SL(2, $\mathbb{C}$)-structure.

Applying the Lie derivative $\mathcal{L}_v$ to the orthonormality equations (1.11) gives

\[
\rho.\alpha = 0, \quad \rho.\omega = 0.
\]

So, $\rho$ is a section of the rank 4 bundle $\Lambda^{1,1}_K \subset \Lambda^2 H^*$, the real form of the bundle of 2-forms on $H$ which have type (1,1) with respect to the almost complex structure $K$ from (1.15). The tensor $\rho$ up to an action of SL(2, $\mathbb{C}$) is an invariant of the 2-jet of the structure $\psi$ at a given point. It has the scalar invariant $Q = \rho.\rho$.

If $Q = 1$, then $(\omega, \alpha, \rho)$ is an orthonormal triple, and thus, $(\vartheta, \omega, \alpha, \rho)$ is an SU(2)-structure such that $(\omega, \alpha, \rho)$ spans the bundle $\Lambda^{+,+}_H \subset \Lambda^2 H^*$ of self-dual 2-forms.
We can also consider a third-order invariant $L_v \rho$. When $Q = 1$, the orthogonality conditions imply that

$$L_v \rho. \omega = 0, \quad L_v \rho. \rho = 0, \quad L_v \rho. \alpha = -1.$$  

Thus $L_v \rho = -\alpha \mod \Omega_H$. Therefore, the conditions $Q = 1$ and $L_v \rho. L_v \rho = 1$ imply that $\alpha$ and $\beta = \rho$ solve Equations (1.16) and thus characterise Sasaki–Einstein structures in the setting of a strongly pseudoconvex 3-form on a 5-manifold.

### 1.4 Example: The Minkowski problem

Let $Z = \mathbb{C}^3/i\mathbb{Z}^3$ with holomorphic 3-form induced by $idz_1dz_2dz_3$ on $\mathbb{C}^3$, where by concatenations such as $dz_1dz_2dz_3$, we mean the wedge product. Thus, in standard co-ordinates $z_a = x_a + iy_a$, the real 3-form is

$$\Psi = dy_1dy_2dy_3 - \sum dy_adx_bdx_c,$$

where $(abc)$ run over cyclic permutations. In the quotient, we divide by integral translations in the $y$ coordinates. Let $M = \Sigma \times \mathbb{R}^3/\mathbb{Z}^3$ where $\Sigma$ is a compact, connected, oriented surface and consider a 3-form of the shape

$$\psi = dt_1dt_2dt_3 - \sum \lambda_a dt_a,$$

where $\lambda_1, \lambda_2, \lambda_3$ are 2-forms on $\Sigma$. We consider maps $F : M \to Z$ of the form $F(p, t) = f(p) + it$, where $f : \Sigma \to \mathbb{R}^3$. Our problem is to find $f$ such that

$$f^*(dx_bdx_c) = \lambda_a,$$  

for $(abc)$ cyclic. Certainly, a necessary condition is that

$$\int_\Sigma \lambda_a = 0.$$  

The problem is invariant under special-affine transformations of $\mathbb{R}^3$ but it is convenient, for exposition, to use the standard Euclidean metric and unit sphere $S^2 \subset \mathbb{R}^3$. Clearly, the condition that $\psi$ has maximal rank is equivalent to the condition that the 2-forms $\lambda_a$ do not simultaneously vanish at any point. Thus, we can define a map $L = (L_1, L_2, L_3) : \Sigma \to S^2 \subset \mathbb{R}^3$ and a 2-form $\Omega$ on $\Sigma$ such that $\Omega$ is positive with respect to the orientation of $\Sigma$ and the $\mathbb{R}^3$-valued 2-form $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ can be written $\lambda = L\Omega$.

We claim that the condition that $\psi$ is a strongly pseudoconvex form on $M$ is equivalent to the condition that $L$ is an oriented local diffeomorphism. To see this, at a given point $p$ in $\Sigma$, we can assume (by rotating axes) that $\lambda_2, \lambda_3$ vanish and $\lambda_1$ is non-zero. Take oriented local co-ordinates $(u, v)$ on $\Sigma$ near $p$ so that $\Omega = dudv$. Thus, $\lambda_a = L_adudv$. Then, near $p$,

$$\psi = \left( \sum L_adt_a \right) \left( L^{-1}_1dt_2dt_3 - dudv \right).$$
so \( \theta_0 = \sum L_a dt_a \) defines the subbundle \( H \) and \( \psi = \theta_0 \wedge \alpha_0 \) with \( \alpha_0 = L_1^{-1} dt_2 dt_3 - dudv \). At the point \( p \), the restriction of \( d\theta_0 \) to \( H \) is

\[
\omega_0|_H = \frac{\partial L_2}{\partial u} du dt_2 + \frac{\partial L_3}{\partial u} du dt_3 + \frac{\partial L_2}{\partial v} dv dt_2 + \frac{\partial L_3}{\partial v} dv dt_3
\]

so \( \omega_0^2|_H = -J du dv dt_2 dt_3 \) where

\[
J = \frac{\partial L_2}{\partial u} \frac{\partial L_3}{\partial v} - \frac{\partial L_2}{\partial v} \frac{\partial L_3}{\partial u},
\]

while \( \alpha_0^2|_H = -2(du dv dt_2 dt_3) \), since \( L_1 = 1 \). The claim now follows because \( J \) is the determinant of the derivative of \( L \) at \( p \) with respect to the area forms \( du dv \) on \( \Sigma \) and the standard area form \( dA_{S^2} \) on \( S^2 \).

For a map \( f: \Sigma \rightarrow \mathbb{R}^3 \), the condition (1) is equivalent to the three statements.

1. \( f \) is an immersion, so the image is an immersed surface \( X \subset \mathbb{R}^3 \).
2. The oriented normal to \( X \) at \( f(p) \) is \( L(p) \).
3. The pull-back by \( f \) of the (oriented) area form \( dA_X \) on \( X \) at \( p \) is \( \Omega(p) \).

Since \( S^2 \) is simply connected, the local diffeomorphism \( L \) from \( \Sigma \) to \( S^2 \) is a global diffeomorphism (so to have a pseudoconvex form \( \psi \) of this shape, we must suppose that \( \Sigma \) is diffeomorphic to \( S^2 \)). Define a positive function \( K \) on \( S^2 \) by

\[
\Omega = L^*(K^{-1} dA_{S^2}).
\]

Thus, the function \( K \) on \( S^2 \) is determined by the original data \( \lambda \). Let \( g = f \circ L^{-1}: S^2 \rightarrow X \subset \mathbb{R}^3 \). The above statements about \( f \) are equivalent to the statements that for all \( \nu \in S^2 \), the normal to \( X \) at \( g(\nu) \) is \( \nu \) and the Gauss curvature of \( X \) at \( g(\nu) \) is \( K(\nu) \). This is the usual formulation of the Minkowski problem for the map \( g \), with prescribed Gauss curvature \( K \) as a function of the normal direction. The solution of the Minkowski problem tells us that for pseudoconvex data \( \lambda_a \) satisfying the obvious conditions (1.19), there is a solution \( f \), unique up to translations.

2 | LINEAR ANALYSIS

In this section, we describe the linear analysis on a closed 5-manifold which carries an oriented contact structure \( H \subset TM \) with contact 1-form \( \theta \) and Reeb vector field \( v \), and a Euclidean metric \( g_H \) on \( H \), which we extend to a Riemannian metric \( g = \theta^2 + g_H \) on \( M \). Set \( \omega := d\theta \). In particular, we can apply this theory to contact hyperkähler structures.

We start by describing the main differential operators. As in (1.7) denote by \( d_H^0 : \Omega_H^0 \rightarrow \Omega_H^{0+1} \) the projection of the exterior derivative to \( H \). The Riemannian metric \( g \) has an associated \( L^2 \)-inner product

\[
\langle \gamma, \tau \rangle = \int_M g(\gamma, \tau) dVol_g, \quad \gamma, \tau \in \Omega^*(M).
\]  

Henceforth, we will write \( \langle \gamma, \tau \rangle \) for \( g(\gamma, \tau) \) when we work with a fixed metric. Denote by \( d_H^0 \) the \( L^2 \)-adjoint of \( d_H^0 \) with respect to the metric \( g \). \( d_H^0 \) explicitly is given by a formula analogous to the Riemannian setting.
Lemma 2.2. The adjoint of $d_H$ with respect to $g$ is given by $d_H^* = -\ast d_H \ast$, where $\ast$ denotes the Hodge star operator on $H$.

Proof. For $\eta \in \Omega^{k-1}_H$ and $\zeta \in \Omega^k_H$, we have

$$(\eta, d_H^* \zeta) = (d_H^* \eta, \zeta) = \int_M (d_H \eta, \zeta) \text{Vol} = \int_M d_H \eta \wedge \ast \zeta \wedge \theta = \int_M d \eta \wedge \ast \zeta \wedge \theta$$

$$= (-1)^k \int_M \eta \wedge d \ast \zeta \wedge \theta = (-1)^k \int_M \eta \wedge d_H \ast \zeta \wedge \theta$$

$$= - \int_M \eta \wedge \ast (d_H \ast) \zeta \wedge \theta$$

$$= (\eta, - \ast d_H \ast \zeta). \quad \Box$$

This allows us to define a Laplacian

$$\Delta_H := d_H d_H^* + d_H^* d_H : \Omega^p_H \to \Omega^p_H.$$

As in (1.14), we have a splitting $\Lambda^2_H = \Lambda^+_H \oplus \Lambda^-_H$ with respect to $g_H$. The main operator in this article is the derivative

$$d^-_H := \frac{1}{2} (d_H - \ast d_H) : \Omega^1_H \to \Omega^-_H,$$

the projection of $d_H : \Omega^1_H \to \Omega^2_H$ to anti-self-dual forms. By Lemma 2.2, its $L^2$-adjoint is given by

$$(d^-_H)^* = \frac{1}{2} (d_H - \ast d_H)^* = d_H^*.$$ (2.3)

In Section 3, we will see that to solve the linearisation of the embedding problem, we need to solve an equation of the form

$$d^-_H \eta = \sigma$$ (2.4)

for a given right-hand side $\sigma \in \Omega^-_H$. We will study this equation by considering the Laplacian

$$\Box_H := d^-_H d_H^* : \Omega^-_H \to \Omega^-_H,$$ (2.5)

which equals $\frac{1}{2} \Delta_H$ restricted to $\Omega^-_H$.

### 2.1 Adapted connections

To work with the operators introduced above, it will be useful to choose a metric connection which preserves $H$. The Levi–Civita connection $\nabla^{LC}$ does not preserve $H$, so choosing such a connection comes at the cost of introducing torsion. We wish to use a connection which ‘looks’ torsion-free on $H$, that is, the torsion tensor does not have a component in $\Lambda^2_H \otimes H$. 
Lemma 2.6. There exists a connection $\nabla$ on $TM$ with the following properties:

(a) $\nabla$ is metric, that is, $\nabla g = 0$.
(b) $\nabla$ preserves $H$, that is, $\nabla_X Y \in \Gamma(H)$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$.
(c) the torsion tensor $T$ of $\nabla$ has no component in $\Lambda^2_H \otimes H$.
(d) The Reeb vector field $v$ is parallel, that is, $\nabla v = 0$.

Moreover, $\nabla$ is unique up to a skew-symmetric endomorphism of $H$ and satisfies

(i) If $X, Y \in \Gamma(H)$ and $\gamma \in \Omega^p_H$, then

$$\nabla_X Y = \pi_H(\nabla^LC_X Y), \quad \nabla_X \gamma = \nabla^LC_X \gamma |_H.$$  

(ii) If $e_1, \ldots, e_4$ is a local orthonormal frame for $H$ with dual orthonormal co-frame $e_1, \ldots, e_4$, then

$$d_H = \sum_{i=1}^4 e_i \wedge \nabla_i, \quad d_H^* = -\sum_{i=1}^4 e_i \lrcorner \nabla_i,$$

where we write $\nabla_i$ for $\nabla_{e_i}$.

(iii) Define $B$ by

$$\nabla^LC_X Y = \nabla_X Y + B(X, Y)v, \quad X, Y \in \Gamma(H).$$

Then,

$$B(X, Y) = -\frac{1}{2}(\mathcal{L}_v g)(X, Y) - \frac{1}{2} \omega(X, Y).$$

(iv) For $X, Y \in \Gamma(H)$, we have

$$T(X, Y) = \omega(X, Y)v.$$  

(v) Write $T_v := T(v, \cdot)$ for the contraction of the torsion tensor with $v$. Then, $T_v \in \text{End}(H)$. Furthermore, if we decompose $T_v = T^s_v + T^a_v$, where $T^s_v$ is symmetric and $T^a_v$ is skew-symmetric with respect to $g$, then

$$g(T^s_v X, Y) = \frac{1}{2} \mathcal{L}_v g(X, Y)$$

for all $X, Y \in \Gamma(H)$.

Proof. We first assume that $\nabla$ exists and derive the properties (i)–(v) from the properties (a)–(d).

(i) If $X, Y, Z \in \Gamma(H)$, by (c), we have

$$g(\nabla_X Y - \nabla_Y X, Z) = g([X, Y], Z) + g(T(X, Y), Z)$$

$$= g([X, Y], Z).$$
Thus, we can use $\nabla g = 0$ as in the standard Riemannian setting to get the Koszul formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) = 2g(\nabla^LC_X Y, Z).$$

Because $\nabla$ preserves $H$, this means $\nabla_X Y = \pi_H(\nabla^LC_X Y)$. For $\gamma \in \Omega^p_H$ and $Y_1, \ldots, Y_p \in \Gamma(H)$, we have

$$(\nabla_X \gamma)(Y_1, \ldots, Y_p) = X(\gamma(Y_1, \ldots, Y_p)) - \sum_{j=1}^p \gamma(Y_1, \ldots, \nabla_X Y_j, \ldots, Y_p) = X(\gamma(Y_1, \ldots, Y_p)) - \sum_{j=1}^p \gamma(Y_1, \ldots, \nabla^LC_X Y_j, \ldots, Y_p) = (\nabla^LC_X \gamma)(Y_1, \ldots, Y_p).$$

(ii) The usual formula for the exterior derivative in terms of the torsion-free connection $\nabla^LC$ is

$$d = \sum_{i=1}^4 e^i \wedge \nabla^LC_i + \vartheta \wedge \nabla^LC_v.$$

By (i), we get

$$d_H = \sum_{i=1}^4 e^i \wedge \nabla^LC_i|_H = \sum_{i=1}^4 e^i \wedge \nabla_i.$$

Because $\nabla$ is metric, the above formula for $d_H$ implies the formula for $d^*_H$ in the usual way.

(iii) From (i), we have

$$\nabla^LC_X Y = \nabla_X Y + \vartheta(\nabla^LC_X Y)v.$$

The standard Koszul formula for the Levi–Civita connection gives

$$2B(X, Y) = 2\vartheta(\nabla^LC_X Y) = -v g(X, Y) + g([X, Y], v) - g([Y, v], X) - g([X, v], Y)$$

$$= -(\mathcal{L}_v g)(X, Y) + \vartheta([X, Y]) = -(\mathcal{L}_v g)(X, Y) - \omega(X, Y).$$

(iv) By (iii), the torsion is

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$$= \nabla^LC_X Y - \nabla^LC_Y X - [X, Y] + \frac{1}{2}((\mathcal{L}_v g)(X, Y) + \omega(X, Y))v - \frac{1}{2}((\mathcal{L}_v g)(Y, X) + \omega(Y, X))v$$

$$= \omega(X, Y)v.$$
Because $\nabla v = 0$, for $X \in \Gamma(H)$, we have
$$\nabla_X v = \nabla_X v + [v, X] + T(v, X) = \mathcal{L}_v X + T_v X,$$  \hfill (2.7)

Because $\nabla$ and $\mathcal{L}_v$ preserve $H$, we get $T_v \in \text{End}(H)$. $\nabla g = 0$ and formula (2.7) imply for $X, Y \in \Gamma(H)$
$$0 = \nabla_v g(X, Y) = \mathcal{L}_v g(X, Y) - g(T_v X, Y) - g(X, T_v Y) = \mathcal{L}_v g(X, Y) - 2g(T_v^a X, Y).$$

We now come to the existence of $\nabla$. By (a)–(d) and (i)–(v), we need to define $\nabla$ as
$$\nabla X Y = \pi_H(\nabla_{LC_X Y}), \quad \nabla v = 0, \quad g(\nabla v X, Y) = g(\mathcal{L}_v X, Y) + \frac{1}{2} \mathcal{L}_v g(X, Y) + g(T_v^a X, Y),$$

where $X, Y \in \Gamma(H)$. It is clear that this defines a connection which satisfies properties (a)–(d) and that the only freedom in the construction is the choice of $T_v^a$. $\square$

A connection as in Lemma 2.6 allows us to compute a Weitzenböck formula for $\Delta_H$. The choice of $T_v^a$ only influences the curvature term.

**Lemma 2.8.** Let $e_1, \ldots, e_4$ be a local orthonormal frame for $H$ with dual co-frame $e^1, \ldots, e^4$. For $\gamma \in \Omega^{\ast}_H$, denote by $\nabla_H \gamma := \sum_{i=1}^4 e^i \otimes \nabla_{e^i} \gamma$ the covariant derivative in the 'H-direction'. Denote by $e^k \wedge e^l$ the wedge product with $e^k$ and by $\iota^k$ the contraction with $e_k$. Then, for $\Delta_H$, we have the Weitzenböck formula
$$\Delta_H = \nabla^*_H \nabla_H \gamma + \sum_{k, l} \omega_{kl} \iota^k \iota^l \nabla v + \sum_{k, l, m, n} \iota^k \iota^l \iota^m \iota^n R^V_{klmn}, \hfill (2.9)$$

where $R^V$ denotes the curvature tensor of the connection $\nabla$.

**Proof.** For a given $p \in M$, we can choose the frame $\{e_i\}$ such that $\nabla_{e_i}|_p = 0, i = 1, \ldots, 4$. This implies $\nabla_k (\iota^l \gamma)|_p = \iota^l \nabla_k \gamma|_p$ and $\nabla_k \iota^l \gamma|_p = \nabla^2_k \gamma|_p$. Lemma 2.6 and the Ricci formula give
$$d_H^* d_H \gamma|_p = - \sum_{k, l=1}^4 \varepsilon^k \varepsilon^l \nabla_k (\iota^l \gamma)|_p = - \sum_{k, l=1}^4 \varepsilon^k \varepsilon^l \nabla_k \iota^l \gamma|_p,$$
$$d_H^* d_H \gamma|_p = - \sum_{k, l=1}^4 \iota^k \iota^l \nabla_k (\iota^l \gamma)|_p = - \sum_{k, l=1}^4 \iota^k \iota^l \nabla_k \iota^l \gamma|_p,$$
$$\Delta \gamma|_p = - \sum_{k, l=1}^4 (\varepsilon^k \iota^l + \iota^k \varepsilon^l) \nabla^2_{k, l} \gamma|_p = - \sum_{k=1}^4 \nabla_k \nabla_k \gamma|_p - \sum_{k, l=1}^4 (\varepsilon^k \iota^l + \iota^k \varepsilon^l) (\nabla^2_k \gamma|_p - \nabla^2_l \gamma|_p)$$
$$= \nabla^*_H \nabla_H \gamma|_p - \sum_{k, l} (\varepsilon^k \iota^l + \iota^k \varepsilon^l) (R^V(e_k, e_l), \gamma|_p - \nabla_{T(e_k, e_l)} \gamma|_p)$$
$$= \nabla^*_H \nabla_H \gamma|_p - \sum_{k, l} (\varepsilon^k \iota^l + \iota^k \varepsilon^l) (R^V(e_k, e_l), \gamma|_p - \omega(e_k, e_l) \nabla_v \gamma|_p)$$
$$= \nabla^*_H \nabla_H \gamma|_p + \sum_{k, l=1}^4 \omega(e_k, e_l) \varepsilon^k \iota^l \nabla_v \gamma|_p - \sum_{k, l=1}^4 \varepsilon^k \iota^l R^V(e_k, e_l), \gamma|_p.$$
2.2 | Sub-ellipticity

In this section, we describe the analytic properties of the operators $d_H^-$ and $\Box$. We adopt the convention from [6] that

\[ f(x) \lesssim g(x) \quad \text{means} \quad \exists C > 0 \text{ such that } f(x) \leq C g(x) \forall x. \]

By choosing an atlas for $M$ and a subordinate partition of unity, we can define as usual Sobolev spaces $L^2_s$ for sections of $TM$ and its associated bundles. We denote the norm of $L^2_s$ by $\| \cdot \|_s$. Here, $s$ is the number of derivatives if it is an integer, but we also need to consider non-integral $s$. For details on fractional Sobolev spaces, we refer to [6, Appendix 1 and 2]. We also use $C^k$-norms which we denote by $[ [ \cdot ] ]_k$. Here, $k$ is an integer. The splitting (1.4) allows us to consider $\Lambda_H^∗$ and $\Lambda_H^±$ as subbundles of $\Lambda^∗ T^∗ M$, and thus, we also have norms for sections of these bundles.

For $\sigma \in \Omega_H^−$, introduce the bilinear form

\[ Q(\sigma, \sigma) = ( (\Box_H + 1)\sigma, \sigma) = (d_H^∗ d_H^\ast \sigma, \sigma) + (\sigma, \sigma) = \| d_H^\ast \sigma \|^2 + \| \sigma \|^2, \quad (2.10) \]

where $(\cdot, \cdot)$ denotes the $L^2$-inner product (2.1). Because $d_H^\ast = \ast d_H$ on $\Omega_H^−$ and $\ast$ acts isometrically, we also have the identity

\[ Q(\sigma, \sigma) = \| d_H \sigma \|^2 + \| \sigma \|^2. \quad (2.11) \]

The operator (2.9) is not elliptic as it does not see second derivatives in the direction of $v$. In particular, the bilinear form $Q$ is not coercive, that is, there is no estimate of the form

\[ \| \sigma \|^2 \lesssim Q(\sigma, \sigma). \]

However, because $H$ is a contact structure, the Reeb vector field $v$ locally can be written as a commutator of sections of $H$. In other words, a local frame of $H$ satisfies the Hörmander condition. This leads to the fundamental ‘1/2-estimate’ for the ‘rough Laplacian’ $\nabla_H^\ast \nabla_H$.

**Proposition 2.12.** For $\phi \in \Omega_H^∗$, we have

\[ \| \phi \|^2 \lesssim (\nabla_H^\ast \nabla_H \phi, \phi) + \| \phi \|^2. \quad (2.13) \]

**Proof.** Because $H$ is a contact structure, the local frame $e_1, \ldots, e_4$ for $H$ satisfies the Hörmander condition. Working locally in coordinate charts, we can therefore apply the 1/2-estimate for functions on $\mathbb{R}^5$ [6, Theorem 5.4.7] to each component to obtain the result. \qed

Analysing the Weitzenböck formula (2.9) now shows that $Q$ also satisfies a sub-elliptic estimate.

**Proposition 2.14.** For all $\sigma \in \Omega_H^−$, we have the sub-elliptic estimate

\[ \| \sigma \|^2 \lesssim Q(\sigma, \sigma). \quad (2.15) \]
Proof. Because $\nabla$ is metric, $\nabla_v$ preserves $\Omega_H^-$. The action of $\omega$ on $\Lambda_H^-$ vanishes. Thus, the first-order term in (2.9) drops out and $\Box_H$ differs from $\frac{1}{2} \nabla^* \nabla H$ only by the curvature term, which is an algebraic operator. Therefore, (2.15) follows from (2.13). \hfill \Box

Equation (2.15) and the Cauchy–Schwarz inequality imply

\[ \|\sigma\|_{\frac{1}{2}}^2 \lesssim \|\Box_H \sigma\|_{\frac{1}{2}}^2 + \|\sigma\|_{\frac{1}{2}}^2. \tag{2.16} \]

From this, Kohn–Nirenberg [12, Lemma 3.1] derive higher order estimates: we obtain for every $k \in \mathbb{N}$ and $\sigma \in \Omega_H^-$

\[ \|\sigma\|_{k+\frac{1}{2}}^2 \lesssim \|\Box_H \sigma\|_{k}^2 + \|\sigma\|_{k}^2. \tag{2.17} \]

An adapted proof can be found in [8, section 3.6]. We give a more detailed outline of Hamilton’s proof in the next section when we discuss uniform estimates.

After establishing the higher order estimates, the method of elliptic regularisation and standard arguments from functional analysis show that $\Box_H$ behaves like the standard Laplace-operator on a Riemannian manifold [12, Theorem 4 (ii), (6.3)].

Proposition 2.18.

- $\Box_H$ is hypo-elliptic, that is, if $\zeta$ is a distributional solution to the equation $\Box_H \zeta = \sigma$, where $\sigma$ is a smooth section of $\Lambda_H^-$, then $\zeta$ is smooth.
- $\ker \Box_H$ is finite-dimensional and there is an $L^2$-orthogonal ‘Hodge’ decomposition

\[ \Omega_H^- = \ker \Box_H \oplus \im \Box_H. \tag{2.19} \]

- The vanishing of $\ker \Box_H$ is an open condition.

Proof. We briefly outline the proof of the third item. The estimate (2.16) depends only on finitely many derivatives of the contact structure and metric, so it holds with a uniform constant in a neighbourhood of $\theta$ and $g$ which is open in the Fréchet topology. Suppose that we have a sequence $(\theta_i, g_i)$ converging in that topology to $(\theta, g)$ and for each $i$, the $\Box$ operator defined by $(\theta_i, g_i)$ has non-trivial kernel. We choose elements $\sigma_i$ of these kernels with $L^2$ norm 1. Then, by the compactness of the inclusion of $L^2_{1/2}$ in $L^2$, we can suppose that these converge in $L^2$ to some non-zero limit $\sigma$ and the regularity statement above implies that $\sigma$ is smooth and lies in the kernel of $\Box_H$ for $(\theta, g)$. \hfill \Box

Define

\[ \mathcal{H} := \ker \Box_H = \{ \sigma \in \Omega_H^- : d_H \sigma = 0 \}. \tag{2.20} \]

The finite-dimensional vector space $\mathcal{H}$ is the obstruction space to solve Equation (2.4). Decomposition (2.19) gives
Proposition 2.21. Let $\sigma \in \Omega^-$. Then, the equation

$$d^-H \eta = \sigma$$

has a solution $\eta \in \Omega^1$ if and only if $\sigma \perp \mathcal{H}$. In particular, if $\mathcal{H} = 0$, then $d^-H$ is surjective onto $\Omega^-\mathcal{H}$ with right inverse $R = d^+H \square^{-1}$.

In the case $\mathcal{H} = 0$, (2.16) gives the estimate

$$\|R\sigma\| \lesssim 2 \approx \|\sigma\|.$$  

This is not optimal. By using pseudo-differential operators, one can improve this estimate to

$$\|R\sigma\| \lesssim 2 \approx \|\sigma\|. \quad (2.22)$$

2.3 Uniform estimates for the perturbed $d^-H$-equation

Any rank 3 subbundle of $\Lambda^2\mathcal{H}$ which is positive definite with respect to the wedge product pairing and close to $\Lambda^+\mathcal{H}$ can be written as a graph

$$\Lambda^+\mathcal{H,\mu} = \text{graph}(\mu) = \{\zeta + \mu(\zeta) | \zeta \in \Lambda^+\mathcal{H}\}$$

of a map

$$\mu : \Lambda^+\mathcal{H,\mu} \to \Lambda^-\mathcal{H}.$$  

Denote by $\Lambda^0\mathcal{H,\mu}$ the orthogonal complement of $\Lambda^+\mathcal{H,\mu}$ in $\Lambda^2\mathcal{H}$ with respect to the wedge product pairing. This rank 3 bundle is the graph of $\mu^\ast$:

$$\Lambda^0\mathcal{H,\mu} = \text{graph}(\mu^\ast) = \{\zeta + \mu^\ast(\zeta) | \zeta \in \Lambda^-\mathcal{H}\}.$$  

Indeed, if $\sigma \in \Lambda^+\mathcal{H}$ and $\zeta \in \Lambda^-\mathcal{H}$, then

$$(\sigma + \mu(\sigma)) \wedge (\zeta + \mu^\ast(\zeta)) = \mu(\sigma) \wedge \zeta + \sigma \wedge \mu^\ast(\zeta) = \{ -\langle \mu(\sigma), \zeta \rangle + \langle \sigma, \mu^\ast(\zeta) \rangle \} \text{Vol} = 0.$$  

Because the wedge product of any elements of $\Lambda^+\mathcal{H,\mu}$ and $\text{graph}(\mu^\ast)$ vanishes, we have $\text{graph}(\mu^\ast) \subseteq \Lambda^0\mathcal{H,\mu}$. Equality follows because the dimensions are equal. The projection $\pi_+ : \Lambda^+\mathcal{H,\mu} \to \Lambda^+\mathcal{H}$ is an isomorphism. Indeed, for $\zeta + \mu(\zeta)$, we have $\pi_+(\zeta + \mu(\zeta)) = \zeta$. Therefore, $\pi_+$ is injective and thus an isomorphism. Similarly, $\pi_- : \Lambda^-\mathcal{H,\mu} \to \Lambda^-\mathcal{H}$ is an isomorphism. To sum up, we have bundle isomorphisms

$$\Lambda^+\mathcal{H} \xrightarrow{\pi_+ \mu} \Lambda^+\mathcal{H,\mu}, \quad \Lambda^-\mathcal{H} \xrightarrow{\pi_- \mu^\ast} \Lambda^-\mathcal{H,\mu}. \quad (2.23)$$
By keeping the volume form fixed, $\Lambda^+_H,\mu$ defines a new metric $(\cdot,\cdot)_\mu$ on $H$ and its associated bundles, where induced $L^2$-inner product we denote by $(\cdot,\cdot)_\mu$. Via the isomorphism $1 + \mu^*$, we can pull back $(\cdot,\cdot)_\mu$ to a metric $\langle \cdot, \cdot \rangle_{\mu}$ on $\Lambda_H^+$ with induced $L^2$-inner product $\langle \cdot, \cdot \rangle_{\mu}$. An explicit calculation shows that on $\Lambda_H^-$ the inner products are related by $\langle \cdot, \cdot \rangle_{\mu} = \langle \cdot, (1 - \mu \mu^*) \cdot \rangle$.

The decomposition $\Lambda^2_H = \Lambda^+_H,\mu \oplus \Lambda^-_H,\mu$ gives rise to a perturbation $d^-_H,\mu$ of $d^-_H$ and a corresponding perturbation of Equation (2.4). We want to view differential operators arising from the perturbed decomposition of $\Lambda^2_H$ as operators acting between sections of fixed bundles. The isomorphisms (2.23) allow us to identify $d^-_H,\mu$ with the operator

$$D_\mu := \pi_\mu \circ d^-_H,\mu : \Omega^1_H \to \Omega^-_H.$$  

(2.24)

With respect to $(\cdot,\cdot)_\mu$, we have an adjoint $(d^-_H,\mu)^\mu$, which by Lemma 2.2 coincides with $d^\mu$. Set $D^\mu := (d^-_H,\mu)^\mu \circ (1 + \mu^*) : \Omega^-_H \to \Omega^1_H$. This notation is justified as we have $\langle (D_\mu \eta, \sigma) \rangle_\mu = \langle \eta, D^\mu \sigma \rangle_\mu$ for $\eta \in \Omega^1_H$ and $\sigma \in \Omega^-_H$. The relevant second-order operator is

$$E_\mu := D_\mu \circ D^\mu = \pi_\mu \circ d^-_H,\mu \circ (d^-_H,\mu)^\mu \circ (1 + \mu^*) = \pi_\mu \circ \Box_H,\mu \circ (1 + \mu^*) : \Omega^-_H \to \Omega^-_H.$$  

By Proposition 2.18, $E_\mu$ is sub-elliptic and ker $\Box_H = 0$ implies that ker $E_\mu$ vanishes if $\mu$ is sufficiently small. To apply the Nash–Moser implicit function theorem to the perturbative embedding problem, we need to carefully check how the higher order estimates (2.17) depend on the parameter $\mu$. For example an estimate of the form

$$\|\sigma\|_{k+1/2} \lesssim \|E_\mu \sigma\|_k + ([\mu])_2k + 1 \|\sigma\|_2$$

would not be enough [9, Counterexample I.5.5.4].

We start by deriving an uniform version of the ‘$1/2$’-estimate (2.15). Analogously to (2.10), define a quadratic form

$$Q_\mu(\sigma, \sigma) = \langle (E_\mu + 1) \sigma, \sigma \rangle_\mu = (D^\mu_\mu \sigma, D^{\mu^*}_\mu \sigma) + \langle (\cdot, \cdot) \rangle_\mu.$$  

The analogue of formula (2.11) is

$$Q_\mu(\sigma, \sigma) = (d_H(1 + \mu^*) \sigma, d_H(1 + \mu^*) \sigma) + (\sigma, \sigma)_\mu.$$  

In the following, we will assume a $C^0$-bound on $\mu$. This then implies that we have a uniform equivalence of $L^2$-products

$$(\cdot, \cdot) \lesssim (\cdot, \cdot)_\mu \lesssim (\cdot, \cdot)$$

Proposition 2.25. Under a $C^1$-bound on $\mu$ for all $\sigma \in \Omega^-_H$, we have

$$\|\sigma\|^2_{k+1/2} \lesssim Q_\mu(\sigma, \sigma).$$  

(2.26)
Proof. We have
\[
Q_\mu(\sigma, \sigma) = (d_H(1 + \mu^*) \sigma, d_H(1 + \mu^*) \sigma)_\mu + (\sigma, \sigma)_\mu
\geq (d_H(1 + \mu^*) \sigma, d_H(1 + \mu^*) \sigma) + (\sigma, \sigma)
= Q_0(\sigma, \sigma) + 2(d_H \sigma, d_H \mu^* \sigma) + \|d_H \mu^* \sigma\|^2
\geq Q_0(\sigma, \sigma) - 2\|d_H \sigma\|\|d_H \mu^* \sigma\|.
\]

By Lemma 2.6 (ii), we have
\[
d_H \mu^* \sigma = \sum_{i=1}^{4} e_i \wedge \nabla_i (\mu^* \sigma) = \sum_{i=1}^{4} e_i \wedge (\nabla_i \mu^*) \sigma - \sum_{i=1}^{4} e_i \wedge \mu^* \nabla_i \sigma.
\]

Thus, with Proposition 2.14 and formula (2.11),
\[
\|d_H \mu^* \sigma\| \leq \|\mu\|_1 \|\sigma\| + \|\mu\|_0 \|\nabla H \sigma\| \leq \|\mu\|_1 (\|\nabla H \sigma\| + \|\sigma\|)
\leq \|\mu\|_1 \sqrt{Q_0(\sigma, \sigma)} \leq \|\mu\|_1 (\|d_H \sigma\| + \|\sigma\|).
\]

If we denote the constant in the last inequality by \(C\), with (2.11), this gives
\[
Q_\mu(\sigma, \sigma) \geq Q_0(\sigma, \sigma) - 2C\|\mu\|_1 \|d_H \sigma\| (\|d_H \sigma\| + \|\sigma\|)
\geq Q_0(\sigma, \sigma) - 3C\|\mu\|_1 (\|d_H \sigma\|^2 + \|\sigma\|^2)
\geq Q_0(\sigma, \sigma) - 3C\|\mu\|_1 Q_0(\sigma, \sigma).
\]

Then, if \(\|\mu\|_1 < \frac{1}{6C}\), we get with (2.15)
\[
\|\sigma\|_{\frac{3}{2}}^2 \leq Q_0(\sigma, \sigma) \leq Q_\mu(\sigma, \sigma).
\]

Next, we explain how to obtain uniform higher order estimates from the uniform ‘1/2-estimate’. The proof is due to Hamilton [8].

**Proposition 2.27.** For all \(\mu\) in a sufficiently small neighbourhood of 0, there is an estimate
\[
\|\sigma\|_{k+\frac{1}{2}} \leq \|E_\mu \sigma\|_k + \|\mu\|_{k+2} + 1)\|\sigma\|_2.
\] (2.28)

for each \(k \in \mathbb{N}\).

**Proof.** We use induction on \(k\). For \(k = 0\), the estimate follows from (2.26) and the Cauchy–Schwarz inequality:
\[
\|\sigma\|_{\frac{3}{2}}^2 \leq (E_\mu \sigma, \sigma) + \|\sigma\|^2 \leq \|E_\mu \sigma\|^2 + \|\sigma\|^2.
\]
In the derivation of the higher order estimates, we want to avoid dealing with mixed partial derivatives. We use the following trick from [8, p.420]: for each \( k \in \mathbb{N} \), there exist \( N \) (which is allowed to depend on \( k \) and can be large) vector fields \( X_1, \ldots, X_N \) such that

\[
\left\| \sigma \right\|_{k+\frac{1}{2}}^2 \leq \sum_{l=0}^{k} \sum_{j=1}^{N} \left\| \nabla_{X_j}^l \sigma \right\|_{\frac{1}{2}}^2.
\]

On \( \mathbb{R}^n \), this follows from the statement from algebra that each homogeneous polynomial \( q(\bar{\partial}) \) of degree \( k \) depending on \( n \) formal variables \( \bar{\partial} = (\partial_1, \ldots, \partial_n) \) can be written as a linear combination of the \( k \)th powers of linear polynomials \( p_1(\bar{\partial}), \ldots, p_N(\bar{\partial}) \). For example, we have

\[
6\partial_1\partial_2^2 = \partial_3^2 - 6\partial_3^2 - 2(\partial_1 + \partial_2)^3 + (\partial_1 + 2\partial_2)^3.
\]

On the manifold \( M \), we can find the vector fields \( X_j, j = 1, \ldots, N \), by choosing a partition of unity to reduce to the local case. In the following, fix \( j \) and just write \( \nabla \) for \( \nabla_{X_j} \). Inserting \( \nabla^k \sigma \) into (2.26) gives

\[
\left\| \nabla^k \sigma \right\|_{\frac{1}{2}}^2 \lesssim Q_\mu(\nabla^k \sigma, \nabla^k \sigma) \lesssim |Q_\mu(\sigma, \nabla^{2k} \sigma)| + |Q_\mu(\nabla^k \sigma, \nabla^k \sigma) - (-1)^k Q_\mu(\sigma, \nabla^{2k} \sigma)|.
\] (2.29)

Because \( Q_\mu \) depends on \( \mu \) and its first derivative, by the ‘uniform Kohn–Nirenberg Lemma’ [8, p. 448], we can estimate the error term as

\[
|Q_\mu(\nabla^k \sigma, \nabla^k \sigma) - (-1)^k Q_\mu(\sigma, \nabla^{2k} \sigma)| \lesssim \left\| \sigma \right\|_{k+1}^2 + [\left\| \mu \right\|_{k+2}^2] \left\| \sigma \right\|_{k+1}^2 + (sc)\left\| \sigma \right\|_{k-1}^2 + [\left\| \mu \right\|_{k+2}^2] \left\| \sigma \right\|_{k-1}^2,
\] (2.30)

where we can make the ‘small’ constant (sc) arbitrarily small by choosing the ‘large’ constant (lc) sufficiently large. The adjoint \( \nabla^* \) of \( \nabla \) with respect to \( (\cdot, \cdot)_\mu \) has the form \( \nabla^* = -\nabla + a(\mu) \), where \( a(\mu) \) is a linear partial differential operator of degree 0 whose coefficients depend on \( \mu \) and its first derivative. Thus, with [8, Lemma on bottom of p. 448], we get

\[
|Q_\mu(\sigma, \nabla^{2k} \sigma)| = |((E_\mu + 1)\sigma, \nabla^{2k} \sigma)_\mu| = |((\nabla^k(E_\mu + 1)\sigma, \nabla^k \sigma))_\mu|
\lesssim \left\| \nabla^k(E_\mu + 1)\sigma \right\|_k^2 + \left\| \sigma \right\|_k^2
\lesssim \left\| (E_\mu + 1)\sigma \right\|_k^2 + [\left\| \mu \right\|_{k+2}^2] \left\| (E_\mu + 1)\sigma \right\|_k^2 + \left\| \sigma \right\|_k^2
\lesssim \left\| E_\mu \sigma \right\|_k^2 + [\left\| \mu \right\|_k^2] \left\| \sigma \right\|_k^2 + [\left\| \mu \right\|_{k+2}^2] \left\| E_\mu \sigma \right\|_k^2 + \left\| \sigma \right\|_k^2.
\]

Under a \( C^2 \)-bound on \( \mu \), we have \( \left\| E_\mu \sigma \right\| \lesssim \left\| \sigma \right\|_2 \). Using this, we then get

\[
|Q_\mu(\sigma, \nabla^{2k} \sigma)| \lesssim \left\| E_\mu \sigma \right\|_{k+\frac{1}{2}}^2 + [\left\| \mu \right\|_{k+2}^2] \left\| \sigma \right\|_{k+\frac{1}{2}}^2 + (sc)\left\| \sigma \right\|_{k-\frac{1}{2}}^2 + (lc)\left\| \sigma \right\|_{k-\frac{1}{2}}^2.
\] (2.31)

Combining (2.29), (2.30) and (2.31) gives

\[
\left\| \nabla^k \sigma \right\|_{\frac{1}{2}}^2 \lesssim \left\| E_\mu \sigma \right\|_{k+\frac{1}{2}}^2 + [\left\| \mu \right\|_{k+2}^2] \left\| \sigma \right\|_{k+\frac{1}{2}}^2 + (sc)\left\| \sigma \right\|_{k-\frac{1}{2}}^2 + (lc)\left\| \sigma \right\|_{k-\frac{1}{2}}^2.
\]
Summing over all vector fields $X_j$, rearranging and using the induction hypothesis gives the desired estimate. □

To derive uniform estimates for the right inverse $R_\mu$ of $D_\mu$, we will use the ‘second Moser estimate’ [8, p. 439]: If the operator $L(m)f$ depends on derivatives of $m$ up to order $r$, possibly in a non-linear way, and is linear and of order $s$ in $f$, then under a $C^r$-bound on $m$, we have a uniform estimate

$$
\|L(m)f\|_k \lesssim \|f\|_{s+k} + [[m]]_{k+r} \|f\|_s.
$$

(2.32)

**Corollary 2.33.** There exists $l \in \mathbb{N}$, such that for all $\mu$ in a sufficiently small neighbourhood of 0, the right inverse $R_\mu := D^\ast_\mu E^{-1}$ of $D_\mu$ satisfies an estimate

$$
\|R_\mu \sigma\|_k \lesssim \|\sigma\|_{k+1} + ([[\mu]]_{k+3} + 1) \|\sigma\|_l
$$

(2.34)

for every $k \in \mathbb{N}$.

**Proof.** With a contradiction argument [8, Lemma on p. 453], one can conclude from the estimates (2.28) that there exists $l \in \mathbb{N}$ such that for all $\mu$ in some neighbourhood of 0 and $\sigma \in \Omega^{-}_{H}$, we have

$$
\|\sigma\|_2 \lesssim \|E_\mu \sigma\|_l.
$$

(2.35)

By (2.28) and (2.35), we have

$$
\|E^{-1}_\mu \sigma\|_{k+\frac{1}{2}} \lesssim \|\sigma\|_k + ([[\mu]]_{k+2} + 1) \|\sigma\|_l.
$$

(2.36)

Applying the second Moser estimate (2.32) to the operator $D^\ast_\mu$, which depends on $\mu$ and its first derivative, gives for sufficiently small $\mu$

$$
\|R_\mu \sigma\|_k = \|D^\ast_\mu E^{-1}_\mu \sigma\|_k \lesssim \|E^{-1}_\mu \sigma\|_{k+1} + [[\mu]]_{k+1} \|E^{-1}_\mu \sigma\|_1,
$$

which together with (2.35) leads to the estimate

$$
\|R_\mu \sigma\|_k \lesssim \|E^{-1}_\mu \sigma\|_{k+\frac{3}{2}} + [[\mu]]_{k+1} \|\sigma\|_l.
$$

(2.37)

With (2.36), we get the desired estimate (2.34). □

### 2.4 The obstruction space and the tangential Cauchy–Riemann operator

Here, we briefly review the $\bar{\partial}_b$-operator associated with the CR-structure $I$, that is, the one induced on $M$ from the complex structure of the ambient manifold $Z$. For details, we refer to [2]. The action of $I$ on $H \otimes \mathbb{C}$ has eigenvalues $\pm i$ with eigenspace decomposition $H \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$, where $H^{0,1} = H^{1,0}$. This has a dual decomposition $H^* \otimes \mathbb{C} = (H^*)^{1,0} \oplus (H^*)^{0,1}$. Set $\Lambda^{p,0}_H = \Lambda^p (H^*)^{1,0} \otimes \Lambda^q (H^*)^{0,1}$ and $\Lambda^{p,q}_M = \mathcal{E} \Lambda^{p-1,q}_H \oplus \Lambda^{p,q}_H$. Write $\Omega^{p,q}_H$ for sections of $\Lambda^{p,q}_H$ and
\( \Omega^p,q_M \) for sections of \( \Lambda^p,q_M \). We have decompositions \( \Lambda^k_H = \bigoplus_{p+q=k} \Lambda^p,q_H \) and \( \Lambda^k_M = \bigoplus_{p+q=k} \Omega^p,q_M \). Write \( \pi^p,q_H \) for the projection onto \( \Lambda^p,q_H \) and \( \pi^p,q_M \) for the projection onto \( \Lambda^p,q_M \).

The \( \bar{\partial}_b \)-operator is given by
\[
\bar{\partial}_b = \pi^{p,q+1}_M \circ d : \Omega^p,q_M \to \Omega^{p,q+1}_M
\]
and satisfies \( \bar{\partial}^2_b = 0 \), that is, leads to a complex. For \( \gamma \in \Omega^p,q_H \), \( d_H \gamma \) takes values in \( \Omega^{p+1,q}_H \oplus \Omega^{p,q+1}_H \), that is, we have a splitting \( d_H = \partial_H + \bar{\partial}_H \).

Under the identification \( \Omega^p,q_M \cong \Omega^{p-1,q}_H \oplus \Omega^p,q_H \), in matrix notation, \( \bar{\partial}_b \) is given by
\[
\bar{\partial}_b = \begin{pmatrix} \Omega^p,q_H & \Omega^{p-1,q+1}_H \\ \Omega^{p-1,q}_H & \Omega^p,q_H \end{pmatrix}
\]
where \( S := \pi^{p-1,q+1}_H \circ \mathcal{L}_v \). The adjoint \( \bar{\partial}^*_b : \Omega^{p,q+1}_M \to \Omega^{p,q}_M \) of \( \bar{\partial}_b : \Omega^{p,q}_M \to \Omega^{p,q+1}_M \) is given by
\[
\bar{\partial}^*_b = \begin{pmatrix} \Omega^{p-1,q+1}_H & \Lambda \vphantom{\pi}^{p,q}_H \\ -\bar{\partial}^*_H \vphantom{\pi}^{p,q}_H & S^* \end{pmatrix}
\]
where \( \Lambda \) is the adjoint of \( \omega \wedge \cdot \). A similar computation as in the proof of Lemma 2.2 shows that \( \bar{\partial}^*_H = \pm \star \bar{\partial}_H \star \). The decomposition \( \Lambda^2_H = \Lambda^+_H \oplus \Lambda^-_H \) from (1.14) relates to the decomposition in \((p,q)\)-forms as
\[
\Lambda^{1,1}_H = \mathbb{C} \omega \oplus \Lambda^+_H \otimes \mathbb{C}, \quad \Lambda^{2,0}_H = \mathbb{C} \alpha + i \beta.
\]

**Lemma 2.38.** If \( \sigma \in \mathcal{H} \), then \( \bar{\partial}^*_b \sigma = 0 \).

**Proof.** By the decomposition (2.37), the map \( \Lambda \) vanishes on \( \Lambda^-_H \). Therefore, for \( \sigma \in \mathcal{H} \), we have
\[
\Lambda \sigma = 0, \quad \bar{\partial}^*_H \sigma = \pm \star \bar{\partial}_H \star \sigma = \pm \star \bar{\partial}_H \sigma = \pm \star \pi^{2,1}_H d_H \sigma = 0.
\]
This implies \( \bar{\partial}^*_b \sigma = 0 \).

If the ambient space \( Z \) is a Stein manifold, the cohomology groups of the \( \bar{\partial}_b \)-complex vanish and there are no non-trivial \( \bar{\partial}_b \)-harmonic forms on \( M \) (see Appendix A). This has two consequences. Firstly, every \( \sigma \in \mathcal{H} \) is \( \bar{\partial}^*_b \)-exact. Secondly, if \( \bar{\partial}_b \sigma = 0 \), then \( \sigma = 0 \). With this in mind, in the following, we compute \( \| \bar{\partial}_b \sigma \| \) for \( \sigma \in \mathcal{H} \).

**Proposition 2.39.** For \( \sigma \in \mathcal{H} \), we have
\[
|\bar{\partial}_b \sigma|^2 = \frac{1}{4} (\langle \sigma, \mathcal{L}_v \alpha \rangle^2 + \langle \sigma, \mathcal{L}_v \beta \rangle^2).
\]
In particular, if \((\theta, \omega, \alpha, \beta)\) is a Sasaki–Einstein structure, this implies with the equations (1.16) that \(\bar{\delta}_b \sigma = 0\).

**Proof.** Because \(d_H \sigma = 0\), we have \(\bar{\delta}_b \sigma = \theta \wedge \pi^{0,2} \mathcal{L}_u \sigma\). Write \(\pi^{0,2} \mathcal{L}_u \sigma = u(\alpha - i \beta)\) for some complex valued function \(u\) on \(M\). Then,

\[
\mathcal{L}_u \sigma \wedge (\alpha + i \beta) = 4u \text{Vol}_H.
\]

Now we have

\[
0 = \mathcal{L}_u (\sigma \wedge \alpha) = \mathcal{L}_u \sigma \wedge \alpha + \sigma \wedge \mathcal{L}_u \alpha.
\]

This implies \(\mathcal{L}_u \sigma \wedge \alpha = -\sigma \wedge \mathcal{L}_u \alpha = \langle \sigma, \mathcal{L}_u \alpha \rangle \text{Vol}_H\). Combining the above gives

\[
u = \frac{1}{4} (\langle \sigma, \mathcal{L}_u \alpha \rangle + i \langle \sigma, \mathcal{L}_u \beta \rangle)
\]

and

\[
|\bar{\delta}_b \sigma|^2 = |\pi^{0,2} \mathcal{L}_u \sigma|^2 = 4u \bar{u} = \frac{1}{4} (\langle \sigma, \mathcal{L}_u \alpha \rangle^2 + \langle \sigma, \mathcal{L}_u \beta \rangle^2).
\]

Lemma 2.38, Proposition 2.39 and Corollary A.3 prove the third item in Theorem 1.

**Corollary 2.40.** If \(F : M \to Z\) is a strongly pseudoconvex embedding of \(M\) into a Stein manifold \(Z\) such that the induced \(SU(2)\)-structure is Sasaki–Einstein, then \(H = 0\).

### 3 THE PERTURBATIVE EMBEDDING PROBLEM

In this section, we will prove the second item in Theorem 1. Denote by \(\mathcal{E}(M, Z)\) the set of smooth embeddings of \(M\) into \(Z\). Because \(M\) is compact, this is a tame Fréchet manifold [9, Corollary II.2.3.2]. If \(F : M \hookrightarrow Z\) is an embedding, a chart \(u : U \subset \mathcal{E}(M, Z) \to C^\infty(M, F^*TZ)\) around \(F\) is obtained as follows: Choose a Riemannian metric on \(Z\) with associated exponential map, which maps a neighbourhood of the zero section in \(TZ\) to \(Z\). Then for \(\tilde{F} \in \mathcal{E}(M, Z)\) close to \(F\), we define \(u(\tilde{F}) \in C^\infty(M, F^*TZ)\) by \(\exp(u(\tilde{F})) = \tilde{F}\).

Let \(F \in \mathcal{E}(M, Z)\) be a strongly pseudoconvex embedding with induced 3-form \(\psi := F^*\Psi\) and contact hyperkähler \(SU(2)\)-structure \((\theta, \omega, \alpha, \beta)\). Denote by \(\mathcal{E}_0(M, Z)\) the connected component containing \(F\). To prove the existence part in Theorem 1, we want to show that the map

\[
P : \mathcal{E}_0(M, Z) \to \psi + d\Omega^2(M),
\]

\[
\tilde{F} \mapsto \tilde{F}^*\Psi.
\]

is surjective onto a neighbourhood of \(\psi\). With the aim of applying a type of inverse function theorem, we study the properties of the derivative of \(P\) at \(F\), which is a map

\[
DP(F) : T_F \mathcal{E}(M, Z) = C^\infty(M, F^*TZ) \to d\Omega^2(M).
\]
By the decomposition (1.3), we have the identification

\[ F^*TZ = \mathbb{R}v \oplus \mathbb{R}Iv \oplus H, \]

where \( I \) is the complex structure of the ambient space \( Z \). Indeed, because \( H \) is the real part of \( T^{1,0}Z|_M \cap T_C M \), which is preserved by \( I \), the vector field \( Iv \) is transversal to \( M \). Consider the map

\[ K(F) : C^\infty(M, F^*TZ) = C^\infty(M) \oplus C^\infty(M) \oplus C^\infty(M, H) \to \Omega^2(M), \]

\[ (h_\alpha, h_\beta, w_H) \mapsto F^*((h_\alpha v + h_\beta Iv + w_H) \cdot \Psi) = h_\alpha \alpha + h_\beta \beta + (w_H \cdot \alpha) \wedge \theta. \]

Then, \( DP(F) = d \circ K(F) \). If \( h \in H \), then \( d(\partial \wedge h) = \omega \wedge h - \theta \wedge d_H h = 0 \). Thus, \( \partial \wedge h \) is closed and we can define the map

\[ \Pi_M : H \to H^3(M, \mathbb{R}) \]

which maps \( h \in H \) to the class of \( \partial \wedge h \). In the following, we will prove the following.

**Proposition 3.1.** We have \( \text{coker} \, DP(F) \cong \ker \, \Pi_M \).

Define \( H^\perp \) to be the \( L^2 \)-orthogonal complement of \( H \) in \( \Omega^2(M) \). By Proposition 2.21, we have

\[ H^\perp = \partial \wedge \Omega^1_H \oplus \Omega^1_H \oplus \text{im } d_H. \]

**Lemma 3.2.** We have \( dH^\perp \subset \text{im } DP(F) \).

*Proof.* Let \( \sigma \in H^\perp \) and decompose

\[ \sigma = \sigma_H + \chi \wedge \theta = \sigma_+ + \sigma_- + \chi \wedge \theta, \]

where \( \sigma_H \in \Omega^2_H, \sigma_\pm = \pi_\pm(\sigma_H) \in \Omega^2_H \) and \( \chi \in \Omega^1_H \). We will show that we can find \( \lambda \in \Omega^1(M) \) and \( w \in \Gamma(F^*TZ) \) such that

\[ \sigma - d\lambda = K(F)w. \quad (3.3) \]

Then, \( d\sigma = dK(F)w = DP(F)w \). By assumption, there is a solution \( \eta \in \Omega^1_H \) of the equation

\[ d^-_H \eta = \sigma_- \quad (3.4) \]

Thus, we have

\[ \sigma - d\eta = \sigma_+ - d^+_H \eta + (\chi + \mathcal{L}_H \eta) \wedge \theta =: \sigma'_+ + \chi' \wedge \theta. \]
Next, decompose
\[ \sigma'_+ = h_\omega \omega + h_\alpha \alpha + h_\beta \beta. \]

We eliminate the term involving \( \omega \) by subtracting \( d(h_\omega \theta) \). We get
\[
\sigma - d(\eta + h_\omega \theta) = h_\alpha \alpha + h_\beta \beta + (\chi' - dH h_\omega) \wedge \theta =: h_\alpha \alpha + h_\beta \beta + \chi'' \wedge \theta.
\]

The expression on the right lies in the image of \( K(F) \). Therefore, we have solved (3.3). □

Set
\[
\mathcal{H}_0 = \{ h \in \mathcal{H} : dh \in \text{im} DP(F) \}.
\]

This is a linear subspace of \( \mathcal{H} \). Define \( \mathcal{H}_1 \) to be the \( L^2 \)-orthogonal complement of \( \mathcal{H}_0 \) in \( \mathcal{H} \), so that we have
\[
\Omega^2(M) = \mathcal{H}^\perp \oplus \mathcal{H}_0 \oplus \mathcal{H}_1. \tag{3.5}
\]

Let \( h \) be a non-zero element of \( \mathcal{H}_1 \). If \( dh = 0 \), then \( dh \in \text{im} DP(F) \) trivially, which is a contradiction. Thus, \( d : \mathcal{H}_1 \to d\Omega^2(M) \) is injective. Therefore, with Lemma 3.2, we have
\[
\text{coker} DP(F) \cong \mathcal{H}_1.
\]

The next lemma completes the proof of Proposition 3.1.

**Lemma 3.6.** We have \( \mathcal{H}_1 = \ker \Pi_M \).

**Proof.** We first show \( \mathcal{H}_1 \subset \ker \Pi_M \). If \( H^3(M, \mathbb{R}) = 0 \), then this is clear, so that we can suppose that \( H^2(M, \mathbb{R}) \neq 0 \). Let \( \tau \) be any closed 2-form on \( M \). According to (3.5), we can decompose
\[
\tau = \sigma + h_0 + h_1.
\]

By Lemma 3.2 and the definition of \( \mathcal{H}_0 \), we have
\[
dh_1 = -d\sigma - dh_0 \in \text{im} DP(F),
\]

which implies \( h_1 = 0 \). Thus, \( \mathcal{H}_1 \) is \( L^2 \)-orthogonal to the space of closed 2-forms on \( M \), and if \( h \in \mathcal{H}_1 \), we have
\[
0 = (h, \tau)_{L^2} = -\int_M \theta \wedge h \wedge \tau = -\langle [\theta \wedge h] \cup [\tau], [M] \rangle
\]

for all closed 2-forms \( \tau \). Because by assumption \( H^2(M, \mathbb{R}) \neq 0 \), this implies \( [\theta \wedge h] = 0 \).

Next, we show \( \ker \Pi_M \subset \mathcal{H}_1 \). Let \( h \in \ker \Pi_M \) and decompose \( h = h_0 + h_1 \) with \( h_0 \in \mathcal{H}_0 \) and \( h_1 \in \mathcal{H}_1 \). From the inclusion \( \mathcal{H}_1 \subset \ker \Pi_M \), we get \( h_0 \in \ker \Pi_M \). By the definition of \( \mathcal{H}_0 \), there exists some \( w \in \Gamma(F^*TZ) \) such that \( dh_0 = dK(F)w \). Thus, \( \tau := h_0 - K(F)w \) is a closed 2-form.
We get
\[0 = \langle [\theta \wedge h_0] \cup [\tau], [M] \rangle = \int_M \theta \wedge h_0 \wedge \tau = -\|h_0\|^2.\]
Thus, \(h_0 = 0\) and \(h \in \mathcal{H}_1\).

**Example 1.** Let \(X\) be a complex \(K3\) surface with holomorphic 2-form \(\Theta\) and let \(L\) be a holomorphic line bundle over \(X\). Let \(Z\) be the total space of \(L\) minus its zero section. This is a non-compact complex 3-fold with a holomorphic 3-form \(\Psi\) given in a local trivialisation by \(\Theta \wedge \zeta^{-1} d\zeta\), where \(\zeta\) is a co-ordinate along the fibre. A choice of Hermitian metric on \(L\) defines a unit-circle bundle \(M \subset Z\) and this is pseudoconvex (with a suitable orientation) if and only if the metric has positive curvature \(\omega\), which defines a Kähler metric on \(X\). Yau’s solution of the Calabi conjecture gives a metric on \(L\) such that our Reeb vector field coincides with the generator of the circle action, up to a factor. It is straightforward to show that \(\mathcal{H}\) can be identified with the anti-self dual forms on \(X\). On the other hand, from the Serre spectral sequence, \(H^3(M)\) can be identified with \([\omega] \perp \subset H^2(X)\).

**Proposition 3.1.** If the obstruction space \(\mathcal{H}\) vanishes, then \(\mathcal{D}^P(F)\) is surjective. Because the vanishing of \(\mathcal{H}\) is an open condition, \(\mathcal{D}^P(\tilde{F})\) is then surjective for all embeddings \(\tilde{F}\) in an open neighbourhood of \(F\) in \(\mathcal{E}(M, Z)\).

**Proposition 3.7.** Suppose that \(\mathcal{H} = 0\) for the SU(2)-structure induced by the strongly pseudoconvex embedding \(F \in \mathcal{E}(M, Z)\). Then, the map \(P\) is surjective onto an open neighbourhood of \(\psi\).

**Proof.** The conditions in Definition 1.1 depend on two derivatives of \(F\). Thus, for any \(\tilde{F}\) which is \(C^2\)-close to \(F\) the induced 3-form \(\tilde{\psi} := P(\tilde{F})\) is strongly pseudoconvex as well. Denote by \((\tilde{\partial}, \tilde{\omega}, \tilde{\alpha}, \tilde{\beta})\) the SU(2)-structure induced by \(\tilde{F}\). We observe here that this SU(2)-structure depends on \(\tilde{F}\) and its first three derivatives. One derivative is needed to determine \(\tilde{\psi}\). Another derivative is needed to normalise as in Section 1.1. Third derivatives come in by computing \(\omega = \partial \Theta\).

To describe a right inverse for \(D^P(\tilde{F})\), we will distinguish three cases.

1. \(\tilde{H} = H\) and \(\tilde{\partial} = \partial\) (and thus, \(\tilde{\psi} = \psi\)).
2. \(\tilde{H} = H\) but \(\tilde{\partial} \neq \partial\) (and thus, \(\tilde{\psi} \neq \psi\)).
3. The contact distributions \(\tilde{H}\) and \(H\) are different.

We start by explaining case (1). In this case, the splitting (1.4) remains the same. However, \((\omega, \tilde{\alpha}, \tilde{\beta})\) defines a new bundle of self-dual 2-forms, so the splitting (1.14) changes. If \(\tilde{F}\) is sufficiently close to \(F\), then as in the beginning of Section 2.3, we can write \(\text{span}\{\omega, \tilde{\alpha}, \tilde{\beta}\} = \Lambda^+_{H, \mu}\) for a map \(\Lambda^+_{H} \to \Lambda^+_{H, \mu}\). To construct a right inverse \(V(P)(\tilde{F})\) of \(D^P(\tilde{F})\), let \(\tilde{\psi} \in \mathcal{G}(M) = d\Omega^2(M)\). By using the Hodge decomposition with respect to some background Riemannian metric, we find a preferred primitive \(d\sigma = \tilde{\psi}\). More precisely, \(\sigma = d^* \xi\) where \(\xi\) is the unique solution of \(\Delta \xi = \tilde{\psi}\). In particular, we have \(\|\sigma\|_{s+1} \leq \|\tilde{\psi}\|_s\) for all \(s\). To construct a pre-image of \(d\sigma\) with respect to the map \(D^P(\tilde{F})\),
we proceed as in the proof of Proposition 3.1. Decompose $\sigma$ as

$$\sigma = \sigma_H + \chi \wedge \theta = \sigma_+ + \sigma_- + \chi \wedge \theta,$$

where $\sigma_H \in \Omega^2_H$, $\sigma_\pm = \pi_\pm(\sigma_H) \in \Omega^\pm_H$ and $\chi \in \Omega^1_H$. In place of (3.4), we now need to solve the perturbed equation

$$D_{\mu} \eta = \sigma_-,$$

where $D_\mu$ is the operator from (2.24). Under the hypothesis $H = 0$, the right inverse $R_\mu$ of $D_\mu$ from Corollary 2.33 gives the solution $R_\mu \sigma_-$ if $\tilde{F}$ is in a sufficiently small neighbourhood of $F$.

Denote by $L_1(\tilde{F})$ the projection of $\Lambda^2_H$ to $\mathbb{R} \omega$, by $L_2(\tilde{F})$ the projection to $\mathbb{R} \tilde{\alpha}$ and by $L_3(\tilde{F})$ the projection to $\mathbb{R} \tilde{\beta}$. $\tilde{\alpha}$ defines an isomorphism $H \to \Lambda^1_H$ via $w \to w \tilde{\alpha}$. Denote the inverse by $L_4(\tilde{F})$.

Tracing through the proof of Proposition 3.1, we find $\gamma \in \Omega^1(M)$, such that

$$\sigma - d\gamma = K(\tilde{F})(h_\alpha, h_\beta, w_H),$$

where

$$\sigma = d^a \Delta^{-1} \tilde{\psi},$$

$$h_\omega = L_1(\tilde{F})(\sigma_H - d_H R_\mu \sigma_-),$$

$$h_\alpha = L_2(\tilde{F})(\sigma_H - d_H R_\mu \sigma_-),$$

$$h_\beta = L_3(\tilde{F})(\sigma_H - d_H R_\mu \sigma_-),$$

$$w_H = L_4(\tilde{F})\{\chi + L_v R_\mu \sigma_- - d_H h_\omega\}.$$

We define the right inverse $VP(\tilde{F})$ of $DP(\tilde{F})$ to be

$$VP(\tilde{F})\tilde{\psi} := (h_\alpha, h_\beta, w_H).$$

Next, we need to show that the map $VP(\tilde{F})$ satisfies a tame estimate. Because the SU(2)-structure depends on three derivatives of $\tilde{F}$, the operators $L_i(\tilde{F})\sigma$, $i = 1, \ldots, 4$, also depend on three derivatives of $\tilde{F}$, and are linear and of order 0 in $\sigma$. By (2.32), for each $k \in \mathbb{N}$, we have under a $C^3$-bound on $\tilde{F}$

$$\|L_i(\tilde{F})\sigma\|_k \lesssim \|\sigma\|_k + \|[u(\tilde{F})]\|_{k+3}\|\sigma\|. \quad (3.8)$$

Together with (2.34), this gives

$$\|h_\alpha\|_k = \|L_2(\tilde{F})(\sigma_H - d_H R_\mu \sigma_-)\|_k$$

$$\lesssim \|\sigma\|_k + \|R_\mu \sigma_-\|_{k+1} + \|[u(\tilde{F})]\|_{k+3}(\|\sigma\| + \|R_\mu \sigma_-\|_1)$$

$$\lesssim \|\sigma\|_{k+2} + (\|[\mu]\|_{k+4} + 1)\|\sigma\|_l + \|[u(\tilde{F})]\|_{k+3}\|\sigma\|_l$$

$$\lesssim \|\tilde{\psi}\|_{k+1} + ([u(\tilde{F})]\|_{k+7} + 1)\|\tilde{\psi}\|_{l-1}.$$
We get analogous estimates for $h_\beta$ and $h_\omega$. Thus, we get

$$
\|\chi''\|_k \lesssim \|\chi\|_k + \|R_\mu\sigma_-\|_{k+1} + \|h_\omega\|_{k+1}
$$

$$
\lesssim \|\psi\|_{k+2} + (\|u(\tilde{F})\|_{k+8} + 1)\|\psi\|_{l-1}.
$$

Similarly to (3.8) under a $C^3$-bound on $F$, we have

$$
\|L_4(\tilde{F})\chi\|_k \lesssim \|\chi\|_k + \|u(\tilde{F})\|_{k+3}\|\chi\|.
$$

Using this gives

$$
\|w_H\|_k \lesssim \|\chi''\|_k + \|u(\tilde{F})\|_{k+3}\|\chi''\|,
$$

$$
\lesssim \|\psi\|_{k+2} + (\|u(\tilde{F})\|_{k+8} + 1)\|\psi\|_{l-1} + \|u(\tilde{F})\|_{k+3}\|\psi\|_{l-1}
$$

$$
\lesssim \|\psi\|_{k+2} + (\|u(\tilde{F})\|_{k+8} + 1)\|\psi\|_{l-1},
$$

if $\tilde{F}$ is sufficiently close to $F$. Thus, we get

$$
\|VP(\tilde{F})\psi\|_k \lesssim \|\psi\|_{k+2} + (\|u(\tilde{F})\|_{k+8} + 1)\|\psi\|_{l-1}.
$$

(3.9)

Case (2): In this case, the splitting (1.4) changes and $\text{span}\{\tilde{\omega}, \tilde{\alpha}, \tilde{\beta}\}$ is not a subspace of $\Lambda^2_H$ any more. However, instead of normalising as in Section 1.1, we still can write

$$
\tilde{F}^n\Psi = \theta \wedge \alpha', \quad \tilde{F}^n\tilde{\Psi} = \theta \wedge \beta',
$$

where $\alpha'.\alpha' = \beta'.\beta' > 0$ and $\omega, \alpha', \beta'$ are mutually orthogonal. Thus, $(\omega, \alpha', \beta')$ spans a positive definite rank 3 subbundle of $\Lambda^2_H$. The estimate follows in case (2) if we replace $\text{span}\{\omega, \tilde{\alpha}, \tilde{\beta}\}$ by $\text{span}\{\omega, \alpha', \beta'\}$ in the derivation for the estimate in case (1).

Case (3): Now suppose $\tilde{H} \neq H$. By a result of Gray [7], there exists a diffeomorphism of $M$ such that $\tilde{H} = \Phi_*H$. A manifold chart around the identity in the group $D(M)$ of diffeomorphisms is given by the inverse $\nu : V \subset D(M) \to C^\infty(M, TM)$ of the exponential map. The map $\tilde{H} \mapsto \Phi$ is tame [9, Theorem III.2.4.6], which implies that there exists $r$ such that

$$
\|u(\tilde{F})\|_n \lesssim (\|u(\tilde{F})\|_{n+r} + 1)
$$

(3.10)

for all $n$. Furthermore, the composition

$$
\mathcal{E}(M, Z) \times D(M) \to \mathcal{E}(M, Z),
$$

$$(\tilde{F}, \Phi) \mapsto \tilde{F} \circ \Phi
$$

is a smooth tame map [9, II.4.4.5], so there exists an $s$ such that

$$
\|u(\tilde{F} \circ \Phi)\|_n \lesssim (\|\nu(\Phi)\|_{n+s} + (\|u(\tilde{F})\|_{n+s} + 1 \lesssim (\|u(\tilde{F})\|_{n+r+s} + 1).
$$

(3.11)
By the discussion below Definition 1.1, \((\tilde{F} \circ \Phi)^* \Psi = \Phi^* \psi\) is a strongly pseudoconvex 3-form which determines the contact distribution \(\Phi^{-1} \tilde{H} = H\). Furthermore, if \(\tilde{F}\) is sufficiently close to \(F\), then \(\Phi\) must be sufficiently close to the identity. Thus, we can apply the result from cases (1) and (2) to \(\tilde{F} \circ \Phi\), that is, that there exists a right inverse \(VP(\tilde{F} \circ \Phi)\) to \(DP(\tilde{F} \circ \Phi)\) which satisfies the estimate (3.9). The calculation
\[
d(X \Phi^* \psi) = d(\Phi^* ((\Phi^* X)_\psi)) = \Phi^* d((\Phi^* X)_\psi)
\]
shows that \(VP(\tilde{F}) := \Phi^* \circ VP(\tilde{F} \circ \Phi) \circ \Phi^*\) is a right inverse for \(DP(\tilde{F})\). A repeated application of the second Moser estimate and (3.9) gives
\[
\|VP(\tilde{F}) \psi\|_k \leq \|\psi\|_{k+2} + ([u(\tilde{F} \circ \Phi))]_{k+8} + 1)\|\psi\|_{l-1}
\]
for some \(d\). Putting together the results for the three cases, we have constructed a right inverse \(VP(\tilde{F})\) of \(DP(\tilde{F})\) for all \(\tilde{F}\) in some neighbourhood of \(F\) in \(\mathcal{E}(M, Z)\) which for all \(k\) satisfies an estimate of the form (3.12). The result follows from [9, III.1.1.3].

Local uniqueness

To complete the proof of Theorem 1, we are left to prove the local uniqueness statements for embeddings into \(\mathbb{C}^3\). We start by proving local uniqueness for Problem 1.

**Proposition 3.13.** Let \((\theta, \omega, \alpha, \beta)\) be a contact hyperkähler SU(2)-structure on \(M\) with \(\mathcal{H} = 0\). Then there exists a neighbourhood of \(\beta\) in \(\Omega^2_H\) in which \(\beta\) is the only solution of the system
\[
\omega \cdot \beta = 0, \quad \alpha \cdot \beta = 0, \quad \tilde{\beta} \cdot \beta = 1, \quad d_H \tilde{\beta} = 0.
\]

**Proof.** Suppose that there is another solution \(\tilde{\beta}\). Then the first two equations imply that we can decompose \(\tilde{\beta} = h \beta + \beta_-\), where \(\beta_- \in \Omega^2_H\). The third condition gives \(\tilde{\beta} \cdot \beta_- = 1 - h^2\). If we choose the neighbourhood of \(\beta\) sufficiently small such that \(h > -1\), then \(\text{span}\{\omega, \alpha, (1 + h)\beta + \tilde{\beta}_-\}\) is a positive definite subspace which defines a perturbation \(\Lambda^+_{H, \mu}\) of \(\Lambda^+_{H}\). It is constructed such that the difference \(\gamma := \tilde{\beta} - \beta\) is a section of \(\Lambda^-_{H, \mu}\). The last equation implies \(d_H \gamma = 0\), and thus,
γ ∈ ker □_{H, 𝜇}. Because the vanishing of ker □ is an open condition, γ must vanish if 𝜇 is sufficiently small.

Suppose that \( \hat{F} \) is another embedding of \( M \) into \( C^3 \) which is close to \( F \) and satisfies \( \hat{F}^\ast \Psi = F^\ast \Psi = \psi \). Then, by Proposition 3.13, we also have \( \hat{F}^\ast \Psi = F^\ast \Psi \). Denote by \( \Phi \) the diffeomorphism \( \hat{F} \circ F^{-1} \) from \( F(M) \) to \( \hat{F}(M) \). Because \( \Phi \) preserves \( \alpha + i\beta \), it is an isomorphism of CR-manifolds. In particular, the component functions of \( \Phi = (\phi_1, \phi_2, \phi_3) \) are CR functions, that is, are annihilated by \( \bar{\partial}_b \). Because \( F(M) \) is strongly pseudoconvex, \( \phi_1, \phi_2 \) and \( \phi_3 \) extend to holomorphic functions \( \Phi_1, \Phi_2, \Phi_3 \) on the domain \( U \) which is bounded by \( F(M) \) [6, Theorem 5.3.2]. We are left to prove that \( \Phi = (\Phi_1, \Phi_2, \Phi_3) \) preserves \( \Psi \). We can write \( \Phi^\ast(\Psi + i\bar{\partial} \Psi) = h(\Psi + i\bar{\partial} \Psi) \) for some holomorphic function on \( U \). In particular, the real and imaginary parts of \( h \) are harmonic with respect to the Euclidean metric on \( C^3 \). On the boundary, we have \( \text{Re} \ h = 1 \) and \( \text{Im} \ h = 1 \). By the maximum principle, we have \( \text{Re} \ h = 1 \) and \( \text{Im} \ h = 1 \) on all of \( U \). Thus, \( \Phi^\ast(\Psi + i\bar{\partial} \Psi) = \Psi + i\bar{\partial} \Psi \).

**APPENDIX A: VANISHING OF COHOMOLOGY FOR PSEUDOCONVEX SUBMANIFOLDS OF STEIN MANIFOLDS**

Let \( Z \) be a complex manifold with complex dimension \( n \) and boundary \( M := \partial Z \). Denote by \( \mathcal{A}^p,q(Z) \) the space of \( (p,q) \)-forms on \( Z \) which are smooth up to the boundary. Choose a hermitian metric on \( Z \), which then also induces a hermitian metric on the bundles \( \Lambda^p,qZ \). Denote by \( \Lambda^p,qZ|_M \) the restriction of the bundle \( \Lambda^p,qZ \) to \( M \), that is, the collection of the spaces \( \Lambda^p,qZ|_z \), where \( z \) varies over \( M \). A \( (p,q) \)-form at \( z \in M \) on the boundary is called complex normal if it is of the form \( \bar{\partial}_b \wedge z \), where \( z \in \Lambda^{p,q-1}Z|_M \) and \( \bar{\partial}_b \) is a boundary defining function in a neighbourhood of \( z \). Denote by \( \Lambda^p,q_M|_z \subset \Lambda^p,qZ|_z \) the space of all complex normal \( (p,q) \)-forms at \( z \), \( \Lambda^p,q_M|_z \) does not depend on the choice of \( r \). Denote by \( \Lambda^p,q_M|_z \) the orthogonal complement of \( \Lambda^p,q_M|_z \) in \( \Lambda^{p,q}_Z|_z \). Forms in \( \Lambda^p,q_M|_z \) are called complex tangential. We note that \( \Lambda^p,q_M|_z \) is not a subspace of \( \Lambda^{p+q}_Z|_z M \otimes \mathbb{C} \). By \( \Lambda^p,q_M \) and \( \Lambda^p,q_M \), we denote the bundles over \( M \) given by the union of the \( \Lambda^p,q_M|_z \) and \( \Lambda^p,q_M|_z \), respectively. We will denote the spaces of smooth sections of \( \Lambda^p,q_M \) and \( \Lambda^p,q_M \) by \( \mathcal{A}^p,q(M) \) and \( \mathcal{A}^p,q(M) \) respectively. \( \mathcal{A}^p,q(Z) \) are those \( (p,q) \)-forms on \( Z \) which restricted to the boundary lie in \( \mathcal{A}^p,q(M) \) and \( \mathcal{A}^p,q(Z) \) are those \( (p,q) \)-forms on \( Z \) which restricted to the boundary lie in \( \mathcal{A}^p,q(M) \).

Because \( \bar{\partial}_b^2 = 0 \), we have for each \( p = 0, \ldots, n \) a chain complex \( (\mathcal{A}^p,q(Z), \bar{\partial}_b)_q \) whose cohomology groups we denote by \( H^{p,q}(Z) \). A form in \( \mathcal{A}^p,q(Z) \) can be written as \( \bar{\partial}_b r \wedge \sigma + r \xi \), where \( \sigma \in \mathcal{A}^{p,q-1}(Z) \) and \( \xi \in \mathcal{A}^{p,q}(Z) \). The calculation

\[
\bar{\partial}_b(\bar{\partial}_b r \wedge \sigma + r \xi) = \bar{\partial}_b r \wedge (\xi - \bar{\partial}_b \sigma) + r \bar{\partial}_b \xi \tag{A.1}
\]

shows that \( \bar{\partial}_b \) maps \( \mathcal{A}^{p,q}_N(Z) \) to \( \mathcal{A}^{p,q+1}_N(Z) \). Denote the cohomology groups of the complex \( (\mathcal{A}^p,q(Z), \bar{\partial}_b)_q \) by \( H^{p,q}_N(Z) \).

Next, we define the \( \bar{\partial}_b \)-complex on the boundary \( M \). If \( \zeta \in \mathcal{A}^p,q(M) \), let \( \zeta \) be any extension to \( \mathcal{A}^p,q(Z) \). Set \( \bar{\partial}_b \zeta := \pi^T_M(\bar{\partial}_b \zeta) \), where \( \pi^T_M \) denotes the projection from \( \mathcal{A}^p,q(Z) \) to \( \mathcal{A}^p,q(M) \). Any two extensions of \( \zeta \) differ by an element in \( \mathcal{A}^{p,q}_N(Z) \). By (A.1) \( \pi^T_M \circ \bar{\partial}_b \) vanishes on \( \mathcal{A}^{p,q}_N(Z) \), so that \( \bar{\partial}_b \) is well defined. \( \bar{\partial}_b^2 = 0 \) implies \( \bar{\partial}_b \bar{\partial}_b = 0 \), and thus, we get a chain complex \( (\mathcal{A}^p,q(M), \bar{\partial}_b)_q \), whose cohomology groups we denote by \( H^{p,q}(M) \). By the construction of \( \bar{\partial}_b \), we have a short exact sequence
of chain complexes

\[
0 \longrightarrow A_N^{p,q+1}(Z) \longrightarrow A_T^{p,q+1}(Z) \xrightarrow{\Delta_M} A_T^{p,q+1}(M) \longrightarrow 0
\]

\[
0 \longrightarrow A_N^{p,q}(Z) \longrightarrow A_T^{p,q}(Z) \xrightarrow{\Delta_M} A_T^{p,q}(M) \longrightarrow 0
\]

This induces a long exact sequence of cohomology groups

\[
\cdots \longrightarrow H_N^{p,q}(Z) \longrightarrow H_T^{p,q}(Z) \longrightarrow H_T^{p,q}(M) \longrightarrow H_N^{p,q+1}(Z) \longrightarrow \cdots
\]

**Proposition A.2** [6, Proposition 5.1.5]. If the Levi form of \( M \) has at least \( n - q \) positive eigenvalues or at least \( q + 1 \) negative eigenvalues (certainly true on a strongly pseudoconvex domain), then \( H^{p,q}(Z) \cong H_N^{n-p,n-q}(Z)^* \).

**Corollary A.3.** Let \( Z \) be a strongly pseudoconvex domain in a Stein manifold of complex dimension 3 with boundary \( M \). Then, \( H^{1,1}(M) = 0 \).

**Proof.** Because \( Z \) is a strongly pseudoconvex domain in a Stein manifold, we have \( H^{1,1}(Z) = 0 \) and \( H_N^{1,2}(Z) \cong H^{2,1}(Z)^* = 0 \) [11, Corollary 5.2.6]. The long exact sequence gives \( H^{1,1}(M) = 0 \).

**APPENDIX B: ALTERNATIVE PROOF THAT THE VANISHING OF \( H \) IS AN OPEN CONDITION**

In this section, we prove in a more quantitative way that the obstruction space vanishes for an open set of Euclidean metrics on \( H \). The advantage of this approach is that it can give an estimate on the size of the neighbourhood of \( \eta_H \) in which the obstruction space vanishes. We will derive an explicit such bound in the case of the standard embedding of \( S^5 \) in \( C^3 \).

Suppose that \( H = 0 \), so that \( \Box_H \) is invertible and \( d^-_H \) has right inverse \( R = d^+_H \Box_H^{-1} \) as in Proposition 2.21. As in Section 2.3, we parametrise nearby metrics on \( H \) by a bundle map \( \mu : \Lambda^+_H \rightarrow \Lambda^-_H \). The vanishing of \( \ker \Box_H,\mu \) is equivalent to the surjectivity of \( d^-_H,\mu \). We have the explicit formula

\[
(1 - \mu \mu^*) \circ \pi^- \circ d^-_H,\mu = d^-_H - \mu \circ d^+_H.
\]

\( 1 - \mu \mu^* \) is an isomorphism if \( |\mu| < 1 \). Thus, under this bound on \( \mu \), it is enough to show that

\[
d^-_H - \mu \circ d^+_H : \Omega^1_H \rightarrow \Omega^1_H
\]

is surjective. To prove this, we will show that

\[
(d^-_H - \mu \circ d^+_H) \circ R = 1 - \mu \circ d^+_H \circ R : L^2(\Lambda^-_H) \rightarrow L^2(\Lambda^-_H)
\]
is an isomorphism if $\mu$ is sufficiently small. If $\kappa$ is a constant such that, for all $\sigma$,

$$\|d^+_H R \sigma\| \leq \kappa \|\sigma\|, \quad (B.1)$$

then if $|\mu| < \kappa^{-1}$ everywhere the operator

$$\mu d^+_H R : L^2(\Lambda^-_H) \to L^2(\Lambda^-_H)$$

has operator norm less than 1 and it follows from the usual Neumann series that $d^-_{H,\mu}$ is surjective.

On a compact, oriented Riemannian 4-manifold, we have the identity $\|d^+ \eta\| = \|d^- \eta\|$ for any 1-form $\eta$. In the contact case, there is an extra term:

$$\|d^+_H \eta\|^2 - \|d^-_H \eta\|^2 = \int_M d_H \eta \wedge H \eta \wedge \theta = \int_M d\eta \wedge d\eta \wedge \theta = \int_M d\eta \wedge \eta \wedge d\theta$$

$$= \int_M d\eta \wedge \eta \wedge \omega = \int_M \theta \wedge \mathcal{L}_v \eta \wedge \eta \wedge \omega = - \int_M \langle I\mathcal{L}_v \eta, \eta \rangle \, \text{Vol}$$

$$= -(I\mathcal{L}_v \eta, \eta).$$

Setting $\eta = R \sigma$, this gives

$$\|d^+_H R \sigma\|^2 \leq \|\sigma\|^2 + |(I\mathcal{L}_v \eta, \eta)|. \quad (B.2)$$

The existence of a constant $\kappa$ in (B.1) follows from formula (B.2) and our analysis in Section 2. Let $P$ be the standard self-adjoint second-order operator $P = 1 + \nabla^* \nabla$. Then

$$|(I\mathcal{L}_v \eta, \eta)| = |(P^{-1/4}I\mathcal{L}_v \eta, P^{1/4} \eta)| \leq c_1 \|\eta\|_2^2,$$

since $P^{-1/4}I\mathcal{L}_v$ and $P^{1/4}$ are pseudo-differential operators of order $1/2$. Now the estimate (2.22) for the operator $R$ gives

$$|(I\mathcal{L}_v \eta, \eta)| \leq c_2 \|\sigma\|^2, \quad (B.3)$$

for some $c_2$. Then, we can take $\kappa = \sqrt{1 + c_2}$.

We now study the case of the standard contact structure on $S^5$, with standard metric. Regarding $S^5$ as the principle circle bundle over $\mathbb{C}P^2$ corresponding to the Hopf line bundle $L \to \mathbb{C}P^2$, the contact structure is the field of horizontal subspaces for the standard connection. Passing to the complexified bundles, we can write any $\eta \in \Omega^1_H$ as a sum $\eta = \sum_k \eta_k$ of components in the weight spaces for the circle action. A component $\eta_k$ can be regarded as a 1-form on $\mathbb{C}P^2$ with values in the line bundle $L^k$. Similarly, for sections of $\Lambda^-_H$, the operator $d^-_H$ is a sum of components

$$d^-_k : \Omega^1_{\mathbb{C}P^2}(L^k) \to \Omega^-_{\mathbb{C}P^2}(L^k),$$

and similarly, $\Box = \sum_k \Box_k$. These operators are very familiar in four-dimensional Riemannian geometry and in particular the fact that $\mathbb{C}P^2$ is a self-dual manifold and that the curvature of $L$ is
a self-dual form means that there are simple Weitzenbock formulae

\[ \square_k = \frac{1}{2} \nabla^*_k \nabla_k + S/6, \]

where \( S \) is the scalar curvature of \( \mathbb{C}P^2 \). Because \( S \) is positive, this gives an alternative proof that the obstruction space \( \text{ker} \; \square \) vanishes for this example.

To get favourable bounds on the operators \( \nabla^*_k \nabla_k \), we review some general theory. Let \( V \) be a complex vector bundle with Hermitian connection over a compact Kähler manifold \( X \). The connection defines operators

\[ \partial_V : \Omega^0(V) \to \Omega^{1,0}(V) \quad \overline{\partial}_V : \Omega^0(V) \to \Omega^{0,1}(V), \]

in the usual way (note that we not supposing that \( \overline{\partial}_V \) defines a holomorphic structure). Then, we have the identity, for any section \( s \) of \( V \):

\[ \| \partial_V s \|^2 - \| \overline{\partial}_V s \|^2 = (i(F_V \omega)(s), s). \]

where \( F_V \) is the curvature of the connection. In our application, we take \( V = \Lambda^+_{\mathbb{C}P^2} \otimes L^k \). The fact that \( \mathbb{C}P^2 \) is an Einstein manifold implies that the curvature of the bundle \( \Lambda^+_{\mathbb{C}P^2} \) is anti-self-dual and so orthogonal to the self-dual form \( \omega \). Thus, the only contribution to \( F_V \cdot \omega \) comes from \( L^k \) and we get

\[ \| \partial_V s \|^2 - \| \overline{\partial}_V s \|^2 = k \| s \|^2. \]

This implies that

\[ \| \nabla_k s \|^2 \geq |k| \| s \|^2. \]

Combining with the previous discussion, and using the fact that the scalar curvature is positive, we get

\[ (\square_k s, s) \geq \frac{1}{2} |k| \| s \|^2. \]

This implies that, for \( k \neq 0 \),

\[ (\square_k^{-1} s, s) \leq 2 |k|^{-1} \| s \|^2. \]

Now if \( \sigma \in \Omega^-_H \) has components \( \sigma_k \) and \( \eta_k = d\star \square_k^{-1} \sigma_k \), we have

\[ \| \eta_k \|^2 = (d^* \square_k^{-1} \sigma_k, d^* \square_k^{-1} \sigma_k) = (\square_k^{-1} \sigma_k, \sigma_k) \leq 2 |k|^{-1} \| \sigma_k \|^2. \]

One can check that the Reeb vector field \( v \) is \( \sqrt{2} \) times the generator of the circle action. Therefore, the Lie derivative term is \( \mathcal{L}_v \eta_k = i \sqrt{2} k \eta_k \), so

\[ |(\mathcal{L}_v \eta, \eta)| = \left| \sum \sqrt{2} k l \eta_k, \eta_k \right| \leq \sum \sqrt{2} |k| \| \eta_k \|^2. \]
Thus, we get
\[ |\langle I L_v \eta, \eta \rangle| \leq \sum_k 2 \sqrt{2} \| \sigma_k \|^2 = 2 \sqrt{2} \| \sigma \|^2. \]
Therefore, in this case, we can take \( c_2 = 2 \sqrt{2} \) for the constant in (B.3) and we see that \( d_{H, \mu}^{-} \) is invertible if \( |\mu| < 1/\sqrt{1 + 2 \sqrt{2}} \).

**APPENDIX C: A WEITZENBÖCK FORMULA FOR AN SU(2)-HOLONOMY CONNECTION**

In Lemma 2.6, we have shown that there exist metric connections \( \nabla \) on \( TM \) which respect the contact structure in a suitable way. From Lemma 2.8 and the proof of Proposition 2.14, we know that for each such a connection, there is a self-adjoint endomorphism \( R^\nabla \) of \( \Lambda^{-}_H \) such that
\[ \square_H = \frac{1}{2} \nabla^\nabla_H \nabla_H + R^\nabla. \]
This gives us a criterion for the vanishing of the obstruction space \( \mathcal{H} \).

**Proposition C.1.** If \( R^\nabla \) is a positive operator, then \( \mathcal{H} = 0 \).

In four-dimensional Riemannian geometry, the curvature term in the Weitzenböck formula is
\[ W^- + \frac{S}{3}, \tag{C.2} \]
where \( W^- \) is the anti-self-dual part of the Weyl tensor and \( S \) is the scalar curvature. The goal of this appendix is to pick a particularly well-suited connection from Lemma 2.6 and show that for this connection, \( R^\nabla \) essentially is a contact analogue of (C.2). The choice in Lemma 2.6 is \( T^a_u \), the skew-symmetric part of \( T(u, \cdot) \), the torsion tensor contracted with \( u \). In Section C.1, we show that the additional condition that the full SU(2)-structure is parallel completely determines the component of \( T^a_u \) in \( \mathfrak{so}^+(H) \). We then set the component in \( \mathfrak{so}^-(H) \) equal to 0 to pick a specific such connection. In Section C.2, we describe the curvature tensor of this connection, and in Section C.3, we compute the Weitzenböck formula.

C.1 | Determining the torsion

By (1.17), we know that \( L_v \alpha \in C^\infty(M) \beta \oplus \Omega^-_H \). Define the function \( f \) by
\[ L_v \alpha = f \beta \mod \Omega^-_H. \]
In this section, we prove the following.

**Proposition C.3.** \( (\omega, \alpha, \beta) \) are parallel with respect to a connection \( \nabla \) on \( TM \) from Lemma 2.6 if and only if
\[ T^a_u = -\frac{1}{2} f I \mod \Gamma(\mathfrak{so}^-(H)). \]
Remark. $T^u_v = 0$ leads to the Tanaka–Webster connection \[14\] and \[15\], which preserves $g$ and $I$ but not the full SU(2)-structure.

By property (ii) of Lemma 2.6, we have

$$\text{Alt}(\nabla_H \omega) = d_H \omega = 0, \quad \text{Alt}(\nabla_H \alpha) = d_H \alpha = 0, \quad \text{Alt}(\nabla_H \beta) = d_H \beta = 0.$$ 

With a similar representation-theoretic argument as for hyperkähler structures \[13\], one can conclude

$$\nabla_H \omega = 0, \quad \nabla_H \alpha = 0, \quad \nabla_H \beta = 0.$$ 

Thus, we need to determine $T^u_v$ such that

$$\nabla_v \omega = 0, \quad \nabla_v \alpha = 0, \quad \nabla_v \beta = 0.$$ 

Via the metric $g_H$, the 2-forms $\omega, \alpha, \beta$ correspond to the triple of almost complex structures $I, J, K$ as in (1.15). It will be convenient to express the Lie derivatives of $\omega, \alpha, \beta$ as endomorphism fields of $H$.

**Lemma C.4.** As endomorphism fields, the Lie derivatives of $g, \alpha$ and $\beta$ in the direction of $v$ correspond to

$$\mathcal{L}_v g \sim \mathcal{L}_v I \circ I = -I \circ \mathcal{L}_v I,$$ 

$$\mathcal{L}_v \alpha \sim \mathcal{L}_v I \circ K + \mathcal{L}_v J = -I \circ \mathcal{L}_v K,$$ 

$$\mathcal{L}_v \beta \sim -\mathcal{L}_v I \circ J + \mathcal{L}_v K = I \circ \mathcal{L}_v J.$$

**Proof.** Because $\mathcal{L}_v \omega = 0$, for any $X, Y \in \Gamma(H)$, we get

$$0 = \mathcal{L}_v \omega(X, Y) = v(g(IX, Y)) - g(I \mathcal{L}_v X, Y) - g(IX, \mathcal{L}_v Y)$$

$$= v(g(IX, Y)) - g(\mathcal{L}_v (IX), Y) - g(IX, \mathcal{L}_v Y) + g(\mathcal{L}_v (IX) - I \mathcal{L}_v X, Y)$$

$$= \mathcal{L}_v g(IX, Y) + g((\mathcal{L}_v I)X, Y).$$

Thus, $\mathcal{L}_v g$ corresponds to the endomorphism $\mathcal{L}_v I \circ I$. An analogous computation for $\alpha$ and $\mathcal{L}_v g_H \sim \mathcal{L}_v I \circ I$ gives

$$\mathcal{L}_v \alpha(X, Y) = \mathcal{L}_v g(JX, Y) + g((\mathcal{L}_v J)X, Y)$$

$$= g((\mathcal{L}_v I \circ I \circ J)X, Y) + g((\mathcal{L}_v J)X, Y)$$

$$= g((\mathcal{L}_v I \circ K + \mathcal{L}_v J)X, Y).$$

Similarly for $\beta$, we get

$$\mathcal{L}_v \beta(X, Y) = \mathcal{L}_v g(KX, Y) + g((\mathcal{L}_v K)X, Y)$$
\[ = g((\mathcal{L}_v I \circ I K)X, Y) + g((\mathcal{L}_v K)X, Y) \]
\[ = g((-\mathcal{L}_v I \circ J + \mathcal{L}_v K)X, Y). \]

The remaining identities follow from

\[ 0 = \mathcal{L}_v (-1) = \mathcal{L}_v I^2 = \mathcal{L}_v I \circ I + I \circ \mathcal{L}_v I, \]
\[ \mathcal{L}_v J = -\mathcal{L}_v (I \circ K) = -\mathcal{L}_v I \circ K - I \circ \mathcal{L}_v K, \]
\[ \mathcal{L}_v K = \mathcal{L}_v (I \circ J) = \mathcal{L}_v I \circ J + I \circ \mathcal{L}_v J. \]

\[ \square \]

**Lemma C.8.** The conditions

\[ \nabla_v \omega = 0, \quad \nabla_v \alpha = 0, \quad \nabla_v \beta = 0, \]

are equivalent to

\[ [I, T^a_v] = 0, \tag{C.9a} \]
\[ [J, T^a_v] = \frac{1}{2}[J, \mathcal{L}_v K \circ K], \tag{C.9b} \]
\[ [K, T^a_v] = \frac{1}{2}[K, \mathcal{L}_v J \circ J]. \tag{C.9c} \]

**Proof.** If \( B \) is any field of bilinear forms on \( H \) and \( X, Y \in \Gamma(H) \), formula 2.7 implies

\[ (\nabla_v B)(X, Y) = \nu(B(X, Y)) - B(\nabla_v X, Y) - B(X, \nabla_v Y) \]
\[ = \nu(B(X, Y)) - B(\mathcal{L}_v X, Y) - B(X, \mathcal{L}_v Y) - B(T_v X, Y) - B(X, T_v Y) \]
\[ = \mathcal{L}_v B(X, Y) - T_v B(X, Y), \]

where we write

\[ T_v B(X, Y) = B(T_v X, Y) + B(X, T_v Y) \]

for the action of \( T_v \) on tensor fields. Thus, we need to determine \( T^a_v \) such that

\[ T_v \omega = \mathcal{L}_v \omega = 0, \quad T_v \alpha = \mathcal{L}_v \alpha, \quad T_v \beta = \mathcal{L}_v \beta. \]

The first equation gives

\[ 0 = T_v \omega(X, Y) = g(IT_v X, Y) + g(I X, T_v Y) = g(IT_v X, Y) - g(X, IT_v Y) = 2g(\text{Alt}(I \circ T_v)X, Y). \]

We have

\[ 2\text{Alt}(I \circ T_v) = (I \circ T_v) - (I \circ T_v)^* = I \circ T_v + T_v^* \circ I = I \circ T^a_v + T_v^* \circ I + I \circ T^a_v - T_v^* \circ I \]
\[ = \frac{1}{2} \mathcal{L}_v I - \frac{1}{2} \mathcal{L}_v I + I \circ T^a_v - T_v^a \circ I = [I, T^a_v]. \]
Thus, $\nabla_v \omega = 0$ gives condition (C.9a). $\nabla_v \alpha = 0$ leads to the equation

$$\mathcal{L}_v \alpha(X, Y) = T_v \alpha(X, Y) = g(JT_v X, Y) + g(JX, T_v Y)$$

$$= g(JT_v X, Y) - g(X, JT_v Y)$$

$$= 2g(\operatorname{Alt}(J \circ T_v)X, Y),$$

and analogously $\nabla_v \beta = 0$ gives

$$\mathcal{L}_v \beta(X, Y) = 2g(\operatorname{Alt}(K \circ T_v)X, Y).$$

Substituting formulas (C.6) and (C.7) for $\mathcal{L}_v \alpha$ and $\mathcal{L}_v \beta$ in the above equations gives

$$[J, T^a_v] = 2\operatorname{Alt}(J \circ T^a_v) = \mathcal{L}_v I \circ K + \mathcal{L}_v J - 2\operatorname{Alt}(J \circ T^s_v)$$

$$= \mathcal{L}_v I \circ K + \mathcal{L}_v J + \frac{1}{2} J \circ I \circ \mathcal{L}_v I - \frac{1}{2} \mathcal{L}_v I \circ I \circ J$$

$$= \frac{1}{2} \mathcal{L}_v I \circ K - \frac{1}{2} K \circ \mathcal{L}_v I + \mathcal{L}_v J$$

$$= \frac{1}{2}(\mathcal{L}_v J \circ K + J \circ \mathcal{L}_v K) \circ K + \frac{1}{2} K \circ (\mathcal{L}_v K \circ J + K \circ \mathcal{L}_v J) + \mathcal{L}_v J$$

$$= \frac{1}{2} J \circ \mathcal{L}_v K \circ K + \frac{1}{2} K \circ \mathcal{L}_v K \circ J$$

$$= \frac{1}{2} [J, \mathcal{L}_v K \circ K]$$

and

$$[K, T^a_v] = 2\operatorname{Alt}(K \circ T^a_v) = -\mathcal{L}_v I \circ J + \mathcal{L}_v K - 2\operatorname{Alt}(K \circ T^s_v)$$

$$= -\mathcal{L}_v I \circ J + \mathcal{L}_v K - (K \circ T^s_v + T^s_v \circ K)$$

$$= -\mathcal{L}_v I \circ J + \mathcal{L}_v K + \frac{1}{2} K \circ I \circ \mathcal{L}_v I - \frac{1}{2} \mathcal{L}_v I \circ I \circ K$$

$$= \frac{1}{2} J \circ \mathcal{L}_v I - \frac{1}{2} \mathcal{L}_v I \circ J + \mathcal{L}_v K$$

$$= \frac{1}{2} J \circ (\mathcal{L}_v J \circ K + J \circ \mathcal{L}_v K) + \frac{1}{2} (\mathcal{L}_v K \circ J + K \circ \mathcal{L}_v J) \circ J + \mathcal{L}_v K$$

$$= \frac{1}{2} J \circ \mathcal{L}_v J \circ K + \frac{1}{2} K \circ \mathcal{L}_v J \circ J$$

$$= \frac{1}{2} [K, \mathcal{L}_v J \circ J].$$

Therefore, $\nabla_v \alpha = 0$ and $\nabla_v \beta = 0$ are equivalent to (C.9b) and (C.9c), respectively. □
We now compute the right-hand sides in (C.9b) and (C.9c). The identity $\mathcal{L}_v \alpha = f \beta \mod \Omega_H^{-1}$ and (C.6) give

$$\mathcal{L}_v \alpha \sim -I \circ \mathcal{L}_v K = f K \mod \Gamma(\mathfrak{so}^{-}(H)),$$

and thus,

$$\mathcal{L}_v K \circ K = -f I \mod J \circ \Gamma(\mathfrak{so}^{-}(H)).$$

Because $I$ commutes with $\mathfrak{so}^{-}(H)$, we get $[J, \mathcal{L}_v K \circ K] = 2f K$. Analogously, $\mathcal{L}_v \beta = -f \alpha \mod \Omega_H^{-1}$ and (C.7) give

$$\mathcal{L}_v \beta \sim I \circ \mathcal{L}_v J = -f J \mod \Gamma(\mathfrak{so}^{-}(H))$$

and thus,

$$\mathcal{L}_v J \circ J = -f I \mod K \circ \Gamma(\mathfrak{so}^{-}(H)),$$

which implies $[K, \mathcal{L}_v J \circ J] = -2f J$. Thus, to determine $T^a_v$, we need to solve the system

$$[I, T^a_v] = 0, \quad (C.10a)$$

$$[J, T^a_v] = f K, \quad (C.10b)$$

$$[K, T^a_v] = -f J. \quad (C.10c)$$

The solution is $T^a_v = -\frac{1}{2}f I \mod \Gamma(\mathfrak{so}^{-}(H))$, which proves Proposition C.3.

### C.2 Structure of the curvature tensor

From now on, we fix a connection on $TM$ with the properties from Lemma 2.6 and Proposition C.3 by setting the component of $T^a_v$ in $\mathfrak{so}^{-}(H)$ equal to zero and get $T_v = \frac{1}{2} \mathcal{L}_v I \circ I - \frac{1}{2} f I$. Denote by $R_g$ and $R^\nabla$ the curvature tensors of the Levi–Civita connection of $g$ and of $\nabla$, respectively. The following lemma gives the analogue of the Gauss equations in submanifold theory, relating $R_g$ and $R^\nabla$ in terms of the ‘second fundamental form’ $B$ introduced in Lemma 2.6.

**Lemma C.11.** For $X, Y, Z, W \in \Gamma(H)$, we have

$$\pi_H(R_g(X, Y)Z) = R^\nabla(X, Y)Z + B(Y, Z)\nabla^L_X v - B(X, Z)\nabla^L_Y v$$

$$+ \omega(X, Y)\nabla^L_Z v - \omega(X, Y)T_v Z,$$

$$R_g(X, Y, Z, W) = R^\nabla(X, Y, Z, W) - B(Y, Z)B(X, W) + B(X, Z)B(Y, W)$$

$$+ \frac{f + 1}{2} \omega(X, Y)\omega(Z, W).$$

**Proof.** We compute

$$\nabla^L_X \nabla^L_Y Z = \nabla^L_X (\nabla_Y Z + B(Y, Z)v)$$

$$= \nabla_X \nabla_Y Z + B(Y, Z)\nabla^L_X v + (B(X, \nabla_Y Z) + X(B(Y, Z))v,$$
and thus,

\[ \pi_H(\nabla^\text{LC}_X \nabla^\text{LC}_Y Z) = \nabla_X \nabla_Y Z + B(Y, Z) \nabla^\text{LC}_X v. \]

Interchanging \( X \) and \( Y \) gives

\[ \pi_H(\nabla^\text{LC}_Y \nabla^\text{LC}_X Z) = \nabla_Y \nabla_X Z + B(X, Z) \nabla^\text{LC}_Y v. \]

The identity \([X, Y] = \pi_H([X, Y]) - \omega(X, Y)v\) gives

\[
\nabla^\text{LC}_{[X,Y]} Z = \nabla^\text{LC}_{\pi_H([X,Y])} Z - \omega(X, Y) \nabla^\text{LC}_v [v, Z] + \text{multiple of } v \\
= \nabla_{[X,Y]} Z - \omega(X, Y) \nabla^\text{LC}_Z v + \omega(X, Y)T_v Z + \text{multiple of } v.
\]

Thus, we get

\[
R_g(X, Y)Z = R^\nabla(X, Y)Z + B(Y, Z) \nabla^\text{LC}_X v - B(X, Z) \nabla^\text{LC}_Y v \\
+ \omega(X, Y) \nabla^\text{LC}_Z v - \omega(X, Y) T_v Z + \text{multiple of } v.
\]

Now we have

\[
g(\nabla^\text{LC}_v v, W) = -g(v, \nabla^\text{LC}_v W) = -B(X, W).
\]

Using the above equation gives

\[
R_g(X, Y, Z, W) \\
= R^\nabla(X, Y, Z, W) - B(Y, Z)B(X, W) + B(X, Z)B(Y, W) \\
- \omega(X, Y)B(Z, W) - \omega(X, Y)g(T_v Z, W) \\
= R^\nabla(X, Y, Z, W) - B(Y, Z)B(X, W) + B(X, Z)B(Y, W) \\
+ \frac{1}{2} \omega(X, Y)(L_v g(Z, W) + \omega(Z, W)) - \frac{1}{2} \omega(X, Y)L_v g(Z, W) + \frac{f}{2} \omega(X, Y)\omega(Z, W) \\
= R^\nabla(X, Y, Z, W) - B(Y, Z)B(X, W) + B(X, Z)B(Y, W) + \frac{f + 1}{2} \omega(X, Y)\omega(Z, W). \]

\[ \square \]

Lemma C.12. For \( X, Y, Z, W \in \Gamma(H) \), the curvature tensor \( R^\nabla \) of \( \nabla \) satisfies the following identities:

\[
R^\nabla(X, Y, Z, W) - R^\nabla(Z, W, X, Y) = -\frac{1}{2} L_v g(X, Z)\omega(Y, W) - \frac{1}{2} \omega(X, Z)L_v g(Y, W) \\
+ \frac{1}{2} L_v g(X, W)\omega(Y, Z) + \frac{1}{2} \omega(X, W)L_v g(Y, Z),
\]

\[
R^\nabla(X, Y)Z + R^\nabla(Y, Z)X + R^\nabla(Z, X)Y = \omega(X, Y)T_v Z + \omega(Y, Z)T_v X + \omega(Z, X)T_v Y.
\]
Proof. By the formula from Lemma C.11 and the symmetry of the standard Riemannian curvature tensor $R_g$, we have
\[ R^\nabla(X, Y, Z, W) - R^\nabla(Z, W, X, Y) \]
\[ = B(Y, Z)B(X, W) - B(X, Z)B(Y, W) - B(W, X)B(Z, Y) + B(Z, X)B(W, Y). \]
Splitting $B = B^s + B^a$ into a symmetric and alternating part, we get
\[ B(Y, Z)B(X, W) - B(Z, Y)B(W, X) \]
\[ = B^s(Y, Z)B^a(X, W) + B^a(Y, Z)B^s(X, W) \]
\[ - B^s(Z, Y)B^a(W, X) - B^a(Z, Y)B^s(W, X) \]
\[ = 2B^s(Y, Z)B^a(X, W) + 2B^a(Y, Z)B^s(X, W). \]
Interchanging $X$ and $Y$ gives
\[ B(X, Z)B(Y, W) - B(Z, X)B(W, Y) \]
\[ = 2B^s(X, Z)B^a(Y, W) + 2B^a(X, Z)B^s(Y, W). \]
Thus, we have
\[ R^\nabla(X, Y, Z, W) - R^\nabla(Z, W, X, Y) \]
\[ = 2B^s(Y, Z)B^a(X, W) + 2B^a(Y, Z)B^s(X, W) - 2B^s(X, Z)B^a(Y, W) - 2B^a(X, Z)B^s(Y, W) \]
\[ = -\frac{1}{2}\omega(X, Z)\mathcal{L}_v g(Y, W) - \frac{1}{2}\omega(Y, W)\mathcal{L}_v g(X, Z) + \frac{1}{2}\omega(X, W)\mathcal{L}_v g(Y, Z) + \frac{1}{2}\omega(Y, Z)\mathcal{L}_v g(X, W). \]
This gives the first identity. To explain the second identity, denote by $\mathfrak{S}_{X, Y, Z}$ the cyclic sum in $X, Y, Z$. We have the general formula [1, Theorem 1.24]
\[ \mathfrak{S}_{X, Y, Z} R^\nabla(X, Y)Z = \mathfrak{S}_{X, Y, Z}(T(T(X, Y), Z) + (\nabla_X T)(Y, Z)). \]
We have
\[ T(T(X, Y), Z) = \omega(X, Y)T_v Z \]
and
\[ (\nabla_X T)(Y, Z) = \nabla_X (T(Y, Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z) \]
\[ = \nabla_X (\omega(Y, Z)v) - \omega(\nabla_X Y, Z)v - \omega(Y, \nabla_X Z)v \]
\[ = ((\nabla_X \omega)(Y, Z))v = 0. \]
This gives the second identity. \qed

Denote by $R^H$ the restriction of $R^\nabla$ to $H$. $R^H$ is a section of
\[ \Lambda^2_H \otimes \Lambda^2_H = S^2(\Lambda^2_H) \oplus \Lambda^2(\Lambda^2_H), \]
and thus decomposes as

\[ R^H = \pi_s(R^H) + \pi_a(R^H) \]

into a symmetric and alternating part. To further discuss the structure of \( R^H \), we need the Kulkarni–Nomizu product. For \( h, k \in H^* \otimes H^* \), define \( h \otimes k \in \Lambda^2_H \otimes \Lambda^2_H \) by

\[ (h \otimes k)(X, Y, Z, W) = h(X, W)k(Y, Z) + h(Y, Z)k(X, W) - h(X, Z)k(Y, W) - h(Y, W)k(X, Z). \]

The Kulkarni–Nomizu product allows to interpret the first identity in Lemma C.12 as

\[ \pi_a(R^H) = \frac{1}{4} \omega \otimes \mathcal{L}_v g. \] (C.13)

For \( R \in \Lambda^2_H \otimes \Lambda^2_H \), define the Ricci contraction \( c(R) \) to be

\[ c(R)(X, Y) = \text{tr}(R(\cdot, X, Y, \cdot)) = \sum_{i=1}^4 R(e_i, X, Y, e_i), \]

and the Bianchi map \( b \) by

\[ b(R)(X, Y, Z, W) = \frac{1}{3}(R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W)). \]

For the Bianchi map restricted to \( S^2(\Lambda^2_H) \), we have \( \text{im} \, b = \Lambda^4_H \), which gives the decomposition

\[ S^2(\Lambda^2_H) = CH \oplus \Lambda^4_H, \]

where \( CH := \ker b|_{S^2(\Lambda^2_H)} \) is the space of algebraic curvature tensors on \( H \). Thus, we have the decomposition

\[ \Lambda^2_H \otimes \Lambda^2_H = CH \oplus \Lambda^4_H \oplus \Lambda^2(\Lambda^2_H), \]

and because \( b \) is an idempotent, \( R^H \) decomposes as

\[ R^H = C^H + b(\pi_s(R^H)) + \pi_a(R^H), \] (C.14)

where

\[ C^H = \pi_s(R^H) - b(\pi_s(R^H)). \]

A key property of the Kulkarni–Nomizu product is that for \( h \in S^2H^* \), we have \( b(h \otimes g_H) = 0 \) and that the resulting map \( \cdot \otimes g_H : S^2H^* \to CH \) is the adjoint of \( 4c \). Together with the formula \( c(h \otimes g_H) = 2h + \text{tr}(h)g_H \), this leads to the decomposition

\[ C^H = W^H + \frac{1}{2} r^H \otimes g_H - \frac{1}{12} S_H (g_H \otimes g_H), \]

where we call the trace-less part \( W^H \) the Weyl tensor of \( R^H \), and write \( r^H := c(C^H) \) and \( S_H := \text{tr}(c(C^H)) \), the contact analogues of Ricci and scalar curvature.
C.3 Computation of Weitzenböck remainder

In this section, we derive a Weitzenböck identity for the Laplacian $\Delta_H$ acting on $\Omega^-_H$, which is related by $\Delta_H = 2\Box_H$ to the operator $\Box_H$ defined in (2.5). Denote the component of $W^H$ in $\Lambda^-_H \otimes \Lambda^-_H$ by $(W^H)^-$. We will prove the following.

**Proposition C.15.** We have

$$2\Box_H = \Delta_H|_{\Omega^-_H} = \nabla^*_H \nabla_H + (W^H)^-_{klmn} \varepsilon^k l e^m n + \frac{S_H}{3} - \frac{2}{3} f. \quad \text{(C.16)}$$

We first derive a general formula for the curvature term in Equation (2.9).

**Lemma C.17.** $R \in \Lambda^2_H \otimes \Lambda^2_H$ acts on 2-forms as

$$R_{klmn} \varepsilon^k l e^m n e^{ab} = c(R)_{ka} e^{kb} + c(R)_{kb} e^{ak} - R_{balm} e^{km} + 3b(R)_{balm} e^{km}.$$

**Proof.** We compute

$$R_{klmn} \varepsilon^k l e^m n e^{ab}$$

$$= R_{klma} e^{kb} + R_{klmb} e^{ak}$$

$$= R_{klla} e^{kb} - R_{kmba} e^{km} + R_{kamba} e^{km} - R_{klbb} e^{ka}$$

$$= c(R)_{ka} e^{kb} - c(R)_{kb} e^{ak} + (R_{kbam} + R_{akbm}) e^{km}$$

$$= c(R)_{ka} e^{kb} + c(R)_{kb} e^{ak} - R_{balm} e^{km} + 3b(R)_{balm} e^{km}. \quad \square$$

The Kulkarni–Nomizu product with $\omega$ behaves differently than that with $g_H$.

**Lemma C.18.** Let $h \in S^2 H^*$. Then, we have

$$b(\omega \otimes h)(X, Y, Z, W) = 2b(\omega \otimes h)(X, Y, Z, W) = \frac{2}{3} \Theta_{X,Y,Z}(\omega(X, Y)h(Z, W)),$$

and

$$3b(\omega \otimes h)_{balm} e^{km} = (\omega \otimes h)_{balm} e^{km}.$$

**Proof.** To prove the first formula, we calculate

$$3b(\omega \otimes h)(X, Y, Z, W)$$

$$= (\omega \otimes h)(X, Y, Z, W) + (\omega \otimes h)(Y, Z, X, W) + (\omega \otimes h)(Z, X, Y, W)$$

$$= \omega(X, W)h(Y, Z) + \omega(Y, Z)h(X, W) - \omega(X, Z)h(Y, W) - \omega(Y, W)h(X, Z)$$

$$+ \omega(Y, W)h(Z, X) + \omega(Z, X)h(Y, W) - \omega(Y, X)h(Z, W) - \omega(Z, W)h(Y, X)$$

$$+ \omega(Z, W)h(X, Y) + \omega(X, Y)h(Z, W) - \omega(Z, Y)h(X, W) - \omega(X, W)h(Z, Y)$$

$$= 2\{\omega(X, Y)h(Z, W) + \omega(Y, Z)h(X, W) + \omega(Z, Y)h(X, W)\}.$$
For the second formula we have
\[
(\omega \otimes h)_{b a k m} e^{k m} = (\omega_{b m} h_{a k} + \omega_{a k} h_{b m} - \omega_{b k} h_{a m} - \omega_{a m} h_{b k}) e^{k m} = 2(\omega_{b m} h_{a k} + \omega_{a k} h_{b m}) e^{k m}
\]
and
\[
3b(\omega \otimes h)_{b a k m} e^{k m} = (\omega_{b a} h_{k m} + \omega_{a k} h_{b m} + \omega_{k b} h_{a m}) e^{k m} = (\omega_{a k} h_{b m} + \omega_{b m} h_{a k}) e^{k m}.
\]
Thus proving
\[
3b(\omega \otimes h)_{b a k m} e^{k m} = \frac{1}{2}(\omega \otimes h)_{b a k m} e^{k m}.
\]
The statement follows from the first formula.

We will now compute the contribution to the Weitzenböck identity of each component in the decomposition (C.14).

**Contribution of $C^H$**

Here, the algebra is the same as on a four-dimensional Riemannian manifold. The action of $W^H$ preserves the decomposition $\Lambda^2_H = \Lambda_+^H \oplus \Lambda_-^H$. Denote the component of $W^H$ in $\Lambda_-^H \otimes \Lambda_-^H$ by $(W^H)^-$. Then, the total contribution of $C^H$ is
\[
(W^H)^-_klm n \varepsilon^k l \varepsilon^m t^n + \frac{S_H}{3}.
\]

**Contribution of $b(\pi_s(R^H))$**

Because
\[
T_v = \frac{1}{2} L_v \omega I - \frac{f}{2} I
\]
and $L_v g \sim L_v \omega I$, by Lemma C.12, we have

\[
b(R^H)(X, Y, Z, W) = \frac{1}{6} \Theta_{X, Y, Z}(\omega(X, Y)L_v g(Z, W)) - \frac{f}{6} \Theta_{X, Y, Z}(\omega(X, Y)\omega(Z, W)),
\]
or
\[
b(R^H) = \frac{1}{2} b(\omega \otimes L_v g) - \frac{f}{2} b(\omega \otimes \omega).
\]

Therefore, by (C.13) and Lemma C.18,
\[
b(\pi_s(R^H)) = b(R^H) - b(\pi_s(R^H)) = \frac{1}{2} b(\omega \otimes L_v g) - \frac{f}{2} b(\omega \otimes \omega) - \frac{1}{4} b(\omega \otimes L_v g)
\]
\[
= -\frac{f}{2} b(\omega \otimes \omega).
\]
Because $b$ is idempotent and the contraction is zero on $b(S^2\Lambda^2) = \Lambda^4$, by Lemma C.17, $b(\pi_s(R_H))$ contributes

$$2b(\pi_s(R_H))_{bkm}e^{km}$$

to the Weitzenböck formula.

We have

$$3b(\omega \otimes \omega)_{bkm}e^{km} = \omega_{ba}\omega_{km}e^{km} + \omega_{ak}\omega_{bm}e^{km} + \omega_{kb}\omega_{am}e^{km}$$

$$= -g_H(e^a,\omega)e^b + (Ie^a) \wedge (Ie^b) - (Ie^b) \wedge (Ie^a) = -g_H(e^a,\omega)e^b + 2(I \wedge I)e^{ab}.$$

**Lemma C.20.** $I \wedge I$ acts as the identity on $\Lambda^2_\omega$.

**Proof.**

$$(I \wedge I)\omega_1^- = (I \wedge I)(e^0 \wedge e^1 - e^2 \wedge e^3) = e^1 \wedge (-e^0) - e^3 \wedge (-e^2) = e^0 \wedge e^1 - e^2 \wedge e^3 = \omega_1^-,$$

$$(I \wedge I)\omega_2^- = (I \wedge I)(e^0 \wedge e^2 - e^3 \wedge e^1) = e^1 \wedge e^3 - (-e^0) \wedge (-e^2) = e^0 \wedge e^2 - e^3 \wedge e^1 = \omega_2^-,$$

$$(I \wedge I)\omega_3^- = (I \wedge I)(e^0 \wedge e^3 - e^1 \wedge e^2) = e^1 \wedge (-e^2) - (-e^0) \wedge e^3 = e^0 \wedge e^3 - e^1 \wedge e^2 = \omega_3^-.$$ 

Therefore, $b(\pi_s(R_H))$ contributes

$$-\frac{2}{3}f$$

to the Weitzenböck formula.

**Contribution of $\pi_a(R_H)$**

**Lemma C.22.** We have

$$c(\omega \otimes L_vg) = \text{tr}(L_vg)\omega.$$

**Proof.** We have

$$\sum_i \omega(X,e_i)e_i = \sum_i g(IX,e_i)e_i = IX,$$

and thus, for $h \in S^2$,

$$c(\omega \otimes h)(X,Y) = \sum_i (\omega \otimes h)(e_i,X,Y,e_i)$$

$$= \sum_i \omega(e_i,e_i)h(X,Y) + \omega(X,Y)h(e_i,e_i) - \omega(e_i,Y)h(X,e_i) - \omega(X,e_i)h(e_i,Y)$$

$$= \text{tr}(h)\omega(X,Y) + h(X,MY) - h(IX,Y).$$

By (C.5), for $h = L_vg$, we have $h(X,MY) - h(IX,Y) = 0.$
Thus, the contribution of the Ricci term to the Weitzenböck formula is

\[ c(\pi_a(R_H))_{ka}e^{kb} + c(\pi_a(R_H))_{kb}e^{ak} = \frac{1}{4} \text{tr}(\mathcal{L}_v g)(\omega_{ka}e^{kb} + \omega_{kb}e^{ak}) = \frac{1}{4} \text{tr}(\mathcal{L}_v g)I(e^{ab}). \]

But \( I \) acts trivially on \( \Lambda^-_H \). Thus, the contribution is

\[ -\frac{1}{4}(\omega \otimes \mathcal{L}_v g)_{baks}e^{km} + \frac{3}{4}b(\omega \otimes \mathcal{L}_v g)_{baks}e^{km} = 0 \quad (C.23) \]

by Lemma C.18(ii). Therefore, the total contribution of \( \pi_a(R_H) \) is zero.

Adding the contributions (C.19), (C.21) and (C.23) gives the identity (C.16).