Inverse medium scattering problems with Kalman filter techniques

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Abstract
We study the inverse medium scattering problem to reconstruct the unknown inhomogeneous medium from the far field patterns of scattered waves. The inverse scattering problem is generally ill-posed and nonlinear, and the iterative optimization method is often adapted. A natural iterative approach to this problem is to place all available measurements and mappings into one long vector and mapping, respectively, and to iteratively solve the linearized large system equation using the Tikhonov regularization method, which is called Levenberg–Marquardt scheme. However, this is computationally expensive because we must construct the larger system equations when the number of available measurements is increasing. In this paper, we propose two reconstruction algorithms based on the Kalman filter. One is the algorithm equivalent to the Levenberg–Marquardt scheme, and the other is inspired by the extended Kalman filter. For the algorithm derivation, we iteratively apply the Kalman filter to the linearized equation for our nonlinear equation. By applying the Kalman filter, our proposed algorithms sequentially update the state and the weight of the norm for the state space, which avoids the construction of large system equation, and retains the information of past updates. Finally, we provide numerical examples to demonstrate the proposed algorithm.

Keywords: inverse acoustic scattering, inhomogeneous medium, far field pattern, Tikhonov regularization, Kalman filter, Levenberg–Marquardt, extended Kalman filter

(Some figures may appear in colour only in the online journal)

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1. Introduction

The inverse scattering problem is used to determine unknown scatterers by measuring scattered waves generated by sending incident waves far away from the scatterers. This is important in many applications, such as in medical imaging, nondestructive testing, remote exploration, and geophysical prospecting. Owing to its many applications, the inverse scattering problem has been studied from various perspectives. For further readings, we refer to the following books [9, 10, 13, 27, 34], which include reviews of classical and recent developments of the inverse scattering problem.

We begin with the mathematical formulation of the scattering problem. Let \( k > 0 \) be the wave number and \( \theta \in S^1 \) be the incident direction. We denote the incident field \( u^{inc}(\cdot, \theta) \) with direction \( \theta \) by the plane wave of the form

\[
u^{inc}(x, \theta) := e^{ikx \cdot \theta}, \quad x \in \mathbb{R}^2.
\]

Let \( Q \subset \mathbb{R}^2 \) be a bounded open set with the smooth boundary and its exterior \( \mathbb{R}^2 \setminus Q \) be connected. We assume that \( q \in L^\infty(\mathbb{R}^2) \), which refers to the inhomogeneous medium, satisfies \( \text{Re}(1 + q) > 0, \text{Im} q \geq 0 \), and its support \( \text{supp} q \) is embedded in \( Q \), that is, \( \text{supp} q \subseteq Q \). Then, the direct scattering problem concerns determining the total field \( u = u^{sca} + u^{inc} \) such that

\[
\Delta u + k^2(1 + q)u = 0 \quad \text{in} \quad \mathbb{R}^2,
\]

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^{sca}}{\partial r} - iku^{sca} \right) = 0,
\]

where \( r = |x| \). The Sommerfeld radiation condition (1.3) holds uniformly in all directions \( \hat{x} := \frac{x}{|x|} \). Furthermore, the problem (1.2)–(1.3) is equivalent to the Lippmann–Schwinger integral equation

\[
u(x, \theta) = u^{inc}(x, \theta) + k^2 \int_Q q(y)u(y, \theta)\Phi(x, y)dy,
\]

where \( \Phi(x, y) \) denotes the fundamental solution to the Helmholtz equation in \( \mathbb{R}^2 \), that is,

\[
\Phi(x, y) := \frac{i}{4}H^{(1)}_{0}(k|x - y|), \quad x \neq y,
\]

where \( H^{(1)}_{0} \) is the Hankel function of the first kind of order one. It is well known that there exists a unique solution \( u^{sca} \) of the problem (1.2) and (1.3), and it has the following asymptotic behaviour,

\[
u^{sca}(x, \theta) = e^{ikx \cdot \theta} \sqrt{\frac{r}{8\pi k}} \left\{ u^\infty(\hat{x}, \theta) + O(1/r) \right\}, \quad r \to \infty, \quad \hat{x} := \frac{x}{|x|}.
\]

The function \( u^\infty \) is called the far field pattern of \( u^{sca} \), and it has the form

\[
u^\infty(\hat{x}, \theta) = \frac{k^2e^{i\hat{x} \cdot \theta}}{8\pi k} \int_Q e^{-ik\hat{x} \cdot y}u(y, \theta)q(y)dy =: F_\theta q(\hat{x}),
\]

where the far field mapping \( F_\theta : L^2(Q) \to L^2(S^1) \) is defined in the second equality for each incident direction \( \theta \in S^1 \). For more details on these direct scattering problems, see chapter 8 of [13].
We consider the inverse scattering problem to reconstruct function $q$ from the far field pattern $u^\infty(\hat{x}, \theta)$ for all directions $\hat{x} \in S^1$ and several directions $\{\theta_n\}_{n=1}^N \subset S^1$ with some large $N \in \mathbb{N}$, and one fixed wave number $k > 0$. It is well known that function $q$ is uniquely determined from the far field pattern $u^\infty(\hat{x}, \theta)$ for all $\hat{x}, \theta \in S^1$ and one fixed $k > 0$ (see, e.g., [8, 35, 37]). However, the uniqueness for several incident plane waves is an open question. For the impenetrable obstacle scattering case, if we assume that the shape of the scatterer is a polyhedron or ball, the uniqueness for a single incident plane wave is proved (see [2, 11, 32, 33]). Recently, in [1], they showed the Lipschitz stability for the inverse medium scattering with finite measurements $\{u^\infty(\hat{x}_i, \theta_j)\}_{i,j=1}^N$ for large $N \in \mathbb{N}$ under the assumption that the true function belongs to a compact and convex subset of finite-dimensional subspace. From these results, we expect that a large number of measurements $N$ is necessary to reconstruct the general shape of the scatterer.

Our inverse problem (1.7) is not only ill-posed, but also nonlinear. Existing methods for solving the nonlinear inverse problem can be roughly categorized into two groups: iterative optimization and qualitative. Iterative optimization methods (see, e.g., [4, 13, 15, 20, 26]) do not require many measurements, but an initial guess, which is the starting point of the iteration, is required. It must be appropriately chosen by a priori knowledge of the unknown function $q$, otherwise, the iterative solution cannot converge to the true function. Moreover, the qualitative methods such as the linear sampling method [12], no-response test [21], probe method [22], factorization method [29], and singular source method [36], do not require an initial guess, and they are computationally faster than iterative methods. However, the disadvantage of the qualitative method is that it requires an uncountable number measurements. For a survey of qualitative methods, we refer to [34]. Recently, in [23, 31], they suggested a reconstruction method from a single incident plane wave, although rigorous justifications are lacking. Another approach is the Born approximation, which approximates the total field $u$ in (1.7) with the incident field $u^\text{inc}$, and the nonlinear equation is then transformed into the linear equation. Such an approximation is appropriate when the wavenumber $k > 0$ and value of $q$ are small (see the second term in the right-hand side of (1.4)). For further readings on the Born approximation, see [4, 6, 13, 28, 38]. Nowadays, data driven approaches based on deep learning are being developed, we refer to [3] for their survey.

Here, we focus on the iterative optimization method. It is basically based on the Newton method (see, e.g., [14, 25, 34]), which is a classical method for constructing an iterative solution by the first-order linearization. A natural iterative approach to our problem is to place all available measurements $\{u^\infty(\cdot, \theta_n)\}_{n=1}^N$ and far field mappings $\{F_{\theta_n}\}_{n=1}^N$ into one large vector $\vec{u}^\infty$ and mapping $\vec{F}$, respectively, and to iterative solve the linearized large system for $\vec{u}^\infty = \vec{F}q$ by the Tikhonov regularization method (see, e.g., [9, 19, 30, 34]), which is called Levenberg–Marquardt scheme (see, e.g., [18, 26]). However, this is computationally expensive because we must construct the larger vector $\vec{u}^\infty$ and mapping $\vec{F}$ when the number of measurements $N$ is increasing.

In this paper, we propose two reconstruction algorithms based on the Kalman filter without using $\vec{u}^\infty$ and $\vec{F}$. The contributions of this paper are the following.

(A) We propose a reconstruction algorithm equivalent to the Levenberg–Marquardt scheme (see (3.23)–(3.25) and theorem 3.1).

(B) We propose a reconstruction algorithm inspired by the extended Kalman filter (EKF) (see (4.22)–(4.24)).

The Kalman filter (see, e.g., [14, 25, 34]) is an algorithm that estimates the unknown state in a linear dynamical system. It sequentially updates the state and the weight of the norm for
the state space, which avoids constructing the large system equation, and retains the information of past updates. For the algorithm derivation, we iteratively apply the Kalman filter to the linearized equation for our nonlinear equation (1.7). We call algorithm (A) the *Kalman filter Levenberg–Marquardt* (KFL) and algorithm (B) the *iterative EKF*. KFL is based on the linearization at the initial state for each iteration step, whereas EKF is based on the linearization at the current state for every iteration step, implying that the update of KFL is slower than that of EKF (see remark 4.1). For both algorithms, we can select either the initialization or update of the weight of the norm for each iteration step depending on the situation (see remark 4.2), which is a beneficial result from viewing the iterative algorithm as a Kalman filter. We provide numerical examples to demonstrate the proposed algorithms and observed that, in both noise-free and noise cases, the error of EKF decreases faster than that of KFL if the regularization parameter is chosen appropriately (see figures 4–7). However, EKF is more sensitive to the regularization parameter and noise than KFL (see figures 8 and 9). We also observed that, in both algorithms, if we select the update of the weight in (4.27), the algorithms become more robust to the regularization parameter and noise, although the error slowly decreases (see figures 8 and 9).

The remainder of this paper is organized as follows. In section 2, we briefly review the far field mapping and Kalman filter. In section 3, we propose the KFL reconstruction algorithm. In section 4, we propose the EKF reconstruction algorithm. Finally, in section 5, we provide numerical examples to demonstrate the algorithms.

2. Preliminary

2.1. Far field mapping

In this section, we briefly review the Fréchet derivative of the far field mapping, the Lipschitz stability, and Levenberg–Marquardt scheme for inverse scattering. We redefine the far field mapping

\[ F_\theta : L^2(Q) \rightarrow L^2(S^1) \]

by

\[ F_\theta q(\hat{x}) = \frac{k^2}{8\pi} \int_Q e^{-ik\hat{x} \cdot y} u_q(y, \theta) q(y) dy, \quad \hat{x} \in S^1, \quad (2.1) \]

where the total field \( u_q(\cdot, \theta) \) is given by solving the integral equation of (1.4). In addition, we denote

\[ L^\infty_+ (Q) := \{ q \in L^\infty(Q) : \exists q_0 > 0, \text{Im } q \geq q_0 \text{ a.e on } Q \} \subset L^2(Q). \]

The following lemma is proved by the same argument in section 2 of [5].

**Lemma 2.1.**

(a) \( F_\theta \in C^1(L^\infty_+(Q), L^2(S^1)) \), that is, for any \( q \in L^\infty_+(Q) \), \( F_\theta \) is Fréchet differentiable at \( q \), and by denoting the Fréchet derivative by \( F'_\theta[q] : L^2(Q) \rightarrow L^2(S^1) \), the mapping \( q \in L^\infty_+(Q) \mapsto F'_\theta[q] \in \mathcal{L}(L^2(Q), L^2(S^1)) \) is continuous, and its derivative \( F''_\theta[q] \) at \( q \) is given by

\[ F''_\theta[q][m] = v_{q,m}^\infty, \quad (2.2) \]

where \( v_{q,m}^\infty \) is the far field pattern of the radiating solution \( v = v_{q,m} \) such that

\[ \Delta v + k^2(1 + q)v = -k^2 mu_q(\cdot, \theta) \text{ in } \mathbb{R}^2. \quad (2.3) \]

(b) \( F'_\theta[\cdot] \) is locally bounded.
We denote the far field mappings for all incident directions $\mathcal{F} : L^2_+(Q) \subset L^2(Q) \to L^2(S^1 \times S^1)$ by $\mathcal{F}q(x, \theta) := \mathcal{F}_{\theta}q(x)$. The following lemma is proved by the same argument as in section 11 of [13] and section 2 of [5].

**Lemma 2.2.**

(a) $\mathcal{F} : \{q \in L^\infty(Q) : \text{Im } q \geq 0 \text{ a.e. on } Q \} \to L^2(S^1 \times S^1)$ is injective.

(b) $\mathcal{F} \in C^1(L^2_+(Q), L^2(S^1 \times S^1))$, and its derivative $\mathcal{F}'[q] : L^2(Q) \to L^2(S^1 \times S^1)$ at $q$ is injective.

By applying theorem 2.1 of [7] with lemma 2.2, we obtain the following Lipschitz stability.

**Lemma 2.3.** Let $W$ be a finite dimensional subspace of $L^2(Q)$ and $K$ be a compact and convex subset of $W \cap L^\infty_+(Q)$. Then, there exists a constant $C > 0$ such that

$$
|q_1 - q_2|_{L^2_+(Q)} \leq C\|\mathcal{F}q_1 - \mathcal{F}q_2\|_{L^2(S^1 \times S^1)}, \quad q_1, q_2 \in K.
$$

Let $\{\theta_i : i \in \mathbb{N}\}$ be dense in $S^1$. We denote $\mathcal{F}_N : L^2(Q) \to L^\infty(S^1)^N$ by

$$
\mathcal{F}_N(q) := \begin{pmatrix}
\mathcal{F}_{\theta_1}q \\
\vdots \\
\mathcal{F}_{\theta_N}q
\end{pmatrix}.
$$

Although lemma 2.3 showed the Lipschitz stability with infinite dimensional measurements, by applying theorem 7 of [1], we have the Lipschitz stability with finitely many measurements.

**Lemma 2.4.** Let $W$ be a finite dimensional subspace of $L^2(Q)$ and $K$ be a compact and convex subset of $W \cap L^\infty_+(Q)$. Then, for large $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that

$$
|q_1 - q_2|_{L^2_+(Q)} \leq C_N\|\mathcal{F}_Nq_1 - \mathcal{F}_Nq_2\|_{L^2(S^1)^N}, \quad q_1, q_2 \in K.
$$

Finally, we recall the Levenberg–Marquardt scheme for our problem as follows:

$$
q_{i+1} = q_i + \left(\alpha_i I + \mathcal{F}_N[q_i]\mathcal{F}_N[q_i]^*\right)^{-1}\mathcal{F}_N[q_i]\mathcal{F}_N[q_i]^*\left(\hat{u}^\infty - \mathcal{F}_N(q_i)\right),
$$

for $i \in \mathbb{N}_0$ (see, e.g., [18, 26]). We remark that the regularization parameter $\alpha_i > 0$ in (2.7) must be chosen appropriately. One of the choices of $\alpha_i$ is based on the Morozov discrepancy principle, that is, $\alpha_i > 0$ is chosen such that it satisfies

$$
\|\hat{u}^\infty - \mathcal{F}_N(q_i) - \mathcal{F}_N[q_i]\{q_{i+1}(\alpha_i) - q_i\}\| = \rho\|\hat{u}^\infty - \mathcal{F}_N(q_i)\|,
$$

for some fixed $\rho \in (0, 1)$, where $q_{i+1}(\alpha_i)$ is defined by (2.7). By applying theorem 4.2 of [26] with lemma 2.4, we obtain the following convergence for (2.7).

**Lemma 2.5.** Let $W$ be a finite dimensional subspace of $L^2(Q)$ and $0 < r < 1$ be small such that $B_r(q_0) \subset W \cap L^\infty_+(Q)$. Let $N \in \mathbb{N}$ be large such that (2.6) holds, let $0 < \rho < 1$, and let $q_i \in B_r(q_0)$ be a solution of $\hat{u}^\infty = \mathcal{F}_N(q_i)$. Then, there exists a constant $C_N > 0$ such that if $\|q_0 - q^i\| < \rho/C_N$, then the Levenberg–Marquardt iteration $q_i$ in (2.7) with the $\{\alpha_i\}_{i \in \mathbb{N}_0}$ defined in (2.8) converges to the solution $q^i$ as $i \to \infty$. 


2.2. Kalman filter

In this section, we review the Kalman filter in a general functional analytic setting. Let $X$ and $Y$ be Hilbert spaces over complex variables $\mathbb{C}$, $f_n \in Y$ ($n = 1, \ldots, N$) be a measurement, and $A_n : X \to Y$ ($n = 1, \ldots, N$) be a linear operator from $X$ to $Y$. We consider the problem of determining $\varphi \in X$ such that

$$A_n \varphi = f_n,$$

for all $n = 1, \ldots, N$. Now, we assume that we have the initial guess $\varphi_0 \in X$, which is the starting point of the algorithm, and that it was appropriately determined by a priori information of the true solution $\varphi_{\text{true}}$. Then, we consider the minimization problem of the following functional

$$J_{\text{Full},N}(\varphi) := \alpha \| \varphi - \varphi_0 \|_X^2 + \| f - \bar{A} \varphi \|_{Y^{\mathbb{R}^{-1}}}^2,$$

where $\bar{A} := \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix}$ and $\bar{f} := \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$. The norm $\| \cdot \|_{Y^{\mathbb{R}^{-1}}}^2 := \langle \cdot, R^{-1} \cdot \rangle_Y$ is a weighted norm with a positive definite symmetric invertible operator $R : Y \to Y$. The minimizer of (2.10) is given by

$$\varphi_{\text{FT}}^N := \varphi_0 + \left( \alpha I + \bar{A}^* \bar{A} \right)^{-1} \bar{A}^* \left( \bar{f} - \bar{A} \varphi_0 \right).$$

We call this the full data Tikhonov. Here, $\bar{A}^*$ is the adjoint operator with respect to usual scalar products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_{Y^{\mathbb{R}^{-1}}}$. We calculate

$$\langle \bar{f}, \bar{A} \varphi \rangle_{Y^{\mathbb{R}^{-1}}} = \sum_{n=1}^N \langle f_n, R^{-1} A_n \varphi \rangle_Y$$

$$= \sum_{n=1}^N \langle \bar{A}_n^H R^{-1} f_n, \varphi \rangle_X = \langle \bar{A}^H R^{-1} \bar{f}, \varphi \rangle_X,$$

which implies that

$$\bar{A}^* = \bar{A}^H R^{-1},$$

where $\bar{A}_n^H$ and $\bar{A}^H$ are the adjoint operator with respect to usual scalar products $\langle \cdot, \cdot \rangle_X$, $\langle \cdot, \cdot \rangle_Y$ and $\langle \cdot, \cdot \rangle_X$, $\langle \cdot, \cdot \rangle_{Y^{\mathbb{R}^{-1}}}$, respectively. Then, the full data Tikhonov solution in (2.11) is of the form

$$\varphi_{\text{FT}}^N = \varphi_0 + \left( \alpha I + \bar{A}^H R^{-1} \bar{A} \right)^{-1} \bar{A}^H R^{-1} \left( \bar{f} - \bar{A} \varphi_0 \right).$$

However, algorithm (2.14) of the full data Tikhonov is computationally expensive because we must construct the larger vector $\bar{f}$ and large operator $\bar{A}$ when the number of measurements $N$ is increasing. Accordingly, we consider the alternative approach based on the Kalman filter (see, e.g., [16, 17, 24]), which is the algorithm given by the following algorithm:
\( \phi_{KF}^{n} := \phi_{n-1}^{KF} + K_{n} \left( f_{n} - A_{n} \phi_{n-1}^{KF} \right), \) \hfill (2.15)

\( K_{n} := B_{n-1} A_{n}^{H} \left( R + A_{n} B_{n-1} A_{n}^{H} \right)^{-1}, \) \hfill (2.16)

\( B_{n} := (I - K_{n} A_{n}) B_{n-1}, \) \hfill (2.17)

for \( n = 1, \ldots, N, \) where \( \phi_{0}^{KF} := \phi_{0} \) and \( B_{0} := \frac{1}{\alpha} I. \) Here, \( \phi_{KF}^{n} \) is the unique minimizer of the following functional (see section 7 of [14] and section 5 of [34]):

\[
J_{KF,n}(\varphi) := \| \varphi - \phi_{n-1}^{KF} \|_{X_{B_{n-1}}^{-1}}^{2} + \| f_{n} - A_{n} \varphi \|_{Y_{R^{-1}}}^{2}. \tag{2.18}
\]

We observe that the Kalman filter algorithm updates state \( \varphi \) every \( n \) with measurement \( f_{n} \) and one operator \( A_{n} \), and it does not require large vectors or operators. Instead, it updates both the state \( \varphi \) in (2.15) and weight \( B \) of the norm in (2.17), which plays the role of retaining the information of past updates. By the same argument in theorem 5.4.7 of [34], we can prove the equivalence of the full data Tikhonov and Kalman filter when all observation operators \( A_{n} \) are linear.

**Lemma 2.6.** For measurements \( f_{1}, \ldots, f_{N} \), linear operators \( A_{1}, \ldots, A_{N} \), and the initial guess \( \varphi_{0} \in X \), the final state of the Kalman filter given by (2.15)–(2.17) is equivalent to the state of the full data Tikhonov given by (2.14), that is,

\[
\phi_{KF}^{N} = \phi_{FT}^{N}. \tag{2.19}
\]

### 3. Kalman filter Levenberg–Marquardt

In this section, we propose a reconstruction algorithm based on the Kalman filter that is equivalent to the Levenberg–Marquardt algorithm. We solve the following problem with respect to \( q \):

\[
\mathcal{F}_{0_{n}} q = u_{n}^{\infty}, \quad n = 1, \ldots, N. \tag{3.1}
\]

It is convenient to employ the vector notation as follows:

\[
\tilde{\mathcal{F}} q = \bar{u}^{\infty}, \tag{3.2}
\]

where \( \tilde{\mathcal{F}} q := \begin{pmatrix} \mathcal{F}_{\theta_{1}} q \\ \vdots \\ \mathcal{F}_{\theta_{n}} q \end{pmatrix} \) and \( \bar{u}^{\infty} := \begin{pmatrix} u_{\theta_{1}}^{\infty} \\ \vdots \\ u_{\theta_{n}}^{\infty} \end{pmatrix}. \)

First, we review the derivation of Levenberg–Marquardt scheme (2.7) as follows. We assume we have an initial guess \( q_{0} \) and consider the Taylor expansion at \( q = q_{0} \)

\[
\tilde{\mathcal{F}} q = \tilde{\mathcal{F}} q_{0} + \tilde{\mathcal{F}}'[q_{0}](q - q_{0}) + r(q - q_{0}). \tag{3.3}
\]

We forget the high order term \( r(q - q_{0}) \) and solve the linearized problem for (3.2):

\[
\tilde{\mathcal{F}}'[q_{0}] q = \bar{u}^{\infty} - \tilde{\mathcal{F}} q_{0} + \tilde{\mathcal{F}}'[q_{0}] q_{0}, \tag{3.4}
\]
where \( \tilde{F}^* [q_0] q \) is given by
\[
\begin{pmatrix}
    \tilde{F}_0^* [q_0] q \\
    \vdots \\
    \tilde{F}_N^* [q_0] q
\end{pmatrix}.
\]
Then, the Tikhonov regularization solution is given by
\[
q_1 := q_0 + \left( \alpha I + \tilde{F}^* [q_1] \tilde{F}^* [q_0] \right)^{-1} \tilde{F}^* [q_0] \left( \tilde{u}^\infty - \tilde{F} q_0 \right),
\]
(3.5)
where \( \alpha > 0 \) is a regularization parameter.

Next, we solve the linearized problem for (3.2) with initial guess \( q_1 \):
\[
\tilde{F}^* [q_1] q = \tilde{u}^\infty - \tilde{F} q_1 + \tilde{F}^* [q_1] q_1.
\]
(3.6)
Then, the Tikhonov regularization solution is given by
\[
q_2 := q_1 + \left( \alpha I + \tilde{F}^* [q_1] \tilde{F}^* [q_1] \right)^{-1} \tilde{F}^* [q_1] \left( \tilde{u}^\infty - \tilde{F} q_1 \right),
\]
(3.7)
where \( \alpha > 0 \) is a regularization parameter. Repeating the above arguments (3.3)–(3.7), we have the iteration scheme for \( i \in \mathbb{N}_0 \):
\[
q_{i+1}^{FLM} := q_i^{FLM} + \left( \alpha I + \tilde{F}^* [q_i^{FLM}] \tilde{F}^* [q_i^{FLM}] \right)^{-1} \tilde{F}^* [q_i^{FLM}] \left( \tilde{u}^\infty - \tilde{F} q_i^{FLM} \right),
\]
(3.8)
where \( \{ q_i \}_{i \in \mathbb{N}_0} \) is a sequence of regularization parameters. We call this the full data Levenberg–Marquardt (FLM). Here, \( \tilde{F}^* [q_i^{FLM}]^* \) is an adjoint operator of \( \tilde{F}^* [q_i^{FLM}] \) with respect to the usual scalar product \( \langle \cdot , \cdot \rangle_{L^2 (\mathcal{G})} \) and weighted scalar product \( \langle \cdot , \cdot \rangle_{L^2 (\mathcal{G}) \otimes R^d} \), where \( R : L^2 (\mathcal{G}) \rightarrow L^2 (\mathcal{G}) \) is the positive definite symmetric invertible linear operator. By the same calculation in (2.12), we have
\[
\tilde{F}^* [q_i^{FLM}]^* = \tilde{F}^* [q_i^{FLM}]^H R^{-1},
\]
(3.9)
where \( \tilde{F}^* [q_i^{LM}]^H \) is an adjoint operator of \( \tilde{F}^* [q_i^{LM}] \) with respect to usual scalar products \( \langle \cdot , \cdot \rangle_{L^2 (\mathcal{G})} \) and \( \langle \cdot , \cdot \rangle_{L^2 (\mathcal{G}) \otimes R^d} \). Then, (3.8) can be of the form
\[
q_{i+1}^{FLM} = q_i^{FLM} + \left( \alpha I + \tilde{F}^* [q_i^{LM}]^H R^{-1} \tilde{F}^* [q_i^{LM}] \right)^{-1} \tilde{F}^* [q_i^{LM}]^H R^{-1} \left( \tilde{u}^\infty - \tilde{F} q_i^{FLM} \right),
\]
(3.10)
As stated in section 2.2, algorithm (3.10) is computationally expensive when the number of measurements \( N \) is increasing. Accordingly, we consider the alternative approach based on the Kalman filter.

We denote
\[
q_{0,0} := q_0 \quad \text{and} \quad B_{0,0} := \frac{1}{\alpha_0} I.
\]
(3.11)
We solve the linearized problem for (3.1) with initial guess \( q_{0,0} \):
\[
F^* \theta_0 [q_{0,0}] q = \theta_0^\infty - F_0 \theta_0 q_{0,0} + F^* \theta_0 [q_{0,0}] q_{0,0},
\]
(3.12)
for \( n = 1, \ldots, N \). Applying the Kalman filter update (2.15)–(2.17) as
\[
q_n = q_{n-1} - F_n \theta_0 q_{n-1} + F^* \theta_0 [q_{n-1}], \quad \theta_n = F^* \theta_0 [q_n],
\]
\[
\varphi_0 = q_{0,0} \quad \text{and} \quad B_0 = B_{0,0},
\]
(3.13)
we obtain the algorithm for \( n = 1, \ldots, N \),
\[
q_{0,n} := q_{0,n-1} + K_{0,n} \left( u_{n}^\infty - F_{B_0,q_0,0} q_{0,0} + F_{B_0,q_0,0}^T [q_{0,0}] q_{0,0} - F_{B_0,q_0,0} [q_{0,0}] q_{0,0-1} \right),
\]
(3.14)
\[
K_{0,n} := B_{0,n-1} F_{B_0,q_0,0}^T [q_{0,0}] \left( R + F_{B_0,q_0,0} B_{0,n-1} F_{B_0,q_0,0}^T [q_{0,0}] \right)^{-1},
\]
(3.15)
\[
B_{0,n} := (I - K_{0,n} F_{B_0,q_0,0} [q_{0,0}]) B_{0,n-1}.
\]
(3.16)

Next, we denote
\[
q_{1,0} := q_{0,N} \text{ and } B_{1,0} := \frac{1}{\alpha_1} I.
\]
(3.17)

We solve the linearized problem for (3.1) with initial guess \( q_{1,0} \):
\[
F_{B_0,q_0,1}^T [q_{1,0}] q = u_{n}^\infty - F_{B_0,q_1,0} + F_{B_0,q_1,0}^T [q_{1,0}] q_{1,0},
\]
(3.18)
for \( n = 1, \ldots, N \). Applying the Kalman filter update (2.15)–(2.17) as
\[
f_n = u_{n}^\infty - F_{B_0,q_1,0} + F_{B_0,q_1,0}^T [q_{1,0}] q_{1,0}, \quad A_n = F_{B_0,q_1,0},
\]
(3.19)
\[
\varphi_0 = q_{1,0}, \quad \text{and } B_0 = B_{1,0},
\]
we obtain the algorithm for \( n = 1, \ldots, N \):
\[
q_{1,n} := q_{1,n-1} + K_{1,n} \left( u_{n}^\infty - F_{B_0,q_1,0} + F_{B_0,q_1,0}^T [q_{1,0}] q_{1,0} - F_{B_0,q_1,0} [q_{1,0}] q_{1,n-1} \right),
\]
(3.20)
\[
K_{1,n} := B_{1,n-1} F_{B_0,q_1,0}^T [q_{1,0}] \left( R + F_{B_0,q_1,0} B_{1,n-1} F_{B_0,q_1,0}^T [q_{1,0}] \right)^{-1},
\]
(3.21)
\[
B_{1,n} := (I - K_{1,n} F_{B_0,q_1,0} [q_{1,0}]) B_{1,n-1}.
\]
(3.22)

Repeating the above arguments (3.11)–(3.22), we obtain the following algorithm for \( n = 1, \ldots, N \) and \( i \in \mathbb{N}_0 \):
\[
q_{i,0}^{\text{KFL}} := q_{i-1,0}^{\text{KFL}} + K_{i,n} \left( u_{n}^\infty - F_{B_0,q_0,0}^{\text{KFL}} + F_{B_0,q_0,0}^{\text{KFL}}^T [q_{0,0}] q_{0,0}^{\text{KFL}} - F_{B_0,q_0,0}^{\text{KFL}} [q_{0,0}] q_{0,0-1}^{\text{KFL}} \right),
\]
(3.23)
\[
K_{i,n} := B_{i,n-1} F_{B_0,q_0,0}^{\text{KFL}}^T [q_{i,0}^{\text{KFL}}] \left( R + F_{B_0,q_0,0}^{\text{KFL}} B_{i,n-1} F_{B_0,q_0,0}^{\text{KFL}}^T [q_{i,0}^{\text{KFL}}] \right)^{-1},
\]
(3.24)
\[
B_{i,n} := (I - K_{i,n} F_{B_0,q_0,0}^{\text{KFL}} [q_{i,0}^{\text{KFL}}]) B_{i,n-1}.
\]
(3.25)

where
\[
q_{i,0}^{\text{KFL}} := q_{i-1,0}^{\text{KFL}},
\]
(3.26)
\[
B_{i,0} := \frac{1}{\alpha_i} I.
\]
(3.27)

We call this the Kalman filter Levenberg–Marquardt (KFL). We remark that the algorithm has indexes \( i \) and \( n \), where \( i \) is associated with the iteration step and \( n \) with the measurement step, respectively.
Finally, we show that the KFL is equivalent to the FLM.

**Theorem 3.1.** For the initial guess $q_0 \in L^2(Q)$ and sequence $\{\alpha_i\}_{i \in \mathbb{N}_0}$ of the regularization parameters, the Kalman filter Levenberg–Marquardt (3.23)–(3.27) is equivalent to the full data Levenberg–Marquardt given by (3.10), that is, for all $i \in \mathbb{N}_0$, we have

$$q_{iN}^{\text{KFL}} = q_{i+1}^{\text{FLM}}.$$  

**Proof.** We prove (3.28) by induction. Applying lemma 2.6 to the linearized problem (3.12) with the regularization parameter $\alpha_0$, we have

$$q_{0N}^{\text{KFL}} = q_1^{\text{FLM}},$$

which is the base case $i = 0$.

Assume that (3.28) $i - 1$ holds, that is, we have

$$q_{i-1N}^{\text{KFL}} = q_i^{\text{FLM}} =: q_i.$$  

Again, applying lemma 2.6 to the linearized problem ($n = 1, \ldots, N$)

$$\mathcal{F}_{\partial_n}^i[q_i]q = u_n^\infty - \mathcal{F}_{\partial_n}q_i + \mathcal{F}_{\partial_n}^i[q_i]q_i,$$

with the regularization parameter $\alpha_{i-1}$, we have

$$q_{iN}^{\text{KFL}} = q_i^{\text{FLM}}.$$  

Theorem 3.1 has been shown. $\square$

**Remark 3.2.** We remark that by the equivalence with the FLM (theorem 3.1) and convergence of Levenberg–Marquardt (lemma 2.5), KFL algorithm (3.23)–(3.25) converges under some assumption.

Figure 1 illustrates the FLM and KFL. While the FLM only moves horizontally, the KFL first moves vertically. Once it was moved up to $n = N$, it moves horizontally and linearization is then complete.
4. Iterative extended Kalman filter

In this section, we propose the algorithm inspired by the EKF (see, e.g., [16, 17, 24]). We denote

\[ q_{0,0} := q_0 \quad \text{and} \quad B_{0,0} := \frac{1}{\alpha_0} I. \]  

(4.1)

We solve the linearized problem for the equation

\[ u_{\theta_1}^\infty = \mathcal{F}_{\theta_1} q, \]  

(4.2)

with respect to \( n = 1 \) and initial guess \( q_{0,0} \), that is,

\[ \mathcal{F}_{\theta_1} q_{0,0} [q] = u_{\theta_1}^\infty - \mathcal{F}_{\theta_1} q_{0,0} + \mathcal{F}_{\theta_1} [q_{0,0}] q_{0,0}, \]  

(4.3)

which is equivalent to solving the minimization problem of the following functional:

\[ J_{0,0}(q) := \| q - q_{0,0} \|_{X \setminus B_{0,0}^{-1}}^2 + \| u_{\theta_1}^\infty - \mathcal{F}_{\theta_1} q_{0,0} + \mathcal{F}_{\theta_1} [q_{0,0}] q_{0,0} - \mathcal{F}_{\theta_1} [q_{0,0}] q \|_{Y \setminus R_{0,0}^{-1}}^2. \]  

(4.4)

By the same argument in section 7 of [14], the Tikhonov regularization solution has the following form:

\[ q_{0,1} := q_{0,0} + K_{0,1} \left( u_{\theta_1}^\infty - \mathcal{F}_{\theta_1} q_{0,0} \right), \]  

(4.5)

\[ K_{0,1} := B_{0,0} \mathcal{F}_{\theta_1} [q_{0,0}] H (R + \mathcal{F}_{\theta_1} [q_{0,0}] B_{0,0} \mathcal{F}_{\theta_1} [q_{0,0}] H)^{-1}, \]  

(4.6)

\[ B_{0,1} := (I - K_{0,1} \mathcal{F}_{\theta_1} [q_{0,0}]) B_{0,0}. \]  

(4.7)

Next, we solve the linearized problem for the equation

\[ u_{\theta_2}^\infty = \mathcal{F}_{\theta_2} q, \]  

(4.8)

with respect to \( n = 2 \) and initial guess \( q_{0,1} \), that is,

\[ \mathcal{F}_{\theta_2} q_{0,1} [q] = u_{\theta_2}^\infty - \mathcal{F}_{\theta_2} q_{0,1} + \mathcal{F}_{\theta_2} [q_{0,1}] q_{0,1}, \]  

(4.9)

which is equivalent to solving the minimization problem of the following functional:

\[ J_{0,1}(q) := \| q - q_{0,1} \|_{X \setminus B_{0,1}^{-1}}^2 + \| u_{\theta_2}^\infty - \mathcal{F}_{\theta_2} q_{0,1} + \mathcal{F}_{\theta_2} [q_{0,1}] q_{0,1} - \mathcal{F}_{\theta_2} [q_{0,1}] q \|_{Y \setminus R_{0,1}^{-1}}^2. \]  

(4.10)

The Tikhonov regularization solution has the following form:

\[ q_{0,2} := q_{0,1} + K_{0,2} \left( u_{\theta_2}^\infty - \mathcal{F}_{\theta_2} q_{0,1} \right), \]  

(4.11)

\[ K_{0,2} := B_{0,1} \mathcal{F}_{\theta_2} [q_{0,1}] H (R + \mathcal{F}_{\theta_2} [q_{0,1}] B_{0,1} \mathcal{F}_{\theta_2} [q_{0,1}] H)^{-1}, \]  

(4.12)

\[ B_{0,2} := (I - K_{0,2} \mathcal{F}_{\theta_2} [q_{0,1}]) B_{0,1}. \]  

(4.13)
We solve the linearized problem for $u_{q_i}^\infty = \mathcal{F}_{q_i} q$ with respect to $n = 1$ and initial guess $q_{1,0}$, that is,
\[ \mathcal{F}_{q_i}^r[q_{1,0}]q = u_{q_i}^\infty - \mathcal{F}_{q_i} q_{1,0} + \mathcal{F}_{q_i}^r[q_{1,0}]q_{1,0}. \]

The Tikhonov regularization solution has the following form:
\[ q_{1,1} := q_{1,0} + K_1(u_{q_i}^\infty - \mathcal{F}_{q_i} q_{1,0}), \]
\[ K_1 := B_{1,0} \mathcal{F}_{q_i}^r[q_{1,0}]^H \left( R + F_{q_i}^r[q_{1,0}]B_{1,0} \mathcal{F}_{q_i}^r[q_{1,0}]^H \right)^{-1}, \]
\[ B_{1,1} := (I - K_1 \mathcal{F}_{q_i}^r[q_{1,0}]B_{1,0}). \]

By solving the linearized problem for $u_{q_i}^\infty = \mathcal{F}_{q_i} q$ up to $n = N$, we obtain $q_{1,N}$, $B_{1,N}$, and denote $q_{2,0} := q_{1,N}$, $B_{2,0} := \frac{1}{\alpha_1}I$. Repeating the above arguments, we finally obtain the following algorithm for $n = 1, \ldots, N$ and $i \in \mathbb{N}_0$:
\[ q_{i,0}^{\text{EKF}} := q_{i-1,0}^{\text{EKF}} + K_{i,n-1} u_{q_i}^\infty - \mathcal{F}_{q_i} q_{i,n-1}, \]
\[ K_{i,n} := B_{i,n-1} \mathcal{F}_{q_i}^r[q_{i,n-1}]^H \left( R + F_{q_i}^r[q_{i,n-1}]B_{i,n-1} \mathcal{F}_{q_i}^r[q_{i,n-1}]^H \right)^{-1}, \]
\[ B_{i,1} := (I - K_{i,1} \mathcal{F}_{q_i}^r[q_{i,1}]B_{i,1}). \]

where
\[ q_{i,0}^{\text{EKF}} := q_{i-1,N}^{\text{EKF}}, \]
\[ B_{i,0} := \frac{1}{\alpha_i}I. \]

We call this the iterative EKF. As remarked in section 3, the algorithm has indexes $i$ and $n$, where $i$ is associated with the iteration step and $n$ with the measurement step, respectively. Figure 2 illustrates the EKF. EKF always moves diagonally because linearization is performed every measurement step.

**Remark 4.1.** We compare the KFL with the EKF. KFL is based on the linearization at the initial state for each iteration step, whereas EKF is based on the linearization at the current state for every iteration step, implying that the update of the KFL is slower than that of the EKF.
Remark 4.2. In both the KFL and EKF algorithms, instead of initializations (3.27) and (4.26), we can update the weight of the norm for each iteration step, that is,

$$B_{i,0} := B_{i-1,N},$$

(4.27)

which plays a role in retaining the information of past updates as iteration step $i$ proceeds.

5. Numerical examples

In this section, we provide numerical examples to demonstrate the algorithms. The inverse scattering problem concerns solving the nonlinear integral equation for $n = 1, \ldots, N$

$$\mathcal{F}_{\theta_n} q = u^\infty(\cdot, \theta_n),$$

(5.1)
Figure 4. Reconstruction for the noise-free case: $q_{\text{true}}^\text{noise}$, $\alpha = 100$.

where the operator $F_{\theta_n} : L^2(Q) \rightarrow L^\infty(S^1)$ is defined by

$$F_{\theta_n}q(x) = \frac{k^2 e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_Q e^{i k \cdot y} u_q(y, \theta_n) q(y) dy,$$

(5.2)

where the incident direction is denoted by $\theta_n := (\cos(2\pi n/N), \sin(2\pi n/N))$. Here, $u_q(\cdot, \theta_n)$ is the solution of the Lippmann–Schwinger integral equation (1.4), which is numerically
computed based on Vainikko’s method \[39, 40\]. This is a fast solution method for the Lippmann–Schwinger equation based on periodization, fast Fourier transform techniques, and multi-grid methods. We assume that the support of function \(q\) is included in \([-S, S]^2\) with some \(S > 0\), and function \(q\) is discretized by a piecewise constant on \([-S, S]^2\) decomposed by squares with length \(\frac{1}{M}\), that is,
Figure 6. Reconstruction for the noise-free case: $q_{\text{true}}^2$, $\alpha = 3000$.

$$q \approx \left( q(y_{m_1,m_2}) \right)_{M \leq m_1,m_2 \leq M-1} \in \mathbb{C}^{(2M)^2},$$ (5.3)

where $y_{m_1,m_2} := \left( \frac{(2m_1+1)S}{2M}, \frac{(2m_2+1)S}{2M} \right)$, and $M \in \mathbb{N}$ is a number of the division of $[0,S]$. Furthermore, the function $u^{\infty}(\cdot, \theta_n)$ is discretized by
Figure 7. Reconstruction for the noise-free case: \( q_j^{\text{true}}, \alpha = 10000, \sigma = 0.5 \).

\[
u_\infty^n(\cdot, \theta) \approx (u_\infty^n(\hat{x}_j, \theta_n))_{j=1, \ldots, J} \in \mathbb{C}^J, \tag{5.4}
\]

where \( \hat{x}_j := (\cos(2\pi j/J), \sin(2\pi j/J)) \), and \( J \in \mathbb{N} \) is a number of the division of \([0, 2\pi]\).

We always fix the following parameters as \( J = 60, M = 6, S = 3, N = 60, \) and \( k = 7 \). We consider true functions as the characteristic function

\[
q_j^{\text{true}}(x) := \begin{cases} 
1.0 & \text{for } x \in B_j \\
0 & \text{for } x \notin B_j
\end{cases}, \tag{5.5}
\]
where the support $B_j$ of the true function is considered in the following two types:

$$B_1 := \{(x_1, x_2) : x_1^2 + x_2^2 < 1.0\},$$  \hfill (5.6)

$$B_2 := \{(x_1, x_2) : (x_1 - a)^2 + (x_2 - b)^2 < (0.5)^2, \ a, b \in \{-1.5, 0, 1.5\}\}.$$ \hfill (5.7)

In figure 3, the closed blue curve is the boundary $\partial B_j$ of the support $B_j$, and the green brightness indicates the value of the true function on each cell divided into $(2M)^2$ in the sampling domain $[-S, S]^2$. Here, we always employ the initial guess $q_0$ as

$$q_0 \equiv 0.$$ \hfill (5.8)

We demonstrate four algorithms: the extended Kalman filter (4.22)–(4.26) with initialization (4.26) (EKF-initialization), extended Kalman filter (4.22)–(4.26) with update (4.27) (EKF-update), Kalman filter Levenberg–Marquardt (3.23)–(3.25) with initialization (3.27) (KFL-initialization), and Kalman filter Levenberg–Marquardt (3.23)–(3.25) with update (4.27) (KFL-initialization). Figures 4–7 show the reconstruction of $B_1$ and $B_2$ by the four algorithms. Figures 4–7 correspond to the noise-free and noise cases, respectively. For the noise case, we add a random sampling from a normal distribution with a mean of zero and standard deviation $\sigma > 0$ to our measurement $u^*(\cdot, \theta_n)$, that is,

$$u^*(\cdot, \theta_n) + \epsilon_n, \quad \epsilon_n \sim \mathcal{N}(0, \sigma^2 I).$$ \hfill (5.9)
Figure 9. Graph of MSE for different regularization parameters $\alpha$: $q_1^{\text{true}}$, $\sigma = 1$.

For the noise-free case in figures 4 and 6, we choose the regularization parameter $\alpha = 100$ and $\alpha = 3000$, respectively. For the noise case in figures 5 and 7, we choose the standard deviation $\sigma = 0.5$ and regularization parameters with $\alpha = 2000$, and $\alpha = 10000$, respectively. The first and second rows correspond to a visualization of the four algorithms, and the third row is the graph of the mean square error (MSE) defined by

$$e_i := \left\| q_i^{\text{true}} - q_i \right\|_2^2,$$

(5.10)

where $q_i$ is associated with the state of the $i$th iteration step. The horizontal and vertical axes correspond to the number of iterations and MSE value, respectively. Figures 8 and 9 show the graph of the MSE in the noise case for different standard deviations $\sigma$ and regularization parameters $\alpha$, respectively.

We observe that the error of EKF (or initialization) decreases faster than that of KFL (or update) when we appropriately choose the regularization parameter; however, EKF (or initialization) is more sensitive to the choice of regularization parameter $\alpha$ and standard deviation $\sigma$ than KFL (or update), that is, EKF-initialization converges fastest and KFL-update is robustest to parameters.

6. Conclusions and future works

In this paper, we proposed two reconstruction algorithm called the Kalman filter Levenberg–Marquardt (KFL) and iterative EKF that are categorized as iterative optimization methods for the inverse medium scattering problem. We numerically observed that the error
of EKF decreases faster when the regularization parameter is appropriately chosen, whereas, EKF is more sensitive to the regularization parameter and the noise than the KFL. In addition, we observed that, by selecting the update of weight of the norm for each iteration step, both algorithm becomes robust to parameters. Since the KFL and EKF algorithms are derived by iterations of the linearization and Kalman filter, they can be applied to various nonlinear inverse problems, and other applications will be developed in the future.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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