Dynamics of the Taylor shift on Bergman spaces

Jürgen Müller and Maike Thelen

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Abstract

The Taylor (backward) shift on Bergman spaces $A^p(\Omega)$ for general open sets $\Omega$ in the extended complex plane shows rich variety concerning its dynamical behaviour. Different aspects are worked out, where in the case $p < 2$ a recent result of Bayart and Matheron plays a central role.

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1 Introduction

Let $\Omega$ be an open subset of the Riemann sphere $\mathbb{C}_\infty$, where $\mathbb{C}_\infty$ is equipped with the spherical metric. Moreover, let $H(\Omega)$ denote the Fréchet space of functions holomorphic in $\Omega$ and vanishing at $\infty$, endowed with the topology of compact convergence. If $0 \in \Omega$, the Taylor (backward) shift $T : H(\Omega) \to H(\Omega)$ is defined by

$$(T f)(z) := \begin{cases} (f(z) - f(0))/z, & z \neq 0 \\ f'(0), & z = 0 \end{cases}.$$

It is easily seen that $T$ is a continuous operator on $H(\Omega)$. Moreover, the $n$-th iterate $T^n$ is given by

$$(T^n f)(z) := \begin{cases} (f - S_{n-1} f)(z)/z^n, & z \neq 0 \\ a_n, & z = 0 \end{cases},$$

where $(S_n f)(z) := \sum_{\nu=0}^n a_{\nu} z^{\nu}$ denotes the $n$-th partial sum of the Taylor expansion of $f$ about 0. In particular, for $|z| < \text{dist}(0, \partial \Omega)$ we have

$$(T^n f)(z) = \sum_{\nu=0}^\infty a_{\nu+n} z^{\nu},$$

that is, locally at 0 the Taylor shift acts as backward shift on the Taylor coefficients.
An important feature of the Taylor shift is that the spectrum is easily determined: Setting $A^* = 1/(\mathbb{C}_\infty \setminus A)$ for $A \subset \mathbb{C}_\infty$, the set $\Omega^*$ is compact in the complex plane $\mathbb{C}$ if and only if $\Omega$ is open in $\mathbb{C}_\infty$ with $0 \in \Omega$. For $\alpha \in \mathbb{C}$, we define $\gamma(\alpha) : \{\alpha\}^* \to \mathbb{C}$ by

$$
\gamma(\alpha)(z) := \frac{1}{1 - \alpha z} \quad (z \in \mathbb{C}_\infty \setminus \{1/\alpha\}).
$$

(1)

Since $\gamma(\alpha) \in H(\Omega)$ is an eigenfunction to the eigenvalue $\alpha$ for all $\alpha \in \Omega^*$, the point spectrum contains $\Omega^*$. Moreover, the corresponding eigenspace is one-dimensional. On the other hand, a calculation shows that for $1/\alpha \in \Omega$ the operator $S_\alpha : H(\Omega) \to H(\Omega)$ defined by

$$(S_\alpha g)(z) = \frac{zg(z) - g(1/\alpha)/\alpha}{1 - z\alpha} \quad (z \in \Omega \setminus \{1/\alpha\})$$

(and appropriately extended at $1/\alpha$) is the continuous inverse to $T - \alpha I$ and hence the spectrum and the point spectrum both equal $\Omega^*$.

The Taylor shift may also be considered as an operator on Banach spaces of functions holomorphic in $\Omega$ as e.g. Bergman spaces, that is, subspaces of $H(\Omega)$ of functions which are $p$-integrable with respect to the two-dimensional Lebesgue measure. In the case of the open unit disc $\Omega = \mathbb{D}$ there is an elaborated theory about invariant subspaces and cyclic vectors for Hardy- and Bergman spaces (see e.g. [7], cf. also [10]). Since we are interested also – and in particular – in the case of open sets $\Omega$ containing $\infty$ and in order to avoid difficulties according to local integrability at $\infty$, we modify the usual Bergman spaces and consider the surface measure on the sphere $\mathbb{C}_\infty$ instead. We denote the normalized surface measure by $m_2$ and, correspondingly, the normalized arc length measure on the unit circle $T$ by $m_1$ of briefly $m$.

For $1 \leq p < \infty$ and $\Omega \subset \mathbb{C}_\infty$ open we define the Bergman space $A^p(\Omega) = A^p(\Omega, m_2)$ as the set of all functions $f \in H(\Omega)$ which fulfil

$$
\|f\|_p := \left( \int_{\Omega} |f|^p \, dm_2 \right)^{1/p} < \infty.
$$

Then $(A^p(\Omega), \|\cdot\|_p)$ is a Banach space. If $\Omega$ is open and bounded in $\mathbb{C}$, the above norm and the classical $p$-norm with respect to Lebesgue measure are equivalent.

In case $0 \in \Omega$, the Taylor shift turns out to be a continuous operator on $A^p(\Omega)$. For $\alpha \in (\Omega^*)^\circ$ the functions $\gamma(\alpha)$ belong to $A^p(\Omega)$ for all $p$ and it is clear that $\gamma(\alpha)$ is an eigenfunction to the eigenvalue $\alpha$. Again, for $1/\alpha \in \Omega$, the operator $S_\alpha$ from above, now defined on $A^p(\Omega)$, turns out to be the continuous inverse to $T - \alpha I$. Moreover, in the case $p < 2$ the functions $\gamma(\alpha)$ belong to $A^p(\Omega)$ also for $\alpha \in \partial \Omega^*$, which yields that in this case the point spectrum equals $\Omega^*$. Thus, we obtain:

1. $(\Omega^*)^\circ \subset \sigma_0(T)$ and $(\Omega^*)^\circ \subset \sigma(T) \subset \Omega^*$ for all $p \geq 1$.
2. $\sigma_0(T) = \sigma(T) = \Omega^*$ for $1 \leq p < 2$.
This gives high flexibility in prescribing spectra. In particular, each compact plane set $K$ appears as spectrum and point spectrum of $T$ on $A^p(K^*)$ for $1 \leq p < 2$. For $p \geq 2$ the situation is more delicate. In general $\gamma(\alpha)$ does not belong to $A^p(\Omega)$ for $\alpha \in \partial \Omega^*$. If, however, $\Omega$ is "sufficiently small" near a boundary point $1/\alpha$ of $\Omega$, it may happen that $\gamma(\alpha)$ does belong to $A^p(\Omega)$. A simple example is the crescent-shaped region $\Omega = \mathbb{D} \setminus \{ z : |z - 1/2| \leq 1/2 \}$, where $\gamma(1) \in A^2(\Omega)$. This opens the possibility to choose $\Omega$ in such a way that eigenvalues are placed at certain points.

In [4], [5] and [23], the behaviour of the Taylor shift with respect to topological dynamics was studied. We recall that an operator $T$ on a separable Fréchet space $X$ is called hypercyclic if $T$ has a dense orbit. This is equivalent to $T$ being topologically transitive, that is, for any two nonempty open sets $U, V \subset X$ the images $T^n(U)$ meet $V$ infinitely often. Moreover, $T$ is topologically mixing if $T^n(U)$ meets $V$ for all sufficiently large $n$. Concerning these and further notions from topological (linear) dynamics we refer the reader to [2] and [13].

The main result from [4] states that the following are equivalent:

- $T$ is topologically mixing
- $T$ is hypercyclic
- Each component of $\Omega^*$ meets the unit circle $\mathbb{T}$.

The situation changes drastically if we consider Bergman spaces. If $T$ is hypercyclic on $A^p(\Omega)$, for some $p < 2$, then $\Omega^*$ has to be perfect. In [2] it is shown that $T$ is mixing on $A^p(\Omega)$ if $\Omega \supset \mathbb{D}$ is a Jordan domain such that $\Omega^* \cap \mathbb{T}$ contains an arc.

In Section 2 we study the Taylor shift operator for its metric dynamical properties. For $H(\Omega)$ and in the case of $A^p(\Omega)$ with $p < 2$, the sufficient supply of eigenvectors $\gamma(\alpha)$ allows the application of a recent deep result of Bayart and Matheron (Theorem 1.1 from [3]) which in many respects finishes a line of investigations concerning relations between the existence of unimodular eigenvectors and the dynamics of a linear operator.

This is no longer possible for $p \geq 2$. In Section 3 we show that the Taylor shift is topologically mixing on $A^p(\Omega)$ for arbitrary $p$ if each component of $\Omega^*$ is sufficiently large near the unit circle $\mathbb{T}$.

## 2 Metric dynamics of $T$

In this section, we investigate the Taylor shift on $H(\Omega)$ and $A^p(\Omega)$ for $p < 2$ with respect to its metric dynamical behaviour. We recall that a measure-preserving transformation $T$ on a probability space $(X, \Sigma, \mu)$ is called weakly mixing (with respect to $\mu$) if for any $A, B \in \Sigma$

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(A \cap T^{-n}(B)) - \mu(A)\mu(B)| = 0
$$

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and it is called strongly mixing (with respect to $\mu$) if for any $A, B \in \Sigma$
\[\mu(A \cap T^{-n}(B)) \to \mu(A)\mu(B) \quad (n \to \infty).\]

For further notions from ergodic theory we refer the reader e.g. to [24]. Consider now $X$ to be a complex separable Fréchet space. Each operator $T$ which is weakly mixing with respect to some measure of full support is frequently hypercyclic (see e.g. [2, Corollary 5.5]) and then also hypercyclic. Moreover, strong mixing with respect to some measure of full support implies topological mixing.

The operator $T$ is called weakly (resp. strongly) mixing in the Gaussian sense if it is weakly (resp. strongly) mixing with respect to some Gaussian probability measure $\mu$ having full support. The definition of Gaussian probability measures and related results can be found in [2] and [3]. For the notion of the cotype of a Banach space we refer to [1].

Let now $T$ be the Taylor shift on $H(\Omega)$ or $A^p(\Omega)$, where $1 \le p < 2$. In order to treat both cases simultaneously, we write $A^0(\Omega) := H(\Omega)$. Then, for $D \subset \mathbb{T}$,
\[\text{span} \bigcup_{\alpha \in \mathbb{T} \setminus D} \ker(T - \alpha I) = \text{span}(\gamma(\Omega^* \cap T \setminus D)).\]

For $\Lambda \subset \mathbb{T}$ we say that $\gamma(\Lambda)$ is perfectly spanning in $A^p(\Omega)$ if the span of $\gamma(\Lambda \setminus D)$ is dense in $A^p(\Omega)$ for all countable $D \subset \mathbb{T}$. Similarly, we say that $\gamma(\Lambda)$ is $\mathcal{U}_0$-perfectly spanning if this holds for all $D \in \mathcal{U}_0$, where $\mathcal{U}_0$ denotes class of sets of extended uniqueness (see e.g. [16, p. 76]). We recall that all sets of extended uniqueness have vanishing arc length measure.

Since for $1 \le p \le 2$ the Bergman space $A^p(\Omega)$ as closed subspace of $L^2(\Omega, m_2)$ is of cotype 2, we obtain as an immediate consequence of the Bayart-Matheoren theorem mentioned in the introduction (Theorem 1.1 from [2])

**Theorem 2.1.** Let $0 \in \Omega \subset C_\infty$ be an open set and let $T$ be the Taylor shift on $A^p(\Omega)$, where $p \in \{0\} \cup [1, 2)$.

1. If $\gamma(\Omega^* \cap \mathbb{T})$ is perfectly spanning in $A^p(\Omega)$ then $T$ is weakly mixing in the Gaussian sense.

2. If $\gamma(\Omega^* \cap \mathbb{T})$ is $\mathcal{U}_0$-perfectly spanning in $A^p(\Omega)$ then $T$ is strongly mixing in the Gaussian sense.

If $p \in [1, 2)$ then in both cases the converse implication is true.

We say that a point $z \in \mathbb{C}$ is a perfect limit point of $A \subset \mathbb{C}$ if $U \cap A$ is uncountable for each neighbourhood $U$ of $z$, that is, if $z$ is a limit point of $A \cap U \setminus D$ for each countable set $D$. Similarly, we say that $z$ is a $\mathcal{U}_0$-perfect limit point if $z$ is a limit point of $A \cap U \setminus D$ for each neighbourhood $U$ of $z$ and each $D \in \mathcal{U}_0$. If $A \subset \mathbb{T}$ has locally positive arc length measure at $z$ then $z$ is a $\mathcal{U}_0$-perfect limit point. Applying an appropriate version of Runge's theorem which can be found e.g. in [17, Theorem 10.2] we obtain from Theorem 2.1

**Corollary 2.2.** Let $0 \in \Omega \subset C_\infty$ be an open set and let $T$ be the Taylor shift on $H(\Omega)$.
1. If each component of $\Omega^*$ contains a perfect limit point of $\Omega^* \cap \mathbb{T}$, then $T$ is weakly mixing in the Gaussian sense.

2. If each component of $\Omega^*$ contains a $U_0$-perfect limit point of $\Omega^* \cap \mathbb{T}$, then $T$ is strongly mixing in the Gaussian sense.

**Remark 2.3.** By separating singularities it is easily seen from [23, Corollary 1] that $\Omega^*$ necessarily has to be perfect if the Taylor shift on $H(\Omega)$ is weakly mixing (or merely frequently hypercyclic). Note that $\Omega^* \cap \mathbb{T}$ not necessarily has to be perfect: If $B$ is some closed arc on $\mathbb{T}$ symmetric to the real axis and

$$\Omega = \mathbb{C} \setminus (B \cup (-\infty, -1] \cup [1, \infty))$$

then $\Omega^* = B \cup [-1, 1]$ satisfies the assumption of Corollary 2.2, hence $T$ is strongly mixing on $H(\Omega)$.

We turn to Bergman spaces. Theorem 2.1 shows that the question whether $T$ is (strongly or weakly) mixing completely reduces to a question about mean approximation by rational functions with simple poles in appropriate subsets of $\mathbb{T}$. The following result on separation of singularities implies that the general question may be reduced to special cases.

**Proposition 2.4.** Let $p \geq 1$ and let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open sets in $\mathbb{C}_\infty$ with $\Omega_1 \cup \Omega_2 = \mathbb{C}_\infty$. Then $A^p(\Omega_1 \cap \Omega_2) = A^p(\Omega_1) \oplus A^p(\Omega_2)$.

**Proof.** It is known that, by separation of singularities of holomorphic functions,

$$H(\Omega_1 \cap \Omega_2) = H(\Omega_1) \oplus H(\Omega_2)$$

as topological direct sum. Since convergence in $A^p(\Omega)$ implies convergence in $H(\Omega)$ (see e.g. [11, Chapter 1, Theorem 1]), it suffices to show that for $f \in A^p(\Omega_1 \cap \Omega_2)$ decomposed as $f = f_1 + f_2 \in H(\Omega_1) \oplus H(\Omega_2)$ we have $f_j \in A^p(\Omega_j)$ for $j = 1, 2$.

Let $f \in A^p(\Omega_1 \cap \Omega_2)$ and $f = f_1 + f_2$ with $f_j \in H(\Omega_j)$. Since the boundary of $\Omega_1 \cap \Omega_2$ is the union of the (compact) boundaries $\partial \Omega_j \subset \Omega_{3-j}$, for $j = 1, 2$, we can find compact disjoint neighbourhoods $U_j \subset \Omega_{3-j}$ of $\partial \Omega_j$. Then

$$\int_{\Omega_j} |f_j|^p \, dm_2 = \int_{\Omega_j \setminus U_j} |f_j|^p \, dm_2 + \int_{\Omega_j \cap U_j} |f_j|^p \, dm_2 < \infty.$$

This yields $f_j \in A^p(\Omega_j)$ for $j = 1, 2$. 

An immediate consequence is the fact that hypercyclicity of $T$ on $A^p(\Omega)$, for some $1 \leq p < 2$, implies that $\Omega^*$ is perfect: Suppose that $\zeta$ is an isolated point of $\Omega^*$. Then we have

$$A^p(\Omega) = A^p(\Omega \cup \{1/\zeta\}) \oplus A^p(\mathbb{C}_\infty \setminus \{1/\zeta\}).$$

By [13, Proposition 2.25] it follows that $T$ is also hypercyclic on $A^p(\mathbb{C}_\infty \setminus \{1/\zeta\})$. Since $A^p(\mathbb{C}_\infty \setminus \{1/\zeta\})$ reduces to the span of $\gamma(\zeta)$ and is thus one-dimensional we get a contradiction.

In order to be able to reduce the case of open sets $\Omega$ containing $\mathbb{C}$ to the case of bounded open sets in $\mathbb{C}$ we recall
Proposition 2.5. Let $X$ a Fréchet space and let $L$ be complemented in $X$. If $X = L \oplus M$ and if $A \subset L$ and $B \subset M$ with $\text{span}(A+B)$ dense in $X$ then $\text{span}(A)$ is dense in $L$.

Proof. Let $a \in L$. Then a sequence $(x_n)$ in $\text{span}(A+B)$ exists with $x_n \to a$ in $X$ as $n$ tends to $\infty$. We write $x_n = a_n + b_n$ with $a_n \in \text{span} A$ and $b_n \in \text{span} B$. Since $a$ belongs to $L$ and since the projection of $X$ to $L$ along $M$ is continuous (see e.g. [21, Theorem 5.16]), the sequence $(a_n)$ converges to $a$. \qed

Remark 2.6. Let $\Omega$ be open with $\infty \in \Omega$ and $\rho > \max_{z \in \mathbb{C} \setminus \Omega} |z|$. If we put $\Omega_p := \Omega \cap \rho \mathbb{D}$ then

$$A^p(\Omega_p) = A^p(\Omega) \oplus A^p(\rho \mathbb{D})$$

and $\gamma(\rho^{-1} \mathbb{D}) \subset A^p(\rho \mathbb{D})$ for all $p \geq 1$. If $B \subset A^p(\Omega)$ is so that $B + \gamma(\rho^{-1} \mathbb{D})$ densely spans $A^p(\Omega_p)$ then $B$ has dense span in $A^p(\Omega)$ by Proposition 2.4.

For $K \subset \mathbb{C}$ compact, a set $\Lambda \subset K$ is called a $K$-uniqueness set if every continuous function on $K$ which is holomorphic in the interior of $K$ and vanishes on $\Lambda$ vanishes identically. Obviously, if $K$ is nowhere dense then $\Lambda$ is a $K$-uniqueness set if and only if $\Lambda$ is dense in $K$. More generally, it is easily seen that $\Lambda \subset K$ is a $K$-uniqueness set if and only if $K \setminus \overline{K^c} \subset \overline{\Lambda}$ and for every component $C$ of $K^c$ the set $\Lambda \cap \overline{C}$ is a uniqueness set for $\overline{C}$.

With that notion, we have the following result on rational approximation. For the case $p = 1$ and Lebesgue measure instead of surface measure the result is due to Bers ([6]).

Theorem 2.7. Let $1 \leq p < 2$ and $0 \in \Omega \subset \mathbb{C}_\infty$ be an open set which is either bounded in $\mathbb{C}$ or contains $\infty$. Moreover, suppose $\Lambda$ to be a subset of $\Omega^*$.

1. If $\Lambda$ is a $\Omega^*$-uniqueness set then the span of $\gamma(\Lambda)$ is dense in $A^p(\Omega)$.

2. If $m_2(\Omega^*) = 0$ then the span of $\gamma(\Lambda)$ is dense in $A^p(\Omega)$ if and only if $\Lambda$ is dense in $\Omega^*$.

Proof. 1. We first assume that $\Omega$ is bounded in $\mathbb{C}$. Let $\ell \in A^p(\Omega)^*$ with $\ell(\gamma(\alpha)) = 0$ for all $\alpha \in \Lambda$. Since $A^p(\Omega)$ is a subspace of $L^p(\Omega)$ the Hahn-Banach theorem yields that $\ell$ can be extended to a continuous linear functional on $L^p(\Omega)$. Thus, there exists a function $g \in L^q(\Omega)$, where $q$ is the conjugated exponent, such that

$$\ell(f) = \int_{\Omega} \overline{f} \overline{g} \, dm_2$$

for all $f \in A^p(\Omega)$. For the measure $1_\Omega g dm_2 \in M(\overline{\Omega})$ the Cauchy transform

$$(Vg)(\alpha) := \int_{\Omega} \frac{\overline{f}(\zeta)}{1 - \overline{\zeta} \alpha} \, dm_2(\zeta) = \frac{1}{\alpha} \int_{\Omega} \frac{\overline{f}(\zeta)}{1/\alpha - \zeta} \, dm_2(\zeta)$$

of $1_\Omega g dm_2$ is holomorphic in the interior of $\Omega^*$ and continuous in $\mathbb{C}$ as the convolution of $w \mapsto 1/w \in A_p(\mathbb{C}_\infty \setminus \{0\})$ and the function $1_\Omega g \in L^q(\mathbb{C})$. Since

$$(Vg)(\alpha) = \ell(\gamma(\alpha)) = 0$$
for all $\alpha \in \Lambda$ and since $\Lambda$ is a $\Omega^*$-uniqueness set we have that $V g|_{\Omega^*} = 0$ and thus $\ell(\gamma(\alpha)) = 0$ for all $\alpha \in \Omega^*$. So $\ell$ vanishes on the set of rational functions with simple poles in $\mathbb{C} \setminus \Omega$. According to (the proof of) [14, Theorem 1], the set of these functions is dense in $A^p(\Omega)$. This yields that $\ell = 0$ and then the Hahn-Banach theorem implies the assertion.

Now, let $\Omega$ be open with $\infty \in \Omega$ and $\Omega_\rho$ as in Remark 2.6. Then $\Omega^*_\rho = \Omega^* \cup \rho^{-1}\mathbb{D}$. Since $\Lambda \cup \rho^{-1}\mathbb{D}$ is a $\Omega^*_\rho$-uniqueness set, by the previous considerations we have that the span of $\gamma(\Lambda \cup \rho^{-1}\mathbb{D})$ is dense in $A^p(\Omega_\rho)$. By Remark 2.6 the span of $\gamma(\Lambda)$ is dense in $A^p(\Omega)$.

2. It is easily seen that in case $m_2(\Omega^*) = 0$ the denseness of $\Lambda$ in $\Omega^*$ is necessary for $\gamma(\Lambda)$ to be densely spanning in $A^p(\Omega)$. Conversely, since $\Omega^*$ is nowhere dense, denseness of $\Lambda$ in $\Omega^*$ implies $\Omega^*$-uniqueness.

If $\Omega^*$ has interior points then $\Omega^*$-uniqueness of $\Lambda$ is in general not necessary for $\gamma(\Lambda)$ to be (even perfectly) spanning in $A^p(\Omega)$:

**Example 2.8.** Let $0 < \delta < 1$ and $E_\delta := (1 + iC_\delta)$, where $C_\delta$ is the convex hull of the closed curve bounded by $\{t + i\varphi(t) : -\delta \leq t \leq \delta\}$ with

$$\varphi(t) := e^{-1/|t|} + 1 - \sqrt{1 - t^2} \quad (-\delta \leq t \leq \delta)$$

(where $e^{-\infty} := 0$) and the horizontal line $\{t + i\varphi(\delta) : -\delta \leq t \leq \delta\}$. Since each dense subset of $\mathbb{T}$ is a $\mathbb{D}^\alpha$-uniqueness set, $\gamma(\mathbb{T})$ is $U_0$-perfectly spanning in $A^p(\mathbb{D})$ for $p < 2$. For the crescent-shaped domain $\Omega := \mathbb{D} \setminus E_\delta$, however, $\mathbb{T}$ is no $\Omega^*$-uniqueness set. On the other hand, the domain $\Omega$ is so "sharp" near the point 1 that the polynomials form a dense subspace of $A^2(\Omega)$ (see Theorem 12.1 in [18]; cf. also [11, p. 29]) and thus of $A^p(\Omega)$ for $p < 2$. In particular, $A^p(\mathbb{D})$ is dense in $A^p(\Omega)$. But then $\gamma(\mathbb{T})$ is also $U_0$-perfectly spanning in $A^p(\Omega)$ for $p < 2$ and, according to Theorem 2.7, the Taylor shift on $A^p(\Omega)$ is strongly mixing in the Gaussian sense.

From the second part of Theorem 2.7 we obtain a quite complete characterization of the metric dynamics of $T$ for the case of open sets $\Omega$ with $m_2(\Omega^*) = 0$. We recall that for any perfect set $A \subset \mathbb{T}$ each point in $A$ is a perfect limit point. A closed set $A \subset \mathbb{T}$ is said to be $U_0$-perfect if $U \cap A \not\subseteq U_0$ for all open sets $U$ that meet $A$. In particular, closed sets $A \subset \mathbb{T}$ which have locally positive arc length measure are $U_0$-perfect. For any $U_0$-perfect set $A \subset \mathbb{T}$ each point in $A$ is a $U_0$-perfect limit point.

**Theorem 2.9.** Let $\Omega \subset \mathbb{C}_\infty$ be open with $0, \infty \in \Omega$ and $m_2(\Omega^*) = 0$. Furthermore, let $1 \leq p < 2$ and $T$ be the Taylor shift on $A^p(\Omega)$.

1. $T$ is weakly mixing in the Gaussian sense if and only if $\Omega^*$ is a perfect subset of $\mathbb{T}$,

2. $T$ is strongly mixing in the Gaussian sense if and only if $\Omega^*$ is a $U_0$-perfect subset of $\mathbb{T}$.
Proof. If $\Omega^* \subset T$ is perfect, then $\Omega^* \setminus D$ is dense in $\Omega^*$ for all countable sets $D$. Theorem 2.1 and Theorem 2.7 show that $T$ is weakly mixing in the Gaussian sense. In the same way, Theorem 2.1 and Theorem 2.7 show that $T$ is strongly mixing in the Gaussian sense if $\Omega^* \subset T$ is $U_0$-perfect.

On the other hand, as noted above, already hyperclicity of $T$ requires perfectness of $\Omega^*$. Since $m_2(\Omega^*) = 0$, Theorem 2.1 and Theorem 2.7 show that the set $\Omega^*$ has to be a subset of $T$. Moreover, $\Omega^* \ni \alpha \rightarrow \gamma(\alpha) \in A_p(\Omega)$ defines a continuous eigenvector field for $T$. The same arguments as in Example 2 of [3] show that $U_0$-perfectness of $\Omega^* \cap T$ is necessary for $T$ to be strongly mixing on $A_p(\Omega)$ for any $1 \leq p < 2$.

Example 2.10. Theorem 2.9 implies that for each set $B \subset T$ which has locally positive arc length measure (as e.g. a nontrivial arc) the Taylor shift $T$ on $A_p(C_\infty \setminus B)$ is strongly mixing in the Gaussian sense for $p < 2$. If $B$ is perfect but not $U_0$-perfect then $T$ is weakly mixing but not strongly mixing in the Gaussian sense.

For the case that $\Omega^*$ has interior points we can show

**Theorem 2.11.** Let $0 \in \Omega \subset C_\infty$ be an open set which is either bounded in $\mathbb{C}$ or contains $\infty$. If each component $K$ of $\Omega^*$ is the closure of a simply connected domain $G$ such that the harmonic measure $\omega(\cdot, K \cap T, G)$ is positive or $G$ meets $T$ then the Taylor shift on $A_p(\Omega)$ is strongly mixing in the Gaussian sense for all $p < 2$.

**Proof.** From the two-constant-theorem (see e.g. [20]) it follows that for each domain $G$ with non polar boundary sets $A \subset \partial G$ of positive harmonic measure $\omega(\cdot, A, G)$ are uniqueness sets for $G$. If $K$ is a component of $\Omega^*$ then, according to our assumptions, the local F. and M. Riesz theorem (see [12, p. 415]) shows that $m(K \cap T)$ is positive. Since each $U_0$-set $D \subset \Omega^* \cap T$ has vanishing arc length measure and, again by the local F. and M. Riesz theorem, also vanishing harmonic measure, $(\Omega^* \cap T) \setminus D$ is a $\Omega^*$-uniqueness set for all $D \in U_0$. Hence, according to Theorem 2.1 and Theorem 2.7 the Taylor shift on $A_p(\Omega)$ is strongly mixing in the Gaussian sense for all $p < 2$. □

**Remark 2.12.** Let $\Omega$ with $0 \in \Omega$ be the exterior of a rectifiable Jordan curve $\Gamma$. Then the interior $G = (\Omega^*)^c$ of $1/\Gamma$ is a Jordan domain with rectifiable boundary and, according to the (global) F. and M. Riesz theorem (see e.g. [12, p. 202]), the harmonic measure of a set $A \subset \partial G$ is positive if and only if the linear measure is positive. For $A \subset T$ this in turn is equivalent to $A$ having positive arc length measure. Hence, if $m(\Omega^* \cap T) > 0$, then $T$ is strongly mixing in the Gaussian sense for all $p < 2$.

Let $\lambda_2$ denote the two-dimensional Lebesgue measure and let

$$D_q(G) := \{h \in H(G) : \int_G |h'|^q \, d\lambda_2 < \infty\}$$
be the Dirichlet space of order \( q \) with respect to \( G \). In a similar way as in the proof of Theorem 1 in [19], by applying Theorem 3, Chapter II, Section 4, from [22], it can be shown that for Cauchy transforms \( Vg \) of functions \( g \in L^q(\Omega) \) as considered in the proof of Theorem 2.7 the restrictions \( Vg|_G \) belong to \( D_q(G) \) and thus in particular to \( D_2(G) \). It is known that for the Dirichlet space \( D_2(D) \) perfect uniqueness sets of vanishing arc length measure exist (see [10, Corollary 4.3.4]). By conformal invariance (cf. [10, Theorem 1.4.1]), the rectifiable Jordan curve \( \Gamma \) can be chosen in such a way that \( m(\Omega^* \cap T) = 0 \) and that \( T \) is weakly mixing in the Gaussian sense for all \( p < 2 \).

Example 2.13. Let \( C \subset T \) be a closed set and consider \( \Gamma \) to be a rectifiable Jordan curve in \( C \setminus D \) with \( \Gamma \cap T = C \) and so that the exterior \( \Omega \) of \( \Gamma \) contains 0. If \( C \) has positive arc length measure then Remark 2.12 shows that the Taylor shift \( T \) on \( A^p(\Omega) \) is strongly mixing in the Gaussian sense for \( p < 2 \). Note that \( C \) may be chosen to be a totally disconnected and that \( C \) may have isolated points. Moreover, according to the proof of [10, Corollary 4.3.4], for an appropriate countable union \( C \) of the circular Cantor middle-third set the Taylor shift weakly mixing in the Gaussian sense for \( p < 2 \).

3 Topological dynamics of \( T \)

If \( \Omega \) is an open set such that no point of \( T \) is an interior point of \( \Omega \) and if \( p \geq 2 \), the Taylor shift \( T \) on \( A^p(\Omega) \) may have no unimodular eigenvalues. This is e.g. the case for \( \Omega = D \). Since \( A^2(\Omega) \) is of cotype 2, Theorem 2.1 shows that weak mixing in the Gaussian sense is excluded. We recall that an operator \( T \) on a Fréchet space \( X \) is called frequently hypercyclic if the orbit of some point \( x \) meets each nonempty open set with positive lower density. Each operator that is weakly mixing with respect to some measure of full support is frequently hypercyclic.

The space \( A^2(D) \) is isometrically isomorphic to the weighted sequence space \( \ell^2(1/(n+1)) \) and the Taylor shift is conjugated to the backward shift on \( \ell^2(1/(n+1)) \) (see [13, Example 4.4.(b)]). As a consequence, the Taylor shift is topologically mixing but not frequently hypercyclic on \( A^2(D) \) (see Example 9.18). It turns out that a similar result holds for the Taylor shift on more general domains \( \Omega \) and for arbitrary \( p \).

**Theorem 3.1.** Let \( 1 \leq p < \infty \) and 0 \( \in \Omega \subset C_\infty \) be a domain which is either bounded in \( C \) or contains \( \infty \). If each component \( K \) of \( \Omega^* \) is the closure of a simply connected domain containing a rectifiable Jordan curve \( \Gamma \) such that the linear measure of \( \Gamma \cap T \) is positive, then the Taylor shift on \( A^p(\Omega) \) is topologically mixing. If, in addition, \( D \subset \Omega \) then \( T \) is not frequently hypercyclic for any \( p \geq 2 \).

**Remark 3.2.** Theorem 3.1 extends a corresponding result in [5], where \( \Omega \) is a Carathéodory domain that contains a (nontrivial) subarc of \( T \). It shows, in particular, that in the situation of Example 2.13 the Taylor shift on \( A^p(\Omega) \) is topologically mixing for all \( p \geq 1 \) and not frequently hypercyclic for any \( p \geq 2 \).
According to Example 2.10, for each (nontrivial) arc $B \subset T$ the Taylor shift on $A^p(C_\infty \setminus B)$ is topologically mixing for $p < 2$. We do not know if this is still the case for $p \geq 2$.

The remaining part of the section is devoted to the proof of Theorem 3.1. Our aim is to apply a version of Kitai’s Criterion (see [13, Remark 3.13]).

Let $E \subset \mathbb{C}$ be compact and let $M(E)$ denote the set of complex measures on the Borel sets of $\mathbb{C}$ with support in $E$. It turns out that the Cauchy transforms of measures $\mu \in M(\mathbb{C}^*)$ are of particular interest for analysing the Taylor shift on $A^p(\Omega)$. For $\mu \in M(E)$ the Cauchy transform $C\mu \in H(E^*)$ of $\mu$ is defined (in terms of vector valued integration) by

$$C\mu := \int_\gamma (\zeta) \, d\mu(\zeta) = \int \frac{1}{1 - \zeta} \, d\mu(\zeta).$$

We write $|\mu|$ for the total variation of the measure $\mu$ and set

$$M_p(\Omega) = \left\{ \mu \in M(\mathbb{C}^*) : \int |\gamma(\zeta)| \, d|\mu|(\zeta) \in L^p(\Omega) \right\}$$

as well as

$$C_p(\Omega) = \{ C\mu : \mu \in M_p(\Omega) \}.$$

For $f \in C_p(\Omega)$ we denote by $C^{-1}(f) = \{ \mu \in M_p(\Omega) : C\mu = f \}$ the set of representing measures for $f$. Note that for $C_p(\Omega) \subset A^p(\Omega)$ since Cauchy transforms of measures $\mu \in M_p(\Omega)$ are holomorphic in $\Omega$ and

$$|C\mu| \leq \int |\gamma(\zeta)| \, d|\mu|(\zeta) \in L^p(\Omega).$$

**Lemma 3.3.** Let $1 \leq p < \infty$ and $0 \in \Omega \subset \mathbb{C}_\infty$ be an open set. Then $T(C_p(\Omega)) \subset C_p(\Omega)$ and for $R: M_p(\Omega) \to M_p(\Omega)$, defined by $d(R\mu)(\zeta) = \overline{\zeta} d\mu(\zeta)$, the diagram

$$\begin{array}{ccc}
M_p(\Omega) & \xrightarrow{R} & M_p(\Omega) \\
c \downarrow & & \downarrow c \\
C_p(\Omega) & \xrightarrow{T} & C_p(\Omega)
\end{array}$$

commutes.

**Proof.** Let $1 \leq p < \infty$ and $\mu \in M_p(\Omega)$. We first show that $R$ is a self map. For $c := \max_{\zeta \in \partial \Omega} |\zeta|$ we obtain

$$\int |\gamma(\zeta)| d|R\mu|(\zeta) \leq \int |\gamma(\zeta)| \, d|\mu|(\zeta) \leq c \int |\gamma(\zeta)| \, d|\mu|(\zeta).$$

It follows that $R\mu \in M_p(\Omega)$. Now, let $f \in C_p(\Omega)$ with $\mu \in C^{-1}(f)$. Since we can interchange integration and $T$ (cf. [21, Exercise 3.24]), we obtain

$$Tf = \int T(\gamma(\zeta)) \, d\mu(\zeta) = \int \zeta \gamma(\zeta) \, d\mu(\zeta) = \int \frac{\zeta}{1 - \zeta} \, d\mu(\zeta).$$
i.e. \( T f = CR\mu \). Since \( R \) is a self map on \( \mathcal{M}_p(\Omega) \), it follows that \( T f \in \mathcal{C}_p(\partial\Omega^*) \).

Inductively, from Lemma 3.3 we obtain

\[
T^n f = \int \zeta^n \gamma(\zeta) d\mu(\zeta)
\]

for \( f \in \mathcal{C}_p(\Omega), \mu \in \mathcal{C}^{-1}(f) \) and \( n \in \mathbb{N} \). In view of Kitai’s criterion, our aim is to find measures \( \mu \) such that \( T^n(C\mu) \) converges to 0 in \( \mathcal{A}^p(\Omega) \). We shall see that this is the case if \( \mu \in \mathcal{M}_p(\Omega) \) is supported on \( \Omega^* \cap T \) and a Rajchman measure. We recall that a Borel measure \( \nu \) supported on \( T \) is called a Rajchman measure if the Fourier-Stieltjes coefficients \( \hat{\nu}(k) = \int \zeta^k d\nu(\zeta) \) tend to 0 as \( k \) tends to \( \pm \infty \) (see e.g. [16]).

Again according to Kitai’s criterion, we also need a kind of right inverse of \( T \): If \( \mu \in \mathcal{M}_p(\Omega) \) is a measure with support in \( T \) we define

\[
S_n\mu := \int \gamma(\zeta) d\mu(\zeta) = \int \frac{d\mu(\zeta)}{\zeta^n(1-\zeta^n)}
\]

for all \( n \in \mathbb{N} \). As in the proof of Lemma 3.3 it is seen that \( S_n\mu \in \mathcal{C}_p(\Omega) \).

**Lemma 3.4.** Let \( 1 \leq p < \infty \) and \( 0 \in \Omega \subset \mathbb{C}_\infty \) be an open set which is either bounded in \( \mathbb{C} \) or contains \( \infty \). Furthermore, let \( T \) be the Taylor shift operator on \( \mathcal{A}^p(\Omega) \). If \( f \in \mathcal{C}_p(\Omega) \) such that \( f \) is represented by a Rajchman measure \( \mu_f \in \mathcal{C}^{-1}(f) \) supported on \( \Omega^* \cap T \) then \( T^n f \to 0 \) and \( S_n\mu_f \to 0 \) in \( \mathcal{A}^p(\Omega) \) as \( n \to \infty \).

**Proof.** Let \( f \in \mathcal{C}_p(\Omega) \) and \( \mu_f \in \mathcal{C}^{-1}(f) \) such that \( \mu_f \) is a Rajchman measure supported on \( B := \Omega^* \cap T \). We fix \( z \in \Omega \). By (2) we have

\[
T^n f(z) = \int_{\partial\Omega^*} \frac{\zeta^n}{1-\zeta^n} d\mu_f(\zeta) = \int_{B} \frac{\zeta^n}{1-\zeta^n} d\mu_f(\zeta)
\]

for all \( n \in \mathbb{N} \). Because \( \mu_f \in \mathcal{M}_p(\Omega) \) is supported on \( B \subset \mathbb{T} \), the function \( \gamma(z) \) belongs to \( \mathcal{L}^1(\mathbb{T},|\mu_f|) \). Since \( \mu_f \) is a Rajchman measure and \( \mu_{f,z} \) with

\[
d\mu_{f,z} := \gamma(z) d\mu_f
\]

is absolutely continuous with respect to \( \mu_f \) [16, Lemma 4, p. 77] yields that \( \mu_{f,z} \) is a Rajchman measure as well. Thus, we have

\[
T^n f(z) = \int \zeta^n \gamma(\zeta) d\mu_f(\zeta) = \int \zeta^n d\mu_{f,z}(\zeta) = \hat{\mu}_{f,z}(-n) \to 0
\]

and

\[
S_n\mu_f(z) = \int \zeta^{-n} \gamma(\zeta) d\mu_f(\zeta) = \int \zeta^{-n} d\mu_{f,z}(\zeta) = \hat{\mu}_{f,z}(n) \to 0
\]
as $n$ tends to $\infty$. Furthermore, for all $n \in \mathbb{N}$ we have

$$|T^nf(z)| \leq \int |\gamma(\zeta)| d|\mu_f|(\zeta) \quad \text{and} \quad |S_n\mu_f(z)| \leq \int |\gamma(\zeta)| d|\mu_f|(\zeta)$$

where $\int_B |\gamma(\zeta)| d|\mu_f|(\zeta)$ is $p$-integrable on $\Omega$ by assumption. Lebesgue’s theorem of dominated convergence yields that $\|T^nf\|_p \to 0$ and $\|S_n\mu_f\|_p \to 0$ as $n$ tends to $\infty$.

As noted in the introduction, for $p \geq 2$ and $\zeta \in \partial \Omega^*$ the functions $\gamma(\zeta)$ are in general not $p$-integrable. We introduce appropriate means of the $\gamma(\zeta)$ which turn out to be integrable for all $p$.

Remark 3.5. It is easily seen (see e.g. [15, Theorem 1.7]) that

$$\int_T \frac{dm(\alpha)}{|1 - \alpha z|} = O \left( \log \frac{1}{1 - |z|} \right) \quad (|z| \to 1^-).$$

Since, for all $p \geq 1$,

$$\int_D \left| \log \frac{1}{1 - |z|} \right|^p dm_2(z) \leq \frac{1}{\pi} \int_0^1 \left| \log \frac{1}{1 - r} \right|^p dr < \infty,$$

by symmetry we obtain that

$$\int |\gamma(\alpha)| dm(\alpha) \in L^p(\mathbb{C}_\infty \setminus \mathbb{T}, m_2).$$

For a Borel set $B \subset \mathbb{T}$ we define $dm_B = 1_B dm$ and

$$f_B := C m_B = \int \gamma(\alpha) dm_B(\alpha) = \int_B \frac{dm(\alpha)}{|1 - \alpha|} \in H(\overline{B}). \quad (4)$$

Let now $\Omega \subset \mathbb{C}_\infty$ be an open set and $\Omega^* \cap \mathbb{T} \neq \emptyset$. Then, for all Borel sets $B \subset \Omega^* \cap \mathbb{T}$ and for all $1 \leq p < \infty$

$$\int |\gamma(\alpha)| dm_B(\alpha) \in L^p(\Omega, m_2),$$

which yields that $m_B$ is a measure in $\mathcal{M}_p(\Omega)$ supported on $B$ and hence $f_B \in \mathcal{C}_p(\Omega)$. Since $1_B \in L^1(\mathbb{T})$ and the arc length measure is a Rajchman measure, Theorem [10, Lemma 4, p. 77] yields that $m_B$ is a Rajchman measure as well.

The following result shows that under the conditions of Theorem 3.1 the functions $f_B$ densely span $A^p(\Omega)$.

Theorem 3.6. Let $1 \leq p < \infty$ and $0 \in \Omega \subset \mathbb{C}_\infty$ be a domain which is either bounded in $\mathbb{C}$ or contains $\infty$. If each component $K$ of $\Omega^*$ is the closure of a simply connected domain containing a rectifiable Jordan curve $\Gamma$ such that the linear measure of $\Gamma \cap \mathbb{T}$ is positive, then the span of $\{f_B : B \subset \Omega^* \cap \mathbb{T}\}$ is dense in $A^p(\Omega)$.
Proof. We first assume that $\Omega$ is bounded in $\mathbb{C}$ and fix $\ell \in A^p(\Omega)'$ with $\ell(f_B) = 0$ for all Borel sets $B \subset \Omega^* \cap T$. Again, according to the Hahn-Banach theorem there exists a function $g \in L^q(\Omega)$, where $q$ is the conjugated exponent, such that

$$\ell(f) = \int_{\Omega} f g \, dm_2$$

for all $f \in A^p(\Omega)$, and for $1_{\Omega} gdm_2 \in M(\Omega)$ the Cauchy transform

$$(Vg)(\alpha) = \int_{\Omega} \frac{\overline{g}(\zeta)}{1 - \zeta \alpha} \, dm_2(\zeta)$$

of $1_{\Omega} gdm_2$ is holomorphic in the interior of $\Omega^*$. However, since $q \leq 2$ for $p \geq 2$, it is no longer guaranteed that the Cauchy integral is defined and continuous on $\mathbb{C}$.

Since $\int_{\Omega^* \cap T} |\gamma(\alpha)| \, dm(\alpha) \in L^p(\Omega)$, Hölder’s inequality yields

$$\int_{\Omega} \int_{\Omega^* \cap T} \frac{|g(\zeta)|}{1 - \zeta \alpha} \, dm(\alpha) \, dm_2(\zeta) \leq \|g\|_q \cdot \|\gamma(\alpha) \, dm(\alpha)\|_p < \infty.$$ 

Hence the maximal Cauchy transform

$$\int_{\Omega} \frac{|g(\zeta)|}{1 - \zeta \alpha} \, dm_2(\zeta)$$

is finite for $m$-almost all on $\alpha \in \Omega^* \cap T$ and $Vg$ exists $m$-almost everywhere on $\Omega^* \cap T$. Moreover, for all Borel sets $B \subset \Omega^* \cap T$ we may apply Fubini’s theorem to get

$$0 = \ell(f_B) = \int_{\Omega} \int_{B} \frac{dm(\alpha)}{1 - \zeta \alpha} \overline{g}(\zeta) \, dm_2(\zeta) = \int_{B} Vg(\alpha) \, dm(\alpha).$$

This implies that $Vg = 0$ $m$-almost everywhere on $\Omega^* \cap T$.

Let $G$ be a bounded simply connected domain in $\mathbb{C}$ and let $D_q(G)$ denote the Dirichlet space of order $q$ defined as in Remark 2.12. Fixing a point $\beta \in G$, we equip $D_q(G)$ with the (complete) norm

$$\|h\|_q = |h(\beta)| + \left( \int_{G} |h'(\zeta)|^q \, dA_2 \right)^{1/q}.$$ 

If $\varphi$ is the conformal mapping from $D$ to $G$ with $\varphi(0) = \beta$ and $\varphi'(0) > 0$ then

$$h \mapsto (h \circ \varphi)(\varphi')^{2-q}$$

defines an isomorphism between $D_q(G)$ and the Dirichlet space $D_q := D_q(\mathbb{D})$ on the unit disc. It is known that $D_q \subset H^q$, where $H^q$ denotes the Hardy space of order $q$ (see e.g. [4, p. 88]). In particular, for $h \in D_q(G) \subset D_1(G)$ we have $(h \circ \varphi)\varphi' \in H^1$, which in turn implies that $h$ belongs to the Hardy-Smirnov space $E^1(G)$ (see [3, Corollary to Theorem 10.1]).
Let now $K$ be a component of $\Omega^*$ and $G$ the interior of $\Gamma$. Then the harmonic measure $\omega(\cdot, K \cap \mathbb{T}, G)$ is positive or $G$ meets $\mathbb{T}$. In a similar way as in the proof of Theorem 1 in [19], by applying Theorem 3, Chapter II, Section 4, from [22] it can be shown that $Vg|_G$ belongs to $D_p(G)$. Since $Vg = 0$ $m$-almost everywhere on $\Omega^* \cap \mathbb{T}$, in the case of positive harmonic measure $\omega(\cdot, K \cap \mathbb{T}, G)$ the local F. and M. Riesz theorem implies that $Vg$ vanishes on a subset of $K \cap \mathbb{T}$ of positive harmonic measure (cf. Remark 2.12).

Since $\Gamma := \partial G$ is a rectifiable Jordan curve, the set of cone points of $\Gamma$ has full linear measure (see [12, Corollary 1.3 and p. 207] or [8, Section 3.5]). Hence $m$-almost every point in $K \cap \mathbb{T}$ is a cone point. It with similar arguments as in the proof of Theorem 3.2.4 in [10] it can be shown that the non-tangential limit $\ell(\gamma(\alpha)) = (Vg)(\alpha) = 0$

for all $\alpha \in (\Omega^*)^\circ$. Since, by assumption, $\Omega$ is a domain, it follows that the inner boundary of $\Omega$ is empty, which allows to apply [14, Corollary p. 162] showing that the rational functions with (simple) poles in $\mathbb{C} \setminus \Omega$ are dense in $A^p(\Omega)$. This implies that $\ell = 0$ and thus the denseness of $\text{span}\{f_B : B \subset \Omega^* \cap \mathbb{T}\}$ in $A^p(\Omega)$.

Along the same lines, we get in case of $\Omega$ containing $\infty$ and $\Omega_p := \Omega \cap \rho \mathbb{D}$ that the span of $\{f_B : B \subset \Omega^* \cap \mathbb{T}\} \cup \gamma(\rho^{-1} \mathbb{D})$ is dense in $A^p(\Omega_p)$. According to Remark 2.9 the span of $\{f_B : B \subset \Omega^* \cap \mathbb{T}\}$ is dense in $A^p(\Omega)$.

**Proof of Theorem 3.7** By Theorem 3.6 the span $L$ of $\{f_B : B \subset \Omega^* \cap \mathbb{T}\}$ is dense in $A^p(\Omega)$. Furthermore, for $f = \sum_B \lambda_B f_B \in L$ let $(S_n \mu_f)_{n \in \mathbb{N}}$ be the sequence defined in (3) with $\mu_f = \sum_B \lambda_B m_B$. Then Lemma 3.3 yields that $\|T^n f\|_p$ and $\|S_n \mu_f\|_p$ converge to 0 as $n \to \infty$. Hence, applying Lemma 3.3 and interchanging integration and $T^n$ we obtain for $n \in \mathbb{N}$

$$T^n S_n \mu_f = T^n \left( \int \gamma(\zeta) \zeta^{-n} \, dm(\zeta) \right) = \int T^n \gamma(\zeta) \zeta^{-n} \, dm(\zeta) = f.$$ 

The Kitai criterion in the version [13, Exercise 3.1.1 or Remark 3.13] yields the assertion.

Finally, the denseness of the span of $\{f_B : B \subset \Omega^* \cap \mathbb{T}\}$ implies that in the case $\mathbb{D} \subset \Omega$ and $p \geq 2$ the space $A^p(\Omega)$ is (continuously and) densely embedded in $A^2(\mathbb{D})$. Since the Taylor shift is not frequently hypercyclic on $A^2(\mathbb{D})$ ([13, Example 9.18]) it is also not frequently hypercyclic on $A^p(\Omega)$.

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Address: University of Trier, FB IV, Mathematics, D-54286 Trier, Germany

e-mail: jmueller@uni-trier.de; maikethelen@web.de