Variations of gwistor space

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Abstract

We study variations of the $G_2$ structure on the unit tangent sphere bundle, introduced in [4, 5, 6] and now called gwistor space. We analyze the equations of calibration and cocalibration, as well as those of $W_3$ pure type or nearly-parallel type.

Key Words: calibration, Einstein manifold, $G_2$-structure, gwistor space.

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1 Introduction

In [4, 5, 6] it was shown how a natural $G_2$ structure is associated to the unit tangent sphere bundle $\pi : SM \to M$ of any given oriented Riemannian 4-manifold $M$. The techniques are twistorial so we have chosen to give the name of gwistors to the theory.

One starts by a construction of the octonions over the 3-sphere fibre bundle. The Levi-Civita connection of the base induces a canonical splitting of the tangent bundle of $TM$. Both vertical and horizontal subbundles $V, H$ become isometric to $\pi^* TM$ with the pull-back metric. On the space $SM = \{ u \in TM : \| u \| = 1 \}$ each point $u$ becomes the identity
element, the generator of the real line in $\mathbb{O}$. Then we use the volume form coupled with $u = U_u \in V$, to induce a cross-product on $u^\perp \subset V$. This gives a quaternionic structure on $V$ and then, applying the well-known Cayley-Dickson process, we obtain the $\mathbb{O}$-structure on $V \oplus H$. The pull-back of $TM$ also inherits a metric connection $\nabla^* = \pi^*\nabla$ and hence parallel identifications of horizontals and verticals, passing through $\pi^*TM$, cf. loc. cit. and [14]. The manifold $SM$ is endowed with the induced metric from the canonical or Sasaki metric on $TM$. Clearly $TSM$ coincides with $V_1 \oplus H$ where $V_1 = \{v \in V : \langle u, v \rangle = 0\}$ at each point $u$. Since $u$ is pointing outwards, our space $SM$ inherits a $G_2$-structure, for which it receives the name of gwistor space. Recall $G_2 = \text{Aut} \mathbb{O}$. Of course the structure is the extension of an $SO(3)$ structure. The connection induces a projection $\nabla^*U : TSM \rightarrow V$ with kernel $H$, where the section $U$ is the tautological vertical vector field.

It is known, by a Theorem of Y. Tashiro, that $SM$ has an almost contact structure in any dimension of the base. As rigid geometrical objects these are, the contact structure is bound to be K-contact if and only if $M$ is locally a radius 1 sphere. Then it is also Sasakian, cf. [7]. The model space is the trivial fibration $SO(5)/SO(3)$.

If we leave aside the Cayley-Dickson process and concentrate on the five invariant 3-forms which are naturally defined on $SM$, then we may try to find other interesting $G_2$ structures. This article is devoted to them, the variations of gwistor space, which should also be called $g$-natural $G_2$-structures on the unit tangent sphere bundle, in analogy with the terms used by [1, 2] and many references therein. On the other hand, the terms deformation or perturbation are also used in similar context by other authors, so we made an option.

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1.1 The basic 3-forms

We start by abbreviating the notation and write $SM = \mathcal{G}$. There is, as we have seen, an isometry map connecting $H$ with $V$, which we denote by $\theta$. We extend it by 0 to $V$. Therefore the tangent vector field $\theta U$ generates a real line bundle, contained in $T\mathcal{G}$. We now pass to the language of differential forms. We may write a splitting:

$$T^*\mathcal{G} = \mathbb{R}\mu \oplus H_1^* \oplus V_1^*$$

where $\mu = (\theta U)^b$ and $H_1 = \theta'V_1$. This 1-form is the aforementioned contact structure, satisfying:

$$\mu_u(v) = \langle u, d\pi(v) \rangle, \quad \forall u \in \mathcal{G}, \ v \in T\mathcal{G}.$$  \hspace{1cm} (2)

The usual pull-back (horizontal) of the volume form of $M$ is also denoted by $\text{vol}$. The vertical pull-back of $\text{vol} \in \Omega^4(M)$ coupled with $U$ is denoted by $\alpha$; then we define analogously
a 3-form $\alpha_3 = (\theta^t U) \cdot \text{vol}$. Of course (we omit the wedge product symbol throughout the text),

$$\mu \alpha_3 = \text{vol}, \quad \text{vol} \alpha = \text{Vol}_G. \quad (3)$$

As shown in [4], it is possible to find an ‘adapted’ direct orthonormal frame $e_0, e_1, \ldots, e_6$ such that

$$\mu = e^0, \quad \alpha_3 = e^{123}, \quad \alpha = e^{456}. \quad (4)$$

It is also known that $d\mu = e^{41} + e^{52} + e^{63}$, which restricts to a symplectic 2-form on the vector bundle $H_1 \oplus V_1$.

The endomorphism $\theta$ allows one to find two other 3-forms (see [4] for the invariant definition):

$$\alpha_1 = e^{156} + e^{264} + e^{345} \quad (5)$$

and

$$\alpha_2 = e^{126} + e^{234} + e^{315}. \quad (6)$$

One can prove the five 3-forms $\alpha, \ldots, \alpha_3, \mu d\mu$ correspond to a basis for the space of invariants in $\Lambda^3(\mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3)$ under $SO(3)$, the underlying structure group of $G$, i.e. there are five irreducible 1-dimensional submodules.

The natural $G_2$ structure on $G$ to which we have referred is given by the 3-form

$$\sigma_0 = \alpha_2 - \alpha + \mu d\mu. \quad (7)$$

Its integrability was studied first in the case of the torsion free metric connection on $M$ and then in the case of metric connections with torsion (which clearly allow the same construction as the Levi-Civita). We know that the structure is co-calibrated, i.e. $d \ast \phi = 0$, if and only if the base $M$ is an Einstein manifold.

### 1.2 Stability of $G_2$ structures

Let us recall the definition of stable forms from the theory of $G_2$-manifolds, [8, 9].

Let $\sigma$ denote a linear $G_2$ structure on a 7-dimensional oriented vector space $V$. A consequence of the study of the Lie group $G_2 = \text{Aut} \sigma \subset SO(7)$ is that it is connected and 14 dimensional; henceforth, that the orbit of $\sigma$ under $GL(7, \mathbb{R})$ is an open set inside the module $\Lambda^3 V^\ast$. This orbit is denoted $\Lambda^3 \mathbb{R}$ and known as the space of stable $G_2$-structures on $V$. We somehow detect the boundaries of such stability by the non-degeneracy of the induced Euclidean product. Indeed, the inner product is given by the map $(v, w) \mapsto v \cdot \sigma \wedge w \cdot \sigma \wedge \sigma$, required to be a positive multiple of the chosen orientation on the diagonal of $V$. The given

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1The author acknowledges I. Agricola and Th. Friedrich for this computation.

2Actually the structure was given first by the opposite, $-\sigma_0$, but we take the opportunity here to change. The reason is that it gives the right canonical representation theory without changing the canonical orientation of $G$; namely the $G_2$-modules $\Lambda^2, \Lambda^4$, which appear from opposite highest weights in [4].
σ satisfies this condition by assumption. Letting σ vary, we have a $GL(7, \mathbb{R})$-equivariant map

$$V \otimes V \otimes \Lambda^3 V^* \rightarrow \Lambda^7 V^*.$$ 

Then of course $\Lambda^3$ is the reunion of two open orbits under the subgroup $GL^+(7, \mathbb{R})$, identified 1-1 by a $-$ sign as it is easy to see. Moreover, the orientation in $V$ induced by the first map itself is preserved in each of these orbits.

Now we return to gwistor space $G \rightarrow M$ and admit a variation of the ‘canonical’ structure $\sigma_0$. We let $f_0, \ldots, f_4$ be scalar functions on $G$ and define

$$\sigma = f_0 \alpha + f_1 \alpha_1 + f_2 \alpha_2 + f_3 \alpha_3 + f_4 \mu d \mu.$$ \hspace{1cm} (8)

Clearly, at least for sufficiently close values to the preferred, we obtain new $G_2$-structures. For the fixed orientation $\text{Vol}_G = e^{0-6}$, induced by the Sasaki structure on $TM$ and the vector field $U$, we have that on any two vectors $v, w$:

$$v \lrcorner \sigma \wedge w \lrcorner \sigma \wedge \sigma = 6 \langle v, w \rangle_\sigma \text{Vol}_\sigma = 6 \langle v, w \rangle_{\sigma_0} m \text{Vol}_G.$$ \hspace{1cm} (9)

This identity defines the scalar function $m > 0$, already assumed to be positive—as we may without loss of regularity or significant generality.

Detailed computations of the metric matrix on the adapted frame yield

$$[(e_i, e_j)_\sigma] = t \begin{bmatrix} f_4^2 & x & z \\ x & z & x \\ z & y & y \\ z & y & z \\ x & z & x \\ y & z & y \\ z & y & z \end{bmatrix}$$ \hspace{1cm} (10)

where

$$t = \frac{f_4}{m}, \quad x = f_2^2 - f_1 f_3, \quad y = f_1^2 - f_0 f_2, \quad z = f_1 f_2 - f_0 f_3.$$ \hspace{1cm} (11)

Notice $\sigma_0$ corresponds to the identity $1_7$. Computing determinants, the metric is positive-definite if $f_4 > 0$, $x > 0$ and $xy - z^2 > 0$. This proves the following result.

**Theorem 1.1.** If a set of scalar functions $f_0, \ldots, f_4$ induces a $G_2$ structure on $G$, then it satisfies $f_4 > 0$, $f_2^2 - f_1 f_3 > 0$ and

$$3 f_0 f_1 f_2 f_3 - f_0 f_2^3 - f_0^2 f_3^2 - f_3 f_1^3 > 0.$$ \hspace{1cm} (12)

**Remarks.** 1. The homogeneous fourth degree polynomial is irreducible and has no critical values in the domain. 2. The metrics obtained are all natural metrics in the sense of [1, 2] and other references therein.
Now, by Gram-Schmidt process on the new metric, we obtain the direct orthonormal frame, 
\[ i = 1, 2, 3, \quad \tilde{e}_0 = \frac{1}{f_4 \sqrt{t}} e_0, \quad \tilde{e}_i = \frac{1}{\sqrt{tx}} e_i, \quad \tilde{e}_{i+3} = \sqrt{\frac{x}{th}} (e_{i+3} - \frac{z}{x} e_i), \] 
(13)
where \( h = xy - z^2 \), the polynomial in (12). A dual co-frame is then
\[ \tilde{e}_0 = f_4 \sqrt{t} e^0, \quad \tilde{e}_i = \sqrt{tx} e^i + z \sqrt{\frac{t}{x}} e_{i+3}, \quad \tilde{e}_{i+3} = \sqrt{\frac{th}{x}} e_{i+3}. \] 
(14)
We obtain also the useful formulas
\[ e^0 = f_4 \sqrt{t} e^0, \quad e^i = \frac{1}{\sqrt{txh}} (\sqrt{h} e^i - z e_{i+3}), \quad e^{i+3} = \sqrt{\frac{x}{th}} e^{i+3}. \] 
(15)
Indeed the frame (13) is direct, i.e. \( \tilde{e}_0^123456 \) is a positive multiple of the chosen orientation.

1.3 Exterior derivatives for \( \sigma \) preserving the Sasaki metric

Let \( \sigma \) be a variation of \( \sigma_0 \).

**Proposition 1.1.** The metric induced by \( \sigma \) coincides with the Sasaki metric on \( G \) if and only if
\[ f_0^2 + f_1^2 = 1, \quad f_2 = -f_0, \quad f_3 = -f_1, \quad f_4 = 1. \] 
(17)

The orbit under \( SO(7) \) of 3-forms which can be written in the form (8) is a circle \( S^1 \).

**Proof.** By hypothesis, we have \( tf_4^2 = tx = ty = 1 \) and \( z = 0 \). Hence \( f_4^2 = f_4 x = f_4 y = m \) and \( h = xy = f_4^4 \). By (16) we get all these equal to 1, except \( z \). Now solving the system (11) we deduce the equivalence in the first part of the result. The second follows from the orbit of \( \sigma_0 = \alpha_2 - \alpha + \mu d\mu \) intersected with our set of 3-forms, observed through typical methods. Indeed already \( U(3) \subset SO(7) \) acts as a real group on the vector space \( E = H_1 \oplus V_1 \), which has a natural complex structure, and fixing \( \epsilon_0 \). We notice
\[ (e^1 + \sqrt{-1}e^4)(e^2 + \sqrt{-1}e^5)(e^3 + \sqrt{-1}e^6) = \alpha_3 - \alpha_1 + \sqrt{-1}(\alpha_2 - \alpha) =: \eta \in \Lambda^3 E_{(1,0)} \]
As \( SU(3) \subset G_2 \) we only have to consider maps \( g = e^{is} 1_E \) for \( s \in \mathbb{R} \) (restricted to \( E \)). Immediately we see such \( g \) fixes the 3-form \( \mu d\mu = e^{041+052+063} \). Finally \( g \cdot \eta = g^3 \eta \). Letting \( g \) be such that \( g^3 = f_0 + \sqrt{-1} f_1 \in S^1 \) we find that the real map \( g \) solves
\[ g \cdot \sigma_0 = -f_0 \alpha - f_1 \alpha_1 + f_0 \alpha_2 + f_1 \alpha_3 + \mu d\mu. \]
The invariant statement follows (relevant due to \( SO(7)/G_2 \) being 7-dimensional). \( \blacksquare \)

For the following computations we apply formulas which have been deduced in [4, 5, 6].

We start by the particular case found above, when the Sasaki metric is preserved. The manifold \( M \) is assumed connected.
Theorem 1.2. Let $\sigma = -f_0\alpha - f_1\alpha_1 + f_0\alpha_2 + f_1\alpha_3 + \mu d\mu$ with $(f_0, f_1): G \to S^1$ smooth.
1. Always $d\sigma \neq 0$.
2. If $(f_0, f_1) \neq (\pm 1, 0)$, then $d*\sigma = 0$ if and only if the functions $f_0, f_1$ are constant and the Riemannian base $M$ has constant sectional curvature.
3. If $(f_0, f_1) = (\pm 1, 0)$, then $d*\sigma = 0$ if and only if $M$ is Einstein.

The proof follows by recalling the list of derivatives of the fundamental 3-forms given as before, with
$$R_{ijpq} = k(\delta^i_l\delta^p_j - \delta^p_l\delta^i_j)$$
for constant sectional curvature metrics. By definitions in \[32, 33\] below, we have $R^U\alpha = -k\mu\alpha_1$, $R^U\alpha_1 = -2k\mu\alpha_2$. Now, we know $d*\sigma = 0$ and thence $d\sigma = \lambda*\sigma + *\tau_3$, with $\tau_3$ the so-called $W_3$ part characterized by $\tau_3\sigma = \tau_3*\sigma = 0$. The condition $\lambda = 0 \in \mathbb{R}$ resumes to $(d\sigma)\sigma = 0$ by a simple duality argument. Computing from the formulas and repeatedly using $f_0^2 + f_1^2 = 1$, we find $k = -2$.

The following formula is used in the proof:
$$d\sigma = \mu(-3f_1\alpha + f_0(k + 2)\alpha_1 + f_1(2k + 1)\alpha_2 - 3f_0k\alpha_3) + (d\mu)^2. \quad (18)$$

The Proposition recovers, in particular, the result in \[4\ Corollary 3.1\] for the preferred $\sigma_0 = \alpha_2 - \alpha + \mu d\mu$ on hyperbolic space of curvature $-2$. However, the result now is independent of the pair $(f_0, f_1) \in S^1$, just as the result $\|d\sigma\|^2 = 48$.

1.4 Exterior derivatives for $\sigma$ in the general case

Suppose $(f_0, \ldots, f_4) : G \to \mathbb{R}^5$ is a vectorial function satisfying the conditions in Theorem \[3] We study the possibly $G_2$-structure on $G \to M$
$$\sigma = f_0\alpha + f_1\alpha_1 + f_2\alpha_2 + f_3\alpha_3 + f_4\mu d\mu. \quad (19)$$
From the formulas in (15) we deduce
\[ \mu = \frac{1}{f_4 t^\frac{1}{2}} \tilde{\mu}, \quad d\mu = \frac{1}{th^\frac{1}{2}} \tilde{d}\mu, \quad \alpha = \frac{x^2}{(th)^{\frac{3}{2}}} \tilde{\alpha}, \quad (20) \]
\[ \alpha_1 = \frac{x^\frac{1}{2}}{t^\frac{1}{2}h} (\tilde{\alpha}_1 - \frac{z}{h} \tilde{\alpha}), \quad \alpha_2 = \frac{1}{x^\frac{1}{2}(th)^{\frac{3}{2}}} (h\tilde{\alpha}_2 - 2h^\frac{1}{2}z \tilde{\alpha}_1 + 3z^2 \tilde{\alpha}), \quad (21) \]
\[ \alpha_3 = \frac{1}{(txh)^{\frac{3}{2}}} (h^\frac{3}{2} \tilde{\alpha}_3 - hz \tilde{\alpha}_2 + h^\frac{1}{2}z^2 \tilde{\alpha}_1 - z^3 \tilde{\alpha}). \quad (22) \]
The forms with a tilde are defined algebraically using the orthonormal basis for \( \sigma \), formally introduced on the respective \( \mu, d\mu, \alpha, \ldots, \alpha_3 \) (it is the \( \text{SO}(3) \) structure of the tangent sphere bundle revealing itself). In particular, we may use the so called first structure equations from [4] but with a tilde. We also need the inversed formulas of the above:
\[ \tilde{\mu}d\mu = f_4 t^\frac{1}{2} h^\frac{1}{2} d\mu, \quad \tilde{\alpha} = \frac{(th)^\frac{3}{2}}{x^\frac{3}{2}} \alpha, \quad (23) \]
\[ \tilde{\alpha}_1 = \frac{ht^\frac{3}{2}}{x^\frac{3}{2}} (x\alpha_1 + 3z\alpha), \quad \tilde{\alpha}_2 = \frac{h^\frac{3}{2} t^\frac{3}{2}}{x^\frac{3}{2}} (x^2 \alpha_2 + 2xz\alpha_1 + 3z^2 \alpha), \quad (24) \]
\[ \tilde{\alpha}_3 = \frac{t^\frac{3}{2}}{x^\frac{3}{2}} (x^3 \alpha_3 + x^2 z \alpha_2 + x^3 \alpha_1 + z^3 \alpha). \quad (25) \]
Using the ‘first structure equations’ in [4] Proposition 2.1], but for the Hodge operator of the metric and orientation induced by \( \sigma \), and writing back in terms of the usual frame, we obtain:
\[ *_{\sigma} (\mu d\mu) = \frac{t^\frac{1}{2} h^\frac{1}{2}}{2f_4} (d\mu)^2, \quad (26) \]
\[ *_{\sigma} \alpha = \frac{1}{h^\frac{3}{2}} \mu \left( x^3 \alpha_3 + x^2 z \alpha_2 + x^2 \alpha_1 + z^3 \alpha \right), \quad (27) \]
\[ *_{\sigma} \alpha_1 = -\frac{f_4 t^\frac{1}{2}}{x h^\frac{3}{2}} \mu \left( 3x^3 z \alpha_3 + x^2 (h + 3z^2) \alpha_2 + x(2hz + 3z^3) \alpha_1 + (3hz^2 + 3z^4) \alpha \right), \quad (28) \]
\[ *_{\sigma} \alpha_2 = \frac{f_4 t^\frac{1}{2}}{x^2 h^\frac{3}{2}} \mu \left( 3x^3 z^2 \alpha_3 + x^2 (2hz + 3z^3) \alpha_2 + x(h^2 + 4hz^2 + 3z^4) \alpha_1 + (3h^2 z + 6hz^3 + 3z^5) \alpha \right), \quad (29) \]
\[ *_{\sigma} \alpha_3 = -\frac{f_4 t^\frac{1}{2}}{x^3 h^\frac{3}{2}} \mu \left( x^3 z^3 \alpha_3 + x^2 (hz^2 + z^4) \alpha_2 + x(h^2 z + 2hz^3 + z^5) \alpha_1 + (h^3 + 3h^2 z^2 + 3hz^4 + z^6) \alpha \right). \quad (30) \]
Now we recall the formulas from [4] Proposition 2.3):
\[ d\alpha = R^\nu \alpha, \quad d\alpha_1 = 3\mu \alpha + R^\nu \alpha_1, \quad d\alpha_2 = 2\mu \alpha_1 - z \text{vol}, \quad d\alpha_3 = \mu \alpha_2 \quad (31) \]
where $\mathcal{R}^U\alpha, \mathcal{R}^U\alpha_1$ are linearly independent forms depending on the curvature of $M$, and $\mathcal{L} = r(u,u)$ is a function on $\mathcal{G}$ (of course $R$ and $r$ are the usual Riemannian and Ricci curvature tensors). Concretely, cf. [4, formulas 25 and 26],

$$\mathcal{R}^U\alpha = \sum_{0 \leq i < j \leq 3} R_{ij01} e^{ij56} + R_{ij02} e^{ij64} + R_{ij03} e^{ij45},$$
(32)

$$\mathcal{R}^U\alpha_1 = \sum_{0 \leq i < j \leq 3} R_{ij01}(e^{ij26} + e^{ij53}) + R_{ij02}(e^{ij61} + e^{ij34}) + R_{ij03}(e^{ij15} + e^{ij42}).$$
(33)

In particular $\mu \mathcal{R}^U\alpha_1 = -\rho \text{vol}$ where $\rho = \sum_{i=1}^{3} r(e_i, e_0)e^{i+3}$.

**Theorem 1.3.** For any functions $f_0, \ldots, f_4$, we have $d\sigma \neq 0$.

**Proof.** Indeed, since $(d\mu)\alpha_i = 0, \forall i = 0, 1, 2, 3$, $\alpha_0 = \alpha$, a moments thought gives

$$\mu(d\mu)d\sigma = (6f_4 + f_0(R_{2301} + R_{3102} + R_{1203}))\text{Vol}_G = 6f_4\text{Vol}_G$$

by Bianchi identity. However, we saw $f_4$ must be positive. \qed

From now on we assume the functions $f_0, \ldots, f_4$ are constant.

A metric almost contact structure is said to be K-contact if the characteristic vector field is Killing. In the case of the Sasaki metric, $(\mathcal{G}, \mu, \theta^tU)$ is K-contact if and only if $M$ is locally isometric to $S^4$ of radius 1, a result due to Y. Tashiro. In general, since our metrics turn out to be natural metrics, we have the question in the larger setting solved in [1].

Another feature of gwistor theory is that never a $G_2$-structure varying from the usual is preserved by the vector field $\theta^tU$ (known both as the geodesic spray or the geodesic flow vector field, cf. [13, 14]). Indeed, computations for constant $f_i$ have shown that $\mathcal{L}_{\theta^tU}\sigma \neq 0$.

Returning to the Hodge duals, then we have by simple reasons

$$d(*)d\mu) = 0,$$

$$d(*)\alpha = -\frac{f_4 t^\frac{1}{2}}{h^\frac{3}{2}}\mu(xz^2\mathcal{R}^U\alpha_1 + z^3\mathcal{R}^U\alpha),$$

$$d(*)\alpha_1 = \frac{f_4 t^\frac{1}{2}}{xh^\frac{3}{2}}\mu(x(2hz + 3z^3)\mathcal{R}^U\alpha_1 + (3hz^2 + 3z^4)\mathcal{R}^U\alpha),$$
(34)

$$d(*)\alpha_2 = -\frac{f_4 t^\frac{1}{2}}{x^2h^\frac{3}{2}}\mu(x(h^2 + 4hz^2 + 3z^4)\mathcal{R}^U\alpha_1 + (3h^2z + 6hz^3 + 3z^5)\mathcal{R}^U\alpha),$$

$$d(*)\alpha_3 = \frac{f_4 t^\frac{1}{2}}{x^3h^\frac{3}{2}}\mu(x(h^2z + 2hz^3 + z^5)\mathcal{R}^U\alpha_1 + (h^3 + 3h^2z^2 + 3hz^4 + z^6)\mathcal{R}^U\alpha).$$

Hence the vanishing of the two polynomials

$$- f_0 x^3z^2 + f_1 x^2(2hz + 3z^3) - f_2 x(h^2 + 4hz^2 + 3z^4) + f_3 (h^2z + 2hz^3 + z^5),$$
(35)

$$f_0 x^3z^3 - f_1 x^2(3hz^2 + 3z^4) + f_2 x(3h^2z + 6hz^3 + 3z^5) - f_3(h^3 + 3h^2z^2 + 3hz^4 + z^6)$$
(36)
is a sufficient condition for the vanishing of $d(*_\sigma \sigma)$. Multiplying the first by $z$, adding to the second and factoring out a $h(>0)$ from the result, we obtain:

$$-f_1x^2z^2 + 2f_2xhz + 2f_2z^3x - f_3h^2 - 2f_3hz^2 - f_3z^4. \quad (37)$$

Finally recurring to some computer algebra software, we are able to find two independent expressions in the original parameters $f_0, \ldots, f_3$:

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\begin{align*}
-f_0 \left( f_1^2 - f_0 f_2 \right) \left( -f_2^2 + f_1 f_3 \right)^2 \quad (= (35)), \\
(f_2^2 - f_1 f_3)^3 \left( -2f_0 f_1^3 f_2^3 + 3f_0^2 f_1 f_2^4 - f_0^3 f_3 + 6f_0 f_1^4 f_2 f_3 - 6f_0^2 f_1^2 f_2 f_3^2 \\
-2f_0^3 f_3 f_3 - 3f_0^2 f_3^2 - 3f_0^2 f_3^2 + 6f_0 f_1 f_2 f_3^2 - f_0^3 f_3^2 \right) \quad (= (36)).
\end{align*}

(38)

(39)

Notice they are homogeneous, as expected, and notice the factor $y = f_1^2 - f_0 f_2$ in the second polynomial and the common factor $x = f_2^2 - f_1 f_3$, which must both be positive. From equivalence we get the simple expression

$$\left( f_1^3 - 2f_0 f_1 f_2 + f_0^2 f_3 \right) \left( f_2^2 - f_1 f_3 \right)^3 \quad (= (37)). \quad (40)$$

**Theorem 1.4.** A 3-form $\sigma$ as above, with $f_0, \ldots, f_4$ constant, satisfies $d(*_\sigma \sigma) = 0$ if and only if any of the following occurs:

(i) the polynomial (32) vanishes and $M$ is Einstein.

(ii) $M$ has constant sectional curvature.

**Proof.** Notice first that, if $f_0 = 0$, then neither $f_1$ or $f_3$ can vanish (otherwise we would get $y = 0$ or $h = 0$ from definition). So the two main polynomials cannot vanish simultaneously, as we see directly, or from the implied equation (40).

Now, if the polynomial (39) vanishes, then we may conclude $f_0 \neq 0$, ie. the first polynomial (38) does not vanish. So the cocalibration equation is equivalent to the vanishing of $\mu R^U \alpha = -\rho \text{vol}$, which happens if and only if $M$ is Einstein. On the contrary, if the polynomial does not vanish, then the equation is on metrics such that $\mu R^U \alpha = 0$; equivalently, $R_{1201} = R_{2301} = 0$, etc. This is the same as $M$ having constant sectional curvature.

In particular, being Einstein. ■

For example, if $f_0 = 0$, then we are certainly bound to the second case.

Noteworthy is the case when $f_1 f_2 = f_0 f_3$ (or $z = 0$), which generalizes Proposition 1.2.

A question put to the author by colleagues was: if we could always find, invariant of the metric on $M$, a natural $G_2$ structure which would be co-closed. The answer is no, because the two polynomials do not vanish altogether.

We thus stress the relevance of $G_2$ cocalibration goes much beyond the known cases and examples.
1.5 Nearly-parallel $G_2$-structures

Nearly-parallel $G_2$-structures on 7-dimensional manifolds are defined by $\delta \sigma = 0$ and $d \sigma = c \ast \sigma$ for some constant $c$. Clearly, if $c \neq 0$, the condition is simply the latter equation.

We consider a variation of the $G_2$ structure on $\mathcal{G}$, as in \cite{19}. In order to find a nearly-parallel structure $\sigma$, we may assume already it is cocalibrated ($c \neq 0$). We notice the Hodge $\ast$ operator is homogeneous of degree $1/3$ on 3-forms seen as a map $\sigma \mapsto \ast \sigma$ (the simplest way to see this is by \cite{26}, but from the definition will also do). Hence if we find a solution to the above in our subspace of $\sigma \in \Lambda^3_+$, we find a line of solutions: $d(s\sigma) = \frac{c}{s^2} \ast s \sigma$, $s \in \mathbb{R}^+$.

We restrict here to the case $z = f_1 f_2 - f_0 f_3 = 0$, the less ‘prohibitive’ condition.

**Theorem 1.5.** Under the previous condition, the only metric on an oriented Riemannian 4-manifold $M$ for which a $(\mathcal{G}, \sigma)$ is nearly-parallel is the constant sectional curvature 1 metric. Then there are two classes of solutions, represented by the following two $G_2$-structures:

$$\sigma_{\pm} = \pm \frac{\sqrt{2}}{2} (\alpha_2 - \alpha + \alpha_3 - \alpha_1) + \sqrt{\frac{3}{2}} \mu d \mu,$$

both satisfying $d \sigma = \sqrt{6} \ast \sigma$.

**Proof.** Since we assume $z = 0$ and this is maintained on the line $\mathbb{R}^+ \sigma$, there exists a positive multiple of $\sigma$ such that $(f_0, f_1)$ is in the unit circle. Then we easily deduce $x = y = 1$ and $f_2 = -f_0, f_3 = -f_1$. Hence $h = 1 = t$ and $m = f_4$, cf. \cite{10}.

From formulas \cite{26,30} and the hypothesis of $\sigma$ being nearly-parallel, we see the 4-form $d \sigma$ is again $SO(3)$-invariant. Then we easily deduce the curvature restriction: it must be of the constant kind. The equation $d \sigma = c \ast \sigma$ is solved using those same formulas, with $z = 0$ proving a major advantage. Looking at components, we find a system ($k$ is the sectional curvature)

$$\begin{cases}
  c = 2 f_4 \\
  f_0 f_1 - k f_0^2 = 0 \\
  2 f_0 f_1 k + f_0 f_1 - 3 f_1^2 = 0 \\
  3 f_1 - 2 f_0 f_4^2 = 0 \\
  2 f_0 + k f_0 - 2 f_0 f_4^2 = 0
\end{cases}$$

This yields $f_0 = f_1$, which occurs twice in the circle; and $k = 1, f_4 = \sqrt{3/2}, c = \sqrt{6}$. The given 3-forms satisfy the equation and are genuine $G_2$-structures. $\blacksquare$

Notice the metric on $\mathcal{G}$ is the same on both solutions. Now we recall the classification of nearly-parallel $G_2$ structures in \cite{11}. The ones we got correspond to the Stiefel manifold $V_{5,2} = SO(5)/SO(3)$ in their Table 2, which is of course the unit tangent sphere bundle of $S^4$. The $G_2$ is constructed as a $U(1)$-bundle over the complex quadric $G_{5,2}$, the Grassmannian of 2-planes, with a Kähler-Einstein metric. The resulting nearly parallel $G_2$ is said to be Einstein-Sasakian for some homogeneous $SO(5)$-invariant metric. We have thus found
just some more details of this case. It is also most interesting to see that our result gives a metric coinciding precisely with the Einstein metric on $V_{5,2}$ deduced in [2, Theorem 4]. It has Riemannian scalar curvature $\frac{63}{4}$, by a formula there.

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