Weak uniqueness and partial regularity for the composite membrane problem

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1 Introduction

Our main consideration will be the physical problem proposed in [CGI+00] which can be stated as:

Problem (P). Build a body of prescribed shape out of given materials of varying density, in such a way that the body has prescribed mass and so that the basic frequency (with fixed boundary) is as small as possible.

This problem by virtue of Theorem 13 in [CGI+00] can be converted into the following minimization problem. Given a bounded domain $\Omega \subset \mathbb{R}^n$ with

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smooth boundary, fix $\alpha > 0$ and $A \in [0, |\Omega|]$. For any measurable subset $D \subset \Omega$, denote by $\lambda_{\Omega}(\alpha, D)$ the first Dirichlet eigenvalue for the problem,

$$
-\Delta u + \alpha \chi_D u = \lambda_{\Omega}(\alpha, D) u, \text{ on } \Omega
$$

$$
\begin{align*}
& u = 0, \text{ on } \partial \Omega. \\
\end{align*}
$$

(1.1)

Define,

$$
\Lambda_{\Omega}(\alpha, A) = \inf_{D \subset \Omega, |D| = A} \lambda_{\Omega}(\alpha, D).
$$

(1.2)

A minimizer $D$ to (1.2) will be called an optimal configuration for the data $(\Omega, \alpha, A)$. For this $D$ we denote the associated eigenfunction solution to (0.1) by $u$. The pair $(u, D)$ will be called an optimal pair solution to the composite problem or for short a solution to the composite problem.

A variational formulation of our problem is also possible and is given by (see [CGI+00]),

$$
\begin{align*}
& \Lambda_{\Omega}(\alpha, A) = \inf_{u \in H^1_0(\Omega), \|D\| = A, \|u\|_2 = 1} \int_{\Omega} (|\nabla u|^2 + \alpha \chi_D u^2). \\
\end{align*}
$$

(1.3)

Theorem 1 in [CGI+00] establishes the basic properties of the existence and regularity of optimal pairs.

**Theorem 1.1.** [CGI+00] For any $\alpha > 0$ and $A \in [0, |\Omega|]$, there exists an optimal pair $(u, D)$. Moreover, the optimal pair $(u, D)$ has the property,

(a) $u \in C^{1,\gamma}(\overline{\Omega}) \cap H^2(\overline{\Omega})$, for every $\gamma < 1$.

(b) $D$ is a sub-level set of $u$, that is there exists $c \geq 0$ such that,

$$
D = \{ u \leq c \}.
$$

(c) If $\alpha \neq \Lambda_{\Omega}(\alpha, A)$, then every level set $\{ u = s \}$ has measure zero.

See Remark 2.2 for additional comments regarding (c). From Theorem 13 in [CGI+00] we also know that the physical problem (P) stated earlier is equivalent to the variational problem (1.3) provided,

$$
\alpha < \Lambda_{\Omega}(\alpha, A).
$$

(1.4)
In the sequel we shall always assume (1.4). Now putting together Theorem 1.1 and the variational characterization of the problem (0.3) we see that the Euler-Lagrange equation of our problem is:

$$-\Delta u + \alpha \chi_{\{u \leq c\}} u = \Lambda_\Omega(\alpha, A) u, \text{ in } \Omega$$

$$u = 0, \text{ on } \partial \Omega. \quad (1.5)$$

In Section 2 we first turn to the problem of uniqueness of optimal pairs \((u, D)\). A principal result of [CGI+00] is that even in domains that exhibit symmetry, the optimal pair need not be unique, and in fact uniqueness is known without any assumptions only if \(\Omega\) is the ball. Nevertheless, we establish that generically there is a sort of weak uniqueness in the problem.

**Theorem 1.2** (Weak Uniqueness). Assume (0.4). For almost every value of \(A \in (0, |\Omega|)\), there exists \(c > 0\) such that for all optimizing pairs \((u_i, D_i)\),

\[ D_i = \{x|u_i(x) \leq c\}. \]

Thus though there is non-uniqueness in the problem, the level height where one must cut-off the eigenfunction to get \(D_i\) must generically be the same for all eigenfunctions.

Under additional assumptions, that is if eigenfunctions agree at one point to infinite order or if \(\Omega\) is convex in \(\mathbb{R}^2\) with additional assumptions, the assertion of weak uniqueness can be turned into a statement of true uniqueness. See for example, Lemma 2.9 and Theorem 2.1 in Section 2.

In Section 3 we turn to the regularity of the free boundary \(\mathcal{F}\), defined by,

\[ \mathcal{F} = \{x|u(x) = c\}. \quad (1.6) \]

We recall an initial result, Theorem 8 in [CGK00],

**Theorem 1.3.** Let \(x_0 \in \mathcal{F}\). Assume \(\nabla u(x_0) \neq 0\). That is \(x_0\) is a regular point of the free boundary. Then there exists a ball \(B(x_0, r)\) of radius \(r > 0\) centered at \(x_0\), and a real-analytic function \(\phi(x_1, x_2, \cdots, x_{n-1})\) such that,

\[ \mathcal{F} \cap B(x_0, r) = \{(x_1, x_2, \cdots, x_n)|x_n = \phi(x_1, x_2, \cdots, x_{n-1})\}. \]

That is the free boundary in the neighborhood of a regular point is a hypersurface given by the graph of a real-analytic function.
Subsequently Blank [Bla04] performed a blow-up analysis in dimension 2 to classify the singular points of $F$, that is those points on $F$ where $\nabla u = 0$. This analysis in dimension 2 was completed in the paper by Shahgholian [Sha], who also obtained a condition that guarantees when singular points of $F$ in dimension 2 are isolated.

The free boundary problem for the composite problem can be easily converted to an equivalent problem (see for e.g. [Sha]), given by,

$$\Delta v = f\chi_{\{v \geq 0\}} - g\chi_{\{v \leq 0\}},$$

$$f, g \in C^1, f > 0, g < 0, f + g < 0.$$  \tag{1.7}

Our main result concerning the structure of $F$ in Section 3 is:

**Theorem 1.4.** [Structure of the Free Boundary of Solutions (1.7)] For $\Omega \subset \mathbb{R}^n$, there is a decomposition,

$$F = F_0 \cup S^1_v \cup S^2_v,$$

where $S^2_v$ has Hausdorff dim $\leq n-2$, $\mathcal{H}^{n-1}(S^1_v) \leq C$, and for all $x_0 \in F_0$, there exists a ball $B(x_0, r)$ centered at $x_0$ such that, $F \cap B(x_0, r)$ is a hypersurface given by the graph of a real-analytic function.

The principal tool we use to perform our blow-up analysis and thereby get Theorem 1.4 is an energy functional introduced by Weiss [Wei98]. Set, $(f \equiv f_0, g \equiv g_0)$

$$W(r) = \frac{1}{r^{n+2}} \int_{B(x_0, r)} (|\nabla v|^2 + 2(f_0 v^+ + g_0 v^-)) - \frac{2}{r^{n+3}} \int_{\partial B(x_0, r)} u^2. \tag{1.8}$$

Weiss showed that $W(r)$ is monotonically increasing. We offer an alternative proof based in part on the Rellich-Pohozhaev identity which explicitly shows that no structural assumptions are needed to get the monotonicity.

Next we proceed to classify the blow-up limits in the spirit of the paper by Monneau-Weiss [MW05]. Two points are to be noted in contrast to [MW05]. First that in our case blow-up limits are non-degenerate, and second that we have two types of blow-up limit solutions that are homogeneous of degree 2. This is already evident in the work in dimension 2 by Blank [Bla04] and Shahgholian [Sha].
Lastly we address the question of $C^{1,1}$ bounds. In general such bounds are not available for the composite problem if we only analyze the Euler-Lagrange equation (1.7). So-called cross solutions arise from homogeneous harmonic polynomials of degree 2 with corresponding failure of $C^{1,1}$ bounds in dimension 2 as has been exhibited by Andersson and Weiss [AW05] in the case $f \equiv -1, g \equiv 0$. The example of Andersson-Weiss can be easily extended to all dimensions by the addition of dummy variables. We show that the [AW05] construction extends to our setting (Remark 3.20). Our regularity result proved in Section 3 is:

**Theorem 1.5.** The free boundary $F = G \cup B$, where in $G$ we have pointwise $C^{1,1}$ bounds and $B$ has Hausdorff dim $\leq n - 2$. (See Theorem 3.4, Definition 3.16 and Remark 3.19).

It remains open whether proceeding from the variational problem instead of (1.7) allows one to get $C^{1,1}$ bounds. It is readily seen that global assumptions on the boundary of $\Omega$ do ensure that $C^{1,1}$ bounds and full regularity are achieved. A result of this type proved in Section 3 is, (Proposition 3.7)

**Proposition 1.6.** Assume $\Omega \subset \mathbb{R}^2$ has two axes of symmetry. Then the free boundary $F$ is a real-analytic curve and $u \in C^{1,1}$.

## 2 Uniqueness and Weak uniqueness

Our goal is to prove Theorem 1.2 of the introduction in this section. We shall also show that a weak uniqueness assertion like in Theorem 1.2 can be converted to a uniqueness assertion on convex domains with additional assumptions. Let $\Omega \subset \mathbb{R}^n$, be a bounded domain with $\partial \Omega$ smooth. For $\alpha > 0$, $A \in [0, |\Omega|]$, and $D \subset \Omega$, let $\lambda_\Omega(\alpha, D) = \lambda$, be the lowest eigenvalue to,

\[
\begin{cases}
-\Delta v + \alpha \chi_D v = \lambda v & \text{on } \Omega \\
v|_{\partial \Omega} = 0
\end{cases}
\]

(2.1)

The variational characterization of (2.1) gives,

\[
\lambda = \inf_{u \in H_0^1(\Omega)} \frac{\int_{\Omega} (|\nabla u|^2 + \alpha \chi_D u^2)}{\int_{\Omega} u^2}
\]

(2.2)
Lemma 2.1. There exists a unique minimizer \( v \in H^1_0 \) of (2.2) with \( \|v\|_2 = 1 \), which is non-negative.

Proof. By Theorem 8.38 in [GT83], the eigenvalue \( \lambda \) is simple and the eigenspace is spanned by a non-negative eigenfunction. Since \( \|v\|_2 = 1 \), we have a unique non-negative eigenfunction, with the property, \( \|v\|_2 = 1 \). \qed

Define
\[
\Lambda = \Lambda_\Omega(\alpha, A) = \inf_\{D \subset \Omega \atop |D| = A\} \lambda(\alpha, A).
\]

Remark 2.2. Assume \( \alpha < \Lambda \). Then for the solution to the composite problem \((u, D)\) stated in the introduction, \( |\{u = s\}| = 0 \), for all \( s \). This is Theorem 1(c) in [CGI⁺00]. Note that \( s = 0 \) is not covered by the proof in [CGI⁺00] but is easily ruled out by superharmonicity of \( u \).

Lemma 2.3. Let \( F = \{u : u = c\} \), where \( D = \{x \in \Omega : u \leq c\} \) and \((u, D)\) is the solution of our composite problem. Then \( \nabla u \not\equiv 0 \) on \( F \). (In fact in the boundary of each connected component of \( C D \), \( \nabla u \) cannot be identically 0).

Proof. Assume \( \nabla u|_F \equiv 0 \). Consider the open set \( O = \{x : u > c\} \). Since \( A < |\Omega| \), by remark 2.2, \( |O| > 0 \). Let \( U \) be a connected component of \( O \), and let
\[
\begin{cases}
-\Delta w = \mu_U w & \text{in } U \\
w|_{\partial U} = 0
\end{cases}
\]
(2.3)
where \( \mu_U \) is the first Dirichlet eigenvalue of \( U \). We claim \( \Lambda \leq \mu_U \). To check this extend \( w \) to \( U^c \), by setting \( w \equiv 0 \) in \( U^c \). The extended function will still be denoted by \( w \) and we may normalize it so that \( \|w\|_2 = 1 \). Then,
\[
\Lambda \leq \int_\Omega |\nabla w|^2 + \alpha \int_D w^2 = \int_\Omega |\nabla w|^2 = \mu_U.
\]
If \( \Lambda = \mu_U \), by Lemma 2.1, \( u = w \). Since \( w \equiv 0 \) on \( D \), and since \( u \) is superharmonic, so is \( w \), so \( w = u = 0 \), a contradiction. So, \( \Lambda < \mu_U \). Let \( v = \partial_{x_j} u \), for some fixed \( j \).

In \( U \),
\[
-\Delta u = \Lambda u,
\]
(2.4)
so that on differentiating (2.4), \( v \) satisfies,
\[
\begin{aligned}
-\Delta v &= \Lambda v \quad \text{in } U \\
v|_{\partial U} &\equiv 0 \quad v \in C^\infty(\bar{U})
\end{aligned}
\]  \hfill (2.5)

We claim \( v \equiv 0 \). This will imply \( u \equiv c \) in \( U \), which will contradict Remark 2.2. Since \( \Lambda < \mu_U \), we may solve using the Fredholm alternative,

\[-\Delta f - \Lambda f = -\Lambda \text{ in } U, \quad f \in H_0^1(U)\]  \hfill (2.6)

Let \( h = f^+ = \max(f,0) \). Clearly \( h \in H_0^1(U) \). Multiplying (2.6) by \( h \) and integrating by parts,

\[\int_U |\nabla h|^2 - \Lambda \int_U h^2 = -\int_U \Lambda h \leq 0.\]

Thus,

\[\int_U |\nabla h|^2 \leq \Lambda \int_U h^2. \]  \hfill (2.7)

If \( \int_U h^2 \neq 0 \), then from (2.7), \( \mu_U \leq \Lambda \). This is a contradiction. Hence \( \int_U h^2 = 0 \) and \( h \equiv 0 \) in \( U \). Thus \( f \leq 0 \). Set \( \psi = 1 - f \). Thus \( \psi \geq 1 \), and from (2.6),

\[-\Delta \psi - \Lambda \psi = 0.\]

By elliptic regularity, \( \psi \in C^\infty(U) \). Now find \( U_j \Subset U \), with \( \text{dist}(\partial U_j, \partial U) \to 0 \), and \( \partial U_j \) smooth. So if \( x \in U \), then \( x \in U_j \) for large enough \( j \). Let \( \phi = v/\psi \), where \( v \) is defined in (2.5). Note, because \( \psi \geq 1 \), and by (2.5) again,

\[\sup_{\partial U_j} |\phi| \leq \sup_{\partial U_j} |v| \to 0 \text{ as } j \to \infty\]  \hfill (2.8)

Now,

\[\nabla \phi = \frac{\nabla v}{\psi} - \frac{v \nabla \psi}{\psi^2} = \frac{\psi \nabla v - v \nabla \psi}{\psi^2},\]

and

\[\Delta \phi = \frac{\nabla \psi \cdot \nabla v + \psi \Delta v - \nabla v \cdot \nabla \psi - v \Delta \psi}{\psi^2} - \frac{2}{\psi^3} (\psi \nabla v - v \nabla \psi) \cdot \nabla \psi\]

\[= \frac{\psi \Delta v - v \Delta \psi}{\psi^2} - \frac{2}{\psi} \nabla \psi \cdot \nabla \phi = -\frac{2}{\psi} \nabla \psi \cdot \nabla \phi.\]
Thus \( \phi \) satisfies
\[
\Delta \phi + \frac{2}{\psi} \nabla \psi \cdot \nabla \phi = 0 \quad \text{in } U_j
\]

Thus by the maximum principle, and (2.8),
\[
\sup_{\bar{U}_j} |\phi| \leq \sup_{\partial U_j} |\phi| \to 0 \quad \text{as } j \to \infty
\]

Thus \( \phi \equiv 0 \) in \( U \) and hence \( v \equiv 0 \) in \( U \). Our proposition is proved. \( \square \)

Combining Theorem 8 in [CGK00] and Lemma 2.3, we have,

**Lemma 2.4.** If \((u, D)\) is a minimizing pair with \( \alpha < \Lambda_\Omega \), then there exists \( x_0 \in \mathcal{F} = \{u = c\} \) and a ball \( B(x_0, r) = B \) centered at \( x_0 \), so that \( B \subset \Omega \) and
\[
\mathcal{F} \cap B = \{(x, \phi(x)), x \in \mathbb{R}^{n-1}, \phi : U \subset \mathbb{R}^{n-1} \to \mathbb{R}\},
\]
with \( \phi(x_0) = 0, \nabla \phi(x_0) = 0 \) and \( \phi(x) \) real-analytic. Furthermore,
\[
D \cap B = \{(x, y) : y < \phi(x)\} \cap B
\]
\[
^c D \cap B = \{(x, y) : y > \phi(x)\} \cap B
\]

**Lemma 2.5.** Let \( \psi(x') : U \subset \mathbb{R}^{n-1} \to \mathbb{R} \) be smooth; where \( U \) is open and \( U \supset B(0, r) \). Assume \( \psi(0) = 0, \nabla \psi(0) = 0 \), and let
\[
D = \{(x', y) : y < \psi(x')\} \cap B, \quad B = B(0, r).
\]
Then there exists \( \epsilon_0 > 0 \), and a smooth function, \( x = (x', y) \)
\[
\Phi(t,x) = \Phi_t(x) : \{|t| \leq \epsilon_0\} \times B \to B
\]
such that,
(a) for all fixed \( t \),
\[
\Phi_t : B \to B
\]
is a diffeomorphism, with \( \Phi(0,x) = \Phi_0(x) = x. \)
(b) for all \( t, |t| \leq \epsilon_0 \), and some \( \delta > 0, \delta < r/50 \)
\[
\Phi_t|_{\overline{B \setminus B(0,2\delta)}} = x.
\]
(c) Let \( \chi(D_t)(x) = \chi_D(\Phi_{-t}(x)) \). Then
\[
\frac{d}{dt} \left( |D_t| \right) |_{t=0} = 1.
\]

Proof. Let \( f(x) \) be a smooth cut-off function, \( f \in C_0^\infty(B(0, \delta/10)), f \geq 0 \). Let \( \nu(x') \) denote the unit outward normal to \( y = \psi(x') \). We extend \( \nu(x') \) smoothly as a vector field \( X \) to all points in \( B(0, \delta/10) \). Now define
\[
\frac{d\Phi_t}{dt}(x) = \frac{X(x)f(x)}{\int_{\partial D \cap B(0, \delta/10)} f(\sigma) \, d\sigma} = V(x), \quad \Phi_0(x) = x \quad (2.9)
\]

(a), (b) follows from (2.9). Note that a simple degree argument is needed to show that \( \Phi_t \) is a diffeomorphism. (c) follows from the Appendix 1, by noting that
\[
V|_{\partial D} = \frac{\nu(x')f(x)}{\int_{\partial D} f(\sigma) \, d\sigma}.
\]
Hence
\[
\int_{\partial D} \left< V, \nu \right> = 1.
\]

\[\Box\]

**Lemma 2.6.** Construct \( \Phi_t(x) \) as in Lemma 2.5, \( x_0 = 0 \) as in Lemma 2.4. Define
\[
\phi_t(x) : \Omega \to \Omega, \text{ by}
\]
\[
\phi_t(x) = \begin{cases} 
\Phi_t(x) & x \in B(0, 3\delta) \\
x & x \in \Omega \setminus B(0, 3\delta).
\end{cases}
\]

(a) Then \( \phi_t(x) \) is a diffeomorphism of \( \Omega \).

(b) If \( D_t = \{ \phi_t(x), x \in D \} \), then
\[
\frac{d}{dt} \left( |D_t| \right) = 1.
\]

(c) If \( (u, D) \) is a solution to the composite problem and
\[
-\Delta u_t + \alpha \chi_{D_t} u_t = \lambda(t) u_t
\]

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\[ u_t \big|_{\partial \Omega} = 0, \]

where \( D = \{ x \in \Omega, u \leq c \} = D_0, \ u_0 = u, \ \lambda(0) = \Lambda. \) Then
\[ \lambda'(0) = \alpha c^2. \]

Proof. Using (b) in (A1.10) and Lemma 2.5(c) we get (c). (b) follows from Lemma 2.5. (a) follows from the definition of \( \phi_t(x) \) and Lemma 2.5(a). \( \square \)

Lemma 2.7. Assume that \( \Lambda_\Omega(\alpha, A) \) is differentiable at \( A = A_0 \). Let \( (u, D) \) be a minimizer. Construct domains \( D_t \) as in Lemma 2.6, where \( B = B(x_o, r) \) is supplied by Lemma 2.4. Then
\[ \frac{d}{dt} \Lambda(\alpha, A)|_{A=A_0} = \alpha c^2. \]

Proof. Let \( |D_t| = m(t); \ D_t \) as in Lemma 2.6. Let \( f(t) = \Lambda(\alpha, m(t)) \). Then, \( f \) is differentiable at \( t = 0 \).
\[ f'(0) = \frac{d\Lambda}{dA}(\alpha, A) \bigg|_{A=A_0} \cdot m'(0) = \frac{d\Lambda}{dA}(\alpha, A) \bigg|_{A=A_0}. \quad (2.10) \]

Next for \( t > 0 \), by the definition of \( \Lambda \),
\[ \frac{f(t) - f(0)}{t} \leq \frac{\lambda(t) - \lambda(0)}{t}. \]

Letting \( t \downarrow 0 \), we get \( f'(0) \leq \lambda'(0) \). Arguing similarly for \( t < 0 \), letting \( t \uparrow 0 \), using the differentiability at \( t = 0 \) of \( f \) and \( \lambda \) we get \( f'(0) = \lambda'(0) = \alpha c^2 \) by Lemma 2.6. Thus from (2.10),
\[ \frac{d\Lambda}{dA}(\alpha, A) \bigg|_{A=A_0} = \alpha c^2. \]

Proof of Theorem 1.2. \( \Lambda(\alpha, A) \) is strictly increasing in \( A \) and Lipschitz in \( A \), Prop. 10, [CGI+00]. Thus \( \Lambda'(\alpha, A) \) exists, a.e. \( A_0 \), and \( \Lambda'(\alpha, A) = \alpha c^2 \) by Lemma 2.7. Hence if \( (u_1, D_1), (u_2, D_2) \) are two configurations \( |D_i| = A \), with \( D_i = \{ x : u_i < c_i \} \), then \( \alpha c_i^2 = \alpha c_j^2 \). Hence \( c_1 = c_2 \). \( \square \)

We shall now show that under some conditions, the weak uniqueness conclusion of Theorem 1.2 can be turned into a uniqueness result. We will restrict our attention to domains \( \Omega \subset \mathbb{R}^2. \)
Lemma 2.8. Let $\Omega \subset \mathbb{R}^2$, and let
\[
\begin{aligned}
-\Delta u + \alpha \chi_{\{u \leq c\}}(x) u &= \lambda u \\
u|_{\partial \Omega} &= 0, \|u\|_2 = 1.
\end{aligned}
\tag{2.11}
\]
Then for any $x_0 \in \mathbb{R}^2$,
\[
\frac{1}{2} \int_{\partial \Omega} \langle x - x_0, \nu \rangle \left( \frac{\partial u}{\partial \nu} \right)^2 = \lambda - \alpha c^2 |D^c| - \alpha \int_D u^2
\]
where $D = \{ x : u \leq c \}$.

**Proof.** We use the Rellich-Pohozhaev identity. Now,
\[
-\langle x - x_0, \nabla u \rangle \Delta u = -\nabla \cdot (\langle x - x_0, \nabla u \rangle \nabla u) + |\nabla u|^2 + \frac{1}{2} (x - x_0) \cdot \nabla (|\nabla u|^2).
\]
Thus, integrating the identity above over $\Omega$,
\[
- \int_{\Omega} \langle x - x_0, \nabla u \rangle \Delta u
\]
\[
= - \int_{\Omega} \langle x - x_0, \nu \rangle \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\sigma + \frac{1}{2} \int_{\Omega} \langle x - x_0, \nu \rangle \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\sigma
\]
\[
= \frac{1}{2} \int_{\Omega} \langle x - x_0, \nu \rangle \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\sigma \quad (2.12)
\]
From (2.11),
\[
-\Delta u = \lambda u - \alpha \chi_D u. \quad (2.13)
\]
Substituting (2.13) into the left side of (2.12) we get,
\[
\int_{\Omega} \alpha \chi_D u \langle x - x_0, \nabla u \rangle - \int_{\Omega} \langle x - x_0, \nabla u \rangle \lambda u = \int_{\Omega} \langle x - x_0, \nu \rangle \left( \frac{\partial u}{\partial \nu} \right)^2.
\]
Thus,
\[
\frac{1}{2} \int_{\Omega} \langle x - x_0, \nu \rangle \left( \frac{\partial u}{\partial \nu} \right)^2
\]
\[
= - \frac{\lambda}{2} \int_{\Omega} \langle x - x_0, \nabla (u^2) \rangle + \frac{\alpha}{2} \int_{\Omega} \langle x - x_0, \nabla (u^2) \rangle. \quad (2.14)
\]
The first integral on the right by integration by parts is

$$\lambda \int_\Omega u^2 = \lambda. \quad (2.15)$$

For the second integral, since $D = \{ x : u(x) \leq c \}$, there exists $c_j \uparrow c$, such that by Sard’s Theorem $c_j$ is a regular value. Let $D_j = \{ x : u < c_j \}$. Now by integration by parts,

$$\int_\Omega \langle x - x_0, \nabla (u^2) \rangle = -2 \int_{D_j} u^2 + \int_{\partial D_j \cap \Omega} \langle x - x_0, \nu \rangle u^2$$

$$= -2 \int_{D_j} u^2 + \int_{\partial D_j \cap \Omega} c_j^2 \langle x - x_0, \nu \rangle$$

$$= -2 \int_{D_j} u^2 + c_j^2 |c_j D_j|.$$

Letting $j \to \infty$,

$$\int_D < x - x_0, \nabla (u^2) > = -2 \int_D u^2 + c^2 |D| \quad (2.16)$$

Inserting (2.16), (2.15) into (2.14) we get our result. \qed

To obtain a true uniqueness assertion we first need a preliminary lemma which is valid in all dimensions. We shall assume in the sequel that our solutions are normalized by the condition $\|u\|_2 = 1$.

**Lemma 2.9.** Let $(u_i, D_i), \ i = 1, 2$ be two solutions of our composite problem. Assume that $D_1$ is connected. Assume furthermore we have weak uniqueness that is $D_i = \{ x \in \Omega : u_i \leq c \}$ and $u_1 - u_2$ vanishes at a single point $x_0 \in D_1$ to infinite order. Then $u_1 \equiv u_2$ in $\Omega$.

**Proof.** First we note $u_1(x_0) = u_2(x_0) < c$. Thus there is a ball $B$ centered at $x_0$ where $u_i(x) < c$, $i = 1, 2$. Thus in this ball we have,

$$-\Delta u_i + \alpha u_i = \Lambda u_i, \ i = 1, 2 \quad (2.17)$$

Thus, $w = u_1 - u_2$ also satisfies the equation (2.17) and $w$ vanishes at $x_0$ to infinite order. Thus, $w$ vanishes identically in $B$. Now consider the set,

$$W = \text{int}\{ x \in D_1, \ u_1 = u_2 \}.$$
We have established that $W$ is non-empty. We shall now show that $W$ is both open and closed in the relative topology of $D_1$. Since $D_1$ is connected we then get $W = D_1$. Since $u_1 = u_2 < c$ on $\overset{\circ}{D}_1$ we obtain that, $D_1 \subset D_2$. Since $|D_1| = |D_2|$ we see right away that $D_1 = D_2$.

Now by definition $W$ is open. So let $z_0 \in W = F \cap D_1$ where $F$ is closed. Thus $u_1(z_0) = u_2(z_0) < c$ and thus there is a ball $B$ centered at $z_0$ where (2.17) is satisfied. Again $w$ satisfies (2.17) with $w$ vanishing on some open set in $B$. This is because $z_0$ is a boundary point to $W$. Again by unique continuation $w$ vanishes in $B$. Thus $z_0 \in W$. We have checked $W$ is also closed. Since now $D_1 = D_2$, applying Lemma 2.1 we obtain the conclusion of our lemma.

**Remark 2.10.** The same result holds if $x_0 \in \partial \Omega$. The proof is similar, but slightly more complicated.

**Theorem 2.1.** Assume $\Omega \subset \mathbb{R}^2$ with smooth boundary. Assume that $\Omega$ is strictly convex. Let $(u_i, D_i)$ be two solutions to the composite problem with eigenvalue $\Lambda$. Assume that,

(a)  
$$\int_{D_1} u_1^2 = \int_{D_2} u_2^2.$$  
(b) Weaker uniqueness holds, $D_i = \{x \in \Omega | u_i(x) \leq c\}$.

(c) The sets $\{x | u_1(x) < u_2(x)\}$ and $\{x | u_1(x) > u_2(x)\}$ are both connected.

Then $u_1 \equiv u_2$.

**Proof.** Since $\Omega$ is convex, it is simply connected and since $\alpha < \Lambda$, by Theorem 2 [CGI+00] the sets $D_i$ are connected. Writing Lemma 2.8 for $u_i$ and subtracting the expression for $u_2$ from that of $u_1$, we get after using the hypothesis (a), (b) above that,

$$\int_{\partial \Omega} \langle x - x_0, \nu \rangle \left[ \left( \frac{\partial u_1}{\partial \nu} \right)^2 - \left( \frac{\partial u_2}{\partial \nu} \right)^2 \right] = 0.$$  
We re-write this expression to get,

$$\int_{\partial \Omega} \langle x - x_0, \nu \rangle \frac{\partial}{\partial \nu} (u_1 + u_2) \frac{\partial}{\partial \nu} (u_1 - u_2) = 0. \quad (2.18)$$
Now in a tubular neighborhood of $\partial \Omega$ both $u_1, u_2$ satisfy (2.17). Thus $u_1 + u_2$ also satisfies (2.17) with $u_1 + u_2 > 0$ in $\Omega$ and vanishing on $\partial \Omega$. Thus by Hopf’s boundary point lemma,

$$\frac{\partial}{\partial \nu}(u_1 + u_2) < 0$$

(2.19)

Now set $\psi = u_1 - u_2$. Let,

$$E_1 = \left\{ x \in \partial \Omega \mid \frac{\partial \psi}{\partial \nu} > 0 \right\}, \quad E_2 = \left\{ x \in \partial \Omega \mid \frac{\partial \psi}{\partial \nu} < 0 \right\}.$$

We show both sets are empty. If we establish this result we have the conclusion of the lemma. The reason is that if $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial \Omega$, since $\psi = 0$ on $\partial \Omega$ we conclude by the Cauchy-Kovalevskaya Theorem that $\psi$ vanishes in a neighborhood of a boundary point and thus applying Lemma 2.9 we conclude $u_1 = u_2$ in $\Omega$.

**Case 1:** Assume w.l.o.g. that $E_2$ is empty and $E_1$ is non-empty. Pick any $x_0 \in \Omega$. Then by the strict convexity of $\partial \Omega$, $\langle x - x_0, \nu \rangle > 0$. Thus by (2.19) and the choice of $x_0$ we conclude that the integral in (2.18) is negative. This contradicts the identity (2.18).

**Case 2:** We may now assume that both $E_1$ and $E_2$ are non-empty. Consider the components of $E_1$ and $E_2$ on $\partial \Omega$. These are intervals. We claim that the hypothesis (c) rules out interlacing of intervals. That is the intervals that make up the components of $E_1$ must share at least one boundary point and likewise for the intervals that make up the components of $E_2$. For assume there exist two intervals $I_1, I_2$ which are components of $E_1$ and two intervals $J_1, J_2$ which are components of $E_2$. Now we shall obtain a contradiction if we assume that $I_1, I_2$ lie in different components of $\partial \Omega \setminus (J_1 \cup J_2)$. Taking interior points in $I_1, I_2$ we can connect the points by a curve that lies entirely in $\Omega$ and in the set $\{ u_1 < u_2 \}$. Now it is easily seen that $\{ u_1 > u_2 \}$ is disconnected. This contradicts (c). Thus we have shown that $\partial \Omega$ consists of two arcs $\gamma_1, \gamma_2$ such that, $\gamma_1$ and $\gamma_2$ have common endpoints $P, Q$ and such that, on $\gamma_1$, $\frac{\partial \psi}{\partial \nu} \geq 0$, with $\frac{\partial \psi}{\partial \nu} > 0$ on some sub-interval of $\gamma_1$. Likewise, on $\gamma_2$, $\frac{\partial \psi}{\partial \nu} \leq 0$, with $\frac{\partial \psi}{\partial \nu} < 0$ on some sub-interval of $\gamma_2$. Now consider tangent lines to $\partial \Omega$ at $P, Q$. 

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If the tangent lines intersect at $x_0$, apply (2.18) with this choice $x_0$. Notice then by the strict convexity of $\partial \Omega$, $\langle x - x_0, \nu \rangle > 0$ (except possibly at $P, Q$) on $\gamma_1$ and $\langle x - x_0, \nu \rangle < 0$ on $\gamma_2$. Thus using (2.19) and the behavior of $\psi$ on $\gamma_1, \gamma_2$ we easily see that the integral in (2.18) is negative. This is a contradiction.

Assume thus that the tangent lines at $P, Q$ are parallel and with no loss of generality assume they are parallel to the $x_1$ axis, $x = (x_1, x_2)$. Set $v(x) = (n_1(x), n_2(x))$. Now (2.18) holds for every $x_0$. Set $x_0 = (x_0^1, x_0^2)$. We may now differentiate (2.18) with respect to $x_0^1$ and we obtain,

$$
\int_{\partial \Omega} n_1(x) \frac{\partial}{\partial \nu} (u_1 + u_2) \frac{\partial \psi}{\partial v} = 0.
$$

Now we may assume that $n_1(x) > 0$ on $\gamma_1$ and $n_1(x) < 0$ on $\gamma_2$ except at $P, Q$ by the strict convexity of $\partial \Omega$. Thus the integrand in the integral in (2.18) is non-positive by the use of (2.19). Furthermore, from (2.19) and the behavior of $\psi$ on the arcs $\gamma_i$ there are arcs on $\partial \Omega$ where the integrand in (2.18) is negative. This again is a contradiction to (2.18). Thus both sets $E_1$ and $E_2$ are empty. Our Theorem is established.

\section{Partial Regularity}

Our goal is to prove Theorems 1.4 and 1.5 of the introduction in this section. We follow the works of Blank [Bla04], Shahgolian [Sha], Weiss [Wei98] and Monneau-Weiss [MW05], with some necessary variants and extensions.

\textbf{The set-up: } Let $\Omega \subset \mathbb{R}^n$, be a bounded domain with $\partial \Omega$ smooth. For $\alpha > 0$, $A \in (0, |\Omega|)$, we let $(u, D)$ be a solution of the composite problem, so that

$$
\begin{cases}
-\Delta u + \alpha \chi_{\{u \leq c\}} u = \Lambda u & \text{on } \Omega \\
u|_{\partial \Omega} = 0, \int_{\Omega} u^2 = 1
\end{cases}
$$

(3.1)

where $D = \{u \leq c\}$. Recall that $u \geq 0$ in $\Omega$ and that we are assuming throughout that $\alpha < \Lambda$. Note that $u \in W^{2,p}(\Omega) \forall 1 \leq p < \infty$, $u \in C^{1,\gamma}(\overline{\Omega})$, $0 \leq \gamma < 1$, with norm depending only on $A$, $n$, $\Omega$, $p$, $\gamma$, $\alpha$ and $\Lambda$. Note also that $c > 0$ since if $u(x_0) = 0$, by superharmonicity of $u$, $x_0 \in \partial \Omega$, and $|\{u \leq c\}| = A > 0$. Note also that $|\{u = c\}| = 0$, by Remark 2.2. We
next let $v = c - u$ and write the equation for $v$, namely
\[ \Delta v = f\chi_{\{v\geq 0\}} - g\chi_{\{v<0\}} \quad (3.2) \]
where $f = (\Lambda - \alpha)u$, $g = -\Lambda u$. Fix a neighborhood $U$ of $\mathcal{F} = \{u = c\}$, the free boundary, so that $f > 0$, $g < 0$ and $f + g < 0$ in $\overline{U}$. We thus have a solution $v$ of (3.2), in $U$ open, and functions $f, g \in C^{1,\gamma}(\overline{U})$, with norm bounded by $\tilde{B}_1 = \tilde{B}_1(\gamma, u, \alpha, \Lambda, A, \Omega)$ in $\overline{U}$, with $f, g \in W^{2,p}(U)$, with norm bounded by $\tilde{B}_2 = \tilde{B}_2(p, n, \alpha, \Lambda, A, \Omega)$ and with $|\Delta f|, |\Delta g|$ bounded by $\tilde{B}_3 = \tilde{B}_3(\alpha, \Lambda)$, and such that, for some $\eta_0 > 0$, $\eta_0 = \eta_0(\alpha, \Lambda, A, n, \Omega)$, we have $f \geq \eta_0 > 0, g \leq -\eta_0, (f + g) \leq \eta_0$ in $\overline{U}$. We also have $\|v\|_{C^{1,\gamma}(\overline{U})} + \|v\|_{W^{2,p}(\overline{U})} \leq N, N = N(\gamma, p, n, x, \Lambda, A, \Omega)$. Finally, we fix $r_0$ so small that for all $x_0 \in \mathcal{F}$ we have that $B(x_0, r_0) \subset U$. We still study the behavior of $S_u = \{x \in \mathcal{F} : \nabla u(x) = 0\} = S_v = \{x \in \mathcal{F} : \nabla v(x) = 0\}$, where $\mathcal{F} = \{v = 0\}$. Note that, by [CGK00] (see Lemma 8 and Theorem (0.3) here) for each $x_0 \in \mathcal{F} \setminus S_v$, there exists a neighborhood $V_{x_0}$ around $x_0$ so that $\mathcal{F}$ is real analytic in it and $v$ (and $u$) are real analytic in $V_{x_0} \cap \overline{D}, V_{x_0} \cap \overline{\mathcal{D}}$. One of our main tools in this section is an energy functional introduced by Weiss:
\[ W(r) = \frac{1}{r^{n+2}} \int_{B(x_0, r)} (|\nabla v|^2 + 2(fv + gv)) - \frac{2}{r^{n+3}} \int_{\partial B(x_0, r)} v^2 \quad (3.3) \]
In the next Lemma we compute $W'(r)$. (See [Wei98], where the computation is also carried out).

**Lemma 3.1.** Let $x_0 \in S_v$, $0 < r < r_0$. Then, for $0 < r < r_0$,
\[ W'(r) = \frac{2}{r^{n+2}} \int_{\partial B_r} \left[ \frac{\partial v}{\partial \nu} - 2\frac{v}{r} \right]^2 d\sigma + e(r), \quad (3.4) \]
where for $0 \leq \gamma < 1$ we have, for $0 < r < r_0$,
\[ |e(r)| \leq F(n, \gamma, \|\nabla f\|_\infty, \|\nabla g\|_\infty, N) r^{\gamma-1}, \quad (3.5) \]
with $F(-, -, 0, 0, -) \equiv 0$. (Here $\nu$ is the outward unit normal to $\partial B_r$ and $B_r$ stands for $B(x_0, r)$).

**Proof.** We can assume, without loss of generality, that $x_0 = 0$. We have:
\[ \frac{\partial}{\partial r} \left( \frac{1}{r^{n+2}} \int_{B_r} |\nabla v|^2 \right) = -\frac{n - 2}{r^{n+3}} \int_{B_r} |\nabla v|^2 + \frac{1}{r^{n+2}} \int_{\partial B_r} |\nabla v|^2. \]
Moreover, the Rellich-Pohozaev identity gives:

\[
div (x|\nabla v|^2) = 2div (x \cdot \nabla v \nabla v) + (n - 2)|\nabla v|^2 - 2x \cdot \nabla \Delta v,
\]

we also have the identity

\[
(f \chi_{\{v \geq 0\}} - g \chi_{\{v < 0\}}) \nabla v = \nabla (fv^+ + gv^-) - \nabla f v^+ - \nabla g v^-,
\]

so that

\[
\int_{B_r} x \cdot \nabla (fv^+ + gv^-) = r \int_{\partial B_r} (fv^+ + gv^-) - n \int_{B_r} (fv^+ + gv^-),
\]

so that

\[
\int_{\partial B_r} |\nabla v|^2 = 2 \int_{\partial B_r} \left( \frac{\partial v}{\partial \nu} \right)^2 + \frac{n - 2}{r} \int_{\partial B_r} |\nabla v|^2 - 2 \int_{B_r} (fv^+ + gv^-)
\]

\[
+ \frac{2n}{r} \int_{B_r} (fv^+ + gv^-) + \frac{2}{r} \int_{B_r} [(x \cdot \nabla f) v^+ + (x \cdot \nabla g) v^-]
\]

and hence

\[
\frac{\partial}{\partial r} \left( \frac{1}{r^{n+2}} \int_{B_r} |\nabla v|^2 \right) = - \frac{4}{r^{n+3}} \int_{\partial B_r} v \frac{\partial v}{\partial \nu}
\]

\[
+ \frac{2(n + 2)}{r^{n+3}} \int_{B_r} (fv^+ + gv^-) + \frac{2}{r^{n+2}} \int_{\partial B_r} \left( \frac{\partial v}{\partial \nu} \right)^2 - \frac{2}{r^{n+2}} \int_{\partial B_r} (fv^+ + gv^-)
\]

\[
+ \frac{2}{r^{n+3}} \int_{B_r} [(x \cdot \nabla f) v^+ + (x \cdot \nabla g) v^-],
\]  

(3.6)

where we have also used the identity

\[
- \frac{4}{r^{n+3}} \int_{B_r} |\nabla v|^2 = - \frac{2}{r^{n+3}} \int_{B_r} [\Delta (v^2) - 2v \Delta v]
\]

\[
= - \frac{4}{r^{n+3}} \int_{\partial B_r} v \frac{\partial v}{\partial \nu} + \frac{4}{r^{n+3}} \int_{\partial B_r} (fv^+ + gv^-).
\]

Since

\[
\frac{\partial}{\partial r} \left( \frac{2}{r^{n+2}} \int_{B_r} (fv^+ + gv^-) \right)
\]

\[
= - \frac{2(n + 2)}{r^{n+3}} \int_{B_r} (fv^+ + gv^-) + \frac{2}{r^{n+3}} \int_{\partial B_r} (fv^+ + gv^-) \ d\sigma
\]

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and
\[ \frac{\partial}{\partial r} \left( \frac{2}{r^{n+3}} \int_{\partial B_r} v^2 \right) = -\frac{8}{r^{n+3}} \int_{\partial B_r} v^2 + \frac{4}{r^{n+3}} \int_{\partial B_r} v \frac{\partial v}{\partial \nu}, \]

(3.4) follows, with
\[ e(r) = \frac{2}{r^{n+3}} \int_{B_r} \left[ (x \cdot \nabla f) v^+ + (x \cdot \nabla g) v^- \right]. \]

(3.5) is an immediate consequence of this formula and the fact that \( x_0 \in S_v, \ v \in C^{1, \gamma} \).

**Corollary 3.2.** If \( f = f_0, g = g_0 \), both constants, and \( W'(r) = 0 \) for \( 0 < r < r_0 \), \( v(x_0 + x) \) is homogeneous of degree 2 in \( x \).

**Proof.** From the formula for \( W' \) and the fact that \( e \equiv 0 \) in this case. \( \square \)

**Corollary 3.3.** \( W_1(r) = W(r) + Dr^\gamma \) (where \( D = D(n, \gamma, \|\nabla f\|_\infty, \|\nabla g\|_\infty, N) \geq 0, \ D(-, -, 0, 0, -) \equiv 0 \)) is increasing for \( 0 < r < r_0 \).

For further use we will recall Kato’s inequality:

**Lemma 3.4** (Kato [Kat73]). Assume that \( w \in W^{2,2}_{\text{loc}}(U) \). Then, \( \Delta |w| \geq (\text{sign } w) \Delta w \) in the \( H^1_{\text{loc}}(U) \) sense, i.e. for all \( \theta \in C_0^\infty(U), \theta \geq 0 \), we have
\[ -\int \nabla |w| \cdot \nabla \theta \geq \int (\text{sign } w) \Delta w \theta. \]

**Lemma 3.5.** For \( 0 < r < r_0, \ x_0 \in S_v \), we have
\[ \frac{\partial}{\partial r} \left( \frac{1}{2r^{n+3}} \int_{\partial B_r} v^2 \right) = \frac{1}{r} \left[ W_1(r) - \frac{1}{r^{n+3}} \int_{B_r} \left[ (fv^+ + gv^-) \right] - Dr^\gamma \right]. \]

**Proof.** Recall from the proof of Lemma 3.1, that
\[ \frac{\partial}{\partial r} \left( \frac{1}{2r^{n+3}} \int_{\partial B_r} v^2 \right) = -\frac{2}{r^{n+4}} \int_{\partial B_r} v^2 + \frac{1}{r^{n+3}} \int_{\partial B_r} v \frac{\partial v}{\partial \nu}. \]

But,
\[ \int_{\partial B_r} v \frac{\partial v}{\partial \nu} = \frac{1}{2} \int_{\partial B_r} \frac{\partial}{\partial r} (v^2) = \frac{1}{2} \int_{B_r} \Delta (v^2) = \int_{B_r} v \Delta v + |\nabla v|^2 = \int_{B_r} |\nabla v|^2 + \int_{B_r} [fv^+ + gv^-] \]

and the Lemma follows. \( \square \)
We now let, for $0 < r < r_0$, $v_r(x) = \frac{v(rx + x_0)}{r^2}$, $f_r(x) = f(rx + x_0)$, $g_r(x) = g(rx + x_0)$, where $x_0 \in S_r$. Note that $\Delta v_r = f_r \chi_{\{v_r \geq 0\}} - g_r \chi_{\{v_r < 0\}}$ in $B_1 = B(0, r) (x_0 = 0)$.

**Lemma 3.6.** Let $v_r^{(1)} = f_r v_r^+(x) + (g_r + \eta_0/2) v^-(x) - a_1 |x|^2$, $v_r^{(2)} = v_r^- + a_2 |x|^2$, where $a_i = a_i(n, B_1, B_2, B_3, N, \alpha, \Lambda) \geq 0$. Then:

(i) $v_r^{(1)}$ is superharmonic in $B_1$.

(ii) $v_r^{(2)}$ is subharmonic and non-negative in $B_1$.

(iii) $v_r^+$ is subharmonic in $B_1$.

**Proof.** All functions are continuous, so we just need to check the sign of the distributional Laplacian. Note that

$$v_r^{(1)} = \frac{f_r + g_r + \eta_0/2}{2} v_r(x) + \frac{f_r - g_r - \eta_0/2}{2} v_r(x) - a_1 |x|^2$$

$$\Delta v_r^{(1)} = \frac{f_r + g_r + \eta_0/2}{2} \Delta (|v_r(x)|) + \frac{f_r - g_r - \eta_0/2}{2} \Delta v_r(x)$$

$$+ 2 \frac{\nabla (f_r + g_r)}{r} \nabla (|v_r|) + 2 \frac{\Delta (f_r - g_r)}{r} \cdot v_r$$

$$+ \frac{\Delta (f_r + g_r)}{2} |v_r| + \frac{\Delta (f_r - g_r)}{r} v_r - a_1 n$$

$$\leq \frac{f_r + g_r + \eta_0/2}{2} (\text{sign } v_r) (f_r \chi_{\{v_r \geq 0\}} - g_r \chi_{\{v_r < 0\}})$$

$$+ \frac{f_r - g_r - \eta_0/2}{2} (\text{sign } v_r) (f_r \chi_{\{v_r \geq 0\}} - g_r \chi_{\{v_r < 0\}}) + 2 \tilde{B}_2 N + 2 \tilde{B}_3 N - a_1 2n$$

and (i) follows. (Here we have used that $(f_r + g_r + \eta_0/2) < 0$.) Also,

$$v_r^{(2)}(x) = \frac{|v_r(x)| - v_r(x)}{2} + a_2 |x|^2,$$

so that

$$\Delta v_r^{(2)}(x) = \frac{(\text{sign } v_r) v_r(x) - \Delta v_r(x)}{2} + 2na_2$$

$$= \frac{(f_r \chi_{\{v_r \geq 0\}} - g_r \chi_{\{v_r < 0\}}) - (f_r \chi_{\{v_r \geq 0\}} - g_r \chi_{\{v_r < 0\}})}{2} + 2na_2$$

$$\geq g_r \chi_{\{v_r < 0\}} + 2na_2$$

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and (ii) follows. For (iii) note that $v^+_r = |v^+_r| + v^+_r/2$, so that

$$\Delta v^+_r(x) \geq \frac{(\text{sign } v_r)\Delta v_r(x) + \Delta v_r(x)}{2} + 2na_2$$

$$= \frac{(f_r \chi_{\{v_r \geq 0\}} + g_r \chi_{\{v_r < 0\}}) + (f_r \chi_{\{v_r \geq 0\}} - g_r \chi_{\{v_r < 0\}})}{2}$$

$$= f_r \chi_{\{v_r \geq 0\}} \geq 0.$$

Corollary 3.7. $-\int_{B_1} [f_r v^+_r + (g_r + \eta_0/2) v^-_r] \geq -a_3$, where $a_3 > 0$ has the same dependance as $a_i$ in Lemma 3.6.

Proof. $v_r^{(1)}$ is superharmonic in $B_1$, $v_r^{(1)}(0) = 0$. Then

$$v_r^{(1)}(0) \geq \int_{B_1} f_r v^+_r + (g_r + \eta_0/2) v^-_r - a_1|x|^2$$

and the Corollary follows.

We now define, for $x_0 \in S_v$, $0 < r < r_0$, $S(r) = \left(\int_{\partial B_r} v^2\right)^{1/2}$.

Lemma 3.8 (Non-degeneracy). $\liminf_{r \to 0} \frac{S(r)}{r^2} > 0$

Proof. Assume without loss of generality, that $x_0 = 0$. If the conclusion fails, we can find $r_i \to 0$ such that $\frac{S(r_i)}{r_i^2} \to 0$. Let $v_i(x) = \frac{v(r_i x)}{r_i^2}$, so that $\int_{\partial B_1} v_i^2 \to 0$. Note that, in $B_1$, $\Delta v_i = f_r \chi_{\{v_i \geq 0\}} - g_r \chi_{\{v_i < 0\}} \geq \eta_0 > 0$. Also, $|\Delta v_i| \leq 2\tilde{B}_1$, $v_i(0) = 0$. By subharmonicity of $v_i$, $v_i(0) = 0$ we see that $\int_{B_1} v^-_i \leq \int_{B_1} v^+_i$. Since, by (iii) in Lemma 3.6 $v^+_i$ is subharmonic, $\int_{B_1} v^+_i \leq c_n \int_{\partial B_1} v^+_i \leq c_n \left(\int_{\partial B_1} (v^+_i)^2\right)^{1/2}$. Thus,

$$\int_{B_1} |v_i| \leq \int_{B_1} v^+_i + \int_{B_1} v^-_i \leq 2 \int_{B_1} v^+_i \leq 2c_n \left(\int_{\partial B_1} (v^+_i)^2\right)^{1/2} \to 0.$$

After passing to a subsequence $v_i \to v_0$, where the convergence is uniform on compact subsets of $B_1$ and in $W^{2,2}_{loc}(B_1)$. But then $\Delta v_0 \geq \eta_0 > 0$, but $\int_{B_1} |v_0| = 0$, a contradiction.
Remark 3.9. Note that the above proof shows that, if $S^+(r) = \left(\int_{\partial B_r} (v^+)^2\right)^{1/2}$, then $\liminf_{r \to 0} \frac{S^+(r)}{r^2} > 0$.

We now turn to the classification of blow-up points, following the ideas of Monneau-Weiss [MW05].

Lemma 3.10. Let $-M = \lim_{r \to 0} W_1(r)$. Assume that $x_0 \in S_v$ is such that $M < \infty$. Then, there exists $G = G(n, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3, N, M)$ such that $\sup_{0 < r < r_0} S(r) \leq G$.

Proof. Note that, in view of Lemma 3.5, if $0 < r < r_0$ is such that $-1 \frac{n}{n+2} \int_{B_r} [fv^+ + gv^-] > M + Dr^\gamma$, then $\frac{\partial}{\partial r} \left( \frac{1}{r^{n+2}} \int_{\partial B_r} v^2 \right) > 0$. Note that the last inequality is equivalent with $\frac{\partial}{\partial r} \left( \int_{B_1} v_r^2 \right) > 0$. Our first step in the proof is to show that there exists $C_1 = C_1(n, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3, N, \Omega)$ such that, for $0 < r < r_0$ we have

$$\int_{\partial B_1} (v^+_r)^2 \leq C_1 \left\{ 1 + \int_{\partial B_1} (v^-_r)^2 \right\}$$

In order to establish (3.7), we first prove an auxiliary Claim:

Claim 3.11. For each $R > 0$, there exists $\epsilon_0 = \epsilon_0(R, n)$ such that if $w(0) = 0$, $\Delta w^+ \geq 0$, $0 \leq \Delta w \leq \epsilon_0$, $\int_{B_1} |\nabla w|^2 \leq R$ and $\int_{\partial B_1} (w^-)^2 \leq \epsilon_0$, then $\left( \int_{\partial B_1} w^2 \right)^{1/2} \leq 2$, but $\int_{\partial B_1} w^2 \geq 1/2$.

Proof of Claim 3.11. If not, we can find $R > 0$ and functions $w_j$ with $w_j(0) = 0$, $0 \leq \Delta w_j \leq 1/j$, $\int_{B_1} |\nabla w_j|^2 \leq R$, $\left( \int_{\partial B_1} w_j^+ \right)^{1/2} \leq 2$, $\int_{\partial B_1} (w^-_j)^2 \leq 1/j$ but $\int_{\partial B_1} w_j^+ \geq 1/2$. Since the $w_j^+$ are subharmonic, $\int_{B_1} w_j^+ \leq c_n \int_{\partial B_1} w_j^+ \leq 2c_n$. Since $w_j$ are subharmonic, $w_j(0) = 0$, $\int_{B_1} w_j^- \leq \int_{B_1} w_j^+ \leq 2c_n$. Hence, by Poincare’s inequality $\int_{B_1} w_j^2 \leq (R + 4c_n) \alpha_n$. Hence, we can find a subsequence, (still called $j$) such that $w_j \to w$, uniformly on compact sets and $\int_{B_1} |\nabla w|^2 \leq R$. Moreover, by compactness in the trace theorem, we have $\int_{\partial B_1} w^2 \geq 1/2$. We also have $\Delta w = 0$, $w(0) = 0$, $\left( \int_{\partial B_1} w^2 \right)^{1/2} \leq 2$ and
\[ \int_{\partial B_1} w^- = 0. \] But then \( w \geq 0, w(0) = 0 \) and \( \Delta w = 0 \) imply \( w \equiv 0 \), a contradiction. \[ \square \]

Suppose now that (3.7) fails for some fixed \( C_1 > 1 \), to be determined. Then, there exists a sequence \( \{r_m\} \), \( 0 < r_m < r_0 \) so that \( \int_{\partial B_1} (v_{r_m}^+)^2 \geq C_1 \left\{ 1 + \int_{\partial B_1} (v_{r_m}^-)^2 \right\} \). Using Corollary 3.3, we see that

\[ \int_{B_1} |\nabla v_{r_m}|^2 - 2 \int_{\partial B_1} v_{r_m}^2 \leq W_1(r_0) + 2Dr_0^\gamma - 2 \int_{B_1} (fv_{r_m}^+ + gw_{r_m}^-) \leq W_1(r_0) + 2Dr_0^\gamma - 2 \int_{B_1} gw_{r_m}^- . \]  

(3.8)

Consider now \( w_n = v_{r_m}/ \left( \int_{\partial B_1} (v_{r_m}^+)^2 \right)^{1/2} \). Note that \( w_n(0) = 0, \Delta w_n \geq 0 \) and by (iii) in Lemma 3.6, we have \( \Delta w_n^+ \geq 0 \). Also \( \left( \int_{\partial B_1} (w_n^+)^2 \right)^{1/2} \leq (1+1/C_1^{1/2}) \leq 2, \int_{\partial B_1} w_n^2 \geq \int_{\partial B_1} (w_n^+)^2 = 1 \) and \( |\Delta w_n| \leq C/C_1^{1/2} \), where \( C = C(B_1) \), since \( \int_{\partial B_1} (v_{r_m}^+)^2 \geq C_1 \). Moreover, \( \left( \int_{\partial B_1} (w_n^-)^2 \right)^{1/2} \leq 1/C_1^{1/2} \).

But, (3.8) shows that

\[ \int_{B_1} |\nabla w_n|^2 \leq 2 \int_{\partial B_1} w_n^2 + \frac{W_1(r_0)}{\int_{\partial B_1} (v_{r_m}^+)^2} + \frac{2Dr_0^\gamma}{\int_{\partial B_1} (v_{r_m}^+)^2} \]

\[ + 2cnB_1 \cdot \left( \int_{\partial B_1} (v_{r_m}^-) + a_2 \right) / \int_{\partial B_1} (v_{r_m}^+)^2 , \]

in view of Lemma 3.6, (ii). Finally, since \( \int_{\partial B_1} (v_{r_m}^+)^2 \geq C_1 \geq 1, \left( \int_{\partial B_1} w_n^2 \right)^{1/2} \leq 2 \) and \( \int_{\partial B_1} (v_{r_m}^-)^2 \leq \left( \int_{\partial B_1} (v_{r_m}^-)^2 \right)^{1/2} \leq \frac{1}{C_1^{1/2}} \left( \int_{\partial B_1} (v_{r_m}^+)^2 \right)^{1/2} \), we see that

\[ \int_{B_1} |\nabla w_n|^2 \leq R, R = R(B_1, \tilde{B}_2, \tilde{B}_3, n, N, r_0), \] for all \( C_1 \geq 1 \). But if we now choose \( C/C_1 \leq \epsilon_0 \), \( \frac{1}{C_1^{1/2}} \leq \epsilon_0 \), where \( \epsilon_0 \) is as in Claim 3.11, we reach a contradiction to Claim 3.11, establishing (3.7).

We now proceed to the completion of the proof of Lemma 3.10. For \( 0 < r < r_0, \tilde{r} \in (r/2, r) \), we have: \( W_1(r) - W_1(\tilde{r}) \leq W_1(r_0) + M \). But, by
Lemma 3.1,

\[ W_1(r) - W_1(\tilde{r}) = \int_{\tilde{r}}^r W'_1(s) \, ds \]

\[ = \int_{\tilde{r}}^r 2s \int_{\partial B_1} (\partial_s v_s)^2 \, ds + \int_{\tilde{r}}^r e(s) \, ds + \gamma D \int_{\tilde{r}}^r s^{\gamma-1} \, ds \]

\[ \geq \int_{\tilde{r}}^r 2s \int_{\partial B_1} (\partial_s v_s)^2 \, ds, \]

by our choice of \( D \) and the fact that \( \partial_s v_s = \frac{x \cdot \nabla v(sx + x_0)}{s^3} - \frac{2v(sx + x_0)}{s^3} \).

The right hand side is bigger than \( r \int_{\tilde{r}}^r \int_{\partial B_1} (\partial_s v_s)^2 \, d\sigma ds \), which by Cauchy-Schwarz is bigger than \( \int_{\partial B_1} (v_r - v_{\tilde{r}})^2 \, d\sigma \). Hence, for \( 0 < r < r_0 \), \( \tilde{r} \in (r/2, r) \),

we have

\[ \int_{\partial B_1} (v_r - v_{\tilde{r}})^2 \leq (W_1(r_0) + M). \]  

(3.9)

We next show:

**Claim 3.12.** There exists \( \tilde{M} = \tilde{M}(n, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3, N, M, r_0, \eta_0) \) such that if \( \int_{\partial B_1} v_r^2 > \tilde{M} \), then \( \frac{\partial}{\partial r} \left( \int_{\partial B_1} v_r^2 \right) > 0 \), for \( 0 < r < r_0 \).

To establish the Claim note that in light of the Remark at the beginning of the proof of Lemma 3.10, we only need to show that \( \int_{\partial B_1} v_r^2 \geq \tilde{M} \) implies that

\[ - \int_{\tilde{B}_1} [f_r v_r^+ + g_r v_r^-] > M + Dr^\gamma. \]

By Corollary 3.7, \( - \int_{\tilde{B}_1} [f_r v_r^+ + (g_r + \eta_0/2) v_r^-] \geq -a_3 \), so it is enough to show that

\[ \eta_0 \int_{\tilde{B}_1} v_r^- > M + Dr^\gamma + a_3. \]  

(3.10)

From (ii) in Lemma 3.6, we have (by interior estimates),

\[ \left( \int_{3/4 < |x| < 3/4} (v_r^- + a_2 |x|^2)^2 \right)^{1/2} \leq c_n \int_{B_1} (v_r^- + a_2 |x|^2), \]

so that

\[ \int_{B_1} v_r^- \geq \frac{1}{c_n} \left( \int_{3/4 < |x| < 3/4} (v_r^-)^2 \right)^{1/2} - c_n. \]  

(3.11)
But, \[ \int_{1/2 < |x| < 3/4} (v_{-r})^2 \geq a_n \int_{1/2}^{3/4} \int_{\partial B_1} (v_{rs})^2 \, d\sigma ds \] and

\[ \int_{\partial B_1} (v_{rs})^2 = \int_{\partial B_1} (v_{rs})^2 - (v_{rs}^+)^2 \geq \int_{\partial B_1} (v_{rs})^2 - C_1 - C_1 \int_{\partial B_1} (v_{rs}^{-})^2, \]

from (3.7). Thus, \[ \int_{\partial B_1} (v_{rs})^2 \geq \frac{1}{1 + C_1} \int_{\partial B_1} (v_{rs})^2 - \frac{C_1}{C_1 + 1}, \]
and so, from (3.11) we obtain

\[ \int_{B_1} v_{-r}^2 \geq d_n \left( \int_{1/2}^{3/4} \int_{\partial B_1} v_{rs}^2 d\sigma ds \right) - C_2, \]

with \( C_2 \) having the same dependence as \( C_1 \). If we now use (3.8) with \( \tilde{r} = rs \), we see using (3.9),

\[ \int_{B_1} v_{-r} \geq \tilde{d}_n \left( \int_{\partial B_1} v_{rs}^2 \, d\sigma ds \right) - b_n (W_1(r_0) + M)^{1/2} - C_2 \]

and (3.10) holds for \( \tilde{M} \) large enough.

We can now conclude the proof of Lemma 3.10: if \( S(r)/r^2 \leq \tilde{M} \) for \( 0 < r < r_0 \), we are done. If for all \( 0 < r < r_0 \), \( S(r)/r^2 > \tilde{M} \), then by Claim 3.12 we have \( S(r)/r^2 = \int_{\partial B_1} v_r^2 < S(r_0)/r_0^2 \) for \( 0 < r < r_0 \) and we are also done. Note that, if for some \( 0 < r_1 < r_0 \) we have \( S(r_1)/r_1^2 > \tilde{M} \), then for all \( r_1 < r < r_0 \) we have \( S(r)/r^2 > \tilde{M} \), by virtue of Claim 3.12. It is now easy to show that \( S(r)/r^2 \leq \max \left( \tilde{M}, S(r_0)/r_0 \right) \), \( 0 < r < r_0 \). Thus, Lemma 3.10 follows.

\[ \square \]

**Corollary 3.13.** Let \( M, G \) be as in Lemma 3.10. Then there exists \( \tilde{G} \), with the same dependence as \( G \) such that, for \( 0 < r < r_0/2 \), we have

\[ \sup_{|x| \leq 1} |v_r(x)| + \left( \int_{B_1} |\nabla v_r|^2 \right)^{1/2} \leq \tilde{G}. \]

**Proof.** By Corollary 3.3, for \( 0 < r < r_0 \) we have

\[ \int_{B_1} |\nabla v_r|^2 \leq 2 \int_{\partial B_1} v_r^2 - 2 \int_{B_1} (fv_r^+ + gv_r^-) + 2Dr_0^\gamma + W_1(r_0) \]

\[ \leq 2 \int_{\partial B_1} v_r^2 - 2 \int_{B_1} gv_r^- + 2Dr_0^\gamma + W_1(r_0). \]
Using now Lemma 3.6 (ii) and Lemma 3.10, the gradient estimate follows. For the $L^\infty$ estimate, we use Lemma 3.6 (i) and (ii) and the fact that for non-negative subharmonic functions, the $L^2$ spherical averages are increasing. Thus, for instance

$$
\sup_{|x| \leq 1} |v_r^+(x)| \leq \sup_{|x| = 1} |v_r^+(x)| \leq c_n \left( \int_{\partial B_1} |v_r^+|^2 \right)^{1/2} 
\leq c_n \left( \int_{1/2 < |x| < 3/2} |v_r^+|^2 \right)^{1/2} \leq \tilde{c}_n \left( \int_{\partial B_1} (v_{2r}^+)^2 \right)^{1/2}
$$

and correspondingly for $v_r^-$.

We are now ready, in analogy with [MW05], to state our classification of blow-up points.

**Theorem 3.1.** Assume that $x_0 \in S_v$ and $W_1(r)$ is defined in Corollary 3.3.

(i) If $\lim_{r \downarrow 0} W_1(r) = -M$, $M < +\infty$, then $S(r)/r^2$ and $\|v_r\|_{W^{2,p}(B_1)}$, $1 < p < \infty$ remain bounded for $0 < r < r_0/2$. Moreover, if $\{r_j\}$ is a sequence tending to 0, after passing to a subsequence $\{r_{j'}\}$, the functions $v_{r_{j'}}$ converge in $C^{1,\gamma}(B_1)$, $0 \leq \gamma < 1$ and $W^{2,p}(B_1)$, $1 \leq p < \infty$ to a function $\bar{v}$. The function $\bar{v}$ solves the equation

$$
\Delta \bar{v} = f_0 \chi_{\{\bar{v} \geq 0\}} - g_0 \chi_{\{\bar{v} < 0\}} \text{ in } \mathbb{R}^n,
$$

with $f_0 = f(x_0)$, $g_0 = g(x_0)$ and is homogeneous of degree 2.

(ii) If $\lim_{r \downarrow 0} W_1(r) = -\infty$, then $\lim_{r \downarrow 0} \frac{S(r)}{r^2} = +\infty$. Let $r_j \downarrow 0$ and define $w_j(x) = \frac{v(r_j x + x_0)}{S(r_j)}$, $T_j = \frac{S(r_j)}{r_j^2}$. Then, after passing to a subsequence $\{r_{j'}\}$, $w_j$ converge in $C^{1,\gamma}(B_1)$ and $W^{2,p}(B_1)$, $0 \leq \gamma < 1$, $1 \leq p < \infty$ to a harmonic function $\bar{w}$, with $\bar{w}(0) = \nabla \bar{w}(0) = 0$, which is non-zero and homogeneous of degree 2.

**Proof.** From Corollary 3.13, in case (i) it only remains to show that $\bar{v}$ is homogeneous of degree 2. But, for any $0 < s < 1$ we have $W(s) = W(s; 0; \bar{v}) = \ldots$
\[
\lim_{j' \to \infty} W_1(s r_{j'}, x_0, v) = -M. \text{ Then, from Corollary 3.2 } \bar{v} \text{ is homogeneous of degree 2.}
\]

For case (ii), we must have
\[
\lim_{r \to 0} \int_{B_1} |\nabla v_r|^2 + 2 \int_{B_1} f v_r^2 + 2 \int_{B_1} g v_r^2 - 2 \int_{\partial B_1} v_r^2 = -\infty
\]
But then, since \(f > 0, g < 0\), we must have that
\[
\lim_{r \to 0} 2 \int_{\partial B_1} v_r^2 - 2 \int_{B_1} g v_r^2 = +\infty. \quad (3.12)
\]
By Lemma 3.6 (ii),
\[
\int_{B_1} v_r^2 \leq c_n a_2 + c_n \int_{\partial B_1} v_r^2 \leq c_n a_2 + \left( \int_{\partial B_1} (v_r^2)^2 \right)^{1/2}.
\]
But then, since \((-g) \geq \eta_0\), and \(-g \leq \bar{B}_1\), we conclude from (3.12) that
\[
\lim_{r \to 0} \int_{\partial B_1} v_r^2 = +\infty, \text{ which in turn implies } \lim_{r \to 0} \int_{\partial B_1} v_r^2 = +\infty, \text{ or }
\]
\[
\lim_{r \to 0} \frac{S(r)}{r^2} = +\infty. \text{ By Corollary 3.3, dividing by } T_j^2, \text{ we obtain }
\]
\[
\int_{B_1} |\nabla w_j|^2 \leq \frac{W_1(r_0)}{T_j^2} + \frac{2}{T_j} \int_{B_1} \left[ f_{r_j} w_j^+ + g_{r_j} w_j^\gamma \right] + 2 \int_{\partial B_1} w_j^2 - \frac{D_{r_j}}{T_j^2}. \quad (3.13)
\]
Also, for \(j\) large, \(|\Delta w_j| \leq 1 \text{ in } B_1\), \(\int_{\partial B_1} w_j^2 = 1, \Delta w_j^+ \geq 0, \Delta w_j \geq 0, w_j(0) = 0\). Then, \(\int_{B_1} (w_j)^2 \leq C\) and from the formulae above \(\int_{B_1} |\nabla w_j|^2 \leq 3\), for \(j\) large. Thus, the \(w_j\), after passing to a subsequence \(j'\), converge uniformly on compacts and in \(C^{1, \gamma}(B_1)\), \(W^{2, p}(B_1)\) to a \(\bar{w}\) which is harmonic in \(B_1\), \(\bar{w}(0) = \nabla \bar{w}(0) = 0\). Also, by compactness of the trace operator, \(\int_{\partial B_1} \bar{w}^2 = 1\), so that \(\bar{w}\) is not zero. But, from (3.13), we conclude that \(\int_{B_1} |\nabla \bar{w}|^2 \leq 2 \int_{\partial B_1} |\bar{w}|^2\). But then, by the Almgren monotonicity formula (see for example Lemma 4.2 in [MW05]) \(w\) is homogeneous of degree 2.

**Corollary 3.14 (No mixed asymptotics).** We cannot have for two sequences \(\{r_j\}, \{\tilde{r}_j\}\), both tending to 0 that \(\lim_{j \to \infty} \frac{S(r_j)}{r_j^2} = +\infty, \text{ but } \sup_j \frac{S(\tilde{r}_j)}{\tilde{r}_j^2} < +\infty.\)
Proof. If \( \lim_{r \downarrow 0} W_1(r) = -\infty \), then for all such sequences the limit is \(+\infty\). On the other hand if \( \lim_{r \downarrow 0} W_1(r) > -\infty \), we have boundedness near \( r = 0 \). In either case the mixed asymptotic assumption leads to a contradiction. \( \square \)

We will next use these results to study partial regularity of the free boundary \( \mathcal{F} \). We start with a 2-dimensional result, due to Shahgholian ([Sha])

**Theorem 3.2** ([Sha]). Let \( v \) be the solution of (3.2), when \( n = 2 \), under our assumptions. Assume that \( x_0 \in S_v \) is such that \( |\{v < 0\} \cap B(x_0, r)| \geq c_0 r^2 \), for \( 0 < r < r(x_0) \), with \( c_0 > 0 \). Then \( x_0 \) is an isolated point of \( S_v \).

We will provide a proof of this Theorem, (following [Sha]), for the reader’s convenience. The key point is the following

**Lemma 3.15** ([Sha]). Assume that \( \bar{v} \) is a homogeneous of degree 2 solution to (3.2) in \( \mathbb{R}^2 \), with \( f = f_0, g = g_0 \), both constants. (As before, \( f_0 > 0, g_0 < 0, f_0 + g_0 < 0 \)). Then, \( S_{\bar{v}} = \{0\} \), or, after rotation, \( S_{\bar{v}} = \{(x_1, x_2) = (0, x_2) : x_2 \in \mathbb{R}\} \). In this case \( \bar{v} = \frac{f_0}{2} x_1^2 \).

**Proof.** Recall that \( \Delta \bar{v} \geq \eta_0 > 0 \) (\( \eta_0 = \min(f_0, g_0) \)). Assume that \( S_{\bar{v}} \neq \{0\} \). After rotation we can assume that, by the homogeneity of \( \bar{v} \), \( (0, 1) \in S_{\bar{v}} \), so that \( \lambda(0, 1) \in S_{\bar{v}}, \lambda > 0 \). Assume first that \( \bar{v} \geq 0 \) in a neighborhood of \( (0, 1) \). Then, in an angle, \( \Delta \bar{v} = f_0 \). Consider \( w = \bar{v} - \frac{f_0}{2} x_1^2 \). Then, in this angle, by uniqueness in the Cauchy problem, \( w \equiv 0 \). But this argument can be continued all around, so that \( \bar{v} = \frac{f_0}{2} x_1^2 \). Thus, if not, there exists a neighborhood of \( (0, 1) \) in which \( \bar{v} < 0 \) is non-empty. Assume, for instance that the negative point is in the top right quadrant. By homogeneity, the point can be taken on the unit circle. But then, all the points in the unit circle between this point and the vertical axis are points where \( \bar{v} \) is negative, otherwise we would have a local maximum, contradicting the subharmonicity of \( \bar{v} \). But then, if we consider a small half-ball in the top right quadrant, centered at \( (0, 1) \), the Hopf maximum principle yields a contradiction to \( \bar{v}(0, 1) = 0, \nabla \bar{v}(0, 1) = 0 \). \( \square \)

**Proof of Theorem 3.2.** We can assume, without loss of generality, that \( x_0 = 0 \). Suppose we have \( x_j \in S_v, x_j \to 0 \). Let \( r_j = |x_j| \). Assume first that \( \lim_{r_j \downarrow 0} W_1(r_j) = -\infty \). Then, by Theorem 3.1 (ii), \( \frac{v(r_j, x)}{s(r_j)} \), after passing to a subsequence, converges in \( C^{1,\gamma}(B_1), L^2(\partial B_1) \) to a harmonic polynomial \( \tilde{w} \) homogeneous of degree 2 and non-zero. Moreover, \( \frac{x_j}{|x_j|} \to \bar{x} \in \partial B_1 \), and
\[ \bar{w}(\bar{x}) = 0, \nabla \bar{w}(\bar{x}) = 0. \] But, when \( n = 2 \), \( \bar{w} \) must be a rotate of \( a(x_1^2 - x_2^2) \) and hence \( S_{\bar{w}} = \{0\} \), a contradiction. If \( \lim_{r \to 0} W_1(r) > -\infty \), by Theorem 3.2 (i), \( \frac{v(r,x)}{r^k} \) converges, after passing to a subsequence to a \( \bar{v} \), a homogeneous of degree 2 solution, \( f = f_0, g = g_0 \). Clearly \( |\{\bar{v} < 0\} \cap B_1| \geq c_0 \). Also, \( \bar{x} \in S_{\bar{v}} \), so that by Lemma 3.15, \( \bar{v} = \frac{f_0}{2} x_1^2 \), after a rotation, which is a contradiction.

We will extend next Theorem 3.2 to \( n > 2 \). The argument is a standard one from the theory of minimal surfaces (see Chapter 11 of [Giu84], whose notation for Hausdorff measures and Hausdorff dimension we adopt). Similar arguments have been used by Weiss [Wei98] and Monneau-Weiss [MW05] in the context of free boundary problems. Our result here is:

**Theorem 3.3.** Let \( v \) be a solution of (3.2), \( n \geq 2 \), under our assumptions. Let \( \bar{S}_v = \{x_0 \in S_v : |\{v < 0\} \cap B(x_0, r)| \geq c_0 r^n \text{, for } 0 < r < r_0(x_0)\} \). Then, for each fixed \( c_0 > 0 \), the Hausdorff dimension of \( \bar{S}_v \) is at most \( n - 2 \).

**Proof.** Fix \( k > n - 2 \), we need to show that \( H_k(\bar{S}_v) = 0 \). Assume not, so that \( H_k(\bar{S}_v) > 0 \). Consider the sets

\[ \bar{S}_v^j = \{x_0 \in S_v : |\{v \leq 0\} \cap B(x_0, r)| \geq c_0 r^n \text{, for } 0 < r < 1/j\}. \]

Then, \( \bar{S}_v = \bigcup_{j=j_0}^{\infty} \bar{S}_v^j \), where \( 1/j_0 < r_0 \). Then, for some \( j \geq j_0 \) we have \( H_k(\bar{S}_v^j) > 0 \). Hence by Proposition 11.3 in [Giu84], for \( H_k \)-almost all \( x_0 \in \bar{S}_v^j \), we have

\[ \limsup_{r \to 0} \frac{H_k^\infty(\bar{S}_v^j \cap B(x_0, r))}{\omega_k r^k} \geq 2^{-k}. \] (3.14)

Fix such an \( x_0 \), which we assume, without loss of generality, to be 0. Choose a sequence \( r_n \to 0 \) such that

\[ \frac{H_k^\infty(\bar{S}_v^j \cap B_{r_n})}{\omega_k r_n^k} \geq 2^{-k}. \]
Consider $v_n(x) = \frac{v(x)}{S(x)}$ and let $\tilde{v}(x)$ be a blow-up limit of a subsequence of $v_n$, in the sense of Theorem 3.1. Fix a compact set $K$ in $B_1$, $U$ open $\subset B_1$, with $U \supset K \cap \tilde{S}_v^j$. Assume that $x_n \in \tilde{S}_v^j$, $x_n \in K \setminus U$ and after passing to a subsequence, assume that $x_n \to \bar{x} \in K \setminus U$. Then, $v_n(x_n) \to \tilde{v}(\bar{x})$, $\nabla v_n(x_n) \to \nabla \tilde{v}(\bar{x})$, so that $\bar{x} \in S_\psi$. Also, fix $0 < r < 1/j$. Then

$$
|\{\tilde{v} \leq 0\} \cap B(\bar{x}, r)| = |\{\tilde{v} < 0\} \cap B(\bar{x}, r)| = \lim_{n \to \infty} |\{v_n < 0\} \cap B(x_n, r)| \geq c_0 r^n,
$$

and so $\bar{x} \in \tilde{S}_v^j$, but $\bar{x} \in K \setminus U$, and $K \cap \tilde{S}_v^j \subset U$, which is a contradiction. Thus, we have shown that there exists $n_0$ so that, for $n > n_0$, we have

$$U \supset K \cap \tilde{S}_v^j$$

(3.15)

Then, the proof of Lemma 11.5 in [Giu84] shows that for all $K \subset B_1$, we have

$$H_k^\infty \left( K \cap \tilde{S}_v^j \right) \geq \limsup_{n \to \infty} H_k^\infty \left( K \cap \tilde{S}_{v_n}^j \right).$$

(3.16)

We next claim that

$$\left\{ x/r_n : x \in \tilde{S}_v^j \right\} \subset \tilde{S}_{v_n}^j.$$

(3.17)

In fact, clearly $v_n(x/r_n) = 0$, $\nabla v_n(x/r_n) = 0$. Consider now $\{y : v_n(y) < 0\} \cap B(x/r_n, r)$, $0 < r < 1/j$. This equals $\{y : v_n(y) < 0\} \cap \{y : |y - x/r_n| < r\}$. By the transformation $y = z/r_n$, this set equals

$$\{z : v(z) < 0\} \cap \left\{ z : \left| \frac{z}{r_n} - \frac{x}{r_n} \right| < r \right\} = \{z : v(z) < 0\} \cap \{z : |z - x| < r r_n \}.$$

Also, if $0 < r < 1/j$, $r r_n < 1/j$, $n$ large. The Lebesgue measure of the set of $y$'s equals $(r_n)^{-n}$ times that Lebesgue measure of the set of $z$'s, which is then bigger than $\frac{1}{(r_n)^n} \cdot c_0 (r n)^n = c_0 r^n$, so that $x/r_n \in \tilde{S}_{v_n}^j$. But then,

$$H_k^\infty \left( B_1 \cap \tilde{S}_{v_n}^j \right) \geq \frac{H_k^\infty \left( B_{r_n} \cap \tilde{S}_{v_n}^j \right)}{\omega_k r_n} \geq 2^{-k},$$

by our choice of $r_n$. Hence, using (3.16), we see that

$$H_k^\infty \left( B_1 \cap \tilde{S}_v^j \right) > 0$$

(3.18)
We now consider our classification of blow-ups. If \( \lim_{r \to 0} W_1(r) = -\infty \), then, by (ii) \( \bar{v} \) is a non-zero, homogeneous of degree 2 harmonic polynomial. But then, as is well-known \( H_{n-2}(S_0) < \infty \), \( S_0 \supset \bar{S}_0^j \), which contradicts (3.18) since \( k > n-2 \). If \( \lim_{r \to 0} W_1(r) > -\infty \), in view of Theorem 3.1 (i) and Lemma 3.8, after passing to a further sequence, we can assume that \( r_n^2 S(r_n) \to \alpha \), \( \alpha \in (0, \infty) \).

Hence, \( \alpha \bar{v} = \bar{v}_1 \), where \( \bar{v}_1 \) is a homogeneous of degree 2 solution to (3.2) with \( f = f_0 \), \( g = g_0 \), both constants. We can now do the dimension reduction.

From (3.18), we know that \( H_{k-1}^\infty (B_1 \cap \bar{S}_0^j) > 0 \). Using Lemmas 11.2 and 11.3 in [Giu84], we can find \( \bar{x} \in \bar{S}_0^j \setminus \{0\} \) such that \( \lim_{r \to 0} \frac{H_{k-1}^\infty (\bar{S}_0^j \cap B(\bar{x}, r))}{\omega_{k-1} r^k} \geq 2^{-k} \).

By homogeneity of \( \bar{v}_1 \), we can assume that \( \bar{x} \in \partial B_1 \). We can pick a sequence \( r_n \to 0 \), and consider a blow-up limit \( \bar{v}_{1,0} \), at \( \bar{x} \), with respect to \( r_n \). By the homogeneity of \( \bar{v}_1 \), it is easy to see that \( \bar{v}_{1,0} \) is constant in the \( \bar{x} \) direction. After rotation, we can assume this direction to be the \( x_n \) direction. But, it is easy to see that \( (x_1, x_2, \cdots, x_{n-1}, x_n) \in \bar{S}_0^j \{|x_n| = 1\} \) and that \( H_{k-1}^\infty (\bar{S}_0^j \{|x_n| = 1\}) > 0 \).

Proceeding in this way \( n-2 \) times, we find a contradiction to Theorem 3.2, which concludes the proof.

We are now ready to establish partial \( C^{1,1} \) bounds.

**Definition 3.16.** Let \( f \) be a \( C^{1,\gamma} \), \( 0 \leq \gamma < 1 \) function defined in a neighborhood of a point \( x_0 \). We say that \( f \) satisfies \( C^{1,1} \) bounds at \( x_0 \) if

\[
\lim_{r \to 0} \sup_{|x-x_0| \leq r} \frac{|f(x) - (x-x_0) \nabla f(x_0) - f(x_0)|}{r^2} < +\infty.
\]

We call the above limit “the \( C^{1,1} \) norm of \( f \) at \( x_0 \).”

Our next task is to show that our solutions \( v \) verify \( C^{1,1} \) bounds at all \( x_0 \in \mathcal{F} \), except for a set of Hausdorff dimension at most \( n - 2 \). We start out with some preliminary results.

**Lemma 3.17.** There exists a constant \( c_n \) such that for all homogeneous of degree 2 harmonic polynomials \( p, p \neq 0 \), we have

\[
|\{p < 0\} \cap B_1| \geq c_n.
\]
Proof. We can assume \( \int_{B_1} p^2 = 1 \). If the conclusion fails, we can find a sequence \( p_j \), \( \int_{B_1} p_j^2 = 1 \), \( p_j \) a harmonic polynomial, homogeneous of degree 2, with \( |\{p_j < 0\} \cap B_1| \xrightarrow{j \to 0} 0 \). After passing to a subsequence, \( p_j \to p_0 \), \( p_0 \) a harmonic polynomial, homogeneous of degree 2, \( \int_{B_1} p_0 = 1 \) and such that \( |\{p_0 < 0\} \cap B_1| = 0 \). By homogeneity, \( p_0 \geq 0 \), but \( p_0(0) = 0 \), so that \( p_0 \equiv 0 \), a contradiction.

Lemma 3.18. Let \( c_n \) be as in Lemma 3.17. Assume that \( v \) is a solution, \( x_0 \in S_v \). Assume that for some sequence \( r_j \to 0 \), \( \sup_{|x-x_0|<r_j} \frac{|v(x)|}{r_j^2} \to \infty \). Then,

\[
|\{v < 0\} \cap B(x_0, r)| \geq \frac{c_n}{2} r^n, \quad \text{for } 0 < r < r_0(x_0).
\]

Proof. If not, there exists \( \tilde{r}_j \to 0 \), such that

\[
|\{v < 0\} \cap B(x_0, \tilde{r}_j)| < \frac{c_n}{2} (\tilde{r}_j)^n.
\]

But, by the proof in Corollary 3.13, we see that \( \frac{S(2\tilde{r}_j)}{(2\tilde{r}_j)^2} \to +\infty \). By Corollary 3.14 we have \( \frac{S(\tilde{r}_j)}{\tilde{r}_j^2} \to +\infty \). But then, by Theorem 3.1 (ii), \( \frac{v(x_0+x_0)}{S(\tilde{r}_j)} \) converges, after passing to a subsequence, to a \( \bar{w} \) which is a non-zero harmonic polynomial homogeneous of degree 2. But then, \( |\{\bar{w} < 0\} \cap B_1| \leq \frac{c_n}{2} \), which contradicts Lemma 3.17.

Theorem 3.4 (Pointwise \( C^{1,1} \) bounds on \( S_v \)). Let \( v \) be a solution. Consider the set \( B_v = \{x_0 \in S_v : v \) does not have pointwise \( C^{1,1} \) bounds at \( x_0 \} \). Then, the Hausdorff dimension of \( B_v \) is at most \( n-2 \).

Proof. Combine Lemma 3.18 with Theorem 3.3.

Remark 3.19. If \( x_0 \in \mathcal{F}, \nabla v(x_0) \neq 0 \), then by [CGK00] \( \mathcal{F} \) is real analytic in a neighborhood of \( x_0 \) and by boundary elliptic regularity we obtain \( C^{1,1} \) bounds at \( x_0 \). Thus, the set of points in \( \mathcal{F} \) for which \( v \) does have pointwise \( C^{1,1} \) bounds has Hausdorff dimension at most \( (n-2) \).

Remark 3.20. The results in Theorems 3.2, 3.3, 3.4 and in Remark 3.19 are sharp. We show this for the case \( f = f_0, g = g_0 \) constants. In order to show this, we make some preliminary comments, in the case \( n = 2 \). In this case, Blank ([Bla04]) found all homogeneous of degree 2 solutions, for
which \( \{v < 0\} \neq \emptyset \). The calculation in Appendix 2 shows that, for these solutions, \( W(1) > -A \), where \( A \) depends only on \( f_0, g_0 \). Shahgholian ([Sha]) observed that there are other homogeneous of degree 2 solutions, which are non-negative. In fact, any such solution \( \tilde{v} \) verifies \( \Delta \tilde{v} = f_0, \tilde{v} \geq 0 \) in \( \mathbb{R}^2 \). Let \( w = \tilde{v} - \frac{f_0}{4}(x_1^2 + x_2^2) \). This is a harmonic polynomial, homogeneous of degree 2, so that, after rotation \( w = a(x_1^2 - x_2^2) \) or \( \tilde{v} = a + \frac{f_0}{4}x_1^2 + \left(\frac{f_0}{4} - a\right)x_2^2 \).

Since \( \tilde{v} \geq 0 \), we must have \(-\frac{f_0}{4} \leq a \leq \frac{f_0}{4}\). For those solutions we also find \( W(1) > -A \), \( A \) depending only on \( f_0, g_0 \). Combining these comments with Theorem 3.1, we see that, for \( n = 2 \) there exists \( A = A(f_0, g_0) \) such that, if for \( v \) we have \( \lim_{r \downarrow 0} W_1(r) < -A \), then \( \lim_{r \downarrow 0} W_1(r) = -\infty \) and \( \lim_{r \downarrow 0} \frac{S(r)}{r^2} = +\infty \).

One can then use the argument in [AW05] to see that, by the Anderson-Weiss construction we can find solutions (taking \( M \) large in [AW05]) so that \( W_1(1) < -A \), and hence, solutions which don’t have \( C^{1,1} \) bounds towards 0. In light of Lemma 3.17, this shows the sharpness of Theorem 3.2 and of Theorem 3.4 when \( n = 2 \). To create higher dimensional examples, one just adds \( n - 2 \) dummy variables. It remains a challenging problem to see if such pathology can hold for solutions of (3.2).

We now turn to the issue of uniform pointwise \( C^{1,1} \) bounds.

**Theorem 3.5.** Let \( S_v^{(1)} = S_v / S_v^{(2)} \), where

\[
S_v^{(2), j} = \{x_0 \in S_v : |\{v < 0\} \cap B(x, r)| \geq \frac{1}{j} r^n, 0 < r < r_{0,j}(x_0)\}, \quad S_v^{(2)} = \bigcup_{j=1}^{\infty} S_v^{(2), j}.
\]

Note that Theorem 3.3 shows that the Hausdorff dimension of \( S_v^{(2)} \) is at most \( n - 2 \). Then, for \( x_0 \in S_v^{(1)} \) we have uniform \( C^{1,1} \) estimates, i.e. there exists \( C = C(\bar{B}_1, \bar{B}_2, \bar{B}_3, n, \eta_0, r_0, N) > 0 \) such that for all \( x_0 \in S_v^{(1)} \),

\[
\sup_{|x-x_0| \leq r \atop 0 < r < r_0/2} \frac{|v(x)|}{r^2} \leq C.
\]

**Proof.** In light of Theorem 3.1, Lemma 3.17 and Corollary 3.13 it suffices to show that for such \( x_0 \lim_{r \downarrow 0} W_1(r) > -A \), where \( A \) has the right dependence. Let \( \tilde{v} \) be a blow-up limit at such an \( x_0 \). Clearly, \( \tilde{v} \geq 0 \). Thus, it suffices to show that, for such \( \tilde{v}, W(1, \tilde{v}) > -A \). But, \( \Delta \tilde{v} = f_0, \int_{B_1} \Delta \tilde{v} = w_n f_0 = \int_{\partial B_1} \frac{\partial}{\partial v \tilde{v}} = 2 \int_{\partial B_1} \tilde{v}, \) since \( \tilde{v} \) is homogeneous of degree 2. Thus, \( \int_{\partial B_1} \tilde{v} = \frac{w_n f_0}{2} \).

Since \( \tilde{v} \) is non-negative and subharmonic, \( \int_{B_1} \tilde{v} \leq c_n f_0 w_n / 2 \). The rest of the proof follows easily from interior estimates and homogeneity. \( \square \)
Remark 3.21. Similarly, if \( K \in \{ x_0 \in \mathcal{F} : \nabla v(x_0) \neq 0 \} \) we also have uniform pointwise \( C^{1,1} \) bounds on \( K \). (See Remark 3.19).

Our final result is a partial regularity result for \( \mathcal{F} \).

**Theorem 3.6.** Let \( v \) be a solution of (3.2) satisfying our assumptions. Then \( \mathcal{F} = \mathcal{F}_0 \cup S^{(1)}_v \cup S^{(2)}_v \), where \( S^{(2)}_v \) has Hausdorff dimension at most \( (n-2) \), \( S^{(1)}_v \) is \( (n-1) \) regular i.e. \( H_{n-1}(S^{(1)}_v) \leq C \), with \( C = C(\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, N, \eta_0, r_0, n) \) and \( \mathcal{F}_0 \) is relatively open and for each \( x_0 \in \mathcal{F} \) there exists a neighborhood \( U_{x_0} \) such that \( \mathcal{F} \cap U_{x_0} \) is a real-analytic hypersurface.

**Proof.** \( \mathcal{F}_0 = \{ x_0 \in \nabla v(x_0) \neq 0 \} \) and \( S^{(1)}_v, S^{(2)}_v \) are defined in Theorem 3.5. From Theorem 3.5 we know that the Hausdorff dimension of \( S^{(2)}_v \) is at most \( n-2 \), so it remains to show that \( S^{(1)}_v \) is \( (n-1) \) regular, (in light of Theorem 8 in [CGK00], which shows the desired property of \( \mathcal{F}_0 \)). In order to show this, we make some preliminary claims.

**Claim 3.22.** If \( x_0 \in S^{(1)}_v \) (without loss of generality, we take \( x_0 = 0 \)) we have, for \( 0 < r < r_0/4 \), \( x \in B_r \), \( |\nabla v(x)| \leq Cr \), with \( C \) as in the statement of Theorem 3.6.

In order to establish the Claim, note that for \( x \in B_{2r} \), we have \( |v(x)| \leq C|x|^2 \), by Theorem 3.5. Next, we use Lemma 3.6 (ii) and (iii) to obtain:

\[
\int_{B_r} |\nabla v^+|^2 \leq c_n C r^{n+2} \quad \text{and} \quad \int_{B_r} |\nabla v^-|^2 \leq c_n \{ C + a_2 \} r^{n+2}
\]

so that \( \int_{B_r} |\nabla v|^2 \leq c_n (C + a_2) r^{n+2} \). Next, consider \( v_r(x) \) on \( B_1 \). We have \( \int_{B_1} |v_r|^2 \leq C, \int_{B_1} |\nabla v_r|^2 \leq C, \) and \( |\Delta v_r| \leq C \). From this it is easy to see that, for \( |x| \leq 1/2 \) we have \( |\nabla v_r| \leq C \), which is our claim.

The next step is:

**Claim 3.23.** Let \( x_0 \in S_v, e_i \) be a fixed coordinate direction, \( v_{e_i} = e_i \cdot \nabla v \). Then, for \( 0 \leq h, \) small, we have

\[
\int_{B(x_0, r_0/2) \cap \{ x : |\nabla v| \leq h \}} |\nabla v_{e_i}|^2 \leq Ch.
\]
To establish Claim 3.23, we first introduce a truncation of \( v_{e_i} \in W^{1,2}(U) \cap C^\gamma (\overline{U}) \), to obtain \( \bar{v}_{e_i} \), where

\[
\bar{v}_{e_i} = \begin{cases} 
  v_{e_i} & \text{if } -h < v_{e_i} < -\delta \text{ or } \delta < v_{e_i} < h, \\
  0 & \text{if } |v_{e_i}| \leq \delta, \\
  h & \text{if } |v_{e_i}| \geq h.
\end{cases}
\]

Let \( \psi \) be a standard mollifier and for \( 0 < \epsilon \ll \delta \), consider the mollifier \( v_{e_i} \ast \psi_\epsilon \). We will apply Green’s Theorem to

\[
\int_{B_r} \nabla \bar{v}_{e_i} \cdot \nabla (v_{e_i} \ast \psi_\epsilon) , \text{ for } \frac{r_0}{2} < r < r_0,
\]

where we have assumed, without loss of generality that \( x_0 = 0 \). Since \( |F| = 0 \) (see Theorem 1.1 (c)), this integral equals

\[
\int_{B_r \cap \{ v > 0 \}} \nabla \bar{v}_{e_i} \cdot \nabla (v_{e_i} \ast \psi_\epsilon) + \int_{B_r \cap \{ v < 0 \}} \nabla \bar{v}_{e_i} \cdot \nabla (v_{e_i} \ast \psi_\epsilon).
\]

On \( S_v \), \( \nabla v = 0 \), so that \( \bar{v}_{e_i} \) will vanish on a neighborhood of \( S_v \). In fact, if \( |v_{e_i}(x)| \geq \delta \), \( z_0 \in S_v \), then \( \delta \leq |v_{e_i}(x) - v_{e_i}(z_0)| \leq C|x - z_0|^{\gamma} \). In \( F \setminus \text{nbd}(S_v) \), we have analyticity of \( F \) and a well-defined normal, so that we can integrate by parts in the above integrals, using Green’s Theorem. We obtain for the above sum,

\[
- \int_{B_r \cap \{ v > 0 \}} \bar{v}_{e_i} \Delta (v_{e_i} \ast \psi_\epsilon) - \int_{B_r \cap \{ v < 0 \}} \bar{v}_{e_i} \Delta (v_{e_i} \ast \psi_\epsilon) + \int_{\partial B_r} \bar{v}_{e_i} \frac{\partial}{\partial v} (v_{e_i} \ast \psi_\epsilon)
\]

\[
+ \int_{B_r \cap \{ v > 0 \}} \bar{v}_{e_i} \frac{\partial}{\partial v} (v_{e_i} \ast \psi_\epsilon) + \int_{B_r \cap \{ v < 0 \}} \bar{v}_{e_i} \frac{\partial}{\partial v} (v_{e_i} \ast \psi_\epsilon).
\]

The last two integrals cancel each other since the normals point in opposite directions, in pieces of a real analytic surface. Thus, we have obtained:

\[
\int_{B_r} \nabla \bar{v}_{e_i} \cdot \nabla (v_{e_i} \ast \psi_\epsilon) = - \int_{B_r \cap \{ v > 0 \}} \bar{v}_{e_i} \Delta (v_{e_i} \ast \psi_\epsilon)
\]

\[
- \int_{B_r \cap \{ v < 0 \}} \bar{v}_{e_i} \Delta (v_{e_i} \ast \psi_\epsilon) + \int_{\partial B_r} \bar{v}_{e_i} \frac{\partial}{\partial v} (v_{e_i} \ast \psi_\epsilon).
\]
We next average this identity in $r$, for $r \in (\frac{r_0}{2}, \frac{3r_0}{4})$. We estimate first the averaged last term. Its absolute value is bounded by

$$c_n h \int_{\frac{r_0}{2} \leq |x| \leq \frac{3r_0}{4}} |\nabla v_{ei} * \psi_e| \leq c_n Ch.$$ 

We next consider the absolute value of the averaged term of the left-hand side, as $\epsilon \to 0$. It converges to

$$\frac{4}{r_0} \int_{r_0/2}^{3r_0/4} \int_{B_r} \nabla \tilde{v}_{ei} \cdot \nabla v_{ei} \bigg|_{\delta \to 0} \frac{4}{r_0} \int_{r_0/2}^{3r_0/4} \int_{B_r} \nabla \tilde{v}_{ei} \cdot \nabla v_{ei} \bigg|$$

where

$$\tilde{v}_{ei} = \begin{cases} v_{ei} & \text{if } |v_{ei}| \leq h, \\ h & \text{otherwise}. \end{cases}$$

This last expression is bounded below by $c_n \int_{B_{r_0/2}} |\nabla \tilde{v}_{ei}|^2$.

The absolute value of the sum on the averaged first two terms in the right hand side converges (first letting $\epsilon \to 0$ and then $\delta \to 0$) to

$$\frac{4}{r_0} \int_{r_0/2}^{3r_0/4} \int_{B_r \cap \{v > 0\}} \tilde{v}_{ei} \Delta v_{ei} \bigg| + \frac{4}{r_0} \int_{r_0/2}^{3r_0/4} \int_{B_r \cap \{v < 0\}} \tilde{v}_{ei} \Delta v_{ei} \bigg|.$$ 

But on $\{v > 0\}$, $\Delta v_{ei} = \partial_{ei} f$, on $\{v < 0\}$, $\Delta v_{ei} = -\partial_{ei} g$. Hence, the above sum is bounded by

$$h \left( \int_{B_{r_0} \cap \{v > 0\}} |\Delta v_{ei}| + \int_{B_{r_0} \cap \{v < 0\}} |\Delta v_{ei}| \right) \leq Ch.$$ 

Finally, gathering terms and using that

$$\int_{B_{r_0/2}} |\nabla \tilde{v}_{ei}|^2 = \int_{B_{r_0/2} \cap \{|v_{ei}| \leq h\}} |\nabla v_{ei}|^2,$$

Claim 3.23 follows.

We next complete the proof of the bound $H_{(n-1)} \left( S_{\nu}^{(1)} \right) \leq C$. Fix $z_0 \in S_{\nu}^{(1)}$ and consider $S_{\nu}^{(1)} \cap B(z_0, r_0/4)$. It suffices to prove our bound for this intersection. For each $x_0$ in $S_{\nu}^{(1)} \cap B(z_0, r_0/4)$, and each $0 < r < r_0/100$, we
consider the cover of \( S_v^{(1)} \cap B(z_0, r_0/4) \) by the balls \( B(x_0, r) \). We can cover \( S_v^{(1)} \cap B(z_0, r_0/4) \) by finitely many such balls, and by the Vitali covering Lemma, we can find \( \tilde{N} \) disjoint balls \( B(x_i, r) \), \( x_i \in S_v^{(1)} \cap B(z_0, r_0/4) \) so that \( S_v^{(1)} \cap B(z_0, r_0/4) \subset \bigcup_{i=1}^{\tilde{N}} B(x_i, 5r) \). The disjointness of \( \{B(x_i, r)\} \) gives

\[
\sum_{i=1}^{\tilde{N}} \chi_{B(x_i, 5r)}(x) \leq c_n. \]

By Claim 3.22, \( |\nabla v(x)| \leq Cr \) in \( B(x_i, 5r) \). By (3.2) \( |\Delta v| \geq C \). We then have:

\[
c_n \tilde{N} r^n \leq \sum_{i} \int_{B(x_i, 5r)} (\Delta v)^2 \leq \int_{\{|\nabla v(x)| \leq cr\}} (\Delta v)^2 \sum_{i=1}^{\tilde{N}} \chi_{B(x_i, 5r)} \leq c_n \int_{B(z_0, r_0/2) \cap \{|\nabla v(x)| \leq cr\}} (\Delta v)^2 \leq Cr,
\]

by Claim 3.23. Thus, \( \tilde{N} r^{n-1} \leq C \), which gives our Hausdorff measure bound.

To conclude this paper we give a simple result in the direction of showing that better regularity results can hold for solutions of the composite problem than for solutions of (3.2) (see the end of Remark 3.20). We will show that geometric assumptions on \( \Omega \) can ensure that for all solutions of the composite problem, \( S_u = \emptyset \) and thus \( \mathcal{F} \) is real analytic and \( u \) is \( C^{1,1} \).

**Proposition 3.7.** Let \( \Omega \subset \mathbb{R}^2 \) have two axis of symmetry. Then for all solutions \( u \) of the composite problem (1.1), (1.2) we have \( S_u = \emptyset \) and hence \( \mathcal{F} \) is real analytic and \( u \in C^{1,1} \).

**Proof.** We recall (see [CGI+00]) that we say that \( \Omega \) has an axis of symmetry \( L \) (which we take to be \( \{x_1 = 0\} \)) if whenever \( (x_1, x_2) \) belongs to \( \Omega \), so does \( (-x_1, x_2) \) and the set \( \{x_1 : (x_1, x_2) \in \Omega\} \) is either \( \emptyset \) or an interval \( (-c, c) \) for each \( x_2 \). Let us give the proof, for simplicity, in the case when the two axis \( L_1, L_2 \) are the \( x_1 \) and \( x_2 \) axis. It is shown in [CGI+00], Theorem 4, that any solution \( u \) is symmetric with respect to \( x_1 \) (and \( x_2 \)) and \( u \) is strictly decreasing in \( x_1 \), for \( x_1 \geq 0 \) (in \( x_2 \), for \( x_2 \geq 0 \)). (The strict decrease follows from \( \alpha < \Lambda \), see [CGI+00], the bottom of page 326). Because of the strict decrease, \( \frac{\partial}{\partial x_1} u(x_1, x_2) \neq 0, x_1 \neq 0 \) and \( \frac{\partial}{\partial x_2} u(x_1, x_2) \neq 0 \) for \( x_2 \neq 0 \). Thus, the only possible point in \( S_u \) is \( (0, 0) \). But, by the increase and decrease described
before $u(0, 0) = \sup_{\Omega} u$. Recall that $D = \{0 \leq u \leq c\}$, $\mathcal{F} = \{u = c\}$. If $c = \sup_{\Omega} u$, $D = \Omega$, which contradicts $|D| = A < |\Omega|$. Thus, $(0, 0) \notin \mathcal{F}$ and the Proposition follows. \hfill \square

### Appendix I

The results (A1.9), (A1.10) are to be found in [CP]. They are reproduced here for the reader’s benefit.

We have the equation

$$-\Delta u_t + \alpha \chi_{D_t} u_t = \lambda(t) u_t$$

(A1.1)

and the corresponding one for $u_0 = u$, given by

$$-\Delta u + \alpha \chi_D u = \lambda u$$

(A1.2)

where $\lambda(0) = \lambda$. We also note that by our definition of $D_t$, $\chi_{D_t}(x) = \chi_D(\phi_t(x))$. (A1.3)

We will set,

$$V(x) = \frac{d\phi_t(x)}{dt} \bigg|_{t=0}$$

and assume the vector field $V \in C^2(\Omega)$ and that $V$ is supported in a compact set $S$.

Multiplying (A1.1) by $u$, (A1.2) by $u_t$ and subtracting we get,

$$u_t \Delta u - u \Delta u_t + \alpha(\chi_D(\phi_t(x)) - \chi_D(x)) uu_t = (\lambda(t) - \lambda) uu_t. \quad (A1.4)$$

We integrate A1.4 over $\Omega$. Since, $u = u_t = 0$ on $\partial \Omega$ we get

$$\int_{\Omega} u_t \Delta u - u \Delta u_t = 0$$

Thus the integral over $\Omega$ of (A1.4) becomes,

$$\int_{\Omega} \alpha(\chi_D(\phi_t(x)) - \chi_D(x)) uu_t = (\lambda(t) - \lambda) \int_{\Omega} uu_t. \quad (A1.5)$$
Now from (A1.1) we notice that if we normalize our functions \( \| u_t \|_2 = 1 \) as we certainly can, we always have \( \| u_t \|_{2,2} \leq C \). Now,

\[
\left| \int_\Omega (uu_t - u^2) \right| \leq \int_\Omega |u - u_t|
\]

In a tubular neighborhood of \( \partial \Omega, \mathcal{U} \) we have,

\[
\int_\mathcal{U} |u - u_t| \leq C \left( \int_\mathcal{U} u^2 \right)^{1/2} \leq \epsilon
\]

Outside \( \mathcal{U} \) by the uniform \( W^{2,2} \) bounds of \( u_t \) we have strong convergence of \( u_t \) to \( u \) in \( L^2 \). Thus we have,

\[
\lim_{t \to 0} \int_\Omega uu_t = \int_\Omega u^2 = 1. \quad \text{(A1.6)}
\]

Now we change variables in the left side of (A1.5). We set \( \phi_{-1}(x) = y \). Thus, \( x = \phi_{-1}^{-1}(y) \). Thus the left side of (A1.5) becomes,

\[
\alpha \int_\Omega \chi_D(x)(h_t(\phi_{-1}^{-1}(x)))J_t(x) - h_t(x))dx.
\]

Here we have set \( h_t(x) = uu_t \) and \( J_t(x) \) is the determinant of the Jacobian matrix of the transformation \( y = \phi_{-1}^{-1}(x) \). Since \( \phi_0(x) = I \) the identity, it is well-known that,

\[
J_t(x) = 1 + t \text{ div } V + O(t^2). \quad \text{(A1.7)}
\]

See for example Lemma 1(pg 69) in [Arn97], in fact (A1.7) is an elementary consequence of the fact that for a \( n \times n \) matrix \( B \), \( \det(I - tB)^{-1} = 1 + t \text{ trace } B + O(t^2) \). Since \( h_t \in C^{1,\beta} \), we see that,

\[
h_t \left( \phi_{-1}^{-1}(x) \right) J_t(x) - h_t(x) = t ((V \cdot \nabla)h_t(x) + h_t(x)\text{div } V) + o(t)
\]

Thus on division by \( t \) and letting \( t \to 0 \) we see easily,

\[
\lim_{t \to 0} \frac{h_t \left( \phi_{-1}^{-1}(x) \right) J_t(x) - h_t(x)}{t} = (V \cdot \nabla)(u^2) + u^2\text{div } V.
\]

The term on the right above is,

\[
\text{div } (Vu^2).
\]
Thus dividing (A1.5) by $t$ and using (A1.6) we easily get,

$$\lambda'(0) = \alpha \int_D \text{div} (Vu^2) = \alpha \int_{D \cap S} \text{div} (Vu^2).$$

By the hypothesis that the part of the boundary of $\partial D$ that lies inside the support of $V$ is regular enough to have a bonafide unit outer normal $\nu$, and Green’s theorem, the last integral above yields,

$$\lambda'(0) = \alpha \int_{S \cap \partial D} \langle V, \nu \rangle u^2.$$ (A1.8)

Now consider,

$$|D_t| - |D| = \int_{\Omega} (\chi_D(\phi_{-t}(x)) - \chi_D(x)) \, dx$$

Change variables in the integral above as before to get,

$$\int_D (J_t(x) - 1) \, dx.$$ By (A1.7) again we see the integral above is,

$$t \int_D \text{div} V \, dx + O(t^2).$$

Thus we easily get,

$$\frac{d}{dt} (|D_t|)_{t=0} = \int_D \text{div} V \, dx = \int_{S \cap \partial D} \langle V, \nu \rangle \, d\sigma.$$ (A1.9)

If $u = c$ along $\partial D$, combining (A1.8) and (A1.9) we get,

$$\lambda'(0) = \alpha c^2 \frac{d}{dt} (|D_t|)_{t=0} = \alpha c^2 \int_{\partial D} \langle V, \nu \rangle \, d\sigma.$$ (A1.10)

**Appendix II**

We use Blank’s [Bla04] notation. We have,

$$f_1(\theta) = C_+ \sin(2\theta + D_+) \gamma, \quad f_1 > 0$$
and also,
\[ f_2(\theta) = C_- \sin(2\theta + D_-) + \mu, \quad f_2 < 0 \]

Now we focus on the interval \([0, 2\pi/3]\). First for \(\theta_0 \in (0, 2\pi/3)\), we know \(f_1(\theta_0) = f_2(\theta_0) = 0\) and \(f_1'(\theta_0) = f_2'(\theta_0) = 0\). We get,
\[ C_+ \sin(2\theta_0 + D_+) + \gamma = C_- \sin(2\theta_0 + D_-) + \mu = 0 \]

and,
\[ C_+ \cos(2\theta_0 + D_+) = C_- \cos(2\theta_0 + D_-) \]

This leads after squaring and adding both equations to,
\[ C^2_+ - C^2_- = \gamma^2 - \mu^2. \tag{A2.1} \]

Next because \(f_1(0) = f_1(\theta_0) = 0\), we get,
\[ C_+ \sin(D_+) = -\gamma, \quad D_+ = \arcsin(-\gamma/C_+) \tag{A2.2} \]

and also we have,
\[ \theta_0 = \pi/2 + \arcsin(\gamma/C_+). \tag{A2.3} \]

Since \(f_2(\theta_0) = 0\), inserting the value of \(\theta_0\) from (A2.3) in the expression for \(f_2\), we see,
\[ D_- = \arcsin(\mu/C_-) - 2 \arcsin(\gamma/C_+). \tag{A2.4} \]

Now lastly \(f_2(2\pi/3) = 0\), so,
\[ C_- \sin(4\pi/3 + D_-) = -\mu \]

We get,
\[ C_- = \mu/\sin(\pi/3 + D_-) \tag{A2.5} \]

Now assume \(|C_+| > 10^6 (|\gamma| + |\mu|)\), then by (A2.1), \(|C_-| > 10^6 (|\gamma| + |\mu|)\). Thus, from (A2.4), \(|D_-| \leq \pi/20\). From (A2.5) we get,
\[ |C_-| \leq 2|\mu| \]

And we get a bound on \(|C_+|\) from (A2.1) again.
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