Mechanism of entanglement preservation

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We study the entanglement preservation of two qubits locally interacting with their reservoirs. We show that the existence of a bound state of the qubit and its reservoir and the non-Markovian effect are two essential ingredients and their interplay plays a crucial role in preserving the entanglement in the steady state. When the non-Markovian effect is neglected, the entanglement sudden death (ESD) is reproduced. On the other hand, when the non-Markovian is significantly strong but the bound state is absent, the phenomenon of the ESD and its revival is recovered. Our formulation presents a unified picture about the entanglement preservation and provides a clear clue on how to preserve the entanglement in quantum information processing.

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I. INTRODUCTION

Entanglement is not only of fundamental interest to quantum mechanics but also of great importance to quantum information processing [1]. However, due to the inescapable interaction of qubits with their environments, entanglement always experiences degradation. Entanglement sudden death (ESD), a phenomenon in which the entanglement between two qubits may completely disappear in a finite time, has been predicted theoretically [2] and subsequently been verified experimentally [3], indicating specific behavior of entanglement that differs from that of coherence. From the point of view of applications, ESD is apparently disadvantageous to quantum information processing.

Recently, Bellomo et al. [4] found that the entanglement can revive after some time interval of ESD and thus extends significantly the entangled time of the qubits. This remarkable phenomenon, which has been experimentally observed [5], is physically due to the dynamical back action (that is, the non-Markovian effect) of the memory environments [4, 6]. However, in many cases the finite extension of the entangled time is not enough and thus it is desired to preserve a significant fraction of the entanglement in the longtime limit. Indeed, it was shown [7] that some noticeable fraction of entanglement can be obtained by engineering structured environment such as photonic band-gap materials [8, 9]. According to these works, it is still unclear whether the residual entanglement is fundamentally due to the specific structured materials or due to certain physical mechanisms. Is there any essential relationship between ESD and/or its revival phenomena and the residual entanglement?

In this work we focus on these questions and elucidate the physical nature of the residual entanglement. Before proceeding, it is helpful to recall the physics of quantum electrodynamics of a single two-level atom placed in a dielectric with a photonic band gap [8, 9]. The coupling between the excited atom and electromagnetic vacuum in the dielectric leads to a novel photon-atom bound state [9], in which the fractional atomic population on the excited state occurs, also known as population trapping [10]. This result has been verified experimentally for quantum dots embedded in a photonic band-gap environment [11]. The population trapping has been directly connected to the entanglement trapping due to the structured environment [12]. Here we reveal that there are two essential conditions needed to preserve the entanglement. One is the existence of the bound state between the system and its environment, which provides the ability to preserve the entanglement, and the other is the non-Markovian effect, which provides a way to preserve the entanglement. Our result can reproduce ESD when the non-Markovian effect is neglected. The phenomenon of ESD and its revival discussed in Ref. [4] results from the non-Markovian effect when the bound state is not available. The interplay between the availability of the bound state and the non-Markovian effect can lead to a significant fraction of the entanglement preserved in the steady state. We verify these results by considering two reservoirs modeled by the super-Ohmic and Lorentzian spectra, respectively. The result provides a general method on how to protect the entanglement by engineering the environment.

This paper is organized as follows. In Sec. III the model of two independent qubits in two reservoirs is introduced. By exploring the eigen-spectrum of the model, we derive the condition for the formation of bound states between the qubits and their respective reservoirs and discuss its profound consequence on the dynamics of the open two-qubit system. In Sec. IV the entanglement preservation caused by the formation of a bound state and the non-Markovian effect is studied numerically in different cases of the environmental spectral density. Finally, we end with a short conclusion and a discussion.
about the experimental feasibility of our result in Sec. IV.

II. MODEL AND DECOHERENCE DYNAMICS

We consider a system consisting of two independent subsystems, each of which contains a qubit coupled to a zero-temperature reservoir. Due to the dynamical independence between the two subsystems, we can first investigate the single subsystem, then extend our studies to the double-system case. The Hamiltonian of each subsystem is

\[ H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k b_k^\dagger b_k + \sum_k (g_k \sigma_+ b_k + g_k^\ast \sigma_- b_k^\dagger), \]

where \( \sigma_\pm \) are the inversion operators and transition frequency of the qubit, \( b_k^\dagger \) and \( b_k \) are the creation and annihilation operators of the \( k \)th mode with frequency \( \omega_k \) of the reservoir, and \( g_k \) denotes the coupling strength between the atom and the radiation field.

To check the spectrum of the Hamiltonian we first solve the eigenvalue equation

\[ H |\varphi_E\rangle = E |\varphi_E\rangle. \]

If only one excitation is present in the system at zero temperature initially, then \( |\varphi_E\rangle = c_0 |+, \{0_k\}\rangle + \sum_k c_k |-, \{1_k\}\rangle \), where \( |+\rangle \) is the atomic excited (or ground) state, and \( |\{0_k\}\rangle \) and \( |\{1_k\}\rangle \) are the vacuum state and the state with only one photon in the \( k \)th mode of the reservoir, respectively. Substituting Eq. (1) into Eq. (2), one has

\[ y(E) \equiv \omega_0 - \int_0^\infty \frac{J(\omega)}{\omega - E} d\omega = E, \]

where \( J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k) \) is the spectral density of the reservoir. The solution of Eq. (3) highly depends on the explicit form of \( J(\omega) \). If the reservoir contains only one mode \( \omega' \), then \( J(\omega) = g^2 \delta(\omega - \omega') \). This is the ideal Jaynes-Cummings (JC) model [12], in which two bound states in one excitation sector are formed and as a result the dynamics of the system displays a lossless oscillation. When the reservoir contains infinite modes, one can model \( J(\omega) \) by some typical spectrum functions such as the super-Ohmic or Lorentzian forms.

We first consider the super-Ohmic spectrum \( J(\omega) = \eta \omega^\gamma e^{-\omega/\omega_c}, \) where \( \eta \) is a dimensionless coupling constant and \( \omega_c \) characterizes the frequency regime in which the power law is valid [13]. It corresponds to the case in which the reservoir consists of a vacuum radiation field, where \( g_k \propto \sqrt{\omega_k} \) [12]. The existence of a bound state requires that Eq. (3) has at least a real solution for \( E < 0 \). It is easy to check that the solution always exists if the condition \( y(0) < 0 \) (i.e. \( \omega_0 - 2\eta \omega_c^\gamma < 0 \)) is satisfied. Otherwise, no bound state exists. This condition can be fulfilled easily by engineering the environment. For the Lorentzian spectrum it is found that a criterion for the existence of a bound state cannot be obtained analytically. In this case one can use the diagrammatic technique shown later.

The existence of a bound state has a profound implication on the dynamics of the single-qubit system such as the inhibition of spontaneous emission [8, 9, 14]. Furthermore, it also has an important impact on the entanglement dynamics of a two-qubit system, which is governed by the master equation [12]

\[ \dot{\rho}(t) = \sum_{n=1}^{2} \{ -i\Omega(t)[\sigma_+^n \sigma_-^n, \rho(t)] + \Gamma(t)[2\sigma_-^n \rho(t) \sigma_+^n - \sigma_+^n \sigma_-^n \sigma_-^n \sigma_+^n] \}, \]

where \( \Omega(t) = -\text{Im}[\dot{c}_0(t)/c_0(t)] \), and \( \Gamma(t) = -\text{Re}[\dot{c}_0(t)/c_0(t)] \). It is shown that \( c_0(t) \) satisfies

\[ \dot{c}_0(t) + i\omega_0 c_0(t) + \int_0^t c_0(\tau) f(t - \tau) d\tau = 0, \]

where \( f(t - \tau) = \int_{-\infty}^\infty J(\omega) e^{-i\omega(t-\tau)} d\omega \). The time-dependent parameters \( \Omega(t) \) and \( \Gamma(t) \) play the role of Lamb-shifted frequency and decay rate of the qubits, respectively. The integro-differential equation (5) contains the memory effect of the reservoir registered in the time-nonlocal kernel function \( f(t - \tau) \) and thus the dynamics of the qubits displays a non-Markovian effect. If \( f(t - \tau) \) is replaced by the time-local function, then Eq. (4) recovers the conventional Born-Markovian master equation, where the parameters become constants [16], that is, \( \Gamma_0 = \pi J(\omega_0), \Omega_0 = \omega_0 - P \int_{-\infty}^\infty J(\omega) d\omega/\omega_0 \) with \( P \) denoting the Cauchy principal value.

The dynamical consequence of the bound state is decoherence suppression [10]. If a bound state is absent, then Eq. (3) has only complex solutions. Physically this means that the corresponding eigenstate experiences decay from the imaginary part of the eigenvalue during the time evolution, which causes the excited-state population to approach zero asymptotically and the decoherence of the reduced qubit system. While if a bound state is formed, then the population of the atomic excited state in the bound state is constant in time because a bound state is actually a stationary state with a vanishing decay rate during the time evolution. So there will be some residual excited-state population in the long-time limit. Due to this decoherence suppression, we expect that the formation of a bound state plays a constructive role in entanglement preservation under the non-Markovian dynamics, as shown in the following section.

III. ENTANGLEMENT PRESERVATION

To study the entanglement dynamics of the bipartite system, we use the concurrence to quantify entanglement [17]. The concurrence is defined as

\[ C(\rho) = \frac{\max(0, \gamma_{\max})}{\sqrt{\gamma_1} + \sqrt{\gamma_2}} \]

where \( \gamma_{\max} \) is the maximum eigenvalue of the matrix \( \rho_{12} \) and \( \gamma_1, \gamma_2 \) are the second-largest eigenvalues of the partial transpose of \( \rho_{12} \) with respect to subsystem 1 and 2, respectively.
where ics of the two-qubit system. Consider first the super-

an obvious oscillation. When the bound state is avail-

ment approaches zero in a long enough time, as shown by

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for the last one it is available, as shown in Fig. 1(a).

two parameter sets the bound state is absent, while

an initially entangled state

be proved that the concurrence varies from 0 for a separa-

rameter $C$ qubit system with local super-Ohmic reservoirs. (a) Diagram-

Markovian approximation has also been presented by using

Now we are ready to study the entanglement dynam-

max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where the decreasing-

order-arranged quantities $\lambda_i$ are the eigenvalues of the

matrix $\rho(A \otimes B)\rho^*(A \otimes B)$. Here $\rho^*$ means the com-

plex conjugation of $\rho$, and $\sigma_y$ is the Pauli matrix. It can be

proved that the concurrence varies from 0 for a separa-

ble state to 1 for a maximally entangled state. Consider

an initially entangled state $|\psi(0)\rangle = \alpha|++\rangle + \beta|--\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. Then $C(t)$ can be calculated as

$C(t) = \max(0, Q(t))$, where

\begin{equation}
Q(t) = 2|\alpha|\beta|c_0(t)|^2 - 2|\beta|^2|c_0(t)|^2[1 - |c_0(t)|^2],
\end{equation}

which indicates that the time-dependent factor of the

excited state population $|c_0(t)|^2$ determines solely the

entanglement dynamics.

Next we are ready to study the entanglement dynam-

ics of the two-qubit system. Consider first the super-

Ohmic case. Figure 1 shows the entanglement dynamics in different parameter regimes [i.e., $(\omega_c, \eta) = (0.7 \omega_0, 0.2), (0.7 \omega_0, 1.0)$ and $(3.0 \omega_0, 0.2)$]. For the first two parameter sets the bound state is absent, while for the last one it is available, as shown in Fig. 1(a). Whether the bound state exists or not plays a key role in the entanglement preservation in the longtime limit. When the bound state is absent, the residual entanglement approaches zero in a long enough time, as shown by the non-Markovian lines in Figs. 1(b) and 1(c). The difference between these two cases is that Fig. 1(b) shows the weak-coupling regime, where the non-Markovian effect is weak, while Fig. 1(c) shows the strong-coupling regime, where the strong non-Markovian effect leads to an obvious oscillation. When the bound state is available, the situation is quite different, as shown in Fig.

\begin{figure}[ht]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{(Color online) Entanglement dynamics of the two qubit system with local super-Ohmic reservoirs. (a) Diagrammatic solutions of Eq. (6) with different parameters. C(t) as a function of time is shown in (b): $(\omega_c, \eta) = (0.7 \omega_0, 0.2)$, (c): $(\omega_c, \eta) = (0.7 \omega_0, 1.0)$ and (d): $(\omega_c, \eta) = (3.0 \omega_0, 0.2)$. The parameter $\alpha$ is taken as 0.7. For comparison, C(t) under the Markovian approximation has also been presented by using the same parameters.}
\end{figure}

\begin{figure}[ht]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{(Color online) The decay rate $\Gamma(t)$ as a function of time in the non-Markovian and Markovian cases. The parameters used are $\omega_c = 3.0 \omega_0$ and $\eta = 0.2$.}
\end{figure}

which shows a finite disentanglement time when $|\alpha| < |\beta|$. In short, different to the need for a structured environment as emphasized in Ref. [7], our discussion clearly reveals two essential conditions to preserve the entanglement: the availability of the bound state and the non-Markovian effect.

This discussion focused on an almost maximally entan-
gled initial state by taking $\alpha = 0.7$. In Fig. 2 we show the results for different initial states with different initial entanglement. With decreasing initial entanglement, the residual entanglement also decreases in the longtime limit and, finally, ESD happens for $\alpha = 0.3$. The result can be understood from Eq. (6). On the one hand, the residual entanglement is determined by $c_0(\infty)$, which is directly related to the property of the bound state. On the other hand, the residual entanglement is also determined by the competition between the first and the second terms in Eq. (6), which is dependent of the initial state.
In order to make a comparative study and confirm our observations we consider the Lorentzian spectrum if the reservoir is composed of a lossy cavity,

\[ J(\omega) = \frac{1}{2\pi} \frac{\gamma \lambda^2}{(\omega - \omega_0)^2 + \lambda^2}, \]

where \( \gamma \) is the coupling constant and \( \lambda \) is the spectrum width. This model has also been studied in Ref. [3], where the lower limit of the frequency integral in \( f(t - \tau) \) was extended from zero to negative infinity. This extension is mathematically convenient but the availability of the bound state is missed. Here we follow the original definition of the frequency integral ranges.

Our model with the Lorentzian spectral density corresponds exactly to the extended damping J-C model [13]. It is noted that the strong coupling of the J-C model has been achieved in circuit QED [19] and quantum dot [20] systems. Figure 4 shows the entanglement dynamics of the qubits under the Lorentzian reservoir for different spectral widths in the strong-coupling regime. When \( \lambda = 0.1\omega_0 \), Eq. (3) lacks the bound state. According to our discussion, there is no residual entanglement in the longtime limit. This is indeed true, as shown in Fig. 4(b). However, it is noted that before becoming zero the entanglement exhibits “sudden death” and revives for several times. This is an analog of the central result found in Ref. [4], that is, the phenomenon of ESD and revival. Apparently, this is due to the non-Markovian effect with the revival being a result of back action of the memory reservoir. The situation changes with increasing the spectral width, and the bound states become available. A significant fraction of the entanglement initially present is preserved in the longtime limit, where the decay rates shown in the insert of Fig. 4(b) approach zero in these cases. Likewise, the physical nature of the entanglement preservation is still the interplay between the bound state and the non-Markovian effect. The stronger the coupling is, the more striking the entanglement oscillates as a function of time and, consequently, the more noticeable the non-Markovian effect is, as shown in Fig. 5. For \( \gamma = 0.2\omega_0 \), the system is in the weak-coupling regime, where the bound state is also not available. As a result, ESD is reproduced in this case.

In Fig. 6 we present a phase diagram of the entanglement in the steady state for the Lorentzian spectral density. In the large-\( \gamma \) and small-\( \lambda \) regime, the system approaches the J-C model. In this situation the strong back action effect of the reservoirs makes it difficult for the qubit system to form a steady state. The entanglement oscillates with time but has no dissipation. In the small-\( \gamma \) and large-\( \lambda \) regime, the non-Markovian effect is extremely weak and our results reduce to the Markovian case. In the limit of a flat spectral density, the Born-Markovian approximation is applicable and the system has no bound state. This is the case of ESD [2].

**IV. CONCLUSIONS AND DISCUSSION**

In summary, we have studied the entanglement protection of two qubits in two uncorrelated reservoirs. Two essential conditions for preserving the entanglement are explored: the existence of the bound state of the system and its reservoir and the non-Markovian effect. The bound state provides the ability of the entanglement preservation and the non-Markovian effect provides the way to...
Engineer the non-Markovian regime via the potential usage of reservoir engineering \[21\] \[23\]. Many experimental platforms (e.g., mesoscopic ion traps \[21\], cold atom BECs \[22\], and the photonic crystal materials \[14\]) have exhibited the controllability of decoherence behavior of relevant quantum systems through optimally designing the size (i.e., modifying the spectrum) of the reservoir and/or the coupling strength between the system and the reservoir. It is also worth mentioning that a proposal aimed at simulating the spin-Boson model, which is relevant to the one considered in this paper, has been reported in a trapped ion system \[24\]. On the other hand many practical systems can now be engineered to show the novel non-Markovian effect \[5\] \[25\] \[28\]. All these achievements demonstrate that the recent advances have paved the way to experimentally simulate the paradigmatic models of open quantum system, which is one part of the newly emerging field of quantum simulators \[29\]. Our work sheds new light on the way to indirectly control and manipulate the dynamics of a quantum system in these experimental platforms. It provides a clue to preserving the entanglement in quantum information processing.

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