Scalar scattering in Schwarzschild spacetime: Integral equation method

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ABSTRACT: An integral equation method for scalar scattering in Schwarzschild spacetime is constructed. The zeroth-order and first-order scattering phase shift is obtained.

KEYWORDS: Scattering, Schwarzschild spacetime, Scalar field, Integral equation method

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1 Introduction

Scattering in a curved spacetime plays an important role in gravity theory [1, 2]. There are many studies on scattering [3–5]. Approximate methods are important in the study of black-hole scattering, such as the Born approximation [6] and the WKB approximation [7]. Various kinds of fields, e.g., scalar fields [8], spinor fields [9–13], and vector fields [14–16] scattered on black holes are systematically studied. Besides the Schwarzschild spacetime, scattering in other kinds of spacetime are also discussed, such as the Reissner-Nordström spacetime [17–19], the Kerr spacetime [20, 21], and a deformed non-rotating black hole [22]. Some exact results are also obtained [23–25].

Solving a scattering problem on the Schwarzschild spacetime is to solve the scattering solution of the scalar field equation with the Schwarzschild metric. An effective way to solve a differential equation is to convert the differential equation to an integral equation with the help of the Green function. The integral equation then can be solved by the iterative method [26]. The radial equation in the scalar scattering in the Schwarzschild spacetime has both second-order derivative terms and first-order derivative terms, and moreover, there is also a singularity on the horizon. In this paper, we develop an integral equation method for solving the scattering problem of a massive scalar particle in the Schwarzschild spacetime.

Using the integral equation method, we calculate the zeroth-order and first-order contributions of the scattering wave function and the scattering phase shift. The integral equation method constructed in the present paper is a systematic method for solving scattering in curved spacetime and in principle can be applied to other scattering problems in gravity theory.

In section 2, we construct the integral equation for a scalar field in the Schwarzschild spacetime. In section 3, we calculate the phase shift by solving the integral equation. The conclusions are summarized in section 4.
2 Integral equation

In this section, we convert the radial differential equation of a scalar field in the Schwarzschild spacetime into an integral equation. Then we solve the scattering wave function and the scattering phase shift from this integral equation.

2.1 Integral equation

To solve a scalar scattering problem in the Schwarzschild spacetime, technically speaking, is to solve the scalar equation

\[
\left(\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} - \mu^2\right) \Phi = 0
\]

under the Schwarzschild metric

\[
ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2
\]

with \( \mu \) the mass of the particle \([2]\). The corresponding radial equation is

\[
\frac{1}{r^2} \left(1 - \frac{2M}{r}\right) \frac{d}{dr} \left(1 - \frac{2M}{r}\right) \frac{d}{dr} \Phi - \left(1 - \frac{2M}{r}\right) \omega^2 - \left(1 - \frac{2M}{r}\right) \frac{l(l+1)}{r^2} \phi_l (r) = 0,
\]

(2.1)

where \( \phi_l (r) \) is the radial wave function and \( r \geq 2M \).

Introducing \( u_l (r) \) by

\[
\phi_l (r) = \frac{u_l (r)}{r}
\]

and substituting Eq. (2.2) into Eq. (2.1) with a variable substitution \( \rho = r/(2M) \) give an equation of \( u_l (\rho) \):

\[
\left(1 - \frac{1}{\rho}\right) \frac{d}{d\rho} \left(1 - \frac{1}{\rho}\right) \frac{d}{d\rho} u_l (\rho) + \left(2M \eta\right)^2 \left(1 - \frac{1}{\rho}\right) \left[l(l+1)\rho^2 + \frac{1}{\rho^3}\right] + (2M \mu)^2 \rho u_l (\rho) = 0,
\]

(2.3)

where \( \eta = \sqrt{\omega^2 - \mu^2} \). By introducing an effective potential

\[
V_{l}^{eff} (\rho) = \left(1 - \frac{1}{\rho}\right) \left[l(l+1)\rho^2 + \frac{1}{\rho^3}\right] - \frac{(2M \mu)^2}{\rho},
\]

(2.4)

we rewrite Eq. (2.3) as

\[
\left(1 - \frac{1}{\rho}\right) \frac{d}{d\rho} \left(1 - \frac{1}{\rho}\right) \frac{d}{d\rho} u_l (\rho) + \left(2M \eta\right)^2 u_l (\rho) = V_{l}^{eff} (\rho) u_l (\rho).
\]

(2.5)

In order to solve the equation (2.5) by the Green function method, we first solve Eq. (2.5) with \( V_{l}^{eff} (\rho) = 0 \), i.e.,

\[
\left(1 - \frac{1}{\rho}\right) \frac{d^2}{d\rho^2} y_l (\rho) + \left(1 - \frac{1}{\rho}\right) \frac{1}{\rho^2} \frac{d}{d\rho} y_l (\rho) + (2M \eta)^2 y_l (\rho) = 0,
\]

(2.6)

where \( y_l (\rho) \) is \( u_l (\rho) \) with \( V_{l}^{eff} (\rho) = 0 \).

For \( 1 < \rho < \infty \), Eq. (2.6) has two linearly independent solutions:

\[
y_l^{(1)} (\rho) = \sin \left(2M \eta \left[\rho + \ln (\rho - 1)\right]\right),
\]

(2.7)

\[
y_l^{(2)} (\rho) = \cos \left(2M \eta \left[\rho + \ln (\rho - 1)\right]\right).
\]

(2.8)
The Green function can be constructed as

\[ G (\rho, \rho') = C_1 (\rho') y_1^{(1)} (\rho) + C_2 (\rho') y_2^{(2)} (\rho), \quad \rho > \rho', \quad (2.9) \]
\[ G (\rho, \rho') = 0, \quad \rho < \rho', \quad (2.10) \]

in order to satisfy the boundary condition that the Green function must be finite at \( \rho = 1 \) [27].

Continuity requires that [27]

\[ \lim_{\epsilon \to 0^+} G (\rho, \rho') \bigg|_{\rho=\rho'+\epsilon} = \lim_{\epsilon \to 0^+} G (\rho, \rho') \bigg|_{\rho=\rho'-\epsilon}, \quad (2.11) \]
\[ \lim_{\epsilon \to 0^+} \left[ \frac{\partial}{\partial \rho} G (\rho, \rho') \bigg|_{\rho=\rho'+\epsilon} - \frac{\partial}{\partial \rho} G (\rho, \rho') \bigg|_{\rho=\rho'-\epsilon} \right] = \frac{1}{(1-1/\rho)^2}. \quad (2.12) \]

Then we have

\[ C_1 (\rho') \sin (2M\eta [\rho' + \ln (\rho' - 1)]) + C_2 (\rho') \cos (2M\eta [\rho' + \ln (\rho' - 1)]) = 0, \quad (2.13) \]
\[ 2M\eta \left( 1 + \frac{1}{\rho' - 1} \right) C_1 (\rho') \cos (2M\eta [\rho' + \ln (\rho' - 1)]) - 2M\eta \left( 1 + \frac{1}{\rho' - 1} \right) C_2 (\rho') \sin (2M\eta [\rho' + \ln (\rho' - 1)]) = \frac{1}{(1-1/\rho)^2}. \quad (2.14) \]

Solving \( C_1 (\rho') \) and \( C_2 (\rho') \) from Eqs. (2.13) and (2.14) and substituting \( C_1 (\rho'), C_2 (\rho') \), and Eqs. (2.7) and (2.8) into Eq. (2.9) give the Green function,

\[ G (\rho, \rho') = \frac{\rho' \cos (2M\eta [\rho' + \ln (\rho' - 1)]) \sin (2M\eta [\rho + \ln (\rho - 1)])}{2M\eta (\rho' - 1)} - \frac{\rho' \sin (2M\eta [\rho' + \ln (\rho' - 1)])}{2M\eta (\rho' - 1)} \cos (2M\eta [\rho + \ln (\rho - 1)]), \quad \rho > \rho'. \quad (2.15) \]

In order to construct the general solution of the inhomogeneous equation (2.5), we start with the general solution of the corresponding homogeneous equation which is the inhomogeneous equation (2.5) without the effective potential (2.4). The general solution of the homogeneous equation is \( A y_1^{(1)} (\rho) + B y_2^{(2)} (\rho) \). Then the general solution of the homogeneous equation (2.5) can be constructed by the general solution of the homogeneous equation and the Green function \( G (\rho, \rho') \) [27]. Concretely, by the Green function (2.15), we can establish an integral equation for \( u_i (\rho) \):

\[ u_i (\rho) = A y_1^{(1)} (\rho) + B y_2^{(2)} (\rho) + \int_1^\rho G (\rho, \rho') V_i^{\text{eff}} (\rho') u_i (\rho') d\rho' \]
\[ = A \sin (2M\eta [\rho + \ln (\rho - 1)]) + B \cos (2M\eta [\rho + \ln (\rho - 1)]) + \frac{\sin (2M\eta [\rho + \ln (\rho - 1)])}{2M\eta} \int_1^\rho \cos (2M\eta [\rho' + \ln (\rho' - 1)]) \rho' d\rho' \]
\[ - \frac{\cos (2M\eta [\rho + \ln (\rho - 1)])}{2M\eta} \int_1^\rho \sin (2M\eta [\rho' + \ln (\rho' - 1)]) \rho' d\rho'; \quad (2.16) \]
or, equivalently,

\[
    u_l (\rho) = A \sin (2M \eta [\rho + \ln (\rho - 1)]) + B \cos (2M \eta [\rho + \ln (\rho - 1)])
        - \frac{1}{2M \eta} \int_1^\rho d\rho' \frac{\rho'}{\rho' - 1} \sin \left( 2M \eta \left[ \rho' - \rho + \ln \left( \frac{\rho' - 1}{\rho - 1} \right) \right] \right) V_{eff}^{l} (\rho') u_l (\rho').
\]

(2.17)

### 2.2 Boundary condition at horizon

In the Schwarzschild spacetime, there is a boundary condition at the horizon \( r = 2M \) [28–31]

\[
    \phi_l (r) \overset{r \to 2M}{\sim} e^{\pm i \omega r_*},
\]

(2.18)

with \( r_* = r + 2M \ln \left| \frac{r}{2M} - 1 \right| \) the tortoise coordinate. This boundary condition can be equivalently expressed as

\[
    u_l (\rho) \overset{\rho \to 1^+}{\sim} e^{\pm 2M \eta \rho_*}
\]

(2.19)

with the tortoise coordinate \( \rho_* = \int d\rho \frac{1}{1 - \rho/\rho} = \rho + \ln (\rho - 1) \) [32].

The integral equation (2.17) can be rewritten as

\[
    u_l (\rho) = A \sin (2M \eta \rho_*) + B \cos (2M \eta \rho_*) - \frac{1}{2M \eta} \int_1^\rho d\rho' \sin (2M \eta (\rho_*' - \rho_*)) V_{eff}^{l} (\rho') u_l (\rho') d\rho'.
\]

(2.20)

Near the outer horizon, i.e., \( \rho \to 1^+ \) (corresponding to \( r \to 2M \)), Eq. (2.20) reduces to

\[
    u_l (\rho) \bigg|_{\rho \to 1^+} = C \lim_{\rho \to 1^+} \sin (2M \eta \rho_* + \phi),
\]

(2.21)

where \( \cos \phi = A/\sqrt{A^2 + B^2} \), \( \sin \phi = B/\sqrt{A^2 + B^2} \), and \( C = \sqrt{A^2 + B^2} \). The superscript \(+\) denotes that \( \rho \) tends to the horizon from outside.

Rewrite the wave function (2.21) as

\[
    u_l (\rho) \bigg|_{\rho \to 1^+} = \frac{C}{2i} \lim_{\rho \to 1^+} \left[ e^{i(2M \eta \rho_* + \phi)} - e^{-i(2M \eta \rho_* + \phi)} \right],
\]

(2.22)

which includes two parts:

\[
    u_l (\rho) \bigg|_{\rho \to 1^+}^{out} = \frac{C}{2i} e^{i(2M \eta \rho_* + \phi)}, \quad \text{outgoing wave},
\]

(2.23)

\[
    u_l (\rho) \bigg|_{\rho \to 1^+}^{in} = \frac{C}{2i} e^{-i(2M \eta \rho_* + \phi)}, \quad \text{ingoing wave}.
\]

(2.24)

Clearly, this satisfies the boundary condition (2.19).

### 3 Scattering phase shift

In this section, we calculate the scattering phase shift for high energy scattering, i.e., \( \mu/\eta \ll 1 \), based on the integral equation constructed above.
3.1 Scattering phase shift

The radial equation, Eq. (2.3), under the replacement $2\rho - 1 \to \rho$, is the Heun equation. The $\rho \to \infty$ asymptotic solution, for $\mu/\eta \ll 1$, is [33]

$$u_i(\rho) \overset{\rho \to \infty}{\sim} \sin \left(2M\eta(\rho + \ln(\rho - 1)) + \delta_i - \frac{\pi}{2} - \eta M + 2M\eta \ln 2 \right)$$

$$= \sin \left(2M\eta[\rho + \ln(\rho - 1)]\right) \cos (\delta_i + \Delta(\eta, M)) + \cos \left(2M\eta[\rho + \ln(\rho - 1)]\right) \sin (\delta_i + \Delta(\eta, M)) ,$$  \hspace{1cm} (3.1)

where $\rho = \frac{r}{2M}$ and $\Delta(\eta, M) = -\frac{\pi}{2} - \eta M + 2M\eta \ln 2$ are used.

It can be seen that the asymptotic wave function, Eq. (3.1), is determined by the phase shift $\delta_i$: once the phase shift is obtained, the asymptotic wave function is obtained. This is the same as the quantum-mechanical scattering: all information of the scattering wave function is embodied in the phase shift. Therefore, in a scattering problem, the main task is to find the phase shift.

In order to compare with (3.1), we rewrite Eq. (2.16) as

$$u_i(\rho) = \sin \left(2M\eta[\rho + \ln(\rho - 1)]\right) \left[ A + \frac{1}{2M\eta} \int_1^\infty \frac{\cos \left(2M\eta[\rho' + \ln(\rho' - 1)]\right)}{\rho' - 1} V_i^{\text{eff}}(\rho') u_i(\rho') \rho' d\rho' \right]$$

$$+ \cos \left(2M\eta[\rho + \ln(\rho - 1)]\right) \left[ B - \frac{1}{2M\eta} \int_1^\infty \frac{\sin \left(2M\eta[\rho' + \ln(\rho' - 1)]\right)}{\rho' - 1} V_i^{\text{eff}}(\rho') u_i(\rho') \rho' d\rho' \right] ,$$  \hspace{1cm} (3.2)

and, then, take $\rho \to \infty$ asymptotics:

$$u_i(\rho) \overset{\rho \to \infty}{\sim} \alpha(\eta, M) \sin \left(2M\eta[\rho + \ln(\rho - 1)]\right) + \beta(\eta, M) \cos \left(2M\eta[\rho + \ln(\rho - 1)]\right)$$

$$= C \sin \left(2M\eta[\rho + \ln(\rho - 1)] + \phi \right) ,$$  \hspace{1cm} (3.3)

where $\alpha(\eta, M) = A + \frac{1}{2M\eta} \int_1^\infty \frac{\cos \left(2M\eta[\rho' + \ln(\rho' - 1)]\right)}{\rho' - 1} V_i^{\text{eff}}(\rho') u_i(\rho') \rho' d\rho'$, $\beta(\eta, M) = B - \frac{1}{2M\eta} \int_1^\infty \frac{\sin \left(2M\eta[\rho' + \ln(\rho' - 1)]\right)}{\rho' - 1} V_i^{\text{eff}}(\rho') u_i(\rho') \rho' d\rho'$, $\cos \phi = \alpha(\eta, M) / \sqrt{\alpha^2(\eta, M) + \beta^2(\eta, M)}$, and the normalization constant $C = \sqrt{\alpha^2(\eta, M) + \beta^2(\eta, M)}$.

Comparing Eqs. (3.1) and (3.3), we have

$$\tan (\delta_i + \Delta(\eta, M)) = \frac{\beta(\eta, M)}{\alpha(\eta, M)}$$

$$= B - \frac{1}{2M\eta} \int_1^\infty \frac{\sin \left(2M\eta[\rho' + \ln(\rho' - 1)]\right)}{\rho' - 1} V_i^{\text{eff}}(\rho') u_i(\rho') \rho' d\rho'$$

$$A + \frac{1}{2M\eta} \int_1^\infty \frac{\cos \left(2M\eta[\rho' + \ln(\rho' - 1)]\right)}{\rho' - 1} V_i^{\text{eff}}(\rho') u_i(\rho') \rho' d\rho' ,$$  \hspace{1cm} (3.4)

where

$$\phi = \arctan \left( \frac{\beta(\eta, M)}{\alpha(\eta, M)} \right)$$

$$= B - \frac{1}{2M\eta} \int_1^\infty \frac{\sin \left(2M\eta[\rho' + \ln(\rho' - 1)]\right)}{\rho' - 1} V_i^{\text{eff}}(\rho') u_i(\rho') \rho' d\rho'$$

$$A + \frac{1}{2M\eta} \int_1^\infty \frac{\cos \left(2M\eta[\rho' + \ln(\rho' - 1)]\right)}{\rho' - 1} V_i^{\text{eff}}(\rho') u_i(\rho') \rho' d\rho' ,$$  \hspace{1cm} (3.5)
Then we arrive at an expression of the scattering phase shift,\
\[
\delta_l = \arctan \left( \frac{B - \frac{1}{2M\eta} \int_1^\infty \frac{\sin(2M\eta\rho' + \ln(\rho'-1))}{\rho'-1} V_{l}^{\text{eff}} (\rho') u_l (\rho') \rho' \, d\rho'}{A + \frac{1}{2M\eta} \int_1^\infty \frac{\cos(2M\eta\rho' + \ln(\rho'-1))}{\rho'-1} V_{l}^{\text{eff}} (\rho') u_l (\rho') \rho' \, d\rho'} \right) - \Delta (\eta, M). \tag{3.6}
\]
This is a relation between the scattering phase shift \(\delta_l\) and the scattering wave function \(u_l (\rho)\).

Now we determine the constants \(A\) and \(B\).

When \(V_{l}^{\text{eff}} (\rho) = 0\), Eq. (2.20) gives
\[
u_l (\rho) = A \sin (2M\eta\rho_*) + B \cos (2M\eta\rho_*)
= A \sin \left( \eta \left[ r + 2M \ln \left( \frac{r}{2M} - 1 \right) \right] \right) + B \cos \left( \eta \left[ r + 2M \ln \left( \frac{r}{2M} - 1 \right) \right] \right). \tag{3.7}
\]
For \(M = 0\),
\[
u_l (r) = A \sin (\eta r) + B \cos (\eta r)
\]
and the horizon is at \(r = 2M = 0\). The boundary condition requires that \(\phi_l (r) = \frac{u_l (r)}{r}\) must be finite at the horizon, i.e., \(u_l (0) = 0\). This gives \(B = 0\). From Eqs. (3.6) and (2.20), we can see that the constant \(A\) will be eliminated finally, i.e.,
\[
\delta_l = - \arctan \left( \frac{\frac{1}{2M\eta} \int_1^\infty \frac{\sin(2M\eta\rho' + \ln(\rho'-1))}{\rho'-1} V_{l}^{\text{eff}} (\rho') u_l (\rho') \rho' \, d\rho'}{1 + \frac{1}{2M\eta} \int_1^\infty \frac{\cos(2M\eta\rho' + \ln(\rho'-1))}{\rho'-1} V_{l}^{\text{eff}} (\rho') u_l (\rho') \rho' \, d\rho'} \right) - \Delta (\eta, M) \tag{3.8}
\]
That is, the constant \(A\) can take any nonzero value; here and after, we take \(A = 1\):
\[
u_l (\rho) = \sin (2M\eta\rho_*) - \frac{1}{2M\eta} \int_1^{\rho} \sin \left( 2M\eta (\rho_* - \rho) \right) V_{l}^{\text{eff}} (\rho') u_l (\rho') \, d\rho'. \tag{3.9}
\]
Next, in order to obtain the scattering phase shift by Eq. (3.6), we need to iteratively solve the wave function \(u_l (\rho)\).

### 3.2 Zeroth-order and first-order phase shifts

By iteratively solving the integral equation of the wave function \(u_l (\rho)\), Eq. (2.17), we can obtain various orders of \(u_l (\rho)\). In this section, we solve the zeroth-order and first-order scattering phase shifts.

**Zeroth-order phase shift.** The zeroth-order contribution of Eq. (3.8) is
\[
[tan (\delta_l + \Delta (\eta, M))]^{(0)} = 0. \tag{3.10}
\]
The zeroth-order scattering phase shift then reads
\[
\delta_l^{(0)} = - \Delta (\eta, M) = \frac{l\pi}{2} + \eta M - 2M\eta \ln 2. \tag{3.11}
\]
First-order phase shift. The first-order phase shift $\delta_l^{(1)}$ can be obtained by substituting the zeroth-order scattering wave function $u_l^{(0)}(\rho) = \sin(2M\eta [\rho + \ln (\rho - 1)])$ into Eq. (3.8): 

$$
\delta_l^{(1)} = - \arctan \left( \frac{\int_1^\infty \sin^2 \left(2M\eta [\rho + \ln (\rho - 1)]\right) \frac{1}{\rho V_l^{e,f}}(\rho) \rho d\rho}{2M\eta + \frac{1}{2} \int_1^\infty \sin \left(4M\eta [\rho + \ln (\rho - 1)]\right) \frac{1}{\rho V_l^{e,f}}(\rho) \rho d\rho} \right). \quad (3.12)
$$

The integral in Eq. (3.12) can be worked out analytically, but it is too complicated to be listed here.

4 Conclusion

An integral equation method for solving scattering of a scalar field in the Schwarzschild spacetime is constructed. By solving the integral equation, we obtain the zeroth-order and first-order scattering phase shifts.

Scattering in curved spacetime is an important issue and has been discussed in many literatures. In this letter, we calculate the zeroth-order and first-order scattering wave functions and scattering phase shifts of scalar scattering in the Schwarzschild spacetime. There are some authors also consider the scalar scattering in the Schwarzschild spacetime. In Ref. [3], the author consider the late-time evolution in the Schwarzschild background. In Ref. [6], the author calculate the scattering amplitude; they also calculate the phase shift, but the phase shift is a large $r$ Coulomb-like phase shift. In Ref. [15], the author concentrates on the reflection coefficient through the asymptotic solution. Based on the confluent Heun function, some authors calculate the quasinormal modes of nonrotating black holes [23], resonant frequencies, Hawking radiation, and scattering of scalar waves [24], and the angular and radial solutions [25]. Other types of fields scattered in a curved spacetime are also considered, such as spinor fields [7, 9–11, 13] and vector fields [14, 16].

Beyond the Schwarzschild spacetime, there are many discussions devote to other types of spacetime, such as the Reissner–Nordström spacetime [4, 8, 17–19], the Kerr spacetime [20], deformed black hole [22], and the AdS spacetime [12].

The integral equation method suggested in the present paper is a method for solving the radial equations with various effective potentials. The field equations in different space-times correspond to different effective potentials. Besides the Schwarzschild spacetime, the integral equation method can be applied to more general cases, such as the charged RN black hole and the spinning Kerr black hole. Concretely, for the charged RN black hole, the field equation can be separated into two parts: the radial equation and the angular equation. The angular equation can be solved exactly and the solution, the same as the Schwarzschild case, is the spherical harmonics function. Then the remaining task is to solve the radial equation. For the spinning Kerr black hole, the angular equation can also be separated and solved exactly, and the solution is the confluent Heun function. Again, the remaining task then is to solve the radial equation.

It is worthy to note that scattering by Schwarzschild spacetime is a long-range potential scattering [1], i.e., this is an integral equation method for long-range potential scattering. The long-range scattering is more difficult than short-range scattering [31, 34–36].
The method developed in the present paper is an integral equation method for solving scattering phase shifts. This method can be applied to the scattering spectral method [37, 38] and heat kernel method [39–43]. Furthermore, through the scattering spectral method and the heat kernel method, the method for scattering phase shift can be used to quantum field theory [37, 44–46].

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