Energy and helicity conservation for the generalized quasi-geostrophic equation

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Abstract

In this paper, we consider the 2-D generalized surface quasi-geostrophic equation with the velocity \( v \) determined by \( v = R^\perp \Lambda^{\gamma - 1} \theta \). It is shown that the \( L^p \) type energy norm of weak solutions is conserved provided \( \theta \in L^{p+1}(0, T; B^{\frac{\gamma}{p} + 1}_{p+1,c}(\mathbb{N})) \) for \( 0 < \gamma < \frac{3}{2} \) or \( \theta \in L^{p+1}(0, T; B^{\alpha}_{p+1,c}(\mathbb{N})) \) for any \( \gamma - 1 < \alpha < 1 \) with \( \frac{3}{2} \leq \gamma < 2 \). Moreover, we also prove that the helicity of weak solutions satisfying \( \nabla \theta \in L^3(0, T; \dot{B}^{\frac{3}{2}}_{3,\infty}(\mathbb{N})) \) for \( 0 < \gamma < \frac{3}{2} \) or \( \nabla \theta \in L^3(0, T; \dot{B}^{\frac{\gamma}{3}}_{3,\infty}(\mathbb{N})) \) for any \( \gamma - 1 < \alpha < 1 \) with \( \frac{3}{2} \leq \gamma < 2 \) is invariant. Therefore, the accurate relationships between the critical regularity for the energy (helicity) conservation of the weak solutions and the regularity of velocity in 2-D generalized quasi-geostrophic equation are presented.

MSC(2020): 35Q30, 35Q35, 76D03, 76D05
Keywords: quasi-geostrophic equation; energy conservation; helicity conservation;

1 Introduction

In this paper, we consider the 2-D generalized surface quasi-geostrophic (SQG) equation in \((0, T) \times \mathbb{R}^2\) below

\[
\begin{align*}
\theta_t + v \cdot \nabla \theta &= 0, \\
v &= R^\perp \Lambda^{\gamma - 1} \theta = (-R_2 \Lambda^{\gamma - 1} \theta, R_1 \Lambda^{\gamma - 1} \theta), \quad \gamma \in [0, 2], \\
\theta|_{t=0} &= \theta_0,
\end{align*}
\]

where the unknown scalar function \( \theta(x, t): \mathbb{R}^2 \to \mathbb{R} \) stands for the temperature and \( v \) is the velocity field. The Riesz transforms \( R_j \) are defined by \( \hat{R_j} f = -\frac{i\xi_j}{|\xi|} \hat{f}(\xi) \) with \( j = 1, 2 \), where \( \hat{f}(\xi) = \frac{1}{(2\pi)^{2}} \int_{\mathbb{R}^2} f(x) e^{-i\xi \cdot x} \, dx \). \( \Lambda^s f \) is defined via \( \Lambda^s \hat{f}(\xi) = |\xi|^s \hat{f}(\xi) \). This model was introduced in \([10, 12, 14]\) and includes many classical hydrodynamic equations. In particular, there hold

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1. \((1.1)\) with \(\gamma = 0\) reduces to the 2-D vorticity equation of the incompressible Euler equations \((1.2)\) below,
\[
\begin{align*}
v_t + v \cdot \nabla v + \nabla \Pi &= 0, \\
\text{div } v &= 0, \\
v|_{t=0} &= v_0(x).
\end{align*}
\]

2. \((1.1)\) with \(\gamma = 1\) is the following standard surface quasi-geostrophic equation \((1.3)\);
\[
\begin{align*}
\theta_t + \text{div} (v \otimes \theta) &= 0, \\
v(x, t) &= (-R \theta, R \theta), \\
\theta|_{t=0} &= \theta_0,
\end{align*}
\]

3. \((1.1)\) with \(\gamma = 2\) becomes the magneto-geostrophic equations \((1.4)\).

The generalized quasi-geostrophic equation \((1.1)\) attracted a lot of attention and important progress has been made (see e.g. \cite{10, 12, 14, 22–24, 32–34}). The goal of this paper is to examine the relationships between critical regularity for weak solutions keeping energy (helicity) conservation and the regularity of velocity in 2-D generalized quasi-geostrophic equation. A classical question involving energy conservation in incompressible fluid is the Onsager conjecture. In particular, Onsager \cite{27} conjectured that the weak solutions of incompressible Euler equations \((1.2)\) with Hölder continuity exponent \(\alpha > \frac{1}{3}\) do conserve energy. In \cite{16}, Constantin-E-Titi successfully solved this Onsager’s conjecture, where it is shown that the energy is conserved if a weak solution \(v\) is in the Besov space \(L^3(0, T; B_{3,\infty}^{\frac{1}{3}}(\mathbb{T}^3))\) with \(\alpha > 1/3\). Subsequently, the result due to Cheskidov-Constantin-Friedlander-Shvydkoy \cite{13} complemented the one of Constantin-E-Titi by studying in the critical space \(L^3(0, T; B_{3,c(\mathbb{N})}^{1/3})\), where \(B_{3,c(\mathbb{N})}^{1/3} = \{v \in B_{3,\infty}^{1/3}, \lim_{q \to \infty} 2^q ||\Delta_q v||_{L^3} = 0\}\) and \(\Delta_q\) stands for a smooth restriction of \(v\) into Fourier modes of order \(2^q\). The spaces \(B_{3,c(\mathbb{N})}^{1/3}\) is usually called as the Onsager’s critical spaces. Along this direction, there are some progresses recently, one can refer to \cite{3, 19} for details.

We turn our attention back to the persistence of energy in surface quasi-geostrophic equation. A parallel of Constantin-E-Titi’s result for the 2-D standard surface quasi-geostrophic equation \((1.3)\) was obtained by Zhou in \cite{36}, where he showed that the \(L^2\) type energy norm of weak solutions is conserved provided \(\theta \in L^3(0, T; B_{3,\infty}^{\alpha}(\mathbb{T}^3))\) with \(\alpha > \frac{1}{3}\). Chae \cite{8} proved that the \(L^p\) type energy norm of \(\theta\) is preserved if the weak solution \((\theta, v)\) satisfy
\[
v \in L^{\alpha r_1}(0, T; \dot{B}_{p+1,\infty}^\alpha) \quad \text{and} \quad \theta \in L^{\alpha r_2}(0, T; \dot{B}_{p+1,\infty}^\alpha), \quad \frac{1}{r_1} + \frac{p}{r_2} = 1, \quad \alpha > \frac{1}{3}. \tag{1.4}
\]

Very recently, Akramova-Wiedemann \cite{1} present the following sufficient conditions implying \(L^p\) norm conservation
\[
\theta \in L^{\alpha p_1}(0, T; \dot{B}_{3,\infty}^\alpha), \quad \alpha > \frac{1}{3}, \quad p_1 \leq \frac{6}{2 - 3\alpha},
\]
for a weak solution for the 2-D standard surface quasi-geostrophic equation \((1.3)\). We note that all above results are in Onsager’s subcritical space other than the Onsager’s critical space, which means the regularity of space is required to satisfy \(\alpha > \frac{1}{3}\) not \(\alpha = \frac{1}{3}\) exactly. Hence, our first objective is to show the regularity criterion for the energy conservation of weak solutions of 2-D generalized surface quasi-geostrophic equation \((1.1)\) in Onsager’s critical space. Now, we formulate our first result as follows.
**Theorem 1.1.** Let $p \in [2, \infty)$ and $\theta \in C([0, T]; L^p(\mathbb{R}^2))$ be a weak solution of the 2-D generalized surface quasi-geostrophic equation (1.1) in the sense of Definition 2.1 then the $L^p$ type energy norm of $\theta$ is preserved, that is, for any $t \in [0, T]$,

$$\|\theta(x, t)\|_{L^p(\mathbb{R}^2)} = \|\theta(x, 0)\|_{L^p(\mathbb{R}^2)},$$

provided one of the following conditions is satisfied

$$\theta \in L^{p+1}(0, T; \dot{B}_{p+1,c(N)}^{\gamma}) \text{ with } 0 < \gamma < \frac{3}{2};$$

or

$$\theta \in L^{p+1}(0, T; \dot{B}_{p+1,\infty}^{\alpha}) \text{ for any } \gamma - 1 < \alpha < 1 \text{ with } \frac{3}{2} \leq \gamma < 2.$$  \hfill (1.5)

\hfill (1.6)

**Remark 1.1.** An analogue of Cheskidov-Constantin-Friedlander-Shvydkoy’s theorem [13] is established for the 2-D generalized surface quasi-geostrophic equation (1.1). Even though for the standard 2-D surface quasi-geostrophic equation (1.3), a special case of this theorem with $p = 2$ and $\gamma = 1$ is novel and improves the corresponding result in [36]. This theorem reveals how the regularity of the velocity field influences the critical regularity of the weak solutions preserving the energy in generalized surface quasi-geostrophic equation (1.1).

**Remark 1.2.** As pointed in [12], the situation in model (1.1) for $1 < \gamma \leq 2$ is more singular than the classical quasi-geostrophic equation (1.3), hence we only get the subcritical criterion for energy conservation for $\frac{3}{2} \leq \gamma < 2$. It is an interesting problem to study the persistence of energy in (1.1) in Onsager’s critical space for the case $\frac{3}{2} \leq \gamma \leq 2$.

Moreover, when $p = 2$, the condition $\theta \in L^3(0, T; L^3(\mathbb{R}^2))$ in Theorem 1.1 can be removed. Precisely, we have

**Corollary 1.2.** Let $0 < \gamma < \frac{3}{2}$. Assume that $\theta \in C([0, T]; L^2(\mathbb{R}^2))$ is a weak solution of the 2D quasi-geostrophic equation (1.3) in the sense of Definition 2.1 satisfying $\theta \in L^3(0, T; \dot{B}_{3,c(N)}^{\gamma})$, then the $L^2$ type energy norm of $\theta$ is preserved, that is, for any $t \in [0, T]$,

$$\|\theta(x, t)\|_{L^2(\mathbb{R}^2)} = \|\theta(x, 0)\|_{L^2(\mathbb{R}^2)}.$$

Inspired by the persistence of energy criterion (1.4), we have

**Theorem 1.3.** Let $p \in [2, \infty)$ and $r_1 \in [1, \infty), r_2 \in [p, \infty]$ be given, satisfying $\frac{1}{r_1} + \frac{p}{r_2} = 1$. Assume that $\theta \in C([0, T]; L^p(\mathbb{R}^2))$ is a weak solution of the 2-D SQG equation (1.1) in the sense of Definition 2.1 with $v \in L^{r_1}(0, T; \dot{B}_{p+1,c(N)}^{\frac{\gamma}{p}})$ and $\theta \in L^{r_2}(0, T; \dot{B}_{p+1,\infty}^{\frac{\gamma}{p}})$, then the $L^p$ type energy norm of $\theta$ is preserved, that is, for any $t \in [0, T]$,

$$\|\theta(t)\|_{L^p(\mathbb{R}^2)} = \|\theta(0)\|_{L^p(\mathbb{R}^2)}.$$

**Remark 1.3.** A slightly modified the proof of this theorem means that the same result also holds if $v \in L^{r_1}(0, T; \dot{B}_{p+1,\infty}^{\frac{\gamma}{p}})$ and $\theta \in L^{r_2}(0, T; \dot{B}_{p+1,\infty}^{\frac{\gamma}{p}})$. This and Theorem 1.3 refine the criterion (1.4).

**Remark 1.4.** Owing to the boundedness of Riesz transforms in homogeneous Besov spaces, Theorem 1.3 guarantees that the $L^2$ type energy norm of weak solutions satisfying $\theta \in L^3(0, T; \dot{B}_{3,c(N)}^{\gamma})$ are preserved in the standard quasi-geostrophic equation (1.3).
We would like to mention that Dai [17] showed that the energy of any viscosity solution of the quasi-geostrophic equation (1.3) with supercritical dissipation $\Lambda^{\alpha} \theta$ satisfying $\theta \in L^2(0, T; \dot{B}_{\frac{3}{2}, c(N)}^{\frac{1}{2}})$ is invariant.

Beside the persistence of energy of weak solutions in the quasi-geostrophic equation (1.3), the general helicity defined as

$$\int \theta \partial_i \theta dx, i = 1, 2,$$

is also conserved, which is observed by Zhou in [36]. It is shown that the general helicity of weak solutions for the 2-D standard surface quasi-geostrophic equation (1.3) is conserved if $\nabla \theta \in C([0, T]; \dot{L}^{\frac{4}{3}}(\mathbb{R}^2)) \cap L^{3}(0, T; \dot{B}_{\frac{3}{2}, c(N)}^{\frac{1}{2}})$ with $\alpha > 1/3$ in [36]. Recently, the authors [30] improves this to $\nabla \theta \in C([0, T]; \dot{L}^{\frac{4}{3}}(\mathbb{R}^2)) \cap L^{3}(0, T; \dot{B}_{\frac{3}{2}, c(N)}^{\frac{1}{2}})$. The helicity of flow was originated from Moffatt’s work [26]. The helicity is important at a fundamental level in relation to flow kinematics because it admits topological interpretation in relation to the linkage or linkages of vortex lines of the flow (see [25, 26]). See [8, 9, 18, 30] for the study of the helicity conservation of weak solutions of the Euler equations. Now we state our rest results involving helicity conservation of weak solutions for the generalized quasi-geostrophic equation (1.1) as follows:

**Theorem 1.4.** Let $\theta$ be a weak solution of the 2-D generalized quasi-geostrophic equation (1.1) in the sense of Definition 2.1, then the helicity conservation

$$\int_{\mathbb{R}^2} \theta(x, t) \partial_i \theta(x, t) dx = \int_{\mathbb{R}^2} \theta_0(x) \partial_i \theta_0(x) dx, i = 1, 2,$$

is valid provided one of the following conditions is satisfied

1. $\nabla \theta \in L^{3}(0, T; \dot{B}_{\frac{3}{2}}^{\frac{3}{2}, c(N)}(\mathbb{R}^2)) \cap C([0, T]; \dot{L}^{\frac{4}{3}}(\mathbb{R}^2))$ with $0 < \gamma < \frac{3}{2}$;

2. $\nabla \theta \in L^{3}(0, T; \dot{B}_{\frac{3}{2}}^{\frac{3}{2}, \infty}(\mathbb{R}^2)) \cap C([0, T]; \dot{L}^{\frac{4}{3}}(\mathbb{R}^2))$ for any $\gamma - 1 < \alpha < 1$ with $\frac{3}{2} \leq \gamma < 2$.

**Remark 1.5.** By the Bernstein inequality, $\nabla \theta \in L^{3}(0, T; \dot{B}_{\frac{3}{2}}^{\frac{3}{2}, c(N)}(\mathbb{R}^2))$ may be replaced by $\theta \in L^{3}(0, T; \dot{B}_{\frac{3}{2}}^{\frac{3}{2} + \gamma, c(N)}(\mathbb{R}^2))$ in this theorem.

**Remark 1.6.** The required regularity $\nabla \theta \in C([0, T]; \dot{L}^{\frac{4}{3}}(\mathbb{R}^2))$ is used to ensure the helicity conservation (1.7) make sense.

**Remark 1.7.** Compare the results in Theorem 1.4 and Corollary 1.2, we see that there may exist a weak solution of the 2-D generalized surface quasi-geostrophic equation (1.1) that keep the $L^2$ type energy rather than the helicity. This was also previously pointed out by Chae in [8] for 2-D standard surface quasi-geostrophic equation (1.3).

**Remark 1.8.** Our result in Theorem 1.4 covers and generalizes the recent result obtained for 2-D standard surface quasi-geostrophic equation (1.3) in [30].

We will provide two approaches to show Theorem 1.1-1.4. One is an application of the Littlewood-Paley theory developed by Cheskidov-Constantin-Friedlander-Shvydkoy in [13].
The second one relies on the Constantin-E-Titi type commutator estimates in physical Onsager type spaces (see Lemma 2.3). It seems that the arguments in this paper can be applicable to other fluid models such as the surface growth model without dissipation

\[ h_t + \partial_{xx}(h_x)^2 = 0, \]  

(1.8)

where \( h \) stands for the height of a crystalline layer. The background of the surface growth model (1.8) can be found in [3, 28, 29, 31]. The energy conservation in the Besov space \( L^3(0, T; B^{\alpha}_{3,\infty}(T^3)) \) with \( \alpha > 1/3 \) was considered in [31]. One can establish the persistence of energy criterion in the Onsager’s critical spaces for the inviscid surface growth model (1.8).

The rest of the paper is organized as follows. In Section 2, we present some notations and auxiliary lemmas which will be frequently used throughout this paper. The energy conservation of weak solutions of the surface quasi-geostrophic equation is considered in Section 3. Section 4 is devoted to the helicity conservation of weak solutions of the generalized surface quasi-geostrophic equation. Concluding remarks are given in Section 5.

2 Notations and some auxiliary lemmas

**Sobolev spaces:** First, we introduce some notations used in this paper. For \( p \in [1, \infty] \), the notation \( L^p(0, T; X) \) stands for the set of measurable functions on the interval \((0, T)\) with values in \( X \) and \( \|f(t, \cdot)\|_X \) belonging to \( L^p(0, T) \). The classical Sobolev space \( W^{k,p}(R^2) \) is equipped with the norm \( \|f\|_{W^{k,p}(R^2)} = \sum_{|\alpha| = 0}^{k} \|D^\alpha f\|_{L^p(R^2)} \).

**Besov spaces:** \( S \) denotes the Schwartz class of rapidly decreasing functions, \( S' \) the space of tempered distributions, \( S'/P \) the quotient space of tempered distributions which modulo polynomials. We use \( \mathcal{F}f \) or \( \hat{f} \) to denote the Fourier transform of a tempered distribution \( f \). To define Besov spaces, we need the following dyadic unity partition (see e.g. [2]). Choose two nonnegative radial functions \( \rho, \varphi \in C_\infty(R^d) \) supported respectively in the ball \( B = \{\xi \in R^d : |\xi| \leq \frac{3}{4}\} \) and the shell \( C = \{\xi \in R^d : \frac{3}{4} \leq |\xi| \leq \frac{5}{4}\} \) such that

\[ g(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \in R^d; \quad \sum_{j \in Z} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \neq 0. \]

Then for every \( \xi \in R^d \), \( \varphi(\xi) = g(\xi/2) - g(\xi) \). Write \( h = \mathcal{F}^{-1} \varphi \) and \( \tilde{h} = \mathcal{F}^{-1} g \), then nonhomogeneous dyadic blocks \( \Delta_j \) are defined by

\[ \Delta_j u := 0 \text{ if } j \leq -2, \quad \Delta_{-1} u := g(D) u = \int_{R^d} \tilde{h}(y) u(x - y)dy, \]

and

\[ \Delta_j u := \varphi(2^{-j} D) u = 2^{jd} \int_{R^d} h(2^j y) u(x - y)dy \text{ if } j \geq 0. \]

The nonhomogeneous low-frequency cut-off operator \( S_j \) is defined by

\[ S_j u := \sum_{k \leq j - 1} \Delta_k u = g(2^{-j} D) u = 2^{jd} \int_{R^d} \tilde{h}(2^j y) u(x - y)dy, \quad j \in N \cup 0. \]
The homogeneous dyadic blocks $\dot{\Delta}_j$ and homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for $\forall j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u := \varphi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^d} h(2^j y)u(x - y)dy, \ j \in \mathbb{Z}$$

and

$$\dot{S}_j u := \varrho(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^d} \tilde{h}(2^j y)u(x - y)dy, \ j \in \mathbb{Z}$$

Now we introduce the definition of Besov spaces. Let $(p, r) \in [1, \infty]^2$, $s \in \mathbb{R}$, the nonhomogeneous Besov space $B^s_{p, r}$ is defined as

$$B^s_{p, r} := \left\{ f \in \mathcal{S}' \left( \mathbb{R}^d \right) : \| f \|_{B^s_{p, r}} := \| 2^{js} \| \dot{\Delta}_j f \|_{L^p(\mathbb{R}^d)} < \infty \right\}$$

and the homogeneous space

$$\dot{B}^s_{p, r} := \left\{ f \in \mathcal{S}' \left( \mathbb{R}^d \right) / \mathcal{P} \left( \mathbb{R}^d \right) : \| f \|_{\dot{B}^s_{p, r}} := \| 2^{js} \| \dot{\Delta}_j f \|_{L^p(\mathbb{R}^d)} < \infty \right\}.$$ 

Moreover, for $s > 0$ and $1 \leq p, q \leq \infty$, we may write the equivalent norm below in the nonhomogeneous Besov norm $\| f \|_{B^s_{p, q}}$ of $f \in \mathcal{S}'$ as

$$\| f \|_{B^s_{p, q}} = \| f \|_{L^p} + \| f \|_{\dot{B}^s_{p, q}}.$$ 

Motivated by [13], we define $\dot{B}^\alpha_{p, q}(\mathbb{N})$ to be the class of all tempered distributions $f$ for which

$$\| f \|_{\dot{B}^\alpha_{p, \infty}} < \infty \text{ and } \lim_{j \to \infty} 2^{j\alpha} \| \dot{\Delta}_j f \|_{L^p} = 0, \text{ for any } 1 \leq p \leq \infty. \quad (2.1)$$

It is clear that the Besov spaces $\dot{B}^\alpha_{p, q}$ are included in $\dot{B}^\alpha_{p, q}(\mathbb{N})$ for any $1 \leq q < \infty$. Likewise, one can define the Besov spaces $B^\alpha_{p, q}(\mathbb{N})$ similarly.

**Mollifier kernel:** Let $\eta_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ be a standard mollifier i.e. $\eta(x) = C_0 \varepsilon^{-\frac{|x|}{\varepsilon}}$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \geq 1$, where $C_0$ is a constant such that $\int_{\mathbb{R}^d} \eta(x)dx = 1$. For $\varepsilon > 0$, we define the rescaled mollifier $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta(\frac{x}{\varepsilon})$ and for any function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_{\mathbb{R}^d} f(x - y)\eta_\varepsilon(y)dy, \ x \in \mathbb{R}^d.$$ 

Next, we collect some Lemmas which will be used in the present paper.

**Lemma 2.1.** *(Bernstein inequality [3]) Let $B$ be a ball of $\mathbb{R}^d$, and $C$ be a ring of $\mathbb{R}^d$. There exists a positive constant $C$ such that for all integer $k \geq 0$, all $1 \leq a \leq b \leq \infty$ and $u \in L^a(\mathbb{R}^d)$, the following estimates are satisfied:

$$\sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^b(\mathbb{R}^d)} \leq C^{k+1} \lambda^{b(a-b)} \| u \|_{L^a(\mathbb{R}^d)}, \ \text{supp} \tilde{u} \subset \lambda B,$$

$$C^{-(k+1)} \lambda^k \| u \|_{L^a(\mathbb{R}^d)} \leq \sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^a(\mathbb{R}^d)} \leq C^{k+1} \lambda^k \| u \|_{L^a(\mathbb{R}^d)}, \ \text{supp} \tilde{u} \subset \lambda C.$$ 

**Lemma 2.2.** *(30) Let $\Omega$ denote the whole space $\mathbb{R}^d$ or the periodic domain $\mathbb{T}^d$. Suppose that $\alpha, \beta \in (0, 1)$, $p, q \in [1, \infty]$, and $k \in \mathbb{N}^+$. Assume that $f \in L^p(0, T; \dot{B}^\alpha_{q, \infty})$, $g \in L^p(0, T; \dot{B}^\beta_{q, c(\mathbb{N})})$, then there holds that
developed by Cheskidov-Constantin-Friedlander-Shvydkoy in [13] and the Constantin-E-Titi
quasi-geostrophic equation (1.1) and 2-D standard surface quasi-geostrophic equation (1.3).

3 Energy conservation of weak solutions for 2-D surface

Next, we will state the Constantin-E-Titi type commutator estimates in physical Onsager type spaces (see also [35]).

Lemma 2.3. ([36]) Let \( \Omega \) denote the whole space \( \mathbb{R}^d \) or the periodic domain \( \mathbb{T}^d \). Assume that \( 0 < \alpha, \beta < 1 \), \( 1 \leq p, q, p_1, p_2 \leq \infty \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then, there holds

\[
\| (fg)^\varepsilon - f^\varepsilon g^\varepsilon \|_{L^p(0,T;L^q(\Omega))} \leq C(\varepsilon^{\alpha + \beta}),
\]

provided one of the following three conditions holds

(1) \( f \in L^{p_1}(0,T;\dot{B}^\alpha_{q_1,c}(\Omega)), g \in L^{p_2}(0,T;\dot{B}^\beta_{q_2,c}(\Omega)), 1 \leq q_1, q_2 \leq \infty, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \);

(2) \( \nabla f \in L^{p_1}(0,T;\dot{B}^\alpha_{q_1,c}(\Omega)), \nabla g \in L^{p_2}(0,T;\dot{B}^\beta_{q_2,c}(\Omega)), \frac{2}{q} + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \); \( 1 \leq q_1, q_2 < d \);

(3) \( f \in L^{p_1}(0,T;\dot{B}^\alpha_{q_1,c}(\Omega)), \nabla g \in L^{p_2}(0,T;\dot{B}^\beta_{q_2,c}(\Omega)), \frac{1}{q} + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \); \( 1 \leq q_2 < d, 1 \leq q_1 \leq \infty \).

For the convenience of readers, we present the definition of the weak solutions of the surface quasi-geostrophic equation (1.1).

Definition 2.1. A vector field \( \theta \in C^{\text{weak}}_{\text{loc}}([0,T];L^p(\mathbb{R}^2)) \) is called a weak solution of the 2-D quasi-geostrophic equation with initial data \( \theta_0 \in L^p(\mathbb{R}^2) \) with \( p \in [2, \infty) \) if there holds

\[
\int_{\mathbb{R}^2} [\theta(x,t)\varphi(x,t) - \theta(x,0)\varphi(x,0)] dx = \int_0^t \int_{\mathbb{R}^2} \theta(x,s)(\partial_t \varphi(x,s) + v(x,s) \cdot \nabla \varphi(x,s)) dx ds \tag{2.3}
\]

and

\[
v(x,t) = \mathcal{R}^\perp \Lambda^{\gamma-1} \theta, \tag{2.4}
\]

for any test function \( \varphi \in C^\infty_0([0,T];C^\infty(\mathbb{R}^2)) \).

3 Energy conservation of weak solutions for 2-D surface quasi-geostrophic equation

In this section, we are concerned with the energy conservation for 2-D generalized surface quasi-geostrophic equation (1.1) and 2-D standard surface quasi-geostrophic equation (1.3). To prove theorem (1.1) we will give two different approaches due to Littlewood-Paley theory developed by Cheskidov-Constantin-Friedlander-Shvydkoy in [13] and the Constantin-E-Titi type commutator estimates in physical Onsager type spaces (see Lemma 2.3), respectively.
Proof of Theorem 1.1.

**Approach 1: Littlewood-Paley theory:** Multiplying the surface quasi-geostrophic equation (1.1) by $S_N(S_N\theta|S_N\theta|^{p-2})$ with $p \geq 2$ (see the notations in Section 2), together with the incompressible condition and using integration by parts, we see that

$$\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |S_N\theta|^p dx = (p - 1) \int_{\mathbb{R}^2} S_N(v_j\theta) \partial_j S_N\theta |S_N\theta|^{p-2} dx.$$ 

Since the divergence-free condition of the velocity field $v(x, t)$ helps us to derive that

$$\int_{\mathbb{R}^2} S_N v_j \partial_j S_N\theta |S_N\theta|^{p-2} dx = 0,$$

thus we conclude that

$$\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |S_N\theta|^p dx = (p - 1) \int_{\mathbb{R}^2} [S_N(v_j\theta) - S_N v_j S_N\theta] \partial_j S_N\theta |S_N\theta|^{p-2} dx.$$ 

Recall the Constantin-E-Titi identity

$$S_N(fg) - S_N f S_N g = 2^{2N} \int_{\mathbb{R}^2} \tilde{h}(2^N y)[f(x - y) - f(x)][g(x - y) - g(x)] dy - (f - S_Nf)(g - S_Ng), \quad (3.1)$$

where we used $2^{2N} \int_{\mathbb{R}^2} \tilde{h}(2^N y) dy = \mathcal{F}(\tilde{h}(|\cdot|))|_{\xi=0} = 1$.

Taking advantage of the Hölder inequality, we discover that

$$\left| \int_{\mathbb{R}^2} [S_N(v_j\theta) - S_N v_j S_N\theta] \partial_j S_N\theta |S_N\theta|^{p-2} dx \right| \leq C \left[ S_N(v_j\theta) - S_N v_j S_N\theta \right]_{L^{p+1}(\mathbb{R}^2)} \left[ \partial_j S_N\theta \right]_{L^{p+1}(\mathbb{R}^2)} \left| S_N\theta \right|_{L^{p+1}(\mathbb{R}^2)} \left( 3.2 \right)$$

$$\leq C \left[ S_N(v_j\theta) - S_N v_j S_N\theta \right]_{L^{p+1}(\mathbb{R}^2)} \left[ \partial_j S_N\theta \right]_{L^{p+1}(\mathbb{R}^2)} \left| S_N\theta \right|_{L^{p+1}(\mathbb{R}^2)}.$$

With the help of (3.1) and the Minkowski inequality, we write

$$\left[ S_N(v_j\theta) - S_N v_j S_N\theta \right]_{L^{p+1}(\mathbb{R}^2)} \leq 2^{2N} \int_{\mathbb{R}^2} \left[ \tilde{h}(2^N y) \right] \left| v_j(x - y) - v_j(x) \right|_{L^{p+1}(\mathbb{R}^2)} \left[ \theta(x - y) - \theta(x) \right]_{L^{p+1}(\mathbb{R}^2)} dy$$

$$+ \left| v_j - S_N v_j \right|_{L^{p+1}(\mathbb{R}^2)} \left[ \theta - S_N\theta \right]_{L^{p+1}(\mathbb{R}^2)} = I + II.$$

In view of the mean value theorem and the Bernstein inequality, we know that

$$\left| v_j(x - y) - v_j(x) \right|_{L^{p+1}(\mathbb{R}^2)} \leq C \left( \sum_{j \leq N} 2^j |\theta| \left[ \hat{\Delta}_j v \right]_{L^{p+1}(\mathbb{R}^2)} + \sum_{j > N} \left[ \hat{\Delta}_j v \right]_{L^{p+1}(\mathbb{R}^2)} \right).$$

(3.3)

Using the Bernstein inequality again and the boundedness of Riesz transforms on Lebesgue spaces, we see that

$$\left[ \hat{\Delta}_j v \right]_{L^{p+1}(\mathbb{R}^2)} = \left[ \mathcal{R}_j A^{\gamma-1} \hat{\Delta}_j \theta \right]_{L^{p+1}(\mathbb{R}^2)} \leq C 2^{j(\gamma-1)} \left[ \hat{\Delta}_j \theta \right]_{L^{p+1}(\mathbb{R}^2)},$$

for $0 < p < \infty$. 

(4.4)
In light of the Bernstein inequality, we infer that

$$
\|v_j(x - y) - v_j(x)\|_{L^{p+1}(\mathbb{R}^2)} 
\leq C\left(2^{N(\gamma - \alpha)}|y| \sum_{j \leq N} 2^{-(N-j)(\gamma - \alpha)} 2^{j\alpha} \|\tilde{\Delta} \theta\|_{L^{p+1}(\mathbb{R}^2)} + 2^{(\gamma - \alpha)N} \sum_{j > N} 2^{(N-j)(\alpha + 1 - \gamma)} 2^{j\alpha} \|\tilde{\Delta} \theta\|_{L^{p+1}(\mathbb{R}^2)}\right). 
$$

(3.4)

Before going further, in the spirit of [13], we set the following localized kernel

$$
K_1(j) = \begin{cases} 
2^{j(\alpha + 1 - \gamma)}, & \text{if } j \leq 0, \\
2^{-(\gamma - \alpha)j}, & \text{if } j > 0,
\end{cases}
$$

(3.5)

and we denote $\hat{d}_j = 2^{j\alpha}\|\tilde{\Delta} \theta\|_{L^{p+1}(\mathbb{R}^2)}$. As a consequence, we get

$$
\|v_j(x - y) - v_j(x)\|_{L^{p+1}(\mathbb{R}^2)} \leq C\left(2^{N(\gamma - \alpha)}|y| + 2^{(\gamma - \alpha)N} \left(K_1 * \hat{d}_j\right)(N)\right) \leq C(2^N|y| + 1)2^{(\gamma - \alpha)N} \left(K_1 * \hat{d}_j\right)(N).
$$

To bound $\|\theta(x - y) - \theta(x)\|_{L^{p+1}(\mathbb{R}^2)}$, just as [13], we denote

$$
K_2(j) = \begin{cases} 
2^{j\alpha}, & \text{if } j \leq 0, \\
2^{-(1-\alpha)j}, & \text{if } j > 0.
\end{cases}
$$

(3.6)

A slightly modified proof of (3.3) and (3.4) gives

$$
\|\theta(x - y) - \theta(x)\|_{L^{p+1}(\mathbb{R}^2)} 
\leq C\left(\sum_{j \leq N} 2^j|y|\|\tilde{\Delta} \theta\|_{L^{p+1}(\mathbb{R}^2)} + \sum_{j > N} \|\tilde{\Delta} \theta\|_{L^{p+1}(\mathbb{R}^2)}\right) 
\leq C\left(2^{N(1-\alpha)}|y| \sum_{j \leq N} 2^{-(N-j)(1-\alpha)} 2^{j\alpha} \|\tilde{\Delta} \theta\|_{L^{p+1}(\mathbb{R}^2)} + 2^{-\alpha N} \sum_{j > N} 2^{(N-j)\alpha} 2^{j\alpha} \|\tilde{\Delta} \theta\|_{L^{p+1}(\mathbb{R}^2)}\right) 
\leq C\left(2^{N(1-\alpha)}|y| + 2^{-\alpha N}\right) \left(K_2 * \hat{d}_j\right)(N) 
\leq C(2^N|y| + 1)2^{-\alpha N} \left(K_2 * \hat{d}_j\right)(N).
$$

(3.7)

Notice that

$$
\sup_N 2^{2N} \int_{\mathbb{R}^2} |\hat{h}(2^N y)|(2^N|y| + 1)^2 dy < \infty.
$$

Hence, we deduce from (3.4) and (3.7) that

$$
I \leq C2^{(\gamma - \alpha)N} \left(K_1 * \hat{d}_j\right)(N)2^{-\alpha N} \left(K_2 * \hat{d}_j\right)(N).
$$

In light of the Bernstein inequality, we infer that

$$
\|v_j - S_Nv_j\|_{L^{p+1}(\mathbb{R}^2)} \leq \sum_{j \geq N} \|\tilde{\Delta} \theta\|_{L^{p+1}} \leq C2^{(\gamma - \alpha)N} \left(K_1 * \hat{d}_j\right)(N),
$$

(3.8)
where we used $N > 0$.

Likewise,
\[ \| \theta - S_N \theta \|_{L^{p+1}(\mathbb{R}^2)} \leq C 2^{-\alpha N} \left( K_2 * \hat{d}_j \right)(N), \]
from which it follows that
\[ II \leq C 2^{(\gamma - 1 - \alpha)N} \left( K_1 * \hat{d}_j \right)(N) 2^{-\alpha N} \left( K_2 * \hat{d}_j \right)(N). \]

Consequently, we know that
\[ \left\| S_N(v_j \theta) - S_N v_j S_N \theta \right\|_{L^{p+1}(\mathbb{R}^2)} \leq C 2^{(\gamma - 1 - \alpha)N} \left( K_1 * \hat{d}_j \right)(N) 2^{-\alpha N} \left( K_2 * \hat{d}_j \right)(N). \] (3.8)

We conclude by some straightforward calculations that
\[ \left\| \partial_j S_N \theta \right\|_{L^{p+1}(\mathbb{R}^2)} \leq \sum_{j \leq N} 2^j \left\| \Delta_j \theta \right\|_{L^{p+1}(\mathbb{R}^2)} \leq 2^{N(1-\alpha)} \left( K_2 * d_j \right)(N), \] (3.9)

where $d_j = 2^{j\alpha} \left\| \Delta_j \theta \right\|_{L^{p+1}(\mathbb{R}^2)}$.

Inserting (3.8) and (3.9) into (3.2) gives
\[ \left\| S_N(v_j \theta) - S_N v_j S_N \theta \right\|_{L^{p+1}(\mathbb{R}^2)} \leq C 2^{(\gamma - 1 - \alpha)N} \left( K_1 * \hat{d}_j \right)(N) 2^{-\alpha N} \left( K_2 * \hat{d}_j \right)(N) \] (3.10)

To ensure that $K_1, K_2 \in l^1(\mathbb{Z})$, we need
\[ \begin{cases} 
\alpha + 1 - \gamma > 0, \\
\gamma - \alpha > 0, \\
0 < \alpha < 1, \\
\gamma - 3\alpha \leq 0, 
\end{cases} \] (3.11)

which lead to $\alpha \geq \frac{\gamma}{3}$ and $\alpha < \gamma < \alpha + 1$.

Then substituting (3.10) into (3.2) and using the Young inequality, we arrive at
\[ \left\| \int \left[ S_N(v_j \theta) - S_N v_j S_N \theta \right] \partial_j S_N \theta | S_N \theta |^{p-2} dx \right\|^{p-2}_{L^{p+1}(\mathbb{R}^2)} \leq C 2^{(\gamma - 3\alpha)N} \left( K_1 * \hat{d}_j \right)(N) \left( K_2 * \hat{d}_j \right)(N) \| S_N \theta \|_{L^{p+1}(\mathbb{R}^2)} \] (3.12)

where $d_N = 2^{N\alpha} \left\| \Delta_N \theta \right\|_{L^{p+1}(\mathbb{R}^2)}$ and $d_N = 2^{N\alpha} \left\| \Delta_N \theta \right\|_{L^{p+1}(\mathbb{R}^2)}$.

Case 1: if $\alpha = \frac{\gamma}{3}$ with $0 < \gamma < \frac{3}{4}$, it follows from (3.12) and the dominated convergence theorem that
\[ \left\| \int \left[ S_N(v_j \theta) - S_N v_j S_N \theta \right] \partial_j S_N \theta | S_N \theta |^{p-2} dx \right\|^{p-2}_{L^{p+1}(\mathbb{R}^2)} \leq C \left( K_1 * \hat{d}_j \right)(N) \left( K_2 * \hat{d}_j \right)(N) \sup_{N} (d_N) \| \theta \|_{L^{p+1}(\mathbb{R}^2)} \] (3.13)

where $\| \theta \|_{L^{p+1}(\mathbb{R}^2)} \rightarrow 0$, as $N \rightarrow +\infty$. 

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This together with \( \theta \in L^{p+1}(0, T; B^{\frac{\gamma}{p}+1, c(N)}) \), we conclude by the dominated convergence theorem once again that
\[
\limsup_{N \to +\infty} \int_0^T \left| \int_{\mathbb{R}^2} \left[ S_N(v_j \theta) - S_N v_j S_N \theta \right] \partial_j S_N \theta |S_N \theta|^{p-2} dx \right| dt 
\leq C \int_0^T \left( K_1 * \dot{d}_j \right) (N) \left( K_2 * \dot{d}_j \right) (N) \|\theta\|_{B^{p+1,1}(\mathbb{R}^2)}^{p-1} dt \to 0,
\]

Case 2: if \( \frac{\gamma}{2} \leq \gamma - 1 < \alpha < 1 \) with \( \frac{3}{2} \leq \gamma < 2 \), then from [3.12] and taking \( N \to +\infty \), we have
\[
\left| \int_{\mathbb{R}^2} \left[ S_N(v_j \theta) - S_N v_j S_N \theta \right] \partial_j S_N \theta |S_N \theta|^{p-2} dx \right| 
\leq C 2^{(\gamma-3)\alpha} N \|\theta\|_{B^{p+1,1}(\mathbb{R}^2)} \|\theta\|_{L^{p+1}(\mathbb{R}^2)}^{p-2} 
\leq C 2^{(\gamma-3)\alpha} N \|\theta\|_{B^{p+1,1}(\mathbb{R}^2)}^{p-1} \to 0.
\]

which in turn gives
\[
\limsup_{N \to +\infty} \int_0^t \left| \int_{\mathbb{R}^2} \left[ S_N(v_j \theta) - S_N v_j S_N \theta \right] \partial_j S_N \theta |S_N \theta|^{p-2} dx \right| ds 
\leq \limsup_{N \to +\infty} 2^{(\gamma-3)\alpha} N C \int_0^t \|\theta\|_{B^{p+1,1}(\mathbb{R}^2)}^{p-1} ds \to 0.
\]

Hence, no matter in which case, we have
\[
\left| \int_0^t \int_{\mathbb{R}^2} \left[ S_N(v_j \theta) - S_N v_j S_N \theta \right] \partial_j S_N \theta |S_N \theta|^{p-2} dx ds \right| \to 0, \text{ as } N \to +\infty.
\]

Then we have completed the proof of Theorem 1.1.

**Approach 2: Constantin-E-Titi type commutator estimates in physical Onsager type spaces:** Mollifying the surface quasi-geostrophic equation (1.1) in space (see the notations in Section 2) and using the divergence-free condition, we know that
\[ \theta_t^\varepsilon + \text{div}(v \theta)^\varepsilon = 0, \]
which yields that
\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |\theta^\varepsilon|^p dx = (p - 1) \int_{\mathbb{R}^2} (v_j \theta)^\varepsilon \partial_j \theta^\varepsilon |\theta^\varepsilon|^{p-2} dx.
\]
The incompressible condition allows us to formulate the above equation as
\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |\theta^\varepsilon|^p dx = (p - 1) \int_{\mathbb{R}^2} \left[ (v_j \theta)^\varepsilon - v_j^\varepsilon \theta^\varepsilon \right] \partial_j \theta^\varepsilon |\theta^\varepsilon|^{p-2} dx,
\]
which immediately means
\[
\frac{1}{p} \left( \|\theta^\varepsilon(x, t)\|_{L^p(\mathbb{R}^2)} - \|\theta^\varepsilon(x, 0)\|_{L^p(\mathbb{R}^2)} \right) = (p - 1) \int_0^t \int_{\mathbb{R}^2} \left( (v_j \theta)^\varepsilon - v_j^\varepsilon \theta^\varepsilon \right) \partial_j \theta^\varepsilon |\theta^\varepsilon|^{p-2} dx ds.
\]
The Hölder inequality enables us to get
\[
\left| \int_0^t \int_{\mathbb{R}^2} \left[ (v_j \theta)^\varepsilon - v_j^\varepsilon \right] \partial_j \theta^\varepsilon \theta^{|p-2} \, dx \, ds \right|
\leq C \left\| (v_j \theta)^\varepsilon - v_j^\varepsilon \right\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))} \left\| \partial_j \theta^\varepsilon \right\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))} \left\| \theta^{|p-2} \right\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))}.
\]
(3.14)
Since \( B_{p,q}^s = \hat{B}_{p,q}^s \cap L^p \) for \( s > 0 \), the hypothesis \( \theta \in L^{p+1}(0,T;B_{p,\infty}^0(\mathbb{R}^N)) \) means \( \theta \in L^{p+1}(0,T;\hat{B}_{p,\infty}^0(\mathbb{R}^N)) \). This and the boundedness of Riesz transforms in homogeneous Besov spaces, we obtain
\[
v = \mathcal{R}^{\frac{1}{2}} \Lambda^{-1} \theta \in L^{p+1}(0,T;\hat{B}_{p+1,\infty}^0(\mathbb{R}^N)).
\]
Combining \( \theta \in L^{p+1}(0,T;\hat{B}_{p+1,\infty}^0(\mathbb{R}^N)) \) with \( v \in L^{p+1}(0,T;\hat{B}_{p+1,\infty}^0(\mathbb{R}^N)) \) and invoking Lemma 2.3, we see that
\[
\left\| (v_j \theta)^\varepsilon - v_j^\varepsilon \right\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))} \leq o(\varepsilon^{2\alpha-1+1}),
\]
(3.15)
where we require \( 0 < \alpha < 1 \) and \( 0 < \alpha - \gamma + 1 < 1 \).
Using Lemma 2.2, we know that
\[
\left\| \partial_j \theta^\varepsilon \right\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))} \leq o(\varepsilon^{\alpha-1}).
\]
(3.16)
Moreover, in view of the definition of Besov spaces, we have
\[
\left\| \theta^{|p-2} \right\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))} \leq C \left\| \theta \right\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))} \leq C \left\| \theta \right\|_{L^{p+1}(0,T;B_{p,\infty}^0(\mathbb{R}^N))}.
\]
(3.17)
Then substituting (3.15)-(3.17) into (3.14), setting \( \alpha = \frac{\gamma}{3} \) and choosing \( \varepsilon \to 0 \) with \( 0 < \gamma < \frac{3}{2} \), we have
\[
\left| \int_0^t \int_{\mathbb{R}^2} \left[ (v_j \theta)^\varepsilon - v_j^\varepsilon \right] \partial_j \theta^\varepsilon \theta^{|p-2} \, dx \, ds \right| \leq o(\varepsilon^{2\alpha-1} \left\| \theta \right\|_{L^{p+1}(0,T;B_{p+1,\infty}^0(\mathbb{R}^N))} \to 0.
\]
Then we have completed the proof of the first part of Theorem 1.1. By a similar argument to (3.15)-(3.17), we can conclude the second part of Theorem 1.1 for \( \theta \in L^{p+1}(0,T;B_{p+1,\infty}^0). \)

**Proof of Corollary 1.2.** It is enough to notice that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |S_N \theta|^2 \, dx = \int_{\mathbb{R}^2} \left[ S_N v_j \theta - S_N v_j S_N \theta \right] \partial_j S_N \theta \, dx.
\]
Exactly as in the above derivation in the Theorem 1.1 the proof of this Corollary.

Next, we present the proof of Theorem 1.3. In the proof of Theorem 1.1, it suffices to replace (3.4) by
\[
\left\| v_j(x - y) - v_j(x) \right\|_{L^{p+1}(\mathbb{R}^2)} \leq C \left( 2^{N(1-\alpha)} \left| y \right| \sum_{j \leq N} 2^{-(N-j)(1-\alpha)} 2^{j\alpha} \left\| \hat{\Delta}_j y \right\|_{L^{p+1}(\mathbb{R}^2)} + 2^{-\alpha N} \sum_{j > N} 2^{(N-j)(\alpha)} 2^{j\alpha} \left\| \hat{\Delta}_j y \right\|_{L^{p+1}(\mathbb{R}^2)} \right)
\leq C \left[ 2^{N(1-\alpha)} \left| y \right| + 2^{-\alpha N} \right] \left( K_1 * d_j \right) (N)
\leq C (2^N |y| + 1) 2^{-\alpha N} \left( K_1 * d_j \right) (N),
\]
where
\[ K_1(j) = \begin{cases} 2^{ja}, & \text{if } j \leq 0, \\ 2^{-(1-\alpha)j}, & \text{if } j > 0, \end{cases} \]
and \( \hat{d}_1(j) = 2^{ja}\|\Delta_j v\|_{L^{p+1}} \). We omit the details here. We only outline its proof by Constantin-E-Titi type commutator estimates in physical Onsager type spaces in the following.

**Proof of Theorem 1.3.** Based on the second proof of Theorem 1.1, we just give the key estimates. It follows from the Hölder inequality, we discover that

\[
\left| \int_0^t \int_{\mathbb{R}^2} \left[ (v_j \theta)^{\varepsilon} - v_j^{\varepsilon} \theta^{\varepsilon} \right] \partial_t \theta^{\varepsilon} |\theta^{\varepsilon}|^{p-2} dxds \right| \\
\leq C \| (v_j \theta)^{\varepsilon} - v_j^{\varepsilon} \theta^{\varepsilon} \|_{L^{\frac{p+1}{2}}(0, T; L^{\frac{p+1}{p}}(\mathbb{R}^2))} \| \partial_t \theta^{\varepsilon} \|_{L^2(0, T; L^{p+1}(\mathbb{R}^2))} \| \theta^{\varepsilon} |^{p-2} \|_{L^{p+1}(0, T; L^{p+1}(\mathbb{R}^2))},
\]

where \( \frac{r_1}{r_2} + \frac{1}{r_4} = 1 \).

From \( v \in L^{r_1}(0, T; \dot{B}^{\frac{1}{p}+1,c}_{p+1}(\mathbb{R}^2)) \) and \( \theta \in L^q(0, T; \dot{B}^{\frac{1}{p}+1}_{p+1,\infty}) \), we deduce from Lemma 2.3 that

\[
\| (v_j \theta)^{\varepsilon} - v_j^{\varepsilon} \theta^{\varepsilon} \|_{L^{\frac{p+1}{2}}(0, T; L^{\frac{p+1}{p}}(\mathbb{R}^2))} \leq C \varepsilon^{\frac{2}{q}}.
\]

From Lemma 2.2 we infer that

\[
\| \partial_t \theta^{\varepsilon} \|_{L^2(0, T; L^{p+1}(\mathbb{R}^2))} \leq C \varepsilon^{-\frac{2}{q}}.
\]

According to the definition of Besov spaces, we have

\[
\| \theta |^{p-2} \|_{L^{p+1}(0, T; L^{p+1}(\mathbb{R}^2))} \leq C \| \theta |^{p-2} \|_{L^{p+2}(0, T; L^{p+1}(\mathbb{R}^2))} \leq C \| \theta |^{p-2} \|_{L^{p+2}(0, T; \dot{B}^{\frac{1}{p}+1}_{p+1,\infty})},
\]

where we used \( p_4(p - 2) = r_2 \), which means \( \frac{1}{r_2} + \frac{1}{r_1} = 1 \) and \( p \geq 2 \).

Then substituting (3.19)-(3.21) into (3.18) and letting \( \varepsilon \to 0 \), we have

\[
\left| \int_0^t \int_{\mathbb{R}^2} \left[ (v_j \theta)^{\varepsilon} - v_j^{\varepsilon} \theta^{\varepsilon} \right] \partial_t |\theta^{\varepsilon}|^{p-2} dxds \right| \leq C(1) \| \theta |^{p-2} \|_{L^{p+2}(0, T; \dot{B}^{\frac{1}{p}+1}_{p+1,\infty})} \to 0.
\]

Then we have completed the proof of Theorem 1.3.

4 General helicity conservation for 2-D surface quasi-geostrophic equations

In this section, we are concerned with the helicity conservation of weak solutions for 2-D generalized surface quasi-geostrophic equation (1.1). We also show two different approaches to prove Theorem 1.3.
Proof of Theorem 1.4. Approach 1: Littlewood-Paley theory First, due to the divergence free of velocity $v(x, t)$ and applying the operator $S_N$ to the surface quasi-geostrophic equation (1.1), we get

$$S_N \theta_t + S_N \partial_j (v_j \theta) = 0,$$

and

$$\partial_i S_N \theta_t + \partial_j S_N (\partial_i v_j \theta) + \partial_j S_N (v_j \partial_i \theta) = 0.$$

Straightforward calculations show that

$$\frac{d}{dt} \int_{\mathbb{R}^2} S_N \theta \partial_i S_N \theta dx$$

$$= \int_{\mathbb{R}^2} S_N \theta \partial_i \partial_i S_N \theta dx + \int_{\mathbb{R}^2} \partial_i S_N \theta \partial_i S_N \theta dx$$

$$= - \int_{\mathbb{R}^2} S_N \theta [\partial_j S_N (\partial_i v_j \theta) + \partial_j S_N (v_j \partial_i \theta)] dx - \int_{\mathbb{R}^2} \partial_i S_N (v_j \theta) \partial_i S_N \theta dx, \quad i = 1, 2.$$

Thanks to $\int_{\mathbb{R}^2} \partial_j (\partial_i S_N v_j S_N \theta) S_N \theta dx = 0$, we may write

$$\frac{d}{dt} \int_{\mathbb{R}^2} S_N \theta \partial_i S_N \theta dx$$

$$= - \int_{\mathbb{R}^2} S_N \theta [\partial_j S_N (\partial_i v_j \theta) - \partial_j (\partial_i S_N v_j S_N \theta)] dx - \int_{\mathbb{R}^2} S_N \theta [\partial_j S_N (v_j \partial_i \theta) - \partial_j (S_N v_j \partial_i S_N \theta)] dx$$

$$= \int_{\mathbb{R}^2} \partial_j S_N \theta [S_N (\partial_i v_j \theta) - (\partial_i S_N v_j S_N \theta)] dx + \int_{\mathbb{R}^2} [S_N (v_j \partial_i \theta) - (S_N v_j \partial_i S_N \theta)] \partial_j S_N \theta dx$$

$$+ \int_{\mathbb{R}^2} [S_N (v_j \theta) - S_N \partial_i S_N v_j] \partial_i \partial_j S_N \theta dx$$

$$= I + II + III.$$

To control $I$, we deduce from the Hölder inequality that

$$|I| \leq \|S_N (\partial_i v_j \theta) - (\partial_i S_N v_j S_N \theta)\|_{L^4} \|\partial_j S_N \theta\|_{L^6}.$$

Taking advantage of Constantin-E-Titi identity (3.1), Minkowski inequality and the Sobolev inequality, we infer that

$$\|S_N (\partial_i v_j \theta) - (\partial_i S_N v_j S_N \theta)\|_{L^4}$$

$$\leq C 2^N \int_{\mathbb{R}^2} |h(2^N y)| \|\partial_i v_j (x - y) - \partial_i v_j (x)\|_{L^4} \|\theta (x - y) - \theta (x)\|_{L^6} dy$$

$$+ C \|\partial_i v_j - S_N \partial_i v_j\|_{L^4} \|\theta - S_N \theta\|_{L^6}$$

$$\leq C 2^N \int_{\mathbb{R}^2} |h(2^N y)| \|\partial_i v_j (x - y) - \partial_i v_j (x)\|_{L^4} \|\nabla \theta (x - y) - \nabla \theta (x)\|_{L^4} dy$$

$$+ C \|\partial_i v_j - S_N \partial_i v_j\|_{L^4} \|\nabla \theta - \nabla S_N \theta\|_{L^4}. \quad (4.2)$$
Arguing in the same manner as (3.4), we observe that
\[
\| \partial_t v_j(x - y) - \partial_t v_j(x) \|_{L^2_y(R^2)} \leq C \left( \sum_{j \leq N} 2^j |y| \| \hat{\Delta}_j \nabla v \|_{L^2_y(R^2)} + \sum_{j > N} \| \hat{\Delta}_j \nabla v \|_{L^2_y(R^2)} \right)
\]
\[
\leq C \left( 2^{N(\gamma - \alpha)} |y| \sum_{j \leq N} 2^{-(N-j)(\gamma - \alpha)j} \| \hat{\Delta}_j \nabla \theta \|_{L^2_y(R^2)} + 2^{(1-\alpha)N} \sum_{j > N} 2^{(N-j)(\alpha + 1 - \gamma)j} \| \hat{\Delta}_j \nabla \theta \|_{L^2_y(R^2)} \right)
\]
\[
\leq C \left( 2^{N(\gamma - \alpha)} |y| + 2^{(1-\alpha)N} \right) \left( K_1 * d_j \right)(N)
\]
\[
\leq C(2^N|y| + 1)2^{(\gamma - \alpha)N} \left( K_1 * d_j \right)(N).
\] (4.3)

where $K_1$ is defined in (3.5) and $\dot{d}_j = 2^j \alpha \| \hat{\Delta}_j \nabla \theta \|_{L^2_y(R^2)}$.

Similar to the derivation of (3.7) and using (3.6), we get
\[
\| \nabla \theta(x - y) - \nabla \theta(x) \|_{L^2_y(R^2)} \leq C \left( \sum_{j \leq N} 2^j |y| \| \hat{\Delta}_j \nabla \theta \|_{L^2_y(R^2)} + \sum_{j > N} \| \hat{\Delta}_j \nabla \theta \|_{L^2_y(R^2)} \right)
\]
\[
\leq C \left( 2^{N(1-\alpha)} |y| \sum_{j \leq N} 2^{-(N-j)(1-\alpha)j} \| \hat{\Delta}_j \nabla \theta \|_{L^2_y(R^2)} + 2^{-\alpha N} \sum_{j > N} 2^{(N-j)\alpha + 2j} \| \hat{\Delta}_j \nabla \theta \|_{L^2_y(R^2)} \right)
\]
\[
\leq C \left( 2^{N(1-\alpha)} |y| + 2^{-\alpha N} \right) \left( K_2 * \dot{\theta} \right)(N)
\]
\[
\leq C(2^N|y| + 1)2^{-\alpha N} \left( K_2 * d_j \right)(N).
\] (4.4)

Some straightforward computations yields
\[
\| \nabla \theta - \nabla S_N \theta \|_{L^2_y(R^2)} \leq C2^{-\alpha N} \left( K_2 * d_j \right)(N),
\] (4.5)
and
\[
\| \partial_j S_N \theta \|_{L^2_y(R^2)} \leq C\| \nabla \partial_j S_N \theta \|_{L^2_y(R^2)} \leq C \sum_{j \leq N} 2^j \| \Delta_j \nabla \theta \|_{L^2_y(R^2)} \leq 2^{N(1-\alpha)} \left( K_2 * d_j \right)(N),
\] (4.6)

where the Sobolev embedding was used and $\dot{d}_j = 2^{j\alpha} \| \hat{\Delta}_j \nabla \theta \|_{L^2_y(R^2)}$.

As a consequence, we know
\[
I \leq 2^{(\gamma - \alpha)N} \left( K_1 * \dot{d}_j \right)(N)2^{-\alpha N} \left( K_2 * \dot{d}_j \right)(N)2^{N(1-\alpha)} \left( K_2 * \dot{d}_j \right)(N).
\]

Repeating the deduction process of $I$, we have
\[
II \leq 2^{(\gamma - 3\alpha)N} \left( K_1 * \dot{d}_2 \right)(N) \left( K_2 * \dot{d}_2 \right)(N) \left( K_2 * \dot{d}_2 \right)(N).
\]

Taking advantage of the Hölder inequality, we infer that
\[
III \leq \| S_N(v_j \theta) - S_N S_N v_j \|_{L^2_y(R^2)} \| \partial_t S_N \theta \|_{L^2_y(R^2)}.
\] (4.7)
Following the path of (4.3), we arrive at
\[
\|S_N(v_j \theta) - S_N \theta S_N v_j\|_{L^3(R^2)} \\
\leq C 2^{2N} \int | \hat{h}(2^N y) \|v_j(x-y) - v_j(x)\|_{L^5(R^2)} \|\theta(x-y) - \theta(x)\|_{L^5(R^2)} dy \\
+ C \|v_j - S_N v_j\|_{L^6(R^2)} \|\theta - S_N \theta\|_{L^6(R^2)} \\
\leq C 2^{2N} \int | \hat{h}(2^N y) \|\nabla v_j(x-y) - \nabla v_j(x)\|_{L^2(R^2)} \|\nabla \theta(x-y) - \nabla \theta(x)\|_{L^2(R^2)} dy \\
+ C \|\nabla v_j - S_N \nabla v_j\|_{L^2(R^2)} \|\nabla \theta - \nabla S_N \theta\|_{L^2(R^2)}.
\]

From (4.3)-(4.4), we have
\[
\|S_N(v_j \theta) - S_N \theta S_N v_j\|_{L^3(R^2)} \leq 2^{(\gamma - 1 - \alpha)N} \left( K_1 \ast \hat{d}_j \right)(N) 2^{-\alpha N} \left( K_2 \ast \hat{d}_j \right)(N). \tag{4.8}
\]

It follows from (4.6) that
\[
\|\nabla \partial_j S_N \theta\|_{L^4(R^2)} \leq C \sum_{j \leq N} 2^j \|\Delta_j \nabla \theta\|_{L^4(R^2)} \leq 2^{N(1-\alpha)} \left( K_2 \ast \hat{d}_j \right)(N). \tag{4.9}
\]

Substituting (4.8) (4.9) into (4.7), we conclude that
\[
III \leq C 2^{(\gamma - 3\alpha)N} \left( K_1 \ast \hat{d}_j \right)(N) \left( K_2 \ast \hat{d}_j \right)(N) \left( K_2 \ast \hat{d}_j \right)(N).
\]

Finally, we end up with
\[
\frac{d}{dt} \int S_N \theta \partial_i S_N \theta dx \leq C 2^{(\gamma - 3\alpha)N} \left( K_1 \ast \hat{d}_j \right)(N) \left( K_2 \ast \hat{d}_j \right)(N) \left( K_2 \ast \hat{d}_j \right)(N). \tag{4.10}
\]

At this stage, the rest proof of this theorem is the same as the one of Theorem 1.1. \hfill \Box

**Proof of Theorem 1.4** Approach 2: Constantin-E-Titi type commutator estimates in physical Onsager type spaces

It is obvious that
\[
\theta_i^\varepsilon + \partial_j(v_j \theta)^\varepsilon = 0,
\]
and
\[
\partial_i \theta_i^\varepsilon + \partial_j(\partial_i v_j \theta)^\varepsilon + \partial_j(v_j \partial_i \theta)^\varepsilon = 0.
\]

Thus, it follows from the direct computation that
\[
\frac{d}{dt} \int_{R^2} \theta^\varepsilon \partial_i \theta^\varepsilon dx = \int_{R^2} \theta^\varepsilon \partial_i \partial_i \theta^\varepsilon dx + \int_{R^2} \partial_i \theta^\varepsilon \partial_i \theta^\varepsilon dx \\
= - \int_{R^2} \theta^\varepsilon [\partial_j(\partial_i v_j \theta)^\varepsilon + \partial_j(v_j \partial_i \theta)^\varepsilon] dx - \int_{R^2} \partial_j(v_j \theta)^\varepsilon \partial_i \theta^\varepsilon dx, i = 1, 2. \tag{4.11}
\]
Since \( \int_{\mathbb{R}^2} \partial_j (\partial_i v_j^\varepsilon) \theta^\varepsilon \, dx = 0 \), we can rewrite the above equation \((4.11)\) as

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \theta^\varepsilon \partial_t \theta^\varepsilon \, dx = - \int_{\mathbb{R}^2} \theta^\varepsilon \left[ \partial_j (\partial_i v_j \theta^\varepsilon) - \partial_j (\partial_i v_j^\varepsilon \theta^\varepsilon) \right] \, dx - \int_{\mathbb{R}^2} \theta^\varepsilon \left[ \partial_j (v_j \partial_i \theta) \right] \, dx - \int_{\mathbb{R}^2} \theta^\varepsilon \partial_j (v_j^\varepsilon \partial_i \theta^\varepsilon) \, dx - \int_{\mathbb{R}^2} \theta^\varepsilon \partial_j (v_j^\varepsilon \theta^\varepsilon) \, dx
\]

\[
= \int_{\mathbb{R}^2} \partial_j \theta^\varepsilon \left[ (\partial_i v_j \theta^\varepsilon) - (\partial_i v_j^\varepsilon \theta^\varepsilon) \right] \, dx + \int_{\mathbb{R}^2} \left[ (v_j \partial_i \theta) \theta^\varepsilon - (v_j^\varepsilon \partial_i \theta^\varepsilon) \right] \, dx - \int_{\mathbb{R}^2} \theta^\varepsilon v_j^\varepsilon \partial_i \theta^\varepsilon \, dx + \int_{\mathbb{R}^2} (v_j^\varepsilon \theta^\varepsilon) \partial_i \theta^\varepsilon \, dx
\]

\[
+ \int_{\mathbb{R}^2} \left[ (v_j \theta^\varepsilon - \theta^\varepsilon v_j^\varepsilon) \right] \partial_i \partial_j \theta^\varepsilon \, dx,
\]

which implies

\[
\int_{\mathbb{R}^2} \theta^\varepsilon (x, t) \partial_t \theta^\varepsilon (x, t) \, dx - \int_{\mathbb{R}^2} \theta^\varepsilon (x, 0) \partial_t \theta^\varepsilon (x, 0) \, dx
\]

\[
= \int_0^t \int_{\mathbb{R}^2} \partial_j \theta^\varepsilon \left[ (\partial_i v_j \theta^\varepsilon) - (\partial_i v_j^\varepsilon \theta^\varepsilon) \right] \, dxds + \int_0^t \int_{\mathbb{R}^2} \left[ (v_j \partial_i \theta) \theta^\varepsilon - (v_j^\varepsilon \partial_i \theta^\varepsilon) \right] \, dxds + \int_0^t \int_{\mathbb{R}^2} \left[ (v_j \theta^\varepsilon - \theta^\varepsilon v_j^\varepsilon) \right] \partial_i \partial_j \theta^\varepsilon \, dxds
\]

\[
= I + II + III.
\]

Taking advantage of the Hölder inequality, we get

\[
|I| \leq \| (\partial_i v_j) \theta^\varepsilon - (\partial_i v_j^\varepsilon \theta^\varepsilon) \|_{L^2(0, T; L^q(\mathbb{R}^2))} \| \partial_j \theta^\varepsilon \|_{L^3(0, T; L^6(\mathbb{R}^2))}. \tag{4.12}
\]

Due to the hypothesis \( \nabla \theta \in L^3(0, T; \dot{B}_{\frac{3}{2}, c}^{\alpha-\gamma+1}(\mathbb{R}^2)) \) and the boundedness of Riesz transforms in homogeneous Besov spaces, we obtain that

\[
\nabla v = R^+ \Lambda^\gamma \nabla \theta \in L^3(0, T; \dot{B}_{\frac{3}{2}, c}^{\alpha-\gamma+1}(\mathbb{R}^2)).
\]

Then we employ \((3)\) in Lemma \(2.3\) with \( q = \frac{6}{5}, d = 2, q_1 = \frac{3}{2}, q_2 = \frac{3}{2} < 2 \) and use \( \nabla \theta \in L^3(0, T; \dot{B}_{\frac{3}{2}, c}^{\alpha-\gamma+1}(\mathbb{R}^2)) \) and \( \nabla v \in L^3(0, T; \dot{B}_{\frac{3}{2}, c}^{\alpha-\gamma+1}(\mathbb{R}^2)) \) to derive that

\[
\| (\partial_i v_j) \theta^\varepsilon - (\partial_i v_j^\varepsilon \theta^\varepsilon) \|_{L^2(0, T; L^q(\mathbb{R}^2))} \leq o(\varepsilon^{2\alpha-\gamma+1}), \tag{4.13}
\]

where \( \frac{1}{2} + \frac{5}{6} = \frac{1}{q_1} + \frac{1}{q_2} \) and \( 0 < \alpha - \gamma + 1 < 1 \) were utilized.

By means of Sobolev embedding theorem, Lemma \(2.2\) and \( \nabla \theta \in L^3(0, T; \dot{B}_{\frac{3}{2}, c}^{\alpha-\gamma+1}(\mathbb{R}^2)) \), we conclude that

\[
\| \partial_j \theta^\varepsilon \|_{L^3(0, T; L^6(\mathbb{R}^2))} \leq C \| \nabla \theta^\varepsilon \|_{L^3(0, T; L^\frac{3}{2} (\mathbb{R}^2))} \leq C(\varepsilon^{\alpha-1}). \tag{4.14}
\]

Plugging \((4.13)\) and \((4.14)\) into \((4.12)\), we arrive at

\[
|I| \leq C(\varepsilon^{3\alpha-\gamma}).
\]
Arguing as above, we deduce that
\[ |II| \leq C \alpha (\varepsilon^{3 \alpha - \gamma}). \]

It is enough to estimate the term III. Applying the Hölder inequality once again, we get
\[ |III| \leq \| (v_j \theta^\varepsilon - \theta^\varepsilon v_j^\varepsilon) \|_{L^2(0,T;L^3(\mathbb{R}^2))} \| \partial_i \partial_j \theta^\varepsilon \|_{L^3(0,T;L^3(\mathbb{R}^2))}. \tag{4.15} \]

Invoking (2) in Lemma 2.3 with \( q = 3, d = 2, q_1 = \frac{3}{2}, q_2 = \frac{3}{2} < 2 \), we find
\[ \| (v_j \theta^\varepsilon - \theta^\varepsilon v_j^\varepsilon) \|_{L^2(0,T;L^3(\mathbb{R}^2))} \leq C \alpha (\varepsilon^{2 \alpha - \gamma + 1}), \tag{4.16} \]

where we used \( \nabla \theta \in L^3(0,T;B^{\frac{3}{2}}_{\frac{3}{2},\infty}(\mathbb{R}^2)) \) and \( \nabla v \in L^3(0,T;B^{\alpha - \gamma + 1}_{\frac{3}{2},\infty}(\mathbb{R}^2)). \)

In the light of Lemma 2.2, we infer that
\[ \| \partial_i \partial_j \theta^\varepsilon \|_{L^3(0,T;L^3(\mathbb{R}^2))} \leq C \alpha (\varepsilon^{\alpha - 1}). \tag{4.17} \]

Substituting (4.16) and (4.17) into (4.15), we see that
\[ |III| \leq C \alpha (\varepsilon^{3 \alpha - \gamma}). \]

Since we need \( 0 < \alpha - \gamma + 1 < 1 \), we discuss in two cases \( 0 < \gamma < \frac{3}{2} \) and \( \frac{3}{2} \leq \gamma < 2 \) as Theorem 1.1. This enables us to complete the proof.

5 Conclusion

We apply the Littlewood-Paley theory as [13] and the Constantin-E-Titi type commutator estimates in physical Onsager type spaces to study the energy (helicity) conservation of weak solutions for the 2-D generalized quasi-geostrophic equation with the velocity \( v \) determined by \( v = R^\perp \Lambda^{\gamma-1} \theta \) with \( 0 < \gamma < 2 \), respectively. For the case \( 0 < \gamma < \frac{3}{2} \), the sufficient conditions for the energy (helicity) conservation of weak solutions of this equation in Onsager's critical space are derived. For the more singular case \( \frac{3}{2} \leq \gamma < 2 \), we obtain the corresponding results in critical spaces. Since the Littlewood-Paley decomposition and Besov space and Lemma 2.2 and 2.3 are known for periodic domain, the main results are also valid for periodic case.

A natural question is to extend our results to other models which modifies the velocity. A possible candidate is the inviscid Leary-\( \alpha \) or Euler-\( \alpha \) system. After we completed the main part of this paper, we learned the energy conservation of these models recently studied by Boutros-Titi in [6] and Beekie-Novack in [4]. Compared with their results, the results here give how the critical regularity for the energy conservation of the weak solutions depends on the the parameter \( \alpha \) of the velocity.

The non-uniqueness of weak solutions to the standard surface quasi-geostrophic equation (1.3) can be found in [1, 21]. It would be interesting to show the weak solutions to the generalized quasi-geostrophic equation (1.1) are not unique.
Acknowledgement

Wang was partially supported by the National Natural Science Foundation of China under grant (No. 11971446, No. 12071113 and No. 11601492). Ye was partially supported by the National Natural Science Foundation of China under grant (No.11701145) and China Postdoctoral Science Foundation (No. 2020M672196). Yu was partially supported by the National Natural Science Foundation of China (NNSFC) (No. 11901040), Beijing Natural Science Foundation (BNSF) (No. 1204030) and Beijing Municipal Education Commission (KM202011232020).

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