Symmetry analysis of an elastic beam with axial load

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Abstract
We construct the closed form solution of the elastic beam with axial load using Lie symmetry method. Here we consider the beam of spatially varying physical properties such as mass and second moment of area. The equation of motion of the elastic beam with axial load is a partial differential equation of fourth order with variable coefficient. Closed form solution of this system, which is hard and sometimes, not possible to derive with the common techniques, is found here. We also incorporate the boundary conditions. The corresponding Lie algebras are also given for several different cases.
Introduction.— An elastic beam is a three dimensional structure whose axial extension is more than any other dimension orthogonal to it. It is a fundamental model which pervades every corner of the physics and engineering field. However, the integrability in finite numbers of terms or exact solution of the governing equation of this model is an open question. The basic equation of motion of an elastic beam is the relationship between the applied load and deflection of the beam. The deflection is dependent not only on the mass of the beam or external applied force but also on the elastic properties of the material and geometry of the beam. The non-uniform distribution of geometry is spatially varying and mass is also spatially varying. Hence, the equation of motion of the beam is a fourth-order linear partial differential equation with variable coefficients. The equation can be derived from generalised Hooke’s law and force balance or by minimizing the energy of the system. Though the equation is linear in nature, due to the presence of variable coefficients, it is very arduous to get the solution analytically. To the best of our knowledge, in this work, for the first time, Euler-Bernoulli beam with axial force is studied using the Lie-method. Here we follow the Lie-group method to get the similarity solution. We also want to mention that the variable flexural stiffness and mass consideration is also very special and emergent topic. This model plays the primary role in the dynamical system of the helicopter, the wind turbine and musical instruments also.

The main motivation to use the Lie method is to get the solution of the mathematical model in different forms which are very easy to use. First, Sophus Lie applied this method to partial differential equations and later this method was further developed by Ovsjannikov [1] and Matschat and Müller [2]. This method has been applied to various kinds of ordinary differential equations (ODE) and partial differential equations (PDE). A systematic approach for applying Lie method to ODE and PDE is found in [3, 4]. Bluman et al. used this method for different types of mathematical physics problems such as wave equation, diffusion equation etc. [2, 5]. Torrisi et al. studied diffusion equation by equivalence transformation [5]. N. H. Ibragimov applied this method to some real life problems [6, 7] related to tumour growth model and metallurgical industry mathematical models. The celebrated Lie-method is also available in recent research addressing equations in mathematical physics [11-13]. The symmetry analysis for a physical problem has great importance. In Lorentz transformation used in special relativity, Yang-Mills theory [14, 15], and Schrödinger equation, the Lie symmetry analysis has been used extensively. In [16], Lie symmetries and canonical transformations are applied to construct the explicit solutions of Schrödinger equation with a spatially inhomogeneous nonlinearity from those of the homogeneous nonlinear Schrödinger equation. In [17], symmetry analysis is used to understand the fluid flow in a pipe or channel structure.

The Lie-method has also been employed in the field of elastic problems and structural mechanics [18]. In [19], a beam moving with time-dependent axial velocity is studied using equivalence transformation. Célestin Wafo discussed the Euler-Bernoulli beam from symmetry standpoint [20]. The general beam equation is studied by Bokhari et al. for symmetries and integrability with Lie method [21]. Bokhari et al. also found the complete Lie symmetry classification of the fourth-order dynamic Euler-Bernoulli beam equation with load dependent on normal displacement. Johnpillai et al. studied the Euler-Bernoulli beam equation from the Noether symmetry viewpoint [22].

We search for one-parameter group of transformation which leaves the governing PDE invariant and we get the corresponding Lie-algebras. Using the infinitesimal generator of this transformation, we solve the newly reduced differential equation. In Section 2, we describe our problem. The mathematical theories for the Lie-symmetry approach and the procedure to get the invariant solution are written in Section 3. We apply the Lie method to our problem in Section 4 of this Letter.

The Formulation.— We fix the coordinate axes $X, Y, Z$ along the length, breadth and height of the beam, respectively. We consider that the beam is slender and of length $l$. Here, $u(x, t)$ and $M(x, t)$ are the out-plane bending displacement along $Z$ and bending moment, respectively, at the point $x$ and the instant $t$ where $x \in [0, l]$ and $t \in \mathbb{R}$.

We assume the stiffness functions $EI : [0, l] \rightarrow \mathbb{R}$ and $m : [0, l] \rightarrow \mathbb{R}$ are continuously differentiable functions.

Based on the assumptions considered by Euler and Jacob Bernoulli, the equation of motion of the Euler-Bernoulli beam with axial force is given by

$$\frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) + m(x) \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left( T(x) \frac{\partial u}{\partial x} \right) = 0. \quad (1)$$

For rotating beam $T(x) = \int_x^l m(x) \Omega^2 x dx$ where $\Omega$ is the rotating speed. Another important practical case is the gravity loaded beam. For this problem $T(x) = \int_x^l m(x) g dx$ where $g$ is the gravitational force acting on the system. In the case of a stiff-string or piano string $T$ is constant [23]. In general, for this type of beam, cantilever boundary condition at the left end and free at the right end is considered.
The Euler Bernoulli beam with axial force is

\[
\frac{\partial^2}{\partial x^2} [EI(x) \frac{\partial^2}{\partial x^2} u(x,t)] + m(x) \frac{\partial^2}{\partial t^2} [u(x,t)] - \frac{\partial}{\partial x} \left[ T(x) \frac{\partial}{\partial x} u(x,t) \right] = 0. \tag{2}
\]

Hence \( M \subset X \times U \) where \( X = \{(x,t)|x,t \in \mathbb{R}\} \) and \( U \subset \mathbb{R} \). So \( M \) is a manifold of dimension three or \( M \cong \mathbb{R}^3 \). As this is a PDE of order four, we need the prolongation of order four, i.e. \( pr^{(4)} \mathfrak{v} \). For a PDE having one dependent variable and two independent variables, the one-parameter Lie group of transformations is

\[
x^* = X(x,t,u; \epsilon) = x + \epsilon \xi(x,t,u) + O(\epsilon^2)
\]

\[
t^* = T(x,t,u; \epsilon) = t + \tau \xi(x,t,u) + O(\epsilon^2)
\]

\[
u^* = U(x,t,u; \epsilon) = u + \eta(x,t,u) + O(\epsilon^2)
\]

From the Fundamental theorem \([3]\) of the transformations \([3]\) admitted by the PDE \([2]\), we will apply \( pr^{(4)} \mathfrak{v} \) on our equation and we get

\[
\xi(x,t,u(x,t)) \left( EI^{(3)}(x) u_{xx}(x,t) + 2EI''(x)u_{xxx}(x,t) \right) + EI'(x)u_{xxxx}(x,t) + m'(x)u_{tt}(x,t) - T''(x)u_{x}(x,t) - T'(x)u_{xx}(x,t) + \eta u_{xxxx}EI'(x) + m(x) - \eta E^{xxx}EI'(x) + \eta^t xEI(x) + \eta^t m(x) - \eta^T x - \eta^z T(x) = 0 \tag{4}
\]

To get the admissible transformations we require to satisfy the equation \([2]\) also. We substitute the value of \( u_{xxxx}(x,t) \) from \([2]\) to the above equation \([4]\). Now substituting the values of \( \eta^t, \eta^t, \eta^t x, \eta^t x, \eta^t x, \eta^t x \) from \([24]\) Theorem 32.3.5, we simplify the equation \([4]\) and we will get a polynomial equation in all possible derivatives \( u^{i,j} \) of \( u \) upto order four with respect to \( x, t \) where

\[
u^{i,j} = \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \tag{5}
\]

This polynomial equation should be satisfied for arbitrary \( x, t \) and \( u^{i,j} \). So to satisfy this polynomial equation we need the coefficients of all \( u^{i,j} \) and all its product terms to be equal to zero. Hence we obtain the determining equations for \((\xi, \tau, \eta)\).

\[
\frac{\partial \tau}{\partial x} = 0 \tag{6}
\]

\[
\frac{\partial \tau}{\partial u} = 0 \tag{7}
\]

\[
\frac{\partial \xi}{\partial t} = 0 \tag{8}
\]

\[
\frac{\partial \xi}{\partial u} = 0 \tag{9}
\]

\[
\frac{1}{EI(x)} \left[ m(x) \frac{\partial^2 \eta}{\partial t^2} - T'(x) \frac{\partial \eta}{\partial x} - T(x) \frac{\partial^2 \eta}{\partial x^2} + EI''(x) \frac{\partial^2 \eta}{\partial x^2} + 2EI'(x) \frac{\partial^3 \eta}{\partial x^3} + EI(x) \frac{\partial^4 \eta}{\partial x^4} \right] = 0 \tag{10}
\]

\[
\frac{1}{EI(x)} \left[ T(x) \xi(x,t,u)EI'(x) - \xi(x,t,u)T'(x) - \frac{\xi(x,t,u)}{EI(x)}EI''(x) + \frac{\xi(x,t,u)}{EI(x)}EI''(x) \right] = 0 \tag{11}
\]

\[
\frac{1}{EI(x)} \left[ T(x) \xi(x,t,u)EI'(x) - \xi(x,t,u)T'(x) - \frac{\xi(x,t,u)}{EI(x)}EI''(x) + \frac{\xi(x,t,u)}{EI(x)}EI''(x) \right] = 0 \tag{12}
\]
\[ T'(x)\xi(x, t, u)EI'(x) + T''(x)\xi(x, t, u) - m(x)\frac{\partial^2 \xi}{\partial x^2} - 3T'(x)\frac{\partial \xi}{\partial x} - 2T(x)\frac{\partial^2 \eta}{\partial x \partial u} + 2EI''(x)\frac{\partial^2 \eta}{\partial x^2} \]
\[ + T(x)\frac{\partial^2 \xi}{\partial x^2} - E\frac{\partial^2 \xi}{\partial x^2} + 6EI'(x)\frac{\partial^2 \xi}{\partial x^4} - 2EI'(x)\frac{\partial^2 \eta}{\partial x^4} + 4EI(x)\frac{\partial^4 \eta}{\partial x^3 \partial u} + EI(x)\frac{\partial^4 \xi}{\partial x^4} = 0 \] (13)
\[ \frac{m(x)\xi(x, t, u)EI'(x)}{EI(x)} + \xi(x, t, u)m'(x) - 2m(x)\frac{\partial \tau}{\partial t} + 4\frac{\partial \xi}{\partial x} = 0 \] (14)
\[ 2m(x)\frac{\partial^2 \eta}{\partial t \partial u} - m(x)\frac{\partial^2 \tau}{\partial t^2} + T'(x)\frac{\partial \tau}{\partial x} + T(x)\frac{\partial^2 \tau}{\partial x^2} - EI''(x)\frac{\partial^2 \tau}{\partial x^2} - 2EI'(x)\frac{\partial^4 \tau}{\partial x^4} - EI(x)\frac{\partial^4 \tau}{\partial x^4} = 0 \] (15)
\[ \frac{\partial^2 \eta}{\partial u^2} = 0 \] (16)

From (6) \( \tau(x, t, u) = \tau(t) \), from (8) \( \xi(x, t, u) = \xi(x) \) and from (10) \( \eta(x, t, u) = A(x, t)u + B(x, t) \) for some arbitrary functions \( A(x, t), B(x, t) \). We assume \( \frac{\partial \eta}{\partial t} = 0 \) and \( \frac{\partial^2 \eta}{\partial t^2} = 0 \), i.e., \( B(x, t) = d_1 + d_2 t \) where \( d_1, d_2 \) are constants.

From the equation (11), we observe that \( \frac{\partial^2 \eta}{\partial x \partial u} = \frac{\partial A}{\partial x} \) should be free from time variable \( t \). Hence, \( A(x, t) = f_1(x) + f_2(t) \).

Now from (11), we see that
\[ 2m(x)\frac{\partial^2 \eta}{\partial t \partial u} - m(x)\frac{\partial^2 \tau}{\partial t^2} = 0 \] (17)

Again the equation (11), \( \frac{\partial \tau}{\partial t} \) should be some constant. Assume, \( 2\frac{\partial \tau}{\partial t} = \omega \) which implies \( \tau(t) = \frac{\omega}{2} t + t_0 \). As from (17),
\[ 2f_2(t) = \frac{\partial \tau}{\partial t} = \frac{\omega}{2} \] (18)

So, \( f_2(t) = \frac{\omega}{2} t \) and \( \eta(x, t, u) = (f_1(x) + \frac{\omega}{2} t)u + d_1 + d_2 t \). Now we consider two cases; (a) when \( f_1(x) \) is constant, (b) when \( f_1(x) \) is not constant. For the first case, there are two subcases; (a.1) when \( \frac{d^2 \xi}{dx^2} = k \) for some constant \( k \) i.e., \( \frac{d^2 \xi}{dx^2} = 0, \frac{d^2 \eta}{dx^2} = 0 \) and \( \frac{\partial^2 \xi}{\partial x \partial u} \) is not any constant.

Case (a) when \( f_1(x) \) is constant. For the case (a.1), from (11), it can be shown that
\[ \xi(x) = f_0(EI(x))^{\frac{1}{2}} \] (19)

and for the assumption \( \frac{d^2 \xi}{dx^2} = k, EI(x) = \frac{k^3x^3}{8\nu} \). It can be shown that from other equations, \( T(x) = T_0x^2 - \frac{3k^3x^4}{4\nu} \) and \( m(x) = m_0x \). Hence the rule of coordinate transformation is:
\[ \xi(x) = \frac{kx^2}{2} \] (20)
\[ \tau(t) = \frac{\omega}{2} t + t_0 \] (21)
\[ \eta(x, t, u) = (\alpha + \frac{\omega}{4} t)u + d_1 + d_2 t, \text{ where } f_1(x) = \alpha. \] (22)

For the case, \( \frac{d^2 \xi}{dx^2} \) is not any constant. In this case, we assume the coefficients of \( \xi(x) \) are zero in the equations (11), (12), (13) and the form of \( EI(x) \) and \( T(x) \) are evaluated which are
\[ EI(x) = a_1e^{a_0x} \] (23)
\[ T(x) = -\frac{2}{9}a_0^2a_1e^{a_0x} \] (24)

Using these form, from (11) we get
\[ \xi(x) = 3f_0e^{a_0x} \] (25)
\[ m(x) = m_0e^{\frac{a_0x}{a_0}} \] (26)
So, the rule of coordinate transformation is:

$$\xi(x) = \frac{3f_0e^{a_0x}}{a_0}$$  \hspace{1cm} (27)

$$\tau(t) = \frac{\omega}{2}t + t_0$$  \hspace{1cm} (28)

$$\eta(x,t,u) = (\alpha + \frac{\omega}{4})u + d_1 + d_2t, \text{ where } f_1(x) = \alpha.$$  \hspace{1cm} (29)

Case (b) when $f_1(x)$ is not a constant. – In this case also, if we assume the coefficients of $\xi(x)$ are zero in the equations (11), (12), (13) and the form of $EI(x)$ and $T(x)$ are evaluated which are

$$EI(x) = a_1e^{-vx}$$  \hspace{1cm} (30)

$$T(x) = 2a_1\nu^2e^{-vx}$$  \hspace{1cm} (31)

$$m(x) = m_0e^{v(4c^2e^{-vx} - 5x)}$$  \hspace{1cm} (32)

and

$$\xi(x) = e^{vx} \frac{v}{2\nu^2}$$  \hspace{1cm} (33)

$$\tau(t) = \frac{\omega}{2}t + t_0$$  \hspace{1cm} (34)

$$\eta = \left(\frac{e^{vx}}{v} + \frac{\omega}{4}\right)u + d_1 + d_2t$$  \hspace{1cm} (35)

Based on the assumptions, there may be more than these above three combination of physical characteristics and coordinate transformations which can lead us to find a closed form solution. In the Table, the possible list of these combinations are listed. It is observed that the transformation rule for the spatial coordinate is strongly dependent on the stiffness of the beam, i.e., the geometry of beam of certain material. It can be shown that for a beam of polynomial varying stiffness $EI(x) = (a_0 + a_1x)^n$ where $n \neq 2$ is a nonzero positive integer, then the coordinate transformation will be

$$\xi(x) = \frac{(a_0 + a_1x)}{a_1n}$$  \hspace{1cm} (36)

$$\tau(t) = \frac{\omega}{2}t + t_0$$  \hspace{1cm} (37)

$$\eta = \frac{\omega}{4}u + d_1 + d_2t$$  \hspace{1cm} (38)

| Case | Stiffness | $\xi(x)$ | $\tau(t)$ | $\eta(x,t,u)$ |
|------|-----------|----------|----------|---------------|
| (a.1)| $e^{vx}$  | $\frac{e^{vx}}{s_0}$ | $\frac{\omega}{2}t + t_0$ | $(\alpha + \frac{\omega}{4})u + d_1 + d_2t$ |
| (a.2)| $a_1e^{a_0x}$ | $\frac{e^{vx}}{a_0}$ | $\frac{\omega}{2}t + t_0$ | $(\alpha + \frac{\omega}{4})u + d_1 + d_2t$ |
| (b) | $a_1e^{-vx}$ | $\frac{e^{vx}}{a_0}$ | $\frac{\omega}{2}t + t_0$ | $(\alpha + \frac{\omega}{4})u + d_1 + d_2t$ |
| (c) | $(a_0 + a_1x)^n$ | $\frac{e^{vx}}{(a_0 + a_1x)} a_1n$ | $\frac{\omega}{2}t + t_0$ | $(\alpha + \frac{\omega}{4})u + d_1 + d_2t$ |
Closed form solution.— For the case (a.1), the acting vector field is

\[ v = \frac{kx^2}{2} \frac{\partial}{\partial x} + \frac{\omega^2}{2} t_0 \frac{\partial}{\partial t} + \left((\alpha + \frac{\omega}{4})u + d_1 + d_2 t\right) \frac{\partial}{\partial u} \]  

(39)

If we choose, \((\alpha + \frac{\omega}{4}) = k\), then for vector field \(X = \frac{x^2}{2} \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}\) the characteristic equation is

\[ \frac{dx}{x^2} = \frac{dt}{0} = \frac{du}{u} \]  

(40)

which implies

\[ u(x, t) = \exp \left( -\frac{2}{x} \right) F(t) \]  

(41)

for some arbitrary function \(F(t)\). Here for \(EI(x) = \frac{k^3 a^6}{8 \sigma_0^2}\), \(T(x) = T_0 x^2 - \frac{3k^3 a^4}{4 \sigma_0^2}\) and \(m(x) = m_0 \frac{-\frac{a^2}{x^4}}{x^4}\) from the equation it can be shown that the invariant solution is

\[ u(x, t) = \exp \left( -\frac{2}{x} \right) \left( A_1 e^{\frac{-3a^2}{a_0}} (A_1 \cos qt + A_2 \sin qt) \right) \]  

(42)

Following the similar way, we get the invariant solutions for other cases given in the Table 2.

| \(EI(x)\) | \(T(x)\) | \(m(x)\) | \(u(x, t)\) |
|----------|----------|----------|----------|
| \(a_1 e^{a_0 x}\) | \(-\frac{2}{5} a_0^2 a_1 e^{a_0 x}\) | \(m_0 e^{\frac{6a_0^2}{a_0f_0} - \frac{a_0^2}{a_0f_0} - a_0}\) | \(e^{\frac{-3a^2}{a_0}} (A_1 \cos qt + A_2 \sin qt)\) |
| \(a_1 e^{-v x}\) | \(2a_1 v^2 e^{-v x}\) | \(m_0 e^{\frac{-4a_0^2}{a_0f_0} - \frac{a_0^2}{a_0f_0} - a_0^2}\) | \(e^{2v x} (A_1 + A_2 t)\) |
| \((a_0 + a_1 x)^n\) | \(\frac{\sqrt{a_1}}{\sqrt[3]{m_0}} \sqrt{-8a_0^3 + 36a_0^2 e^{\frac{a_0^2}{a_0f_0}} - 36a_0 e^{\frac{2a_0^2}{a_0f_0}} + 9e^{a_0 x}} \exp \left( \frac{1}{2} \left( \frac{4a_0^2}{3} - \frac{a_0}{a_0f_0} \right) \frac{\left( -\frac{\alpha^2}{a_0f_0} - x \right)}{m_0} \right) \) |

where \(q(x) = \frac{\sqrt{a_1}}{\sqrt[3]{m_0}} \sqrt{-8a_0^3 + 36a_0^2 e^{\frac{a_0^2}{a_0f_0}} - 36a_0 e^{\frac{2a_0^2}{a_0f_0}} + 9e^{a_0 x}} \exp \left( \frac{1}{2} \left( \frac{4a_0^2}{3} - \frac{a_0}{a_0f_0} \right) \frac{\left( -\frac{\alpha^2}{a_0f_0} - x \right)}{m_0} \right) \),

(43)

\[ G(x) = (a_0 + a_1 x)^{-n} \left( -2 \sqrt{\frac{a_1}{a_1^2} \frac{\sqrt{n - 2}}{n - 3} \frac{a_0 + a_1 x}{\sqrt{n - 2}} + \frac{\sqrt{a_1^2 (n - 1)^2 - 4T_1}}{a_1^2 \sqrt{n - 2}}} \right) \]  

(44)

Boundary condition imposition.— Some problems may come when we want to impose some boundary conditions. For a given PDE, an invariant solution corresponding to an admitted infinitesimal generator solves the same PDE with some boundary conditions if all the boundary conditions will be unchanged under the infinitesimal generator. The details can be found in section 4.4.1 [4]. Here we impose cantilever boundary condition at the left end of the beam. Following the same procedures, we get the same set of equations (61 66). But, according to [4], now we also require to satisfy

\[ \xi(x = 0) = 0 \text{ at the left boundary } x = 0 \]  

(45)
FIG. 1. Deflection at different times along the beam length

\[ u(x, t) = x^2 \left( A_1 \cos \left( \sqrt{21}tx^2 \right) + A_2 \sin \left( \sqrt{21}tx^2 \right) \right) \]  

For \( A_1 = 1, A_2 = 2 \) we plot [17] in Figure 1 to get the deflection of the beam at different times.

Conclusion.— In the conclusion, we have applied Lie symmetry method to analyze the symmetry and the exact form of the solution of the Euler-Bernoulli beam with axial load. We have found some combinations of the coordinate transformations dependent on the system properties which provide us with the closed form solution. As discussed before, the closed form or exact form of this kind of beam problem is the much-awaited solution for the scientific community. Here, we have found different kind of stiffness, mass and axial force which are feasible to possess a closed form solution. It is observed that the crucial spatial transformations are dependent on the stiffness of the beam. This method can be applied on more exact model of the beam such as Timoshenko or Rayleigh beam or any nonlinear model. We hope the form of the solutions derived here will be useful for academic and industry both. In future, the work contained in this Letter will be effective for the frequency analysis of vibrating beam with axial load.

[1] L. Ovsjannikov and G. Bluman, *Group properties of differential equations* (Siberian Section of the Academy of Science of USSR, 1962)
[2] E. Müller and K. Matschat (1962, p 190)
[3] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Vol. 107 (Springer Science & Business Media, New York, 2012)
[4] G. Bluman and S. Kumei, *Symmetries and differential equations*, Vol. 154 (Springer Science & Business Media, New York, 2013)
[5] N. H. Ibragimov, *CRC Handbook of Lie group analysis of differential equations*, Vol. 3 (CRC press, Boca Raton, FL, 1995)
[6] G. Bluman and S. Kumei, Journal of Mathematical Physics 21, 1019 (1980)
[7] G. Bluman and S. Kumei, Journal of mathematical physics 28, 307 (1987)
[8] M. Torrisi, R. Tracina, and A. Valenti, Journal of Mathematical Physics 37, 4758 (1996)
[9] N. H. Ibragimov and N. Säfström, Communications in Nonlinear Science and Numerical Simulation 9, 61 (2004)
[10] N. H. Ibragimov, Journal of Nonlinear Mathematical Physics 18, 143 (2011)
[11] J. Kang and C. Qu, Journal of Mathematical Physics 53, 023509 (2012)
[12] K. Singla and R. Gupta, Journal of Mathematical Physics 58, 051503 (2017)
[13] J.-N. Hau, M. Oberlack, and G. Chagelishvili, Journal of Mathematical Physics 58, 043101 (2017)
[14] T. Strobl, Physical review letters 93, 211601 (2004)
[15] N. Beisert, A. Garus, and M. Rosso, Physical Review Letters 118, 141603 (2017)
[16] J. Belmonte-Beitia, V. M. Pérez-García, V. Vekslerchik, and P. J. Torres, Physical Review Letters 98, 064102 (2007)
[17] N. B. Budanur, P. Cvitanović, R. L. Davidchack, and E. Siminos, Physical review letters 114, 084102 (2015)
[18] J. Bocko, V. Nohajová, and T. Harčarik, Procedia Engineering 48, 40 (2012)
[19] E. Özkaya and M. Pakdemirli, Acta Mechanica 155, 111 (2002)
[20] C. W. Soh, Journal of Mathematical Analysis and Applications 345, 387 (2008)
[21] A. H. Bokhari, F. Mahomed, and F. Zaman, Journal of Mathematical Physics 51, 053517 (2010)
[22] A. Johnpillai, K. Mahomed, C. Harley, and F. Mahomed, Zeitschrift für Naturforschung A 71, 447 (2016)
[23] J. B. Gunda and R. Ganguli, Journal of Applied Mechanics 75, 024502 (2008)
[24] S. Hassani, Mathematical physics: a modern introduction to its foundations (Springer Science & Business Media, 2013)