A global implicit function theorem and its applications to functional equations

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Abstract

The main result of the paper is a global implicit function theorem. In the proof of this theorem, we use a variational approach and apply Mountain Pass Theorem. An assumption guaranteeing existence of an implicit function on the whole space is a Palais-Smale condition. Some applications to differential and integro-differential equations are given.

1 Introduction

In the paper, we derive a global implicit function theorem for a map $F : X \times Y \to H$ where $X$, $Y$ are real Banach spaces and $H$ is a real Hilbert space. In the proof, we use a variational approach and apply Mountain Pass Theorem. Such a method has been used in paper [3] to prove a theorem on the global diffeomorphism between Banach and Hilbert spaces (global inverse function theorem). The main assumption guaranteeing the existence of an implicit function $\lambda : Y \to X$ described by the equation $F(x, y) = 0$ is a Palais-Smale condition connected with $F$, with respect to $x \in X$.

In the literature, some extensions of the local implicit function theorem to the global ones are known. In [7], the case of Banach spaces $X$, $Y$, $H$ is considered. Author use a concept of "continuation property" which is equivalent to the so-called "path-lifting property". In [2], some variants of global implicit function theorems in the case of Banach spaces $X$, $Y$ and $H = Y$, have been obtained. The main assumptions are some inequalities involving Jacobians $F_x(x, y)$, $F_y(x, y)$ and a function $\omega : [0, \infty) \to [0, \infty)$ such that $\int_{1}^{\infty} \frac{ds}{\omega(s)} = \infty$. In paper [9], the authors consider the case of $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, $H = \mathbb{R}^m$. As a condition guaranteeing existence of an implicit function on the whole space they propose a "lower bound" condition imposed...
on the Jacobian $F_x(x, y)$. The comparison of the above results to our ones seems to be not so easy and remains an open question.

Our paper consists in two parts. In the first part, we derive a global implicit function theorem. The second part is devoted to some applications. First, we study the classical nonlinear ordinary Cauchy problem involving a functional parameter $u$ (nonlinearly). Next, we analyze an integro-differential Cauchy problem of Volterra type involving two functional parameters $u$ (nonlinearly) and $v$ (linearly). Problem of such a type but without $u$, was investigated in [3]. In both cases, we obtain existence, uniqueness and global, differentiable dependence of solutions on parameters.

2 Existence of a global implicit function

Let $X$ be a real Banach space and $I : X \to \mathbb{R}$ - a functional differentiable in Gateaux sense. We say that $I$ satisfies Palais-Smale (PS) condition if any sequence $(x_m)$ satisfying conditions:

- $|I(x_m)| \leq M$ for all $m \in \mathbb{N}$ and some $M > 0$,
- $I'(x_m) \to 0$,

admits a convergent subsequence ($I'(x_m)$ denotes the Gateaux differential of $I$ at $x_m$).

A point $x_* \in X$ is called the critical point of $I$ if $I'(x_*) = 0$. In such a case $I(x_*)$ is called the critical value of $I$.

In [5, Corollary 3.3] the following theorem is proved.

**Theorem 1** Let $X$ be a real Banach space. If $I : X \to \mathbb{R}$ is lower semicontinuous, bounded below and differentiable in Gateaux sense functional satisfying (PS) condition, then there exists a critical point $x_*$ of $I$.

Using the above theorem we obtain

**Theorem 2** Let $X, Y$ be real Banach spaces, $H$ - a real Hilbert space. If $F : X \times Y \to H$ is differentiable with respect to $x \in X$ in Gateaux sense and

- $F(x, y) \in F'_x(x, y)X$ for any $(x, y) \in X \times Y$ ($F'_x(x, y)$ denotes the Gateaux differential of $F$ at $(x, y)$ with respect to $x$)
the functional
\[ \varphi : X \ni x \mapsto \frac{1}{2} \| F(x, y) \|^2 \in \mathbb{R} \] (1)
is lower semicontinuous and satisfies (PS) condition for any \( y \in Y \),
then, for any \( y \in Y \), there exists \( x_y \in X \) such that \( F(x_y, y) = 0 \).

**Proof.** Let us fix a point \( y \in Y \). Functional \( \varphi \), being a superposition of the mapping \((1/2) \| \cdot \|^2\) differentiable in Frechet sense on \( H \) and the mapping \( F(\cdot, y) \) differentiable in Gateaux sense on \( X \), is differentiable in Gateaux sense on \( X \) and its Gateaux differential \( \varphi'(x) \) at \( x \in X \) is given by
\[ \varphi'(x)h = \langle F(x, y), F'_x(x, y)h \rangle \]
for \( h \in X \). Moreover, \( \varphi \) is bounded below and, by assumption, lower semicontinuous and satisfies (PS) condition. So, by Theorem 1, there exists a point \( x_y \in X \) such that
\[ \langle F(x_y, y), F'_x(x_y, y)h \rangle = 0 \]
for \( h \in X \). Since \( F'_x(x_y, y)X = H, F(x_y, y) = 0 \). □

**Remark 3** The assumption on lower semicontinuity of \( \varphi \) can be replaced by a more restrictive one but concerning directly \( F \), namely - continuity of \( F \) with respect to \( x \).

## 3 Uniqueness of a global implicit function

Let \( d \neq 0 \) be a point of \( X \). By \( W_d \) we denote the set
\[ W_d = \{ U \subset X; U \text{ is open}, 0 \in U \text{ and } d \notin \overline{U} \} \].

We have ([1], [6])

**Theorem 4 (Mountain Pass Theorem)** Let \( I : X \to \mathbb{R} \) be a functional which is continuously differentiable in Gateaux sense, satisfies (PS) condition and \( I(0) = 0 \). If there exist constants \( \rho, \alpha > 0 \) such that \( I|_{\partial B(0, \rho)} \geq \alpha \) and \( I(e) \leq 0 \) for some \( e \in X \setminus \overline{B(0, \rho)} \), then
\[ b := \sup_{U \in W_d} \inf_{u \in \partial U} I(u) \]
is the critical value of $I$ and $b \geq \alpha$ (\[\text{1}\]).

We have

**Theorem 5** Let $X$, $Y$ be real Banach spaces, $H$ - a real Hilbert space. If $F : X \times Y \to H$ is continuously differentiable with respect to $x \in X$ in Gateaux (equivalently, in Frechet) sense and

- $F'_x(x, y) : X \to Y$ is bijective for any $(x, y) \in X \times Y$ such that $F(x, y) = 0$ and $F(x, y) \in F'_x(x, y)X$ for the remaining $(x, y) \in X \times Y$

- the functional $\varphi$ given by (1) satisfies (PS) condition for any $y \in Y$,

then, for any $y \in Y$, there exists a unique $x_y \in X$ such that $F(x_y, y) = 0$.

**Proof.** Let us fix $y \in Y$. From Theorem 2 it follows that there exists a point $x_y \in X$ such that $F(x_y, y) = 0$. Let us suppose that there exist $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $F(x_1, y) = F(x_2, y) = 0$. Let us put $e = x_2 - x_1$ and

$$g(x) = F(x + x_1, y)$$

for $x \in X$. Of course,

$$g(x) = g'(0)x + o(x) = F'_x(x_1, y)x + o(x)$$

for $x \in X$, where $o(x)/\|x\|_X \to 0$ in $H$ when $x \to 0$ in $X$. So,

$$\beta \|x\|_X \leq \|F'_x(x_1, y)x\|_H \leq \|g(x)\|_H + \|o(x)\|_H \leq \|g(x)\|_H + (1/2)\beta \|x\|_X$$

for sufficiently small $\|x\|_X$ and some $\beta > 0$ (existence of such an $\beta$ follows from the bijectivity of $F'_x(x_1, y)$). Thus, there exists $\rho > 0$ such that

$$(1/2)\beta \|x\|_X \leq \|g(x)\|_H$$

for $x \in B(0, \rho)$. Without loss of the generality one may assume that $\rho < |e|$. Put

$$\psi(x) = (1/2)\|g(x)\|^2 = (1/2)\|F(x + x_1, y)\|^2 = \varphi(x + x_1)$$

\[\text{1}\] It is known (cf. [4]) that if a mapping is continuously differentiable at a point in Gateaux sense then it is differentiable in this point in Frechet sense and both differentials coincide.
for \( x \in X \). Of course, \( \psi \) is continuously differentiable on \( X \) in Gateaux sense and
\[
\psi'(x) = \varphi'(x + x_1).
\]
Consequently, since \( \varphi \) satisfies (PS) condition, \( \psi \) has this property, too. Moreover, 
\[
\psi(0) = \psi(e) = 0, \ e \notin B(0, \rho) \text{ and } \psi(x) \geq \alpha \text{ for } x \in \partial B(0, \rho)
\]
with \( \alpha = (1/8)\beta^2 \rho^2 > 0 \).

Thus, the Mountain Pass Theorem implies that 
\[
b = \sup_{U \in \mathcal{W} e} \inf_{x \in \partial U} \psi(x)
\]

is the critical value of \( \psi \) and \( b \geq \alpha \), i.e. there exists a point \( x_* \in X \) such that 
\[
\psi(x_*) = b > 0 \text{ and } \psi'(x_*)h = \langle F(x_* + x_1, y), F_x'(x_*)h \rangle = 0
\]
for \( h \in X \). The first condition means that \( F(x_* + x_1, y) \neq 0 \). The second one and relation 
\[
F(x_* + x_1, y) \in F_x'(x_* + x_1, y)X
\]

imply that \( F(x_* + x_1, y) = 0 \). The obtained contradiction completes the proof. \( \blacksquare \)

**Remark 6** When \( X = \mathbb{R}^n \), \( Y = \mathbb{R}^m \) and \( H = \mathbb{R}^n \), assumption on \( \varphi \) can be replaced by the following equivalent one: 

\( \varphi \) is coercive for any \( y \in \mathbb{R}^m \), i.e. \( \varphi(x) \to \infty \) when \( |x| \to \infty \).

### 4 Global implicit function theorem

From Theorems 2, 5 and classical local implicit function theorem we immediately obtain the following global implicit function theorem.

**Theorem 7** Let \( X, Y \) be real Banach spaces, \( H \) - a real Hilbert space. If \( F : X \times Y \to H \) is continuously differentiable in Gateaux (equivalently, in Frechet) sense with respect to \( (x, y) \in X \times Y \) and

- differential \( F'_x(x, y) : X \to H \) is bijective for any \( (x, y) \in X \times Y \) such that \( F(x, y) = 0 \) and \( F(x, y) \in F'_x(x, y)X \) for the remaining \( (x, y) \in X \times Y \)

- the functional \( \varphi \) given by (1) satisfies the (PS) condition for any \( y \in Y \),

then there exists a unique function \( \lambda : Y \to X \) such that \( F(\lambda(y), y) = 0 \) for any \( y \in Y \) and this function is continuously differentiable in Gateaux (equivalently, in Frechet) sense on \( Y \) with differential \( \lambda'(y) \) at \( y \) given by

\[
\lambda'(y) = -[F'_x(\lambda(y), y)]^{-1} \circ F'_y(\lambda(y), y).
\]
Proof. Of course, it is sufficient to put $\lambda(y) = x_y$ where $x_y$ is a solution to $F(x, y) = 0$, given by Theorem 5.

5 Applications

In this section, we give two examples illustrating Theorem 7. First of them concerns a nonlinear Cauchy problem containing a functional parameter.

Example 8 Let us consider the following control system

$$x'(t) = f(t, x(t), u(t)), \ t \in J = [0, 1] \text{ a.e.},$$

where $f : J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $x \in AC^2_0 = AC^2_0(J, \mathbb{R}^n) = \{x : J \to \mathbb{R}^n; \ x \text{ is } \text{absolutely continuous}, \ x(0) = 0, \ x' \in L^2(J, \mathbb{R}^n)\}$, $u \in L^\infty = L^\infty(J, \mathbb{R}^m)$. On the function $f$ we assume that

(A1) $f(\cdot, x, u)$ is measurable on $J$ for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; $f(\cdot, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m$ for $t \in J$ a.e.

(A2) there exist functions $a, b \in L^2(J, \mathbb{R}^+_0)$ such that

$$|f(t, x, u)| \leq a(t)|x| + b(t)|u|$$

for $t \in J$ a.e., $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and

$$\int_0^1 (a(t))^2tdt < 1/8$$

(A3) there exist functions $c, d \in L^2(J, \mathbb{R}^+_0)$, $\alpha, \beta \in C(\mathbb{R}^+_0, \mathbb{R}^+_0)$ such that

$$|f_x(t, x, u)|, \ |f_u(t, x, u)| \leq c(t)\alpha(|x|) + d(t)\beta(|u|)$$

for $t \in J$ a.e., $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.

We shall check that the mapping

$$F : AC^2_0 \times L^\infty \to L^2(J, \mathbb{R}^n),$$

$$F(x, u) = x'(t) - f(t, x(t), u(t)),$$
satisfies assumptions of global implicit function theorem with $X = \mathcal{AC}_0^2$, $Y = L^\infty$, $H = L^2(J, \mathbb{R}^n)$. In a standard way one can check that it is continuously differentiable in Gateaux (equivalently, in Frechet) sense on $\mathcal{AC}_0^2 \times L^\infty$ and the differentials with respect to $x$ and $u$ are

$$F_x'(x, u) : \mathcal{AC}_0^2 \ni h \mapsto h' - f_x(\cdot, x(\cdot), u(\cdot))h(\cdot) \in L^2(J, \mathbb{R}^n),$$

$$F_u'(x, u) : L^\infty \ni g \mapsto f_u(\cdot, x(\cdot), u(\cdot))g(\cdot) \in L^2(J, \mathbb{R}^n),$$

respectively. It is clear that $F_x'(x, u)$ is "one-one" and "onto".

Now, let us fix a function $u \in L^\infty$ and consider the functional

$$\varphi : \mathcal{AC}_0^2 \ni x \mapsto (1/2) \|F(x, u)\|^2 = (1/2) \int_0^1 |x'(t) - f(t, x(t), u(t))|^2 \in \mathbb{R}.$$  

It is easy to see that, for any $x \in \mathcal{AC}_0^2$,

$$|\varphi(x)| \geq (1/2) \|x\|^2_{\mathcal{AC}_0^2} - \int_0^1 x'(t)f(t, x(t), u(t))dt$$

and

$$\left| \int_0^1 x'(t)f(t, x(t), u(t))dt \right| \leq \|x\|_{\mathcal{AC}_0^2} \sqrt{\int_0^1 |f(t, x(t), u(t))|^2 dt}.$$ 

Since

$$\int_0^1 |f(t, x(t), u(t))|^2 dt \leq \int_0^1 (a(t) |x(t)| + b(t) |u(t)|)^2 dt$$

$$\leq 2 \int_0^1 ((a(t))^2 |x(t)|^2 + (b(t))^2 |u(t)|^2) dt$$

$$\leq 2(\int_0^1 (a(t))^2 t dt) \int_0^1 |x'(t)|^2 dt + \int_0^1 (b(t))^2 |u(t)|^2 dt$$

$$\leq 2(\int_0^1 (a(t))^2 t dt \|x\|^2_{\mathcal{AC}_0^2} + \int_0^1 (b(t))^2 |u(t)|^2 dt),$$
therefore

\[ |\varphi(x)| \geq (1/2) \|x\|^2_{AC^2} - \|x\|_{AC^2} \sqrt{2 \int_0^1 (a(t))^2 dt \|x\|^2_{AC^2} + \int_0^1 (b(t))^2 |u(t)|^2 dt} \]

\[ = \frac{((1/4) - 2 \int_0^1 (a(t))^2 dt) \|x\|^4_{AC^2} - 2 \int_0^1 (b(t))^2 dt \|x\|^2_{AC^2}}{\sqrt{2 \int_0^1 (a(t))^2 dt \|x\|^2_{AC^2} + \int_0^1 (b(t))^2 |u(t)|^2 dt}} \]

for \( x \in AC^2_0, x \neq 0 \). This means that \( \varphi \) is coercive.

In a standard way, we check that the differential \( \varphi'(x) \) of \( \varphi \) at \( x \) is given by

\[ \varphi'(x)h = \int_0^1 (x'(t) - f(t, x(t), u(t)))(h'(t) - f_x(t, x(t), u(t))h(t))dt \]

for \( h \in AC^2_0 \). Consequently, for any \( x_m, x_0 \in AC^2_0 \),

\[ (\varphi'(x_m) - \varphi'(x_0))(x_m - x_0) = \|x_m - x_0\|^2_{AC^2} + \sum_{i=1}^5 \psi_i(x_m) \]

(4)

where

\[ \psi_1(x_m) = \int_0^1 x_m'(t) (f_x(t, x_m(t), u(t))(x_m(t) - x_0(t))dt, \]

\[ \psi_2(x_m) = \int_0^1 x_o'(t) (f_x(t, x_0(t), u(t))(x_m(t) - x_0(t))dt, \]

\[ \psi_3(x_m) = \int_0^1 f(t, x_m(t), u(t))(f_x(t, x_m(t), u(t))(x_m(t) - x_0(t))dt, \]

\[ \psi_4(x_m) = \int_0^1 f(t, x_0(t), u(t))(f_x(t, x_0(t), u(t))(x_m(t) - x_0(t))dt, \]

\[ \psi_5(x_m) = \int_0^1 (f(t, x_0(t), u(t)) - f(t, x_m(t), u(t)))(x_m'(t) - x_0'(t))dt. \]

Now, we are in a position to prove that \( \varphi \) satisfies \((PS)\) condition. Indeed, if \( (x_m) \) is a \((PS)\) sequence for \( \varphi \), then the coercivity of \( \varphi \) implies its boundedness. Consequently, there exists a subsequence \( (x_{m_k}) \) which is weakly convergent in \( AC^2_0 \) to some \( x_0 \) (we recall that the weak convergence of functions
in $AC^2_0$ implies the uniform convergence of these functions and weak convergence of their derivatives in $L^2(I, \mathbb{R}^n)$. Let us observe that

$$\left| \psi_1(x_{m_k}) \right|^2 \leq \int_0^1 \left| x'_{m_k}(t) \right|^2 dt \int_0^1 \left| f_x(t, x_{m_k}(t), u(t)) \right|^2 \left| x_{m_k}(t) - x_0(t) \right|^2 dt$$

$$\leq \int_0^1 \left| x'_{m_k}(t) \right|^2 dt \int_0^1 \left( c(t) \alpha(|x_{m_k}(t)|) + d(t) \beta(|u(t)|) \right)^2 \left| x_{m_k}(t) - x_0(t) \right|^2 dt$$

$$\leq C \max\{|x_{m_k}(t) - x_0(t)|^2; t \in [0, 1]\}$$

where $C > 0$ is a constant depending on the sequence $(x_{m_k})$ and control $u$. Thus, from the uniform convergence of $(x_{m_k})$ to $x_0$ the convergence of $(\psi_1(x_{m_k}))$ to $0$ follows. In the same way, one can check that sequences $(\psi_i(x_{m_k}))$, $i = 2, 3, 4$, converge to $0$. Convergence of the sequence $(\psi_5(x_{m_k}))$ to $0$ follows from the weak convergence of $(x_{m_k})$ to $x_0$ in $L^2(J, \mathbb{R}^n)$ and from the convergence of the sequence $(f(t, x_{m_k}(t), u(t)))$ to $(f(t, x_0(t), u(t)))$ in $L^2(J, \mathbb{R}^n)$ (the last convergence follows from the Lebesgue dominated convergence theorem). Thus, from [4] it follows that $(x_{m_k})$ converges to $x_0$ in $AC^2_0$, i.e. $\varphi$ satisfies (PS) condition.

So, all assumptions of the global implicit function theorem are satisfied. Consequently, for any $u \in L^\infty$ there exists a unique solution $x_u \in AC^2_0$ of the equation (3) and the mapping

$$\lambda : L^\infty \ni u \longmapsto x_u \in AC^2_0$$

is continuously differentiable in Gateaux (equivalently, in Fréchet) sense on $L^\infty$, with the differential $\lambda'(u)$ at a point $u \in L^\infty$, given by

$$L^\infty \ni g \longmapsto z_g \in AC^2_0$$

where $z_g$ is such that

$$z'_g(t) - f_x(t, x_u(t), u(t))z_g(t) = f_u(t, x_u(t), u(t))g(t)$$
a.e. on $J$.

An example of a function satisfying conditions (A1), (A2), (A3) is the function $f(t, x, u) = \frac{1}{4}t^5 \sin x + \sqrt{t} \sin^2 x u$.

Second example concerns a nonlinear integro-differential Cauchy problem containing two functional parameters. It is a generalization of the Cauchy problem considered in [3] and illustrating the global inverse function theorem.
Example 9 Let us consider a nonlinear integro-differential control system of Volterra type

\[ x'(t) + \int_0^t \Phi(t, \tau, x(\tau), u(\tau))d\tau = v(t), \ t \in J \ a.e., \tag{5} \]

where \( \Phi : P_{\Delta} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) \((P_{\Delta} = \{(t, \tau) \in J \times J; \tau \leq t\})\), \( x \in AC^2_0 \), \( u \in L^2(J, \mathbb{R}^m) \), \( v \in L^2(J, \mathbb{R}^n) \). On the function \( \Phi \) we assume that

(B1) \( \Phi(\cdot, \cdot, x, u) \) is measurable on \( P_{\Delta} \) for any \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \); \( \Phi(t, \tau, \cdot, \cdot) \) is continuously differentiable on \( \mathbb{R}^n \times \mathbb{R}^m \) for \( (t, \tau) \in P_{\Delta} \) a.e.

(B2) there exist a function \( a \in L^2(P_{\Delta}, \mathbb{R}^+_{0}) \) and a constant \( b > 0 \) such that

\[ |\Phi(t, \tau, x, u)| \leq a(t, \tau) |x| + b |u| \]

for \( (t, \tau) \in P_{\Delta} \) a.e., \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and

\[ \int_{P_{\Delta}} a^2(t, \tau)dtd\tau < 1/2 \]

(B3) there exist functions \( c, e \in L^2(P_{\Delta}, \mathbb{R}^+_{0}) \), \( \alpha, \beta \in C(\mathbb{R}^+_0, \mathbb{R}^+_0) \) and constants \( d, p, C > 0 \) such that

\[ |\Phi_x(t, \tau, x, u)| \leq c(t, \tau)\alpha(|x|) + d |u|, \]
\[ |\Phi_u(t, \tau, x, u)| \leq e(t, \tau)\beta(|x|) + p |u| \]

for \( (t, \tau) \in P_{\Delta} \) a.e., \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and

\[ \int_0^t e^2(t, \tau)d\tau \leq C, \ t \in J \ a.e. \]

We shall show that the mapping

\[ F : AC^2_0 \times L^2(J, \mathbb{R}^{m+n}) \rightarrow L^2(J, \mathbb{R}^n), \]

\[ F(x, u, v) = x'(t) + \int_0^t \Phi(t, \tau, x(\tau), u(\tau))d\tau - v(t) \]

satisfies assumptions of the global implicit function theorem with \( X = AC^2_0 \), \( Y = L^2(J, \mathbb{R}^{m+n}) \), \( H = L^2(J, \mathbb{R}^n) \).
In a standard way, one can check that $F$ is of class $C^1$ and the mappings

$$F_x(x, u, v) : AC^2_0 \rightarrow L^2(J, \mathbb{R}^n),$$

$$F_x(x, u, v)h = h'(t) + \int_0^1 \Phi_x(t, \tau, x(\tau), u(\tau))h(\tau)d\tau$$

$$F_{u,v}(x, u, v) : L^2(J, \mathbb{R}^m) \times L^2(J, \mathbb{R}^n) \rightarrow L^2(J, \mathbb{R}^n),$$

$$F_{u,v}(x, u, v)(f, g) = \int_0^1 \Phi_u(t, \tau, x(\tau), u(\tau))f(\tau)d\tau - g(t)$$

are the differentials of $F$ in $x$ and $(u, v)$, respectively.

Let us fix a function $(u, v) \in L^2(J, \mathbb{R}^{m+n})$. The mapping

$$\bar{\Phi} : P_\Delta \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\bar{\Phi}(t, \tau, x) = \Phi(t, \tau, x, u(t))$$

satisfies assumptions of Theorem 4.1 from the paper [3]. In the same way as in the proof of the mentioned theorem, one can show that the functional

$$\varphi : AC^2_0 \rightarrow \mathbb{R},$$

$$\varphi(x) = (1/2) \|F(x, u, v)\|_{L^2(J, \mathbb{R}^n)}^2$$

$$= (1/2) \int_0^1 \left| x'(t) + \int_0^t \bar{\Phi}(t, \tau, x(\tau))d\tau - v(t) \right|^2 dt$$

satisfies (PS) condition. Moreover, since

$$h'(t) + \int_0^t \Phi_x(t, \tau, x(\tau), u(\tau))h(\tau)d\tau = h'(t) + \int_0^t \bar{\Phi}_x(t, \tau, x(\tau))h(\tau)d\tau,$$

therefore, just as in [3], one can show that the mapping $F_x(x, u, v)$ is "one-one" and "onto".

Thus, from the global implicit function theorem it follows that for any $(u, v) \in L^2(I, \mathbb{R}^{m+n})$ there exists a unique solution $x_{u,v} \in AC^2_0$ of the equation (5) and the mapping

$$\lambda : L^2(J, \mathbb{R}^{m+n}) \ni (u, v) \mapsto x_{u,v} \in AC^2_0$$
is continuously differentiable in Gateaux (equivalently, in Frechet) sense on $L^2(J, \mathbb{R}^{m+n})$, with the differential $\lambda(u,v)$ at a point $(u,v) \in L^2(J, \mathbb{R}^{m+n})$, given by

$$L^2(J, \mathbb{R}^{m+n}) \ni (f,g) \mapsto z_{f,g} \in AC^0_0$$

where $z_{f,g}$ is such that

$$z'_{f,g}(t) + \int_0^t \Phi_x(t, \tau, x_{u,v}(\tau), u(\tau))z_{f,g}(\tau)d\tau$$

$$= -\int_0^t \Phi_u(t, \tau, x_{u,v}(\tau), u(\tau))f(\tau)d\tau + g(t)$$

a.e. on $J$. An example of a function satisfying conditions (B1), (B2), (B3) is the function $\Phi(t, \tau, x, u) = \frac{1}{3}t^5 \sqrt{\tau}\sin x + \sqrt{\tau}t^3 (\sin^2 x) \sin u$.

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