$k$-Strong Shortest Path Union Cover for Certain Graphs and Networks

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Abstract

The $k$-distance strong shortest path union cover of a graph is the minimum cardinality among all strong shortest path union cover at distance $k$ of $G$ where $1 \leq k \leq d$. In this paper we determine the 2-strong shortest path union cover for certain graphs, also we prove that the $k$-strong shortest path union cover problem in general is NP-complete.

Keywords: Shortest paths, Strong shortest paths, Shortest path union cover, Strong shortest path union cover, Networks, Sierpiński graphs and Sierpiński gasket graphs

1 Introduction

For a graph $G = (V, E)$, where $V$ and $E$ denote the vertex and the edge sets of a graph $G$, respectively. For vertices $u$ and $v$ in a connected graph $G$, the distance $d(u, v)$ is the length of a shortest $u-v$ path in $G$. The degree of a vertex $v$ in $G$ is the number of edges connecting it. The maximum distance between a vertex to all other vertices in a graph $G$ is considered as the eccentricity of the vertex. The diameter of a graph is the maximum eccentricity of any vertex in the graph. That is, the greatest distance between any pair of vertices. For more information on preliminaries we refer the reader to [1]. Shortest path union cover problem was introduced by Peter Boothe et al [3]. If $G$ is a graph, then a set $S$ of its vertices is called shortest path union cover, if the shortest paths that start at the vertices of $S$ cover all the edges of $G$. The shortest path union cover problem is to find a shortest path union cover of minimum cardinality [3]. Strong shortest path union cover problem is a new concept introduced by Xavier et al [5]. Xavier et al [5, 4, 6] proved that strong shortest path union cover problem is $NP$-complete and obtained the results for certain graphs, networks, Sierpiński Graphs and product graphs.

This manuscript is organized as follows. In section 2, the basic definitions and general results on $k$-strong shortest path union cover problem for certain graphs are obtained. In Section 3, we discuss
the Complexity of the $k$-strong shortest path union cover. In section 4, we determine the 2-strong shortest path union cover problem for certain graphs. In section 5, we discuss about the 2-strong shortest path union cover problem for various networks and Sierpiński Graphs.

2 General Results

Definition 2.1. Let $G$ be a graph, then a set $S$ of its vertices is called $k$-shortest path union cover, if the shortest paths of length at most $k$ that start at the vertices of $S$ cover all the edges of $G$. The minimum cardinality of $k$-shortest path union cover is denoted as $SPC_{kU}(G)$.

Definition 2.2. For a fixed vertex $u$, $ar{P}_k(u,v)$ be the set of all edges of a fixed shortest path between $u$ and $v$ where $v \in V(G)$ and $d(u,v) \leq k$. Let $ar{P}_k(u) = \bigcup_{v \in V(G)} \bar{P}_k(u,v)$. For any vertex set $S \subset V(G)$, define $ar{P}_k(S) = \bigcup_{u \in S} \bar{P}_k(u)$. If $ar{P}_k(S) = E(G)$, then $S$ is called the $k$-strong shortest path union cover of $G$. The minimum cardinality of $k$-strong shortest path union cover is denoted as $SSPC_{kU}(G)$.

Example 2.3. In Figure 1, $S = \{v_1, v_5, v_7\}$ forms 2-strong shortest path union cover and $T = \{v_1, v_7\}$ forms 2-shortest path union cover.

Figure 1: Graph $G$

Theorem 2.4. For any graph $G$, $\gamma_k(G) \leq SPC_{kU}(G)$.

Proof. Every $k$-strong shortest path union cover of $G$ is a $k$-dominating set of $G$. Hence $\gamma_k(G) \leq SPC_{kU}(G)$.

Observation 2.5. For any graph $G$ with $\delta \geq 1$, $SSPC_{2U}(G) \leq \frac{n}{2}$

Theorem 2.6. For any connected graph $G$, $SPC_U(G) \leq SPC_{kU}(G) \leq SSPC_{kU}(G)$.
Proof. Every $k$-shortest path union cover is a shortest path union cover of $G$. Therefore, $SPC_U(G) \leq SPC_{kU}(G)$. Every $k$-strong shortest path union cover is a $k$-shortest path union cover. Therefore, $SPC_{kU}(G) \leq SSPC_{kU}(G)$.
Hence $SPC_U(G) \leq SPC_{kU}(G) \leq SSPC_{kU}(G)$.

**Lemma 2.7.** For any connected graph $G$ of order $n$, $SSPC_U(G) \leq SPC_{kU}(G) \leq n - 1$.

Proof. Every $k$-strong shortest path union cover of $G$ is a strong shortest path union cover of $G$. Also $V(G)/\{u\}$ where $u \in V(G)$ is a $k$-strong shortest path union cover of $G$. Therefore $SSPC_{kU}(G) \leq n - 1$. Thus $SSPC_U(G) \leq SPC_{kU}(G) \leq n - 1$.

**Remark 2.8.** If $G$ is any connected graph with $\delta \geq 2$, then $SSPC_{kU}(G) \geq 2$.

**Remark 2.9.** If $G$ is a graph with no pendent vertices then $SSPC_{kU}(G) \geq 2$.

**Theorem 2.10.** For a graph $G$, with $SPC_{kU}(G) \geq |E| \Delta((\Delta - 1)^{k-1})^{\Delta - 2}$.

Proof. Let $S \subseteq V(G)$ be the $k$-shortest path union cover of $G$. This implies that every edge of $G$ lies on a path with distance at most $k$ from the vertices in $S$. Consider a vertex $u \in S$. All possible paths of length at most $k$ with one end $u$ can cover at most $\Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 + \ldots + \Delta(\Delta - 1)^{k-1}$ edges. That is the vertex $u$ can cover at most $\Delta(\Delta - 1)^{k-1}$ edges. Therefore $|E| \leq |S| \Delta(\Delta - 1)^{k-1}$. Hence $|S| \geq \frac{|E|((\Delta - 2)}{(\Delta - 2)^{k-1}}$.

**Remark 2.11.** The bound in Theorem 2.10 is sharp for the graph given in Figure 2 when $k = 3$.

![Figure 2: Graph G](image)

**Theorem 2.12.** If $G$ is a nontrivial connected graph of order $n$ and diameter $d$, then $SSPC_{kU}(G) \leq n - k + 1$ where $k \leq d$. 
Proof. Since \( \text{diam}(G) = d \), there exists a \( u - v \) geodesic of length \( k \). Let \( u = v_0, v_1, v_2, \ldots, v_k = v \) be the \( u - v \) geodesic. Let \( S = V(G) - \{v_1, v_2, \ldots, v_{k-1}\} \). Then all the edges of \( G \) are covered by the vertices in \( S \). Hence \( \text{SSPC}_{kU}(G) \leq |S| = n - k + 1. \)

**Theorem 2.13.** For a graph \( G(V,E) \) with diameter \( d \geq 2 \), the \( k \)-strong shortest path union cover \( \text{SSPC}_{kU}(G) \leq n - (d + 1) + \left( \frac{d+1}{2k+1} \right) \).

Proof. Consider a graph \( G(V(G), E(G)) \) with diameter \( d \geq 2 \). Let \( u \) and \( v \) be the vertices of \( G \) for which \( d(u,v) = d \). Assume that \( S \subseteq V(G) \) forms a \( k \)-strong shortest path union cover set for \( G \). Let \( u = v_k, v_{3k}, \ldots, v_{(d+1)k} = v \) be a \( u - v \) path of length at distance \( k \). Let \( S = V(G) - \{v_k, v_{3k}, \ldots, v_{(d+1)k}\} \). Thus \( \text{SSPC}_{kU}(G) \leq |S| \leq |V(G)| - (d + 1) + \left( \frac{d+1}{2k+1} \right). \)

**Remark 2.14.** The bound in Theorem 2.13 is sharp for the graph given in Figure 3 when \( k = 2 \).

![Figure 3: Graph G](image)

**Theorem 2.15.** \( G \) has cliques \( K_{n_1}, \ldots, K_{n_n} \) each cliques has \( n_i' \) simplicial vertices then \( \text{SSPC}_{kU}(G) \geq \sum (n_i' - 1) \), where \( n_i' \geq 2 \).

Proof. Consider the clique \( K_{n_i} \) with simplicial vertices \( n_i' \). Let \( K_{n_i}' \) be the graph spanned by \( n_i' \) simplicial vertices. It is straightforward to see that any geodesic of length at most \( k \) through the non simplicial vertices of \( K_{n_i} \) will not cover any edge of \( K_{n_i}' \). To cover the edges of \( K_{n_i}' \), requires \( n_i' - 1 \) simplicial vertices of \( K_{n_i}' \). Therefore \( \text{SSPC}_{kU}(G) \geq \sum (n_i' - 1). \)

**Remark 2.16.** The bound in Theorem 2.15 is sharp for the graph given in Figure 4.

![Figure 4: Graph G](image)
3 Complexity Results

The proof for the NP-completeness of the $k$-strong shortest path union cover problem for general graphs can be reduced from the vertex cover problem which is already proved to NP-complete [2]. A vertex cover in an undirected graph $G = (V, E)$ is a subset of vertices $V' \subseteq V$ such that every edge in $G$ has at least one endpoint in $V'$.

![Figure 5: NP-complete illustration for a general graph](image)

**Theorem 3.1.** The $k$-strong shortest path union cover problem is NP-Complete.

*Proof.* There are two cases to prove that the $k$-strong shortest path union cover problem is NP-Complete, when $k \geq 3$ and $k = 2$.

Case (i): When $k \geq 3$

Given a graph $G = (V, E)$. We construct a graph $G' = (V', E')$ from $G$ by subdividing every edge in $E(G)$ by a path of length 3. All the new vertices that are created in this step are called path vertices. Now create two new $K_3$ and introduce two new vertices $b_1$ and $c_1$ such that all the vertices of $V(G)$ in $G'$ will be joined with $b_1$ and all the path vertices in $G'$ will be joined with vertex $c_1$. Join any one vertex from anyone $K_3$ to $b_1$ with the path $b_1b_2\ldots b_{k-2}$ and anyone vertex of other $K_3$ to $c_1$ with the path $c_1c_2\ldots c_{k-2}$ of length $k - 3$ as shown in the Figure 5. Thus the graph $G'$ is obtained.
Let $S$ be the vertex cover of $G$. Let $A = \{x, y/x = a_1 \text{ or } a_2 \rightarrow y = d_1 \text{ or } d_2\}$.

It is straightforward to see that $S \cup A$ is a $k$-strong shortest path union cover of $G'$ and $|S \cup A| = |S| + 2$. From the construction of $G'$, any $k$-strong shortest path union cover of $G$ contains one vertex each from $\{a_1, a_2\}$ and $\{d_1, d_2\}$ to cover the edges $a_1a_2$ and $d_1d_2$.

Conversely let $T$ be a $k$-strong shortest path union cover of $G'$. Let $T' = T \setminus \{b_1, b_2, \ldots, b_{k-2}\} \cup \{c_1, c_2, \ldots, c_{k-2}\}$. Note that $|T'| \leq |T|$. Construct $T''$ as follows. If $T''$, contains $a_1$ and $a_2$, then consider either $a_1$ or $a_2$ in $T''$. If $T'$ contains only one vertex from $\{a_1, a_2\}$, then consider the same vertex in $T''$. Similarly, if $T'$ contains $d_1$ and $d_2$, then consider either $d_1$ or $d_2$ in $T''$. If $T'$ contains only one vertex from $\{d_1, d_2\}$, then consider the same vertex in $T''$. It is easy to verify that $|T''| \leq |T'|$.

Now if $T''$ contains a path vertex $x'$, then replace $x'$ with the adjacent vertex $x \in V(G)$. Repeat this replacement procedure until there are no more path vertices and the resulting set be $T'''$. Also $|T'''| \leq |T''|$ and $|T'''| = |T''' \cap V(G)| + 2$. It is straightforward to see that $T'''$ is a $k$-strong shortest path union cover of $G'$ and $T''' \cap V(G)$ is a vertex cover of $G$.

Case (ii): When $k = 2$

Figure 6: NP-complete illustration for a general graph

For $k = 2$, Given a graph $G = (V, E)$. We construct a graph $G' = (V', E')$ from $G$ by subdividing every edge in $E(G)$ by a path of length 3. All the new vertices that are created in this step are called path vertices. Now create two new $K_3$ and introduce two new vertices $u_1$ and $w_1$ such that
all the vertices of $V(G)$ in $G'$ will be joined with $u_1$ and all the path vertices in $G'$ will be joined with vertex $w_1$. Join any one vertex from each $K_3$ to $u_1$ and $w_1$ as shown in the Figure 6. Thus the graph $G'$ is obtained.

Let $S$ be the vertex cover of $G$. Then $S \cup \{a_1, u_1, d_1, w_1\}$ is a 2-strong shortest path union cover. Conversely let $T$ be the 2-strong shortest path union cover. Let $T' = T \setminus \{a_1, a_2, u_1, u_2, w_1, w_2, d_1, d_2\} \cup \{a_1, u_1, d_1, w_1\}$. Clearly $T'$ is a 2-strong shortest path union cover and $|T'| \leq |T|$. If $T'$ contains a path vertex $x'$, then replace $x'$ with adjacent vertex $x \in V$. Repeat this replacement procedure until there are no more path vertices and the resulting set be $T''$. Also $|T''| \leq |T'|$ and $|T''| = |T'' \cap V(G)| + 4$. It is straightforward to see that $T''$ is a 2-strong shortest path union cover of $G'$ and $|T'' \cap V(G)|$ is a vertex cover of $G$. \hfill \Box

**Remark 3.2.** The construction for $k = 3$, $k = 4$ and $k = 5$ are illustrated in Figure 7.

![Figure 7: NP-complete illustration for a general graph](image)

4 2-Strong shortest path union cover for certain graphs

The following observations are easily verifiable.
Observation 4.1. For any path $P_m$, $SSPC_{2U}(P_m) = \lceil \frac{m}{5} \rceil$.

Proof. Let $P_m$ be the Path with order $m$. Because the maximum degree of every vertex on $P_m$ is 2, then degree 2 vertex can cover at most 4 edges at distance 2. Therefore the minimum number of vertices that can be covered by $m$ vertices are $\lceil \frac{m}{5} \rceil$. Thus, $SSPC_{2U}(P_m) \geq \lceil \frac{m}{5} \rceil$. Also it can be easily verified that the set $S = \{v_3, v_7, \ldots, v_{m-1}\}$ form the strong shortest path union cover set where $|S| = \lceil \frac{m}{5} \rceil$. Therefore $SSPC_{2U}(P_m) = \lceil \frac{m}{5} \rceil$. □

Observation 4.2. For any cycle $C_n$, $SSPC_{2U}(C_n) = \lceil \frac{n}{5} \rceil$, $n \geq 5$.

Proof. Let $C_n$ be the Cycle with order $n$. Cycle is a 2 regular graph. Because the maximum degree of every vertex on $C_n$ is 2, then the 2 degree vertex can cover at most 4 edges at distance 2. Therefore the minimum number of vertices that can be covered by $n$ vertices are $\lceil \frac{n}{5} \rceil$. Thus, $SSPC_{2U}(C_n) \geq \lceil \frac{n}{5} \rceil$. Also it can be easily verified that the set $S = \{v_1, v_5, \ldots, v_{n-3}\}$ form the strong shortest path union cover set where $|S| = \lceil \frac{n}{5} \rceil$. Therefore $SSPC_{2U}(C_n) = \lceil \frac{n}{5} \rceil$. □

Observation 4.3. Let $A(m, n)$ be the Actinia graph for $m \geq 2$ and $n \geq 1$, then the $SSPC_{2U}(A(m, n)) = \lceil \frac{n}{5} \rceil$, when $m$ is odd or even.

Lemma 4.4. For any Complete bipartite graph $K_{m,n}$, $2 \leq m \leq n$, $SSPC_{2U}(K_{m,n}) = m$.

Proof. Let $G= (M, N)$ be the complete bipartite graph $K_{m,n}$. Let $2 \leq m \leq n$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of the bipartite graph $K_{m,n}$. $|U| = m$, $|W| = n$. Assume that $T \subseteq V(G)$ is the 2-strong shortest path union cover of $G$ and $|T| = m - 1$. case(i): $|T| \subseteq U$.

Without loss of generality, $u_m \notin T$. Let $T = \{u_1, u_2, \ldots, u_{m-1}\}$. Clearly the path between the vertices of $T$ and $u_m$ can cover at most $(m - 1)$ edges at distance 2 adjacent with $u_m$. These paths will not cover $n - (m - 1)$ edges incident with $u_m$. Also it is straightforward to see that these $n - (m - 1)$ edges incident with $u_m$ are not covered at distance 2 by any other shortest path between the vertices of $T$ and $N$. Therefore $T$ is not the strong shortest path union cover of $K_{m,n}$.

case(ii) : $T \subseteq W$.

Case(iii): $T = T_1 \cup T_2$.

The proof of Case (ii) and (iii) are similar to case (i).

Again $T$ is not the 2-strong shortest path union cover of $G$ by a similar argument. Clearly the vertices of $M$ form the strong shortest path union cover. Hence $SSPC_{U}(K_{m,n}) = |M|$. □

Proposition 4.5. For any Wheel graph $W_n$, $n \geq 5$, $SPC_{2U}(W_n) = SPC_{2U}(W_n) = \lceil \frac{n}{5} \rceil$. 

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Proof. Let \( v_0 \) be the core vertex of \( W_n \). Since \( v_0 \) is adjacent to all the vertices and the diameter of the wheel graph is 2 for \( n \geq 5 \). For \( n \geq 2 \) the graph \( K_1 + C_n \) is called wheel graph of order \( n + 1 \) with \( 2n \) edges. The vertex on \( K_1 \) is called the core vertex denoted by \( v_0 \). The vertices that lie on the cycle \( C_n \) is denoted by \( v_1, v_2, \ldots, v_n \) in clockwise. Label the interior edges in anticlockwise \( v_0v_1, v_0v_n, v_0v_{n-1}, \ldots, v_0v_2 \) by 1, 3, 5, \ldots, \( 2n-1 \). Label the cycle edges in clockwise \( v_1v_2, v_2v_3, v_3v_4, \ldots, v_nv_1 \) by 1, 2, 3, \ldots, \( n \). The 2-strong shortest path union cover problem is to cover every edge of a graph at distance 2 by the unique shortest path from a subset of vertices in the graph. Let \( S \subseteq V(W_n) \) be the 2-strong shortest path union cover of \( W_n \). \( S = \{v_1, v_5, v_9, v_{13}, \ldots, v_{n-3}\} \). Consider a set \( T = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\} \subseteq V(G), 1 \leq i \leq n \), then the vertex \( v_{i+2} \subseteq S \) should be in \( S \) such that the edges connected \( v_{i+2} \) to the remaining vertices in \( T/\{v_{i+2}\} \) are covered in a unique path at distance 2. Continuing this for every distinct set of 5 vertices in \( C_n \) which is \( V(W_n)/\{v_0\} \), such that one vertex ought to be chosen to cover all the edges at distance 2. The \( SSPC_{2U}(W_n) \) set of graph \( W_n, n \geq 5 \) is a set with vertices \( \lceil \frac{n}{5} \rceil \) from the set \( V(W_n)/\{v_0\} \). Hence the 2-strong shortest path union cover for Wheel graph \( W_{1,n} \) is \( \lceil \frac{n}{5} \rceil \).

Proposition 4.6. For any Double Wheel graph \( DW_n, n \geq 5 \), \( SPC_{2U}(DW_n) = SSPC_{2U}(DW_n) = 2 \lceil \frac{n}{5} \rceil \).

Proof. Follows from proposition 4.5

Proposition 4.7. For any Crown graph \( H_{n,n}, n \geq 3 \), \( SPC_{2U}(H_{n,n}) = 2 \).

Proof. Let \( H_{n,n}, n \geq 3 \) be a crown graph with vertex set \( \{(a_i, b_j); 0 \leq i, j \leq n - 1, i \neq j\} \) of degree \( n - 1 \). To cover all the edges at distance 2 in shortest path require 2 vertices. Figure 9 shows that the vertices \( \{a_0, b_0\} \) form the 2-strong shortest path union cover for \( H_{n,n} \). If the 2-strong shortest path union cover set contains only any one vertex, there exists \( n \) edges that are left uncovered. Thus \( SPC_{2U}(H_{n,n}) = 2 \).
Lemma 4.8. For any Crown graph $H_{n,n}$, $n \geq 3$, $SSPC_{2U}(H_{n,n}) = n - 1$

Proof. Let $G$ be a crown graph $H_{n,n}$, where $n \geq 3$. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex sets of a crown graph $H_{n,n}$. Assume that $T \subseteq V(G)$ is the 2-strong shortest path union cover of $G$ and $|T| = n - 2$.

Case(a): $|T| \subseteq U$.

Without loss of generality, $u_n \notin T$. Let $T = \{u_1, u_2, \ldots, u_{n-2}\}$. Clearly the path at distance 2 between the vertices of $T$ and $u_n$ cover at most $n - 2$ edges adjacent with $u_n$. The path between $T$ and $u_n$ will not cover 2 edges incident with $u_n$. Also it is straightforward to see that these 2 edges incident with $u_n$ are not covered at distance 2 by any other strong shortest path between $T$ and $V$.

Therefore $T$ is not a strong shortest path union cover of $H_{n,n}$.

Case(b) : $T \subseteq V$.

Case(c): $T = T_1 \cup T_2$.

The proof of Case (b) and (c) are similar to case (a).

Clearly $(n - 1)$ vertices form the 2-strong shortest path union cover for $H_{n,n}$. Hence the proof. \qed

Lemma 4.9. Let $G(n,k)$ be a Petersen graph, then $SSPC_{2U}(G_{n,k}) = 3$.

Proof. Let $G(n,k)$ be a Petersen graph. Let $a, b, c, d$ and $e$ be the outer vertices and $f, g, h, i$ and $j$ be the corresponding inner vertices. Refer Figure 10.

Assume that $T \subseteq V(G)$ is the 2-strong shortest path union cover of $G$, and $|T| = 2$.

Case (i): $T$ contains any one inner and outer vertex

When $a \in T$, the edge $cd$ is left uncovered. To cover the edge $cd$ either $h$ or $i \in T$. Without loss of generality, $h \in T$, then the edge $cd$ is uncovered by $a$. If $h \in T$, then $h$ will not cover the edge $gi$. This implies that $T$ is not the 2-strong shortest path union cover for $G(n,k)$.
Case (ii): $T$ contains any two inner vertices.
Let $T = \{g, f\}$ where $g$ and $f$ be any two vertices from inner cycle. There exists at least one edge $cd$ from outer cycle is left uncovered by $T$. This implies that $T$ is not the 2-strong shortest path union cover for $G(n, k)$.

Case (iii): $T$ contains any two outer vertices
The proof is similar to the case (ii).

In these three cases, $T$ is not the 2-strong shortest path union cover for $G(n, k)$. Clearly $S = \{a, c, g\}$ forms the 2-strong shortest path union cover for $G(n, k)$. And moreover by Remark 2.8, $SSPC_{U}G(n, k) = 3$. Hence it implies $SSPC_{kU}G(n, k) = 3$.

Lemma 4.10. For any Fan graph $F_{1,n}$, $SPC_{2U}(F_{1,n}) = SSPC_{2U}(F_{1,n}) = \lceil \frac{n}{5} \rceil$, $n \geq 5$.

Proof. A fan graph $F_{1,n}$ is defined as the graph $K_1 + P_n$, where $K_1$ is with single vertex and $P_n$ is a path on n vertices. Let $v_o$ be the central vertex and $v_1, v_2, \ldots, v_n$ be the vertices in $P_n$. Let $S \subseteq V(G)$ be the 2-strong shortest path union cover of $G$. Then $S$ contains any one vertex out of every 5 vertices from $P_n$ which form the 2-strong shortest path union cover of $G$. It is straightforward to see any $\lceil \frac{n}{5} \rceil - 1$ vertices will not cover $E(F_{1,n})$. Hence $SPC_{2U}(F_{1,n}) = \lceil \frac{n}{5} \rceil$. \hfill \Box

Lemma 4.11. For any Double Fan graph $DF_n$, $SPC_{2U}(DF_n) = SSPC_{2U}(DF_n) = 1 + \lceil \frac{n}{5} \rceil$.

Proof. The double fan graph $F_{2,n}$ consists of $n + 2$ vertices and $3n - 1$ edges. The path vertices $v_1, v_2, \ldots, v_n$ are adjacent to $v_o$ and $v'_o$. Let $S \subseteq V(G)$ be the 2-strong shortest path union cover of $G$. Then $S$ contains $v_o$ or $v'_o$ and any one vertex out of every 5 vertices from $P_n$ which form a 2-strong shortest path union cover of $G$. Thus $SPC_{2U}(DF_n) = SSPC_{2U}(DF_n) = 1 + \lceil \frac{n}{5} \rceil$. \hfill \Box

Theorem 4.12. For any Generalised Friendship graph $F_{k,n}$, $k, n \geq 4$, $SSPC_{2U}(F_{k,n}) \leq 1 + n \lceil \frac{k}{5} \rceil$.

Proof. Let $v_0$ be the central vertex and $v_1, v_2, \ldots, v_n$ be the outer vertices of the petals. The central or core vertex $v_0$ covers the edges at distance 2 in each petal. Still there exists edges within the petals left uncovered. By choosing a vertex out of every five vertices in each petal would cover the
edges at distance 2. Hence there exists \(1 + n[^{k-1}\over 5]\) vertices for 2-strong shortest path union cover in \(F_{k,n}\).

\[\square\]

**Remark 4.13.** For any Generalised Friendship graph \(F_{k,n}\), \(k, n \geq 4\), \(SPC_{2U}(F_{k,n}) \leq 1 + n[^{k-1}\over 5]\).

## 5 2-Strong shortest path union cover for Networks

Interconnection networks play a key role in the design and implementation of communication networks and the recent advent of optic technology add more design problems \[12\]. In general, an interconnection network may be modeled by a simple graph whose nodes represent components of the network and whose links represent physical communication links. The vertex which connects the two degree vertices in \((BF(2))\) known as a binding vertex. In this section we determine the 2-shortest path and strong shortest path union cover for various networks consist of Butterfly network, Augmented Butterfly network, Enhanced butterfly network, Benes network, Silicate network \(SL(n)\), Hypercube \(Q_n\), Sierpiński graphs and Sierpiński gasket graphs.

**Theorem 5.1.** Let \(G\) be an \(r\)-dimensional butterfly network. Then

\[SPC_{2U}(BF(r)) \leq \begin{cases} 2^{r-1}, & \text{when } r \leq 4 \\ 2^r, & \text{when } r > 4 \end{cases}\]

\[\text{Figure 11: (a) Pairs of binding vertices form 2-shortest path union cover for } BF(2) \text{ (b) Pairs of binding vertices in each } BF(2) \text{ form 2-shortest path union cover for } BF(4)\]

**Proof.** case(i) : when \(r \leq 4\)

Consider butterfly network \(BF(2)\). \(BF(2)\) is a bipartite graph with 4 vertices in each part, which
are independent sets called as levels $L_0$ and $L_1$. The removal of level 0 vertices of $BF(2)$ leaves 2 disjoint copies of $BF(1)$. The binding vertex in each copy of $BF(1)$ forms the 2-shortest path union cover for $BF(2)$. Similarly for $BF(3)$, we have 2 copies of $BF(2)$ where the binding vertex in each copy of $BF(2)$ forms the 2-shortest path union cover for $BF(3)$. As the butterfly network has dual symmetry, in $BF(4)$ there are 2 copies of $BF(3)$ and 4 copies of $BF(2)$. The pairs of binding vertices in each $BF(2)$ form the 2-shortest path union cover for $BF(4)$. Hence $SPC_{2U}(BF(4)) = 8$. Therefore $SPC_{2U}(BF(r)) \leq 2^r - 1$.

**case(ii) :** When $r > 4$

Consider the butterfly graph $BF(5)$. By the recursive construction $BF(5)$ has 2 copies of $BF(4)$ and a new level $L_4$ with $2^5$ vertices. Let $S_1$ and $S_2$ denote 2-shortest path union cover sets in the left and the right copy of $BF(4)$ respectively. Since $SPC_{2U}(BF(4)) = 8$, select all 16 binding vertices from left copy of $BF(5)$ into $S_1$ and 16 binding vertices from right copy of $BF(5)$ into $S_2$. Let $S_1 = \{(1,i)(2,i)/i=0 \text{ to } 15\}$. Then all the edges in left copy of $L_1$ are covered at distance 2 by the binding vertices in $S_1$. Using the symmetry of butterfly graphs, mirror image compliment set of $S_1$ in the right copy of $BF(5)$ will cover all the edges at distance 2 in that copy and it is given by $S_2 = \{(1,i)(2,i)/i=16 \text{ to } 31\}$. Hence $S = S_1 \cup S_2$ form the 2-shortest path union cover for $BF(5)$ with $|S| = 2.2^4 = 2^5$. By symmetry, the result holds true for $n > 5$. Therefore $SPC_{2U}(BF(r)) \leq 2^r$, when $r \geq 5$. 

![Figure 12: Binding vertices in each $BF(2)$ form 2- strong shortest path union cover for $BF(4)$](image)

**Theorem 5.2.** Let $G$ be an $r$-dimensional butterfly network. Then $SSPC_{2U}(BF(r)) \leq \left\lceil \frac{r}{2} \right\rceil 2^{r-1}$, $n \geq 3$.

**Proof.** Let $G$ be an $r$-dimensional butterfly network. By recursive construction, $BF(r)$ has 2 copies of $BF(r-1)$. Let $S_1$ and $S_2$ denote the 2-strong shortest path union cover sets in the bottom and top copy of $B(r-1)$ respectively. In $BF(3)$ there exists 2 layers of bottom and top layers. The binding vertices between two consecutive layers in bottom and top cover all the edges at distance 2 and form the 2-strong shortest path union cover for $BF(3)$. Similarly in $BF(4)$, there exists four
bottom and top layers. The binding vertices between two consecutive layers both in bottom and top cover all the edges at distance two and form the 2-strong shortest path union cover for $BF(4)$. Therefore $SSPC_{2U}(BF(4)) = 2.2^3 = \lceil \frac{r}{2} \rceil 2^{r-1}$.

Proceeding like this, for $BF(r)$, there exists bottom and top copy of $BF(r-1)$. The binding vertices between two consecutive layers cover all the edges at distance 2 in bottom and top copies of $BF(r-1)$. Hence $S = S_1 \cup S_2$ becomes the 2-strong shortest path union cover of $BF(r)$ with $|S| \leq \lceil \frac{r}{2} \rceil 2^{r-1}$. Therefore $SSPC_{2U}(BF(r)) \leq \lceil \frac{r}{2} \rceil 2^{r-1}$.

**Theorem 5.3.** Let $G$ be an 3-dimensional Augmented butterfly network. Then $SSPC_{2U}(ABF(3)) = 12$.

![Figure 13: (a) and (b) One vertex from each diamond form 2-strong shortest path union cover for $ABF(2)$ and $ABF(3)$.](image)

**Proof.** Consider the $r$-dimensional butterfly network $BF(r)$. Place an new edge on the antipodal vertices in a cycle. The resulting graph is known as Augmented butterfly network.

Let $G$ be an 3-dimensional Augmented butterfly network consists of diamond representation as shown in Figure 13. Let $S$ be the 2-strong shortest path union cover for $ABF(3)$. Let $a, b \in S$. The vertices $a, b$ can cover all the edges of the diamond $cedf$, except the edge $ef$. Therefore to cover the edge $ef$ we require at least one vertex from the diamond $cedf$. Thus for each diamond of $ABF(3)$ we have to take at least one vertex in the 2-strong shortest path union cover set. Hence $SSPC_{2U}(ABF(3)) = 12$.

**Corollary 5.4.** Let $G$ be an $r$-dimensional Augmented butterfly network. Then the $SSPC_{2U}(ABF(r)) \leq r(2^{r-1})$. 


Proof. By Theorem 5.3, for $SSPC_{2U}(ABF(3)) = 12$. Similarly for $SSPC_{2U}(ABF(4)) = 28$. Proceeding like this for $r$-dimension of augmented butterfly network, there exists $r(2^r - 1)$ diamond in $(ABF(r))$. Also it is straightforward to note that a set $S \subseteq V(G)$ that contains one vertex from each diamond of $ABF(r)$ forms the 2-strong shortest path union cover. Hence $SSPC_{2U}(ABF(r)) \leq r(2^r - 1)$.

Remark 5.5. Let $G$ be an $r$-dimensional augmented butterfly network. Then $SPC_{2U}(ABF(r)) \leq r(2^r - 1)$.

**Theorem 5.6.** Let $G$ be an 3-dimensional Enhanced butterfly network. Then $SSPC_{2U}(EBF(3)) = 12$.

![Figure 14](image.png)

Figure 14: (a) and (b) The core vertices form 2-strong shortest path union cover for $EBF(2)$ and $EBF(3)$

Proof. Consider the $r$-dimensional butterfly network $BF(r)$. Place a new vertex in each 4-cycle of $BF(r)$ and join this vertex to the four vertices of the 4-cycle. The resulting graph is called an enhanced butterfly network $EBF(r)$.

Let $G$ be an 3-dimensional enhanced butterfly network consists of diamond representation as shown in Figure 14. The center vertex in each diamond is known as core vertex $o$. The centre vertex with four degree in diamond structure $D_1 : cbda$ known as core vertex $o_1$ as shown in Figure 14(a). Let $S$ be the 2-strong shortest path union cover for $EBF(2)$. Let $o_2, o_3 \in S$. The core vertices $o_2, o_3$ can cover all the edges of the diamond $cbda$, except the edges $(ao_1)$ and $(o_1b)$ in $D_1$. These edges is not covered by any other vertices in the diamond $cbda$, except the core vertex $o_1$. Hence in each diamond structure of $EBF(2)$, core vertex should be there in 2-strong shortest path union cover set. Therefore for $SSPC_{2U}(EBF(2)) = 4$. Similarly for 3 dimensional enhanced butterfly network, we require a core vertex in each diamond. Hence $SSPC_{2U}(EBF(3)) = 12$. 


Theorem 5.7. Let $G$ be an $r$-dimensional Enhanced butterfly network. Then $SSPC_{2U}(EBF(r)) \leq r(2^r-1)$.

Proof. By Theorem 5.6, $SSPC_{2U}(EBF(3)) = 12$. Similarly for $SSPC_{2U}(EBF(4)) = 28$. Proceeding like this for $r$-dimension of enhanced butterfly network, there exists $r(2^r-1)$ diamond in $EBF(r)$. Also it is straightforward to note that a set $S \subseteq V(G)$ that contains core vertex from each diamond of $EBF(r)$ forms the 2-strong shortest path union cover. Hence $SSPC_{2U}(EBF(r)) \leq r(2^r-1)$.

Remark 5.8. Let $G$ be an $r$-dimensional enhanced butterfly network. Then $SC_{2U}(EBF(r)) \leq r(2^r-1)$.

Corollary 5.9. Let $G$ be an $r$-dimensional benes network. Then $SC_{2U}(B(r)) \leq 2^r$ for $r \geq 3$.

Proof. Let $G$ be an $r$-dimensional benes network. The removal of level 0 vertices $v_1, v_2, \ldots, v_n$ of $B(r)$ gives two subgraphs $G_1$ and $G_2$ of $B(r)$, each isomorphic to $B(r-1)$. The pairs of binding vertices in each $B(2)$ covers all the edges in $B(r)$. No other vertices can cover the edges in the layers except the binding vertices. Hence the pairs of binding vertices beween layers form 2-shortest path union cover for $B(r)$. Hence $SC_{2U}(B(r)) \leq 2^r$ for $r \geq 3$.

![Diagram of Binding vertices in each $B(2)$ form 2-shortest path union cover for $B(4)$](image)

Figure 15: Binding vertices in each $B(2)$ form 2-shortest path union cover for $B(4)$

Remark 5.10. For an 2-dimensional benes network, the $SC_{2U}(B(r)) = 2$.

The two antipodal vertices $u_1, u_2$ form the 2-shortest path union cover for $B(2)$.

Theorem 5.11. Let $G$ be an $r$-dimensional benes network. Then $SSPC_{2U}(B(r)) \leq \left[\frac{r}{2}\right]2^r$.
Figure 16: Binding vertices between two layers form 2-strong shortest path union cover for $B(4)$

Proof. The benes network consists of back to back butterflies. Let $G$ be an $r$-dimensional benes network. By recursive construction, $B(r)$ has 2 copies of $B(r-1)$. Let $S_1$ and $S_2$ denote 2-strong shortest path union cover sets in the bottom and top copy of $B(r-1)$ respectively. In $B(2)$ there exists 2 layers of bottom and top layers. There are two pairs antipodal vertices in bottom and top layers which cover all the edges at distance 2 and form the 2-strong shortest path union cover for $B(2)$. Similarly in $B(3)$ and $B(4)$, the binding vertices between two consecutive layers in bottom and top copy of $B(3)$ cover all the edges at distance 2, thus form the 2-strong shortest path union cover. Therefore $SSPC_{2U}(B(4)) = 2.2^4 = \lceil \frac{r}{2} \rceil 2^r$.

Proceeding like this, for $B(r)$, there exists bottom and top copy of $B(r-1)$. The binding vertices between two consecutive layers cover all the edges at distance 2 in bottom and top copies of $B(r-1)$. Hence $S = S_1 \cup S_2$ becomes the 2-strong shortest path union cover of $B(r)$ with $|S| \leq \lceil \frac{r}{2} \rceil 2^r$. Therefore $SSPC_{2U}(B(r)) \leq \lceil \frac{r}{2} \rceil 2^r$.

Theorem 5.12. Let $G$ be the silicate network $SL(n)$ of dimension $n$, then $SSPC_{2U}(G) = 6n^2$

Proof. Let $SL(n)$ be the silicate network with $15n^2+3n$ vertices and $36n^2$ edges. Let $S \subseteq V(SL(n))$ be the 2-strong shortest path union cover set for $SL(n)$. In each $K_4$ of $SL(1)$, there exists an edge $(ae)$ which is uncovered by any other vertices in $SL(1)$. Hence the 3 degree simplicial vertices in each $K_4$ should be in 2-strong shortest path union cover set. Therefore $S$ contains only 3 degree simplicial vertices in each $K_4$ of $SL(n)$ as shown in Figure 17(a). Therefore $S$ contains the $6n^2$ vertices in $SL(n)$ and form the 2-strong shortest path union cover set for $SL(n)$. Refer Figure 17(b). Hence by Theorem 2.15 $SSPC_{2U}(G) = 6n^2$. □
Figure 17: (a) and (b) Three degree vertices in each $K_4$ form the 2-shortest and strong shortest path union cover for $SL(1)$ and $SL(3)$

**Remark 5.13.** Let $G$ be the silicate network $SL(n)$ of dimension $n$, then $SPC_{2U}(G) = 6n^2$.

Figure 18: (a) and (b) The set of vertices marked in dark form the 2-strong shortest path union cover for $Q_3$ and $Q_5$

**Theorem 5.14.** For the $n$ cube $Q_n$, $n \geq 3$, then $SSPC_{2U}(Q_n) \leq 2^{n-2}$.

*Proof.* Let $Q_n$ be the hypercube network with $2^n$ vertices. For $n = 3$, $Q_3$ has two copies of $Q_2$. It can be easily verified that in each copy of $Q_2$ at least one vertex is sufficiently enough to cover all the edges at distance 2 to form the 2-strong shortest path union cover in $Q_3$. For $Q_4$, there exists four copies of $Q_2$, one vertex in each copy of $Q_2$ is more than enough to cover all the edges at distance 2 in $Q_4$. For $Q_5$, there exists eight copies of $Q_2$, at least one vertex in each copy of $Q_2$ is more than enough to cover all the edges at distance 2 in $Q_5$ as shown in Figure. Similarly proceeding by induction method for $Q_n$, there exists $2^{n-2}$ copies of $Q_2$. Choosing at least one vertex in each copy of $Q_2$ cover all the edges at distance 2 and form the 2-strong shortest path union cover for $Q_n$. Thus
it can easily be verified that $2^{n-2}$ vertices are sufficient to form the 2-strong shortest path union cover for $Q_n$. Hence by Remark 2.8, 2.9 $SSPC_{2U}(Q_n) \leq 2^{n-2}$.

Remark 5.15. For the $n$ cube $Q_n$, $n \geq 3$, then $SPC_{2U}(Q_n) \leq 2^{n-2}$.

Remark 5.16. The result in Theorem 5.14 and Remark 5.15 is sharp for $n = 3$.

Theorem 5.17. Let $G$ be the Sierpiński Graph $S(n,3)$, $n \geq 2$, then $SSPC_{2U}(G) \leq 3^{n-1}$.

Proof. Let $G(V, E)$ be the $n$-dimensional Sierpiński Graph $S(n,3)$, $n \geq 2$. There exists $3^{(n-1)}$ copies of $S(1,3)$ in $S(n,3)$. In $S(2,3)$, the three copies of $S(1,3)$ are connected to each other by an edge as depicted in Figure 19 (a). In $S(3,3)$, the three copies of $S(2,3)$ are connected to each other by an edge. Similarly in $S(n,3)$, the three copies of $S(n-1,3)$ are connected by an edge. In each $S(2,3)$, choose the alternative vertices of $C_6$ except the extreme vertices in $S(2,3)$ which cover the edges at distance 2 and form a 2-strong shortest path union cover as shown in Figure 19 (a). The similar pattern of choosing the vertices in each $S(2,3)$ is followed in $S(n,3)$. Proceeding like this, there exists $3^{(n-2)}$ copies of $S(2,3)$ in $S(n,3)$. Let $S$ be the 2-strong shortest path union cover of $G$ and $S$ contains $3^{(n-1)}$ vertices which cover the edges at distance 2 in $S(n,3)$.

Assume that $T \subseteq V(G)$ such that $|T| \leq |S|$ forms the 2-shortest and strong shortest path union cover. $T$ contains two alternative vertices of $C_6$ in $S(2,3)$.

There exists 2 edges left uncovered in each copy of $S(2,3)$ in $S(n,3)$ by the vertices in $T$ as shown in Figure 19 (b). This implies that $T$ does not form 2-strong shortest path union cover for $S(n,3)$. Hence $S$ forms a 2-strong shortest path union cover and by Remark 2.8, 2.9 $SSPC_{2U}(G) \leq 3^{n-1}$.

Remark 5.18. Let $G$ be the Sierpiński Graph $S(n,3)$, $n \geq 2$, then $SPC_{2U}(G) \leq 3^{n-1}$.

Figure 19: (a) $S$ forms 2-shortest and strong shortest path union cover for $S(3,3)$ (b) The dotted lines represent the edges that are not covered at distance 2 by $T$ in $S(3,3)$.
Figure 20: (a) $S$ forms the 2-shortest path union cover for $S_4$ (b) The dotted lines represent the edges that are not covered at distance 2 by $T$ in $S_4$.

**Theorem 5.19.** Let $G$ be the Sierpiński gasket Graph $S_n$, $n \geq 3$, then $SPC_{2U}(S_n) \leq 3^{n-2}$.

**Proof.** Let $S_n$ be the $n$-dimensional Sierpiński gasket Graph $S_n$, $n \geq 3$. In $S_3$, the three copies of $S_2$ are merged to each other. In $S_4$, the three copies of $S_3$, are merged to each other as depicted in Figure 20(a). Similarly in $S_n$, the three copies of $S_{n-1}$, are merged to each other. In $S_3$, choosing any three vertices except the merged vertices would leave at least one edge in each $K_3$ uncovered. Hence choose three merged vertices $\{x, y, z\}$ which cover the edges at distance 2 and form the 2-shortest path union cover as shown in 20(a). The similar pattern of choosing the vertices in each $S_3$ is followed in $S_n$. Proceeding like this, there exists $3^{(n-3)}$ copies of $S_3$, in $S_n$. Let $S$ be the 2-shortest path union cover for $G$ and $S$ contains three merged vertices in each copy of $S_3$ which cover the edges at distance 2 in $S_n$. Hence $3^{n-2}$ vertices in $S$ form the 2-shortest path union cover for $S_n$. Assume that $T \subseteq V(G)$ such that $|T| \leq |S|$ forms the 2-shortest path union cover. $T$ contains any two merged vertices in each $S_3$ of $S_n$. There exists 7 edges left uncovered in each $S_3$ of $S_n$ by the vertices in $T$ as shown in 20(b). This implies that $T$ does not form the 2-shortest path union cover for $S_n$. Hence $S$ forms the 2-shortest path union cover for $S_n$ and by Remark 2.8 2.9, $SPC_{2U}(S_n) \leq 3^{n-2}$. 

**Theorem 5.20.** Let $G$ be the Sierpiński gasket Graph $S_n$, $n \geq 3$, then $SSPC_{2U}(S_n) \leq 6(3^{n-3})$.

**Proof.** Let $S_n$ be the $n$-dimensional Sierpiński gasket Graph $S_n$, $n \geq 3$. In $S_3$, the three copies of $S_2$ are merged to each other and in $S_4$, the three copies of $S_3$, are merged to each other. Similarly in $S_n$, the three copies of $S_{n-1}$, are merged to each other. In each $S_3$, choose three merged vertices and the vertices between the merged vertices which cover the edges at distance 2 and form the 2-strong shortest path union cover as shown in 21(b). The similar pattern of choosing the vertices in each copy of $S_3$ is followed in $S_n$. Proceeding like this, there exists $3^{(n-3)}$ copies of $S_3$, in $S_n$. Let $S$ be
the 2-strong shortest path union cover of $G$ and $S$ contains three merged vertices and the vertices between the merged vertices in each copy of $S_3$ which cover the edges at distance 2 in $S_n$. Assume that $T \subseteq V(G)$ such that $|T| \leq |S|$ forms the 2 shortest and strong shortest path union cover. $T$ contains only the merged vertices in each $S_3$ of $S_n$. There exists at least one edge left uncovered in each copy of $S_2$ in $S_n$ by the vertices in $T$ as shown in Figure 21(a). This implies that $T$ does not form the 2-strong shortest path union cover for $S_n$. Hence $S$ forms the 2-strong shortest path union cover for $S_n$ and by Remark 2.8, 2.9, $SSPC_{2U}(S_n) \leq 6(3^n-3) \, \square$

**Remark 5.21.** Let $G$ be the Sierpiński gasket Graph $S_n$, $n = 2$, then $SPC_{2U}(S_n) = S_{SSPC_{2U}(S_n)} = 2$.

### 6 Conclusion

In this manuscript we have determined the complexity results for $k$-strong shortest path union cover, the 2-strong shortest path union cover for general graphs, various networks, Sierpiński graphs and Sierpiński gasket graphs. Further $k$-strong shortest path union cover for other product graphs and networks are under consideration.

### References

[1] Introduction to graph theory, Discrete Mathematics, 37, 133, 1981.

[2] Lewis, R. Harry, *Computers and intractability. A guide to the theory of NP-completeness*, (1983): 498–500.

[3] Peter Boothe, Zdeněk Dvořák Arthur M. Farley Andrzej Proskurowski, *Graph Covering via Shortest Paths*, Congr.Number. **87** (2007) 145–155.
[4] S. Theresal, A. Xavier, Deepa Mathew *Strong Shortest Path Union Cover for some Networks and Sierpiński graphs*, Malaya Journal of Matematik. 1 (2020) 104–110.

[5] A. Xavier, S. Theresal, Deepa Mathew *Strong Shortest Path Union Cover for Certain Graphs*, Adalya Journal. 9 (2) (2020) 1086–1100.

[6] S. Theresal, A. Xavier, Deepa Mathew, *Strong Shortest Path Union Cover Problem for Product graphs*, AIP Conf. Proc 2261 (1) (2020) 030009-1 – 030009-14.

[7] Xu, J., Topological Structures and Analysis of Interconnection Networks, Kluwer Academic Publishers (2001).