Kernel-based collocation methods for Zakai equations

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Abstract

We examine an application of the kernel-based interpolation to numerical solutions for Zakai equations in nonlinear filtering, and aim to prove its rigorous convergence. To this end, we find the class of kernels and the structure of collocation points explicitly under which the process of iterative interpolation is stable. This result together with standard argument in error estimation shows that the approximation error is bounded by the order of the square root of the time step and the error that comes from a single step interpolation. Our theorem is well consistent with the results of numerical experiments.

Key words: Zakai equations, kernel-based interpolation, stochastic partial differential equations, radial basis functions.

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1 Introduction

We are concerned with numerical methods for Zakai equations, linear stochastic partial differential equations of the form

\[ du(t, x) = L_0 u(t, x)dt + \sum_{k=1}^{m} L_k u(s, x)dW_k(t), \quad 0 \leq t \leq T, \]

with initial condition \( u(0, x) = u_0(x) \), where the process \( \{W(t) = (W_1(t), \ldots, W_m(t))\}_{0 \leq t \leq T} \) is an \( m \)-dimensional standard Wiener process on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Here, for each \( k = 0, 1, \ldots, m \), the partial differential operator \( L_k \) is given by

\[ L_0 f(x) = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x)f(x)) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (b_i(x)f(x)), \]

\[ L_k f(x) = \beta_k(x)f(x) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (\gamma_{ik}(x)f(x)), \quad k = 1, \ldots, m, \]
where \( a = (a_{ij}) \) is \( \mathbb{R}^{d \times d} \)-valued, \( b = (b_i) \) is \( \mathbb{R}^d \)-valued, \( \beta = (\beta_k) \) is \( \mathbb{R}^m \)-valued, \( \gamma = (\gamma_{ik}) \) is \( \mathbb{R}^{d \times m} \)-valued, and \( u_0 \) is \( \mathbb{R} \)-valued, all of which are defined on \( \mathbb{R}^d \). The conditions for these functions are described in Section 2 below.

It is well known that solving Zakai equations is amount to computing the optimal filter for diffusion processes. We refer to Rozovskii [19], Kunita [15], Liptser and Shiryaev [16], Bensoussan [3], Bain and Crisan [2], and the references therein for Zakai equations and their relation with nonlinear filtering. It is also well known that for linear diffusion processes the optimal filters allow for finite dimensional realizations, i.e., they can be represented by some stochastic and deterministic differential equations in finite dimensions. For nonlinear diffusion processes, it is difficult to obtain such realizations except some special cases (see Beneš [3] and [4]). Thus one may be led to numerical approach to Zakai equations for computing the optimal filter. Several efforts have been made to obtain approximation methods for the equations during the past several decades. For example, the finite difference method (see Yoo [23], Gyöngy [11] and the references therein), the particle method (see Crisan et al. [6]), a series expansion approach (Lototsky et al. [17]), Galerkin type approximation (Ahmed and Yoo [23], Gyöngy [11] and the references therein), the particle method (see Crisan et al. [6]), a series expansion approach (Lototsky et al. [17]), Galerkin type approximation (Ahmed and Radaideh [11] and Frey et al. [8]) and the splitting up method (Bensoussan et al. [5]).

In the present paper, we examine the approximation of \( u(t,x) \) by a collocation method with kernel-based interpolation. Given a points set \( \Gamma = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d \) and a positive definite function \( \Phi : \mathbb{R}^d \to \mathbb{R} \), the function

\[
I(f)(x) := \sum_{j=1}^{N} (A^{-1} f|\Gamma)_j \Phi(x - x_j), \quad x \in \mathbb{R}^d,
\]

interpolates \( f \) on \( \Gamma \). Here, \( A = \{\Phi(x_j - x_\ell)\}_{j,\ell=1,\ldots,N} \) is the column vector composed of \( f(x_j) \), \( j = 1, \ldots, N \), and \( (A^{-1} z)_j \) denotes the \( j \)-th component of \( A^{-1} z \) for \( z \in \mathbb{R}^N \). Thus, with time grid \( \{t_0, \ldots, t_n\} \), the function \( u^h \) recursively defined by

\[
u^h(t_i, x) = u^h(t_{i-1}, x) + L_0 I(u^h(t_{i-1}, \cdot))(x)(t_i - t_{i-1}) + \sum_{k=1}^{m} L_k I(u^h(t_{i-1}, \cdot))(x)(W_k(t_i) - W_k(t_{i-1})), \quad i = 0, \ldots, n, \quad x \in \mathbb{R}^d,
\]

is a good candidate for an approximate solution of (1.1). The approximation above can be seen as a kernel-based (or meshfree) collocation method for stochastic partial differential equations. The meshfree collocation method is proposed by Kansa [13], where deterministic partial differential equations are concerned. Since then many studies on numerical experiments and practical applications for this method are generated. As for rigorous convergence, Schaback [20] and Nakano [18] study the case of deterministic linear operator equations and fully nonlinear parabolic equations, respectively. However, at least for parabolic equations, there is little known about explicit examples of the grid structure and kernel functions that ensure rigorous convergence. An exception is Hon et al. [12], where an error bound is obtained for a special heat equation in one dimension. A main difficulty lies in handling the process of the iterative kernel-based interpolation. A straightforward estimates for \( |I(f)(x)| \) involves the condition number of the matrix \( A \), which in general rapidly diverges to infinity (see Wendland [22]). Thus we need to take a different route. Our main idea is to introduce a

\[\text{(continued on next page)}\]
condition on the decay of the cardinal function with respect to the interpolant and to choose
an appropriate approximation domain whose radius goes to infinity such that the interpolation
is still effective. From this together with standard argument in error estimation we find
that the approximation error is bounded by the order of the square root of the time step
and the error that comes from a single step interpolation. See Lemma 3.5 and Theorem 3.3
below.

The structure of this paper is as follows: Section 2 introduces some notation, and describes
the basic results for Zakai equations and the kernel-based interpolation, which are used in this
paper. We derive an approximation method for Zakai equations and prove its convergence
in Section 3. Numerical experiments are performed in Section 4.

2 Preliminaries

2.1 Notation
Throughout this paper, we denote by $a^T$ the transpose of a vector or matrix $a$. For $a = (a_i) \in \mathbb{R}^d$ we set $|a| = (\sum_{i=1}^{d} (a_i)^2)^{1/2}$. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ of nonnegative integers, the differential operator $D^\alpha$ is defined as usual by

$$D^\alpha = \frac{\partial^{|\alpha|_1}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

with $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$. For an open set $\mathcal{O} \subset \mathbb{R}^d$, we denote by $C^\kappa(\mathcal{O})$ the space of continuous real-valued functions on $\mathcal{O}$ with continuous derivatives up to the order $\kappa \in \mathbb{N}$, with the norm

$$\|f\|_{C^\kappa(\mathcal{O})} = \max_{|\alpha|_1 \leq \kappa} \sup_{x \in \mathcal{O}} |D^\alpha f(x)|.$$

Further, we denote by $C_0^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact supports. For any $p \in [1, \infty)$ and any open set $\mathcal{O} \subset \mathbb{R}^d$, we denote by $L^p(\mathcal{O})$ the space of all measurable functions $f : \mathcal{O} \to \mathbb{R}$ such that

$$\|f\|_{L^p(\mathcal{O})} := \left\{ \int_{\mathcal{O}} |f(x)|^p dx \right\}^{1/p} < \infty.$$

For $\kappa \in \mathbb{N}$, we write $H^\kappa(\mathcal{O})$ for the space of all measurable functions $f$ on $\mathcal{O}$ such that the generalized derivatives $D^\alpha f$ exist for all $|\alpha|_1 \leq \kappa$ and that

$$\|f\|_{H^\kappa(\mathcal{O})} := \sum_{|\alpha|_1 \leq \kappa} \|D^\alpha f\|^2_{L^2(\mathcal{O})} < \infty.$$

In addition, for $0 < r < 1$, we write $H^{\kappa+r}(\mathcal{O})$ for the space of all measurable functions $f$ on $\mathcal{O}$ such that the generalized derivatives $D^\alpha u$ exist for all $|\alpha|_1 \leq \kappa$ and that $\|f\|^2_{H^{\kappa+r}(\mathcal{O})} := \|f\|^2_{H^\kappa(\mathcal{O})} + |f|^2_{H^{\kappa+r}(\mathcal{O})} < \infty$ with

$$|f|^2_{H^{\kappa+r}(\mathcal{O})} = \sum_{|\alpha|_1 = \kappa} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|D^\alpha f(x) - D^\alpha f(y)|^2}{|x - y|^{d+2r}} dx dy.$$
For $x \in \mathbb{R}$ we use the notation $|x| = \max\{n \in \mathbb{Z} : n \leq x\}$. By $C$ we denote positive constants that may vary from line to line and that are independent of $h$ introduced below.

### 2.2 Zakai equations

We impose the following conditions on the coefficients of the equation (1.1):

**Assumption 2.1.** (i) All components of the functions $a$, $b$, $\beta$, $\gamma$, and $u_0$ are infinitely differentiable with bounded continuous derivatives of any order.

(ii) For any $x \in \mathbb{R}^d$,

$$
\xi^T(a(x) - \gamma(x)\gamma(x)^T)\xi \geq 0, \quad \xi \in \mathbb{R}^d.
$$

It follows from Assumption 2.1 and Gerencsér et al. [10, Theorem 2.1] that there exists a unique predictable process $\tilde{u}(t)$ such that the following are satisfied:

(i) $u(t, \cdot, \omega) \in H^\nu(\mathbb{R}^d)$ for any $(t, \omega) \in [0, T] \times \Omega_0$, where $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ and for any $\nu \in \mathbb{N};$

(ii) for $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$(2.1)$$

$$(u(t), \varphi) = (u_0, \varphi) + \int_0^t (u(s, \cdot), L^*_0\varphi)ds + \sum_{k=1}^m \int_0^t (u(s, \cdot), L^*_k\varphi)dW_k(s), \quad 0 \leq t \leq T, \quad \text{a.s.}$$

Here, $(\cdot, \cdot)$ denotes the inner product in $L^2(\mathbb{R}^d)$, and for each $k = 0, 1, \ldots, m$, the partial differential operator $L^*_k$ is the formal adjoint of $L_k$. Moreover, $u(t, x)$ satisfies

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|u(t, \cdot)\|^2_{H^\nu(\mathbb{R}^d)} \right] \leq C\|u_0\|^2_{H^\nu(\mathbb{R}^d)}, \quad \nu \in \mathbb{N}.
$$

Further, as in [10, Proposition 3, Section 1.3, Chapter 4], there exists a version $\tilde{u}$, with respect to $x$, of $u$ such that $\tilde{u}(t, x, \omega) \in C^\infty(\mathbb{R}^d)$ for $(t, \omega) \in [0, T] \times \Omega$ and that for any $\kappa \in \mathbb{N}$ and $|\alpha|_1 \leq \kappa$,

$$(2.2)$$

$$
D^\alpha \tilde{u}(t, x) = D^\alpha u_0(x) + \int_0^t D^\alpha L_0 \tilde{u}(s, x)ds + \sum_{k=1}^m \int_0^t D^\alpha L_k \tilde{u}(s, x)dW_k(s), \quad \text{a.s.,} \quad (t, x) \in [0, T] \times \mathbb{R}^d.
$$

In particular, $\tilde{u}$ is a solution to the Zakai equation in the strong sense, i.e., $\tilde{u}$ satisfies

$$
\tilde{u}(t, x) = \tilde{u}_0(x) + \int_0^t L_0 \tilde{u}(s, x)ds + \sum_{k=1}^m \int_0^t L_k \tilde{u}(s, x)dW_k(s), \quad \text{a.s.,} \quad (t, x) \in [0, T] \times \mathbb{R}^d.
$$

We remark that in (2.2) the stochastic integral is taken to be a continuous version with respect to $(t, x)$. With this version, (2.2) holds with probability one uniformly on $[0, T] \times \mathbb{R}^d$. 

\[4\]
2.3 Kernel-based interpolation

In this subsection, we recall the basis of the interpolation theory with positive definite functions. We refer to [22] for a complete account. Let \( \Phi : \mathbb{R}^d \rightarrow \mathbb{R} \) be a radial and positive definite function, i.e., \( \Phi(\cdot) = \phi(\| \cdot \|) \) for some \( \phi : [0, \infty) \rightarrow \mathbb{R} \) and for every \( \ell \in \mathbb{N} \), for all pairwise distinct \( y_1, \ldots, y_\ell \in \mathbb{R}^d \) and for all \( \alpha = (\alpha_i) \in \mathbb{R}^\ell \setminus \{0\} \), we have

\[
\sum_{i,j=1}^\ell \alpha_i \alpha_j \Phi(y_i - y_j) > 0.
\]

Let \( \Gamma = \{x_1, \cdots, x_N\} \) be a finite subset of \( \mathbb{R}^d \) and put \( A = \{\Phi(x_i - x_j)\}_{1 \leq i, j \leq N} \). Then \( A \) is invertible and thus for any \( g : \mathbb{R}^d \rightarrow \mathbb{R} \) the function

\[
I(g)(x) = \sum_{j=1}^N (A^{-1}(g|_\Gamma))_j \Phi(x - x_j), \quad x \in \mathbb{R}^d,
\]

interpolates \( g \) on \( \Gamma \). If \( \Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \), then

\[
\mathcal{N}_\Phi(\mathbb{R}^d) := \left\{ f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) : \frac{\hat{f}/\sqrt{\Phi}}{\Phi} \in L^2(\mathbb{R}^d) \right\},
\]

called the native space, is a real Hilbert space with the inner product

\[
(f, g)_{\mathcal{N}_\Phi(\mathbb{R}^d)} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{f}(\xi) \frac{\hat{g}(\xi)}{\Phi(\xi)} d\xi
\]

and the norm \( \|f\|_{\mathcal{N}_\Phi(\mathbb{R}^d)} := (f, f)_{\mathcal{N}_\Phi(\mathbb{R}^d)} \). Here, for \( f \in L^1(\mathbb{R}^d) \), the function \( \hat{f} \) is the Fourier transform of \( f \), defined as usual by

\[
\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi^T x} dx, \quad \xi \in \mathbb{R}^d.
\]

Moreover, \( \mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto \Phi(x - y) \) is a reproducing kernel for \( \mathcal{N}_\Phi(\mathbb{R}^d) \). If \( \Phi \) satisfies

\[
c_1 (1 + |\xi|^2)^{-\kappa} \leq \Phi(\xi) \leq c_2 (1 + |\xi|^2)^{-\kappa}, \quad \xi \in \mathbb{R}^d,
\]

for some constants \( c_1, c_2 > 0 \) and \( \kappa > d/2 \), then we have from Corollary 10.13 in [22] that \( H^\kappa(\mathbb{R}^d) = \mathcal{N}_\Phi(\mathbb{R}^d) \) and

\[
c_1 \|f\|_{H^\kappa(\mathbb{R}^d)} \leq \|f\|_{\mathcal{N}_\Phi(\mathbb{R}^d)} \leq c_2 \|f\|_{H^\kappa(\mathbb{R}^d)}, \quad f \in H^\kappa(\mathbb{R}^d).
\]

Namely, the native space \( \mathcal{N}_\Phi(\mathbb{R}^d) \) coincides with the Sobolev space \( H^\kappa(\mathbb{R}^d) \) with equivalent norm. Further, we mention that (2.4) and Corollary 10.25 in [22] implies

\[
\|I(g)\|_{H^\kappa(\mathbb{R}^d)} \leq C\|g\|_{H^\kappa(\mathbb{R}^d)}, \quad \|g - I(g)\|_{H^\kappa(\mathbb{R}^d)} \leq C\|g\|_{H^\kappa(\mathbb{R}^d)}, \quad g \in H^\kappa(\mathbb{R}^d).
\]
The so-called Wendland kernel is a typical example of $\Phi$ satisfying $(2.3)$–$(2.5)$, which is defined as follows: for a given $\tau \in \mathbb{N}$, set the function $\Phi_{d,\tau}$ satisfying $\Phi_{d,\tau}(x) = \phi_{d,\tau}(|x|)$, $x \in \mathbb{R}^d$, where

$$
\phi_{d,\tau}(r) = \int_{r}^{\infty} \phi_{d,\tau}(r \tau) \phi_{d,\tau}(r \tau - 1) \cdots \phi_{d,\tau}(r \tau_{r-1}) \int_{0}^{\infty} r_1 \max\{1 - r_1, 0\}^{\nu} dr_1 dr_2 \cdots dr_{\tau}, \quad r \geq 0
$$

with $\nu = [d/2] + \tau + 1$. For example,

$$
\phi_{1,2}(r) = \max\{1 - r, 0\}^5(8r^2 + 5r + 1),
\phi_{1,3}(r) = \max\{1 - r, 0\}^7(21r^3 + 19r^2 + 7r + 1),
\phi_{1,4}(r) = \max\{1 - r, 0\}^9(384r^4 + 453r^3 + 237r^2 + 63r + 7),
\phi_{2,4}(r) = \max\{1 - r, 0\}^{10}(429r^4 + 450r^3 + 210r^2 + 50r + 5),
\phi_{2,5}(r) = \max\{1 - r, 0\}^{12}(2048r^5 + 2697r^4 + 1644r^3 + 566r^2 + 108r + 9),
$$

where $\hat{=}$ denotes equality up to a positive constant factor.

Then, $\Phi_{d,\tau} \in C^2(\mathbb{R}^d)$ and $N_{\Phi_{d,\tau}}(\mathbb{R}^d) = H^{\nu+(d+1)/2}(\mathbb{R}^d)$. Furthermore, $\Phi_{d,\tau}$ satisfies $(2.3)$–$(2.5)$ with $\kappa = \tau + (d + 1)/2$.

3 Collocation method for Zakai equations

Let us describe the collocation methods for $(2.1)$. In what follows, we always consider the version of $u$, and thus by abuse of notation, we write $u$ for $\tilde{u}$. Moreover, we restrict ourselves to the class of Wendland kernels $\Phi = \Phi_{d,\tau}$ described in Section 2.2. We choose a set $\Gamma = \{x_1, \ldots, x_N\}$ consisting of pairwise distinct points such that

$$
\Gamma = \{x_1, \ldots, x_N\} \subset (-R, R)^d,
$$

for some $R > 0$.

To construct an approximate solution of Zakai equation, we first take a set $\{t_0, \ldots, t_n\}$ of time discretized points such that $0 = t_0 < t_1 < \cdots < t_n = T$. The solution $u$ of the Zakai equation approximately satisfies

$$
u(t_i, x) \approx u(t_{i-1}, x) + \sum_{k=0}^{m} L_ku(t_{i-1}, x) \Delta W_k(t_i),
$$

where $W_0(t) = t$ and $\Delta W_k(t_i) = W_k(t_i) - W_k(t_{i-1})$. Since $L_ku(t_{i-1}, x) \approx L_kI(u(t_{i-1}, \cdot))(x)$, we see

$$
u(t_i, x) \approx u(t_{i-1}, x) + \sum_{k=0}^{m} L_kI(u(t_{i-1}, \cdot))(x) \Delta W_k(t_i).
$$

Thus, we define the function $u^h$, a candidate of an approximate solution parametrized with a parameter $h > 0$, by

$$
u(t_0, x) = u_0(x), \quad x \in \mathbb{R}^d,
$$

$$(3.1) \quad u^h(t_i, x) = u^h(t_{i-1}, x) + \sum_{k=0}^{m} L_kI(u^h(t_{i-1}, \cdot))(x) \Delta W_k(t_i), \quad x \in \mathbb{R}^d, \quad i = 1, \ldots, n.$$
With this definition, the $N$-dimensional vector $u^h_i = (u^h_{i,1}, \ldots, u^h_{i,N})^T$ of the collocation points satisfies
\[ u^h_0 = (u_0(x_1), \ldots, u_0(x_N))^T, \]
\[ u^h_i = u^h_{i-1} + \sum_{k=0}^{m} (A_k A^{-1} u^h_{i-1}) \Delta W_k(t_i), \quad i = 1, \ldots, n. \]

Here, we have set $A_k = (A_{k,j}\ell_{1,\ell})_{j,\ell=1}^N$ with $A_{k,j}\ell x_{\ell} | x_j$. This follows from
\[ L_k u^h(t_i, x_j) = \sum_{\ell=1}^{N} (A^{-1} u^h_{i})_\ell L_k \Phi(x - x_{\ell})|_{x=x_j} = (A_k A^{-1} u^h_{i})_j. \]

To discuss the error of the approximation above, set $\Delta t = \max_{1 \leq i \leq n} (t_i - t_{i-1})$ and consider the Hausdorff distance $\Delta x$ between $\Gamma$ and $(-R, R)^d$ by
\[ \Delta x = \sup_{x \in (-R, R)^d} \min_{j=1, \ldots, N} |x - x_j|. \]

Then suppose that $\Delta t$, $R$, $N$, and $\Delta x$ are functions of $h$. Let $\tilde{\Gamma}$ be a finite subset of $(-R, R)^d$, the set of points at which the approximate solution is to be evaluated, which may be different from $\Gamma$. For $j = 1, \ldots, N$, we write $Q_j$ for the cardinal function defined by
\[ Q_j(x) = \sum_{i=1}^{N} (A^{-1})_{ij} \Phi(x - x_i), \quad x \in \mathbb{R}^d, \quad j = 1, \ldots, N. \]

In what follows, $\# \mathcal{K}$ denotes the cardinality of a finite set $\mathcal{K}$.

**Assumption 3.1.** (i) The parameters $\Delta t$, $R$, $N$, and $\Delta x$ satisfy $\Delta t \to 0$, $R \to \infty$, $N \to \infty$, and $\Delta x \to 0$ as $h \to 0$.

(ii) There exist $c_1, c_2, c_3$, positive constants independent of $h$, such that for any $|\alpha|_1 \leq 2$,
\[ \max_{x \in \Gamma \cup \tilde{\Gamma}} \# \left\{ j \in \{1, \ldots, N\} : |D^\alpha Q_j(x)| > \frac{c_1}{N} \right\} \leq c_2 R^{1/2} \leq c_3 (\Delta x)^{-1/2}. \]

**Remark 3.2.** Suppose that $\Gamma$ is quasi-uniform in the sense that
\[ c_4 R N^{-1/d} \leq \Delta x \leq c_5 R N^{-1/d} \]
hold for some positive constants $c_4, c_5$. In this case, a sufficient condition for which the latter inequality in Assumption 3.1 (ii) holds is
\[ R \leq c_6 N^{(1-d/(d+2\tau-3))^{1/2}} \]
for some positive constant $c_6$.

The approximation error for the Zakai equation is estimated as follows:
Theorem 3.3. Suppose that Assumptions [2.1] and [3.1] hold. Suppose moreover that \( \tau \geq 3 \).
Then, there exists \( h_0 > 0 \) such that

\[
\max_{i=1,\ldots,n} \max_{x \in \Gamma \cup \tilde{\Gamma}} \mathbb{E} \left[ |u(t_i,x) - u^h(t_i,x)|^2 \right] \leq C \left( \Delta t + (\Delta x)^{2\tau-3} \right), \quad h \leq h_0.
\]

The rest of this section is devoted to the proof of Theorem 3.3. We start with the following lemma:

Lemma 3.4. Suppose that Assumption [3.1] (i) and \( \tau \geq 3 \) hold. Then, there exists \( h_0 > 0 \) such that for any multi-index \( \alpha \) with \( |\alpha|_1 \leq 2 \) and \( f \in H^{\tau+(d+1)/2}(\mathbb{R}^d) \), we have

\[
\|D^\alpha f - D^\alpha I(f)\|_{L^\infty(-R,R)^d} \leq C(\Delta x)^{\tau+1/2-|\alpha|_1} \|f\|_{H^{\tau+(d+1)/2}(\mathbb{R}^d)}, \quad h \leq h_0.
\]

Proof. This result is reported in [22, Corollary 11.33] for more general domains. However, a simple application of that result leads to an ambiguity of the dependence of the constant \( C \) on \( R \). Here we will confirm that we can take \( C \) to be independent of \( R \).

Let \( f \in H^{\tau+(d+1)/2}(\mathbb{R}^d) \) with \( f|\Gamma = 0 \). Set \( \tilde{\Gamma} = \{x_1/R, \ldots, x_N/R\} \) and \( \tilde{f}(z) = f(Rz), \ z \in (-1,1)^d \). Then, \( \tilde{f}|_{\Gamma} = f|_{\Gamma} = 0 \) and

\[
\sup_{z \in (-1,1)^d} \min_{\xi \in \tilde{\Gamma}} |\xi - y| = \sup_{y \in (-R,R)^d} \min_{j=1,\ldots,N} \left| \frac{x_j}{R} - \frac{y}{R} \right| = \frac{\Delta x}{R}.
\]

Since \( \Delta x/R \to 0 \) as \( h \searrow 0 \) and \( \tau \geq 3 \), we can apply [22, Theorem 11.32] to \( \tilde{f} \) to obtain

\[
|D^\alpha \tilde{f}(z)| \leq C(\Delta x/R)^{\tau+1/2-|\alpha|_1} |\tilde{f}|_{H^{\tau+(d+1)/2}((-1,1)^d)}, \quad h \leq h_0
\]

for some \( h_0 > 0 \). It is straightforward to see that

\[
D^\alpha \tilde{f}(z) = R^{|\alpha|_1} (D^\alpha f)(Rz), \quad |\tilde{f}|_{H^{\tau+(d+1)/2}((-1,1)^d)} = R^{\tau+1/2} |f|_{H^{\tau+(d+1)/2}((-R,R)^d)}.
\]

Substituting these relations into (3.2), we have

\[
|D^\alpha f(y)| \leq C(\Delta x)^{\tau+1/2-|\alpha|_1} |f|_{H^{\tau+(d+1)/2}((-R,R)^d)}, \quad y \in (-R,R)^d.
\]

This and (2.5) yield

\[
\|D^\alpha f - D^\alpha I(f)\|_{L^\infty((-R,R)^d)} \leq C(\Delta x)^{\tau+1/2-|\alpha|_1} |f - I(f)|_{H^{\tau+(d+1)/2}((-R,R)^d)} \leq C(\Delta x)^{\tau+1/2-|\alpha|_1} \|f\|_{H^{\tau+(d+1)/2}(\mathbb{R}^d)}.
\]

Thus the lemma follows. \( \square \)

Observe that for any \( f : \mathbb{R}^d \to \mathbb{R} \),

\[
I(f)(x) = \sum_{j=1}^N (A^{-1}f|_\Gamma)_j \Phi(x - x_j) = \sum_{j=1}^N f(x_j)Q_j(x), \quad x \in \mathbb{R}^d.
\]

The following result tells us that the process of iterative kernel-based interpolation is stable on \( \Gamma \cup \tilde{\Gamma} \), which is a key to our convergence analysis.

8
Lemma 3.5. Suppose that Assumption 3.1 and \( \tau \geq 3 \) hold. Then, there exists \( h_0 > 0 \) such that

\[
\sup_{0 < h \leq h_0} \max_{x \in \Gamma \cup \hat{\Gamma}} N \sum_{j=1}^{\infty} |D^\alpha Q_j(x)| < \infty, \quad |\alpha|_1 \leq 2.
\]

Proof. Fix \( \tilde{x} \in \Gamma \cup \hat{\Gamma} \) and \( |\alpha|_1 \leq 2 \). First consider the set

\[
\mathcal{J}(\tilde{x}) := \{ j \in \{1, \ldots, N\} : |D^\alpha Q_j(\tilde{x})| \leq \frac{c_1}{N} \}.
\]

Then of course we have

\[
\sum_{j \in \mathcal{J}(\tilde{x})} |D^\alpha Q_j(\tilde{x})| \leq c_1.
\]

By Assumption 3.1 (ii), there exists \( \tilde{\nu} \in \mathbb{N} \) such that

\[
\#\{ j : j \notin \mathcal{J}(\tilde{x}) \} \leq \tilde{\nu}[R^{1/2}].
\]

Then, by Kergin interpolation (see Kergin [14]) there exists a polynomial \( p \) on \( \mathbb{R}^d \) with degree at most \( \tilde{\nu}[R^{1/2}] \) that interpolates \( \text{sgn}(D^\alpha Q_j(\tilde{x})) \) at \( x_j \) for all \( j \notin \mathcal{J}(\tilde{x}) \). This leads to

\[
\sum_{j \notin \mathcal{J}(\tilde{x})} |D^\alpha Q_j(\tilde{x})| = \sum_{j \notin \mathcal{J}(\tilde{x})} \text{sgn}(D^\alpha Q_j(\tilde{x})) D^\alpha Q_j(\tilde{x}) = \sum_{j \notin \mathcal{J}(\tilde{x})} p(x_j) D^\alpha Q_j(\tilde{x}).
\]

Bernstein inequality (see Proposition 11.6 in [22]) and Assumption 3.1 (ii) implies that

\[
\max_{|\alpha| \leq \nu_1 + 1} \sup_{x \in (-R, R)^d} |D^\alpha p(x)| \leq C \sup_{x \in (-R, R)^d} |p(x)|,
\]

where \( \nu_1 = \tau + \min\{\kappa \in \mathbb{Z} : \kappa \geq (d + 1)/2\} \). Thus, for \( x \in (-R, R)^d \), take a nearest \( x_j \) to observe

\[
|p(x)| \leq \max_{y \in (-R, R)^d} |p(y)|, \quad \text{and} \quad |p(x) - p(x_j)| \leq 1 + \Delta x \sup_{y \in (-R, R)^d} |p(y)|,
\]

from which and \( \Delta x \to 0 \) this polynomial \( p \) satisfies \( |p(x)| \leq 2 \) for \( x \in (-R, R)^d \). Further, we have

\[
\max_{|\alpha| \leq \nu_1 + 1} \sup_{x \in [-R, R]^d} |D^\alpha p(x)| \leq C_0 \quad \text{for some} \quad C_0 > 0 \quad \text{that is independent of} \quad R \quad \text{and} \quad \tilde{x}.
\]

Then, by Whitney’s extension theorem (see, e.g., Stein [21]), there exists a function \( \tilde{p} \) on \( \mathbb{R}^d \) such that \( \tilde{p} = p \) on \( [-R, R]^d \) and

\[
\max_{|\alpha| \leq \nu_1} \sup_{x \in [-R, R]^d} |D^\alpha \tilde{p}(x)| \leq C_0.
\]

Then consider the function \( \hat{p} \in H^{\nu_1}(\mathbb{R}^d) \) defined by \( \hat{p}(x) = \tilde{p}(x)\zeta(x/R), \) \( x \in \mathbb{R}^d \) where \( \zeta \) is a \( C^\infty \)-function such that \( 0 \leq \zeta(x) \leq 1 \) for \( x \in \mathbb{R}^d \), \( \zeta(x) = 1 \) for \( |x| \leq 1 \), and \( \zeta(x) = 0 \) for \( |x| > 1 + \hat{c} \) for some \( \hat{c} > 0 \).
With these modifications and in view of \((3.4)\) and \((3.5)\), we obtain
\[
\sum_{j=1}^{N} |D^\alpha Q_j(\tilde{x})| \leq \sum_{j \notin \mathcal{J}(\tilde{x})} p(x_j)D^\alpha Q_j(\tilde{x}) + C = \sum_{j=1}^{N} p(x_j)D^\alpha Q_j(\tilde{x}) - \sum_{j \notin \mathcal{J}(\tilde{x})} p(x_j)D^\alpha Q_j(\tilde{x}) + C
\]
\[
\leq \sum_{j=1}^{N} p(x_j)D^\alpha Q_j(\tilde{x}) + 2 \sum_{j \notin \mathcal{J}(\tilde{x})} |D^\alpha Q_j(\tilde{x})| + C
\]
\[
\leq \sum_{j=1}^{N} \hat{p}(x_j)D^\alpha Q_j(\tilde{x}) + C.
\]
Moreover, by Lemma 3.4,
\[
|D^\alpha I(\hat{p})(\tilde{x})| \leq |D^\alpha I(\hat{p})(\tilde{x}) - D^\alpha \hat{p}(\tilde{x})| + |D^\alpha \hat{p}(\tilde{x})|
\leq C(\Delta x)^{-3/2} \|\hat{p}\|_{H^{\nu+d+1/2}(\mathbb{R}^d)} + C
\leq C(\Delta x)^{-3/2} \|\hat{p}\|_{H^{\nu+((1+\tilde{c})R,(1+\tilde{c})R_\Gamma^d)} + C
\leq C \|\hat{p}\|_{C^\nu(\mathbb{R}^d)} R^{d/2}(\Delta x)^{-3/2} + C.
\]
Assumption 3.1 (ii) and the boundedness of \(\|\hat{p}\|_{C^\nu(\mathbb{R}^d)}\) now lead to the conclusion of the lemma. \(\square\)

Remark 3.6. One might ask if Assumption 3.1 (ii) can be simplified in some sense. This problem, however, seems to be nontrivial. For example, the classical result by Demko et al \([7]\) tells us that if a matrix \(A\) is \(m\)-banded, symmetric and positive definite then we have
\[
|\langle (A^{-1})_{ij} \rangle | \leq \frac{2}{\lambda_{\min}} \left( 1 - \frac{2}{\sqrt{r} + 1} \right) \frac{q^r}{r^{|i-j|}}.
\]
Here, \(\lambda_{\min}\) and \(r\) are the minimum eigenvalue and the condition number of \(A\), respectively.

Our interpolation matrix satisfies \(\lambda_{\min} \geq C q^{2r+1}\) and \(r \leq C q^{-2r-d-1}\), where \(q = \min_{i \neq j} |x_i - x_j|\). Moreover, if the matrix is banded, then it is necessarily \(Cq^{-d}\)-banded. So a sufficient condition for which \(|\langle (A^{-1})_{ij} \rangle | \leq C q^d/N\) holds is
\[
q^{-2r-1} \exp(-C q^{r+3d+1/2} |i - j|) \leq C q^d/N.
\]
This is equivalent to \(|i - j| \geq C q^{r-(3d+1)/2} \log(N q^{-2r-d-1})\). The arguments in the proof of Lemma 3.7 then leads to the condition
\[
C q^{-r-1-3d+1/2} \log(N q^{-2r-d-1}) \leq \sqrt{R} \leq C(\Delta x)^{-2r-d-1},
\]
which is similar to the latter part of Assumption 3.1 (ii). If this condition holds, then we must have \((\tau - 3/2)/d \geq \tau + (5d + 1)/2\) and this is of course impossible.

Proof of Theorem 3.3. First, for \(i = 0, \ldots, n-1\) and \(x \in \Gamma \cup \tilde{\Gamma}\), we have
\[
(u(t_{i+1}, x) - u^h(t_{i+1}, x))^2 = (u(t_i, x) - u^h(t_i, x))^2 + (S_{i+1}(x))^2 + (\Theta_{i+1}(x))^2
+ 2(u(t_i, x) - u^h(t_i, x)) S_{i+1}(x) + 2\Theta_{i+1}(x) S_{i+1}(x)
+ 2(u(t_i, x) - u^h(t_i, x)) \Theta_{i+1}(x).
\]
Here, for $i = 0, \ldots, n - 1$ and $x \in \Gamma \cup \bar{\Gamma}$,

$$S_{i+1}(x) = \sum_{k=0}^{m} L_k I(u(t_i)) - u^h(t_i))(x) \Delta W^k_{t_{i+1}},$$

$$\Theta_{i+1}(x) = \sum_{k=0}^{m} \int_{t_i}^{t_{i+1}} (L_k u(s, x) - L_k I(u(t_i))(x)) dW^k_s.$$ 

It is straightforward to see that

$$\mathbb{E}(S_{i+1}(x))^2 = \mathbb{E}[L_0 I(u(t_i)) - u^h(t_i))(x)]^2(t_{i+1} - t_i)^2 + \sum_{k=1}^{m} \mathbb{E}[L_k I(u(t_i)) - u^h(t_i))(x)]^2(t_{i+1} - t_i) \leq C \sum_{|\alpha| \leq 2} \mathbb{E}|D^\alpha I(u(t_i)) - u^h(t_i))(x)|^2 \Delta t.$$

By Cauchy-Schwartz inequality and Lemma 3.5,

$$\mathbb{E}|D^\alpha I(u(t_i)) - u^h(t_i))(x)|^2 = \mathbb{E} \left| \sum_{j=1}^{N} (u(t_i, x_j) - u^h(t_i, x_j)) D^\alpha Q_j(x) \right|^2$$

$$= \sum_{j, \ell=1}^{N} \mathbb{E} [(u(t_i, x_j) - u^h(t_i, x_j))(u(t_i, x_\ell) - u^h(t_i, x_\ell))] D^\alpha Q_j(x) D^\alpha Q_\ell(x)$$

$$\leq \sum_{j, \ell=1}^{N} \left( \mathbb{E}|u(t_i, x_j) - u^h(t_i, x_j)|^2 \right)^{1/2} \left( \mathbb{E}|u(t_i, x_\ell) - u^h(t_i, x_\ell)|^2 \right)^{1/2} |D^\alpha Q_j(x)||D^\alpha Q_\ell(x)|$$

$$\leq \max_{y \in \Gamma \cup \bar{\Gamma}} \mathbb{E}|u(t_i, y) - u^h(t_i, y)|^2 \left( \sum_{j=1}^{N} |D^\alpha Q_j(x)| \right)^2$$

$$\leq C \max_{y \in \Gamma \cup \bar{\Gamma}} \mathbb{E}|u(t_i, y) - u^h(t_i, y)|^2.$$

Hence, (3.6)

$$\mathbb{E}(S_{i+1}(x))^2 \leq C \max_{y \in \Gamma \cup \bar{\Gamma}} \mathbb{E}|u(t_i, y) - u^h(t_i, y)|^2 \Delta t.$$

Next, it follows from Itô isometry that

$$\mathbb{E}|\Theta_{i+1}(x)|^2 \leq 2 \mathbb{E} \left| \int_{t_i}^{t_{i+1}} (L_0 u(s, x) - L_0 I(u(t_i))(x)) ds \right|^2 + 2 \mathbb{E} \left| \int_{t_i}^{t_{i+1}} (L_k u(s, x) - L_k I(u(t_i))(x)) dW^k_s \right|^2$$

$$\leq 2 \Delta t \mathbb{E} \int_{t_i}^{t_{i+1}} |L_0 u(s, x) - L_0 I(u(t_i))(x)|^2 ds + 2 \sum_{k=1}^{m} \mathbb{E} \int_{t_i}^{t_{i+1}} |L_k u(s, x) - L_k I(u(t_i))(x)|^2 ds$$

$$\leq C \sum_{|\alpha| \leq 2} \mathbb{E} \int_{t_i}^{t_{i+1}} |D^\alpha u(s, x) - D^\alpha I(u(t_i))(x)|^2 ds.$$
Again by Itô isometry,
\[
\mathbb{E}|D^αu(s, x) - D^αu(t_i, x)|^2 = \mathbb{E}\left[\sum_{k=0}^{m} \int_{t_i}^{s} D^αu(r, x)dW_r^k\right]^2 \leq C \mathbb{E}\left[\sum_{k=0}^{m} |D^αL_ku(r, x)|^2dr\right]
\]
\[
\leq C\mathbb{E}\sup_{0\leq t \leq T} \|u(t)\|_{C^4(\mathbb{R}^d)}^2 (s - t_i).
\]

Further, Lemma 3.4 means
\[
\mathbb{E}|D^αu(t_i, x) - D^αI(u(t_i))(x)|^2 \leq C(\Delta x)^{2(\tau - 3)}\mathbb{E}\|u\|^2_{H^{\tau + (d + 1)/2}(\mathbb{R}^d)}.
\]

Thus,
\[
(3.7) \quad \mathbb{E}|\Theta_{i+1}(x)|^2 \leq C(\Delta t)^2 + C(\Delta x)^{2(\tau - 3)}\Delta t.
\]

The arguments used in the estimations above yield
\[
\mathbb{E}(u(t_i, x) - u^h(t_i, x))S_{i+1}(x)
= \mathbb{E}(u(t_i, x) - u^h(t_i, x))L_0I(u(t_i) - u^h(t_i))(x)(t_{i+1} - t_i)
\leq (\mathbb{E}|u(t_i, x) - u^h(t_i, x)|^2)^{1/2}(\mathbb{E}|L_0I(u(t_i) - u^h(t_i))(x)|^2)^{1/2}\Delta t
\leq C \max_{y \in \Gamma \cup \Gamma} \mathbb{E}|u(t_i, y) - u^h(t_i, y)|^2\Delta t.
\]

Here we have again used the boundedness of the coefficients of $L_0$ and Lemma 3.5 to derive the last inequality. Furthermore, we obtain
\[
\mathbb{E}(u(t_i, x) - u^h(t_i, x))\Theta_{i+1}(x)
= \mathbb{E}(u(t_i, x) - u^h(t_i, x))\int_{t_i}^{t_{i+1}} (L_0u(s, x) - L_0I(u(t_i))(x))ds
\leq (\mathbb{E}|u(t_i, x) - u^h(t_i, x)|^2)^{1/2}\left(\mathbb{E}\left[\int_{t_i}^{t_{i+1}} (L_0u(s, x) - L_0I(u(t_i))(x))ds\right]^2\right)^{1/2}
\leq (\mathbb{E}|u(t_i, x) - u^h(t_i, x)|^2)^{1/2}\left(\Delta t\mathbb{E}\int_{t_i}^{t_{i+1}} |L_0u(s, x) - L_0I(u(t_i))(x)|^2ds\right)^{1/2}
= (\mathbb{E}|u(t_i, x) - u^h(t_i, x)|^2\Delta t)^{1/2}\left(\mathbb{E}\int_{t_i}^{t_{i+1}} |L_0u(s, x) - L_0I(u(t_i))(x)|^2ds\right)^{1/2}
\leq 2\mathbb{E}|u(t_i, x) - u^h(t_i, x)|^2\Delta t + 2\mathbb{E}\int_{t_i}^{t_{i+1}} |L_0u(s, x) - L_0I(u(t_i))(x)|^2ds.
\]

and so
\[
(3.9) \quad \mathbb{E}(u(t_i, x) - u^h(t_i, x))\Theta_{i+1}(x)
\leq 2 \max_{y \in \Gamma \cup \Gamma} \mathbb{E}|u(t_i, y) - u^h(t_i, y)|^2\Delta t + C(\Delta t)^2 + C(\Delta x)^{2(\tau - 3)}\Delta t.
\]
Then, from (3.6)–(3.9) we have, for \( i = 0, \ldots, n - 1 \),
\[
\max_{x \in \Gamma \cup \tilde{\Gamma}} E|u(t_{i+1}, x) - u^h(t_{i+1}, x)|^2 \\
\leq (1 + C\Delta t) \max_{x \in \Gamma \cup \tilde{\Gamma}} E|u(t_i, x) - u^h(t_i, x)|^2 + C(\Delta t)^2 + C(\Delta x)^{2r-3} \Delta t.
\]

A simple application of the discrete Gronwall lemma now leads to what we aim to prove.

4 Numerical experiments

In this section, we apply our collocation method to the one-dimensional Zakai equation

\[
\begin{align*}
\text{(4.1)} \\
du(t, x) &= \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x) - \frac{\partial}{\partial x} (\tanh(x) u(t, x)) \right) dt + u(t, x) dW(t), \quad 0 \leq t \leq 1, \\
u(0, x) &= \frac{1}{\sqrt{2\pi}} \cosh(x) e^{-|x|^2/2}.
\end{align*}
\]

The unique solution \( u(t, x) \) to (4.1) is given by
\[
u(t, x) = \frac{1}{\sqrt{2\pi}} \cosh(x) \exp \left( W(t) - \frac{3t}{2} - \frac{|x|^2}{2(1+t)} \right).
\]

We use the Wendland kernel \( \phi_{1,4} \) scaled by some positive constant for the performance test. We choose the time grid as a uniform one in \([0, 1]\), and as suggested in Remark 3.2 we define \( \Gamma \) by the uniform spatial grid points on \([-R + 2R/(N + 1), R - 2R/(N + 1)]\) where \( R = (1/5)N^{1/2(2r-2)} \), while the set of evaluation points \( \tilde{\Gamma} = \{ \xi_1, \ldots, \xi_{41} \} \) by the equi-spaced grid points on \([-2, 2]\).

To check the validity of Assumption 3.1 (ii), we plot
\[
\tau(N) = \max_{|\alpha| \leq 2} \max_{x \in \Gamma \cup \tilde{\Gamma}} \# \{ j : |D^\alpha Q_j(x)| > 6/N \}
\]
in Figure 4.1. We can see that \( \tau(N) < 12\sqrt{R} \) for all \( N \leq 1000 \). Thus, Assumption 3.1 (ii) seems to be satisfied with \( c_1 = 6 \) and \( c_2 = 12 \) for the sequence of the tuning parameters defined by \( N \) from 1 at least to 1000.

Figure 4.2 plots sample paths of \( u(\cdot, x) \) and \( u^h(\cdot, x) \) for several spatial points. Figure 4.3 plots snapshots of the time evolutions of \( u(t, \cdot) \) and \( u^h(t, \cdot) \). The both show that our collocation method yields a good approximation as well as the accumulation of the error near the time maturity cannot be negligible. To compare an averaged performance, we compute the root mean squared errors averaged over 10000 samples, defined by

\[
\text{RMSE} := \left( \frac{1}{10000 \times 41(n + 1)} \sum_{\ell=0}^{n} \sum_{j=1}^{41} \sum_{t=1}^{10000} |u_{\ell}(t_i, \xi_j) - u^h_{\ell}(t_i, \xi_j)|^2 \right)^{1/2},
\]

for several values of \( N \) and \( n \). Here, \( u_{\ell} \) and \( u^h_{\ell} \) are the exact solution and approximate solution at \( \ell \)-th trial, respectively.
Figure 4.1: Plotting $\iota(N)$ and $12\sqrt{R}$ for $N = 1, 2, \ldots, 1000$.

Figure 4.2: Comparing the exact solution (solid line) and the approximated one (dashed line) at $x = -1, -1/2, 1/2, 1$, in the case of $N = 2^5$ and $n = 2^8$. 
Figure 4.3: Comparing the exact solution (solid line) and the approximated one (dashed line) at time $= 2^1 \Delta t, 2^3 \Delta t, 2^5 \Delta t, 2^7 \Delta t$, in the case of $N = 2^5$ and $n = 2^8$. 
As another comparison, we compute numerical solutions by the implicit Euler finite difference method for the test equation, which is described as follows:

\[
\begin{align*}
\tilde{u}(0, x) &= u(0, x), \quad x \in \tilde{\Gamma}, \\
\tilde{u}(t_i, \pm R) &= 0, \quad i = 0, \ldots, n, \\
\tilde{u}(t_{i+1}, x) &= \tilde{u}(t_i, x) + \tilde{L}\tilde{u}(t_{i+1}, x) + \tilde{u}(t_i, x)\Delta W(t_{i+1}), \quad i = 0, \ldots, n - 1, \quad x \in \tilde{\Gamma}.
\end{align*}
\]

Here, \( \tilde{\Gamma} = \{-R+j\Delta x : j = 1, \ldots, N\} \) with \( \Delta x = 2R/(N+1) \) and \( \tilde{L} \) denotes the corresponding finite difference operator. Then \( \tilde{u} \) converges to \( u \) as \( R \to \infty, \Delta t \to 0, \) and \( \Delta x \to 0. \) See Gerencsér and Gyöngy [9].

Table 4.1 shows that the resulting RMSE’s are sufficiently small for all pairs \((N, n)\) although its decrease is nonmonotonic. Here \( \text{RMSE}_{\text{fd}} \) denotes the corresponding root mean squared errors for the finite difference method, where the set of evaluation points is taken to be \( \tilde{\Gamma} \) itself. We can conclude that Theorem 3.3 is well consistent with the results of our experiments, and that the kernel-based collocation method outperforms the implicit Euler finite difference method when the length \( R \)'s are set to be the ones that guarantee the convergence of the former.

| \(N\) | \(R\) | \(\Delta x^{-3/2}\) | \(n\) | \(\sqrt{\Delta t}\) | RMSE | \(\text{RMSE}_{\text{fd}}\) |
|---|---|---|---|---|---|---|
| \(2^4\) | 2.0159 | 0.0274 | \(2^6\) | 0.1250 | 0.0714 | 0.2163 |
| | | | \(2^8\) | 0.0625 | 0.0726 | 0.2171 |
| | | | \(2^{10}\) | 0.0312 | 0.0745 | 0.2267 |
| \(2^5\) | 3.5919 | 0.0221 | \(2^6\) | 0.1250 | 0.0323 | 0.1981 |
| | | | \(2^8\) | 0.0625 | 0.0252 | 0.2017 |
| | | | \(2^{10}\) | 0.0312 | 0.0234 | 0.1991 |
| \(2^6\) | 6.4000 | 0.0172 | \(2^6\) | 0.1250 | 0.0333 | 0.2020 |
| | | | \(2^8\) | 0.0625 | 0.0261 | 0.2088 |
| | | | \(2^{10}\) | 0.0312 | 0.0241 | 0.2067 |

Table 4.1: The resulting root mean squared errors for several pairs \((N, n)\).

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