A new class of $f$-deformed charge coherent states and their nonclassical properties

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Abstract

Two-mode charge (pair) coherent states have been introduced previously by using the $\langle \eta |$ representation. In this paper, we reobtain these states by a rather different method. Then, using the nonlinear coherent state approach and based on a simple manner by which the representation of two-mode charge coherent states is introduced, we generalize the bosonic creation and annihilation operators to the $f$-deformed ladder operators and construct a new class of $f$-deformed charge coherent states. Unlike the (linear) pair coherent states, our presented structure has the potential to generate a large class of pair coherent states with various nonclassicality signs and physical properties which are of interest. For this purpose, we use a few well-known nonlinearity functions associated with particular quantum systems as some physical appearances of our presented formalism. After introducing the explicit form of the above correlated states in the two-mode Fock space, several nonclassicality features of the corresponding states (as well as the two-mode linear charge coherent states) are numerically investigated by calculating quadrature squeezing, the Mandel parameter, the second-order correlation function, the second-order correlation function between the two modes and the Cauchy–Schwartz inequality. Also, the oscillatory behaviour of the photon count and the quasi-probability (Husimi) function of the associated states will be discussed.

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1. Introduction

Coherent states are venerable objects in many areas of physical research in recent decades [1, 2], with a special place in quantum optics [3, 4]. Therefore, various generalizations have been proposed up to now. In this paper, we confine ourselves to the two-mode type of generalization of coherent states. Along the latter subject, we start with a brief review of the charge (pair) coherent states. Horn and Silver [5] defined the so-called charge coherent state $|\alpha, q\rangle$, which is the common eigenvector of the number difference operator, sometimes named the charge operator, given by

$$ \hat{Q} = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} $$

and the pair annihilation operator $\hat{a}\hat{b}$ with eigenvalues $q$ and $\alpha$, respectively, i.e.

$$ \hat{Q}|\alpha, q\rangle = q|\alpha, q\rangle, $$

$$ \hat{a}\hat{b}|\alpha, q\rangle = \alpha|\alpha, q\rangle, $$

where $\hat{a}(\hat{a}^\dagger)$ and $\hat{b}(\hat{b}^\dagger)$ are the bosonic annihilation (creation) operators and $q$ is an integer that has been named the ‘charge number’ which is indeed the photon number difference between the two modes of the field. These states, sometimes called pair coherent states, have been used for the description of the production of pions [6] and other problems in quantum field theory [1]. The explicit form of $|\alpha, q\rangle$, which represents a well-known class of states within the general theory of coherent states, reads

$$ |\alpha, q\rangle = N_q(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!n+|q|!}} |n + \frac{q + |q|}{2}, n - \frac{q - |q|}{2}\rangle, $$

where $\alpha \in \mathbb{C}$, the kets $|m, n\rangle$ are the two-mode number states and $N_q$ is an appropriate normalization factor that may be determined. The states in (4) include two distinct sets of charge coherent states corresponding to $q \geq 0$ and $q \leq 0$. In (4), the following definition has been assumed:

$$ [n + |q|]! = (1 + |q|)(2 + |q|) \cdots (n + |q|). $$

An experimental scheme for generation of the states in (4) has been proposed by Agarwal [7, 8]. Recently, by using the nonlinear coherent state method [9, 10], one of the present authors, along with his co-author, has introduced the nonlinear charge coherent states in a general structure which will be briefly discussed here [11]. Consider $f$-deformed ladder operators

$$ \hat{A} = \hat{a}f(\hat{\eta}_a), \quad \hat{A}^\dagger = f^\dagger(\hat{\eta}_a)\hat{a}^\dagger, $$

$$ \hat{B} = \hat{b}f(\hat{\eta}_b), \quad \hat{B}^\dagger = f^\dagger(\hat{\eta}_b)\hat{b}^\dagger, $$

where $\hat{a}(\hat{a}^\dagger)$, $\hat{b}(\hat{b}^\dagger)$ and $\hat{\eta}_a = \hat{a}^\dagger \hat{a}$ ($\hat{\eta}_b = \hat{b}^\dagger \hat{b}$) are, respectively, bosonic annihilation, creation and number operators of mode $a(b)$, and $f(n)$ is an operator-valued function of the intensity of the radiation field (from now on assumed to be real) that characterizes the nonlinear nature of physical systems. The pair of $f$-deformed annihilation operator $\hat{A}\hat{B}$ commutes with the charge operator, i.e. $[\hat{Q}, \hat{A}\hat{B}] = 0$. Thus, the latter two operators should satisfy the following eigenvalue equations:

$$ \hat{Q}|\xi, q, f\rangle = q|\xi, q, f\rangle, $$

$$ \hat{A}\hat{B}|\xi, q, f\rangle = \xi|\xi, q, f\rangle, \quad \xi \in \mathbb{C}, $$

$$ \xi \in \mathbb{C}, $$

$$ n \in \mathbb{N}, $$

$$ [n + |q|]! = (1 + |q|)(2 + |q|) \cdots (n + |q|). $$
where \( \hat{Q} \) and \( q \) keep their previous definitions as expressed in (1). These eigenstates have in general two distinct representations which can be put into a single expression as follows:

\[
|\xi, q, f\rangle = N(|\xi|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{|\xi|^n}{\sqrt{n!(n+|q|)!}[f(n)!][f(n+|q|)!]} \left| n + \frac{q + |q|}{2}, n - \frac{q - |q|}{2} \right\rangle,
\]

with the normalization constant given by

\[
N(|\xi|^2) = \sum_{n=0}^{\infty} \frac{|\xi|^{2n}}{n!(n+|q|)!}[f(n)!][f(n+|q|)!]^2.
\]

Note that, in obtaining (9) and (10), we have used the conventional definitions

\[
[f(n)!] = f(n)f(n-1)f(n-2)\cdots f(1), \quad [f(0)!] = 1,
\]

and

\[
[f(n+|q|)!] = f(n+|q|)f(n-1+|q|)f(n-2+|q|)\cdots f(1+|q|), \quad [f(|q|)!] = 1.
\]

On the other hand, Fan et al [12] introduced \(|q, \lambda\rangle\) [13], i.e. the common eigenstate of \( \hat{Q} \) defined in (1), and the (Hermitian) \( \hat{g} \) operator is given by

\[
\hat{g} = (\hat{a} + \hat{b}^\dagger)(\hat{a}^\dagger + \hat{b}),
\]

with the eigenvalues \( q \) and \( \lambda \), respectively, i.e.

\[
\hat{Q}|q, \lambda\rangle = q|q, \lambda\rangle, \quad (14)
\]

\[
\hat{g}|q, \lambda\rangle = \lambda|q, \lambda\rangle, \quad \lambda \geq 0, \quad (15)
\]

by using the \(|\eta\rangle\) representation. This is due to the fact that \([\hat{Q}, \hat{g}] = 0\). The explicit form of \(|q, \lambda\rangle\) in the two-mode Fock space has been deduced by the authors, which reads

\[
|q, \lambda\rangle = e^{-\frac{1}{2}} \sum_{n=\max(0, -q)}^{\infty} H_{n+q,n}(\sqrt{\lambda}, \sqrt{\lambda}) \frac{1}{(n+q)!n!} |n + q, n\rangle, \quad (16)
\]

where \( H_{m,n} \) is the two-variable Hermite polynomial, which has been defined as

\[
H_{m,n}(z, z^*) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k m! n!}{k! (m-k)! (n-k)!} z^{m-k} z^*^{n-k}. \quad (17)
\]

In this paper, our main aims may be expressed as follows: (i) reobtaining the explicit form of \(|q, \lambda\rangle\) in the two-mode Fock space by using a rather different method other than the \(|\eta\rangle\) representation that has been followed in [12], (ii) generalizing the \( \hat{g} \) operator (combination of bosonic annihilation and creation operators in (13)) to the \( \hat{G} \) operator (combination of \( f \)-deformed ladder operators) and obtaining the common eigenstates of \( \hat{Q} \) and \( \hat{G} \) that have been called by us\'{f} \-deformed charge coherent states\footnote{We have selected this name to distinguish from the previous nonlinear charge coherent states in [11], whereas it is clear that both classes of states are in fact nonlinear or \( f \)-deformed.}, and (iii) investigating some of the nonclassical features and quantum statistical properties of the \( f \)-deformed charge coherent states associated with a few quantum systems with particular nonlinearity functions, in addition to the state \(|q, \lambda\rangle\) which is indeed a special case of our \( f \)-deformed charge coherent states with \( f(n) = 1 \). Obviously, our new type of \( f \)-deformed charge coherent states in this paper is substantially different from the states in (9) that have been introduced in [11].
This paper is organized as follows. In section 2, we reobtain the two-mode (linear) charge coherent state, which is the common eigenvector of $\hat{Q}$ and $\hat{g}$ operators. Then, in section 3, we generalize the creation and annihilation operators to the $f$-deformed operators and construct the $f$-deformed charge coherent states. Next, in section 4, as some physical realizations of the formalism, we consider a few particular nonlinearity functions and then study some of the nonclassical features and quantum statistical properties of the two-mode (linear) charge coherent states and $f$-deformed charge coherent states associated with those physical systems. Finally, in section 5, we present a summary and conclusion.

2. The two-mode (linear) charge coherent state: a common eigenstate of $\hat{Q}$ and $\hat{g}$ operators

The two-mode (linear) charge coherent state is a common eigenstate of the charge operator introduced in (1) and $\hat{g}$ operator defined in (13), respectively, with the eigenvalues $q$ and $\xi$ that have been expressed in (14) and (15). It is worth noting that we start our discussion by imposing a little modification in the notation of [12] by changing the eigenstate $|q, \lambda\rangle$ to $|\xi, q\rangle$.

The form of the state in the two-mode Fock space is considered to be

$$|\xi, q\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m}|n, m\rangle.$$  \hspace{1cm} (18)

Substituting (18) into (14), one readily finds

$$n = m + q.$$  \hspace{1cm} (19)

With the latter result in mind, the explicit form of the state may be rewritten as

$$|\xi, q\rangle^{(+)} = \sum_{n=0}^{\infty} c_{n+q,n}^{(+)}|n+q, n\rangle, \quad q \geq 0,$$

$$|\xi, q\rangle^{(-)} = \sum_{n=0}^{\infty} c_{n-n-q,n}^{(-)}|n, n-q\rangle, \quad q \leq 0.$$  \hspace{1cm} (20)

Now, by substituting (20), for instance, into (15), we find the recursion relation

$$c_{n+q,n}^{(+)} = \frac{1}{\sqrt{(n+q)n}} \left[ c_{n+q-1,n-1}^{(+)}(\xi - (n+q) - (n-1)) - c_{n+q-2,n-2}\sqrt{(n+q-1)(n-1)} \right].$$  \hspace{1cm} (22)

By a straightforward but lengthy procedure, the expansion coefficients are then obtained in terms of $c_{q,0}^{(+)}$ as follows:

$$c_{n+q,n}^{(+)} = c_{q,0}^{(+)} \sqrt{(n+q)!n!} \sum_{k=0}^{n} \frac{(-1)^k \xi^{n-k}}{k!(n+q-k)!(n-k)!},$$  \hspace{1cm} (23)

and thus, the explicit form of the state for $q \geq 0$ may be expressed as

$$|\xi, q\rangle^{(+)} = c_{q,0}^{(+)} \sum_{n=0}^{\infty} \sqrt{(n+q)!n!} \sum_{k=0}^{n} \frac{(-1)^k \xi^{n-k}}{k!(n+q-k)!(n-k)!}|n+q, n\rangle.$$  \hspace{1cm} (24)

A similar procedure can be performed for $q \leq 0$, which led us to the explicit form of (linear) charge coherent state for $q \leq 0$ as

$$|\xi, q\rangle^{(-)} = c_{q,0}^{(-)} \sum_{n=0}^{\infty} \sqrt{n!|n-q|!} \sum_{k=0}^{n} \frac{(-1)^k \xi^{n-k}}{k!(n-q-k)!(n-k)!}|n, n-q\rangle.$$  \hspace{1cm} (25)
In (24) and (25), $\xi \in \mathbb{C}$, and the normalization constants can be easily calculated from

$$c^{(\pm)}_{q,0} = \left[ \sum_{n=0}^{\infty} |n + |q|!|n! \left( \sum_{k=0}^{n} \frac{(-1)^k \xi^{n-k}}{k!|n+|q|!-(n-k)!} \right) \right]^{1/2}.$$  \hspace{1cm} \text{(26)}

By setting $q \geq 0$ and $q \leq 0$ in (26), one obtains the exact form of normalization factors of states in (24) and (25), respectively. It is worth mentioning that in deriving the relations (23)–(26), we have used the following definition:

$$[n + |q| - k]! \equiv (1 + |q|)(2 + |q|) \cdots (n + |q| - k)$$

$$= \frac{(n + |q| - k)!}{|q|!},$$  \hspace{1cm} \text{(27)}

which is a slightly different from the conventional definition of $[f(n)]!$ that has frequently been used in the literature and devotes $f$-deformed coherent states. Noting that the state in (16) that has been introduced in [12] is un-normalized, it is easy to check that our final states in (24) and (25) are in exact consistence with (16), when one sets $\xi = \xi^* = \sqrt{\lambda}$ in (24) and (25) with $\lambda \in \mathbb{R}$. Also, note that the domain of the states obtained by us is the entire space of a complex plane, while in [12] $\lambda$ is real and positive. We mention that our states, which were obtained by a simpler, and at the same time more general manner, are normalized to unity, while the states introduced in (16) are not normalized. In addition to this fact, to the best of our knowledge, the statistical properties and nonclassical features of these states have not yet been discussed in the literature. Henceforth, we will pay attention to this subject in the continuation of the paper, since any generalization scheme without this discussion seems to be poor from physical points of view.

3. Introducing the $f$-deformed charge coherent states

In this section, using the nonlinear coherent state method and based on the procedure by which the charge coherent states are derived in the previous section, we first generalize the bosonic creation and annihilation operators to the $f$-deformed operators defined in (6), and then, $\hat{g}$ that has been defined in (13) will be converted to $\hat{G}$, which is given by

$$\hat{G} = (\hat{A} + \hat{B}^\dagger)(\hat{A}^\dagger + \hat{B}).$$  \hspace{1cm} \text{(28)}

Now, we introduce $f$-deformed charge coherent states as the simultaneous eigenstates of charge operator in (1) and $\hat{G}$ in (28) with the eigenvalues $q$ and $\xi$, respectively, i.e.

$$\hat{Q}|\xi, q, f\rangle = q|\xi, q, f\rangle,$$  \hspace{1cm} \text{(29)}

$$\hat{G}|\xi, q, f\rangle = \xi|\xi, q, f\rangle,$$  \hspace{1cm} \text{(30)}

since it can be easily checked that $[\hat{Q}, \hat{G}] = 0$.\footnote{It is worth mentioning that we can also deform the bosonic charge $\hat{Q}$ to the $f$-deformed charge operator; however, it will then not commute with any of the operators $\hat{a}\hat{b}, \hat{A}\hat{B}, \hat{g}$, or $\hat{G}$. So, no common eigenstate may be expected.} The explicit form of the common eigenstate in the two-mode Fock space is described as

$$|\xi, q, f\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m}|n, m\rangle.$$  \hspace{1cm} \text{(31)}

Substituting (31) into (29), one obtains the same expression as in (19). With this result in mind, the explicit form of the state can be rewritten as

$$|\xi, q, f\rangle^{(+)} = \sum_{n=0}^{\infty} c^{(+)}_{n+q, n}|n + q, n\rangle,$$  \hspace{1cm} q \geq 0.$$  \hspace{1cm} \text{(32)}
\[
|\xi, q, f|^{-}(\xi) = \sum_{n=0}^{\infty} c_{n, n-q}^{(\xi)}|n, n-q), \quad q \leq 0, \tag{33}
\]

where the superscript \((+)(-))\) indicates the positive (negative) values of the \(q\) parameter. Now, substituting the state (32), for instance, into (30), one finds a complicated recursion relation for \(q \geq 0\) as follows:

\[
c_{n+q,n}^{(+)} = \frac{1}{\sqrt{(n+q)n}f(n+q)f(n)} \times \left[ c_{n+q-1,n-1}^{(+)}(\xi - (n+q)f^2(n+q) - (n-1)f^2(n-1)) - c_{n+q-2,n-2}^{(+)}(\sqrt{(n+q-1)(n-1)}f(n+q-1)f(n-1)) \right]. \tag{34}
\]

However, by defining

\[
B_{n}^{(+)}, B_{n}^{(-)} = \frac{c_{n+q,n}^{(+)}}, c_{n+q-1,n-1}^{(+)} \tag{35}
\]

the recursion relation (34) can be inverted to the form

\[
B_{n}^{(+)} = \frac{1}{\sqrt{(n+q)n}f(n+q)f(n)} \left[ \frac{\xi - (n+q)f^2(n+q) - (n-1)f^2(n-1)}{\sqrt{(n+q-1)(n-1)}f(n+q-1)f(n-1)} \right]. \tag{37}
\]

From the above equation (37), we may obtain an explicit expression for \(B_{n-1}^{(+)}\) in terms of \(B_{n}^{(+)}\). Setting the obtained \(B_{n}^{(+)}\) in (37), we arrive at \(B_{n}^{(+)}\) in terms of \(B_{n-1}^{(+)}\). Continuing this procedure, we can obtain \(B_{n}^{(+)}\) in terms of \(B_{n-1}^{(+)}\), \(B_{n-2}^{(+)}\), \ldots Finally, we arrive at the complicated expression for \(B_{n}^{(+)}\) in terms of \(B_{0}^{(+)}\) which is assumed to be 1:

\[
B_{n}^{(+)} = \frac{1}{\sqrt{(n+q)n}f(n+q)f(n)} \left[ \frac{\xi - (n+q)f^2(n+q) - (n-1)f^2(n-1)}{\sqrt{(n+q-1)(n-1)}f(n+q-1)f(n-1)} \right] \cdots \frac{\xi - (1+q)f^2(1+q)}{\xi - (n+q-2)f^2(n+q-2)f^2(n-2)) \cdots (n+q-2)(n-2)f^2(n-2))} \tag{38}
\]

Note that the following relations hold:

\[
c_{n+q,n}^{(+)} = B_{n}^{(+)} c_{n+q-1,n-1}^{(+)} = B_{n}^{(+)} B_{n-1}^{(+)} c_{n+q-2,n-2}^{(+)} = B_{n}^{(+)} B_{n-1}^{(+)} B_{n-2}^{(+)} c_{n+q-3,n-3}^{(+)} \cdots B_{n}^{(+)} B_{n-1}^{(+)} B_{n-2}^{(+)} \cdots B_{n}^{(+)} c_{n,0}^{(+)} = B_{0}^{(+)} \equiv 1, \tag{39}
\]

or in a compact form, we have

\[
c_{n+q,n}^{(+)} = \left[B_{n}^{(+)}\right] c_{n,0}^{(+)}. \tag{40}
\]
Adding all of the above results, one readily deduces the explicit form of $f$-deformed charge coherent states for $q \geq 0$ as follows:

\[ |\xi, q, f\rangle^{(+)} = c_{q,0}^{(+)} \sum_{n=0}^{\infty} [B_n^{(+)}]^n|n + q, n\rangle. \]  

(41)

The normalization constant in (41) is given by

\[ c_{q,0}^{(+)} = \left( \sum_{n=0}^{\infty} |[B_n^{(+)}]^n| \right)^{-1/2}. \]  

(42)

A similar procedure may be followed for $q \leq 0$, which leads one to the explicit form of $f$-deformed charge coherent states as

\[ |\xi, q, f\rangle^{(-)} = c_{q,0}^{(-)} \sum_{n=0}^{\infty} [B_n^{(-)}]^n|n, n - q\rangle, \]  

(43)

where $B_n^{(-)}$ may be expressed as

\[
B_n^{(-)} = \frac{1}{\sqrt{n(n - q)f(n)f(n - q)}} \left[ (\xi - (n)f^2(n) - (n - q - 1)f^2(n - q - 1) \right] \\
- \frac{(n - 1)(n - q - 1)f^2(n - 1)f^2(n - q - 1)}{[\xi - (n - 1)f^2(n - 1) - (n - q - 2)f^2(n - q - 2)] - \frac{1}{\sum_{n=0}^{\infty} [B_n^{(-)}]^n} \]  

(44)

The normalization constant in (43) can be simply obtained as

\[ c_{q,0}^{(-)} = \left( \sum_{n=0}^{\infty} |[B_n^{(-)}]^n| \right)^{-1/2}. \]  

(45)

Note that in the above formula we have replaced $B_n^{(\pm)}(\xi, q, f)$ with $B_n^{(\pm)}$ for simplicity. It can be easily checked that setting $f(n) = 1$ in (41) and (43) recovers (24) and (25), respectively.

4. Physical properties of the introduced states

As is required, we briefly review some of the criteria which will be used in the continuation of the paper for investigating the nonclassicality of the introduced states, as the Mandel parameter, second-order correlation function, second-order correlation function between the two modes, the Cauchy–Schwartz inequality and, finally, the quasi-probability function $Q(\alpha)$.

4.1. Nonclassicality criteria

- Using the definitions of position and momentum quadratures as $x = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$ and $p = (\hat{a} - \hat{a}^\dagger)/i\sqrt{2}$, it is well known that squeezing, respectively, occurs in $x$ or $p$ if $(\Delta x)^2 < 0.5$ or $(\Delta p)^2 < 0.5$. For instance, for position quadrature, one has

\[ (\Delta x)^2 = (x^2) - (\langle x \rangle)^2 \]

\[ = \frac{1}{2} [\langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle]. \]  

(46)

By calculating the necessary mean values over the states $|\xi, q, f\rangle^{(+)}$, it can be easily seen that $\langle \hat{a} \rangle = \langle \hat{a}^\dagger \rangle = \langle \hat{a}^2 \rangle = \langle \hat{a}^\dagger^2 \rangle = 0$. All of these arise from the fact that
The second-order correlation function between the two modes of the radiation field has been defined as \[ g_0^{(2)} = \frac{\langle a_1^+ a_2 a_1 a_2^+ \rangle}{\langle a_1^+ a_1 a_2^+ a_2 \rangle} \] where \( i \) stands for the two modes \( a \) and \( b \) and so \( \hat{n}_a = \hat{a}^+ \hat{a}, \hat{n}_b = \hat{b}^+ \hat{b} \). The states for which \( Q_{a(b)} = 0, Q_{a(b)} < 0 \) and \( Q_{a(b)} > 0 \), respectively, correspond to Poissonian (standard coherent states), sub-Poissonian (nonclassical states) and super-Poissonian (classical states) statistics.

Though there are quantum states in which super-/sub-Poissonian statistical behaviour appears simultaneously with bunching/antibunching effects, this is not absolutely true [15]. Therefore, to investigate bunching or antibunching effects, the second-order correlation function, which is defined as \[ g_0^{(2)} = \frac{\langle a_1^+ a_2 a_1 a_2^+ \rangle}{\langle a_1^+ a_1 a_2^+ a_2 \rangle} \] is widely used. Depending on the particular nonlinearity function \( f(\alpha) \), which has been chosen for the construction of any class of \( f \)-deformed charge coherent states, \( g_0^{(2)}(0) > 1 \) and \( g_0^{(2)}(0) < 1 \), respectively, indicate the bunching and antibunching effects. The case \( g_0^{(2)}(0) = 1 \) corresponds particularly to the canonical coherent states.

The second-order correlation function between the two modes of the radiation field has been defined as \[ g_0^{(2)}(0) = \frac{\langle a_1^+ a_2 a_1 a_2^+ \rangle}{\langle a_1^+ a_1 a_2^+ a_2 \rangle} \] This quantity shows that the two modes are correlated and the state is classical if \( g_0^{(2)}(0) > 1 \); otherwise, it is nonclassical.

As another quantity, recall that the Cauchy–Schwartz inequality [17] has been defined as
\[ I_0 = \frac{\langle a_1^+ a_1 a_2^+ a_2 \rangle}{\langle a_1^+ a_1 a_2^+ a_2 \rangle} \leq 1 < 0 \]
If this inequality is violated, then the state is nonclassical.

It is to be mentioned that each of the above signs are sufficient, not necessary, for a state to be nonclassical, so we do not pay more attention to other nonclassicality criteria.

Different quasi-probability functions have been proposed in quantum optics [18]. Among them we pay attention to the Husimi function \( Q(\alpha) \) which is defined for the single-mode quantum state \( |\psi\rangle \) as \( Q(\alpha) = \langle \alpha | |\psi\rangle |^2 / \pi \), where \( |\alpha\rangle \) is the canonical coherent states. This positive-definite function, which is defined over the phase space, can be constructed in the homodyne experiments [19]. By this function, the quantum interference effects in phase space have been illustrated [20]. Seemingly, it is possible to generalize this definition to our two-mode introduced states as follows:
\[ Q(\alpha_1, \alpha_2, \xi) = \frac{1}{\pi} |\langle \alpha_1, \alpha_2 | \psi(\xi) \rangle |^2 \]
we have used the explicit form of the associated types of well-known nonlinearity functions. In this section, we want to investigate the nonclassical properties of the reobtained two-mode states defined by Penson and Solomon [21] by using the nonlinear coherent state method [22].

(i) The Penson–Solomon nonlinearity function is defined by

\[ f_{PS}(n) = p^{1-n}, \]

where \(0 \leq p \leq 1\). This function, indeed, may be executed from the special class of coherent states defined by Penson and Solomon [21] by using the nonlinear coherent state method [22].

(ii) A nonlinearity of the type \(f(n) = \sqrt{n}\) This function appears in a natural way in the Hamiltonian describing the interaction between a two-level atom and the electromagnetic field with intensity-dependent coupling [23, 24].

(iii) The q-deformed nonlinearity associated with the well-known q-coherent states, defined as \(f_{q}(n) = \sqrt{\frac{q^{-n} - q^{-n+1}}{nq^{-q^{-1}}}}\) [10]. Note that since charge is denoted usually by q, therefore, we have used q notation for defining the q-nonlinearity function.

Inserting each of these three nonlinearity functions into (41) and (43), one may readily obtain the explicit form of the associated f-deformed charge coherent states.

- Evaluating the Mandel parameter for the f-deformed positive charge coherent states requires the following expectation values:

\[
\langle \hat{n}_a \rangle^{(+)} = \langle \xi, q, f | \hat{n}_a | \xi, q, f \rangle^{(+)}
\]

\[
= |c_{q,0}^{(+)}|^2 \sum_{n=0}^{\infty} |(B_n^{(+)}|)^2 (n + q),
\]

and

\[
\langle \hat{n}_a^2 \rangle^{(+)} = \langle \xi, q, f | \hat{n}_a^2 | \xi, q, f \rangle^{(+)}
\]

\[
= |c_{q,0}^{(+)}|^2 \sum_{n=0}^{\infty} |(B_n^{(+)}|)^2 (n + q)^2,
\]

where \(\hat{n}_a = \hat{a}^\dagger \hat{a}\). In figure 1, the Mandel parameter for f-deformed charge coherent states is shown for the Penson–Solomon nonlinearity function with \(p = 0.5\) and fixed charge \(q = 1\), as well as the q-deformation function with \(q = 7\) and fixed charge \(q = 2\). It is seen that, for both cases, this parameter is always negative, and so the sub-Possonian behaviour is visible. Also, note that its value is nearly \(\approx -1\), which indicates the high strength nonclassicality of the considered states. It ought to be mentioned that we have also calculated the Mandel parameter for \(f(n) = \sqrt{n}\) and \(f(n) = 1\), but in both the cases, we obtain a very high positive value relative to the previous ones, such that displaying them in this figure would make the graphs unclear. Moreover, we were sure about the lack of this nonclassicality criterion for the other two functions.
To calculate the second-order correlation function for mode $a$, defined in (48), the following relation is needed:

$$
\langle \hat{a}^\dagger \hat{a}^2 \rangle = \langle \xi, q, f | \hat{a}^\dagger \hat{a}^2 | \xi, q, f \rangle
$$

$$
= \epsilon_{q,0}^{(+)2} \sum_{n=0}^{\infty} |B_n^{(q)}|^{2} (n + q) (n + q - 1),
$$

and $\langle \hat{a}^\dagger \hat{a} \rangle$ is also obtained in (53). From figure 2, it is visible that, the inequality $g^{(2)}(0) > 1$ holds for two-mode (linear) charge coherent states with $f(n) = 1$, as well as the
Figure 3. The second-order correlation function between two modes \( g_{12}^{(2)}(0) \), as a function of \( \xi \), for \( f \)-deformed charge coherent states. The continuous line is for the two-mode (linear) charge coherent state \( f(n) = 1 \) with the fixed parameter \( q = -1 \), the dotted line is for \( f_{PS}(n) \) with the fixed parameters \( q = -2 \) and \( p = 0.5 \), the dashed line is for \( f_{q}(n) \) with the fixed parameters \( q = 2 \) and \( q = 7 \), and the dot–dashed line is for \( f(n) = \sqrt{n} \) with the fixed parameter \( q = 1 \).

\( f \)-deformed charge states with \( f(n) = \sqrt{n} \) with, respectively, charge parameters \( q = 1, 3 \), which indicate the bunching effect of the corresponding states. Interestingly, the \( q \)-deformation plot shows that it becomes exactly 1, like the canonical coherent states, where \( q = 1, 7 \). This criterion obtains the values less than 1, only for the Penson–Solomon nonlinearity function when we considered charge a \( q = -1 \), i.e. it shows antibunching (nonclassical) effect in a finite region of \( \xi \in \mathbb{R} \). Recalling that this nonclassicality sign becomes 2 for thermal light, we observe from figure 2 that for the cases \( f(n) = \sqrt{n} \) and \( f_{PS}(n) \) this function takes values greater than 2 in some regions of \( \xi \), i.e. in view of this criterion the corresponding states behave like a super-thermal light.

For the second-order correlation function between the two modes, the following mean values are necessary:

\[
\langle \hat{b}^\dagger \hat{b} \rangle^{(+)} = \langle \xi, q, f | \hat{b}^\dagger \hat{b} | \xi, q, f \rangle^{(+)} = |c_{q,0}^{(+)}|^2 \sum_{n=0}^{\infty} n! [B_n^{(+)}]^2, \tag{56}
\]

and

\[
\langle \hat{a}^\dagger \hat{a} \rangle^{(+)} = \langle \xi, q, f | \hat{a}^\dagger \hat{a} | \xi, q, f \rangle^{(+)} = |c_{q,0}^{(+)}|^2 \sum_{n=0}^{\infty} [B_n^{(+)}]^2 n(n + q), \tag{57}
\]

and \( \langle \hat{a}^\dagger \hat{a} \rangle \) is also obtained in (53). Figure 3 shows \( g_{12}^{(2)}(0) \) for two-mode linear and \( f \)-deformed charge coherent states. From the figure, it is clear that again for the \( q \)-deformation function with charge \( q = 2 \) and \( q = 7 \), this criterion obtains the exact value 1, like the canonical coherent states; while for all other nonlinearity functions and the chosen charge parameters \( q \), the value of \( g_{12}^{(2)}(0) \) becomes greater than 1, indicating the bunching (classical) effect of the corresponding field.
Figure 4. The plot of the Cauchy–Schwartz inequality, as a function of $\xi$, for $f$-deformed charge coherent states. The continuous line is for the two-mode (linear) charge coherent state ($f(n) = 1$) with the fixed parameter $q = 1$, the dotted line is for $f_{PS}(n)$ with the fixed parameters $q = 1$ and $p = 0.5$, the dashed line is for $f_q(n)$ with the fixed parameters $q = 3$ and $q = 7$ and the dot–dashed line is for $f(n) = \sqrt{n}$ with the fixed parameter $q = 2$.

Figure 5. The plot of probability of finding $n + q$ photons in mode $a$ and $n$ photons in mode $b$ in $|\xi, q, f\rangle$. ($P(n + q, n)$), as a function of $n$, for $f$-deformed charge coherent states. The continuous line is for the two-mode (linear) charge coherent state ($f(n) = 1$) with the fixed parameter $\xi = 5$ and $q = 2$, the dotted line is for $f_{PS}(n)$ with the fixed parameters $\xi = 10$ and $q = -1$, $p = 0.5$, the dashed line is for $f_q(n)$ with the fixed parameters $\xi = 5$ and $q = -2$, $q = 7$, and the dot–dashed line is for $f(n) = \sqrt{n}$ with the fixed parameters $\xi = 10$ and $q = 1$.

- For the Cauchy–Schwartz inequality, defined in (50), the following mean value is necessary:

$$\langle \hat{b}^{\dagger 2} \hat{b}^2 \rangle^{(+)} = \langle \xi, q, f | \hat{b}^{\dagger 2} \hat{b}^2 | \xi, q, f \rangle^{(+)} = |c_{q,0}^{(+)}|^2 \sum_{n=0}^{\infty} |c_{n}^{(+)}|^2 n(n - 1),$$

(58)
Figure 6. The plot of $Q(\alpha_1, \alpha_2)$ as a function of the amplitudes $x = \text{Re}(\alpha_1)$ and $y = \text{Im}(\alpha_1)$ for different classes of states. (a) The two-mode (linear) charge coherent state ($f(n) = 1$), with the fixed parameters $\xi = 10, q = 1$ and $\alpha_2 = 1 + i$. (b) The two-mode (linear) charge coherent state ($f(n) = 1$), with the fixed parameters $\xi = 10, q = -1$ and $\alpha_2 = 1 + i$. (c) The $f$-deformed charge coherent state with $f_{PS}(n)$, and the fixed parameters are $\xi = 10, q = 2, p = 0.5$ and $\alpha_2 = 1 + i$. (d) The $f$-deformed charge coherent state with $f_{PS}(n)$, and the fixed parameters are $\xi = 10, q = -2, p = 0.5$ and $\alpha_2 = 1 + i$. (e) The $f$-deformed charge coherent state with $f_q(n)$, and the fixed parameters are $\xi = 10, q = 3, q = 7$ and $\alpha_2 = 1 + i$. (f) The $f$-deformed charge coherent state with $f_q(n)$, and the fixed parameters are $\xi = 10, q = -3, q = 7$ and $\alpha_2 = 1 + i$. (g) The $f$-deformed charge coherent state with $f(n) = \sqrt{n}$, and the fixed parameters are $\xi = 10, q = 4$ and $\alpha_2 = 1 + i$. (h) The $f$-deformed charge coherent state with $f(n) = \sqrt{n}$, and the fixed parameters are $\xi = 10, q = -4$ and $\alpha_2 = 1 + i$.  

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and $|\hat{a}^\dagger \hat{a}^2\rangle$ and $|\hat{a}^\dagger \hat{a} \hat{b} \hat{b}^\dagger\rangle$ are obtained in (55) and (57), respectively. Figure 4 shows the Cauchy–Schwartz inequality of mode $a$ for two-mode linear and the three types of $f$-deformed charge coherent states. It is seen from the figure that in all cases this criterion obtains a negative value for the chosen charge parameters, indicating the nonclassicality feature of the associated states. Note that the graph of $f_q(n)$ coincides with that of $f_{PS}(n)$ at the value $-0.01$ in figure 4.

- In addition, the probability of finding $n + q$ photons in mode $a$ and $n$ photons in mode $b$ for our introduced positive charge states $|\xi, q, f\rangle^+$ reads as

$$P^{(+)}(n + q, n) = |\langle n + q, n|\xi, q, f\rangle^{(+)}|^2 = |c_{q, 0}^{(+)}|^2|B_n^{(+)}|^2.$$  

(59)

Figure 5 shows photon number distribution for all of the considered nonlinearity functions including the linear one ($f(n) = 1$). From the figure one may observe that, in all cases, the oscillatory behaviour of the photon count is visible for the chosen $q$ and $\xi$ parameters. As is well known, this behaviour is one of the nonclassicality signs of quantum states. Therefore, in view of this point, all of the considered charge states are nonclassical.

- The quasi-distribution (Husimi) function for the $f$-deformed charge coherent states with $q \geq 0$ reads as

$$Q(\alpha_1, \alpha_2, \xi) = \frac{1}{\pi} |\langle \alpha_1, \alpha_2|\xi, q, f\rangle^{(+)}|^2$$

$$= e^{-|\alpha_1|^2} e^{-|z_2|^2} \sum_{n=0}^{\infty} \frac{\alpha_1^{2(n+q)} |\alpha_2|^{2n}}{n!(n + q)!} |c_{q, 0}^{(+)}|^2 |B_n^{(+)}|^2.$$  

(60)

We have plotted the three-dimensional graphs of quasi-probability distribution $Q(\alpha_1, \alpha_2)$ in terms of the amplitudes $x = \text{Re}(\alpha_1)$ and $y = \text{Im}(\alpha_1)$ for various nonlinearity functions including $f(n) = 1$ (linear) with the fixed parameters $\xi$, $q$ and $\alpha_2$. Generally, irrespective of the selected nonlinearity functions, the plots corresponding to negative values of $q$ are similar, as is the case for positive $q$'s. Specifically, from figure 6, it is obvious that, in all cases, for the chosen parameters that have been used in our numerical calculations, when $q$ obtains positive values, there exists a hole (centred at the origin of the complex plane) in the three-dimensional graphs, while this hole will disappear when the $q$ parameter obtains negative values.

We end this section by mentioning the following two points. First, although we brought the necessary mean values for our numerical calculations only for positive charge coherent states $|\xi, q, f\rangle^{(+)}$, the calculation of similar quantities for the $|\xi, q, f\rangle^{(-)}$ states, which have been used in our numerical results may easily be done, too. Second, our interpretations on the plotted graphs are specifically concerned with the chosen fixed parameters that have been indicated in the related figure captions. So, obviously, variation of the related parameters may lead one to enter states with various physical behaviour and different nonclassicality features.

5. Summary and conclusion

In summary, we reobtained two-mode charge coherent states by a standard method rather than the $|\eta\rangle$ representation. Then, we extended the latter approach to the nonlinear coherent state method and introduced the representation of two-mode $f$-deformed charge coherent states. The construction is valid for a large class of generalized nonlinear oscillators. However, in this paper, we only used a few well-known nonlinearity functions associated with particular quantum systems as some physical appearances of our presented formalism. After introducing the explicit form of the associated states in the two-mode Fock space, we established that
the $f$-deformed charge coherent state recovers two-mode charge coherent states when one sets $f(n) = 1$. As a clear fact, it is observed that different $f(n)$'s lead to various classes of deformed charge coherent states, obviously with different physical properties, by tuning the parameters which exist in any case, i.e. $\xi$, $q$, and sometimes like for the Penson–Solomon and $q$-deformation the excess parameters $p$ and $q$, respectively. Anyway, due to the common motivation in the introduction of any generalized coherent states, i.e. investigating the nonclassicality features of the states, we pay attention to this matter by evaluating some of the nonclassicality properties of both two-mode linear and $f$-deformed charge coherent states.

As some criteria, we computed the Mandel parameter, second-order correlation function, second-order correlation function between the two modes, Cauchy–Schwartz inequality and probability distribution function in addition to the quasi-probability Husimi function $Q(\alpha_1, \alpha_2)$ of the states, using three types of well-known nonlinearity functions, numerically. Summing up, all of the corresponding states have enough nonclassicality features to be classified in the nonclassical states, i.e. the states with no classical analogue. Although our work mainly possesses mathematical-physics structure, we mention that the pair coherent states have found a few experimental schemes for their generations, so we hope that the two-mode charge coherent states in (24) and (25), which were originally introduced in [12], and our $f$-deformed counterpart in (41) and (43) also find their appropriate experimental generation proposals in the near future.

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