PT-symmetry and Transparency

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Abstract

It is known that the perfect absorption of two identical waves incident on a complex potential from left and right can occur at a fixed real energy and that the time-reversed setting of this system would act as a laser at threshold at the same energy. Here, we argue and show that PT-symmetric potentials are exceptional in this regard which do not allow Coherent Perfect Absorption without lasing as the modulus of the determinant of the S-matrix, $|\det S|$, becomes 1, for all positive energies. Next we show that in the parametric regimes where the PT-symmetry is unbroken, the eigenvalues, $s_{\pm}$ of $S$ can become unitary (uni-modular) for all energies. Then the potential becomes coherent perfect emitter on both sides for any energy of coherent injection. We call this property Transparency.

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Recently, there has been a considerable revival of interest in the scattering from complex one dimensional potentials [1-17]. Phenomena like (handedness) non-reciprocity [1] of reflectivity when the potential is spatially asymmetric, spectral singularity (SS) [pole in transmitivity and reflectivity at a real energy] [7], coherent perfect absorption (CPA) without [9,12] and with [10] lasing (spectral singularity) have been proposed. Here in this communication, we re-examine the phenomenon of CPA without lasing and show that complex PT-symmetric potentials are exceptional in this regard. Here \( P \) means Parity transformation \((x \rightarrow -x)\) and \( T \) means Time reversal \((i \rightarrow -i, k \rightarrow -k)\), \( k \) is wavenumber defined as \( k = \sqrt{E} \left( \frac{\hbar^2}{2m} = 1 = 2m \right) \).

It is now well known [18,19] that complex PT-symmetric potentials can have real discrete spectrum below or above the critical value of a potential parameter. In these cases the energy eigenstates are simultaneous eigenstates of PT and the PT symmetry remains exact (unbroken) otherwise it is spontaneously broken. Here we show that under unbroken PT symmetry, there can be a parametric regime where the eigenvalues of the \( S \)-matrix (see (6) below) of scattering from left and right can have both eigenvalues, \( s_\pm \), unitary (unimodular) for all energies such that two coherent waves injected on the (optical medium) potential will be emitted perfectly on both sides for any energy of injection. We call this phenomenon Transparency.

A potential which vanishes asymptotically has both kinds of spectrum real discrete and real continuous. For such complex PT-symmetric potentials recently a very interesting criterion using the properties of scattering (reflection and transmission amplitudes) has been proposed ([11], see (11) below) to detect whether the potential preserves PT-symmetry. This has also been suggested to be the condition for unitarity of the eigenvalues \( s_\pm \). In this Letter we show that this condition is not sufficient.

When a one-dimensional complex potential (vanishing asymptotically) is spatially asymmetric, the reflectivity is sensitive to the side of incidence of wave whether it is left or right. It has been proved that [1]
\[
T_{\text{left}}(k) = T_{\text{right}}(k) = T(k) \text{ but } R_{\text{left}}(k) \neq R_{\text{right}}(k),
\]
Also see Refs. [3-7]. Following the same proof it has also been proposed [14] that for PT-symmetric potentials
\[
T(-k) = T(k), \text{ and } R_{\text{left}}(-k) = R_{\text{right}}(k).
\]
For non-PT-symmetric cases we have [14]
\[
T(-k) \neq T(k), R(-k) \neq R(k) \text{ and } R_{\text{left}}(-k) \neq R_{\text{right}}(k).
\]
Let \( r \) and \( t \) be reflection and transmission (complex) amplitudes with phases as \( \phi \) and \( \theta \), respectively. Then reflectivity, \( R = |r|^2 \), and transmitivity, \( T = |t|^2 \). For complex PT-symmetric structures, it has been proved that [13]
\[
\theta - \phi_{\text{left}} = \pi/2 = \theta - \phi_{\text{right}}, \text{ if } T < 1 \text{ and } \theta - \phi_{\text{left}} = \pi/2 = \phi_{\text{right}} - \theta, \text{ if } T > 1.
\]
When two waves identical in all respects are incident on a complex scattering potential from left and right the determinant of \( S \)-matrix is given as [7]
\[
S = \begin{pmatrix} t & r_{\text{left}} \\ r_{\text{right}} & t \end{pmatrix}, \det S = t^2 - r_{\text{left}}r_{\text{right}} = \frac{m_{11}}{m_{22}}.
\]
Here, $m_{11}$ and $m_{22}$ are the entries of $M_{2 \times 2}$ transfer matrix for two-channel scattering. Using (4) in (5) it can be seen that

$$| \det S | = T \pm \sqrt{R_{\text{left}} R_{\text{right}} } = 1, \quad (6)$$

following sub(super)-unitarity as proposed in [13,15]. On the contrary, the condition for CPA without lasing is [9]

$$| \det S | = 0, \quad (7)$$

at a real positive energy. One can therefore see the impossibility of CPA for complex PT-symmetric cases.

However, the novel possibility of PT-symmetric potentials to display CPA with lasing is distinct and different. It happens when at $E = E_*$, $T$ becomes infinity and $| \det S(E_* \pm \epsilon) | = 1$. Here $\epsilon$ is as small as you please. Also the constancy of $| \det S |$ in (6) indicates its invariance under time-reversal as

$$| \det S(-k) | = | \det S(k) |, \quad (8)$$

confirming the proposal (2).

Nevertheless, let us point out that the possibility of CPA without lasing is due to the variance of $| \det S(k) |$ under time-reversal for non-PT-symmetric potentials which in turn is due to (3). Thus, the following conditions may be met at $k = k_c$

$$| \det S(k_c) | = 0, \quad T(-k_c) = R_{\text{left}}(-k_c) = R_{\text{right}}(-k_c) = \infty. \quad (9)$$

and CPA alone (without lasing) is observed. This is how coherent perfect absorbers are also called [9] time-reversed lasers. In this regard, the claim of this Letter is that these potentials can not be PT-symmetric.

We would like to remark that unlike the first proposal for the general CPA [9], the authors in [12] have been cautious in choosing the optical medium as P-symmetric. They set less general, yet more simple and intuitive condition for CPA at a real energy as $t + r_{\text{left}} = 0 = t + r_{\text{right}}$. For P-symmetric complex potentials the reciprocity ($r_{\text{left}} = r_{\text{right}}$) [1,3] works and that the result $\theta - \phi = \pi/2$ [20] of real Hermitian P-symmetric potentials is defied favourably due to the presence of non-Hermiticity (dissipation) so the CPA is feasible. This phenomena has been called controlled CPA which is a special case of the more general condition (8) [9].

Thus, we conclude that complex PT-symmetric potentials do not display coherent perfect absorption alone (without lasing).

The complex eigenvalues of two-port $S$-matrix (6) are given as

$$s_{\pm} = [t(k) \pm \sqrt{r_{\text{left}}(k) r_{\text{right}}(k)} ], \quad (10)$$

which from theory of matrices follow $s_+ s_- = \det S$. Then in view of the argument leading to (6), for PT-symmetric potentials we have $| s_- || s_+ | = 1$. So it may also happen that these two eigenvalues are are uni-modular, representing Coherent Perfect Emission from both sides of the potential, we propose to call this property transparency. Though not a necessity however transparency is a possibility like other phenomena [7,9-13] discussed here. We find that in several parametric regimes of a PT-symmetric potentials transparency can actually happen at any energy of injection of coherent waves on the potential from left and right.
Recently, an interesting condition [11]

$$B(k) = \left| \frac{r_{\text{left}} - r_{\text{right}}}{t} \right| \leq 2, \quad \forall E > 0,$$

has been proposed [11] to detect whether a complex PT-symmetric potential possesses unbroken PT-symmetry and whether it would give rise to unitary eigenvalues ($s_{\pm}$). The latter result is being re-examined in the sequel.

The Scarf II potential

$$V(x) = P \text{sech}^2 x + Q \text{sech} x \tanh x$$

is a versatile potential entailing several interesting parametric regimes in both PT-symmetric and non-PT-symmetric domains. By virtue of the available beautiful complex transmission and reflection amplitudes [21], Scarf II has helped in giving simple expressions for SS [8,14]. Recently, it has revealed [16] a rare (accidental) phenomena like reciprocity despite complex PT-symmetry and unitarity ($R + T = 1$) despite non-Hermiticity. In the following we invoke three parametric regimes of complex Scarf II by complexifying $P$ and $Q$ in various ways to demonstrate the novel [7,9-13] aspects of scattering from optical potentials discussed above.

### NPT: Non-PT-symmetric domain (CPA alone)

Let us consider the non-PT-symmetric domain of Scarf II as

$$V_{d}(x) = (d^2 - id) \text{sech}^2 x, \quad d \in \mathbb{R}.$$  

This is an absorptive P-symmetric potential ($d > 0$) and it would be ideal for demonstrating controlled CPA [11]. Using the transmission and reflection amplitudes [21] and by eliminating the Gamma functions with complex argument in them, in this case (13), we find

$$T(k) = \frac{k - d}{k + d} \frac{\sinh \pi k}{\cosh^2 \pi k - \cosh^2 \pi d},$$

$$f_{\text{left}}(k) = -\frac{\sinh \pi d}{\sinh \pi k} = f_{\text{right}}(k),$$

the reflection amplitudes are calculated as $r(k) = t(k) f(k)$, the equivalence of left/ right in (15) is by virtue of the P-symmetry of the potential (13) see [1,3]. Next we derive

$$|\det S(k)| = \left| \frac{k - d}{k + d} \frac{\cosh^2 \pi k - \cosh^2 \pi d}{\cosh^2 \pi k - \cosh^2 \pi d} \right|.$$  

Notice that in (14), $k = -d$ is a pole (SS) and $k = +d$ is not a pole, use L'Hospital rule to see this limit $\lim_{k \to +d} T(k) = \frac{\tanh \pi d}{\pi d}$, which is finite. So there is only one SS. The non-variances (3) can be verified readily. More interestingly at $k = d$, $|\det S(d)| = \frac{d}{6}$ but $\lim_{k \to +d} |\det S(k)| = 0$. Unlike the case of CPA with lasing [10] where it ought to be 1. This however presents the scenario of CPA [9]. The second aspect of CPA is met by noticing that $k = -d$ is clearly an SS [equivalently SS at $k = d$ in time-reversed transmittivity, $T(-k)$ (14)]. Moreover, unlike the case of CPA with lasing for PT-symmetric case or CPA, at $k = d$ here we get $\lim_{k \to -d} |\det S(k)| = \infty$. Interestingly, the simple non-PT-symmetric version
of Scarf II in (13) captures all essential features of its more general cases not considered here.

**PT1: The occurrence of CPA with lasing**

However, the situation changes dramatically when there is a spectral singularity present in the potential. Now let us consider the following parametrization of Scarf II potential for $c \in \mathbb{R}$

$$V_c(x) = [2c^2 - 1/4]\text{sech}^2 x - i[2c^2 + 1/2]\text{sech} x \tanh x.$$  

(17)

For this case, we obtain

$$T(k) = \frac{\sinh^2 \pi k \cosh^2 \pi k}{(\cosh^2 \pi k - \cosh^2 \pi c)^2}.$$  

(18)

and

$$f_{\text{left}}(k) = i [e^{-\pi k} - e^{\pi k} \cosh 2\pi c] \text{cosech}2\pi k, \quad f_{\text{right}}(k) = i [e^{\pi k} - e^{-\pi k} \cosh 2\pi c] \text{cosech}2\pi k.$$  

(19)

One can readily notice self-dual SS [15] in transmission co-efficient (18) (poles at $k = \pm c$), i.e., at $E = c^2$ both $T(-c)$ and $T(c)$ are infinity. This potential entails an essential anisotropic transmission resonance (ATR [13]) at $k = k_0$, such that $R_{\text{left}}(k_0) = 0, T(k_0) = 0$ but $R_{\text{right}}(k_0) \neq 0$. Using (19) we obtain $k_0 = \frac{1}{2\pi} \log \cosh 2\pi c$

Next, using (18,19) we obtain

$$|\det S| = \left(\frac{\cosh^2 \pi k - \cosh^2 \pi c}{\cosh^2 \pi k - \cosh^2 \pi c}\right)^2.$$  

(20)

Thus $|\det S|$ becomes $0$ at $k = \pm c$ meaning that $|\det S| = 1$, for $k \neq \pm c$ and limit$_{k\to \pm c}|\det S| = 1$. This completes the simplest demonstration of the phenomenon called CPA with lasing [10].

Further, we find the expression for eigenvalues $s_{\pm}(k)$ as

$$s_{\pm}(k) = t(k) \left[1 \pm \sqrt{1 - \left(\frac{\cosh 2\pi k - \cosh 2\pi c}{\sinh 2\pi k}\right)^2}\right].$$  

(21)

At $E = c$ (SS), $s_-$ is zero and $s_+ \to \infty$ as it can also be re-written as $s_+ = \sqrt{T} + \sqrt{T - 1}$ using (18). The expression (21) can also be re-written in terms of $k_0$ wherein it follows that $|s_{\pm}(k < k_0)| = 1$ and after the corresponding energy, $E_0$, these eigenvalues are non-unitary. So the potential (17) can not display transparency.

The potential (17) is a scattering potential without entailing real discrete eigenvalues- a fact that can be confirmed by seeing that there is no negative energy pole in (18). Therefore this potential essentially breaks PT-symmetry as

$$B_c(k) = 2 \cosh^2 \pi c \text{ sech}\pi k > 2,$$  

(22)

at lower energies. So, for this domain of PT-symmetry, we observe simultaneous occurrence of CPA with lasing at the same energy ($E = c$) and existence of an ATR and non existence of transparency as the $s_{\pm}$ are not unitary for all real positive energies.

**PT2: PT-symmetric domain (Transparency and no CPA)**
When in (12), \( P = -V_1, V_1 > 0 \) and \( Q = iV_2 \) (both \( V_1, V_2 \in R \)), it has been shown [19] that if \( |V_2| \leq V_1 + 1/4 \), the potential entails real discrete spectrum wherein the energy eigenstates are also eigenstates PT (PT-symmetry exact(unbroken)[18]), otherwise the real discrete eigenvalues disappear and make transition to non-real complex conjugate pairs and the PT symmetry is said to be spontaneously broken. Therefore, for all real values of \( a, b \) the potential

\[
V_{a,b}(x) = -(a^2 + b^2 + a) \text{sech}^2 x - ib(2a + 1)\text{sech} x \tanh x
\]  

(23)
can be verified to have finite number of real discrete eigenvalues and the PT-symmetry remains unbroken. As the beautiful complex transmission and reflection amplitudes of this versatile potential are already available [21], the following results follow from there [16].

\[
T(k) = \frac{\sinh^2 \pi k \cosh^2 \pi k}{(\sinh^2 \pi k + \sin^2 \pi a)(\sinh^2 \pi k + \cos^2 \pi b)},
\]

(24)

and

\[
f_{a,b}(k) = i \left[ -\frac{\cos \pi a \sin \pi b}{\cosh \pi k} + \frac{\sin \pi a \cos \pi b}{\sinh \pi k} \right].
\]

(25)

\[
R_{left}(k) = T(k)|f_{a,b}(k)|^2, R_{right}(k) = T(k)|f_{a,-b}(k)|^2
\]

(26)

Verify that the reflection and transmission (24) coefficients have common relevant poles at real discrete energies:

\[
E_n = -(n - a)^2, E_m = -(m - 1/2 - b)^2,
\]

(27)

where \( 0 \leq n < a \) and \( 0 \leq m < b + 1/2 < m \) which are two branches of the well known discrete eigenvalues [14,22] of (23). The in-variances given in (2) can be readily checked using (24-26). From (25), we can find the only zero of left-reflectivity as

\[
k_0 = \frac{1}{\pi} \tanh^{-1} \left( \frac{\tan \pi a}{\tan \pi b} \right), \text{ if } |\tan \pi a| < |\tan \pi b|.
\]

(28)

Using (25), it can be analytically verified that at this real wave-number, \( k_0 \), we have \( T(k_0) = 1 \) and \( R_{left}(k_0) = 0 \), but \( R_{right}(k_0) \neq 0 \). This is called ATR [13] at \( E = E_0 = k_0^2 \). Further, we can write

\[
|\det S(k)| = T(k) \left[ 1 - f_{a,b}(k)f_{a,-b}(k) \right].
\]

(29)

Using (24,26), we eventually find that

\[
|\det S(k)| = \frac{\sinh^4 \pi k + \sinh^2 \pi k(\sin^2 \pi a + \cos^2 \pi b) + \sin^2 \pi a \cos^2 \pi b}{(\sinh^2 \pi k + \sin^2 \pi a)(\sinh^2 \pi k + \cos^2 \pi b)} = 1.
\]

(30)

One can at once check that \( T(k) \) (24) does not have any pole at a real \( k \) and it can not become infinity (absence of SS) at any positive or negative real value of \( k \). Additionally the resulting \( |\det S| = 1 \) (30), at every real energy does not allow the potential (23) to become a coherent perfect absorber, as it requires \( |\det S| \) to be zero at one real positive energy (7) [9].

Now let us see the behaviour of eigenvalues \( s_{\pm} \) (10) for the (23). These can be expressed as

\[
s_{\pm} = t(k) \left[ 1 \pm \sqrt{f_{a,b}(k) f_{a,-b}(k)} \right].
\]

(31)
For (23), we obtain

\[
s_{\pm}(k) = t(k) \left[ 1 \pm \sqrt{-\sin(a-b)\pi \sin(a+b)\pi \cosh^2 \pi k - \cos^2 \pi a \sin^2 \pi b \over \sinh \pi k \cosh \pi k} \right]
\]  

(32)

We observe that when \( a \pm b = n \in I \) the above expression simplifies and even more so if \( a = b = \alpha \) or \( a = -b = \alpha \). Then from (24) and (32), for (23) we obtain

\[
s_{\pm}(k) = \sinh 2\pi k \sqrt{2 \cosh \pi k + \sin^2 2\pi \alpha} \left[ 1 \pm i \sin 2\pi \alpha \cosech 2\pi k \right].
\]  

(33)

in addition to (24,25) and (30). The s-eigenvalues (33) are indeed unitary as we get \(|s_{\pm}(k)| = 1\). There are several other parametric regimes where \( s_{\pm} \) will be unitary (uni-modular). However, there will be parametric regimes where \( s_{\pm} \) are non-unitary for all real positive energies. For instance, when \( a = n + 3/4, b = n + 1/2, n \in I^+ \{0\} \) or vice-versa, the potential (23) supports bound states, from (24) and (32) we get non-unitary eigenvalues

\[
s_{\pm}(k) = \frac{\sqrt{2 \cosh \pi k}}{\sqrt{2 \cosh^2 \pi k - 1}} \left[ 1 \pm \frac{\text{sech} \pi k}{\sqrt{2}} \right],
\]  

(34)

as \(|s_{\pm}(k)| \neq 1\), however \(|s_{+}(k)||s_{-}(k)| = 1\). Else, there could be parametric regimes where \( s_{\pm} \) may be unitary only upto \( E_0[E_0 = k_0^2 \text{ (ATR)}] \). To appreciate this, we need to rewrite (32) in terms of \( k_0 \) as

\[
s_{\pm}(k) = t(k) \left[ 1 \pm \sqrt{(\sin^2 \pi b - \sin^2 \pi a)(\cosh^2 \pi k - \cosh^2 \pi k_0)} \over \sinh \pi k \cosh \pi k} \right]
\]  

(35)

Thus, the complex PT-symmetric potential (23) which preserves PT-symmetry [by virtue of its real discrete spectrum (27)] and which is devoid of spectral singularity or ATR, both of its s-eigenvalues of S-matrix can be uni-modal at any energy. This will result in perfect emission of coherent waves injected with any energy on both sides of the potential. We have called this property transparency.

In contrast to the just discussed transparency, we see that in the presence of ATR the potential will be transparent only for \( E < E_0\)\((= \pi k_0^2)\). We remark that this situation has already been observed experimentally and reported in [11]. For the potential (23) using (24-26) we obtain \( B_{a,b}(k) = |2 \sin \pi a \cos \pi b \sech \pi k| \) and \( B_{a}(k) = |\sin 2\pi \alpha \sech \pi k| \), it follows readily that in these cases the proposed [11] condition of exact (unbroken) PT-symmetry that \( B(k) < 2 \) is met well irrespective of whether the eigenvalues \( s_{\pm} \) are uni-modal or not [see Eqs. (31-35)].

The various aspects of both PT-symmetry and non-PT-symmetry demonstrated here are no way the special features of Scarf II potential. We have confirmed this by solving other potentials using numerical integration of Schrödinger equation. The smooth complex potentials like \( V(x) = V_1e^{-x^2} + iV_2xe^{-x^2} \) behave like Scarf II, but rectangular wells in some parametric domains show multiple ATRs and multiple bands of unitarity or non-unitarity of eigenvalues \( s_{\pm} \); the end-energies of bands and energies of ATRs show some interesting concurrences. However, its other parametric domains entail the qualitatively similar aspects
as presented here, including the transparency for all positive energies of coherent injection. We hope that the simplicity and the pedagogic value of complex Scarf II has been noted well.

Lastly, we conclude by asserting that complex PT-symmetric potentials do not display coherent perfect absorption without lasing. Instead, these potentials in some regime of unbroken PT-symmetry can display transparency for any energy of injection of coherent waves on a complex medium from left and right. These potentials have both eigenvalues of the $S$-matrix enjoying unitarity: $|s_{\pm}(k)| = 1$. The unitarity (uni-modularity) of eigenvalues, $s_{\pm}$, for all positive energies manifests in transparency however the question of the condition governing this remains open.

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