Convex Fairness Measures: Theory and Optimization

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Abstract

We propose a new parameterized class of fairness measures, convex fairness measures, suitable for optimization contexts. This class includes our new proposed order-based fairness measure and several popular measures (e.g., deviation-based measures, Gini deviation). We provide theoretical analyses and derive a dual representation of these measures. Importantly, this dual representation renders a unified mathematical expression and a geometric characterization for convex fairness measures through their dual sets. Moreover, we propose a generic framework for optimization problems with a convex fairness measure objective, including reformulations and solution methods. Finally, we provide a stability analysis on the choice of convex fairness measures in the objective of optimization models.

Keywords: Fairness in optimization, convexity, decomposition, stability analysis

1. Introduction

Fairness concerns arise naturally in various decision-making contexts and application domains including, but not limited to, healthcare scheduling, facility location, resource allocation, public service provision, and humanitarian operations (Bertsimas et al., 2013; Breugem et al., 2022; Filippi et al., 2021; Gutjahr and Fischer, 2018; Li et al., 2022; Sun et al., 2022). Thus, a detailed understanding of how to incorporate fairness in optimization contexts is necessary. Unfortunately, quantifying fairness is challenging because there is no single definition or measure of fairness that is universally accepted. Instead, there is a wide variety of notions and measures of fairness in the economics, decision theory, and operations research literature. In addition, different measures of fairness may produce remarkably different conclusions. For instance, in a study of emergency room placement problem, Gupta et al. (2020) showed that employing different measures of fairness in the objective could lead to different optimal placement locations. Moreover, it is well-known that focusing on fairness alone may degrade efficiency. Despite the importance of the subject, to date, we still do not have a unified framework for incorporating fairness criteria in optimization models (Chen and Hooker, 2021; Karsu and Morton, 2015). In this paper, we address this knowledge gap...
by proposing a new unified framework for optimization problems with a convex fairness measure objective, which includes the new notion of convex fairness measures and their theoretical properties, as well as solution approaches to these problems.

We start by introducing an axiomatic definition of fairness measures in Section 3. Then, we analyze a well-known and commonly employed set of deviation-based fairness measures in Section 4. We show that these measures are not equivalent in general (e.g., they could yield different optimal solutions with varying outcomes). This motivates us to propose a new framework that unifies different fairness measures into a general class of convex fairness measures suitable for optimization contexts. Specifically, we first propose a new order-based fairness measure in Section 5, which serves as the building block of our proposed class of convex fairness measures in Section 6. Next, we derive the dual representation of convex fairness measures, which allows us to investigate the equivalence of convex fairness measures from a geometric perspective. In Section 7, we use this dual representation to derive equivalent reformulations of optimization models with a convex fairness measure objective and develop decomposition methods to solve the reformulations. Finally, we conduct a stability analysis on the choice of convex fairness measures in optimization models via their dual representations in Section 8.

Notation. For two integers \(a\) and \(b\) with \(a < b\), we define the sets \([a] := \{1, \ldots, a\}\) and \([a,b]_Z := \{a, a+1, \ldots, b\}\). We use boldface letters to denote vectors. In particular, \(\mathbf{0}\) and \(\mathbf{1}\) are vectors of zeros and ones respectively, where the dimension will be clear in the context. For a given vector \(\mathbf{x}\), we denote \(x(i)\) as the \(i\)th smallest entry of \(\mathbf{x}\). For two vectors \(\mathbf{x} \in \mathbb{R}^N\) and \(\mathbf{y} \in \mathbb{R}^N\), we say \(\mathbf{x}\) is majorized by \(\mathbf{y}\), denoted as \(\mathbf{x} \preccurlyeq \mathbf{y}\), if \(\sum_{i=1}^N x_i = \sum_{i=1}^N y_i\) and \(\sum_{i=k}^N x(i) \leq \sum_{i=k}^N y(i)\) for \(k \in [2,N]_Z\). We use \(\mathbb{R}^N_+\) to denote the set of \(\mathbb{R}^N\) vectors with entries in ascending order, i.e., \(\mathbb{R}^N_+ = \{\mathbf{x} \in \mathbb{R}^N | x_1 \leq \cdots \leq x_N\}\). Finally, we use \(\text{conv}(C)\) to denote the convex hull of a set \(C\).

2. Literature Review

The importance of fairness has been recognized and well studied in various settings and application domains. As a result, a large body of the literature has invested in understanding and axiomatically characterizing what might constitute a measure of fairness (Barbati and Piccolo, 2016; Lan et al., 2010; Marsh and Schilling, 1994). Another stream, most relevant to our work, focuses on how to mathematically represent a measure of fairness in a way that is suitable for an optimization context. We refer readers to Chen and Hooker (2021) and Karsu and Morton (2015) for comprehensive surveys of existing fairness measures and their use in optimization models. Next, we briefly review several categories of fairness measures reported in these surveys and other literature. The first category measures the degree of equality in the distribution of utilities (e.g., resources allocated to different entities). Several well-known measures such as relative range and mean deviation belong to this category (Cowell, 2011). It also includes the well-known Gini index (Gini,
(1912), commonly used to measure income inequality in the economic literature. Another category focuses on the disadvantaged entities. This includes the famous Rawlsian principle (Rawls, 1999) which focuses on improving the worst-off entity (e.g., the minimum outcome level). As a result, the Rawlsian approach ignores the outcomes of all other entities. To remedy this issue, the Rawlsian approach is extended to a maximum lexicographic approach which sequentially maximizes the welfare of the worst-off, then the second worst-off, then the third worst-off, and so on (Kostreva et al., 2004; Ogryczak and Śliwiński, 2006).

Statistical fairness measures are considered in machine learning literature. Specifically, they are often employed in classification problems. For a given a set of sensitive groups (e.g., age groups), they measure the extent of differences in classification errors from two different groups (i.e., measure the fairness of a classification rule). Multiple definitions of fairness have been proposed such as demographic parity (Dwork et al., 2012) and equalized odds (Hardt et al., 2016). We refer readers to Mehrabi et al. (2021) for a recent survey. We note that this type of measures are typically designed for classification problems with only 0-1 outcomes, which is not the focus of this paper.

Note that efficiency (e.g., a classical objective such as the total utility) and fairness are commonly recognized as competing objectives. Indeed, various authors have shown that optimizing fairness alone may produce unrealistic (and potentially costly) solutions (Barbati and Piccolo, 2016; Filippi et al., 2021; Gupta et al., 2020; Karsu and Morton, 2015; Marsh and Schilling, 1994). There are two common approaches to co-optimize efficiency and fairness. The first approach is to choose the objective function as a linear or convex combination of an efficiency measure such as total costs and a fairness measure (Chen and Hooker, 2021; Filippi et al., 2021; Holguín-Veras et al., 2013). The weights on the two objectives represent the trade-off between efficiency and fairness. The second approach is constructing a new objective function that combines the two. The parameter of this new objective function typically controls the trade-off between efficiency and fairness. For example, Mo and Walrand (2000) proposed the alpha fairness measure, which combines the sum of utilities as the efficiency measure and the Rawlsian principle as the fairness measure. Williams and Cookson (2000) and Hooker and Williams (2012) proposed threshold models, where the objective function represents the efficiency or fairness criterion based on some threshold value. As pointed out in Chen and Hooker (2021), these objective functions are non-linear and hence, solution methods need to be tailored for each objective function.

In this paper, we first study the equivalence of a set of common deviation-based fairness measures. According to recent surveys by Chen and Hooker (2021) and Karsu and Morton (2015), there are no such analysis except a recent work by Gupta et al. (2020) that considers only two entities. We show that these deviation-based fairness measures are not equivalent in general (specifically, when there are more than two entities), motivating us to develop a framework that unifies different fairness measures into a general, parameterized class of convex fairness measures. We introduce a new order-based fairness measure, which serves as the building block of convex fairness measures.
Our proposed convex fairness measures include popular fairness measures such as Gini deviation and mean absolute deviation. We derive the dual representation of convex fairness measures, which provides a new geometric perspective in understanding the equivalence of convex fairness measures. Moreover, it allows us to propose a unified framework, including reformulations and solution approaches, for optimization problems with a convex fairness measure objective. In addition, we conduct stability analysis on the choice of convex fairness measures in optimization models via their dual representations.

### 3. Fairness Measures

In this section, we provide an axiomatic definition of fairness measures to lay the foundation for discussions in the following sections. Let \( \mathbf{u} = (u_1, \ldots, u_N)^\top \) be a utility vector of \( N \) different subjects (or groups), i.e., \( u_i \) is the utility of subject \( i \). We define a fairness measure through the following six axioms.

**Definition 3.1 (Fairness Measure).** A function \( \nu : \mathbb{R}^N \to \mathbb{R} \) is a fairness measure if it satisfies the following axioms.

1. **Continuity:** \( \nu \) is continuous on \( \mathbb{R}^N \);
2. **Normalization:** \( \nu(\mathbf{u}) \geq 0 \) for any \( \mathbf{u} \in \mathbb{R}^N \) and \( \nu(\mathbf{u}) = 0 \) if and only if \( \mathbf{u} = \alpha \mathbf{1} \) for some \( \alpha \in \mathbb{R} \);
3. **Symmetry:** \( \nu(\mathbf{u}) = \nu(P\mathbf{u}) \) for any \( \mathbf{u} \in \mathbb{R}^N \) and permutation matrix \( P \);
4. **Location invariance:** \( \nu(\mathbf{u} + \alpha \mathbf{1}) = \nu(\mathbf{u}) \);
5. **Positive homogeneity:** \( \nu(\alpha \mathbf{u}) = \alpha \nu(\mathbf{u}) \) for any \( \alpha > 0 \);
6. **Schur convexity:** if \( \mathbf{u}^1 \preceq \mathbf{u}^2 \), then \( \nu(\mathbf{u}^1) \leq \nu(\mathbf{u}^2) \).

Axiom (a) ensures that a small perturbation of the utility vector \( \mathbf{u} \) results in a small change in the fairness measure. Axiom (b) requires that the fairness measure is always non-negative and equal zero if every subject has the same utility. This implies a smaller value of \( \nu \) is preferable since utilities among subjects are less dispersed. Axiom (c) ensures that the value of \( \nu \) is invariant to the permutation of \( \mathbf{u} \), i.e., the identities of the subjects are irrelevant. Axiom (d) says that increasing the utility of each subject by \( \alpha \) will not affect \( \nu \). This is reasonable since fairness measures quantify the variability of the utilities. Axiom (e) requires that if the utility vector is multiplied by a factor \( \alpha \) (and hence, the pairwise difference between utilities is enlarged by a factor of \( \alpha \)), the value of \( \nu \) also scales with the same factor. Finally, axiom (f) is a desirable property of a fairness measure. Note that the condition \( \mathbf{u}^1 \preceq \mathbf{u}^2 \) implies \( \mathbf{u}^2 \) is less equitable than \( \mathbf{u}^1 \). Indeed, if \( \mathbf{u}^1 \preceq \mathbf{u}^2 \), one can obtain \( \mathbf{u}^1 \) from \( \mathbf{u}^2 \) by a finite number of Robin Hood transfers (a.k.a. Dalton transfers): replacing two entries \( u_i^2 \) and \( u_j^2 \) of \( \mathbf{u}^2 \) such that \( u_i^2 < u_j^2 \) by the values \( u_i^2 + \varepsilon \) and \( u_j^2 - \varepsilon \), respectively, for some \( \varepsilon \in (0, u_j^2 - u_i^2) \) (Marshall et al., 2011). Therefore, axiom (f) ensures that if \( \mathbf{u}^1 \preceq \mathbf{u}^2 \), \( \nu(\mathbf{u}^2) \) is no less than \( \nu(\mathbf{u}^1) \). The Schur convexity axiom is also known as the Pigou-Dalton condition (Dalton, 1920; Moulin, 2004).
### Table 1: Existing deviation-based fairness measures

| Index | Metric | Name |
|-------|--------|------|
| i.    | $\max_{i \in [N]} u_i - \min_{i \in [N]} u_i$ | Range |
| ii.   | $\sum_{i=1}^{N} \sum_{j=1}^{N} |u_i - u_j|$ | Gini deviation |
| iii.  | $\max_{i \in [N]} \max_{j \in [N]} |u_i - u_j|$ | Maximum pairwise deviation |
| iv.   | $\sum_{i=1}^{N} |u_i - \bar{u}|$ | Absolute deviation from mean |
| v.    | $\left[ \sum_{i=1}^{N} (u_i - \bar{u})^2 \right]^{1/2}$ | Standard deviation |
| vi.   | $\max_{i \in [N]} \sum_{j=1}^{N} |u_i - u_j|$ | Maximum absolute deviation from mean |
| vii.  | $\sum_{i=1}^{N} \max_{j \in [N]} |u_i - u_j|$ | Maximum sum of pairwise deviation |
| viii. | $\sum_{i=1}^{N} \max_{j \in [N]} |u_i - u_j|$ | Sum of maximum pairwise deviation |

**Remark** 1. Axioms and properties related to (a), (c), and (f) are also discussed in Lan et al. (2010); Karsu and Morton (2015). Note that we define fairness measures in an absolute sense (with the same unit as the utility), which could be observed from conditions (d) and (e). As pointed out in Mostajabdaveh et al. (2019), if one employs a linear combination of an efficiency measure and a fairness measure in the objective of an optimization model, it is natural that the fairness measure has the same unit as the efficiency measure as opposed to fairness measures in a relative sense (i.e., unitless).

### 4. Equivalence of Existing Fairness Measures

Recently, Gupta et al. (2020) investigated the equivalence relationships of various well-known fairness metrics when there are only $N = 2$ subjects (or groups of interest). They show that a subset of these measures is equivalent to the absolute difference and the remaining are equivalent to the relative difference. However, it unclear whether similar equivalence relationships hold when $N > 2$, which motivates us to extend the analysis to high-dimensional cases. For illustrative purposes, we focus on the set of eight common deviation-based fairness metrics in Table 1. First, in Proposition 1, we verify that these metrics are fairness measures satisfying Definition 3.1 (see Appendix Appendix A for a proof).

**Proposition 1.** The metrics (i) to (viii) in Table 1 are fairness measures.

Next, we investigate the equivalence between these measures. We say two fairness measures $\nu^1$ and $\nu^2$ are equivalent if there exists $\beta > 0$ such that $\nu^1(u) = \beta \nu^2(u)$ for all $u \in \mathbb{R}^N$. Hence, replacing $\nu^2$ by $\nu^1$ as a fairness criterion in the objective function essentially scales the weight on the fairness criterion by a factor of $\beta > 0$ (i.e., from $\nu^2$ to $\beta \nu^2$). Recall that Gupta et al. (2020) showed that all fairness measures in Table 1 are equivalent to $|u_1 - u_2|$ when $N = 2$. In Proposition 2, we show that only some of these measures are equivalent when $N > 2$ (see Appendix Appendix A for a proof). In Appendix Appendix B, we provide tables summarizing the equivalence or examples showing in-equivalence of these fairness measures.
Proposition 2. Only the following equivalence relationships between fairness measures shown in Table 1 hold: (a) for any $N \geq 3$, (i) and (iii) are equivalent; (b) for any $N \geq 3$, (vi) and (vii) are equivalent; (c) when $N = 3$, (i), (ii), and (iii) are equivalent; (d) when $N = 3$, (iv), (vi), and (vii) are equivalent. The remaining pairs of fairness measures are not equivalent.

5. Order-based Fairness Measures

The results in Proposition 2 emphasize that most fairness measures are not equivalent when $N \geq 3$, and thus, customized optimization frameworks are needed for each class of equivalent fairness measures (because non-equivalent measures have different mathematical expressions and characteristics). This inspires this paper’s main question: is there a general class of fairness measures that is rich enough to include these existing fairness measures and have a unified mathematical expression suitable for optimization contexts?. To provide an affirmative answer to this question, in this section, we introduce a new notion called order-based fairness measure. This order-based fairness measure is the building block of our proposed general class of convex fairness measures (presented in Section 6), which admits a unified mathematical expression.

Definition 5.1 (Order-based Fairness Measure). Let $W = \{ w \in \mathbb{R}^N \mid \sum_{i=1}^N w_i = 0, w_1 \leq \cdots \leq w_N, w_1 < 0, w_N > 0 \}$. We call a fairness measure $\nu : \mathbb{R}^N \to \mathbb{R}$ order-based if there exists $w \in W$ such that $\nu(u) = \nu_w(u) = \sum_{i=1}^N w_i u_i$.

Note that each order-based fairness measure is associated with a weight vector $w \in W$, where the weight $w_i$ associated with the $i$th smallest entry $u_i$ can be considered as the relative priority. Intuitively, from Definition 5.1, increasing the $i$th smallest utility by a small amount $\delta > 0$ changes $\nu_w(u)$ to $\nu_w(u) + \delta w_i$. Since $w_i \leq w_j$ when $i < j$, this implies that improving the $i$th smallest utility would lead to a smaller increase or larger decrease in $\nu_w(u)$ (depending on the signs of $w_i$ and $w_j$), and thus promoting fairness. To illustrate, we provide the following example.

Example 5.1 (Fair resource allocation). Consider the problem of allocating $R$ resources fairly to $N$ individuals, where we use $i \in [N]$ to denote each individual. Let $x_i$ be the number of resources allocated to $i$. The utility of $i$ is measured as $u_i = a_i x_i$, where $a_i$ may represent the efficiency per unit resource allocated to $i$. Moreover, there is a limit $K \leq R$ on the number of resources allocated to each individual. Let $N = 6$ and $a_i = i$, for all $i \in [6]$. Suppose we use the order-based fairness measure $\nu_w(u)$ to ensure fair allocations with $w_i = 2(2i - 7)$, for $i \in [6]$. Then, our fair resource allocation optimization problem can be stated as

$$\begin{align*}
\text{minimize} & \quad -10u_{(1)} - 6u_{(2)} + 2u_{(3)} + 2u_{(4)} + 6u_{(5)} + 10u_{(6)} \\
\text{subject to} & \quad u_1 = x_1, u_2 = 2x_2, \ldots, u_6 = 6x_6, \sum_{i=1}^6 x_i = R, 0 \leq x_i \leq K, \forall i \in [6].
\end{align*}$$
Clearly, our optimization problem prioritizes resources to less advantaged individuals (i.e., those with lower utility, and thus, efficiency). To illustrate, we solve (1) numerically (see Section 7) with \( R = 25 \). Figure 1 shows the optimal allocation decisions with different values of \( K \). It is clear that more resources are allocated to less advantaged individuals. Even when \( K \) decreases from 10 to 7, i.e., when a smaller upper bound is imposed on \( x_i \), more resources are allocated to the less advantaged individuals with a priority to individual 2, followed by 3 to 6. (Note that the resources allocated to individual 1 decrease because of the decrease in the imposed upper bound \( K \) on \( x_i \)).

In Proposition 3, we show that \( \nu_w(u) \) is a fairness measure satisfying Definition 3.1.

**Proposition 3.** For any \( w \in W \), the function \( \nu_w : \mathbb{R}^N \to \mathbb{R} \) defined as \( \nu_w(u) = \sum_{i=1}^{N} w_i u(i) \) is a fairness measure satisfying Definition 3.1.

**Proof.** (a) **Continuity.** First, we claim that the sorting operator \( S : \mathbb{R}^N \to \mathbb{R}^N \) that maps a vector \( u \) to \( u^\uparrow \in \mathbb{R}^N \) with entries in ascending order is continuous. Consider two vectors \( u^1 \) and \( u^2 \) with \( \| u^1 - u^2 \|_\infty = \epsilon \), i.e., \( |u^1_i - u^2_i| \leq \epsilon \) for all \( i \in [N] \). We claim that \( |u^1(i) - u^2(i)| \leq \epsilon \) for all \( i \in [N] \). To show this, suppose, on the contrary, that \( |u^1(i) - u^2(i)| > \epsilon \) for some \( i \in [N] \). Consider the following two cases. First, if \( u^2(i) < u^1(i) - \epsilon \), define \( J_i^- = \{ \pi_2(1), \ldots, \pi_2(i) \} \), where \( \pi_2(k) \) is the index such that \( u^2_{\pi_2(k)} = u^2(k) \) for \( k \in [N] \). For all \( j \in J_i^- \), we have

\[
    u^1_j \leq u^2_j + \epsilon \leq u^2_{\pi_2(j)} + \epsilon < u^1_{\pi_2(j)},
\]

where the first inequality follows from \( \epsilon = \| u^1 - u^2 \|_\infty \), and the second inequality follows from \( j \in J_i^- \). This contradicts that \( u^1_{\pi_2(j)} \) is the \( j \)th smallest entry in \( u^\uparrow \). Similarly, if \( u^2(i) > u^1(i) + \epsilon \), define \( J_i^+ = \{ \pi_2(i), \ldots, \pi_2(N) \} \). Then, following a similar argument in (2), for all \( j \in J_i^+ \), we have

\[
    u^1_j \geq u^2_j - \epsilon \geq u^2(i) - \epsilon > u^1(i),
\]
which leads to the contradiction that \( u^1_{(i)} \) is the \( i \)th smallest entry in \( u^1 \). Therefore,

\[
|\nu_w(u^1) - \nu_w(u^2)| = \left| \sum_{i=1}^{N} w_i u^1_{(i)} - \sum_{i=1}^{N} w_i u^2_{(i)} \right| \leq \sum_{i=1}^{N} |w_i| |u^1_{(i)} - u^2_{(i)}| \\
\leq ||w||_1 ||u^1 - u^2||_{\infty}.
\]

This shows that \( \nu_w \) is continuous.

(b) **Normalization.** We first show that \( \nu_w(u) \geq 0 \) for any \( u \in \mathbb{R}^N \). Letting \( u(0) = 0 \), we can write

\[
\nu_w(u) = \sum_{i=1}^{N} w_i u_{(i)} = \sum_{i=1}^{N} w_i \sum_{j=1}^{i} [u(j) - u(j-1)] = \sum_{j=1}^{N} \left( \sum_{i=j}^{N} w_i \right) [u(j) - u(j-1)].
\]

Since \( \sum_{i=1}^{N} w_i = 0 \), we have

\[
\nu_w(u) = \sum_{j=2}^{N} \left( \sum_{i=j}^{N} w_i \right) [u(j) - u(j-1)].
\]

Since \( u(j) - u(j-1) \geq 0 \), to show that \( \nu_w(u) \geq 0 \), it suffices to show that \( \sum_{i=j}^{N} w_i > 0 \) for all \( j \in [2, N] \). We show \( \sum_{i=1}^{N} w_i > 0 \) by induction. When \( j = 2 \), since \( \sum_{i=1}^{N} w_i = 0 \) and \( w_1 < 0 \), we have \( \sum_{i=1}^{N} w_i = \sum_{i=1}^{N} [i - w_1 > 0] \). Next, suppose that \( \sum_{i=j-1}^{N} w_i > 0 \). If \( w_j \geq 0 \), then it is trivial that \( \sum_{i=j}^{N} w_j > 0 \) since \( 0 \leq w_j \leq \cdots \leq w_N \) and \( w_N > 0 \). If \( w_j \leq 0 \), then \( \sum_{i=j}^{N} w_i = \sum_{i=j}^{N} w_i - w_{j-1} > 0 \) by induction hypothesis. This completes the induction step and shows that \( \nu_w(u) \geq 0 \). Finally, we show that \( \nu_w(u) = 0 \) if and only if \( u = \alpha 1 \) for some \( \alpha \in \mathbb{R} \). If \( u = \alpha 1 \), then it trivial that \( \nu_w(u) = \alpha \sum_{i=1}^{N} w_i = 0 \). If \( \nu_w(u) = 0 \), by (3) and \( \sum_{i=j}^{N} w_i > 0 \), we must have \( u(j) - u(j-1) = 0 \) for all \( j \in [2, N] \), which in turn implies \( u(1) = \cdots = u(N) \).

(c) **Symmetry.** Symmetry follows directly from the definition of \( \nu_w \) that depends only on the order of utilities.

(d) **Location invariant.** It is straightforward to verify that

\[
\nu_w(u + \alpha 1) = \sum_{i=1}^{N} w_i [u(i) + \alpha] = \sum_{i=1}^{N} w_i u(i) + \alpha \sum_{i=1}^{N} w_i = \nu_w(u) + \alpha \nu_w(1).
\]

(e) **Positive homogeneity.** It is straightforward to verify that

\[
\nu_w(\alpha u) = \sum_{i=1}^{N} w_i [\alpha u(i)] = \alpha \sum_{i=1}^{N} w_i u(i) = \alpha \nu_w(u).
\]

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(f) **Schur convexity.** Letting \( w_0 = 0 \), we can write

\[
\nu_w(u) = \sum_{i=1}^{N} w_i u(i) = \sum_{i=1}^{N} \left[ \sum_{j=1}^{i} (w_j - w_{j-1}) \right] u(i) = \sum_{j=1}^{N} (w_j - w_{j-1}) \left[ \sum_{i=j}^{N} u(i) \right]
\]

\[
= w_1 \left[ \sum_{i=1}^{N} u(i) \right] + \sum_{j=2}^{N} (w_j - w_{j-1}) \left[ \sum_{i=j}^{N} u(i) \right].
\]

By definition, \( u^1 \leq u^2 \) implies \( \sum_{i=1}^{N} u_i^1 = \sum_{i=1}^{N} u_i^2 \) and \( \sum_{i=j}^{N} u_i^1 \leq \sum_{i=j}^{N} u_i^2 \) for all \( j \in [N] \).

Since \( w_j - w_{j-1} \geq 0 \), we have \( \nu_w(u^1) \leq \nu_w(u^2) \).

This shows that \( \nu_w \) is a fairness measure satisfying Definition 3.1.

Remark 2. Our definition of order-based fairness measure resembles the form of the ordered weighted average (OWA) operator introduced by Yager (1988) (see Csiszar, 2021 for a review).

Specifically, given a utility vector \( u \in \mathbb{R}^N \), the OWA operator is a mapping \( F : \mathbb{R}^N \rightarrow \mathbb{R} \) defined as \( F(u) = \sum_{i=1}^{N} w_i u(i) \), where the weight vector \( w \) satisfies \( w_i \in [0,1] \) and \( \sum_{i=1}^{N} w_i = 1 \). In contrast, the weight vector \( w \) in our order-based fairness measure satisfies \( \sum_{i=1}^{N} w_i = 0 \) with \( w \in \mathbb{R}^N \uparrow \), \( w_1 < 0 \), and \( w_N > 0 \) (see Definition 5.1). As detailed in Yager (1988), the OWA operator is a way to aggregate different values of \( u_i \), which was designed for multi-criteria decision-making problems.

Next, in Theorem 4, we show that \( \nu_w \) is a supremum of linear functions (in \( u \)) over a set of permutations of \( w \). As we show in Section 7, this characterization helps us incorporate order-based fairness measures \( \nu_w \) into optimization models.

**Theorem 4.** Let \( \Pi \) be the set of permutation functions \( \pi : \mathbb{R}^N \rightarrow \mathbb{R}^N \) on \([N]\). For any \( u \in \mathbb{R}^N \), we have \( \nu_w(u) = \sup_{\pi \in \Pi} \sum_{i=1}^{N} w_{\pi(i)} u(i) \), where \( \tilde{W} = \{ \tilde{w} \in \mathbb{R}^N \mid \tilde{w}_i = w_{\pi(i)}, i \in [N], \pi \in \Pi \} \).

**Proof.** From the rearrangement inequality (Marshall et al., 2011), for any \( \pi \in \Pi \), we have

\[
\sum_{i=1}^{N} w_{\pi(i)} u_i \leq \sum_{i=1}^{N} w(i) u(i) = \sum_{i=1}^{N} w_i u(i) = \nu_w(u),
\]

where the first equality follows from \( w_1 \leq \cdots \leq w_N \).

Remark 3. Note, from Theorem 4, that \( \nu_w \) is a maximum of linear functions. It follows that \( \nu_w \) is convex.
6. Convex Fairness Measures

In the section, we present our proposed class of fairness measures, convex fairness measures. We also derive a dual representation of convex fairness measures. With this dual representation, one can express any convex fairness measure as a worst-case order-based fairness measure (over a set of weights $w$). In addition, this dual representation allows us to investigate the equivalence of convex fairness measure from a geometric perspective. In Section 7, we use this dual representation to propose a unified framework for optimization models with a convex fairness measure objective.

We first define our proposed class of fairness measures. We call a function $\nu : \mathbb{R}^N \to \mathbb{R}$ a convex fairness measure if it is convex and satisfies Definition 3.1. Next, in Theorem 5, we provide one of the key results of this paper, which is the dual representation of convex fairness measures.

**Theorem 5.** Let $\nu : \mathbb{R}^N \to \mathbb{R}$ be a fairness measure and $S^N = \{w \in \mathbb{R}_+^N \mid 1^T w = 0\}$. The following statements are equivalent: (a) $\nu$ is a convex fairness measure; (b) there exists a convex compact set $W_\nu \subseteq S^N$ with $W_\nu \neq \{0\}$ such that $\nu(u) = \sup_{w \in W_\nu} \nu_w(u)$; (c) there exists a compact set $W_\nu \subseteq S^N$ with $W_\nu \neq \{0\}$ such that $\nu(u) = \sup_{w \in W_\nu} \nu_w(u)$.

**Proof.** Note that it is trivial that (b) implies (c). We first prove that (c) implies (a). Suppose there exists a compact set $W_\nu \subseteq \{w \in \mathbb{R}^N_+ \mid 1^T w = 0\}$ such that $\nu(u) = \sup_{w \in W_\nu} \nu_w(u)$. We want to verify that $\nu(u)$ is a convex fairness measure.

- **Continuity.** From the proof of Proposition 3, for any $\{u^1, u^2\} \subset \mathbb{R}^N$, we have $|\nu_w(u^1) - \nu_w(u^2)| \leq \|w\|_1 \|u^1 - u^2\|_\infty$. Hence,

  $$|\nu(u^1) - \nu(u^2)| = \left| \sup_{w \in W} \nu_w(u^1) - \sup_{w \in W} \nu_w(u^2) \right| \leq \sup_{w \in W} |\nu_w(u^1) - \nu_w(u^2)| \leq \sup_{w \in W} \|w\|_1 \cdot \|u^1 - u^2\|_\infty,$$

  where $\sup_{w \in W} \|w\|_1 < \infty$ since $W$ is compact. Hence, $\nu$ is continuous in $u$.

- **Normalization.** Since $\nu_w(u) \geq 0$ for any $w \in W$ by Proposition 3, we have $\nu(u) \geq 0$. Moreover, note that $\nu(u) = 0$ is equivalent to $\nu_w(u) = 0$ for all $w \in W$. By Proposition 3, $\nu_w(u) = 0$ if and only if $u = \alpha 1$ for some $\alpha \in \mathbb{R}$.

- **Symmetry.** Symmetry holds since each $\nu_w(u)$ is symmetric by Proposition 3.

- **Location invariance.** By Proposition 3, for any $\alpha \in \mathbb{R}$,

  $$\nu(u + \alpha 1) = \sup_{w \in W} \nu_w(u + \alpha 1) = \sup_{w \in W} \nu_w(u) = \nu(u).$$

- **Positive homogeneity.** By Proposition 3, for any $\alpha > 0$,

  $$\nu(\alpha u) = \sup_{w \in W} \nu_w(\alpha u) = \alpha \sup_{w \in W} \nu_w(u) = \alpha \nu(u).$$
\textbullet{} \textit{Schur convexity/Convexity.} Since $\nu_w$ is convex for all $w \in \mathcal{W}$ from Remark 3 and $\nu$ is a supremum of convex functions, $\nu$ is also convex.

Next, we prove the (a) implies (b). Note that by definition of convex fairness measures, $\nu$ is proper, continuous, and convex. By Fenchel–Moreau theorem, $\nu(u) = \sup_{w \in \mathbb{R}^N} \{ u^T w - \nu^*(w) \}$, where $\nu^*(w) = \sup_{u \in \mathbb{R}^N} \{ u^T w - \nu(u) \}$ is the convex conjugate of $\nu$ (Bertsekas, 2009). We divide the proof into the following four steps.

\textit{Step 1.} By positive homogeneity of $\nu$, we have for any $\alpha > 0$,

$$
\nu^*(w) = \sup_{u \in \mathbb{R}^N} \{ u^T w - \nu(u) \} = \alpha \sup_{u \in \mathbb{R}^N} \left\{ \left( \frac{u}{\alpha} \right)^T w - \nu \left( \frac{u}{\alpha} \right) \right\} = \alpha \sup_{u' \in \mathbb{R}^N} \{ (u')^T w - \nu(u') \} = \alpha \nu^*(w),
$$

where we apply a change of variable $u' = u/\alpha$. Thus, $\nu^*(w)$ equals 0 if $w \in \text{dom}(\nu^*)$, and $\infty$ otherwise. Note that $\text{dom}(\nu^*) = \{ w \in \mathbb{R}^N \mid \nu^*(w) \leq 0 \} = \{ w \in \mathbb{R}^N \mid u^T w - \nu(u) \leq 0, \forall u \in \mathbb{R}^N \} = \partial \nu(0)$, where $\nu(0)$ is the subdifferential of $\nu$ at $u = 0$. Therefore, the set $\text{dom}(\nu^*)$ is closed, bounded, and convex (see Proposition 5.4.2 of Bertsekas, 2009). Thus, we have $\nu(u) = \sup_{w \in \text{dom}(\nu^*)} \{ u^T w \}$.

\textit{Step 2.} Since $\nu$ is location invariant, we have for any $\tilde{u} \in \mathbb{R}^N$,

$$
\nu^*(w) = \sup_{u \in \mathbb{R}^N} \{ u^T w - \nu(u) \} \geq \sup_{\alpha \in \mathbb{R}} \{ (\tilde{u} + \alpha 1)^T w - \nu(\tilde{u} + \alpha 1) \} = \sup_{\alpha \in \mathbb{R}} \{ \tilde{u}^T w - \nu(\tilde{u}) + \alpha 1^T w \}.
$$

Therefore, if $w \in \text{dom}(\nu^*)$, we must have $1^T w = 0$.

\textit{Step 3.} Since $\nu$ is symmetric, for any permutation matrix $P \in \mathcal{P}$ (the set of all $N \times N$ permutation matrices),

$$
\nu^*(Pw) = \sup_{u \in \mathbb{R}^N} \{ u^T (Pw) - \nu(u) \} = \sup_{u \in \mathbb{R}^N} \{ (P^T u)^T w - \nu(u) \} = \sup_{u' \in \mathbb{R}^N} \{ (u')^T w - \nu(Pu') \} = \sup_{u' \in \mathbb{R}^N} \{ (u')^T w - \nu(u') \} = \nu^*(w),
$$

where we apply a change of variable $u' = P^T u$. Thus, $\nu^*$ is also symmetric. That is, if $w \in \text{dom}(\nu^*)$, then $Pw \in \text{dom}(\nu^*)$ for any $P \in \mathcal{P}$. As a result, $\text{dom}(\nu^*) = \bigcup_{P \in \mathcal{P}} \{ \tilde{w} \in \mathbb{R}^N \mid \tilde{w} = Pw, w \in \mathcal{W}_u \}$ with $\mathcal{W}_u = \text{dom}(\nu^*) \cap \mathbb{R}^N_1$ still being compact and convex. Hence, we can write

$$
\nu(u) = \sup_{w \in \text{dom}(\nu^*)} \{ u^T w \} = \sup_{w \in \mathcal{W}_u} \sup_{P \in \mathcal{P}} \{ u^T (Pw) \} = \sup_{w \in \mathcal{W}_u} \nu_w(u),
$$

where the last equality follows from Theorem 4.
Step 4. By the axioms of continuity, convexity, positive homogeneity, and location invariance, we have shown that $\nu(u) = \sup_{w \in W_\nu} \nu_w(u)$. Therefore, we immediately have $\nu(u) \geq 0$ since $\nu_w(u) \geq 0$. Moreover, since $\nu$ equals zero if and only if $u = \alpha 1$ for some $\alpha \in \mathbb{R}$. This implies that $W_\nu \neq \{0\}$ (otherwise, $\nu(u) = 0$ for any $u \in \mathbb{R}^N$).

To conclude, if $\nu$ is a convex fairness measure, then $\nu(u) = \sup_{w \in W_\nu} \nu_w(u)$, where $W_\nu = \text{dom}(\nu^*) \cap \mathbb{R}_+^N \subseteq \{w \in \mathbb{R}^N \mid 1^T w = 0\}$ and $W_\nu \neq \{0\}$. \hfill \Box

**Remark 4.** Note that $\nu^*(0) = \sup_{u \in \mathbb{R}^N} \{u^T 0 - \nu(u)\} = -\inf_{u \in \mathbb{R}^N} \nu(u) = 0$, which implies that $0 \in \text{dom}(\nu^*)$.

**Remark 5.** Let $\tilde{W} \subseteq \mathbb{R}^N$ be a compact set and consider the convex fairness measure $\nu(u) = \sup_{w \in \tilde{W}} \nu_w(u)$. Then, $\nu(u) = \sup_{w \in \text{conv} \tilde{W}} \nu_w(u)$. Indeed, letting $\mathcal{P}$ be the set of permutation matrices, we have

$$
\sup_{w \in \tilde{W}} \nu_w(u) = \sup_{w \in \tilde{W}} \sup_{P \in \mathcal{P}} u^T (P^T w) = \sup_{P \in \mathcal{P}} \sup_{w \in \tilde{W}} (P^T u)^T w = \sup_{P \in \mathcal{P}} \sup_{w \in \text{conv} \tilde{W}} (P^T u)^T w,
$$

where the first equality follows from Theorem 4 and the last equality follows from the linearity of the objective.

Theorem 5 provides a dual representation of any convex fairness measures characterized by a **dual set**, i.e., $W_\nu \subseteq \{w \in \mathbb{R}_+^N \mid 1^T w = 0\}$. Note that the dual set may not be unique in the sense that $\nu$ can equal $\sup_{w \in W_1} \nu_w$ and $\sup_{w \in W_2} \nu_w$ but $W_1 \neq W_2$ (see Remark 5). Also, Theorem 5 shows that any convex fairness measure can be expressed as a robustified order-based fairness measure (i.e., worst-case over the dual set $W_\nu$). Moreover, one can construct any convex fairness measure by defining a dual set. In particular, when preference information on $w$ is incomplete or ambiguous, one can construct a set of potential weight vectors (i.e., the dual set) instead of using a single (biased) weight vector. This is useful in practice where the decision-maker is concerned about fairness but cannot articulate one’s preference on $w$ (see, e.g., Armbruster and Delage, 2015; Hu et al., 2018, for similar discussions in preference robust optimization). Next, in Theorem 6, we show that one can use the dual representation in Theorem 5 to verify whether two convex fairness measures are equivalent.

**Theorem 6.** Let $\nu_1$ and $\nu_2$ be two convex fairness measures with dual sets $W_1 = \text{dom}(\nu_1^*) \cap \mathbb{R}_+^N$ and $W_2 = \text{dom}(\nu_2^*) \cap \mathbb{R}_+^N$, respectively. Then, $\nu_1$ is equivalent to $\nu_2$ if and only if $W_1 = \beta W_2$ for some $\beta > 0$.

**Proof.** First, if $W_1 = \beta W_2$ for some $\beta > 0$, then we have

$$
\nu_1(u) = \sup_{w \in W_1} \nu_w(u) = \sup_{w \in \beta W_2} \nu_w(u) = \beta \sup_{w \in W_2} \nu_w(u) = \beta \nu_2(u),
$$

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implying the equivalence of $\nu_1$ and $\nu_2$. Next, if $\nu_1$ is equivalent to $\nu_2$, then $\nu_1(u) = \beta \nu_2(u)$ for some $\beta > 0$. From the proof of Theorem 5, we have

$$
\sup_{w \in \text{dom}(\nu_1^*)} u^\top w = \nu_1(u) = \beta \nu_2(u) = \beta \sup_{w \in \text{dom}(\nu_2^*)} u^\top w = \sup_{w \in \beta \text{dom}(\nu_2^*)} u^\top w. \tag{4}
$$

As a result, (4) implies that the support functions of the convex compact sets $\text{dom}(\nu_1^*)$ and $\beta \text{dom}(\nu_2^*)$ are the same. Therefore, we have $\text{dom}(\nu_1^*) = \beta \text{dom}(\nu_2^*)$ by Theorem 13.2 of Rockafellar (1970), and thus, $W_1 = \text{dom}(\nu_1^*) \cap \mathbb{R}_+^N = \beta \text{dom}(\nu_2^*) \cap \mathbb{R}_+^N = \beta W_2$. \hfill \Box

**Example 6.1** (Equivalence of two-dimensional convex fairness measures). Figure 2a shows the set $S^N = \{ w \in \mathbb{R}_+^N \mid 1^\top w = 0 \}$ with $N = 2$. By Remark 4, the dual set $W_\nu = \text{dom}(\nu^*) \cap \mathbb{R}_+^N$ of any convex fairness measure $\nu$ is a line segment joining $0$ and a point on $S^2$. Thus, Theorem 6 implies that all two-dimensional convex fairness measures are equivalent. On the other hand, Figure 2b shows the set $S^3$, which is a surface. Since two different dual sets may not necessarily be proportional, two convex fairness measures may not be equivalent when $N > 2$.

From Theorem 5, a convex fairness measure can be characterized using a dual set $W_\nu$, which may have an infinite number of weight vectors. In Proposition 7, we show that it suffices to consider only the non-zero extreme points of $W_\nu$. Thus, this characterization could help establish theoretical convergence guarantees of solution approaches that exploit the extreme points of the dual set $W_\nu$ when solving optimization problems with a convex fairness measure in the objective. We propose such algorithms in Section 7.

**Proposition 7.** Let $\nu$ be a convex fairness measure with a convex dual set $W_\nu$. Then, $\nu(u) = \sup_{w \in \mathcal{E}(W_\nu) \setminus \{0\}} \nu_w(u)$, where $\mathcal{E}(W_\nu)$ is the set of extreme points of $W_\nu$.

**Proof.** Since $W_\nu$ is compact and convex, we have $w \in \text{conv}(\mathcal{E}(W_\nu))$. For any $w \in \text{conv}(\mathcal{E}(W_\nu))$,
Finally, since \( \max (\nu_{i}) \) is consistent with Proposition 2, implying that it is not proportional to that of (iv) and (vi). Thus, (v) is not equivalent to (iv) and which is also proved algebraically in Proposition 2. However, dual set of (v) has a curved boundary, the dual set of (iv) is proportional to that of (vi). Thus, by Theorem 6, (iv) and (vi) are equivalent, and (iii) are order-based with \( w_{i} = 2(2i - 1 - N) \), \( \forall i \in [N] \).

That is, \( \nu_{w}(u) \) is a convex combination of \( \nu_{\tilde{w}^{j}}(u) \) for \( j \in [K] \). As a result, we have \( \nu_{w}(u) \leq \max_{j \in [K]} \nu_{\tilde{w}^{j}}(u) \). Hence, it suffices to consider the supremum over the set of extreme points \( E(W_{\nu}) \). Finally, since \( \nu_{0} \equiv 0 \) is always dominated by \( \nu_{w} \) for any \( w \neq 0 \), we can consider the supremum over the set \( E(W_{\nu}) \setminus \{0\} \).

Since all the fairness measures in Table 1 are convex, they admit the dual representation in Theorem 5. Next, in Proposition 8, we derive the dual sets of these fairness measures, which enable us to investigate the equivalence of these fairness measures from a geometric perspective (see Appendix Appendix A for a proof).

**Proposition 8.** Let \( S^{N} = \{ w \in \mathbb{R}^{N} | 1^{T}w = 0 \} \). The dual sets of fairness measures in Table 1 are as follows.

(a) For (i) and (iii), \( W_{\nu} = \{(-1,0,\ldots,0,1) \in \mathbb{R}^{N}\} \).

(b) For (ii), \( W_{\nu} = \{ w' \in \mathbb{R}^{N} | w'_i = 2(2i - 1 - N), \forall i \in [N]\} \).

(c) For (iv)–(vi), \( W_{\nu} = S^{N} \cap \{ w \in \mathbb{R}^{N} | w = w' - \overline{w}1, \overline{w}' = (1/N) \sum_{j=1}^{N} w'_j, \|w'\|_{1} \leq 1 \} \), where \( q = \infty \), \( q = 2 \), and \( q = 1 \) for (iv), (v), and (vi), respectively.

(d) For (vii), \( W_{\nu} = S^{N} \cap \{ w \in \mathbb{R}^{N} | w = w' - \overline{w}1, \overline{w}' = (1/N) \sum_{j=1}^{N} w'_j, \|w'\|_{1} \leq N \} \).

(e) For (viii), \( W_{\nu} = \{ w \in \mathbb{R}^{N} | w_1 = -(N-k) - 1, w_2 = \cdots = w_k = -1, w_{k+1} = \cdots = w_{N-1} = 1, w_N = k + 1, k \in [N]\} \).

**Remark 6.** If follows from Proposition 8 that metrics (i)–(iii) are order-based since the dual sets of metrics (i)–(iii) have only one weight vector satisfying Definition 5.1. Specifically, metrics (i) and (iii) are order-based with \( w = (-1,0,\ldots,0,1) \in \mathbb{R}^{N} \) and metric (ii) is order-based with \( w' \), where \( w'_i = 2(2i - 1 - N) \) for all \( i \in [N] \). However, dual sets of metrics (iv)–(viii) consist of more than one non-zero extreme point, and hence, (iv)–(viii) do not satisfy Definition 5.1, i.e., (iv)–(viii) are not order-based.

**Example 6.2** (Dual sets in \( \mathbb{R}^{3} \)). Figure 3 shows the dual sets for (iv)–(vi) when \( N = 3 \). Note that the dual set of (iv) is proportional to that of (vi). Thus, by Theorem 6, (iv) and (vi) are equivalent, which is also proved algebraically in Proposition 2. However, dual set of (v) has a curved boundary, implying that it is not proportional to that of (iv) and (vi). Thus, (v) is not equivalent to (iv) and (vi), consistent with Proposition 2.
7. Optimization with Convex Fairness Measures

In this section, we propose solution approaches to solve optimization problems with our proposed convex fairness measures. We focus on a welfare maximization problem (Chen and Hooker, 2021). That is, our goal is to maximize the social welfare function \( W(u) = -\nu(u) \). For simplicity, we only consider a fairness measure in the objective, although our optimization approaches can be applied to problems that also include efficiency criteria. Specifically, our problem of interest is

\[
\min_{x, u} \left\{ \nu(u) \left| u = U(x), x \in X \right. \right\},
\]

where \( x \) is the decision, \( X \) is the set of feasible decisions, and \( U \) is the utility function.

7.1. Minimizing Order-based Fairness Measures

We first consider problem (5) with an order-based fairness measure objective \( \nu_w \) for some \( w \in W \) (see Section 5). In Theorem 9, we use the characterization of \( \nu_w \) in Theorem 4 to derive an equivalent reformulation of problem (5) with \( \nu = \nu_w \), which is a linear program if \( U \) is linear and \( X \) is characterized by linear constraints.

**Theorem 9.** Problem (5) with an order-based fairness measure \( \nu_w \) is equivalent to

\[
\min_{x, u, \lambda, \theta} \begin{align*}
1^T (\lambda + \theta) \\
\lambda_i + \theta_j \geq u_i w_j, & \quad \forall i \in [N], j \in [N], \\
u = U(x), & \quad x \in X.
\end{align*}
\]

**Proof.** From Theorem 4, we can write \( \nu(u) = \max_{P \in \mathcal{P}} u^T (Pw) \), where \( \mathcal{P} \) is the set of all permutation matrices given by

\[
\mathcal{P} = \left\{ P \in \mathbb{R}^{N \times N} \left| \sum_{i=1}^{N} P_{ij} = 1, \sum_{j=1}^{N} P_{ij} = 1, P_{ij} \in \{0, 1\}, \forall i \in [N], j \in [N] \right. \right\}.
\]
Note that the objective \( u^\top (Pw) \) is linear in \( P \), and the constraint matrix formed by the assignment constraints in \( P \) is totally unimodular (Martello and Toth, 1987). Hence, \( \max_{P \in \mathcal{P}} u^\top (Pw) \) is a linear program in variables \( P_{ij} \) and we can take its dual as

\[
\max_{P \in \mathcal{P}} u^\top (Pw) = \min_{\lambda \in \mathbb{R}^N, \theta \in \mathbb{R}^N} \left\{ \begin{bmatrix} 1 \top (\lambda + \theta) \right\} \lambda_i + \theta_j \geq u_iw_j, \forall i \in [N], j \in [N] \right\}.
\]

(8)

Combining (8) with the outer minimization over \( x \) and \( u \) in problem (5), we obtain the reformulation in (6).

\[ \square \]

Note that variables \( \lambda \) and \( \theta \) in (6) are unrestricted in sign. In Proposition 10, we derive lower and upper bounds on \( \lambda \) and \( \theta \), which reduce the search space of these variables (see Appendix A for a proof).

**Proposition 10.** Let \( U_{\max} = \sup_{x \in \mathcal{X}} \max_{i \in [N]} \{ U(x) \} \), and \( U_{\min} = \inf_{x \in \mathcal{X}} \min_{i \in [N]} \{ U(x) \} \), where \( U(x) \) is the \( i \)th entry of \( U(x) \). Without loss of optimality, we can impose the following bounds on variables \( \lambda \) and \( \theta \) in (6). For all \( i \in [N] \),

\[
0 \leq \lambda_i \leq (U_{\max} - U_{\min}) \| w \|_\infty =: \bar{\lambda},
\]

(9)

\[
\min \left\{ U_{\max}w_i, U_{\min}w_i \right\} - \bar{\lambda} \leq \theta_i \leq \max \left\{ U_{\max}w_i, U_{\min}w_i \right\}.
\]

(10)

**Example 7.1** (Gini deviation reformulation). Consider the problem of minimizing the Gini deviation, i.e., \( \sum_{i=1}^{N} \sum_{j=1}^{N} |u_i - u_j| \). Existing literature (Chen and Hooker, 2021; Lejeune and Turner, 2019; Shehadeh and Snyder, 2021) introduce auxiliary variables \( z_{i,i'} = |u_i - u_{i'}| \) and use the following equivalent reformulation:

\[
\min_{x, u, z} \left\{ \sum_{i=1}^{N} \sum_{i'=1}^{N} z_{i,i'} \left| z_{i,i'} \geq u_i - u_{i'}, z_{i,i'} \geq u_{i'} - u_i, \forall \{i, i'\} \subseteq [N], u = U(x), x \in \mathcal{X} \right. \right\}.
\]

(11)

In addition to variables \( (x, u) \) and the constraints in (5), this reformulation requires \( N^2 \) variables \( z \) and \( N^2 \) constraints on \( z \), while our reformulation (6) involves only \( 2N \) variables \( (\lambda, \theta) \) and \( N^2 \) constraints on \( (\lambda, \theta) \). Thus, our reformulation has a significantly smaller number of variables, especially when \( N \) is large.

To illustrate the potential computational gains when using (6) instead of (11), we consider a \( p \)-median facility location problem. Given a set of customer locations \( I \) and potential facility locations \( J \), we want to decide where to open \( p < |J| \) facilities to minimize the sum of the total transportation cost (i.e., an efficiency measure) and the Gini deviation in the transportation cost across all customers. The demand at \( i \in I \) is \( d_i \) and the transportation cost, per unit demand, between \( i \in I \) and \( j \in J \) is \( c_{ij} \). Let \( x_j \) be a binary variable taking value 1 if facility \( j \) is open, and \( y_{ij} \) be a binary variable taking value 1 if customer \( i \) is assigned to facility \( j \), and are 0 otherwise. Also, let \( r_i \) be the transportation cost at \( i \in I \). With this notation, we formulate this problem as

\[
\text{minimize} \quad x \in \{0,1\}^J, y \in \{0,1\}^{I \times J}, r \quad \gamma \sum_{i \in I} r_i + (1 - \gamma) \nu(r)
\]

(12a)
Table 2: Solution time (in second) over 5 randomly generated instances with $\gamma = 0.2$.

| $|I| = 40$ | $|I| = 50$ |
|-----------|-----------|
| $p = 13$  | $p = 13$  | $p = 10$  | $p = 10$  | $p = 8$  | $p = 8$  |
| Our Reformulation (6) | 27 | 37 | 39 | 48 | 74 |
| Linear Reformulation (11) | 37 | 52 | 53 | 48 | 71 |

subject to

$$r_i = \sum_{j \in J} d_{ij}y_{ij}, \quad \forall i \in I,$$

$$\sum_{j \in J} x_j = p, \quad \sum_{j \in J} y_{ij} = 1, \quad y_{ij} \leq x_j, \quad \forall i \in I, j \in J,$$

where \((12c)\) ensures that only \(p\) facilities are open, and each customer is assigned to exactly one open facility. For the objective, we use a convex combination of the total transportation cost and the Gini deviation $\nu(r) = |I|^{-1} \sum_{i \in I} \sum_{i' \in I} |r_i - r_{i'}|$.

We use data from Daskin (2013) to generate 5 random instances for each combination of $|I| \in \{40, 50\}$ and $p \in \{|I|/3, |I|/4, |I|/5\}$, rounded to the nearest integer (see Appendix C for details). We solve these instances using the linear reformulation (11) and our reformulation (6) with (9)–(10). Table 2 shows the minimum (min), average (avg), and maximum (max) computational time (in second) over 5 generated instances. In particular, we consider $\gamma = 0.2$, where more emphasis is put on the fairness measure. It is clear from Table 2 that our reformulation can solve all the generated instances significantly faster. Moreover, (11) cannot solve instances with $|I| = 50$ and $p = 10$ within 2 hours, while our reformulation can solve these instances within 40 minutes. These results demonstrate the computational advantages of our reformulation.

7.2. Minimizing Convex Fairness Measures

Next, we consider problem (5) with a convex fairness measure objective (see Section 6). By the dual representation in Theorem 5, we can reformulate problem (5) as

$$\min_{x, u} \left\{ \sup_{\nu \in \mathcal{W}_\nu} \nu_u(u) \left| \begin{array}{c} \nu = U(x), \quad x \in \mathcal{X} \end{array} \right. \right\}$$

$$= \min_{x, u, \delta} \left\{ \delta \left| \begin{array}{c} \delta \geq \sum_{i=1}^{N} w_iu_{i(i)}, \quad \forall \nu \in \mathcal{W}_\nu, \quad u = U(x), \quad x \in \mathcal{X} \end{array} \right. \right\}. \quad (13)$$

Problem (13) is challenging to solve because of the following two reasons. First, $\nu_u(u)$ is an order-based fairness measure, which is non-linear in $u$. Second, $\mathcal{W}_\nu$ may not be finite, i.e., (13) is a semi-infinite program. However, from Proposition 7, we know that it suffices to consider the non-zero extreme points of the dual set $\mathcal{W}_\nu$ to solve problem (13). Leveraging this fact, we next propose two decomposition methods to solve problem (13): (a) a column-and-constraint generation algorithm (C&CG), and (b) an alternating minimax algorithm (AMM). Both algorithms share the
Algorithm 1: Decomposition Methods

Initialization: Set $LB = 0, UB = \infty$, $\varepsilon > 0$, $j = 1$, $w^0 \in W_\nu$.

1. Master problem.
   (a) C&CG: Solve master problem (16) with weights $\{w^0, \ldots, w^{j-1}\}$.
   (b) AMM: Solve (6) with weight $w^{j-1}$.

   Record the optimal solution $(x^j, u^j)$ and optimal value $\delta^j$.
   Set $LB \leftarrow \max\{LB, \delta^j\}$.

2. Subproblem. Solve subproblem (15) for fixed $u = u^j$.
   Record the optimal solution $w^j$ and value $D^j$. Set $UB \leftarrow \min\{UB, D^j\}$.
   If $(UB - LB)/UB < \varepsilon$, terminate and return the solution with best objective.

3. Scenario set enlargement.
   Update $j \leftarrow j + 1$ and go back to step 1.

same three steps summarized in Algorithm 1, where we solve a master problem and subproblem at each iteration. However, as described next, the master problems in C&CG and AMM are different. We first discuss the details of C&CG. At iteration $j$ of C&CG, we aim to solve the following master problem

$$
\min_{x, u, \delta} \left\{ \delta \middle| \delta \geq \sum_{i=1}^{N} w_i u_{(i)}, \forall w \in \{w^0, \ldots, w^{j-1}\}, u = U(x), x \in X \right\},
$$

(14)

where $\{w^0, \ldots, w^{j-1}\} \subseteq W_\nu$. However, (14) cannot be solved directly because of the non-linearity of $\nu_w(u)$. In Proposition 11, we provide an equivalent solvable reformulation of (14). Since only a set of weight vectors in $W_\nu$ is considered, the master problem is a relaxation of the original problem (13), and thus its optimal value $\delta^j$ provides a lower bound to (13). With the optimal solution $w^j$ from the master problem, we solve the following subproblem

$$
D^j = \max \left\{ \sum_{i=1}^{N} w_i u_{(i)} \middle| w \in W_\nu \right\},
$$

(15)

and record the optimal solution $w^j \in W_\nu$ of the subproblem. Since $w^j$ obtained from the master problem is feasible, $D^j$ provides an upper bound to (13). Note that subproblem (15) is always feasible since $W_\nu$ is non-empty. Since $W_\nu$ is convex (see Remark 5) and the objective $\sum_{i=1}^{N} w_i u_{(i)}$ is linear, subproblem (15) can be efficiently solved using convex optimization algorithms. In particular, if $W_\nu$ is a polytope, subproblem (15) reduces to a linear program. Finally, if the gap between the lower and upper bounds is smaller than a pre-specified tolerance $\varepsilon$, C&CG terminates and returns the solution with the best objective value $UB$. Otherwise, we proceed to the next iteration and solve the master problem with an enlarged subset of weights $\{w^0, \ldots, w^j\}$. Note that one can set the initial weight $w^0$ as the zero vector 0 (see Remark 4).
Proposition 11. The master problem (14) in C&CG is equivalent to

\[
\begin{align*}
\text{minimize} & \quad \delta \\
\text{subject to} & \quad \delta \geq 1^T (\lambda^k + \theta^k), \quad \forall k \in [0, j - 1]_Z, \quad (16a) \\
& \quad \lambda^k_i + \theta^k_{i'} \geq u^k_i w^k_i, \quad \forall i \in [N], \ i' \in [N], \ k \in [0, j - 1]_Z, \quad (16b) \\
& \quad u = U(x), \ x \in X. \quad (16d)
\end{align*}
\]

Proof. By the LP reformulation of \(\nu_{w^k}(u)\) in (8), \(\delta \geq \nu_{w^k}(u)\) if and only if there exist \(\lambda^k\) and \(\theta^k\) such that \(\lambda^k_i + \theta^k_{i'} \geq u^k_i w^k_i\) for all \(i \in [N], \ i' \in [N]\) and \(\delta \geq 1^T (\lambda^k + \theta^k)\). Therefore, by introducing the variables \(\lambda^k\) and \(\theta^k\), we can reformulate (14) into (16). \(\square\)

Note that the size of the master problem (16) in C&CG increases with the number of iterations. Specifically, at each iteration of C&CG, the master problem is enlarged by \(2N\) additional variables \((\lambda^k, \theta^k)\) and \(N^2 + 1\) additional constraints (16b)–(16c). Hence, after several iterations of C&CG, the size of the master problem could be huge. This motivates us to propose the AMM algorithm. In AMM, we do not solve problem (16) with all weight vectors from previous iterations. Instead, at iteration \(j\), we solve (16) with the most recent weight vector \(w^j\) from iteration \(j - 1\), which reduces to solving problem (6) with \(w = w^j\). As a result, the master problem in AMM is smaller (i.e., has fewer variables and constraints) and hence, could be easier to solve than that of C&CG. It follows that AMM could have a better computational performance when \(W_\nu\) has a large number of non-zero extreme points, in which case C&CG may require a large number of iterations before convergence. The remaining steps of AMM are the same as C&CG.

Next, we discuss the convergence of the proposed algorithms. Note that \(W_\nu\) is compact and convex. If, in addition, \(X\) is compact (which is a mild assumption in practice), Proposition 2 of Bertsimas and Shtern (2018) ensures that any accumulation point of the sequence \(\{x^j\}\) generated from C&CG is an optimal solution to (13). Moreover, if \(W_\nu\) is a polyhedron and a vertex is always returned when solving the subproblem, C&CG terminates in a finite number of iterations by Proposition 7. Note that the convergence of C&CG does not depend on the utility function \(U\).

In Theorem 12, we show that our AMM algorithm converges for any continuous utility function \(U\).

Theorem 12. If \(X\) is compact and \(U\) is continuous, then any accumulation points of \(\{(x^j, u^j)\}\) generated from the AMM algorithm is an optimal solution to (13).

Proof. Since \(X\) is compact, \(W_\nu\) is compact (by Theorem 5), and \(u^j = U(x^j)\), we can always find a convergent subsequence of \(\{(x^j, u^j, w^j)\}\) as \(j \to \infty\). Without loss of generality, assume that \((x^j, u^j, w^j) \to (x^*, u^*, w^*)\) as \(j \to \infty\), where \(u^* = U(x^*)\) by continuity of \(U\). We claim that \((x^*, w^*)\) is a saddle point of \(\nu_{w^*}(U(x))\), i.e., \(\nu_{w^*}(U(x^*)) \leq \nu_{w^*}(U(x)) \leq \nu_{w^*}(U(x^*))\) for any \((x, w) \in X \times W_\nu\). To prove this claim, first, from step 1 of AMM, we have \(\nu_{w^{j-1}}(U(x^j)) \leq \nu_{w^{j-1}}(U(x))\) for all \(x \in X\). Note that
by Cauchy-Schwarz inequality, we have $|\nu_{w^{j-1}}(U(x)) - \nu_{w^*}(U(x))| \leq \|w^{j-1} - w^*\|_2 \|U(x)\|_2 \to 0$.

Moreover,

$$
\left| \nu_{w^{j-1}}(U(x^j)) - \nu_{w^*}(U(x^*)) \right|
\leq \left| \nu_{w^{j-1}}(U(x^j)) - \nu_{w^{j-1}}(U(x^*)) \right| + \left| \nu_{w^{j-1}}(U(x^*)) - \nu_{w^*}(U(x^*)) \right|
\leq \|w^{j-1}\|_1 \|U(x^j) - U(x^*)\|_\infty + \|w^{j-1} - w^*\|_2 \|U(x^*)\|_2
\leq \sup_{w \in W} \|w\|_1 \cdot \|U(x^j) - U(x^*)\|_\infty + \|w^{j-1} - w^*\|_2 \|U(x^*)\|_2 \to 0,
$$

where the second inequality follows from the proof of Proposition 3 and Cauchy-Schwarz inequality, and the convergence follows from compactness of $W$, $\|U(x^*)\|_2 < \infty$, and continuity of $U$. Thus, we obtain $\nu_{w^*}(U(x^*)) \leq \nu_{w^*}(U(x))$ for all $x \in \mathcal{X}$. On the other hand, from step 2, we have $\nu_w(U(x^j)) \leq \nu_{w^j}(U(x^j))$ for all $w \in W_j$. Since $\nu_w(U(x^j)) \to \nu_w(U(x^*))$ by continuity of $U$ and $\nu_{w^j}(U(x^j)) \to \nu_{w^*}(U(x^*))$ by a similar argument in (17), we obtain $\nu_w(U(x^*)) \leq \nu_{w^*}(U(x^*))$ for all $w \in W_j$. Hence, $\nu_w(U(x^*)) \leq \nu_{w^*}(U(x^*)) \leq \nu_{w^*}(U(x))$ for any $(x, w) \in \mathcal{X} \times W$, showing that $(x^*, w^*)$ is a saddle point of $\nu_w(u)$. Thus, $(x^*, u^*)$ with $u^* = U(x^*)$ is an optimal solution to (13).

8. Stability Analysis

Given that there are many different convex fairness measures, it is crucial to quantify how the choice of convex fairness measure in the objective of an optimization problem would affect the optimal value and solution. In this section, we study the stability of the optimal value and solution with respect to the choice of the convex fairness measure in the objective of the following problem

$$
\min_{x, u} \left\{ f(u) + \nu(u) \mid u = U(x), x \in \mathcal{X} \right\},
$$

where $f(u)$ is some efficiency measure. To facilitate the analysis, we define the following notation.

We define the distance between a point $w \in \mathbb{R}^N$ and a set $W$ as $d(w, W) := \inf_{\tilde{w} \in W} \|w - \tilde{w}\|_2$. The Hausdorff distance between two sets $W_1$ and $W_2$ is defined as

$$
d_H(W_1, W_2) := \max \left\{ \sup_{w_2 \in W_2} d(w_2, W_1), \sup_{w_1 \in W_1} d(w_1, W_2) \right\}.
$$

First, in Lemma 13, we show that the difference between two convex fairness measures is bounded by the Hausdorff distance between their dual sets.

**Lemma 13.** Let $\nu_1$ and $\nu_2$ be two convex fairness measures with dual sets $W_1$ and $W_2$, respectively. For any $u \in \mathbb{R}^N$, we have $|\nu_1(u) - \nu_2(u)| \leq d_H(W_1, W_2) \cdot \|u\|_2$.

**Proof.** First, note that

$$
\sup_{w_1 \in W_1} \nu_{w_1}(u) - \sup_{w_2 \in W_2} \nu_{w_2}(u) = \sup_{w_2 \in W_2} \inf_{w_1 \in W_1} \left\{ \nu_{w_1}(u) - \nu_{w_2}(u) \right\}
$$

and

$$
\sup_{w_1 \in W_1} \nu_{w_1}(u) - \sup_{w_2 \in W_2} \nu_{w_2}(u) = \sup_{w_1 \in W_1} \nu_{w_1}(u) - \nu_{w_2}(u) = \nu_{w_1}(u) - \nu_{w_2}(u).
$$
\[ \leq \sup_{w_1 \in W_1} \inf_{w_2 \in W_2} \|w_1 - w_2\| \cdot \|u\|_2 \leq d_H(W_1, W_2) \cdot \|u\|_2, \]

where the first inequality follows from Cauchy-Schwarz inequality, and the second inequality follows from the definition of \(d_H\). Next, using the same argument, we have

\[ \sup_{w_1 \in W_1} \nu_{w_1}(u) - \sup_{w_2 \in W_2} \nu_{w_2}(u) = \inf_{w_2 \in W_2} \sup_{w_1 \in W_1} \left\{ \nu_{w_1}(u) - \nu_{w_2}(u) \right\} \]

\[ \geq - \sup_{w_2 \in W_2} \inf_{w_1 \in W_1} \|w_1 - w_2\| \cdot \|u\|_2 \geq -d_H(W_1, W_2) \cdot \|u\|_2. \]

Hence, we obtain \(|\nu_1(u) - \nu_2(u)| \leq d_H(W_1, W_2) \cdot \|u\|_2. \)

Note that if \(W_1 = W_2\), then \(d_H(W_1, W_2) = 0\), implying that \(\nu_1 \equiv \nu_2\) from Lemma 13. Next, in Theorem 14, we use the results in Lemma 13 to show that the differences in optimal value and solution of (18) under two different convex fairness measures are bounded by the Hausdorff distance between their dual sets. Hence, the optimal value and solution would not deviate significantly if the two convex fairness measures are close enough (i.e., the Hausdorff distance between their dual sets is small).

**Theorem 14.** Let \(\nu_1\) and \(\nu_2\) be two convex fairness measures with dual sets \(W_1\) and \(W_2\), respectively. Let \(\nu_k^*\) and \(x_k^*\) be the optimal value and an optimal solution of (18) with \(\nu = \nu_k\) for \(k \in \{1, 2\}\). Assume that \(X\) is compact. Then, the following statements hold.

(a) We have \(|\nu_1^* - \nu_2^*| \leq U_{\max} d_H(W_1, W_2)\), where \(U_{\max} = \max_{x \in X} \|U(x)\|_2 < \infty\).

(b) Let \(w_k^* \in \arg \max_{w \in W_k} \nu_w(U(x_k^*))\) for \(k \in \{1, 2\}\). Assume that the following quadratic growth condition hold for \(k \in \{1, 2\}\): there exists \(\tau_k > 0\) such that

\[ f(U(x)) + \nu_{w_k^*}(U(x)) - f(U(x_k^*)) - \nu_{w_k^*}(U(x_k^*)) \geq \tau_k \|x - x_k^*\|_2^2 \quad (19) \]

for all \(x \in X\). Then, we have

\[ \|x_1^* - x_2^*\|_2 \leq \sqrt{\frac{2U_{\max}}{\min\{\tau_1, \tau_2\}}} \cdot d_H^{1/2}(W_1, W_2). \quad (20) \]

**Proof.** First, we prove part (a). Note that

\[ \nu_2^* - \nu_1^* = \min_{x \in X} \left\{ f(U(x)) + \nu_2(U(x)) \right\} - \min_{x \in X} \left\{ f(U(x)) + \nu_1(U(x)) \right\} \]

\[ \leq \left[ f(U(x_1^*)) + \sup_{w \in W_2} \nu_w(U(x_1^*)) \right] - \left[ f(U(x_1^*)) + \sup_{w \in W_1} \nu_w(U(x_1^*)) \right] \quad (21a) \]

\[ \leq U_{\max} d_H(W_1, W_2), \quad (21b) \]

where (21a) follows from \(x_1^* \in X\) and the optimality of \(x_1^*\), and (21b) follows from Lemma 13. Using the same logic, it is easy to verify that \(\nu_1^* - \nu_2^* \leq U_{\max} d_H(W_1, W_2)\). This completes the proof showing that \(|\nu_1^* - \nu_2^*| \leq U_{\max} d_H(W_1, W_2)\).
Next, we proceed to prove part (b). Note that

\[ v_2^* - v_1^* = \min_{x \in X} \left\{ f(U(x)) + \nu_2(U(x)) \right\} - \min_{x \in X} \left\{ f(U(x)) + \nu_1(U(x)) \right\} \]

\[ = \left[ f(U(x_2^*)) + \nu_{w_2^*}(U(x_2^*)) \right] - \left[ f(U(x_1^*)) + \nu_{w_1^*}(U(x_1^*)) \right] \]

\[ = \left\{ \left[ f(U(x_2^*)) + \nu_{w_2^*}(U(x_2^*)) \right] - \left[ f(U(x_1^*)) + \nu_{w_1^*}(U(x_1^*)) \right] \right\} \]

\[ + \left\{ \left[ f(U(x_1^*)) + \nu_{w_2^*}(U(x_1^*)) \right] - \left[ f(U(x_1^*)) + \nu_{w_1^*}(U(x_1^*)) \right] \right\} \]

\[ \leq -\tau_2 \| x_2^* - x_1^* \|_2^2 + \left\{ \sup_{w \in W_2} \nu_w(U(x_1^*)) - \sup_{w \in W_1} \nu_w(U(x_1^*)) \right\} \] \hspace{1cm} (22a)

\[ \leq -\tau_2 \| x_2^* - x_1^* \|_2^2 + U_{\text{max}} d_H(W_1, W_2), \] \hspace{1cm} (22b)

where (22a) follows from (19), \( w_2^* \in W_2 \), and the optimality of \( w_1^* \), and (22b) follows from Lemma 13. Using the same logic, it is easy to verify that \( v_1^* - v_2^* \leq -\tau_1 \| x_1^* - x_2^* \|_2^2 + U_{\text{max}} d_H(W_1, W_2) \). Thus, we have

\[ 0 \leq \left| v_1^* - v_2^* \right| \leq -\min\{\tau_1, \tau_2\} \| x_1^* - x_2^* \|_2^2 + U_{\text{max}} d_H(W_1, W_2), \]

which directly implies the desired inequality (20).

\[ \square \]

**Remark 7.** The quadratic growth condition in Theorem 14 is a standard assumption in stability analysis of stochastic programs and distributionally robust optimization problems (see, e.g., Liu et al., 2019; Pichler and Xu, 2018; Shapiro, 1994). For example, if \( f \) is strongly convex and \( U \) is affine of the form \( U(x) = Ax + b \) with \( A \) being a matrix of rank \( N \), then \( f \circ U + \nu \circ U \) is also strongly convex in \( x \) (where \( \circ \) denotes the composition of two functions). Thus, in this case, the quadratic growth condition is satisfied (see, e.g., discussions in Chang et al., 2018).

9. Conclusion

In this paper, we propose a new framework for optimization problems with a convex fairness measure objective. This framework includes the new notion of convex fairness measures and their theoretical properties, as well as solution approaches for optimization problems with such fairness measures. As a starting point, we show that some well-known deviation-based fairness measures, each having a unique mathematical expression, are generally not equivalent. This motivates us to propose our new parameterized class of convex fairness measures with a unified mathematical expression suitable for optimization contexts. Specifically, we first introduce a new order-based fairness measure that serves as the building block of our proposed convex fairness measures. Then, we define our convex fairness measures and derive their dual representations. Importantly, this dual representation shows that any convex fairness measure can be expressed as a robustified order-based fairness measure. It also allows us to investigate the equivalence of convex fairness measures from
a geometric perspective based on their dual sets. Moreover, using the dual representation, we propose a generic framework for optimization problems with a convex fairness measure, including reformulations and solution approaches. Finally, we conduct a stability analysis on the choice of convex fairness measure in the objective of optimization models.

Our future research steps include extending the proposed framework to the case when the utility vector \( \mathbf{u} \) is random. This extension will allow us to address fairness concerns in various application domains (e.g., facility location, scheduling, humanitarian logistics) where the utility function \( U \) depends on random factors such as random travel time, demand, and service time. Also, we aim to propose new classes of stochastic optimization approaches with fairness criterion that combine our work with different methodologies such as (distributionally) robust optimization and their solution approaches.
Appendix A. Mathematical Proofs

Proof of Proposition 1. It is easy to verify that metrics (i)–(viii) satisfy axioms (a)–(e) in Definition 3.1. Next, we verify that these metrics are Schur convex. Note that if \( \nu \) is convex and symmetric, then \( \nu \) is Schur convex (Marshall et al., 2011). Thus, it suffices to show that metrics \( \nu \) defined in (i)–(viii) are convex. In the following, we assume that \( \{u_1, u_2\} \subseteq \mathbb{R}^N \) and \( \lambda \in [0, 1] \). For metric (i), note that \( u_i \) is a linear function in \( u \). Since a maximum (minimum) of linear functions is convex (concave), the metric \( \nu \) is also convex. For metric (ii), we have

\[
\nu(\lambda u^1 + (1 - \lambda) u^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left| \lambda u_i^1 + (1 - \lambda) u_j^2 - \lambda u_j^1 + (1 - \lambda) u_i^2 \right|
\]

\[
\leq \lambda \sum_{i=1}^{N} \sum_{j=1}^{N} |u_i^1 - u_j^1| + (1 - \lambda) \sum_{i=1}^{N} \sum_{j=1}^{N} |u_j^2 - u_i^2|
\]

\[
= \lambda \nu(u^1) + (1 - \lambda) \nu(u^2).
\]

For metric (iii), following a similar argument in (ii), we have

\[
\nu(\lambda u^1 + (1 - \lambda) u^2) = \max_{i \in [N]} \max_{j \in [N]} \left| \lambda u_i^1 + (1 - \lambda) u_j^2 - \lambda u_j^1 + (1 - \lambda) u_i^2 \right|
\]

\[
\leq \max_{i \in [N]} \max_{j \in [N]} \left\{ \lambda |u_i^1 - u_j^1| + (1 - \lambda) |u_j^2 - u_i^2| \right\}
\]

\[
\leq \lambda \max_{i \in [N]} \max_{j \in [N]} |u_i^1 - u_j^1| + (1 - \lambda) \max_{i \in [N]} \max_{j \in [N]} |u_j^2 - u_i^2|
\]

\[
= \lambda \nu(u^1) + (1 - \lambda) \nu(u^2).
\]

For metric (iv), note that \( \pi = \lambda \pi^1 + (1 - \lambda) \pi^2 \), where \( \pi^k = (1/N) \sum_{i=1}^{N} u_i^k \) for \( k \in \{1, 2\} \). Convexity follows from a similar argument for metric (ii). For metric (v), we have

\[
\nu(u) = \sqrt{\sum_{i=1}^{N} (u_i - \bar{u})^2} = \sqrt{\sum_{i=1}^{N} \left( u_i - \frac{1}{N} 1^\top u \right)^2} = \left\| u - \frac{1}{N} 11^\top u \right\|_2
\]

\[
= \left\| \left( I - \frac{1}{N} 11^\top \right) u \right\|_2 =: \|Au\|_2,
\]

where \( I \in \mathbb{R}^{N \times N} \) is the identity matrix. Since \( \ell_2 \) norm is convex, it follows that \( \nu \) is also convex (Bertsekas, 2015). For metric (vi), one can easily verify its convexity by following the same logic used to verify the convexity of (iv). Similarly, one can verify the convexity of (vii) and (viii) by following a similar argument as in metrics (ii) and (iii).

Proof of Proposition 2. We first prove (a)–(d). Without loss of generality, we assume that the utility vector \( u \) is sorted in ascending order, i.e., \( u_1 \leq u_2 \leq \cdots \leq u_N \).
(a) Note that the maximum pairwise difference (iii) is equal to $u_N - u_1$, which is the same as the range (i).

(b) We claim that $\max_{i \in [N]} \sum_{j=1}^N |u_i - u_j| = N \max_{i \in [N]} |u_i - \bar{u}|$. To prove this claim, note that we can write metric (vii) as

$$\max_{i \in [N]} \sum_{j=1}^N |u_i - u_j| = \max \left\{ \sum_{i=2}^N u_i - (N - 1)u_1, (N - 1)u_N - \sum_{i=1}^{N-1} u_i \right\} =: \max\{C_1, C_2\},$$

where $C_1$ and $C_2$ represent the first and second expressions in the max operator, respectively. Consider the case when $C_1 \leq C_2$. This implies that $2 \sum_{j=2}^{N-1} u_i \leq (N - 2)(u_N + u_1)$. Adding $2(u_1 + u_N)$ on both sides of the inequality results in $2\bar{u} \leq u_1 + u_N$, implying that $\bar{u} - u_1 \leq u_N - \bar{u}$. Hence, we have

$$\max_{i \in [N]} \sum_{j=1}^N |u_i - u_j| = (N - 1)u_N - \sum_{i=1}^{N-1} u_i = N(u_N - \bar{u}) = N \max_{i \in [N]} |u_i - \bar{u}|.$$

A similar argument holds for the case when $C_1 \geq C_2$.

(c) By (a), it suffices to show that (i) and (ii) are equivalent when $N = 3$. By Mesa et al. (2003), we can write (ii) as

$$\sum_{i=1}^3 \sum_{j=1}^3 |u_i - u_j| = \sum_{i=1}^3 2(2i - 4)u_i = 4(u_3 - u_1),$$

which shows the equivalence between (i) and (ii).

(d) By (b), it suffices to show that (iv) and (vi) are equivalent when $N = 3$. First, we claim that if $\bar{u} \in [u_2, u_3]$, then $u_3 - \bar{u} \geq \bar{u} - u_1$. Indeed, since $\bar{u} \geq u_2$, we have $u_1 + u_3 \geq 2u_2$. Adding $2(u_1 + u_3)$ on both sides of the inequality results in $u_1 + u_3 \geq 2\bar{u}$, implying that $u_3 - \bar{u} \geq \bar{u} - u_1$. Therefore, when $\bar{u} \in [u_2, u_3]$, we have

$$\sum_{i=1}^3 |u_i - \bar{u}| = (\bar{u} - u_1) + (\bar{u} - u_2) + (u_3 - \bar{u}) = \frac{1}{3}(u_1 + u_2 + u_3) - u_1 - u_2 + u_3$$

$$= \frac{2}{3}(2u_3 - u_1 - u_2) = 2(u_3 - \bar{u}) = 2 \max_{i=1,2,3} |u_i - \bar{u}|.$$

Similarly, in the case when $\bar{u} \in [u_1, u_2]$, we have $\bar{u} - u_1 \geq u_3 - \bar{u}$. Thus,

$$\sum_{i=1}^3 |u_i - \bar{u}| = (\bar{u} - u_1) + (u_2 - \bar{u}) + (u_3 - \bar{u}) = -\frac{1}{3}(u_1 + u_2 + u_3) - u_1 + u_2 + u_3$$

$$= \frac{2}{3}(-2u_1 + u_2 + u_3) = 2(\bar{u} - u_1) = 2 \max_{i=1,2,3} |u_i - \bar{u}|.$$

This proves the equivalence between (iv) and (vi).
Table A.3: Examples that some fairness measures are not equivalent

| E.g. | Utility vectors (in $\mathbb{R}^N$) | (i) | (ii) | (iv) | (v) | (vi) | (viii) |
|------|------------------------------------|-----|------|------|-----|------|--------|
| A    | $u^1 = (1, 2, 2, \ldots, 2, 5, 4.5)$ | 3.5 | 14 + 8(N − 3) | 4 | $\sqrt{6.5}$ | 2 | 9.5 + 2(N − 3) |
|      | $u^2 = (1, 1, 2, \ldots, 2, 4)$   | 3   | 12 + 8(N − 3) | 4 | $\sqrt{6}$   | 2 | 9 + 2(N − 3)  |
| B    | $u^1 = (2, 5, 5, \ldots, 5, 9)$   | 7   | 28 + 14(N − 3) | / | $\sqrt{25 - \frac{1}{3}}$ | / | 18 + 4(N − 3) |
|      | $u^2 = (2, 2, 4, \ldots, 4, 8)$   | 6   | 24 + 16(N − 3) | / | $\sqrt{24}$   | / | 18 + 4(N − 3) |
| B'   | $u^1 = (2, 5, 5, 6, 9) \in \mathbb{R}^5$ | /   | /    | /   | /   | /   | 26     |
|      | $u^2 = (2, 2, 4, 4, 8) \in \mathbb{R}^5$ | /   | /    | /   | /   | /   | 26     |
| C    | $u^1 = (2, 5, \frac{13}{3}, \ldots, \frac{13}{3}, 9)$ | 7   | 28 + $\frac{44}{3}(N − 3)$ | / | $\sqrt{23}$   | / | /      |
|      | $u^2 = (2, 2, \frac{13}{3}, \ldots, \frac{13}{3}, 9)$ | 7   | 28 + $\frac{52}{3}(N − 3)$ | / | $\sqrt{28}$   | / | /      |
| D    | $u^1 = (1, 2, 3, \ldots, 3, 6)$   | 7   | 20 + 12(N − 3) | / | $\sqrt{14}$   | / | /      |
|      | $u^2 = 3 + (0, 0, \sqrt{\frac{21}{3}}, \ldots, \sqrt{\frac{21}{3}}, \sqrt{21})$ | /   | $4\sqrt{21} + \frac{8\sqrt{21}}{3}(N − 3)$ | / | $\sqrt{14}$   | / | /      |
| E    | $u^1 = (1, 7, 7, \ldots, 7, 8, 12)$ | /   | /    | 12  | /   | 6    | /      |
|      | $u^2 = (5, 10, 10.5, \ldots, 10.5, 13, 14)$ | /   | /    | 12  | /   | 5.5  | /      |

Finally, we show that the remaining pairs of fairness measures are not equivalent. To prove two measures $\nu$ and $\tilde{\nu}$ are not equivalent, it suffices to find utility vectors $u^1$ and $u^2$ such that $\nu(u^1) = \nu(u^2)$ but $\tilde{\nu}(u^1) \neq \tilde{\nu}(u^2)$. In Table A.3, we provide examples showing that the remaining pairs of fairness measures are not equivalent. Specifically, example A shows that the pairs $\{(\nu, \nu)\}$ is not equivalent even when $\nu$ is defined. In example B, fairness measure (ii) at $u^1$ is not equivalent; example E shows that the pair (iv, vi) is not equivalent when $\nu$ is defined. Therefore, a dual set $W_\nu$ is given by the singleton $\{−1, 0, \ldots, 0, 1\} \in \mathbb{R}^N$.

**Proof of Proposition 8.** (a) From Proposition 2, both (i) and (iii) are equivalent to $\max_{i \in [N]} u_i - \min_{i \in [N]} u_i = −u(1)+u(N)$. Therefore, a dual set $W_\nu$ is given by the singleton $\{−1, 0, \ldots, 0, 1\} \in \mathbb{R}^N$.

(b) From Mesa et al. (2003), we can write the Gini deviation (ii) as

$$\sum_{i=1}^{N} \sum_{j=1}^{N} |u_i - u_j| = \sum_{i=1}^{N} 2(2i - 1 - N)u_{(i)} = \sum_{i=1}^{N} w'_i u_{(i)},$$

where we let $w'_i = 2(2i - 1 - N)$. Note that $w'_1 = 2(1 - N) < 0$, $w'_{N} = 2(N - 1) > 0$ and $w'_i$ is increasing in $i$. Moreover,

$$\sum_{i=1}^{N} w'_i = 2 \sum_{i=1}^{N} (2i - 1 - N) = 2[N(N + 1) - N(N + 1)] = 0.$$

Therefore, a dual set $W_\nu$ is given by the singleton $\{w'\}$. 

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(c) Consider \( \| u - \overline{v}1 \|_p \) for some \( p \in [1, \infty] \). Let \( q \) be such that \( 1/p + 1/q = 1 \), where we let \( q = \infty \) if \( p = 1 \), and \( q = 1 \) if \( p = \infty \). Since \( \ell_p \) norm is the dual of \( \ell_q \) norm,

\[
\| u - \overline{v}1 \|_p = \sup_{w': \| w' \|_q \leq 1} (u - \overline{v}1)^\top w' = \sup_{w': \| w' \|_q \leq 1} u^\top (w' - \overline{v}1),
\]

\[ (A.1) \]

where \( \overline{w'} = (1/N)1^\top w' \). Note that (iv)-(vi) can be written as \( \| u - \overline{v}1 \|_1 \), \( \| u - \overline{v}1 \|_2 \), and \( \| u - \overline{v}1 \|_\infty \), respectively. The desired dual sets \( W_\nu \) follow directly from \( (A.1) \).

(d) From Proposition 2, (vii) is equivalent to \( N\| u - \overline{v}1 \|_{\infty} \). The same argument in \( (A.1) \) shows that

\[
N\| u - \overline{v}1 \|_p = \sup_{w': \| w' \|_q \leq 1} (u - \overline{v}1)^\top (Nw') = \sup_{w': \| w' \|_q \leq N} u^\top (w' - \overline{w}1),
\]

which gives the desired dual set \( W_\nu \).

(e) Finally, for (viii), let \( k = k(u) \) be the number of entries in \( u \) that are closer to \( u(1) \), i.e., \( u(i) - u(1) < u(N) - u(i) \) for \( i \in [k] \). Then,

\[
\sum_{i=1}^{N} \max_{j \in [N]} |u_i - u_j| = \sum_{i=1}^{k} [u(N) - u(i)] + \sum_{i=k+1}^{N} [u(i) - u(1)] \\
= [-(N-k) - 1]u(1) + \sum_{i=2}^{k} (-1)u(i) + \sum_{i=k+1}^{N-1} u(i) + (k+1)u(N).
\]

Note that \( k \) takes value in \([N]\) only, it suffices to consider \( w^k \) with entries \( w_1^k = -(N-k) - 1 \), \( w_2^k = \cdots = w_k^k = -1 \), \( w_{k+1}^k = \cdots = w_{N-1}^k = 1 \), and \( w_N^k = k + 1 \). Moreover, it is easy to verify that if there are \( k \) entries in \( u \) that are closer to \( u(1) \), then \( \nu_{w^k}(u) \geq \nu_{w^h}(u) \) for \( h \neq k \). Indeed, if \( h < k \), then

\[
\nu_{w^k}(u) - \nu_{w^h}(u) \\
= \left\{ \sum_{i=1}^{k} [u(N) - u(i)] + \sum_{i=k+1}^{N} [u(i) - u(1)] \right\} - \left\{ \sum_{i=1}^{h} [u(N) - u(i)] + \sum_{i=h+1}^{N} [u(i) - u(1)] \right\} \\
= \sum_{i=h+1}^{k} \left\{ [u(N) - u(i)] - [u(i) - u(1)] \right\} > 0.
\]

Following a similar argument, if \( h > k \), we also have \( \nu_{w^k}(u) - \nu_{w^h}(u) > 0 \). Hence, a dual set \( W_\nu \) is given by \( \{ w^k \}_{k=1}^{N} \).

This completes the proof. \( \square \)

Proof of Proposition 10. First, note that we can relax the set of permutation matrices \( P \) in (7) as

\[
P = \left\{ P \in \mathbb{R}^{N \times N} \left| \sum_{i=1}^{N} P_{ij} = 1, \sum_{j=1}^{N} P_{ij} \leq 1, P_{ij} \in \{0, 1\}, \forall i \in [N], j \in [N] \right. \right\}.
\]

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Indeed, the first constraint requires that every column of \( P \) has exactly one entry with value 1 and the second constraint requires that every row of \( P \) has at most one entry with value 1. Since \( P \) is an \( N \)-by-\( N \) matrix, this immediately ensures that \( P \) has exactly one entry with value 1 in every row and column. As a result, the dual variable \( \lambda_i \) associated with the second constraint is non-negative.

Next, for a given \( u \in \mathbb{R}^N \), let \( P^* \) be an optimal solution to the primal problem \( \max_{P \in \mathcal{P}} u^T (Pw) \) and \((\lambda^*, \theta^*)\) be an optimal dual solution. Let \( \{(i, \pi(i))\}_{i \in [N]} \) be the set of pairs of indices such that \( P_{i, \pi(i)}^* = 1 \) (and is 0 otherwise). Without loss of generality, we can assume that there exists \( i' \in [N] \) such that \( \lambda^*_{i'} = 0 \). Indeed, if \( \varepsilon := \min_{i \in [N]} \lambda^*_{i} > 0 \), then \((\tilde{\lambda}, \tilde{\theta})\) defined by \( \tilde{\lambda}_i = \lambda_i^* - \varepsilon \) and \( \tilde{\theta}_i = \theta_i^* + \varepsilon \) for all \( i \in [N] \) is another optimal solution to (6).

Now, we derive an upper bound on \( \theta^*_1 \). By the complementary slackness condition, if \( P^*_{i, \pi(i)} = 1 \), then we have \( \lambda_i^* + \theta_{\pi(i)}^* = u_i w_{\pi(i)} \) (i.e., with a zero slack variable), implying that \( \theta_{\pi(i)}^* = u_i w_{\pi(i)} - \lambda_i^* \). Since \( \lambda_i^* \geq 0 \), we immediately have \( \theta_{\pi(i)}^* \leq u_i w_{\pi(i)} \leq u_{\max} w_{\pi(i)} \), where \( u_{\max} = \max_{i \in [N]} u_i \). Now, we derive an upper bound on \( \lambda_i \). Using \( \theta_{\pi(i)}^* = u_i w_{\pi(i)} - \lambda_i^* \) and letting \( i = \pi^{-1}(j) \), constraints (6b) implies that \( \lambda_i^* + (u_{\pi(i)} - \lambda_i^*) \geq u_j w_j \) for all \( i \in [N] \) and \( j \in [N] \), which is equivalent to \( \lambda_i^* + u_i w_{\pi(j)} - \lambda_j^* \geq u_j w_{\pi(j)} \) for all \( i \in [N] \) and \( j \in [N] \). Hence, we have \( \lambda_i^* - \lambda_j^* \geq (u_i - u_j) w_{\pi(j)} \) for all \( i \in [N] \) and \( j \in [N] \). Setting \( i = i' \), since \( \lambda^*_{i'} = 0 \), the inequalities imply that \( \lambda_j^* \leq (u_{i'} - u_j) w_{\pi(j)} \leq (u_{\max} - u_{\min}) \|w\|_{\infty} \) for all \( j \in [N] \setminus \{i'\} \), where \( u_{\min} = \min_{i \in [N]} u_i \). Finally, the lower bound of \( \theta_i \) follows from the upper bound of \( \lambda_i \) that \( \theta_{\pi(i)}^* = u_i w_{\pi(i)} - \lambda_i^* \geq u_i w_{\pi(i)} - (u_{\max} - u_{\min}) \|w\|_{\infty} \).

Note that the above lower and upper bounds on \( \lambda \) and \( \theta \) are obtained by fixing a utility vector \( u \in \mathbb{R}^N \). Thus, the desired lower bound follows from taking the infimum over all feasible \( u \in \mathcal{U} := \{u \in \mathbb{R}^N \mid u = U(x), x \in \mathcal{X}\} \) and the desired upper bounds follow from taking the supremum over all \( u \in \mathcal{U} \). Thus, we obtain the bounds on \( \lambda \)

\[
0 \leq \lambda_i \leq \sup_{u \in \mathcal{U}} \left\{ u_{\max} - u_{\min} \right\} \cdot \|w\|_{\infty} \leq (U_{\max} - U_{\min}) \|w\|_{\infty} =: \bar{\lambda},
\]

the lower bound on \( \theta \)

\[
\theta_j \geq \inf_{u \in \mathcal{U}} \left\{ u_{\pi^{-1}(j)} w_j - (u_{\max} - u_{\min}) \|w\|_{\infty} \right\} \geq \min \left\{ U_{\max} w_j, U_{\min} w_j \right\} - \bar{\lambda},
\]

and the upper bound on \( \theta \)

\[
\theta_j \leq \sup_{u \in \mathcal{U}} \left\{ u_{\max} w_j \right\} \leq \max \left\{ U_{\max} w_j, U_{\min} w_j \right\}.
\]

**Appendix B. Equivalence of fairness measures**

Tables B.4–B.5 summarize the equivalence of the fairness measures shown in Table 1. The two groups of equivalent fairness measures proved in Proposition 2 are highlighted in red and blue with ‘Equiv.’ representing equivalence in the tables. If a given pair of fairness measure is not equivalent, one of the corresponding counterexamples from (A) to (E) in the proof of Proposition 2 is stated (see Table A.3).

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Table B.4: Equivalence of fairness measures or counterexamples when \( N = 3 \)

| \( N = 3 \) | i | ii | iii | iv | v | vi | vii | viii |
|-------------|---|----|-----|----|---|----|-----|-----|
| i           | / | Equiv. | Equiv. | A | C | A | A | B |
| ii          | / | Equiv. | A | C/D | A | A | B |
| iii         | / | A | C | A | A | B |
| iv          | / | A | Equiv. | Equiv. | A |
| v           | / | A | A | B |
| vi          | / | Equiv. | A |
| vii         | / | A |
| viii        | / |

Table B.5: Equivalence of fairness measures or counterexamples when \( N > 3 \)

| \( N > 3 \) | i | ii | iii | iv | v | vi | vii | viii |
|-------------|---|----|-----|----|---|----|-----|-----|
| i           | / | C | Equiv. | A | C | A | A | B |
| ii          | / | C | A | D | A | A | B, B’ |
| iii         | / | A | C | A | A | B |
| iv          | / | A | E | E | A |
| v           | / | A | A | B |
| vi          | / | Equiv. | A |
| vii         | / | A |
| viii        | / |

Appendix C. Implementation Details of Example 7.1

To generate the instances, for a given \(|I|\), we randomly select \(|I|\) different locations in the “2010 County Sorted 250” data from Daskin (2013), where we set \( I = J \) as these locations. We compute the transportation cost using the Euclidean distance based on the locations’ latitude and longitude, and we set the demand as the location’s population. We implement the two reformulations in AMPL modeling language and use CPLEX (version 20.1.0.0) as the solver with default settings. We conduct all the experiments on a computer with an Intel Xeon Silver processor with a 2.10 GHz CPU and 128 Gb memory.
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