Mapping of the $2 + 1$ q-deformed Dirac oscillator onto the q-deformed Jaynes-Cummings model

Parisa Majari$^1$, Alfredo Luis$^2$ and Mohammad Reza Setare$^1$

1 Department of Science, University of Kurdistan - Sanandaj, Iran
2 Department of Optics, University Complutense - Madrid, Spain

Received 25 August 2017; accepted in final form 15 January 2018

Published online 5 February 2018

PACS 42.50.Pq - Cavity quantum electrodynamics; micromasers
PACS 03.65.Pm - Relativistic wave equations
PACS 03.65.Ca - Formalism

Abstract - We develop the equivalence between the two-dimensional Dirac oscillator and the anti–Jaynes-Cummings model within a q-deformed scenario. We solve the Hamiltonian spectrum and the time evolution for number and coherent initial states. We show the lack of preservation of the q-deformed versions of the total angular momentum that reproduces a collapse-revival structure. We provide suitable relations for the non-relativistic limit.

Copyright © EPLA, 2018

Introduction. – The introduction and proper combination of suitable physical models provide a significant advance in many different areas of physics. In this work we combine the relativistic Dirac oscillator, the Jaynes-Cummings model, and the idea of q-deformation.

The Dirac oscillator provides a generalization to the relativistic domain of one of the most applied models in classical and quantum physics [1,2]. This is a notably simple solvable equation, linear both in momentum and position, formally looking as a simple minimal coupling of a charged particle with a magnetic field. Accordingly, it has found applications in nuclear physics, quantum chromodynamics, many-body theories, supersymmetric relativistic quantum mechanics and quantum optics [3–6]. Also as another application, the q-deformation has been used to study the thermostatistics of q-bosons. Some authors have considered the case of $q = e^{i\theta}$ with a unit modulus and there resulted that the partition function diverges for the q-deformed oscillators [7].

This relativistic generalization is accompanied by new effects such as the unavoidable emergences of spin, negative energies and rather unexpected behaviors such as the Klein tunneling and the Zitterbewegung as a result of the interference between positive- and negative-energy components. A further key feature, is that the Dirac oscillator contains in itself another germane model in quantum physics: the Jaynes-Cummings model (JC), or, more rigorously, the anti-Jaynes-Cummings version (AJC) [8,9]. This describes the coupling of a discrete spin-like variable with a bosonic degree of freedom [10, 11]. The powerfulness of the model relies on its simplicity and its capacity to embrace very different physical situations such as light-matter interaction, dynamics of trapped ions, or Bose-Einstein condensates [12]. In particular this model allows to use the quantum-optics–based technology or trapped ions technology for simulating relativistic quantum mechanics [3,4].

On the other hand, the q-deformation provides a quite simple but rich and powerful enough way to describe complex situations such as electronic conductance in disordered metals and doped semiconductors, phonon spectrum in $^4$He, oscillatory-rotational spectra of diatomic and multi-atomic molecules [13–17]. Deformation is accomplished by entering a dimensionless parameter $q$ in the normal Weyl algebra. This grants a more general case of the ancient theory at the same time it replicates the primary theory for $q = 1$.

For a linear system, the frequency is independent of amplitude, but in the nonlinear q-oscillator, the frequency is a dependent quantity. An interesting physical example of the q-nonlinear systems usually used for modelling the Kerr medium. We would like to study the JC and AJC models in a nonlinear optical medium (Kerr-like medium). For this purpose, we use the q-deformed scenario, for the description of a system where a two-level atom is surrounded by a Kerr medium [18,19]. Also for describing a relativistic electron which moves in the nonlinear harmonic-oscillator potential, we can use a q-deformed Dirac oscillator. This goal
is motivated by the fact that the $q$-oscillator belongs to the system of the subclass corresponding to the specific non-linearity [20].

Thus, in this work we investigate the $q$-deformed perspective of the interplay between the Dirac oscillator and the Jaynes-Cummings model following ref. [21], both in the relativistic domain and in the non-relativistic limit. In particular we focus on the preservation in the $q$-deformed scenario of peculiar relativistic features such as the Zitterbewegung effect.

**The $q$-deformed model.** — Let us present the main results regarding the $q$-deformation and the equivalence proposed.

$q$-Deformed observables and states. A general case of modified Weyl algebra holds as follows [16,22]:

$$a_q a_q^+ - q a_q^+ a_q = 1, \quad \text{for } q \leq 1,$$

where $a_q$ and $a_q^+$ are the annihilation and creation operators, that operate such that

$$a_q |n\rangle = \sqrt{n} |n-1\rangle, \quad a_q^+ |n\rangle = \sqrt{n+1} |n+1\rangle,$$

where the number basis $|n\rangle$ is actually defined by the eigenvalue equation

$$[\hat{n}] |n\rangle = [n] |n\rangle, \quad [n] \equiv a_q^+ a_q,$$

and $|n\rangle$ satisfies this relation [17]:

$$[n] = 1 - q^n / 1 - q.$$

For $q \to 1$ naturally $|n\rangle \to n$ while for $q < 1$ we have $|n\rangle < n$, with $|n\rangle \to 1$ as $q \to 0$.

Alternatively to the $q$-number operator $[n]$ in eq. (3) we can define also an operator $\hat{n}$ satisfying the standard commutation relation with $a_q$, i.e.,

$$\hat{n} |n\rangle = n |n\rangle, \quad [a_q, \hat{n}] = a_q.$$

We can present some alternative realizations of the $q$-deformed algebra. On the one hand, we have the following differential form for $a_q$:

$$a_q = e^{-2\alpha z} - e^{i\alpha z} / -i\sqrt{1 - e^{-2\alpha z}},$$

where $\alpha = \sqrt{-\ln q / 2}$ and $z$ is a Cartesian coordinate. An alternative differential realization is given by

$$a = 1 / 1 - q D_q, \quad a^\dagger = 1 - z - z(1 - q) D_q,$$

where $D_q$ is the Jackson derivative [23]

$$D_q f(z) = f(qz) - f(z) / z(q - 1),$$

which links $q$-deformation with generalized entropies [24]. On the other hand, after [22] we may express $a_q$ also as

$$a_q = F(a^\dagger a) a, \quad F(n) = \frac{[n + 1]}{n + 1},$$

where naturally $a = a_{q=1}$. Weyl algebra modifications like $q$-deformations are known to be particular cases of a more general framework called $f$-deformations (or $f$-oscillators) introduced in ref. [20], for which the first are obtained from the latter using a particular deformed function of the number operator [22].

In particular this means that $q$-deformation implies a highly nonlinear interaction. The degree of nonlinearity may be reduced to more simple levels if $q \to 1$. Thus, considering $q = \exp(-\epsilon)$, a power series expansion as $\epsilon \to 0$ leads to

$$F(a^\dagger a) \simeq 1 - \left( \frac{\epsilon}{4} \right)^2 a^\dagger a.$$

Therefore, from a quantum-optical perspective $q$-deformation corresponds to optics in nonlinear media [25,26]. Also, the limit (10) recalls the dynamics of trapped ions, where $a$ describes the motion of the center of mass, as shown for example in ref. [27,28]. Moreover, this connection between $q$-deformation and nonlinearity suggest applications in the field of nonlinear quantum metrology [29,30].

On the other hand, in the opposite limit of large nonlinearity $q \to 0$ we get

$$F(a^\dagger a) \to \frac{1}{\sqrt{1 + a^\dagger a}}, \quad a_q \to \frac{1}{\sqrt{1 + a^\dagger a}},$$

so that

$$a_q |n\rangle \to |n - 1\rangle, \quad a_q |0\rangle = 0.$$

This is to say that in this limit $a_q$ becomes the Susskind-Glogower phase operator [31-34]. That is that $a_q$ tends to represent the complex exponential of the oscillator phase.

Besides the number states (3) we consider some analogs of the coherent states defined via the eigenvalue equation [35,36]:

$$a_q |\alpha\rangle = \alpha |\alpha\rangle,$$

being

$$|\alpha\rangle = \frac{1}{\sqrt{\epsilon_\alpha(|\alpha|^2)}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{|n|!}} |n\rangle, \quad |\alpha|^2 < \frac{1}{1 - q},$$

where $|\alpha|! = |n|! \cdots |1\rangle$ with $|0\rangle! = 1$ and the following relation has been used:

$$\epsilon_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{|n|!}.$$

These relations hold provided that $|\alpha|^2 < 1/(1 - q)$ as a necessary condition such that the number coefficients in eq. (14) tends to zero as $n$ tends to infinity.

Regarding the number statistics let us show that this state is sub-Poissonian for the $q$-deformed number operator $[\hat{n}]$ since

$$\langle [\hat{n}] | \alpha \rangle |\alpha\rangle = |\alpha|^2, \quad \langle [\hat{n}] | \alpha \rangle^2 |\alpha\rangle^2 = |\alpha|^2 + q |\alpha|^4.$$
We can introduce the Mandel Q parameter in the $q$-deformed scenario as

$$Q_q = \frac{\Delta^2[\hat{n}]}{\langle \hat{n} \rangle} - 1 = (q - 1)|\alpha|^2,$$

so that $Q_q < 0$ if $q < 1$. On the other hand, regarding the number operator $\hat{n}$ in eq. (5) it has been shown that the $q$-deformed coherent states (14) are always super-Poissonian [37], that is

$$Q = \frac{\Delta^2[\hat{n}]}{\langle \hat{n} \rangle} - 1 > 0.$$  

This is illustrated in fig. 1 where we plot both Mandel parameters $Q_q, Q$ as functions of $|\alpha|^2$ for $q = 0.25$ (solid line) and $q = 0.75$ (dashed line). Note that there is no straightforward connection between sub-Poissonian statistics for $\langle \hat{n} \rangle$ and nonclassical behavior, as illustrated by the case of undeformed classical-like coherent states $a|\beta\rangle = \beta|\beta\rangle$ for which

$$Q_q = \frac{e^{-2(1-q)|\beta|^2} - e^{-2(1-q)|\alpha|^2}}{(1-q) [1 - e^{-2(1-q)|\beta|^2}] - 1},$$

and we get $Q_q < 0$ as illustrated in fig. 2.

The $q$-deformed Dirac oscillator. The Hamiltonian of the Dirac oscillator in two spatial coordinates $x, y$ is given by [21]

$$H_D = \left( \begin{array}{cc} mc^2 & i\sqrt{4mc^2\hbar}a_L^+ \\ -i\sqrt{4mc^2\hbar}a_L & -mc^2 \end{array} \right),$$

where $m$ is the rest mass of the electron, $\omega$ is the Dirac-oscillator frequency, $c$ is the speed of light, and $a_L$ is the left-handed combination of $x, y$ dynamics [3]

$$a_L = \frac{a_x + i\omega a_y}{\sqrt{2}}.$$  

Here, we would like to consider the Dirac oscillator from a general viewpoint, where the linear vector potential is replaced with a nonlinear one. So, we can write the $q$-deformed version of the above Hamiltonian [21]:

$$H^D_q = \left( \begin{array}{cc} mc^2 & i\sqrt{4mc^2\hbar}a_L^+ \\ -i\sqrt{4mc^2\hbar}a_L & -mc^2 \end{array} \right),$$

or equivalently,

$$H^D_q = mc^2\sigma_z + i\sqrt{4mc^2\hbar} \left( \sigma^+ a_L^+ - \sigma^- a_L \right),$$

where $\sigma_i$ with $i = x, y, z$ refers to the corresponding Pauli matrices. Alternatively,

$$H^D_q = mc^2\sigma_z + i\sqrt{2mc^2\hbar} \left( \begin{array}{cc} 0 & a_{x,q} - ia_{y,q} \\ -a_{x,q} - ia_{y,q} & 0 \end{array} \right).$$

Within the $q$-deformed scenario we can define the $z$ components of the orbital $L$, spin $S$, and total $J$ angular-momentum operators:

$$L_z = \hbar (a_{r,q}a_{r,q} - a_{r,q}a_{r,q}), \quad S_z = \frac{\hbar}{2} \sigma_z, \quad J_z = L_z + S_z,$$

where the right-handed mode $a_{r,q}$ is defined as

$$a_{r,q} = \frac{a_{x,q} - ia_{y,q}}{\sqrt{2}},$$

and we will always assume that this mode is in the vacuum state.

Next we compare the $q$-deformed Dirac oscillator $H^D_q$ with the $q$-deformed Jaynes-Cummings and anti-Jaynes-Cummings models.

The $q$-deformed JC and AJC Hamiltonians. The JC model is an example of Rabi model in the framework of quantum electrodynamics that presents the coupling of two energy levels, described by the Pauli matrices $\sigma$, with a harmonic oscillator, described by the complex-amplitude or annihilation operator $a$. As two typical realizations the oscillator we can mention a quantum electromagnetic field mode, i.e., photons, or the motion of the center of mass of a trapped ion, that is phonons. This last case is specially interesting since it allows to simulate both JC and AJC models simply by properly selecting the frequency tuning of a laser field illuminating the ion, leading to the AJC and JC Hamiltonians [12]:

$$H^{AJC} = \hbar\delta \sigma_z + \hbar\eta (\sigma^+ a_L^+ e^{i\phi} + \sigma^- a_L e^{-i\phi}),$$

$$H^{JC} = \hbar \delta' \sigma_z + \hbar \eta' (\sigma^+ a_L^+ e^{i\phi} + \sigma^- a_L e^{-i\phi}).$$
where $\eta, \eta'$ are constant, $\delta, \delta'$ represent detuning, and $\phi, \phi'$ are arbitrary phases. In the Lamb-Dicke regime, the ionic center-of-mass position is well-localised with respect to the laser frequencies of the laser fields. On the other hand, the $q$-deformed AJC and JC models can be implemented in trapped ions in the vicinity of the Lamb-Dicke regime. In this case the ionic center-of-mass position is not well-localised with respect to the laser wavelengths. So, nonlinear effects emerge from the overlap of light and matter waves, allowing one to realize the $q$-deformed and JC models.

Now we $q$-deform the above JC and AJC Hamiltonians simply by substituting $a, a^\dagger$ by their the $q$-deformed versions of creation and annihilation operators:

$$H_q^{\text{JC}} = \hbar \delta \sigma_z + \hbar \eta (\sigma^+ a_q^\dagger e^{i\phi} + \sigma^- a_q e^{-i\phi}), \quad (29)$$

$$H_q^{\text{AJC}} = \hbar \delta \sigma_z + \hbar \eta (\sigma^+ a_q^\dagger e^{i\phi} + \sigma^- a_q e^{-i\phi}'. \quad (30)$$

In general terms, $q$-deformed JC Hamiltonians have been studied in [39].

**Equivalence.** A ready inspection of eqs. (23) and (29) reveals an exact mapping between them if we properly identify the complex-amplitude operators and parameters. Alternatively, if we consider explicitly the two-dimensional nature of our Dirac oscillator, the equivalence reads

$$H_q^D = H_{q,x}^{\text{JC}} + H_{q,y}^{\text{AJC}}, \quad (31)$$

with

$$h\eta_x = h\eta_y = \sqrt{2m^2 \hbar \omega}, \quad \phi_x = \pi/2, \quad \phi_y = 0, \quad h\delta_x = h\delta_y = \frac{mc^2}{2}. \quad (32)$$

**Energy spectrum of the $q$-deformed Dirac oscillator.** Next we derive the energy spectrum of the $q$-deformed Dirac oscillator $H_q^D$. The spectrum must be calculated from scratch since the commutation relations have changed in comparison with the $q = 1$ case. The $q$-deformed time-independent Dirac equation is

$$H_q^D \psi = E \psi, \quad (33)$$

with $H_q^D$ in eq. (23). Looking for spinor solutions,

$$|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}, \quad (34)$$

we get that eq. (33) is equivalent to the pair of equations

$$mc^2|\psi_1\rangle + i\sqrt{4mc^2 \hbar \omega a_{\ell,q}^\dagger}|\psi_2\rangle = E|\psi_1\rangle,$$

$$-mc^2|\psi_2\rangle - i\sqrt{4mc^2 \hbar \omega a_{\ell,q}}|\psi_1\rangle = E|\psi_2\rangle. \quad (35)$$

We have a readily solution with $E = E_0 = mc^2, |\psi_2\rangle = 0$, and $|\psi_1\rangle = |0\rangle$, that is

$$|E_0\rangle = |0\rangle|\chi_1\rangle, \quad (36)$$

where $|\chi_1\rangle = (1, 0)^\dagger$ and also $|\chi_1\rangle = (0, 1)^\dagger$ are known as the Pauli spinors. Whenever $E \neq mc^2$ combining eqs. (35) we get

$$E^2 - m^2c^4 |\psi_1\rangle = 4mc^2 \hbar \omega a_{\ell,q}^\dagger a_{\ell,q}|\psi_1\rangle, \quad (37)$$

$$E^2 - m^2c^4 |\psi_2\rangle = 4mc^2 \hbar \omega a_{\ell,q} a_{\ell,q}^\dagger|\psi_2\rangle, \quad (38)$$

and we recall that in terms of $q$-deformed number operators $|\bar{n}\rangle = a_{\ell,q}^\dagger a_{\ell,q}$ and $|\bar{n} + 1\rangle = a_{\ell,q} a_{\ell,q}^\dagger$. From the above equations the energies can be easily obtained as follows:

$$E = \pm E_n = \pm mc^2\sqrt{1 + 4\xi |n|}, \quad \xi = \frac{\hbar \omega}{mc^2}, \quad (39)$$

for natural numbers $n \neq 0$, where $n = n_\ell$ means the excitation number in mode $a_{\ell,q}$. The energy eigenstates with positive and negative energy values $|\pm E_n\rangle$ are

$$|\pm E_n\rangle = \begin{pmatrix} \sqrt{E_n + mc^2} |n\rangle \\ \mp i \sqrt{E_n - mc^2} |n - 1\rangle \end{pmatrix}. \quad (40)$$

The above result can be rewritten in terms of Pauli spinors, as

$$|\pm E_n\rangle = c_\pm |n\rangle|\chi_1\rangle = i c_\mp |n - 1\rangle|\chi_1\rangle, \quad (41)$$

where $c_\pm$ are

$$c_\pm = \sqrt{E_n \pm mc^2 \over 2E_n}. \quad (42)$$

Let us present an alternative picture of this diagonalization and time evolution as an interference effect [4]. To this end, we express the total Hilbert space as direct sum of one- and two-dimensional subspaces as

$$H = \oplus_{n = 0}^\infty H_n, \quad (43)$$

where the subspaces $H_n$ are defined as

$$H_n = \text{span}\{|n\rangle|\chi_1\rangle, |n - 1\rangle|\chi_1\rangle\}, \quad H_0 = \text{span}\{|0\rangle|\chi_1\rangle\}. \quad (44)$$

Leaving aside the trivial subspace $H_0$, within each of the two-dimensional spaces $H_n$ the Dirac oscillator (23) can be written simply as, in the basis in eq. (44),

$$H_{n,q}^D = mc^2(\sigma_x - \sqrt{4\xi |n|}\sigma_y). \quad (45)$$

This is equivalent to

$$H_{n,q}^D = E_n e^{-i\theta_n\sigma_x}\sigma_x e^{i\theta_n\sigma_x}, \quad (46)$$

where $\tan(2\theta_n) = \sqrt{4\xi |n|}$. Thus the unitary time evolution operator within each subspace $H_n$ is given by

$$U_{n,q} = e^{-iH_{n,q}^D t/h} = e^{-i\theta_n\sigma_x} e^{-iE_n t/\hbar} e^{i\theta_n\sigma_x}. \quad (47)$$

This allows us to interpret the whole process as a series of Mach-Zehnder interferometers in parallel, one for each subspace $H_n$. In this picture the basis vectors $|n\rangle|\chi_1\rangle$, 

44002-p4
\(|n - 1\rangle\langle \chi_1|\) play the role of input/output beams, the operators \(\exp(\pm i\sigma_2)\) play the role of input/output beam splitters, while \(\exp(-iE_n\sigma_z/\hbar)\) represents the phase difference acquired within the two arms of the interferometer. In other words, the beam splitters provide the eigenstates of the Hamiltonian, while the phase difference is given by the eigenvalues. This also allows suitable simple expressions regarding the approximations involved in the non-relativistic limit to be considered below.

**Zitterbewegung effect.** – The Zitterbewegung effect is a trembling motion of relativistic particles that occurs due to the interference between positive and negative energies [40]. Here we want to investigate this effect for the following \(q\)-deformed initial state:

\[
|\psi_0\rangle = |n - 1\rangle|\chi_1\rangle = ic_+|n\rangle + ic_-|E_n\rangle - ic_+|E_n\rangle,
\]

for \(n \geq 1\). The actual existence of these superposition states might be debated after the charge super-selection rule introduced in refs. [41,42]. The evolution of this state can be obtained by

\[
|\psi_t\rangle = ic_+e^{-i\omega_nt}|n\rangle + ic_-e^{i\omega_nt}|E_n\rangle,
\]

where

\[
\omega_n = \frac{E_n}{\hbar} = \frac{1}{\hbar}mc^2\sqrt{1 + 4\xi|n|},
\]

is the frequency of Zitterbewegung oscillation. The oscillation can be well appreciated for example in the mean value of the spin and angular-momentum operators (25):

\[
\langle L_z \rangle_t = -\hbar[n - 1] - 4\xi\hbar[n] + \frac{4\hbar[n]}{1 + 4\xi|n|}\sin^2(\omega_nt),
\]

\[
\langle S_z \rangle_t = -\frac{\hbar}{2} + \frac{4\hbar[n]}{1 + 4\xi|n|}\sin^2(\omega_nt),
\]

and

\[
\langle J_z \rangle_t = -\frac{\hbar}{2}(1 + 2|n - 1|) + \frac{4\hbar[n]}{1 + 4\xi|n|}\sin^2(\omega_nt)(1 - q^{n-1}),
\]

or, equivalently,

\[
\langle J_z \rangle_t = -\frac{\hbar}{2}(1 + 2|n - 1|) - \frac{4\hbar[n]}{1 + 4\xi|n|}\sin^2(\omega_nt)(1 - q^{n-1}) - \frac{4\hbar[\xi|n|]}{1 + 4\xi|n|}\sin^2(\omega_nt). (54)
\]

Several interesting features can be noticed in these results:

i) As an interference effect we can note that the visibility of the time evolution on \(\langle L_z \rangle\) and \(\langle S_z \rangle\) depends on \(q\) explicitly as well as implicitly thorough the factor \([n]\).

ii) At difference with the undeformed case \(q = 1\), in the \(q\)-deformed scenario \(J_z\) is not a constant of the motion except for \(n = 1\), as clearly shown in eqs. (53) and (54).

\(\quad\text{This is because after the commutation relation (1) we have } [H^q_D, J_z] \neq 0.\)

iii) We can recover the constancy of \(\langle J_z \rangle\) provided that we consider an alternative expression for \(L_z\), this is that \(L_z \sim -\hbarn\), where \(n\) is defined in eq. (5). Now the commutation relation (5) grants that \([H^q_D, J_z] = 0\) and \(J_z\) is a constant of the motion as in the undeformed case.

As a further example of Zitterbewegung we continue by considering a coherent initial state (14) so that \(|\psi_0\rangle = \langle \alpha |\chi_1\rangle\) leading to

\[
\langle L_z \rangle_t = -\hbar|\alpha|^2 - \frac{\hbar}{\omega_q(|\alpha|^2)}\sum_{n=0}^{\infty} |\alpha|^{2n}S_{n+1}(t),
\]

\[
\langle S_z \rangle_t = -\frac{\hbar}{2} + \frac{\hbar}{\omega_q(|\alpha|^2)}\sum_{n=0}^{\infty} |\alpha|^{2n}S_{n+1}(t),
\]

\[
\langle J_z \rangle_t = -\frac{\hbar}{2}(1 + 2|\alpha|^2) + \frac{1}{\omega_q(|\alpha|^2)}\sum_{n=0}^{\infty} |\alpha|^{2n}|n|S_{n+1}(t),
\]

where

\[
S_n(t) = \frac{4\xi|n|}{1 + 4\xi|n|}\sin^2(\omega_nt). (58)
\]

All the above quantities satisfy the proper \(q = 1\) limit, and in particular we have that \(\langle J_z \rangle_t\) is constant for \(q = 1\). On the other hand, for \(q \neq 1\) these expressions have the typical structure leading to a scenario of collapse and revivals [25,26]. This is clearly shown in fig. 3.

**Non-relativistic limit.** – Since typically \(\xi|n| \ll mc^2\) it is worth checking the non-relativistic limit of the Dirac oscillator, and for that we may use the quasi-degenerate perturbation theory [43]. In this spirit and given that we know the exact solutions for energies (39) and states (40), the non-relativistic limit can be achieved from a series expansion in powers of \(\xi\). For the first order on \(\xi\) of the eigenvalues we get

\[
E_n \approx mc^2(1 + 2\xi|n|),
\]

while the order \(\xi^0\) of the eigenvectors is

\[
| + E_n \rangle = |n\rangle|\chi_1\rangle, \quad | - E_n \rangle = |n - 1\rangle|\chi_1\rangle. (60)
\]
Thus, within this level of approximation we have $H^D_q \simeq H^D_{\text{eff}, q}$ with

$$H^D_{\text{eff}, q} = \begin{pmatrix} mc^2 + 2\hbar \omega a^\dagger_{\ell,q} a_{\ell,q} & 0 \\ 0 & -mc^2 - 2\hbar \omega a^\dagger_{\ell,q} a_{\ell,q} \end{pmatrix}.$$  

(61)

This result is quite natural since from an optical perspective in this limit the beam splitters $\exp(\pm i\theta \sigma_z)$ are replaced by the identity and the phases are approximate linearly in $\xi$.

* * *

**Zitterbewegung in the non-relativistic limit.** Let us pursue the existence of Zitterbewegung in the non-relativistic limit. We can examine it in several steps. On the one hand, after the form (61) for the effective Hamiltonian we get that in this non-relativistic limit

$$[H^D_{\text{eff}, q}, S_z] = 0, \quad [H^D_{\text{eff}, q}, L_z] = 0,$$

(62)

so that $S_z$ and $L_z$ are constants of the motion and display no Zitterbewegung in this strict limit.

As a further alternative we may examine the evolution of more sophisticated observables such as $M = \sigma^r a_{\ell,q} + \sigma^l a^\dagger_{\ell,q}$. Since the key point of this effect is the interference between negative and positive energies, after eq. (60) we have that the initial state must be different from (48). Thus let us consider instead for example the following initial state:

$$|\psi_0\rangle = c_1 |n\rangle |\chi_1\rangle + c_4 |n-1\rangle |\chi_1\rangle,$$

(63)

where $c_{1,4}$ are real constants. Its time evolution results in

$$|\psi_t\rangle = c_1 e^{-\omega_n t} |n\rangle |\chi_1\rangle + c_4 e^{\omega_n t} |n-1\rangle |\chi_1\rangle,$$

(64)

where here $\omega_n = mc^2(1 + 2\xi |n|)/\hbar$. So we may easily calculate $\langle M \rangle$ leading to

$$\langle M \rangle_t = 2c_1 c_4 \sqrt{|n|} \cos (2\omega_n t),$$

(65)

displaying the desired oscillation fully analogous to eqs. (51) to (53).

On the other hand, we may consider also further corrections to the non-relativistic limit. For example, regarding Zitterbewegung we may consider the same initial state (48) including also the first approximation in powers of $\xi$ in $\exp(\pm i\theta \sigma_z)$, leading to

$$\langle J_z \rangle_t = -\frac{\hbar}{2} (1 + 2|n-1|) + 4\xi \hbar |n| (1 - q^{a-1}) \sin^2 (\omega_n t),$$

(66)

for the same $\omega_n$ above.

**Conclusions.** – We have shown that the equivalence between the two-dimensional Dirac oscillator and the anti-Jaynes-Cummings model extends to a $q$-deformed scenario. After this fundamental equivalence we have investigated from first principles the Hamiltonian spectrum and time evolution for initial number and coherent states, focusing on the appearance of the Zitterbewegung effect. We have highlighted the algebraic modifications introduced by the $q$-deformation. In particular we have shown the lack of preservation of the $q$-deformed versions of the total angular momentum. Actually we have shown it leads to a time evolution mimicking the well-known collapse-revival structure of the population inversion or spin in standard undeformed Jaynes-Cummings model. Moreover, we have provided suitable relations for the non-relativistic limit.

**REFERENCES**

[1] Itô D., Mori K. and Carrieri E., Nuovo Cimento A, 51 (1967) 1119.
[2] Mosincky M. and Szczepaniak A., J. Phys. A, 22 (1989) L817.
[3] Bermudez A., Martin-Delgado M. A. and Solano E., Phys. Rev. A, 76 (2007) 041801(R).
[4] Bermudez A., Martin-Delgado M. A. and LUIS A., Phys. Rev. A, 77 (2008) 033832.
[5] Longhi S., Opt. Lett., 35 (2010) 1302.
[6] Franco-Villafañe J. A., Sadurní E., Barkhofen S., Kuh U., Mortessagne F. and Seligman T. H., Phys. Rev. Lett., 111 (2013) 170405.
[7] MARTIN-DELGADO M. A., J. Phys. A: Math. Gen., 24 (1991) 1285.
[8] Luo Y., Cui Y., Long Z. and Jing J., Int. J. Theor. Phys., 50 (2011) 2992.
[9] Jellal A., El Mouhafid A. and Daoud M., J. Stat. Mech. (2012) P01021.
[10] Jaynes E. T. and Cummings W. F., Proc. IEEE, 51 (1963) 89.
[11] Shore B. W. and Knight P. L., J. Mod. Opt., 40 (1993) 1195.
[12] Leibfried D., Blatt R., Monroe C. and Wineland D., Rev. Mod. Phys., 75 (2003) 281.
[13] Macfarlane A. J., J. Phys. A, 22 (1989) 4581.
[14] Biedenharn L. S., J. Phys. A, 22 (1989) L873.
[15] KULSHI P. and DAMASKINSKY E. V., J. Phys. A, 23 (1990) L415.
[16] LOREK A., RUFFING A. and WESS J., Z. Phys. C, 74 (1997) 369.
[17] EREMIN V. V. and MELDIANOV A. A., Theor. Math. Phys., 147 (2006) 709.
[18] Sanchez O. S. and Re´CAMIER J., J. Phys. B: At. Mol. Opt. Phys., 45 (2012) 015502.

44002-p6
\[ q \]-deformed Dirac oscillator and Jaynes-Cummings model

[19] Črnugelj J., Martinis M. and Mikuta-Martinis V., Phys. Lett. A, 188 (1994) 347.
[20] Man'ko V. I., Marmo G., Sudarshan E. C. G. and Zaccaria F., Phys. Scr., 55 (1997) 528.
[21] Hatami N. and Setare M. R., Phys. Lett. A, 380 (2016) 3469.
[22] Dodonov V. V., J. Opt. B: Quantum Semiclass. Opt., 4 (2002) R1.
[23] Dey S., Fring A., Gouba L. and Castro P. G., Phys. Rev. D, 87 (2013) 084033.
[24] Baez J. C., arXiv:1102.2098 [quant-ph].
[25] Cordero S. and Récamier J., J. Phys. B: At. Mol. Opt. Phys., 44 (2011) 135502.
[26] de los Santos-Sánchez O. and Récamier J., J. Phys. B: At. Mol. Opt. Phys., 45 (2012) 015502.
[27] Vogel W. and de Matos Filho R. L., Phys. Rev. A, 52 (1995) 4214.
[28] de Matos Filho R. L. and Vogel W., Phys. Rev. A, 54 (1996) 4560.
[29] Berrada K., Laser Phys., 26 (2016) 075201.
[30] Luis A., SPIE Rev., 1 (2010) 018006.
[31] Lynch R., Phys. Rep., 256 (1995) 367.
[32] Pešínová V., Lukš A. and Pešina J., Phase in Optics (World Scientific, Singapore) 1998.
[33] Carruthers P. and Nieto M. M., Rev. Mod. Phys., 40 (1968) 411.
[34] Pegg D. T. and Barnett S. M., J. Mod. Opt., 44 (1997) 225.
[35] Fivel D. I., J. Phys. A, 24 (1991) 3575.
[36] Spiridonov V., Lett. Math. Phys., 35 (1995) 179.
[37] Solomon A. I., Phys. Lett. A, 196 (1994) 29.
[38] Wallentowitz S. and Vogel W., Phys. Rev. A, 55 (1997) 4438.
[39] Črnugelj J., Martinis M. and Mikuta-Martinis V., Phys. Rev. A, 50 (1994) 1785.
[40] Greiner W., Relativistic Quantum Mechanics: Wave Equations (Springer, Berlin) 2000.
[41] Wick G. C., Wightman A. S. and Wigner E. P., Phys. Rev., 88 (1952) 101.
[42] Schwbeber S. S., An Introduction to Relativistic Quantum Field Theory (Dover Books on Physics) 2005.
[43] Cohen-Tannoudji C., Dupont-Roc J. and Grynberg G., Atom-Photon Interactions, Basic Processes and Applications (Wiley-VCH, Weinheim) 2004.