Born Rule and Finkelstein-Hartle Frequency Operator Revisited

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(Dated: January 23, 2013)

Abstract

Character of observables in classical physics and quantum theory is reflected upon. Born rule in the context of measurement being an interaction between two quantum systems is discussed. A pedagogical introduction to Finkelstein-Hartle frequency operator is presented.
INTRODUCTION

According to quantum theory, the physical state of a system is represented by a normalized state-vector \( |\psi> \) lying in a Hilbert space corresponding to the system. In the absence of any measurement performed on the system, as time rolls out, this state-vector \( |\psi(t)> \) is found to evolve unitarily, ordained by the Hamiltonian of the system. Such an evolution, led by Schrodinger equation, is causal and deterministic. Physical quantities associated with a system that can be measured are termed as observables, and are represented by self-adjoint operators defined on the Hilbert space. Hamiltonian is a special linear operator corresponding to the energy observable of the system that is responsible for dynamics.

There is a subtle difference between the roles played by observables in classical physics and those in quantum mechanics (QM), other than the measurement limitation imposed by Heisenberg uncertainty principle on a pair of non-commuting observables in QM. In the realm of classical theories, the physical state of a system is completely specified by a set of independent observables of the system. There is no fundamental distinction between the physical state and a complete list of measured values of observables. Generically, the operational definition of a physical observable associated with a classical particle involves directly its state, and the definition itself suggests a method by which one can measure it.

As an illustration, if one were to measure the velocity of a particle at time \( t \), one could just measure the positions at times \( t \) and \( t + \Delta t \), given an infinitesimally tiny \( \Delta t \) (note that the position variable is associated with the state of the particle). The ratio of difference between the two positions to \( \Delta t \) is the measured velocity of the particle, since that is how velocity is defined in the first place. As an aside, this is one of the primary reasons why velocity is ill-defined in QM, because a precise measurement of position of the particle at an instant renders its momentum completely uncertain, making the subsequent position measurement impossible (the other reason being that special relativity along with uncertainty principle imply that localization of a particle to a region of size less than its Compton wavelength necessitates creation of other particles, thereby raising doubts about single particle QM and position being a fundamental observable).

On the other hand, in quantum physics, an observable has an abstract representation in terms of a hermitian operator defined independently of the state. This fundamental separation between states and observables is a quantum feature. Now, the form of a hermitian
operator by itself does not suggest a way to measure it. Instead, one frequently invokes classical concepts and physical intuition to arrange for a suitable interaction between the system and the apparatus such that the observable (or, a related operator like its canonical conjugate, etc.) gets coupled with a ‘pointer variable’ associated with the measuring apparatus, for the purpose of measurement. To cite an example, the expression for the all too familiar spin angular momentum operator of an electron is given by,

\[ \hat{\vec{S}} = \frac{1}{2}\hbar\vec{\sigma}, \]

with \( \sigma_i, i = 1, 2, 3 \) being the Pauli matrices. Evidently, \( \hat{\vec{S}} \) does not involve the general state of the electron, and its mathematical form (or, the commutator brackets of components of the spin) does not give us any clue as to how to go about measuring it.

For the measurement of electron spin in a Stern-Gerlach like contraption, one relies on the classical picture that a spinning charge particle has a magnetic moment \( \vec{\mu} \), so that in the presence of an inhomogeneous magnetic field \( \vec{B}(\vec{r}) \), the interaction energy \( -\vec{\mu}.\vec{B} \) leads to a coupling between the spin degree of freedom and the position (‘pointer variable’) of the electron. Then, assuming that the magnetic moment operator is related to the spin,

\[ \hat{\vec{\mu}} \propto \hat{\vec{S}}, \]

one obtains an interaction Hamiltonian,

\[ \hat{H}_{\text{int}} = -\hat{\vec{\mu}}.\vec{B} \propto -\hat{\vec{S}}.\vec{B} \]

If the initial spin state is orthogonal to \( \vec{B} \), time evolution due to the above interaction Hamiltonian causes, by virtue of Schrodinger equation, an entanglement between spin and position degrees such that the state of the electron is described by a superposition of correlated electron’s spin and position states.

However, when a device like a photographic plate is deployed to measure the position of an electron, one observes a definite position correlated with a spin state, and not a superposition of correlated states predicted by the unitary evolution (this is the central mystery of QM). The spin state, thereafter, is inferred from the observed position. Attempts to understand how the entangled state breaks into two distinct branches corresponding to spin up and spin down have kept the QM community active for the last eight decades or so (see for example [1]).
In general, the outcome of a measurement is always one of the eigenvalues $a_i, i = 1, 2, ...$ of the self-adjoint operator $\hat{A}$ that represents the measured observable. But which one? It is at this point that the apparition of indeterminism makes its presence felt. Normally, one cannot predict with certainty the outcome of a measurement.

More importantly, Max Born had discovered that the probability of finding a particular eigenvalue $a_i$ to be the outcome at time $t$ is given by $|<i|\psi(t)>|^2$ and, according to standard QM, it is associated with a ‘collapse’ of the state-vector $|\psi(t)> > |i>$, an eigenstate of $\hat{A}$ with eigenvalue $a_i$. The probabilities of the potential outcomes can definitely be predicted with certainty. How is the Born rule verified in practice?

The standard probabilistic nature of QM emerges from measurements conducted on an ensemble consisting of a large number $N$ of identical systems that are isolated from each other and placed in identical surroundings, with each being described by an identical state-vector $|\psi(t_0)> >$ at time $t_0$. Thereafter, the state of each system evolves unitarily in an identical manner, governed by the Schrodinger equation. When one performs at some time $t$ measurements of an observable, say $\hat{A}$, on each system belonging to the ensemble, one finds a random distribution of measured values with each outcome being one of the eigenvalues $a_i, i = 1, 2, ..$, corresponding to a measurement. Then, it turns out that the frequency of occurrence of the eigenvalue $a_i$ is $|<i|\psi(t)>|^2$, in the limit $N$ tending to a very large number.

Although from a practical point of view quantum mechanics is very successful, issues related to measurements have continued to perplex one since the inception of the theory. Any measurement after all is a physical interaction between the system and the apparatus (which too is governed by laws of QM) involving a definite interaction Hamiltonian, so that one expects the combined system to evolve with time according to Schrodinger equation in a deterministic manner. In that case, why are outcomes of measurement acausal and unpredictable? Stated differently, from where does Born rule spring from?

In the framework of Many-Worlds Interpretation (MWI) of QM, the state of the combined system continues to be an entanglement of system-apparatus-observer correlated states as ordained by unitary evolution, and there never occurs a wavefunction collapse in this picture [2]. Can the observed indeterminism in QM be explained by MWI? There have been humongous amount of research work on the issue of obtaining Born rule without invoking probabilistic ‘collapse of the state-vector to an eigenstate’ in MWI, as well as other
approaches like, for instance, decoherence framework (see [1-5] and the references therein).

In this lecture, we ask a different question. Can we have Born probabilities as eigenvalues of a frequency observable that is defined over an ensemble? It is in this context that Hartle’s paper assumes significance since it had attempted to deduce Born probabilities from frequencies of occurrences of eigenvalues in measurements in a framework describing an ensemble of large number of identical systems as an individual quantum system over which a frequency operator is defined [6]. In fact, the form of the frequency observable was introduced even earlier by Finkelstein [7], and therefore is referred to as the Finkelstein-Hartle frequency operator [8]. In the following section, we present a pedagogical description of Hartle’s approach.

**FINKELSTEIN-HARTLE FREQUENCY OPERATOR**

The basic idea is to devise an observable whose eigenvalues are the Born probabilities (without invoking the Born rule). If the collection of measurements over an ensemble is viewed as a grand measurement of this frequency observable, then it follows from laws of QM that only its eigenvalues are the outcomes. Suppose the eigenvalues turn out to be $| < i|\psi(t) > |^2$, then one has moved one step forward in understanding the origin of indeterminism in QM.

We begin with a quantum system described by a Hilbert space $H$ and an observable $\hat{A}$ defined on it. For simplicity, we assume that $\hat{A}$ has a discrete spectrum of eigenvalues so that,

$$\hat{A}|i\rangle = a_i|i\rangle, i = 1, 2, 3, \ldots$$

(1a)

with eigenstates $|i\rangle$ forming a complete orthonormal basis, satisfying the inner product orthonormality,

$$\langle |i\rangle , |j\rangle \rangle = \langle i|j \rangle = \delta_{ij}$$

(1b)

and completeness,

$$\sum_{i=1}^{\infty} |i\rangle \langle i| = 1 ,$$

(1c)

where $a_i$ are the eigenvalues that are observed when $\hat{A}$ is measured.

A physical state of the system is described by $|\psi\rangle$, which is an element of $H$, and can be
linearly expanded in terms of the orthonormal basis vectors \( \{ |i\rangle, \ i = 1, 2 \ldots \} \),

\[ |\psi\rangle = \sum_{i=1}^{\infty} c_i |i\rangle, \tag{2a} \]

with,

\[ \langle \psi | \psi \rangle = 1, \tag{2b} \]

and,

\[ c_i = \langle i | \psi \rangle, \tag{2c} \]

which follows from eqs.(1b) and (2a).

Since, the probabilistic character of QM (concerning outcomes of measurements) have been tested by making use of large number of identically prepared systems, we need to formulate the measurement problem accordingly. An ensemble of \( N \) identical systems in QM is represented by the Hilbert space formed out of the tensor product of individual spaces \( H \times H \times \ldots \times H \equiv H^N \).

For convenience, we label the systems belonging to the ensemble using the index \( \alpha \) with \( \alpha = 1, 2, ..., N \). If each system is specified by the state-vector \( |\psi\rangle \), the physical state describing the ensemble is then represented by an element of \( H^N \) given by the direct product of \( |\psi\rangle \)s,

\[ |(\psi)^N\rangle \equiv |\psi\rangle_1 |\psi\rangle_2 \ldots |\psi\rangle_N, \tag{3a} \]

where the subscript on \( |\psi\rangle \) indicates the system to which the state-vector corresponds.

In the limit \( N \to \infty \), the ensemble state-vector tends to,

\[ |(\psi)^\infty\rangle \equiv |\psi\rangle_1 |\psi\rangle_2 \ldots |\psi\rangle_N |\psi\rangle_{N+1} \ldots, \tag{3b} \]

assuming that the limit is well-defined (one has to be careful with such limits, as most of the peculiarities which we come across later, stem from such infinities and associated measures [8]).

We use a notation in which, for the \( \alpha^{th} \) system of the ensemble, \( |i_{\alpha}\rangle \) denotes the eigenstate of \( \hat{A} \) with eigenvalue \( a_{i_{\alpha}} \), so that \( \{ |i_{\alpha}\rangle, i_{\alpha} = 1, 2, \ldots \} \) is an orthonormal basis corresponding to the \( \alpha^{th} \) system (see eqs.(1a)-(1c)). The Hilbert space \( H^N \) is therefore spanned by the direct product of orthonormal vectors \( \{ |i_1\rangle |i_2\rangle \ldots |i_N\rangle, \ i_1, i_2, \ldots = 1, 2, \ldots \} \).
These direct product of eigenstates and their dual can be used to construct a frequency operator $\hat{F}_{N}^{j}$ for the eigenvalue $a_j$ of $\hat{A}$ as follows,

$$\hat{F}_{N}^{j} \equiv \sum_{i_1, i_2, \ldots, i_N} f_j |i_1\rangle |i_2\rangle \cdots |i_N\rangle \langle i_{N-1}| \cdots \langle i_2| |i_1| ,$$

where,

$$f_j \equiv \frac{1}{N} \sum_{\alpha=1}^{N} \delta_{ji_\alpha}$$

is clearly the frequency of $i_\alpha$ being equal to $j$ in the set $\{i_1, i_2, \ldots, i_N\}$.

It is easy to see that $|i'_1\rangle |i'_2\rangle \cdots |i'_N\rangle$ is an eigenstate of $\hat{F}_{N}^{j}$ corresponding to the eigenvalue being the frequency of $i'_\alpha$ equal to $j$ for $\alpha = 1, 2, \ldots, N$ in $\{i'_1, i'_2, \ldots, i'_N\}$, since from eqs. (4a-b) and the orthonormality of $\{|i_\alpha\rangle, i_\alpha = 1, 2, \ldots\}$ (eq.(1b)) we get,

$$\hat{F}_{N}^{j} |i'_1\rangle |i'_2\rangle \cdots |i'_N\rangle = \sum_{i_1, i_2, \ldots, i_N} \frac{1}{N} \sum_{\alpha=1}^{N} \delta_{ji_\alpha} |i_1\rangle |i_2\rangle \cdots |i_N\rangle \delta_{ii'_1} \delta_{ii'_2} \cdots \delta_{ii'_N}$$

$$= \frac{1}{N} \sum_{\alpha=1}^{N} \delta_{ji'_\alpha} \left(|i'_1\rangle |i'_2\rangle \cdots |i'_N\rangle\right),$$

thus, vindicating that $\hat{F}_{N}^{j}$ indeed is a frequency operator.

For later purposes, it is useful to express the frequency operator as,

$$\hat{F}_{N}^{j} = \frac{1}{N} \sum_{i_1, i_2, \ldots, i_N} |i_1\rangle |i_2\rangle \cdots |i_N\rangle \left(\delta_{ji_1} + \delta_{ji_2} + \cdots + \delta_{ji_N}\right) \langle i_{N-1}| \cdots \langle i_2| |i_1|$$

$$= \frac{1}{N} \left\{|j\rangle_{11} \langle j| + \sum_{i_2, i_3, \ldots, i_N} |i_2\rangle \langle i_2| |i_3\rangle \langle i_3| \cdots \langle i_N| \langle i_N| + \cdots +

+ |j\rangle_{NN} \langle j| \sum_{i_1, i_2, \ldots, i_{N-1}} |i_1\rangle \langle i_1| |i_2\rangle \langle i_2| \cdots \langle i_{N-1}| \langle i_{N-1}|\right\}$$

$$= \frac{1}{N} \left\{|j\rangle_{11} + |j\rangle_{22} + \cdots + |j\rangle_{NN}\right\}$$

The last step follows from eq.(1c). In eq.(6), the ket $|j\rangle_{\alpha}$ and its dual represent the eigenstate $|j\rangle$ and the eigenbra, respectively, corresponding to the eigenvalue $a_j$ for the $\alpha^{th}$ system.

The operation of $\hat{F}_{N}^{j}$ on $|\langle\psi\rangle^\infty\rangle$ is defined by,

$$\hat{F}_{N}^{j} |\langle\psi\rangle^\infty\rangle \equiv (\hat{F}_{N}^{j} |\langle\psi\rangle^N\rangle) |\langle\psi\rangle^N_{+1}\rangle |\psi\rangle^N_{+2} \cdots$$

\[7a\]
Using eq. (6) and eq. (2c), we obtain,

\[ \hat{F}_N^j|\psi^N\rangle = \frac{c_j}{N} \left\{ |j\rangle_1|\psi\rangle_2|\psi\rangle_3 \ldots |\psi\rangle_N + |\psi\rangle_1|j\rangle_2|\psi\rangle_3 \ldots |\psi\rangle_N + \ldots + |\psi\rangle_1|\psi\rangle_2 \ldots |\psi\rangle_{N-1}|j\rangle_N \right\}, \]  

(7b)

so that, from eq. (7a), we have,

\[ \hat{F}_N^j|\psi^\infty\rangle = \frac{c_j}{N} \left\{ |j\rangle_1|\psi\rangle_2|\psi\rangle_3 \ldots |\psi\rangle_N + |\psi\rangle_1|j\rangle_2|\psi\rangle_3 \ldots |\psi\rangle_N + \ldots + |\psi\rangle_1|\psi\rangle_2 \ldots |\psi\rangle_{N-1}|j\rangle_N \right\}|\psi\rangle_{N+1}|\psi\rangle_{N+2} \ldots \]  

(7c)

Following Hartle [6], we may ask how close is the state-vector \( \hat{F}_N^j|\psi^\infty\rangle \) to \( |c_j|^2|\psi^\infty\rangle \) in the limit \( N \to \infty \)? Now, to address this question, we may use the norm that is induced by the inner product, in order to obtain,

\[ ||\hat{F}_N^j|\psi^\infty\rangle - |c_j|^2|\psi^\infty\rangle||^2 = \left( \hat{F}_N^j|\psi^\infty\rangle - |c_j|^2|\psi^\infty\rangle, \hat{F}_N^j|\psi^\infty\rangle - |c_j|^2|\psi^\infty\rangle \right) \]  

(8a)

\[ = \left( \hat{F}_N^j|\psi^\infty\rangle, \hat{F}_N^j|\psi^\infty\rangle \right) - 2|c_j|^2 \left( \left|\langle \psi^\infty|\psi^\infty\rangle \right|, \hat{F}_N^j|\psi^\infty\rangle \right) + |c_j|^4 \]  

(8b)

The last term in eq. (8b) arises from the fact that \( \langle \psi^\infty|\psi^\infty\rangle = 1 \) because of eqs. (2b) and (3b). Again, eqs. (2b), (2c), (3b) and (7c) lead to the inner product,

\[ \left( \langle \psi^\infty|, \hat{F}_N^j|\psi^\infty\rangle \right) = \frac{c_j}{N} \left\{ 1\langle \psi|j\rangle_1 + 2\langle \psi|j\rangle_2 + \ldots + N\langle \psi|j\rangle_N \right\} = |c_j|^2 \]  

(8c)

From eqs. (7c) and (2b), we get,

\[ \left( \hat{F}_N^j|\psi^\infty\rangle, \hat{F}_N^j|\psi^\infty\rangle \right) = \frac{|c_j|^2}{N^2} \left\{ 1\langle j|2\langle \psi|3\langle \psi \ldots N\langle \psi \right| + + \left\{ \langle j\rangle_1|\psi\rangle_2|\psi\rangle_3 \ldots |\psi\rangle_N + |\psi\rangle_1|j\rangle_2|\psi\rangle_3 \ldots |\psi\rangle_N + \ldots + |\psi\rangle_1|\psi\rangle_2 \ldots |\psi\rangle_{N-1}|j\rangle_N \right\} \times \left\{ |j\rangle_1|\psi\rangle_2|\psi\rangle_3 \ldots |\psi\rangle_N + |\psi\rangle_1|j\rangle_2|\psi\rangle_3 \ldots |\psi\rangle_N + \ldots + |\psi\rangle_1|\psi\rangle_2 \ldots |\psi\rangle_{N-1}|j\rangle_N \right\} \]  

(9a)

\[ = \frac{|c_j|^2}{N^2} \left\{ N + N(N-1)|c_j|^2 \right\} \]  

(9b)

Hence, employing eqs. (8c) and (9b) in eq. (8b), give rise to,

\[ ||\hat{F}_N^j|\psi^\infty\rangle - |c_j|^2|\psi^\infty\rangle||^2 = \frac{|c_j|^2}{N} \left\{ 1 - |c_j|^2 \right\} \]  

(9c)

From eq. (9c), it is evident that as \( N \to \infty \), we have \( ||\hat{F}_N^j|\psi^\infty\rangle - |c_j|^2|\psi^\infty\rangle||^2 \to 0 \).
This is a remarkable result in the sense that no matter what $|\psi\rangle$ is, for every eigenvalue $a_j$ of the observable $\hat{A}$, the distance between the state $\hat{F}_N^j|\langle\psi\rangle^\infty\rangle$ and the Born probability times $|\langle\psi\rangle^\infty\rangle$ can be made arbitrarily small by considering sufficiently large ensemble. But this by no means implies that as $N \to \infty$, the state-vector $\hat{F}_N^j|\langle\psi\rangle^\infty\rangle \to |c_j|^2|\langle\psi\rangle^\infty\rangle$.

In fact, it can be demonstrated that the vanishing of the left hand side of eq.(9c) does not entail that $\hat{F}_N^j|\langle\psi\rangle^\infty\rangle = |c_j|^2|\langle\psi\rangle^\infty\rangle$ as $N \to \infty$ [8,9]. As much is hinted by the expression in the right hand side of eq.(7b).

The hope articulated in the beginning of this section of obtaining Born probabilities $|c_j|^2$, $j = 1, 2, \ldots$ as eigenvalues of the frequency operator remains unfulfilled in this approach (For a detailed discussion on this issue please refer to the papers by Squires [9], Caves and Schack [8] as well as N. D. Hari Dass’ lecture in this meeting).

**SUMMARY**

The preceding section draws our attention to some very interesting points. From eqs.(8c) and (9c), we find that the expectation value of the frequency operator and its uncertainty corresponding to the state $|\langle\psi\rangle^\infty\rangle$ are given by,

$$\langle F_N^j \rangle = |c_j|^2$$

and,

$$\Delta F_N^j = \sqrt{\frac{|c_j|^2 - |c_j|^4}{N}},$$

respectively. Although in the limit $N$ tending to infinity, $\Delta F_N^j$ vanishes, the state-vector $|\langle\psi\rangle^\infty\rangle$ describing the ensemble does not become an eigenstate of the frequency operator with the Born probability $|c_j|^2$ as the eigenvalue (For a thorough critical analysis, please see the paper by Caves and Schack [8]).

The frequency operator approach is silent about how to measure the corresponding observable. If one employs the obvious method of measuring $\hat{A}$ for every system in the ensemble, to find the frequency of occurrence of an eigenvalue, then this approach does not throw much light on the measurement problem as to whether $|\psi\rangle$ collapses to one of the eigenstates in individual measurements. The enigma of Born rule continues to be wrapped in a riddle inside a mystery!
ACKNOWLEDGEMENTS

It is a pleasure to thank Professor N.D. Hari Dass and Professor R. Srikanth for providing a stimulating atmosphere for debates and discussions during the meeting, and also for their generous hospitality at PPISR, Bangalore.

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