SCATTERING THEORY FOR SCHRÖDINGER OPERATORS ON
STEPLIKE, ALMOST PERIODIC INFINITE-GAP
BACKGROUNDS

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Abstract. We develop direct scattering theory for one-dimensional Schrödinger
operators with steplike potentials, which are asymptotically close to different
Bohr almost periodic infinite-gap potentials on different half-axes.

1. Introduction

One of the main tools for solving various Cauchy problems, since the seminal
work of Gardner, Green, Kruskal, and Miura [10] in 1967, is the inverse scattering
transform and therefore, since then, a large number of articles has been devoted to
direct and inverse scattering theory.

Given two (in general different) one-dimensional background Schrödinger oper-
ators $L_\pm$ with real finite-gap potentials $p_\pm(x)$, i.e.

\begin{equation}
L_\pm = -\frac{d^2}{dx^2} + p_\pm(x), \quad x \in \mathbb{R},
\end{equation}

one can consider the perturbed one-dimensional Schrödinger operator

\begin{equation}
L = -\frac{d^2}{dx^2} + p(x), \quad x \in \mathbb{R},
\end{equation}

where $p(x)$ satisfies a second moment condition, i.e.

\begin{equation}
\pm \int_0^{\pm \infty} (1 + x^2) |p(x) - p_\pm(x)| dx < \infty.
\end{equation}

Then one of the main tools when considering the scattering problem for the Schrödinger
operator $L$, are the transformation operators which map the background Weyl solu-
tions of the operators $L_\pm$ to the Jost solutions of $L$. In particular, if the background
operators are well-understood, the transformation operators enable us to perform
the direct scattering step, which means to characterize the scattering data and to
derive the Gel’fand-Levitan-Marchenko equation. The starting point for the inverse
scattering step is the Gel’fand-Levitan-Marchenko equation together with the scatter-
ing data, from which one deduces the kernels of the transformation operators
and recovers the potential $p(x)$.

In much detail the scattering problem has been studied in the case where $p(x)$
is asymptotically close to $p_\pm(x) = 0$. For a complete investigation and discussions
on the history of this problem we refer to the monographs of Levitan [17] and
Marchenko [21]. Taking this as a starting point, two natural extension have been

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considered. On the one hand the case of steplike constant asymptotics $p_{\pm}(x) = c_{\pm}$, where $c_{+} \neq c_{-}$ denote some constants, has been investigated by Buslaev and Fomin [2], Cohen and Kappeler [4], and Davies and Simon [3]. On the other hand Firsova [9] studied the case of equal periodic, finite-gap potentials $p_{+}(x) = p_{-}(x)$. Rather recently, the combination of these two cases, namely the case that the initial condition is asymptotically close to steplike, quasi-periodic, finite-gap potentials $p_{-}(x) \neq p_{+}(x)$, has been investigated by Boutet de Monvel, Egorova, and Teschl [1]. Trace formulas in the case of one periodic background were given by Mikikits-Leitner and Teschl [22] and a Paley–Wiener theorem in Egorova and Teschl [6].

Of course the inverse scattering theory is also the main ingredient for solving the Cauchy problem of the Korteweg–de Vries (KdV) equation via the inverse scattering transform [21]. Moreover, scattering theory is also the basic ingredient for setting up the associated Riemann–Hilbert problem from which the long-time asymptotics can be derived via the nonlinear steepest descent analysis (see [14] for an overview). In the case of finite-gap backgrounds the Cauchy problem was solved by Grunert, Egorova, and Teschl [13, 8]. Note that the analogous Cauchy problem for the modified KdV equations can be obtained via the Miura transform [7]. The long-time asymptotics in case of one finite-gap background were recently derived by Mikikits-Leitner and Teschl [23].

Of much interest is also the case of asymptotically periodic solutions, which has been first considered by Firsova [9]. In the present work we propose a complete investigation of the direct scattering theory for Bohr almost periodic infinite-gap backgrounds, which belong to the so–called Levitan class. It should be noticed, that this class, as a special case, includes the set of smooth, periodic infinite–gap operators.

To set the stage, we need:

**Hypothesis H.1.1.** Let

$$0 \leq E_{0}^{\pm} < E_{1}^{\pm} < \cdots < E_{n}^{\pm} < \cdots$$

be two increasing sequences of points on the real axis which satisfy the following conditions:

1. For a certain $l^{\pm} > 1$, $\sum_{n=1}^{\infty} (E_{2n-1}^{\pm} - E_{2n-2}^{\pm}) < \infty$ and
2. $E_{2n+1}^{\pm} - E_{2n-1}^{\pm} > C^{\pm}n^{\alpha^{\pm}}$, where $C^{\pm}$ and $\alpha^{\pm}$ are some fixed, positive constants.

We will call, in what follows, the intervals $(E_{2j-1}^{\pm}, E_{2j}^{\pm})$ for $j = 1, 2, \ldots$ gaps. In each closed gap $[E_{2j-1}^{\pm}, E_{2j}^{\pm}]$, $j = 1, 2, \ldots$, we choose a point $\mu_{j}^{\pm}$ and an arbitrary sign $\sigma_{j}^{\pm} \in \{-1, 1\}$.

Next consider the system of differential equations for the functions $\mu_{j}^{\pm}(x)$, $\sigma_{j}^{\pm}(x)$, $j = 1, 2, \ldots$, which is an infinite analogue of the well-known Dubrovin equations, given by

$$\frac{d\mu_{j}^{\pm}(x)}{dx} = -2\sigma_{j}^{\pm}(x)\sqrt{-(\mu_{j}^{\pm}(x) - E_{0}^{\pm})\mu_{j}^{\pm}(x) - E_{2j-1}^{\pm}}\mu_{j}^{\pm}(x) - E_{2j}^{\pm}) \times \prod_{k=1, k \neq j}^{\infty} \frac{\sqrt{\mu_{j}^{\pm}(x) - E_{2k-1}^{\pm}}\sqrt{\mu_{j}^{\pm}(x) - E_{2k}^{\pm}}}{\mu_{j}^{\pm}(x) - \mu_{k}^{\pm}(x)}$$
with initial conditions $\mu_j^\pm(0) = \mu_j^\pm$ and $\sigma_j^\pm(0) = \sigma_j^\pm$, $j = 1, 2, \ldots$ [1]. Levitan [17], [18], and [19], proved, that this system of differential equations is uniquely solvable, that the solutions $\mu_j^\pm(x)$, $j = 1, 2, \ldots$ are continuously differentiable and satisfy $\mu_j^\pm(x) \in [E_j^\pm, E_{j+1}^\pm]$ for all $x \in \mathbb{R}$. Moreover, these functions $\mu_j^\pm(x)$, $j = 1, 2, \ldots$ are Bohr almost periodic [2]. Using the trace formula (see for example [17])

\begin{equation}
\sigma_j^\pm = [E_0^\pm, E_1^\pm] \cup \ldots \cup [E_j^\pm, E_{j+1}^\pm] \cup \ldots,
\end{equation}

we see that also $p_j^\pm(x)$ are real Bohr almost periodic. The operators

\begin{equation}
L_j^\pm := -\frac{d^2}{dx^2} + p_j^\pm(x), \quad \text{dom}(L_j^\pm) = H^2(\mathbb{R}),
\end{equation}

in $L^2(\mathbb{R})$, are then called almost periodic infinite-gap Schrödinger operators of the Levitan class. The spectra of $L_j^\pm$ are purely absolutely continuous and of the form

\begin{equation}
\sigma_j^\pm = [E_0^\pm, E_1^\pm] \cup \ldots \cup [E_j^\pm, E_{j+1}^\pm] \cup \ldots,
\end{equation}

and have spectral properties analogous to the quasi-periodic finite-gap Schrödinger operator. In particular, they are completely defined by the series $\sum_{j=1}^{\infty} (\mu_j^+, \sigma_j^+)$, which we call the Dirichlet divisor. These divisors are associated to Riemann surfaces of infinite genus, which are connected with the functions $Y_j^{1/2}(z)$, where

\begin{equation}
Y_j^\pm(z) = -(z - E_0^\pm) \prod_{j=1}^{\infty} \frac{(z - E_{2j-1}^\pm) (z - E_{2j}^\pm)}{E_{2j-1}^\pm E_{2j}^\pm},
\end{equation}

and where the branch cuts are taken along the spectrum. It is known, that the Schrödinger equations

\begin{equation}
\left( -\frac{d^2}{dx^2} + p_j^\pm(x) \right) y(x) = z y(x)
\end{equation}

with any continuous, bounded potential $p_j^\pm(x)$ have two Weyl solutions $\psi_j^\pm(z, x)$ and $\tilde{\psi}_j^\pm(z, x)$, which satisfy

\begin{equation}
\psi_j^\pm(z, \cdot) \in L^2(\mathbb{R}_\pm), \quad \text{resp.} \quad \tilde{\psi}_j^\pm(z, \cdot) \in L^2(\mathbb{R}_\mp),
\end{equation}

for $z \in \mathbb{C} \setminus \sigma_j^\pm$ and which are normalized by $\psi_j^\pm(z, 0) = \tilde{\psi}_j^\pm(z, 0) = 1$. In our case of Bohr almost periodic potentials of the Levitan class, these solutions have complementary properties similar to the properties of the Baker-Akhiezer functions in the finite-gap case. We will briefly discuss them in the next section.

The object of interest, for us, is the one-dimensional Schrödinger operator $L$ in $L^2(\mathbb{R})$

\begin{equation}
L := -\frac{d^2}{dx^2} + q(x), \quad \text{dom}(L) = H^2(\mathbb{R}),
\end{equation}

with the real potential $q(x) \in C(\mathbb{R})$ satisfying the following condition

\begin{equation}
\pm \int_0^{\pm \infty} (1 + |x|^2) q(x) - p_j^\pm(x) dx < \infty,
\end{equation}

We will use the standard branch cut of the square root in the domain $\mathbb{C} \setminus \mathbb{R}_+$ with $\text{Im} \sqrt{\tau} > 0$. For informations about almost periodic functions we refer to [20].
for which we will characterize the corresponding scattering data and derive the Gel’fand-Levitan-Marchenko equation with the help of the transformation operator, which has been investigated in [12].

2. The Weyl solutions of the background operators

In this section we want to summarize some facts for the background Schrödinger operators $L_{\pm}$ of Levitan class. We present these results, obtained in [12], [17], [25], and [26], in a form, similar to the finite-gap case used in [1] and [11].

Let $L_{\pm}$ be the quasi-periodic one-dimensional Schrödinger operators associated with the potentials $p_{\pm}(x)$. Let $s_{\pm}(z,x)$, $c_{\pm}(z,x)$ be sine- and cosine-type solutions of the corresponding equation

$$
(2.1) \quad \left( -\frac{d^2}{dx^2} + p_{\pm}(x) \right) y(x) = z y(x), \quad z \in \mathbb{C},
$$

associated with the initial conditions

$$
s_{\pm}(z,0) = c'_{\pm}(z,0) = 0, \quad c_{\pm}(z,0) = s'_{\pm}(z,0) = 1,
$$

where prime denotes the derivative with respect to $x$. Then $c_{\pm}(z,x)$, $c'_{\pm}(z,x)$, $s_{\pm}(z,x)$, and $s'_{\pm}(z,x)$ are entire with respect to $z$. Moreover, they can be represented in the following form

$$
c_{\pm}(z,x) = \cos(\sqrt{z}x) + \int_{0}^{x} \frac{\sin(\sqrt{z}(x-y))}{\sqrt{z}} p_{\pm}(y)c_{\pm}(z,y)dy,
$$

$$
s_{\pm}(z,x) = \frac{\sin(\sqrt{z}x)}{\sqrt{z}} + \int_{0}^{x} \frac{\sin(\sqrt{z}(x-y))}{\sqrt{z}} p_{\pm}(y)s_{\pm}(z,y)dy.
$$

The background Weyl solutions are given by

$$
\psi_{\pm}(z,x) = c_{\pm}(z,x) + m_{\pm}(z,0)s_{\pm}(z,x),
$$

resp.

$$
\tilde{\psi}_{\pm}(z,x) = c_{\pm}(z,x) + \tilde{m}_{\pm}(z,0)s_{\pm}(z,x),
$$

where

$$
m_{\pm}(z,x) = \frac{H_{\pm}(z,x) \pm Y^{1/2}_{\pm}(z)}{G_{\pm}(z,x)}, \quad \tilde{m}_{\pm}(z,x) = \frac{H_{\pm}(z,x) \mp Y^{1/2}_{\pm}(z)}{G_{\pm}(z,x)},
$$

are the Weyl functions of $L_{\pm}$ (cf [17]), where $Y_{\pm}(z)$ are defined by (1.7),

$$
(2.4) \quad G_{\pm}(z,x) = \prod_{j=1}^{\infty} \frac{z - \mu_{\pm}^{j}(x)}{E_{2j-1}}, \quad \text{and} \quad H_{\pm}(z,x) = \frac{1}{2} \frac{d}{dx} G_{\pm}(z,x).
$$

Using (1.4) and (2.4), we have

$$
(2.5) \quad H_{\pm}(z,x) = \frac{1}{2} \frac{d}{dx} G_{\pm}(z,x) = G_{\pm}(z,x) \sum_{j=1}^{\infty} \frac{\sigma_{\pm}^{j}(x) Y^{1/2}_{\pm}(\mu_{\pm}^{j}(x))}{dz G_{\pm}(\mu_{\pm}^{j}(x),x)(z - \mu_{\pm}^{j}(x))}.
$$

The Weyl functions $m_{\pm}(z,x)$ and $\tilde{m}_{\pm}(z,x)$ are Bohr almost periodic.

Lemma 2.1. The background Weyl solutions, for $z \in \mathbb{C}$, can be represented in the following form

$$
(2.6) \quad \psi_{\pm}(z,x) = \exp \left( \int_{0}^{x} m_{\pm}(z,y)dy \right) = \left( \frac{G_{\pm}(z,x)}{G_{\pm}(z,0)} \right)^{1/2} \exp \left( \pm \int_{0}^{x} \frac{Y^{1/2}_{\pm}(z)}{G_{\pm}(z,y)}dy \right),
$$

$$
\tilde{\psi}_{\pm}(z,x) = \exp \left( \int_{0}^{x} \frac{Y^{1/2}_{\pm}(z)}{G_{\pm}(z,y)}dy \right),
$$

where $\sigma_{\pm}^{j}(x)$ are the characteristic functions of the corresponding quantum operators.
We set \( \mu_j \).

If for some \( \varepsilon > 0 \), \(|z - \mu_j^\pm(x)| > \varepsilon\) for all \( j \in \mathbb{N} \) and \( x \in \mathbb{R} \), then the following holds: For any \( 1 > \delta > 0\) there exists an \( R > 0 \) such that

\[
|\psi_\pm(z,x)| \leq e^\mp(1-\delta)x \text{Im}(\sqrt{\varepsilon}) \left( 1 + \frac{D_R}{|z|} \right), \text{ for any } |z| \geq R, \quad \pm x > 0,
\]

and

\[
|\tilde{\psi}_\pm(z,x)| \leq e^\pm(1-\delta)x \text{Im}(\sqrt{\varepsilon}) \left( 1 + \frac{D_R}{|z|} \right), \text{ for any } |z| \geq R, \quad \pm x < 0,
\]

where \( D_R \) denotes some constant dependent on \( R \).

As the spectra \( \sigma_\pm \) consist of infinitely many bands, let us cut the complex plane along the spectrum \( \sigma_\pm \) and denote the upper and lower sides of the cuts by \( \sigma_\pm^+ \) and \( \sigma_\pm^- \). The corresponding points on these cuts will be denoted by \( \lambda^u \) and \( \lambda^l \), respectively. In particular, this means

\[
f(\lambda^u) := \lim_{\varepsilon \downarrow 0} f(\lambda + i\varepsilon), \quad f(\lambda^l) := \lim_{\varepsilon \downarrow 0} f(\lambda - i\varepsilon), \quad \lambda \in \sigma_\pm.
\]

Defining

\begin{equation}
(2.7) \quad g_\pm(\lambda) = -\frac{G_\pm(\lambda,0)}{2Y^{1/2}_\pm(\lambda)},
\end{equation}

where the branch of the square root is chosen in such a way that

\begin{equation}
(2.8) \quad \frac{1}{i}g_\pm(\lambda^u) = \text{Im}(g_\pm(\lambda^u)) > 0 \quad \text{for } \lambda \in \sigma_\pm,
\end{equation}

it follows from Lemma \[2.1\] that

\begin{equation}
(2.9) \quad W(\tilde{\psi}_\pm(z),\psi_\pm(z)) = m_\pm(z) - \tilde{m}_\pm(z) = \mp g_\pm(z)^{-1},
\end{equation}

where \( W(f,g)(x) = f(x)g'(x) - f'(x)g(x) \) denotes the usual Wronskian determinant.

For every Dirichlet eigenvalue \( \mu_j^\pm = \mu_j^\pm(0) \), the Weyl functions \( m_\pm(z) \) and \( \tilde{m}_\pm(z) \) might have poles. If \( \mu_j^\pm \) is in the interior of its gap, precisely one Weyl function \( m_\pm \) or \( \tilde{m}_\pm \) will have a simple pole. Otherwise, if \( \mu_j^\pm \) sits at an edge, both will have a square root singularity. Hence we divide the set of poles accordingly:

\[
M_\pm = \{ \mu_j^+ \mid \mu_j^+ \in (E_{2j-1}^\pm, E_{2j}^\pm) \text{ and } m_\pm \text{ has a simple pole} \},
\]

\[
\tilde{M}_\pm = \{ \mu_j^\pm \mid \mu_j^\pm \in (E_{2j-1}^\pm, E_{2j}^\pm) \text{ and } \tilde{m}_\pm \text{ has a simple pole} \},
\]

\[
\check{M}_\pm = \{ \mu_j^\pm \mid \mu_j^\pm \in \{ E_{2j-1}^\pm, E_{2j}^\pm \} \},
\]

and we set \( M_{\tau,\pm} = M_\pm \cup \tilde{M}_\pm \cup \check{M}_\pm \).

In particular, we obtain the following properties of the Weyl solutions (see, e.g. \[3, 12, 17, 27\], and \[28\]):

**Lemma 2.2.** The Weyl solutions have the following properties:
(i) The function \( \psi_{\pm}(z, x) \) (resp. \( \tilde{\psi}_{\pm}(z, x) \)) is holomorphic as a function of \( z \) in the domain \( \mathbb{C} \setminus (\sigma_{\pm} \cup M_{\pm}) \) (resp. \( \mathbb{C} \setminus (\sigma_{\pm} \cup \tilde{M}_{\pm}) \)), real valued on the set \( \mathbb{R} \setminus \sigma_{\pm} \), and have simple poles at the points of the set \( M_{\pm} \) (resp. \( \tilde{M}_{\pm} \)). Moreover, they are continuous up to the boundary \( \sigma_{\pm}^u \cup \sigma_{\pm}^l \) except at the points from \( \tilde{M}_{\pm} \) and

\[
\psi_{\pm}(\lambda^u) = \tilde{\psi}_{\pm}(\lambda^l) = \frac{\psi_{\pm}(\lambda^l)}{\psi_{\pm}(\lambda^l)}, \quad \lambda \in \sigma_{\pm}.
\]

For \( E \in \tilde{M}_{\pm} \) the Weyl solutions satisfy

\[
\psi_{\pm}(z, x) = O\left(\frac{1}{\sqrt{|z - E|}}\right), \quad \tilde{\psi}_{\pm}(z, x) = O\left(\frac{1}{\sqrt{|z - E|}}\right), \quad \text{as } z \to E \in \tilde{M}_{\pm},
\]

where the \( O((|z - E|)^{-1/2}) \)-term is independent of \( x \).

The same applies to \( \psi'_{\pm}(z, x) \) and \( \tilde{\psi}'_{\pm}(z, x) \).

(ii) At the edges of the spectrum the Weyl solutions satisfy

\[
\lim_{z \to E} \psi_{\pm}(z, x) - \tilde{\psi}_{\pm}(z, x) = 0 \quad \text{for } E \in \partial \sigma_{\pm} \setminus \tilde{M}_{\pm},
\]

and

\[
\psi_{\pm}(z, x) + \tilde{\psi}_{\pm}(z, x) = O(1) \quad \text{for } z \text{ near } E \in \tilde{M}_{\pm},
\]

where the \( O(1) \)-term depends on \( x \).

(iii) The functions \( \psi_{\pm}(z, x) \) and \( \tilde{\psi}_{\pm}(z, x) \) form an orthonormal basis on the spectrum with respect to the weight

\[
d\rho_{\pm}(z) = \frac{1}{2\pi i} g_{\pm}(z)dz,
\]

and any \( f(x) \in L^2(\mathbb{R}) \) can be expressed through

\[
f(x) = \int_{\sigma_{\pm}} \left( \int_{\mathbb{R}} f(y)\psi_{\pm}(z, y)dy \right) \psi_{\pm}(z, x) d\rho(z).
\]

Here we use the notation

\[
\int_{\sigma_{\pm}} f(z) d\rho_{\pm}(z) := \int_{\sigma_{\pm}^u} f(z) d\rho_{\pm}(z) - \int_{\sigma_{\pm}^l} f(z) d\rho_{\pm}(z).
\]

Proof. For a proof of (i) and (iii) we refer to [12] Lemma 2.2.

(ii) We only prove the claim for the + case (the − case can be handled in the same way) and drop the + in what follows. In [12] Lemma 2.2 we showed that

\[
\lim_{z \to E} \exp \left( \int_{0}^{z} \frac{Y^{1/2}(\tau)}{G(z, \tau)} d\tau \right) = \begin{cases} 
\pm 1, & \mu_j(0) \neq E, \mu_j(x) \neq E, \\
\pm 1, & \mu_j(0) = E, \mu_j(x) = E, \\
\pm i, & \mu_j(0) = E, \mu_j(x) \neq E, \\
\pm i, & \mu_j(0) \neq E, \mu_j(x) = E,
\end{cases}
\]

for any \( E \in \partial \sigma \).

Thus assuming that \( \mu_j(0) \neq E \), we can write along the spectrum

\[
\psi(z, x) - \tilde{\psi}(z, x) = 2i \left( \frac{G(z, x)}{G(z, 0)} \right)^{1/2} \sin \left( \int_{0}^{z} \frac{Y^{1/2}(\tau)}{G(z, \tau)} d\tau \right).
\]

(i) If in addition \( \mu_j(x) \neq E \), then by (2.13) we have \( \lim_{z \to E} \sin \left( \int_{0}^{z} \frac{Y^{1/2}(\tau)}{G(z, \tau)} d\tau \right) = 0 \) and \( \lim_{z \to E} \frac{G(z, x)}{G(z, 0)} \) exists. Thus we end up with

\[
\lim_{z \to E} \left( \psi(z, x) - \tilde{\psi}(z, x) \right) = 0.
\]
(ii) If \( \mu_j(x) = E \), then by \((2.13)\) we get \( \lim_{z \to E} \sin \left( \int_0^x \frac{Y^{1/2}(z)}{G(z, \tau)} \, d\tau \right) = \pm 1 \) and \( \lim_{z \to E} \frac{G(z, x)}{G(z, 0)} = 0 \). Hence

\[
\lim_{z \to E} (\psi(z, x) - \tilde{\psi}(z, x)) = 0.
\]

To prove the second claim, assume that \( \mu_j(0) = E \) and write

\[
\psi(z, x) - \tilde{\psi}(z, x) = 2 \left( \frac{G(z, x)}{G(z, 0)} \right)^{1/2} \cos \left( \int_0^x \frac{Y^{1/2}(z)}{G(z, \tau)} \, d\tau \right).
\]

(i) If \( \mu_j(x) = E \), then by \((2.13)\) we get \( \lim_{z \to E} \cos \left( \int_0^x \frac{Y^{1/2}(z)}{G(z, \tau)} \, d\tau \right) = \pm 1 \) and \( \lim_{z \to E} \frac{G(z, x)}{G(z, 0)} \) exists. Therefore \( \lim_{z \to E}(\psi(z, x) - \tilde{\psi}(z, x)) \) exists and especially

\[
\psi(z, x) - \tilde{\psi}(z, x) = O(1) \quad \text{for} \quad z \text{ near } E = \mu_j(0).
\]

(ii) If \( \mu_j(x) \neq E \), we cannot conclude as before, because \( \lim_{z \to E} \frac{G(z, x)}{G(z, 0)} \) does not exist. Assume that \( E = E_{2j} \) (the case \( E = E_{2j} \) can be handled in a similar way). Then we can separate for fixed \( x \in \mathbb{R} \) the interval \([0, x]\) into smaller intervals \([x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{2j}, x]\) such that \( x_0 = 0 \), \( x_j \in \{E_{2j-1}, E_{2j}\} \) for \( k = 0, 1, 2, \ldots, 2l \), \( \mu_j(x) \) is monotone increasing or decreasing on every interval \([x_k, x_{k+1}]\) and \( \mu_j(x_0) = E_{2j} = \mu_j(x_{2j}) \). Following the proof of \([12\text{, Lemma } 2.2 \text{ (ii)}]\), one obtains for \( E = E_{2j} \) that

\[
\int_0^x \frac{Y^{1/2}(z)}{G(z, \tau)} \, d\tau = i\sigma_j 2(l + 1) \arctan \left( \frac{\sqrt{E_{2j} - E_{2j-1}}}{\sqrt{z} - E_{2j}} \right) + iO(\sqrt{z} - E_{2j}),
\]

where the \( O(\sqrt{z} - E_{2j}) \) term depends on \( x \). Using now that \( \arctan(x) = \frac{\pi}{2} + O \left( \frac{1}{x} \right) \) for \( x \to \infty \), we have

\[
\arctan \left( \frac{\sqrt{E_{2j} - E_{2j-1}}}{\sqrt{z} - E_{2j}} \right) = \frac{\pi}{2} + O \left( \frac{\sqrt{z} - E_{2j}}{E_{2j} - E_{2j-1}} \right),
\]

and

\[
\arctan \left( \frac{\sqrt{z} - \mu_j(x)}{\sqrt{z} - E_{2j}} \right) = \frac{\pi}{2} + O \left( \frac{\sqrt{z} - E_{2j}}{E_{2j} - \mu_j(x)} \right),
\]

which gives

\[
\cos \left( \int_0^x \frac{Y^{1/2}(z)}{G(z, \tau)} \, d\tau \right) = O(\sqrt{z} - E_{2j}),
\]

where the \( O(\sqrt{z} - E_{2j}) \) term depends on \( x \). Plugging this into \((2.14)\) yields

\[
\psi(z, x) - \tilde{\psi}(z, x) = O(1) \quad \text{for} \quad z \text{ near } E = \mu_j(0),
\]

where the \( O(1) \)-term depends on \( x \).

\[\square\]
Consider the Schrödinger equation
\[
\left(-\frac{d^2}{dx^2} + q(x)\right)y(x) = zy(x), \quad z \in \mathbb{C},
\]
with a potential \(q(x)\) satisfying the following condition
\[
\pm \int_0^{\pm \infty} (1 + x^2) |q(x) - p_\pm(x)| dx < \infty.
\]

Then there exist two solutions, the so-called Jost solutions \(\phi_\pm(z, x)\), which are asymptotically close to the background Weyl solutions \(\psi_\pm(z, x)\) of equation (2.1) as \(x \to \pm \infty\) and they can be represented as
\[
\phi_\pm(z, x) = \psi_\pm(z, x) \pm \int_x^{\pm \infty} K_\pm(x, y) \psi_\pm(z, y) dy.
\]
Here \(K_\pm(x, y)\) are real-valued functions, which are continuously differentiable with respect to both parameters and satisfy the estimate
\[
|K_\pm(x, y)| \leq C_\pm(x) Q_\pm(x + y) = \pm C_\pm(x) \int_{x+y}^{\pm \infty} |q(t) - p_\pm(t)| dt,
\]
where \(C_\pm(x)\) are continuous, positive, monotonically decreasing functions, and therefore bounded as \(x \to \pm \infty\). Furthermore,
\[
\left| \frac{dK_\pm(x, y)}{dx} \right| + \left| \frac{dK_\pm(x, y)}{dy} \right| \leq C_\pm(x) \left( \left| q_\pm \left( \frac{x + y}{2} \right) \right| + Q_\pm(x + y) \right)
\]
and
\[
\pm \int_0^{\pm \infty} (1 + x^2) \left| \frac{d}{dx} K_\pm(x, x) \right| dx < \infty, \quad \forall a \in \mathbb{R}.
\]

For more information we refer to [12].

Moreover, for \(\lambda \in \sigma^+_\pm \cup \sigma^-_\pm\), a second pair of solutions of (3.1) is given by
\[
\phi_\pm(\lambda, x) = \tilde{\psi}_\pm(\lambda, x) \pm \int_x^{\pm \infty} K_\pm(x, y) \tilde{\psi}_\pm(\lambda, y) dy, \quad \lambda \in \sigma^+_\pm \cup \sigma^-_\pm.
\]
Note \(\tilde{\psi}_\pm(\lambda, x) = \psi_\pm(\lambda, x)\) for \(\lambda \in \sigma_\pm\).

Unlike the Jost solutions \(\phi_\pm(z, x)\), these solutions only exist on the upper and lower cuts of the spectrum and cannot be continued to the whole complex plane. Combining (2.9), (3.3), (3.4), and (3.7), one obtains
\[
W(\phi_\pm(\lambda), \phi_\pm(\lambda)) = \pm g(\lambda)^{-1}.
\]

In the next lemma we want to point out, which properties of the background Weyl solutions are also inherited by the Jost solutions.

**Lemma 3.1.** The Jost solutions \(\phi_\pm(z, x)\) have the following properties:

(i) The function \(\phi_\pm(z, x)\) considered as a function of \(z\), is holomorphic in the domain \(\mathbb{C} \setminus (\sigma^+_\pm \cup M_\pm)\), and has simple poles at the points of the set \(M_\pm\). It is continuous up to the boundary \(\sigma^+_\pm \cup \sigma^-_\pm\) except at the points from \(M_\pm\). Moreover, we have
\[
\phi_\pm(z, x) \in L^2(\mathbb{R}_\pm), \quad z \in \mathbb{C} \setminus \sigma_\pm
\]
For $E \in \hat{M}_\pm$ they satisfy
\[ \phi_\pm(z, x) = O\left(\frac{1}{\sqrt{z - E}}\right), \quad \text{as} \quad z \to E \in \hat{M}_\pm, \]
where the $O((z - E)^{-1/2})$-term depends on $x$.

(ii) At the band edges of the spectrum we have the following behavior:
\[ \lim_{z \to E} \phi_\pm(z, x) - \phi_\pm(z, x) = 0 \quad \text{for} \quad E \in \partial\sigma \setminus \hat{M}_\pm, \]
and
\[ \phi_\pm(z, x) + \phi_\pm(z, x) = O(1) \quad \text{for} \quad z \text{ near } E \in \hat{M}_\pm, \]
where the $O(1)$-term depends on $x$.

Proof. Everything follows from the fact that these properties are only dependent on $z$ and therefore the transformation operator does not influence them. \qed

Now we want to characterize the spectrum of our operator $L$, which consists of an (absolutely) continuous part, $\sigma = \sigma_+ \cup \sigma_-$ and an at most countable number of discrete eigenvalues, which are situated in the gaps, $\sigma_d \subset \mathbb{R} \setminus \sigma$. In particular every gap can only contain a finite number of discrete eigenvalues (cf. [15], [16], and [24, Thm. 6.12]) and thus they cannot cluster. For our purposes it will be convenient to write
\[ \sigma = \sigma^{(1)}_+ \cup \sigma^{(1)}_- \cup \sigma^{(2)}, \]
with
\[ \sigma^{(2)} := \sigma_+ \cap \sigma_-, \quad \sigma^{(1)}_\pm = \text{clos}(\sigma_\pm \setminus \sigma^{(2)}). \]

It is well-known that a point $\lambda \in \mathbb{R} \setminus \sigma$ corresponds to the discrete spectrum if and only if the two Jost solutions are linearly dependent, which implies that we should investigate
\[ (3.9) \quad W(z) := W(\phi_-(z, .), \phi_+(z, .)), \]
the Wronskian of the Jost solutions. This is a meromorphic function in the domain $\mathbb{C} \setminus \sigma$, with possible poles at the points $M_+ \cup M_- \cup (\hat{M}_+ \cap \hat{M}_-)$ and possible square root singularities at the points $(\hat{M}_+ \cup \hat{M}_-) \setminus (\hat{M}_+ \cap \hat{M}_-)$. For investigating the function $W(z)$ in more detail, we will multiply the possible poles and square root singularities away. Thus we define locally in a small neighborhood $U_j^\pm$ of the $j$th gap $[E_{2j-1}^\pm, E_{2j}^\pm]$, where $j = 1, 2, \ldots$

(3.10) $\tilde{\phi}_{j, \pm}(z, x) = \delta_{j, \pm}(z) \phi_{\pm}(z, x),$
where

(3.11) $\delta_{j, \pm}(z) = \begin{cases} z - \mu_{j, \pm}^+, & \text{if } \mu_{j, \pm}^+ \in M_+, \\ 1, & \text{else} \end{cases}$

and

(3.12) $\hat{\phi}_{j, \pm}(z, x) = \hat{\delta}_{j, \pm}(z) \phi_{\pm}(z, x),$
where

(3.13) $\hat{\delta}_{j, \pm}(z) = \begin{cases} z - \mu_{j, \pm}^+, & \text{if } \mu_{j, \pm}^+ \in M_+, \\ \sqrt{z - \mu_{j, \pm}^+}, & \text{if } \mu_{j, \pm}^+ \in M_+, \\ 1, & \text{else}. \end{cases}$
Correspondingly, we set

\begin{equation}
\tilde{W}(z) = W(\tilde{\phi}_-(z, \cdot), \tilde{\phi}_+(z, \cdot)), \quad \bar{W}(z) = W(\bar{\phi}_-(z, \cdot), \bar{\phi}_+(z, \cdot)).
\end{equation}

Here we use the definitions

\begin{equation}
\tilde{\phi}_\pm(z, x) = \begin{cases} \phi_{j, \pm}(z, x), & \text{for } z \in U^\pm_j, j = 1, 2, \ldots, \\ \phi_\pm(z, x), & \text{else}, \end{cases}
\end{equation}

\begin{equation}
\hat{\phi}_\pm(z, x) = \begin{cases} \hat{\phi}_{j, \pm}(z, x), & \text{for } z \in U^\pm_j, j = 1, 2, \ldots, \\ \hat{\phi}_\pm(z, x), & \text{else}. \end{cases}
\end{equation}

and we will choose \( U^+_j = U^-_m \), if \([E^+_{2j-1}, E^+_{2j}] \cap [E^-_{2m-1}, E^-_{2m}] \neq \emptyset \). Analogously, one can define \( \delta_\pm(z) \) and \( \hat{\delta}_\pm(z) \).

Note that the function \( \tilde{W}(z) \) is holomorphic in the domain \( U^\pm_j \cap (\mathbb{C} \setminus \sigma) \) and continuous up to the boundary. But unlike the functions \( W(z) \) and \( \bar{W}(z) \) it may not take real values on the set \( \mathbb{R} \setminus \sigma \) and complex conjugated values on the different sides of the spectrum \( \sigma^u \cup \sigma^l \) inside the domains \( U^\pm_j \). That is why we will characterize the spectral properties of our operator \( L \) in terms of the function \( \tilde{W}(z) \) which can have poles at the band edges.

Since the discrete spectrum of our operator \( L \) is at most countable, we can write it as

\[ \sigma_d = \bigcup_{n=1}^{\infty} \sigma_{n} \subset \mathbb{R} \setminus \sigma, \]

where

\[ \sigma_{n} = \{ \lambda_{n,1}, \ldots, \lambda_{n,k(n)} \}, \quad n \in \mathbb{N} \]

and \( k(n) \) denotes the number of eigenvalues in the \( n \)’th gap of \( \sigma \).

For every eigenvalue \( \lambda_{n,m} \) we can introduce the corresponding norming constants

\begin{equation}
(\gamma_{n,m}^\pm)^{-2} = \int_{\mathbb{R}} \hat{\phi}_\pm^2(\lambda_{n,m}, x)dx.
\end{equation}

Now we begin with the study of the properties of the scattering data. Therefore we introduce the scattering relations

\begin{equation}
T_\pm(\lambda) \phi_\pm(\lambda, x) = \bar{\phi}_\pm(\lambda, x) + R_\pm(\lambda) \phi_\pm(\lambda, x), \quad \lambda \in \sigma^u, \quad \lambda \in \sigma^l,
\end{equation}

where the transmission and reflection coefficients are defined as usual,

\begin{equation}
T_\pm(\lambda) := \frac{W(\phi_\pm(\lambda), \bar{\phi}_\pm(\lambda))}{W(\phi_\pm(\lambda), \phi_\pm(\lambda))}, \quad R_\pm(\lambda) := \frac{W(\phi_\pm(\lambda), \phi_\pm(\lambda))}{W(\phi_\pm(\lambda), \bar{\phi}_\pm(\lambda))}, \quad \lambda \in \sigma^u, \quad \lambda \in \sigma^l.
\end{equation}

**Theorem 3.2.** For the scattering matrix the following properties are valid:

(i) \( T_\pm(\lambda^u) = T_\pm(\lambda^l) \) and \( R_\pm(\lambda^u) = R_\pm(\lambda^l) \) for \( \lambda \in \sigma_{\pm} \).

(ii) \( \frac{T_\pm(\lambda)}{R_\pm(\lambda)} = R_\pm(\lambda) \) for \( \lambda \in \sigma_{\pm}^{(1)} \).

(iii) \( 1 - |R_\pm(\lambda)|^2 = \frac{g_\pm(\lambda)}{g_\mp(\lambda)} |T_\pm(\lambda)|^2 \) for \( \lambda \in \sigma_{\pm}^{(2)} \).

(iv) \( R_\pm(\lambda)T_\pm(\lambda) + R_\mp(\lambda)T_\mp(\lambda) = 0 \) for \( \lambda \in \sigma_{\pm}^{(2)} \).
Proof. (i) and (iv) follow from (3.13), (3.7), (3.19), and Lemma 2.2.

For showing (ii) observe that $\tilde{\phi}_+(\lambda, x) \in \mathbb{R}$ as $\lambda \in \text{int}(\sigma_+^{(1)})$, which implies (ii).

To show (iii), assume $\lambda \in \text{int}(\sigma_+^{(2)})$, then by (3.18)

$$|T_{\pm}|^2 \phi_{\pm}(\lambda, x) = (|R_{\pm}|^2 - 1)\phi_{\pm}(\lambda, x).$$

Thus using (3.8) finishes the proof. \[ \Box \]

**Theorem 3.3.** The transmission and reflection coefficients have the following asymptotic behavior, as $\lambda \to \infty$ for $\lambda \in \sigma_+^{(2)}$ outside a small $\varepsilon$ neighborhood of the band edges of $\sigma_+^{(2)}$:

$$R_{\pm}(\lambda) = O(|\lambda|^{-1/2}),$$

$$T_{\pm}(\lambda) = 1 + O(|\lambda|^{-1/2}).$$

Proof. The asymptotics can only be valid for $\lambda \in \sigma_+^{(2)}$ outside an $\varepsilon$ neighborhood of the band edges, because the Jost solutions $\phi_{\pm}$ might have square root singularities there. At first we will investigate $W(\phi_{-}(\lambda, 0), \phi_{+}(\lambda, 0))$:

$$\phi_{-}(\lambda, 0)\phi_{+}(\lambda, 0) = \left(1 + \int_{-\infty}^{0} K_-(0, y)\psi_{-}(\lambda, y)dy\right) \times \left(m_+(\lambda) - K_+(0, 0) + \int_{0}^{\infty} K_{+, x}(0, y)\psi_{+}(\lambda, y)dy\right).$$

Using (cf. 2.20)

$$\psi_\pm(\lambda, x) = m_\pm(\lambda, x)\psi_\pm(\lambda, x),$$

we can write

$$\int_{-\infty}^{0} K_-(0, y)\psi_{-}(\lambda, y)dy = \int_{-\infty}^{0} \frac{K_-(0, y)}{m_-(\lambda, y)}\psi_-'(\lambda, y)dy.$$  

Hence

$$\int_{-\infty}^{0} K_-(0, y)\psi_{-}(\lambda, y)dy = \frac{K_-(0, 0)}{m_-(\lambda)} + I_1(\lambda),$$

$$I_1(\lambda) = - \int_{-\infty}^{0} \left(K_{-, y}(0, y)\frac{\psi_{-}(\lambda, y)}{m_-(\lambda, y)} - K_-(0, y)\psi_{-}(\lambda, y)\frac{m_+'(\lambda, y)}{m_-(\lambda, y)^2}\right)dy.$$  

Here it should be noticed that $m_{\pm}(\lambda, y)^{-1}$ has no pole, because (see e.g. [17])

$$G_{\pm}(\lambda, y)N_{\pm}(\lambda, y) + H_{\pm}(\lambda, y)^2 = Y_{\pm}(\lambda),$$

where

$$N_{\pm}(\lambda, y) = - (\lambda - \nu_0^{\pm}(y))\prod_{j=1}^{\infty} \frac{\lambda - \nu_j^{\pm}(y)}{E_{2j-1}^{\pm}},$$

with $\nu_0^{\pm}(y) \in (-\infty, E_{0}^{\pm})$ and $\nu_j^{\pm}(y) \in [E_{2j-1}^{\pm}, E_{2j}^{\pm}]$. Thus we obtain

$$m_{\pm}(\lambda, y)^{-1} = \frac{G_{\pm}(\lambda, y)}{H_{\pm}(\lambda, y) \pm Y_{\pm}(\lambda)^{1/2}} = \frac{H_{\pm}(\lambda, y) \mp \psi_{\pm}(\lambda)^{1/2}}{N_{\pm}(\lambda, y)},$$

and therefore $K_{-, (0, 0)}(\lambda, y) = O(\sqrt{\lambda})$. 
Moreover $I_1(\lambda) = O\left(\frac{1}{\sqrt{\lambda}}\right)$ as the following estimates show:

$$|I_1(\lambda)| \leq \int_{-\infty}^{0} |K_{-,\psi}(0, y)\psi_-(\lambda, y)| dy + \int_{-\infty}^{0} |K_{-}(0, y)\psi_-(\lambda, y)\frac{m'_-(\lambda, y)}{m_-(\lambda, y)^2}| dy$$

$$\leq \frac{C}{\sqrt{\lambda}} \int_{-\infty}^{0} (|q(y) - p_-(y)| + Q_-(y)) dy,$$

where we used that $|\psi_+(\lambda, y)| = |\frac{\psi_+(\lambda, y)}{|\psi_+(\lambda, y)|}| = O(1)$ and $m_{\pm}^{-1}(\lambda, y) = O\left(\frac{1}{\sqrt{\lambda}}\right)$ for all $y$ by the quasi-periodicity, together with (1.8) and (3.8) and

$$\psi_+(\lambda, x) = m_+(\lambda, x)^2 \psi_+(\lambda, x) + m'_+(\lambda, x)\psi_+(\lambda, x).$$

Making the same conclusions as before, one obtains

$$\int_{0}^{\infty} K_{+,\psi}(0, y)\psi_+(\lambda, y) dy = O(1).$$

In a similar manner one can investigate

$$\phi_-'(\lambda, 0)\phi_+'(\lambda, 0) = \left( m_-(\lambda) + K_{-}(0, 0) + \int_{-\infty}^{0} K_{-,\psi}(0, y)\psi_-(\lambda, y) dy \right)$$

$$\times \left( 1 + \int_{0}^{\infty} K_{+}(0, y)\psi_+(\lambda, y) dy \right),$$

where

$$\int_{-\infty}^{0} K_{-,\psi}(0, y)\psi_-(\lambda, y) dy = O(1),$$

$$\int_{0}^{\infty} K_{+}(0, y)\psi_+(\lambda, y) dy = -\frac{K_{+}(0, 0)}{m_{+}(\lambda)} + I_2(\lambda),$$

$$I_2(\lambda) = -\int_{0}^{\infty} \left( K_{+,\psi}(0, y)\frac{\psi_+(\lambda, y)}{m_{+}(\lambda, y)} - K_{+}(0, y)\psi_+(\lambda, y)\frac{m'_+(\lambda, y)}{m_{+}(\lambda, y)^2} \right) dy,$$

and $I_2(\lambda) = O\left(\frac{1}{\sqrt{\lambda}}\right)$. Thus combining all the informations we obtained so far yields

$$W(\phi_-(\lambda), \phi_+(\lambda)) = m_+(\lambda) - m_-(\lambda) + K_{-}(0, 0) \left( \frac{m_+(\lambda) - m_-(\lambda)}{m_-(\lambda)} \right)$$

$$+ K_{+}(0, 0) \left( \frac{m_-(\lambda) - m_+(\lambda)}{m_+(\lambda)} \right) + O(1),$$

and therefore, using (3.8),

$$T_{\pm}(\lambda) = 1 + O\left(\frac{1}{\sqrt{\lambda}}\right).$$

Analogously one can investigate the behavior of $W(\phi_{\pm}(\lambda), \phi_{\pm}(\lambda))$ to obtain $R_{\pm}(\lambda) = O\left(\frac{1}{\sqrt{\lambda}}\right)$.

**Theorem 3.4.** The functions $T_{\pm}(\lambda)$ can be extended analytically to the domain $\mathbb{C}\setminus(\sigma \cup M_{\pm} \cup \bar{M}_{\pm})$ and satisfy

$$-1 \frac{1}{T_{\pm}(z)g_{\pm}(z)} = -1 \frac{1}{T_{\pm}(z)g_{\pm}(z)} = W(z),$$

where $W(z)$ possesses the following properties:
Proof.

(i) Except for (3.23) everything follows from the corresponding properties of $\hat{\phi}_\pm (z,x)$. Therefore assume $\hat{W}(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{C} \setminus \sigma$, then

$$
\hat{\phi}_\pm (\lambda_0, x) = c_\pm \hat{\phi}_\mp (\lambda_0, x),
$$

for some constants $c_\pm$, which satisfy $c_- c_+ = 1$. Moreover, every zero of $\hat{W}$ (or $\hat{W}$) outside the continuous spectrum, is a point of the discrete spectrum of $L$ and vice versa.

Denote by $\gamma_\pm$ the corresponding norming constants defined in (3.17) for some fixed point $\lambda_0$ of the discrete spectrum. Proceeding as in [21] one obtains

$$
W (\hat{\phi}_\pm (\lambda_0,0), \frac{d}{d\lambda} \hat{\phi}_\mp (\lambda_0,0)) = \int_0^{\pm \infty} \hat{\phi}_\mp^2 (\lambda_0,x) dx.
$$

Thus using (3.25) and (3.26) yields

$$
\gamma_\pm^{-2} = \mp c_\pm^2 \int_0^{\mp \infty} \hat{\phi}_\pm^2 (\lambda_0,x) dx \pm \int_0^{\pm \infty} \hat{\phi}_\mp^2 (\lambda_0,x) dx
$$

$$
= \mp c_\pm^2 W (\hat{\phi}_\mp (\lambda_0,0), \frac{d}{d\lambda} \hat{\phi}_\mp (\lambda_0,0)) \pm W (\hat{\phi}_\pm (\lambda_0,0), \frac{d}{d\lambda} \hat{\phi}_\pm (\lambda_0,0))
$$

$$
= c_\pm \frac{d}{d\lambda} W (\hat{\phi}_\mp (\lambda_0,0), \hat{\phi}_\pm (\lambda_0)).
$$

Applying now $c_- c_+ = 1$, we obtain (3.24).

(ii) The continuity of $\hat{W}(z)$ up to the boundary follows immediately from the corresponding properties of $\hat{\phi}_\pm (z,x)$. Now we will investigate the possible zeros of $\hat{W}(\lambda)$ for $\lambda \in \sigma$.

Assume $\hat{W}(\lambda_0) = 0$ for some $\lambda_0 \in \text{int}(\sigma^{(2)})$. Then $\hat{\phi}_+(\lambda_0, x) = c_+ \hat{\phi}_-(\lambda_0, x)$ and $\hat{\phi}_-(\lambda_0, x) = \overline{c} \hat{\phi}_+(\lambda_0, x)$. Thus $W(\phi_+, \phi_-) = c |c|^2 W(\phi_-, \phi_-)$ and therefore $\text{sign} g_+(\lambda_0) = - \text{sign} g_-(\lambda_0)$ by (3.8), contradicting (2.8).

Next let $\lambda_0 \in \text{int}(\sigma^{(1)}_\pm)$ and $\hat{W}(\lambda_0) = 0$, then $\hat{\phi}_\pm (\lambda_0, x)$ and $\overline{\phi}_\pm (\lambda_0, x)$ are linearly independent and bounded, moreover $\hat{\phi}^*_\mp (\lambda_0, x) \in \mathbb{R}$. Therefore $\hat{W}(\lambda_0) = 0$ implies that $\phi^*_\mp = c^*_\pm \phi^*_\mp = c^*_\pm \phi^*_\mp$ and thus $W(\phi^*_\pm, \phi^*_\pm) = 0$, which is impossible by (3.8). Note that in this case $\lambda_0$ can coincide with a pole $\mu \in M_\sigma$.

Since $\hat{W}(\lambda) \neq 0$ for $\lambda \in \text{int}(\sigma^{(2)}) \cup \text{int}(\sigma^{(1)}) \cup \text{int}(\sigma^{(1)})$, it is left to investigate the behavior at the band edges of $\sigma_+$ and $\sigma_-$. Therefore introduce
the local parameter $\tau = \sqrt{z - E}$ in a small neighborhood of each point $E \in \partial \sigma_{\pm}$ and define $\hat{y}(z, x) = \frac{d}{d\tau}y(z, x)$. A simple calculation shows that $\frac{d\hat{y}}{d\tau}(E) = 0$, hence for every solution $y(z, x)$ of (3.1) its derivative $\hat{y}(E, x)$ is again a solution of (3.1). Therefore, the Wronskian $W(y(E), \hat{y}(E))$ is independent of $x$.

For each $x \in \mathbb{R}$ in a small neighborhood of a fixed point $E \in \partial \sigma_{\pm}$ we introduce the function

$$
\hat{\psi}_{\pm, E}(z, x) = \begin{cases} 
\psi_{\pm}(z, x), & E \in \partial \sigma_{\pm} \setminus \tilde{M}_{\pm}, \\
\tau \psi_{\pm}(z, x), & E \in \tilde{M}_{\pm}.
\end{cases}
$$

Proceeding as in [1, Lem. B.1] one obtains

$$(3.28) \quad W\left(\hat{\psi}_{\pm, E}(E), \frac{d}{d\tau}\hat{\psi}_{\pm, E}(E)\right) = \pm \lim_{z \to E} \frac{\alpha \tau^{\alpha}}{2g_{\pm}(z)},$$

where $\alpha = -1$ if $E \in \partial \sigma_{\pm} \setminus \tilde{M}_{\pm}$ and $\alpha = 1$ if $E \in \tilde{M}_{\pm}$.

Using representation (2.6) for $\hat{\psi}_{\pm}(z, x)$ one can show (cf [12]),

$$
\psi_{\pm}(E, x) = \left(\frac{G_{\pm}(E, x)}{G_{\pm}(E, 0)}\right)^{1/2} \exp\left(\pm \lim_{z \to E} \int_{0}^{x} \frac{Y_{\pm}(z)\tau^{1/2}}{G_{\pm}(z, \tau)} d\tau\right), \quad E \in \partial \sigma
$$

where

$$
\exp\left(\pm \lim_{z \to E} \int_{0}^{x} \frac{Y_{\pm}(z)\tau^{1/2}}{G_{\pm}(z, \tau)} d\tau\right) = \begin{cases} 
i^{2s+1}, & \mu_j \neq E, \mu_j(x) = E, \\
\mu_j = E, \mu_j(x) \neq E, \\
i^{2s}, & \mu_j = E, \mu_j(x) = E, \\
i^{2s}, & \mu_j \neq E, \mu_j(x) \neq E,
\end{cases}
$$

for $s \in \{0, 1\}$. Defining

$$
\hat{\phi}_{\pm, E}(\lambda, x) = \begin{cases} 
\phi_{\pm}(\lambda, x), & E \in \partial \sigma_{\pm} \setminus \tilde{M}_{\pm}, \\
\tau \phi_{\pm}(\lambda, x), & E \in \tilde{M}_{\pm},
\end{cases}
$$

we can conclude using (3.3) that

$$(3.29) \quad \hat{\phi}_{\pm}(E, x) = \phi_{\pm}(E, x), \quad \text{for } E \in \partial \sigma_{\pm} \setminus \tilde{M}_{\pm}. $$

Moreover, for $E \in \tilde{M}_{\pm}$,

$$(3.30) \quad \begin{cases} 
\hat{\phi}_{\pm, E}(E, x) = -\hat{\phi}_{\pm, E}(E, x), & \text{a left band edge from } \sigma_{\pm}, \\
\hat{\phi}_{\pm, E}(E, x) = \hat{\phi}_{\pm, E}(E, x), & \text{a right band edge from } \sigma_{\pm},
\end{cases}$$

If $\lambda_0 = E \in \partial \sigma^{(2)} \cap \text{int}(\sigma_{\pm}) \subset \text{int}(\sigma_{\pm})$, then $\hat{W}(E) = 0$ if and only if $W(\psi_{\pm}, \hat{\psi}_{\pm}(E, E)) = 0$. Therefore, as $\hat{\phi}_{\pm, E}(E, x)$ are either pure real or pure imaginary, $W(\phi_{\pm}, \hat{\phi}_{\pm, E}(E, E)) = 0$, which implies that $\phi_{\pm}(E, x)$ and $\hat{\phi}_{\pm}(E, x)$ are linearly dependent, a contradiction.

Thus the function $\hat{W}(z)$ can only be zero at points $E$ of the set $\partial \sigma \cup (\partial \sigma_{\pm}^{(1)} \cap \partial \sigma_{\pm}^{(1)})$. We will now compute the order of the zero. First of all note that the function $\hat{W}(\lambda)$ is continuously differentiable with respect to the local parameter $\tau$. Since $\frac{d}{d\tau}(\delta_{\pm} \delta_{-})(E) = 0$, the function $W(\hat{\phi}_{\pm}, E, \hat{\phi}_{\pm}(E, E))$ has the same order of zero at $E$ as $\hat{W}(\lambda)$. Moreover, if $\hat{\delta}_{\pm}(E) \neq 0$, then $\frac{d}{d\tau} \hat{\delta}_{\pm}(E) = 0$ and if $\delta_{\pm}(E) = \hat{\delta}_{\pm}(E) = 0$, then $\frac{d}{d\tau}(x^{-2} \delta_{\pm} \delta_{-})(E) = 0$. Hence $\frac{d}{d\tau} \hat{W}(E) = 0$ if and only if $\frac{d}{d\tau}W(\hat{\phi}_{\pm}, E, \hat{\phi}_{\pm, E}) = 0$. 

Combining now all the informations we obtained so far, we can conclude
as follows: if $W(E) = 0$, then $\tilde{\phi}_{\pm,E}(E, \cdot) = c_\pm \phi_{\mp,E}(E, \cdot)$, with $c_- c_+ = 1$. Moreover we can write

$$W(\tilde{\phi}_{+,E}, \tilde{\phi}_{-,E})(E) = W(\frac{d}{dt} \tilde{\phi}_{+,E}, \tilde{\phi}_{-,E})(E) - W(\frac{d}{dt} \tilde{\phi}_{-,E}, \tilde{\phi}_{+,E})(E)$$

$$= c_- W(\frac{d}{dt} \tilde{\phi}_{+,E}, \tilde{\phi}_{+,E})(E) - c_+ W(\frac{d}{dt} \tilde{\phi}_{-,E}, \tilde{\phi}_{-,E})(E)$$

Using (3.28), (3.29), (3.30), and distinguishing several cases as in [1, Lem. B.1] finishes the proof.

\[\square\]

**Theorem 3.5.** The reflection coefficient $R_\pm(\lambda)$ satisfies:

(i) The reflection coefficient $R_\pm(\lambda)$ is a continuous function on the set $\text{int}(\sigma^{u,l}_\pm)$.

(ii) If $E \in \sigma_+ \cap \partial \sigma_-$ and $W(E) \neq 0$, then the function $R_\pm(\lambda)$ is also continuous at $E$. Moreover,

$$R_\pm(E) = \begin{cases} -1 & \text{for } E \notin \hat{M}_\pm, \\ 1 & \text{for } E \in \hat{M}_\pm. \end{cases}$$

**Proof.** (i) At first it should be noted that by Lemma 3.2 the reflection coefficient is bounded, as $\frac{d}{dt} \tilde{\phi}_{\pm}(\lambda) > 0$ for $\lambda \in \text{int}(\sigma^{(2)})$. Thus, using the corresponding properties of $\phi_{\pm}(z, x)$, finishes the first part.

(ii) We proceed as in the proof of [1, Lemma 3.3 III.(b)]. By (3.19) the reflection coefficient can be represented in the following form:

$$R_\pm(\lambda) = -\frac{W(\phi_\pm(\lambda), \phi_{\mp}(\lambda))}{W(\phi_{\mp}(\lambda), \phi_\pm(\lambda))} = \pm \frac{W(\phi_\pm(\lambda), \phi_{\mp}(\lambda))}{W(\lambda)}, \tag{3.31}$$

and is therefore continuous on both sides of the set $\text{int}(\sigma_\pm) \setminus (\hat{M}_\mp \cup \hat{M}_\mp)$. Moreover,

$$|R_\pm(\lambda)| = \left| \frac{W(\phi_{\pm}(\lambda), \phi_{\mp}(\lambda))}{W(\lambda)} \right|,$$

where the denominator does not vanish, by assumption and hence $R_\pm(\lambda)$ is continuous on both sides of the spectrum in a small neighborhood of the band edges under consideration.

Next, let $E \in \{E_{2j-1}^\pm, E_{2j}^\pm\}$ with $W(E) \neq 0$. Then, if $E \notin \hat{M}_\pm$, we can write

$$R_\pm(\lambda) = -1 \mp \frac{\delta_{j_\pm}(\lambda)W(\phi_\pm(\lambda) - \phi_{\mp}(\lambda), \phi_{\mp}(\lambda))}{W(\lambda)},$$

which implies $R_\pm(\lambda) \to -1$, since $\phi_{\pm}(\lambda) - \phi_{\mp}(\lambda) \to 0$ by Lemma 3.1 as $\lambda \to E$. Thus we proved the first case.

If $E \in \hat{M}_\pm$ with $W(E) \neq 0$, we use (3.31) in the form

$$R_\pm(\lambda) = 1 \pm \frac{\delta_{j_\pm}(\lambda)W(\phi_{\mp}(\lambda) + \phi_{\pm}(\lambda), \phi_{\pm}(\lambda))}{W(\lambda)},$$
which yields $R_\pm(\lambda) \to 1$, since $\hat{\delta}_{\pm}(\lambda) \to 0$ and $\phi_\pm(\lambda) + \bar{\phi}_\pm(\lambda) = O(1)$ by Lemma 3.1 as $\lambda \to E$. This settles the second case.

\begin{proof}
\end{proof}

4. The Gel’fand-Levitan-Marchenko Equation

The aim of this section is to derive the Gel’fand-Levitan-Marchenko (GLM) equation, which is also called the inverse scattering problem equation and to obtain some additional properties of the scattering data, as a consequence of the GLM equation.

Therefore consider the function

\[ G_\pm(z, x, y) = T_\pm(z)\phi_\pm(z, x)v_\pm(z, y)g_\pm(z) - \psi_\pm(z, x)v_\pm(z, y)g_\pm(z) =: G'_\pm(z, x, y) + G''_\pm(z, x, y), \quad \pm y > \pm x, \]

where $x$ and $y$ are considered as fixed parameters. As a function of $z$ it is meromorphic in the domain $\mathbb{C}\setminus\sigma$ with simple poles at the points $\lambda_k$ of the discrete spectrum. It is continuous up to the boundary $\sigma^+ \cup \sigma^-$, except for the points of the set, which consists of the band edges of the background spectra $\partial\sigma_+$ and $\partial\sigma_-$, where

\begin{equation}
G_\pm(z, x, y) = O((z - E)^{-1/2}) \quad \text{as} \quad E \in \partial\sigma_+ \cup \partial\sigma_-.
\end{equation}

Outside a small neighborhood of the gaps of $\sigma_+$ and $\sigma_-$, the following asymptotics as $z \to \infty$ are valid:

\begin{align*}
\phi_\pm(z, x) &= e^{\mp i\sqrt{z}x + O(1/2)} \left(1 + O(z^{-1/2})\right), \quad g_\pm(z) = \frac{-1}{2i\sqrt{z}} + O(z^{-1}), \\
\psi_\pm(z, x) &= e^{\mp i\sqrt{z}x + O(1/2)} \left(1 + O(z^{-1})\right), \quad T_\pm(z) = 1 + O(z^{-1/2}), \\
\psi_\pm(z, y) &= e^{\pm i\sqrt{z}y + O(1/2)} \left(1 + O(z^{-1})\right),
\end{align*}

and the leading term of $\phi_\pm(z, x)$ and $\psi_\pm(z, x)$ are equal, thus

\begin{equation}
G_\pm(z, x, y) = e^{\mp i\sqrt{z}(y-x)(1+O(1/2))}O(z^{-1}), \quad \pm y > \pm x.
\end{equation}

Consider the following sequence of contours $\Gamma_{\epsilon, n, \pm}$, where $\Gamma_{\epsilon, n, \pm}$ consists of two parts for every $n \in \mathbb{N}$ and $\epsilon \geq 0$:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{contours.png}
\caption{Contours $\Gamma_{\epsilon, n}$}
\end{figure}
(i) $C_{\varepsilon,n,\pm}$ consists of a part of a circle which is centered at the origin and has as radii the distance from the origin to the midpoint of the largest band of $[E_{2n\varepsilon}, E_{2n+1}]$, which lies inside $\sigma^{(2)}$, together with a part wrapping around the corresponding band of $\sigma$ at a small distance, which is at most $\varepsilon$, as indicated by figure 1.

(ii) Each band of the spectrum $\sigma$, which is fully contained in $C_{\varepsilon,n,\pm}$, is surrounded by a small loop at a small distance from $\sigma$ not bigger than $\varepsilon$.

W.l.o.g. we can assume that all the contours are non-intersecting.

Using the Cauchy theorem, we obtain

$$\frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n,\pm}} G_{\pm}(z, x, y) dz = \sum_{\lambda_k \in \text{int}(\Gamma_{\varepsilon,n,\pm})} \text{Res} G_{\pm}(z, x, y), \quad \varepsilon > 0.$$  

By (4.1) the limit value of $G_{\pm}(z, x, y)$ as $\varepsilon \to 0$ is integrable on $\sigma$, and the function $G_{\pm}''(z, x, y)$ has no poles at the points of the discrete spectrum, thus we arrive at

$$\frac{1}{2\pi i} \oint_{\Gamma_{0,n,\pm}} G_{\pm}(z, x, y) dz = \sum_{\lambda_k \in \text{int}(\Gamma_{0,n,\pm})} \text{Res} G_{\pm}''(z, x, y), \quad \pm y > \pm x.$$  

Estimate (4.2) allows us now to apply Jordan’s lemma, when letting $n \to \infty$, and we therefore arrive, up to that point only formally, at

$$\frac{1}{2\pi i} \int_{\sigma} G_{\pm}(\lambda, x, y) d\lambda = \sum_{\lambda_k \in \sigma_d} \text{Res} G_{\pm}''(\lambda, x, y), \quad \pm y > \pm x.$$  

Next, note that the function $G_{\pm}''(\lambda, x, y)$ does not contribute to the left part of (4.4), since $G_{\pm}''(\lambda^+, x, y) = G_{\pm}''(\lambda^-, x, y)$ for $\lambda \in \sigma^{(1)}_\pm$ and, hence $\int_{\sigma^{(1)}_\pm} G_{\pm}''(\lambda, x, y) d\lambda = 0$. In addition, $\int_{\sigma^{(2)}_\pm} G_{\pm}''(\lambda, x, y) d\lambda = 0$ for $x \neq y$ by Lemma 2.2 (iv).

Therefore we arrive at the following equation,

$$\frac{1}{2\pi i} \int_{\sigma} G_{\pm}'(\lambda, x, y) d\lambda = \sum_{\lambda_k \in \sigma_d} \text{Res} G_{\pm}''(\lambda, x, y), \quad \pm y > \pm x.$$  

To make our argument rigorous we have to show that the series of contour integrals along the parts of the spectrum contained in $C_{0,n,\pm}$ on the left hand side of (4.3) converges as $n \to \infty$ and that the contribution of the integrals along the circles $C_{0,n,\pm}$ converges against zero as $n \to \infty$, by applying Jordan’s lemma. This will be done next.

Using (2.12), (3.3), (3.7), and (3.18), we obtain

$$\frac{1}{2\pi i} \int_{\sigma_\pm} G_{\pm}'(\lambda, x, y) d\lambda = \int_{\sigma_\pm} T_{\pm}(\lambda) \phi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda)$$

$$= \int_{\sigma_\pm} \left( R_{\pm}(\lambda) \phi_{\pm}(\lambda, x) + \frac{\phi_{\pm}(\lambda, x)}{2} \right) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda)$$

$$= \int_{\sigma_\pm} R_{\pm}(\lambda) \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda) + \int_{\sigma_\pm} \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda)$$

$$\pm \int_{x}^{\pm \infty} dt K_{\pm}(x, t) \left( \int_{\sigma_\pm} R_{\pm}(\lambda) \psi_{\pm}(\lambda, t) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda) + \delta(t - y) \right)$$

$$= F_{\varepsilon,\pm}(x, y) \pm \int_{x}^{\pm \infty} K_{\pm}(x, t) F_{\varepsilon,\pm}(t, y) dt + K_{\pm}(x, y),$$
where
\[ F_{r,\pm}(x, y) = \int_{\sigma_{\pm}} R_{\pm}(\lambda) \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda). \]

Now properties (ii) and (iii) from Lemma 3.2 imply that
\[ |R_{\pm}(\lambda)| < 1 \quad \text{for} \quad \lambda \in \text{int}(\sigma^{(2)}), \quad |R_{\pm}(\lambda)| = 1 \quad \text{for} \quad \lambda \in \sigma^{(1)}. \]
and by (2.6) we can write
\[ F_{r,\pm}(x, y) = \int_{\sigma_{\pm}} R_{\pm}(\lambda) \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda) \]
\[ = - \int_{\sigma_{\pm}} R_{\pm}(\lambda) \frac{G_{\pm}(\lambda, x)G_{\pm}(\lambda, y)}{2Y_{\pm}(\lambda)^{1/2}} \exp(\eta_{\pm}(\lambda, x) + \eta_{\pm}(\lambda, y)) d\lambda, \]
with
\[ \eta_{\pm}(\lambda, x) := \pm \int_{0}^{x} Y_{\pm}(\lambda, \tau) d\tau \in i\mathbb{R}. \]

We are now ready to prove the following lemma.

**Lemma 4.1.** The sequence of functions
\[ F_{r,n,\pm}(x, y) = \int_{\sigma_{\pm} \cap V_{n,n,\pm}} R_{\pm}(\lambda) \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda), \]
is uniformly bounded with respect to \( x \) and \( y \), that means for all \( n \in \mathbb{N} \), \( |F_{r,n,\pm}(x, y)| \leq C \). Moreover, \( F_{r,n,\pm}(x, y) \) converges uniformly as \( n \to \infty \) to the function
\[ F_{r,\pm}(x, y) = \int_{\sigma_{\pm}} R_{\pm}(\lambda) \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda), \]
which is again uniformly bounded with respect to \( x \) and \( y \). In particular, \( F_{r,\pm}(x, y) \) is continuous with respect to \( x \) and \( y \).

**Proof.** For \( \lambda \in \sigma_{\pm} \) as \( \lambda \to \infty \) we have the following asymptotic behavior

(i) in a small neighborhood \( V_{n,\pm}^{\pm} \) of \( E = E_{n}^{\pm} \)
\[ |R_{\pm}(\lambda) \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) g_{\pm}(\lambda)| = O\left(\frac{\sqrt{E_{n}^{\pm} - E_{n+1}^{\pm}}}{\sqrt{\lambda(\lambda - E)}}\right), \]

(ii) in a small neighborhood \( W_{n}^{\pm} \) of \( E = E_{n}^{\pm} \), if \( E \in \sigma_{\pm} \)
\[ R_{\pm}(\lambda) \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) g_{\pm}(\lambda) = O\left(\frac{1}{\lambda^{2}}\right), \]

(iii) and for \( \lambda \in \sigma_{\pm} \setminus \bigcup_{n \in \mathbb{N}} (V_{n,\pm}^{\pm} \cup W_{n}^{\pm}) \)
\[ (4.6) \quad R_{\pm}(\lambda) \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) g_{\pm}(\lambda) = \exp(\pm i \sqrt{\lambda(|x| + |y|)}(1 + O\left(\frac{1}{\lambda}\right)))\left(C_{\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right)\right). \]

These estimates are good enough to show that \( F_{r,\pm}(x, y) \) exists, if we choose \( V_{n,\pm}^{\pm} \) and \( W_{n}^{\pm} \) in the following way: We choose \( V_{n,\pm}^{\pm} \subset \sigma_{\pm}^{(1)} \cup \sigma_{\pm}^{(2)} \), if \( E_{n}^{\pm} \) is a band edge of \( \sigma_{\pm}^{(1)} \), such that \( V_{n}^{\pm} \) consists of the corresponding band of \( \sigma_{\pm}^{(1)} \) together with the following part of \( \sigma_{\pm}^{(2)} \) with length \( E_{n+1}^{\pm} - E_{n-1}^{\pm} \), if \( n \) is even and \( E_{n+1}^{\pm} - E_{n}^{\pm} \), if \( n \) is odd. If \( E_{n}^{\pm} \) is a band edge of \( \sigma_{\pm}^{(2)} \), we choose \( V_{n,\pm}^{\pm} \subset \sigma_{\pm}^{(2)} \), where the length of \( V_{n}^{\pm} \) is equal to the length of the gap \( \rho_{\pm} \) next to it. We set \( W_{n,\pm}^{\pm} \subset \sigma_{\pm}^{(2)} \) with length \( 3(E_{n+1}^{\pm} - E_{n-1}^{\pm}) \), if \( n \) is even and \( 3(E_{n+1}^{\pm} - E_{n}^{\pm}) \), if \( n \) is odd, centered at the
midpoint of the corresponding gap in $\sigma_\pm$. As we are working in the Levitan class and we therefore know that $\sum_{n=1}^{\infty} (E_{2n}^\pm - E_{2n-1}^\pm) < \infty$ for some $l^\pm > 1$, we obtain that the sequences belonging to $V_n^\pm$ and $W_n^\pm$ converge.

As far as the behavior along the spectrum away from the band edges of $\sigma_+$ and $\sigma_-$ is concerned observe that

$$|\exp(\pm i\sqrt{\lambda}(x+y)O(\frac{1}{\lambda}))| \leq 1 + (x+y)O(\frac{1}{\sqrt{\lambda}}), \quad \lambda \to \infty.$$ and therefore

$$R_\pm(\lambda) \psi_\pm(\lambda, x) \psi_\pm(\lambda, y) g_\pm(\lambda) = \exp(\pm i\sqrt{\lambda}(x+y)) \left(\frac{C}{\lambda} + (1+x+y)O\left(\frac{1}{\lambda^{3/2}}\right)\right).$$

To show the convergence of the series $F_{r,\pm}(x, y)$ for fixed $x$ and $y$, we split the integral along the spectrum $\sigma$ up into three integrals along $\bigcup_{n\in\mathbb{N}} V_n^\pm$, $\bigcup_{n\in\mathbb{N}} W_n^\pm$, and $\sigma \setminus \bigcup_{n\in\mathbb{N}} (V_n^\pm \cup W_n^\pm)$ respectively.

As far as the integral along $\bigcup_{n\in\mathbb{N}} V_n^\pm m$ is concerned observe that the integrand has a square root singularity at the boundary and is therefore integrable along $V_n^\pm$ for all $n \in \mathbb{N}$. Since we are working within the Levitan class the sum over all $n \in \mathbb{N}$ converges.

The integrand can be uniformly bounded for all $\lambda \in W_n^\pm$ such that $\lambda \geq 1$. Since there are only finitely many $n \in \mathbb{N}$ such that $W_n^\pm \subset [0, 1]$, the corresponding series converges by the definition of the Levitan class.

Thus it is left to consider the integral along $\sigma \setminus \bigcup_{n\in\mathbb{N}} (V_n^\pm \cup W_n^\pm)$: $\int_a^b$. Direct computation yields

$$\int_a^b \exp(\pm i\sqrt{\lambda}(x+y)) \frac{C}{\lambda} d\lambda = \pm \exp(\pm i\sqrt{\lambda}(x+y)) \frac{2C}{\lambda^{1/2}(x+y)} \bigg|_a^b \pm \int_a^b \exp(\pm i\sqrt{\lambda}(x+y)) \frac{C}{\lambda^{3/2}(x+y)} d\lambda,$$

which is finite since by assumption $\pm x \leq \pm y$. Hence one possibility to see that the corresponding series of integrals converges is to integrate first the function describing the asymptotic behavior along $[E_0^\pm, \infty]$ and subtract from it the series of integrals corresponding to the $[E_0^\pm, \infty] \cap I^c$. Since every interval belonging to the complement belongs to a neighborhood of the gaps of $\sigma^{(2)}$ and the integrand can be uniformly bounded, the definition of the Levitan class implies that this series converges.

Similarly we conclude

$$\int_a^b \exp(\pm i\sqrt{\lambda}(x+y)) (x+y)O(\frac{1}{\lambda^{3/2}}) d\lambda = \exp(\pm i\sqrt{\lambda}(x+y)) O(\frac{1}{\lambda}) \bigg|_a^b$$

$$+ \int_a^b \exp(\pm i\sqrt{\lambda}(x+y)) O(\frac{1}{\lambda^2}) d\lambda$$

Note that since we are working within the Levitan class all estimates are independent of $x$ and $y$.

For investigating the other terms, we will need the following lemma, which is taken from [9]:

$$\text{SCATTERING THEORY FOR SCHRÖDINGER OPERATORS 19}$$
Lemma 4.2. Suppose in an integral equation of the form
\[
(4.8) \quad f_\pm(x, y) \pm \int_x^{\pm \infty} K_\pm(x, t)f_\pm(t, y)dt = g_\pm(x, y), \quad \pm y > \pm x,
\]
the kernel \( K_\pm(x, y) \) and the function \( g_\pm(x, y) \) are continuous for \( \pm y > \pm x \),
\[
|K_\pm(x, y)| \leq C_\pm(x)Q_\pm(x + y),
\]
and for \( g_\pm(x, y) \) one of the following estimates hold
\[
(4.9) \quad |g_\pm(x, y)| \leq C_\pm(x)Q_\pm(x + y), \quad \text{or}
\]
\[
(4.10) \quad |g_\pm(x, y)| \leq C_\pm(x)(1 + \max(0, \pm x)).
\]
Furthermore assume that
\[
\pm \int_0^{\pm \infty} (1 + |x|^2)|q(x) - p_\pm(x)|dx < \infty.
\]
Then (4.8) is uniquely solvable for \( f_\pm(x, y) \). The solution \( f_\pm(x, y) \) is also continuous in the half-plane \( \pm y > \pm x \), and for it the estimate (4.9) respectively (4.10) is reproduced.

Moreover, if a sequence \( g_{n, \pm}(x, y) \) satisfies (4.9) or (4.10) uniformly with respect to \( n \) and pointwise \( g_{n, \pm}(x, y) \to 0 \), for \( \pm y > \pm x \), then the same is true for the corresponding sequence of solutions \( f_{n, \pm}(x, y) \) of (4.8).

Proof. For a proof we refer to [9, Lemma 6.3]. \( \square \)

Remark 4.3. An immediate consequence of this lemma is the following. If \( |g_\pm(x, y)| \leq C_\pm(x) \), where \( C_\pm(x) \) denotes a bounded function, then \( |g_\pm(x, y)| \leq C_\pm(x)(1 + \max(0, \pm x)) \) and therefore \( |f_\pm(x, y)| \leq C_\pm(x)(1 + \max(0, \pm x)) \). Rewriting this integral equation as follows
\[
f_\pm(x, y) = g_\pm(x, y) \mp \int_x^{\pm \infty} K_\pm(x, t)f_\pm(t, y)dy,
\]
we obtain that the absolute value of the right hand side is smaller than a bounded function \( C_\pm(x) \) by using (4.10) and (3.4), and hence the same is true for the left hand side. In particular if \( C_\pm(x) \) is a decreasing function the same will be true for \( C_\pm(x) \).

We will now continue the investigation of our integral equation.

Lemma 4.4. The sequence of functions
\[
F_{h,n, \pm}(x, y) = \int_{\sigma_{\pm}^{(1)}} |T_\pm(\lambda)|^2 \psi_\pm(\lambda, x)\psi_\pm(\lambda, y)d\rho_x(\lambda)
\]
is uniformly bounded, that means for all \( n \in \mathbb{N} \), \( |F_{h,n, \pm}(x, y)| \leq C_\pm(x) \), where \( C_\pm(x) \) are monotonically decreasing functions as \( x \to \pm \infty \). Moreover, \( F_{h,n, \pm}(x, y) \) converges uniformly as \( n \to \infty \) to the function
\[
F_{h, \pm}(x, y) = \int_{\sigma_{\pm}^{(1)}} |T_\pm(\lambda)|^2 \psi_\pm(\lambda, x)\psi_\pm(\lambda, y)d\rho_x(\lambda),
\]
which is again bounded by some monotonically increasing function. In particular, \( F_{h, \pm}(x, y) \) is continuous with respect to \( x \) and \( y \).
Proof. On the set $\sigma^{(1)}_\pm$ both the numerator and the denominator of the function $G^\prime_\pm(\lambda, x, y)$ have poles (resp. square root singularities) at the points of the set $\sigma^{(1)}_\pm \cap (M_\pm \cup \partial \sigma^{(1)}_\pm \cap \partial \sigma^{(1)})$ (resp. $\sigma^{(1)}_\pm \cap (M_\pm \setminus (M_\pm \cap M_\pm))$, but multiplying them, if necessary away, we can avoid singularities. Hence, w.l.o.g., we can suppose $\sigma^{(1)}_\pm \cap (M_{\pm, +} \cup M_{\pm, -}) = \emptyset$. Thus we can write

$$
\frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} G^\prime_\pm(\lambda, x, y) d\lambda = \frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} T_\pm(\lambda) \phi_\pm(\lambda, x) \psi_\pm(\lambda, y) g_\pm(\lambda) d\lambda.
$$

For investigating this integral we will consider, using (3.18),

$$
\frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} T_\pm(\lambda) \phi_\pm(\lambda, x) \phi_\pm(\lambda, y) g_\pm(\lambda) d\lambda = \frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} \phi_\pm(\lambda, x) \left( \frac{\phi_\pm(\lambda, y) + R_\pm(\lambda) \phi_\pm(\lambda, y)}{g_\pm(\lambda)} \right) g_\pm(\lambda) d\lambda.
$$

First of all note that the integrand, because of the representation on the right hand side, can only have square root singularities at the boundary $\partial \sigma^{(1)}_\pm$ and we therefore have

$$
\int_{\sigma^{(1)}_\pm \cap [E^\pm_{2n-1}, E^\pm_{2n}]} |\phi_\pm(\lambda, x) \left( \frac{\phi_\pm(\lambda, y) + R_\pm(\lambda) \phi_\pm(\lambda, y)}{g_\pm(\lambda)} \right) g_\pm(\lambda)| d\lambda \\
\leq 2 \int_{\sigma^{(1)}_\pm \cap [E^\pm_{2n-1}, E^\pm_{2n}]} |\phi_\pm(\lambda, x) \phi_\pm(\lambda, y) g_\pm(\lambda)| d\lambda \\
\leq C_\pm(y) C_\pm(x) \left( \frac{E^\pm_{2n} - E^\pm_{2n-1}}{\sqrt{\lambda - E^\pm_0}} + \frac{E^\pm_{2n} - E^\pm_{2n-1}}{\sqrt{\lambda - E^\pm_0}} \right),
$$

where $E^\pm_{2n-1}$ and $E^\pm_{2n}$ denote the edges of the gap of $\sigma_\pm$ in which the corresponding part of $\sigma^{(1)}_\pm$ lies and $C_\pm(x)$ denote monotonically decreasing functions from now on. Therefore as we are working in the Levitan class and by separating $\sigma^{(1)}_\pm$ into the different parts, one obtains that

$$
| \frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} T_\pm(\lambda) \phi_\pm(\lambda, x) \phi_\pm(\lambda, y) g_\pm(\lambda) d\lambda | \leq C_\pm(y) C_\pm(x).
$$

Thus we can now apply Lemma 4.2 and hence

$$
| \frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} T_\pm(\lambda) \phi_\pm(\lambda, x) \phi_\pm(\lambda, y) g_\pm(\lambda) d\lambda | \leq C_\pm(y) C_\pm(x)(1 + \max(0, \pm y)).
$$

Note that we especially have, because of (3.4),

$$
| \frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} T_\pm(\lambda) \phi_\pm(\lambda, x) \phi_\pm(\lambda, y) g_\pm(\lambda) d\lambda | \leq C_\pm(x)
$$

Therefore we can conclude that for fixed $x$ and $y$ the left hands side of (4.5) exists and satisfies

$$
| \frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} T_\pm(\lambda) \phi_\pm(\lambda, x) \psi_\pm(\lambda, y) g_\pm(\lambda) d\lambda | \leq C_\pm(x),
$$

Note that we especially have, because of (3.4),

$$
| \frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} T_\pm(\lambda) \phi_\pm(\lambda, x) \phi_\pm(\lambda, y) g_\pm(\lambda) d\lambda | \leq C_\pm(x)
$$

Therefore we can conclude that for fixed $x$ and $y$ the left hands side of (4.5) exists and satisfies

$$
| \frac{1}{2\pi i} \oint_{\sigma^{(1)}_\pm} T_\pm(\lambda) \phi_\pm(\lambda, x) \psi_\pm(\lambda, y) g_\pm(\lambda) d\lambda | \leq C_\pm(x),
$$
and hence
\[ \left| \frac{1}{2\pi i} \oint_{\sigma^{(1)}_+} G'_{\pm}(z, x, y) dz \right| \leq C_{\pm}(x). \]

We will now rewrite the integrand in a form suitable for our further purposes. Namely, since \( \psi_{\pm}(\lambda, x) \in \mathbb{R} \) as \( \lambda \in \sigma^{(1)}_+ \), we have
\[ \frac{1}{2\pi i} \oint_{\sigma^{(1)}_+} G'_{\pm}(\lambda, x, y) d\lambda = \frac{1}{2\pi i} \int_{\sigma^{(1)}_+} \psi_{\pm}(\lambda, y) \left( \frac{\phi_+(\lambda, x)}{W(\lambda)} - \frac{\phi_-(\lambda, x)}{W(\lambda)} \right) d\lambda \]
Moreover, (3.18) and Lemma 3.2 (ii) imply
\[ \phi_{\mp}(\lambda, x) = T_{\mp}(\lambda) \phi_{\pm}(\lambda, x) - \frac{T_{\mp}(\lambda)}{T_{\mp}(\lambda)} \]
Therefore,
\[ \frac{\phi_{\pm}(\lambda, x)}{W(\lambda)} - \frac{\phi_{\mp}(\lambda, x)}{W(\lambda)} = \phi_{\pm}(\lambda, x) \left( \frac{1}{W(\lambda)} + \frac{T_{\mp}(\lambda)}{T_{\pm}(\lambda)W(\lambda)} \right) - \frac{T_{\mp}(\lambda)\phi_{\pm}(\lambda, x)}{W(\lambda)} \]
But by (3.22)
\[ T_{\mp}^{-1}(\lambda)W(\lambda) = |W(\lambda)|^2 g_{\mp}(\lambda) \in i\mathbb{R}, \quad \text{for} \ \lambda \in \sigma^{(1)}_+, \]
and therefore the first summand of (4.11) vanishes. Using now \( W = (T_{\mp} g_{\mp})^{-1} \) we arrive at
\[ \frac{\phi_{\pm}(\lambda, x)}{W(\lambda)} - \frac{\phi_{\mp}(\lambda, x)}{W(\lambda)} = |T_{\mp}(\lambda)|^2 g_{\mp}(\lambda) \phi_{\pm}(\lambda, x) \]
and hence
\[ \frac{1}{2\pi i} \oint_{\sigma^{(1)}_+} G'_{\pm}(\lambda, x, y) d\lambda = F_{h, \pm}(x, y) \pm \int_{x}^{\pm \infty} K_{\pm}(x, t) F_{h, \pm}(t, y) dt, \]
where
\[ F_{h, \pm}(x, y) = \int_{\sigma^{(1)}_+} |T_{\mp}(\lambda)|^2 \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\mp}(\lambda), \]
and
\[ |F_{h, \pm}(x, y)| \leq C_{\pm}(x)C_{\pm}(y) \]
by Lemma 4.2. The partial sums \( F_{h, n, \pm}(x, y) \) can be investigated similarly

We will now investigate the r.h.s. of (4.3) and (4.5). Therefore we consider first the question of the existence of the right hand side:

To prove the boundedness of the corresponding series on the left hand side, it is left to investigate the series, which correspond to the circles. We will derive the necessary estimates only for the part of the n'th circle \( K_{R_n, \pm} \), where \( R_{n, \pm} \)
denotes the radius, in the upper half plane as the part in the lower half plane can be considered similarly. We have

\[ | \int _{K \cap n, \pm} G_\pm (z, x, y) dz | \leq \int _0 ^{\pi} C e ^{\pm \sqrt{R(x - y)(1 - \nu)}} \sin (\theta / 2) d\theta \]

\[ \leq \int _0 ^{\pi / 2} C e ^{\pm \sqrt{R(x - y)(1 - \nu)}} \frac{1}{2} \pi \eta d\eta \]

\[ \leq C \sqrt{R(x - y)(1 - \nu)} e ^{\pm \sqrt{R(x - y)(1 - \nu)}} \frac{1}{2} \pi \eta , \]

where \( C \) and \( \nu \) denote some constant, which are dependent on the radius (cf. Lemma 2.1). Therefore as already mentioned the part belonging to the circles converges against zero and hence the same is true for the corresponding series, by Jordan's lemma.

Thus we obtain that the sequence of partial sums on the right hand side of (4.3) and (4.5) is uniformly bounded and we are therefore ready to prove the following result.

**Lemma 4.5.** The sequence of functions

\[ F_{d, n, \pm}(x, y) = \sum _{\lambda_k \in \sigma_d \cap \Gamma _{d, n, \pm}} (\gamma _k ^\pm)^2 \tilde{\psi} _\pm (\lambda_k, x) \tilde{\psi} _\pm (\lambda_k, y) \]

is uniformly bounded, that means for all \( n \in \mathbb{N} \), \( |F_{d, n, \pm}(x, y)| \leq C_n(x) \), where \( C_n(x) \) are monotonically increasing functions. Moreover, \( F_{d, n, \pm}(x, y) \) converges uniformly as \( n \to \infty \) to the function

\[ F_{d, \pm}(x, y) = \sum _{\lambda_k \in \sigma_d} (\gamma _k ^\pm)^2 \tilde{\psi} _\pm (\lambda_k, x) \tilde{\psi} _\pm (\lambda_k, y) , \]

which is again bounded by some monotonically increasing function. In particular, \( F_{d, \pm}(x, y) \) is continuous with respect to \( x \) and \( y \).

**Proof.** Applying (3.3), (3.14), (3.15), (3.25), and (3.27) to the right hand side of (4.5), yields

\[ \sum _{\lambda_k \in \sigma_d} \text{Res} G' _\pm (\lambda, x, y) = - \sum _{\lambda_k \in \sigma_d} \text{Res} \frac{\tilde{\phi} _\pm (\lambda, x) \tilde{\psi} _\pm (\lambda, y)}{W(\lambda)} \]

\[ = - \sum _{\lambda_k \in \sigma_d} \frac{\phi _\pm (\lambda_k, x) \tilde{\psi} _\pm (\lambda_k, y)}{W'(\lambda_k) c_{k, \pm}} \]

\[ = - \sum _{\lambda_k \in \sigma_d} (\gamma _k ^\pm)^2 \phi _\pm (\lambda_k, x) \tilde{\psi} _\pm (\lambda_k, y) \]

\[ = - F_{d, \pm}(x, y) \mp \int _{x} ^{\pm \infty} K _\pm (x, t) F_{d, \pm}(t, y) dt , \]

where

\[ F_{d, \pm}(x, y) := \sum _{\lambda_k \in \sigma_d} (\gamma _k ^\pm)^2 \tilde{\psi} _\pm (\lambda_k, x) \tilde{\psi} _\pm (\lambda_k, y) . \]
Thus we obtained the following integral equation,

\[
F_{d,\pm}(x, y) = -K_{\pm}(x, y) - F_{c,\pm}(x, y) \mp \int_{x}^{\pm \infty} K_{\pm}(x, t) F_{d,\pm}(t, y) dt
\]

\[
+ \int_{x}^{\pm \infty} K_{\pm}(x, t) F_{d,\pm}(t, y) dt,
\]

which we can now solve for \( F_{d,\pm}(x, y) \) using again Lemma 4.2 and hence \( F_{d,\pm}(x, y) \) exists and satisfies the given estimates. The corresponding partial sums can be investigated analogously using the considerations from above.

Putting everything together, we see that we have obtained the GLM equation.

**Theorem 4.6.** The GLM equation has the form

\[
K_{\pm}(x, y) + F_{\pm}(x, y) \pm \int_{x}^{\pm \infty} K_{\pm}(x, t) F_{\pm}(t, y) dt = 0, \quad \pm(y - x) > 0,
\]

where

\[
F_{\pm}(x, y) = \int_{\sigma_{\pm}} R_{\pm}(\lambda) \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda)
\]

\[
+ \int_{x}^{\pm \infty} |T_{\pm}(\lambda)|^2 \psi_{\pm}(\lambda, x) \psi_{\pm}(\lambda, y) d\rho_{\pm}(\lambda)
\]

\[
+ \sum_{k=1}^{\infty} (\gamma_{k}^{\pm})^2 \tilde{\psi}_{\pm}(\lambda_k, x) \tilde{\psi}_{\pm}(\lambda_k, y).
\]

Moreover, we have

**Lemma 4.7.** The function \( F_{\pm}(x, y) \) is continuously differentiable with respect to both variables and there exists a real-valued function \( q_{\pm}(x) \), \( x \in \mathbb{R} \) with

\[
\pm \int_{a}^{\pm \infty} (1 + x^2)|q_{\pm}(x)| dx < \infty, \quad \text{for all } a \in \mathbb{R},
\]

such that

\[
|F_{\pm}(x, y)| \leq \tilde{C}_{\pm}(x) Q_{\pm}(x + y),
\]

\[
\left| \frac{d}{dx} F_{\pm}(x, y) \right| \leq \tilde{C}_{\pm}(x) \left( \left| q_{\pm}\left(\frac{x + y}{2}\right) \right| + Q_{\pm}(x + y) \right),
\]

\[
\pm \int_{a}^{\pm \infty} \left| \frac{d}{dx} F_{\pm}(x, x) \right| (1 + x^2) dx < \infty,
\]

where

\[
Q_{\pm}(x) = \pm \int_{x}^{\pm \infty} |q_{\pm}(t)| dt,
\]

and \( \tilde{C}_{\pm}(x) > 0 \) is a continuous function, which decreases monotonically as \( x \to \pm \infty \).
Proof. Applying once more Lemma 4.2 one obtains (4.15). Now, for simplicity, we will restrict our considerations to the + case and omit + whenever possible. Set $Q_1(u) = \int_a^\infty Q(t)dt$. Then, using (3.2), the functions $Q(x)$ and $Q_1(x)$ satisfy

$$
\int_a^\infty Q_1(t)dt < \infty, \quad \int_a^\infty Q(t)(1 + |t|)dt < \infty.
$$

Differentiating (4.13) with respect to $x$ together with the estimates

$$
K(x, y) = \int_0^\infty K(x, t)F(t, y)dt = 0.
$$

We already know that the functions $Q(x), Q_1(x), C(x)$, and $\hat{C}(x)$ are monotonically decreasing and positive. Moreover,

$$
\int_x^\infty \left( q\left( \frac{x + t}{2} \right) + Q(x + t) \right)Q(t + y)dt \leq (Q(2x) + Q_1(2x))Q(x + y),
$$

thus we can estimate $F_x(x, y)$ and $F_y(x, y)$ can be estimates using (3.5) and the method of successive approximation. It is left to prove (4.17). Therefore consider (4.13) for $x = y$ and differentiate it with respect to $x$:

$$
\frac{dF(x, x)}{dx} + \frac{dK(x, x)}{dx} - K(x, x)F(x, x) + \int_x^\infty (K_x(x, tF(t, x) + K(x, t)F_y(t, x))dt = 0.
$$

Next (3.4) and (4.5) imply

$$
|K(x, y)F(x, x)| \leq \hat{C}(a)C(a)Q^2(2x), \quad \text{for } x > a,
$$

where $\int_a^\infty (1 + x^2)Q^2(2x)dx < \infty$. Moreover, by (3.5) and (4.6)

$$
|K_x(x, t)F(t, x)| + |K(x, t)F_y(t, x)| \leq 4\hat{C}(a)\hat{C}(a)\left\{ q\left( \frac{x + t}{2} \right) |Q(x + t) + Q^2(x + t) \right\},
$$

together with the estimates

$$
\int_a^\infty dx \int_x^\infty Q(x + t)dt \leq \int_a^\infty |x|Q(2x)dx \sup_{x \geq a} \int_x^\infty |x + t|Q(x + t)dt < \infty,
$$

$$
\int_a^\infty x^2 \int_x^\infty |q\left( \frac{x + t}{2} \right) Q(x + t)dt \leq \int_a^\infty Q(2x)dx \sup_{x \geq a} \int_x^\infty q\left( \frac{x + t}{2} \right) (1 + (x + t)^2)dt < \infty,
$$

and (3.6), we arrive at (4.17).

In summary, we have obtained the following necessary conditions for the scattering data:

**Theorem 4.8.** The scattering data

$$
S = \left\{ R_+(\lambda), T_+(\lambda), \lambda \in \sigma^+; R_-(\lambda), T_-(\lambda), \lambda \in \sigma^-; \lambda_1, \lambda_2, \cdots \in \mathbb{R} \setminus (\sigma_+ \cup \sigma_-); \gamma^+_1, \gamma^+_2, \cdots \in \mathbb{R}^+ \right\}
$$

(4.18)
possess the properties listed in Theorem 3.2, 3.3, 3.4, and 3.5, and Lemma 4.1, 4.4, and 4.5. The functions $F_\pm(x,y)$ defined in (4.14), possess the properties listed in Lemma 4.7.

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References

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