ECCENTRICITY EVOLUTION OF EXTRASOLAR MULTIPLE PLANETARY SYSTEMS DUE TO THE DEPLETION OF NASCENT PROTOstellAR DISKS

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Received 2002 April 18; accepted 2002 December 7

ABSTRACT

Most extrasolar planets are observed to have eccentricities much larger than those in the solar system. Some of these planets have sibling planets, with comparable masses, orbiting around the same host stars. In these multiple planetary systems, eccentricity is modulated by the planets’ mutual secular interaction as a consequence of angular momentum exchange between them. For mature planets, the eigenfrequencies of this modulation are determined by their mass and semimajor axis ratios. However, prior to the disk depletion, self-gravity of the planets’ nascent disks dominates the precession eigenfrequencies. We examine here the initial evolution of young planets’ eccentricity due to the apsidal libration or circulation induced by both the secular interaction between them and the self-gravity of their nascent disks. We show that as the latter effect declines adiabatically with disk depletion, the modulation amplitude of the planets’ relative phase of periapsis is approximately invariant despite the time-asymmetrical exchange of angular momentum between planets. However, as the young planets’ orbits pass through a state of secular resonance, their mean eccentricities undergo systematic quantitative changes. For applications, we analyze the eccentricity evolution of planets around $\upsilon$ Andromedae and HD 168443 during the epoch of protostellar disk depletion. We find that the disk depletion can change the planets’ eccentricity ratio. However, the relatively large amplitude of the planets’ eccentricity cannot be excited if all the planets had small initial eccentricities.

Subject headings: celestial mechanics — planetary systems: formation — planetary systems: protoplanetary disks — stars: individual ($\upsilon$ Andromedae, HD 168443)

1. INTRODUCTION

Many extrasolar planetary systems have been discovered recently using the radial velocity technique (Marcy & Butler 2000). The basic assumption is that the spectroscopic variations observed in some targeted stars are due to the Doppler shift associated with the reflex motion of stars with unseen companions. The mass of these companions depends on the poorly known inclination of these systems. However, unless the orbits of these systems are highly inclined (Stepinski & Black 2001), the inferred masses of the companions are comparable to that of Jupiter. The observed spectra of some targeted stars indicate the presence of multiple companions around them. For example, three companions are found to orbit around $\upsilon$ Andromedae. The long-term stability of this system requires their inclination to be sufficiently small such that the masses of these companions are no more than a few Jupiter masses ($M_J$) and much smaller than that of their host star (Laughlin & Adams 1999; Rivera & Lissauer 2000; Stepinski, Malhotra, & Black 2000; Ito & Miyama 2001; Lissauer & Rivera 2001). Two companions are also found around HD 168443 with minimum masses that are an order of magnitude larger than $M_J$. Although the masses of these companions would be a modest fraction of a solar mass ($M\odot$) if this system is viewed nearly face-on, such compact hierarchical stellar systems have not been seen before. Thus, we follow the conventional practice to refer to these multiple companions as planets.

In the limit that the planets’ masses are substantially smaller than that of their host star, their mutual secular perturbation induces them to exchange angular momentum while preserving their energy. (Planets in mean motion resonances also exchange energy on comparable timescales.) Around $\upsilon$ And and HD 168443, the planets’ orbits are not in mean motion resonances so that their semimajor axes are conserved while their eccentricity and longitude of periapsis modulate over some characteristic secular timescale (Murray & Dermott 1999).

However, the secular perturbation between the planets has not always been sustained at the present level. During the epoch of their formation, protoplanets are embedded in protostellar disks that have been found around most young stellar objects (Haisch, Lada, & Lada 2001). The mass and temperature distribution of these disks are very similar to those inferred from the minimum mass nebula model for the solar system (Beckwith 1999). The self-gravity of these disks can induce the orbits of planets formed within them to precess at a rate faster than that due to their mutual perturbation. Consequently, the rate of angular momentum transfer between the interacting planets is suppressed. Jiang & Ip (2001) calculated the evolution of massless planets around $\upsilon$ And with the potential of a depleting axisymmetric protostellar disk. Because of the assumption of massless planets, there is no angular momentum transfer between the planets.

In the solar system, the initial contribution of the solar nebula to the total gravitational potential dominates the precession frequency of asteroids and comets over that due to the secular perturbation induced on them by the giant...
The planets also undergo precession induced by the disk gravity and other planets' secular perturbation. In general the precession frequencies of these celestial bodies do not equal each other, but as gas is depleted in the solar nebula, its self-gravity weakens. The total precession frequencies of both the asteroids and the planets decline, although at a different rate. When the precession frequency of asteroids with some semimajor axes coincides with that of a major planet, they enter into a state of secular resonance. In this resonance, the eccentricity of the asteroids is either excited or damped monotonically, depending on their relative longitude of periastron passage with respect to that of the planet. As this secular resonance sweeps across the solar system, large eccentricity may be excited among some small celestial bodies (Ward, Colombo, & Franklin 1976; Heppenheimer 1980; Nagasawa, Tanaka, & Ida 2000; Nagasawa & Ida 2000).

In this paper we examine the effects of disk depletion on the eccentricity evolution of the planetary systems around \( \upsilon \) And and HD 168443. Through such an investigation, we hope to infer the kinematic properties that these planets are born with and thereby cast constraints on their formation process. In \( \S \) 2 we briefly discuss the precession due to the secular interaction between planets and that due to the self-gravity of the protostellar disk. The conditions for secular resonance are discussed. We introduce in \( \S \) 3 a working model and describe the method we used to analyze the eccentricity evolution. In \( \S \) 4 we present the results of some calculations. We use these numerical results to demonstrate the evolution of these systems with Hamiltonian contour maps. Based on these results, we infer in \( \S \) 5 some implications on the planets' orbital eccentricity shortly after their formation and while they are still embedded in their nascent disks. Finally, we summarize our results in \( \S \) 6.

### 2. SECULAR INTERACTION IN MULTIPLE EXTRASOLAR PLANETARY SYSTEMS

#### 2.1. Current Orbital Properties

In this paper we focus our discussion on the planets around \( \upsilon \) And (c and d) and those around HD 168443. The decomposition of \( \upsilon \) And’s spectra indicates that there are three planets orbiting around it (Butler et al. 1999). Inferred orbital elements\( \textsuperscript{4} \) of the \( \upsilon \) And planets are shown in Table 1A. The semimajor axis, eccentricity, longitude of periastron, mass of the planet, and periastron passage time (JD) are denoted by \( a \), \( e \), \( \omega \), \( M \), and \( T_{\text{peri}} \), respectively. The subscripts b, c, and d represent the values for the individual planets. Throughout this analysis, we assume that these systems are viewed edge-on and their masses correspond to their minimum values. Provided that their orbits are coplanar, our analysis is independent of the planets' inclination. Our approach becomes inadequate for the limiting cases of nearly face-on orbits where the masses of the companions are comparable to their host stars.

Around \( \upsilon \) And, the eccentricity of the innermost planet b is essentially undetectable. The orbit of planet b is likely to be circularized during the main-sequence life span of the host star \( \upsilon \) And by the tidal dissipation within the planet’s interior as expected for Jupiter-like extrasolar planets with semimajor axis less than 0.05 AU (Rasio et al. 1996). With its low mass and small semimajor axis, planet b does not contribute significantly to the dynamical evolution of the system (Mardling & Lin 2003). Although the outer planets (c and d) still exert secular perturbation on planet b, the cumulative effect on its eccentricity modulation is limited. Under the present configuration, the secular interaction of

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\( \textsuperscript{4} \) See http://exoplanets.org/almanacframe.html.

#### Table 1

| Planets | \( a \) (AU) | \( e \) | \( \omega \) (rad) | \( M \sin i \) (\( M_\oplus \)) | \( T_{\text{peri}} \) |
|---------|---------------|----------|-----------------|----------------|-------------|
| A. \( \upsilon \) And (\( M_\oplus = 1.3 \ M_\oplus \)) | | | | | |
| Planet b.............. | 0.059 | 0.01 | 5.252 | 0.69 | 2,450,000.6383 |
| Planet c.............. | 0.827 | 0.23 | 4.314 | 2.06 | 2,450,399.0 |
| Planet d.............. | 2.56 | 0.35 | 4.374 | 4.10 | 2,451,348.9 |
| B. HD 168443 (\( M_\oplus = 1.01 \ M_\oplus \)) | | | | | |
| Planet b.............. | 0.295 | 0.53 | 3.018 | 7.73 | 2,450,047.58 |
| Planet c.............. | 2.87 | 0.20 | 1.098 | 17.15 | 2,450,250.6 |
| C. HD 74156 (\( M_\oplus = 1.05 \ M_\oplus \)) | | | | | |
| Planet b.............. | 0.276 | 0.649 | 3.206 | 1.56 | 2,451,981.4 |
| Planet c.............. | 3.47 | 0.395 | 4.189 | >7.5 | 2,450,649 |
| D. 47 UMa (\( M_\oplus = 1.03 \ M_\oplus \)) | | | | | |
| Planet b.............. | 2.09 | 0.061 ± 0.014 | 3.00 ± 0.26 | 2.54 | 2,453,622±34 |
| Planet c.............. | 3.73 | 0.1 ± 0.1 | 2.22 ± 0.98 | 0.76 | 2,453,365.15 ±493 |
| E. HD 12661 (\( M_\oplus = 1.07 \ M_\oplus \)) | | | | | |
| Planet b.............. | 0.804 | 0.35 | 5.11 | 2.21 | 2,460,208.393 |
| Planet c.............. | 2.652 | 0.11 | 2.29 | 1.58 | 2,459,398.08 |

Note.—See http://exoplanets.org/almanacframe.html.
planet b with planets c and d is weakened by the rapid precession due to the post-Newtonian relativistic correction in the gravitational potential of the host star (Mardling & Lin 2003). Neglecting the secular perturbation due to planet b, the orbital evolution of planets c and d obtained from the direct orbital integration of the full equation of motion is shown in Figure 1a. In the numerical integration, the calculation is started with the eccentricity ratio $e_c/e_d$ and the relative longitude of periapsis between planets c and d, $\eta = \varpi_c - \varpi_d$, based on their observed values 0.66 and roughly $-0.06$ rad, respectively. We compute the evolution of the system for over $2 \times 10^4$ yr. The equi-Hamiltonian contours (see Appendix D) are also shown. For planets c and d around $\upsilon$ And, the massive planet d has larger eccentricity than smaller planet c. The longitudes of periapsion of planets c and d are always close to each other. When the planets are in librating (closed) track in the $(x, \eta)$-diagram, the planetary system tends to be stable for a long time. The stabilities of this system around $\upsilon$ And are well investigated (Laughlin & Adams 1999; Rivera & Lissauer 2000; Stepiński et al. 2000; Barnes & Quinn 2001; Ito & Miyama 2001; Lissauer & Rivera 2001; Chiang, Tabachnik, & Tremaine 2001; Mardling & Lin 2003).

Around HD 168443, two massive planets are inferred from the radial velocity curves (Marcy et al. 2001). The best-fit orbital elements are shown in Table 1B. We also numerically integrate the present-day orbital evolution of the planets b and c around HD 168443. In contrast to the planetary system around $\upsilon$ And, the lower mass planet b around the HD 168443 system has an eccentricity that is more than twice that of the massive planet c. The orbits are integrated over $2.5 \times 10^4$ yr, starting with $x = 2.65$ and $\eta \sim 1.92$ rad. These planets evolve in circulating (open) track in the $(x, \eta)$-diagram (Fig. 1b), and $\eta$ evolves from $-\pi$ to $\pi$ over a period of about $1.6 \times 10^4$ yr. This system is stable despite the large magnitude of eccentricity of planet b. The origin of such large mass and eccentricity of planets in this system has not been addressed previously.

2.2. Secular Perturbation between Two Planets

The secular interaction induces eccentricity modulation between planets c and d around $\upsilon$ And (Laughlin & Adams 1999; Rivera & Lissauer 2000; Ito & Miyama 2001; Mardling & Lin 2003). For presentation purposes, it is useful to briefly recapitulate the analysis of secular interaction between two planets. It is customary to consider the secular (long-term) evolution of the interacting planets’ orbits using a disturbing function (Murray & Dermott 1999). To the lowest order, the modulation of the eccentricity ($e_{c,d}$) and longitude of periapsis ($\varpi_{c,d}$) of some planets c and d can be approximated by

$$\frac{de_{c,d}}{d\tau} = \gamma_{c,d} C \left[ \frac{e_{c,d}}{\varpi_{c,d} - \varpi_{c,d}} + \Lambda_{c,d} \right],$$

(1)

$$\frac{d\varpi_{c,d}}{d\tau} = \gamma_{c,d} \left[ 1 + C \left( \frac{e_{c,d}}{\varpi_{c,d} - \varpi_{c,d}} \right) \cos(\varpi_{c,d} - \varpi_{c,d}) \right] + \Lambda_{c,d} + N_{c,d},$$

(2)

where $\tau = t/t_c$, $t$ is the real time; $t_c = (4/n_c)(M_{c}/M_{d}){(a_{c}/a_{d})}^2b_{c,d}^{[3]}$; $b_{c,d}^{[3]}$ are the Laplace coefficients with $\alpha = a_{c}/a_{d}$; $M_{c}$, $M_{d}$, and $M_{*}$ are the mass of planets c, d, and the host star, respectively; $a_{c}$ and $a_{d}$ are the semimajor axes of planets c and d, respectively; $n_{c,d} = (GM_{*}/a_{c,d}^3)^{1/2}$ is the mean motion of planet c or d; $\gamma_{c,d} = (a_{c}/a_{d})^{1/2}(M_{c}/M_{d})$ so that $\gamma_{c} = 1$; $C = -b_{c,d}^{[2]}/b_{c,d}^{[3]}$ (Mardling & Lin 2002); $\Lambda_{c,d}^T$ is the change rate of eccentricity due to the nonaxisymmetric torque; $\Lambda_{c,d}^N$ is the change rate of $\varpi$ due to the nonaxisymmetric torque; and $N_{c,d}$ is axisymmetric modification of the gravitational potential. When $a_{c}/a_{d} \ll 1$, $t_c = (4/3n_c)(M_{c}/M_{d})(a_{c}/a_{d})^3$ and $C = -5a_{c}/4a_{d}$. Frequently used variables are tabulated in Table 2. The planets’ secular interaction induces both axisymmetric and nonaxisymmetric perturbations on each other. While only the nonaxisymmetric component of the planets’ secular perturbation on each other can lead to angular momentum exchange and $e_{c,d}$ modulations in equation (1), both axisymmetric and nonaxisymmetric perturbations
contribute to the evolution of \( \omega \) through the first and second terms on the right-hand side of equation (2), respectively. These linearized equations of secular motion, although only accurate to first order, are useful for obtaining analytic approximation solutions that highlight the dominant physical effects. For the numerical calculation of the planets’ orbits (see § 4), we use the exact equation of planets’ motion.

Planet-disk interaction also leads to both axisymmetric modification of the gravitational potential \((N_{c,d})\) and nonaxisymmetric torque \((A_{c,d}; Goldreich \& Tremaine 1980, 1982; Lin \& Papaloizou 1986a, 1986b, 1993). For the latter effect, planets excite waves in the disk that carry angular momentum. The dissipation of these waves, anywhere in the disk, would result in a finite torque (Papaloizou \& Lin 1984). In principle, the net torque vanishes in the inviscid limit. However, their amplitude grows and steepens into nonlinear shocks as the waves propagate away from the location where they are launched, leading to an effective torque (Savonije, Papaloizou, \& Lin 1994). The evolution of the axisymmetric potential modifies the precession of \( \omega \) but does not directly influence the eccentricity of the orbits (see eq. [2]). While the nonaxisymmetric torque may induce monotonic changes in \( e \) (Chiang \& Murray 2002) and \( \omega \) (see eq. [1]), the magnitude and sign of \( N_{c,d} \) depend sensitively on the disk structure. Interaction between the embedded planets with disk gas through corotation resonances damps the eccentricity, while that through Lindblad resonances excites the eccentricity (Goldreich \& Tremaine 1980). For protoplanets with mass less than a few times that of Jupiter and modest eccentricity, gas may flow in the vicinity of their orbits such that the eccentricity damping effect of the corotation resonances is stronger than the excitation effect of the Lindblad resonances (Goldreich \& Tremaine 1980). However, protoplanets with masses an order of magnitude larger than that of Jupiter may open relatively wide gaps in protostellar disks. In this limit, the protoplanets’ corotation resonances may be cleared of disk gas such that their eccentricity may be excited (Artymowicz 1993; Papaloizou, Nelson, \& Masset 2001; Goldreich \& Sari 2003). For precession, the contribution from the nonaxisymmetric torque may be weaker than that due to the disk’s axisymmetric contribution to the total gravitational potential in the limit that the torque resulting from the planets’ interaction with the interior and exterior regions of the disk is balanced or in low-viscosity and thin disks where planets with relatively low masses can open wide gaps. However, the nonaxisymmetric torque may lead to significant contributions over time in relatively massive disks.

In order to analyze the contribution of each effect, we adopt a piecemeal approach. In this paper we focus our attention on the evolution of planetary orbits due to the changes in the axisymmetric disk potential. For the departure from a point-mass potential, the apsidal motions of planets \( c \) and \( d \) are included in \( N_{c,d} \) (see Appendix A). In this first step, we neglect the effects due to the nonaxisymmetric torque by setting \( A_{c,d} = 0 \) in equations (1) and (2). The extension of our discussion to the limit of finite nonaxisymmetric torque will be presented in a future contribution.

### 2.3. Precession Frequencies

To the lowest order, the solution of equations (1) and (2) can be expressed as

\[
\varepsilon_{c,d} \exp \left( i \omega_{c,d} \right) = A_{c,d} \exp \left[ i \left( g_1 t + \beta_1 \right) \right] \\
+ B_{c,d} \exp \left[ i \left( g_2 t + \beta_2 \right) \right],
\]

where \( A_{c,d} \) and \( B_{c,d} \) are the oscillation amplitudes and \( \beta_1 \) and \( \beta_2 \) are the phase angles (e.g., Brouwer \& Clemence 1961). The individual planet’s longitudes of periastra precess with two independent eigenfrequencies,

\[
g_{1,2} \equiv \frac{1}{2} \left( g_c + g_d \mp \left[ \left( g_c - g_d \right)^2 + 4g_{cd}^2 \right]^{1/2} \right),
\]

where

\[
g_{cd} \equiv \frac{g_{cd}}{g_c + g_d} \quad \text{and} \quad g_{cd} \equiv \frac{C_{\gamma}^{1/2}}{I_c} \propto (M_c M_d)^{1/2}.
\]

From equation (2), we find that \( g_c \) would be the precession frequency of planet \( c \) if it is massless and subjected to the secular perturbation of a massive planet \( d \) with a circular orbit. Similarly, \( g_d \) would be the precession frequency of planet \( d \) if it is massless and subjected to the secular perturbation of a massive planet \( c \) with a circular orbit.

In the limit that \( A_{c,d} \) and \( B_{c,d} \) have comparable magnitudes and the two eigenfrequencies \( g_{1,2} \) have sufficiently large differences, the relative longitude of the two planets would circulate. Precessional degeneracy occurs when \( g_1 = g_2 \) so that the two planets precess synchronously regardless of the amplitude of their \( e \) and \( \omega \). In this degenerate state, the relative phase of the two planets’ longitude of periaapsis passage, \( \eta \equiv \omega_c - \omega_d \), retains a constant value, and the two planets are in a state of secular resonance. In this secular resonance, angular momentum is monotonically transferred from one planet to another because the relative orientation of their periastra is maintained indefinitely. To lowest order, the planets’ individual energy is conserved. With the exception of the special configuration in which \( \eta = 0 \) or \( \pm \pi \), this monotonic transfer generally leads to eccentricity excitation of the angular momentum losing bodies and eccentricity damping of the angular momentum gaining bodies. In celestial mechanics, secular resonance is thought to be important in the current eccentricity distribution of the asteroids (Williams 1969).

From equation (4), we find that the magnitude of the difference between \( g_1 \) and \( g_2 \) is

\[
\Delta g_{12} \equiv |g_1 - g_2| = \left[ \left( g_c - g_d \right)^2 + 4g_{cd} \right]^{1/2} \\
= \left( \frac{1 - \gamma_d + \Delta N}{\tau_c} \right)^2 + 4g_{cd} \right]^{1/2},
\]
which vanishes on the exact center of secular resonance (where \(g_1 = g_2\)) only if \(y_3 = g_3\) and \(y_{3d} = 0\). The former requirement corresponds to a necessary resonance condition \(A_1 = 0\) where

\[
A_1 \equiv 1 - \gamma_d + \Delta N
\]

is the precession rate induced by the axisymmetric component of the perturbed potential (see eq. [9]). The latter requirement is satisfied if either \(M_e = 0\) or \(M_d = 0\) (see eq. [5]). Equation (6) implies that planets with comparable masses cannot have precessional degeneracy with \(g_1 = g_2\) (Kinoshita & Nakai 2000).

The present value of \(\gamma_d\) is 0.29 for planets c and d around v. And, today, in the absence of any residual disk \(N_{r,d} = 0\), \(g_2 \approx g_1 < g_1 \approx g_1\) and the outer two planets of v. And are not in a state of precessional degeneracy. Nevertheless, their present orbits can be approximated by equation (3) with the magnitude of \(A_{r,d}\) much smaller than that of \(B_{r,d}\). Thus, the planets primarily precess with eigenfrequency \(g_1\) and their relative longitudes ofperiastron librate over a restricted range of phases. This phase lock is equivalent to a state of secular resonance. This resonant interaction is the result of a dynamical feedback through the planets’ secular interaction.

### 2.4. Three Classes of Relative Orbits

In order to illustrate the importance of this feedback effect, we substitute \(x \equiv e_1/e_{d}\) and \(\eta \equiv \omega_1 - \omega_d\). Equations (1) and (2) reduce to

\[
\frac{d\eta}{d\tau} = C(1 + \gamma_d x^2) \sin \eta, \quad (8)
\]

\[
\frac{dx}{d\tau} = (1 - \gamma_d) + \left(\frac{C}{x}\right)(1 - \gamma_d x^2) \cos \eta + \Delta N = A_1 + A_2(x) \cos \eta, \quad (9)
\]

where \(\Delta N \equiv N_e - N_{d}\), \(A_1\) is given in equation (7), and \(A_2(x) \equiv (C/x)(1 - \gamma_d x^2)\) is introduced for notational simplicity. There are three families of relative orbits that can be illustrated below with the linearized approximation solutions of equations (8) and (9) (see Appendix B). We show below that the relative magnitudes of \(A_1\) and \(A_2\) determine the nature of the orbits.

Circulation.—In these solutions, the magnitude of \(x\) modulates about some values \(x_0\) such that \(x = x_0 + \delta x(\tau)\) with an amplitude \(|\delta x(\tau)| \ll x_0\) for all values of \(\eta\), which ranges between 0 and 2\(\pi\). In the limit that \(|A_1| > |A_2|\), \(\eta\) decreases monotonically (because \(A_1 < 0\)) while \(\delta x\) oscillates (see solutions in eqs. [B7] and [B8] in § B2). To the lowest order these solutions reduce to

\[
\eta \simeq A_1 \tau, \quad \delta x \simeq -\frac{C}{A_1} \cos A_1 \tau. \quad (10)
\]

Near the secular resonance where \(|A_1|\) is relatively small, the amplitude of \(\delta x\) becomes large.

Libration.—In the opposite limit that \(|A_1| \ll |A_2|\), the nonaxisymmetric secular interaction between the planets is important. There are stationary points in the \((x, \eta)\)-plane that center on the values of \(x = x_m\) and \(\eta = 0\) or \(\pi\) (see § B1). Around these points, there are orbits with small-amplitude modulation such that

\[
\epsilon \equiv \frac{x - x_m}{x_m} = \epsilon_0 \sin \omega \tau, \quad \eta = \eta_0 \cos \omega \tau, \quad (11)
\]

where

\[
\eta_0 = \pm \epsilon_0, \quad \omega = \pm \frac{C}{x_m} (1 + \gamma_d x_m^2) \quad (12)
\]

is the oscillation frequency (see § B1). The dimensionless amplitude of these librational orbits \(\epsilon_0 \ll 1\). For these orbits, although \(g_1 \neq g_2\), secular interaction induces them to precess at similar frequencies so that their relative longitude of periapsis passage is always approximately aligned or anti-aligned. Thus, these planets are effectively in a state of secular resonance. Note that because the libration is centered around \(\eta = 0\) or \(\pi\), there is no effective angular momentum transfer despite the phase lock.

Excitation.—For systems with \(|A_1| < |A_2|\), there are also orbits in which \(x\) and \(\eta\) have very different values as \(x_m\) and 0 (or \(\pi\)), respectively. In these cases, \(\eta\) would evolve rapidly to a phase angle

\[
\eta_1 \simeq \cos^{-1} \left(\frac{-A_1}{A_2}\right) = \cos^{-1} \left[\frac{-A_1 x}{C(1 - \gamma_d x^2)}\right], \quad (13)
\]

such that \(d\eta_1/d\tau\) is reduced to zero (see eq. [9]) and the two planets become phase locked.

For all nonzero (or \(\pi\)) values of \(\eta_1\), the monotonic increases (decreases) of \(x\) correspond to eccentricity excitation (damping), analogous to the situation of precessional degeneracy. In this state of near secular resonances, angular momentum is monotonically transferred from one planet to the other, resulting in a monotonic evolution of \(x\). The modification of \(x\) in turn leads to an evolution in \(\eta\). Along the path of \(x\) and \(\eta_1\) evolution, \(g_1 \neq g_2\) and the planets would librate about their evolving guiding center. For the special cases in which the two planets enter the resonance, the second-order solution reduces to that in equation (11) and all orbits become instantaneously librational, even though \(\epsilon_0\) and \(\eta_0\) may become arbitrarily large. At the center of resonance where \(A_1, A_2,\) and \(A_1/A_2\) all vanish, \(\eta_1 = \pi/2\) and \(x_m\) evolves exponentially. We discuss the evolution of these systems in Appendix C.

### 3. MODELS

The above analytic approximation is useful for isolating the three families of orbits along with two stationary points in the \((x, \eta)\)-diagram. These orbits generally follow the contours of the equi-Hamiltonian map (see Appendix D). However, the topological evolution of the Hamiltonian map alone is insufficient for the determination of the planets’ orbits during the depletion of the disk. We carry out, below, numerical integration of the full equation of motion for the planets subject to the potential of their host star and nascent disks. In this section we briefly describe a model prescription with which we examine the passage of librational degeneracy during the epoch of disk depletion.

#### 3.1. Planetary Formation Scenarios and Disk Model

We first discuss the physical process of disk depletion. According to conventional theories, planets are formed through the condensation of grains that grow to
planetary systems via cohesive collisions (Hayashi, Nakazawa, & Nakagawa 1985; Lissauer 1987; Wetherill 1990). Upon attaining a sufficiently large mass, planetesimals accrete gas (Mizuno 1980; Bodenheimer & Pollack 1986; Pollack et al. 1996). Eventually, their growth is terminated when protoplanets can tidally induce the formation of a gap near their orbit (Goldreich & Tremaine 1980; Lin & Papaloizou 1980, 1993; Takeuchi, Miyama, & Lin 1996).

Thereafter, gas in the inner region of the disk continues to diffuse inward as it loses angular momentum to the planet. Since gas replenishment is cut off by the formation of the gap, the inner region is depleted well before the outer region of the disk. For the present discussions, we assume that the mass in the inner regions of the disk becomes negligible by the time when the second planet is fully grown. Thus, in our analysis of planets’ dynamical evolution, we only need to consider their interaction with the disk region extended outside the outer planet. Both the gravitational and tidal influences of the disk on the outer planet are much more intense than on the inner planet because the former is much closer to where most of the gas is distributed.

When the planets’ orbits lie in the same plane as their nascent disks and their distance to the nearest edge of their nascent disks is greater than the disk’s scale height, a thin-disk approximation is sufficient for the computation of the disks’ gravitational potential. Following the approach of Ward (1981), the self-gravitating potential for a disk model with a surface density profile \( \Sigma = \Sigma_0 (r_0/r)^k \) (where \( \Sigma_0 \) and \( r_0 \) being some fiducial values) can be expressed as

\[
V(r) = 2\pi G \Sigma_0 r \sum_{n=0}^{k} \frac{A_n}{2n + k - 1} \left( \frac{r}{r_{\text{edge}}} \right)^{2n+k-1}.
\]

In the above expression, \( r \) is the instantaneous location (rather than the semimajor axis) of the planet. We also assume that the disk has an inner edge that is located at \( r_{\text{edge}}, \) interior to which \( \Sigma = 0 \). The coefficient \( A_n = \left[ \left( 2n+1 \right)! / 2^{2n} n! \right]^2 \). For illustration purposes, we consider the minimum mass solar nebula model (Hayashi 1981), in which \( k = 3/2 \). The general applicability of this phenomenological model for protostellar disks around different stars is questionable. In the absence of a reliable prescription, we parameterize disk models with a range of values in \( k \) (see § 4.3). We assume that the disk is extended to infinity and the semimajor axes of planets do not change as a consequence of disk depletion or planets’ secular interaction with each other. The results to be presented below show that while the overall dynamical evolution of the planets’ orbit is insensitive to the detailed disk model, the planets’ extrapolated initial eccentricity ratio does depend on the mass of the disk gas near their orbits.

3.2. Planetary System and Prescription for Disk Depletion

For computational simplicity, we consider that a planetary system consists of two planets with coplanar orbits. In the case of the \( v \) And system, we are concerned with gravitational interaction between planets c and d and neglect the contribution by planet b. We also neglect the influence of the planet’s tidal interaction with the disk. Only the axisymmetric component of the disk potential is taken into account. Mutual inclinations between the two planets and between the planet and the disk are neglected. The inclination of the orbital plane is denoted by \( i \). We take surface density of the disk as 5 \( \sin i \) times that of the minimum mass solar nebula model. Only the ratios, rather than the individual values, of the planets’ and the disks’ masses determine the planets’ motion. Thus, our results do not depend on the inclination of the system (except for small changes in the mean motion and the long-term dynamical stability for exceptionally small values of \( i \)).

For models with different values of \( k \), we scale \( \Sigma \) such that its value at the disks’ inner edge \( r_{\text{edge}} \) remains invariant. As we have indicated above, the inner region of the disk is likely to be severely depleted prior to the emergence of the second planet. Thus, we assume that the disk depletion proceeds in an inside-out manner by adopting a model in which the location of the nebula inner edge \( r_{\text{edge}} \) is prescribed to expand outward as \( r_{\text{edge}}(t) = r_{\text{edge}}(t = 0) + r_{\text{t}}/\Delta N \) with \( r_{\text{t}} = 1 \text{ AU} \) while \( \Sigma_0 \) is kept constant. In order to demonstrate that the passage of the secular resonance does not depend on the detailed prescription of the depletion process or the structure of the disk, we also consider a model in which the surface density of the disk is assumed to decline by an identical reduction factor everywhere such that \( \Sigma_0(t) = \Sigma_0(t = 0) \exp[-t/\tau_{\text{X}}] \) with \( r_{\text{edge}} = \text{const} \). In the limit that \( \alpha_i/\alpha_d \ll 1 \) and \( \alpha_{d,c}/r_{\text{edge}} \ll 1 \), we find that the two prescriptions of the disk depletion do not lead to a significant difference in the numerical results because \( \Delta N \approx \Sigma_0/r_{\text{edge}} \). For the approximate prescription of \( \Delta N \) (see eq. [D2]), the magnitude of \( A_1 = 1 + \Delta N - \gamma_d \) changes on a timescale \( \tau_{\text{X}} \approx \tau_{\text{X}} \) and the condition of the secular resonance \( A_1 = 0 \) is satisfied when

\[
\frac{\Sigma_0 r_{\text{edge}}^{3/2}}{\bar{r}_{\text{edge}}^{5/2}} \approx \frac{5(\gamma_d - 1) M_d}{4\pi a_d^3} \left( \frac{1}{n_c/n_d} \right)^{-1} = \frac{1}{a_d^3}.
\]

This condition can be attained with both prescriptions.

The orbital evolution of the planets can be directly integrated numerically, including the point-mass potential of the host stars and sibling planets as well as the contribution from the disk potential where the gravity can be expressed as

\[
\frac{dV}{dt} = 2\pi G \Sigma(r, t) \sum_{n=0}^{4} \frac{4n}{4n+4} A_n \left[ \frac{r}{r_{\text{edge}}(t)} \right]^{2n+1/2},
\]

where \( k = 3/2 \) is assumed.

4. RESULT OF NUMERICAL COMPUTATION

4.1. Evolution of Liberating Planets around \( v \) And

Using the equi-Hamiltonian contour maps, we illustrate the orbital evolution of planets c and d during disk depletion. Figure 2 shows the time evolution of the equi-Hamiltonian map of the \( v \) And system. In these calculations, we adopt the mass ratio \( M_c/M_d = 0.502 \) based on the assumption that they occupy the same orbital plane. The nebula edge retreats from inside to outside (from panel 1 to panel 5). When the nebula edge is at 4.2 AU (panel 1), the orbits librate around \( (x = 10, \eta = 0) \) or \( (x = 0.4, \eta = \pm 3\pi) \). The precession speed of the longitude of planet c’s periastron is slower than that of planet d’s periastron except for the occasional regressions. Embedded within such a disk, a pair of planets with the present observed values of \( x \) and \( \eta \) would not follow a closed librating track. When the nebula edge is
at 5 AU (panel 2), the two eigenfrequencies of the system become closer to each other. The eccentricities of planets modulate with large amplitudes. The magnitude of $x$ librates on a closed equi-Hamiltonian track, about a central point at $x = \gamma_d^{1/2}$. From the estimation of equation (D2), we find that $A_1 = 1 + \Delta N - \gamma_d$ vanishes when the disk is retreated to $r_{\text{edge}} = 5.5$ AU, namely, secular resonance occurs at $r_{\text{edge}} = 5.5$ AU. Panel 3 represents the epoch of secular resonance passage. At this stage, the precession frequencies of the longitudes of periastra nearly match with each other (i.e., $\dot{\eta} = 0$). Consequently, $\eta$ varies slowly (with finite values other than 0 or $\pi$) in comparison with the nonnegligible evolution of $x$ such that the eccentricities can change significantly. The equi-Hamiltonian map is locally symmetric about the lines of $\eta = -\pi$, $-\pi/2$, $\pi/2$, and $\pi$ and that of $x = \gamma_d^{1/2} \sim 2$.

When the nebula edge is beyond 5.5 AU, the precession speed of longitude of planet c’s periastron is faster than that of planet d’s periastron (except for the occasional regressions). Panel 4 shows the case that the nebula edge has retreated to 7 AU. The centers of the closed libration tracks are reversed from the situation in panels 1 and 2. The planetary configuration with $e_c > e_d$ and $\omega_c \sim \omega_d$ in panel 2 becomes that with $e_c < e_d$ and $\omega_c \sim \omega_d$ in panel 4. When the entire protoplanetary disk is depleted (panel 5), stable orbits librate around $(x = 0.5, \eta = 0)$ or $(x = 7, \eta = \pm \pi)$. The shape of the equi-Hamiltonian map is turned upside down with $x = \gamma_d^{1/2}$. Figure 3 is the time evolution of the equi-Hamiltonian map for the uniform-depletion prescription with $r_{\text{edge}} = 4.5$ AU. In this model, the secular resonance occurs when $\Sigma(t)/\Sigma_0(t = 0) = 0.5$. Although the time of the secular resonance passage is slightly modified from that obtained with the model of inside-out depletion, the general evolutionary pattern is not qualitatively changed.

We also numerically integrate the orbits of planets c and d using the same inside-out disk depletion prescription as above (i.e., we arbitrarily set the initial disk edge to be at 4.2 AU and specify its retreating speed to be $10$ AU yr$^{-1}$). We consider two sets of initial conditions. Figures 4a and 4c illustrate the evolution of a model with initial values of $x = 8$ and $\eta = \pi/4$. We illustrate the orbital evolution of planets c and d with the $(x, \eta)$-diagram (see Fig. 4a) and with $e_c$ and $e_d$ as a function of time (see Fig. 4c). These results show that the planets’ orbits initially librate relative to each other. The planets remain on librating closed tracks in the $(x, \eta)$-diagram after the disk is totally depleted. However, as a consequence of the secular resonance passage, the eccentricity of planet c becomes smaller than that of planet d. The time-averaged eccentricities after the disk depletion are $\langle e_c \rangle \simeq 0.12$ and $\langle e_d \rangle \simeq 0.2$.
We also consider a second set of initial conditions with $x = 8$ ($e_c = 0.4$, $e_d = 0.05$) and $\eta = 3\pi/2$. In this case, the planets’ orbits initially circulate relative to each other, but they become trapped in librating orbits during the passage through the secular resonance (see Fig. 4b). The mean eccentricities after the total depletion of the disk are very similar to that obtained in the case of Figure 4a because they are determined by the conservation of angular momentum (eq. [D6]).

We also carried out numerical integration of the full equations with the uniform disk depletion prescription with $t_{AN} = 10^5 \text{ yr}$, $r_{edge} = 4.5 \text{ AU}$. When the same initial condition is used (i.e., $x = 8$ and $\eta = \pi/4$), the pattern of the orbital evolution with this prescription (see Fig. 5a) is not significantly changed from that in Figure 4a. In Figures 5b and 5c we show the evolution from $(x = 8, \eta = \pi)$ and $(x = 5.6, \eta = 0)$. The pair of planets in Figure 4b has an initially wide open circulating track. During the disk depletion, they temporarily enter into closed librating track. However, their orbital configuration becomes wide open again when most of the disk material is depleted. The results of these numerical calculations show that as long as the depletion timescale of the disk is longer than the oscillation period of eccentricity, the planets’ orbits evolve adiabatically. Those systems with librating orbits in the $(x, \eta)$-diagram prior to the disk depletion usually remain on the closed librating tracks after the disk is totally depleted despite large changes in magnitude of $x$ as in the case of $v$ And (see below). Those pairs of planets with the widely open circulating tracks initially generally remain on open circulating tracks after the disk depletion is completed as in the case of HD 168443 (see below). Under some circumstances, it is also possible for planets with marginally circulating/librating initial orbits to undergo transition after the disk depletion.

4.2. Orbital Evolution of Circulating Planets around HD 168443

Next we consider the case of the planetary system around HD 168443 for which we assume the mass ratio to be $M_p/M_\star = 0.451$. Figure 6 shows the time evolution of the equi-Hamiltonian contour map of the HD 168443 system. The nebula edge retreats from inside to outside (from panel
The secular resonance occurs when $\Delta N = -0.86$ with $r_{\text{edge}} = 7.2$ AU. Similar to the case of the planetary system around $\nu$ And, the librating orbit with $|\eta| < \pi/2$ is confined in regions with $e_b > e_c$ prior to disk depletion. However, in the case of the HD 168443 system, the domain of closed librating tracks is small. [In contrast, for small $M_b/M_c(a_b/a_c)^{1/2}$, the domain of open circulating tracks is large as is the case for planets c and d around $\nu$ And.] The domain of closed librating tracks with $|\eta| < \pi/2$ moves downward in the Hamiltonian map as disk depletion proceeds. If planets of HD 168443 initially follow open circulating tracks prior to the disk depletion, they would remain on the open circulating tracks after the disk depletion. In this case, during the onset of secular resonance, planets b and c may briefly attain a librating track before moving onto an open circulating track as the disk continues to deplete. Throughout the epoch of disk depletion, the planets’ longitudes of periastra are widely separated such that little angular momentum may be exchanged between them. Consequently, the net change of eccentricities is limited.

Figure 7 shows the numerically computed orbital evolution. The disk potential is calculated with the uniform depletion prescription in which we specify $t_{\text{AN}} = 10^6$ yr. Figures 7a and 7b represent the first model with $x = 6$ and $\eta = 0$ initially, and Figures 7c and 7d show the second model with $x = 1$ and $\eta = 0$ initially. For the first model, the eccentricity of outer planet c becomes larger than that of planet b after the disk depletion in contrast to the observed eccentricities of planets b and c. With the initial $x$ smaller than $\gamma_d^{-1/2}$, the results for the second model (in Fig. 7c) reproduce the present eccentricities of both planets. [For presentation simplicity, $\gamma_d = (a_b/a_c)^{1/2}(M_b/M_c)$, which represents planets b and c.] The extent of the open circulating track is large both before and after the disk depletion. During the passage through the secular resonance, the planets’ orbits are temporarily trapped in the libration cycles around $\eta = \pi$, but the two planets break free from their librational state to follow a circulation path after the disk is totally depleted.

4.3. Dependence on Disk Model
4.3.1. Single External Disks

The initial eccentricities of planets, prior to the disk depletion, can be inferred from the conservation of the total angular momentum and energy of the planets. We represent the mean eccentricities $(\langle e_c \rangle$ and $\langle e_d \rangle$) of planets c and d around $\nu$ And by their values near the libration center (i.e., with $x = x_0$ and $\eta = 0$). In Figure 8a we plot the values of $\langle e_c \rangle$ and $\langle e_d \rangle$ for various $x_0$ and $r_{\text{edge}}$. The eccentricities $\langle e_c \rangle$ and $\langle e_d \rangle$ are shown by filled circles for six different predepletion disk surface densities ($5 \sin i, 4 \sin i, 3 \sin i, 2 \sin i, 1 \sin i,$ and zero times that of the minimum mass model at $r_{\text{edge}}$ where zero corresponds to the present state). The model with $k = 3/2$ has the same power-law index for the $\Sigma$ distribution as that of the minimum mass nebula model so that these scaling factors for $\Sigma(r_{\text{edge}})$ also represent that for the disk mass.

We also adopt three different predepletion locations of the disk [$\log(r_{\text{edge}}/a_d) = 0.14, 0.2, 0.3$]. The predepletion locations of the edge correspond to 3.53, 4.06, and 5.11 AU, respectively. The apogee distance of planet d [$a_d(1 + e_d)$] is unlikely to exceed the inner edge of the outer disk ($r_{\text{edge}}$) because its eccentricity may be effectively damped and the disk may be strongly perturbed (Papaloizou et al. 2001). From this requirement, we note that $r_{\text{edge}} > 3.5$ AU for the present value of $e_d' = 0.35$. Prior to the disk depletion, it is possible that $e_d$ may be close to zero. Nevertheless, the tidal force of planet d can induce the formation of a gap with a width that is $\Delta r \sim (12)^{1/2}R_{\text{Roche}}$, where $R_{\text{Roche}} = (M_d/3M_*)^{1/2}a_d \sim 0.35a_0$ is its Roche radius (Bryden et al. 1999). Thus, we adopt the minimum value of $r_{\text{edge}} \sim a_d + \Delta r \approx 3.5$ AU (see top panel of Fig. 8a).

The solid line shows an ellipse that is delineated by the conservation of total angular momentum (see eq. [D6] in Appendix D). To second order in eccentricity, the conservation of the total angular momentum (eq. [D4] in Appendix D) implies that

$$e_d' - e_c' = M_c \sin i \sqrt{a_c} \quad e_c' - e_d' = M_d \sin i \sqrt{a_d} = \gamma_d,$$

where $e_d'$ and $e_c'$ are the predepletion eccentricities of planets c and d, respectively. The value of $\gamma_d \sim 0.286$ is obtained in the case of planets around $\nu$ And. A maximum value of $e_d' = 0.37$ is attained for a minimum value of $e_c' = 0$. However, our results in Figure 4 indicate that $x$ decreases during the disk depletion so that $e_c' > e_c$ and the maximum value of $e_d'$ is unattainable.

When the disk edge is relatively close to the planet (top panel), $e_c' > 0.65$ and $e_d' < 0.1$ if the predepletion disk mass
is greater than \(2 \sin t\) times that of the minimum mass nebula model. But for larger values of \(r_{\text{edge}}\), the predepletion eccentricities approach their present values as the contribution of the disk mass to the total gravity becomes vanishingly small.

The secular resonance occurs at \(x_m = \frac{a}{C^2}\) in the case of \(v\ And.\) As long as there was the disk that corresponds to a predepletion \(e_c > 0.51\), the planets pass through the secular resonance. Note that with an arbitrarily large disk mass, \(e'_d\) vanishes while \(e'_c\) attains a maximum value of 0.69. Even under these extreme conditions, the orbits of the two planets do not cross and remain stable. During the disk depletion, the eccentricities of planets c and d coincide with each other at about 0.33. For the predepletion eccentricity of planet c to be greater than 0.33 (which is generally the case with the exception of the three least massive initial disk models), the inversion of the eccentricity ratio between planets c and d around \(v\ And.\) occurs.

Because the influence of the disk on the planet is mainly due to the gas near \(r_{\text{disk.}}\) the above results are insensitive to changes to the surface density distribution at large radii. In Figure 8b we set \(r_{\text{edge}} = 4.057\) AU \([\log(r_{\text{edge}}/a) = 0.2]\) and considered three sets of \(\Sigma\) distributions with \(k = 2, 1,\) and \(\frac{1}{2}\) (Fig. 8b). We adopt six sets of values of the surface density \(\Sigma(r_{\text{edge}})\) of each model at \(r_{\text{edge}}\). For a disk with a similar surface density scaling factor, there are no significant differences between the three panels. The detailed disk structure is not important for the evolution of the planetary eccentricities in our model.

### 4.3.2. Interplanetary Rings and Outer Disks

After the emergence of multiple planets, gap formation effectively cuts off gas flow from the external disk across the outermost planet’s orbit. Since both dynamical and viscous timescales are a rapidly increasing function of the disk radius, the disk interior to the gap is depleted rapidly (Lin & Papaloizou 1986b). As the result of a shepherding effect induced by the multiple planets (Goldreich & Tremaine 1979), the gaseous interplanetary rings may be preserved for a much longer timescale than the internal disk. Nevertheless, the dissipation of the competing tidal perturbations by the planets can lead to ring dispersal well before the depletion of the external disk (Bryden et al. 1999).

In the present context, the evolution of the eccentricity ratio \((x_{n0})\) at the libration center around \(\eta = 0\) depends intricately on the depletion pattern of both the ring and disk as well as the locations of the edges of the ring and disk. For example, the value of \(x_{n0}\) is smaller for interplanetary rings with inner edge \(d_{\text{in}}\) closer to the inner planet. In contrast, the value of \(x_{n0}\) is larger for interplanetary rings with outer edge \(d_{\text{out}}\) closer to the outer planet. The balance between these
effects determines the evolution of the eccentricity ratio. In order to illustrate these dependencies, we consider the epoch when planets c and d around \( \nu \) And were surrounded by an interplanetary ring and an external disk. We adopt a surface density distribution that is \( 5 \sin i \) times that of the minimum mass solar model. A hole is cut out interior to \( d_{\text{in}} \). An annular strip is also removed between \( d_{\text{in}} \) and \( r_{\text{edge}} \). By fixing the inner edge of the external disk to be \( r_{\text{edge}} = 4 \) AU, we compute the value of \( x_{m0} \) as a function of \( d_{\text{in}} \) (see Fig. 9). In these calculations, we used eight values of the outer edge of the ring \( d_{\text{out}} \), ranging from 1.2 to 1.9 AU in intervals of 0.1 AU.

The results in Figure 9 indicate that when \( d_{\text{out}} < 1.66 \) AU, the gravitational perturbation of the ring on the inner planet always surpasses that on the outer planet. Consequently, the value of \( x_{m0} \) is reduced below that of \( x_{m0} \) in the ringless cases. \( x_{m0} \) approaches 12 for relatively large \( d_{\text{in}} \) as the ring’s influence weakens. However, in the limit \( d_{\text{out}} > 1.66 \) AU, \( x_{m0} \) becomes larger than \( x_{\text{disk}} \) for relatively large \( d_{\text{in}} \). For example, when \( d_{\text{out}} = 1.8 \) AU, \( x_{m0} \) becomes larger than \( x_{\text{disk}} \) at \( d_{\text{in}} > 1.5 \) AU because the ring’s perturbation on the outer planet is enhanced. Also in this large \( d_{\text{out}} \) limit, \( x_{m0} \) becomes smaller than the current observed eccentricity ratio of 0.66 at \( d_{\text{in}} < 1.33 \) AU. For all values of \( d_{\text{out}} \), the predepletion values of \( x_{m0} \) in this model are always larger than the current observed eccentricity ratio.

In Figure 10 we show the evolution of equi-Hamiltonian contours in a model with both a ring between planets c and d and an external disk beyond planet d. We adopt a prescription in which the ring is depleted uniformly prior to the uniform depletion of the external disk. For our model, we specify \( \log(d_{m}/a_{c}) = \log(a_{d}/d_{\text{out}}) = 0.2 \) for the ring and \( \log(r_{\text{edge}}/a_{d}) = 0.2 \) for the external disk. Our numerical results indicate that \( x_{m0} \) is initially smaller than the currently observed eccentricity ratio (panel 1). The ring proceeds to be depleted during the epochs represented by panels 2 and 3. When the ring mass is reduced to half of its original value, \( x_{m0} \) becomes \( \sim 2 \) (panel 2); i.e., the system passes through a state of secular resonance. The entire ring is depleted while the external disk is preserved during the epoch represented by panel 3. After that, the contours again evolve through a
state of secular resonance (when \( \frac{1}{3} \) of the external disk is depleted in panel 4) to current values (in panel 5) as in the case of Figure 3.

Although the same pattern of evolution occurs in the case of planets around HD 168443, there is little change in the eccentricities of either planet since the contour of their eccentricity ratio always retains the values around \( \frac{C_1}{C_0} = 2 \). We realize that the evolution of the interplanetary ring is determined by the process of gap formation and planet-disk tidal interaction; its influence on the eccentricity evolution of the planets should be analyzed along with its nonaxisymmetric contributions. The results of such analyses will be presented elsewhere.

5. IMPLICATIONS ON THE PROTOPLANETS’ ORBITAL ECCENTRICITY PRIOR TO THE DEPLETION OF THEIR NASCENT DISKS

5.1. Planets around \( \upsilon \) And

First, we consider the evolution of planets c and d around \( \upsilon \) And. We infer from their present librating orbits and the results in Figure 2 that these planets followed a close librating track prior to the disk depletion. Our results also indicate that when the disk dominated the planets’ apsidal precession, the domain of close librating tracks with \( |\eta| < \pi/2 \) resided in the parameter region with \( e_c > e_d \). Thus, the present observed librating orbits of planets c and d would be attainable if \( e_c > e_d \) prior to the disk depletion. Since \( e_c < e_d \) today, the passage through the secular resonance must have led to the inversion of this ratio. In Figure 11a we map out, in the \( (x, \eta) \) plane, the most likely initial domain that may have led to the present planetary orbits in the \( \upsilon \) And system. The periapsides of those pairs of planets that initially occupied region 1 maintain their near alignment despite the large changes in \( x \) during the disk depletion. These systems enter the domain of closed librating tracks in the \( (x, \eta) \)-plane after the disk is completely depleted. Their orbits become very similar to the present orbits of planets c and d. Planets that initially occupy region 2 have open circulating tracks prior to the disk depletion. Among those pairs of planets that originated from both regions 1 and 2.
(\(x > \gamma_d^{-1/2} \sim 2\)), the eccentricity ratio is reduced after the disk depletion. Thus, we infer that those pairs of planets that currently occupy the region \(x (= e_c/e_d) \leq 1\) originate from the domain \(x \approx 0.5\) prior to the disk depletion. Those pairs of planets with \(x < 2\) initially cannot attain the present observed values of \(x \sim 0.6\) because the magnitude of their \(x\) increases during the disk depletion.

In the absence of a strong nonaxisymmetric torque associated with the planet-disk interaction, secular interaction between the planets does not alter the total angular momentum and energy of the system. In contrast to the eccentricity ratio \((x)\) between planets c and d, the magnitude of eccentricities cannot be excited by the sweeping secular resonances alone. Equation (17) indicates that if both planets had nearly circular orbits initially, their eccentricity would remain small after the disk is completely depleted. Thus, around \(v\) And, at least planet c must have acquired some initial eccentricity prior to the epoch of planet formation.

Up to now, we have neglected the effect of nonaxisymmetric torque associated with the planet-disk interaction. During their formation, the eccentricities of isolated protoplanets are damped and excited as a result of their interaction with their nascent disks at the corotation and Lindblad resonances, respectively. For protoplanets with masses comparable to that of Jupiter, the damping effect of the corotation resonances is stronger than the excitation effect of the Lindblad resonances (Goldreich & Tremaine 1980). However, protoplanets with masses an order of magnitude larger than that of Jupiter may open relatively wide gaps in protostellar disks. In this limit, the protoplanets’ corotation resonances may be cleared of disk gas such that their eccentricity may be excited (Artymowicz 1993; Papaloizou et al. 2001). For modest inclination angle between their orbital plane and the line of sight, neither planet c nor planet d has sufficiently large mass to clear a wide gap. However, planet c is likely to form prior to planet d. The tidal torque of planet d may truncate the disk well beyond the region that contains any corotation resonance for planet c so that the latter’s eccentricity is excited whereas that of planet d is damped.

The actual mass of planets c and d depends on the value of \(\sin i\). If \(i < 20^\circ\), the mass of planet d would be sufficiently large for its interaction with the disk to excite eccentricity. The constraint on the inclination angle inferred from the dynamical stability of the planetary system around \(v\) And indicates that \(i > 12^\circ–45^\circ\) (Lissauer & Rivera 2001; Ito & Miyama 2001). In a multiple system, however, the rate of angular momentum transfer between planet d and the outer disk may be faster than that between it and planet c. We shall consider in a future contribution whether planet c will be able to attain a larger initial eccentricity.

It is also possible that the planets around \(v\) And were closer to each other in the past and dynamical instability may have led them to undergo orbit crossing (e.g., Chambers, Wetherill, & Boss 1996). Rasio et al. (1996) suggested that such a process may have led to the scattering of a planet close to the surface of its host star and subsequent tidal circularization may have caused it to become a short-period planet. Around \(v\) And, regardless of whether such a process can lead to the formation and survival of the short-period planet b, planetary scattering may also have excited the eccentricity and modified the semimajor axes of both planets c and d (Weidenschilling & Marzari 1996; Lin & Ida 1997). During such encounters, planets with lower mass generally acquire larger eccentricity as a result of the conservation of the system’s total energy and angular momentum. Although these speculations provide arguments for planet c to have initially acquired more eccentricity than planet d, rigorous analysis is needed to quantitatively verify these conjectures.

5.2. Planets around HD 168443

In the case of planets around HD 168443, the present orbit follows an open circulating track. In Figure 11b we outline the necessary initial domain in the \((x, \eta)\)-diagram, which would lead planets b and c to acquire their present orbital parameters. Planets that were initially in region 2 would enter into open circulating tracks after their nascent disk is completely depleted. However, their eccentricity ratio would not generally match those inferred from the observations of HD 168443. The most likely initial domain of planets around HD 168443 (with their current dynamical properties) is mapped out as region 1. During the passage through secular resonance, the eccentricity of planet c decreases while that of planet b increases slightly. The geometry of the \((x, \eta)\)-diagram is reversed during the passage through the secular resonances, i.e., at the point \(x = \gamma_d^{-1/2}\). This small change is due to the present value of \(x\) being close to that of \(\gamma_d^{-1/2}\) in the HD 168443 system. In contrast, the present value of \(x\) of planets around \(v\) And is much smaller than \(\gamma_d^{-1/2}\).

5.3. Other Multiple-Planet Systems

We comment here about another system of planets around HD 74156 (Table 1C). In this system, the outer planet is more massive than the inner planet. Since their orbits are circulating (similar to those of the planets around HD 168443), the evolution of this system during the disk depletion would be very similar to that described in § 4.2 for the planets around HD 168443. However, the present value of \(x\) is not close to that of \(\gamma_d^{-1/2}\). The inversion of the eccentricity ratio is expected to occur in the case of planets around HD 74156 (similar to those of the planets around \(v\) And).

Our own solar system also contains multiple gaseous giant planets. In this case, the inner planet, Jupiter, is more massive than the outer planet, Saturn. Their longitudes of periastron are not aligned with each other. The ratio of the jovian eccentricity to the Saturnian eccentricity \((e_J/e_S)\) varies between 0.3 and 5 with time. In this case, the orbits of Jupiter and Saturn do not pass through the secular resonance during the depletion of the primordial solar nebula. Therefore, the initial orbits attained by Jupiter and Saturn during their formation would not be significantly changed as a consequence of the disk depletion. This evolution pattern is similar to that of the two-planet system around 47 Uma. Those two planets have masses 2.0 and 0.67 \(M_J\) and semimajor axes 2.1 and 3.8 AU, respectively (Table 1D).

In general, in those systems where the outer planets are more massive than the inner planets (such as the planetary systems around \(v\) And, HD 168443, and HD 74156), the depletion of the outer disks induces a secular resonance that sweeps through both planets. However, in the opposite situation where the inner planets are more massive (more precisely, where \(\gamma_d^{-1/2} < 1\)) such as the planetary systems around the Sun and 47 Uma, the depletion of the disk does not lead to the secular resonances between the planets.
Thus, the changes of the eccentricities are not so large. The planets around HD 12661 (Table 1E) have librating orbits with \( \eta \sim \pi \). During the disk depletion, the secular resonance passes through this system (i.e., \( x_m \) attains a value of \( \gamma_d^{-1/2} \)) because its \( \gamma_d^{-1/2} > 1 \) (the inner planet is more massive than the outer planet). Today, the value of \( x_m \) around the libration centers at \( \pi \) is slightly larger than \( \gamma_d^{-1/2} \). Thus, the pre-depletion value of \( x_m \) must be smaller than \( \gamma_d^{-1/2} \). Nevertheless, the magnitude of the eccentricity change is relatively small because the present value of \( x_m \) is close to that of \( \gamma_d^{-1/2} \).

6. SUMMARY AND DISCUSSIONS

In this paper we consider the secular interaction between pairs of coplanar planets. Provided that these planets are not locked in a mean motion resonance, their secular interaction leads to angular momentum exchange without energy transfer between them. Consequently, both planets undergo apsidal precession and eccentricity modulation about some mean values. Thus, dynamical evolution of these systems of planets is best illustrated by the ratio of their eccentricity \( x \) and the relative longitude of their periastron \( \eta \). For example, based on their observed orbital elements, we find that gravitational perturbation between the outer two planets around \( \nu \) And causes both \( \eta \) and \( x \) to modulate, with small amplitude, along a close librating track around \( \eta = 0 \) and \( x \sim 0.66 \) (see Fig. 1).

Although the mean eccentricities of these planets are constant today, they may have undergone significant changes during the epoch when their nascent disks were depleted. In order to extrapolate the initial dynamical properties of planetary systems, we consider the precession caused by the self-gravity of these disks. We show that contribution from the disks initially dominates the precession frequency of the planets and regulates the angular momentum exchange rate between them. As their influence fades during the disks’ depletion, the precession eigenfrequencies of the planets’ longitude of periastron pass through a state of secular resonance.

As a result of this transition, the magnitude of both planets’ eccentricities changes with their ratio inverted. However, provided that the disk depletion proceeds on a timescale long compared with the precession frequencies associated with the secular interaction between the planets, the libration amplitude of \( \eta \) is not significantly changed as a result of the preservation of an adiabatic invariance. Thus, if a pair of planets started out with a close librating orbit prior to the disk depletion, they would generally retain the librating nature of their secular interaction after the disk depletion. Planets with open circulating tracks initially would generally evolve to attain other open circulating tracks after the disk depletion. However, the circulating track may temporarily enter into closed librating track near the secular resonance. Planets with marginally circulating orbits (close to librating orbits) prior to the disk depletion may sometimes attain librating orbits after the disk depletion provided that the difference between the libration and resonant centers (i.e., \( [x_m - \gamma_d^{-1/2}] \)) is reduced during the transition. Inversely, planets with marginally librating orbits prior to the disk depletion may also sometimes attain circulating orbits after the disk depletion provided that the difference between the libration and resonant centers is increased.

Using these results, we infer the kinematic properties of planetary systems around both \( \nu \) And and HD 168443. During the passage of the secular resonances, the libration timescales of the planets around these two stars are \( 2 \times 10^4 \) and \( 10^3 \) yr, respectively. Provided that the external axisymmetric disk is depleted on a timescale longer than these libration timescales, transition through the secular resonance is adiabatic.

In this contribution, we show how the process of disk depletion may lead to significant modifications in the eccentricity distribution within some extrasolar multiple planetary systems. These changes are the result of secular perturbation that the planets exerted on each other. However, planetary secular perturbation and the axisymmetric potential of evolving disks alone do not lead to eccentricity excitation. We have not quantitatively addressed the issue of initial eccentricity excitation among isolated planets or at least one planet in a multiple planetary system. We will discuss these problems in the future.

M. N. would like to thank R. Mardling for her helpful suggestions. M. N. used the parallel computer (Silicon Graphics Origin 2000) of Earthquake Information Center of Earthquake Research Institute, University of Tokyo. This work was partly performed while M. N. held a National Research Council Associateship at NASA Ames. This work is supported in part by NASA through grant NAG 5-10727 and NSF through AST 99-87417 to D. N. C. L.

APPENDIX A

PRECESSION DUE TO THE DISKS’ GRAVITY

In this section we evaluate \( \Delta N \) in equation (9). After protoplanets have acquired a sufficiently large mass, they induce the formation of a gap near the orbit through tidal interaction with the disk (Goldreich & Tremaine 1980; Lin & Papaloizou 1980, 1993). While the disk material can no longer be rapidly accreted onto the protoplanet (Bryden et al. 1999), it continues to contribute to the gravitational potential. Although the disk does not have a sufficiently large mass to modify the angular frequency of the planet’s orbit, its gravity can induce precession (Ward 1981). The disk’s contribution to the gravitational force in the radial direction at \( r \) is

\[
F_r(r) = - \frac{d \Psi}{dr} = -2G \int_0^\infty y \Sigma(y) \left[ \frac{E(\psi)}{1 - y} + \frac{K(\psi)}{1 + y} \right] dy,
\]

where \( \Sigma \) is the surface density of the disk, \( \Psi \) is the potential, \( y \equiv r'/r \), \( \psi \equiv 2y^{1/2}/(1 + y) \), \( K(\psi) \) and \( E(\psi) \) are elliptic integrals of
first and second kind, and \( G \) is the gravitational constant (eq. [2-146] in Binney & Tremaine 1987). The associated precession frequencies for planets c and d can be obtained from

\[
N_{c,d} = (\Omega_{c,d} - \kappa_{c,d})t_c = \frac{n_{c,d} t_c}{2GM_\ast} \left( \frac{\partial}{\partial r} r^3 F_r \right)_{r = a_{c,d}}, \tag{A2}
\]

where \( \Omega_{c,d} \) and \( \kappa_{c,d} \) are, respectively, the angular and epicyclic frequencies at semimajor axis of planets c and d \((a_{c,d})\). Depending on the \( \Sigma \) distribution, \( N_{c,d} \sim O(\epsilon_{c,d} t, M_D / M_\ast) \), where \( M_D = \Sigma a_{c,d}^2 \) is the characteristic disk mass at \( r = a_{c,d} \). From equation (A2), we find

\[
\Delta N = N_c - N_d = \frac{t_c}{2GM_\ast} \left[ n_c \left( \frac{\partial}{\partial r} r^3 F_r \right)_{r = a_c} - n_d \left( \frac{\partial}{\partial r} r^3 F_r \right)_{r = a_d} \right], \tag{A3}
\]

with its sign determined by the \( \Sigma \) distribution. For disks with finite \( \Sigma \) only outside the planets’ orbit, the addition of the disk self-gravity leads to both \( N_c \) and \( N_d \) being positive. Since planet d is closer to the external disk, \( N_d > N_c \), so that \( \Delta N \) is negative. For disks that engulf both planets’ orbits, \( \Delta N \) can still be negative provided that \( \Sigma \) does not decrease too rapidly with \( r \). For example, if we use the minimum mass nebula model (Hayashi et al. 1985) as a fiducial prescription for \( \Sigma \) such that

\[
\Sigma = 1.7 \times 10^3 \left( \frac{r}{1 \text{ AU}} \right)^{-1.5} \text{ g cm}^{-2}, \tag{A4}
\]

\( \Delta N \) is negative.

APPENDIX B

ORBITS OF LIBRATING AND CIRCULATING PLANETS

Using linearized equations (8) and (9), we consider here three families of orbits in systems that contain two planets. We assume that the two planets orbit on the same plane as the disk.

B1. THE ORBITS OF LIBRATING PLANETS

As an example, we consider planets c and d around \( \nu \) And. For planets c and d around \( \nu \) And, \( C = -0.4 \), \( \gamma_d = 0.29 \). The longitudes of these two planets are aligned such that their \( \eta \approx 0 \) today. They have the present values of \( x = 0.66 \) and \( t_c = 1.1 \times 10^3 \text{ yr} \). With these values, we show below that secular interaction between the two planets induces \( x \) and \( \eta \) to undergo small-amplitude oscillation around stationary points. These points correspond to extrema for \( x \) and \( \eta \). From the linearized equations (8) and (9) of secular motion, we find that these extrema occur at \( \eta = 0 \) and \( \pi \).

For \( \eta = 0 \), a stationary solution can be obtained for \( x = x_m(> 0) \), where \( x_m \) is a root of the quadratic equation

\[
x_m^2 - \left( \frac{\Delta N + 1 - \gamma_d}{\gamma_d C} \right)x_m - \frac{1}{\gamma_d} = 0. \tag{B1}
\]

With \( x = x_m \) and \( \eta = 0 \), longitudes of the periapsis of both planets are aligned and precess together, even though \( \eta_1 \neq \eta_2 \). Consequently, there is no angular momentum exchange and the eccentricities of both planets are conserved. For solutions near the stationary point with \( \eta = \pi \), \( x_m \) can be obtained from a similar equation where

\[
x_m^2 + \left( \frac{\Delta N + 1 - \gamma_d}{\gamma_d C} \right)x_m - \frac{1}{\gamma_d} = 0. \tag{B2}
\]

In the \((x, \eta)\)-diagram, these orbits follow the equi-Hamiltonian contours (see Appendix D). Angular momentum is being continually exchanged between the planets, leading to an eccentricity modulation. However, the libration is centered around \( \eta = 0 \) or \( \pm \pi \) where there is no net torque between the two planets. Although the range of \( \eta \) modulation \( (\Delta \eta) \) is limited, the two planets are not in the center of secular resonance because \( \eta \) modulates with a frequency \( \omega \) that does not vanish with arbitrarily small \( \Delta \eta \).

We now consider a small perturbation about the stationary point. For small values of \( \eta \) and \( \epsilon \equiv x/x_m - 1 \), equations (8) and (9) can be linearized in \( \eta \) and \( \epsilon \) such that

\[
\frac{d\epsilon}{d\tau} = \frac{C}{x_m \left( 1 + \gamma_d x_m^2 \right)} \eta, \tag{B3}
\]

\[
\frac{d\eta}{d\tau} = -\frac{C}{x_m \left( 1 + \gamma_d x_m^2 \right)} \epsilon. \tag{B4}
\]

Combining these two equations, we find the solution

\[
\epsilon = \epsilon_0 \sin(\omega \tau), \quad \eta = \eta_0 \cos(\omega \tau), \quad \text{with} \quad \eta_0 = \pm \epsilon_0, \tag{B5}
\]
where the oscillation frequency is

\[ \omega = \pm \frac{C}{x_m} (1 + \gamma_d^2 x_m^2) \, , \]  

(B6)

where the signs of the expression for \( \eta_0 \) and \( \omega \) are the same. Near the stationary point \( x = x_m \) and \( \eta = 0 \), \( \omega \) is also approximately the difference of the precession frequencies of the two planets because \( \pi \) for both precess in the same direction.

### B2. ORBIT OF CIRCULATING PLANETS

As another example, we use the planets around HD 168443. For planets b and c around HD 168443, \( C = -0.13 \), \( \gamma_d = 0.14 \), and \( t_c = 1.9 \times 10^3 \) yr. For notational consistency, we use \( c \) for the subscription for the inner planet and \( d \) for that for the outer planet. With these parameters, \( \Delta N = 0 \) for \( x_m = 0.15 \) (or 46) and \( \eta = 0 \) (or \( \pi \)). The observed value is \( x = 2.65 \), \( \eta = 1.92 \), which is far from the stationary point \( x = x_m \), \( \eta = 0 \) (or \( \pi \)). With these parameters, \( |A_1| > |A_2| \) (see eqs. [7] and [9]), we expect the orbits of planets b and c to circulate with respect to each other. In this case, the linearized equations (B3) and (B4) are no longer applicable. The small magnitude of \( C \) in equation (8) implies that the modulation amplitude of \( x \) is small compared with its average value. The dominance of the term of \( 1 - \gamma_d + \Delta N \) in equation (9) implies that, to zeroth order in \( \tau \), \( x \simeq x_0 \) and \( \eta \simeq (1 - \gamma_d) \tau \), where \( x_0 \) is a constant. Expansion to the next order in \( \tau \) yields the solution that

\[ \eta \simeq (1 - \gamma_d + \Delta N) \tau + \frac{C}{x_0} \left( 1 - (1 - \gamma_d) x_0^2 \right) \sin(1 - \gamma_d + \Delta N) \tau \, , \]  

(B7)

\[ x \simeq x_0 - \frac{C}{1 - \gamma_d + \Delta N} \cos(1 - \gamma_d + \Delta N) \tau \, , \]  

(B8)

so that the circulation period is \( t_c \simeq 2\pi t_c / (1 - \gamma_d + \Delta N) \sim 1.4 \times 10^4 \) yr for \( \Delta N = 0 \).

### B3. NONPERIODIC ORBITS NEAR SECULAR RESONANCE

In addition to the librating and circulating orbits, there is another family of nonperiodic orbits for the case in which \( |A_1| \leq |A_2| \). In the limit that both \( x \) and \( \eta \) have values that are very different from \( x_m \), and \( \eta = 0 \) (or \( \pi \)), \( \eta \) would evolve rapidly to a phase angle

\[ \eta \simeq \cos^{-1} \left( -\frac{A_1}{A_2} \right) = \cos^{-1} \left[ \frac{-A_1 x}{C(1 - \gamma_d x^2)} \right] \, , \]  

(B9)

such that \( d\eta/d\tau \) is, to within the first-order approximation, reduced to zero (see eq. [9]) and the two planets become phase locked. Within the same order approximation, \( x \) becomes \( x_1 \) and the evolution of \( x \) is determined by

\[ \frac{dx}{d\tau} = B_1 \eta_2 + B_2 x_2 \, , \]  

(B11)

\[ \frac{d\eta_2}{d\tau} = B_0 (\tau - \tau_0) + B_3 x_2 + B_4 \eta_2 \, , \]  

(B12)

The coefficients are

\[ B_0 = \Delta N \, , \quad B_1 = C(1 + \gamma_d x_1^2) \cos \eta_1 \, , \]  

\[ B_2 = 2C \gamma_d x_1 \sin \eta_1 \, , \quad B_3 = -\left( \frac{C}{x_1} \right) (1 + \gamma_d x_1^2) \cos \eta_1 \, , \]  

\[ B_4 = -\left( \frac{C}{x_1} \right) (1 - \gamma_d x_1^2) \sin \eta_1 \, , \]  

(B13)

where \( \Delta N = d\Delta N/d\tau \) and the fiducial dimensionless time \( \tau_0 \) corresponds to the instance when the two planets first enter into a
phase lock, i.e., when their \( x = x_1 \) and \( \eta = \eta_1 \). The solution of equations (B11) and (B12) can be expressed as

\[
x_2 = x_{20} \exp[\omega_2(\tau - \tau_0)] + x_{21} \exp[\omega_2(\tau - \tau_0)] + B_5 \tau + B_6,
\]

\[
\eta_2 = \frac{1}{B_1} [(\omega_2 - B_2)x_{20} \exp[\omega_2(\tau - \tau_0)] + (\omega_2 - B_2)x_{21} \exp[\omega_2(\tau - \tau_0)] + \frac{1}{B_1} B_5(1 - B_2 \tau) - B_2 B_6],
\]

where \( x_{20} \) and \( x_{21} \) are the amplitudes of oscillations and

\[
B_5 \equiv \frac{B_0 B_{11} - B_1 B_5}{B_2 B_4 - B_1 B_5}, \quad B_6 \equiv \frac{(B_2 + B_4) B_5 - B_0 B_{11} \tau_0}{B_2 B_4 - B_1 B_5},
\]

with the growth rate

\[
\omega_2 = \omega_{20,21} = \frac{C}{2 x_1} [(3 \gamma_d x_1^2 - 1) \sin \eta_1 \pm (1 + \gamma_d x_1^2)(5 \sin^2 \eta_1 - 4)^{1/2}] .
\]

Note that when \( \eta_1 \) pass through the value \( \sin^{-1}(4/\sqrt{5}) \), \( \omega_20 = \omega_{21} \) and the solution of equations (B11) and (B12) becomes slightly modified from that in equation (B14). Also note that for \( \eta_1 = 0 \) or \( \pi \), equations (B11) and (B12) reduce to equations (B3) and (B4), respectively. In addition, \( \omega_2 \) becomes purely imaginary \((\pm i \omega)\), so all the orbits undergo libration about the stationary point. However, for a more general value of \( \eta_1 \), \( \omega_2 \) is complex, and oscillations about the evolving \( x_1 \) values either grow or decay depending on the value of \( \eta_1 \). At the center of resonance where \( A_1 = 0 \) and \( \eta_1 = \pi/2 \), \( \omega_2 = 2 C \gamma_d x_1 \) or \((C/x_1)(\gamma_d x_1^2 - 1) \). Both roots are real so that the \( x_2 \) either converge or diverge (if \( x_1 > \gamma^{-1/2} \)) without any oscillation. Note that oscillation about \( x_1 \) and \( \eta_1 \) only occurs if the magnitude of \( \omega_2 \) is larger than \( B_5 \) such that \( B_5 \) changes in \( \Delta N \) and \( x_1 \) occur in an adiabatic manner. For further discussions on the evolution of planetary orbits during the epoch of disk depletion, see Appendix C.

**APPENDIX C**

**THE EVOLUTION OF MULTIPLE-PLANET SYSTEMS DURING THE EPOCH OF DISK DEPLETION**

During the depletion of the disk, changes in the magnitude of \( \Delta N \) can cause \( A_1 \) to change sign. In § 4 we presented numerical results of orbital evolution during the epoch of disk depletion. Here we present analytical solutions to interpret the numerical results. Following the discussions in Appendix B, we consider three initial orbits separately.

**C1. THE EVOLUTION OF TWO-PLANET SYSTEMS WITH INITIALLY LIBRATING ORBITS**

In § B1 we show that the condition for libration is \(|A_1| \leq |A_2| \) and \( \epsilon_0 \ll 1 \). For planets c and d around \( \nu \) And, \( \Delta N = 0 \) today so that \( x_m \sim -C/(1 - \gamma_d) \sim 0.5 \) and \( \omega \sim 0.83 \). Since the observed values of \( x \) and \( \eta \) are close to those of \( x_m \) and 0, respectively, the small-amplitude oscillator approximation is valid. The necessary condition for libration, i.e., \(|A_1| \leq |A_2| \), is marginally satisfied. From equation (12), we find that the timescale for one complete libration cycle today is \( \tau = \tau_{lib} \sim 2 \pi \omega / \omega = 8.5 \times 10^3 \) yr.

Prior to its depletion, 5 times the minimum mass nebula model would lead to \( M_D/M_J = \Sigma (a_J^2)/M_J \sim 0.015 \) and \( \Delta N \sim 1.4 \) (eqs. [A3] and [D2]). In this case, the necessary condition for libration is also marginally satisfied and it is possible to find a real positive solution to the quadratic equation (B1). Around the stationary point \( x_m \sim 6.5 \), there are also solutions that correspond to small-amplitude libration between planets c and d \((|\eta| \leq \pi/2)\) despite the rapid precession induced by the disk’s self-gravity. The corresponding \( \omega \sim 0.80 \) for such a protoplanetary system is very similar to the present disk-free value of \( x_m \).

During the depletion of the protostellar disk, the magnitude of \( \Delta N \) gradually vanishes so that both \( x_m \) and \( \omega \) change with time. For \(-1.5 < \Delta N < 0 \), the condition for libration \(|A_1| \leq |A_2| \) is satisfied. Provided that \( \tau_{lib} < \tau_{A_1} \), \( \tau_{lib} < \tau_\omega \), and \( \tau_{lib} < \tau_{x_m} \), where

\[
\tau_{lib} \equiv \frac{2 \pi \omega}{C(1 + \gamma_d x_1^2)} , \quad \tau_{A_1} \equiv \frac{A_1}{A_1} = \frac{1 - \gamma_d + \Delta N}{\Delta N} , \quad \tau_{x_m} \equiv \frac{x_m - x_m}{x_m} , \quad \tau_\omega \equiv \frac{\omega}{\omega} \frac{(1 + \gamma_d x_1^2)^2}{(1 - \gamma_d x_1^2)^2} \tau_{A_1} ,
\]

the libration of the planets’ orbits is preserved while the librational center and frequency evolve adiabatically.

At the special phase when \( A_1 = 0 \), the second term in equations (B1) and (B2) reduces to zero and the stationary points with \( \eta = 0 \) and \( \eta = \pi \) coincide with each other with \( x_m \equiv \gamma_d^{-1/2} \). We refer to this special configuration as librational degeneracy. In this state, all orbits librate either around \( x_m \equiv \gamma_d^{-1/2} \), \( \eta = 0 \) or \( \pi \) with two separatrices located on \( \eta = \pi/2 \) and \( 3\pi/2 \) (see Hamiltonian contour maps). From equation (6), we note that in this degenerate state \( \Delta \eta_1 \) attains a minimum value with respect to variations in \( \Delta N \). The oscillation frequency around the stationary points \( \omega \) attains a minimum magnitude \( 2\gamma_d^{1/2} C \) with \( x_m = \gamma_d^{-1/2} \) (see eq. [12]), which is \(-1.8 \) for planets c and d around \( \nu \) And.

Note that in this state of librational degeneracy, \( \eta_1 \neq \eta_2 \) in general. However, if planet c has negligible mass, \( \Delta \eta_1 \) would vanish and would enter into a state of exact secular resonance and \( x_m \) would diverge. This asymptotic state represents the eccentricity excitation of asteroids by the Jovian planets. For finite values of \( \gamma_d \), this state of librational degeneracy can also be
considered as a state of secular resonance despite the finite value of $\Delta g_{12}$. The secular interaction between the planets forces them to librate so that their relative longitude of periapsis passage is confined to a limited amplitude.

We now consider the evolution of the librational amplitude. The action associated with the simple oscillator equations (B3) and (B4),

$$J = \int \dot{e} \, d\epsilon = e^2 \omega \int_0^{2\pi} \cos^2 \theta \, d\theta = \pi e^2 \omega^2,$$

(C3)
is invariant to first order in $\tau_{ib}/\tau_A$ or $\tau_{ib}/\tau_c$ (Goldstein 1980), whichever is larger. (For values of $\tau_A$ and $\tau_c$, see eqs. [C1] and [C2].) Based on this invariance, we can also determine the changes in $\epsilon_0$, which is $\infty \omega^{-1/2}$. For planets $c$ and $d$ around $\nu$ And, we adopted a model in which $\omega \sim 0.8$ both during the embedded phases and after the disk is completely depleted. Thus, we expect both $e_0$ and $\eta_0$ to remain invariant despite large changes in $x_{\nu m}$. The numerical results in § 4.1 are in agreement with this analytic derivation.

However, the condition for adiabatic evolution may not always be satisfied especially when the two planets pass through a phase of secular resonance with $x_{\nu m} = \gamma_1^{-1/2}$ when the libration period takes the maximum value $t_l = -\pi t_d/\left(C_4 x_1^{-1} \right) = 1.7 \times 10^4$ yr and $\omega$ attains a minimum value of $\pm 2 \gamma_1^{1/2} = \pm 0.43$ (compared with its initial value of $\sim 0.8$). During this phase, the libration timescale $\tau_{ib}$ reaches a maximum. Although at the center of secular resonance $\tau_{ib}$ vanishes with $A_1$, the more appropriate value for $\tau_{ib}$ to be preserved is $\tau_{ib}$ when $\omega$ is $\infty \omega^{-1/2}$. Nevertheless, it is possible for $\tau_{ib}$ to be $\tau_{ib}$ if $\omega$ is $\infty \omega^{-1/2}$. Note that $\tau_{ib}$ reduces to $\pm 2 \Delta N/\Delta N$, which remains finite, Nevertheless, it is possible for $\tau_{ib}$ to be larger than $\tau_{ib}$. Note that $\tau_{ib}$ reduces to $\pm 2 \Delta N/\Delta N$ in the resonance. In accordance with equation (C2), $\tau_{ib}$ diverges. However, the characteristic timescale for $\omega$ evolution based on a second-order expansion remains finite. The numerical results in § 4.1 show a slight increase in $e_0$ with a $\sim 60\%$ reduction in $\eta_0$. In contrast, a $40\%$ increase in $e_0$ is inferred from equation (C3). Nevertheless, for planets $c$ and $d$ around $\nu$ And, the duration of nonadiabatic evolution is brief so that the small-amplitude libration around $x = x_{\nu m}$ and $\eta = 0$ is preserved despite the large net changes in the magnitude of $x_{\nu m}$.

Note that we have only considered the axisymmetric contribution of the disk to the total potential. Therefore, the total angular momentum of the two planets is conserved during the disk depletion even though there is a net transfer of angular momentum between them that leads to the decline in $e_c$ and the growth in $e_d$.

**C2. THE EVOLUTION OF TWO-PLANET SYSTEMS WITH INITIALLY CIRCULATING ORBITS: ASTEROIDS**

We now consider the orbital evolution of multiple planets with initial values of $|A_1| \geq |A_2|$. In § B2 we showed that these systems have initially circulating orbits. There are two possible branches of solutions. First, we consider the limiting case of small $\gamma_d (\ll x^{-2})$, which is appropriate in the context of secular interaction between the asteroids and Jupiter for which $M_c \lesssim M_d$.

During the epoch of disk depletion, as the magnitude of $A_1$ in equation (9) is reduced below that of $A_2$, $\eta$ evolves to $\eta_1$ (see eq. [B9]), which reduces to $\eta_1 = \cos^{-1}(-A_1 x/C)$ in the limit $\gamma_d \ll x^{-2}$. Since $A_1$, $A_2$, and $C$ are negative, $\eta_1 = \pi$ when the planets first enter into a secular resonance. Furthermore, equation (B10) implies that $x_1$ decreases (increases) for $\eta_1$ slightly less (greater) than $\pi$.

Let us first consider the case that $\eta_1$ is slightly less than $\pi$. Continual decline in $x$ is attainable provided that $d\eta_1/d\tau \leq 0$. The reduction in $x$ leads to an increase in the magnitude of $A_2$, while the disk depletion causes a decrease in that of $A_1$. Consequently, the magnitude of $A_1 / A_2$ decreases in equation (B9), leading to a reduction in $\eta_1$. This state of phase lock can only be maintained if $\eta_1$ can adjust to $\eta_1$ and decrease to $\sim \pi/2$. As long as this state of resonant phase locking is maintained, the magnitude of $x$ (i.e., $x_1$) can decrease indefinitely. When $A_1$ passes through the zero value and becomes positive, $x$ continues to decline. If the magnitude of $x$ cannot be reduced as fast as that of $\Delta N$, $|A_1| > |A_2|$ and the planets would attain circulating orbits. Even in the limit that the magnitude of $x$ can be reduced as fast as that of $\Delta N$, the planets are unlikely to attain librating orbits because their $\eta_1 \sim \pi/2$. Eventually, $\eta_1$ terminates its evolution with values slightly less than $\pi/2$ while the orbits of the massless particles are approximately circularized.

If the planets enter into the secular resonance with $\eta$ slightly larger than $\pi$, $x$ would increase. Further excitation of $x$ requires $d\eta_1/d\tau \geq 0$. In equation (9), the growth in $x$ reduces the magnitude of $A_2$ while that of $A_1$ is also decreasing. Phase lock can only be maintained if $|A_1| < |A_2|$ such that $x$ cannot grow faster than $C/|A_1| \simeq C/(1 - \Delta N)$ during the epoch of disk depletion. If $x$ grows slower than $C/|A_1|$, $d\eta_1/d\tau > 0$ and $\eta_1$ would increase to enhance the $x$ growth rate. The magnitude of $\eta_1$ is determined by the ratio of $A_1$ and $A_2$ (see eq. [B9]). The evolution timescale of $A_2$ equals $\tau_{ec}$. Equations (C1) and (C2) imply that $\tau_{ec} \approx \tau_{ib}$ in the limit of small $\gamma_d x^2$. Thus, the magnitude of $A_1 / A_2$ is regulated to be less than unity as phase lock causes $x$ to increase (see eq. [B10]).

As the resonant condition $1 - \Delta N \simeq 0$ is approached, $\eta_1$ increases to $3\pi/2$ and $x$ grows at a maximum rate $dx/d\tau = -C$. As long as this state of resonant phase locking is maintained, the magnitude of $x$ can increase indefinitely. On the resonance, $\eta = \eta_2$ such that the growth occurs regardless of the value $\eta$. Increases in $x$ are equivalent to eccentricity excitation for the less massive body $c$. When $A_1$ passes through the zero value and becomes positive, $x$ can have a sufficiently large value such that $e_c$ may approach and even exceed unity. This eccentricity excitation process has been noted by Heppenheimer (1980) as a potential cause for the existence of highly eccentric asteroids.

As $x$ increases in the resonances, there are three eventual outcomes: (1) $x$ becomes sufficiently large for $e_c$ to exceed unity and body $c$ to escape, (2) the planets would attain circulating orbits as the magnitude of $A_1$ exceeds that of $A_2$ after $|\Delta N|$ decreases below $\sim 1$ if $\tau_{ib}$ becomes much smaller than $\tau_{x_1}$, and (3) in systems with finite $\gamma_d$, the above approximation becomes inappropriate when $x$ exceeds $\gamma_d x^2$ (see below).
C. THE EVOLUTION OF TWO-PLANET SYSTEMS WITH INITIALLY CIRCULATING ORBITS: TWO PLANETS WITH COMPARABLE MASSES

We now consider the second branch of solutions in which \( \gamma_d \) is finite. These parameters are more appropriate for the extra-solar multiple-planet systems in which the masses of the planets are comparable. The main difference introduced by the finite value of \( \gamma_d \) is the possibility of a transition as the value of \( x \) passes through \( x_m = x_m^{1/2} \). This branch of solution is applicable for planets b and c around HD 168443. We use the analytic solutions in this subsection to study the numerical results in § 4.

For planets b and c around HD 168443, \( C = -0.13, \gamma_d = 0.13 \), and \( t_{c} = 1.9 \times 10^3 \) yr. In this system, the value of \( x_m \) depends sensitively on \( \Delta N \) as a result of the small magnitude of \( C \) and \( \gamma_d \). Within the uncertainty of the analytic estimate, we estimate \( M_D/M_J \sim 0.007 \) and \( \Delta N \approx -4 \) for 5 times the minimum mass nebula model. Despite this relatively small value of \( M_D(< M_d < M_\ast) \), the corresponding \( x_m \approx 180 \). At this initial stage, \( |A_1| > |A_2| \) and the planets follow circulating orbits, which is well approximated by equations (B7) and (B8).

However, during the depletion of the disk, the decline of \( \Sigma \) reduces the magnitude of \( \Delta N \), which passes through a stage of secular resonance in which \( \Delta N \approx \gamma_d - 1 = -0.86 \) with the corresponding value of \( x_m = x_m^{1/2} = 2.6 \) similar to the present value of \( x \). In this limit, \( |A_1| < |A_2| \) and the leading-order approximation with the solutions in equations (B7) and (B8) is no longer valid. Since the planets’ orbits circulate relative to each other prior to entering this stage, the small oscillation amplitude approximation that led to the solutions in equation (B5) may also be inappropriate. Nevertheless, when the planets first enter into the resonance, \( |A_1| = |A_2|, \eta_1 = \pi, \) and \( \omega_2 \) in equation (B16) becomes purely imaginary. The discussions in § 3 indicate that during the passage through the resonance, all orbits are librating regardless of the amplitude of the libration cycles.

Within the resonance where \( |A_1| < |A_2| \), the two planets become phase locked such that \( x_1 \) and \( \eta_1 \) evolve in accordance with equations (B9) and (B10). Depending on the rate of disk depletion, there are three potential outcomes for passage through the secular resonance:

1. In the limit that \( \tau_{d1} < \tau_{db} \), the system evolves in an impulsive manner. Such a situation is relevant for rapid clearing of disks that contain long-period, low-mass, widely separated planets with large \( t_e \). In this case, there is insufficient time for the longitude of periapsis to significantly circulate before the effect of secular resonance fades with the rapid disk depletion. During this rapid passage through the secular resonance, both \( x \) and \( \eta \) essentially retain the values they had when they first entered into the resonance \( (x_1, \eta_1) \). These values of \( \eta_1 \) and \( x_1 \) become the initial values of circulating orbits in the new disk-free potential. Since \( \eta_1 \) and \( x_1 \) are random parameters, the actual variational amplitude of the circulating orbits is unpredictable.

2. For systems such as planets b and c around HD 168443, \( \tau_{d1} < \tau_{db} \) so that the planets adiabatically evolve into a phase lock. The relevant governing equations become equations (B9) and (B10). In contrast to the case for the massless asteroids, the system first enters or finally exits the secular resonances with \( \eta_1 = 0 \) if \( x > x_{b1} = x_{b2} \) or \( \pi \) if \( x < x_{b1} \). In the middle of the resonant zone where \( \gamma_d - 1 - \Delta N = 0 \), although \( x_m = x_m^{1/2} \), \( x \) generally does not equal \( x_m \) so that \( \eta \neq \pm \pi/2 \). (Even in the limit that \( x = x_m \) at the center of resonances, \( A_1/A_2 \) also vanishes at the center of the resonance.) Our analysis in § 3 indicates that, at the center of the resonance, both roots of \( \omega_2 = -2C\gamma_d x_1 \) or \( (C/\gamma)(1 - \gamma_d x_1^2) \) are real so the amplitudes of small perturbations either converge or diverge (if \( x_1 > x_{b1} \)) from the stationary points without any oscillation. Note that in contrast to the case of massless asteroids, \( \omega_2 \) changes sign as \( x_1 \) passes through \( x_{b1} \) so the system can be neither damped nor excited indefinitely. This sign reversal of \( \omega_2 \) limits the changes in \( x_1 \). Physically, this limit is due to the comparable momentum of inertia carried by planets of similar masses. In such systems, the exchange of angular momentum cannot lead to vastly different eccentricity changes for the two planets.

After passing through the center of the resonance, \( \eta_1 \) continues to evolve toward 0 or \( \pi \) in accordance with equation (B9) while \( x_1 \) evolves in accordance with equation (B10). When \( \eta_1 \) reduces below \( \sin^{-1} 0.8 \), \( \omega_1 \) once again become complex and libration around \( x_1 \) and \( \eta_1 \) would resume if the magnitude of \( \omega_2 \) is larger than \( B_5 \) such that changes in \( \Delta N \) and \( x_1 \) occur in an adiabatic manner. In equation (B15), however, the magnitude of \( B_5 \) is determined by \( \Delta N \) or equivalently \( \tau_{d1} \). Thus, in the modestly rapid depletion case in which \( \tau_{d1} \leq \omega_2^{-1} \), the difference between \( x_1 \) and \( x_m \) at the center of resonance is preserved. For small \( C \) magnitude (as in the case of HD 168442), the width of the libration is small and \( x_m \) is also not changed significantly during the passage through the resonance. Thus, planets would not be able to enter into a librating state and the circulation pattern of their orbits is retained.

3. Finally, in the slow depletion limit where \( \tau_{d1} > \omega_2^{-1} \), the protracted passage through the secular resonance can greatly modify both \( x \) and \( \eta \). As the planets exit the resonance, they can become trapped onto librating orbits. Large changes in \( x \) imply that at least one planet attains a relatively large eccentricity. In relatively compact multiple-planet systems, large eccentricity can cause mean motion resonances to overlap and dynamical instability. We shall address the issue of dynamical instability elsewhere.

APPENDIX D

HAMILTONIAN CONTOURS

These phase-space evolutions of the planets’ orbits may be compared with equi-Hamiltonian contour maps. By expanding the disk potential in equation (14) for planets with eccentric orbits and then averaging it over their orbital period, Ward (1981) obtained the disturbing function due to the disk potential as follows:

\[
\langle R_{\text{disk}} \rangle_{c,d} = c_{c,d} R G \sum_{d,c} \left( \frac{a_0}{a_{c,d}} \right)^k \sum_{n=1}^{\infty} A_n \frac{n(2n+1)}{2n+k-1} \left( \frac{a_{c,d}}{\ell_{\text{edge}}} \right)^{2n+k-1} \equiv c_{c,d} T_{c,d}. \tag{D1}
\]
Using this potential, $\Delta N$ of equation (A3) can be written as

$$\Delta N = \frac{8a_d^2}{Ga_d^3} \left[ T_c - T_d \left( \frac{a_c}{a_d} \right)^{1/2} \right].$$

When the disk depletion timescale is longer than the characteristic librational timescale associated with equations (8) and (9) (timescales are given in Appendix C), $\Delta N$ can be regarded as quasi-static. In this limit, the planets’ orbits follow a single trajectory on the $(x, \eta)$-plane depending on the magnitude of their Hamiltonian and angular momentum. Unless the orbits of the interacting planets are locked in mean motion resonances, the orbital energy and semimajor axes of each planet are conserved to the lowest order. The orbit-averaged Hamiltonian ($H$) can be calculated semianalytically using the disturbing functions of planets (e.g., Brouwer & Clemence 1961) and equation (D1). Expanding to the second order of the eccentricities, we find

$$H = \frac{M_n a_d^2 c^2}{2} + \frac{M_d a_d^2 d^2}{2} + \frac{n_d a_d^2 N_c}{2t_c} + \frac{n_d a_d^2 N_d}{2t_c} + \frac{GM_d M_a c}{8a_d^2} \left[ b_{3/2}^1 (c^2 + d^2) - 2b_{3/2}^2 c d \cos \eta \right].$$

From the conservation of the total angular momentum, a relation

$$J \equiv M_n a_d^2 \sqrt{1 - e_c^2} + M_d a_d^2 \sqrt{1 - e_d^2} = \text{const}$$

holds provided that their orbital planes are not inclined from each other. Thus, to second order in eccentricity,

$$\frac{n_d a_d^2 N_c}{t_c} + \frac{n_d a_d^2 N_d}{t_c} + \frac{GM_d M_a c}{4a_d^2} \left[ b_{3/2}^1 (c^2 + d^2) - 2b_{3/2}^2 c d \cos \eta \right] = \text{const},$$

$$M_n a_d^2 c^2 + M_d a_d^2 d^2 = \text{const},$$

for any given values of $H$ and $J$. Dividing equation (D5) with equation (D6), we find that

$$\frac{x^2(1 + N_c) + 2Cx \cos \eta + (\gamma_d^{-1} + N_d)}{\gamma_d x^2 + 1} = \text{const}$$

and obtain the equi-Hamiltonian contour lines for various values of the constants of integration. In equation (D7), the masses of the disk and planets are contained only in the form of their ratios. Therefore, as long as $M_s \propto M_{d,c}$, the equi-Hamiltonian contours are not affected by the obliquity $i$ of the system. We show in 2.1 and 4 that other than small deviations due to nonlinear effects, the planets’ orbits closely follow the contours in the equi-Hamiltonian map.

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