LUZIN’S CONDITION (N) AND MODULUS OF CONTINUITY

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Abstract. In this paper, we establish Luzin’s condition (N) for mappings in certain Sobolev-Orlicz spaces with certain moduli of continuity. Further, given a mapping in these Sobolev-Orlicz spaces, we give bounds on the size of the exceptional set where Luzin’s condition (N) may fail. If a mapping violates Luzin’s condition (N), we show that there is a Cantor set of measure zero that is mapped to a set of positive measure.

1. Introduction

In this introduction, we assume for simplicity that our mappings are continuous. In the entire paper, we assume that $n \geq 2$.

As seen by the constructions of J. Malý and O. Martio, for each $n \geq 2$ there exists a continuous mapping $f: \Omega \to \mathbb{R}^n$ in $W^{1,n}(\Omega; \mathbb{R}^n)$ such that $f$ maps a set of $n$-dimensional Lebesgue measure zero onto an $n$-dimensional cube, see [MM95, Section 5]. However, if we require better integrability for the differential, then $f$ satisfies Luzin’s condition (N): the image of each set of zero $n$-dimensional Lebesgue measure is also of measure zero. More precisely, if additionally

\begin{equation}
\int_{\Omega} |Df|^n \log(\lambda(e + |Df|)) \, dy < \infty
\end{equation}

for some $\lambda > n - 1$, then $f$ satisfies Luzin’s condition (N), see [KKM99, Example 5.3]. For $\lambda = n - 1$, there exist continuous mappings in $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ satisfying (1) and violating Luzin’s condition (N), see [KKM99, Theorem 5.2]. Such a mapping $f$ cannot be Hölder continuous. Indeed, in [MM95] the authors showed that Hölder continuous mappings in $W^{1,n}(\Omega; \mathbb{R}^n)$ satisfy Luzin’s condition (N), see their Theorem C. Based on this result, the authors further showed in Theorem G that for each (continuous) mapping in $W^{1,n}(\Omega; \mathbb{R}^n)$, there is a zero-dimensional set such that the mapping satisfies Luzin’s condition (N) outside this set.

In this paper, we consider mappings that satisfy (1) for $0 \leq \lambda \leq n - 1$ and have a modulus of continuity that is slightly weaker than Hölder continuity. Our first result reads as follows.

Theorem 1.1. Let $0 \leq \lambda \leq n - 1$ and $\alpha = 1 - \frac{1}{n+1}$. Suppose $f \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^d)$,

\[ \int_{\Omega} |Df|^n \log(e + |Df|) \, dy < \infty, \]

2010 Mathematics Subject Classification. 26B15, 26B35, 46E35.

The first and third authors acknowledge the support by the Academy of Finland grant numbers 131477 and 251650, respectively. The research of the second author was supported by the grant GA ČR P201/12/0436.
and \( f \) has the modulus of continuity
\[
|f(x) - f(y)| \leq \begin{cases} 
\exp(-\mu \log^\alpha(1/|x - y|)), & \alpha > 0, \\
\log^{-\mu}(1/|x - y|), & \alpha = 0, 
\end{cases}
\]
for some \( \mu > 0 \). Then \( f \) satisfies Luzin’s condition (N) in the sense that sets of zero \( n \)-dimensional Lebesgue measure get mapped to sets of zero \( n \)-dimensional Hausdorff measure.

We prove a corresponding result about the size of the exceptional set as well. Our continuity assumption below can actually be removed provided we choose a quasicontinuous representative, see Section 5.

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^n \) open, \( n \geq 2 \). Suppose \( f \in W^{1,n}(\Omega; \mathbb{R}^d) \) is continuous, \( 0 \leq \lambda \leq n - 1 \), and \( \alpha = 1 - \frac{\lambda}{n-1} \). If
\[
\int_\Omega |Df|^n \log(e + |Df|)^\lambda \, dy < \infty,
\]
then there exists a set \( E \) such that \( f \) satisfies Luzin’s condition (N) in \( \Omega \setminus E \), and \( \mathcal{H}^\varphi(E) = 0 \) whenever \( \varphi \) is of the form
\[
\varphi(r) = \begin{cases} 
\exp(-\gamma \log^\alpha(\frac{1}{r})), & \alpha > 0, \\
\log^{-\gamma}(\frac{1}{r}), & \alpha = 0,
\end{cases}
\]
for some \( \gamma > 0 \).

Regarding necessity of our modulus of continuity, we construct the following example.

**Example 1.3.** Let \( 0 \leq \lambda < n - 1 \) and \( 0 < \alpha < \frac{n-1-\lambda}{n} \). Then there is a mapping \( f \in W^{1,1}(\mathbb{R}^d; \mathbb{R}^d) \) such that
\[
\int_{\mathbb{R}^n} |Df|^n \log^\lambda(e + |Df|) \, dy < \infty
\]
and \( f \) has modulus of continuity no worse than
\[
\Psi(t) = C \exp(-\log 2 \log^\alpha(\frac{1}{t})),
\]
and violates Luzin’s condition (N).

The example above shows that the logarithmic scale in Theorem 1.1 is essentially sharp. However, we have a slight mismatch between the exponent \( \alpha \) in Theorem 1.1 and the one in the example. As \( n \) gets larger, the gap gets smaller. We do not know if the positive result or the example could be improved. For the endpoint case \( \lambda = n - 1 \) see the discussion at the end of Section 4.

In Example 1.3 a compact, perfect, totally disconnected set of measure zero is mapped onto a set of positive measure. We further show that this phenomenon is ubiquitous, i.e. if we have a continuous mapping that maps a set of measure zero onto a set of positive measure, then there is a compact, perfect, totally disconnected set that is blown up as well. Since the verification of Luzin’s condition (N) then only needs to focus on these Cantor sets, we would like to know if additional regularity can be required for the blown up sets. If \( f \in W^{1,n}(\Omega; \mathbb{R}^d) \), then we can actually find a zero-dimensional Cantor set that is blown up. We would like to know if one could require the set to be porous as well, and in dimension two if the set could
be required to lie on a quasicircle. For the significance of these questions see [KK], [KZ].

Some of our results were announced in [KMZ12]. The paper is organized as follows. The following section contains notations and basic definitions. Section 3 deals with Luzin’s condition (N). Especially, we prove a slightly stronger statement than Theorem 1.1. We continue by constructing Example 1.3 in Section 4. In the penultimate section, we study the size of the exceptional set. Our reasoning is closed by regularity considerations in Section 6.

2. Notation and basic definitions

We will use open and closed balls. To distinguish them, we use an overbar for closed balls and the plain symbol for open balls.

Definition 2.1 (gauge function). A (Hausdorff) gauge function \( h: [0, \infty] \to [0, \infty] \) is a nondecreasing function that is positive for \( t > 0 \) and continuous from the right.

Example 2.2. The function \( \varphi: [0, \infty] \to [0, \infty] \) defined as
\[
\varphi(t) = \begin{cases} 
0, & t = 0, \\
D \exp(-C \log^\alpha(\frac{1}{t})), & 0 < t < 1, \\
\infty, & t \geq 1,
\end{cases}
\]
where \( \alpha, C, D > 0 \), is a gauge function.

Definition 2.3 (Hausdorff measure). Let \( A \subset \mathbb{R}^n \), \( h \) be a gauge function, and \( 0 < \delta \leq \infty \). Define
\[
\mathcal{H}_h^\delta := \inf \left\{ \sum_{j=1}^{\infty} h(\text{diam } C_j) : A \subset \bigcup_{j=1}^{\infty} C_j, \text{ diam } C_j \leq \delta \right\},
\]
and the Hausdorff \( \mathcal{H}^h \)-measure as
\[
\mathcal{H}^h(A) := \lim_{\delta \to 0} \mathcal{H}^h_\delta(A).
\]
If \( h(t) = t^s \) for some \( s \geq 0 \), we simply write \( \mathcal{H}^s \) instead of \( \mathcal{H}^{t^s} \). The Hausdorff dimension \( \dim_H(A) \) of a set \( A \subset \Omega \) is defined as \( \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} \).

In what follows \( \Omega \) is an open set in \( \mathbb{R}^n \).

Definition 2.4 ((median) modulus of continuity). Let \( f: \Omega \to \mathbb{R}^d \) be a measurable function and \( x \in \Omega \). We define the modulus of continuity
\[
\omega_f^*(x, r) = \inf\{s \geq 0 : |f(x) - f(y)| \leq s \text{ on } B(x, r)\}
\]
and the median modulus of continuity
\[
\omega_f(x, r) = \inf\{s \geq 0 : \{y \in B(x, r) : |f(y) - f(x)| \geq s\} < \frac{1}{2} |B(x, r)|\}.
\]
Obviously \( \omega_f(x, r) \leq \omega_f^*(x, r) \), so that results using \( \omega_f \) are better.
3. Pointwise modulus of continuity and Luzin’s condition (N)

The following result is a more general version of Theorem 1.1. The current section is devoted to its proof.

**Theorem 3.1.** Let \(0 \leq \lambda \leq n-1\) and \(\alpha = 1 - \frac{\lambda}{n-1}\). Let \(f \in W^{1,1}_{loc}(\Omega; \mathbb{R}^d)\) and \(N \subset \Omega\) be a Lebesgue null set. Suppose that for each \(x \in N\) there exist \(R = R(x) > 0\) and \(\mu = \mu(x) > 0\) such that \(B(x, R) \subset \Omega\) and

\[ \omega_f(x, \rho) \leq \begin{cases} \exp(-\mu \log^\alpha(1/\rho)), & \alpha > 0, \\ \log^{-\mu}(1/\rho), & \alpha = 0 \end{cases} \quad 0 < \rho \leq R. \]

If

\[ \int_\Omega |Df|^n \log^\lambda(e + |Df|) \, dy < \infty, \]

then \(\mathcal{H}^n(f(N)) = 0\).

3.1. Key estimates.

**Lemma 3.2.** Let \(u \in W^{1,1}(B(0, R))\). Let \(M\) be the median of \(u\) in \(B(0, R)\) and \(m\) be the median of \(u\) in \(B(0, r)\), where \(0 < r \leq R\). Then

\[(2) \quad |M - m| \leq C(n)r^{1-n} \int_{B(0, r)} |Du| \, dy + C(n) \int_{B(0, R) \setminus B(0, r)} |y|^{1-n} |Du| \, dy. \]

**Proof.** We find \(j \in \mathbb{N}\) such that \(2^{j-1} \leq R/r < 2^j\) and consider a chain of balls \(B(0, r_k), k = 0, \ldots, j\), where \(r_0 = R\) and \(r_k = r 2^{j-k}\) for \(k = 1, \ldots, j\). Let \(m_k\) denote the median of \(u\) in \(B(0, r_k)\). For each \(k = 0, \ldots, j-1\) and \(c \in \mathbb{R}\) we have

\[
\frac{|B(0, r_{k+1})|}{2} |m_k - m_{k+1}| \leq \frac{|B(0, r_k)|}{2} |m_k - c| + \frac{|B(0, r_k)|}{2} |m_{k+1} - c| \\
\leq \int_{B(0, r_k)} |u - c| \, dy.
\]

Then we apply the Poincaré inequality with the choice \(c = u_{B(0, r_k)}\) and obtain

\[
|m_k - m_{k+1}| \leq Cr_k^{-n} \int_{B(0, r_k)} |u - c| \, dy \leq Cr_k^{-n} \int_{B(0, r_k)} |Du| \, dy \\
= Cr_k^{-n} \left( \sum_{i=k}^{j-1} \int_{B(0, r_i) \setminus B(0, r_{i+1})} |Du| \, dy + \int_{B(0, r_j)} |Du| \, dy \right).
\]

Summing over \(k\) we obtain

\[
|M - m| \leq C \sum_{k=0}^j \left( \sum_{i=k}^{j-1} r_k^{1-n} \int_{B(0, r_i) \setminus B(0, r_{i+1})} |Du| \, dy + r_k^{1-n} \int_{B(0, r_j)} |Du| \, dy \right) \\
= C \sum_{i=0}^j \sum_{k=0}^{j-1} r_k^{1-n} \int_{B(0, r_i) \setminus B(0, r_{i+1})} |Du| \, dy + C \sum_{k=0}^j r_k^{1-n} \int_{B(0, r_j)} |Du| \, dy.
\]

Since

\[
\sum_{k=0}^i r_k^{1-n} \leq C |y|^{1-n}, \quad y \in B(0, r_i) \setminus B(0, r_{i+1}), \ i = 0, \ldots, j - 1
\]
Proof. We apply Lemma 3.2 to \( m \) and \( v \) and obtain the estimate follows.

**Lemma 3.3.** Let \( f \in W^{1,n}(B(0,R);\mathbb{R}^d) \). Let \( M \) be the median of \(|f|\) in \( B(0,R) \) and \( m \) be the median of \(|f|\) in \( B(0,r) \), where \( 0 < r \leq R \). Suppose \( m \leq M \). Then

\[
(M - m)^n \leq C(n)(1 + \log(R/r))^{n-1} \int_{B(0,R) \cap \{|f| < M\}} |Df|^n \, dy.
\]

**Proof.** We apply Lemma 3.2 to \( u = \min\{|f|, M\} \) and obtain

\[
M - m \leq C \int_{B(0,R)} v^{1-n} |Du| \, dy \leq C \int_{B(0,R) \cap \{|f| < M\}} v^{1-n} |Df| \, dy,
\]

where \( v = \max\{r, |x|\} \). By the Hölder inequality we conclude that

\[
M - m \leq C \left( \int_{B(0,R)} v^{-n} \, dy \right)^{\frac{1}{n}} \left( \int_{B(0,R) \cap \{|f| < M\}} |Df|^n \, dy \right)^{\frac{1}{n}}.
\]

Now, it remains to notice that

\[
\int_{B(0,R)} v^{-n} \, dy \leq C(1 + \log(R/r)).
\]

**Lemma 3.4.** Let \( q \) be a positive integer, \( 0 \leq \alpha < 1 \), and \( \tau > 0 \). Then there exists a constant \( C = C(\tau, \alpha) > 0 \) with the following property: Let \( \{a_j\}_{j=q}^{\infty} \) be a nondecreasing sequence of positive real numbers and

\[
b_j = \begin{cases} 
\tau^j, & \alpha > 0, \\
e^{\tau j}, & \alpha = 0.
\end{cases}
\]

Suppose

\[
(3) \quad j + 1 \leq a_j \leq b_j, \quad j = q, q+1, q+2, \ldots.
\]

Then there exists an integer \( k \geq q \) such that

\[
(4) \quad a_{k+1} - a_k \leq C(a_k - k)^{1-\alpha}.
\]

**Proof.** Suppose that (4) fails for a fixed \( C \). Write \( \nu = \tau + 1 \).

First, let \( 0 < \alpha < 1 \). Towards a contradiction with (3), it suffices to verify

\[
(5) \quad a_k \geq \nu(k - q)^{1/\alpha} + k + 1, \quad k = q, q+1, q+2, \ldots.
\]

Assume (5) is true for some \( k \); it clearly holds for \( k = q \). Then

\[
(6) \quad a_{k+1} \geq C(a_k - k)^{1-\alpha} + a_k \geq C(\nu(k - q)^{1/\alpha} + 1)^{1-\alpha} + \nu(k - q)^{1/\alpha} + k + 1.
\]

We see that we succeed in proving (5) if

\[
C(\nu(k - q)^{1/\alpha} + 1)^{1-\alpha} + \nu(k - q)^{1/\alpha} + k + 1 \geq \nu(k + 1 - q)^{1/\alpha} + k + 2.
\]

We are fine if there is \( 0 < C < \infty \) such that

\[
C \geq \sup_{x \geq q} \frac{\nu(x + 1 - q)^{1/\alpha} - \nu(x - q)^{1/\alpha} + 1}{(\nu(x - q)^{1/\alpha} + 1)^{1-\alpha}} = \sup_{x \geq 0} \frac{\nu(x + 1)^{1/\alpha} - \nu x^{1/\alpha} + 1}{(\nu x^{1/\alpha} + 1)^{1-\alpha}}.
\]
This requires the supremum to be finite. Applying the mean value theorem in the nominator and arguing that we take the supremum of a continuous function, we only need to show that the limit as \( x \) tends to infinity is finite. This can easily be seen.

Now, suppose \( \alpha = 0 \). We set \( C = e^\nu \) and head for
\[
a_k \geq k + e^{\nu(k-q)}
\]
instead of (5). The induction step is
\[
a_{k+1} \geq k + e^{\nu(k-q)} + C(e^{\nu(k-q)}) \geq k + 1 + e^\nu e^{\nu(k-q)} = k + 1 + e^{\nu(k+1-q)}.
\]
As in the previous case, we obtain a contradiction with (3) for \( k \) big enough. \( \square \)

3.2. A criterion for Luzin’s condition (N). We provide a criterion with a standard proof for Luzin’s condition (N) that we will apply subsequently.

**Proposition 3.5.** Assume that \( \Omega \subset \mathbb{R}^n \) is open and \( \Phi: [0, \infty) \to [0, \infty] \). We suppose that \( f: \Omega \to \mathbb{R}^d \) is in \( W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^d) \) and such that
\[
\int_K \Phi(|Df|) \, dy < \infty
\]
for each compact set \( K \subset \Omega \). Assume that \( S \subset \Omega \) is such that for each point \( x \in S \) there exists a natural number \( L(x) \), two sequences \( (r_k(x))_k \) and \( (R_k(x))_k \), where the first one converges to zero. We further stipulate the existence of a sequence \( (A_k(x))_k \) of sets \( A_k(x) \) such that \( A_k(x) \subset B(x, r_k) \), \( f(A_k(x)) \subset B(f(x), R_k(x)) \), and
\[
R_k(x)^n \leq L(x) \int_{A_k(x)} \Phi(|Df|) \, dy, \quad k \in \mathbb{N}.
\]
Then \( f \) satisfies Luzin’s condition (N) in \( S \).

**Proof.** We may assume that \( S \) is relatively compact and \( x \mapsto L(x) \) is bounded on \( S \). Let \( G \subset \Omega \) be an open set that contains \( S \). Using Vitali covering theorem on the image side, see e.g. [Fal86, Theorem 1.10 (a)], we find (finite or infinite, but of the same length) sequences \( (B(x_j, r_j))_j \) of balls in \( G \), \( (A_j)_j \) of subsets of \( B(x_j, r_j) \), and \( (B(y_j, R_j))_j \) of balls in \( \mathbb{R}^n \) such that \( y_j = f(x_j) \), the balls \( B(y_j, R_j) \) are pairwise disjoint, cover \( f(S) \) up to a set of measure zero, and for each \( j \) there exist \( k \in \mathbb{N} \) such that \( r_j = r_k(x_j) \), \( R_j = R_k(x_j) \) and \( A_j = A_k(x_j) \). Then
\[
\mathcal{H}^n(f(S)) \leq C \sum_j R_j^n \leq C \sum_j \int_{A_j} \Phi(|Df|) \, dy \leq C \int_G \Phi(|Df|) \, dy.
\]
By specifying the choice of \( G \), the integral on the right can be made arbitrarily small, and thus \( \mathcal{H}^n(f(S)) = 0 \). \( \square \)

3.3. Luzin’s condition (N) for the weakly approximately Hölder continuous part. Here we take care of the case \( \alpha = 1 \) in Theorem 3.1. As a subcase, we will also need it for \( \alpha < 1 \). In [Mal94], approximately Hölder continuous mappings were considered. Here, we need as slight generalization.

**Definition 3.6** ((\( \zeta, \vartheta \))-weakly approximately Hölder continuous). Let \( \Omega \subset \mathbb{R}^n \) be open and \( (\rho_k)_k \) a decreasing sequence. For \( \zeta \), \( \vartheta \in (0, 1] \), we say that a mapping \( f: \Omega \to \mathbb{R}^d \) is \((\zeta, \vartheta)\)-weakly approximately Hölder continuous at \( x \in \Omega \) with respect
to \((\rho_k)\), if there is a sequence \((H_k)\) of measurable sets \(H_k \subset B_k := B(x, \rho_k)\) such that

\[
\limsup_{k \to \infty} \sup_{y \in H_k} \frac{|f(y) - f(x)|}{\rho_k^\zeta} < \infty,
\]

and

\[
\limsup_{k \to \infty} \frac{|B_k \cap H_k|}{|B_k|} \geq \vartheta.
\]

We say that \(f\) is weakly approximately Hölder continuous at \(x\) with respect to \((\rho_k)\) if there exist \(\zeta, \vartheta \in [0, 1)\) such that \(f\) is \((\zeta, \vartheta)\)-weakly approximately Hölder continuous at \(x \in \Omega\) with respect to \((\rho_k)\).

**Theorem 3.7.** Suppose that \(\Omega \subset \mathbb{R}^n\) is open and \(f\) is a mapping in \(W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^d)\). Assume that for every point \(x \in H\) there exist \(\zeta(x), \vartheta(x)\) and a sequence \((\rho_k(x))\) converging to zero such that \(f\) is \((\zeta(x), \vartheta(x))\) weakly approximately Hölder continuous at \(x\) with respect to \((\rho_k(x))\). Then \(f\) satisfies Luzin’s condition \((N)\) in \(H\).

**Proof.** For simplicity assume that \(\vartheta(x) \geq 1/2\), otherwise we would be forced to alter the definition of the modulus of continuity. Consider \(\tilde{r}_k(x) = e^{-k}\) and \(\tilde{R}_k(x) = \omega_f(x, \tilde{r}_k)\). Then Lemma 3.3 yields sets \(\tilde{A}_k \subset B(x, \tilde{r}_k)\) such that

\[
\tilde{R}_k - \tilde{R}_{k+1} \leq C \left( \int_{\tilde{A}_k} |Df|^{n} \, dy \right)^{1/n}.
\]

In order to apply Proposition 3.5 for a subsequence \((r_k, \tilde{R}_k, A_k)\) of \((\tilde{r}_k, \tilde{R}_k, \tilde{A}_k)\), we need the inequality

\[
\tilde{R}_k \leq C(x)(\tilde{R}_k - \tilde{R}_{k+1})
\]

to hold for infinitely many \(k\). However, if it fails for \(k \geq k_0\), then by iteration we obtain a contradiction with the weak approximate Hölder continuity for an appropriate choice of \(C(x)\) depending on \(\zeta(x)\).

**3.4. Proof of Theorem 3.1.** We split the set \(N\) into two parts. The result then follows from Proposition 3.8 and Proposition 3.9.

**Proposition 3.8.** Suppose the same assumptions as in Theorem 3.1. Then \(f\) satisfies Luzin’s condition \((N)\) in

\[
S = \{ x \in \Omega : \omega_f(x, e^{-k-1}) \leq e^{-k} \text{ for infinitely many } k \}.
\]

**Proof.** Let \(N \subset S\) be a set of measure zero. For \(x \in N\), we define \(r_k := e^{-k-1}\) and \(R_k := e^{-k}\).

Our goal is to show that \(f\) is weakly approximately Hölder continuous at \(x\) (for \(\zeta = 1, \vartheta = 1/2\)) with respect to some subsequence of \((r_k)\). We set

\[
H_k := \{ y \in B(x, r_k) : |f(y) - f(x)| \leq R_k \}.
\]

Set

\[
J = \{ k \in \mathbb{N} : \omega_f(x, e^{-k-1}) \leq e^{-k} \}.
\]

Then

\[
\limsup_{k \to \infty} \sup_{y \in H_k} \frac{|f(y) - f(x)|}{r_k} \leq \limsup_{k \to \infty} \frac{e^{-k}}{e^{-k-1}} < \infty.
\]
and

$$|H_k| \geq \frac{1}{2} |B_k|, \quad k \in J.$$  

Since $J$ is infinite, we may apply Theorem 3.7. \hfill $\square$

**Proposition 3.9.** Assume that the assumptions in Theorem 3.1 hold. Then $f$ satisfies Luzin’s condition (N) in the set

$$S = \{x \in \Omega : \omega(e^{-k-1}) \geq \omega_f(x, e^{-k-1}) > e^{-k} \text{ for almost all } k\}.$$  

where

$$\omega(r) = \begin{cases} \exp(-\mu \log^\alpha (1/r)), & \alpha > 0, \\ \log^{-\mu} (1/r), & \alpha = 0. \end{cases}$$  

**Proof.** The goal is to apply Proposition 3.5. Assume $N \subset S$ is a set of measure zero. We may assume that $\mu$ is constant on $N$. If $x \in N$, we choose $q \in \mathbb{N}$ such that $e^{-q} < R$ and such that $\omega_f(x, e^{-k-1}) > e^{-k}$ for all $k \geq q$. We define

$$r_k := r_k(x) := \inf \{t > 0 : \omega_f(x, t) \geq e^{-k}\}, \quad k = q, q+1, q+2, \ldots.$$  

It is clear that $e^{-k}$ is a median of $|f - f(x)|$ in $B(x, r_k)$. Since Sobolev functions have unique medians, we have

$$R_k := R_k(x) := \omega_f(x, r_k) = e^{-k}.$$  

As $x \in S$, using Lemma 3.3, we obtain

$$R_k^n = e^{-kn} \leq C \left(1 + \log(r_k/r_{k+1})\right)^{n-1} \int_{A_k} |Df|^n \, dy,$$  

where

$$A_k := A_k(x) := \{y \in B(x, r_k) : |f(y) - f(x)| \leq R_k(x)\}.$$  

From now on, $C$ may vary from line to line but does not depend on $k$. Our goal is to exploit the full integrability of $|Df|$, i.e. we want to prove that

$$R_k^n \leq \frac{L}{n \lambda} \int_{A_k} |Df|^n \log^\lambda (e + |Df|) \, dy,$$  

for infinitely many $k$, where $L$ is independent of $k$. Towards applying Jensen’s inequality with $\Phi(t) = t \log^\lambda (e + t) \sim t \log^\lambda (e + t^{1/n})$, we rewrite (9) as

$$\frac{R_k^n}{Cr_k^n (1 + \log(r_k/r_{k+1})^{n-1})} \leq \int_{B(x, r_k)} |Df|^n \chi_{A_k} \, dy.$$  

We conclude that

$$\frac{R_k^n}{Cr_k^n (1 + \log(r_k/r_{k+1})^{n-1})} \log^\lambda \left(e + \frac{e^{-kn}}{Cr_k^n (1 + \log(r_k/r_{k+1})^{n-1})}\right) \leq \frac{1}{r_k^n} \int_{A_k} \Phi(|Df|^n) \, dy.$$  

Hence, it suffices to show that

$$\limsup_{k \to \infty} \frac{\log^\lambda \left(e + \frac{e^{-kn}}{Cr_k^n (1 + \log(r_k/r_{k+1})^{n-1})}\right)}{(1 + \log(r_k/r_{k+1})^{n-1})} > 0.$$  

To this end, we first consider the inequality

$$1 + \log(r_k/r_{k+1})^{n-1} \leq C \log^\lambda \left(e + \frac{e^{-k}}{r_k}\right).$$
In fact, setting $a_j = \log(1/r_j)$, we denote the set of all indices that satisfy inequality (4) by $J$. Then $k \in J$ satisfy

$$\log(r_k/r_{k+1}) \leq C\left(\log \frac{e^{-k}}{r_k}\right)^{1-\alpha},$$

and (12) follows. To prove that $J$ is infinite, it suffices to verify the assumptions of Lemma 3.4.

We see that $r_j \leq e^{-j-1}$. Obviously, $(a_j)_j$ forms a nondecreasing sequence and $a_j \geq j + 1$.

If $\alpha > 0$, we estimate

$$e^{-j} = \omega_f(x, r_j) \leq \exp(-\mu \log^\alpha(1/r_j)) = \exp(-\mu a_j^\alpha),$$

so that

$$j \geq \mu a_j^\alpha, \quad a_j \leq (j/\mu)^{1/\alpha}.$$  

If $\alpha = 0$, we estimate

$$e^{-j} = \omega_f(x, r_j) \leq \log^{-\mu}(1/r_j) = a_j^{-\mu},$$

so that

$$a_j \leq e^{j/\mu}.$$  

The application of Lemma 3.4 is permitted, and we see that (12) holds for infinitely many $k$, where, in the notation of Lemma 3.4, $C = C(\mu^{-1/\alpha}, \alpha)$ if $\alpha > 0$ and $C = C(1/\mu, \alpha)$ if $\alpha = 0$.

Inequality (12) shows that $(1 + \log(r_k/r_{k+1}))^{n-1} \leq C(e^{-k}/r_k)^{n-1}$ for all $k \in J$; Indeed, for $s \geq e$ we have

$$\log(e + s) = \int_1^{e+s} \frac{dt}{t} \leq \int_s^{e+s} \frac{dt}{t} \leq s - 1 + \frac{e}{s} \leq s - 1 + 1 = s,$$

and $s := \frac{e^{-k}}{r_k} \geq e$. Thus,

$$\log^{\lambda}\left( e + \frac{e^{-k}}{r_k} \right) \leq \left( \frac{e^{-k}}{r_k} \right)^{\lambda} \leq \left( \frac{e^{-k}}{r_k} \right)^{n-1}.$$  

Hence there exists $C' > 0$ such that for every $k \in J$ we have

$$\log^{\lambda}\left( e + \frac{e^{-kn}}{Cr_k^n(1 + \log(r_k/r_{k+1}))^{n-1}} \right) \geq \log^{\lambda}\left( e + \frac{e^{-k}}{Cr_k} \right).$$  

Using (12) once more, (11) reduces to

$$\limsup_{k \to 0} \frac{\log^{\lambda}\left( e + \frac{e^{-k}}{Cr_k} \right)}{\log^{\lambda}\left( e + \frac{e^{-k}}{r_k} \right)} > 0,$$

which is easily verified.

Thus we have found a constant $L$ depending only on $x$, $\mu$, $\alpha$, $n$, and $f$ such that (10) is true. We apply Proposition 3.5 to finish. \qed
4. Example

Here, we construct an example as required in Example 1.3.

We split the construction and the verification of the properties of the example over the next subsections. First, we note that it suffices to consider the case \( d = n \); if \( d > n \) and we have an example \( f : \mathbb{R}^n \to \mathbb{R}^n \), we define \( F : \mathbb{R}^n \to \mathbb{R}^d \) as \( F(x) = (f_1(x), \ldots, f_n(x), 0, \ldots, 0) \).

4.1. Definition of the mapping. Let us start with the construction. We denote \( \beta = 1/\alpha > 1 \) and set

\[
\begin{align*}
\beta_j &= \frac{1}{\alpha} > 1 \\
r_j &= 2^n \exp(-(j + 1)\beta), \\
R_j &= \exp(-j\beta).
\end{align*}
\]

Notice that \( r_j/R_{j+1} = 2^n \), so that each ball of radius \( r_j \) contains at least \( 2^n \) pairwise disjoint balls of radius \( R_{j+1} \). Let us choose a natural number \( j_0 \geq 2 \) such that

\[
\begin{align*}
(\beta - 1)j_0^{\beta-1} &> n \log 2, \\
\sqrt{n} &< 2^{j_0 - 1} \quad \text{and} \quad 2eR_{j_0} < 1.
\end{align*}
\]

Using the mean value theorem, for \( j > j_0 \) we obtain

\[
\log \left( \frac{R_j}{r_j} \right) = \log \left( \frac{\exp(-j\beta)}{2^n \exp(-(j + 1)\beta)} \right) = (j + 1)\beta - j\beta - n \log 2 \geq \beta j^{\beta-1} - n \log 2 \geq j^{\beta-1},
\]

in particular \( r_j < R_j \). In the first generation, we choose \( 2^n \) balls \( B(a_{i,j_0+1}, R_{j_0+1}) \) of radius \( R_{j_0+1} \) and the same number of concentric balls with radius \( r_{j_0+1} \). In each subsequent generation with \( j > j_0 + 1 \), we choose in every ball of radius \( r_{j-1} \) of the previous generation \( 2^n \) pairwise disjoint open balls of radius \( R_j \) and in each of the new balls a concentric closed ball with radius \( r_j \). We denote by \( a_{i,j} \) the centers, \( i = 1, \ldots, 2^{(j-j_0)n} \).

The function

\[
\tilde{\eta}_j(r) = \frac{\log R_j - \log r}{\log R_j - \log r_j}
\]

attains the values \( \tilde{\eta}_j(R_j) = 0, \tilde{\eta}_j(r_j) = 1 \). By truncation and smoothing we find a smooth function \( \eta_j : \mathbb{R} \to \mathbb{R} \) such that

\[
\begin{align*}
\eta_j(r) &= 0, \quad r \geq R_j, \\
\eta_j(r) &= 1, \quad r \leq r_j, \\
0 &\leq -r\eta'_j(r) \leq 2\log^{-1} \left( \frac{R_j}{r_j} \right), \quad r_j < r < R_j.
\end{align*}
\]

By (14) we have

\[
(15) \quad 0 \leq -r\eta'_j(r) \leq 2j^{1-\beta} \frac{1}{r}, \quad j > j_0.
\]

We define

\[
\Psi_{ij}(x) := \eta_j(|a_{i,j} - x|), \quad i = 1, \ldots, 2^{(j-j_0)n}
\]

and given a bijection \( \sigma : \{1, \ldots, 2^n\} \to \{-1, 1\}^n \), we set \( v_k = \sigma(k) \) if \( 1 \leq k \leq 2^n \) and \( v_k := v_{k-2^n} \) for \( k > 2^n \). Further, we define

\[
w_{ij} := 2^{-j}v_i, \quad i = 1, \ldots, 2^{(j-j_0)n}.
\]
Finally, we let

\[ f := \sum_{j=j_0+1}^{\infty} f_j \quad \text{with} \quad f_j = \sum_{i=1}^{2^{(j-j_0)n}} w_{ij} \Psi_{ij}(x). \]

The mapping \( f \) will serve as the required example.

4.2. The mapping blows up a set of measure zero. Let

\[ N_j := \bigcup_{i=1}^{2^{(j-j_0)n}} \overline{B}(a_{i,j}, r_j), \quad N := \bigcap_{j=j_0+1}^{\infty} N_j. \]

Then

\[ |N_j| \leq C2^{n j} r_j^n \leq C2^{n j} \exp(-n(j + 1)^\beta) \to 0 \quad \text{as} \quad j \to \infty \]

and thus \( |N| = 0 \). On the other hand, each dyadic cube intersecting \([-2^{-j_0}, 2^{-j_0}]^n\) also intersects \( f(N) \). The continuity of \( f \), which we will verify in Subsection 4.4, together with the fact that \( N \) is compact, now shows that \( f(N) \) contains \([-2^{-j_0}, 2^{-j_0}]^n\) and hence has positive measure.

4.3. The integrability of the derivative. We write \( A_{i,j} \) for the closed annulus with center \( a_{i,j} \) and inner and outer radii \( r_j \) and \( R_j \), respectively, and set

\[ A_j = \bigcup_{i=1}^{2^{(j-j_0)n}} A_{i,j} \]

Then \( Df_j = 0 \) outside \( A_j \). Since the annuli \( A_{i,j} \) are pairwise disjoint and \( |w_{ij}|/\sqrt{n} = 2^{-j} \) we have

\[ |Df(x)| = |Df_j(x)| = |w_{ij} D\Psi_{ij}(x)| \leq \sqrt{n} 2^{-j} |\eta_j'(|x - a_{i,j}|)|, \quad x \in A_{i,j}. \]

Using (15) we obtain

\[ \int_{\mathbb{R}^n} |Df_j| \, dy \leq C2^{n j} \int_{r_j}^{R_j} r^{n-1} 2^{-j} |\eta_j'(r)| \, dr \leq C2^{n j} j 2^{-j} \int_{r_j}^{R_j} r^{n-2} \, dr \]

\[ \leq C2^{n j} j 2^{-j} R_{j-1} = C2^{n j} j 2^{-j} \exp(-(n - 1) j \beta). \]

It follows that the partial sums form a fundamental sequence in \( W^{1,1} \) and the total sum is a Sobolev function. Since \( |N| = 0 \), we have

\[ \int_{\mathbb{R}^n} |Df|^n \log^\lambda(e + |Df|) \, dy = \sum_{j=j_0+1}^{\infty} I_j, \]

where

\[ I_j = \int_{A_j} |Df|^n \log^\lambda(e + |Df|) \, dy \]

\[ \leq C2^{n j} 2^{-n j} \int_{r_j}^{R_j} r^{n-1} |\eta_j'(r)|^n \log^\lambda(e + \sqrt{n} 2^{-j} |\eta_j'(r)|) \, dr. \]

By (13) and (15), we have

\[ e + \sqrt{n} 2^{-j} |\eta_j'(r)| \leq \frac{1}{2R_{j_0}} + \frac{1}{2r} \leq \frac{1}{r}, \quad r_j < r < R_j, \]
We estimate

\[
I_j \leq C_j^{\beta(1 - \beta)} \left( \int_{r_j}^{R_j} \log \left( \frac{1}{r} \right) \frac{dr}{r} \right) \leq C_j^{\beta(1 - \beta)} \left( \frac{\log \lambda + 1}{r_j} - \frac{\log \lambda + 1}{R_j} \right)
\]

\[
\leq C_j^{\beta(1 - \beta)} \left( (j + 1)^{\beta(\lambda + 1)} - j^{\beta(\lambda + 1)} \right) \leq C_j^{\lambda \beta - 1 + n - n \beta}.
\]

We conclude that the integrability of the derivative is as required.

4.4. Modulus of continuity. We seek for an estimate of the modulus of continuity of \( \tilde{f} = f_{j_0} + \cdots + f_j \) which does not depend on \( J \). The same estimate then holds for \( f \). Without loss of generality, choose \( x, y \in \mathbb{R}^n \setminus N \) such that \( f(x) \neq \hat{f}(y) \). Then there exist chains of balls,

\[
\overline{B}(a_{j_0 + 1}(x), R_{j_0 + 1}) \supset \overline{B}(a_{j_0 + 1}(x), R_{j_0 + 2}) \supset \cdots \supset \overline{B}(a_{j_0 + 1}(x), R_{j_0}) \ni x,
\]

\[
\overline{B}(a_{j_0 + 1}(y), R_{j_0 + 1}) \supset \overline{B}(a_{j_0 + 1}(y), R_{j_0 + 2}) \supset \cdots \supset \overline{B}(a_{j_0 + 1}(y), R_{j_0}) \ni y,
\]

such that \( a_j(x), a_j(y) \in \{ a_{i,j} : i = 1, \ldots, 2^{(j-j_0)n} \} \) and

\[
x \notin \bigcup_{j > j_0} \bigcup_{i = 1}^{2^{(j-j_0)n}} \overline{B}(a_{i,j}, R_j), \quad y \notin \bigcup_{j > j_0} \bigcup_{i = 1}^{2^{(j-j_0)n}} \overline{B}(a_{i,j}, R_j).
\]

We denote by \( A_j(x) \) and \( A_j(y) \) the corresponding annuli \( \overline{B}(a_j(x), R_j) \setminus \overline{B}(a_j(x), R_j) \) and \( \overline{B}(a_j(y), R_j) \setminus \overline{B}(a_j(y), R_j) \), respectively. We may assume that \( j_x \leq j_y \). Let \( L \) be the line segment connecting \( x \) and \( y \). Our next step is to find \( x', y' \in L \) such that

\[
|\tilde{f}(y) - \hat{f}(x)| \leq C|\tilde{f}(y') - \hat{f}(x')|,
\]

and \( x' \) and \( y' \) belong both to the same annulus \( A_{i,j} \). This is possible if \( \tilde{f}(y) \neq \hat{f}(x) \), but otherwise there is nothing to be estimated.

We distinguish several cases.

Case 1. There exists \( a \in \mathbb{R}^n \) such that \( a_{j_0}(x) = a_{j_0}(y) = a \) and \( j_y = j_x \). We may assume that \( |x - a| \geq |y - a| \). Then \( x \in A_{j_y}(x) \), as otherwise we would get \( \hat{f}(x) = \hat{f}(y) \). We set \( x' = x \) and find \( y' \in L \cap A_{j_y}(x) \) such that \( \tilde{f}(y') = \hat{f}(y') \). We have \( |\tilde{f}(x') - \hat{f}(x')| = |\tilde{f}(x) - \hat{f}(y)| \).

Case 2. There exist \( a, b \in \mathbb{R}^n \) such that \( a_{j_x}(x) = a_{j_x}(y) = a \) and \( j_y = j_x + 1 \). Then we find \( z \in L \cap \partial B(b, R_{j_x + 1}) \). If \( |\tilde{f}(y) - \hat{f}(z)| \geq |\tilde{f}(x) - \hat{f}(z)| \), we find \( y' \in L \cap A_{j_x}(y) \) such that \( \tilde{f}(y') = \hat{f}(y) \) and set \( x' = z \). If \( |\tilde{f}(y) - \hat{f}(z)| < |\tilde{f}(x) - \hat{f}(z)| \), we find \( y' \in L \cap \partial B(a, R_{j_x}) \) and set \( x' = x \). We obtain \( |\tilde{f}(x') - \hat{f}(y')| \geq \frac{1}{2} |\tilde{f}(x) - \hat{f}(y)| \).

Case 3. There exist \( a, b \in \mathbb{R}^n \) such that \( a_{j_x}(x) = a_{j_x}(y) = a \) and \( j_y > j_x + 1 \). We find \( x' \in L \cap \partial B(b, R_{j_x+1}) \) and \( y' \in L \cap \partial B(b, R_{j_x+1}) \). Then

\[
|\tilde{f}(y) - \hat{f}(x)| \leq \sqrt{n} \left( (2^{-j_x} + 2^{-j_x-1} + \cdots) \right) \leq 4\sqrt{n} 2^{-j_x-1} \leq 4|\tilde{f}(y') - \hat{f}(x')|.
\]

Case 4. There exists \( j_1 \leq j_y \) such that \( a_{j_1}(x) \neq a_{j_1}(y) \), and \( a_j(x) = a_j(y) \) whenever \( j_0 < j < j_1 \). Then we find \( x'' \in L \cap \partial B(a_j(x), R_j), y'' \in L \cap \partial B(a_j(y), R_j) \) and estimate

\[
|\tilde{f}(x) - f(x)| \leq |\tilde{f}(y) - \hat{f}(y'')| + |\tilde{f}(x'') - \hat{f}(x)|.
\]

The relation of \( x \) and \( x'' \) and of \( y \) and \( y'' \), respectively, is one of these described in the preceding steps. Hence, we find \( x', y' \in L \) such that \( x' \) and \( y' \) belong both to
the same annulus \( A_{i,j} \) and
\[
|\tilde{f}(y) - \tilde{f}(x)| \leq 8|\tilde{f}(y') - \tilde{f}(x')|.
\]

Therefore, in either case we obtain (17) with \( C = 8 \). Letting \( J \to \infty \), we observe that
\[
|f(y) - f(x)| \leq 8 \sup_i \sup_j \{|f(\tilde{y}) - f(\tilde{x})| : \tilde{x}, \tilde{y} \in A_{i,j}, |\tilde{y} - \tilde{x}| \leq |y - x|\}.
\]
Since \( \omega \) is increasing, we can reduce the estimate to the case that \( x \) and \( y \) belong both to the same annulus \( A_{i,j} \). We have
\[
|f(x) - f(y)| = |f_j(x) - f_j(y)|
\]
\[
= |w_{ij} \Psi_j(x) - w_{ij} \Psi_j(y)| = 2^{-j} \sqrt{n} |\eta_j(|a_{i,j} - x|) - \eta_j(|a_{i,j} - y|)|.
\]
We assume that \( |a_{i,j} - x| \leq |a_{i,j} - y| \) and set \( r := |a_{i,j} - x|, h = |a_{i,j} - y| - r \). Then \( 0 \leq h \leq \min\{|x - y|, R_j - r_j\} \). Thus
\[
\frac{|f(x) - f(y)|}{\sqrt{n}} \leq 2^{-j} \int_r^{r+h} |\eta'(t)| \, dt \leq \frac{2j^{1-\beta}}{2^j} \log \left( 1 + \frac{h}{r} \right) \leq j^{1-\beta} \log \left( 1 + \frac{h}{r_j} \right).
\]
Set
\[
s = \log \frac{\beta}{\log 2} \left( \frac{1}{h} \right).
\]
Then
\[
e^{-j \beta} = R_j \geq h = e^{-s^\beta},
\]
and thus \( s \geq j \). We will distinguish two cases.

**Case 1.** \( j \leq s \leq j + 1 \). Then
\[
j^{1-\beta} 2^{1-j} \log \left( 1 + \frac{e^{-s^\beta}}{2ne^{-(j+1)s}} \right)
\]
\[
\leq j^{1-\beta} 2^{1-j} \log \left( 2 + \frac{e^{-s^\beta}}{e^{-(j+1)s}} \right) \leq j^{1-\beta} 2^{1-j} (\log 2 + (j + 1)^\beta - j^\beta)
\]
\[
\leq j^{1-\beta} 2^{1-j} (\log 2 + \beta(j + 1)^{\beta - 1}) \leq 2^{1-j} (\log 2 + \beta 2^{\beta - 1}) \leq C 2^{-s}.
\]

**Case 2.** \( s > j + 1 \). Then
\[
s^\beta - (j + 1)^\beta \geq \beta(s - j - 1)(j + 1)^{\beta - 1} \geq (s - j - 1).
\]
Hence
\[
j^{1-\beta} 2^{1-j} \log \left( 1 + \frac{e^{-s^\beta}}{2ne^{-(j+1)s}} \right)
\]
\[
\leq 2^{1-j-n} \frac{e^{-s^\beta}}{e^{-(j+1)s}} \leq 2^{1-j-n} e^{j+1-s} \leq 2^{1-j-n} 2^{j+1-s} \leq C 2^{-s}.
\]
In both cases we have
\[
|f(y) - f(x)| \leq C 2^{-s} = C 2^{-\log \beta \left( \frac{1}{h} \right)} \leq C 2^{-\log \beta \left( \frac{1}{|x - y|} \right)},
\]
which is the required modulus of continuity.
4.5. The case $\lambda=n-1$. We may proceed as in the case $0 \leq \lambda < n-1$ and construct a mapping $f \in W^{1,1}(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} |Df|^n \log^{n-1}(e + |Df|) \, dy < \infty,$$

$f$ has modulus of continuity no worse than

$$\Psi(t) = C \exp(- \log 2(\log \log(1/t))^\frac{1}{\beta}),$$

and violates Luzin's condition (N).

In this case, we fix $\beta > \frac{n}{n-1}$ and set

$$r_j := 2^n \exp(- \exp(j+1)\beta),$$
$$R_j := \exp(- \exp(j\beta)).$$

The only other substantial change to the construction that we gave above is that we set

$$\tilde{\eta}_j(r) = \frac{\log \log(1/r) - \log \log(1/R_j)}{\log \log(1/r_j) - \log \log(1/R_j)}.$$

We leave the details to the reader.

5. Size of the exceptional set

Marcus and Mizel have shown in [MM73] that continuous mappings in $W^{1,p}(\Omega;\mathbb{R}^n)$ satisfy Luzin’s condition (N) for $p > n$. From [MM95], we know that given an $n$-quasicontinuous mapping $f$ in $W^{1,n}(\Omega;\mathbb{R}^n)$, we find a zero-dimensional set $E$ (depending on $f$) such that $f$ satisfies Luzin’s condition (N) in $\Omega \setminus E$. Here, we give an upper bound for the size of the exceptional set when we have an additional logarithmic term in the integrability condition for $|Df|$. We will use the following result to estimate the size of the set where $f$ fails to satisfy Luzin’s condition.

**Lemma 5.1.** Let $v: \Omega \to [0, \infty)$ be an integrable function and $\varphi$ a gauge function. Let $E \subset \Omega$. Assume that

$$\limsup_{r \to 0^+} (\varphi(2r))^{-1} \int_{B(x,r)} v \, dy > 1, \quad x \in E.$$

Then there exists a constant $C = C(n)$ such that

$$\mathcal{H}^{\varphi}(E) \leq C \int_U v \, dy$$

for each open set $U \subset \Omega$ containing $E$.

Moreover, if $|E| = 0$, then $\mathcal{H}^{\varphi}(E) = 0$.

**Proof.** Let $U$ be an open set containing $E$ and fix $\delta > 0$. Choose for each $x \in E$ a radius $r_x < \delta/2$ such that $B(x,2r_x) \subset U$ and

$$\int_{B(x,4r_x)} v \, dy \geq \varphi(2r_x).$$

By Besicovitch’s Theorem, see for example Theorem 1.5.2 in [EG92], there exist a constant $N = N(n)$ and subcollections $\mathcal{G}_1, \ldots, \mathcal{G}_m$, $m \leq N$, such that each of these
collections consists of disjoint balls, and the union of all collections covers $E$. Then

$$\mathcal{H}^\varphi_\delta(E) \leq \sum_{i=1}^N \sum_{B(x,r_x) \in G_i} \varphi(2r_x) \leq \sum_{i=1}^N \sum_{B(x,r_x) \in G_i} \int_{B(x,r_x)} v \, dy \leq N \int_U v \, dy.$$  

Since $\delta > 0$ was arbitrary, the first claim follows.

Assume now that $|E| = 0$. Given $\varepsilon > 0$, by the absolute continuity of the integral and the fact that $v$ is integrable, we can choose the open set $U$ so small that

$$N \int_U v \, dy < \varepsilon.$$  

This implies $\mathcal{H}^\varphi(E) < \varepsilon$ and letting $\varepsilon$ tend to zero, we obtain the claim. \(\square\)

Recall that $f \in L^1_{\text{loc}}(\Omega; \mathbb{R}^d)$ is said to be a precise representative if for each $x \in \Omega$ and $c \in \mathbb{R}^d$ we have

$$\lim_{r \to 0^+} \int_{B(x,r)} |f - c| \, dy = 0 \implies c = f(x).$$

To investigate moduli of continuity of precise representatives, we need a joint estimate for integral means and medians.

For the rest of this section, we fix $0 \leq \lambda \leq n - 1$ and $\mu > 0$, write $\alpha = 1 - \frac{\lambda}{n - 1}$ and set

$$\varphi(r) = \begin{cases} 
\exp(- (n + 1) \mu \log^\alpha(\frac{1}{r})), & \alpha > 0, \\
2^{-n} \mu^n \log^{-n(1 + \mu)}(\frac{1}{r}), & \alpha = 0,
\end{cases}$$

$$\psi(r) = \begin{cases} 
\exp(- \mu \log^\alpha(\frac{1}{r})), & \alpha > 0, \\
\log^{-\mu}(\frac{1}{r}), & \alpha = 0.
\end{cases}$$

**Lemma 5.2.** There exists a radius $R_0 = R_0(\alpha, \mu)$ such that

$$\psi(2r) - \psi(r) \geq (\varphi(r))^{1/n}, \quad 0 < r < R_0.$$

**Proof.** Assume first that $\alpha > 0$. We have

$$\psi(2r) - \psi(r) = \exp(- \mu \log^\alpha(\frac{1}{2r})) - \exp(- \mu \log^\alpha(\frac{1}{r})) 
\geq \mu \exp(- \mu \log^\alpha(\frac{1}{r})) \left( \log^\alpha(\frac{1}{r}) - \log^\alpha(\frac{1}{2r}) \right).$$

Now, it is enough to find $R_0$ such that

$$\mu \left( \log^\alpha(\frac{1}{r}) - \log^\alpha(\frac{1}{2r}) \right) \geq \exp(- \mu \log^\alpha(\frac{1}{r})), \quad 0 < r < R_0.$$  

If $\alpha = 0$, we estimate

$$\psi(2r) - \psi(r) = \log^{-\mu}(\frac{1}{r}) - \log^{-\mu}(\frac{1}{2r}) 
\geq \mu \log^{-\mu - 1}(\frac{1}{r}) \left( \log(\frac{1}{r}) - \log(\frac{1}{2r}) \right) = \mu \log 2 \log^{-\mu - 1}(\frac{1}{r}).$$  

\(\square\)

**Lemma 5.3.** Let $f \in W^{1,n}(\Omega; \mathbb{R}^d)$ and $x \in \Omega$. Suppose that

$$\limsup_{r \to 0^+} \frac{\int_{B(x,r)} |Df|^n \, dy}{\psi(2r)} \leq 1.$$
Then there is a constant $C = C(\alpha, \gamma, n)$ and $R = R(x) > 0$ such that for each $0 < t < r$ we have
\[
|f_{B(x,r)} - f_{B(x,t)}| + |\omega_f(x,r) - \omega_f(x,t)| \leq C\psi(r).
\]
\[\tag{18}
(18)
\]
Proof. Let $R_0$ be the radius from Lemma 5.2. Choose $0 < R < R_0$ such that $B(x,R) \subset \Omega$ and
\[
\left(\int_{B(x,r)} |Df|^n \, dy\right) \leq 2\varphi(2r) \leq C\varphi(r/2), \quad 0 < r < R.
\]
\[\tag{19}
(19)
\]
If $\varphi$ is concave on a right neighborhood of 0, then we may choose $C = 8$. This fails (and calls for a different constant) only in the case $\alpha = 1$, $(n+1)\mu > 1$.

First consider the case $0 < t < r < 2t$. Then
\[
|f_{B(x,r)} - f_{B(x,t)}| + |\omega_f(x,r) - \omega_f(x,t)| \leq C\left(\int_{B(x,r)} |Df|^n \, dy\right)^{\frac{1}{n}}.
\]
\[\tag{20}
(20)
\]
The estimate of $|f_{B(x,r)} - f_{B(x,t)}|$ is a standard use of the Poincaré inequality; for the estimate of $|\omega_f(x,r) - \omega_f(x,t)|$ we apply the Poincaré inequality (to $u = |f-f(x)|$) like in the proof of Lemma 3.2. We continue by applying (19) and Lemma 5.2 and obtain
\[
|f_{B(x,r)} - f_{B(x,t)}| + |\omega_f(x,r) - \omega_f(x,t)| \leq C(\varphi(r/2))^{\frac{1}{2}}
\]
\[\leq C(\psi(r) - \psi(r/2)).
\]
In the general case $0 < t < r < R$, we find $j \in \mathbb{N}$ such that $2^{j-1} \leq r/t < 2^j$ and consider a chain of balls $B(0,r_k)$, $k = 0, \ldots, j$, where $r_0 = r$ and $r_k = t2^{j-k}$ for $k = 1, \ldots, j$. Then we apply (20) to each ball in the chain. Summing over $k$ and using the cancellation effect of (20) we obtain
\[
|f_{B(x,r)} - f_{B(x,t)}| + |\omega_f(x,r) - \omega_f(x,t)| \leq C\psi(r).
\]
\[\square
\]
\[\tag{20}
\]
Lemma 5.4. Let $f \in W^{1,n}(\Omega; \mathbb{R}^d)$ be a precise representative and $x \in \Omega$. Suppose that
\[
\limsup_{r \to 0_+} \frac{\int_{B(x,r)} |Df|^n \, dy}{\varphi(2r)} \leq 1.
\]
\[\tag{21}
(21)
\]
Then there exists a constant $C = C(\alpha, \gamma)$ and $R = R(x) > 0$ such that
\[
\omega_f(x,r) \leq C\psi(r), \quad 0 < r < R.
\]
\[\tag{22}
(22)
\]
Proof. By Lemma 5.3, there exists $R > 0$ such that
\[
|f_{B(x,r)} - f_{B(x,t)}| \leq C\psi(r), \quad 0 < t < r < R.
\]
\[\tag{23}
(23)
\]
This guarantees the existence of the limit
\[
c = \lim_{r \to 0_+} f_{B(x,r)}.
\]
By the Poincaré inequality, we have
\[
\int_{B(x,r)} |f - c| \, dy \leq |f_{B(x,r)} - c| + \int_{B(x,r)} |f - f_{B(x,r)}| \, dy
\]
\[\leq |f_{B(x,r)} - c| + C\left(\int_{B(x,r)} |Df|^n \, dy\right)^{1/n} \to 0 \quad \text{as } r \to 0_+.
\]
Thus $c = f(x)$ and $x$ is a Lebesgue point for $f$. It easily follows that
\[
\lim_{t \to 0_+} \omega_f(x,t) = 0.
\]
Now, using (18) again we have
\[
|\omega_f(x,r) - \omega_f(x,t)| \leq C \psi(r), \quad 0 < t < r < R.
\]
We pass to the limit for $t \to 0_+$ and obtain the required estimate. □

Proof of Theorem 1.2. Without loss of generality, we may assume that $\gamma < n$. Set
\[
E = \left\{ x \in \Omega : \limsup_{r \to 0_+} \frac{\int_{B(x,r)} |Df|^n \, dy}{\varphi(2r)} > 1 \right\}.
\]
Then $E$ does not contain any Lebesgue point of $|Df|^n$ and thus $|E| = 0$. Lemma 5.1 gives
\[
\mathcal{H}^n(E) = 0.
\]
By Lemma 5.4, for each $x \in \Omega \setminus E$ we have
\[
\omega_f(x,r) \leq C \psi(r), \quad 0 < r < R(x).
\]
Hence we may apply Theorem 3.1 to obtain the claim. □

6. Regularity of the blown up set

As compact, perfect, totally disconnected sets are exactly the sets homeomorphic to the ternary Cantor set, cf. Corollary 2-98 in [HY61], we refer to them as Cantor sets. In Example 1.3, a Cantor set gets mapped onto a set of positive measure. Our next result shows all examples must exhibit such a behavior.

Theorem 6.1. Let $\Omega \subset \mathbb{R}^n$ be open and $f: \Omega \to \mathbb{R}^d$ be continuous. If for some gauge function $\varphi$ there is a set $N \subset \Omega$ with $\mathcal{H}^n(N) = 0$ and $\mathcal{H}^n(f(N)) > 0$, then there is a Cantor set $K$ such that $\mathcal{H}^n(K) = 0$ and $\mathcal{H}^n(f(K)) > 0$.

First, we need a result about the image of an intersection.

Lemma 6.2. Let $A \subset \mathbb{R}^n$ and $f: A \to \mathbb{R}^d$ be a continuous mapping. We suppose that $K_1 \supset K_2 \supset \cdots$ is a sequence of nested compact subsets of $A$. Then
\[
f\left(\bigcap_{m=1}^{\infty} K_m\right) = \bigcap_{m=1}^{\infty} f(K_m).
\]

Proof. This is a standard application of compactness, see e.g. Exercises 4.28 and 4.29vi in [Kec95]. □

Proof of Theorem 6.1. Since Hausdorff measures are Borel regular, we may assume that $N$ is a Borel set. By, for example, Lemma 423G in [Fre06], we see that $f(N)$ is analytic. We apply Theorem 57 in [Rog70] to obtain a compact set $M \subset f(N)$ with positive and finite $\mathcal{H}^n$-measure. We look at the closed set $f^{-1}(M)$ and by decomposing if necessary, we may further assume that $N$ is a set of $\mathcal{H}^n$-measure zero contained in a compact set $K_0$ (not necessarily of measure zero) with $\mathcal{H}^n(f(N)) \leq \mathcal{H}^n(f(K_0)) < \infty$.

We start with verifying the existence of a compact set $K \subset \Omega$ with $\mathcal{H}^n(K) = 0$ and $\mathcal{H}^n(f(K)) > 0$.

Let us fix a sequence $(a_m)_m$ decreasing to zero such that $a_1 < 1$ and $\prod_{m=1}^{\infty} (1 - a_m) \geq \frac{1}{2}$. 

We may cover \( N \) with countably many closed balls \( \mathcal{B}_i \) such that \( \operatorname{diam} \mathcal{B}_i < a_1 \) and \( \sum_i \varepsilon(\operatorname{diam} \mathcal{B}_i) < a_1 \).

We set
\[
L_i := K_0 \cap \bigcup_{k=1}^i \mathcal{B}_k
\]
and note that
\[
\mathcal{H}^n \left( \bigcup_i f(L_i \cap N) \right) = \mathcal{H}^n(f(N)) > 0.
\]

Since the sequence of the sets \( L_i \) is increasing, we find \( i_1 \) such that
\[
\mathcal{H}^n \left( f \left( \bigcup_{i=1}^{i_1} L_i \cap N \right) \right) \geq (1 - a_1)\mathcal{H}^n(f(N)).
\]

We set
\[
K_1 := \bigcup_{i=1}^{i_1} L_i.
\]

Then \( K_1 \) is clearly compact.

Let us argue now that we can find a sequence \( K_1 \supset K_2 \supset \cdots \) of nested compact sets with the property that \( \mathcal{H}^n(K_m) < a_m \) for \( m \geq 1 \) and
\[
\mathcal{H}^n(f(K_m)) \geq \mathcal{H}^n(f(N \cap K_m)) \geq \prod_{m=1}^M (1 - a_m)\mathcal{H}^n(f(N)) \geq \frac{1}{2}\mathcal{H}^n(f(N)).
\]

Having constructed \( K_1, \ldots, K_M \), we obtain \( K_{M+1} \) by applying above procedure for \( N \cap K_M \) and \( a_{M+1} \) instead of \( N \) and \( a_1 \), respectively, and intersecting the obtained set by \( K_M \).

Let us now define the compact set \( K := \bigcap_{k=1}^\infty K_k \). For all \( k \in \mathbb{N} \), we have
\[
\mathcal{H}^n_{a_k}(K) \leq \mathcal{H}_{a_k}^n(K_k) < a_k.
\]

Thus \( \mathcal{H}^n(K) = 0 \).

To show that \( \mathcal{H}^n(f(K)) > 0 \), we apply Lemma 6.2 and conclude that
\[
\mathcal{H}^n(f(K)) = \mathcal{H}^n \left( \bigcap_{m=1}^\infty f(K_m) \right) = \lim_{m \to \infty} \mathcal{H}^n(f(K_m)) \geq \frac{1}{2}\mathcal{H}^n(f(N)) > 0.
\]

Next, we want to show that we can choose \( K \) to be totally disconnected. The compact set constructed in the preceding step will be relabelled as \( K' \). We may assume that \( K' \) is a subset of a dyadic cube \( I_n \). We claim first that we can find \( 1 \leq m \leq n \) and a closed dyadic \( m \)-dimensional cube \( I \) such that

(a) \( \mathcal{H}^n(f(I \cap K')) > 0 \), and

(b) \( \mathcal{H}^n(f(\partial I \cap K')) = 0 \) for every \( m \)-dimensional dyadic subcube of \( J \) of \( I \).

Indeed, letting \( m = n \), if \( I = I_m \) does not satisfy (b), then there exists a \( m \)-dimensional dyadic subcube \( J_m \) and a \((m-1)\)-dimensional face \( I_{m-1} \) of \( J_m \) such that \( \mathcal{H}^n(f(\partial J_m \cap K')) > 0 \). We continue this process with \( I = I_{m-1} \) until (b) holds for \( I = I_k \) for some \( k \). It is clear that the process has to stop for if \( m = 1 \), then the boundaries of the subcubes under consideration are points and therefore mapped to sets of \( \mathcal{H}^n \)-measure zero.

We may and will in the following assume that \( K' = I \cap K' \). Let \((J_i)\) be an enumeration of all dyadic subcubes of \( I \). We fix \( 0 < \varepsilon < \mathcal{H}^n(f(K')) \). The set \( K' \setminus \bigcup_i \partial J_i \) is totally disconnected but not necessarily compact as the sets \( \partial J_i \) are...
not open. However, choosing $J = J_i$, we can write $\partial J_i$ as intersections $\bigcap_m U_m = \bigcap_m \overline{U_m}$ where $U_m$ is the $1/m$-neighborhood of $J_i$. Now

$$\partial J \cap K' = \bigcap_m U_m \cap K' = \bigcap_m (\overline{U_m} \cap K').$$

and thus, by Lemma 6.2, we see that

$$0 = H^n(f(\partial J \cap K')) = H^n\left(f\left(\bigcap_m (\overline{U_m} \cap K')\right)\right) = H^n\left(\bigcap_m f(\overline{U_m} \cap K')\right).$$

We can thus choose an open set $O_i$ such that $\partial J \subset O_i$ and $H^n(f(K' \cap O_i)) < \varepsilon/2^i$.

By Cantor-Bendixson's theorem, see for example Theorem XIV.5.3 in [Kur72], we may write any closed set as union of a perfect and a countable set. We extract such a perfect set from $K' \setminus \bigcup_i O_i$ and call it $K$.

It is clear that $K$ is compact and that $H^n(K) = 0$. By the choice of the sets $O_i$, we further have $H^n(f(K)) > 0$, and by above choice, $K$ is perfect.

To show that $K$ is totally disconnected, let us assume that $A \subset K$ and $x$ and $y$ are two points in $A$. Then there is a dyadic cube $J$ containing $x$ but not $y$. Then the set $A \setminus \partial J = A$ is the union of two disjoint open sets one containing $x$ and the other $y$. Thus $A$ is not connected. We conclude that $K$ is totally disconnected. □

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