Review of Rotational Symmetry Breaking in Baby Skyrme Models

Marek Karliner and Itay Hen

Raymond and Beverly Sackler School of Physics and Astronomy
Tel-Aviv University, Tel-Aviv 69978, Israel.
marek@proton.tau.ac.il

We discuss one of the most interesting phenomena exhibited by baby skyrmions – breaking of rotational symmetry. The topics we will deal with here include the appearance of rotational symmetry breaking in the static solutions of baby Skyrme models, both in flat as well as in curved spaces, the zero-temperature crystalline structure of baby skyrmions, and finally, the appearance of spontaneous breaking of rotational symmetry in rotating baby skyrmions.

1.1. Breaking of Rotational Symmetry in Baby Skyrme Models

The Skyrme model\(^1,2\) is an SU(2)-valued nonlinear theory for pions in (3+1) dimensions with topological soliton solutions called skyrmions. Apart from a kinetic term, the Lagrangian of the model contains a ‘Skyrme’ term which is of the fourth order in derivatives, and is used to introduce scale to the model.\(^3\) The existence of stable solutions in the Skyrme model is a consequence of the nontrivial topology of the mapping \(M\) of the physical space into the field space at a given time, \(M : S^3 \rightarrow SU(2) \cong S^3\), where the physical space \(\mathbb{R}^3\) is compactified to \(S^3\) by requiring the spatial infinity to be equivalent in each direction. The topology which stems from this one-point compactification allows the classification of maps into equivalence classes, each of which has a unique conserved quantity called the topological charge.

The Skyrme model has an analogue in (2+1) dimensions known as the baby Skyrme model, also admitting stable field configurations of a solitonic nature.\(^4\) Due to its lower dimension, the baby Skyrme model serves as a simplification of the original model, but nonetheless it has a physical significance in its own right, having several applications in condensed-matter physics,\(^5\) specifically in ferromagnetic quantum Hall systems.\(^6-9\) There, baby skyrmions describe the excitations relative to ferromagnetic quantum Hall states, in terms of a gradient expansion in the spin density, a field with properties analogous to the pion field in the 3D case.\(^10\)

The target manifold in the baby model is described by a three-dimensional vector \(\phi = (\phi_1, \phi_2, \phi_3)\) with the constraint \(\phi \cdot \phi = 1\). In analogy with the (3+1)D case, the domain of this model \(\mathbb{R}^2\) is compactified to \(S^2\), yielding the topology required for

\(^1\)to appear in: G. Brown and M. Rho, Eds., *Multifaceted Skyrmions*, (World Scientific, Singapore, 2009).

\(^2\)A version of this manuscript with higher-resolution figures is available at [www.tau.ac.il/~itayhe/SkReview/SkReview.rar](http://www.tau.ac.il/~itayhe/SkReview/SkReview.rar)
the classification of its field configurations into classes with conserved topological charges. The Lagrangian density of the baby Skyrme model is given by:

\[ L = \frac{1}{2} \partial_{\mu} \phi \cdot \partial_{\mu} \phi - \frac{\kappa^2}{2} \left[ \left( \partial_{\mu} \phi \cdot \partial_{\nu} \phi \right)^2 - \left( \partial_{\mu} \phi \cdot \partial_{\nu} \phi \right) \cdot \left( \partial_{\mu} \phi \cdot \partial_{\nu} \phi \right) \right] - \mu^2 \left( 1 - \phi^3 \right), \]

(1.1)

and consists of a kinetic term, a Skyrme term and a potential term.

While in (3+1) dimensions the latter term is optional, its presence in the (2+1)D model is necessary for the stability of the solutions. However, aside from the requirement that the potential vanishes at infinity for a given vacuum field value (normally taken to be \( \phi^{(0)} = (0, 0, 1) \)), its exact form is arbitrary and gives rise to a rich family of possible baby-Skyrme models, several of which have been studied in detail in the literature. The simplest potential is the ‘holomorphic’ model with \( U(\phi_3) = \mu^2 (1 - \phi_3)^4 \). It is known to have a stable solution only in the charge-one sector (the name refers to the fact that the stable solution has an analytic form in terms of holomorphic functions). The model with the potential \( U(\phi_3) = \mu^2 (1 - \phi_3^3) \) (commonly referred to as the ‘old’ model) has also been extensively studied. This potential gives rise to very structured non-rotationally-symmetric multi-skyrmions. Another model with \( U(\phi_3) = \mu^2 (1 - \phi_3^2) \) produces ring-like multi-skyrmions. Other double-vacuum potentials which give rise to other types of solutions have also been studied.

Clearly, the form of the potential term has a decisive effect on the properties of the minimal energy configurations of the model. It is then worthwhile to see how the multisols of the baby Skyrme model look like for the one-parametric family of potentials \( U = \mu^2 (1 - \phi_3)^s \) which generalizes the ‘old’ model \((s = 1)\) and the holomorphic model \((s = 4)\). As it turns out, the value of the parameter \( s \) has dramatic effects on the static solutions of the model, both quantitatively and qualitatively, in the sense that it can be viewed as a ‘control’ parameter responsible for the repulsion or attraction between skyrmions, which in turn determines whether or not the minimal-energy configuration breaks rotational symmetry.

The Lagrangian density is now:

\[ L = \frac{1}{2} \partial_{\mu} \phi \cdot \partial_{\mu} \phi - \frac{\kappa^2}{2} \left( \left( \partial_{\mu} \phi \cdot \partial_{\nu} \phi \right)^2 - \left( \partial_{\mu} \phi \cdot \partial_{\nu} \phi \right) \cdot \left( \partial_{\mu} \phi \cdot \partial_{\nu} \phi \right) \right) - \mu^2 \left( 1 - \phi_3 \right)^s, \]

(1.2)

and contains three free parameters, namely \( \kappa, \mu \) and \( s \). Since either \( \kappa \) or \( \mu \) may be scaled away, the parameter space of this model is in fact only two dimensional. Our main goal here is to study the effects of these parameters on the static solutions of the model within each topological sector.

The multi-skyrmions of our model are those field configurations which minimize the static energy functional within each topological sector. In polar coordinates the
energy functional is given by

\[ E = \int r \, dr \, d\theta \left( \frac{1}{2} (\partial_r \phi \cdot \partial_r \phi + \frac{1}{r^2} \partial_\theta \phi \cdot \partial_\theta \phi) + \frac{\kappa^2}{2} \frac{(\partial_r \phi \times \partial_\theta \phi)^2}{r^2} + \mu^2 (1 - \phi^s) \right). \]

(1.3)

The Euler-Lagrange equations derived from the energy functional (1.3) are non-linear PDE’s, so in most cases one must resort to numerical techniques in order to solve them. In our approach, the minimal energy configuration of a baby skyrmion of charge \( B \) and a given set of values \( \mu, \kappa, s \) is found by a full-field relaxation method, which we describe in more detail in the Appendix.

1.1.1. Results

Applying the minimization procedure, we obtain the static solutions of the model for \( 1 \leq B \leq 5 \). Since the parameter space of the model is effectively two dimensional (as discussed earlier), without loss of generality we fix the potential strength at \( \mu^2 = 0.1 \) throughout, and the \( s-\kappa \) parameter space is scanned in the region \( 0 < s \leq 4 \), \( 0.01 \leq \kappa^2 \leq 1 \).

1.1.1.1. Charge-one skyrmions

In the charge-one sector, the solutions for every value of \( s \) and \( \kappa \) are stable rotationally-symmetric configurations. Figure 1.1 shows the obtained profile functions of the \( B = 1 \) solution for different values of \( s \) with \( \kappa \) fixed at \( \kappa^2 = 0.25 \). Interestingly, the skyrmion energy as a function of \( s \) is not monotonic, but acquires a minimum at \( s \approx 2.2 \), as is shown in Fig. 1.2.

![Fig. 1.1. Profile functions of the \( B = 1 \) (left) and \( B = 2 \) (right) skyrmions for \( s = 0.5 \) (solid), \( s = 1 \) (dotted) and \( s = 2 \) (dot-dashed). Here \( \kappa \) is fixed at \( \kappa^2 = 0.25 \).]
Fig. 1.2. Total energies (divided by $4\pi B$) of the charge-one (♦) charge-two (■) and charge-three (◇) skyrmions as a function of the parameter $s$ for $\kappa^2 = 0.05$. Each of the energy graphs attains a minimal value at some $s$. At $s \approx 2$ the energy-per-topological-charge of the charge-two and charge-three solutions reaches the charge-one energy (from below), and stable solutions are no longer observed.

1.1.1.2. Charge-two skyrmions

Stable solutions also exist in the $B = 2$ sector, but only for $s < 2$. They are rotationally-symmetric and ring-like, corresponding to two charge-one skyrmions on top of each other. Figure 1.1b shows the profile functions of the stable solutions for different values of $s$ and $\kappa^2 = 0.25$.

As in the $B = 1$ case, the energy of the charge-two skyrmion as a function of $s$ is non-monotonic and has a minimum around $s = 1.3$. As shown in Fig. 1.2 at $s \approx 2$ the energy of the ring-like configuration reaches the value of twice the energy of the charge-one skyrmion and stable configurations cease to exist. At this point, the skyrmion breaks apart into its constituent charge-one skyrmions, which in turn start drifting away from each other, thus breaking the rotational symmetry of the solution. Contour plots of the energy distribution of the $B = 2$ skyrmion are shown in Fig. 1.3 for $\kappa^2 = 1$ and for two $s$ values. While for $s = 1.5$ a ring-like stable configuration exists (Fig. 1.3a), for $s = 2.6$ the skyrmion breaks apart. The latter case is shown in Fig. 1.3b which is a “snapshot” taken while the distance between the individual skyrmions kept growing.

These results are in accord with corresponding results from previously known
studies of both the ‘old’ ($s = 1$) model in which ring-like configurations have been observed$^{4,15}$ and the holomorphic model for which no stable solutions have been found.$^{12,13}$

![Contour plots of the energy distributions](image)

**Fig. 1.3.** Contour plots of the energy distributions (ranging from violet – low density to red – high density) of the $B = 2$ skyrmion for $\kappa^2 = 1$. In the $s < 2$ regime, ring-like rotationally-symmetric configurations exist, corresponding to two charge-one skyrmions on top of each other (left), whereas for $s > 2$, the charge-two skyrmion splits into two one-charge skyrmions drifting infinitely apart (right).

Rotationally-symmetric charge-two locally stable solutions may also be observed in the large $s$ regime, including the ‘holomorphic’ $s = 4$ case, in which case the global minimum in this regime corresponds to two infinitely separated charge-one skyrmions. The total energy of the rotationally symmetric solutions is larger than twice the energy of a charge-one skyrmion, indicating that the split skyrmion is an energetically more favorable configuration. We discuss this issue in more detail in the section $^{12,22}$.

### 1.1.1.3. Charge-three and higher-charge skyrmions

As with the $B = 2$ skyrmion, the existence of stable charge-three skyrmions was also found to be $s$ dependent. For any tested value of $\kappa$ in the range $0.01 \leq \kappa^2 \leq 1$, we have found that above $s \approx 2$, no stable charge-three solutions exist; in this region the skyrmion breaks apart into individual skyrmions drifting further and further away from each other.

In the $s < 2$ region, where stable solutions exist, the energy distribution of the charge-three skyrmion turns out to be highly dependent on both $s$ and $\kappa$. Keeping $\kappa$ fixed and varying $s$, we find that in the small $s$ regime, ring-like rotationally-symmetric configurations exist. Increasing the value of $s$ results in stable minimal
energy configurations with only $\mathbb{Z}(2)$ symmetry, corresponding to three partially-overlapping charge-one skyrmions in a row, as shown in Figs 1.4a and 1.4c. The energy of the charge-three skyrmion also has a minimum in $\alpha$, at around $\alpha = 1.5$ (as shown in Fig. 1.2). At $\alpha \approx 2$ the energy of the skyrmion becomes larger than three times the energy of a charge-one skyrmion and stable configurations are no longer obtainable. This is illustrated in Fig. 1.4 which shows contour plots of the energy distribution of the $B = 3$ skyrmion for different values of $\alpha$ and fixed $\kappa$. For $\alpha = 0.5$ (Fig. 1.4a), the solution is rotationally symmetric and for $\alpha = 0.75$ and $\alpha = 1$ (Figs 1.4b and 1.4c respectively) the rotational symmetry of the solution is broken and only $\mathbb{Z}(2)$ symmetry remains. At $\alpha = 3$, no stable solution exists. The latter case is shown in Fig. 1.4d which is a “snapshot” taken while the distance between the individual skyrmions kept growing.

The dependence of the skyrmion solutions on the value of $\kappa$ with fixed $\alpha$ show the following behavior: While for small $\kappa$ the minimal energy configurations are rotationally-symmetric, increasing its value results in an increasingly larger rotational symmetry breaking. This is illustrated in Fig. 1.5.

![Energy Density](image)

**Fig. 1.4.** Energy densities and corresponding contour plots (ranging from violet – low density to red – high density) of the $B = 3$ skyrmion for fixed $\kappa (\kappa^2 = 0.01)$ and varying $\alpha$. In the $\alpha = 0.5$ case, the minimal energy configuration is rotationally symmetric, corresponding the three one-skyrmions on top of each other. For $\alpha = 0.75$ and $\alpha = 1$ the solutions exhibit only $\mathbb{Z}(2)$ symmetry, corresponding to partially-overlapping one-skyrmions. For $\alpha = 3$ no stable solution exists and the individual skyrmions are drifting apart.
Fig. 1.5. Energy densities and corresponding contour plots (ranging from violet – low density to red – high density) of the $B = 3$ skyrmion for fixed $s$ ($s = 0.5$) and varying $\kappa$. At low $\kappa$, the minimal energy configuration is rotationally symmetric. As $\kappa$ is increased, breaking of rotational symmetry appears, and only $Z(2)$ symmetry remains.

The $B = 4$ and $B = 5$ skyrmion solutions behave similarly to the $B = 3$ solutions. This is illustrated in Fig. 1.6, which shows the stable solutions that have been obtained in the $s = 0.9$ case and the splitting of these skyrmions into their constituents in the $s = 4$ case.

1.2. The Lattice Structure of Baby Skyrmions

The Skyrme model may also be used to describe systems of a few nucleons, and has also been applied to nuclear and quark matter. One of the most complicated aspects of the physics of hadrons is the behavior of the phase diagram of hadronic matter at finite density at low or even zero temperature. Particularly, the properties of zero-temperature skyrmions on a lattice are interesting, since the behavior of nuclear matter at high densities is now a focus of considerable interest. Within the standard zero-temperature Skyrme model description, a crystal of nucleons turns into a crystal of half nucleons at finite density.

To study skyrmion crystals one imposes periodic boundary conditions on the Skyrme field and works within a unit cell. The first attempted construction of a crystal was by Klebanov, using a simple cubic lattice of skyrmions whose symmetries maximize the attraction between nearest neighbors. Other symmetries were
Fig. 1.6. Contour plots of the energy distributions (ranging from violet – low density to red – high density) of the $B = 4$ and $B = 5$ skyrmions for $s = 0.9$ and $s = 4$ ($\kappa^2 = 0.1$). In the lower $s$ region stable solutions exist; the upper figures show a $B = 4$ skyrmion in a bound state of two charge-two skyrmions (left), and a $B = 5$ skyrmion in a two-one-two configuration. For values of $s$ higher than 2, the multi-skyrmions split into individual one-skyrmions constantly drifting apart (lower figures).

proposed which lead to crystals with slightly lower, but not minimal energy. It is now understood that it is best to arrange the skyrmions initially as a face-centered cubic lattice, with their orientations chosen symmetrically to give maximal attraction between all nearest neighbors.

The baby Skyrme model too has been studied under various lattice settings and in fact, it is known that the baby skyrmions also split into half-skyrmions when placed inside a rectangular lattice. However, as we shall see, the rectangular
periodic boundary conditions do not yield the true minimal energy configurations over all possible lattice types.\textsuperscript{32} This fact is particularly interesting both because of its relevance to quantum Hall systems in two-dimensions, and also because it may be used to conjecture the crystalline structure of nucleons in three-dimensions.

In two dimensions there are five lattice types, as given by the crystallographic restriction theorem.\textsuperscript{33} In in all of them the fundamental unit cell is a certain type of a parallelogram. To find the crystalline structure of the baby skyrmions, we place them inside different parallelograms with periodic boundary conditions and find the minimal energy configurations over all parallelograms of fixed area (thus keeping the skyrmion density fixed). As we show later, there is a certain type of parallelogram, namely the hexagonal, which yields the minimal energy configuration. In particular, its energy is lower than the known ‘square-cell’ configurations in which the skyrmion splits into half-skyrmions. As will be pointed out later, the hexagonal structure revealed here is not unique to the present model, but also arises in other solitonic models, such as Ginzburg-Landau vortices,\textsuperscript{34} quantum Hall systems,\textsuperscript{6,7} and even in the context of 3D skyrmions.\textsuperscript{35} The reason for this will also be discussed later.

In what follows we review the setup of our numerical computations, introducing a systematic approach for the identification of the minimal energy crystalline structure of baby skyrmions. In section 1.2.2 we present the main results of our study and in section 1.2.3 a somewhat more analytical analysis of the problem is presented.

1.2.1. Baby skyrmions inside a parallelogram

We find the static solutions of the model by minimizing the static energy functional:

$$E = \frac{1}{2} \int_{\Lambda} dxdy \left( (\partial_x \phi)^2 + (\partial_y \phi)^2 + \kappa^2 (\partial_x \phi \times \partial_y \phi)^2 + 2\mu^2 (1 - \phi^3) \right),$$

within each topological sector. In this example, we use the ‘old’ model potential term. In our setup, the integration is over parallelograms, denoted here by $\Lambda$:

$$\Lambda = \{ \alpha_1(L,0) + \alpha_2(sL \sin \gamma, sL \cos \gamma) : 0 \leq \alpha_1, \alpha_2 < 1 \}.$$  \hspace{1cm} (1.5)

Here $L$ is the length of one side of the parallelogram, $sL$ with $0 < s \leq 1$ is the length of its other side and $0 \leq \gamma < \pi/2$ is the angle between the ‘$sL$’ side and the vertical to the ‘$L$’ side (as sketched in Fig. 1.7).

Each parallelogram is thus specified by a set $\{L, s, \gamma\}$ and the skyrmion density inside a parallelogram is $\rho = B/(sL^2 \cos \gamma)$, where $B$ is the topological charge of the skyrmion. The periodic boundary conditions are taken into account by identifying each of the two opposite sides of a parallelogram as equivalent:

$$\phi(x) = \phi(x + n_1(L,0) + n_2(sL \sin \gamma, sL \cos \gamma)),$$  \hspace{1cm} (1.6)

with $n_1, n_2 \in \mathbb{Z}$. We are interested in static finite-energy solutions, which in the language of differential geometry are $T_2 \mapsto S_2$ maps. These are partitioned into
homotopy sectors parameterized by an invariant integral topological charge $B$, the degree of the map, given by:

$$B = \frac{1}{4\pi} \int_{\Lambda} \text{d}x\text{d}y \left( \phi \cdot (\partial_x \phi \times \partial_y \phi) \right).$$

(1.7)

The static energy $E$ can be shown to satisfy

$$E \geq 4\pi B,$$

(1.8)

with equality possible only in the ‘pure’ $O(3)$ case (i.e., when both the Skyrme and the potential terms are absent). We note that while in the baby Skyrme model on $\mathbb{R}^2$ with fixed boundary conditions the potential term is necessary to prevent the solitons from expanding indefinitely, in our setup it is not required, due to the periodic boundary conditions. We study the model both with and without the potential term.

The problem in question can be simplified by a linear mapping of the parallelograms $\Lambda$ into the unit-area two-torus $T_2$. In the new coordinates, the energy functional becomes

$$E = \frac{1}{2s\cos\gamma} \int_{T_2} \text{d}x\text{d}y \left( s^2 (\partial_x \phi)^2 - 2s \sin\gamma \partial_x \phi \partial_y \phi + (\partial_y \phi)^2 \right)$$

$$+ \frac{\kappa^2}{2B} \int_{T_2} \text{d}x\text{d}y (\partial_x \phi \times \partial_y \phi)^2 + \frac{\mu^2}{\rho} \int_{T_2} \text{d}x\text{d}y (1 - \phi_3).$$

(1.9)

We note that the dependence of the energy on the Skyrme parameters $\kappa$ and $\mu$ and the skyrmion density $\rho$ is only through $\kappa^2/\rho$ and $\mu^2/\rho$. 
In order to find the minimal energy configuration of skyrmions over all parallelograms with fixed area (equivalently, a specified $\rho$), we scan the parallelogram parameter space $\{s, \gamma\}$ and find the parallelogram for which the resultant energy is minimal over the parameter space. An alternative approach to this problem, which is of a more analytical nature, may also been implemented. We discuss it in detail in section 1.2.3.

1.2.2. Results

In what follows, we present the minimal energy static skyrmion configurations over all parallelograms, for various settings: The ‘pure’ $O(3)$ case, in which both $\kappa$, the Skyrme parameter, and $\mu$, the potential coupling, are set to zero, the Skyrme case for which only $\mu = 0$, and the general case for which neither the Skyrme term nor the potential term vanish.

In each of these settings, we scanned the parameter space of parallelograms, while the skyrmion density $\rho$ was held fixed, yielding for each set of $\{s, \gamma\}$ a minimal energy configuration. The choice as to how many skyrmions are to be placed inside the unit cells was made after some preliminary testing in which skyrmions of other charges (up to $B = 8$) were also examined. The odd-charge minimal-energy configurations turn out to have substantially higher energies than even-charge ones, where among the latter, the charge-two skyrmion is found to be the most fundamental, as it is observed that the charge-two configuration is a ‘building-block’ of the higher-charge configurations. This is illustrated in Fig. 1.8 in which the typical behavior of the multi-skyrmion energies as a function of topological charge is shown.

1.2.2.1. The pure $O(3)$ case ($\kappa = \mu = 0$)

The pure $O(3)$ case corresponds to setting both $\kappa$ and $\mu$ to zero. In this case, analytic solutions in terms of Weierstrass elliptic functions may be found\textsuperscript{29–31} and the minimal energy configurations, in all parallelogram settings, saturate the energy bound in (1.8) giving $E = 4\pi B$. Thus, comparison of our numerical results with the analytic solutions serves as a useful check on the precision of our numerical procedure. The agreement is to 6 significant digits. Contour plots of the charge densities for different parallelogram settings for the charge-two skyrmions are shown in Fig. 1.9 all of them of equal energy $E/8\pi = 1$.

1.2.2.2. The Skyrme case ($\kappa \neq 0, \mu = 0$)

As pointed out earlier, for $\mu = 0$ the dependence of the energy functional on the Skyrme parameter $\kappa$ is only through $\kappa^2 \rho$, so without loss of generality we vary $\rho$ and fix $\kappa^2 = 0.03$ throughout (this particular choice for $\kappa$ was made for numerical convenience). Minimization of the energy functional (1.9) over all parallelograms yields the following. For any fixed density $\rho$, the minimal energy is obtained for
Fig. 1.8. Energy per charge of the multi-skyrmion configurations as a function of topological charge. The horizontal dashed line was added to guide the eye. (Here, $\kappa^2 = 0.03$, $\mu = 0$, $\rho = 1$, $s = 1$ and $\gamma = \pi/6$.)

$s = 1$ and $\gamma = \pi/6$. This set of values corresponds to the ‘hexagonal’ or ‘equilateral triangular’ lattice. In this configuration, any three adjacent zero-energy loci (or ‘holes’) are the vertices of equilateral triangles, and eight distinct high-density peaks are observed (as shown in Fig. 1.10b). This configuration can thus be interpreted as the splitting of the two-skyrmion into eight quarter-skyrmions. This result is independent of the skyrmion density $\rho$.

In particular, the well-studied square-cell minimal energy configuration (Fig. 1.10a), in which the two-skyrmion splits into four half-skyrmions, has higher energy than the hexagonal case. Figure 1.10 shows the total energies (divided by $8\pi$) and the corresponding contour plots of charge densities of the hexagonal, square and other configurations (for comparison), all of them with $\rho = 2$.

The total energy of the skyrmions in the hexagonal setting turns out to be linearly proportional to the density of the skyrmions, reflecting the scale invariance of the model (Fig. 1.11). In particular, the global minimum of $E = 4\pi B = 8\pi$ is reached when $\rho \to 0$. This is expected since setting $\rho = 0$ is equivalent to setting the Skyrme parameter $\kappa$ to zero, in which case the model is effectively pure $O(3)$ and inequality (1.8) is saturated.
1.2.2.3. The general case ($\kappa \neq 0$, $\mu \neq 0$)

The hexagonal setting turns out to be the energetically favorable also in the general case. Moreover, since in this case the skyrmion has a definite size (as is demonstrated by the $\rho$ dependence in the energy functional), the skyrmion structure is different at low and at high densities and a phase transition occurs at a certain critical density. While at low densities the individual skyrmions are isolated from each other, at high densities they fuse together, forming the quarter-skyrmion crystal, as in the Skyrme case reported above. As the density $\rho$ decreases, or equivalently the value of $\mu$ increases, the size of the skyrmions becomes small compared to the cell size. The exact shape of the lattice loses its effects and the differences in energy among the various lattice types become very small. This is illustrated in Fig. 1.11.

Due to the finite size of the skyrmion, there is an optimal density for which the energy is minimal among all densities. Figure 1.11 shows the contour plots of the charge density of the charge-two skyrmion for several densities with $\kappa^2 = 0.03$ and $\mu^2 = 0.1$. The energy of the skyrmion is minimal for $\rho \approx 0.14$ (Fig. 1.11).
Fig. 1.10. Charge-two skyrmions in the Skyrme case with $\kappa^2 = 0.03$ and $\rho = 2$: Contour plots of the charge densities for the hexagonal, square and other fundamental cells ranging from violet (low density) to red (high density). As the captions of the individual subfigures indicate, the hexagonal configuration has the lowest energy.

1.2.3. Semi-analytical approach

The energy functional (1.9) depends both on the Skyrme field $\phi$ and on the parallelogram parameters $\gamma$ and $s$. Formally, the minimal energy configuration over all parallelograms may be obtained by functional differentiation with respect to $\phi$ and regular differentiation with respect to $\gamma$ and $s$. However, since the resulting equations are very complicated, a direct numerical solution is quite hard. Nonetheless, some analytical progress may be achieved in the following way. As a first step, we differentiate the energy functional (1.9) only with respect to $\gamma$ and $s$:

$$
\begin{align*}
\frac{\partial E}{\partial \gamma} &= \frac{1}{2s \cos^2 \gamma} \left( \sin \gamma (E_{yy} + s^2 E_{xx}) - 2s E_{xy} \right) = 0, \\
\frac{\partial E}{\partial s} &= \frac{1}{2s^2 \cos \gamma} (E_{yy} - s^2 E_{xx}) = 0,
\end{align*}
$$

(1.10)
Fig. 1.11. Total energy $E$ (divided by $8\pi$) of the charge-two skyrmion in the hexagonal lattice (■ – Skyrme case and ♦ – general case) and in the square lattice (■ – Skyrme case and ♦ – general case) as function of the skyrmion density (in the Skyrme case, $\kappa^2 = 0.03$ and in the general case $\kappa^2 = 0.03$ and $\mu^2 = 0.1$). Note the existence of an optimal density (at $\rho \approx 0.14$) in the general case, for which the energy attains a global minimum. Figure (b) is an enlargement of the lower left corner of figure (a).

where $E^{ij} = \int_{T^2} dxdy (\partial_i \phi \cdot \partial_j \phi)$ and $i, j \in \{x, y\}$. Solving these equations for $\gamma$ and $s$ yields

\[ s = \sqrt{\frac{E_{yy}}{E_{xx}}}, \]
\[ \sin \gamma = \sqrt{\frac{E_{xy}}{E_{xx}E_{yy}}}. \]

Substituting these expressions into the energy functional, we arrive at a ‘reduced’ functional

\[ E = \sqrt{E_{xx}E_{yy} - (E_{xy})^2} + \frac{\kappa^2 \rho}{2B} E_{sk} + \frac{\mu^2 B}{\rho} E_{pot}, \]

where $E_{sk} = \int_{T^2} dxdy (\partial_x \phi \times \partial_y \phi)^2$ is the Skyrme energy and $E_{pot} = \int_{T^2} dxdy (1 - \phi_3)$ is the potential energy. Now that both $\gamma$ and $s$ are eliminated from the resultant expression, and the conditions for their optimization are built into the functional, the numerical minimization is carried out. We note here, however, that the procedure presented above should be treated with caution. This is since Eqs. (1.11) are only extremum conditions, and may correspond to a maximum or saddle-point. Hence, it is important to confirm any results obtained using this
method by comparing them with corresponding results obtained from the method described in the previous section.

It is therefore reassuring that numerical minimization of the reduced functional (1.12) gives $\sin \gamma = 0.498$ ($\gamma \approx \pi/6$) and $s = 1$ (both for the Skyrme case and the general case), confirming the results presented in the previous section.

In the general ($\mu \neq 0$) case, the energy functional (1.12) may be further differentiated with respect to the skyrmion density $\rho$ to obtain the optimal density for which the skyrmion energy is minimal. Differentiating with respect to $\rho$, and substituting the obtained expression into the energy functional, results in the functional

$$E = \sqrt{E_{xx}E_{yy} - (E_{xy})^2} + \kappa \mu \sqrt{2E_{sk}E_{pot}}.$$  \hspace{1cm} (1.13)

Numerical minimization of the above expression for $\kappa^2 = 0.03$ and various $\mu$ values ($0.1 \leq \mu^2 \leq 10$) yields the hexagonal setting as in the Skyrme case. In particular, for $\mu^2 = 0.1$ the optimal density turns out to be $\rho \approx 0.14$, in accord with results presented in Sec. 1.2.2.3.

1.2.4. Further remarks

As pointed out earlier, the special role of the hexagonal lattice revealed here is not unique to the baby Skyrme model, but in fact arises in other solitonic models. In the context of Skyrme models, the existence of a hexagonal two dimensional structure of 3D skyrmions has also been found by Battye and Sutcliffe,\(^{35}\) where it has already been noted that energetically, the optimum infinite planar structure of 3D skyrmions is the hexagonal lattice, which resembles the structure of a graphite...
sheet, the most stable form of carbon thermodynamically.  
Other examples in which the hexagonal structure is revealed are Ginzburg-Landau vortices which are known to have lower energy in a hexagonal configuration than on a square lattice.  
Thus, it should not come as a surprise that the hexagonal structure is found to be the most favorable in the baby Skyrme model.  

As briefly noted in the opening paragraphs of this section, a certain type of baby skyrmions also arise in quantum Hall systems as low-energy excitations of the ground state near ferromagnetic filling factors (notably 1 and 1/3).  
It has been pointed out that this state contains a finite density of skyrmions, and in fact the hexagonal configuration has been suggested as a candidate for their lattice structure.  
Our results may therefore serve as a supporting evidence in that direction, although a more detailed analysis is in order.  

Our results also raise some interesting questions. The first is how the dynamical properties of baby skyrmions on the hexagonal lattice differ from their behavior in the usual rectangular lattice. Another question has to do with their behavior in non-zero-temperature.  
One may also wonder whether and how these results can be generalized to the 3D case. Is the face-centered cubic lattice indeed the minimal energy crystalline structure of 3D skyrmions among all parallelepiped lattices? If not, what is the minimal energy structure, and how do these results depend on the presence of a mass term? These questions await a systematic study.  

1.3. Baby Skyrmions on the Two-Sphere

Although skyrmions were originally introduced to describe baryons in three spatial dimensions, they have been shown to exist for a very wide class of geometries, specifically cylinders, two-spheres and three-spheres.  
Here, we consider a baby Skyrme model on the two-sphere.  
We compute the full-field minimal energy solutions of the model up to charge 14 and show that they exhibit complex multi-skyrmion solutions closely related to the skyrmion solutions of the 3D model with the same topological charge. To obtain the minimum energy configurations, we apply two completely different methods. One is the full-field relaxation method, with which exact numerical solutions of the model are obtained, and the other is a rational map approximation scheme, which as we show yields very good approximate solutions.  
In an exact analogy to the 3D Skyrme model, the charge-one skyrmion has a spherical energy distribution, the charge-two skyrmion is toroidal, and skyrmions with higher charge all have point symmetries which are subgroups of O(3). As we shall see, it is not a coincidence that the symmetries of these solutions are the same as those of the 3D skyrmions.  

\footnote{This type of model has been studied before, although only rotationally-symmetric configurations have been considered.}
1.3.1. The baby Skyrme model on the two-sphere

The model in question is a baby Skyrme model in which both the domain and target are two-spheres. The Lagrangian density here is simply

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi + \frac{\kappa^2}{2} \left[ (\partial_\mu \phi \cdot \partial^\mu \phi)^2 - (\partial_\mu \phi \cdot \partial_\nu \phi)(\partial^\mu \phi \cdot \partial^\nu \phi) \right], \quad (1.14)$$

with metric $d^2 = dt^2 - d\theta^2 - \sin^2 \theta \, d\phi^2$, where $\theta$ is the polar angle $\in [0, \pi]$ and $\phi$ is the azimuthal angle $\in [0, 2\pi]$. The Lagrangian of this model is invariant under rotations in both domain and the target spaces, possessing an $O(3)_{\text{domain}} \times O(3)_{\text{target}}$ symmetry. As noted earlier, in flat two-dimensional space an additional potential term is necessary to ensure the existence of stable finite-size solutions. Without it, the repulsive effect of the Skyrme term causes the skyrmions to expand indefinitely. In the present model, however, the finite geometry of the sphere acts as a stabilizer, so a potential term is not required. Furthermore, stable solutions exist even without the Skyrme term. In the latter case, we obtain the well known $O(3)$ (or $\mathbb{CP}^1$) nonlinear sigma model.

As before, the field $\phi$ in this model is an $S^2 \to S^2$ mapping, so the relevant homotopy group is $\pi_2(S^2) = \mathbb{Z}$, implying that each field configuration is characterized by an integer topological charge $B$, the topological degree of the map $\phi$. In spherical coordinates $B$ is given by

$$B = \frac{1}{4\pi} \int d\Omega \frac{\phi \cdot (\partial_\theta \phi \times \partial_\phi \phi)}{\sin \theta}, \quad (1.15)$$

where $d\Omega = \sin \theta \, d\theta \, d\phi$.

Static solutions are obtained by minimizing the energy functional

$$E = \frac{1}{2} \int d\Omega \left[ (\partial_\theta \phi)^2 + \frac{1}{\sin^2 \theta} (\partial_\phi \phi)^2 \right] + \frac{\kappa^2}{2} \int d\Omega \left( \frac{(\partial_\theta \phi \times \partial_\phi \phi)^2}{\sin^2 \theta} \right), \quad (1.16)$$

within each topological sector. Before proceeding, it is worthwhile to note that setting $\kappa = 0$ in Eq. (1.16) yields the energy functional of the $O(3)$ nonlinear sigma model. The latter has analytic minimal energy solutions within every topological sector, given by

$$\phi = (\sin f(\theta) \cos(B\phi), \sin f(\theta) \sin(B\phi), \cos f(\theta)), \quad (1.17)$$

where $f(\theta) = \cos^{-1}(1-2(1+\lambda \tan \theta/2)^{2B})^{-1}$ with $\lambda$ being some positive number. These solutions are not unique, as other solutions with the same energy may be obtained by rotating (1.17) either in the target or in the domain spaces. The energy distributions of these solutions in each sector have total energy $E_B = 4\pi B$.

Analytic solutions also exist for the energy functional (1.16) with the Skyrme term only. They too have the rotationally symmetric form (1.17) with $f(\theta) = \theta$ and total energy $E_B = 4\pi B^2$. They can be shown to be the global minima by the
following Cauchy-Schwartz inequality:

\[
\left( \frac{1}{4\pi} \int d\Omega \frac{\phi \cdot (\partial_\theta \phi \times \partial_\phi \phi)}{\sin \theta} \right)^2 \leq \left( \frac{1}{4\pi} \int d\Omega \phi^2 \right) \cdot \left( \frac{1}{4\pi} \int d\Omega (\frac{\partial_\theta \phi \times \partial_\phi \phi}{\sin \theta})^2 \right).
\]

(1.18)

The left-hand side is simply \( B^2 \) and the first term in parenthesis on the right-hand side integrates to 1. Noting that the second term in the RHS is the Skyrme energy (without the \( \kappa^2/2 \) factor), the inequality reads \( E \geq 4\pi B^2 \), with an equality for the rotationally-symmetric solutions.

### 1.3.2. Static solutions

In general, if both the kinetic and Skyrme terms are present, static solutions of the model cannot be obtained analytically. This is with the exception of the \( B = 1 \) skyrmion which has an analytic “hedgehog” solution

\[
\phi_{[B=1]} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]

(1.19)

with total energy \( E_{4\pi} = 1 + \frac{\kappa^2}{2} \).

For skyrmions with higher charge, we find the minimal energy configurations by utilizing the full-field relaxation method described earlier. In parallel, we also apply the rational map approximation method, originally developed for the 3D Skyrme model and compare the results with the relaxation method. Let us briefly discuss the rational map approximation method: computing the minimum energy configurations using the full nonlinear energy functional is a procedure which is both time-consuming and resource-hungry. To circumvent these problems, the rational map ansatz scheme has been devised. First introduced by Houghton, Manton and Sutcliffe,\(^{44}\) this scheme has been used in obtaining approximate solutions to the 3D Skyrme model using rational maps between Riemann spheres. Although this representation is not exact, it drastically reduces the number of degrees of freedom in the problem, allowing computations to take place in a relatively short amount of time. The results in the case of 3D Skyrme model are known to be quite accurate.

Application of the approximation, begins with expressing points on the base sphere by the Riemann sphere coordinate \( z = \tan \frac{\theta}{2} e^{i\varphi} \). The complex-valued function \( R(z) \) is a rational map of degree \( B \) between Riemann spheres

\[
R(z) = \frac{p(z)}{q(z)},
\]

(1.20)

where \( p(z) \) and \( q(z) \) are polynomials in \( z \), such that \( \max[\deg(p), \deg(q)] = B \), and \( p \) and \( q \) have no common factors. Given such a rational map, the ansatz for the field triplet is

\[
\phi = \left( \frac{R + \bar{R}}{1 + |R|^2}, \frac{R - \bar{R}}{1 + |R|^2}, \frac{1 - |R|^2}{1 + |R|^2} \right).
\]

(1.21)
It can be shown that rational maps of degree $B$ correspond to field configurations with charge $B$.\cite{14} Substitution of the ansatz (1.21) into the energy functional (1.16) results in the simple expression

$$
\frac{1}{4\pi} E = B + \frac{\kappa^2}{2} \mathcal{I},
$$

(1.22)

with

$$
\mathcal{I} = \frac{1}{4\pi} \int \left( \frac{1 + |z|^2}{1 + |R|^2} \frac{dR}{dz} \right)^4 \frac{2i \, dz \, d\bar{z}}{(1 + |z|^2)^2}.
$$

(1.23)

Note that in the $\kappa \to 0$ limit, where our model reduces to the $O(3)$ nonlinear sigma model, the rational maps become exact solutions and the minimal energy value $E = 4\pi B$ is attained. Furthermore, the minimal energy is reached independently of the specific details of the map (apart from its degree), i.e., all rational maps of a given degree are minimal energy configurations in the topological sector corresponding to this degree. This is a reflection of the scale- and the rotational invariance of the $O(3)$ model.

In the general case where $\kappa \neq 0$, the situation is different. Here, minimizing the energy (1.22) requires finding the rational map which minimizes the functional $\mathcal{I}$. As we discuss in the next section, the expression for $\mathcal{I}$ given in Eq. (1.23) is encountered in the application of the rational map in the context of 3D skyrmions, where the procedure of minimizing $\mathcal{I}$ over all rational maps of the various degrees has been used.\cite{44-46} Here we redo the calculations, utilizing a relaxation method: to obtain the rational map of degree $B$ that minimizes $\mathcal{I}$, we start off with a rational map of degree $B$, with the real and imaginary parts of the coefficients of $p(z)$ and $q(z)$ assigned random values from the segment $[-1, 1]$. Solutions are obtained by relaxing the map until a minimum of $\mathcal{I}$ is reached.

### 1.3.3. Relation to the 3D Skyrmie model

In the 3D Skyrmie model, the rational map ansatz can be thought of as taken in two steps. First, the radial coordinate is separated from the angular coordinates by taking the SU(2) Skyrmie field $U(r, \theta, \varphi)$ to be of the form

$$
U(r, \theta, \varphi) = \exp(\mathbf{i} f(r) \, \phi(\theta, \varphi) \cdot \mathbf{\sigma}),
$$

(1.24)

where $\mathbf{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices, $f(r)$ is the radial profile function subject to the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$, and $\phi(\theta, \varphi) : S^2 \mapsto S^2$ is a normalized vector which carries the angular dependence of the field. In terms of the ansatz (1.24), the energy of the Skyrmie field is

$$
E = \int 4\pi f^2 \, r^2 \, dr + \int 2(f^2 + 1) \sin^2 \theta \, f \, dr \int \left( \left( \partial_\theta \phi \right)^2 + \frac{1}{\sin^2 \theta} \left( \partial_\varphi \phi \right)^2 \right) \, d\Omega \\
+ \int \frac{\sin^4 \theta}{r^2} \, f \, dr \int \left( \frac{\partial_\theta \phi \times \partial_\varphi \phi}{\sin^2 \theta} \right)^2 \, d\Omega.
$$

(1.25)
Note that the energy functional (1.25) is actually the energy functional of our model (1.16) once the radial coordinate is integrated out. Thus, our 2D model can be thought of as a 3D Skyrme model with a ‘frozen’ radial coordinate.

The essence of the rational map approximation is the assumption that $\phi(\theta, \varphi)$ takes the rational map form (1.21), which in turn leads to a simple expression for the energy

$$E = 4\pi \int \left( r^2 f'^2 + 2B(f'^2 + 1) \sin^2 f + I \frac{\sin^4 f}{r^2} \right) \, dr,$$

where $I$ is given in Eq. (1.23). As in the baby model on the two-sphere, minimizing the energy functional requires minimizing $I$ over all maps of degree $B$, which is then followed by finding the profile function $f(r)$.

Since the symmetries of the 3D skyrmions are determined solely by the angular dependence of the Skyrme field, it should not be too surprising that the solutions of the model discussed here share the symmetries of the corresponding solutions of the 3D Skyrme model.

1.3.4. Results

The configurations obtained from the full-field relaxation method are found to have the same symmetries as corresponding multi-skyrmions of the 3D model with the same charge. The $B = 2$ solution is axially symmetric, whereas higher-charge solutions were all found to have point symmetries which are subgroups of $O(3)$. For $B = 3$ and $B = 12$, the skyrmions have a tetrahedral symmetry. The $B = 4$ and $B = 13$ skyrmions have a cubic symmetry, and the $B = 7$ is dodecahedral. The other skyrmion solutions have dihedral symmetries. For $B = 5$ and $B = 14$ a $D_{2d}$ symmetry, for $B = 6, 9$ and 10 a $D_{4d}$ symmetry, for $B = 8$ a $D_{6d}$ symmetry and for $B = 11$ a $D_{3h}$ symmetry. In Fig. [1.13] we show the energy distributions of the obtained solutions for $\kappa^2 = 0.05$.

While for solutions with $B < 8$ the energy density and the charge density are distributed in distinct peaks, for solutions with higher charge they are spread in a much more complicated manner. The total energies of the solutions (divided by $4\pi B$) are listed in Table [1.11] along with the symmetries of the solutions (again with $\kappa^2 = 0.05$).

Application of the rational map ansatz yields results with only slightly higher energies, only about 0.3% to 3% above the full-field results. The calculated values of $I$ are in agreement with results obtained in the context of 3D skyrmions.\textsuperscript{45} For $9 \leq B \leq 14$, the rational map approximation yielded slightly less symmetric solutions than the full-field ones. Considering the relatively small number of degrees of freedom, this method all-in-all yields very good approximations. The total energies of the solutions obtained with the rational map approximation is also listed in Table [1.11].
Fig. 1.13. The energy distributions of the multi-skyrmion solutions for charges $2 \leq B \leq 14$ ($\kappa^2 = 0.05$).
Table 1.1. Total energies (divided by $4\pi B$) of the multi-solitons of the model for $\kappa^2 = 0.05$.

| Charge B | Total energy Full-field | Total energy Rational maps | Difference in % | Symmetry of the solution |
|----------|------------------------|---------------------------|-----------------|-------------------------|
| 2        | 1.071                  | 1.073                     | 0.177           | Toroidal                |
| 3        | 1.105                  | 1.113                     | 0.750           | Tetrahedral             |
| 4        | 1.125                  | 1.129                     | 0.359           | Cubic                   |
| 5        | 1.168                  | 1.179                     | 0.958           | $D_{2d}$                |
| 6        | 1.194                  | 1.211                     | 1.426           | $D_{4d}$                |
| 7        | 1.209                  | 1.217                     | 0.649           | Icosahedral             |
| 8        | 1.250                  | 1.268                     | 1.406           | $D_{4d}$                |
| 9        | 1.281                  | 1.304                     | 1.771           | $D_{4d}$                |
| 10       | 1.306                  | 1.332                     | 1.991           | $D_{4d}$                |
| 11       | 1.337                  | 1.366                     | 2.224           | $D_{3h}$                |
| 12       | 1.360                  | 1.388                     | 2.072           | Tetrahedral             |
| 13       | 1.386                  | 1.415                     | 2.137           | Cubic                   |
| 14       | 1.421                  | 1.459                     | 2.712           | $D_2$                   |

1.3.5. Further remarks

As we have just seen, the baby Skyrme model on the two-sphere shares very significant similarities with the 3D model, especially in terms of multi-skyrmion symmetries. The fact that the model discussed here is two-dimensional makes it simpler to study and perform computations with, when compared with the 3D Skyrme model.

Some of the results presented above may, at least to some extent, also be linked to the baby skyrmions which appear in two-dimensional electron gas systems, exhibiting the quantum Hall effect. As briefly noted in the Introduction, baby skyrmions arise in quantum Hall systems as low-energy excitations of the ground state, near ferromagnetic filling factors (notably 1 and 1/3). There, the skyrmion is a twisted two-dimensional configuration of spin, and its topological charge corresponds to the number of time the spin rotates by $2\pi$. While for the electron gas, the stability of the soliton arises from a balance between the electron-electron Coulomb energy and the Zeeman energy, in our model the repulsive Skyrme-term energy is balanced by the underlying geometry (i.e., the sphere). The connection between these two models suggests possible existence of very structured spin textures in quantum Hall systems, although a more detailed analysis of this analogy is in order.

1.4. Rotating Baby Skyrmions

We now turn to analyze the phenomenon of spontaneous breaking of rotational symmetry (SBRS) as it appears in rotating baby skyrmions. In general, SBRS refers to cases where physical systems which rotate fast enough deform in a manner which breaks their rotational symmetry, a symmetry they posses when static or rotating slowly. The recognition that rotating physical systems can yield solutions with less symmetry than the governing equations is not new. One famous example which
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dates back to 1834 is that of the equilibrium configurations of a rotating fluid mass. It was Jacobi who was first to discover that if rotated fast enough, a self-gravitating fluid mass can have equilibrium configurations lacking rotational symmetry. In modern terminology, Jacobi’s asymmetric equilibria appear through a symmetry breaking bifurcation from a family of symmetric equilibria as the angular momentum of the system increases above a critical value (a “bifurcation point”).\textsuperscript{48,49} Above this critical value, rotationally-symmetric configurations are no longer stable, and configurations with a broken rotational symmetry become energetically favorable.

By now it is widely recognized that symmetry-breaking bifurcations in rotating systems are of frequent occurrence and that this is in fact a very general phenomenon, appearing in a variety of physical settings, among which are fluid dynamics, star formation, heavy nuclei, chemical reactions, plasmas, and biological systems, to mention some diverse examples.

Recently, SBRS has also been observed in self-gravitating $N$-body systems,\textsuperscript{50,51} where the equilibrium configurations of an $N$-body self-gravitating system enclosed in a finite 3 dimensional spherical volume have been investigated using a mean-field approach. It was shown that when the ratio of the angular momentum of the system to its energy is high, spontaneous breaking of rotational symmetry occurs, manifesting itself in the formation of double-cluster structures. These results have also been confirmed with direct numerical simulations.\textsuperscript{52}

It is well-known that a large number of phenomena exhibited by many-body systems have their counterparts and parallels in field theory, which in some sense is a limiting case of $N$-body systems in the limit $N \to \infty$. Since the closest analogues of a lump of matter in field theories are solitons, the presence of SBRS in self-gravitating $N$-body systems has led us to expect that it may also be present in solitonic field theories.

Our main motivation towards studying SBRS in solitons is that in hadronic physics Skyrme-type solitons often provide a fairly good qualitative description of nucleon properties (See, e.g.,\textsuperscript{53,54}). In particular, it is interesting to ask what happens when such solitons rotate quickly, because this might shed some light on the non-spherical deformation of excited nucleons with high orbital angular momentum, a subject which is now of considerable interest.

We shall see that the baby Skyrme model on the two-sphere indeed exhibits SBRS, and we will try to understand why this is so.\textsuperscript{55} First, we give a brief account for the occurrence of SBRS in physical systems in general, and then use the insights gained from this discussion to infer the conditions under which SBRS might appear in solitonic models and in that context we study its appearance in baby Skyrme models. Specifically, we shall show that SBRS emerges if the domain manifold of the model is a two-sphere, while if the domain is $\mathbb{R}^2$, SBRS does not occur.
1.4.1. **SBRS from a dynamical point of view**

The onset of SBRS may be qualitatively understood as resulting from a competition between the static energy of a system and its moment of inertia. To see this, let us consider a system described by a set of degrees of freedom $\phi$, and assume that the dynamics of the system is governed by a Lagrangian which is invariant under spatial rotations. When the system is static, its equilibrium configuration is obtained by minimizing its static energy $E_{\text{static}}$ with respect to its degrees of freedom

$$\frac{\delta E}{\delta \phi} = 0 \quad \text{where} \quad E = E_{\text{static}}(\phi). \quad (1.27)$$

Usually, if $E_{\text{static}}(\phi)$ does not include terms which manifestly break rotational symmetry, the solution to (1.27) is rotationally-symmetric (with the exception of degenerate spontaneously-broken vacua, which are not of our concern here). If the system rotates with a given angular momentum $J = J\hat{z}$, its configuration is naturally deformed. Assuming that the Lagrangian of the system is quadratic in the time derivatives, stable rotating configurations (if such exist) are obtained by minimizing its total energy $E_J$

$$\frac{\delta E_J}{\delta \phi} = 0 \quad \text{where} \quad E_J = E_{\text{static}}(\phi) + \frac{J^2}{2I(\phi)}, \quad (1.28)$$

where $I(\phi)$ is the ratio between the angular momentum of the system and its angular velocity $\omega = \omega\hat{z}$ (which for simplicity we assume is oriented in the direction of the angular momentum). $I(\phi)$ is the (scalar) moment of inertia of the system.

The energy functional (1.28) consists of two terms. The first, $E_{\text{static}}$, increases with the asymmetry. This is simply a manifestation of the minimal-energy configuration in the static case being rotationally-symmetric. The second term $J^2/2I$, having the moment of inertia in the denominator, decreases with the asymmetry. At low values of angular momentum, the $E_{\text{static}}$ term dominates, and thus asymmetry is not energetically favorable, but as the value of angular momentum increases, the second term becomes dominant, giving rise to a possible breaking of rotational symmetry.

1.4.2. **SBRS in baby Skyrme models**

In what follows, we show that the above mechanism of SBRS is present in certain types of baby Skyrme models.

As already discussed in previous sections, the static solutions of the baby Skyrme model (1.1) have rotationally-symmetric energy and charge distributions in the charge-one and charge-two sectors. The charge-one skyrmion has an energy peak at its center which drops down exponentially. The energy distribution of the charge-two skyrmion has a ring-like peak around its center at some characteristic distance. The rotating solutions of the model in $\mathbb{R}^2$ are also known. Rotation at low angular velocities slightly deforms the skyrmion but it remains rotationally-symmetric.
For larger values of angular velocity, the rotationally-symmetric configuration becomes unstable but in this case the skyrmion does not undergo symmetry breaking. Its stability is restored through a different mechanism, namely that of radiation. The skyrmion radiates out the excessive energy and angular momentum, and as a result begins slowing down until it reaches equilibrium at some constant angular velocity, its core remaining rotationally-symmetric. Moreover, if the Skyrme fields are restricted to a rotationally-symmetric (hedgehog) form, the critical angular velocity above which the skyrmion radiates can be obtained analytically. It is simply the coefficient of the potential term $\omega_{\text{crit}} = \mu$. Numerical full-field simulations also show that the skyrmion actually begins radiating well below $\omega_{\text{crit}}$, as radiation itself may be non-rotationally-symmetric. The skyrmion’s core, however, remains rotationally-symmetric for every angular velocity.

The stabilizing effect of the radiation on the solutions of the model has lead us to believe that models in which radiation is somehow inhibited may turn out to be good candidates for the occurrence of SBRS. In what follows, we study the baby Skyrme model on the two-sphere, whose static solutions were presented in the previous section. Within this model, energy and angular momentum are not allowed to escape to infinity through radiation, and as a consequence, for high enough angular momentum the mechanism responsible for SBRS discussed in the previous section takes over, revealing solutions with spontaneously broken rotational symmetry.

1.4.3. The baby Skyrme model on the two-sphere

In order to find the stable rotating solutions of the model, we assume for simplicity that any stable solution would rotate around the axis of angular momentum (which is taken to be the $z$ direction) with some angular velocity $\omega$. The rotating solutions thus take the form $\phi(\theta, \varphi, t) = \phi(\theta, \varphi - \omega t)$. The energy functional to be minimized is

$$E = E_{\text{static}} + \frac{J^2}{2I},$$

(1.29)

where $I$ is the ratio of the angular momentum of the skyrmion to its angular velocity, or its “moment of inertia”, given by

$$I = \frac{1}{4\pi^2B} \int d\Omega \left( (\partial_\varphi \phi)^2 + \kappa^2 (\partial_\theta \phi \times \partial_\varphi \phi)^2 \right).$$

(1.30)

1.4.4. Results

In what follows we present the results obtained by the minimization scheme applied to the rotating solutions of the model in the charge-one and charge-two sectors, which as mentioned above are rotationally-symmetric. For simplicity, we fix the parameter $\kappa$ at $\kappa^2 = 0.01$ although other $\kappa$ values were tested as well, yielding qualitatively similar solutions.
1.4.4.1. Rotating charge-one solutions

The rotating charge-one skyrmion has spherically-symmetric energy and charge distributions in the static limit (Fig. 1.14a). When rotated slowly, its symmetry is reduced to $O(2)$, with the axis of symmetry coinciding with the axis of rotation (Fig. 1.14b). At some critical value of angular momentum (which in the current settings is $J_{\text{crit}} \approx 0.2$), the axial symmetry is further broken, yielding an ellipsoidal energy distribution with three unequal axes (Fig. 1.14c). Any further increase in angular momentum results in the elongation of the skyrmion in one horizontal direction and its shortening in the perpendicular one. The results are very similar to those of the rotating self-gravitating ellipsoid.

Fig. 1.14. Baby skyrmions on the two-sphere ($\kappa^2 = 0.01$): The charge distribution $B(\theta, \phi)$ of the charge-one skyrmion for different angular momenta. In the figure, the vector $B(\theta, \phi)\hat{r}$ is plotted for the various $\theta$ and $\phi$ values.
1.4.4.2. Rotating charge-two solutions

SBRS is also observed in rotating charge-two skyrmions. The static charge-two skyrmion has only axial symmetry (Fig. 1.15a), with its symmetry axis having no preferred direction. Nonzero angular momentum aligns the axis of symmetry with the axis of rotation. For small values of angular momentum, the skyrmion is slightly deformed but remains axially-symmetric (Fig. 1.15b). Above $J_{\text{crit}} \approx 0.55$ however, its rotational symmetry is broken, and it starts splitting to its ‘constituent’ charge-one skyrmions (Fig. 1.15c and 1.15d). As the angular momentum is further increased, the splitting becomes more evident, and the skyrmion assumes a string-like shape. This is somewhat reminiscent of the well-known elongation, familiar from high-spin hadrons which are also known to assume a string-like shape with the constituent quarks taking position at the ends of the string.\[58,59\]

Fig. 1.15. Baby skyrmions on the two-sphere ($\kappa^2 = 0.01$): The charge distribution $B(\theta, \phi)$ of the charge-two skyrmion for different angular momenta. In the figure, the vector $B(\theta, \phi)\hat{\mathbf{r}}$ is plotted for the various $\theta$ and $\phi$ values.

1.4.5. The rational map ansatz

A somewhat more analytical analysis of this system may be achieved by the use of the rational maps approximation scheme,\[44\] which as was shown earlier provides quite accurate results for the static solutions of the model.\[57\]

In its implementation here, we simplify matters even more and reduce the degrees of freedom of the maps by a restriction only to those maps which exhibit the symmetries observed in the rotating full-field solutions. This allows the isolation of those parameters which are the most critical for the minimization of the energy functional.
As shown in Fig. 1.14, the charge and energy densities of the charge-one skyrmion exhibit progressively lower symmetries as $J$ is increased. The static solution has an $O(3)$ symmetry, while the slowly-rotating solution has an $O(2)$ symmetry. Above a certain critical $J$, the $O(2)$ symmetry is further broken and only an ellipsoidal symmetry survives. Rational maps of degree one, however, cannot produce charge densities which have all the discrete symmetries of an ellipsoid with three unequal axes. Nonetheless, approximate solutions with only a reflection symmetry through the $xy$ plane (the plane perpendicular to the axis of rotation) and a reflection through one horizontal axis may be generated by the following one-parametric family of rational maps

$$R(z) = \frac{\cos \alpha}{z + \sin \alpha}, \quad (1.31)$$

which has the charge density

$$B(\theta, \varphi) = \left( \frac{\cos \alpha}{1 + \sin \alpha \sin \theta \cos \varphi} \right)^2. \quad (1.32)$$

Here, $\alpha \in [-\pi, \pi]$ is the parameter of the map, with $\alpha = 0$ corresponding to a spherically-symmetric solution and a non-zero value of $\alpha$ corresponding to a nonrotationally-symmetric solution. Results of a numerical minimization of the energy functional (1.29) for fields constructed from (1.31) for different values of angular momentum $J$ are shown in Fig. 1.16a. While for angular momentum less than $J_{\text{crit}} \approx 0.1$, $\alpha = 0$ minimizes the energy functional (a spherically-symmetric solution), above this critical value bifurcation occurs and $\alpha = 0$ is no longer a minimum; the rotational symmetry of the charge-one skyrmion is broken and it becomes nonrotationally-symmetric.

A similar analysis of the charge-two rotating solution yields the one-parametric map

$$R(z) = \frac{\sin \alpha + z^2 \cos \alpha}{\cos \alpha + z^2 \sin \alpha}, \quad (1.33)$$

with corresponding charge density

$$B(\theta, \varphi) = \left( \frac{2 \cos 2\alpha \sin \theta}{2 + \sin^2 \theta (\sin 2\alpha \cos 2\varphi - 1)} \right)^2. \quad (1.34)$$

In this case, $\alpha = 0$ corresponds to a torodial configuration, and a non-zero value of $\alpha$ yields solutions very similar to those shown in Fig. 1.15 having the proper discrete symmetries. The results in this case are summarized in Fig. 1.16b, indicating that above $J_{\text{crit}} \approx 0.57$ the minimal energy configuration is no longer axially-symmetric.

The discrepancies in the critical angular momenta $J_{\text{crit}}$ between the full-field method (0.2 for charge-one and 0.55 for charge-two) and the rational maps scheme (0.1 for charge-one and 0.57 for charge-two) are of course expected, as in the latter method, the solutions have only one degree of freedom. Nonetheless, the qualitative similarity in the behavior of the solutions in both cases is strong.
1.4.6. Further remarks

We have seen that SBRS appears not only in rotating classical-mechanical systems but also in the baby Skyrme model on the two-sphere. We have argued that this is so because the phenomenon originates from general principles, and hence it is a universal one.

The results presented above may, at least to some extent, also be linked to recent advances in the understanding the non-sphericity of excited nucleons with of large orbital momentum. Non-spherical deformation of the nucleon shape is now a focus of considerable interest, both experimental\textsuperscript{60,61} and theoretical.\textsuperscript{62–64} As skyrmions are known to provide a good qualitative description of many nucleon properties, the results presented here may provide some corroboration to recent results on this subject (e.g., \textsuperscript{64}), although a more detailed analysis of this analogy is in order.

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A.1. Obtaining Baby Skyrmion Solutions — The Relaxation Method

As a large part of the studies presented above is based on numerically obtaining the baby skyrmion configurations, in the following we describe the relaxation method that was used to obtain the solutions.

The multi-solitons of the baby Skyrme model are those field configurations which minimize the static energy functional within each topological sector. The energy
functional is given by

\[ E = \int d^2x \left( \frac{1}{2} \partial_x \phi \cdot \partial_x \phi + \partial_y \phi \cdot \partial_y \phi + \kappa^2 \left( \partial_x \phi \times \partial_y \phi \right)^2 + U(\phi_3) \right). \] (A.1)

As already noted, the baby Skyrme model is a nonintegrable system, so in general, explicit analytical solutions to its Euler-Lagrange equations are nearly impossible to find. Hence, one must resort to numerical techniques.

Generally speaking, there are two main approaches to finding the baby skyrmion solutions numerically. One approach is to employ standard techniques to numerically solve the Euler-Lagrange equations which follow from the energy functional (A.1). The other approach – the one taken here – is to utilize relaxation methods to minimize the energy of the skyrmion within any desired topological sector. In what follows, we describe in some detail the relaxation method we have used all throughout this research. This method is based on the work of Hale, Schwindt and Weidig. We assume for simplicity that the base space is discretized to a rectangular grid. The implementation of this method in the case of curved spaces or for a polar grid is straightforward.

The relaxation method begins by defining a grid with \( N^2 \) points, where at each point a field triplet \( \phi(x_m, y_n) \) is defined. All measurable quantities such as energy density or charge density are calculated at the centers of the grid squares, using the following expressions for the numerical derivatives, also evaluated at these points:

\[ \frac{\partial \phi}{\partial x}(x_m, y_n) = \frac{1}{\Delta x} \times \left( \frac{\phi(x_{m+1}, y_n) + \phi(x_{m+1}, y_{n+1})}{2} \right) \text{normed} - \left( \frac{\phi(x_m, y_n) + \phi(x_{m+1}, y_{n+1})}{2} \right) \text{normed}, \] (A.2)

with the \( y \)-derivatives analogously defined, and the “normed” subscript indicates that the averaged fields are normalized to one. If the field itself has to calculated at that center of a grid square, we use the prescription

\[ \phi(x_m, y_n) = \left( \frac{1}{4} \left( \phi(x_m, y_n) + \phi(x_{m+1}, y_n) + \phi(x_m, y_{n+1}) + \phi(x_{m+1}, y_{n+1}) \right) \right) \text{normed}. \] (A.3)

The basic updating mechanism of the relaxation process consists of the following two steps: A point \((x_m, y_n)\) on the grid is chosen at random, along with one of the three components of the field \( \phi(x_m, y_n) \). The chosen component is then shifted by a value \( \delta \phi \) chosen uniformly from the segment \([-\Delta \phi, \Delta \phi]\) where \( \Delta \phi = 0.1 \) initially. The field triplet is then scaled and the change in energy is calculated. If the energy decreases, the modification of the field is accepted and otherwise it is discarded.

The relaxation process, through which the energy of the baby skyrmion is minimized, is as follows:
(1) Initialize the field triplet $\phi$ to a rotationally–symmetric configuration
\[ \phi_{\text{initial}} = (\sin f(r) \cos B\theta, \sin f(r) \sin B\theta, \cos f(r)) \quad \text{(A.4)} \]
In our setup, we have chosen the profile function $f(r)$ to be $f(r) = \pi \exp(-r)$, $r$ and $\theta$ being the usual polar coordinates.

(2) Perform the basic updating mechanism for $M \times N^2$ times (we took $M = 100$), and then calculate the average rate of acceptance. If it is smaller than 5%, decrease $\Delta \phi$ by half.

(3) Repeat step (2) until $\Delta \phi < 10^{-9}$, meaning no further decrease in energy is observed.

This procedure was found to work very well in practice, and its accuracy and validity were verified by comparison of our results to known ones. There is however one undesired feature to this minimization scheme, which we note here: it can get stuck at a local minimum. This problem can be resolved by using the “simulated annealing” algorithm,\textsuperscript{65,66} which in fact has been successfully implemented before, in obtaining the minimal energy configurations of three dimensional skyrmions.\textsuperscript{67} The algorithm is comprised of repeated applications of a Metropolis algorithm with a gradually decreasing temperature, based on the fact that when a physical system is slowly cooled down, reaching thermal equilibrium at each temperature, it will end up in its ground state. This algorithm, however, is much more expensive in terms of computer time. We therefore employed it only in part, just as a check on our method, which corresponds to a Metropolis algorithm algorithm at zero temperature. We found no apparent changes in the results.
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