Polymer quantization, stability and higher-order time derivative terms

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The stability of higher-order time derivative theories using the polymer extension of quantum mechanics is studied. First, we focus on the well-known Pais-Uhlenbeck model and by casting the theory into the sum of two decoupled harmonic oscillators we show that the energy spectrum is composed with positive and negative energy parts. The Schrödinger quantization of the model with creation and annihilations operators leads to a theory with unbounded Hamiltonian that can be interpreted in terms of normal particles and Lee-Wick-like particles responsible for the instability. We investigate whether the fundamental discreteness implicit in the polymer quantization can regularize the effects of the negative energies introduced by the Lee-Wick-like particles which are associated to a high-energy scale. Precisely, we show that the polymer quantization leads to a positive defined Hamiltonian whose stability is improved as the number of Lee-Wick-like particles grows.

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I. INTRODUCTION

The Lagrangians of the standard model are constructed with renormalizable operators of mass dimension $d \leq 4$. The equations of motion that follows are normal-order time derivative equations of motion, that is first-order for fermions and second-order for bosons. In general, within the framework of local quantum field theory with normal-order operators it is straightforward to satisfy both the ingredients of stability and unitarity without any extension [1]. However, when ascending to higher energy scales close to the Planck scale where quantum gravity effects start to appear, it is expected that higher-order operators could describe some of these new physics. This is a certain possibility considering that higher-order operators allow to describe plane wave solutions with frequencies associated to a high-energy scale. Recently, higher-order theories have been investigated in loop quantum gravity [2–4], Lorentz symmetry violation [5, 6], causality [7], fine tuning [8, 9], the hierarchy problem [10], radiative corrections [11] and nonminimal couplings [12, 13]. In this work we are specially interested in considering higher-order operators in the language of effective theory.

The variational formalism for higher-order time derivative theories were developed long time ago by Ostrogradsky [14]. The formulation is based on an extended phase space in which besides $\dot{x}$ also $\dot{x}, x^{(3)}, \ldots, x^{(n)}$ are treated as configuration variables having their corresponding canonically conjugated momenta. It has been largely studied the effect of linear phase variable terms in the Hamiltonian which produces a classical instability. In some models with constraints the instability can be avoided [15]. However, this is not the case for a generic theory and usually the Hamiltonian is unbounded from below.

Pais and Uhlenbeck studied the quantization of higher-order time derivative theories [16]. They considered an harmonic oscillator with a correction term depending on the second-order time derivative of the position. This is the well known Pais-Uhlenbeck (P-U) model whose equations of motion depend on fourth-order time derivatives. Unfortunately, the quantization of such model leads to an unstable theory using a particular vacuum state. The instability of the Hamiltonian has been approached from different perspectives, such as phase space reduction [17, 18], complex canonical transformations [19], $PT$ symmetry [20], and modified amplitudes [21]. In general the problem of stability can be exchanged with the problem of having negative norm states leading to the loss of unitarity. This is easily seen by defining a new vacuum state and hence swapping the role of creation and annihilation operators. In the context of quantum field theory it was shown by Lee and Wick that it is possible to preserve unitarity order by order in perturbation theory by defining a new prescription to compute amplitudes [22]. The preservation of unitarity has also been studied in models containing higher-order Lorentz symmetry violations [23, 24].

In this work we study the stability of higher-order time derivative theories in the framework of polymer quantization [25]. The polymer representation is a modification of usual Schrödinger quantum mechanics inspired in some ideas that arise in loop quantum gravity due to possible discreteness. The polymer representation has been considered in several studies, such as for propagators [26], cosmology [27], central forces [28], higher space derivatives [29], thermodynamics [30] and low energy limi-
its\cite{31}. Here we explore the effects of polymer quantizing the P-U model taking into consideration the difficulties of Schrödinger quantization due to the Ostrogradsky instability introduced at the quantum level. Our main goal is to study whether the fundamental discreetness in the polymer representation can stabilize the theory with higher-order time derivative terms.

The organization of this work is as follows. In section II we introduce the P-U model and we show how the instability arises in the Schrödinger quantization motivating the polymer representation. In the third section we give a background review of the polymer formalism. In section IV we solve the eigenvalue equation for the P-U polymer Hamiltonian and emphasize the enhanced stability of the model. In the last section we give the conclusions.

II. THE PAIS-UHLENBECK MODEL

Let us start with the Pais-Uhlenbeck Lagrangian

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 - g x^{(4)} ,$$

(1)

which describes a standard harmonic oscillator with a correction higher-order term and where $g$ is a small parameter.

The equations of motion are

$$-k x - m \dot{x} - g x^{(4)} = 0 ,$$

(2)

which contain the term $x^{(4)} = \frac{d^4 x}{dt^4}$. Inserting the plane wave ansatz $x(t) = x_0 e^{-\imath \omega t}$ in the above equation produces a quartic equation whose solutions are

$$\omega = \pm \sqrt{\frac{m \pm \sqrt{m^2 - 4kg}}{2g}} .$$

(3)

To simplify the notation, let us consider $k = m \omega_0^2$ and $g = \frac{m \xi}{\omega_0}$ with $\xi \ll 1$ (dimensionless) such to rewrite the positive frequencies as

$$\omega = \omega_0 \sqrt{\frac{1 - \sqrt{1 - 4\xi}}{2\xi}} ,$$

(4)

and

$$W = \omega_0 \sqrt{\frac{1 + \sqrt{1 - 4\xi}}{2\xi}} .$$

(5)

We note both solutions have a different behavior in the limit $\xi \to 0$. For instance, we recover the usual harmonic solution $\omega \to \omega_0$, but we obtain a divergency in the second solution $W \approx \frac{m \omega_0}{\sqrt{\xi}}$, exhibiting the non perturbative form of the solution. This last solution can be interpreted in terms of a high-energy scale and quantum mechanically in terms of Lee-Wick like particles as we will see below.

Following the Ostrogradsky formulation we define momenta conjugated to $x$

$$p = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) ,$$

(6)

and to $\dot{x}$

$$\pi = \frac{\partial L}{\partial \dot{x}} .$$

(7)

They are explicitly given by the expressions

$$p = m \dot{x} + g x^{(3)} ,$$

(8)

and

$$\pi = -g \dot{x} .$$

(9)

The Hamiltonian is defined by

$$H = p \dot{x} + \pi \ddot{x} - L ,$$

(10)

which replacing produces

$$H = -\frac{\pi^2}{2g} - \frac{m \dot{x}^2}{2} + \dot{x} p + \frac{k x^2}{2} .$$

(11)

Now we turn to the quantization of the theory. As usual one imposes the commutation relations

$$[\hat{x}, \hat{p}] = i\hbar ,$$

(12)

$$[\hat{\pi}, \hat{x}] = i\hbar .$$

(13)

Next, we switch to the new variables

$$\hat{a}_+ = \sqrt{\frac{M \omega}{2\hbar}} \left( \hat{x}_+ + i \frac{1}{M \omega} \hat{p}_+ \right) ,$$

(14)

$$\hat{a}_- = \sqrt{\frac{MW}{2\hbar}} \left( \hat{x}_- + i \frac{1}{MW} \hat{p}_- \right) ,$$

(15)

with

$$\hat{x}_+ = \frac{W^2 x - \pi/g}{W^2 - \omega^2} ,$$

(16)

$$\hat{x}_- = \frac{\omega^2 x - \pi/g}{W^2 - \omega^2} ,$$

and

$$\hat{p}_+ = \hat{p} - (m - gW^2) \hat{x} ,$$

(17)

$$\hat{p}_- = \hat{p} - (m - g\omega^2) \hat{x} ,$$

where $M = \sqrt{m^2 - 4kg}$.

The new position and momenta variables defines a canonical transformation

$$[\hat{x}_+, \hat{p}_+] = i\hbar .$$

(18)
Using the above relations one can check that the creation and annihilation operators satisfy

\[ [\hat{a}_+, \hat{a}_+] = 1, \quad [\hat{a}_-, \hat{a}_-^\dagger] = 1. \tag{20} \]

The Hamiltonian operator turns out to be

\[ \hat{H} = \frac{\hbar \omega}{2} (\hat{a}_+ \hat{a}_+^\dagger + \hat{a}_+^\dagger \hat{a}_+) - \frac{\hbar W}{2} (\hat{a}_- \hat{a}_-^\dagger + \hat{a}_-^\dagger \hat{a}_-). \tag{21} \]

The vacuum of the theory can be defined as the one annihilated by \( \hat{a}_+ \) and \( \hat{a}_- \) denoted by \( \Psi_0 \). Indeed, using the explicit form of the operators

\[ \hat{x} = x, \quad \hat{p} = -i\hbar \frac{\partial}{\partial x}, \tag{22} \]

and

\[ \hat{x}' = \hat{x}, \quad \hat{p}' = -i\hbar \frac{\partial}{\partial \hat{x}}, \tag{23} \]

one can find the vacuum that fulfills

\[ \hat{a}_+ \Psi_0 = 0, \quad \hat{a}_- \Psi_0 = 0. \tag{24} \]

After some calculation we arrive at

\[ \Psi_0 = N_0 e^{\frac{\hbar \omega - W}{2\hbar} (x^2 + \omega W x^2) + \frac{i\hbar \omega W x^2}{2}}, \tag{25} \]

where \( N_0 \) is a normalization constant. In this case the Hamiltonian is not bounded from below leading to an unstable theory. The second alternative is to define the vacuum state to satisfy

\[ \hat{a}_+ \Psi_0 = 0, \quad \hat{a}_-^\dagger \Psi_0 = 0, \tag{26} \]

where one can show that the new vacuum state is obtained changing \( W \to -W \). In this case the commutator involving \( a_- \) in Eq. \([20]\), changes to a minus sign introducing ghost states but leading to a stable theory.

### III. POLYMER QUANTUM MECHANICS

To review the polymer representation of quantum mechanics we follow closely the references \([23, 26]\). Let us start from the Weyl-Heisenberg algebra which is generated by finite linear combinations of one-parameter families of operators \( \hat{U}_\alpha \) and \( \hat{V}_\beta \) equipped with the product law

\[ \hat{U}_\alpha \hat{U}_{\alpha'} = \hat{U}_{\alpha + \alpha'}, \quad \hat{V}_\beta \hat{V}_{\beta'} = \hat{V}_{\beta + \beta'}, \quad \hat{U}_\alpha \hat{V}_\beta = e^{-i\alpha \beta} \hat{V}_\beta \hat{U}_\alpha, \tag{27} \]

where \( \alpha \) and \( \beta \) are parameters with momentum and length dimensions respectively.

The quantization of a system can be achieved by finding a unitary representation of the Weyl-Heisenberg algebra, that is to say, by looking for a Hilbert space and unitary operators satisfying \([27]\) in that space.

In the Schrödinger representation for a system consisting of one degree of freedom the standard choice is \( H_{Sch} = L^2(\mathbb{R}) \), the set of complex-valued square integrable functions over \( \mathbb{R} \), and the self-adjoint operators \( \hat{x} \) and \( \hat{p} \) satisfying the canonical commutation relations

\[ [\hat{x}, \hat{p}] = i\hbar. \tag{28} \]

However, alternatively one can define the unitary operators \( \hat{U}_\alpha \) and \( \hat{V}_\beta \) defined by their action as

\[ \hat{U}_\alpha \psi(x) = e^{i\alpha x} \psi(x), \quad \hat{V}_\beta \psi(x) = \psi(x + \beta), \tag{29} \]

for all \( \psi \in H_{Sch} \), which correspond to the exponentiated versions of \( \hat{x} \) and \( \hat{p} \) respectively. The two operators above satisfy the algebra which is equivalent to the canonical commutation relations \([28]\).

The two operators \( \hat{U}_\alpha \) and \( \hat{V}_\beta \) solely defined by Eq. \([29]\) are well-defined operators on the whole Hilbert space \( H_{Sch} \), and since \( \hat{U}_\alpha \) and \( \hat{V}_\beta \) are weakly continuous with respect to their parameters, the Stone-Von Neumann theorem tells us that any representation of the algebra \([27]\), modulo unitary transformations is equivalent to choosing

\[ \hat{U}_\alpha = e^{i\alpha \hat{x}}, \quad \hat{V}_\beta = e^{i\beta \hat{p}/\hbar}, \tag{30} \]

as in the Schrödinger case.

On the other hand, the polymer quantization is based on a non-separable Hilbert space called the polymer Hilbert space. This means that there is no dense subset in the polymer Hilbert space implying the absence of any countable basis. This is the origin of the inequivalence of the polymer representation with respect to the Schrödinger representation.

To be more precise, the polymer representation is defined with functions in the space \( L^2(\mathbb{R}_{disc}) \), where \( \mathbb{R}_{disc} \) is the real line dotted with the discrete topology and \( L^2 \) equipped with the discrete measure. Although less intuitive, certainly the collection of singletons defines a genuine topology. In this way one chooses a Hilbert space denoted by \( \hat{H}_{poly} \), which has an uncountable orthonormal basis being characterized by abstract kets \( |x| \) labeled by real numbers \( x \), such that

\[ \langle x|x' \rangle = \delta_{x,x'} \tag{31} \]

where \( \langle x|x' \rangle \) stands for the inner product and \( \delta_{x,x'} \) is the Kronecker delta.

Any state \( \psi \in \hat{H}_{poly} \) is then expanded as \( \sum_{i=1}^{\infty} \langle x_i|\psi| \rangle |x_i \rangle \), for some countable sequence of basis kets \( |x_i \rangle \). The unitary operators, satisfying the commutation relations \([27]\), are then defined by their action on the basis vectors as:

\[ \hat{U}_\alpha |x \rangle = e^{i\alpha x} |x \rangle, \quad \hat{V}_\beta |x \rangle = |x - \beta \rangle, \tag{32} \]
in analogy with (30). Their action is obviously extended by linearity on \( H_{poly} \). The position operator \( \hat{x} \), defined by its action on basis kets as \( \hat{x}\mu = \mu \mid \mu \rangle \) is such that the left-hand side of (30) is verified just as in the Schrödinger representation. However, one of the key differences between the Schrödinger and the polymer representations is that in the polymer Hilbert space there is no self-adjoint operator \( \hat{p} \) such that the right-hand side of (30) is satisfied, that is to say, the momentum operator \( \hat{p} \) is not defined on \( H_{poly} \). This is due to the fact that the \( \hat{V}_\beta \) operator is not weakly continuous in the parameter \( \beta \), as can be verified using the modified product with the \( \hat{p} \) operator.

As mentioned before, unlike the Schrödinger representation of \( H_{poly} \), ones substitutes integration of functions on \( \mathbb{R} \) by discrete sums, so each wave function \( \psi \in H_{poly} \) is a (complex-valued) function defined on \( \mathbb{R} \) which is supported on at most a countable set of points \( \{ x_\ell \}_{\ell \in \mathbb{N}} \), and such that \( \sum_{i=1}^{\infty} | \psi(x_i) |^2 < \infty \). Since each wave function \( \psi \in H_{poly} \) is supported on a different countable set of points, then the basis of \( H_{poly} \) is constituted by kets \( \mid \mu \rangle \) labeled by all the real numbers \( \mu \). Each vector \( \mid \mu \rangle \in H_{poly} \) of the uncountable basis can be identified with the function \( \phi_\mu \in H_{poly} \) defined by

\[ \phi_\mu(x) = \langle x | \mu \rangle = \delta_{\mu,x} \]  

so that any arbitrary wave function \( \psi \in H_{poly} \) can be expressed as \( \psi(x) = \sum_{i=1}^{\infty} \psi(x_i) \phi_{x_i}(x) \). With this identification, the basic operators act on the basis vectors as \( \hat{\phi}_\mu(x) = \mu \phi_\mu(x) \) and \( \hat{V}_\beta \phi_\mu(x) = \phi_{\mu-\beta}(x) \) such to arrive at

\[ \hat{x}\psi(x) = \sum_{i=1}^{\infty} x_i \psi(x_i) \phi_{x_i}(x) \]  
\[ \hat{V}_\beta\psi(x) = \sum_{i=1}^{\infty} \psi(x_i) \phi_{x_i-\beta}(x) \]  

As mentioned before, unlike the Schrödinger representation, there is no self-adjoint momentum operator \( \hat{p} \), defined on \( H_{poly} \), such that the right-hand side of (30) is verified. However, since by construction any state in the polymer Hilbert space \( H_{poly} \) has support on at most countably many points, then one can approximate the operator \( \hat{p} \) as

\[ \hat{p} = -\frac{i\hbar}{\mu_0} \left( \hat{V}_{\mu_0/2} - \hat{V}_{-\mu_0/2} \right) \]  

where \( \mu_0 \) is a fundamental length scale associated with a possible discreteness of space, coming from a more fundamental theory. The above approximation is natural, at least in the distributional sense, since if we take the limit as \( \mu_0 \to 0 \) we recover the usual momentum operator in \( H_{Sch} \).

IV. STABILITY AND HIGHER-ORDER TIME DERIVATIVES

Recall from section III that the P-U Hamiltonian can be expressed as the difference of two decoupled harmonic oscillators, one corresponding to normal particles and the other associated to a very high energy scale. From the expressions (11), (16), (17) we can rewrite the Hamiltonian as

\[ H = H_+ - H_- \]  

where

\[ H_+ = \frac{1}{2} k_+ x_+^2 + \frac{1}{2 M_+ p_+^2} \]  
\[ H_- = \frac{1}{2} k_- x_-^2 + \frac{1}{2 M_- p_-^2} \]  

and

\[ k_+ = M_\omega^2 \]  
\[ k_- = M_\omega^2 \]  

Since our Hamiltonian is the difference of two normal oscillators, it is convenient to perform the polymer quantization of a single harmonic oscillator given in the reference [25]. For this, let us consider the Schrödinger equation

\[ \left( \frac{\hat{p}^2}{2m} + \frac{1}{2} k_\omega ^2 \right) \psi = E \psi \]  

for \( \psi \in H_{poly} \). Using equation (36), we have

\[ \hat{p}^2 = \frac{\hbar^2}{\mu_0^2} \left( 2 - \hat{V}_0 - \hat{V}_{-\mu_0} \right) \]  

and recalling the action of the basic operators \( \hat{x} \) and \( \hat{V}_0 \) in the polymer Hilbert space, (42) becomes the difference equation

\[ \psi(x_\ell + \mu_0) + \psi(x_\ell - \mu_0) = \left( 2 - \frac{2E \mu_0^2}{\hbar \omega d^2} + \frac{\mu_0^2 x_\ell^2}{d^4} \right) \psi(x_\ell) \]  

where the notation is \( k = \frac{\hbar \omega}{m} \) and \( m = \frac{\hbar}{2 \pi} \).

The previous equation suggests that one can find a solution supported at the uniformly spaced points \( x_\ell = x_0 + \ell \mu_0 \), for some \( x_0 \in [0, \mu_0) \). Indeed, given the parameters of the equation \( \{ \psi(x_\ell) \}_{\ell = -\infty}^{\infty} \) and by using (44), one can construct a unique solution \( \psi \) supported on the lattice

\[ \gamma_{x_0,\mu_0} := \{ x_\ell \in \mathbb{R} | x_\ell = x_0 + \ell \mu_0, \ell \in \mathbb{Z} \} \]  

More precisely, let \( \psi \in H_{poly} \) be defined by

\[ \psi(x) = \sum_{\ell = -\infty}^{\infty} \psi(x_\ell) \phi_{x_\ell}(x) \]  

Replacing this ansatz in (44) we arrive at
\[ \psi(x_{t+1}) + \psi(x_{t-1}) = \left[ 2 - \frac{2E\mu^2}{\hbar \omega d^2} + \frac{\mu^2 (x_0 + \ell \mu_0)^2}{d^4} \right] \psi(x_t), \] (47)
Without loss of generality we can set \( x_0 = 0 \) so \( x_\ell = \ell \mu_0 \), and consider the (discrete) Fourier transform of \( \psi \)
\[ f(k) = \sum_{\ell = -\infty}^{\infty} \psi(x_\ell) e^{-ik\ell \mu_0}. \] (48)
Now, by multiplying the equation (47) by \( e^{-ik\ell \mu_0} \) and summing over \( \ell \), we obtain
\[ 2 \cos(k\mu_0) f(k) = 2 \left( 1 - \frac{E\mu^2}{\hbar \omega d^2} \right) f(k) - \frac{\mu^2}{d^4} f''(k), \] (49)
where \( k \in (\frac{\pi}{\mu_0}, \frac{\pi}{\mu_0}) \).
Thus, normalizing and arranging the terms we get
\[ f''(k) + 2d^2 \left[ \frac{E}{\hbar \omega} + \frac{d^2}{\mu^2} (\cos(k\mu_0) - 1) \right] f(k) = 0. \] (50)
By making the change of variables \( z = \frac{k\mu_0 + \pi}{2} \in (0, \pi) \), we finally have
\[ f''(z) + [a(q) - 2q \cos(2\phi)] f(z) = 0, \] (51)
where
\[ q = 4 \left( \frac{d}{\mu_0} \right)^4, \] (52)
and
\[ a(q) = 4 \sqrt{q} \left( \frac{E}{\hbar \omega} - \frac{\sqrt{q}}{2} \right). \] (53)
Equation (51) is the well-known Mathieu equation in its canonical form. Defining
\[ h = \sqrt{q} = 2 \left( \frac{d}{\mu_0} \right)^2, \] (54)
it is known that the eigenvalues of Mathieu equation satisfy the following asymptotic expansion
\[ a_n(h) = -2h^2 + 2(2n + 1)h - \frac{1}{4} (2n^2 + 2n + 1) + O\left( \frac{1}{h} \right), \quad \text{for } n = 0, 1, 2, \ldots. \] (55)
provided that \( h \gg 1 \).
By replacing this asymptotic expansion in (53) and neglecting the lower-order terms \( O(1/h) \), the energy turns out to be
\[ E_n = \frac{\hbar \omega}{2} \left[ (2n + 1) - \frac{1}{16} \left( \frac{\mu_0}{d} \right)^2 (2n^2 + 2n + 1) \right], \] (56)
Thus, in the limit \( \mu_0/d \to 0 \), the \( E_n \) reduce to the usual Schrödinger eigenvalues, but the fact that \( \mu_0/d \) is actually different from zero means that there is a correction term introduced by the discreteness coming from a more fundamental theory. Thus, these corrections may produce that the system becomes unstable for very high-energy modes. For example, consider the vibrational oscillations of a carbon monoxide molecule, which are well described by a harmonic oscillator with parameters
\[ m \approx 10^{-26} \text{Kg}, \quad \omega_0 \approx 10^{15} \text{s}^{-1}, \] (57)
and
\[ d_0 = \sqrt{\frac{\hbar}{m\omega_0}} \approx 10^{-12} \text{m}. \] (58)
On the other hand, using the very conservative value \[ \mu_0 = 10^{-19} \text{m}, \] (59)
for the lattice parameter, one obtains that \( \hbar \gg 1 \) which yields that expression (56) is then valid (by taking \( \omega = \omega_0 \) and \( d = d_0 \)) to describe the energy of this physical system. Combining (58) and (59) we get that \( \mu_0/d_0 \approx 10^{-7} \), which yields that the correction term becomes significant for very high \( n \), as can be shown in Figure 1 in which we have depicted the energy of a carbon monoxide molecule with and without the correction term. Thus, the corrections introduced within the polymer framework are conceptually important, despite that in the domain of validity of non-relativistic quantum mechanics they are too small to have been observed \[ 25].

FIG. 1: Comparison of the energy of a usual harmonic oscillator (filled line) with the corrected energy by the polymeric quantization (dots)

Let us return to the P-U model, and consider the previous construction for the polymer quantization of the difference of two single harmonic oscillators. It is straight-
forward to obtain the result given in terms of the contribution of positive and negative energy parts

\[ E_n^{(+)} = \frac{\hbar \omega}{2} \left[ (2n + 1) - \left( \frac{\mu_0}{d_+} \right)^2 \frac{2n^2 + 2n + 1}{16} \right], \]

\[ E_n^{(-)} = \frac{\hbar W}{2} \left[ (2m + 1) - \left( \frac{\mu_0}{d_-} \right)^2 \frac{2m^2 + 2m + 1}{16} \right], \] (60)

where we have used the same lattice parameter \( \mu_0 \) for each harmonic oscillator and where it can be shown consistently that above one has

\[ d_+^2 = \frac{\hbar}{M \omega}, \] (61)

\[ d_-^2 = \frac{\hbar}{M W}. \] (62)

Recalling that for \( \xi \ll 1 \) we have that \( \omega \approx \omega_0, \ W \approx \omega_0/\sqrt{\xi} \) and \( M \approx m \), and using them above we obtain

\[ d_+^2 \approx \frac{\hbar}{m \omega_0} = d_0^2, \] (63)

\[ d_-^2 \approx \frac{\hbar}{m \omega_0/\sqrt{\xi}} = \sqrt{\xi} d_0^2. \] (64)

Replacing the above variables in both energies, and considering the same occupation number, we get

\[ E_n^{(+)} = \frac{\hbar \omega_0}{2} \left[ (2n + 1) - \left( \frac{\mu_0}{d_+} \right)^2 \frac{2n^2 + 2n + 1}{16} \right], \]

\[ E_n^{(-)} = \frac{\hbar \omega_0}{\xi} \left[ \sqrt{\xi}(2n + 1) \right. \]

\[ - \left( \frac{\mu_0}{d_0} \right)^2 \frac{2n^2 + 2n + 1}{16} \]. (65)

It can be seen that for \( \xi \) small enough, the part of the energy \( E_n^{(-)} \) becomes negative and moreover it reaches very high values with respect to the other part \( E_n^{(+)} \). Hence, the energy \( E_n^{PU} = E_n^{(+)} - E_n^{(-)} \) of the P-U model turns out to be

\[ E_n^{PU} = \frac{\hbar \omega_0}{2} \left[ \left( 1 - \frac{1}{\sqrt{\xi}} \right) (2n + 1) \right. \]

\[ + \left. \left( \frac{\mu_0}{d_0} \right)^2 \frac{2n^2 + 2n + 1}{16} \right]. \] (66)

Consequently, for \( \xi \) small enough the energy of the P-U model is negative only for small \( n \) and becomes positive from \( n \) moderately high. In Figure 2 we have depicted \( E_n^{(+)}, E_n^{(-)} \) and the difference \( E_n^{PU} \) for the same parameters as for the carbon monoxide molecule and for a value of \( \xi \) which is not as small as to appreciate both energies and its difference in the same plot. From Figure 2 it can be observed that \( E_n^{(-)} \) grows much faster than \( E_n^{(+)} \), after which \( E_n^{(-)} \) decays also much faster and more sharply than \( E_n^{(+)} \), which produces that the difference \( E_n^{PU} \) becomes positive for \( n \) high enough, that is to say, in the ultraviolet regime.

![Figure 2](image-url)

**FIG. 2:** Plot of \( E_n^{(+)} \) (dots), \( E_n^{(-)} \) (segmented line) and \( E_n^{PU} \) (filled line) for \( \xi = 5 \cdot 10^{-2} \).

Therefore the negative part of the energy, \( E_n^{(-)} \), serves to soften the stability problem, contrary to what happens within the Dirac quantization. To higher energies in the standard model one should introduce with this correction some kind of instability. Contrarily, we have shown that the correction term stabilizes the theory.

V. CONCLUSIONS

In this work we have studied the stability issue associated to the P-U model. With the help of a canonical transformation, the Hamiltonian of the P-U model has been casted in terms of positive and negative energy spectrum of harmonic oscillators. In this way the positive energy is associated to normal particles and the negative energy to Lee-Wick like type particles coming from a high energy scale.

We have shown that the quantization of the P-U model in the polymer representation avoids the Ostrogradsky instability. This is specially important, since in the case of a single harmonic oscillator, the polymer representation introduces a correction that may lead to a instability at very high-energy modes. The discrete nature of the polymer Hilbert space introduces a correction term in the energy spectrum which stabilizes the P-U model.

In general, higher-order time derivative models lead to a quantum theory which is unstable or contains negative norm states. In this work we have shown that a higher-order term on the contrary serves to soften the stability problem in the framework of polymer quantization containing a level of fundamental discreetness.
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