A gravitating global $k$-monopole

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Abstract
A gravitating global $k$-monopole produces a tiny gravitational field outside the core in addition to a solid angular deficit in $k$-field theory. As a new feature, the gravitational field can be attractive or repulsive depending on the non-canonical kinetic term.

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1. Introduction

The phase transition in the early universe could have produced different kinds of topological defects which have some important implications in cosmology [1]. Domain walls are two-dimensional defects, and strings are one-dimensional defects. Point-like defects which undergo spontaneous symmetry breaking also arise in some theories and they appear as monopoles. The global monopole, which has divergent mass in flat spacetime, is one of the most interesting defects. The idea that monopoles ought to exist has proven to be remarkably durable. Barriola and Vilenkin [2] first researched the characteristic of the global monopole in curved spacetime or, equivalently, its gravitational effects. When one considers gravity, the linearly divergent mass of the global monopole has an effect analogous to that of a deficit solid angle plus that of a tiny mass at the origin. Harari and Lousto [3], and Shi and Li [4] have shown that this small gravitational potential is actually repulsive. Furthermore, Li and co-workers [5–7] have proposed a new class of cold stars called D-stars (defect stars). One of the most important features of such stars, compared to Q-stars, is that the theory has monopole solutions when the matter field is absent, which makes D-stars behave very differently from Q-stars. Topological defects are also investigated in Friedmann–Robertson–Walker spacetime [8]. It is shown that the properties of global monopoles in asymptotically

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de Sitter/anti-de Sitter (ds/AdS) spacetime [9] and Brans–Dicke theory [10] are very different from those of ordinary monopoles.

The huge attractive force between a global monopole $M$ and an anti-monopole $\bar{M}$ implies that the monopole over-production problem does not exist, because pair annihilation is very efficient. Barriola and Vilenkin have shown that the radiative lifetime of the pair is very short as they lose energy by Goldstone boson radiation [2]. No serious attempt has made to develop an analytical model of the cosmological evolution of a global monopole, so we are limited to the numerical simulations of evolution by Bennett and Rhie [11]. In the $\sigma$-model approximation, the average number of monopoles per horizon is $N_H \sim 4$. The gravitational field of global monopoles can lead to clustering of matter, and can later evolve into galaxies and clusters. The scale-invariant spectrum of fluctuations has been given in [11]. Furthermore, one can numerically obtain the microwave background anisotropy $(\delta T/T)_{\text{rms}}$ patterns [12]. Comparing the theoretical value to the observed rms fluctuation, one can find the constraint of parameters in the global monopole.

On the other hand, non-canonical kinetic terms are rather ordinary for effective field theories. $k$-field theory, in which the non-canonical kinetic terms are introduced in the Lagrangian, has been recently investigated to serve as the inflaton in the inflation scenario, which is so-called $k$-inflation [13], and to explain the current acceleration of the universe and the cosmic coincidence problem, $k$-essence [14]. Some authors [15, 16] have discussed gravitationally bound static and spherically symmetric configurations of $k$-essence fields. Another interesting application of $k$-fields is topological defects, called $k$-defects [17]. Monopoles [18] and vortices [19] of a tachyon field, which as examples of $k$-fields come from string/M-theory, have also been investigated. The mass of the global $k$-monopole diverges in flat spacetime, as does that of the standard global monopole; therefore, it is of more physical significance to consider the gravitational effects of a global $k$-monopole.

In this paper, we study the gravitational field of a global $k$-monopole and derive the solutions numerically and asymptotically. We find that the topological condition of the vacuum manifold for the formation of a $k$-monopole is identical to that of an ordinary monopole, but their physical properties are different. In particular, we show that the mass of the $k$-monopole can be positive in some form of the non-canonical kinetic term. In other words, the gravitational field can be attractive or repulsive depending on the non-canonical kinetic term.

2. Equations of motion

We shall work within a particular model in units $c = 1$, where a global $O(3)$ symmetry is broken down to $U(1)$ in $k$-field theory. Its action is given by

$$S = \frac{1}{\kappa} \int d^4 \tilde{x} \sqrt{-\tilde{g}} \left[ M^4 K(\tilde{X}/M^4) - \frac{1}{4} \lambda^2 (\tilde{\phi}^a \tilde{\phi}^a - \tilde{\sigma}_0^2)^2 \right],$$

where $\kappa = 8\pi G$ and $\lambda$ is a dimensionless constant. In action (1), $\tilde{X} = \frac{1}{2} (\tilde{\partial}_\mu \tilde{\phi}^a \tilde{\partial}^\mu \tilde{\phi}^a)$, where $\tilde{\phi}^a$ is the $SO(3)$ triplet of the Goldstone field and $\tilde{\sigma}_0$ is the symmetry-breaking scale with the dimension of mass. After setting the dimensionless quantities: $x = M \tilde{x}$, $\phi_a = \tilde{\phi}_a/M$ and $\sigma_0 = \tilde{\sigma}_0/M$, action (1) becomes

$$S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} [K(X) - V(\phi)],$$

where $V(\phi) = \frac{1}{2} \lambda^2 (\phi^a \phi^a - \sigma_0^2)^2$. The hedgehog configuration describing a global $k$-monopole is

$$\phi^a = \sigma_0 f(\rho) \frac{x^a}{\rho},$$
where \( x^a x^a = \rho^2 \) and \( a = 1, 2, 3 \), so that we shall actually have a global \( k \)-monopole solution if \( f \to 1 \) at spatial infinity and \( f \to 0 \) near the origin.

The static spherically symmetric metric can be written as
\[
\mathrm{d}s^2 = B(\rho) \mathrm{d}t^2 - A(\rho) \mathrm{d}\rho^2 - \rho^2 (\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\phi^2)
\]
(4)
with the usual relation between the spherical coordinates \( \rho, \theta, \phi \) and the ‘Cartesian’ coordinate \( x^a \). Introducing a dimensionless parameter \( r = \sigma_0 \rho \), from (2) and (3), we obtain the equations of motion for \( f \) as
\[
\ddot{K} \left\{ \frac{1}{A} f'' + \left[ \frac{2}{Ar} + \frac{1}{2B} \left( \frac{B}{A} \right)' \right] f' - \frac{2}{r^2} f \right\} + \dot{K} X f' - \lambda^2 f (f^2 - 1) = 0,
\]
(5)
where the prime denotes the derivative with respect to \( r \), the dot denotes the derivative with respect to \( X \) and \( X = -f^2/r^2 - f'^2/2A \). Since we only consider the static solution, positive \( X \) and negative \( X \) are irrelevant to each other. In this paper, we will assume \( K(X) \) to be valid for negative \( X \).

The Einstein equation for a \( k \)-monopole is
\[
G_{\mu\nu} = \kappa T_{\mu\nu}
\]
(6)
where \( T_{\mu\nu} \) is the energy–momentum tensor for action (2). The \( tt \) and \( rr \) components of the Einstein equations can now be written as
\[
-\frac{1}{A} \left( \frac{1}{r^2} - \frac{1}{r} \frac{A'}{A} \right) + \frac{1}{r^2} = \epsilon^2 T^0_0
\]
(7)
\[
-\frac{1}{A} \left( \frac{1}{r^2} + \frac{1}{r} \frac{B'}{B} \right) + \frac{1}{r^2} = \epsilon^2 T^1_1,
\]
(8)
where
\[
T^0_0 = -K + \frac{\lambda^2}{4} (f^2 - 1)^2
\]
(9)
\[
T^1_1 = -K + \frac{\lambda^2}{4} (f^2 - 1)^2 - K \frac{f'^2}{A}
\]
(10)
and \( \epsilon^2 = \kappa \sigma_0^2 = 8\pi G \sigma_0^2 \) is a dimensionless parameter.

3. \( k \)-monopole

Although the existence of the global \( k \)-monopole, as well as the standard one, is guaranteed by the symmetry-breaking potential, there exists a non-canonical kinetic term in the \( k \)-monopole which certainly leads to the appearance of a new scale in the action and the mass parameter in the potential term. However, the non-canonical kinetic term is non-trivial. At small gradients, it can be chosen to have the same asymptotical behaviour as the standard one, so that it ensures the standard manner of a small perturbations. At large gradients we can choose it to have a different form the standard one.

In the small \( X \) case, we assume that the kinetic term has asymptotically canonical behaviour, which can avoid the ‘zero-kinetic problem’. If \( |X| \ll 1 \), we have \( K(X) \sim X^\alpha \), and \( \alpha < 1 \), then there is a singularity at \( X = 0 \); and if \( \alpha > 1 \) then the system becomes non-dynamical at \( X = 0 \). For the monopole solution, it is easily found that \( K(X) \sim X \) at \( r \gg 1 \). On the other hand, we assume that the modifying kinetic term \( K(X) \sim X^\alpha \) and \( \alpha \neq 1 \) at \( |X| \gg 1 \). One can easily obtain the equation of motion inside the core of a global
monopole after assuming that $|X| \gg 1$ in the core of the global monopole. The equations of motion are highly nonlinear and cannot be solved analytically. Next, we investigate the asymptotic behaviours of the global monopole with kinetic term nonlinear in $X$. To be specific, we consider the following type of kinetic term:

$$K(X) = X - \beta X^2, \quad (11)$$

where $\beta$ is a parameter of the global $k$-monopole. It is easy to find that the global $k$-monopole will reduce to the standard one when $\beta = 0$. It is easy to check whether the kinetic term (11) satisfies the condition for hyperbolicity [16, 17, 20]

$$\frac{K}{2XX + K} > 0, \quad (12)$$

which leads to a positive definite speed of sound for the small perturbations of the field. The stability of solutions shows that for the case $\beta < 0$, the range $1 \beta > X > 1/2\beta$ must be excluded. However, this will not destroy the results carried out from the case $\beta > 0$. Here we only consider the cases for $\beta > 0$.

Using equations (5)–(10), we obtain the asymptotic expression for $A(r)$, $B(r)$, and $f(r)$ which is valid near $r = 0$,

$$f(r) = f_0 r + \frac{f_0^2}{60(1 + 5\beta f_0^2)} \left[ 2\lambda^2 \left(-3 + \epsilon^2\right) + 42\beta \epsilon^2 f_0^4 \right] r^3$$

$$+ \frac{f_0^2}{60(1 + 5\beta f_0^2)} \left[ 9 + 7\beta \lambda^2 \epsilon^2 f_0^2 + 36\beta^2 \epsilon^2 f_0^6 \right] r^3 + O(r^4) \quad (13)$$

$$A(r) = 1 + \frac{\epsilon^2 (\lambda^2 + 6 f_0^2 + 9 \beta f_0^4)}{12} r^2 + O(r^3) \quad (14)$$

$$B(r) = 1 + \frac{\epsilon^2 (\lambda^2 - 9 \beta f_0^4)}{12} r^2 + O(r^3). \quad (15)$$

where the undetermined coefficient $f_0$ is characterized as the mass of the $k$-monopole, which can be determined in the numerical calculation.

Similarly, in the region of $r \gg 1$, we can expand $f(r)$, $A(r)$ and $B(r)$ as

$$f(r) = 1 - \frac{1}{\lambda^2} \left( \frac{1}{r} \right)^2 - \frac{3 - 2\epsilon^2 + 4\beta \lambda^2}{2\lambda^2} \left( \frac{1}{r} \right)^4 + O(r^{-5}) \quad (16)$$

$$A(r) = 1 - \frac{M_\infty}{1 - \epsilon^2} - \frac{1}{(1 - \epsilon^2)^2} \frac{M_\infty^2}{r} + \left[ \frac{\epsilon^2 (1 - \beta \lambda^2)}{(1 - \epsilon^2)^2 \lambda^2} + \frac{M_\infty^2}{(1 - \epsilon^2)^3} \right] \left( \frac{1}{r} \right)^2 + O(r^{-3}) \quad (17)$$

$$B(r) = (1 - \epsilon^2) + M_\infty - \frac{\epsilon^2 (1 - \beta \lambda^2)}{\lambda^2} \left( \frac{1}{r} \right)^2 + O(r^{-3}), \quad (18)$$

where the constant $M_\infty$ will be discussed later.

Using a shooting method for boundary value problems, we get the numerical results of the function $f(r)$ which describes the configuration of the global $k$-monopole. In figure 1 we show the function $f(r)$ for $\beta = 0$, $\beta = 1$, $\beta = 5$ and $\beta = 10$ respectively and for given values of $\lambda$ and $\epsilon$. Obviously, the configuration of field $f$ is not influenced heavily by the choice of the parameter $\beta$.

From equations (13) and (16), it is easy to construct a global $k$-monopole which has the same asymptotic condition as the standard global monopole, i.e., $f$ will approach zero when $r \ll 1$ and unity when $r \gg 1$. 


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Figure 1. The plot of $f(r)$ as a function of $r$. Here we choose $\lambda = 1$, $G = 1$ and $\epsilon = 0.001$. The four curved lines are plotted when $\beta = 0$, $\beta = 1$, $\beta = 5$ and $\beta = 10$ respectively.

There is actually a general solution to the Einstein equation with energy–momentum tensor $T_{\mu\nu}$ which takes the form as (9) and (10) for spherically-symmetric metric (4)

\[
A(r)^{-1} = 1 - \frac{\epsilon^2}{r} \int_0^r \left[ -K + \frac{\lambda^2}{4} (f^2 - 1)^2 \right] r^2 \, dr \tag{19}
\]

\[
B(r) = A(r)^{-1} \exp \left[ \epsilon^2 \int_{r}^{\infty} \left( \frac{\dot{f}^2}{r^2} \right) dr \right]. \tag{20}
\]

In terms of the dimensionless quantity $\epsilon$, the metric coefficients $A(r)$ and $B(r)$ can be formally integrated and read

\[
A(r)^{-1} = 1 - \epsilon^2 - \frac{2G\sigma_0 M_A(r)}{r} \tag{21}
\]

\[
B(r) = 1 - \epsilon^2 - \frac{2G\sigma_0 M_B(r)}{r}. \tag{22}
\]

The small dimensionless parameter $\epsilon$ arises naturally from Einstein equations and $\epsilon^2$ clearly describes a solid angular deficit of spacetime.

A global $k$-monopole solution $f$ should approach unity when $r \gg 1$. If this convergence is fast enough, then $M_A(r)$ and $M_B(r)$ will also rapidly converge to finite values. Therefore from equations (16)–(18) we have the asymptotic expansions:

\[
M_A(r) = M_\infty + 4\pi \sigma_0 \left( -\beta + \frac{1}{\lambda^2} \right) \frac{1}{r} + \frac{8\pi \sigma_0 (-1 + \epsilon^2 + 2\beta \lambda^2)}{3\lambda^4} \left( \frac{1}{r} \right)^3 + O(r^{-5}) \tag{23}
\]

\[
M_B(r) = M_\infty + 4\pi \sigma_0 \left( -\beta + \frac{1}{\lambda^2} \right) \frac{1}{r} - \frac{4\pi \sigma_0 (-1 + \epsilon^2 - 4\beta \lambda^2)}{3\lambda^4} \left( \frac{1}{r} \right)^3 + O(r^{-5}). \tag{24}
\]
Figure 2. The plot of $M_A(r)/\sigma_0$ as a function of $r$. Here we choose $\lambda = 1$, $G = 1$ and $\epsilon = 0.001$. The four curved lines are plotted when $\beta = 0$, $\beta = 1$, $\beta = 5$ and $\beta = 10$ respectively.

where $M_\infty \equiv \lim_{r \to \infty} M_A(r)$. One can easily find that the dependence on $\epsilon$ of the asymptotic expansion for $f(r)$ is very weak, in other words, the asymptotic behaviour is quite independent of the scale of symmetry breakdown $\sigma_0$ up to values as large as the Planck scale. In contrast, $M_A(r)$ obviously depends on $\sigma_0$.

The numerical results of $M_A(r)/\sigma_0$ are shown in figure 2 by a shooting method for boundary value problems where we choose $\lambda = 1$, $G = 1$ and $\epsilon = 0.001$. From the figure, we find that the mass of the global $k$-monopole decreases to a negative asymptotic value when $r$ approaches infinity in the case that $\beta = 0$ and $\beta = 1$. While the mass will be positive if $\beta = 5$ or $\beta = 10$. The asymptotic mass for the cases above are $-19.15\sigma_0$, $-13.83\sigma_0$, $3.62\sigma_0$ and $22.14\sigma_0$ respectively. It is clear that the presence of parameter $\beta$, which measures the degree of deviation of the kinetic term from canonical one, affects the effective mass of the global $k$-monopole significantly. It is not difficult to understand this property. From equations (11), (19) and (21), the mass function $M_A(r)$ can be expressed explicitly as

$$\frac{M_A(r)}{4\pi \sigma_0} = -r + \int_0^r \left[ \beta X^2 - X + \frac{\lambda^2}{4} (f^2 - 1)^2 \right] r^2 \, dr. \quad (25)$$

Obviously, the $\beta$-term in the integration has a positive contribution for the mass function. From figure 1, $f$ (thus $X$) is not sensitive to the value of $\beta$, so for the greater parameter $\beta$ chosen, the larger value $M_A(r)$ takes for a given $r$, and if $\beta$ is greater than some value, $M_A(r)$ will be positive for large $r$ as figure 2 shows. However, inside the core, $X$ varies slowly but $f$ varies quickly with respect to $r$, therefore, the $\lambda^2$-term in the integration will become dominant as $r$ decreases. This leads to two characteristics of the mass curves which is also shown in figure 2: (i) the mass curves with different $\beta$ converge gradually in the region near $r = 0$; (ii) in the case that $\beta$ is large enough, $M_A(r)$ has a minimum.

To show the effect of a solid defect angle, we then investigated the motion test particles around a global $k$-monopole. It is a good approximation to take $M_A(r)$ as a constant in the region distant from the core of the global $k$-monopole, since the the mass $M_A(r)$ approaches
very quickly to its asymptotic value. Therefore we can consider the geodesic equation in the metric (4) with
\[ A(r)^{-1} = B(r) = 1 - \epsilon^2 - \frac{2GM}{r} \]  
where \( M = \sigma_0 M_\infty \). Solving the geodesic equation and introducing a dimensionless quantity \( u = GM/r \), one will obtain the second-order differentiating equation of \( u \) with respect to \( \varphi \) [9, 21]
\[ \frac{d^2 u}{d\varphi^2} + (1 - \epsilon^2)u = \left( \frac{GM}{L} \right)^2 + 3u^2, \]  
where \( L \) is the angular momentum per unit of mass. When \( (GM/L)^2 \varphi \ll 1 \), one has the approximate solution of \( u \)
\[ u \approx \left( \frac{GM}{L} \right)^2 \left\{ \frac{1}{1 - \epsilon^2} + e \cos \left[ \left( 1 - \frac{3}{\sqrt{(1 - \epsilon^2)^3}} \left( \frac{GM}{L} \right)^2 \right) \varphi \right] \right\}, \]  
where \( e \) denotes the eccentricity. When a test particle rotates one loop around the global \( k \)-monopole, the precession of it will be
\[ \Delta \varphi = 6\pi \left( \frac{GM}{L} \right)^2 \frac{1}{\sqrt{(1 - \epsilon^2)^3}} \approx 6\pi \left( \frac{GM}{L} \right)^2 + 9\pi \left( \frac{GM}{L} \right)^2 \epsilon^2. \]  
The last term in equation (29) is the modification comparing this result with that for the precession around an ordinary star.

4. Conclusion

In summary, a \( k \)-monopole could have arisen during the phase transition in the early universe. We calculate the asymptotic solutions of a global \( k \)-monopole in static spherically symmetric spacetime, and find the behaviour of a \( k \)-monopole similar to that of a standard one. Although the choice of the parameter \( \beta \), which measures the degree of deviation of the kinetic term from the canonical one, has little influence on the configuration of \( k \)-field \( \phi \), the effective mass of a global \( k \)-monopole is significantly affected. The mass may be negative or positive when different parameters \( \beta \) are chosen. This shows that the gravitational field of the global \( k \)-monopole could be attractive or repulsive depending on the non-canonical kinetic term.

The configuration of a global \( k \)-monopole is more complicated than that of a standard one. As for its cosmological evolution, we should not attempt to obtain the analytical mode; we can only use numerical simulation. However, the energy dominance of the global \( k \)-monopole is in the region outside the core. We can roughly estimate that global \( k \)-monopoles will result in the clustering of matter and the evolution into galaxies and clusters in a way similar to that of standard monopoles.

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