A COMBINATORIAL PROOF OF THE SUPPER SYMMETRIC PROPERTY OF HOOK LENGTH

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1. INTEGER PARTITION

Let \( n \) be a positive integer. A partition \( \lambda \) is an integer sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) satisfying \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 1 \). We call \( \ell(\lambda) := \ell \) the length of \( \lambda \), \( |\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i \) the size of \( \lambda \), and each \( \lambda_i \) a part of \( \lambda \). We let \( \mathcal{P} \) denote the set of partitions, \( \mathcal{P}(n) \) the set of partitions size \( n \). After this, \( "(n)" \) means the restriction of size \( n \). For a partition \( \lambda \), we let \( m_i(\lambda) \) denote the multiplicity of \( i \) as its part. \( (1^{m_1(\lambda)}2^{m_2(\lambda)}\ldots) \) is another representation of \( \lambda \).

Example. For \( n = 5 \),
\[
\mathcal{P}(5) = \{ (5), (41), (32), (311), (221), (2111), (11111) \}
\]
\[
= \{ (5), (41), (32), (31^2), (2^21), (21^3), (1^5) \}.
\]

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) be a partition. The Young diagram of \( \lambda \) is defined by
\[
Y(\lambda) := \{ (i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i \}.
\]

Example. For \( \lambda = (7552) \),
\[
Y(\lambda) = \begin{array}{|c|c|c|c|}
\hline
| & | & | & |
\hline
| & | & | & |
\hline
| & | & | & |
\hline
| & | & | & |
\hline
\end{array}
\]

Let \( (i, j) \in Y(\lambda) \). The \((i, j)\)-hook of \( \lambda \) is defined by
\[
H_{(i,j)}(\lambda) := \{ (a, b) \in Y(\lambda) \mid (a = i \land b \geq j) \lor (a \geq i \land b = j) \}.
\]

And we call \( h_{(i,j)}(\lambda) := \#H_{(i,j)}(\lambda) \) the \((i, j)\)-hook length of \( \lambda \). We also define \( a_{(i,j)}(\lambda) := \lambda_i - j \) the \((i, j)\)-arm length of \( \lambda \), \( b_{(i,j)}(\lambda) := \sum_{k \geq j} m_k(\lambda) - i \) the \((i, j)\)-leg length of \( \lambda \).

Example. For \( \lambda = (7552) \), \( (2, 2) \in Y(\lambda) \),

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Shaded area is the \((2, 2)\)-hook of \(\lambda\). Then, \(h_{(2, 2)}(7552) = 6\). And \(a_{(2, 2)}(7552) = 3\), \(b_{(2, 2)}(7552) = 2\).

We call the operation that removing the hook from Young diagram and sliding the divided part to the upper left, simply remove the hook. And it is represented by \(Y \setminus H\).

**Example.**

\[
\begin{array}{c}
\lambda = (7552) \\
\downarrow \\
Y(\lambda) \setminus H_{(2, 2)}(\lambda)
\end{array}
\]

2. **The product of hook lengths in Young diagrams**

Let \(\sigma \in \mathfrak{S}_n\). When the cycle type of \(\sigma\) is \(\lambda = (1^{m_1}2^{m_2} \ldots n^{m_n})\), the size of centralizer of \(\sigma\) is

\[
z_\lambda = 1^{m_1}2^{m_2} \ldots n^{m_n} \times m_1!m_2! \ldots m_n!.
\]

The following result is classical.

**Theorem 2.1.** We put the first half of \(z_\lambda\) as \(a_\lambda\), the latter half as \(b_\lambda\). The following is hold.

\[
\prod_{\lambda \in \mathcal{P}(n)} a_\lambda = \prod_{\lambda \in \mathcal{P}(n)} b_\lambda.
\]

**Example.** \(n = 5\).

\[
\mathcal{P}(5) = \{(5), (41), (32), (311), (221), (2111), (11111)\}.
\]

In figure, hook lengths appear only on the right end of Young diagrams. Here, the numbers appearing in \(k\)-part rows are the factors of \(m_k!\). There are four 2 in Young diagrams. And there are four part 2 in
partitions. For each numbers, the number appearing in Young diagrams equals the number appearing in partitions. Let extend from right end to entire.

Example. \( n = 5 \).

\[
\begin{array}{cccccc}
5 & 4 & 3 & 2 & 1 \\
1 & 2 & 1 & 1 & 1
\end{array}
\]

\[
\begin{array}{cccccc}
5 & 3 & 2 & 1 & 2 & 1 \\
2 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

\[
\begin{array}{cccccc}
4 & 3 & 1 & 2 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

\[
\begin{array}{cccccc}
5 & 2 & 1 & 3 & 2 & 1 \\
3 & 2 & 1 & 1 & 1 & 1
\end{array}
\]

\[
\begin{array}{cccccc}
4 & 2 & 2 & 3 & 3 & 2 \\
2 & 2 & 2 & 1 & 1 & 1
\end{array}
\]

\[
\begin{array}{cccccc}
5 & 1 & 5 & 4 & 3 & 2 \\
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

\[1; 12, 2; 8, 3; 6, 4; 4, 5; 5\]

The number of \( k \) has increased by \( k \) times.

Theorem 2.2 (\[2\]). For any positive integer \( n \),

\[
\prod_{\lambda \in \mathcal{P}(n)} 1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots n^{m_n(\lambda)} = \prod_{\lambda \in \mathcal{P}(n)} \prod_{c \in \mathcal{Y}(\lambda)} h_c(\lambda).
\]

There are \( k \) kinds of length \( k \) hooks with different arm length. Actually, this \( k \) kinds appear uniformly in Young diagrams of size \( n \). This is the explanation of the trick. The property “appear uniformly” is called super symmetric [1].

3. A COMBINATORIAL PROOF OF THE SUPER SYMMETRIC
PROPERTY OF HOOK LENGTH

The following definitions are in [3].

We consider double infinite sequence \( \Lambda \) of 0 and 1. When all entries to the left of a certain point are 0 and all entries to the right of a certain point are 1, we call \( \Lambda \) partition sequence. For a partition sequence \( \Lambda \), all zeros are numbered from right to left. And, all ones are numbered from left to right. We call these numberings natural numbering.

Example. For \( \Lambda = \ldots 001001101001 \ldots \) = \( 0100110100_{\downarrow} \).

\[
\begin{array}{cccccccc}
\ldots & 5 & 4 & 3 & 2 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}
\]

Natural numbering of the zeros

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 0 & 1 & 0 & 0 & 1
\end{array}
\]

Natural numbering of the ones

And, we define another numbering \( \beta \)-numbering, we give number 0 to the first 1, then increase the number to the right from there. (This is a special case of definition in [3].)

Example. For \( \Lambda = 0100110100_{\downarrow} \).

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \ldots & \beta \text{-numbering}
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}
\]
For a partition sequence $\Lambda$, we consider a path that replace 0 to $→$ and 1 to $↑$. And we define $P(\Lambda)$ the partition which rim is this path.

**Example.** For $\Lambda = \underline{01001101001}$, $P(\Lambda) = (4^231^2)$.

![Diagram](image)

**Proposition 3.1.** For any partition sequence $\Lambda$, a 0 and an 1 are numbered $i, j$ in natural numbering. Then, $h_{(i,j)}(P(\Lambda))$ equals the difference of $\beta$-numberings of that 0, 1. And, the operation to swap that 0 and 1 correspond to remove that hook.

**Example.** For $\Lambda = \underline{01001101001}$.

Select 0, 1.

![Diagram](image)

The (3, 1)-hook length equals the difference of $\beta$-numbering of 0, 1. And swap that 0, 1.

![Diagram](image)

$Y(P(\Lambda')) = Y(P(\Lambda)) \setminus H_{(3,1)}(P(\Lambda))$.

**Definition 3.2.** We cut partition sequence $\Lambda$, $\Lambda = (\Lambda_2|\Lambda_1)$. And, we put $g$ the number of zeros in $\Lambda_1$, $h$ the number of ones in $\Lambda_2$. When $h - g = i$, we call $i$ the position of this cut. We denote $g_i(\Lambda) := g, h_i(\Lambda) := h$. 
Example. For a cut $\Lambda = (01001|101001)$, $g = 3, h = 2$. Then the position of this cut is $-1$.

Proposition 3.3. When cut is shifted to the right 1, position will be increased by just 1.

Definition 3.4. For a cut $\Lambda = (\Lambda_2|\Lambda_1)$ position $i$, we define $0(\Lambda; i, k) := g_i(\Lambda) - g_{i+k}(\Lambda)$. This is the number of 0 out of the first $k$ numbers of $\Lambda_1$.

Proposition 3.5. For sufficiently small $i$, $0(\Lambda; i, k) = k$. And, for sufficiently large $i$, $0(\Lambda; i, k) = 0$.

Proof. From the definition of partition sequence.

Proposition 3.6. For any partition sequence $\Lambda$, integer $i$ and positive integer $k$,

$$|0(\Lambda; i, k) - 0(\Lambda; i + 1, k)| \leq 1$$

Proof. $k - 1$ numbers are common among first $k$ numbers of both $\Lambda_1$. Therefore, the difference is at most 1.

Definition 3.7. For any partition $\lambda$, we define the number of “addable” $k$-hooks that arm length is $s$ as $a_{k,s}(\lambda)$. Similarly, we define the number of “removable” $k$-hooks that arm length is $s$ as $r_{k,s}(\lambda)$. That is

$$a_{k,s}(\lambda) = \sharp \{ \mu \in P(|\lambda|+k) \mid \exists c \in Y(\mu), h_c(\mu) = k, a_c(\mu) = s, Y(\mu) \setminus H_c(\mu) = Y(\lambda) \}.$$  

$$r_{k,s}(\lambda) = \sharp \{ c \in Y(\lambda) \mid h_c(\lambda) = k, a_c(\lambda) = s \}.$$

Proposition 3.8.

$$a_{k,s}(\lambda) = r_{k,s}(\lambda) + 1.$$  

Proof. We consider the graph of $0(\Lambda; i, k)$.

Example. $\Lambda = \underline{01001101001}$, $k = 3$.

Here, the number of $\searrow$ is one more than the number of $\nearrow$ at each height. When the distance between 0 and 1 is $k$, $\searrow$ appears. Then, $\searrow$ corresponds to an addable $k$-hook. Similarly, $\nearrow$ corresponds to a removable $k$-hook. And the height of arrow corresponds the arm length of that hook.
\[
\sum_{\lambda \in \mathcal{P}(n)} r_{k,s}(\lambda) = \sum_{\lambda \in \mathcal{P}(n-k)} a_{k,s}(\lambda)
\]
\[
= \sum_{\lambda \in \mathcal{P}(n-k)} (r_{k,s}(\lambda) + 1)
\]
\[
= \sharp \mathcal{P}(n-k) + \sum_{\lambda \in \mathcal{P}(n-k)} r_{k,s}(\lambda)
\]
\[
= \sharp \mathcal{P}(n-k) + \sharp \mathcal{P}(n-2k) + \cdots
\]

Therefore, \( \sum_{\lambda \in \mathcal{P}(n)} r_{k,s}(\lambda) \) does not depend on arm length \( s \). The super symmetric property of hook length was proved.

**References**

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