A simple proof for monotone CLT

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Abstract

In the case of monotone independence, the transparent understanding of the mechanism to validate the central limit theorem (CLT) has been lacking, in sharp contrast to commutative, free and Boolean cases. We have succeeded in clarifying it by making use of simple combinatorial structure of peakless pair partitions.

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1 Introduction

As a generalization of probability spaces, we define the notion of algebraic probability space.

Definition 1.1. (Algebraic probability space) An algebraic probability space is a pair \((A, \varphi)\) consisting of a unital \(*\)-algebra \(A\) and of a state \(\varphi\).

An element \(a \in A\) is called an algebraic random variable. For algebraic random variables, quantities such as \(\varphi(a_1a_2...a_n)\) are called mixed moments.

In quantum probability, the notion of independence is understood as a reduction rule in calculation of mixed moments. Among many different rules, commutative, free [13], Boolean [2, 12] and monotone independence [6, 7, 8] are known as basic types [4, 9, 10, 3]. In the present paper, we focus on monotone independence, which is defined later.

We first introduce a key notion, a peak for a mapping (not necessarily homomorphism) between finite ordered set.

Definition 1.2. Let \(S, T\) are finite ordered sets. An element \(s \in S\) is said to be a peak of a map \(f: S \to T\) if \(f(s) > f(s')\) holds for any \(s' (\neq s)\) which is next to \(s\) (that is, there is no elements between \(s\) and \(s'\) with respect to the order on \(S\)).

Now we introduce the notion of monotone independence.

Definition 1.3. (Monotone independence). Let \(\{A_\lambda; \lambda \in \Lambda\}\) be a family of \(*\)-subalgebras of \(A\), where the index set \(\Lambda\) is equipped with a linear order \(<\).

\(\{A_\lambda\}\) is said to be monotone independent if

\[\varphi(a_1...a_i...a_n) = \varphi(a_i)\varphi(a_1...a_{i-1}a_{i+1}...a_n)\]

holds for \(a_i \in A_\lambda \setminus C1\), whenever \(i\) is a peak of the mapping \(j \mapsto \lambda_j\).
For a monotone (or commutative, free, Boolean) independent family of \(*\)-subalgebras \(\{A_\lambda\}\), the following impotant property holds, which is known as the singleton condition discussed by von Waldenfels in 1970s.

**Definition 1.4.** (Singleton condition). Let \((A, \varphi)\) be an algebraic probability space. If a family of \(*\)-subalgebras \(\{A_\lambda\}\) satisfies the singleton condition if for any finite sequence \(\lambda_1, \ldots, \lambda_n \in \Lambda\) with a singleton \(\lambda_s\), i.e. \(\lambda_s \neq \lambda_i\) for all \(i \neq s\),

\[
\varphi(a_1 \ldots a_s \ldots a_n) = \varphi(a_s)\varphi(a_1 \ldots a_{s-1}a_{s+1} \ldots a_n)
\]

holds when \(a_i \in A_{\lambda_i}\).

This condition is essential for understanding asymptotic behavior of algebraic random variables. In fact, the following theorem holds [1, 11].

**Theorem 1.5.** Let \((A, \varphi)\) be an algebraic probability space and \((a_n)\) a sequence of elements of \(A\) which satisfies the following:

i) \(a_i = a_i^*\),

ii) \(\varphi(a_i) = 0\),

iii) \(\varphi(a_i^2) = 1\),

iv) \(\{a_i\}\) has uniform mixed moments, i.e., for \(m \geq 1\),

\[
\sup \{|\varphi(a_{n_1}a_{n_2} \ldots a_{n_m})| \mid n_1, n_2, \ldots, n_m \in \mathbb{N}\} < \infty
\]

v) \(\{C[a_i]\}\) satisfies the singleton condition.

Then,

\[
M_m := \lim_{N \to \infty} \varphi\left(\left(\frac{1}{\sqrt{N}} \sum_{n=1}^{\infty} a_n\right)^m\right)
\]

can be computed as follows:

\[
M_{2m+1} = 0, \quad M_{2m} = \lim_{N \to \infty} \frac{1}{N^m} \sum_{n \in \Pi(\{1,2,\ldots,2m\},\{1,2,\ldots,N\})} \varphi(a_{n_1}a_{n_2} \ldots a_{n_{2m}})
\]

Here,

\[
\Pi(S, T) := \{f : S \to T \text{ such that } |f^{-1}(t)| = 2 \text{ or } 0\}
\]

for finite sets \(S\) and \(T\).

## 2 A simple proof for the monotone CLT

In the case of commutative, free and Boolean independence, the corresponding central limit theorems (CLTs) can be proved directly from Theorem 1.5 by using some simple combinatorics. The purpose of this section is to derive the monotone CLT(Theorem 2.4) from Theorem 1.5 by using simple combinatorial argument.

To do this, we define a subset of \(\Pi(S, T)\) for finite ordered sets \(S\) and \(T\), which we call the set of peakless pair partitions \(\Pi_0(S, T)\) as follows:

\[
\Pi_0(S, T) := \{f \in \Pi(S, T) \mid \text{there is no peak of } f\}.
\]
Remark 2.1. For those who are familiar with the notion of “monotone partition” defined by Muraki to classify quasi-universal products [9], we note that a mapping \( f : S \to T \) which is an element of \( \Pi(S, T) \) belongs to \( \Pi_0(S, T) \) if and only if it is a monotone partition, when \( S, T \) are finite linearly ordered sets.

It is easy to see that for the case of monotone independence, only peakless pair partitions contribute and each contributions are equal to 1, in the expression of moments in Thm.1.5. Hence the problem is reduced to counting the number of the elements in
\[
\Pi_0(\{1, 2, \cdots, 2m\}, \{1, 2, \cdots, N\}).
\]

**Lemma 2.2.**
\[
|\Pi_0(\{1, 2, \cdots, 2m\}, \{1, 2, \cdots, N\})| = \binom{N}{m} \times (2m - 1)!!
\]
for \( N \geq m \geq 1 \).

**Proof.** First note that \( f \) is an element of \( \Pi_0(\{1, \cdots, 2m\}, \{1, \cdots, m\}) \) if and only if \( f|_{\{1, \cdots, 2m\}\setminus\{i, i+1\}} \) is an element of \( \Pi_0(\{1, \cdots, 2m\} \setminus \{i, i + 1\}, \{1, \cdots, m - 1\}) \) for \( i \) which satisfies \( f^{-1}(m) = \{i, i + 1\} \). Then we have
\[
|\Pi_0(\{1, \cdots, 2m\}, \{1, \cdots, m\})| = \sum_i |\Pi_0(\{1, \cdots, 2m\} \setminus \{i, i + 1\}, \{1, \cdots, m - 1\})| \\
= (2m - 1) \times |\Pi_0(\{1, \cdots, 2m - 1\}, \{1, \cdots, m - 1\})|
\]
for \( N \geq m \geq 2 \), and hence, \(|\Pi_0(\{1, \cdots, 2m\}, \{1, \cdots, m\})| = (2m - 1)!! \) holds.

It is easy to see that \(|\Pi_0(\{1, \cdots, 2m\}, \{1, \cdots, N\})| = \binom{N}{m} \times |\Pi_0(\{1, \cdots, 2m\}, \{1, \cdots, m\})| \), and we obtain the lemma.

\[\square\]

The essence of the proof above can be understood as follows:

First note that \( \Pi(\{1, 2, \cdots, 2m\}, \{1, 2, \cdots, N\}) \) represents all the possible ways to devide \( 2m \) balls into \( m \) pairs and to paint each pairs by different \( m \) colours chosen from \( N \) colours. Then, every peakless partition corresponding to the way how to paint balls can be generated uniquely by the following algorithm. Put \( 2m \) balls in line, choose \( m \) colours from \( N \) colours. Paint a couple of neighbouring balls by the highest colour. Then repeat this procedure, treating all the balls already painted and all the colours already used.

Now it is easy to count the way of such colourings. The number of the choices of \( m \) colours from \( N \) colours is \( \binom{N}{m} \). The number of the choices of a couple of neighbouring balls painted by the highest colour is \( 2m - 1 \), and the number of the choices of a couple of neighbouring (ignoring the balls which have painted ) balls painted by the second highest colour is \( (2m - 2) - 1 = 2m - 3, \cdots \), and so on.

**Remark 2.3.** The proof of the lemma is the prototype of combinatorial argument in [5].

Then we can prove the monotone CLT [3 7 8 11 11].
Theorem 2.4. Let \((A, \varphi)\) be an algebraic probability space and \((a_n)\) a sequence of elements of \(A\) which satisfies the following:

i) \(a_i = a_i^*\),

ii) \(\varphi(a_i) = 0\),

iii) \(\varphi(a_i^2) = 1\),

iv) \(\{C[a_i]\}\) has uniform mixed moments,

v) \(\{C[a_i]\}\) is monotone independent.

Then

\[
\lim_{N \to \infty} \varphi \left( \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{\infty} a_n \right)^m \right) = \frac{1}{\pi} \int_{-\sqrt{2}}^{\sqrt{2}} \frac{x^m}{\sqrt{2 - x^2}} dx
\]

for \(m = 0, 1, 2, \ldots\).

Proof. By the lemma 2.2

\[
M_{2m} = \lim_{N \to \infty} N^{-m} \binom{N}{m} (2m - 1)!! = \lim_{N \to \infty} N^{-m} \binom{N}{m} m! \times \frac{(2m - 1)!!}{m!} = \frac{(2m - 1)!!}{m!}
\]

This is nothing but the \(2m\)-th moment of the standard arcsine law. \(\square\)

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