Load balancing with heterogeneous schedulers

Urtzi Ayesta, Manu K. Gupta,* and Ina Maria Verloop
IRIT, 2 rue C. Camichel, Toulouse, France

October 19, 2018

Abstract

Load balancing is a common approach in web server farms or inventory routing problems. An important issue in such systems is to determine the server to which an incoming request should be routed to optimize a given performance criteria. In this paper, we assume the server’s scheduling disciplines to be heterogeneous. More precisely, a server implements a scheduling discipline which belongs to the class of limited processor sharing (LPS-\(d\)) scheduling disciplines. Under LPS-\(d\), up to \(d\) jobs can be served simultaneously, and hence, includes as special cases First Come First Served (FCFS \((d = 1)\) and Processor Sharing \((d = \infty)\).

In order to obtain efficient heuristics, we model the above load-balancing framework as a multi-armed restless bandit problem. Using the relaxation technique, as first developed in the seminal work of Whittle, we derive Whittle’s index policy for general cost functions and obtain a closed-form expression for Whittle’s index in terms of the steady-state distribution. Through numerical computations, we investigate the performance of Whittle’s index with two different performance criteria: linear cost criterion and a cost criterion that depends on the first and second moment of the throughput. Our results show that \((i)\) the structure of Whittle’s index policy can strongly depend on the scheduling discipline implemented in the server, i.e., on \(d\), and that \((ii)\) Whittle’s index policy significantly outperforms standard dispatching rules such as Join the Shortest Queue (JSQ), Join the Shortest Expected Workload (JSEW), and Random Server allocation (RSA).

keywords: Queuing, load balancing, restless bandits, limited processor sharing, index policies.

1 Introduction

We address the problem of load balancing in a multi-server system with heterogeneous service disciplines. In such systems, a dispatcher seeks to balance the assignment of service requests (jobs) across many servers in the system in order to optimize the performance. Such models are ubiquitous in applications, for instance in clusters of web-server nodes, database systems, grid computing and inventory routing (see for example [19, 21, 56]). The literature on load-balancing is vast, with many papers published every year (see Section 2 for a brief summary), but a common assumption in most of the literature is that all the servers implement the same scheduling policy. In particular, the overwhelming majority of works assume that servers implement FCFS, even though Processor-Sharing (PS) has also received some attention. This is the motivation for our work, where we aim at studying, to the best of our knowledge for the first time, how a job dispatcher should operate in the presence of heterogeneous servers that might vary in terms of speed as well as in terms of scheduling discipline.

In order to do so, we formulate the above load-balancing problem as a discrete Markov decision process (MDP). New jobs arrive to the system according to a Bernoulli process of mean \(p\), and upon arrival of a new job, the dispatcher must decide to which server to dispatch the new job, with the objective of optimizing the long-run performance.

Within a server, we assume that jobs are served according to the Limited Processor Sharing (LPS) discipline. In LPS-\(d\), \(d\) jobs (if present) are served simultaneously, that is, the server equally distributes its attention to each of these \(d\) jobs. First Come First Served (FCFS) and Processor Sharing (PS) disciplines are special cases of LPS-\(d\) with \(d = 1\) and \(d = \infty\), respectively. Thus, every server is characterized by the scheduling discipline it deploys, i.e., the value of \(d \in \mathbb{N}\), and its capacity is measured by \(q \in (0, 1]\), the averaged number of jobs that he can serve in a time unit. The evolution of every server is independent of each other, and their behavior is coupled only through the policy that dispatches jobs among them. This allows us to cast the problem within the so-called Multi-Armed Restless Bandit problems (MARB), where a bandit models a server, a class of MDP’s that has been widely studied in the literature for its multiple applications [16]. The multi-armed restless bandit problem has been shown to be \(PSPACE\)-complete [42], and closed-form solutions can only be obtained in toy examples.

*Corresponding author: Manu K. Gupta (manu-kumar.gupta@irit.fr)
We thus consider the relaxed version of the problem, as pioneered by Whittle in [49], in which on average $p$ jobs are dispatched. Under the technical condition known as indexability, Whittle showed that the solution to the relaxed problem is of index type: for every server there exists a function, the index, that depends only on its own state, and the optimal policy dispatches to the servers with highest current index. Whittle then defined a heuristic for the original problem, referred to as Whittle's index policy, where in every decision epoch the bandit with highest Whittle index is selected. It has been shown that the Whittle’s index policy performs strikingly well, see [38] for a discussion, and is asymptotically optimal under certain conditions, see [47] and [48].

Unfortunately, Whittle’s index can not always be calculated in closed form. For instance, Borkar et al. [6] consider a similar problem to ours with $d = \infty$, and Whittle’s index is calculated via an iterative scheme. In our main methodological contribution, we provide a closed-form expression for Whittle’s index for a LPS-$d$ server, which allows us to investigate the performance and properties of Whittle’s index policy as a function of server’s heterogeneity, that is, the values of $d$ and $q$. In order to do so we consider two specific cost functions: (i) a linear cost function, and (ii) a cost criterion that depends on the first and second moment of the throughput. A linear cost function has often been used as a cost criteria in the literature in the context of computer services (see [32] for a discussion), whereas the weighted sum of linear cost and second order throughput captures the mean-variance trade-off (see [45]). Under the linear cost criterion, we observe numerically that $q$ is the key parameter characterizing the structure of Whittle’s index, while the impact of $d$ on the dispatching decision is negligible. Under the mean-variance trade-off cost criterion, we observe that Whittle’s index policy strongly depends on the value of $d$. In both cases, Whittle’s index policy depends on the arrival probability $p$, which is key in making Whittle’s index policy more efficient than dispatching rules that are oblivious to $p$, such as Join the Shortest Queue (JSQ), Join the Shortest Expected Workload (JSEW), and Random Server allocation (RSA). Our numerical results also indicate that Whittle’s index policy is close-to-optimal under a wide range of parameters.

In summary, the main contributions of this paper are:

- We characterize in closed-form Whittle’s index for a LPS-$d$ server and apply it to study the problem of dispatching jobs to heterogeneous servers.
- Investigate the performance of Whittle’s index policy as a function of the scheduling discipline of servers parameterized by $d$, the speed as measured by $q$, and the arrival probability given by $p$.
- Conclude that the impact of $d$ and $q$ vary depending on the objective. For linear cost criterion, the speed $q$ is the main parameter while $d$ does not have a big impact on the structure of Whittle’s index policy. However, for a cost criterion that depends on the first and second moment, the dispatching policy strongly depends on both $d$ and $q$.
- Observe that Whittle’s index policy can outperform known dispatching policies like JSQ (oblivious to $d$, $q$, and $p$) and JSEW (oblivious to $d$ and $p$).

The remainder of the paper is organized as follows. Section 2 presents related work. Section 3 presents the load-balancing problem and explains our solution approach from a restless bandit perspective. Section 4 is dedicated to develop a closed-form expression for Whittle’s index in terms of stationary probabilities. Section 5 simplifies the expression in the case of FCFS schedulers and provides insights on the index for the linear cost criterion. In Section 6, we discuss the details of numerically finding an optimal policy and dispatching rules. We present our extensive numerical findings in Section 7 for two different cost criteria. A concluding remark can be found in Section 8.

2 Related literature
In this section we provide an overview of the most important related work to the problem under consideration.

As mentioned in the introduction, literature on load balancing is vast and keeps growing motivated by application domains like cloud computing and parallel computing systems. A textbook covering load balancing is [23, Chapter 24], and a survey covering recent developments is [46]. The large majority of papers consider FCFS implemented in servers, and a few works consider PS. As mentioned in the introduction, we are not aware of any work that analyzes how to dispatch jobs when servers deploy different scheduling disciplines. All the works mentioned in this overview assume FCFS, unless otherwise stated. A classical result shows that in case the dispatcher has precise information on the queue length in each server, and the servers are homogeneous, JSQ minimizes the total number of jobs in the system, see [50]. Similarly, when the expected workload in each server is known to the dispatcher, JSEW has been shown to be optimal, see for example [31, 35]. A very well-known policy is the so-called Power-of-$d$ policy, according to which upon the arrival of a new job, the dispatcher will probe $d$ servers, and dispatch the job to the shortest queue among these $d$ servers, see [34]. In another stream of work, researchers have investigated the performance of systems
in which idle servers send a notification back to the dispatcher, see [13]. In size-based dispatching, the dispatching decision is based on the size of the incoming job, see for example [24, 12]. With the so-called pull/push mechanism, servers that are idle can claim jobs from busy servers, and vice versa, servers with long queues can transfer jobs to idle servers, see [33] for an analysis. In recent years, researchers have also investigated redundancy models, in which multiple copies of the same job are dispatched to a subset of servers, see [15]. The optimal size-based dispatching policy with PS servers is characterized in [1].

Restless bandits are a popular framework in various application domains, including inventory routing [2], machine maintenance [20], cloud computing [7], sensor scheduling [40], etc. We refer to [18] for a recent survey on the application of index policies in scheduling. Book length treatments of restless bandits can be found in [26] and [44]. In the particular area of load balancing, there has been several previous works that have applied the restless banding framework, and who have calculated Whittle’s index either in closed-form or via an iterative scheme (see [3, 6, 17, 37, 39]). For example, in [6], servers are PS and an iterative scheme is reported to compute Whittle’s index in the case of linear cost functions and no blocking. The load balancing problem in [3, 17] deals with FCFS servers with dedicated arrivals for each queue. Dedicated arrivals have priority over new arrivals in the load balancing model of [17]. In [37, section 8.1] the author considers finite buffers and FCFS servers and determines an index policy. The model in [39] considers instead the objective of minimizing the average job loss rate.

An important modeling assumption we make is that time is discrete. In addition to the obvious mathematical tractability, from a practical point of view, it can be argued that decisions are often made in regular time moments instead of continuously. Example application areas include web servers, distributed caching systems, large data stores and grid computing (see [14, 21]). Discrete-time queues have been used to investigate the behavior of communication and computer systems in which time is slotted (see [4, 10, 8, 30]). For a comprehensive treatment of discrete time queues, we refer to recent books (see [9, 25]). The load balancing problem has been previously studied in discrete time for distributed systems in the presence of time delays (see [11]).

The single server under the LPS-$d$ policy has been widely studied for the particular cases of $d = 1$ (FCFS) and $d = \infty$ (PS). For $1 < d < \infty$, analysis is scarce due to its complexity. Avi-Itzhak & Hallin [5] propose an approximation for the mean response time assuming Poisson arrivals. A computational analysis based on matrix geometric methods is developed by Zhang & Lipsky [52, 51]. Some stochastic ordering results are derived in [41]. Zhang, Dai & Zwart [53, 54, 55] develop fluid, diffusion and heavy traffic approximations. Gupta & Harchol-Balter [22] consider approximation methods and Markov decision techniques to determine the optimal level $d$ when the system is not work-conserving. The sojourn time tail asymptotics for the LPS-$d$ queue for both heavy-tailed and light-tailed job size distributions are recently explored in [36]. However, none of the prior work has focused on load balancing in a system with LPS-$d$ servers, which is the central theme of this work.

3 Model description

We consider a slotted-time model with decision epochs $t \in T := \{0, 1, 2, \ldots\}$. Time epoch $t$ corresponds to the beginning of time period $t$. Jobs arrive according to a Bernoulli arrival process with arrival probability $p \in (0, 1)$. New arrival must be dispatched to at most one of the $K$ servers, or must be blocked (see Figure 1). Server $k$ serves the jobs in his queue according to the LPS-$d_k$ service discipline (defined below), where $d_k$ is a parameter determining the scheduling discipline. Since $d_i$ can be different from $d_j$, $i, j = 1, \ldots, K$, this models heterogeneous scheduling disciplines. We assume that the servers are independent, and have capacity $q_k$.

![Figure 1: Abstraction of load balancing problem in a multi-server system with heterogeneous service disciplines.](image-url)
Under the LPS-\(d_k\) service discipline, the first \(d_k\) jobs are served simultaneously and equally share the capacity of the server \(k\). More precisely, the probability of departure in server \(k\) for the first \(\min(d_k, n)\) jobs, where \(n\) is the total number of jobs present in the queue of server \(k\), is given by \(q_k / \min(d_k, n)\). The departure process from server \(k\) will be a binomial process, with the following properties. If there is only one job in the server, the probability of it being completed in a given time slot is \(q_k\). However, if there are \(m\) jobs in the server, by the egalitarian scheme, the probability of departure will be reduced to \(q_k / m\) for each of the (up to \(d_k\)) jobs in the server. The number of departures from the server will be a binomial random variable with departure probability \(q_k/m\), and the mean number of departures in a time slot remains fixed at \(q_k\). The departure process from server \(k\) can jump down up to \(n - d_k\) in one step. The transition probability for \(i\) departures in an LPS-\(d_k\) system when there are in total \(n\) jobs, is given by,

\[
q_{k}(i;n) := \left(\min\{n, d_k\}\right) \left(\frac{q_k}{\min\{n, d_k\}}\right)^i \left(1 - \frac{q_k}{\min\{n, d_k\}}\right)^{\min\{n, d_k\} - i},
\]

for \(i \leq \min\{n, d_k\}\), and is equal to 0 for \(i > \min\{n, d_k\}\).

It can be easily seen that FCFS and PS are special cases of the LPS-\(d_k\) scheduling scheme with \(d_k = 1\) and \(\infty\), respectively. Under FCFS, a job in front of the queue is always served with full capacity, hence \(d_k = 1\). If the capacity of server \(k\) is \(q_k\), the probability that a job in front of the queue will be completed in a given time slot is \(q_k\). Hence, in one unit of time, the process describing the number of jobs in the server can jump down at most by one. Under PS, all jobs in the system are given the same processing power, hence \(d_k = \infty\). Thus, there is a strictly positive probability that the process describing the number of jobs in the server goes in one time unit to zero, that is, all \(n\) jobs present in the server depart.

A policy \(\phi\) decides how new arriving jobs are dispatched. We focus on policies which base their decision only on the current states of the servers. For policy \(\phi\), \(N_k^\phi(t)\) denotes the state of server \(k\) at time epoch \(t\) and \(\vec{N}^\phi(t) = (N_1^\phi(t), ..., N_K^\phi(t))\). Let \(S_k^\phi(\vec{N}^\phi(t)) \in \{0, 1\}\) represent whether or not an arrival is routed to server \(k\) at time \(t\) under policy \(\phi\). Since a job can be dispatched to at most one out of \(K\) servers at each stage, we have \(\sum_{k=1}^{K} S_k^\phi(\vec{N}) \leq 1\), which can be re-written as

\[
\sum_{k=1}^{K} (1 - S_k^\phi(\vec{N})) \geq K - 1.
\]

If \(\sum_{k=1}^{K} S_k^\phi(\vec{N}) = 0\), the job is blocked. The dynamics of the queue length process is then described as follows:

\[
N_k^\phi(t + 1) = N_k^\phi(t) + S_k^\phi(\vec{N}^\phi(t))\psi(t + 1) - R_k^\phi(t + 1),
\]

where \(\psi(t)\) represents the number of jobs arriving in time period \(t\) and \(R_k^\phi(t)\) represents the process describing the number of jobs that departed in time period \(t\) for server \(k\) under the scheduling discipline LPS-\(d_k\).

Let us denote by \(U\) the set of policies that satisfy the constraint (2) at each decision epoch and make the system ergodic. Throughout this paper, we assume that \(p < \sum_{k=1}^{K} q_k\), which is the maximum stability condition. Hence, under this assumption there exist feasible load balancing policies that make the system stable, i.e., \(U \neq \phi\).

**Remark 3.1.** In the model description, we assumed that at most one job can arrive per time period. A natural generalization is to consider uniformly bounded batch arrivals that need to be allocated to one server. We note here that the relaxation and decomposition technique that are developed in this paper will also go through in the batch setting, that is left out for the sake of tractability.

**Stochastic optimal control**

For server \(k\), let \(C_k(n)\) denote the cost of being in state \(n\). We assume that \(C_k(n)\) is non-decreasing and bounded by a polynomial of finite degree. Let \(D\) be the cost of blocking a job. The objective is to find a scheduling policy, \(\phi \in U\), that minimizes the long run average-cost criterion,

\[
C^\phi := \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T} \left( \sum_{k=1}^{K} C_k(N_k^\phi(t)) + pD \left(1 - \sum_{k=1}^{K} S_k^\phi(\vec{N}^\phi(t))\right)\right).
\]

Due to the hard constraint (2), \(\sum_{k=1}^{K} S_k^\phi(\vec{N}^\phi(t))\) is either zero (when blocked) or one (when routed). Since \(\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T} \left(1 - \sum_{k=1}^{K} S_k^\phi(\vec{N}^\phi(t))\right)\) represents the fraction of time that an arriving job is blocked, the second term in (4) represents
the average cost of blocking. For very large blocking cost, i.e., \( D \to \infty \), in order to minimize (4), one will take
\[
\sum_{k=1}^{K} S_k^\phi (\hat{N}^\phi(t)) = 1.
\]
Thus, when setting \( D = \infty \), one retrieves the load balancing model without blocking.

The above problem can be seen as a particular case of Markov decision process (MDP) and an exact dynamic programming formulation is possible (see [43]). For certain MDPs, it is feasible to explicitly characterize an optimal stochastic control. An important class of such problems is the multi-armed bandit problem (MABP). In a MABP, only one bandit can be made active and only the active bandit can change state while the state of all other bandits remain frozen. In MABP, an optimal solution has a simple structure, known as Gittin’s index policy (see [16]). In brief, there exists functions \( G_k(n_k) \), depending only on the parameters of bandit \( k \), such that an optimal policy in state \( n \) is to serve the bandit having currently the highest index \( G_k(n_k) \).

However, the optimal scheduling policies for restless bandit problems is typically out of reach. We note that the transition probability matrix of \( N_k^\phi(t) \) is action dependent. In particular the state of the server can evolve both when the job is dispatched to the server or not. Hence, each of the servers can be considered as a restless bandit, and the load balancing problem can be seen as a restless bandit problem. The analytical solution of restless problems is inaccessible because of the sample path constraint (2). Even the numerical resolution of this problem via dynamic programming becomes quickly intractable because of the curse of dimensionality. The restless bandit problem has been reported to be PSPACE complete (see [42]).

Therefore, we deploy the relaxation approach pioneered by Whittle in [49]. The main idea is to relax the hard constraint (2) where the constraint has to be respected only in average but not at every decision epoch. Whittle showed that under the so-called indexability property, the solution to the relaxed problem is fully characterized by Whittle’s index policy, that allocates an incoming job to the schedulers with index larger than the Lagrange multiplier associated to the relaxation. Whittle then proposed a heuristic for the original problem with hard constraint, in which an incoming job is dispatched to the server with the highest index. The latter heuristic is nowadays referred to as Whittle’s index policy. Whittle’s index policy is in general not optimal for the problem with hard constraint, however known to be asymptotically optimal as the number of bandits grows to infinity (see [48]), and it has been reported that its performance is nearly-optimal for different problems.

In the next section, we will carry out the above research agenda, by first considering the relaxed problem, and then establishing indexability. We will then obtain in Equation (9) the main result of the paper, i.e., a closed-form expression for the Whittle Index.

## 4 Derivation of Whittle’s index

In this section we derive Whittle’s index. In Section 4.1 we describe how the relaxation decomposes the original \( K \) dimensional optimization problem into \( K \) one-dimensional subproblems. In Section 4.2 we show that the optimal solution to such a subproblem is of threshold type, which allows us to show in Section 4.3 that the relaxed problem satisfies the indexability property. The latter justifies the derivation of Whittle’s index, which is stated in Section 4.4. An optimal policy for the relaxed problem, which is described by Whittle’s index, will then serve as a heuristic for the original optimization problem, as described in Section 4.5.

### 4.1 Relaxation and decomposition

In [49], Whittle proposed to replace the infinite set of sample-path constraints (2) by its time-average version, that is, on average at most \( K - 1 \) servers are kept passive:
\[
\lim \sup_{T \to \infty} \frac{1}{T} \mathbb{E} \left( \sum_{t=0}^{T} \sum_{k=1}^{K} (1 - S_k^\phi (\hat{N}^\phi(t))) \right) \leq K - 1.
\]
(5)

We denote by \( \mathcal{U}^{REL} \) the set of stationary policies for which the Markov chain is ergodic and that satisfy (5). We note that \( \mathcal{U} \subset \mathcal{U}^{REL} \). The objective of the relaxed problem is hence to minimize (4) among all policies in \( \mathcal{U}^{REL} \).

The relaxed problem can be solved by considering the following unconstrained problem: find a policy \( \phi \) that minimizes
\[
\lim \sup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T} \left( \sum_{k=1}^{K} G_k(N_k^\phi(t)) + pD \left( \frac{1}{K} - S_k^\phi (\hat{N}^\phi(t)) \right) \right) \right] + W \left( K - 1 - \sum_{k=1}^{K} (1 - S_k^\phi (\hat{N}^\phi(t))) \right),
\]
(6)

where \( W \) is the Lagrange multiplier. The key observation made by Whittle is that the above relaxed problem can be
decomposed into $K$ subproblems, one for each server $k$, that is, minimize

$$
C_k^{\phi_k} := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T} (C_k(N_k^{\phi_k}(t)) - (W - pD)((1 - S_k^{\phi_k}(N_k^{\phi_k}(t)))) \right].
$$

The solution to (6) is then obtained by combining the solution to the $K$ separate subproblems (7). Under a stationarity assumption, we can invoke ergodicity to show that (7) is equivalent to minimizing

$$
\mathbb{E}(C_k(N_k^{\phi_k}, S_k^{\phi_k}(N_k^{\phi_k}))) - (W - pD) \mathbb{E}(1_{S_k^{\phi_k}(N_k^{\phi_k}) = 0}),
$$

where $N_k^{\phi_k}$ is distributed as the stationary distribution of the state of server $k$ under policy $\phi_k$.

### 4.2 Threshold optimality

In this section, we establish that the optimal solution of problem (7) is of threshold type. That is, there is a threshold $n_k(W)$ such that when there are $n_k$ users in server $k$, $n_k \leq n_k(W)$, then accepting a job is optimal, and otherwise blocking a job is optimal. We let policy $\phi_k = n_k$ denote a threshold policy with threshold $n_k$.

**Proposition 1.** Threshold policies are optimal for the relaxed load balancing problem, i.e., there exists an $n_k \in \{-1, 0, 1, \ldots\}$ such that the threshold policy $n_k$ optimally solves problem (8), for all $k = 1, \ldots, K$.

**Proof.** Let us drop the dependency on $k$ throughout the proof. Since there exists $\phi \in U_{REL}$, there exists a stationary optimal policy $\phi^*$ that optimally solves problem (8). Define $n^* = \inf\{m \in \{0, 1, \ldots\} : S^{\phi^*}(m) = 0\}$. This implies $S^{\phi^*}(m) = 1, \forall m < n^*$ and $S^{\phi^*}(n^*) = 0$. It follows from the evolution of the Markov chain (see Equation (3)) that all states $m > n^*$ are transient. Thus, $\pi^{\phi^*}(m) = 0, \forall m > n^*$. Thus, the average cost as given by (8) under the optimal policy $\phi^*$ then reduces to

$$
\mathbb{E}(C(N^{\phi^*}) - (W - pD) \mathbb{E}(1_{S^{\phi^*}(N^{\phi^*}) = 0}) = \sum_{m=0}^{n^*-1} C(m)\pi^{\phi^*}(m) + C(n^*)\pi^{\phi^*}(n^*) - (W - pD)\pi^{\phi^*}(n^*)
$$

that is, a threshold policy with threshold $n^*$ gives the optimal performance.

The above proposition implies that an optimal policy for problem (8) is fully characterized by a threshold $n$. We let $\pi_k^n(m)$ denote the steady state probability of being in state $m$ for bandit $k$ under the threshold policy $n$. Equation (8) under policy $\phi_k = n$ can be written as

$$
g_k^n(W) := \sum_{m=0}^{\infty} C_k(m)\pi_k^n(m) - (W - pD) \sum_{m=n+1}^{\infty} \pi_k^n(m).
$$

We hence conclude that the optimal solution of problem (8) is given by:

$$
g_k(W) = \min_n g_k^n(W).
$$

### 4.3 Indexability

Indexability is the property that allows us to develop a heuristic for the original problem. This property requires to establish that as the Lagrange multiplier, or equivalently the subsidy for passivity, $W$, increases, the collection of states in which the optimal action is passive increases. It was first introduced by Whittle [49] and we formalize it in the following definition.

**Definition 4.1.** A server is indexable if the set of states in which passive is an optimal action in (7) (denoted by $D_k(W)$) increases in $W$, that is, $W' < W \Rightarrow D_k(W') \subseteq D_k(W)$.

If indexability is satisfied, Whittle’s index in state $N_k$ is defined as follows:

**Definition 4.2.** When server $k$ is indexable, Whittle’s index in state $N_k$ is defined as the smallest value for the subsidy such that actions active and passive are equally attractive in state $N_k$. The Whittle’s index is denoted by $W_k(N_k)$. 6
Given that the indexability property holds, Whittle established in [49] that the solution to the relaxed control problem (6) will be to activate all servers that are in state \( n_k \) such that their Whittle’s index exceeds the Lagrange multiplier, i.e., \( W_k(n_k) > W \).

In order to prove indexability for our problem, we will make use of the following result (see A for its proof).

**Lemma 4.3.** \( \sum_{m=0}^{n} \pi_k^n(m) \) is non-decreasing in \( n \).

We can now prove indexability of the problem. Some ideas in this proof are adopted from [19].

**Proposition 2.** The load balancing problem is Whittle indexable.

**Proof.** Let us drop the dependency on \( k \) throughout the proof for ease of notation. Since an optimal policy for (8) is of threshold type, for a given subsidy \( W \) the optimal average cost under threshold \( n \) will be \( g(W) := \min_n \{ g^n(W) \} \), where

\[
\begin{aligned}
g^n(W) &:= \sum_{m=0}^{\infty} C(m)\pi^n(m) - (W - pD) \sum_{m=n+1}^{\infty} \pi^n(m) \\
&= \sum_{m=0}^{\infty} C(m)\pi^n(m) + (W - pD) \sum_{m=0}^{n} \pi^n(m) - (W - pD).
\end{aligned}
\]

Let \( n(W) \) be the minimizer of \( g(W) \). Note that \( g^n(W) \) is an affine non-increasing function of \( W \). Thus, the function \( g(W) \) is a lower envelope of affine non-increasing functions of \( W \). It thus follows that \( g(W) \) is a concave non-increasing function.

It directly follows that the right derivative of \( g(W) \) in \( W \) is given by \( \sum_{m=0}^{n(W)} \pi^n(W)(m) - 1 \). Since \( g(W) \) is concave in \( W \), the right derivative is non-increasing in \( W \). But \( \sum_{m=0}^{n} \pi^n(m) \) is non-decreasing in \( n \), from Lemma 4.3. It hence follows that \( n(W) \) is non-increasing in \( W \). Together with \( D(W) = \{ m : m \geq n(W) \} \) and by Definition 4.1, indexability follows.

**4.4 Whittle’s index**

Recall that Whittle’s index is the smallest value of \( W \) such that we are indifferent of the action taken in state \( n \). Under the optimality of threshold policies, one is indifferent of the action taken in state \( n \) if the performance under threshold policies \( n - 1 \) and \( n \) are equal. The following proposition presents a closed form expression of Whittle’s index for the load-balancing problem.

**Proposition 3.** Assume \( \sum_{m=0}^{n} \pi_k^n(m) \) is strictly increasing in \( n \). The Whittle index is given by

\[
W_k(n) = pD - \frac{\sum_{m=0}^{n} C_k(m)[\pi_k^n(m) - \pi_k^{n-1}(m)] + C_k(n+1)\pi_k^n(n+1)}{\sum_{m=0}^{n} \pi_k^n(m) - \sum_{m=0}^{n-1} \pi_k^{n-1}(m)},
\]

if \( W_k(n) \) is non-increasing in \( n \).

Note that we require \( \sum_{m=0}^{n} \pi_k^n(m) \) to be strictly increasing in \( n \). In Lemma 4.3 we proved that this function is non-decreasing. Numerical evidence shows that the function is in fact strictly increasing.

**Proof.** We drop the subscript \( k \) throughout the proof. Let \( \tilde{W}(n) \) be the value for the subsidy such that the average cost under threshold policy \( n \) is equal to that under policy \( n - 1 \). Hence, using (8), we obtain \( \mathbb{E}(C(N^n)) - (\tilde{W}(n) - pD)\mathbb{E}(1_{S^n(N^n) = 0}) = \mathbb{E}(C(N^{n-1})) - (\tilde{W}(n) - pD)\mathbb{E}(1_{S^{n-1}(N^{n-1}) = 0}), \) for all \( n \geq 1 \). Together with \( \mathbb{E}(1_{S^n(N^n) = 0}) = \sum_{m=n+1}^{\infty} \pi^n(m) = 1 - \sum_{m=0}^{n} \pi^n(m), \) and the fact that \( \sum_{m=0}^{n} \pi_k^n(m) \) is strictly increasing in \( n \), we obtain

\[
\tilde{W}(n) = pD - \frac{\mathbb{E}(C(N^n)) - \mathbb{E}(C(N^{n-1}))}{\sum_{m=0}^{n} \pi^n(m) - \sum_{m=0}^{n-1} \pi^{n-1}(m)}.
\]
Below we will show that \( \tilde{W}(n) \) is non-increasing in \( n \). This then implies that \( g(\tilde{W}(n)) = g(n)(\tilde{W}(n)) = g(n-1)(\tilde{W}(n)) \).

Now, recall that \( \frac{d g^{(n)}(W)}{dW} = - \sum_{m=n+1}^{\infty} \pi^n(m) \) is non-decreasing in \( n \). Hence, \( g(W) = g(n)(W) \), for \( \tilde{W}(n) \leq W \leq \tilde{W}(n-1) \). This implies that Whittle’s index is given by \( W(n) = \tilde{W}(n) \). Note that \( \mathbb{E}(C(N^n)) = \sum_{m=0}^{n+1} C(m)\pi^n(m) \), hence \( \tilde{W}(n) \) simplifies to the Whittle’s index as stated in the proposition.

Remark 4.4. For a load balancing problem with homogeneous PS servers, an iterative scheme to approximate Whittle’s index was reported in [6]. The iterative scheme to compute Whittle’s index in [6] adjusts the current guess for the index in the direction of decreasing the discrepancy in the active and passive values which should agree for the exact index. Further, a linear interpolation is used after computing the index for sufficiently large number of states which makes the indices approximate in nature in [6].

Our approach results in a closed form expression for Whittle’s index in terms of steady-state probabilities for LPS-d schedulers. Though, we could not theoretically argue the non-increasing nature of the index (9), we numerically note that this is indeed the case for a wide set of parameters and for different cost functions (see Figures 3, 11 and 12). We note that Whittle’s index (9) has a common term \( pD \). In case no blocking is allowed, Whittle’s index is obtained by simply dropping the term \( pD \).

4.5 Heuristic for load-balancing problem

In this section, we describe how the optimal solution to the relaxed optimization problem is used to obtain a heuristic for the original model. The optimal solution to the relaxed problem, that is, activate all servers that are in a state \( n_k \) such that \( W_k(n_k) > W \), might be infeasible for the original model where a job can be dispatched to at most one server. Hence, Whittle [49] proposed the following heuristic, which is referred as Whittle’s index policy.

Definition 4.5 (Whittle’s index policy). Assume at time \( t \) we are in state \( \tilde{N}(t) = \bar{n} \).

- In case \( D < \infty \) and hence blocking is allowed, Whittle’s index policy sends a new arriving job to the server having currently the highest non-negative Whittle’s index value \( W_k(n_k) \). If all Whittle’s indices are negative, the job is blocked.
- In case no blocking is allowed (and hence \( D = \infty \)), Whittle’s index policy sends a new arriving job to the server having currently the highest (possibly negative) Whittle’s index value \( W_k(n_k) \).

In case blocking is allowed, and all servers are in a state such that their Whittle’s index is negative, all servers are kept passive, i.e., the job is not dispatched to any of the servers and it is blocked. The latter is a direct consequence of the relaxed optimization problem: when the Whittle’s index is negative for a server in state \( \bar{n} \), this means that it is made active only if \( W < W_k(\bar{n}) < 0 \), that is, when a cost is paid for being passive.

Remark 4.6. The closed form expressions of Whittle’s index in Proposition 3 facilitate a major computational saving on the computation of the true optimal policy.

5 Computation of Whittle’s index

Whittle’s index depends on the stationary distribution of the threshold policies, see Proposition 3. In this section we give further details on how these stationary probabilities can be calculated. In general, the closed form expression of the stationary distribution is intractable. In Section 5.1, we present details of the transition probability matrix and how this can be used to compute Whittle’s index numerically. In Section 5.2, we focus on \( d = 1 \) (FCFS), in which case a closed form expression for the steady state distribution, and hence for Whittle’s index, can be given.

5.1 LPS-d discipline

In this section we describe the transitions of the one-dimensional problem of the relaxed optimization problem. Hence, we are given one single server, which implements LPS-d, and who accepts customers until threshold \( n \). The one-step state transitions of this DTMC of server \( k \) are shown in Figure 2 for a state \( m < n \). Recall that \( q_k^m(i|m) \) was defined in (1) and describes the probability of \( i \) departures when \( m \) jobs are in server \( k \).
The normalizing condition (where \( b_k \))

We can now check the condition needed in supposition of Proposition 3, i.e., obtaining a closed-form expression for the stationary distribution for arbitrary \( d_k \) seems infeasible, and hence, we will consider the case \( d_k = 1 \) and derive a closed-form expression.

### 5.2 FCFS scheduling discipline

Under FCFS, a job in front of the queue departs with probability \( q_k \) in a given time slot. Thus, the evolution of DTMC for server \( k \) is according to the following transition probability matrix under the threshold policy \( n \):

\[
P_k = \begin{pmatrix}
1-p & p & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & d_k & r_k & b_k & 0 & \cdots & 0 & 0 \\
0 & 0 & d_k & r_k & b_k & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & d_k & r_k & b_k \\
0 & 0 & 0 & 0 & \cdots & 0 & q_k & 1- q_k \\
\end{pmatrix}_{(n+2) \times (n+2)}
\]

where \( b_k = p(1 - q_k) \), \( d_k = q_k(1 - p) \) and \( r_k = pq_k + (1 - p)(1 - q_k) \).

The stationary distribution \( \pi_k(\cdot) \) of the above DTMC is given by solving a set of linear equations \( \pi_k^n = \pi_k^n P_k \) with the normalizing condition \( \sum_i \pi_k^n(i) = 1 \). The solution can be written in closed form and is given by:

\[
\pi_k^n(0) = \frac{d_k}{p} \pi_k^n(1), \quad (10)
\]

\[
\pi_k^n(1) = \frac{\frac{p}{q_k}(1 - \frac{p}{q_k})}{(1 - p)^n - \frac{p}{q_k} + 1 - q_k} \pi_k^n(1), \quad (11)
\]

\[
\pi_k^n(m) = \left( \frac{b_k}{d_k} \right)^{m-1} \pi_k^n(1), \quad m = 2, 3, ..., n, \quad (12)
\]

\[
\pi_k^n(n+1) = \left( \frac{b_k}{q_k} \frac{d_k}{b_k} \right)^{n-1} \pi_k^n(1). \quad (13)
\]

We can now check the condition needed in supposition of Proposition 3, i.e., \( \sum_{i=0}^{n} \pi_k^n(i) \) is strictly increasing in \( n \), or equivalently, \( \pi_k^n(n+1) < \pi_k^n(n) \). After some algebra, the latter simplifies to \( \frac{b_k}{d_k} \pi_k^n(1) < \pi_k^{n-1}(1) \). Using the stationary probabilities, it can be easily verified that the latter holds true. In addition, we will prove that the expression (9) in Proposition 3 is non-increasing. This provides us with a closed form expression for Whittle’s index under FCFS (see Appendix A.2. for the proof).
5.2.1 Linear cost criterion

was proved for a set of load balancing policies that made sure that smaller queues were sent more traffic.

index policy is maximum stable as it will not dispatch jobs to relatively crowded servers. In [57] maximum stability
below some function which is linear in \( N \) in Whittle’s index policy is determined by some linear switching curve (see Figures 7, 8 and 13). That is, when
\( N \) arriving jobs.

\[ q - \infty \]

\[ W \]

\[ D \]

which is a non-increasing function. In addition,

\[ W_k(0) = pD + C_k(1)p q_k + p - 1 \]

and if \( \lim_{n \to \infty} C_k(n) \to \infty \), then \( \lim_{n \to \infty} W_k(n) \to -\infty \).

Remark 5.1. Load balancing problems with FCFS schedulers have been previously modeled as multi-armed restless bandit problems and closed form expressions for Whittle’s index are available in the literature for the continuous time setting. For example, Niño-mora studied a load balancing problem with finite buffer queues and derived a closed form expression (see [37, section 8.1]) whereas [3] and [19] derived Whittle’s index for load-balancing problems with dedicated arrivals and abandonments, respectively. Note that these indices are different from ours, as we consider a discrete-time setting.

Recall that Whittle’s index policy is proposed as an efficient heuristic. It is however not known in general whether Whittle’s index policy will indeed make the system stable. However, an immediate consequence from \( \lim_{n \to \infty} W_k(n) \to -\infty \) is that the Whittle index policy is stable for the systems with blocking: As the Whittle index (for any \( d \) and \( q \)) approaches to \( -\infty \), there is a finite threshold for each server above which Whittle’s index policy will block new arriving jobs.

Without blocking, every job needs to be routed to some server. Numerically we observed that the optimal action in Whittle’s index policy is determined by some linear switching curve (see Figures 7, 8 and 13). That is, when \( N_k \) is below some function which is linear in \( N_i \), \( i \neq k \), then we send to server \( k \). One would therefore expect that Whittle’s index policy is maximum stable as it will not dispatch jobs to relatively crowded servers. In [57] maximum stability was proved for a set of load balancing policies that made sure that smaller queues were sent more traffic.

5.2.1 Linear cost criterion

For the linear cost function, that is, \( C_k(m) = mC_k \), the Whittle index (as stated in Proposition 4) can be rewritten as

\[ W_k(n) = pD + \frac{C_k p^2 (1 - p)}{q_k (p - q_k)^2} \frac{C_k p (1 - q_k)}{q_k} - \frac{nC_k p}{q_k (p - q_k)^2} \left( \frac{p (1 - q_k)}{q_k (1 - p)} \right)^n . \]

The details can be found in A.

In case \( p < q_k \), it follows from (15) that \( W(n) \) behaves as

\[ \frac{W(n)}{n} = C_k \frac{p}{q_k - p} + o(1), \text{ for large } n. \]

That is, when there are many jobs in the servers, under Whittle’s index policy, the job is routed to the server having highest value \( C_k n_k q_k / p \), with \( n_k \) as number of jobs in the server \( k \). Instead, under JSEW, the job is routed to the server with the least expected workload, or in other words, to the server with the largest \( n_k / q_k \) index. Hence, JSEW is greedy, and only sees what lies ahead of an incoming job, i.e., it ignores the impact of jobs in the future. Whittle’s index policy differs from JSEW (in case \( C_k = 1 \)) by the multiplicative term \( p / (1 - p / q) \), that is, it also takes into account the impact of future arrivals \( p \). In the numerical section we will see that Whittle’s index policy indeed performs better than JSEW.

6 Other load balancing policies

In the numerical section we compare the performance of Whittle’s index policy with various dispatching policies that we describe here.
6.1 Optimal load balancing policy

We numerically solve Bellman’s optimality equation by value iteration to compute the optimal performance. Consider a system with \(K\) servers and let \(m_1, m_2, \cdots, m_K\) denote the number of jobs in these servers. Bellman’s optimality equation for the average cost criterion is given by:

\[
g + V_{t+1}(m_1, m_2, \cdots, m_K) = \sum_{i=1}^{K} C_i(m_i) + \min_{i \in \{1, 2, \ldots, K\}} \left\{ E'(V_i(.)|m_1, m_2, \cdots, m_K) \right\},
\]  

where \(E'(\cdot)\) denotes the expectation with respect to the transition probability when an arriving job is dispatched to server \(i\) and \(g\) represents the average cost under an optimal policy. An optimal server to send a new arriving job is chosen according to the value of the right-hand side of (16). We will focus on \(K = 2\). The term \(E'(V_i(.)|m_1, m_2)\) then simplifies to

\[
E'(V_i(.)|m_1, m_2) = p \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} q_1^{d_1}(i_1|m_1)q_2^{d_2}(i_2|m_2) V_i(m_1 + 1_{(A=1)} - i_1, m_2 + 1_{(A=2)} - i_2) + (1-p) \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} q_1(i_1)q_2(i_2) V_i(m_1 - i_1, m_2 - i_2),
\]

where \(q_l(i|m)\) is given in (1) and denotes the probability of \(i\) jobs departing in one time slot, when there are \(m\) jobs present in server \(l\) with LPS-\(d\) service discipline and \(1_{(A=j)}, j = 1, 2\) presents the indicator function of the event that an arrival is dispatched to queue \(j\). Due to the curse of dimensionality, the optimality equation (16) cannot be solved for a moderate number of servers. We therefore use the value iteration algorithm to find the optimal policy numerically, which we describe below:

**Step 0:** Initialization: \(V_0(m) = 0\) for all states \(m = (m_1, \ldots, m_K) \in S\), where \(S\) is state space.

**Step 1:** Evaluate \(V_{t+1}(m) \ \forall \ m \in S\) using Bellman’s optimality equation (16).

**Step 2:** Stopping criterion: If \(\max_{m \in S} \{V_{t+1}(m) - V_t(m)\} - \min_{m \in S} \{V_{t+1}(m) - V_t(m)\} \leq \epsilon\), stop. Otherwise increase \(t\) by 1 and go to step 1.

6.2 RSA, JSQ and JSEW

- **Random server allocation (RSA):** An incoming job is dispatched to a server chosen according to a uniform distribution. Information regarding the number of jobs in the server, server scheduling disciplines, service rates, and cost of service is not taken into account while making a decision to choose a server.

- **Join the shortest queue (JSQ):** An incoming job is routed to a server with the least number of jobs, \(n_k\), and ties are broken randomly. Thus, JSQ strives to balance load across the servers, reducing the probability of one server having several jobs while another server sits idle. JSQ is blind in terms of using the information of server speed and scheduling discipline.

- **Join the shortest expected workload (JSEW)** An incoming job is dispatched to the queue with the least expected workload, i.e., \(n_k/q_k\) for server \(k\). This dispatching rule uses the information of server speed but is independent of the scheduling discipline of the server.

7 Characteristics and Performance of index policy\(^1\)

We consider a linear cost criterion in Section 7.1 and a weighted second order throughput based criterion (motivated by [45]) capturing the mean variance trade-off in Section 7.2. Our main observations are that Whittle index policy is close to optimal under a wide range of parameter settings, and that with the weighted second order throughput criterion, the qualitative properties of Whittle’s index policy strongly depend on the value of \(d\) used in servers.

7.1 Linear cost criterion

In this section we assume a linear cost structure, i.e., \(C_k(n) = n\) for all \(k\).

\(^1\)All numerical codes are available in github repository: https://github.com/manugupta-or/Limited_processor_sharing
7.1.1 Characteristics of index policy

We numerically examine the pattern of indices as a function of states. We consider three servers each with different scheduling policies \((d = \infty \text{ (PS)}, LPS - d, d = 1 \text{ (FCFS)})\). An instance of indices patterns is shown in Figure 3 which confirms the non-increasing nature of Equation (9) in Proposition 3. We chose \(D = 300, p = 0.55, q_1 = q_2 = q_3 = 0.6, d_1 = \infty \text{ (PS)}, d_2 = 2 \text{ (LPS - 2)}\) and \(d_3 = 1 \text{ (FCFS)}\). We took equal capacity for the three servers in order to assess the importance of scheduling disciplines employed at different servers when dispatching jobs. It can be observed that the service discipline has a small impact on the Whittle index. We magnify the index pattern of Figure 3 and present the instances where FCFS, LPS-2 and PS have the highest Whittle’s index in Figures 4, 5, and 6, respectively.

It is evident from these figures that FCFS is preferred for small queues \((n \leq 2)\), LPS-2 is preferred for medium size queues \((2 < n < 7)\) and PS is preferred for large queue sizes \((n \geq 7)\). This phenomenon is intuitive: when there are not many jobs in the server, a new arriving job does not mind waiting, and receiving the full service once in service (as under FCFS). On the other hand, jobs will prefer to receive the service immediately upon arrival when the queue length is large (as under PS). Further, note from Figure 6 that PS queues with \(n = 13\) (higher Whittle’s index) is preferred over the FCFS queue with \(n = 12\). Thus, the index policy accounts for the trade-off between waiting for the job’s turn and getting the full dedication of the server against sharing the server but relatively less waiting in queue.

7.1.2 Comparison with JSEW/JSQ

We consider two heterogeneous servers with service discipline parameters \(d_1\) and \(d_2\), respectively.

In Figures 7 and 8, we present the actions taken under Whittle’s index policy. Figure 7 (Figure 8) is produced while varying the server speeds (service-disciplines) and keeping the same service disciplines (server speeds) in both servers. Parameter settings are as mentioned in the caption. In the area with “.” (“*”), queue 1 (queue 2) is prioritized
in dispatching the jobs and in the white area jobs are blocked. In each of these figures, the straight line represents the switching curve for JSEW. Jobs are dispatched to queue 1 (queue 2) for the states above (below) the switching curve under JSEW.

From the Figures 7 and 8, we see that Whittle’s index policy differs from JSEW when \( q_1 \neq q_2 \). For instance, having \( q_1 > q_2 \) makes that Whittle’s index policy dispatches more jobs to queue 1, see Figure 7. On the other hand, from Figure 8 we observe that the scheduling discipline has hardly any impact on the load balancing decision: the actions taken under Whittle’s index policy coincide with those under JSEW when \( d_1 = 1 \) and \( d_2 = 10 \). In general, we concluded from extensive numerical experiments, that the impact of \( d \) on the Whittle index policy is negligible for linear cost functions. This is not surprising as it is well known that in the continuous time setting, the evolution of the Markov process will be independent on how the capacity is shared among the jobs, that is, is independent on the value of \( d \).

### 7.1.3 Performance comparison

In this section we compare the performance under our Whittle index policy with that of JSQ, JSEW and RSA. We do not allow any blocking in the system.

In Figure 9 and 10 we plot the relative difference of the performance compared to that under Whittle’s index policy.
and that under an optimal policy, respectively, and let $p$ run from 0 till $q_1 + q_2$ (the stability region). The relative difference (%) (compared to the index policy) is computed as follows:

$$\text{Relative difference} = \frac{\mathbb{E}(N^\psi) - \mathbb{E}(N^{Index})}{\mathbb{E}(N^{Index})} \times 100,$$

(17)

where $\mathbb{E}(N^\psi)$ is the expected number of jobs under policy $\psi \in \{\text{JSQ, JSEW}\}$. Similarly, the relative difference (%) (compared to the optimal policy) is computed as follows:

$$\text{Relative difference} = \frac{\mathbb{E}(N^\psi) - \mathbb{E}(N^{Opt})}{\mathbb{E}(N^{Opt})} \times 100,$$

(18)

where $\mathbb{E}(N^\psi)$ is the expected number of jobs under policy $\psi \in \{\text{RSA, JSQ, JSEW, Index}\}$. We implement the value iteration algorithm as described in Section 6.1 to compute the optimal performance. First of all, we notice that Whittle’s index policy provides a stable system. Second, over a wide set of parameters, we notice the following general pattern. The Whittle index policy outperforms all the standard dispatching rules and the random server allocation has by far the worst performance for higher values of $p$.

From Figure 9, we notice a significant relative difference (upto 75%) in performance with respect to JSQ/JSEW. Thus, one of the important conclusions of these numerical experiments is that the Whittle index policy significantly outperforms all these standard dispatching rules. We further note from Figure 10 that relative difference of Whittle’s index policy compared to the optimal policy is very small (less than 3%). We note that in other related load balancing problems, it has been established that Whittle’s index policy is asymptotically optimal in light traffic, see for example [19].

### 7.2 Second order throughput cost criterion

In this section, we consider the problem of variance minimization that has been used in Markov decision processes to regularize systems (see [27, 29]). More specifically, we consider a cost criterion that consists of the linear cost and mean-variance trade off cost as in [45]:

$$C_k(n) = \beta n + (1 - \beta) \sum_{i=1}^{\min(n,d_k)} (i^2 - i\theta)q_k^{d_k}(i|n),$$

where $\theta$ is a parameter to tune the trade-off between mean throughput and the service regularity (second moment of throughput). By varying $\theta$, one can explore the entire Pareto frontier of mean-variance tradeoffs (see [45]). And $\beta$ is the weight associated with the linear cost term. Note that $\beta = 0$ would imply that the cost function, $C_k(n)$, is independent of $n$ (constant) for $n > d_k$.

We can show that $q_k^{d_k}(i|n)$ is a non-decreasing function of $n$ (see A.4). Thus, $C_k(n)$ is a non-decreasing function in $n$, hence, the Whittle index is as stated in (9), under the condition that $\sum_{m=0}^{n} \psi_k^m(m)$ is strictly increasing in $n$.

#### 7.2.1 Characteristics of index policy

In Figure 11 and 12 we plot the Whittle index functions for an LPS-4 server and an LPS-6 server. We do this both for homogeneous server speeds (Figure 11) and heterogeneous server speeds (Figure 12), respectively. We set $\beta = 0.001$ and $\theta = 0.9$.

In Figure 11, it can be seen that when there are $n < 4$ jobs in the system, the index is the same for both servers. An explanation for this is that in such states, both servers equally share their capacity among all $n$ jobs. However, note that under LPS-6, there is a possibility that two more jobs enter and they will also receive a fair share of the capacity. Therefore, for $n < 4$ we observe that both servers are equally attractive, however, when $n = 4$, LPS-4 is preferred over LPS-6, due to a possible new arrival of a job.

We further observe in Figure 11 that $W^{LPS-4(10)} > W^{LPS-6(5)}$, that is, Whittle’s index policy chooses an LPS-4 server with 15 (or less) jobs over an LPS-6 server with 5 (or more) jobs. This preference for a higher loaded server comes from the fact that the LPS-4 system has a smaller variation in terms of second-order throughput and the cost criterion consist of such term. Similar observations can be made in Figure 12 for heterogeneous server speeds.

#### 7.2.2 Performance comparison

We present the scheduling pattern in Figure 13 (left) for homogeneous server speeds with parameter settings mentioned in the caption. That is, the servers only differ in their implemented service policies. We observe that this difference
Figure 11: Whittle’s indices with homogeneous server speeds ($p = 0.25, q_1 = q_2 = 0.3$).

Figure 12: Whittle’s indices with heterogeneous server speeds ($p = 0.25, q_1 = 0.25, q_2 = 0.5$).

Figure 13: Scheduling pattern (left) under index policy with $p = 0.65$, percentage relative difference (middle and right) as compare to index and optimal (respectively) policy for homogeneous server speeds ($q_1 = q_2 = 0.5$) and $d_1 = 1, d_2 = 6$. 
has a large impact on the scheduling pattern: Whittle’s index policy might send jobs to LPS-1, even though this server is higher loaded than the LPS-6 server and vice-versa depending on the state of the system. This is unlike with linear costs, where we saw in Figure 8 that under Whittle’s index policy one would send the job to the server with the least number of jobs, even though the servers have different implemented policies. We also plot the switching curve for JSQ and JSEW in Figure 13 (left) (straight line with 45° slope). We observe that the scheduling pattern in Figure 13 is quite different from JSQ/JSEW. In Figure 13 (middle and right), we plot the relative difference (in %) when comparing to Whittle’s index policy and an optimal policy, respectively. We notice that losses under the Whittle index policy are uniformly minimum across different load factors as compare to other dispatching rules (JSQ, JSEW, RSA).

We perform another set of experiments with heterogeneous server speeds for the same parameters setting as in Figure 12 and notice a similar phenomenon (see Figure 14).

Figure 14: Scheduling pattern (left) under index policy with \( p = 0.4 \), percentage relative difference (middle and right) as compare to index and optimal (respectively) policy for heterogeneous server speeds \((q_1 = 0.2 \text{ and } q_2 = 0.5)\) and \(d_1 = 1, d_2 = 6\).

8 Concluding remark

We have considered the load-balancing problem for LPS-d type of schedulers with a possibility of blocking. Our solution approach is from a restless bandit perspective and one of the important take away messages of the study is that a well performing dispatcher can depend on the scheduling discipline deployed in the servers.

Several possible extensions suggest themselves. The impact of abandonment and batch arrivals on the load-balancing problem can be an immediate follow up. The performance of Whittle’s dispatching rule in a many-server regime, also known as mean-field regime, would be interesting, as it might yield closed-form expressions. The stability of Whittle’s index policy for load balancing problem is another interesting future avenue. Extending the results of this work to load balancing systems with multiple dispatchers, as it happens in modern data centers, is also relevant. In particular, this would require to capture the impact that one dispatcher’s action has upon others.

Acknowledgement

This research is partially supported by the French Agence Nationale de la Recherche (ANR) through the project ANR-15-CE25-0004 (ANR JCJC RACON) and by ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-02.

References

[1] E. Altman, U. Ayesta, and B. Prabhu. Load balancing in processor sharing systems. *Telecommunication Systems*, 47(1–2):35–48, 2011.
[2] Thomas W Archibald, DP Black, and Kevin D Glazebrook. Indexability and index heuristics for a simple class of inventory routing problems. *Operations research*, 57(2):314–326, 2009.
[3] Nilay Tanik Argon, Li Ding, Kevin D Glazebrook, and Serhan Ziya. Dynamic routing of customers with general delay costs in a multiserver queuing system. *Probability in the Engineering and Informational Sciences*, 23(2):175–203, 2009.
[4] Jesus R Artalejo and Onésimo Hernández-Lerma. Performance analysis and optimal control of the Geo/Geo/c queue. Performance Evaluation, 52(1):15–39, 2003.

[5] Benjamin Avi-Itzhak and Shlomo Hallín. Expected response times in a non-symmetric time sharing queue with limited number of service positions. In Proceedings of the 12th International Teletraffic Congress, 1988.

[6] Vivek S Borkar and Sarath Pattathil. Whittle indexability in egalitarian processor sharing systems. Annals of Operations Research, pages 1–21, 2017.

[7] Vivek S Borkar, K Ravikumar, and Krishnakant Saboo. An index policy for dynamic pricing in cloud computing under price commitments. Applicationes Mathematicae, 44:215–245, 2017.

[8] Herwig Bruneel. Performance of discrete-time queueing systems. Computers & Operations Research, 20(3):303–320, 1993.

[9] Herwig Bruneel and Byung G Kim. Discrete-time models for communication systems including ATM, volume 205. Springer Science & Business Media, 2012.

[10] Herwig Bruneel and Ilse Wuyts. Analysis of discrete-time multiserver queueing models with constant service times. Operations Research Letters, 15(5):231–236, 1994.

[11] Sagar Dhakal, BS Paskaleva, Majeed M Hayat, Edl Schamiloglu, and Chaouki T Abdallah. Dynamical discrete-time load balancing in distributed systems in the presence of time delays. In Decision and Control, 2003. Proceedings. 42nd IEEE Conference on, volume 5, pages 5128–5134. IEEE, 2003.

[12] H. Feng, V. Misra, and D. Rubenstein. Optimal state-free, size-aware dispatching for heterogeneous M/G/-type systems. Performance Evaluation, 62(1–4):36–39, 2005.

[13] Sergey Foss and Alexander L. Stolyar. Large-scale join-idle-queue system with general service times. J. Applied Probability, 54(4):995–1007, 2017.

[14] Ian Foster, Yong Zhao, Ioan Raicu, and Shiyong Lu. Cloud computing and grid computing 360-degree compared. In Grid Computing Environments Workshop, 2008. GCE’08, pages 1–10. IEEE, 2008.

[15] Kristen Gardner, Mor Harchol-Balter, Alan Scheller-Wolf, Mark Velednitsky, and Samuel Zbarsky. Redundancy-d: The power of d choices for redundancy. Operations Research, 2017.

[16] John Gittins, Kevin Glazebrook, and Richard Weber. Multi-armed bandit allocation indices. John Wiley & Sons, 2011.

[17] KD Glazebrook and Christopher Kirkbride. Index policies for the routing of background jobs. Naval Research Logistics (NRL), 51(6):856–872, 2004.

[18] Kevin D Glazebrook, David J Hodge, Christopher Kirbride, and RJ Minty. Stochastic scheduling: A short history of index policies and new approaches to index generation for dynamic resource allocation. Journal of Scheduling, 17(5):407–425, 2014.

[19] Kevin D Glazebrook, Christopher Kirbride, and Jamal Ouenniche. Index policies for the admission control and routing of impatient customers to heterogeneous service stations. Operations Research, 57(4):975–989, 2009.

[20] Kevin D Glazebrook, HM Mitchell, and PS Ansell. Index policies for the maintenance of a collection of machines by a set of repairmen. European Journal of Operational Research, 165(1):267–284, 2005.

[21] Varun Gupta, Mor Harchol Balter, Karl Sigman, and Ward Whitt. Analysis of join-the-shortest-queue routing for web server farms. Performance Evaluation, 64(9-12):1062–1081, 2007.

[22] Varun Gupta and Mor Harchol-Balter. Self-adaptive admission control policies for resource-sharing systems. ACM SIGMETRICS Performance Evaluation Review, 37(1):311–322, 2009.

[23] M. Harchol-Balter. Performance Modeling and Design of Computer Systems: Queueing Theory in Action. Cambridge University Press, 2013.

[24] M. Harchol-Balter and R. Vesilo. To balance or unbalance load in size-interval task allocation. Probability in the Engineering and Informational Sciences, 24:219–244, 2010.
[25] Jeffrey J Hunter. *Mathematical Techniques of Applied Probability: Discrete Time Models: Basic Theory*, volume 1. Academic Press, 2014.

[26] Peter Jacko. Dynamic priority allocation in restless bandit models, 2010.

[27] Hajime Kawai. A variance minimization problem for a Markov decision process. *European Journal of Operational Research*, 31(1):140–145, 1987.

[28] Masaaki Kijima. *Markov processes for stochastic modeling*. Springer, 2013.

[29] Masami Kurano. Markov decision processes with a minimum-variance criterion. *Journal of mathematical analysis and applications*, 123(2):572–583, 1987.

[30] Koenraad Laevens and Herwig Bruneel. Discrete-time multiserver queues with priorities. *Performance Evaluation*, 33(4):249–275, 1998.

[31] Zhen Liu and Rhonda Righter. Optimal load balancing on distributed homogeneous unreliable processors. *Operations Research*, 46(4):563–573, 1998.

[32] Haim Mendelson. Pricing computer services: queueing effects. *Communications of the ACM*, 28(3):312–321, 1985.

[33] Wouter Minnebo, Tim Hellemans, and Benny Van Houdt. On a class of push and pull strategies with single migrations and limited probe rate. *Perform. Eval.*, 113:42–67, 2017.

[34] Michael Mitzenmacher. The power of two choices in randomized load balancing. *IEEE Trans. Parallel Distrib. Syst.*, 12(10):1094–1104, October 2001.

[35] Pascal Moyal. A pathwise comparison of parallel queues. *Discrete Event Dynamic Systems*, 27(3):573–584, 2017.

[36] Jayakrishnan Nair, Adam Wierman, and Bert Zwart. Tail-robust scheduling via limited processor sharing. *Performance Evaluation*, 67(11):978–995, 2010.

[37] José Nino-Mora. Dynamic allocation indices for restless projects and queueing admission control: a polyhedral approach. *Mathematical programming*, 93(3):361–413, 2002.

[38] José Niño-Mora. Dynamic priority allocation via restless bandit marginal productivity indices. *Top*, 15(2):161–198, 2007.

[39] José Niño-Mora. Towards minimum loss job routing to parallel heterogeneous multiserver queues via index policies. *European Journal of Operational Research*, 220(3):705–715, 2012.

[40] José Niño-Mora and Sofía S Villar. Sensor scheduling for hunting elusive hiding targets via Whittle’s restless bandit index policy. In *Network Games, Control and Optimization (NetGCooP), 2011 5th International Conference on*, pages 1–8. IEEE, 2011.

[41] Misja Nuyens and Wenke van der Weij. Monotonicity in the limited processor-sharing queue. *Stochastic Models*, 25(3):408–419, 2009.

[42] Christos H Papadimitriou and John N Tsitsiklis. The complexity of optimal queuing network control. *Mathematics of Operations Research*, 24(2):293–305, 1999.

[43] Martin L Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.

[44] D Ruiz-Hernandez. Indexable restless bandits, 2008.

[45] Rahul Singh, Xueying Guo, and Panganamala Ramana Kumar. Index policies for optimal mean-variance trade-off of inter-delivery times in real-time sensor networks. In *Computer Communications (INFOCOM), 2015 IEEE Conference on*, pages 505–512. IEEE, 2015.

[46] Mark van der Boor, Sem C. Borst, Johan S. H. van Leeuwaarden, and Debankur Mukherjee. Scalable load balancing in networked systems: A survey of recent advances. *CoRR*, abs/1806.05444, 2018.

[47] Ina Maria Verloop. Asymptotically optimal priority policies for indexable and nonindexable restless bandits. *The Annals of Applied Probability*, 26(4):1947–1995, 2016.
A Proof of propositions and lemma

We present various proofs in this section.

A.1 Proof of Lemma 4.3

Some ideas in this proof are borrowed from [6]. We drop the dependency on $k$ for ease of notations. We prove this result by using the idea of stochastic dominance in Markov Chains. Consider two Markov chains, $\{X_t^1\}$, $\{X_t^2\}$ corresponding to threshold $n$ and $n+1$. Label the positive recurrent states in the reverse order, i.e., if the threshold is $x$, state with $(x+1)$ jobs is labeled as state 0, that with $x$ jobs is labeled as state 1, and this is repeated till the state with 0 jobs is labeled $(x+1)$. We will show that $\{X_t^2\}$ stochastically dominates $\{X_t^1\}$ for LPS-d scheduling discipline.

Let the transition matrices of the Markov chains be $P_1$ and $P_2$ for $\{X_t^1\}$ and $\{X_t^2\}$ respectively. Their elements $p_k(i,j)$, $k=1,2$, give the corresponding probability of going from state $i$ (i.e., $(x+1-i)$ jobs in the server) to state $j$ (i.e., $(x+1-j)$ jobs in the server). Extend $P_1$ to a $(n+3) \times (n+3)$ matrix by adding a column and row of zeros:

$$
\begin{pmatrix}
  p_{n+1,n+1} & p_{n+1,n} & p_{n+1,n-1} & \cdots & p_{n+1,1} & p_{n+1,0} & 0 \\
p_{n,n+1} & p_{n,n} & p_{n,n-1} & \cdots & p_{n,1} & p_{n,0} & 0 \\
0 & p_{n-1,n} & p_{n-1,n-1} & \cdots & p_{n-1,1} & p_{n-1,0} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & p_{1,1} & p_{1,0} & 0 \\
0 & 0 & 0 & \cdots & 0 & 1-p & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
$$

and the matrix $P_2$ is:

$$
\begin{pmatrix}
p_{n+2,n+2} & p_{n+2,n+1} & p_{n+2,n} & \cdots & p_{n+2,1} & p_{n+2,0} & 0 \\
p_{n+1,n+2} & p_{n+1,n+1} & p_{n+1,n} & \cdots & p_{n+1,1} & p_{n+1,0} & 0 \\
0 & p_{n,n+1} & p_{n,n} & \cdots & p_{n,2} & p_{n,1} & p_{n,0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & p_{2,2} & p_{2,1} & p_{2,0} \\
0 & 0 & 0 & \cdots & p_{1,2} & p_{1,1} & p_{1,0} \\
0 & 0 & 0 & \cdots & 0 & 1-p & 0
\end{pmatrix}
$$
where each element in above matrices is according to the Bernoulli arrival Process and a departure process driven by LPS-$d$ scheduling scheme (See section 5.1 for the precise definition of $P_{i,j}$ in above matrices). Now consider the following lower triangular matrix of dimension $(n + 2) \times (n + 2)$:

$$U = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}$$

It follows from above matrices that

$$P_1 U \leq P_2 U.$$ 

This shows that $\{X^2_t\}$ stochastically dominates $\{X^1_t\}$ (see [28, Definition 3.12]). Further, using Theorem 3.31 in [28, Page 158], we have

$$\bar{\pi}^{n+1}(0) \leq \bar{\pi}^n(0)$$

where $\bar{\pi}^n$ (resp., $\bar{\pi}^{n+1}$) is the stationary distribution for the threshold $n$ (resp., $n + 1$) with relabeled states. In view of our relabeling of the positive recurrent states, this translates into the following in the original notation:

$$\pi^{n+1}(n + 2) \leq \pi^n(n + 1)$$

This shows that the stationary probability of the only passive state with positive stationary probability decreases as the threshold increases. However,

$$\sum_{j=0}^{n+1} \pi^n(j) = 1$$

and $\pi^n(n + 1)$ decreases with the threshold $n$. This shows that $n \sum_{m=0}^{\pi^n(m)}$ is non-decreasing in $n$.

### A.2 Proof of Proposition 4

We drop the subscript $k$ for ease of notation. The Whittle’s index is given by Equation (9) if it is non-increasing in $n$:

$$pD = \frac{\sum_{m=0}^{n} C(m)[\pi^n(m) - \pi^{n-1}(m)] + C(n+1)\pi^n(n+1)}{\sum_{m=0}^{n} \pi^n(m) - \sum_{m=0}^{n-1} \pi^{n-1}(m)}$$

Above index can be rewritten as:

$$pD = \frac{\sum_{m=0}^{n-1} C(m)[\pi^n(m) - \pi^{n-1}(m)] + C(n)[\pi^n(n) - \pi^{n-1}(n)] + C(n+1)\pi^n(n+1)}{\sum_{m=0}^{n} \pi^n(m) - \sum_{m=0}^{n-1} \pi^{n-1}(m)}$$

The sum of stationary probability simplifies to the following by using Equations (10) - (13):

$$\sum_{m=0}^{n} \pi^n(m) = \frac{q^{n+2}(1-p)^n - p^{n+1}(1-q)^n}{q^{n+2}(1-p)^n - p^{n+2}(1-q)^n},$$

and the term, $\sum_{m=0}^{n} \pi^n(m) - \sum_{m=0}^{n-1} \pi^{n-1}(m)$, simplifies to

$$\frac{(1-p)^{n-1}(1-q)^{n-1}p^nq^{n+1}(p-q)^2}{(q^{n+2}(1-p)^n - p^{n+2}(1-q)^n)(q^{n+1}(1-p)^n - p^{n+1}(1-q)^n)},$$

by using the stationary distribution for FCFS discipline (derived in Section 5.2). We have

$$\frac{\pi^n(m) - \pi^{n-1}(m)}{\sum_{m=0}^{n} \pi^n(m) - \sum_{m=0}^{n-1} \pi^{n-1}(m)} = \frac{p^2 (\frac{k}{2})^{m-1}}{(1-p)q^2} \forall m = 1, 2, \cdots, n - 1,$$
Further simplification results in
\[ \sum_{m=0}^{n} \pi(n) - \sum_{m=0}^{n-1} \pi(n-1)(m) = \frac{q}{p} \left[ p + \left( \frac{p}{q} \right)^{n+1} \left( \frac{1-q}{1-p} \right)^{n-1} \left( \frac{p}{q} - 1 \right) \right], \]
and
\[ \sum_{m=0}^{n} \pi(n+1) - \sum_{m=0}^{n-1} \pi(n-1)(m) = \frac{p(1-q)}{q-p} \left[ 1 - \left( \frac{p}{q} \right)^{n+1} \left( \frac{1-q}{1-p} \right)^{n-1} \right] \]

Using above expressions, the Whittle’s index simplifies to the expression as stated in the proposition. Note that we are left to argue the non-increasing nature of the index. We argue this in the following steps: After sum algebra we can write
\[ W_k(n+1) - W_k(n) = \frac{C_k(n)pq_k}{q_k-p} \left[ 1 - \left( \frac{p}{q} \right)^2 \left( \frac{p(1-q_k)}{q_k(1-p)} \right)^n \right] + \frac{C_k(n+1)p(1-2q_k)}{q_k-p} \left[ 1 - \left( \frac{p}{q} \right)^2 \left( \frac{p(1-q_k)}{q_k(1-p)} \right)^n \right] \]

Further simplification results in
\[ W_k(n+1) - W_k(n) = \left( \frac{pq_k(C_k(n) - C_k(n+1) + p(1-q_k)C_k(n+1) - C_k(n+2))}{q_k-p} \right) \left[ 1 - \left( \frac{p}{q} \right)^2 \left( \frac{p(1-q_k)}{q_k(1-p)} \right)^n \right]. \]

Note that the term
\[ \frac{1}{q_k-p} \left[ 1 - \left( \frac{p}{q} \right)^2 \left( \frac{p(1-q_k)}{q_k(1-p)} \right)^n \right] > 0. \]
Thus, the sign of the term \( W(n+1) - W(n) \) is decided by \( pq_k(C_k(n) - C_k(n+1) + p(1-q_k)C_k(n+1) - C_k(n+2)) \). Note that this term is negative from the assumption that the cost function is non-decreasing. Thus, \( W(n+1) - W(n) \leq 0 \), \( \forall n \). This completes the proof for the non-increasing nature of the index.

We now give a proof for the second part of the proposition. It immediately follows with some simplification from the expression of the index that
\[ W_k(0) = pD + \frac{C_k(1)p}{q_k+p-1}, \]

We consider the following two cases to determine \( \lim_{n \to \infty} W_k(n) \).

**Case \( q_k < p \):** It follows from Equation (19) and the fact that the cost function is non-decreasing, that \( \lim_{n \to \infty} W_k(n+1) - W_k(n) \to -\infty \). Since \( W_k(n) \) is non-increasing, we then have \( \lim_{n \to \infty} W_k(n) \to -\infty \).

**Case \( q_k > p \):** From (20), we have that
\[ \lim_{n \to \infty} W_k(n) = pD + \frac{p^2(1-p)}{q_k(1-p)^2} \sum_{m=0}^{\infty} C_k(m) \left( \frac{b_k}{d_k} \right)^{m-1} - \lim_{n \to \infty} \frac{C_k(n)(q_k)p}{(q_k-p)} - \lim_{n \to \infty} \frac{C_k(n+1)(1-q_k)p}{(q_k-p)}. \]

Since \( b < d \), \( C_k(n) \) is bounded by a polynomial and \( \lim_{n \to \infty} C_k(n) \to \infty \), the above limit converges to \(-\infty \). This completes the proof.

### A.3 Proof of Equation (15)
Consider the linear cost, i.e., \( C_k(m) = mC \forall m \). Using Equation (20), the index term simplifies as stated below:
\[ pD + \frac{Cp^2(1-p)}{(q_k-p)^2} \left[ 1 - n \left( \frac{b}{d} \right)^n \right] + (n-1) \left( \frac{b}{d} \right)^n - \lim_{n \to \infty} \frac{nCq_k}{(q_k-p)} \left[ p + \left( \frac{p}{q_k} \right)^{n+1} \left( \frac{1-q_k}{1-p} \right)^{n-1} \left( \frac{p}{q_k} - 1 \right) \right] - \frac{(n+1)Cp(1-q_k)}{(q_k-p)} \left[ 1 - \left( \frac{p}{q_k} \right)^{n+1} \left( \frac{1-q_k}{1-p} \right)^{n-1} \right]. \]
Above, further simplifies to:

\[ pD + \frac{Cp^2(1-p)}{(q_k-p)^2} \left[ 1 - n \left( \frac{b_k}{d_k} \right)^{n-1} + (n-1) \left( \frac{b_k}{d_k} \right)^n \right] \]

\[ - \frac{Cp(1-q_k)}{q_k-p} \cdot \frac{nCp}{q_k-p} + \frac{Cp(1-q_k)}{q_k-p} \left( \frac{p}{q_k} \right)^2 \left( \frac{b_k}{d_k} \right)^{n-1} + \frac{nCq_k}{q_k-p} \left( \frac{p}{q_k} \right)^2 \left( \frac{b_k}{d_k} \right)^{n-1} \].

On rearranging the terms, above can be written as:

\[ pD + \frac{Cp^2(1-p)}{(q_k-p)^2} - \frac{Cp(1-q_k)}{q_k-p} \cdot \frac{nCp}{q_k-p} + \frac{Cp^3(1-p)}{q_k(q_k-p)^2} \left( \frac{b_k}{d_k} \right)^n \].

It can be easily verified that above term is non-increasing in \( n \) in either of the cases \( q < p \) or \( q > p \). Thus, the Whittle's index

\[ W_k(n) = pD + \frac{Cp^2(1-p)}{(q_k-p)^2} - \frac{Cp(1-q_k)}{q_k-p} \cdot \frac{nCp}{q_k-p} + \frac{Cp^3(1-p)}{q_k(q_k-p)^2} \left( \frac{b}{d} \right)^n \].

Further, it immediately follows from above expression at \( n = 0 \) that

\[ W_k(0) = pD + \frac{Cp^2(1-p)}{(q_k-p)^2} - \frac{Cp(1-q_k)}{q_k-p} + \frac{Cp^3(1-p)}{q_k(q_k-p)^2} \]

which simplifies to \( W_k(0) \) stated in the proposition.

**A.4** \( q^{d_k}_k(i|n) \) is non-decreasing function of \( n \).

By definition,

\[ q^{d_k}_k(i|n) = \binom{\min\{n,d_k\}}{i} \left( \frac{q_k}{\min\{n,d_k\}} \right)^i \left( 1 - \frac{q_k}{\min\{n,d_k\}} \right)^{\min\{n,d_k\}-i} \],

note that it is straight forward to see that above function is constant in \( n \) for \( n > d_k \). We prove below non-decreasing nature of \( q^{d_k}_k(i|n) \) for \( n \leq d_k \). For \( n \leq d_k \),

\[ q^{d_k}_k(i|n) = \binom{n}{i} \left( \frac{q_k}{n} \right)^i \left( 1 - \frac{q_k}{n} \right)^{n-i} \].

Above further simplifies to

\[ q^{d_k}_k(i|n) = \binom{n}{i} \frac{n(n-1)(n-2)\ldots(n-i+1)}{n^i} \left( 1 - \frac{q_k}{n} \right)^{n-i} \].

It can be easily argued that the terms \( \frac{n(n-1)(n-2)\ldots(n-i+1)}{n^i} \) and \( \left( 1 - \frac{q_k}{n} \right)^{n-i} \) are positive and increasing in \( n \). Hence, \( q^{d_k}_k(i|n) \) is increasing in \( n \).