A GILLET–WALDHAUSEN THEOREM FOR CHAIN COMPLEXES OF SETS

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Abstract. The (A)CGW categories of Campbell and Zakharevich show how finite sets and varieties behave like the objects of an exact category for the purpose of algebraic $K$-theory. We further develop that program by defining chain complexes and quasi-isomorphisms for any category with suitably nice coproducts. In particular, chain complexes of finite sets satisfy an analogue of the Gillet–Waldhausen Theorem: their $K$-theory agrees with the classical $K$-theory of finite sets. Along the way, we define new double categorical structures that modify those of Campbell and Zakharevich to include the data of weak equivalences. These ECGW categories produce $K$-theory spectra which satisfy analogues of the Additivity and Fibration Theorems. The weak equivalences are determined by a subcategory of acyclic objects satisfying minimal conditions, resulting in a Localization Theorem that generalizes previous versions in the literature.

Contents

Introduction 2

Part 1. ECGW categories
1. Double categorical preliminaries 5
2. g-CGW categories 8
3. $\star$-CGW categories 14
4. Adding weak equivalences 19

Part 2. $K$-theory of ECGW categories
5. $S_{\star}$-construction 22
6. Additivity Theorem 26
7. Delooping 30
8. Fibration Theorem 32
9. Localization Theorem 35

Part 3. Chain complexes of finite sets
10. Chain complexes 36
11. Exact complexes 45
12. Gillet–Waldhausen Theorem 47

Appendix. Functoriality constructions
Appendix A. Properties of $\star$-pushouts 50
Appendix B. ECGW categories of functors 59

References 64
INTRODUCTION

In recent work [CZ], Campbell and Zakharevich introduced CGW categories, which are double categories satisfying a list of additional axioms that seek to extract the properties of exact categories\(^1\) which make them so particularly well-suited for algebraic $K$-theory. Their key insight lies in the fact that the only morphisms that a $K$-theory machinery for exact categories truly sees are the admissible monomorphisms and epimorphisms, and moreover, that these are not required to interact with each other outside of the short exact sequences — or more generally, the bicartesian squares. This suggests that these monomorphisms and epimorphisms could form the horizontal and vertical morphisms in a double category, with squares the bicartesian squares, and that one should be able to axiomatize in the language of double categories any remaining crucial properties in order to obtain a $K$-theory machinery analogous to the $Q$-construction.

The main appeal of these double categories is that they generalize the structure of exact sequences to key non-additive settings such as finite sets and varieties, where the notion of complements replaces that of kernels and cokernels. This makes it possible to study finite sets and varieties as if they were the objects of an exact category, for the purposes of $K$-theory. Aside from setting the stage for further study of derived motivic measures and of the $K$-theory of varieties, this new framework has already been used by Haesemeyer and Weibel [HW21] and by Coley and Weibel [CW21] to develop the $K$-theory of partially cancellative $A$-sets.

In order to recover analogues of Quillen’s classical results [Qui73] such as the Localization and Devissage theorems, one needs to pass from CGW categories to a structure with additional information: ACGW categories. This is reminiscent of the restriction from exact to abelian categories — indeed, the “A” stands for “Abelian” — and just like in the classical case, exact categories are no longer an example. Varieties also fail to form an ACGW category, though unlike exact categories this is not a notable loss, as the equivalent $K$-theory of reduced schemes is modeled by an ACGW category.

As well as these classical foundational theorems, (A)CGW categories admit an $S\cdot$-construction in the flavor of Waldhausen’s [Wal85]. However, the $S\cdot$-construction of an ACGW category cannot be iterated more than once, as the double category $S\cdot A$ is CGW but not ACGW. This means that $S\cdot$ of an ACGW category yields a space but not necessarily a spectrum, since the classical delooping machine constructed by iterating $S\cdot$ is not available. In addition, the Additivity Theorem is only proven for ACGW categories arising from subtractive categories [Cam19].

In this article, we expand on the work of [CZ] to allow for the addition of homotopical information; in other words, we set out to define a notion of “ACGW categories with weak equivalences.” Much like Quillen’s $Q$-construction, the $K$-theory of ACGW categories is not equipped to handle settings where certain non-invertible morphisms should be treated as equivalences between objects. Following Waldhausen’s approach, this requires us to instead construct the $K$-theory space using an $S\cdot$-construction, so we further modify the ACGW category axioms in order for $S\cdot$ to have the expected behavior.

In doing so, we find that the obstructions for ACGW categories to encompass all of the motivating examples, to have the functoriality properties required for a general proof of the Additivity theorem, and to allow for the $S\cdot$-construction to be iterated are all essentially the same. Campbell and Zakharevich use pullback diagrams as the morphisms between arrows and demand that kernels, cokernels, and restricted pushouts apply to all pullbacks. We argue that this role should be played by a class of “good squares” which are potentially more restrictive

\(^1\)In this work, “exact categories” refers to Quillen-exact categories, the additive structure used in $K$-theory and homological algebra, as opposed to the Barr-exact categories used in regular logic.
than pullbacks. The intuition behind these stems from Waldhausen’s original construction, which tells us that in order for a square

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

to admit a cofiber (when considered as a morphism in the category whose objects are cofibrations), it must satisfy an additional property: the induced map \( B \cup_A C \rightarrow D \) must be a cofibration as well. Our “good squares” are defined in nearly perfect analogy with Waldhausen’s, and we discuss the comparison in Remark 3.2. We call our modified version of ACGW categories based on “good squares” a \( \star \)-CGW category.

In order to introduce the (horizontal and vertical) weak equivalences, we borrow intuition from an archetypal algebraic category with weak equivalences: chain complexes with quasi-isomorphisms. A quick study finds that among monomorphisms (resp. epimorphisms), the quasi-isomorphisms are those whose cokernel (resp. kernel) is an exact chain complex. Mirroring this scenario, the weak equivalences in a \( \star \)-CGW category are determined by a choice of “acyclic objects” satisfying certain closure properties, much like the weak equivalences considered in [Sar20]. Our main structure, which we call an ECGW category, then consists of a pair \((\mathcal{C}, \mathcal{W})\), where \(\mathcal{C}\) is a \( \star \)-CGW category and \(\mathcal{W}\) is the full double subcategory of acyclic objects.

Restricting the weak equivalences to isomorphisms, our more robust axioms allow us to recover the examples of exact categories and varieties. These are not ACGW categories, but they do form ECGW categories, so they can now be studied using the full force of our foundational theorems of \( K \)-theory. We also show that the double category \( \mathcal{C}^\mathcal{D} \) of \( \mathcal{D} \)-shaped diagrams in an ECGW category \( \mathcal{C} \), for any double category \( \mathcal{D} \), is ECGW. We interpret this as an “exponentiality” property, which inspires the “E” in ECGW. Notably, this allows us to show that each \( S_n \mathcal{C} \) is ECGW which implies that this \( S_\bullet \)-construction can be iterated, and thus \( K(\mathcal{C}) \) is an infinite loop space, as shown in Theorem 7.5.

Our main motivating example, and the driving force behind this generalization, is that of chain complexes. Aside from being the building blocks of homological algebra, chain complexes over an exact category also play a crucial role in algebraic \( K \)-theory. When endowed with quasi-isomorphisms as the class of weak equivalences, they form a Waldhausen category, and the Gillet–Waldhausen Theorem tells us that the \( K \)-theory spectrum of an exact category \( \mathcal{C} \) — with isomorphisms — is equivalent to the \( K \)-theory spectrum of bounded chain complexes on \( \mathcal{C} \) — with quasi-isomorphisms. Chain complexes provide an often more convenient model for the \( K \)-theory of exact categories.

Our aim is to construct a similar chain complex model for the \( K \)-theory of non-additive categories such as sets and varieties. In this spirit, we define an ECGW category of chain complexes of finite sets, while in future work with Inna Zakharevich we plan to do the same for varieties. This construction of non-additive chain complexes in fact applies more generally to any category which is extensive, meaning it has coproducts which interact nicely with pullbacks. Extensive categories include the categories of sets, \( M \)-sets for a monoid \( M \), and finite variants thereof.

Over sets, the differentials in our chain complexes are given by partial functions, which correspond to basepoint-preserving functions between pointed sets. Under this correspondence, our chain complexes agree with the “naive” notion of a chain complex of finite pointed sets: a sequence of basepoint-preserving functions such that any two that are adjacent compose to the constant function. The familiarity of these objects is an appealing part of our theory, though the morphisms and weak equivalences between them which determine their \( K \)-theory are more
subtle, obtained through a different analogy with classical chain complexes more natural to the ECGW formalism.

These chain complexes satisfy an analogue of the Gillet–Waldhausen Theorem [TT90, Theorem 1.11.7], thus forming a new model for the $K$-theory of finite sets, and more generally for the symmetric monoidal $K$-theory of any extensive category. It also provides further evidence that most classical results of algebraic $K$-theory can be adapted to the ECGW setting, which we see as the theme of this work. When applied to the particular case of finite sets, Theorem 12.3 states the following.

**Theorem (Gillet–Waldhausen).** There exists a homotopy equivalence

$$K(\text{FinSet}) \simeq K(\text{Ch}^b(\text{FinSet}), \text{Ch}^b_b(\text{FinSet}))$$

between the $K$-theory of finite sets (with isomorphisms) and the $K$-theory of the ECGW category of bounded chain complexes of finite sets (with weak equivalences determined by bounded exact complexes).

Just as [CZ] captures the essential features required to carry out Quillen’s major foundational theorems, our ECGW categories allow us to obtain many of Waldhausen’s structural results. Chief among them are the Additivity Theorem (Theorem 6.11) — which as the modern perspective shows us [BGT13, Bar16], characterizes algebraic $K$-theory — and the Fibration Theorem (Theorem 8.1), which compares the $K$-theory of a category equipped with two classes of weak equivalences by constructing a homotopy fiber.

**Theorem (Fibration).** Let $\mathcal{V}$ and $\mathcal{W}$ be two acyclicity structures on a $\star$-CGW category $\mathcal{C}$, such that $\mathcal{V} \subseteq \mathcal{W}$. Then, there exists a homotopy fiber sequence

$$K(\mathcal{W}, \mathcal{V}) \longrightarrow K(\mathcal{C}, \mathcal{V}) \longrightarrow K(\mathcal{C}, \mathcal{W})$$

Unlike the Fibration Theorem for Waldhausen categories, our theorem requires neither the existence of a cylinder functor nor any special factorizations, instead relying on the symmetry between the two types of morphisms in an ECGW category, analogous to the dual properties of mono- and epimorphisms in an exact category.

As a consequence of this result in the case where $\mathcal{V}$ is trivial, we obtain a Localization Theorem (Theorem 9.1) that generalizes many of those existing in the literature; this includes Quillen’s original theorem for abelian categories [Qui73], Schlichting’s [Sch04] and Cardenas’ [Car98] Localization Theorems for exact categories, the first author’s Localization Theorem obtained from cotorsion pairs [Sar20], and the Localization Theorem for ACGW categories of [CZ]. In the setting of ECGW categories arising from exact categories, it reads as follows:

**Theorem (Localization).** Let $\mathcal{B}$ be an exact category and $\mathcal{A} \subseteq \mathcal{B}$ a full subcategory such that if any two terms in an exact sequence in $\mathcal{B}$ are in $\mathcal{A}$, then the third term is as well. Then there exists an ECGW category $(\mathcal{B}, \mathcal{A})$ such that

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}, \mathcal{A})$$

is a homotopy fiber sequence.

This version of the Localization Theorem has fewer requirements than any of those mentioned above, which with the exception of [Sar20] all require the subcategory $\mathcal{A}$ to be closed under subobjects and quotients.\(^2\) However, the localization result in [BGT13] for stable $(\infty, 1)$-categories requires only a full subcategory. There is then a trade-off to be made between the requirements on the subcategory $\mathcal{A}$ and the strictness of the cofiber model that is obtained; we believe our Localization Theorem is the optimal one before passage to a higher categorical setting.

\(^2\)This excludes, for example, the subcategory of projective modules for most rings.
Outline. The first part of this work introduces the main protagonists. After a brief tour through the world of double categories in Section 1, we define g-CGW categories in Section 2 as double categories with some additional structure and properties. In Section 3, we introduce ⋆-CGW categories, which satisfy stronger axioms that allow us to prove our foundational $K$-theory results in Part 2. Finally, Section 4 contains the definition of our principal structures of interest: ECGW categories. These are ⋆-CGW categories that allow for a notion of weak equivalences, defined from a class of acyclic objects.

The second part contains our main results regarding the $K$-theory of ECGW categories. First, Section 5 introduces an $S_\bullet$-construction for ECGW categories. We support this definition by showing that $K_0$ admits the expected explicit description as a Grothendieck group, and that this $K$-theory agrees with that of the known examples of exact categories with weak equivalences which form ECGW categories, and with the $K$-theory of their underlying CGW categories as defined in [CZ] when the weak equivalences are simply isomorphisms.

The next sections are dedicated to several foundational results. Section 6 shows that our $K$-theory machinery satisfies the Additivity Theorem, and, in Section 7, we show how our $S_\bullet$ construction produces a spectrum. Section 8 proves our version of Waldhausen’s Fibration Theorem, which relates the $K$-theory spectra of a ⋆-CGW category equipped with two comparable classes of weak equivalences by constructing a homotopy fiber. In a similar vein, we obtain a Localization Theorem in Section 9 that allows us to relate the $K$-theories of an inclusion of ⋆-CGW categories by constructing a homotopy cofiber; we then compare this to previous Localization Theorems in the literature.

In the third part, we construct our main novel example of ECGW categories: chain complexes of sets, with a notion of quasi-isomorphisms. Section 10 is devoted to proving that chain complexes of finite sets form a ⋆-CGW category. Section 11 further gives an ECGW structure by considering exact chain complexes as acyclics. In turn, Section 12 contains a Gillet–Waldhausen Theorem that establishes these chain complexes as an alternate model for the $K$-theory of sets.

Finally, the appendix deals with a collection of technical results building up to the proofs that each level of the $S_\bullet$ construction and the grids used to prove the Fibration Theorem form ECGW categories.

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Part 1. ECGW categories

1. Double categorical preliminaries

Double categories, originally defined as categories internal to categories, describe categorical settings with two different types of morphisms, related by higher cells called squares. In this section, we recall the well-known notions of double categories, double functors, and the natural transformations between them, as well as the space associated to a double category. We also introduce a notion of double categories with shared isomorphisms and discuss a natural notion of equivalence between them that will be useful in later sections.

Definition 1.1. A double category $C$ consists of:

- a set of objects $\text{Ob}(C)$
two categories $\mathcal{M}$ and $\mathcal{E}$ with the same objects as $\mathcal{C}$. We call their maps \textbf{m-morphisms} ($\rightarrowtail$) and \textbf{e-morphisms} ($\rightarrow$), respectively.

- a set of squares of the form

\[
\begin{array}{c}
A \xrightarrow{f} B \\
g \downarrow \bigcirc \downarrow g' \\
C \xrightarrow{f'} D
\end{array}
\]

- categories $\text{Ar}_\circ \mathcal{M}$, $\text{Ar}_\circ \mathcal{E}$ with objects the m-morphisms (resp. e-morphisms) and maps from $f$ to $f'$ (resp. $g$ to $g'$) given by the squares above, such that

- composite and identity squares respect those of the e-morphisms (resp. m-morphisms) along their sides, and satisfy the interchange law: in a grid

\[
\begin{array}{c}
\bullet \longrightarrow \bullet \longrightarrow \bullet \\
\bigcirc \downarrow \bigcirc \downarrow \bigcirc \\
\bullet \longrightarrow \bullet \longrightarrow \bullet \\
\bigcirc \downarrow \bigcirc \downarrow \bigcirc \\
\bullet \longrightarrow \bullet \longrightarrow \bullet 
\end{array}
\]

applying the composition operations in either order yields the same result.

\textit{Remark 1.2.} In the definition above, we use the symbol $\bigcirc$ to denote that there exists a square having the depicted boundary; this should not be interpreted as the square being a commutative diagram, especially since m- and e-morphisms need not compose among each other.

\textbf{Definition 1.3.} Let $\mathcal{C}$ and $\mathcal{D}$ be double categories. A \textbf{double functor} $F: \mathcal{C} \longrightarrow \mathcal{D}$ consists of an assignment on objects, m-morphisms, e-morphisms, and squares, which are compatible with domains and codomains and preserve all double categorical compositions and identities.

\textbf{Definition 1.4.} A double functor is \textbf{full} (resp. \textbf{faithful}) if it is surjective (resp. injective) on each set of m-morphisms and e-morphisms with fixed source and target, and on each set of squares with fixed boundary.

We say a double subcategory $\mathcal{C} \subseteq \mathcal{D}$ is full if the inclusion is a full double functor.

The category of double categories is cartesian closed, and thus there exists a double category whose objects are the double functors. We briefly describe the horizontal morphisms, vertical morphisms, and squares of this double category; the reader unfamiliar with double categories is encouraged to see [Gra20, §3.2.7] for more explicit definitions.

\textbf{Definition 1.5.} Let $F, G: \mathcal{C} \longrightarrow \mathcal{D}$ be double functors. A horizontal natural transformation $\mu: F \Rightarrow G$, which we henceforth call \textbf{m-natural transformation}, consists of

- an m-morphism $\mu_A: FA \rightarrowtail GA$ in $\mathcal{D}$ for each object $A \in \mathcal{C}$, and

- a square

\[
\begin{array}{c}
FA \xrightarrow{\mu_A} GA \\
Ff \downarrow \bigcirc \downarrow gf \\
FB \xrightarrow{\mu_B} GB
\end{array}
\]

in $\mathcal{D}$ for each e-morphism $f: A \rightarrow B$ in $\mathcal{C}$,
such that the assignment of squares is functorial with respect to the composition of e-morphisms, and that these data satisfy a naturality condition with respect to m-morphisms and squares.

Dually, one defines a vertical natural transformation, which we call e-natural transformation.

**Definition 1.6.** Given m-natural transformations \( \mu: F \Rightarrow G \), \( \mu': F' \Rightarrow G' \) and e-natural transformations \( \eta: F \Rightarrow F' \), \( \eta': G \Rightarrow G' \) between double functors \( C \longrightarrow D \), a modification \( \alpha \) shown below left

\[
\begin{array}{ccc}
F & \longrightarrow & G \\
\downarrow\alpha & & \downarrow\eta' \\
F' & \longrightarrow & G'
\end{array}
\quad
\begin{array}{ccc}
FA & \overset{\mu A}{\longrightarrow} & GA \\
\downarrow\eta_A \circ \alpha_A & & \downarrow\eta'_A \\
F'A & \overset{\mu'_A}{\longrightarrow} & G'A
\end{array}
\]

consists of a square in \( D \) as above right for each object \( A \in C \), satisfying horizontal and vertical coherence conditions with respect to the squares of the transformations \( \mu, \mu', \eta, \) and \( \eta' \).

The double categories of interest to this paper arise from taking m- and e-morphisms to be certain classes of morphisms in some category, and squares from certain commuting squares in the ambient category. For these, it will be convenient for the two classes of maps in the double category to have a common class of isomorphisms. To that purpose, we introduce the following notion.

**Definition 1.7.** A double category \( C \) has shared isomorphisms if:

- there is a groupoid \( I \) with identity-on-objects functors \( \mathcal{M} \leftarrow I \rightarrow \mathcal{E} \) which create isomorphisms. For a morphism \( f \) in \( I \), we write \( f \) for both the corresponding m-isomorphism and e-isomorphism, which we distinguish in diagrams by the different arrow shapes
- for isomorphisms \( f, f' \) and m-morphisms \( g, g' \) there is a (unique) square as below left if and only if the square below right commutes in \( \mathcal{M} \)

\[
\begin{array}{ccc}
\bullet & \overset{g}{\longrightarrow} & \bullet \\
\downarrow f & & \downarrow f' \\
\bullet & & \bullet
\end{array}
\quad
\begin{array}{ccc}
\bullet & \overset{g}{\longrightarrow} & \bullet \\
\downarrow f & & \downarrow f' \\
\bullet & \overset{g'}{\longrightarrow} & \bullet
\end{array}
\]

- the analogous correspondence holds between squares in \( C \) and commuting squares in \( \mathcal{E} \) for isomorphisms \( f, f' \) and e-morphisms \( g, g' \)

In our double categories of interest, squares between fixed m- and e-morphisms will be unique when they exist, so the uniqueness of the squares above will be inconsequential.

The unification of m- and e-isomorphisms extends to natural isomorphisms between double functors as well, which allows us to define a canonical notion of equivalence of double categories with shared isomorphisms.

**Definition 1.8.** Let \( F,G:C \longrightarrow D \) be double functors, where \( D \) has shared isomorphisms. A natural isomorphism \( \alpha: F \cong G \) consists of an isomorphism \( \alpha_A: FA \cong GA \) for each object \( A \) in \( C \), such that when we regard all \( \alpha_A \) as m-morphisms (resp. e-morphisms), \( \alpha \) is an m- (resp. e-) natural transformation.

**Remark 1.9.** Note that any natural isomorphism will be such that the component squares of the m- and e-natural transformation \( \alpha \) are invertible (horizontally or vertically, as it corresponds), by the uniqueness of the squares in Definition 1.7. Definition 1.7 also shows that the naturality condition can be reduced to checking that the components of \( \alpha \) form a natural transformation.
in the 1-categorical sense between the underlying functors on $m$-morphisms and $e$-morphisms, so it is not necessary here to provide naturality squares in the data of $\alpha$.

We can use these natural isomorphisms to define a notion of equivalence between double categories with shared isomorphisms. A careful study of these equivalences is beyond the scope of this paper; our goal is simply to show that they induce homotopy equivalences of spaces after realization.

**Definition 1.10.** Let $\mathcal{C}, \mathcal{D}$ be double categories with shared isomorphisms. An **equivalence** between $\mathcal{C}$ and $\mathcal{D}$ is a pair of double functors $F: \mathcal{C} \leftrightarrow \mathcal{D}: G$ equipped with natural isomorphisms $1_\mathcal{C} \cong GF$ and $FG \cong 1_\mathcal{D}$.

A definition of this form is not possible for general double categories without making arbitrary choices for whether the natural isomorphisms are $m$- or $e$-transformations. This is appropriate for the double categories we consider which arise from categories, and has the following convenient characterization.

**Proposition 1.11.** Let $F: \mathcal{C} \to \mathcal{D}$ be a double functor between double categories with shared isomorphisms. Then, $F$ belongs to an equivalence if and only if it is fully faithful and essentially surjective.

Here essentially surjective means that every object in $\mathcal{D}$ is isomorphic to $FC$ for some object $C$ in $\mathcal{C}$, just as for ordinary categories. The proof of this characterization, which we omit for brevity, uses the analogous result for 1-categories applied to both the $m$- and $e$-morphisms, which is straightforward to extend to squares as well.

Finally, we recall that the process of constructing a space from a category by taking the geometric realization of its nerve has an analogue in double categories, as defined for example in [FP10, Definition 2.14]. This is an especially important construction for us, as it will be used to define the $K$-theory space of our double categories of interest.

**Definition 1.12.** The double nerve, or **bisimplicial nerve**, of a double category $\mathcal{C}$ is the bisimplicial set $N\Box \mathcal{C}$ whose $(m,n)$-simplices are the $m \times n$-matrices of composable squares in $\mathcal{C}$.

We let $|\mathcal{C}|$ denote the geometric realization of the bisimplicial set $N\Box \mathcal{C}$, or, equivalently, the geometric realization of its diagonal simplicial set $n \mapsto N\Box \mathcal{C}_{n,n}$. Going forward, we abuse notation and use these two spaces interchangeably.

**Lemma 1.13.** Let $\mathcal{C}, \mathcal{D}$ be double categories with shared isomorphisms. If there exists an equivalence between $\mathcal{C}$ and $\mathcal{D}$, then $|\mathcal{C}|$ and $|\mathcal{D}|$ are homotopy equivalent.

**Proof.** This follows by noting that an $m$-natural transformation can be equivalently described as a double functor $\mathcal{C} \times \mathbb{H}(\Delta^1) \to \mathcal{D}$, where $\mathbb{H}(\Delta^1)$ is the double category with a single non-identity $m$-morphism and whose geometric realization is the interval. \(\square\)

## 2. g-CGW categories

In this section, we introduce g-CGW categories and establish the necessary categorical yoga. Pre-$\star$-CGW categories are almost identical to the pre-ACGW categories of [CZ], as their name suggests. The differences are that we begin with pseudo-commutative squares and define distinguished squares among them by a property, replace pullback squares of $m$- and $e$-morphisms with a more flexible notion of “good” squares, and don’t require axioms (S) or (A) involving pushouts and sums. Pushouts (and consequently sums) will be axiomatized in the following section on $\star$-CGW categories.
All names aside, the purpose of these double categories is to capture the essential features of exact categories that make them so suitable for $K$-theory, while allowing for non-additive examples. First of all, they have two classes of maps that mimic the role of admissible monomorphisms and admissible epimorphisms (reversing the direction of the latter): these will be the $m$- and $e$-morphisms in the double category. They also contain associated notions of (co)kernels and short exact sequences, but instead of defining these as certain (co)limits that would require an additive setting, their relevant features are axiomatized. This allows one to expand the classical intuition from exact categories to other settings such as sets and varieties, as done in [CZ].

**Notation 2.1.** Following the ACGW categories of [CZ], from now on the squares in a double category will be called “mixed” or “pseudo-commutative” squares. This last nomenclature was inspired by the fact that, when working with abelian categories, the role of the pseudo-commutative squares is played by the commutative squares between monomorphisms and epimorphisms in the category.

Throughout this paper, we work with several different categories with objects the $m$- or $e$-morphisms of $\mathcal{C}$, such as $\text{Ar}_\triangle \mathcal{M}$, $\text{Ar}_\triangle \mathcal{E}$ introduced in Definition 1.1. We also recall the following notation from [CZ, Definition 2.4].

**Definition 2.2.** Given a double category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$, let $\text{Ar}_\triangle \mathcal{M}$ denote the category whose objects are morphisms $A \rightarrow B \in \mathcal{M}$, and where

$$\text{Hom}_{\text{Ar}_\triangle \mathcal{M}}(A \rightarrow B, A' \rightarrow B') = \begin{cases} \text{commutative squares} & A \rightarrow f \rightarrow B \equiv A' \rightarrow f' \rightarrow B' \\ \text{equivalences} & A \rightarrow B \rightarrow A' \rightarrow B' \end{cases}.$$  

Similarly, we have a category $\text{Ar}_\triangle \mathcal{E}$ defined analogously.

We can imitate this definition for more general types of squares.

**Definition 2.3.** Given a category $\mathcal{A}$, a **class of good squares** is a subcategory $\text{Ar}_g \mathcal{A}$ of the category $\text{Ar} \mathcal{A}$ with objects arrows in $\mathcal{A}$ and morphisms commuting squares between them. Good squares in $\text{Ar}_g \mathcal{A}$ are denoted by

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\]

Examples of classes of good squares include the weak triangles of $\text{Ar}_\triangle \mathcal{A}$ and the pullback squares denoted $\text{Ar}_\times \mathcal{A}$.

We now define g-CGW categories. The reader unfamiliar with (A)CGW categories is strongly encouraged to read each axiom together with its counterpart in exact categories, explained below in Example 2.6.

**Definition 2.4.** A **g-CGW category** is a double category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ with shared isomorphisms, equipped with

- classes of good squares $\text{Ar}_g \mathcal{M}$ and $\text{Ar}_g \mathcal{E}$
- equivalences of categories $k: \text{Ar}_\triangle \mathcal{E} \rightarrow \text{Ar}_g \mathcal{M}$ and $c: \text{Ar}_\triangle \mathcal{M} \rightarrow \text{Ar}_g \mathcal{E}$

such that
(Z) $\mathcal{M}, \mathcal{E}$ each have initial objects which agree
(M) All morphisms in $\mathcal{M}, \mathcal{E}$ are monic
(G) $\text{Ar}_\Delta \mathcal{M} \subseteq \text{Ar}_g \mathcal{M} \subseteq \text{Ar}_\times \mathcal{M}$ and $\text{Ar}_\Delta \mathcal{E} \subseteq \text{Ar}_g \mathcal{E} \subseteq \text{Ar}_\times \mathcal{E}$
(D) $k$ sends a pseudo-commutative square to $\text{Ar}_\Delta \mathcal{M} \subset \text{Ar}_g \mathcal{M}$ if and only if $c$ sends the square to $\text{Ar}_\Delta \mathcal{E} \subset \text{Ar}_g \mathcal{E}$. In this case the square is called distinguished and is denoted as follows:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & D
\end{array}
\]

(K) For any m-morphism $f: A \rightarrowtail B$ there is a distinguished square as below left, and for any e-morphism $g: A \leftarrowtail B$ there is a distinguished square as below right.

\[
\begin{array}{ccc}
\varnothing & \xrightarrow{f} & B/A \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\quad
\begin{array}{ccc}
\varnothing & \xrightarrow{g} & A \\
\downarrow & & \downarrow \\
B \setminus A & \xrightarrow{k(g)} & B
\end{array}
\]

The notation $B/A, B \setminus A$ will only be used when the defining maps $f$ and $g$ are clear from context. Otherwise the cokernel and kernel objects will be denoted $\text{coker } f, \text{ker } g$ respectively.

**Remark 2.5.** The double subcategory of distinguished squares of any g-CGW category forms a CGW category\(^3\) by restricting the functors $k$ and $c$ to this subcategory, where axiom (I) of CGW categories follows from the properties of shared isomorphisms in Definition 1.7. Conversely, any CGW category satisfying these stronger isomorphism conditions gives a g-CGW category where the only squares are the distinguished ones, and the good squares are given by $\text{Ar}_\Delta \mathcal{M}$ and $\text{Ar}_\Delta \mathcal{E}$.

Therefore, it is not surprising that all of the basic examples of interest agree with those of [CZ, Section 3]. We include them here as well, since they illustrate the ideas behind the axioms; in particular, the first example illustrates the motivation behind good squares, which are new to our formulation.

**Example 2.6.** Let $\mathcal{A}$ be an exact category, and let $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ be the double category with the same objects as $\mathcal{A}$, and where

\[\mathcal{M} = \{\text{admissible monomorphisms}\}\]

We want the functors $k$ and $c$ to be the usual kernel and cokernel functors, and the cokernel of an admissible monomorphism $i: A \rightarrowtail B$ is an admissible epimorphism $B \twoheadrightarrow \text{coker } i$. Keeping axioms (M) and (K) in mind, this suggests we should let $\mathcal{E}$ be the admissible epimorphisms pointing in the opposite direction; i.e.,

\[\mathcal{E} = \{\text{admissible epimorphisms}\}^{\text{op}}\]

We must now define the good squares and the pseudo-commutative squares in the double category accordingly. Given a pullback square of admissible monomorphisms as below, the induced map on cokernels is always a monomorphism.

---

\(^3\)A careful reader might observe that axiom (A) of CGW categories is missing in our formulation. However, this will hold in all examples of interest (⋆-CGW-categories) as we discuss in Remark 3.3.
We claim that this monomorphism will be admissible precisely when the induced morphism out of the pushout $B \cup_A A' \rightarrow B'$ is an admissible monomorphism. Indeed, one can factor the diagram above as follows, where all rows are exact:

\[ \begin{array}{ccc}
A & \xrightarrow{i} & B & \xleftarrow{\coker i} \\
\downarrow & & \downarrow & & \downarrow \\
A' & \xrightarrow{i'} & B' & \xleftarrow{\coker i'} \\
\end{array} \]

Applying the Snake Lemma to the bottom part of the diagram, we see that

\[ \coker(B \cup_A A') \cong \coker(\coker i \rightarrow \coker i'); \]

thus, one of these monomorphisms is admissible if and only if the other one is.

This leads us to define the good squares in $\mathcal{M}$ as the pullback squares of maps in $\mathcal{M}$ with this pushout property, which include weak triangles as pushouts preserve isomorphisms. The pseudo-commutative squares are then the squares who commute in $\mathcal{A}$, and such that the morphism induced on kernels (which is always a monomorphism) is admissible. One can show that the dual notion of good squares in $\mathcal{E}$ is also compatible with this class of pseudo-commutative squares.

Once the structure has been determined, the axioms are not hard to check. Axiom (Z) holds since $0$ is both initial and terminal. Axiom (M) is immediate, since monomorphisms are monics, epimorphisms are epics, and epics become monic in the opposite category. Axiom (D) is also satisfied, and one finds that distinguished squares are the bicartesian squares. Axiom (K) is the familiar statement that any admissible monomorphism (resp. epimorphism) determines a short exact sequence by taking its cokernel (resp. kernel), which is constructed as the pushout (resp. pullback) along the unique map to (resp. from) the $0$ object.

That this double category has shared isomorphisms follows immediately, as a map in an exact category is an isomorphism if and only if it is both an admissible monomorphism and an admissible epimorphisms, and pseudo-commutative squares are defined to agree with commuting squares, where a square with parallel isomorphisms always induces an isomorphism on kernels (resp. cokernels).

Remark 2.7. If the exact category $\mathcal{A}$ in the previous example is abelian, then all monomorphisms and epimorphisms are admissible and the g-CGW structure is somewhat simplified. In this case, the good squares are precisely the pullbacks of monomorphisms or pushouts of epimorphisms, and the pseudo-commutative squares are simply the commuting squares.

Example 2.8. We can define a double category of finite sets $\text{FinSet} = (\mathcal{M}, \mathcal{E})$ by setting

\[ \mathcal{M} = \mathcal{E} = \{ \text{injective functions} \} \]

and letting both good and pseudo-commutative squares be the pullback squares. Both of the functors $k$ and $c$ take an injection $A \rightarrow B$ to the inclusion of the complement of its image $B \setminus A \rightarrow B$. With $\emptyset$ as the initial object and good squares also the pullbacks, this gives a
g-CGW category. The distinguished squares are then the pushout squares of injections, so this agrees with [CZ, Example 3.3].

The category of (finite) sets has the property that all injective functions are "coproduct inclusions" of the form \( A \hookrightarrow A \sqcup B \) up to isomorphism. Coproduct inclusions come with a natural choice of complement, the opposing coproduct inclusion \( A \sqcup B \leftarrow B \), which lets us construct a g-CGW category of coproduct inclusions in a category with suitably nice coproducts.

**Definition 2.9 ([CLW93]).** A category \( \mathcal{X} \) is **finitely extensive** (henceforth simply **extensive**) if it has finite coproducts such that

- Coproduct inclusions are monic and the cospan \( A \hookrightarrow A \sqcup B \leftarrow B \) has a pullback given by the initial object \( \emptyset \).
- For any morphism \( X \rightarrow A \sqcup B \), we have pullback squares as below with \( X \cong Y \sqcup Z \):

\[
\begin{array}{ccc}
Y & \rightarrow & X & \rightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
A & \hookrightarrow & A \sqcup B & \hookrightarrow & B
\end{array}
\]

Extensive categories include finite sets, finite \( M \)-sets for a monoid \( M \), functors from any category to \( \text{Set} \), finite and small categories, topological spaces, and many more.

**Example 2.10.** Any extensive category \( \mathcal{X} \) defines a g-CGW category with

\[
\mathcal{M} = \mathcal{E} = \{ \text{coproduct inclusions} \}
\]

and letting both good and pseudo-commutative squares be the pullback squares. Both of the functors \( k \) and \( c \) take a coproduct inclusion to its complementary coproduct inclusion, and by the second axiom for extensive categories this assignment extends appropriately to squares as in the previous example. The first three axioms follow immediately, axiom (D) follows by observing that the distinguished squares are those of the form below,

\[
\begin{array}{ccc}
A & \rightarrow & A \sqcup B \\
\downarrow & & \downarrow \\
C \sqcup A & \rightarrow & C \sqcup A \sqcup B
\end{array}
\]

and axiom (K) follows from the first axiom for extensive categories.

**Example 2.11.** We can define a double category \( \text{Var} \) whose objects are varieties, with \( m \)- and \( e \)-morphisms given by

\[
\mathcal{M} = \{ \text{closed immersions} \} \quad \text{and} \quad \mathcal{E} = \{ \text{open immersions} \}
\]

Like the example above, pseudo-commutative and good squares are given by (all) pullback squares (as varieties are closed under pullbacks), and the functors \( k \) and \( c \) take a morphism to the inclusion of its complement. This example is identical to [CZ, Example 3.4], except we swap open and closed immersions when defining \( m \)- and \( e \)-morphisms. The reason for this is explained in Example 3.7.
Axioms (Z), (M), and (G) are easily checked, and this is clearly a double category with shared isomorphisms. For axiom (D), one can verify that the distinguished squares

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow f \\
C & \to & D
\end{array}
\]

are the pullback squares in which \( \text{im} f \cup \text{im} g = D \). Then, axiom (K) holds directly as well.

We conclude this section with a collection of useful technical results. For the sake of completeness, we first recall three lemmas from [CZ] which only rely on the underlying CGW category, and whose proofs apply verbatim in our setting.

**Lemma 2.12.** [CZ, Lemma 2.9] For any diagram \( A \xrightarrow{f} B \xleftarrow{g} C \) there is a unique (up to unique isomorphism) distinguished square

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow g \\
D & \to & C
\end{array}
\]

The analogous statement holds for any diagram \( A \xleftarrow{f} B \xrightarrow{g} C \).

**Remark 2.13.** As a corollary, we obtain a key consequence of axiom (K): the functors \( k \) and \( c \) are inverses on objects. It also invites us to consider distinguished squares of the form below as extensions of \( A \) by \( B \), which is exactly what they are in Example 2.6.

\[
\begin{array}{ccc}
\emptyset & \to & B \\
\downarrow & & \downarrow \cdot \\
A & \to & C
\end{array}
\]

**Lemma 2.14.** [CZ, Lemma 2.10] Given any composition \( C \to B \to A \), there is an induced map \( B/A \to C/A \) such that the composite \( B/A \to C/A \to A \) agrees with the map \( C/A \to A \). The same holds when the roles of m- and e-morphisms are reversed.

**Lemma 2.15.** [CZ, Lemma 5.12] In a pseudo-commutative square as below, if \( f' \) is an isomorphism then so is \( f \).

\[
\begin{array}{ccc}
\bullet & \to & \bullet \\
\downarrow & & \downarrow f' \\
\bullet & \to & \bullet
\end{array}
\]

The same holds when the roles of m- and e-morphisms are reversed.

We will also make repeated use of the following straightforward generalization of [CZ, Lemma 2.8].

**Lemma 2.16.** An m-morphism (resp. e-morphism) in a g-CGW category is an isomorphism if and only if its cokernel (resp. kernel) has initial domain.
Lemma 2.17. Given a composite square of two pseudo-commutative squares, if two of the three squares are distinguished, then so is the third.

Proof. This follows from the 2-out-of-3 property of isomorphisms for the induced maps between the kernels of the e-morphisms. □

Lemma 2.18. In a g-CGW category, if there exists a square as below right completing the mixed cospan below left, then it is unique up to unique isomorphism.

\[
\begin{array}{c}
\bullet & \bullet \\
\downarrow & \downarrow \\
g & f
\end{array}
\quad\quad
\begin{array}{c}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bigcirc & g
\end{array}
\]

Proof. Given any such square, applying the inverse equivalence \(c^{-1}\) yields the pullback square of \(g\) and \(c(f)\), as seen in the following diagram.

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
\bigcirc & g & f & c(f)
\end{array}
\]

Since pullbacks and kernel-cokernel pairs are unique up to unique isomorphism, the same must be true of this pseudo-commutative square. □

In particular, the above lemma implies that a pseudo-commutative square (if it exists) is unique relative to its boundary. Then, for a given square of m- and e-morphisms, the existence of a pseudo-commutative square filler can be treated as a property rather than data. When such a pseudo-commutative filler exists, we say that the square is pseudo-commutative.

3. \(\ast\)-CGW Categories

Thanks to Remark 2.5, g-CGW categories admit the same \(Q\)-construction introduced in [CZ] for CGW categories. However, we are interested in a model similar to Waldhausen’s \(S\_\ast\) construction, which naturally lends itself to iteration, as well as eventually allowing us to incorporate weak equivalences into our structures.

In this section we introduce \(\ast\)-CGW categories, together with several technical results that will allow us to prove the necessary functoriality to iterate the \(S\_\ast\) construction and prove the Additivity and Fibration theorems. Key among these is a way to define a \(\ast\)-CGW structure on certain double categories of diagrams over a \(\ast\)-CGW category. This proof is quite long, and will be deferred to Appendix B.

Definition 3.1. An \(\ast\)-CGW category is a g-CGW category satisfying the following additional axioms:

(GS) A square in \(\mathcal{M}\) as below is a good square from \(f\) to \(k\) if and only if it is a good square from \(g\) to \(h\).

\[
\begin{array}{c}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bullet & \bullet \\
\end{array}
\quad
\begin{array}{c}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bullet & \bullet \\
\end{array}
\]

In particular this means that good squares are closed under composition in both directions.
(⋆) For every diagram \( C \xleftrightarrow{g} A \xrightarrow{B} B \), if the category of good squares as below left (with morphisms maps \( D \xrightarrow{g} D' \) commuting under \( B \) and \( C \)) is non-empty, then it has an initial object which we write \( D = B \star_A C \).

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & D
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{g} & B & \xleftarrow{B/A} \\
\downarrow & & \downarrow & \downarrow \\
C & \xrightarrow{g} & B \star_A C & \xleftarrow{B \star_A C/C}
\end{array}
\]

Furthermore, the induced maps \( B/A \xrightarrow{g} B \star_A C/C \) and \( C/A \xrightarrow{g} B \star_A C/B \) are isomorphisms (above right). The dual statement holds for spans of e-morphisms as well.

(PO) For every diagram \( C \xleftrightarrow{g} A \xrightarrow{B} B \), the category of good squares as in axiom (PO) is non-empty. The dual statement need not hold for spans of e-morphisms.

(PBL) Pseudo-commutative squares satisfy the “pullback lemma”: if the outer composite below is a pseudo-commutative square, then so is the square on the left.

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B & \xleftarrow{C} \\
\downarrow & & \downarrow & \downarrow \\
A' & \xrightarrow{g} & B' & \xleftarrow{C'}
\end{array}
\]

The analogous statement holds for composites in the e-direction.

(POL) If the outer square in the commutative diagram below is good, then the right square is good.

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B & \xleftarrow{D} \\
\downarrow & & \downarrow & \downarrow \\
C & \xrightarrow{g} & B \star_A C & \xleftarrow{E}
\end{array}
\]

The same property holds for e-morphisms when the \( \star \)-pushout exists.

This definition warrants some explanation. Axiom (GS) is a categorical technicality that allows us to treat good squares in a symmetrical way. Axioms (PBL) and (POL) are in a way dual to each other, and they mean to capture the “pullback lemma” and “pushout lemma” which are known to hold in a category with pullbacks and pushouts. Axioms (PO) and (⋆) deal with the existence of certain initial objects among good squares, which are intended to behave as pushouts do in an exact category. From this perspective, axiom (PO) then says that any span of morphisms in \( \mathcal{M} \) admits a “pushout”. This is not required of the maps in \( \mathcal{E} \), where instead we only expect a “pushout” if the given span is already known to be part of a good square\(^4\).

The need for these pushouts arises when studying the classical proofs of the Additivity Theorem (see, for example, [McC93], [Wal85, Section 1.4], [Wei13, Chapter V, Theorem 1.3]). We will see that \( \star \)-pushouts are adequately functorial and allow for a construction of \( \star \) in categories of diagrams; in particular, this will allow us to define an \( S \) construction that can

\(^4\)While this distinction is not necessary in an exact category where we have all pullbacks of admissible epimorphisms, the reader curious about this asymmetry is directed to Example 3.7 and Section 10 for examples of where this asymmetry may arise. This is the only asymmetry between m- and e-morphisms in our definition.
be iterated. Indeed, the “F” in $\star$-CGW stands for Functorial. A more detailed study of the properties of the $\star$-pushout can be found in Appendix A.

Remark 3.2. The good squares are meant to behave like the cofibrations in Waldha usesn’s category $F_1C$. Recall that, given a Waldhausen category $C$, $F_1C$ is the subcategory of $\text{Ar}C$ whose objects are the cofibrations. Here, a morphism

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

is a cofibration if the maps $A \longrightarrow C$, $B \longrightarrow D$ and $B \cup_A C \longrightarrow D$ are cofibrations.

In our setting, the pushout is replaced by the $\star$-pushout, and by axiom $(\star)$ all good squares are such that there is an induced m-morphism $B \star_A C \longrightarrow D$. Moreover, the converse also holds, and so this property characterizes good squares. Indeed, given a commutative square as below left

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\hspace{1cm}
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & B \star_A C
\end{array}
\hspace{1cm}
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

Together with an m-morphism $B \star_A C \longrightarrow D$ over $D$, we can rewrite it as the composite above right, which implies the square is good.

Remark 3.3. Our $\star$-CGW categories are very similar in nature to the ACGW categories of [CZ]. The key distinctions are that we do not require all pullback squares to participate in the equivalences $k$ and $c$ as in [CZ, Definition 5.3], but rather consider the class of good squares which specialize pullbacks; and our requirements of $\star$-pushouts are more relaxed than axioms (S) and (PP) of [CZ], reducing the necessary $\star$-pushouts — we then prove the extra functoriality properties asserted in those axioms as consequences of ours (Lemma A.6, Proposition A.4). These distinctions turn out to be crucial both when iterating the process of the $S_\star$ construction, and for including new examples such as exact categories and varieties which are not ACGW categories.

The reader might also notice that we do not require an analogue to axiom (A) in [CZ]. This is due to the fact that a stronger, functorial version of this notion (which is intended to axiomatize the existence of a trivial extension) can be recovered from our axioms by taking the star pushout of the span below.

$$
\begin{array}{ccc}
A & \leftarrow & \emptyset & \longrightarrow & B
\end{array}
$$

For our first example, recall that an exact category is called weakly idempotent complete when every monomorphism that admits a retraction is admissible, or equivalently, every epimorphism that admits a section is admissible.

Example 3.4. Given an exact category $\mathcal{A}$ which is weakly idempotent complete, the $\mathfrak{g}$-CGW structure described in Example 2.6 can be upgraded to a $\star$-CGW structure by defining $\star$-pushouts as the pushouts. This is well-defined and satisfies axiom (PO), as admissible monomorphisms (resp. epimorphisms) are stable under pushout (resp. pullback). Axiom $(\star)$ is easily checked, as pushouts of admissible monomorphisms preserve cokernels, and dually for pullbacks of epimorphisms. Axiom (POL) is satisfied as pushouts in an exact category (unlike
⋆-pushouts in the full generality of a ⋆-CGW category) have a universal property with respect to commutative (and not necessarily good) squares.

Weak idempotent completeness plays a role when verifying axiom (PBL). Given a pasting as in axiom (PBL), we can take kernels to obtain the following diagram

\[ \begin{array}{ccc}
A & \longrightarrow & B & \longrightarrow & C \\
\downarrow a & & \downarrow b & & \downarrow c \\
A' & \longleftarrow & B' & \longleftarrow & C'
\end{array} \]

\[ \begin{array}{cc}
\ker a & \ker b \\
\ker c & \end{array} \]

where the bottom outer diagram is a good square.

Since good squares are pullbacks, there exists an induced morphism \( i: \ker a \to \ker b \) such that \( k = ji \). Thus \( i \) is a monomorphism, but in a general exact category, there is no way to ensure that it is admissible. This property is guaranteed by the fact that \( A \) is weakly idempotent complete, as proven in [B10, Proposition 7.6]. Similarly, the vertical pasting in axiom (PBL) uses the fact that, given a composite \( r = qp \) where \( p, r \) are admissible epimorphisms, the weak idempotent completeness implies that \( q \) is also an admissible epimorphism.

In fact, using this same property, one can easily observe that in weakly idempotent complete categories, all commutative squares of mixed type are pseudo-commutative.

**Remark 3.5.** (Weakly idempotent complete) exact categories do not in general have all pullbacks, and so they are not examples of ACGW (or pre-ACGW) categories in [CZ].

Even when pullbacks exist, our restriction from pullback squares to good squares is not vacuous, as we now illustrate. Let \( \mathcal{C} \) denote the exact category of finitely generated projective (i.e., free) abelian groups. This category is idempotent complete, and thus it is in particular weakly idempotent complete. If we consider the square below

\[ \begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow d \\
\mathbb{Z} & \longrightarrow & \mathbb{Z} \otimes \mathbb{Z}
\end{array} \]

where \( d \) is the diagonal map \( d(x) = (x, x) \) and \( f \) is given by \( f(x) = (x, -x) \), we see that this is a pullback square in \( \mathcal{C} \) which is not good. Indeed, the map induced on cokernels is the monomorphism \( i: \mathbb{Z} \to \mathbb{Z} \) given by \( i(x) = 2x \), which is not admissible since its cokernel \( \text{coker } i = \mathbb{Z}/2\mathbb{Z} \) is not free.

**Example 3.6.** The g-CGW structure for an extensive category \( \mathcal{X} \) described in Example 2.10 can be upgraded to a ⋆-CGW structure by defining the ⋆-pushout of a span \( \mathcal{C} \sqcup A \leftarrow A \to \mathcal{C} \sqcup B \) to be the triple coproduct \( \mathcal{C} \sqcup A \sqcup B \). This is clearly the categorical pushout in \( \mathcal{X} \), but one must further verify that, when restricted to the case of pullback squares, the induced maps out of \( \mathcal{C} \sqcup A \sqcup B \) given by the universal property of the pushout are indeed m- or e-morphisms, as corresponds. This fact, together with the rest of the claims in axioms (⋆), (PO), and (POL), can be easily deduced from the axioms of an extensive category (Definition 2.9). Axioms (PBL) and (GS) are immediate as the good and pseudo-commutative squares are both given by pullbacks.

In particular, this establishes a ⋆-CGW structure for the category of finite sets.
Example 3.7. The g-CGW category of varieties of Example 2.11 can be upgraded to the structure of a ⋆-CGW category, by letting ⋆-pushouts be the pushouts of varieties.

Axiom (GS) is immediate in this setting, and axiom (PB) is satisfied as pseudo-commutative squares are pullbacks. Axiom (PO) holds, since pushouts of closed immersions exist, and the resulting square is a pullback. We note that this does not hold for e-morphisms, as the pushout of open immersions need not exist. However, it does when the span of open immersions is known to belong to a pullback square, and thus ⋆-pushouts of both m- and e-spans satisfy axiom (⋆). Finally, axiom (POL) can be verified in a similar manner as either of the previous examples.

Remark 3.8. Just as Example 3.4, varieties give another example that fits our axioms, and not those of ACGW categories (although, unlike exact categories, varieties are pre-ACGW). In this case, this is due to the fact that our ⋆-pushouts need not exist in the case of e-morphisms, while ⋆-pushouts of both classes of morphisms are required in axiom (PP) of [CZ, Definition 5.4].

As usual, ⋆-CGW categories have natural notions of functors and subcategories.

Definition 3.9. An ⋆-CGW functor is a double functor that preserves all of the relevant structure up to natural isomorphism.

Definition 3.10. A double subcategory $D$ of a ⋆-CGW category $C$ is an ⋆-CGW subcategory if it inherits a ⋆-CGW structure from $C$.

For full double subcategories of a ⋆-CGW category, many of the axioms are automatically preserved, so it is easy to check whether they are ⋆-CGW.

Lemma 3.11. A full double subcategory of a ⋆-CGW category $C$ is a ⋆-CGW subcategory if it is closed under $k, c, ⋆$, and contains $∅$.

The most common way for us to construct new ⋆-CGW categories from familiar ones will be through functor categories. Given a ⋆-CGW category $C$ and any double category $D$, we wish to describe a ⋆-CGW structure on a double subcategory of the double category $[D, C]$ of double functors described in Definition 1.5.

Definition 3.12. For $C$ a ⋆-CGW category and $D$ any double category, we define the double subcategory $C^D \subset [D, C]$ as follows:

- objects are all double functors $D \longrightarrow C$
- $M$ consists of the “good” m-natural transformations: these are the ones whose naturality squares of m-morphisms are good
- $E$ is given by the “good” e-natural transformations: these are the ones whose naturality squares of e-morphisms are good
- mixed squares consist of all modifications between the m- and e-morphisms; note that these are pointwise pseudo-commutative squares in $C$

Note that $M$ and $E$ here are in fact categories, as good squares are closed under identities and composition and there are no restrictions placed on the mixed naturality squares of these transformations.

As we saw in Example 2.6, it is not enough to consider squares whose sides are all in $M$, and instead we need to work with a more well-behaved notion of good square. Similarly, when working with m-natural transformations, it will not suffice to ask that all the squares involved are good, but instead we need a stronger notion of “good cube”. In order to do this, we present the following definition, which adapts the good cubes of [Zak18, Definition 2.3] to our setting.
Definition 3.13. Let $C$ be a $\star$-CGW category. A commutative cube of morphisms in $\mathcal{M}$ is a **good cube** if each face is a good square, and if the induced $m$-morphism between $\star$-pushouts is such that the square below right is good.

We call this the “southern square”. Good cubes in $\mathcal{E}$ are defined in the same way.

Remark 3.14. A priori, it seems as if our definition of good cube is subject to a choice of direction. Indeed, we could have taken $\star$-pushouts of the back and front faces, instead of the left and right faces, and induced a different southern square. However, as we show in Remark A.8, if any of these induced squares are good, then all of them are. Moreover, it is possible to define a “southern arrow” $m$-morphism of the entire cube as in [Zak18, Definition 2.3] and show that any of the southern squares of a cube is good if and only if this southern arrow exists.

Theorem 3.15. For $C$ a $\star$-CGW category and $D$ any double category, the functor double category $C \to D$ admits the structure of a $\star$-CGW category as follows:

- $\text{Ar}_g \mathcal{M}$ are the commutative squares of $m$-natural transformations whose component cubes of naturality squares between $m$-morphisms are good cubes. $\text{Ar}_g \mathcal{E}$ is defined dually
- the functors $k$ and $c$ are defined pointwise from those of $C$, as is $\star$ in the sense that the $\star$-pushout of a span of $D$-shaped diagrams in $C$ is the $D$-shaped diagram of pointwise $\star$-pushouts

Showing that this defines a $\star$-CGW structure is nontrivial, especially for $\star$-pushouts, but the axioms of $\star$-CGW categories were designed to enable this kind of construction. As the technical details of this proof are not needed to describe our main results, we defer it to Appendix B, along with several helpful corollaries providing FCGW structures on more specialized subcategories of $C \to D$.

4. Adding weak equivalences

One of the benefits of Waldhausen’s $S_\star$-construction over Quillen’s $Q$-construction is that it allows us to incorporate homotopical data in the form of weak equivalences. In practice, when a Waldhausen category has additional algebraic structure (such as that of an exact or abelian category), the weak equivalences often interact nicely with that structure.

In particular, one often finds that the class of weak equivalences can be completely determined by the acyclic monomorphisms and epimorphisms, and that in turn, these can be characterized by having acyclic (co)kernels. Such is the case, for example, in the category of bounded chain complexes over an exact category, with quasi-isomorphisms as weak equivalences.

In this section, we borrow this intuition and define $m$- and $e$-equivalences on a $\star$-CGW category, constructed from a given class of acyclic objects.

---

5Such a morphism always exists; see Proposition A.3.
Definition 4.1. An acyclicity structure on a ⋆-CGW category \( C \) is a class of objects of \( C \), which we call acyclic objects, such that:

1. (IA) any initial object is acyclic
2. (A23) for any kernel-cokernel pair \( A \to B \to C \), if any two of \( A, B, C \) are acyclic then so is the third

If we let \( W \) denote the full double subcategory of acyclic objects, we refer to the pair \((C, W)\) as an FCGWA category, which stands for Functorial CGW category with Acyclics.

Definition 4.2. An ECGW functor \((C, W) \to (C', W')\) is a ⋆-CGW functor \( C \to C' \) that preserves acyclic objects.

Definition 4.3. An m-morphism (resp. e-morphism) in an ECGW category \((C, W)\) is a weak equivalence if its cokernel (resp. kernel) is acyclic.

Notation 4.4. We will refer to the m-morphisms (resp. e-morphisms) which are weak equivalences as m-equivalences (resp. e-equivalences), and denote them by \( \sim \) (resp. \( \sim \)). When it is not relevant whether the weak equivalence is horizontal or vertical, we denote them by \( \sim \).

ECGW categories can be equivalently defined in terms of the weak equivalences rather than their acyclic objects, but as we now show, the desired properties of weak equivalences are more easily expressed in terms of acyclic objects. This is reminiscent of the construction of Waldhausen structures on exact categories via cotorsion pairs of \([\text{Sar20}]\). Much of the theory we develop holds equally well in a more general setting in which weak equivalences are not determined by acyclic objects, but this complicates the proofs significantly and is not necessary for any of our examples.

Example 4.5. In any ⋆-CGW category \( C \), acyclic objects can be chosen to be the initial objects. By Lemma 2.16, they satisfy Definition 4.1 and weak equivalences are precisely the isomorphisms. The \( K \)-theory of this ECGW category as defined in Section 5 is the same as that of the underlying CGW category of \( C \) defined in \([\text{CZ}]\) (for more details, see Proposition 5.8).

Example 4.6. For any ECGW category \((C, W)\) and \( C' \subset C \) a ⋆-CGW subcategory, \((C', W \cap C')\) forms an ECGW category.

Example 4.7. As explained in Example 3.4, weakly idempotent complete exact categories can be given the structure of a ⋆-CGW category. Let \( C \) be such a category, which in addition has a Waldhausen structure. If we denote by \( W \) the class of objects \( X \in C \) such that \( 0 \to X \) is a weak equivalence, then \((C, W)\) will be an ECGW category whenever \( W \) has 2-out-of-3.

For example, this will be the case when \( C \) is a Waldhausen category constructed from a cotorsion pair and any such class \( W \) of acyclic objects as in \([\text{Sar20}]\), when \( C \) is a biWaldhausen category satisfying the extension and saturation axioms (such as the complicial biWaldhausen categories of \([\text{TT90}, 1.2.11])\), and when \( C \) satisfies the saturation axiom and is both left and right proper (like the complicial exact categories with weak equivalences of \([\text{Sch11}, \text{Definition 3.2.9}]\)).

In particular, our construction recovers the classical (epi and mono) quasi-isomorphisms for the case of chain complexes on an exact category, where acyclic objects are given by the exact complexes.

The following results can be easily deduced for any ECGW category from Definition 4.1.

Lemma 4.8. All isomorphisms are weak equivalences.
Lemma 4.9. Given a weak equivalence $X \sim Y$, if either $X$ or $Y$ is acyclic, then both are.

Lemma 4.10. Any map between acyclic objects is a weak equivalence.

In particular, all morphisms in the full double subcategory $W$ are weak equivalences, and an object in $C$ is acyclic if and only if both the $m$- and $e$-morphisms from $\emptyset$ are weak equivalences.

Additionally, we can prove the following.

Lemma 4.11. $m$- and $e$-equivalences each satisfy 2-out-of-3. In particular, they form subcategories of $M$ and $E$.

Proof. We prove this for $m$-morphisms, the argument for $e$-morphisms is dual.

Given $m$-morphisms $f: A \to B$ and $g: B \to C$, we consider the following diagram:

\[
\begin{array}{ccc}
\text{coker } f & \to & \text{coker } gf \\
\downarrow & & \downarrow \\
B & \to & \text{coker } g \\
\downarrow & & \downarrow \\
A & \to & A
\end{array}
\]

By Lemma 4.9 $D$ is acyclic if and only if $\text{coker } g$ is, so if any two of $f, g, gf$ are weak equivalences, then two of $\text{coker } f$, $\text{coker } g$, $\text{coker } gf$ are acyclic, and hence so is the third by Definition 4.1. Together with Lemma 4.8, this shows that weak equivalences form subcategories of $M$ and $E$. □

Lemma 4.12. In a kernel-cokernel pair of squares, if any two of the three parallel maps are weak equivalences then so is the third.

Proof. Consider the kernel-cokernel pair of squares depicted in the left column of the diagram below, with parallel $m$-morphisms $f, g, h$:

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \to & D \\
\downarrow & & \downarrow \\
E & \to & F
\end{array}
\]

Taking cokernels of both squares we get a kernel-cokernel sequence

\[
\begin{array}{ccc}
\text{coker } f & \to & \text{coker } g & \to & \text{coker } h
\end{array}
\]

as shown in the diagram, so by Definition 4.1 if any two of $f, g, h$ are weak equivalences then so is the third. □

Lemma 4.13. Acyclic objects are closed under $\star$-pushouts (when these exist).
Proof. Consider a span of m-morphisms $B \leftarrow A \rightarrow C$ where $A, B, C$ are acyclic. By Lemma 4.10 these morphisms are weak equivalences, hence $B/A$ is acyclic. By axiom $(\ast)$, $(B \ast_A C)/C \cong B/A$, so the map $C \leftarrow B \ast_A C$ is a weak equivalence. Therefore, $B \ast_A C$ is acyclic by Lemma 4.10. The same argument holds for spans of e-morphisms whose $\ast$-pushout exists. □

Remark 4.14. The definition of acyclicity structures, along with Lemma 4.13 above, imply that $W$ forms a $\ast$-CGW category by Lemma 3.11. Conversely, given a $\ast$-CGW category $C$, any full $\ast$-CGW subcategory that is closed under extensions provides an acyclicity structure.

**Definition 4.15.** A $\ast$-CGW subcategory $C'$ of $C$ is **closed under extensions** if, for any kernel-cokernel sequence

$$A \longrightarrow B \leftarrow \rightarrow C$$

in $C$ such that $A, C$ are in $C'$, $B$ is also in $C'$.

A $\ast$-CGW category often admits more than one natural choice of acyclicity structure; in fact, Section 8 provides a tool for comparing the two resulting ECGW structures when one is a refinement of the other.

**Definition 4.16.** A refinement of an ECGW category $(C, W)$ is a subclass $V \subseteq W$ of acyclic objects such that $(C, V)$ also forms an ECGW category.

**Example 4.17.** The poset of refinements of $(C, W)$ ordered by inclusion has both minimal and maximal elements, given by initial objects in $C$ and $W$ itself, respectively.

The following is immediate from our definitions, along with Remark 4.14.

**Lemma 4.18.** For any refinement $(C, V)$ of an ECGW category $(C, W)$, the pair $(W, V)$ is an ECGW subcategory of $(C, W)$.

**Part 2. $K$-theory of ECGW categories**

**5. $S_\ast$-construction**

We are now equipped to define the $K$-theory of an ECGW category by translating the $S_\ast$-construction into our setting. The construction is similar to that of [CZ, Definition 7.10], but we also accommodate weak equivalences, and moreover the variants in our construction (mostly, the restriction to good cubes) allow for this process to be iterated. In other words, given an ECGW category $C$, we construct a simplicial double category $S_\ast C$ which is furthermore a simplicial ECGW category.

The following double category will be useful for defining our $S_\ast$ construction.

**Definition 5.1.** For each $n$, let $S_n$ denote the double category generated by the following objects, horizontal morphisms, vertical morphisms, and squares.
Definition 5.2. Given an ECGW category $\mathcal{C}$, we define a simplicial double category $S_*\mathcal{C}$ as follows:

- for each $n$, $S_n\mathcal{C}$ is the full double subcategory of $\mathcal{C}^{S_n}$ given by the functors $F$ such that $F(A_{i,i}) = \emptyset$ for all $i$, and that $F$ sends all squares in $S_n$ to distinguished squares in $\mathcal{C}$.
- for the simplicial structure, the face map $d_i: S_n\mathcal{C} \to S_{n-1}\mathcal{C}$, $0 \leq i \leq n$, deletes the objects $F(A_{j,i})$ and $F(A_{i,j})$ for all $j$, where what remains after discarding or composing the affected squares is a diagram of shape $S_{n-1}$; the degeneracy map $s_i: S_n\mathcal{C} \to S_{n+1}\mathcal{C}$ inserts a row and column of identity morphisms above and to the right of $F(A_{i,i})$.

We will often refer to the objects of $S_n\mathcal{C}$ as “staircases”.

Proposition 5.3. $S_n\mathcal{C}$ is an ECGW category, with $\star$-CGW structure inherited from that of $\mathcal{C}^{S_n}$ as described in Theorem 3.15, and acyclic objects defined as the pointwise acyclics in $\mathcal{C}$.

Proof. We show in Proposition B.2 that $S_n\mathcal{C}$ is a $\star$-CGW subcategory of $\mathcal{C}^{S_n}$, and pointwise acyclic diagrams clearly form an acyclicity structure. \hfill \box

Definition 5.4. For an ECGW category $(\mathcal{C}, W)$, define

$$K(\mathcal{C}, W) = \Omega |wS_*\mathcal{C}|$$ and $K_n(\mathcal{C}, W) = \pi_n K(\mathcal{C}, W),$$

where $wS_*\mathcal{C}$ is the simplicial double category obtained by restricting the $m$-morphisms and $e$-morphisms in $S_*\mathcal{C}$ to the $m$-equivalences and $e$-equivalences.

As usual, we start by studying $K_0$ and showing that it agrees with the intuitive Grothendieck group. Similarly to [CZ, Theorem 4.3], most of the relations will be given by the distinguished squares, except that we get additional relations induced by the weak equivalences.

Proposition 5.5. For any ECGW category $(\mathcal{C}, W)$, $K_0(\mathcal{C}, W)$ is the free abelian group generated by the objects of $\mathcal{C}$, modulo the relations that, for any distinguished square

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & \downarrow & \downarrow \\
C & \xrightarrow{g} & D
\end{array}$$

we have $[A] + [D] = [B] + [C]$, and that for any horizontal or vertical weak equivalence $A \xrightarrow{\sim} B$ we have $[A] = [B]$. 

Proof. By definition, $K_0(C, W) = \pi_0\Omega|wS_0C| = \pi_1|wS_0C|$. Since $|wS_0C|$ is path-connected (as $|wS_0C| = \ast$), it follows from the Van-Kampen Theorem that $\pi_1|wS_0C|$ is the free group on $\pi_0|wS_1C|$, modulo the relations $\delta_1(x) = \delta_2(x)\delta_0(x)$ for each $x \in \pi_0|wS_2C|$.

Let us describe what these conditions entail. The elements of $|wS_1C|$ are the objects of $C$, and two objects $A, B$ are in the same connected component precisely when there exists a pseudo-commutative square as below left.

These squares include those above right, which shows that weakly equivalent objects are identified. Conversely, the relation that weakly equivalent objects are identified implies that objects $A$ and $B$ in the same connected component of $|wS_1C|$ are identified, so these two relations are equivalent.

Elements of $|wS_2C|$ are kernel-cokernel sequences in $C$ represented by distinguished squares $x$ as below left, where $\delta_1(x) = B$, $\delta_0(x) = B/A$ and $\delta_2(x) = A$.

To show that our distinguished square relation is always satisfied in $K_0(C, W)$, we recall that distinguished squares induce isomorphisms on cokernels. We can then see that for any distinguished square as above right we have

$$[B] = [A] + [B/A]$$

$$= [A] + [D/C]$$

$$= [A] + [D] - [C]$$

which yields the desired relation. Conversely, if we start from the relation on generic squares, we obtain the splitting $[B] = [A] + [B/A]$ by restricting to the objects of $|wS_2C|$, so that relation is equivalent to ours.

Finally, note that $K_0(C)$ is abelian because, as explained in Remark 3.3, we have trivial extensions

$$A \longrightarrow A + B \longleftarrow B,$$

$$B \longrightarrow A + B \longleftarrow A$$

and so $[A] + [B] = [A + B] = [B] + [A]$.

Having established a new $K$-theory machinery, we now wish to show that it agrees with the existing ones for all the relevant examples. We start by stating the following, analogous to [Wal85, 1.4.1 Corollary (2)].

**Definition 5.6.** Given an ECGW category $C$, let $s_\bullet C$ denote the simplicial set given by $s_nC = \text{ob} S_nC$.

**Lemma 5.7.** For a $\ast$-CGW category $C$, we have $iS_\bullet C \simeq s_\bullet C$, where $i$ denotes the class of isomorphisms in $C$. 

Proof. Since $\mathcal{C}$ has shared isomorphisms, as does each $S_n\mathcal{C}$ by Proposition 5.3, the double subcategory $iS_n\mathcal{C}$ is isomorphic to the double category of commutative squares in the groupoid $I(S_n\mathcal{C})$ of isomorphisms in $S_n\mathcal{C}$. By Waldhausen’s Swallowing Lemma ([Wal85, 1.5.6]), $iS_n\mathcal{C}$ is then homotopy equivalent to the groupoid $I(S_n\mathcal{C})$ itself, and from this point the proof proceeds exactly as in [Wal85, 1.4.1]. □

Using this lemma, we see that the $K$-theory of an ECGW category with isomorphisms as weak equivalences agrees with its $K$-theory as constructed in [CZ].

Proposition 5.8. For an ECGW category $\mathcal{C}$, $K(\mathcal{C}, \emptyset)$ agrees with the $K$-theory of its underlying CGW category as defined in [CZ].

Proof. By Lemma 5.7, $K(\mathcal{C}, \emptyset)$ is homotopy equivalent to $\Omega|\mathcal{C}|$, which is precisely $K^S$ of the underlying CGW category of $\mathcal{C}$ as defined in [CZ, Definition 7.4]. □

Remark 5.9. In particular, this implies that the $K$-theory of the $*$-CGW categories given by exact categories, finite sets, and varieties of Examples 3.4 to 3.7 agree with their existing counterparts in the literature.

Remark 5.10. The only caveat if one wishes to model the $K$-theory of exact categories through our formalism is that, as explained in Example 3.4, they need to be weakly idempotent complete. However, this does not present a real obstruction for $K$-theoretic purposes, as any exact category $\mathcal{C}$ satisfies $K(\mathcal{C}) \simeq K(\mathcal{C})$, where $\mathcal{C}$ denotes the full exact subcategory of the idempotent completion of $\mathcal{C}$ consisting of the objects $A$ such that $[A] \in K_0(\mathcal{C})$. In particular, $\mathcal{C}$ is weakly idempotent complete.

It is natural to ask whether our notion of $K$-theory also agrees with the existing ones when working with an exact category with weak equivalences, such as chain complexes with quasi-isomorphisms. Due to the way it was constructed, our $K$-theory machinery is only designed to take as input a category whose weak equivalences are defined through a class of acyclics. That is, if there is any hope of a comparison, the exact category must be such that an admissible monomorphism (resp. epimorphism) is a weak equivalence if and only if its cokernel (resp. kernel) is weakly equivalent to 0.

Furthermore, since our double-categorical perspective only deals with admissible monomorphisms and epimorphisms, it must be the case that m- and e-equivalences encode the data of all weak equivalences. This is the case, for example, when weak equivalences can be expressed as composites of admissible monomorphisms and epimorphisms which are themselves weak equivalences.

Fortunately, this seems to be the case for the vast majority of exact categories with weak equivalences that arise in practice.

Proposition 5.11. Let $\mathcal{C}$ be an exact category with a class of weak equivalences $w$, and let $W$ be the class of objects $X \in \mathcal{C}$ such that $0 \rightarrow X$ is in $w$. If $(\mathcal{C}, w)$ is either

- a complicial exact category with weak equivalences as in [Sch11, Definition 3.2.9],
- a complicial biWaldhausen category as in [TT90, 1.2.11] closed under canonical homotopy pushouts and pullbacks ([TT90, 1.1.2]), or
- an exact category with weak equivalences constructed from a cotorsion pair as in [Sar20] and such that $W$ has 2-out-of-3

then the $K$-theory of $(\mathcal{C}, w)$ as a Waldhausen category is homotopy equivalent to the $K$-theory of $(\mathcal{C}, W)$ as an ECGW category.

Proof. In all the specified cases, there exists a homotopy fiber sequence of $K$-theory spectra of Waldhausen categories

\[
K(W, i) \longrightarrow K(\mathcal{C}, i) \longrightarrow K(\mathcal{C}, w)
\]
However, by Proposition 5.8 the two leftmost terms are equivalent to the $K$-theory spaces of the ECGW categories $(W, \emptyset)$ and $(C, \emptyset)$, and by Theorem 8.1, there exists a homotopy fiber sequence of $K$-theory spaces of ECGW categories

$$K(W, \emptyset) \longrightarrow K(C, \emptyset) \longrightarrow K(C, W)$$

These are furthermore shown to be spectra in Theorem 7.5, and so we conclude that their cofibers must be homotopy equivalent. □

Our definition also agrees with the existing notion of $K$-theory for an extensive category $X$, namely its symmetric monoidal $K$-theory with the coproduct monoidal structure. Recall that the $K$-theory of a symmetric monoidal category is given by the group completion of the realization of its underlying monoidal groupoid, which is a topological monoid.

**Proposition 5.12.** For an extensive category $X$, $K(X, \emptyset)$ is homotopy equivalent to the symmetric monoidal $K$-theory of $(X, \sqcup)$.

**Proof.** The simplicial set $s_*X$ is evidently isomorphic to the $s_*$ construction of a Waldhausen category $X_*$ defined as follows: the objects are those of $X$, the morphisms from $A$ to $B$ are isomorphism classes of spans $A \leftarrow C \longrightarrow B$, where the map on the left is a coproduct inclusion, and composites are defined in the usual way for spans as $X$ has pullbacks of coproduct inclusions. The initial object in $X$ is a zero object in $X_*$, and the coproduct on $X$ is a coproduct on $X_*$ as well. The cofibrations are forward coproduct inclusions $A = A \hookrightarrow A \sqcup B$, and their cofiber maps are the backward coproduct inclusions $A \sqcup B \leftarrow B = B$. Weak equivalences are taken to be the isomorphisms, which are spans with both arrows invertible.

By [MM21, Proposition 2.16], the $K$-theory of $X_*$ agrees with the corresponding Segal $K$-theory, so long as cofiber maps in $X_*$ are split, meaning that for each cofiber sequence $A = A \hookrightarrow A \sqcup B \leftarrow B = B$ there is a morphism from $B$ to $A \sqcup B$ which composes to the identity on $B$. These splittings can be taken to be the forward inclusions $B = B \hookrightarrow A \sqcup B$. This completes the proof, as Segal $K$-theory is known to agree with symmetric monoidal $K$-theory [Wei13, IV.4.5.1], and the underlying monoidal groupoid of $X_*$ is equivalent to that of $X$ so their symmetric monoidal $K$-theories agree. □

### 6. Additivity Theorem

The purpose of this section is to show that our $K$-theory construction satisfies the Additivity Theorem. Aside from being a fundamental result that any $K$-theory machinery is expected to satisfy, it will be useful in the next sections when we establish the Fibration Theorem and discuss a version of the Gillet–Waldhausen Theorem.

In order to state the Additivity Theorem, we define extension categories in our setting.

**Definition 6.1.** Let $A, B \subseteq C$ be full $*$-CGW subcategories of a $*$-CGW category $C$. We define the **extension double category** $E(A, C, B)$ as the full double category of $S_2(C)$ whose objects are determined by kernel-cokernel sequences in $C$ of the form

$$A \xrightarrow{f} C \xleftarrow{g} B$$

with $A \in A, B \in B$ and $C \in C$. Explicitly, an m-morphism in $E(A, C, B)$ is a triple of pointwise m-morphisms in $A, C, B$ respectively, related by good and pseudo-commutative squares as follows
and e-morphisms are defined analogously. Pseudo-commutative squares in $E(A,C,B)$ are given by triples of pseudo-commutative squares in $A,C,B$ respectively, natural in the appropriate sense.

Interestingly, any commuting square as above left is automatically good by an argument that recurs throughout the rest of the paper.

**Lemma 6.2.** For any two extensions related by a pseudo-commutative square and a commuting square as above, the commuting square is always the (co)kernel of the pseudo-commutative square, and thus a good square. This holds when the vertical maps are either m- or e-morphisms.

**Proof.** As the top and bottom row in the diagram are kernel-cokernel pairs and the square on the right is pseudo-commutative, there exists a good square in $M$ which agrees with the left square everywhere except possibly $h_A$. However, as both squares commute and $f'$ is a monomorphism, the remaining map in the good square must indeed be $h_A$, so the square agrees with the good square kernel of the right square. The argument for e-morphisms is entirely dual. □

**Lemma 6.3.** $E(A,C,B)$ is a $\star$-CGW category, with the structure inherited from $S_2(C)$ of Proposition B.2. Furthermore if $C$ is ECGW, then pointwise acyclic objects give $E(A,C,B)$ an ECGW structure.

**Proof.** When $A = B = C$, we have that $E(C,C,C) = S_2(C)$ and the result is shown in Proposition B.2. It is then straightforward to check that $E(A,C,B) \subseteq E(C,C,C)$ is an FCGW($A$) subcategory by Lemma 3.11, as $A,B$ are $\star$-CGW subcategories. □

In several instances, it will be useful to recognize when a certain $\star$-CGW category is equivalent (in the sense of Definition 1.10) to an extension category. We study this in the following lemma.

**Lemma 6.4.** Let $A,B \subseteq C$ be full $\star$-CGW subcategories of an ECGW category $C$ with inclusion functors $i_A, i_B$. $C$ is equivalent to $E(A,C,B)$ if we have the following:

- ECGW functors $F:C \rightarrow A$, $G:C \rightarrow B$,
- an m-natural transformation $\phi: i_AF \Rightarrow 1_C$,
- an e-natural transformation $\psi: i_BG \Rightarrow 1_C$,
- for each object $C$ in $C$, $FC \overset{\phi_C}{\rightarrow} C \overset{\psi_C}{\leftarrow} GC$ is a kernel-cokernel pair,
- every extension in $C$ is isomorphic to one of the above form

**Proof.** The data above, excluding the last property, determine a ECGW functor $C \rightarrow E(A,C,B)$ left inverse to the forgetful functor $E(A,C,B) \rightarrow C$ so long as $\phi, \psi$ are good natural transformations, which is automatic by Lemma 6.2.

It remains then to show that this functor is an equivalence by checking the conditions of Proposition 1.11. Essential surjectivity holds by our last assumption. Fullness and faithfulness for m-morphisms follows from Lemma 2.18 and the analogous uniqueness of pullback squares, as any m-morphism in $E(A,C,B)$ as above is uniquely determined by its source and target extensions and the map $f$. The same properties follow dually for e-morphisms and similarly for pseudo-commutative squares, which are uniquely determined by their boundaries. □
Corollary 6.5. Under the conditions of Lemma 6.4, we have a homotopy equivalence
\[ wS_\bullet C \simeq wS_\bullet E(A, C, B). \]

We now return to the goal of this section: to prove the Additivity Theorem stated below.

Theorem 6.6 (Additivity). Let \( \mathcal{C} \) be an ECGW category. Then, the map
\[ wS_\bullet E(\mathcal{C}, \mathcal{C}, \mathcal{C}) \longrightarrow wS_\bullet \mathcal{C} \times wS_\bullet \mathcal{C} \]
induced by
\[ (A \rightarrow C \leftarrow B) \mapsto (A, B) \]
is a homotopy equivalence.

The proof of Additivity proceeds in a manner almost identical to McCarthy’s [McC93]. Just as in [Wal85, Theorem 1.4.2], the first step is to reduce the proof of Additivity to the case when the equivalences considered are isomorphisms. In the classical case, this is done by showing that the bisimplicial set \( (m, n) \mapsto s_m \mathcal{C}(m, w) \) is equivalent to the bisimplicial set \( (m, n) \mapsto w_m S_n C \), or, in other words, that staircases of sequences of weak equivalences in \( \mathcal{C} \) are the same as sequences of weak equivalences of staircases in \( \mathcal{C} \). We now introduce the double categorical version of this statement.

Definition 6.7. Let \( (\mathcal{C}, W) \) be an ECGW category, and let \( D \) denote the free double category on an \( l \times m \) grid of squares. The double category of w-grids \( w_{l,m} \mathcal{C} \) is the full double subcategory of \( \mathcal{C} D \) consisting of the grids whose morphisms are all weak equivalences.

Proposition 6.8. Let \( (\mathcal{C}, W) \) be an ECGW category. Then \( w_{l,m} \mathcal{C} \) is a \( \star \)-CGW category with structure inherited from that of \( \mathcal{C} D \) in Theorem B.1. Moreover, if \( V \) a refinement of \( W \), then the double subcategory of grids in \( V \) forms an acyclicity structure on \( w_{l,m} \mathcal{C} \).

We defer the proof of this proposition to Proposition B.3. With this structure in hand, we can see the following.

Lemma 6.9. There is an isomorphism of simplicial sets
\[ s_\bullet w_{l,m} \mathcal{C} \cong w_{l,m} S_\bullet \mathcal{C}, \]
simplicial in both \( l \) and \( m \). More generally, for any refinement \( V \subseteq W \),
\[ vS_\bullet w_{l,m} \mathcal{C} \cong v w_{l,m} S_\bullet \mathcal{C}. \]

Proof. This follows immediately from the definitions, and it amounts to saying that staircases of w-grids in \( \mathcal{C} \) are the same as w-grids of staircases in \( \mathcal{C} \). \( \square \)

Like in the classical case, this allows us to show that weak equivalences are not an integral part of the Additivity Theorem.

Proposition 6.10. If the map
\[ s_\bullet E(A, A, A) \longrightarrow s_\bullet A \times s_\bullet A \]
is a homotopy equivalence for every \( \star \)-CGW category \( A \), then the map
\[ wS_\bullet E(\mathcal{C}, \mathcal{C}, \mathcal{C}) \longrightarrow wS_\bullet \mathcal{C} \times wS_\bullet \mathcal{C} \]
is a homotopy equivalence for every ECGW category \( (\mathcal{C}, W) \).
Proof. Let \((C, W)\) be an ECGW category, and consider the \(\ast\)-CGW category of w-grids \(w_{l,m}C\) of Proposition 6.8. Note that for each \(l, m, n\), we have by Lemma 6.9 an isomorphism
\[
s_n w_{l,m}C \cong w_{l,m}S_n C.
\]
Moreover, there is a homotopy equivalence
\[
s_\ast w_{l,m}E(C, C, C) \cong s_\ast E(w_{l,m}C, w_{l,m}C, w_{l,m}C)
\]
for each \(l, m\) arising via Lemma 1.13 from the evident equivalence of double categories. Applying the assumption of the lemma to each \(A = w_{l,m}C\) gives homotopy equivalences of simplicial sets
\[
s_\ast w_{l,m}S \ast E(C, C, C) \cong s_\ast E(w_{l,m}C, w_{l,m}C, w_{l,m}C)
\]
which assemble into a levelwise homotopy equivalence of trisimplicial sets, and thus a homotopy equivalence
\[
wS_\ast E(C, C, C) \cong wS_\ast C \times wS_\ast C.
\]

We are now ready to prove Additivity. Our proof is nearly identical to [Cam19, Section 4], which in turn follows McCarthy [McC93]; we outline the details in the proof that require some attention when translated to our setting.

Proof of Theorem 6.6. By Proposition 6.10, it suffices to show that Additivity holds for \(\ast\)-CGW categories (with isomorphisms as weak equivalences). Note that all definitions and results up to (and including) [Cam19, Proposition 4.13] can be readily adapted to our setting. Using McCarthy’s notation, it remains to show that the map of simplicial sets
\[
\Gamma_n : S.F|C^2(\ast, n) \longrightarrow S.F|C^2(\ast, n)
\]
is homotopic to the identity, where \(F : E(C, C, C) \longrightarrow C \times C\) denotes the additivity functor\(^6\).

This is achieved by defining a simplicial homotopy \(h\) as follows: for each \(m\), and each \(0 \leq i \leq m\), the map
\[
h_i : S.F|C^2(m, n) \longrightarrow S.F|C^2(m + 1, n)
\]
takes a generic element \(e \in S.F|C^2(m, n)\) of the form
\[
\emptyset = A_0 \gg A_1 \gg \ldots \gg A_m
\]
\[
\emptyset = C_0 \gg C_1 \gg \ldots \gg C_m
\]
\[
\emptyset = B_0 \gg B_1 \gg \ldots \gg B_m
\]
\[
\emptyset = A_0 \gg A_1 \gg \ldots \gg A_m \gg S_0 \gg S_1 \gg \ldots \gg S_n
\]
\[
\emptyset = B_0 \gg B_1 \gg \ldots \gg B_m \gg T_0 \gg T_1 \gg \ldots \gg T_n
\]
to the element \(h_i(e) \in S.F|C^2(m + 1, n)\) given by

\(^6\)The reader should keep in mind that McCarthy’s use of the term \(S.F\) is simply notation and does not denote an \(S_\ast\)-construction.
$\emptyset = A_0 \rightrightarrows A_1 \rightrightarrows \ldots \rightrightarrows A_i \rightrightarrows S_0 \rightrightarrows \ldots \rightrightarrows S_0$

\[\begin{array}{ccc}
\emptyset &=& C_0 \rightrightarrows C_1 \rightrightarrows \ldots \rightrightarrows C_i \rightrightarrows C_i \star A_i \rightrightarrows S_0 \rightrightarrows \ldots \rightrightarrows C_m \star A_m \rightrightarrows S_0 \\
\emptyset &=& B_0 \rightrightarrows B_1 \rightrightarrows \ldots \rightrightarrows B_i \rightrightarrows B_i \rightrightarrows S_0 \rightrightarrows S_0 \rightrightarrows S_1 \rightrightarrows \ldots \rightrightarrows S_n \\
\emptyset &=& B_0 \rightrightarrows B_1 \rightrightarrows \ldots \rightrightarrows B_i \rightrightarrows B_i \rightrightarrows B_i \rightrightarrows \ldots \rightrightarrows B_m \rightrightarrows T_0 \rightrightarrows T_1 \rightrightarrows \ldots \rightrightarrows T_m
\end{array}\]

where the maps and squares between $\star$-pushouts are given by Proposition A.3. Just as in [Cam19], one can check that $h$ defines a simplicial homotopy from $\Gamma_n$ to id.

Even though they are not pictured in the above diagrams, we must make choices of staircases, and verify that the maps pictured above truly give kernel-cokernel pairs in $C$. Details of how to construct these staircases using the properties of $\star$-pushouts from Appendix A can be found in [?]. □

It will also be useful to have this more general version of the Additivity Theorem at hand.

**Theorem 6.11.** Let $A, B \subseteq C$ be full $\star$-CGW subcategories of an ECGW category $(C, \mathcal{W})$. Then, the map

$$wS_\bullet E(A, C, B) \longrightarrow wS_\bullet A \times wS_\bullet B$$

induced by

$$(A \rightrightarrows C \rightrightarrows B) \mapsto (A, B)$$

is a homotopy equivalence.

**Proof.** The proof is identical to the relevant part of [Wal85, Proposition 1.3.2], since by Remark 3.3 our $\star$-CGW categories always admit trivial extensions of the form

$$A \rightrightarrows A \star \emptyset B \rightrightarrows B$$

□

7. Delooping

In this section, we show that for any ECGW category $(C, \mathcal{W})$, $K(C, \mathcal{W})$ is a spectrum. This is done by defining a notion of relative $K$-theory and following the same outline as in [Wal85, Section 1.5]; we include the proofs here for completeness.

**Definition 7.1.** Let $F: A \longrightarrow B$ be an ECGW functor between ECGW categories. For each $n$, we define the double category $S_n(F)$ as the pullback

$$
\begin{CD}
S_n(F) \longrightarrow & S_{n+1}B \\
@VVV \quad \downarrow d_0 \quad \downarrow \quad \downarrow \quad \downarrow \\
S_nA \longrightarrow & S_nB
\end{CD}
$$

$S_n(F)$ is then the double category of staircases in $S_{n+1}B$ which are equipped with a lift of all but the top row to $S_nA$ along $F$.

**Lemma 7.2.** $S_\bullet(F)$ is a simplicial ECGW category.
Proof. The fact that each $S_n(F)$ is an ECGW category follows directly from the ECGW structures on $S_{n+1}B$ and $S_nA$ given by Proposition 5.3. The face and degeneracy maps are given by shifting those of $S_nB$; that is, $d^*_i(F) := d^*_{i+1}$, and $s^*_i(F) := s^*_{i+1}$.

Just as in [Wei13, Chapter IV, 8.5.4], we have the following.

Lemma 7.3. If $A = B$, $wS\cdot S\cdot (id_B)$ is contractible.

Proof. Note that in this case, $S_n(id_B)$ is defined via the pullback $S_n(id_B)$, and thus $S_n(id_B) \cong S_{n+1}B$; in other words, $S\cdot(id_B)$ is the simplicial path space of $S\cdot B$. Similarly, one can see that for each $n$, $wS_nS\cdot(id_B)$ is the simplicial path space of $wS_nS\cdot B$. Then, we have a homotopy equivalence $wS_nS\cdot(id_B) \simeq wS_nS_0B \simeq *$ for each $n$, from which we conclude our result.

Proposition 7.4. Let $F: A \to B$ be an ECGW functor. Then, we have a homotopy fiber sequence

$$wS\cdot B \longrightarrow wS\cdot S\cdot (id_B) \longrightarrow wS\cdot S\cdot A$$

Proof. First, we have a homotopy equivalence $wS\cdot S\cdot n(F) \simeq wS\cdot E(B, S\cdot n(F), S\cdot nA)$, as the conditions in Corollary 6.5 are easily checked. Then, by the Additivity Theorem 6.11, we have a homotopy equivalence

$$wS\cdot S\cdot n(F) \simeq wS\cdot B \times wS\cdot S\cdot nA$$

for each $n$, from which we deduce the existence of the homotopy fiber sequence in the statement.

We can finally deduce the main result in this section.

Theorem 7.5. Let $(C, W)$ be an ECGW category. Then, $K(C, W) = \Omega|wS\cdot C|$ is an infinite loop space.

Proof. Using Proposition 7.4 for $A = B = C$ yields a homotopy fiber sequence

$$wS\cdot C \longrightarrow wS\cdot S\cdot (id_C) \longrightarrow wS\cdot S\cdot C$$

But $wS\cdot S\cdot (id_C)$ is contractible by Lemma 7.3, and so we conclude that there exists a homotopy equivalence $|wS\cdot C| \simeq \Omega|wS\cdot S\cdot C|$. Iterating this process yields the desired delooping

$$|wS\cdot C| \simeq \Omega|wS\cdot S\cdot C| \simeq \Omega\Omega|wS\cdot S\cdot S\cdot C| \simeq \cdots \simeq \Omega^n|wS\cdot^{n+1}C| \simeq \cdots$$

\[\square\]
8. Fibration Theorem

This section is dedicated to our primary tool for comparing ECGW categories: the analogue of Waldhausen’s Fibration Theorem, which relates the $K$-theory spectra of a $\ast$-CGW category equipped with two comparable classes of weak equivalences. The statement is as follows.

**Theorem 8.1** (Fibration). Let $\mathcal{V}$ and $\mathcal{W}$ be two acyclicity structures on a $\ast$-CGW category $\mathcal{C}$, such that $\mathcal{V} \subseteq \mathcal{W}$. Then, there exists a homotopy fiber sequence

$$K(\mathcal{W}, \mathcal{V}) \rightarrow K(\mathcal{C}, \mathcal{V}) \rightarrow K(\mathcal{C}, \mathcal{W})$$

Our proof largely follows that of Waldhausen, but avoids the rather burdensome assumptions that go into proving that the category of weak equivalences is homotopy equivalent to that of trivial cofibrations. Indeed, the reader might have noticed we do not require any additional conditions on our structures in order for our Fibration Theorem to hold. In contrast, the classical version due to Waldhausen (see [Wal85, Theorem 1.6.4]) asks for the saturation and extension axioms, and for the existence of a cylinder functor satisfying the cylinder axiom. Even the more relaxed version of Waldhausen’s Fibration due to Schlichting (see [Sch06, Theorem A.3]) only goes as far as replacing cylinders by factorizations: every map must factor as a cofibration followed by a weak equivalence.

The reason behind this apparent clash is that our ECGW categories were, in a way, constructed so that all of these properties are already incorporated. Namely, the saturation axiom (in our case, the fact that m- and e-equivalences satisfy 2-out-of-3) is an easy consequence of the definition of m- and e-equivalences, as seen in Lemma 4.11. Similarly, the extension axiom is required in the classical setting in order to prove that trivial cofibrations can be characterized by having acyclic cokernels; this is precisely how all our m-equivalences are defined in Definition 4.3.

As for the absence of a cylinder or factorization requirement, the reason is that all of the maps that our constructions see are already “simple enough” and do not need to be decomposed any further; this is a feature of the double-categorical approach. Concretely, this amounts to considering only admissible monomorphisms and epimorphisms in an exact category as opposed to working with arbitrary morphisms.

As a consequence, our proof departs from Waldhausen’s in that it does not need to go through the subcategory of trivial cofibrations, which he denotes $\mathfrak{wS}C$. Instead, we rely on the following result, which exploits the symmetry of our setting, where vertical maps have equally convenient properties to horizontal ones.

**Proposition 8.2.** For any refinement $(\mathcal{C}, \mathcal{V})$ of $(\mathcal{C}, \mathcal{W})$ and any $l, m$, we have homotopy equivalences of simplicial double categories

$$vS_{w_{l,m}C} \simeq vS_{w_{0,m}C} \times vS_{w_{l-1,m}W}$$

and

$$vS_{w_{l,m}C} \simeq vS_{w_{l,0}C} \times vS_{w_{l,m-1}W}$$

*Proof.* We prove the first statement; the second is entirely dual. The strategy will be to show that $w_{l,m}C$ is equivalent (in the sense of Lemma 6.4) to the extension ECGW category $E(w_{l-1,m}W, w_{l,m}C, w_{0,m}C)$; then, we deduce the desired statement from Corollary 6.5 and the Additivity Theorem 6.11.

For this, consider an object $A$ in $w_{l,m}C$ pictured below left, and associate to it the object in $E(w_{l-1,m}W, w_{l,m}C, w_{0,m}C)$ pictured below right (where $l, m$ are pictured as 2 and 1 respectively for convenience). We henceforth abuse notation and identify $w_{0,m}C$ with its image under the
inclusion \( w_{0,m}C \hookrightarrow w_{l,m}C \), and similarly for \( w_{l-1,m}W \).

First of all, we check that the diagram above right truly is an object of \( E(\omega_{l-1,m}W, w_{l,m}C, w_{0,m}C) \). Indeed, all of the squares are either good or pseudo-commutative, it is clearly a kernel-cokernel pair since these are constructed pointwise, and the grid on the right is an element of \( w_{0,m}C \). Lastly, the grid on the left is comprised of objects in \( W \) since they are all kernels of \( e \)-equivalences, and then the maps between them must be \( w \)- and \( e \)-equivalences by Lemma 4.10; thus, this grid is an object of \( w_{l-1,m}W \).

Now, to use Lemma 6.4, we need to define ECGW functors \( R: w_{l,m}C \rightarrow w_{0,m}C \) and \( L: w_{l,m}C \rightarrow w_{l-1,m}W \) together with an \( e \)-natural transformation \( \eta: R \Rightarrow \text{id} \) and an \( m \)-natural transformation \( \mu: R \Rightarrow \text{id} \). Let \( R \) and \( L \) respectively send an object \( A \) as above left to the right and left grids in the pictured extension, and let the components of \( \eta \) and \( \mu \) be given by the horizontally depicted \( e \)- and \( m \)-morphisms. The mixed naturality squares of \( \eta \) are given by composites of the squares in the grid \( A \), and \( \mu \) is the kernel transformation of \( \eta \).

\( R \) is evidently an ECGW functor and \( \eta \) an \( e \)-natural transformation whose component squares are good. To see that \( L \) is an ECGW functor, we must check that it preserves the remaining relevant structure. The fact that \( L \) preserves good squares is ensured by the converse in Proposition A.7, and it also preserves \( \ast \)-pushouts, since by Remark A.5 the \( \ast \)-pushout of the kernels is the kernel of the \( \ast \)-pushouts. To see that \( L \) preserves cokernels, let \( A \rightarrowtail B \) be an \( m \)-morphism in \( w_{l,m}C \) and construct the following diagram

where all columns and rows are kernel-cokernel pairs. Then, we have that \( \bullet \) must be both the kernel of \( R(B/A) \rightarrowtail B/A \) (which is by definition \( L(B/A) \)) and the cokernel of \( LA \rightarrowtail LB \) (which is \( LB/LA \)). This shows that \( L \) preserves cokernels; the proof for kernels is analogous. Lastly, \( L \) preserves acyclic objects, as \( V \) is closed under kernels.
As to the last condition of Lemma 6.4, in order to see that every object $B \Rightarrow A \Leftarrow C$ in $E(w_{l-1,m}W, w_{l,m}C, w_{0,m}C)$ is of the form $LA \Rightarrow A \Leftarrow RA$ up to isomorphism, note that as $B \in w_{l-1,m}W$, it has initial objects in the top row, and so the top components of $C \Rightarrow A$ are necessarily isomorphisms by Lemma 2.16. Hence, up to isomorphism, each row of $C$ must agree with the top row of $A$, and we get that $C \cong RA$. As $k$ preserves isomorphisms, this implies that $B \cong LA$, completing the proof. □

We can now proceed to the proof of the Fibration Theorem.

Proof. (Theorem 8.1) To obtain the desired homotopy fiber sequence on $K$-theory, it is enough to show that $vS\mathcal{W} \longrightarrow vS\mathcal{C} \longrightarrow wS\mathcal{C}$ is a homotopy fiber sequence. For this, let $vwS\mathcal{C}$ denote the simplicial quadruple category which has $w$-$m$-equivalences and $w$-$e$-equivalences in the first and second directions, $v$-$m$-equivalences and $v$-$e$-equivalences in the third and fourth directions, with pseudo-commutative or commuting squares between them as appropriate for the higher cells.

Note that we can include $vS\mathcal{C}$ into $vwS\mathcal{C}$ by considering identities in the $w$-directions. Similarly, we have an inclusion of $wS\mathcal{C}$ into $vwS\mathcal{C}$ which, as $V \subseteq W$, is furthermore a homotopy equivalence by the 2-dimensional analogue of Waldhausen’s Swallowing Lemma ([Wal85, Lemma 1.6.5]), proven easily by applying the original Lemma twice. We will abuse notation and write $vwS\mathcal{C} \longrightarrow wS\mathcal{C}$ for the homotopy inverse, which formally only exists at the level of spaces.

In order to show that the sequence pictured above is a homotopy fiber sequence, it suffices to prove that the outer rectangle below is a homotopy pullback, as each category $w_{l-1,m}S_nW$ has an initial object and so $wS\mathcal{W}$ is contractible.

\[
\begin{array}{ccc}
 vS\mathcal{W} & \longrightarrow & vS\mathcal{C} \\
 \downarrow & & \downarrow \\
 vS\mathcal{C} & \longrightarrow & wS\mathcal{C}
\end{array}
\]

Since the horizontal maps in the square above right are homotopy equivalences by the Swallowing Lemma, this is equivalent to showing that the square above left is a homotopy pullback.

Up to this point, our proof is virtually identical (albeit higher-dimensional) to [Wal85, Theorem 1.6.4]. The conclusion, however, diverges from Waldhausen’s approach and instead exploits the symmetry in our $\ast$-CGW categories.

Recall that we have homotopy equivalences

\[
v_{w_{l,m}}S\mathcal{C} \cong vS\mathcal{C} \times vS\mathcal{w_{0,m}C} \cong (vS\mathcal{w_{0,m}C}) \times (vS\mathcal{w_{l-1,m}W}) \\
\cong (vS\mathcal{w_{0,0}C} \times vS\mathcal{w_{0,m-1}W}) \times (vS\mathcal{w_{l-1,m}W})
\]

where the first equivalence (in fact, isomorphism) is due to Lemma 6.9, and the others are obtained from Proposition 8.2. Then, we have

\[
v_{w_{l,m}}S\mathcal{W} \cong vS\mathcal{W} \times vS\mathcal{w_{0,m-1}W} \times vS\mathcal{w_{l-1,m}W},
\]

and using the same reasoning for the $\ast$-CGW category $W$ in place of $\mathcal{C}$, we see that

\[
v_{w_{l,m}}S\mathcal{W} \cong vS\mathcal{W} \times vS\mathcal{w_{0,m-1}W} \times vS\mathcal{w_{l-1,m}W}.
\]
Writing $X$ for the trisimplicial double category with

$$X_{l,m} = vS_{l,m}W_0 \times vS_{l-1,m}W,$$

the argument above shows that the relevant square is homotopy equivalent to the following:

$$
\begin{array}{ccc}
  vS_0W & \rightarrow & vS_0W \times X \\
  \downarrow & & \downarrow \\
  vS_0C & \rightarrow & vS_0C \times X
\end{array}
$$

which is a homotopy pullback, as the homotopy cofibers of the horizontal maps agree. 

9. Localization Theorem

In the previous section, we saw how the Fibration Theorem 8.1 allows us to compare the $K$-theory spectra $K(C,W)$ and $K(C,V)$ of a $\ast$-CGW category $C$ with two classes of weak equivalences when $V \subseteq W$; namely, they differ by a homotopy fiber $K(W,V)$. Interestingly, as an immediate consequence of our Fibration Theorem, we obtain a Localization Theorem that allows us to compare the $K$-theory spectra of two different $\ast$-CGW categories $A \subseteq B$ by finding a homotopy cofiber. Unlike most previous localization theorems, ours requires only that $A$ is closed under kernels, cokernels, and extensions in $B$, compared to the much stronger classical Serre condition.

**Theorem 9.1 (Localization).** Let $A \subseteq B$ be a full inclusion of $\ast$-CGW categories, such that $A$ is closed under cokernels of m-morphisms, kernels of e-morphisms, and extensions in $B$. Then, there exists an ECGW category $(B,A)$ such that

$$K(A) \rightarrow K(B) \rightarrow K(B,A)$$

is a homotopy fiber sequence.

**Proof.** This is a direct application of Theorem 8.1 for $C = B$, $W = A$, $V = \emptyset$, as any full $\ast$-CGW subcategory $A \subseteq B$ which is closed under extensions forms an acyclicity structure in $B$. 

This generalizes many Localization Theorems in the literature when restricted to $\ast$-CGW categories arising from exact categories. For example, any inclusion of abelian categories $A \subseteq B$ satisfying the hypotheses of Quillen’s original Localization Theorem [Qui73, Theorem 5], or any inclusion of exact categories $A \subseteq B$ satisfying the conditions of either Schlichting’s Localization Theorem [Sch04, Theorem 2.1], Cárdenas’ Localization Theorem [Car98], or the first author’s Localization Theorem [Sar20, Theorem 6.1] will satisfy the conditions of Theorem 9.1.

Notably, Theorem 9.1 only requires that $A$ has 2-out-of-3 for short exact sequences in $B$, and thus provides a wider field for applications than the previously existing results. In particular, it can also be used to compare $K(R)$ and $G(R)$ for certain classes of rings as in [Sar20, Section 8]. However,

In each case, passing to a more general setting broadens the scope of the theorem but also reduces the tractability of the resulting cofiber. The inclusion of an exact subcategory closed under extensions, subobjects, and quotients (among other technical conditions) has an exact category as its cofiber. If it is only closed under extensions, kernels, and cokernels but also has enough injective objects, the cofiber is a Waldhausen category [Sar20]. Without enough injectives, our theorem produces an ECGW category whose weak equivalences may not satisfy the axioms of a Waldhausen category. And lastly for any full subcategory, the cofiber can be modeled by a stable $(\infty,1)$-category, as seen in [BGT13].
We believe that our localization theorem is significant in spite of the existence of a more
general result. Modeling the cofiber as an ECGW category rather than a stable \((\infty, 1)\)-category
is similar to modeling a homotopy theory using a Quillen model structure rather than, say, a
quasicategory: the presentation in terms of a strict category with weak equivalences can be very
useful in practice. Note that, by [Wal85, Corollary 1.5.7], any exact functor admits a cofiber
modeled by a simplicial Waldhausen category. Then, in a way, producing a more convenient
model for the cofiber is the entire goal of Localization theorems.

In the non-additive setting, we can compare our result to the Localization Theorem of
Campbell and Zakharevich: any inclusion of ACGW categories \( \mathcal{A} \subseteq \mathcal{B} \) satisfying the conditions
of [CZ, Theorem 8.6] will be under the hypotheses of Theorem 9.1. In this case, the ECGW
perspective of adding weak equivalences as additional structure in a cofiber \((\mathcal{B}, \mathcal{A})\), as opposed
to the ACGW perspective of strictly inverting them in a cofiber \(\mathcal{B} \backslash \mathcal{A}\), lets us avoid several of
their conditions including the often tedious process of checking that the double category \(\mathcal{B} \backslash \mathcal{A}\)
is CGW.

Part 3. Chain complexes of finite sets

10. Chain complexes

One of the main motivations for developing the theory of \(\star\)-CGW categories is to allow
for more general mathematical objects to be analyzed “algebraically” in the mold of exact
categories. A very powerful tool in the algebraic world is that of chain complexes: these
provide a convenient model one can use to do homological algebra, homotopy theory, and even
\(K\)-theory. In short, chain complexes over an exact category generalize its objects and allow for
more combinatorial manipulations, without changing its \(K\)-theory, according to the classical
Gillet–Waldhausen Theorem.

In this section, we seek to generalize this approach, and use the unifying language of \(\star\)-CGW
categories to motivate a definition of chain complexes in a new setting: the \(\star\)-CGW category
of finite sets, or more generally for any extensive category (Examples 2.10 and 3.6)). While
much of the theory of chain complexes can be imitated for general \(\star\)-CGW categories, these
chain complexes do not themselves form a \(\star\)-CGW category without introducing additional
information, particularly for the construction of \(\star\)-pushouts of spans in \(\mathcal{M}\). In future work with
Inna Zakharevich, we expect this approach to generalize to other examples of interest, such as
varieties.

We begin by recalling the usual definition of a chain complex on an abelian category, cast in
the light of \(\star\)-CGW categories.

**Definition 10.1.** Let \(\mathcal{A}\) be an abelian category, considered as a \(\star\)-CGW category in the standard way. A **chain complex over the abelian category** \(\mathcal{A}\) is a diagram in \(\mathcal{A}\) of the form

\[
\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow X_{i-1} \longrightarrow \cdots
\]

where \(i\) ranges over the integers, satisfying the **chain condition**: for each \(i\), the following is a
pseudo-commutative square.

\[
\begin{array}{c}
\emptyset \longrightarrow X_i \\
\downarrow \hspace{1cm} \downarrow \\
X_{i+1} \longrightarrow X_i
\end{array}
\]
A monomorphism (resp. epimorphism) of chain complexes is a collection \( \{ f_i, \overline{f_i} \} \) of monomorphisms (resp. epimorphisms) in \( A \) that form commutative diagrams

\[
\begin{array}{ccc}
X_i \leftarrow \overline{X}_i \rightarrow X_{i-1} & & X_i \leftarrow \overline{X}_i \rightarrow X_{i-1} \\
\scriptstyle f_i & \scriptstyle \circ & \scriptstyle \overline{f}_i & \scriptstyle \circ & \scriptstyle f_{i-1} & \scriptstyle \circ & \scriptstyle \overline{f}_{i-1} \\
Y_i \leftarrow \overline{Y}_i \rightarrow Y_{i-1} & & Y_i \leftarrow \overline{Y}_i \rightarrow Y_{i-1}
\end{array}
\]

Note that this notion of chain complex agrees with the classical one. Here, a differential \( X_i \leftarrow \overline{X}_i \rightarrow X_{i-1} \) is simply the epi-mono factorization of a general map \( d_i : X_{i+1} \rightarrow X_i \), and we have \( \overline{X}_{i+1} = \text{im}d_i \). Furthermore, given a diagram \( X_{i+1} \rightarrow X_i \leftarrow \overline{X}_i \), we can complete it to a pseudo-commutative square as done in Lemma 2.18. In this case, the pseudo-commutative completion always exists since abelian categories have pullbacks of monomorphisms and epimorphisms, and the process yields the epi-mono factorization of the composite \( \overline{X}_{i+1} \leftarrow X_i \rightarrow \overline{X}_i \) in the abelian category. Then, the chain condition says that this composite must factor through the zero object, which is equivalent to the classical condition on differentials \( d^2 = 0 \). We henceforth refer to these pseudo-commutative completions as “mixed pullbacks”, following the convention in [CZ]. As for the morphisms, recall that pseudo-commutative squares are the commutative squares, and so the maps \( \overline{f}_i \) simply denote the induced maps on the images of the differentials.

Since the \( \star \)-CGW category of finite sets (Examples 2.8 and 3.6) also has all mixed pullbacks, we could easily use the above definition to obtain a notion of chain complex of sets. These admit a simple notion of homology where \( H_i \) is defined as the total complement in \( X_i \) of the pair of injections \( \overline{X}_{i+1} \rightarrow X_i \leftarrow \overline{X}_i \); that is, \( H_i = X_i \setminus (\overline{X}_i \cup \overline{X}_{i+1}) \). Moreover, we recover classical results from homological algebra, such as the Snake Lemma and the long exact sequence in homology.

However, these chain complexes of sets do not form a \( \star \)-CGW category, as they fail to have the necessary \( \star \)-pushouts. The reason for this obstruction is that, even though any span of inclusions between finite sets admits a pushout, a natural transformation between two such spans induces a function between their pushouts which is not in general an inclusion, even if the transformation is a pointwise inclusion.

In order to remedy this, we relax the m-morphisms in our differentials to instead include all functions of sets in that direction, which we denote by \( \overrightarrow{} \). In doing so, we find that this discussion applies not just to sets, but in fact to any extensive category, as introduced in Definition 2.9. Hence, the rest of our work deals with a construction of chain complexes over any extensive category and their \( K \)-theoretical features, which the reader may easily specialize to finite sets if desired.

**Definition 10.2.** For \( X \) an extensive category, a chain complex over \( X \) is a diagram in \( X \) of the form

\[
\cdots X_{i+1} \leftarrow \overline{X}_{i+1} \rightarrow X_i \leftarrow \overline{X}_i \rightarrow X_{i-1} \cdots
\]

where \( i \) ranges over the integers, satisfying the chain condition: for each \( i \), the following is a pseudo-commutative square.

\[
\begin{array}{ccc}
\emptyset & \rightarrow & \overline{X}_i \\
\downarrow & \circ & \downarrow \\
\overline{X}_{i+1} & \rightarrow & X_i
\end{array}
\]
The objects \( \{X_i\} \) are called the **degrees** of \( X \), \( \{\overline{X}_i\} \) are called the **images** of \( X \), and each \( X_i \leftrightarrow \overline{X}_i \rightarrow X_{i-1} \) is called a **differential** of \( X \). As we will show, it is these complexes which form a \( \ast \)-CGW category satisfying our version of the Gillet–Waldhausen Theorem.

**Remark 10.3.** Since we departed from the FCGW translation of the classical (abelian) chain complexes, a word must be said in order to justify the legitimacy of our choice of differentials.

In our motivating example of finite sets, the differentials in our chain complexes agree with partial functions, where \( X_i \leftrightarrow \overline{X}_i \) represents the inclusion of the domain into \( X_i \). Then, our chain condition amounts to the fact that the composite of two consecutive partial function differentials must be the partial function with empty domain. Moreover, note that partial functions from \( X_i \) to \( X_{i-1} \) are in bijective correspondence with basepoint-preserving functions \((X_i)_* \rightarrow (X_{i-1})_*\), obtained by sending every element outside the domain in \( X_i \) to the basepoint. From this perspective, our choice of differentials is no longer arbitrary since this type of maps is particularly relevant: a model for the \( K \)-theory of finite sets is given by the Waldhausen category of pointed sets and basepoint preserving maps, with isomorphisms as the weak equivalences.

However, the intuition behind this correspondence does not extend to the \( m \)- and \( e \)-morphisms of chain complexes defined below, which are essential to the construction of the \( K \)-theory space and more easily expressed in the language of unbased finite sets.

The remainder of this subsection is devoted to the construction of the \( \ast \)-CGW structure.

**Definition 10.4.** An \( m \)-morphism \( f \) of chain complexes over \( \mathcal{X} \), or **chain \( m \)-morphism**, is a collection \( \{f_i, \overline{f}_i\} \) of \( m \)-morphisms in \( \mathcal{X} \) that form diagrams as below left, where the square in \( \mathcal{X} \) commutes.

\[
\begin{array}{ccc}
X_i & \longrightarrow & X_{i-1} \\
\downarrow f_i & & \downarrow g_i \\
\overline{X}_i & \longrightarrow & \overline{X}_{i-1}
\end{array}
\]

Similarly, a **chain \( e \)-morphism** is a collection \( \{g_i, \overline{g}_i\} \) of \( e \)-morphisms in \( \mathcal{X} \) that form diagrams as above right, where the square in \( \mathcal{E} \) commutes.

A **pseudo-commutative square** between such morphisms is a levelwise pseudo-commutative square, meaning a pseudo-commutative square at each degree and each image, which commutes with all the squares in the surrounding \( m \)- and \( e \)-morphisms.

Similarly, a **good square** of chain \( m \)-morphisms (resp. \( e \)-morphisms) is a levelwise good commuting square of chain \( m \)-morphisms (resp. \( e \)-morphisms).

**Example 10.5.** For any chain complex \( X \), there are unique chain \( m \)- and \( e \)-morphisms from the constant complex at \( \emptyset \):

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & \emptyset \\
\downarrow & & \downarrow \\
X_i & \longrightarrow & X_{i-1}
\end{array}
\]

Before discussing chain complexes in more detail, we prove several basic results that will be useful for extending the \( \ast \)-CGW structure of \( \mathcal{X} \) to include arbitrary maps in the \( m \)-direction, and verifying the chain condition more easily.

**Lemma 10.6.** Let \( \text{Ar}_g \mathcal{X} \) denote the category with objects \( m \)-morphisms and morphisms pullback squares between them in \( \mathcal{X} \), and \( \text{Ar}_g \mathcal{E} \) denote the category with objects \( e \)-morphisms and morphisms pullback squares between them in \( \mathcal{X} \). Then, the following hold:
• $\mathcal{M}$ (resp. $\mathcal{E}$) is closed under base change in $\mathcal{X}$; that is, the pullback of a span

$$B \longleftarrow A \hookrightarrow C \quad \text{(resp. } B \twoheadrightarrow A \hookrightarrow C)$$

exists and we get $B \times_A C \twoheadrightarrow C$ (resp. $B \times_A C \rightarrow C$),

• $k: \text{Ar}_\mathcal{E} \longrightarrow \text{Ar}_\mathcal{M} \text{ extends to an equivalence } \text{Ar}_\mathcal{E} \longrightarrow \text{Ar}_\mathcal{X}$ between squares as below

\begin{align*}
A & \longrightarrow B \\
\downarrow & \downarrow \\
C & \longrightarrow D
\end{align*}

\begin{align*}
C & \longrightarrow D \\
\uparrow & \uparrow \\
E & \longrightarrow F
\end{align*}

• any cospan as below can be completed to a unique mixed pullback as below right

\begin{align*}
B & \longrightarrow A \\
\downarrow & \downarrow \\
C & \longrightarrow D
\end{align*}

\begin{align*}
A & \longrightarrow B \\
\downarrow & \downarrow \\
C & \longrightarrow D
\end{align*}

• If morphisms $f$ and $fg$ in $\mathcal{X}$ are coproduct inclusions, then so is $g$

$\textbf{Proof.}$ The first three claims are immediate from Definition 2.9. For the last claim, since $f$ is monic, the following square is a pullback in $\mathcal{X}$, which completes the proof as coproduct inclusions are closed under pullbacks in $\mathcal{X}$.

\begin{align*}
\begin{array}{c}
g \\
\downarrow \\
f \downarrow \\
g \downarrow
\end{array}
\end{align*}

$\square$

$\textbf{Lemma 10.7.}$ Given a cospan $B \longrightarrow f A \longrightarrow g C$, its mixed pullback has $\emptyset$ in the remaining corner if and only if $f$ factors through the kernel of $g$.

$\textbf{Proof.}$ We can find the mixed pullback by taking the pullback of $f$ and $k(g)$ in $\mathcal{X}$, and applying $k^{-1}$.

\begin{align*}
\begin{array}{c}
B/P \longrightarrow C \\
\downarrow \\
B \longrightarrow A \\
\downarrow \\
P \longrightarrow A \setminus C
\end{array}
\end{align*}

Then, $B/P = \emptyset$ if and only if the map $P \rightarrow B$ is an isomorphism by Lemma 2.16, which happens if and only if $f$ is the composite $B \cong P \rightarrow A \setminus C \twoheadrightarrow A$. $\square$
Proposition 10.8. Let \( Y \) be a chain complex, and \( X \) be a diagram containing the data of a chain complex, possibly without the chain condition. If we have either the data of a chain \( m \)-morphism \( X \to Y \) or that of a chain \( e \)-morphism \( X \leftarrow Y \), then \( X \) must satisfy the chain condition.

Proof. Assume we have the data of a chain \( m \)-morphism \( f: X \to Y \) as pictured below left.

Applying \( k \) to the given pseudo-commutative square, we get an induced good square on kernels as pictured above right. Since \( Y \) is a chain complex, it satisfies the chain condition, and so Lemma 10.7 ensures that \( m' \) factors through \( \ker e' \). Finally, since good squares are pullbacks, we get an induced map \( m: X \to \ker e' \) such that \( m_{i+1} = k(e_i)m \) and using Lemma 10.7 again we conclude that \( X \) satisfies the chain condition.

Now assume instead that we have the data of a chain \( e \)-morphism \( g: X \leftarrow Y \) as below left.

We describe the steps that need to be taken to construct the diagram above right. First, take the mixed pullback of \( g \) and \( k(e'_i) \) to produce \( \bullet \) and the pseudo-commutative square on the right. Taking the mixed pullback of the new map \( \bullet \to \ker e'_i \) and of the map \( Y \to \ker e' \) from Lemma 10.7, we produce the left pseudo-commutative square whose new \( e \)-morphism must agree with \( X \to Y \) by the uniqueness of pseudo-commutative squares of Lemma 10.6, since the original square involving the vertices \( X, X, Y, Y \) is pseudo-commutative.

Now apply \( c \) to the first pseudo-commutative square we constructed, to produce the good square on its right. Since good squares are pullbacks, we get an induced map \( \overline{X} \to \ker h \). This in turn induces a map \( m: \bullet \to \ker e_i \) such that \( h = k(e_i)m \) by Lemma 2.14, which concludes the proof, as now \( m_{i+1} = hf = k(e_i)m f \) and we can apply Lemma 10.7. \( \square \)

The chain \( m \)- and \( e \)-morphisms between chain complexes, together with the pseudo-commutative and good squares of Definition 10.4, form a double category \( \mathcal{Ch}(X) = (\mathcal{M}_{\mathcal{Ch}}, \mathcal{E}_{\mathcal{Ch}}) \), which we now endow with the structure of a \( g \)-CGW category. First, we deal with isomorphisms.

Lemma 10.9. A chain \( m \)-morphism (resp. \( e \)-morphism) is an isomorphism in \( \mathcal{M}_{\mathcal{Ch}} \) (resp. \( \mathcal{E}_{\mathcal{Ch}} \)) if and only if it is a degreewise isomorphism.
Proof. An isomorphism in $\mathcal{M}_{\text{Ch}}$ necessarily consists of isomorphisms on each degree and each image, as the identity in $\mathcal{M}_{\text{Ch}}$ is given by levelwise identity $m$-morphisms. For the converse, we note that if $f_i: X_i \to Y_i$ is an isomorphism, then $\overline{f}_i: \overline{X}_i \to \overline{Y}_i$ is an isomorphism by Lemma 2.15, and the chain $m$-morphism has an inverse by the condition on pseudo-commutative squares in Definition 1.7.

We now construct the cokernel (resp. kernel) of a chain $m$-morphism (resp. $e$-morphism).

**Proposition 10.10.** Given a chain $m$-morphism $X \to Y$ (or $e$-morphism $Z \to Y$) as in the diagram below, we can construct the pictured $e$-morphism (resp. $m$-morphism) and its domain complex as the cokernel (resp. kernel) in each degree.

\[ X_i \leftarrow \overline{X}_i \rightarrow X_{i-1} \]
\[ Y_i \leftarrow \overline{Y}_i \rightarrow Y_{i-1} \]
\[ Z_i \leftarrow \overline{Z}_i \rightarrow Z_{i-1} \]

**Proof.** Given such an $m$-morphism of chain complexes, we can apply $c$ to the top left pseudo-commutative square to get the good square as pictured, with vertices $\bullet$, $Z_i$, $\overline{Y}_i$, $Y_i$. We then take the pullback of the top right commuting square, and apply $k^{-1}$ to get the pseudo-commutative square below right (thus defining $Z_i := \text{coker}(\ast \to \overline{Y}_i)$). The map from $\overline{X}_i$ to the pullback is necessarily in $\mathcal{M}$ by Lemma 10.6 and by Lemma 2.14 it induces a map from $\overline{Z}_i$ to the bottom right pullback, completing the construction of the differential in $Z$. That $Z$ satisfies the chain condition follows immediately from Proposition 10.8.

The converse construction is entirely dual.

These constructions define the kernel and cokernel of chain $e$- and $m$-morphisms (respectively), which are inverse to one another by construction (up to isomorphism). We further show that these correspondences extend to functors $k$ and $c$ by defining their action on maps.

**Lemma 10.11.** The kernel and cokernel constructions defined above above extend to equivalences of categories

\[ k: \text{Ar}_{\odot} \mathcal{E}_{\text{Ch}} \longrightarrow \text{Ar}_{\odot} \mathcal{M}_{\text{Ch}} \quad c: \text{Ar}_{\odot} \mathcal{M}_{\text{Ch}} \longrightarrow \text{Ar}_{\odot} \mathcal{E}_{\text{Ch}} \]

**Proof.** Start with a morphism in $\text{Ar}_{\odot} \mathcal{E}_{\text{Ch}}$, that is, a pseudo-commutative square between two chain $e$-morphisms; we show that there exists an induced chain $m$-morphism between their kernels which forms a good square.

\[ X \xleftarrow{f} Y \leftarrow \ker f \]
\[ X' \xleftarrow{f'} Y' \leftarrow \ker f' \]

To do this, consider the more detailed picture below, where the chain complexes $\ker f$ and $\ker f'$ are constructed as in Proposition 10.10.
Since pseudo-commutative squares in chain complexes are levelwise pseudo-commutative, and cokernels are constructed degreewise, we immediately get the dashed morphisms in the picture above, which form good squares in $\mathcal{X}$.

To construct the blue map, take the mixed pullback of the cospan

$$
\xymatrix{ & \ker f_i \ar[dll] \ar[d] \ar[drr] & \\
X_i \ar[r]_{f_i} & Y_i & Z_i \ar[l]^{\ker f_i}.
}$$

By Lemma 2.18, the composite of this new pseudo-commutative square with the pseudo-commutative square of vertices $Z_i$, $\ker f_i$, $Y_i$, $Y_i'$ must agree with the composite of the pseudo-commutative squares of $Z_i$, $\ker f_i$, $Y_i$, $Y_i$ and $Y_i$, $\ker f_i$, $Y_i'$, $Y_i'$. Thus, the mixed pullback we constructed must have $Z_i$ as the new vertex, and a map $Z_i \leftrightarrow Z_i'$ which is the desired blue map.

To show that the morphisms we constructed form a chain $m$-morphism and that the square of $m$-morphisms is good, we must check that the involved squares are of the correct type. By construction, the square involving $Z_i$, $Z_i'$, $\ker f_i$, $\ker f_i'$ is pseudo-commutative, and the square $Z_i$, $Z_i'$, $Y_i$, $Y_i'$ commutes; in a moment we will show that it is actually good. Note that the square $Z_i$, $Z_i'$, $\ker f_i$, $\ker f_i'$ now commutes, as it does when post-composed with the monic coker $f_i'$, $Y_i'$. Finally, the aforementioned square is good by appealing to the pullback lemma, since good squares in $\mathcal{X}$ are pullbacks.

This proves we have a functor $k: \text{Ar}_{\mathcal{E}_{\text{Ch}}} \rightarrow \text{Ar}_{\mathcal{M}_{\text{Ch}}}$, as the construction above is evidently functorial. Furthermore, this functor is faithful since the maps constructed are unique. To see that it is full, it suffices to start with a good square of chain $m$-morphisms and prove that we get an induced pseudo-commutative square after taking $c$ on objects; this proof is entirely dual to the one above, as is the statement about the functor $c$. $k$ is essentially surjective by the invertibility on objects apparent from Proposition 10.10, and thus we conclude that $k$ is an equivalence as desired. □

The above construction also reveals the following.

**Lemma 10.12.** A pseudo-commutative square of chain complexes induces an isomorphism on kernels (and cokernels) if and only if it is degreewise distinguished.

**Proof.** Since chain isomorphisms are characterized by being degreewise isomorphisms by Lemma 10.9, we obtain the desired correspondence by recalling that pseudo-commutative squares of chain
complexes are in particular degreewise pseudo-commutative in \( \mathcal{X} \), and that the induced morphisms on degrees on (co)kernels are the induced maps from these pseudo-commutative squares in \( \mathcal{X} \), as we can see in the proof of Lemma 10.11.

\[ \square \]

**Lemma 10.11.** Good squares in \( \mathcal{M}_{\text{Ch}} \) (resp. \( \mathcal{E}_{\text{Ch}} \)) are pullbacks.

*Proof.* We prove the statement for \( \mathcal{M}_{\text{Ch}} \); the one for \( \mathcal{E}_{\text{Ch}} \) is identical. Suppose that we have a good square in \( \mathcal{M}_{\text{Ch}} \), and another commutative square as depicted below; we wish to show there exists a unique chain m-morphism \( Z \to X \) making the diagram commute.

\[
\begin{array}{ccc}
Z & \to & Y \\
\downarrow & & \downarrow \\
X' & \to & Y'
\end{array}
\]

Recall that good squares of chain complexes are levelwise good by definition, and so we get induced maps \( Z_i \to X_i \) and \( Z_i \to X_i \) for every \( i \). It remains to show that these form a chain m-morphism, but this is immediate: the required squares will be pseudo-commutative by appealing to axiom (PBL) in \( \mathcal{X} \), and the remaining squares in \( \mathcal{X} \) commute, since they do when post-composed with the monics \( X_i \to X'_i \).

\[ \square \]

**Theorem 10.14.** \( \text{Ch}(\mathcal{X}) \) is a \( g \)-CGW category.

*Proof.* \( \text{Ch}(\mathcal{X}) \) has shared isomorphisms by Lemma 10.9. Good squares are pullbacks by Lemma 10.13, and they include weak triangles by definition as these are levelwise pullbacks. In addition, the functors \( k: \text{Ar} \otimes \mathcal{E}_{\text{Ch}} \to \text{Ar}_g \mathcal{M}_{\text{Ch}} \) and \( c: \text{Ar} \otimes \mathcal{M}_{\text{Ch}} \to \text{Ar}_g \mathcal{E}_{\text{Ch}} \) are equivalences by Lemma 10.11.

For the axioms, note that \( \mathcal{M}_{\text{Ch}} \) and \( \mathcal{E}_{\text{Ch}} \) have a shared initial object \( \emptyset \) by Example 10.5, and all morphisms monic by the same property of the levelwise morphisms. Axiom (D) follows from Lemma 10.12, and axiom (K) follows from the same property in each degree.

\[ \square \]

In order to upgrade this \( g \)-CGW structure to a full \( \ast \)-CGW structure, we need to construct \( \ast \)-pushouts of chain complexes. We do so in the next two results.

**Proposition 10.15.** Any span of chain m-morphisms admits a \( \ast \)-pushout, which is a levelwise \( \ast \)-pushout. Furthermore, it has a universal property with respect to good squares in \( \mathcal{M}_{\text{Ch}} \).

*Proof.* Let \( Y \leftarrow X \to Z \) be a span of chain m-morphisms, and consider the levelwise \( \ast \)-pushouts in \( \mathcal{X} \), which we denote by \( \ast_i, \pi_i \). For each \( i \), there exists a map \( \pi_i \leftarrow \ast_i \) such that the squares below are pseudo-commutative, by Proposition A.4.

\[
\begin{array}{ccc}
Y_i & \leftarrow & Y_i \\
\downarrow & \circ & \downarrow \\
\ast_i & \leftarrow & \pi_i
\end{array}
\]

Also, since \( \ast \)-pushouts in \( \mathcal{X} \) are (categorical) pushouts, there exists a map \( \pi_i \leftarrow \ast_{i-1} \) such that the squares below commute, by the universal property of \( \pi_i \).

\[
\begin{array}{ccc}
Y_i & \rightarrow & Y_{i-1} \\
\downarrow & \downarrow & \downarrow \\
\pi_i & \rightarrow & \ast_{i-1}
\end{array}
\]

\[
\begin{array}{ccc}
Z_i & \rightarrow & Z_{i-1} \\
\downarrow & \downarrow & \downarrow \\
\pi_i & \rightarrow & \ast_{i-1}
\end{array}
\]
One can check that the data above determines a chain complex $\star$, together with chain m-morphisms $Y \xrightarrow{\star} \star$ and $Z \xrightarrow{\star} \star$ that complete the span to a good square, since it is levelwise good.

For the universal property, consider a good square as below.

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\star} & W
\end{array}
$$

By construction, we get induced levelwise maps $\star \iota_i \xrightarrow{} W_i$ and $\overline{\star} \iota_i \xrightarrow{} \overline{W}_i$ that make the relevant levelwise diagrams commute. It remains to show that they assemble into a chain m-morphism; that is, that in the diagram

$$
\begin{array}{ccc}
\star \iota_i & \xrightarrow{} & \star_{i-1} \\
\downarrow & & \downarrow \\
W_i & \xrightarrow{} & W_{i-1}
\end{array}
$$

the square on the left is pseudo-commutative, and the one on the right commutes in $\mathcal{X}$. But the first assertion is the content of Corollary A.9, and the second is a consequence of the uniqueness in the universal property of the pushout for $\overline{\star} \iota_i$. Clearly the map $\star \iota W$ is unique (up to unique isomorphism), since it is constructed using the levelwise universal properties in $\mathcal{X}$. □

**Proposition 10.16.** Any span of chain e-morphisms which is part of a good square in $\mathcal{E}_{\text{Ch}}$ admits a $\star$-pushout, which is a levelwise $\star$-pushout. Furthermore, it has a universal property with respect to good squares in $\mathcal{E}_{\text{Ch}}$.

**Proof.** Consider the following good square in $\mathcal{E}_{\text{Ch}}$

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\star} & W
\end{array}
$$

If we take degree-wise $\star$-pushouts, we get induced maps $\star_i \iota_i \xrightarrow{} W_i$. For the images, let $P_i$ denote the mixed pullback of the cospan

$$
\begin{array}{ccc}
\overline{W}_i & \xrightarrow{} & W_{i-1} \\
\downarrow & & \downarrow \\
\overline{W}_i' & \xrightarrow{} & \star_{i-1}
\end{array}
$$

which comes equipped with a morphism $P_i \xrightarrow{} \star_{i-1}$. It is straightforward to check that $P_i = \overline{Y}_i \xrightarrow{\star_i} Z_i$, as by the axioms of an extensive category (Definition 2.9) both of these definitions of $P_i$ unwind to the same object. Then as $P_i$ is a pushout, we get an induced map $P_i \xrightarrow{} \star_i$. This induced map commutes with the maps $\star_i \iota_i \xrightarrow{} W_i$ and $P_i \xrightarrow{} \overline{W}_i \xrightarrow{} W_i$ by the universal property of the pushout $P_i$, which shows it is in $\mathcal{E}$ by Lemma 10.6.

It is immediate, either by construction or by applying axiom (PBL) for $\mathcal{X}$, that we get the data of chain e-morphisms $Y \xrightarrow{} \star$, $Z \xrightarrow{} \star$ and $\star \iota W$; in addition, the latter is unique by construction. Finally, we see that $\star$ is indeed a chain complex by Proposition 10.8. □

**Theorem 10.17.** $\text{Ch}(\mathcal{X})$ is a $\star$-CGW category.
Proof. By Theorem 10.14, we know that $\text{Ch}(\mathcal{X})$ forms a $g$-CGW category. We now check the axioms of Definition 3.1. Axiom (PO) holds by Proposition 10.15. Similarly, axiom ($\ast$) holds by Propositions 10.15 and 10.16, where the isomorphism on (co)kernels is consequence of the fact that $\ast$-pushouts of chain complexes are degree-wise $\ast$-pushouts in $\mathcal{X}$, together with Lemma 10.9. Finally, axioms (GS), (PBL) and (POL) follow immediately from the same properties for $\mathcal{X}$, as all structures involved are defined or constructed levelwise.

Remark 10.18. Although all of the constructions in this section have been for chain complexes indexed by the integers, it is easy to see that every result holds if one restricts to bounded chain complexes; that is, chain complexes of sets with a finite number of non-empty degrees and images; we denote this $\ast$-CGW category by $\text{Ch}(\mathcal{X})^b$. Similarly, we denote by $\text{Ch}(\mathcal{X})_{[a,b]}$ the $\ast$-CGW category of chain complexes $X$ such that $X_i = \emptyset$ for $i \not\in [a,b]$, for any $a \leq b$.

11. Exact complexes

Classically, the class of weak equivalences between chain complexes we consider are the quasi-isomorphisms. Using homological algebra methods, one can characterize the monomorphisms (resp. epimorphisms) that are quasi-isomorphisms as the ones whose cokernel (resp. kernel) are exact complexes. We now define exact chain complexes over $\mathcal{X}$ in analogy with the classical algebraic case, and show that they form a class of acyclic objects in $\text{Ch}(\mathcal{X})$, thus providing us with a notion of quasi-isomorphism in this setting.

Definition 11.1. A chain complex over $\mathcal{X}$ is exact if it is of the form

$$X_{i+1} \overset{\sim}{\longrightarrow} \overline{X}_{i+1} \overset{\sim}{\longrightarrow} X_i \overset{\sim}{\longrightarrow} \overline{X}_i \overset{\sim}{\longrightarrow} X_{i-1}$$

in the sense that all of the forward-pointing maps from $\mathcal{X}$ are $m$-morphisms, and additionally each mixed cospan $\overline{X}_{i+1} \rightarrowtail X_i \leftarrowtail \overline{X}_i$ is a kernel-cokernel pair. In other words, for all $i$ the pseudo-commutative square expressing the chain condition has the form:

$$\emptyset \overset{\sim}{\longrightarrow} \overline{X}_i \quad \square \quad X_i \overset{\sim}{\longrightarrow} \overline{X}_{i+1}$$

We write $\text{Ch}_E(\mathcal{X})$ for the full double subcategory of exact complexes in $\text{Ch}(\mathcal{X})$.

Remark 11.2. Just like with general chain complexes over $\mathcal{X}$, this definition should be compared to exact complexes in abelian categories. In fact, in this case the two are completely analogous: this definition could have been formulated for an abelian category $\mathcal{A}$ instead of $\mathcal{X}$, and it would recover the classical notion.

An exact complex then amounts to a coproduct decomposition $X_i \cong \overline{X}_{i+1} \sqcup \overline{X}_i$ for all $i$, by both restricting the maps $\overline{X}_{i+1} \rightarrowtail X_i$ to be coproduct inclusions and insisting by the exactness condition that they are complementary to $X_i \leftarrowtail \overline{X}_i$. The latter condition is also equivalent to the homology $H_i$ mentioned previously being $\emptyset$.

As expected, exact complexes form a class of acyclic objects in our chain complexes over $\mathcal{X}$.

Proposition 11.3. $(\text{Ch}(\mathcal{X}), \text{Ch}_E(\mathcal{X}))$ forms an ECGW category.

Proof. As the constant complex at $\emptyset$ is always exact, it remains only to show that exact complexes are closed under kernels, cokernels, and extensions. To see that they are closed under kernels and cokernels, consider the following kernel-cokernel pair in $\text{Ch}(\mathcal{X})$: 
If $X$ and $Y$ are exact, then the top left square is necessarily good by Lemma 6.2. By the construction of the cokernel in Proposition 10.10, this implies the leftmost column above is a kernel-cokernel sequence (which in particular means that the map $Z_{i+1} \to Z_i$ is in $\mathcal{M}$), and by the same argument with indices shifted, so is the rightmost column. This shows that the bottom left and right squares form a kernel-cokernel pair, so $Z$ is exact. The dual argument shows that kernels also preserve exact complexes, so it only remains to show that they are closed under extensions.

Consider an extension of exact complexes in $\text{Ch}(\mathcal{X})$ as depicted above; note that by exactness of $X$ and $Z$, the maps $X_{i+1} \to X_i$ and $Z_{i+1} \to Z_i$ must be m-morphisms. It follows from the definition of kernel and cokernel of chain morphisms that $Y_i \cong X_i \sqcup Z_i$ and $Y_i \cong X_i \sqcup Z_i \sqcup V_i$ for all $i$ and some objects $V_i$, where the components of the differential of $Y$ agree with those of $X$ and $Z$ on $X_i$ and $Z_i$ and the inclusions from $X$ and $Z$ are the canonical coproduct inclusions at each level. As the top right square is a pullback, the e-morphism in the differential of $Y$ restricted to $Z_i \sqcup V_i$ must factor through $Z_i$. But as $Z_i \cong Z_{i+1} \sqcup Z_i$ by exactness of $Z$, the chain condition for $Y$ and Lemma 10.7 ensure that $Z_i \sqcup V_i \to Y_i$ must further factor through $Z_i$.

We therefore have a coproduct inclusion $Z_i \sqcup V_i \to Z_i$ which restricts to the identity on $Z_i$. By the first axiom for extensive categories (Definition 2.9) and the fact that coproduct inclusions are monic, the squares below are pullbacks.

\[
\begin{array}{ccc}
\emptyset & \to & Z_i \\
\downarrow & & \downarrow \\
V_i & \to & V_i \sqcup Z_i \to Z_i
\end{array}
\]

Therefore, as pullbacks preserve isomorphisms, $V_i \cong \emptyset$. Therefore, $Y$ is isomorphic to $X \sqcup Z$ and therefore exact as $X$ and $Z$ are. □

Remark 11.4. From the first part of the proof of Proposition 11.3, we can also observe that in the special case of exact chain complexes, the (co)kernel construction of Proposition 10.10 is done by taking (co)kernels levelwise, not just degreewise.

Exact chain complexes determine classes of m- and e-equivalences which, mirroring the classical algebraic setting, we call quasi-isomorphisms. A chain map (of either type) from $X$ to $Y$ is a quasi-isomorphism when its complement is an exact complex, which for finite sets is equivalent to the map inducing an isomorphism on the homology sets and “covering” the part of the differentials $Y_{i+1} \to Y_i$ which are non-injective. In full generality, we can express the quasi-isomorphism condition as follows.
Proposition 11.5. A chain map \( f \) from \( X \) to \( Y \) (of either type) is a quasi-isomorphism if and only if the commuting squares in \( X \) below are bicartesian.

\[
\begin{array}{ccc}
X_{i+1} \sqcup X_i & \longrightarrow & X_i \\
\downarrow & & \downarrow \\
Y_{i+1} \sqcup Y_i & \longrightarrow & Y_i
\end{array}
\]

Proof. Let \( Z := Y \setminus X \) with respect to \( f \). It is tedious but straightforward to show that the square above is a pullback if and only if the induced map on complements of the vertical arrows is \( \overline{Z}_{i+1} \sqcup \overline{Z}_i \longrightarrow Z_i \), which holds when \( Z \) is exact. Furthermore, \( Z \) is exact precisely when these maps are isomorphisms, which is the case if and only if the square above is also a pushout. \( \square \)

Indeed, in the case of finite sets, this square being bicartesian expresses that the induced map on complements in the horizontal direction (the homology sets) is an isomorphism, and that all of the non-injective parts of \( Y_{i+1} \) can also be found in the map \( X_{i+1} \longrightarrow X_i \).

12. Gillet–Waldhausen Theorem

The aim of this final section is to prove a version of the Gillet–Waldhausen Theorem; this will show that our new notion of chain complexes over an extensive category \( \mathcal{X} \) with quasi-isomorphisms provides an alternate model for the \( K \)-theory of \( \mathcal{X} \) (see Proposition 5.12). In particular, when \( \mathcal{X} = \text{FinSet} \) this provides a new model for the classical \( K \)-theory of finite sets, which agrees with the sphere spectrum.

Our proof of the Gillet–Waldhausen Theorem follows the same outline as the classical proof in [TT90, Theorem 1.11.7]; nevertheless, we include it here, adapted to our setting. We first show two lemmas that will be crucial for the proof of the theorem. In both lemmas, whenever we allude to the \( K \)-theory of a category of chain complexes, we do so by considering chain complexes as a \( \ast \)-CGW category (with isomorphisms).

Lemma 12.1. The \( \ast \)-CGW functor

\[
\text{Ch}(\mathcal{X})_{[a,b]} \longrightarrow \prod_{b-a+1} \mathcal{X}
\]

sending a chain complex \( X \) to the tuple \((X_{b-1}, X_{b-2}, \ldots, X_a, X_b)\) induces a homotopy equivalence on \( K \)-theory.

Proof. First of all, note that this correspondence (the projection of a chain complex to its degrees) is indeed a \( \ast \)-CGW functor, as all the structure on chain complexes is defined degreewise.

The proof then proceeds by induction on \( b-a \). If \( b = a \), the assertion is trivial since the two \( \ast \)-CGW categories in question are the same. For the inductive step, it suffices to show that the \( \ast \)-CGW functor

\[
\text{Ch}(\mathcal{X})_{[a,b]} \longrightarrow \text{Ch}(\mathcal{X})_{[a,b-1]} \times \mathcal{X}
\]

sending a chain complex \( X \) to the pair

\[
(X_{b-1} \longrightarrow X_{b-2} \longrightarrow \ldots \longrightarrow X_a, X_b)
\]

induces a homotopy equivalence on \( K \)-theory.

By the Additivity Theorem 6.11, we have a homotopy equivalence

\[
K(E(\text{Ch}(\mathcal{X})_{[a,b-1]}, \text{Ch}(\mathcal{X})_{[a,b]}; \mathcal{X})) \simeq K(\text{Ch}(\mathcal{X})_{[a,b-1]}) \times K(\mathcal{X}).
\]
On the other hand, we can consider the $\star$-CGW functors

$$F: \text{Ch}(\mathcal{X})_{[a,b]} \to \text{Ch}(\mathcal{X})_{[a,b-1]}, \quad G: \text{Ch}(\mathcal{X})_{[a,b]} \to \mathcal{X}$$

that truncate a chain complex, where $F$ removes $X_b$ and $G$ removes everything except for $X_b$. Clearly these satisfy the hypotheses of Corollary 6.5, as every extension in $\text{Ch}(\mathcal{X})_{[a,b]}$ is, up to isomorphism, of the form

$$FX \xleftarrow{\varnothing} \varnothing \xrightarrow{} X_{b-1} \xleftarrow{} X_{b-2} \xrightarrow{} \cdots \xleftarrow{} X_{a+1} \xrightarrow{} \cdots \xrightarrow{} X_a$$

$$X \xleftarrow{X_b} X_{b-1} \xrightarrow{} X_{b-2} \xrightarrow{} \cdots \xrightarrow{} X_{a+1} \xrightarrow{} \cdots \xrightarrow{} X_a$$

$$GX \xleftarrow{X_b} \varnothing \xrightarrow{} \varnothing \xrightarrow{} \varnothing \xrightarrow{} \cdots \xrightarrow{} \varnothing \xrightarrow{} \varnothing$$

and so we get a homotopy equivalence

$$K(\text{Ch}(\mathcal{X})_{[a,b]}) \simeq K(E(\text{Ch}(\mathcal{X})_{[a,b-1]}, \text{Ch}(\mathcal{X})_{[a,b]})),$$

which proves the claim. $\square$

**Lemma 12.2.** The $\star$-CGW functor

$$\text{Ch}_{E}(\mathcal{X})_{[a,b]} \to \prod^{b-a} \mathcal{X}$$

sending an exact chain complex $X$ to the tuple $(\overline{X}_b, \overline{X}_{b-1}, \ldots, \overline{X}_{a+1})$ induces a homotopy equivalence on $K$-theory.

**Proof.** First, note that this correspondence (the projection of an exact chain complex to its images) is a $\star$-CGW functor, since all the structure on exact chain complexes is defined levelwise, as noted in Remark 11.4.

The proof then proceeds by induction on $b-a$. If $b = a$, the result follows trivially as an exact complex concentrated in a single degree is trivial. For the inductive step, it suffices to show that the $\star$-CGW functor

$$\text{Ch}_{E}(\mathcal{X})_{[a,b]} \to \text{Ch}_{E}(\mathcal{X})_{[a+1,b]} \times \mathcal{X}$$

sending an exact chain complex $X$ to the pair

$$(X_b \xleftarrow{\sim} \overline{X}_b \xrightarrow{} X_{b-1} \xrightarrow{} \cdots \xrightarrow{} X_{a+2} \xleftarrow{} \overline{X}_{a+2} \xrightarrow{id} \overline{X}_{a+2}, \overline{X}_{a+1})$$

induces a homotopy equivalence on $K$-theory. Consider the $\star$-CGW functors

$$F: \text{Ch}_{E}(\mathcal{X})_{[a,b]} \to \text{Ch}_{E}(\mathcal{X})_{[a+1,b]}, \quad G: \text{Ch}_{E}(\mathcal{X})_{[a,b]} \to \mathcal{X}$$

that respectively send an exact chain complex to

$$X_b \xleftarrow{\sim} \overline{X}_b \xrightarrow{} X_{b-1} \xrightarrow{} \cdots \xrightarrow{} X_{a+2} \xleftarrow{} \overline{X}_{a+2} \xrightarrow{id} \overline{X}_{a+2}$$

and

$$\overline{X}_{a+1} \xrightarrow{\sim} \overline{X}_{a+1} \xrightarrow{\sim} X_a$$
Clearly these satisfy the hypotheses of Corollary 6.5, as every extension in \( \text{Ch}_E(\mathcal{X})_{[a,b]} \) is, up to isomorphism, of the form

\[
\begin{array}{ccccccc}
FX & X_b & X_b & \cdots & X_{a+2} & \cdots & X_a \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \downarrow \\
X & X_b & X_b & \cdots & X_{a+2} & \cdots & X_a \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \downarrow \\
GX & \emptyset & \emptyset & \cdots & \emptyset & \cdots & \emptyset
\end{array}
\]

and so we get a homotopy equivalence

\[
K(\text{Ch}_E(\mathcal{X})_{[a,b]}) \simeq K(E(\text{Ch}_E(\mathcal{X})_{[a+1,b]}, \text{Ch}_E(\mathcal{X})_{[a,b]})),
\]

which proves the claim by the Additivity Theorem 6.11.

We now use the lemmas above to prove the main result of this section: the Gillet–Waldhausen Theorem.

**Theorem 12.3** (Gillet–Waldhausen). There exists a homotopy equivalence

\[
K(\mathcal{X}) \simeq K(\text{Ch}(\mathcal{X})^{b}, \text{Ch}_E(\mathcal{X})^{b})
\]

between the \( K \)-theory of \( \mathcal{X} \) with isomorphisms, and the \( K \)-theory of the ECGW category of bounded chain complexes over \( \mathcal{X} \) with quasi-isomorphisms.

**Proof.** Our goal is to show that for all \( a \leq b \), there is a homotopy fiber sequence

\[
K(\text{Ch}_E(\mathcal{X})_{[a,b]}) \longrightarrow K(\text{Ch}(\mathcal{X})_{[a,b]}) \longrightarrow K(\mathcal{X}),
\]

and then take colimits on all intervals of the form \([a - i, a + i]\) to obtain the fiber sequence

\[
K(\text{Ch}_E(\mathcal{X})^{b}) \longrightarrow K(\text{Ch}(\mathcal{X})^{b}) \longrightarrow K(\mathcal{X}).
\]

Recalling that the Localization Theorem 9.1 gives a homotopy fiber sequence

\[
K(\text{Ch}_E(\mathcal{X})^{b}) \longrightarrow K(\text{Ch}(\mathcal{X})^{b}) \longrightarrow K(\text{Ch}(\mathcal{X})^{b}, \text{Ch}_E(\mathcal{X})^{b}),
\]

whose terms are spectra by Proposition 7.4, we must have a homotopy equivalence

\[
K(\mathcal{X}) \simeq K(\text{Ch}(\mathcal{X})^{b}, \text{Ch}_E(\mathcal{X})^{b}),
\]

as in the stable case the homotopy fiber sequences are also homotopy cofiber sequences and their cofibers are uniquely determined up to homotopy.

By Lemmas 12.1 and 12.2, we have the following diagram whose vertical maps are homotopy equivalences

\[
\begin{array}{ccc}
K(\text{Ch}_E(\mathcal{X})_{[a,b]}) & \longrightarrow & K(\text{Ch}(\mathcal{X})_{[a,b]}) \\
\simeq & \quad & \simeq \\
\prod_{b-a} K(\mathcal{X}) & \longrightarrow & \prod_{b-a+1} K(\mathcal{X})
\end{array}
\]
We now define the map

\[
\prod^{b-a} \mathcal{X} \longrightarrow \prod^{b-a+1} \mathcal{X}
\]

\((A_b, A_{b-1}, \ldots, A_{a+1}) \mapsto (A_b, A_b \sqcup A_{b-1}, A_{b-1} \sqcup A_{b-2}, \ldots, A_{a+2} \sqcup A_{a+1}, A_{a+1})\)

which makes the following diagram commute (up to isomorphism).

Indeed, if we start with a chain complex \(X \in \text{Ch}_E(\mathcal{X})_{[a,b]}\), the left vertical map sends \(X\) to the tuple \((\overline{X}_b, \overline{X}_{b-1}, \ldots, \overline{X}_{a+1})\), which is then sent to

\[(\overline{X}_b, \overline{X}_b \sqcup \overline{X}_{b-1}, \overline{X}_{b-1} \sqcup \overline{X}_{b-2}, \ldots, \overline{X}_{a+2} \sqcup \overline{X}_{a+1}, \overline{X}_{a+1})\]

in \(\prod^{b-a+1} \mathcal{X}\). On the other side, the composite of the top inclusion with the right vertical map sends \(X\) to \((X_b, X_{b-1}, X_{b-2}, \ldots, X_{a+1}, X_a)\). As \(X\) is an exact chain complex, these two are isomorphic.

Then, in order to obtain the desired homotopy fiber sequence, it suffices to show that

\[
\prod^{b-a} K(\mathcal{X}) \longrightarrow \prod^{b-a+1} K(\mathcal{X}) \longrightarrow K(\mathcal{X})
\]

is a homotopy cofiber sequence; which is a standard argument and proceeds just like in the classical case. \(\square\)

Appendix. Functoriality constructions

**Appendix A. Properties of \(\star\)-pushouts**

We establish some technical results concerning \(\star\)-pushouts. All of the results in this section assume a \(\star\)-CGW category.

**Lemma A.1.** For any good square in \(\mathcal{M}\) as below inducing an isomorphism on cokernels, the induced map \(B \star_A C \rightarrow D\) is an isomorphism.

\[
\begin{array}{ccc}
A & \longrightarrow & B \leftarrow & \cdots & B/A \\
\downarrow & & \downarrow \circ & \cdots & \approx \\
C & \longrightarrow & D \leftarrow & \cdots & D/C
\end{array}
\]

**Proof.** By the definition of \(\star\)-pushouts, we have the following diagram
where the map $B \star A C/C \rightarrow D/C$ is an isomorphism as the composite $B/A \cong B \star A C/C \rightarrow D/C$ is an isomorphism. Then, since distinguished squares induce isomorphisms on cokernels, Lemma 2.16 implies that the map $B \star A C \rightarrow D$ is an isomorphism.

**Corollary A.2.** Given a diagram $C \leftarrow A \rightarrow B \rightarrow B'$, we have $B' \star_B (B \star A C) \cong B' \star A C$. In other words, the composite of $\star$-pushouts below is the $\star$-pushout of the outer span.

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & B \star_A C \\
\downarrow & & \downarrow \\
C & \rightarrow & D \\
\end{array}
\]

\[
\begin{array}{ccc}
\cong & & \\
\downarrow & & \downarrow \\
\cong & & \downarrow \\
\end{array}
\]

**Proof.** The induced map on cokernels of the vertical $m$-morphisms is a composite of isomorphisms, so by Lemma A.1 the composite is a $\star$-pushout.

**Proposition A.3.** Given a black commutative diagram as below, where the top face is a good square, there exists an induced blue $m$-morphism between $\star$-pushouts such that the two squares created commute, and the bottom one is a good square.

\[
\begin{array}{ccc}
A & \rightarrow & A' \\
\downarrow & & \downarrow \\
B & \rightarrow & B' \\
\downarrow & & \downarrow \\
C & \rightarrow & C' \\
\downarrow & & \downarrow \\
B \star_A C & \rightarrow & B' \star_{A'} C'
\end{array}
\]

Moreover, this assignment is functorial, and if all the original faces are good squares then the two squares created are good, and this is a good cube. The analogous statement for $e$-morphisms also holds, if both $\star$-pushouts exist.

**Proof.** In order to obtain the desired blue $m$-morphism such that the two squares created commute, it suffices to note that the square
is good, and invoke the universal property of the $\star$-pushout $B \star_A C$. The bottom square is
good by axiom (POL), and functoriality follows from uniqueness of the maps induced by the
$\star$-pushout. Finally, if all faces are good, then this is a good cube, since the southern square is
an identity square.

\[ \begin{array}{c}
A \rightarrow C \\
A \rightarrow A' \rightarrow C' \\
B \rightarrow B' \rightarrow B' \star_A C'
\end{array} \]

\[ \begin{array}{c}
A \rightarrow A' \\
B \rightarrow B' \\
C \rightarrow C'
\end{array} \]

Moreover, this assignment is functorial, and if one of the pseudo-commutative squares is dis-
tinguished, then so is the parallel new square. The analogous statement for $\epsilon$-morphisms also
holds, if we start from a black diagram as above right.

\[ \begin{array}{c}
A \rightarrow A' \\
B \rightarrow B' \\
C \rightarrow C'
\end{array} \]

\[ \begin{array}{c}
B \star_A C \rightarrow B' \star_A C'
\end{array} \]

\[ \begin{array}{c}
A \rightarrow A' \\
B \rightarrow B' \\
C \rightarrow C'
\end{array} \]

\[ \begin{array}{c}
B \star_A C \rightarrow B' \star_A C'
\end{array} \]

Moreover, this assignment is functorial, and if one of the pseudo-commutative squares is dis-
tinguished, then so is the parallel new square. The analogous statement for $\epsilon$-morphisms also
holds, if we start from a black diagram as above right.

\[ \begin{array}{c}
A \rightarrow A' \\
B \rightarrow B' \\
C \rightarrow C'
\end{array} \]

\[ \begin{array}{c}
B \star_A C \rightarrow B' \star_A C'
\end{array} \]

\[ \begin{array}{c}
D \rightarrow D'
\end{array} \]

Proof. The constructions necessary for the proof are represented in the diagram below, where
the black arrows are given in the data, and the ones we construct are dashed. We proceed to
explain the steps in order.
First, consider the kernels of the given horizontal e-morphisms, and construct the \( \star \)-pushout of the induced span between them. By Proposition A.3, there exists an m-morphism

\[
(B' \setminus B) \star (A' \setminus A) \hookrightarrow (C' \setminus C)
\]

such that all squares on the top right cube are good.

We can now consider \( \text{coker} \, f \) and form the cube on the top left, which uses all of the original data except for \( B \star_A C \), placing \( \text{coker} \, f \) in its stead. Note that all the squares in this cube are either good or pseudo-commutative (by construction, together with axiom (PBL)).

Taking cokernels of the vertical m-morphisms yields the bottom left cube, where all squares are either good or pseudo-commutative (again by construction, together with axiom (PBL)). By definition of \( B' \star_A C' \), the map \( C'/A' \hookrightarrow B' \star_A C'/B' \) is an isomorphism. Then, by Lemma 2.15, the map \( C/A \twoheadrightarrow \text{coker} \, f/B \) is an isomorphism as well, and by Lemma A.1 we get that the induced m-morphism \( B \star_A C \twoheadrightarrow \text{coker} \, f/B \) must also be an isomorphism, which concludes the proof of the first statement.

Now suppose the given top square is distinguished. This implies that the map \( A' \setminus A \twoheadrightarrow B' \setminus B \) is an isomorphism; then, so is \( C'/A' \twoheadrightarrow (B' \setminus B) \star (A' \setminus A) \) \( C'/C \), and thus the bottom square of the top left cube must be distinguished as well. \( \square \)

**Remark A.5.** From the kernel-cokernel sequence

\[
B \star_A C \cong \text{coker} \, f \twoheadrightarrow B' \star_A C' \twoheadleftarrow (B' \setminus B) \star (A' \setminus A) \twoheadrightarrow (C' \setminus C)
\]

crafted in the proof above, we see that the kernel of the induced e-morphism is precisely the \( \star \)-pushout of the kernels of the three given e-morphisms in the data.

**Lemma A.6.** Given a good square between objects \( A, B, C, D \) as in the diagram below, where \( \star \) denotes \( B \star_A C \), the maps in blue form a kernel-cokernel pair.
Proof. First, note that both maps are unique, as the blue m-morphism is the unique map from the $\star$-pushout from axiom (PO), and the blue e-morphism is the composite of the good square formed by applying $k^{-1}$ followed by $c$ to the original good square, equivalently in either direction by Lemma 2.18.

Now, we can factor the left column of the diagram above as below left:

We then have the diagram of horizontal kernel-cokernel pairs above right, where the lower square is pseudo-commutative by Definition 1.7 and distinguished by Lemma 2.16. Therefore, $D/\star \cong \bullet$, so $\star \rightarrow D \leftarrow \bullet$ is a kernel-cokernel sequence.

Let us say a cube is an m-m-e cube if it has m-morphisms in two directions and e-morphisms in the remaining direction; similarly, we have e-e-m cubes, m-m-m cubes, etc.

**Proposition A.7.** Given a good m-m-m cube, taking cokernels of the m-morphisms and squares in any of the three directions produces an m-m-e cube whose faces are all good or pseudo-commutative squares. Conversely, given such an m-m-e cube, taking kernels produces a good m-m-m cube. The same is also true with the roles of m- and e-morphisms reversed.

Proof. Consider a good m-m-m cube, whose faces and a choice of southern square are all good squares, and let $\star, \star'$ denote the $\star$-pushouts of the relevant spans. We first take cokernels in the direction of the southern square, as pictured below.
By Remark A.5, \( A' / \star \) is the \( \star \)-pushout of \( B' / B \leftarrow A' / A \rightarrow C' / C \), so Remark 3.2 ensures that the square involving \( A' / A, B' / B, C' / C, D' / D \) is good. As all of the mixed squares in this m-m-e cube are pseudo-commutative by construction, we have showed that the cokernel cube in this direction is of the desired form.

We now take cokernels of the m-m-m cube in the remaining two directions, as depicted below. This diagram can be further completed by taking cokernels of the m-m-e cubes and producing the black dashed e-morphisms; note that both squares of e-morphisms created are good.
Now, these m-m-e cubes are such that their remaining face is a good square if and only if there exists an induced dashed blue m-morphism as in the picture such that the square

\[
\bullet, \bullet', D, D'
\]
is pseudo-commutative. Indeed, the square with vertices

\[
B/A, B'/A', D/C, D'/C'
\]
is a good square if and only if taking its cokernel produces the induced dashed blue m-morphism such that the square

\[
\bullet, \bullet', D/C, D'/C'
\]
is pseudo-commutative. This, by axiom (PBL), is equivalent to the square

\[
\bullet, \bullet', D, D'
\]
being pseudo-commutative, which again by axiom (PBL) is equivalent to the square

\[
\bullet, \bullet', C'/A', D'/B'
\]
being pseudo-commutative. But that, in turn, happens if and only if its kernel square

\[
C/A, D/B, C'/A', D'/B'
\]
is good.

Finally, as \( \ast \) denotes \( B \ast_A C \) and \( \ast' \) denotes \( B' \ast_{A'} C' \), the existence of the induced dashed blue m-morphism such that the square

\[
\bullet, \bullet', D, D'
\]
is pseudo-commutative is equivalent to the southern square of the m-m-m cube being good, since these squares form a kernel-cokernel pair by Lemma A.6.

For the converse, to show that the kernel of an m-m-e cube with all faces good or pseudo-commutative is always good, first observe that given such an m-m-e cube pictured as the lower left cube in the diagram above, taking cokernels we get the lower right cube with all faces good or pseudo-commutative, either by construction or in the case of the rightmost face by axiom (PBL). This shows, by Lemma A.6, that in the kernel m-m-m cube pictured as the top left cube in the diagram, the southern square is good.

It then remains only to show that the topmost square of the m-m-m cube is good. This follows by constructing the top right m-m-e cube as the kernel of the bottom right cube. Its topmost square is pseudo-commutative by axiom (PBL), and forms a kernel-cokernel pair with the topmost square of the m-m-m cube, which is therefore good.

Remark A.8. In particular, this implies that there is no need to specify a direction for the good southern square when dealing with good cubes, as claimed in Remark 3.14, since the “goodness” of an m-m-m cube can be equivalently determined from any of its m-m-e cokernel cubes.

We can further deduce the following, which can be interpreted as the statement that all m-m-e and e-e-m cubes with good and pseudo-commutative faces are “good cubes”.

\[ \textbf{Corollary A.9.} \text{Consider an m-m-e cube whose faces are either good or pseudo-commutative squares, together with the induced cube to the } \ast \text{-pushouts as constructed in Proposition A.4,} \]
The analogous statement holds for e-e-m cubes when the ⋆-pushouts exist.

Example A.10. This result illustrates an interesting difference between our motivating examples. In a weakly idempotent complete exact category, where pseudo-commutative squares are simply commuting squares between admissible monomorphisms and epimorphisms, this follows immediately from the universal property of the pushout. In finite sets, however, where the pseudo-commutative squares are pullbacks, this result is precisely the distributivity of intersections over unions among subsets of $D'$: $D \cap (B' \cup C') = (D \cap B') \cup (D \cap C')$.

We now show that ⋆-pushouts preserve pseudo-commutative and distinguished squares.

Proposition A.11. Given an m-span of pseudo-commutative squares, where all the other mixed squares involved are pseudo-commutative and the squares in one of the cube-legs of the span are good, the induced square between the ⋆-pushouts is pseudo-commutative.
The same statement holds for e-spans when the $\star$-pushouts exist.

Proof. The gray and dashed $m$-morphisms are obtained from applying Proposition A.3 to the diagrams of $m$-morphisms on the “top” and “bottom” rows respectively in the diagram above. In turn, the dashed $e$-morphism $A' \star_A A'' \hookrightarrow C' \star_C C''$ is obtained by applying Proposition A.4 to the sub-diagram involving the objects

$$A, \hspace{0.1cm} C, \hspace{0.1cm} A', \hspace{0.1cm} C', \hspace{0.1cm} A'', \hspace{0.1cm} A' \star_A A'', \hspace{0.1cm} C' \star_C C''.$$  

Similarly, we get a map $B' \star_B B'' \hookrightarrow D' \star_D D''$.

The result then follows from applying Corollary A.9 to the following cube of good and pseudo-commutative squares, where the resulting pseudo-commutative southern square is precisely the desired induced square of $\star$-pushouts.

\begin{center}
\begin{tikzpicture}

\node (A) at (0,2) {$A$};
\node (C) at (4,2) {$C$};
\node (A') at (0,1) {$A'$};
\node (B) at (4,1) {$B'$};
\node (A'') at (0,0) {$A''$};
\node (C'') at (4,0) {$C''$};
\node (C') at (2,1) {$C'$};
\node (C'') at (2,0) {$C''$};

\draw[->] (A) -- (C);
\draw[->] (A') -- (A'');
\draw[->] (A') -- (C');
\draw[->] (A'') -- (C'');
\draw[->] (B) -- (C');
\draw[->] (B') -- (C'');
\draw[->] (A') -- (A'');
\draw[->] (A'') -- (C'');
\draw[->] (A') -- (C');
\draw[->] (B') -- (C'');

\end{tikzpicture}
\end{center}

Proposition A.12. If the three initial squares in Proposition A.11 are distinguished, then so is the induced square between the $\star$-pushouts.

Proof. By Proposition A.11, we know that the square between the $\star$-pushouts is pseudo-commutative. To show it is distinguished, first consider the particular case where $A = A' = A'' = \emptyset$; note that then we have $A' \star_A A'' = \emptyset$. In this case, we see that $C \twoheadrightarrow D$ is the kernel of $B \hookrightarrow D$ (and similarly for the other two distinguished squares). Then, by Remark A.5, $C' \star_C C'' \twoheadrightarrow D' \star_D D''$ must be the kernel of $B' \star_B B'' \hookrightarrow D' \star_D D''$, which shows that the desired square is distinguished.

For the general case, we paste distinguished squares besides the given squares as follows

$$\begin{array}{c}
\emptyset \twoheadrightarrow A \twoheadrightarrow B \\
\emptyset \twoheadrightarrow A' \twoheadrightarrow B' \\
\emptyset \twoheadrightarrow A'' \twoheadrightarrow B'' \\
C \setminus A \twoheadrightarrow C \twoheadrightarrow D \\
C' \setminus A' \twoheadrightarrow C' \twoheadrightarrow D' \\
C'' \setminus A'' \twoheadrightarrow C'' \twoheadrightarrow D''
\end{array}$$

and obtain a diagram between $\star$-pushouts

$$\begin{array}{c}
\emptyset \twoheadrightarrow A' \star_A A'' \twoheadrightarrow B' \star_B B'' \\
(C' \setminus A') \star_{(C \setminus A)} (C'' \setminus A'') \twoheadrightarrow C' \star_C C'' \twoheadrightarrow D' \star_D D''
\end{array}$$

The particular case guarantees that both the left square and the composite are distinguished; then, by Lemma 2.17, the desired square on the right is also distinguished. □
Appendix B. ECGW categories of functors

The aim of this subsection is to show that double categories of functors over a $\ast$-CGW category $C$ admit a $\ast$-CGW structure themselves. In particular, this allows us to restrict to the special cases of interest: the double categories of staircases $S_nC$ and the double categories of w-grids $w_{l,m}C$.

Theorem B.1. For any $\ast$-CGW category $C$ and double category $D$, the double category $C^D$ with structure described in Definition 3.12 and Theorem 3.15 is a $\ast$-CGW category.

Proof. We begin by checking the conditions in Definition 2.4. First of all, note that $C^D$ is a double category with shared isomorphisms, since these are defined pointwise, and $C$ has shared isomorphisms.

We now show that $k: \text{Ar}_E \rightarrow \text{Ar}_M$ is well-defined; the argument for $c$ proceeds analogously. To see that $k$ takes an object in $\text{Ar}_E$ to an object in $\text{Ar}_M$, we must check that taking pointwise kernels of an $e$-natural transformation $\eta: A \Rightarrow B$ whose squares between $e$-morphisms are good produces a functor $C \in C^D$, together with an $m$-natural transformation $\mu: C \Rightarrow B$ whose squares between $m$-morphisms are good.

For an object $i \in D$, $C_i$ and $\mu_i$ are defined as the kernel of $\eta_i: A_i \hookrightarrow B_i$. For an $m$-morphism $f: i \Rightarrow j$ in $D$, let $Cf$ be the induced morphism on kernels

\[
\begin{array}{c}
A_i \xhookrightarrow{\eta_i} B_i \\
A_j \xhookrightarrow{\eta_j} B_j
\end{array}
\]

where the pseudo-commutative square on the left exists since $\eta$ is an $e$-natural transformation. Similarly, given an $e$-morphism $g: i \hookrightarrow j$ in $D$, let $Cg$ be the induced morphism on kernels

\[
\begin{array}{c}
A_i \xhookrightarrow{\eta_i} B_i \\
A_j \xhookrightarrow{\eta_j} B_j
\end{array}
\]

and $\mu_g$ be the induced pseudo-commutative square on the right, where the square on the left commutes by naturality of $\eta$, and is good by the additional assumption on $\eta$.

Finally, we must check that taking pointwise kernels of the leftmost cube below (whose faces are all good or pseudo-commutative) produces a cube as the one on the right (whose faces are all good or pseudo-commutative).

\[
\begin{array}{c}
A_i \xhookrightarrow{\eta_i} B_i \xhookrightarrow{\eta_j} C_i \\
A_j \xhookrightarrow{\eta_j} B_j \xhookrightarrow{\eta_k} C_j \\
A_k \xhookrightarrow{\eta_k} B_k \xhookrightarrow{\eta_l} C_k \\
A_l \xhookrightarrow{\eta_l} B_l \xhookrightarrow{\eta_i} C_i
\end{array}
\]

Most of these faces are of the correct type by construction; indeed, the only face one needs to check is the rightmost square between the $C_i$'s, which is pseudo-commutative by axiom (PBL).
The fact that \( k \) takes a morphism in \( \text{Ar}_\Delta \mathcal{E} \) to a morphism in \( \text{Ar}_g \mathcal{M} \) is further ensured by Proposition A.7.

Since \( k \) is defined pointwise from the kernel functor in \( \mathcal{C} \), it is clear that it is faithful. Furthermore, the fact that \( k \) and \( c \) are inverses on objects up to isomorphism, together with Proposition A.7, show that \( k \) is essentially surjective and full.

Axioms (Z) and (M) are trivially satisfied, since \( m \)- and \( e \)-morphisms in \( \mathcal{C} \) are pointwise \( m \)- and \( e \)-morphisms in \( \mathcal{C} \). For axiom (G), note that good squares in \( \mathcal{C} \) are composed of faces which are good squares in \( \mathcal{C} \); in particular, all faces are pullbacks in \( \mathcal{C} \), and so they are pullbacks in \( \mathcal{C} \). To see that \( \text{Ar}_\Delta \mathcal{M} \subseteq \text{Ar}_g \mathcal{M} \), it suffices to note that the southern square of a cube in \( \text{Ar}_\Delta \mathcal{M} \) agrees (up to isomorphism) with one of the faces of the cube, which is a good square.

Finally, axioms (D) and (K) are immediate, since the functors \( k \) and \( c \) are defined pointwise. This shows that \( \mathcal{C}^D \) is a \( g \)-CGW category.

We now check the axioms in Definition 3.1. Axiom (GS) holds, as it is true pointwise in \( \mathcal{C} \), and good cubes are symmetric by Remark A.8. Axiom (PBL) is satisfied, since a square in \( \mathcal{C} \) is pseudo-commutative precisely if it is pointwise pseudo-commutative in \( \mathcal{C} \). For axiom (⋆), given a span of \( m \)-morphisms \( B \leftarrow A \rightarrow C \) in \( \mathcal{C}^D \), we can construct their pointwise \( * \)-pushouts using axiom (⋆) for \( \mathcal{C} \). By Propositions A.3 and A.4, \( * \)-pushouts preserve \( m \)- and \( e \)-morphisms in the appropriate manner. Furthermore, by Proposition A.11, they preserve pseudo-commutative squares. Thus, pointwise \( * \)-pushouts are double functors \( \mathcal{D} \rightarrow \mathcal{C} \).

Propositions A.3 and A.4 also imply that the induced maps \( B \rightarrow B *_A C \) and \( C \rightarrow B *_A C \) are \( m \)-morphisms in \( \mathcal{C}^D \), and that the square below is good.

Similarly, we can construct the \( * \)-pushout of a span of \( e \)-morphisms \( B \leftarrow A \rightarrow C \) in \( \mathcal{C}^D \) when we already know the span is part of some good square.

It remains to show the universal property in axiom (PO), since \( * \)-pushouts will preserve \( (co) \)kernels as \( * \), \( k \), and \( c \) are all defined pointwise. Consider a good square in \( \mathcal{C}^D \) as below left.

In particular, for each \( i \in \mathcal{D} \) we have a good square in \( \mathcal{C} \) as above right, which induce pointwise maps \( B_i \leftarrow A_i \rightarrow C_i \rightarrow D_i \), which are unique up to unique isomorphism. We need to show that for
each $i \rightarrow j$ and $i \leftarrow j$ in $D$, the induced squares below are either good or pseudo-commutative.

\[
\begin{array}{ccc}
B_i \star_A C_i & \rightarrow & D_i \\
\downarrow & & \downarrow \\
B_j \star_A C_j & \rightarrow & D_j \\
\end{array}
\]

For the first statement, note that the square above left is the southern square of the cube

\[
\begin{array}{ccc}
A_i & \rightarrow & A_j \\
\downarrow & & \downarrow \\
B_i & \rightarrow & B_j \\
\downarrow & & \downarrow \\
B_i \star_A C_i & \rightarrow & B_j \star_A C_j \\
\downarrow & & \downarrow \\
C_i & \rightarrow & C_j \\
\downarrow & & \downarrow \\
D_i & \rightarrow & D_j \\
\end{array}
\]

which was assumed to be a good cube; thus, the square must be good. For the second, note that the square above right is the “southern square” of the cube

\[
\begin{array}{ccc}
A_i & \rightarrow & A_j \\
\downarrow & & \downarrow \\
B_i & \rightarrow & B_j \\
\downarrow & & \downarrow \\
B_i \star_A C_i & \rightarrow & B_j \star_A C_j \\
\downarrow & & \downarrow \\
C_i & \rightarrow & C_j \\
\downarrow & & \downarrow \\
D_i & \rightarrow & D_j \\
\end{array}
\]

which, by Corollary A.9, is always pseudo-commutative.

Finally, for axiom (POL), it suffices to check that in any diagram

\[
\begin{array}{ccc}
A_i & \rightarrow & B_i \rightarrow C_i \\
\downarrow & & \downarrow & & \downarrow \\
A_j & \rightarrow & B_j \rightarrow C_j \\
\downarrow & & \downarrow & & \downarrow \\
A_k \star A_{i} & \rightarrow & B_j \star A_{i} & \rightarrow & C_k \\
\downarrow & & \downarrow & & \downarrow \\
A_l & \rightarrow & *_{2} \rightarrow C_l \\
\end{array}
\]

whose outer cube is good, the right cube must be good. Here $*_{1}$ denotes $B_i \star_A A_k$, and $*_{2}$ denotes $B_j \star_A A_l$. Indeed, the back and front faces of the right cube must be good squares due to $C$ satisfying axiom (POL), and the southern square of the right cube can easily be seen to agree with the southern square of the outer cube, which is good. □
We can further show that we get a $\ast$-CGW structure when restricting the squares in our $D$-shaped diagrams to be distinguished in $C$ and requiring certain objects in $D$ to be sent to $\emptyset$, as in the double subcategory $S_nC \subset C^{S_n}$ of Definition 5.2.

**Proposition B.2.** $S_nC$ is a $\ast$-CGW subcategory of $C^{S_n}$.

**Proof.** By Lemma 3.11, in order to show that this is a $\ast$-CGW subcategory, it suffices to prove that it is closed under $k$, $c$, $\ast$, and that it contains the initial object. The latter is trivial, as any square whose boundary consists of isomorphisms is distinguished. Furthermore, since $k$, $c$ and $\ast$ are computed pointwise, it is clear that they preserve the condition of sending the objects $A_{i,j}$ to $\emptyset$. It remains to show that each of these preserves distinguished squares.

We first show that $k$ preserves distinguished squares; for this, we show that in the following diagram, where the right cube is the kernel of the left one, the rightmost square is distinguished in $C$.

\[
\begin{array}{ccc}
A_{i,j} & \rightarrow & B_{i,j} & \rightarrow & C_{i,j} \\
A_{i+1,j} & \rightarrow & B_{i+1,j} & \rightarrow & C_{i+1,j} \\
A_{i,j+1} & \rightarrow & B_{i,j+1} & \rightarrow & C_{i,j+1} \\
A_{i+1,j+1} & \rightarrow & B_{i+1,j+1} & \rightarrow & C_{i+1,j+1} \\
\end{array}
\]

Note that the square is known to be pseudo-commutative, since it is a face in a kernel cube in the ECGW category $C^{S_n}$. To prove it is distinguished, we take the kernel of the right cube in the vertical direction.

\[
\begin{array}{ccc}
B' & \rightarrow & C' \\
B'' & \rightarrow & C'' \\
B_{i,j} & \rightarrow & C_{i,j} \\
B_{i+1,j} & \rightarrow & C_{i+1,j} \\
B_{i,j+1} & \rightarrow & C_{i,j+1} \\
B_{i+1,j+1} & \rightarrow & C_{i+1,j+1} \\
\end{array}
\]

Since the indicated square is distinguished, the induced m-morphism on kernels is an isomorphism. But the top cube is a good cube; in particular, the top face is good, and thus a pullback. This implies that the m-morphism $C' \rightarrow C''$ must be an isomorphism, which in turn proves that the desired square is distinguished. The proof that $S_nC$ is closed under $c$ proceeds dually.

Finally, we prove that $S_nC$ is closed under $\ast$. For this, we need to show that for any span of m-morphisms
the resulting square of $\ast$-pushouts below is distinguished,

$$
\begin{array}{c}
A_{i,j} \ast B_{i,j} \ast C_{i,j} \\
\Downarrow \quad \Downarrow \quad \Downarrow
\end{array}
$$

which is ensured by Proposition A.12.

Lastly, we show that the double category of w-grids $w_{l,m}C \subset C^D$ of Definition 6.7 is also a $\ast$-CGW category.

**Proposition B.3.** $w_{l,m}C$ is a $\ast$-CGW subcategory of $C^D$, where $D$ denotes the free double category on an $l \times m$ grid of squares. Moreover, if $V$ a refinement of $W$, then the double subcategory of grids in $V$ forms an acyclicity structure on $w_{l,m}C$.

**Proof.** Once again, by Lemma 3.11, it suffices to prove that $w_{l,m}C$ is closed under $k$, $c$, $\ast$, and that it contains the initial object. The latter is trivial, as identity morphisms are always m- and e-equivalences.

In order to prove that $w_{l,m}C$ is closed under $\ast$, we must show that in the following diagram, where the right cube is the kernel of the left one, the maps in the rightmost square are m- and e-equivalences.

$$
\begin{array}{c}
A_i \ast B_i \ast C_i \\
\Downarrow \quad \Downarrow \quad \Downarrow
\end{array}
$$

This is a direct consequence of Lemma 4.12; the statement for $c$ is analogous.

To show that $w_{l,m}C$ is closed under $\ast$, we need to prove that for any m-span as below left
the resulting square of ⋆-pushouts pictured above right is distinguished. But by Proposition A.12, we know that ⋆-pushouts preserve kernel-cokernel sequences; in other words, we have that

\[ k(A_k \star_B C_k \hookrightarrow A_i \star_B C_i) = (A_k \setminus A_i) \star_B (C_k \setminus C_i), \]
\[ c(A_i \star_B C_i \twoheadrightarrow A_j \star_B C_j) = (A_j / A_i) \star_B (C_j / C_i), \]

and similarly for the other two maps. We then conclude that the square above right is made of m- and e-equivalences due to Lemma 4.13. \qed

REFERENCES

[Bü10] Theo Bühler. Exact categories. *Expo. Math.*, 28(1):1–69, 2010.
[Bar16] Clark Barwick. On the algebraic K-theory of higher categories. *J. Topol.*, 9(1):245–347, 2016.
[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013.
[Cam19] Jonathan A. Campbell. The K-theory spectrum of varieties. *Trans. Amer. Math. Soc.*, 371(11):7845–7884, 2019.
[Car98] Manuel Enrique Cardenas. Localization for exact categories, 1998. Thesis (Ph.D.)–State University of New York at Binghamton.
[CLW93] Aurelio Carboni, Stephen Lack, and R.F.C Walters. Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84(2):145–158, 1993.
[CW21] Ian Coley and Charles A. Weibel. Localization, monoid sets and K-theory. Preprint on arXiv:2109.03193, 2021.
[CZ] Jonathan A. Campbell and Inna Zakharevich. Dévissage and localization for the Grothendieck spectrum of varieties. Preprint available at.
[FPP10] Thomas M. Fiore and Simona Paoli. A Thomason model structure on the category of small n-fold categories. *Algebr. Geom. Topol.*, 10(4):1933–2008, 2010.
[Gra20] Marco Grandis. *Higher dimensional categories*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020. From double to multiple categories.
[HW21] Christian Haesemeyer and Charles A. Weibel. The K'-theory of monoid sets. *Proc. Amer. Math. Soc.*, 149(7):2813–2824, 2021.
[McC93] Randy McCarthy. On fundamental theorems of algebraic K-theory. *Topology*, 32(2):325–328, 1993.
[MM21] Cary Malkiewich and Mona Merling. The equivariant parametrized h-cobordism theorem, the non-manifold part, 2021. Preprint available at.
[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In: *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341, 1973.
[Sar20] Maru Sarazola. Cotorsion pairs and a K-theory localization theorem. *J. Pure Appl. Algebra*, 224(11):106399, 29, 2020.
[Sch04] Marco Schlichting. Delooping the K-theory of exact categories. *Topology*, 43(5):1089–1103, 2004.
[Sch06] Marco Schlichting. Negative K-theory of derived categories. *Math. Z.*, 253(1):97–134, 2006.
[Sch11] Marco Schlichting. Higher algebraic K-theory. In: *Topics in algebraic and topological K-theory*, volume 2008 of *Lecture Notes in Math.*, pages 167–241. Springer, Berlin, 2011.
[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In: *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.
[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In: *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.
[Wei13] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory.
[Zak18] Inna Zakharevich. The category of Waldhausen categories is a closed multicategory. In: *New directions in homotopy theory*, volume 707 of *Contemp. Math.*, pages 175–194. Amer. Math. Soc., Providence, RI, 2018.