ON RATIONAL PERIODIC POINTS OF $x^d + c$

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Abstract. We consider the polynomials $f(x) = x^d + c$, where $d \geq 2$ and $c \in \mathbb{Q}$. It is conjectured that if $d = 2$, then $f$ has no rational periodic point of exact period $N \geq 4$. In this note, fixing some integer $d \geq 2$, we show that the density of such polynomials with a rational periodic point of any period among all polynomials $f(x) = x^d + c$, $c \in \mathbb{Q}$, is zero. Furthermore, we establish the connection between polynomials $f$ with periodic points and two arithmetic sequences. This yields necessary conditions that must be satisfied by $c$ and $d$ in order for the polynomial $f$ to possess a rational periodic point of exact period $N$, and a lower bound on the number of primitive prime divisors in the critical orbit of $f$ when such a rational periodic point exists. The note also introduces new results on the irreducibility of iterates of $f$.

1. Introduction

An arithmetic dynamical system over a number field $K$ consists of a rational function $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ of degree at least 2 with coefficients in $K$ where the $n$th iterate of $f$ is defined recursively by $f^1(x) = f(x)$ and $f^m(x) = f(f^{m-1}(x))$ when $m \geq 2$. A point $P \in \mathbb{P}^n(K)$ is said to be a periodic (preperiodic) point for $f$ if the orbit $P, f(P), f^2(P), \ldots, f^m(P), \ldots$ of $P$ is periodic (eventually periodic). If $N$ is the smallest positive integer such that $f^N(P) = P$, then the periodic point $P$ is said to be of exact period $N$.

The following conjecture was proposed by Morton and Silverman. There exists a bound $B(D, n, d)$ such that if $K/\mathbb{Q}$ is a number field of degree $D$, and $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ is a morphism of degree $d \geq 2$ defined over $K$, then the number of $K$-rational preperiodic points of $f$ is bounded by $B(D, n, d)$, see [11]. When $f$ is taken to be a quadratic polynomial over $\mathbb{Q}$, the following conjecture was suggested in [13]. If $N \geq 4$, then there is no quadratic polynomial $f(x) \in \mathbb{Q}[x]$ with a rational point of exact period $N$. The conjecture has been proved when $N = 4$, see [12], and $N = 5$, see [7]. A conditional proof for the case $N = 6$ was given in [15].

We consider the polynomial $f(x) = x^d + c$ over a number field $K$. If $c = c_1/c_2$ where $c_1$ and $c_2$ are relatively prime in the ring of integers $O_K$ of $K$, we investigate the divisibility of the coefficients of the iterates $f^m(x)$, $m \geq 2$, by the prime divisors of $c_1$ and $c_2$. Using these divisibility criteria, we approach three questions concerning the arithmetic dynamical system of $f(x) = x^d + c$: (i) When is $f(x)$ stable over $K$? (ii) Fixing $d$, what is the density
of such polynomials with periodic points? (iii) Given that \( f(x) \) possesses a rational periodic point of period \( n \), should this yield necessary conditions satisfied by \( d \) and \( c \)?

The stability question in arithmetic dynamical systems concerns the irreducibility of the iterates of \( f(x) \) over \( K \). More precisely, a polynomial \( f(x) \) is said to be stable over a field \( K \) if \( f^n(x) \) is irreducible over \( K \) for every \( n \geq 1 \). In [1], the authors showed that most monic quadratic polynomials in \( \mathbb{Z}[x] \) are stable over \( \mathbb{Q} \). One may find sufficient conditions for an irreducible monic quadratic polynomial in \( \mathbb{Z}[x] \) to be stable over \( \mathbb{Q} \) in [3]. It was shown that \( f(x) = x^2 + c \in \mathbb{Z}[x] \) is stable over \( \mathbb{Q} \) if \( f(x) \) is irreducible itself, see [14]. Further, the polynomial \( f(x) = x^d + c \in \mathbb{Z}[x], d \geq 2 \), is known to be stable over \( \mathbb{Q} \) if \( f(x) \) is irreducible, see [6].

Unlike the situation over \( O_K \), \( f(x) = x^d + c \in K[x] \) can be irreducible over \( K \) whereas \( f^n(x) \) is reducible over \( K \) for some \( n > 1 \). In this note, if \( c = c_1/c_2 \) where \( c_1 \) and \( c_2 \) are relatively prime in \( O_K \), we show that the existence of a prime divisor \( p \) of \( c_1 \) such that \( \gcd(\nu_p(c_1), d) = 1 \), where \( \nu_p \) is the valuation of \( K \) at the prime \( p \), implies the stability of \( f(x) \). For instance, if \( d \) is prime and \( c_1 \) is not a \( d \)-th power modulo units in \( O_K \), then \( f(x) \) is stable.

Assuming that \( u_1/u_2 \) is a periodic point of \( f(x) \) of exact period \( n \), where \( u_1 \) and \( u_2 \) are relatively prime in \( O_K \), we give several results on the divisibility of the coefficients of the iterate \( f^n(x) \) by prime divisors of \( u_1 \) and \( u_2 \). This enables us to show that if \( f(x) \) has a \( K \)-rational periodic point, then \( c_2 \) must be a \( d \)-th power modulo units in \( O_K \). More precisely, \( c_2 = u_2^d \) modulo units. Fixing \( d \), a hight argument, then, yields that the density of such polynomials with periodic points among all polynomials \( f(x) = x^d + c \) is zero. In particular, almost all polynomials \( f(x) = x^d + c \) satisfy the conjecture of Morton and Silverman.

We establish the connection between a periodic point \( u_1/u_2 \) of \( f(x) = x^d + c \in \mathbb{Q}[x] \) of period \( n \) and the sequence \( u_1^n - u_2^n, m = 1, 2, \ldots \). In fact, we show that \( c_1 \) divides \( u_1^{dn} - u_2^{dn} \), yet none of the prime divisors of \( c_1 \) divide \( u_1 - u_2 \). This provides us with necessary conditions on \( c_1 \) in order for \( f(x) \) to have such a periodic point. For instance, one knows that if \( p \) is a prime divisor of \( c_1 \) such that \( \gcd(p - 1, d^n - 1) = 1 \), then \( f(x) \) has no periodic points of period \( n \).

Finally, we display the relation between rational periodic points of the polynomials \( f(x) = x^d + c \in \mathbb{Q}[x] \) and another sequence, namely the sequence of the iterates, \( f^n(0) \), evaluated at 0. One may consult [10] for several results on the existence of primitive prime divisors of such sequences. In this note, we show that the existence of a periodic point of \( f(x) \) of exact period \( n \) implies a lower bound on the number of primitive prime divisors of \( f^n(0) \).

2. Valuations of the coefficients of the iterates of \( f \)

In this section, we assume that \( K \) is an arbitrary field unless otherwise stated.
Lemma 2.1. Let \( f(x) = x^d + c \), \( d \geq 2 \), \( c \in K \). One has \( f^n(0) = c + c^d g_n(c) \) where \( g_n \in \mathbb{Z}[x] \) is a polynomial of degree \( d^{n-1} - d \), \( n \geq 2 \).

Proof: Since \( f^2(0) = c + c^d \), the statement is true when \( n = 2 \) by taking \( g_2(x) = 1 \). Now, an induction argument will yield the statement. Assume that \( f^n(0) = c + c^d g_n(c) \) where \( g_n(x) \in \mathbb{Z}[x] \) is of degree \( d^{n-1} - d \). One has that \( f^{n+1}(0) = c + (f^n(0))^d \). One observes that

\[
\begin{align*}
f^{n+1}(0) &= c + (c + c^d g_n(c))^d = c + c^d \left(1 + c^{d-1} g_n(c)\right)^d.
\end{align*}
\]

We set \( g_{n+1}(x) = (1 + x^{d-1} g_n(x))^d \). The polynomial \( g_{n+1}(x) \in \mathbb{Z}[x] \). Moreover, since \( g_n \) has degree \( d^{n-1} - d \) by assumption, one gets that the degree of \( g_{n+1} \) is \( d(d^{n-1} - d + d - 1) = d^n - d \).

The following lemma gives an explicit description of the coefficients of \( f^n(x) \).

Proposition 2.2. Let \( f(x) = x^d + c \), \( d \geq 2 \), \( c \in K \). Assume that \( f^n(x) = f_0 + f_1 x^d + f_2 x^{2d} + \ldots + f_{d^n-1} x^{d^n} \). The following statements are correct.

a) \( f_{d^{n-1}-1} = 1 \).
b) \( f_i \in c\mathbb{Z}[c] \) for every \( 0 \leq i < d^{n-1} \).
c) \( \deg f_i = d^{n-1} - i \) for \( 0 \leq i \leq d^{n-1} \).

Proof: That \( f_0 \in c\mathbb{Z}[c] \) and \( \deg f_0 \) in \( \mathbb{Z}[c] \) is \( d^{n-1} \) is implied by Lemma 2.1.

We now follow an induction argument. For the polynomial \( f^2(x) \), one has

\[
\begin{align*}
f^2(x) &= (x^d + c)^d + c = x^{d^2} + \sum_{i=0}^{d-1} \binom{d}{i} c^{d-i} x^{id} + c
\end{align*}
\]

\[
\begin{align*}
= x^{d^2} + c \sum_{i=1}^{d-1} \binom{d}{i} c^{d-1-i} x^{id} + c + c^d.
\end{align*}
\]

Since \( f_i = \binom{d}{i} c^{d-i} \in c\mathbb{Z}[c] \), \( 1 \leq i < d - 1 \), is of degree \( < d \), the statement is correct for \( f^2(x) \).

Assume the statement holds for \( f^n(x) \). One obtains the following equalities

\[
\begin{align*}
f^{n+1}(x) &= (f^n(x))^d + c = \left[f_0 + f_1 x^d + f_2 x^{2d} + \ldots + f_{d^n-1} x^{d^n-d} + x^{d^n}\right]^d + c
\end{align*}
\]

\[
\begin{align*}
= \left[c \left(f'_0 + f'_1 x^d + f'_2 x^{2d} + \ldots + f'_{d^n-1} x^{d^n-d}\right) + x^{d^n}\right]^d + c
\end{align*}
\]

where \( f'_i = f_i / c \in \mathbb{Z}[c] \) and \( \deg f'_i < d^{n-1} - 1 \) by assumption. Setting \( f'(x) = f'_0 + f'_1 x^d + f'_2 x^{2d} + \ldots + f'_{d^n-1} x^{d^n-1} \), one obtains

\[
\begin{align*}
f^{n+1}(x) &= x^{d^{n+1}} + \sum_{i=1}^{d} \binom{d}{i} c^i f'(x)^i x^{d^{n+i}} + c.
\end{align*}
\]
It is obvious that each coefficient of \( f^{n+1}(x) - x^{dn+1} \) is in \( c\mathbb{Z}[c] \).

For part c), one sees that

\[
f^{n+1}(x) = (f^n(x))^d + c = (f_0 + f_1 x^d + f_2 x^{2d} + \ldots + f_{dn-1} x^{dn-d} + x^{dn})^d + c.
\]

We are looking for the degree of the coefficient of \( x^{ld} \) in the latter expansion where \( 0 \leq l \leq dn \). Using an induction argument, we assume that \( \deg f_i = d^{n-1} - i \) in \( \mathbb{Z}[c] \). In view of the multinomial expansion, the latter expansion is given by

\[
f^{n+1}(x) = \sum_{k_0+k_1+\ldots+k_{dn}=d} \binom{d}{k_0, \ldots, k_{dn}} \prod_{t=0}^{dn} (f_t x^{ld})^{k_t} + c.
\]

Using the induction assumption, the degree of the coefficient of \( x^{ld} \) in \( f^{n+1}(x) \) is obtained as follows

\[
\sum_{t=0}^{dn} k_t (d^{n-1} - t) = d^{n-1} \sum_{t=0}^{dn} k_t - \sum_{t=0}^{dn} tk_t
\]

where \( \sum_{t=0}^{dn} k_t = d \) and \( \sum_{t=0}^{dn} tk_t = ld \). \( \square \)

The following corollary is a straightforward result of the proposition above.

**Corollary 2.3.** Let \( K \) be a discrete valuation field with ring of integers \( O_K \). Let \( f(x) = x^d + c, d \geq 2 \), where \( c = c_1/c_2 \) is such that \( c_1 \) and \( c_2 \) are relatively prime in \( O_K \). Assume that \( f^n(x) = f_0 + f_1 x^d + f_2 x^{2d} + \ldots + f_{dn-1} x^{dn} \). Then

\[
c_2^{dn-1} f^n(x) = F_0(c_1, c_2) + F_1(c_1, c_2) x^d + F_2(c_1, c_2) x^{2d} + \ldots + F_{dn-1}(c_1, c_2) x^{dn-d} + F_{dn-1}(c_1, c_2) x^{dn}
\]

where \( F_i(c_1, c_2) = c_2^{dn-1} f_i \in \mathbb{Z}[c_1, c_2] \) is a homogeneous polynomial of degree \( d^{n-1} \). Moreover, \( F_i(c_1, c_2) \in c_1 c_2^d \mathbb{Z}[c_1, c_2] \) if \( i \neq d^{n-1} \), and \( F_{dn-1}(c_1, c_2) = c_2^{dn-1} \).

**Proof:** Since \( f_i \in c\mathbb{Z}[c] \), \( i \neq d^{n-1} \), and \( \deg f_i = d^{n-1} - i \) for \( 0 \leq i \leq d^{n-1} \), see Proposition \( 2.2 \), we may clear the denominators of the coefficients \( f_i \)'s by multiplying throughout by \( c_2^{dn-1} \), hence the result is obtained. \( \square \)

3. **The stability of** \( f(x) = x^d + c \)

Let \( K \) be a field with valuation \( \nu \) whose value group is \( \mathbb{Z} \). Let \( F[x] \in K[x] \) be the polynomial \( F_0 + F_1 x + \ldots + F_k x^k \) where \( F_0 \neq 0 \) and \( F_k \neq 0 \).

The Newton polygon of \( F \) over \( K \) is constructed as follows. We consider the following points in the real plane: \( A_i = (i, \nu(F_i)) \) for \( i = 0, \ldots, k \). If \( F_i = 0 \) for some \( i \), then we omit the corresponding point \( A_i \). The **Newton polygon** of \( F \) over \( K \) is defined to be the lower convex hull of these points. More precisely, we consider the broken line \( P_0 P_1 \ldots P_1 \) where
\[ P_0 = A_0, \quad P_1 = A_{i_1} \text{ where } i_1 \text{ is the largest integer such that there are no points } A_i \text{ below the line segment } P_0P_1. \] Similarly, \( P_2 \) is \( A_{i_2} \) where \( i_2 \) is the largest integer such that there are no point \( A_i \) below the line segment \( P_1P_2 \). In a similar fashion, we may define \( P_i, i = 2, \ldots, l, \) where \( P_l = A_k \). If some line segments of the broken line \( P_0P_1 \ldots P_l \) pass through points in the plane with integer coordinates, then such points in the plane will be also considered as vertices of the broken line. Therefore, we may add \( s \geq 0 \) more points to the vertices \( P_0P_1 \ldots P_l \). The Newton polygon of \( F \) over \( K \) is the polygon \( Q_0Q_1 \ldots Q_{l+s} \) obtained after relabelling all these points from left to the right, where \( Q_0 = P_0 \) and \( Q_{l+s} = P_l \).

The following theorem generalizes Eisenstein’s criterion of irreducibility, see for example [9, Theorem 9.1.13].

**Theorem 3.1** (Eisenstein-Dumas Criterion). Let \( K \) be a field with valuation \( \nu \) whose value group is \( \mathbb{Z} \). Let \( F(x) = F_0 + F_1x + \ldots + F_kx^k \in K[x] \) with \( F_0F_k \neq 0 \). If the Newton polygon of \( F \) over \( K \) consists of the only line segment from \( (0, m) \) to \( (k, 0) \) and if \( \gcd(k, m) = 1 \), then \( F \) is irreducible over \( K \).

We recall that \( x^d + c \) is irreducible over a field \( K \) if and only if for every prime \( p \) dividing \( d \), \(-c\) is not a \( p^{th} \)-power in \( K \); and if \( 4 \mid d \) then \( c \) is not 4 times a \( 4^{th} \)-power in \( K \), see [9, Theorem 8.1.6].

**Theorem 3.2.** Let \( K \) be a number field with ring of integers \( O_K \). Let \( f(x) = x^d + c, \ d \geq 2, \) be such that \( c = c_1/c_2 \) is such that \( c_1 \) and \( c_2 \) are relatively prime in \( O_K \). Assume that there is a prime \( p \) in \( O_K \) such that \( \gcd(\nu_p(c_1), d) = 1 \) where \( \nu_p \) is the valuation of \( K \) at the prime \( p \). Then \( f(x) \) is stable over \( K \).

**Proof:** Let \( K_p \) be the completion of \( K \) with respect to the prime \( p \) and \( \nu_p \) be the corresponding valuation. In view of Corollary 2.3 one has \( f^n(x) = \frac{H_n(x)}{c_2^{d^n-1}} \) where

\[ H_n(x) = F_0(c_1, c_2) + F_1(c_1, c_2)x + F_2(c_1, c_2)x^2 + \ldots + F_{d^n-1}(c_1, c_2)x^{d^n-1} + F_{d^n}(c_1, c_2)x^{d^n} \]

and \( F_i(c_1, c_2) = c_2^{d^n-1}f_i \). Now we consider the Newton polygon of the polynomial \( H_n(x) \in \mathbb{Z}[c_1, c_2][x] \) over \( K_p \). According to Lemma 2.1 one has \( \nu_p(F_0(c_1, c_2)) = \nu_p(c_1) \). Proposition 2.2 indicates that \( \nu_p(F_i(c_1, c_2)) \geq \nu_p(c_1) \) if \( 1 \leq i < d^n \) and \( \nu_p(F_{d^n}(c_1, c_2)) = \nu_p \left( c_2^{d^n-1} \right) = 0 \) where the latter equality follows from the fact that \( c_1 \) and \( c_2 \) are relatively prime. Therefore, the Newton polygon of \( H_n(x) \) consists of one line segment joining the two points \((0, \nu_p(c_1))\) and \((d^n, 0)\). Since \( \gcd(\nu_p(c_1), d^n) = 1 \) by assumption, Theorem 3.11 yields that \( H_n(x) \) is irreducible over \( K_p \), hence over \( K \). This implies that \( f(x) \) is stable.

**Corollary 3.3.** Let \( K \) be a number field and \( f(x) = x^d + c, \ d \geq 2, \) where \( c = c_1/c_2 \) is such that \( c_1 \) and \( c_2 \) are relatively prime in the ring of integers \( O_K \) of \( K \). Assume that \( c_1 \) is not
of the form $uv^p$ for any prime divisor $p$ of $d$, where $v \in O_K$ and $u$ is a unit of $O_K$. Then $f(x)$ is stable over $K$.

In particular, if $f(x) = x^d + c$ where $d$ is prime, then $f(x)$ is stable over $K$ if $c_1$ is not a $d$th-power modulo units in $O_K$.

In what follows, we see some examples of polynomials $f(x)$ violating the relative primality condition $\gcd(\nu_p(c_1), d) = 1$ in Theorem 3.2. We remark that these polynomials are not stable.

**Example 3.4.** If one considers the polynomial $f(x) = x^d - c$, $c \in K$, over a field $K$, then $f(x)$ is not stable as $f^1(x) = f(x)$ is reducible. The polynomial $f(x) = x^2 - 4/3$ is irreducible over $\mathbb{Q}$ since $4/3$ is not a square in $\mathbb{Q}$, yet $f^2(x) = \left(x^2 - 2x + \frac{2}{3}\right)\left(x^2 + 2x + \frac{2}{3}\right)$.

4. Periodic points

From now on $K$ is a number field with ring of integers $O_K$. We will write $O_K^\times$ for the group of units in $O_K$. If $p$ is a prime in $O_K$, then $\nu_p$ is the valuation of $K$ at $p$.

We consider $f(x) = x^d + c$ where $c = c_1/c_2$ such that $c_1 \in O_K$ and $c_2 \in O_K/O_K^\times$ are relatively prime in $O_K$. Given $u \in K$, the orbit of $u$ under $f$ is the set $O_f(u) = \{u, f(u), f^2(u), \ldots\}$. By a periodic point $u$ of exact period $n$, we mean that $f^n(u) = u$ and that $n$ is the smallest such positive integer. In particular, the polynomial $f^n(x) - x$ has a zero at $u$ and $O_f(u)$ is a finite set with exactly $n$ elements. Moreover, any point in the orbit $O_f(u)$ is a periodic point with period $n$. In particular, $f^n(x) - x$ has at least $n$ linear factors.

In accordance with Corollary 2.3, one recalls that

$$f^n(x) = F_0(c_1, c_2) + F_1(c_1, c_2)x^d + F_2(c_1, c_2)x^{2d} + \ldots + F_{d^{n-1}}(c_1, c_2)x^{d^{n-1}}d + F_{d^n-1}(c_1, c_2)x^{d^n}/c_2^{d^{n-1}}.$$ 

Finding the zeros of $f^n(x) - x$ is equivalent to finding the zeros of the following polynomial

$$G^n(x) = F_0(c_1, c_2) - c_2^{d^{n-1}}x + F_1(c_1, c_2)x^d + F_2(c_1, c_2)x^{2d} + \ldots + F_{d^{n-1}}(c_1, c_2)x^{d^{n-1}}d + F_{d^n-1}(c_1, c_2)x^{d^n}.$$ 

Given that $u_1/u_2$ is a periodic point of period $n$ of $f(x)$, where $u_1$ and $u_2$ are relatively prime in $O_K$ and $u_2 \in O_K/O_K^\times$, one multiplies throughout times $u_2^{d^n}$ to get

$$F_0u_2^{d^n} - c_2^{d^{n-1}}u_1u_2^{d^{n-1}} + F_1u_1^{d^n}u_2^{d^n-d} + F_2u_1^{2d}u_2^{d^{n-2}} + \ldots + F_{d^{n-1}}u_1^{d^{n-d}}u_2^{d} + F_{d^n-1}u_1^{d^n} = 0$$ (1)

where $F_i := F_i(c_1, c_2)$. 
4.1. The denominators $c_2$ and $u_2$ of $c$ and $u$.

**Proposition 4.1.** Let $f(x) = x^d + c_1/c_2$ such that $c_1 \in O_K$ and $c_2 \in O_K/O_K^\times$ are relatively prime in $O_K$. Let $u_1/u_2$ be a periodic point of $f(x)$ with period $n$ where $u_1, u_2 \in O_K$ are relatively prime. The following properties hold.

- (a) $u_2^n | F_{d^{n-1}} = c_2^{d^{n-1}}$.
- (b) $c_2$ and $F_0$ are relatively prime in $O_K$.
- (c) $c_2 | u_2^d$.
- (d) $c_2$ and $u_2$ have exactly the same prime divisors.

**Proof:** (a) follows directly from eq (1) and the fact that $u_1$ and $u_2$ are relatively prime in $O_K$.

For (b), Lemma 2.1 yields that

$$F_0 = c_1 c_2^{d^{n-1}-1} + c_2^{d^{n-1}-d} c_1 g_n(c_1/c_2), \quad g_n(x) = \sum_{i=0}^{d^{n-1}-d} g_{n,i} x^i, g_{n,i} \in \mathbb{Z}$$

$$= c_1 c_2^{d^{n-1}-1} + c_2^{d^{n-1}-d} c_1 \sum_{i=0}^{d^{n-1}-d} g_{n,i} (c_1/c_2)^i$$

$$= c_1 c_2^{d^{n-1}-1} + \sum_{i=0}^{d^{n-1}-d} g_{n,i} c_1^{d+i} c_2^{d^{n-1}-d-i} \in c_1 \mathbb{Z}[c_1, c_2].$$

Every term in the latter expansion of $F_0$ is divisible by $c_2$ except for the term whose coefficient is $g_{n,d^{n-1}-d} = 1$. Since $c_1$ and $c_2$ are relatively prime, it follows that $c_2$ and $F_0$ are relatively prime in $O_K$.

For (c), since $F_i \in c_2 \mathbb{Z}[c_1, c_2]$ except when $i = 0$, see Corollary 2.3, this yields that $c_2 | F_0 u_2^d$, see eq (1). Since by (c), one knows that $c_2$ and $F_0$ are relatively prime, it follows that $c_2 | u_2^d$. Part (d) follows from (a) and (c). \hfill \Box

**Corollary 4.2.** Let $c \in O_K$. If $f(x) = x^d + c$, $d \geq 2$, has a periodic point $u$, then $u \in O_K$.

**Proof:** This follows from Proposition 4.1 (d). \hfill \Box

**Theorem 4.3.** Let $f(x) = x^d + c_1/c_2$, $d \geq 2$, such that $c_1 \in O_K$ and $c_2 \in O_K/O_K^\times$ are relatively prime in $O_K$. Let $u_1/u_2$ be a periodic point of $f(x)$ where $u_1, u_2 \in O_K$ are relatively prime. One has $c_2 = u_2^d$.

**Proof:** We assume that $u_1/u_2$ is of period $n$. Let $p$ be a prime divisor of $u_2$. Proposition 4.1 (d) implies that $p$ divides $c_2$. Considering eq (1), one sets $\alpha := \nu_p (c_2^{d^{n-1}} u_1 u_2^{d^{n-1}}) =$
\[ d^{n-1}\nu_p(c_2) + (d^n - 1)\nu_p(u_2). \] We also set
\[
\alpha_l : = \nu_p(F_l u_1^{d\cdot u_2^d} u_2^{d^n - ld}) = \nu_p(F_1) + (d^n - ld)\nu_p(u_2), \quad 0 < l < d^n-1
\]
\[
\geq \nu_p(c_2) + (d^n - ld)\nu_p(u_2)
\]
\[
= d^n\nu_p(u_2) + l(\nu_p(c_2) - d\nu_p(u_2)),
\]
see Corollary 2.3. Furthermore, we define
\[
\alpha_{d^n-1} := \nu_p(F_{d^n-1}) = d^n\nu_p(c_2), \quad \alpha_0 := \nu_p(F_0 u_2^d) = d^n\nu_p(u_2),
\]
see Corollary 2.3 and Proposition 4.1 b), respectively. If \( \nu_p(c_2) \geq \nu_p(u_2) \), then
\[
\min_{0 \leq l < d^n-1} \alpha_l > d^n\nu_p(c_2) = \alpha_{d^n-1}.
\]
In this case, either \( \alpha_{d^n-1} = \alpha_r \) for some \( r \neq d^n-1 \), which is impossible, or \( \alpha_{d^n-1} = \alpha \) which is again impossible as \( \nu_p(u_2) > 0 \).

If \( \nu_p(c_2) > \nu_p(u_2) \), then
\[
\min_{0 < l \leq d^n-1} \alpha_l > d^n\nu_p(u_2) = \alpha_0.
\]
In the latter case, since \( \alpha_0 \neq \alpha_r \) for any \( r \neq 0 \), one must have \( \alpha_0 = \alpha \). It follows that \( \nu_p(u_2) = d^n\nu_p(c_2) \) which contradicts our assumption that \( \nu_p(c_2) > \nu_p(u_2) \).

One concludes that it must be the case that \( \nu_p(c_2) = \nu_p(u_2) \) for any common prime divisor of \( c_2 \) and \( u_2 \). Therefore, assuming that \( u_2 \in O_K/O_K^* \), one obtains that \( c_2 = u_2^d \).

**Remark 4.4.** If \( u_1/u_2 \) is a periodic point of \( f(x) = x^d + c_1/c_2 \) where \( c_i \) and \( u_i \) are as in Theorem 4.3 then \( c_2 = u_2^d \). In other words, a periodic point of \( f(x) \) of any period will have the same denominator. In particular, if \( f^j(u_1/u_2) = v_{1,j}/v_{2,j}, \ j = 1, 2, \ldots, \) are the elements in the orbit \( O_f(u_1/u_2) \) of \( u_1/u_2 \), where \( v_{1,j} \) and \( v_{2,j} \) are relatively prime in \( O_K \), then one may assume that \( v_{2,j} = u_2 \) for every \( j \). In fact, since \( c_2 = u_2^d \), one has \( f(u_1/u_2) = (u_1^d + c_1)/u_2^d \). Therefore, \( u_2^{d-1} \mid (u_1^d + c_1) \).

The following is a direct consequence of Theorem 4.3

**Corollary 4.5.** If \( f(x) = x^d + c_1/c_2, \ d \geq 2, \) where \( c_1 \) and \( c_2 \) are relatively prime and \( c_2 \) is not a \( d^\nu \)-power in \( O_K \), then \( f \) has no periodic points of any period. In particular, there are infinitely many polynomials \( f(x) = x^d + c \) that have no periodic points of any period.

**Corollary 4.6.** Let \( u_1/u_2 \) be a periodic point of exact period \( n \) of \( f(x) = x^d + c_1/c_2, \) where \( c_i \) and \( u_i \) are as above. If \( g(x) = x/u_2^{d-1} \) and \( h(x) = x^d + c_1 \), then \( u_1 \) is a periodic point of the polynomial \( g \circ h \in K[x] \) of exact period \( n \).
Fixing $d = c \in \mathbb{Q}$ for an integer $c$, the statement follows by a simple induction argument to show that $f^2(u_1/u_2) = (g \circ h)(u_1)/u_2$. Now the statement of the corollary holds because $f^n(u_1/u_2) = u_1/u_2$. □

Corollary 4.5 can be strengthened in the following manner over $\mathbb{Q}$. We recall that for $c = a/b \in \mathbb{Q}$ where $\gcd(a, b) = 1$, one may define the height of $c$ to be $h(c) = \max\{|a|, |b|\}$. Fixing $d \geq 2$, we define the following two subsets in $\mathbb{Q}$

$$S(N) = \left\{ \frac{\alpha}{\beta} : \alpha \in \mathbb{Z}, \beta \in \mathbb{Z}^+, \gcd(\alpha, \beta) = 1, h\left(\frac{\alpha}{\beta}\right) \leq N \right\},$$

$$S_d(N) = \left\{ \frac{\alpha}{\beta} : \alpha \in \mathbb{Z}, \beta \in \mathbb{Z}^+, h\left(\frac{\alpha}{\beta}\right) \leq N, \beta \text{ is a } d\text{-th power} \right\}.$$

We will show that $\lim_{N \to \infty} \frac{|S_d(N)|}{|S(N)|} = 0$. This implies the following consequence. Fixing $d \geq 2$, if $f(x) = x^d + c_1/c_2 \in \mathbb{Q}[x]$, where $c_1 \in \mathbb{Z}$ and $c_2 \in \mathbb{Z}^+$ are relatively prime in $\mathbb{Z}$, has a periodic point, then $c_2$ is a $d$-th power. In other words, if we consider the set of such polynomials with periodic points such that the height of $c_1/c_2$ is less than $N$, then according to Theorem 4.3 the set of those $c_1/c_2$ is contained in $S_d(N)$. This means that the density of polynomials $x^d + c$ which have periodic points among all polynomials of the form $x^d + c$, $c \in \mathbb{Q}$, is zero. This can be restated as follows: Fixing $d \geq 2$, almost all polynomials $x^d + c$, $c \in \mathbb{Q}$, have no periodic points.

**Proposition 4.7.** For an integer $d \geq 2$, one has the following asymptotic formula

$$\frac{|S_d(N)|}{|S(N)|} \sim \frac{\pi^2}{6N^{(d-1)/d}} \text{ as } N \to \infty.$$ 

**Proof:** It is clear that $|S_d(N)|$ is asymptotically $2N^{(d+1)/d}$. A standard analytic number theory exercise shows that

$$\sum_{0 < \alpha, \beta \leq N, \gcd(\alpha, \beta) = 1} 1$$

is asymptotically $6N^2/\pi^2$. It follows that $|S_d(N)|/|S(N)|$ is asymptotically $2\pi^2/12N^{(d-1)/d}$. □

Fixing $d \geq 2$, we set

$$P(N) = \{c \in \mathbb{Q} : h(c) \leq N\},$$

$$P_d(N) = \{c \in \mathbb{Q} : x^d + c \text{ has a periodic point, } h(c) \leq N\}.$$ 

According to Theorem 4.3 one has $|P_d(N)|/|P(N)| < |S_d(N)|/|S(N)|$. Now, the following result holds as a direct consequence of Proposition 4.7.
Theorem 4.8. One has the following limit \( \lim_{N \to \infty} \frac{P_d(N)}{P(N)} = 0. \)

The above limit holds if one replaces \( \mathbb{Q} \) with a number field. The proof is similar but the height function has to be changed appropriately.

4.2. The numerators \( c_1 \) and \( u_1 \) of \( c \) and \( u \). We now deduce some divisibility conditions on the numerators of \( c \) and \( u \). Recall that

\[
G^n(x) = F_0 - c_2^{d-1}x + F_1x^d + F_2x^{2d} + \ldots + F_{d^n-1}x^{d^n-d} + F_{d^n-1}x^{d^n},
\]

and eq (1) is given by

\[
F_0u_2^n - c_2^{d-1}u_1u_2^{d-1} + F_1u_1u_2^{d-2} + F_2u_2^{d-2} + \ldots + F_{d^n-1}u_1^{d-1}u_2 + F_{d^n-1}u_1 = 0.
\]

In the following lemma, we list some of the divisibility criteria satisfied by the numerator \( u_1 \) of a periodic point \( u_1/u_2 \) of \( f(x) = x^d + c_1/c_2 \) of exact period \( n > 1 \).

**Lemma 4.9.** The following statements hold.

a) If \( p \) is a prime such that \( \nu_p(u_1) = a \), then \( \nu_p(F_0) = a \). In particular, \( u_1 \parallel F_0 \).

b) \( c_1 \) and \( u_1 \) are relatively prime in \( O_K \).

c) \( u_1 \parallel \frac{F_0}{c_1} \), and \( \frac{F_0}{c_1} \) and \( c_1 \) are relatively prime in \( O_K \).

d) \( c_1 \mid (u_1^{d-1} - u_2^{d-1}) \).

**Proof:** We will mainly considering eq (1) above. For (a), that \( \nu_p(F_0) \geq a \) is a direct consequence of eq (1) and the fact that \( u_1 \) and \( u_2 \) are relatively prime. If \( p^{a+1} \mid F_0 \), then this will imply that \( p \) divides the coefficient of the linear term in \( u_1 \), namely, \( c_2^{d-1}u_2^{d-2} \), which is a contradiction.

For (b), according to Corollary 4.6, the linear factor \( (x - (g \circ h)^j(u_1)/u_2) \), \( 1 \leq j \leq n \), divides \( G^n(x) \). In other words, \( u_2x - (g \circ h)^j(u_1) \) divides \( u_2^n G^n(x/u_2) \). In particular, one sees that \( u_1(u_1^d + c_1)/u_2^{d-1} \) divides \( F_0 \). It follows that if there is a common prime divisor \( p \) of \( c_1 \) and \( u_1 \) such that \( \nu_p(u_1) = a \), then \( \nu(F_0) > a \) which contradicts (a).

Since \( F_0 = c_1c_2^{d-1} + \sum_{i=0}^{d^n-1-d} g_{n,i}c_1^{d+i}c_2^{d^n-1-d-i} \in c_1\mathbb{Z}[c_1,c_2] \), see Lemma 2.1, part (c) follows directly from (a) and (b) and the condition that \( c_1 \) and \( c_2 \) are relatively prime in \( O_K \).

Since \( F_1 \in c_1\mathbb{Z}[c_1,c_2] \), \( i \neq d-1 \), it follows that

\[
c_1 \mid F_{d^n-1}u_1^{d^n-1} - c_2^{d-1}u_1u_2^{d-1} = c_2^{d-1}u_1(u_1^{d-1} - u_2^{d-1}).
\]

Since \( c_1 \) is relatively prime to both \( u_1 \) and \( u_2 \) in \( O_K \), where the latter relative primality holds because \( c_2 = u_2^d \), this yields that \( c_1 \mid (u_1^{d-1} - u_2^{d-1}) \). 

\[\square\]
5. Periodic points and divisors of arithmetic sequences

In the rest of this note, we illustrate the connection between periodic points of the polynomial $f(x) = x^d + c \in \mathbb{Q}[x]$ and two arithmetic sequences.

Let $c = c_1/c_2$ be such that $c_1 \in \mathbb{Z}$ and $c_2 \in \mathbb{Z}^+$ are relatively prime. Given that $u_1/u_2$ is a periodic point of exact period $n$ of $x^d + c$, the orbit of $u_1/u_2$ is the set $O_f(u_1/u_2) = \{f^j(u_1/u_2) : j = 1, 2, 3, \ldots \}$. We recall that $f^j(u_1/u_2) = (g \circ h)^j(u_1)/u_2$ where $h(x) = x^d + c_1$ and $g(x) = x/u_2^{d-1}$, $j = 1, 2, \ldots$, see Remark 4.4 and Corollary 4.6. We set $u_{1,j} = (g \circ h)^j(u_1)$.

In this section, fixing $i$ and $j$, we consider the sequence $\frac{u_{1,i}^k - u_{1,j}^k}{u_{1,i} - u_{1,j}}$, $k = 1, 2, 3, \ldots$. We investigate the divisibility of the terms of the latter sequence by prime divisors of $c_1$. In fact, according to Lemma 4.9(d), if $p$ is a prime divisor of $c_1$, then $p \mid (u_{1,i}^{d^n-1} - u_{1,j}^{d^n-1})$ for every $l$. Therefore, $p \mid (u_{1,i}^{d^n-1} - u_{1,j}^{d^n-1})$ for any $i$ and $j$.

We first prove the coprimality of $u_{1,i}$ and $u_{1,j}$ for any choice of $i$ and $j$, $i \neq j$.

**Lemma 5.1.** Let $f(x) = x^d + c_1/c_2 \in K[x]$ where $c_1 \in O_K$ and $c_2 \in O_K/O_K^\times$ are relatively prime. If $u_1/u_2$ is a periodic point of exact period $n$, where $u_1$ and $u_2$ are relatively prime in $O_K$, then $u_{1,i}$ and $u_{1,j}$ are relatively prime for any $i \neq j$.

**Proof:** Let $p$ be a common prime divisor of $u_{1,i}$ and $u_{1,j}$. Assume that $\nu_p(u_{1,k}) = a_k$, $k = i, j$. According to Lemma 4.9, one has $\nu_p(F_0) = a_i = a_j$ where $F_0$ is defined as before. Since both $u_{1,i}/u_2$ and $u_{1,j}/u_2$ are periodic points of $f(x)$, it follows that they are zeros of the polynomial $G^n(x)$ defined in §4. In particular, $u_{1,i}u_{1,j}$ divides $F_0$. Therefore, if $p$ was a prime divisor of both $u_{1,i}$ and $u_{1,j}$, this would contradict the fact that $\nu_p(F_0) = a_i$. \hfill \Box

**Theorem 5.2.** Let $u_1/u_2$ be a periodic point of $f(x) = x^d + c \in \mathbb{Q}[x]$ of exact period $n$ where $c = c_1/c_2$ is as above. Assume, moreover, that there is a prime $p \mid c_1$ such that $\gcd(p, d^n - 1) = 1$, then $p \nmid (u_{1,i} - u_{1,j})$, for all $i \neq j$. In particular, $p \mid \frac{u_{1,i}^{d^n-1} - u_{1,j}^{d^n-1}}{u_{1,i} - u_{1,j}}$.

**Proof:** Let $p$ be a prime such that $p \mid c_1$ and $\gcd(p, d^n - 1) = 1$. We assume on the contrary that $\nu_p(u_{1,i} - u_{1,j}) = \alpha > 0$. We set $b_{i,j}(m) = \frac{u_{1,i}^m - u_{1,j}^m}{u_{1,i} - u_{1,j}}$. We recall that

$$\gcd(b_{i,j}(k), b_{i,j}(l)) = b_{i,j}(g), \quad g = \gcd(k, l),$$

see [4, Theorem VI].

Since $\nu_p(u_{1,i} - u_{1,j}) = \alpha$, one has $\nu_p(u_{1,i}^p - u_{1,j}^p) \geq \alpha + 1$, see [3, Theorem III]. Noting that $\gcd(b_{i,j}(m), b_{i,j}(p)) = b_{i,j}(1) = 1$ whenever $\gcd(m, p) = 1$ and that $\nu_p(u_{1,i}^m - u_{1,j}^m) \geq \alpha$ for all $k \geq 1$, one has $\nu_p(u_{1,i}^m - u_{1,j}^m) = \nu_p(u_{1,i} - u_{1,j}) = \alpha$ whenever $\gcd(m, p) = 1$.
Since \( u_{1,i}/u_2 \) is a point in the orbit of \( u_1/u_2 \), hence a periodic point of period \( n \), one has \( f^n(u_{1,i}/u_2) = u_{1,i}/u_2 \). Thus, eq (1) may be written for \( u_{1,i}/u_2 \) as follows
\[
F_0u_2^{d^n} + F_1u_1^d u_2^{d^n-d} + F_2u_1^{2d} u_2^{d^n-2d} + \ldots + F_{d^n-1}u_1^{d^n-d} u_2 + F_{d^n-1}u_1 = c_2^{d^n-1} u_{1,i} u_2^{d^n-1}.
\] (2)

Similarly,
\[
F_0u_2^{d^n} + F_1u_1^d u_2^{d^n-d} + F_2u_1^{2d} u_2^{d^n-2d} + \ldots + F_{d^n-1}u_1^{d^n-d} u_2 + F_{d^n-1}u_1 = c_2^{d^n-1} u_{1,j} u_2^{d^n-1}.
\] (3)

Multiplying (2) and (3) times \( u_{1,j}^{d^n} \) and \( u_{1,i}^{d^n} \), respectively, and subtracting the two resulting equations, one obtains
\[
F_0u_2^{d^n} (u_{1,i}^{d^n} - u_{1,j}^{d^n}) + F_1 (u_{1,i}^{d^n-d} - u_{1,j}^{d^n-d}) u_{1,i}^d u_{1,j}^{d^n-d} u_2^{d^n-d} + F_2 (u_{1,i}^{d^n-2d} - u_{1,j}^{d^n-2d}) u_{1,i}^{2d} u_{1,j}^{d^n-2d} + \ldots
\]
\[
+ F_{d^n-1} (u_{1,i}^{d^n-d} - u_{1,j}^{d^n-d}) u_{1,i}^{d^n-d} u_2 = c_2^{d^n-1} (u_{1,i}^{d^n-1} - u_{1,j}^{d^n-1}) u_{1,i} u_{1,j} u_2^{d^n-1}.
\] (4)

One recalls that \( F_i \in c_1\mathbb{Z}[c_1, c_2] \) for \( i \neq d^n-1 \), see Corollary 2.3 and \( p^2 || (u_{1,i} - u_{1,j}) \). This yields that the left hand side of eq (4) is divisible by \( p^{n+1} \). Now since \( c_1 \) is relatively prime to each of \( c_2, u_2, u_{1,i} \) and \( u_{1,j} \), it follows that \( p^{n+1} \) divides \( (u_{1,i}^{d^n-1} - u_{1,j}^{d^n-1}) \) on the right hand side of eq (4), which is a contradiction as \( \gcd(p, d^n - 1) = 1 \).

**Corollary 5.3.** Let \( u_1/u_2 \) be a periodic point of \( x^d + c \) of exact period \( n \) where \( c = c_1/c_2 \) is as above. If there is a prime \( p \) such that \( p \mid c_1 \) and \( \gcd(p, d^n - 1) = 1 \), then \( \gcd(p-1, d^n - 1) > 1 \).

**Proof:** Since \( \gcd(p, d^n - 1) = 1 \), one knows that \( p \nmid (u_{1,i} - u_{1,j}) \); see Theorem 5.2. We recall that
\[
\gcd(b_{i,j}(k), b_{i,j}(l)) = b_{i,j}(g), \quad g = \gcd(k, l).
\]
Since \( \nu_p (u_{1,i}^{p-1} - u_{1,j}^{p-1}) > 0 \) by Fermat’s Little Theorem, one knows that \( \nu_p (b_{i,j}(p-1)) > 0 \). Furthermore, as \( c_1 \mid (u_{1,i}^{d^n-1} - u_{1,j}^{d^n-1}) \), one has \( \nu_p (b_{i,j}(d^n - 1)) > 0 \). It follows that \( \gcd(p-1, d^n - 1) > 1 \).

If \( d^n - 1 \) is prime, then \( d^n - 1 \) is the order of \( u_1 u_2^{-1} \mod p \). This implies that \( (d^n - 1) \mid p-1 \).

**Remark 5.4.** Let \( p \) be a prime divisor of \( c_1 \) such that \( \gcd(p, d^n - 1) = 1 \). In view of Corollary 5.3, if \( \gcd(p-1, d^n - 1) = 1 \), then \( x^d + c_1/c_2 \) has no periodic points of period \( n \). Furthermore, if \( d^n - 1 \) is prime, then \( d^n - 1 \) divides \( p - 1 \) for every prime divisor \( p \) of \( c_1 \). Finally, if \( p \mid (u_{1,i}^{m} - u_{1,j}^{m}) \) for some \( m < (d^n - 1) \), then \( \gcd(m, p - 1) > 1 \). In particular, if \( \gcd(m, p - 1) = 1 \) for any \( m < d^n - 1 \), then \( p \) is a primitive prime divisor of \( u_{1,i}^{d^n-1} - u_{1,j}^{d^n-1} \).
Example 5.5. Let $m > 1$. Let the polynomial $f(x) = x^2 + 2^m$ be such that $2^m - 1$ is prime. If $n > 1$ is an integer such that $\gcd(m, n) = 1$, then $\gcd(2^m - 1, 2^n - 1) = 1$. Thus, Corollary 5.3 implies that $f(x) = x^2 + 2^m$ has no periodic point of period $n$ when $\gcd(m, n) = 1$.

6. A remark on primitive prime divisors of $f^n(0)$

We recall that if $x_i, i = 1, 2, \ldots$, is a sequence in the ring of integers $O_K$ of a number field $K$, then the term $x_n$ is said to have a primitive prime divisor $p$ if $p$ is a prime such that $\nu_p(x_n) > 0$, and $\nu_p(x_m) = 0$ for any $m < n$.

Set $f(x) = x^d + c_1/c_2 \in K[x]$, $c_1 \in O_K$, $c_2 \in O_K/O_K^\times$, $d \geq 2$. In this section, we write $F_0^n$ for $c_2^{n-1} f^n(0)$. It is known that the sequence $F_0^n$ is a divisibility sequence. In particular, $F_0^m | F_0^n$ whenever $m | n$. Several results were proved concerning the existence of primitive prime divisors for each term of the sequence $F_0^n$, see for example [10].

Lemma 6.1. Let $K$ be a number field with ring of integers $O_K$. Let $g(x) \in O_K[x]$ and $u \in O_K$ be such that there is a prime $p$ dividing $g^m(u)$ and $g^n(u)$, $n > m$. Then $p$ divides $g^{n-m}(0)$.

PROOF: This follows directly by observing that $g^n(u) = g^{n-m}(g^m(u))$. □

Theorem 6.2. If $u_1/u_2$ is a periodic point of $f(x) = x^d + c_1/c_2 \in K[x]$ of exact period $n$, where $u_i, c_i$ are as before, then every prime divisor of $u_1$ is a primitive prime divisor of $F_0^n$, $n > 1$.

PROOF: One knows that $u_1 | (F_0^n/c_1)$, see Lemma 4.9(c). We assume that $p$ is a prime divisor of $u_1$ such that $p | F_0^m$ for $m < n$. According to Lemma 6.1, one has $\nu_p(F_0^{n-m}) > 0$. Let $m$ be the smallest such positive integer. One knows that $m \geq 2$ since $\gcd(c_1, u_1) = 1$, see Lemma 4.9(b). By successive application of the division algorithm, one has $m | n$.

Therefore, if $n$ is prime, then it is impossible for $p$ to divide $F_0^m$ for $m < n$.

Now, we assume $n$ is composite. Let $q_1$ and $q_2$ be two distinct prime divisors of $n$ where $n = q_k k_i$. We consider the polynomial $g_i(x) = f^{k_i}(x)$. One has $g_i(0), g_i^2(0) = f^{2k_i}(0), g_i^3(0) = f^{3k_i}(0), \ldots, g_i^n(0) = f^n(0)$. Since $f^n(0) = g_i^n(0)$, Lemma 4.3 implies that $\nu_p(g_i^{n}(0)) > 0$. Since $q_i$ is prime, it follows that the smaller possible integer $l$ such that $\nu_p(g_i^{l}(0)) > 0$ is $l = 1$. In other words, $\nu_p(f^{k_1}(0)), \nu_p(f^{k_2}(0)) > 0$. This yields that either $k_1 | k_2$ or $k_2 | k_1$, a contradiction. □

Corollary 6.3. If $f(x) = x^d + c_1/c_2 \in \mathbb{Q}[x]$ has a periodic point of period $n$, then $F_0^n$ has at least $n - 1$ distinct primitive prime divisors.

PROOF: This follows immediately from Theorem 6.2 and Lemma 5.1. □
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