Semiclassical Dynamics, Berry Curvature and Spiral Holonomy in Optical Quasicrystals

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We describe the theory of the dynamics of atoms in two-dimensional quasicrystalline optical lattices. We focus on a regime of shallow lattice depths under which the applied force can cause Landau-Zener tunnelling past a dense hierarchy of gaps in the quasiperiodic energy spectrum. We derive conditions on the external force that allow for a "semi-adiabatic" regime in which semiclassical equations of motion can apply, leading to Bloch oscillations between the edges of a pseudo-Brillouin-zone. We verify this semiclassical theory by comparing to the results of an exact numerical solution. Novel features appear in the semiclassical dynamics for the quasicrystal for a particle driven in a cyclic trajectory around the corner of the pseudo-Brillouin-zone; the particle fails to return to its initial state, providing a realization of a "spiral holonomy" in the dynamics. We show that there can appear anomalous velocity contributions, associated with non-zero Berry curvature. We relate these to the Berry phase associated with the spiral holonomy, and show how the Berry curvature can be accessed from the semiclassical dynamics.

I. INTRODUCTION

Quasicrystals are an interesting class of materials, in which the delicate mix of long range order and lack of translational symmetry provides a setting that is intermediate between periodic and random systems. Recent work has shown that quasicrystals can lead to unconventional dynamical and topological properties. Novel experimental settings have allowed these properties to be explored with an unparalleled level of control in recent years compared to conventional condensed matter systems. A particularly flexible setting in which quasicrystals have begun to be studied is in ultracold gases – referred to as optical lattices – including a variety of one-dimensional (1D) and two-dimensional (2D) quasicrystals, that are essentially free from disorder and also highly tunable.

The lack of disorder in optical lattices offers an advantage over solid state in allowing for the study of phase coherent transport phenomena without scattering. The classic example is the demonstration of Bloch oscillations in an optical lattice, a phenomenon which has not been observed for bulk crystalline electrons. The theory that describes these phenomena is semiclassical dynamics. This says that under the influence of a weak external force a particles motion is determined by the band structure and by the geometrical properties of its eigenstates encoded in the Berry curvature – a quantity that is intimately related to the topological properties of the band structure. The ability to access these properties cleanly in cold atoms has been exploited experimentally to measure geometrical and topological features of energy bands of fundamental models.

Here we explore the nature of semiclassical dynamics in an optical quasicrystal. We develop this for lattices of shallow depth, corresponding to the nearly-free electron limit of solid state terminology. Our approach exploits the idea that within this limit, and due to the quasiperiodicity, there is a unending fractal hierarchy of gaps in the band structure controlled by perturbation theory. For any finite external force, Landau-Zener tunnelling will make all but a finite number of these gaps relevant within the semiclassical dynamics. The resulting theory is closely analogous to that of a periodic system except that the unconventional rotational symmetries – disallowed for periodic systems – can lead to exotic band structures. As a surprising result of this, we find a realisation of a spiral holonomy, involving a permutation between bands under an adiabatic cyclic trajectory. This phenomena is a generalisation of Berry’s phase and the Wilczek-Zee holonomy. A comparison against an exact solution to the time-dependent Schrödinger equation verifies that the semiclassical theory works well within the shallow-lattice limit. We show under what conditions Berry curvature effects can appear for semiclassical dynamics in quasicrystals, at least within the shallow-lattice limit. Finally we discuss how these ideas are generalised to arbitrary rotational symmetries.

II. MODEL

We consider a two-dimensional optical lattice quasicrystal shown in Fig. 1h, with potential

\[ V(\mathbf{r}) = 2V_0 \sum_{j=1}^{5} \cos(\mathbf{G}_j \cdot \mathbf{r} + \theta_j), \]  

where \( V_0 \) sets the overall strength of the potential, \( \mathbf{G}_j \) are wavevectors given by

\[ \mathbf{G}_j = \kappa (\cos(2\pi j/5), \sin(2\pi j/5)), \]  

with \( \kappa \) being an overall strength factor.
and $\theta_j$ are arbitrary phase offsets. This optical lattice could be generated using standard experimental methods using a laser arrangment shown in Fig. 1b, consisting of five mutually incoherent laser standing waves set at an angle of $2\pi/5$ with respect to one another.

We highlight that this potential satisfies the definition of a quasicrystal[3] in that the minimum number of basis vectors needed to span its Fourier transform (four) is more than the dimension of the space (two). These basis vectors can be chosen as any four of the five vectors $G_j$, Eqn. (2). (The reduction from five to four arises from the linear dependence $\sum_{j} G_j = 0$.) In general, the eigenstates for the Hamiltonian

$$\hat{H} = \frac{\hbar^2 k^2}{2m} + V(r),$$

(3)
can be found by expanding in a basis of plane wave states $|k + G\rangle$ where

$$G = \sum_i n_i G_i,$$

(4)
runs over all possible vectors formed from the four linearly independent basis vectors, as $n_i$ run over all integers. For crystalline lattices, $G$ forms the reciprocal lattice. For the quasicrystal, the key difference is that this set of vectors fills reciprocal space densely, as shown in Fig. 2a.

An important assumption we work with throughout the paper is the shallow-lattice limit

$$V_0 \ll E_R,$$

(5)
where $E_R \equiv \frac{\hbar^2 \kappa^2}{2m}$ is the recoil energy. In this limit the band structure and eigenstates for the Hamiltonian (3) can be found by applying perturbation theory. Away from lines of degeneracy between free particle states (Bragg planes), the energy spectrum is given by

$$E(k) = \frac{\hbar^2 k^2}{2m} + \sum_{\text{w} \in \{G\}} \frac{|\langle k | V | k' \rangle|^2}{E_0(k) - E_0(k')},$$

(6)
and the effect of $V$ is just a second order correction. (We have used $\langle k | V | k \rangle = 0$.) On the other hand, along any two-fold degeneracy – at the crossing of the free particle energies for $k$ and $k'$ say – degenerate perturbation theory must be used. This opens a gap proportional to the matrix element between the two degenerate states

$$\Delta_{\text{gap}} = 2|\langle k | \hat{V} | k' \rangle|,$$

(7)
with matrix elements given by the Fourier coefficients

$$V_{k-k'} \equiv \langle k | \hat{V} | k' \rangle = \int dr V(r) e^{-i(k-k') \cdot r}.$$

(8)
The only non-zero Fourier coefficients, and therefore non-zero gaps to first order in $V$, are those shown in Fig. 2b corresponding to $\pm G_j$. These define a region known as the pseudo-Brillouin zone (PBZ)[45–48].

These gaps represent Bragg scattering processes to first order in $V$. To higher orders of perturbation theory, gaps will open along all lines of degeneracy, corresponding to effective multiple scattering processes. Therefore the initial free particle dispersion develops a dense hierarchy of gaps[40, 41, 49]. However, in the shallow-lattice limit[3] these gaps in the hierarchy have rapidly decreasing sizes with order of perturbation theory. Thus, under suitable conditions, the hierarchy can be truncated in their contributions to physical observables. Indeed we make this idea explicit in the “semi-adiabatic” limit which we now define, and which allows access to a description based on semiclassical dynamics.
weak, they can give a very accurate description of the dynamics over long times.\cite{37,38}

This theory describes the motion of a wavepacket centred at $\mathbf{k}$ in reciprocal space and $\mathbf{r}$ in real space under the influence of an external force $\mathbf{F}$. In solid state this force arises from the electric or magnetic fields acting on the electron, whereas, because atoms are neutral, for ultracold atomic gases this force typically arises from tilting or accelerating the lattice. For a sufficiently weak external force, such that the typical evolution time is sufficiently long compared to the inverse of the gap, the wavefunction will remain in an eigenstate throughout the evolution and the resulting dynamics will be accurately described by the semiclassical equations of motion.\cite{51,52}

\[
\begin{align*}
\dot{\mathbf{k}} &= \frac{1}{\hbar} \mathbf{F} \tag{9} \\
\dot{\mathbf{r}} &= \frac{1}{\hbar} \frac{\partial E(\mathbf{k})}{\partial \mathbf{k}} - (\dot{\mathbf{k}} \times \hat{z}) \Omega(\mathbf{k}). \tag{10}
\end{align*}
\]

The first equation describes the trajectory of $\mathbf{k}$ through reciprocal space under the external force $\mathbf{F}$. Whilst the second relates the motion in real space to the dispersion relation $E(\mathbf{k})$\cite{60} and an additional term\cite{53} (often referred to as the anomalous velocity) proportional to the Berry curvature $\Omega(\mathbf{k})$ defined by\cite{53}

\[
\Omega(\mathbf{k}) \equiv \nabla_{\mathbf{k}} \times [\langle \psi_{\mathbf{k}} | \nabla_{\mathbf{k}} \psi_{\mathbf{k}} \rangle] \cdot \hat{z}. \tag{11}
\]

Applying these equations to a quasicrystal presents a number of difficulties. The central issue is the interpretation of $\mathbf{k}$. In a periodic system $\mathbf{k}$ is the crystal momentum and is thereby only defined up to the addition of a reciprocal lattice vector. This encourages one to restrict $\mathbf{k}$ to the Brillouin zone, ensuring that each $\mathbf{k}$ labels a unique eigenstate. A similar approach for quasicrystals is inappropriate as here the Brillouin zone is infinitesimally small (since there is no lower limit on the size of a reciprocal lattice vector). Instead throughout the following we essentially use an extended zone scheme in which $\mathbf{k}$ is allowed to take any value in reciprocal space.

Closely related to the issue of how to interpret $\mathbf{k}$ is the problem of defining $E(\mathbf{k})$ and $\Omega(\mathbf{k})$. Generically the spectrum will be dense and cannot be assigned a conserved crystal momentum in a Brillouin zone. Nevertheless if one considers the shallow-lattice limit (as discussed in the previous section), the spectrum simplifies to a free particle dispersion which is broken into a dense hierarchy of gaps. Then in a dynamical picture, in which the particle is driven by a finite external force, there will be Landau-Zener tunnelling past almost all gaps in the hierarchy. Indeed, each gap will locally consist of an avoided crossing between two free particle states, $| \mathbf{k} - \mathbf{G} \rangle$ and $| \mathbf{k} - \mathbf{G}' \rangle$, and the Landau-Zener formula then gives the probability to non-adiabatically tunnel past this gap as\cite{51}

\[
P_{\text{LZ}} = e^{-\alpha \Delta_{\text{gap}}^2 / F} \tag{12}
\]

with $F = | \mathbf{F} |$, $\alpha = \pi \kappa^2 / 4 E_R \delta$ and $\delta = | \mathbf{F} \cdot (\mathbf{G} - \mathbf{G'}) |$. For all gaps that satisfy

\[
\alpha \Delta_{\text{gap}}^2 \ll F \tag{13}
\]
the probability of Landau-Zener tunnelling will go to one, $P_{LZ} \to 1$, and these gaps will be essentially ignored in the semiclassical dynamics. For the remaining gaps, the evolution can be made adiabatic past these if the force is chosen so that $P_{LZ} \to 0$. In this case the semiclassical equations of motion (9)-(10) will provide an accurate description of the dynamics, with $E(k)$ and $Ω(k)$ interpreted as the remaining part of the spectrum which is relevant in the semiclassical dynamics for some particular choice of external force. These ideas highlight that the particular semiclassical dynamics found in a quasicrystal will depend on the magnitude of the external force. With increasingly weaker regimes of force resulting in a growing number of gaps becoming relevant (11).

Throughout the following we focus on a particularly simple regime of forcing which we refer to as the “semi-adiabatic limit”. We define this as the regime in which the dynamics are adiabatic with respect to the largest gaps — those of order $V_0$ which form the boundary of the PBZ, but non-adiabatic with respect to the gaps of order $V_0^2/E_R$ (as well as all smaller gaps in the hierarchy), as shown in Fig. 3. Therefore the dynamics are semi-adiabatic when $F$ satisfies

$$\left(\frac{V_0}{E_R}\right)^4 \ll \frac{F}{\kappa E_R} \ll \left(\frac{V_0}{E_R}\right)^2.$$  \hspace{1cm} (14)

The form of $E(k)$ and $Ω(k)$ in the semi-adiabatic limit falls into two cases depending on the location of $k$ in the PBZ. Away from the boundary of the PBZ, $V(r)$ has little effect and to leading order one has free particle dispersion $E(k) = \epsilon_k$, with $Ω(k)$ zero. Whereas nearby the boundary, $E(k)$ and $Ω(k)$ are determined by considering mixing between the free particle states that are degenerate there. Along a straight edge, this involves just two states, whereas at a corner we have the more interesting case of mixing between five degenerate states. These can be identified by considering a series of scatterings at a corner, as shown in Fig. 4. For example, if we consider $k$ nearby the topmost corner, the state $|k\rangle$ will be coupled to the states $|k-G_1\rangle$ and $|k+G_4\rangle$, and these to the states $|k-G_1+G_3\rangle$ and $|k+G_2+G_4\rangle$ respectively, with the final two states coupled to each other. The Hamiltonian that describes the mixing between these five states is given by,

$$H^\text{corner}_k = \begin{pmatrix}
\epsilon_k & V_{G_1} & V_{-G_4} & 0 & 0 \\
V_{-G_1} & \epsilon_{k-G_1} & 0 & V_{G_3} & 0 \\
V_{G_4} & 0 & \epsilon_{k+G_4} & 0 & V_{-G_2} \\
0 & V_{-G_3} & 0 & \epsilon_{k-G_1+G_3} & V_{G_5} \\
0 & 0 & V_{G_2} & V_{-G_5} & \epsilon_{k+G_2+G_4}
\end{pmatrix}$$  \hspace{1cm} (15)

with $V_{G_j} = V_0 e^{i\theta_j}$.

IV. BLOCH OSCILLATIONS

An immediate result of the above discussion is that, within the semi-adiabatic limit, a constant external force will drive Bloch oscillations in a manner closely analogous to those in periodic systems. The possibility of Bloch oscillations in a quasicrystal was first identified in a number of numerical studies [56]. There the Bloch oscillations were found to be quasiperiodic whereas, within the semi-adiabatic limit defined here, it is possible to have approximately periodic oscillations if the force is directed along certain high symmetry directions. For arbitrary directions, the resulting evolution can be highly complicated, as indeed is also the case for periodic crystals [57].
A surprising result of semiclassical dynamics of quasicrystals in the semi-adiabatic limit is found by considering a cyclic variation of the momentum around a corner of the PBZ. Such dynamics could be induced for example by applying a force that changes in direction with time in such a way that the net impulse imparted vanishes, such that one expects the momentum to return to its initial value. In this case we find that a eigenstate does not return to its original form. Instead, the system is left in a different energy eigenstate, orthogonal to its initial state. (Naturally, this result will have a direct impact on how we understand the Berry phase and Berry curvature in later discussions.)

The origin of this phenomena can be attributed to the geometry of the PBZ. Consider following the set of Bragg scatterings, as depicted on the left of Fig. 6 along one cyclic path around a corner in which the momentum changes direction by $2\pi$ to encircle the corner just once. After this single cycle, the wavepacket finishes at a different corner of the PBZ. Although the net external impulse is zero, the set of Bragg scatterings are imbalanced in such a way that there is a net momentum transfer from the quasicrystalline lattice. It is only after performing a second $2\pi$ cycle that the particle returns to its initial location. This unusual geometrical property manifests in the band structure local to a corner, given by Eq. (15) and as shown on the right in Fig. 6. This appears as a series of transitions between the two lowest bands which is zero, the set of Bragg scatterings are imbalanced in such a way that there is a net momentum transfer from the quasicrystalline lattice. It is only after performing a second $2\pi$ cycle that the particle returns to its initial location. This unusual geometrical property manifests in the band structure local to a corner, given by Eq. (15) and as shown on the right in Fig. 6. This appears as a series of transitions between the two lowest bands which finishes in a different band to which it started. Such behaviour is referred to as a “spiral holonomy”[12, 13]. We emphasise that the appearance of this phenomena is a necessary consequence of working in the semi-adiabatic limit for the quasicrystal.

V. SPIRAL HOLONOMY

To our knowledge, similar phenomena to what we see here – the key feature being a change in energy level after a cyclic parameter variation – have been described only in two, very different, settings for energy bands. One setting concerns the 2D surface states of a 3D Weyl semimetal. Here there appears a helicoidal band structure around the projection of the Weyl point[59], that is at the edges of the Fermi arcs of the surface metal[57, 58]. The other setting concerns energy bands in lossy (non-Hermitian) systems. These can show “exceptional points” at which
FIG. 6. A cyclic trajectory around a corner of the PBZ leads to the surprising result of a spiral holonomy in which after a cyclic variation of the parameter $k$ the system fails to return to its initial eigenstate. This result appears in two ways: (left) the geometry of the path encircling the corner and (right) as transitions between the two lowest bands (of the Hamiltonian in Eq. (15)) at a corner.

the (complex) energy eigenvalue has a square root singularity between two energy levels as a function of a 2D parameter that results in the state returning to itself after two cycles. The energy level structure in both examples can be naturally thought of in terms of Riemann surfaces.

VI. BERRY PHASE AND CURVATURE

Topological and geometrical properties of the energy bands of crystalline systems are of a central interest in a large amount of fascinating recent research. Naturally some of these ideas have been extended to quasicrystalline systems with these works focusing on tight-binding models. Here we exploit our description based on semiclassical dynamics, to explore two fundamental quantities: the Berry phase and curvature. In the following we will focus on the properties nearby a corner of the PBZ as it is here where the Berry phase and curvature can be non-zero.

A. Berry Phase

The usual consideration for the Berry phase asks what geometrical phase is acquired for a cyclic parameter variation. However, as discussed in Sec. V, a cyclic trajectory that encircles the corner of the PBZ returns to an orthogonal state and in this case the Berry phase cannot be defined. However for a trajectory that encircles the corner twice, the state does return to its initial form. It is this situation which address here.

We can find the Berry phase for a two-fold trajectory by using a simple argument based on the phase acquired after a series of Bragg scatterings between the edges of the PBZ. In the local band structure picture of Fig. 6 as a certain state $|k\rangle$ adiabatically traverses an avoided crossing into a state $|k'\rangle$, it acquires a phase equal to that of the matrix element which opened that gap between these states, $\langle k'|\hat{V}|k\rangle$. For a path that encircles the corner twice, five such adiabatic crossings are traversed – one for each scattering in Fig. 6 – each contributing one of the five phases $\theta_i$. Therefore the Berry phase acquired for this trajectory is given by

$$\gamma = \sum_{i=1}^{5} \theta_i.$$  \hspace{1cm} (18)

A caveat to this argument is that the second order gaps that are irrelevant far from the corner open into a first order gap as they approach the centre. Therefore this argument only applies to trajectories that remain sufficiently far from the corner.

It is important to highlight that each of the phases $\theta_i$ in the previous argument are gauge dependent since each is equal to the phase of the matrix element $\langle k'|\hat{V}|k\rangle$ which is changed by redefining the phases of the each basis element, $|k\rangle \rightarrow e^{i\delta_k}|k\rangle$. However their total, $\gamma$, is gauge invariant, as can be seen by looking at the structure of the off-diagonal couplings in (15). As shown in Fig. 6, this set forms a closed loop in reciprocal space which ensures that any gauge transformation leaves the sum around this loop invariant.
In the current section we will explore the properties of the Berry curvature of the Hamiltonian $H_{\text{corner}}$ from Eq. (15) which describes mixing at a corner of the PBZ. However, first we outline some general properties of the Berry curvature based on symmetries of the system and use these ideas to derive a condition on the phases $\theta_i$ to allow for non-zero Berry curvature. A symmetry which is present here is time reversal symmetry, which results in $\Omega(k)$ being an odd function of $k$. The presence of inversion symmetry would also mean that $\Omega(k)$ must be an even function of $k$ and therefore both symmetries would result in zero Berry curvature. To determine whether such a point of inversion exists for the quasiperiodic potential (1), we search for a point $R$ such that

$$V(R + r) = V(R - r).$$

(19)

It is straightforward to show that this equality is equivalent to the following set of equations

$$G_i \cdot R + \theta_i = 0 \mod \pi.$$

(20)

By taking the sum of these and using the property

$$\sum_{i=1}^{5} G_i = 0,$$

(21)

one can show that the following equation must hold

$$\sum_{i=1}^{5} \theta_i = 0 \mod \pi.$$

(22)

If this final equality fails to hold, the assumption that there exists an $R$ such that $V(r)$ satisfies (19) must be incorrect: there cannot exist a point of inversion symmetry and the Berry curvature can be non-zero. The sum in Eq. (22) is just equal to the previously found Berry phase (18). Thus, the results are consistent: if the Berry phase (18) is zero or $\pi$ then the Berry curvature must be zero. The fact that the Berry phase can be equal to $\pi$ (and therefore non-zero) whilst the Berry curvature is zero everywhere except at the Dirac points where it is singular.

It is simple to find the exact form of the Berry curvatures $\Omega^{(n)}(k)$ for each of the five bands, labelled by $n$, of (15) by using standard numerical methods. However there is a subtlety here in that calculating the Berry curvature we have assumed adiabaticity with respect to all gaps in the band structure. For the lowest band, there are gaps of order $V^2_0/E_R$ (cf. the discussion on the spiral holonomy of Sec. V and Fig. 6), which would be tunnelled past non-adiabatically in the semi-adiabatic limit. Therefore, although $H_{\text{corner}}$ was motivated by the semi-adiabatic limit, in order to calculate the Berry curvature we must work outside of this regime. The Berry curvature calculated here is simply that associated with adiabatic transport for the band structure described by $H_{\text{corner}}$.

We plot the Berry curvature of (15) for the lowest two bands as well as their sum in Fig. 7 since generally the dynamics here will visit both bands. A striking feature of the separate Berry curvatures $\Omega^{(1)}$ and $\Omega^{(2)}$ are the five sharp peaks associated with the near degeneracy between the two bands. As discussed above, their relevance to the semiclassical dynamics in the semi-adiabatic regime is obscured due to transitions between the bands. On the other hand, their sum $\Omega^{(1)} + \Omega^{(2)}$ is highly relevant within the semi-adiabatic limit and can be cleanly mapped out from the semiclassical dynamics. To do so one can simply perform two evolutions, one for the particle starting in each of the two bands and then summing the separate anomalous velocities as shown in Fig. 8. Numerically this procedure works well up to the same parameter values used in the Bloch oscillations discussion and will therefore require similar evolution times experimentally.
Aside from its appearance in the semiclassical dynamics, the Berry curvature is fundamentally related to the Berry phase via a surface integral over the region enclosed by the cyclic trajectory for which the Berry phase is defined. Making a similar statement here is subtle since for a generic trajectory one encounters transitions between the bands which means the separate adiabatic Berry curvatures are insufficient to describe the semi-adiabatic Berry phase. Nevertheless for the two-fold trajectory discussed in Sec. VI A one can associate the Berry phase here to the integral of the sum of the Berry curvatures, i.e.

\[
\gamma = \int \! dS (\Omega^{(1)} + \Omega^{(2)}). \tag{23}
\]

This result is easily confirmed numerically by integrating over the peak in the summed Berry curvatures from Fig. 7.

VII. GENERALISATIONS

The semi-classical approach we have presented in Sec. III is quite general. The only assumption it relies on is that the hierarchy of gaps can be clearly separated in terms of their sizes. For this condition to be satisfied two criteria must be met: the first is that the Fourier components of the potential must fall off sufficiently quickly (in our case only ten were non-zero). The second is that these components must also be sufficiently weak so that higher order effective couplings can be neglected (here this meant working in the shallow-lattice limit). Both conditions can be satisfied in an optical lattice setting, since the potentials are often formed by a small number of standing waves and the lattice depth is freely tunable. Surprisingly these conditions could also be satisfied for a solid state quasicrystal, as a number of ARPES studies on various icosahedral and decagonal solid state quasicrystals have demonstrated that these have an almost free electron-like dispersion\cite{65, 62, 63}. Of course disorder plays a key role in these materials, likely obscuring the semiclassical dynamics. However there are situations – like in quantum oscillations – where semiclassical dynamics remain highly relevant. Indeed, related ideas to those presented here were already used in Ref. 41 to explain quantum oscillations in incommensurate charge density waves.

Many of the novel results presented here can be simply extended to systems with arbitrary rotational symmetries. These include the spiral holonomy and the possibility of non-trivial Berry phases and curvature. Essentially these only depend on the overall geometry of the PBZ, so that as long as a PBZ can be well defined one can ask such questions. The results naturally split into two cases which we refer to as odd and even. The number of sides of the PBZ is necessarily even, so the two cases are given by half the number of sides. In our case the PBZ had ten sides so is considered odd (and referred to as five-fold) and one with eight sides is even (referred to as four-fold). For simplicity we only discuss the cases of seven-fold and four-fold, with results for higher rotational symmetries left as a simple generalisation.

For the spiral holonomy, the same geometrical picture used in Sec. V A and shown in Fig. 6 to find the number of cycles around a corner before returning can be applied here. For both cases of seven-fold and four-fold, the result is found to be three cycles, with this number incrementing for higher symmetries. Therefore in these cases the number of bands local to the corner that one visits...
FIG. 9. Same as Fig. 4 but for a PBZ with eight sides. Here the couplings again form a closed loop meaning the total phase is gauge invariant, however the sum is now zero and therefore the Berry phase and curvature are also zero.

is three. This result also implies a chirality, since going clockwise or anti-clockwise produces different results. In addition this result highlights that systems with an arbitrary number of cycles can be found by going to higher rotational symmetries.

An interesting difference between odd and even cases appears by asking whether one can find non-zero Berry curvature. The seven-fold case is essentially the same as the five-fold case in this respect. Half the corners are coupled in such a way that the off-diagonal terms again form a closed loop allowing for non-zero Berry curvature. However, in the four-fold case all eight corners couple, forcing the Berry curvature to be the same at all corners. Since we know that time-reversal symmetry forces $\Omega(k)$ to be odd, the only possible Berry curvature at a corner is zero. Therefore the even cases do not allow for any non-trivial Berry phases or curvature.

VIII. CONCLUSION

We have demonstrated that for a two-dimensional shallow-lattice optical quasicrystal, it is possible to identify a regime in which the dynamics is accurately described by the semiclassical equations of motion. By comparing the prediction of Bloch oscillations against an exact numerical solution we determined the maximum potential depth allowed in order for the semiclassical description to apply and related this to experimental parameters.

A surprising result was the appearance of a spiral holonomy around a corner of the PBZ – a phenomena which has been described in a few, very different, settings for energy bands. We also demonstrated that it is possible to have non-trivial Berry phase and curvature at a corner – with both having an unconventional structure due to the spiral holonomy. A method of extracting the Berry curvature from the semiclassical dynamics is provided and its overall properties are related to time reversal and inversion symmetries.

The semiclassical approach can be applied to a generic quasicrystal and can be applicable in solid state quasicrystals with a nearly-free-electron dispersion which have been observed experimentally. We also extend the findings of the spiral holonomy and Berry curvature to systems with arbitrary rotational symmetries by relating these to the properties of the PBZ. We showed that Berry curvature effects appear for certain ‘odd’ arrangements but disappear for ‘even’ arrangements.

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