TERNARY ALGEBRAIC STRUCTURES
AND THEIR APPLICATIONS IN PHYSICS

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Abstract.

We discuss certain ternary algebraic structures appearing more or less naturally in various domains of theoretical and mathematical physics. Far from being exhaustive, this article is intended above all to draw attention to these algebras, which may find more interesting applications in the years to come.

1. INTRODUCTION.

Ternary algebraic operations and cubic relations have been considered, although quite sporadically, by several authors already in the XIX-th century, e.g. by A. Cayley (1) and J.J. Sylvester (2). The development of Cayley’s ideas, which contained a cubic generalization of matrices and their determinants, can be found in a recent book by M. Kapranov, I.M. Gelfand and A. Zelevinskii (3). A discussion of the next step in generality, the so called \( n \)-ary algebras, can be found in (4). Here we shall focus our attention on the ternary and cubic algebraic structures only.

We shall introduce the following distinction between these two denominations: we shall call a ternary algebraic structure any linear space \( V \) endowed with one or more ternary composition laws:

\[
m_3 : V \otimes V \otimes V \Rightarrow V \quad \text{or} \quad m'_3 : V \otimes V \otimes V \Rightarrow \mathbb{C},
\]

the second law being an analogue of a scalar product in the usual (binary) case.

We shall call a cubic structure or an algebra generated by cubic relations, an ordinary algebra with a binary composition law:

\[
m_2 : V \otimes V \Rightarrow V
\]

with cubic (third order) defining relations for the generators: e.g. \( (abc) = e^{2\pi i/3} (bca) \)

Some of ternary operations and cubic relations are so familiar that we don’t pay much attention to their special character. We can cite as example the triple product of vectors in 3-dimensional Euclidean vector space:

\[
\{a, b, c\} = \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})
\]

which is a tri-linear mapping from \( E \otimes E \otimes E \) onto \( \mathbb{R}^1 \), invariant under the cyclic group \( Z_3 \).
Curiously enough, it is in the 4-dimensional Minkowskian space-time $M_4$ where a natural ternary composition of 4-vectors can be easily defined:

$$(X, Y, Z) \to U(X, Y, Z) \in M_4$$

with the resulting 4-vector $U^\mu$ defined via its components in a given coordinate system as follows:

$$U^\mu (X, Y, Z) = g^{\mu\sigma} \eta_{\sigma\lambda\rho} X^\nu Y^\lambda Z^\rho, \quad \text{with} \quad \mu, \nu, ... = 0, 1, 2, 3.$$  

where $g^{\mu\nu}$ is the metric tensor, and $\eta_{\mu\nu\lambda\rho}$ is the canonical volume element of $M_4$.

Other examples of “ternary ideas” that we should cite here are:

- **cubic matrices** and a generalization of the determinant, called the “hyperdeterminant”, first introduced by Cayley in 1840, then found again and generalized by Kapranov, Gelfand and Zelevinskii in 1990 (8). The simplest example of this (non-commutative and non-associative) ternary algebra is given by the following composition rule:

$$\{a, b, c\}_{ijk} = \sum_{l,m,n} a_{nil} b_{ljm} c_{kmn}, \quad i, j, k... = 1, 2, ..., N$$

Other ternary rules can be obtained from this one by taking various linear combinations, with real or complex coefficients, of the above 3-product, e.g.

$$[a, b, c] = \{a, b, c\} + \omega \{b, c, a\} + \omega^2 \{c, a, b\} \quad \text{with} \quad \omega = e^{2\pi i/3}.$$  

- the algebra of “nonions”, introduced by Sylvester as a ternary analog of Hamilton’s quaternions. The “nonions” are generated by two matrices:

$$\eta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}$$

and all their linearly independent powers; the constitutive relations are of cubic character:

$$\sum_{\text{perm.}(ikm)} \Gamma_i \Gamma_k \Gamma_m = \delta_{km} \mathbf{1}$$

where $\delta_{km}$ is equal to 1 when $i = k = m$ and 0 otherwise.

- cubic analog of Laplace and d’Alembert equations, first considered by Himbert (9) in 1934: the third-order differential operator that generalized the Laplacian was

$$\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial x} + \omega \frac{\partial}{\partial y} + \omega^2 \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial x} + \omega^2 \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial z} \right)$$

$$= \frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3} - 3 \frac{\partial^3}{\partial x \partial y \partial z}$$

Other ternary and cubic algebras have been studied by Ruth Lawrence, L. Dabrowski, F. Nesti and P. Siniscalco (8), Plyushchay and Rausch de Traubenberg (9), and other authors.
2. IMPORTANT TERNARY RELATIONS IN PHYSICS.

The quark model inspired a particular brand of ternary algebraic systems, intended to explain the non-observability of isolated quarks as a phenomenon of “algebraic confinement”. One of the first such attempts has been proposed by Y. Nambu (1973), and known under the name of “Nambu mechanics” since then.

Consider a 3-dimensional real space parametrized by Cartesian coordinates, with \( \vec{r} = (x, y, z) \in \mathbb{R}^3 \). Introducing two smooth functions \( H(x, y, z) \) and \( G(x, y, z) \), one may define the following ternary analog of the Poisson bracket and dynamical equations:

For a given function \( f(x, y, z) \) defined on our 3-dimensional space, its time derivative is postulated to be:

\[
\frac{d\vec{r}}{dt} = (\vec{\nabla}H) \times (\vec{\nabla}G) \tag{1}
\]

or more explicitly, because we have

\[
\frac{dx}{dt} = \text{det} \left( \frac{\partial(H, G)}{\partial(y, z)} \right), \quad \frac{dy}{dt} = \text{det} \left( \frac{\partial(H, G)}{\partial(z, x)} \right), \quad \frac{dz}{dt} = \text{det} \left( \frac{\partial(H, G)}{\partial(x, y)} \right),
\]

we can write

\[
\frac{df}{dt} = (\vec{\nabla}) \cdot (\vec{\nabla}H) \times (\vec{\nabla}G) = \text{det} \left( \frac{\partial(f, G, H)}{\partial(x, y, z)} \right) = [f, H, G] \tag{2}
\]

The so defined “ternary Poisson bracket” satisfies obvious relations:

a) \([A, B, C] = -[B, A, C] = [B, C, A] \)

b) \([A_1 A_2, B, C] = [A_1, B, C] A_2 + A_1 [A_2, B, C] \)

c) \(\vec{\nabla} \cdot \left( \frac{d\vec{r}}{dt} \right) = \vec{\nabla} \cdot (\vec{\nabla}H \times \vec{\nabla}G) = 0 \).

A canonical transformation \((x, y, z) \Rightarrow (x', y', z')\) is readily defined as a smooth coordinate transformation whose determinant is equal to 1:

\[
[x', y', z'] = \text{det} \left( \frac{\partial(x', y', z')}{\partial(x, y, z)} \right) = 1,
\]

so that one automatically has:

\[
\frac{df}{dt} = \text{det} \left( \frac{\partial(f, G, H)}{\partial(x, y, z)} \right) = \text{det} \left( \frac{\partial(f, H, G)}{\partial(x', y', z')} \right).
\]

It is easily seen that linear canonical transformations leaving this ternary Poisson bracket invariant form the group \( SL(3, \mathbb{R}) \).

The dynamical equations describing the Euler top can be cast into this new ternary mechanics scheme, if we identify the vector \( \vec{r} \) with the components of the angular momentum \( \vec{L} = [L_x, L_y, L_z] \), and the two “Hamiltonians” with the following functions of the above:

\[
H = \frac{1}{2} \left[ L_x^2 + L_y^2 + L_z^2 \right], \quad G = \frac{1}{2} \left[ \frac{L_x^2}{J_x} + \frac{L_y^2}{J_y} + \frac{L_z^2}{J_z} \right]. \tag{3}
\]

Recently R. Yamaleev has found an interesting link between the Nambu mechanics and ternary \( \mathbb{Z}_3 \)-graded algebras (14).
The Yang-Baxter equation provides another celebrated cubic relation imposed on the bilinear operators named \( \tilde{R} \)-matrices: for \( \tilde{R}_{km} : V \otimes V \rightarrow V \otimes V \), one has
\[
\tilde{R}_{23} \circ \tilde{R}_{12} \circ \tilde{R}_{23} = \tilde{R}_{12} \circ \tilde{R}_{23} \circ \tilde{R}_{12},
\]
where the indices refer to various choices of two out of three distinct specimens of the vector space \( V \).

An alternative formulation of this formula is more widely used. Let \( P \) be the operator of permutation, \( P : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1 \) and let us introduce another \( R \)-matrix by defining \( \tilde{R} = P \circ R \). Then the same relation takes on the following form:
\[
R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}.
\]
(4)

Applications of this equation are innumerable indeed; they serve to solve many integrable systems, such as Toda lattices; they also give the representations of braid groups, etc.

In a given local basis of \( V \otimes V \), \( e_i \otimes e_k \), we can write, for \( X = X^i e_i \), \( Y = Y^k e_k \),
\[
R(X,Y) = R_{km}^{ij} X^i Y^j e_k \otimes e_m
\]
(5)

An interesting ternary aspect of these \( R \)-matrices has been discovered by S. Okubo in search for new solutions of Yang-Baxter equations. Introducing a supplementary real parameter \( \theta \), we can write this equation as follows:
\[
R^{b'}_{a_1 b_1} (\theta) R^{c' a_2 c_1} (\theta') R^{b_2 c_1} (\theta'') = R^{c'}_{b_1 c_1} (\theta') R^{c_2 a' c'} (\theta') R^{a_2 b} (\theta'),
\]
(6)
with \( \theta' = \theta + \theta'' \).

An entire class of solutions of Yang-Baxter equation, including the ones found by de Vega and Nicolai, can be obtained in terms of triple product systems if the matrix \( R \) satisfies an extra symmetry condition:
\[
R^{b a}_{c d} (\theta) = R^{a b}_{c d} (\theta).
\]
(7)

Okubo considered the following symplectic and orthogonal triple systems, i.e. vector spaces (denoted by \( V \)) endowed simultaneously with a non-degenerate bi-linear form
\[
\langle x, y \rangle : V \otimes V \rightarrow \mathbb{C}, \quad x, y \in V
\]
and a triple product
\[
\{ x, y, z \} : V \otimes V \otimes V \rightarrow V, \quad x, y, z \in V
\]
The fundamental assumptions about the relationship between these two products are:

a) \( \langle y, x \rangle = \varepsilon \langle x, y \rangle \);  \hspace{1cm} b) \( \{ y, x, z \} = -\varepsilon \{ x, y, z \} \),

If \( \varepsilon = -1 \), the system is called symplectic; if \( \varepsilon = 1 \), it is called orthogonal.

c) \( \langle \{ u, v, x \}, y \rangle = -\langle x, \{ u, v, y \} \rangle \);  \hspace{1cm} d) \( \{ u, v \{ y, x, z \} \} = \{ \{ u, v, x \}, y, z \} + \{ x, \{ u, v, y \}, z \} + \{ x, y, \{ u, v, z \} \} \)

e) \( \{ x, y, z \} + \varepsilon \{ x, z, y \} = 2 \lambda_0 \langle y, z \rangle x - \lambda_0 \langle x, y \rangle z - \lambda_0 < z, x > y. \)

with a free real parameter \( \lambda_0 \).
In a chosen basis of $V$, $(e_1, e_2, \ldots, e_N)$, one can write
\[
\langle e_i, e_k \rangle = g_{ik} = \varepsilon g_{ki}, \quad \text{and} \quad \{e_i, e_k, e_m\} = C^j_{ikm} e_j
\]
where the coefficients $C^j_{ikm}$ play the rôle of ternary structure constants.

With the help of the inverse metric tensor, $g^{jk}$, we can now raise the lower-case indeces, defining the contravariant basis $e^b = g^{km} e_m$. If a one-parameter family of triple products is defined, $\{e_i, e_k, e_m\}_\theta$, then we may define an $R$-matrix depending on the same parameter $\theta$:
\[
R^{ij}_{\ km} = \langle e^i, \{e^j, e_k, e_m\}_\theta \rangle
\]
or equivalently,
\[
\{e^b, e_c e_d\}_\theta = R^{ab}_{\ cd} e_a.
\] (8)

The symmetry condition $R^{ab}_{\ dc}(\theta) = R^{ab}_{\ cd}(\theta)$ can be now written as
\[
\langle u, \{x, y, z\}_\theta \rangle = \langle z, \{y, x, u\}_\theta \rangle
\]
and the Yang-Baxter equation becomes equivalent with an extra condition imposed on the ternary product:
\[
\sum_a \{v, \{u, e_a, z\}_\theta, \{e^a, x, y\}_\theta \} \theta = \sum_a \{u, \{v, e_a, x\}_\theta, \{e^a, z, y\}_\theta \} \theta
\] (9)

Using thus encoded form of the Yang-Baxter equation S. Okubo was able to find a series of new solutions just by finding 1-parameter families of ternary products satisfying the above constraints.

This original approach suggests another possibility of introducing ternary structures in the very fabric of traditional quantum mechanics. As we know, any bounded linear operator acting in Hilbert space $\mathcal{H}$ can be represented as
\[
A : \mathcal{H} \to \mathcal{H}, \quad A = \sum_{k, m} a_{km} \langle e_k | e_m \rangle;
\] (10)
where $| e_k \rangle$ is a basis in Hilbert space. If now
\[
| x \rangle = \sum_k c_m | e_m \rangle,
\]
then one has
\[
A | x \rangle = \sum_{i, k, m} a_{ik} | e_i \rangle \langle e_k | c_m | e_m \rangle = \sum_{i, k, m} a_{ik} c_m \delta_{km} | e_i \rangle = \sum_k a_{km} c_m | e_m \rangle
\] (11)

Each item in this sum can be considered as a result of ternary multiplication defined in the Hilbert space of states:
\[
m(| e_i \rangle, | e_j \rangle, | e_k \rangle) = | e_i \rangle \langle e_j | e_k \rangle = \delta_{jk} | e_i \rangle = \sum_n \delta_{jk} \delta^n_i | e_n \rangle,
\] (12)
with the structure constants defined as $C^n_{\ ijk} = \delta_{jk} \delta^n_i$. Using this interpretational scheme, the states and the observables (operators) are no more separate entities, but can interact with each other: by superposing triplets of states, we arrive at the result which amounts to changing both the state and the observable simultaneously.
Similar constructions, often referred to as *algebraic confinement*, were considered by many authors, in particular by H.J. Lipkin quite a long time ago ([12]).

Consider an algebra of operators $O$ acting on a Hilbert space $H$ which is a free module with respect to the algebra $O$, endowed with Hilbertian scalar product. Let us introduce tensor products of the algebra and the module with the following $Z_3$-graded matrix algebra $A$ over the complex field $C$:

$$A = A_0 \oplus A_1 \oplus A_2, \quad A \in \text{Mat} (3, C).$$

The three linear subspaces of $A$, of which only $A_0$ forms a subalgebra, are defined as follows:

$$A_0 := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad A_1 := \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 0 & \gamma \\ \beta & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}.$$

It is easy to check that under matrix multiplication the degrees 0, 1 and 2 add up modulo 3: a product of two elements of degree 1 belongs to $A_2$, the product of two elements of degree 2 belongs to $A_1$, and the product of an element of degree 1 with an element of degree 2 belongs to $A_0$, etc.

With this in mind, a generalized state vector can now belong to one of these subspaces, e.g., $|\Psi\rangle$ of degree one, its hermitian conjugate being automatically of $Z_3$-degree 2:

$$|\Psi\rangle := \begin{pmatrix} 0 & |\psi_1\rangle & 0 \\ 0 & 0 & |\psi_2\rangle \\ |\psi_3\rangle & 0 & 0 \end{pmatrix}; \quad \langle\Psi| := \begin{pmatrix} \langle\psi_1| & 0 & 0 \\ 0 & \langle\psi_2| & 0 \\ 0 & 0 & \langle\psi_3| \end{pmatrix}$$

with $|\psi_k\rangle \in H$. The scalar product obviously generalizes as follows:

$$\langle\Phi|\Psi\rangle := \text{Tr} \left[ \begin{pmatrix} 0 & \langle\psi_3| \\ \langle\phi_1| & 0 & 0 \\ 0 & \langle\phi_2| & 0 \end{pmatrix} \begin{pmatrix} 0 & |\psi_1\rangle & 0 \\ 0 & 0 & |\psi_2\rangle \\ |\psi_3\rangle & 0 & 0 \end{pmatrix} \right] = \langle\phi_1|\psi_1\rangle + \langle\phi_2|\psi_2\rangle + \langle\phi_3|\psi_3\rangle. \quad (13)$$

With this definition of scalar product any expectation value of an operator of degree 1 or 2 (represented by the corresponding traceless matrices) will identically vanish, e.g. for an operator of degree 1: $(D_i \in O)$

$$\text{Tr} \left[ \begin{pmatrix} 0 & 0 & \langle\psi_3| \\ \langle\psi_1| & 0 & 0 \\ 0 & \langle\psi_2| & 0 \end{pmatrix} \begin{pmatrix} 0 & D_1 & 0 \\ 0 & 0 & D_2 \\ D_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & |\psi_1\rangle & 0 \\ 0 & 0 & |\psi_2\rangle \\ |\psi_3\rangle & 0 & 0 \end{pmatrix} \right] = 0.$$

It is clear that only the operators whose $Z_3$-degree is 0 may have non-vanishing expectation values, because the operators of degrees 1 and 2 are traceless. Denoting the operator of degree 1 by $Q$, and the operators of degree 2 by $\bar{Q}$, the only combinations that can be observed, i.e. that can lead to non-vanishing expectation values no matter what the nature of the operator and the observable it is supposed to represent, are the following products:

$$QQQ; \quad \bar{Q}\bar{Q}\bar{Q}; \quad Q\bar{Q} \quad \text{and} \quad \bar{Q}Q \quad (14)$$

which correspond to the observable combinations (tensor products) of the fields supposed to describe the quarks. This particular realisation of “*algebraic confinement*” suggests the importance of ternary and cubic relations in algebras of observables.
3. CUBIC GRASSMANN AND CLIFFORD ALGEBRAS.

A general 3-algebra (or ternary algebra) is defined as internal ternary multiplication in a vector space $V$. Such a multiplication must be of course 3-linear, but not necessarily associative:

$$m : V \otimes V \otimes V \rightarrow V; \quad m(X, Y, Z) \in V$$  \hspace{1cm} (15)

Such a 3-product is said to be strongly associative if one has

$$m(X, m(S, Y, T), Z) = m(m(X, S, Y), T, Z) = m(X, S, m(Y, T, Z))$$  \hspace{1cm} (16)

Of course, any associative binary algebra can serve as starting point for introduction of a (not necessarily associative) ternary algebra, by defining its ternary product:

*(X, Y, Z) = XYZ \quad \text{(trivial);}  

**(X, Y, Z) = XYZ + YZX + ZXY \quad \text{(symmetric);}  

***[X, Y, Z] = XYZ + \omega YZX + \omega^2 ZXY \quad \text{(} \omega \text{ - skew-symmetric)} \hspace{1cm} (17)

where we set $\omega = e^{2\pi i/3}$, the primitive cubic root of unity. It is worthwhile to note that the last cubic algebra, which is a direct generalization of the $Z_2$-graded skew-symmetric product $[X, Y] = XY - YX$, which defines the usual Lie algebra product, contains it as a special substructure if the underlying associative algebra is unital. Indeed, if 1 is the unit of that algebra, one easily checks that substituting it in place of the second factor of the skew-symmetric ternary product, one gets:

\[
\{X, 1, Z\} := X1Z + \omega 1ZX + \omega^2 ZX1 = XZ + (\omega + \omega^2)ZX = XZ - ZX ,
\]

because of the identity $\omega + \omega^2 + 1 = 0$, so that the usual Lie-algebraic structure is recovered as a special case.

In general, a ternary algebra cannot be derived from an associative binary algebra. Indeed, suppose that we have, on one side, a ternary multiplication law defined by its structural constants with respect to a given basis $\{e_k\}$:

$$m(e_i, e_j, e_k) = \sum_{l=1}^{N} m^l_{ijk} e_l,$$

and on the other hand, a binary multiplication law, defined in the same basis by

$$p(e_i, e_k) = \sum_{m=1}^{N} p^m_{ik} e_m;$$

and suppose that we want to interpret the ternary multimlication as two consecutive binary multiplications:

$$m(a, b, c) = p(a, p(b, c)) = p(p(a, b), c)$$

(supposing that the binary algebra is associative). Then, after projection on the basis vectors $e_k$ we should have

$$m^i_{jkm} = \sum_{r=1}^{N} p^r_{km} p^i_{jr} .$$  \hspace{1cm} (19)

Even in the simplest case of dimension $N = 2$, we get $2^4 = 16$ equations for $2^3 = 8$ unknowns (the coefficients $p^i_{jk}$), which can not be solved in general, except maybe for some very special cases.
Recently, A. Sitarz ([13]) proved that any associative \( n \)-ary algebra can be generated by a part of the \( A_1 \), i.e. the grade 1 subspace of certain \( Z_{n-1} \)-graded associative ordinary (binary) algebra. The simplest example of this situation is given by the groups of the symmetry group \( S_3 \). It contains two subspaces, which are naturally \( Z_2 \)-graded. The even subspace (of degree 0) is spanned by the cyclic subgroup \( Z_3 \), while the odd subspace is spanned by three involutions, corresponding to odd permutations. As the square of each involution is the unit element, the product of three involutions gives another involution, which defines a ternary algebra (without unit element). The full ternary multiplication table contains 27 independent products.

Just as binary products can be divided into different classes reflecting their behavior under the permutation group \( Z_2 \), so all ternary products can be divided into classes according to their behavior under the actions of the permutation group \( S_3 \). These in turn are naturally separated into symmetric cubic and skew-symmetric cubic subsets.

There are four possible ternary generalizations of the symmetric binary product:

\[
S_0 : x^j x^k x^m = x^{\pi(j)} x^{\pi(k)} x^{\pi(m)}, \quad \text{any permutation } \pi \in S_3;
\]

\[
S_1 : x^j x^k x^m = x^k x^m x^j \quad \text{(cyclic permutations only)};
\]

\[
S : \quad x^k x^m x^n + \omega x^m x^k x^n + 2 x^m x^k x^n = 0;
\]

\[
\bar{S} : \quad x^k x^m x^n + \omega^2 x^m x^k x^n + \omega x^m x^k x^n = 0.
\]

Obviously, the spaces \( S \) and \( \bar{S} \) are isomorphic, and there exist surjective homomorphisms from \( S \) and \( \bar{S} \) onto \( S_1 \), and a surjective homomorphism from \( S_1 \) onto \( S_0 \). Similarly, the skew-symmetric cubic algebras can be defined as a direct generalisation of Grassmann algebras:

\[
\Lambda_0 : \theta^A \theta^B \theta^C + \theta^B \theta^C \theta^A + \theta^C \theta^A \theta^B + \theta^C \theta^B \theta^A + \theta^A \theta^B \theta^C + \theta^A \theta^C \theta^B = 0,
\]

\[
\Lambda_1 : \quad \theta^A \theta^B \theta^C + \theta^B \theta^C \theta^A + \theta^C \theta^A \theta^B = 0,
\]

\[ \Lambda : \quad \theta^A \theta^B \theta^C = \omega \theta^B \theta^C \theta^A; \quad \bar{\Lambda} : \quad \bar{\theta}^A \bar{\theta}^B \bar{\theta}^C = \omega^2 \bar{\theta}^B \bar{\theta}^C \bar{\theta}^A. \]

Here again, a surjective homomorphism exists from \( \Lambda_0 \) onto \( \Lambda_1 \), then two surjective homomorphisms can be defined from \( \Lambda_1 \) onto \( \Lambda \) or onto \( \bar{\Lambda} \).

The natural \( Z_3 \)-grading attributes degree 1 to variables \( \theta^A \) and degree 2 to the variables \( \bar{\theta}^B \); the degrees add up modulo 3 under the associative multiplication. Then the algebras \( \Lambda \) and \( \bar{\Lambda} \) can be merged into a bigger one if we postulate the extra binary commutation relations between variables \( \theta^A \) and \( \bar{\theta}^B \):

\[
\theta^A \bar{\theta}^B = \omega \bar{\theta}^B \theta^A, \quad \bar{\theta}^B \theta^A = \omega^2 \theta^A \bar{\theta}^B.
\]

If \( A, B, C, \ldots = 1, 2, \ldots N \), then the total dimension of this algebra is

\[
D(N) = 1 + 2N + 3N^2 + \frac{2(N^3 - N)}{3} = \frac{2N^3 + 9N^2 + 4N + 3}{3}.
\]

These algebras are the most natural \( Z_3 \)-graded generalizations of usual \( Z_2 \)-graded algebras of fermionic (anticommuting) variables. Similarly, cubic Clifford algebras can be defined if their generators \( Q^a \) are supposed to satisfy the following ternary commutation relations:

\[
Q^a Q^b Q^c = \omega Q^b Q^c Q^a + \omega^2 Q^c Q^a Q^b + 3 \rho^{abc} 1. \quad (20)
\]
instead of usual binary constitutive relations

\[ \gamma^\mu \gamma^\lambda = (-1) \gamma^\lambda \gamma^\mu + g^{\mu\lambda} \mathbf{1}. \]

A conjugate ternary Clifford algebra isomorphic with the above is readily defined if we introduce the conjugate matrices \( \bar{Q}^a \) satisfying similar ternary condition with \( \omega \) replacing \( \omega^2 \) and vice versa ([14]). Applying cyclic permutation operator \( \pi \) to all triplets of indices on both sides of the definition (20), one easily arrives at the condition that must be satisfied by the tensor \( \rho^{abc} \), corresponding to the symmetry condition on the metric tensor \( g^{\mu\lambda} \) in the usual (binary) case:

\[ \rho^{abc} + \omega \rho^{bca} + \omega^2 \rho^{cab} = 0. \]

This equation has two independent solutions,

\[ \rho^{abc} = \rho^{bca} = \rho^{cab}, \quad \text{and} \quad \rho^{abc} = \omega^2 \rho^{bca} = \omega \rho^{cab}. \]

The second, non-trivial solution defines a cubic matrix \( \rho^{abc} \); its conjugate, satisfying complex conjugate ternary relations, provides a \( \mathbb{Z}_3 \)-conjugate matrix \( \bar{\rho}^{abc} \).

These two non-trivial solutions, denoted by \( \rho^{(1)} \) and \( \rho^{(2)} \), form an interesting non-associative ternary algebra with ternary multiplication rule defined as follows ([15], [4]):

\[ (\rho^{(i)} * \rho^{(k)} * \rho^{(m)})_{abc} = \sum_{d,e,f} \rho^{(i)}_{fad} \rho^{(k)}_{dbe} \rho^{(m)}_{ecf}. \] (21)

A \( \mathbb{Z}_3 \)-graded analogue of usual commutator as readily defined as

\[ \{\rho^{(i)}, \rho^{(j)}, \rho^{(k)}\} := \rho^{(i)} * \rho^{(j)} * \rho^{(k)} + \omega \rho^{(j)} * \rho^{(k)} * \rho^{(i)} + \omega^2 \rho^{(k)} * \rho^{(i)} * \rho^{(j)}; \] (22)

It has been shown in ([15]) that this ternary algebra spanned by two generators \( \rho^{(1)} \) and \( \rho^{(2)} \) can be represented by ordinary matrices (which are nothing else but two arbitrarily chosen Pauli matrices) with ternary multiplication defined as

\[ \{\sigma^1, \sigma^2, \sigma^1\} := \sigma^1 \sigma^2 \sigma^1 + \omega \sigma^2 \sigma^1 \sigma^1 + \omega^2 \sigma^1 \sigma^1 \sigma^2 = -2 \sigma^2, \quad \text{etc.} \]

which is an illustration of the observation made by A. Sitarz ([13]).

4. CUBIC ROOTS OF LINEAR DIFFERENTIAL OPERATORS.

The existence and particular properties of cubic Grassmann and Clifford algebras suggest that they can be used in order to define cubic roots of linear differential operators, in the same sense as the Dirac operator is said to represent a square root of the Klein-Gordon operator, and the generators of the supersymmetric translations are said to represent square root of the Dirac operator.

The search for the “cubic root” of linear differential operator (which need not to be the Dirac operator \( \gamma^\mu \partial_\mu + m \)) consists in defining (pseudo)-differential operators whose third power, or a special cubic combination, will yield the linear differential operator we started with.

Such tentatives have been made some time ago ([15], [16], [17]), and were only partially successful. Formal solutions have been found, but their relation with the Lorentz transformations remains unclear.
The construction is based on the analogy with the supersymmetry generators, which are realized as pseudo-differential operators acting on the fields which are functions of the space-time variables \( x^\mu \) and of the anti-commuting spinorial variables \( \theta^\alpha \) and \( \bar{\theta}^\dot{\alpha} \). These variables anticommute with each other following the rule

\[
\theta^\alpha \theta^\beta + \theta^\beta \theta^\alpha = 0, \quad \bar{\theta}^\dot{\alpha} \bar{\theta}^\dot{\beta} + \bar{\theta}^\dot{\beta} \bar{\theta}^\dot{\alpha} = 0, \quad \theta^\alpha \bar{\theta}^\dot{\beta} + \bar{\theta}^\dot{\beta} \theta^\alpha = 0, \quad (23)
\]

A formal partial derivation can be defined, satisfying the following \textit{anti-Leibniz} rule:

\[
\partial_\alpha \theta^\beta = \delta^\beta_\alpha, \quad \partial_\beta \bar{\theta}^\dot{\alpha} = \delta^\dot{\alpha}_\beta; \quad \partial_\alpha \bar{\theta}^\dot{\alpha} = 0, \quad \partial_\beta \theta^\alpha = 0; \quad \partial_\alpha (\theta^\beta \theta^\gamma) = \delta^\beta_\alpha \theta^\gamma - \delta^\gamma_\alpha \theta^\beta, \quad \text{etc}. \quad (24)
\]

The generators of the supersymmetry translations are defined then as

\[
\mathcal{D}_\alpha = \partial_\alpha + \bar{\theta}^\dot{\beta} \sigma^\mu_{\alpha \dot{\beta}} \partial_\mu, \quad \bar{\mathcal{D}}_\dot{\beta} = \partial_\dot{\beta} + \theta^\alpha \sigma^\lambda_{\alpha \dot{\beta}} \partial_\lambda \quad (25)
\]

and are supposed to act on the space of generalized functions of space-time points and Grassmann variables, i.e. formal \textit{hermitian} series

\[
\Phi(x^\mu, \theta^\alpha, \bar{\theta}^\dot{\alpha}) = \phi(x^\mu) + \psi_\alpha (x^\mu) \theta^\alpha + \bar{\psi}_\dot{\beta} (x^\mu) + W_\mu (x^\lambda) \theta^\alpha \sigma^\mu_{\alpha \dot{\beta}} \bar{\theta}^\dot{\beta} + \ldots
\]

Therefore, a special quadratic combination of the supersymmetry translations yields a linear combination of space-time translations, combined with Pauli matrices as coefficients, which enables us to generate the full Poincaré group.

The existence of ternary generalization of Grassmann variables displayed in the previous Section suggests how to construct the operators acting on formal polynomial series spanned by this \( Z_3 \)-graded algebra. Introducing partial derivations with respect to these variables as follows:

\[
\partial_A \theta^B = \delta^B_A, \quad \partial_C \bar{\theta}^D = \delta^D_C, \quad \partial_A (\theta^B \theta^C) = \delta^B_A \theta^C + \omega \delta^C_A \theta^B, \quad \text{etc.,}
\]

Thus defined \( Z_3 \)-graded derivations satisfy ternary commutation relations

\[
\partial_A \partial_B \partial_C = \omega \partial_B \partial_C \partial_A, \quad \partial_A \partial_B \partial_C = \omega^2 \partial_B \partial_C \partial_A.
\]

It is easy to prove (13) that any polynomial in variables \( \theta \) of order four must vanish; if one extends the commutation rules to the entire algebra (treating, for example, all products of two \( \theta \) variables, \( \theta^A \theta^B \), as variables of degree 2, all the products of the type \( \theta^A \bar{\theta}^B \) as variables of \( Z_3 \)-degree 0, and so on, then the only surviving combinations are

\[
\mathcal{A}_1 : \{ \theta, \bar{\theta} \}, \quad \mathcal{A}_2 : \{ \bar{\theta}, \theta \}, \quad \mathcal{A}_0 : \{ 1, \theta \bar{\theta}, \bar{\theta} \theta, \theta \theta \theta, \bar{\theta} \bar{\theta} \theta \bar{\theta} \}.
\]

The (pseudo) differential operators whose third powers yield a linear differential operator should have the following form (17):

\[
\mathcal{D}_A = \partial_A + \rho^A_{\alpha BC} \bar{\theta}^B \bar{\theta}^C \mathcal{D}_\alpha + \pi^\mu_{\alpha B} \bar{\theta}^B \partial_\mu, \quad (26)
\]

\[
\mathcal{D}_B = \partial_B + \rho^B_{\dot{\alpha} AC} \theta^A \bar{\theta}^C \mathcal{D}_\dot{\alpha} + \pi^\lambda_{\dot{\alpha} B} \theta^A \partial_\lambda, \quad \mathcal{D}_\dot{\alpha} = \partial_\dot{\alpha} + \rho^\dot{\alpha}_{\dot{\beta} AC} \theta^A \bar{\theta}^B \mathcal{D}_\dot{\beta} + \pi^\lambda_{\dot{\alpha} \dot{\beta} C} \theta^A \partial_\lambda, \quad (27)
\]
where we have introduced three different types of $\mathbb{Z}_3$-graded variables $\theta$, and their conjuga-
tes. The above three combinations do not exhaust all the possibilities of construction of such operators; there are nine other combinations. It has been shown in ([15], [17]), how, under some conditions imposed on scalars (i.e. zeroth $\mathbb{Z}_3$-degree polynomials in $\theta$’s), special ternary or binary expressions in the operators $\mathcal{D}_A, \mathcal{D}_B$ and $\mathcal{D}_C$ lead to linear expres-
sions in supersymmetry generators $\mathcal{D}_{\alpha}$ and $\mathcal{D}_{\dot{\beta}}$ or in the Poincaré transla-
tions $\partial_\mu$.

A ternary linearization of the Klein-Gordon equation has been proposed recently by M. Plyushchay and M. Rausch de Traubenberg ([7]. In order to obtain a first-order differential relativistic equation of the form

$$\left(i \Gamma^\mu \partial_\mu + m \tilde{\Gamma}\right) \psi := \mathcal{D} \psi = 0,$$

such that $\mathcal{D}$ would satisfy

$$\mathcal{D}^3 \psi = -m \left(\eta^{\mu\nu} \partial_\mu \partial_\nu + m^2\right) \psi,$$

where $\eta^{\mu\nu}$ denotes the Minkowskian space-time metric, $\mu, \nu = 0, 1, 2, 3$, one has to intro-
duce a new cubic algebra, called Clifford algebra of polynomials, which is defined as follows. Let us denote by $S_3(a, b, c)$ the sum of all permutations of the product $abc$,

$$S_3(a, b, c) = \frac{1}{6} (abc + bca + cab + acb + cba + bac);$$

then we require the following identities to hold:

$$S_3(\tilde{\Gamma}, \tilde{\Gamma}, \tilde{\Gamma}) = \tilde{\Gamma}^3 = -1; \quad S_3(\Gamma^\mu, \tilde{\Gamma}, \tilde{\Gamma}) = 0;$$

$$S_3(\Gamma^\mu, \Gamma^\nu, \tilde{\Gamma}) = \frac{1}{3} \eta^{\mu\nu}, \quad S_3(\Gamma^\mu, \Gamma^\rho, \Gamma^\lambda) = 0.$$

The group of outer automorphisms of this algebraic structure is $SO(3, 1) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

The two factors $\mathbb{Z}_2$ correspond to the $PT$-invariance of the constitutive relations, whereas the factor $\mathbb{Z}_3$ is associated with the obvious automorphism generated by the substitution

$$(\Gamma^\mu, \tilde{\Gamma}) \rightarrow (\omega \Gamma^\mu, \omega \tilde{\Gamma}).$$

With the generators $\Gamma^\mu$ and $\tilde{\Gamma}$, the following Minkowskian 4-vectors can be constructed:

$$W^\mu = \tilde{\Gamma}^2 \Gamma^\mu, \quad \tilde{W}^\mu = \Gamma^\mu \tilde{\Gamma}^2, \quad \hat{W}^\mu = \tilde{\Gamma} \Gamma^\mu \tilde{\Gamma}.$$

It is easy to check that one has $W^\mu + \tilde{W}^\mu + \hat{W}^\mu = 0$.

The generators of the Lorentz transformations have been constructed in a few particular representations only, and with supplementary constraints. For example:

$$J^{(1)}_{\mu\lambda} = \hat{W}_\mu W_\lambda + W_\mu \hat{W}_\lambda + \hat{W}_\mu W_\lambda = \tilde{\Gamma}\Gamma_\mu \Gamma_\lambda + \Gamma_\mu \tilde{\Gamma} \Gamma_\lambda + \Gamma_\mu \Gamma_\lambda \tilde{\Gamma};$$

$$J^{(2)}_{\mu\lambda} = W_\mu W_\lambda + \hat{W}_\mu \hat{W}_\lambda + \hat{W}_\mu W_\lambda = \tilde{\Gamma}^2 \Gamma_\mu \tilde{\Gamma}^2 \Gamma_\lambda + \Gamma_\mu \tilde{\Gamma}^2 \Gamma_\lambda \tilde{\Gamma}^2 + \tilde{\Gamma}^2 \Gamma_\mu \Gamma_\lambda \tilde{\Gamma}^2.$$

It can be shown that the operators $\frac{i}{2} J^{(1)}_{\mu\lambda}$ satisfy the commutation rules of the Lorentz group if the following extra condition is imposed on the matrices $\Gamma_\mu$ and $\tilde{\Gamma}$:

$$\left[ (\Gamma_\mu \Gamma_\rho \Gamma_\lambda + \Gamma_\mu \Gamma_\lambda \Gamma_\rho + \Gamma_\lambda \Gamma_\mu \Gamma_\rho), \tilde{\Gamma} \right] = 0$$

The obvious aim of all these constructions is to produce a model of algebraic quark confinement.
5. $Z_3$-GRADED EXTERIOR DIFFERENTIALS.

The $Z_3$-graded ternary analogue of Grassmann algebra suggests the existence of a generalized exterior differential calculus based on cubic commutation relations. As a matter of fact, such differential calculus has been developed in a series of papers in the 90-ties (\cite{13}, \cite{18}).

The starting point for the introduction of exterior differentials is the observation that while the first differentials of local coordinates on a manifold transform naturally as vectors, this does not remain true for the second and higher order differentials. As a matter of fact, consider formal first, then second-order differentials of a function $f(x^k)$ defined on a manifold with local coordinates $x^k$:

$$df = (\partial_k f) \, dx^k; \quad d^2 f = (\partial^2_{ik} f) \, dx^i \, dx^k + (\partial_k f) \, d^2 x^k,$$

(33)

It becomes obvious that in order to ensure the nilpotency of the operator $d$, i.e. $d^2 = 0$, one has to assume that the product of 1-forms $dx^i$ is antisymmetric, $dx^i \wedge dx^k = -dx^k \wedge dx^i$. However, if this condition is not imposed, then $d^2 x^k \neq 0$, and it combines with the symmetric part of the product $dx^i x^k$. Then it is not difficult to impose third-order nilpotency, $d^3 = 0$.

Let $M$ be a smooth $n$-dimensional manifold and let $\omega$ be a 3-rd primitive root of unity, $\omega = e^{2\pi i/3}, \omega^3 = 1$. Let $U$ be an open subset of $M$ with local coordinates $x_1, x_2, \ldots, x_n$. Our aim is to construct an analogue of the exterior algebra of differential forms with exterior differential $d$ satisfying the $\omega$-Leibniz rule

$$d(\phi \theta) = d\phi \, \theta + \omega^{|\phi|} \, \phi \, d\theta,$$

(34)

where $\phi, \theta$ are complex valued differential forms, $|\phi|$ is the degree of $\phi$, and $d^3 = 0$. We shall also assume that as in the usual $Z_2$-graded case, $d$ is a linear operator that raises the degree of any exterior form by one. As in the usual exterior differential calculus, we identify the vector space of differential forms of degree zero, denoted by $\Omega^0(M)$, with the space of smooth functions on the manifold $M$. We shall assume that there is no difference between the vector space of differential forms of degree one in our case (denoted by $\Omega^1(M)$) and the same vector space in the case of the classical exterior algebra. Thus $\Omega^1(M) = \{ \omega_i \, dx^i \}$: where $\omega_i = \omega_i(x^k)$, $i = 1, 2, \ldots, n$ are smooth functions on $M$.

The assumption $d^2 \neq 0$ implies that there is no reason to use only the first order differentials $dx_i$ in the construction of the algebra of differential forms induced by $d$; one can also add a set of formal second-order differentials, in which case the algebra will be generated by

$$dx^1, \ldots, dx^n, \ldots, d^2 x^1, \ldots, d^2 x^n.$$

In order to endow the algebra of differential forms with appropriate $Z_3$-grading we shall associate the degree $k$ to each differential $d^k x^i$. As usual, the degree of the product of differentials is the sum of the degrees of its components modulo 3. Given any smooth function $f$ and successively applying to it the exterior differential $d$ one obtains the following expressions for the first three steps:

$$df = (\partial_i f) \, dx^i,$$

$$d^2 f = (\partial^2_{ij} f) \, dx^i dx^j + (\partial_i f) \, d^2 x^i,$$

(35)

$$d^3 f = (\partial^3_{ijk} f) \, dx^i dx^j dx^k + (\partial^2_{ij} f) \, (d^2 x^i, dx^j)_\omega + (\partial_i f) \, d^3 x^i.$$

(36)
Because the partial derivatives of a smooth function do commute, only the totally symmetric combinations of indices are relevant here. This is why in the above formula the parentheses mean the symmetrization with respect to the superscripts they contain, i.e.

\[ dx^i(dx^j + dx^j dx^i) = \frac{1}{2!} (dx^i dx^j + dx^j dx^i), \]  

(37)

\[ dx^i(dx^j dx^k) = \frac{1}{3!} \sum_{\pi \in S_3} dx^{\pi(i)} dx^{\pi(j)} dx^{\pi(k)} \]  

(38)

and \( (d^2 x^i, dx^j)\omega = d^2 x^{(i} dx^{j)} + (1 + \omega) dx^{(i} d^2 x^{j)} \).

(39)

In order to guarantee that \( d^3 f = 0 \) for any smooth function on \( M \), the following three conditions have to be satisfied:

\[ dx^i(dx^j dx^k) = 0, \quad d^2 x^{(i} dx^{j)} + (1 + \omega) dx^{(i} d^2 x^{j)} = 0, \quad d^3 x^i = 0. \]  

(40)

These relations represent the minimal set of conditions that should be imposed on the differentials in order to ensure \( d^3 = 0 \). From the first condition it is obvious that first differentials are always 3-nilpotent, \( (dx^k)^3 = 0 \). On the other hand the equations \( \text{(40)} \) demonstrate clearly that generally there are no relations implying the nilpotency of any power for the differentials of higher order. Therefore though the algebra generated by the relations \( \text{(40)} \) is finite-dimensional with respect to the first order differentials because of \( (dx^k)^3 = 0 \), it remains infinite-dimensional with respect to the entire set of differentials.

We solve the first condition in \( \text{(40)} \) by assuming that each cyclic permutation of any three differentials of first order is accompanied by the factor \( \omega \) which in this case is a primitive cubic root of unity and satisfies the identity \( 1 + \omega + \omega^2 = 0 \).

Thus we assume that each triple of differentials of first order \( dx^i, dx^j, dx^k \) is subjected to ternary commutation relations

\[ dx^i dx^j dx^k = \omega dx^j dx^k dx^i. \]  

(41)

These ternary commutation relations can not be made compatible with binary commutation relations of any kind. Therefore we suppose that all binary products \( dx^i dx^j \) are independent quantities. The second condition in \( \text{(40)} \) can be easily solved by assuming the following commutation relations:

\[ dx^i d^2 x^l = \omega d^2 x^l dx^i. \]  

(42)

Note that from \( \text{(11)} \) and \( \text{(12)} \) it follows that the above ternary and binary commutation relations are coherent with the \( Z_3 \)-grading, i.e. the quantities \( dx^k dx^m \) and \( d^2 x^3 \) behave as elements of degree 2 and could be interchanged in the formulae \( \text{(11)} \) and \( \text{(12)} \).

The ternary commutation relations \( \text{(11)} \) are much stronger than the cubic nilpotence which follows from the first relation of \( \text{(40)} \). It has been proved in \( \text{(15)} \) that if the generators of an associative algebra obey ternary commutation relations such as \( \text{(11)} \) then all the expressions containing four generators should vanish. This means that the highest degree monomials which can be made up of the first order differentials have the form \( dx^i dx^j dx^k, dx^i(dx^j)^2 \). In order to construct an algebra with self-consistent structure we shall extend this fact to the higher order differentials supposing that all differential forms of fourth or higher degree vanish.
If we assume that the functions commute with the first differentials, i.e. if

\[ x^k dx^m = dx^m x^k, \]

then by virtue of the \( \omega \)-Leibniz rule the second order differentials do not commute with smooth functions, because we get by differentiating the above equality we obtain

\[ d(x^k dx^m) = dx^k dx^m + x^k d^2 x^m = d(dx^m x^k) = d^2 x^m x^k + \omega dx^m dx^k \]

which leads to the identity

\[ x^k d^2 x^m - d^2 x^m x^k = \omega (dx^k dx^m - \omega^2 dx^m dx^k) \quad (43) \]

In what follows, we shall consider only the expressions in which the forms of different degrees are multiplied on the left by smooth functions of the coordinates \( x^k \), which means that we consider the algebra \( \Omega(M) \) as a free finite-dimensional left module over the algebra of smooth functions.

It is quite easy to evaluate the dimension of the module \( \Omega(M) \), which is \( \mathcal{N} = (n^2 + 6n^2 + 5n)/3 \).

This \( Z_3 \)-graded of exterior differential calculus has been also realized in other representations, among others, as a differential algebra of operators acting on a generalized Clifford algebra (\cite{19}), or in other matrix representations; a covariant formulation of this calculus, including naturally the notions of generalized connections and curvatures, has been elaborated recently (\cite{18}, \cite{20}, \cite{19} \cite{21}).

The homological content of the theory becomes richer than in ordinary case, because now one can define not only the spaces \( \text{Ker}(d) \), \( \text{Im}(d) \), but also \( \text{Ker}(d^2) \) and \( \text{Im}(d^2) \), with obvious inclusions \( \text{Im}(d^2) \subset \text{Im}(d) \) and \( \text{Ker}(d) \subset \text{Ker}(d^2) \), and various quotients of those; for the general case of differential calculus based on the postulate \( d^N = 0 \), the full theory is exposed in \( \cite{22} \cite{23} \cite{24} \).

An interesting application of these cohomologies has been recently found by M. Dubois-Violette and I.T. Todorov (\cite{25}) in relation with the WZNW model and a generalization of the corresponding BRS-symmetry operator \( A \) satisfying \( A^h = 0 \) with \( h = 2n + 1, n = 1, 2, ... \).

An alternative way of realizing exterior differential calculus with \( d^3 = 0 \) has been proposed by M. Dubois-Violette and M. Henneaux (\cite{26}). Instead of \( Z_3 \)-grading, one considers all possible tensor fields whose Young diagrams have no more than \( two \) columns. By differentiating these fields and then using the appropriate symmetrization procedure, we can define a coherent differential calculus with \( d^3 = 0 \), which may prove useful in handling higher spins, in particular, the graviton field. As a matter of fact, in order to arrive at physically relevant field, represented in General Relativity by Riemann tensor, starting from the metric field \( g_{\mu\nu} \), we have to differentiate \( twice \). Subsequently, the field equations can be cast into the form of \( d^3(g) = 0 \).

We have given here a very shortened overview of “ternary ideas” in Mathematical Physics; we believe that many interesting applications are still ahead of us.

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