Measurement of the radial mode spectrum of photons through a phase-retrieval method

Saumya Choudhary∗
The Institute of Optics, University of Rochester, Rochester, NY 14627, USA

Rachel Sampson
CREOL, The College of Optics and Photonics at UCF, Orlando, FL 32816, USA

Yoko Miyamoto
Department of Engineering Science, The University of Electro-Communications,
1-5-1 Chofugaoka, Chofu, Tokyo 182-8585, Japan

Omar S. Magaña-Loaiza
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA

Seyed Mohammad Hashemi Rafsanjani
Department of Physics, University of Miami, Coral Gables, Florida 33146, USA

Mohammad Mirhosseini
Department of Applied Physics and Material Science,
California Institute of Technology, Pasadena, CA 91125, USA

Robert W. Boyd
The Institute of Optics, University of Rochester, Rochester, NY 14627, USA and
Department of Physics, University of Ottawa, Ottawa, ON KIN 6N5, Canada

(Dated: September 24, 2018)

We propose and demonstrate a simple and easy-to-implement projective-measurement protocol to determine the radial index $p$ of a Laguerre-Gaussian ($LG_l^p$) mode. Our method entails converting any specified high-order $LG_0^p$ mode into a near-Gaussian distribution that matches the fundamental mode of a single-mode fiber (SMF) through the use of two phase-screens (unitary transformations) obtained by applying a phase-retrieval algorithm. The unitary transformations preserve the orthogonality of modes and guarantee that our protocol can, in principle, be free of crosstalk. We measure the coupling efficiency of the transformed radial modes to the SMF for different pairs of phase-screens. Because of the universality of phase-retrieval methods, we believe that our protocol provides an efficient way of fully characterizing the radial spatial profile of an optical field.

Laguerre-Gaussian (LG) modes, $LG_l^p$, characterized by the radial index $p$ (a non-negative integer) and by an azimuthal index $l$ (an integer), are solutions of the paraxial wave equation in cylindrical coordinates [1, 2]. The LG modes are orthonormal and form a complete basis set. Both the azimuthal and radial degrees of freedom are theoretically unbounded (unlike polarization, which is 2-dimensional). This unbounded nature is potentially useful for applications such as classical and quantum communication [3–7], including quantum key distribution (QKD) [8, 9] by allowing for high-dimensional encoding, which leads to increased information capacity [10, 11]. To date, most of these applications have employed the $LG_0^l$, that is, the OAM modes of lowest radial order, for which efficient means of detection and characterization are available [12–17]. The radial modes, $LG_l^p$, have been under-utilized for such applications because of the absence of methods for detecting and measuring the radial index.

Some methods for measuring the radial index have recently been reported [18–21]. Based on the principle of a universal quantum sorter [22], Zhou et al. [18] demonstrated the sorting of even- and odd-order radial modes into separate output ports of an interferometer. However, further classification of a large number of modes would require cascading several such interferometers, thereby increasing the implementation complexity. Projective measurement presents one method for mode-sorting, and generally entails mapping a specified input mode to a particular output mode (typically a Gaussian). This output mode can then be coupled to a SMF that selects only the fundamental Gaussian component. The projective measurement method for radial modes demonstrated in [19] employs flattening the phase-front of the incoming field. However, the incomplete projection onto the SMF mode results in low (input mode-dependent) detection efficiency as well as crosstalk, which are detrimental for most quantum applications. Although the crosstalk can be reduced by selecting the central (flat intensity) portion of the phase-flattened mode, it would also decrease...
LG modes retain their transverse structure as they propagate through free space, through lenses and as they reflect off of mirrors. As a consequence, one cannot use these elements to losslessly project high-order radial modes onto a Gaussian. Morizur et al. proposed \cite{24} that unitary transformation of any given mode to another can be achieved by repeated reflection off a deformable mirror (which introduces a spatially varying transverse phase) and Fourier transformation through a lens, over several iterations. However, their protocol was only tested for the first four Hermite-Gaussian modes and required several optimization steps to arrive at the correct configuration of the deformable mirror required for high conversion efficiency. Repeated phase transformations introduced by either reflecting off a spatial light modulator (SLM) or by multiple scattering have been used in \cite{24} and in \cite{25}, respectively, to map an LG mode to a mode index-dependent position at the output. However, while the former configuration cannot be used to extract the relative phases between constituent modes in a coherent superposition, the latter suffers from high scattering losses (approximately 0.2% of input power was collected at the output).

Here, we present a simple and easy-to-implement protocol for measuring the radial mode spectrum of an optical field by transforming high-order radial modes to the fundamental mode of a SMF. In our approach, we use two phase-screens (to introduce a spatially varying transverse phase structure), placed at the object plane and at the Fourier plane of a lens respectively, to transform a fundamental mode of a SMF. In our approach, we use the Gerchberg-Saxton (GS) algorithm, [27, 28] to calculate the required phase-screens \( \Phi_1(x, y) \) and \( \Phi_2(x, y) \). The SLM1, placed at the object plane of a lens, introduces a phase \( \Phi_1(x, y) \) on the input field \( LG^p_0 \) (or \( f(x, y) \)). On propagation through the lens, the intensity distribution at the Fourier plane (or equivalently at the far-field) becomes similar to the intensity distribution of the \( LG^p_0 \) mode (or \( g(x, y) \)). However, this transformation often results in an incorrect phase distribution at the Fourier plane. The SLM2, placed at the Fourier plane, subsequently introduces another phase, \( \Phi_2(x, y) \), which corrects these residual phase errors and flattens the phase. As phase-transformations in the absence of losses are unitary, the orthogonality of these projected modes remains preserved at the SMF. Due to restrictions on the aperture size of our SLM, we choose the first three high-order radial modes \( LG^p_0 \) with \( p = \{1, 2, 3\} \) to test our protocol.

For our two-dimensional problem, the uniqueness of a phase-retrieval solution is ensured due to the non-factorability of polynomials of two or more complex variables \cite{29}. We define an error metric \( \epsilon \) in the Fourier domain to study the convergence of the GS algorithm to this solution with the number of iterations \( N_{iter} \). In many cases, if a solution exists, the algorithm converges within a few iterations. However, the algorithm can also stagnate close to the local minimum of \( \epsilon \) without converging any further \cite{26}. To study the efficiency of the algorithm at calculating the required pair of phase-screens for each input mode, we define \( \epsilon \) in terms of \( C \), the coherent mode-overlap efficiency between the projected \( LG^p_0 \) mode and the mode of the SMF, as follows

\[
\epsilon = 1 - C = 1 - |2\pi \sum_{\rho} \rho F^*_n(\rho) g_n(\rho)|^2.
\]

Here, \( \rho \) is the Fourier domain radial coordinate, \( F_n(\rho) \) is the normalized radial mode at the Fourier plane after the application of the relevant pair of phase-screens \( \Phi_1 \) and \( \Phi_2 \), and \( g_n(\rho) \) is the desired mode at the Fourier plane.
plane (normalized Gaussian). The modes are normalized such that \( 2\pi \sum_{n} (\rho E_{n}(\rho)^{2}) = 2\pi \sum_{n} (\rho g_{n}(\rho)^{2}) = 1 \). \( C \) also represents the fraction of input power that couples into the SMF after the phase-corrections. Figure 1(b) shows the variation of \( \epsilon \) with \( N_{\text{iter}} \), where \( N_{\text{iter}} \) is the number of iterations, for the different input radial modes (\( LG_{1} \) to \( LG_{3} \)). Without the phase-corrections, or when \( N_{\text{iter}} = 0 \), \( \epsilon \) is unity as the unconverted high-order radial modes are orthogonal to the SMF mode. As \( N_{\text{iter}} \) increases, \( \epsilon \) (\( C \)) decreases (increases) for all modes until it stagnates close to 0.18 after approximately 35 iterations. This non-negligible error represents an overall loss of power coupled into the SMF. However, as we show later in the paper, this loss of power is not as consequential as there is negligible crosstalk due to the preserved orthogonality of modes in our protocol (see figure 3(a)).

For convenience and efficient resource utilization, the first phase-screen (see figures 3(a)-(d)) is implemented along with the mode generation on SLM1 itself. The azimuthally-averaged intensity distributions for each input radial mode at the image plane and the Fourier plane, recorded using a CCD camera, were found to differ from the theoretical intensities by a root-mean-square (rms) error of less than 10% (see figure 3 in the appendix), thereby confirming that the generated radial modes are of good fidelity. After the second phase-correction impressed by SLM2 (see figures 3(e)-(h)), the transformed modes are coupled via an imaging system and a microscope objective to a SMF-coupled photodetector. Figures 3(i)-(l) show the recorded intensities at the Fourier plane (or SLM2) of the various input modes after the application of the corresponding pair of phase-screens.

The coupling efficiency of a given mode to the SMF is taken to be the ratio of the power coupled into the fiber to the total power incident on the microscope objective. This method of calculation allows us to account for any losses prior to the objective. The coupling efficiencies of different input radial modes (\( LG_{0} \) to \( LG_{3} \)) for various pairs of phase-screens are obtained similarly, forming the ‘crosstalk matrix’. We note that although the \( LG_{0} \) mode simply requires a rescaling of the waist size for efficient coupling to the SMF, we include it in our basis set for completeness and for examining the crosstalk.

Figure 3 shows the calculated intensities and phase-fronts of different input modes (\( LG_{0} \), \( LG_{1} \), \( LG_{2} \) and \( LG_{3} \)) at the Fourier plane after the application of phase-screens calculated for the \( LG_{2} \) mode. Essentially, one obtains a nearly-Gaussian amplitude and a flat-phase at the Fourier plane only when the applied phase-screens corre-
approximately 81% for LG with increasing radial index, from 99% for agonal elements of figure 5(a) decreases monotonically figure 1(b), the conversion efficiency of radial modes (diagonal elements of the matrices in figures 5(b) and 5(a), we see that the coupling efficiencies for all the modes, with the correct pair of phase-screens is lower than the calculated result. Also, except for LG (or \( p_{in} = 3 \)), the measured crosstalk values, given by the off-diagonal terms, are less than 15%.

The lower coupling efficiency measured overall, and the observed crosstalk for radial indices larger than 1 could be due to a combined effect of the following factors: (1) calibration errors at each pixel on the SLMs (Correct calibration of each pixel is a stringent requirement for SLM2. SLM1 has a grating for mode generation. Therefore, any calibration errors therein would manifest in the diffraction efficiency, and not as significantly in the profile of the generated mode.); (2) imperfect optics, and the presence of aberrations such as astigmatism and spherical aberration in the imaging systems before and after SLM2; (3) crosstalk between the pixels on the SLMs (called the fringing effect in [31]), which becomes significant whenever the phase wraps from 2\( \pi \) to 0 gradually over a few pixels instead of sharply. A possible solution for improving the coupling efficiency and lowering the crosstalk could be to use a genetic algorithm similar to the one used in [20] to calibrate and correct for the phase errors due to the SLMs as well as the imaging systems. Also, a hologram with high spatial resolution should reduce the crosstalk between adjacent pixels during phase-wrapping.

To summarize, we have proposed and provided a proof-of-principle demonstration of a new protocol for determining the radial mode decomposition of an optical field. The protocol utilizes two phase transformations, one at the object and the other in the Fourier plane of a lens, to convert high-order radial modes to the fundamental mode of a SMF. The required phase transformations were calculated using the Gerchberg-Saxton phase-retrieval algorithm. The implementation is straightforward and does not require complicated setups and optimization. Also, as it utilizes only phase-corrections, our procedure for maximizing the coherent mode overlap between the high-order LG modes and the Gaussian mode is intrinsically non-lossy. We believe that by improving on the implementation, as suggested in the preceding paragraph, the performance can be improved further.

The universal nature of the phase-retrieval algorithm suggests the future use of this protocol for measurements of the azimuthal mode structure in addition to the radial modes, or for measurements in other bases including the Hermite-Gaussian basis. This possibility opens up a plethora of applications pertaining to classical and quantum communication, and to quantum computation that benefit from the increased information capacity obtained by accessing the entire transverse degree of freedom of photons. In addition, this protocol may be useful for applications such as super-resolution and quantum state tomography.
The authors thank N. Treps, Y. Zhou, J. Zhao, T. Gerrits and L. Hernandez for helpful discussions. The authors also acknowledge funding support from the U.S. Office of Naval Research (grant number: ONR N00014-17-1-2443). Y. M. acknowledges funding support from JSPS KAKENHI (grant number: JP16K05499).

*schoudha@ur.rochester.edu

[1] Peter W. Milonni and Joseph H. Eberly. *Laser Physics*, pages 269–329. John Wiley & Sons, Inc., 2010.

[2] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman. Orbital angular momentum of light and the transformation of laguerre-gaussian laser modes. *Phys. Rev. A*, 45:8185–8189, Jun 1992.

[3] J. Wang, J. Y. Yang, I. M Fazal, N. Ahmed, Y. Yan, H. Huang, Y. Ren, Y. Yue, S. Dolinar, M. Tur, et al. Terabit free-space data transmission employing orbital angular momentum multiplexing. *Nat. Photon.*, 6(7):488–496, 2012.

[4] N. Bozinovic, Y. Yue, M. Ren, Y. and Tur, P. Kristensen, H. Huang, A. E. Willner, and S. Ramachandran. Terabit-scale orbital angular momentum mode division multiplexing in fibers. *Science*, 340(6140):1545–1548, 2013.

[5] A. E. Willner, H. Huang, Y. Yan, Y. Ren, N. Ahmed, G. Xie, C. Bao, L. Li, Y. Cao, Z. Zhao, J. Wang, M. P. J. Lavery, M. Tur, S. Ramachandran, A. F. Molisch, N. Ashrafii, and S. Ashrafii. Optical communications using orbital angular momentum beams. *Adv. Opt. Photon.*, 7(1):66–106, Mar 2015.

[6] G. Xie, Y. Ren, Y. Yan, H. Huang, N. Ahmed, L. Li, Z. Zhao, C. Bao, M. Tur, S. Ashrafii, and A. E. Willner. Experimental demonstration of a 200-gbit/s free-space optical link by multiplexing laguerre-gaussian beams with different radial indices. *Optics letters*, 41(15):3447–3450, 2016.

[7] G. Gibson, J. Courtial, M. J. Padgett, M. Vassetsos, V. Pas’ko, S. M. Barnett, and S. Franke-Arnold. Free-space information transfer using light beams carrying orbital angular momentum. *Opt. Exp.*, 12(22):5448–5456, Nov 2004.

[8] M. Mirhosseini, O. S. Magaña-Loaiza, M. N O’Sullivan, B. Rodenburg, M. Malik, M. P. J. Lavery, M. J. Padgett, D. J Gauthier, and R. W. Boyd. High-dimensional quantum cryptography with twisted light. *J. of Phys.*, 17(3):033033, 2015.

[9] Alicia Sit, Frédéric Bouchard, Robert Fickler, Jérémie Gagnon-Bischoff, Hugo Larocque, Khabat Heshami, Dominique Esler, Christian Peuntinger, Kevin Günther, Bettina Heim, Christoph Marquardt, Gerd Leuchs, Robert W. Boyd, and Ebrahim Karimi. High-dimensional intrinsic quantum cryptography with structured photons. *Optica*, 4(9):1006–1010, Sep 2017.

[10] Mohamed Bourennane, Anders Karlsson, Gunnar Björk, Nicolas Gisin, and Nicolas J Cerf. Quantum key distribution using multilevel encoding: security analysis. *Journal of Physics A: Mathematical and General*, 35(47):10065, 2002.

[11] Mario Krenn, Marcus Huber, Robert Fickler, Radek Lapkiewicz, Sven Ramelow, and Anton Zeilinger. Generation and confirmation of a (100 x 100)-dimensional entangled quantum system. *Proceedings of the National Academy of Sciences*, 111(17):6243–6247, 2014.

[12] M. Mirhosseini, M. Malik, Z. Shi, and R. W. Boyd. Efficient separation of the orbital angular momentum eigenstates of light. *Nat. Comm.*, 4, 2013.

[13] G. C. G Berkhout, M. P. J. Lavery, J. Courtial, M. W. Beijersbergen, and M. J. Padgett. Efficient sorting of orbital angular momentum states of light. *Phys. Rev. Lett.*, 105(15):153601, 2010.

[14] J. Leach, J. Courtial, K. Skeldon, S. M. Barnett, S. Franke-Arnold, and M. J. Padgett. Interferometric methods to measure orbital and spin, or the total angular momentum of a single photon. *Phys. Rev. Lett.*, 92(1):013601, 2004.

[15] E. Karimi, B. Piccirillo, E. Nagali, L. Marrucci, and E. Santamato. Efficient generation and sorting of orbital angular momentum eigenmodes of light by thermally tuned q-plates. *App. Phys. Lett.*, 94(23):231124, 2009.

[16] T. Su, R. P. Scott, S. S. Djordjevic, N. K. Fontaine, D. J. Geisler, X. Cai, and S. J. B. Yoo. Demonstration of free space coherent optical communication using integrated silicon photonic orbital angular momentum devices. *Opt. Exp.*, 20(9):9396–9402, 2012.

[17] A. Mair, A. Vaziri, G. Weihs, and A. Zeilinger. Entanglement of the orbital angular momentum states of photons. *Nature*, 412(6844):313–316, 2001.

[18] Yiyu Zhou, Mohammad Mirhosseini, Dongzhi Fu, Jinping Zhao, Seyed Mohammad Hashemi Rafsanjani, Alan E. Willner, and Robert W. Boyd. Sorting photons by radial quantum number. *Phys. Rev. Lett.*, 119:263602, Dec 2017.

[19] H. Qassim, F. M. Miatto, J. P. Torres, M. J. Padgett, E. Karimi, and R. W. Boyd. Limitations to the determination of a laguerre-gauss spectrum via projective, phase-flattening measurement. *J. Opt. Soc. Am. B*, 31(6):A20–A23, Jun 2014.

[20] R. Fickler, M. Ginoya, and R. W. Boyd. Custom-tailored spatial mode sorting by controlled random scattering. *Phys. Rev. B*, 95:161108, Apr 2017.

[21] G. Xie, Y. Ren, H. Huang, N. Ahmed, L. Li, Z. Zhao, C. Bao, M. Tur, S. Ashrafii, and A. E. Willner. Experimental demonstration of a 200-gbit/s free-space optical link by multiplexing laguerre-gaussian beams with different radial indices. *Optics letters*, 41(15):3447–3450, 2016.

[22] G. Gibson, J. Courtial, M. J. Padgett, M. Vassetsos, V. Pas’ko, S. M. Barnett, and S. Franke-Arnold. Free-space information transfer using light beams carrying orbital angular momentum. *Opt. Exp.*, 12(22):5448–5456, Nov 2004.

[23] M. Mirhosseini, O. S. Magaña-Loaiza, M. N O’Sullivan, B. Rodenburg, M. Malik, M. P. J. Lavery, M. J. Padgett, D. J Gauthier, and R. W. Boyd. High-dimensional quantum cryptography with twisted light. *J. of Phys.*, 17(3):033033, 2015.

[24] Radu Ionicioiu. Sorting quantum systems efficiently. *Scientific Reports*, 6:25356 EP –, 05 2016.

[25] F. Bouchard, N. H. Valencia, F. Brandt, R. Fickler, M. Huber, and M. Malik. Measuring azimuthal and radial modes of photons. *arXiv preprint arXiv:1808.03533*, 2018.

[26] J. F. Morizur, L. Nicholls, P. Jian, S. Armstrong, N. Treps, B. Hage, M. Hsu, W. Bowen, J. Janousek, and H.A. Bachor. Programmable unitary spatial mode manipulation. *J. of Opt. Soc. of Am. A*, 27(11):2524–2531, 2010.

[27] N. K. Fontaine, R. Ryu, H. Chen, D.T. Neilson, K. Kim, and J. Carpenter. Optical spatial mode sorter of azimuthal and radial components. *arXiv preprint arXiv:1803.04126*, 2018.

[28] J. R. Fienup. Reconstruction and synthesis applications of an iterative algorithm. In *Transformations in Optical Signal Processing*, volume 373, pages 147–161. International Society for Optics and Photonics, 1984.

[29] R. W. Gerchberg and W. O. Saxton. A practical algo-
rithm for the determination of the phase from image and
diffraction plane pictures. Optik (Jena), 35:237, 1972.
[28] J. R. Fienup. Phase retrieval algorithms: a comparison. 
App. Opt., 21(15):2758–2769, 1982.
[29] J.H. Seldin and J. R. Fienup. Numerical investigation of 
the uniqueness of phase retrieval. JOSA A, 7(3):412–427, 
1990.
[30] V. Arrizón, U. Ruiz, R. Carrada, and L. A. González. 
Pixelated phase computer holograms for the accurate en-
coding of scalar complex fields. J. of Opt. Soc. of Am. 
A, 24(11):3500–3507, 2007.
[31] T. Lu, M. Pivnenko, B. Robertson, and D. Chu. Pixel-
level fringing-effect model to describe the phase profile 
and diffraction efficiency of a liquid crystal on silicon de-
vice. Applied optics, 54(19):5903–5910, 2015.

[32] M. A. Nielsen and I. L. Chuang. Quantum computation 
and quantum information. Cambridge university press, 
2010.
[33] M. Tsang, R. Nair, and X. M. Lu. Quantum theory of 
superresolution for two incoherent optical point sources. 
Physical Review X, 6(3):031033, 2016.
[34] B Jack, J Leach, H Ritsch, SM Barnett, MJ Padgett, 
and S Franke-Arnold. Precise quantum tomography of 
photon pairs with entangled orbital angular momentum. 
N. J. of Phys., 11(10):103024, 2009.
[35] William N. Plick and Mario Krenn. Physical meaning of 
the radial index of laguerre-gauss beams. Phys. Rev. A, 
92:063841, Dec 2015.
APPENDIX

Fidelity of generated radial modes

The Laguerre-Gaussian (LG) modes propagating along \( z \) with waist size \( w_0 \) can be written in the position representation in cylindrical coordinates as \([35]\)

\[
LG^l_p(r, \phi, z) = \sqrt{\frac{2p!}{\pi(p + |l|)!}} \frac{1}{w_z} \left( \frac{\sqrt{2r}}{w_z} \right)^{|l|} L^{|l|}_{p+|l|} \left( \frac{2r^2}{w_z^2} \right) \cdot \exp \left( -\frac{r^2}{w_z^2} + i \left( l\phi + \frac{kr^2}{2R_z} - (2p + |l| + 1)\phi_g \right) \right).
\]

where \( w_z = w_0 \sqrt{1 + \left( \frac{z}{z_0} \right)^2} \) is the waist size of the mode after propagating a distance \( z \), \( z_0 = \pi w_0^2 / \lambda \) is the Rayleigh range, \( R_z = z + z_0^2 / z \) is the radius of curvature of the wavefront, and \( \phi_g = \tan^{-1} z / z_0 \) is the Guoy phase shift. The radial modes considered here have a zero azimuthal index, or \( l = 0 \).

As a first step, the fidelity of the generated modes from SLM1 was verified. Without adding any correcting phases on the SLM1, the intensities of different \( LG^0_p \) modes were captured by a CCD camera at the Fourier plane, as well as the image plane of SLM1. The mean intensity distributions were then obtained from the recorded intensities by averaging over the azimuthal coordinates. The average measured intensity distributions were then compared with the theoretical intensities obtained by taking the squared modulus of the field distribution of \( LG^0_p(r, \phi, z = 0) \).

Figures 6(a-d) and (e-h) compare the averaged intensity distributions of \( LG^0_p \) modes, obtained experimentally (blue) and theoretically (red), at the Fourier plane and at the image plane respectively. The intensity of the modes recorded on a CCD camera are shown in the insets. The root-mean-squared error between the theoretical cross-section and the cross-section obtained from the experiment is less than 10% for both the image plane and the Fourier plane (as indicated in the inset), which confirms that the generated \( LG^0_p \) modes are of high fidelity.

Tables of coupling efficiencies

Tables I and II show the calculated and the measured coupling efficiencies \((C_{pin,p_{screen}})\) respectively, of the various \( LG^0_{pin} \) modes for the different pairs of added phase-screens (denoted by the radial index \( p_{screen} \)). The coupling efficiencies are plotted in matrix form as the two crosstalk matrices shown in figure 5(a) and 5(b). The conditional probability \((P_{c_{pin,p_{screen}}})\) corresponds to the probability of detecting the \( LG^0_{pin} \) mode provided that the Hilbert space comprises \( \{LG^0_0, LG^0_1, LG^0_2, LG^0_3\} \). The conditional probability for both simulations and the experiment is given by

\[
P_{c_{pin,p_{screen}}} = \frac{C_{pin,p_{screen}}}{\sum_{pin} C_{pin,p_{screen}}}
\]
FIG. 6. Azimuthally averaged intensity distributions of the generated radial modes at the (a-d) Fourier plane, and at the (e-h) image plane of SLM1 recorded in the experiment (blue), and from theory for radial modes $LG_{0}^0, LG_{0}^1, LG_{0}^2$ and $LG_{0}^3$ (from top to bottom). Insets show the field intensities recorded by a CCD camera. The goodness-of-fit is represented as the percent root-mean-square-error (RMSE) on the plots.

| $p_{\text{screen}}$ | $p_{\text{in}}$ | Coupling Efficiency | Conditional Probability |
|---------------------|----------------|---------------------|-------------------------|
| 0                   | 0              | 0.9947              | 0.994501                |
| 0                   | 1              | 0.0001              | 0.00009998              |
| 0                   | 2              | 0.0018              | 0.00179964              |
| 0                   | 3              | 0.0036              | 0.00359928              |
| 1                   | 0              | 0.0000              | 0.0000                  |
| 1                   | 1              | 0.8310              | 0.996761                |
| 1                   | 2              | 0.0001              | 0.000119947             |
| 1                   | 3              | 0.0026              | 0.00311863              |
| 2                   | 0              | 0.0000              | 0.0000                  |
| 2                   | 1              | 0.0025              | 0.0030499               |
| 2                   | 2              | 0.8170              | 0.996706                |
| 2                   | 3              | 0.0002              | 0.000243992             |
| 3                   | 0              | 0.0000              | 0.0000                  |
| 3                   | 1              | 0.0173              | 0.0209468               |
| 3                   | 2              | 0.0004              | 0.00048432              |
| 3                   | 3              | 0.8082              | 0.978569                |

TABLE I. Table of calculated coupling efficiencies and conditional probabilities of the $LG_{p_{\text{in}}}^0$ modes for the pairs of phase-screens corresponding to radial index $p_{\text{screen}}$. 
| $p_{\text{screen}}$ | $p_{\text{in}}$ | Coupling Efficiency $C_{p_{\text{in}},p_{\text{screen}}}$ | Conditional Probability $P_{C_{p_{\text{in}},p_{\text{screen}}}}$ |
|-----------------|----------------|---------------------|---------------------|
| 0               | 0              | 0.30837             | 0.950736            |
| 0               | 1              | 0.00686248          | 0.0213656           |
| 0               | 2              | 0.00497067          | 0.0154757           |
| 0               | 3              | 0.00390011          | 0.0124228           |
| 1               | 0              | 0.00399687          | 0.0151793           |
| 1               | 1              | 0.245118            | 0.93091             |
| 1               | 2              | 0.00608885          | 0.0231242           |
| 1               | 3              | 0.00810648          | 0.0307868           |
| 2               | 0              | 0.0111205           | 0.0538439           |
| 2               | 1              | 0.0192632           | 0.0932698           |
| 2               | 2              | 0.150514            | 0.728766            |
| 2               | 3              | 0.0256349           | 0.124121            |
| 3               | 0              | 0.0153835           | 0.0718829           |
| 3               | 1              | 0.00634927          | 0.0296683           |
| 3               | 2              | 0.0480315           | 0.224437            |
| 3               | 3              | 0.144244            | 0.674011            |

TABLE II. Table of measured coupling efficiencies and conditional probabilities of the $L_{\nu}^{0}$ modes for the pairs of phase-screens corresponding to radial index $p_{\text{screen}}$. 