Axion Gravitodynamics, geodetic effects and gravitational waves

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INTRODUCTION

We investigate physical implications of a gravitational analog of axion electrodynamics with a parity violating gravitoelectromagnetic theta term. This is related to the Nieh-Yan topological invariant in gravity with torsion, in contrast to the well-studied gravitational Chern-Simons term quadratic in curvature, coupled via a dynamical axion-like scalar field. Axion gravitodynamics is the corresponding linearized theory. We find that potentially observable effects are of order of magnitude stronger for its Chern-Simons counterpart and could thus be in reach for detection by experiments in the near future. For a near-earth scenario, we derive corrections to the Lense-Thirring geodetic effect and compare them to data from satellite-based experiments (Gravity Probe B). For gravitational waves we find modified dispersion relations, derive the corresponding polarization-dependent modified group and phase velocities and compare them to data from neutron star mergers (GW170817) to derive even stronger bounds.

The linearized limit of general relativity plays a central role in both cases above. Notably, Einstein’s equations in this weak field limit are similar to Maxwell’s equations for electrodynamics, with the electric and magnetic field replaced by the gravitational and the gravitomagnetic field. The latter is generated by rotating masses and it is the backbone of the Lense-Thirring geodetic effect [3]. Having this analogy in mind, a natural question regards the existence and the potential physical effects of a gravitational theta term, the gravitational counterpart of the electrodynamical field. The latter is generated by rotating charges and it is the backbone of the Peccei-Quinn mechanism in QCD, we promote the parameter \( \theta \) in the notation of the fields from now on until the conclusion, since we will always refer to the gravitational and never to the electromagnetic ones. Remarkably, this term has been used in the recent past to study the gravitational response of topological materials [9–12]. Its nonlinear origin was proposed in Ref. [13] to be the teleparallel equivalent of general relativity with the Weitzenböck connection, modified with the Nieh-Yan term [14], the quadratic in torsion topological invariant in four dimensions [15-16]. It is worth noting here that the role of torsion in the gravitational response of materials is already implicit in Luttinger’s study of thermal transport coefficients [17] (see also [18] for further developments regarding torsion in condensed matter systems). On the other hand, the Nieh-Yan term has found numerous applications in recent years, for instance in modified Einstein-Cartan [19] or teleparallel gravity [20] and in cosmological scenarios [21-22] and moreover it is directly related to the Holst term that has played a pivotal role in loop quantum gravity [23-27].

Our purpose in this paper is to study the effect of axion gravitodynamics on the precession of gyroscopes (geodetic effect) and on the propagation of gravitational waves. In a similar spirit to the Peccei-Quinn mechanism in QCD, we promote the parameter \( \theta \) to an axion-like field \( \theta(t,\mathbf{x}) \) [28]. This introduces a new dimensionful scale, essentially the decay constant of the axion-like field, which should be bounded by observations.

The setting is analogous to the Chern-Simons modification of general relativity [29], which was used in Ref. [30] to calculate corrections to the general-relativistic gravitomagnetic field around massive spin-
ning bodies in case \( \theta \) is only time-dependent (as in the quintessence scenario). Although a parity-violating interaction between the gravitoelectric and gravitomagnetic fields is introduced in both settings \([31]\), the crucial difference is that the Chern-Simons modification is based on higher derivatives (since the curvature and torsion tensors differ by one derivative) and therefore an \( \vec{E} \cdot \vec{B} \) term is not sufficient for a nontrivial coupling to \( \theta \). On the contrary, in axion gravitodynamics it is precisely the \( \vec{E} \cdot \vec{B} \) term that couples to \( \theta \), which leads to different quantitative predictions. What is more, we argue that the effect of this term in axion gravitodynamics is much stronger and therefore if it exists it becomes much more relevant than its Chern-Simons counterpart. Comparing with the Gravity Probe B experiment, we find that the coupling is very weak and therefore it should either be detected in upcoming experiments or it will be ruled out soon.

In a similar spirit to the modification induced by the electromagnetic theta term to the propagation of light \([32]\), we find that the dispersion relation of gravitational waves is modified in axion gravitodynamics. This leads to a splitting of the group and phase velocities of the two polarizations of the gravitational wave, which is a direct consequence of parity and Lorentz violation. Using the observed discrepancy between the group velocity of the gravitational wave event GW170817 and the speed of light \([33]\), we are able to place an additional bound on the coupling of the gravitational theta term. This bound is stricter than the one obtained from geodetic effects by five orders of magnitude.

**AXION GRAVITODYNAMICS**

The starting point of our analysis is the gravitational action functional (we work in units where \( c = 1 \))

\[
S = \frac{1}{2\kappa^2} \int d^4 x e \left( -\mathcal{T} + \frac{\ell \theta}{4} \epsilon^{\mu\nu\rho\sigma} T_{\mu \nu \lambda} T_{\rho \sigma}^{\lambda} \right) - \int d^4 x \left( \frac{1}{2}(\partial \theta)^2 + V(\theta) \right) + S_M , \tag{1}
\]

where \( \kappa^2 = 8\pi G \), \( \epsilon_{\mu\nu\rho\sigma} \) is the Levi-Civita symbol and \( \mathcal{T} \) is the following linear combination of the three parity-preserving Weitzenböck invariants:

\[
\mathcal{T} = \frac{1}{4} T_{\mu \nu \rho} T^{\mu \nu \rho} + \frac{1}{2} T_{\mu \nu \rho} T^{\rho \mu \nu} - T_{\mu \nu} \epsilon^{\rho \mu \nu} . \tag{2}
\]

Here we have denoted the components of the torsion tensor by \( T_{\mu \nu} = 2 \epsilon_{\mu \nu} \partial [e^a \wedge e^b \wedge T_{ab}] \), written in terms of the vierbein and its inverse. It is known that the scalar combination \([2]\) underlies the teleparallel equivalent to general relativity. The remaining term quadratic in torsion is closely related to the Nieh-Yan topological invariant 4-form, for which an arbitrary connection reads as

\[
\mathcal{N} = T_a \wedge T^a - e^a \wedge e^b \wedge R_{ab} = d(e^a \wedge T_a) \tag{3}
\]

in terms of the curvature and torsion 2-forms, \( R_{ab} \) and \( T_a \). In a coordinate basis, this equation implies that the local components of the objects involved satisfy

\[
\epsilon^{\mu\nu\rho\sigma} \left( \frac{1}{2} T_{\mu \nu \lambda} T_{\rho \sigma}^{\lambda} - R_{\mu \nu \rho \sigma} \right) = \epsilon^{\mu\nu\rho\sigma} \partial_\mu T_{\nu \rho \sigma} . \tag{4}
\]

In absence of torsion (e.g. for the Levi-Civita connection) the Nieh-Yan 4-form \([2]\) vanishes identically due to the algebraic Bianchi identity for the Riemann tensor. On the other hand, choosing the curvature-free but torsion-full Weitzenböck connection, we directly observe that the corresponding term in the action \([1]\) is the Nieh-Yan invariant coupled to a (pseudo)scalar field \( \theta(x) \), which in turn has a kinetic term and a potential \( V(\theta) \). Since \( \theta \) has mass dimension 1, a scale \( \ell \) with dimensions of length is introduced in the theory.

Alternatively, making use of the fact that the T-sector of the action is equal to the Einstein-Hilbert action in the vierbein formulation up to a total derivative, one may consider as a starting point the action

\[
S' = \frac{1}{2\kappa^2} \int \left( \epsilon^{abcd} e_a \wedge e_b \wedge R_{cd} + \ell \theta \, d e^a \wedge d e^d \right) , \tag{5}
\]

along with the kinetic and potential terms for \( \theta \) and the matter action, where \( R(\omega) \) is the curvature 2-form in terms of the torsion-free spin connection \( \omega = \omega(\epsilon) \), provided that one works on a parallelizable manifold or patch where the vierbein is globally well-defined \([34]\).

The field equations obtained from this theory after variation with respect to the vierbein and the field \( \theta \) are the modified Einstein and Klein-Gordon equations

\[
G_{\mu \nu} + C_{\mu \nu} = \kappa^2 T_{\mu \nu} , \tag{6}
\]

\[
\square \theta = \frac{1}{2\kappa^2} \frac{\ell}{4} \epsilon^{\mu\nu\rho\sigma} T_{\mu \nu \lambda} T_{\rho \sigma}^{\lambda} . \tag{7}
\]

Here \( G_{\mu \nu} \) in the usual Einstein tensor and \( C_{\mu \nu} \) is obtained from the variation of the Nieh-Yan (\( \theta \)-)term with respect to the vierbein,

\[
C_{\mu \nu} = \ell \epsilon_{\mu \nu \rho \sigma} \partial^\rho \partial^\sigma T_{\nu \mu} . \tag{8}
\]

As usual, \( T_{\mu \nu} \) is the total energy-momentum tensor containing contributions both from the matter fields and from \( \theta(x) \). Simple inspection of the modified Einstein equation \([6]\) leads to two remarks: First, the antisymmetric part of \( C_{\mu \nu} \) vanishes, which can be seen as the on-shell constraint \( \epsilon^{\lambda \sigma \nu \rho} [T_{\rho \theta}^{\nu}] = 0 \) for the torsion tensor. Second, one may confirm that the divergence of \( C_{\mu \nu} \) is equal to the divergence of the energy-momentum tensor for \( \theta(x) \). This is a useful consistency check for \([6]\).

For the purposes of the present paper, we are interested in the weak field limit of this theory. Since we are not going to delve in the dynamics of the field \( \theta \), hence we consider it effectively nondynamical and treat it as a background field. As shown in \([13]\), for the metric perturbation \( h_{\mu \nu} \) and in the harmonic gauge \( \partial^\mu h_{\mu \nu} = 0 \),
where $\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h$, the action and field equations reduce to

$$S \simeq \frac{1}{8 \kappa^2} \int d^4 x \, h_{\mu\nu} (\Box \tilde{h}^{\mu\nu} - \ell \varepsilon_{\mu\rho\sigma} \partial_{\rho} \theta \, \partial_{\sigma} \tilde{h}^{\nu}) + S_M, \quad \Box \tilde{h}_{\mu\nu} - \ell \varepsilon_{\mu\rho\sigma} \partial_{\rho} \theta \, \partial_{\sigma} \tilde{h}_{\nu}^{\lambda} = -\kappa^2 T_{\mu\nu}. \quad (9)$$

The linearized field equation in vacuum is consistent with the harmonic gauge condition provided the second derivatives of $\theta$ are negligible, a requirement that is justified a posteriori below. The energy-momentum tensor has the form $T_{\mu\nu} = \rho v_{\mu} v_{\nu}$, where $\rho$ is the mass density of the source and we will use the notation $J_1 = \rho \psi_i$ for the components of the mass current vector with Latin indices denoting spatial directions.

The theory given by the action and field equations (9) is what we call axion gravitodynamics. To see why, recall that the components of $\tilde{h}_{\mu\nu}$ may be related to the Newtonian potential and the gravitomagnetic potential of a rotating mass according to

$$\Phi = -\frac{1}{4} \tilde{h}_{00} \quad \text{and} \quad A_i = \frac{1}{2} \tilde{h}_{i0}. \quad (10)$$

Subsequently, the gravitational field $\vec{E} := -\vec{\nabla} \Phi - \frac{1}{2} \vec{A}$ and the gravitomagnetic field $\vec{B} := \vec{\nabla} \times \vec{A}$ satisfy the field equations

$$\vec{\nabla} \cdot \vec{E} = -\frac{\kappa^2}{2} \rho - \frac{\ell}{2} \vec{\nabla} \theta \cdot \vec{B},$$
$$\vec{\nabla} \cdot \vec{B} = 0,$$
$$\vec{\nabla} \times \vec{E} + \frac{1}{2} \vec{B} = \vec{0},$$
$$\vec{\nabla} \times \vec{B} - 2 \vec{E} = -\kappa^2 \vec{J} + \ell \vec{\nabla} \theta \times \vec{E} + \frac{\ell}{2} \theta \vec{B}. \quad (11)$$

In the coming sections we will solve these equations in specific physical settings. It is worth mentioning that formally they have the same form as the ones found by Sikivie in the context of axion models [35] and by Wilczek in axion electrodynamics [4].

**EFFECT ON GYROSCOPE PRECESSION**

We would now like to use axion gravitodynamics to compute the gravitomagnetic field of a spinning spherical body, for instance a planet, in case $\theta$ is spatially homogeneous and slowly-varying, such as a quintessence field [30-37]. We further assume that the source is stationary, i.e. $\dot{\phi} = A^i = E^i = 0$. Under these assumptions, one can act on the fourth equation in (11) with the operator $\left( \frac{\ell \dot{\theta}}{2} \hat{1} + \vec{\nabla} \times \right)$ and, using $\vec{\nabla} \cdot \vec{B} = 0$, obtain

$$\left( \vec{\nabla}^2 + \frac{\ell^2 \dot{\theta}^2}{4} \right) \vec{B} = \kappa^2 \left( \frac{\ell \dot{\theta}}{2} \vec{J} + \vec{\nabla} \times \vec{J} \right). \quad (12)$$

This is an inhomogeneous Helmholtz equation for the gravitomagnetic field $\vec{B}$. To solve it, we consider the case of a homogeneous spherical source of radius $R$ with constant density $\rho$ and angular velocity $\vec{\omega}$. The mass current of such a matter distribution is given by

$$\vec{J}(\vec{r}) = -\rho \theta (R - r) \hat{r} \times \vec{\omega}. \quad (13)$$

The general solution of equation (12) for such a source reads as

$$\vec{B} = \kappa^2 \rho R^2 \left[ f_1(r) \vec{\omega} + f_2(r) \hat{r} \times \vec{\omega} + f_3(r) \hat{r} \times (\hat{r} \times \vec{\omega}) \right],$$

in terms of the functions

$$f_1(r) = \frac{2}{\ell \dot{\theta} R^2} + \frac{2 R}{r} y_2 \left( \frac{\ell \dot{\theta}}{2} R \right) j_1 \left( \frac{\ell \dot{\theta}}{2} r \right),$$
$$f_2(r) = -\frac{r}{\ell \dot{\theta} R^2} - \frac{2 R}{r} y_2 \left( \frac{\ell \dot{\theta}}{2} R \right) j_1 \left( \frac{\ell \dot{\theta}}{2} r \right),$$
$$f_3(r) = \frac{\ell \dot{\theta}}{2} R y_2 \left( \frac{\ell \dot{\theta}}{2} R \right) j_2 \left( \frac{\ell \dot{\theta}}{2} r \right) \quad (14)$$

for the region inside the source ($r \leq R$) and

$$f_1(r) = \frac{2 R}{r} j_2 \left( \frac{\ell \dot{\theta}}{2} R \right) y_1 \left( \frac{\ell \dot{\theta}}{2} r \right),$$
$$f_2(r) = -\frac{\ell \dot{\theta}}{2} R j_2 \left( \frac{\ell \dot{\theta}}{2} R \right) y_1 \left( \frac{\ell \dot{\theta}}{2} r \right),$$
$$f_3(r) = \frac{\ell \dot{\theta}}{2} R j_2 \left( \frac{\ell \dot{\theta}}{2} R \right) y_2 \left( \frac{\ell \dot{\theta}}{2} r \right) \quad (15)$$

for the exterior region ($r \geq R$). The elementary functions $j$ and $y$ correspond to spherical Bessel functions of the first and second kind, respectively. This solution is continuous on the boundary of the sphere and in the GR limit $\ell \to 0$ it reproduces the well-known result for the gravitomagnetic field generated by a rotating spherical body

$$\vec{B}_{GR} = \begin{pmatrix} -\kappa^2 \rho R^2 \left[ \left( \frac{1}{3} \right) - \frac{r^2}{3 R^2} \vec{\omega} \right] + \frac{r^2}{3 R^2} \hat{r} \times (\hat{r} \times \vec{\omega}) \\ -\kappa^2 \rho R^2 \left[ \frac{2 R^3}{15 r^3} \vec{\omega} + \frac{R^2}{br^3} \hat{r} \times (\hat{r} \times \vec{\omega}) \right] \end{pmatrix}.$$

inside (upper) and outside (lower) the source.

This result highlights an important qualitative feature of the gravitational $\theta$ term, which was also found in Chern-Simons modified gravity in Ref. [30]. The gravitomagnetic field has an additional component parallel to the vector $\hat{r} \times \vec{\omega}$. This toroidal component is absent in GR and appears as a consequence of the parity violation introduced by the topological terms in each case. However, quantitatively the two results differ significantly as we discuss below.

Following the analogous discussion in Ref. [30], we can now compute the correction induced by the $\theta$ term in
the precession of a gyroscope, while it performs a circular polar orbit around a rotating spherical body. This effect is also known as Schiff precession \[38\]. This will allow us to put a bound on \(\ell \dot{\theta}\) and the decay constant \(f_\theta\) using observational results from the Gravity Probe B experiment. Using the standard relation \[38\]

\[
\dot{\mathbf{S}} = 2\mathbf{B} \times \mathbf{S},
\]

between the time variation of the spin \(\mathbf{S}\) of the gyroscope and the gravitomagnetic field generated by the spherical body, one can show that the ratio \(\alpha := |\langle \dot{\mathbf{S}}_\theta \rangle| / |\langle \dot{\mathbf{S}}_{GR} \rangle|\) reads as

\[
\alpha = 1 + \frac{15\rho^2}{R^2} j_2 \left( \frac{\ell}{2} \right) \left( y_1 \left( \frac{\ell}{2} r \right) + \ell \left( \frac{\ell}{2} r \right) y_0 \left( \frac{\ell}{2} r \right) \right),
\]

where \(r\) and \(R\) are the radii of the gyroscope orbit and the spherical body, respectively. In this formula, we have denoted by \(\dot{\mathbf{S}}_\theta\) the contribution of \(\theta\) to the change in gyroscope’s spin and by \(\dot{\mathbf{S}}_{GR}\) the standard change predicted by GR, averaging these vectors over one orbital period. This result is qualitatively similar to the one obtained in Ref. \[32\] for Chern-Simons modified gravity, \(\alpha_{CS}\), obtained by solving the corresponding fourth equation of the system \[11\] in that case, which crucially differs from ours in that the last term on the right hand side contains \(\nabla \mathbf{B}\) instead of \(\mathbf{B}\). However, the two ratios \(\alpha\) and \(\alpha_{CS}\) are quantitatively different and observational results place a different upper bound on \(\ell \dot{\theta}\).

Measurement of the Schiff precession for gyroscopes orbiting the Earth at an altitude of 642 km was one of the primary goals of the Gravity Probe B mission \[2\]. The final results of the mission reported a verification of the general-relativistic result at an accuracy of 19%. The laser-ranged satellites LAGEOS, LAGEOS 2, and LARES further improve the accuracy to about 5% \[39\], but here we will work with the more conservative result. If we assume that the Earth is a perfect sphere with radius \(R = 6368\) km, then the satellite’s orbit has a radius \(r = 7010\) km. Using these values, we can then plot the ratio \(\alpha\) as depicted in Figure 1. From this plot we conclude that a 19% verification of the general-relativistic effect places the following generic upper bound:

\[
|\ell \dot{\theta}| \lesssim 10^{-4} \text{ km}^{-1} \simeq 2 \times 10^{-23} \text{ GeV}.
\]

One should also note that for certain non generic values of the fundamental parameters it can be relaxed by 1-2 orders of magnitude. At this stage, it is useful to compare the strengths of the predicted effects between Axion Gravitodynamics and Chern-Simons modified gravity. In the latter case and in terms of the same parameters \(\ell\) and \(\dot{\theta}\), Ref. \[30\] find

\[
|\ell_{CS} \dot{\theta}| \lesssim 10^{60} \text{ GeV}.
\]

Thus we observe that the bound obtained from axion gravitodynamics is immensely stricter that the Chern-Simons one. For a field \(\theta(t) = \theta_0 e^{-\ell t/\ell} \) and, therefore \(\ell \dot{\theta} \approx \theta\), the above bounds imply that the coupling of the topological term in axion gravitodynamics is weaker than the Chern-Simons one by approximately 83 orders of magnitude. The bound \(\alpha\) will be further improved in the next section.

**EFFECT ON GRAVITATIONAL WAVES**

Let us now study the gravitational wave solution of the system of equations \[11\], for the case of a slowly varying and spatially homogeneous axion field obeying \(\nabla \theta = 0\). For gravitational waves propagating in vacuum, we also have to consider the source-free case of vanishing \(\rho\) and \(J\). From \[11\], we see that this implies \(\nabla \cdot \mathbf{E} = 0\). Acting on the fourth equation in \[11\] with a time derivative, neglecting second time derivatives of \(\theta\), and using the third equation in \[11\] to eliminate \(\ddot{\mathbf{B}}\) from the resulting equation leads to

\[
\Box \dot{\mathbf{E}} = -\frac{\ell}{2} \mathbf{\nabla} \times \dot{\mathbf{E}}.
\]

This PDE is solved by a transversal circular polarized wave \(\dot{\mathbf{E}}\) with frequency \(\omega\), wave vector \(\ell\) and dispersion relation

\[
\omega^2 - k^2 = \pm \frac{\ell \dot{\theta}}{2} k,
\]

where \(k = |\ell|\). The positive and negative signs correspond to left and right helicity respectively. This is in full formal agreement with the results of Ref. \[32\] for the case of electromagnetic waves in Chern-Simons modified gravity.
electromagnetism. For $k > \frac{\ell \theta}{\alpha}$, this dispersion relation has the four real solutions
\begin{equation}
\omega_1 = \sqrt{k^2 + \frac{\ell \theta}{2} k}, \quad \omega_2 = -\sqrt{k^2 + \frac{\ell \theta}{2} k}, \\
\omega_3 = \sqrt{k^2 - \frac{\ell \theta}{2} k}, \quad \omega_4 = -\sqrt{k^2 - \frac{\ell \theta}{2} k},
\end{equation}
two of which are positive and the other two negative. The requirement $k > \frac{\ell \theta}{\alpha}$ holds by default for gravitational waves, since their typical wavelengths are $\lambda_{GW} = \frac{2\pi}{k} \simeq 10^9\text{km} \simeq 5 \times 10^{21}\text{GeV}^{-1}$ and $|\ell \theta|$ should respect the bound \[18\]. As explained in Ref. \[40\], the solutions $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$ define the two distinct double hypersurfaces
\begin{equation}
\omega^2 - k_1^2 - k_2^2 - k_3^2 - \frac{\ell \theta}{2} \sqrt{k_1^2 + k_2^2 + k_3^2} = 0,
\end{equation}
\begin{equation}
\omega^2 - k_1^2 - k_2^2 - k_3^2 + \frac{\ell \theta}{2} \sqrt{k_1^2 + k_2^2 + k_3^2} = 0,
\end{equation}
where $k_i, i = 1, 2, 3$, are the wave vector components. As pictured in Figure 2, the first hypersurface is topologically equivalent to the lightcone and, therefore, respects causality. On the other hand, the second hypersurface plotted in Figure 3 clearly has different topology (is of genus 1). This implies that it violates causality since the upper and lower parts (future and past) are always connected \[11\].

Let us now discuss the consequences of the modified dispersion relation \[21\] to the phase and group velocities of a gravitational wave. These are found to be
\begin{equation}
v_p = \frac{\omega}{k} = \sqrt{1 \pm \frac{\ell \theta}{2k}} \simeq 1 \pm \frac{\ell \theta}{4k} + \mathcal{O}(\ell^2),
\end{equation}
\begin{equation}
v_g = \frac{d \omega}{dk} = \frac{1 \pm \frac{\ell \theta}{2k}}{\sqrt{1 \pm \frac{\ell \theta}{2k}}} \simeq 1 + \frac{\ell^2 \theta^2}{32k^2} + \mathcal{O}(\ell^3),
\end{equation}
respectively. The front velocities ($k \to \infty$ limit) are 1, i.e. equal to the speed of light. Similarly to the electromagnetic case of \[32\], we observe that the two polarizations propagate with different velocities, which are also different from the speed of light. The group velocity is larger than the speed of light and polarization-independent up to $\mathcal{O}(\ell^2)$. These are direct consequences of the parity and Lorentz violation, respectively, which are introduced through the gravitational theta term. Moreover, we observe that the group velocity is modified at order $\mathcal{O}(\ell^3)$, while the phase velocity is already affected at order $\mathcal{O}(\ell)$.

The group velocity of the neutron star merger gravitational wave event GW170817 was bounded by observation to be \[33\]
\[3 \times 10^{-15} \leq v_g - 1 \leq 7 \times 10^{-16}. \tag{25}\]
This is a bound on the ratio of the speed of gravitational waves over the speed of light, which is taken to be $c = 1$ in our units. The group velocity \[24\] predicted by axion gravitodynamics can be compared to the observed one \[25\]. As we already mentioned, the typical wavelengths of gravitational waves are $\lambda_{GW} \simeq 5 \times 10^{21}\text{GeV}^{-1}$. Ignoring terms of order $\mathcal{O}(\ell^3)$, we can therefore obtain the upper bound
\[|\ell \dot{\theta}| \lesssim 2 \times 10^{-28}\text{GeV}. \tag{26}\]
This is stricter than the one in \[18\] by five orders of magnitude. We discuss this further in the conclusions.

**CONCLUSIONS**

Axion gravitodynamics is the theory that incorporates in the weak field limit of general relativity a gravitational $\theta$ term proportional to the product $\vec{E}_G \cdot \vec{B}_G$ of the gravitational and gravitomagnetic fields. It arises naturally as the linearization of the teleparallel equivalent of general relativity with a Nieh-Yan term. Promoting $\theta$ to a field and introducing an associated length scale $\ell$, leads to a set of gravitational field equations for $\vec{E}_G$ and $\vec{B}_G$ that include first derivatives of $\theta(x)$ in striking similarity to axion electrodynamics. Motivated by the variety of uses of the latter in physics, it is natural to ask what is the
effect of axion gravitodynamics in certain physical phenomena, such as geodetic effects and the propagation of gravitational waves in vacuum.

In this paper, under the assumption that the field $\theta$ is spatially homogeneous and depends only on cosmic time, we solved the gravitational field equations of the theory in these two settings. First we computed the gravitomagnetic field around a spinning spherical mass and its corrections to the general relativistic prediction. At a qualitative level, we found additional components to the gravitomagnetic field with no analogue in general relativity, which were first reported in the study of Chern-Simons modified gravity [30]. Comparing with the observational data of the Gravity Probe B satellite mission, a bound on the parameters of the theory is found, $|\ell \dot{\theta}| \lesssim 2 \times 10^{-23}$ GeV, which is much stronger than the one in the Chern-Simons case ($\lesssim 2 \times 10^{60}$ GeV). Therefore, we conclude that the effect of axion gravitodynamics is stronger and if it exists it could very well be discovered or ruled out by near future observations [12, 43].

On the other hand, the presence of $\theta(t)$ has an effect on the propagation of gravitational waves too, leading to a modification of their dispersion relation which in turn influences the phase and group velocities of gravitational waves in a polarization-dependent way (birefringence). Comparison to the reported bounds on the group velocity from the neutron star merger GW170817 allowed us to place an even stricter bound to the parameters of axion gravitodynamics, $|\ell \dot{\theta}| \lesssim 2 \times 10^{-28}$ GeV. Together with the already mentioned geodetic effects, the modified gravitational wave propagation can serve as observational signatures of Axion Gravitodynamics.

Although we have not discussed the dynamics of the field $\theta$, appealing to its (quintessential) axionic nature prompts us to interpret this as a bound on its decay constant $|f_\theta| > 10^{28}|\dot{\theta}|$ GeV. The decay constant for a quintessence field should be approximately equal to (and definitely not higher than) the reduced Planck mass $M_P \simeq 3 \times 10^{18}$ GeV where quantum gravitational effects become relevant. Assuming that $f_\theta \lesssim M_P$ directly implies that $|\ell| \lesssim M_P^{-1}$. This would mean that $|\dot{\theta}| \lesssim 6 \times 10^{-10}$ GeV$^2$, a reasonable outcome given that to explain dark energy the kinetic energy of quintessence should be much lower than its potential energy. More detailed study on the dynamics of $\theta$ is required to test these statements more precisely. Finally, it would be interesting to explore the possibility of a field that is not spatially homogeneous. In that case, finding solutions to the field equations becomes more demanding and we plan to report on this in future work.

**Note added.** While this work was being finalised, we learned of the preprint [45] which has some overlap with our discussion on gravitational waves and a proposal for a stronger bound on the parameters of the theory based on phase velocities.

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