ON THE PRODUCT IN BESOV-LORENTZ-MORREY SPACES AND EXISTENCE OF SOLUTIONS FOR THE STATIONARY BOUSSINESQ EQUATIONS

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Abstract. This paper is devoted to the Boussinesq equations that models natural convection in a viscous fluid by coupling Navier-Stokes and heat equations via a zero order approximation. We consider the problem in \( \mathbb{R}^n \) and prove the existence of stationary solutions in critical Besov-Lorentz-Morrey spaces. For that, we prove some estimates for the product of distributions in these spaces, as well as Bernstein inequalities and Mihlin multiplier type results in our setting. Considering in particular the decoupled case, our existence result provides a new class of stationary solutions for the Navier-Stokes equations in critical spaces.

1. Introduction. The Boussinesq equations of hydrodynamics arises from zero order approximation to the coupling between the Navier-Stokes equations and the heat equation, modeling the fluid movement by the natural convection. The stationary Boussinesq system is given by the following system of PDEs:

\[
\begin{align*}
-\nu \Delta u + u \cdot \nabla u + \nabla \pi &= \theta f + F \quad \text{in} \ \mathbb{R}^n, \\
\nabla \cdot u &= 0 \quad \text{in} \ \mathbb{R}^n, \\
-\kappa \Delta \theta + u \cdot \nabla \theta &= G \quad \text{in} \ \mathbb{R}^n,
\end{align*}
\]

where \( u = (u_j)_{j=1}^n \), \( \pi \) and \( \theta \) are respectively the velocity of the fluid, scalar pressure and temperature. The fields \( F \) and \( f \) denote given external forces and \( G \) represents a given reference temperature. The coefficient \( \nu \) denotes the dynamic viscosity and \( \kappa \) is the molecular diffusivity; throughout this paper, they are assumed to be positive and will be taken to be 1. The dynamic model consists in the heat advection-diffusion equation of the temperature coupled with the incompressible

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Navier-Stokes equations through the gravitational term $\theta f$. The coupling term $\theta f$ arises from the Boussinesq approximation which establishes that the density variation can be neglected in the system except for a buoyancy force proportional to the local temperature in the momentum balance [9]. As usual, system (1.1) can be rewritten as

$$
\begin{align*}
-\Delta u + P (u \cdot \nabla u) &= P (\theta f) + PF \quad \text{in } \mathbb{R}^n, \\
-\Delta \theta + u \cdot \nabla \theta &= G \quad \text{in } \mathbb{R}^n,
\end{align*}
$$

(1.2)

where the so-called Leray-Hopf projector $P$ is defined as $(P_{i,j})_{n \times n}$, where $P_{i,j} := \delta_{i,j} + R_i R_j$ and $R_i = (-\Delta)^{-1/2} \partial_i$ is the $i$-th Riesz transform. The issues of existence and long-time behavior of solutions for the corresponding non-stationary Boussinesq equations have attracted the attention of many authors (see for instance, [1, 6, 7, 11] and references therein). Most of the existence results for the stationary Boussinesq equations have been obtained in classical Sobolev spaces and considering bounded domains [20, 22]. The existence of a class of stable stationary solutions for the Boussinesq equations in the scaling invariant class $L^{(n,\infty)}$ was obtained in [12] in the whole space $\mathbb{R}^n$ and in [14] in exterior domains.

For $f$ and $G$ satisfying the scale relation $f(\lambda x) = \lambda^2 f(\lambda x)$ and $G(\lambda x) = \lambda^3 G(\lambda x)$, for all $\lambda > 0$, it holds that if $(u, \theta)$ solves (1.2) in $\mathbb{R}^n$, then so does $(u(x), \theta(x)) \rightarrow (\lambda u(\lambda x), \lambda^2 \theta(\lambda x))$. A Banach space $X$ with the norm $\| \cdot \|_X$ is called a critical space for (1.2) if $\| (u(x), \theta(x))\|_X \approx \lambda^{-1} \| (u(\lambda x), \theta(\lambda x))\|_X$, for all $\lambda > 0$. Typical examples of critical spaces for (1.2) are the Lebesgue space $L^n(\mathbb{R}^n)$, the weak-$L^p$ space $L^{(n,\infty)}(\mathbb{R}^n)$, the homogeneous Besov space $B^{p,1}_{\infty \infty}$, and others. A natural problem is to find large critical spaces in which PDEs presents a good existence theory.

In this paper, we obtain existence of stationary solutions in the class of Besov-Lorentz-Morrey spaces $BM^{l,s}_{(p,d),r}$, where $BM^{l,s}_{(p,d),r}$ is defined by

$$
\hat{BM}^{l,s}_{(p,d),r} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}; \| f \|_{BM^{l,s}_{(p,d),r}} < \infty \right\},
$$

(1.3)

with

$$
\| f \|_{BM^{l,s}_{(p,d),r}} = \begin{cases} 
\left( \sum_{j \in \mathbb{Z}} 2^{jsr} \| \Delta_j f \|_{M^{l,s}_{(p,d)}}^r \right)^{\frac{1}{r}} & \text{if } r < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{jsr} \| \Delta_j f \|_{M^{l,s}_{(p,d)}} & \text{if } r = \infty,
\end{cases}
$$

(1.4)

for some parameters $s, l, p, d, r$, where $M^{l,s}_{(p,d)}$ stands for the Lorentz-Morrey spaces (see definitions in Section 2). Considering in particular $d = \infty$ and $r = \infty$, we desire to analyze the existence of solutions in the larger critical spaces $BM^{l,n/l-1}_{(p,\infty),\infty}$ in (1.3). The case $d = \infty$ corresponds to the Besov-weak-Morrey space $BM^{l,s}_{(p,\infty),r}$ and is denoted by $BW_{(p,d),r}$.

Using the inverse of $\Delta$ and the identities $u \cdot \nabla u = \text{div} (u \otimes u)$ and $u \cdot \nabla \theta = \text{div} (\theta u)$, we rewrite system (1.2) as follows

$$
\begin{align*}
\frac{d}{dt} u &= \Delta^{-1} P \text{div} (u \otimes u) - \Delta^{-1} P (\theta f) - \Delta^{-1} P F, \\
\frac{d}{dt} \theta &= \Delta^{-1} \text{div} (\theta u) - \Delta^{-1} G.
\end{align*}
$$

(1.5)

In order to analyze the existence of solution in Besov-Lorentz-Morrey spaces for (1.2), observing the system (1.5), one needs to establish product estimates that permit us to deal with $u \otimes u$ and the coupling terms $\theta u$ and $\theta f$. Naturally, a
multiplier result of Mihlin is also needed to handle $\mathbb{P}$, $\Delta^{-1}$ and derivatives terms in (1.5) (see Lemmas 2.7 and 2.8).

Explicitly, our first aim is to prove the following product estimates. For related estimates in other types of Besov spaces, see e.g. [16, 19]. To simplify the notation, we denote $s_l = \frac{n}{2} - 1$ with $1 \leq l \leq \infty$.

**Proposition 1.1.** Let $n \geq 3$, $1 \leq r < \infty$, $1 < p \leq l$, $\frac{n}{2} < l < n$, $1 < p_2 \leq l_2$ and $1 \leq d, d_2 \leq \infty$ be such that $d_2 \leq d$, $d \leq \tilde{d}$ with $\frac{1}{d_2} = \frac{1}{d} - \frac{1}{d_2}$, where $\frac{n}{2} < l_2$ and $\frac{1}{l_2} \geq \frac{1}{r} - \frac{1}{l}$. Then, we have that

$$\left\|fg\right\|_{\dot{B}^{s_l}_{r,\infty} - 1} \leq C \left\|f\right\|_{\dot{B}^{s_l}_{r,\infty}} \left\|g\right\|_{\dot{B}^{s_l}_{r,\infty} - 1},$$

for all $f \in \dot{B}^{s_l}_{(p,d),\infty}$ and $g \in \dot{B}^{s_l}_{(p,d_2),r}$, where $C > 0$ is a universal constant.

**Proposition 1.2.** Let $n = 3$, $\frac{n}{2} < l < n$, $1 < l_1$ and $1 < l_2$ be such that $l_1 \leq 2l$, $l_2 \leq 2l$, $l_1 < \tilde{l}_1$ and $l_2 \leq \tilde{l}_2$ for some $\tilde{l}_1, \tilde{l}_2$ such that $\frac{1}{l_1} + \frac{1}{l_2} = 1$. Then, there is a universal constant $C > 0$ such that

$$\left\|fg\right\|_{\dot{B}^{s_l}_{(p,\infty)} - 2} \leq C \left\|f\right\|_{\dot{B}^{s_l}_{(p,\infty)} - 1} \left\|g\right\|_{\dot{B}^{s_l}_{(p,\infty)} - 1},$$

for all $f \in \dot{B}^{s_l}_{(p,\infty),\infty}$ and $g \in \dot{B}^{s_l}_{(p,\infty),r}$.

**Proposition 1.3.** Let $n \geq 4$, $1 < p \leq l$, $1 < p_1 \leq l_1$ and $1 < p_2 \leq l_2$ be such that $\frac{n}{2} = \frac{p_1}{p_2} = \frac{l_1}{l_2}$ and $\frac{n}{2} < l < n$. Moreover, suppose that $2l \geq l_1$, $2l \geq l_2$ and $\frac{1}{l_1} \geq \frac{1}{r} - \frac{1}{l}$. Then, we have that

$$\left\|fg\right\|_{\dot{B}^{s_l}_{(p,\infty)} - 2} \leq C \left\|f\right\|_{\dot{B}^{s_l}_{(p,\infty)} - 1} \left\|g\right\|_{\dot{B}^{s_l}_{(p,\infty)} - 1},$$

for all $f \in \dot{B}^{s_l}_{(p,\infty),\infty}$ and $g \in \dot{B}^{s_l}_{(p,\infty),r}$, where $C > 0$ is a universal constant.

**Remark 1.**

1. Note that the conditions in Proposition 1.1 are not empty. For example, they are verified when $\frac{n}{2} < l = l_2 < n$ and $l < 2l < n$. Also, we can consider $n \leq l_2 < n + \epsilon$ by taking $\tilde{l}$ close enough to $\frac{n}{2}$; even more, $\epsilon$ goes to $\infty$ provided $l$ and $\tilde{l}$ go to $\frac{n}{2}$.

2. Let $1 < p \leq l$, $1 < p_1 \leq l_1$, $1 < p_2 \leq l_2$, $\frac{2}{n} = \frac{1}{l} - \frac{1}{l_2}$ and $\frac{1}{l} = \frac{p_1}{p_2}$. Then, the conditions in Proposition 1.3 can be verified if $\tilde{l}$ is close to $\frac{n}{2}$ (so $l_2 < n$ is close to $n$).

3. Let $n \geq 3$ and $1 < p_1 \leq l_1 \leq \infty$. If $\frac{n}{2} \leq l$, then the spaces $\dot{B}^{s_l}_{(p,\infty),\infty}$ and $\dot{B}^{s_l}_{(p,\infty),r}$ contain homogeneous functions of degree $-2$. In fact, using the Hölder and Bernstein inequalities in Morrey spaces, it follows that

$$|x|^{-2} \in L^{\tilde{r},\infty} = W^{\tilde{r},\infty}_{\tilde{r},\infty} \hookrightarrow \dot{B}^{s_l}_{\frac{n}{2},\infty} \hookrightarrow \dot{B}^{s_l}_{(p,\infty),\infty} \hookrightarrow \dot{B}^{s_l}_{(p,\infty),r} \hookrightarrow \dot{B}^{s_l}_{(p,\infty),\infty}$$

In what follows, we state our existence result and afterwards, in connection with the above product estimates, we present the main examples in two corollaries.

**Theorem 1.4.** Let $n \geq 3$, $1 \leq r \leq \infty$, $1 < p \leq l$, $1 < p_2 \leq l_2$ and $1 \leq d, d_2 \leq \infty$ be such that $\frac{p_2}{p} = \frac{n}{2} < l < n$, $\frac{n}{2} < l_2$, $\frac{1}{l_2} \geq \frac{1}{r} - \frac{1}{l}$ for some $1 < \tilde{l} < \frac{n}{2}$, and
$$\frac{1}{d_2} = \frac{1}{d_2} + \frac{1}{d}$$ with $d_2 \leq \tilde{d}_2$ and $d \leq \tilde{d}$. Moreover, suppose that $f \in X$ where $X$ is a normed space such that

$$\|fg\|_{B^{s_1}_{p,2}} \leq C \|f\|_X \|g\|_{B^{s_2}_{p_2,2}},$$

Then, there exist positive constants $A_1, A_2, A_3$ (independent of $f$ and $0 < \epsilon \ll 1$ such that if $\|G\|_{B^{s_1}_{p_1,2}} \leq \frac{A_1}{\|f\|_X}$ and $\|F\|_{B^{s_2}_{p_2,2}} \leq A_2\epsilon$, we have that there exists a unique solution $(u, \theta) \in \hat{B}M^{1,sl}_{(p,d),\infty} \times \hat{B}M^{2,sl}_{(p,d),\infty}$ for (1.5) satisfying

$$\|u\|_{B^{s_1}_{p,2}} < \epsilon \text{ and } \|\theta\|_{B^{s_2}_{p_2,2}} < \frac{A_3}{\|f\|_X}.$$  

Corollary 1.5. Let $n = 3$, $\frac{2}{3} < l < n$, $1 < l_1, l_2 < 2l$, $l_1 \leq l_2$, $l_1 < l_2$, $\tilde{l}_2 < \tilde{l}_2$ and $\frac{1}{l} \geq \frac{1}{l} - \frac{1}{l}$ for some $l, \tilde{l}_1, \tilde{l}_2$ such that $\frac{1}{l_1} + \frac{1}{l_2} = 1$ and $1 < \tilde{l} < \frac{n}{2}$. Moreover, suppose that $f \in \hat{B}M^{1,sl}_{1,\infty}$. Then, there are constants $A_1, A_2, A_3 > 0$ and $0 < \epsilon \ll 1$ as in Theorem 1.4 with $X = \hat{B}M^{1,sl}_{1,\infty}$ such that if $\|G\|_{\hat{B}M^{2,sl}_{1,\infty}} \leq \frac{A_1}{\|f\|_X}$ and $\|F\|_{\hat{B}M^{2,sl}_{1,\infty}} \leq A_2\epsilon$, we have that (1.5) has a unique solution $(u, \theta) \in \hat{B}M^{1,sl}_{1,\infty} \times \hat{B}M^{2,sl}_{1,\infty}$ satisfying

$$\|u\|_{\hat{B}M^{1,sl}_{1,\infty}} < \epsilon \text{ and } \|\theta\|_{\hat{B}M^{2,sl}_{1,\infty}} < \frac{A_3}{\|f\|_X}.$$  

Corollary 1.6. Let $n \geq 4$, $1 < p \leq l$, $1 < p_1 \leq l_1$ and $1 < p_2 \leq l_2$ be such that $\frac{1}{p_1} = \frac{1}{p_2} = \frac{1}{p}, \frac{2}{3} < l < n$, $\frac{2}{3} < l_2$ and $\frac{1}{l} \geq \frac{1}{l} - \frac{1}{l}$ for some $1 < \tilde{l} < \frac{n}{2}$. Moreover, suppose that $2l \geq l_1, 2l \geq \tilde{l}_2$ and $\frac{1}{l} \geq \frac{1}{l} - \frac{1}{l}$ for some $l_1 \leq \tilde{l}_1$ and $1 < \tilde{l}_1 < \frac{n}{3}$. Then, for $f \in \hat{B}M^{1,sl}_{1,\infty}$, there are constants $A_1, A_2, A_3 > 0$ and $0 < \epsilon \ll 1$ as in Theorem 1.4 with $X = \hat{B}M^{1,sl}_{1,\infty}$, such that if $\|G\|_{\hat{B}M^{2,sl}_{1,\infty}} \leq \frac{A_1}{\|f\|_X}$ and $\|F\|_{\hat{B}M^{2,sl}_{1,\infty}} \leq A_2\epsilon$, we have that (1.5) has a unique solution $(u, \theta) \in \hat{B}M^{1,sl}_{p,\infty} \times \hat{B}M^{2,sl}_{p_2,\infty}$ satisfying

$$\|u\|_{\hat{B}M^{1,sl}_{p,\infty}} < \epsilon \text{ and } \|\theta\|_{\hat{B}M^{2,sl}_{p_2,\infty}} < \frac{A_3}{\|f\|_X}.$$  

Some further comments about our results are in order.

Remark 2.

1. (Bénard problem) Corollaries 1.5 and 1.6 provide large classes of forces $f$ for which we guarantee the existence of solution in Besov-Lorentz-Morrey spaces. In particular, since $\hat{B}M^{1,sl}_{1,\infty}$ contains homogeneous functions of degree -2, we are able to consider $f$ as the gravitational field $f(x) = G\nabla_x(\frac{1}{|x|^l}) = -\frac{G}{|x|^{l+2}}$, where $G$ is the gravitational constant. This case can be regarded as a mathematical version in the whole space of the Bénard problem (see [9]).

2. (Large $f$) In Theorem 1.4 and Corollaries 1.5 and 1.6, no smallness condition is needed on the factor $f$ of the coupling term $\theta f$. Here, after obtaining the needed estimates, we use an iterative scheme (see (4.1)-(4.2)) that is equivalent to the contraction mapping argument for continuous bilinear mappings, when the problem is formulated in the variables $u$ and $\|f\|_X \theta$ and with the data $F$ and $\|f\|_X G$. 
3. (Navier-Stokes equations) Considering \( \theta_0 = 0 \) and \( \theta = 0 \), the system (1.2) becomes the stationary Navier-Stokes equations (S-NS). Existence results for (S-NS) in \( \mathbb{R}^n \) are known in the framework of Sobolev [13, 24], pseudomeasure [8], weak-\( L^p \) [18] and Morrey spaces [17] (see also [16] for solutions in Besov-Morrey spaces in the non-stationary case). In comparison with previous results, our existence results provided a new class of existence-uniqueness of solutions for (S-NS) in \( \mathbb{R}^n \). For results in other types of domains \( \Omega \) (e.g. bounded and exterior ones), the reader is referred to [5, 13, 18, 24].

4. (Pressure) After obtaining the solution \((u, \theta)\) for (1.5), the pressure \(\pi\) in (1.2) can be found by solving the elliptic problem \(-\Delta \pi = \nabla \cdot [u \cdot \nabla u - \theta f - F]\) in the sense of distributions.

This paper is organized as follows. In Section 2, we give some preliminaries about Lorentz, Lorentz-Morrey and Besov-Lorentz-Morrey spaces. Section 3 is devoted to the proof of the product estimates stated in Propositions 1.1, 1.2 and 1.3. In Section 4, we prove our results about existence of stationary solutions for the Boussinesq system.

2. Function spaces.

2.1. Lorentz and Lorentz-Morrey spaces. Briefly, we introduce some preliminaries about Lorentz spaces \(L^{(p,q)}(\Omega), \Omega \subset \mathbb{R}^n\). The reader interested in more details about these spaces is referred to [2]. A measurable function \(f\) defined on \(\Omega\) belongs to the Lorentz space \(L^{(p,q)}(\Omega)\) if the quantity

\[
\|f\|_{(p,q)} = \left\{ \begin{array}{ll}
\left( \int_0^\infty \left[ \frac{t^{p-1} f^{**}(t)}{t} \right]^q dt \right)^{\frac{1}{q}}, & \text{if } 1 < p < \infty, 1 \leq q < \infty, \\
\sup_{t>0} t^{p-1} f^{**}(t), & \text{if } 1 < p \leq \infty, q = \infty,
\end{array} \right.
\]

is finite, where

\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds \quad \text{and} \quad f^*(t) = \inf \{s > 0 : m(\{x \in \Omega : |f(x)| > s\}) \leq t\}, \ t > 0,
\]

with \(m\) denoting the Lebesgue measure in \(\mathbb{R}^n\).

The space \(L^{(p,q)}\) with the norm \(\|f\|_{(p,q)}\) is a Banach space. In particular, \(L^p(\Omega) = L^{(p,p)}(\Omega)\) and \(L^{(p,\infty)}(\Omega)\) is called the Marcinkiewicz spaces or weak-\(L^p\) space \((q = \infty)\). Moreover, \(L^{(p,q_1)}(\Omega) \subset L^p(\Omega) \subset L^{(p,q_2)}(\Omega) \subset L^{(p,\infty)}(\Omega)\) for \(1 \leq q_1 \leq p \leq q_2 \leq \infty\).

The following two propositions are Hölder and convolution inequalities in Lorentz spaces, respectively.

**Proposition 2.1** ([15, 23]). Let \(\Omega \subseteq \mathbb{R}^n, 1 < p_1, p_2, p \leq \infty\) and \(1 \leq d_1, d_2, d \leq \infty\) be such that \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\) and \(\frac{1}{d} = \frac{1}{d_1} + \frac{1}{d_2}\). If \(f \in L^{(p_1,d_1)}(\Omega)\) and \(g \in L^{(p_2,d_2)}(\Omega)\), then \(h = fg \in L^{(p,d)}(\Omega)\) and

\[
\|h\|_{L^{(p,d)}(\Omega)} \leq C(r) \|f\|_{L^{(p_1,d_1)}(\Omega)} \|g\|_{L^{(p_2,d_2)}(\Omega)}.
\]

**Proposition 2.2** ([3]). Let \(\Omega = \mathbb{R}^n, 1 < p \leq \infty, f \in L^{(p,d)}\) and \(g \in L^1\). Then, \(f * g \in L^{(p,s)}\) for \(s \geq d, \) and

\[
\|f * g\|_{L^{(p,s)}} \leq C \|g\|_{L^1} \|f\|_{L^{(p,d)}}.
\]
Moreover, if \( f \in L^{(p_1,d_1)} \) and \( g \in L^{(p_2,d_2)} \) where \( \frac{1}{p_1} + \frac{1}{p_2} > 1 \), \( 1 < p_i < \infty \), \( 1 \leq d_i \leq \infty \) (\( i = 1,2 \)), then \( f \ast g \in L^{(r,s)} \) where \( \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1 \) and \( s \geq 1 \) is such that \( \frac{1}{d_1} + \frac{1}{d_2} \geq \frac{1}{s} \). Moreover,

\[
\|f \ast g\|_{L^{(r,s)}} \leq C \|f\|_{L^{(p_1,d_1)}} \|g\|_{L^{(p_2,d_2)}}.
\]

Now we recall the definition of Lorentz-Morrey spaces \( \mathcal{M}^{l}_{(p,d)} \) that are natural generalizations of Morrey spaces \( \mathcal{M}^{l}_{(p,d)} = \mathcal{M}^{l}_{(p,p)} \).

**Definition 2.3.** Let \( 1 < p \leq l \leq \infty \) and \( 1 \leq d \leq \infty \) (\( d = \infty \) if \( p = \infty \)). The homogeneous Lorentz-Morrey space \( \mathcal{M}^{l}_{(p,d)} = \mathcal{M}^{l}_{(p,d)}(\mathbb{R}^n) \) is defined as the set of all measurable functions such that

\[
\|f\|_{\mathcal{M}^{l}_{(p,d)}} := \sup_{x_0 \in \mathbb{R}^n} \sup_{R > 0} R^{\frac{d}{d} - \frac{p}{p}} \|f\|_{L^{p,q}(D(x_0,R))} < \infty,
\]

where \( D(x_0,R) = \{x \in \mathbb{R}^n; |x - x_0| < R\} \).

We also have a version of the Hölder inequality in the framework of homogeneous Lorentz-Morrey spaces. To be more specific, let \( 1 < p, p_0, p_1 \leq \infty \), \( 1 \leq d_0, d_1 \leq \infty \), \( p \leq l \), and \( p_i \leq l_i \) (\( i = 0,1 \)) be such that \( \frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} \), \( \frac{1}{l} = \frac{1}{l_0} + \frac{1}{l_1} \) and \( \frac{1}{d_0} + \frac{1}{d_1} \geq \frac{1}{d} \). Then, we have that

\[
\|fg\|_{\mathcal{M}^{l}_{(p,d)}} \leq C \|f\|_{\mathcal{M}^{l}_{(p_0,d_0)}} \|g\|_{\mathcal{M}^{l}_{(p_1,d_1)}},
\]

where \( C > 0 \) is a universal constant.

The following lemma is a convolution estimate in Lorentz-Morrey spaces (see [10]).

**Lemma 2.4 (Convolution in Lorentz-Morrey spaces).** Let \( 1 < p \leq l \leq \infty \), \( 1 \leq d \leq \infty \) and \( \theta \in L^{1}(\mathbb{R}^n) \). Then, there exists \( C > 0 \) (independent of \( \theta \)) such that

\[
\|\theta \ast f\|_{\mathcal{M}^{l}_{(p,d)}} \leq C \|\theta\|_{L^{1}} \|f\|_{\mathcal{M}^{l}_{(p,d)}},
\]

for all \( f \in \mathcal{M}^{l}_{(p,d)} \).

For the remainder of this paper \( \varphi \) denotes a radially symmetric function such that

\[
\varphi \in C_{c}^{\infty}(\mathbb{R}^n \setminus \{0\}) \text{, sup} \varphi \subset \left\{ x; \frac{3}{4} \leq |x| \leq \frac{8}{3} \right\},
\]

and

\[
\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \text{ where } \varphi_j(\xi) := \varphi(\xi 2^{-j}).
\]

The localization operators \( \Delta_j \) and \( S_j \) are defined by

\[
\Delta_j f = \varphi_j(D) f = (\varphi_j)\wedge f \quad \text{and} \quad S_k f = \sum_{j \leq k} \Delta_j f.
\]

One can check easily the identities

\[
\Delta_j \Delta_k f = 0 \text{ if } |j - k| \geq 2 \text{ and } \Delta_j (S_{k-2} g \Delta_k f) = 0 \text{ if } |j - k| \geq 5.
\]

Moreover, we have the Bony’s decomposition (see [4])

\[
f g = T_{fg} + T_{g} f + R(f g),
\]

where

\[
T_{fg} = \sum_{j \in \mathbb{Z}} S_{j-2} f \Delta_j g, \quad R(f g) = \sum_{j \in \mathbb{Z}} \Delta_j f \tilde{\Delta}_j g \quad \text{and} \quad \tilde{\Delta}_j g = \sum_{|j-j^-| \leq 1} \Delta_j g.
\]
For simplicity, in some calculations we also denote \( \tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1} \) and \( D_j = D_{j-1} \cup D_j \cup D_{j+1} \) where \( j \in \mathbb{Z} \) and \( D_j = \{ x : \frac{1}{4} 2^j \leq |x| \leq \frac{8}{3} 2^j \} \). Notice that \( \tilde{\varphi}_j = 1 \) in \( D_j \), so we always have \( \varphi_j = \varphi_j \tilde{\varphi}_j \).

The following lemma corresponds to a Bernstein type inequality in Lorentz-Morrey spaces.

**Lemma 2.5** (Bernstein type inequality in Lorentz-Morrey spaces). Let \( 1 \leq d_0, d \leq \infty, 1 < p_0, p \leq \infty, p_0 \leq l_0, p \leq l, p_0 < p \) and \( \frac{1}{p} = \frac{p_0}{l_0} \). Then, there exists a constant \( C > 0 \) such that

\[
\| f \|_{\mathcal{M}^1_{(p,d)}} \leq C 2^{jn} \left( \frac{1}{p} - \frac{1}{d} \right) \| f \|_{\mathcal{M}^0_{(p_0,d_0)}} , \tag{2.5}
\]

for all \( f \in \mathcal{M}^0_{(p_0,d_0)} \) such that \( \text{supp} \hat{f} \subset D_j \). The estimate (2.5) also holds true if \( 1 < p < \infty \) and \( p_0 = d_0 = 1 \).

**Proof.** Note that if \( \text{supp} \hat{f} \subset D_j \), then \( \hat{f} = \tilde{\varphi}_j \hat{f} \) and \( f = (\tilde{\varphi}_j)^* f \).

Consider first the case \( p = d = l = \infty \). Let \( x \in \mathbb{R}^n \) and \( \rho \in \mathbb{R} \). Defining \( \mu(\rho) = \int_{D(0,\rho)} \tau_x \hat{f} \ dy \), where \( \tau_x \hat{f}(y) = f(x-y) \), we have

\[
|f(x)| = |((\tilde{\varphi}_j)^* f)(x)| \leq \left| \int (\tilde{\varphi}_j)^*(y) f(x-y) \ dy \right| \leq \left| \int_0^\infty 2^{jn} (\tilde{\varphi}_j)^*(2^j \rho) \ d\mu(\rho) \right| \leq \left| \int_0^\infty 2^{jn} \frac{d}{d\rho} ((\tilde{\varphi}_j)^*(2^j \rho)) \ d\rho \right| \mu(\rho) \ |d\rho|.
\]

Taking \( 1 \leq d_0 < \infty \), we can estimate

\[
|\mu(\rho)| \leq \int_{\mathbb{R}^n} |\tau_x \hat{f} (y) 1_{D(0,\rho)}| \ dy \leq \int_{\mathbb{R}^n} \left( \tau_x \hat{f} 1_{D(0,\rho)} \right)^*(s) \ ds
\]

\[
\leq \rho^n \left( \int_{0}^s (1 - \frac{s}{\rho^n} ) \left( \tau_x \hat{f} 1_{D(0,\rho)} \right)^* (s) \ ds \right) \frac{1}{s^n}
\]

\[
\leq \left( \int_{0}^{\rho^n} (1 - \frac{s}{\rho^n} ) \ ds \right) \frac{1}{s^n} \left( \int_{0}^\infty (1 - \frac{s}{\rho^n} ) \left( \tau_x \hat{f} 1_{D(0,\rho)} \right)^* (s) \ ds \right) \frac{1}{s^n}
\]

\[
\leq C \rho^{n - \frac{n}{p_0}} \left\| \tau_x \hat{f} 1_{D(0,\rho)} \right\|_{L(p_0,d_0)} \leq C \rho^{n - \frac{n}{p_0}} \left\| \tau_x \hat{f} \right\|_{L(p_0,d_0)} \leq C \rho^{n - \frac{n}{p_0}} \left\| \tau_x \hat{f} \right\|_{\mathcal{M}^0_{(p_0,d_0)}} \leq C \rho^{n - \frac{n}{p_0}} \| f \|_{\mathcal{M}^0_{(p_0,d_0)}} . \tag{2.6}
\]

Note that the estimate (2.6) is also true in the case \( p_0 = l_0 = d_0 = 1 \). For \( d_0 = \infty \), it follows that

\[
|\mu(\rho)| \leq \left\| \tau_x \hat{f} 1_{D(0,\rho)} \right\|_{L(p_0,\infty)} \int_{0}^{\rho^n} \frac{1}{s^n} \ ds \leq C \rho^{n - \frac{n}{p_0}} \rho^{\frac{n}{p_0} - \frac{n}{p}} \| f \|_{\mathcal{M}^0_{(p_0,d_0)}}
\]

\[
\leq C \rho^{n - \frac{n}{p_0}} \| f \|_{\mathcal{M}^0_{(p_0,d_0)}} .
\]
In any case, we obtain that for any \( x \in \mathbb{R}^n \) and \( R > 0 \)

\[
\|f\|_{L^{\infty}(D(x,R))} \leq \|f\|_{L^{\infty}} \leq C \|f\|_{M^0_{(p_0,d_0)}} \int_0^\infty 2^{jn} \left| \frac{d}{d\rho} \left( \tilde{\varphi} \right) \right|^{(n-\frac{p_0}{n})} d\rho
\]

\[
\leq C \|f\|_{M^0_{(p_0,d_0)}} 2^{jn} 2^{j} \left( \frac{n}{p_0} - n \right) \int_0^\infty \left| \frac{d}{d\rho} \left( \tilde{\varphi} \right) \right| \left( \frac{2^j}{2^j} \right)^{(n-\frac{p_0}{n})} d\rho
\]

\[
\leq C \|f\|_{M^0_{(p_0,d_0)}} 2^{jn} R^n.
\] (2.7)

Now consider the case \( 1 \leq p_0 < p < \infty \) and \( \frac{p}{p_0} = \frac{p_0}{d_0} \) \( (d_0 = 1 \text{ if } p_0 = 1) \). For \( x \in \mathbb{R}^n \)
and \( R > 0 \), we interpolate the estimate \( \|f\|_{L^{p_0,d_0}(D(x,R))} \leq CR^n \|f\|_{M^0_{(p_0,d_0)}} \) and (2.7), in order to get

\[
\|f\|_{L^{p,d}(D(x,R))} \leq C2^j \left( \frac{n}{p_0} - \frac{n}{p} \right) R^\frac{p_0}{p} \|f\|_{M^0_{(p_0,d_0)}} = C2^j \left( \frac{n}{p_0} - \frac{n}{p} \right) R^n \|f\|_{M^0_{(p_0,d_0)}}.
\]

Therefore

\[
\|f\|_{M^j_{(p,d)}} \leq C2^j \left( \frac{n}{p_0} - \frac{n}{p} \right) \|f\|_{M^0_{(p_0,d_0)}},
\]

which concludes the proof of the lemma. \( \square \)

2.2. Besov-Lorentz-Morrey spaces. Below we give a general definition of Besov-type spaces based on a Banach space \( E \). This definition has been used by some authors, see e.g. \([21, 19]\).

**Definition 2.6.** Let \( E \subset S' \) be a Banach space, \( 1 \leq r \leq \infty \) and \( s \in \mathbb{R} \). The homogeneous Besov-\( E \) space \( \dot{B}E^s_{r} \) is defined as

\[
\dot{B}E^s_{r} = \left\{ f \in S'(\mathbb{R}^n)/\mathcal{P}; \|f\|_{\dot{B}E^s_{r}} < \infty \right\},
\]

where

\[
\|f\|_{\dot{B}E^s_{r}} := \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jr} \|\Delta_j f\|^r_E \right)^{\frac{1}{r}}, & \text{if } r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{jr} \|\Delta_j f\|^r_E, & \text{if } r = \infty. \end{cases}
\] (2.8)

There exist other notations for the Besov-\( E \) space \( \dot{B}E^s_{r} \), for example \( \dot{B}E^s_{r} \) or \( \dot{B}_{E,r}^s \) (see [19]). If we take \( E = M^p_{(p,p)} = L^p \) then we obtain the usual homogeneous Besov space \( \dot{B}_{p,r}^s \). Taking \( E = M^p_{(p,p)} = M^p_s \) we obtain the homogeneous Besov-Morrey space \( \dot{BM}^s_{(p,d),r} = N_{l_{p,r}}^s \) introduced in [16]. Here we use \( E = M^p_{(p,d)} \) that corresponds to the homogeneous Besov-Lorentz-Morrey space \( \dot{B}E^s_{r} = \dot{BM}^s_{(p,d),r} \).

As already pointed in the Introduction, in the case \( d = \infty \) we obtain the Besov-weak-Morrey space and use the notation \( \dot{BM}^s_{(p,\infty),r} = BW_{p,r}^s \).

**Remark 3.** In [16], it was proved the inclusion relation \( \dot{BM}^s_{(p,p),r} = N_{l_{p,r}}^s \hookrightarrow \dot{B}_{\infty,r}^{s-n/l} \). Proceeding analogously, we can show that

\[
\dot{BM}^s_{(p,d),r} \hookrightarrow BW_{p,r}^s \hookrightarrow \dot{B}_{\infty,r}^{s-n/l}.
\] (2.9)
In particular, we obtain
\[ \hat{B} \mathcal{M}^{l,s}_{p,\infty} \hookrightarrow \hat{B}^{-1,\infty}, \]
and
\[ \hat{B} \mathcal{M}^{l,s}_{p,1} \hookrightarrow \hat{B}^{0,1} \hookrightarrow L^\infty. \]

**Remark 4.** Let \( s - n/l < 0 \) and \( r > 1 \), or \( s - n/l \leq 0 \) and \( r = 1 \). For \( f \in B \mathcal{M}^{l,s}_{(p,d),r} \) it follows that \( \sum_{j=-\infty}^{\infty} \Delta_j f \) converges in \( S' \) to the canonical representative of \( f \) in \( S'/\mathcal{P} \) (see e.g. [19]). Then, the space \( B \mathcal{M}^{l,s}_{(p,d),r} \) can be regarded as a subspace of \( S' \). From now on, we say that \( f \in S' \) belongs to \( B \mathcal{M}^{l,s}_{(p,d),r} \) when \( f = \sum_{j=-\infty}^{\infty} \Delta_j f \) in \( S' \), that is, if \( f \) is the canonical representative of its class in \( S'/\mathcal{P} \).

**Lemma 2.7.** Let \( 1 < p \leq l \leq \infty, 1 \leq d \leq \infty \) and \( m \in \mathbb{R} \). Suppose that \( P \) is a \( C^n \)-function on \( \hat{D}_j \) such that \( \| \partial_\xi^\beta P(\xi) \| \leq C 2^{j(m-|\beta|)} \) for all \( \xi \in \hat{D}_j \) and \( |\beta| \leq n \). Then
\[ \| (P\hat{f}) \|_{\mathcal{M}^{l,d}_{(p,d)}} \leq C 2^{jm} \| f \|_{\mathcal{M}^{l,d}_{(p,d)}}, \]
for all \( f \in \mathcal{M}^{l,d}_{(p,d)} \) such that \( \text{supp } \hat{f} \subset D_j \).

**Proof.** Define \( K(x) = (P\hat{\varphi}_j)^{-1}(x) \). Since \( \text{supp } \hat{f} \subset D_j \) it follows that \( \hat{f}(\xi) = \hat{\varphi}_j(\xi) \hat{\varphi}(\xi) \) and \( P(\xi) \hat{f}(\xi) = P(\xi) \hat{\varphi}_j(\xi) \hat{\varphi}(\xi) \), therefore \( (P\hat{f})^{-1} = (P\hat{\varphi}_j \hat{\varphi})^{-1} = K \ast f \).

It follows from Lemma 2.4 that
\[ \| (P\hat{f}) \|_{\mathcal{M}^{l,d}_{(p,d)}} \leq C \| K \|_{L^1} \| f \|_{\mathcal{M}^{l,d}_{(p,d)}}. \]

In order to finish the proof, it is enough to show that \( \| K \|_{L^1} \leq C 2^{mj} \). Taking \( N \in \mathbb{N} \) such that \( 2^N < N \leq n \), we can estimate
\[ \| K \|_{L^1} = \int_{D(0,2^{-j})} K(y) + \int_{|y| \geq 2^{-j}} K(y) \]
\[ \leq \left( \int_{D(0,2^{-j})} 1 \right)^{1/2} \left( \int_{D(0,2^{-j})} |K(y)|^2 \right)^{1/2} \]
\[ + \left( \int_{|y| \geq 2^{-j}} |y|^{-2N} \right)^{1/2} \left( \int_{|y| \geq 2^{-j}} |y|^{2N} |K(y)|^2 \right)^{1/2} \]
\[ \leq C 2^{-j/2} \| P\hat{\varphi}_j \|_{L^2} + C 2^{-j(-N + \frac{s}{2})} \sum_{|\beta|=N} \left\| \partial_\beta \hat{\varphi}_j \right\|_{L^2} \]
\[ \leq C 2^{-j/2} \| P\hat{\varphi}_j \|_{L^2} + C 2^{-j(-N + \frac{s}{2})} \sum_{|\beta|=N} \left\| \partial_\beta \hat{P}\varphi_j \right\|_{L^2} \]
\[ \leq C 2^{-j/2} C 2^{mj} 2^{j/2} + C 2^{-j(-N + \frac{s}{2})} C 2^{j(m-N)2j/2} \]
\[ \leq C 2^{mj}. \]

\( \square \)
The following lemma is a multiplier result of Mihlin type. This can be regarded as an extension of [16, Theorem 2.9] to our setting. As a direct consequence (taking \( m = 0 \)), we obtain the boundedness of the Leray-Hopf projector \( \hat{P} \) in Besov-Lorentz-Morrey spaces.

**Lemma 2.8.** Let \( 1 < p \leq l \leq \infty, 1 \leq d, r \leq \infty \) and \( m, s \in \mathbb{R} \). Suppose that \( P \) is a \( C^m(\mathbb{R}^n \setminus \{0\}) \) function such that \( |\partial_\xi^\beta P| \leq C|\xi|^{m-|\beta|}\) for all multi-index \( \beta \) satisfying \( |\beta| \leq n \). Then, there exists a constant \( C > 0 \) such that
\[
\|P(D) f\|_{\dot{B}M_{(p,d),i}^{1,s-m}} \leq C \|f\|_{\dot{B}M_{(p,d),i}^{1,s}},
\]
for all \( f \in \dot{B}M_{(p,d),i}^{1,s} \).

**Proof.** Note that for all \( \xi \in \hat{D}_j \) and \( j \in \mathbb{Z} \) we have \( |\xi|^{m-|\beta|} \leq C2^{j(m-|\beta|)} \). Thus \( |\partial_\xi^\beta P| \leq C2^{j(m-|\beta|)} \). Moreover, since \( \text{supp} \Delta_j f \subset D_j \) we can use Lemma 2.7 in order to get
\[
\|\Delta_j (P(D) f)\|_{\mathcal{M}_{(p,d)}^{l,s}} = \|P(D) (\Delta_j f)\|_{\mathcal{M}_{(p,d)}^{l,s}} \leq C2^{jm} \|\Delta_j f\|_{\mathcal{M}_{(p,d)}^{l,s}}.
\]
Multiplying (2.11) by \( 2^{j(s-m)} \) and taking the \( l' \)-norm we obtain the result. \( \square \)

3. Product estimates.

3.1. Proof of Proposition 1.1.

**Proof of Proposition 1.1.** Using Bony’s decomposition (2.4), we have
\[
\Delta_j (fg) = \sum_{|k-j| \leq 4} \Delta_j (S_{k-2} f \Delta_k g) + \sum_{|k-j| \leq 4} \Delta_j (S_{k-2} g \Delta_k f) + \sum_{k \geq j-2} \Delta_j (\Delta_k f \Delta_k g)
= I_1^j + I_2^j + I_3^j.
\]
Now we estimate \( I_1^j, I_2^j \) and \( I_3^j \) separately. For \( I_1^j \) we have
\[
2^j \left( \frac{\mu}{2} - 2 \right) \|I_1^j\|_{\mathcal{M}_{(p_2,d_2)}^{l_2}} \leq C2^j \left( \frac{\mu}{2} - 2 \right) \sum_{|k-j| \leq 4} \|S_{k-2} f\|_{\mathcal{M}_{(p_2,d_2)}^{l_2}} \|\Delta_k g\|_{\mathcal{M}_{(p_2,d_2)}^{l_2}}
\leq C2^j \left( \frac{\mu}{2} - 2 \right) \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} \|\Delta_m f\|_{\mathcal{M}_{(p_2,d_2)}^{l_2}} \|\Delta_k g\|_{\mathcal{M}_{(p_2,d_2)}^{l_2}} \right)
\leq C \|f\|_{\dot{B}M_{(p,d),i}^{1,s-m}} 2^j \left( \frac{\mu}{2} - 2 \right) \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( \frac{\mu}{2} - \frac{\mu}{2} \right) \|\Delta_k g\|_{\mathcal{M}_{(p_2,d_2)}^{l_2}} \right)
\leq C \|f\|_{\dot{B}M_{(p,d),i}^{1,s-m}} 2^j \left( \frac{\mu}{2} - 2 \right) \sum_{|k-j| \leq 4} 2^k \left( \frac{\mu}{2} - \frac{\mu}{2} \right) \|\Delta_k g\|_{\mathcal{M}_{(p_2,d_2)}^{l_2}}
\leq C \|f\|_{\dot{B}M_{(p,d),i}^{1,s-m}} 2^j \left( \frac{\mu}{2} - 2 \right) \sum_{|k-j| \leq 4} 2^k \|\Delta_k g\|_{\mathcal{M}_{(p_2,d_2)}^{l_2}}.
\]
Since \( j \approx k \), we can take the \( l' \)-norm in order to obtain

\[
\left( \sum_{j \in \mathbb{Z}} 2^j \left( \frac{p}{2} - 2 \right) r \left\| I_j \right\|_{\mathcal{M}_{(2p_2,2p_2)}^2} \right)^{\frac{1}{r}} \leq C \|f\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{1+p_1}}} \|g\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{1+p_2}}}, \tag{3.2}
\]

with the obvious modification in the case \( r = \infty \).

In order to estimate \( I_2^j \) we use the change \( z = m - k \) and \( j \approx k \) to obtain

\[
2^j \left( \frac{p}{2} - 2 \right) \left\| I_j^2 \right\|_{\mathcal{M}_{(2p_2,2p_2)}^2} \leq C 2^j \left( \frac{p}{2} - 2 \right) \sum_{|k-j| \leq 4} \| S_{k-2} g \|_{\mathcal{M}_{(2p_2,2p_2)}^{2i_1}} \| \Delta_k f \|_{\mathcal{M}_{(2p_2,2p_2)}^{2i_2}} \\
\leq C 2^j \left( \frac{p}{2} - 2 \right) \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( \frac{p}{2} - \frac{m}{2} \right) \| \Delta_m g \|_{\mathcal{M}_{(2p_2,2p_2)}^{2i_1}} \right) 2^k \left( \frac{p}{2} - \frac{m}{2} \right) \| \Delta_k f \|_{\mathcal{M}_{(p,d)}^{i_2}} \\
\leq C \|f\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{1+p_1}}} \left( \sum_{z \leq 0} 2^z \left( \frac{p}{2} - \frac{m}{2} \right) 2^{(z+j) \left( \frac{p}{2} - \frac{z}{2} \right)} \| \Delta_z + g \|_{\mathcal{M}_{(2p_2,2p_2)}^{2i_2}} \right) \\
\times 2^j \left( \frac{p}{2} - \frac{m}{2} \right) \\
\leq C \|f\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{1+p_1}}} \left( \sum_{z \leq 0} 2^z \left( \frac{p}{2} - \frac{m}{2} \right) 2^{(z+j) \left( \frac{p}{2} - \frac{m}{2} \right)} \| \Delta_z + g \|_{\mathcal{M}_{(2p_2,2p_2)}^{2i_2}} \right).
\]

Now we take the \( l' \)-norm and use the Fubini theorem to get (with the usual modification in the case \( r = \infty \))

\[
\left( \sum_{j \in \mathbb{Z}} 2^j \left( \frac{p}{2} - 2 \right) r \left\| I_j \right\|_{\mathcal{M}_{(2p_2,2p_2)}^2} \right)^{\frac{1}{r}} \leq C \|f\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{1+p_1}}} \|g\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{1+p_2}}} \sum_{z \leq 0} 2^z \left( \frac{p}{2} - \frac{m}{2} \right) \| \Delta_z + g \|_{\mathcal{M}_{(2p_2,2p_2)}^{2i_2}} \\
\leq C \|f\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{1+p_1}}} \|g\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{1+p_2}}} - \infty \tag{3.3}
\]

Finally, we turn to \( I_3^j \). Let \( \tilde{p}, \tilde{l}_2 \) and \( \tilde{p}_2 \) be such that \( \tilde{p} \equiv \frac{2}{\tilde{l} + 1}, \tilde{p} = \frac{1}{\tilde{l}} + \frac{1}{\tilde{l}_2} \) and \( \frac{1}{\tilde{p}} = \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}_2} \). Using the change \( z = k - j \) we have

\[
2^j \left( \frac{p}{2} - 2 \right) \left\| I_j^3 \right\|_{\mathcal{M}_{(2p_2,2p_2)}^2} \leq C 2^j \left( \frac{p}{2} - 2 \right) \left( \frac{p}{2} - \frac{m}{2} \right) \left\| I_j^3 \right\|_{\mathcal{M}_{(\tilde{l},\tilde{l}_2)}^2} \\
\leq C 2^j \left( \frac{p}{2} - 2 \right) \sum_{k \geq j-2} \| \Delta_k f \|_{\mathcal{M}_{(\tilde{l},\tilde{l}_2)}^2} \| \Delta_k g \|_{\mathcal{M}_{(\tilde{l},\tilde{l}_2)}^2} \\
\leq C 2^j \left( \frac{p}{2} - 2 \right) \sum_{k \geq j-2} \| \Delta_k f \|_{\mathcal{M}_{(p,d)}^{i_1}} 2^k \left( \frac{p}{2} - \frac{m}{2} \right) \| \Delta_k g \|_{\mathcal{M}_{(p_2,d_2)}^{i_2}} \\
\leq C \|f\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{i_1}}} \sum_{k \geq j-2} 2^k \left( \frac{p}{2} - \frac{m}{2} \right) \| \Delta_k g \|_{\mathcal{M}_{(p_2,d_2)}^{i_2}} \\
\leq C \|f\|_{\dot{B}_{\mathcal{M}_{(p,d)}^{i_1}}} \sum_{k \geq j-2} 2^k \left( \frac{p}{2} - \frac{m}{2} \right) \| \Delta_k g \|_{\mathcal{M}_{(p_2,d_2)}^{i_2}}.
The term

\[ \leq C \| f \|_{\mathcal{B}M_{l_2}^{1,\infty}} \left( \sum_{j \in \mathbb{Z}} 2^{j \left( \frac{n}{2} - 2 \right)} \sum_{|z| \geq -2} 2^{(z+j)(1-\frac{n}{4})} \left\| \Delta_{z+j} g \right\|_{M_{l_2}^{1,\infty}} \right) \]

As before, we take the \( l' \)-norm and use the Fubini theorem to get

\[ \left( \sum_{j \in \mathbb{Z}} 2^{j \left( \frac{n}{2} - 2 \right)} \left\| I_j^3 \right\|_{M_{l_2}^{1,\infty}} \right)^{\frac{1}{l'}} \leq C \| f \|_{\mathcal{B}M_{l_2}^{1,\infty}} \| g \|_{\mathcal{B}M_{l_2}^{1,\infty}} \sum_{|z| \geq -2} 2^{-z \left( \frac{n}{4} - 2 \right)} \leq C \| f \|_{\mathcal{B}M_{l_2}^{1,\infty}} \| g \|_{\mathcal{B}M_{l_2}^{1,\infty}}. \]  

Using the estimates (3.2), (3.3) and (3.4) we obtain the result. \( \square \)

3.2. Proof of Propositions 1.2 and 1.3.

Proof of Propositions 1.2 and 1.3. The proof of Proposition 1.3 follows similarly to that of Proposition 1.1 and is left to the reader. In what follows, we prove Proposition 1.2.

As before, we have that

\[ \Delta_j (fg) = \sum_{|k-j| \leq 4} \Delta_j (S_{k-2} f \Delta_k g) + \sum_{|k-j| \leq 4} \Delta_j (S_{k-2} \Delta_k f) + \sum_{k \geq j} \Delta_j \left( \Delta_k f \Delta_k g \right) \]

\[ = I_j^1 + I_j^2 + I_j^3. \tag{3.5} \]

The term \( I_j^1 \) can be estimated as

\[ \left\| I_j^1 \right\|_{W_{l_1}} \leq C \sum_{|k-j| \leq 4} \| S_{k-2} f \|_{W_{l_1}} \| \Delta_k g \|_{W_{l_2}} \]

\[ \leq C \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} \| \Delta_m f \|_{W_{l_1}} \right) \| \Delta_k g \|_{W_{l_2}} \]

\[ \leq C \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^{2m \left( \frac{n}{4} - \frac{1}{3} \right)} \| \Delta_m f \|_{W_{l_1}} \right) \left( \sum_{m \leq k-2} 2^{2m \left( \frac{n}{4} - \frac{1}{3} \right)} \right) \| \Delta_k g \|_{W_{l_2}} \]

\[ \leq C \| f \|_{BW_{l_1}^{1,\infty}} \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^{2m \left( \frac{n}{4} - \frac{1}{3} \right)} \right) \| \Delta_k g \|_{W_{l_2}} \]

\[ \leq C \| f \|_{BW_{l_1}^{1,\infty}} \sum_{|k-j| \leq 4} 2^{k(1-\frac{n}{4})} \| \Delta_k g \|_{W_{l_2}}. \]

and then,

\[ \left\| I_j^1 \right\|_{W_{l_1}} \leq C \| f \|_{BW_{l_1}^{1,\infty}} \| g \|_{BW_{l_2}^{1,\infty}} 2^{-j \left( \frac{n}{4} - 3 \right)}. \tag{3.6} \]

For \( I_j^2 \), we proceed similarly in order to estimate

\[ \left\| I_j^2 \right\|_{W_{l_1}} \leq C \sum_{|k-j| \leq 4} \| S_{k-2} g \|_{W_{l_1}} \| \Delta_k f \|_{W_{l_2}}. \]
Proposition 1.1, we have

Existence of solutions.

Proof of Theorem 1.4.

Next, we deal with $I_3^j$. In this case, we have

Next, we observe that the result follows directly from the estimates (3.6), (3.7) and (3.8).

4. Existence of solutions.

4.1. Proof of Theorem 1.4.

Proof of Theorem 1.4. We consider the following iterative algorithm. Indeed, this scheme is equivalent to the contraction mapping argument for continuous bilinear mappings (see Remark 2). Set $u^0 = 0, \theta^0 = 0$ and for $n \geq 0$ define $\theta^{n+1}, u^{n+1}$ as

$$\theta^{n+1} = \Delta^{-1} \text{div} (u^n \theta^n) - \Delta^{-1} F,$$

and

$$u^{n+1} = \Delta^{-1} \text{div} (u^n \otimes u^n) - \Delta^{-1} \text{div} (\theta^{n+1} f) - \Delta^{-1} F.$$

We are going to show that there exists $(u, \theta) \in \tilde{B}M_{(p,d_1)}^{l_1,s_1} \times \tilde{B}M_{(p_2,d_2)}^{l_2,s_2}$ such that $u^n \rightarrow u$, $\theta^{n} \rightarrow \theta$ and $(u, \theta)$ is solution of (1.5). From Lemma 2.8 and Proposition 1.1, we have

$$\|g^{n+1}\|_{\tilde{B}M_{(p_1,d_2)}^{l_2,s_2}} \leq \|\Delta^{-1} \text{div} (u^n \theta^n)\|_{\tilde{B}M_{(p_1,d_2)}^{l_2,s_2}} + \|\Delta^{-1} \text{div} (\theta^{n+1} f)\|_{\tilde{B}M_{(p_1,d_2)}^{l_2,s_2}}$$

$$\leq C_1 \|u^n \theta^n\|_{\tilde{B}M_{(p_1,d_2)}^{l_2,s_2}} + C_2 \|G\|_{\tilde{B}M_{(p_1,d_2)}^{l_2,s_2}}$$

$$\leq K \left( \|u^n\|_{\tilde{B}M_{(p_1,d_2)}^{l_2,s_2}} \|\theta^n\|_{\tilde{B}M_{(p_1,d_2)}^{l_2,s_2}} + \|G\|_{\tilde{B}M_{(p_1,d_2)}^{l_2,s_2}} \right)$$
and

\[
\|u^{n+1}\|_{B_M^{1,\infty}} \leq \|\Delta^{-1}\mathbb{P}\text{div} (u^n \otimes u^n)\|_{B_M^{1,\infty}} + \|\Delta^{-1}\mathbb{P} (\theta^{n+1} f)\|_{B_M^{1,\infty}} \\
+ \|\Delta^{-1}F\|_{B_M^{1,\infty}} \\
\leq C_4 \|u^n \otimes u^n\|_{B_M^{1,\infty}} + C_5 \|\theta^{n+1} f\|_{B_M^{1,\infty}} + C_5 \|F\|_{B_M^{1,\infty}} \\
\leq K \left( \|u^n\|^2_{B_M^{1,\infty}} + \|\theta^{n+1}\|_{B_M^{1,\infty}} \|f\|_X + \|F\|_{B_M^{1,\infty}} \right).
\]

Let \( \epsilon > 0 \) be such that \( K \epsilon + \frac{1}{2} < 1 \) and \( K \epsilon^2 + \frac{\epsilon}{3} < \epsilon \), and let \( G, F \) be such that

\[
\|G\|_{B_M^{1,\infty}} < \epsilon \frac{A_1}{\|f\|_X} \quad \text{and} \quad \|F\|_{B_M^{1,\infty}} < \epsilon A_2,
\]

where \( A_1 = \frac{1}{6K^2} \) and \( A_2 = \frac{1}{3K^2} \). Then

\[
\|\theta^1\|_{B_M^{1,\infty}} \leq K \|G\|_{B_M^{1,\infty}} < \frac{\epsilon}{6 \|f\|_X K} < \frac{\epsilon}{3 \|f\|_X K},
\]

\[
\|u^1\|_{B_M^{1,\infty}} \leq K \left( \|\theta^1\|_{B_M^{1,\infty}} \|f\|_X + \|F\|_{B_M^{1,\infty}} \right) \\
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon,
\]

\[
\|\theta^2\|_{B_M^{1,\infty}} \leq K \left( \|u^1\|_{B_M^{1,\infty}} \|\theta^1\|_{B_M^{1,\infty}} + \|G\|_{B_M^{1,\infty}} \right) \\
\leq K \left( \frac{\epsilon}{3 \|f\|_X K} + \frac{\epsilon}{6 \|f\|_X K^2} \right) = \frac{\epsilon}{3 \|f\|_X K} \left( K \epsilon + \frac{1}{2} \right) \\
\leq \frac{\epsilon}{3 \|f\|_X K},
\]

\[
\|u^2\|_{B_M^{1,\infty}} \leq K \left( \|u^1\|^2_{B_M^{1,\infty}} + \|\theta^2\|_{B_M^{1,\infty}} \|f\|_X + \|F\|_{B_M^{1,\infty}} \right) \\
\leq K \left( \epsilon^2 + \frac{\epsilon}{3K} + \|F\|_{B_M^{1,\infty}} \right) \leq K \epsilon^2 + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.
\]

Thus, proceeding inductively and taking \( A_3 = \frac{1}{3K} \), for each \( n \in \mathbb{N} \) we have

\[
\|\theta^n\|_{B_M^{1,\infty}} < \epsilon \frac{A_3}{\|f\|_X} \quad \text{and} \quad \|u^n\|_{B_M^{1,\infty}} < \epsilon.
\]

Now we prove that in fact the sequences \( (\theta^n)_{n \in \mathbb{N}} \) and \( (u^n)_{n \in \mathbb{N}} \) are Cauchy in the respective spaces. From (4.1) and (4.2), we have

\[
\theta^{n+1} - \theta^n = \Delta^{-1} \text{div} \left( (u^n - u^{n-1}) \theta^n + u^{n-1} (\theta^n - \theta^{n-1}) \right),
\]

and

\[
u^{n+1} - u^n = \Delta^{-1} \mathbb{P}\text{div} \left( (u^n - u^{n-1}) \otimes u^n + u^{n-1} \otimes (u^n - u^{n-1}) \right) \\
- \Delta^{-1} \mathbb{P} \left( (\theta^{n+1} - \theta^n) f \right).
\]
Thus,
\[
\| \theta^{n+1} - \theta^n \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2)}, r} \leq \| \Delta^{-1} \text{div} \left( (u^n - u^{n-1}) \theta^n + u^{n-1} (\theta^n - \theta^{n-1}) \right) \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2)}, r} \\
\leq C_1 \| (u^n - u^{n-1}) \theta^n + u^{n-1} (\theta^n - \theta^{n-1}) \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2)}, r} \\
\leq K \| u^n - u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \| \theta^n \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}} \\
+ K \| u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \| \theta^n - \theta^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}} \\
\leq \| u^n - u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \| \mathbf{f} \|_X + \epsilon K \| \theta^n - \theta^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}}, \quad (4.3)
\]
and
\[
\| u^{n+1} - u^n \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \\
\leq \| \Delta^{-1} \text{div} \left( (u^n - u^{n-1}) \otimes u^n + u^{n-1} \otimes (u^n - u^{n-1}) \right) \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \\
+ \| \Delta^{-1} \text{div} \left( (\theta^{n+1} - \theta^n) \mathbf{f} \right) \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \\
\leq C_4 \| (u^n - u^{n-1}) \otimes u^n + u^{n-1} \otimes (u^n - u^{n-1}) \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \\
+ C_5 \| (\theta^{n+1} - \theta^n) \mathbf{f} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \\
\leq K \left( 2\epsilon \| u^n - u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} + \| \theta^{n+1} - \theta^n \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}} \| \mathbf{f} \|_X \right). \quad (4.4)
\]
Now, using (4.3) in (4.4) we obtain
\[
\| u^{n+1} - u^n \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \\
\leq 2\epsilon K \| u^n - u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} + \epsilon \| \mathbf{f} \|_X K^2 \| \theta^n - \theta^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}}, \quad (4.5)
\]
Adding (4.3) to (4.5), we get
\[
\| \theta^{n+1} - \theta^n \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}} + \| u^{n+1} - u^n \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \\
\leq \| u^n - u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \left( \frac{\epsilon}{3 \| \mathbf{f} \|_X} + \epsilon K \| \theta^n - \theta^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}} \right) \\
+ 2\epsilon K \| u^n - u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \\
+ \epsilon K \| u^n - u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} + \epsilon \| \mathbf{f} \|_X K^2 \| \theta^n - \theta^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}} \\
\leq \left( \frac{\epsilon}{3 \| \mathbf{f} \|_X} + 2\epsilon K + \epsilon K \right) \left( \frac{\epsilon}{3 \| \mathbf{f} \|_X} \right) \| u^n - u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \\
+ \left( \epsilon K + \epsilon \| \mathbf{f} \|_X K^2 \right) \| \theta^n - \theta^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}} \\
\leq C(\epsilon) \left( \| \theta^n - \theta^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}} + \| u^n - u^{n-1} \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \right) \\
\leq (C(\epsilon))^n \left( \| \theta^1 \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), r}} + \| u^1 \|_{\hat{B}M^{l_2, r_2}_{(p_d, d_2), \infty}} \right), \quad (4.6)
\]
with \( C(\epsilon) \to 0 \) when \( \epsilon \to 0 \). Therefore, by taking \( 0 < \epsilon \ll 1 \) we have \( 0 < C(\epsilon) < 1 \), and from (4.6) we conclude that the sequences \( (\theta^n)_{n \in \mathbb{N}} \) and \( (u^n)_{n \in \mathbb{N}} \) are Cauchy in \( \dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right) \) and \( \dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right) \), respectively.

Let \( u \) and \( \theta \) be such that \( u^n \to u \) and \( \theta^n \to \theta \). Now we are going to prove that \( (u, \theta) \) is solution of (1.5). For that, we estimate
\[
0 \leq \| u - \Delta^{-1} P \text{div} (u \otimes u) + \Delta^{-1} P (\theta f) + \Delta^{-1} P F^{\prime} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} \\
= \| u - \Delta^{-1} P \text{div} (u \otimes u) + \Delta^{-1} P (\theta f) + \Delta^{-1} P F - u^{n+1} + u^{n+1} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} \\
\leq \| u - u^{n+1} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} \\
+ \| -\Delta^{-1} P \text{div} (u \otimes u) + \Delta^{-1} P (\theta f) + \Delta^{-1} P F + u^{n+1} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} \\
\leq C \left( \| u - u^{n+1} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} + \| u - u^{n} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} + \| \theta - \theta^{n+1} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} \right) \\
\to 0
\]

and
\[
0 \leq \| \theta - \Delta^{-1} \text{div} (u \theta) + \Delta^{-1} G \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} \\
= \| \theta - \Delta^{-1} \text{div} (u \theta) + \Delta^{-1} G - \theta^{n+1} + \theta^{n+1} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} \\
\leq \| \theta - \theta^{n+1} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} + \| -\Delta^{-1} \text{div} (u \theta) + \Delta^{-1} G + \theta^{n+1} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} \\
\leq C \left( \| \theta - \theta^{n+1} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} + \| u - u^{n} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} + \| \theta - \theta^{n} \|_{\dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right)} \right) \\
\to 0,
\]

which concludes the proof of the existence part.

Finally, notice that the third estimate in (4.6), with slight modifications to consider two possible solutions \( (u, \theta) \) and \( (\tilde{u}, \tilde{\theta}) \), also implies the statement about uniqueness of solutions.

**4.2. Proof of Corollaries 1.5 and 1.6.**

**Proof of Corollaries 1.5 and 1.6.** The proof of Corollary 1.5 follows by using Proposition 1.2 and Theorem 1.4 with \( X = \dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right) \). Also, Corollary 1.6 follows by using Proposition 1.3 and Theorem 1.4 with \( X = \dot{BM}_{l,s_{1},2}^{1} \left( (p_{1},d_{2}), \right) \).

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