1. Introduction

Landau-Lifshitz-Gilbert equation (LLGE) coupled to Maxwell equations provides a fundamental mathematical model for physical properties of ferromagnetic materials, and it has been intensely investigated by physicists since the seminal work by Landau and Lifschitz \cite{26}. The exact form of the equation is determined by the energy functional that may include varying number of terms, so that in fact we have to deal with the whole family of equations. The first mathematical analysis of the LLGE corresponding to the full energy functional and coupled to the time dependent Maxwell equations was provided by Visintin in \cite{32}. It has been noticed by physicists a long time ago, that the noise must be included into the deterministic LLGE, see \cite{28, 7, 6}. Including the noise into LLGE requires sophisticated tools from the theory of quasilinear stochastic PDEs that have been missing for some time and are still not well developed. A rigorous mathematical theory of stochastic LLGE was initiated in \cite{10} and intensely studied since then, see \cite{5, 9, 11, 21, 23}. In all these papers a simplified version of the energy functional is considered and so far the stochastic LLG equation associated to the full energy functional and coupled to the time dependent Maxwell equations has never been studied. This is a serious deficiency since coupling with the Maxwell equations is fundamental for many physical phenomena, such as emergence and movement of boundary vortices, and movement of the domain walls, see \cite{27}. Even for deterministic systems, the case with time dependent Maxwell equations is not well understood and after the seminal paper \cite{32} most of the effort was focused on the so-called quasi-static case. Recently, the interest in the full time-dependent case has been renewed, see for example \cite{17, 25, 33}. In the stochastic case, the only work in this direction, we are aware of, is the paper \cite{22} but it imposes strong simplifying assumptions on the noise and the energy functional.
In this paper we are concerned with the stochastic Landau-Lifschitz-Gilbert equation coupled to time dependent Maxwell equations and we assume that the evolution of spins is driven by the full energy functional described below. To be more precise, given the time horizon $T > 0$ and a bounded open domain $\mathcal{D} \subset \mathbb{R}^3$, the magnetization field $M : [0, T] \times \mathcal{D} \rightarrow \mathbb{R}^3$ satisfies the Landau-Lifschitz-Gilbert equation:

\[
\frac{dM(t, x)}{dt} = \lambda_1 M(t, x) \times \rho(t, x) - \lambda_2 M(t, x) \times (M(t, x) \times \rho(t, x)),
\]

subject to the constraint

\[
|M(t, x)| = |M_0(x)|,
\]

where $\rho$ is the effective field defined by

\[
\rho = -\nabla_M \mathcal{E}.
\]

Here $\mathcal{E}$ is the total electro-magnetic energy including anisotropy energy, exchange energy, magnetic field energy and electronic energy.

Since the magnetic field energy and electronic energy are related to the magnetic field and electric field, we also consider the magnetic field $H : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and the electric field $E : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ in this paper. We denote

\[
B := H + \overline{M},
\]

where

\[
\overline{M}(x) := \begin{cases} 
M(x), & x \in \mathcal{D}; \\
0, & x \notin \mathcal{D}.
\end{cases}
\]

Then $B$ and $E$ are related by the Maxwell’s equation:

\[
dB = \nabla \times E \, dt.
\]
\[
dE = \nabla \times [B - \overline{M}] \, dt - [1_{\partial \mathcal{D}} E + \overline{f}(t)] \, dt.
\]

Summarising, the equation we are going to study in this paper has the following form:

\[
dM(t) = \left[\lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho)\right] \, dt \\
+ \sum_{j=1}^{\infty} \left\{ [\alpha M \times h_j + \beta M \times (M \times h_j)] \circ dW_j(t) \right\}.
\]

\[
dB(t) = \nabla \times E(t) \, dt.
\]
\[
dE(t) = \nabla \times [B(t) - \overline{M}(t)] \, ds - [1_{\partial \mathcal{D}} E(t) + \overline{f}(t)] \, dt.
\]
\[
\frac{\partial M}{\partial r} \big|_{\partial \mathcal{D}} = 0.
\]

$M(0) = M_0$, \quad $B(0) = B_0$, \quad $E(0) = E_0$.

This paper is constructed as follows. In section 2, we first give all the formal definitions of all the energies and state the problem we consider. Secondly we give the definition of the solution of the stochastic differential equation. And at last we formulate the main result (Theorem 2.9) of the whole paper. In section 3, we construct a series of some auxiliary equations (3.12), with all the elements in a finite dimensional space and proved the existence and uniqueness of the global solution of the finite dimensional equations. In section 4, we get some a’priori estimates of the series of solutions of equations (3.12). In section 5, we show the laws of the finite dimensional solutions are tight on some spaces. In section 6, we construct a new probability space by the Skorohod Theorem in which there exist limit processes $\overline{M}, \overline{B}, \overline{E}$ of the solutions of (3.12). In section 7, we prove that the $\overline{M}, \overline{B}, \overline{E}$ which we got in section 6 are actually the weak solution of our original problem. In section 8, we show some more regularities of the weak solution and complete the proof of the main result, i.e. the Theorem 2.9.
2. Statement of the Problem and Formulation of the Main Result

Assumption 2.1. Throughout this paper we assume \( D \subset \mathbb{R}^3 \) to be a bounded open domain with \( C^2 \) boundary.

Notation 2.2. (1) We use the following notations for the classical functional spaces:

\[
L^p := L^p(D; \mathbb{R}^3) \quad \text{or} \quad L^p(D; \mathbb{R}^{3 \times 3}), \quad L^\infty := L^\infty(\mathbb{R}^3; \mathbb{R}^3)
\]

\( W^{k,p} := W^{k,p}(D; \mathbb{R}^3), \quad \mathbb{H}^k := H^k(D; \mathbb{R}^3) = W^{k,2}(D; \mathbb{R}^3), \) and \( \mathbb{V} := W^{1,2}, \mathbb{H} := L^2. \)

(2) The duality between a Banach space \( X \) and its dual \( X' \) will be denoted by \( X \langle \cdot, \cdot \rangle_X \). The notations \( \langle \cdot, \cdot \rangle_X \) and \( \| \cdot \|_X \) stand for the scalar product and its associated norm in a given Hilbert space \( K \) respectively. The norm of a vector \( x \in \mathbb{R}^d \) will be denoted by \( |x| \) for any \( d \).

(3) For a function \( \varphi : \mathbb{R}^3 \to \mathbb{R} \) we will write

\[
\varphi' := \nabla \varphi, \quad \text{and} \quad \varphi'' := \nabla^2 \varphi.
\]

(4) For a function \( u : D \to \mathbb{R}^3 \), we denote

\[
\overline{u}(x) := \begin{cases} u(x), & x \in D, \\ 0, & x \notin D. \end{cases}
\]

(5) Let us recall that for \( u \in L^2(\mathbb{R}^3) \), we define the distribution \( \nabla \times u \) by

\[
\langle \nabla \times u, v \rangle_\mathbb{H} := \langle u, \nabla \times v \rangle_\mathbb{H}, \quad v \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3).
\]

Then we define the Hilbert space

\[
\mathbb{Y} := \left\{ u \in L^2(\mathbb{R}^3) : \nabla \times u \in L^2(\mathbb{R}^3) \right\},
\]

with the inner product

\[
\langle u, v \rangle_\mathbb{Y} := \langle u, v \rangle_{L^2(\mathbb{R}^3)} + \langle \nabla \times u, \nabla \times v \rangle_{L^2(\mathbb{R}^3)}.
\]

Definition 2.3 (Magnetic Induction). Given a magnetization field \( M : \mathbb{D} \to \mathbb{R}^3 \) and a magnetic field \( H : \mathbb{R}^3 \to \mathbb{R}^3 \), we define the magnetic induction as a vector field \( B : \mathbb{R}^3 \to \mathbb{R}^3 \) by

\[
B := H + M.
\]

Definition 2.4. (The energy)

(i) Suppose that \( \varphi \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^+) \). For a magnetization field \( M \in \mathbb{Y} \), we define the anisotropy energy of \( M \) by:

\[
E_{\text{an}}(M) := \int_D \varphi(M(x)) \, dx.
\]

(ii) We define the exchange energy of \( M \) by:

\[
E_{\text{ex}}(M) := \frac{1}{2} \int_D |\nabla M(x)|^2 \, dx = \frac{1}{2} \|\nabla M\|_{L^2}^2.
\]

(iii) For a magnetic field \( H \in L^2(\mathbb{R}^3) \), we define the Zeeman energy by:

\[
E_{\text{z}}(H) := \frac{1}{2} \int_{\mathbb{R}^3} |H(x)|^2 \, dx = \frac{1}{2} \|H\|_{L^2(\mathbb{R}^3)}^2 = \frac{1}{2} \|B - M\|_{L^2(\mathbb{R}^3)}^2.
\]

Finally, given an electric field \( E \in L^2(\mathbb{R}^3) \), a magnetization field \( M \in \mathbb{Y} \) and a magnetic field \( H \in L^2(\mathbb{R}^3) \), (hence the magnetic induction \( B \in L^2(\mathbb{R}^3) \)) we define the total electro-magnetic energy by

\[
E(M, B, E) = \int_D \varphi(M(x)) \, dx + \frac{1}{2} \|\nabla M\|_{L^2}^2 + \frac{1}{2} \|B - M\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \|E\|_{L^2(\mathbb{R}^3)}^2.
\]

Now let us investigate some properties of the total energy \( E \).

Definition 2.5 (Effective field). We define the effective field \( \rho \in \mathbb{Y}^* \) as

\[
\rho := \varphi'(M) - (1 + B - M) - \Delta M, \quad \text{in} \ \mathbb{Y}^*.
\]
Lemma 1. For \( M \in \mathcal{V} \), if we define \( \Delta M \in \mathcal{V}^* \) by
\[
\langle \Delta M, u \rangle_{\mathcal{V}} := -\langle \nabla M, \nabla u \rangle_{L^2}, \quad \forall u \in \mathcal{V}.
\]
Then the total energy \( E : \mathcal{V} \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \rightarrow \mathbb{R} \) defined in (2.4) has partial derivatives of 2nd order with respect to \( M, B \) and \( E \) well defined and:
\[
\frac{\partial E}{\partial M}(M, B, E) = \varphi'(M) - (1_2B - M) - \Delta M, \quad \text{in } \mathcal{V}^*,
\]
for \( u, v \in \mathcal{V} \),
\[
\frac{\partial^2 E}{\partial M^2}(M, B, E)(u, v) = \int_\Omega \varphi''(M(x))(u(x), v(x)) \, dx + \langle u, v \rangle_{\mathcal{V}} ,
\]
\[
\frac{\partial E}{\partial B}(M, B, E) = B - \mathcal{M} ,
\]
\[
\frac{\partial E}{\partial E}(M, B, E) = E ,
\]
for a fixed \( h \in \mathbb{L}^\infty \) and \( \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_2 > 0 \) we define a mapping \( G_h \) by
\[
\mathbb{L}^\infty \ni u \longmapsto G_h(u) = \lambda_1 u \times h - \lambda_2 u \times (u \times h) \in \mathbb{L}^2 .
\]
For a given sequence \( \{h_j\}_{j=1}^\infty \subset \mathbb{L}^\infty \) we will use the notation \( G_j(u) := G_{h_j}(u) \). We are now ready to formulate the problem we are going to study in this paper.

Problem 2.6. Let \((\Omega, \mathcal{F}, \mathbb{P} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space, and let \( W = (W_j)_{j=1}^\infty \) be independent, real valued, \( \mathbb{P} \)-Wiener processes. Let
\[
M_0 \in \mathcal{V} \text{ with } |M_0(x)| = 1 \text{ for all } x \in \mathcal{D};
\]
\[
B_0 \in L^2(\mathbb{R}^3); \quad \nabla \cdot B_0 = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^3; \mathbb{R}); \quad E_0 \in L^2(\mathbb{R}^3); \quad c^2 = \sum_{j=1}^\infty \|h_j\|_{\mathbb{L}^\infty} + \sum_{j=1}^\infty \|h_j\|_{\mathbb{L}^1(\mathbb{R}^3)}^2 < \infty ,
\]
\[
f \in L^2(0, T; \mathbb{H}); \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{R}^*);
\]
\[
\lambda_1 \in \mathbb{R}, \quad \lambda_2 > 0 .
\]
Our aim is to show that the following system of stochastic PDEs has a solution in the sense made precise below:
\[
\begin{cases}
dM(t) = [\lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho)] \, dt + \sum_{j=1}^\infty G_j(M) \circ dW_j(t), \\
\quad dB(t) = \nabla \times E(t) \, dt , \\
\quad dE(t) = \nabla \times [B(t) - \mathcal{M}(t)] \, ds - [1_2E(t) + \mathcal{T}(t)] \, dt .
\end{cases}
\]
with the boundary conditions
\[
\frac{\partial M(t)}{\partial \nu} \bigg|_{\partial \mathcal{D}} = 0, \quad t \geq 0, \quad \text{where } \nu \text{ is the exterior normal vector on } \partial \mathcal{D}.
\]
and the initial conditions
\[
M(0) = M_0, \quad B(0) = B_0, \quad E(0) = E_0 .
\]
The Stratonovich equation in (2.12) can be rewritten as an Itô equation

\[
dM = \left[ \lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho) + \frac{1}{2} \sum_{j=1}^{\infty} G_j(M) G_j(M) \right] dt + \sum_{j=1}^{\infty} G_j(M) dW_j
\]

(2.13) = \left\{ \lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho) + \frac{1}{2} \sum_{j=1}^{\infty} \left[ \lambda_1^2 \left[ (M \times h_j) \times h_j \right] + \lambda_1 \lambda_2 \left[ M \times (M \times h_j) \right] \times h_j \right] + \lambda_2^2 \left[ M \times (M \times h_j) \times h_j \right] \right\} dt + \sum_{j=1}^{\infty} \left\{ \lambda_1 \left[ M \times h_j \right] + \lambda_2 \left[ M \times (M \times h_j) \right] \right\} dW_j.

**Remark 2.7.** We can understand the noise as a $Q$-Wiener process $W_\theta(t) := \sum_{j=1}^{\infty} W_j(t) h_j \sim N(0, tQ)$ on $\mathbb{H}$ for some operator $Q$ which is nonnegative, symmetric and with finite trace. In fact, $W_\theta(0) = 0$ a.s. is obvious.

By [15, Proposition 3.18], $W_j h_j$ can be viewed as a random variable taking values on $C([0,T]; \mathbb{H})$ for each $j$. And by our assumption of $h_j$ and using Doob’s maximal inequality, we can show $\{\sum_{j=1}^{\infty} W_j(t) h_j\}_t$ is a Cauchy sequence in $L^2(\Omega; C([0,T]; \mathbb{H}))$, therefore their limit $W_\theta \in L^2(\Omega; C([0,T]; \mathbb{H}))$.

Hence $W_\theta$ has continuous trajectory almost surely.

The independence of increment of $W_\theta$ follows from $W_j$ are independent for all different $j$ and they all have independent increments for each $j$.

So it only remains to check the distribution of $W_\theta$ on $\mathbb{H}$. Since $W_\theta$ is sum of independent normal random variables with mean 0, so it has normal distribution with mean 0 as well.

Next we try to find its covariance operator. For any $u, v \in \mathbb{H}$, we have

\[
\mathbb{E} \left( \langle W_\theta(t), u \rangle_{\mathbb{H}} \langle W_\theta(t), v \rangle_{\mathbb{H}} \right) = \mathbb{E} \left( \sum_{j=1}^{\infty} \langle W_j(t) h_j, u \rangle_{\mathbb{H}} \langle W_j(t) h_j, v \rangle_{\mathbb{H}} \right) = t \sum_{j=1}^{\infty} \langle h_j, u \rangle_{\mathbb{H}} \langle h_j, v \rangle_{\mathbb{H}}
\]

So the covariance operator $Q$ is uniquely determined by

\[
\langle Qu, v \rangle_{\mathbb{H}} = \sum_{j=1}^{\infty} \langle h_j, u \rangle_{\mathbb{H}} \langle h_j, v \rangle_{\mathbb{H}}, \quad u, v \in \mathbb{H}.
\]

We can check that the operator $Q$ is nonnegative, symmetric and with finite trace.

So $W_\theta$ is really a $Q$-Wiener process for some nonnegative, symmetric and with finite trace operator $Q$.

Hence it has a representation

\[
W_\theta(t) = \sum_{j=1}^{\infty} \tilde{W}_j(t) \tilde{h}_j,
\]

Where $\{\tilde{W}_j\}$ are independent 1-dimensional Brownian motions and $\{\tilde{h}_j\}$ is an ONB of $\mathbb{H}$ which consisted by eigenvectors of $Q$.

Therefore we can actually assume that $\{h_j\}$ is an ONB of $\mathbb{H}$.

**Definition 2.8 (Weak martingale solution of equation (2.12)).** Given $T > 0$, a weak martingale solution of equation (2.12) is a system consisting of a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, an $t$-dimensional $\mathbb{F}$-Wiener process $W = (W_t)_{t=1}^{\infty}$ and $\mathbb{F}$-progressively measurable processes

\[
\tilde{M} : [0,T] \times \tilde{\Omega} \longrightarrow \mathbb{V}, \quad \tilde{B} : [0,T] \times \tilde{\Omega} \longrightarrow L^2(\mathbb{R}^3), \quad \tilde{E} : [0,T] \times \tilde{\Omega} \longrightarrow L^2(\mathbb{R}^3)
\]
such that for all the test functions \( u \in \mathcal{V} \), \( v \in \mathcal{Y} \) and \( t \in [0, T] \), we have the following equalities holds \( \mathbb{P} \)-a.s.:

\[
\int_D \left( \tilde{M}(t) - M_0, u \right) dx = \int_0^t \int_D \left( \tilde{B} - \tilde{M} - \varphi'(\tilde{M}), \lambda_1 u \times \tilde{M} - \lambda_2 (u \times \tilde{M}) \right) dx ds \\
- \sum_{i=1}^3 \left( \nabla \tilde{M}, \lambda_i \nabla u \times \nabla \tilde{M} - \lambda_1 \left( \nabla u \times \tilde{M} + u \times \nabla \tilde{M} \right) \right) dx ds \\
+ \sum_{j=1}^N \int_0^t \left( G_j (\tilde{M}), u \right) \circ d\tilde{W}(s);
\]

\[
\int_{\mathbb{R}^3} \left( \tilde{B}(t) - B_0, v \right) dx = - \int_0^t \int_{\mathbb{R}^3} \left( \tilde{E}, \nabla \times v \right) dx ds;
\]

\[
\int_{\mathbb{R}^3} \left( \tilde{E}(t) - E_0, v \right) dx = \int_0^t \int_{\mathbb{R}^3} \left( \tilde{B} - \tilde{M}, \nabla \times v \right) dx ds - \int_0^t \int_{\mathbb{R}^3} \left( \tilde{E} + f, v \right) dx ds.
\]

Next we would like to formulate the main result of this paper:

**Theorem 2.9.** There exists a weak martingale solution of Problem 2.6 with the following stronger regularity properties:

(i) \( \tilde{M} \in L^2(\tilde{\Omega}; L^\infty(0, T; \mathcal{V})) \), \( \forall r > 0 \); \( \tilde{B}, \tilde{E} \in L^2(\tilde{\Omega}; L^2(0, T; L^2(\mathbb{R}^3))) \);

(ii) For every \( t \in [0, \infty) \), the equation

\[
\tilde{M}(t) = M_0 + \int_0^t \left( \lambda_1 \tilde{M} \times \tilde{\rho} - \lambda_2 \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \right) ds + \sum_{j=1}^N \int_0^t G_j (\tilde{M}) \circ d\tilde{W}(s),
\]

holds in \( L^2(\tilde{\Omega}; \mathcal{H}) \).

(iii)

\[
\tilde{B}(t) = B_0 - \int_0^t \nabla \times \tilde{E} ds \in \mathcal{Y}, \quad \mathbb{P} - a.s.
\]

\[
\tilde{E}(t) = E_0 + \int_0^t \nabla \times [\tilde{B} - \tilde{M}] ds - \int_0^t [1_{\partial \mathcal{D}} \tilde{E} + \tilde{f}] ds \in \mathcal{Y}, \quad \mathbb{P} - a.s.
\]

(iv) For every \( \theta \in \left( 0, \frac{1}{2} \right) \), \( \tilde{M} \in C^\theta([0, T]; \mathcal{H}) \), \( \tilde{B} - a.s. \)

**Remark 2.10.** In (2.18) \( \tilde{\rho} \) is a distribution, but \( \tilde{M} \times \tilde{\rho} \in L^2(\tilde{\Omega}; L^2(0, T; \mathcal{H})) \). Precise definition is provided in Notation 6.20.

**Remark 2.11.** Equality (2.19) implies \( \nabla \cdot B(t) = 0 \), for all \( t \in [0, T] \).
3. Galerkin approximation

In this section we start to prove the existence of the martingale solution to Problem 2.6. We begin with the classical Galerkin approximation. Let $A$ denote the negative Laplace operator in $\mathcal{D}$ with the homogeneous Neumann boundary condition:
\[
D(A) := \left\{ u \in \mathbb{H}^2 : \frac{\partial u}{\partial \nu}|_{\partial D} = 0 \right\}, \quad A := -\Delta,
\]
where $\nu$ stands for the outer normal to the boundary of $\mathcal{D}$. The operator $A$ is self-adjoint and there exists an orthonormal basis $\{e_k : k \geq 1\} \subset C^0(\mathcal{D}, \mathbb{R}^3) \cap D(A)$ of $\mathbb{H}$ that consists of eigenvectors of $A$. We set $\mathbb{H}_n = \text{linspan}(e_1, e_2, \ldots, e_n)$ and denote by $\pi_n$ the orthogonal projection from $\mathbb{H}$ to $\mathbb{H}_n$. We also note that $V = D(A^{1/2})$ for $A_1 := I + A$, and $\|u\|_V = \|A^{1/2}u\|_{\mathbb{H}}$ for $u \in V$.

The following properties of the operator $A$ will be frequently used later: for any $u \in D(A)$ and $v \in V$,
\[
\langle Au, v \rangle_H = \int_D \left( \nabla u(x), \nabla v(x) \right)_{\mathbb{R}^3} \, dx
\]
and
\[
\langle u \times Au, v \rangle_H = \sum_{i=1}^3 \langle \nabla u_i, \nabla v \times u \rangle_H.
\]

Let $\{\varphi_n\}_{n=1}^{\infty} \subset C_0^0(\mathbb{R}^3; \mathbb{R}^3)$ be an orthonormal basis of $L^2(\mathbb{R}^3)$. We define $\mathcal{Y}_n := \text{linspan}(\varphi_1, \ldots, \varphi_n)$ and the orthogonal projections
\[
\pi_n^\varphi : L^2(\mathbb{R}^3) \longrightarrow \mathcal{Y}_n \quad \text{and} \quad \pi_n^\varphi|_V : V \longrightarrow \mathcal{Y}_n, \quad n \in \mathbb{N}.
\]

On $\mathbb{H}_n$ and $\mathcal{Y}_n$, we consider the scalar product inherited from $\mathbb{H}$ and $V$ respectively. Let us denote by $E_n$ the restriction of the total energy functional $E$ to the finite dimensional space $\mathbb{H}_n \times \mathcal{Y}_n \times \mathcal{Y}_n$, i.e.
\[
E_n : \mathbb{H}_n \times \mathcal{Y}_n \times \mathcal{Y}_n \longrightarrow \mathbb{R},
\]
\[
E_n(M, B, E) = \int_D \varphi(M(x)) \, dx + \frac{1}{2} \|\nabla M\|_H^2 + \frac{1}{2} \|B - \pi_n^\varphi M\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \|E\|_{L^2(\mathbb{R}^3)}^2.
\]

**Lemma 3.1.** The function $E_n$ is of class $C^2$ and for $M \in \mathbb{H}_n$, $B, E \in \mathcal{Y}_n$ we have:

(i) \quad \langle \nabla_M E_n(M, B, E), (M, B, E) \rangle_H = \pi_n(\varphi(M) - \varphi'(M)) - \Delta M,

(ii) \quad \langle \nabla_B E_n(M, B, E), (M, B, E) \rangle_H = B - \pi_n^\varphi M,

(iii) \quad \langle \nabla_E E_n(M, B, E), (M, B, E) \rangle_H = E,

(iv) \quad \frac{\partial^2 E_n}{\partial M^2}(M, B, E)(u, v) = \int_D \varphi''(M(x))(u(x), v(x)) \, dx + \langle u, v \rangle_V, \quad u, v \in V.

**Notation 3.2.** Let us define the function $\rho_n : \mathbb{H}_n \times \mathcal{Y}_n \times \mathcal{Y}_n \longrightarrow \mathbb{H}_n$ which corresponds to $\rho$ by:
\[
\rho_n := -\langle \nabla_M E_n(M_n, B_n, E_n), (M_n, B_n, E_n) \rangle_H = \pi_n(-\varphi(M_n) + \varphi'(M_n)) + \Delta M_n \in \mathbb{H}_n.
\]

We will also need a function $\psi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $\psi \in C^1(\mathbb{R}^3)$,
\[
\psi(x) = \begin{cases} 
1, & |x| \leq 3, \\
0, & |x| \geq 5,
\end{cases}
\]
and $|\nabla \psi| \leq 1$.

**Remark 3.3.** The $\psi$ defined above is a truncation on $\mathcal{M}$ which to make sure we can get the estimates in Proposition 4.1 below. The setting of $|\nabla \psi| \leq 1$ is also necessary, for instance in the proof of Lemma 7.2. By Theorem 8.1, we will prove that $|\mathcal{M}(t, x)| = 1$ for almost every $x \in \mathcal{D}$, therefore we can remove this $\psi$ by the end.
It also will be convenient to define mappings

\[ F_n : \mathbb{H}_n \times \mathbb{Y}_n \times \mathbb{Y}_n \to \mathbb{H}_n \text{ and } G_m : \mathbb{H}_n \to \mathbb{H}_n, \quad j = 1, 2, \ldots \text{ by} \]

\[ F_n(M_n, B_n, E_n) := \lambda_1 \pi_n[M_n \times \rho_n] - \lambda_2 \pi_n[M_n \times (M_n \times \rho_n)] + \frac{1}{2} \sum_{j=1}^{\infty} G'_m(M_n) G_m(M_n), \]

where

\[ G'_m(M_n) \left[ G_m(M_n) \right] := \lambda_1^2 \pi_n \left[ \pi_n(M_n \times h_1) \times h_1 \right] \]

\[ + \lambda_1 \lambda_2 \pi_n \left[ \psi(M_n)(M_n \times (M_n \times h_1)) \times h_1 \right] + \lambda_2^2 \pi_n \left[ \psi(M_n)(M_n \times [(M_n \times (M_n \times h_1)) \times h_1]) \right] \]

\[ + \lambda_1 \lambda_2 \pi_n \left[ \psi(M_n)(M_n \times [(M_n \times h_1) \times h_1]) \right] + \lambda_2^2 \pi_n \left[ \pi_n(M_n \times (M_n \times h_1)) \times (M_n \times h_1) \right] \]

Note that because of the \( \psi \), (3.9) is only a notation, not the Fréchet derivative of \( G_m \).

Similar as (3.9), we will also use the following notations

\[ G^\psi_n(M) := \lambda_1 M \times h_1 + \lambda_2 \psi(M) M \times (M \times h_1), \]

and

\[ (G^\psi_n)'(M) \left[ G^\psi_n(M) \right] := \lambda_1^2 \left[ (M \times h_1) \times h_1 \right] \]

\[ + \lambda_1 \lambda_2 \left[ \psi(M)(M \times (M \times h_1)) \times h_1 \right] + \lambda_2^2 \left[ \psi(M)(M \times [(M \times (M \times h_1)) \times h_1]) \right] \]

\[ + \lambda_1 \lambda_2 \left[ \psi(M)(M \times [(M \times h_1) \times h_1]) \right] + \lambda_2^2 \left[ \psi(M)(M \times (M \times h_1)) \times (M \times h_1) \right] \]

Remark 3.4. It may looks like there are too many \( \pi_n \) in (3.9), but all of them are necessary because it is not only we want all the terms of (3.9) are in \( \mathbb{H}_n \), but we also want to get the a’priori estimates in Proposition 4.1.

To solve Problem 2.6, we first consider the following system of equations in \( \mathbb{H}_n \times \mathbb{Y}_n \times \mathbb{Y}_n \):

**Problem 3.5.**

\[ \begin{cases} 
  \frac{dM_n(t)}{dt} = F_n(M_n(t), B(t), E(t)) \, dt + \sum_{j=1}^{\infty} G_m(M_n(t)) \, dW_j \\
  \frac{dE_n(t)}{dt} = -\pi_n^\psi[1_2(E_n(t) + \overline{f}(t))] \, dt + \pi_n^\psi[\nabla \times (B_n(t) - \pi_n^\psi \overline{M}_n(t))] \, dt \\
  \frac{dB_n(t)}{dt} = -\pi_n^\psi[\nabla \times E_n(t)] \, dt 
\end{cases} \]

with the initial conditions

\[ M_n(0) = \pi_n M_0, \quad E_n(0) = \pi_n^\psi E_0, \quad B_n(0) = \pi_n^\psi B_0. \]

**Lemma 3.6.** There exists a unique global strong solution \((M_n, B_n, E_n)\) of Problem 3.5. In particular, \((M_n, B_n, E_n) \in C^1([0, \infty); \mathbb{H}_n \times \mathbb{Y}_n \times \mathbb{Y}_n), \mathbb{P}\) almost surely.

**Proof.** We define mappings

\[ \overline{F}_n : \mathbb{H}_n \times \mathbb{Y}_n \times \mathbb{Y}_n \to \mathbb{H}_n \times \mathbb{Y}_n \times \mathbb{Y}_n \]

\[ \overline{G}_m : \mathbb{H}_n \times \mathbb{Y}_n \times \mathbb{Y}_n \to \mathbb{H}_n \times \mathbb{Y}_n \times \mathbb{Y}_n \]

putting

\[ \overline{F}_n(u, v, w) = \begin{pmatrix} F_n(u, v, w) \\ -\pi_n^\psi[1_2(w + \overline{f})] \, dt + \pi_n^\psi[\nabla \times (v - \pi_n^\psi \overline{M})] \\ -\pi_n^\psi[\nabla \times w] \end{pmatrix} \]

and

\[ \overline{G}_m(u, v, w) = \begin{pmatrix} G_m(u) \\ 0 \\ 0 \end{pmatrix} \]
Then system (3.12) takes the form of a stochastic differential equation

\[ dX_n = \widehat{F}_n(X_n) \, dt + \sum_{j=1}^{\infty} \widehat{G}_{jm}(X_n) \, dW_j \]

where \( X_n = (M_n, B_n, E_n) \). The mapping \( \widehat{F}_n \) defined in (3.14) is Lipschitz on balls. For the mapping \( \widehat{G}_{jm} \) defined in (3.15), note that \( G_{jm} \) are Lipschitz and we have

\[ \sum_{j=1}^{\infty} \| G_{jm}(u) - G_{jm}(v) \|_{[1]} \leq c_n^2 \| u - v \|_{[1]} , \]

where \( c_n \) was defined in (2.11). Since \( \widehat{F}_n \) are of one sided linear growth, the claim follows by standard arguments, see for example Theorem 3.1 in [1]. \( \square \)

4. A priori estimates

Next we will get some a priori estimates of the solution to equation (3.12).

**Proposition 4.1.** For any \( T > 0 \), \( p > 0 \) and \( b > \frac{1}{2} \), there exists a constant \( C = C(p, b) > 0 \) independent of \( n \) such that:

(4.1) \( \| M_n \|_{L^\infty(0,T;[3])} \leq \| M_0 \|_{[3]} \).  
(4.2) \( \mathbb{E} \| B_n - \pi_n^2 \overrightarrow{M}_n \|_{L^p(0,T;L^2(\mathbb{R}^3))} \leq C \).  
(4.3) \( \mathbb{E} \| E_n \|_{L^p(0,T;L^2(\mathbb{R}^3))} \leq C \).  
(4.4) \( \mathbb{E} \| M_n \|_{L^p(0,T;H^b)} \leq C \).  
(4.5) \( \mathbb{E} \| M_n \times \rho_n \|_{L^p(0,T;H^b)} \leq C \).  
(4.6) \( \mathbb{E} \| B_n \|_{L^p(0,T;L^2(\mathbb{R}^3))} \leq C \).  
(4.7) \( \mathbb{E} \left( \int_0^T \| M_n(t) \times (M_n(t) \times \rho_n(t)) \|_{L_b^2}^2 \, dt \right)^{\frac{1}{2}} \leq C \).  
(4.8) \( \mathbb{E} \| \pi_n [ M_n(t) \times (M_n(t) \times \rho_n(t)) ] \|_{L^2(0,T;\mathcal{X}^{-b})} \leq C \).  
(4.9) \( \mathbb{E} \left\| \frac{dE_n}{dt} \right\|_{L^p(0,T;\mathcal{X}^{-b})} \leq C \).  
(4.10) \( \mathbb{E} \left\| \frac{dB_n}{dt} \right\|_{L^p(0,T;\mathcal{X}^{-b})} \leq C \).

where \( \mathcal{X}^{-b} \) is the dual space of \( \mathcal{X}^b = D(A^b) \).

**Proof of (4.1).** By the Itô formula and straightforward calculus we have

\[
\begin{align*}
\int_0^t \frac{d\| M_n \|_{[3]}^2}{dt} &= \sum_{j=1}^{\infty} 2 \left( \langle M_n, G_{jm}(M_n) \rangle_{[3]} + \left( \sum_{j=1}^{\infty} \| G_{jm}(M_n) \|_{[3]}^2 \right) \right) \frac{dW_j}{dt} + \int_0^t \left( 2 \langle M_n, F_n(M_n) \rangle_{[3]} + \sum_{j=1}^{\infty} \| G_{jm}(M_n) \|_{[3]}^2 \right) \, dt \\
&= 0
\end{align*}
\]

hence

\[ \| M_n(t) \|_{[3]}^2 = \| M_n(0) \|_{[3]}^2 = \| \pi_n M_0 \|_{[3]}^2 \leq \| M_0 \|_{[3]}^2, \quad t \geq 0. \]

\( \square \)
Proof of (4.2), (4.3), (4.4), (4.5). By the Itô formula we get:
\[
\mathcal{E}_n(t) - \mathcal{E}_n(0) = \int_0^t \left\{ -\rho_n(s), F_n(M_n(s)) \right\} \text{d}s + \frac{1}{2} \sum_{j=1}^\infty \left( \rho_{n}''(s) G_{n}(M_n(s)) \right) \text{d}s
\]
\[
+ \frac{1}{2} \sum_{j=1}^\infty \left( \left\{ \nabla G_{n}(M_n(s)) \right\}^2 + \frac{1}{2} \left\| B_{n}(s) - \nabla \nabla M_{n}(s) \right\|_{L^2(R^3)}^2 + \frac{1}{2} \left\| E_{n}(s) \right\|_{L^2(R^3)}^2 \right) \text{d}s + \int_0^t \left( E_n(s), \nabla \nu \right) \text{d}W(s).
\]

Now let's consider each term in the equality (4.11).

For the term on the left hand side of (4.11),
\[
\mathcal{E}_n(t) - \mathcal{E}_n(0) = \int_0^t \left\{ -\rho_n(s), F_n(M_n(s)) \right\} \text{d}s + \frac{1}{2} \sum_{j=1}^\infty \left( \rho_{n}''(s) G_{n}(M_n(s)) \right) \text{d}s
\]
\[
+ \frac{1}{2} \sum_{j=1}^\infty \left( \left\{ \nabla G_{n}(M_n(s)) \right\}^2 + \frac{1}{2} \left\| B_{n}(s) - \nabla \nabla M_{n}(s) \right\|_{L^2(R^3)}^2 + \frac{1}{2} \left\| E_{n}(s) \right\|_{L^2(R^3)}^2 \right) \text{d}s + \int_0^t \left( E_n(s), \nabla \nu \right) \text{d}W(s).
\]

For the 1st term on the right hand side of (4.11), by (3.7),
\[
- \left\{ \rho_n, F_n(M_n) \right\} \text{d}s = -A_1 \left\langle \rho_n, \pi_n[M_n \times \rho_n] \right\rangle \text{d}s + A_2 \left\langle \rho_n, \pi_n[M_n \times (M_n \times \rho_n)] \right\rangle \text{d}s
\]
\[
- \frac{1}{2} \sum_{j=1}^\infty \left\langle \rho_n, G_{n}^{\nu}(M_n) \left( G_{n}^{\nu}(M_n) \right) \right\rangle \text{d}s.
\]

Since
\[
\left\langle \rho_n, \pi_n[M_n \times \rho_n] \right\rangle = \left\langle \rho_n, M_n \times \rho_n \right\rangle = 0,
\]
and
\[
\left\langle \rho_n, \pi_n[M_n \times (M_n \times \rho_n)] \right\rangle = \left\langle \rho_n, M_n \times (M_n \times \rho_n) \right\rangle = -\|M_n \times \rho_n\|_{L^2(R^3)}^2,
\]
we find that
\[
- \left\{ \rho_n, F_n(M_n) \right\} \text{d}s = -A_2 \|M_n \times \rho_n\|_{L^2(R^3)}^2 - \frac{1}{2} \sum_{j=1}^\infty \left\langle \rho_n, G_{n}^{\nu}(M_n) \left( G_{n}^{\nu}(M_n) \right) \right\rangle \text{d}s.
\]

For the 4th and 5th terms on the right hand side of (4.11), we notice that
\[
- \left\{ B_n(s) - \nabla \nabla M_n(s), \nabla \nu \times E_n(s) \right\} \text{d}s + \left\{ E_n(s), \nabla \nu \times (B_n(s) - \nabla \nabla M_n(s)) \right\} \text{d}s
\]
\[
= - \left\{ B_n(s) - \nabla \nabla M_n(s), \nabla \nu \times E_n(s) \right\} \text{d}s + \left\{ E_n(s), \nabla \nu \times (B_n(s) - \nabla \nabla M_n(s)) \right\} \text{d}s = 0.
\]
Therefore,

\[
\left( B_n(s), -\pi_n^\ast \mathbf{M}_n(t), \pi_n^\ast (\nabla \times E_n(s)) \right)_{L^2(\mathbb{R}^3)}^2
\]

\[
+ \left( E_n(s), \pi_n^\ast \left[ \nabla \times (B_n(s) - \pi_n^\ast \mathbf{M}_n(t)) \right] - \pi_n^\ast \left[ 1_{D_2}(E_n(s) + \mathbf{f}(s)) \right] \right)_{L^2(\mathbb{R}^3)}^2
\]

\[
= - \left( E_n(s), 1_{D_2}(E_n(s)) + \mathbf{f}(s) \right)_{L^2(\mathbb{R}^3)}^2 = -\|1_{D_2}E_n\|_{L_\beta}^2 - \langle f, 1_{D_2}E_n \rangle_{L_\beta}.
\]

By (4.12) and (4.13), equality (4.11) takes the form

\[
\int_D \varphi(M_n(t, x)) \, dx + \frac{1}{2} \left\| \nabla M_n(t) \right\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \left\| B_n(t) - \pi_n^\ast \mathbf{M}_n(t) \right\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \left\| E_n(t) \right\|_{L^2(\mathbb{R}^3)}^2
\]

\[
+ \lambda_2 \int_0^t \int [M_n \times \rho_n]^2 \, ds + \frac{1}{2} \sum_{j=1}^n \int_0^t \left\langle \rho_n, G_{jn}^\ast (M_n) \left( G_{jm}^\ast (M_n) \right) \right\rangle \, ds
\]

\[
+ \frac{1}{2} \sum_{j=1}^n \int_0^t \left( \nabla G_{jn}^\ast (M_n) \right)^2 \, ds - \frac{1}{2} \sum_{j=1}^n \int_0^t \left\| \nabla G_{jn}^\ast (M_n) \right\|_{L^2(\mathbb{R}^3)}^2
\]

\[
+ \sum_{j=1}^n \int_0^t \left\langle \rho_n, G_{jm}^\ast (M_n) \right\rangle \, dW_j(s)
\]

\[
= \int_D \varphi(M_n(0, x)) \, dx + \frac{1}{2} \left\| \nabla M_n(0) \right\|_{L^2(\mathbb{R}^3)}^2
\]

\[
+ \frac{1}{2} \left\| B_n(0) - \pi_n^\ast \mathbf{M}_n(0) \right\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \left\| E_n(0) \right\|_{L^2(\mathbb{R}^3)}^2, \quad \forall t \in (0, T).
\]

Now let us consider some terms in the equality (4.14).

By (3.6) we have

\[
\left\langle \rho_n, G_{jm}^\ast (M_n) \right\rangle
\]

\[
= - \left\langle \pi_n \varphi(M_n), G_{jm}^\ast (M_n) \right\rangle + \left\langle \Delta M_n, G_{jm}^\ast (M_n) \right\rangle
\]

\[
+ \left\langle \pi_n[B_n - \pi_n^\ast \mathbf{M}_n], G_{jm}^\ast (M_n) \right\rangle.
\]

We also have

\[
\left\langle \Delta M_n, G_{jm}^\ast (M_n) \right\rangle
\]

\[
= - \left\langle \nabla M_n, \nabla G_{jm}^\ast (M_n) \right\rangle
\]

\[
= - \lambda_1 \left\langle \nabla M_n, \nabla K_{jm} (M_n) \right\rangle
\]

\[
\leq \left\| \nabla M_n \right\|_{L^2(\mathbb{R}^3)}^2 \left\| \mathbf{K}_{jm} \right\|_{L^\infty(\mathbb{R}^3)}^2 + 2 \left\| \nabla M_n \right\|_{L^2(\mathbb{R}^3)} \left\| \mathbf{M}_n \right\|_{L^\infty(\mathbb{R}^3)} \left\| \mathbf{K}_{jm} \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla M_n \right\|_{L^\infty(\mathbb{R}^3)}^2
\]

\[
\leq \left\| \mathbf{M}_n \right\|_{C^0(\mathbb{R}^3)} \left\| \mathbf{K}_{jm} \right\|_{C^0(\mathbb{R}^3)} \left\| \mathbf{M}_n \right\|_{L^\infty(\mathbb{R}^3)} \left\| \mathbf{K}_{jm} \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla M_n \right\|_{L^\infty(\mathbb{R}^3)}^2 + 2 \left\| \mathbf{M}_n \right\|_{C^0(\mathbb{R}^3)} \left\| \mathbf{K}_{jm} \right\|_{C^0(\mathbb{R}^3)} \left\| \mathbf{M}_n \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla M_n \right\|_{C^0(\mathbb{R}^3)}.$

Next we have

\[
\left\| \left( \pi_n[B_n - \pi_n^\ast \mathbf{M}_n], G_{jm}^\ast (M_n) \right) \right\|_{L^2(\mathbb{R}^3)} \leq C \left( \left\| \mathbf{h} \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla h \right\|_{L^\infty(\mathbb{R}^3)} \left\| \mathbf{M}_n \right\|_{L^\infty(\mathbb{R}^3)} \right)^2
\]

Since we assume that \( \varphi \) is bounded we obtain

\[
\left\| \left( \pi_n[\varphi(M_n)], G_{jm}^\ast (M_n) \right) \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| h \right\|_{L^\infty(\mathbb{R}^3)}.
\]

Note that we also have,

\[
\int_0^t \int_D \left( f, E_n \right) \, dx \, ds \leq \frac{1}{2} \int_0^t \int_D \left( \left| f \right|^2 + \left| E_n \right|^2 \right) \, dx \, ds.
\]

Hence by (4.1) and (4.14) we infer that there exists a constant \( C(\alpha, \beta, \mathcal{D}) > 0 \) independent of \( n \) such that
We are going to estimate the stochastic term in the above inequality (4.15). We will show first that the infinite sum of stochastic integrals

(4.16) \[
\sum_{j=1}^{\infty} \int_0^t \left\langle \rho_n, G_{j,n} (M_n) \right\rangle \, dW_j(s)
\]
is well defined. We have

(4.17) \[
\left\| \left\langle \rho_n, G_{j,n} (M_n) \right\rangle \right\| _{\mathcal{H}} \\
\leq \left\| -\varphi (M_n) + \sigma \left( B_n - \pi _n^{\gamma} \overline{M}_n \right), G_{j,n} (M_n) \right\| _{\mathcal{H}} + \left\| \Delta M_n, G_{j,n} (M_n) \right\| _{\mathcal{H}} \\
\leq C \left( \| h_j \| _{L^\infty} + \| 1_D (B_n - \pi _n^{\gamma} \overline{M}_n) \| _{L^2} + \| \nabla h_j \| _{L^2} + \| h_j \| _{L^\infty} \right) \\
\leq C \left( \| \nabla h_j \| _{L^2} + \| h_j \| _{L^\infty} \right) \left( 1 + \| 1_D (B_n - \pi _n^{\gamma} \overline{M}_n) \| _{L^2} + \| \nabla M_n \| _{L^2} \right)
\]
hence

\[
\mathbb{E} \sum_{j=1}^{\infty} \int_0^t \left\langle \rho_n, G_{j,n} (M_n) \right\rangle ^2 \, ds \leq c_0 C \int_0^t \left( 1 + \| 1_D (B_n - \pi _n^{\gamma} \overline{M}_n) \| _{L^2} + \| \nabla M_n \| _{L^2} \right)^2 \, ds
\]
and the Itô integral (4.16) is a well defined square-integrable martingale. Secondly, we do some preparation before using the Burkholder-Davis-Gundy inequality on the stochastic term of (4.15). Taking supremum over \( r \in [0, t] \) on both sides of (4.15) we obtain

\[
\frac{1}{2} \sup_{r \in [0,t]} \left\{ \| B_n (r) - \pi _n^{\gamma} \overline{M}_n (r) \| _{L^2 (\mathbb{R}^3)} ^2 + \| E_n (r) \| _{L^2 (\mathbb{R}^3)} ^2 \right\} \\
+ A_2 \sum_{0}^{\infty} \int_0^t \| M_n (s) \times \rho_n (s) \| _{L^2 (\mathbb{R}^3)} \, ds + \sup_{r \in [0,t]} \left( \int_{\mathcal{D}} \varphi (M_n (r)) \, dx + \frac{1}{2} \| M_n (r) \| _{L^2 (\mathbb{R}^3)} ^2 \right) \\
\leq \frac{1}{2} \| B_n (0) - \pi _n^{\gamma} \overline{M}_n (0) \| _{L^2 (\mathbb{R}^3)} ^2 + \frac{1}{2} \| E_n (0) \| _{L^2 (\mathbb{R}^3)} ^2 + \frac{1}{2} \int_0^t \| f (s) \| _{L^2 (\mathbb{R}^3)} \, ds \\
+ \int_{\mathcal{D}} \varphi (M_n (0, x)) \, dx + \frac{1}{2} \| \nabla M_n (0) \| _{L^2 (\mathbb{R}^3)} ^2 \\
+ C c_0 \int_0^t \left( \| M_n (s) \| _{L^2 (\mathbb{R}^3)} ^2 + \| 1_D (B_n (s) - \pi _n^{\gamma} \overline{M}_n (s)) \| _{L^2 (\mathbb{R}^3)} ^2 \right) \, ds + C c_0 t \\
+ \sup_{r \in [0,t]} \sum_{j=1}^{\infty} \int_0^t \left\langle \rho_n, G_{j,n} (M_n) \right\rangle \, dW_j(s).
\]
Let $p \geq 2$. Then using the Jensen inequality we find that for some constant $C$ which includes the initial data, we have

\begin{equation}
\begin{aligned}
&\mathbb{E} \left( \sup_{r \in [0, l]} \left( \| [B_n - \pi_n^* \mathcal{M}_n](r) \|_{L^2(\Omega)}^2 + \| E_n(r) \|_{L^2(\Omega)}^2 + \| M_n(r) \|_{\mathbb{H}}^2 \right) + 2l \| M_n \times \rho_n \|_{\mathbb{H}}^2 \right)^p \\
&\leq C C_{t}^{p-1} \mathbb{E} \int_0^l \left( \| M_n(s) \|_{\mathbb{H}}^2 + \| [B_n - \pi_n^* \mathcal{M}_n](s) \|_{L^2(\Omega)}^2 \right)^p ds \\
&+ 3^{p-1} \mathbb{E} \left( \sup_{r \in [0, l]} \sum_{j=1}^\infty \left( \| \rho_n \|_{\mathbb{H}}^2 \right) \, dW_j(s) \right)^p + C C_t^p. 
\end{aligned}
\end{equation}

Finally, by the Burkholder-Davis-Gundy inequality, the Jensen’s inequality again and (4.17), there exists a $n$-independent constants $K = K(p) > 0$ and $C > 0$ such that:

\begin{equation}
\begin{aligned}
&\mathbb{E} \left( \sup_{r \in [0, l]} \left[ \rho_n \cdot (M_n) \right]_{\mathbb{H}} \right)^p \\
&\leq K \mathbb{E} \left| \sum_{j=1}^{\infty} \left( \| \rho_n \|_{\mathbb{H}}^2 \right) \, dW_j(s) \right|^p \\
&\leq C t^{-1} \mathbb{E} \int_0^l \sup_{r \in [0, l]} \left( \| M_n(r) \|_{\mathbb{H}}^2 + \| [B_n - \pi_n^* \mathcal{M}_n](r) \|_{\mathbb{H}}^2 \right) ds + Ct^p.
\end{aligned}
\end{equation}

Hence by (4.18) and (4.19) there exists $C > 0$ independent of $n$ such that,

\begin{equation}
\begin{aligned}
&\sup_{r \in [0, l]} \left( \| [B_n - \pi_n^* \mathcal{M}_n](r) \|_{L^2(\Omega)}^2 + \| E_n(r) \|_{L^2(\Omega)}^2 + \| M_n(r) \|_{\mathbb{H}}^2 \right) + \int_0^l \| M_n \times \rho_n \|_{\mathbb{H}}^2 \, dt \right)^p \\
&\leq C (p^{-1} + t^{p-1}) \mathbb{E} \sup_{r \in [0, l]} \left( \| M_n(r) \|_{\mathbb{H}}^2 + \| [B_n - \pi_n^* \mathcal{M}_n](r) \|_{L^2(\Omega)}^2 \right) ds + Ct^p.
\end{aligned}
\end{equation}

Hence by the Gronwall inequality, with $C = CT^p e^{C(T^p + t^2)}$, we get the following four a’priori estimates,

\begin{align*}
\mathbb{E} \| B_n - \pi_n^* \mathcal{M}_n \|^p_{L^0(0, t; L^2(\Omega) = C,} \\
\mathbb{E} \| E_n \|^p_{L^0(0, t; L^2(\Omega) = C,} \\
\mathbb{E} \| M_n \|^p_{L^0(0, t; \mathbb{H} = C,} \\
\mathbb{E} \| M_n \times \rho_n \|^p_{L^0(0, t; \mathbb{H} = C.}
\end{align*}

And since $L^2(\Omega) \hookrightarrow L^q(\Omega)$ continuously for all $q < 2p$, these four inequalities imply the inequalities: (4.2), (4.3), (4.4), (4.5) for all $p > 0$.

We continue with the proof of Proposition 4.1.

**Proof of (4.6).** For fixed $p \geq 1$, we have

\begin{equation}
\begin{aligned}
&\mathbb{E} \| B_n \|^p_{L^0(0, t; \mathbb{H}(\Omega))} \leq 2^p \left( \mathbb{E} \| [B_n - \pi_n^* \mathcal{M}_n] \|^p_{L^0(0, t; \mathbb{H}(\Omega))} + \mathbb{E} \| M_n \|^p_{L^0(0, t; \mathbb{H}(\Omega))} \right).
\end{aligned}
\end{equation}

By the a’priori estimates (4.2) and (4.4), there exists some $C > 0$ independent of $n$ such that

\begin{equation}
\begin{aligned}
\mathbb{E} \| B_n \|^p_{L^0(0, t; \mathbb{H}(\Omega))} \leq C.
\end{aligned}
\end{equation}

Together with the fact $L^2(\Omega) \hookrightarrow L^q(\Omega)$ continuously for all $q < 2p$, we complete the proof of (4.6). \qed
Proof of (4.7). By the Sobolev imbedding theorem, there is a constant $C$ such that
\[
\|M_n\|_{L^2} \leq C\|M_n\|_Y,
\]
therefore by the Hölder inequality, we have
\[
\left\|M_n(t) \times (M_n(t) \times \rho_n(t)) \right\|_{L^2} \leq \|M_n(t)\|_{L^2} \|M_n(t) \times \rho_n(t)\|_{L^2} \leq C\|M_n(t)\|_Y\|M_n(t) \times \rho_n(t)\|_{L^2}.
\]
Hence, by the Cauchy-Schwarz inequality,
\[
\mathbb{E}\left[\left(\int_0^T \|M_n(t) \times (M_n(t) \times \rho_n(t))\|_{L^2}^2 \, dt\right)^{\frac{1}{2}}\right] \leq C\mathbb{E}\left[\sup_{t \in [0,T]} \|M_n(t)\|_{L^2}^p\left(\int_0^T \|M_n(t) \times \rho_n(t)\|_{L^2}^2 \, dt\right)^{\frac{1}{2}}\right] \leq C\mathbb{E}\left[\sup_{t \in [0,T]} \|M_n(t)\|_{L^2}^p\right]^{\frac{1}{2}} \left[\left(\int_0^T \|M_n(t) \times \rho_n(t)\|_{L^2}^2 \, dt\right)^{\frac{1}{2}}\right].
\]
Then by (4.4) and (4.5), we get (4.7). \qed

Proof of (4.8). Since $\|\cdot\|_{L^p} = \|A_{1,1}^{\alpha} \cdot \|^p_{L_h}$, $X^b \leftrightarrow L^1$ compactly for $b > \frac{1}{2}$. Hence $L^2$ is compactly embedded in $X^{b,\alpha}$. Thus there is a constant $C$ independent of $n$ such that
\[
\mathbb{E}\int_0^T \|\pi_n [M_n(t) \times (M_n(t) \times \rho_n(t))]\|_{L^2}^p \, dt \leq \mathbb{E}\int_0^T \|M_n(t) \times (M_n(t) \times \rho_n(t))\|_{L^2}^p \, dt \leq C\mathbb{E}\int_0^T \|M_n(t) \times (M_n(t) \times \rho_n(t))\|_{L^2}^p \, dt.
\]
Then by (4.7), we get (4.8). \qed

Remark 4.2. From now on we will keep assuming that $b > \frac{1}{2}$.

Proof of (4.9) and (4.10). By the second equation in (3.12), we have
\[
\mathbb{E}\left\|\frac{dE_n}{dt}\right\|_{L^p(0,T;H)}^p = \mathbb{E}\|\pi_n^y (\nabla \times [B_n - \pi_n^y M_n]) - \pi_n^y [1\rho(E_n + \tilde{T})]\|_{L^p(0,T;H)}^p \\
\leq C_p\mathbb{E}\sup_{t \in [0,T]} \|\nabla \times [B_n - \pi_n^y M_n(t)]\|_{L^p(y)}^p + C_{\rho}\mathbb{E}\sup_{t \in [0,T]} \|1\rho(E_n(t) + \tilde{T}(t))\|_{L^p(y)}^p \\
\leq C_p\mathbb{E}\sup_{t \in [0,T]} \left\|\left\{B_n(t) - \pi_n^y M_n(t), \nabla \times \chi \right\}_{L^p(\mathbb{R}^3)} \right\|_{L^p(y)}^p + C_{\rho}\mathbb{E}\|E_n\|_{L^p(0,T;L^p(\mathbb{R}^3))}^p + C_{\rho}\|f\|_{L^p(0,T;L^p(\mathbb{R}^3))}^p.
\]
Hence, since $f \in L^1(0,T;H)$, by (4.2) and (4.3), we get (4.9) and similarly (4.10). \qed

After so many pages of long calculation, the proof of Proposition 4.1 has been finished. Next let us consider the estimate of the stochastic term in the finite dimensional system (3.12).

Lemma 4.3. For $a \in [0,\frac{1}{2})$ and $p \geq 2$, there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$,
\[
\mathbb{E}\left\|\sum_{j=1}^n \int_0^T G_{n,j}(M_\omega(s)) \, dW_j(s)\right\|_{L^p(0,T;H)}^p \leq C.
\]
To prove Lemma 4.3, we will use the Lemma 2.1 from Flandoli and Gatarek’s paper [19].
Proof of Lemma 4.3. By Lemma 9.2, there exists constant $C_1 > 0$, such that
\[
\mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^T G_{j\mu} (M_n) \, dW_j(s) \right\|_{W^{p,q}(0,T;\mathbb{R}^d)}^p \leq C_1 \mathbb{E} \left( \int_0^T \left\| G_{j\mu} (M_n) \right\|_{L^2} \, dt \right)^{\frac{p}{2}} \leq 2^{p-1} C_1 \left( \sum_{j=1}^{\infty} \| h_j \|_{L^2}^2 \right)^{\frac{p}{2}} \mathbb{E} \int_0^T \left( 1 + \| M_n \|_{L^2}^p \right) \, dt \leq C \leq C,
\]
where the last inequality followed by (4.4). This completes the proof of the estimate (4.20). □

Remark 4.4. From now on we will always assume $a \in [0, \frac{1}{2})$, $b > \frac{1}{4}$ and $p \geq 2$.

Lemma 4.5. For $a \in [0, \frac{1}{2})$, $b > \frac{1}{4}$, $p \geq 2$, there exists $C > 0$ such that for all $n \in \mathbb{N}$,
\[
\mathbb{E} \left\| M_n \right\|_{W^{p,q}(0,T;\mathbb{R}^d)}^2 \leq C.
\]

Proof. By (3.12),
\[
\mathbb{E} \left\| M_n \right\|_{W^{p,q}(0,T;\mathbb{R}^d)}^2 \leq \mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^T G_{j\mu} (M_n) \, dW_j \right\|_{W^{p,q}(0,T;\mathbb{R}^d)}^2 + \sum_{j=1}^{\infty} \int_0^T G_{j\mu} (M_n) \, dW_j \right\|_{W^{p,q}(0,T;\mathbb{R}^d)}^2.
\]

By our assumption, $a \in [0, \frac{1}{2})$, so $L^1(0,T;X^\#) \hookrightarrow W^{p,q}(0,T;\mathbb{R}^d)$ compactly for all $p > 0$. And since $\mathbb{H} \hookrightarrow X^\#$ continuously, there is a constant $C$ independent of $n$ such that
\[
\mathbb{E} \left\| M_n \right\|_{W^{p,q}(0,T;\mathbb{R}^d)}^2 \leq C \left\| \sum_{j=1}^{\infty} \int_0^T \pi_n \left\{ L_1 M_n \times \rho_n - L_2 M_n \times (M_n \times \rho_n) + \frac{1}{2} \sum_{j=1}^{\infty} G_{j\mu} (M_n) \left[ G_{j\mu} (M_n) \right] \right\} \, ds \right\|_{L^1(0,T;\mathbb{H})}^2 + C \left\| \sum_{j=1}^{\infty} \int_0^T \pi_n \left\{ L_2 \pi_n \left[ M_n \times (M_n \times \rho_n) \right] \right\} \, ds \right\|_{L^1(0,T;\mathbb{H})}^2 + C \left\| \sum_{j=1}^{\infty} \int_0^T G_{j\mu} (M_n) \, dW_j \right\|_{W^{p,q}(0,T;\mathbb{R}^d)}^2.
\]

To prove (4.21), it is enough to consider each term on the right hand side of the above inequality. By (4.5), (4.8) and (4.20), we can conclude (4.21). □

5. Tightness results

In this subsection we will use the a’ priori estimates (4.1)-(4.10) to show that the laws $\{\mathcal{L}(M_n, B_n, E_n) : n \in \mathbb{N}\}$ are tight on a suitable path space. Then we will use Skorohod’s theorem to obtain another probability space and an almost surely convergent sequence defined on this space whose limit is a weak martingale solution of the Problem 2.6.

To do so, we will use Lemma 9.1-Lemma 9.3 about compact embedding from Flandoli and Gatarek’s paper [19]. We will also need the following Lemma about tightness.

Lemma 5.1. Let $X, Y$ be separable Banach spaces and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, we assume that $i : X \hookrightarrow Y$ compact and the random variables $u_n : \Omega \rightarrow X, n \in \mathbb{N}$, satisfy the following condition: there is a constant $C > 0$, such that
\[
\mathbb{E} \left( \| u_n \|_X \right) \leq C, \quad n \in \mathbb{N}.
\]

Then the family of laws $\{\mathcal{L}(i \circ u_n)\}_{n \in \mathbb{N}}$ is tight on $Y$.

Now let’s state and prove our tightness results.
Lemma 5.2. For any $p \geq 2$, $q \in [2, 6)$ and $b > \frac{1}{q}$ the set of laws $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ on the Banach space $L^p(0,T;\mathbb{L}^q) \cap C([0,T];X^b)$ is tight.

Proof. Firstly, let us prove $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ is tight on $L^p(0,T;\mathbb{L}^q)$ for all $p \geq 2$ and $q \in [2, 6)$. To this end fix $p \geq 2$, $a \in (0,\frac{1}{q})$, $b > \frac{1}{q}$ and $q \in [2,6)$. Since $q < 6$ and the embedding $\mathbb{V} = D(A^q) \hookrightarrow X^r = D(A^r)$ is compact for $r < \frac{1}{q}$, we can choose $y \in (\frac{q}{a} - \frac{q}{2r},\frac{1}{r})$, such that, Lemma 9.1 yields a compact embedding $L^p(0,T;\mathbb{V}) \cap W^{a,p}(0,T;\mathbb{X}^b) \hookrightarrow L^p(0,T;X^r)$.

Therefore

$$\mathbb{P}(\|M_n\|_{L^p(0,T;\mathbb{V})} > r) = \mathbb{P}(\|M_n\|_{L^p(0,T;\mathbb{V})} > r) \leq \mathbb{P}(\|M_n\|_{L^p(0,T;X^r)} > \frac{r}{\gamma}) + \mathbb{P}(\|M_n\|_{L^p(0,T;X^r)} > \frac{r}{\gamma}) \leq \frac{C}{r^2} \mathbb{E}(\|M_n\|_{L^p(0,T;X^r)}^2) \mathbb{E}(\|M_n\|_{L^p(0,T;X^r)}^2) \mathbb{E}(\|M_n\|_{L^p(0,T;X^r)}^2).$$

Let $X_r := L^p(0,T;\mathbb{V}) \cap W^{a,p}(0,T;\mathbb{X}^b)$. By estimates (4.21) and (4.4), there exists a constant $C$, such that

$$\mathbb{P}(\|M_n\|_{X_r} > r) \leq C \mathbb{E}, \quad \forall r, n.$$

By Lemma 5.1, the family of laws $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ is tight on $L^p(0,T;X^r)$. For $r > \frac{1}{q} - \frac{1}{2r}$, we have $X^r = \mathbb{V}^{\mathbb{V}^q(D)} \hookrightarrow \mathbb{V}^q$ continuously. Hence $L^p(0,T;X^r) \hookrightarrow L^p(0,T;\mathbb{L}^q)$ continuously and $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ is also tight on $L^p(0,T;\mathbb{L}^q)$.

Secondly, we prove the laws $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ are tight on $C([0,T];X^b)$ for all $b > \frac{1}{q}$. To do this, we fix some $b > \frac{1}{q}$ and choose $b' \in (\frac{1}{q},b)$. Since $b' < b$, by Lemma 9.3 we have $W^{a,p}(0,T;X^{b'}) \hookrightarrow C([0,T];\mathbb{X}^{b'})$ compactly for $a \in (0,\frac{1}{q})$ and $p > 2$ satisfying $a > \frac{1}{p}$. Therefore by estimate (4.21) and Lemma 5.1 again, we conclude that $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ is tight on $C([0,T];\mathbb{X}^{b'})$.

Therefore $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ is tight on $L^p(0,T;\mathbb{L}^q) \cap C([0,T];\mathbb{X}^{b'})$ and the proof is complete.

To prove the tightness results about $\{\mathcal{L}(E_n)\}$ and $\{\mathcal{L}(B_n)\}$, we need the version ([12], Def. 3.7) of Aldous Condition ([2]), i.e. Definition 9.4 and the tightness criterion Lemma 9.5.

Lemma 5.3. The sets of laws $\{\mathcal{L}(E_n)\}$ and $\{\mathcal{L}(B_n)\}$ are tight on the space $L^p(0,T;\mathbb{L}^{2}(\mathbb{R}^d))$.

Proof. Here we will only prove the result about $\{\mathcal{L}(E_n)\}$, the proof about $\{\mathcal{L}(B_n)\}$ is exactly the same. In order to use Lemma 9.5, let us set $H = \mathbb{L}^{2}(\mathbb{R}^d)$ and choose an auxiliary Hilbert space $U$ such that the embedding $U \hookrightarrow \mathbb{V}$ is compact.

Since the embedding $\mathbb{V} \hookrightarrow \mathbb{L}^{2}(\mathbb{R}^d)$ is bounded, the embedding $U \hookrightarrow \mathbb{L}^{2}(\mathbb{R}^d)$ is also compact. Next we will check the condition (a) and (b) in Lemma 9.5.

Firstly, let us observe that by estimate (4.3), condition (a) of the Lemma 9.5 is satisfied. Secondly, we will check the Aldous condition (Definition 9.4) in the space $U^*$. To this end, fix $\varepsilon > 0$, $\eta > 0$ and a sequence of $\mathbb{P}$-stopping times $\tau_n$. The embedding $\mathbb{V} \hookrightarrow U^*$ is compact so it is bounded and thus there exists a constant $C_1 > 0$ such that $\|\cdot\|_{U^*} \geq C_1\|\cdot\|_{U^*}$. Hence together with the Chebyshev inequality and estimate (4.9), we have

$$\mathbb{P}(\|E_n(\tau_n + \theta) - E_n(\tau_n)\|_{U^*} \geq \eta) \leq \mathbb{P}(\|E_n(\tau_n + \theta) - E_n(\tau_n)\|_{U^*} \geq \eta) \leq \frac{1}{C_1 \eta} \mathbb{E} \left(\int_{\tau_n}^{\tau_n + \theta} \|E_n(s)\|_{U^*} \right) d\varepsilon \leq \frac{C_1 \eta}{C_1 \eta} \theta > 0.$$

Hence for $\delta \leq \frac{\varepsilon}{C_1 \eta}$, we have

$$\sup_{n \in \mathbb{N}} \mathbb{P}(\|E_n(\tau_n + \theta) - E_n(\tau_n)\|_{U^*} \geq \eta) \leq \varepsilon.$$
On the product space $\mu$ follows.

Proposition 5.4. There exists a subsequence $((M_{i_n}, B_{i_n}, E_{i_n}))$ of $((M_i, B_i, E_i))$, such that the laws $L(M_{i_n}, B_{i_n}, E_{i_n}, W_{i_n})$ converge weakly to a probability measure $\mu$ on $[L^p(0, T; L^q) \cap C([0, T]; X^{-k})] \times L^2_c(0, T; L^2(\mathbb{R}^3)) \times L^2_c(0, T; L^2(\mathbb{R}^3)) \times C([0, T]; \mathbb{H})$, where $p \in [2, \infty)$, $q \in [2, 6)$ and $b > \frac{3}{4}$.

6. Construction of new probability space and processes

Now we are going to use the Skorokhod Theorem to construct our new probability space and processes as the weak solution of Problem 2.6.

Theorem 6.1. For $p \in [2, \infty)$, $q \in [2, 6)$, $b > \frac{3}{4}$, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence $((\tilde{M}_n, \tilde{E}_n, \tilde{B}_n, \tilde{W}_{nh}))$ of $L^p(0, T; L^q(D)) \cap C([0, T]; X^{-k}) \times L^2_c(0, T; L^2(\mathbb{R}^3)) \times L^2_c(0, T; L^2(\mathbb{R}^3)) \times C([0, T]; \mathbb{H})$-valued random variables defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

(a) On the product space

$$L(M_n, E_n, B_n, W_n) = L(\tilde{M}_n, \tilde{E}_n, \tilde{B}_n, \tilde{W}_{nh}), \quad \forall n \in \mathbb{N}$$

(b) There exists a random variable $(\tilde{M}, \tilde{E}, \tilde{B}, \tilde{W}) : (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \rightarrow [L^p(0, T; L^q(D)) \cap C([0, T]; X^{-k})] \times L^2_c(0, T; L^2(\mathbb{R}^3)) \times L^2_c(0, T; L^2(\mathbb{R}^3)) \times C([0, T]; \mathbb{H})$

such that

(i) On the product space

$$[L^p(0, T; L^q(D)) \cap C([0, T]; X^{-k})] \times L^2_c(0, T; L^2(\mathbb{R}^3)) \times L^2_c(0, T; L^2(\mathbb{R}^3)) \times C([0, T]; \mathbb{H})$$

$$L(\tilde{M}, \tilde{E}, \tilde{B}, \tilde{W}_n) = \mu,$$

where $\mu$ is same as in Proposition 5.4. Moreover, the following results hold $\tilde{\mathbb{P}}$-a.s.,

(ii) $M_n \rightarrow M$ in $L^p(0, T; L^q(D)) \cap C([0, T]; X^{-k})$.

(iii) $E_n \rightarrow \tilde{E}$ in $L^2_c(0, T; L^2(\mathbb{R}^3))$.

(iv) $B_n \rightarrow \tilde{B}$ in $L^2_c(0, T; L^2(\mathbb{R}^3))$.

(v) $W_{nh} \rightarrow \tilde{W}_n$ in $C([0, T]; \mathbb{H})$.

To prove Theorem 6.1, we need the standard Skorohod theorem [16, Thm 11.7.2] for separable metric spaces as well as the following Jakubowski’s version of Skorohod theorem:

Lemma 6.2 (124, 13, Thm A.1). Let $X$ be a topological space such that there exists a sequence of continuous functions $f_m : X \rightarrow \mathbb{R}, m = 1, 2, \ldots$ which separates points of $X$. Let us denote by $\mathcal{F}$ the $\sigma$-algebra generated by the maps $\{f_m\}$. Then

(i) every compact subset of $X$ is metrizable,

(ii) if $\{\mu_m\}$ is a tight sequence of probability measures on $(X, \mathcal{F})$, then there exists a subsequence $(m_k)$, a probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], 2\mathbb{B}(0, 1), \text{Leb})$ with $X$-valued random variables $\xi_k, \xi$ such that $\mu_m$ is the law of $\xi_k$ and $\xi_k$ converges to $\xi$ almost surely. Moreover, the law of $\xi$ is a Radon measure.
Proof of Theorem 6.1. Let \( L^p(0, T; \mathbb{L}^q(D)) \cap C([0, T]; \mathbb{X}^-) \) and \( C([0, T]; \mathbb{H}) \) be separable metric spaces, so by the Skorohod Theorem for the separable metric spaces \([16, \text{Thm 11.7.2}]\), there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and corresponding random variables take values in

\[
\left[ L^p(0, T; \mathbb{L}^q(D)) \cap C([0, T]; \mathbb{X}^-) \right] \times C([0, T]; \mathbb{H})
\]

such that the related results in Theorem 6.1 hold.

To prove the results relate to the space \( L^p(0, T; \mathbb{L}^2) \) in Theorem 6.1, we will use the Proposition 5.4 and Lemma 6.2. Let us recall, that for any separable Hilbert space \( H \), the elements of \( H^\prime \) separate points in \( H \), so the countable dense subset of \( H^\prime \) also separate points in \( H \). We also have that the Borel \( \sigma \)-algebras generated from strong and weak topologies are coincide, so \([\mu_m]\) is tight on \((H, \mathscr{B}(H))\).

Then the product probability space and the corresponding random variables of above two related results are the aims we are looking for and this completes the proof of Theorem 6.1. \( \square \)

Remark 6.3. We set \( \mathbb{F} \) to be the filtration generated from \( \tilde{W}_t \) and \( \tilde{M} \). So now we have a filtered new probability space \((\Omega, \mathbb{F}, \mathbb{P}, \mathbb{F})\).

Remark 6.4. As stated in Theorem 6.1, \( \tilde{W}_t \) has same distribution on \( C([0, T]; \mathbb{H}) \) as \( W_t \). Hence it can be proved that \( \{t \circ \tilde{W}_t\}_{t \geq 0} \) is also a \( \mathbb{F} \)-Wiener process on \( \mathbb{H} \), where

\[ i_t : C([0, T]; \mathbb{H}) \ni f \mapsto f(t) \in \mathbb{H}. \]

And for convenience, we will use \( \tilde{W}_t(t) \) to denote \( i_t \circ \tilde{W}_t \).

Since we assumed that \([h_j]\) is an ONB of \( \mathbb{H} \), as in Remark 2.7, \( \tilde{W}_t \) has the following representation:

\[
\tilde{W}_t(t) = \sum_{j=1}^{\infty} \tilde{W}_j(t) h_j, \quad t \in [0, T],
\]

where

\[
\tilde{W}_j(t) := \frac{\langle \tilde{W}_j(t), h_j \rangle_{\mathbb{H}}}{\|h_j\|_{\mathbb{H}}}.
\]

It can be shown that \( \tilde{W}_j(t) \) is \( N(0, t) \) distributed for each \( j \) and form a Gaussian family and so are independent for all \( j = 1, \ldots, \infty \).

The map:

\[
t \mapsto \frac{\langle \tilde{W}_j(t), h_j \rangle_{\mathbb{H}}}{\|h_j\|_{\mathbb{H}}} = \tilde{W}_j(t)
\]

is continuous almost surely. So \( \tilde{W}_j(t) \) has continuous trajectory almost surely for every \( j \).

The independence of increments of \( \tilde{W}_j \) for each \( j \) follows from the independence of increments of \( \tilde{W}_t \). Therefore \( \tilde{W}_j, j = 1, 2, \ldots \) are independent 1-dimensional \( \mathbb{F} \)-Brownian motions.

Similarly, we also have

\[
\tilde{W}_m(t) = \sum_{j=1}^{\infty} \tilde{W}_j(t) h_j, \quad t \in [0, T],
\]

for some independent 1-dimensional \( \mathbb{F} \)-Brownian motions \( \tilde{W}_j, j = 1, 2, \ldots \).

Let \( \tilde{M}_a, \tilde{B}_a, \tilde{E}_a \) be as in Theorem 6.1, we have the following result:

**Proposition 6.5.** The processes \( \tilde{M}_a, \tilde{B}_a, \tilde{E}_a \) have the following properties:

(i) \( \tilde{M}_a \in C^4([0, T]; \mathbb{H}_a) \) almost surely and \( \mathcal{L}(\tilde{M}_a) = \mathcal{L}(M_a) \) on \( C^4([0, T]; \mathbb{H}_a) \);

(ii) \( \tilde{E}_a \in C^4([0, T]; \mathbb{Y}_a) \) almost surely and \( \mathcal{L}(\tilde{E}_a) = \mathcal{L}(E_a) \) on \( C^4([0, T]; \mathbb{Y}_a) \);

(iii) \( \tilde{B}_a \in C^4([0, T]; \mathbb{Y}_a) \) almost surely and \( \mathcal{L}(\tilde{B}_a) = \mathcal{L}(B_a) \) on \( C^4([0, T]; \mathbb{Y}_a) \).

**Proof of Proposition 6.5.** (i) Since \( C^4([0, T]; \mathbb{H}_a) \subset L^p(0, T; \mathbb{L}^4(D)) \cap C([0, T]; \mathbb{X}^-) \), if we take \( \varphi \) to be the embedding map, then by the Kuratowski Theorem 9.6, the Borel sets in \( C^4([0, T]; \mathbb{H}_a) \) are the Borel sets in \( L^p(0, T; \mathbb{L}^4(D)) \cap C([0, T]; \mathbb{X}^-) \). On the other hand, by Theorem 6.1, \( \mathcal{L}(\tilde{M}_a) = \mathcal{L}(M_a) \) on \( L^p(0, T; \mathbb{L}^4(D)) \cap C([0, T]; \mathbb{X}^-) \), so \( \mathcal{L}(\tilde{M}_a) = \mathcal{L}(M_a) \) on \( C^4([0, T]; \mathbb{H}_a) \). By Lemma 3.6, \( \mathbb{P}(M_a \in C^4([0, T]; \mathbb{H}_a)) = 1 \). Hence \( \mathbb{P}(\tilde{M}_a \in C^4([0, T]; \mathbb{H}_a)) = 1 \).
By the Kuratowski Theorem 9.6, the Borel sets in $C^1([0,T];Y_n)$ are Borel sets in $L^2(0,T;Y_n)$. And since $L^2(0,T;Y_n)$ is closed in $L^2(0,T;L^2(\mathbb{R}^3))$, by the Lemma 9.7, $L^2(0,T;Y_n)$ is also closed in the space $L^2_s(0,T;L^2(\mathbb{R}^3))$. Hence the Borel sets in $L^2(0,T;Y_n)$ are also Borel sets in $L^2_s(0,T;L^2(\mathbb{R}^3))$. Therefore the Borel sets in $C^1([0,T];Y_n)$ are the Borel sets in $L^2_s(0,T;L^2(\mathbb{R}^3))$. By Theorem 6.1, $L(E_n) = L(E_n)$ on $L^2_s(0,T;L^2(\mathbb{R}^3))$, so $L(E_n) = L(E_n)$ on $C^1([0,T];Y_n)$. By Lemma 3.6, $P(E_n \in C^1([0,T];Y_n)) = 1$. Hence $\overline{P}(E_n \in C^1([0,T];Y_n)) = 1$.

(iii) Exactly the same as the proof of (ii).

This complete the proof of Proposition 6.5.

The next result shows that the sequence $(\tilde{M}_n, \tilde{B}_n, \tilde{E}_n)$ satisfies the similar a priori estimates as $(M_n, B_n, E_n)$ in Proposition 4.1.

**Proposition 6.6.** Let us define

$$\tilde{\rho}_n := \pi_n[ - \varphi'(\tilde{M}_n) + 1_B(\tilde{B}_n - \pi_n\tilde{M}_n)] + \Delta \tilde{M}_n,$$

Then for all $p \geq 0$, $b > \frac{1}{2}$, there exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$\|\tilde{M}_n\|_{L^p(0,T;\mathbb{R}^3)} \leq \|M_n\|_{p}, \quad \overline{P} - a.s.,$$

$$\mathbb{E}\|\tilde{B}_n - \pi_n\tilde{M}_n\|_{L^p(0,T;L^2(\mathbb{R}^3))} \leq C,$$

$$\mathbb{E}\|\tilde{E}_n\|_{L^p(0,T;L^2(\mathbb{R}^3))} \leq C,$$

$$\mathbb{E}\|\tilde{M}_n\|_{L^p(0,T;\mathbb{R}^3)} \leq C,$$

$$\mathbb{E}\|\tilde{M}_n \times \tilde{\rho}_n\|_{L^p(0,T;\mathbb{R}^3)} \leq C,$$

$$\mathbb{E}\|\tilde{B}_n\|_{L^p(0,T;L^2(\mathbb{R}^3))} \leq C,$$

$$\mathbb{E}\left(\int_0^T \left\|\tilde{M}_n(t) \times (\tilde{M}_n(t) \times \tilde{\rho}_n(t))\right\|^2_{L^2(\mathbb{R}^3)} \, dt\right)^{\frac{1}{2}} \leq C,$$

$$\mathbb{E}\left(\int_0^T \left\|\pi_n(\tilde{M}_n(t) \times (\tilde{M}_n(t) \times \tilde{\rho}_n(t)))\right\|^2_{L^2(\mathbb{R}^3)} \, dt\right) \leq C,$$

$$\mathbb{E}\left\|\frac{d\tilde{E}_n}{dt}\right\|_{L^p(0,T;\mathbb{R}^3)} \leq C,$$

$$\mathbb{E}\left\|\frac{d\tilde{B}_n}{dt}\right\|_{L^p(0,T;\mathbb{R}^3)} \leq C.$$

**Proof.** Note that all the maps, $\pi_n \circ \varphi$, $\pi_n \circ 1_B \circ \pi_n^\Delta$, $\Delta$ are measurable maps on the corresponding spaces. Therefore by the Proposition 6.5 and Proposition 4.1, we get the estimates (6.1)-(6.10).

**Remark 6.7.** We will use $\overline{P}$ to denote the filtration generated by $\tilde{M}$ and $\tilde{W}_h$ in the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \overline{P})$. Since $(M_n, W_n)$ and $(\tilde{M}_n, \tilde{W}_n)$ have same distribution, and the increment $W_h(t) - W_h(s)$ is independent of $\sigma(M_n(r) : r \leq s)$. It is straight forward to see that $\tilde{W}_h(t) - \tilde{W}_h(s)$ is independent of $\overline{P}$, for all $t > s$. 

\[\square\]
Remark 6.8. From now on we will set $p = q = 4$ and $b = \frac{1}{2}$ in Theorem 6.1. That will be enough to show the existence of the solution of the Problem 2.6.

Proposition 6.9. As defined in Theorem 6.1, the $\tilde{M}$ satisfies the following estimates:

\[(6.11) \quad \text{ess sup}_{t \in [0,T]} \|\tilde{M}(t)\|_4 \leq \|M_0\|_4, \quad \tilde{P} - \text{a.s.,}\]

And for some constant $C > 0$,

\[(6.12) \quad \text{ess sup}_{t \in [0,T]} \|\tilde{M}(t)\|_{L^\infty} \leq C \|M_0\|_4, \quad \tilde{P} - \text{a.s.}\]

Proof. The results follows from Theorem 6.1 (b) (ii), $L^4 \hookrightarrow L^\infty$ continuously and the estimate (6.1).\[
\]

We continue to investigate properties of the process $\tilde{M}$, the next result and it’s proof are related to the estimate (6.4).

Theorem 6.10. The process $\tilde{M}$ defined in Theorem 6.1 satisfies the following estimate:

\[(6.13) \quad \mathbb{E} \left[ \text{ess sup}_{t \in [0,T]} \|\tilde{M}(t)\|_r^r \right] < \infty \quad r \geq 0.\]

Proof. Since $L^2(\tilde{\Omega}; L^\infty(0, T; \mathbb{V}))$ is isomorphic to $[L^2(\tilde{\Omega}; L^1(0, T; X^{-\frac{1}{2}}))]^*$, by the estimate (6.4) and the Banach-Alaoglu Theorem we infer that the sequence $\{\tilde{M}_n\}$ contains a subsequence (which denoted in the same way as the full sequence) and there exists an element $v \in L^2(\tilde{\Omega}; L^\infty(0, T; \mathbb{V}))$ such that $\tilde{M}_n \to v$ weakly* in $L^2(\tilde{\Omega}; L^\infty(0, T; \mathbb{V}))$. So it remains to show that $\tilde{M} = v$.

We have

\[
\langle \tilde{M}_n, \varphi \rangle \to \langle v, \varphi \rangle, \quad \varphi \in \mathcal{D}(\tilde{\Omega}; (L^1(0, T; X^{-\frac{1}{2}}))),
\]

which means that

\[
\int_{\tilde{\Omega}} \int_0^T \langle \tilde{M}_n(t, \omega), \varphi(t, \omega) \rangle \, dt \, d\tilde{P}(\omega) \to \int_{\tilde{\Omega}} \int_0^T \langle v(t, \omega), \varphi(t, \omega) \rangle \, dt \, d\tilde{P}(\omega).
\]

On the other hand, if we fix $\varphi \in L^4(\tilde{\Omega}; L^2(0, T; \mathbb{L}^\infty))$, we have

\[
\sup_n \int_{\tilde{\Omega}} \int_0^T \left\| \tilde{M}_n(t, \varphi(t)) \right\|_{L^2}^4 \, d\tilde{P}(\omega) \leq \sup_n \int_{\tilde{\Omega}} \int_0^T \left\| \widetilde{M}_n \right\|_{L^2} \left\| \varphi \right\|_{L^4}^4 \, d\tilde{P}(\omega) \leq \sup_n \left\| \widetilde{M}_n \right\|^2_{L^2(\tilde{\Omega}; L^2(0, T; \mathbb{L}^\infty))} \left\| \varphi \right\|^2_{L^4(\tilde{\Omega}; L^2(0, T; \mathbb{L}^\infty))} < \infty.
\]

So the sequence $\int_{\tilde{\Omega}} \int_0^T \langle \tilde{M}_n(t, \varphi(t)) \rangle_{L^2}^4 \, d\tilde{P}(\omega)$ is uniformly integrable on $\tilde{\Omega}$. Moreover, by the $\tilde{P}$ almost surely convergence of $\tilde{M}_n$ to $\tilde{M}$ in $L^4(0, T; \mathbb{L}^\infty)$, we infer that $\int_{\tilde{\Omega}} \int_0^T \langle \tilde{M}(t, \varphi(t)) \rangle_{L^2}^4 \, d\tilde{P}(\omega) \to \int_{\tilde{\Omega}} \int_0^T \langle \tilde{M}(t, \varphi(t)) \rangle_{L^2}^4 \, d\tilde{P}(\omega)$ almost surely. Thus for $n \to \infty$, we have

\[
\int_{\tilde{\Omega}} \int_0^T \langle \tilde{M}_n(t, \omega), \varphi(t, \omega) \rangle \, dt \, d\tilde{P}(\omega) \to \int_{\tilde{\Omega}} \int_0^T \langle \tilde{M}(t, \omega), \varphi(t, \omega) \rangle \, dt \, d\tilde{P}(\omega).
\]

Hence we deduce that

\[
\int_{\tilde{\Omega}} \int_0^T \langle \varphi(t, \omega), \varphi(t, \omega) \rangle_{L^2}^4 \, d\tilde{P}(\omega) = \int_{\tilde{\Omega}} \int_0^T \langle \tilde{M}(t, \omega), \varphi(t, \omega) \rangle_{L^2}^4 \, d\tilde{P}(\omega)
\]

By the arbitrariness of $\varphi$ and denseness of $L^4(\tilde{\Omega}; L^2(0, T; \mathbb{L}^\infty))$ in $L^\infty(\tilde{\Omega}; L^1(0, T; X^{-\frac{1}{2}}))$, we infer that $\tilde{M} = v$ and since $v$ satisfies (6.13) we infer that $\tilde{M}$ also satisfies (6.13). In this way the proof of (6.13) is complete.\[
\]

We also investigate the following property of $\tilde{B}$ and $\tilde{E}$. 

Proposition 6.11.

\[ \mathbb{E} \int_0^T \| \tilde{B}(t) \|_{L^2([0,1])}^2 \, dt < \infty. \]

\[ \mathbb{E} \int_0^T \| \tilde{E}(t) \|_{L^2([0,1])}^2 \, dt < \infty. \]

**Proof.** The proof of (6.14) and (6.15) is similar as the proof of (6.13). \( \square \)

Next we will strengthen part (ii) and (iv) of Theorem 6.1 (b) about the convergence.

Proposition 6.12.

\[ \lim_{n \to \infty} \mathbb{E} \int_0^T \| \tilde{M}_n(t) - \tilde{M}(t) \|^4_{L^2} \, dt = 0. \]

**Proof of (6.16).** By the Theorem 6.1, \( \tilde{M}_n(t) \to \tilde{M}(t) \) in \( L^4(0, T; \mathbb{L}^4) \cap C([0, T]; \mathbb{X}^{-b}) \) \( \mathbb{P} \)-almost surely, \( \tilde{M}_n(t) \to \tilde{M}(t) \) in \( L^3(0, T; L^3) \) \( \mathbb{P} \)-almost surely, that is

\[ \lim_{n \to \infty} \int_0^T \| \tilde{M}_n(t) - \tilde{M}(t) \|^4_{L^2} \, dt = 0, \quad \mathbb{P} \text{-a.s.}, \]

and by (6.4) and (6.13),

\[ \sup_n \mathbb{E} \left( \int_0^T \| \tilde{M}_n(t) - \tilde{M}(t) \|^4_{L^2} \, dt \right)^{\frac{1}{2}} \leq 2^2 \sup_n \left( \| \tilde{M}_n \|_{L^4(0, T; L^4)}^8 + \| \tilde{M} \|_{L^4(0, T; L^4)}^8 \right) < \infty, \]

hence,

\[ \lim_{n \to \infty} \mathbb{E} \int_0^T \| \tilde{M}_n(t) - \tilde{M}(t) \|^4_{L^2} \, dt = \mathbb{E} \left( \lim_{n \to \infty} \int_0^T \| \tilde{M}_n(t) - \tilde{M}(t) \|^4_{L^2} \, dt \right) = 0. \]

This completes the proof. \( \square \)

**Corollary 6.13.** \( \tilde{M}_n \to \tilde{M} \) almost everywhere in \( \tilde{\Omega} \times [0, T] \times D \) as \( n \to \infty. \)

Proposition 6.14.

\[ \lim_{n \to \infty} \mathbb{E} \int_0^T \| \pi_x \phi' \left( \tilde{M}_n(s), \tilde{M}(s) \right) \|^2_{L^2} \, ds = 0. \]

**Proof of (6.17).** By Corollary 6.13, \( \tilde{M}_n \to \tilde{M} \) almost everywhere in \( \tilde{\Omega} \times [0, T] \times D. \) And since \( \phi' \) is continuous,

\[ \lim_{n \to \infty} \left| \phi' \left( \tilde{M}_n(s) \right) - \phi' \left( \tilde{M}(s) \right) \right|^2 = 0, \]

almost everywhere in \( \tilde{\Omega} \times [0, T] \times D. \) Moreover, \( \phi' \) is bounded, so there exists some constant \( C > 0 \) such that \( |\phi'(x)| \leq C \) for all \( x \in \mathbb{R}^l. \) Therefore for almost every \( (\omega, s) \in \tilde{\Omega} \times [0, T], \)

\[ \int_0^1 \left| \phi' \left( \tilde{M}_n(\omega, s, x) \right) - \phi' \left( \tilde{M}(\omega, s, x) \right) \right|^4 \, dx \leq 16C^4 m(D) < \infty. \]

Hence \( \left| \phi' \left( \tilde{M}_n(\omega, s) \right) - \phi' \left( \tilde{M}(\omega, s) \right) \right|^2 \) is uniformly integrable on \( D, \) so

\[ \lim_{n \to \infty} \left\| \phi' \left( \tilde{M}_n(\omega, s) \right) - \phi' \left( \tilde{M}(\omega, s) \right) \right\|_{L^2}^2 = 0, \quad \tilde{\Omega} \times [0, T] - a.e. \]

Therefore for almost every \( (\omega, s) \in \tilde{\Omega} \times [0, T], \)

\[ \left\| \pi_x \phi' \left( \tilde{M}_n(\omega, s) \right) - \phi' \left( \tilde{M}(\omega, s) \right) \right\|_{L^2}^2 \leq 2 \left\| \phi' \left( \tilde{M}_n(\omega, s) \right) - \phi' \left( \tilde{M}(\omega, s) \right) \right\|_{L^2}^2 + 2 \left\| \pi_x \phi' \left( \tilde{M}(\omega, s) \right) - \phi' \left( \tilde{M}(\omega, s) \right) \right\|_{L^2}^2 \to 0. \]

Moreover since

\[ \mathbb{E} \int_0^T \left\| \pi_x \phi' \left( \tilde{M}_n(\omega, s) \right) - \phi' \left( \tilde{M}(\omega, s) \right) \right\|_{L^2}^2 \, ds \leq 16TC^4 m(D) < \infty, \]
Moreover, by (Proposition 6.17).

Next we will define \( \pi_n \). By the estimate (Proof).

The proof of (iv) of Theorem 6.1, we have

Moreover, by (6.6) and (6.14) we have

Hence \( \int_{\tilde{\Omega}} \left| \left| u(s), \pi_n 1_D(\tilde{B}_n - \tilde{B})(s) \right| \right|_{\tilde{H}} , \) is uniformly integrable on \( \tilde{\Omega} \), so

The proof of (6.18) has been complete.

Proposition 6.16.

(6.19)

\( \nabla \tilde{M}_n \rightharpoonup \nabla \tilde{M} \) weakly in \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{L})) \), \( i = 1, 2, 3 \).

Proof. Let us fix \( \varphi \in L^2(\tilde{\Omega}; L^2(0, T; \mathbb{V})) \), by (6.16) \( \tilde{M}_n \rightharpoonup \tilde{M} \) in \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H})) \), so we have:

By the estimate (6.4), \( \{M_n\}_{n=1}^\infty \) is bounded in \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{V})) \), so the limit of the right hand side of above equation exists. Hence the result follows.

Next we will define \( \tilde{M} \times \tilde{\rho} \) and show that the limits of the sequences \( \{\tilde{M}_n \times \tilde{\rho}_n\}_n, \{\tilde{M} \times (\tilde{M}_n \times \tilde{\rho}_n)\}_n \) and \( \{\pi_n(\tilde{M} \times \tilde{M}_n \times \tilde{\rho}_n)\}_n \) are actually \( \tilde{M} \times \tilde{\rho}, \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \) and \( \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \).

Proposition 6.17. For \( p \geq 1 \) and \( b > \frac{1}{2} \), there exist \( Z_1 \in L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H})) \), \( Z_2 \in L^2(\tilde{\Omega}; L^2(0, T; \mathbb{L}^2)) \) and \( Z_3 \in L^2(\tilde{\Omega}; L^2(0, T; \mathbb{X}^{-b})) \), such that

(6.20) \( \tilde{M}_n \times \tilde{\rho}_n \rightharpoonup Z_1 \) weakly in \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H})) \),

(6.21) \( M_n \times (M_n \times \rho_n) \rightharpoonup Z_2 \) weakly in \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{L}^2)) \),

(6.22) \( \pi_n(M_n \times (M_n \times \rho_n)) \rightharpoonup Z_3 \) weakly in \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{X}^{-b})) \).

Proof. The spaces \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H})) \), \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{L}^2)) \) and \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{X}^{-b})) \) are reflexive. Then by equations (6.5), (6.7), (6.8) and by the Banach-Alaoglu Theorem, we get equations (6.20), (6.21) and (6.22).

Proposition 6.18.

\( Z_2 = Z_3 \) in the space \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{X}^{-b})) \).
Proof. Notice that \((L^b)^* = L^1\), and \(X^b = \mathbb{H}^{1b}\), \(X^b \subset L^1\) for \(b > \frac{1}{2}\), hence \(L^b \subset X^{-b}\), so
\[
L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H}^{1b})) \subset L^2(\tilde{\Omega}; L^2(0, T; X^{-b})).
\]
Therefore \(Z_2 \in L^2(\tilde{\Omega}; L^2(0, T; X^{-b}))\) as well as \(Z_3\).
Since by definition \(X^b = D(A^b)\) and \(A\) is self-adjoint, we can define
\[
X^b_n := \left\{ x_\eta e = \sum_{j=1}^n \langle x, e_j \rangle \eta^j : x \in \mathbb{H}, \sum_{j=1}^n A^{1b} \langle x, e_j \rangle^2 \eta^j < \infty \right\}.
\]
Then \(X^b = \bigcup_{n=1}^\infty X^b_n, L^2(\tilde{\Omega}; L^2(0, T; X^b)) = \bigcup_{n=1}^\infty L^2(\tilde{\Omega}; L^2(0, T; X^b_n))\).
Firstly, we prove the result for each \(u_n \in L^2(\tilde{\Omega}; L^2(0, T; X^b_n))\). To do this, let us fix \(n\) and take \(u_n \in L^2(\tilde{\Omega}; L^2(0, T; X^b_n))\), then for any \(m \geq n\), we have
\[
l^2(\tilde{\Omega}; L^2(0, T; X^b)) \left( (Z_3, u_n), l^2(\tilde{\Omega}; L^2(0, T; X^b)) \right) = l^2(\tilde{\Omega}; L^2(0, T; X^b)) \left( (Z_2, u_n), l^2(\tilde{\Omega}; L^2(0, T; X^b)) \right),
\]
\forall u_n \in L^2(\tilde{\Omega}; L^2(0, T; X^b)).
Secondly, for any \(u \in L^2(\tilde{\Omega}; L^2(0, T; X^b))\), there exists \(L^2(\tilde{\Omega}; L^2(0, T; X^b)) \ni u_n \rightarrow u\) as \(n \rightarrow \infty\), hence for all \(u \in L^2(\tilde{\Omega}; L^2(0, T; X^b))\), we have
\[
l^2(\tilde{\Omega}; L^2(0, T; X^b)) \left( (Z_3, u), l^2(\tilde{\Omega}; L^2(0, T; X^b)) \right) = \lim_{n \rightarrow \infty} l^2(\tilde{\Omega}; L^2(0, T; X^b)) \left( (Z_3, u_n), l^2(\tilde{\Omega}; L^2(0, T; X^b)) \right) = \lim_{n \rightarrow \infty} l^2(\tilde{\Omega}; L^2(0, T; X^b)) \left( (Z_2, u_n), l^2(\tilde{\Omega}; L^2(0, T; X^b)) \right) = l^2(\tilde{\Omega}; L^2(0, T; X^b)) \left( (Z_2, u), l^2(\tilde{\Omega}; L^2(0, T; X^b)) \right)
\]
Therefore \(Z_2 = Z_3 \in L^2(\tilde{\Omega}; L^2(0, T; X^{-b}))\) and this concludes the proof. \(\square\)

In next Lemma, we look into \(Z_1\).

Theorem 6.19. \(Z_1\) is the unique element in \(L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H}^{1b}))\) such that \(\forall u \in L^2(\tilde{\Omega}; L^2(0, T; \mathbb{W}^{1, b})),\)
the following equality holds
\[
\lim_{n \rightarrow \infty} \mathbb{E} \int_0^T \langle \tilde{M}_n(s), u(s) \rangle ds = \mathbb{E} \int_0^T \langle Z_1(s), u(s) \rangle ds
\]
\[= \mathbb{E} \int_0^T \langle \tilde{M}(t) \times (\varphi'(\tilde{M}(t)) + 1_2(B - \tilde{M}(t)), u(t) \rangle d\tilde{s} + \sum_{i=1}^3 \mathbb{E} \int_0^T \langle \nabla \tilde{M}(t), \tilde{M}(t) \times \nabla u(t) \rangle ds dt.
\]
Proof. Let us recall that
\[
\tilde{\rho}_n := \frac{1}{\rho_n}(\varphi(\tilde{M}_n) + 1_2(\tilde{B}_n - \tilde{M}_n)) = \Delta \tilde{M},
\]
so we take three parts to prove the desired result.
Firstly we show that
\[
\lim_{n \rightarrow \infty} \mathbb{E} \int_0^T \langle \tilde{M}_n(t) \times \Delta \tilde{M}_n(t), u(t) \rangle ds = \sum_{i=1}^3 \mathbb{E} \int_0^T \langle \nabla \tilde{M}(t), \tilde{M}(t) \times \nabla u(t) \rangle ds dt.
\]
Finally, we will show that

\[(6.23) \quad \left\langle \tilde{M}_n(t) \times \nabla \tilde{M}_n(t), u(t) \right\rangle_{L^2} = \sum_{i=1}^3 \left\langle \nabla \tilde{M}_n(t), \tilde{M}_n(t) \times \nabla u(t) \right\rangle_{L^2}\]

for almost every \(t \in [0, T]\) and \(\tilde{F}\) almost surely. Moreover, by the results: \((6.19), (6.4)\) and \((6.16)\), we have for \(i = 1, 2, 3,\)

\[
\begin{align*}
&\left| \mathbb{E} \int_0^T \left\langle \nabla \tilde{M}, \tilde{M} \times \nabla v \right\rangle_{L^2} \, dt - \mathbb{E} \int_0^T \left\langle \nabla \tilde{M}_n, \tilde{M}_n \times \nabla v \right\rangle_{L^2} \, dt \right| \\
&\leq \left| \mathbb{E} \int_0^T \left\langle \nabla \tilde{M} - \nabla \tilde{M}_n, \tilde{M} \times \nabla v \right\rangle_{L^2} \, dt \right| + \left| \mathbb{E} \int_0^T \left\langle \nabla \tilde{M}_n, (\tilde{M} - \tilde{M}_n) \times \nabla v \right\rangle_{L^2} \, dt \right| \\
&\leq \left( \mathbb{E} \int_0^T \|\nabla \tilde{M}_n\|^2_{L^2} \, dt \right)^{1/2} \left( \mathbb{E} \int_0^T \|\tilde{M} - \tilde{M}_n\|^4_{L^4} \, dt \right)^{1/4} \left( \mathbb{E} \int_0^T \|\nabla v\|^4_{L^4} \, dt \right)^{1/4} \\
&+ \left( \mathbb{E} \int_0^T \|\nabla \tilde{M}_n\|^2_{L^2} \, dt \right)^{1/2} \left( \mathbb{E} \int_0^T \|\tilde{M} - \tilde{M}_n\|^4_{L^4} \, dt \right)^{1/4} \left( \mathbb{E} \int_0^T \|\nabla v\|^4_{L^4} \, dt \right)^{1/4} \\
&\to 0, \quad \text{as } n \to \infty.
\end{align*}
\]

Secondly we show that

\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \left\langle \tilde{M}_n(t) \times \pi_n \varphi' (\tilde{M}_n(t)), u(t) \right\rangle_{L^2} \, dt = \mathbb{E} \int_0^T \left\langle \tilde{M}(t) \times \varphi' (\tilde{M}(t)), u(t) \right\rangle_{L^2} \, dt.
\]

Proof of the above equality: By \((6.16)\) and \((6.17)\), we have

\[
\begin{align*}
&\left| \mathbb{E} \int_0^T \left\langle \tilde{M}_n(s) \times \pi_n \varphi' (\tilde{M}_n(s)) - \tilde{M}(s) \times \varphi' (\tilde{M}(s)), u(s) \right\rangle_{L^2} \, ds \right| \\
&\leq \mathbb{E} \int_0^T \left| \left[ \tilde{M}_n(s) - \tilde{M}(s) \right] \times u(s), \pi_n \varphi' (\tilde{M}_n(s)) \right|_{L^2} \, ds \\
&\quad + \mathbb{E} \int_0^T \left| \left[ \tilde{M}_n(s) \times u(s), \pi_n \varphi' (\tilde{M}_n(s)) - \varphi' (\tilde{M}(s)) \right] \right|_{L^2} \, ds \\
&\leq \left( \mathbb{E} \int_0^T \|\tilde{M}_n(s) - \tilde{M}(s)\|^4_{L^4} \, ds \right)^{1/4} \left( \mathbb{E} \int_0^T \|u(s)\|^4_{L^4} \, ds \right)^{1/4} \left( \mathbb{E} \int_0^T \|\varphi' (\tilde{M}_n(s))\|^2_{L^2} \, ds \right)^{1/2} \\
&\quad + \left( \mathbb{E} \int_0^T \|\tilde{M}_n(s)\|^4_{L^4} \, ds \right)^{1/4} \left( \mathbb{E} \int_0^T \|u(s)\|^4_{L^4} \, ds \right)^{1/4} \left( \mathbb{E} \int_0^T \|\pi_n \varphi' (\tilde{M}_n(s)) - \varphi' (\tilde{M}(s))\|^2_{L^2} \, ds \right)^{1/2} \\
&\to 0, \quad \text{as } n \to \infty.
\end{align*}
\]

Finally, we will show that

\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \left\langle \tilde{M}_n(t) \times \pi_n 1_2 (\tilde{B}_n - \pi_n \tilde{M}_n)(t), u(t) \right\rangle_{L^2} \, dt = \mathbb{E} \int_0^T \left\langle \tilde{M}(t) \times 1_2 (\tilde{B} - \tilde{M})(t), u(t) \right\rangle_{L^2} \, dt.
\]
Proof of the above equality: By (6.16) and (6.18), we have
\[
\left| \mathbb{E} \int_0^T \langle \tilde{M}_n(s) \times \pi_n 1_{\Omega}(\tilde{B}_n - \pi_n \tilde{M}_n)(s) - \tilde{M}(s) \times 1_{\Omega}(\tilde{B} - \tilde{M})(s), u(s) \rangle_{H} ds \right|
\]
\[
\leq \mathbb{E} \int_0^T \left| \langle \tilde{M}_n(s) - \tilde{M}(s) \rangle \times u(s), \pi_n 1_{\Omega}(\tilde{B}_n - \pi_n \tilde{M}_n)(s) \rangle_{H} ds \right|
\]
\[
+ \mathbb{E} \int_0^T \left| \langle \tilde{M}(s) \times u(s), \pi_n 1_{\Omega}(\tilde{B}_n - \pi_n \tilde{M}_n)(s) - 1_{\Omega}(\tilde{B} - \tilde{M})(s) \rangle_{H} ds \right|
\]
\[
\leq \left( \mathbb{E} \int_0^T \| \tilde{M}_n(s) - \tilde{M}(s) \|^2_{L^2} ds \right)^\frac{1}{2} \left( \mathbb{E} \int_0^T \| u(s) \|^2_{L^2} ds \right)^\frac{1}{2} \left( \mathbb{E} \int_0^T \| \pi_n 1_{\Omega}(\tilde{B}_n - \pi_n \tilde{M}_n)(s) \|^2_{L^2} ds \right)^\frac{1}{2}
\]
\[
+ \left( \mathbb{E} \int_0^T \| \tilde{M}(s) \|^2_{L^2} ds \right)^\frac{1}{2} \left( \mathbb{E} \int_0^T \| u(s) \|^2_{L^2} ds \right)^\frac{1}{2} \left( \mathbb{E} \int_0^T \| \pi_n 1_{\Omega}(\tilde{B}_n - \pi_n \tilde{M}_n)(s) - 1_{\Omega}(\tilde{B} - \tilde{M})(s) \|^2_{L^2} ds \right)^\frac{1}{2}
\]
\[
+ \mathbb{E} \int_0^T \left| \langle \tilde{M}(s) \times u(s), \pi_n 1_{\Omega}(\tilde{B}_n - \tilde{B})(s) \rangle_{H} ds \right| \longrightarrow 0, \quad \text{as } n \to \infty.
\]

So far we have shown that
\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \langle \tilde{M}_n(s) \times \tilde{\rho}_n(s), u(s) \rangle_{H} ds = \mathbb{E} \int_0^T \langle \tilde{M}(t) \times (\varphi'(\tilde{M}(t)) + 1_{\Omega}(\tilde{B} - \tilde{M})(t)), u(t) \rangle_{H} dt + \sum_{j=1}^{3} \left\langle \nabla \tilde{M}_n(t), \tilde{M}_n(t) \times \nabla u(t) \right\rangle_{H},
\]

for all \( u \in L^2(\tilde{\Omega}; L^2(0, T; \mathbb{W}^{1,1})) \). Since \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{W}^{1,1})) \) is dense in \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H})) \), we also have
\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \langle \tilde{M}_n(s) \times \tilde{\rho}_n(s), u(s) \rangle_{H} ds = \mathbb{E} \int_0^T \langle Z_1(s), u(s) \rangle_{H} ds, \quad u \in L^2(\tilde{\Omega}; L^2(0, T; \mathbb{W}^{1,1})),
\]

and such \( Z_1 \) is unique in \( L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H})) \). This completes the proof. \( \square \)

Notation 6.20. We will denote \( \tilde{M} \times \tilde{\rho} := Z_1 \).

Remark 6.21. By the Notation 6.20, the Theorem 6.19 shows that \( \tilde{M} \times \tilde{\rho} \in L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H})) \) and
\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \langle \tilde{M}_n(s) \times \tilde{\rho}_n(s), u(s) \rangle_{H} ds = \mathbb{E} \int_0^T \langle \tilde{M}(s) \times \tilde{\rho}(s), u(s) \rangle_{H} ds,
\]

for all \( u \in L^2(\tilde{\Omega}; L^2(0, T; \mathbb{H})) \). By (6.13), we also have
\[
\tilde{M} \times (\tilde{M} \times \tilde{\rho}) \in L^4(\tilde{\Omega}; L^4(0, T; L^4(\mathcal{D}))).
\]

Lemma 6.22. For any \( \eta \in L^4(\tilde{\Omega}; L^4(0, T; L^4)) \) we have
\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \langle \tilde{M}_n(s) \times \tilde{\rho}_n(s), \eta(s) \rangle_{L_2^{1,1}(D)} ds = \mathbb{E} \int_0^T \langle \tilde{M}(s) \times \tilde{\rho}(s), \eta(s) \rangle_{L_2^{1,1}(D)} ds
\]
\[
(6.24)
\]
\[
\int_0^T \frac{1}{2} \left\| Z_2(s), \eta(s) \right\|_{L_2^{1,1}(D)} ds
\]
\[
(6.25)
\]

Proof. Let us denote \( Z_n := \tilde{M}_n \times \tilde{\rho}_n \) for each \( n \in \mathbb{N} \). \( L^4(\tilde{\Omega}; L^4(0, T; L^4)) \subset L^2(\tilde{\Omega}; L^2(0, T; L^*) \cap L^2(\tilde{\Omega}; L^2(0, T; L^*)) \) which is the dual space of \( L^2(\tilde{\Omega}; L^2(0, T; L^*)) \). Hence (6.21) implies that (6.24) holds.

Next we are going to prove (6.25).
By (6.16), $\widetilde{M} \in L^4(\Omega; L^4(0, T; L^4))$, hence by the Hölder inequality, we have
\[
\mathbb{E} \int_0^T |\eta \times \widetilde{M}|_{L^4}^2 \, dt \leq \mathbb{E} \int_0^T |\eta|_{L^4}^p |\widetilde{M}|_{L^4}^q \, dt \leq 2 \mathbb{E} \int_0^T |\eta|_{L^4}^p \, dt + \mathbb{E} \int_0^T |\widetilde{M}|_{L^4}^q \, dt < \infty.
\]
So $\eta \times \widetilde{M} \in L^2(\Omega; L^2(0, T; L^2))$ and similarly $\eta \times \tilde{M}_n \in L^2(\Omega; L^2(0, T; L^2))$.

By (6.20), $Z_{1n} \in L^2(\Omega; L^2(0, T; L^2))$. And $\eta \times \tilde{M}_n \in L^2(\Omega; L^2(0, T; L^2))$.

Hence
\[
\mathbb{E} \int_0^T \langle \tilde{M}_n \times Z_{1n}, \eta \rangle_{L^2} \, dt = \mathbb{E} \int_0^T \langle \tilde{M}_n(x) \times Z_{1n}(x), \eta(x) \rangle \, dx = (\tilde{M}_n, \eta \times \tilde{M}_n)_{L^2}.
\]

By (6.20), $Z_1 \in L^2(\Omega; L^2(0, T; L^2))$.

So
\[
\mathbb{E} \int_0^T \langle \tilde{M} \times Z_1, \eta \rangle_{L^2} \, dt = \mathbb{E} \int_0^T \langle \tilde{M}(x) \times Z_1(x), \eta(x) \rangle \, dx = (\tilde{M}, \eta \times \tilde{M})_{L^2}.
\]

By (6.20) and (6.27),
\[
\mathbb{E} \int_0^T \langle Z_{1n}(s) - Z_1(s), \eta(s) \rangle_{L^2} \, ds \to 0.
\]

By the Cauchy-Schwartz inequality,
\[
\mathbb{E} \int_0^T \langle Z_{1n}, \eta \rangle_{L^2} \langle \tilde{M}_n - \hat{M}, \eta \rangle_{L^2} \leq \mathbb{E} \int_0^T \langle Z_{1n} \rangle_{L^2} |\eta|_{L^4}^p \langle \tilde{M}_n - \hat{M} \rangle_{L^4}^q \to 0, \quad \text{as } n \to \infty.
\]

Hence
\[
\mathbb{E} \int_0^T \langle Z_{1n}(s), \eta \rangle_{L^2} \langle \tilde{M}_n(s) - \hat{M}(s) \rangle_{L^2} \, ds = 0.
\]

Therefore,
\[
\mathbb{E} \int_0^T \langle Z_{1n}(s), \eta \rangle_{L^2} \langle \tilde{M}_n(s) \times \tilde{\rho}_n(s), \eta(s) \rangle_{L^2} \, ds = \mathbb{E} \int_0^T \langle \tilde{M}(s) \times Z_1(s), \eta(s) \rangle_{L^2} \, ds.
\]

This completes the proof of the Lemma 6.22. \hfill \Box

Remark 6.23. By the notation 6.20, the Lemma 6.22 has proved that
\[
Z_2 = \tilde{M} \times (\tilde{M} \times \tilde{\rho})
\]
in $L^2(\Omega; L^2(0, T; L^2))$. So
\[
\tilde{M}_n \times (\tilde{M}_n \times \tilde{\rho}_n) \rightharpoonup \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \quad \text{weakly in } L^2(\Omega; L^2(0, T; L^2)).
\]

The next result will be used to show that the process $\tilde{M}$ satisfies the condition $|\tilde{M}(t, x)| = 1$ for all $t \in [0, T], x \in \Omega$ and $\bar{\Omega}$-almost surely.

Lemma 6.24. For any bounded measurable function $\varphi : \Omega \to \mathbb{R}$, we have
\[
\mathbb{E} \int_{[0, T]} \langle Z_1(s, \omega), \varphi(\tilde{M}(s, \omega)) \rangle_{L^4} = 0,
\]
for almost every $(s, \omega) \in [0, T] \times \Omega$. 

Proof. Let $B \subset [0, T] \times \tilde{\Omega}$ be a measurable set and $1_B$ be the indicator function of $B$. Then

$$\mathbb{E} \int_0^T ||1_B \varphi \tilde{M}_n(t) - 1_B \varphi \tilde{M}(t)||_{L^2}^2 \ dt = \mathbb{E} \int_0^T ||1_B \varphi \tilde{M}_n(t) - \tilde{M}_n(t)||_{L^2}^2 \ dt \leq ||\varphi||_r \mathbb{E} \int_0^T ||\tilde{M}_n(t) - \tilde{M}(t)||_{L^2}^2 \ dt \leq C ||\varphi||_r \mathbb{E} \int_0^T ||\tilde{M}_n(t) - \tilde{M}(t)||_{L^2}^2 \ dt,$$

for some constant $C > 0$. Hence by (6.16), we have

$$\lim_{n \to \infty} \mathbb{E} \int_0^T ||1_B \varphi \tilde{M}_n(t) - 1_B \varphi \tilde{M}(t)||_{L^2}^2 \ dt = 0.$$

Together with the fact that $\tilde{M}_n \times \tilde{\rho}_n$ converges to $Z_t$ weakly in $L^2(\tilde{\Omega}; L^2(0, T; \mathbb{R}^2))$ we can infer that

$$0 = \lim_{n \to \infty} \mathbb{E} \int_0^T 1_B(s) \left( \tilde{M}_n(s) \times \tilde{\rho}_n(s) - \tilde{M}(s) \right)_{L^2} \ ds = \mathbb{E} \int_0^T 1_B(s) \left( Z_t(s) - \tilde{M}(s) \right)_{L^2} \ ds.$$

This complete the proof. □

7. The existence of a weak solution

In this section, we will prove that the process $(\tilde{M}, \tilde{B}, \tilde{E})$ from Theorem 6.1 is a weak solution of the Problem 2.6.

To do this, let us define

$$(7.1) \quad \xi_n(t) := M_n(t) - M_n(0) - \int_0^t \left\{ \lambda_1 \pi_n \{ M_n \times \rho_n \} - \lambda_2 \pi_n \{ M_n \times (M_n \times \rho_n) \} + \frac{1}{2} \sum_{j=1}^{\infty} G_{\mu_j} (M_n) \left[ G_{\mu_j} (M_n) \right] \right\} \ ds$$

$$= \sum_{j=1}^{\infty} \int_0^t G_{\mu_j} (M_n) \ dW_j(s).$$

the second "=" in (7.1) holds is because $M_n$ satisfies (3.12).

The proof will consists in two steps:

Step 1 : We are going to find some $\bar{\xi}$ as a limit of $\xi_n$, which is similar as $\bar{\xi}$ defined in (7.1).

Step 2 : We will show the second "=" in (7.1) holds for the limit process $\bar{\xi}$, but with $\tilde{M}$ instead of $M_n$ and $\tilde{W}_j$ instead of $W_j$, etc.

7.1. Step 1. Let

$$(7.1) \quad \bar{\xi}_n(t) := \tilde{M}_n(t) - \tilde{M}_n(0) - \int_0^t \left\{ \pi_n \{ \tilde{M}_n \times \tilde{\rho}_n \} - \lambda_2 \pi_n \{ \tilde{M}_n \times (\tilde{M}_n \times \tilde{\rho}_n) \} \right\} \ ds$$

$$+ \frac{1}{2} \sum_{j=1}^{\infty} G_{\mu_j} (\tilde{M}_n) \left[ G_{\mu_j} (\tilde{M}_n) \right] \ ds.$$

Lemma 7.1. For each $t \in [0, T]$ the sequence of random variables $\bar{\xi}_n(t)$ converges weakly in $L^2(\tilde{\Omega}; \mathbb{X}^{-b})$ to the limit

$$\bar{\xi}(t) := \tilde{M}(t) - M_0 - \int_0^t \left\{ \left[ \lambda_1 \tilde{M} \times \tilde{\rho} \right] - \lambda_2 \left[ \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \right] \right\} \ ds$$

$$+ \frac{1}{2} \sum_{j=1}^{\infty} (G_{\mu_j} (\tilde{M})) \left[ G_{\mu_j} (\tilde{M}) \right] \ ds.$$
Proof. The dual space of $L^2(\mathbb{Q}, X^0)$ is $L^2(\mathbb{Q}, X^0)$. Let $t \in (0, T]$ and $U \in L^2(\mathbb{Q}, X^0)$. We have

$$L^2(\mathbb{Q}, X^0) \left\langle \tilde{e}_n(t), U \right\rangle_{L^2(\mathbb{Q}, X^0)} = \mathbb{E} \left[ \int_0^T \left\langle \tilde{e}_n(t), U \right\rangle_{X^0} \right]$$

$$= \mathbb{E} \left[ \left\langle M_n(t), U \right\rangle - \left\langle M_n(0), U \right\rangle - A_1 \int_0^T \left\langle M_n(s) \times \tilde{p}_n(s), \pi_n U \right\rangle_{\mathbb{H}} \, ds \right.$$ 

$$+ A_2 \int_0^T \left\langle \left( M_n(s) \times (M_n(s) \times \tilde{p}_n(s)) \right), \pi_n U \right\rangle_{\mathbb{H}} \, ds - \frac{1}{2} \sum_{j=1}^\infty \int_0^T \left\langle G_{\mu_j}(\tilde{M}_n) \left| G_{\mu_j}(\tilde{M}_n) \right|, \pi_n U \right\rangle \, ds \right]$$

Next we are going to consider the right hand side of above equality term by term.

By the Theorem 6.1, $M_n \rightarrow \tilde{M}$ in $C([0, T]; X^0)$ $\mathbb{P}$-a.s., so

$$\sup_{t \in [0, T]} \| M_n(t) - M(t) \|_{X^0} \rightarrow 0, \quad \mathbb{P} - a.s.$$  

and $X^0 \left\langle \cdot, U \right\rangle_{X^0}$ is a continuous function on $X^0$, hence

$$\lim_{n \to \infty} \left\langle \tilde{M}_n(t), U \right\rangle_{X^0} = \left\langle \tilde{M}(t), U \right\rangle_{X^0}, \quad \mathbb{P} - a.s.$$  

By (6.1), $\sup_{t \in [0, T]} |M_n(t)| \leq |M_0|$, so that we can find a constant $C$ such that

$$\sup_n \mathbb{E} \left\| \left\langle \tilde{M}_n(t), U \right\rangle \right\|_{X^0} \leq \sup_n \mathbb{E} \left\| U \right\|_{X^0}^2 \mathbb{E} \| M_n(t) \|_{X^0}^2$$

$$\leq C \mathbb{E} \| U \|_{X^0}^2 \mathbb{E} \| M_0 \|_{X^0}^2 < \infty.$$  

Therefore, the sequence $X^0 \left\langle \tilde{M}_n(t), U \right\rangle_{X^0}$ is uniformly integrable. So the almost surely convergence and uniform integrability implies that

$$\lim_{n \to \infty} \mathbb{E} \left[ X^0 \left\langle \tilde{M}_n(t), U \right\rangle_{X^0} \right] = \mathbb{E} \left[ X^0 \left\langle \tilde{M}(t), U \right\rangle_{X^0} \right].$$

By (6.20),

$$\lim_{n \to \infty} \mathbb{E} \left\{ \int_0^T \left\langle \tilde{M}_n(s) \times \tilde{p}_n(s), \pi_n U \right\rangle_{\mathbb{H}} \, ds \right\} = \mathbb{E} \left\{ \int_0^T \left\langle Z_1(s), U \right\rangle_{\mathbb{H}} \, ds \right\}.$$  

By (6.22)

$$\lim_{n \to \infty} \mathbb{E} \left\{ \int_0^T \left\langle \pi_n (\tilde{M}_n(s) \times (\tilde{M}_n(s) \times \tilde{p}_n(s))), U \right\rangle_{X^0} \, ds \right\} = \mathbb{E} \left\{ \int_0^T \left\langle Z_2(s), U \right\rangle_{X^0} \, ds \right\}.$$  

By the Hölder’s inequality,

$$\| \tilde{M}_n(t) - \tilde{M}(t) \|_{X^0}^2 \leq \| \tilde{M}_n(t) - \tilde{M}_n(t) \|_{X^0}^2.$$  

We will show that for any $U \in L^2(\mathbb{Q}, L^2(0, T; \mathbb{H})$

$$\mathbb{E} \left\{ \sum_{j=1}^\infty \int_0^T \left\langle G_{\mu_j}(\tilde{M}_n) \left| G_{\mu_j}(\tilde{M}_n) \right|, \pi_n U \right\rangle_{\mathbb{H}} \, ds \right\} = \mathbb{E} \left\{ \sum_{j=1}^\infty \int_0^T \left\langle G_{\mu_j}(\tilde{M}) \left| G_{\mu_j}(\tilde{M}) \right|, U \right\rangle_{\mathbb{H}} \, ds \right\}.$$  

Using (6.1) we can prove

$$\left\| G_{\mu_j}(\tilde{M}_n) \left| G_{\mu_j}(\tilde{M}_n) \right|, \pi_n U \right\|_{\mathbb{H}} \leq C \left\| h_j \|_{L^\infty} \| U \|_{\mathbb{H}} \right\|,$$

it remains to show that

$$\lim_{n \to \infty} \left\{ \left\langle G_{\mu_j}(\tilde{M}_n) \left| G_{\mu_j}(\tilde{M}_n) \right|, \pi_n U \right\rangle_{\mathbb{H}} \right\} = \left\{ \left\langle G_{\mu_j}(\tilde{M}) \left| G_{\mu_j}(\tilde{M}) \right|, U \right\rangle_{\mathbb{H}} \right\}.$$  

This follows immediately from the convergence of $\tilde{M}_n(t)$ to $\tilde{M}(t)$ for every $t \in [0, T]$ $\mathbb{P}$-a.s. Hence

$$\lim_{n \to \infty} \mathbb{E} \left[ X^0 \left\langle \tilde{e}_n(t), U \right\rangle_{L^2(\mathbb{Q}, X^0)} \right] = \mathbb{E} \left[ X^0 \left\langle \tilde{M}(t), U \right\rangle_{X^0} \right] - \mathbb{E} \left[ X^0 \left\langle M_0, U \right\rangle_{X^0} \right] - A_1 \int_0^T \left\langle Z_1(s), U \right\rangle_{\mathbb{H}} \, ds$$

$$+ A_2 \int_0^T \left\langle Z_2(s), U \right\rangle_{X^0} \, ds - \frac{1}{2} \sum_{j=1}^\infty \mathbb{E} \left[ \left\langle G_{\mu_j}(\tilde{M}) \left| G_{\mu_j}(\tilde{M}) \right|, U \right\rangle \right] \, ds.$$
Since by Theorem 6.19 and Lemma 6.22, we have $Z_1 = \tilde{M} \times \tilde{\rho}$ and $Z_2 = \tilde{M} \times \tilde{\rho}$. Therefore for any $U \in L^2(\Omega; X)$,
\[
\lim_{n \to \infty} L^2(\tilde{\Omega}; X^*) \langle \tilde{\xi}_n(t), U \rangle_{L^2(\tilde{\Omega}; X^*)} = L^2(\tilde{\Omega}; X^*) \langle \tilde{\xi}(t), U \rangle_{L^2(\tilde{\Omega}; X^*)}.
\]
This concludes the proof. \hfill \Box

### 7.2. Step 2.

In this step we are going to show that
\[
(7.3) \quad \tilde{\xi}(t) = \sum_{j=1}^{\infty} \int_0^t G_{ij}^j(\tilde{\mathcal{M}}(s)) \, d\tilde{W}_j(s).
\]

We will finish this step by the approximation method. To do this, we need the next Lemma for preparation. Let us define, for each $m \in \mathbb{N}$, a partition $\{s^n_i := \frac{t}{m}, i = 0, \ldots, m\}$ of $[0,T]$. It will be convenient to define on $[0,T]$ processes
\[
(7.4) \quad \sigma_{jm}(s) = G_{jm} \left( \tilde{\mathcal{M}}_m(s) \right),
\]
and
\[
(7.5) \quad \sigma_{jm}^m(s) = \sum_{i=0}^{m-1} G_{jm} \left( \tilde{\mathcal{M}}_m(s^n_i) \right) 1_{(s^n_i, s^n_{i+1})}(s) = \sum_{i=0}^{m-1} \sigma_{jm}(s^n_i) 1_{(s^n_i, s^n_{i+1})}(s).
\]

**Lemma 7.2.** For any $\varepsilon > 0$, we can choose $m \in \mathbb{N}$ sufficiently large such that:

(i) \[
\lim_{n \to \infty} \mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^t \left[ \sigma_{jm}(s) - \sigma_{jm}^m(s) \right] \, d\tilde{W}_j(s) \right\|^2_{L^2(\mathbb{R}^d)} < \frac{\varepsilon^2}{4},
\]

(ii) \[
\lim_{n \to \infty} \mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^t \sigma_{jm}(s) \, d\tilde{W}_j(s) - \sum_{j=1}^{\infty} \int_0^t \sigma_{jm}^m(s) \, d\tilde{W}_j(s) \right\|^2_{L^2(\mathbb{R}^d)} = 0,
\]

(iii) \[
\lim_{n \to \infty} \mathbb{E} \left\| \sum_{j=1}^{\infty} \left( \sigma_{jm}(s) - \sigma_{jm}(s) \right) \, d\tilde{W}_j(s) \right\|^2_{L^2(\mathbb{R}^d)} < \frac{\varepsilon^2}{4},
\]

(iv) \[
\lim_{n \to \infty} \mathbb{E} \left\| \sum_{j=1}^{\infty} \left( \sigma_{jm}(s) - G_{jm}^j \left( \tilde{\mathcal{M}}(s) \right) \right) \, d\tilde{W}_j(s) \right\|^2_{L^2(\mathbb{R}^d)} = 0.
\]

**Proof.** (i) By Itô isometry, our assumptions on $\psi$ and $h_j$, there exists some constants $C > 0$, such that
\[
\mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^t \left[ \sigma_{jm}(s) - \sigma_{jm}^m(s) \right] \, d\tilde{W}_j(s) \right\|^2_{L^2(\mathbb{R}^d)}
\]
\[
= \sum_{j=1}^{\infty} \mathbb{E} \int_0^t \left\| \sigma_{jm}(s) - \sigma_{jm}^m(s) \right\|^2_{L^2(\mathbb{R}^d)} \, ds
\]
\[
\leq C \left( \sum_{j=1}^{\infty} \|h_j\|_{L^2(\mathbb{R}^d)} \left( \sum_{i=0}^{m-1} \mathbb{E} \int_{s^n_i}^{s^n_{i+1}} \left\| \tilde{\mathcal{M}}_m(s) - \tilde{\mathcal{M}}_m(s^n_i) \right\|^2_{L^2(\mathbb{R}^d)} \, ds \right) \right)
\]
\[
\leq C \sum_{i=0}^{m-1} \mathbb{E} \int_{s^n_i}^{s^n_{i+1}} \left\| \tilde{\mathcal{M}}_m(s) - \tilde{\mathcal{M}}_m(s^n_i) \right\|^2_{L^2(\mathbb{R}^d)} \, ds + C \sum_{i=1}^{m-1} \mathbb{E} \int_{s^n_i}^{s^n_{i+1}} \left\| \tilde{\mathcal{M}}(s) - \tilde{M}(s^n_i) \right\|^2_{L^2(\mathbb{R}^d)} \, ds
\]
\[
+ C \sum_{i=1}^{m-1} \mathbb{E} \int_{s^n_i}^{s^n_{i+1}} \left\| \tilde{M}(s^n_i) - \tilde{\mathcal{M}}_m(s^n_i) \right\|^2_{L^2(\mathbb{R}^d)} \, ds
\]
\[
= C (I_1(n) + I_2(m) + I_3(n, m)) .
\]

By the estimate (6.16), $\lim_{n \to \infty} I_1(n) = 0$. 

---

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Since $\bar{M} \in C([0, T]; \mathbb{R}^d)$, for every $\epsilon > 0$ we can find $m_0$ such that

$$I_2(m) < \frac{\epsilon^2}{4C}, \quad \text{for } m > m_0.$$ 

For any $m \geq 1$

$$I_3(n, m) \leq T \mathbb{E} \sup_{s \in [0, T]} \left\| \bar{M}(s) - \bar{M}_n(s) \right\|_{L_{\infty, \gamma}}^2.$$ 

By Theorem 6.1 (ii),

$$\lim_{n \to \infty} \sup_{s \in [0, T]} \left\| \bar{M}(s) - \bar{M}_n(s) \right\|_{L_{\infty, \gamma}}^2 = 0, \quad \text{P.a.s.},$$

so by the dominated convergence theorem,

$$\lim_{n \to \infty} I_3(m, n) = 0 \text{ for every } m.$$ 

Finally, combining these facts together we obtain (i).

(ii) By the estimate (6.1), remark 6.4 and Jensen inequality we have

$$\mathbb{E} \left< \sum_{j=1}^{\infty} \int_{0}^{t} \sigma^m_{j\mu}(s) d\bar{W}_j(s) - \sum_{j=1}^{\infty} \int_{0}^{t} \sigma^m_{\mu j}(s) d\bar{W}_j(s) \right>^2 \leq \mathbb{E} \left[ \sum_{j=1}^{\infty} \left\| h_j \right\|_{L_\infty} \sum_{i=1}^{m-1} \left\| \bar{W}_j(t \land s_{\mu j}^i) - \bar{W}_j(t \land s_{i}) \right\|_{L_{\infty, \gamma}}^2 \right]^2 \leq C \mathbb{E} \left[ \sum_{j=1}^{\infty} \left\| h_j \right\|_{L_\infty} \sum_{i=1}^{m-1} \left\| \bar{W}_j(t \land s_{\mu j}^i) - \bar{W}_j(t \land s_{i}) \right\|_{L_{\infty, \gamma}}^2 \right]^2 \leq C \mathbb{E} \left[ \sum_{j=1}^{\infty} \left\| h_j \right\|_{L_\infty} \sum_{i=1}^{m-1} \left\| \bar{W}_j(t \land s_{\mu j}^i) - \bar{W}_j(t \land s_{i}) \right\|_{L_{\infty, \gamma}}^2 \right]^2$$

For $m$ fixed we have

$$\sup_n \mathbb{E} \left< \sum_{j=1}^{\infty} \left( \int_{0}^{t} \sigma^m_{j\mu}(s) d\bar{W}_j(s) - \int_{0}^{t} \sigma^m_{\mu j}(s) d\bar{W}_j(s) \right)^2 \right> < \infty.$$ 

Therefore, we can pass with $n$ to the limit under the sum and expectation above and since $\bar{W}_n$ converges to $\bar{W}$ in $C\left([0, T]; \mathbb{R}^d\right)$ we obtain

$$\lim_{n \to \infty} \mathbb{E} \left< \sum_{j=1}^{\infty} \int_{0}^{t} \sigma^m_{j\mu}(s) d\bar{W}_j(s) - \int_{0}^{t} \sigma^m_{\mu j}(s) d\bar{W}_j(s) \right>^2 = 0.$$ 

(iii) The proof of (iii) is same as the proof of (i).

(iv) By Itô isometry, we have

$$\mathbb{E} \left< \int_{0}^{t} \left( \sum_{j=1}^{\infty} \sigma^m_{j\mu}(s) - G^\mu_j \left( \bar{M}(s) \right) \right)^2 ds \right> = \sum_{j=1}^{\infty} \mathbb{E} \left< \int_{0}^{t} \left( G^\mu_j \left( \bar{M}_n(s) \right) - G^\mu_j \left( \bar{M}(s) \right) \right)^2 ds \right> = \sum_{j=1}^{\infty} \mathbb{E} \left< \int_{0}^{t} \left( \left\| G^\mu_j \left( \bar{M}_n(s) \right) - G^\mu_j \left( \bar{M}(s) \right) \right\|_{L_{\infty, \gamma}}^2 ds \right> = \sum_{j=1}^{\infty} \mathbb{E} \left< \int_{0}^{t} \left( \left\| G^\mu_j \left( \bar{M}_n(s) \right) - G^\mu_j \left( \bar{M}(s) \right) \right\|_{L_{\infty, \gamma}}^2 ds \right>$$

By our assumption on $h_j$, the estimates (6.1) and (6.12), we have

$$\sup_n \sum_{j=1}^{\infty} \mathbb{E} \left< \int_{0}^{t} \left\| G^\mu_j \left( \bar{M}_n(s) \right) - G^\mu_j \left( \bar{M}(s) \right) \right\|_{L_{\infty, \gamma}}^2 ds \right> \leq \sum_{j=1}^{\infty} \mathbb{E} \left< \int_{0}^{t} \left( \left\| G^\mu_j \left( \bar{M}(s) \right) \right\|_{L_{\infty, \gamma}}^2 + \left\| \bar{M}(s) \right\|_{L_{\infty, \gamma}}^2 \right) ds \right> < \infty$$

and

$$\left\| G^\mu_j \left( \bar{M}_n(s) \right) - G^\mu_j \left( \bar{M}(s) \right) \right\|_{L_{\infty, \gamma}}^2 \leq 2 \left\| G^\mu_j \left( \bar{M}_n(s) \right) - G^\mu_j \left( \bar{M}_n(s) \right) \right\|_{L_{\infty, \gamma}}^2 + 2 \left\| G^\mu_j \left( \bar{M}(s) \right) - G^\mu_j \left( \bar{M}(s) \right) \right\|_{L_{\infty, \gamma}}^2 \leq C \left\| h_j \right\|_{L_{\infty}} \left\| \bar{M}_n(s) - \bar{M}(s) \right\|_{L_{\infty, \gamma}}^2 + 2 \left\| G^\mu_j \left( \bar{M}(s) \right) - G^\mu_j \left( \bar{M}(s) \right) \right\|_{L_{\infty, \gamma}}^2.$$
By the Theorem 6.1, \( \tilde{M}_n \rightarrow \tilde{M} \) in \( C([0,T]; \mathbb{X}^\beta) \) \( \mathbb{P} \)-a.s., therefore
\[
\lim_{n \to \infty} \left\| G_{j^*} \left( \tilde{M}_n(s) \right) - G_{j^*} \left( \tilde{M}(s) \right) \right\|_{L^2} = 0, \quad \mathbb{P} - a.s.
\]
and (iv) follows by the uniform integrability.

After the above preparation, now we can finish the Step 2 by the following Lemma.

**Lemma 7.3.** For each \( t \in [0,T] \), we have
\[
\bar{\xi}(t) = \sum_{j=1}^{\infty} \int_0^t G_{j^*} \left( \tilde{M} \right) \ d\tilde{W}_j(s),
\]
in \( L^2(\tilde{\Omega}; \mathbb{X}^\beta) \).

**Proof.** Firstly, we show that
\[
\bar{\xi}_n(t) = \sum_{j=1}^{\infty} \int_0^t G_{j^*} \left( \tilde{M}_n(s) \right) \ d\tilde{W}_j(s)
\]
\( \mathbb{P} \) almost surely for each \( t \in [0,T] \) and \( n \in \mathbb{N} \).

Let us fix that \( t \in [0,T] \) and \( n \in \mathbb{N} \). For each \( m \in \mathbb{N} \) we define the partition \( \{s^m_i := \frac{i}{m}, i = 0, \ldots, m\} \) of \([0,T]\). By Theorem 6.1 and Proposition 6.5, \( (\tilde{M}_n, \tilde{B}_n, \tilde{E}_n \tilde{W}_n) \) and \( (M_n, B_n, E_n, W_n) \) have the same distribution on \( C^1([0,T]; \mathbb{H}_n) \times C^1([0,T]; \mathbb{X}_n) \times C^1([0,T]; \mathbb{Y}_n) \times C^1([0,T]; \mathbb{E}) \), so for each \( m \), the \( \mathbb{H} \)-valued random variables:
\[
\tilde{\xi}_n(t) = \sum_{j=1}^{m-1} \sum_{i=0}^{m-1} G_{j^*} \left( M_n(s^m_i) \right) \left( W_j(t \wedge s^m_i) - W_j(t \wedge s^m_{i+1}) \right)
\]
and
\[
\tilde{\xi}_n(t) = \sum_{j=1}^{m-1} \sum_{i=0}^{m-1} G_{j^*} \left( \tilde{M}_n(s^m_i) \right) \left( \tilde{W}_j(t \wedge s^m_i) - \tilde{W}_j(t \wedge s^m_{i+1}) \right)
\]
have the same distribution. For each \( j \), we have
\[
\lim_{m \to \infty} \mathbb{E} \left\| \sum_{j=1}^{m-1} \int_0^t G_{j^*} \left( M_n(s^m_i) \right) \left( W_j(t \wedge s^m_i) - W_j(t \wedge s^m_{i+1}) \right) \ dW_j(s) \right\|_{L^2}^2 = 0
\]
and
\[
\lim_{m \to \infty} \mathbb{E} \left\| \sum_{j=1}^{m-1} \int_0^t G_{j^*} \left( \tilde{M}_n(s^m_i) \right) \left( \tilde{W}_j(t \wedge s^m_i) - \tilde{W}_j(t \wedge s^m_{i+1}) \right) \ d\tilde{W}_j(s) \right\|_{L^2}^2 = 0,
\]
so
\[
\int_0^t G_{j^*} \left( M_n(s) \right) \ dW_j(s) \quad \text{and} \quad \int_0^t G_{j^*} \left( \tilde{M}_n(s) \right) \ d\tilde{W}_j(s)
\]
have the same distribution. Hence
\[
\sum_{j=1}^{\infty} \int_0^t G_{j^*} \left( M_n(s) \right) \ dW_j(s) \quad \text{and} \quad \sum_{j=1}^{\infty} \int_0^t G_{j^*} \left( \tilde{M}_n(s) \right) \ d\tilde{W}_j(s)
\]
have the same distribution. Therefore
\[
\bar{\xi}_n(t) - \sum_{j=1}^{\infty} \int_0^t G_{j^*} \left( M_n(s) \right) \ dW_j(s)
\]
and
\[
\bar{\xi}_n(t) - \sum_{j=1}^{\infty} \int_0^t G_{j^*} \left( \tilde{M}_n(s) \right) \ d\tilde{W}_j(s)
\]
have the same distribution. But
\[ \xi_n(t) = \sum_{j=1}^{\infty} \int_0^t G_{j,m}(M_n(s)) \, dW_j(s), \quad \mathbb{P} - a.s. \]
and thereby
\[ \tilde{\xi}_n(t) = \sum_{j=1}^{\infty} \int_0^t G_j(\tilde{M}(s)) \, d\tilde{W}_j(s), \quad \mathbb{P} - a.s. \]
We will show that \( \tilde{\xi}_n(t) \) converges in \( L^2(\tilde{\Omega}; \mathbb{X}^{-b}) \) to
\[ \sum_{j=1}^{\infty} \int_0^t G_j^0(\tilde{M}(s)) \, d\tilde{W}_j(s) \]
as \( n \to \infty \). Indeed, using notation (7.4), and (7.5) we obtain for a certain \( C > 0 \)
\[
C \mathbb{E} \left\| \tilde{\xi}_n(t) - \sum_{j=1}^{\infty} \int_0^t G_j^0(\tilde{M}(s)) \, d\tilde{W}_j(s) \right\|^2_{\mathbb{X}^{-b}} \\
= \mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^t (\sigma_{j,m}(s) - \sigma_{j,m}(s)) \, d\tilde{W}_j(s) \right\|^2_{\mathbb{X}^{-b}} \\
\leq \mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^t \sigma_{j,m}(s) \, d\tilde{W}_j(s) - \sum_{j=1}^{\infty} \int_0^t \sigma_{j,m}(s) \, d\tilde{W}_j(s) \right\|^2_{\mathbb{X}^{-b}} \\
+ \mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^t \sigma_{j,m}(s) \, d\tilde{W}_j(s) \right\|^2_{\mathbb{X}^{-b}} \\
+ \mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^t \sigma_{j,m}(s) \, d\tilde{W}_j(s) \right\|^2_{\mathbb{X}^{-b}} + \mathbb{E} \left\| \sum_{j=1}^{\infty} \int_0^t \sigma_{j,m}(s) \, d\tilde{W}_j(s) \right\|^2_{\mathbb{X}^{-b}} ,
\]
and invoking Lemma 7.2, we conclude the proof.

\[ \square \]

**Corollary 7.4.**
\[ \tilde{M}(t) = M_0 + \int_0^t \left\{ \Lambda_1 \tilde{M} \times \tilde{\rho} - \Lambda_2 \tilde{M} \times \tilde{\rho} \right\} \right\} ds \\
+ \sum_{j=1}^{\infty} \int_0^t G_j^0(\tilde{M}(s)) \, d\tilde{W}_j(s), \]
in \( L^2(\tilde{\Omega}; \mathbb{X}^{-b}) \).

**Proof.** The corollary follows immediately from Lemma 7.1 and Lemma 7.3.

\[ \square \]

8. **CONTINUE TO PROVE THE EXISTENCE OF THE WEAK SOLUTION AND FURTHER REGULARITIES**

In order to prove the main Theorem 2.9 in a series of lemmas. We start with the proof of the constraint condition of \( \tilde{M} \), i.e. condition (iii) of the main Theorem 2.9.

**Lemma 8.1.** Let \( \tilde{M} \) be a process defined in Theorem 6.1. Then for each \( t \in [0, T] \), we have \( \mathbb{P} \)-almost surely
\[ |\tilde{M}(t,x)| = 1, \quad \text{for a.e. } x \in \mathcal{D}. \]

**Proof.** We will use a version of the Itô formula proved in Pardoux’s paper [29], see 9.8. Let \( \eta \in C_0^\infty(D, \mathbb{R}) \) and let \( \gamma \) denote a function
\[ \gamma : H \ni M \mapsto \langle M, \eta M \rangle_H \in \mathbb{R}. \]
Then $\gamma \in C^2(\mathbb{R})$, $\gamma'(M) = 2\eta M$ and $\gamma''(M)(v) = 2\eta v$ for $M, v \in \mathbb{R}$.

In view of definition of the problem and (6.4), (6.5) and (6.8), all the assumptions 9.8 are satisfied. Therefore, Lemma 9.8 yields for $t \in [0, T]$ $\tilde{P}$-a.s.

$$
\left< \tilde{M}(t), \eta \tilde{M}(t) \right>_{H} - \left< M_0, \eta M_0 \right>_{H} = \int_{0}^{T} \left[ \lambda_1 \tilde{M} \times \tilde{\rho} - \lambda_2 \tilde{M} \times (\tilde{M} \times \tilde{\rho}) + \frac{1}{2} \sum_{j=1}^{\infty} \left( G_{ij}^{\rho} \tilde{M} \right) \left( G_{ij}^{\rho} \tilde{M} \right) \right] ds
$$

$$+ \sum_{j=1}^{\infty} \int_{0}^{T} \left< 2\eta \tilde{M}(s), G_{ij}^{\rho} (\tilde{M}) \right>_{H} d\tilde{W}_i(s) + \sum_{j=1}^{\infty} \int_{0}^{T} \left< \eta G_{ij}^{\rho} (\tilde{M}), G_{ij}^{\rho} (\tilde{M}) \right>_{H} ds = 0.
$$

Hence we have

$$
\left< \eta, (\tilde{M}(t))^2 - |M_0|^2 \right>_{L^2(D; \mathbb{R})} = \left< \tilde{M}(t), \eta \tilde{M}(t) \right>_{H} - \left< M_0, \eta M_0 \right>_{H} = 0.
$$

Since $\eta$ is arbitrary and $|M_0(x)| = 1$ for almost every $x \in D$, we infer that $|\tilde{M}(t, x)| = 1$ for almost every $x \in D$ as well.

Finally we are ready to prove the main result of the paper. Note that, if $|\tilde{M}(t, x)| = 1$ then $\psi(\tilde{M}(t, x)) = 1$, so we can get rid of it, which means that now we have the following equalities:

$$
G^{\rho}_{ij} (\tilde{M}(t, x)) = G_{ij} (\tilde{M}(t, x)), \quad \left( G^{\rho}_{ij} (\tilde{M}(t, x)) \right) \left( G^{\rho}_{ij} (\tilde{M}(t, x)) \right) = G_{ij} (\tilde{M}(t, x)) \left( G_{ij} (\tilde{M}(t, x)) \right) .
$$

**Lemma 8.2.** The process $(\tilde{M}, \tilde{E}, \tilde{B})$ is a weak martingale solution of Problem 2.6, that is, $(\tilde{M}, \tilde{E}, \tilde{B})$ satisfies (2.14), (2.19) and (2.20).

**Proof of (2.14).** By Lemma 7.3 and Lemma 8.1, we have $\psi(\tilde{M}(t)) \equiv 1$ for $t \in [0, T]$. Hence we deduce that for $t \in [0, T]$, the following equation holds in $L^2(\Omega; \mathbb{R}^d)$.

$$
\tilde{M}(t) = M_0 + \int_{0}^{T} \left[ \lambda_1 \tilde{M} \times \tilde{\rho} - \lambda_2 \tilde{M} \times (\tilde{M} \times \tilde{\rho}) + \frac{1}{2} \sum_{j=1}^{\infty} \left( G_{ij} (\tilde{M}) \right) \left( G_{ij} (\tilde{M}) \right) \right] ds
$$

$$+ \sum_{j=1}^{\infty} \int_{0}^{T} G_{ij} (\tilde{M}) d\tilde{W}_j(s)
$$

$$= M_0 + \int_{0}^{T} \left[ \lambda_1 \tilde{M} \times \tilde{\rho} - \lambda_2 \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \right] ds + \sum_{j=1}^{\infty} \left\{ \int_{0}^{T} G_{ij} (\tilde{M}) d\tilde{W}_j(s) \right\}.
$$

Then (2.14) follows from our explanation of $\tilde{M} \times \tilde{\rho}$, see Theorem 6.19.

**Proof of (2.19).** By Theorem 6.1 and the equation (3.12), we have

$$
\tilde{B}_n(t) - \tilde{B}_n(0) = - \int_{0}^{T} \pi_n^{\nu} [\nabla \times \tilde{E}_n(s)] ds, \quad \tilde{B}_n \sim \tilde{P} - \text{a.s.}
$$

We also have

(a) $\tilde{E}_n \rightarrow \tilde{E}$ in $L^2(0, T; L^2(\mathbb{R}^d))$ $\tilde{P}$ almost surely, and

(b) $\tilde{B}_n \rightarrow \tilde{B}$ in $L^2(0, T; L^2(\mathbb{R}^d))$ $\tilde{P}$ almost surely.

Hence for any $u \in H^1(0, T; \mathbb{R}^d)$,

$$
\int_{0}^{T} \left< \tilde{B}_n(s), \frac{du(s)}{ds} \right>_{L^2(\mathbb{R}^d)} ds = \lim_{n \rightarrow \infty} \int_{0}^{T} \tilde{B}_n(s), \frac{du(s)}{ds} \right>_{L^2(\mathbb{R}^d)} ds
$$

$$= - \lim_{n \rightarrow \infty} \int_{0}^{T} \frac{d\tilde{B}_n(s)}{ds} u(s) ds,
$$

$$= \lim_{n \rightarrow \infty} \int_{0}^{T} \nabla \times \tilde{E}_n(s), \pi_n^{\nu} u(s) \right>_{L^2(\mathbb{R}^d)} ds = \lim_{n \rightarrow \infty} \int_{0}^{T} \tilde{E}_n(s), \nabla \times \pi_n^{\nu} u(s) \right>_{L^2(\mathbb{R}^d)} ds.
Since
\[
\lim_{n \to \infty} \int_0^T \left( \| \nu_{\Pi} - u(s) \|_{L^2(\mathbb{R}^3)}^2 \right) \, ds = 0,
\]
we have
\[
\lim_{n \to \infty} \int_0^T \left( \| \nu_{\Pi} - \bar{u}(s) \|_{L^2(\mathbb{R}^3)}^2 \right) \, ds = 0.
\]
Therefore
\[
\int_0^T \left( \| \nu_{\Pi} - \bar{u}(s) \|_{L^2(\mathbb{R}^3)}^2 \right) \, ds = 0,
\]
for all \( u \in H^1(0, T; \mathbb{Y}) \).

Hence for \( t \in [0, T] \),
\[
\bar{B}(t) = B_0 - \int_0^t \nabla \times \bar{E}(s) \, ds, \quad \in \mathbb{Y}, \quad \mathbb{P} - a.s.
\]

\[\square\]

**Proof of (2.20).** Similar as in the proof of (2.19). Let \( p = q = 2 \) in Theorem 6.1, we have
(a) \( \bar{M} \to \bar{M} \) in \( L^2(0, T; \mathbb{H}) \), \( \mathbb{P} \) almost surely,
(b) \( \bar{E} \to \bar{E} \) in \( L^2(0, T; L^2(\mathbb{R}^3)) \), \( \mathbb{P} \) almost surely, and
(c) \( \bar{B} \to \bar{B} \) in \( L^2(0, T; L^2(\mathbb{R}^3)) \), \( \mathbb{P} \) almost surely.

Hence by (3.12) we have for all \( u \in H^1(0, T; \mathbb{Y}) \),
\[
\int_0^T \left( \| \nu_{\Pi} - \bar{u}(s) \|_{L^2(\mathbb{R}^3)}^2 \right) \, ds = \lim_{n \to \infty} \int_0^T \left( \| \nu_{\Pi} - \bar{u}(s) \|_{L^2(\mathbb{R}^3)}^2 \right) \, ds
\]
\[= - \lim_{n \to \infty} \int_0^T \left( \| \nu_{\Pi} - \bar{u}(s) \|_{L^2(\mathbb{R}^3)}^2 \right) \, ds
\]
\[= \int_0^T \left( \| \nu_{\Pi} - \bar{u}(s) \|_{L^2(\mathbb{R}^3)}^2 \right) \, ds.
\]

Hence for \( t \in [0, T] \),
\[
\bar{E}(t) = E_0 + \int_0^t \nabla \times \left( \bar{B}(s) - \bar{M}(s) \right) \, ds - \int_0^t \left( \| \nu_{\Pi} - \bar{u}(s) \|_{L^2(\mathbb{R}^3)}^2 \right) \, ds, \quad \in \mathbb{Y}, \quad \mathbb{P} - a.s.
\]

\[\square\]

Next we will show some regularity of \( \bar{M} \).

**Lemma 8.3.** For \( t \in [0, T] \) the following equation holds in \( L^2(\mathbb{R}; \mathbb{H}) \).
\[
\bar{M}(t) = M_0 + \int_0^t \left\{ \lambda_1 \bar{M} \times \bar{p} - \lambda_2 \left[ \bar{M} \times (\bar{M} \times \bar{p}) \right] + \sum_{j=1}^m G_j' \left( \bar{M} \right) \right\} \, ds + \int_0^t \sum_{j=1}^m G_j \left( \bar{M} \right) \, d\bar{W}_j(s)
\]
\[= M_0 + \int_0^t \left\{ \lambda_1 \bar{M} \times \bar{p} - \lambda_2 \bar{M} \times (\bar{M} \times \bar{p}) \right\} \, ds + \int_0^t \sum_{j=1}^m G_j \left( \bar{M} \right) \, d\bar{W}_j(s).
\]
Proof. We will only show the following two terms of (8.3) are in $L^2(\widetilde{\Omega}; \mathbb{H})$, the other terms can be dealt similarly. Firstly, we consider the term $\int_0^t \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \, ds$. Making use of Jensen’s inequality, (8.1) and Remark 6.21, we have

\[
\left\| \int_0^t \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \, ds \right\|_{L^2(\widetilde{\Omega}; \mathbb{H})}^2 = \mathbb{E} \left\| \int_0^t \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \, ds \right\|_{L^2}^2 \\
\leq C \mathbb{E} \int_0^t \int_D |\tilde{M} \times (\tilde{M} \times \tilde{\rho})|^2 \, dx \, ds \leq C \mathbb{E}\|\tilde{M} \times \tilde{\rho}\|^2_{L^2(0,T; \mathbb{H})} < \infty.
\]

So for all $t \in [0, T]$, $\int_0^t \tilde{M} \times (\tilde{M} \times \tilde{\rho}) \, ds \in L^2(\tilde{\Omega}; \mathbb{H})$.

Secondly, we consider the term $\sum_{j=1}^\infty \int_0^t \tilde{M} \times (\tilde{M} \times h_j) \, dW_j$. Making use of Burkholder-Davis-Gundy inequality, Jensen’s inequality, (8.1) and our assumption on $h_j$, we have

\[
\left\| \sum_{j=1}^\infty \int_0^t \tilde{M} \times (\tilde{M} \times h_j) \, dW_j \right\|_{L^2(\widetilde{\Omega}; \mathbb{H})} \\
\leq \sum_{j=1}^\infty \left\| \int_0^t \tilde{M} \times (\tilde{M} \times h_j) \, dW_j \right\|_{L^2(\widetilde{\Omega}; \mathbb{H})} \\
= \sum_{j=1}^\infty \left( \mathbb{E} \int_0^t \left| \int_D \tilde{M} \times (\tilde{M} \times h_j) \, dW_j \right|^2 \, dx \right)^{\frac{1}{2}} \\
\leq C \sum_{j=1}^\infty \left( \mathbb{E} \int_0^t \left| \int_D \tilde{M} \times (\tilde{M} \times h_j) \right|^2 \, dx \, ds \right)^{\frac{1}{2}} \\
\leq C \sum_{j=1}^\infty \|h_j\|_{\mathbb{H}} < \infty.
\]

So for all $t \in [0, T]$, $\sum_{j=1}^\infty \int_0^t \tilde{M} \times (\tilde{M} \times h_j) \, dW_j \in L^2(\tilde{\Omega}; \mathbb{H})$. The proof is complete. \hfill \square

Lemma 8.4. The process $\tilde{M}$ introduced in Theorem 6.1 satisfies: For $\theta \in [0, \frac{1}{2})$,

$$\tilde{M} \in C^\theta(0,T; \mathbb{H}), \quad \tilde{\rho} - a.s..$$

Proof. By Lemma 8.3, we have

$$\tilde{M}(t) - \tilde{M}(s) = \int_s^t \left\{ A_1 \tilde{M} \times \tilde{\rho} - A_2 \tilde{M} \times (\tilde{M} \times \tilde{\rho}) + \frac{1}{2} \sum_{j=1}^\infty G_j(\tilde{M}) \left[ G_j(\tilde{M}) \right] \right\} \, dr \\
+ \sum_{j=1}^\infty \int_s^t G_j(\tilde{M}) \, d\tilde{W}_j(\tau) \\
= \int_s^t F(s) \, ds + \sum_{j=1}^\infty \int_s^t G_j(\tilde{M}) \, d\tilde{W}_j(\tau)$$

for $0 \leq s < t \leq T$. By the constraint (8.1), the estimate (6.11) and the regularity of $\tilde{M} \times \tilde{\rho}$ as in (6.20), we have

$$\int_0^T \|F(t)\|_{\mathbb{H}}^2 \, dr < \infty, \quad \tilde{\rho} - a.s.,$$

hence the process

$$V(t) = \int_0^t F(s) \, ds \in \mathbb{H}$$

has trajectories in $C^{1/2}([0,T]; \mathbb{H})$. Putting

$$N(t) = \sum_{j=1}^\infty \int_0^t G_j(\tilde{M}) \, d\tilde{W}_j(\tau)$$

and invoking the Burkholder-Davis-Gundy inequalities we obtain for any $p \geq 2$

$$\mathbb{E}\|N(t) - N(s)\|_{\mathbb{H}}^p \leq C_p \mathbb{E} \left( \sum_{j=1}^\infty \int_s^t \|G_j(\tilde{M})\|_{\mathbb{H}}^p \, dr \right)^{\frac{p}{2}} \leq C \left( \sum_{j=1}^\infty \|h_j\|_{L^p}^p \right) (t-s)^p.$$
Then the Kolmogorov continuity test, see Lemma 9.9, yields
\[ N \in C^θ([0, T]; \mathbb{H}), \quad θ \in \left(0, \frac{1}{2}\right). \]
since \( \tilde{M}(t) = V(t) + N(t) \), the lemma follows.  

\[ \square \]

9. Appendix

Lemma 9.1. [19] Let \( B_0 \subset B \subset B_1 \) be Banach spaces, \( B_0 \) and \( B_1 \) being reflexive and the embedding \( B_0 \hookrightarrow B \) to be compact. Let \( p \in (1, \infty) \) and \( α \in (0, 1) \) be given. Then the embedding
\[ L^p(0, T; B_0) \cap W^{α,p}(0, T; B_1) \hookrightarrow L^p(0, T; B) \]
is compact.

Lemma 9.2. [19] Let \( p \geq 2 \) and \( α \in [0, \frac{1}{2}) \) be given. There exists a constant \( C(p, α) > 0 \) such that for any progressively measurable process \( ξ = \sum_{j=1}^∞ \xi_j \in L^p(\Omega \times [0, T]; \mathbb{H}) \) with \( \sum_{j=1}^∞ \|ξ_j\|_{\mathbb{H}}^p < \infty \), we have
\[ \mathbb{E} \left[ \sum_{j=1}^∞ \left( \int_0^T \xi_j(t) dW_j(t) \right)^p \right] \leq C(p, α) \mathbb{E} \left[ \sum_{j=1}^∞ \|ξ_j\|_{\mathbb{H}}^p \right] \]
for any \( α \geq 1 \).

Lemma 9.3. [19] Assume that \( B_1 \subset B_2 \) be two Banach spaces with compact embedding, and \( α \in (0, 1) \), \( p > 1 \) satisfying \( α > \frac{1}{p} \). Then the space \( W^{α,p}(0, T; B_1) \) is compactly embedded into \( C([0, T]; B_2) \).

Definition 9.4 (Aldous condition). Let \( (Ω, \mathcal{F}, P) \) be a probability space with a filtration \( \mathbb{F} \). Let \( (S, ρ) \) be a separable metric space with the metric \( ρ \). We say that \( \{X_t(\omega)\}, t \in [0, T], \) of \( S \)-valued processes satisfies the Aldous condition iff \( \forall ε > 0, \forall η > 0, \exists θ > 0 \) such that for every sequence \( \{τ_n\} \) of \( P \)-stopping times with \( τ_n \leq T \) a.s. one has:
\[ \sup_{n∈\mathbb{N}} \sup_{0\leq ε≤θ} \mathbb{P}[ρ(X_{τ_n}(\omega) + ε, X_n(\omega)) \geq η] ≤ ε. \]
We will also need the following Tightness Criterion.

Lemma 9.5 (Tightness Criterion). ([12], Cor 3.10) Let \( (Ω, \mathcal{F}, P) \) be a probability space with the filtration \( \mathbb{F} \). Let \( H \) be a separable Hilbert space, \( U \) be another Hilbert space such that the embedding \( U \hookrightarrow H \) is compact and dense, \( U^∗ \) be the dual space of \( U \). Let \( \{X_t(\omega)\}_{t∈[0,T]} \), \( t \in [0, T] \) be a sequence of continuous \( \mathbb{F} \)-adapted \( U^∗ \)-valued process such that
\begin{enumerate}[(a)]  
  \item there exists a positive constant \( C \) such that
  \[ \sup_{n∈\mathbb{N}} \mathbb{E} \sup_{0≤t≤T} \|X_n(t)\|_H \leq C. \]
\end{enumerate}
\begin{enumerate}[(b)]  
  \item \( \{X_n\}_{n∈\mathbb{N}} \) satisfies the Aldous condition (9.1) in \( U^∗ \).
\end{enumerate}

Then the laws of \( X_n \) on \( C([0, T]; U^∗) \cap L^2_2([0, T]; H) \) are tight.

Theorem 9.6 (Kuratowski Theorem). ([30]) Let \( X_1, X_2 \) be Polish spaces equipped with their Borel σ-field \( \mathcal{B}(X_1), \mathcal{B}(X_2) \), and \( ϕ : X_1 \hookrightarrow X_2 \) be a one to one Borel measurable map, then for any \( E \in \mathcal{B}(X_1), \varnothing(E) \in \mathcal{B}(X_2). \)

Lemma 9.7 ([31], Page 66, Thm 3.12). Suppose \( E \) is a convex subset of a locally convex space \( X \). Then the weak closure \( E^w \) of \( E \) in \( X \) is equal to its original closure \( E \).

Lemma 9.8. [29] (Th. 1.2) Let \( V \) and \( H \) be two separable Hilbert spaces, such that \( V \hookrightarrow H \) continuously and densely. We identify \( H \) with its dual space. And let \( M^2(0, T; \mathbb{E}) \) denote the space of \( H \)-valued measurable process with the filtered probability space \( (Ω, (\mathcal{F}_t)_{t∈[0,T]}, P) \) which satisfy:
\[ ϕ \in M^2(0, T; \mathbb{E}) \text{ if and only if} \]
\begin{enumerate}[(i)]  
  \item \( ϕ(t) \) is \( \mathcal{F}_t \)-measurable for almost every \( t \);
  \item \( \mathbb{E} \int_0^T |ϕ(t)|^2 dt < ∞. \)
\end{enumerate}
We suppose that
\[ u \in M^2(0, T; V), \quad u_0 \in \mathbb{H}, \quad v \in M^2(0, T; V'), \]
with
\[ \mathbb{E} \int_0^T \sum_{j=1}^{\infty} |z_j(t)|^2 \, dt < \infty, \]
\[
\gamma \left( u(t) \right) = u(t) + \int_0^t v(s) \, ds + \sum_{j=1}^{\infty} \int_0^t z_j(s) \, dW_j(s).
\]

Let \( \gamma \) be a twice differentiable functional on \( H \), which satisfies:

(i) \( \gamma, \gamma' \) and \( \gamma'' \) are locally bounded.
(ii) \( \gamma \) and \( \gamma' \) are continuous on \( H \).
(iii) Let \( \mathcal{L}^1(H) \) be the Banach space of all the trace class operators on \( H \). Then \( \forall Q \in \mathcal{L}^1(H) \), \( \text{Tr}(Q \circ \gamma'') \) is a continuous functional on \( H \).
(iv) If \( u \in V \), \( \gamma'(u) \in V \); \( u \mapsto \gamma'(u) \) is continuous from \( V \) (with the strong topology) into \( V \) endowed with the weak topology.
(v) \( \exists k \) such that \( \|\gamma'(u)|_V \leq k(1 + \|u\|_V) \), \( \forall u \in V \).

Then \( \mathbb{P} \) almost surely,
\[
\gamma(u(t)) = \gamma(u_0) + \int_0^t \langle v(s), \gamma'(u(s)) \rangle_V \, ds + \sum_{j=1}^{\infty} \int_0^t H \left( \gamma'(u(s)), z_j(s) \right)_H \, dW_j(s) + \frac{1}{2} \sum_{j=1}^{\infty} \int_0^t H \left( \gamma''(u(s)) z_j(s), z_j(s) \right)_H \, ds.
\]

**Lemma 9.9** (Kolmogorov continuity). Let \( \{u(t)\}_{t \in [0, T]} \) be a stochastic process with values in a separable Banach space \( \mathcal{X} \), such that for some \( C > 0, \epsilon > 0, \delta > 1 \) and all \( t, s \in [0, T] \),
\[
\mathbb{E} \|u(t) - u(s)\|_\mathcal{X}^\epsilon \leq C |t - s|^{1+\epsilon}.
\]

Then there exists a version of \( u \) with \( \mathbb{P} \) almost surely trajectories being Hölder continuous functions with an arbitrary exponent smaller than \( \frac{\epsilon}{\delta} \).

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