Maximum Quadratic Assignment Problem:  
Reduction from Maximum Label Cover and  
LP-based Approximation Algorithm

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Abstract

We show that for every positive $\varepsilon > 0$, unless $NP \subset BPQP$, it is impossible to approximate the maximum quadratic assignment problem within a factor better than $2^{\log^{1-\varepsilon} n}$ by a reduction from the maximum label cover problem. Our result also implies that Approximate Graph Isomorphism is not robust and is in fact, $1-\varepsilon$ vs $\varepsilon$ hard assuming the Unique Games Conjecture.

Then, we present an $O(\sqrt{n})$-approximation algorithm for the problem based on rounding of the linear programming relaxation often used in the state of the art exact algorithms.

1 Introduction

In this paper we consider the Quadratic Assignment Problem. An instance of the problem, $\Gamma = (G, H)$ is specified by two weighted graphs $G = (V_G, w_G)$ and $H = (V_H, w_H)$ such that $|V_G| = |V_H|$ (we denote $n = |V_G|$). The set of feasible solutions consists of bijections from $V_G$ to $V_H$. For a given bijection $\varphi$ the objective function is

$$\text{value}_{QAP}(\Gamma, \varphi) = \sum_{(u,v) \in V_G \times V_G} w_G(u,v)w_H(\varphi(u), \varphi(v)).$$  (1)

There are two variants of the problem the Minimum Quadratic Assignment Problem and the Maximum Quadratic Assignment Problem ($\text{MaxQAP}$) depending on whether the objective function $\text{(1)}$ is to be minimized or maximized. The problem was first defined by Koopmans and Beckman [26] and sometimes this formulation of the problem is referred to as Koopmans-Beckman formulation of the Quadratic Assignment Problem. Both variants of the problem model an astonishingly large number of combinatorial optimization problems such as traveling salesman, maximum acyclic subgraph, densest subgraph and clustering problems to name a few. It also generalizes many practical problems that arise in various areas such as modeling of backboard wiring [35], campus and hospital layout [17] [19], scheduling [23] and many others [18] [27]. The surveys and books [2] [11] [12] [28] [31] contain an in-depth treatment of special cases and various applications of the Quadratic Assignment Problem.

The Quadratic Assignment Problem is an extremely difficult optimization problem. The state of the art exact algorithms can solve instances with approximately 30 vertices, so a lot of research effort was concentrated on constructing good heuristics and relaxations of the problem.

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Previous Results. The Minimum Quadratic Assignment Problem is known to be hard to approximate even under some very restrictive conditions on the weights of graphs $G$ and $H$. In particular, even when $H$ induces a line metric, any polynomial factor approximation (in polynomial time) implies that $P = NP$ \cite{14}. Polynomial time exact \cite{14} and approximation algorithms \cite{24} are known for very specialized instances.

In contrast, MaxQAP seem to be more tractable. Barvinok \cite{9} constructed an approximation algorithm with performance guarantee $\varepsilon n$ for any $\varepsilon > 0$. Nagarajan and Sviridenko \cite{30} designed $O(\sqrt{n} \log^2 n)$-approximation algorithm by utilizing approximation algorithms for the minimum vertex cover, densest $k$-subgraph and star packing problems. For the special case when one of the edge weight functions ($w_G$ or $w_H$) satisfy the triangle inequality there are combinatorial 4-approximation \cite{3} and LP-based 3.16-approximation algorithms \cite{30}. Another tractable special case is the so-called dense Quadratic Assignment Problem \cite{4}. This special case admits a sub-exponential time approximation scheme and in some cases it could be implemented in polynomial time \cite{4, 22}. On the negative side, APX-hardness of MaxQAP is implied by the APX-hardness of its special cases, e.g. Traveling Salesman Problem with Distances One and Two \cite{32}.

An interesting special case of MaxQAP is the Densest $k$-Subgraph Problem. The best known algorithm by Bhaskara, Charikar, Chlamtac, Feige, and Vijayaraghavan \cite{10} gives a $O(n^{1/4})$ approximation. However, the problem is not even known to be APX-hard (under standard complexity assumptions). Feige \cite{20} showed that the Densest $k$-Subgraph Problem does not admit a $\rho$-approximation (for some universal constant $\rho > 1$) assuming that random 3-SAT formulas are hard to refute. Khot \cite{25} ruled out PTAS for the problem under the assumption that $NP$ does not have randomized algorithms that run in sub-exponential time.

Our Results. Our first result is the first superconstant non-approximability for MaxQAP. We show that for every positive $\varepsilon > 0$, unless $NP \subseteq BPQP$ ($BPQP$ is the class of problems solvable in randomized quasipolynomial-time), it is impossible to approximate the maximum quadratic assignment problem with the approximation factor better than $2^{\log^{1-\varepsilon} n}$. Particularly, there is no polynomial time poly-logarithmic approximation algorithms for MaxQAP under the above complexity assumption. It is an interesting open question if our techniques can be used to obtain a similar result for the Densest $k$-Subgraph Problem.

Our second result is an $O(\sqrt{n})$-approximation algorithm based on rounding of the optimal solution of the linear programming relaxation. The LP relaxation was first considered by Adams and Johnson \cite{1} in 1994. As a consequence of our result we obtain a bound of $O(\sqrt{n})$ on the integrality gap of this relaxation that almost matches a lower bound of $\Omega(\sqrt{n} / \log n)$ of Nagarajan and Sviridenko \cite{30}. Note, that the previous $O(\sqrt{n} \log^2 n)$-approximation algorithm \cite{30} was not based on the linear programming relaxation, and therefore no non-trivial upper bound on the integrality gap of the LP was known.

Note Added in Proof. Suppose that the graphs $G$ and $H$ have the same number of edges. Then, $G$ and $H$ are isomorphic if and only if the optimal value of the unweighted Maximum Quadratic Assignment problem equals 1. This observation gives another name to the problem: The unweighted version of Maximum Quadratic Assignment is also known as Approximate Graph Isomorphism. In Approximate Graph Isomorphism, it is natural to divide the objective function by $|E_G| = |E_H|$, then for isomorphic graphs $G$ and $H$, the optimal objective value is 1. We do not know the complexity of the (exact) Graph Isomorphism problem, and hence we do not know whether finding the exact solution for satisfiable instances of Approximate Graph Isomorphism (i.e., instances of value 1) is $NP$-hard. In several recent works \cite{8, 16} (published after the conference
version of this paper appeared at ICALP 2010), the authors asked what can be done if the instance is almost satisfiable i.e., the value of the optimal solution is at least \((1 - \varepsilon)\). The immediate corollary of our result is that it is not possible to distinguish instances of value at least \((1 - \varepsilon)\) and instances of value at most \(\delta\) in randomized polynomial time for every positive \(\varepsilon\) and \(\delta\). This result holds assuming that the (randomized) Unique Games Conjecture holds. In other words, we assume that for every positive \(\varepsilon\) and \(\delta\), there is no randomized polynomial-time algorithm that distinguishes \((1 - \varepsilon)\) satisfiable instances of Unique Games and \(\delta\) satisfiable instances of Unique Games. To get the result, in the reduction we present below, we need to use an instance of \(\text{MAX } \Gamma\text{-Lin}(k)\) instead of an arbitrary instance of Label Cover, the graph \(G\) contains \(k\) copies of the constraint graph of the \(\text{MAX } \Gamma\text{-Lin}(k)\) instance, the graph \(H\) is the label-extended graph of the \(\text{MAX } \Gamma\text{-Lin}(k)\) instance.

2 Hardness of Approximation

A weighted graph \(G = (V, w)\) is specified by a vertex set \(V\) along with a weight function \(w : V \times V \rightarrow \mathbb{R}\) such that for every \(u, v \in V\), \(w(u, v) = w(v, u)\) and \(w(u, v) \geq 0\). An edge \(e = (u, v)\) is said to be present in the graph \(G\) if \(w(u, v)\) is non-zero.

We prove the inapproximability of the \(\text{MaxQAP}\) problem via an approximation preserving poly-time randomized reduction from the Label Cover problem defined below.

**Definition 2.1** (Label Cover Problem). An instance of the label cover problem denoted by \(\Upsilon = (G = (V_G, E_G), \pi, [k])\) consists of a graph \(G\) on \(V_G\) with edge set \(E_G\) along with a set of labels \([k] = \{0, 1, \ldots, k - 1\}\). For each edge \((u, v) \in E_G\), there is a constraint \(\pi_{uv}\), a subset of \([k] \times [k]\) defining the set of accepted labelings for the end points of the edge. The goal is to find a labeling of the vertices, \(\Lambda : V_G \rightarrow [k]\) maximizing the total fraction of the edge constraints satisfied. We will denote the optimum of an instance \(\Upsilon\) by \(\text{OPT}_{LC}(\Upsilon)\). In other words,

\[
\text{OPT}_{LC}(\Upsilon) \overset{\text{def}}{=} \max_{\Lambda : V_G \rightarrow [k]} \frac{1}{|E_G|} \sum_{(u,v) \in E} I((\Lambda(u), \Lambda(v)) \in \pi_{uv}),
\]

where \(I(\cdot)\) is the indicator of an event. We denote the optimum by \(\text{OPT}_{\text{QAP}}(\Gamma)\). We will denote the fraction of edges satisfied by a labeling \(\Lambda\) by \(\text{value}_{LC}(\Upsilon, \Lambda)\).

The PCP theorem [6, 7], along with the Raz parallel repetition theorem [33], shows that the label cover problem is hard to approximate within a factor of \(2^{\log^{1-\varepsilon} n}\).

**Theorem 2.2** (see e.g., Arora and Lund [5]). If \(\mathcal{NP} \not\subseteq \mathcal{QP}\), then for every positive \(\varepsilon > 0\), it is not possible to distinguish satisfiable instances of the label cover problem from instances with optimum at most \(2^{-\log^{1-\varepsilon} n}\) in polynomial time.

We will show an approximation preserving reduction from a label cover instance to a \(\text{MaxQAP}\) instance such that: If the label cover instance \(\Upsilon\) is completely satisfiable, then the \(\text{MaxQAP}\) instance \(\Gamma\) will have optimum 1; on the other hand, if \(\text{OPT}_{LC}(\Upsilon)\) is at most \(\delta\), then no bijection \(\varphi\) obtains a value greater than \(O(\delta)\).

Strictly speaking, the problem is not well defined when the graphs \(G\) and \(H\) do not have the same number of vertices. However, in our reduction, we will relax this condition by letting \(G\) have fewer vertices than \(H\), and allowing the map \(\varphi\) to be only injective (i.e., \(\varphi(u) \neq \varphi(v), \text{ for } u \neq v\)). The reason is that we can always add enough isolated vertices to \(G\) to satisfy \(|V_G| = |V_H|\). We
also assume that the graphs are unweighted, and thus given an instance $\Gamma$ consisting of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, the goal is to find an injective map $\varphi : V_G \rightarrow V_H$, so as maximize

$$\text{value}_{\text{QAP}}(\Gamma, \varphi) = \sum_{(u,v) \in E_G} I((\varphi(u), \varphi(v)) \in E_H).$$

Informally, our reduction does the following. Given an instance $\Upsilon = (G = (V_G, E_G), \pi, [k])$ of the label cover problem, consider the label extended graph $H$ on $V_G \times [k]$ with edges $((u,i) - (v,j))$ for every $(u,v) \in E_G$ and every accepting label pair $(i,j) \in \pi_{uv}$. Every labeling $\Lambda$ for $\Upsilon$ naturally defines an injective map, $\varphi$ between $V_G$ and $V_G \times [k]$: $\varphi(u) = (u, \Lambda(u))$. Note that $\varphi$ maps constraint edges in $G$ onto edges of $H$. Conversely, given an injection $\varphi : V_G \rightarrow V_G \times [k]$ such that $\varphi(u) \in \{u\} \times [k]$ for every $u \in V_G$, we can construct a labeling $\Lambda$ for $\Upsilon$ satisfying exactly the constraint edges in $G$ which were mapped on to edges of $H$. However, the requirement that $\varphi(u) = (u, \Lambda(u))$ is crucial for the converse to hold: an arbitrary injective map might not correspond to any labeling of the label cover $\Upsilon$.

To overcome the above shortcoming, we modify the graphs $G$ and $H$ as follows. We replace each vertex $u$ in $G$ with a “cloud” of vertices $\{(u,i) : i \in [N]\}$ and each vertex $(u,x)$ in $H$ with a cloud of vertices $\{(u,x,i) : i \in [N]\}$, each index $i$ is from a significantly large set $[N]$. Call the new graphs $\tilde{G}$ and $\tilde{H}$ respectively.

For every edge $(u,v) \in E_G$, the corresponding clouds in $\tilde{G}$ are connected by a random bipartite graph where each edge occurs with probability $\alpha$. We do this independently for each edge in $E_G$. For every accepting pair $(x,y) \in \pi_{uv}$, we copy the “pattern” between the clouds $(u,x,\star)$ and $(v,y,\star)$ in $\tilde{H}$.

As before, every solution of the label cover problem $u \mapsto \Lambda(u)$ corresponds to the map $(u,i) \mapsto (u,\Lambda(u),i)$ which maps every “satisfied” edge of $G$ to an edge of $H$. However, now, we may assume that every $(u,i)$ is mapped to some $(u,x,i)$, since, loosely speaking, the pattern of edges between $(u,\star)$ and $(v,\star)$ is unique for each edge $(u,v)$: there is no way to map the cloud of $u$ to the cloud of $u'$ and the cloud of $v$ to the cloud of $v'$ (unless $u = u'$ and $v = v'$), so that more than an $\alpha$ fraction of the edges of one cloud are mapped on edges of the other cloud. We will make the above discussion formal in the rest of this section.

**Hardness Reduction**

**Input:** A label cover instance $\Upsilon = (G = (V_G, E_G), \pi, [k])$.

**Output:** A MAXQAP instance $\Gamma = (\tilde{G}, \tilde{H})$; $\tilde{G} = (V_{\tilde{G}}, E_{\tilde{G}})$, $\tilde{H} = (V_{\tilde{H}}, E_{\tilde{H}})$.

**Parameters:** Let $N = [n^4|E_G|k^5]$ and $\alpha = 1/n$.

- Define $V_{\tilde{G}} = V_G \times [N]$ and $V_{\tilde{H}} = V_G \times [k] \times [N]$.
- For every edge $(u,v)$ of $G$ pick a random set of pairs $E_{uv} \subset [N] \times [N]$. Each pair $(i,j) \in [N] \times [N]$ belongs to $E_{uv}$ independently with probability $\alpha$.
- For every edge $(u,v)$ of $G$ and every pair $(i,j)$ in $E_{uv}$, add an edge $((u,i),(v,j))$ to $\tilde{G}$. Then $E_{\tilde{G}} = \{((u,i),(v,j)) : (u,v) \in E_G$ and $(i,j) \in E_{uv}\}$. 


For every edge \((u, v)\) of \(G\), every pair \((i, j)\) in \(\mathcal{E}_{uv}\), and every pair \((x, y)\) in \(\pi_{uv}\), add an edge \(((u, x, i), (v, y, j))\) to \(\tilde{H}\). Then
\[
E_{\tilde{H}} = \{((u, x, i), (v, y, j)) : (u, v) \in E_G, (i, j) \in \mathcal{E}_{uv} \text{ and } (x, y) \in \pi_{uv}\}.
\]

It is easy to see that the reduction runs in polynomial time. In our reduction, both \(k\) and \(N\) are polynomial in \(n\).

We will now show that the reduction is in fact approximation preserving with high probability. In the rest of the section, we will assume \(\Gamma = (\tilde{G}, \tilde{H})\) is a MaxQAP instance obtained from a label cover instance \(\Upsilon\) using the above reduction with parameters \(N\) and \(\alpha\). Note that \(\Gamma\) is a random variable.

We will first show that if the label cover instance has a good labeling, then the MaxQAP instance output by the above reduction has a large optimum. The following claim, which follows from a simple concentration inequality, shows that the graph \(\tilde{G}\) has, in fact, as many edges as expected.

**Claim 2.3.** With high probability, \(\tilde{G}\) contains at least \(\alpha|E_G|N^2/2\) edges.

**Lemma 2.4** (Completeness). Let \(\Upsilon\) be a satisfiable instance of the Label Cover Problem. Then there exists a map of \(\tilde{G}\) to \(H\) that maps every edge of \(\tilde{G}\) to an edge of \(H\). Thus, \(\text{OPT}_{\text{MaxQAP}}(\Gamma) = |E_{\tilde{G}}|\).

**Proof.** Let \(u \mapsto \Lambda(u)\) be the solution of the label cover that satisfies all constrains. Define the map \(\varphi : V_{\tilde{G}} \to V_{\tilde{H}}\) as follows \(\varphi(u, i) = (u, \Lambda(u), i)\). Suppose that \(((u, i), (v, j))\) is an edge in \(\tilde{G}\). Then \((u, v) \in E_G\) and \((i, j) \in \mathcal{E}_{uv}\). Since the constraint between \(u\) and \(v\) is satisfied in the instance of the label cover, \(((\Lambda(u), \Lambda(v))) \in \pi_{uv}\). Thus, \(((u, \Lambda(u), i), (v, \Lambda(v), j)) \in E_{\tilde{H}}\).

Next, we will bound the optimum of \(\Gamma\) in terms of the value of the label cover instance \(\Upsilon\). We do this in two steps. We will first show that for a fixed map \(\varphi\) from \(V_{\tilde{G}}\) to \(V_{\tilde{H}}\) the expected value of \(\Gamma\) can be bounded as a function of the optimum of \(\Upsilon\). Note that this is well defined as \(V_{\tilde{G}}\) and \(V_{\tilde{H}}\) are determined by \(\Upsilon\) and \(N\) (and independent of the randomness used by the reduction). Next, we show that the value is, in fact, tightly concentrated around the expected value. Then, we do a simple union bound over all possible \(\varphi\) to obtain the desired result. In what follows, \(\varphi\) is a fixed injective map from \(V_{\tilde{G}}\) to \(V_{\tilde{H}}\). Denote the first, second and third components of \(\varphi\) by \(\varphi_V, \varphi_{\text{label}}\) and \(\varphi_{[N]}\) respectively. Then, \(\varphi(u, i) = (\varphi_V(u, i), \varphi_{\text{label}}(u, i), \varphi_{[N]}(u, i))\).

**Lemma 2.5.** For every injective map \(\varphi : V_{\tilde{G}} \to V_{\tilde{H}},\)
\[
\mathbb{E}[\text{value}_{\text{MaxQAP}}(\Gamma, \varphi)] \leq \alpha|E_G|N^2 \times (\text{OPT}_{\text{LC}}(\Upsilon) + \alpha).
\]

**Proof.** Define a probabilistic labeling of \(G\) as follows: for every vertex \(u\), pick a random \(i \in [N]\), and assign label \(\varphi_{\text{label}}(u, i)\) to \(u\) i.e., set \(\Lambda(u) = \varphi_{\text{label}}(u, i)\). The expected value of the solution to the Label Cover problem equals
\[
\mathbb{E}_\Lambda[\text{value}_{\text{LC}}(\Upsilon, \Lambda)] = \frac{1}{|E_G|} \sum_{(u,v) \in E_G} \mathbb{E}_\Lambda[I((\Lambda(u), \Lambda(v)) \in \pi_{uv})]
= \frac{1}{|E_G|} \sum_{(u,v) \in E_G} \frac{1}{N^2} \sum_{i,j \in [N]} I((\varphi_{\text{label}}(u, i), \varphi_{\text{label}}(v, j)) \in \pi_{uv}).
\]

5
Since \(\text{value}_{\text{LC}}(\Upsilon, \Lambda) \leq \text{OPT}_{\text{LC}}(\Upsilon)\) for every labeling \(u \mapsto \Lambda(u)\),
\[
\sum_{(u,v) \in E_G} \sum_{i,j \in [N]} I((\varphi_{\text{label}}(u,i), \varphi_{\text{label}}(v,j)) \in \pi_{uv}) \leq |E_G| \cdot N^2 \cdot \text{OPT}_{\text{LC}}(\Upsilon). 
\]

(2)

On the other hand,
\[
\mathbb{E}[\text{value}_{\text{QAP}}(\Gamma, \varphi)] = \mathbb{E}\left[ \sum_{(u,i,v,j) \in E_G} I((\varphi(u,i), \varphi(v,j)) \in E^H) \right]
= \sum_{(u,v) \in E_G} \sum_{i,j \in [N]} \Pr\{(i,j) \in E_{uv} \text{ and } (\varphi(u,i), \varphi(v,j)) \in E^H\}. 
\]

(3)

Recall, that the goal of the whole construction was to force the solution to map each \((u, i)\) to \((u, \varphi_{\text{label}}(u,i), i)\). Let \(C_{\varphi}\) denote the set of quadruples that satisfy this property:
\[
C_{\varphi} = \{(u, i, v, j) : (u, v) \in E_G \text{ and } \varphi(u,i) = (u, \varphi_{\text{label}}(u,i), i), \ \varphi(v,j) = (v, \varphi_{\text{label}}(v,j), j)\}. 
\]

If \((u, i, v, j) \in C_{\varphi}\), then
\[
\Pr\{(i,j) \in E_{uv} \text{ and } (\varphi(u,i), \varphi(v,j)) \in E^H\}
= \Pr\{(i,j) \in E_{uv} \text{ and } (\varphi_{\text{label}}(u,i), \varphi_{\text{label}}(v,j)) \in \pi_{uv}\}
= \Pr\{(i,j) \in E_{uv}\} \cdot I((\varphi_{\text{label}}(u,i), \varphi_{\text{label}}(v,j)) \in \pi_{uv})
= \alpha \cdot I((\varphi_{\text{label}}(u,i), \varphi_{\text{label}}(v,j)) \in \pi_{uv}).
\]

If \((u, v) \in E_G\), but \((u, i, v, j) \notin C_{\varphi}\), then either \((i,j) \neq (\varphi_{[N]}(u,i), \varphi_{[N]}(v,j))\) or \((u, v) \neq (\varphi_{V}(u,i), \varphi_{V}(v,j))\), and hence the events \(\{(i,j) \in E_{uv}\}\) and \(\{(\varphi_{[N]}(u,i), \varphi_{[N]}(v,j)) \in E_{\varphi V}(u,i,\varphi_{V}(v,j))\}\) are independent. We have
\[
\Pr\{(i,j) \in E_{uv} \text{ and } (\varphi(u,i), \varphi(v,j)) \in E^H\} \leq \Pr\{(i,j) \in E_{uv} \text{ and } (\varphi_{[N]}(u,i), \varphi_{[N]}(v,j)) \in E_{\varphi V}(u,i,\varphi_{V}(v,j))\} \leq \alpha^2.
\]

Now, splitting summation (3) into two parts depending on whether \((u, i, v, j) \in C_{\varphi}\), we have
\[
\mathbb{E}[\text{value}_{\text{QAP}}(\Gamma, \varphi)] \leq \alpha |E_G| N^2 \text{OPT}_{\text{LC}}(\Upsilon) + \alpha^2 |E_G| N^2.
\]

We use the following concentration inequality for Lipschitz functions on the boolean cube.

**Theorem 2.6** (McDiarmid [29], Theorem 3.1, p. 206). Let \(X_1, \ldots, X_T\) be independent random variables taking values in the set \(\{0, 1\}\). Let \(f : \{0,1\}^T \rightarrow \mathbb{R}\) be a \(K\)-Lipschitz function i.e., for every \(x, y \in \{0, 1\}^T\), \(|f(x) - f(y)| \leq K \|x - y\|_1\). Finally, let \(\mu = \mathbb{E}[f(X_1, \ldots, X_T)]\). Then for every positive \(\varepsilon\),
\[
\Pr\{f(X_1, \ldots, X_n) - \mu \geq \varepsilon\} \leq e^{-\frac{2\varepsilon^2}{K^2n}}.
\]

**Lemma 2.7.** For every injective map \(\varphi : V_G \rightarrow V_H\),
\[
\Pr\{\text{value}_{\text{QAP}}(\Gamma, \varphi) - \mathbb{E}[\text{value}_{\text{QAP}}(\Gamma, \varphi)] \geq \alpha N^2\} \leq e^{-n^2Nk}.
\]


Proof. The presence of edges in the random graphs \( \tilde{G} \) and \( \tilde{H} \) is determined by the random sets \( E_{uv} \) (where \((u, v) \in E_G\)). Thus, we can think of the random variable \( \text{value}_{QAP}(\Gamma, \varphi) \) as a function of the indicator variables \( X_{uivj} \), where \( X_{uivj} \) equals 1, if \((i, j) \in E_{uv}\); and 0, otherwise. To be precise, \( \text{value}_{QAP}(\Gamma, \varphi) \) equals

\[
\sum_{(u,v) \in E_G} X_{uivj} \varphi_{v}(u,i) \varphi_{v}(v,j) \varphi_{v}(v,j) \varphi_{v}(v,j) I((\varphi_{\text{label}}(u,i), \varphi_{\text{label}}(v,j)) = \pi_{\text{label}}(u,i) \varphi_{v}(v,j), (v,j)).
\]

Observe, that variables \( X_{uivj} \) are mutually independent (we identify \( X_{uivj} \) with \( X_{uivj} \)). Each \( X_{uivj} = 1 \) with probability \( \alpha \). Finally, \( \text{value}_{QAP}(\Gamma, \varphi) \) is \((k^2 + 1)\)-Lipschitz as a function of the variables \( X_{uivj} \). That is, if we change one of the variables \( X_{uivj} \) from 0 to 1, or from 1 to 0, then the value of the function may change by at most \( k^2 + 1 \). This follows from the expression above, since for every fixed \( \varphi \), each \( X_{uivj} \) may appear in at most \( k^2 + 1 \) terms (reason: there is one term \( X_{uivj} \varphi_{v}(u,i) \varphi_{v}(v,j) \varphi_{v}(v,j) \varphi_{v}(v,j) \) and at most \( k^2 \) terms \( X_{uivj} \varphi_{v}(u,i) \varphi_{v}(v,j) \varphi_{v}(v,j) \varphi_{v}(v,j) \), such that \( \varphi(u', i') = (u, x, i) \) and \( \varphi(v', j') = (v, y, j) \) for some \( x, y \in [k] \), since \( \varphi \) is an injective map). McDiarmid’s inequality with \( T = N^2 \cdot |E_G|, K = (k^2 + 1), \) and \( \varepsilon = \alpha N^2 \), implies the statement of the lemma.

Corollary 2.8 (Soundness). With high probability, the reduction outputs an instance \( \Gamma \) such that

\[
\text{OPT}_{QAP}(\Gamma) \leq \alpha |E_G| N^2 \times (\text{OPT}_{LC}(\Upsilon) + 2\alpha)
\]

Remark 2.9. It is instructive to think, that \( 2\alpha \ll \text{OPT}_{LC}(\Upsilon) \).

Proof. The total number of maps from \( V_G \) to \( V_H \) is \((nNk)^nN\). Thus, by the union bound, with probability \( 1 - o(1) \), for every injective mapping \( \varphi : V_G \rightarrow V_H \):

\[
\text{value}_{QAP}(\Gamma, \varphi) - \mathbb{E}[\text{value}_{QAP}(\Gamma, \varphi)] \leq \alpha N^2.
\]

Plugging in the bound for the expected value from Lemma 2.5 gives

\[
\text{OPT}_{QAP}(\Gamma) \leq \alpha |E_G| N^2 \text{OPT}_{LC}(\Upsilon) + \alpha^2 |E_G| N^2 + \alpha N^2.
\]

Theorem 2.10. For every positive \( \varepsilon > 0 \), there is no polynomial time approximation algorithm for the Maximum Quadratic Assignment problem with the approximation factor less than \( D = 2^{\log^{1-\varepsilon} n} \) (where \( n \) is the number of vertices in the graph) unless \( \mathcal{NP} \subset \mathcal{BPQP} \).

Proof. Assume to the contrary that there exists a polynomial time algorithm \( A \) with the approximation factor less than \( D = 2^{\log^{1-\varepsilon} n} \) for some positive \( \varepsilon \). We use this algorithm to distinguish satisfiable instances of the label cover from at most \( 1/(4D) \)-satisfiable instances in randomized polynomial time, which is not possible (if \( \mathcal{NP} \not\subset \mathcal{BPQP} \)) according to Theorem 2.2.

Let \( \Upsilon \) be an instance of the label cover. Using the reduction described above transform \( \Upsilon \) to an instance of \( \text{MAXQAP} \ \Gamma \). Run the algorithm \( A \) on \( \Gamma \). Accept \( \Upsilon \), if the value \( A(\Gamma) \) returned by the algorithm is at least \( |E_G|/D \). Reject \( \Upsilon \), otherwise. By Lemma 2.4 if \( \Upsilon \) is satisfiable, then \( \text{OPT}_{QAP}(\Gamma) = |E_G| \) and, hence \( A(\Gamma) \geq |E_G|/D \). Thus we always accept satisfiable instances. On
the other hand, if the instance Υ is at most $1/(4D)$-satisfiable, then, by Corollary 2.8 with high probability

$$\text{OPT}_{QAP}(\Gamma) \leq \alpha |E_G| N^2 (\text{OPT}_{LC}(Υ) + 2\alpha) < |\tilde{E}_G|/D,$$

the second inequality follows from $|\tilde{E}_G| \geq \alpha |E_G| N^2/2$ (see Claim 2.3). Therefore, with high probability, we reject Υ.

\[ \square \]

3 LP Relaxation and Approximation Algorithm

We now present a new $O(\sqrt{n})$ approximation algorithm slightly improving on the result of Nagarajan and Sviridenko [30]. The new algorithm is surprisingly simple. It is based on a rounding of a natural LP relaxation. The LP relaxation is due to Adams and Johnson [1]. Thus we show that the integrality gap of the LP is $O(\sqrt{n})$.

Consider the following integer program. We have assignment variables $x_{up}$ between vertices of the two graphs that are indicator variables of the events “$u$ maps to $p$”, and variables $y_{upeq}$ that are indicator variables of the events “$u$ maps to $p$ and $v$ maps to $q$”. The LP relaxation is obtained by dropping the integrality condition on variables.
LP Relaxation

\[
\max \sum_{u,v \in V} w_G(u,v)w_H(p,q)y_{upvq} \\
\sum_{p \in V_H} x_{up} = 1, \quad \text{for all } u \in V_G; \\
\sum_{u \in V_G} x_{up} = 1, \quad \text{for all } p \in V_H; \\
\sum_{u \in V_G} y_{upvq} = x_{vq}, \quad \text{for all } v \in V_G, p, q \in V_H; \\
\sum_{p \in V_H} y_{upvq} = x_{vq}, \quad \text{for all } u, v \in V_G, q \in V_H; \\
y_{upvq} \in [0,1], \quad \text{for all } u \in V_G, p \in V_H; \\
x_{up} \in [0,1], \quad \text{for all } u \in V_G, \ p \in V_H. 
\]

Approximation Algorithm

1. We solve the LP relaxation and obtain an optimal solution \((x^*, y^*)\). Then we pick random subsets of vertices \(L_G \subset V_G\) and \(L_H \subset V_H\) of size \(\lfloor n/2 \rfloor\). Let \(R_G = V_G \setminus L_G\) and \(R_H = V_H \setminus L_H\). In the rest of the algorithm, we will care only about edges going from \(L_G\) to \(R_G\) and from \(L_H\) to \(R_H\); and we will ignore edges that completely lie in \(L_G\), \(R_G\), \(L_H\) or \(R_H\).

2. For every vertex \(u\) in the set \(L_G\), we pick a vertex \(p\) in \(L_H\) with probability \(x_{up}^*\) and set \(\tilde{\varphi}(u) = p\) (recall that \(\sum_p x_{up}^* = 1\), for all \(u\); with probability \(1 - \sum_p x_{up}^*\) we do not choose any vertex for \(u\)). Then for every vertex \(p \in L_H\), which is chosen by at least one element \(u\), we pick one of these \(u\)'s uniformly at random; and set \(\varphi(u) = p\) (in other words, we choose a random \(u \in \tilde{\varphi}^{-1}(p)\) and set \(\varphi(u) = p\)). Let \(\bar{L}_G \subset L_G\) be the set of all chosen \(u\)'s.

3. We now find a bijection \(\psi : R_G \to R_H\) so as to maximize the contribution we get from edges from \(\bar{L}_G\) to \(R_G\) i.e., to maximize the sum

\[
\sum_{u \in \bar{L}_G, v \in R_G} w_G(u,v)w_H(\varphi(u), \psi(v)). \quad (4)
\]

This can be done, since the problem is equivalent to the maximum matching problem between the sets \(R_G\) and \(R_H\) where the weight of the edge from \(v\) to \(q\) equals

\[
\sum_{u \in \bar{L}_G} w_G(u,v)w_H(\varphi(u), q).
\]

4. Output the union of the maps \(\varphi\), \(\psi\) and an arbitrary bijection from \(L_G \setminus \bar{L}_G\) to \(L_H \setminus \varphi(\bar{L}_G)\).
3.1 Analysis of the Algorithm

**Theorem 3.1.** The approximation ratio of the algorithm is $O(\sqrt{n})$.

While the algorithm is really simple, the analysis is more involved. Let $LP^*$ be the value of the LP solution. To prove that the algorithm gives $O(\sqrt{n})$-approximation, it suffices to show that

$$
\mathbb{E} \left[ \sum_{u \in L_G} \sum_{v \in R_G} w_G(u,v) w_H(\varphi(u), \psi(v)) \right] \geq \frac{LP^*}{O(\sqrt{n})}.
$$

(5)

We split all edges of graph $G$ into two sets: heavy edges and light edges. For each vertex $u \in V_G$, let $W_u$ be the set of $\lceil \sqrt{n} \rceil$ vertices $v \in V_G$ with the largest weight $w_G(u,v)$. Then,

$$
LP^* = \sum_{u \in V_G} \sum_{v \in G \setminus W_u} y^*_{upvq} w_G(u,v) w_H(p,q) + \sum_{u \in V_G} \sum_{p,q \in V_H} y^*_{upvq} w_G(u,v) w_H(p,q).
$$

Denote the first term by $LP^*_I$ and the second by $LP^*_II$. Instead of working with $\psi$, we explicitly define two new bijective maps $\nu_I$ and $\nu_{II}$ from $R_G$ to $R_H$ and prove that

$$
\mathbb{E} \left[ \sum_{u \in L_G} \sum_{v \in R_G} w_G(u,v) w_H(\varphi(u), \nu_I(v)) \right] \geq \frac{LP^*_I}{O(\sqrt{n})}; \quad \text{and} \quad \mathbb{E} \left[ \sum_{u \in L_G} \sum_{v \in R_G} w_G(u,v) w_H(\varphi(u), \nu_{II}(v)) \right] \geq \frac{LP^*_II}{O(\sqrt{n})}.
$$

These two inequalities imply the bound we need, since the sum (5) is greater than or equal to each of the sums above (by the choice of $\psi$; see (4)). Before we proceed, we state two simple lemmas we need later (see the appendix for the proofs).

**Lemma 3.2.** Let $S$ be a random subset of a set $V$. Suppose that for $u \in V$, all events $\{u' \in S\}$ where $u' \neq u$ are jointly independent of the event $\{u \in S\}$. Let $s$ be an element of $S$ chosen uniformly at random (if $S = \emptyset$, then $s$ is not defined). Then $\Pr \{u = s\} \geq \Pr \{u \in S\} / (\mathbb{E} |S| + 1)$.

**Lemma 3.3.** Let $S$ be a random subset of a set $L$, and $T$ be a random subset of a set $R$. Suppose that for $(l,r) \in L \times R$, all events $\{l' \in S\}$ where $l' \neq l$ and all events $\{r' \in T\}$ where $r' \neq r$ are jointly independent of the event $\{(l,r) \in S \times T\}$. Let $s$ be an element of $S$ chosen uniformly at random, and let $t$ be an element of $T$ chosen uniformly at random. Then,

$$
\Pr \{(l,r) = (s,t)\} \geq \frac{\Pr \{(l,r) \in S \times T\}}{(\mathbb{E} |S| + 1) \times (\mathbb{E} |T| + 1)}
$$

(here $(s,t)$ is not defined if $S = \emptyset$ or $T = \emptyset$).

The first map $\nu_I$ is a random permutation between $R_G$ and $R_H$. Observe, that given subsets $L_G$ and $L_H$, the events $\{\tilde{\varphi}(u) = p\}$ are mutually independent for different $u$'s and the expected size of $\tilde{\varphi}^{-1}(p)$ is at most 1, here $\tilde{\varphi}^{-1}(p)$ is the preimage of $p$ (recall the map $\tilde{\varphi}$ may have collisions, and hence $\tilde{\varphi}^{-1}(p)$ may contain more than one element). Thus, by Lemma 3.2, applied to the set $\tilde{\varphi}^{-1}(p) \subset L_G$,

$$
\Pr \{\varphi(u) = p \mid L_G, L_H\} \geq \Pr \{\tilde{\varphi}(u) = p \mid L_G, L_H\} / 2 = \begin{cases} 
\frac{x_{up}}{2}, & \text{if } u \in L_G \text{ and } p \in L_H; \\
0, & \text{otherwise.}
\end{cases}
$$
For every \( u, v \in V_G \) and \( p, q \in V_H \), let \( \mathcal{E}_{upvq} \) be the event \( \{ u \in L_G, v \in R_G, p \in L_H, q \in R_H \} \). Then,

\[
\Pr \{ \mathcal{E}_{upvq} \} = \Pr \{ u \in L_G, v \in R_G, p \in L_H, q \in R_H \} \geq \frac{1}{16}.
\]

Thus, the probability that \( \varphi(u) = p \) and \( \nu_I(v) = q \) is \( \Omega(x_{up}/n) \). We have

\[
\mathbb{E} \left[ \sum_{u \in L_G, v \in R_G} w_G(u, v)w_H(\varphi(u), \nu_I(v)) \right] \geq \Omega(1) \times \sum_{p, q \in V_H} \sum_{u \in V_G} x_{up}^* \sum_{v \in W_u} \frac{w_G(u, v)}{n}
\]

On the other hand, using \( \sum_{v \in V_G} y_{upvq}^*/x_{up}^* = 1 \), we get

\[
LP_I^* = \sum_{p, q \in V_H} w_H(p, q) \sum_{u \in V_G} x_{up}^* \left( \sum_{v \in W_u} \frac{y_{upvq}^*}{x_{up}^*} w_G(u, v) \right)
\]

We now define \( \nu_I \). For every \( v \in V_G \), let

\[
l(v) = \arg\max_{u \in V_G} \left\{ \sum_{p, q \in V_H} w_G(u, v)w_H(p, q)y_{upvq}^* \right\}.
\]

We say that \((l(v), v)\) is a heavy edge. For every \( u \in L_G \), let

\[
\mathcal{R}_u = \{ v \in R_G : l(v) = u \}.
\]

All sets \( \mathcal{R}_u \) are disjoint subsets of \( R_G \). Note, that \( l(v) \) does not depend on the partitioning \( V_G = L_G \cup R_G \) and \( V_H = L_H \cup R_H \), but \( \mathcal{R}_u \) depends on \( R_G \). We now define a map \( \tilde{\nu}_I : \mathcal{R}_u \to R_H \) independently for each \( u \) for which \( \varphi(u) \) is defined (even if \( \varphi(u) \) is not defined). For every \( v \in \mathcal{R}_u \), and \( q \in R_H \), define

\[
z_{vq} = \frac{y_{u\tilde{\varphi}(u)vq}}{x_{u\tilde{\varphi}(u)}}.
\]

Observe, that \( \sum_{v \in \mathcal{R}_u} z_{vq} \leq 1 \) for each \( q \in R_H \) and \( \sum_{q \in R_H} z_{vq} \leq 1 \) for each \( v \in \mathcal{R}_u \). Hence, for a fixed \( \mathcal{R}_u \), the vector \((z_{vq} : v \in \mathcal{R}_u, q \in R_H)\) lies in the convex hull of integral partial matchings between \( \mathcal{R}_u \) and \( R_H \). Thus, the fractional matching \((z_{vq} : v \in \mathcal{R}_u, q \in R_H)\) can be represented as a convex combination of integral partial matchings. Pick one of them with the probability
proportional to its weight in the convex combination. Call this matching $\tilde{\nu}^{-1}_{II}$. Note, that $\tilde{\nu}^{-1}_{II}$ is injective and that the supports of $\tilde{\nu}^{u'}_{II}$ and $\tilde{\nu}^{u''}_{II}$ do not intersect if $u' \neq u''$ (since $R_{u'} \cap R_{u''} = \emptyset$).
Let $\tilde{\nu}_{II}$ be the union of $\tilde{\nu}^{u}_{II}$ for all $u \in L_{G}$. The partial map $\tilde{\nu}_{II}$ may not be injective and may map several vertices of $R_{G}$ to the same vertex $q$. Thus, for every $q$ in the image of $R_{G}$, we pick uniformly at random one preimage $v$ and set $\nu_{II}(v) = q$. We define $\nu_{II}$ on the rest of $R_{G}$ arbitrarily.

Fix $L_{G}, L_{H}, R_{G} = V_{G} \setminus L_{G}, R_{H} = V_{H} \setminus L_{H}$, and also $R_{u} = \{ v \in R_{G} : l(v) = u \}$ (for all $u \in L_{G}$).
Let $u \in L_{G}, v \in R_{u}, p \in L_{H}$ and $q \in R_{H}$. We want to estimate the probability that $(\phi(u) = p$ and $\nu_{II}(v) = q$. Observe, that given sets $L_{G}$ and $L_{H}$, the event $\{ \tilde{\phi}(u) = p$ and $\tilde{\nu}_{II}(v) = q \}$ is independent of all events $\{ \tilde{\nu}_{II}(v') = q \}$ for $v' \neq u$ and all events $\{ \nu_{II}(v') = q \}$ for $v' \notin R_{u}$. The expected size of $\tilde{\nu}_{II}^{-1}(q)$ is at most 1, since
\[
\sum_{u' \in L_{G}} \sum_{v' \in R_{u'}} \Pr\{ \tilde{\nu}_{II}^{u'}(v') = q \} \leq \sum_{u' \in L_{G}} \sum_{v' \in R_{u'}} \sum_{p' \in L_{H}} x_{u'p'}^{*} y_{u'p'v'q}/x_{u'p'}^{*} \leq \sum_{v' \in V_{G}} \sum_{p' \in L_{H}} y_{(v')p'q}^{*} = \sum_{v' \in V_{G}} x_{v'q}^{*} \leq 1.
\]

Therefore, by Lemma 333
\[
\Pr\{ \phi(u) = p$ and $\nu_{II}(v) = q \mid L_{G}, L_{H}, u \in L_{G}, v \in R_{u}, p \in L_{H}, q \in R_{H} \} \geq \\
\Pr\{ \tilde{\phi}(u) = p$ and $\tilde{\nu}_{II}(v) = q \mid L_{G}, L_{H}, u \in L_{G}, v \in R_{u}, p \in L_{H}, q \in R_{H} \} / 4 = y_{upvq}/4.
\]

We are now ready to estimate the value of the solution:
\[
\mathbb{E} \left[ \sum_{u \in L_{G}} \sum_{v \in R_{G}} w_{G}(u, v) w_{H}(\phi(u), \nu_{II}(v)) \right] \geq \mathbb{E}_{L_{G}, L_{H}} \left[ \sum_{u \in L_{G}} \sum_{v \in R_{u}} \sum_{q \in R_{H}} y_{upv}(4) w_{G}(u, v) w_{H}(p, q) \right] \]
\[
= \frac{1}{4} \mathbb{E}_{L_{G}, L_{H}} \left[ \sum_{v \in R_{G}} \sum_{l(v) \in L_{G}} \sum_{p \in L_{H}} \sum_{q \in R_{H}} y_{l(v)pq}^{*} w_{G}(l(v), v) w_{H}(p, q) \right] \]
\[
= \frac{1}{4} \sum_{v \in V_{G}} \sum_{p, q \in V_{H}} \Pr\{ E_{l(v)pq} \} y_{l(v)pq}^{*} w_{G}(l(v), v) w_{H}(p, q) \]
\[
= \frac{1}{64} \sum_{v \in V_{G}} \sum_{p, q \in V_{H}} y_{l(v)pq}^{*} w_{G}(l(v), v) w_{H}(p, q) \]
\[
\geq \frac{1}{64} \sum_{v \in V_{G}} \max_{u \in V_{G}} \left\{ \sum_{p, q \in V_{H}} y_{upvq}^{*} w_{G}(u, v) w_{H}(p, q) \right\} \]
\[
\geq \frac{1}{64} \sum_{v \in V_{G}} \frac{1}{|W_{v}|} \sum_{u \in W_{v}} \left( \sum_{p, q \in V_{H}} y_{upvq}^{*} w_{G}(u, v) w_{H}(p, q) \right) \]
\[
= \frac{1}{64} \times \frac{L_{P}^{*}}{\sqrt{n}}.
\]

This finishes the proof.
3.2 De-randomized algorithm

We now give a de-randomized version of the approximation algorithm. The algorithm will iteratively find partial mappings of $V_G$ to $V_H$ and remove vertices for which the mapping is defined. We want the LP to be valid even after we removed some vertices from $V_G$ and $V_H$. To this end, we slightly modify the LP. We new LP is slightly weaker than the original LP.

**LP Relaxation**

\[
\begin{align*}
\text{max} & \quad \sum_{u,v \in V_G} w_G(u,v)w_H(p,q)y_{u,vq} \\
& \quad \sum_{p \in V_H} x_{up} \leq 1, \quad \text{for all } u \in V_G; \\
& \quad \sum_{u \in V_G} x_{up} \leq 1, \quad \text{for all } p \in V_H; \\
& \quad \sum_{u \in V_G} y_{u,vpq} \leq x_{vp}, \quad \text{for all } u, v \in V_G, p, q \in V_H; \\
& \quad \sum_{p \in V_H} y_{u,vpq} \leq x_{vp}, \quad \text{for all } u, v \in V_G, q \in V_H; \\
& \quad y_{u,vpq} = y_{v,up}, \quad \text{for all } u, v \in V_G, p, q \in V_H; \\
x_{up} & \in [0,1], \quad \text{for all } u \in V_G, p \in V_H; \\
y_{u,vpq} & \in [0,1], \quad \text{for all } u \in V_G, p \in V_H.
\end{align*}
\]

This LP is obtained from the original LP by replacing equalities “=” with inequalities “≤” in the first four constraints. The integrality gap of the new LP is the same as of the original LP. In fact, given a feasible solution $x^*, y^*$ of the new LP we can always increase the values of some variables to get a feasible solution $x^{**}, y^{**}$ of the original LP (then $x^{**} \geq x^*$ and $y^{**} \geq y^*$ component-wise).

**Theorem 3.4.** There exists a polynomial time (deterministic) algorithm that given an instance $\Gamma$ of MaxQAP consisting of two weighted graphs $G = (V_G, w_G)$, $H = (V_H, w_H)$ and a solution $(x^*, y^*)$ to the LP, of cost $LP^*$, outputs a bijection $\varphi : V_G \to V_H$ such that

\[
\text{value}_{QAP}(\Gamma, \varphi) \geq \frac{LP^*}{O(\sqrt{n})}.
\]

**Proof.** The existence of the map $\varphi$ follows from Theorem 3.1. We have already established that either $LP^*_I \geq LP^*_II$ (see Theorem 3.1 for definitions) and then

\[
\sum_{u,v \in V_G} \sum_{p,q \in V_H} \frac{x_{up}}{n}w_G(u,v)w_H(p,q) \geq C_{r.alg} \frac{LP^*}{\sqrt{n}};
\]

or $LP^*_II \geq LP^*_I$ and then there exists a map $\varphi_{r.alg} : V_G \to V_H$ (returned by the randomized algorithm) and a disjoint set of stars $S = \{(u, R_u)\}$ (each with the center in the vertex $u \in V_G$ and leaves $R_u \subset V_G$) such that

\[
\sum_{(u, R_u) \in S} \sum_{v \in R_u} w_G(u,v)w_H(\varphi_{r.alg}(u), \varphi_{r.alg}(v)) \geq C_{r.alg} \frac{LP^*}{\sqrt{n}},
\]

for some universal constant $C_{r.alg}$. We consider these cases separately.
I. First, assume that \( LP_1^* \geq LP_{II}^* \). Our approach is similar to the approach we used in Theorem 3.1. However, instead of peaking random sets \( L_G \), \( L_H \) and random maps \( \varphi \) and \( \nu \) we pick them deterministically. We first find \( \varphi \) and \( \nu \) to maximize the fractional value:
\[
\sum_{u \in V_G} \sum_{v \in V_G} w_G(u,v)w_H(\varphi(u),\nu(v)).
\]
Then, we pick \( L_G \) and \( L_H \) greedily to maximize
\[
\sum_{\substack{u \in L_G: \varphi(u) \in L_H \varepsilon \in R_G: \nu(v) \in H}} w_G(u,v)w_H(\varphi(u),\nu(v)).
\]
We map \( L_G \) according to \( \varphi \) and \( R_G = V_G \setminus L_G \) according to \( \nu \). The details are below.

Find a bijection \( \varphi : V_G \to V_H \) that maximizes
\[
E_{\nu} \sum_{u \in V_G} \sum_{v \in V_G} w_G(u,v)w_H(\varphi(u),\nu(v)) = \sum_{u \in V_G} \left[ \frac{1}{n} \sum_{v \in V_G} \sum_{q \in V_H} w_G(u,v)w_H(\varphi(u),q) \right],
\]
here \( \nu : V_G \to V_H \) is a random bijection chosen uniformly from the set of all bijections. We find the bijection by solving the maximum matching problem between \( V_G \) and \( V_H \), where the cost of mapping \( u \mapsto p \) equals
\[
\frac{1}{n} \sum_{v \in V_G} w_G(u,v)w_H(p,q).
\]
Then we find a bijection \( \nu : V_G \to V_H \) that maximizes
\[
\sum_{u,v \in V_G} w_G(u,v)w_H(\varphi(u),\nu(v)).
\]
Again, we do this by solving the maximum matching problem, where now the cost of mapping \( v \mapsto q \) equals
\[
\sum_{u \in V_G} w_G(u,v)w_H(\varphi(u),q).
\]
Since for a random permutation \( \nu_I \) the maximum is at least \( C_{r,\text{alg}}LP^*/\sqrt{n} \), we get
\[
\sum_{u \in V_G} \sum_{v \in V_G} w_G(u,v)w_H(\varphi(u),\nu(v)) \geq C_{r,\text{alg}} LP^*/\sqrt{n}.
\] (6)

We now use the greedy deterministic MAX CUT approximation algorithm\(^1\) to partition \( V_G \) into two sets \( L_G \) and \( R_G \) so as to maximize
\[
\sum_{u \in L_G} \sum_{v \in R_G} w_G(u,v)w_H(\varphi(u),\nu(v)).
\]

\(^1\)The greedy MAX CUT algorithm picks vertices from the set \( V_G \) in an arbitrary order and puts them in the sets \( L_G \) or \( R_G \). Thus, at every step \( t \) all vertices are partitioned into three groups \( L_G(t), R_G(t) \) and a group of not yet processed vertices \( U_G(t) \). If the weight of edges going from \( v \) to \( R_G(t) \) is greater than the weight of edges going from \( v \) to \( L_G(t) \), then the algorithm adds \( v \) to \( L_G \), otherwise to \( R_G \). The algorithm maintain the following invariant: at every step the weight of cut edges is greater than or equal to the weight of uncut edges. Thus, in the end, the weight of cut edges is at least a half of the total weight of all edges.
The cost of cutting an edge \((u,v)\) is \(w_G(u,v)w_H(\varphi(u),\nu(v))\). The cost of the obtained solution is at least a half of (6). We now use the greedy deterministic MAX DICUT (directed maximum cut) approximation algorithm to partition \(V_H\) into sets \(L_H\) and \(R_H\) so as to maximize

\[
\sum_{u \in L_G} \sum_{v \in R_G} w_G(u,v)w_H(\varphi(u),\nu(v)) = \sum_{p \in L_H} \sum_{q \in R_H} w_G(\varphi^{-1}(p),\nu^{-1}(q))w_H(p,q).
\]

The cost of a directed edge \((p,q)\) is \(w_G(\varphi^{-1}(p),\nu^{-1}(q))w_H(p,q)\), if \(\varphi^{-1}(p) \in L_G, \nu^{-1}(q) \in R_G\); and 0 otherwise. The cost of the obtained solution is at least 1/8 of (6). Thus

\[
\sum_{u \in L_G: \varphi(u) \in L_H} \sum_{v \in R_G: \nu(v) \in R_H} w_G(u,v)w_H(\varphi(u),\nu(v)) \geq \frac{C_{r.alg}}{8} \frac{\text{LP}^*}{\sqrt{n}}.
\]

Note that we do not require that \(|L_G| = |L_H|\) or that \(|R_G| = |R_H|\). We output the map that maps \(u \in L_G\) to \(\varphi(u)\) if \(\varphi(u) \in L_H\); and \(v \in R_G\) to \(\nu(v)\) if \(\nu(v) \in R_H\). It maps the remaining vertices in an arbitrary way. The cost of the solution is at least (7).

II. We now assume that there exists a collection of disjoint stars \(S = \{(u, \mathcal{R}_u)\}\) (each with the center in the vertex \(u \in V_G\) and leaves \(\mathcal{R}_u \subseteq V_G\)) and a map \(\varphi_{r.alg}: V_G \rightarrow V_H\) such that

\[
\sum_{(u,\mathcal{R}_u) \in S} \sum_{v \in \mathcal{R}_u} w_G(u,v)w_H(\varphi_{r.alg}(u),\varphi_{r.alg}(v)) \geq C_{r.alg} \frac{\text{LP}^*}{\sqrt{n}}.
\]

Define the LP volume of sets \(S \subseteq V_G, \ T \subseteq V_H\) as follows:

\[
\text{vol}_{LP}(S,T) = \sum_{u \in S} \sum_{p,q \in V_H} w_G(u,v)w_H(p,q)y^u_{upvq} + \sum_{u,v \in V_G} \sum_{p \in T} w_G(u,v)w_H(p,q)y^u_{upvq}.
\]

If \(S_1, \ldots, S_k\) is a partition of \(V_G\) and \(T_1, \ldots, T_k\) is a partition of \(V_H\), then

\[
\sum_{i=1}^k \text{vol}_{LP}(S_i,T_k) = 2\text{LP}^*,
\]

since on the left hand side every term of the LP is counted twice. Particularly,

\[
\sum_{(u,\mathcal{R}_u) \in S} \text{vol}_{LP}((\{u\} \cup \mathcal{R}_u, \varphi_{r.alg}(\{u\} \cup \mathcal{R}_u)) = 2\text{LP}^*.
\]

Plugging in (8), we get

\[
\sum_{(u,\mathcal{R}_u) \in S} \left( 2 \sum_{v \in \mathcal{R}_u} w_G(u,v)w_H(\varphi_{r.alg}(u),\varphi_{r.alg}(v)) - \frac{C_{r.alg}}{\sqrt{n}} \text{vol}_{LP}((\{u\} \cup \mathcal{R}_u, \varphi_{r.alg}(\{u\} \cup \mathcal{R}_u)) \right) \geq 0.
\]

\[\text{The greedy MAX DICUT algorithm first finds an undirected maximum cut } (A_G, B_G) \text{ using the greedy MAX CUT algorithm. The cost of the undirected maximum cut is at least a half of the total weight of all edges. Then, it outputs the cut } (A_G, B_G) \text{, if more edges are directed from } A_G \text{ to } B_G \text{ than from } B_G \text{ to } A_G \text{, it outputs the cut } (B_G, A_G) \text{, otherwise. The cost of the directed cut is at least a quarter of the total weight of all directed edges.}\]
This inequality implies that there exists one star \((u^*, R_u^*)\) such that

\[
2 \sum_{v \in R_u^*} w_G(u^*, v)w_H(\varphi_{r.alg}(u^*), \varphi_{r.alg}(v)) \geq \frac{C_{r.alg}}{\sqrt{n}} \cdot \text{vol}_{LP}(\{u^*\} \cup R_{u^*}, \varphi_{r.alg}(\{u^*\} \cup R_{u^*})).
\]

We find a star \((u^*, R^*)\) and an injective map \(\varphi : \{u\} \cup R \to V_H\) satisfying this inequality. We do this as follows: For every choice of \(u\) and \(\varphi(u)\), we solve the maximum partial matching problem where the cost of assigning \(v \mapsto q\) equals

\[
2w_G(u, v)w_H(\varphi(u), q) - \frac{C_{r.alg}}{\sqrt{n}} \sum_{u', v' \in V_G} \sum_{p, q' \in V_H} w_G(u', v)w_H(p', q')y_{u'p'v'q}^* + \sum_{u', v' \in V_G} \sum_{p' \in V_H} y_{u'p'v'q}^*.
\]

The set of matched vertices \(v\) is the set of leaves of the star; \(u\) is the center.

We fix the solution to be \(\varphi\) on \((u^*, R^*)\). We remove the star \((u^*, R^*)\) from the graph \(G\) and its image \((\varphi(u^*), \varphi(R^*))\) from the graph \(H\). We repeat the algorithm recursively for the remaining graphs (we do not resolve the LP, but we again consider two cases: \(LP_1^* \geq LP_1^*\) and \(LP_1^* \leq LP_1^*\)). To estimate the cost of the solution, observe that the value of the LP decreases by

\[
\sum_{u,v \in V_G \atop p,q \in V_H} w_G(u,v)w_H(p,q)y_{upvq}^* - \sum_{u,v \in V_G \atop p,q \in V_H \setminus \{u^*\cup R^*\}} w_G(u,v)w_H(p,q)y_{upvq}^*
\]

\[
\leq \sum_{u \in \{u^*\cup R^*\} \atop p,q \in V_H} w_G(u,v)w_H(p,q)y_{upvq}^* + \sum_{p \in \{\varphi(u^*)\cup R^*)\} \atop q \in V_H} w_G(u,v)w_H(p,q)y_{upvq}^*
\]

\[
+ \sum_{u \in V_G \atop p,q \in V_H \setminus \{u^*\cup R^*\}} w_G(u,v)w_H(p,q)y_{upvq}^* + \sum_{u \in V_G \atop p \in V_H \setminus \{\varphi(u^*)\cup R^*)\} \atop q \in V_H} w_G(u,v)w_H(p,q)y_{upvq}^*
\]

\[
= 2 \cdot \text{vol}_{LP}(\{u^*\} \cup R^*, \varphi(\{u^*\} \cup R^*)),
\]

while the profit we get from mapping \((u^*, R^*) \mapsto (\varphi(u^*), \varphi(R^*))\) is at least

\[
\frac{C_{r.alg}}{2\sqrt{n}} \cdot \text{vol}_{LP}(\{u^*\} \cup R^*, \varphi(\{u^*\} \cup R^*)).
\]

Hence, the approximation ratio is at least \(C_{r.alg}/(4\sqrt{n})\). \(\square\)

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A Appendix

Lemma 3.2 Let $S$ be a random subset of a set $V$. Suppose that for $u \in V$, all events $\{u' \in S\}$ where $u' \neq u$ are jointly independent of the event $\{u \in S\}$. Let $s$ be an element of $S$ chosen uniformly at random (if $S = \emptyset$, then $s$ is not defined). Then $\Pr\{u = s\} \geq \Pr\{u \in S\} / (\mathbb{E}[|S|] + 1)$.

Proof. We have
\[
\Pr\{u = s\} = \Pr\{u \in S\} \times \mathbb{E}\left[\frac{1}{|S|} \mid u \in S\right].
\]
By Jensen’s inequality $\mathbb{E}[1/|S| \mid u \in S] \geq 1/\mathbb{E}[|S| \mid u \in S]$. Moreover,
\[
\mathbb{E}[|S| \mid u \in S] = \mathbb{E}[|S \setminus \{u\}| \mid u \in S] + 1 = \mathbb{E}[|S \setminus \{u\}|] + 1 \leq \mathbb{E}[|S|] + 1.
\]

Lemma 3.3 Let $S$ be a random subset of a set $L$, and $T$ be a random subset of a set $R$. Suppose that for $(l,r) \in L \times R$, all events $\{l' \in S\}$ where $l' \neq l$ and all events $\{r' \in T\}$ where $r' \neq r$ are jointly independent of the event $\{(l,r) \in S \times T\}$. Let $s$ be an element of $S$ chosen uniformly at random, and let $t$ be an element of $T$ chosen uniformly at random. Then,
\[
\Pr\{(l,r) = (s,t)\} \geq \frac{\Pr\{(l,r) \in S \times T\}}{(\mathbb{E}[|S|] + 1) \times (\mathbb{E}[|T|] + 1)}
\]
(here $(s,t)$ is not defined if $S = \emptyset$ or $T = \emptyset$).

Proof. We have
\[
\Pr\{(l,r) = (s,t)\} = \Pr\{(l,r) \in S \times T\} \times \mathbb{E}\left[\frac{1}{|S| \cdot |T|} \mid (l,r) \in S \times T\right].
\]
Note, that if \((l, r) \in S \times T\), then \(S \neq \emptyset\) and \(T \neq \emptyset\) and hence \(1/(|S| \cdot |T|)\) is well defined. By Jensen’s inequality (for the convex function \(t \mapsto (1/t)^2\)),

\[
E \left[ \frac{1}{|S| \cdot |T|} \mid (l, r) \in S \times T \right] =
E \left[ \left( \frac{1}{\sqrt{|S| \cdot |T|}} \right)^2 \mid (l, r) \in S \times T \right] \geq \left( \frac{1}{E \left[ \sqrt{|S| \cdot |T|} \mid (l, r) \in S \times T \right]} \right)^2.
\]

Then,

\[
E \left[ \sqrt{|S| \cdot |T|} \mid (l, r) \in S \times T \right] =
E \left[ \sqrt{(|S \setminus \{l\}| + 1)(|T \setminus \{r\}| + 1)} \mid (l, r) \in S \times T \right]
= E \left[ \sqrt{|S \setminus \{l\}| + 1)(|T \setminus \{r\}| + 1)} \right]
\leq E \left[ \sqrt{|S| + 1)(|T| + 1)} \right]
\leq \sqrt{E [|S| + 1]E [|T| + 1]},
\]

where the last inequality follows from the Cauchy-Schwarz inequality. This finishes the proof. \(\square\)