A GENERALIZED MAXIMUM PRINCIPLE FOR 
YAU'S SQUARE OPERATOR, WITH APPLICATIONS 
TO THE STEADY STATE SPACE

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Abstract. We derive, for the square operator of Yau, an analogue of the Omori-Yau maximum principle for the Laplacian. We then apply it to obtain nonexistence results concerning complete spacelike hypersurfaces with constant higher order mean curvature in the Steady State space.

1. Introduction

The interest in the study of spacelike hypersurfaces in Lorentz manifolds (space-times) has increased very much in recent years, from both the physical and mathematical points of view. A basic question on this topic is the existence and uniqueness of spacelike hypersurfaces with some reasonable geometric properties, like the vanishing of the mean curvature, for instance. A first relevant result in this direction was the proof of the Calabi-Bernstein conjecture for maximal hypersurfaces (that is, hypersurfaces with vanishing mean curvature) in Lorentz-Minkowski space, given by Cheng an Yau in [10]. As for the case of de Sitter space, Goddard in [12] conjectured that every complete spacelike hypersurface with constant mean curvature in de Sitter space should be totally umbilical. Although the conjecture turned out to be false in its original form, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriated additional hypotheses (see, for example, [11] and [16]).

More recently, Alías, Brasil and Colares, in [2], developed general Minkowski-type formulae for compact spacelike hypersurfaces immersed into conformally stationary spacetimes, that is, spacetimes endowed with a timelike conformal vector field; then, they applied these formulae to the study of the umbilicity of compact spacelike hypersurfaces under appropriate conditions on their $r$-mean curvatures. Furthermore, the first author in [8] computed $L_r(S_r)$ for a spacelike hypersurface $\Sigma^n$ immersed in a spacetime $\mathbb{M}^{n+1}$ of constant sectional curvature, applying the resulting formula to study both $r$-maximal spacelike hypersurfaces of $\mathbb{M}$, and, in the presence of a constant higher order mean curvature, constraints on the sectional curvature of $\Sigma$ that also suffice to guarantee the umbilicity of it. Here, by $L_r$ we mean the linearization of the second order differential operator associated to the $r$-th elementary symmetric function $S_r$ on the eigenvalues of the second fundamental

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form of such immersion (cf. section 2). Let us also remark that Alías and Colares in [3] studied the problem of uniqueness for spacelike hypersurfaces with constant higher order mean curvature in Generalized Robertson-Walker spacetimes. Their approach is based on the use of the Newton transformations \( P_r \) (and their associated differential operators \( L_r \)) and the abovementioned Minkowski formulae for spacelike hypersurfaces.

Returning to the case of complete noncompact spacelike hypersurfaces, the first author in [9] used the standard formula for the Laplacian of the squared norm of the second fundamental form and the Omori-Yau maximum principle to classify complete spacelike hypersurfaces with constant mean curvature in a spacetime of nonnegative constant sectional curvature, under appropriate bounds on the scalar curvature. For the de Sitter space, Brasil Jr., Colares and Palmas also used the Omori-Yau maximum principle in [7] to characterize the hyperbolic cylinders as the only complete hypersurfaces in the de Sitter space with constant mean curvature, nonnegative Ricci curvature and having at least two ends (see also [6] for the case of the scalar curvature).

The discussion of related questions involving higher order mean curvatures faces a first difficulty: there is no corresponding version of maximum principle for the appropriate second order partial differential operators. Therefore, we begin this paper by overcoming this obstacle. More precisely, if \( \Phi \) is a field of self-adjoint linear maps on a spacelike hypersurface \( \Sigma \) of a spacetime \( M^{n+1} \), and \( f \in D(\Sigma) \), we consider the square operator

\[
\Box f = \text{tr}(\Phi \text{Hess} f)
\]

of Yau [11]; when \( \Sigma \) is complete, and under certain conditions on \( \Phi \), we develop for \( \Box \) an analogue of the Omori-Yau maximum principle for the Laplacian (see Corollary 3.3). With the aid of a suitable corollary of it (cf. Proposition 4.1), we obtain nonexistence results on complete noncompact spacelike hypersurfaces \( \Sigma \) into the half \( \mathcal{H}^{n+1} \) of the de Sitter space having one constant higher order mean curvature (cf. theorems 4.4, 4.7 and 4.8).

We observe that \( \mathcal{H}^{n+1} \), the so-called Steady State space, appears naturally in physical context as an exact solution for the Einstein equations, being a cosmological model where matter is supposed to travel along geodesics normal to horizontal hyperplanes (slices); these, in turn, serve as the initial data for the Cauchy problem associated to those equations (cf. [14], chapter 5).

This paper is organized in the following manner: in section 2 we set notation and recall a few results which will be needed later; section 3 is devoted to the statement and proof of the maximum principle and its corollaries; applications are collected in section 4.

2. Preliminaries

Let \( \overline{M}^{n+1} \) be a connected Lorentz manifold with metric \( \overline{g} = \langle , \rangle \), and Levi-Civita connection \( \nabla \). We recall (cf. [10]) that a vector field \( X \in \mathcal{X}(M) \) is said to be timelike if \( \langle X, X \rangle < 0 \) on \( \overline{M} \); spacelike if \( \langle X, X \rangle > 0 \) on \( \overline{M} \); a unit vector field if \( \langle X, X \rangle = \pm 1 \) on \( \overline{M} \).

A vector field \( V \) on \( \overline{M}^{n+1} \) is said to be conformal if

\[
(2.1) \quad \mathcal{L}_V \langle , \rangle = 2\phi \langle , \rangle
\]
for some function \( \phi \in C^\infty(M) \), where \( L \) stands for the Lie derivative of the metric of \( M \). The function \( \phi \) is called the conformal factor of \( V \).

Since \( L_{\phi}(X) = [V, X] \) for all \( X \in \mathcal{X}(M) \), it follows from the tensorial character of \( L_{\phi} \) that \( V \in \mathcal{X}(M) \) is conformal if and only if

\[
(\nabla_X V, Y) + (X, \nabla_Y V) = 2\phi(X, Y),
\]

for all \( X, Y \in \mathcal{X}(M) \). In particular, \( V \) is a Killing vector field relatively to \( \mathcal{G} \) if and only if \( \phi \equiv 0 \).

In all that follows, we consider spacelike immersions \( \psi : \Sigma^n \to \overline{M}^{n+1} \), namely, isometric immersions from a connected, \( n \)-dimensional orientable Riemannian manifold \( \Sigma \) into \( \overline{M} \). We let \( \nabla \) denote the Levi-Civita connection of \( \Sigma \).

Let us orient \( \Sigma \) by the choice of a unit normal vector field \( N \) on it, and \( A \) denote the corresponding shape operator. At each \( p \in \Sigma \), \( A \) restricts to a self-adjoint linear map \( A_p : T_p \Sigma \to T_p \Sigma \). For \( 1 \leq r \leq n \), let \( S_r(p) \) denote the \( r \)-th elementary symmetric function on the eigenvalues of \( A_p \); this way one gets \( n \) smooth functions \( S_r : \Sigma^n \to \mathbb{R} \), such that

\[
\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},
\]

where \( S_0 = 1 \) by definition. If \( p \in \Sigma \) and \( \{e_k\} \) is a basis of \( T_p \Sigma \) formed by eigenvectors of \( A_p \), with corresponding eigenvalues \( \{\lambda_k\} \), one immediately sees that

\[
S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),
\]

where \( \sigma_r \in \mathbb{R}[X_1, \ldots, X_n] \) is the \( r \)-th elementary symmetric polynomial on the indeterminates \( X_1, \ldots, X_n \). In particular, if \( |A|^2 \) stands for \( \text{tr}(A^2) \) then it is immediate to check that

\[
2S_2 + |A|^2 = S_1^2.
\]

For \( 1 \leq r \leq n \), one defines the \( r \)-th mean curvature \( H_r \) of \( \psi \) by

\[
H_r = \frac{(-1)^r}{\binom{n}{r}} S_r = \frac{1}{\binom{n}{r}} \sigma_r(-\lambda_1, \ldots, -\lambda_n).
\]

In particular, \( H_1 = H \) is the mean curvature of \( x \). It is a classical fact that such functions satisfy a very useful set of inequalities, usually referred to as Newton’s inequalities (see \( \text{[13]} \)). It turns out, however, that such inequalities remain true for arbitrary real numbers. For future reference, we collect them here. A proof can be found in \( \text{[3]} \), proposition 1.

**Proposition 2.1.** Let \( n > 1 \) be an integer, and \( \lambda_1, \ldots, \lambda_n \) be real numbers. Define, for \( 0 \leq r \leq n \), \( S_r = S_r(\lambda_1) \) as above, and \( H_r = H_r(\lambda_1) = \binom{n}{r}^{-1} S_r(\lambda_1) \).

(a) For \( 1 \leq r < n \), one has \( H_r^2 \geq H_{r-1} H_{r+1} \). Moreover, if equality happens for \( r = 1 \) or for some \( 1 < r < n \), with \( H_{r+1} \neq 0 \) in this case, then \( \lambda_1 = \cdots = \lambda_n \).

(b) If \( H_1, H_2, \ldots, H_r > 0 \) for some \( 1 < r \leq n \), then \( H_1 \geq \sqrt{H_2} \geq \sqrt[r]{H_3} \geq \cdots \geq \sqrt[n]{H_r} \). Moreover, if equality happens for some \( 1 \leq j < r \), then \( \lambda_1 = \cdots = \lambda_n \).

(c) If, for some \( 1 \leq r < n \), one has \( H_r = H_{r+1} = 0 \), then \( H_j = 0 \) for all \( r \leq j \leq n \). In particular, at most \( r - 1 \) of the \( \lambda_i \) are different from zero.
When the ambient space $\mathbb{M}$ has constant sectional curvature $c$, Gauss equation allows one to immediately check that the scalar curvature $R$ of $\Sigma$ relates to $H_2$ in the following manner:

$$R = n(n - 1)(c - H_2).$$

For $0 \leq r \leq n$ one defines the $r$-th Newton transformation $P_r$ on $\Sigma$ by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = (-1)^r S_r I + AP_{r-1}.$$ 

A trivial induction shows that

$$P_r = (-1)^r(S_r I - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r),$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since $P_r$ is a polynomial in $A$ for every $r$, it is also self-adjoint and commutes with $A$. Therefore, all bases of $T_p\Sigma$ diagonalizing $A$ at $p \in \Sigma$ also diagonalize all of the $P_r$ at $p$. Let $\{e_k\}$ be such a basis. Denoting by $A_i$ the restriction of $A$ to $\langle e_i \rangle^\perp \subset T_p\Sigma$, it is easy to see that

$$\det(t I - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{1 \leq j_1 < \cdots < j_k \leq n, j_1, \ldots, j_k \neq i} \lambda_{j_1} \cdots \lambda_{j_k}.$$ 

With the above notations, it is also immediate to check that $P_r e_i = (-1)^r S_r(A_i) e_i$, and hence (lemma 2.1 of [5])

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$;
(b) $\tr(P_r) = (-1)^r \sum_{i=1}^{n} S_r(A_i) = (-1)^r(n - r) S_r = b_r H_r$;
(c) $\tr(AP_r) = (-1)^r \sum_{i=1}^{n} \lambda_i S_r(A_i) = (-1)^r(r + 1) S_{r+1} = -b_{r+1} H_{r+1}$;
(d) $\tr(A^2 P_r) = (-1)^r \sum_{i=1}^{n} \lambda_i^2 S_r(A_i) = (-1)^r(S_1 S_{r+1} - (r + 2) S_{r+2}).$

where $b_r = (n - r)\binom{n}{r}$.

The next two results will be extremely useful in section 4.

**Proposition 2.2** (proposition 1.5 of [16]). With respect to a spacelike immersion $\psi : \Sigma^n \to \mathbb{M}^{n+1}$,

(a) if $H_r = 0$ on $\Sigma$, then $P_{r-1}$ is semi-definite on $\Sigma$.
(b) if $H_r = 0$ and $H_{r+1} \neq 0$ on $\Sigma$, then $P_{r-1}$ is definite on $\Sigma$.

If $p \in \Sigma$ is such that all eigenvalues of $A_p$ are negative, we say that $p$ is an **elliptic point** of $\Sigma$.

**Proposition 2.3** (proposition 3.2 of [5]). With respect to a spacelike immersion $\psi : \Sigma^n \to \mathbb{M}^{n+1}$, if $H_r > 0$ on $\Sigma$ and $\psi$ has an elliptic point, then $P_{r-1}$ is positive definite on $\Sigma$.

Associated to each Newton transformation $P_r$ one has the second order linear differential operator $L_r : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$, given by

$$L_r(f) = \tr(P_r \Hess f).$$

Therefore, for $f, g \in \mathcal{D}(\Sigma)$, it follows from the properties of the Hessian of functions that

$$L_r(fg) = f L_r(g) + g L_r(f) + 2(P_r \nabla f, \nabla g).$$
3. The generalized maximum principle

Let $\Sigma^n$ be a complete $n$–dimensional Riemannian manifold. Let also $\Phi : T\Sigma \to T\Sigma$ denote a field of self adjoint linear transformations on $\Sigma$. We consider the second order linear differential operator $\Box : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$ by setting

$$\Box f = \text{tr}(\phi \text{Hess } f).$$

For fixed $p \in \Sigma$, let $\rho(x) = \rho_p(x) = d(x, p)$ be the distance function from $p$ and $C_m(p)$ denote the cut locus of $p$. Set also

$$K(x) = s'_c(\rho(x))s_c(\rho(x))\text{tr}(\Phi_x),$$

where $s_c : [0, +\infty] \to \mathbb{R}$ is defined by

$$s_c(t) = \begin{cases} \frac{\sinh(\sqrt{-ct})}{\sqrt{-ct}}, & \text{if } c < 0 \\ t, & \text{if } c = 0 \\ \frac{\sin(\sqrt{ct})}{\sqrt{ct}}, & \text{if } c > 0 \end{cases}.$$

**Lemma 3.1.** If $\Phi$ is positive semi-definite on $\Sigma$ and $\Sigma$ has sectional curvature $K_\Sigma \geq c$ then, for all $x \in \Sigma \setminus C_m(p)$, one has $\Box \rho(x) \leq K(x)$.

**Proof.** Let $\gamma : [0, l] \to \Sigma$ be the only minimizing normalized geodesic joining $p$ to $x$, with length $l = \rho(x)$. Decompose any unit vector $u \in T_x\Sigma$ as $u = v + w$, where $u$ is collinear with $\gamma'(l)$ and $w \perp \gamma'(l)$. Then $|v|^2 + |w|^2 = 1$ and, at $x$,

$$\text{Hess } \rho(u, u) = \text{Hess } \rho(v, v) + 2\text{Hess } \rho(v, w) + \text{Hess } \rho(w, w)$$

$$= \langle \nabla_v \gamma', v \rangle + 2\langle \nabla_v \gamma', w \rangle + \text{Hess } \rho(w, w)$$

$$= \text{Hess } \rho(w, w).$$

It follows from the Hessian comparison theorem and from the characterization of Jacobi fields in spaces of constant sectional curvature that if $K_\Sigma \geq c$ then, at $x$,

$$\text{Hess } \rho(w, w) \leq s'_c(\rho)s_c(\rho)|w|^2 \leq s'_c(\rho)s_c(\rho).$$

Now take a moving frame $\{e_1, \ldots, e_n\}$ on a neighborhood of $x$, diagonalizing $\Phi$ at $x$, with $\Phi(e_i) = \lambda_i e_i$. Then, one has at $x$

$$\Box \rho = \text{tr}(\Phi \text{Hess } \rho) = \sum_i \lambda_i \text{Hess } \rho(e_i, e_i)$$

$$\leq \sum_i \lambda_i s'_c(\rho)s_c(\rho) = s'_c(\rho)s_c(\rho)\text{tr}(\Phi).$$

$$\Box \rho \leq K(x).$$

**Theorem 3.2.** Let $\Sigma$ be a complete Riemannian manifold with sectional curvature $K_\Sigma \geq c$, and $f \in \mathcal{D}(\Sigma)$ be a function bounded from above. If $\Phi$ is positive semi-definite at every $x \in \Sigma$ then, for every $p \in \Sigma$, there exists a sequence $(p_k)_{k \geq 1}$ in $\Sigma$ such that

$$\lim_{k \to +\infty} f(p_k) = \sup_{\Sigma} f,$$

$$|\nabla f(p_k)| = \frac{2(f(p_k) - f(p) + 1)\rho(p_k)}{k(\rho(p_k)^2 + 2)\log(\rho(p_k)^2 + 2)}$$
and

\[ (3.5) \quad \square f(p_k) \leq \frac{4\text{tr}(\Phi_{p_k})\rho(p_k)^2(f(p_k) - f(p) + 1)}{k^2(\rho(p_k)^2 + 2)^2 \log(\rho(p_k)^2 + 2)^2} + \frac{2(f(p_k) - f(p) + 1)}{k(\rho(p_k)^2 + 2) \log(\rho(p_k)^2 + 2)} \left\{ \text{tr}(\Phi_{p_k}) + \rho(p_k)K(p_k) \right\}, \]

where \( K \) is given as in (3.2).

Proof. The proof parallels that of the classical Omori-Yau maximum principle in [21]. For positive integer \( k \), let

\[ g(x) = \frac{f(x) - f(p) + 1}{(\log(\rho(x)^2 + 2))^{1/k}}. \]

One has that \( g \) is continuous, \( g(p) = \frac{1}{(\log 2)^{1/k}} > 0 \) and, since \( f \) is bounded above,

\[ \limsup_{\rho(x) \to +\infty} g(x) \leq 0. \]

Therefore, \( g \) attains its maximum at some \( p_k \in \Sigma \). In particular, \( f(p_k) - f(p) + 1 > 0 \). One now has to consider two cases separately: \( p_k \in C_m(p) \) and \( p_k \notin C_m(p) \).

Here, we treat only the first case; for the second one and the conclusion of the proof of the theorem, copy the corresponding steps in [21].

Suppose \( p_k \notin C_m(p) \). Since (omitting \( x \) for clarity)

\[ (3.6) \quad v(g) = \frac{v(f)}{(\log(\rho^2 + 2))^{1/k}} - \frac{2(f - f(p) + 1)\rho v(\rho)}{k(\rho^2 + 2)(\log(\rho^2 + 2))^{1/k + 1}}, \]

one gets at \( p_k \)

\[ 0 = \nabla g = \frac{\nabla f}{(\log(\rho^2 + 2))^{1/k}} - \frac{2(f - f(p) + 1)\rho \nabla \rho}{k(\rho^2 + 2)(\log(\rho^2 + 2))^{1/k + 1}}, \]

from where (3.5) follows.

For the estimate on \( \square f \), it follows from (3.6) that

\[ v(v(g)) = \frac{v(v(f))}{(\log(\rho^2 + 2))^{1/k}} - \frac{2pv(f)v(\rho)}{k(\rho^2 + 2)(\log(\rho^2 + 2))^{1/k + 1}} - \frac{2\{pv(f)v(\rho) + (f - f(p) + 1)[v(\rho)^2 + pv(v(\rho))]\}}{k(\rho^2 + 2)(\log(\rho^2 + 2))^{1/k + 1}} + \frac{4(f - f(p) + 1)\rho^2 v(\rho)^2}{k(\rho^2 + 2)^2(\log(\rho^2 + 2))^{1/k+2}} \left( \frac{1}{k} + 1 + \log(\rho^2 + 2) \right). \]

Now take a moving frame \( \{e_1, \ldots, e_n\} \) on a neighborhood of \( p_k \), geodesic at \( p_k \) and diagonalizing \( \Phi \) at \( p_k \), with \( \Phi(e_i) = \lambda_i e_i \). Then, one has at \( p_k \)

\[ \square f = \sum_i \lambda_i e_i(e_i(f)). \]
On the other hand, since $\text{Hess} f_{p_k} \leq 0$ and $\Phi_{p_k} \geq 0$, one has $\Box g = \text{tr}(\Phi \text{Hess} g) \leq 0$ at $p_k$, and it follows at once from the above computations that
\begin{equation}
0 \geq \Box g = \frac{\Box f}{\log(\rho^2 + 2)}/k - \frac{4\rho(\Phi \nabla f, \nabla \rho)}{k(\rho^2 + 2)\log(\rho^2 + 2)}/k^{1/k+1} - \frac{2(f - f(p) + 1)((\Phi \nabla \rho, \nabla \rho) + \rho \Box \rho)}{k(\rho^2 + 2)\log(\rho^2 + 2)}/k^{1/k+1} + \frac{4(f - f(p) + 1)\rho^2(\Phi \nabla \rho, \nabla \rho)}{k(\rho^2 + 2)^2\log(\rho^2 + 2)}/k^{1/k+2} \left(\frac{1}{k} + 1 + \log(\rho^2 + 2)\right).
\end{equation}
One also has at $p_k$ that
\begin{equation}
\langle \Phi \nabla f, \nabla \rho \rangle = \frac{2(f - f(p) + 1)\rho(\Phi \nabla \rho, \nabla \rho) + \rho K}{k(\rho^2 + 2)\log(\rho^2 + 2)},
\end{equation}
from where, substituting into the above and taking into account lemma 3.1, we get at $p_k$
\begin{equation}
\Box f \leq \frac{8(f - f(p) + 1)\rho^2(\Phi \nabla \rho, \nabla \rho) + 2(f - f(p) + 1)((\Phi \nabla \rho, \nabla \rho) + \rho K)}{k(\rho^2 + 2)^2\log(\rho^2 + 2)} - \frac{4(k + 1)(f - f(p) + 1)\rho^2(\Phi \nabla \rho, \nabla \rho)}{k(\rho^2 + 2)^2}\left[2 - (k + 1) - k\log(\rho^2 + 2)\right] + \frac{4(f - f(p) + 1)\rho^2(\Phi \nabla \rho, \nabla \rho)}{k(\rho^2 + 2)^2}\left[2 - (k + 1) - k\log(\rho^2 + 2)\right]
\end{equation}
Now, since $|\nabla \rho| = 1$ and $\Phi$ is positive semi-definite, one has $\langle \Phi \nabla \rho, \nabla \rho \rangle \leq \text{tr}(\Phi)$, so that the desired estimate follows. $\Box$

**Corollary 3.3.** Let $\Sigma$ be a complete Riemannian manifold with sectional curvature $K_{\Sigma} \geq 0$, and $f \in \mathcal{D}(\Sigma)$ be a function bounded from above. If $\Phi$ is positive semi-definite and $\text{tr}(\Phi)$ is bounded from above on $\Sigma$, then there exists a sequence $(p_k)_{k \geq 1}$ in $\Sigma$ such that
\begin{equation}
f(p_k) > \sup_M f - \frac{1}{k}, \quad |\nabla f(p_k)| < \frac{1}{k}, \quad \Box f(p_k) < \frac{1}{k}.
\end{equation}

**Proof.** Letting $C_1 = \sup_{\Sigma} f$, it follows from (3.1) that
\begin{equation}
|\nabla f(p_k)| \leq \frac{2(C_1 - f(p) + 1)}{k} \cdot \frac{\rho(p_k)}{\rho(p_k)^2 + 2} \cdot \frac{1}{\log(\rho(p_k)^2 + 2)} \leq \frac{2(C_1 - f(p) + 1)}{k} \cdot \frac{1}{2\sqrt{2}} \cdot \frac{1}{\log 2},
\end{equation}
so that
\begin{equation}
\lim_{k \to +\infty} |\nabla f(p_k)| = 0.
\end{equation}

If $f$ attains its maximum at some point of $\Sigma$, there is nothing to do. Otherwise, since $(\Sigma, d)$ is a metric space, the sequence $(p_k)_{k \geq 1}$ whose existence is assured by the previous theorem is such that $\lim_{k \to +\infty} \rho(p_k) = +\infty$. Hence, since $K_{\Sigma} \geq 0$, it
follows from lemma 3.1 that, for sufficiently large $k$, one has $K(p_k) \leq \rho(p_k)tr(\Phi_{p_k})$. Therefore, (3.5) gives

$$\Box f(p_k) \leq \frac{2tr(\Phi_{p_k})(C_1 - f(p) + 1)}{k} \left( \frac{\rho(p_k)^2 + 1}{\rho(p_k)^2 + 2} \right) \frac{1}{\log(\rho(p_k)^2 + 2)}$$

$$+ \frac{4tr(\Phi_{p_k})(C_1 - f(p) + 1)}{k^2} \left( \frac{\rho(p_k)}{\rho(p_k)^2 + 2} \right)^2 \frac{1}{\log(\rho(p_k)^2 + 2)^2}$$

$$\leq \frac{2C_2(C_1 - f(p) + 1)}{k \log 2} + \frac{C_2(C_1 - f(p) + 1)}{2k^2 \log^2 2},$$

so that

$$(3.9) \quad \lim_{k \to +\infty} \Box f(p_k) = 0.$$

The statement of the corollary follows from (3.4), (3.8) and (3.9), passing to a subsequence, if necessary.

**Corollary 3.4.** Let $\Sigma$ be a complete Riemannian manifold with sectional curvature $K_{\Sigma} \geq 0$, and $f \in D(\Sigma)$ be a function bounded from below. If $\Phi$ is positive semi-definite and $tr(\Phi)$ is bounded from above on $\Sigma$, then there exists a sequence $(p_k)_{k \geq 1}$ in $\Sigma$ such that

$$(3.10) \quad f(p_k) < \inf_{\Sigma} f + \frac{1}{k}, \quad |\nabla f(p_k)| < \frac{1}{k}, \quad \Box f(p_k) > -\frac{1}{k}.$$ 

**Proof.** Apply the previous corollary to $-f$. \qed

4. Applications

Throughout this section, $\psi : \Sigma^n \to M^{n+1}$ denotes, as before, a spacelike immersion into a Lorentz manifold $M$. In all that follows we set $\Phi = H_{r-1}P_{r-1}$, where $H_{r-1}$ and $P_{r-1}$ are as in section 2. If $H_r = 0$ on $\Sigma$, or else $H_r > 0$ on $\Sigma$ and $\psi$ has an elliptic point, then propositions 2.2 and 2.3 assure the semi-definiteness of $P_{r-1}$ (actually, $P_{r-1}$ is definite when $H_r > 0$). Moreover, since

$$(4.1) \quad tr \Phi = b_{r-1}H_{r-1}^2 \geq 0,$$

$\Phi$ is positive semi-definite in each of the above cases. In addition, if $H_{r-1}$ is bounded on $\Sigma$, then the same is true of $tr \Phi$, so that we can apply corollaries 3.3 and 3.4 to such a $\Phi$.

The following proposition is the analogue, in our context, of a lemma due to K. Akutagawa (cf. [1]).

**Proposition 4.1.** Let $M^{n+1}$ be a Lorentz manifold and $\psi : \Sigma^n \to M^{n+1}$ a spacelike immersion from a complete Riemannian manifold $\Sigma$ of sectional curvature $K_{\Sigma} \geq 0$ into $M$. Suppose that, for some $0 < r \leq n$, $H_{r-1}$ is bounded on $\Sigma$ and one of the following is true:

(a) $H_r = 0$ on $\Sigma$.

(b) $H_r > 0$ on $\Sigma$ and $\psi$ has an elliptic point.

If $f \in D(M)$ is nonnegative and such that $\Box f \geq af^\beta$, for some $a > 0, \beta > 1$, then $f \equiv 0$. 

Proof. Let \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a smooth function to be chosen later, and \( g = \phi \circ f \). Then \( \nabla g = \phi' \nabla f \) and

\[
\Box g = \text{tr}(\Phi \text{Hess } g) = H_{r-1}L_{r-1}(g) = H_{r-1} \text{ div } (P_{r-1} \nabla g)
\]

\[
= \phi'(f)H_{r-1}L_{r-1}(f) + \phi''(f)H_{r-1}(P_{r-1} \nabla f, \nabla f)
\]

\[
= \phi'(f) \Box f + \phi''(f)\langle \Phi \nabla f, \nabla f \rangle
\]

so that

\[
- \frac{\phi''(f)}{\phi'(f)^2} \langle \Phi \nabla g, \nabla g \rangle + \Box g = \phi'(f) \Box f.
\]

Letting \( \phi(t) = \frac{1}{(1+tf)^{\alpha}} \), \( \alpha > 0 \), one gets

\[
\phi'(t) = -\alpha \phi(t) \frac{\alpha + 1}{\alpha^2}, \quad \frac{\phi''(f)}{\phi'(f)^2} = \left( \frac{\alpha + 1}{\alpha} \right) \frac{1}{\phi(t)},
\]

and hence

\[
\left( \frac{\alpha + 1}{\alpha} \right) \langle \Phi \nabla g, \nabla g \rangle - \phi(f) \Box g = \alpha \phi(f) \frac{\alpha + 1}{\alpha^2} \Box f \geq \alpha \alpha \frac{f^2}{(1+f)^{2\alpha+1}}.
\]

If one now takes \( \alpha = \frac{\beta-1}{2} > 0 \), we arrive at

\[
(4.2) \quad \left( \frac{\alpha + 1}{\alpha} \right) \langle \Phi \nabla g, \nabla g \rangle - g \Box g \geq \alpha \left( \frac{f}{1+f} \right)^{\beta}.
\]

Since \( g \) is bounded from below, by corollary 3.4 we get a sequence \( \{p_k\} \) of points in \( M \) such that

\[
g(p_k) < \inf_M g + \frac{1}{k}, \quad |\nabla g|(p_k) < \frac{1}{k}, \quad \Box g(p_k) > -\frac{1}{k}.
\]

Therefore, \( f(p_k) \to \sup_M f \), and taking into account that

\[
\langle \Phi \nabla g, \nabla g \rangle \leq (\text{tr } \Phi) |\nabla g|^2 = b_{r-1}H^{2}_{r-1} |\nabla g|^2,
\]

we get from (4.2) that

\[
b_{r-1}H^{2}_{r-1} \left( \frac{\alpha + 1}{\alpha k^2} \right) - \frac{1}{k} \left( \inf_M g + \frac{1}{k} \right) \geq \alpha \alpha \left( \frac{f(p_k)}{1+f(p_k)} \right)^{\beta}.
\]

Making \( k \to +\infty \), we get \( \sup_M f = 0 \), and since \( f \geq 0 \) this gives \( f \equiv 0 \). \( \Box \)

Let \( M^n \) be a connected, \( n \)-dimensional oriented Riemannian manifold and \( I \subset \mathbb{R} \) an interval. In the product manifold \( \mathbb{M}^{n+1} = I \times M^n \), let \( \pi_I \) and \( \pi_M \) denote the projections onto the \( I \) and \( M \) factors, respectively. If \( g : I \to \mathbb{R} \) is a positive smooth function, we obtain a particular class of Lorentz metrics in \( \mathbb{M}^{n+1} \) by setting

\[
\langle v, w \rangle_p = -\langle (\pi_I)_* v, (\pi_I)_* w \rangle + (f \circ \pi_I)(p)^2 \langle (\pi_M)_* v, (\pi_M)_* w \rangle,
\]

for all \( p \in \mathbb{M} \) and all \( v, w \in T_p\mathbb{M} \). Furnished with such a metric, \( \mathbb{M} \) is called a Generalized Robertson-Walker (GRW) spacetime, and will be denoted by writing \( \mathbb{M}^{n+1} = -I \times_g M^n \). In Cosmology, a GRW gives a simple, physically plausible relativistic model (cf. [13]), so a natural space to work with.

In a GRW spacetime \( \mathbb{M}^{n+1} = -I \times_g M^n \) one has the globally defined conformal vector field \( V = g \partial_t \), which is even closed, in the sense that its dual 1-form is
closed; moreover, one can easily prove that \( \text{div} \, V = (n + 1)g' \). If \( \psi : \Sigma^n \to \overline{M}^{n+1} \) is a spacelike immersion, we oriented \( \Sigma \) by choosing a timelike unit normal vector field \( N \). For future use, we quote lemma 5.4 of [2], where the reader can also find a thorough discussion of a class of spacetimes more general than that of GRW’s.

**Lemma 4.2.** Let \( \overline{M}^{n+1} = -I \times_g M^n \) be a GRW spacetime, and \( \psi : \Sigma^n \to \overline{M}^{n+1} \) a spacelike immersion. If the restriction of \( g \circ \pi_I \) to \( \psi(\Sigma) \) attains a local minimum at some \( p \in \psi(\Sigma) \), such that \( g'(\pi_I(p)) \neq 0 \), then \( p \) is an elliptic point for \( \Sigma \).

The following proposition is due to L.J. Alías and A.G. Colares, as lemma 4.1 of preprint [3]. Here, and for the sake of completeness, we present a more direct proof.

**Proposition 4.3.** Let \( \overline{M}^{n+1} = -I \times_g M^n \) be a GRW spacetime, and \( \psi : \Sigma^n \to \overline{M}^{n+1} \) a spacelike immersion. If \( h = \pi_{I|\Sigma} : \Sigma^n \to I \) is the height function of \( \Sigma \), then

\[
L_r(h) = -(\log f)'\left(b_rH_r + \langle P_r(\nabla h), \nabla h \rangle \right) - b_rH_{r+1} \langle N, \partial_t \rangle.
\]

**Proof.** One has

\[
\nabla h = \nabla(\pi_{I|\Sigma}) = (\nabla \pi_I)\nabla = -\partial_t - (N, \partial_t)N
\]

where \( \nabla \) denotes the gradient with respect to the metric of the ambient space and \( X^\top \) the tangential component of a vector field \( X \in \mathcal{X}(\overline{M}) \) in \( \Sigma \). Now fix \( p \in M, v \in T_pM \) and let \( A \) denote the Weingarten map with respect to \( N \). Write \( v = w - \langle v, \partial_t \rangle \partial_t \), so that \( w \in T_p\overline{M} \) is tangent to the fiber of \( \overline{M} \) passing through \( p \). By repeated use of the formulas of item (2) of proposition 7.35 of [19], we get

\[
\nabla_v \partial_t = \nabla_w \partial_t - \langle v, \partial_t \rangle \nabla \partial_t \partial_t = \nabla_w \partial_t = (\log f)'w = (\log f)'(v + \langle v, \partial_t \rangle \partial_t).
\]

Thus,

\[
\nabla_v \nabla h = \nabla_v \nabla h + \langle Av, \nabla h \rangle N
\]

\[
= \nabla_v (\langle \partial_t - (N, \partial_t) N \rangle + \langle Av, \nabla h \rangle N)
\]

\[
= -(\log f)'w - \langle (N, \partial_t) N \rangle N + \langle (N, \partial_t) Av + \langle Av, \nabla h \rangle N \rangle N
\]

\[
= -(\log f)'w + (\langle Av, \partial_t \rangle - \langle N, \nabla \partial_t \rangle) N + (N, \partial_t) Av + \langle Av, \nabla h \rangle N
\]

\[
= -(\log f)'w + (\langle Av, \partial_t \rangle - \langle N, (\log f)'w \rangle) N + \langle N, \partial_t \rangle Av + \langle Av, \nabla h \rangle N
\]

\[
= -(\log f)'(v - \langle v, \partial_t \rangle (-\partial_t - (N, \partial_t) N)) + \langle N, \partial_t \rangle Av
\]

\[
= (\log f)'(v + \langle v, \partial_t \rangle) \nabla h + (N, \partial_t) Av
\]

\[
= -(\log f)'(v + \langle v, \nabla h \rangle \nabla h) + (N, \partial_t) Av.
\]
Now, by fixing \( p \in \Sigma \) and an orthonormal frame \( \{ e_i \} \) at \( T_p \Sigma \), one gets

\[
L_r h = \text{tr}(\text{Hess } h) = \sum_{i=1}^{n} \langle \nabla e_i, \nabla h, P_r e_i \rangle \\
= \sum_{i=1}^{n} \langle -(\log f)'(e_i + \langle e_i, \nabla h \rangle \nabla h) + \langle N, \partial_t \rangle A e_i, P_r e_i \rangle \\
= -(\log f)' \langle \text{tr}(P_r) + \langle P_r(\nabla h), \nabla h \rangle \rangle + \langle N, \partial_t \rangle \text{tr}(AP_r).
\]

The result follows from the formulas for the traces of \( P_r \) and \( AP_r \). \( \Box \)

Now we consider a particular model of Lorentzian GRW, the Steady State space, namely

\begin{equation}
\mathcal{H}^{n+1} = -\mathbb{R} \times_{\epsilon^t} \mathbb{R}^n.
\end{equation}

This spacetime corresponds to the steady state model of the universe proposed by Bondi, Gold and Hoyle (cf. [14], chapter 5).

A spacelike immersion \( \psi : \Sigma^n \rightarrow \mathcal{H}^{n+1} \) such that \( H_r = 0 \) on \( \Sigma \) is said to be \( r \)-maximal. If \( h \geq t_0 \) on \( \Sigma \), \( \psi \) is said to be a spacelike hypersurface over the slice \( M_{t_0} = \{ t_0 \} \times M \).

**Theorem 4.4.** There exists no \( r \)-maximal complete spacelike hypersurface \( \psi : \Sigma^n \rightarrow \mathcal{H}^{n+1} \) over the slice \( M_{t_0} \) of \( \mathcal{H}^{n+1} \), with sectional curvature \( K_{\Sigma} \geq 0 \) and such that \( C_1 \leq |H_{r-1}| \leq C_2 \), for some positive constants \( C_1, C_2 \).

**Proof.** Suppose, by contradiction, the existence of such a hypersurface. Given a smooth function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \), a straightforward computation shows that

\[
L_r(\varphi \circ h) = \varphi''(h) \langle P_r(\nabla h), \nabla h \rangle + \varphi'(h)L_r(h),
\]

so that equation (1.15) gives

\[
L_{r-1}(e^{-h+t_0}) = e^{-h+t_0} \{ \langle P_{r-1}(\nabla h), \nabla h \rangle + b_{r-1}H_{r-1} \}.
\]

Consequently, since \( \Phi = H_{r-1}P_{r-1} \), we have that

\[
\Box(e^{-h+t_0}) = \text{tr}(\Phi \text{Hess}(e^{-h+t_0})) = H_{r-1}L_{r-1}(e^{-h+t_0}) \\
= e^{-h+t_0} \{ 2\langle \Phi(\nabla h), \nabla h \rangle + b_{r-1}H_{r-1}^2 \}.
\]

Thus, since \( \Phi \) is positive semi-definite and \( h - t_0 \geq 0 \), we get

\[
\Box(e^{-h+t_0}) \geq C_1^2 b_{r-1}e^{\beta(-h+t_0)}, \quad \forall \beta > 1.
\]

Therefore, from proposition 1.1 we conclude that \( e^{-h+t_0} \equiv 0 \), which is an absurd. \( \Box \)

**Remark 4.5.** As a consequence of Bonnet-Myers theorem, a complete spacelike hypersurface \( \psi : \Sigma^n \rightarrow \mathcal{H}^{n+1} \) having (not necessarily constant) mean curvature \( H \) satisfying \( |H| \leq \rho < 2\sqrt{n-1}/n \) (\( \rho \) a real constant), has to be compact; in fact, for such a bound on \( H \), Gauss' equation would give

\[
\text{Ric}_M \geq (n-1) - n^2 \rho^2/4 > 0,
\]

where \( \text{Ric}_\Sigma \) denotes the Ricci curvature of \( \Sigma \). However, since the Steady State space is not spatially closed, i.e., since its Riemannian fiber is not compact, such a hypersurface does not exist (cf. proposition 3.2(i) of [1]).
Remark 4.6. As a special case of the reasoning of the above remark, we see that there are no complete maximal spacelike hypersurfaces in $\mathcal{H}^{n+1}$. Theorem 4.8 can thus be seen as a sort of generalization of this situation for higher order mean curvatures.

In what follows, we say that a spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ has the same time-orientation of $\partial_t$ if $\Sigma$ is oriented by the choice of a timelike unit normal vector field $N$, such that $\langle N, \partial_t \rangle \leq -1$; otherwise we say that $\Sigma$ has time-orientation opposite to that of $\partial_t$.

Theorem 4.7. Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface over a slice $\Sigma_0$ of $\mathcal{H}^{n+1}$, with sectional curvature $K_{\Sigma} \geq 0$ and time-orientation opposite to that of $\partial_t$. If $H_r > 0$ and $C_1 \leq H_{r-1} \leq C_2$ for some positive constants $C_1$ and $C_2$, then the height function $h = \pi_{R,\Sigma}$ does not attain a local minimum on $\Sigma$.

Proof. Suppose that, for some such hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$, the height function $h$ attains a local minimum, at $p \in \psi(\Sigma)$, say. Since $g = e^h$ on $\psi(\Sigma)$, $p$ is also a local minimum for $g \circ \pi_{t,1}$, and hence lemma 1.2 assures the existence of an elliptic point for $\psi(\Sigma)$; therefore, by proposition 2.1 $P_{r-1}$ is positive definite.

Now, equation (4.7) gives

$$L_{r-1}(e^{-h+t_0}) = e^{-h+t_0}\{\langle P_{r-1} \nabla h, \nabla h \rangle + b_{r-1}[H_{r-1} + H_r \langle N, \partial_t \rangle]\}.$$  

Thus, taking once more $\Phi = H_{r-1}P_{r-1}$ we get

$$\Box(e^{-h+t_0}) = \text{tr}(\Phi \text{Hess}(e^{-h+t_0})) = H_{r-1}L_{r-1}(e^{-h+t_0})$$

$$= e^{-h+t_0}\{2\langle \nabla h, \nabla h \rangle - b_{r-1}[H_{r-1}^2 + H_{r-1}H_r \langle N, \partial_t \rangle]\}. $$

Since $\Phi$ is positive definite, $\langle N, \partial_t \rangle \geq 1$ and $h - t_0 \geq 0$, we finally obtain

$$\Box(e^{-h+t_0}) \geq C_2^2 b_{r-1} e^{\beta(-h+t_0)}, \ \forall \beta > 1.$$  

Therefore, by proposition 4.4 we conclude that $e^{-h+t_0} \equiv 0$, which is an absurd. □

When $r = 2$ and $\Sigma$ has time-orientation opposite to that of $\partial_t$, lemma 3.2 of [16] assures the ellipticity of $L_1$ whenever $H_2 > 0$. Since, by Gauss’ equation, this is the same as asking that $\Sigma$ has scalar curvature $R < n(n-1)$, one can reason as in the previous result to obtain the following

Theorem 4.8. There exists no complete spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ over a slice $\Sigma_0$ of $\mathcal{H}^{n+1}$, with sectional curvature $K_{\Sigma} \geq 0$ and satisfying the following conditions:

(a) $\Sigma$ has scalar curvature $R < n(n-1)$;
(b) if the time-orientation of $\Sigma$ is opposite to that of $\partial_t$, then its mean curvature $H$ is such that $C_1 \leq H \leq C_2$, for some positive constants $C_1$ and $C_2$.

References

1. K. Akutagawa. On spacelike hypersurfaces with constant mean curvature in the de Sitter space, Math. Z. 196, (1987) 13-19.
2. L. J. Alías, A. Brasil Jr. and A. G. Colares. Integral Formulae for Spacelike Hypersurfaces in Conformally Stationary Spacetimes and Applications, Proc. of Edinburgh Math. Soc. 46 (2003), 465-488.
3. L. J. Alías and A.G. Colares. Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in Generalized Robertson-Walker spacetimes, preprint.
4. L.J. Álias, A. Romero and M. Sánchez. Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes, Gen. Relativity Gravitation 27 (1995), 71-84.
5. J. L. M. Barbosa and A. G. Colares, Stability of Hypersurfaces with Constant r–Mean Curvature, Ann. Global Anal. Geom., 15 (1997), 277-297.
6. A. Brasil Jr., A. G. Colares and O. Palmas, Complete spacelike hypersurfaces with constant mean curvature in the de Sitter space: a gap theorem, Illinois J. of Math., 47(3) (2003), 847-866.
7. A. Brasil Jr., A. G. Colares and O. Palmas, A gap theorem for complete constant scalar curvature hypersurfaces in the de Sitter space, J. Geom. and Physics, 37 (2001), 237-250.
8. A. Caminha, On spacelike hypersurfaces of constant sectional curvature lorentz manifolds, J. of Geom. and Physics, 56 (2006), 1144-1174.
9. A. Caminha, A rigidity theorem for complete CMC hypersurfaces in lorentz manifolds, to appear in Diff. Geom. and Applications.
10. S. Y. Cheng and S. T. Yau, Maximal Spacelike Hypersurfaces in the Lorentz-Minkowski Space, Ann. of Math., 104 (1976), 407-419.
11. S. Y. Cheng and S. T. Yau, Hypersurfaces with Constant Scalar Curvature, Math. Ann., 225 (1977), 195-204.
12. A. J. Goddard, Some remarks on the existence of spacelike hypersurfaces with constant mean curvature, Math. Proc. Camb. Phil. Soc., 82 (1977), 489-495.
13. G. Hardy, J. E. Littlewood and G. Pólya. Inequalities. Cambridge Mathematical Library, Cambridge, 1989.
14. S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Spacetime, Cambridge Univ. Press, Cambridge (1973).
15. J. Hounie and M. L. Leite. Two-Ended Hypersurfaces with Zero Scalar Curvature. Indiana Univ. Math. J. 48, 867-882 (1999).
16. S. Montiel. An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature, Indiana Univ. Math. J. 37, (1988) 909-917.
17. S. Montiel. Unicity of Constant Mean Curvature Hypersurfaces in Some Riemannian Manifolds, Indiana Univ. Math. J. 48, (1999) 711-748.
18. S. Montiel. Uniqueness of Spacelike Hypersurfaces of Constant Mean Curvature in foliated Spacetimes, Math. Ann. 314, (1999) 529-553.
19. B. O’Neill. Semi-Riemannian Geometry with Applications to Relativity, London, Academic Press (1983).
20. H. Rosenberg, Hypersurfaces of Constant Curvature in Space Forms, Bull. Sc. Math., 117 (1993), 217-239.
21. S. T. Yau, Harmonic Functions on Complete Riemannian Manifolds, Comm. in Pure and Appl. Math., 28 (1975), 201-228.

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