Rotational threshold in global numerical dynamo simulations.
M. Schrinner

To cite this version:
M. Schrinner. Rotational threshold in global numerical dynamo simulations.. Monthly Notices of the Royal Astronomical Society, 2013, 431, pp.L78-L82. 10.1093/mnrasl/slt012. insu-03581793

HAL Id: insu-03581793
https://insu.hal.science/insu-03581793
Submitted on 21 Feb 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution 4.0 International License
Rotational threshold in global numerical dynamo simulations

M. Schrinner

MAG(ENS/IPGP), LRA, École Normale Supérieure, 24 Rue Lhomond, F-75252 Paris Cedex 05, France

ABSTRACT
Magnetic field observations of low-mass stars reveal an increase of magnetic activity with increasing rotation rate. The so-called activity–rotation relation is usually attributed to changes in the underlying dynamo processes generating the magnetic field. We examine the dependence of the field strength on rotation in global numerical dynamo models and interpret our results on the basis of energy considerations. In agreement with the scaling law proposed by Christensen and Aubert, the field strength in our simulations is set by the fraction of the available power used for the magnetic field generation. This is controlled by the dynamo efficiency calculated as the ratio of ohmic to total dissipation in our models. The dynamo efficiency grows strongly with increasing rotation rate at a Rossby number of 0.1 until it reaches its upper bound of 1 and becomes independent of rotation. This gain in efficiency is related to the strong rotational dependence of the mean electromotive force in this parameter regime. For multipolar models at Rossby numbers clearly larger than 0.1, on the other hand, we do not find a systematic dependence of the field strength on rotation. Whether the enhancement of the dynamo efficiency found in our dipolar models explains the observed activity–rotation relation needs to be further assessed.

Key words: dynamo – methods: numerical – stars: magnetic fields.

1 INTRODUCTION
There is considerable observational evidence that magnetic activity on low-mass stars increases with increasing stellar rotation rate until it saturates and reaches a constant level for very fast rotating stars (e.g. Reiners 2012). This so-called activity–rotation relation is primarily based on observations of chromospheric or coronal magnetic activity indicators, i.e. on chromospheric or coronal emission (Skumanich 1972; Noyes et al. 1984a; Delfosse et al. 1998; Pizzolato et al. 2003). Furthermore, it is supported by magnetic flux measurements in stars of spectral types G–M (Saar 2001; Reiners, Basri & Browning 2009). Noyes et al. (1984a) pointed out that chromospheric emission and thus magnetic activity depending on spectral type and rotation is well described by a single parameter, the Rossby number $\text{Ro} = \frac{P_{\text{rot}}}{\tau_c}$, with $P_{\text{rot}}$ being the observed rotation period of a star and $\tau_c$ a convective turnover time derived from mixing-length theory. The observed increase of magnetic activity with decreasing Rossby number is usually attributed to changes in the underlying dynamo processes generating the magnetic field (Donati & Landstreet 2009; Reiners 2012). However, until now, theoretical arguments explaining the activity–rotation relation are poorly developed and formulated only on a heuristic level.

The standard argument worked out in the framework of linear mean-field theory refers to the so-called dynamo number, $D$, a dimensionless parameter which determines the distance to the dynamo threshold of a given dynamo model. If $D$ exceeds some critical value, $D > D_c$, a small initial magnetic perturbation may grow exponentially, i.e. the field-free state is linearly unstable. The larger $D$, the larger the growth rates of the magnetic field expected in this scenario. Under some simplifying assumptions, $D$ scales inversely proportional to the square of the Rossby number (Noyes, Weiss & Vaughan 1984b). Therefore, one might argue that dynamo action can be easily excited and thus leads to larger field strengths if the Rossby number decreases.

The principal objection against this argument is that it is entirely based on linear, kinematic theory. For stellar dynamos with $D \gg D_c$, the magnetic field would grow rapidly until the Lorentz force changes the velocity field and as a result the magnetic field saturates. In this dynamical regime, linear theory is in general no longer applicable. Even if the dynamo number predicted the onset of dynamo action properly, its predictive power for the saturation level of the magnetic field would remain uncertain.

A successful scaling law for the field strength of fast rotating stars and planets, on the other hand, was presented by Christensen, Holzwarth & Reiners (2009). They showed that a scaling originally derived from geodynamo models in the Boussinesq approximation (Christensen & Aubert 2006) also applies to certain classes of stars. The scaling law reads:

$$\frac{B^2}{(2\mu_0)} \sim f_{\text{ohm}} q_c \frac{L}{H_f}^{2/3},$$

(1)

© 2013 The Author
Published by Oxford University Press on behalf of the Royal Astronomical Society
and is based on the available energy flux $q_r$, which sets the field strength apart from the density $\rho$, and a convective length-scale relative to the temperature scaleheight $L/H$. The coefficient $f_{\text{ohm}} \leq 1$ gives the ratio of ohmic to total dissipation and was set to unity in this context. Christensen et al. (2009) highlighted two remarkable findings. First, the flux-based scaling law is at first glance independent of rotation, and secondly, it is only applicable to fast rotating stars and planets.

In the following, we will argue that the flux-based scaling law (equation 1) is in fact valid for a wide range of rotation rates, but it is not independent of rotation. On the contrary, we claim that the rotational dependence is captured by $f_{\text{ohm}}$ and simply eliminated by setting $f_{\text{ohm}}$ to unity. This indeed seems to be justified only for stars in the rotationally saturated regime. Our analysis is based on 30 numerical dynamo models in the Boussinesq approximation. The modelling strategy and the models are briefly described in the next section. Hereafter, we present results for $f_{\text{ohm}}$ revealing its dependence on the rotation rate and discuss the implications of our finding for the activity–rotation relation.

### 2 Dynamo Calculations

Our dynamo models are solutions of the magnetohydrodynamic equations for a conducting Boussinesq fluid in a rotating spherical shell. Convection is driven by an applied temperature difference $\Delta T$ between the inner boundary at radius $r_1$ and the outer boundary at $r_o$. The governing equations for the velocity $\mathbf{v}$, the magnetic field $\mathbf{B}$ and the temperature $T$ written in a dimensionless form proposed by Olson, Christensen & Glatzmaier (1999) are

$$
E \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} - \nabla^2 \mathbf{v} \right) + 2z \times \mathbf{v} + \nabla P = \frac{\mathbf{r}}{r_o} T + \frac{1}{Pm} (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (2)
$$

$$
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \frac{1}{Pr} \nabla^2 T, \quad (3)
$$

$$
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{Pm} \nabla^2 \mathbf{B}, \quad (4)
$$

$$
\nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{B} = 0. \quad (5)
$$

They are controlled by four dimensionless parameters, the Ekman number $E = v/\Omega L^2$, the (modified) Rayleigh number $Ra = \alpha g_0 \Delta T L v/\Omega$, the Prandtl number $Pr = v/\kappa$ and the magnetic Prandtl number $Pm = v/\eta$. In these definitions, $L$ denotes the width of the convective zone, $\Omega$ stands for the angular velocity, $\alpha$ is the thermal expansion coefficient, $g_0$ is the gravitational acceleration at the outer boundary and $v, \eta, \kappa$ are the kinematic viscosity, the magnetic and the thermal diffusivity. The mechanical boundary conditions are no slip and the magnetic field continues as a potential field outside the fluid shell.

In the following paragraph, we need to define some further quantities which are used throughout in this Letter. The time-averaged ratio of ohmic to total dissipation, $f_{\text{ohm}}$, is computed as:

$$
f_{\text{ohm}} = \frac{W_f}{W_o}, \quad (6)
$$

with the rate of ohmic dissipation

$$
W_f = \frac{1}{Pm} E \int (\nabla \times B)^2 \, dv, \quad (7)
$$

and the power $W_o$ generated by buoyancy forces,

$$
W_o = \frac{Ra}{E} \int \frac{r}{r_o} v_i T \, dv. \quad (8)
$$

Moreover, we define a Rossby number for our models similar to the observational one. It is given by the ratio of the Rossby radius to a typical convective length-scale $\ell_c$, $Ro = v_{rms}/(\Omega \ell_c)$, where $v_{rms}$ stands for the rms velocity of the flow and $\ell_c$ is derived from the kinetic energy spectrum (Christensen & Aubert 2006; Schrinner, Petitdemange & Dormy 2012). A non-dimensional measure for the convective energy flux is the Nusselt number, $Nu$, defined as the ratio of the total heat flow to the conducted heat flow. Finally, we note that the temperature scaleheight $H_T$ for our models is given by $H_T = c_p/(\rho g_0)$ with the heat capacity $c_p$.

We aim at comparing models which differ only in their rotation rates. Therefore, we keep the thermodynamically available energy flux and the diffusivities for a sequence of models constant and vary only the angular velocity. Translated in non-dimensional quantities, this means that we keep (in a first approximation) the Rayleigh number over some critical Rayleigh number $Ra_c$, the Prandtl number and the magnetic Prandtl number constant and change successively the Ekman number. More precisely, we try to keep the Nusselt number $Nu$ for a sequence of models constant. Because $Nu$ is an output parameter in our simulations and only roughly determined by $Ra$ normalized by its critical value, $Ra/Ra_c$ has to be adjusted accordingly. Some more details are given in Appendix A.

Due to computational limitations all current numerical dynamo simulations run in a parameter regime which is not appropriate for stellar interiors. Moreover, Boussinesq models do not account for the strong density variation in stars and thus certainly do not reproduce stellar dynamo processes in realistic detail. However, our models are adequate to study the flux-based scaling law which was originally derived from Boussinesq models.

### 3 Results

We considered sequences of models with $Nu \approx 2.2, 3.5$ and 7, and $Pm$ varying between 3 and 7. The Prandtl number $Pr$ was always set to unity in our simulations. For given $Pm$ and $Nu$, we obtained a sequence of models by varying the Ekman number between $E = 10^{-3}$ and $E = 10^{-5}$; some models with $E = 3 \times 10^{-3}$ and $E = 3 \times 10^{-6}$ could also be added to our sample. On the other hand, simulations with $Nu \approx 7$ and $E \leq 3 \times 10^{-5}$ were numerically not feasible. The magnetic Reynolds number of our models, $Rm = v_{rms} L/\eta$, is always larger than 100 and thus far above the minimum value of $Rm = 40$ needed to obtain dynamo action in this setting (Olson & Christensen 2006). We show in Appendix A that there is a slight but systematic increase of $Rm$ with rotation rate for a sequence of models at constant Nusselt and Prandtl numbers.

Following Christensen & Aubert (2006), a non-dimensional form of the flux-based scaling law is obtained by dividing relation (1) by $q \Omega^2 L^2$.

$$
B^2/(2\mu_0 q \Omega^2 L^2) \sim f_{\text{ohm}}(q_r/(q \Omega^2 L^2)) / H_T(2/3). \quad (9)
$$

With the Lorentz number $Lo = B/(\mu_0 q \Omega^2 L^2)^{1/2}$, the flux-based Rayleigh number $Ra_q = q \Omega g_0/(4\pi q c_p \Omega^2 L^2)$, and $L/H_T = L a g/\mu c_p$, relation (9) may simply be written as

$$
Lo/f_{\text{ohm}} = c Ra_q^{1/3}. \quad (10)
$$

Christensen & Aubert (2006) found the prefactor $c$ to be 0.92 for models with a predominantly dipolar field geometry, and a slightly
lower value of $c = 0.48$ was given by Christensen (2010) for multipolar models. In Fig. 1, we plotted $L_0/(c f_{\text{dynamo}}^{1/2})$ against $Ra_Q$ in logarithmic scales for our sample of models. Independent of their Ekman number, all models are in agreement with the scaling (10) indicated in Fig. 1 by the solid line. Apparently, the flux-based scaling law holds for all models independently of their rotation rates.

However, the ratio of ohmic to total dissipation, $f_{\text{ohm}}$, may increase drastically with rotation rate, as demonstrated in Fig. 2. Shown is $f_{\text{ohm}}$ versus the Rossby number for sequences of models with $Nu = 2.2$ and various $Pm$. At $Ro_l \approx 0.12$, the rate of ohmic diffusion increases rapidly until the steep slope flattens and $f_{\text{ohm}}$ saturates for lower Rossby numbers. On the other hand, a strong dependence of $f_{\text{ohm}}$ on $Pm$ cannot be inferred from Fig. 2.

The strong increase of $f_{\text{ohm}}$ at $Ro_l \lesssim 0.12$ coincides with a transition from multipolar dynamo models at higher Rossby number to models with a dipole-dominated magnetic field. In fact, the dependence of $f_{\text{ohm}}$ on the rotation rate changes crucially at the regime boundary as shown in Fig. 3. For a sequence of predominantly dipolar models with a slightly higher Nusselt number, $Nu = 3.5$, the increase of $f_{\text{ohm}}$ is qualitatively reproduced. However, the amplified convective energy flux shifts the sequence towards higher Rossby numbers. For an even higher Nusselt number, $Nu = 7$, the sequence of models falls entirely in the multipolar regime and a systematic increase of $f_{\text{ohm}}$ with rotation rate is no longer observed.

4 DISCUSSION AND CONCLUSIONS

In an equilibrium state, the energy released by buoyancy in our models is dissipated by viscous dissipation and ohmic diffusion. The latter requires that a magnetic field is built up by dynamo action and the rate of ohmic dissipation determines the fraction of the available power used for the magnetic field generation. For fast rotators $f_{\text{ohm}}$ increases, this means that a larger fraction of the available power is converted to magnetic energy and dynamo action becomes more efficient. According to relation (1), the growth of $f_{\text{ohm}}$ visible in Fig. 2 leads to an increase of the average magnetic field strength by an order of magnitude. Because $f_{\text{ohm}}$ is bound by 1, the field strength saturates for even higher rotation rates and then becomes independent of rotation. Both, the increase at $Ro_l \approx 0.1$ and the saturation of the field strength are in good agreement with observations. Yet, not much is known about the saturation level of the magnetic field in slowly rotating stars, except that it falls below the value predicted by (1) with $f_{\text{ohm}} = 1$ (Christensen et al. 2009). With a Rossby number of $Ro_l > 0.5$ (Reiners, Joshi & Goldman 2012), the Sun is an example for a slow rotator. Assuming an energy flux of $q_e = 63 \text{ MW m}^{-2}$, a mean density of $\rho = 1.4 \text{ kg m}^{-3}$ and an average internal field of $B = 0.063 \text{T}$ (Christensen et al. 2009), relation (1) requires $f_{\text{ohm}} \approx 0.07$. This would be consistent with the range of $f_{\text{ohm}}$ presented in this study. We note, however, that the flux-based scaling law is somewhat at odds with the estimate of $f_{\text{ohm}} = 10^{-3}$ derived by Kopp (2006) from dynamic flux-transport solar dynamo models.

The decline of $f_{\text{ohm}}$ at $Ro_l \approx 0.1$ visible in Figs 2 and 3 is related to a rotational dynamo threshold. It is characterized by a minimum magnetic Prandtl number, $Pm_{\text{crit}}$, below which self-sustained dynamo action does not occur. Christensen, Olson & Glatzmaier (1999) found that $Pm_{\text{crit}}$ is a function of only the Ekman number and varies in the dipolar regime as:

$$Pm_{\text{crit}} = 450 E^{0.75}.$$  \hfill (11)
Models with given diffusivities approach this dynamo threshold if their rotation rate is decreased and $f_{\text{ohm}}$ drops to zero.

Equation (11) is independent of any velocity amplitude and does not relate the rotational threshold to a given Rossby number. Hence, particular low values for $f_{\text{ohm}}$ could in principle be found at any Rossby number (and at any Rm). However, we could not confirm equation (11) for the multipolar dynamo regime where $f_{\text{ohm}}$ remains low and does not change significantly with rotation rate (see Fig. 3). Attempts to identify a similar decline of $f_{\text{ohm}}$ close to a rotational threshold at much lower Rossby numbers also failed. Models in this parameter regime with a lower Ekman and magnetic Prandtl often bifurcate subcritically (Morin & Dormy 2009). Consequently, the magnetic field strength and $f_{\text{ohm}}$ remain high and collapse abruptly to zero only beyond the dynamo threshold.

What explains the rotational threshold and the high rotational sensitivity of $f_{\text{ohm}}$ for models in the dipolar regime close to $R_\Omega = 0.1$? Dipolar models typically exhibit different characteristic length-scales for their velocity and their magnetic field with $\ell_v < \ell_B$. The balance of the advection and the diffusion term in the induction equation then leads to a modified magnetic Reynolds number, $Rm = v_{\text{rms}} \ell_B^2 / (\eta \ell_v)$, which needs to exceed a critical value for the onset of dynamo action. For given diffusivities, $\ell_B$ is inversely proportional to $v_{\text{rms}}$ (Christensen & Tilmann 2004). Therefore, an increase of the velocity amplitude is compensated by smaller $\ell_B$ and does not change Rm. This heuristic argument might explain why equation (11) is independent of any velocity amplitude and holds only in the dipolar regime. The strong dependence of the dynamo efficiency on rotation rate, however, requires some further explanation.

Dynamo models in the dipolar regime with $R_\Omega \lesssim 0.1$ may be adequately described in the framework of mean-field theory (Krause & Rädler 1980; Schrinner et al. 2007; Schrinner 2011; Schrinner, Schmitt & Hoyng 2011). The mean-field formalism provides useful concepts to better understand the influence of rotation on the dynamo processes in our models. It is usually set up by splitting the velocity and the magnetic field in a mean and a residual component varying on different length-scales, $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'$ and $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'$. Mean quantities denoted here by an overbar may be thought of as a result obtained from direct numerical simulations in a particular parameter regime. The mean flow $\overline{\mathbf{v}}$ is negligible for the models considered here (Olson et al. 1999; Schrinner et al. 2007). Therefore, the induction equation (2) separated for the mean and the residual component may be written as:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \frac{1}{Pm} \nabla \times \overline{\mathbf{E}} = \nabla \times (\overline{\mathbf{v}} \times \mathbf{B}') \tag{12}$$

$$\wedge \frac{\partial \mathbf{B}'}{\partial t} - \mathbf{v} \times (\mathbf{v} \times \mathbf{B}') - \frac{1}{Pm} \nabla \times \mathbf{B}' = \nabla \times (\mathbf{v} \times \overline{\mathbf{B}}). \tag{13}$$

The residual magnetic field $\mathbf{B}'$ is diffusive on the length-scale $\ell_v$ and would rapidly decay without the source term on the right-hand side of equation (13). Thus, the magnetic field generation depends decisively on $\overline{\mathbf{B}}$ and in particular on the so-called mean electromotive force, $\mathbf{E} = \overline{\mathbf{v}} \times \mathbf{B}'$. Similarly, the magnetic energy density is dominated by the mean field. The energy equation for $\overline{\mathbf{B}}$ reads

$$\frac{d}{dt} \int_{\infty} \frac{\overline{\mathbf{B}}^2}{2} \, dv = - \int \mathbf{J} \cdot \overline{\mathbf{E}} \, dv, \tag{14}$$

where $\overline{\mathbf{E}}$ is the mean electrical field, $\mathbf{J}$ is the mean current density and $\nabla \times$ denotes the volume of the fluid shell. With Ohm’s law,

$$\mathbf{J} = Pm (\overline{\mathbf{E}} + \mathbf{E}), \tag{15}$$

equation (14) yields

$$\frac{1}{Pm} \int_{V} \nabla^2 j^2 \, dv = \int_{V} \mathbf{J} \cdot \mathbf{E} \, dv, \tag{16}$$

for an equilibrium state. Hence, also the mean ohmic diffusion is controlled by the electromotive force, i.e. by the correlation of the residual velocity and the residual magnetic field, and thus linked to rotation. It is expected that rotation strengthens the correlation between $\mathbf{v}$ and $\mathbf{B}'$. Indeed, $\mathbf{E}$ grows with rotation rate for a sequence of models with fixed $\text{Nu}$, as shown in Fig. 4. Therefore, also $W_j$ and eventually the dynamo efficiency $f_{\text{ohm}}$ increase with decreasing Rossby number. We note, however, that also $W_k$ (in units of $\rho \mathbf{v}^2/L$) increases for this sequence of models, though somewhat slower than $W_j$.

For clarification, we stress that the decrease of the dynamo efficiency with increasing Rossby number in our models is not caused by an $Rm$-dependent quenching of the electromotive force, which is sometimes called catastrophic quenching (see Brandenburg & Subramanian 2005, and references therein). In contrast to the catastrophic quenching scenario, the mean electromotive force increases with $Rm$ in our simulations (see also Appendix A).

In summary, the field strength of our models is set by the available energy flux and via $f_{\text{ohm}}$ by the rotation rate. The dynamo efficiency $f_{\text{ohm}}$ increases strongly with rotation rate at $R_\Omega \approx 0.1$ and saturates at smaller Rossby numbers. The high rotational sensitivity of $f_{\text{ohm}}$ is related to a rotational dynamo threshold and finally to the strong dependence of the mean electromotive force on rotation in this parameter regime. For multipolar dynamos at higher Rossby numbers, however, the dynamo efficiency seems to be almost independent of rotation. Similarities with the observed activity–rotation relation are encouraging and need to be further assessed.

ACKNOWLEDGEMENTS

This study was initiated by an interesting discussion with E. Dormy, J. Morin and J.F. Donati. MS is grateful for financial support from the DFG fellowship SCHr 1299/1-1. Computations were performed at CINES and CEMAG computing centres.
APPENDIX A: SCALING OF RA/RAc AND Rm AT CONSTANT NUSSELT NUMBER

We use scaling laws given by Christensen & Aubert (2006) to show that the ratio Ra/Rac decreases slightly with rotation rate for a sequence of models at constant Nusselt and Prandtl numbers, whereas the magnetic Reynolds number increases.

The Nusselt number scaling proposed by Christensen & Aubert (2006) may be written as:

\[ \text{Nu} - 1 \sim Ra, \quad (A1) \]

provided that Nu > 1 and convection is sufficiently supercritical. Moreover, the critical Rayleigh number varies as Ra_c \sim E^{-4/3} (Busse 1970), and we finally find Ra/Ra_c \sim E^{1/3} for models at constant Nusselt number. For Nu \approx 2.2, 3.5 and 7, we considered a maximum Rayleigh number of 6, 15 and 50 times its critical value.

The Rossby number scaling from Christensen & Aubert (2006) together with the Nusselt number scaling yields

\[ \text{Ro} \sim ((\text{Nu} - 1)/\text{Pr})^{0.77} \quad (A2) \]

With Rm = Ro Pm/E we conclude that the magnetic Reynolds number increases slightly with increasing rotation rate, Rm \sim E^{-0.23}. For Nu \approx 2.2 and Pm = 4, for instance, we obtained Rm \approx 130 at E = 10^{-3} and Rm \approx 270 at E = 10^{-5}.

This paper has been typeset from a \TeX/\LaTeX file prepared by the author.