4D Flat-space scattering amplitude \(/CFT_3\) correlator correspondence revisited

Sachin Jain, Abhishek Mehta

Indian Institute of Science Education and Research, Homi Bhabha Road, Pashan, Pune 411 008, India
E-mail: sachin.jain@iiserpune.ac.in,
abhishek.mehta@students.iiserpune.ac.in

ABSTRACT: We point out a mismatch between a number of independent structures between 3D CFT correlator of conserved currents and 4D flat space covariant vertex of massless higher spin fields. To account for this mismatch, we construct a new parity odd structure in CFT by doing what we call epsilon transform of the known free bosonic and free fermionic correlation function of conserved currents. It turns out that this new parity odd CFT correlator is not constructed entirely out of conserved currents as expected, however, they are consistent with the position space OPE limit.

The same story in spinor helicity variables is much richer as 4D flat-space amplitudes in spinor helicity variables have many more structures than flat space covariant vertex. We show that one can construct missing CFT correlators in spinor helicity variables for conserved currents which in the flat-space limit reproduce all possible flat-space spinor helicity amplitudes; however they do not give rise to any consistent covariant vertex. We also show that these extra CFT structures are not consistent with the position space OPE limit. We further comment on the connection to AdS amplitudes in spinor helicity variables and light cone variables.
1 Introduction

Conformal field theory (CFT) plays an important role in physics. One of the main quantities of interest in CFT is its correlation function of operators. In the last decade, CFT has seen tremendous progress in position space, however very little progress has been made in momentum space. Even though relatively less explored, exploration of CFT correlation function in momentum space \([1–7]\) in three dimension \([8–17]\) has already led to new understanding of structures of correlation function which were not understood in position space \([13, 15, 16, 18]\). Momentum space correlation function also find application in cosmology \([8, 9, 19–22]\). Another remarkable feature of the momentum space CFT correlation function is its connection to flat-space amplitude \([23, 24]\). Any \(d\)-dimensional three-point CFT correlator can be shown to give rise to \(d + 1\)-dimensional three-point flat-space amplitude in the flat-space limit. The connection between momentum space helicity structure and three-point coupling in higher spin AdS theory was also explored in \([25–27]\). The relation between CFT correlator and AdS amplitude were discussed in \([28]\).

Any three-point CFT correlator of conserved current has a maximum of three structures, two parity-even and one parity-odd structure \([29, 30]\). Two parity-even structures can be obtained from free bosonic or free fermionic theory whereas the parity-odd structures can also be obtained from local Lagrangian such as Chern-Simons matter theories \([31, 32]\). On the other hand, there are two parity-even amplitude structures and two parity-odd covariant amplitude structures \([33, 34]\). Interestingly, it is known that in spinor helicity variables there are four parity even and four parity odd amplitudes for massless spinning fields \([35, 36]\). In AdS as well there are four parity even and four parity odd amplitudes in both spinor helicity variables and in light cone gauge \([25–27]\). This is summarised in the following table.
Table 1. Comparison of flat space/AdS amplitude and CFT correlator. Special cases involving one or more scalar and two or three equal spins are discussed in the main text.

The table immediately makes it clear that there is gross mismatch in number of independent structures of amplitude in flat space, AdS and CFT correlator. The mismatch between flat space covariant vertex and spinor helicity amplitude in 4D was already pointed out in [34, 37]. Amplitudes that cannot be written as local Lorentz covariant vertices were written in [38] in flat space. Interestingly, this puzzle was reconsidered in light-cone gauge in $AdS_4$ in [25] where it was shown that in the light cone gauge one can construct as many amplitudes as in spinor helicity variables.

As can be seen in the table 1, the number of CFT correlators is less than the number of flat space covariant amplitudes or the flat space spinor helicity amplitude. This immediately raises the question on validity of the flat space amplitude/ CFT correlator correspondence. Aim of this article is to show that one can construct extra CFT correlators which in the flat space limit match with spinor helicity or covariant amplitude. To do this we first resolve the mismatch of numbers of CFT correlators with flat space covariant vertex. Let us concentrate on the case of $s_3 \leq s_1 + s_2$. In this case we see there is one less parity odd structure in CFT as compared to covariant vertex. To resolve this issue we first show that parity odd flat space covariant amplitude can be constructed starting from parity even flat space covariant amplitude by what we call as epsilon transformation. We show that by using same epsilon transform on some combination of parity even free fermion and free boson CFT correlator of conserved currents, we can construct a new parity odd CFT correlator which in the flat space limit reproduces the correct flat space covariant parity odd vertex. The mismatch in number of CFT structure and flat space spinor helicity amplitude is even larger. To account for the mismatch with spinor helicity variable amplitude structure, we construct CFT correlation function which in the flat space limit does not give rise to covariant vertex but gives correct flat space spinor helicity amplitude. These extra CFT correlators have some unusual properties that they are not consistent with position space OPE limit.

The paper is organized as follows. In Section 2, we give a brief review of various flat-space vertices in covariant notation. In Section 2.2, we introduce the epsilon transformation that allows us to go from parity-even to parity-odd structures for covariant vertex while preserving gauge-invariance. In section 3 we discuss non-perturbative spinor helicity amplitudes and discuss the mismatch with covariant vertex. In section 4, we show how the CFT correlator maps to the various flat-space covariant vertices. In section 5, we make use of the epsilon transformation to propose a new CFT structure that in the flat-space limit gives the extra parity-odd amplitude and discuss several examples. In subsection section 6,
we discuss the mismatch between the flat-space limit of CFT correlator in spinor helicity variable and four-dimensional flat space amplitude in spinor helicity variables and discuss how CFT correlators can be constructed which in the flat space limit does not give rise to covariant vertex but gives correct flat space spinor helicity amplitude. In section 7, after reviewing some known results in amplitude in $\text{AdS}_4$ and their connection to CFT correlator, we discuss connection to CFT correlator in the light of results presented in this paper. In section 8 we discuss the results obtained in this paper, some future directions, and double copy relations. In Appendix A, we discuss some explicit examples of flat-space amplitudes. In Appendix B, we discuss various CFT correlation functions and their flat-space limits. In Appendix C we discuss the flat space limit of CFT correlators in spinor helicity variables. Appendix D is devoted to understanding more on parity odd CFT correlators that we constructed by doing epsilon transformation. Appendix E is devoted to understanding the missing CFT structures in spinor helicity variables for the example of $\langle JJJ\rangle$. In Appendix F, we discuss various identities which are useful in the main text. The last Appendix G discusses the special case of flat space amplitude and CFT correlator in spinor helicity variables with net helicity zero configuration.

2 Brief review of flat-space covariant vertex

In this section, we review known results for flat-space amplitude in four dimensions for massless particles. We follow closely the notation and discussion in [34].

2.1 Covariant vertex

Consider a generic action of the form

$$S^{(3)} = \int d^4x C(\partial x_1, \partial u_I)\phi^1(x_1, u_1)\phi^2(x_2, u_2)\phi^3(x_3, u_3) \bigg|_{u_I = 0, x_I = x}$$

(2.1)

where

$$\phi(x, u) = \phi_{\mu_1 \cdots \mu_s}(x) u^{\mu_1} \cdots u^{\mu_s}$$

(2.2)

are higher spin fields and $u_I$ are some some auxiliary variables and $C$ is some operator which makes the higher-spin fields contract with each other with various derivatives forming a cubic interaction. Under higher spin symmetry and gauge invariance, $C$ can be made up of only certain parity-even and parity-odd structures namely
Flat-space amplitudes Building blocks

| Parity   | Position space                                           | Momentum space                                      |
|----------|----------------------------------------------------------|-----------------------------------------------------|
| Even     | $Y_I = \partial u_I, \partial_x u_{I+1}$               | $Y_I = z_I p_{I+1}$                                 |
|          | $Z_I = \partial u_{I+1}, \partial_x u_{I-1}$            | $Z_I = z_{I+1} z_{I-1}$                              |
| Odd      | $V_I = \epsilon^{\mu \nu \rho \sigma} \partial_{u_{I+1}}^\mu \partial_{x_{I+1}}^\nu \partial_{u_{I-1}}^\rho \partial_{x_{I-1}}^\sigma$ | $V_I = \epsilon (z_{I+1} p_{I+1} z_{I-1} p_{I-1})$  |
|          | $W_I = \epsilon^{\mu \nu \rho \sigma} \partial_{u_1}^\mu \partial_{u_2}^\nu \partial_{u_3}^\rho \partial_{x_I}^\sigma$ | $W_I = \epsilon (z_1 z_2 z_3 p_I)$                  |

Table 2. Building blocks for flat-space amplitude in 4d. $\epsilon$ is 4D totally antisymmetric tensor and the notation $\epsilon(xyzw) = \epsilon^{\mu \nu \rho \sigma} x^\mu y^\nu z^\rho w^\sigma$.

The gauge symmetry and higher spin symmetry, constrains the possible amplitudes to be

| Parity   | Minimal                                              | Non-Minimal                                           |
|----------|------------------------------------------------------|-------------------------------------------------------|
| parity-even | $g_{m,e} G^{s_1} Y_2^{s_2-s_1} Y_3^{s_3-s_1}$       | $g_{nm,e} Y_1^{s_1} Y_2^{s_2} Y_3^{s_3}$             |
| parity-odd | $g_{m,o} V_1 G^{s_1} Y_2^{s_2-s_1} s_3^{s_3-s_1-1}$ | $g_{nm,o} V_1 Y_1^{s_1} Y_2^{s_2} Y_3^{s_3-1}$       |

Table 3. Amplitudes in 4d. The subscript "m" means minimal, "nm" means non-minimal, "e" means parity even, "o" means parity odd.

with $G = Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3$. We have also assumed that $s_3 > s_2, s_1$ with out loss of generality. Minimal and non-minimal distinction comes from number of derivatives or number of momentum factors present in the amplitudes.

Notice that for the odd case there is an issue when spins are coincident i.e. $s_i = s_j$, one obtains negative powers of derivatives which is not possible due to locality. For the case of $s_1 = s_2 < s_3$, one may use the identity

$$V_1 G^{s_1} Y_2^{s_2-1} Y_3^{s_3-1} \approx \frac{1}{2} [V_1 Z_2 - V_2 Z_1 + (W_2 - W_1) Y_3] G^{s_1-1} Y_3^{s_3-s_1-1}$$

(2.3)

to remove the negative powers. Such identities are derived using Schouten identities and momentum conservation. For the case of coincidence of $s_1 = s_2 < s_3$, notice that (2.3) is anti-symmetric under the 1 $\leftrightarrow$ 2 exchange. This implies for parity odd minimal amplitude to be non-zero for this case, we do require Chan-Paton factor which is antisymmetric in exchange of 1 $\leftrightarrow$ 2 indices. Together with Chan-Paton factor and (2.3), the amplitude becomes symmetric.

For the case $s_1 = s_2 = s_3$, it can be shown that the negative powers for parity odd minimal amplitude cannot be removed [34]. In this case the cubic vertices with negative powers are simply dropped. The final results for special cases are summarized below in the table.

Let us note that, the discussion involving scalar is simpler and can be obtained from results in Table 3. Some explicit examples are worked in detail in the appendix A.
Table 4. parity-odd special cases of amplitudes in 4d. The parity-even part of the amplitude is as given in Table 3.

In the next section we show an interesting relation between parity even and parity odd part of the covariant vertex discussed till now.

2.2 Epsilon transformation

In this section, we introduce what we call epsilon transform which maps parity-even amplitude to parity-odd amplitude and vice-versa. In 4D one may work with the following choice of momenta and polarization

\[ p^\mu = (k, k^i) \quad z^\mu = (0, z^i) \]  

(2.4)

with this choice the momentum space expressions in Table 2 takes the form

\[
\begin{align*}
V_I &= \epsilon(z_{I+1}z_{I-1}k_{I-1})k_{I+1} - \epsilon(z_{I+1}z_{I-1}k_{I+1})k_{I-1}, \quad Y_I = z_I k_{I+1} \\
Z_I &= z_{I+1}z_{I-1}, \quad p_I.p_J = -k_I k_J + k_I k_J = 0
\end{align*}
\]  

(2.5)

where we have used three dimensional epsilon in the last expression, more precisely \( \epsilon^{0\mu\nu\rho} = \epsilon^{\mu\nu\rho} \) where index 0 is time direction. Let us define an epsilon transformation [16] which is given by

\[ z^i \rightarrow \epsilon^{zki \over k} \]  

(2.6)

The \( \epsilon \)-transformation can also be implemented by a differential operator

\[
[O_\epsilon]_I = \frac{1}{k_I} \epsilon(z_I k_I) \frac{\partial}{\partial z_I}
\]  

(2.7)

which acts on parity-even gauge-invariant structures to give parity-odd gauge invariant structures i.e

\[
O_\epsilon : \mathcal{M}_{m,e} \rightarrow \mathcal{M}_{m,o} \\
O_\epsilon : \mathcal{M}_{nm,e} \rightarrow \mathcal{M}_{nm,o}
\]  

(2.8)

We show this explicitly below. Consider the epsilon transformation of the even minimal amplitude

\[
[O_\epsilon]_2 \mathcal{M}_{m,e} = s_1 G^{s_1-1} Y_2^{s_2-s_1} Y_3^{s_3-s_1} [O_\epsilon]_2 G + (s_2 - s_1) G^{s_1} Y_2^{s_2-s_1-1} Y_3^{s_3-s_1} [O_\epsilon]_2 Y_2
\]  

(2.9)
Now, it can be shown \(^1\) that

\[
Y_2Y_3[O_e]_2G = -GV_1 \tag{2.10}
\]
\[
Y_3[O_e]_2Y_2 = -V_1 \tag{2.11}
\]

which when used in (2.9) gives

\[
[O_e]_2\mathcal{M}_{m,e} = -s_2V_1G^{s_1}Y_2^{s_2-s_1-1}Y_3^{s_3-s_1-1} = \mathcal{M}_{m,o} \tag{2.12}
\]

which is precisely the odd minimal amplitude. Similarly, for the even non-minimal amplitude we have

\[
[O_e]_2\mathcal{M}_{nm,e} = s_2Y_1^{s_1}Y_2^{s_2-1}Y_3^{s_3}[O_e]_2Y_2 = -s_2V_1Y_1^{s_1}Y_2^{s_2-1}Y_3^{s_3-1} = \mathcal{M}_{nm,o} \tag{2.13}
\]

where in the second equality (2.11) was used and we have precisely obtained the odd non-minimal amplitude given in 3. Let us now consider some special examples to illustrate this.

**All equal spin:** \( s_1 = s_2 = s_3 \)

Notice when \( s_1 = s_2 = s_3 = s \) for minimal even amplitude, from (2.12) we get

\[
[O_e]_2\mathcal{M}_{sss}^{m,e} = -sG^{s-1}Y_2^{-1}Y_3^{-1} \tag{2.14}
\]

These negative powers cannot be removed by any Schouten identities or degeneracies, therefore, we conclude that parity odd minimal gauge-invariant vertex does not exist. This is consistent with table 4.

For the case of \( s_1 = s_2 = s_3 = 2 \), from (3.8) we have

\[
\mathcal{M}_{m}^{222} = g_{m,e}G^2
\]
\[
\mathcal{M}_{nm}^{222} = g_{nm,e}Y_1^2Y_2^2Y_3^2 + g_{nm,o}V_1Y_1^2Y_2Y_3 \tag{2.15}
\]

It is easy to show that non-minimal parity even and parity odd terms are related by epsilon transform. However, if we look at the epsilon transform of the minimal even amplitude we get

\[
[M]^{222} = ([O_e]_1 + [O_e]_2 + [O_e]_3)G^2 = -\frac{G}{k_1k_2k_3}(k_2k_3\epsilon(z_1k_1k_2)z_2.z_3 + \text{cyclic terms}) \tag{2.16}
\]

which cannot be re-written as a 4D Lorentz invariant structure. Hence it is a not a valid covarint amplitude. This is consistent with the fact that there can not exist any parity odd minimal covariant vertex for an equal spin case.

\(^1\)Refer to the Appendix C for its derivation.
Two equal spin: $s_1 = s_2 \neq s_3$

Another coincidence point of concern is $s_1 = s_2 = s$, where we obtain

$$[O_{s_1}][M^{sss}_{m,e}] = -sV_{1}GY_2^{-1}Y_3^{-s_3-s-1}$$  \hspace{1cm} (2.17)

These negative powers can be removed by using the Schouten identities $[13]$ and re-write them as

$$[O_{s_1}][M^{sss}_{m,e}] = -\frac{1}{2} [V_1 Z_2 - V_2 Z_1 + (W_2 - W_1) Y_3] G^{s_3-s_1-s-1}$$  \hspace{1cm} (2.18)

Due to the nature of the Schouten identities in 4D momentum space, an anti-symmetrization in epsilon transforms at 1, 2 was not necessary to derive the above, but one can in principle still use anti-symmetrization and obtain the same result using (F.3)

$$([O_{s_1}][M^{sss}_{m,e}]) = s [V_1 Z_2 - V_2 Z_1 + (W_2 - W_1) Y_3] G^{s_3-s_1-s-1}$$  \hspace{1cm} (2.19)

However, as we will see later, this anti-symmetrization of the epsilon transforms is necessary at the level of the CFT correlator because not all Schouten identities in 4D momentum space carry on to the 3D momentum space. The epsilon transform can be done at any operator in the correlator, however, for the case of $s_1 = s_2$, some care needs to be taken. Notice that (2.18) is anti-symmetric under $1 \leftrightarrow 2$ while its minimal even counterpart is symmetric under the exchange. This is due to the presence of Chan-Paton factors for the case of $s_1 = s_2$. Therefore, an epsilon transform of minimal even for $s_3$ will give zero. Hence, when Chan-Paton factors are involved, the epsilon transform of $s_3$ must be avoided.

3 Amplitude in spinor helicity variables

Let us now review expressions of amplitude written directly in spinor helicity variables. The most general cubic amplitude in 4D with massless particles of arbitrary spin is given by

$$A_{h_{1},h_{2},h_{3}}^{s_{1},s_{2},s_{3}} = \begin{cases} (1,2)^{h_{3}s_{3}+h_{3}s_{1}+h_{2}s_{2}} \langle 3,1 \rangle^{h_{2}s_{2}+h_{3}s_{3}+h_{1}s_{1}}(2,3)^{h_{1}s_{1}+h_{2}s_{2}+h_{3}s_{3}} \text{ when } h_{1}s_{1}+h_{2}s_{2}+h_{3}s_{3} < 0 \\ (1,2)^{h_{3}s_{3}+h_{3}s_{1}+h_{2}s_{2}} \langle 3,1 \rangle^{h_{2}s_{2}+h_{3}s_{3}+h_{1}s_{1}}(2,3)^{h_{1}s_{1}+h_{2}s_{2}+h_{3}s_{3}} \text{ when } h_{1}s_{1}+h_{2}s_{2}+h_{3}s_{3} > 0 \\ \end{cases}$$  \hspace{1cm} (3.1)

The parity even and odd parts are given by the same expression in spinor helicity variables. Interestingly, it can be shown that the epsilon transformation in section 2.2 keeps the form of amplitude in spinor helicity variables same upto some imaginary number of $i$. In (3.1) there are a total eight independent helicity components $-\cdots-,-\cdots+,-\cdots-$ and their conjugate $\cdots+\cdots,\cdots+\cdots,\cdots-\cdots,-\cdots+\cdots$. For each helicity component together with its conjugate, one can construct one parity even and one parity odd amplitude. This implies one can define total four parity even and four parity odd amplitudes in spinor helicity variables. Let us consider a few examples.
The amplitude for this case is given by
\[ A^{00s}_+ = (g_e - ig_o) (1, 2)^s (2, 3)^{s-1} [3, 1]^{-s} \]
\[ A^{00s}_- = (g_e + ig_o) (1, 2)^s (2, 3)^{s-1} [3, 1]^{-s} \] (3.2)
which has one parity odd and one parity even amplitude.

**graviton-graviton-graviton**

Three graviton amplitude in spinor helicity variables are given by
\[ A_{++-}^{222} = (g_{nm,e} - ig_{nm,o}) [1, 2]^2 [2, 3]^2 [3, 1]^2 \]
\[ A_{++-}^{222} = (g_{m,e} - ig_m^o) \frac{[2, 3]^6}{[1, 2]^2 [3, 1]^2} \]
\[ A_{++-}^{222} = (g_{m,o} + ig_{m,o}) \frac{(2, 3)^2 (3, 1)^2}{(1, 2)^2} \] (3.3)
where superscript e, o stands for even or odd and subscript m, nm stands for minimal and non-minimal coupling respectively. Other spinor helicity components can be obtained by permutation.

**photon-photon-graviton**

Two photon one graviton amplitude is given by
\[ A^{112}_+ = (g_{nm,e} - ig_{nm,o}) [2, 3]^2 [3, 1]^2 \]
\[ A^{112}_- = (g_{m,e} + ig_{m,o}) (2, 3)^2 (3, 1)^2 \]
\[ |A_m|^{112}_{++} = (g_{m,e} - ig_{m,o}) \frac{[3, 1]^4}{[1, 2]^2} \]
\[ |A_m|^{112}_{--} = (g_{m,e} + ig_{m,o}) \frac{(3, 1)^4}{(1, 2)^2} \] (3.4)
where the last two lines have identical couplings because of 1 ↔ 2 exchange symmetry. One can also naively add net helicity amplitudes $^2A_{++-}^{112}, A_{++-}^{112}$ but as is argued in Appendix G, they do not lead to any consistent amplitude and hence can be ignored.

### 3.1 Cubic vertices in spinor-helicity variables

From (34), the minimal-coupling vertices with $s_1 \leq s_2 \leq s_3$ in 4D is given by
\[ \mathcal{M}_m^{s_1,s_2,s_3} = g_{m,e} G^{s_1} Y_{s_2}^{s_3} Y_{s_3}^{-s_1} + g_{nm,o} V Y_{s_1}^{s_2} Y_{s_3}^{-s_1} \]
\[ \mathcal{M}_m^{s_1,s_2,s_3} = g_{m,o} V Y_{s_1}^{s_2} (3.6) \]
\[ g_{nm,e} V Y_{s_1}^{s_2} Y_{s_3}^{-s_1} \]
\[ \mathcal{M}_m^{s_1,s_2,s_3} = g_{nm,o} V Y_{s_1}^{s_2} Y_{s_3}^{-s_1} \]
\[ A^{112}_{++} = (g_{m,e} - ig_{m,o}) \frac{[2, 3]^4}{[1, 2]^2 [3, 1]^2} \]
\[ A^{112}_{--} = (g_{m,e} + ig_{m,o}) \frac{(2, 3)^4}{(1, 2)^2 [3, 1]^2} \] (3.5)
which in the spinor-helicity variables give

\[ [M_{nm}]_{++--}^{1,2,3} = g_{A,nm} [1,2]^{s_1+s_2+s_3} [2,3]^{s_2+s_3-s_1} [3,1]^{s_1+s_3-s_2} \]

\[ [M_{nm}]_{++--}^{1,2,3} = g_{H,nm} (1,2)^{s_1+s_2+s_3} (2,3)^{s_2+s_3-s_1} (3,1)^{s_1+s_3-s_2} \]

\[ [M_m]_{++--}^{1,2,3} = g_{A,m} [2,3]^{s_1+s_2+s_3} \]

\[ [M_m]_{++--}^{1,2,3} = g_{H,m} (1,2)^{s_1+s_2+s_3} \]

\[ [M_m]_{++--}^{1,2,3} = g_{A,m} f_2 [3,1]^{s_1+s_2+s_3} \]

\[ [M_m]_{++--}^{1,2,3} = g_{H,m} f_2 (1,2)^{s_1+s_2+s_3} \]

\[ [M_m]_{++--}^{1,2,3} = g_{A,m} f_3 [2,3]^{s_1+s_2+s_3} [3,1]^{s_1+s_3-s_2} \]

\[ [M_m]_{++--}^{1,2,3} = g_{H,m} f_3 (2,3)^{s_1+s_2+s_3} (3,1)^{s_1+s_3-s_2} \]

where

\[ f_2 = \left( \frac{(1,2)[1,2][2,3][3,1]}{(3,1)[3,1]} \right)^{s_2-s_1} \]

\[ f_3 = \frac{(3,1)[3,1][2,3][3,1]}{(1,2)[1,2]} \]

\[ (3.8) \]

\[ g_{A,m} = g_{m,e} + ig_{m,o} \]

\[ g_{H,m} = g_{m,e} - ig_{m,o} \]

\[ g_{A,nm} = g_{nm,e} + ig_{nm,o} \]

\[ g_{H,nm} = g_{nm,e} - ig_{nm,o} \]

\[ (3.9) \]

For the case \( s_1 \neq s_2 \neq s_3 \), \( f_2 = f_3 = 0 \) and hence only giving a total four amplitudes. For the special case of \( s_1 = s_2 < s_3 \), expression for \( f_3 \) differs slightly from the above and is given by

\[ f_3 = \frac{(3,1)[3,1][2,3][3,1]}{(1,2)[1,2]} \]

\[ (3.11) \]

Let us note that \( f_2, f_3 \) are zero due to momentum conservation, since \( p_i p_j = \langle ij \rangle [ij] = 0 \). This implies many of the components of \( (3.8) \) are zero. For special cases such as \( s_1 = s_2 \neq s_3 \) factor \( f_2 \) drops out whereas for \( s_1 = s_2 = s_3 \) factor \( f_2, f_3 \) dropout. Also for \( s_1 = s_2 = s_3 \) we have \( g_m^2 = 0 \), that is parity odd minimal cubic vertex does not exist. Here we present some examples.

**graviton-graviton-graviton**

\[ [M_{nm}]^{222}_{++--} = (g_{nm,e} + ig_{nm,o}) [1,2]^{2}[2,3]^{2}[3,1]^{2} \]

\[ [M_{nm}]^{222}_{++--} = (g_{nm,e} - ig_{nm,o}) (1,2)^{2}[2,3]^{2}[3,1]^{2} \]

\[ [M_m]^{222}_{++--} = g_{m,e} [2,3]^{0} \]

\[ [M_m]^{222}_{++--} = g_{m,e} [1,2]^{2}[3,1]^{2} \]

\[ (3.12) \]

and permutations. We notice that minimal amplitude does not have any parity odd contribution in contrast to \( (3.3) \).

**photon-photon-graviton**

\[ [M_{nm}]^{112}_{++--} = g_{A,nm} [2,3]^{2}[1,2]^{2} \]

\[ [M_{nm}]^{112}_{++--} = g_{H,nm} (2,3)^{2}[1,2]^{2} \]

\[ [M_m]^{112}_{++--} = g_{A,m} [2,3]^{4} \]

\[ [M_m]^{112}_{++--} = g_{H,m} (2,3)^{4} \]

\[ [M_m]^{112}_{++--} = g_{A,m} [1,2]^{4} \]

\[ [M_m]^{112}_{++--} = g_{H,m} (3,1)^{4} \]

\[ (3.13) \]

\(^3\)In spinor helicity variables, parity even and parity odd amplitudes are identical up overall factor.
where we have not considered net helicity zero amplitudes\textsuperscript{4}. Let us note that for this case minimal and non-minimal amplitude contains both parity even and odd parts.

### 3.2 Mismatch between spinor helicity amplitude and covariant vertex

It turns out that (3.7) doesn’t reproduce the full amplitude (3.1). The mismatch is most clearly understood for the case of unequal spin. Let us consider an example.

#### 3.2.1 \( s_1 \neq s_2 \neq s_3 \) with \( s_3 \geq s_3 \geq s_1 \)

The covariant amplitude in spinor helicity variables is given by (3.8)

\[
[M_{mn}]_{\pm\mp\pm\mp} = g_{A,mn}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

\[
[M_{nn}]_{\pm\mp\pm\mp} = g_{H,nn}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

\[
[M_m]_{\pm\mp\pm\mp} = g_{A,m}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

\[
[M_m]_{\pm\mp\pm\mp} = g_{H,m}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

Writing explicitly the spinor helicity amplitude we obtain

\[
[A_{mm}]_{\pm\mp\pm\mp} = g_{A,mm}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

\[
[A_{mn}]_{\pm\mp\pm\mp} = g_{H,mm}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

\[
[A_m]_{\pm\mp\pm\mp} = g_{A,m}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

\[
[A_m]_{\pm\mp\pm\mp} = g_{H,m}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

\[
[A_m]_{\pm\mp\mp\mp} = g_{A,m}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

\[
[A_m]_{\pm\mp\mp\mp} = g_{H,m}[1,2]^{s_1+s_2-s_3}[2,3]^{s_2+s_3-s_1}[3,1]^{s_1+s_3-s_2}
\]

Comparing (3.14) with (3.15), we see that there is a clear mismatch between the two. We have introduced \( g', g'' \) to indicate that they are independent structures. One can convert the spinor helicity amplitude to momentum space variables and show that these extra structures do not correspond to any covariant 4d vertex\textsuperscript{5}. We will encounter explicit examples when we consider CFT correlators in spinor helicity variables. We shall see that there are CFT correlators which in the flat space limit reproduce the correct spinor helicity amplitude but do not correspond to any covariant vertex.

Let us consider another example.

\textsuperscript{4}It is easy to check that they are zero any way for this case.

\textsuperscript{5}See [39] for recent discussion on this issue. For the case of spin-\( s \)-s-2 amplitude, it was shown that \(- - - - - + ++\) and their conjugates can be covariantized but \(- - + - - +\) and its conjugate helicity components can not be covariantized with the help of symmetric tensors. However they can be covariantized with respect to field variables which originate from twisters. Interestingly, the story is the same in AdS as well. We thank E. Skvortsov for emphasizing this to us.
3.2.2 photon-photon-spin

The spinor helicity amplitude is given by

\[ A_{114}^{+--+} = \langle 12 \rangle^6 \langle 23 \rangle^4 \langle 12 \rangle^4 \]

\[ A_{114}^{++-} = \langle 12 \rangle^6 \langle 23 \rangle^4 \langle 12 \rangle^4 \quad (3.16) \]

\[ A_{114}^{-+-} = \langle 23 \rangle^6 \langle 31 \rangle^4 \langle 23 \rangle^4 \]

\[ A_{114}^{+-+-} = \langle 23 \rangle^6 \langle 31 \rangle^4 \langle 12 \rangle^4 \quad (3.17) \]

The cubic vertex in spinor helicity variables is given by

\[ [M_{nm}]^{114}_{114} = [2, 3]^2 \langle 1, 2 \rangle^2 \]

\[ [M_{nm}]^{114}_{114} = [2, 3]^6 \langle 3, 1 \rangle^2 \langle 1, 2 \rangle^4 \]

\[ [M_{m}]^{114}_{m} = [M_{m}]^{114}_{m} = 0 \quad (3.19) \]

This implies, this cubic amplitude is not consistent with the full non-perturbative amplitude (3.18), since, we have

\[ M^{114}_{++} \neq A^{114}_{--}, M^{114}_{++} \neq A^{114}_{--}. \]

The mismatch is summarised in the following Table.

| Covariant vertex vs spinor helicity amplitude |
|-----------------------------------------------|
| \( s_1 \leq s_2 \leq s_3 \) | Covariant Cubic vertex | Spinor Helicity Amplitude |
| \( s_1 = s_2 = 0, s_3 = s \) | 1 even | 1 even and 1 odd |
| \( s_1 = 0, s_2 = s_3 = s \) | 1 even, 1 odd | 1 even and 1 odd |
| \( s_1 = s_2 = 3, s_3 \) | 2 even, 1 odd | 2 even, 2 odd |
| \( s_1 = s_2 = s_3 \) | 2 even, 2 odd | 2 even, 2 odd |
| \( s_1 = s_2 = \frac{s_3}{2} \) | 2 even, 2 odd | 2 even, 2 odd |
| \( s_1 = s_2 \) | 2 even, 2 odd | 3 even, 3 odd |
| \( s_1 \neq s_2 \neq s_3 \) | 2 even, 2 odd | 4 even, 4 odd |

**Table 5.** Mismatch between spinor helicity amplitude and covariant vertex

3.3 Resolving the mismatch?

Table 5 summarizes the mismatch between covariant amplitude and spinor helicity amplitude. In this section, we discuss briefly that demanding covariant or local structure for the four-dimensional vertex can not lead to the resolution of this mismatch. Let us take an example to illustrate this.

3.3.1 graviton-graviton-graviton

The covariant vertex in (3.12) has no parity odd minimal term whereas amplitude in spinor helicity variables (3.3) do get contribution from minimal odd amplitude. However this mismatch can be readily resolved if we consider parity odd minimal amplitudes defined in
If we convert this parity odd minimal amplitude in spinor helicity variables we get

\[
[M'_n]^{222}_{+-} = i \frac{12}{(23)^2(31)^2} \quad [M'_n]^{222}_{++} = -i \frac{12}{(23)^2(31)^2} \\
[M'_n]^{222}_{--} = [M'_n]^{222}_{++} = 0
\]

which are precisely the missing term in (3.12) as compared to (3.3). Let us emphasize again that minimal odd amplitude defined in (2.16) can not be converted into a covariant 4D amplitude even though it matches with spinor helicity amplitude. One can show that this is a general feature. In other words, it is easy to convert all the extra spinor helicity amplitudes in (3.1) into momentum space variables and show that all these extra spinor helicity amplitudes lead to a vertex that is not covariant.

Interestingly, in AdS space it was shown in light-cone formalism, see [25], that one can construct vertex equal in number to amplitude in spinor helicity variables in AdS [26]. These extra light-cone vertices do not have any covariant analogue. One can take the flat-space limit as well of the AdS amplitude to connect between light-cone vertex and spinor-helicity amplitudes. This can be schematically represented by

\[\begin{array}{c}
\text{4D Minkowski} \\
\text{light-cone} \\
\downarrow \\
\text{AdS light-cone} \\
\downarrow \\
\text{4D Minkowski} \\
\text{spinor-helicity} \\
\downarrow \\
\text{AdS spinor-helicity}
\end{array}\]

We'll see below that we can construct CFT correlator that reproduces correct amplitude in spinor helicity variables but does not correspond to any covariant vertex in the flat space limit.

4 Flat-space limit of CFT correlators and mismatch with flat space covariant vertex and spinor helicity amplitude

This section aims to discuss the general map of the CFT correlation function to the amplitude that we discussed in the previous sections. As we shall demonstrate, in the flat space limit, the CFT correlator obtained in [2, 4, 14, 16] in general gives rise to flat space covariant amplitude discussed in section 2 but not the full spinor helicity amplitude in (3.1). In the process, we shall see that there is no analogue of the parity odd minimal amplitude in the case of the CFT correlator.
For a general CFT correlator \( \langle J_{s_1}(k_1)J_{s_2}(k_2)J_{s_3}(k_3) \rangle \) we have over all delta function \( \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \) where \( \vec{k} \) is three dimensional momentum vector. However four dimensional amplitude has four dimensional delta function. This missing delta function can be recovered if we set zeroth component of the 4D momentum vector to be \( k^0 = |\vec{k}| \) and work in the limit \( E = |k_1| + |k_2| + |k_3| \to 0 \). We call \( E \) to be total energy and it is clear that setting \( E \to 0 \) provides us with the extra delta function \(^6\). For simplicity of notation we simply denote \( |\vec{k}_i| \) simply by \( k_i \).

### 4.1 Flat space limit of CFT correlator of exactly conserved currents: No parity odd minimal analogue

CFT correlator of conserved currents can be split up into homogeneous and non-homogeneous pieces as follows \([14]\)

\[
\langle J_{s_1}J_{s_2}J_{s_3} \rangle = \langle J_{s_1}J_{s_2}J_{s_3} \rangle_h + \langle J_{s_1}J_{s_2}J_{s_3} \rangle_{nh}.
\]

(4.1)

These satisfy following conformal ward identity

\[
\tilde{K}^\kappa \begin{pmatrix} J_{s_1} & J_{s_2} & J_{s_3} \\ \frac{k_{s_1} - 1}{k_{s_1} - 1} & \frac{k_{s_2} - 1}{k_{s_2} - 1} & \frac{k_{s_3} - 1}{k_{s_3} - 1} \end{pmatrix}_{nh} = \text{Ward identity}
\]

\[
\tilde{K}^\kappa \begin{pmatrix} J_{s_1} & J_{s_2} & J_{s_3} \\ \frac{k_{s_1} - 1}{k_{s_1} - 1} & \frac{k_{s_2} - 1}{k_{s_2} - 1} & \frac{k_{s_3} - 1}{k_{s_3} - 1} \end{pmatrix}_h = 0
\]

(4.2)

where \( \tilde{K}^\kappa \) is a special conformal generator. It was further shown in \([16, 17]\) that homogeneous and non-homogeneous pieces can further be identified as

\[
\langle J_{s_1}J_{s_2}J_{s_3} \rangle_h = \frac{\langle J_{s_1}J_{s_2}J_{s_3} \rangle_{FB} - \langle J_{s_1}J_{s_2}J_{s_3} \rangle_{FF}}{2}
\]

\[
\langle J_{s_1}J_{s_2}J_{s_3} \rangle_{nh} = \frac{\langle J_{s_1}J_{s_2}J_{s_3} \rangle_{FB} + \langle J_{s_1}J_{s_2}J_{s_3} \rangle_{FF}}{2}
\]

(4.3)

where FB and FF stand for free bosonic and free fermionic theory respectively. It was shown in \([29]\) that for exactly conserved currents satisfying triangle inequality \( s_i \leq s_j + s_k \), there is also a parity odd contribution. It was shown in \([14]\) that parity odd contribution is always homogeneous and further it was shown in \([16]\) that the parity odd term can be calculated using free bosonic and free fermion answer using some epsilon transformation as follows

\[
\langle J_{s_1}J_{s_2}J_{s_3} \rangle_{h,o} = \mathcal{O}_\epsilon \left( \frac{\langle J_{s_1}J_{s_2}J_{s_3} \rangle_{FB} - \langle J_{s_1}J_{s_2}J_{s_3} \rangle_{FF}}{2} \right)
\]

(4.4)

In the last line of (4.4) we have used the fact that epsilon transform of parity-even H term produces parity-odd H term, see \([16]\) for more details. In the flat-space limit one obtains

---

\(^6\)There are other ways to get the flat space amplitude starting from CFT correlation function. For example, see \([40]\) for a recent discussion on the flat-space limit of the CFT correlator, where one of the scaling dimensions of the CFT correlator was taken to be large.
the following map

\[ \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{n h, e} \rightarrow M_{m, e}^{s_1 s_2 s_3} \]
\[ \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{h, e} \rightarrow M_{m, e}^{s_1 s_2 s_3} \]
\[ \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{h, o} \rightarrow M_{m, o}^{s_1 s_2 s_3} \] (4.5)

This implies that parity-odd minimal amplitude in section 2 is not reproduced by CFT correlators of conserved currents in the flat space limit. When the spin violates the triangle inequality \( s_i > s_j + s_k \), as was shown in [41] the only contribution to correlation function comes from non-homogeneous pieces. Furthermore, for exactly conserved currents there is no parity-odd contribution for this configuration. One obtains the following map in this case

\[ \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{n h, e, 1} = \frac{\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{F B} + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{F F}}{2} \rightarrow M_{m, e}^{s_1 s_2 s_3} \]
\[ \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{n h, e, 2} = \frac{\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{F B} - \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{F F}}{2} \rightarrow M_{m, e}^{s_1 s_2 s_3} \] (4.6)

It is very clear that, there is a mismatch with CFT correlator with flat space covariant vertex as well as with flat space spinor helicity amplitude. We summarise the mismatch in the following table.
Mismatch between amplitude and CFT correlator

| $s_1 \leq s_2 \leq s_3$ | Covariant Cubic vertex | Spinor Helicity Amplitude | CFT Correlator of conserved currents |
|-------------------------|------------------------|---------------------------|-------------------------------------|
| $s_1 = s_2 = 0, s_3 = s$| 1 even                 | 1 even and 1 odd          | 1 even                             |
| $s_1 = 0, s_2 = s_3 = s$| 1 even, 1 odd          | 4 even, 4 odd             | 1 even, 1 odd                      |
| $s_1 = 0, s_2 \neq s_3$ | 1 even, 1 odd          | 2 even, 2 odd             | 1 even, 1 odd                      |
| $s_1 \leq s_2 \leq s_3$ with $s_3 \leq s_1 + s_2$ | 2 even, 2 odd          | 4 even, 4 odd             | 2 even, 1 odd                      |
| $s_1 \leq s_2 \leq s_3$ with $s_3 > s_1 + s_2$ | 2 even, 2 odd          | 4 even, 4 odd             | 2 even                             |

Table 6. Mismatch between spinor helicity amplitude and covariant vertex and CFT correlator.

The parity-odd part for $s_3 > s_1 + s_2$ is generated when we consider the correlation function of slightly-broken higher spin currents.

CFT correlator for weakly broken HS current and flat space limit

When we consider $s_3 > s_1 + s_2$, with $s_3 > s_1 \geq s_2$, the parity even part of CFT correlator and its flat space limit is considered in (4.6). The parity odd part is non-zero for weakly broken HS current and is given by

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o} = [O_\epsilon] \frac{\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FB} - \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FF}}{2}. \quad (4.7)$$

One obtains the following map

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o} \rightarrow M_{nn,o}^{s_1 s_2 s_3}. \quad (4.8)$$

Explicit examples are worked out in the appendix B. In the following section, we construct a parity-odd CFT correlator which in the flat-space limit produces correct parity-odd minimal amplitude. Correlation function involving one or more scalars maps trivially to the flat-space limit. For details see Appendix B. Let us note that there is no analogue of parity-odd minimal amplitude in CFT as can be seen in (4.5), (4.8) when we consider exactly conserved currents or weakly broken currents. In the next section, we construct such a parity odd CFT correlator which in the flat space limit reproduces missing parity odd covariant amplitude.

5 CFT Correlator/ covariant vertex correspondence: A new parity-odd CFT correlation function

In this section, we construct parity-odd CFT correlation function which in the flat-space limit goes over to parity-odd minimal amplitude listed in section 2. In the case of flat-space amplitude, we explicitly showed in section 2.2 that the parity-odd minimal amplitude is obtained from parity-even minimal amplitude by doing what is called epsilon transform. Our proposal for the CFT correlation function is exactly the analogue of the amplitude case.

We propose

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o} = [O_\epsilon] \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,e} = [O_\epsilon] \frac{\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FB} + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FF}}{2}. \quad (5.1)$$
which by construction produces the correct flat-space minimal parity-odd amplitude\(^7\). Since this is non-homogeneous, this CFT correlator satisfies Ward-Takahashi (WT) identity and is given by

\[
WT[(J_{s_1}J_{s_2}J_{s_3})_{nh,o}] = [O_i] \frac{WT_{FB} + WT_{FF}}{2}. \tag{5.2}
\]

When the spins satisfy triangle inequality \(s_i \leq s_j + s_k\), we have \(WT_{FB} = WT_{FF}\) \([41]\) which gives

\[
WT[(J_{s_1}J_{s_2}J_{s_3})_{nh,o}] = [O_i]WT_{FB}. \tag{5.3}
\]

For correlators with spin which violates the triangle inequality, we have \(WT_{FB} \neq WT_{FF}\) and hence we have to use (5.1). The explicit form of WT identity for general spins can be very complicated. Below we work out a few simple examples of this correlator (5.1) and their flat-space limit. We also discuss the WT identity (5.2).

### 5.1 Example: \(\langle JJT \rangle\)

Let us consider the simplest example of two spin-1 and one spin-2 operators. The parity even part of the CFT correlator is given by [2, 14]

\[
\langle J(z_1, k_1)J(z_2, k_2)T(z_3, k_3) \rangle = \langle J(z_1, k_1)J(z_2, k_2)T(z_3, k_3) \rangle_{nh,e} + \langle J(z_1, k_1)J(z_2, k_2)T(z_3, k_3) \rangle_{nh,o}
\]

\[
= \frac{2(3k_3 + E)}{E^4}(z_3, k_2)^2(z_2, k_3)(z_1, k_3) + \frac{2k_3}{E^3} - \frac{2(k_3 + E)}{E^2}(z_3, k_2)^2(z_1, z_2)
\]

\[
+ \left[ \frac{1}{E^3}(-2k_3^2 - k_2^2 + k_1^2 - 3k_3k_2 + 3k_3k_1) - \frac{2(k_3 + E)}{E^2} \right](z_3, z_2)(z_3, z_1)
\]

\[
+ \left[ \frac{1}{E^3}(-2k_3^2 - k_2^2 + k_1^2 - 3k_3k_2 + 3k_3k_1) - \frac{2(k_3 + E)}{E^2} \right](z_3, z_2)(z_2, z_1)
\]

\[
+ \frac{(k_3 + E)(k_3^2 - (k_2 + k_1)^2 + 4k_2k_1)}{2E^2} + \frac{2k_3}{E} - k_2 - k_1 \right](z_1, z_3)(z_2, z_2) \tag{5.4}
\]

where term proportional to \(c_J\) is non-homogeneous parity even part and rest is parity even homogeneous contribution. In the flat-space limit we get

\[
\lim_{E \to 0} \langle J(z_1, k_1)J(z_2, k_2)T(z_3, k_3) \rangle = \frac{6k_3}{E^4}(z_3, k_2)^2(z_2, k_3)(z_1, k_2) + O(\frac{1}{E^3})
\]

\[
+ \frac{2k_3}{E^2}(z_3, z_2)(z_3, k_1) \tag{5.5}
\]

which perfectly match with parity even minimal and non-minimal vertices

\[
M_{n,m,e} = (z_3, k_1)^2(z_2, k_3)(z_1, k_2) \quad M_{m,e} = (z_1, z_2)(z_3, k_1) \tag{5.6}
\]

\[
\]
The non-minimal parity odd amplitude is obtained by taking the flat space limit of parity odd homogeneous contribution. The parity odd CFT correlator can be found in [14]. It is easy to show that in the flat space limit
\[
\lim_{E \to 0} \langle JJT \rangle_{h,o} \to M_{nm,o}.
\] (5.7)

However as it is clear, there is no analogue of the CFT correlator which in the flat space limit reproduces correct flat space minimal parity odd amplitude.

We now show using definition (5.1) we get a CFT correlator which in the flat space reproduces correct parity odd minimal amplitude. As was discussed below (2.18), in this case also we need to introduce Chan-Paton factors and anti symmetric with respect to two spin-1 currents to get parity-odd non-homogeneous results. Using the proposal (5.1), we see that for \(\langle TJJ \rangle_{nh,odd} \), we get
\[
\langle JJT \rangle_{nh,o} = \left( \left[ O_\epsilon \right]_2 - \left[ O_\epsilon \right]_1 \right) \frac{(JJT)_{FB} + (JJT)_{FF}}{2}
\]
\[ = A(k_1, k_2, k_3)z_1z_2z_3\epsilon(z_1k_1k_3) + B(k_1, k_2, k_3)z_1z_2z_3\epsilon(z_1k_1k_3)
\]
\[ + C(k_1, k_2, k_3)z_1z_2z_3\epsilon(z_1k_1k_3) + D(k_1, k_2, k_3)(z_1k_1k_3)^2\epsilon(z_1k_1z_2)
\]
\[ - (2 \leftrightarrow 1) \] (5.8)

where
\[
D = C = - A = \frac{E + k_3}{k_2E^2} \quad B = - \frac{1}{2k_2} \left( -k_2 - k_1 + \frac{2k_3^2}{E} \right). \quad (5.9)
\]

This new parity odd CFT correlator has a pole only in the total energy \(E\). The ward identity can be obtained using the proposal (5.2)
\[
\langle JJk_3,T \rangle_{nh,o} = \left( \left[ O_\epsilon \right]_2 - \left[ O_\epsilon \right]_1 \right) \langle JJk_3,T \rangle_{nh,e}
\]
\[ = \frac{k_1}{k_2} [(z_1k_2)(z_2k_3) - (z_2k_2)(z_3k_1)] - z_3k_1(2k_2z_1)
\]
\[ - (2 \leftrightarrow 1) \] (5.10)

One can confirm that (5.8) and (5.10) are consistent with each other by going to Spinor-Helicity variables and checking conformal ward identity.

In the flat-space limit we obtain
\[
\lim_{E \to 0} \langle JJT \rangle_{nh,o} \sim \frac{1}{E^2} \left( - (z_2z_3k_2) k_3 - (z_2z_3k_3) k_2 (z_3 \cdot z_1) + \epsilon(z_1z_2z_3) k_2 (z_3 \cdot k_1) \right) + \mathcal{O}\left( \frac{1}{E} \right)
\] (5.11)

which matches with minimal odd amplitude in (A.3) and converting it in 4D notation we obtain amplitude given in (A.1).

Even though the correlator that has been constructed has a nice behaviour that it has only total energy singularity, it also has some other unusual properties which become clear in position space.
5.1.1 In position space

In position space, the epsilon transform is given by

\[ [O_\epsilon]_I : (O_1(x_1) \cdots O^{\mu_1 \cdots \mu_r}(x_I) \cdots O_n(x_n)) \rightarrow \epsilon_{\sigma \alpha} \int \frac{d^3y_I}{|x_I - y_I|^2} \partial^\sigma_O (O_1(x_1) \cdots O^{\mu_1 \cdots \mu_r}(y_I) \cdots O(x_n)) \]

(5.12)

From the above, it is clear that the transformation is not so straightforward in position space, unlike the momentum space.

Let us look into the simplest example \( \langle TJJ \rangle \). To start with, let us consider the epsilon transform of ward identity. The ward identity for \( \langle TJJ \rangle \) in position space is given by

\[
\partial^\rho_3 \langle J_\rho(x_1) J_\sigma(x_2) T_{\mu \nu}(x_3) \rangle = \partial_{3 \nu} \delta^{(3)}(x_3 - x_2) \langle J_\sigma(x_3) J_\rho(x_1) \rangle - \partial_{3 \mu} \delta^{(3)}(x_3 - x_2) \delta_{\nu \sigma} \langle J_\mu(x_3) J_\rho(x_1) \rangle
\]

+ \( \partial_{\nu} \delta^{(3)}(x_1 - x_3) \langle J_\sigma(x_2) J_\rho(x_3) \rangle - \partial_{\mu} \delta^{(3)}(x_1 - x_3) \delta_{\nu \rho} \langle J_\sigma(x_2) J_\mu(x_3) \rangle \)

(5.13)

Using (5.12) in the above equation, we get

\[
\partial^\rho_3 \langle J_\rho(x_1) J_\sigma(x_2) T_{\mu \nu}(x_3) \rangle_{nh, o} = \left( [O_\epsilon]_{12} - [O_\epsilon]_1 \right) \partial^\rho_3 \langle J_\rho(x_1) J_\sigma(x_2) T_{\mu \nu}(x_3) \rangle
\]

\[
= \epsilon_{\sigma \alpha} \partial_{2 \mu} \partial_{3 \nu} \frac{1}{|x_2 - x_3|^2} (J^\mu(x_3) J^\nu(x_1)) - \epsilon_{\sigma \alpha \nu} \partial_{2 \mu} \partial_{3 \nu} \frac{1}{|x_2 - x_3|^2} (J^\mu(x_3) J^\nu(x_1))
\]

+ \( \partial_{3 \nu} \delta^{(3)}(x_1 - x_3) \langle J^\nu(x_2) J^\rho(x_3) \rangle_o - \partial_{3 \mu} \delta^{(3)}(x_1 - x_3) \delta_{\nu \rho} \langle J^\nu(x_2) J^\rho(x_3) \rangle_o \)

\[- [(2, \sigma) \leftrightarrow (1, \rho)] \]

(5.14)

In the first line, we made use of the properties of the delta function and we also have made use of

\[
\langle J_\mu(x) J_\nu(y) \rangle_o = \epsilon_{\mu \sigma \alpha} \int \frac{d^3x_1}{|x - x_1|^2} \partial^\rho_3 \langle J^\rho(x_1) J_\nu(y) \rangle_o \]

(5.15)

in the second line. The RHS of (5.14) gives

\[
\partial^\rho_3 \langle J_\rho(x_1) J_\sigma(x_2) T_{\mu \nu}(x_3) \rangle_{nh, o}
\]

\[
= \epsilon_{\sigma \alpha} \left( -2 \delta^\rho_\mu \frac{2x_3^\rho x_1^\mu}{x_3^2} + 8x_3^\rho x_1^\mu x_2^\sigma x_3^\nu \right) \left( \delta^\nu_\sigma \frac{x_2^\nu}{x_3^2} - 2x_2^\nu x_3^\rho x_1^\mu x_3^\sigma \right)
\]

\[- \epsilon_{\sigma \alpha \nu} \left( -2 \delta^\rho_\mu \frac{2x_3^\rho x_1^\mu}{x_3^2} + 8x_3^\rho x_1^\mu x_2^\sigma x_3^\nu \right) \left( \delta^\nu_\sigma \frac{x_2^\nu}{x_3^2} - 2x_2^\nu x_3^\rho x_1^\mu x_3^\sigma \right)
\]

\[- \partial_{3 \nu} \delta^{(3)}(x_1 - x_3) \epsilon_{\sigma \rho \nu} \partial^\sigma_2 \delta^{(3)}(x_2 - x_3) - \partial_{3 \mu} \delta^{(3)}(x_1 - x_3) \delta_{\nu \rho} \epsilon_{\sigma \mu \tau} \partial^\rho_2 \delta^{(3)}(x_2 - x_3)
\]

\[- [(2, \sigma) \leftrightarrow (1, \rho)] \]

(5.16)

where in the last line we have removed contact terms. These contact terms will give rise to a contact term in correlation function, see \cite{14} for similar discussion. However, it is important to note that, unlike in (5.13), no delta function appears in (5.16). This implies that spin-2 current is not conserved even away from coincident points. This implies we can’t identify this spin-2 current as a stress tensor. This is as expected as for exactly conserved current we can’t have more than three structures.
WT identity for \(\langle JT, J_3\rangle\)

The fact that Ward-Takahashi identity is non zero even away from contact points, is a universal fact for non-homogeneous parity-odd terms, which can be checked easily. Consider now the WT identity for \(\langle J_1, J_2, J_3\rangle\)

\[
\partial_3^2 \langle J_1(x_1)T_{\nu\rho}(x_2)J_{\alpha\beta\gamma}(x_3)\rangle \sim \partial_3 \delta^{(3)}(x_3 - x_1)\langle T_{\alpha\beta}(x_3)T_{\nu\rho}(x_2)\rangle + \partial_3(\alpha \delta^{(3)}(x_3 - x_1)\langle T_{\beta\mu}(x_3)T_{\nu\rho}(x_2)\rangle
\]

\[
+ (3\partial_\alpha \partial_\beta - \delta_\alpha\beta \Box_3)\partial_3(\delta^{(3)}(x_3 - x_2)\langle J_\rho(x_3)J_{\mu}(x_1)\rangle + (3\partial_\alpha \partial_\beta - \delta_\alpha\beta \Box_3)\partial_3(\delta^{(3)}(x_3 - x_2)\langle J_\beta(x_3)J_{\mu}(x_1)\rangle)
\]

(5.17)

After an epsilon transform we get

\[
[O_{|1}\partial_3^2 \langle J_1(x_1)T_{\nu\rho}(x_2)J_{\alpha\beta\gamma}(x_3)\rangle \sim \partial_1 \mu \frac{1}{|x_1 x_2|^2} \langle T_{\alpha\beta}(x_3)T_{\nu\rho}(x_2)\rangle + \partial_1(\alpha \delta^{(3)}(x_3 - x_1)\langle T_{\beta\mu}(x_3)T_{\nu\rho}(x_2)\rangle
\]

\[
+ (3\partial_\alpha \partial_\beta - \delta_\alpha\beta \Box_3)\partial_3(\delta^{(3)}(x_3 - x_2)\langle J_\rho(x_3)J_{\mu}(x_1)\rangle) + (3\partial_\alpha \partial_\beta - \delta_\alpha\beta \Box_3)\partial_3(\delta^{(3)}(x_3 - x_2)\langle J_\beta(x_3)J_{\mu}(x_1)\rangle)
\]

(5.18)

where

\[
\langle J_\alpha(x_1)J_{1\beta}(x_2)\rangle = \epsilon_{\alpha\beta\lambda}(x_1 - x_2)
\]

(5.19)

Just like \(\langle J T T \rangle\), we see that an epsilon transform at \(x_1\) gives rise to contact terms in the second line of (5.18) and is therefore dropped. We also get terms that do not vanish at non-coincident points. Hence, we see that the epsilon transform of conserved current correlations gives rise to correlations that are not conserved. Let us consider another example, all equal spin \(TTT\).

5.2 \(\langle TTT\rangle\)

Consider now the \(\langle TTT\rangle_e\) correlator in momentum space

\[
\langle T(z_1,k_1)T(z_2,k_2)T(z_3,k_3)\rangle_e = c_1 \frac{k_1 k_2 k_3}{E^6} [2z_1 k_2 z_2 k_3 z_3 k_1 + E(k_1 z_1 z_2 z_3 k_1 + k_2 z_2 z_3 z_1 k_2 + k_3 z_3 z_1 z_2 k_3)]
\]

\[
+ c_T \left( \frac{k_1 k_2 k_3}{E^2} + \frac{k_1 k_2 + k_2 k_3 + k_3 k_1}{E} - E \right) (z_1 z_2 z_3 k_1 + z_2 z_3 z_1 k_2 + z_3 z_1 z_2 k_3)^2
\]

(5.20)

which in the flat-space limit gives

\[
\lim_{E \to 0} \langle T(z_1,k_1)T(z_2,k_2)T(z_3,k_3)\rangle_e = c_1 \frac{k_1 k_2 k_3}{E^6} [2z_1 k_2 z_2 k_3 z_3 k_1 + O(\frac{1}{E^5})]
\]

\[
+ c_T \frac{k_1 k_2 k_3}{E^2} (z_1 z_2 z_3 k_1 + z_2 z_3 z_1 k_2 + z_3 z_1 z_2 k_3)^2 + c_T O(\frac{1}{E})
\]

(5.21)
which perfectly matches the minimal and non-minimal parity even vertices in (2.15). For parity-odd case \(^8\), we have

\[
\langle T(z_1, k_1)T(z_2, k_2)T(z_3, k_3) \rangle_{h.o} = (k_1k_2k_3) \frac{1}{E^3} \left[ \left( \bar{k}_1 \cdot \bar{z}_3 \right) \left( \epsilon^{k_1z_1z_2k_1} - \epsilon^{k_1z_1z_3k_1} \right) + \left( \bar{k}_3 \cdot \bar{z}_2 \right) \left( \epsilon^{k_1z_1z_3k_2} - \epsilon^{k_1z_1z_3k_1} \right) \\
- (\bar{z}_2 \cdot \bar{z}_3) \epsilon^{k_1z_1z_3k_2} E + \frac{k_1}{2} \epsilon^{z_1z_2z_3} E (E - 2k_1) \right] + \text{cyclic perm}
\]

\[
\left[ \frac{1}{E^3} \left\{ 2 \left( \bar{z}_1 \cdot \bar{k}_2 \right) \left( \bar{z}_2 \cdot \bar{k}_3 \right) \left( \bar{z}_3 \cdot \bar{k}_1 \right) + E \left\{ k_3 (\bar{z}_1 \cdot \bar{k}_2) + \text{cyclic} \right\} \right\} \right] (5.23)
\]

In the flat-space limit, which becomes

\[
\lim_{E \to 0} \langle TTT \rangle_{h,o} = \frac{k_1k_2k_3}{E^6} \left[ \left( \bar{k}_1 \cdot \bar{z}_3 \right) \left( \epsilon^{k_1z_1z_2k_1} - \epsilon^{k_1z_1z_3k_1} \right) + \text{cyclic perm} \right] \left( \bar{z}_1 \cdot \bar{k}_2 \right) \left( \bar{z}_2 \cdot \bar{k}_3 \right) \left( \bar{z}_3 \cdot \bar{k}_1 \right) (5.24)
\]

which is precisely the non-minimal parity-odd cubic vertex mentioned in (2.15). Since the three-point function of conserved currents at maximum can only have three structures, we see that just like cubic vertex we do not have any analogue of parity odd-minimal amplitude at the level CFT correlation function.

However, let us define another parity-odd structure namely

\[
\langle TTT \rangle'_o = \left[ [O_1]_1 + [O_2]_2 + [O_3]_3 \right] \langle TTT \rangle_{n.h.e}
\]

\[
= \frac{E^3 - E (k_1k_2 + k_2k_3 + k_3k_1) - k_1k_2k_3 (z_1.k_2z_2.z_3 + z_1.z_2z_3.k_1 + z_2.k_3z_3.z_1)}{E^2k_1k_2k_3}
\]

\[
[k_2k_3z_2.z_3\epsilon(z_1k_1k_2) + \text{cyclic terms} - k_1k_2k_3 E \epsilon(z_1z_2z_3)] (5.25)
\]

which in the flat-space limit gives

\[
\lim_{E \to 0} \langle TTT \rangle'_o \sim -\frac{c_{123}}{E^2} |V|^2 + O\left( \frac{1}{E} \right) (5.26)
\]

which is precisely what we computed in (2.16). As mentioned before, this flat-space limit cannot be recast as a 4D flat-space amplitude. It is easy to show that, spin two current that appears in (5.25) is not conserved. To show this we work in position space.

\[\text{Following [14], one can write parity-odd non-homogeneous piece as follows}
\]

\[
\langle TTT \rangle_{n.h.o} = \frac{1}{24} \left[ \epsilon(z_1z_2k_1)(z_1.z_2)(z_3.k_1)^2 - \epsilon(z_1z_2k_2)(z_1.z_3)(z_2.z_3) \right] \\
+ \frac{1}{12} \left[ (z_1.z_3)(z_2.z_3) \epsilon(z_1z_2k_1)(k_1^2 + \frac{7}{4}k_2^2 + \frac{7}{4}k_3^2) \right] - (z_1.z_2)(z_3.k_1)^2 \epsilon(z_1z_2k_2)(k_2^2 + \frac{7}{4}k_1^2 + \frac{7}{4}k_3^2) + \text{cyclic terms} (5.22)
\]

However, one can see that \(\langle TTT \rangle_{n.h.o} \) is just a contact term. This does not correspond to any cubic vertex.
5.2.1 Epsilon transform WT identity of $\langle TTT \rangle$ in position space

Consider the $\langle TTT \rangle$ ward identity in position space

$$\partial^\mu \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}\rangle = \partial_\nu \delta^{(3)}(x-y)\langle T_{\sigma\rho}(x)T_{\alpha\beta}(z)\rangle + \{ \partial_\sigma \delta^{(3)}(x-y)\langle T_{\rho\nu}(x)T_{\alpha\beta}(z)\rangle + \sigma \leftrightarrow \rho \} + \partial_\alpha \delta^{(3)}(x-z)\langle T_{\sigma\rho}(y)T_{\alpha\beta}(x)\rangle + \{ \partial_\alpha \delta^{(3)}(x-z)\langle T_{\beta\nu}(x)T_{\sigma\rho}(y)\rangle + \alpha \leftrightarrow \beta \}$$

(5.27)

where we now perform an epsilon transform and just like for the case of $\langle JJT \rangle$ we find that

$$[O]_y \partial^\mu \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}\rangle 
= \epsilon_{\sigma\eta\varsigma} \partial^\varsigma \partial_\nu \left( \frac{1}{|x-y|^2} \langle T_{\eta\rho}(x)T_{\alpha\beta}(z)\rangle + \partial_\rho (\epsilon_{\sigma\eta\varsigma} \partial^\varsigma \frac{1}{|x-y|^2} \langle T_{\eta\rho}(x)T_{\alpha\beta}(z)\rangle) + \epsilon_{\sigma\eta\varsigma} \partial^\varsigma \frac{1}{|x-y|^2} \langle T_{\eta\rho}(x)T_{\alpha\beta}(z)\rangle \right) + \partial_\alpha \delta^{(3)}(x-z)\langle T_{\sigma\rho}(y)T_{\alpha\beta}(x)\rangle \text{odd} + \{ \partial_\alpha \delta^{(3)}(x-z)\langle T_{\beta\nu}(x)T_{\sigma\rho}(y)\rangle \text{odd} + \alpha \leftrightarrow \beta \} + \epsilon_{\sigma\eta\varsigma} \partial^\varsigma \partial_\nu \left( \frac{1}{|x-y|^2} \langle T_{\eta\rho}(x)T_{\alpha\beta}(z)\rangle + \partial_\rho (\epsilon_{\sigma\eta\varsigma} \partial^\varsigma \frac{1}{|x-y|^2} \langle T_{\eta\rho}(x)T_{\alpha\beta}(z)\rangle) + \epsilon_{\sigma\eta\varsigma} \partial^\varsigma \frac{1}{|x-y|^2} \langle T_{\eta\rho}(x)T_{\alpha\beta}(z)\rangle \right)$$

(5.28)

Again, the Ward identity has terms that survive at non-coincident points and therefore, showing that the epsilon transform leads to a non-conserved spin-2 current.

We consider a few more explicit examples of the proposal in (5.1) in the Appendix.

D. We also discuss converting this new parity odd CFT correlator in spinor helicity variables and connecting to spinor helicity amplitude in the flat-space limit. To summarize this subsection, we have shown that the new parity odd CFT correlator defined in (5.1) reproduces minimal parity odd flat space amplitude. However, as the table 6 indicate, even though we have been able to deal with a mismatch of CFT correlator and covariant vertex, the mismatch of counting between CFT correlator and flat space spinor helicity amplitude remains. We summarize this by the following diagram.

In the next section, we address this issue of mismatch of CFT correlator with spinor helicity variable amplitude.
6 CFT correlator/spinor helicity amplitude correspondence: Resolving the mismatch

The table 6 highlights the mismatch between spinor helicity amplitude and CFT correlator of conserved currents. This section aims to show that one can construct an extra CFT correlator of conserved currents which in the flat space limit goes over to flat space spinor helicity amplitude. Interestingly it turns out that these new CFT correlators do not give rise to any consistent covariant 4d amplitude. Let us start the discussion with the simplest of the cases.

6.1 Two scalar one spinning three point function

Let us consider CFT correlator of the form \( \langle J^s_1 O O \rangle \). In the table 6 it is summarised that there is only one parity even CFT correlator whereas amplitude has one parity even and one parity odd component. To take care of this mismatch we define epsilon transformed correlator

\[
\langle J^s_1 O O \rangle^o = [O_\epsilon] \langle J^s_1 O O \rangle^e.
\] (6.1)

As a illustrative example let us consider \( \langle T O_2 O_2 \rangle \) which is given by

\[
z^\mu_3 z^{\nu}_3 \langle O_2(k_1) O_2(k_2) T_{\mu\nu}(k_3) \rangle_{nh,e} = \frac{k_1 + k_2 + 2k_3}{(k_1 + k_2 + k_3)^2} (k_1 \cdot z_3)^2 \] (6.2)

where \( O_2 \) is scalar operator with scaling dimension 2 and \( z_3 \) is transverse polarization tensor.

The solution in (6.2) satisfies ward identity given by

\[
\langle O_{Ok_3}.T \rangle = k_2 \cdot z_3 \left( \langle O(k_1)O(-k_1) \rangle - \langle O(k_2)O(-k_2) \rangle \right) \] (6.3)

Using proposal (6.1) we obtain

\[
z^\mu_3 z^{\nu}_3 \langle O_2(k_1) O_2(k_2) T_{\mu\nu}(k_3) \rangle^o = \frac{k_1 + k_2 + 2k_3}{k_3 (k_1 + k_2 + k_3)^2} (k_1 \cdot z_3) \epsilon(z_3 k_1 k_3) \] (6.4)

In the flat space limit we obtain

\[
\lim_{E\rightarrow0} z^\mu_3 z^{\nu}_3 \langle O_2(k_1) O_2(k_2) T_{\mu\nu}(k_3) \rangle^o \sim \frac{1}{E^2} (k_1 \cdot z_3) \epsilon(z_3 k_1 k_3) + \text{subleading terms} \] (6.5)

which implies \( A \sim (k_1 \cdot z_3) \epsilon(z_3 k_1 k_3) \) which can not be converted to a four dimensional covariant expression \(^9\). However it is easy to show that in spinor helicity variables (6.2), (6.4) can be written as

\[
\langle OOT \rangle = (c_e + ic_o) \frac{(k_1 + k_2 + 2k_3) (k_3 - k_1 - k_2)^2}{k_3^2 (k_1 + k_2 + k_3)^2} \left( \frac{\langle 23 \rangle \langle 13 \rangle}{\langle 12 \rangle} \right)^2 \] (6.6)

and its conjugate. It is clear that in the flat space limit (6.6) reproduces both parity even and parity odd amplitude correctly that appears in (3.1). Again it is easy to show by going

---

\(^9\)Four dimensional covariant epsilon will have four free indices. Because of momentum conservation and only one polarization tensor \( z_3 \) we can’t write the expression in terms of four dimensional epsilon tensor.
to position space that the result in (6.4) corresponds to spin two non conserved current as we shaw in the last section 10.

It turns out that story in CFT side is much more richer. One can also find more solution to conformal ward identity (4.2) for \(\langle O O J \rangle\). Solution (6.2) is non-homogeneous, one can also find homogeneous solutions

\[
\langle O O T \rangle = (c'_e + ic'_o)\frac{k_3}{E^2} \left(\frac{\langle 23\rangle \langle 13\rangle}{\langle 12\rangle}\right)^2. \tag{6.9}
\]

If we convert the solution (6.9) in momentum space we obtain

\[
\langle O O T \rangle = \frac{k_3^3}{(k_1 + k_2 + k_3)^2 (k_3 - k_1 - k_2)^2} (z_3 k_1)^2 \tag{6.10}
\]

which has bad pole at \(k_3 = k_1 + k_2\) and is not consistent with position space OPE limit [8, 41]. However, the flat space limit produces correct spinor helicity amplitude.

We now consider another example \(\langle J J J J \rangle\). In this case, we have considered correlators that saturate the WT identity, see Appendix D for details. To construct a new CFT correlator we need to construct a homogeneous solution that does not contribute to WT identity as we did above.

### 6.2 General spin \(s_1 \neq s_2 \neq s_3 \neq 0\)

It can be shown that for a general spin as well the flat space limit, in general, reproduces the correct covariant vertex but not the full amplitude in spinor helicity variables. This implies the flat space limit of the CFT correlator can not reproduce all the spinor helicity amplitudes. As is discussed earlier, in this case, we have four CFT correlators, three coming from conserved currents and one constructed in (5.1). However, spinor helicity amplitude has a total of eight independent structures. It is easy to establish that the mismatch is in the helicity components \(+++,--,-+-,---,+++,+++\) as is the case in (3.14) and (3.15). In the following, we show that this can be remedied by considering homogeneous solutions 11.

Consider the following ansatz for \(\langle J_{s_1} J_{s_2} J_{s_3} \rangle\) in spinor-helicity variables [41]

\[
\langle J_{s_1}^{h_1} J_{s_2}^{h_2} J_{s_3}^{h_3} \rangle = f_{h_1, h_2, h_3} (k_1, k_2, k_3) \langle 12 \rangle^{h_2 s_2 - h_3 s_3 - h_1 s_1} (3.14)^{h_1 s_1 - h_2 s_2 - h_3 s_3} \langle 31 \rangle^{h_2 s_2 - h_3 s_3 - h_1 s_1} \tag{6.11}
\]

10More precisely, the \((TOO)\) ward identity in position space is given by

\[
\partial_\mu \langle T_{\mu \nu}(x_1) O(x_2) O(x_3) \rangle = \partial_\nu \delta^{(4)}(x_1 - x_2) \langle O(x_1) O(x_2) O(x_3) \rangle + \partial_\nu \delta^{(3)}(x_1 - x_3) \langle O(x_1) O(x_2) O(x_3) \rangle \tag{6.7}
\]

The epsilon transform of the above yields

\[
\langle O_1 | \epsilon^\mu \langle T_{\mu \nu}(x_1) O(x_2) O(x_3) \rangle = \epsilon_{\nu \alpha \tau} \left( \partial_\alpha \frac{1}{|x_{12}|^2} \partial_\tau (O(x_2) O(x_3)) + \partial_\alpha \frac{1}{|x_{13}|^2} \partial_\tau (O(x_1) O(x_2)) \right) \tag{6.8}
\]

which is non-zero even away from insertion of the operators.

11Let us note that, we can’t have more non-homogeneous solutions as we have already saturated the WT identity with non-homogeneous parity even solution.
Using the homogeneous ward identity, one finds that the most general solution to \( f \) is given by \([41]\)

\[
f_{h_1,h_2,h_3}(k_1,k_2,k_3) = k_1^{s_1} - \frac{k_2^{s_2} - 1}{E^{-h_1s_1 - h_2s_2 - h_3s_3}}
\]

(6.12)

Except in all \(-\) or all \(+\) helicity, both the solutions in (6.12) have bad poles and are not consistent with the position space OPE limit, see \([41]\) for more discussion. For all \(-\) or all \(+\) helicity also only the first solution in (6.12) is consistent. We can use the solution in (6.12) in the flat space limit to reproduce missing helicity components \(+ - +, - + -\) and \(- - +, + + -\). However as mentioned, it can be shown easily that they don’t reproduce any local covariant vertex. Also, the CFT correlator corresponding to these helicities are not consistent with the position space OPE limit because of the presence of bad poles \([41]\).

Notice that in the flat-space limit we have

\[
\lim_{E \to 0} \left\langle J_{h_1}^{s_1} J_{h_2}^{s_2} J_{h_3}^{s_3} \right\rangle \sim \langle 12 \rangle^{h_3s_3 - h_1s_1 - h_2s_2} \langle 23 \rangle^{h_1s_1 - h_2s_2 - h_3s_3} \langle 31 \rangle^{h_2s_2 - h_3s_3 - h_1s_1}
\]

(6.13)

we obtain the non-perturbative amplitude. See Appendix C for more details of flat space limit of CFT correlators in spinor helicity variables. The above result implies that the CFT correlator has more solutions and in the flat space limit it reproduces all possible spinor helicity amplitudes. The following table summarizes the counting.

| CFT correlator vs spinor helicity amplitude |
|---------------------------------------------|
| \(s_1 \leq s_2 \leq s_3 \neq 0\)             | CFT correlator of conserved currents | Spinor Helicity Amplitude |
| \(s_1 \neq s_2 \neq s_3\)                  | 4 h even + 4 h odd + 1 nh even + 1 nh odd | 4 even , 4 odd |

Table 7. This table summarizes CFT correlator counting and flat space spinor helicity amplitude in most general cases. Let us note that we have ignored the second homogeneous solution in (6.12). We have also ignored non-homogeneous parity odd contact terms analogous to (5.22). In the table \(h\) stands for homogeneous solution and \(nh\) stands for non-homogeneous solution. Except for the \(nh\) odd solution in this table, all the other solutions are for conserved currents. The \(nh\) odd solution is defined in (5.1).

We also work out an explicit example Appendix E.

7 Connection to AdS amplitude

Three-point amplitudes in AdS was calculated in spinor helicity variables in \([26, 27]\) and in light cone variables gauge in \([25]\). This section aims to make a connection with CFT results presented in previous sections with results in AdS\(^{12}\). One of the main issues with

\(^{12}\)We thank D. Ponomarev, E. Skvortsov for their insightful comments which led to better understanding of the content of this section.
identification is it is not clear which amplitude in AdS is identified with a homogeneous/non-homogeneous CFT correlator. One way to get around this difficulty is to make use of the connection between AdS amplitude and flat space amplitude. Since we also know which flat space amplitude corresponds to which CFT correlator, this gives us an indirect way to connect the CFT correlator to AdS amplitudes. Let us take an example to illustrate this connection. In [26, 27] a simple relation between flat space and AdS amplitude in spinor helicity variables was pointed out. For example for the case of spin $-0-0-2$ the explicit relation is given in eq.(7.23) of [27]

$$A_- \sim \left(\frac{\langle 23\rangle\langle 13\rangle}{\langle 12\rangle}\right)^2 \left(1 + \frac{\Box_k}{R_{AdS}^2}\right) \delta^4(k),$$  \hspace{1cm} (7.1)

which $A_-$ stands for amplitude in minus helicity. It is clear that in the limit $R_{AdS} \to \infty$ we get the flat space amplitude\(^{13}\). Let us note that the corresponding CFT correlator is as given in (6.6), (6.9).\(^{13}\)

Amplitudes in AdS in light-cone was calculated in [25]. In [28] AdS results were related to CFT amplitudes. Below we describe briefly the results that are important for us. Consider the $AdS_4$ metric in the light-cone coordinates

$$ds^2 = \frac{2dx^+dx^- + dz^2 + dx_1^2}{z^2}$$  \hspace{1cm} (7.3)

where

$$x^\pm = \frac{x_2 \pm x_0}{\sqrt{2}}$$  \hspace{1cm} (7.4)

For a massless arbitrary spin-$s$ field, the fourier transform implemented in these coordinates is [25]

$$\phi_\lambda(x^+, x^-, x_1, z) = \int \frac{dk_1 d\beta}{2\pi} e^{ik_1(x_1 + \beta x^-)} \phi_\lambda(x^+, \beta, k_1, z), \quad \lambda = \pm s$$  \hspace{1cm} (7.5)

The fields also satisfy

$$\phi_\lambda(x^+, x^-, x_1, z) = \phi_{-\lambda}(x^+, x^-, x_1, z)$$  \hspace{1cm} (7.6)

The most general cubic vertices made out of $\phi_\lambda(x^+, \beta, k_1, z)$ is given by

$$V_{\lambda_1,\lambda_2,\lambda_3} = \{ V_{R_{\lambda_1,\lambda_2,\lambda_3}}^{\lambda_1,\lambda_2,\lambda_3}, \quad H > 0 \\
V_{L_{\lambda_1,\lambda_2,\lambda_3}}^{\lambda_1,\lambda_2,\lambda_3}, \quad H < 0$$  \hspace{1cm} (7.7)

\(^{13}\)Let us note that corresponding CFT correlator is as given in (6.6), (6.9). It is not clear if the result in (7.1) corresponds to a non-homogenous solution in (6.6) or homogeneous solution in (6.9). Let us note that (7.1) can be generalized for other correlators and in general can be stated as

$$A_{AdS} \sim A_{flat\ space} \left(1 + \frac{\Box_k}{R_{AdS}^2}\right) \delta^4(k).$$  \hspace{1cm} (7.2)

For general spin, the connection between the CFT correlator and AdS is more clear in the helicity component $++-, --+$ and it’s conjugate as for CFT correlator, only homogeneous solution that exists in this case.
where \( H = h_1 s_1 + h_2 s_2 + h_3 s_3 \). We refer interested readers to eq.(1.19) of [28] for detailed discussions. In the flat-space limit, this leads to

\[
\begin{align*}
V_{R}^{\lambda_1, \lambda_2, \lambda_3} & \sim [12] \lambda_1 + \lambda_2 - \lambda_3 [23] \lambda_2 + \lambda_3 - \lambda_1 [13] \lambda_1 + \lambda_3 - \lambda_2, \quad H > 0, \\
V_{L}^{\lambda_1, \lambda_2, \lambda_3} & \sim (12) - \lambda_1 - \lambda_2 + \lambda_3 (23) - \lambda_2 - \lambda_3 + \lambda_1 (13) - \lambda_1 - \lambda_3 + \lambda_2, \quad H < 0,
\end{align*}
\]

which is equivalent to the statement made in (6.13) where we took the flat space limit of the CFT correlator to get the flat space helicity amplitude. The AdS vertex can formally be related to CFT correlator by following identification [28]

\[
\int_{AdS_4} V^{\lambda_1, \lambda_2, \lambda_3} = \sum_{n=0}^{\lfloor |H| \rfloor - 1} Q^n \Gamma(|H| - n) \left( \frac{1}{3\sqrt{2}E} \left( \sum_a \tilde{\beta}_a \left( |k_a| \pm k_a^1 \right) \right) \right)^{|H| - n} \sim \langle J_{\lambda_1} J_{\lambda_2} J_{\lambda_3} \rangle
\]

see eq.(2.15) of [28] for details. What is important here is appearance of the pole in \( \frac{1}{g} \). There are 8 different structures in the above equation\(^{14}\). This equation is very similar to the solution obtained in (6.12). Even though there is a \( 1/E \) pole in the result (7.9), when we convert this to covariant momentum space expressions as in previous section, it can have bad poles as we saw in (6.10)\(^{15}\) so they in general are not consistent with position space OPE limit. Some particular combination of the correlator will correspond to acceptable homogeneous and non-homogeneous results\(^{16}\).

For example, in [28] spin-2 case \( \langle TTT \rangle \) was discussed explicitly. For CFT, \( \langle TTT \rangle \) results are presented in spinor helicity variables in (D.12). As is clear the homogeneous part appears only in helicity \(- - - , + + + \) where as non-homogeneous piece has non-trivial contribution also in mixed helicity such as \(- - +, + + - \). The homogeneous parity even and odd parts arises due to weyl tensor cubed \( W^3 \) and from \( \bar{W} W^2 \) respectively where as non-homogeneous parity even contribution is given by Einstein term \( \sqrt{g} R \) from \( AdS \) or \( dS \) perspective. It was shown in [28] that \( V_{-2, -2} ^{+2, +2} + V_{-2, +2} ^{-2, +2} \) corresponds to non-homogeneous or Einstein gravity term where as \( V_{-2} ^{-2} \pm V_{-2} ^{+2} \) contributes to parity even and parity odd homogeneous term or Weyl cubed term. This is consistent with results in spinor helicity variables in (D.12). Using this correspondence, one can in general for \( \langle J_{s_1} J_{s_2} J_{s_3} \rangle \) one can argue that\(^ {17}\) all negative or all positive helicity components in (7.9) will correspond to parity even and parity odd acceptable homogeneous solution. Some of the mixed helicity components will lead to non-homogeneous acceptable CFT correlator and

\(^{14}\)Namely, \( (\lambda_1, \lambda_2, \lambda_3) = (- - - , - - +, + + - , - + -) \) and their conjugates. Each helicity structure and their conjugate gives rise to a parity even and a parity odd amplitude.

\(^{15}\)The relation in (7.9) is very formal and needs to be understood more properly. For example, it is not very clear how to use the relation in (7.9) to obtain what we define various homogeneous or non-homogeneous CFT correlators.

\(^{16}\)Interestingly, study of mismatch between covariant vertex and spinor helicity/light cone amplitude in was explored also in \( AdS_4 \) and it eventually lead to understanding of Chiral HS theory [39, 42]. It would be interesting to understand analogous statement in CFT side.

\(^{17}\)At least when spin satisfy triangle inequality \( s_i \leq s_j + s_k \). For correlator violating triangle inequality, things are little more complicated. See [41] for more details.
other mixed helicity component will lead to homogeneous CFT correlator with bad poles. For specific cases this relation can be made more precise.

To conclude, we observe that relation in (7.9) gives rise to in general known CFT correlators which are consistent with position space OPE limit as well as some other CFT correlator which contains bad poles and hence are not consistent with position space OPE limit.

8 Summary and Discussion

In this paper we have discussed the connection between 4D flat space covariant vertex, spinor helicity amplitude and three dimensional CFT correlation function of conserved currents. The known results are summarised in Table 6. We also have discussed a similar issue with $AdS_4$ amplitude in spinor helicity variables and light cone gauge. We first resolved the issue of mismatch of counting the number of independent covariant amplitudes with CFT correlator of conserved currents by constructing extra parity odd CFT correlator which in the flat space limit goes over to correct flat space covariant vertex. However we showed that this extra parity odd CFT correlator can not be constructed out of conserved currents as expected. Interestingly, this extra CFT correlator is consistent with the position space OPE limit. We then addressed the issue of mismatch of CFT correlator in spinor helicity variables with amplitude in spinor helicity variables. We constructed extra CFT correlators which in the flat space limit gives rise to correct spinor helicity amplitudes but do not give rise to any new covariant vertex. We showed these extra CFT structures are not consistent with the position space OPE limit. We also briefly discussed connection of newly constructed CFT structures with amplitude in $AdS_4$ in light cone gauge as well as in spinor helicity variables.

One can also establish a double copy relation involving a new parity-odd non-homogeneous CFT correlator that we defined using epsilon transformation. Let us briefly discuss this here.

Double copy relation

For the homogeneous piece we have in spinor helicity variables

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{h},e} \propto \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{h},o}.$$  \hfill (8.1)

Using such properties it is easy to establish in momentum space [43]

$$\langle J_{2s_1} J_{2s_2} J_{2s_3} \rangle_{\text{h},e} \propto \left( \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{even},\text{h}} \right)^2 \propto \left( \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{h},o} \right)^2$$

$$\langle J_{2s_1} J_{2s_2} J_{2s_3} \rangle_{\text{h},o} \propto \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{e},\text{h}} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{h},o}$$ \hfill (8.2)

using which we get

$$\langle J_{2s_1} J_{2s_2} J_{2s_3} \rangle_{\text{h},o} + \langle J_{2s_1} J_{2s_2} J_{2s_3} \rangle_{\text{o},\text{h}} \propto \left( \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{h},e} + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{h},o} \right)^2.$$  \hfill (8.3)
It was also shown that non-homogeneous structures separately satisfy double copy relation

$$\langle J_{2s_1} J_{2s_2} J_{2s_3} \rangle_{nh,e} \propto (\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,e})^2$$  \hspace{1cm} (8.4)

As discussed in the main text, even for non-homogeneous parity-odd and even pieces defined in (5.1) we have analogue of (8.1) in spinor helicity variables

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,e} \propto \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o}.$$  \hspace{1cm} (8.5)

This again implies [43]

$$\langle J_{2s_1} J_{2s_2} J_{2s_3} \rangle_{nh,e} \propto (\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,e})^2 \propto (\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o})^2$$

$$\langle J_{2s_1} J_{2s_2} J_{2s_3} \rangle_{nh,o} \propto \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,e} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o}$$  \hspace{1cm} (8.6)

using which we get

$$\langle J_{2s_1} J_{2s_2} J_{2s_3} \rangle_{nh,e} + \langle J_{2s_1} J_{2s_2} J_{2s_3} \rangle_{nh,o} \propto (\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,e} + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o})^2.$$  \hspace{1cm} (8.7)

It would be interesting to generalize the discussion for four point functions. However, this will be a significantly harder problem to address. In [11, 12] using higher spin equations in momentum space, CFT three-point functions of higher spin operators and four point functions of few simple cases were explored. It is easy to show that the three point covariant vertex, the minimal even, non-minimal even and non-minimal odd amplitudes satisfy the same higher spin equations. Infact, one uses these higher spin equations to calculate them. It will be interesting to use this strategy to calculate four point covariant amplitudes of higher spin fields as well as four point CFT correlators of higher spin operators. We should also be able to check easily if the epsilon transform that we have discussed in this paper can be used at the level of amplitudes which has been classified recently in [44]. In general the study of flat space amplitude/dS or AdS amplitude/ CFT correlator correspondence is an important one has led to important developments and there are many important avenues open to be explored.

Acknowledgments

The work of S.J is supported by the Ramanujan Fellowship. AM would like to acknowledge the support of CSIR-UGC (JRF) fellowship (09/936(0212)/2019-EMR-I). We thank S.C-Hout, E. Joung, K. Mkrtchyan for valuable email exchanges explaining their earlier works and comments on a previous version. We specially thank D. Ponomarev, E. Skvortsov for extensive email exchange explaining their works, related issues and extensive comments on a previous version of the draft. We would like to thank S. Ananth for the valuable discussion. We acknowledge our debt to the people of India for their steady support of research in basic sciences.

\[18\] See [45–51] and references therein for some recent development.
A Flat-space amplitudes: Examples

In this section, we give some simple examples of flat-space 4D scattering amplitude. We define two sets of amplitudes, one that satisfies $s_i \leq s_j + s_k$ is called inside the triangle and one that violates this is called outside the triangle. This distinction becomes very important for momentum space CFT correlators as was discussed in [41].

A.1 Inside the triangle inequality

We take a simple example of two photon and graviton scattering. The results are given by four structures, two parity-even and one parity-odd

$$
M^{112}_{\text{even}} = g_{m,e}(z_1.p_2 z_2 z_3 + z_2.p_3 z_2 z_1 + z_3.p_1 z_1 z_2) (z_3.p_1) + g_{nm,e}(z_1.p_2)(z_2.p_3)(z_3.p_1)^2
$$

$$
M^{112}_{\text{odd}} = g_{m,o}[(\epsilon(z_2.p_2 z_3 p_3)(z_3.z_1) + \epsilon(z_3.p_3 z_1 p_1)(z_2.z_3) + (\epsilon(z_1.z_2 z_3 p_2) - \epsilon(z_1.z_2 z_3 p_1))(z_3.k_1)]
$$

$$ + g_{nm,o}[(\epsilon(z_2.p_2 z_3 p_3)(z_1.p_2)(z_3.p_1)]

(A.1)

(A.2)

Both the minimal and non-minimal amplitudes are present for both parity-even and parity-odd cases. Notice that the odd minimal amplitude is antisymmetric under $1 \leftrightarrow 2$ exchange. Therefore, one needs to introduce Chan-Paton factors for that amplitude. In fact, it turns out Chan-Paton factors must be introduced for amplitudes with $s_1 = s_2 < s_3$ for even $s_3$

We rewrite the odd amplitude in 3D momentum space variables (2.4)

$$
M^{112}_{\text{odd}} = g_{m,o}[-(\epsilon(z_2.z_3 k_2) k_3 - \epsilon(z_2.z_3 k_3) k_2) (z_1 \cdot z_3) + \epsilon(z_1.z_2 z_3 k_2 (z_3 \cdot k_1)]
$$

$$ + g_{nm,o}[-(\epsilon(z_2.z_3 k_2) k_3 - \epsilon(z_2.z_3 k_3) k_2)](z_1.k_2)(z_3.k_1)

(A.3)

It is easy to show under epsilon transform that the parity-even amplitude in (A.1) maps to parity-odd amplitude (A.3). Let us for completeness show this below explicitly. Consider

$$
|O|_2 M^{112}_e = g_{m,e}(z_1.k_2 \epsilon(z_2.k_2 z_3) k_2 + z_3.z_1 \epsilon(z_2.k_2 z_3) k_2 + z_3.k_1 \epsilon(z_2.k_2 z_3) k_2)
$$

$$ + g_{nm,e}(z_1.k_2) \epsilon(z_2.k_2 z_3) k_2 (z_3.k_1)^2

(A.4)

Using the Schouten identities

$$
(z_1.k_2)\epsilon(z_2.k_2 z_3) = k_2^2 \epsilon(z_2.z_1 z_3) - z_3.k_1 \epsilon(z_2.k_2 z_3)
$$

$$
(z_3.k_1)\epsilon(z_2.k_2 z_3) = -k_2^2 \epsilon(z_2.z_3 k_3) - k_3.k_3 \epsilon(z_2.k_2 z_3)

(A.5)

(A.6)

in the above we exactly get (A.3). We have also made use of $k_I k_J = k_J.k_I$ in the above. A more abstract derivation of the same is given in Section 2.2. From here on-wards, we write all the flat-space amplitudes in 3D momentum space variables.

Three spin-s amplitude

For general three spin-s amplitude we have

$$
M^{s s s} = g_{m,e}(z_1.k_2 z_2 z_3 + z_2.k_3 z_2 z_1 + z_3.k_1 z_1 z_2)^s + g_{nm,e}(z_1.k_2)^s(z_2.k_3)^s(z_3.k_1)^s

M^{sss}_o = g_{nm,o}[-\epsilon(z_2.z_3 k_2) k_3 + \epsilon(z_2.z_3 k_3) k_2](z_1.k_2)^s(z_2.k_1)^s-1(z_3.k_1)^s-1

(A.7)

(A.8)

(A.9)
Notice no minimal amplitude present for the parity-odd case. This is as mentioned before in Table 4. The minimal term was dropped as the negative powers appearing due to $s_1 = s_2 = s_3$. For example, three photon amplitude is given by

\[
\mathcal{M}_e^{111} = g_{m,e}(z_1.k_2z_2,z_3 + z_2.k_3z_3) + g_{nm,e}(z_1.k_2)(z_2.k_3)(z_3.k_1)
\]
\[
\mathcal{M}_o^{111} = g_{nm,o}[\epsilon(z_2.z_3k_2)k_3 + \epsilon(z_2.z_3k_3)k_2](z_1.k_2)
\]

(A.8)

Another simple example is of three graviton scattering where we have

\[
\mathcal{M}_e^{222} = g_{m,e}(z_1.k_2z_2,z_3_z_1 + z_3.k_1z_1.k_2) + g_{nm,e}(z_1.k_2)(z_2.k_3)(z_3.k_1)^2
\]
\[
\mathcal{M}_o^{222} = g_{nm,o}[-\epsilon(z_2.z_3k_2)k_3 + \epsilon(z_2.z_3k_3)k_2](z_1.k_2)(z_3.k_1)
\]

(A.9)

**One scalar two spin-s amplitude**

For two photon and one scalar we have only two structures

\[
\mathcal{M}_e^{011} = g_{nm,e}(z_2.k_3)(z_3.k_1), \quad \mathcal{M}_o^{011} = g_{nm,o}[-\epsilon(z_2.z_3k_2)k_3 + \epsilon(z_2.z_3k_3)k_2]
\]

(A.10)

this can be generalised to two spin-s and one scalar.

\[
\mathcal{M}_e^{0ss} = g_{nm,e}(z_2.k_3)^s(z_3.k_1)^s, \quad \mathcal{M}_o^{0ss} = g_{nm,o}[-\epsilon(z_2.z_3k_2)k_3 + \epsilon(z_2.z_3k_3)k_2](z_2.z_3)^{s-1}
\]

(A.11)

**A.2 Outside the triangle inequality**

Let us consider one spin-s two scalar amplitude

\[
\mathcal{M}_e^{0ss} = g_e(z_3.k_1)^s.
\]

(A.12)

This only has parity-even contribution. For one scalar one spin $s_2$ and one spin $s_3$ amplitude we have

\[
\mathcal{M}_e^{0s_2s_3} = g_e(z_2.k_1)^{s_2}(z_3.k_1)^{s_3}, \quad \mathcal{M}_o^{0s_2s_3} = g_o[-\epsilon(z_2.z_3k_2)k_3 + \epsilon(z_2.z_3k_3)k_2](z_2.k_1)^{s_2-1}(z_3.k_1)^{s_3-1}
\]

(A.13)

with $s_3 > s_2$. As an example let us consider spin-3 spin-1 scalar amplitude which is given by

\[
\mathcal{M}_e^{013} = g_e(z_2.k_1)(z_3.k_1)^3, \quad \mathcal{M}_o^{013} = g_o[-\epsilon(z_2.z_3k_2)k_3 + \epsilon(z_2.z_3k_3)k_2](z_3.k_1)^2.
\]

(A.14)

Now let us consider two spin-1 and one spin-4 particles which will be useful for our purposes. We have

\[
\mathcal{M}_e^{114} = g_{m,e}(z_1.k_2z_2.z_3 + z_2.k_3z_3.z_1 + z_3.k_1z_1.z_2)(z_3.k_1)^3 + g_{nm,e}(z_1.k_2)(z_2.k_3)(z_3.k_1)^3
\]
\[
\mathcal{M}_o^{114} = g_{nm,o}[-\epsilon(z_2.z_3k_2)k_3 - \epsilon(z_2.z_3k_3)k_2](z_1.k_2)(z_3.k_1)^2
\]
\[
+ g_{nm,o}[-\epsilon(z_2.z_3k_2)k_3 + \epsilon(z_2.z_3k_3)k_2](z_1.k_2)(z_3.k_1)^3.
\]

(A.15)
In this section, we shall discuss with some examples how the flat-space amplitude can be obtained from the CFT correlator in momentum space. We again break up the discussion into inside the triangle and outside the triangle. The details of the CFT correlators can be found in [14].

B.1 Within triangle inequality

Let us start with the simplest of examples.

\( \langle O J_s J_s \rangle \)

In momentum space, we have

\[
\langle J_s J_s O \Delta \rangle_e = (k_1 k_2)^{s-1} I_{\frac{1}{2}+2s\{\frac{1}{2}, \frac{1}{2}, \Delta - \frac{3}{2}\}} \left[ 2 \left( \vec{z}_1 \cdot \vec{k}_2 \right) \left( \vec{z}_2 \cdot \vec{k}_1 \right) + E (E - 2 k_3) \vec{z}_1 \cdot \vec{z}_2 \right]^s \quad (B.1)
\]

\[
\langle J_s J_s O \Delta \rangle_o = (k_1 k_2)^{s-1} I_{\frac{1}{2}+2s\{\frac{1}{2}, \frac{1}{2}, \Delta - \frac{3}{2}\}} \left[ k_2 e^{k_1 z_1 z_2} - k_1 e^{k_2 z_1 z_2} \right] \times \left[ 2 \left( \vec{z}_1 \cdot \vec{k}_2 \right) \left( \vec{z}_2 \cdot \vec{k}_1 \right) + E (E - 2 k_3) \vec{z}_1 \cdot \vec{z}_2 \right]^{s-1} \quad (B.2)
\]

which in the flat-space limit \( E \to 0 \) leads to

\[
\lim_{E \to 0} \langle J_s J_s O \Delta \rangle_e \sim \frac{(k_1 k_2)^{s-1} k_3^{\Delta - 2}}{E^{2s}} (z_1, k_2 z_2, k_1)^s \quad (B.3)
\]

\[
\lim_{E \to 0} \langle J_s J_s O \Delta \rangle_o \sim \frac{(k_1 k_2)^{s-1} k_3^{\Delta - 2}}{E^{2s}} [\epsilon(z_1 z_2 k_1) k_2 - \epsilon(z_1 z_2 k_2) k_1] (z_1, k_2 z_2, k_1)^{s-1} \quad (B.4)
\]

where we can immediately identify \( A_{even}^{ss0} \) and \( A_{odd}^{ss0} \) amplitudes where in the last line we have used (2.4) to convert the amplitude to four dimensional language.

| 4D Flat-space Amplitudes | 3D CFT correlator | Expected CFT Pole structure |
|--------------------------|-------------------|----------------------------|
| Even                     | Free Boson        | \( E^{-2s} \)              |
| Odd                      | Free Fermion      | \( E^{-2s} \)              |

Table 8. \( \langle O J_s J_s \rangle \) has only Homogeneous component and the flat-space amplitude can be obtained from free theories.

\( \langle J_s J_s J_s \rangle \)

This correlator has homogeneous and non-homogeneous contribution [14]. Homogeneous piece has both parity-even and parity-odd contribution whereas non-homogeneous piece has only parity-even contribution. The parity-even homogeneous and non-homogeneous piece can be obtained from

\[
\langle J_s J_s J_s \rangle_{e,h} = \langle J_s J_s J_s \rangle_{e,FB} - \langle J_s J_s J_s \rangle_{e,FB} \\
\langle J_s J_s J_s \rangle_{e,h} = \langle J_s J_s J_s \rangle_{e,FB} + \langle J_s J_s J_s \rangle_{e,FB} \quad (B.5)
\]
The homogeneous piece answer can be written down as follows

\[
\langle J_s J_s J_s \rangle_{e,h} = (k_1 k_2 k_3)^{s-1} \left[ \frac{1}{E^3} \left\{ 2 \left( \vec{z}_1 \cdot \vec{k}_2 \right) \left( \vec{z}_2 \cdot \vec{k}_3 \right) \left( \vec{z}_3 \cdot \vec{k}_1 \right) + E \left\{ k_3 \left( \vec{z}_1 \cdot \vec{z}_2 \right) \left( \vec{z}_3 \cdot \vec{k}_1 \right) + \text{cyclic} \right\} \right\} \right]^s \tag{B.6}
\]

\[
\langle J_s J_s J_s \rangle_{\text{odd}, h} = (k_1 k_2 k_3)^{s-1} \frac{1}{E^3} \left\{ \left( \vec{k}_1 \cdot \vec{z}_3 \right) \left( k_3 z_1 z_2 k_1 - \epsilon k_1 z_2 z_3 k_3 \right) + \left( \vec{k}_3 \cdot \vec{z}_2 \right) \left( \epsilon k_1 z_1 z_3 k_2 - \epsilon k_2 z_1 z_3 k_1 \right) \right. \\
- \left( \vec{z}_2 \cdot \vec{z}_3 \right) \epsilon^{k_1 k_2 z_1} E + \frac{k_1}{2} \epsilon^{z_1 z_2 z_3} E (E - 2k_1) \left\} + \text{cyclic perm} \right\}
\times \left[ \frac{1}{E^3} \left\{ 2 \left( \vec{z}_1 \cdot \vec{k}_2 \right) \left( \vec{z}_2 \cdot \vec{k}_3 \right) \left( \vec{z}_3 \cdot \vec{k}_1 \right) + E \left\{ k_3 \left( \vec{z}_1 \cdot \vec{z}_2 \right) \left( \vec{z}_3 \cdot \vec{k}_1 \right) + \text{cyclic} \right\} \right\} \right]^{s-1} \tag{B.7}
\]

which in the flat-space limit \( E \to 0 \) becomes

\[
\lim_{E \to 0} \langle J_s J_s J_s \rangle_{e,h} \sim (k_1 k_2 k_3)^{s-1} (z_1, k_2)^s (z_2, k_3)^s (z_3, k_1)^s
\]

\[
\lim_{E \to 0} \langle J_s J_s J_s \rangle_{\text{odd}, h} \sim (k_1 k_2 k_3)^{s-1} [\epsilon (z_1 z_2 k_1) k_2 - \epsilon (z_1 z_2 k_2) k_1] (z_3, k_1)^s (z_1, k_2)^s (z_2, k_3)^s \tag{B.8}
\]

where in the last line we have used (2.4) to convert the amplitude to four dimensional language. As usual, the minimal part of \( A_{\text{even}}^{s,s,s} \) and \( A_{\text{odd}}^{s,s,s} \) appear in the flat-space limit. One does similar computation for the non-homogeneous piece. In general non-homogeneous pieces are very complicated. However, we can look into a few examples such as \( \langle TTT \rangle_{nh} \).

We summarize the findings in a few tables.

| 4D flat-space Amplitudes | 3D CFT correlator | Expected CFT Pole structure |
|--------------------------|------------------|-----------------------------|
| Non-minimal (Even)       | Homogenous        | \( E^{-s_1-s_2-s_3} \)     |
| Minimal (Even)           | Non-Homogenous    | \( E^{-s_1-s_2-s_3+2} \)   |
| Non-minimal (Odd)        | Homogenous        | \( E^{-s_1-s_2-s_3} \)     |
| Minimal (Odd) for \( s_1 \neq s_2 \neq s_3 \) | No corresponding CFT correlator | \( \times \) |

Table 9. Flat-space limit of CFT correlator.

### B.2 Outside triangle inequality

Let us now consider a few examples of correlators outside the triangle. One very important aspect which is different for correlators outside the triangle as compared to inside is that all the correlator outside the triangle is non-homogeneous [41].

\( \langle J_a O O \rangle \)

In momentum space, we have

\[
\langle J_a O O \rangle e = c_1 k_1^{2s-1} I_{\frac{1}{2}+s,\frac{1}{2}} \left( \frac{1}{2}-s_1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) (k_2 \cdot z_1)^s \tag{B.9}
\]
which in the flat-space limit $E \to 0$ becomes

$$
\lim_{E \to 0} \langle J_s O \Delta O \Delta \rangle_{\text{even}} \sim \frac{k_1^{s-1}(k_2k_3)^{\Delta - 2}}{E^s} (z_1, k_2)^s
$$

(B.10)

where again we see $A_e^{100}$ appear in the flat-space limit.

The $\langle J_3 J_1 O \rangle$ correlator in momentum space is given by

$$
\langle J_3 J_1 O \rangle = \left( -\frac{E^2 + 2k_1(E + k_1)}{E^4} \right) (z_1, k_2)^3 z_2 \cdot k_1 \\
+ \left( \frac{E^3 - 2k_1^2(E + 2k_1) - 2(3E^2 + 3Ek_1 + 2k_1^2)k_2}{2E^3} \right) (z_1, k_2)^2 z_1 \cdot z_2
$$

(B.11)

Now, we take the flat-space limit $E \to 0$, where we see that

$$
\lim_{E \to 0} \langle J_3 J_1 O \rangle = \frac{2k_1^2}{E^4} (z_1 \cdot k_2)^3 z_2 \cdot k_1
$$

(B.12)

which matches with flat-space amplitude. The parity-odd term can be obtained from free fermion theory and the amplitude can be identified in a similar way.

It is interesting to note that correlators involving exactly conserved currents outside the triangle do not have any parity-odd contribution [29]. To get the parity-odd non-minimal coupling, we need to consider theories with weakly broken Higher-Spin symmetry [52] which we consider in the next section.

The general correspondence is listed below.

| 4D Flat-space Amplitudes | 3D CFT correlator | Expected CFT Pole structure |
|--------------------------|-------------------|-----------------------------|
| Non-minimal (Even)       | Non-Homogenous $(b - f)$ | $E^{-s_1 - s_2 - s_3}$ |
| Minimal (Even)           | Non-Homogenous $(b + f)$ | $E^{-s_1 - s_2 - s_3 + 2}$ |
| Non-minimal (Odd)        | No corresponding CFT correlator | $\times$ |
| Minimal (Odd)            | No corresponding CFT correlator | $\times$ |

Table 10. Correspondence outside the triangle inequality. Here $b - f$ implies subtraction of free bosonic correlator and free fermionic correlator. Let us also note that there is no parity-odd CFT correlator for exactly conserved current outside the triangle.

Weakly broken Higher-spin theory

As discussed above, there is no parity-odd contribution to the correlator outside the triangle for exactly conserved currents. However, it is known that for weakly broken currents it is possible to have a parity-odd contribution for such cases [30, 52]. Recently, in [41], the
Table 11. Parity-even and odd part of the amplitude can be obtained from the flat-space limit of the free bosonic and free fermionic CFT correlator respectively.

| 4D flat-space Amplitudes | 3D CFT correlator          | Expected CFT Pole structure |
|--------------------------|-----------------------------|----------------------------|
| Even                     | Non-Homogeneous free bosonic | $E^{-s_2-s_3}$             |
| Odd                      | Non-Homogeneous free fermionic | $E^{-s_2-s_3}$             |

Table 12. For this case there is only one structure and can be obtained by considering free bosonic or free fermionic theory alone.

parity-odd part was computed in momentum space. Let us consider $\langle J_1 J_1 J_4 \rangle$ for simplicity. This correlator was computed explicitly in [41]. The parity-odd piece is given by a non-homogeneous piece

$$\langle J_1 J_1 J_4 \rangle_{o,nh} = ([O_1]_1 + [O_2]_2)\langle J_1 J_1 J_4 \rangle_{b-f}. \quad (B.13)$$

It is easy to show that in the flat-space limit this gives rise to non-minimal parity-odd amplitude

$$\lim_{E \to 0} ([O_1]_1 + [O_2]_2)\langle J_1 J_1 J_4 \rangle_{b-f} \sim [M_0^{114}]_{nm}. \quad (B.14)$$

Table 13. Flat-space limits for outside the triangle correlators.

| 4D flat-space Amplitudes | 3D CFT correlator          | Expected CFT Pole structure |
|--------------------------|-----------------------------|----------------------------|
| Non-minimal (Even)       | Non-Homogenous (b – f)      | $E^{-s_1-s_2-s_3}$         |
| Minimal (Even)           | Non-Homogenous (b + f)      | $E^{-s_1-s_2-s_3+2}$       |
| Non-minimal (Odd)        | $s_1 \neq 0$, $\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{odd,nh}$ | $E^{-s_1-s_2-s_3}$         |
| Minimal (Odd)            | No corresponding correlator | ×                          |

C Flat space limit in spinor helicity variable

The most general form of the CFT correlator in spinor helicity is given by

$$\langle J_{s_1}^{h_1} J_{s_2}^{h_2} J_{s_3}^{h_3} \rangle = f_{h_1,h_2,h_3}\langle k_1,k_2,k_3 \rangle \langle 12 \rangle^{h_3 s_3-h_1 s_1-h_2 s_2} \langle 23 \rangle^{h_1 s_1-h_3 s_2-h_4 s_3} \langle 31 \rangle^{h_2 s_2-h_3 s_3-h_1 s_1}$$

(C.1)
However, for clarity let us consider two different class of CFT structures based on their net helicities $H$ and rewrite the most general CFT correlator in spinor-helicity as

$$
\left\langle j_{h_1}^h j_{h_2}^h j_{h_3}^h \right\rangle = \begin{cases} 
\frac{1}{2} f_{h_1 h_2 h_3} \langle 12 \rangle^{h_3 s_3 - h_1 s_1 - h_2 s_2} \langle 31 \rangle^{h_2 s_2 - h_3 s_3 - h_1 s_1} \langle 23 \rangle^{h_1 s_1 - h_2 s_2 - h_3 s_3} & H < 0 \\
\frac{1}{2} f_{h_1 h_2 h_3} \langle 12 \rangle^{h_1 s_1 + h_2 s_2 - h_3 s_3} \langle 31 \rangle^{h_3 s_3 + h_1 s_1 - h_2 s_2} \langle 23 \rangle^{h_2 s_2 + h_3 s_3 - h_1 s_1} & H > 0 
\end{cases}
$$

where $H = h_1 s_1 + h_2 s_2 + h_3 s_3$. We also have

$$
\langle ij \rangle \langle ij \rangle = -E(k_i + k_j - k_k) \quad (C.3)
$$

in three dimensional CFT which implies the barred spinor brackets and the unbarred spinor brackets are related, therefore, (C.1) and (C.2) are equivalent\(^{19}\). In the limit of $E \to 0$, we see that

$$
\langle ij \rangle \langle ij \rangle = 0 \quad (C.4)
$$

If the momenta are real, then $(ij)^* = (ij)$, then both barred and unbarred brackets are zero. If the momenta are complex, then the barred brackets and the unbarred brackets are independent and in the flat-space limit only one can be non-zero i.e $\langle ij \rangle = 0$ or $\langle ij \rangle = 0$. Also in the flat space limit, barred and un barred brackets are independent of each other. One can see that the flat-space limit of (C.2) reproduces (3.1) where $\langle ij \rangle$ must be identified with $[ij]$, therefore,

$$
\lim_{E \to 0} \left\langle j_{h_1}^h j_{h_2}^h j_{h_3}^h \right\rangle = A_{h_1 h_2 h_3}^{s_1 s_2 s_3} \quad (C.5)
$$

Let us see the above happen with the example of $\langle JJO \rangle$ which in spinor-helicity variables is given by

$$
\langle J - J - O \rangle_{\text{even}} = \frac{\langle 12 \rangle^2}{E^2} \quad \langle J - J - O \rangle_{\text{odd}} = i \frac{\langle 12 \rangle^2}{E^2} \quad H < 0 \quad (C.6)
$$

$$
\langle J + J + O \rangle_{\text{even}} = \frac{\langle 12 \rangle^2}{E^2} \quad \langle J + J + O \rangle_{\text{odd}} = -i \frac{\langle 12 \rangle^2}{E^2} \quad H > 0 \quad (C.7)
$$

$$
\langle J + J - O \rangle_{\text{even}} = \langle J + J - O \rangle_{\text{odd}} = 0 \quad \text{for } H = 0 \quad (C.8)
$$

In the flat space limit we obtain

$$
\lim_{E \to 0} \langle J - J - O \rangle = \frac{\langle 12 \rangle^2}{E^2} = \frac{A_{-110}^{110}}{E^2} \quad H < 0 \quad (C.9)
$$

$$
\lim_{E \to 0} \langle J + J + O \rangle = \frac{\langle 12 \rangle^2}{E^2} = \frac{A_{110}^{110}}{E^2} \quad H > 0 \quad (C.10)
$$

$$
\lim_{E \to 0} \langle J + J - O \rangle = 0 \quad H = 0 \quad (C.11)
$$

which precisely matches all the helicity components of $A^{110}$.

\(^{19}\)Except when $h_1 s_1 + h_2 s_2 + h_3 s_3 = 0$. This case discussed below separately.
More on new parity odd CFT correlator

In this Appendix we consider another example of new parity odd CFT correlator proposed in section 5. We also discuss the same in spinor helicity variables.

### D.1 $\langle J_1J_1J_4 \rangle$

In the momentum space the CFT correlator is given by \[41\]

$$
\langle J_1J_1J_4 \rangle_{nh,b+f} = \left( - \frac{5E^3 + k_3(5E^2 + 2k_3(2E + k_3))}{512E^4} \right) \left[ (z_3,k_2)^4z_2z_1 + (z_3 \cdot k_2)^3z_3z_2z_1 \cdot k_3 - (z_3 \cdot k_2)^3z_3z_1z_2 \cdot k_3 \right]
$$

(D.1)

Now, we take the flat-space limit $E \to 0$, where we see that

$$
\lim_{E \to 0} \langle J_1J_1J_4 \rangle_{b+f} = \frac{k_3^3}{256E^3} \left( z_3,k_2 \right)^4z_2z_1 + (z_3 \cdot k_2)^3z_3z_2z_1 \cdot k_3 - (z_3 \cdot k_2)^3z_3z_1z_2 \cdot k_3
$$

(D.2)

which is precisely the parity-even minimal amplitude obtained in (A.15). Now, we look at the difference

$$
\langle J_1J_1J_4 \rangle_{b-f} = \left( - \frac{5E^3 + k_3(15E^2 + 4k_3(6E + 5k_3))}{2560E^6} \right) (z_3,k_2)^4z_2z_1 \cdot k_3 + O\left( \frac{1}{E^5} \right)
$$

(D.3)

which in the flat-space limit $E \to 0$ gives

$$
\lim_{E \to 0} \langle J_1J_1J_4 \rangle_{b-f} = - \frac{k_3^3}{128E^6} (z_3,k_2)^4z_2z_1 \cdot k_3
$$

(D.4)

which is precisely the parity-even non-minimal amplitude obtained in (A.15). To obtain the parity-odd vertex, we consider the epsilon transform of the above. Since, this correlator is outside the triangle, there will be no parity-odd contribution for exactly conserved currents, therefore, we consider weakly broken current correlation to generate parity-odd terms. The parity-odd contributions in momentum space were computed explicitly in \[41\]. Using parity odd result in \[41\] it is easy to show that

$$
\lim_{E \to 0} \left[ O_1 \right]_{1} + \left[ O_2 \right]_{2} \langle J_1J_1J_4 \rangle_{b-f} \sim M_{nm,o}^{114}
$$

(D.5)

Using (5.1) we can define parity odd CFT correlator which in the flat space limit gives parity odd minimal amplitude. More precisely,

$$
\lim_{E \to 0} \left[ O_1 \right] - \left[ O_2 \right] \langle J_1J_1J_4 \rangle_{b+f} \sim M_{m,o}^{114}
$$

(D.6)

generates the desired parity-odd minimal cubic vertex in (A.15). The anti-symmetry is due to the Chan-Paton factors discussed earlier near (2.18).

We now consider spinor helicity variables to describe the new parity odd CFT correlators. This will give us some more understanding of the same correlator.

\[20\]Correlators which are out-side the triangle, has only non-homogeneous piece. Here we talked about nh component obtained by adding free boson and free fermion $\langle J_4J_1J_1 \rangle$ correlator.
D.2 In spinor helicity variables

Let us consider \( \langle JJT \rangle \) in the spinor-helicity variables

\[
\langle JJT \rangle = (g_e + ig_o) \frac{(23)^2(31)^2k_3}{E^4}
\]

\[
\langle J_+ J_- T_- \rangle = \langle J_- J_+ T_+ \rangle_{nh} = 0
\]

\[
\langle J_+ J_- T_- \rangle = c_J \frac{(23)^4(E - 2k_3)^2(E + k_3)}{(12)^2 k_3^2 E^2}
\]

and its conjugate.\(^{21}\)

Since, in the flat space limit, we have

\[
\langle ij \rangle \langle \bar{i}\bar{j} \rangle = E (k_i + k_j - k_k) \rightarrow 0
\]

Therefore, either \( \langle ij \rangle = 0 \) or \( \langle \bar{i}\bar{j} \rangle = 0 \) in the flat-sapce limit. If we work with \( \langle \bar{i}\bar{j} \rangle = 0 \), then

\[
\lim_{E \to 0} \langle J_- J_- T_- \rangle = \frac{(23)^2(31)^2k_3}{E^4} \equiv \frac{[M_{nm}^{[112]}]_{- -} k_3}{E^4}
\]

\[
\lim_{E \to 0} \langle J_+ J_- T_- \rangle = c_J \frac{(23)^4k_3}{(12)^2 E^2} + c_J O(\frac{1}{E^2}) \equiv \frac{[M_{m}^{[112]}]_{+ -} k_3}{E^2}
\]

where we have identified with the flat-space photon-photon-graviton vertices computed in (3.13). If we work with \( \langle \bar{i}\bar{j} \rangle = 0 \) then

\[
\lim_{E \to 0} \langle J_+ J_+ T_+ \rangle = \frac{(23)^2(31)^2k_3}{E^4} \equiv \frac{[M_{nm}^{[112]}]_{++} k_3}{E^4}
\]

\[
\lim_{E \to 0} \langle J_- J_+ T_+ \rangle = c_J \frac{(23)^4k_3}{(12)^2 E^2} + c_J O(\frac{1}{E^2}) \equiv \frac{[M_{m}^{[112]}]_{- +} k_3}{E^2}
\]

where again we have identified the flat-space photon-photon-graviton vertices in (3.13). One can also convert (5.8) in spinor helicity variables and show that it is consistent with (3.13).

It is interesting to note that, if we ignore net helicity zero amplitude, then there is no mismatch between spinor helicity (3.4) and cubic vertex (3.19) answers. Similarly, for the CFT correlator in spinor helicity variables, there is no net helicity zero CFT correlation which is consistent with locality. See appendix G for more discussion.

\(^{21}\)Here the conjugate is a means to generate the barred spinors bracket and doesn’t actually mean the complex conjugate of the CFT correlators. For example, the conjugate of \( \langle T^- T^- T^- \rangle \) is \( \langle T^+ T^+ T^+ \rangle \sim (12)^2 (23)^2 (31)^2 \)
\( \langle TTT \rangle \) in spinor helicity variables

Let us now look at \( \langle TTT \rangle \) in spinor-helicity variables:

\[
\langle T^- T^- T^- \rangle = \left( (c_1 + i c_1') \frac{c_{123}}{E^6} + c_T \frac{E^3 - Eb_{123} - c_{123}}{c_{123}^2} \right) \langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2
\]

\[
\langle T^- T^- T^+ \rangle = c_T \frac{(E - 2k_3)^2 (E^3 - Eb_{123} - c_{123})(E - 2k_2)^2(E - 2k_1)^2}{E^2 c_{123}^2} \langle 12 \rangle^6 \langle 23 \rangle^2 \langle 31 \rangle^2
\]

Its conjugate and cyclic permutation. We have not displayed parity-even or parity-odd correlators separately. The parity odd homogeneous part is proportional to \( c_1' \). The term proportional to \( c_T \) is parity even non-homogeneous and does not have parity odd analogue for conserved current. In the flat space limit we have

\[
\lim_{E \to 0} \langle T^- T^- T^- \rangle = (c_1 + c_1') \frac{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2}{E^6} + \text{subleading terms}
\]

\[
\lim_{E \to 0} \langle T^- T^- T^+ \rangle = c_T \lim_{E \to 0} \frac{k_1 k_2 k_3}{E^2} \frac{\langle 12 \rangle^6}{\langle 31 \rangle^2 \langle 23 \rangle^2} + \mathcal{O}\left( \frac{1}{E} \right)
\]

which matches with both cubic vertex and spinor helicity amplitude (3.12), (3.3). To produce correct flat space parity odd spinor helicity minimal amplitude, we look at (5.25) in the spinor-helicity variables

\[
\langle T^- T^- T^- \rangle'_o = i \frac{E^3 - Eb_{123} - c_{123}}{c_{123}^2} \langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2
\]

\[
\langle T^- T^- T^+ \rangle'_o = i \frac{(E - 2k_3)^2 (E^3 - Eb_{123} - c_{123})(E - 2k_2)^2(E - 2k_1)^2}{E^2 c_{123}^2} \langle 12 \rangle^6 \langle 23 \rangle^2 \langle 31 \rangle^2
\]

which in the flat space limit gives

\[
\lim_{E \to 0} \langle T^- T^- T^+ \rangle'_o \sim \frac{1}{E^2} \frac{\langle 12 \rangle^6}{\langle 31 \rangle^2 \langle 23 \rangle^2} + \mathcal{O}\left( \frac{1}{E} \right)
\]

which matches precisely with missing parity odd term \( A_{222}^{22 \pm} \) in (3.3). This analysis of \( \langle TTT \rangle \) can be generalized for any arbitrary equal spin correlator \( \langle J_s J_s J_s \rangle \). So we conclude that, for equal spin case, we are able to construct CFT correlator which in the flat space limit reproduces correct amplitude in spinor helicity variables.

---

\(^{23}\)The contact term in (5.22) takes the form

\[
\langle T^- T^- T^- \rangle = c_T \frac{E^3 - Eb_{123} - c_{123}}{c_{123}^2} \langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2
\]

\[
\langle T^- T^- T^+ \rangle = c_T \frac{(E - 2k_3)^2 - (E - 2k_3) (b_{123} - 2k_3 a_{12}) + c_{123}(E - 2k_2)^2(E - 2k_1)^2}{c_{123}^2} \langle 12 \rangle^6 \langle 23 \rangle^2 \langle 31 \rangle^2
\]

\(^{24}\)Interestingly (D.11) has the missing the terms in spinor helicity variables which can be identified with \( A_{222}^{22 \pm} \) that appears in (3.3). However, this is a contact term and can not be thought as reproducing flat space amplitude.
Parity odd results in (D.5) in spinor helicity variable is identical to above answer in spinor helicity angle-brackets of (3.18) while the conjugate correlator reproduces the square-brackets. We can see that the above can be identified with
\[ \langle J_1 J_2 J_3 J_4 \rangle \] in spinor helicity variables

Consider the correlator \( \langle J_1 J_2 J_3 J_4 \rangle \) in spinor-helicity, in two separate linear combinations
\[
\langle J_+ J_+ J_+ J_+ \rangle_{b-f} = (\frac{21}{12})^2 \frac{E^3 + k_3^3}{\epsilon^2} \]
\[
\langle J_+ J_+ J_+ J_+ \rangle_{f-f} = (\frac{21}{12})^2 \frac{E^3 + k_3^3}{\epsilon^2} \]
\[
\langle J_+ J_+ J_+ J_+ \rangle_{b-f} = (\frac{21}{12})^2 \frac{E^3 + k_3^3}{\epsilon^2} \]
\[
\langle J_+ J_+ J_+ J_+ \rangle_{f-f} = (\frac{21}{12})^2 \frac{E^3 + k_3^3}{\epsilon^2} \]
Parity odd results in (D.5) in spinor helicity variable is identical to above answer in spinor helicity up to some factor of \( i \) and the other combination
\[
\langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
and its conjugate. Parity odd results in (D.6) in spinor helicity variable is identical to above answer in spinor helicity up to some factor of \( i \). We look at the flat-space limits of the above
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{f-f} = \langle J_+ J_+ J_+ J_+ \rangle_{f-f} = 0
\]
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{f-f} = \langle J_+ J_+ J_+ J_+ \rangle_{f-f} = 0
\]
where \( \approx \) means that there are no singularities in \( E \). Now let us look at the other combination
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
Notice that the flat-space limits correspond to the photon-photon-spinor cubic vertices (3.19). We observe that in both the combination, in the leading order there is no contribution in \( --+ \) or \( ++- \) helicity\(^{25}\). We have seen that covariant vertex answer in (3.8), the helicity component \( M_--+ = M_++- = 0 \) whereas spinor helicity amplitude \( A_--+ \neq 0, A_++- \neq 0 \). In (D.18) we ignored subleading terms as they are not singular in the limit \( E \to 0 \). Neglecting subleading terms the flat space limit, we conclude that the CFT correlator \( \langle J_1 J_2 J_3 J_4 \rangle \) reproduces correct flat space covariant vertex but not the full spinor helicity amplitude in (3.8). This implies that in general flat space limit of the CFT correlator, in general, reproduces the correct cubic vertex but not the full spinor helicity amplitude.

\(^{25}\)However, if we include non-singular contributions as well, namely in(D.19),
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
\[
\lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = \lim_{E \to 0} \langle J_+ J_+ J_+ J_+ \rangle_{b-f} = 0
\]
We can see that the above can be identified with \( A_+-+ \) and therefore, the full correlator reproduces the angle-brackets of (3.18) while the conjugate correlator reproduces the square-brackets.
E Constructing missing CFT correlator for $\langle JJ_J4 \rangle$

If we ignore subleading terms, it is shown in (D.18), (D.19) that $\langle JJ_J4 \rangle$ does not produce $- - +$ or $++-$ helicity amplitude in the flat space limit.

It turns out that one can construct more homogeneous solutions such that

$$K^\kappa \frac{\langle J^-J^-J_4^+ \rangle}{k_3} = 0, \quad K^\kappa \frac{\langle J^+J^+J_4^- \rangle}{k_3} = 0.$$  \hspace{1cm} (E.1)

One can explicitly solve the above equation to obtain

$$\langle J_1^-J_1^-J_4^+ \rangle = E^2 \frac{(12)_6}{(23)_4(31)_4}$$  \hspace{1cm} (E.2)

and its complex conjugate. Converting the into momentum space gives

$$\langle J_1(z_1,k_1)J_1(z_2,k_2)J_4(z_3,k_3) \rangle = A(z_3,k_3)^4 z_2.k_1 z_1.k_3 + B z_1.z_2(z_3,k_3)^4 + C(z_3.z_2)(z_3,k_3)^3(z_1,k_3) + C(k_2 \leftrightarrow k_1)z_3.z_1(z_3,k_3)^3(z_2,k_1) \hspace{1cm} (E.3)$$

where

$$A = \frac{8k_3^6}{E(E-2k_1)^4(E-2k_2)^4}, \quad B = -\frac{4(E-2k_3)k_3^7}{E(E-2k_1)^4(E-2k_2)^4}, \quad C = \frac{4k_1k_3^4(E-2k_3)}{E(E-2k_1)^4(E-2k_2)^4}. \hspace{1cm} (E.4)$$

It is easy to see that this CFT correlator has a bad pole, that is it has poles at other momentum configuration other than $E \rightarrow 0$.

One can check that the flat space limit cannot be re-written as 4D Lorentz invariant structure hence even though it reproduces correct flat space spinor helicity amplitude, it does not lead to any new covariant vertex.

F Various-Identities

In this section, we derive

$$Y_2Y_3[O_\epsilon]G = -GV_1 \hspace{1cm} (F.1)$$

$$Y_3[O_\epsilon]Y_2 = -V_1 \hspace{1cm} (F.2)$$

$$Y_2Y_3[O_\epsilon]G = GV_1 \hspace{1cm} (F.3)$$

Consider first

$$Y_2Y_3[O_\epsilon]G = (z_2,k_3)(z_3,k_1) \left[ \frac{\epsilon(z_2k_2z_3)}{k_2} + (z_3,z_1) \frac{\epsilon(z_2k_2k_3)}{k_2} + (z_3,k_1) \frac{\epsilon(z_2k_2z_1)}{k_2} \right] \hspace{1cm} (F.4)$$

$$= (z_3,k_1) \left[ \frac{\epsilon(z_2k_2k_3)}{k_2} + (z_3,z_1) \frac{\epsilon(z_2k_2k_3)}{k_2} + (z_3,k_1) \frac{\epsilon(z_2k_2z_1)}{k_2} \right] \hspace{1cm} (F.5)$$
where in the second equality we take $z_2.k_3$ inside the bracket and make use of the Schouten identities

\[
\epsilon(z_2k_2z_3)Y_2 = \epsilon(z_2k_2z_3)(z_2,k_3) = \epsilon(z_2k_2k_3)(z_2,z_3) = \epsilon(z_2k_2k_3)Z_1 \quad \text{(F.6)}
\]

\[
\epsilon(z_2k_2z_1)Y_2 = \epsilon(z_2k_2z_1)(z_2,k_3) = \epsilon(z_2k_2k_3)(z_1,z_2) = \epsilon(z_2k_2k_3)Z_3 \quad \text{(F.7)}
\]

to obtain

\[
Y_2Y_3[O_\epsilon]_2G = (z_3,k_1) \frac{\epsilon(z_2k_2k_3)}{k_2} G \quad \text{(F.8)}
\]

Now, consider the Schouten identity

\[
\epsilon(z_2k_2k_3)(z_3,k_2) = -\epsilon(z_2z_3k_2)k_2.k_3 + \epsilon(z_2z_3k_3)k_3^2 \quad \text{(F.9)}
\]

since, $p_1.p_2 = k_1k_2 - k_1,k_2 = 0$, the above Schouten identity becomes

\[
\epsilon(z_2k_2k_3)(z_3,k_2) = -\epsilon(z_2z_3k_2)k_2k_3 + \epsilon(z_2z_3k_3)k_3^2 = -k_2\epsilon(z_2z_3p_2p_3) = k_2V_1 \quad \text{(F.10)}
\]

which we now use in (F.8) to get

\[
Y_2Y_3[O_\epsilon]_2G = -GV_1 \quad \text{(F.11)}
\]

Similarly one can show

\[
Y_2Y_3[O_\epsilon]_3G = -GV_1 \quad \text{(F.12)}
\]

Similarly, we derive (F.2). Consider the epsilon transform of $Y_2$ as follows

\[
Y_3[O_\epsilon]_2Y_2 = (z_3,k_1) \frac{\epsilon(z_2k_2k_3)}{k_2} \quad \text{(F.13)}
\]

Now, by using (F.10) in the above, we immediately see

\[
Y_3[O_\epsilon]_2Y_2 = -V_1 \quad \text{(F.14)}
\]

Similarly, one can show

\[
Y_2[O_\epsilon]_3Y_3 = -V_1 \quad \text{(F.15)}
\]

To derive (F.3), consider

\[
Y_2Y_3[O_\epsilon]_2G = (z_2,k_3)(z_3,k_1) \left[ (z_2.z_3) \frac{\epsilon(z_1k_1k_2)}{k_1} + (z_2,k_3) \frac{\epsilon(z_1k_1z_3)}{k_1} + (z_3,k_1) \frac{\epsilon(z_1k_1z_2)}{k_1} \right] \quad \text{(F.16)}
\]

Now, we take $z_3.k_1$ inside the bracket and use the following Schouten identities

\[
Y_3\epsilon(z_1k_1z_2) = (z_3,k_1)\epsilon(z_1k_1z_2) = -k_1^2\epsilon(z_1z_2z_3) - (z_2,k_3)\epsilon(z_1k_1z_3) \quad \text{(F.17)}
\]
\[
Y_3\epsilon(z_1k_1k_2) = (z_3,k_1)\epsilon(z_1k_1k_2) = k_1^2\epsilon(z_1z_3k_2) + k_1k_2\epsilon(z_1k_1z_3) = -k_1V_2 \quad \text{(F.18)}
\]
to obtain
\[ Y_2 Y_3 [O_e]_1 G = -Y_2 (Z_1 V_2 + Y_3 W_1) \] (F.19)

Now we use the Schouten identities \[34\]
\[ W_1 Y_2 Y_3 + V_1 (G + Y_1 Z_1) = 0 \quad V_1 Y_1 = V_2 Y_2 = V_3 Y_3 \] (F.20)
in (F.19) and simplify to obtain
\[ Y_2 Y_3 [O_e]_1 G = GV_1. \] (F.21)

\[ \text{G Net-helicity zero amplitudes and CFT correlator} \]

The amplitudes in (3.1) were determined by imposing little group scaling and vanishing of the amplitude in case of real momenta. This left out the case of
\[ h_1 s_1 + h_2 s_2 + h_3 s_3 = 0 \]
which we consider here
\[ A_{h_1, h_2, h_3}^{s_1, s_2, s_3} = \left\{ \begin{array}{ll}
(1, 2)^{2h_3 s_3} (3, 1)^{2h_2 s_2} (2, 3)^{2h_1 s_1} & \text{when } [ij] = 0 \\
[1, 2]^{-2h_3 s_3} [3, 1]^{-2h_2 s_2} [2, 3]^{-2h_1 s_1} & \text{when } \langle ij \rangle = 0
\end{array} \right. \] (G.1)

Notice how this case is ambiguous as it can be written in terms of either angle-brackets or square brackets. However, due to momentum conservation, we either have \([ij] = 0\) or \(\langle ij \rangle = 0\), which in the above case leads to \(0^0\) for one of the representations which is ill-defined. However, this is due to the fact that momentum conservation is imposed in the end, if one were to impose momentum conservation before the little group scaling, then we have

\[ A_{h_1, h_2, h_3}^{s_1, s_2, s_3} = \left\{ \begin{array}{ll}
(1, 2)^{2h_3 s_3} (3, 1)^{2h_2 s_2} (2, 3)^{2h_1 s_1} & \text{when } [ij] = 0 \\
[1, 2]^{-2h_3 s_3} [3, 1]^{-2h_2 s_2} [2, 3]^{-2h_1 s_1} & \text{when } \langle ij \rangle = 0
\end{array} \right. \] (G.2)

which are well-defined. This means that the momentum conservation and little group scaling do not commute. But again, in the limit where the brackets vanish, we get a \(0^0\), which means that such amplitudes may not vanish for real momenta. However, such amplitudes have been shown to be inconsistent with locality and unitarity \[53\].

Let us consider an example of net helicity zero CFT correlator.

\[ G.1 \quad \langle JJT \rangle \]

In (D.10) it was shown that in spinor helicity variables the following CFT correlators vanishes
\[ \langle J_- J_- T_+ \rangle = \langle J_+ J_+ T_- \rangle = 0 \] (G.3)

However, corresponding flat space spinor helicity amplitude is non-zero. It is interesting to note that one can write down additional homogeneous structures for this CFT correlator and they are given by
\[ \langle J_- J_- T_+ \rangle = k_3 \frac{(12)^4}{(23)^2 (31)^2} \] (G.4)

– 42 –
and its conjugate. It can be verified that they satisfy

\[
K^* \frac{\langle J_- J_+ T_+ \rangle}{k_3} = 0 \quad K^* \frac{\langle J_+ J_- T_- \rangle}{k_3} = 0
\]

This in the flat space limit gives

\[
\lim_{E \to 0} \langle J_- J_+ T_+ \rangle = k_3 \frac{(12)^4}{(23)^2(31)^2},
\]

which precisely reproduces the extra spinor-helicity amplitudes. Converting (G.4) in momentum space using the ansatz

\[
\langle J(z_1, k_1) J(z_3, k_3) T(z_3, k_3) \rangle = A(z_3, k_2)^2 z_2 z_1 - k_3 + B(z_2, z_1(z_3, k_2)^2 + C z_3 z_2 z_3 z_2 z_3 z_1 \quad \text{G.6}
\]

we get

\[
A = \frac{k_3^2}{8E(E-2k_2)^2(E-2k_1)^2}, \quad B = -\frac{k_3^3(E-2k_3)}{16E(E-2k_2)^2(E-2k_1)^2}, \quad C = \frac{k_3^3 k_1(E-2k_3)}{16E(E-2k_2)^2(E-2k_1)^2}
\]

In the flat space, limit (G.7) gives

\[
\lim_{E \to 0} \langle J(z_1, k_1) J(z_2, k_2) T(z_3, k_3) \rangle = \frac{k_3^2 z_3 z_2}{128E^2 k_3^2 k_2^2} \left( -k_3 z_2 z_1 z_3 z_2 k_1 + k_1 k_3 z_3 z_2 z_1 k_2 \\
+ z_3 z_2 z_2 z_1 k_3^2 - z_3 z_2 z_3 z_3 z_1 k_2 \right) + O(E^0)
\]

which cannot be re-written as a 4D Lorentz invariant structure. Another issue is that this CFT structure (G.7), (G.8) has poles of the form $k_i + k_j - k_k$ which is unphysical for a local CFT. Therefore, if one abandons locality in the CFT one can write down CFT structures that reproduces the full amplitude in the flat-space limit.

References

[1] C. Coriano, L. Delle Rose, E. Mottola and M. Serino, *Solving the Conformal Constraints for Scalar Operators in Momentum Space and the Evaluation of Feynman’s Master Integrals*, *JHEP* 07 (2013) 011 [1304.6944].

[2] A. Bzowski, P. McFadden and K. Skenderis, *Implications of conformal invariance in momentum space*, *JHEP* 03 (2014) 111 [1304.7760].

[3] A. Bzowski, P. McFadden and K. Skenderis, *Scalar 3-point functions in CFT: renormalisation, beta functions and anomalies*, *JHEP* 03 (2016) 066 [1510.08442].

[4] A. Bzowski, P. McFadden and K. Skenderis, *Renormalised 3-point functions of stress tensors and conserved currents in CFT*, *JHEP* 11 (2018) 153 [1711.09105].

[5] A. Bzowski, P. McFadden and K. Skenderis, *Renormalised CFT 3-point functions of scalars, currents and stress tensors*, *JHEP* 11 (2018) 159 [1805.12100].
[6] M. Gillioz, Conformal 3-point functions and the Lorentzian OPE in momentum space, *Commun. Math. Phys.* **379** (2020) 227 [1909.00878].

[7] H. Isono, T. Noumi and T. Takeuchi, Momentum space conformal three-point functions of conserved currents and a general spinning operator, *JHEP* **05** (2019) 057 [1903.01110].

[8] J. M. Maldacena and G. L. Pimentel, On graviton non-Gaussianities during inflation, *JHEP* **09** (2011) 045 [1104.2846].

[9] D. Baumann, C. Duaso Pueyo, A. Joyce, H. Lee and G. L. Pimentel, The cosmological bootstrap: weight-shifting operators and scalar seeds, *JHEP* **12** (2020) 204 [1910.14051].

[10] D. Baumann, C. Duaso Pueyo, A. Joyce, H. Lee and G. L. Pimentel, The Cosmological Bootstrap: Spinning Correlators from Symmetries and Factorization, *SciPost Phys.* **11** (2021) 071 [2005.04234].

[11] S. Jain, R. R. John and V. Malvimat, Momentum space spinning correlators and higher spin equations in three dimensions, *JHEP* **11** (2020) 049 [2005.07212].

[12] S. Jain, R. R. John and V. Malvimat, Constraining momentum space correlators using slightly broken higher spin symmetry, *JHEP* **04** (2021) 231 [2008.08610].

[13] S. Jain, R. R. John, A. Mehta, A. A. Nizami and A. Suresh, Momentum space parity-odd CFT 3-point functions, *JHEP* **08** (2021) 089 [2101.11635].

[14] S. Jain, R. R. John, A. Mehta, A. A. Nizami and A. Suresh, Higher spin 3-point functions in 3d CFT using spinor-helicity variables, *JHEP* **09** (2021) 041 [2106.00016].

[15] S. Caron-Huot and Y.-Z. Li, Helicity basis for three-dimensional conformal field theory, *JHEP* **06** (2021) [2102.08160].

[16] S. Jain and R. R. John, Relation between parity-even and parity-odd CFT correlation functions in three dimensions, *JHEP* **12** (2021) 067 [2107.00695].

[17] Y. Gandhi, S. Jain and R. R. John, Anyonic correlation functions in Chern-Simons matter theories, 2106.09043.

[18] J. A. Farrow, A. E. Lipstein and P. McFadden, Double copy structure of CFT correlators, *JHEP* **02** (2019) 130 [1812.11129].

[19] I. Mata, S. Raju and S. Trivedi, CMB from CFT, *JHEP* **07** (2013) 015 [1211.5482].

[20] N. Kundu, A. Shukla and S. P. Trivedi, Constraints from Conformal Symmetry on the Three Point Scalar Correlator in Inflation, *JHEP* **04** (2015) 061 [1410.2606].

[21] C. Sleight and M. Taronna, Bootstrapping Inflationary Correlators in Mellin Space, *JHEP* **02** (2020) 098 [1907.01143].

[22] E. Pajer, Building a Boostless Bootstrap for the Bispectrum, *JCAP* **01** (2021) 023 [2010.12818].

[23] J. Penedones, Writing CFT correlation functions as AdS scattering amplitudes, *JHEP* **03** (2011) 025 [1011.1485].

[24] S. Raju, New Recursion Relations and a Flat Space Limit for AdS/CFT Correlators, *Phys. Rev. D* **85** (2012) 126009 [1201.6449].

[25] R. R. Metsaev, Light-cone gauge cubic interaction vertices for massless fields in AdS(4), *Nucl. Phys. B* **936** (2018) 320 [1807.07542].
B. Nagaraj and D. Ponomarev, *Spinor-Helicity Formalism for Massless Fields in AdS$_4$, Phys. Rev. Lett. 122 (2019) 101602 [1811.08438].

B. Nagaraj and D. Ponomarev, *Spinor-helicity formalism for massless fields in AdS$_4$. Part II. Potentials, JHEP 06 (2020) 068 [1912.07494].

E. Skvortsov, *Light-Front Bootstrap for Chern-Simons Matter Theories, JHEP 06 (2019) 058 [1811.12333].

S. Giombi, S. Prakash and X. Yin, *A Note on CFT Correlators in Three Dimensions, JHEP 07 (2013) 105 [1104.4317].

J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry, Class. Quant. Grav. 30 (2013) 104003 [1204.3882].

S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia and X. Yin, *Chern-Simons Theory with Vector Fermion Matter, Eur. Phys. J. C 72 (2012) 2112 [1110.4386].

O. Aharony, G. Gur-Ari and R. Yacoby, *d=3 Bosonic Vector Models Coupled to Chern-Simons Gauge Theories, JHEP 03 (2012) 037 [1110.4382].

A. K. H. Bengtsson, I. Bengtsson and N. Linden, *Interacting Higher Spin Gauge Fields on the Light Front, Class. Quant. Grav. 4 (1987) 1333.

E. Conde, E. Joung and K. Mkrtchyan, *Spinor-Helicity Three-Point Amplitudes from Local Cubic Interactions, JHEP 08 (2016) 040 [1605.07402].

S. J. Parke and T. R. Taylor, *Amplitude for n-gluon scattering, Phys. Rev. Lett. 56 (1986) 2459.

H. Elvang and Y.-t. Huang, *Scattering Amplitudes, 1308.1697.

A. K. H. Bengtsson, *A Riccati type PDE for light-front higher helicity vertices, JHEP 09 (2014) 105 [1403.7345].

P. Benincasa and E. Conde, *Exploring the S-Matrix of Massless Particles, Phys. Rev. D 86 (2012) 025007 [1108.3078].

K. Krasnov, E. Skvortsov and T. Tran, *Actions for Self-dual Higher Spin Gravities, 2105.12782.

Y.-Z. Li, *Notes on flat-space limit of AdS/CFT, JHEP 09 (2021) 027 [2106.04606].

S. Jain, R. R. John, A. Mehta and D. K. S, *Constraining momentum space CFT correlators with consistent position space OPE limit and the collider bound, 2111.08024.

E. D. Skvortsov, T. Tran and M. Tsulaia, *Quantum Chiral Higher Spin Gravity, Phys. Rev. Lett. 121 (2018) 031601 [1805.00048].

S. Jain, R. R. John, A. Mehta, A. A. Nizami and A. Suresh, *Double copy structure of parity-violating CFT correlators, JHEP 07 (2021) 033 [2104.12803].

S. D. Chowdhury, A. Gadde, T. Gopalka, I. Halder, L. Janagal and S. Minwalla, *Classifying and constraining local four photon and four graviton S-matrices, JHEP 02 (2020) 114 [1910.14392].

N. Arkani-Hamed, P. Benincasa and A. Postnikov, *Cosmological Polytopes and the Wavefunction of the Universe, 1709.02813.

S. Albayrak, S. Kharel and D. Meltzer, *On duality of color and kinematics in (A)dS momentum space, JHEP 03 (2021) 249 [2012.10460].
[47] M. Gillioz, M. Meineri and J. Penedones, *A scattering amplitude in Conformal Field Theory*, *JHEP* 11 (2020) 139 [2003.07361].

[48] L. Eberhardt, S. Komatsu and S. Mizera, *Scattering equations in AdS: scalar correlators in arbitrary dimensions*, *JHEP* 11 (2020) 158 [2007.06574].

[49] K. Roehrig and D. Skinner, *Ambitwistor Strings and the Scattering Equations on AdS$_3 \times$S$^3$, 2007.07234*. 

[50] A. Herderschee, R. Roiban and F. Teng, *On the Differential Representation and Color-Kinematics Duality of AdS Boundary Correlators*, 2201.05067.

[51] C. Cheung, J. Parra-Martinez and A. Sivaramakrishnan, *On-shell Correlators and Color-Kinematics Duality in Curved Symmetric Spacetimes*, 2201.05147.

[52] S. Giombi, V. Gurucharan, V. Kirilin, S. Prakash and E. Skvortsov, *On the Higher-Spin Spectrum in Large N Chern-Simons Vector Models*, *JHEP* 01 (2017) 058 [1610.08472].

[53] D. A. McGady and L. Rodina, *Higher-spin massless S-matrices in four-dimensions*, *Phys. Rev. D* 90 (2014) 084048 [1311.2938].