Stochastic maximum principle for optimal control problem of backward systems with terminal condition in $L^1$*

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Abstract
We consider a stochastic control problem, where the control domain is convex and the system is governed by a nonlinear backward stochastic differential equation. With a $L^1$ terminal data, we derive necessary optimality conditions in the form of stochastic maximum principle.

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1 Introduction
We consider a stochastic control problem where the control domain is convex and the system is governed by a backward stochastic differential equation (BSDE for short) of the type
\[
\begin{aligned}
\left\{
\begin{array}{l}
  dy^v_t = b(t, y^v_t, z^v_t, v_t) dt + z^v_t dW_t,
  \\
y^v_T = \xi,
\end{array}
\right.
\end{aligned}
\]

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where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$, satisfying the usual conditions. The control variable $v$ is an $\mathcal{F}_t$-adapted process with values in a convex closed subset $U$ of $\mathbb{R}^m$. The terminal condition $\xi$ is an $n$-dimensional $\mathcal{F}_T$-measurable random vector such that $\mathbb{E} \lvert \xi \rvert < \infty$.

The objective of the control problem, is to choose $u$ in such a way as to minimize a functional cost of the type

$$J(v) = \mathbb{E} \left[ g(y^v_0) + \int_0^T h(t, y^v_t, z^v_t, v_t) \, dt \right].$$

A control process that solves this problem is called optimal.

Stochastic control problems for the backward and forward-backward systems have been studied by many authors including Peng [21], Xu [24], El-Karoui et al [12], Wu [23], Dokuchaev and Zhou [9], Peng and Wu [22], Bahlali and Labed [1], Bahlali [2, 3]. Approaches based on dynamic programming have been studied by Fuhrman and Tessitore [14]. All these papers consider BSDEs with $L^p$ terminal condition, $p \geq 2$.

The aim of the present paper is to derive necessary optimality conditions, in the form of stochastic maximum principle. The terminal condition is assumed in $L^1$. This is the first version which covers the control of backward systems in $L^1$. Our result extend all the previous works in the subject.

Since the control domain is convex, a classical way of treating such a problem consists to use the convex perturbation method. More precisely, if $u$ is an optimal control and $v$ is arbitrary, we define, for each $t \in [0, T]$, a perturbed control as follows

$$u^\theta = u + \theta (v - u).$$

With a sufficiently small $\theta > 0$, we derive the variational equation from the fact that

$$0 \leq J(u^\theta) - J(u).$$

The paper is organized as follows. In Section 2, we formulate the problem and give the various assumptions used throughout the paper. Section 3 is devoted to some preliminary results, which will be used in the sequel. In the last Section, we derive our main result, the necessary optimality conditions.

Along this paper, we denote by $C$ some positive constant and for simplicity, we need the following matrix notation. We denote by $\mathcal{M}_{n \times d} (\mathbb{R})$ the space of $n \times d$ real matrix and $\mathcal{M}_{n \times n}^d (\mathbb{R})$ the linear space of vectors $M = (M_1, ..., M_d)$ where $M_i \in \mathcal{M}_{n \times n} (\mathbb{R})$. 

For any $M, N \in \mathcal{M}_{n \times n}^d(\mathbb{R})$, $L, S \in \mathcal{M}_{n \times d}(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^d$, we use the following notations

\[ \alpha \beta = \sum_{i=1}^{n} \alpha_i \beta_i \in \mathbb{R}^n, \]

\[ L \gamma = \sum_{i=1}^{d} L_i \gamma_i \in \mathbb{R}^n, \]

\[ S \gamma = \sum_{i=1}^{d} S_i \gamma_i \in \mathbb{R}^n, \]

\[ M \alpha = \sum_{i=1}^{n} M_i \alpha_i \in \mathbb{R}^n, \]

\[ MN = \sum_{i=1}^{d} M_i N_i \in \mathcal{M}_{n \times n}(\mathbb{R}), \]

\[ ML = \sum_{i=1}^{d} M_i L_i \in \mathbb{R}^n, \]

\[ MLN = \sum_{i=1}^{d} M_i LN_i \in \mathcal{M}_{n \times n}(\mathbb{R}), \]

\[ M\alpha \gamma = \sum_{i=1}^{n} (M_i \alpha) \gamma_i \in \mathbb{R}^n, \]

\[ M\alpha \gamma = \sum_{i=1}^{d} (M_i \alpha) \gamma_i \in \mathbb{R}^n, \]

\[ MN = \sum_{i=1}^{d} M_i N_i \in \mathcal{M}_{n \times n}(\mathbb{R}), \]

\[ ML = \sum_{i=1}^{d} M_i L_i \in \mathbb{R}^n, \]

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We denote by $L^*$ the transpose of the matrix $L$ and $M^* = (M_1^*, ..., M_d^*)$.

2 Formulation of the problem

Let $T$ be a fixed strictly positive real number and $\left( \Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P} \right)$ be a filtered probability space satisfying the usual conditions, on which a $d$-dimensional Brownian motion $W = (W_t)_{t \in [0,T]}$ is defined. We assume that $(\mathcal{F}_t)_{t \in [0,T]}$ is the $\mathbb{P}$- augmentation of the natural filtration of $(W_t)_{t \in [0,T]}$.

Definition 1 Let $U$ be a closed convex subset of $\mathbb{R}^m$. An admissible control $v$ is an $\mathcal{F}_t$-adapted process with values in $U$ such that

\[ \sup_{t \in [0,T]} \mathbb{E} |v_t|^2 < \infty. \]

We denote by $\mathcal{U}$ the set of all admissible controls.

For any $v \in \mathcal{U}$, we consider the following controlled BSDE

\[ \begin{align*}
\left\{ \begin{array}{l}
\, dy_t^v = b(t, y_t^v, z_t^v, v_t) \, dt + z_t^v \, dW_t, \\
\, y_T^v = \xi,
\end{array} \right. \tag{1}
\end{align*} \]
where \( b : [0, T] \times \mathbb{R}^n \times \mathcal{M}_{n \times d} (\mathbb{R}) \times U \rightarrow \mathbb{R}^n \) and \( \xi \) is an \( n \)-dimensional \( \mathcal{F}_T \)-measurable random vector such that \( \mathbb{E} |\xi| < \infty \).

The aim of the control problem is to minimize, over the class \( \mathcal{U} \) of admissible controls, a functional cost of the form

\[
J(v) = \mathbb{E} \left[ g(y_0^v) + \int_0^T h(t, y_t^v, z_t^v, v_t) \, dt \right],
\]

(2)

where \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( h : [0, T] \times \mathbb{R}^n \times \mathcal{M}_{n \times d} (\mathbb{R}) \times U \rightarrow \mathbb{R} \).

A control \( u \in \mathcal{U} \) is called optimal, if that solves the problem

\[
J(u) = \inf_{v \in \mathcal{U}} J(v).
\]

(3)

Our goal in this paper is to establish necessary optimality conditions, in the form of stochastic maximum principle.

To study this kind of problem, we need reasonable conditions which ensure the existence and uniqueness of solutions of BSDEs with \( L^1 \) terminal condition. This is given by the results of Briand et al [5, page 124-128].

Miming [5], we use the following notations.

Let us denote by \( \sum_T \) the set of all stopping times \( \tau \) such that \( \tau \leq T \). A process \( Y = (Y_t)_{t \in [0,T]} \) belongs to class \( (D) \), if the family \( \{Y_\tau, \tau \in \sum_T\} \) is uniformly integrable.

For a process \( Y \) in class \( (D) \), we put

\[
\|Y\|_1 = \sup \left\{ \mathbb{E}|Y_\tau|, \, \tau \in \sum_T \right\}.
\]

The space of progressively measurable continuous processes which belong to class \( (D) \) is complete with this norm, see Dellacherie and Meyer [7, page 90].

For any real \( p > 0 \), \( S^p = S^p (\mathbb{R}^n) \) denotes the set of \( \mathbb{R}^n \)-valued, adapted cadlag processes \( \{X_t\}_{t \in [0,T]} \) such that

\[
\|X\|_{S^p} = \mathbb{E} \left[ \sup_t |X_t|^p \right]^{1/p} < +\infty.
\]

If \( p \geq 1 \), \( \|\cdot\|_{S^p} \) is a norm on \( S^p \) and if \( p \in (0, 1) \), \( (X, X') \mapsto \|X - X'\|_{S^p} \) defines a distance on \( S^p \). Under this metric, \( S^p \) is complete.
\[ M^p = M^p (\mathbb{R}^n) \] denotes the set of (equivalent classes of) predictable processes \( \{X_t\}_{t \in [0,T]} \) with values in \( \mathbb{R}^n \) such that
\[
\|X\|_{M^p} = \mathbb{E} \left( \left( \int_0^T |X_t|^2 \, dt \right)^{p/2} \right)^{1/p} < +\infty.
\]

For \( p \geq 1 \), \( M^p \) is a Banach space endowed with this norm and for \( p \in (0, 1) \), \( M^p \) is a complete metric space with the resulting distance.

We assume,

(4.1) \( b, g, h \) are continuously differentiable with respect to \((y, z, v)\).

(4.2) The derivatives \( b_y, b_z, b_v, h_y, h_z, h_v \) and \( g_y \) are continuous in \((y, z, v)\) and uniformly bounded.

(4.3) \( g \) is bounded by \( C (1 + |y|) \).

(4.4) \( \forall r > 0 \), we have (for \( f = b, h \))
\[
\phi_r (t) := \sup_{|y| \leq r} |f(t, y, 0, 0, v) - f(t, 0, 0, v)| \in L^1 ([0, T] \times \Omega, \mathcal{M} \otimes \mathcal{P}).
\]

(4.5) There exists two constants \( C \geq 0, \alpha \in (0, 1) \) and a non-negative progressively measurable processes \( \{\varphi_t\}_{t \in [0,T]} \) and \( \{\psi_t\}_{t \in [0,T]} \) such that \( \forall (t, y, z, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{M}_{n \times d} (\mathbb{R}) \times U \), \( |f(t, y, z, v) - f(t, y, 0, v)| \leq C (\varphi_t + |y| + |z| + |v|)^\alpha \), for \( f = b, h \).
\[
\mathbb{E} \left[ |\xi| + \int_0^T (\varphi_t + \psi_t) \, dt \right] < +\infty.
\]

(4.6) \( \forall (t, y, z_1, v), (t, y, z_2, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{M}_{n \times d} (\mathbb{R}) \times U \),
\[
|f(t, y, z_1, v) - f(t, y, z_2, v)| \leq C |z_1 - z_2| ,
\]
for \( f = b_y, b_z, b_v, h_y, h_z, h_v \).

The above assumptions imply those of Briand et al [5]. Hence from [5, Th 6.2, p 125 and Th 6.3, p 126], for every \( v \in \mathcal{U} \), equation (1) admits a unique adapted solution.

We note that for the uniqueness, the solution \( y \) belongs to the class \((D)\) and \( z \) belongs to the space \( \bigcup_{\beta \geq \alpha} M^\beta \), \( \alpha \in (0, 1) \). For the existence, the solution \( y \) belongs to the class \((\overline{D})\) and for each \( \beta \in (0, 1) \), \((y, z)\) belongs to the space \( S^\beta \times M^\beta \).

More details are given in Briand et al [5, page 124-128].

To enclose the formulation of the problem, it remains us to prove that the cost \( J \) is well defined. This is given by the following lemma.

**Lemma 2** The functional cost \( J \) is well defined from \( \mathcal{U} \) into \( \mathbb{R} \).
Proof. Consider the following controlled one dimensional BSDE

\[
\begin{cases}
    dx^v_t = h(t, y^v_t, z^v_t, v_t) \, dt + k^v_t \, dW_t, \\
x^v_T = \eta.
\end{cases}
\]

where \( k^v = (k^v_1, \ldots, k^v_d) \) is an \((1 \times d)\) real matrix, \((y^v, z^v)\) is the solution of equation (1) and \( \eta \) is a one dimensional \( \mathcal{F}_T \)-measurable random variable such that \( \mathbb{E} |\eta| < \infty \).

Under assumptions (4), the above one dimensional BSDE admits a unique adapted solution \((x^v, k^v)\).

We put
\[
\tilde{y} = \begin{pmatrix} y^v \\ x^v \end{pmatrix},
\]
and consider now the following \((n + 1)\)-dimensional BSDE

\[
\begin{cases}
    d\tilde{y}_t = \tilde{b}(t, \tilde{y}_t, \tilde{z}_t, v_t) \, dt + \tilde{z}_t \, dW_t, \\
\tilde{y}_T = \begin{pmatrix} \xi \\ \eta \end{pmatrix},
\end{cases}
\]

where the function \( \tilde{b} \) is defined from \([0, T] \times \mathbb{R}^{n+1} \times \mathcal{M}_{(n+1) \times d}(\mathbb{R}) \times U \) into \( \mathbb{R}^{n+1} \) by
\[
\tilde{b}(t, \tilde{y}_t, \tilde{z}_t, v_t) = \begin{pmatrix} b(t, y^v_t, z^v_t, v_t) \\ h(t, y^v_t, z^v_t, v_t) \end{pmatrix},
\]
and \( \tilde{z} \) is a \((n + 1) \times d\) real matrix given by
\[
\tilde{z} = \begin{pmatrix} z^v \\ k^v \end{pmatrix} = \begin{pmatrix} z^v_{11} & z^v_{12} & \cdots & z^v_{1d} \\ z^v_{21} & z^v_{22} & \cdots & z^v_{2d} \\ \vdots & \vdots & & \vdots \\ z^v_{n1} & z^v_{n2} & \cdots & z^v_{nd} \\ k^v_1 & k^v_2 & \cdots & k^v_d \end{pmatrix},
\]

It is obvious that \( \tilde{b} \) satisfies hypothesis (4), then the above \((n + 1)\)-dimensional BSDE admits a unique adapted solution \((\tilde{y}_t, \tilde{z}_t)\).

Define now the function \( \tilde{g} \) from \( \mathbb{R}^{n+1} \) into \( \mathbb{R} \) by
\[
\tilde{g}(\tilde{y}_t) = g(y^v_t) - x^v_t,
\]
and the new functional cost from \( \mathcal{U} \) into \( \mathbb{R} \) by
\[
\tilde{J}(v) = \mathbb{E}[\tilde{g}(\tilde{y}_0)] + \mathbb{E}[\eta].
\]
It’s easy to see that for every \( v \in U \)
\[
\tilde{J}(v) = J(v).
\]

By (4.3) the cost \( \tilde{J} \) is well defined from \( U \) into \( \mathbb{R} \) and since \( \tilde{J}(v) = J(v) \), for every \( v \in U \), the cost \( J \) is well defined from \( U \) into \( \mathbb{R} \).

The proof is completed. ■

Let us now state and prove an alternative result that we will be used along this paper. This result said that the difference between two solutions of BSDEs with the same terminal condition in \( L^1 \) is a solution of BSDE in \( L^2 \), and it is given by the following lemma.

**Lemma 3** Let \((y^v, z^v)\) and \((y^w, z^w)\) be the solutions of (1) associated respectively with the controls \( v \) and \( w \). Then the following BSDE
\[
\begin{aligned}
\left\{
\begin{array}{l}
\d y_t^v - y_t^w = \left[ b_t(s, y_s^v, z_s^v, v_s) - b_t(s, y_s^w, z_s^w, w_s) \right] dt + (z_t^v - z_t^w) dW_t, \\
y_T^v - y_T^w = 0,
\end{array}
\right.
\end{aligned}
\]

admits a unique adapted solution \((y^v - y^w, z^v - z^w)\) such that
\[
\sup_{t \in [0, T]} \mathbb{E} |y_t^v - y_t^w|^2 + \mathbb{E} \int_0^T |z_t^v - z_t^w|^2 dt < +\infty. \quad (5)
\]

**Proof.** We have
\[
y_t^v - y_t^w = -\int_t^T \left[ b(s, y_s^v, z_s^v, v_s) - b(s, y_s^w, z_s^w, w_s) \right] ds - \int_t^T (z_s^v - z_s^w) dW_s.
\]

Then
\[
y_t^v - y_t^w = -\int_t^T \left( \int_0^1 b_y(s, y_s^w + \lambda (y_s^v - y_s^w), z_s^w + \lambda (z_s^v - z_s^w), w_s + \lambda (v_s - w_s)) d\lambda \right) (y_s^v - y_s^w) ds
\]
\[
- \left( \int_0^1 b_z(s, y_s^w + \lambda (y_s^v - y_s^w), z_s^w + \lambda (z_s^v - z_s^w), w_s + \lambda (v_s - w_s)) d\lambda \right) (z_s^v - z_s^w) ds
\]
\[
- \int_t^T \left( \int_0^1 b_v(s, y_s^w + \lambda (y_s^v - y_s^w), z_s^w + \lambda (z_s^v - z_s^w), w_s + \lambda (v_s - w_s)) d\lambda \right) (v_s - w_s) ds
\]
\[
- \int_t^T (z_s^v - z_s^w) dW_s.
\]

The above equation is a linear BSDE. Since \( b_y, b_z, b_v \) are bounded, the terminal condition \( y_T^v - y_T^w = 0 \) and the controls are in \( L^2 \), then by a classical result on BSDEs (see Pardoux-Peng [19], El Karoui et al [12]), we have the desired results. ■
3 Preliminary results

Since the control domain $U$ is convex, the classical way consists to use the convex perturbation method. More precisely, let $u$ be an optimal control minimizing the cost $J$ over $U$ and $(y^u_t, z^u_t)$ the solution of (1) controlled by $u$. Define a perturbed control as follows

$$u^\theta_t = u_t + \theta (v_t - u_t),$$

where $\theta > 0$ is sufficiently small and $v$ is an arbitrary element of $U$.

It’s clear that $u^\theta$ is an element of $U$ (admissible control).

Denote by $(y^\theta_t, z^\theta_t)$ the solution of (1) associated with $u^\theta$.

Since $u$ is optimal, the variational inequality follows from the fact that

$$0 \leq J(u^\theta) - J(u).$$

This is can be proved by using the following lemmas.

Lemma 4 Under assumptions (4), we have

$$\lim_{\theta \to 0} \left( \sup_{t \in [0,T]} \mathbb{E} \left| y^\theta_t - y^u_t \right|^2 + \mathbb{E} \int_0^T \left| z^\theta_t - z^u_t \right|^2 dt \right) = 0. \quad (6)$$

Proof. By (5), we have

$$\sup_{t \in [0,T]} \mathbb{E} \left| y^\theta_t - y^u_t \right|^2 + \mathbb{E} \int_0^T \left| z^\theta_t - z^u_t \right|^2 dt < +\infty.$$

Applying the Ito formula to $(y^\theta_t - y^u_t)^2$, we get

$$\mathbb{E} \left| y^\theta_t - y^u_t \right|^2 + \mathbb{E} \int_0^T \left| z^\theta_s - z^u_s \right|^2 ds$$

$$= 2\mathbb{E} \int_0^T \left( y^\theta_s - y^u_s \right) \left( b \left( s, y^\theta_s, z^\theta_s, u^\theta_s \right) - b \left( s, y^u_s, z^u_s, u_s \right) \right) ds$$

$$\leq 2\mathbb{E} \int_0^T \left( y^\theta_s - y^u_s \right) \left( b \left( s, y^\theta_s, z^\theta_s, u^\theta_s \right) - b \left( s, y^u_s, z^u_s, u_s \right) \right) ds$$

$$+ 2\mathbb{E} \int_0^T \left( y^\theta_s - y^u_s \right) \left( b \left( s, y^u_s, z^u_s, u^\theta_s \right) - b \left( s, y^u_s, z^u_s, u_s \right) \right) ds.$$
Applying the Young's formula to the first term in the right hand side of the above inequality, we have for every $\varepsilon > 0$
\[
\mathbb{E} |y_t^\theta - y_t^u|^2 + \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds \leq \frac{1}{\varepsilon} \mathbb{E} \int_t^T |y_s^\theta - y_s^u|^2 \, ds + \varepsilon \mathbb{E} \int_t^T |b(s, y_s^\theta, z_s^\theta, u_s^\theta) - b(s, y_s^u, z_s^u, u_s^u)|^2 \, ds
\]
\[
+ 2\mathbb{E} \int_t^T |(y_s^\theta - y_s^u) (b(s, y_s^\theta, z_s^\theta, u_s^\theta) - b(s, y_s^u, z_s^u, u_s^u))| \, ds.
\]

By (4.2), $b$ is uniformly Lipschitz with respect $(y, z, v)$. Then
\[
\mathbb{E} |y_t^\theta - y_t^u|^2 + \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds \leq \left(\frac{1}{\varepsilon} + C\varepsilon\right) \mathbb{E} \int_t^T |y_s^\theta - y_s^u|^2 \, ds + C\varepsilon \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds
\]
\[
+ C \theta \int_t^T \mathbb{E} \left[|y_s^\theta - y_s^u| |v_s - u_s|\right] \, ds.
\]

Applying the Cauchy-Schwarz inequality to the third term in the right hand side of the above inequality, we get
\[
\mathbb{E} |y_t^\theta - y_t^u|^2 + \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds \leq \left(\frac{1}{\varepsilon} + C\varepsilon\right) \mathbb{E} \int_t^T |y_s^\theta - y_s^u|^2 \, ds + C\varepsilon \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds
\]
\[
+ C \theta \left(\int_t^T \mathbb{E} |y_s^\theta - y_s^u|^2 \, ds\right)^{1/2} \left(\int_t^T \mathbb{E} |v_s - u_s|^2 \, ds\right)^{1/2}.
\]

Using definition 1 and (6), we have
\[
\mathbb{E} |y_t^\theta - y_t^u|^2 + \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds \leq \left(\frac{1}{\varepsilon} + C\varepsilon\right) \mathbb{E} \int_t^T |y_s^\theta - y_s^u|^2 \, ds + C\varepsilon \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds
\]
\[
+ C \varepsilon \int_t^T \mathbb{E} |z_s^\theta - z_s^u|^2 \, ds + C \theta.
\]

Choose $\varepsilon = \frac{1}{2C}$, then we get
\[
\mathbb{E} |y_t^\theta - y_t^u|^2 + \frac{1}{2} \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds \leq \left(2C + \frac{1}{2}\right) \mathbb{E} \int_t^T |y_s^\theta - y_s^u|^2 \, ds + C \theta.
\]
From this above inequality, we deduce two inequalities

\[ \mathbb{E} |y_t^{\theta} - y_t^u|^2 \leq \left( 2C + \frac{1}{2} \right) \int_t^T \mathbb{E} |y_s^{\theta} - y_s^u|^2 \, ds + C\theta. \]  

(7)

\[ \mathbb{E} \int_t^T |z_s^{\theta} - z_s^u|^2 \, ds \leq (4C + 1) \int_t^T \mathbb{E} |y_s^{\theta} - y_s^u|^2 \, ds + C\theta. \]  

(8)

By (7), Gronwall lemma and Buckholers-Davis-Gundy inequality, we have

\[ \lim_{\theta \to 0} \left( \sup_{t \in [0,T]} \mathbb{E} |y_t^{\theta} - y_t^u|^2 \right) = 0. \]

Finally, by (8) and the above result, we obtain

\[ \lim_{\theta \to 0} \mathbb{E} \int_t^T |z_s^{\theta} - z_s^u|^2 \, ds = 0. \]

The lemma is proved. □

**Lemma 5** For every \( v \in \mathcal{U} \), the following linear BSDE

\[
\begin{cases}
    dY_t = \left[ b_y(t, y_t^u, z_t^u, u_t) Y_t + b_z(t, y_t^u, z_t^u, u_t) Z_t \right] dt \\
    b_v(t, y_t^u, z_t^u, u_t) (v_t - u_t) dt + Z_t dW_t, \\
    Y_T = 0,
\end{cases}
\]

(9)

admits a unique adapted solution \((Y, Z)\) such that

\[ \sup_{t \in [0,T]} \mathbb{E} |Y_t|^2 + \mathbb{E} \int_0^T |Z_t|^2 \, dt < \infty. \]  

(10)

\[ \lim_{\theta \to 0} \left( \mathbb{E} \left| Y_t - \frac{y_t^{\theta} - y_t^u}{\theta} \right|^2 + \mathbb{E} \int_0^T \left| Z_t - \frac{z_t^{\theta} - z_t^u}{\theta} \right|^2 \, dt \right) = 0. \]  

(11)

**Proof.** i) Assertion (10) is obvious since the BSDE (9) is linear, \( b_y, b_z, b_v \) are bounded and the terminal condition \( Y_T = 0 \).

ii) Let us prove (11).

Put

\[ \Phi_t^{\theta} = Y_t - \frac{y_t^{\theta} - y_t^u}{\theta}, \]

\[ \Psi_t^{\theta} = Z_t - \frac{z_t^{\theta} - z_t^u}{\theta}. \]
We have
\[ d\Phi_t^\theta = [B_y^\theta (t) \Phi_t^\theta + B_z^\theta (t) \Psi_t^\theta + \rho_t^\theta] \, dt + \Psi_t^\theta dW_t, \]
where
\[
B_y^\theta (t) = \int_0^1 b_y \left( t, y_i^u + \lambda (y_i^\theta - y_i^u), z_i^u + \lambda (z_i^\theta - z_i^u), u_t + \lambda \theta (v_t - u_t) \right) d\lambda, \\
B_z^\theta (t) = \int_0^1 b_z \left( t, y_i^u + \lambda (y_i^\theta - y_i^u), z_i^u + \lambda (z_i^\theta - z_i^u), u_t + \lambda \theta (v_t - u_t) \right) d\lambda, \\
\rho_t^\theta = \int_0^1 \left[ b_y \left( t, y_i^u + \lambda (y_i^\theta - y_i^u), z_i^u + \lambda (z_i^\theta - z_i^u), u_t + \lambda \theta (v_t - u_t) \right) - b_y \left( t, y_i^u, z_i^u, u_t \right) \right] Y_t d\lambda \\
+ \int_0^1 \left[ b_z \left( t, y_i^u + \lambda (y_i^\theta - y_i^u), z_i^u + \lambda (z_i^\theta - z_i^u), u_t + \lambda \theta (v_t - u_t) \right) - b_z \left( t, y_i^u, z_i^u, u_t \right) \right] Z_t d\lambda \\
+ \int_0^1 \left[ b_v \left( t, y_i^u + \lambda (y_i^\theta - y_i^u), z_i^u + \lambda (z_i^\theta - z_i^u), u_t + \lambda \theta (v_t - u_t) \right) - b_v \left( t, y_i^u, z_i^u, u_t \right) \right] (u_t - v_t) d\lambda.
\]
By (5) and (10), it is easy to see that
\[ \mathbb{E} |\Phi_t^\theta|^2 + \int_0^T |\Psi_t^\theta|^2 \, dt < +\infty. \quad (12) \]

Applying the Ito formula to \((\Phi_t^\theta)^2\), we get
\[ \mathbb{E} |\Phi_t^\theta|^2 + \mathbb{E} \int_t^T |\Psi_s^\theta|^2 \, ds \leq 2 \mathbb{E} \int_t^T |\Phi_s^\theta (B_y^\theta (s) \Phi_s^\theta + B_z^\theta (s) \Psi_s^\theta + \rho_s^\theta)| \, ds. \]

By the Young's formula and using the fact that \(B_y^\theta\) and \(B_z^\theta\) are bounded, we have for every \(\varepsilon > 0\)
\[ \mathbb{E} |\Phi_t^\theta|^2 + \mathbb{E} \int_t^T |\Psi_s^\theta|^2 \, ds \leq \left( \frac{1}{\varepsilon} + C \varepsilon \right) \mathbb{E} \int_t^T |\Phi_s^\theta|^2 \, ds + C \varepsilon \mathbb{E} \int_t^T |\Psi_s^\theta|^2 \, ds + C \varepsilon \mathbb{E} \int_t^T |\rho_s^\theta|^2 \, ds. \]

Choose \(\varepsilon = \frac{1}{2C}\), then we get
\[ \mathbb{E} |\Phi_t^\theta|^2 + \frac{1}{2} \mathbb{E} \int_t^T |\Psi_s^\theta|^2 \, ds \leq \left( 2C + \frac{1}{2} \right) \mathbb{E} \int_t^T |\Phi_s^\theta|^2 \, ds + \frac{1}{2} \mathbb{E} \int_t^T |\rho_s^\theta|^2 \, ds. \]
From this above inequality, we deduce two inequalities

$$\mathbb{E} |\Phi_t^\theta|^2 \leq \left( 2C + \frac{1}{2} \right) \mathbb{E} \int_t^T |\Phi_s^\theta|^2 \, ds + \frac{1}{2} \mathbb{E} \int_t^T |\rho_s^\theta|^2 \, ds.$$

(13)

$$\mathbb{E} \int_t^T |\Psi_s^\theta|^2 \, ds \leq (4C + 1) \mathbb{E} \int_t^T |\Phi_s^\theta|^2 \, ds + \mathbb{E} \int_t^T |\rho_s^\theta|^2 \, ds.$$

(14)

Let us prove now that \( \lim_{\theta \to 0} \mathbb{E} \int_t^T |\rho_s^\theta|^2 \, ds = 0 \).

We have

$$\mathbb{E} \int_t^T |\rho_s^\theta| \, ds \leq \mathbb{E} \int_t^T \int_0^1 \left| b_y (s, y_s^u + \lambda (y_s^\theta - y_s^u), z_s^u, u_s + \lambda \theta (v_s - u_s)) - b_y (s, y_s^u + \lambda (y_s^\theta - y_s^u), z_s^u, u_s + \lambda \theta (v_s - u_s)) \right| Y_s d\lambda ds$$

$$+ \mathbb{E} \int_t^T \int_0^1 \left| b_z (s, y_s^u + \lambda (y_s^\theta - y_s^u), z_s^u, u_s + \lambda \theta (v_s - u_s)) - b_z (s, y_s^u + \lambda (y_s^\theta - y_s^u), z_s^u, u_s + \lambda \theta (v_s - u_s)) \right| Z_s d\lambda ds$$

$$+ \mathbb{E} \int_t^T \int_0^1 \left| b_v (s, y_s^u + \lambda (y_s^\theta - y_s^u), z_s^u, u_s + \lambda \theta (v_s - u_s)) - b_v (s, y_s^u + \lambda (y_s^\theta - y_s^u), z_s^u, u_s + \lambda \theta (v_s - u_s)) \right| (u_s - v_s) \, ds.$$

Applying the Cauchy-Schwarz inequality, then by using (4.6) and (10),
we get
\[
\mathbb{E} \int_t^T |\rho_s^\theta| \, ds \leq C \left( \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds \right)^{1/2} + C \left( \mathbb{E} \int_t^T |z_s^\theta - z_s^u|^2 \, ds \right)^{1/2}
\]
\[
\quad + C \left( \mathbb{E} \int_t^T \int_0^1 \left| b_y \left( s, y_s^u + \lambda (y_s^\theta - y_s^u), z_s^u, u_s + \lambda \theta (v_s - u_s) \right) \right|^2 \, d\lambda ds \right)^{1/2}
\]
\[
\quad + C \left( \mathbb{E} \int_t^T \int_0^1 \left| b_z \left( s, y_s^u + \lambda (y_s^\theta - y_s^u), z_s^u, u_s + \lambda \theta (v_s - u_s) \right) \right|^2 \, d\lambda ds \right)^{1/2}
\]
\[
\quad + C \left( \mathbb{E} \int_t^T \int_0^1 \left| b_v \left( s, y_s^u + \lambda (y_s^\theta - y_s^u), z_s^u, u_s + \lambda \theta (v_s - u_s) \right) \right|^2 \, d\lambda ds \right)^{1/2}.
\]

(15)

By (6), the first and second terms in the right hand side of the above inequality tends to 0 as \( \theta \) go to 0.

On the other hand, since \( b_y, b_z \) and \( b_v \) are continuous and bounded, then from (6) and the dominated convergence theorem, we show that the third, fourth and fifth terms in the right hand side tends to 0 as \( \theta \) go to 0.

Then, we get
\[
\lim_{\theta \to 0} \mathbb{E} \int_t^T |\rho_s^\theta| \, ds = 0.
\]

Moreover, from (15), (5) and the fact that \( b_y, b_z \) and \( b_v \) are bounded, we show that
\[
\mathbb{E} |\rho_s^\theta| \, ds < +\infty.
\]

Using the dominated convergence theorem, we have
\[
\lim_{\theta \to 0} \mathbb{E} \int_t^T |\rho_s^\theta|^2 \, ds = 0.
\]

By (13) and Gronwall lemma, we deduce that
\[
\lim_{\theta \to 0} \mathbb{E} |\Phi_t^\theta|^2 = 0.
\]

Finally, by (14) we have
\[
\lim_{\theta \to 0} \mathbb{E} \int_0^T |\Psi_t^\theta|^2 \, dt = 0.
\]

Lemma 5 is proved. ■
Lemma 6 Let $u$ be an optimal control minimizing the cost $J$ over $\mathcal{U}$ and $(y^u_t, z^u_t)$ the solution of (1) controlled by $u$. Then for any $v \in \mathcal{U}$, we have

\[
0 \leq E \left[ g_y (y^0_0) Y_0 \right] + E \int_0^T h_y (t, y^u_t, z^u_t, u_t) Y_t dt
\]
\[
+ E \int_0^T h_z (t, y^u_t, z^u_t, u_t) Z_t dt + E \int_0^T h_v (t, y^u_t, z^u_t, u_t) (v_t - u_t) dt.
\]

Proof. We use the same notations that in lemma 5 for $\Phi^\theta_t$ and $\Psi^\theta_t$.

Since $u$ is optimal, we have

\[
0 \leq J (u^\theta) - J (u)
\]
\[
\leq E \left[ g (y^\theta_0) - g (y^u_0) \right] + E \int_0^T \left[ h (t, y^\theta_t, z^\theta_t, u^\theta_t) - h (t, y^u_t, z^u_t, u_t) \right] dt
\]
\[
\leq E \int_0^1 g_y (y^\theta_0 + \lambda (y^\theta_0 - y^u_0)) \left( \frac{y^\theta_0 - y^u_0}{\theta} \right) d\lambda
\]
\[
+ E \int_0^T \int_0^1 h_y (t, y^u_t + \lambda (y^\theta_t - y^u_t), z^u_t + \lambda (z^\theta_t - z^u_t), u_t + \lambda \theta (v_t - u_t)) \left( \frac{y^\theta_t - y^u_t}{\theta} \right) d\lambda dt
\]
\[
+ E \int_0^T \int_0^1 h_z (t, y^u_t + \lambda (y^\theta_t - y^u_t), z^u_t + \lambda (z^\theta_t - z^u_t), u_t + \lambda \theta (v_t - u_t)) \left( \frac{z^\theta_t - z^u_t}{\theta} \right) d\lambda dt
\]
\[
+ E \int_0^T \int_0^1 h_v (t, y^u_t + \lambda (y^\theta_t - y^u_t), z^u_t + \lambda (z^\theta_t - z^u_t), u_t + \lambda \theta (v_t - u_t)) (v_t - u_t) d\lambda dt.
\]

Then

\[
0 \leq E \left[ g_y (y^0_0) Y_0 \right] + E \int_0^T h_y (t, y^u_t, z^u_t, u_t) Y_t dt
\]
\[
+ E \int_0^T h_z (t, y^u_t, z^u_t, u_t) Z_t dt
\]
\[
+ E \int_0^T h_v (t, y^u_t, z^u_t, u_t) (v_t - u_t) dt + \delta^\theta_t.
\]
where $\delta_\theta^t$ is given by

$$
\delta_\theta^t = \mathbb{E}\left[(g_y (y_0^\theta + \lambda (y_0^\theta - y_0^u)) - g_y (y_0^u)) Y_0\right] - \mathbb{E}\int_0^1 g_y (y_0^\theta + \lambda (y_0^\theta - y_0^u)) \Phi_\theta^0 d\lambda \\
- \mathbb{E}\int_0^T \int_0^1 h_y (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) \Phi_\theta^0 d\lambda d\lambda dt \\
+ \mathbb{E}\int_0^1 \int_0^T [h_y (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) \\
- h_y (t, y_t^u, z_t^u, u_t)] Y_t d\lambda dt \\
- \mathbb{E}\int_0^T \int_0^1 h_z (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) \Psi_\theta^0 d\lambda d\lambda dt \\
+ \mathbb{E}\int_0^1 \int_0^T [h_z (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) \\
- h_z (t, y_t^u, z_t^u, u_t)] Z_t d\lambda dt \\
+ \mathbb{E}\int_0^T [h_v (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) \\
- h_v (t, y_t^u, z_t^u, u_t)] (v_t - u_t) d\lambda dt.
$$

Let us show that $\lim_{\theta \to 0} \delta_\theta^t = 0.$
We have

\[
\delta_t^\theta = \mathbb{E} \left[ (g_y (y_t^u + \lambda (y_0^\theta - y_0^u)) - g_y (y_0^u)) Y_0 \right] - \mathbb{E} \int_0^1 g_y (y_0^u + \lambda (y_0^\theta - y_0^u)) \Phi_0^t d\lambda.
\]

\[
- \mathbb{E} \int_0^1 \int_0^1 h_y (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) \Phi_t^u d\lambda dt
\]

\[
+ \mathbb{E} \int_0^1 \int_0^1 \left[ h_y (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) - h_y (t, y_t^u, z_t^u, u_t) \right] Y_t^u d\lambda dt
\]

\[
- \mathbb{E} \int_0^1 \int_0^1 h_z (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) \Psi_t^u d\lambda dt
\]

\[
+ \mathbb{E} \int_0^1 \int_0^1 \left[ h_z (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) - h_z (t, y_t^u, z_t^u, u_t) \right] Z_t d\lambda dt
\]

\[
+ \mathbb{E} \int_0^1 \int_0^1 [h_w (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u + \lambda (z_t^\theta - z_t^u), u_t + \lambda \theta (v_t - u_t)) - h_w (t, y_t^u, z_t^u, u_t)] (v_t - u_t) d\lambda dt
\]

\[
+ \mathbb{E} \int_0^1 \int_0^1 \left[ h_w (t, y_t^u + \lambda (y_t^\theta - y_t^u), z_t^u, u_t + \lambda \theta (v_t - u_t)) - h_w (t, y_t^u, z_t^u, u_t) \right] (v_t - u_t) d\lambda dt
\]

Applying the Cauchy-Schwarz inequality, then by using (10), (4.6), def-
inition 1 and the fact that \( g_y, h_y, h_z \) are bounded, we get

\[
|\delta_t^\theta| \leq C \left( \mathbb{E} \left| \Phi_t^\theta \right|^2 \right)^{1/2} + C \left( \mathbb{E} \int_0^T \left| \Phi_t^\theta \right|^2 \, dt \right)^{1/2} + C \left( \mathbb{E} \int_0^T \left| \Psi_t^\theta \right|^2 \, dt \right)^{1/2} \\
+ C \left( \mathbb{E} \int_0^T \left| z_t^\theta - z_t^\lambda \right|^2 \, dt \right)^{1/2} + C \left( \mathbb{E} \left| g_y \left( y_0^u + \lambda \left( y_t^\theta - y_t^\lambda \right) \right) - g_y (y_0^\lambda) \right|^2 \right)^{1/2} \\
+ C \left( \mathbb{E} \int_0^T \int_0^T \left| h_y \left( t, y_t^u + \lambda \left( y_t^\theta - y_t^\lambda \right) , z_t^u, u_t + \lambda \theta (v_t - u_t) \right) - h_y (t, y_t^u, z_t^u, u_t) \right|^2 \, d\lambda dt \right)^{1/2} \\
+ C \left( \mathbb{E} \int_0^T \int_0^T \left| h_z \left( t, y_t^u + \lambda \left( y_t^\theta - y_t^\lambda \right) , z_t^u, u_t + \lambda \theta (v_t - u_t) \right) - h_z (t, y_t^u, z_t^u, u_t) \right|^2 \, d\lambda dt \right)^{1/2} \\
+ C \left( \mathbb{E} \int_0^T \int_0^T \left| h_v \left( t, y_t^u + \lambda \left( y_t^\theta - y_t^\lambda \right) , z_t^u, u_t + \lambda \theta (v_t - u_t) \right) - h_v (t, y_t^u, z_t^u, u_t) \right|^2 \, d\lambda dt \right)^{1/2}.
\]

By (6) and (11), the first, second, third and fourth terms in the right hand side of the above inequality tend to 0 as \( \theta \) go to 0.

On the other hand, since \( g_y, h_y, h_z \) and \( h_v \) are continuous and bounded, then by (6) and the dominated convergence theorem, the fifth, sixth, seventh and eighth terms in the right hand side tend to 0 as \( \theta \) to 0.

Consequently, \( \lim_{\theta \to 0} \delta_t^\theta = 0 \) and by letting \( \theta \) go to 0 in (17), the proof is completed. ■

\section{4 Necessary optimality conditions}

Starting from the variational inequality (16), we can now state and prove our main result in this paper, the necessary optimality conditions.

\textbf{Theorem 7} (Necessary optimality conditions). Let \( (u, y^u, z^u) \) be an optimal solution of the control problem \{ (1), (2), (3) \}. Then, there exists a unique adapted process

\[
p^u \in L_\mathcal{F}^2 ([0, T] ; \mathbb{R}^n),
\]

which is solution of the following forward stochastic differential equation (called adjoint equation)

\[
\begin{cases}
-dp_t^u = H_y (t, y_t^u, z_t^u, u_t, p_t^u) \, dt + H_z (t, y_t^u, z_t^u, u_t, p_t^u) \, dW_t, \\
p_0^u = g_y (y_0^u),
\end{cases}
\]

(18)
such that for every \( v \in U \)

\[
H_v (t, y_t^u, z_t^u, u_t, p_t^u) (u_t - v_t) \geq 0, \text{ as }, ae, \tag{19}
\]

where the Hamiltonian \( H \) is defined from \([0, T] \times \mathbb{R}^n \times M_{n \times d} (\mathbb{R}) \times U \times \mathbb{R}^n \) into \( \mathbb{R} \) by

\[
H (t, y, z, v, p) = p b(t, y, z, v) - h(t, y, z, v).
\]

Proof. Since \( p_0^u = g(y_0^u) \), then by the variational inequality (16), we have

\[
0 \leq \mathbb{E} [p_0^u Y_0] + \mathbb{E} \int_0^T h_y (t, y_t^u, z_t^u, u_t) Y_t dt
\]

\[
+ \mathbb{E} \int_0^T h_z (t, y_t^u, z_t^u, u_t) Z_t dt + \mathbb{E} \int_0^T h_v (t, y_t^u, z_t^u, u_t) (v_t - u_t) dt.
\]

where \((Y, Z)\) is the solution of (9).

Applying the Ito formula to \( p_t^u Y_t \), we get

\[
\mathbb{E} [p_0^u Y_0] = -\mathbb{E} \int_0^T h_y (t, y_t^u, z_t^u, u_t) Y_t dt - \mathbb{E} \int_0^T p_t^u b_v (t, y_t^u, z_t^u, u_t) (v_t - u_t) dt
\]

\[
- \mathbb{E} \int_0^T h_z (t, y_t^u, z_t^u, u_t) Z_t dt + \mathbb{E} [S_T].
\]

where \( S_T \) is given by

\[
S_T = \int_0^T [H_z (t, y_t^u, z_t^u, u_t, p_t^u) Y - p_t^u Z] dW_t.
\]

By replaces \( \mathbb{E} [p_0^u Y_0] \) by it’s value in (20), we have

\[
0 \leq \mathbb{E} \int_0^T H_v (t, y_t^u, z_t^u, u_t) (u_t - v_t) dt + \mathbb{E} [S_T]. \tag{21}
\]

The adjoint equation (18) is a linear forward stochastic differential equation with bounded coefficients and bounded initial condition, then it admits a unique adapted solution \( p^u \) such that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |p_t^u|^2 \right] < +\infty. \tag{22}
\]

By the Cauchy-Schwarz inequality, and using (10), (22), the fact that \( b_z, h_z \) are bounded and the dominated convergence theorem, we show that \( S \) is a \( L^2 \)-martingale.

Hence, \( \mathbb{E} [S_T] = 0 \) and the result follows immediately from (21). \( \blacksquare \)
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