The Chiral Lagrangian parameters, $\ell_1, \ell_2$, are determined by the $\rho$–resonance

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Abstract
The all–important consequence of Chiral Dynamics for $\pi\pi$ scattering is the Adler zero, which forces $\pi\pi$ amplitudes to grow asymptotically. The continuation of this subthreshold zero into the physical regions requires a $P$–wave resonance, to be identified with the $\rho$. It is a feature of $\pi\pi$ scattering that convergent dispersive integrals for the $I = 1$ channel are essentially saturated by the $\rho$–resonance and are much larger than those with $I = 2$ quantum numbers. These facts predict the parameters $\ell_1, \ell_2$ of the Gasser–Leutwyler Chiral Lagrangian, as well as reproducing the well–known KSFR relation and self-consistently generating the $\rho$–resonance.
1 Introduction

Interest in low energy pion dynamics has been rekindled by developments in Chiral Perturbation Theory ($\chi$PT) \[1, 2, 3\] in the past decade. At lowest order in $\chi$PT, pion amplitudes are determined by just two constants: the pion decay constant, $f_\pi$, and the scale of explicit chiral symmetry breaking which is, of course, set by the pion mass, $m_\pi$. At the next order, new parameters, $\ell_i$ ($i = 1, 4$) enter. These have been fixed by Gasser and Leutwyler \[2\] by appeal to detailed phenomenology. However, low energy hadron processes are for the most part dominated by resonances. Thus, low energy $\pi\pi$ dynamics is determined by the $P$–wave $\rho$–resonance and by a strong $S$–wave interaction, often called the $\sigma$ or $\epsilon$, now the $f_0(1300)$. Of course, Chiral Dynamics and resonance contributions are not in contradiction. Indeed, the crucial link between these is provided by the continuity of zero contours, Fig. 1. The Adler zero \[4\] that controls near threshold $\pi\pi$ scattering becomes the Legendre zero of the $\rho$ \[4\], that ensures that it is a spin–one resonance. The dominance of the $I = 1$ $\pi\pi$ cross–section by the $\rho$–resonance at low energies and the weakness of $I = 2$ $\pi\pi$ interaction means that the only parameters needed to determine low energy $\pi\pi$ physics are the mass and width of the $\rho$.

It has long been known that the pion decay constant and the $\rho$–parameters are connected by the KSFR relation \[6\]. Here, using dispersion relations and the continuity of zero contours we show that the same $\rho$–parameters not only provide relationships with the $\ell_1, \ell_2$ of Gasser and Leutwyler as previously known \[2, 7, 8\], but lead to a self-consistent generation of the $\rho$–resonance.

2 The high–low connection

Chiral Dynamics is often thought of as imposing constraints on low momentum processes and so attention is focussed on these in isolation. However, our purpose is to emphasize that there is a unity in hadron reactions that means that Chiral Dynamics affects even high energy behaviour. Since this is germane to our discussion, we begin by recalling this, allowing us to set up the relevant machinery.

Consider the amplitude, $F(s, t)$, for $\pi^-\pi^+ \to \pi^-\pi^+$ in the $s$–channel. Chiral Dynamics imposes an Adler zero at low energies in the scattering amplitude. This is, in fact, a line of zeros through the Mandelstam triangle, Fig. 1, here obtained from $O(p^4)$ $\chi$PT \[2\] but experimental data give a very similar contour \[5\]. This line imposes zeros in both the $s$
and $t$ channel $S$–wave amplitudes typically at $s \simeq m_{\pi}^2/2$ and $t \simeq m_{\pi}^2$, respectively. This line means, for example, that at some fixed values of $t$ the amplitude, $F(s,t)$, which is real within the Mandelstam triangle, changes sign as $s$ increases, Fig. 1.

Now this amplitude is known to have the appropriate cut plane analyticity to satisfy fixed–$t$ dispersion relations for a range of $t$, in particular for $0 \leq t \leq 4m_{\pi}^2$. Let us first assume the asymptotic behaviour of the amplitude is such that for $s \to \infty$, with $t \in [0, 4m_{\pi}^2]$, $|F(s,t)| < \text{const}$. We can then write an unsubtracted dispersion relation for this $s–u$ symmetric amplitude, viz.

$$F(s,t) = \frac{1}{\pi} \int_{4m_{\pi}^2}^{\infty} ds' \Im F(s',t) \left( \frac{1}{s' - s} + \frac{1}{s' - u} \right).$$  \hspace{1cm} (1)$$

Since this amplitude describes the physical processes $\pi^+\pi^- \to \pi^+\pi^-$ in the $s$ and $u$–channels, it has a positive imaginary part for $s' \geq 4m_{\pi}^2, t \in [0, 4m_{\pi}^2]$. Thus the dispersive integral is positive definite for $0 \leq s, t, u \leq 4m_{\pi}^2$ and the amplitude cannot have a zero in the Mandelstam triangle as Adler requires.

Consequently, Chiral Dynamics requires the amplitude $F(s,t)$ must grow asymptotically, so that its dispersive representation must have subtractions. Rigorously we know $\Im |F(s,t)| < s^2$ as $s \to \infty$ for $t \in [0, 4m_{\pi}^2]$, then once again making use of the $s–u$ symmetry of $F(s,t)$, we have :

$$F(s,t) = a(t) + \frac{(s-u)^2}{4\pi} \int_{4m_{\pi}^2}^{\infty} \frac{ds'}{(s' - 2m_{\pi}^2 + t/2)^2} \left( \frac{1}{s' - s} + \frac{1}{s' - u} \right) \Im F(s',t)$$ \hspace{1cm} (2)

where

$$a(t) = F(s = u, t).$$ \hspace{1cm} (3)$$

Now we see that for those values of $t$ for which $a(t)$ (the amplitude on the line $s = u$) is negative the amplitude will have a zero as $s$ increases. Thus asymptotically growing amplitudes may be regarded as a consequence of, or at the very least are consistent with, Chiral Dynamics.

### 3 Calculating $\bar{\ell}_1$, $\bar{\ell}_2$

Let $F^{xI}$ denote the $\pi\pi$ amplitude with isospin $I$ in channel $x$, where $x = s$ or $t$. These amplitudes can be decomposed into partial waves, $f^I_\ell(t)$, in the $t$-channel, for example,

$$F^{xI}(t,s) = \sum_{\ell=0}^{\infty} (2\ell + 1) f^I_\ell(t) P_\ell \left( 1 + \frac{2s}{t - 4m_{\pi}^2} \right).$$ \hspace{1cm} (4)$$
The scattering lengths $a_I^{\ell}$ are defined by the threshold limit:

$$a_I^{\ell} = \lim_{t \to 4m^2_{\pi}} \frac{f_I^{\ell}(t)}{t} \left(\frac{t}{4 - m^2_{\pi}}\right)^{\ell}.$$  \hspace{1cm} (5)

Our starting point is the fixed-$t$ dispersion relation for each isospin amplitude, which rigorously needs no more than two subtractions for $t \leq 4m^2_{\pi}$. Partial wave projecting these and taking $t \to 4m^2_{\pi}$, we find the $D$–wave scattering lengths given by the Froissart–Gribov representation

$$a_I^{2} = \frac{16}{15\pi} \int_{4m^2_{\pi}}^{\infty} \frac{ds'}{s'^3} \text{Im} F_I^{\ell}(s', 4m^2_{\pi})$$  \hspace{1cm} (6)

for $I = 0, 2$. This representation for the $D$–wave scattering lengths makes it clear that the isospin combinations, for which the integral over the absorptive parts, $\text{Im} F(s', 4m^2_{\pi})$, is positive, must themselves be positive. Thus, we have

$$a_0^2 - a_2^2 \geq 0 \quad , \quad a_0^2 + 2a_2^2 \geq 0.$$  \hspace{1cm} (7)

In terms of $\mathcal{O}(p^4) \chi$PT these scattering lengths are combinations of the constants $\overline{t}_1$ and $\overline{t}_2$, that appear in the Gasser–Leutwyler Lagrangian [2]:

$$a_0^2 = \frac{1}{1440\pi^3 F^4} \left[ \overline{t}_1 + 4\overline{t}_2 - \frac{53}{8} \right],$$  \hspace{1cm} (8)

$$a_2^2 = \frac{1}{1440\pi^3 F^4} \left[ \overline{t}_1 + \overline{t}_2 - \frac{103}{40} \right].$$

Then Eq. (7) leads to the straightforward constraints:

$$\overline{t}_1 + 2\overline{t}_2 \geq \frac{157}{40}, \quad \overline{t}_2 \geq \frac{27}{20}.$$  \hspace{1cm} (9)

These are displayed in Fig. 2 together with the previous phenomenological values

$$\overline{t}_1 = -2.3 \pm 3.7, \quad \overline{t}_2 = 6.0 \pm 1.3,$$  \hspace{1cm} (10)

determined by Gasser and Leutwyler [2] using $a_0^2, a_2^2$ of [10]. The simple bounds, Eq. (8), are close to the limits presented by Wanders [11] from the far more complicated conditions on the shape of the subthreshold $\pi^0\pi^0$ $S$–wave that rigorously also follow from three–channel crossing symmetry and positivity.

Because of the explicit $1/s^3$ factor, the integrals in Eq. (6) are dominated by the low energy region, since they converge rapidly. Now each isospin amplitude, $F^{\ell I}$ ($I = 0, 1, 2$),
can be re-expressed in terms of $F^s_1$, $F^s_2$ and $F^t_2$ by the isospin crossing matrix, for instance
\[ F^{t0} = \frac{3}{2} F^{s1} + \frac{3}{2} F^{s2} + F^{t2} \]  
(11)

Thus
\[ a^0_2 = \frac{16}{15\pi} \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^3} \left[ \frac{3}{2} \text{Im} F^{s1}(s', 4m^2_\pi) + \frac{3}{2} \text{Im} F^{s2}(s', 4m^2_\pi) + \text{Im} F^{t2}(s', 4m^2_\pi) \right] \]  
(12)

We then use the weakness of $I = 2$ channels for physical mass pions to mean
\[ \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^3} \left( \text{Im} F^{s2}, \text{Im} F^{t2} \right) \ll \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^3} \text{Im} F^{s1} \]  
(13)

so that
\[ a^0_2 \simeq \frac{16}{15\pi} \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^3} \frac{3}{2} \text{Im} F^{s1}(s', 4m^2_\pi) \]  
(14)

In the real world with physical pion mass this integral is dominated by the $\rho$-contribution, which can be reliably evaluated in the narrow resonance approximation by
\[ \text{Im} F^{s1}(s, t) = 3\pi \sqrt{\frac{s}{s - 4m^2_\pi}} m_\rho \Gamma_\rho \left( 1 + \frac{2t}{s - 4m^2_\pi} \right) \delta(s - m^2_\rho) \]  
(15)

so that
\[ a^0_2\big|_\rho = \frac{24}{5} \frac{\Gamma_\rho(m^2_\rho + 4m^2_\pi)}{m^4_\rho(m^2_\rho - 4m^2_\pi)^{3/2}} \]  
(16)

For this approximation to make sense, clearly $|a^2_2| \ll a^0_2$. We will check the consistency of this later.

While we are primarily concerned with the real world with physical mass pions, it will be useful to compare our results with those of $\chi$PT. Consequently, we need to discuss what happens when the pion mass goes to zero. To distinguish between the physical pion mass and a variable mass, we denote the former by $M_\pi$ and the latter by $m_\pi$. The inequalities of Eq. (13), of course, hold for physical pions —each side is logarithmically divergent when $m_\pi \to 0$. We must therefore consider the dispersive integral from $s' = 4m^2_\pi$ to $4M^2_\pi$ separately. This threshold contribution is readily evaluated using Weinberg’s model amplitude [12], so that
\[ \text{Im} f^I_\ell(s) \simeq \sqrt{\frac{s - 4m^2_\pi}{s}} (\text{Re} f^I_\ell(s))^2 \]  
(17)
with

$$\text{Re} f_0^0(s) = \frac{3}{2} a_1^1(s - m^2/2) \quad , \quad \text{Re} f_1^1(s) = \frac{1}{4} a_1^1(s - 4m^2) ,$$  

$$\text{Re} f_0^2(s) = \frac{3}{4} a_1^1(2m^2 - s) \quad ,$$

where $a_1^1$ is the $P$-wave scattering length. Then we easily deduce that the “chiral” components of $a_2^0$ and $a_2^2$ are respectively:

$$a_2^0 |_\chi = \frac{2}{\pi} (a_1^1)^2 \left[ \ln \left( \frac{1 + X}{1 - X} \right) - 2X - \frac{91X^3}{240} - \frac{X^5}{80} \right] \quad ,$$

$$a_2^2 |_\chi = \frac{4}{5\pi} (a_1^1)^2 \left[ \ln \left( \frac{1 + X}{1 - X} \right) - 2X - \frac{13X^3}{96} - \frac{11X^5}{160} \right] \quad ,$$

where $X^2 = 1 - m^2/M^2$. These threshold contributions are to be added to the $\rho$–component of Eq. (16), for example. However, for physical mass pions, i.e. with $m_\pi = M_\pi$, $X = 0$, these “chiral” components vanish.

Because the intercept of the $\rho$–Regge trajectory is below one, the $I = 1$ $t$–channel amplitude divided by $(s - u)$ satisfies an unsubtracted dispersion relation for $t \leq 4m^2_\pi$.

Projecting out the $P$–wave and expanding for $t \simeq 4m^2_\pi$ gives

$$\frac{f_1^1(t)}{4m^2_\pi} = \frac{4}{3\pi} \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^2} \left[ \frac{1}{s'^2} - \frac{(t - 4m^2_\pi)}{s'^3} + \ldots \right] \text{Im} F^{s1}(s', t)$$

$\simeq a_1^1 + b_1^1 \left( \frac{t}{4} - m^2_\pi \right) + \ldots$

then

$$a_1^1 = \frac{4}{3\pi} \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^2} \text{Im} F^{s1}(s', 4m^2_\pi)$$

$$b_1^1 = \frac{16}{3\pi} \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^2} \left[ \frac{\partial}{\partial t} \text{Im} F^{s1}(s', 4m^2_\pi) - \frac{1}{s} \text{Im} F^{s1}(s', 4m^2_\pi) \right] \quad .$$

Note that integral of Eq. (21) has an explicit factor of $1/s'^2$, while that of Eq. (22), like Eq. (3), has $1/s'^3$. Consequently, the integral for $a_1^1$ is not so dominated by the low energy $\rho$–contribution. Nevertheless, this $\rho$–contribution gives, using $F^{s1} = F^{s1} - F^{s2} + F^{v2}$

$$a_1^1 \bigg|_\rho = \frac{4\Gamma_\rho (m^2_\rho + 4m^2_\pi)}{m^2_\rho (m^2_\rho - 4m^2_\pi)^{3/2}} \quad ,$$
\[ b_1^1|_\rho = \frac{16 \Gamma_\rho}{m_\rho^4 (m_\rho^2 - 4m_\pi^2)^{1/2}} \quad \cdot (24) \]

In the chiral limit \( f_1^1(t) \) displays no logarithms. To show that the effective range \( b_1^1 \) also has no logarithms of \( m_\pi^2 \), care must be taken in the order of limits \( m_\pi \to 0 \) and \( t \to 4m_\pi^2 \).

Now in the chiral limit when \( m_\pi \to 0 \), the \( \rho \)–contributions give:

\[ a_0^0, a_0^2 \to 0 \quad , \quad a_1^1|_\rho \to \frac{4 \Gamma_\rho}{m_\rho^3} \quad , \quad b_1^1|_\rho \to \frac{16 \Gamma_\rho}{m_\rho^4} \quad , \quad a_2^0|_\rho \to \frac{24 \Gamma_\rho}{3 m_\rho^5} \quad , \quad (25) \]

while the near threshold contribution of Eq. (19) yields

\[ a_0^0|_\chi = \frac{2}{\pi} (a_1^1)^2 \ln \left( \frac{4 M_\pi^2}{m_\pi^2} \right) \quad , \quad a_2^0|_\chi = \frac{4}{5 \pi} (a_1^1)^2 \ln \left( \frac{4 M_\pi^2}{m_\pi^2} \right) \quad , \quad (26) \]

when \( m_\pi^2 \ll M_\pi^2 \). In \( \chi \)PT in the same limit

\[ a_1^1 \to \frac{1}{24 \pi F^2} \quad , \quad b_1^1 \to \frac{1}{288 \pi^3 F^4} \left[ -\bar{\ell}_1 + \bar{\ell}_2 + \frac{97}{120} \right] \quad , \quad (27) \]

\[ a_2^0 \to \frac{1}{1440 \pi^3 F^4} \left[ \bar{\ell}_1 + 4 \bar{\ell}_2 - \frac{53}{8} \right] \quad . \]

Simply equating and combining these results gives a straightforward idea of the size of \( \bar{\ell}_1, \bar{\ell}_2 \). We find

\[ \frac{1}{96 \pi F^2} = \frac{\Gamma_\rho}{m_\rho^3} \quad , \quad \bar{\ell}_1 = \frac{1183}{600} - \frac{\pi m_\rho}{4 \Gamma_\rho} + \bar{\ell}_1|_\chi \quad , \quad (28) \]

\[ \bar{\ell}_2 = \frac{349}{300} + \frac{\pi m_\rho}{4 \Gamma_\rho} + \bar{\ell}_2|_\chi \quad , \]

where

\[ \bar{\ell}_1|_\chi = \ln \left( \frac{4 M_\pi^2}{m_\pi^2} \right) \quad , \quad \bar{\ell}_2|_\chi = \ln \left( \frac{4 M_\pi^2}{m_\pi^2} \right) \quad . \quad (29) \]

— in Eq. (29), we drop the constants implied by Eq. (19), since if \( m_\pi^2 \ll M_\pi^2 \), these are irrelevant.

The first of the relations in Eq. (28) yields the long established KSFR relation [1]. Moreover, Eqs. (28, 29) reproduce the chiral logs of \( \chi \)PT [2], but with their renormalization scale \( \mu \) fixed by Eq. (13). Of course, for the real world \( \bar{\ell}_1|_\chi = \bar{\ell}_2|_\chi = 0 \) (cf. Eq. (19) with \( X = 0 \)) and then these simplified relations give with the physical \( \rho \) mass and width [13]

\[ F = 99.6 \text{ MeV} \quad , \quad \bar{\ell}_1 = -2.01 \quad , \quad \bar{\ell}_2 = 5.15 \quad (30) \]
and
\[ \frac{a_2^2}{a_2^0} = \frac{56 \Gamma_\rho}{75 \pi m_\rho} = 0.047 \quad (31) \]
which is reassuringly small.

To have an idea of how certain these numbers are, we could, at the same level of approximation, regard Eq. (14) as a calculation of \( a_2^0 - a_2^2 \) rather than \( a_2^0 \). Then with \( \overline{\ell}_1 \big|_\chi = \overline{\ell}_2 \big|_\chi = 0 \)

\[ \overline{\ell}_1 = \frac{259}{120} - \frac{\pi m_\rho}{4 \Gamma_\rho} = -1.82, \quad (32) \]

\[ \overline{\ell}_2 = \frac{27}{20} + \frac{\pi m_\rho}{4 \Gamma_\rho} = 5.33 \]

and
\[ \frac{a_2^2}{a_2^0} = 0.072 \quad (33) \]

We now turn to the full results with physical pion mass. Eqs. (14, 22) are dominated by the \( \rho \)-resonance to an accuracy of better than 20%. To take into account this uncertainty we introduce a factor \( \lambda = 1.0 \pm 0.2 \). Then, with \( f_\pi = (90 \pm 2) \text{ MeV} \) and \( m_\pi = 138 \text{ MeV} \) (both the mean of their charged and neutral values), we have from Eqs. (16, 24, 27)

\[ \overline{\ell}_1 + 4 \overline{\ell}_2 - \frac{53}{8} = 6912 \pi^3 \frac{f_\pi}{m_\rho^5} \left( \frac{1 + \frac{4m_\pi^2}{m_\rho^2}}{1 - \frac{4m_\pi^2}{m_\rho^2}} \right)^{3/2} \lambda \]

\[ = 10.9 \pm 2.4 \quad (34) \]

and

\[ - \overline{\ell}_1 + \overline{\ell}_2 + \frac{97}{120} = 4608 \pi^3 \frac{f_\pi}{m_\rho^5} \left( \frac{\lambda}{1 - \frac{4m_\pi^2}{m_\rho^2}} \right)^{1/2} \]

\[ = 5.6 \pm 1.2 \quad (35) \]

These yield our main result:

\[ \overline{\ell}_1 = -0.3 \pm 1.1, \quad \overline{\ell}_2 = 4.5 \pm 0.5 \quad (36) \]
which are in agreement with the values deduced by Gasser and Leutwyler [2] given in Eq. (11), but with considerably reduced errors, which in Eqs. (34,35) remain on the conservative side. These results are shown in Fig. 2 together with the determinations by Riggenbach et al. from a fit of calculations in χPT with experimental ππ and Kπ parameters [14] and with the estimate by Beldjoudi and Truong [15] found by fitting the ππ P–wave and the isoscalar S–wave with one loop χPT unitarized by [1,1] Padé approximant [15,16].

An explanation of why we do not use the Froissart–Gribov representation of the P–wave scattering length, Eq. (21), to determine the $\bar{t}_i$ is in order. Despite the fact that $\rho$–dominance of the integral of Eq. (21) does lead to the KSFR relation (the first relation in Eq. (28)) in the chiral limit, nonetheless, the sum rule of Eq. (21) does have two sources of appreciable corrections both resulting from its slower convergence than the integrals of Eqs. (13,22): one is the high energy contribution, the other are corrections to the assumed weakness of $I = 2$ components (i.e. the analogue of Eq. (13) with $s^2$ in the denominator). It seems these corrections unexpectedly cancel. The $\rho$–contribution of Eq. (23) gives $a_1^{\rho} = 0.035$. With the values of $\bar{t}_1, \bar{t}_2$ we have obtained substituted into Eq. (18.5) of Ref. 2, $O(p^4)$ χPT gives $a_1^{\rho} |_{\chi PT} = 0.036 \pm 0.002$ — a very similar value. However, because of the corrections to Eq. (23), $a_1^{\rho}$ provides a consistency check rather than a vehicle for determining the $\bar{t}_i$.

In χPT it is the chiral logarithms, typically with a scale of $\mu = m_\pi$ [2], that give the major contribution to the D-wave scattering length $a_0^0$, for instance, while the explicit $\rho$–resonance component is smaller. In contrast, here the “chiral logarithms” play no role when the pion mass is 138 MeV, see Eq. (19), and the whole answer, Eqs. (8,34) is given by the $\rho$–component. This is a direct consequence of the physical assumption that Eq. (13) holds for $m_\pi = M_\pi$.

That $\bar{t}_1, \bar{t}_2$ are directly relatable to the $\rho$–resonance has already been considered in [2,8] using vector meson dominance. Their idea is to couple the $\rho$–meson to the pion in a chirally symmetric way and then to assume that elastic ππ scattering is dominated by $\rho$–exchange. By comparing the effective Lagrangian obtained in the limit of $p^2 \ll m_\rho^2$ (i.e. neglecting the momentum dependence in the $\rho$–propagator) with the $SU(2) \times SU(2)$ χPT Lagrangian at $O(p^4)$, Gasser and Leutwyler are able to determine the $\rho$–contribution to $\bar{t}_1, \bar{t}_2$, that happens to saturate the phenomenologically determined values. An extension of this procedure to $SU(3) \times SU(3)$ including the lightest (non-Goldstone) meson spectrum: scalars, pseudoscalars, vectors and axial vectors, has been studied in Ref. [4].
As we have stressed above, our procedure is also based on $\rho$-dominance of the $I = J = 1$ $\pi\pi$ channel, together with the well established phenomenological fact that exotic ($I = 2$) channels have comparatively small absorptive parts. In the next section we shall show that the zero contours of the $\pi\pi$ amplitude allow us to check the consistency of these twin assumptions.

4 Zeros connected

As already emphasized, a key feature of $\pi\pi \rightarrow \pi\pi$ scattering is the appearance of the Adler zero on-shell \cite{3}. Moreover, amplitudes being analytic functions of several complex variables, this zero is not isolated but lies on a line that passes through the Mandelstam triangle, Fig. 1. This zero may (depending on the channel) continue into the scattering regions and thereby generate a dip in the angular distribution that can be measured. Thus, if we consider the amplitude for $\pi^+\pi^- \rightarrow \pi^0\pi^0$ in the $t$–channel, the zero enters the $s$ and $u$–channel $\pi^-\pi^0 \rightarrow \pi^-\pi^0$ physical regions. That this zero contour curves down, Fig. 1, is a consequence of the $D$–wave scattering length, $a_2^0 - a_2^2$ being positive, Eq. (37).

Now, well into the $s$ and $u$–channels, in the $\rho$–region, if the amplitude is dominated by the $P$–wave, the angular distribution would have a marked dip at $\cos \theta = 0$, reflecting the spin–one nature of the $\rho$–resonance:

$$F(\pi^-\pi^0 \rightarrow \pi^-\pi^0) \simeq \frac{3}{2} \frac{m_\rho \Gamma(s)}{m_\rho^2 - s - im_\rho \Gamma(s)} \cos \theta_s + \ldots$$

(37)

In reality, the $\rho$ has an $S$–wave background, which means the minimum in the differential cross–section is not exactly zero nor exactly at $\cos \theta_s = 0$. Nevertheless, there is a dip and this corresponds to the zero of the amplitude having moved into the complex plane. However, it does not move far away, since, in this channel, the $S$–wave background has isospin two and so is small. This zero follows the track shown in Fig. 1 for $\Re e s$, $\Re e t$. Clearly it connects to the Adler zero.

In $\chi$PT, a resonance, like the $\rho$, is not generated at any finite order and so the predictions of $\chi$PT are not realistically continuable much above $\pi\pi$ threshold without some technique, typically Padé approximants \cite{15}, for estimating the all orders sum from the known low order predictions. However, there is an alternative procedure, which we present here, that provides a consistency check on the values of $\ell_1, \ell_2$, we have just deduced from the $\rho$–parameters.

It is a feature of zeros of amplitudes that they generally continue smoothly from one region to another \cite{17}, even if the corresponding amplitudes are not well determined.
Thus, we assume that though the one loop prediction from $\chi$PT for the $\pi\pi$ amplitude is not to be believed beyond $\sim 600$ MeV, the zeros of this amplitude are more reliably given — even up to 900 MeV. In Fig. 1, we have shown the track of the corresponding minimum of the differential cross-section in the $s$ and $u\, \pi^-\pi^o \to \pi^-\pi^o$ channels whether from $O(p^4)$ $\chi$PT or experiment. Taking the $I = 2$ $S$–wave, through this region, to be that parameterized by Schenk [18] which matches on to $\chi$PT near threshold, one can, knowing the zero, predict the corresponding behaviour of the $P$–wave. Thus from the minimum at $\cos \theta_s = z(s)$, shown in Fig. 1, we have, assuming elastic unitarity,

$$
\tan \delta_1^1(s) = \frac{-1}{2} \frac{\sin 2\delta_0^2(s)}{3z(s) + \sin^2 \delta_0^2(s)}
$$

(38)

where $\delta_0^2$ and $\delta_1^1$ are the $I = 2$ $S$–wave and $I = 1$ $P$–wave phase shifts. Note that since the imaginary part of the $I = 2$ amplitude is small, the $P$–wave phase $\delta_1^1 \to \pi/2$, where $z(s) \to 0$, i.e. where the zero is exactly in the middle of the angular distribution of Eq. (37). With Schenk’s parameterization for $\delta_0^2$, we predict the $P$–wave phase shown in Fig. 3, using the zero determined by one loop $\chi$PT with $\ell_1 = -0.3, \ell_2 = 5.0$, within the range we expect, Eq. (38). This prediction is compared with the LBL [19] and CERN-Munich phase–shifts, the latter from both the analysis by Ochs [20] and that by Estabrooks and Martin [21]. We also display in the same Fig. 3 the prediction of $O(p^4)\, \chi$PT using $\delta_1^1 = \sqrt{1 - 4m^2/\pi s \, Re f_1^1}$ agreed as the correct interpretation of the chiral prediction for $\delta_1^1$ from $Re f_1^1$ [22, 23]. The contrast is marked. This comparison illustrates

(i) the consistency of our determination of $\ell_1, \ell_2$, and

(ii) that though the $P$–wave in one loop $\chi$PT is not reliable beyond $\sim 600$ MeV or so, the zero $\chi$PT predicts is much less affected by as yet uncalculated higher order corrections. Consequently, this provides a physically motivated unitarization procedure, bringing good agreement with experiment.

5 Conclusion

$\rho$–dominance of the $I = 1$ $\pi\pi$ cross–section and the relative weakness of $I = 2$ interactions leads to values for the parameters of the $\chi$PT Lagrangian [4] $\ell_1 = -0.3 \pm 1.1, \ell_2 = 4.5 \pm 0.5$. These values give a zero of the $\pi^-\pi^o \to \pi^-\pi^o$ amplitude that continues from the Adler zero below threshold to the Legendre zero of the $\rho$–resonance. Such a smooth continuation of the Adler zero demands the existence of a spin–one resonance —
a resonance we know as the $\rho$. Consequently, it is natural that the constants $\ell_1, \ell_2$ of the Chiral Lagrangian should be fixed self-consistently by the $\rho$-parameters, as we have shown.
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Figure Captions

Fig. 1

The track of the zero in the amplitude for $\pi^- \pi^0 \rightarrow \pi^- \pi^0$ (in the $s$-channel) in the Mandelstam plane obtained from $\mathcal{O}(p^4) \chi$PT [2]. Where the zero is at complex $s$ (in the physical regions), the $\Re s$ is plotted.

Fig. 2

Predictions for $\ell_1, \ell_2$ of the Gasser–Leutwyler Chiral Lagrangian. The dashed lines mark the positivity bounds of Eq. (9). The shaded ellipse defines the present results. ◆ is the first evaluation by Gasser and Leutwyler [2], ○ is the updated fit by Riggenbach et al. [14] and □ the central value from the fit by Beldjoudi and Truong [13] for which no errors were determined.

Fig. 3

The $\pi \pi I = 1$, $P$-wave phase shift, $\delta_1^1$, below 1GeV. The solid line is the present prediction based on the zero contour of Fig. 1, as described in the text. The dashed line is the $\mathcal{O}(p^4) \chi$PT result for $\sqrt{1 - 4m_\pi^2/s} \Re f_1^1$, which is the agreed definition of $\delta_1^1$ at this order [22, 23]. The data are from the energy-independent analyses by Protopopescu et al. [19] (□) of the LBL experimental results, by Ochs [20] (○) and by Estabrooks and Martin [21] (△) of the CERN–Munich data.
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