Hyperbolic Problems: Theory, Numerics, Applications

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NON–UNIQUENESS OF ENTROPY–CONSERVING SOLUTIONS TO THE IDEAL COMPRESSIBLE MHD EQUATIONS

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ABSTRACT. In this note we consider the ideal compressible magneto–hydrodynamics (MHD) equations in a special two dimensional setting. We show that there exist particular initial data for which one obtains infinitely many entropy–conserving weak solutions by using the convex integration technique. Finally this is applied to the isentropic case.

1. Introduction. We consider the ideal compressible magneto–hydrodynamics (MHD) equations

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) &= 0, \\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - (\text{curl} \mathbf{B}) \times \mathbf{B} &= 0, \\
\frac{\partial \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e(\rho, p) + \frac{1}{2} |\mathbf{B}|^2 \right)}{\partial t} + \text{div} \left[ \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e(\rho, p) + p + |\mathbf{B}|^2 \right) \mathbf{u} \right] - \text{div} ((\mathbf{B} \cdot \mathbf{u}) \mathbf{B}) &= 0, \\
\frac{\partial \mathbf{B}}{\partial t} + \text{curl}(\mathbf{B} \times \mathbf{u}) &= 0, \\
\text{div} \mathbf{B} &= 0.
\end{align*}
\]  

(1)

The unknown functions in (1) are the density \( \rho > 0 \), the pressure \( p > 0 \), the velocity \( \mathbf{u} \in \mathbb{R}^3 \) and the magnetic field \( \mathbf{B} \in \mathbb{R}^3 \), which are all functions of the time \( t \in [0, T) \) and the spatial variable \( \mathbf{x} = (x, y, z)^T \in \mathbb{R}^3 \). The internal energy \( e \) is a given function of the density \( \rho \) and the pressure \( p \).

In this note we consider a special two dimensional setting. Let \( \Omega \subset \mathbb{R}^2 \) a bounded two dimensional spacial domain. We consider \( \mathbf{u} = (u, v, 0)^T \) and \( \mathbf{B} = (0, 0, b)^T \) and furthermore we let all the unknowns only depend on \( (x, y) \in \Omega \). From now on we write \( \mathbf{u} = (u, v)^T \in \mathbb{R}^2 \) and \( \mathbf{x} = (x, y)^T \in \Omega \subset \mathbb{R}^2 \) for the corresponding two

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dimensional vectors. Then the MHD system (1) turns into
\[ \partial_t \varrho + \text{div} (\varrho \mathbf{u}) = 0, \]
\[ \partial_t (\varrho \mathbf{u}) + \text{div} (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \left( p + \frac{1}{2} b^2 \right) = 0, \]
\[ \partial_t \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, p) + \frac{1}{2} b^2 \right) + \text{div} \left[ \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, p) + p + b^2 \right) \mathbf{u} \right] = 0, \]
\[ \partial_t b + \text{div} (b \mathbf{u}) = 0. \] (2)

Note that in (2) div, \( \nabla \) are two dimensional differential operators in contrast to (1), where they are three dimensional differential operators.

We endow system (2) with initial conditions
\[ (\varrho, p, \mathbf{u}, b)(0, \cdot) = (\varrho_0, p_0, \mathbf{u}_0, b_0) \] (3)
and impermeability boundary conditions
\[ \mathbf{u} \cdot \mathbf{n} |_{\partial \Omega} = 0. \] (4)

Definition 1.1. A 4-tuple \((\varrho, p, \mathbf{u}, b)\) \(\in L^\infty([0, T) \times \Omega; (0, \infty) \times (0, \infty) \times \mathbb{R}^2 \times \mathbb{R})\) is a weak solution to (2), (3), (4) if the following equations hold for all test functions \(\varphi, \psi \in C^\infty_c([0, T) \times \mathbb{R}^2)\) and \(\varphi \in C^\infty_c([0, T) \times \mathbb{R}^2; \mathbb{R})\) with \(\varphi \cdot \mathbf{n} |_{\partial \Omega} = 0\):
\[ \int_0^T \int_\Omega \varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla \varphi \, dx \, dt + \int_\Omega \varrho_0 \varphi(0, \cdot) \, dx = 0; \] (5)
\[ \int_0^T \int_\Omega \left[ \varrho \mathbf{u} \cdot \partial_t \varphi + (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi + \left( p + \frac{1}{2} b^2 \right) \text{div} \varphi \right] \, dx \, dt + \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \, dx = 0; \] (6)
\[ \int_0^T \int_\Omega \left[ \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, p) + \frac{1}{2} b^2 \right) \partial_t \varphi + \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, p) + p + b^2 \right) \mathbf{u} \cdot \nabla \varphi \right] \, dx \, dt + \int_\Omega \left( \frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, p_0) + \frac{1}{2} b_0^2 \right) \varphi(0, \cdot) \, dx = 0; \] (7)
\[ \int_0^T \int_\Omega \left[ b \partial_t \psi + b \mathbf{u} \cdot \nabla \psi \right] \, dx \, dt + \int_\Omega b_0 \psi(0, \cdot) \, dx = 0. \] (8)

Remark 1.2. The impermeability boundary condition is represented by the choice of the test functions.

Remark 1.3. Note that we exclude vacuum for our consideration, i.e. in this note \(\varrho > 0, \ p > 0\).

It is a well-known fact that there may exist physically non-relevant weak solutions to conservation laws. Hence one has to introduce additional selection criteria in order to single out the physically relevant weak solutions. A common approach is to impose an entropy inequality. However for the MHD system (1) there is no known entropy.
Note that for the Euler system the functions
\[ \eta = -\rho s(\rho, p) \quad \text{and} \quad q = -\rho s(\rho, p) u \]
form an entropy pair. Here the specific entropy \( s = s(\rho, p) \) is a given function as well as the internal energy \( e \) and note that these functions are interrelated by Gibbs’ relation.

It is a straightforward computation to show that a strong solution to the MHD system \((1)\) fulfills
\[ \partial_t (\rho s(\rho, p)) + \text{div} (\rho s(\rho, p) u) = 0. \] (9)
Although this suggests that \((\eta, q)\) is an entropy pair for the MHD system, too, \((\eta, q)\) is not an entropy pair for MHD, cf. [2]. However \((\eta, q)\) is still used as a selection criterion in the literature for example if Riemann problems are considered and one wants to find out whether or not a shock is physical, see e.g. [9]. We misuse terminology and call \(\eta\) and \(q\) still entropy, entropy flux respectively.

The weak solutions, whose existence we will prove in this note, fulfill the entropy equation (9) in the weak sense. We call such solutions entropy–conserving.

**Definition 1.4.** A weak solution \((\rho, p, u, b)\) to \((2), (3), (4)\) is called entropy–conserving, if for all test functions \(\varphi \in C^0_c([0, T) \times \mathbb{R}^2)\) the entropy equation
\[ \int_0^T \int_\Omega (\rho s(\rho, p) \partial_t \varphi + \rho s(\rho, p) u \cdot \nabla \varphi) \, dt + \int_\Omega (\rho_0 s(\rho_0, p_0) \varphi(0, \cdot)) \, dx = 0 \] (10)
holds.

The following theorem is our main result:

**Theorem 1.5.** Let \(\rho_0, p_0 \in L^\infty(\Omega; (0, \infty))\) and \(b_0 \in L^\infty(\Omega)\) be arbitrary piecewise constant functions. Then there exists \(u_0 \in L^\infty(\Omega; \mathbb{R}^2)\) such that there are infinitely many entropy–conserving weak solutions to \((2)\) with initial data \(\rho_0, p_0, u_0, b_0\) and impermeability boundary condition. These solutions have the property that \(\rho, p\) and \(b\) do not depend on time; in other words \(\rho \equiv \rho_0, p \equiv p_0\) and \(b \equiv b_0\).

The proof of Theorem 1.5 relies on the non–uniqueness proof for the full Euler system provided in [7] and consists of two main ideas. The first one is to make use of a result (see Proposition 2.1 below) which was proved by Feireisl [6] and also by Chiodaroli [3]. This result is based on the convex integration method, that was developed by De Lellis and Székelyhidi [4, 5] in the context of the pressureless incompressible Euler equations. The second idea is the fact that \(\rho, p\) and \(b\) can be chosen piecewise constant, what was observed originally by Luo, Xie and Xin [8].

Note that non–uniqueness of weak solutions fulfilling an entropy inequality (even in one space dimension) is well–known: There exist Riemann initial data for which one has more than one solutions, see e.g. Torrilhon [9] and references therein.

Note furthermore that there is also a convex integration result to incompressible ideal MHD by Bronzi et al. [1]. There the same two dimensional setting as in the present note is considered. In contrast to this note, where a convex integration result for Euler is used, Bronzi et al. apply the convex integration technique directly to an incompressible version of (2).

**2. Proof of the main result.** In order to prove Theorem 1.5 we will make use of the following proposition whose proof is based on convex integration.
Proposition 2.1. Let \( Q \subset \mathbb{R}^2 \) a bounded domain, \( \varrho > 0 \) and \( C > 0 \) positive constants. Then there exists \( m_0 \in L^\infty(Q; \mathbb{R}^2) \) such that there are infinitely many functions

\[
m \in L^\infty((0, T) \times Q; \mathbb{R}^2) \cap C_{\text{weak}}([0, T]; L^2(Q; \mathbb{R}^2))
\]

satisfying

\[
\int_0^T \int_Q m \cdot \nabla \varphi \, dx \, dt = 0, \quad (11)
\]

\[
\int_0^T \int_Q \left[ m \cdot \partial_t \varphi + \left( \frac{m \otimes m}{\varrho} - \frac{1}{2} \left| m \right|^2 \right) : \nabla \varphi \right] \, dx \, dt
+ \int_Q m_0 \cdot \varphi(0, \cdot) \, dx = 0, \quad (12)
\]

for all test functions \( \varphi \in C^\infty_c([0, T) \times \mathbb{R}^2) \) and \( \varphi \in C^\infty_c([0, T) \times \mathbb{R}^2; \mathbb{R}^2) \), and additionally

\[
E_{\text{kin}} = \frac{1}{2} \frac{|m|^2}{\varrho} = C \quad \text{a.e. in } (0, T) \times Q, \quad E_{0,\text{kin}} = \frac{1}{2} \frac{|m_0|^2}{\varrho} = C \quad \text{a.e. in } Q.
\]

For the proof of Proposition 2.1 we refer to [6, Theorem 13.6.1].

Now we are able to prove Theorem 1.5.

Proof of Theorem 1.5. Let \( \varrho_0, p_0 \in L^\infty(\Omega; (0, \infty)) \) and \( b_0 \in L^\infty(\Omega) \) given piecewise constant functions. Then there exist finitely many \( Q_i \subset \Omega \) open and pairwise disjoint, such that \( \Omega = \bigcup_i Q_i \) and \( \varrho_0 \mid_{Q_i} = \varrho_i \), \( p_0 \mid_{Q_i} = p_i \) and \( b_0 \mid_{Q_i} = b_i \) with constants \( \varrho_i, p_i, b_i > 0 \) and \( b_i \in \mathbb{R} \). We apply Proposition 2.1 on each \( Q_i \) to \( \varrho = \varrho_i \) and \( C = \Lambda - p_i - \frac{1}{2} |b_i|^2 \), where \( \Lambda \) is a constant with \( \Lambda > \max_i (p_i + \frac{1}{2} |b_i|^2) \). This yields \( m_{0,i} \in L^\infty(Q_i; \mathbb{R}^2) \) and infinitely many \( m_i \in L^\infty((0, T) \times Q_i; \mathbb{R}^2) \) with the properties given in Proposition 2.1. We then piece together the \( m_{0,i} \in L^\infty(Q_i; \mathbb{R}^2) \) to \( m_0 \in L^\infty(\Omega; \mathbb{R}^2) \) and the \( m_i \in L^\infty((0, T) \times Q_i; \mathbb{R}^2) \) to \( m \in L^\infty((0, T) \times \Omega; \mathbb{R}^2) \).

We define \( u_0 := \frac{m_0}{\varrho_0} \in L^\infty(\Omega; \mathbb{R}^2) \) and for each momentum field \( m \) we define a corresponding velocity field \( u := \frac{m}{\varrho_0} \in L^\infty((0, T) \times \Omega; \mathbb{R}^2) \). Furthermore we define \( (\varrho, p, b) \in L^\infty((0, T) \times \Omega; (0, \infty) \times (0, \infty) \times \mathbb{R}) \) by \( \varrho \equiv \varrho_0 \), \( p \equiv p_0 \) and \( b \equiv b_0 \). We claim that \( (\varrho, p, u, b) \) is an entropy-conserving weak solution to (2) with initial data \( \varrho_0, p_0, u_0, b_0 \).

Let \( \varphi, \phi, \psi \in C^\infty_c((0, T) \times \mathbb{R}^2) \) and \( \varphi \in C^\infty_c((0, T) \times \mathbb{R}^2; \mathbb{R}^2) \) with \( \varphi \cdot n_{\mid_{\partial \Omega}} = 0 \) arbitrary test functions. Using (11) and (12), we obtain the following.

\[
\int_0^T \int_\Omega \left[ \varrho \partial_t \varphi + \varrho u \cdot \nabla \varphi \right] \, dx \, dt + \int_\Omega \varrho_0 \varphi(0, \cdot) \, dx
= \sum_i \varrho_i \int_{Q_i} \left( \int_0^T \partial_t \varphi \, dt + \varphi(0, \cdot) \right) \, dx + \sum_i \int_0^T \int_{Q_i} m_i \cdot \nabla \varphi \, dx \, dt = 0;
\]
\[
\int_0^T \int_\Omega \left[ \rho \mathbf{u} : \nabla \varphi + (\rho \mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{v} + \left( p + \frac{1}{2} \mathbf{b}^2 \right) \operatorname{div} \varphi \right] \, dx \, dt + \int_\Omega \rho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \, dx \\
= \sum_i \left( \int_0^T \int_{Q_i} \left[ \mathbf{m}_i : \nabla \varphi + \left( \frac{\mathbf{m}_i \otimes \mathbf{m}_i}{\rho_i} - \frac{1}{2} \frac{\mathbf{m}_i^2}{\rho_i^2} \right) : \nabla \mathbf{v} \right] \, dx \, dt \\
+ \int_{Q_i} \mathbf{m}_{0,i} \cdot \varphi(0, \cdot) \, dx \right) + \sum_i \int_0^T \int_{Q_i} \left[ \frac{1}{2} \frac{\mathbf{m}_i^2}{\rho_i} + \left( p_i + \frac{1}{2} \mathbf{b}_i^2 \right) \right] \operatorname{div} \varphi \, dx \, dt \\
= \Lambda \int_0^T \int_{Q_i} \operatorname{div} \varphi \, dx \, dt = 0;
\]

\[
\int_0^T \int_\Omega \left[ \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e(\rho, p) + \frac{1}{2} \mathbf{b}^2 \right] \partial_t \phi + \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e(\rho, p) + p + \mathbf{b}^2 \right) \mathbf{u} \cdot \nabla \phi \right] \, dx \, dt \\
+ \int_{\Omega} \left( \frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \rho_0 e(\rho_0, p_0) + \frac{1}{2} \mathbf{b}_0^2 \right) \phi(0, \cdot) \, dx \\
= \sum_i \left( \Lambda + \rho_i e(\rho_i, p_i) - p_i \right) \int_{Q_i} \left( \int_0^T \partial_t \phi \, dt + \phi(0, \cdot) \right) \, dx \\
+ \sum_i \left( \Lambda + \rho_i e(\rho_i, p_i) + \frac{1}{2} \mathbf{b}_i \right) \int_0^T \int_{Q_i} \mathbf{m}_i \cdot \nabla \phi \, dx \, dt = 0;
\]

\[
\int_0^T \int_{\Omega} \left[ b \partial_t \psi + \mathbf{u} \cdot \nabla \psi \right] \, dx \, dt + \int_{\Omega} b_0 \psi(0, \cdot) \, dx \\
= \sum_i b_i \int_{Q_i} \left( \int_0^T \partial_t \psi \, dt + \psi(0, \cdot) \right) \, dx + \sum_i \frac{b_i}{\rho_i} \int_0^T \int_{Q_i} \mathbf{m}_i \cdot \nabla \psi \, dx \, dt = 0.
\]

We have shown that the equations (5) - (8) hold. Hence \((\rho, p, \mathbf{u}, \mathbf{b})\) is indeed a weak solution. It remains to show that this solution is entropy-conserving. In other words we have to show that (10) holds. Let \(\varphi \in C^\infty_c([0, T) \times \mathbb{R}^2)\) be an arbitrary test function. We obtain

\[
\int_0^T \int_\Omega \left[ g_0 s(\rho, p) \partial_t \varphi + g_0 s(\rho, p) \mathbf{u} \cdot \nabla \varphi \right] \, dx \, dt + \int_\Omega \rho_0 s(\rho_0, p_0) \varphi(0, \cdot) \, dx \\
= \sum_i g_i s(\rho_i, p_i) \int_{Q_i} \left( \int_0^T \partial_t \varphi \, dt + \varphi(0, \cdot) \right) \, dx \\
+ \sum_i s(\rho_i, p_i) \int_0^T \int_{Q_i} \mathbf{m}_i \cdot \nabla \psi \, dx \, dt = 0.
\]

Thus \((\rho, p, \mathbf{u}, \mathbf{b})\) is an entropy-conserving weak solution. Since there are infinitely many \(\mathbf{m}\) from Prop. 2.1, there are infinitely many entropy-conserving solutions \((\rho, p, \mathbf{u}, \mathbf{b})\). \qed
3. Isentropic MHD. In this section we apply our result to isentropic MHD equations. The isentropic MHD system reads

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) - (\text{curl} \mathbf{B}) \times \mathbf{B} &= 0, \\
\partial_t \mathbf{B} + \text{curl} (\mathbf{B} \times \mathbf{u}) &= 0, \\
\text{div} \mathbf{B} &= 0.
\end{align*}
\]

(13)

The unknown functions are the density \(\rho > 0\), the velocity \(\mathbf{u} \in \mathbb{R}^3\) and the magnetic field \(\mathbf{B} \in \mathbb{R}^3\). In contrast to the MHD system (1) the pressure \(p\) in (13) is not an unknown but a given function of the density, where \(p(\rho) > 0\) for all \(\rho > 0\).

Again we consider a two dimensional setting. Let \(\Omega \subset \mathbb{R}^2\) a bounded two dimensional spacial domain. We consider \(\mathbf{u} = (u, v, 0)^T\) and \(\mathbf{B} = (0, 0, b)^T\) and furthermore we let all the unknowns only depend on \((x, y)\) \(\in \Omega\). Then the isentropic MHD system (13) turns into

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \left(p(\rho) + \frac{1}{2} b^2\right) &= 0, \\
\partial_t b + \text{div} (b \mathbf{u}) &= 0.
\end{align*}
\]

(14)

For the isentropic Euler system, the energy

\[\eta = \frac{1}{2} \rho |\mathbf{u}|^2 + P(\rho) + \frac{1}{2} |\mathbf{B}|^2\]

is an entropy. Here \(P(\rho)\) is called pressure potential and is given by

\[P(\rho) = \rho \int_1^\rho \frac{p(r)}{r} \, dr.\]

Similar to the full MHD system considered above, one can show that the energy is not an entropy for (13) but strong solutions fulfill the corresponding energy equation

\[
\begin{align*}
\partial_t \left( \frac{1}{2} \rho |\mathbf{u}|^2 + P(\rho) + \frac{1}{2} |\mathbf{B}|^2 \right)
+ \text{div} \left[ \left( \frac{1}{2} \rho |\mathbf{u}|^2 + P(\rho) + p(\rho) + |\mathbf{B}|^2 \right) \mathbf{u} \right] - \text{div} ((\mathbf{B} \cdot \mathbf{u}) \mathbf{B}) &= 0.
\end{align*}
\]

(15)

Hence we will look for energy–conserving weak solutions. In the considered setting the energy equation (15) turns into

\[
\begin{align*}
\partial_t \left( \frac{1}{2} \rho |\mathbf{u}|^2 + P(\rho) + \frac{1}{2} b^2 \right) + \text{div} \left[ \left( \frac{1}{2} \rho |\mathbf{u}|^2 + P(\rho) + p(\rho) + b^2 \right) \mathbf{u} \right] &= 0.
\end{align*}
\]

Definition 3.1. A triple \((\rho, \mathbf{u}, b) \in L^\infty([0, T) \times \Omega; (0, \infty) \times \mathbb{R}^2 \times \mathbb{R})\) is a weak solution to (14) with initial data \(\rho_0, \mathbf{u}_0, b_0\) and impermeability boundary condition if the following equations hold for all test functions \(\varphi, \psi \in C_c^\infty([0, T) \times \mathbb{R}^2)\) and \(\varphi \in C_c^\infty([0, T) \times \mathbb{R}^2; \mathbb{R}^2)\) with \(\varphi \cdot \mathbf{n}\big|_{\partial \Omega} = 0;\)

\[
\int_0^T \int_\Omega \left[ \rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla \varphi \right] \, dx \, dt + \int_\Omega \varrho_0 \varphi(0, \cdot) \, dx = 0;
\]

(16)
Acknowledgement. The authors thank Bruno Despres for fruitful discussions.

The Cauchy problem for the isentropic MHD equations is ill-posed, too:

**Corollary 3.2.** Let \( q_0 \in L^\infty(\Omega; (0, \infty)) \) and \( b_0 \in L^\infty(\Omega) \) be arbitrary piecewise constant functions. Then there exists \( u_0 \in L^\infty(\Omega; \mathbb{R}^2) \) such that there are infinitely many energy-conserving weak solutions to (14) with initial data \( q_0, u_0, b_0 \) and impermeability boundary condition. These solutions have the property that \( q \) and \( b \) do not depend on time; in other words \( q \equiv q_0 \) and \( b \equiv b_0 \).

**Proof of Corollary 3.2.** Let \( q_0 \in L^\infty(\Omega; (0, \infty)) \) and \( b_0 \in L^\infty(\Omega) \) given piecewise constant functions. Set furthermore \( p_0 := p(q_0) \). Then \( p_0 \in L^\infty(\Omega; (0, \infty)) \) is a piecewise constant function. Additionally we can choose the function \( e(q, p) \) in such a way that \( q_0 e(q_0, p_0) = P(q_0) \). We know from Theorem 1.5 that there exists an initial velocity \( u_0 \in L^\infty(\Omega; \mathbb{R}^2) \) such that there are infinitely many entropy-conserving weak solutions \( (q \equiv q_0, p \equiv p_0, u, b \equiv b_0) \) to (2) with initial data \( q_0, p_0, u_0, b_0 \). It is easy to check that for each of these solutions, the triple \( (q \equiv q_0, u, b \equiv b_0) \) is an energy-conserving weak solution to the isentropic MHD equations (14) with initial data \( q_0, u_0, b_0 \) in the sense of Definition 3.1.

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Hyperbolic Problems: Theory, Numerics, Applications

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The contributions collected in this volume cover a wide range of topics. Some of these represent the latest developments on classical multi-dimensional problems, dealing with shock reflections and with the stability of vortices and boundary layers. Other contributions provide sharp results on the structure and regularity of solutions to conservation laws, or discuss the fine line between well-posedness and ill-posedness for transport equations with rough coefficients, and for the equations of inviscid fluid flow. Further progress is reported at the interface between hyperbolic and kinetic models, including the hydrodynamic limit of the Boltzmann equation. Kinetic and macroscopic models for collective dynamics of many-body systems, which have attracted much interest in recent years, are also covered in this volume. Finally, a large number of papers are devoted to advances in computational methods, with diverse applications such as: submarine avalanches, tsunami waves, chemically reacting flows, solitary waves, gas flow on a network of pipelines, traffic flow with multiple types of vehicles, etc.

The present volume provides a timely survey of the state of the art, which will be of interest to researchers, students and practitioners, with interest in the theoretical, computational and applied aspects of hyperbolic problems.

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