Spiral galaxies and their dark halos are self-organized patterns

SHANKAR C. VENKATARAMANI and ALAN C. NEWELL

1 Department of Mathematics, University of Arizona, 627 N. Santa Rita Ave., Tucson, AZ 85721, USA

Abstract – Motivated by and generalizing the Cross-Newell energy functional for pattern forming systems, we propose a modification of the Einstein-Hilbert action on galactic scales. We introduce additional terms that reflect the presence of patterns, like fingerprints, with the same length scales as the clumping instabilities of baryonic matter. The resulting pattern structures carry an additional energy density which, remarkably, accounts for many of the phenomena that are generally attributed to dark matter.

Mass discrepancies and dark matter. – Many independent lines of reasoning have unequivocally established that all the visible (baryonic) mass in the universe falls well short of the amount of matter inferred from the Newtonian/General relativistic dynamics of stars in galaxies, of galaxies in clusters, and the expansion history of the universe [1].

In the early 1930s, Oort [2] analyzed stars in the solar neighborhood and concluded that visible stars only accounted for about half of the mass needed to explain their observed vertical displacements from the galactic plane. Contemporaneously, Zwicky [3] studied the velocity dispersion of nebulae in the Coma cluster and deduced that the cluster required more than 100 times the mass in the luminous galaxies in order to stay bound. This led him to propose that the gravity of unseen dunkle materie (dark matter) was holding the cluster together. Widespread acceptance of the existence of dark matter took a few more decades and followed the important findings of Rubin, Ford and Thonnard [4] on the rotation curves of galaxies.

A simple model in which the force experienced by the star executing a circular orbit at radius \( r \) about the galactic center due to some large central mass \( M \) leads to a Keplerian rotation curve \( v = \sqrt{GM/r} \). Observations, however, consistently demonstrate that the galactic rotation curves flatten out and the orbital velocity of distant stars is roughly constant [4]. This is illustrated, for example, in fig [1] generated from rotation velocity data for the galaxy NGC 3198 (van Albada et al [5]).

Turning the argument around by balancing \( GM/r^2 \) with \( v^2/r \) with \( v \) constant, gives a mass \( M \) of \( v^2/rG \) which is more mass than the galaxy would seem to contain. Dark matter (DM) was invented to resolve this and other discrepancies [6][7]. For example, the data in Fig. [1] can be explained by a spherical distribution of dark matter in the galactic halo [5]. Unfortunately, to date, the source of this extra mass has not yet been definitively identified. Also, there are countervailing view points. Milgrom [7] and others have argued that the observations should be interpreted as the need to modify Newton’s second law at small accelerations (MOND), and that the dynamics on galactic scales can be explained without any appeal to missing mass. Indeed, there are a slew of observations which indicate, for quasi-steady systems, the dynamically inferred DM halo is strongly correlated with the baryon distribution [8]. These include the baryonic Tully-Fisher relation (BTFR) [9] and the radial acceleration relation (RAR) [10]. Why, then, do DM halos conspire to arrange themselves in a manner that mimics MOND [11][12]?
These arguments are often the starting point for a debate on the correctness of MOND and/or the existence/nature of dark matter \cite{11}. Our viewpoint is somewhat different. We recognize that, while MOND has difficulties explaining dynamics on cluster and larger scales \cite{11}, and detailed features of CMB anisotropies \cite{13}, it quite accurately describes the “inner” behaviors of galaxies, out to several scale lengths. The inner regions of spiral galaxies are self-organized structures \cite{14}, since, in this regime, nominally independent entities, the DM halo and the distribution of baryons, are strongly tied to each other \cite{8}. The main goal of this letter is to identify physical feedback mechanisms connecting baryons to (effective) DM and to derive mathematical models for this phenomenon.

Our ideas derive from the study of pattern forming systems \cite{15} and universal behaviors in phase transitions \cite{16}. Patterns arise in far from equilibrium systems as they evolve through a sequence of phase transitions – from homogenous states to increasingly symmetry-broken, patterned microstructures with preferred length scales. Such patterns with a nearly periodic microstructure are ubiquitous, from fingerprints to the spiral arms of galaxies (see Fig.\[2\]).

Patterns are born at critical values of a stress parameter as a pre-existing state becomes unstable. Instabilities with preferred shapes and configurations are amplified breaking some but not all of the existing symmetries of the original state. For example, in a horizontal layer of fluid with high Prandtl number and heated from below, the homogeneous conduction state destabilizes to a state of convection rolls with a preferred wavelength. The continuous translational symmetry is broken but not the rotational symmetry. Therefore in large aspect ratio systems, where the overall size is much greater than the roll wavelength, the orientation of the roll patches is determined by local biases and the pattern consists of a metastable mosaic of patches of the preferred planform with different orientations which meet and the pattern field. We derive this coupling based representing a locally periodic structure \cite{17,18}.

Our ideas derive from the study of pattern forming systems \cite{15} and universal behaviors in phase transitions \cite{16}. Patterns arise in far from equilibrium systems as they evolve through a sequence of phase transitions – from homogenous states to increasingly symmetry-broken, patterned microstructures with preferred length scales. Such patterns with a nearly periodic microstructure are ubiquitous, from fingerprints to the spiral arms of galaxies (see Fig.\[2\]).

Patterns are born at critical values of a stress parameter as a pre-existing state becomes unstable. Instabilities with preferred shapes and configurations are amplified breaking some but not all of the existing symmetries of the original state. For example, in a horizontal layer of fluid with high Prandtl number and heated from below, the homogeneous conduction state destabilizes to a state of convection rolls with a preferred wavelength. The continuous translational symmetry is broken but not the rotational symmetry. Therefore in large aspect ratio systems, where the overall size is much greater than the roll wavelength, the orientation of the roll patches is determined by local biases and the pattern consists of a metastable mosaic of patches of the preferred planform with different orientations which meet and the pattern field. We derive this coupling based representing a locally periodic structure \cite{17,18}.

Patterns are born at critical values of a stress parameter as a pre-existing state becomes unstable. Instabilities with preferred shapes and configurations are amplified breaking some but not all of the existing symmetries of the original state. For example, in a horizontal layer of fluid with high Prandtl number and heated from below, the homogeneous conduction state destabilizes to a state of convection rolls with a preferred wavelength. The continuous translational symmetry is broken but not the rotational symmetry. Therefore in large aspect ratio systems, where the overall size is much greater than the roll wavelength, the orientation of the roll patches is determined by local biases and the pattern consists of a metastable mosaic of patches of the preferred planform with different orientations which meet and the pattern field. We derive this coupling based representing a locally periodic structure \cite{17,18}.

Patterns are born at critical values of a stress parameter as a pre-existing state becomes unstable. Instabilities with preferred shapes and configurations are amplified breaking some but not all of the existing symmetries of the original state. For example, in a horizontal layer of fluid with high Prandtl number and heated from below, the homogeneous conduction state destabilizes to a state of convection rolls with a preferred wavelength. The continuous translational symmetry is broken but not the rotational symmetry. Therefore in large aspect ratio systems, where the overall size is much greater than the roll wavelength, the orientation of the roll patches is determined by local biases and the pattern consists of a metastable mosaic of patches of the preferred planform with different orientations which meet and the pattern field. We derive this coupling based representing a locally periodic structure \cite{17,18}.
Patterns, defects and a Lagrangian for dark matter. — A curved space-time generalization Cross-Newell energy [15] can be obtained from the ‘minimal coupling’ assumption [22] as

\[ S_p = \frac{\rho_0 c^2}{k_0^2} \int \left\{ (\nabla^\mu \psi \nabla \mu \psi - k_0^2)^2 + (\nabla^\rho \nabla \rho \psi)^2 \right\} \sqrt{-g} \, d^4x, \]  

where the metric \( g_{ab} \) has signature \((- + + +)\) and \( \nabla \) is the corresponding covariant derivative. Eq. (1) gives the natural covariant generalization [17] of the universal averaged energy for nearly periodic stripe patterns [23], and is thus expected to describe the macroscopic behavior of phase hyper-surfaces in curved spacetimes for a variety of microscopic models [17]. The phase \( \psi \) is dimensionless, \( k_0 \) is the preferred wavenumber, and \( \rho_0 c^2 \) is the energy density in the pattern field.

Assuming spherical symmetry and weak fields, we can work on a flat “background” spacetime \( g_{ab} \approx \delta_{ab} \) and obtain target patterns \( \psi(r) \approx k_0 r \) in the far field \( r \gg k_0^{-1} \), and \( \phi(r) \approx 0 \) in the near field \( r \ll k_0^{-1} \) [18]. The (dominant) bending energy in the pattern field \( \psi \) is equivalent to a mass distribution \( M_{f}(r) \approx 32\pi\rho_0 R^3/(b^2 + k_0^2 R^2) \), with \( b \approx O(1) \), corresponding to a quasi-isothermal halo, rather than the more commonly invoked “cuspy” NFW halo. We model the baryon distribution by a Plummer sphere [24] \( M(R) = M_B R^3 / (R^2 + z_0^2)^{3/2} \) where \( z_0 \) denotes the scale length of the baryon distribution. Equating the total gravitational acceleration \( G(M_B + M_f(R))/R^2 \) with the centripetal acceleration \( v^2/R \) of a circular orbit, yields

\[ v^2 = \frac{G M_B R^2}{(R^2 + z_0^2)^{3/2}} + \frac{32 G \rho_0 R^2}{b^2 + k_0^2 R^2} \]  

demonstrating the flattening of the rotation curve with an asymptotic velocity given by \( v_\infty^2 = 32 G \rho_0 k_0^2 \).

As a first approximation, we will assume that the rotation curve in (2) also holds for spiral galaxies. As illustrated in Fig. 1 for a typical high surface brightness (HSB) spiral, the rotation curve \( \nu(r) \) has the following universal features – (i) in the inner region, the velocity increases roughly linearly; (ii) in the intermediate region, the velocity is essentially a constant; and (iii) the rotation curve is “smooth” and “featureless”. It is (essentially) of the form \( \nu(r) = v_\infty f(z_0/r, \alpha) \) with dimensional velocity scale \( v_\infty \) and length scale \( r_0 \). \( O(1) \) dimensionless parameter(s) \( \alpha \) determine the shape of the curve and (potentially) vary between galaxies. There is an outer length scale, a few times \( r_0 \), beyond which the \( \nu(r) \) starts to decay [25]. \( v_\infty \) is the velocity in the flat portion of the rotation curve, and not the asymptotic velocity in the strict sense of \( r \to \infty \). Indeed, we neither expect, nor require that our theory apply to the “far-field”, \( r/r_0 \gg O(1) \), behavior of spiral galaxies, where interactions with other galaxies in a cluster and/or the Hubble flow cause effects comparable to those from the self-organized disk/halo structure.

In MOND, the baryonic parameters \( M_B \) and \( z_0 \) determine the entire rotation curve. The rotation curve in (2), however, has 2 additional dimensionless parameters \( \alpha_L = z_0/\rho_0 = k_0 z_0/b \) the ratio of the baryonic and DM scale lengths, and \( \alpha_M = 3 M_B / 4 \pi \rho_0 R_0^3 \), the ratio of density scales. For our theory to have a comparable predictive power, instead of merely serving as a parametrization for fitting rotation curves, we need to determine the dimensionless parameters \( \alpha_L \) and \( \alpha_M \), or equivalently the quantities \( \rho_0 \) and \( k_0 \) in terms of \( M_B \) and \( z_0 \).

The first relation between \( \rho_0 \), \( k_0 \) and the baryonic parameters comes from demanding that the disk and the halo contributions to the rotation velocity (2) are comparable for \( r \sim r_0 \), so that

\[ \rho_0 \approx \frac{b^2 M_B}{16 \pi (r_0^2 + z_0^2)^{3/2}}, \quad \alpha_M \approx \frac{12}{b^2} (1 + \alpha_L^2)^{3/2}. \]  

The second relation comes from stability considerations of a differentially rotating fluid disk. A WKB analysis gives the dispersion relation for density waves [24]

\[ (\omega - m \Omega)^2 = c_s^2 (k - k_0(r))^2 + \kappa^2 - c_3^2 k_0(r)^2, \]  

\[ \Omega(r) = r^{-1} v(r), \quad \kappa^2 = r^2 \partial_r (\Omega(r)^2) + 4 \Omega(r)^2, \]  

where \( c_s \) is the effective sound speed (relating restoring pressures to density perturbations), \( \kappa \) is the epicyclic frequency and \( k_0(r) = \pi G \Sigma c_s^{-2} \) is the “locally” most unstable (i.e. pattern forming) wavenumber and \( \Sigma \) is the local surface density. The action in (1) can only account for a constant wavenumber \( k_0 \), which we identify with the instability wavenumber near the center \( \pi G \Sigma c_s^{-2} \), since the phase \( \psi \) in our model is (conjecturally) connected with the clumping instability in baryons.

Stability for a fluid disk requires \( Q = \frac{c_3}{c_s} \gg 1 \). For a stellar disk \( c_s \) is replaced by \( \sigma \), the radial velocity dispersion (Toomre’s criterion). A positive \( \kappa^2 \) means that the angular momentum is increasing with \( r \), the distance to the center of the galaxy, and stabilizes the long wavelengths \( k \to 0 \). We assume that disk galaxies are self-organized through the following stabilization feedback mechanism – if \( Q < 1 \), the disk/gas heats up until it reaches marginal stability \( Q = 1 \).

For a rotation curve \( v = v_\infty f(r/r_0) \), a homogeneous disk is unstable to spiral density waves between the inner and the outer Lindblad resonances \( r_0 < r < r_\ast \) [24]. As argued above, the disk evolves to marginal stability \( Q = 1 \) at \( r = r_\ast \). By dimensional analysis \( r_\ast = c_s \omega_0 / \kappa \), where the constants \( c_s \) only depend on the scaling function \( f \) and the pattern speed \( \omega_0 \). As we show elsewhere, averaging the radial momentum equation for a differentially rotating disk, and numerically evolving a density wave pattern, show that the second term in (4) is proportional to \( \sigma^2 \), the square of the radial velocity dispersion, and \( r \approx c_s / \kappa \).

Combining this expression with Eq. (4), \( v_\infty^2 = 32 G \rho_0 k_0^2 \), we get \( v_\infty^2 \sim \pi G \rho_0 \sigma^2 \sim G M_B \sigma_0^2 \left( \Sigma_0 \right)^2 \). For the Milky Way, the central dispersion \( \sigma \approx 100 \text{km/s}^{-1} \) and \( r \approx z_0 \approx 3 \text{kpc} \), so we get \( v_\infty^2 / G M_B \approx O(1) \times 10^{-10} \text{ms}^{-2} \). By way of comparison, the BTFR gives \( v_\infty^2 / G M_B = a_0 \approx 1.2 \times 10^{-10} \text{ms}^{-2} \), a universal value with very low intrinsic scatter for a wide range of galaxies [9].

Turning our argument around, i.e assuming the BTFR as a “natural law” and deducing consequences from it, we get that \( \rho_0 k_0^2 \) in tightly constrained by the total baryonic mass \( M_B \). The argument leading to (3) can be recast as \( \alpha_L, \alpha_M \sim O(1) \), with some scatter between different galaxies. This weakly constrains \( \rho_0 k_0^2 \) in terms of \( M_B \). Somewhat surprisingly, it also relates the baryonic mass \( M_B \) to the scale length \( z_0 \) and the cen-
tral dispersion \( \sigma \), yielding the relations
\[
\frac{32\pi\rho_0}{k_0^2} = \sqrt{\frac{M_{B,0}}{G}}, \quad \rho_0 \sim M_B \frac{M}{2\pi \sigma_0^2} \approx a_0 \frac{M_B}{G}, \quad \sigma^2 \sim a_0 k_0^{-1} \sim M_B^{1/2}.
\]
The first expression calibrates the constants in [1], the second, as we argue below, gives the Freeman limit [20], and the third expression is the \( M_B \sim \sigma^4 \) relation for spirals [26], where \( M_B \) is a proxy for the mass of the central (dynamically hot) bulge.

We thus get a lot of mileage from the BTFR along with “classical physics” arguments. We interpret the successes of these simple arguments as evidence that disk galaxies are indeed (nonlinearly) self-organized systems, with very few independent “state” variables, and are thus amenable to analysis using universal models for phase transitions/pattern formation.

Non-relativistic spherical galaxies. – For a typical galaxy the rotational velocity scale \( v_\infty \ll c \) motivating the definition,
\[
\epsilon \equiv \frac{(v_\infty/c)^2}{32\pi\rho_0} = 32\epsilon^{-2}k_0^{-2}G\rho_0 = \epsilon^{-2}(GM_{B,0}\sigma_0)^{1/2},
\]
of our fundamental small parameter \( \epsilon \).

As before, \( r_0 \) is a characteristic length scale from the rotation curve. We will non-dimensionally by choosing units such that \( r_0 = G = c = 1 \). Eqs. (5) and (3) together imply that \( \rho_0, M_B \) and \( a_0 \) are all \( \mathcal{O}(\epsilon) \), while \( z_0 \) and \( k_0 \) are \( \mathcal{O}(1) \) in these units.

Defining scaled coordinates \( \xi = r/r_0 \) and \( \tau = ct/r_0 \), we have the following asymptotic dependences in \( \epsilon \) – the baryonic density is \( \epsilon \rho_B(\xi) \), the “spatial 3-velocity” \( \frac{dv}{d\xi} \) is \( \sqrt{\epsilon} cv(\xi) \), the pattern field is given by \( \psi(\xi) = k_0^{-1}\psi = \chi(\xi) + \epsilon\chi(\xi) + \cdots \) and spherical galaxies are described by the weak-field metric
\[
g = -(1 + 2\epsilon\phi(\xi))d\tau^2 + (1 + 2\epsilon\lambda(\xi))d\xi^2 + \xi^2 d\Omega^2.
\]
(6)
All the \( \epsilon \) dependences are explicit and the functions \( \rho_B, v, \chi, \phi \) and \( \lambda \) are \( \mathcal{O}(1) \) smooth functions of their argument \( \xi \). With these redefinitions, the Lagrangian density \( \mathcal{L}_p \) for the pattern action [1] is formally \( \mathcal{O}(\epsilon) \),
\[
\mathcal{L}_p = \frac{\epsilon}{32\pi G} \left[ k_0^2 \left( \psi^2 (1 + 2\epsilon\lambda(\xi) ) - 1 \right)^2 + \left( \psi'' + \frac{2}{\xi} \psi \right)^2 \right].
\]
(7)
The Einstein-Hilbert action [22] in this weak-field limit is
\[
S_{EH} = \int R \sqrt{-g} d^4x = \frac{\epsilon^2}{8\pi G} \int \left( \frac{\lambda^2}{2} - \xi \psi' \right) d\xi d\tau + \cdots,
\]
where the terms that are not explicitly exhibited are higher order and/or constitute a null Lagrangian, so they do not contribute to the Euler-Lagrange equations at \( O(\epsilon^2) \).

We model the baryonic matter as dust. For non-interacting discrete masses \( \epsilon m_i \), with trajectories \( x_i(\tau) \), the matter action is
\[
S_M = -\sum_i \int \epsilon m_i c^2 \sqrt{1 + 2\phi(x_i(\tau)) - \psi^2} d\tau
= \epsilon^2 \sum_i \int m_i \left( \frac{g^{(3)}(c_{vi}, c_{v_i})}{2} - \phi(x_i(\tau)) \right) d\tau + \cdots
\]
where \( \frac{d}{d\tau} x_i = v_i , g^{(3)} \) is the “spatial” metric on the constant \( \tau \) slices [22], and we use \( \sum_i \epsilon m_i = M_B \) is independent of \( \tau \).

We have exhibited all the terms that contribute to the Euler-Lagrange equation up to \( O(\epsilon^2) \). Expanding the total action \( S = S_{EH} + S_M + S_P = c\epsilon^2 + c^2 S_2 + \cdots \) in powers of \( \epsilon \), dropping null Lagrangians and taking the continuum limit via the replacement \( \sum_i \int d\tau m_i f(x_i(\tau)) \rightarrow \epsilon \int P_B f \sqrt{-g} d^4x \), we obtain
\[
S_1 = \frac{1}{8G} \int \left[ k_0^2 \left( \psi^2 - 1 \right)^2 + \left( \psi' + 2\epsilon\lambda \right)^2 \right] d\xi d\tau.
\]
(8)
\( S_1 \) is independent of the weak-field potentials \( \phi, \lambda \) and the baryonic distribution \( \rho_B \), so we can solve for \( \chi(\xi) \) without further knowledge of the full solution. The solutions with spherical symmetry are target patterns with \( \chi(\xi) = \pm 1 \) in the far-field \( \xi \sim 1 \). This is a common feature for phase patterns with a preferred wavenumber. Away from the defects the phase contours have constant separation so the phase is given by the Eikonal equation \( |\nabla \psi| = 1 \). Using the solution \( \chi(\xi) = \xi \) all the way down to the defect at \( \xi = 0 \), we get
\[
S_2 = \int \left[ \frac{1}{2G} \left( \frac{\lambda^2}{2} - \xi \psi' \right) + 4\epsilon \rho_B \epsilon^2 \left( \frac{\psi^2}{2} - \phi \right) \right] d\xi d\tau.
\]
(9)
The corresponding variational equations are
\[
\begin{align*}
\partialnf(\xi, \lambda) - 4\pi G \rho_B \epsilon^2 & = 0, \\
\lambda - \xi \psi' & = 1 = 0
\end{align*}
\]
(10)
For circular orbits \( v = v_\phi(\xi) \epsilon \phi \) we obtain \( v^2 = \xi \psi', \psi' = GM_B(\xi)\xi^{-2} + \xi^{-1} \). Introducing the dimensional scales \( v_0(R) = \sqrt{v^2_0 + GM_B(R)/R} \). This expression captures the asymptotic velocity but is, of course, inaccurate as \( R \rightarrow 0 \). One way to understand this result is that, in the non-relativistic limit, the effective DM is approximated by the pattern Lagrangian density \( \mathcal{L}_p \) defined in (7). For the approximate extremizer of \( S_1, \chi(\xi) = \xi, \mathcal{L}_p = \frac{\epsilon}{8\pi G} \) in dimensional units, corresponding to a “cuspy” isothermal halo. More accurate extremizers for the phase \( \psi \) give solutions that have a core of size \( k_0^{-1} \sim r_0 \) and correspond to quasi-isothermal halos. Analysis along these lines [18] predicts the (more accurate) rotation curves in (2).

Towards a universal theory for spiral galaxies. – While the action \( S_{2} \) in [1] naturally gives a target pattern and the action \( S_{1} \) in [9] describes the flattening of rotation curves, they have a serious lacuna. The phase \( \psi \) is a “background field”, determined by symmetry considerations, and is independent of the baryonic density \( \rho_B, \rho_0 \) ought to be coupled to \( \psi \) at leading order (i.e. in Eq. [8]). In this section, we use physical considerations to derive this coupling.

Our first idea comes from MOND [7][8]. In MOND, the Newtonian gravitational potential \( \phi_\infty \) is only sourced by the baryonic matter density \( \rho_B, \Delta \phi_\infty = 4\pi G \rho_B \), while the dynamics are given by \( \ddot{x} = -\mu (\frac{\Delta \phi_\infty}{\Delta m_0}) \nabla \phi_\infty \) for an appropriate transition function \( \mu \) and \( \Delta m_0 \sim 10^{-10} m/ s^2 \) is a universal acceleration scale.

The claim that \( \ddot{x} \) is determined locally by \( \nabla \phi_\infty \), has strong observational support in the radial acceleration relationship
The Freeman “law” is a selection effect. i.e. it holds for high surface brightness (HSB) galaxies, but low surface brightness (LSB) galaxies have a wide distribution of central densities, supported on \( \Sigma_p(0) < \Sigma^* \), defining the Freeman limit [20, 29].

To us, this suggests that the Kuzmin disk with a central density \( \Sigma_p(0) = \Sigma^* \) is a self-organized critical state, and is “more scale-invariant”, i.e. has fewer “relevant” dimensionless constants than a generic baryonic distribution in a disk.

Motivated by these observations, our construction of the coupling between the phase field \( \tilde{\psi} \) and the baryonic density \( \rho_B \) will be predicated on requiring that, for the critical Kuzmin disk, the phase gradient \( \nabla \tilde{\psi} \) (a proxy for “dark matter”) is determined locally by the Newtonian acceleration \( \nabla \Phi_N \). Consequently, the spherical background target pattern \( \tilde{\psi} = (r^2 + z^2)^{1/2} \) for a point mass ought to be replaced by \( \tilde{\psi} = (r^2 + (|z| + z_0)^2)^{1/2} \) for the critical Kuzmin disk. Here and henceforth \( r = \sqrt{x^2 + y^2} \).

Away from \( z = 0 \), the bending energy in this pattern is \( 4\epsilon/(r^2 + (|z| + z_0)^2) \approx 4\epsilon/(r^2 + z^2) \) for \( |z| \gg z_0 \), and gives an asymptotically flat rotation curve with \( v_\infty = c = (GM_0)^{1/2} \).

The surface \( z = 0 \), however, is a phase grain boundary corresponding to a sharp jump in \( \nabla \tilde{\psi} \). The tangential component of \( \nabla \tilde{\psi} \) is, of course, continuous at the jump. The discontinuity arises from insisting \( \nabla \tilde{\psi} = 1 \) everywhere. It is regularized by a boundary layer that matches the stretching and bending energies in (1) with energy density proportional to the third power of the jump in \( \nabla \tilde{\psi} \) [15]. Writing \( \nabla \tilde{\psi}^2 = \nabla \tilde{\psi}^0 + \nabla \tilde{\psi} \tilde{\psi} \) in components along and perpendicular to the grain boundary, and using \( \nabla \tilde{\psi}^2 = 1 \), we get \( \nabla \tilde{\psi}^0 = \nabla \tilde{\psi} = 2(1 - |\nabla \tilde{\psi}|^2)^{1/2} \). Assuming the Freeman Law [13] a direct computation for the pattern energy density \( \Sigma_p \) along the grain boundary yields

\[
\Sigma_p \propto \sin^3 \theta(s) = (1 - |\nabla \tilde{\psi}|^2)^{3/2} = \frac{z_0^3}{(r^2 + z_0^2)^{3/2}} = \frac{\Sigma_B}{\Sigma^*},
\]

a remarkable relation between pattern dark matter (through \( \nabla \tilde{\psi} \)) and the baryonic surface density \( \Sigma_B \). Although we obtained this relation for the critical Kuzmin disk, it generalizes to all disk galaxies and furthermore, it naturally explains the existence of a maximum surface density, the Freeman limit \( \Sigma^* \).

There are several ways to obtain [14] from an action that is defined through a Lagrangian 3-form \( L_\psi dAdt \) where \( dA \) is the area element on the plane \( z = 0, t = \text{constant} \), e.g. \( L_\psi = \frac{1}{2} (1 - |\nabla \tilde{\psi}|^2)^2 \Sigma^* - (1 - |\nabla \tilde{\psi}|^2)^2 \Sigma_B \). We, however, require a Lagrangian 4-form that “behaves nicely” under the replacement \( \rho_B \nabla \tilde{\psi}^2 d^2 x \to \Sigma_B d^2 x \) (the thin-disk limit). The action

\[
S_\psi = -4\epsilon^2 \int \rho_B \log \left( 1 - \nabla \tilde{\psi} \nabla \psi \right) \sqrt{-g} d^2 x,
\]

is (essentially) the unique choice such that (i) the coupling between \( \tilde{\psi} \) and \( \rho_B \) has “natural” 3d and (limiting) 2d representations, and (ii) extremizing over the choice of \( \psi \) on the plane \( z = 0 \) for the 2d limiting action recovers Eq. [13]. \( S_\psi \) is \( O(\epsilon) \) and, in the thin-disk limit, we have the convergences \( |\nabla \tilde{\psi}| \to 1 \) off the galactic plane, and, with \( \Sigma^* = 4\rho_0 k^{-1} - \alpha_0/G \).

Ignoring bulk bending and extremizing \( S_\psi + S_\rho \) yields Eq. [14].

Eq. [14] determines the phase \( \psi \) on the galactic plane \( z = 0 \) through \( |\nabla \tilde{\psi}| = \cos(\theta(s)) \). Off the plane, \( \psi \) satisfies the Eikonal equation \( |\nabla \tilde{\psi}| = 1 \) and is given by Huygens’ principle:

\[
\psi(r, z) = \min_{s \geq 0} \left[ \psi(s, 0) + \sqrt{(r - s)^2 + z^2} \right] \Rightarrow \psi[s + t \cos \theta(s), \pm t \sin \theta(s)] = \psi(s, 0) + t.
\]

where the second line follows for regions with non-intersecting characteristics \((s + t \cos \theta(s)) \pm t \sin \theta(s)\). To ensure that characteristics do not cross and cover all of space, we henceforth assume that \( \theta(0) = \pi/2 \), \( \theta'(s) \leq 0 \). Computing the bending energy of the phase \( \psi \), we get the Pattern DM distribution

\[
e(\Delta \tilde{\psi})^2 = 4\epsilon(t - \sin \theta(s)/\theta'(s))^{-2}.
\]

Eqs. [14], [16] and [17] give a prescription for computing the azimuthal velocities in rotation supported (cold) disks. We illustrate the method for (cored) exponential disks. The phase contours have constant separation, so their centers of curvature must lie on a common evolute. Conversely, the contours are the involutes of a master curve. This is illustrated in Fig. [4] for the curve \( z = -\frac{A_0}{r} \exp(-r/t) \), corresponding to a surface density

\[
\frac{\Sigma(s)}{\Sigma^*} = \left\{ \begin{array}{ll}
1 + A^{-2} e^2/|l|^2 & , s \leq l_0, \\
1 + A^{-2} e^2/|l|^2 & , s \geq l_0.
\end{array} \right.
\]

The disks in [18] are characterized by a scale length \( z_0 = A_0/e \) and a dimensionless parameter \( A \). Fig. [5] shows the numerically obtained rotation curves with \( A = 1 \), (representing a LSB galaxy) and \( A = 10 \) (HSB) with the choice \( \Sigma^* = a_0/G \).
We will present the details of the computations elsewhere. Here we only note that we obtain the entire curve \( v(r) \) and that the curves are in remarkable agreement with observed rotation curves (see Fig. 1) and inferred disk/halo decompositions [21].

In Fig. 5 \( v_{\text{disk}} \) is the Keplerian velocity for the baryonic distribution in (18) and \( v_{\text{halo}} \) is the contribution to the rotation curve from the pattern field \( \psi \). The rotation curve is \( v_{\text{total}} = (v_{\text{disk}} + v_{\text{halo}})^{1/2} \) (cf. Eq. (2)). After non-dimensionalizing using \((r_0, v_{\text{Kep}})\) corresponding to the peak of \( v_{\text{disk}}(r) \), we see that (i) the baryonic contributions to the rotation curve are indistinguishable and any differences are due to the phase \( \psi \), (ii) the HSB galaxy is consistent with a maximum disk decomposition, and \( v_\infty \approx v_{\text{Kep}} \), and (iii) for the LSB galaxy \( v(r) \) continues to rise beyond \( r_0 \), and has \( v_{\text{halo}} \lesssim v_{\text{disk}} \) all the way down to \( r = 0 \), inconsistent with a maximum disk decomposition [21]. Also, \( v_\infty > v_{\text{Kep}} \), so LSB galaxies are dominated by the energy in \( \psi \).

Discussion. – We have proposed a novel theory, derived from a non-relativistic action principle, that combines ideas from pattern formation along with observed astrophysical regularities—the RAR and the existence of the Freeman limit, to explain and predict dynamical features of spiral galaxies. Our model is very parsimonious and by varying a single parameter, it naturally describes HSB as well as LSB rotation curves, and the corresponding “DM halos” in their fine details.

Ours is an effective, long wave theory, applicable on scales \( \gtrsim k_0^{-1} \), rather than a fundamental theory, since the parameters \( \rho_0 \) and \( k_0 \) in [1] are not universal, but explicitly depend on the baryonic mass \( M_B \) of the host galaxy. This is to be expected. Indeed, the action \( S_2 \) in [9] describes a (purely) metric theory since it only depends on \( \lambda \) and \( \phi \) and is already extremized over the choice of \( \phi \). Any metric theory consistent with flattening rotation curves and BTFR has to be nonlocal [30]. Our theory is “minimally” nonlocal through the dependence of its action on \( \epsilon \) and thus on the global quantity \( M_B \).

We have introduced an additional “dark field” \( \psi \) that plays the role of DM. In our theory, no structures are formed on scales smaller than \( k_0^{-1} \). The resulting “DM halo” is therefore naturally cored in contrast to the cuspy halos formed by CDM.

Our theory is based on “universal” equations and can thus describe a variety of physical mechanisms. We think of \( \psi \) as the order parameter for a broken translational symmetry, with a characteristic scale \( k_0^{-1} \) given by the clumping of baryons, but this is just one possible interpretation. Indeed, at the heart of our theory is the contention that (exponential) disk galaxies are self organized systems whose state is (very well) described by two independent parameters. We have identified the important feedback mechanisms that lead to this self-organization. They are the evolution of a differentially rotating disk to marginal stability, as given by Eq. (4), and a direct coupling between the effective dark matter and the baryonic matter in the disk, as given by Eq. (15). Any fundamental theory with these features can potentially recover our results, and conversely, we expect that these features should be present in any fundamental theory of dark matter and the structure of (cold) disk galaxies.

Our theory uncovers a surprising “critical scaling” for disk galaxies. In our theory, as in MOND, the state of a disk galaxy, and the associated “DM halo”, is determined by two parameters \( M_B \) and \( z_0 \). In MOND, \( M_B \) and \( z_0 \) can be arbitrary, but in our theory it is necessary that \( M_B \leq 2\pi \Sigma \xi_{z_0}^2 \). The disks in [18] converge, in the scaling limit \( A \to 0, l_0 \to \infty, Al_0 \to e z_0 \) to a one-parameter family of critical Kuzmin disks with mass \( M_B = 2\pi \Sigma \xi_{z_0}^2 \). This “loss” of a degree of freedom is naturally associated with a symmetry, that is broken for \( A > 0 \).

A subclass of transformations, \( A \to 0, l_0 \to \lambda l_0 \), corresponds to convergence to a “scale-invariant Kuzmin disk” (i.e. point mass) since \( z_0 \to 0 \). This set of transformations corresponds to the regime of vanishing mass \( M_B = 2\pi \Sigma \xi_{z_0}^2 \to 0 \), with a space-time scale invariance \( x \to \lambda x \). Invariance under this subclass of transformations characterizes the deep MOND limit [31], so our theory includes the deep MOND regime as a limiting case.

Our full theory has a rich structure including an entire 1-parameter set of critical states with \( M_B > 0 \). Its critical scaling behavior suggests there should be a renormalization group
Galaxies, dark matter and patterns

(RG) flow for the coupled baryon/pattern system. Interestingly, we know the fixed points for the RG flow [16], the critical Kuzmin disks, although the flow itself remains to be determined. Linearizing the flow at its fixed points will yield valuable information including the number of independent state variables, and the appropriate shape of the “master curve” (see Fig. [3]), which was picked somewhat arbitrarily.

Even without the full RG flow, we can deduce that the light curve and the rotation curve are in the scaling forms \( L \propto r_0 \mu (r, r_0, A) \) and \( v = \sqrt{\alpha_0 f(r, r_0, A)} \) where the functions \( \mu, f \) are “universal”. Fig. [5] displays the curves \( f(\xi, A) \) for \( A = 1, 10 \) under the assumption that Eq. (18) gives a reasonable approximation to the true density profile. This argument indicates how we should collapse observational data for \( L(r) \) and \( v(r) \) from various galaxies. The fitting problem for \( L \) and \( f \) is overdetermined, and without the various bulge-disk-halo degeneracies that plague the standard approach of inferring a DM halo [8][21]. The ability to fit observations will thus serve as a good test of our theory. This approach will allow us to infer \( r_0 \) and \( A \) and hence also \( M_B \) and extra-galactic distances using the entire rotation + light curves and thus improve distance measurements that rely on the (B)TFR.

Our theory does predict novel effects. If the jump in \( \nabla \psi \) at the galactic plane is too sharp, the pattern is unstable to the generation of codimension 2 defects called disclinations [15][32]. These defects take the form of spirals, for a traveling wave of disclinations in 3-space. Spiral disclinations create a periodic forcing in the gravitational potential which can excite and sustain density waves in the baryonic matter that will be correlated with the disclinations in the underlying pattern. This is a fully nonlinear, far from threshold phenomenon and might be relevant to understanding the surprising nonlinearity of spiral arms over many rotation periods [24][33].

We do not expect that the pattern Lagrangian [1] with a single field \( \psi \) and two constants \( \rho_0 \) and \( k_0 \) can describe interacting galaxies or clusters. For starters, Eq. (4) implies that \( k_0 \) should vary in space. We also need more order parameters, since clusters have an additional broken symmetry – the orientations of the galactic planes (angular momenta). Defects in the orientation order parameter will give additional energies that might help account for the additional DM (over the DM in individual galaxies) that clusters seem to contain. Finally, galactic mergers are unsteady phenomena that can generate new defects and are potentially accompanied by large disruptions of existing quasi-steady patterns/changes in \( \rho_0 \) and \( k_0 \) on a short time scale. Further work is therefore needed to understand the physics of interacting galaxies and clusters.

More generally, these ideas will inform the bottom up theory of large scale structure formation. An initial “clumpy” distribution of mass density \( \rho \) and specific angular momentum \( j \) will set up order parameter fields \( \psi_i \) with defects. \( \rho, j \) and \( \alpha \) will evolve jointly towards a universal spectrum governed by the accretion of mass into larger and larger clumps, with a simultaneous coarsening of the defect field, a process driven by nonlinear interactions in a manner discussed by Press and Schechter [34]. We are working on turning this physical picture into concrete predictions for the large scale structure of the universe.

SCV is supported by the Simons Foundation through awards 524875 and 560103. We are grateful to Raman Venkataramani and Peter Behroozi for many stimulating discussions. Portions of this work were carried out when SCV was visiting the University of Bonn and RWTH Aachen.

REFERENCES

[1] TRIMBLE V., Annu. Rev. Astron. Astrophys., 25 (1987) 425.
[2] OORT J. H., Bull. Astr. Inst. Neth., 6 (1932) 249.
[3] ZWICKY F., Helvetica Physica Acta, 6 (1933) 110.
[4] RUBIN V. C., THONNARD N. and FORD, Jr. W. K., Astrophys. J. Lett., 225 (1978) L107.
[5] VAN ALBADA T. S. et al., Astrophys. J., 295 (1985) 305.
[6] RUBIN V. C., FORD, Jr. W. K. and THONNARD N., Astrophys. J., 238 (1980) 471.
[7] MILGROM M., Astrophys. J., 270 (1983) 365.
[8] FAMAey B. and McGAUgh S. S., Living Reviews in Relativity, 15 (2012) 10.
[9] McGAUgh S. S. et al., Astrophys. J., 533 (2000) L99.
[10] LELLI F. et al., Astrophys. J., 836 (2017) 152.
[11] FAMAey B. and McGAUgh S., Journal of Physics: Conference Series, 437 (2013) 012001.
[12] NAVARRO J. F. et al., Mon. Not. Roy. Astron. Soc., 471 (2017) 1841.
[13] Doddelson S., Int. J. Mod. Phys. D, 20 (2011) 2749.
[14] ASCHWANDEn M. J. et al., Space Science Reviews, 214 (2018) 55.
[15] NEWELL A. C., PASSOT T., Bowman C., Ercolani N. and IndRk R., Physica D: Nonlinear Phenomena, 97 (1996) 185.
[16] KADANOFF L. P., Physica A: Statistical Mechanics and its Applications, 163 (1990) 1.
[17] NEWELL A. C. and VenKATARAmAni S. C., Studies in Applied Mathematics, 139 (2017) 322.
[18] NEWELL A. C. and VenKATARAmAni S. C., Comptes Rendus MéganeQue, 347 (2019) 318.
[19] Freeman K. C., Astrophys. J., 160 (1970) 811.
[20] McGAUgh S. S., BOTHn G. D. and SCHOMBjERT M., Astron. J., 110 (1995) 573.
[21] de BLOK W. J. G. and McGAUgh S. S., Mon. Not. Roy. Astron. Soc., 290 (1997) 533.
[22] MISNER C. W., THORNE K. S. and WHEELER J. A., Gravitation (W.H. Freeman and Co., San Francisco) 1973.
[23] PASSOT T. and NEWELL A. C., Physica D: Nonlinear Phenomena, 74 (1994) 301.
[24] BERTIN G., Dynamics of Galaxies (Cambridge University Press) 2014.
[25] SOFUE Y., Pub. Astron. Soc. Japan, 64 (2012) 75.
[26] GEBBARDT K. et al., Astrophys. J. Lett., 539 (2000) L13.
[27] Bekenstein J. and MILGROM M., Astrophys. J., 286 (1984) 7.
[28] BRADA R. and MILGROM M., Mon. Not. Roy. Astron. Soc., 276 (1995) 453.
[29] FATHI K., Astrophys. J. Lett., 722 (2010) L120.
[30] DEFAYEY C., ESPOSITO-FARES G. and WOODARD R. P., Phys. Rev. D, 84 (2011) 124054.
[31] MilGrom M., Astrophys. J., 698 (2009) 1630.
[32] Ercolani N. M. and VenKATARAmAni S. C., J. Nonlinear Sci., 19 (2009) 267.
[33] Toomre A., Annu. Rev. Astron. Astrophys., 15 (1977) 437.
[34] PRESS W. H. and Schechter P., Astrophys. J., 187 (1974) 425.