Hamiltonian formulation of the effective kinetic theory for superfluid Fermi liquids

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Abstract

We present in a local form the time dependent effective description of a superfluid Fermi liquid which includes Landau damping effects at $T \neq 0$. This is achieved by the introduction of an additional variable, the quasiparticle distribution function, which obeys a simple kinetic equation. The transport equation is coupled with first order equations for the Goldstone mode and the particle density. We prove that a main feature of this formulation is its Hamiltonian structure relative to a certain Poisson bracket. We construct the Hamiltonian to quadratic order.

1. Introduction

The effective description at finite temperature of the broken symmetry phase of a Fermi gas in the BCS-BEC crossover is complicated by the effect of Landau damping, which results in a highly nonlocal time-dependent Ginzburg-Landau theory [1]. Apart from the fact that the derivation of such an effective Lagrangian at $T \neq 0$ for either the complex order parameter or only the Goldstone mode is rather tricky even at quadratic level [2, 3, 4, 5, 6], this nonlocal formulation makes it difficult to simulate numerically the real time evolution of non-equilibrium processes, such as oscillations of trapped Fermi gases.

Indeed, a similar situation arises in dealing with collective effects and dynamical screening of Abelian and non-Abelian plasmas at high temperature or high density. To leading order in the coupling constant, consistency
requires the inclusion of a set of one-loop diagrams termed “hard thermal loops”, which are derived from a nonlocal effective Lagrangian [7]. Fortunately, the equations of motion can be written in a local form by introducing auxiliary fields [8, 9]. In the Abelian case, the auxiliary field obeys a linearized Vlasov equation corresponding to the collisionless regime in the plasma. Remarkably, it turns out that the resulting equations form a Hamiltonian system with a noncanonical bracket structure [10, 11].

In this paper, we turn to the question of the derivation of the low-energy dynamics of the phase of the order parameter in the superfluid phase, in connection with the above analogy. We present a simple derivation of the linearized equations of motion in a local form. Moreover, we prove that this local formulation is Hamiltonian, and $H$ and the Poisson structure are completely identified.

2. The equations of motion

We begin by introducing the action for the system. In terms of the Nambu spinor $\Psi^\dagger = (\psi^\dagger, \psi_\downarrow)$ the action for the two-component balanced Fermi system is written as

$$S = \int dx \int dt \Psi^\dagger(X) \left[ i \partial_t + \tau^3 \frac{1}{2m} \nabla^2 + \tau^3 \mu - \tau^3 V_{\text{ext}}(X) + \tau^+ \Delta(X) + \tau^- \Delta^*(X) \right] \Psi(X) - \int dx \int dt \frac{1}{g_\Lambda} \Delta^* \Delta,$$

(1)

where $\tau^\pm = \frac{1}{2} (\tau^1 \pm i \tau^2)$, and the $\tau^j$'s are Pauli matrices. Here $\mu$ is the chemical potential, $g_\Lambda$ is the bare coupling parameter, and we have included an arbitrary trapping potential $V_{\text{ext}}(X)$. The complex field $\Delta(X)$ performs the Hubbard-Stratonovich decoupling of the quartic interaction between fermions in the BCS channel.

In order to derive an effective theory when the symmetry $U(1)$ is broken to $Z_2$, it is convenient to express the above Lagrangian in terms of a Goldstone field $\theta(X)$ and “heavy” fields, $\tilde{\Psi}(X)$ and $\tilde{\Delta}(X)$ to be integrated out

$$\Psi(X) = e^{i \tau^3 \theta(X)} \tilde{\Psi}(X),$$

(2)

$$\Delta(X) = e^{2 i \theta(X)} \tilde{\Delta}(X).$$

(3)

Let us choose the non-zero expectation value of the scalar field as real, $\langle \Delta \rangle = \frac{\mu}{\sqrt{g_\Lambda}}$. 

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The condition that \( \Delta(X) \) does not contain the Goldstone mode \( \tilde{\Delta} \) turns out to be \( \text{Im} \left( \Delta(X)^* \Delta_0 \right) = 0 \), so the heavy field \( \tilde{\Delta}(X) \) needs to be real with arbitrary sign. In terms of the covariant derivative
\[
D_\mu \tilde{\Psi}(X) = \partial_\mu \tilde{\Psi}(X) + i \partial_\mu \theta(X) \tau^3 \tilde{\Psi}(X),
\]
the Lagrangian becomes
\[
\mathcal{L} = \tilde{\Psi} i D_t \tilde{\Psi} - \frac{1}{2m} \left( D_i \tilde{\Psi} \right)^\dagger \tau^3 D_i \tilde{\Psi} + \tilde{\Psi}^\dagger \tau^1 \tilde{\Psi} \tilde{\Delta} - \frac{1}{g_\Delta} \tilde{\Delta}^2 + \tilde{\Psi}^\dagger \tau^3 \tilde{\Psi} \left( \mu - V_{\text{ext}} \right),
\]
and the Noether current of the \( U(1) \) symmetry, which now is realized as \( \delta \tilde{\Psi} = \delta \tilde{\Delta} = 0 \) and \( \delta \theta = \epsilon \), is given by
\[
J^0 = n = \tilde{\Psi}^\dagger \tau^3 \tilde{\Psi},
\]
\[
J^k = \frac{1}{2mi} \left( \tilde{\Psi}^\dagger \partial_k \tilde{\Psi} - \partial_k \tilde{\Psi}^\dagger \tilde{\Psi} \right) + \tilde{\Psi}^\dagger \tau^3 \tilde{\Psi} \frac{\partial_k \theta}{m}.
\]
To effectively integrate out the fermionic degrees of freedom we note that the total Hamiltonian for these may be viewed as \( H = H_0 + H_{\text{ext}} \), where
\[
H_0 = \int d\mathbf{x} \tilde{\Psi}^\dagger \left( -\tau^3 \frac{\nabla^2}{2m} - \tau^3 \mu - \tau^1 \Delta_0 \right) \tilde{\Psi},
\]
and
\[
H_{\text{ext}} = \int d\mathbf{x} \left( \tilde{\Psi}^\dagger \tau^3 \tilde{\Psi} (\partial_k \theta + V_{\text{ext}}) - \tilde{\Psi}^\dagger \tau^1 \tilde{\Psi} \sigma \right.
\]
\[
\left. + \frac{1}{2mi} \left( \tilde{\Psi}^\dagger \nabla \tilde{\Psi} - \nabla \tilde{\Psi}^\dagger \tilde{\Psi} \right) \cdot \nabla \theta + \tilde{\Psi}^\dagger \tau^3 \tilde{\Psi} \frac{(\nabla \theta)^2}{2m} \right).
\]
Therefore \( H_{\text{ext}} \) couples the system to an applied perturbation given by \( \sigma(X) \equiv \tilde{\Delta}(X) - \Delta_0 \), the gradients of \( \theta \), and \( V_{\text{ext}} \). Following the procedure reviewed in Ref. \cite{13} in the framework of the high temperature regime of QCD, the idea is to compute the induced changes in the expectation values of \( n, J \)

\footnote{For a detailed discussion on how to construct effective Lagrangians in the case of spontaneously broken symmetries, see Weinberg \cite{12}.}
and the pairing field $\tilde{\Psi}^\dagger \tau^1 \tilde{\Psi}$ by the external perturbations $\{\partial_\mu \theta, \sigma, V_{\text{ext}}\}$. By ignoring nonlinear corrections, we need the retarded response functions $\chi_{AB}(Q) = -i\langle[A(X), B(0)]\rangle \theta(t)$ whose Fourier transforms are denoted by $\chi_{AB}(Q)$. In the linear response approximation the induced changes take the form \[\delta \langle n(Q) \rangle = -\chi_{nn}(Q)v(Q) + \chi_{nk}(Q)iq^k \theta(Q) - \chi_{n1}(Q)\sigma(Q), \] \[\delta \langle J^k(Q) \rangle = -\chi_{kn}(Q)v(Q) + \chi_{kk}(Q)iq^k \theta(Q) - \chi_{k1}(Q)\sigma(Q), \]
\[\delta \langle \tilde{\Psi}^\dagger \tau^1 \tilde{\Psi} \rangle = -\chi_{1n}(Q)v(Q) + \chi_{kk}(Q)iq^k \theta(Q) - \chi_{11}(Q)\sigma(Q), \]
where \[v(Q) = i\omega \theta(Q) - V_{\text{ext}}(Q). \]

From a functional point of view, if $\Gamma^{(2)}[\theta, \sigma]$ is the quadratic approximation for the effective action that one obtains after integrating out the fermions, and $S[\sigma] = -g^{-1}_A \int \sigma^2$ is the bare action, then the integration of the heavy field $\sigma$ is simply gaussian. This produces an effective action for the Goldstone mode of the form $\Gamma^{(2)}[\theta, \bar{\sigma}] + S[\bar{\sigma}] - \frac{1}{2} \ln \left( -2g^{-1}_A - \chi_{11} \right)$, where $\bar{\sigma}$ is the solution to the saddle point condition $\delta(S[\sigma] + \Gamma^{(2)}[\theta, \sigma])/\delta\sigma = 0$. The last term, related to the determinant, does not depend on $\theta$, but gives rise to important corrections to the thermodynamic properties evaluated in the mean field approximation \[15, 16\]. However, for the purpose of deriving an effective description only in terms of the phase $\theta$, such a term will be ignored. Thus, in this approximation, the $\sigma$ field may be effectively integrated out by simply adjusting its value to the solution of the gap equation \[\frac{2\sigma}{g_A} = \delta \langle \tilde{\Psi}^\dagger \tau^1 \tilde{\Psi} \rangle. \]

Exploiting the fact that $n$ and $\theta$ are canonically conjugated variables, and imposing conservation of the Noether current once $\sigma$ has been eliminated, we may derive a set of equations for the time derivatives of $\delta \langle n \rangle$ and $\theta$. These equations encode the effective dynamics.

The explicit expressions of the response functions are easily computed at small frequency and momentum. By keeping the $O(\lambda^0)$ terms after the scalings $\omega \rightarrow \lambda \omega$, $q \rightarrow \lambda q$, the general form of $\chi_{AB}(Q)$ when $T \neq 0$ is the sum a regular piece independent of $Q$, and a contribution due to Landau damping which is non-analytical at $Q = 0$:

\[ \chi_{AB}(Q) = \tilde{\chi}_{AB} + 2 \int \frac{d^3k}{(2\pi)^3} F_{AB}(k) \frac{q \cdot \nabla_k E_k n'_F(E_k)}{\omega + i\eta - q \cdot \nabla_k E_k}. \]
The non-vanishing regular parts are given by

\[ \bar{\chi}_{nn} = -\int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0^2}{E_k} \tanh \frac{\beta E_k}{2}, \quad (16) \]

\[ \bar{\chi}_{1n} = \bar{\chi}_{n1} = -\int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0 \xi_k}{E_k^3} \tanh \frac{\beta E_k}{2}, \quad (17) \]

\[ \bar{\chi}_{11}^{(A)} = -\int^A \frac{d^3k}{(2\pi)^3} \frac{\xi_k^2}{E_k^3} \tanh \frac{\beta E_k}{2}, \quad (18) \]

\[ \bar{\chi}_{k l}^{JJ} = -\frac{q^k q^l}{m} \int \frac{d^3k}{(2\pi)^3} \left( \frac{\xi_k}{E_k} \tanh \frac{\beta E_k}{2} - 1 \right) = \langle n \rangle m \frac{q^k q^l}{E_k}, \quad (19) \]

with the standard notation, \( E_k = \sqrt{\xi_k^2 + \Delta_0^2}, \xi_k = k^2/2m - \mu \). The form of \( \bar{\chi}_{JJ}^{k l} \) is entirely due to the last term in Eq. (17). On the other hand, the factors in the integrand of the Landau damping contributions are given by

\[ F_{nn}(k) = -\frac{\xi_k^2}{E_k^2}, \quad (20) \]

\[ F_{1n}(k) = F_{1n}(k) = \frac{\Delta_0 \xi_k}{E_k^2}, \quad (21) \]

\[ F_{11}(k) = -\frac{\Delta_0}{E_k^2}, \quad (22) \]

\[ F_{k l}^{JJ}(k) = -\frac{k^k k^l}{m^2}, \quad (23) \]

\[ F_{Jn}(k) = F_{Jn}(k) = -\frac{\xi_k k^l}{mE_k}, \quad (24) \]

\[ F_{J1}(k) = F_{J1}(k) = \frac{\Delta_0 k^l}{mE_k}. \quad (25) \]

These last results determine the nonlocal part of the departure from the equilibrium values of the quantities of interest. For instance, the nonlocal contribution to \( \delta \langle n(X) \rangle \) in space-time is obtained by combining the above terms according to Eq. (10)

\[ \delta \langle n(Q) \rangle^{\text{nonlocal}} = -2 \int \frac{d^3k}{(2\pi)^3} \frac{\xi_k}{E_k^3} \frac{q \cdot \nabla_k E_k}{E_k} \frac{n_F(E_k)}{\omega + i\eta - q \cdot \nabla_k E_k} \delta E(Q, k), \quad (26) \]
where

\[
\delta E(Q, k) = \frac{\partial E_k}{\partial \mu} \nu(Q) + \frac{\partial E_k}{\partial \Delta_0} \sigma(Q) + \frac{i k \cdot q}{m} \theta(Q).
\]

(27)

In order to present the equations in a local form, it is natural to introduce a new variable \( w(Q; k) \), defined as

\[
\delta \langle n(Q) \rangle_{\text{nonlocal}} \equiv 2 \int \frac{d^3 k}{(2\pi)^3} \frac{\xi_k}{E_k} w(Q; k),
\]

(28)

which measures the departure of the distribution function from equilibrium when \( T \neq 0 \). From this definition and Eq. (26) it follows that

\[
(\omega - q \cdot \nabla_k E_k) w(Q; k) = -n'_F(E_k) \nabla_k E_k \cdot q \delta E(Q, k),
\]

(29)

and the equation of motion for distribution function \( w(X; k) \) takes the form of a transport equation without collision term resembling a linearized Vlasov equation

\[
(\partial_t + \nabla_k E_k \cdot \nabla_x) w(X, k) = n'_F(E_k) \nabla_k E_k \cdot \nabla_x \delta E(X, k),
\]

(30)

where \( \delta E \) is the induced change in the energy of the quasiparticle due to the applied perturbation

\[
\delta E(X, k) = -\frac{\xi_k}{E_k} \nu(X) + \frac{\Delta_0}{E_k} \sigma(X) + \frac{k \cdot \nabla \theta(X)}{m}.
\]

(31)

Thus, by combining the expressions for \( F_{AB}(k) \) with the definition of \( w \), one finds the total changes in a local form

\[
\delta \langle n(X) \rangle = -\bar{\chi}_{nn} \nu(X) - \bar{\chi}_{n1} \sigma(X) + 2 \int \frac{d^3 k}{(2\pi)^3} \frac{\xi_k}{E_k} w(X; k),
\]

(32)

\[
\delta \langle J^i(X) \rangle = \langle n \rangle m \nabla_i \theta + 2 \int \frac{d^3 k}{(2\pi)^3} \frac{k^i}{m} w(X; k),
\]

(33)

\[
\delta \langle \psi^\dagger \tau^1 \psi(X) \rangle = -\bar{\chi}^{(A)}_{11} \sigma(X) - \bar{\chi}_{n1} \nu(X) - 2 \int \frac{d^3 k}{(2\pi)^3} \frac{\Delta_0}{E_k} w(X; k).
\]

(34)

The transport equation (30) was derived long time ago by Betbeder-Matibet and Nozières [17], and more recently by Urban and Schuck [18]. In
these approaches the starting point is the Heisenberg equation of motion for a matrix distribution function, whose diagonalization leads to the above kinetic equation. As Leggett [19, 20] and Betbeder-Matibet and Nozières [17] have shown, it is possible to consider Fermi-liquid effects by adding to $\delta E(X, \mathbf{k})$ an extra term, $\sum_{\mathbf{k}' f_{\mathbf{kk}'}} w(X, \mathbf{k}')$, where $f_{\mathbf{kk}'}$ describes interactions of two elementary excitations.

To eliminate the $\sigma$ field we use the gap equation (14). The relation between the $s$-wave scattering length $a_s$ and the bare coupling constant $g_\Lambda$

$$-\frac{m}{4\pi a_s} = \frac{1}{g_\Lambda} - \int_{-\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k},$$

(35)

together with the gap equation for $\Delta_0$ at $T \neq 0$,

$$-\frac{m}{2\pi a_s} = \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{E_k} \tanh \frac{\beta E_k}{2} - \frac{1}{\epsilon_k} \right),$$

(36)

produce $\chi_{11}^{(A)} = -2g_\Lambda^{-1} - \chi_{nn}$. Combining this result with Eqs. (14) and (34) we obtain

$$\sigma(X) = \frac{\chi_{nn}}{\chi_{nn}} w(X) + 2 \int \frac{d^3k}{(2\pi)^3} \Delta_0 \frac{\Delta}{E_k} w(X; \mathbf{k}).$$

(37)

Finally, the conservation of the particle number

$$\partial_t \delta \langle n(X) \rangle + \nabla \cdot \delta \langle J(X) \rangle = 0,$$

(38)
yields the equation of motion satisfied by $v$, thus completing the set of dynamical equations for the variables $\{w(X; \mathbf{k}), \theta(X), v(X)\}$. This is found to be

$$-\chi_{nn} \left(1 + \frac{\chi_{nn}^2}{\chi_{nn}^2} \right) \frac{\partial v}{\partial t} - 2 \int \frac{d^3k}{(2\pi)^3} \left( \frac{\xi_k}{E_k} - \frac{\chi_{nn} \Delta_0}{\chi_{nn} E_k} \right) \nabla_k E_k \cdot \nabla_x w(X, \mathbf{k})$$

$$+ 2 \int \frac{d^3k}{(2\pi)^3} \left( \frac{\xi_k}{E_k} - \frac{\chi_{nn} \Delta_0}{\chi_{nn} E_k} \right) n_F(E_k) \nabla_k E_k \cdot \nabla_x \left( \frac{k \cdot \nabla \theta(X)}{m} \right)$$

$$+ \langle n \rangle \nabla^2 \theta + 2 \int \frac{d^3k}{(2\pi)^3} \frac{k \cdot \nabla_x w(X, \mathbf{k})}{m} = 0,$$

(39)

The term proportional to $v$ was omitted in [17].
where we have used Eqs. (30) and (37), as well as the vanishing of the angular integration \( \int d\Omega_k \nabla_k E_k f(k) \) for any isotropic function \( f(k) \) such as \( \xi_k / E_k \) and \( \Delta_0 / E_k \). Thus, the dynamical equations (30), (39), together with

\[
\frac{\partial \theta}{\partial t} = -v(X) - V_{\text{ext}}(X),
\]

and the equation (37) for \( \sigma(X) \) form a closed set of local equations for the effective low energy theory of the superfluid Fermi liquid at the one-loop level, when Fermi liquid effects are ignored.

3. Hamiltonian formulation

As we have pointed out before, a similar situation to the one posed by the Landau damping terms in the present context also appears when one attempts to derive a local time-dependent effective Lagrangian for the soft degrees of freedom of a gauge theory at high temperature or density. In that case, at the expense of introducing a new kind of degrees of freedom, one can reformulate the equations of motion as a system of local equations. The new variables represent the fluctuations of the charged (or coloured) particle distributions, and satisfy kinetic equations with external and induced fields due to these fluctuations \[8\]. It turns out that the complete set of dynamical equations are Hamiltonian with respect to certain Poisson brackets \[10, 11\].

It is natural to ask whether a similar formulation can be given in this case, and if so, what the Hamiltonian structure describing the low energy theory of the Fermi superfluid would be.

To address this question, it is convenient to replace the variable \( v(X) \) by the more natural \( n_1(X) \equiv \delta \langle n(X) \rangle \), and to consider \( \{ w(X; k), \theta(X), n_1(X) \} \) as the set of dynamical variables. This choice exploits the fact that the particle density and the Goldstone mode are canonically conjugated, as follows from the role of the particle number operator as the generator of the \( U(1) \) symmetry, and the inhomogeneous transformation law for \( \theta \), \( \delta \theta \propto i[N, \theta] \). Therefore the Poisson bracket of these variables may be written as

\[
\{ n_1(t, x), \theta(t, y) \} = \delta(x - y).
\]

The remainder Poisson structure may be guessed by noting that a kinetic equation of Vlasov type usually takes the Hamiltonian form

\[
\partial_t f(t, x, k) = \{ f, H \}.
\]
Here the relevant bracket is the Poisson-Vlasov bracket \[21\]

\[
\{ F[f], G[f] \} = \int d^3x \int d^3k \ f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\}_{xk}, \tag{42}
\]

where \(\{,\}_{xk}\) is the canonical bracket on the single particle phase space \(\mathbb{R}^6\) spanned by \((x, k)\),

\[
\{ f, g \}_{xk} \equiv \nabla_x f \cdot \nabla_k g - \nabla_x g \cdot \nabla_k f. \tag{43}
\]

The functions \(f, g\) are distribution functions on \(\mathbb{R}^6\). Since we are interested in the linearized equations around equilibrium, where \(f = n_F(E_k) + w(x, k)\), the Poisson structure will be chosen as

\[
\{ F, G \} = P \int d^3x \int d^3k n_F(E_k) \left\{ \frac{\delta F}{\delta w}, \frac{\delta G}{\delta w} \right\}_{xk} + \int d^3x \left( \frac{\delta F}{\delta n_1} \frac{\delta G}{\delta \theta} - \frac{\delta G}{\delta n_1} \frac{\delta F}{\delta \theta} \right). \tag{44}
\]

The first term corresponds to a frozen Lie-Poisson bracket \[21\] which is obtained from (42) by linearization; the value of the factor \(P\) will be determined shortly. The Jacobi identity for the frozen bracket follows from the general properties of Lie-Poisson brackets \[21\]. The second line corresponds to the canonical Poisson structure of Eq. (41).

It remains to see that the equations of motion we have found before are Hamilton’s equations

\[
\partial_t w = \{ w, H \} = P n_F'(E_k) \nabla_k E_k \cdot \nabla_x \left( \frac{\delta H}{\delta w(X, k)} \right), \tag{45}
\]

\[
\partial_t \theta = \{ \theta, H \} = -\frac{\delta H}{\delta n_1(X)}; \tag{46}
\]

\[
\partial_t n_1 = \{ n_1, H \} = \frac{\delta H}{\delta \theta(X)}. \tag{47}
\]
for some quadratic functional $H[w, \theta, n_1]$. With the aid of the relations

$$
\nu(X) = \bar{\chi}_{nn} \bar{\chi}_{2n} \bar{\chi}_{2n} - \bar{\chi}_{n1} n_1(X)
$$

we may express the change in the energy of the quasiparticle in terms of the new variables

$$
\delta E(X, k) = \bar{\chi}_{nn} \bar{\chi}_{2n} \bar{\chi}_{2n} - \bar{\chi}_{n1} n_1(X) + k \cdot \nabla \theta(X) + \delta E^{(1)}(X, k),
$$

where $\delta E^{(1)}$ denotes the contribution linear in $w$

$$
\delta E^{(1)}(X, k) = \frac{2 \bar{\chi}_{nn} \bar{\chi}_{2n} \bar{\chi}_{2n}}{\bar{\chi}_{nn}^2 + \bar{\chi}_{n1}^2} \left[ -\frac{\xi_k}{E_k} \int \frac{d^3k'}{(2\pi)^3} \left( \frac{\xi_{k'}}{E_{k'}} - \frac{\bar{\chi}_{n1} \Delta_0}{\bar{\chi}_{nn} E_k} \right) w(X, k')
\right. + \frac{\Delta_0}{E_k} \int \frac{d^3k'}{(2\pi)^3} \left( \frac{\Delta_0}{E_{k'}} + \frac{\bar{\chi}_{n1} \xi_{k'}}{\bar{\chi}_{nn} E_{k'}} \right) w(X, k') \left. \right]
$$

By comparing Eq. (47) for $\partial t n_1$ and Eq. (33), it follows that the Hamiltonian must contain exactly the term

$$
\int d^3x \int \frac{d^3k}{(2\pi)^3} 2k \cdot \nabla \theta(X) w(X, k).
$$

On the other hand, its derivative with respect to $w$ when inserted into the RHS of Eq. (45) produces

$$
P \frac{2}{(2\pi)^3} n_F(E_k) \nabla_k E_k \cdot \nabla_x \left( \frac{k \cdot \nabla \theta(X)}{m} \right),
$$

so, in order to match the corresponding piece of $\partial_t w$ in Eq. (30), $P$ must assume the value $P = (2\pi)^3/2$. Now, since

$$
\frac{\delta}{\delta w(X, k)} \int d^3y d^3q \delta E^{(1)}(t, y, q) w(t, y, q) = 2 \delta E^{(1)}(X, k),
$$

(52)
one can check immediately by functional derivation that the Hamiltonian

\[ H[w, \theta, n_1] = \int d^3x \left( -\frac{\bar{\chi}_{nn}}{\bar{\chi}_{nn}' + \bar{\chi}_{n1}'} \frac{n_1(X)^2}{2} \right. \\
+ n_1(X)V_{\text{ext}}(X) + \frac{\langle n \rangle}{2m} (\nabla \theta(X))^2 \left. \right) \\
+ \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{2k \cdot \nabla \theta(X)}{m} w(X, k) \\
+ \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{2\bar{\chi}_{nn} \bar{\chi}_{n1}}{\bar{\chi}_{nn} + \bar{\chi}_{n1}} \left( \frac{\xi_k}{E_k} - \frac{\bar{\chi}_{n1} \Delta_0}{\bar{\chi}_{nn} E_k} \right) n_1(X)w(X, k) \\
+ \int d^3x \int \frac{d^3k}{(2\pi)^3} \left( -\frac{w(X, k)^2}{n_F'(E_k)} + \delta E^{(1)}(X, k)w(X, k) \right), \quad (53) \\
\]

and the bracket (44) yield the equation of motion for \( \{w, \theta, n_1\} \). Although the Poisson structures are decoupled, the effective Hamiltonian contains terms mixing all variables. As \( n_F'(E) \) is a monotonic function the regularity of the integrand is guaranteed.

When \( \Delta_0 \to 0 \) the coefficients \( \bar{\chi}_{nn} \) and \( \bar{\chi}_{n1} \) vanish, and the most singular terms of the Hamiltonian are grouped as

\[-\frac{\bar{\chi}_{nn}}{\bar{\chi}_{nn}' + \bar{\chi}_{n1}'} \frac{1}{2} \int d^3x \left( n_1 - 2 \int \frac{d^3k}{(2\pi)^3} \text{sign}(\xi_k)w \right)^2. \]

To keep the energy finite, the change in the particle density for vanishing \( \bar{\chi}' \)'s is restricted to \( n_1 = 2 \int \text{sign}(\xi_k)w \), according with Eq. (32). Thus the equation of motion for \( \theta \) yields \( \partial_t \theta = 0 \), and this variable in no longer time dependent. Now the Hamiltonian becomes

\[ H[w, \theta(x)] = -\int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{w(X, k)^2}{n_F'(\xi_k)} \\
+ 2 \int d^3x V_{\text{ext}}(X) \int \frac{d^3k}{(2\pi)^3} \text{sign}(\xi_k)w(X, k) \\
+ \int d^3x \frac{\langle n \rangle}{2m} (\nabla \theta(x))^2 \\
+ 2 \int d^3x \int \frac{d^3k}{(2\pi)^3} k \cdot \frac{\nabla \theta(x)}{m} w(X, k), \quad (54) \]

where we have used \( n_F'(|\xi_k|) = n_F'(\xi_k) \). The equation of motion for \( \partial_t n_1 \) obtained by variation of \( \theta \) is not an independent equation as it corresponds
exactly to the integration of $2 \int \xi_k \text{sign}(\xi_k) \partial_t w$. If the initial condition is chosen $\theta(t = 0, \mathbf{x}) = \text{constant}$, the $\nabla \theta(\mathbf{x})$-terms in the Hamiltonian may be ignored, and one recovers the Hamiltonian of a non-interacting Fermi gas in an external potential. By contrast, in the limit of zero temperature when $n'_{F}(E_K) \to 0$, the distribution function $w$ must vanish in order to keep the energy finite. Therefore one recovers a description in terms of $n_1$ and $\theta$ alone.

4. Conclusion

To conclude, we have provided the Poisson structure and an effective Hamiltonian for the low energy description of a superfluid Fermi gas in the collisionless regime. The basic dynamical variables are the quasiparticle distribution function, the Goldstone mode, and the particle number density. Apart from the limitations inherent to the linear approximation to the equations of motion we have made, the most serious limitation of this approach is that it neglects Fermi liquid effects. A more accurate treatment would require the addition of the contribution $1/2 \int d\mathbf{k} d\mathbf{k'} f_{kk'w}(X, \mathbf{k})w(X, \mathbf{k'})$ in terms of the appropriate Landau parameters to the above effective Hamiltonian.

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$^3$The term proportional to $w^2$ in Eq. (53) is similar to one used in [22] to express the variations of the energy of a plasma in equilibrium.
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