Acoustics of early universe. II. 
Lifshitz vs. gauge-invariant theories

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Abstract

Appealing to classical methods of order reduction, we reduce the Lifshitz system to a second order differential equation. We demonstrate its equivalence to well known gauge-invariant results. For a radiation dominated universe we express the metric and density corrections in their exact forms and discuss their acoustic character.

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1 Introduction

The density perturbations affect the microwave background temperature. The theory of gravitational instability describes how these inhomogeneities propagate throughout the radiational era, and foresee the temperature image they “paint” on the last scattering surface. Classical perturbation theory formulated half a century ago by Lifshitz and Khalatnikov [1, 2, 3] has nowadays been replaced by more appropriate gauge-invariant descriptions [4]–[11]. These formalisms introduce some new measures of inhomogeneity. They do not appeal to the metric tensor, so they easily avoid spurious perturbations arising from an inappropriate choice of the equal time hypersurfaces. They guarantee that the space structures they describe are real physical objects.

On the other hand, the interpretation of the microwave background temperature fluctuations [12] is based on the Sach-Wolfe effect, where the metric corrections play a key role [13]. Therefore, data obtained from COBE is mostly referred to the classical concepts of Lifshitz and Khalatnikov, and only in a minor part to gauge invariant measures, which are more precise but difficult to observe [14]. Both theories in their original formulations differ essentially. Lifshitz theory provides the two parameter family of increasing solutions for the density contrast ([3] formula (115.19)), while all the gauge-invariant approaches foresee in concert only a single growing density mode. Thus the interpretation of the microwave temperature map as the initial data for cosmic structure formation is fairly ambiguous.

In this paper we attempt to reconcile both types of theories. We appeal to simple and classical methods of order reduction of differential equations [15]. By use of these techniques we remove the pure-gauge perturbations from Lifshitz theory in the radiation dominated universe. In consequence we reduce the Lifshitz system to a second order differential equation, exactly the same as obtained earlier on the ground of gauge-invariant formalisms. Applying well known solutions, we express corrections to the metric tensor, the density contrast and the peculiar velocity in exact form. We show that in the early universe, scalar perturbations of any length-scale form acoustic waves propagating with the velocity $1/\sqrt{3}$.

2 Order reduction

Relativistic perturbations of a Friedman universe, described in synchronous coordinates [1, 2, 3] form a system of two second order differential equations with variable coefficients. In contrast, the similar Newtonian problem is expressed by only one second-order equation [16, 17, 18]. Obviously, the two additional degrees of freedom appearing in the relativistic case must correspond to pure coordinate transformations (gauge freedom) [4], and should be removed from the theory.

Removing pure gauge modes we reduce the Lifshitz equations with pressure $p = \rho/3$ to Bessel equation. The procedure is as follows: 1) we raise the equations order to fourth, in
order to separate the $\mu_n(\eta)$ and $\lambda_n(\eta)$ coefficients, and then 2) we reduce the order of each of the separated equations back by eliminating gauge degrees of freedom. The resulting equations have exact solutions in the form of Hankel functions $H_{3/2}$ and their integrals.

In the synchronous system of reference, the metric corrections $h_{\mu\nu}$ ($\mu, \nu = 1, 2, 3$) to the homogeneous and isotropic, spatially flat universe fulfill the partial differential equations \[5\] \((8\pi G = c = 1)\)

\[h^\beta\gamma_{\alpha} + 2\frac{a'}{a} h_{\alpha}^\beta + (h_{\gamma}^{\beta:\alpha} + h_{\alpha}^{\beta:\gamma} - h_{\alpha:}^{\beta} - h_{\alpha}^{\beta:\gamma}) = 0, \quad (1)\]

\[2 \left[1 + 3 \frac{dp}{d\epsilon} \right]^{-1} \left( h'' + \frac{a'}{a} \left[ 2 + 3 \frac{dp}{d\epsilon} \right] h' \right) + (h_{\gamma}^{\delta:\gamma} - h_{\gamma}^{\beta:\gamma}) = 0. \quad (2)\]

These equations are usually solved by means of the Fourier transform

\[h_{\mu\nu} = \int \mathcal{A}(\mathbf{n}) \left[ \lambda_n(\eta) \left( \frac{\delta_{\mu\nu}}{3} - \frac{n_\mu n_\nu}{n^2} \right) + \frac{1}{3} \mu_n(\eta) \delta_{\mu\nu} \right] e^{in\mathbf{x}d^3\mathbf{n}} + c.c. \quad (3)\]

The Fourier transform \[(3)\] is defined for absolute integrable functions (the case of least interest for cosmology), for nonintegrable functions in the framework of distribution theory, or can be understood as a stochastic integral if the initial conditions are given at random \[19, 20\]. When the barotropic fluid \((p/\rho = \delta p/\delta \rho = w = \text{const})\) is the matter content of the universe, the functions $\lambda_n(\eta)$ and $\mu_n(\eta)$ obey ordinary, second order equations

\[-n^2 w(\lambda_n(\eta) + \mu_n(\eta)) + 2 \frac{a'(\eta)}{a(\eta)} \lambda_n'(\eta) + \lambda_n''(\eta) = 0, \quad (4)\]

\[-n^2 w(1 + 3w)(\lambda_n(\eta) + \mu_n(\eta)) + (2 + 3w) \frac{a'(\eta)}{a(\eta)} \mu_n'(\eta) + \mu_n''(\eta) = 0, \quad (5)\]

where prime denotes differentiation with respect to the conformal time $\eta$ and $a$ is the scale factor for the background metric tensor. In order to separate the variable $\lambda_n(\eta)$, we differentiate \[(4)\] twice and eliminate terms containing $\mu_n(\eta)$ or its derivatives by help of eq. \[(3)\]. We obtain the fourth order differential equation

\[\left( n^2 w \frac{a'(\eta)}{a(\eta)} \right)^3 - 6w \left( \frac{a'(\eta)}{a(\eta)} \right)^3 + 2(-1 + 3w) \frac{a'(\eta)}{a(\eta)} \lambda_n''(\eta) + 2 \frac{a^{(3)}(\eta)}{a(\eta)} \lambda_n'(\eta) \]

\[+ \left( n^2 w + 6w \left( \frac{a'(\eta)}{a(\eta)} \right)^2 + 4 \frac{a''(\eta)}{a(\eta)} \right) \lambda_n''(\eta) + (4 + 3w) \frac{a'(\eta)}{a(\eta)} \lambda_n^{(3)}(\eta) + \lambda_n^{(4)}(\eta) = 0. \quad (6)\]

In the same way one can treat \[(4)\] to find the equation for $\mu_n(\eta)$

\[\left( n^2 w \frac{a'(\eta)}{a(\eta)} - (2 + 3w) \frac{a'(\eta)}{a(\eta)} \frac{a''(\eta)}{a(\eta)} + (2 + 3w) \frac{a^{(3)}(\eta)}{a(\eta)} \right) \mu_n(\eta) \]
\[ + \left( n^2 w + 2(2 + 3w) \frac{a''(\eta)}{a(\eta)} \right) \mu_n''(\eta) + (4 + 3w) \frac{a'(\eta)}{a(\eta)} \mu_n(3)(\eta) + \mu_n(4)(\eta) = 0. \] (7)

In the following part of this paper we restrict ourselves to a universe filled with relativistic particles, where both \( w = p_0/\rho_0 = \frac{1}{3} \) and \( M = \rho_0 a^3 \) are constants of motion, and the scale factor \( a \) is a linear function of the conformal time \( a(\eta) = \sqrt{M/3} \eta \). In the flat universe the expansion rate \( \theta(\eta) = 3a'(\eta)/a(\eta)^2 \) and the energy density \( \rho_0(\eta) \) relate to each other by

\[ \rho_0(\eta) = \frac{\theta(\eta)^2}{3}, \]

so the equations for \( \lambda_n(\eta) \) and \( \mu_n(\eta) \) take fairly legible form, both prior to

\[ -\frac{1}{3} n^2 (\lambda_n(\eta) + \mu_n(\eta)) + \frac{2}{\eta} \lambda_n'(\eta) + \lambda_n''(\eta) = 0, \] (8)

\[ \frac{2}{3} n^2 (\lambda_n(\eta) + \mu_n(\eta)) + \frac{3}{\eta} \mu_n'(\eta) + \mu_n''(\eta) = 0, \] (9)

and after separation

\[ \left( \frac{n^2}{3\eta} - \frac{2}{\eta^3} \right) \lambda_n'(\eta) + \left( \frac{n^2}{3} + \frac{2}{\eta^2} \right) \lambda_n(\eta) + \frac{5}{\eta} \lambda_n(3)(\eta) + \lambda_n(4)(\eta) = 0, \] (10)

\[ \frac{n^2}{3\eta} \mu_n'(\eta) + \frac{n^2}{3} \mu_n''(\eta) + \frac{5}{\eta} \mu_n(3)(\eta) + \mu_n(4)(\eta) = 0. \] (11)

We start with equation (10). The two well known gauge solutions \[ f_1(\eta) = 1, \] (12)

\[ f_2(\eta) = -\sqrt{M/3} \int \frac{1}{a(\eta)} d\eta = -\log(\eta). \] (13)

We expect to obtain solutions for (10) in the form \[ f_1(\eta) \]

\[ \lambda_n(\eta) = f_1(\eta) \left( \int A(\eta) d\eta \right), \] (14)

\[ A(\eta) = \frac{d}{d\eta} \left( \frac{f_2(\eta)}{f_1(\eta)} \right) \left( \int \frac{B(\eta)}{\eta} d\eta \right). \] (15)

where \( A(\eta) \) and \( B(\eta) \) are some auxiliary functions. Inserting (14, 15) into (10) we obtain the Bessel equation in its canonical form

\[ \left( \frac{n^2}{3} - \frac{2}{\eta^2} \right) B(\eta) + B''(\eta) = 0. \] (16)

Equation (16) is already free of gauge modes, as one can see from simple heuristic considerations. Let us assume that there exist a third linearly independent solution of equation (1), which
corresponds to a pure coordinate transformation. Then, the linear space of gauge modes would be 3-dimensional, leaving only a single degree of freedom for the real, physical perturbations. Such a theory has no proper Newtonian limit.

Equation (16) is identical to the Sakai equation ([21] formula 5.1), the equation for density perturbations in orthogonal gauge ([5] formula (4.9), [8] formulae (16–17)), the equation for gauge invariant density gradients ([9] formula (38)) or Laplacians ([4] formulae (8–9), [11] formula (22)) after transforming these equations to their canonical form (see [22]). It is interesting to note that equation (16) is also identical to the propagation equation for gravitational waves [23] (except for gravitational waves moving with the speed of light). This means that the solutions to equation (16) represent waves travelling with the phase velocity \( \frac{1}{\sqrt{3}} \) (we show this explicitly in the next section). This picture also is consistent\(^1\) with the phonon approach [25], as the transformation \( \phi(\eta) = B(\eta)/\eta + B'(\eta) \) to the Field-Shepley variable [26, 27] reduces (16) to the harmonic oscillator \( \phi''(\eta) + \frac{n^2}{3} \phi(\eta) = 0 \).

### 3 Solutions

The general solution for (16) is a combination of

\[
B(\eta) = e^{-i\omega \eta} \left( 1 + \frac{1}{i\omega \eta} \right)
\]  

and its complex conjugate, with the frequency \( \omega = \frac{n}{\sqrt{3}} \). These solutions are proportional to Hankel functions \( H_{3/2} \), but more frequently are presented as a combination of Bessel and Neumann functions \( b = a_1 J + a_2 N \) [3]. Performing integrations (14–15) we determine the solution for \( \lambda_n(\eta) \) and find the correction \( \mu_n(\eta) \) by solving equation (4) algebraically

\[
\lambda(\omega \eta) = -\frac{e^{-i\omega \eta}}{i\omega \eta} - Ei(-i\omega \eta),
\]

\[
\mu(\omega \eta) = \left( 1 + \frac{1}{i\omega \eta} \right) \frac{e^{-i\omega \eta}}{i\omega \eta} + Ei(-i\omega \eta).
\]

Obviously, equation (11) is automatically fulfilled. As a result we obtain the metric corrections \( h_{\mu\nu} \) expanded into planar waves with the frequency constant in conformal time \( \eta \) and with varying amplitude

\[
h_{\mu\nu} = - \int A(\mathbf{n}) \left( \delta_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \right) \left( \frac{e^{i(n \cdot x - \omega \eta)}}{i\omega \eta} + e^{i\mathbf{n} \cdot \mathbf{x} Ei(-i\omega \eta)} \right) d^3 \mathbf{n}
\]

\(^1\)The procedure we present here may also be treated as a method to reconstruct metric corrections and hydrodynamic quantities in their explicit form, out of the Field and Shepley variables.

\(^2\)For similar solutions in the gravitational waves theory see [24].
\[ + \int \mathcal{A}(n) \frac{\delta \mu}{3} \left( \left( 1 + \frac{1}{i \omega \eta} \right) \frac{e^{i(n \cdot x - \omega \eta)}}{i \omega \eta} + e^{i n \cdot x} \text{Ei}(-i \omega \eta) \right) d^3 n + c.c. \]

The density perturbation and peculiar velocity can be inferred from formulae (8.2-8.3) of [2] and expressed as

\[ \frac{\delta \rho}{\rho} = \int \mathcal{A}(n) u_\rho(n \cdot x, \omega \eta) d^3 n + c.c. \]  
\[ \delta v = \int \mathcal{A}(n) u_v(n \cdot x, \omega \eta) d^3 n + c.c. \]

where the Fourier modes form travelling waves

\[ u_\rho(n \cdot x, \omega \eta) = \frac{2}{3} \left( 1 + \frac{1}{i \omega \eta} + \frac{i \omega \eta}{2} \right) \frac{e^{i(n \cdot x - \omega \eta)}}{i \omega \eta}, \]  
\[ u_v(n \cdot x, \omega \eta) = \frac{1}{2 \sqrt{3}} \left( 1 + \frac{i \omega \eta}{2} \right) \frac{e^{i(n \cdot x - \omega \eta)}}{i \omega \eta}. \]

A generic scalar perturbation in the early universe is a superposition of acoustic waves. Its amplitude decreases to reach a constant and positive value at late times. This decrease is substantial in the low frequency (early times) limit \( \omega \eta \ll 1 \). Solutions are formally divergent at \( \eta = 0 \), nevertheless evaluating the cosmic structure backward in time beyond its stochastic initiation \( \eta \) has no well defined physical sense.

The only perturbations, which are regular at \( \eta = 0 \), and growing near the initial singularity, consist of standing waves \( u(n \cdot x, \omega \eta) + u(-n \cdot x, \omega \eta) \) (compare [21, 28, 30] or similar effect in the gravitational waves theory [23]). They form a one-parameter family in the 2-parameter space of all solutions, so they are non-generic. This property has been confirmed by use of other techniques in the gauge-invariant theories [29]. In the stochastic approach nongeneric solutions are of marginal interest since they contribute with the zero probability measure.

4 Summary and conclusions

It is a matter of dispute whether cosmic structure was created solely by gravity forces [1] or initiated by other, non-gravitational phenomena manifesting themselves as stochastic processes [19, 20] in some early epochs. For the first hypothesis regular and growing solutions are indispensible, while in the second one the generic perturbations play a key role. In a radiation dominated universe these properties exclude each other.

Lifshitz theory and the gauge-invariant theories differ less than usually expected. Both types of theories, when properly written, lead to the same perturbation equation of the wave-equation form. Generic scalar perturbations are superpositions of acoustic waves. Solutions depend on
the product $n\eta$ (equivalently on $\omega\eta$). Everything which concerns early epochs refers also to long waves, and vice versa\(^3\). The perturbation scale does not divide solutions into different classes. Perturbations propagate with the same speed $1/\sqrt{3}$, which does not depend on the wave vector. This confirms the wave nature of scalar perturbations in the radiation dominated universe (an important property already pointed out by Lukash \[24\], but hardly discussed elsewhere) and compels one to use the complete metric corrections \(\text{(20)}\) in the Sachs-Wolfe procedure (not only the non-generic growing solutions) at the end of radiational era.

The reduction technique we apply in this paper can be used for other equations of state. For $p/\rho = \text{const} \neq 1/3$ solutions can be expressed in terms of hypergeometric functions. In other cases solutions may not reduce to any known elementary or special functions, although the reduced equation \((\text{16)}\) can be always found.

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### Appendix: Lifshitz “synchronous” gauge

The original Lifshitz approach \[1\], \[2\] \[3\] provides solutions which are different from \((\text{18–19)}\), and also inconsistent with the gauge-invariant theories. To explain these differences in detail, we appeal to the complete solution to \((\text{10–11)}\) containing both physical and spurious inhomogeneities. All the gauge freedom within synchronous system is limited to the choice of the integral constants in \((\text{15)}\). Actually each of these “constants” can be defined as an arbitrary function of the wave number $n$ (equivalently $\omega$). We write them explicitly as $A(n)$ and $G(n)$ satisfying

\[\lambda(\omega\eta) = f_1(\eta) \left( A(n) \int A(\eta) \, d\eta - G(n) \log(i\omega) \right) \quad (25)\]

\[A(\eta) = \frac{d}{d\eta} \left( \frac{f_2(\eta)}{f_1(\eta)} \right) \left( \int B(\eta) \frac{d\eta}{\eta} + \frac{G(n)}{A(n)} \right). \quad (26)\]

so they are equal to the Fourier coefficients in the integral

\[h_{\mu\nu} = \int A(n) \left( \frac{n_\mu n_\nu}{n^2} - \frac{\delta_{\mu\nu}}{3} \right) \left( \frac{e^{i(n\cdot x - \omega\eta)}}{i\omega\eta} + e^{i n \cdot x} \text{Ei}(-i\omega\eta) \right) \, d^3n\]

\(^3\)This is a peculiar property of the spatially flat radiation-filled universe.
\[
\begin{align*}
&+ \int \mathcal{A}(n) \frac{\delta_{\mu\nu}}{3} \left[ \left(1 + \frac{1}{i\omega \eta}\right) e^{i(n \cdot x - \omega \eta)} + e^{i n \cdot x} \text{Ei}(-i\omega \eta) \right] d^3 n \\
&+ \int \mathcal{G}(n) \left[ \left( \frac{n_\mu n_\nu}{n^2} - \frac{\delta_{\mu\nu}}{3} \right) \log(i\omega \eta) + \frac{\delta_{\mu\nu}}{3} \left( \log(i\omega \eta) - \frac{1}{\omega^2 \eta^2} \right) \right] e^{i n \cdot x} d^3 n + \text{c.c.} 
\end{align*}
\] (27)

Each coefficient \( \mathcal{A}(n) \), \( \mathcal{G}(n) \), can be defined independently. The gauge freedom is carried by \( \mathcal{G}(n) \) what follows directly from (13). Also knowing the gauge invariant methods one can \textit{a posteriori} check that \( \mathcal{A}(n) \) affects the gauge invariant inhomogeneity measures, while \( \mathcal{G}(n) \) does not. Now, the density contrast and the peculiar velocity, as inferred from formulae (8.2-8.3) of [2]

\[
\begin{align*}
\delta \rho &= \int \left[ \mathcal{A}(n) u_\rho(n \cdot x, \omega \eta) + \mathcal{G}(n) \tilde{u}_\rho(n \cdot x, \omega \eta) \right] d^3 n + \text{c.c.} \\
\delta v &= \int \left[ \mathcal{A}(n) u_v(n \cdot x, \omega \eta) + \mathcal{G}(n) \tilde{u}_v(n \cdot x, \omega \eta) \right] d^3 n + \text{c.c.} 
\end{align*}
\] (28) (29)

consists of the physical modes \( u_\rho \), \( u_v \) already found in (23-24) and the pure gauge modes equal to

\[
\begin{align*}
\tilde{u}_\rho(n \cdot x, \omega \eta) &= \frac{2}{3} \frac{1}{\eta^2 \omega^2} e^{i n \cdot x} \\
\tilde{u}_v(n \cdot x, \omega \eta) &= \frac{i}{2 \sqrt{3} \omega \eta} e^{i n \cdot x} 
\end{align*}
\] (30) (31)

We expand integrals (28) and (29) in the early times limit (with the accuracy to \( \eta^2 \)), to obtain

\[
\begin{align*}
\frac{\delta \rho}{\rho} &= \int \left[ \frac{2 \mathcal{A}(n) + \mathcal{G}(n)}{3 \omega^2 \eta^2} + \left( \frac{1}{9} i \omega \eta + \frac{1}{12} \omega^2 \eta^2 \right) \mathcal{A}(n) \right] e^{i n \cdot x} d^3 n + \text{c.c.} \\
\delta v &= \frac{1}{2 \sqrt{3}} \int \left[ \frac{\mathcal{A}(n) + \mathcal{G}(n)}{i \omega \eta} + \frac{1}{2} \mathcal{A}(n) \left( 1 + \frac{\omega^2 \eta^2}{6} \right) \right] e^{i n \cdot x} d^3 n + \text{c.c.} 
\end{align*}
\] (32) (33)

Both physical and gauge perturbations manifest identical singular behaviour at \( \eta = 0 \). Therefore, one cannot distinguish between them solely on the grounds of their asymptotic forms. On the other hand, one is able to regularize perturbations by the gauge choice \( \mathcal{G}(n) = -\mathcal{A}(n) \). Then, the equal time hypersurfaces follow the hypersurfaces of equal density at early epochs. This gauge\footnote{commonly known as the synchronous gauge} has been actually employed by Lifshitz and Khalatnikov \cite{1,2}, where divergent terms \( 1/\omega^2 \eta^2 \) are cancelled by the exactly opposite pure-gauge corrections\footnote{This does not refer to the metric correction where \( 1/\eta \)-divergence is still present.}. In consequence, perturbations described there form a mixture of both the physical and the gauge modes.
In the Lifshitz gauge, the mode amplitude \( u_{\rho}(n \cdot x, \omega \eta) u_{\rho}(n \cdot x, \omega \eta) \)^{1/2} grow with time, therefore, the two independent solutions for the density contrast increase. The same concerns the peculiar velocity. In the low \( \omega \eta \) limit the density contrast and peculiar velocity form the two-parameter linear spaces of growing solutions. As a consequence, a generic inhomogeneity increases, which is in conflict with the gauge-invariant theories \[4, 5, 9\].

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