FRACTIONAL SOBOLEV SPACES WITH VARIABLE EXPONENTS AND FRACTIONAL $P(X)$-LAPLACIANS

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Abstract. In this article we extend the Sobolev spaces with variable exponents to include the fractional case, and we prove a compact embedding theorem of these spaces into variable exponent Lebesgue spaces. As an application we prove the existence and uniqueness of a solution for a nonlocal problem involving the fractional $p(x)$-Laplacian.

1. Introduction

Our main goal in this paper is to extend Sobolev spaces with variable exponents to cover the fractional case.

For a smooth bounded domain $\Omega \subset \mathbb{R}^n$ we consider two variable exponents, that is, we let $q : \Omega \rightarrow (1, \infty)$ and $p : \Omega \times \Omega \rightarrow (1, \infty)$ be two continuous functions. We assume that $p$ is symmetric, $p(x, y) = p(y, x)$, and that both $p$ and $q$ are bounded away from 1 and $\infty$, that is, there exist $1 < q^- < q^+ < \infty$ and $1 < p^- < p^+ < \infty$ such that $q^- \leq q(x) \leq q^+$ for every $x \in \Omega$ and $p^- \leq p(x, y) \leq p^+$ for every $(x, y) \in \Omega \times \Omega$.

We define the Banach space $L^{q(x)}(\Omega)$ as usual,

$$L^{q(x)}(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} : \exists \lambda > 0 : \int_{\Omega} \left| f(x)^{q(x)} \right| dx < \infty \right\},$$

with its natural norm

$$\|f\|_{L^{q(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| f(x)^{q(x)} \right| dx < 1 \right\}.$$

Now for $0 < s < 1$ we introduce the variable exponent Sobolev fractional space as follows:

$$W = W^{s, p(x, y)}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} : u \in L^{q(x)}(\Omega) : \right.$$

$$\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^{p(x, y)}}{\lambda^{p(x, y)}|x - y|^{n + sp(x, y)}} < \infty, \text{ for some } \lambda > 0 \right\},$$

and we set

$$[f]^{s, p(x, y)}(\Omega) := \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^{p(x, y)}}{\lambda^{p(x, y)}|x - y|^{n + sp(x, y)}} < 1 \right\}.$$
as the variable exponent seminorm. It is easy to see that $W$ is a Banach space with the norm
\[ \|f\|_W := \|f\|_{L^{r(x)}(\Omega)} + \|f\|_{s,p(x,y)}(\Omega); \]
in fact, one just has to follow the arguments in [18] for the constant exponent case. For general theory of classical Sobolev spaces we refer the reader to [1, 4] and for the variable exponent case to [6].

Our main result is the following compact embedding theorem into variable exponent Lebesgue spaces.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $s \in (0, 1)$. Let $q(x)$, $p(x,y)$ be continuous variable exponents with $sp(x,y) < n$ for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and $q(x) > p(x,x)$ for $x \in \overline{\Omega}$. Assume that $r : \overline{\Omega} \to (1, \infty)$ is a continuous function such that
\[ p^*(x) := \frac{np(x,x)}{n - sp(x,x)} > r(x) \geq r_+ > 1, \]
for $x \in \overline{\Omega}$. Then, there exists a constant $C = C(n,s,p,q,r,\Omega)$ such that for every $f \in W$, it holds that
\[ \|f\|_{L^{r(x)}(\Omega)} \leq C\|f\|_W. \]
That is, the space $W^{s,q(x),p(x,y)}(\Omega)$ is continuously embedded in $L^{r(x)}(\Omega)$ for any $r \in (1, p^*)$. Moreover, this embedding is compact.

In addition, when one considers functions $f \in W$ that are compactly supported inside $\Omega$, it holds that
\[ \|f\|_{L^{r(x)}(\Omega)} \leq C[f]_{s,p(x,y)}(\Omega). \]

**Remark 1.2.** Observe that if $p$ is a continuous variable exponent in $\overline{\Omega}$ and we extend $p$ to $\overline{\Omega} \times \overline{\Omega}$ as $p(x,y) := \frac{p(x)+p(y)}{2}$, then $p^*(x)$ is the classical Sobolev exponent associated with $p(x)$, see [6].

**Remark 1.3.** When $q(x) \geq r(x)$ for every $x \in \overline{\Omega}$ the main inequality in the previous theorem, $\|f\|_{L^{r(x)}(\Omega)} \leq C\|f\|_W$, trivially holds. Hence our results are meaningful when $q(x) < r(x)$ for some points $x$ inside $\Omega$.

With the above theorem at hand one can readily deduce existence of solutions to some nonlocal problems. Let us consider the operator $\mathcal{L}$ given by
\begin{equation}
\mathcal{L}u(x) := p.v. \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)} - 2(u(x) - u(y))}{|x - y|^{n + sp(x,y)}} dy.
\end{equation}
This operator appears naturally associated with the space $W$. In the constant exponent case it is known as the fractional $p$-Laplacian, see [2, 3, 5, 7, 8, 9, 11, 12, 15, 16, 17] and references therein. On the other hand, we remark that (1.1) is a fractional version of the well-known $p(x)$-Laplacian, given by $\text{div}(|\nabla u|^{p(x) - 2}\nabla u)$, that is associated with the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. We refer for instance to [6, 10, 13, 14].

Let $f \in L^{p(x)}(\Omega)$, $a(x) > 1$. We look for solutions to the problem
\begin{equation}
\begin{cases}
\mathcal{L}u(x) + |u(x)|^{p(x) - 2}u(x) = f(x), & x \in \Omega, \\
u(x) = 0, & x \in \partial\Omega.
\end{cases}
\end{equation}
Associated with this problem we have the following functional (1.3)
\[ \mathcal{F}(u) := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)p(x,y)}} \, dx \, dy + \int_{\Omega} \frac{|u(x)|^{q(x)}}{q(x)} \, dx - \int_{\Omega} f(x)u(x) \, dx. \]

To take into account the boundary condition in (1.2) we consider the space \( W_0 \) that is the closure in \( W \) of compactly supported functions in \( \Omega \). In order to have a well defined trace on \( \partial \Omega \), for simplicity, we just restrict ourselves to \( sp_\gamma > 1 \), since then it is easy to see that \( W \subset W^{1,p}((\Omega) \subset W^{s-1/p-}(\partial \Omega)\), with \( \tilde{sp}_\gamma > 1 \), see [18, 1]. Concerning problem (1.2), we shall prove the following existence and uniqueness result.

**Theorem 1.4.** Let \( s \in (1/2, 1) \), and let \( q(x) \) and \( p(x,y) \) be continuous variable exponents as in Theorem 1.1 with \( sp_\gamma > 1 \). Let \( f \in L^{a(x)}(\Omega) \), with \( 1 < a_\gamma \leq a(x) \leq a_+ < +\infty \) for every \( x \in \Omega \), such that
\[ \frac{np(x,x)}{n - sp(x,x)} > \frac{a(x)}{a(x) - 1} > 1. \]
Then, there exists a unique minimizer of (1.3) in \( W_0 \) that is the unique weak solution to (1.2).

The rest of the paper is organized as follows: In Section 2 we collect previous results on fractional Sobolev embeddings; in Section 3 we prove our main result, Theorem 1.1, and finally in Section 4 we deal with the elliptic problem (1.2).

2. Preliminary results.

In this section we collect some results that will be used along this paper.

**Theorem 2.1** (Holder’s inequality). Let \( p, q, r : \Omega \to (1, \infty) \) with \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \). If \( f \in L^r(x) \) and \( g \in L^q(x) \), then \( fg \in L^p(x) \) and
\[ \|fg\|_{L^p(x)} \leq c\|f\|_{L^r(x)}\|g\|_{L^q(x)}. \]

For the constant exponent case we have a fractional Sobolev embedding theorem.

**Theorem 2.2** (Sobolev embedding, [18]). Let \( s \in (0, 1) \) and \( p \in [1, +\infty) \) such that \( sp < n \). Then, there exists a positive constant \( C = C(n, p, s) \) such that, for any measurable and compactly supported function \( f : \mathbb{R}^n \to \mathbb{R} \), we have
\[ \|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n + sp}} \, dx \, dy \right)^{1/p}, \]
where
\[ p^* = p^*(n, s) = \frac{np}{n - sp} \]
is the so-called “fractional critical exponent”.

Consequently, the space \( W^{s,p}(\mathbb{R}^n) \) is continuously embedded in \( L^q(\mathbb{R}^n) \) for any \( q \in [p, p^*] \).

Using the previous result together with an extension property, we also have an embedding theorem in a domain.
Theorem 2.3 ([18]). Let $s \in (0,1)$ and $p \in [1, +\infty)$ such that $sp < n$. Let $\Omega \subset \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n, p, s, \Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have
\[
\|f\|_{L^q(\Omega)} \leq C\|f\|_{W^{s,p}(\Omega)}
\]
for any $q \in [p, p^\ast]$; i.e., the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, p^\ast]$.

If, in addition, $\Omega$ is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1, p^\ast]$. Moreover, this embedding is compact for $q \in [1, p^\ast)$.

3. Fractional Sobolev spaces with variable exponents.

Proof of Theorem 1.1. Being $p$, $q$ and $r$ continuous, and $\Omega$ bounded, there exist two positive constants $k_1$ and $k_2$ such that
\[
q(x) - p(x, x) \geq k_1 > 0
\]
and
\[
\frac{np(x, x)}{n - sp(x, x)} - r(x) \geq k_2 > 0,
\]
for every $x \in \Omega$.

Let $t \in (0, s)$. Since $p$, $q$ and $r$ are continuous, using (3.1) and (3.2) we can find a constant $\epsilon = \epsilon(p, r, q, k_2, k_1, t)$ and a finite family of disjoint sets $B_i$ such that
\[
\Omega = \bigcup_{i=1}^N B_i \quad \text{and} \quad \text{diam}(B_i) < \epsilon,
\]
that verify that
\[
\frac{np(z, y)}{n - tp(z, y)} - r(x) \geq \frac{k_2}{2},
\]
\[
q(x) \geq p(z, y) + \frac{k_1}{2},
\]
for every $x \in B_i$ and $(z, y) \in B_i \times B_i$.

Let
\[
p_i := \inf_{(z, y) \in B_i \times B_i} (p(z, y) - \delta).
\]

From (3.3) and the continuity of the involved exponents we can choose $\delta = \delta(k_2)$, with $p_- - 1 > \delta > 0$, such that
\[
\frac{np_i}{n - tp_i} \geq \frac{k_2}{3} + r(x)
\]
for each $x \in B_i$.

It holds that
\[
(1) \quad \text{if we let } p_i^\ast = \frac{np_i}{n - tp_i}, \text{ then } p_i^\ast \geq \frac{k_2}{3} + r(x) \text{ for every } x \in B_i,
\]
\[
(2) \quad q(x) \geq p_i + \frac{k_1}{2} \text{ for every } x \in B_i.
\]

Hence we can apply Theorem 2.3 for constant exponents to obtain the existence of a constant $C = C(n, p_i, t, \epsilon, B_i)$ such that
\[
\|f\|_{L^{p_i^\ast}(B_i)} \leq C\left(\|f\|_{L^{p_i}(B_i)} + [f]_{L^{p_i^\ast}(B_i)}\right).
\]

Now we want to show that the following three statements hold.
(A) There exists a constant $c_1$ such that
\[ \sum_{i=0}^{N} \| f \|_{L^{p_i^*}(B_i)} \geq c_1 \| f \|_{L^{r(x)}(\Omega)}. \]

(B) There exists a constant $c_2$ such that
\[ c_2 \| f \|_{L^{q(x)}(\Omega)} \geq \sum_{i=0}^{N} \| f \|_{L^{p_i}(B_i)}. \]

(C) There exists a constant $c_3$ such that
\[ c_3 [f]_{s,p(x,y)}(\Omega) \geq \sum_{i=0}^{N} [f]_{t,p_i}^{t_p}(B_i). \]

These three inequalities and (3.5) imply that
\[ \| f \|_{L^{r(x)}(\Omega)} \leq C \sum_{i=0}^{N} \| f \|_{L^{p_i^*}(B_i)} \]
\[ \leq C \left( \sum_{i=0}^{N} \left( \| f \|_{L^{p_i}(B_i)} + [f]_{t,p_i}(B_i) \right) \right) \]
\[ \leq C \left( \| f \|_{L^{q(x)}(\Omega)} + [f]_{s,p(x,y)}^{s_p}(\Omega) \right) \]
\[ = C \| f \|_{W}, \]
as we wanted to show.

Let us start with (A). We have
\[ |f(x)| = \sum_{i=0}^{N} |f(x)| \chi_{B_i}. \]
Hence
\[ (3.6) \quad \| f \|_{L^{r(x)}(\Omega)} \leq \sum_{i=0}^{N} \| f \|_{L^{r(x)}(B_i)}. \]
and by item (1), for each $i$, $p_i^* > r(x)$ if $x \in B_i$. Then we take $a_i(x)$ such that
\[ \frac{1}{r(x)} = \frac{1}{p_i^*} + \frac{1}{a_i(x)}. \]

Using Theorem 2.1 we obtain
\[ \| f \|_{L^{r(x)}(B_i)} \leq c \| f \|_{L^{p_i}(B_i)} \| f \|_{L^{a_i}(B_i)} \]
\[ = C \| f \|_{L^{p_i}(B_i)}. \]

Thus, recalling (3.6) we get $|A|$. To show (B) we argue in a similar way using that $q(x) > p_i$ for $x \in B_i$.

In order to prove (C) let us set
\[ F(x,y) := \frac{|f(x) - f(y)|}{|x - y|^s}, \]
and observe that
\[
[f]^{*,p_i}(B_i) = \left( \int_{B_i} \int_{B_i} \frac{|f(x) - f(y)|^{p_i}}{|x - y|^{n + \tau p_i + s p_i - s p_i}} \, dx \, dy \right)^{\frac{1}{p_i}}
= \left( \int_{B_i} \int_{B_i} \left( \frac{|f(x) - f(y)|^{p_i}}{|x - y|^s} \right)^{p_i} \frac{dx \, dy}{|x - y|^{n + (t - s)p_i}} \right)^{\frac{1}{p_i}}
\]
(3.7)
\[
\leq C \|F\|_{L^{p(x),y}(\mu, B_i \times B_i)} 1_{L^{h(x,y)}(\mu, B_i \times B_i)}
= C \|F\|_{L^{p(x),y}(\mu, B_i \times B_i)},
\]
where we have used Theorem 2.1 with
\[
\frac{1}{p_i} = \frac{1}{p(x,y)} + \frac{1}{b_i(x, y)},
\]
but considering the measure in $B_i \times B_i$ given by
\[
d\mu(x, y) = \frac{dx \, dy}{|x - y|^{n + (t - s)p_i}}.
\]

Now our aim is to show that
\[
\|F\|_{L^{p(x),y}(\mu, B_i \times B_i)} \leq C[f]^{*,p(x,y)}(B_i)
\]
for every $i$. If this is true, then we immediately derive (C) from (3.7).

Let $\lambda > 0$ be such that
\[
\int_{B_i} \int_{B_i} \frac{|f(x) - f(y)|^{p(x,y)}}{(\lambda |x - y|^{s_t})^{p(x,y)}} \, dx \, dy < 1.
\]
Choose
\[
k := \sup \left\{ 1, \sup_{(x,y) \in \Omega \times \Omega} |x - y|^{s_t} \right\}
\]
and
\[
\lambda := \lambda k.
\]
Then
\[
\int_{B_i} \int_{B_i} \left( \frac{|f(x) - f(y)|^{p(x,y)}}{(\lambda |x - y|^{s_t})} \right)^{p(x,y)} \frac{dx \, dy}{|x - y|^{n + (t - s)p_i}}
= \int_{B_i} \int_{B_i} \frac{|x - y|^{(s_t - 1)p_i}}{k^{p(x,y)}} \frac{|f(x) - f(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + s p(x,y)}} \, dx \, dy
\]
\[
\leq \int_{B_i} \int_{B_i} \frac{|f(x) - f(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + s p(x,y)}} \, dx \, dy < 1.
\]
Therefore
\[
\|F\|_{L^{p(x),y}(\mu, B_i \times B_i)} \leq \lambda k,
\]
which implies the inequality (3.8).

On the other hand, when we consider functions that are compactly supported inside $\Omega$ we can get rid of the term $\|f\|_{L^{q(x)}(\Omega)}$ and it holds that
\[
\|f\|_{L^{q(x)}(\Omega)} \leq C[f]^{*,p(x,y)}(\Omega).
\]
Finally, we recall that the previous embedding is compact since in the constant exponent case we have that for subcritical exponents the embedding is compact.
Hence, for a bounded sequence in $W$, $f_i$, we can mimic the previous proof obtaining that for each $B_i$ we can extract a convergent subsequence in $L^{r(x)}(B_i)$. □

**Remark 3.1.** Our result is sharp in the following sense: if

$$p^*(x_0) := \frac{np(x_0, x_0)}{n - sp(x_0, x_0)} < r(x_0)$$

for some $x_0 \in \Omega$, then the embedding of $W$ in $L^{r(x)}(\Omega)$ cannot hold for every $q(x)$. In fact, from our continuity conditions on $p$ and $r$ there is a small ball $B_\delta(x_0)$ such that

$$\max_{B_\delta(x_0) \times B_\delta(x_0)} \frac{np(x, y)}{n - sp(x, y)} < \min_{B_\delta(x_0)} r(x).$$

Now, fix $q < \min_{B_\delta(x_0)} r(x)$ (note that for $q(x) \geq r(x)$ we trivially have that $W$ is embedded in $L^{r(x)}(\Omega)$). In this situation, with the same arguments that hold for the constant exponent case, one can find a sequence $f_k$ supported inside $B_\delta(x_0)$ such that $\|f_k\|_W \leq C$ and $\|f_k\|_{L^{r(x)}(B_\delta(x_0))} \to +\infty$. In fact, just consider a smooth, compactly supported function $g$ and take $f_k(x) = k^a \delta(x)$ with $a$ such that $ap(x, y) - n + sp(x, y) \leq 0$ and $ar(x) - n > 0$ for $x, y \in B_\delta(x_0)$.

Finally, we mention that the critical case

$$p^*(x) := \frac{np(x, x)}{n - sp(x, x)} \geq r(x)$$

with equality for some $x_0 \in \Omega$ is left open.

4. Equations with the fractional $p(x)$-Laplacian.

In this section we apply our previous results to solve the following problem. Let us consider the operator $\mathcal{L}$ given by

$$\mathcal{L}u(x) := p.v. \int_{\Omega} |u(x) - u(y)|^{p(x,y) - 2} (u(x) - u(y)) \frac{dy}{|x - y|^{n + sp(x,y)}}$$

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}_n$ and $f \in L^{a(x)}(\Omega)$ with $a_+ > a(x) > a_- > 1$ for each $x \in \Omega$. We look for solutions to the problem

$$\begin{cases}
\mathcal{L}u(x) + |u(x)|^{q(x) - 2}u(x) = f(x), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega.
\end{cases}$$

(4.1)

To this end we consider the following functional

$$\mathcal{F}(u) := \int_{\Omega} \int_{\Omega} |u(x) - u(y)|^{p(x,y)} dx dy + \int_{\Omega} |u(x)|^{q(x)} dx - \int_{\Omega} f(x)u(x) dx.$$

(4.2)

Let us first state the definition of a weak solution to our problem (4.1). Note that here we are using that $p$ is symmetric, that is, we have $p(x, y) = p(y, x)$.

**Definition 4.1.** We call $u$ a weak solution to (4.1) if $u \in W_0^{s, q(x), p(x,y)}(\Omega)$ and

$$\int_{\Omega} \int_{\Omega} |u(x) - u(y)|^{p(x,y) - 2} (u(x) - u(y))(v(x) - v(y)) \frac{dy}{|x - y|^{n + sp(x,y)}} dx dy$$

$$+ \int_{\Omega} |u|^{q(x) - 2}u(x)v(x) dx = \int_{\Omega} f(x)v(x) dx,$$

(4.3)
for every \( v \in W_{0}^{s,q(x),p(x,y)}(\Omega) \).

Now our aim is to show that \( \mathcal{F} \) has a unique minimizer in \( W_{0}^{s,q(x),p(x,y)}(\Omega) \). This minimizer shall provide the unique weak solution to the problem (4.1).

**Proof of Theorem 1.4.** We just observe that we can apply the direct method of calculus of variations. Note that the functional \( \mathcal{F} \) given in (4.2) is bounded below and strictly convex (this holds since for any \( x \) and \( y \) the function \( t \mapsto t^{p(x,y)} \) is strictly convex).

From our previous results, \( W_{0}^{s,q(x),p(x,y)}(\Omega) \) is compactly embedded in \( L^{r(x)}(\Omega) \) for \( r(x) < p^{*}(x) \), see Theorem 1.1. In particular, we have that \( W_{0}^{s,q(x),p(x,y)}(\Omega) \) is compactly embedded in \( L^{s,q(x)-1}(\Omega) \).

Let us see that \( \mathcal{F} \) is coercive. We have

\[
\mathcal{F}(u) = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)p(x,y)}} \, dx \, dy + \int_{\Omega} |u(x)|^{q(x)} \, dx - \int_{\Omega} f(x)u(x) \, dx \\
\geq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)p(x,y)}} \, dx \, dy + \int_{\Omega} |u(x)|^{q(x)} \, dx \\
- \|f\|_{L^{s,q(x)}(\Omega)} \|u\|_{W^{s,q(x)}(\Omega)}^{s,q(x)-1}(\Omega) \\
\geq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)p(x,y)}} \, dx \, dy + \int_{\Omega} |u(x)|^{q(x)} \, dx - C\|u\|_{W}.
\]

Now, let us assume that \( \|u\|_{W} > 1 \). Then we have

\[
\frac{\mathcal{F}(u)}{\|u\|_{W}} \geq \frac{1}{\|u\|_{W}} \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)p(x,y)}} \, dx \, dy + \int_{\Omega} |u(x)|^{q(x)} \, dx \right) - C \\
\geq \|u\|_{W}^{\min\{p^{*}(x) - 1\} - 1} - C.
\]

We next choose a sequence \( u_{j} \) such that \( \|u_{j}\|_{W} \to \infty \) as \( j \to \infty \). Then we have

\[
\mathcal{F}(u_{j}) \geq \|u_{j}\|_{W}^{\min\{p^{*}(x) - 1\} - 1}/\|u_{j}\|_{W} \to \infty,
\]

and we conclude that \( \mathcal{F} \) is coercive. Therefore, there is a unique minimizer of \( \mathcal{F} \).

Finally, let us check that when \( u \) is a minimizer to (4.2) then it is a weak solution to (4.1). Given \( v \in W_{0}^{s,q(x),p(x,y)}(\Omega) \) we compute

\[
0 = \frac{d}{dt} \mathcal{F}(u + tv) \bigg|_{t=0} = \int_{\Omega} \int_{\Omega} \frac{d}{dt} \left( \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)p(x,y)}} \right) \, dx \, dy \bigg|_{t=0} \\
+ \int_{\Omega} \frac{d}{dt} \left( \frac{|u(x) + tv(x)|^{q(x)}}{q(x)} \right) \, dx \bigg|_{t=0} - \int_{\Omega} \frac{d}{dt} f(x)(u(x) + tv(x)) \, dx \bigg|_{t=0} \\
= \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n + sp(x,y)p(x,y)}} \, dx \, dy \\
+ \int_{\Omega} |u(x)|^{q(x)-2}u(x)v(x) \, dx - \int_{\Omega} f(x)v(x),
\]

as \( u \) is a minimizer of (4.2). Thus, we deduce that \( u \) is a weak solution to the problem (4.1).

The proof of the converse (that every weak solution is a minimizer of \( \mathcal{F} \)) is standard and we leave the details to the reader. \( \square \)
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