On negative dependence inequalities and maximal scores in round-robin tournaments

Yaakov Malinovsky *
Department of Mathematics and Statistics
University of Maryland, Baltimore County, Baltimore, MD 21250, USA

John W. Moon †
Department of Mathematical and Statistical Sciences
University of Alberta, Edmonton, AB T6G 2G1, Canada

February 7, 2022

Abstract

We extend Huber’s (1963) inequality for the joint distribution function of negative dependent scores in round-robin tournaments. As a byproduct, this extension implies convergence in probability of the maximal score in round-robin tournaments in a more general setting.

Keywords: large deviation, negative correlation, probabilistic inequalities, round-robin tournaments
MSC2020: 60E15, 05C20, 60F10

1 Introduction and Background

In a classical round-robin tournament, each of \( n \) players wins or loses against each of the other \( n - 1 \) players (Moon, 2013). Denote by \( X_{ij} \) the score of player \( i \) after the game with player \( j, j \neq i \). We assume that all \( \binom{n}{2} \) pairs of scores \((X_{12}, X_{21}), \ldots, (X_{1n}, X_{n1}), \ldots, (X_{n-1,n}, X_{n,n-1})\) are independent. Let \( s_i = \sum_{j=1,j\neq i}^{n} X_{ij} \) be the score of player \( i \) \((i = 1, \ldots, n)\) after playing with all \( n - 1 \) opponents. We use a standard notation and denote by \( s(1) \leq s(2) \leq \ldots \leq s(n) \) the order statistics of the random variables \( s_1, s_2, \ldots, s_n \); and

*email: yaakovm@umbc.edu
†email: jwmoon@ualberta.ca
we denote by $s_1^*, s_2^*, \ldots, s_n^*$ normalized scores (zero expectation and unit variance) with corresponding order statistics $s_{(1)}^*, s_{(2)}^*, \ldots, s_{(n)}^*$.

Measuring players strengths in chess tournaments by modeling paired comparisons of strength has a long history and appeared in Zermelo (1929). Zemelro’s model is given by

$$P(\text{player } i \text{ defeats player } j) = P(X_{ij} = 1) = \frac{u_i}{u_i + u_j},$$

where $u_i$ and $u_j$ are unknown strengths of players $i$ and $j$, respectively, and for $i \neq j$, $X_{ij} + X_{ji} = 1, X_{ij} \in \{0, 1\}$. Zermelo (1929) used the maximum likelihood (ML) method to estimate the parameters $\{u_i\}$. This approach was rediscovered by Bradley and Terry (1952) and Ford (1957) and is usually referred to as the Bradley-Terry model; see, e.g. David (1988). Interesting historical comments related to Zermelo’s model can be found in the article “Comments on Zermelo (1929)” of the book by David and Edwards (2001) and in Glickman (2013).

Simons and Yao (1999) estimate $\{u_i\}$ based on the data $X_{ij}, 1 \leq i < j \leq n$, and proved the consistency and asymptotic normality of the ML estimators. Chatterjee et al. (2011) investigated maximum likelihood estimates pertaining to the degree sequences of random graphs generated by what the authors refer to as a “close cousin” of the Bradley-Terry model. Further, Chetrite et al. (2017) investigated the asymptotic probability that the best player wins, assuming that the strengths of the players are random variables.

Let $r_n$ denote the probability that an ordinary round-robin tournament with $n$ labelled vertices has a unique vertex with maximum score, assuming all the $2^{n \choose 2}$ such tournaments are equally likely. Epstein (2013) (Section nine) gave the values $r_4 = .5, r_5 = .586, r_6 = .627, r_7 = .581,$ and $r_8 = .634$ (We remark that it follows from Table 1 in David (1959) that $r_8 = 160, 241, 152/2^{28} = .596\ldots$). Epstein also stated that as $n$ increases indefinitely, $r_n$ approaches unity. However, we are not aware of a proof of the final conclusion.

Material on round-robin tournaments can be found in Harary and Moser (1966), Moon (2013), and Reid (2014). Landau (1953) has given necessary and sufficient conditions for a set of integers to be the score sequence of some tournament. Landau was interested in animal behavior, and his work grew out of dealing with the pecking orders of chickens. Landau’s theorem has been reproved and generalized by a number of authors in a variety of ways; see, for example, Moon (2013) (Section 22), the survey paper Griggs and Reid (1996), and, more recently, Holshauser et al. (2011), and Brualdi and Fritscher (2015).

Huber (1963) was concerned with the asymptotic behavior of the highest score in a
paired comparison experiment when the number \( n \) of treatments (players) is very large. He assumed that the players are all of equal strength, except for a single 'outlier,' which will be preferred with probability \( p > 1/2 \) when compared with any other player. Each pair of players is compared exactly once and no ties are permitted. Huber proved that the probability that the outlier has the maximum score tends to 1 for all fixed \( p > 1/2 \) as \( n \) tends to infinity. A byproduct of Huber’s work when \( p = 1/2 \) is the following result, which he gave as a Corollary.

**Result 1** (Huber (1963)). If \( X_{ij} \in \{0,1\}, X_{ij} + X_{ji} = 1, p_{ij} = P(X_{ij} = 1) = \frac{1}{2} \), and if \( n \to \infty \), then \( s^*_n - \sqrt{2 \log(n-1)} \to 0 \) in probability.

A key step in Huber’s approach was an inequality for the joint cumulative distribution function of the negatively dependent scores \( s_1, \ldots, s_n \), where \( X_{ij} + X_{ji} = 1 \); extending this inequality permits one to estimate the maximal scores in more general tournament settings. A verbatim statement of Huber’s inequality is given below.

**Lemma 1** (Huber (1963)). For any probability matrix \( (p_{ij}) \) and any numbers \( (k_1, \ldots, k_m) \), \( m \leq n \), the joint cumulative distribution function of the scores \( s_1, \ldots, s_m \) satisfies

\[
P(s_1 < k_1, \ldots, s_m < k_m) \leq P(s_1 < k_1) \cdots P(s_m < k_m) .
\]

A similar inequality for negatively correlated normal random variables appears in Slepian (1962), for the multinomial random variables in Mallows (1968), and for other multivariate discrete distributions in Jogdeo and Patil (1975).

In particular, Jogdeo and Patil (1975) showed that if \( A \) denotes a measurable event and \( P(A|X = x) \) is a well-defined nondecreasing function of \( x \), then \( P(A, X \leq a) \leq P(A) P(X \leq a) \) for every \( a \). Their proof is based on Chebyshev’s order inequality. In order to use their result to prove (1) in the tournaments setting we would need to verify that \( P(s_{j+1} \leq k_{j+1}, \ldots, s_n < k_n|s_j = k) \) is a nondecreasing function of \( k \) for \( j = 1, \ldots, n - 1 \), and it would be a hard task.

Lehmann (1966) called two random variables **negative quadrant dependent** if they satisfy (1) (see also Nelsen (2006)). The random variables which satisfy (1) are known as **negative lower orthant dependent (NLOD)** and were investigated in Joag-Dev and Proschan (1983) and references therein. All this is closely related to the property of **negatively associated (NA)** random variables \( X_1, X_2, \ldots, X_n \), where for every pair of disjoint subsets
A_1, A_2 of \{1, 2, \ldots , n\}, Cov \{f(X_i, i \in A_1), g(X_j, j \in A_2)\} \leq 0, for all nondecreasing functions f, g. NA implies NLOD, but not vice versa (Joag-Dev and Proschan, 1983). In addition, Joag-Dev and Proschan (1983) show that negatively correlated normal random variables, which are NLOD (Slepian, 1962), also are NA.

A recent result was published in July 2021 (Ross, 2021), which is closely related to (1). Ross considered a tournament model where \(X_{ij} \sim Bin(n_{ij}, p_{ij})\). The binomial distribution is log-concave, i.e., for all \(u \geq 1\), \((p(u))^2 \geq p(u - 1)p(u + 1)\), where \(p(u) = P(X_{ij} = u)\) (Johnson and Goldschmidt, 2006). Ross (2021) (Proposition 1) used a theorem from Efron (1965) on log-concave distributions to show that in the model he was studying, \(s_{-i} \mid s_i = k\) is stochastically decreasing in \(k\), where \(s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)\); that is, \(E(\Phi(s_{i-1}) \mid s_i = k)\) is a non-increasing function of \(k\) for any a real measurable function \(\Phi(x_1, \ldots, x_{n-1})\) on Euclidean \((n-1)\)-space which is non-decreasing in each of its arguments. Ross (2021) (Corollary 2) then deduced that \(s_{-i} \mid s_i \geq k\) is stochastically smaller than \(s_{-i}\). This implies that \(I_{-i} \mid I_i = 1\) is stochastically smaller than \(I_{-i}\), where \(I_i\) is the indicator function of the event that \(s_i > k\), and \(I_{-i} = (I_1, \ldots, I_{i-1}, I_{i+1}, \ldots, I_n)\). Then, it follows from (Proposition 2, Ross (2016)) that \(P\left(\sum_{i=1}^{n} I_i = 0\right) \leq \prod_{i=1}^{n} P(I_i = 0)\), which is equivalent to \(P(s_1 \leq k, \ldots, s_n \leq k) \leq P(s_1 \leq k) \cdots P(s_n \leq k)\). Joag-Dev and Proschan (1983) (Theorem 2.8) gave another result that made use of the work of Efron (1965), namely, that if \(X_1, \ldots, X_n\) are independent random variables with log-concave densities then the joint conditional distribution of \(X_1, \ldots, X_n\) given \(\sum_{i=1}^{n} X_i\) is NA. Using this result, they show that the multinomial random variables are NA and therefore NLOD.

Malinovsky (2021a,b) considered the chess round-robin tournament model (see Example 3 below) and found an asymptotic distribution of \(s_{(i)}^*\), \(i = 1, \ldots, n\).

In this work, we extend Huber’s lemma to a large class of discrete distributions of \(X_{ij}\) and, as a byproduct, show that this extension implies convergence in probability of the normalized maximal score for generalizations of round-robin tournaments.

## 2 Main Results

Suppose that \(n (\geq 2)\) players participate in a generalized round–robin tournament and that each player is compared with each of the other \(n - 1\) players (one or more times); and that
as a result of the comparison(s) between players $i$ and $j$, these players receive $X_{ij}$ and $X_{ji}$ points, respectively, where $X_{ij}$ and $X_{ji}$ range over the integers $0, 1, \ldots, m$, for some fixed positive integer $m$, and $X_{ij} + X_{ji} = m$. We further assume that for each given ordered pair of distinct integers $(i,j)$, $1 \leq i, j \leq n$, there exist nonnegative numbers $p_0, \ldots, p_m$ such that $p_0 + \cdots + p_m = 1$ and

$$p_u = P(X_{ij} = u) = P(X_{ij} = u, X_{ji} = m - u) = P(X_{ji} = m - u)$$

for $0 \leq u \leq m$.

Notice that it follows from assumption (L) and the fact that $X_{ij} + X_{ji} = m$ that $E(X_{ij}) + E(X_{ji}) = m$ and $\text{Var}(X_{ij}) = \text{Var}(X_{ji})$.

**Theorem 1.** If the probabilities associated with a generalized tournament satisfy condition (L), then for any fixed nonnegative integers $k_1, \ldots, k_n$, the joint distribution function $F(k_1, \ldots, k_n)$ of the scores $s_1, \ldots, s_n$ satisfies the relation

$$F(k_1, \ldots, k_n) = P(s_1 \leq k_1, \ldots, s_n \leq k_n) \leq P(s_1 \leq k_1) \cdots P(s_n \leq k_n).$$

\(2\)

**Proof.** Any two particular scores $s_1$ and $s_2$, say, can be rewritten as $s_1 = s_1' + X_{12}$ and $s_2 = s_2' + X_{21}$, where $s_1' = \sum_{j \neq 1, 2} X_{1j}$ and $s_2' = \sum_{j \neq 2, 1} X_{2j}$. So the expression for $F(k_1, \ldots, k_n)$ can be rewritten as

$$F := F(k_1, \ldots, k_n) = P(s_1' + X_{12} \leq k_1, s_2' + X_{21} \leq k_2, s_3 \leq k_3, \ldots, s_n \leq k_n).$$

\(3\)

We now replace the dependent variables $X_{12}$ and $X_{21}$ by independent variables $Y_{12}$ and $Y_{21}$ such that $P(Y_{12} = u) = p_u$ and $P(Y_{21} = v) = p_{m-v}$ for $0 \leq u, v \leq m$, but we do not require that $Y_{12} + Y_{21} = m$. This gives a new joint distribution function

$$F_1 := F_1(k_1, \ldots, k_n) = P(s_1' + Y_{12} \leq k_1, s_2' + Y_{21} \leq k_2, s_3 \leq k_3, \ldots, s_n \leq k_n).$$

\(4\)

For notational convenience we shall temporarily suppress the $s_i \leq k_i$ terms for $3 \leq i \leq n$ in relations (3) and (4) and in what follows. When we subtract $F$ from $F_1$ and then sum over the possible values of $Y_{12}, Y_{21}, X_{12}$, and $X_{21} = m - X_{12}$, bearing in mind that these variables are independent of the other variables, we find that
\[ F_1 - F = P(s'_1 \leq k_1 - Y_{12}, s'_2 \leq k_2 - Y_{21}) - P(s'_1 \leq k_1 - X_{12}, s'_2 \leq k_2 - X_{21}) \]
\[ = \sum_{u=0}^{m} \sum_{v=0}^{m} P(s'_1 \leq k_1 - u, s'_2 \leq k_2 - v) \{ P(Y_{12} = u, Y_{21} = v) - P(X_{12} = u, X_{21} = v) \} \]
\[ = \sum_{u=0}^{\min(m,k_1)} \sum_{v=0}^{\min(m,k_2)} P(s'_1 \leq k_1 - u, s'_2 \leq k_2 - v)p(u,v), \]  
where
\[ p(u,v) = P(Y_{12} = u, Y_{21} = v) - P(X_{12} = u, X_{21} = v) = \begin{cases} p_u p_{m-v} & \text{if } u + v \neq m \\ p_u^2 - p_u & \text{if } u + v = m, \end{cases} \]  
and where we have appealed to assumption (L) at the last step.

We want to show that
\[ F_1 - F \geq 0. \]  

To establish this, we need to introduce some more notation and Assertions 1 and 2 (with proofs in Appendix A and B) in order to obtain a simpler form of relation (5).

We let
\[ R(g,h) = P(s'_1 = k_1 - g, s'_2 = k_2 - h) \]  
and
\[ W(g,h) = \sum_{u=0}^{\min(m,g)} \sum_{v=0}^{\min(m,h)} p(u,v) \]  
for \( 0 \leq g \leq k_1 \) and \( 0 \leq h \leq k_2 \).

**Assertion 1.** \( F_1 - F = \sum_{g=0}^{k_1} \sum_{h=0}^{k_2} R(g,h)W(g,h). \)

**Assertion 2.** If \( 0 \leq g \leq k_1 \) and \( 0 \leq h \leq k_2 \), then \( W(g,h) \geq 0. \)

Since \( R(g,h) \) and \( W(g,h) \) are each nonnegative, by definition and by Assertion 2, it follows from the relation in Assertion 1 that \( F_1 - F \geq 0 \), as required. To complete the proof of Theorem 1 we proceed as follows. Suppose the pairs \( \{(i,j) : 1 \leq i < j \leq n\} \) are lexicographically ordered and labelled from 1 to \( n(n-1)/2 \). We defined \( F_1 \) as the distribution function obtained from the distribution function \( F \) by replacing the dependent variables \( X_{12} \) and \( X_{21} \) by the independent variables \( Y_{12} \) and \( Y_{21} \); where \( Y_{12} \) and \( Y_{21} \) have
the same distribution as $X_{12}$ and $X_{21}$ except that we do not require that $Y_{12} + Y_{21} = m$. Similarly, if $1 < t \leq n(n-1)/2$ and the pair $(i, j)$ has label $t$, then $F_t$ is defined to be the distribution function obtained from the function $F_{t-1}$ by replacing the dependent variables $X_{ij}$ and $X_{ji}$ by the independent variables $Y_{ij}$ and $Y_{ji}$; where $Y_{ij}$ and $Y_{ji}$ have the same distribution as $X_{ij}$ and $X_{ji}$ except that we do not require that $Y_{ij} + Y_{ji} = m$. The conclusion that $F_{t-1} \leq F_t$ for $1 < t \leq n(n-1)/2$ follows by essentially the same type of argument as was used to show that $F \leq F_1$. When we combine these inequalities, we find that

$$F(k_1, \ldots, k_n) = F \leq F_1 \leq F_2 \leq \cdots \leq F_{n(n-1)/2}$$

$$= P\left(\sum_{j \neq 1} Y_{ij} \leq k_1, \ldots, \sum_{j \neq n} Y_{nj} \leq k_n\right) = P\left(\sum_{j \neq 1} Y_{ij} \leq k_1\right) \cdots P\left(\sum_{j \neq n} Y_{nj} \leq k_n\right)$$

$$= P\left(\sum_{j \neq 1} X_{1j} \leq k_1\right) \cdots P\left(\sum_{j \neq n} X_{nj} \leq k_n\right) = P(s_1 \leq k_1) \cdots P(s_n \leq k_n),$$

since the variables $Y_{ij}$ are independent and the variables $\sum_{j \neq i} Y_{ij}$ and $\sum_{j \neq i} X_{ij}$ have the same distribution for each $i$. This completes the proof of the Theorem 1.

Put $\sigma_i^2(n-1) = \sum_{j=1, j \neq i}^n \text{Var}(X_{ij})$ for $i = 1, \ldots, n$.

**Theorem 2.** If the probabilities associated with a generalized tournament satisfy condition (L) and, for $i = 1, \ldots, n$, $\sigma_i(n-1) \to \infty$ as $n \to \infty$, then $s_{n}^* \sqrt{\log(n-1)} \to 0$ in probability.

**Proof.** The proof differs from the arguments used by Huber (1963) to establish Result 1 in two ways. (i) Instead of using the large deviation result for Bernoulli random variables (stated on p.193 of the 3rd edition of Feller (1968)), we use Cramér-type large deviation results for independent non-identically distributed random variables, as stated below; (ii) we use our Theorem 1, a stronger form of the Lemma 1 that Huber used.

For each fixed $i$ the random variables $X_{ij}, j \neq i$, are independent, but not necessarily identically distributed, such that $0 \leq X_{ij} \leq m$ and $m$ is finite. Hence, it follows from Feller (1971) (p. 553, Theorem 3) that

$$P(s_i^* > x_{n-1}) \sim 1 - \Phi(x_{n-1}), \quad (10)$$
provided that \( x_{n-1} = o \left( \left( \sigma_i(n-1) \right)^{1/3} \right) \), \( \frac{x_{n-1}^3}{\sigma_i(n)} \rightarrow 0 \) and \( \sigma_i(n-1) \rightarrow \infty \), as \( n \rightarrow \infty \), where \( \Phi() \) is the CDF of a standard normal variable.

It is well known (see for example, Lemma 2, p. 175 Feller (1968)) that as \( x_{n-1} \rightarrow \infty \),
\[
1 - \Phi(x_{n-1}) \sim \frac{1}{x_{n-1}} \varphi(x_{n-1}),
\]
where \( \varphi() \) is the PDF of a standard normal random variable.

Let \( \epsilon > 0 \) and put
\[
x_{n-1}^{\pm} = \left[ 2 \log(n-1) - (1 \pm \epsilon) \log(\log(n-1)) \right]^{1/2}.
\]
Combining (10) and (11), we obtain
\[
P(s_{(n)}^* > x_{n-1}^{\pm}) \sim \frac{\left( \log(n-1) \right)^{\pm \epsilon/2}}{\sqrt{4\pi(n-1)}}.
\]

Then, choosing \( c'' < \frac{1}{\sqrt{4\pi}} < c' \), we find that for all sufficiently large \( n \)
\[
P(s_{(n)}^* > x_{n-1}^{-}) \leq \sum_{i=1}^{n} P(s_i^* > x_{n-1}^{-}) < nc' \frac{\left( \log(n-1) \right)^{-\epsilon/2}}{n-1}.
\]

Using Theorem 1 we find that for sufficient large \( n \)
\[
P(s_{(n)}^* \leq x_{n-1}^{+}) \leq \prod_{i=1}^{n} P(s_i^* \leq x_{n-1}^{+}) < \left[ 1 - c'' \frac{\left( \log(n-1) \right)^{\epsilon/2}}{n-1} \right]^n \leq e^{-c'' n \frac{\left( \log(n-1) \right)^{\epsilon/2}}{n-1}}.
\]
Theorem 2 now follows from (13) and (14).

Comment 1. It is not difficult to see that the proofs of Theorem 1 and 2 still go through in the following more general situations: when

(a) the probabilities that appear in condition (L) can take on different values \( (p_u)^{(i,j)} \) for different ordered pairs \( (i, j) \); and

(b) the integer \( m \) that appears in condition (L) and the relation \( X_{ij} + X_{ji} = m \) can take on different values \( m_{ij} = m_{ji} \) for different unordered pairs \( (i, j) \).

Corollary 1. If \( E(s_1) = \cdots = E(s_n), \ Var(s_1) = \cdots = Var(s_n) \), then under the conditions of Theorem 2, we have \( (s_{(n)} - E(s_1))/\sqrt{Var(s_1)} - \sqrt{2\log(n-1)} \rightarrow 0 \) in probability as \( n \rightarrow \infty \).

Comment 2. The assumptions in Corollary 1 imply that in our model \( E(s_i) = E(s_j) = \frac{1}{2} \) (maximum total score possible) and \( Var(s_i) = Var(s_j) \); whereas in Huber’s model, with no outlier, \( E(X_{ij}) = 1/2, Var(X_{ij}) = 1/4 \) for all \( i \neq j \).
3 Examples

We present a few examples where the assumptions of Corollary 1 are satisfied.

Example 1 (Uniform distribution). If \( p_u = \frac{1}{m+1} \), for \( u = 0, 1, \ldots, m \), and if \( n \to \infty \), then

\[
s(n) - \left\lfloor \frac{(n-1)m}{2} + \sqrt{\frac{(n-1)(\log(n-1))m(m+2)}{6}} \right\rfloor \to 0
\]

in probability.

Example 2 (Symmetric Binomial distribution). If \( X_{ij} \sim \text{Bin}(m, p = 1/2) \), and if \( n \to \infty \), then

\[
s(n) - \left\lfloor \frac{(n-1)m}{2} + \sqrt{\frac{(n-1)(\log(n-1))m}{2}} \right\rfloor \to 0
\]

in probability.

Example 3 (Chess round-robin tournament with draws). In this case \( X_{ij} \), the score of player \( i \) after the game with player \( j, j \neq i \), equals 1, 1/2, or 0 accordingly as player \( i \) wins, draws, or loses the game against player \( j \). Therefore, \( X_{ij} + X_{ji} = 1, i \neq j \). For any \( i \neq j \) let \( p = P(X_{ij} = 1/2) \), and assume \( P(X_{ij} = 1) = P(X_{ij} = 0) \). In this case we find that

\[
E(s_1) = \frac{n-1}{2}, \text{Var}(s_1) = \frac{(n-1)(1-p)}{4}, \text{and if } n \to \infty, \text{then}
\]

\[
s(n) - \left\lfloor \frac{n-1}{2} + \sqrt{\frac{(n-1)(\log(n-1))(1-p)}{2}} \right\rfloor \to 0
\]

in probability.

Example 4 (Non-identically distributed scores). Suppose \( n = 2k + 1 \) and let \( p_1, \ldots, p_k \) denote arbitrary probabilities (not equal to 0 or 1). Suppose points labelled 1, 2, \ldots, \( n \) (corresponding to the \( n \) players) are arranged around the circumference of a circle in that order—so it makes sense to talk of one point being the successor or predecessor of another point etc. Consider three vertices labelled \( h, i, \) and \( j \) where \( 1 \leq h, i, j \leq n \) and \( h + d = i \) and \( i + d = j \) for some \( d \) such that \( 1 \leq d \leq k \); we reduce labels modulo \( n \) when necessary. For each such triple, let

\[
P(X_{hi} = u) = P(X_{ij} = u) = C(m, u)(p_d)^u(1-p_d)^{m-u},
\]

\[
P(X_{ih} = u) = P(X_{ji} = u) = C(m, u)(1-p_d)^u(p_d)^{m-u},
\]

(15)
where \( C(m, u) \) denotes the binomial coefficient \( m \)-choose-\( u \).

Then

\[
E(X_{ij}) + E(X_{ih}) = mp_d + m(1 - p_d) = m, \quad \text{Var}(X_{ij}) + \text{Var}(X_{ih}) = 2mp_d(1 - p_d). \quad (16)
\]

Consequently,

\[
E(s_i) = \sum_{j \neq i} E(X_{ij}) = m(n - 1)/2, \quad \text{Var}(s_i) = \sum_{j \neq i} \text{Var}(X_{ij}) = 2m(v_1 + \cdots + v_k) \quad (17)
\]

for all \( i \), where \( v_d = p_d(1 - p_d), d = 1, \ldots, k \); and if \( n \to \infty \), then

\[
s_{(n)} - \left\{ \frac{m(n - 1)}{2} + 2\sqrt{(\log(n - 1)m(v_1 + \cdots + v_k)} \right\} \to 0
\]

in probability.

When \( n = 2k \), we proceed essentially as before as far as the probabilities \( p_d \) are concerned, for \( 1 \leq d \leq k - 1 \). For the remaining case, consider diametrically opposite points \( i \) and \( j \), where \( 1 \leq i \leq k \) and \( k + 1 \leq j \leq n \); if \( j = i + k \), let \( P(X_{ij} = u) = C(m, u)(1/2)^m \); and if \( i = j + k \) let \( P(X_{ji} = u) = C(m, u)(1/2)^m \). Consequently,

\[
E(s_i) = m(k - 1 + 1/2) = m(n - 1)/2, \quad \text{Var}(s_i) = m(2(v_1 + \cdots + v_{k-1}) + 1/4) \quad (18)
\]

for all \( i \); and if \( n \to \infty \), then

\[
s_{(n)} - \left\{ \frac{m(n - 1)}{2} + 2\sqrt{(\log(n - 1)m(v_1 + \cdots + v_{k-1} + 1/8)} \right\} \to 0
\]

in probability.

**Example 5** (Non-identically distributed scores with two different values of \( m, m_w \) and \( m_b \)). Let \( n = 3k \) and suppose the \( n \) competitors are split into three classes \( (1), (2), \) and \( (3) \) of \( k \) competitors each. Let \( p \) and \( q \) denote two constants such that \( 0 < q < p < 1 \). Let \( i \) and \( j \) denote any two competitors. If \( i \) and \( j \) belong to the same class \( -(1), (2), \) or \( (3) \)– and \( i < j \), let \( P(X_{ij} = u) = C(m_w, u)(1/2)^m_w = P(X_{ji} = m_w - u), u = 0, 1, \ldots, m_w \) If \( i \) belongs to class \( (h) \) and \( j \) belongs to class \( h+1 \), where \( 1 \leq h \leq 3 \) and \( h + 1 \) is reduced modulo \( 3\), if necessary, then \( P(X_{ij} = u) = C(m_b, u)(p)^u(q)^{m_b - u} = P(X_{ji} = m_b - u), u = 0, 1, \ldots, m_b \). In this case we find that

\[
E(s_i) = (k - 1)m_w/2 + km_b, \quad \text{Var}(s_i) = (k - 1)m_w/4 + 2km_bpq; \quad (19)
\]
and if \( n \to \infty \), then

\[
    s(n) - \left\{ \frac{(k - 1)m_w}{2} + km_b + \sqrt{2\log(n - 1)\left(\frac{(k - 1)m_w}{4} + 2km_bpq\right)} \right\} \to 0
\]

in probability.

**Remark 1.** Suppose the outcomes of the competitions depend primarily upon three attributes of the participants: strength, speed, and experience. and suppose that participants in classes (1), (2), and (3) excel in strength, speed, and experience, respectively. Our assumptions are that strong players have an advantage when competing against fast players but are at a disadvantage when competing against more experienced players, and similarly for the other combinations. And that players that excel in the same attribute are equally likely to win when competing against each other. Thus the situation here is somewhat similar to that arising in the scissors, paper, stone game except that we have introduced probabilities instead of certainties for the outcomes here.

**Example 6.** If \( m = 2k \), let \( p_u = p_{m-u} = (u + 1)/(k + 1)^2 \) for \( u = 0, 1, \ldots, k \). Then \( E(X_{ij}) = m/2 \) and \( \text{Var}(X_{ij}) = m(m + 4)/24 \) for all distinct \( i \) and \( j \). Hence \( E(s_i) = m(n - 1)/2 \) and \( \text{Var}(s_i) = (n - 1)m(m + 4)/24 \) for all \( i \); and if \( n \to \infty \), then

\[
    s(n) - \left\{ \frac{(n - 1)m}{2} + \sqrt{\frac{(n - 1)(\log(n - 1))m(m + 4)}{12}} \right\} \to 0
\]

in probability.

**Example 7.** If \( m = 4 \), let \( p_0 = p_4 = L^3 \), \( p_1 = p_3 = L^2 \), and \( p_2 = L \) where \( L = .4406197 \ldots \) is the positive root of the equation \( L + 2L^2 + 2L^3 = 1 \). Then \( E(X_{ij}) = L^2 + 2L + 3L^2 + 4L^3 = 4(L^2 + L^2 + L^3) = 4/2 = 2 \) and \( \text{Var}(X_{ij}) = 2(L^2 + 4L^3) = 1.0726468 \ldots \) for all distinct \( i \) and \( j \). Hence, \( E(s_i) = 2(n - 1) \) and \( \text{Var}(s_i) = 1.0726468(n - 1) \) for all \( i \); and if \( n \to \infty \), then

\[
    s(n) - \left\{ 2(n - 1) + \sqrt{2.1452936(n - 1)(\log(n - 1))} \right\} \to 0
\]

in probability.

**Acknowledgements**

We would like to thank the Editor, Associate Editor and referee for the insightful and helpful comments that led to significant improvements in the paper. YM thanks Abram Kagan.
for describing a score issue in chess round-robin tournaments with draws. The research of YM was supported by grant no. 2020063 from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

Appendix

A Proof of Assertion 1

Proof. When we apply definitions (8), and (9) to the last expression in relation (5), we find that

\[
F_1 - F = \sum_{u=0}^{\min(m,k_1)} \sum_{v=0}^{\min(m,k_2)} p(u, v) P(s_1' \leq k_1 - u, s_2' \leq k_2 - v)
\]

\[
= \sum_{u=0}^{\min(m,k_1)} \sum_{v=0}^{\min(m,k_2)} p(u, v) \sum_{g=u}^{k_1} \sum_{h=v}^{k_2} R(g, h)
\]

\[
= \sum_{g=0}^{k_1} \sum_{h=0}^{k_2} R(g, h) \sum_{u=0}^{\min(m,g)} \sum_{v=0}^{\min(m,h)} p(u, v)
\]

\[
= \sum_{g=0}^{k_1} \sum_{h=0}^{k_2} R(g, h) W(g, h), \quad (20)
\]

as required.  

B Proof of Assertion 2

Proof. Let

\[
Q_u = p_0 + \cdots + p_u
\]

for 0 ≤ u ≤ m, where we adopt the convention that Q_{-1} = 0.

We consider various cases separately.

Case (i). 0 ≤ g, h ≤ m and 0 ≤ g + h < m.

If 0 ≤ u ≤ g and 0 ≤ v ≤ h, then u + v ≤ g + h < m, so p(u, v) = p_u p_{m-v} by relation (6). Consequently,

\[
W(g, h) = \sum_{u=0}^{g} \sum_{v=0}^{h} p_u p_{m-v} = (p_0 + \cdots + p_g)(p_m + p_{m-1} + \cdots + p_{m-h}) = Q_g(1 - Q_{m-h-1}) \geq 0.
\]
Case (ii). $0 \leq g, h \leq m$ and $g + h \geq m$.

In this case it follows from relation (6) that

$$W(g, h) = Q_g(1 - Q_{m-h-1}) - \sum' p_t$$

where the sum is, in effect, over all pairs $(t, m-t)$ such that $m - h \leq t \leq g$; hence,

$$W(g, h) = Q_g(1 - Q_{m-h-1}) - (p_{m-h} + \cdots + p_g)$$

$$= Q_g(1 - Q_{m-h-1}) - (Q_g - Q_{m-h-1}) = Q_{m-h-1}(1 - Q_g) \geq 0.$$

Notice, in particular, that

$$W(m, h) = 0 \text{ for } 0 \leq h \leq m \text{ and } W(g, m) = 0 \text{ for } 0 \leq g \leq m. \quad (22)$$

We also point out that if $0 \leq g, h < m$, then $W(g, h) = W(m - 1 - h, m - 1 - g)$.

Case (iii). $0 \leq h \leq m < g \leq k_1$, or $0 \leq g \leq m < h \leq k_2$, or $m < g \leq k_1$ and $m < h \leq k_2$.

If $0 \leq h \leq m < g \leq k_1$, then it follows from (9) and (22) that

$$W(g, h) = \sum_{u=0}^{m} \sum_{v=0}^{h} p(u, v) = W(m, h) = 0.$$ 

If $0 \leq g \leq m < h \leq k_2$, then it follows from (9) and (22) that

$$W(g, h) = \sum_{u=0}^{g} \sum_{v=0}^{m} p(u, v) = W(g, m) = 0.$$ 

If $m < g \leq k_1$ and $m < h \leq k_2$, then it follows from (9) and (22) that

$$W(g, h) = \sum_{u=0}^{m} \sum_{v=0}^{m} p(u, v) = W(m, m) = 0.$$

References

Bradley, R. A., Terry, M. E. (1952). Rank analysis of incomplete block designs. I. The method of paired comparisons. Biometrika 39, 324–345.
Brualdi, R. A., Fritscher, E. (2015). Tournaments associated with multigraphs and a theorem of Hakimi. Discrete Math. 338, 229–235.

Chatterjee, S., Diaconis, P., Sly, A. (2011). Random graphs with a given degree sequence. Ann. Appl. Probab. 21, 1400–1435.

Chetrite, R., Diel, R., Lerasle, M. (2017). The number of potential winners in Bradley-Terry model in random environment. Ann. Appl. Probab. 27, 1372–1394.

David, H. A. (1959). Tournaments and Paired Comparisons. Biometrika 46, 139–149.

David, H. A. (1988). The method of paired comparisons. Second edition. Charles Griffin & Co., Ltd., London; The Clarendon Press, Oxford University Press, New York.

David, H. A., Edwards, A. W. F. (2001). The Evaluation of Tournament Outcomes: Comments on Zermelo (1929). Annotated readings in the history of statistics. Springer-Verlag, New York, 161–166.

Efron, B. (1965). Increasing properties of Pólya frequency functions. Ann. Math. Statist. 36, 272–279.

Epstein, R. A. (2013). The theory of gambling and statistical logic. Special second edition. Elsevier/Academic Press, Amsterdam.

Lehmann, E. L. (1966). Some concepts of dependence. Ann. Math. Statist. 37, 1137–1153.

Feller, W. (1968). An introduction to probability theory and its applications. Vol. I. Third edition. New York-London-Sydney: Wiley.

Feller, W. (1971). An introduction to probability theory and its applications. Vol. II. Second edition. New York-London-Sydney: Wiley.

Ford, L.R., Jr. (1957). Solution of a ranking problem from binary comparisons. Amer. Math. Monthly 64, 28–33.

Glickman, M. E. (2013). Introduction note to 1928 (=1929). Ernst Zermelo collected works. Vol. II. Edited by Heinz-Dieter Ebbinghaus and Akihiro Kanamori. Springer-Verlag, Berlin, 616–621.
Griggs, J. R., Reid, K. B. (1999). Landau’s theorem revisited. Australian J. Comb. 20, 19–24.

Harary, F., Moser, L. (1966). The theory of round robin tournaments. Amer. Math. Monthly 73, 231–246.

Holshauser, A., Moon, J. W., Reiter, H. (2011). Win-loss sequences for generalized round-robin tournaments. Missouri. J. Math. Sci. 23, 142–150.

Huber, P. J. (1963). A remark on a paper of Trawinski and David entitled: Selection of the best treatment in a paired comparison experiment. Ann.Math.Statist. 34, 92–94.

Joag-Dev, K., Proschan, F. (1983). Negative association of random variables, with applications. Ann. Statist. 11, 286–295.

Jogdeo, K., Patil, G. P. (1975). Probability inequalities for certain multivariate discrete distribution. Sankhya Ser. B 37, 158–164.

Johnson, O., Goldschmidt, C. (2006). Preservation of log-concavity on summation. ESAIM Probab. Stat. 10, 206–215.

Landau, H. G. (1953). On dominance relations and the structure of animal societies. III. The condition for a score structure. Math. Biophys. 15, 143–148.

Malinovsky, Y. (2021a). On the distribution of winners’ scores in a round-robin tournament.
Prob. in Eng. and Inf. Sciences.DOI: https://doi.org/10.1017/S0269964821000267. In press.

Malinovsky, Y. (2021b). Correction to "On the distribution of winners' scores in a round-robin tournament.” Prob. in Eng. and Inf. Sciences. In press from December 9, 2021. https://arxiv.org/abs/2201.05018.

Mallows, C. L. (1968). An inequality involving multinomial probabilities. Biometrika 55, 422–424.

Moon, J. W. (2013). Topics on Tournaments. [Publicly available on website of Project Gutenberg https://www.gutenberg.org/ebooks/42833].
Nelsen, R. B. (2006). An introduction to copulas. Second edition. Springer Series in Statistics. Springer, New York.

Reid, K. B. (2004). Tournaments. Handbook of Graph Theory, Second Edition (ed. J. L. Gross, J. Yellen and P. Zhang), CRC Press, Boca Raton, 196–225.

Ross, S. M. (2016). Improved Chen-Stein bounds on the probability of a union. J. Appl. Probab. 53, 1265–1270.

Ross, S. M. (2021). Team’s seasonal win probabilities. Probab. Engrg. Inform. Sci. In press.

Simons, G., Yao, Y-C., (1999). Asymptotics when the number of parameters tends to infinity in the Bradley-Terry model for paired comparisons. Ann. Statist. 27, 1041–1060.

Slepian, D. (1962). The one-sided barrier problem for Gaussian noise. Bell System Tech. J. 41, 463–501.

Zermelo, E. (1929). Die Berechnung der Turnier-Ergebnisse als ein Maximumproblem der Wahrscheinlichkeitsrechnung. Math. Z. 29, 436–460.