DEFORMATIONS OF SEMISIMPLE POISSON PENCILS OF HYDRODYNAMIC TYPE ARE UNOBLSTRUCTED

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ABSTRACT. We prove that the bihamiltonian cohomology of a semisimple pencil of Poisson brackets of hydrodynamic type vanishes for almost all degrees. This implies the existence of a full dispersive deformation of a semisimple bihamiltonian structure of hydrodynamic type starting from any infinitesimal deformation.

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1. Introduction

1.1. Basic setup from the theory of integrable hierarchies. Consider a system of evolutionary PDEs with one spatial variable $x$ and $n$ dependent variables of the form

$$\frac{\partial u^i}{\partial t} = A^i_j(u)u^j_x + \epsilon \left( B^i_j(u)u^j_{xx} + C^i_{jk}u^j_x u^k_x \right) + O(\epsilon^2).$$

(Here, and in the following we use the summation convention.) The right hand side of this equation is represented as a formal power series in $\epsilon$, where
the coefficient of $\epsilon^k$ is a homogeneous differential polynomial, i.e. an homogeneous polynomial in $u^{i,d} := \partial^d u^i$, $i = 1, \ldots, n$, $d = 1, \ldots, k$, of degree $k+1$ whose coefficients are smooth functions of coordinates $u^1, \ldots, u^n$ on some domain $U \subset \mathbb{R}^n$. We can think of the $u^i(x)$ as ($\epsilon$-power series of) smooth functions of $x \in S^1$ or Schwarzian functions of $x \in \mathbb{R}$.

On of the possible ways to define and study an integrable hierarchy of partial differential equations of this type uses so-called bihamiltonian structures with hydrodynamic limit.

A bihamiltonian structure with hydrodynamic limit is given by: a pencil of two compatible Poisson structures on the space of local functionals of the form

$$\{ u^i(x), u^j(y) \}_a = \left( g_a^{ij}(u) \partial_x + \Gamma^{ij}_{k,a}(u) u^k_x \right) \delta(x-y) + O(\epsilon), \quad a = 1, 2$$

where the leading order is given by two Poisson brackets of hydrodynamic type, and the terms of higher order in $\epsilon$ are homogeneous differential operators acting on $\delta(x-y)$, and their coefficients are homogeneous differential polynomials in the dependent variables $u^1, \ldots, u^n$; a system of Hamiltonians of the form

$$H_a[u] = \int dx \cdot (h_a(u) + O(\epsilon)), \quad a = 1, 2$$

with higher order terms in $\epsilon$ given by homogeneous differential polynomials in $u^1, \ldots, u^n$.

The evolutionary PDE above can be written as an Hamiltonian flow w.r.t. both Poisson structures

$$\frac{\partial u^i}{\partial t} = \{ u^i(x), H_a \}_a$$

for $a = 1, 2$.

The natural equivalence relation on these systems, and in particular on the pencils of Poisson structures with hydrodynamic limit, is given by the so-called Miura transformations, which are transformations of the dependent variables of the form

$$u^i \mapsto v^i(u) + O(\epsilon), \quad (1)$$

where higher order terms in $\epsilon$ are homogeneous differential polynomials in $u^1, \ldots, u^n$, and the leading term is a diffeomorphism.

In this context, an important problem is to classify the pencils of Poisson structures with hydrodynamic limit, up to the equivalence given by Miura transformations. In the scalar ($n = 1$) case a complete solution of this classification problem has been obtained, see [13, 10, 11, 12, 4, 5].

In the general $n > 0$ case (see [3, 6, 7, 9, 11]) it is convenient to make the assumption that the pencil of Poisson brackets of hydrodynamic type that we are considering is semisimple. A Poisson pencil of hydrodynamic type

$$\left( g_a^{ij}(u) \partial_x + \Gamma^{ij}_{k,a}(u) u^k_x \right) \delta(x-y), \quad a = 1, 2$$

is semisimple if the polynomial $\det \left( g_1^{ij} - \lambda g_2^{ij} \right)$ of degree $n$ in $\lambda$ has $n$ pairwise distinct non-constant roots on $U \subset \mathbb{R}^n$. In such case [6] one can use the roots as a set of coordinates on $U$, called canonical coordinates. This choice
ensures that both metrics \( g^{ij}_{1,2} \) are diagonal with diagonal entries respectively equal to \( f^i(u) \), \( u^i f^j(u) \), \( i = 1, \ldots, n \) for non-vanishing functions \( f^i(u) \) on \( U \).

In this paper we shall consider the deformation problem of semisimple Poisson pencils of hydrodynamic type by working in canonical coordinates. The change of coordinates to canonical ones is an example of a Miura transformation of the first kind, i.e., a diffeomorphism with the terms \( O(\epsilon) \) in \([1]\) equal to zero. By fixing these coordinates, we are therefore left with the problem of classifying Poisson pencils up to Miura transformations of the second kind, that is, transformations of the dependent coordinates as in \([1]\), with the zeroth order constant in \( \epsilon \) equal to the identity. Since Miura transformations of the first and second kind obviously generate the whole Miura group, this classification problem is equivalent to the original one described above. Let us now give a precise formulation of this deformation problem in canonical coordinates.

1.2. Classification of Poisson pencils and the extension problem.

Let \( \{ , \}_\lambda^0 \) be a semisimple Poisson pencil of hydrodynamic type \([2]\), and let \( u^1, \ldots, u^n \) be the associated canonical coordinates over a domain \( U \subset \mathbb{R}^n \) where \( u^i - u^j \neq 0 \) for \( i \neq j \). The two compatible Poisson brackets \( \{ , \}_1^0, \{ , \}_2^0 \) are of the form

\[
\{ u^i(x), u^j(y) \}_a^0 = g^{ij}_1(u(x)) \delta'(x - y) + \Gamma^{ij}_k(u(x)) u^k(x) \delta(x - y),
\]

with \( a = 1, 2 \), \( i, j = 1, \ldots, n \), where the contravariant metrics are given by

\[
g^{ij}_1 = f^j \delta_{ij}, \quad g^{ij}_2 = u^j f^i \delta_{ij} \quad \text{(no summation over } i)\]

and \( \Gamma^{ij}_k = -g^{lk}_a \Gamma^{ij}_k, \) where \( \Gamma^{ij}_k \) are the Christoffel symbols of the metric \( g^{ij}_1 \), and \( f^1(u), \ldots, f^n(u) \) are non-vanishing functions on \( U \).

A deformation of \( \{ , \}_\lambda^0 \) is given by a pencil

\[
\{ , \}_\lambda = \{ , \}_2 - \lambda \{ , \}_1
\]

where \( \{ , \}_a, a = 1, 2 \) are compatible Poisson brackets of the form

\[
\{ u^i(x), u^j(y) \}_a = \{ u^i(x), u^j(y) \}_a^0 + \sum_{k>0} \frac{\sum_{l=0}^{k+1} A_{k,l}^{ij}(u(x)) \delta^{(l)}(x - y)}{k+1}
\]

with \( A_{k,l}^{ij} \in \mathcal{A} \) and \( \deg A_{k,l}^{ij} = k - l + 1 \). Here \( \mathcal{A} \) denotes the space of formal power series in the variables \( u^1, \ldots, u^n \), i.e., formal power series of the form \( u^{i,s}, i = 1, \ldots, n, s > 0 \), with coefficients that are smooth functions of \( u^1, \ldots, u^n \). The degree is defined by setting \( \deg u^{i,s} = s \).

Two deformations are equivalent if they are related by a Miura transformation (of the second kind \([11]\)), i.e., by a change of variables of the form

\[
u^i \mapsto \tilde{u}^i = u^i + \sum_{k>0} \epsilon^k F_k^i, \quad i = 1, \ldots, n
\]

with \( F_k^i \in \mathcal{A} \) and \( \deg F_k^i = k \).

An infinitesimal deformation \( \{ , \}_{\lambda}^2 \) of \( \{ , \}_\lambda^0 \) is given by a pair of compatible Poisson brackets of the form \([2]\) where terms of \( O(\epsilon^3) \) are disregarded. This means that in the expansion \([2]\) above, we only consider the the coefficients \( A_{k,l}^{ij}(u(x)) \) for \( k \) up to 2, so that the highest derivative \( \delta^{(2)}(x - y) \) appearing
is 3. Two infinitesimal deformations are equivalent iff they are related by a Miura transformation up to $O(\epsilon^3)$. The following theorem, which classifies the deformations of $\{,\}_\lambda^0$, was proved in [9, 6].

**Theorem 1.** Two deformations of $\{,\}_\lambda^0$ are equivalent if and only if the corresponding infinitesimal deformations are equivalent. Given an infinitesimal deformation of $\{,\}_\lambda^0$, the functions, called central invariants, defined by

$$c_i(u) := \frac{1}{3(f^i(u))^2} \left( A_{2,3;2}^{ij} - u^i A_{2,3;1}^{ij} + \sum_{k \neq i} \frac{(A_{1,2;2}^{ij} - u^i A_{1,2;1}^{ij})^2}{f^k(u)(u^k - u^i)} \right),$$

for $i = 1, \ldots, n$, only depend on the single variable $u^i$ are invariant under Miura transformations. Two infinitesimal deformations of $\{,\}_\lambda^0$ are equivalent if and only if they have the same central invariants.

The main problem in the deformation theory of a semisimple Poisson pencil $\{,\}_\lambda^0$ is the problem of extension [12]. Making a choice of central invariants $c_1(u^1), \ldots, c_n(u^n)$ fixes an equivalence class of infinitesimal deformations of $\{,\}_\lambda^0$, but the question is whether there exists a full deformation $\{,\}_\lambda$ that extends an infinitesimal one to all orders in $\epsilon$. The main result of this paper is the affirmative answer to this question.

**Theorem 2.** Let $\{,\}_\lambda^{\leq 2}$ be an infinitesimal deformation of a semisimple Poisson pencil of hydrodynamic type $\{,\}_\lambda^0$. Then there exists a deformation $\{,\}_\lambda$ that extends $\{,\}_\lambda^{\leq 2}$ to all orders in $\epsilon$.

1.3. Methods of proof and organization of the paper. The problem of the description of automorphisms, infinitesimal deformations, and obstructions to the extension of infinitesimal deformations for any algebraic structure can be formulated in terms of some cohomology groups associated to it. In our case, in order to prove that the deformation of a semisimple pencil of Poisson brackets is not obstructed we have to show that certain cohomology groups, called bihamiltonian cohomology, are equal to zero. However, Liu and Zhang have shown that the vanishing of these cohomology groups follows from the vanishing of the cohomology of the auxiliary complex $(\hat{A}[\lambda], D_\lambda)$, defined below, in certain degrees. It is a difficult task to compute the full cohomology of this complex, however we are able to show vanishing of such cohomology in the required degrees using a clever choice of filtrations and the structure of the associated spectral sequences.

The paper is organized as follows. In Section 2 we recall the definition of the auxiliary complex and explain its relation to the problems of deformation of pencils of Poisson structures. In particular, we formulate a statement about the complex $(\hat{A}[\lambda], D_\lambda)$ that implies Theorem 2. In Section 3 we introduce a series of filtrations on the complex $(\hat{A}[\lambda], D_\lambda)$ that allows us to prove the key statement about its cohomology.

1.4. Conventions. Throughout the paper we use the summation convention in the sense that repeated (upper- and lower-)indices should be summed over. However, there are a few exceptions when the metrics $g_{ij}^i = f^i \delta_{ij}$ and
\( g_{ij}^{\alpha \beta} = u^i f^\alpha \delta_{ij}, \) or the tensors derived from them are involved. Such equations always involve the functions \( f^i. \) To determine which indices are to be summed over, it suffices to consider the other side of the equation and the indices that appear in there.

2. Theta formalism, polyvector fields and cohomology

The deformation theory of a pencil of Poisson brackets is controlled by the so-called bihamiltonian cohomology defined on the space of local polyvector fields. In order to show the vanishing of such bihamiltonian cohomology in certain degrees, from which Theorem 2 follows, we consider the cohomology of a related complex \( (\hat{\mathcal{A}}[\lambda], D_\lambda) \), introduced by Liu and Zhang [12]. This approach has a double advantage: first, we can work in the space \( \hat{\mathcal{A}} \), where the identifications imposed by integration are not imposed, making computations simpler; second, we can compute the cohomology on the space \( \hat{\mathcal{A}}[\lambda] \) of polynomials in \( \lambda \), rather than the bihamiltonian cohomology on \( \hat{\mathcal{A}} \), and this allows us to use directly the methods of spectral sequences associated with filtrations.

In this Section we review some basic definitions, mainly from [12], state our main theorem and derive its most important consequences.

2.1. Basic definitions. Consider the supercommutative associative algebra \( \hat{\mathcal{A}} \) defined as

\[
\hat{\mathcal{A}} = C^\infty(U)[[u^{i,1}, u^{i,2}, \ldots; \theta_i^0, \theta_i^1, \theta_i^2, \ldots]],
\]

where \( u^{i,s}, i = 1, \ldots, n, s = 1, 2, \ldots \) are formal even variables and \( \theta_i^s, i = 1, \ldots, n, s = 0, 1, 2, \ldots \) are odd variables. An element in \( C^\infty(U) \) is represented by a function of the coordinates \( u^i, i = 1, \ldots, n \) on the domain \( U \subset \mathbb{R}^n \). We define the standard gradation on \( \hat{\mathcal{A}} \) by assigning the degrees

\[
\deg u^{i,s} = \deg \theta_i^s = s, \quad s = 1, 2, \ldots
\]

and degree zero to both \( \theta_i = \theta_i^0 \) and the elements in \( C^\infty(U) \). The standard degree \( d \) homogeneous component of \( \hat{\mathcal{A}} \) is denoted \( \hat{\mathcal{A}}_d \). The super gradation is defined by assigning degree one to \( \theta_i^s \) for \( s \geq 0 \) and degree zero to the remaining generators of \( \hat{\mathcal{A}} \). The super degree \( p \) homogeneous component is denoted \( \hat{\mathcal{A}}^p \). We also denote

\[
\hat{\mathcal{A}}^p_d = \hat{\mathcal{A}}_d \cap \hat{\mathcal{A}}^p.
\]

The standard derivation on \( \hat{\mathcal{A}} \),

\[
\partial = \sum_{s \geq 0} \left( u^{i,s+1} \frac{\partial}{\partial u^{i,s}} + \theta_i^{s+1} \frac{\partial}{\partial \theta_i^s} \right),
\]

is compatible with the standard and super gradations, in particular it increases the standard degree by one and leaves invariant the super degree. Thanks to the homogeneity of \( \partial \), the space \( \hat{\mathcal{F}} := \frac{\hat{\mathcal{A}}}{\partial \hat{\mathcal{A}}} \) still possesses two gradations, which we keep denoting with indices \( p \) and \( d \). The elements of \( \hat{\mathcal{F}}^p \) are called local \( p \)-vectors and the projection map is denoted by an integral

\[
\int : \hat{\mathcal{A}} \to \hat{\mathcal{F}}.
\]
The space $\hat{\mathcal{F}}$ can be endowed with the Schouten-Nijenhuis bracket

$$[\cdot, \cdot] : \hat{\mathcal{F}}^p \times \hat{\mathcal{F}}^q \rightarrow \hat{\mathcal{F}}^{p+q-1},$$

which satisfies the usual graded skew-symmetry and graded Jacobi identities, see [12, 11] for more details.

A Poisson bivector $P$ is an element of $\hat{\mathcal{F}}^2$ that satisfies $[P, P] = 0$. If $P$ is a Poisson bivector, its adjoint action $d_P = [P, \cdot]$ on $\hat{\mathcal{F}}$ by the graded Jacobi identity squares to zero, hence defines a differential complex $(\hat{\mathcal{F}}, d_P)$. Given a Poisson bivector $P$, the super derivation $D_P$ on $\hat{A}$ is defined by

$$D_P = \sum_{s \geq 0} \left( \partial^s \left( \frac{\delta P}{\delta u^i} \right) \frac{\partial}{\partial u^{i,s}} + \partial^s \left( \frac{\delta P}{\delta \theta^i} \right) \frac{\partial}{\partial \theta^i} \right),$$

where the variational derivatives on $\hat{A}$ w.r.t. $u^i$ and $\theta^i$ are defined as follows

$$\frac{\delta}{\delta u^i} = \sum_{s=0}^{\infty} (-\partial)^s \frac{\partial}{\partial u^{i,s}}, \quad \frac{\delta}{\delta \theta^i} = \sum_{s=0}^{\infty} (-\partial)^s \frac{\partial}{\partial \theta^i}.$$

The super derivation $D_P$ squares to zero, and is such that the integral defines a map of differential complexes

$$\int : (\hat{A}, D_P) \rightarrow (\hat{\mathcal{F}}, d_P).$$

As pointed out in [12] this allows us to work with the complex $(\hat{A}, D_P)$ rather than with the more complicated $(\hat{\mathcal{F}}, d_P)$.

A Poisson pencil is given by two Poisson bivectors $P_1, P_2$ which are compatible, i.e. $[P_1, P_2] = 0$. For each $\lambda$, then, $P_\lambda := P_2 - \lambda P_1$ is also a Poisson bivector. We denote by $d_1$ and $d_2$ the differentials on $\hat{\mathcal{F}}$ corresponding to $P_1$ and $P_2$, respectively. Due to compatibility, $d_\lambda := d_2 - \lambda d_1$ squares to zero. We denote by $D_1$ and $D_2$ the super derivations on $\hat{A}$ associated to $P_1$ and $P_2$, respectively. Their compatibility in this case implies that $D_\lambda := D_2 - \lambda D_1$ also squares to zero. In summary we can define two differential complexes associated to a Poisson pencil

$$(\hat{A}, D_\lambda), \quad (\hat{\mathcal{F}}, d_\lambda).$$

Remark 3. The Poisson brackets $\{,\}_{1,2}$ introduced in [2] are elements of the space $\Lambda^2_{loc}$ of local bivectors written in $\delta$-formalism, see [7]. There is a one to one correspondence between the space of local $p$-vectors $\Lambda^p_{loc}$, written in $\delta$-formalism and the space $\hat{\mathcal{F}}^p$. We will not recall it here in general but rather refer the reader to [12]. For the case of a bivector, written as

$$\{u^i(x), u^j(y)\} = \sum_{s \geq 0} B^{ij}_{s}(x-y),$$

with $B^{ij}_s \in A$, the corresponding element in $\hat{\mathcal{F}}^2$ is

$$P = \frac{1}{2} \int \theta^i \sum_{s \geq 0} B^{ij}_{s} \theta^j.$$
2.2. The complex \((\hat{A}[\lambda], D_\lambda)\). Let us fix a semisimple Poisson pencil of hydrodynamic type \(\{,\}_0^\lambda = \{,\}_0^\lambda - \lambda \{,\}_0\), and denote by \(u^1, \ldots, u^n\) the associated canonical coordinates on \(U \subset \mathbb{R}^n\), see \[11.2\] The compatible Poisson brackets \(\{,\}_1\) are represented in \(\mathcal{F}_2^1\) by two bivectors

\[
P_a = \frac{1}{2} \int \left( g^{ij}_a \theta_i^0 \theta_j^0 + \Gamma_{k,a}^{ij} u^{k,1} \theta_i^0 \theta_j^0 \right), \quad a = 1, 2.
\]

In canonical coordinates the Christoffel symbol of \(g^{ij}_1 = f^i \delta_{ij}\) is

\[
\Gamma_{k,1}^{ij} = \frac{1}{2} \left( \delta_k f^i \delta_{ij} + f^i \frac{\partial f^j}{\partial u^k} \delta_{ij} - f^j \frac{\partial f^i}{\partial u^k} \delta_{ij} \right)
\]

and \(P_1\) is given by

\[
P_1 = \frac{1}{2} \int \left( f^i \theta_i^0 \theta_j^0 + f^i \frac{\partial f^j}{\partial u^k} \theta_i^0 \theta_j^0 \right).
\]

We denote by \(D_1 = D(f^1, \ldots, f^n)\) the super derivation on \(\hat{A}\) corresponding to \(P_1\). A straightforward computation gives us the following formula

\[
D(f^1, \ldots, f^n) = \sum_{s \geq 0} \partial^s \left( f^i \theta_i^0 \right) \frac{\partial}{\partial u^{i,s}}
\]

\[
+ \frac{1}{2} \sum_{s \geq 0} \partial^s \left( \delta_j f^i u^{i,1} \theta_j^0 + f^i \frac{\partial f^j}{\partial u^k} \delta_{ij} - f^j \frac{\partial f^i}{\partial u^k} \delta_{ij} \right) \frac{\partial}{\partial u^{i,s}}
\]

\[
+ \frac{1}{2} \sum_{s \geq 0} \partial^s \left( \delta_j f^i \theta_j^0 \theta_i^0 + f^i \frac{\partial f^j}{\partial u^k} \theta_j^0 \theta_i^0 - f^j \frac{\partial f^i}{\partial u^k} \theta_j^0 \theta_i^0 \right) \frac{\partial}{\partial \theta_i^s}
\]

\[
+ \frac{1}{2} \sum_{s \geq 0} \partial^s \left( f^j \frac{\partial f^i}{\partial u^k} \theta_j^0 \theta_i^0 - f^j \frac{\partial f^i}{\partial u^k} \theta_j^0 \theta_i^0 \right) \frac{\partial}{\partial \theta_i^s}
\]

The super derivation corresponding to \(P_2\) is then given by

\[
D_2 := D(u^1 f^1, \ldots, u^n f^n).
\]

Our aim is to compute the cohomology of the complex \((\hat{A}[\lambda], D_\lambda)\) with \(D_\lambda = D_2 - \lambda D_1\), and \(D_1, D_2\) given above.

2.3. The main theorem. In Section 3 we prove the following vanishing theorem for the cohomology of the complex \((\hat{A}[\lambda], D_\lambda)\).

**Theorem 4.** The cohomology \(H^p_d(\hat{A}[\lambda], D_\lambda)\) vanishes for all bi-degrees \((p, d)\), unless

\[
d = 0, \ldots, n, \quad p = d, \ldots, d + n,
\]

or

\[
d = n + 1, n + 2, \quad p = d, \ldots, d + n - 1.
\]
Remark 5. For \( n = 1 \) the bi-degrees for which we cannot state vanishing according to Theorem 4 are
\[
(p, d) = (0, 0), (1, 1), (2, 2), (3, 3).
\]
In [5] (and in [4] for the KdV case) we compute by a different method the cohomology \( H^p_d(\hat{\mathcal{A}}[\lambda], D_\lambda) \) for all bi-degrees, proving in particular that it vanishes also for
\[
(p, d) = (1, 0), (1, 1), (2, 2).
\]
This shows that the vanishing theorem above can be strengthened. Our result is enough, however, to prove the absence of obstructions to deformations of bihamiltonian structures, which is our main aim.

Remark 6. In the proof of Theorem 4 we will distinguish two sets of indexes for which we do not prove vanishing:
\[
d = 0, \ldots, n, \quad p = d, \ldots, d + n \quad \text{(Case 1)},
\]
\[
d = 2, \ldots, n + 2, \quad p = d, \ldots, d + n - 1 \quad \text{(Case 2)},
\]
which clearly overlap for \( n \geq 2 \). We distinguish the two cases since they have different sources in the complex.

2.4. Vanishing of bihamiltonian cohomology. By the following lemma of Liu and Zhang [12], the bihamiltonian cohomology of \( \hat{\mathcal{F}} \) is isomorphic to the cohomology of the complex \([\hat{\mathcal{F}}[\lambda], d_\lambda]\) in almost all degrees \((p, d)\).

Let \((C, \partial_1, \partial_2)\) be either the double complex \((\hat{\mathcal{A}}, D_1, D_2)\) or \((\hat{\mathcal{F}}, d_1, d_2)\). The bihamiltonian cohomology of the double complex \((C, \partial_1, \partial_2)\) is defined by
\[
BH(C, \partial_1, \partial_2) = \frac{\ker \partial_1 \cap \ker \partial_2}{\text{Im} \partial_1 \partial_2}.
\]

Lemma 7. The natural embedding of \( C \) in \( C[\lambda] \) induces an isomorphism
\[
BH^p_d(C, \partial_1, \partial_2) \cong H^p_d(C[\lambda], \partial_\lambda)
\]
for \( p \geq 0, d \geq 2 \).

As pointed out in [12], the short exact sequence of complexes
\[
0 \to (\hat{\mathcal{A}}[\lambda]/\mathbb{R}[\lambda], D_\lambda) \to (\hat{\mathcal{A}}[\lambda], D_\lambda) \to (\hat{\mathcal{F}}[\lambda], d_\lambda) \to 0
\]
implies a long exact sequence in cohomology, which includes
\[
\cdots \to H^p_d(\hat{\mathcal{A}}[\lambda]) \to H^p_d(\hat{\mathcal{F}}[\lambda]) \to H^{p+1}_d(\hat{\mathcal{A}}[\lambda]) \to \cdots
\]
for \( p, d \geq 0 \). It is clear that if both \( H^p_d(\hat{\mathcal{A}}[\lambda], D_\lambda) \) and \( H^{p+1}_d(\hat{\mathcal{A}}[\lambda], D_\lambda) \) vanish, then \( H^p_d(\hat{\mathcal{F}}[\lambda], d_\lambda) \) vanishes too. Our vanishing result for \( H^p_d(\hat{\mathcal{A}}[\lambda]) \) translates to the following statement for the cohomology of the complex \((\hat{\mathcal{F}}[\lambda], d_\lambda)\).

Corollary 8. The cohomology \( H^p_d(\hat{\mathcal{F}}[\lambda], d_\lambda) \) vanishes for all bi-degrees \((p, d)\), unless
\[
d = 0, \ldots, n, \quad p = d - 1, \ldots, d + n,
\]
or
\[
d = n + 1, n + 2, \quad p = d - 1, \ldots, d + n - 1.
\]
\(^1\)A similar description in terms of a bicomplex was given in [4].
Using the isomorphism of Lemma [7] we obtain the vanishing of the bihamiltonian cohomology of \( \hat{\mathcal{F}} \).

**Corollary 9.** The bihamiltonian cohomology \( BH^2_d(\hat{\mathcal{F}},d_1,d_2) \) vanishes for all bi-degrees \((p,d)\) with \( d \geq 2 \), unless

\[
d = 2, \ldots, n, \quad p = d - 1, \ldots, d + n,
\]
or

\[
d = n + 1, n + 2, \quad p = d - 1, \ldots, d + n - 1.
\]

**Remark 10.** Notice that in particular it follows that

\[
BH^2_{\geq 4}(\hat{\mathcal{F}},d_1,d_2) = 0, \quad BH^3_{\geq 5}(\hat{\mathcal{F}},d_1,d_2) = 0.
\]

The vanishing of the second cohomology for \( d \geq 4 \) has been already proved in \([9, 6]\), together with the results

\[
BH^2_2(\hat{\mathcal{F}},d_1,d_2) = 0, \quad BH^2_3(\hat{\mathcal{F}},d_1,d_2) \cong \bigoplus_{i=1}^n C^\infty(\mathbb{R}).
\]

The vanishing of the third cohomology for \( d \geq 5 \) is new, and is the most relevant for the extension problem of deformation theory.

2.5. **Bihamiltonian cohomology and deformations.** The deformation problem can be formulated in \( \hat{\mathcal{F}} \) as follows. Let \( P^0_\lambda = P^0_2 - \lambda P^0_1 \in \mathcal{F}_1^2[\lambda] \) be a semisimple Poisson pencil of hydrodynamic type. A deformation of \( P^0_\lambda \) is given by \( P_\lambda = P_2 - \lambda P_1 \in \mathcal{F}_1^2[\lambda] \) with \( P_\lambda - P^0_\lambda \in \mathcal{F}_2^2 \) and \([P_\lambda, P_\lambda] = 0\). An infinitesimal deformation of \( P^0_\lambda \) is given by

\[
P^{\leq 2}_\lambda = P^0_\lambda + P^1_\lambda + P^2_\lambda, \quad P^{d-1}_\lambda = P^{d-1}_2 - \lambda P^{d-1}_1 \in \mathcal{F}_1^2[\lambda]
\]
such that

\[
[P^{\leq 2}_\lambda, P^{\leq 2}_\lambda] \in \mathcal{F}_{\geq 5}^3[\lambda].
\]

A deformation \( P_\lambda \) of \( P^0_\lambda \) extends an infinitesimal deformation \( P^{\leq 2}_\lambda \) if \( P_\lambda - P^{\leq 2}_\lambda \in \mathcal{F}_{\geq 4}^3[\lambda] \).

The extension problem can be stated as follows: _given an infinitesimal deformation \( P^{\leq 2}_\lambda \) of \( P^0_\lambda \), there exists a deformation \( P_\lambda \) extending \( P^{\leq 2}_\lambda \)?

As pointed out in [12] the fact that the second bihamiltonian cohomology groups \( BH^2_d(\mathcal{F},d_1,d_2) \) vanish for \( d \geq 4 \) and \( d = 2 \) but not for \( d = 3 \), implies that the vanishing of \( BH^3_{\geq 6}(\mathcal{F},d_1,d_2) \) guarantees that any infinitesimal deformation \( P^{\leq 2}_\lambda \) can be extended to a full deformation \( P_\lambda \).

This result, expressed in the \( \delta \)-formulation of local bivectors, is Theorem [2]

To see that the vanishing of \( BH^3_{\geq 6}(\mathcal{F}) \) implies that the deformations are unobstructed, consider that any infinitesimal deformation can be put in the form

\[
P^{\leq 2}_1 = P^0_1, \quad P^{\leq 2}_2 = P^0_2 + P^2_2
\]

by an appropriate Miura transformation. Clearly \( P^2_2 \) is in the kernel of both \( d_1 \) and \( d_2 \), hence identifies an element of \( BH^2_2(\mathcal{F}) \). Let us look for a deformation of the form

\[
P_1 = P^0_1, \quad P_2 = P^0_2 + P^2_2 + P^4_2 + P^6_2 + \ldots
\]
Let us first show that exist a term $P_{42}$ such that
\[
d_{1}P_{42} = 0, \quad d_{2}P_{42} + \frac{1}{2}[P_{22}, P_{22}] = 0.
\] (5)

Clearly $[P_{22}, P_{22}] \in \hat{F}_{4}^1$ is in Ker $d_{1} \cap$ Ker $d_{2}$, hence the vanishing of $BH_{\hat{F}}^3(\hat{F})$ implies that
\[
d_{1}d_{2}Q_{1} = \frac{1}{2}[P_{22}, P_{22}]
\]
for some $Q_{1} \in \hat{F}_{4}^1$. Then $P_{42} = d_{1}Q_{1}$ gives a solution of (5).

At the next step of the deformation we want to find $P_{62}$ such that
\[
d_{1}P_{62} = 0, \quad d_{2}P_{62} + [P_{22}, P_{42}] = 0.
\] (6)

As before $[P_{22}, P_{42}] \in \hat{F}_{8}^3$ is in Ker $d_{1} \cap$ Ker $d_{2}$, hence the vanishing of $BH_{\hat{F}}^{3}(\hat{F})$ implies that there is an element $Q_{2} \in \hat{F}_{6}^1$ s.t. $d_{1}d_{2}Q_{2} = [P_{22}, P_{42}]$. Then setting $P_{62} = d_{1}Q_{2}$ gives us the required solution of (6).

Since the vanishing of $BH_{\hat{F}}^{3}(\hat{F})$ ensures that this procedure can be continued indefinitely, as proved by induction in [12], the existence of a full deformation extending the infinitesimal deformation $P_{\lambda}^{\leq 2}$ indeed follows.

Notice that this in particular implies that any deformation can be put in the form (4), i.e., with the first Poisson tensor undeformed and the second one having only odd standard degree terms. This fact was also proved independently of the vanishing of the third bihamiltonian cohomology in [6].

### 3. Filtrations and spectral sequences

In this Section we give a proof of Theorem 4. It is a rather technical argument: basically, we introduce a sequence of filtrations and study the associated spectral sequences in order to show the vanishing of the cohomology in some degrees.

#### 3.1. Strategy of the proof of Theorem 4

Before we start with the technical details, let us make some general remarks about the strategy of the proof. The theorem states that the cohomology of the complex $(\hat{A}[\lambda], D_{\lambda})$ vanishes in certain degrees $(p, d)$. We prove this vanishing by introducing several spectral sequences associated to certain filtrations.

Central to this derivation is the following simple principle: suppose we have a cochain complex $(C, d)$ with a bounded decreasing filtration
\[
\ldots \subset F^{p+1}C \subset F^pC \subset \ldots
\]
The associated spectral sequence is bounded and converges to
\[
E_{1}^{p,q} = H^{p+q}(F^{p}C/F^{p+1}C, d_{0}) \Rightarrow H^{p+q}(C, d),
\]
where $d_{0}$ is the induced differential on the zeroth page, i.e., the associated graded complex. Suppose now that $H^{p}(E_{k}, d_{k}) = 0$ for some $k \geq 0$. Since all higher pages are iterated subquotients of $E_{k}$, we see that in this case $H^{p}(C, d) = 0$.

In the proof we will apply this principle inductively: we introduce a filtration on the complex $(\hat{A}[\lambda], D_{\lambda})$ which induces a spectral sequence. To show the vanishing of the cohomology of the first page $E_{1}$, we introduce another filtration on this page, which induces a spectral sequence converging to the
second page $E_2$ of the previous spectral sequence. In this way, we apply in total a sequence of three filtrations. For the convenience of the reader, we list them below:

1. The first filtration is associated with the degree of monomials in $\hat{A}[λ]$ in the variables $u^{i,s}$, $s \geq 1$, i.e. we assign degree 1 to each $u^{i,s}$, $s \geq 1$. The differential on the zeroth page of the associated spectral sequence is the part of $D_λ$ that preserves this degree, whereas on the first page it is the part that decreases it by 1.

2. On the first page of the spectral sequence above, we consider the filtration given by the degree in $θ^i_1$, for all $i = 1, \ldots, n$, i.e., this time $\deg(θ^i_1) = 1$, $i = 1, \ldots, n$. On the zeroth page, the differential is given by the part of the differential at 1 (at the first page) that increases the number of $θ^i_1$ by 1.

3. The complex on the zeroth page in 2 splits as direct sum of complexes $\hat{C}_i$, $i = 1, \ldots, n$. To compute the cohomology of the subcomplex $\hat{C}_i$, we filter by the degree of monomials in $θ^i_1$, where this time $i$ is fixed. On the zeroth page of the spectral sequence we finally find a complex of which we can prove the vanishing of the cohomology in the relevant degrees.

Having obtained the vanishing of the cohomology in 3, we apply the principle stated above to argue that the first page of the spectral sequence in 2 vanishes in certain degree. The same argument gives the vanishing of the second page of the spectral sequence in 1, which in turn proves the vanishing of the cohomology of the original complex $(\hat{A}[λ], D_λ)$ in certain degrees, i.e. Theorem 4.

3.2. Grading and subcomplexes. Since $θ^i_q$ are odd variables, we have a restriction on possible gradings $p$ and $d$ on the space $\hat{A}$. Indeed, the minimal possible standard degree $d$ of a monomial in $θ^i_q$ of super degree $p = nq + r$, $r < n$ is the degree of $θ^i_0 \cdots θ^i_0 \cdots θ^i_{q-1} θ^i_q \cdots \theta^i_q$ equal to $n(0 + \cdots + (q - 1)) + rq = nq(q - 1)/2 + rq$. So, the finitely generated $C^\infty(U)$-module $A_p^d$ is zero for

$$d < nq(q - 1)/2 + rq. \quad (7)$$

Moreover, the operators $D_1$ and $D_2$ (and, correspondingly, $d_1$ and $d_2$) are of bi-degree $(p, d)$ equal to $(1, 1)$. Therefore, the difference $p - d$ is preserved by both operators, and this means that the space $\hat{A}$ can be seen as the completion of an infinite direct sum of subcomplexes indexed by difference $d - p = -n, -n + 1, \ldots$. The inequality (7) implies that each of these subcomplexes is finite, and consequently we can compute the cohomology of each of them separately. For this general reason all the filtrations and spectral sequences introduced below will be bounded, and consequently the spectral sequences will converge in a finite number steps.

3.3. The first filtration. We now introduce the main filtration of the space $A[λ]$ in our computation. For this we consider the degree of monomials in $A[λ]$ in the variables $u^{i,t}$, $i = 1, \ldots, n$, $t \geq 1$. More formally, we introduce
a new grading on $\hat{A}[\lambda]$ by declaring

$$\tilde{\deg}(u^{i,t}) = 1, \quad i = 1, \ldots, n, \ t \geq 1,$$

$$\tilde{\deg}(\theta^t_i) = 1, \quad i = 1, \ldots, n, \ t \geq 0.$$ 

On the $\theta$-variables, this is just the ordinary super degree introduced in section 2.1. With this notation we can define

$$F^r \hat{A}[\lambda] := \{ f \in \hat{A}[\lambda], \ \tilde{\deg}(f) \geq r \}.$$ 

Clearly, this defines a decreasing filtration

$$\cdots \subset F^2 \hat{A}[\lambda] \subset F^1 \hat{A}[\lambda] \subset F^0 \hat{A}[\lambda] = \hat{A}[\lambda].$$

Consider now a monomial $f \in \hat{A}[\lambda]$ of degree $\tilde{\deg}(f) = r$, and write $q$ for the degree in $u^{i,t}$, $i = 1, \ldots, n$, $t \geq 1$ and $p$ for the degree in $\theta^t_i$, $i = 1, \ldots, n$, $t \geq 0$, so that $p + q = r$. We see from equation (3) that the differential $D_\lambda$ increases $p$ by 1 and changes the degree $q$ to some degree $\tilde{q} \geq q - 1$. Therefore, the sum $p + q$ is either preserved or increased by the differential, in other words, $(F^\bullet \hat{A}[\lambda], D_\lambda)$ is a filtered complex. This filtration, when restricted to each subcomplex with fixed difference $d - p$ is bounded.

We now split the differential as

$$D_\lambda = \Delta_{-1} + \Delta_0 + \ldots,$$

where where $\Delta_{-1}$ decreases $q$ by 1, $\Delta_0$ preserves $q$, and $\Delta_t, t \geq 1$, increases $q$ by $t$. Explicitly, we see from equation (3), that there is only one term that lowers degree $q$, so that we have:

$$\Delta_0 = \sum_{s \geq 1} (-\lambda + q^t_i + s) \frac{\partial}{\partial u^{i,s}}.$$  \hspace{1cm} (8)
The terms in $\Delta$ that preserve the degree $q$ are:

$$\Delta_0 = (-\lambda + u^i) f^i \theta^1_{\theta^i} \partial_{\theta^i} + \sum_{s+a+b \geq 1; a,b \geq 0} (-\lambda + u^i) \partial f^i_{\theta^i} u^a \theta^{1+b}_{\theta^{1+b}} \partial_{\theta^{1+b}} + \sum_{s+a+b \geq 1; a,b \geq 0} \left( \frac{s}{b} \right) f^i_{u^a} \theta^{1+b}_{\theta^{1+b}} \partial_{\theta^{1+b}}$$

$$+ \frac{1}{2} \sum_{s+a+b \geq 1; a,b \geq 0} (-\lambda + u^i) \left( \frac{s}{b} \right) \partial_j f^i_{j} u^a \theta^{1+b}_{\theta^{1+b}} \partial_{\theta^{1+b}} + \frac{1}{2} \sum_{s+a+b \geq 1; a,b \geq 0} \left( \frac{s}{b} \right) f^i_{u^a} \theta^{1+b}_{\theta^{1+b}} \partial_{\theta^{1+b}}$$

$$- \frac{1}{2} \sum_{s+a+b \geq 1; a,b \geq 0} (-\lambda + u^i) \left( \frac{s}{b} \right) f^i_{j} \partial_j f^i_{j} u^a \theta^{1+b}_{\theta^{1+b}} \partial_{\theta^{1+b}} - \frac{1}{2} \sum_{s+a+b \geq 1; a,b \geq 0} \left( \frac{s}{b} \right) f^i_{u^a} \theta^{1+b}_{\theta^{1+b}} \partial_{\theta^{1+b}}$$

We now start the computation on the zeroth page of the spectral sequence. Since the filtration of $\hat{A}[\lambda]$ comes from a grading, we obviously have

$$\bigoplus_{p,q} F^{p,q}_0 := \bigoplus_{p,q} F^p \hat{A}^{p+q}[\lambda]/F^{p+1}\hat{A}^{p+q}[\lambda] \cong \hat{A}[\lambda],$$

but the induced differential is given by $\Delta_{-1}$. Therefore, we first compute the cohomology of $\Delta_{-1}$. Introduce the space

$$\hat{C} := C^\infty(U)[[\theta^0_1, \ldots, \theta^0_n, \theta^1_1, \ldots, \theta^1_n]],$$

together with

$$\hat{C}_i := \hat{C}[[u^{i,s}, \theta^{i+1}_s, s \geq 1]]].$$

On $\hat{C}_i$ we have the de Rham differential

$$\hat{d}_i = \sum_{s \geq 1} \theta^{i+1}_s \partial_{u^{i,s}}.$$
**Proposition 11.** The cohomology of $\Delta_{-1}$ is given by:

$$H(\hat{\mathcal{A}}[\lambda], \Delta_{-1}) \cong \hat{C}[\lambda] \oplus \bigoplus_{i=1}^{n} \text{Im} \left( \hat{d}_i : \hat{C}_i \to \hat{C}_i \right)$$

Before we start the proof of this proposition, as a preliminary step, observe that the cohomology of the de Rham complex $(\hat{C}_i, \hat{d}_i)$ is trivial in positive degree, i.e.,

**Lemma 12** ("Poincaré lemma").

$$H^k(\hat{C}_i, \hat{d}_i) = \begin{cases} C & k = 0, \\ 0 & k > 0. \end{cases}$$

**Proof.** A simple proof of this fact can be given in terms of an homotopy map, a procedure that we will use repeatedly in the following. For fixed $i = 1, \ldots, n$ and $s \geq 1$, let

$$h_{i,s} = \frac{\partial}{\partial \theta^{s+1}} \int du^{i,s},$$

where the integration constant is set to zero. We have

$$h_{i,s} \hat{d}_i + \hat{d}_i h_{i,s} = 1 - \pi_{u^{i,s}} \pi_{\theta^{s+1}}$$

where $\pi_p$ denotes the projection that sets the variable $p$ to zero. Given a cocycle $g \in C_i$ representing a cohomology class $[g]$, the previous formula implies that the same cohomology class can be represented by the cycle $\pi_{u^{i,s}} \pi_{\theta^{s+1}} g$. Repeating this process, we can kill all variables $u^{i,s}$, $\theta^{s+1}$ with $s \geq 1$, hence a cohomology class can be always represented by an element in $C$. Since $\text{Im} \hat{d}_i$ always contains $\theta^{s+1}$ with $s \geq 1$, no further simplification is possible. $\square$

**Proof of Proposition 11.** To prove Proposition 11 we introduce two homotopy maps.

Let $\sigma_i$ be the map that acts on a rational function of $\lambda$ by removing the polar part at $\lambda = u^i$. For a polynomial $p(\lambda)$ we have

$$\sigma_i \left( \frac{p(\lambda)}{\lambda - u^i} \right) = \frac{p(\lambda) - p(u^i)}{\lambda - u^i}.$$

Fix $i = 1, \ldots, n$ and $s \geq 1$ and let

$$h_{i,s} = \sigma_i \left( \frac{1}{u^i - \lambda} \frac{1}{\lambda - u^i} \int du^{i,s} \right).$$

Clearly $h_{i,s}$ defines a map on $\hat{\mathcal{A}}[\lambda]$ which satisfies

$$h_{i,s} d_0 + d_0 h_{i,s} = (1 - \pi_{u^{i,s}} \pi_{\theta^{s+1}})(1 - \pi_{\lambda - u^i}) +$$

$$+ \left( \sum_{j \geq 1} \frac{f^j}{f^i \theta^j \theta^{s+1}} \int du^{i,s} \frac{\partial}{\partial u^{j,t}} \pi_{\lambda - u^i} \right). \tag{9}$$

As above $\pi_p$ denotes the projection that sets the variable $p$ to zero, in particular $\pi_{\lambda - u^i}$ sets $\lambda$ to $u^i$. 

It follows from (9) that we can kill the dependence on all the variables \( u^{i,s}, \theta_i^{s+1} \) with \( i = 1, \ldots, n, \ s \geq 1 \), in the \( \lambda \)-dependent part of any cocycle. In other words a cohomology class \( \hat{g} \) can always be represented by a cocycle of the form \( g + \hat{g} \), where \( g \in \mathcal{C}[\lambda] \) and \( \hat{g} \in \hat{\mathcal{A}} \), and we require each monomial in \( \hat{g} \) to have a nontrivial dependence on the variables \( \theta_i^{s+1} \) for \( s \geq 1 \). Notice that \( g_1 \) is a cocycle by itself, and can no longer be simplified by quotienting by \( \text{Im} \, d_0 \).

Let us then consider the cocycle \( \tilde{g} \). Since \( \tilde{g} \) does not depend on \( \lambda \), it is in the kernel of \( d_0 \) iff \( d' \tilde{g} = 0 \) and \( d'' \tilde{g} = 0 \) where \( d_0 = d'' - \lambda d' \). Moreover, an element of the form \( d'' df \) for \( f \in \hat{\mathcal{A}} \) is in the image of \( d_0 \) and does not depend on \( \lambda \), since \( d_0 d'' f = d'' d' f \).

Let us define, for \( s,t \geq 1 \) and \( i \neq j \)
\[
\hat{d}_{i,s;j,t} = \frac{1}{w^i - w^j} \frac{1}{f^i f^j} \frac{\partial}{\partial \theta_i^{s+1}} \frac{\partial}{\partial \theta_j^{t+1}} \int d\hat{u}^{i,s} \int d\hat{u}^{j,t}.
\]

We have
\[
[h_{i,s;j,t}, d'' df] = (1 - \pi_{u^i} \pi_{\theta_i^{s+1}})(1 - \pi_{u^j} \pi_{\theta_j^{t+1}}) + (\ldots) d' + (\ldots) d''.
\]

We did not specify the last two terms since the vanish when applied on elements in \( \text{Ker} d' \cap \text{Ker} d'' \).

It follows that, up to elements in \( \text{Im} d'' \), a cocycle \( \tilde{g} \) is equivalent to the cocycle
\[
(\pi_{u^i} \pi_{\theta_i^{s+1}} + \pi_{u^j} \pi_{\theta_j^{t+1}} - \pi_{u^i} \pi_{\theta_i^{s+1}} \pi_{u^j} \pi_{\theta_j^{t+1}}) \tilde{g}.
\]

This amounts to removing all monomials that contain any quadratic term of the form \( \theta_i^{s+1} \theta_j^{t+1} \) for \( i \neq j, s,t \geq 1 \). Hence \( \tilde{g} \) can be written as sum of \( g_i \in \hat{C}_i \) for \( i = 1, \ldots, n \), where \( g_i \in \text{Ker} \hat{d}_i \) and each monomial in \( g_i \) has a nontrivial dependence on the variables \( \theta_i^{s+1} \) for \( s \geq 1 \). Because of the vanishing of the cohomology \( H(\hat{C}_i, \hat{d}_i) \) we have that \( g_i \in \text{Im} \hat{d}_i \).

To complete the proof we have to show that we cannot further quotient by elements in \( \text{Im} d_0 \) without spoiling the form of \( \tilde{g} \). In principle the cocycle \( g_i \in \text{im} \hat{d}_i \) could be quotiented by an element of the form \( d_0 f(\lambda) \in C_i \) for \( f(\lambda) \in \hat{\mathcal{A}}[\lambda] \). It is easy to see that \( f \in C_i[\lambda] \), otherwise the image \( d_0 f \) would also depend on some variables \( u^{j,s}, \theta_j^{s+1} \) for \( j \neq i \). We have then
\[
d_0 f = (u^i - \lambda) f^i \sum_{s \geq 1} \theta_i^{s+1} \frac{\partial f}{\partial u^{i,s}}.
\]

But such element can be independent of \( \lambda \) iff it vanishes. Proposition 11 is proved.

The above proof of Proposition 11 gives us more than just the computation of the cohomology of \( \Delta_{-1} \), namely it gives us—in principle—the full homotopy retract data:
\[
\kappa \left( \hat{\mathcal{A}}[\lambda], \Delta_{-1} \right) \xrightarrow{p} (\hat{\mathcal{B}}, 0)
\]
where
\[
\hat{\mathcal{B}} := \hat{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^n \text{Im} \left( \hat{d}_i : \hat{C}_i \to \hat{C}_i \right), \quad (10)
\]
and $i$ and $p$ are cochain maps, of which $i$ is a quasi-isomorphism, and $h$ is a degree one map satisfying

$$
p \circ i = 1,
\quad i \circ p = \Delta_{-1} h + h \Delta_{-1}.
$$

In our case, $i$ is just the obvious inclusion $i : \hat{B} \hookrightarrow \hat{A}[\lambda]$, and $p$ is is a projection operator. The explicit description of the projection operator $p$ and the homotopy $h$ is more complicated than $i$, but fortunately, we don’t need that in general when considering the differential on the first page of the spectral sequence: this differential is simply given by $\Delta'_0 = p \Delta_0 i$. In general, we can decompose

$$\hat{A}[\lambda] = \hat{C}[\lambda] \oplus \bigoplus_{i=1}^{n} \hat{C}_i[\lambda] \oplus \hat{C}[[\text{mixed terms in } u^{i,s}, \theta^{s+1}_i, \ s \geq 1]]]. \quad (11)$$

Then the projection is simply the identity on the first component and zero on the last component. On the second component it is more involved to describe as it involves the homotopy $h$, but on the subspace

$$\text{Im} \left( \hat{d}_i : \hat{C}_i[\lambda] \to \hat{C}_i[\lambda] \right) \subset \hat{C}_i[\lambda],$$

the projection is equal to evaluating a polynomial in $\lambda$ at $\lambda = u_i$.

### 3.4. The second filtration.

In the previous section we have seen that on the first page of the spectral sequence, we are left with the problem of computing the cohomology of the operator $\Delta'_0 := p \Delta_0 i$ on the space $\hat{B}$. However, instead of considering the full operator $\Delta'_0$, we split this operator into two summands: $\Delta'_0 = \Delta'_{00} + \Delta'_{01}$, where $\Delta'_{01}$ increases the number of factors of $\theta^1_i$ by 1 and $\Delta'_{00}$ leaves it invariant. This split can be achieved by an auxiliary filtration

$$\ldots \subset F^2 \hat{B} \subset F^1 \hat{B} \subset F^0 \hat{B} = \hat{B},$$

given by

$$F^k \hat{B} := \{ f \in \hat{B}, \text{ with } \# \theta^1_i \geq k \}.$$ Again, this turns $(\hat{B}, \Delta'_0)$ into a filtered complex and on the zeroth page of the induced spectral sequence we have to compute the cohomology of $\Delta'_{01}$ first.
Lemma 13. The differential $\Delta'_{01}$ is given by

$$
\Delta'_{01} = (-\lambda + u^i) f^i \theta^1_j \frac{\partial}{\partial u^i} + \sum_{s \geq 1} \left( \frac{s+2}{2} f^i u^i \theta^1_j \frac{\partial}{\partial u^{i,s}} \right) \\
- \frac{1}{2} \sum_{s \geq 1} (-\lambda + u^j) sf^i f^j \frac{\partial}{\partial u^{i,s}} \cdot \frac{\partial}{\partial u^{j,s}} + \frac{1}{2} \sum_{s \geq 1} f^i u^i \theta^1_j \frac{\partial}{\partial u^{i,s}} \\
+ \frac{1}{2} \sum_{s \geq 0} (-\lambda + u^j) \frac{\partial}{\partial u^{i,s}} \cdot \frac{\partial}{\partial u^{j,s}} - \frac{1}{2} \sum_{s \geq 1} (-\lambda + u^i) \frac{\partial}{\partial u^{i,s}} \cdot \frac{\partial}{\partial u^{j,s}} \\
+ \sum_{s \geq 0} (-\lambda + u^i) \frac{\partial}{\partial u^{i,s}} \cdot \frac{\partial}{\partial u^{j,s}}.
$$

Proof. As explained above, we have $\Delta'_{01} = p \Delta_{01}$, with $\Delta_0$ as in equation (8) and the inclusion $i$ and projection $p$ as described above. We first collect from (8) all the terms $\Delta_0$ that increase the number of $\theta^1_i s$:

$$
\Delta_{01} = (-\lambda + u^i) f^i \theta^1_j \frac{\partial}{\partial u^i} + \sum_{s \geq 1} \left( \frac{s+2}{2} f^i u^i \theta^1_j \frac{\partial}{\partial u^{i,s}} \right) \\
= \frac{1}{2} \sum_{s \geq 1} (-\lambda + u^i) sf^i f^j \frac{\partial}{\partial u^{i,s}} \cdot \frac{\partial}{\partial u^{j,s}} + \frac{1}{2} \sum_{s \geq 1} f^i u^i \theta^1_j \frac{\partial}{\partial u^{i,s}} \\
+ \frac{1}{2} \sum_{s \geq 0} (-\lambda + u^j) \frac{\partial}{\partial u^{i,s}} \cdot \frac{\partial}{\partial u^{j,s}} - \frac{1}{2} \sum_{s \geq 1} (-\lambda + u^i) \frac{\partial}{\partial u^{i,s}} \cdot \frac{\partial}{\partial u^{j,s}} \\
+ \sum_{s \geq 0} (-\lambda + u^i) \frac{\partial}{\partial u^{i,s}} \cdot \frac{\partial}{\partial u^{j,s}}.
$$

In terms of this operator, we have by definition that $\Delta'_{01} = p \Delta_{01}$. Next, observe that by the property of being a multicomplex, we have that $[\Delta_{-1}, \Delta_0] = 0$. Decomposing $\Delta_0 = \Delta_{00} + \Delta_{01}$ using the $\theta^1$-filtration, we see that

$$
[\Delta_{-1}, \Delta_{00}] = 0, \quad [\Delta_{-1}, \Delta_{01}] = 0,
$$

separately, because $\Delta_{-1}$ has degree 0 with respect to this filtration. The second equation shows that since $i$ maps to the kernel of $\Delta_{-1}$, the projection
acting on $\hat{C}_i[\lambda]$ is always given by setting $\lambda = u_i$ in the computation of $\Delta'_0 := p\Delta_0$. Let us now determine which of the terms in the formula for $\Delta'_{01}$ above are mapped to zero when composing with $p$. For example, the second term, when evaluated on the image (10) of $i$, gives either 0 on $\hat{C}[\lambda]$, or maps $\hat{C}_i$ to the third summand in (11). Therefore this term gives no contribution to $\Delta'_0$. The same reasoning shows that the fourth and sixth term do not contribute. The eight term vanishes identically for $s = 1$ and for $s \geq 2$ either maps $\hat{C}_i$ to the third summand in (11), in which case applying $p$ gives zero, or maps $\hat{C}_i$ to $\hat{C}_j$ so that setting $\lambda = u_j$ yields zero. Therefore only the $s = 0$ term contributes.

For the last two summands, observe that the operator $(-\lambda + u^s)\theta^s$, $s \geq 2$ acting on $\hat{C}_j$ gives zero when composing with $p$. Collecting all the remaining terms, we obtain the expression stated in the Lemma. □

3.5. The third filtration. We have arrived at the problem of computing the cohomology of the operator $\Delta'_{01}$ given in Lemma 13 on the space $\hat{B}$ defined in (10). In this section we introduce our final filtration to estimate the possible non-vanishing $(p, d)$-degrees in the cohomology of $\Delta'_{01}$.

We start with the observation that $\Delta'_{01}$ preserves the splitting (10) of the cohomology $\hat{B}$ of $\Delta_{-1}$ in $n + 1$ summands, namely:

$$\Delta'_{01} \left( \hat{C}[\lambda] \right) \subset \hat{C}[\lambda]$$

This means that we can estimate its cohomology on each of these $n + 1$ spaces separately.

The possible $(p, d)$-degrees of the elements of $\hat{C}[\lambda]$ are those of monomials $\theta^{i_1}_{\lambda_1} \cdots \theta^{i_k}_{\lambda_k} \theta^{j_1}_{\lambda_1} \cdots \theta^{j_k}_{\lambda_k}$, $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_k \leq n$. So, for the standard gradation $d$ we have $0 \leq d \leq n$, and, if we fix $d$, then for the super gradation $p$ we have $d \leq p \leq d + n$. So, without any further computation we see that these are the restrictions for the possible $(p, d)$-degrees of the cohomology of $\Delta'_{01}$. This gives us Case 1 in the statement of Theorem 4.

Now we want to estimate the cohomology of $\Delta'_{01}$ in each of the spaces $\text{Im} \left( \hat{d}_i : \hat{C}_i \to \hat{C}_i \right) [\lambda]/(u^i - \lambda)$, $i = 1, \ldots, n$. For the rest of this Section the index $i$ is fixed and refers to the particular space that we consider.

We represent the operator $\Delta'_{01}$ on $\text{Im} \left( \hat{d}_i : \hat{C}_i \to \hat{C}_i \right)$ in the following way:

$$\Delta'_{01} = \sum_{k=1}^n \theta^1_k D_k,$$
where

\[ D_k := (u^k - u^i) f^k \frac{\partial}{\partial u^k} - \sum_{j=1}^{n} (u^k - u^i) f^k \theta_1^j \frac{\partial}{\partial \theta_1^j} + \sum_{s \geq 1} \frac{s + 2}{2} f^k u^k,s \frac{\partial}{\partial u^k,s} - \frac{1}{2} \sum_{s \geq 1} \sum_{j=1}^{n} (u^k - u^i) s f^k \theta_1^j \frac{\partial}{\partial \theta_1^j} u^k,s \frac{\partial}{\partial u^i,s} + \frac{1}{2} \sum_{s \geq 2} f^k (s - 1) \theta_1^s \frac{\partial}{\partial \theta_1^s} f^k \theta_1^i \frac{\partial}{\partial \theta_1^i} - \frac{1}{2} \sum_{j=1}^{n} (u^k - u^i) \partial_j f^k \theta_1^j \frac{\partial}{\partial \theta_1^j} f^k \theta_1^i \frac{\partial}{\partial \theta_1^i} + \frac{1}{2} \sum_{j=1}^{n} (u^j - u^i) f^j \theta_1^j \frac{\partial}{\partial \theta_1^j}.
\]

In particular,

\[ D_i := \sum_{s \geq 1} \frac{s + 2}{2} f^i u^i,s \frac{\partial}{\partial u^i,s} + \sum_{s \geq 2} \frac{s - 1}{2} f^i \theta_1^s \frac{\partial}{\partial \theta_1^s} - \frac{1}{2} f^i \theta_1^0 \frac{\partial}{\partial \theta_1^0} + \frac{1}{2} \sum_{j=1}^{n} (u^j - u^i) f^j \theta_1^j \frac{\partial}{\partial \theta_1^j} f^k \theta_1^i \frac{\partial}{\partial \theta_1^i} \] (12)

As a first step toward computation of the cohomology of \( \Delta_{u_1} \) on the space \( \text{Im} \left( \hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i \right) \) \( \left[ \lambda \right] / (u^i - \lambda) \), we introduce the filtration with respect to the degree of \( \theta_1^i \), in the same way as we used the general \( \theta_1^i \)-filtration before. On the zero page of the associated spectral sequence we have to compute the cohomology of \( \theta_1^i D_i \) on \( \text{Im} \left( \hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i \right) \). The restrictions on the possible non-trivial \( (p,d) \)-degrees that we get in the cohomology of \( \theta_1^i D_i \) on \( \text{Im} \left( \hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i \right) \) are sufficient for the proof of Theorem 4.

**Proposition 14.** For the cohomology of the differential \( \theta_1^i D_i \) on the space \( d_i(\hat{\mathcal{C}}_i) \subset \mathcal{B} \) we have:

\[ H^p_d \left( d_i(\hat{\mathcal{C}}_i), \theta_1^i D_i \right) = 0, \]
unless \( d = 2, 3, \ldots, n + 2, \ p = d, d + 1, \ldots, d + n - 1 \).

**Proof.** In order to compute the cohomology of \( \theta^1_i D_i \) on \( \text{Im} \left( \hat{d}_i : \hat{C}_i \to \hat{C}_i \right) \), we represent this space as a direct sum of subcomplexes indexed by all possible monomials, which is always a positive half-integer, and the gradation is shifted by \((\theta^1_i \cdot D_i) \cdot \hat{d}_i(m)\) into a direct sum of sub-complexes indexed by all monomials in \( \theta_i^{>1} \) and \( \theta_i^{>2} \). Let us denote the set of all non-trivial monomials in these variables by \( M \). Then we have:

**Lemma 15.** The isomorphism

\[
\text{Im} \left( \hat{d}_i : \hat{C}_i \to \hat{C}_i \right) := \bigoplus_{m \in M} \hat{C} \cdot \hat{d}_i(m)
\]

is preserved by the differential \( \theta^1_i D_i \), and decomposes the cochain complex \( (\text{Im}(\hat{d}_i), \theta^1_i D_i) \) into a direct sum of sub-complexes indexed by all monomials in the variables \( \theta_i^{>1} \) and \( \theta_i^{>2} \).

**Proof.** First, a short computation shows that

\[
[D_i, \hat{d}_i] = f^i \hat{d}_i.
\]

Next, consider a monomial \( m \in M \) in the variables \( u_i^{>1} \) and \( \theta_i^{>2} \). Then it follows from the identity above that

\[
\theta^1_i D_i(\hat{d}_i(m)) = (\theta^1_i D_i(C) \cdot \hat{d}_i(m)) + C \cdot \theta^1_i D_i(\hat{d}_i(m))
\]

\[
= (\theta^1_i D_i(C) \cdot \hat{d}_i(m)) + C \cdot \theta^1_i \hat{d}_i(D_i + f^i)(m).
\]

We see from equation (12) that \( (\theta^1_i D_i(C)) \subset C \) and that \( (D_i + f^i) \) acts on \( m \) by multiplication with a scalar \( (\text{depending on } m) \). This shows that \( \theta^1_i D_i \) preserves the subspace \( \hat{C} \cdot \hat{d}_i(m) \) and proves the Lemma.

Because of this Lemma, we can consider an infinite direct sum of complexes, each of which is isomorphic, as a bi-graded vector space, to \( C \) with shifted \((p, d)\) gradation. The gradation is shifted by \((p_m + 1, d_m + 1)\), where \((p_m, d_m)\) in the gradation of \( m \in M \) (and, therefore, \((p_m + 1, d_m + 1)\) is the gradation of \( \hat{d}_i(m) \)).

Let us discuss the action of \( \theta^1_i D_i \). Observe that this operator is linear over the ring of functions in \( u^1, \ldots, u^n \) and \( \theta^1_j, j \neq i \). So, we omit the coefficients from this ring in the computations below, assuming that there can be an arbitrary coefficient that would be preserved.

We denote by \( C(m) \) the eigenvalue of the operator

\[
\sum_{s \geq 1} \frac{s + 2}{2} u^{i,s} \frac{\partial}{\partial u^{i,s}} + \sum_{s \geq 2} \frac{s - 1}{2} \theta^1_i \frac{\partial}{\theta^1_i}
\]

on \( \hat{d}_i(m) \). Observe that \( C(m) \) is always a positive half-integer, and the minimal value of \( C(m) \) is equal to \( 1/2 \) for \( m = u^{i,1} \).

Consider monomials \( \theta^0_j \cdots \theta^0_j \) such that \( 1 \leq j_1 < \cdots < j_\ell \leq n \), and \( j_k \neq i \) for all \( k = 1, \ldots, \ell \). We have:

\[
\theta^1_i D_i: \theta^0_j \cdots \theta^0_j \hat{d}_i(m) \mapsto f^i C(m) \theta^1_i \theta^0_j \cdots \theta^0_j \hat{d}_i(m)
\]

So, since \( C(m) \neq 0 \), we see the subspace spanned by the elements \( \theta^0_j \cdots \theta^0_j \hat{d}_i(m) \) and \( \theta^1_i \theta^0_j \cdots \theta^0_j \hat{d}_i(m) \) forms an acyclic subcomplex of \( \hat{C} \cdot \hat{d}_i(m) \).
Now we consider monomials \( \theta^0_{j_1} \cdots \theta^0_{j_\ell} \) such that \( 1 \leq j_1 < \cdot \cdot \cdot < j_\ell \leq n \), and \( j_k \neq i \) for all \( k = 1, \ldots, \ell \). Modulo the acyclic subcomplex that we introduced in the previous paragraph, we have:

\[
\theta^1_i \mathcal{D}_i \colon \theta^0_{j_1} \cdots \theta^0_{j_\ell} \hat{d}_i(m) \mapsto f^i \left( C(m) - \frac{1}{2} \right) \theta^0_{j_1} \cdots \theta^0_{j_\ell} \hat{d}_i(m).
\]

Note that \( C(m) - \frac{1}{2} \) is not equal to zero if \( m \neq u^{i,1} \). Therefore, if \( m \neq u^{i,1} \), then the quotient of \( \hat{C} \cdot \hat{d}_i(m) \) modulo an acyclic subcomplex is an acyclic subcomplex, and so \( \hat{C} \cdot \hat{d}_i(m) \) is acyclic.

The only possible case when we can have non-trivial cohomology is the case of \( m = u^{i,1} \), that is, the case of the complex \( \hat{C} \cdot \theta^2_i \). In this case, after taking the quotient modulo the acyclic subcomplex, the cohomology of \( \theta^1_i \mathcal{D}_i \) is represented by a product of \( \theta^0_{j_1} \theta^2_i \) by an arbitrary function in \( u^1, \ldots, u^n, \theta^0_1, \ldots, \theta^0_n \) (\( \theta^0_i \) is omitted), and \( \theta^0_{j_1}, \ldots, \theta^0_{j_\ell} \). This means that we have non-trivial cohomology only for gradation \( d = 2, 3, \ldots, 2 + n \), and once we fixed \( d \), the possible values of gradation \( p \) are \( d, d + 1, \ldots, d + n - 1 \). This proves the Proposition. \( \square \)

**Proof of Theorem 4.** Recall from Remark 6 that we have to prove the vanishing of the bihamiltonian cohomology, except for the two cases 1 and 2 specified in that remark. The computations in this subsection show that the cohomology of \( \Delta_{01} \) vanishes, unless \( d = 0, 1, \ldots, n; p = d, d + 1, \ldots, d + n \) (Case 1, coming from the sub-complex \( \hat{C}[\lambda] \)), or \( d = 2, 3, \ldots, n + 2, p = d, d + 1, \ldots, d + n - 1 \) (Case 2, coming from the sub-complex \( \hat{d}_i(\hat{C}) \)). Going back via the second to the first spectral sequence, we conclude the vanishing of the cohomology groups of the complex \( (\hat{A}[\lambda], \mathcal{D}_\lambda) \) in the same \((p,d)\)-degrees: this is exactly the statement of Theorem 4. \( \square \)

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