Publicly verifiable quantum money from random lattices

Andrey Boris Khesin,1,* Jonathan Z. Lu,2,† and Peter W. Shor1,‡

1 Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA
2 Department of Physics, Harvard University, Cambridge, MA

(Dated: August 15, 2022)

Publicly verifiable quantum money is a protocol for the preparation of quantum states that can be efficiently verified by any party for authenticity but is computationally infeasible to counterfeit. We develop a cryptographic scheme for publicly verifiable quantum money based on Gaussian superpositions over random lattices. We introduce a verification-of-authenticity procedure based on the lattice discrete Fourier transform, and subsequently prove the unforgeability of our quantum money under the hardness of the short vector problem from lattice-based cryptography.

Quantum money is a multiparty quantum cryptographic protocol wherein one fixed participant—the mint—can create quantum states which all participants can verify but cannot duplicate. Specifically, the mint may generate the quantum money state $|s⟩$ associated with a serial number $s$, which it releases publicly along with any other relevant information. Any party with the public information can then certify, with a quantum computer, the authenticity of $|s⟩$ in polynomial time up to superpolynomially small error without changing the state by more than a superpolynomially small amount. Moreover, no party in possession of $|s⟩$ can efficiently create two states that both pass the verification test for $|s⟩$. That is, the quantum money is counterfeiting-secure.

The first proposal of quantum money is due to Wiesner [1], but his scheme did not admit public verifiability; rather, money states had to be sent to a central authority for authentication. Since Wiesner’s quantum money, many novel algorithms have been proposed [2–7]. Of these, several [2, 3, 6] have been fully or partially broken [8–12]. The historical vulnerability of quantum money schemes motivates rigorously basing the protocol on a cryptographically secure problem, one that is widely believed to be robust even against a quantum adversary. The most prominent class of such problems are those of lattice-based cryptography, conjectured to be post-quantum secure. We introduce a quantum money protocol based on the short vector problem (SVP) on lattices, a well-studied computational challenge problem in classical cryptography [13]. In particular, our protocol uses a random lattice that contains one known short vector, and we show that anyone who can duplicate a quantum money state can find another, linearly independent, short vector in the lattice. This problem of finding a second short vector in a random lattice is equivalent to the short vector problem in a random lattice (see Appendix).

Broadly, quantum money may be classified into two groups: one where the mint must know a secret key (e.g., a short basis), and one where the mint can create money with no additional knowledge; that is, the money is openly mintable. In the second case, even the mint is unable to, with more than an exponentially small probability, produce two money states with the same serial number. Such a non-collision property is naturally implemented by the randomness of quantum measurements. Since anyone can produce money in the openly-mintable setting, the mint must use an unspoofable (e.g. protected by post-quantum secure public-key cryptographic protocols) channel to broadcast a valid list of serial numbers. Our money scheme can be implemented to fit in either category: the protocol falls into the former if the mint knows no short basis, and otherwise falls into the second, wherein the mint can create arbitrarily many copies of a bill with a fixed serial number. However, openly mintable schemes are more secure, for even the mint has no adversarial advantage.

Aside from its remarkable physical implications—an explicit example of a provably uncloneable quantum state—our quantum money also offers advantages unachievable by classical cryptocurrencies or physical bills. Since our money states are physical, they can serve as tangible yet unforgeable bills, but they could also be transferred through quantum channels as digital money. Moreover, verification of ownership can be done locally and offline, having no need for global synchronization through such mechanisms as blockchains.

Minting quantum money — A quantum money protocol must have efficiently preparable money states, efficient public authentication, and unforgeability. Our scheme uses a lattice $L$ over $\mathbb{Z}_P$, where $P$ is a large prime, with $\lambda$ the shortest separation between lattice vectors. For a random lattice, $\lambda$ is of the order $\sqrt{d}P^{1/d}$ [14]. We define our dual lattice $L^\perp$ as the standard dual lattice, but scaled by a factor of $P$, so that both lattices are subsets of $\mathbb{Z}_P^d$. 


this substantially simplifies the intuition and analysis. Effectively, a vector \( v \) is in \( \mathcal{L}^\perp \) if for all \( x \in \mathcal{L} \), \( v \cdot x = 0 \), where the dot product is taken mod \( P \). The lattice \( \mathcal{L} \) has dimension \( d - 1 \), so the dual lattice \( \mathcal{L}^\perp \) is a one-dimensional lattice generated by a single vector. Finding short vectors in these lattices is as hard as finding short vectors in arbitrary lattices [15].

One can define a lattice discrete Fourier transform (LDFT) over these lattices [16] that takes points of \( \mathcal{L} \) to superpositions of points of \( \mathcal{L} \). This is related to the standard quantum Fourier transform; it is defined and analyzed rigorously in the Appendix.

The basis of our scheme is Gaussian superpositions of points \( x \in \mathcal{L} \) around a point \( u \in \mathcal{L}^\perp \): For a standard deviation \( \sigma \), the Gaussian superposition around \( u \) is given by

\[
|g(u)\rangle = \frac{P^{1/2}}{(2\pi \sigma^2)^{d/4}} \sum_{x \in \mathcal{L}} e^{-(x-u)^2/4\sigma^2} |x\rangle ,
\]

with a small correction to the normalization constant arising from the discrete nature of the lattice. The \( P^{1/2} \) factor comes from the fact that the determinant of the lattice matrix is \( P \). We use two sizes of Gaussian superpositions: small ones, with standard deviation \( \sigma \gg \sqrt{d} P^{1/d} \) (denoted \( |g(u)\rangle \)) and large ones, with standard deviation \( P/\sqrt{4\pi\sigma} \) (denoted \( |G(u)\rangle \)).

For large \( \sigma \), there is a polynomial-time algorithm to sample from a lattice-discretized Gaussian distribution centered around an arbitrary point \( u \) [17]. Note that for this algorithm to work, \( \sigma \) must be chosen to be comparable to the size of a known basis of the lattice. A straightforward adaptation of the classical algorithm generates a quantum superposition over the lattice, with the amplitudes Gaussian centered at \( u \). We choose our parameters so that this algorithm can be used to generate the large Gaussian superpositions \( |G(u)\rangle \); it can also generate the small Gaussian superpositions \( |g(u)\rangle \) if a small basis is known.

We now give some properties of these Gaussian superpositions that we will use. These are proved in the Appendix.

Taking the LDFT of a small Gaussian ball \( g(|u|) \) yields a superposition of all large Gaussian balls, with amplitude on \( |G(v)\rangle \) being \( e^{-2\pi i (u,v)/P} \). Similarly, taking the LDFT of a large Gaussian ball \( G(|u|) \) yields a superposition of small Gaussian balls. The formula for amplitudes is the same as the amplitude in a Fourier transform, and in essence, for a superposition of Gaussian balls, the LDFT implements a Fourier transform on the Gaussian balls; we will use this fact.

Our quantum money state is a superposition of Gaussian balls. Assuming the SVP is hard, it is impossible to create a specific superposition of Gaussian balls, but it is possible to create a random translate of a specific superposition. The vector the superposition is translated by then becomes the serial number of the money. If the LDFT of the superposition of Gaussian balls is measured in the canonical basis, the probability distribution of the measurement outcomes is independent of the translation vector.

For our quantum money, we use two clusters of \( 2k+1 \) Gaussian balls. When we take the LDFT to check the money, it creates a superposition of large Gaussian balls \( |G(jv)\rangle \), where most of the mass is contained in balls with \( j \) even and between \(-P/k\) and \( P/k\). We choose our lattice so we can use the second coordinate to distinguish between even and odd balls. To show that the money is secure, we show that if the counterfeit money state is a superposition of points in one cluster of Gaussian balls, it will not pass the verification test. Thus, if somebody has two states that both pass the verification test, they can measure a random point in each of them, and with non-negligible probability find one point from each cluster. We then show that these two points can be used to find a short vector.

The parameters of the protocol are a large prime \( P \), a lattice dimension \( d \in \mathbb{Z}^+ \), a standard deviation \( \sigma > 0 \), a cluster-size parameter \( k \in \mathbb{Z}^+ \), an odd-valued proximity parameter \( \Delta \in \mathbb{Z}^+ \), a security tolerance \( \epsilon \), and a spread \( t \). We choose \( t \) so the probability of sampling a normal distribution more than \( t \) standard deviations away from the mean is small; \( t = 3 \) is a reasonable value for this parameter. These parameters must satisfy the hierarchical relation

\[
\sqrt{d} P^{1/d} \ll \sigma \lesssim \Delta \lesssim k .
\]

We denote difference by at least a small multiplicative constant via \( \lesssim \) and that of a large amount via \( \ll \). In particular, we require \( 2t\sigma < \Delta \) and \( 3\Delta < k \).

For technical reasons that we discuss in the proof, we also require that

\[
k \ll 2^d \sqrt{d} P^{1/d} ,
\]

\[
k \lesssim t\sigma \Delta ,
\]

\[
\Delta \ll \sqrt{P} ,
\]

\[
2^d \sqrt{d} P^{1/d} \ll \frac{P}{4\pi\sigma} .
\]

It suffices to show that the probability that a valid atomic money state passes the verification procedure is greater than the failure probability by a constant,
for we may then take for any \( \epsilon > 0 \) a tensor product of a sufficiently large \( N_t \) atomic states to ensure the probability of a valid complete money state passing is \( 1 - \epsilon \). For the remainder of this letter, we therefore focus on atomic states.

To create the lattice, the mint will first choose a vector \( v \) such that \( v_1 = 1 \) and \( v_2 = \frac{p + \Delta}{2} \). By randomly sampling the remaining coordinates, it generates a random vector \( v \) which uniquely determines the dual lattice. It then randomly chooses orthogonal vectors to generate a basis for the corresponding lattice \( \mathcal{L} \). Note that this implies that the vector \( s^{(0)} = (\Delta, -2, 0, 0, \ldots, 0) \) is in \( \mathcal{L} \). We assume that \( \Delta \) is sufficiently small that \( s^{(0)} \) is a short vector in \( \mathcal{L} \), i.e., \( \Delta \ll 2^d \sqrt{d} P^{1/d} \). Moreover, there is a unique dual lattice vector \( w \) such that \( w \cdot v = 1 \).

Since \( 2^d \sqrt{d} P^{1/d} \ll \frac{P}{\lambda} \) from Eq. (6), the mint can apply the \( L^2 \) algorithm [18]—which finds a lattice basis of length \( \approx 2^d \sqrt{d} P^{1/d} \) and the Gaussian sampling algorithm to generate a large Gaussian ball around \( 0 \) with standard deviation \( \frac{P}{\lambda} \). The mint takes the LDFT (see Appendix), yielding a superposition of small Gaussian balls around every dual lattice point with standard deviation \( \sigma \):

\[
\mathcal{F}([G(0)]) = \frac{1}{\sqrt{\mathcal{P}}} \sum_{u \in \mathcal{L}^2} |g(u)|.
\]  

Next, the mint uses a technique known as bounded distance decoding (BDD) designed in [19, 20]. With it, if we are given \( |p\rangle \in \mathbb{Z}_P^d \) sufficiently close to a lattice vector (exponentially closer than \( \lambda \)), we can find the lattice vector \( \text{BDD}(|p\rangle) \) nearest to \( |p\rangle \). BDD can also be done coherently on a quantum computer, so long as we keep the vector \( |p\rangle \) in quantum memory. The mint applies BDD over the dual lattice to the LDFT of the state tensored with ancillary qubits, storing the nearest dual lattice point to each element of the Gaussian ball superposition to yield a state

\[
|\text{BDD}\rangle \propto \sum_{u \in \mathcal{L}^2} \sum_{x \in \mathcal{L}} e^{-(x-u)^2/4\sigma^2} |x\rangle |\text{BDD}(x)\rangle
\]

\[
\approx \sum_{u \in \mathcal{L}^2} \sum_{x \in \mathcal{L}} e^{-(x-u)^2/4\sigma^2} |x\rangle |u\rangle
\]  

where \( \approx \) indicates closeness in the \( L^2 \) norm. This allows the mint to determine the Gaussian ball to which each point in the superposition belongs, because the construction satisfies the BDD condition \( \sigma < 2^{-d} \sqrt{d} P^{1-1/d} \) from Eq. (6).

Finally, the mint will perform the following positive operator-valued measurement (POVM) to collapse the state into a superposition of two clusters of \( 2k + 1 \) Gaussian balls, which is the money state.

The mint first replaces the dual lattice point \( |u\rangle \) with an index \( |m\rangle \), where \( u = mw \), and then applies the POVM to the second register. The POVM is parameterized by \( T \), the index of the dual lattice center of the first cluster, and each element is of the form

\[
\sum_{j=-k}^{k} |T+j\rangle \langle T+j| + |CT+j\rangle \langle CT+j|,
\]

where \( CT \) represents the center of the second cluster and is chosen to be the nearest integer to \( T + \frac{P+1}{2} + t\sigma \Delta \) (mod \( P \)). The mint then uncomputes the ancillary qubits to unentangle them from the money state. The atomic money state \( |S_T\rangle \) is then complete, given by

\[
\sum_{j=-k}^{k} |g((T+j)w)) + |g((CT+j)w)|.
\]  

where \( T \) is the serial number.

Verifying quantum money — The verification protocol proceeds by analysis of the LDFT of \( |S_T\rangle \). The LDFT has a useful property when applied to the superposition of Gaussian balls: applying it to \( \sum_{j=0}^{P-1} e^{i\beta_j} |g(iw)\rangle \) gives \( \sum_{j=0}^{P-1} \alpha_j |G(jv)\rangle \), where the sequence \( [\alpha_j] \) is the standard Fourier transform of the sequence \( [\beta_j] \).

For each atomic state in the \( N \)-tensor product, the verifier first checks that it is a superposition of points in the lattice \( \mathcal{L} \). It then performs BDD again, storing the result in an auxiliary register. If in any of the atomic states, the dual lattice points are not within the two \( (2k + 1) \)-clusters computed from the serial number \( T \), the state is rejected. Next, using another auxiliary register, the verifier computes how many atomic states have lattice points within \( t\sigma \) of their nearest dual lattice point. If this number is too small, the verifier also rejects. Next, the verifier uncomputes the auxiliary registers. For an authentic money state, the probability of rejection is exponentially small, so the quantum state is still exponentially close to the original money state.

Next the verifier takes the LDFT of \( |S_T\rangle \), producing a superposition of large Gaussian balls around each dual lattice point. They then apply a delayed measurement (i.e. apply a controlled unitary, measuring at the end of the computation) to the money state to obtain a lattice point \( x \). If \( |x_2| \) is larger than some threshold, which we will compute later, then they reject. The acceptance or rejection of each of the \( N \) atomic money states is inputted coherently.
into a quantum threshold gate, which determines if the fraction of accepts is above 0.5. If so, the verifier accepts, and otherwise they reject. At the end they uncompute everything, which gives a state exponentially close to the original one, assuming that we started with authentic quantum money (see Un-computation in Appendix).

We now show that an authentic quantum money state has a high probability of passing this test. By Lemmas 5 and 6 in the Appendix, the probability of finding a Gaussian ball centered at a dual lattice point \( u \) with \( |u_1| < \frac{P}{2} \) is at least around \( 1 - \frac{1}{e^2} \), or 0.89.

If \( |u_1| < \frac{P}{k} \), we show that with probability close to 1, \( u_1 \) is even. Suppose that the two clusters of Gaussian balls were shifted by exactly \( \frac{P+1}{2} \). We claim that for \( j \) small, the Gaussian balls \( |G(jv)| \) for \( j \) odd would undergo destructive interference, and thus be unlikely to be observed after the LDFT, while the Gaussian balls with \( j \) even undergo constructive interference. This is because for small odd \( j \), \( e^{2\pi i j(P+1)/2P} \approx -1 \) and for small even \( j \), \( e^{2\pi i j(P+1)/2P} \approx 1 \). For the actual offset of \( \frac{P+1}{2} + t\sigma \Delta \), the probability of seeing an even \( j \) is approximately \( \cos^2(\frac{\pi \Delta}{P}) \) and the probability of \( j \) odd is \( \sin^2(\frac{\pi \Delta}{P}) \). These are close to 1 and 0, respectively, by Eq. (5).

We know the ratio \((\text{mod } P)\) between the first two coordinates of any dual lattice point. So, if \( u_1 < \frac{P}{k} \) is even, then the second coordinate \( u_2 \) satisfies \( |u_2| < \frac{P\Delta}{2k \sigma} \leq \frac{P}{k} \) by Eq. (2), whereas if \( u_1 \) is odd, then \( |u_2 - \frac{P}{k}| < \frac{P}{k} \), i.e. \( |u_2| > \frac{P}{k} \).

Finally, the probability that we observe an \( x \) with a second coordinate \( x_2 \) farther than \( t\sigma \) from \( u_2 \) is \( \text{erf}(t) \), which is small. Thus, if the money state is authentic, with probability at least, say, 0.8, we have

\[
|x_2| < \frac{P\Delta}{2k} + t\sigma. \tag{11}
\]

This is the threshold for our acceptance test.

Security of the protocol — We first note that if any appreciable mass is located farther than \( t\sigma \) from a dual lattice point, it will be detected by the LDFT test in the verification protocol.

We now show that if an adversary has a product of two money states with the same serial number that both pass the verification test, they can find a short vector. We will do this by first finding two "magic vectors" which take a lattice point near a dual lattice point in our money state to a lattice point near any other dual lattice point in the same cluster within the money state.

Let \( e_i \) be \( i \)th standard basis vector. Then since \( w \cdot v = 1 \) and \( e_1 \cdot v = 1 \), we have \( (w - e_1) \cdot v = 0 \), and thus

\[
s^{(1)} = w - e_1 \in \mathcal{L}. \tag{12}
\]

This is our first magic vector. An appropriate multiple of \( s^{(1)} \) permits traversal between any two Gaussian balls in the same cluster.

For the second magic vector, we have \( \frac{P+1}{2} w - \frac{P+1}{2} e_1 \in \mathcal{L} \) and \( \frac{P+\Delta}{2} e_1 - e_2 \in \mathcal{L} \). Adding these, we get

\[
s^{(2)} = \frac{P+1}{2} w + \frac{\Delta-1}{2} e_1 - e_2 \in \mathcal{L}. \tag{13}
\]

This vector \( s^{(2)} \) translates a lattice point near \( Tw \in \mathcal{L}^\perp \) to a lattice point near \( (T + \frac{P+1}{2})w \in \mathcal{L}^\perp \). Combining this with \( s^{(1)} \), and starting with a lattice point near some dual lattice point, one can find a lattice point near any dual lattice point in either cluster. Note that this second lattice point will not necessarily be within the Gaussian ball, even if the first one was. This is due to the small offsets between the magic vectors and the dual lattice points.

We first show that a potential counterfeit state cannot have all the mass located in just one cluster of the Gaussian balls. Suppose it does. The adversary can sample a point from the quantum money state to find a point \( x \). They can then produce a state

\[
\sum_j \alpha_j |x + js^{(1)}\rangle
\]

for some set of at most \( 2k+1 \) consecutive values of \( j \). In fact, a counterfeit state of any other form would let the adversary find a second short lattice vector; if there is a point \( y \neq x + js^{(1)} \) in some Gaussian ball, then \( y - x - js^{(1)} \) would be a short vector in the lattice that the adversary could find.

However, for any arbitrary \( m \), there exists a \( y \in \mathcal{L} \) such that \( y_2 = m \) and \( s^{(1)} \cdot y = 0 \). This means that the LDFT of any state of the form in Eq. (14), has a uniform distribution on the second coordinate. Thus, the probability of satisfying Eq. (11) is \( \frac{\sigma}{k} + \frac{2\sigma \Delta}{P} < 0.4 \), so such families of counterfeit states cannot pass the verification test.

Thus, to pass the test in Eq. (11) one needs a significant amount of mass in both clusters of the Gaussian balls.

Now, assume that an adversary has two states that pass the verification test. They can measure random points in both states, and with non-negligible probability (by the above) they can find two lattice points from different clusters. They can then take the first point they found, and use vector \( s^{(2)} \) to move it by approximately \( \frac{P+1}{2} w \mod P \). They can then apply vector \( s^{(1)} \) approximately \( t\sigma \Delta \)}
times to obtain a point near the Gaussian ball containing the second random point they found.

Once they have found a lattice point near a Gaussian ball, they can extend it to a line of lattice points near the Gaussian ball by adding multiples of $s^{(0)}$, as shown in Fig. 1.

FIG. 1. Line of lattice points of slope $-2/\Delta$ near a Gaussian ball. The two coordinates in this figure are the first two coordinates of the lattice points: $x_1$ and $x_2$. The separation between each pair of points is $s^{(0)}$. The adversary could reach these points after measuring a lattice point and adding $s^{(0)}$. After adding $s^{(0)}$ enough times to be near the Gaussian balls in the other cluster, the line of points shown will have moved down to not intersect the Gaussian ball.

When we move from one Gaussian ball $|g(jw)|$ to the adjacent one, $|g((j + 1)w)|$, the lattice points on the line in Fig. 1 move one unit to the left relative to the Gaussian ball, and the line itself moves down by approximately $\frac{2}{\Delta}$. Using the first magic vector to traverse $t\sigma\Delta$ Gaussian balls, we move the line by a distance $2t\sigma$, and so the line no longer intersects any Gaussian balls. Thus, the adversary has identified two lattice points, one within the associated Gaussian ball and the other outside of it, such that their vector difference is not a multiple of $s^{(0)}$. By subtracting these two points, which by the above must necessarily be distinct, they can find a vector that is linearly independent of $s^{(0)}$ whose length is of order $\Delta$. Since $\Delta$ is small, this would be a second short vector, contradicting our hardness assumption.

Our security proof generalizes easily in the case the joint two-money state is entangled rather than a product state. Suppose the adversary has two entangled states, and measures one of them in the canonical basis to obtain a point $|x\rangle$. By the above arguments, in order for them to be unable to find a short lattice vector, the residual state of the other one must be of the form in Eq. (14). However, a state of this form cannot pass the verification test. Since the measurement of the first quantum money state commutes with the verification of the second quantum money state, the second quantum money state will not pass the verification test even if the adversary does not measure the first quantum money state.

**Conclusion and outlook** — We developed a quantum algorithm to generate states with unforgeability properties protected by classical lattice-based cryptography. From a physical standpoint, our algorithm serves as a concrete, cryptographic realization of the ideas in the no-cloning theorem. Moreover, unforgeable states have a natural interpretation as money, since their authenticity can also be efficiently verified by any public party. We generated such states via Gaussian superpositions over a random lattice, and showed that our verification procedure forces an aspiring counterfeiter to solve the short vector problem.

A natural question about quantum money is whether its ideas may be generalized to further goals involving copy protection. Specifically, a next step is to adapt the quantum money algorithm to an anti-piracy protocol that protects quantum computations (i.e. a circuit) from duplication. A quantum copy protection scheme was constructed by Aaronson and Arkhipov [21] but has been broken. One may also explore connections between quantum money and other branches of quantum cryptography, such as zero-knowledge proofs.

**Acknowledgments** — We thank Vinod Vaikuntanath for his helpful discussion on an early version of this paper and for answering our questions about lattices; David Jerison for helping come up with the proof of Lemma 5; and Mark Zhandry for his helpful comments on a draft of this paper. ABK was supported by the National Science Foundation (NSF) under Grant No. CCF-1729369. PWS was supported by the NSF under Grant No. CCF-1729369, by the NSF Science and Technology Center for Science of Information under Grant No. CCF-0939370, by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Co-design Center for Quantum Advantage (C2QA) under contract number DESC0012704., and by NTT Research Award AGMT DTD 9.24.20.

---

* khesin@mit.edu
† lu@mit.edu
‡ shor@math.mit.edu

[1] S. Wiesner, Conjugate coding, ACM SIGACT News, 78 (1983).
[2] S. Aaronson, Quantum copy-protection and quantum money, in CCC ’09: Proceedings of 24th Annual IEEE Conference on Computational Complexity (IEEE Computer Society, 2009) pp. 229–242.
[3] S. Aaronson and P. Christiano, Quantum money
from hidden subspaces, in *STOC ’12: Proceedings of the Forty-Fourth Annual ACM Symposium on Theory of Computing* (ACM, 2012) pp. 41–60.

[4] E. Farhi, D. Gosset, A. Hassidim, A. Lutomirski, and P. Shor, Quantum money from knots, in *ITCS ’12: Proceedings of the 3rd Innovations in Theoretical Computer Science Conference* (2012).

[5] D. Kane, Quantum money from modular forms (2018), arXiv:1809.05925.

[6] M. Zhandry, Quantum lightning never strikes the same state twice, in *Advances in Cryptology – EUROCRYPT 2019*, edited by Y. Ishai and V. Rijmen (Springer, 2019) pp. 408–438.

[7] D. Kane, S. Sharif, and A. Silverberg, Quantum money from quaternion algebras (2021), arXiv:2109.12643.

[8] A. Lutomirski, S. Aaronson, E. Farhi, D. Gosset, A. Hassidim, J. Kelner, and P. Shor, Breaking and making quantum money: Toward a new quantum cryptographic protocol, in *Innovations in Computer Science - ICS 2010* (Tsinghua University Press, 2010) pp. 20–31.

[9] B. Roberts and M. Zhandry, Toward Secure Quantum Money, Senior thesis, Princeton University (2019).

[10] M. Conda Pena, J.-C. Faugère, and L. Perret, Algebraic cryptanalysis of a quantum money scheme: The noise-free case, in *Public Key Cryptography – PKC 2015*, Lecture Notes in Computer Science, Vol. 9020 (Springer, 2015) pp. 194–213.

[11] M. Conde Pena, R. Durán Díaz, J.-C. Faugère, L. Hernández Encinas, and L. Perret, Non-quantum cryptanalysis of the noisy version of aaronson–christiano’s quantum money scheme, IET Information Security 13, 362 (2019).

[12] A. Bilyk, J. Doliskani, and Z. Gong, The systematic normal form of lattices (2022), arXiv:2205.10488.

[13] M. Ajtai, Generating hard instances of lattice problems, in *STOC ’96: Proceedings of the twenty-eighth annual ACM symposium on Theory of computing* (1996) pp. 99–108.

[14] C. Peikert, A decade of lattice cryptography, *Theoretical Computer Science* 10, 283 (2016).

[15] L. Eldar and P. Shor, The systematic normal form of lattices (2016), arXiv:1604.07800.

[16] L. Eldar and P. Shor, A discrete fourier transform on lattices with quantum applications (2017), arXiv:1703.02951.

[17] C. Gentry, C. Peikert, and V. Vaikuntanathan, Trapdoors for hard lattices and new cryptographic constructions, in *STOC ’08: Proceedings of the 40th Annual ACM Symposium on Theory of Computing*, edited by C. Dwork (ACM, 2008) pp. 197–206.

[18] A. K. Lenstra, H. W. Lenstra, and L. Lovasz, Factoring polynomials with rational coefficients, Math. Ann. 261, 515 (1982).

[19] P. N. Klein, Finding the closest lattice vector when it’s unusually close, in *SODA ’00: Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms*, edited by D. B. Shmoys (ACM/SIAM, 2000) pp. 937–941.

[20] Y. Liu, V. Lyubashevsky, and D. Micciancio, On bounded distance decoding for general lattices, in *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, 9th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems, APPROX 2006 and 10th International Workshop on Randomization and Computation, RANDOM 2006*, Lecture Notes in Computer Science, Vol. 4110, edited by J. Díaz, K. Jansen, J. D. P. Rolim, and U. Zwick (Springer, 2006) pp. 450–461.

[21] S. Aaronson and A. Arkhipov, The computational complexity of linear optics, in *STOC ’11; Proceedings of the forty-third annual ACM symposium on Theory of computing* (2011) pp. 333–342.

[22] D. Unruh, Computationally binding quantum commitments, in *Advances in Cryptology — EUROCRYPT 2016*, Lecture Notes in Computer Science, Vol. 9666, edited by M. Fischlin and J.-S. Coron (Springer, 2016) pp. 497–527.

[23] H. Buhrman, R. Cleve, J. Watrous, and R. De Wolf, Quantum fingerprinting, Physical Review Letters 87, 167902 (2001).

[24] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, 2017) second Edition.

Uncomputation in verification

Our verification procedure is a two-outcome (A, B) stochastic test of the general form:

1. if the quantum money state is valid, the test returns A (accept) with probability at least $p + \epsilon$,

2. if the quantum money state was made by an adversary who does not know a short basis for the lattice, the test returns A with probability at most $p - \epsilon$,

where $\epsilon$ is some positive constant.

Using the test in this raw form does not let us produce reusable quantum money, because such a test may irreversibly destroy the quantum money. To measure the quantum state without destroying it, we need to give a verification procedure such that if the quantum money is valid, it will pass the test with probability at least $1 - \epsilon$, where $\epsilon$ is super-polynomially small. We use a quantum technique known as uncomputation to achieve this probability bound, and then show that the bound implies that the integrity of the money is preserved up to super-polynomially small error.

Instead of one atomic quantum money state, we will use $n$ atomic quantum money states, each with a different serial number. Suppose we have quantum money states $|\mu_1\rangle, |\mu_2\rangle, \ldots, |\mu_n\rangle$, with measurements $M_1, M_2, \ldots, M_n$, that yield either an A
answer or a B answer on a result qubit, and if the money is valid the A answer has probability at least $p + \epsilon$. What we can do is make the actual quantum money state be the tensor product of all the atomic quantum money states:

$$|\mu\rangle = |\mu_1\rangle \otimes |\mu_2\rangle \otimes \ldots \otimes |\mu_n\rangle.$$  

(15)

We add an ancillary qubit for each atomic money state, as well as an overall readout qubit. We then compute the answer A or B in all the ancillary qubits. Next, we decide whether there is a sufficiently high proportion of As in the ancillary qubits, and if there is, we record pass in the actual result qubit. Finally, we “uncompute” the verification procedure by applying the hermitian conjugate of the verification test circuit—everything except for the gates that acted to store information in the result qubit. Hence, if the state outputted by the pre-measurement computation lies almost entirely within the subspace of one of the measurement projectors, the post-measurement state will be very close to the original state measurement state. Then, after uncomputation, the resultant state will be very close to the original state inputted into the quantum computation.

We formalize the above notion for our quantum money case. If the money is valid, the probability of finding the right answer in the actual result qubit is $1 - \epsilon$, where $\epsilon$ is exponentially small. We will show that this implies that the state has changed very little, and that this in turn shows that the money can be reused superpolynomially many times. If the money is not valid, its integrity need not be preserved. To simplify the proof, we will use the special structure of our verification test. In fact, the result holds even for arbitrary verification tests, but the proof is substantially more complicated.

In our verification test, we first check that each atomic state is a superposition of lattice points near dual lattice points in our two clusters, and further that the number of atomic states having lattice points within $\tau d$ of the nearest dual lattice point exceeds some threshold. This piece of the test is a projection, which we call $\Pi_\alpha$.

In the next stage of the verification test, we take the Fourier transform, and check that the number of atomic money states satisfying Eq. (11) exceeds some threshold. We then take the inverse Fourier transform. This piece of the test is another projection, which we call $\Pi_\beta$.

If $|\mu\rangle$ is a valid money state, our construction ensures that the projections $\Pi_\alpha |\mu\rangle$ and $\Pi_\beta |\mu\rangle$ are exponentially close to $|\mu\rangle$. Now, let

$$\Pi_\alpha |\mu\rangle = |\mu\rangle - |\phi_\alpha\rangle, \quad \Pi_\beta |\mu\rangle = |\mu\rangle - |\phi_\beta\rangle,$$

(16)

where we know that $|\phi_\alpha\rangle$ and $|\phi_\beta\rangle$ are exponentially small.

Looking at the complete verification test $\Pi_\beta \Pi_\alpha$, we see that

$$\Pi_\beta \Pi_\alpha |\mu\rangle = \Pi_\beta (|\mu\rangle - |\phi_\alpha\rangle) = |\mu\rangle - |\phi_\beta\rangle - \Pi_\beta |\phi_\alpha\rangle,$$

(17)

and since the last two terms are exponentially small, our verification test only changes the money state by an exponentially small amount.

We still need to show that if we apply this test only polynomially many times, the resulting state is still exponentially close to $|\mu\rangle$. The proof of this is essentially the same as the calculation in Eq. (17), and we omit the details.

**Details on lattices**

In this section, we define the specific class of lattices considered in this paper. These lattices are integer lattices in dimension $d$ that are periodic mod $P$, that is, there is some large $P$ such that if $v \in \mathbb{R}^d$ is in the lattice, and $w$ is any integer vector, then the vector $v + Pw$ is also in the lattice. For intuition, you can consider $P$ prime; however, this is not required for our construction to work.

We will generally consider these lattices mod $P$, so we will consider the coordinate $-1$ (for example) to be equivalent to the coordinate $P - 1$ that is, we will work on the $P^d$ torus.

In particular, we will be looking at the class $C_P$ of such lattices that have exactly one lattice vector in each hyper-row, i.e., there is exactly one lattice vector that projects onto every cell on any $(d - 1)$-dimensional face of the cube. These lattices will have a determinant of $\pm P$; that is, the unit cell of such a lattice has volume $P$. This is easily seen because there are $P^{d-1}$ lattice points in the cube of edge length $P$, which has volume $P^d$.

The converse is also true when $P$ is a prime:

**Lemma 1.** If an integer lattice in $d$ dimensions which is not periodic with period 1 along any coordinate has determinant $\pm P$ and rank $d$ then it is periodic in every dimension with period $P$.

**Proof.** Consider one dimension. For convenience, let it be $x_1$. The point $(0, 0, 0, \ldots, 0)$ is in the lattice. Because this is an integer lattice of rank $d$, there must be another point $(T, 0, 0, \ldots, 0)$ on the $x_1$-axis. Otherwise, this would be an $n - 1$-dimensional lattice. Assuming there is such a point, then we can associate a non-overlapping strip of length $T$ to each
point of the lattice so that each of these strips contains exactly one lattice vector. These points lie in some integer fraction $1/K$ of the rows, so in a large finite region of space, there are approximately $TK^2$ integer points for each lattice point. Thus, $TK = P$. Since $P$ is prime, we must have either $T = P$ or $K = P$. From our condition that the lattice is not periodic 1 in any dimensions, it must be the case that $T = P$. \hfill\Box

In [16], it was shown that if we can find short vectors in lattices in $C_P$ for all $P$, then we can find short vectors in an arbitrary lattice (although note that the proof of this theorem does not hold if we insist that $P$ is prime).

In fact, we believe that a random lattice in $C_P$ with $P$ prime is as hard as an arbitrary lattice. We do not have an actual proof of this, but rather we have a plausibility argument. Suppose you have an arbitrary lattice. By rounding each vertex slightly to a very fine integer grid, we will obtain another lattice whose shortest vectors are very close to the shortest vectors of the original lattice. This new lattice will have some determinant. If the determinant is a prime $P$, then by Lemma 1 we have a lattice periodic with period $P$. Solving the short vector problem in this $P$-periodic lattice will give us the short vector problem in the original lattice.

What still would need to be shown to make this a rigorous proof is that there is a reasonable probability that the determinant is prime, and that if the original lattice is a random lattice, the new lattice is a random lattice. While both of these statements seem quite plausible, we do not have a rigorous proof of either one.

The Dual Lattice

Every primal lattice has a dual lattice, which is the set of vectors whose dot product with all the lattice vectors is an integer. For our paper, it is much more convenient to work with a scaled version of this dual lattice, where the dual $L^\perp$ to lattice $L$ is the set of vectors whose dot product with every vector in $L$ is an integer multiple of $P$. That is,

$$L^\perp = \{ v | \langle v, x \rangle = 0 \text{ (mod } P) \forall x \in L \} \quad (18)$$

This is the standard definition of the dual lattice, scaled by a factor of $P$. We use this scaled version because it is very useful to have the dual lattice and the original lattice on the same $P^d$ grid. The original lattice $L$ is a vector space of dimension $d$ over the ring $Z_P$. This means that the dual lattice is a vector space of dimension 1 over the ring $Z_P$, so it has a single vector $v$ that generates it, and all the vectors are just multiples of $v$. The dual lattice is thus much sparser than the primal lattice.

Lattice Discrete Fourier Transform

The Fourier transform in Eldar and Shor [16] is

$$|x\rangle \rightarrow \frac{1}{P(d-1)/2} \sum_{z \in L} e^{-2\pi i (x,z)/P} |z\rangle. \quad (19)$$

Recall from Eq. (1) the Gaussian superposition state $|g(v)\rangle$ around $v \in L^\perp$. Measurement of a lattice point from $|g(v)\rangle$ yields a random lattice point in a discretized Gaussian distribution with variance $\sigma^2$.

We first calculate the LDFT of a Gaussian superposition centered at $v = 0$.

$$\frac{1}{(2\pi\sigma^2)^{d/4} P^{d/2-1}} \sum_{x \in L} \sum_{z \in L} e^{-x^2/4\sigma^2} e^{2\pi i (x,z)/P} |z\rangle. \quad (20)$$

or $z$ near a point in $L^\perp$, $\langle x, z \rangle$ is small. Consequently, by the same argument that shows that Feynman path integrals are dominated by positions of stationary phase, the value in Eq. (20) will be large for any $z$ near a point in $L^\perp$. This gives the intuition behind the structure of the LDFT, which ultimately will give Gaussian superpositions around every lattice point of $L^\perp$, which we now compute more rigorously.

We compute the amplitude on point $|z\rangle$, first by assuming that $z$ is close to the dual lattice point $|0\rangle$. The Fourier term $e^{-2\pi i (x,z)}$ for $x \in R^d$ oscillates wildly unless $z$ is near $|0\rangle$ (reminiscent of the Feynman path integral derivation). Thus, if we replace the sum over $x$ by an integral, contributions when $z$ is far from $|0\rangle$ cancel, for they are of an oscillatory nature. This is not the case for the original lattice sum, so the integral approximation is valid only for $z$ near $|0\rangle$. The integral therefore serves as a localized approximation that computes the sum for terms with $z$ near $|0\rangle$. To calculate it, we complete the square from Eq. (20). Then by the Cauchy integral theorem, we can ignore the $2\pi i z \sigma/P$ term in the exponent as we replace the sum over $x$ by an integral. We thus have
which one can see by comparison with Eq. (1) is a Gaussian superposition with standard deviation $\frac{P}{4\pi\sigma}$. The $\frac{P}{d}$ term in the integral comes from the fact that only $\frac{1}{P}$ of the points are in $L$. The remaining terms are those of the Gaussian integral.

An analogous result may be obtained for $z$ close to any dual lattice point $u$, since $(x,u) = 0 \pmod{P}$. We replace $z$ with $z-u$ in Eq. (21). We then use the fact that $(x,u) = 0 \pmod{P}$ and complete the square using the expression

$$e^{-\frac{4\pi^2\sigma^2(z-u)^2}{P^2}} e^{-\frac{(x/2\sigma-2\pi i\sigma(z-u))/P^2}{2}}.$$ (22)

This calculation shows that the lattice points near $u$ are in the state $\frac{1}{\sqrt{P}} |G(u)\rangle$. Since the superposition $\sum_{u\in L^\perp} \frac{1}{\sqrt{P}} |G(u)\rangle$ has unit length, the amplitudes on any lattice point that is not close to a dual lattice point must be small.

Eq. (21) shows that the LDFT of a small (large) Gaussian ball $|g(v)\rangle$ gives a superposition of large (small) Gaussian balls $|G(v)\rangle$ around every dual lattice point.

The LDFT of Eq. (20) contains $P$ of the Gaussian superpositions, one around each lattice point $z$, which explains why Eq. (1) and Eq. (20) differ by a $\sqrt{P}$ factor, even after the difference in the standard deviations of the Gaussian superpositions are accounted for.

We next generalize to the case of $v \neq 0$. The easiest way of doing this may be to recall that a Fourier transform turns a translation into a phase shift and vice versa. One can approximate the Gaussian superposition centered at $v \in L^\perp$ by translating by a lattice point $s \in L$ where $s$ is near $v$. When this translation is commuted past the LDFT one ends up shifting a point $t \in L$ by the phase $e^{2\pi i (s,t)/P}$. Let us assume that $t$ is near some point $w \in L^\perp$. Since points in $L$ are orthogonal to those in $L^\perp$,

$$\langle s,t \rangle = \langle v+(s-v),w+(t-w) \rangle = \langle v,w \rangle - 2\langle v,w \rangle - \langle s-v,t-w \rangle.$$ (23)

As $s \to v$, the last term becomes negligible and the phase shift is $e^{-2\pi i (v,w)/P}$, which is what we wanted to show. One can also show this more rigorously by going back to the original definition of the LDFT and working with this.

**Finding a Second Short Vector in a Lattice**

In this section, we show that an algorithm for finding a second short vector in a lattice where one short vector is already known can be used to find a short vector in a different lattice. (We are unsure as to whether this proof has been previously discovered.) We will first show the reduction for arbitrary lattices $L$ over $\mathbb{R}$. Suppose we know a short vector $v$. Consider the hyperplane $H$ orthogonal to $v$. Let us project every vector in $L$ onto this hyperplane. Any two vectors that differ by a multiple of $v$ will be projected to the same point, so we obtain a new lattice $\tilde{L}$.

Now, suppose we can find a short vector $w \in L$ with $w$ linearly independent of $v$. This vector projects on some vector $\tilde{w} \in \tilde{L}$. This new vector $\tilde{w}$ is non-zero, because $w$ was not a multiple of $v$. Thus, we have found a short vector in $\tilde{L}$.

If a random lattice $L$ projects onto a random lattice $\tilde{L}$, this shows that finding a second short vector in a random lattice is as hard as finding a short vector in one.

We now give a reduction for the specific case of our lattices, with the short vector $s^{(0)} = (\Delta,-2,0,\ldots,0)$ known and showing that if we can find a second short vector, we can find a
short vector in a different integer lattice. Suppose we have the lattice $L$. For each point $v = (v_1, v_2, v_3, \ldots, v_d) \in L$, we map it to a point $(2v_1 + \Delta v_2, v_3, \ldots, v_d)$ in a new lattice $\hat{L}$. Any two points that differ by a multiple of $2\lambda$ are the same point in $\hat{L}$. Now, suppose we have a short vector $\hat{v} \neq v \in \hat{L}$. It maps to some vector $\hat{v} \in \hat{L}$. Recall that $\lambda$ is the shortest separation between lattice vectors. Since $\Delta \hat{v}$ is exponentially smaller than $2^d \lambda$, so is $\hat{v}$. Furthermore, if $L$ is a random lattice (i.e., generated by vectors with entries randomly chosen in $[0, P - 1]$), so is $\hat{L}$. So finding a second short vector in a random lattice is as hard as the problem of finding a short vector in a random lattice. The converse reduction also holds, though we do not use it in this paper.

**Broken Quantum Money Attempts**

Parts of the scheme given in the main text may seem unmotivated, such as the design of the Gaussian balls on the lattice. In this section, we give the motivation behind the final scheme by building it up to it in phases. We begin with the simplest quantum money proposal that we considered, and iteratively break-then-fix the scheme until it has reached the final point in our paper. Some of the proofs will only be sketched, focusing on intuition with the understanding that a more rigorous calculation is straightforwardly feasible.

Our first attempt was simply the Gaussian superposition of points of $L$ in a small ball around a random point $v$ of the dual lattice $L^\perp$. The point $v$ serves as the serial number. We prepare the money state by choosing a large $\sigma$ and constructing a $\sigma$-Gaussian superposition $|G(0)\rangle$ (we discuss $v = 0$ for simplicity of notation, but one can do this with a random lattice point). Next, we take the LDFT of this state. As shown previously, this gives small Gaussian superpositions $|g(v)\rangle$ around each point of our dual lattice point $L^\perp$:

$$P^{1/2} \frac{1}{(2\pi \sigma^2)^{d/4}} \sum_{v \in L^\perp} \sum_{x \in L} e^{(x-v)^2/4\sigma^2} |x\rangle. \quad (24)$$

We next use bounded distance decoding to measure which vector of $L^\perp$ is closest to our vector $x$ in the superposition. Once we measure the closest lattice vector in $L^\perp$, we have a Gaussian superposition of lattice points around this vector $v$ with standard deviation $\sigma$, and have thus constructed the quantum money state. Note that the vector $v$ is a random vector in $L^\perp$, so this procedure provides a random serial number.

To show unforgeability—suppose an adversary counterfeits the money. Then we would have two copies of the superposition of lattice points near $v$. We could measure a point from each of these copies, and get two points $x_1$ and $x_2$ of the lattice $L$, both near the point $v$ of the dual lattice. With high probability, these two points will be different. Subtracting them yields a short vector $x_1 - x_2$ in the lattice $L$.

Since the mint could make this quantum money state for a random dual vector $v$ without using any special properties of $L$ or $L^\perp$ (such as a short basis), this would give an algorithm for finding a short vector in $L$, contradicting the hardness assumption of the SVP.

Verifiability is the source of vulnerability in this first attempt. We were unable to find a test that distinguishes a Gaussian superposition of lattice point points around a dual lattice point $v \in L^\perp$ from a probabilistic sample of a single lattice point of $L$ near $v$, known as a collapse of the superposition. (States where a superposition cannot be distinguished from its collapse have been studied, and can be useful for cryptographic protocols [22].) Thus, to counterfeit the money, an adversary could simply measure the Gaussian superposition to obtain a single random lattice vector near $v$, and then duplicate this point as many times as desired to obtain a state that will pass the verification procedure. This quantum money thus does not work, unless a better verification procedure can be found.

In preparation for transitioning into the second attempt, we will describe one of our attempts to find a verification procedure for this quantum money, and show where it goes wrong. Recall that the LDFT of a small ball $|g(v)\rangle$ is a superposition of large balls $|G(v)\rangle$ around every dual lattice point. We could now shift the lattice by some vector and use the famous SWAP test [23] to measure its overlap with the original lattice. The original SWAP test was developed to measure the overlap between two unknown quantum states $|\psi\rangle$ and $|\phi\rangle$, but we formulate a variant in which we estimate $\text{Re}(\psi|U|\phi)$ for some unitary $U$. We start with the state $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. We then apply a controlled-$U$, yielding $\frac{1}{\sqrt{2}} (|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle)$. Finally, we measure the first qubit in the $\{|+, -\rangle\}$ basis. After some algebra, we find that

$$\text{Pr}[|\pm\rangle] = \frac{1}{2} \text{Re}(\psi|U|\phi). \quad (25)$$

The verification procedure we considered is one in which we apply a unitary translation of the Fourier transformed Gaussian balls by some lattice vector, and use the SWAP test to see if there is high over-
lap. The money state will have high overlap with the translate, while a spoof (e.g., a collapse of the money state) will have negligible overlap. Because we know how to describe the Fourier-transformed state well, we can predict the exact overlap that this will yield. We can furthermore find an appropriate lattice vectors to translate the lattice by, which will give an overlap strictly between 0 and 1. This is because the distance between close vectors in the dual lattice is so much larger than the distances in the primal lattice that we can find a lattice point close enough to a desired point to translate the lattice.

The problem with this scheme is that our variant of the SWAP test is linear. That is, it satisfies the following theorem.

**Theorem 1.** Suppose our state $|\psi\rangle$ is expressed as the superposition of some set of basis states $\{|x_i\rangle\}$:

$$|\psi\rangle = \sum_i \alpha_i |x_i\rangle,$$

and the $|x_i\rangle$ have the property that

$$\langle x_i|x_j\rangle = \langle x_i|U|x_j\rangle = 0 \quad \text{if} \ i \neq j,$$

for the transformation $U$ we use in the modified SWAP test. Then the expected probability of measuring $|+\rangle$ in the SWAP test for the superposition $|\psi\rangle = \sum_i \alpha_i |x_i\rangle$ is just the probability of measuring it for the mixture $\rho = \sum_i |\alpha_i|^2 \langle x_i|x_i\rangle$, which is the SWAP test output on a random collapse of a superposition.

The proof of this theorem is by direct computation. Hence, the overlap obtained by taking a translate of the LDFT of the small Gaussian superposition is exactly the same as the expected overlap of the same translate of the LDFT of a random point in the collapse of the Gaussian superposition, so an adversary could simply measure the Gaussian superposition to get one lattice vector contained in it, and use that as the counterfeit money. This state, being classical, could be duplicated arbitrarily.

A second attempt we tried was to make the quantum money be two copies of the Gaussian superposition around the dual lattice vector $v$, instead of one:

$$P \sum_{x \in L} e^{-\langle x-v\rangle^2/4\sigma^2} |x\rangle \otimes \sum_{y \in L} e^{-\langle y-v\rangle^2/4\sigma^2} |y\rangle.$$

The serial numbers must be consistent between the copies, and thus cannot be randomly generated by measurement. Instead, the mint can create money with the same serial number if it knows a short basis of the lattice, so the money is not openly mintable.

Since we have two copies, we can use the original SWAP test that finds the overlap of two quantum states. The original SWAP test is not linear, as can be checked by direct computation. Unfortunately, upon further examination, it turns out that an adversary can use almost the same strategy. They sample a point from each of the Gaussian superpositions, say $|\psi^{(1)}\rangle$ and $|\psi^{(2)}\rangle$, and use the state

$$\frac{1}{\sqrt{2}} \left( |\psi^{(1)}\rangle + |\psi^{(2)}\rangle \right).$$

This counterfeit money can be copied arbitrarily many times, and we were unable to find a test that distinguished it from the real quantum money. More generally, if we use a tensor product of $n$ atomic money states, the adversary could use a symmetrization of the collapses to spoof the verification protocol. While symmetrization of arbitrary qubits is not known to be efficient, it can be done in certain cases, a canonical example being boson sampling [21].

The third attempt we describe is fairly easy to break, but it motivates the reason we choose $\Delta > t\sigma$. In this attempt, rather than choosing the lattice $L^\perp$ by picking a random dual vector, we choose a random dual vector with the first two coordinates being equal. Now, we can distinguish between the LDFT of a single point from the quantum money state and the LDFT of the entire Gaussian ball, because the dual lattice point at the center of a Gaussian ball will have the first two coordinates equal. Thus, $x_1$ and $x_2$ are both Gaussians with standard deviation $\frac{\pi}{2\sqrt{2}\sigma}$ distributed around the same point, so the difference of the first two coordinates is a Gaussian with standard deviation $\frac{\pi}{2\sqrt{2}\sigma}$.

Note that the vector $s = (1, -1, 0, 0, \ldots, 0) \in L$. To counterfeit this money, an adversary uses this fact. Suppose that the quantum money state is $|g(u)\rangle$. The adversary measures a random vector $y$ from the quantum money state, and prepares a state proportional to

$$\sum_{\ell=-\ell_0}^{\ell_0} e^{-\langle y+\ell s\rangle^2/4\sigma^2} |y + \ell s\rangle.$$

This is a Gaussian with standard deviation $\sqrt{2}\sigma$ (recall $|s| = \sqrt{2}$), so this means that in the LDFT, the difference of the first two coordinates is distributed as a Gaussian with length $\frac{\pi}{2\sqrt{2}\sigma}$.

A fourth attempt will be the superposition of $2k + 1$ small Gaussian balls (that is, with small $\sigma$). Specifically, we choose $2k + 1$ balls which will produce an interference pattern that will be detectable.
This proposal is the origin of the relation between $k$ and $P$.

Before we go into the details of this fourth proposal, we first look at simplified example in one dimension. To explain the example, we first need to do some calculations about the discrete Fourier transforms over the one-dimensional ring $\mathbb{Z}_P$. In particular, we let $k = \text{polylog}(P)$, and we look at the Fourier transform of the function $\frac{1}{\sqrt{2k+1}} \sum_{j=-k}^{k} |j\rangle$. This Fourier transform is known as the Dirichlet kernel.

The amplitudes of the Fourier transform of this function will have a sharp peak near 0. An example with $k = 10$ is given for intuition in Fig. 2.

![Graph of the function $\frac{1}{\sqrt{11}} \sum_{j=-10}^{10} e^{2\pi ijx}$](image)

**Fig. 2.** Graph of the function $\frac{1}{\sqrt{11}} \sum_{j=-10}^{10} e^{2\pi ijx}$. The $x$-axis must be scaled by $P$ and the $y$-axis by $\frac{1}{\sqrt{P}}$.

There are two key lemmas about the Fourier transform that allow for a verification test based on the resultant interference pattern.

**Lemma 2.** Write the Fourier transform of the $2k+1$ superposition as

$$\frac{1}{\sqrt{P}\sqrt{2k+1}} \sum_{\ell=-\frac{P}{2k}-1}^{\frac{P}{2k}-1} \sum_{j=-k}^{k} e^{2\pi ij\ell/P} |\ell\rangle = \frac{1}{\sqrt{P}} \sum_{\ell=-\frac{P}{2k}-1}^{\frac{P}{2k}-1} \beta_{\ell} |\ell\rangle. \quad (31)$$

Then

$$\sum_{\ell=-P/8k}^{P/8k} |\beta_{\ell}|^2 > 1/4. \quad (32)$$

That is, a substantial amount of the amplitude in the Fourier transform is concentrated near the center.

**Proof.** We group terms together: $e^{2\pi ij\ell/P} + e^{-2\pi ij\ell/P} = 2 \cos(2\pi j\ell/P)$. This means that

$$\beta_{\ell} = \frac{2}{\sqrt{P}\sqrt{2k+1}} \left( \frac{1}{2} + \sum_{j=1}^{k} \cos(2j\ell/P) \right) \quad (33)$$

But if $|\ell| \leq \frac{P}{8k}$, then $|2\pi j\ell/P| \leq \pi/4$, and $\cos(2\pi j\ell/P) \geq 1/\sqrt{2}$. Thus,

$$\beta_{\ell} \geq \frac{2}{\sqrt{P}\sqrt{2k+1}} \left( \frac{1}{2} + k \right) \frac{1}{\sqrt{2}} \geq \sqrt{\frac{2k+1}{\sqrt{2}P}}, \quad (34)$$

and

$$\sum_{\ell=-P/8k}^{P/8k} |\beta_{\ell}|^2 \geq \frac{1}{4}. \quad (35)$$

Some numerical experiments suggest the real value is approximately $0.467$ for $P \gg k \gg 1$.

**Lemma 3.** Suppose that we have a superposition $\sum_{j=-k}^{k} \alpha_j |j\rangle$ and we take the Fourier transform of it over $\mathbb{Z}_P$ to get $\sum_{\ell=-P/2}^{P/2} \beta_{\ell} |\ell\rangle$. Furthermore, suppose that

$$\sum_{\ell=-P/8k}^{P/8k} |\beta_{\ell}|^2 \geq \frac{1}{4}. \quad (36)$$

Then there must be at least $k$ non-zero amplitudes among the $2k+1$ possible $\alpha_j$.

**Proof.** Suppose there are exactly $m$ non-zero amplitudes among the $\alpha_j$. We have

$$\beta_{\ell} = \frac{1}{\sqrt{P}} \sum_{j=-k}^{k} \alpha_j e^{2\pi ij\ell/P}. \quad (37)$$

We can ignore the terms where $\alpha_j = 0$, so by Cauchy-Schwarz,
Thus, roughly,
\[
\frac{P}{sk} \sum_{\ell = P/2k}^P |\beta_\ell|^2 \leq \frac{m}{4k}.
\]  
(39)

This shows that at least roughly half of the \(2k + 1\) \(\alpha_i\)s must be non-zero.

One other lemma is necessary for the verification protocol, which is an immediate consequence of Lemmas 9.1.1 and 9.4.1 from Wilde [24]:

**Lemma 4.** Suppose we have a quantum state \(|\psi\rangle\) and a projector \(\Pi\), so that \(\langle \psi | \Pi | \psi \rangle \geq 1 - \epsilon\), then the probabilities of the outcomes of any measurement made on \(\Pi | \psi \rangle\) and the same measurement made on \(|\psi\rangle\) differ by at most \(\sqrt{\epsilon}\).

Now, suppose that we have a superposition \(\sum_{j=-k}^k \alpha_j |j\rangle\) and we take the Fourier transform of it over \(\mathbb{Z}_P\) to get \(\sum_{\ell = -P}^{P-1/2} \beta_\ell |\ell\rangle\). Furthermore, suppose that
\[
\frac{P}{sk} \sum_{\ell = -P/2k}^{P/2k} |\beta_\ell|^2 \geq \frac{1}{4}.
\]  
(40)

Then there must be at least \(k\) amplitudes among the \(2k + 1\) possible \(\alpha_j\)s with \(\alpha_j > \frac{1}{k}\). We sketch a proof using Lemma 4. Suppose we first project the state onto all the possible \(|j\rangle\) with \(\alpha_j > \frac{1}{k}\). Then we do not change the probability of getting some \(\beta_\ell\) in the range \(\left[-\frac{1}{8k}(P - 1), \frac{1}{8k}(P - 1)\right]\) by more than \(\frac{1}{k}\). Thus, there are \(\Theta(k)\) \(\alpha_i\)s with \(\alpha_i > \frac{1}{\sqrt{k}}\).

Now, recall if you translate a superposition of basis states, then the Fourier transform of the superposition can be obtained by applying a phase to each of the basis states in the Fourier transform. This implies that the Lemma 2 also holds for the state
\[
\frac{1}{\sqrt{2k+1}} \sum_{j=-k}^k |T + j\rangle.
\]

For the next part, the idea is to choose to start with a set of \(2k + 1\) carefully chosen Gaussian balls. These balls will induce the a probability distribution very similar to the one in Fig. 2 in the first coordinate. Let \(v\) be the vector of the dual lattice with coordinate 1. Let \(w\) be the unique dual lattice vector such that \(\langle v, w \rangle = 1\). Consider large Gaussian superpositions around the dual lattice vectors \(\alpha v\) and small Gaussian superpositions around the dual lattice vectors \(\beta w\). We call these superpositions \(|G(\alpha v)\rangle\) and \(|g(\beta w)\rangle\) respectively.

What happens when we take the Fourier transform of \(|g(\beta w)\rangle\)? We get, as argued after Eq. (23),
\[
\frac{1}{\sqrt{P}} \sum_{a=0}^{P-1} e^{-2\pi iab/P} |G(\alpha v)\rangle.
\]  
(41)

This is analogous to the Fourier transform given in Eq. 24.

Thus, if we start with the superposition of a set of Gaussian balls
\[
\frac{1}{\sqrt{2k+1}} \sum_{b=T}^{b=T+2k} |g(\beta w)\rangle,
\]  
(42)

we will end up with the same result as before, with the amplitudes on large Gaussian balls \(|G(\alpha v)\rangle\) rather than on the one-dimensional set of integers from 0 to \(P - 1\).

To get these amplitudes, we need to take \(j\) such that \(-k \leq j \leq k\), and start with a superposition of the \(2k + 1\) small Gaussian balls around the points \(jw\). Now, the idea was that if we measured random points \(|x_1\rangle, |x_2\rangle, |x_3\rangle\) from the quantum money state, and they came from Gaussian balls with \(j_1, j_2, j_3\), then the point
\[
(j_2 - j_1)(x_3 - x_1) - (j_3 - j_1)(x_2 - x_1)
\]  
(43)

would be a short vector in \(\mathcal{L}\). And the only way an adversary could make a state that passes would be if there was a lattice vector of \(\mathcal{L}\) very close to the dual vector \(w\), where \(\langle w, v \rangle = 1\). And since it is possible for anybody with a quantum computer to produce a quantum money state with a random serial number, this shows that any protocol that lets an adversary produce a duplicate counterfeit quantum money state that passes the verification tests can also be used to find a short vector in a lattice.

The problem is that there is a lattice vector of \(\mathcal{L}\) very close to the dual vector \(w\). In fact,
\[
w - e_1 \in \mathcal{L}.
\]  
(44)

So this proof doesn’t work.
The fact that the proof doesn’t work doesn’t necessarily mean that the money protocol can be broken. However, it does appear to be. Suppose that we try to use the above protocol. An adversary can measure a random point \( x \) from the superposition we have above, and then construct the state

\[
\sum_{j=a_1}^{a_2} |x + j\hat{w}\rangle. \tag{45}
\]

He can only do this for a range \( a_2 - a_1 = O(\sigma) \), as otherwise he starts to move beyond the boundaries of the small Gaussian balls. The first coordinate of the Fourier transform will be concentrated between the \(-O(P/\sigma)\) and \(O(P/\sigma)\). However, because the Gaussian balls of the quantum money state have weight \(O(P/\sigma)\), the true money state cannot be distinguished from this state the adversary can construct.

It will be helpful for our proof of security to have a stronger statement than Lemma 2, which showed that the mass of

\[
\frac{1}{\sqrt{2k+1}} \frac{1}{\sqrt{P}} \sum_{\ell=-(P-1)/2}^{(P-1)/2} \sum_{j=-k}^k e^{2\pi i j/\ell} |\ell\rangle \tag{46}
\]

was concentrated near the center.

Because \( P \) is large, we can treat \( \ell \) in Eq. (46) as a continuous parameter. Using \( x \) for \( \ell \) in Eq. (46), we get the function

\[
F(x) = \frac{1}{\sqrt{2k+1}} \sum_{j=-k}^k e^{2\pi ij x}. \tag{47}
\]

Note that \( F(x) \) is periodic with period 1 and that \( \int_{-1/2}^{1/2} F(x)^2 dx = 1 \). We will show that most of the mass of \( F(x) \) is concentrated around 0:

**Lemma 5.**

\[
\int_{-\tau/k}^{\tau/k} F(x)^2 dx \geq 1 - \frac{1}{\pi^2 \tau}. \tag{48}
\]

**Proof.** Since \( F \) is a geometric sum, we have

\[
F(x) = \frac{1}{\sqrt{2k+1}} \sum_{j=-k}^k e^{2\pi ij x} = \frac{1}{\sqrt{2k+1}} e^{2\pi i(k+1)x} - e^{-2\pi ikx} = \frac{1}{\sqrt{2k+1}} \sin(\pi(2k+1)x) / \sin(\pi x) \tag{49}
\]

Now,

\[
\int_{-\tau/k}^{\tau/k} F(x)^2 dx = 1 - 2 \int_{\tau/k}^{1/2} F(x)^2 dx \tag{50}
\]

and

\[
\int_{\tau/k}^{1/2} F(x)^2 dx = \frac{1}{2k+1} \int_{\tau/k}^{1/2} \sin^2(\pi(2k+1)x) dx \leq \frac{1}{2k+1} \int_{\tau/k}^{1/2} \sin^2(\pi x) dx = -\cot(\pi x) \bigg|_{\tau/k}^{1/2} = \cot\left(\frac{\pi}{2k+1}\right) \leq \frac{1}{(2k+1)^2} \leq \frac{1}{2\pi^2 \tau}, \tag{51}
\]

where we have used the facts \( \int \frac{1}{\sin y} dy = -\cot(y) \), \( \cot(\frac{\pi}{2}) = 0 \), and \( \cot(x) \leq \frac{1}{x} \) for \( 0 < x < \frac{\pi}{2} \). Combining Eq. (50) and Eq. (51) gives Lemma 5. \( \square \)

This lemma shows that if we take just one cluster of \( 2k+1 \) Gaussian balls, the mass is concentrated around 0. What we really need for our security proof is that the mass is concentrated for two clusters of Gaussian balls. This is proved in the following Lemma,

**Lemma 6.** Let

\[
|\phi\rangle = \frac{1}{\sqrt{4k+2}} \left( \sum_{j=-k}^k |j\rangle + \sum_{j=-k}^k |C+j\rangle \right), \tag{52}
\]

where \( C = \frac{P+1}{2} + D \). Then a basis state sampled from the 1DFT of \( |\phi\rangle \) has probability at least \( 1 - \frac{1}{\pi^2} - O\left(\frac{P}{P}\right) \) of being between \( -P/\tau \) and \( P/\tau \).

**Proof.** Set \( \tau = 1 \); it is straightforward to generalize the following proof to all \( \tau \). Let \( |\psi\rangle \) be the Fourier transform of \( |\phi\rangle \). Let \( |\psi_A\rangle \) and \( |\psi_B\rangle \) be the Fourier transform of first and second clusters, respectively:

\[
|\psi_A\rangle = \frac{1}{\sqrt{2k+1}} \frac{1}{\sqrt{P}} \sum_{\ell=-(P-1)/2}^{(P-1)/2} \sum_{j=-k}^k e^{-2\pi ij/\ell} |\ell\rangle \tag{53}
\]

\[
|\psi_B\rangle = \frac{1}{\sqrt{2k+1}} \frac{1}{\sqrt{P}} \sum_{\ell=-(P-1)/2}^{(P-1)/2} \sum_{j=-k}^k e^{-2\pi i(j+C)/\ell} |\ell\rangle
\]
We have that $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_A\rangle + |\psi_B\rangle)$.

Let $\Pi_k$ be the projection onto the $\frac{2P}{k}$ coordinates around 0:

$$\Pi_k = \sum_{j=-\lfloor \frac{P}{k} \rfloor}^{\lfloor \frac{P}{k} \rfloor} |j\rangle \langle j|.$$ (54)

Because $|\psi_A\rangle$ and $|\psi_B\rangle$ are the Fourier transforms of two quantum states that only differ by a translation, they differ only in the phases of their amplitudes on $|\ell\rangle$. Thus, $\langle \psi_A | \Pi_k | \psi_A \rangle = \langle \psi_B | \Pi_k | \psi_B \rangle$.

Lemma 5 gives a lower bound for the quantity $\langle \psi_A | \Pi_k | \psi_A \rangle$. However, what we need for our security proof is to show that the LDFT of the state containing two clusters of $2k + 1$ Gaussian balls also has the mass concentrated near the middle. This is what Lemma 6 says. To prove it, we need a lower bound on $\langle \psi | \Pi_k | \psi \rangle$. However,

$$\langle \psi | \Pi_k | \psi \rangle = \frac{1}{2} (\langle \psi_A | + \langle \psi_B |) \Pi_k (|\psi_A\rangle + |\psi_B\rangle)$$
$$= \langle \psi_A | \Pi_k | \psi_A \rangle + \text{Re}(\langle \psi_A | \Pi_k | \psi_B \rangle),$$ (55)

so this follows if $|\langle \psi_A | \Pi_k | \psi_B \rangle| \ll \langle \psi_A | \Pi_k | \psi_A \rangle$.

Now, if $\Pi_k |\psi_A\rangle = \sum_{\ell=-k}^{k} e^{-C\ell/P} \gamma_{\ell} |\ell\rangle$, then $\Pi_k |\psi_B\rangle = \sum_{\ell=-k}^{k} e^{-C\ell/P} \gamma_{\ell} |\ell\rangle$. So

$$\langle \psi_A | \Pi_k | \psi_B \rangle = \sum_{\ell=-k}^{k} C^{-2\pi i \ell/P} \gamma_{\ell}^2$$ (56)

$$\langle \psi_A | \Pi_k | \psi_A \rangle = \sum_{\ell=-k}^{k} \gamma_{\ell}^2$$ (57)

Note also that $\gamma_{\ell}$ and $\gamma_{\ell+1}$ are approximately equal. So adding just the $\ell$th and $(\ell + 1)$st terms of these quantities, we find that these are $e^{-2\pi i (\ell+1)/P} C^{2\pi i /P} \gamma_{\ell} + \gamma_{\ell+1}$ and $\gamma_{\ell} + \gamma_{\ell+1}$. If we take $\gamma_{\ell} = \gamma_{\ell+1}$, the ratio of these is $e^{2\pi i C/P} \approx 2\pi i \frac{D}{P}$. Now, algebraic manipulation can be used to complete the proof of the lemma. \qed