Coherent quantum control of Λ-atoms through the stochastic limit

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We investigate, using the stochastic limit method, the coherent quantum control of a 3-level atom in Λ-configuration interacting with two laser fields. We prove that, in the generic situation, this interaction entangles the two lower energy levels of the atom into a single qubit, i.e. it drives at an exponentially fast rate the atom to a stationary state which is a coherent superposition of the two lower levels. By applying to the atom two laser fields with appropriately chosen intensities, one can create, in principle, any superposition of the two levels. Thus relaxation is not necessarily synonymous of decoherence.

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I. INTRODUCTION

Preparation of atoms and molecules in a predefined state plays an important role in modern atomic and molecular physics, in particular in atom optics and quantum information. The difficulty of the problem consists in the fact that, in order to control a system, one has to interact with it, but interaction introduces dissipative effects and hence, at least generically, decoherence. In this note we prove, in specific but important and physically realizable example, that in some cases dissipation can generate coherence.

One of the ways to drive the system, atom or molecule, to the desired state is to exploit its interaction with laser pulses, i.e., to use the coherent laser control or laser-induced population transfer. In this approach monochromatized and near-resonant with the atomic Bohr frequencies radiation fields are used to force the system to the final state. The laser coherent control techniques can be used in applications to laser cooling based on coherent population trapping, quantum computing with trapped ions, etc.

The stochastic limit method was applied in [7] to study the phenomenon of coherent population trapping in an atom in lambda-configuration with a doubly degenerate ground state. It was found that the action of the field drives the atom to a 1-parameter family of stationary states so that the choice of one state in this family is uniquely determined by the initial state of the atom and by the initial state of the field.

In [7] the initial state of the field was chosen to be an arbitrary mean zero, gauge invariant Gaussian state, e.g. the Fock vacuum or an equilibrium state at any temperature. On the other hand, in the usual experiments on coherent population trapping, the atom is driven by two laser beams, i.e. coherent states, resonant with the atomic frequencies and, up to now, there is no evidence of this 1-parameter family of invariant states.

In the present paper we prove that, for a three-level non-degenerate atom, the field drives the atom to a pure state, which is a superposition of the two lowest energy levels and that, applying to the atom two laser fields with appropriately chosen intensities one obtains a single superposition. In the case of a degenerate atom we show that, if one starts from a coherent state, then the family of invariant states is destroyed and the field drives the atom to the dark state.

The results of the present paper, combined with those of [5] also suggest that the emergence of the dark state could be experimentally realized not only by tuning two lasers, but also in a generic equilibrium state, by preparing the state of the atom in the domain of attraction of the dark state.

The present result is obtained as a consequence of a general property of the stochastic limit, which we find for the first time in the paper, according to which the effect of replacing a mean zero, gauge invariant state of the field by a coherent one, amounts to the addition of a hamiltonian term to the master equation while the dissipative part of the generator remains the same [cf. [12]–[13]].

This general property is proved in sections 2–4 of the present paper for an arbitrary atom. In sections 5 and 6 we specialize to the case of 2- and 3-level atoms, find for these cases the explicit form of the stationary atomic states [17] for a two-level, [21] and [22] for a three-level atom, and prove the results mentioned in the present introduction. The master equation we find in the 3-level case coincides with the optical Bloch equation considered by Arimondo in [1] in the absence of scattering [Eq. (2.7)] with the only difference that we obtain the equation and the explicit formula for the coefficients Ωj [cf. equations [1] and [20] below] without assuming that the field is classical. Instead, we start from the microscopic quantum dynamics of the total atom+radiation system.

The approach of the present paper to study the coherent quantum control or laser induced population trans-
fer is based on the derivation, using the stochastic limit method, of a quantum white noise equation approximating the total dynamics in the weak coupling regime (Eq. [3], for a rigorous treatment see [9]). The program to exploit exponentially fast decoherence as a control tool to drive an atom into a pre-assigned state was formulated in [10, 11], where the control parameter is the interaction Hamiltonian. In the approach of the present paper the control parameter is the state of the radiation, which can be easily controlled in experiments and, instead of decoherence, it uses relaxation, due to dissipative dynamics, to create quantum coherence between the atomic states.

II. AN ATOM IN A LASER FIELD

The dynamics of an atom interacting with radiation is determined on the microscopic level by the interaction Hamiltonian and by the reference state of radiation.

The Hamiltonian describing an atom interacting with radiation is given by the sum of the free and interaction terms

$$H_\lambda = H_{\text{free}} + H_{\text{int}} = H_\lambda \otimes 1 + 1 \otimes H_R + \lambda H_{\text{int}},$$

where $H_\lambda$ and $H_R$ are free Hamiltonians of the atom and radiation, $H_{\text{int}}$ interaction Hamiltonian, and $\lambda$ is the coupling constant. The free Hamiltonian of the atom has discrete spectrum

$$H_\lambda = \sum_n \varepsilon_n P_n,$$

where $\varepsilon_n$ is an eigenvalue and $P_n$ is the corresponding projection. The free Hamiltonian of the radiation is

$$H_R = \int d\mathbf{k} \omega (\mathbf{k}) a^+(\mathbf{k}) a(\mathbf{k}), \quad \omega (\mathbf{k}) = |\mathbf{k}|,$$

where the creation and annihilation operators $a^+(\mathbf{k}), a(\mathbf{k})$ describe creation and annihilation of photons with momentum $\mathbf{k}$. The interaction Hamiltonian has the form

$$H_{\text{int}} = i(D \otimes a^+(g) - D^+ \otimes a(g)),$$

where the operators $D$ and $D^+$ describe the transitions between the atomic levels, $a^+(g) = \int d\mathbf{k} g(\mathbf{k}) a^+(\mathbf{k})$ is the smeared creation operator, and the formfactor $g(\mathbf{k})$ describes the coupling of the atom with the $\mathbf{k}$ mode of the field.

The state of the radiation, which corresponds to a long time laser pulse of frequency $\omega$ and intensity $f$ is a coherent state, which is determined by the coherent vector

$$\tilde{\Psi}_\lambda = W \left( \lambda \int_{S/\lambda^2} d\tau e^{-i\omega \tau} f d\tau \right) |\Phi_0\rangle,$$

where $\Phi_0$ is the vacuum, $W(\cdot)$ the Weyl operator, which is defined as $W(f) = \exp[i\langle a(f) + a^+(f) \rangle]$. The state of the radiation, which corresponds to a long time laser pulse of frequency $\omega$ and intensity $f$ is the coherent vector

$$\langle X_\lambda(t) \rangle := \langle \Psi_\lambda, U_\lambda^+(t) X U_\lambda(t) \Psi_\lambda \rangle$$

and, by duality, it can be equivalently described in terms of the reduced density matrix $\hat{\rho}_\lambda(t)$, defined by the following equality:

$$\text{Tr}_A(\hat{\rho}_\lambda(t) X) = \text{Tr}_A(\hat{\rho} \langle X_\lambda(t) \rangle)$$

$$= \text{Tr}_A \left\{ \text{Tr}_R \left[ U_\lambda(t) \left( \hat{\rho} \otimes |\psi_\lambda\rangle \langle \psi_\lambda| \right) U_\lambda^+(t) \right] X \right\},$$

the one photon free evolution, and $[S/\lambda^2, T/\lambda^2]$ the time interval in which the laser pulse is active. This vector $\tilde{\Psi}_\lambda$ is an eigenvector of the annihilation operator:

$$a(g) \tilde{\Psi}_\lambda = c_\lambda \tilde{\Psi}_\lambda.$$
III. THE STOCHASTIC LIMIT

It is impossible to find, for a general coupling $D$, the explicit form of the dynamics for the reduced density matrix for realistic models, while the weakness of the interaction suggests some approximations. For small time the dynamics can be effectively studied by using the perturbation series. However, the $n$-th term of the series behaves like $(\lambda^2 t)^n$ and the approximation with the lowest order terms of the series becomes invalid for time which is large enough.

Hence the study of the long time dynamics requires another approach. Such an approach to study the dynamics on the long time scale ($\sim \lambda^{-2}$) is the stochastic limit method [8]. In the stochastic limit one considers the long time dynamics (on the time scale of order $\lambda^{-2}$) for a system with weak interaction. Mathematically this means that one takes the limit as the coupling constant goes to zero, $\lambda \to 0$, time goes to infinity, $t \to \infty$, but in such a way that the quantity $\lambda^2 t$ remains fixed. In this limit the dynamics of the total system is given by the solution of a quantum white noise or stochastic differential equation (which is a unitary adapted process). Since the interaction of an atom with radiation is weak one can apply the stochastic limit procedure to study the long time dynamics of this system.

On the long time scale the behavior of the exact reduced density matrix is approximated by the limiting density matrix

$$\hat{\rho}(t) = \lim_{\lambda \to 0} \hat{\rho}_\lambda(t/\lambda^2)$$

so that $\rho_\lambda(t) \simeq \rho(\lambda^2 t)$. The quantum master equation for the limiting density matrix in the case when the radiation is in a Gibbs state was derived and discussed in many papers (see [8] and references therein). The purpose of the present paper is to derive the quantum master equation for the limiting density matrix $\hat{\rho}(t)$ in the case the radiation is in a coherent state and to study this equation for particular, but important cases of two and three-level atoms.

The evolution operator after the time rescaling satisfies the equation

$$\frac{dU_\lambda(t/\lambda^2)}{dt} = \sum_{\omega \in B} \left( D_\omega \otimes a^\dagger_{\lambda,\omega}(t) - D^+_\omega \otimes a_{\lambda,\omega}(t) \right) U_\lambda(t/\lambda^2),$$

where $B$ is the set of all Bohr frequencies of the atom (spectrum of the free atomic Liouvillian $i\hat{H}_A$, i.e. the set of $\omega = \epsilon_n - \epsilon_m$, where $\epsilon_n, \epsilon_m$ are eigenvalues of the free atomic Hamiltonian), for each $\omega \in B$

$$a_{\lambda,\omega}(t) = \frac{1}{\lambda} \int d\mathbf{k} \hat{g}^*(\mathbf{k}) e^{-i(\omega(\mathbf{k}) - \omega)t/\lambda^2} a(\mathbf{k})$$

is the rescaled time dependent annihilation operator, and the operator

$$D_\omega = \sum_{n,m: \epsilon_n - \epsilon_m = \omega} P_m DP_n$$

describes transitions between atom levels with energies $\epsilon_n$ and $\epsilon_m$ with the energy difference $\epsilon_n - \epsilon_m = \omega$.

In the stochastic limit one first proves that the time rescaled creation and annihilation operators converge, as $\lambda \to 0$, to a quantum white noise (for details of the stochastic limit procedure see [8]):

$$\lim_{\lambda \to 0} a_{\lambda,\omega}(t) = b_\omega(t),$$

where the quantum white noise operators $b_\omega(t)$ are $\delta$-correlated in time and satisfy the commutation relations

$$[b_\omega(t), b^+_\omega(t')] = \delta(t' - t) \delta_{\omega,\omega'} 2\text{Re } \gamma_\omega.$$  (4)

The complex number

$$\gamma_\omega = \int d\mathbf{k} \frac{|g(\mathbf{k})|^2}{i(\omega(\mathbf{k}) - \omega - i0)}$$  (5)

is the generalized susceptibility and its real part

$$\text{Re } \gamma_\omega = \pi \int d\mathbf{k} |g(\mathbf{k})|^2 \delta(\omega(\mathbf{k}) - \omega)$$

gives the decay rate of the $\omega$-transition (cf. [8], sect. 4.20). The Kronecker $\delta$-symbol $\delta_{\omega,\omega'}$ in (4) indicates the mutual independence of the white noise operators for different Bohr frequencies.

This allows us to derive the equation for the limiting evolution operator

$$U_t := \lim_{\lambda \to 0} U_\lambda(t/\lambda^2).$$

That is the white noise Schrödinger equation

$$\frac{dU_t}{dt} = -iV(t)U_t$$  (6)

with the white noise Hamiltonian

$$V(t) = t \sum_{\omega \in B} \left( D_\omega \otimes b^+_\omega(t) - D^+_\omega \otimes b_\omega(t) \right)$$

Let $X$ be any observable of the atom and $X_t = U^*_t X U_t = \lim_{\lambda \to 0} X_\lambda(t/\lambda^2)$ its limiting time evolution. Using the stochastic golden rule of the stochastic limit
(cf. [4], sect. 5.9), one gets the quantum Langevin equation for $X$, that is a normally ordered white noise differential equation:

$$\frac{dX_t}{dt} = U_t^+ \Theta(X) U_t + \sum_{\omega} \left( b_{\omega}^+(t) U_t^+ L_\omega^+(X) U_t + U_t^+ L_\omega(X) U_t b_{\omega}(t) \right)$$

(such an equation is equivalent to a certain quantum stochastic differential equation). Here

$$\Theta(X) = \sum_{\omega} \left( 2 \text{Re} \gamma_\omega D_\omega^+ X D_\omega - \gamma_\omega X D_\omega^+ D_\omega - \tilde{\gamma}_\omega D_\omega^+ D_\omega X \right)$$

is the generator of a quantum Markov semigroup and, in the notation [3],

$$L_\omega^+(X) = [X, D_\omega], \quad L_\omega(X) = [D_\omega^+, X].$$

The normally ordered form of the quantum Langevin equation, when the annihilation white noise operators are put on the right of the evolution operator and creation operators are put on the left, is very convenient for study the reduced dynamics in the case the radiation is in a coherent state. In order to get the reduced dynamics one just has to average equation (7) over the state of radiation.

The coherent state $\Psi_\lambda$ of the radiation is approximated by the limiting coherent vector

$$\Psi = \lim_{\lambda \to 0} \Psi_\lambda.$$ 

This vector is an eigenvector of the quantum white noise annihilation operator:

$$b_{\omega_l}(t) \Psi = \chi_{[S_l, T_l]}(t) c_{\omega_l} \Psi.$$  

(8)

Here $\chi_{[S_l, T_l]}(t)$ is the characteristic function of the interval $[S_l, T_l]$, determining the duration of the pulse. The complex number $c_{\omega_l}$ depends on the state of the radiation as follows:

$$c_{\omega_l} = 2\pi \int d\omega' g^*(\omega') f_l(\omega') \delta(\omega' - \omega_l)$$

and determines the Rabi frequencies of the field.

In fact, the coherent vector $\Psi_\lambda$ is an eigenvector of the time rescaled annihilation operator $a_{\lambda, \omega_l}(t)$ (defined by [3]) with eigenvalue

$$\sum_{m=1}^{m=m} \int_{S_m}^{T_m} d\tau \frac{1}{\lambda^2} \int d\omega' g^*(\omega') f_l(\omega') e^{i(\tau - t)(\omega'(k) - \omega_l)/\lambda^2} \times e^{i\tau(\omega_l - \omega_m)/\lambda^2}.$$  

From that, using the limit (see [3] for details)

$$\lim_{\lambda \to 0} \frac{1}{\lambda^2} \delta(t - \tau) \delta(\omega_l, \omega_m) = 2\pi \delta(t - \tau) \delta(\omega_l, \omega_m) \delta(\omega(k) - \omega_l)$$

one obtains the limiting eigenvalue

$$2\pi \sum_{m=1}^{m=m} \int_{S_m}^{T_m} d\tau \delta(t - \tau) \int d\omega' g^*(\omega') f_l(\omega') \delta(\omega(k) - \omega_l) = \chi_{[S_l, T_l]}(t) c_{\omega_l}.$$ 

We will consider the case $S_l = S$, $T_l = T$ for all $l$, which means that all lasers are applied during the same time interval $[S, T]$.

Let $X$ be an observable of the atom and denote by

$$\langle X \rangle_t = \langle \Psi, X \Psi \rangle$$

the average, over the limiting state of the radiation, of its time evolution. Taking the average of both sides of the quantum Langevin equation (7) and using property (8) one gets the following equation for the averaged observable:

$$\frac{d\langle X \rangle_t}{dt} = \langle \Theta(X) \rangle_t$$

This master equation for the case with presence of external field is one of the main results of the present paper. In the next section we will rewrite it in the equivalent form as an equation for the reduced density matrix.

IV. THE QUANTUM MASTER EQUATION

The dynamics of the atom can be described either in terms of observables, or equivalently, in terms of density matrices. The equation for the reduced density matrix can be obtained using the relation

$$\text{Tr} (\hat{\rho}(t) X) = \text{Tr} (\hat{\rho}(0) \langle X \rangle_t)$$

and equation (10). It is a quantum master equation of the form:

$$\frac{d\hat{\rho}(t)}{dt} = \mathcal{L}(\hat{\rho}(t)).$$

(11)

The time dependent generator of this equation is given by the sum of its dissipative and Hamiltonian parts:

$$\mathcal{L}(\hat{\rho}) = \mathcal{L}_{\text{diss}}(\hat{\rho}) - i [H_{\text{eff}}(t), \hat{\rho}].$$

(12)

As already stated in the Introduction, the dissipative part is the same as in the case the reservoir in the vacuum state:

$$\mathcal{L}_{\text{diss}}(\hat{\rho}) = \sum_{\omega} \text{Re} \gamma_\omega \left( 2 D_\omega \hat{\rho} D_\omega^+ - \hat{\rho} D_\omega^+ D_\omega - D_\omega^+ D_\omega \hat{\rho} \right).$$

(13)

The Hamiltonian part is determined by the effective time dependent Hamiltonian

$$H_{\text{eff}}(t) = \sum_{\omega} \left( \text{Im} \gamma_\omega D_\omega^+ D_\omega + \chi_{[S, T]}(t) (c_{\omega_l}^* D_\omega + c_{\omega_l} D_\omega^+) \right).$$

(14)
The first term in the brackets is the standard term which appears in the vacuum state. It commutes with the free atomic Hamiltonian $H_A$. The other terms are new and their presence leads to the important consequence that the effective Hamiltonian at time $t \in [S,T]$ does not commute with the free atom Hamiltonian

$$[H_{\text{eff}}(t), H_A] \neq 0.$$  

Therefore the states of the atom driven by a long time laser pulse, which are approximated by stationary states of the master equation obtained from (11) replacing in the effective Hamiltonian (14) $\chi_{[S,T]}(t)$ by identity, will not be diagonal with respect to $H_A$. Our goal is to investigate the structure of these states. In generic situation the interaction with the laser field drives the atom to a (dynamical) equilibrium state at an exponential rate. In this sense the interaction prepares a state of the atom. Thus our problem can be reformulated as follows: What kind of states can we prepare applying a laser pulse? For a long laser pulse these states are described by the time independent generator $L$ obtained from (11), by setting $S \to -\infty$, $T \to +\infty$, i.e. the characteristic function $\chi_{[S,T]}$ in (10) replaced by the identity.

In the limit $S \to -\infty$, $T \to +\infty$ the effective Hamiltonian becomes time independent

$$H_{\text{eff}} = \sum_\omega \left( \text{Im} \gamma_\omega D_\omega^+ D_\omega + c_\omega^* D_\omega + c_\omega D_\omega^+ \right).$$

In this case the stationary state $\hat{\rho}_{\text{st}}$ is a solution of the following equation

$$L(\hat{\rho}_{\text{st}}) \equiv L_{\text{diss}}(\hat{\rho}_{\text{st}}) - i[H_{\text{eff}}, \hat{\rho}_{\text{st}}] = 0.$$  

(16)

In the following sections we solve explicitly this equation for two and three-level atoms.

V. STATIONARY STATES FOR A TWO-LEVEL ATOM

Let us consider a two-level atom under the rotating wave approximation.

The free Hamiltonian of the atom is

$$H_A = \varepsilon_0 |0\rangle \langle 0| + \varepsilon_1 |1\rangle \langle 1|,$$

where $\varepsilon_0$ and $\varepsilon_1$ are the energies of the ground and the excited states, so that the Bohr frequency $\omega = \varepsilon_1 - \varepsilon_0 > 0$ is positive. The ground state is denoted by $|0\rangle$ and the excited state by $|1\rangle$. The transitions between atomic levels are described by the operators

$$D = |0\rangle \langle 1| \equiv \sigma_-, \quad D^+ = |1\rangle \langle 0| \equiv \sigma_+.$$

The interaction Hamiltonian has the form

$$H_{\text{int}} = i(\sigma_- \otimes a^+(g) - \sigma_+ \otimes a(g)).$$

The first term in the interaction Hamiltonian describes the process of emission of a photon due to transition of the atom from the excited to the ground state. The second term describes the opposite process, that is the absorption of a photon by the atom and transition to the excited state.

The real part $\gamma = \text{Re} \gamma_\omega$ of the susceptibility (4) determines the inverse lifetime of the atom without interaction with external field and the imaginary part $\Delta \omega = \text{Im} \gamma_\omega$ determines energy shift (cf. [4], sect. 5.5). The dissipative part of the generator and the effective Hamiltonian for this system have the forms

$$L_{\text{diss}}(\hat{\rho}) = \gamma(2\sigma_- \rho \sigma_+ - \rho \sigma_+ \sigma_- - \sigma_+ \sigma_- \rho),$$

$$H_{\text{eff}} = \Delta \omega |1\rangle \langle 1| + \Omega^* \sigma_- + \Omega \sigma_+,$$

where $\Omega \equiv c_\omega$ is the complex Rabi frequency of the laser field. Equation (16) for a stationary state $\rho_{\text{st}}$ of the two-level atom equivalent to the following equations on the matrix elements $\rho_{mn} = \langle m|\rho_{\text{st}}|n\rangle$:

$$\gamma \rho_{11} = \text{Im} (\Omega \rho_{01})$$

$$\gamma \omega \rho_{10} = i \Omega (\rho_{11} - \rho_{00}).$$

The general properties $\text{tr} \hat{\rho} = 1$, $\hat{\rho}^+ = \hat{\rho}$ of the density matrix lead to the relations $\rho_{00} + \rho_{11} = 1$, $\rho_{01} = \rho_{10}^*$, $\rho_{11}^* = \rho_{11}$. Using these relations one finds that, with the notation $\alpha = -i \Omega / \gamma \omega$, the stationary state is

$$\hat{\rho}_{\text{st}} \equiv \begin{pmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix} = \frac{1}{1 + 2|\alpha|^2} \begin{pmatrix} |\alpha|^2 & \alpha \\ \alpha^* & 1 + |\alpha|^2 \end{pmatrix}.$$  

(17)

This state is pure only if $\alpha = 0$. For very intense illumination ($|\Omega|/|\gamma \omega| \gg 1$) the quantity $|\alpha|$ becomes very large and the atom becomes saturated with equal probabilities in the upper and lower levels, so that $\rho_{00} = \rho_{11} = 1/2$, $\rho_{10} = 0$. For very low illumination, the stationary state is the ground state.

VI. STATIONARY STATES FOR A THREE-LEVEL ATOM

The free Hamiltonian of a three-level atom is

$$H_A = \varepsilon_0 |0\rangle \langle 0| + \varepsilon_1 |1\rangle \langle 1| + \varepsilon_2 |2\rangle \langle 2|,$$

where $|0\rangle$, $|1\rangle$ and $|2\rangle$ denote the ground state, the first and second excited states. For a non degenerate atom the energies satisfy the inequalities $\varepsilon_2 > \varepsilon_1 > \varepsilon_0$. There are three positive Bohr frequencies $\omega_1 = \varepsilon_1 - \varepsilon_0$, $\omega_2 = \varepsilon_1 - \varepsilon_1$, $\omega_3 = \varepsilon_2 - \varepsilon_0$, which correspond to three possible transitions. The transitions between the atomic levels due to interaction with radiation are described by the operator

$$D = d_0^* |0\rangle \langle 1| + d_1^* |1\rangle \langle 2| + d_2^* |0\rangle \langle 2|,$$

where $d_j$ are some complex numbers. In particular, the contribution of the first term in the sum to the interaction
Hamiltonian corresponds to emission of a photon by the atom and transition from the first excited level $|1\rangle$ to the ground state $|0\rangle$.

For the three-level atom the three possible transitions are described by the operators
\[ D_1 := D_{\omega_1} = d_1^* |0\rangle \langle 1|, \quad D_2 := D_{\omega_2} = d_2^* |1\rangle \langle 2|, \quad D_3 := D_{\omega_3} = d_3^* |0\rangle \langle 2|. \]

Hence, with the notations
\[ \gamma_j := \gamma_{\omega_j} |d_j|^2 = a_j + ib_j, \]
\[ (a_j \text{ and } b_j \text{ are the real and imaginary parts of } \gamma_j) \]
the dissipative part of the generator, in the generic case, can be written as
\[ \mathcal{L}_{\text{diss}}(\hat{\rho}) = 2(a_1\rho_{11} + a_3\rho_{22}) |0\rangle \langle 0| + 2a_2\rho_{22} |1\rangle \langle 1| - a_1(\hat{\rho}) |1\rangle \langle 1| + (|1\rangle \langle 1| \hat{\rho}) - (a_2 + a_3)(\hat{\rho}) |2\rangle \langle 2| + |2\rangle \langle 2| \hat{\rho}, \]
where $\rho_{mn} = \langle m | \hat{\rho} | n \rangle$ are the matrix elements of the density matrix. Introducing the Rabi frequencies of the laser field
\[ \Omega_j := c_{\omega_j} d_j, \]
the effective Hamiltonian can be written as
\[ \mathcal{H}_\text{eff} = b_1 |1\rangle \langle 1| + (b_2 + b_3) |2\rangle \langle 2| + \Omega_1 |1\rangle \langle 0| + \Omega_2 |2\rangle \langle 1| + \Omega_3 |2\rangle \langle 0| + \Omega_1^* |0\rangle \langle 1| + \Omega_2^* |1\rangle \langle 2| + \Omega_3^* |0\rangle \langle 2|. \]

The matrix elements of the stationary state for arbitrary Rabi frequencies $\Omega_j$ satisfy the following system of equations:
\[ a_1\rho_{11} + a_3\rho_{22} = \text{Im} (\Omega_1\rho_{01} + \Omega_3\rho_{02}) \]
\[ (a_2 + a_3)\rho_{22} = \text{Im} (\Omega_2\rho_{12} + \Omega_3\rho_{02}) \]
\[ (\gamma_2^* + \gamma_3^*)\rho_{02} + i\Omega_2^* (\rho_{22} - \rho_{00}) - i\Omega_3^* \rho_{01} + i\Omega_1^* \rho_{12} = 0 \]
\[ \gamma_1^* \rho_{01} + i\Omega_1^* (\rho_{11} - \rho_{00}) - i\Omega_2^* \rho_{02} + i\Omega_3^* \rho_{21} = 0 \]
\[ (\gamma_1 + \gamma_2^* + \gamma_3^*)\rho_{12} + i\Omega_2^* (\rho_{22} - \rho_{11}) + i\Omega_1\rho_{02} = i\Omega_3^* \rho_{10}. \]

Let us consider the case when the radiation, describing laser pulse, is in a coherent state which drives only transitions from the ground state to the highest excited state, i.e., a laser field with frequency near $\omega_3$. In this case $\Omega_1 = \Omega_2 = 0, \Omega_3 = \Omega \neq 0$ is the only nonzero Rabi frequency and the system of equations for the matrix elements of the stationary state becomes simpler:
\[ a_1\rho_{11} + a_3\rho_{22} = \text{Im} (\Omega\rho_{02}) \]
\[ (a_2 + a_3)\rho_{22} = \text{Im} (\Omega\rho_{02}) \]
\[ (\gamma_2^* + \gamma_3^*)\rho_{02} + i\Omega^* (\rho_{22} - \rho_{00}) = 0 \]
\[ \gamma_1^* \rho_{01} + i\Omega^* \rho_{21} = 0 \]
\[ (\gamma_1 + \gamma_2^* + \gamma_3^*)\rho_{12} - i\Omega^* \rho_{10} = 0. \]

This system of equations, together with the conditions $\text{tr} \hat{\rho} = 1, \hat{\rho}^+ = \hat{\rho}$, can be easily solved. With the notations
\[ \alpha = \frac{-i\Omega}{\gamma_2 + \gamma_3}, \quad r = \frac{a_2}{a_1} \]
the solution, which is the stationary state, has the form
\[ \hat{\rho}_{\text{st}} = \begin{pmatrix} \rho_{22} & \rho_{21} & \rho_{20} \\ \rho_{12} & \rho_{11} & \rho_{10} \\ \rho_{02} & \rho_{01} & \rho_{00} \end{pmatrix} \frac{1}{1 + (2 + r)|\alpha|^2} \begin{pmatrix} |\alpha|^2 & 0 & \alpha \\ 0 & r|\alpha|^2 & 0 \\ \alpha^* & 0 & 1 + |\alpha|^2 \end{pmatrix}. \]

The obtained density matrix $\hat{\rho}_{\text{st}}$ is the stationary state of the three-level atom in the laser field driving only transitions from the ground state to the highest energy state of the atom.

If the ratio $r$ is very large, i.e., $a_2 \gg a_1$, then the stationary state describes population inversion between the first two excited atomic levels. In this case the population of the first excited state of the atom is greater than that of the ground state. To find the physical conditions for this effect let us note that the quantity
\[ a_1 = |d_1|^2 \int d\mathbf{k} \left| g(\mathbf{k}) \right|^2 \delta(\omega(\mathbf{k}) - \omega_1) \]
determines the inverse lifetime of the atom at the level $|1\rangle$ in the free space. That is $\tau \sim 1/a_1$. Hence, if the lifetime $\tau$ of the atom in the first excited state is long enough, then we obtain the population inversion between the ground and the first excited level.

As an example, consider a three-level atom in a configuration, such that the transitions between the ground state $|0\rangle$ and the first excited state $|1\rangle$ are forbidden. This means that $d_1 = 0$. In this case $a_1 = 0$. The lifetime of the first excited level is infinite, and the steady state is $|1\rangle |\langle 1|$, i.e., in this case all the population is concentrated on the first excited state.

\section{VII. Three-Level Lambda-Atom}

Let us now consider a three-level atom in a configuration in the general coherent field, that is both Rabi frequencies $\Omega_2$ and $\Omega_3$ are different from zero.

The system of dynamical equations for matrix elements of the density matrix is (matrix elements are time depen-
An important property is that it is a pure state: 
\[ \hat{\rho}_{\text{st}} = \Omega |\psi\rangle\langle\psi|, \]
where
\[ |\psi\rangle = \frac{1}{\sqrt{|\Omega_2|^2 + |\Omega_3|^2}} \left( \Omega_2 |1\rangle - \Omega_3^* |0\rangle \right). \] (22)

In particular, if the ground state is doubly degenerate and if the coupling does not distinguish between the two ground states, then \( \Omega_2 = \Omega_3 \) and the unique stationary state is the dark state (cf. [11] for a discussion).

VIII. CONCLUSIONS

In the present work we derive, starting from exact microscopic dynamics and using the stochastic limit method, the quantum master equation [11] for an arbitrary atom driven by a general laser field. As an application of this equation, we study the cases of two and three-level atoms. The explicit form of the stationary states of such atoms are found. It is shown that for a three-level lambda-atom in two laser fields, which drive both allowed transitions, the stationary state of the atom is a pure state given by a superposition of the two lowest energy levels. Varying the intensities of the lasers, any such a superposition can be obtained. Hence the two lower levels of the atom behave like a single qubit and any state of this qubit can be obtained in a stable way and in exponentially small time.

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