Relaxation periodic solutions of one singular perturbed system with delay

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Abstract. In this paper, we consider a singularly perturbed system of two differential equations with delay, simulating two coupled oscillators with a nonlinear compactly supported feedback. We reduce studying nonlocal dynamics of initial system to studying dynamics of special finite-dimensional mappings: rough stable (unstable) cycles of these mappings correspond to exponentially orbitally stable (unstable) relaxation solutions of initial problem. We show that dynamics of initial model depends on coupling coefficient crucially. Multistability is proved.

1. Introduction
In this paper, we study nonlocal dynamics of a system of two delay differential equations

\[
\begin{align*}
\dot{u}_0 + u_0 &= \lambda F(u_0(t-T)) + \gamma (u_1 - u_0), \\
\dot{u}_1 + u_1 &= \lambda F(u_1(t-T)) + \gamma (u_0 - u_1),
\end{align*}
\]

which simulates two coupled oscillators with nonlinear feedback. Here parameter \(\lambda\), delay time \(T\), and coupling parameter \(\gamma\) are positive. We suppose that \(F(u)\) is some nonlinear function, whose values are concentrated in a bounded domain: for some \(p > 0\) we have equality

\[
F(u) = \begin{cases} f(u), & |u| < p, \\ 0, & |u| \geq p. \end{cases}
\]

Equations with this type of nonlinearity arise in electrical engineering [1] and in radiophysics [2].

Here function \(f(u)\) is smooth and satisfies the following conditions:

\[
\begin{align*}
uf(u) &> 0 \quad \text{if} \quad 0 < |u| < p, \\
f(-p) &= f(0) = f(p) = 0, \\
f(u) &= \text{const}_{1}(p-u)^2 \quad \text{if} \quad 0 < p - u \ll 1, \\
f(u) &= \text{const}_{2}(p+u)^2 \quad \text{if} \quad 0 < p + u \ll 1.
\end{align*}
\]

We suppose that parameter \(\lambda\) is sufficiently large \((\lambda \gg 1)\) and coupling parameter \(\gamma\) is sufficiently small \((0 < \gamma \ll 1)\). We consider three cases: \(\gamma = O((\ln \lambda)^{-1})\); \(\gamma = O(\lambda^{-\alpha}(\ln \lambda)^{-1})\), where \(0 < \alpha \leq 1/2\), and \(\gamma = O(\lambda^{-\alpha})\), where \(1/2 < \alpha < 1\) as \(\lambda \to +\infty\).
We study nonlocal dynamics of system (1) with special method of large parameter [3, 4, 5, 6, 7]. In phase space $C_{[-T,0]}(R^2)$ of system (1), we choose sets $S(x)$ depending on parameter $x$. Then we investigate the asymptotic behavior as $\lambda \to +\infty$ of all solutions of (1) with initial-value conditions from $S(x)$. We obtain that all considered solutions get into a set $S(\bar{x})$ after a certain time. We get representation $\bar{x} = \psi(x) + o(1)$ as $\lambda \to +\infty$. Then the first return operator $\Pi : \Pi S(x) \subset S(\bar{x})$ is constructed. Its iterations are principally determined by dynamics of mapping $\bar{x} = \psi(x)$. Theorem of correspondence is proved: rough stable (unstable) cycle of mapping $\bar{x} = \psi(x)$ corresponds to exponentially orbitally stable (unstable) relaxation periodic solution of initial problem.

2. Dynamics of system (1) under condition $\gamma = \gamma_1(\ln \lambda)^{-1}$

2.1. Building a mapping

Let

$$\gamma = \gamma_1(\ln \lambda)^{-1}, \quad \gamma_1 > 0. \quad (2)$$

Let us choose a set of initial conditions for system (1). Let $x$ be some value such that $|x| \geq 1$. Let value $k$ be equal either to 1 or $-1$. Let parameter $m$ be equal either to 0 or 1. Fix some values $x$, $k$, and $m$. Let $S(x) \subset C_{[-T,0]}(R^2)$ be a set of continuous functions $u_m(s)$ and $u_{1-m}(s)$ ($s \in [-T,0]$) such that

$$|u_m(s)| > p, \quad |u_{1-m}(s)| > p \quad \text{if} \quad s \in [-T,0],$$

$$u_m(0) = kp, \quad u_{1-m}(0) = xp, \quad kx > 0. \quad (3)$$

We build asymptotics of all solutions of system (1) with initial conditions from set $S(x)$ (see [6]). We get that each solution of system (1) with initial conditions from set $S(x)$ belongs to a set $S(\bar{x})$ after time $t_\ast$. It means that

$$|u_m(t)| > p, \quad |u_{1-m}(t)| > p \quad \text{for all} \quad t \in [t_\ast - T, t_\ast),$$

$$u_m(t_\ast) = kp, \quad u_{1-m}(t_\ast) = \bar{x}p, \quad k\bar{x} > 0.$$ 

Consequently, at point $t = t_\ast$ we return to initial situation with replacement $m$ by $\bar{m}$, $x$ by $\bar{x}$.

By asymptotics of solutions we obtain representation $t_\ast = (1 + o(1)) \ln \lambda$ as $\lambda \to +\infty$.

Let

$$R(x) = \begin{cases} 
(1 + \exp(-2\gamma_1))(1 - \exp(-2\gamma_1))^{-1} & \text{if} \quad |x| e^{-T} \geq 1, \\
g(2T)(1 + \exp(-2\gamma_1)) + g(2T, x)(1 - \exp(-2\gamma_1)) & \text{if} \quad |x| e^{-T} < 1,
\end{cases}$$

where

$$g(t) = \int_t^0 \exp(s - t)f(kp \exp(T - s))ds,$$

$$g(t, x) = \int_t^{T + \ln(x)} \exp(s - t)f(xp \exp(T - s))ds.$$

Then from asymptotics of solutions we obtain that

$$\bar{m} = \begin{cases} 
m & \text{if} \quad 0 < R(x) < 1, \\
1 - m & \text{if} \quad R(x) > 1;
\end{cases} \quad (4)$$

$$\bar{x} = k \max\{R(x), 1/R(x)\} + o(1). \quad (5)$$

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2.2. Theorem of correspondence
Let us consider the main part of mapping (5):

\[ x_{n+1} = k \max \{ R(x_n), 1/R(x_n) \}. \] (6)

We have the following theorem.

**Theorem 1.** Suppose that mapping (4), (6) has a cycle \((m_i, x_i)\) of period \(n\) and absolute value of multiplier of cycle \(x_1, x_2, \ldots, x_n\) is not equal to 1. Let (2) hold. Then for all positive and sufficiently large \(\lambda\) system (1) has a periodic relaxation solution of period \(T_*(\lambda) = n(1+o(1)) \ln \lambda\) as \(\lambda \to +\infty\) of the same stability as the cycle \((m_i, x_i)\).

Proof of existence of relaxation periodic solutions of system (1) follows directly from constructed asymptotics and special method of large parameter. One can see in [5] detailed proof (for another model) of the fact that rough cycle of constructed mapping has the same stability as corresponding relaxation periodic solution of initial problem.

It follows from asymptotics of solutions that if \(|x_i|e^{-T} < 1\), then amplitude of both components of solution is of order \(O(\lambda)\) as \(\lambda \to +\infty\) on this "step", and if \(|x_i|e^{-T} \geq 1\), then amplitude of component \(u_{m_i}\) is of order \(O(\lambda)\) and amplitude of component \(u_{1-m_i}\) is of order \(O(\lambda/\ln \lambda)\) as \(\lambda \to +\infty\). On Figure 1 typical view of inhomogeneous relaxation periodic solutions of system (1) is shown.

![Figure 1](image)

(a) Stable inhomogeneous periodic solution of system (1) with peaks of the same order (b) Stable inhomogeneous periodic solution of system (1) with peaks of different orders

2.3. Number of stable coexisting solutions of system (1)
The following result on dynamics of mapping (6) holds.

**Lemma 1.** If \(x_*\) is a fixed point of mapping (6) and multiplier of (6) at \(x = x_*\) is negative, then \(x_*\) is a stable fixed point of (6).

For a given set of initial parameters \((\gamma_1, T, p)\) and different functions \(f(u)\) mapping (6) may have different number of stable fixed points. For example, for \(\gamma_1 = 0.2, T = 1, p = 1\) and function \(F(u)\), shown on Figure 2, there are twelve stable fixed points of mapping (6) (see Figure 3).

Stable fixed point \(x = 1\) \((x = -1)\) of mapping (6) corresponds to exponentially orbitally stable positive (negative) homogeneous periodic relaxation solution of system (1). One stable fixed point \(x\) (where \(|x| > 1\)) of mapping (6) corresponds to two exponentially orbitally stable inhomogeneous periodic relaxation solutions of system (1) (we may take \(m = 0\) or \(m = 1\) in (3) and get two different solutions). Thus if mapping (6) has \(n\) stable fixed points \(x_i\) where \(|x_i| > 1\) \((i = 1, \ldots, n)\), then for all sufficiently large \(\lambda > 0\) system (1) has \(2n\) coexisting exponentially
Figure 2. Example of function $F(u)$. Values of parameters: $p = 1$.

Figure 3. Example of mapping (6) and mapping $x_{n+1} = x_n$ (dashed line) for (a) $x_n \leq -1$ (b) $x_n \geq 1$. Values of parameters: $\gamma_1 = 0.2$, $T = 1$, $p = 1$.

orbitally stable inhomogeneous periodic relaxation solutions. Note that all types of solutions: homogeneous, inhomogeneous with peaks of the same order and inhomogeneous with peaks of different orders at $\lambda \to +\infty$ may be stable simultaneously.

3. Dynamics of system (1) under condition $\gamma = \gamma_1 \lambda^{-\alpha} (\ln \lambda)^{-1}$

3.1. Building a mapping

Let

$$\gamma = \gamma_1 \lambda^{-\alpha} (\ln \lambda)^{-1}, \quad 0 < \alpha \leq \frac{1}{2}. \quad (7)$$

Let us define a set of initial conditions for system (1). Let, as in the previous case, value $k$ be equal either to 1 or $-1$ and parameter $m$ be equal either to 0 or 1. Let $x$ be some nonzero value ($x \neq 0$). Let $\beta$ be some value from $(0, \alpha]$. Fix some values $x$, $k$, $m$, and $\beta$. Denote $S(x) \subset C_{[-T,0]}(R^2)$ as a set of continuous on $s \in [-T,0]$ functions $u_m(s)$ and $u_{1-m}(s)$ such that

$$|u_m(s)| > p, \quad |u_{1-m}(s)| > p \quad \text{if} \quad s \in [-T,0),$$

$$u_m(0) = kp, \quad u_{1-m}(0) = xp\lambda^\beta, \quad kx > 0.$$
Note that now value $u_{1-m}(0)$ is asymptotically large as $\lambda \to +\infty$ (in previous case it had the order of unity).

As in the previous case, we build asymptotics of solutions of system (1) with initial conditions from set $S(x)$ and get that each solution after time $t_*$ belongs to a set $S(x)$.

In this case $t_* = (1-\alpha)(1+o(1))\ln \lambda$ as $\lambda \to +\infty$ and

$$u_m(t_*) = x_0\beta, \quad u_{1-m}(t_*) = kp, \quad k\bar{x} > 0.$$ 

Here

$$\beta = \alpha,$$ 

$$\bar{x} = \begin{cases} \frac{g(2T)}{|x_0\exp(-2T) + \frac{\beta}{2} g(2T)|} + o(1) & \text{if } \beta = \frac{1}{2}, \\ \frac{k[\gamma(1-\alpha)]^{-1} + o(1)}{\beta} & \text{if } \beta < \frac{1}{2}. \end{cases}$$ 

This implies that at point $t = t_*$ we return to initial situation with swap $u_m$ and $u_{1-m}$ and replacing $\beta$, $x_0$ by $\bar{x}$, $\bar{x}$ using (8), (9). Notice that beginning from the second ”step” we have $\beta = \alpha$.

If $\beta = \alpha = 1/2$, then for the leading term of $\bar{x}$ we have the following mapping

$$x_{n+1} = \frac{g(2T)}{|x_n\exp(-2T) + \frac{\beta}{2} g(2T)|} \quad (n = 1, 2, ...).$$

If $x_1 > 0$, then $k = 1$, $g(2T) > 0$ and sequence $x_n$ tends to a stable positive fixed point. If $x_1 < 0$, then $k = -1$, $g(2T) < 0$ and sequence $x_n$ tends to a stable negative fixed point.

If $0 < \alpha < 1/2$, then for the leading term of value $\bar{x}$ we obtain mapping

$$x_{n+1} = k[\gamma(1-\alpha)]^{-1}, \quad (n = 1, 2, ...).$$

Beginning from $x_2$ sequence $x_n$ is constant. This mapping has one stable positive fixed point and one negative stable fixed point.

3.2. Number of stable coexisting solutions of system (1)

Each stable fixed point of mapping (10) (mapping (11)) corresponds to two exponentially orbitally stable relaxation periodic solutions of system (1). Thus we obtain the following result.

Theorem 2. Let (7) hold. Then for all sufficiently large $\lambda > 0$ system (1) has four (two positive and two negative) exponentially orbitally stable relaxation periodic solutions with period $T_*(\lambda) = [2(1-\alpha) + o(1)]\ln \lambda$ as $\lambda \to +\infty$.

It follows from asymptotics of solutions that if $0 < \alpha < 1/2$, then both components of solution have two peaks on period (see Figure 4 (a)). One peak is of order $O(\lambda)$ and another peak is of order $O(\lambda^{1-\alpha}(\ln \lambda)^{-1})$ as $\lambda \to +\infty$. But if $\alpha = 1/2$, then both components have a single peak on period (see Figure 4 (b)). This peak is of order $O(\lambda)$ as $\lambda \to +\infty$.

4. Dynamics of system (1) under condition $\gamma = \gamma_1 \lambda^{-\alpha}$

Let

$$\gamma = \gamma_1 \lambda^{-\alpha}, \quad \gamma_1 > 0, \quad \frac{1}{2} < \alpha < 1.$$ 

Let us define set $S(x) \subset C_{[-T,0]}(R^2)$ of initial conditions. Let, as in the previous cases, parameter $k$ be equal either to 1 or $-1$ and parameter $m$ be equal either to 0 or 1. Let parameter $x$ be nonzero ($x \neq 0$). Let $\beta$ satisfy inequality $0 < \beta < \alpha$. Fix some values of parameters $k$, $x$, $m$, and $\beta$. Denote as $S(x)$ a set of continuous on $s \in [-T, 0]$ functions $u_m(s)$, $u_{1-m}(s)$ such that

$$|u_m(s)| > p, \quad |u_{1-m}(s)| > p \quad \text{if} \quad s \in [-T, 0]$$

$$\text{if} \quad s \in [-T, 0]$$

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Figure 4. (a) Stable inhomogeneous periodic solution of system (1) (case $\alpha = 0.2$) (b) Stable inhomogeneous periodic solution of system (1) (case $\alpha = 0.5$)

and

\[ u_m(0) = kp, \quad u_{1-m}(0) = xp\beta, \quad kx > 0. \]

We build asymptotics of all solutions of system (1) with initial conditions from set $S(x)$ (see [7]). We obtain that after time $t_*$ each considered solution belongs to a set $S(\bar{x})$. It means that

\[ |u_m(t)| > p, \quad |u_{1-m}(t)| > p \quad \text{for all} \quad t \in [t_* - T, t_*) \]

and

\[ u_m(t_*) = \bar{x}p\beta, \quad u_{1-m}(t_*) = kp, \quad k\bar{x} > 0. \]

From asymptotics of solutions we get representation

\[ t_* = \begin{cases} (\beta + o(1)) \ln \lambda & \text{if} \quad 1 - \alpha < \beta < \alpha, \\ (1 - \alpha + o(1)) \ln \lambda & \text{if} \quad 0 < \beta \leq 1 - \alpha. \end{cases} \]

Quantities $\bar{\beta}$ and $\bar{x}$ can be expressed by formulas

\[ \bar{\beta} = \begin{cases} 1 - \beta & \text{if} \quad 1 - \alpha < \beta < \alpha, \\ \alpha & \text{if} \quad 0 < \beta \leq 1 - \alpha; \end{cases} \]

\[ \bar{x} = \begin{cases} \frac{g(2T)}{|x|p\exp(-2T)} & \text{if} \quad 1 - \alpha < \beta < \alpha, \\ k[\gamma_1(1 - \alpha) \ln \lambda]^{-1} & \text{if} \quad 0 < \beta \leq 1 - \alpha \end{cases} \]

up to accuracy $o(1)$.

Thus at $t = t_*$ we return to initial situation with exchange $u_m$ and $u_{1-m}$ and replacing $\beta$ and $x$ by $\bar{\beta}$ and $\bar{x}$ respectively using (12), (13).

For $\beta$ from interval $(1 - \alpha, \alpha)$ we obtain an iterative process with respect to parameters $\beta$ and $x$

\[ \beta_{n+1} = 1 - \beta_n, \quad x_{n+1} = \frac{g(2T)}{|x_n|p\exp(-2T)}, \quad n = 1, 2, \ldots. \]

It follows from (14) that $\beta_{n+2} = \beta_n$ and $x_{n+2} = x_n$. Thus all solutions of mapping (14) are non-rough cycles of period two. For initial system (1) for all sufficiently large $\lambda > 0$ one can assert existence of non-rough inhomogeneous relaxation periodic asymptotic (with respect to the residual) solutions of period $T_*(\lambda) = (1 + o(1)) \ln \lambda$, where the values of parameters $\beta$ and
with an accuracy up to $o(1)$ vary according to (14). Conclusions about existence of exact periodic solutions and their stability cannot be made.

Note that each component of the non-rough asymptotic (with respect to the residual) solutions of system (1) once on period $T_\ast(\lambda)$ in a short time interval (of the order of $O(1)$) increases from a value of the order of unity to a value of the order $O(\lambda)$ and the remaining time it decreases exponentially to a value of the order of unity.

5. Conclusions
In this paper we study nonlocal dynamics of a model of two coupled oscillators with compactly supported feedback function. We show that studying dynamics of initial system of delay differential equations can be reduced to studying dynamics of finite-dimensional mappings: rough stable (unstable) cycles of these mappings correspond to exponentially orbitally stable (unstable) relaxation periodic solutions of initial problem. Dynamics of system (1) depends on the value of coupling coefficient crucially. If $\gamma = O(1)$, then the only stable solution of system (1) with initial conditions from set $S(x)$ is homogeneous relaxation solution; if $\gamma = O((\ln \lambda)^{-1})$, then homogeneous relaxation solution of system (1) may be stable or unstable and many stable inhomogeneous relaxation solutions may coexist; if $\gamma = O(\lambda^{-\alpha}(\ln \lambda)^{-1})$ (where $0 < \alpha \leq 1/2$), then four stable inhomogeneous relaxation periodic solutions coexist; and if $\gamma = O(\lambda^{-\alpha})$ (where $1/2 < \alpha < 1$), then system (1) has a two-parameter family of non-rough inhomogeneous relaxation periodic asymptotic (with respect to the residual) solutions.

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