SINGULAR KÄHLER-EINSTEIN METRICS ON Q-FANO
COMPACTIFICATIONS OF LIE GROUPS

YAN LI*, GANG TIAN† AND XIAOHUA ZHU‡

Abstract. In this paper, we prove an existence result for Kähler-Einstein
metrics on Q-Fano compactifications of Lie groups. As an application, we
classify Q-Fano compactifications of $SO_4(\mathbb{C})$ which admit a Kähler-Einstein
metric with the same volume as that of a smooth Fano compactification of
$SO_4(\mathbb{C})$.

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1. Introduction

Let \( G \) be an \( n \)-dimensional connected, complex reductive Lie group which is the complexification of a compact Lie group \( K \). Let \( T^C \) be a maximal Cartan torus of \( G \) whose dimension is \( r \). Denote by \( \Phi_+ \) a positive roots system associated to \( T^C \). Put

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha.
\]

It can be regarded as a character in \( a^* \), where \( a^* \) is the dual space of real part \( a \) of Lie algebra of \( T^C \). Let \( \pi \) be a function on \( a^* \) defined by

\[
\pi(y) = \prod_{\alpha \in \Phi_+} (\alpha, y)^2, \quad y \in a^*,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the Cartan-Killing inner product on \( a^* \).

Let \( M \) be a \( \mathbb{Q} \)-Fano compactification of \( G \). Since \( M \) contains a closure \( Z \) of \( T^C \)-orbit, there is an associated moment polytope \( P \) of \( Z \) induced by \( (M, -K_M) \) \([3, 4]\). Let \( P_+ \) be the positive part of \( P \) defined by

\[
P_+ = \{ y \in P | \langle \alpha, y \rangle > 0, \forall \alpha \in \Phi_+ \}.
\]

Denote by \( 2P_+ \) a dilation of \( P_+ \) at rate 2. We define the barycenter of \( 2P_+ \) with respect to the weighted measure \( \pi(y)dy \) by

\[
\text{bar}(2P_+) = \frac{\int_{2P_+} y\pi(y) \, dy}{\int_{2P_+} \pi(y) \, dy}.
\]

In \([17]\), Delcroix proved the following theorem for Kähler-Einstein metrics on smooth Fano compactifications of \( G \).

**Theorem 1.1.** Let \( M \) be a smooth Fano \( G \)-compactification. Then \( M \) admits a Kähler-Einstein metric if and only if

\[
\text{bar}(2P_+) \in 4\rho + \Xi,
\]

where \( \Xi \) is the relative interior of the cone generated by \( \Phi_+ \).

Another proof of Theorem 1.1 was given by Li, Zhou and Zhu \([31]\). They also showed that (1.2) is actually equivalent to the \( K \)-stability condition in terms of \([37]\) and \([21]\) by constructing \( \mathbb{C}^* \)-action through piecewisely rationally linear function which is invariant under the Weyl group action. In particular, it implies that \( M \) is \( K \)-unstable if \( \text{bar}(2P_+) \notin 4\rho + \Xi \). A more general construction of \( \mathbb{C}^* \)-action was also discussed in \([18]\).

In the present paper, we extend the above theorem to \( \mathbb{Q} \)-Fano compactifications of \( G \) which may be singular. It is well known that any \( \mathbb{Q} \)-Fano compactification of \( G \) has klt-singularities \([5]\). For a \( \mathbb{Q} \)-Fano variety \( M \) with klt-singularities, there is naturally a class of admissible Kähler metrics induced by the Fubini-Study metric (cf. \([20]\)). In \([10]\), Berman, Boucksom, Eyssidieux, Guedj and Zeriahi introduce a class of Kähler potentials associated to admissible Kähler metrics and refer it as the \( \mathcal{E}^1(M, -K_M) \) space. Then they define the singular Kähler-Einstein metric on \( M \) with the Kähler potential in \( \mathcal{E}^1(M, -K_M) \) via the complex Monge-Ampère equation, which is the usual Kähler-Einstein metric on the smooth part of \( M \). It is

\(^1\)Without of confusion, we also write \( \langle \alpha, y \rangle \) as \( \alpha(y) \) for simplicity.
an natural problem to establish an extension of the Yau-Tian-Donaldson conjecture we have solved for smooth Fano manifolds \([37, 38]\), that is, an equivalence relation between the existence of such singular Kähler-Einstein metrics and the K-stability on a \(\mathbb{Q}\)-Fano variety with klt-singularities. There are many recent works on this fundamental problem. We refer the readers to \([9, 13, 29, 30, 28]\), etc..

In this paper, we will assume that the moment polytope \(P\) of \(Z\) is fine in sense of \([22]\), namely, each vertex of \(P\) is the intersection of precisely \(r\) facets. We will prove

**Theorem 1.2.** Let \(M\) be a \(\mathbb{Q}\)-Fano compactification of \(G\) such that the moment polytope \(P\) of \(Z\) is fine. Then \(M\) admits a singular Kähler-Einstein metric if and only if (1.2) holds.

By a result of Abreu \([1]\), the polytope \(P\) of \(Z\) being fine is equivalent to that the metric induced by the Guillemin function can be extended to a Kähler orbifold metric on \(Z\). It follows from the fineness assumption of \(P\) in Theorem 1.2 that the Guillemin function of \(2P\) induces a \(K \times K\)-invariant singular metric \(\omega_{2P}\) in \(E^1(M, -K_M)\) (cf. Lemma 3.4). Moreover, we can prove that the Ricci potential of \(\omega_{2P}\) on \(M\) is uniformly bounded above. We note that \(P\) is always fine when \(\text{rank}(G) = 2\) \([23, \text{Chapter 3}]\). Thus for a \(\mathbb{Q}\)-Fano compactification of \(G\) with \(\text{rank}(G) = 2\), \(M\) admits a singular Kähler-Einstein metric if and only if (1.2) holds. As an application of Theorem 1.2, we show that there is only one example of non-smooth Gorenstein Fano \(SO_4(\mathbb{C})\)-compactifications which admits a singular Kähler-Einstein metric (cf. Section 7.1).

On the other hand, it has been shown in \([17]\) and \([33]\) that there are only three smooth Fano compactifications of \(SO_4(\mathbb{C})\), i.e., Case-1.1.2, Case-1.2.1 and Case-2 in Section 7.1. The first two manifolds do not admit any Kähler-Einstein metric. By Theorem 1.2, we further prove

**Theorem 1.3.** There is no \(\mathbb{Q}\)-Fano compactification of \(SO_4(\mathbb{C})\) which admits a singular Kähler-Einstein metric with the same volume as Case-1.1.2 or Case-1.2.1 in Section 7.1.

Theorem 1.3 gives a partial answer to a question proposed in \([33]\) about limit of Kähler-Ricci flow on either Case-1.1.2 or Case-1.2.1. It has been proved there that the flow has type II singularities on each of Case-1.1.2 and Case-1.2.1 whenever the initial metric is \(K \times K\)-invariant. By the Hamilton-Tian conjecture \([37, 7, 14]\), the limit should be a \(\mathbb{Q}\)-Fano variety with a singular Kähler-Ricci soliton of the same volume as that of initial metric. However, by Theorem 1.3 the limit cannot be a \(\mathbb{Q}\)-Fano compactification of \(SO_4(\mathbb{C})\) with a singular Kähler-Einstein metric. This implies that the limiting soliton will has less homogeneity than the initial one, which is totally different from the situation of smooth convergence of \(K \times K\)-metrics on a smooth compactification of Lie group \([33]\).

As in \([10]\), we use the variation method to prove Theorem 1.2 more precisely, we will prove that a modified version of the Ding functional \(D(\cdot)\) is proper under the condition (1.2). This functional is defined for a class of convex functions \(E_{K \times K}^1(2P)\) associated to \(K \times K\)-invariant metrics on the orbit of \(G\) (cf. Section 4, 6). The key point is that the Ricci potential \(h_0\) of the Guillemin metric \(\omega_{2P}\) is bounded from above when \(P\) is fine (cf. Proposition 5.1). This enables us to control the

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2It can not be guaranteed that the \(G\)-compactification is smooth even if \(Z\) is smooth \([5]\).
nonlinear part $\mathcal{F}(\cdot)$ of $\mathcal{D}(\cdot)$ by modifying $\mathcal{D}(\cdot)$ as done in [21, 32] (cf. Section 6.1). We shall note that it is in general impossible to get a lower bound of $h_0$ if the compactification is a singular variety (cf. Remark 5.2). On the other hand, we expect that the “fine” condition in Theorem 1.2 can be dropped.

The minimizer of $\mathcal{D}(\cdot)$ corresponds to a singular Kähler-Einstein metric. We will prove the semi-continuity of $\mathcal{D}(\cdot)$ and derive the Kähler-Einstein equation for the minimizer (cf. Proposition 6.6). Our proof is similar to what Berman and Berndtsson studied on toric varieties in [9].

The proof of the necessity part of Theorem 1.2 is same as one in Theorem 1.1. In fact, a Q-Fano compactification of $G$ is $K$-unstable if $\bar{\text{b}} \mathcal{P}(2P_+) \not\in 4\rho + \Xi$ [31]. This will be a contradiction to the semi-stability of Q-Fano variety with a singular Kähler-Einstein metric (cf. [29]). We will omit this part.

The organization of paper is as follows. In Section 2, we recall some notations in [10] for singular Kähler-Einstein metrics on Q-Fano varieties. In Section 3, we introduce a subspace $\mathcal{E}_{K \times K}(M, -K_M)$ of $\mathcal{E}_1(M, -K_M)$ and prove that the Guillemin function lies in this space (cf. Lemma 3.4). In Section 4, we prove that $\mathcal{E}_{K \times K}(M, -K_M)$ is equivalent to a dual space $\mathcal{E}_{K \times K}(2P)$ of Legendre functions (cf. Theorem 4.2). In Section 5, we compute the Ricci potential $h_0$ of $\omega_2$ and show that it is bounded from above (cf. Proposition 5.1). The sufficient part of Theorem 1.2 will be proved in Section 6. In Section 7, we construct many Q-Fano compactifications of $SO_4(\mathbb{C})$ and in particular, we will prove Theorem 1.3.
The MA-measure $\omega^n_\varphi$ with a full MA-mass has no mass on the pluripolar set of $\varphi$ in $M$. Thus we need to consider the measure on $M_{\text{reg}}$. Moreover, $e^{-\varphi}$ is $L^p$-integrable for any $p > 0$ associated to $\omega^0_\varphi$.

**Definition 2.1.** We call $\omega^n_\varphi$ a (singular) Kähler-Einstein metric on $M$ with full MA-mass if $\varphi$ satisfies the following complex Monge-Ampère equation,

$$\omega^n_\varphi = e^{h_0 - \varphi} \omega_0^n.$$  (2.1)

It has been shown in [10] that $\varphi$ is $C^\infty$ on $M_{\text{reg}}$ if it is a solution of (2.1). Thus $\omega^n_\varphi$ satisfies the Kähler-Einstein equation on $M_{\text{reg}}$,

$$\text{Ric}(\omega^n_\varphi) = \omega^n_\varphi.$$  

2.1. The space $\mathcal{E}^1(M, -K_M)$ and the Ding functional. On a smooth Fano manifold, there is a well-known Euler-Langrange functional for Kähler potentials associated to (2.1), often referred as the Ding functional or F-functional, defined by (cf. [19, 36])

$$F(\phi) = -\frac{1}{(n+1)\mathcal{V}} \sum_{k=0}^{n} \int_M \phi \omega^k_\phi \wedge \omega_0^{n-k} - \log \left( \frac{1}{\mathcal{V}} \int_M e^{h_0 - \phi} \omega_0^n \right).$$  (2.2)

In case of $Q$-Fano manifold with klt-singularities, Berman, Boucksom, Eyssidieux, Guedj and Zeriahi [10] extended $F(\cdot)$ to the space $\mathcal{E}^1(M, -K_M)$ defined by

$$\mathcal{E}^1(M, -K_M) = \{ \phi \mid \phi \text{ has a full MA mass and}$$ 

$$\sup_{M} \phi = 0, \quad I(\phi) = \int_M -\phi \omega_0^n < \infty \}.$$ 

They showed that $\mathcal{E}^1(M, -K_M)$ is compact in certain weak topology. By a result of Davas [20], $\mathcal{E}^1(M, -K_M)$ is in fact compact in the topology of $L^1$-distance. It provides a variational approach to study (2.1).

**Definition 2.2.** [37, 10] The functional $F(\cdot)$ is called proper if there is a continuous function $p(t)$ on $\mathbb{R}$ with the property $\lim_{t \to +\infty} p(t) = +\infty$, such that

$$F(\phi) \geq p(I(\phi)), \quad \forall \phi \in \mathcal{E}^1(M, -K_M).$$  (2.3)

In [10], Berman, Boucksom, Eyssidieux, Guedj and Zeriahi proved the existence of solutions for (2.1) under the properness assumption (2.3) of $F(\cdot)$. However, this assumption does not hold in the case of existence of non-zero holomorphic vector fields such as in our case of $Q$-Fano $G$-compactifications. So we need to consider the reduced Ding functional instead to overcome this new difficulty as done on toric varieties [9, 32].

3. Moment polytope and $K \times K$-invariant metrics

Let $M$ be a $Q$-Fano compactification of $G$ with $Z$ being the closure of a maximal complex torus $T^C$-orbit. We first characterize the moment polytope $P$ of $Z$ associated to $(M, K_M^{-1})$. Let $\{F_A\}_{A=1, \ldots, d_0}$ be the facets of $P$ and $\{F_A\}_{A=1, \ldots, d_+}$ be those whose interior intersects $\mathfrak{a}_+^*$. Suppose that

$$P = \cap_{A=1}^{d_0} \{ F_A : = \lambda_A - u_A(y) \geq 0 \}$$  (3.1)

for some prime vector $u_A \in \mathfrak{g}$ and the facet

$$F_A \subseteq \{ l_0^A = 0 \}, \quad A = 1, \ldots, d_0.$$
Let \( W \) be the Weyl group of \((G, T^\mathbb{C})\). By the \( W \)-invariance, for each \( A \in \{1, \ldots, d_0\} \), there is some \( w_A \in W \) such that \( w_A(F_A) \in \{F_B\}_{B=1, \ldots, d_+} \). Denote by \( \rho_A = w_A^{-1}(\rho) \), where \( \rho \in \mathfrak{a}_+^* \) is given by \([1, 1]\). Then \( \rho_A(u_A) \) is independent of the choice of \( w_A \in W \) and hence it is well-defined.

The following is due to \([12]\).

**Lemma 3.1.** Let \( M \) be a \( \mathbb{Q} \)-Fano compactification of \( G \) with \( P \) being the associated moment polytope. Then for each \( A = 1, \ldots, d_0 \), it holds

\[
(3.2) \quad \lambda_A = 1 + 2\rho_A(u_A).
\]

**Proof.** Suppose that \( -mK_M \) is a Cartier divisor for some \( m \in \mathbb{N} \). Then by \([12\text{ Section 3}],\)

\[
-mK_M|_Z = \sum_{A} m(1 + 2\rho_A(u_A))D_A,
\]

where \( D_A \) is the toric divisor of \( Z \) associated to \( u_A \). Thus the associated polytope of \((Z, -mK_M|_Z)\) is given by

\[
P(Z, -mK_M|_Z) = \cap_{A=1}^{d_0} \{m(1 + 2\rho_A(u_A)) - u_A(y) \geq 0\},
\]

which is precisely \( mP \). Thus \((3.2)\) is true. \( \square \)

### 3.1. \( K \times K \)-invariant metrics.

On a \( \mathbb{Q} \)-Fano compactification of \( G \), we may regard the \( G \times G \) action as a subgroup of \( PGL_{N+1}(\mathbb{C}) \) which acts holomorphically on the hyperplane bundle \( L = \mathcal{O}_{\mathbb{C}P^N}(-1) \). Then any admissible \( K \times K \)-invariant Kähler metric \( \omega_\phi \in \mathbb{Z}_c t_1(L) \) can be regarded as a restriction of \( K \times K \)-invariant Kähler metric of \( \mathbb{C}P^N \). Thus the moment polytope \( P \) associated to \((Z, L|_Z)\) is a \( W \)-invariant rational polytope in \( \mathfrak{a}^* \). By the \( K \times K \)-invariance, the restriction of \( \omega_\phi \) on \( T^\mathbb{C} \) is an open toric Kähler metric. Hence, it induces a strictly convex, \( W \)-invariant function \( \psi_\phi \) on \( \mathfrak{a} \) \([6]\) (also see Lemma 3.3 below) such that

\[
(3.3) \quad \omega_\phi = -\frac{i}{2} \partial \overline{\partial} \psi_\phi, \text{ on } T^\mathbb{C}.
\]

By the \( KAK \)-decomposition \([27\text{ Theorem 7.39}],\) for any \( g \in G \), there are \( k_1, k_2 \in K \) and \( x \in \mathfrak{a} \) such that \( g = k_1 x k_2 \). Here \( x \) is uniquely determined up to a \( W \)-action. This means that \( x \) is unique in \( \mathfrak{a}_+^* \). Thus there is a bijection between \( K \times K \)-invariant functions \( \Psi \) on \( G \) and \( W \)-invariant functions \( \psi \) on \( \mathfrak{a} \) which is given by

\[
\Psi(\exp(\cdot)) = \psi(\cdot) : \mathfrak{a} \to \mathbb{R}.
\]

Clearly, when a \( W \)-invariant \( \psi \) is given, \( \Psi \) is well-defined. Without of confusion, we will not distinguish \( \psi \) and \( \Psi \), and call \( \Psi \) convex on \( G \) if \( \psi \) is convex on \( \mathfrak{a} \).

The following \( KAK \)-integral formula can be found in \([27\text{ Proposition 5.28}]\).

**Proposition 3.2.** Let \( dV_G \) be a Haar measure on \( G \) and \( dx \) the Lebesgue measure on \( \mathfrak{a} \). Then there exists a constant \( C_H > 0 \) such that for any \( K \times K \)-invariant, \( dV_G \)-integrable function \( \psi \) on \( G \),

\[
\int_G \psi(g) \ dV_G = C_H \int_{\mathfrak{a}_+} \psi(x) J(x) \ dx,
\]

where

\[
J(x) = \prod_{\alpha \in \Phi_+} \sinh^2 \alpha(x).
\]
Without loss of generality, we may normalize $C_H = 1$ for simplicity.

Next we recall a local holomorphic coordinate system on $G$ used in [17]. By the standard Cartan decomposition, we can decompose $g$ as

$$g = (t \oplus a) \oplus (\oplus_{\alpha \in \Phi} V_\alpha),$$

where $t$ is the Lie algebra of $T$ and $\Phi$ is the set of positive roots and $\alpha_\Phi$ is the Cartan involution of $g$.

Next we recall a local holomorphic coordinate system on $G$ [17, Theorem 1.2]. For a $\alpha \in G$ such that $X_{-\alpha} = -t(X_\alpha)$ and $[X_\alpha, X_{-\alpha}] = \alpha^\vee$, where $t$ is the Cartan involution and $\alpha^\vee$ is the dual of $\alpha$ by the Killing form. Let $E_\alpha = X_\alpha - X_{-\alpha}$ and $E_{-\alpha} = J(X_\alpha + X_{-\alpha})$. Denoted by $\mathfrak{t}_\alpha$, $\mathfrak{t}_{-\alpha}$ the real line spanned by $E_\alpha$, $E_{-\alpha}$, respectively. Then we get the Cartan decomposition of Lie algebra $\mathfrak{t}$ of $K$ as follows,

$$\mathfrak{t} = \mathfrak{t} \oplus (\oplus_{\alpha \in \Phi_+} (\mathfrak{t}_\alpha \oplus \mathfrak{t}_{-\alpha})).$$

Choose a real basis $\{E_1^0, ..., E^n_0\}$ of $\mathfrak{t}$, where $r$ is the dimension of $T$. Then $\{E_1^0, ..., E^n_r\}$ together with $\{E_\alpha, E_{-\alpha}\}_{\alpha \in \Phi_+}$ forms a real basis of $\mathfrak{t}$, which is indexed by $\{E_1, ..., E_n\}$. We can also regard $\{E_1, ..., E_n\}$ as a complex basis of $g$. For any $g \in G$, we define local coordinates $\{z^i(g)\}_{i=1,...,n}$ on a neighborhood of $g$ by

$$(z^i(g)) \rightarrow \exp(z^i(g)E_i)g.$$

It is easy to see that $\theta^i|_g = dz^i(g)|_g$, where the dual $\theta^i$ of $E_i$ is a right-invariant holomorphic 1-form. Thus $\wedge_{i=1}^n (dz^1(g) \wedge dz^n(g))|_g$ is also a right-invariant $(n, n)$-form, which defines a Haar measure $dV_g$.

For a $K \times K$-invariant function $\phi$, Delcroix computed the Hessian of $\phi$ in the above local coordinates as follows [17 Theorem 1.2].

**Lemma 3.3.** Let $\phi$ be a $K \times K$-invariant function on $G$. Then for any $x \in a_+^*$, the complex Hessian matrix of $\phi$ in the above coordinates is diagonal by blocks, and equals to

$$\text{Hess}_C(\phi)(\exp(x)) = \begin{pmatrix}
\frac{1}{2}Hess_R(\phi)(x) & 0 & 0 \\
0 & M_{\alpha(i)}(x) & 0 \\
0 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & & & M_{\alpha(i)\frac{n-2}{2}}(x)
\end{pmatrix},$$

where $\Phi_+ = \{\alpha(1), ..., \alpha(n-2)\}$ is the set of positive roots and

$$M_{\alpha(i)}(x) = \frac{1}{2} \alpha(i)(\partial^2 \psi(x)) \begin{pmatrix}
\coth \alpha(i)(x) & \sqrt{-1} \\
\sqrt{-1} & \coth \alpha(i)(x)
\end{pmatrix}.$$

By (3.4) in Lemma 3.3 we see that $\psi_\phi$ induced by an admissible $K \times K$-invariant Kähler form $\omega_\phi$ is convex on $a^*$. The complex Monge-Ampère measure is given by

$$\omega^n = (\sqrt{-1} \partial \bar{\partial} \psi_\phi)^n = MA_C(\psi_\phi) dV_G,$$

where

$$\text{MA}_C(\psi_\phi)(\exp(x)) = \frac{1}{2^{r+n}} \text{MA}_R(\psi_\phi)(x) \frac{1}{J(x)} \prod_{\alpha \in \Phi_+} \alpha^2(\partial \psi_\phi(x)).$$
In particular, by Proposition 3.2,
\[
\text{Vol}(M) = \int_M \omega_\psi^n = \int_{2P^+} \pi \, dy = V.
\]
Clearly, (3.5) also holds for any Kähler potential in \( \mathcal{E}^1(M, -K_M) \), which is smooth and \( K \times K \)-invariant on \( G \). For the completeness, we introduce a subspace of \( \mathcal{E}^1(M, -K_M) \) by
\[
\mathcal{E}^1_{K \times K}(M, -K_M) = \{ \phi \in \mathcal{E}^1(M, -K_M) \mid \phi \text{ is } K \times K \text{-invariant and convex on } G \}.
\]
(3.7)
\[\mathcal{E}^1_{K \times K}(M, -K_M) \text{ is locally precompact in terms of convex functions on } G. \]
The subspace of \( \mathcal{E}^1_{K \times K}(M, -K_M) \) will also prove its completeness by using the Legendre dual in subsequent Sections 4, 6.

3.2. Fine polytope \( P \). In this subsection, we show that the Legendre dual of Guillemin function \( u_{2P} \) on \( 2P \) lies in \( \mathcal{E}^1_{K \times K}(M, -K_M) \) when \( P \) is fine.

Recall (3.1). For convenience, we set
\[
l_A(y) = 2\lambda_A - u_A(y).
\]
Then
\[
2P = \cap_{A=1}^{d_0} \{ l_A(y) \geq 0 \}.
\]
Thus, \( u_{2P} \) is given by (cf. [1])
\[
u_{2P} = \frac{1}{2} \sum_{A=1}^{d_0} l_A(y) \log l_A(y).
\]
Clearly, it is \( W \)-invariant, so its Legendre function \( \psi_{2P} \) is also \( W \)-invariant, where
\[
\psi_{2P}(x) = \sup_{y \in 2P} (\langle x, y \rangle - u_{2P}(y)), \quad \forall \ x \in a.
\]
(3.8)
Hence, by [1, Theorem 2] and [6] (also see Lemma 3.3),
we can extend
\[
\omega_{2P} = \sqrt{-1} \partial \bar{\partial} \psi_{2P}, \quad \text{on } a,
\]
to a \( K \times K \)-invariant metric on \( G \).

Lemma 3.4. Let \( \psi_0 \) be a Kähler potential of admissible \( K \times K \)-invariant metric \( \omega_0 \) as in (3.3). Assume that \( P \) is fine. Then the Kähler potential \( (\psi_{2P} - \psi_0) \) of \( \omega_{2P} \) lies in \( \varphi \in L^\infty(M) \cap C^\infty(M_{\text{reg}}) \). In particular, \( (\psi_{2P} - \psi_0) \in \mathcal{E}^1_{K \times K}(M, -K_M) \).

Proof. Fix an \( m_0 \in \mathbb{Z}_+ \) such that \(-m_0K_X\) is very ample. We consider the projective embedding
\[
\iota : M \to \mathbb{C}P^N
\]
given by \( | -m_0K_M| \), where \( N = h^0(M, -m_0K_M) - 1 \). By [34, Section 2.3], the pull back of the Fubini-Study metric on \( \mathbb{C}P^N \) gives a \( K \times K \)-invariant, Hermitian metric \( h \) on \( L = O_{\mathbb{C}P^N}(-1)|_M \) (also see [35]). Moreover, we have
\[
h|_{T^C}(x) = \left( \sum_{\lambda \in mP+\mathbb{R}} \bar{n}(\lambda)e^{2\lambda(x)} \right),
\]
where \( \bar{n}(\lambda) \in \mathbb{Z}_+ \). Thus we have a Kähler potential on \( T^C \) by
\[
\psi_{FS} = \frac{1}{m} \log h|_{T^C}.
\]
\[\psi_{FS} \text{ is given by } \frac{1}{2} \nabla \psi_{2P}, \quad \text{whose image is } P.\]
Since $P$ is fine, one can show directly that
\[ \psi_{FS} \in \mathcal{V}(2P) = \{ \psi \in C^0(\mathfrak{a}) \mid \psi \text{ is convex, } W\text{-invariant} \}
\quad \text{and } \max \limits_{\mathfrak{a}} |v_{2P} - \psi| < \infty, \]
where $v_{2P}(\cdot)$ is the support function on $\mathfrak{a}$ defined by
\[ v_{2P}(x) = \sup \limits_{y \in 2P} \langle x, y \rangle. \tag{3.10} \]
Recall that the Legendre function $u_\psi$ of $\psi$ is defined as in (3.8) by
\[ u_\psi(y) = \sup \limits_{x \in a} (\langle x, y \rangle - \psi(x)), \quad y \in 2P. \tag{3.11} \]
It is known that $\psi(x) \in \mathcal{V}(2P)$ if and only if $u_\psi$ is uniformly bounded on $2P$ since the Legendre function of $v_{2P}$ is zero (cf. [34]). Thus the Legendre function $u_h$ of $h|_{T^c}(x)$ is uniformly bounded on $2P$. It follows that
\[ |u_h - u_{2P}| \leq C. \]
Hence, we get
\[ \max \limits_{\mathfrak{a}} |\psi_{FS} - \psi_{2P}| < +\infty. \]
Consequently,
\[ \max \limits_{\mathfrak{a}} |\psi_{2P} - \psi_0| < +\infty. \]
By [3.6], $(\psi_{2P} - \psi_0)$ has full MA-mass, so we have completed the proof. \qed

4. The space $\mathcal{E}^1_{K \times K}(2P)$

In this section, we describe the space $\mathcal{E}^1_{K \times K}(M, -K_M)$ in (3.7) via Legendre functions as in [15] for Q-Fano toric varieties. Let $\psi_0$ be a Kähler potential of admissible $K \times K$-invariant metric $\omega_0$ as in (3.3). Then we can normalize $\psi_0$ by (cf. [33]),
\[ \inf \limits_{\mathfrak{a}} \psi_0 = \psi_0(O) = 0, \tag{4.1} \]
where $O$ is the origin in $\mathfrak{a}$. Thus for any $\phi \in \mathcal{E}^1_{K \times K}(M, -K_M)$, $\psi_\phi = \psi_0 + \phi$ can be also normalized as in (4.1).

The following lemma is elementary.

**Lemma 4.1.** For any $K \times K$-invariant potential $\phi$ normalized as in (4.1), it holds
\[ \partial(\psi_\phi) \subseteq 2P, \quad \text{and } \psi_\phi \leq v_{2P}, \]
where $\partial(\psi_\phi)(\cdot)$ is the normal mapping of $\psi_\phi$.

**Proof.** We choose a sequence of decreasing and uniformly bounded $K \times K$-invariant potential $\phi_i$ normalized as in (4.1) such that
\[ \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_i > 0, \quad \text{in } M_{reg} \]
and
\[ \phi_i \to \phi, \quad \text{as } i \to +\infty. \]
Then
\[ \sqrt{-1} \partial \bar{\partial} \psi_{\phi_i} > 0 \quad \text{in } G. \]
It follows that
\[ \partial \psi_{\phi_i} \subseteq 2P. \]
This implies that $\partial \psi_\phi \subseteq 2P$. By the convexity, we also get $\psi_\phi \leq v_{2P}$. 

It is easy to see that the Legendre function $u_\phi$ of $\psi_\phi$ with $\phi \in \mathcal{E}_{K \times K}^1(M, -K_M)$ satisfies
\begin{equation}
\inf_{2P} u_\phi = u_\phi(O) = 0.
\end{equation}
We set a class of $W$-invariant convex functions on $2P$ by
$$\mathcal{E}_{K \times K}^1(2P) = \{ u | u \text{ is convex, } W\text{-invariant on } 2P \text{ which satisfies } (4.2) \text{ and } \int_{2P} u \pi dy < +\infty \}.$$

The main goal in this section is to prove

**Theorem 4.2.** A Kähler potential $\phi \in \mathcal{E}_{K \times K}^1(M, -K_M)$ with normalized $\psi_\phi$ satisfying (4.1) if and only if the Legendre function $u_\phi$ of $\psi_\phi$ lies in $\mathcal{E}_{K \times K}^1(2P)$. As a consequence, $u_\phi$ is locally bounded in $\text{Int}(2P)$ if $\phi \in \mathcal{E}_{K \times K}^1(M, -K_M)$.

As in [15], we need to establish a comparison principle for the complex Monge-Ampère measure in $\mathcal{E}_{K \times K}^1(M, -K_M)$. For our purpose, we will introduce a weighted Monge-Ampère measure in the following.

**4.1. Weighted Monge-Ampère measure.**

**Definition 4.3.** Let $\Omega \subseteq a$ be a $W$-invariant domain and $\psi$ any $W$-invariant convex function on $\Omega$. Define a weighted Monge-Ampère measure on $\Omega$ by
$$\int_{\Omega'} \text{MA}_{R, \pi}(\psi) dx = \int_{\partial \psi(\Omega')} \pi dy, \ \forall \ W\text{-invariant } \Omega' \subseteq \Omega,$$
where $\partial \psi(\cdot)$ is the normal mapping of $\psi$.

**Remark 4.4.** Let $\psi_k$ be a sequence of convex functions which converge locally uniformly to $\psi$ on $\Omega$, then $\text{MA}_{R, \pi}(\psi_k)$ converge to $\text{MA}_{R, \pi}(\psi)$ (cf. [2, Section 15]). This follows from the fact:
$$\liminf_{k \to +\infty} \partial \psi_k(U) \supseteq \partial \psi(U), \ \forall \ W\text{-invariant open subset } U \subseteq \Omega.$$

**Lemma 4.5.** Let $\omega_\phi = \sqrt{-1} \partial \bar{\partial} \psi_\phi$ with $\phi \in \mathcal{E}_{K \times K}^1(M, -K_M)$. Then for any $K \times K$-invariant continuous uniformly bounded function $f$ on $G$, it holds
\begin{equation}
\int_M f \omega_\phi^n = \int_{a_+} f \text{MA}_{R, \pi}(\psi_\phi) dx.
\end{equation}

**Proof.** First we assume that $f$ is a $K \times K$-invariant continuous function with compact support on $a$. We take a sequence of smooth $W$-invariant convex functions $\psi_k \searrow \psi$ and let $\omega_k = \sqrt{-1} \partial \bar{\partial} \psi_k$. Then for any $W$-invariant $\Omega' \subseteq a$, it holds
$$\int_{\Omega'} \text{MA}_{R, \pi}(\psi_k) dx := \int_{\Omega'} \det(\nabla^2 \psi_k) \pi(\nabla \psi_k) dy.$$
By the standard $KAK$-integration formula, it follows that
$$\int_M f \omega_k^n = \int_{a_+} f \det(\nabla^2 \psi_k) \pi(\nabla \psi_k) dx = \int_{a_+} f \text{MA}_{R, \pi}(\psi_k) dx.$$
Since
$$\int_M f \omega_k^n \to \int_M f \omega^n,$$
it follows from Remark 4.4 that (4.3) is true.

Next we choose a sequence of exhausting $W$-invariant convex domains $\Omega_k$ in $a$ and a sequence of $W$-invariant convex functions with the support on $\Omega_{k+1}$ such that $f_k = f|_{\Omega_k}$. Since $\omega^n$ has full MA-mass, we get

$$\int_M f_k \omega^n = \lim_k \int_M f_k \omega^n = \lim_k \int_{a_+} f_k \text{MA}_{R;\pi}(\psi) dx = \lim_k \int_{a_+} f \text{MA}_{R;\pi}(\psi) dx.$$

□

4.2. Comparison principles. We establish the following comparison principle for the weighted Monge-Ampère measure $\text{MA}_{R;\pi}(\psi)$.

Proposition 4.6. Let $\Omega \subseteq a$ be a $W$-invariant domain and $\varphi, \psi$ be two convex functions on $\Omega$ such that

$$\varphi \geq \psi \text{ and } (\varphi - \psi)|_{\partial \Omega} = 0. \tag{4.4}$$

Then

$$\int_\Omega \text{MA}_{R;\pi}(\varphi) dx \leq \int_\Omega \text{MA}_{R;\pi}(\psi) dx. \tag{4.5}$$

Proof. It is sufficient to prove (4.5) when $\varphi$ and $\psi$ are smooth, since we can approximate general $\varphi$ and $\psi$ by smooth $W$-invariant convex functions by Lemma 4.5. Let

$$\varphi_t = t\varphi + (1-t)\psi.$$

Then

$$\text{MA}_{R;\pi}(\varphi_t) = \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t)$$

and

$$\frac{d}{dt} \int_\Omega \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) dx$$

$$= \int_\Omega (\nabla^2 \varphi_t)^{-1\cdot ij} \nabla^2 \varphi_{t,ij} \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) dx$$

$$+ \int_\Omega \left( \sum_{\alpha \in \Phi_+} \frac{2\alpha(\nabla \varphi_t)}{\alpha(\nabla \varphi_t)} \right) \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) dx. \tag{4.6}$$

Using the fact that

$$(\det(\nabla^2 \varphi_t)(\nabla^2 \varphi_t)^{-1\cdot ij})_j = 0$$
and integration by parts, we have
\[
\int_\Omega \left( \nabla^2 \varphi_t \right)^{-1,ij} \nabla^2 \dot{\varphi}_t,_{ij} \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) dx
\]
\[
= \int_{\partial \Omega} \left( \nabla^2 \varphi_t \right)^{-1,ij} \nabla^2 \dot{\varphi}_t,_{ij} \nu_j \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) d\sigma
\]
\[
- \int_{\partial \Omega} \left[ \left( \nabla^2 \varphi_t \right)^{-1,ij} \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) \right] \dot{\varphi}_t,_{ij} \nu_i d\sigma
\]
(4.7)
\[
+ \int_\Omega \left( \nabla^2 \varphi_t \right)^{-1,ij} \dot{\varphi}_t \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) \dot{\varphi}_t dx.
\]

Also
\[
\int_\Omega \left( \sum_{\alpha \in \Phi_+} \frac{2\alpha(\nabla \varphi_t)}{\alpha(\nabla \varphi_t)} \right) \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) dx
\]
\[
= 2 \int_{\partial \Omega} \sum_{\alpha \in \Phi_+} \frac{\alpha \nu_i}{\alpha(\nabla \varphi_t)} \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) \dot{\varphi}_t d\sigma
\]
(4.8)
\[
= -2 \int_\Omega \left( \det(\nabla^2 \varphi) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi) \sum_{\alpha \in \Phi_+} \frac{\alpha i}{\alpha(\nabla \varphi)} \dot{\varphi}_t \right) dx.
\]

Note that
\[
\left( \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) \right)^{ij}_{,ij} = -2 \sum_{\alpha \in \Phi_+} \frac{\alpha^k \alpha^l \varphi_t,_{ik} \varphi_t,_{jl}}{\alpha^2(\nabla \varphi_t)} + 2 \sum_{\alpha \in \Phi_+} \frac{\alpha^k \varphi_t,_{ijk}}{\alpha(\nabla \varphi_t)} + 4 \sum_{\alpha, \beta \in \Phi_+} \frac{\alpha^k \beta^l \varphi_t,_{i \beta j} \varphi_t,_{l j}}{\alpha(\nabla \varphi_t) \beta(\nabla \varphi_t)}
\]
(4.9)
\[
= 2 \left( \det(\nabla^2 \varphi) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) \sum_{\alpha \in \Phi_+} \frac{\alpha i}{\alpha(\nabla \varphi)} \right) .
\]

Plugging (4.7)-(4.9) into (4.6) and using the boundary condition (4.4), we have
\[
\frac{d}{dt} \int_\Omega \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) dx
\]
\[
= \int_{\partial \Omega} \sum_{\alpha \in \Phi_+} \nabla \varphi_t,_{ij} \nu_j \det(\nabla^2 \varphi_t) \prod_{\alpha \in \Phi_+} \alpha^2(\nabla \varphi_t) d\sigma
\]
\[
\leq 0.
\]
Hence we get (4.5). \qed

By the above proposition, we get the following analogue of [15, Lemma 2.3].

Lemma 4.7. Let $\varphi, \psi$ be two $W$-invariant convex functions on $a$ so that $\varphi \geq \psi$.
and
\[
\lim_{|x| \to +\infty} \varphi(x) = +\infty.
\]

Then
\[
\int_{\mathfrak{a}_+} MA_{\mathbb{R};\pi}(\varphi)dx \geq \int_{\mathfrak{a}_+} MA_{\mathbb{R};\pi}(\psi)dx.
\]

Combining Lemma 4.7 and the argument of [15, Lemma 2.7], we prove

**Lemma 4.8.** Let \( \psi \) be a \( W \)-invariant convex function on \( \mathfrak{a} \) and \( u \) its Legendre function. Suppose that for some constant \( C \),
\[
\psi \leq v_{2P} + C,
\]
where \( v_{2P} \) is the support function of \( 2P \). Then
\[
\int_{\mathfrak{a}_+} MA_{\mathbb{R};\pi}(\psi)dx = \int_{2P} \pi dy,
\]
if \( u < +\infty \) everywhere in the interior of \( 2P \).

The inverse of Lemma 4.8 is also true as an analogue of [15, Theorem 3.6]. In fact, we have

**Proposition 4.9.** Let \( \phi \) be a \( K \times K \)-invariant potential. Then \( \psi_{\phi} \) satisfies (4.11) if and only if \( u_{\phi} \) is finite everywhere in \( \text{Int}(2P) \).

By Proposition 4.9 we will follow the arguments in [15, Proposition 3.9] to prove Theorem 4.2.

### 4.3. Proof of Theorem 4.2

It is easy to see that (4.1) is equivalent to (4.2). Thus, to prove Theorem 4.2 we only need to show that
\[
\phi \in \mathcal{E}_{K \times K}(M, -K_M) \iff \int_{2P} |u_{\phi}| \pi dy < +\infty.
\]

The following lemma can be found in [9, Lemma 2.7] (proved in [9, Appendix]).

**Lemma 4.10.** Let \( \psi \) be a convex function on \( \mathfrak{a} \) and \( u_{\psi} \) its Legendre dual on \( P \).

1. \( u_{\psi} \) is differentiable at \( p \) if and only if \( u_{\psi} \) is attained at a unique point \( x_p \in \mathfrak{a} \) and \( x_p = \nabla u_{\psi}(p) \);
2. Suppose that \( (\psi - \psi_0) \in \mathcal{E}_{K \times K}^1(M, -K_M) \). Let \( p \in P \) at which \( u_{\psi} \) is differentiable. Then for any continuous uniformly bounded function \( v \) on \( \mathfrak{a} \), it holds
\[
\frac{d}{dt} \bigg|_{t=0} u_{\psi+tv}(p) = -v(\nabla u_{\psi}(p)),
\]
where \( u_{\psi+tv} \) is the Legendre function of \( \psi + tv \) as in [3.11] which is well-defined since \( v \) is continuous and uniformly bounded on \( \mathfrak{a} \).

**Remark 4.11.** By Lemma 4.3 and Part (1) in Lemma 4.10 we can prove the following: Let \( \phi \in \mathcal{E}_{K \times K}^1(M, -K_M) \), then for any \( K \times K \)-invariant continuous uniformly bounded function \( f \) on \( G \), it holds
\[
\int_M f \omega_{\phi}^n = \int_{2P} f(\partial u_{\phi}) \pi dy.
\]
Proof of Theorem 4.2. Necessary part. First we show that $\phi$ has full MA-mass by Proposition 4.9. In fact, by a result in [31, Lemma 4.5], we see that for any $W$-invariant convex polytope $2P' \subseteq 2P$, there is a constant $C = C(P')$ such that for any $W$-invariant convex $u_{\phi} \geq 0$,

$$\int_{2P'} u_{\phi} dy \leq C \int_{2P} u_{\phi} \pi dy < +\infty.$$ 

This implies that $u_{\phi}$ is finite everywhere in $\text{Int}(2P)$ by the convexity of $u_{\phi}$. Thus we get what we want from Proposition 4.9.

Next we prove that $\phi$ is $L^1$-integrate associated to the MA-measure $\omega_n^\phi$. Let $\psi_1 = \psi_0 + \phi$ ($\phi$ may be different to a constant). We define a distance between $\psi_0$ and $\psi_1$ for $p \geq 1$,

$$d_p(\psi_0, \psi_1) = \inf_{\phi_t} \int_0^1 \left( \int_M |\dot{u}_t|^p \omega_n^{\phi_t} \right)^{\frac{1}{p}} dt,$$

where $\phi_t \in E^1(M, -K_M)$ ($t \in [0, 1]$) runs over all curves joining 0 and $\phi$ with $\omega_{\phi_t} \geq 0$. Choose a special path $\phi_t$ such that the corresponding Legendre functions of $\psi_t = \psi_0 + \phi_t$ are given by

$$u_t = tu_1 + (1-t)u_0,$$

(4.14)

where $u_1$ and $u_0$ are the Legendre functions of $\psi_1$ and $\psi_0$, respectively. Note that by Lemma 4.10

$$\dot{\psi}_t = -\dot{u}_t = u_0 - u_1,$$

almost everywhere.

Then by Lemma 4.5 (or Remark 4.11), we get

$$d_p(\psi_0, \psi_1) \leq \int_0^1 \left( \int_{2P} |\dot{u}_t|^p \pi dy \right)^{\frac{1}{p}} dt,$$

(4.15)

$$\leq C(p) \left( \int_{2P} |u_1|^p \pi dy \right)^{\frac{1}{p}} + C'(p, \psi_0).$$

On the other hand, by a result of Darvas [16], there are uniform constant $C_0$ and $C_1$ such that for any Kähler potential $\phi$ with full MA-measure it holds,

$$-\int_M \phi \omega_n^\phi \leq C_0 d_1(\psi_0, \psi_1) + C_1.$$

Thus we obtain

$$-\int_M \phi \omega_n^\phi \leq C.$$

Hence, $\phi \in E^1_{K \times K}(M, -K_M)$.

Sufficient part. Assume that $\phi \in E^1_{K \times K}(M, -K_M)$. We first deal with the case of $\phi \in L^\infty(M) \cap C^\infty(G)$. Then

$$v_{2P} - C \leq \psi_\phi \leq v_{2P} \leq \psi_0 + C,$$

and

$$\nabla \psi_\phi : \mathbf{a} \to 2P$$

is a bijection. Thus

$$-\phi = (\psi_0 - \psi_\phi)(\nabla u_\phi)$$

$$\geq v_{2P}(\nabla u_\phi) - \psi_\phi(\nabla u_\phi) - C_2 \geq -C_2.$$
Moreover,
\[
(\psi_0 - \psi_\phi)(\nabla u_\phi) \
\geq v_{2P}(\nabla u_\phi) - \psi_\phi(\nabla u_\phi) - C \
= \sup_{y' \in 2P} \langle \nabla u_\phi, y' \rangle - \psi(\nabla u_\phi) - C \
\geq \langle \nabla u_\phi, y \rangle - \psi(\nabla u_\phi) - C \
= u_\phi(y) - C.
\]

Hence,
\[
\int_{2P^+} u_\phi \pi \, dy \leq \int_{2P^+} (\psi_0 - \psi_\phi)(\nabla u_\phi) \pi \, dy + C 
\]
\[
= \int_M |\phi| \omega^n_\phi + C < +\infty. 
\]

For an arbitrary \( \phi \in E^1_{K \times K}(M, -K_M) \), we choose a sequence of smooth \( K \times K \)-invariant functions \( \{ \phi_j \} \) decreasing to \( \phi \) such that \( \phi_j \in C^\infty(G) \) and
\[
\sqrt{-1} \partial \bar{\partial}(\psi_0 + \phi_j) > 0, \text{ in } G.
\]
Then as in (4.17), we have
\[
\int_{2P^+} u_j \pi \, dy \leq \int_{2P^+} (\psi_0 - \psi_j)(\nabla u_j) \pi \, dy 
\]
\[
= \int_M |\phi_j| \omega^n_j + C;
\]
where \( u_j \) is the Legendre functions of \( \psi_j = \psi_0 + \phi_j \). Note that
\[
\int_M |\phi_j| \omega^n_j \to \int_M |\phi| \omega^n_\phi
\]
and \( u_j \nearrow u_\phi \). Thus by taking the above limit as \( j \to +\infty \), we get (4.17) for \( \phi \). In particular,
\[
\int_{2P^+} u_\phi \pi \, dy < +\infty.
\]

5. Computation of Ricci potential

In this section, we assume that the moment polytope \( P \) of \( Z \) is fine. Then by Lemma 3.4 \( (\psi_{2P} - \psi_0) \in E^1_{K \times K}(M, -K_M) \) is a smooth \( K \times K \)-invariant Kähler potential on \( G \). It follows that
\[
- \log \det(\partial \bar{\partial} \psi_{2P}) - \psi_{2P} = h_0
\]
gives a Ricci potential \( h_0 \) of \( \omega_{2P} \), which is smooth and \( K \times K \)-invariant on \( G \).

The following proposition gives an upper bound on \( h_0 \).

**Proposition 5.1.** The Ricci potential \( h_0 \) of \( \omega_{2P} \) is uniformly bounded from above on \( G \). In particular, \( e^{h_0} \) is uniformly bounded on \( G \).
Proof. As in [31, Sections 3.2 and 4.3], the proof is based on a direct computation of asymptotic behavior of $h_0$ near every point of $\partial (2P_+ )$. Recall that

$$J(x) = \prod_{\alpha \in \Phi_+} \sinh^2 \alpha(x) \text{ and } \pi(y) = \prod_{\alpha \in \Phi_+} \alpha^2(y).$$

Since the Ricci potential of $h_0$ is also $K \times K$-invariant, by (5.1) and (3.5),

$$h_0 = - \log \det (\psi_{2P,ij}) - \psi_{2P} + \log J(x) - \log \prod_{\alpha \in \Phi_+} \alpha^2 (\nabla \psi_{2P})$$

(5.2)

$$= \log \det (u_{2P,ij}) - y_i u_{2P,i} + u_{2P} + \log J(\nabla u_{2P}) - \log \pi(y).$$

Note that

$$u_{2P,i} = \frac{1}{2} \sum_{A=1}^{d_0} (-u_A^i) (1 + \log l_A),$$

$$u_{2P,ij} = \frac{1}{2} \sum_{A=1}^{d_0} \frac{u_{2P,i}^A u_{2P}^j}{l_A}$$

and

$$\log J(t) = 2 \sum_{\alpha \in \Phi_+} \log \sinh(t).$$

Thus we have

$$h_0 = - \sum_{A=1}^{d_0} \log l_A + \frac{1}{2} \sum_{A=1}^{d_0} (u_A^i y_i) \log l_A$$

(5.3)

$$+ 2 \sum_{\alpha \in \Phi_+} \log \sinh( - \frac{1}{2} \sum_{A=1}^{d_0} \alpha(u_A) \log l_A) - 2 \sum_{\alpha \in \Phi_+} \log \alpha(y) + O(1).$$

By (5.3), $h_0$ is locally bounded in the interior of $2P_+$. Thus we need to prove that $h_0$ is bounded from above near each $y_0 \in \partial (2P_+ )$. There will be three cases as follows.

Case-1. $y_0 \in \partial (2P_+ )$ and is away from any Weyl wall. Note that

$$\log \sinh(t) = \begin{cases} t + O(1), & t \to +\infty, \\ \log t + O(1), & t \to 0^+. \end{cases}$$

(5.4)
Then we get as $y \to y_0$,

$$
\sum_{\alpha \in \Phi_+} \log \sinh(-\frac{1}{2} \sum_{A=1}^{d_0} \alpha(u_A) \log l_A) = - \sum_{A} \rho(u_A) \log l_A + O(1).
$$

By (5.3), it follows that

$$
h_0 = - \sum_{\{A|l_A(y_0)=0\}} \left(1 - \frac{1}{2} y_i u_{A,i} + 2 \rho_i u_{A,i}\right) \log l_A(y) + O(1).
$$

However, by Lemma 5.4 we have

$$
h_0 = \frac{1}{2} \sum_{\{A|l_A(y_0)=0\}} l_A(y) \log l_A(y) + O(1).
$$

Hence $h_0$ is bounded near $y_0$.

**Case-2.** $y_0$ lies on some Weyl walls but away from any facet of $2P$. In this case it is direct to see that $h_0$ is bounded near $y_0$ since

$$
\log \det(u_{2P,i}), \ y_i u_{2P,i}, \ \frac{J(u_{2P})}{\pi(y)}
$$

are all bounded.

**Case-3.** $y_0$ lie on the intersection of $\partial(2P)$ with some Weyl walls. In this case, by (3.1), we rewrite (5.3) as

$$
h_0 = 2 \sum_{A=1}^{d_0} \rho_A(u_A) \log l_A + 2 \sum_{\alpha \in \Phi_+} \log \sinh(-\frac{1}{2} \sum_{A=1}^{d_0} \alpha(u_A) \log l_A)
$$

$$
- 2 \sum_{\alpha \in \Phi_+} \log \alpha(y) + O(1)
$$

$$
= \sum_{\alpha \in \Phi_+} \left[ \sum_{A=1}^{d_0} \alpha(u_A) \log l_A + 2 \log \sinh(-\frac{1}{2} \sum_{A=1}^{d_0} \alpha(u_A) \log l_A)
$$

$$
- 2 \log \alpha(y) \right] + O(1), \ y \to y_0.
$$

Here we used a fact that

$$
2 \rho_A(u_A) = \sum_{\alpha \in \Phi_+} |\alpha(u_A)|.
$$
Set

\[ I_\alpha(y) = \sum_{A=1}^{d_0} |\alpha(u_A)| \log l_A + 2 \log \sinh(-\frac{1}{2} \sum_{A=1}^{d_0} \alpha(u_A) \log l_A) - 2 \log \alpha(y) \]

for each \( \alpha \in \Phi_+ \). Then

\[ h_0(y) = \sum_{\alpha \in \Phi_+} I_\alpha(y) + O(1), \quad y \to y_0. \]

(5.5)

Note that each \( I_\alpha(y) \) involves only one root \( \alpha \). Thus, without loss of generality, we may assume that \( y_0 \) lies on only one Weyl wall.

Assume that \( y_0 \in \partial(2P) \cap W_{\alpha_0} \) for some simple Weyl wall \( W_{\alpha_0} \), \( \alpha_0 \in \Phi_+ \) and it is away from other Weyl walls. Now we estimate each \( I_\alpha(y) \) in (5.5). When \( \beta \neq \alpha_0 \), it is easy to see that

\[ \beta(y) \to c_\beta > 0, \quad \text{as} \quad y \to y_0. \]

Then, by (5.4), we have

\[ \log \sinh\left(-\frac{1}{2} \sum_{A=1}^{d_0} |\beta(u_A)| \log l_A\right) = -\frac{1}{2} \sum_{\{A|l_A(y_0) = 0\}} \beta(u_A) \log l_A + O(1), \quad \forall \beta \neq \alpha_0. \]

Note that \( y_0 \in \{ \beta(y) > 0 \} \). Thus any facet \( F_A \) passing through \( y_0 \) lies in \( \{ \beta(y) > 0 \} \) or is orthogonal to \( W_\beta \). Since \( 2P \) is convex and \( s_\beta \)-invariant, where \( s_\beta \) is the reflection with respect to \( W_\beta \), these facets must satisfy

\[ \beta(u_A) \geq 0. \]

Hence, for any \( \beta \neq \alpha_0 \), we get

\[ I_\beta(y) = \sum_{A=1}^{d_0} |\alpha(u_A)| \log l_A - 2 \sum_{A=1}^{d_0} |\beta(u_A)| \log l_A - 2 \log \beta(y) \]

= \( O(1) \), as \( y \to y_0 \).

(5.6)

It remains to estimate the second term in \( I_{\alpha_0}(y) \),

(5.7)

\[ \log \sinh\left(-\frac{1}{2} \sum_{A} \alpha_0(u_A) \log l_A\right). \]

We first consider a simple case that \( y_0 \) lies on the intersection of \( W_{\alpha_0} \) with at most two facets of \( 2P \). Then there will be two subcases: \( y_0 \in W_{\alpha_0} \cap F_1 \) where \( F_1 \) is orthogonal to \( W_{\alpha_0} \), or \( y_0 \in W_{\alpha_0} \cap F_1 \cap F_2 \), where \( F_1, F_2 \) are two facets of \( P \).

Case-3.1. \( y_0 \in W_{\alpha_0} \cap F_1 \) is away from other facet of \( 2P \). Then \( F_1 \) is orthogonal to \( W_{\alpha_0} \). It follows that \( l_A(y_0) \neq 0 \) for any \( A \neq 1 \). Thus

(5.8)

\[ \langle \alpha_0, y \rangle = o(l_A(y)), \quad y \to y_0, A \neq 1. \]

Let \( \{F_1, ..., F_{d_1}\} \) be all facets of \( P \) such that \( \alpha_0(u_A) \geq 0, A = 1, ..., d_1 \). Let \( s_{\alpha_0} \) be the reflection with respect to \( W_{\alpha_0} \). Then by \( s_{\alpha_0} \)-invariance of \( P \), for each \( A' \notin \{1, ..., d_1\} \) there is some \( A \in \{1, ..., d_1\} \) such that

\[ l_{A'} = l_A + 2\frac{\alpha_0(u_A)(\alpha_0, y)}{|\alpha_0|^2}. \]
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It follows that

\[ \alpha_0(\nabla u_{2P}) = -\frac{1}{2} \sum_{A=1}^{d_0} \alpha_0(u_A) \log l_A \]

\[ = \frac{1}{2} \sum_{A=1}^{d_1} \alpha_0(u_A) \log \left( 1 + 2 \frac{\alpha_0(u_A) \langle \alpha_0, y \rangle}{|\alpha_0|^2 l_A(y)} \right). \]

Thus, by (5.8) and the fact that \( \alpha_0(u_1) = 0 \), we obtain

\[ \log \sinh \left( -\frac{1}{2} \sum_{A=1}^{d_0} \alpha_0(u_A) \log l_A \right) \]

\[ = \log \sinh \sum_{A=2}^{d_1} \alpha_0(u_A) \log \left( 1 + 2 \frac{\alpha_0(u_A) \langle \alpha_0, y \rangle}{|\alpha_0|^2 l_A(y)} \right) \]

\[ = \log \langle \alpha_0, y \rangle + O(1). \]

Hence

\[ I_{\alpha_0}(y) = O(1), \text{ as } y \to y_0. \]

Together with (5.6), we see that \( h_0 \) is bounded near \( y_0 \).

**Case-3.2.** \( y_0 \in W_{\alpha_0} \cap F_1 \cap F_2 \) and is away from other facets of \( 2P \). By the

\[ l_1 = l_2 + \frac{2\alpha_0(u_2) \langle \alpha_0, y \rangle}{|\alpha_0|^2}. \]
As \( y \to y_0 \) we have
\[
\alpha_0(y), l_1(y), l_2(y) \to 0,
\]
\[
l_A(y) \not\to 0, \ \forall A \neq 1, 2.
\]
It follows that
\[
\sum_{A=1}^{d_0} |\alpha_0(u_A)| \log l_A = \alpha_0(u_2)(\log l_1 + \log l_2) + O(1).
\]
(5.9)

Then the second term in \( I_{\alpha_0}(y) \) becomes
\[
\log \sinh \left( -\frac{1}{2} \sum_{A=1}^{d_0} \alpha_0(u_A) \log l_A \right) \\
= \log \sinh \frac{1}{2} \left[ \alpha_0(u_2) \log \left( 1 + 2 \frac{\alpha_0(u_2)(\alpha_0, y)}{\alpha_0^2 l_2(y)} \right) \\
+ \sum_{A \neq 2, \alpha_0(u_A) > 0} d_1 \alpha_0(u_A) \log \left( 1 + 2 \frac{\alpha_0(u_A)(\alpha_0, y)}{\alpha_0^2 l_A(y)} \right) \right].
\]

We will settle it down according to the different rate of \( \frac{\alpha_0(y)}{l_2(y)} \) below.

Case-3.2.1. \( \frac{\alpha_0(y)}{l_2(y)} = o(l_2(y)) \). Then
\[
\log \sinh \alpha_0(\nabla u_2 P) = \log \alpha_0(y) - \log l_2(y).
\]
(5.10)

Note that \( s_{\alpha_0}(u_1) = u_2 \in \overline{a}^+, \) we have
\[
\sum_{A=1,2} |\alpha_0(u_A)| \log l_A = \alpha_0(u_2)(\log l_1 + \log l_2).
\]

Using the above relation, (5.9) and (5.10), we get
\[
I_{\alpha_0}(y) = \alpha_0(u_2) \log l_1 + (\alpha_0(u_2) - 2) \log l_2 + O(1)
\]
(5.11)
\[
= 2(\alpha_0(u_2) - 1) \log l_2 + O(1).
\]

Here we used \( l_1 = l_2(1 + o(1)) \) in the last equality.

Note that by our assumption \( \alpha_0(u_2) > 0 \). Then
\[
\alpha_0(u_2) \geq 1,
\]
since \( \alpha_0(u_2) \in \mathbb{Z} \). Hence, as \( l_1(y), l_2(y) \to 0^+ \), by (5.5), (5.6) and (5.11), we see that \( h_0 \) is bounded from above in this case.

Case-3.2.2. \( c \leq \frac{\alpha_0(y)}{l_2(y)} \leq C \) for some constants \( C, c > 0 \). Then
\[
\log \alpha_0(y) = \log l_2 + O(1), \ \log \sinh \alpha_0(\nabla u_2 P) = O(1)
\]
and the right hand side of (5.5) becomes
\[
\alpha_0(u_2)(\log l_1 + \log l_2) - 2 \log \alpha_0(y) + O(1)
\]
(5.12)
\[
= 2(\alpha_0(u_2) - 1) \log l_2 + O(1).
\]

Again \( h_0 \) is also bounded from above.
Thus Guillemin function $E$ to $S$ and Case-3.1. Furthermore, if rank Case-3 uniformly bounded if and only if the following relation holds,

$$h_0(y) = o_0(y)(1 + o(1))$$

and the right hand side of (5.5) becomes

$$\alpha_0(u_2)(\log l_1 + \log l_2) + \alpha_0(u_2)(\log o_0(y) - \log l_2(y))$$

$$= \alpha_0(u_2) \log l_1 + [\alpha_0(u_2) - 2\log o_0(y) + O(1)$$

(5.13)

$$= 2(\alpha_0(u_2) - 1) \log o_0(y) + O(1).$$

Hence $h_0$ is bounded from above as in Case-3.2.1.

Next we consider the case that there are facets $F_1, ..., F_s$ ($s \geq 3$) such that

$$y_0 \in W_{o_0} \cap F_1 \cap ... \cap F_s$$

and it is away from any other facet of $2P$. We only need to control the term (5.7) as above. If $F_1, ..., F_s$ are all orthogonal to $W_{o_0}$ as in Case-3.1, we see that $h_0(y)$ is uniformly bounded. Otherwise, for any $y$ near $y_0$ there is a facet $F = F_i'$ for some $i' \in \{1, ..., s\}$ such that

$$l_{i'}(y) = \min\{l_i(y) \mid i = 1, ..., s \text{ such that } \alpha_0(u_i) \neq 0\}.$$  

As $y \to y_0$, up to passing to a subsequence, we can fix this $i'$. Clearly, $y_0 \in W_{o_0} \cap F_1 \cap F_2$ as in Case-3.2, where $F_2 = F \subseteq \mathbb{R}$ and $F_1 = s_{o_0}(F)$ for the reflection $s_{o_0}$. Hence by following the argument in Case-3.2, we can also prove that $h_0(y)$ is uniformly bounded from above. Therefore, the proposition is true in Case-3. The proof of our proposition is completed. □

Remark 5.2. We note that $h_0$ is always uniformly bounded in Case-1, Case-2 and Case-3.1. Furthermore, if rank$(G) = 2$, there are at most two facets $F_1, F_2$ intersecting at a same point $y_0$ of $W_{o_0}$ as in Cases-3.2.1-3.2.3, thus, by the asymptotic expressions of $h_0$ in (5.11), (5.12) and (5.13), respectively, we see that $h_0$ is uniformly bounded if and only if the following relation holds,

$$\alpha_0(u_2) = 1.$$

In the other words, in Cases-3.2.1-3.2.3,

$$\lim_{y \to y_0} h_0 = -\infty,$$

if (5.14) does not hold.

6. Reduced Ding functional and existence criterion

By Lemma 4.8 and Theorem 4.2 we see that for any $u \in \mathcal{E}^1_{K \times K}(2P)$, its Legendre function

$$\psi_u(x) = \sup_{y \in 2P} \{\langle x, y \rangle - u(y)\} \leq v_{2P}(x)$$

corresponds to a $K \times K$-invariant weak Kähler potential $\phi_u = \psi_u - \psi_0$ which belongs to $\mathcal{E}^1_{K \times K}(M, -K_M)$. Here we can choose $\psi_0$ to be the Legendre function $v_{2P}$ of Guillemin function $u_{2P}$ as in (4.8). As we know, $e^{-\phi_u} \in L^p(\omega_0)$ for any $p > 0$. Thus $\int_a e^{-\psi_u(x)}J(x)dx$ is well-defined.
We introduce the following functional on $E_{K \times K}(2P)$ by
\[ D(u) = L(u) + F(u), \]
where
\[ L(u) = \frac{1}{V} \int_{2P_+} u \pi dy - u(4\rho) \]
and
\[ F(u) = -\log \left( \int_{a_+} e^{-\psi u} J(x) dx \right) + u(4\rho). \]
It is easy to see that on a smooth Fano compactification of $G$,
\[ L(u \phi) + u \phi(4\rho) = -\frac{1}{(n+1)V} \sum_{k=0}^{n} \int_M \phi \omega_{\phi}^k \wedge \omega_0^{n-k} \]
and $D(u \phi)$ is just the Ding functional $F(\phi)$. We note that a similar functional on such Fano manifolds has been studied for Mabuchi solitons in [32, Section 4]). Hence, for convenience, we call $D(\cdot)$ the reduced Ding functional on a $Q$-Fano compactifications of $G$.

In this section, we will use the variation method to prove Theorem 1.2 by verifying the properness of $D(\cdot)$. We assume that the moment polytope $P$ is fine so that the Ricci potential $h_0$ is uniformly bounded above by Proposition 5.1.

6.1. A criterion for the properness of $D(\cdot)$. In this subsection, we establish a properness criterion for $D(u \phi)$, namely,

**Proposition 6.1.** Let $M$ be a $Q$-Fano compactification of $G$. Suppose that the moment polytope $P$ is fine and it satisfies (1.2). Then there are constants $\delta$ and $C_\delta$ such that
\[ D(u) \geq \delta \int_{2P_+} u \pi dy + C_\delta, \quad u \in E_{K \times K}(2P). \]

The proof goes almost the same as in [32]. We sketch the arguments here for completeness. First we note that $u \phi$ satisfies the normalized condition $u \geq u(O) = 0$. Then we have the following estimate for the linear term $L(\cdot)$ as in [32 Proposition 4.5].

**Lemma 6.2.** Under the assumption (1.2), there exists a constant $\lambda > 0$ such that
\[ L(u) \geq \lambda \int_{2P_+} u \pi dy, \quad \forall u \in E_{K \times K}(2P). \]

For the non-linear term $F(\cdot)$, we can also get an analogy of [32 Lemma 4.8] as follows.

**Lemma 6.3.** For any $\phi \in E_{K \times K}(M, -K_M)$, let
\[ \tilde{\psi}_\phi := \psi_\phi - 4\rho x^i, \quad x \in a_+. \]
Then
\[ F(u \phi) = -\log \left( \int_\Phi e^{-(\tilde{\psi}_\phi - \inf_{x^i} \tilde{\psi}_\phi)} \prod_{\alpha \in \Phi_+} \left( \frac{1 - e^{-2\alpha_\alpha x^i}}{2} \right)^2 dx \right). \]

(6.3)
Consequently, for any $c > 0$,
\begin{equation}
\mathcal{F}(u_\varphi) \geq \mathcal{F} \left( \frac{u_\varphi}{1 + c} \right) - n \cdot \log(1 + c).
\end{equation}

Let $\phi_0, \phi_1 \in \mathcal{E}_1^{K \times K}(-K_M)$ and $u_0, u_1$ be two Legendre functions of $\psi_0 + \phi_0$ and $\psi_0 + \phi_1$, respectively. Let $u_t (t \in [0, 1])$ be a linear path connecting $u_0$ to $u_1$ as in (4.14). Then by Theorem 4.2, the corresponding Legendre functions $u_t$ give a path in $\mathcal{E}_1^{K \times K}(-K_M, -K_M)$. The following lemma shows that $\mathcal{F}(\psi_t)$ is convex in $t$.

**Lemma 6.4.** Let
\[ \hat{\mathcal{F}}(t) = -\log \int_{a_+} e^{-\psi_t} J(x) dx, \; t \in [0, 1]. \]

Then $\hat{\mathcal{F}}(t)$ is convex in $t$ and so is $\mathcal{F}(\psi_t)$.

**Proof.** By definition, we have
\[
\psi_t(tx_1 + (1 - t)x_0) = \sup_y \{ \langle y, tx_1 + (1 - t)x_0 \rangle - (tu_1(y) + (1 - t)u_0(y)) \}
\leq t \sup_y \{ \langle y, x_1 \rangle - u_1(y) \}
+ (1 - t) \sup_y \{ \langle y, x_0 \rangle - u_0(y) \}
\leq t \psi_1(x_1) + (1 - t) \psi_0(x_0), \; \forall \; x_0, x_1 \in a.
\]

On the other hand,
\[
\log \mathcal{J}(tx_1 + (1 - t)x_0) \geq t \log \mathcal{J}(x_1) + (1 - t) \log \mathcal{J}(x_0), \; \forall x_0, x_1 \in a_+.
\]
Combining these two inequalities, we get
\[
(e^{-\psi_t} \mathcal{J})(tx_1 + (1 - t)x_0) \geq (e^{-\psi_t} \mathcal{J})(x_1)(e^{-\psi_0} \mathcal{J})^{1-t}(x_0), \; \forall x_0, x_1 \in a_+.
\]
Hence, by applying the Prekopa-Leindler inequality to three functions $e^{-\psi_t} \mathcal{J}, e^{-\psi_t} \mathcal{J}$ and $e^{-\psi_0} \mathcal{J}$ (cf. [39]), we prove
\[
-\log \int_{a_+} e^{-\psi_t} \mathcal{J}(x) dx \leq -t \log \int_{a_+} e^{-\psi_1} \mathcal{J}(x) dx - (1 - t) \log \int_{a_+} e^{-\psi_0} \mathcal{J}(x) dx.
\]
This means that $\hat{\mathcal{F}}(t)$ is convex. \qed

**Proof of Proposition 6.1.** By Proposition 5.1
\[
A(y) = \frac{V}{\int_{a_+} e^{-\psi_0} \mathcal{J}(x) dx} e^{b_0(\nabla \psi_0(y))}
\]
is bounded, where $y(x) = \nabla \psi_0(x)$. Then the functional
\[
\mathcal{D}_A(u) = \mathcal{L}^0_A(u) + \mathcal{F}(u),
\]
is well-defined on $\mathcal{E}_1^{K \times K}(2P)$, where
\[
\mathcal{L}^0_A(u) = \frac{1}{V} \int_{2P_+} uA(y)\pi(y) dy - u(4\rho).
\]
It is easy to see that that \( u_0 \) is a critical point of \( D_A(\cdot) \). On the other hand, by Lemma 6.4, \( F(\cdot) \) is convex along any path in \( E_{K \times K}(M, -K_M) \) determined by their Legendre functions as in (4.14). Note that \( L_A^0(\cdot) \) is convex in \( E_{K \times K}(2P) \). Hence
\[
D_A(u) \geq D_A(u_0), \quad \forall u \in E_{K \times K}(2P).
\]
Now together with Lemma 6.2 and Lemma 6.3, we can apply arguments in the proof of [32 Proposition 4.9] to proving that there is a constant \( C > 0 \) such that for any \( u \in \mathcal{E}_K \times K(2P) \),
\[
D(u) \geq \frac{C\lambda}{1+C} \int_{2P_+} u \pi dy + D_A(u_0) - n \log(1 + C).
\]
Therefore, we get (6.1).

\[\square\]

6.2. Semi-contiuity. Write \( \mathcal{E}_{K \times K}^1(2P) \) as
\[
\mathcal{E}_{K \times K}^1(2P) = \bigcup_{\kappa \geq 0} \mathcal{E}_{K \times K}^1(2P; \kappa),
\]
where
\[
\mathcal{E}_{K \times K}^1(2P; \kappa) = \{ u \in \mathcal{E}_{K \times K}^1(2P) \mid \int_{2P_+} u \pi dy \leq \kappa \}.
\]
By [31 Lemma 6.1] and Fatou’s lemma, it is easy to see that any sequence \( \{u_n\} \subseteq \mathcal{E}_{K \times K}^1(2P; \kappa) \) has a subsequence which converges locally uniformly to some \( u_\infty \) in it. Thus each \( \mathcal{E}_{K \times K}^1(2P; \kappa) \), and so \( \mathcal{E}_{K \times K}^1(2P) \) is complete. Moreover, we have

**Proposition 6.5.** The reduced Ding functional \( D(\cdot) \) is lower semi-continuous on the space \( \mathcal{E}_{K \times K}^1(2P) \). Namely, for any sequence \( \{u_n\} \subseteq \mathcal{E}_{K \times K}^1(2P) \), which converges locally uniformly to some \( u_\infty \), we have \( u_\infty \in \mathcal{E}_{K \times K}^1(2P) \) and it holds
\[
D(u_\infty) \leq \liminf_{n \to \infty} D(u_n).
\]

**Proof.** By Fatou’s lemma, we have
\[
\int_{2P_+} u_\infty \pi dy \leq \liminf_{n \to +\infty} \int_{2P_+} u_n \pi dy < +\infty.
\]
Then \( u_\infty \in \mathcal{E}_{K \times K}^1(2P) \) and
\[
L(u_\infty) \leq \liminf_{n \to +\infty} L(u_n).
\]
It remains to estimate \( F(u_\infty) \). Note that \( u_\infty \) is finite everywhere in \( \text{Int}(2P) \) by the locally uniformly convergence and its Legendre function \( \psi_\infty \leq v_{2P} \). Thus, for any \( \epsilon_0 \in (0, 1) \) there is a constant \( M_{\epsilon_0} > 0 \) such that (cf. [33 Lemma 2.3]),
\[
\psi_\infty(x) \geq (1 - \epsilon_0)v_{2P}(x) - M_{\epsilon_0}, \quad \forall x \in a.
\]
On the other hand, the Legendre function \( \psi_n \) of \( u_n \) also converges locally uniformly to \( \psi_\infty \). Then
\[
\partial \psi_n \rightarrow \partial \psi_\infty
\]
almost everywhere. Since
\[
\psi_n(O) = \psi_\infty(O) = 0, \quad \forall n \in \mathbb{N},
\]
we have
\[
(6.9) \quad \psi_n(x) \geq (1 - \epsilon_0)v_{2P}(x) - M_{\epsilon_0}, \quad \forall x \in a
\]
as long as \( n \gg 1 \). Note that
\[
0 \leq J(x) \leq e^{4\rho(x)}, \forall x \in a_+.
\]
By choosing an \( \epsilon_0 \) such that \( 4\rho \in (1 - \epsilon_0)\text{Int}(2P) \), we get
\[
\int_{a_+} e^{M_{x_0} - (1 - \epsilon_0)\nu_{2P}(x)} J(x) dx < +\infty.
\]
Hence, combining this with (6.8) and (6.9) and using Fatou’s lemma, we derive
\[
-\log \left( \int_{a_+} e^{-\psi} J(x) dx \right) \leq \liminf_{n \to +\infty} \left[ -\log \left( \int_{a_+} e^{-\psi_n} J(x) dx \right) \right].
\]
Therefore, we have proved (6.6) by (6.7).

### 6.3. Proof of Theorem 1.2

Now we prove the sufficient part of Theorem 1.2. Suppose that (1.2) holds. Then by Theorem 6.1 and Proposition 6.5, there is a minimizing sequence \( \{u_n\} \) of \( D(\cdot) \) on \( E_{K	imes K}^1(2P) \), which converges locally uniformly to some \( u_\star \in E_{K	imes K}^1(2P) \) such that
\[
D(u_\star) \leq \lim_{u \in E_{K	imes K}^1(2P)} D(u).
\]
Let \( \psi_\star \) be the Legendre function of \( u_\star \). Then by Theorem 1.2, we have
\[
-\log \left( \int_{a_+} e^{-\psi} J(x) dx \right) \leq -\log \left( \int_{a_+} e^{-\psi_\star} J(x) dx \right).
\]
We need to show that \( \phi_\star \) satisfies the Kähler-Einstein equation (2.1).

**Proposition 6.6.** \( \phi_\star \) satisfies the Kähler-Einstein equation (2.1).

**Proof.** Let \( \{u_t\}_{t \in [0, 1]} \subseteq E_{K	imes K}^1(2P) \) be a family convex functions with \( u_0 = u_\star \) and \( \psi_t \) the corresponding Legendre functions of \( u_t \). Then by Part (2) in Lemma 4.10,
\[
\dot{\psi}_0 = -\dot{u}_0, \text{ almost everywhere}.
\]
Note that
\[
\int_{a_+} e^{-\psi} J(x) dx = V,
\]
Thus by (4.11) in Lemma 4.8, we get
\[
\frac{d}{dt} \bigg|_{t=0} D(u_t) = \frac{1}{V} \int_{2P_+} \dot{u}_0 \pi dy + \int_{a_+} \dot{\psi}_0 e^{-\psi} J(x) dx
\]
\[
= V \int_{a_+} \dot{\psi}_0 [e^{-\psi} J(x) - M_{\mathbb{R};\pi}(\psi_\star)] dx.
\]
For any continuous, compactly supported \( W \)-invariant function \( \eta \in C_0(a) \), we consider a family of functions \( u_\star + t\eta \). In general, it may not be convex for \( t \neq 0 \) since \( u_\star \) is just weakly convex. In the following, we use a trick to modify the function \( D(u_t) \) as in [9, Section 2.6]. Define a family of \( W \)-invariant functions by
\[
\psi_t = \sup_{\phi \in E_{K	imes K}^1(M, -K_M)} \{ \psi_\phi | \psi_\phi \leq \psi_\star + t\eta \}.
\]
Then it is easy to see that the Legendre function \( \dot{\psi}_t \) of \( \dot{\psi}_t \) satisfies
\[
|\dot{u}_t - u_0| \leq C, \forall |t| \ll 1.
\]
By Theorem 4.2 we see that $(\hat{\psi}_t - \psi_0) \in \mathcal{E}^E_{K \times K}(M, -K_M)$. Without loss of generality, we may assume that $\hat{\psi}_0$ satisfies (4.1).

Let
\[ \hat{D}(t) = \mathcal{L}(\hat{u}_t) + \mathcal{F}(\hat{u}_t). \]

Then
\[ \hat{D}(0) = \mathcal{D}(u_\star) \]
and
\[ \hat{D}(t) \geq \mathcal{D}(u_\star). \]

**Claim 6.7.** $\mathcal{L}(\hat{u}_t) + \hat{u}_t(4\rho)$ is differentiable for $t$. Moreover,
\[ \left. \frac{d}{dt} \right|_{t=0} (\mathcal{L}(\hat{u}_t) + \hat{u}_t(4\rho)) = -\frac{1}{V} \int_M \eta \omega^n_{\hat{\phi}_t}, \]

To prove this claim, we let a convex function $g(t) = \hat{u}_t(p)$ for each fixed $y \in 2P$. Then it has left and right derivatives $g'_-(t; p), g'_+(t; p)$, respectively. Moreover, they are monotone and $g'_-(t; p) \leq g'_+(t; p)$. Thus, $g'_-, g'_+ \in L^\infty_{loc}$. It follows that
\[ \left. \frac{d}{dt} \right|_{t=\tau \pm} \int_{2P_\pm} \hat{u}_t \pi dy = \int_{2P_\pm} g'_\pm(\tau; p) \pi dy. \]

Recall that $g'_-(t; p) = g'_+(t; p)$ holds almost everywhere. Thus we see that
\[ \mathcal{L}(\hat{u}_t) + u(4\rho) = \frac{1}{V} \int_{2P_\pm} \hat{u}_t \pi dy \]
is differentiable.

Note that
\[ u_{\psi_t} = u_{\psi_\star + t\eta}, \]
where $u_{\psi_\star + t\eta}$ is the Legendre function of $\psi_\star + t\eta$. It follows from Part (2) in Lemma 4.10 that
\[ \dot{\psi}_0 = -\dot{u}_0 = \eta, \] almost everywhere.

Hence by Lemma 4.5 (or Remark 4.11), we get
\[ \left. \frac{d}{dt} \right|_{t=0} (\mathcal{L}(\hat{u}_t) + \hat{u}_t(4\rho)) = \frac{1}{V} \int_{2P_\pm} \hat{u}_0 \pi dy \]
\[ = -\frac{1}{V} \int_{2P} \eta \pi dy = -\frac{1}{V} \int_{a_\star} \eta \Lambda_{\hat{\phi}_t}(\psi_0) dx \]
\[ = -\frac{1}{V} \int_{M} \eta \omega^n_{\hat{\phi}_t}, \]
where $\phi_\star = \psi_\star - \psi_0$. The claim is proved.

Similar to Claim 6.7, we have
\[ \left. \frac{d}{dt} \right|_{t=0} (\mathcal{F}(\hat{u}_t) - \hat{u}_t(4\rho)) = \frac{1}{V} \int_{a_\star} \eta e^{-\psi_\star} J(x) dx \]
\[ = \int_G \eta e^{-\phi_\star + h_0} \omega^n_{\psi_0}. \]
Thus, by (6.12)-(6.15), we derive

\[
0 = \frac{d}{dt} \bigg|_{t=0} \tilde{D}(t) = \frac{1}{V} \int_G \eta \left[ e^{-\phi + h_0 \omega_0^n - \omega_{\phi^0_n}} \right] dx.
\]

As a consequence,

\[
\omega_{\phi^0_n} = e^{-\psi + h_0 \omega_0^n}, \text{ in } G.
\]

Therefore, by Lemma 4.5 and KAK-integration formula, we prove that \( \phi^* \) satisfies (2.1) on \( G \).

Next we show that \( \omega_{\phi^*} \) can be extended as a singular Kähler-Einstein metric on \( M \). Choose an \( \epsilon_0 > 0 \) such that \( 4 \rho \in \text{Int}(2(1-\epsilon_0)P) \). Since \( u^* \in \mathcal{E}_{K \times K}(2P) \), by Lemma 4.8 there is a constant \( C_* > 0 \) such that

\[
\psi^* \geq (1 - \epsilon_0) \nu_{2P} - C_*.
\]

Thus

\[
e^{-\psi^*(x)} J(x)
\]

is bounded on \( \mathfrak{a}_+ \). Also \( \pi(\partial \psi^*) \) is bounded. Therefore, by (4.11), for any \( \epsilon > 0 \), we can find a neighborhood \( U_\epsilon \) of \( M \setminus G \) such that

\[
\left| \int_{U_\epsilon} \left( \omega_{\phi^*} - e^{\psi - \psi^*} \omega_0^n \right) \right| < \epsilon.
\]

This implies that \( \phi^* \) can be extended to be a global solution of (2.1) on \( M \). The proposition is proved.

\[\square\]

### 7. \( \mathbb{Q}-\)Fano compactification of \( SO_4(\mathbb{C}) \)

In this section, we will construct \( \mathbb{Q} \)-Fano compactifications of \( SO_4(\mathbb{C}) \) as examples and in particular, we will prove Theorem 1.3. Note that in this case \( \text{rank}(G) = 2 \). Thus we can use Theorem 1.2 to verify whether there exists a Kähler-Einstein metric on a \( \mathbb{Q} \)-Fano \( SO_4(\mathbb{C}) \)-compactification by computing the barycenter of their moment polytopes \( P_+ \). For convenience, we will work with \( P_+ \) instead of \( 2P_+ \) throughout this section. Then it is easy to see that the existence criterion 1.2 is equivalent to

\[
\text{bar}(P_+) \in 2\rho + \Xi.
\]

Denote

\[
R(t) = \begin{pmatrix}
\cos t & -\sin t \\
\sin t & \cos t
\end{pmatrix}.
\]

Consider the canonical embedding of \( SO_4(\mathbb{C}) \) into \( GL_4(\mathbb{C}) \) and choose the maximal torus

\[
T^\mathbb{C} = \left\{ \begin{pmatrix} R(z^1) & O \\
O & R(z^2) \end{pmatrix} \mid z^1, z^2 \in \mathbb{C} \right\}.
\]

Choose the basis of \( \mathfrak{m} \) as \( E_1, E_2 \), which generates the \( R(z^1) \) and \( R(z^2) \)-action. Then we have two positive roots in \( \mathfrak{m} \),

\[
\alpha_1 = (1, -1), \alpha_2 = (1, 1).
\]

Also we have

\[
\mathfrak{a}_+^* = \{ (x, y) \mid -x \leq y \leq x \}, \ 2\rho = (2, 0)
\]
and

\[(7.2) \quad 2\rho + \Xi = \{(x, y) \mid -x + 2 \leq y \leq x - 2\}.
\]

7.1. Gorenstein Fano $SO_4(\mathbb{C})$-compactifications. In this subsection, we use Lemma 3.1 to exhaust all polytopes associated to Gorenstein Fano compactifications. Here by Gorenstein, we mean that $K_{M, reg}$ can be extended as a holomorphic vector line bundle on $M$. In this case, the whole polytope $P$ is a lattice polytope. Also, since $2\rho = (2, 0)$, each outer edge $E$ of $P_+$ must lies on some line

\[(7.3) \quad l_{p,q}(x, y) = (1 + 2p) - (px + qy) = 0
\]

for some coprime pair $(p, q)$. Assume that $l_{p,q} \geq 0$ on $P$. By convexity and $W$-invariance of $P$, $(p, q)$ must satisfy

\[p \geq |q| \geq 0.
\]

Let us start at the outer edge $F_1$ of $P$, which intersects the Weyl wall $W_1 = \{x - y = 0\}$.

There are two cases: Case-1. $F_1$ is orthogonal to $W_1$; Case-2. $F_1$ is not orthogonal to $W_1$.

Case-1. $F_1$ is orthogonal to $W_1$. Then $F_1$ lies on

\[\{(x, y) \mid l_{1,1}(x, y) = 3 - x - y = 0\}.
\]

Consider the vertex $A_1 = (x_1, 3 - x_1)$ of $P_+$ on this edge and suppose that the other edge $F_2$ at this point lies on

\[\{(x, y) \mid l_{p_2,q_2}(x, y) = 0\}.
\]

Thus

\[(7.4) \quad 2p_2 + 1 = x_1p_2 + (3 - x_1)q_2,
\]

and by convexity of $P$,

\[p_2 > q_2 \geq 0.
\]

We will have two subcases according to the possible choices $A_1 = (2, 1)$ or $(3, 0)$.

Case-1.1. $A_1 = (2, 1)$. Then by (7.4),

\[2p_2 + 1 = 2p_2 + q_2.
\]

Thus $q_2 = 1$ and $p_2 \geq 2$.

On the other hand, $l_{p_2,q_2}$ must pass another lattice point $A_2 = (x_2, y_2)$ as the other endpoint of $F_2$. It is direct to see that there are only two possible choices $p_2 = 2, 4$ and three choices of $A_2 = (5, -5)$, $(3, -1)$ and $(3, -3)$.

Case-1.1.1. If $A_2 = (5, -5)$ which lies on the other Weyl wall $W_2 = \{x + y = 0\}$. There can not be any other outer edges of $P_+$, and $P_+$ is given by Figure (7-1-1). By Theorem 1.1 (or equivalently (7.1)), this compactification admits no Kähler-Einstein metric.

Case-1.1.2. $A_2 = (3, -1)$. Then we exhaust the third edge $F_3$ which lies on

\[l_{p_3,q_3} = 2p_3 + 1 - p_3x - q_3y,
\]

\footnote{An edge of $P_+$ is called an outer one if it does not lie in any Weyl wall, cf. [31].}
so that

\[ 2p_3 + 1 = 3p_3 - q_3, \]

\[ p_3 > 2q_3 \geq 0. \]

Hence the only possible choice is \( p_3 = 1, q_3 = 0 \) and the other endpoint of \( F_3 \) is \( A_3 = (0, -3) \). Then \( P_3 \) is given by Figure (7-1-2). Again, this compactification admits no Kähler-Einstein metric.

**Case-1.1.3.** If \( A_2 = (3, -3) \) which lies on the other Weyl wall \( W_2 = \{x + y = 0\} \). There can not be any other outer edges of \( P_+ \), and \( P_+ \) is given by Figure (7-1-3). By Theorem 1.2 this compactification admits no Kähler-Einstein metric.

**Case-1.2.** \( A_1 = (3, 0) \). By the same exhausting progress as in **Case-1.1**. There are two possible polytopes \( P_+ \), **Case-1.2.1** and **Case-1.2.2** (see Figure (7-1-4) and Figure (7-1-5)).
| No.       | Edges, except Weyl walls         | Volume     | KE?   | Smoothness |
|-----------|----------------------------------|------------|-------|------------|
| (7-1-1)   | 3-x-y=0; 5-2x-y=0                | $\frac{4}{7}$ | No    | Singular   |
| (7-1-2)   | 3-x-y=0; 5-2x-y=0; 3-x=0         | $\frac{41}{70}$ | No    | Smooth     |
| (7-1-3)   | 3-x-y=0; 9-4x-y=0                | $\frac{103}{72}$ | No    | Singular   |
| (7-1-4)   | 3-x-y=0; 3-x=0                   | $\frac{70}{20}$ | No    | Smooth     |
| (7-1-5)   | 3-x-y=0; 3-x+y=0                 | $\frac{17}{2}$  | Yes   | Singular   |
| (7-1-6)   | 3-x=0                           | $\frac{648}{5}$ | Yes   | Smooth     |

Table 1. Gorenstein Fano $SO_4(\mathbb{C})$-compactifications.

Case 1.2.1. This compactification admits no Kähler-Einstein metric.

Case 1.2.2. This compactification admits a Kähler-Einstein metric.

Case 2. $F_1$ is not orthogonal to $W_1$. Then its intersection $A_1 = (x_1, x_1)$ with $W_1$ is a vertex of $P$. We see that $F_1$ lies on $l_{p_1,q_1}$ and

$$2p_1 = (p_1 + q_1)x_1,$$

$$p_1 > q_1 \geq 0,$$

$$x_1 = 2 + \frac{1 - 2q_1}{p_1 + q_1} \in \mathbb{N}_+.$$

So the only choice is

$$p_1 = 1, q_1 = 0$$

and $A_1 = (3, 1)$. The only new polytope $P_+$ is given by Figure (7-1-6), which admits

Kähler-Einstein metric.

It is known that Case 1.1.2, Case 1.2.1 and Case 2 are the only smooth $SO_4(\mathbb{C})$-compactifications as shown in [33]. We summarize results of this subsection in Table 1.

7.2. Q-Fano $SO_4(\mathbb{C})$-compactifications. In general, for a fixed integer $m > 0$, it may be hard to give a classification of all Q-Fano compactifications such that $-mK_X$ is Cartier. This is because when $m$ is sufficiently divisible, there will be
too many repeated polytopes directly using Lemma 3.1. To avoid this problem, we give a way to exhaust all $\mathbb{Q}$-Fano polytopes according to the intersection point of $\partial P_+$ with $x$-axis.

We will adopt the notations from the previous subsection. We consider the intersection of $P_+$ with the positive part of the $x$-axis, namely $(x_0, 0)$. Then

$$x_0 = 2 + \frac{1}{p_0}$$

for some $p_0 \in \mathbb{N}_+$, and there is an edge which lies on some $\{l_{p_0,q_0} = 0\}$. Without loss of generality, we may also assume that $\{l_{p_0,q_0} = 0\} \cap \{y > 0\} \neq \emptyset$. Thus by symmetry, it suffices to consider the case

$$p_0 \geq q_0 \geq 0.$$

Indeed, by the prime condition, $q_0 \neq 0, \pm p_0$ if $p_0 \neq 1$. Hence, we may assume

$$(7.5) \quad p_0 > q_0 > 0, p_0 \geq 2.$$  

We associate this number $p_0$ to each $\mathbb{Q}$-Fano polytope $P$ (and hence $\mathbb{Q}$-Fano compactifications of $SO_4(\mathbb{C})$). By the convexity, other edges determined by $l_{p,q}$ must satisfy (see the figure below)

$$p \leq p_0,$$

since we assume that

$$(7.6) \quad P_+ \subseteq (\{l_{p_0,q_0} \geq 0\} \cap a_+).$$

Thus, once $p_0$ is fixed, there are only finitely possible $\mathbb{Q}$-Fano compactifications of $SO_4(\mathbb{C})$ associated to it. In the following table, we list all possible $\mathbb{Q}$-Fano compactifications with $p_0 \leq 2$. We also test the existence of Kähler-Einstein metrics on these compactifications. In the appendix we list the nine non-smooth examples above labeled as in Table-2.

7.3. Proof of Theorem 1.3

**Proof.** We introduce some notations for convenience: For any domain $\Omega \subset \overline{a^+_\mathbb{R}}$, define

$$\text{Vol}(\Omega) := \int_{\Omega} \pi dx \wedge dy,$$

$$\bar{x}(\Omega) := \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} x \pi dx \wedge dy,$$

$$\bar{y}(\Omega) := \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} y \pi dx \wedge dy,$$
Table 2. Q-Fano $SO_4(\mathbb{C})$-compactifications of cases $p_0 \leq 2$. 

| No. | $p_0$ | $(p, q)$ of edges, except Weyl walls | Volume | KE? | Smoothness/Multiple |
|-----|-------|------------------------------------|--------|-----|---------------------|
| (1) | 1     | $(1, 0)$                            | $\frac{127}{2}$ | Yes | Smooth              |
| (2) |       | $(1, 0), (1, 1)$                    | $\frac{1097}{20}$ | No  | Smooth              |
| (3) |       | $(1, -1), (1, 1)$                   | $\frac{37}{4}$ | Yes | Multiple=1          |
| (4) | 2     | $(2, 1)$                            | $\frac{2680}{21}$ | No  | Multiple=3          |
| (5) |       | $(2, 1), (1, 1)$                    | $\frac{411}{2}$ | No  | Multiple=1          |
| (6) |       | $(1, 0), (2, 1)$                    | $\frac{2217}{12}$ | No  | Multiple=3          |
| (7) |       | $(2, 1), (1, -1)$                   | $\frac{497}{36}$ | No  | Multiple=3          |
| (8) |       | $(2, -1), (2, 1)$                   | $\frac{10732}{27}$ | No  | Multiple=6          |
| (9) |       | $(2, 1), (1, 0), (1, 1)$            | $\frac{10731}{10}$ | No  | Smooth              |
| (10)|       | $(2, 1), (1, -1), (1, 1)$           | $\frac{133}{21}$ | No  | Multiple=1          |
| (11)|       | $(2, 1), (2, -1), (1, 1)$           | $\frac{120309}{20}$ | No  | Multiple=6          |
| (12)|       | $(2, 1), (2, -1), (1, 1), (1, -1)$  | $\frac{6299}{728}$ | No  | Multiple=6          |

and

$$\bar{c}(\Omega) := \bar{x} + \bar{y}.$$ 

By Theorem 1.2 and (7.2), we have $\bar{c}(P_+) \geq 2$ whenever the Q-Fano compactification of $SO_4(\mathbb{C})$ admits a Kähler-Einstein metric. 

Recall the number $p_0, q_0$ introduced in Section 7.2. By (7.5), it is direct to see that for any $t \geq 0$ such that $P_+\cap \{y = x - 2t\}$ intersects with $\{y = x - 2t, y \geq -x\}$,

$$(7.7) \quad \bar{c}(P_+) \leq \bar{c}(P_+ \cap \{y \geq x - 2t\}) \leq \bar{c}(\{l_{p_0,q_0} \geq 0, 0 \leq x - y \leq 2t, y \geq -x\}).$$

By a direct computation, we have

$$\bar{c}(\{l_{p_0,q_0} \geq 0, 0 \leq x - y \leq 2t, y \geq -x\}) = \frac{3}{35} \left(15kt + 16b + \frac{3b(10b^2 + 10bkt + 3k^2t^2)}{20b^3 + 45b^2kt + 36bk^2t^2 + 10k^3t^3}\right),$$

where $k = \frac{q_0 - p_0}{p_0 + q_0}$ and $b = \frac{2p_0 + 1}{p_0 + q_0}$. Under the condition (7.5), by using software Wolfram Mathematica 8, we get

$$(7.8) \quad \bar{c}(\{l_{p_0,q_0} \geq 0, 0 \leq x - y \leq 2t, y \geq -x\}) \leq \frac{3}{2} = \frac{6p_0 + 3}{2p_0 + 2q_0}. $$
On the other hand, a polytope with Kähler-Einstein metrics must satisfy
\[ \bar{c}(P_+) > 2. \]

Thus by (7.7) and (7.8), we derive
\[ q_0 < \frac{1}{2} p_0 + \frac{3}{4}. \]

By (7.9), we have
\[ \text{Vol}(P_) \leq \text{Vol}(\{l_{p_0, q_0} \geq 0, x \geq y \geq -x\}) \]
\[ = \frac{8(1 + 2p_0)^6}{45(p_0^2 - q_0^2)^3} \]
\[ \leq \frac{8(1 + 2p_0)^6}{45(p_0^2 - ((1/2)p_0 + (3/4))^2)^3}. \]

It turns that for \( p_0 \geq 9, \)
\[ \text{Vol}(P_+) \leq \frac{224755712}{4100625}. \]

However,
\[ \text{Vol}(P_+^{(2)}) = \frac{1701}{20} > \text{Vol}(P_+^{(3)}) = \frac{10751}{180} > \frac{224755712}{4100625}, \]
where Vol\((P_+^{(2)})\) and Vol\((P_+^{(3)})\) are volumes of polytopes in Case-1.1.2 and Case-1.2.1, respectively. Thus there is no desired Kähler-Einstein polytope with its volume equal to Vol\((P_+^{(2)})\) or Vol\((P_+^{(3)})\) when \( p_0 \geq 9. \)

Since \( q_0 \in \mathbb{N}, \) we can improve (7.10) to
\[ \text{Vol}(P_+) \leq \frac{8(1 + 2p_0)^6}{45(p_0^2 - (1/2)p_0 + (3/4)^2)^3}. \]

Here \( [x] = \max_{n \in \mathbb{Z}} \{n \leq x\}. \) By the above estimation, when \( p_0 = 4, 6, 7, 8, \) we have
\[ \text{Vol}(P_+^{(2)}) > \text{Vol}(P_+^{(3)}) > \text{Vol}(P_+). \]

Hence, it remains to deal with the cases when \( p_0 = 3, 5. \) In these two cases, we shall rule out polytopes that may not satisfy (7.11).

When \( p_0 = 5, \) there are three possible choices of \( q_0, i.e. q_0 = 1, 2, 3 \) by (7.9). It is easy to see that (7.11) still holds for the first two cases by the second relation in (7.10). Thus we only need to consider all possible polytopes when \( q_0 = 3. \) In this case, \( \{l_{5,3} = 0\} \) is an edge of \( P_+. \)

Case-7.3.1. \( P_+ \) has only one outer face which lies on \( \{l_{5,3} = 0\}. \) Then
\[ \text{Vol}(P_+) = \frac{1771561}{23040}. \]

Case-7.3.2. \( P_+ \) has two outer edges. Assume that the second one lies on \( \{l_{p_1, q_1} = 0\}. \) Then
\[ |q_1| \leq p_1 \leq 4 \text{ or } p_1 = 5, q_1 = -3. \]

By a direct computation, we see that (7.11) holds except the following two subcases:

Case-7.3.2.1. \( p_1 = 4, q_1 = 3, \)
\[ \text{Vol}(P_+) = \frac{383478671}{5009940}. \]

Case-7.3.2.2. \( p_1 = 2, q_1 = 1, \)
\[ \text{Vol}(P_+) = \frac{567779}{7680}. \]
Case-7.3.3. $P_+$ has three outer edges. Then $P_+$ is obtained by cutting one of polytopes in Case-7.3.2 with adding new edge $\{l_{p_2,q_2} = 0\}$. In fact we only need to consider $P_+$ obtained by cutting Case-7.3.2.1 and Case-7.3.2.2 above, since it obviously satisfies (7.11) in the other cases. By our construction, we can assume that $|q_2| \leq p_2 \leq p_1$. The only possible $P$ which does not satisfy (7.11) is the case that $p_1 = 4, q_1 = 3$ and $p_2 = 2, q_2 = 1$. However,

$$\text{Vol}(P_+) = \frac{92167583}{1250235}.$$  

Case-7.3.4. $P_+$ has four outer edges. We only need to consider $P_+$ which is obtained by cutting Case-7.3.3 with adding new edge $\{l_{p_3,q_3} = 0\}$ with $|q_3| \leq p_3 \leq 2$. One can show that all of these possible $P_+$ satisfy (7.11). Thus we do not need to consider more polytopes with more than four outer edges in case of $p_0 = 5$. Hence we conclude that for all polytopes $P$ with $p_0 = 5,$

$$\text{Vol}(P_+) \neq \text{Vol}(P_+^{(2)}) \text{ or } \text{Vol}(P_+^{(3)}).$$

Theorem 1.3 is true when $p_0 = 5$.

The case $p_0 = 3$ can be ruled out in a same way. We only list the exceptional polytopes such that the volumes of $P_+$ do not satisfy (7.11):

Case-7.3.1’. $P_+$ has only one outer face $\{l_{3,2} = 0\}$. Then

$$\text{Vol}(P_+) = \frac{941192}{5625}.$$  

Case-7.3.2’. $P_+$ has two outer face $\{l_{3,2} = 0\}$ and $\{l_{2,1} = 0\}$. Then

$$\text{Vol}(P_+) = \frac{177064}{1875}.$$  

In summary, when $p_0 \geq 3$, the volume of $P_+$ is not equal to either $\text{Vol}(P_+^{(2)})$ or $\text{Vol}(P_+^{(3)})$. Finally by exhausting all possible compactifications for $p_0 = 1, 2$ (see Table-2), we finish the proof of Theorem 1.3.

\[\square\]

Remark 7.1. If $P_+$ is further symmetric under the reflection with respect to the $x$-axis, it is easy to see its barycenter is $(\bar{x}(P_+), 0)$ and

$$\bar{x}(P_+) \leq \bar{x}(\{-x \leq y \leq x, 0 \leq x \leq 2 + \frac{1}{p_0}\}) = \frac{6}{7}(2 + \frac{1}{p_0}).$$

Thus a Kähler-Einstein polytope of this type must satisfy

$$p_0 \leq 3.$$  

7.4. Appendix: Non-smooth $\mathbb{Q}$-Fano $SO_4(\mathbb{C})$-compactifications with $p_0 \leq 2$. In this appendix we list all polytopes $P_+$ of non-smooth $\mathbb{Q}$-Fano $SO_4(\mathbb{C})$-compactifications with $p_0 \leq 2$, namely, (3)-(7) and (10)-(12) labeled as in Table-2.
$2\rho(10)$

$2\rho(11)$

$2\rho(12)$
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BICMR and SMS, Peking University, Beijing 100871, China.

E-mail address: liyanmath@pku.edu.cn, tian@math.pku.edu.cn, xhzhu@math.pku.edu.cn