A Note on Guaranteed Stable Recovery of Sparse Signal in Compressed Sensing via the RIP of Orders

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Abstract

In this paper, we shall continue a study of the CS-recovery of signals studied in [1]. Under the assumption that a $m \times n$ matrix $A$ obeys the RIP of order $s$ we decompose the space of unknown vectors into sets $M_0, M_1, \cdots, M_7$ defined by a bias function $p_x$ on a good location $T_0 = \{1, 2, \cdots, s\}$ and research a good condition of CS-recovery.

Keywords: Compressed sensing; restricted isometry property; sparse signal recovery.

1 Introduction

This paper introduces the theory of compressed sensing (CS). For a signal $x \in \mathbb{R}^n$, let $\|x\|_0$ be the $l_0$-norm of $x$, which is defined to be the number of nonzero coordinates, $\|x\|_1$ be the $l_1$-norm of $x$ and $\|x\|_2$ be the $l_2$-norm of $x$. Let $x$ be a sparse or nearly sparse vector. Compressed sensing aims to recover a high-dimensional signal (for example: images signal, voice signal, code signal..etc.) from only a few samples or linear measurements. The efficient recovery of sparse signals has been a very active field in applied mathematics, statistics, machine learning and signal processing. Formally, one considers the following model:

$$y = Ax + z,$$

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where $A$ is a $m \times n$ matrix ($m < n$) and $z$ is an unknown noise term.

Our goal is to reconstruct an unknown signal $x$ based on $A$ and $y$ given. Then we consider reconstructing $x$ as the solution $x^*$ to the optimization problem

$$\min_{x} \|x\|_1, \text{ subject to } \|y - Ax\|_2 \leq \varepsilon,$$

where $\varepsilon$ is an upper bound on the the size of the noisy contribution.

In fact, a crucial issue is to research good conditions under which the inequality

$$\|x - x^*\|_2 \leq C_0\|x - x_T\|_1 + C_1\varepsilon,$$

for suitable constants $C_0$ and $C_1$, where $T$ is any location of $\{1, 2, \ldots, n\}$ with number $|T| = s$ of elements of $T$ and $x_T$ is the restriction of $x$ to indices in $T$. One of the most generally known condition for CS theory is the restricted isometry property (RIP) introduced by [2]. When we discuss our proposed results, it is an important notion. The RIP needs that subsets of columns of $A$ for all locations in $\{1, 2, \ldots, n\}$ behave nearly orthonormal system. In detail, a matrix $A$ satisfies the RIP of order $s$ if there exists a constant $\delta$ with $0 < \delta < 1$ such that

$$(1 - \delta)\|a\|_2^2 \leq \|Aa\|_2^2 \leq (1 + \delta)\|a\|_2^2$$

for all $s$-sparse vectors $a$. A vector is said to be an $s$-sparse vector if it has at most $s$ nonzero entries. The minimum $\delta$ satisfying the above restrictions is said to be the restricted isometry constant and is denoted by $\delta_s$.

Many researchers has been shown that the $l_1$ optimization can recover an unknown signal in noiseless cases and in noisy cases under various sufficient conditions on $\delta$, or $\delta_2$, when $A$ obeys the RIP. For example, E.J. Candès and T. Tao have proved that if $\delta_{2s} < \sqrt{2} - 1$, then an unknown signal can be recovered [3]. Later, S. Foucart and M. Lai have improved the bound to $\delta_{2s} < 0.4531$ [4]. Others, $\delta_{2s} < 0.4652$ is used in [5], $\delta_{2s} < 0.4721$ for cases such that $s$ is a multiple of 4 or $s$ is very large in [6], $\delta_{2s} < 0.4734$ for the case such that $s$ is very large in [5] and $\delta_8 < 0.307$ in [7]. In a recent paper, Q. Mo and S. Li have improved the sufficient condition to $\delta_{2s} < 0.4931$ for general case and $\delta_{2s} < 0.6569$ for the special case such that $n \leq 4s$ [8]. J. Ji and J. Peng have improved the sufficient condition to $\delta_8 < 0.308$ [9]. T. Cai and A. Zhang have improved the sufficient condition to $\delta_8 < 0.333$ for general case [10]. T. Cai and A. Zhang have improved the sufficient condition to $\delta_k$ in case of $k \geq \frac{2}{3}n$, in particular, $\delta_{2s} < 0.707$ [11]. By using a rescaling method, H. Inoue has obtained the sufficient conditions of $\delta_s < 0.5$ and $\delta_{2s} < 0.828$ in [12].

Recently, In [1] we have researched good conditions for the recovery of sparse signals by investigating the difference between the $l_{\infty}$-norm of $h \equiv x^* - x$ and the mean $\frac{|h_1| + |h_2| + \cdots + |h_s|}{|h_1| + |h_2| + \cdots + |h_s|}$ of $\{|h_1|, \ldots, |h_s|\}$. In more details, we considered a function $p$ on $T_0 \equiv \{1, 2, \ldots, s\}$ defined by

$$p(r) = \frac{|h_1| + |h_2| + \cdots + |h_r|}{|h_1| + |h_2| + \cdots + |h_s|}, \quad r = 1, 2, \ldots, s,$$

where the index of $h$ is sorted by $|h_1| \geq |h_2| \geq \cdots \geq |h_s|$ and have shown that for $c > 1$ and $\frac{2}{\pi} < \frac{p(1)}{c}$ if $A$ obeys the RIP of order $\frac{2}{\pi}$ and $\delta_{2s} < \frac{1}{1 + \sqrt{\frac{2}{\pi} p(1)}}$, then we have stable recovery of approximately sparse signals, where $r_c$ is a natural number such that $\frac{2}{\pi} (r_c - \frac{1}{2}) < p(r_c) < \frac{2}{\pi} r_c, \ 2 \leq r_c < \frac{\pi}{2}$. But, the function $p$ on $T_0$ and $r_c$ depend on $x$. Furthermore $r_c$ is not easily searched. In this paper, in order to compensate for these defects, we decompose $K_s(y, A) \equiv \{x \in R^n; \ |y - Ax| \leq \varepsilon\}$ into
the following subsets \( \{M_0, M_1, \cdots, M_7\} \):

\[
M_0 = \left\{ x \in K_s(y, A); \: p_x \left( \frac{1}{2} s \right) \leq \frac{2}{5} \right\},
\]

\[
M_1 = \left\{ x \in K_s(y, A); \: p_x \left( \frac{1}{2} s \right) > \frac{2}{5} \text{ and } p_x \left( \frac{1}{4} s \right) \leq \frac{1}{2} \right\},
\]

\[
 \vdots
\]

\[
M_k = \left\{ x \in K_s(y, A); \: p_x \left( \frac{k + 3}{20} s \right) > \frac{k + 3}{10} \text{ and } p_x \left( \frac{k + 4}{20} s \right) \leq \frac{k + 4}{10} \right\}, \: 2 \leq k \leq 6,
\]

\[
M_7 = \left\{ x \in K_s(y, A); \: p_x \left( \frac{1}{2} s \right) = 1 \right\}
\]

by dividing \( T_0 = \{1, 2, \cdots, s\} \) into \( T_0 \cap [1, \frac{s}{2}), T_0 \cap (\frac{s+3}{20}, \frac{s+4}{20})(k = 1, \cdots, 6) \) and \( T_0 \cap (\frac{s}{2}, s] \), and we show for any \( x \in M_k(k = 1, 2, \cdots, 7) \) that if \( A \) obeys the RIP of order \( s \) and \( \delta_s < \frac{1}{1+\sqrt{\frac{s}{m}-1}} \), then the inequality (1.3) holds. We also state in Section 2 the existence of CS-solution.

### 2 CS-Solution

In this section, we discuss the existence of CS-solutions mathematically.

Let a \( m \times n \) matrix \( A \) \((m < n)\) and a data \( y \in \mathbb{R}^m \) be given. We define closed convex subsets of \( \mathbb{R}^n \) by

\[
K_0(y, A) = \{ x \in \mathbb{R}^n; \: y = Ax \},
\]

\[
K_s(y, A) = \{ x \in \mathbb{R}^n; \: \| y - Ax \|_2 \leq \varepsilon \}, \: \varepsilon > 0.
\]

When \( K_0(y, A) \neq 0 \), that is, \( y \in A\mathbb{R}^n \), then \( K_0(y, A) \) and \( K_s(y, A) \) are

\[
K_0(y, A) = x_0 + \ker A
\]

for some vector \( x_0 \in K_0(y, A) \), where \( \ker A = \{ x \in \mathbb{R}^n; \: Ax = 0 \} \). For example, if the rank \( r(A) \) of \( A \) equals \( m \), then \( AA^* \) is invertible and \( A(A^* AA^*)^{-1} y = y \). Hence, \( AA^*(AA^*)^{-1} y \in K_0(y, A) \).

Let \( y \notin A\mathbb{R}^n \). Since \( A\mathbb{R}^n \) is a closed subspace of \( \mathbb{R}^n \), there exists a unique vector \( y_0 \in A\mathbb{R}^n \) such that \( \| y - y_0 \|_2 = \min \{ \| y - Ax \|_2; \: x \in \mathbb{R}^n \} \). Then \( y_0 \) is a vector in \( A\mathbb{R}^n \) such that \( y - y_0 \) is a vector in the orthogonal complement \( (A\mathbb{R}^n)^\perp \) of \( A\mathbb{R}^n \). It is clear that \( K_s(y, A) \neq \emptyset \) if and only if \( \| y - y_0 \|_2 \leq \varepsilon \). In this paper, we assume that \( K_0(y, A) \neq 0 \) in noiseless cases and \( K_s(y, A) \neq 0 \) in noise cases. We show the existence of CS-solutions.

For any \( t \geq 0 \) we put

\[
D_t = \{ x \in \mathbb{R}^n; \: \| x \|_1 \leq t \}.
\]

Then \( AD_t \) is a closed convex subset of \( A\mathbb{R}^n \) such that \( A(\partial D_t) = \partial AD_t \), where \( \partial K \) is a boundary of a set \( K \). Assume that \( y_0 \notin AD_t \). Then there exists a vector \( x_t \in \partial D_t \) such that \( \| y - Ax_t \|_2 = \min \{ \| y_0 - Ax_\|_2; \: x \in D_t \} \). Since

\[
\| y - Ax_t \|_2^2 = \| y - y_0 \|_2^2 + \| y_0 - Ax_t \|_2^2,
\]

we have

\[
\| y - Ax_t \|_2 = \min \{ \| y - Ax \|_2; \: x \in D_t \}.
\]
which implies that there exists a vector $x^*_1$ in $(x + \ker A) \cap D_t$ such that 

$$\|x^*_1\|_1 \leq \|x_1\|_1, \quad \forall x \in \ker A.$$ 

Thus we have the following:

**Proposition 2.1.** Suppose that $K_s(y, A) \neq \emptyset$. Then there exists a positive number $t_0$ such that 

$$\|y_0 - Ax_{t_0}\|^2 = \varepsilon^2 - \|y - y_0\|^2$$

and the vector $x^*_{t_0}$ determined by $x_{t_0}$ equals the CS-solution $x^*$. In particular, in noiseless cases, $x^* = x^*_{t_0}$, where $t_0$ is a positive number satisfying $y_0 = Ax_{t_0}$.

### 3 Recovery of CS

Take an arbitrary $x \in K_s(y, A)$. We denote by $x^T$ a vector obtained by changing coefficients of $x$ as follows;

$$|h_1| \geq |h_2| \geq \cdots \geq |h_n|,$$

where \( h = (h_1, h_2, \cdots h_n) \equiv x^* - x^T \). Let $T_0 = \{1, 2, \cdots, s\}$ and we define a function $p_x(r)$ on $T_0$ depending on $x$ by 

$$p_x(r) = \|\frac{|h_1| + |h_2| + \cdots + |h_s|}{\|h_{T_0}\|_1} - r\|,$$

where \( r \in T_0 \).

By dividing $T_0 = \{1, 2, \cdots, s\}$ into $T_0 \cap [1, \frac{s}{2}]$, $T_0 \cap [\frac{k+3}{20}s, \frac{k+4}{20}s] (k = 1, \cdots, 6)$ and $T_0 \cap (\frac{s}{2}, s]$, we decomposed $K_s(y, A)$ into the following subsets \( \{M_0, M_1, \cdots, M_7\} \):

- $M_0 = \left\{ x \in K_s(y, A); \; p_x\left(\frac{1}{5}s\right) \leq \frac{2}{5}\right\},$
- $M_1 = \left\{ x \in K_s(y, A); \; p_x\left(\frac{1}{5}s\right) > \frac{2}{5} \text{ and } p_x\left(\frac{1}{4}s\right) \leq \frac{1}{2}\right\},$
- $\vdots$
- $M_6 = \left\{ x \in K_s(y, A); \; p_x\left(\frac{k+3}{20}s\right) > \frac{k+3}{10} \text{ and } p_x\left(\frac{k+4}{20}s\right) \leq \frac{k+4}{10}\right\}, \; 2 \leq k \leq 6,$
- $M_7 = \left\{ x \in K_s(y, A); \; p_x\left(\frac{1}{2}s\right) = 1\right\}.$

Then, $K_s(y, A) = \bigcup_{k=0}^{7} M_k$ and $M_i \cap M_j = \emptyset (i \neq j)$. (Figure 1)

Using the function $p_x(r)$ on $T_0$, we obtain a similar result to that of ([1] Theorem 2.1):

**Theorem 3.1.** Take an arbitrary $x \in M_k (k = 1, 2, \cdots, 7)$. Assume that $A$ obeys the RIP of order $s$ and $\delta_s \leq \frac{1}{1 + \sqrt{\frac{\delta_s}{2\pi}s} - 1}$. Then,

$$\|x^* - x\|_2 \leq C_0^{(k)} \|x - x_s\|_1 + C_1^{(k)} \varepsilon,$$
where $\mathbf{x}_s$ is a vector consisting of the $s$-large entries of $\mathbf{x}$ in magnitude and

$$C^{(k)}_0 = \frac{4\sqrt{\frac{20}{k+3}} - 1 \cdot \delta_s}{1 - (1 + \sqrt{\frac{20}{k+3}} - 1) \delta_s},$$

$$C^{(k)}_1 = \frac{2\sqrt{1 + \delta_s} \sqrt{s}}{\sqrt{\frac{k+3}{20}} (1 - (1 + \sqrt{\frac{20}{k+3}} - 1) \delta_s)}.$$

**Proof.** Take an arbitrary $\mathbf{x} \in M_k$. Let $r_k$ be a natural number such that

$$\frac{k+3}{20} s < r_k \leq \frac{k+4}{20} s \quad \text{and} \quad \frac{2}{s} (r_k - 1) < p_x(r_k) \leq \frac{2}{s} r_k.$$  

Then,

$$\frac{k+3}{10} < p_x(r_k) \leq \frac{k+4}{10}. \quad (3.2)$$

We put

$$\alpha = \frac{\|h_{r_k}\|_1 + 2\|\mathbf{x} - \mathbf{x}_s\|_{1}}{s}.$$
Let $T_1 = \{1, 2, \cdots, r_2\}$ and $T_2 = \{r_2 + 1, \cdots, n\}$ be a decomposition of $\{1, 2, \cdots, n\}$. By (3.1) and (3.2) we have

$$\|h_{T_2}\|_1 \leq \frac{p_x(r_2)}{r_2} \|h_{T_0}\|_1 \leq 2\alpha.$$  

(3.3)

By the definition of CS optimization (1.2), we have

$$\|h_{T_2}\|_1 \leq \|h_{T_0}\|_1 + 2\|x - x_s\|_1.$$  

(3.4)

Hence it follows from (3.3) and (3.4) that

$$\|h_{T_2}\|_1 = \|h_{T_0}\|_1 + \|h_{T_0 \cap T_2}\|_1 \leq \alpha s + (1 - p_x(r_k)) \|h_{T_0}\|_1 \leq (2 - p_x(r_k)) \alpha s \leq 2\alpha \left(1 - \frac{k + 3}{20}\right) s,$$

which implies by [1] Lemma 1.1 and the Cai idea [4] that there exist $\{\lambda_i\}_{1 \leq i \leq N}$ and $\{u_i\}_{1 \leq i \leq N}$ such that

$$h_{T_2} = \sum_{i=1}^{N} \lambda_i u_i,$$
where

\[ 0 \leq \lambda_i \leq 1, \quad \sum_{i=1}^{N} \lambda_i = 1, \]
\[ \text{supp } u_i \subset T_2, \quad |\text{supp } u_i| \leq \left( 1 - \frac{k + 3}{20} \right) s \]
\[ \|u_i\|_{1} \leq 2 \alpha. \]  

Hence we have
\[ \|u_i\|_2 \leq \|u_i\|_{\infty} \sqrt{|\text{supp } u_i|} \leq 2 \alpha \sqrt{1 - \frac{k + 3}{20}}, \]
\[ |T_1| + |\text{supp } u_i| \leq r_k + \left( 1 - \frac{k + 3}{20} \right) s \leq s \]

and
\[ \alpha_s = \|h_{T_1}\|_1 + 2\|x - x_s\|_1 \]
\[ = \frac{1}{\rho_s(r_k)} \|h_{T_1}\|_1 + 2\|x - x_s\|_1 \]
\[ \leq \frac{\sqrt{s}}{\rho_s(r_k)} \|h_{T_1}\|_2 + 2\|x - x_s\|_1 \]
\[ \leq \frac{\sqrt{s}}{2\sqrt{1 - \frac{k + 3}{20}}} \|h_{T_1}\|_2 + 2\|x - x_s\|_1, \]

which implies since \( A \) obeys the RIP of order \( s \) that

\[ (1 - \delta_s)\|h_{T_1}\|_2^2 \leq \|Ah_{T_1}\|_2^2 \leq \|Ah_{T_1}\|_1 + \|Ah_{T_2}\|_1 \]
\[ \leq \sqrt{1 + \delta_s} \|h_{T_1}\|_2 \cdot 2\varepsilon + \sum_{i=1}^{N} \lambda_i |\langle Ah_{T_1}, Au_i \rangle| \]
\[ \leq 2\sqrt{1 + \delta_s} \|h_{T_1}\|_2 + 2\|x - x_s\|_1 \sum_{i=1}^{N} \lambda_i \|h_{T_1}\|_2 |u_i|_2 \]
\[ \leq 2\sqrt{1 + \delta_s} \|h_{T_1}\|_2 \]
\[ + \delta_s \|h_{T_1}\|_2 \left( \frac{1}{2\sqrt{1 - \frac{k + 3}{20}}} \|h_{T_1}\|_2 + \frac{2}{\sqrt{s}} \|x - x_s\|_1 \right) \cdot 2\sqrt{1 - \frac{k + 3}{20}} \]
\[ = 2\sqrt{1 + \delta_s} \|h_{T_1}\|_2 + \delta_s \sqrt{\frac{20}{k + 3} - 1}\|h_{T_1}\|_2^2 \]
\[ + \frac{4\delta_s}{\sqrt{s}} \sqrt{1 - \frac{k + 3}{20}} \|x - x_s\|_1 \|h_{T_1}\|_2. \]

Since
\[ \left( 1 + \sqrt{\frac{20}{k + 3} - 1} \right) \delta_s < 1, \]
we have
\[
\|h_{T_1}\|_2 \leq \frac{2\sqrt{1 + \delta_s \varepsilon} + \frac{4\delta_s}{\sqrt{3}} \sqrt{1 - \frac{6 + 3\delta_s}{20} \|x - x_s\|_1}}{1 - \left(1 + \sqrt{\frac{20}{20 + 1}} - 1\right) \delta_s},
\]
which implies that
\[
\|x - x^*\|_2 \leq \|x - x^*\|_1
\]
\[
= \|h_{T_0}\|_1 + \|h_{T_0}\|_2
\]
\[
\leq 2\|h_{T_0}\|_1 + 2\|x - x_s\|_1
\]
\[
\leq \frac{2\sqrt{\tau}}{\varepsilon}\|h_{T_1}\|_2 + 2\|x - x_s\|_1
\]
\[
\leq \frac{\sqrt{\varepsilon}}{\sqrt{\frac{\delta_s}{1 - \sqrt{\frac{20}{20 + 1}}}}}
\]
\[
\left(\frac{2\sqrt{1 + \delta_s \varepsilon} + \frac{4\delta_s}{\sqrt{3}} \sqrt{1 - \frac{6 + 3\delta_s}{20} \|x - x_s\|_1}}{1 - \left(1 + \sqrt{\frac{20}{20 + 1}} - 1\right) \delta_s}
\right)
\]
\[
+ 2\|x - x_s\|_1
\]
\[
= \frac{2\sqrt{1 + \delta_s \varepsilon}}{\sqrt{\frac{\delta_s}{1 - \sqrt{\frac{20}{20 + 1}}}}} + \frac{4 \sqrt{\frac{20}{20 + 1}} \delta_s - 1 \cdot \delta_s}{1 - \left(1 + \sqrt{\frac{20}{20 + 1}} - 1\right) \delta_s} \|x - x_s\|_1.
\]
This completes the proof.

We state concretely the following case:

(i) Take an arbitrary \(x \in M_1\). If \(\delta_s < \frac{1}{3}\), then
\[
\|x^* - x\|_2 \leq \frac{8\delta_s}{1 - 3\delta_s} \|x - x_s\|_1 + \frac{2\sqrt{\frac{5\delta_s}{20}}\sqrt{1 + \delta_s \varepsilon}}{1 - 3\delta_s}.
\]

(ii) Take an arbitrary \(x \in M_2\). If \(\delta_s < \frac{\sqrt{7} - 1}{2} \approx 0.366\), then
\[
\|x^* - x\|_2 \leq \frac{4\sqrt{3}\delta_s}{1 - (1 + \sqrt{3})\delta_s} \|x - x_s\|_1 + \frac{4\sqrt{1 + \delta_s \varepsilon}}{1 - (1 + \sqrt{3})\delta_s}.
\]

(iii) Take an arbitrary \(x \in M_7\). If \(\delta_s < \frac{1}{5}\), then
\[
\|x^* - x\|_2 \leq \frac{4\delta_s}{1 - 2\delta_s} \|x - x_s\|_1 + \frac{2\sqrt{2\sqrt{1 + \delta_s \varepsilon}}}{1 - 2\delta_s}.
\]

Though we have decomposed \(K_s(y, A)\) into \(M_k(k = 0, 1, \ldots, 7)\) in this paper, we may consider the other decompositions of \(K_s(y, A)\).

4 Conclusion

In a previous paper [1], we have discussed sufficient conditions of isometry constant \(\delta\) by investigating a bias function \(p_x\) defined by each unknown vector \(x\). In this paper, we decompose the space of unknown vectors into sets \(M_0, M_1, \ldots, M_7\) defined by the bias function \(p_x\). More precisely, when
$x$ is contained in $M_k$ ($1 \leq k \leq n$), the sufficient condition of $\delta_s$ is improved, and so this method is useful. When $x \in M_0$, the sufficient condition of $\delta_s$ is not improved by this method. We think that this method is more usable than a previous one in [1].

Competing Interests
The author declares that no competing interests exist.

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