On the Bohr’s Inequality for Stable Mappings

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Abstract
We consider the class of stable harmonic mappings \( f = h + \overline{g} \) introduced by Martin, Hernandez and the class of stable logharmonic mappings \( f = zh\overline{g} \) introduced by AbdulHadi, El-Hajj. We determine Bohr’s radius for the classes of stable univalent harmonic mappings, stable convex harmonic mappings and stable univalent logharmonic mappings. We also consider improved and refined versions of Bohr’s inequality and discuss the Bohr–Rogosinski’s radius for this family of mappings.

Keywords Harmonic mappings · Logharmonic mappings · Bohr’s radius · Stable properties · Stable univalent · Stable convex

Mathematics Subject Classification Primary 30C35 · 30C45 · 30C50

1 Introduction

In this paper, we study Bohr’s phenomenon for the class of harmonic mappings and logharmonic mappings under the condition of stability. The classical Bohr’s inequality for the class of analytic mappings states that if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is analytic in the unit disk \( U \) and \(|f(z)| < 1 \) for all \( z \) in \( U \), then \( \sum_{n=0}^{\infty} |a_n z^n| \leq 1 \) for all \( z \in U \) with \(|z| \leq \frac{1}{3} \). This inequality was discovered by Bohr in 1914 (see [18]). Our focus of attention is the class of harmonic and logharmonic mappings. We first recall that a
planar harmonic mapping in the unit disk $U = \{z : |z| < 1\}$ is a complex-valued harmonic function $f$ which maps $U$ onto some planar domain $f(U)$ and satisfies $\Delta f = \partial_\overline{z} f = 0$. The harmonic function $f$ admits the representation $f = h + \overline{g}$, where the functions $h$ and $g$ belong to the linear space $H(U)$ of all analytic functions on $U$. If, in addition, we require $f$ to be orientation-preserving, then $a = \frac{g'}{h'}$, belongs also to $H(U)$ and we have $|a| < 1$ for all $z \in U$. The function $a$ is called the second dilatation function of $f$. Observe that the analytic functions on $U$ form a subclass of the set of all harmonic orientation-preserving maps characterized by the relation $a \equiv 0$.

A logarithmic mapping defined on $U$ is a solution of the nonlinear elliptic partial differential equation
\[
\frac{\overline{z}}{f} = a \frac{f_z}{f},
\]
where $a$ is an analytic function satisfying $|a(z)| < 1$ in $U$.

If $f$ is a non-constant logarithmic mapping of $U$ and vanishes only at $z = 0$, then $f$ admits the representation
\[
f(z) = z^m |z|^{2\beta} h(z) \overline{g(z)},
\]
where $m$ is a nonnegative integer, $\text{Re}(\beta) > -1/2$, and $h$ and $g$ are analytic functions in $U$ satisfying $g(0) = 1$ and $h(0) \neq 0$ (see [3]). Note that $f(0) \neq 0$ if and only if $m = 0$, and that a univalent logarithmic mapping on $U$ vanishes at the origin if and only if $m = 1$, that is, $f$ has the form
\[
f(z) = z|z|^{2\beta} h(z) \overline{g(z)},
\]
where $\text{Re}(\beta) > -1/2$ and $0 \notin (hg)(U)$. This class has been studied extensively (for details see [1, 2, 4], and the recent articles [19, 35, 38]).

In [23], the authors introduced the notion of stable harmonic mappings.

**Definition 1** A (sense preserving) harmonic mapping $f = h + \overline{g}$ is stable harmonic univalent (briefly SHU) in the unit disk [resp. stable harmonic convex (briefly SHC)] if all the mappings $f_{\lambda} = h + \lambda \overline{g}$ with $|\lambda| = 1$ are univalent (resp. convex) in $U$.

Similarly, in [4], the authors introduced the notion of stable logarithmic mappings and proved some interesting properties for this class of functions. We first recall the definition of stable univalent logarithmic mappings and the properties that will be useful for our purposes.

**Definition 2** A logarithmic mapping $f = zh\overline{g}$ such that $f(0) = 0$ and $h(0) = g(0) = 1$, is said to be stable univalent logarithmic (briefly SST$_{Lh}$) in the unit disk, if the mappings $f_{\lambda}(z) = zh(z)\overline{g(z)}^\lambda$ are univalent for all $|\lambda| \leq 1$.

In recent years, a number of researchers revisited the work of Bohr improving and extending this work to more general settings. Various generalizations of the classical Bohr’s inequality have been investigated in different branches of mathematics, see
[5–9, 16, 17, 24, 37], and the papers [11–13, 39] discussed Bohr’s theorem for the case of several complex variables.

We refer the reader to a survey on this topic by Abu-Muhanna et al. [10] and the references therein, which increased the interest in the topic. This was followed by a series of papers by several authors discussing several improved and refined versions of Bohr’s inequality, see for instance [15, 21, 22, 27–31, 36, 39, 40]. An observation shows that Bohr’s inequality, can be written in the following form

\[
\sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |a_0| = d(f(0), \partial U),
\]

for \(|z| = r \leq 1/3\), where \(d\) is the Euclidean distance. It is important to note that the constant \(1/3\) is independent of the coefficients of the Taylor series of \(f(z)\).

Besides the Bohr’s radius, there is also the notion of Rogosinski’s radius (see [34, 42]) which is described as follows: If \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) is an analytic function on \(U\) such that \(|f(z)| < 1\) in \(U\), then for every \(N \geq 1\), we have \(|S_N(z)| < 1\) in the disk \(|z| < 1/2\) and this radius is sharp, where \(S_N(z) = \sum_{n=0}^{N} a_n z^n\) denotes the partial sums of \(f\). In [26, 29], the authors define the Bohr–Rogosinski’s sum \(R_N^f(z)\) by

\[
R_N^f(z) = |f(z)| + \sum_{n=N}^{\infty} |a_n z^n|, \quad |z| = r.
\]

It is worth noting that for \(N = 1\), this quantity is related to the classical Bohr’s sum in which \(f(0)\) is replaced by \(f(z)\). The Bohr–Rogosinski’s radius is defined to be the largest number \(r_0 > 0\) such that \(R_N^f(z) \leq 1\), for \(|z| \leq r_0\). Moreover, we have \(|S_N(z)| \leq R_N^f(z)\), hence Bohr–Rogosinski’s sum is related to Rogosinski’s characteristic and the validity of Bohr-type radius for \(R_N^f(z)\) gives Rogosinski’s radius in the case of bounded analytic functions. Another version of Bohr’s theorem exists with the initial coefficient \(|a_0|^2\) instead of \(|a_0|\). It is well known that the Bohr’s radius with this change of initial coefficient becomes \(1/2\) instead of \(1/3\). We will also look at the analogue in the setting of Bohr–Rogosinski, i.e., with \(|f(z)|^2\) in place of \(|f(z)|\).

This paper is organized as follows: in Sect. 2, we consider the Bohr-type inequalities for the class of stable harmonic mappings and in Sect. 3 we will consider the class of stable logharmonic mappings. In each of these sections, we will show for our classes of stable mapping a version of the Bohr’s inequality along with its improved and refined versions as in [22, 29, 41]. In addition, the Bohr–Rogosinski’s radius as in [26, 28, 32] will be discussed.

### 2 Bohr’s Inequalities for Stable Harmonic Mappings

There has been a lot of interest recently in Bohr’s radius for harmonic mappings. We refer to the survey paper [32] for a nice compilation and exposition of these results.
Namely, results have been obtained for locally univalent harmonic mappings and $k$-quasiconformal harmonic mappings.

The class $S^0_H$ is defined as the family of sense preserving univalent harmonic mappings $f = h + \overline{g}$ in the unit disk with the normalizations $h(0) = g(0) = 1 - h'(0) = g'(0) = 0$. We let

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \quad (2.1)$$

It is worth pointing out that the coefficient conjecture for $f$ in $S^0_H$ remains an open problem even for $a_2$. (See the paper of Clunie Shiel-Small [20]). The best-known bound for $|a_2|$ is $20.9197$ (see [10]). We define the majorant series for $f = h + \overline{g}$ as in [8] to be

$$M_f(r) = \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n = r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n.$$ 

### 2.1 Bohr’s Radius for Stable Harmonic Mappings

We consider the Bohr’s radius for the class of stable univalent harmonic mappings and stable convex harmonic mappings in $S^0_H$, which were introduced in [23]. We will need the following coefficients and distortion theorems for these classes of functions:

**Theorem A** [23]

(i) Assume that $f = h + \overline{g}$ in $S^0_H$ is stable convex harmonic mapping. Then for all nonnegative integers $n$, we have

$$||a_n| - |b_n|| \leq \max\{|a_n|, |b_n|\} \leq |a_n| + |b_n| \leq 1. \quad (2.2)$$

(ii) Assume that $f = h + \overline{g}$ in $S^0_H$ is stable univalent harmonic mapping. Then for all nonnegative integers $n$, we have

$$||a_n| - |b_n|| \leq \max\{|a_n|, |b_n|\} \leq |a_n| + |b_n| \leq n. \quad (2.3)$$

**Theorem B** [23]

(i) Let $f = h + \overline{g}$ in $S^0_H$ be a stable convex harmonic mapping on the unit disk $U$. Then for all $z \in U$, we have:

$$\frac{|z|}{1 + |z|} \leq |f(z)| \leq \frac{|z|}{1 - |z|}.$$ 

(ii) Let $f = h + \overline{g}$ in $S^0_H$ be a stable univalent harmonic mapping on the unit disk $U$. Then for all $z \in U$, we have:

$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}.$$
We are now ready to prove the Bohr’s radius for stable harmonic mappings.

**Theorem 2.1.1**

(i) Let \( f = h + \overline{g} \) in \( S_H^0 \) be a stable convex harmonic mapping on the unit disk \( U \). Then

\[
M_f(r) \leq d(f(0), \partial f(U))
\]

if \( |z| \leq r_0 = 1/3 \).

(ii) Let \( f = h + \overline{g} \) in \( S_H^0 \) be a stable univalent harmonic mapping (or stable starlike) on the unit disk \( U \). Then

\[
M_f(r) \leq 1
\]

for \( |z| \leq r_0 \), where \( r_0 = \frac{3 - \sqrt{5}}{2} \), and

\[
M_f(r) \leq d(f(0), \partial f(U))
\]

if \( |z| \leq r_0 \), where \( r_0 = 3 - \sqrt{8} \). In all of the above, \( r_0 \) is the best possible radius.

**Proof** For \( |z| = r \), in the case where \( f \) is stable convex harmonic we have

\[
M_f(r) \leq \sum_{n=1}^{\infty} r^n = \frac{r}{1 - r}.
\] (2.4)

But,

\[
d(f(0), \partial f(U)) = \lim_{|z| \to 1} \inf |f(z) - f(0)| \geq \lim_{|z| \to 1} \inf \frac{|z|}{1 + |z|} = \frac{1}{2}.
\] (2.5)

Hence, as \( \frac{r}{1 - r} \leq 1/2 \) if and only if \( r \leq 1/3 \), we obtain

\[
M_f(r) < \frac{1}{2} \leq d(f(0), \partial f(U))
\]

if \( |z| \leq r_0 = 1/3 \). A suitable rotation of the analytic mapping \( g(z) = \frac{z}{1 - z} \) shows that \( r_0 \) is the best possible radius, since for this function we have \( M_g(r) = \frac{r}{1 - r} \) and 
\[
d(g(0), \partial g(U)) = \frac{1}{2}.
\]

Similarly for part (ii), we have for \( f \) stable univalent harmonic

\[
M_f(r) \leq \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq \sum_{n=1}^{\infty} nr^n = \frac{r}{(1 - r)^2}.
\] (2.6)

It follows that \( M_f(r) < 1 \) for \( |z| \leq r_0 = \frac{3 - \sqrt{5}}{2} \), where \( r_0 \) is the unique root in (0, 1) of

\[
\frac{r}{(1 - r)^2} = 1,
\]
that is \( r^2 - 3r + 1 = 0 \). We next note that
\[
d(f(0), \partial f(U)) = \lim_{|z|\to 1} \inf \frac{|f(z) - f(0)|}{|z|} \geq \lim_{|z|\to 1} \inf \frac{1}{(1 + |z|)^2} = \frac{1}{4},
\] (2.7)
where the last inequality follows from Theorem B.

Hence, we have
\[
M_f(r) < \frac{1}{4} \leq d(0, \partial f(U)),
\]
if \(|z| \leq r_0\), where \( r_0 = 3 - \sqrt{8} \) is the unique root in \((0, 1)\) of
\[
\frac{r}{(1 - r)^2} = \frac{1}{4},
\]
that is, \( r^2 - 6r + 1 = 0 \). A suitable rotation of the analytic Koebe mapping \( k(z) = \frac{z}{(1 - z)^2} \) shows that \( r_0 \) is the best possible radius. \( \square \)

### 2.2 Improved Bohr’s Inequality for Stable Harmonic Mappings

Kayumov and Ponnusamy [29] improved the classical version of the Bohr theorem in four different formulations. In the same spirit, as with the analytic case, Evdoridis et al. [22] improved the results of [29] for locally univalent harmonic mappings. We refer to [25, 33] for a nice exposition of all these results. In this paper, we show a version of the improved Bohr’s inequality under the stability condition for harmonic mappings by adding a suitable nonnegative term at the left-hand side of the inequality.

**Theorem 2.2.1** (i) Let \( f = h + \overline{g} \) in \( S^0_H \) be a stable convex harmonic mapping on the unit disk \( U \), and let \( S_r \) be the area of the image \( f(D_r) \), with \( D_r = \{|z| = r\} \). Then,
\[
M_f(r) + \left( \frac{S_r}{\pi} \right)^k \leq d(f(0), \partial f(U))
\]
if \(|z| \leq r_0\), where \( r_0 \) is the unique root in \((0, 1)\) of
\[
\frac{r}{1 - r} + \frac{r^{2k}}{(1 - r^2)^{2k}} = \frac{1}{2}.
\]
Note that, for \( k = 1 \), we have \( r_0 \approx 0.287 \), for \( k = 10 \), we have \( r_0 \approx 0.33 \).

(ii) Let \( f = h + \overline{g} \) in \( S^0_H \) be a stable univalent harmonic mapping (or stable starlike) on the unit disk \( U \), then
\[
M_f(r) + \left( \frac{S_r}{\pi} \right)^k \leq d(f(0), \partial f(U))
\]
for $|z| \leq r_0$, where $r_0$ is the unique root in $(0, 1)$ of

$$\frac{r}{(1-r)^2} + \frac{(r^6 + 4r^4 + r^2)^k}{(r^2 - 1)^{4k}} = \frac{1}{4}.\]  

Note that, for $k = 1$, we have $r_0 \approx 0.157$, for $k = 10$, we have $r_0 \approx 0.172$.

**Proof** (i) We first find the bound on $S_r$ under the stability condition. For $f = h + \bar{g}$ given by (2.1), it is well known that

$$S_r = \frac{1}{\pi} \int \int_U J_f dA = \sum_{n=1}^{\infty} n(|a_n| - |b_n|)(|a_n| + |b_n|)r^{2n}.$$

If $f$ is stable convex, we use Eq. (2.2) to get that $S_r \leq \sum_{n=1}^{\infty} nr^{2n} = \frac{r^2}{(1-r^2)^2}$, and it follows that

$$M_f(r) + \left( \frac{S_r}{\pi} \right)^k \leq \frac{r}{1-r} + \frac{r^{2k}}{(1-r^2)^{2k}} \leq d(f(0), \partial f(U))$$

if $|z| \leq r_0$, where $r_0$ is the unique root in $(0, 1)$ of

$$\frac{r}{1-r} + \frac{r^{2k}}{(1-r^2)^{2k}} = \frac{1}{2}.\]  

(ii) In the case where $f$ is stable univalent or stable starlike, we use Eq. (2.3) to get

$$S_r \leq \sum_{n=1}^{\infty} n^3r^{2n} = \frac{r^6 + 4r^4 + r^2}{(r^2 - 1)^4},$$

and the rest of the proof will follow as in part (i). \qed

Another version of the improved Bohr’s inequality is stated in the following theorem.

**Theorem 2.2.2** (i) Let $f = h + \bar{g}$ in $S_H^0$ be a stable convex harmonic mapping on the unit disk $U$. Then

$$M_f(r) + c \sum_{n=1}^{\infty} \left( |a_n|^k + |b_n|^k \right) r^n \leq d(f(0), \partial f(U))$$

if $|z| \leq r_0$, where $k$ and $c$ are constants, and $r_0 = \frac{1}{3 + 4c}$ is the unique root in $(0, 1)$ of

$$(1 + 2c) \frac{r}{1-r} = \frac{1}{2}.\]  

Note that, for $c = 1$, we have $r_0 \approx 0.143$, and for $c = 0.1$, we have $r_0 \approx 0.29$.\footnote{Springer}
(ii) Let \( f = h + \overline{g} \) in \( S^0_H \) be a stable univalent harmonic mapping (or stable starlike) on the unit disk \( U \). Then

\[
M_f(r) + c \sum_{n=1}^{\infty} \left( |a_n|^2 + |b_n|^2 \right) r^n \leq d(f(0), \partial f(U))
\]

for \( |z| \leq r_0 \), where \( r_0 \) is the unique root in \((0, 1)\) of

\[
\frac{r}{(1-r)^2} + 2c \frac{(r(r+1))}{(1-r)^3} = \frac{1}{4}.
\]

Note that, for \( c = 0.1 \), we have \( r_0 \approx 0.144 \), and for \( c = 1 \), we have \( r_0 \approx 0.066 \).

The proof of this Theorem follows along the same lines as Theorem 2.2.1. Here, in part(i), we use the fact that \( |a_n|^k + |b_n|^k \leq 2 \), to get estimates independent of \( k \).

### 2.3 Refined Bohr’s Inequality for Stable Harmonic Mappings

In [41], the authors compared \( \sum_{n=1}^{\infty} |a_n| r^n \) with another functional often considered in function theory, namely, \( \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \) that they abbreviated as \( \|f\|_r^2 \) for the analytic function \( f(z) = \sum_{n=1}^{\infty} a_n z^n \). They thus obtain the following refinement of Bohr’s inequality:

**Theorem C** Suppose that \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) analytic in the unit disk with \( a_0 = 0 \) and \( |f(z)| \leq 1 \). Then

\[
\sum_{n=1}^{\infty} |a_n| r^n + \frac{1}{1-r} \|f\|_r^2 \leq 1
\]

for \( r \leq 1/2 \).

We consider a similar refinement of Bohr’s inequality for the family of stable harmonic mappings \( f = h + \overline{g} \) in \( S^0_H \). We leave the proof to the reader as it uses similar ideas as the previous theorems.

**Theorem 2.3.1** (i) Let \( f = h + \overline{g} \) in \( S^0_H \) be a stable convex harmonic mapping on the unit disk \( U \). Then for \( N \in \mathbb{N} \) we have

\[
M_f(r) + \frac{r^N}{1-r^N} \sum_{n=1}^{\infty} \left( |a_n|^2 + |b_n|^2 \right) r^{2n} \leq d(f(0), \partial f(U))
\]

if \( |z| \leq r_0 \), where \( r_0 \) is the unique root in \((0, 1)\) of the equation \( \psi_N(r) = 0 \), where

\[
\psi_N(r) = \frac{r}{1-r} + \frac{r^{N+2}}{(1-r^2)(1-r^N)} - \frac{1}{2}.
\]
(ii) Let \( f = h + \overline{g} \) in \( S^0_H \) be a stable univalent (or stable starlike) harmonic mapping on the unit disk \( U \). Then for \( N \in \mathbb{N} \) we have

\[
M_f(r) + \frac{r^N}{1 - r^N} \sum_{n=1}^{\infty} \left( |a_n|^2 + |b_n|^2 \right) r^{2n} \leq d(f(0), \partial f(U))
\]

if \( |z| \leq r_0 \), where \( r_0 \) is the unique root in \((0, 1)\) of the equation \( \phi_N(r) = 0 \), where

\[
\phi_N(r) = \frac{r}{(1 - r)^2} - \frac{r^{N+2}(1 + r^2)}{(r^2 - 1)^3(1 - r^N)} - \frac{1}{4}.
\]

**Remark 1** Some computations show that in part(i), for \( N = 1 \), we have \( r_0 \approx 0.311 \) is the root of the equation \( r^3 - r^2 - 3r + 1 = 0 \), and for larger \( N, r_0 \approx 0.33 \) is the root of the equation \( r_{1/r} = \frac{1}{2} \). In part(ii), for \( N \) large enough we have \( r_0 \approx 0.1711 \).

### 2.4 Bohr–Rogosinski’s radius for stable harmonic mappings

In [26], Kayoumov and Ponnusamy investigate the Bohr–Rogosinski’s radii for analytic functions defined for \( |z| < 1 \) discuss the Bohr–Rogosinski’s radius for a class of subordinations. In particular, they connect the Bohr’s radius and the Bohr–Rogosinski’s radius for this class of mappings. (For details, see Theorems 1 and 2 in [26].) Here we show similar results for the classes of stable mappings in the following theorem:

**Theorem 2.4.1** (i) Let \( f = h + \overline{g} \) in \( S^0_H \) be a stable convex harmonic mapping on the unit disk \( U \). Then for each \( m, N \in \mathbb{N} \), we have

\[
|f(z^m)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(U)),
\]

if \( |z| \leq r_{m,N} \), where \( r_{m,N} \) is the unique root in \((0, 1)\) of the equation \( H_{m,N}(r) = 0 \), where

\[
H_{m,N}(r) = \frac{2r^N(1 - r^m)}{1 - r} + 3r^m - 1, \tag{2.8}
\]

and \( r_{m,N} \) cannot be improved. Here \( \lim_{N \to \infty} r_{m,N} = \left( \frac{1}{3} \right)^{1/m} \lim_{m \to \infty} r_{m,N} = r_N \), where \( r_N \) is the solution of the equation \( 2r^N = 1 - r \), and \( \lim_{m,N \to \infty} r_{m,N} = 1 \). Moreover,

\[
|f(z^m)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(U)),
\]

if \( |z| \leq R_{m,N} \), where \( R_{m,N} \) is the unique root in \((0, 1)\) of the equation \( K_{m,N}(r) = 0 \), where

\[
K_{m,N}(r) = \frac{2r^N(1 - r^m)^2}{1 - r} + r^{2m} + 2r^m - 1, \tag{2.9}
\]
and \( \lim_{n \to \infty} R_{m,N} = R_N \), where \( R_N \) is the solution of the equation \( 2r_N = 1 - r \), and \( \lim_{m \to \infty} R_{m,N} = 1 \).

(ii) Let \( f = h + \overline{g} \) in \( S^0_{U} \) be a stable univalent harmonic mapping on the unit disk \( U \). Then for each \( m, N \in \mathbb{N} \) we have

\[
|f(z^m)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(U)),
\]

if \( |z| \leq r_{m,N} \), where \( r_{m,N} \) is the unique root in \((0, 1)\) of the equation \( \psi_{m,N}(r) = 0 \), where

\[
\psi_{m,N}(r) = \frac{r^m}{(1 - r^m)^2} + \frac{r^N(N - Nr + r)}{(1 - r)^2} - \frac{1}{4}, \tag{2.10}
\]

and \( r_{m,N} \) cannot be improved. Here \( \lim_{N \to \infty} r_{m,N} = \left(3 - 2\sqrt{2}\right)^{1/m}, \lim_{m \to \infty} r_{m,N} = r_N \), where \( r_N \) is the solution of the equation

\[
4r^N(N - Nr + r) = (1 - r)^2,
\]

and \( \lim_{m \to \infty} r_{m,N} = 1 \). Moreover,

\[
|f(z^m)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(U)),
\]

if \( |z| \leq R_{m,N} \), where \( R_{m,N} \) is the unique root in \((0, 1)\) of the equation \( \kappa_{m,N}(r) = 0 \), where

\[
\kappa_{m,N}(r) = \frac{r^{2m}}{(1 - r^m)^4} + \frac{r^N(N - Nr + r)}{(1 - r)^2} - \frac{1}{4}, \tag{2.11}
\]

and \( R_{m,N} \) cannot be improved.

**Remark 2** We note that for \( N = 1 \), we have \( r_1 = 1/3 \) for the stable convex harmonic case and \( r_1 = 3 - 2\sqrt{2} \) for the stable univalent harmonic case, which is consistent with Bohr’s inequality obtained in Theorem 2.1.1.

**Proof** If \( f \) is stable convex, Theorems A and B give for \( |z| = r \),

\[
|a_n| + |b_n| \leq 1
\]

and

\[
|f(z^m)| \leq \frac{r^m}{1 - r^m}.
\]

It follows that

\[
|f(z^m)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq \frac{r^m}{1 - r^m} + \sum_{n=N}^{\infty} r^n = \frac{r^m}{1 - r^m} + \frac{r^N}{1 - r}.
\]
Hence, using Eq. (2.5), \(|f(z^n)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(U))\) for \(|z| \leq r_{m,N}\), where \(r_{m,N}\) is the unique root in \((0, 1)\) of the equation \(H_{m,N}(r) = 0\), where \(H_{m,N}(r)\) is defined in (2.8).

Similarly, \(|f(z^n)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq \frac{r_{m}^{2m}}{(1-r_m)^2} + \frac{r_{N}^{2}}{1-r} \leq d(f(0), \partial f(U))\) if \(|z| \leq R_{m,N}\), where \(R_{m,N}\) is the unique root in \((0, 1)\) of the equation \(K_{m,N}(r) = 0\), where \(K_{m,N}(r)\) is defined in (2.9).

We note that both functions \(H_{m,N}(r)\) and \(K_{m,N}(r)\) are continuous in \(r\) and their value at zero is negative (equals to \(-1\)) and at 1 is positive (equals to \(2\)) which guarantees the existence of roots.

The fact that this is the best possible radii is again guaranteed by a suitable rotation of the analytic function \(l(z) = \frac{z}{1-z}\).

Part (ii) follows in a similar fashion using part (ii) of Theorems A and B. Sharpness is obtained by using the analytic Koebe function \(k(z) = \frac{z}{(1-z)^2}\).

As an immediate corollary letting \(m = 1\) we get:

**Corollary 1**

(i) Let \(f = h + \overline{g}\) in \(S_{H}^{0}\) be a stable convex harmonic mapping on the unit disk \(U\). Then for each \(N \in \mathbb{N}\), we have

\[
|f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(U))
\]

if \(|z| \leq r_N\), where \(r_N\) is the unique root in \((0, 1)\) of the equation \(H_N(r) = 0\), where

\[
H_N(r) = 2r^N + 3r - 1,
\]

and \(r_N\) cannot be improved. Moreover,

\[
|f(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(U))
\]

if \(|z| \leq R_N\), where \(R_N\) is the unique root in \((0, 1)\) of the \(K_N(r) = 0\), where

\[
K_N(r) = 2r^N(1-r) + r^2 + 2r - 1,
\]

and \(R_N\) cannot be improved.

(ii) Let \(f = h + \overline{g}\) in \(S_{H}^{0}\) be a stable univalent harmonic mapping on the unit disk \(U\), then for each \(N \in \mathbb{N}\) we have

\[
|f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(U))
\]

if \(|z| \leq r_N\), where \(r_N\) is the unique root in \((0, 1)\) of the \(\psi_N(r) = 0\), where

\[
\psi_N(r) = r + r^N(1-Nr+r) - \frac{1}{4}(1-r)^2.
\]
Moreover,
\[ |f(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(U)) \]
if \(|z| \leq r_N\), where \(r_N\) is the unique root in \((0, 1)\) of the \(\kappa_N(r) = 0\), where
\[ \kappa_N(r) = \frac{r^2}{(1-r)^4} + \frac{r^N(N-Nr+r)}{(1-r)^2} - \frac{1}{4}. \]

**Remark 3** We note that in case (i) for \(N = 1\), we have \(r_1 = 1/5\) for the stable convex harmonic case and \(R_1 = 2 - \sqrt{3}\).

### 3 Bohr’s Phenomenon for Stable Logharmonic Mappings

In this section, we get Bohr’s radius in the case of stable logharmonic mappings. For \(f(z) = z h g\), such that \(f(0) = 0\) and \(h(0) = g(0) = 1\), we let
\[ h(z) = \exp \left( \sum_{n=1}^{\infty} a_n z^n \right) \quad \text{and} \quad g(z) = \exp \left( \sum_{n=1}^{\infty} b_n z^n \right). \]

#### 3.1 Bohr’s Inequality for Stable Logharmonic Mappings

We define the majorant series for \(f = z h g\) as in [14] to be
\[ B_f(r) = |z| \exp \left( \sum_{n=1}^{\infty} |a_n + e^{i\theta} b_n| |z|^n \right) \]
In [14], the authors proved that for \(f(z) = z h g\) starlike logharmonic mappings \(B_f(r) \leq d(0, \partial f(U))\) for \(|z| \leq r_0\), where \(r_0 \approx 0.09078\). We show here that the results improve under the stability assumption.

We will be needing the following distortion theorem and coefficient estimates proved in [4].

**Theorem D** [4] Let \(f = z h g\) be a stable univalent (or resp. stable starlike) logharmonic mapping on the unit disk \(U\), \(0 \notin hg(U)\). Then for all \(z \in U\), the following inequalities hold:
\[ \frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}. \]

Moreover, we have the following coefficient estimates :

**Theorem E** [4] Assume that \(f(z) = z h g\) is stable univalent logharmonic mapping. Then for all nonnegative integers \(n\), we have
\[ ||a_n| - |b_n|| \leq \max \{|a_n|, |b_n|\} \leq |a_n| + |b_n| \leq n. \]
We will next state and prove a version of Bohr’s inequality for stable logharmonic mappings.

**Theorem 3.1.1** Let $f = z h g$ be a stable univalent (or resp. stable starlike) logharmonic mapping on the unit disk $U$, $0 \notin h g(U)$. Then

(i) $B_f(r) < 1$ if $|z| \leq r_0 = 0.378$, where $r_0$ is the unique root in $(0, 1)$ of

$$r \exp \left( \frac{r}{(1-r)^2} \right) = 1.$$ 

(ii) $B_f(r) \leq d(0, \partial f(U))$ if $|z| \leq r_0 = 0.286$, where $r_0$ is the unique root in $(0, 1)$ of

$$r \exp \left( \frac{r}{(1-r)^2} \right) = \frac{1}{4}.$$ 

**Proof** For $|z| = r$, we have

$$B_f(r) \leq r \exp \left( \sum_{n=1}^{\infty} |a_n| + |b_n| \right) r^n \leq r \exp \left( \sum_{n=1}^{\infty} n |r^n| \right) = r \exp \left( \frac{r}{(1-r)^2} \right),$$

which proves part (i) of the theorem.

We next use Theorem D to establish bounds for $d(0, \partial f(U))$ in the case where $f$ is stable univalent or stable starlike.

$$d(0, \partial f(U)) = \lim_{|z| \to 1} \inf |f(z) - f(0)| = \lim_{|z| \to 1} \inf \frac{|f(z) - f(0)|}{|z|} \geq \lim_{|z| \to 1} \inf \frac{|z|}{(1 + |z|)^2} = \frac{1}{4}. \quad (3.1)$$

Moreover, $B_f(r) < \frac{1}{4} \leq d(0, \partial f(U))$ if $|z| \leq r_0 = 0.286$, where $r_0$ is the unique root in $(0, 1)$ of

$$r \exp \left( \frac{r}{(1-r)^2} \right) = \frac{1}{4}.$$

\[\square\]

### 3.2 Improved Bohr’s Inequality for Logharmonic Mappings

We next show an improved version of Bohr’s inequality for stable logharmonic mappings that follows the ideas proposed in [22, 29].

**Theorem 3.2.1** Let $f = z h g$ be a stable univalent (or resp. stable starlike) logharmonic mapping on the unit disk $U$, $0 \notin h g(U)$. Then

$$|z| \exp \left( \sum_{n=1}^{\infty} (|a_n + e^{it} b_n| + |a_n||b_n|)|z|^n \right) \leq d(0, \partial f(U)),$$
if $|z| \leq r_0 = 0.166$, where $r_0$ is the unique root in $(0, 1)$ of

$$r \exp \left( \frac{r(r - 3)}{2(r - 1)^3} \right) = \frac{1}{4}.$$  

Note that $r_0$ is the best possible radius.

**Proof** By Theorem E, we have that

$$|a_n + e^{it} b_n| + |a_n||b_n| \leq |a_n| + |b_n| + |a_n||b_n| \leq n + \frac{n^2}{2}.$$  

It follows that

$$|z| \exp \left( \sum_{n=1}^{\infty} (|a_n + e^{it} b_n| + |a_n||b_n|)|z|^n \right) \leq r \exp \left( \sum_{n=1}^{\infty} nr^n + \sum_{n=1}^{\infty} \frac{n^2}{2} r^n \right)$$

$$= r \exp \left( \frac{r}{(1 - r)^2} - \frac{r(r + 1)}{2(r - 1)^3} \right),$$

$$= r \exp \left( \frac{r(r - 3)}{2(r - 1)^3} \right).$$  

The rest will follow as in the proof of above theorems.  

\[\square\]

### 3.3 Refined Bohr's Inequality for Stable Logharmonic Mappings

We next state a refined version of Bohr’s inequality for stable logharmonic mappings. The proof is omitted as the idea is similar to the previous theorems in this section.

**Theorem 3.3.1** Let $f = zh\overline{g}$ be a stable univalent (or resp. stable starlike) logharmonic mapping on the unit disk $U$, $0 \notin hg(U)$. Then

$$|z| \exp \left( \sum_{n=1}^{\infty} (|a_n + e^{it} b_n||z|^n + |a_n + e^{it} b_n|^2|z|^{2n}) \right) \leq d(0, \partial f(U))$$

if $|z| \leq r_0 = 0.271$, where $r_0$ is the unique root in $(0, 1)$ of

$$r \exp \left( \frac{r}{(1 - r)^2} - \frac{r^4 + r^2}{(r^2 - 1)^3} \right) = \frac{1}{4}.$$  

\[\square\]
3.4 Bohr–Rogosinski’s Radius for Stable Logharmonic Mappings

Theorem 3.4.1 Let \( f = zhg \) be a stable univalent logharmonic mapping on the unit disk \( U \). Then for each \( m, N \in \mathbb{N} \) we have
\[
|f(z^m)| + |z| \exp \left( \sum_{n=N}^{\infty} (|a_n + e^{i\theta}b_n|)r^n \right) \leq d(f(0), \partial f(U))
\]
if \(|z| \leq r_{m,N} \), where \( r_{m,n} \) is the unique root in \((0, 1)\) of the \( \psi_{m,N}(r) = 0 \), where
\[
\psi_{m,N}(r) = \frac{r^m}{(1-r^m)^2} + r \exp \left( \frac{r^N(N-Nr+r)}{(1-r)^2} \right) - \frac{1}{4}. \tag{3.2}
\]
Here \( \lim_{N \to \infty} r_{m,N} = r_m \), where \( r = r_m \) solves the equation
\[
\frac{r^m}{(1-r^m)^2} + r = 1/4
\]
and \( \lim_{m \to \infty} r_{m,N} = r_N \), where \( r = r_N \) is the solution of the equation
\[
r \exp \left( \frac{r^N(N-Nr+r)}{(1-r)^2} \right) = \frac{1}{4},
\]
and \( \lim_{m,N \to \infty} r_{m,N} = \frac{1}{4} \).

Moreover,
\[
|f(z^m)|^2 + |z| \exp \left( \sum_{n=N}^{\infty} (|a_n + e^{i\theta}b_n|)r^n \right) \leq d(f(0), \partial f(U))
\]
if \(|z| \leq R_{m,N} \), where \( R_{m,n} \) is the unique root in \((0, 1)\) of the equation \( \kappa_{m,N}(r) \), where
\[
\kappa_{m,N}(r) = \frac{r^{2m}}{(1-r^m)^4} + r \exp \left( \frac{r^N(N-Nr+r)}{(1-r)^2} \right) - \frac{1}{4}. \tag{3.3}
\]
Here \( \lim_{N \to \infty} R_{m,N} = R_m \), where \( r = R_m \) solves the equation
\[
\frac{r^{2m}}{(1-r^m)^4} + r = 1/4
\]
and \( \lim_{m \to \infty} R_{m,N} = R_N \), where \( r = R_N \) is the solution of the equation
\[
r \exp \left( \frac{r^N(N-Nr+r)}{(1-r)^2} \right) = \frac{1}{4},
\]
and \( \lim_{m,N \to \infty} R_{m,N} = \frac{1}{4} \).
Proof We use Theorems D and E to get that

\[
|f(z^m)| + |z| \exp \left( \sum_{n=N}^{\infty} (|a_n + e^{it}b_n|)r^n \right) \\
\leq \frac{r^m}{(1 - r^m)^2} + r \exp \left( \frac{r^N(N - Nr + r)}{(1 - r)^2} \right) \\
\leq d(f(0), \partial f(U))
\]

if \( |z| \leq r_{m,N} \), where \( r_{m,n} \) is the unique root in \((0, 1)\) of the function defined in Eq. (3.2). Sharpness follows from a suitable rotation of the Koebe function \( k(z) = \frac{z}{(1 - z)^2} \). The second inequality follows in a similar fashion. \( \square \)

Remark 4 We note that for \( N = 1 \), we have for large \( m \), \( r_1 \approx 0.286 \) which is consistent with Theorem 3.1.1.

Taking \( m = 1 \) we get the following corollary

Corollary 2 Let \( f = zh \bar{g} \) be a stable univalent logharmonic mapping on the unit disk \( U \). Then for each \( N \in \mathbb{N} \) we have

\[
|f(z)| + |z| \exp \left( \sum_{n=N}^{\infty} (|a_n + e^{it}b_n|)r^n \right) \leq d(f(0), \partial f(U))
\]

if \( |z| \leq r_N \), where \( r_N \) is the unique root in \((0, 1)\) of the equation \( \psi_N(r) = 0 \), where

\[
\psi_N(r) = \frac{r}{(1 - r)^2} + r \exp \left( \frac{r^N(N - Nr + r)}{(1 - r)^2} \right) - \frac{1}{4}. \tag{3.4}
\]

Moreover,

\[
|f(z)|^2 + |z| \exp \left( \sum_{n=N}^{\infty} (|a_n + e^{it}b_n|)r^n \right) \leq d(f(0), \partial f(U))
\]

if \( |z| \leq R_N \), where \( R_N \) is the unique root in \((0, 1)\) of the \( \kappa_N(r) = 0 \), where

\[
\kappa_N(r) = \frac{r^2}{(1 - r)^4} + r \exp \left( \frac{r^N(N - Nr + r)}{(1 - r)^2} \right) - \frac{1}{4}. \tag{3.5}
\]

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Conflict of interest The authors declare that they have no conflict of interest.
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