The algebra of subspace collections and their association with rational functions of several variables

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Abstract
A natural connection between rational functions of several real or complex variables, and subspace collections is explored.

1 Introduction
Subspace collections have a rich algebraic structure, and a close connection with rational functions of several real or complex variables. Here we are interested in two types of subspace collections: finite dimensional vector spaces $\mathcal{K}$ (over the real or complex numbers) that have the decomposition

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,$$

(1.1)

which we call a $Y(n)$ subspace collection, and finite dimensional vector spaces $\mathcal{H}$ that have the decomposition

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,$$

(1.2)

which we call a $Z(n)$ subspace collection, where the $\mathcal{E}$ and $\mathcal{J}$ entering (1.2) are not to be confused with the subspaces $\mathcal{E}$ and $\mathcal{J}$ entering (1.1). Let us first suppose $\mathcal{V}$ and $\mathcal{U}$ are one dimensional. We will see that there are generally homogeneous (of degree 1) rational functions $Y(z_1, z_2, \ldots, z_n)$ and $Z(z_1, z_2, \ldots, z_n)$ (over the real or complex numbers) of degree 1 that are associated respectively with these $Y(n)$ and $Z(n)$ subspace collections, where $Z(z_1, z_2, \ldots, z_n)$ satisfies the additional constraint that $Z(1, 1, \ldots, 1) = 1$. Conversely, we will see that given any rational functions $Y(z_1, z_2, \ldots, z_n)$ and $Z(z_1, z_2, \ldots, z_n)$ with these properties, then there exists at least one subspace collection realizing these functions as its associated function. There are also operations on subspace collections that correspond to operations on the associated function, such as substitution. When $\mathcal{V}$ and $\mathcal{U}$ have dimension greater than 1, then $Y(z_1, z_2, \ldots, z_n)$ and $Z(z_1, z_2, \ldots, z_n)$ get
replaced by linear operator valued functions $Y(z_1, z_2, \ldots, z_n)$ and $Z(z_1, z_2, \ldots, z_n)$ which map $V$ to $V$ and $U$ to $U$ respectively.

The original motivation for studying subspace collections, and their associated functions, arose from the study of the effective conductivity tensor $Z$ of periodic composite materials. For a composite with $n$ isotropic phases, with scalar conductivities $z_1, z_2, \ldots, z_n$, the effective conductivity was found to be a homogeneous (of degree 1) analytic function $Z(z_1, z_2, \ldots, z_n)$ of the component conductivities with positive definite imaginary part when the component conductivities have positive imaginary part [1, 2, 3, 4] (see also chapter 18 of [5]). It was also recognized [6, 7] that the problem of determining the effective conductivity function could be formulated in terms of three mutually orthogonal spaces in the Hilbert space $H$ of square integrable functions: namely the space $U$ of constant fields, the space $E$ of periodic square integrable electric fields (having zero curl), and the space $J$ of square integrable current fields (having zero divergence), and if the composite had $n$ isotropic phases, with conductivities $z_1, z_2, \ldots, z_n$, then it was also natural to decompose $H$ into the direct sum of $n$ mutually orthogonal subspaces $P_1, P_2, \ldots, P_n$ where $P_i$ consists of those square integrable fields which are non-zero only within component $i$. This formulation, in terms of a $Z(n)$ subspace collection, evolved out of earlier Hilbert space formulations of the problem [8, 9, 10, 11, 12] and can easily be extended to the elastic, thermoelastic, piezoelectric, and poroelastic of multiphase and polycrystalline materials (see, for example, chapter 12 in [5]). The formulation has proved to be particularly important in the theory of exact relations of composite materials [13, 14, 15] (see also chapter 17 in [5]) where one seeks microstructure independent formulas satisfied by effective tensors. For two-dimensional polycrystals a correspondence was established between subspace collections and a representative class of (multiple rank laminate) polycrystal geometries [16].

Curiously the connection between $Z(n)$ subspace collections and the effective conductivity allowed the effective conductivity function $Z(z_1, z_2, \ldots, z_n)$ to be expanded as a new type of continued fraction, involving matrices of increasing dimension as one proceeds down the continued fraction when $n > 2$ [6, 17, 18] (see also chapters 20 and 29 in [5]). The coefficients in matrices entering the continued fraction can be expressed in terms of inner products between fields that enter the perturbation expansion of the solution field in a nearly homogeneous medium (with all the conductivities $z_1, z_2, \ldots, z_n$ being close to one another). One application of the continued fraction expansion has been to obtain bounds on the diagonal elements of the complex effective conductivity tensor of a three phase conducting composite, with complex conductivities $z_1, z_2$ and $z_3$, that were tighter than bounds obtained by any other method (see figure 4 in [17]). This procedure essentially extends to multivariate functions the procedure, using successive fractional linear transformations, that was used to obtain bounds [19] on the values in the complex plane that Stieltjes functions can take when a finite number of Taylor series coefficients are known (see also [4, 20] where essentially the same transformation is used to derive bounds on the complex dielectric constant of two component media using series expansion coefficients, as noted in the appendix in [21], and see [22] where the same set
of bounds is derived using a different procedure, namely the method of variation of poles and zeros.)

In the case \( n = 2 \) the continued fraction reduces to a usual continued fraction expansion, like those continued fractions associated with Padé approximations (see chapter 4 of Part I of \([23]\)). \( Y(n) \) subspace collections enter, for example, if one eliminates from the Hilbert space the constant fields and then reformulates the conductivity equations in terms of the remaining fields: the driving fields are then fields which are constant in each phase, but have zero average value (see chapter 19 in \([5]\) and references therein). The interrelationship between \( Z(n) \) subspace collections and \( Y(n) \) subspace collections is what gives rise to these novel continued fractions.

Finite dimensional \( Z(n) \) and \( Y(n) \) subspace collections also arise naturally in the study of the effective resistance of resistor networks constructed from \( n \) types of resistors having resistances \( z_1, z_2, \ldots, z_n \) (see chapter 20 in \([5]\)). This is not surprising as periodic resistor networks can be seen as discrete approximations to conducting composite materials (see, for example, \([3]\)).

Here we show that the connection between finite dimensional \( Z(n) \) and \( Y(n) \) subspace collections and homogeneous (of degree 1) operator valued rational functions \( Y(z_1, z_2, \ldots, z_n) \) and \( Z(z_1, z_2, \ldots, z_n) \) persists even when the subspaces in each decomposition are not necessarily mutually orthogonal, and indeed even in the absence of an inner product (on the space \( H \) or \( K \)). The results developed in \([6, 17, 18]\) and in chapters 20 and 29 of \([5]\) are extended to the case where there is no inner product. Accordingly some steps in the analysis, and some assumptions, need to be revised. In this more general setting we can generate, from an appropriate \( Z(n) \) subspace collection, any desired scalar valued rational function \( Z(z_1, z_2, \ldots, z_n) \) satisfying the homogeneity property \( Z(1, 1, \ldots, 1) = 1. \)

Our ultimate hope in developing this connection is that the geometrical structure of subspace collections will be reflected in the algebraic geometrical structure of their associated rational functions. If this is the case, understanding the topological features of subspace collections might shed light on the geometrical features of algebraic varieties. While this paper does not directly address this issue, it sheds the first light on the relation between finite dimensional subspace collections and rational functions of several complex variables, in the case where the subspaces are not mutually orthogonal.

## 2 Subspace collections and their associated functions

Let \( K \) be a vector space which has a decomposition into two different direct sums of subspaces

\[
K = E \oplus J = V \oplus H,
\]

where \( H \) itself is a direct sum of \( n \) subspaces

\[
H = P_1 \oplus P_2 \oplus \cdots \oplus P_n.
\]
Any vector $K \in K$ has a unique decomposition into component vectors,

$$K = E + J = v + H = P_1 + P_2 + \cdots + P_n,$$

(2.3)

each in the associated subspaces:

$$E \in \mathcal{E}, \quad J \in \mathcal{J}, \quad v \in \mathcal{V}, \quad H \in \mathcal{H}, \quad P_i \in \mathcal{P}_i \text{ for } i = 1, 2, \ldots, n.$$  

(2.4)

This decomposition serves to define projection operators $\Gamma_1$ and $\Gamma_2$ onto $\mathcal{E}$ and $\mathcal{J}$, projection operators $\Pi_1$ and $\Pi_2$ onto $\mathcal{V}$ and $\mathcal{H}$, and projection operators $\Lambda_i$ onto the subspaces $\mathcal{P}_i$. By definition we have

$$E = \Gamma_1 K, \quad J = \Gamma_2 K, \quad v = \Pi_1 K, \quad H = \Pi_2 K, \quad P_i = \Lambda_i K.$$  

(2.5)

Associated with this subspace collection is a linear operator valued function $Y(z_1, z_2, \ldots, z_n)$ acting on the space $\mathcal{V}$, which is a homogeneous function of degree 1 of the $n$ complex variables $z_1, z_2, \ldots, z_n$. To obtain the function we look for vectors $J$ and $E$ that solve the equations

$$E \in \mathcal{E}, \quad J \in \mathcal{J}, \quad J_2 = LE_2, \quad \text{where } J_2 = \Pi_2 J, \quad E_2 = \Pi_2 E,$$

(2.6)

where

$$L = \sum_{i=1}^{n} z_i \Lambda_i.$$  

(2.7)

The associated operator $Y$, by definition, governs the linear relation

$$J_1 = -YE_1, \quad \text{where } J_1 = \Pi_1 J \text{ and } E_1 = \Pi_1 E.$$  

(2.8)

A necessary condition for $J_1$ to be uniquely defined given $E_1$ is that

$$\mathcal{V} \cap \mathcal{J} = 0,$$

(2.9)

since if $J$ and $E$ solve (2.6) so too will $J + v$ and $E$, for any $v \in \mathcal{V} \cap \mathcal{J}$. Similarly, a necessary condition for $E_1$ to be uniquely defined given $J_1$ is that

$$\mathcal{V} \cap \mathcal{E} = 0.$$  

(2.10)

If $v_1, v_2, \ldots, v_m$ is a basis of $\mathcal{V}$, then the operator $Y$ can be represented by a matrix, also denoted by $Y$ with elements $Y_{ik}$ such that

$$Yv_k = \sum_{i=1}^{m} Y_{ik} v_i.$$  

(2.11)

Another association between subspace collections and functions comes if a vector space $\mathcal{H}$ has the decomposition

$$\mathcal{H} = U \oplus \mathcal{E} \oplus \mathcal{J} = P_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,$$

(2.12)
where $\mathcal{E}$ and $\mathcal{J}$ are not to be confused with the spaces in (2.1). Any vector $H \in \mathcal{H}$ has a unique decomposition into component vectors,

$$H = u + E + J = P_1 + P_2 + \cdots + P_n,$$

(2.13)

each in the associated subspaces:

$$u \in \mathcal{U}, \quad E \in \mathcal{E}, \quad J \in \mathcal{J}, \quad P_i \in \mathcal{P}_i \text{ for } i = 1, 2, \ldots, n.$$ 

(2.14)

This decomposition serves to define projection operators $\Gamma_0, \Gamma_1$ and $\Gamma_2$ onto $\mathcal{U}, \mathcal{E}$ and $\mathcal{J}$, and projection operators $\Lambda_i$ onto the subspaces $\mathcal{P}_i$. Associated with this subspace collection is a linear operator valued function $Z(z_1, z_2, \ldots, z_n)$ acting on the space $\mathcal{U}$, which is a homogeneous function of degree 1 of the $n$ complex variables $z_1, z_2, \ldots, z_n$. To obtain the function we look for vectors $e, j, J$ and $E$ that solve the equations

$$e, j \in \mathcal{U}, \quad E \in \mathcal{E}, \quad J \in \mathcal{J}, \quad j + J = L(e + E), \quad \text{where } L = \sum_{i=1}^{n} z_i \Lambda_i.$$ 

(2.15)

The associated operator $Z$, by definition, governs the linear relation

$$j = Ze.$$ 

(2.16)

If $u_1, u_2, \ldots, u_m$ is a basis of $\mathcal{U}$, then the operator $Z$ can be represented by a matrix, also denoted by $Z$ with elements $Z_{ik}$ such that

$$Zu_k = \sum_{i=1}^{m} Z_{ik} u_i.$$ 

(2.17)

When $z_1 = z_2 = \cdots = z_n = 1$ (2.15) has the trivial solution

$$j = e, \quad J = E = 0,$$

(2.18)

and so we deduce that

$$Z(1, 1, \ldots, 1) = I.$$ 

(2.19)

3 Some simple examples

Consider a $Y(n)$ subspace collection

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,$$

(3.1)

where $\mathcal{E}, \mathcal{V}, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are all one dimensional, and $\mathcal{J}$ is $n$ dimensional. Choose, as our basis for $\mathcal{K}$, $n + 1$ vectors $p_0 \in \mathcal{V}$, and $p_i \in \mathcal{P}_i, i = 1, 2, \ldots n$. Vectors $E \in \mathcal{E}$ and $J \in \mathcal{J}$ can be expanded in this basis:

$$E = \sum_{i=0}^{n} E_i p_i, \quad J = \sum_{i=0}^{n} J_i p_i.$$ 

(3.2)
The relation $\Pi_2 J = \Pi_2 E$ implies

$$ J_i = z_i E_1. \quad (3.3) $$

Let us suppose that $E_0 = 1$. Then $E_1$ and $E_2$ are determined by the orientation of the one-dimensional subspace $E$ with respect to the subspaces $V, P_1, P_2, \ldots P_n$. Also since $J$ has codimension 1, there exist constants $W_0, W_1, W_2, \ldots W_n$ determined by the orientation of the $n$ dimensional subspace $J$ with respect to the subspaces $V, P_1, P_2, \ldots P_n$ such that

$$ \sum_{i=0}^{n} W_i J_i = 0. \quad (3.4) $$

Let us suppose that $W_0 = 1$. Then we have

$$ J_0 = - \sum_{i=1}^{n} W_i J_i = - \sum_{i=1}^{n} W_i E_i z_i, \quad (3.5) $$

which since $E_0 = 1$ implies $J_0 = -Y^* E_0$, with

$$ Y = \sum_{i=1}^{n} \alpha_i z_i \quad \text{where} \quad \alpha_i = W_i E_i. \quad (3.6) $$

As the $E_i$ and $W_i$ are arbitrary constants, we see that $Y$ can be any linear combination of the $z_i$. In particular, with $W_1 E_1 = 1$ and $W_i E_i = 0$ when $i \neq 1$ we obtain

$$ Y = z_1. \quad (3.7) $$

As a second example consider a $Z(2)$ subspace collection

$$ \mathcal{H} = U \oplus E \oplus J = P_1 \oplus P_2, \quad (3.8) $$

where the subspaces $U, E, J$ and $P_2$ are all one dimensional, while $P_1$ is two dimensional. Choose, as our basis for $\mathcal{H}$, 3 vectors $U_0 \in U$, $E_0 \in E$ and $J_0 \in J$, and take a vector $P$ as a basis for $P_2$. The coefficients $P_U, P_E$ and $P_J$ in the expansion

$$ P = P_U U_0 + P_E E_0 + P_J J_0 \quad (3.9) $$

determine the orientation of $P_2$ with respect to the subspaces $U, E$ and $J$. In the basis $U_0, E_0, and J_0$ the equations

$$ e + E = Q + \alpha P, \quad j + J = z_1 Q + z_2 \alpha P, \quad (3.10) $$

with

$$ e, j \in U, \quad E \in E, \quad J \in J, \quad Q \in P_1, \quad (3.11) $$
take the form

$$\begin{pmatrix} e \\ E \\ 0 \\ j \\ 0 \\ J \end{pmatrix} = \begin{pmatrix} Q_U \\ Q_E \\ Q_J \\ Q_U \\ Q_E \\ Q_J \end{pmatrix} + \begin{pmatrix} W_U \\ W_E \\ W_J \\ W_U \\ W_E \\ W_J \end{pmatrix}, \quad (3.12)$$

and since $\mathbf{Q} \in \mathcal{P}_1$ there exist constants $W_U$, $W_E$ and $W_J$, which determine the orientation of $\mathcal{P}_1$ with respect to $\mathcal{U}$, $\mathcal{E}$ and $\mathcal{J}$, such that

$$W_UQ_U + W_EQ_E + W_JQ_J = 0. \quad (3.13)$$

Hence we obtain the equations

$$W_Ue + W_EQ_E = \alpha(W_UP_U + W_EP_E + W_JP_J) \equiv \alpha \mathbf{W} \cdot \mathbf{P},$$

$$0 = z_1(E - \alpha P_E) + z_2\alpha P_E,$$

$$j = z_1(e - \alpha P_U) + z_2\alpha P_U. \quad (3.14)$$

Eliminating $E$ and $\alpha$ from these equations gives $j = Ze$, with

$$Z = z_1 + \frac{(z_2 - z_1)W_UP_U}{\mathbf{W} \cdot \mathbf{P} + W_EP_E(z_2 - z_1)/z_1}. \quad (3.15)$$

In particular if the subspaces are oriented so that

$$\mathbf{W} \cdot \mathbf{P} = W_EQ_E = -W_UP_U, \quad (3.16)$$

then (3.15) gives

$$Z = z_1^2/z_2, \quad (3.17)$$

which with $z_2 = 1$ produces the function $z_1^2$ and with $z_1 = 1$ produces the function $1/z_2$. Also, with $W_EP_E = 0$ we obtain

$$Z = z_1 + \frac{(z_2 - z_1)W_UP_U}{\mathbf{W} \cdot \mathbf{P}}. \quad (3.18)$$

### 4 Formulas for the associated functions

Following section 12.8 of [5] a formula for the effective tensor $\mathbf{Z}$ results by applying the operator $\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2$ (which projects on the space $\mathcal{U} \oplus \mathcal{J}$) to both sides of the constitutive law $\mathbf{e} + \mathbf{E} = \mathbf{L}^{-1}(\mathbf{j} + \mathbf{J})$. Solving the resulting equation,

$$\mathbf{e} = (\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)\mathbf{L}^{-1}(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)(\mathbf{j} + \mathbf{J}), \quad (4.1)$$
for \( j + J \) gives
\[
j + J = [(\Gamma_0 + \Gamma_2)L^{-1}(\Gamma_0 + \Gamma_2)]^{-1}e,
\]
where the last inverse is to be taken on the subspace \( \mathcal{U} \oplus \mathcal{J} \). By applying \( \Gamma_0 \) to both sides of this equation we see that
\[
Z = \Gamma_0[(\Gamma_0 + \Gamma_2)L^{-1}(\Gamma_0 + \Gamma_2)]^{-1}\Gamma_0,
\]
which is the result given in (12.59) of [5].

Another formula for \( Z \) follows from noting that for any arbitrary constant \( z_0 \neq 0 \),
\[
[z_0I - \Gamma_1(L - z_0I)](e + E) = z_0e + z_0E - \Gamma_1J - z_0\Gamma_1E = z_0e.
\]
Solving this for \( e + E \) gives
\[
e + E = z_0[z_0I - \Gamma_1(L - z_0I)]^{-1}e,
\]
and applying \( \Gamma_0L \) to both sides yields
\[
j = z_0\Gamma_0L[z_0I - \Gamma_1(L - z_0I)]^{-1}e.
\]
Thus we have
\[
Z = z_0\Gamma_0L[z_0I - \Gamma_1(L - z_0I)]^{-1}\Gamma_0 = z_0\Gamma_0 + z_0\Gamma_0(L - z_0I)[z_0I - \Gamma_1(L - z_0I)]^{-1}\Gamma_0,
\]
where we have used the identity
\[
\Gamma_0 = z_0\Gamma_0[z_0I - \Gamma_1(L - z_0I)]^{-1}\Gamma_0,
\]
obtained by applying \( \Gamma_0 \) to both sides of (4.5). This formula (4.7) is a special case of the formula (12.60) given in [5], and is well known in different contexts [24].

To obtain a formula for \( Y \) notice that (2.6) and (2.8) imply that
\[
0 = \Gamma_2E' = \Gamma_2E_1 + \Gamma_2E_2 = \Gamma_2E_1 + \Gamma_2L^{-1}\Pi_2\Gamma_2J',
\]
where the inverse of \( L \) is to be taken on the subspace \( \mathcal{H} \). Solving for \( J' \) gives
\[
J' = - (\Gamma_2L^{-1}\Pi_2\Gamma_2)^{-1}\Gamma_2E_1,
\]
where the inverse is to be taken on the subspace \( \mathcal{J} \). Then by applying \( \Pi_1 \) to both sides of this equation and equating \( \Pi_1J' = J_1 \) with \(-YE_1\) we obtain the desired formula
\[
Y = \Pi_1\Gamma_2(\Gamma_2L^{-1}\Pi_2\Gamma_2)^{-1}\Gamma_2\Pi_1,
\]
for \( Y \), as given in formula (19.29) of [5].
5 Pruning the subspace collections

When \( L \) is close to \( z_0I \) we can expand the inverses in (4.5) and (4.7) to obtain the series expansions

\[
E = \sum_{j=1}^{\infty} \left[ (L - z_0I)/z_0 \right]^j e,
\]

\[
Z = z_0 \Gamma_0 + \sum_{j=1}^{\infty} \Gamma_0 (L - z_0I) \left[ (L - z_0I)/z_0 \right]^j \Gamma_0.
\]

(5.1)

From these expansions it is evident that is only those fields in \( H \) that arise from products of the operators \( \Gamma_1, \Lambda_1, \Lambda_2, \ldots, \Lambda_n \) applied to fields in \( U \) have any role in determining \( E \) and the associated function \( Z(z_1, z_2, \ldots, z_n) \) (also \( j \) and \( J \)): so we may as well prune away any other fields from the Hilbert Space \( H \). Thus we can redefine \( H \) as the smallest subspace containing \( U \) that is closed under the action of \( \Gamma_1, \Lambda_1, \Lambda_2, \ldots, \Lambda_n \) and redefine

\[
\mathcal{E} = \Gamma_1 H, \quad \mathcal{J} = \Gamma_2 H, \quad \mathcal{P}_j = \Lambda_j H, \quad j = 1, 2, \ldots, n.
\]

(5.2)

This imposes constraints on the dimensions of these subspaces, as noted in section 29.2 of [5] where the results are given in the case where \( U \) has dimension 1 and where the spaces are orthogonal. Let \( p_j \) be the dimension of \( \mathcal{P}_j \), \( j = 1, 2, \ldots, n \), and let \( m, q_1 \) and \( q_2 \) represent the dimensions of \( U, \mathcal{E} \) and \( \mathcal{J} \). The total dimension of the Hilbert space \( H \) is therefore

\[
h = m + q_1 + q_2 = p_1 + p_2 + \ldots + p_n.
\]

(5.3)

Now the space

\[
[\Lambda_1(U \oplus \mathcal{E})] \oplus [\Lambda_2(U \oplus \mathcal{E})] \oplus \ldots \oplus [\Lambda_n(U \oplus \mathcal{E})]
\]

(5.4)

certainly contains \( U \), and is closed under \( \Gamma_1 \) (because it contains \( \mathcal{E} \)) and is closed under \( \Lambda_j \) for each \( j \). It therefore must be \( H \) and \( \Lambda_j(U \oplus \mathcal{E}) \) which has at most dimension \( m + q_1 \) must be \( \mathcal{P}_j \). Therefore for each \( j \) we have the inequality

\[
p_j \leq m + q_1,
\]

(5.5)

and by summing these over \( j \) we see that

\[
q_2 \leq (n - 1)(m + q_1).
\]

(5.6)

Similarly the subspace

\[
[\Lambda_1(U \oplus \mathcal{J})] \oplus [\Lambda_2(U \oplus \mathcal{J})] \oplus \ldots \oplus [\Lambda_n(U \oplus \mathcal{J})]
\]

(5.7)

can also be identified with \( H \) and we obtain the inequalities

\[
p_j \leq m + q_2, \quad q_1 \leq (n - 1)(m + q_2).
\]

(5.8)
In the particular case when \( n = 2 \) the constraints (5.6) and (5.8) imply that the dimensions of the subspaces \( \mathcal{E} \) and \( \mathcal{J} \) can differ by at most \( m \). Also in the case \( n = 2 \) we have
\[
p_1 = (m + q_2 - p_2) + q_1 = (m + q_1 - p_2) + q_2 \geq \max\{q_1, q_2\}, \tag{5.9}
\]
and similarly for \( p_2 \).

Likewise we can redefine \( \mathcal{K} \) as the smallest subspace containing \( \mathcal{V} \) that is closed under the action of \( \Gamma_1, \Lambda_1, \Lambda_2, \ldots, \Lambda_n \) and redefine
\[
\mathcal{E} = \Gamma_1 \mathcal{K}, \quad \mathcal{J} = \Gamma_2 \mathcal{K}, \quad \mathcal{P}_j = \Lambda_j \mathcal{K}, \quad j = 1, 2, \ldots, n. \tag{5.10}
\]
Let \( v \) be the dimension of \( \mathcal{V} \), \( p_j \) be the dimension of \( \mathcal{P}_j \), \( j = 1, 2, \ldots, n \), and let \( q_1 \) and \( q_2 \) represent the dimensions of \( \mathcal{E} \) and \( \mathcal{J} \). The total dimension of the Hilbert space \( \mathcal{K} \) is therefore
\[
h = q_1 + q_2 = v + p_1 + p_2 + \ldots + p_n. \tag{5.11}
\]
The space
\[
\mathcal{V} \oplus [\Lambda_1(\mathcal{E})] \oplus [\Lambda_2(\mathcal{E})] \oplus \ldots \oplus [\Lambda_n(\mathcal{E})] \tag{5.12}
\]
certainly contains \( \mathcal{V} \), and is closed under \( \Gamma_1 \) (because it contains \( \mathcal{E} \)) and is closed under \( \Lambda_j \) for each \( j \). It therefore must be \( \mathcal{K} \) and \( \Lambda_j(\mathcal{E}) \) which has at most dimension \( q_1 \) must be \( \mathcal{P}_j \). Thus for each \( j \) we have the inequality
\[
p_j \leq q_1, \tag{5.13}
\]
and summing these over \( j \) we obtain
\[
q_2 \leq v + (n - 1)q_1. \tag{5.14}
\]
Similarly since
\[
\mathcal{K} = \mathcal{V} \oplus [\Lambda_1(\mathcal{J})] \oplus [\Lambda_2(\mathcal{J})] \oplus \ldots \oplus [\Lambda_n(\mathcal{J})], \tag{5.15}
\]
we obtain the inequalities
\[
p_j \leq q_2, \quad q_1 \leq v + (n - 1)q_2. \tag{5.16}
\]
When \( n = 2 \) the constraints (5.14) and (5.16) imply that the dimensions of the subspaces \( \mathcal{E} \) and \( \mathcal{J} \) can differ by at most \( v \). Also in the case \( n = 2 \) we have
\[
p_1 = (q_2 - p_2) + q_1 - v = (q_1 - p_2) + q_2 - v \geq \max\{q_1, q_2\} - v, \tag{5.17}
\]
with a similar inequality for \( p_2 \).
6 Extension operations on subspace collections

Let us suppose we have a $Z(n)$ subspace collection
\[ H = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \tag{6.1} \]
and a basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ for $\mathcal{U}$ with respect to which the matrix valued associated function $Z(z_1, z_2, \ldots, z_n)$ is defined. Let us take a new $m$-dimensional space $\mathcal{V}$ with basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and consider the space
\[ \mathcal{K} = \mathcal{V} \oplus \mathcal{H}. \tag{6.2} \]

Define the subspace $\tilde{\mathcal{E}}$ to consist of all vectors spanned by $\mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2, \ldots, \mathbf{u}_m + \mathbf{v}_m$ and the subspace $\tilde{\mathcal{J}}$ to consist of all vectors spanned by $\mathbf{u}_1 - \mathbf{v}_1, \mathbf{u}_2 - \mathbf{v}_2, \ldots, \mathbf{u}_m - \mathbf{v}_m$. Clearly we have
\[ \mathcal{V} \oplus \mathcal{U} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{J}}, \tag{6.3} \]
and consequently we obtain the $Y(n)$ subspace collection
\[ \mathcal{K} = \mathcal{E}' \oplus \mathcal{J}' = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \tag{6.4} \]
in which
\[ \mathcal{E}' = \tilde{\mathcal{E}} \oplus \mathcal{E}, \quad \mathcal{J}' = \tilde{\mathcal{J}} \oplus \mathcal{J}. \tag{6.5} \]

Furthermore given vectors satisfying
\[ j + J = L(\mathbf{u}_i + \mathbf{E}), \quad \mathbf{E} \in \mathcal{E}, \quad J \in \mathcal{J}, \tag{6.6} \]
where
\[ j = \sum_{k=1}^m Z_{ki} \mathbf{u}_k, \quad L = \sum_{\ell=1}^n z_{\ell} \Lambda_{\ell}, \tag{6.7} \]
we can set
\[ \mathbf{E}_2 = \mathbf{u}_i + \mathbf{E} \in \mathcal{H}, \quad \mathbf{E}_1 = \mathbf{v}_i \in \mathcal{V}, \quad J_2 = \sum_{k=1}^m Z_{ki} \mathbf{u}_k + J \in \mathcal{H}, \quad J_1 = - \sum_{k=1}^m Z_{ki} \mathbf{v}_k \in \mathcal{V}. \tag{6.8} \]

Then we have
\[ \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{v}_i + \mathbf{u}_i + \mathbf{E} \in \mathcal{E}', \quad J_1 + J_2 = \sum_{k=1}^m Z_{ki} (\mathbf{u}_k - \mathbf{v}_k) + J \in \mathcal{J}', \tag{6.9} \]
and (6.6) implies $J_2 = LE_2$. Moreover $J_1 = -YE_1$ where, in the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, $Y$ is represented by the matrix $Z$. Thus with an appropriate choice of bases the functions $Z(z_1, z_2, \ldots, z_n)$ and $Y(z_1, z_2, \ldots, z_n)$ are identical. We call the subspace collection (6.4) the extension of the subspace collection (6.1).
7 Changing the reference point and operations on subspace collections which leave the associated function invariant

Given a homogeneous rational function \( Y(z_1, z_2, \ldots, z_n) \) of degree one, an operation which preserves the homogeneity is obviously to multiply the variables by constants to obtain the function

\[
Y'(z'_1, z'_2, \ldots, z'_n) = Y(d_1z'_1, d_2z'_2, \ldots, d_nz'_n).
\]

(7.1)

The associated operation on the \( Y(n) \)-subspace collection \((E, J)\) and \((V, P_1, P_2, \ldots, P_n)\) is found by generalizing the analysis given after (29.3) in [5] and is as follows. Given non-zero constants \( c^E_i \) and \( c^J_i \), \( i = 1, \ldots, n \) we introduce the linear transformations

\[
\psi^E(P) = \Pi_1P + \sum_{i=1}^n c^E_i \Lambda_i P, \quad \psi^J(P) = \Pi_1P + \sum_{i=1}^n c^J_i \Lambda_i P, \quad (7.2)
\]

on fields \( P \in K \), where \( \Lambda_1 \) is the projection onto \( P_1 \). These transformations leave the subspaces \( V \) and \( P_i \) invariant. Define the spaces

\[
E' = \psi^E(E) \quad \text{and} \quad J' = \psi^J(J). \quad (7.3)
\]

Let \((E', J')\) and \((V, P_1, P_2, \ldots, P_n)\) be our new subspace collection. Given a solution to the equations

\[
E \in E, \quad J \in J, \quad (I - \Pi_1)J = \sum_{i=1}^n z_i \Lambda_i E,
\]

(7.4)
in the original subspace collection, in which \( \Pi_1 \) is the projection onto \( V \), the fields \( E' = \psi^E(E) \) and \( J' = \psi^J(J) \) will be a solution to the equations

\[
E' \in E', \quad J' \in J', \quad (I - \Pi_1)J' = \sum_{i=1}^n z'_i \Lambda_i E',
\]

(7.5)
in the new subspace collection with

\[
z'_i = z_i c^J_i / c^E_i. \quad (7.6)
\]

Since \( \Pi_1E' = \Pi_1E \) and \( \Pi_1J' = \Pi_1J \), it follows that \( Y \)-tensor functions of the two subspace collections are related by (7.1) where

\[
d_i = c^E_i / c^J_i. \quad (7.7)
\]

Note that if we chose \( c^J_i = c^E_i \) for all \( i \) then the associated function remains invariant.

More generally if we are interested in leaving the associated function invariant, we could take

\[
E' = C E, \quad J' = C J, \quad V' = C V, \quad H' = C H, \quad P'_i = C P_i, \quad (7.8)
\]

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where \( C \) is a non-singular linear operator which maps \( K \) to itself. Then the fields \( E' = CE \) and \( J' = CJ \) will be a solution to the equations

\[
E' \in \mathcal{E}', \quad J' \in \mathcal{J}', \quad (I - \Pi'_1)J' = \sum_{i=1}^{n} z_i \Lambda'_i E', \quad (7.9)
\]

where

\[
\Pi'_1 = C \Pi C^{-1}, \quad \Lambda'_i = C \Lambda C^{-1} \quad (7.10)
\]

are the projections onto \( \mathcal{V}' \) and \( \mathcal{P}'_i \). If \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \) is a basis for \( \mathcal{V} \) then setting \( \mathbf{v}'_i = C \mathbf{v}_i \) we can take \( \mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_m \) as a basis for \( \mathcal{V}' \). Since multiplying by \( C \) is a linear operation the coefficients in the expansions

\[
\Pi'_1 E' = \sum_{i=1}^{m} E'_i \mathbf{u}'_i, \quad \Pi_1 E = \sum_{i=1}^{m} E_i \mathbf{u}_i, \quad \Pi'_1 J' = \sum_{i=1}^{m} J'_i \mathbf{u}'_i, \quad \Pi_1 J = \sum_{i=1}^{m} J_i \mathbf{u}_i \quad (7.11)
\]

can be equated:

\[
E'_i = E_i, \quad J'_i = J_i, \quad (7.12)
\]

and as a consequence the same matrix \( \mathbf{Y} \) whose coefficients govern the relation

\[
J_i = \sum_{k=1}^{k} Y_{ik} E_k, \quad (7.13)
\]

also govern the relation

\[
J'_i = \sum_{k=1}^{k} Y_{ik} E'_k. \quad (7.14)
\]

There is a similar invariance of matrix functions associated with \( Z(n) \) subspace collections under the linear transformations,

\[
\mathcal{U}' = C \mathcal{U}, \quad \mathcal{E}' = C \mathcal{E}, \quad \mathcal{J}' = C \mathcal{J}, \quad \mathcal{P}'_i = C \mathcal{P}_i. \quad (7.15)
\]

8 Operations on subspace collections

Another familiar operation that we can do with rational functions is to make substitutions. Thus if \( \mathbf{Y}(z_1, z_2, \ldots, z_n) \) is a \( m \times m \) matrix-valued homogeneous function of degree one and \( Z'(z'_1, z'_2, \ldots, z'_p) \) is a scalar-valued function, say normalized with

\[
Z'(1, 1, \ldots, 1) = 1, \quad (8.1)
\]

then

\[
\mathbf{Y}''(z'_1, z'_2, \ldots, z'_p, z_2, \ldots, z_n) = \mathbf{Y}(Z(z'_1, z'_2, \ldots, z'_p), z_2, \ldots, z_n) \quad (8.2)
\]
will be another \(m \times m\) matrix-valued homogeneous function of degree one. What is the analogous operation on subspace collections? It is natural to expect there should be one, just as in a network of \(n\) types of resistors one can replace each resistor of type 1 with a network of \(p\) other resistors.

Extending the treatment given in section 29.1 of [5] let us suppose that we are given a \(Y(n)\)-subspace collection

\[
\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,
\]

and a \((3, p)\)-subspace collection

\[
\mathcal{H}' = \mathcal{U}' \oplus \mathcal{E}' \oplus \mathcal{J}' = \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_p,
\]

in which \(\mathcal{V}\) is \(m\)-dimensional and \(\mathcal{U}'\) is one-dimensional. Let \(Y(z_1, z_2, \ldots, z_n)\) and \(Z'(z'_1, z'_2, \ldots, z'_p)\) denote the functions associated with these subspace collections. We take as our new \((2, n + p)\)-subspace collection,

\[
\mathcal{K}'' = \mathcal{E}'' \oplus \mathcal{J}'' = \mathcal{V}'' \oplus \mathcal{P}''_1 \oplus \mathcal{P}''_2 \oplus \cdots \oplus \mathcal{P}''_n,
\]

where

\[
\mathcal{E}'' = (\mathcal{E} \otimes \mathcal{U}') \oplus (\mathcal{P}_1 \otimes \mathcal{E}'), \quad \mathcal{J}'' = (\mathcal{J} \otimes \mathcal{U}') \oplus (\mathcal{P}_1 \otimes \mathcal{J}'),
\]

and

\[
\mathcal{V}'' = \mathcal{V} \otimes \mathcal{U}',
\]

\[
\mathcal{P}''_i = \mathcal{P}_1 \otimes \mathcal{P}'_i \quad \text{for} \quad 1 \leq i \leq p,
\]

\[
= \mathcal{P}_{i+1-p} \otimes \mathcal{U}' \quad \text{for} \quad p + 1 \leq i \leq n + p - 1.
\]

in which \(\otimes\) denotes the operation of taking the tensor product of two subspaces. Vectors in the space

\[
\mathcal{K}'' = \mathcal{E}'' \oplus \mathcal{J}'' = (\mathcal{K} \otimes \mathcal{U}') \oplus (\mathcal{P}_1 \otimes (\mathcal{E}' \oplus \mathcal{J}'))
\]

spanned by these subspaces are represented as a pair \([\mathbf{P}, \mathbf{u}']\) added to a linear combination of pairs of the form \([\mathbf{P}_1, \mathbf{P}']\), where \(\mathbf{P} \in \mathcal{K}, \mathbf{u}' \in \mathcal{U}', \mathbf{P}_1 \in \mathcal{P}_1,\) and \(\mathbf{P}' \in \mathcal{E}' \oplus \mathcal{J}'\).

Now define

\[
\mathcal{H} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,
\]

\[
\mathcal{H}'' = \mathcal{P}''_1 \oplus \mathcal{P}''_2 \oplus \cdots \oplus \mathcal{P}''_n,
\]

and suppose that we are given solutions to the equations

\[
\mathbf{J}_2 = \sum_{i=1}^{n} z_i \mathbf{A}_i \mathbf{E}_2 \quad \text{with} \quad \mathbf{E}_1 + \mathbf{E}_2 \in \mathcal{E}, \quad \mathbf{J}_1 + \mathbf{J}_2 \in \mathcal{J}, \quad \mathbf{E}_1, \mathbf{J}_1 \in \mathcal{V}, \quad \mathbf{E}_2, \mathbf{J}_2 \in \mathcal{H},
\]

\[
\mathbf{j}' + \mathbf{J}' = \sum_{j=1}^{n} z'_j \mathbf{A}'_j (\mathbf{e}' + \mathbf{E}') \quad \text{with} \quad \mathbf{e}', \mathbf{j}' \in \mathcal{U}', \quad \mathbf{E}' \in \mathcal{E}', \quad \mathbf{J}' \in \mathcal{J}',
\]

\((8.10)\)
where
\[ z_1 = Z(z'_1, z'_2, \ldots, z'_p), \quad (8.11) \]
and \( \Lambda_i \) and \( \Lambda'_j \) are the projections onto \( P_i \) and \( P'_j \). Let us introduce
\[ P_i = \Lambda_i E_2, \quad P'_j = \Lambda'_j (e' + E'), \quad (8.12) \]
and set
\[ z''_i = z'_i \quad \text{for} \quad 1 \leq i \leq p, \]
\[ = z_{i+p} \quad \text{for} \quad p + 1 \leq i \leq n + p - 1. \quad (8.13) \]

Then, in the new subspace collection, the vectors
\[
E''_1 = [E_1, \ e'] \in V'', \quad E''_2 = [E_2, \ e'] + [P_1, \ E'],
\]
\[
J''_1 = [J_1, \ e'] \in V'', \quad J''_2 = [J_2, \ e'] + [P_1, \ J']
\]
satisfy
\[ E''_1 + E''_2 \in \mathcal{E}'', \quad J''_1 + J''_2 \in \mathcal{J}''. \quad (8.15) \]

Additionally, we have
\[ E'_2 = \left[ \sum_{i=1}^{n} P_i, \ e' \right] - [P_1, \ e'] + [P_1, \ \sum_{i=1}^{p} P'_i] = \sum_{i=1}^{n+p-1} P''_i \in \mathcal{H}'', \quad (8.16) \]
where
\[ P''_i = [P_1, \ P'_i] \quad \text{for} \quad 1 \leq i \leq p \]
\[ = [P_{i+p}, \ e'] \quad \text{for} \quad p + 1 \leq i \leq n + p - 1 \quad (8.17) \]
satisfies \( P''_i \in \mathcal{P}''_i \). Similarly, and using the fact implied by \((8.11)\) that \( j' = Ze' = z_1 e' \), we have
\[ J''_2 = \left[ \sum_{i=1}^{n} z_i P_i, \ e' \right] - [P_1, \ j'] + [P_1, \ \sum_{i=1}^{p} z'_i P'_i] = \sum_{i=1}^{n+p-1} z''_i P''_i \in \mathcal{H}''. \quad (8.18) \]

Given a basis \( v_1, v_2, \ldots, v_m \) for \( \mathcal{V} \) and a vector \( u' \) in \( \mathcal{U}' \) it is natural to take \((v_1, u'),(v_2, u'), \ldots, (v_m, u')\) as our basis for \( \mathcal{V}'' \). Choosing \( e' \) so that \( e' = u' \), it is evident that \( Y(Z'(z'_1, z'_2, \ldots, z'_p), z_2, \ldots, z_n) \) is the matrix-valued function associated the new subspace collection, represented in these bases.

There is a similar subspace operation corresponding to substituting the \( Z \)-function \( Z'(z'_1, z'_2, \ldots, z'_p) \) into another \( Z \)-function \( Z(z_1, z_2, \ldots, z_n) \) to obtain
\[ Z''(z'_1, z'_2, \ldots, z'_p, z_2, \ldots, z_n) = Z(Z(z'_1, z'_2, \ldots, z'_p), z_2, \ldots, z_n). \quad (8.19) \]
Given a $Z(n)$-subspace collection

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = P_1 \oplus P_2 \oplus \cdots \oplus P_n,$$  \hspace{1cm} (8.20)

and a $(3, p)$-subspace collection

$$\mathcal{H}' = \mathcal{U}' \oplus \mathcal{E}' \oplus \mathcal{J}' = P'_1 \oplus P'_2 \oplus \cdots \oplus P'_p,$$  \hspace{1cm} (8.21)

in which $\mathcal{U}$ is $m$-dimensional and $\mathcal{U}'$ is one-dimensional, we take as our new $(3, n + p - 1)$-subspace collection,

$$\mathcal{K}'' = \mathcal{U}'' \oplus \mathcal{E}'' \oplus \mathcal{J}'' = P''_1 \oplus P''_2 \oplus \cdots \oplus P''_n,$$  \hspace{1cm} (8.22)

where

$$\mathcal{U}'' = \mathcal{U} \otimes \mathcal{U}', \hspace{1cm} \mathcal{E}'' = (\mathcal{E} \otimes \mathcal{U}') \oplus (P_1 \otimes \mathcal{E}'), \hspace{1cm} \mathcal{J}'' = (\mathcal{J} \otimes \mathcal{U}') \oplus (P_1 \otimes \mathcal{J}'),$$  \hspace{1cm} (8.23)

and

$$P''_i = P_i \otimes P'_i \hspace{1cm} \text{for} \hspace{1cm} 1 \leq i \leq p$$

$$= P_{i+1-p} \otimes \mathcal{U}' \hspace{1cm} \text{for} \hspace{1cm} p + 1 \leq i \leq n + p - 1.$$  \hspace{1cm} (8.24)

Suppose that we are given solutions to the equations

$$j + J = \sum_{i=1}^n z_i \Lambda_i (e + E) \hspace{1cm} \text{with} \hspace{1cm} e, j \in \mathcal{U}, \hspace{1cm} E \in \mathcal{E}, \hspace{1cm} J \in \mathcal{J},$$

$$j' + J' = \sum_{j=1}^n z'_j \Lambda'_j (e' + E') \hspace{1cm} \text{with} \hspace{1cm} e', j' \in \mathcal{U}', \hspace{1cm} E' \in \mathcal{E}', \hspace{1cm} J' \in \mathcal{J'},$$  \hspace{1cm} (8.25)

where $z_1 = Z(z'_1, z'_2, \ldots, z'_p)$ and $\Lambda_i$ and $\Lambda'_j$ are the projections onto $P_i$ and $P'_j$. Let us introduce

$$P_i = \Lambda_i (e + E), \hspace{1cm} P'_j = \Lambda'_j (e' + E'),$$

and define $z''_i$ by (8.13), and $P''_i \in \mathcal{P}''_i$ by (8.17). In the new subspace collection, the vectors

$$e'' = [e, e'] \in \mathcal{U}'', \hspace{1cm} E'' = [E, e'] + [P_1, E'] \in \mathcal{E}'',$$

$$j'' = [j, e'] \in \mathcal{U}'', \hspace{1cm} J'' = [J, e'] + [P_1, J'] \in \mathcal{J}''$$  \hspace{1cm} (8.26)

satisfy

$$e'' + E'' = \left[ \sum_{i=1}^n P_i, \hspace{1cm} e' \right] + [P_1, \hspace{1cm} \sum_{j=1}^p P'_j] - [P_1, \hspace{1cm} e']$$

$$= \left[ \sum_{i=2}^n P_i, \hspace{1cm} e' \right] + [P_1, \hspace{1cm} \sum_{j=1}^p P'_j]$$

$$= \sum_{i=1}^{n+p-1} P''_i,$$  \hspace{1cm} (8.27)
and, using (8.11),
\[
\begin{align*}
J'' + J''' &= \left[ \sum_{i=1}^{n} z_i P_i, \, e' \right] + \left[ P_1, \, \sum_{j=1}^{p} z_j' P_j' \right] - \left[ P_1, \, j' \right] \\
&= \left[ \sum_{i=2}^{n} z_i P_i, \, e' \right] + \left[ P_1, \, \sum_{j=1}^{p} z_j' P_j' \right] \\
&= \sum_{i=1}^{n+p-1} z_i'' P_i''.
\end{align*}
\]

(8.28)

Given a basis \( u_1, u_2, \ldots, u_m \) for \( \mathcal{U} \) and a vector \( u' \) in \( \mathcal{U}' \) it is natural to take \( (u_1, u'),(u_2, u'), \ldots, (u_m, u') \) as our basis for \( \mathcal{U}'' \). Choosing \( e' \) so that \( e' = u' \), it is evident from (8.26) that \( Z(Z'(z'_1, z'_2, \ldots, z'_p), z_2, \ldots, z_n) \) is the matrix-valued function associated the new subspace collection, represented in these bases.

A further operation we can do on functions \( Y(z_1, z_2, \ldots, z_n) \) while retaining the homogeneity of degree 1 in the variables \( z_1, z_2, \ldots, z_n \) is to replace the function by \( Y(1/z_1, 1/z_2, \ldots, 1/z_n)^{-1} \). The analogous operation on the associated \( Y(n) \)-subspace collection is to interchange the subspaces \( \mathcal{E} \) and \( \mathcal{J} \). Similarly in a \( Z(n) \) subspace collection, interchanging the subspaces \( \mathcal{E} \) and \( \mathcal{J} \) corresponds to replacing \( Z(z_1, z_2, \ldots, z_n) \) by \( [Z(1/z_1, 1/z_2, \ldots, 1/z_n)]^{-1} \), as noted in section 29.1 of [5]. We call such a transformation a duality transformation. As a consequence of the duality transformation (4.3) immediately implies the formula
\[
Z^{-1} = \Gamma_0[\Gamma_0 + \Gamma_1)L(\Gamma_0 + \Gamma_1)]^{-1}\Gamma_0.
\]

(8.29)

9 Realizing any scalar rational function using subspace collections

Given any homogeneous rational function of degree 1,
\[
Z(z_1, z_2, \ldots, z_n) = \frac{p(z_1, z_2, \ldots, z_n)}{q(z_1, z_2, \ldots, z_n)},
\]

(9.1)

satisfying the normalization \( Z(1,1, \ldots, 1) = 1 \) where \( p(z_1, z_2, \ldots, z_n) \) and \( q(z_1, z_2, \ldots, z_n) \) are homogeneous polynomials of degree \( k \) and \( k - 1 \) respectively, where \( k \) is a positive integer, our goal is to find a \( Z(n) \) subspace collection
\[
\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,
\]

(9.2)

where \( \mathcal{U} \) is one-dimensional which has \( Z(z_1, z_2, \ldots, z_n) \) as its associated function. Without loss of generality we could set \( z_n = 1 \), and then \( p(z_1, z_2, \ldots, z_{n-1}, 1) \) and \( q(z_1, z_2, \ldots, z_{n-1}, 1) \)
are just arbitrary polynomials of the \( n - 1 \) variables \( z_1, z_2, \ldots, z_{n-1} \). Also without loss of generality we can assume
\[
p(1, 1, \ldots, 1) = q(1, 1, \ldots, 1) = 1. \tag{9.3}
\]
The first step is to realize \( Z(z_1, z_2, 1) = z_1 z_2 \) as an associated \( Z \)-function. Note that (3.17) implies we can realize
\[
Z(z_1, 1) = z_1^2, \tag{9.4}
\]
and (3.18) implies we can realize
\[
Z(z_1, z_2) = cz_1 + (1 - c)z_2, \tag{9.5}
\]
for any constant \( c \). Hence, by substitution we can realize
\[
Z(z_1, z_2, 1) = 9(2z_1/3 + z_2/3)^2/8 - (2z_1 - z_2)^2/8 = z_1 z_2. \tag{9.6}
\]
Making further substitutions, we can realize any product of the variables
\[
Z(z_1, z_2, \ldots, z_{n-1}, 1) = z_1^{a_1} z_2^{a_2} \cdots z_{n-1}^{a_{n-1}}, \tag{9.7}
\]
where the \( a_i \) are non-negative integers. By repeated substitution in (9.5) we can realize any linear combination of such terms with coefficients summing to 1, and thus we can realize the polynomials \( p(z_1, z_2, \ldots, z_{n-1}, 1) \) and \( q(z_1, z_2, \ldots, z_{n-1}, 1) \).

Furthermore (3.17), with the roles of \( z_1 \) and \( z_2 \) interchanged, implies we can realize
\[
Z(z_1, 1) = 1/z_1, \tag{9.8}
\]
which by substitution into (9.6) implies we can realize
\[
Z(z_1, z_2, 1) = z_2/z_1. \tag{9.9}
\]
Substituting \( p(z_1, z_2, \ldots, z_{n-1}, 1) \) for \( z_2 \) and \( q(z_1, z_2, \ldots, z_{n-1}, 1) \) for \( z_1 \) we see we can find a subspace collection which realizes
\[
Z(z_1, z_2, \ldots, z_{n-1}, 1) = \frac{p(z_1, z_2, \ldots, z_{n-1}, 1)}{q(z_1, z_2, \ldots, z_{n-1}, 1)} \tag{9.10}
\]
as its associated \( Z \)-function when \( z_n = 1 \). When \( z_n \) is not 1, the subspace collection will by homogeneity realize the function (9.1).

### 10 Expressions for the numerator and denominator in the rational function

Assume that a \( Z(n) \) subspace collection, with \( m = 1 \) has been pruned. Let \( w_1, w_2, \ldots, w_{q_1+1} \) be a basis for \( U \oplus E \) with \( w_1 \) in \( U \) and \( w_2, w_3, \ldots, w_{q_1+1} \) in \( E \). In this basis \( (\Gamma_0 + \Gamma_1) \Lambda_i(\Gamma_0 + \cdot \cdot \cdot) \)
\[ \sum_{i=1}^{n} A_i = I. \] (10.1)

Also, because the subspace is pruned, \( \Gamma_i(U \oplus E) \) can be identified with \( P_i \) which implies the matrix \( A_i \) must have at most rank \( p_i \). It is exactly \( p_i \) if \( P_i \cap J = 0 \). The formula (8.29) for the \( Z \)-function implies

\[
1/Z(z_1, z_2, \ldots, z_n) = e_1 \cdot (\sum_{i=1}^{n} z_i A_i)^{-1} e_1,
\] (10.2)

where \( e_1 \) is the \( q_1 + 1 \) component unit vector \([1, 0, 0, \ldots, 0]^T\). Hence, following the argument given in section 29.2 of [5], \( Z(z_1, z_2, \ldots, z_n) \) can be expressed in the form (9.1) with numerator

\[
p(z_1, z_2, \ldots, z_n) = \det \left( \sum_{i=1}^{n} z_i A_i \right) = \sum_{a_1, a_2, \ldots, a_n} \alpha_{a_1} a_2 \ldots a_n z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n},
\] (10.3)

of degree \( 1 + q_1 \), in which the sum extends over all \( a_1, a_2, \ldots, a_n \) with

\[
\sum_{i=1}^{n} a_i = 1 + q_1, \quad 0 \leq a_i \leq p_i \quad \text{for } i = 1, 2, \ldots, n.
\] (10.4)

Typically one expects that the maximum power of \( z_i \) in this polynomial will be the rank of \( A_i \). However, for example, note that for the matrices

\[
M_1 = \begin{pmatrix}
0 & 0 \\
1 & 1 \\
0 & 1
\end{pmatrix}, \quad M_2 = I - M_1,
\] (10.5)

the maximum power of \( z_1 \) in

\[
\det[z_1 M_1 + z_2 M_2] = \det[(z_1 - z_2) M_1 + z_2 I] = z_2 [z_2^2 + 2z_2 (z_1 - z_2)]
\] (10.6)

is 1 while \( M_1 \) has rank 2.

Next let \( w_1, w_{q_1 + 2}, \ldots, w_h \) be a basis for \( U \oplus J \) with \( w_1 \) in \( U \) and \( w_{q_1 + 2}, w_{q_1 + 3}, \ldots, w_h \) in \( J \). In this basis \( (\Gamma_0 + \Gamma_2) A_i (\Gamma_0 + \Gamma_2) \) is represented by a \( (q_2 + 1) \times (q_2 + 1) \) matrix \( B_i \), and since the \( A_i \) sum up to the identity operator it follows that

\[
\sum_{i=1}^{n} B_i = I.
\] (10.7)
Also, because the subspace is pruned, $\Gamma_i(U \oplus J)$ can be identified with $P_i$ which implies

the matrix $B_i$ must have rank at most $p_i$. It is exactly $p_i$ if $P_i \cap E = 0$. The formula (1.3) for the $Z$-function implies

$$Z(z_1, z_2, \ldots, z_n) = e_2 \cdot \left[ \sum_{i=1}^{n} B_i/z_i \right]^{-1} e_2,$$

(10.8)

where $e_2$ is the $q_2 + 1$ component unit vector $[1, 0, 0, \ldots, 0]^T$. The denominator of this expression, as a polynomial in the variables $1/z_i$, is

$$\det \left[ \sum_{i=1}^{n} B_i/z_i \right] = \sum_{b_1, b_2, \ldots, b_n} \beta_{b_1 b_2 \ldots b_n} / z_1^{b_1} z_2^{b_2} \ldots z_n^{b_n},$$

(10.9)

in which the sum extends over all $b_1, b_2, \ldots, b_n$ with

$$\sum_{i=1}^{n} b_i = 1 + q_2, \quad 0 \leq b_i \leq p_i \quad \text{for } i = 1, 2, \ldots, n.$$ (10.10)

Consequently we can make the identification

$$q(z_1, z_2, \ldots, z_n) = \sum_{b_1, b_2, \ldots, b_n} \beta_{b_1 b_2 \ldots b_n} z_1^{b_1} z_2^{p_2 - b_2} \ldots z_n^{p_n - b_n},$$

(10.11)

which is a polynomial of degree $h - (1 + q_2) = q_1$. Furthermore the identities (10.1) and (10.7) imply the polynomial $p$ and $q$ satisfy the normalization (9.3), i.e.

$$\sum_{a_1, a_2, \ldots, a_n} \alpha_{a_1 a_2 \ldots a_n} = 1, \quad \sum_{b_1, b_2, \ldots, b_n} \beta_{b_1 b_2 \ldots b_n} = 1.$$ (10.12)

## 11 The correspondence between rational functions of one variable and $Z(2)$ subspace collections with $m = 1$.

In the case $m = 1$ and $n = 2$ there are two cases to consider. When the dimension of $h$ is even, $h = 2d$, then in order to satisfy the inequalities (5.5), (5.6) and (5.8) the subspaces $E$ and $J$ must have dimension $d$ and $d - 1$ or vice-versa and the subspaces $P_1$ and $P_2$ must have dimension $d$. Without loss of generality, by making a duality transformation if necessary, let us suppose $E$ has dimension $d - 1$. Given $u \in U$ let us take as our basis for $H$ the vectors

$$v_{2j-1} = (\Gamma_1 \Lambda_1)^{j-1} u, \quad v_{2j} = (\Lambda_1 \Gamma_1)^{j-1} \Lambda_1 u, \quad j = 1, 2, \ldots, d,$$ (11.1)
so that
\[ v_1 = u, \quad v_{2j} = \Lambda_1 \nu_{2j-1}, \quad j = 1, 2, \ldots, d, \quad v_{2j+1} = \Gamma_1 \nu_{2j-1}, \quad j = 1, 2, \ldots, d-1. \] (11.2)

These fields are independent since if they were not we could prune the subspace collection. The vectors \( v_{2j+1}, j = 1, 2, \ldots, d-1 \), which number \( d-1 \), must form a basis for \( E \) and so it follows that
\[ \Gamma_1 v_{2d} = \sum_{i=1}^{d-1} \gamma_i v_{2i+1}. \] (11.3)

Also we have
\[ \Gamma_0 v_1 = v_1, \quad \Gamma_0 v_{2j} = \delta_j v_1, \quad j = 1, 2, \ldots, d, \quad \Gamma_0 v_{2j+1} = 0, \quad j = 1, 2, \ldots, d-1. \] (11.4)

The \( 2d-1 \) constants \( \gamma_1, \ldots, \gamma_{d-1} \) and \( \delta_1, \ldots, \delta_d \) characterize the geometry of the subspace collection. The field \( e + E \) must have the expansion
\[ e + E = \sum_{i=1}^{d} a_i v_{2i-1}, \] (11.5)

and consequently, setting \( z_2 = 1 \)
\[ j + J = [I + (z_1 - 1)\Lambda_1](e + E) = \sum_{i=1}^{d} a_i v_{2i-1} + (z_1 - 1) \sum_{i=1}^{d} a_i v_{2i}. \] (11.6)

Since \( \Gamma_1 (j + J) = 0 \) we arrive at the equations
\[
0 = \sum_{i=2}^{d} a_i v_{2i-1} + (z_1 - 1) \sum_{i=1}^{d-1} a_i v_{2i+1} + (z_1 - 1) \sum_{i=1}^{d-1} a_d \gamma_i v_{2i+1} \\
= \sum_{i=1}^{d-1} \left[ a_{i+1} + a_i (z_1 - 1) + \gamma_i a_d (z_1 - 1) \right] v_{2i+1}.
\] (11.7)

implying
\[ a_{i+1} + a_i (z_1 - 1) + \gamma_i a_d (z_1 - 1) = 0, \quad i = 1, \ldots, d-1. \] (11.8)

Choosing a normalization with \( a_d = (1 - z_1)^{d-1} \) these equations are solved with
\[ a_i = (1 - z_1)^{i-1} - \sum_{j=i}^{d-1} \gamma_{d-1+i-j} (1 - z_1)^j. \] (11.9)

Since
\[ \Gamma_0 (e + E) = a_1 v_1, \quad \Gamma_0 (j + J) = [a_1 + (z_1 - 1) \sum_{i=1}^{d} \delta_i a_i] v_1, \] (11.10)
we obtain
\[ Z(z_1, 1) = 1 + \frac{(z_1 - 1) \sum_{i=1}^{d} \delta_i a_i}{a_1}. \] (11.11)

Conversely suppose we are given a rational function \( Z(z_1, 1) \) with a denominator of degree at most \( d - 1 \) and a numerator of degree at most \( d \) satisfying \( Z(1, 1) = 1 \). It can be expressed in the form
\[
Z(z_1, 1) = \frac{p(z_1, 1)}{q(z_1, 1)} = 1 - \frac{\sum_{j=0}^{d-1} t_j (1 - z_1)^{j+1}}{1 - \sum_{j=1}^{d-1} s_j (1 - z_1)^j}.
\] (11.12)

Comparing this with (11.11) we can make the identifications
\[
1 - \sum_{j=1}^{d} s_j (1 - z_1)^j = a_1 = 1 - \sum_{j=1}^{d} \gamma_{d-j} (1 - z_1)^j,
\]
\[
- \sum_{j=0}^{d-1} t_j (1 - z_1)^{j+1} = (z_1 - 1) \sum_{i=1}^{d} \delta_i a_i
\]
\[
= - \sum_{j=0}^{d-1} \delta_{j+1} (1 - z_1)^{j+1} + \sum_{j=0}^{d-1} \sum_{i=1}^{j} \delta_i \gamma_{d-1-i-j} (1 - z_1)^{j+1},(11.13)
\]

which imply
\[
s_j = \gamma_{d-j}, \quad t_0 = \delta_1, \quad t_j = \delta_{j+1} - \sum_{i=1}^{j} \delta_i \gamma_{d-1+i-j} \quad j = 1, \ldots, d - 1. \] (11.14)

Given the coefficients \( s \) and \( t \) we can inductively uniquely determine the coefficients \( \gamma \) and \( \delta \):
\[
\gamma_j = s_{d-j}, \quad \delta_1 = t_0, \quad \delta_{j+1} = t_j + \sum_{i=1}^{j} \delta_i s_{1+j-i} \quad j = 1, \ldots, d - 1. \] (11.15)

On the other hand when the dimension of \( h \) is odd, \( h = 2d - 1 \), then in order to satisfy the inequalities (5.5), (5.6) and (5.8) the subspaces \( \mathcal{E} \) and \( \mathcal{J} \) must have dimension \( d - 1 \) and the subspaces \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) must have dimension \( d - 1 \) and \( d \) or vice-versa. Without loss of generality let us suppose \( \mathcal{P}_1 \) has dimension \( d - 1 \). Given \( u \in \mathcal{U} \) let us take as our basis for \( \mathcal{H} \) the vectors
\[
v_{2j-1} = (\Gamma_1 \Lambda_1)^{j-1} u, \quad j = 1, 2, \ldots, d - 1, \quad v_{2j} = (\Lambda_1 \Gamma_1)^{j-1} \Lambda_1 u, \quad j = 1, 2, \ldots, d,
\] (11.16)

which satisfy
\[
v_1 = u, \quad v_{2j} = \Lambda_1 v_{2j-1}, \quad v_{2j+1} = \Gamma_1 v_{2j-1}, \quad j = 1, 2, \ldots, d - 1. \] (11.17)
Again these fields are independent since if they were not we could prune the subspace collection. The vectors $v_{2j}, j = 1, 2, \ldots, d - 1$, which number $d - 1$, must form a basis for $P_1$ and so it follows that

$$
\Lambda_1 v_{2d-1} = \sum_{i=1}^{d-1} \gamma_i v_{2i}.
$$

(11.18)

Also we have

$$
\Gamma_0 v_1 = v_1, \quad \Gamma_0 v_{2j} = \delta_j v_1, \quad j = 1, 2, \ldots, d - 1, \quad \Gamma_0 v_{2j+1} = 0, \quad j = 1, 2, \ldots, d - 1.
$$

(11.19)

The $2d - 2$ constants $\gamma_1, \ldots, \gamma_{d-1}$ and $\delta_1, \ldots, \delta_{d-1}$ characterize the geometry of the subspace collection. The field $e + E$ has the expansion (11.5) and so, with $z_2 = 1$,

$$
(j + J) = [I + (z_1 - 1)\Lambda_1](e + E) = \sum_{i=1}^{d} a_i v_{2i-1} + (z_1 - 1) \sum_{i=1}^{d-1} a_i v_{2i} + (z_1 - 1) \sum_{i=1}^{d-1} a_i \gamma_i v_{2i}.
$$

(11.20)

Since $\Gamma_1(j + J) = 0$ we arrive at the equations

$$
0 = \sum_{i=2}^{d} a_i v_{2i-1} + (z_1 - 1) \sum_{i=1}^{d-1} a_i v_{2i+1} + (z_1 - 1) \sum_{i=1}^{d-1} a_i \gamma_i v_{2i+1}
$$

$$
= \sum_{i=1}^{d-1} [a_{i+1} + a_i(z_1 - 1) + \gamma_i a_d(z_1 - 1)]v_{2i+1},
$$

(11.21)

implying (11.8) which has the solution (11.9). Since

$$
\Gamma_0(e + E) = a_1 v_1, \quad \Gamma_0(j + J) = [a_1 + (z_1 - 1) \sum_{i=1}^{d-1} \delta_i (a_i + a_d \gamma_i)]v_1 = [a_1 - \sum_{i=1}^{d-1} \delta_i a_{i+1}]v_1,
$$

we obtain

$$
Z(z_1, 1) = 1 - \frac{\sum_{i=1}^{d-1} \delta_i a_{i+1}}{a_1}.
$$

(11.22)

Conversely suppose we are given a rational function $Z(z_1, 1)$ with a denominator of degree at most $d - 1$ and a numerator of degree at most $d - 1$. It can be expressed in the form

$$
Z(z_1, 1) = 1 - \frac{\sum_{j=1}^{d-1} t_j (1 - z_1)^j}{1 - \sum_{j=1}^{d-1} s_j (1 - z_1)^j}.
$$

(11.24)
Comparing this with (11.23) we can make the identifications

\[ 1 - \sum_{j=1}^{d-1} s_j (1 - z_1)^j = a_1 = 1 - \sum_{j=1}^{d-1} \gamma_{d-j} (1 - z_1)^j, \]

\[ \sum_{j=1}^{d-1} t_j (1 - z_1)^j = \sum_{i=1}^{d-1} \delta_i a_{i+1} \]

\[ = \sum_{j=1}^{d-1} \delta_j (1 - z_1)^j - \sum_{j=2}^{d-1} \sum_{i=1}^{j-1} \delta_i \gamma_{d+i-j} (1 - z_1)^j, \quad (11.25) \]

which imply

\[ s_j = \gamma_{d-j}, \quad j = 1, \ldots, d-1, \quad t_1 = \delta_1, \quad t_j = \delta_j - \sum_{i=1}^{j-1} \delta_i \gamma_{d+i-j} \quad j = 2, \ldots, d-1. \quad (11.26) \]

Given the coefficients \( s \) and \( t \) we can inductively uniquely determine the coefficients \( \gamma \) and \( \delta \):

\[ \gamma_j = s_{d-j}, \quad j = 1, \ldots, d-1 \quad \delta_1 = t_1, \quad \delta_j = t_j + \sum_{i=1}^{j} \delta_i s_{j-i} \quad j = 2, \ldots, d-1. \quad (11.27) \]

One can see from this analysis that there can be more than one pruned subspace collection associated with a rational function \( Z(z_1, 1) \). It may happen that one pruned \( Z(n) \) subspace collection gives rise to polynomials \( p(z_1, 1) = f(z_1, 1)r(z_1, 1) \) and \( q(z_1, 1) = g(z_1, 1)r(z_1, 1) \) while another pruned \( Z(n) \) subspace collection gives rise to polynomials \( p'(z_1, 1) = f(z_1, 1)r'(z_1, 1) \) and \( q'(z_1, 1) = t(z_1, 1)r'(z_1, 1) \), so that both give rise to the same function \( Z(z_1, 1) \). However there is a one-to-one correspondence if the pruned subspace collection is such that the polynomials \( p(z_1, z_2) \) and \( q(z_1, z_2) \) have no factor in common, and this correspondence is given by the above algorithm.

12 On the correspondence between certain rational functions of two variables and \( Z(3) \) subspace collections with \( m = 1 \)

In the case \( m = 1 \) and \( n = 3 \) can we uniquely recover a generic subspace collection (modulo the linear transformations (7.15)) from knowledge of the rational function \( Z(z_1, z_2, 1) \)? The answer is no, but let us first provide a counting argument which suggests that, at least in the generic case, we can recover the subspace collection up to a finite number of possibilities. The counting argument is similar to that given in section 29.2 of [5] but here we do not assume that the subspaces are orthogonal.
How many independent coefficients $\alpha_{a_1 a_2 a_3}$ are there in a polynomial

$$p(z_1, z_2, 1) = \sum_{a_1, a_2, a_3} \alpha_{a_1 a_2 a_3} z_1^{a_1} z_2^{a_2},$$

(12.1)

that satisfies

$$a_1 + a_2 + a_3 = 1 + q_1, \quad 0 \leq a_i \leq p_i \leq 1 + q_1, \quad i = 1, 2, 3? \quad (12.2)$$

Without loss of generality, following section 29.2 of [5], let us suppose that $p_1 \geq p_2 \geq p_3$. With $a_1$ fixed in the regime $0 \leq a_1 < 1 + q_1 - p_2$, the constant $a_2$ can take integer values from $a_2 = 1 + q_1 - a_1 - p_3$ (where $a_3 = p_3$) to $a_2 = p_2$, that is, a total of $p_2 + p_3 + a_1 - q_1$ different values. With $a_1$ fixed in the regime $1 + q_1 - p_2 \leq a_1 < 1 + q_1 - p_3$, the constant $a_2$ can take integer values from $a_2 = 1 + q_1 - a_1 - p_3$ (where $a_3 = p_3$) to $a_2 = 1 + q_1 - a_1$ (where $a_3 = 0$) that is, a total of $p_3 + 1$ different values. Finally, with $a_1$ fixed in the regime $1 + q_1 - p_3 \leq a_1 \leq p_1$, the constant $a_2$ can take integer values from $a_2 = 0$ to $a_2 = 1 + q_1 - a_1$ (where $a_3 = 0$), that is, a total of $2 + q_1 - a_1$ different values. Therefore the total number of coefficients in the polynomial is

$$\sum_{a_1 = 0}^{q_1 - p_2} (p_2 + p_3 + a_1 - q_1) + \sum_{a_1 = 1 + q_1 - p_2}^{q_1 - p_3} (p_3 + 1) + \sum_{a_1 = 1 + q_1 - p_3}^{p_1} (2 + q_1 - a_1)
$$

$$= (q_1 - p_2 + 1)(p_2 + p_3 - q_1) + \frac{1}{2}(q_1 - p_2 + 1)(q_1 - p_2) + (p_2 - p_3)(p_3 + 1)
$$

$$+(p_1 + p_3 - q_1)(2 + p_3) - \frac{1}{2}((p_1 + p_3 - q_1)(p_1 + p_3 - q_1 + 1)
$$

$$= k_1 + 1, \quad (12.3)$$

where

$$k_1 = [2(1 + q_1)q_2 - p_1^2 - p_2^2 - p_3^2 + h]/2, \quad (12.4)$$

in which $h = p_1 + p_3 + q_2$ and $q_2 = h - 1 - q_1$. These coefficients are not all independent since, from (12.12) the $\alpha_{a_1 a_2 a_3}$ must sum to one. Subtracting this constraint gives $k_1$ independent coefficients.

Similarly in a polynomial

$$q(z_1, z_2, 1) = \sum_{b_1, b_2, b_3} \alpha_{b_1 b_2 b_3} z_1^{p_1 - b_1} z_2^{p_2 - b_2}, \quad (12.5)$$

that satisfies

$$b_1 + b_2 + b_3 = 1 + q_2, \quad 0 \leq a_i \leq p_i \leq 1 + q_2, \quad i = 1, 2, 3, \sum_{b_1, b_2, b_3} \beta_{b_1 b_2 b_3} = 1, \quad (12.6)$$

there are a total of

$$k_2 = [2(1 + q_2)q_1 - p_1^2 - p_2^2 - p_3^2 + h]/2 \quad (12.7)$$
independent coefficients. Hence the total number of independent coefficients in the rational function

\[ Z(z_1, z_2, 1) = \frac{p(z_1, z_2, 1)}{q(z_1, z_2, 1)} \]

is

\[ k_1 + k_2 = (1 + q_1)q_2 + (1 + q_2)q_1 - p_1^2 - p_2^2 - p_3^2 + h = h^2 - p_1^2 - p_2^2 - p_3^2 - q_1^2 - q_2^2. \] (12.9)

Now how many parameters describe a \( Z(n) \) subspace collection, when the spaces \( \mathcal{U}, \mathcal{E}, \mathcal{J}, \mathcal{P}_1, \mathcal{P}_2, \) and \( \mathcal{P}_3 \) have dimensions 1, \( q_1, q_2, p_1, p_2, \) and \( p_3, \) with \( 1 + q_1 + q_2 = p_1 + p_2 + p_3 = h? \) Let \( \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_h \) be a basis for \( \mathcal{H} \) with \( \mathbf{w}_1 \) in \( \mathcal{U}, \mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_{q_1+1} \) in \( \mathcal{E}, \) and \( \mathbf{w}_{q_1+2}, \mathbf{w}_{q_1+3}, \ldots, \mathbf{w}_h \) in \( \mathcal{J}. \) Recall that it requires \( s(d-s) \) parameters to describe the orientation of a subspace of dimension \( s \) in a space of dimension \( d. \) Therefore, it requires

\[ p_1(h - p_1) + (h - p_2)p_2 + (h - p_3)p_3 = h^2 - p_1^2 - p_2^2 - p_3^2 \]

parameters to describe the orientation of the subspaces \( \mathcal{P}_1, \mathcal{P}_2, \) and \( \mathcal{P}_3 \) with respect to this basis. However some of these subspace collections are equivalent, linked through transformations of the form (7.15). If respect to this basis \( \mathbf{C} \) is represented by a matrix with block form

\[
\mathbf{C} = \begin{pmatrix}
c & 0 & 0 \\
0 & \mathbf{C}_1 & 0 \\
0 & 0 & \mathbf{C}_2
\end{pmatrix},
\]

(12.11)

where \( c \) is a scalar and \( \mathbf{C}_1 \) and \( \mathbf{C}_2 \) are \( q_1 \times q_1 \) and \( q_2 \times q_2 \) matrices, then it will leave the subspaces \( \mathcal{U}, \mathcal{E} \) and \( \mathcal{J} \) unchanged. The transformation \( \mathbf{C} = a \mathbf{I} \) leaves all subspaces unchanged for any scalar \( a \neq 0, \) and so to factor out such trivial transformations we should choose \( c = 1. \) The number of remaining independent parameters in \( \mathbf{C} \) is then \( q_1^2 + q_2^2. \) Substracting these from (12.10) we see that the number of parameters describing the \( Z(n) \) subspace collection is

\[ h^2 - p_1^2 - p_2^2 - p_3^2 - q_1^2 - q_2^2 = k_1 + k_2. \] (12.12)

The precise agreement between the number of coefficients in the rational function and the number of parameters describing the \( Z(n) \) subspace collection is curious (since it holds for all \( q_1, q_2, p_1, p_2, \) and \( p_3, \) with \( 1 + q_1 + q_2 = p_1 + p_2 + p_3 = h). \) Despite this coincidence we now show that it is not possible to uniquely recover a generic subspace collection (modulo the linear transformations (7.15)) from knowledge of the associated rational function \( Z(z_1, z_2, 1). \)

Let us consider a subspace collection with \( h = 5, q_1 = q_2 = 2, p_1 = p_2 = 1, p_3 = 3 \) giving \( k_1 + k_2 = 6 \) according to the formula (12.9). Given \( \mathbf{u} \in \mathcal{U} \) we choose as our basis the vectors

\[ \mathbf{v}_0 = \mathbf{u}, \quad \mathbf{v}_1 = \Lambda_1 \mathbf{u}, \quad \mathbf{v}_2 = \Lambda_2 \mathbf{u}, \quad \mathbf{v}_3 = \Gamma_1 \Lambda_1 \mathbf{u}, \quad \mathbf{v}_4 = \Gamma_1 \Lambda_2 \mathbf{u}, \]

(12.13)
with the closure relations
\[
\begin{align*}
\Lambda_1 v_3 &= \gamma_1 v_1, \quad \Lambda_2 v_3 = \gamma_2 v_2, \quad \Lambda_1 v_4 = \gamma_3 v_1, \quad \Lambda_2 v_4 = \gamma_4 v_1, \\
\Gamma_0 v_1 &= \delta_1 v_0, \quad \Gamma_0 v_2 = \delta_2 v_0,
\end{align*}
\]
expressed in terms of the 6 parameters \(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_1, \) and \(\delta_2\) which describe the subspace collection. The question is: can one uniquely recover these six parameters from \(Z(z_1, z_2, 1)\)? Although the following analysis extends easily to the case of arbitrary \(\gamma_1\) and \(\gamma_4\) let us assume, for simplicity, that \(\gamma_1 = \gamma_4 = 0\) and ask whether one can recover the remaining four parameters. The field \(e + E\) must have the expansion
\[
e + E = a_0 v_0 + a_1 v_3 + a_2 v_4,
\]
and consequently, setting \(z_3 = 1\),
\[
j + J = [I + (z_1 - 1)\Lambda_1 + (z_2 - 1)\Lambda_2](e + E)
\]
\[
= a_0 v_0 + a_1 v_3 + a_2 v_4 + (z_1 - 1)(a_0 + a_2\gamma_3) v_3 + (z_2 - 1)(a_0 + a_1\gamma_2) v_4.
\]
Since \(\Gamma_0 (j + J) = 0\) we arrive at the equations
\[
0 = a_1 v_3 + a_2 v_4 + (z_1 - 1)(a_0 + a_2\gamma_3) v_3 + (z_2 - 1)(a_0 + a_1\gamma_2) v_4,
\]
implying
\[
a_1 + (z_1 - 1)(a_0 + a_2\gamma_3) = 0, \quad a_2 + (z_2 - 1)(a_0 + a_1\gamma_2) = 0
\]
These equations have as a solution,
\[
\begin{align*}
a_0 &= 1 - (z_1 - 1)(z_2 - 1)\gamma_2\gamma_3, \\
a_1 &= \gamma_3(z_1 - 1)(z_2 - 1) - (z_1 - 1), \\
a_2 &= \gamma_2(z_1 - 1)(z_2 - 1) - (z_2 - 1).
\end{align*}
\]
Since
\[
\Gamma_0 (e + E) = a_0 v_0, \quad \Gamma_0 (j + J) = [a_0 + (z_1 - 1)(a_0 + a_2\gamma_3)\delta_1 + (z_2 - 1)(a_0 + a_1\gamma_2)\delta_2] v_0,
\]
we obtain
\[
Z(z_1, z_2, 1) = 1 + \frac{(z_1 - 1)(a_0 + a_2\gamma_3)\delta_1 + (z_2 - 1)(a_0 + a_1\gamma_2)\delta_2}{a_0}
\]
\[
= 1 + \frac{\delta_1(z_1 - 1) - \gamma_3\delta_1(z_1 - 1)(z_2 - 1) + \delta_2(z_2 - 1) - \gamma_2\delta_2(z_1 - 1)(z_2 - 1)}{1 - (z_1 - 1)(z_2 - 1)\gamma_2\gamma_3}
\]
Given this function we can uniquely determine \(\delta_1\) and \(\delta_2\) from the coefficients of \((z_1 - 1)\) and \((z_2 - 1)\) in the numerator. Also from the coefficients of \((z_1 - 1)(z_2 - 1)\) in the numerator and denominator we can uniquely determine
\[
t_1 = \gamma_2\gamma_3, \quad t_2 = \gamma_3\delta_1 + \gamma_2\delta_2,
\]

in terms of which there are two possible values of $\gamma_2$, namely

$$\gamma_2 = \frac{t_3 \pm \sqrt{t_3^2 - 3t_1\delta_1\delta_2}}{2\delta_1}. \quad (12.23)$$

Thus we cannot uniquely recover the subspace collection parameters from $Z(z_1, z_2, 1)$. It remains an open question, raised at the end of section 29.2 of [5], as to whether in general one can uniquely recover the subspace collection parameters when, with respect to some inner product, the subspaces $U, E$ and $J$ are mutually orthogonal, and the subspaces $P_1, P_2$ and $P_3$ are mutually orthogonal.

13 Normalization operations on subspace collections

Rational functions of a single variable may be expanded in continued fractions, which incorporate successively higher and higher order terms in the series expansion of the function about a point. The analogous procedure with subspace collections is achieved through normalization and reduction operations, subject to some technical assumptions. The associated functions are then linked, and provided the technical assumptions hold at each level, these links provide continued fractions for multivariate functions $Z(z_1, z_2, \ldots, z_n)$ and $Y(z_1, z_2, \ldots, z_n)$ incorporating matrices of increasingly high dimension at each level in the continued fraction.

The normalization and reduction operations are discussed in this and the next section. For more insight, in the case where the subspaces in the direct sums are orthogonal, see [6, 17] and sections 19.2, 20.6 and 29.5 in [5].

Normalization reverses extension. Given a subspace collection

$$K = E' \oplus J' = V \oplus P_1 \oplus P_2 \oplus \cdots \oplus P_n, \quad (13.1)$$

define

$$H = P_1 \oplus P_2 \oplus \cdots \oplus P_n, \quad E = E' \cap H, \quad J = J' \cap H,$$

$$U = \Pi_2 \Gamma_1' V = \Pi_2 (I - \Gamma_2') V = \Pi_2 \Gamma_2' V, \quad \tilde{E} = \Gamma_1' V, \quad \tilde{J} = \Gamma_2' V, \quad (13.2)$$

where $\Gamma_1'$ and $\Gamma_2'$ are the projections onto $E'$ and $J'$, and $\Pi_2$ is the projection onto $H$.

We assume that the $Y$-problem has a unique solution when $L = I$ for $J_1 \in V$ given $E_1 \in V$. In other words, we assume that the equations

$$E_1 + E_2 \in E', \quad J_1 + J_2 \in J', \quad J_2 = E_2, \quad E_1, J_1 \in V, \quad E_2, J_2 \in H,$$

$$E_1 + \overline{E_2} \in E', \quad J_1 + \overline{J_2} \in J', \quad J_2 = \overline{E_2}, \quad J_1 \in V, \quad E_2, J_2 \in H, \quad (13.3)$$

imply $J_1 = J_2$. Subtracting these equations we see that

$$E = E_2 - \overline{E_2} \in E', \quad J = J_1 + J_2 - J_1 - J_2 \in J', \quad J_2 - J_2 = E. \quad (13.4)$$
These imply
\[ E \in H, \quad E = J - v, \quad \text{where} \quad v = J_1 - J_1. \] (13.5)
The uniqueness assumption means that these equations imply \( v = 0 \) (and if \( v = 0 \) then necessarily \( E = J = 0 \) since \( \mathcal{E}' \) and \( \mathcal{J}' \) have no vector in common). The relation \( E = J - v \) with \( E \in \mathcal{E}' \cap H \) implies
\[ E = -\Gamma_1'v, \] (13.6)
which will only have the trivial solution \( v = 0 \) if and only if
\[ H \cap \tilde{E} = 0 \quad \text{and} \quad V \cap \mathcal{J}' = 0, \] (13.7)
where the latter guarantees that \( \Gamma_1'v = 0 \) implies \( v = 0 \).

We also assume that the \( Y \)-problem has a unique solution when \( L = I \) for \( E_1 \in \mathcal{V} \) given \( J_1 \in \mathcal{V} \). By similar analysis this is satisfied if and only if
\[ H \cap \tilde{J} = 0 \quad \text{and} \quad V \cap \mathcal{E}' = 0. \] (13.8)

We now establish that
\[ W \equiv \tilde{E} \oplus \tilde{J} = V \oplus U. \] (13.9)
First note that \( V \) and \( U \) have no vector in common since \( U \subset H \), and similarly \( \tilde{E} \) and \( \tilde{J} \) have no vector in common since \( \mathcal{E}' \cap \mathcal{J}' = 0 \). Clearly \( W \) contains \( V \). To show it contains \( U \) notice that
\[ U = \Pi_2 \Gamma_1'V = (I - \Pi_1)\Gamma_1'V \subset \Gamma_1'V \oplus \Pi_1 \Gamma_1'V \subset \tilde{E} \oplus V \subset W. \] (13.10)
Together these imply \( V \oplus U \subset W \). Finally we have
\[ \tilde{E} = \Gamma_1'V = (\Pi_1 + \Pi_2)\Gamma_1'V \subset \Pi_1 \Gamma_1'V \oplus \Pi_2 \Gamma_1'V \subset V \oplus U, \] (13.11)
and similarly \( \tilde{J} \subset V \oplus U \). Together these imply \( W \subset V \oplus U \), establishing (13.9).

If \( V \) has dimension \( m \) then \( \tilde{E} \) must also have dimension \( m \) since otherwise \( \Gamma_1'v = 0 \) for some non-zero \( v \in V \), implying \( v = \Gamma_2'v \) which only has the solution \( v = 0 \) since \( V \cap \mathcal{J}' = 0 \). Similarly \( \tilde{J} \) must have dimension \( m \) and (13.9) then implies \( U \) must have dimension \( m \). The first condition in (13.7) implies
\[ W = U \oplus \tilde{E}, \] (13.12)
since \( U \subset H \) and \( \tilde{E} \) have no vector in common and are \( m \)-dimensional spaces contained in the \( 2m \) dimensional space \( W \). Now any vector \( E' \in \mathcal{E}' \) has the unique decomposition
\[ E' = E'_1 + P, \quad E'_1 \in \mathcal{V}, \quad P \in H, \] (13.13)
and according to (13.12) \( E'_1 \) has the unique decomposition
\[ E'_1 = -e + \tilde{E}, \quad e \in U, \quad \tilde{E} \in \tilde{E}. \] (13.14)
So we have the decomposition
\[ E' = \tilde{E} + E, \]  
where
\[ E = P - e = E' - \tilde{E} \in \mathcal{E}' \cap \mathcal{H} = \mathcal{E}. \]

Also the first condition in (13.7) implies \( \tilde{E} \) and \( \mathcal{E} \subset \mathcal{H} \) have no vector in common, so the decomposition is unique. Therefore we conclude that
\[ \mathcal{E}' = \tilde{E} \oplus \mathcal{E}, \]  
and similarly the first condition in (13.8) implies
\[ \mathcal{J}' = \tilde{J} \oplus \mathcal{J}. \]

These and (13.9) imply
\[ \mathcal{K} = \mathcal{V} \oplus \mathcal{H} = \tilde{E} \oplus \mathcal{E} \oplus \tilde{J} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J}, \]  
and since \( \mathcal{U}, \mathcal{E} \) and \( \mathcal{J} \) are all contained in \( \mathcal{H} \) we conclude that
\[ \mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n. \]

Now a given \( E'_1 \in \mathcal{V} \) has the unique decomposition (13.14). This defines the non-singular operator \( K : \mathcal{V} \to \mathcal{U} \) such that \( e = KE'_1 \). (It is non-singular because \( \mathcal{V} \) and \( \tilde{E} \subset \mathcal{E}' \) have no non-zero vector in common.) Now given \( e \), consider the solution to
\[ e, j \in \mathcal{U}, \quad E \in \mathcal{E}, \quad J \in \mathcal{J}, \quad j + J = L(e + E), \quad \text{where} \quad L = \sum_{i=1}^{n} z_i \Lambda_i, \]  
where \( \Lambda_i \) is the projection onto \( \mathcal{P}_i \), and from the definition of \( Z, j = Ze \). Since the second condition in (13.7) implies \( \mathcal{V} \) and \( \tilde{J} \) have no vector in common we have
\[ \mathcal{W} = \mathcal{V} \oplus \tilde{J}, \]  
and consequently any \( j \in \mathcal{U} \) has the decomposition
\[ j = -J'_1 + \tilde{J}, \quad J'_1 \in \mathcal{V}, \quad \tilde{J} \in \tilde{J}, \]  
which defines the non-singular operator \( M : \mathcal{U} \to \mathcal{V} \) such that \( J'_1 = Mj \). Defining
\[ E'_2 = e + E, \quad J'_2 = j + J, \]  
we have
\[ E'_1 + E'_2 = E'_1 + e + E = \tilde{E} + E \in \mathcal{E}', \]
\[ J'_1 + J'_2 = J'_1 + j + J = \tilde{J} + J \in \mathcal{J}'. \]
\[ J'_1 = Mj = MZe = MZE'_1, \]

which by definition of the associated \( Y \)-function implies

\[ Y(z_1, z_2, \ldots, z_n) = MZ(z_1, z_2, \ldots, z_n)K. \]  

(13.27)

This is analogous to the relation (20.29) in [5] obtained in the case where the subspaces are mutually orthogonal.

In particular by letting \( z_1 = z_2 = \ldots = z_n = 1 \) we obtain

\[ Y(1, 1, \ldots, 1) = MK. \]  

(13.28)

If \( v_1, v_2, \ldots, v_m \) are a basis for \( V \), and we choose \( Kv_1, Kv_2, \ldots, Kv_m \) as our basis for \( U \) then with these bases \( K \) is represented by the identity matrix \( K = I \) and (13.27) and (13.28) imply

\[ Y(z_1, z_2, \ldots, z_n) = Y(1, 1, \ldots, 1)Z(z_1, z_2, \ldots, z_n). \]  

(13.29)

14 Reduction operations on subspace collections

Extension is one way to go from a \( Z(n) \) subspace collection to a \( Y(n) \) subspace collection. Another way is through reduction, which has some features in common with normalization. Given a \( Z(n) \) subspace collection

\[ \mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \]

(14.1)

let \( \Gamma_0 \) be the projection onto \( \mathcal{U} \), and let \( \Lambda_j \) be the projection onto \( \mathcal{P}_j \). Define

\[ \mathcal{K} = \mathcal{E} \oplus \mathcal{J}, \quad \mathcal{P}_j' = \mathcal{P}_j \cap \mathcal{K} \quad \text{for} \quad j = 1, 2, \ldots, n, \]

\[ \mathcal{V} = (I - \Gamma_0)[\Lambda_1 \mathcal{U} \oplus \Lambda_2 \mathcal{U} \oplus \cdots \oplus \Lambda_n \mathcal{U}] \subset \mathcal{K}, \quad \tilde{\mathcal{P}}_j = \Lambda_j \mathcal{U}. \]  

(14.2)

We now establish that

\[ \mathcal{W} \equiv \tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2 \oplus \cdots \oplus \tilde{\mathcal{P}}_n = \mathcal{U} \oplus \mathcal{V}. \]  

(14.3)

First note that \( \mathcal{V} \) and \( \mathcal{U} \) have no vector in common since \( \mathcal{V} \subset \mathcal{K} \), and similarly the subspaces \( \tilde{\mathcal{P}}_j \) have no vector in common since \( \tilde{\mathcal{P}}_j \subset \mathcal{P}_j \). Clearly \( \mathcal{W} \) contains \( \mathcal{U} \) since the projections \( \Lambda_j \) sum to the identity. To show it contains \( \mathcal{V} \) note that

\[ \mathcal{V} \subset \Lambda_1 \mathcal{U} \oplus \Lambda_2 \mathcal{U} \oplus \cdots \oplus \Lambda_n \mathcal{U} + \Gamma_0[\Lambda_1 \mathcal{U} \oplus \Lambda_2 \mathcal{U} \oplus \cdots \oplus \Lambda_n \mathcal{U}] \subset \mathcal{W} + \mathcal{U} = \mathcal{W}. \]  

(14.4)

Therefore we have that \( \mathcal{U} \oplus \mathcal{V} \subset \mathcal{W} \). The converse inclusion that \( \mathcal{W} \subset \mathcal{U} \oplus \mathcal{V} \) follows from the inclusion

\[ \tilde{\mathcal{P}}_j = [\Gamma_0 + (I - \Gamma_0)] \Lambda_j \mathcal{U} \subset \mathcal{U} \oplus \mathcal{V}, \]  

(14.5)
which establishes (14.3). Next, to establish that for all $j$,

$$\mathcal{P}_j = \tilde{\mathcal{P}}_j \oplus \mathcal{P}_j', \tag{14.6}$$

we need to assume that for all $j$

$$\tilde{\mathcal{P}}_j \cap \mathcal{K} = 0, \tag{14.7}$$

and that

$$\Lambda_j u = 0, \quad u \in \mathcal{U} \tag{14.8}$$

only has the trivial solution $u = 0$, i.e.

$$\mathcal{U} \cap (\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \ldots \oplus \mathcal{P}_{j-1} \oplus \mathcal{P}_{j+1} \oplus \ldots \oplus \mathcal{P}_n) = 0. \tag{14.9}$$

These conditions imply that

$$\mathcal{U} = \Gamma_0 \Lambda_j \mathcal{U}, \tag{14.10}$$

and hence that

$$\mathcal{U} \subset \Lambda_j \mathcal{U} \oplus (I - \Gamma_0) \Lambda_j \mathcal{U}, \tag{14.11}$$

which in turn implies that

$$\mathcal{U} \subset \tilde{\mathcal{P}}_j + \mathcal{V}. \tag{14.12}$$

Then any vector $\mathbf{P} \in \mathcal{P}_j$ has the unique decomposition

$$\mathbf{P} = u + K, \quad \text{with} \quad u \in \mathcal{U}, \quad K \in \mathcal{K}, \tag{14.13}$$

and according to (14.12), $u$ has the unique decomposition

$$u = v + \tilde{\mathbf{P}} \quad \text{with} \quad v \in \mathcal{V}, \quad \tilde{\mathbf{P}} \in \tilde{\mathcal{P}}_j, \tag{14.14}$$

which is unique because $\mathcal{V} \subset \mathcal{K}$ and $\tilde{\mathcal{P}}_j$ have no non-zero vector in common. Therefore $\mathbf{P}$ has the unique decomposition

$$\mathbf{P} = \tilde{\mathbf{P}} + \mathbf{P}', \tag{14.15}$$

where

$$\mathbf{P}' = v + K = \mathbf{P} - \tilde{\mathbf{P}} \in \mathcal{P}_j \cap \mathcal{K} = \mathcal{P}_j'. \tag{14.16}$$

This decomposition and the fact that (14.7) implies $\tilde{\mathcal{P}}_j$ and $\mathcal{P}_j' \subset \mathcal{K}$ have no vector in common establishes (14.6).

So we deduce that

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2 \oplus \cdots \oplus \tilde{\mathcal{P}}_n \oplus \mathcal{P}_1' \oplus \mathcal{P}_2' \oplus \cdots \oplus \mathcal{P}_n'$$

$$= \mathcal{U} \oplus \mathcal{V} \oplus \mathcal{P}_1' \oplus \mathcal{P}_2' \oplus \cdots \oplus \mathcal{P}_n', \tag{14.17}$$

and since the $\mathcal{P}_j'$, $j = 1, 2, \ldots, n$ are all contained in $\mathcal{K}$ it follows that

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1' \oplus \mathcal{P}_2' \oplus \cdots \oplus \mathcal{P}_n'. \tag{14.18}$$
Now suppose that given \( e \in U \) we can solve the equations

\[
j + J_1 = L(e + E_1), \quad J_1 = -YE_1, \quad e, j \in U, \quad E_1, J_1 \in V,
\]

where \( Y \) is the \( Y \)-operator associated with the subspace collection (14.18). From the \( Y \)-problem we have

\[
E = E_1 + E_2 \in \mathcal{E}, \quad J = J_1 + J_2 \in \mathcal{J}, \quad J_2 = LE_2, \quad E_2, J_2 \in \mathcal{H}', \quad (14.20)
\]

where

\[
\mathcal{H}' = \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_n. \quad (14.21)
\]

Since

\[
j + J_1 + J_2 = L(e + E_1 + E_2), \quad (14.22)
\]

we see that these fields solve the \( Z \)-problem

\[
e, j \in U, \quad E \in \mathcal{E}, \quad J \in \mathcal{J}, \quad j + J = L(e + E),
\]

and by definition \( j = Ze \). To solve (14.19) let \( \Pi_1 \) be the projection onto \( \mathcal{V} \). Then (14.19) implies

\[
-YE_1 = \Pi_1 L(e + \Pi_1 E_1),
\]

giving

\[
E_1 = -\Pi_1 (Y + \Pi_1 L \Pi_1)^{-1} \Pi_1 L e, \quad (14.25)
\]

where the inverse is to be taken on the subspace \( \mathcal{V} \). It follows that

\[
j + J_1 = Le - L \Pi_1 (Y + \Pi_1 L \Pi_1)^{-1} \Pi_1 Le,
\]

implying

\[
Z = \Gamma_0 L \Gamma_0 - \Gamma_0 L \Pi_1 (Y + \Pi_1 L \Pi_1)^{-1} \Pi_1 L \Gamma_0. \quad (14.27)
\]

This formula is analogous to that given in (29.12) of [3].

To obtain a more explicit way of writing (14.27) let us suppose we are given a basis \( u_1, u_2, \ldots, u_m \) of \( U \). Since (14.8) only has the trivial solution \( u = 0 \) each space \( \tilde{\mathcal{P}}_j \) has dimension \( m \). It then follows from (14.3) that \( \mathcal{V} \) has dimension \( m(n - 1) \). Also, for \( i = 1, 2, \ldots, n - 1 \), (14.3) implies \( \Lambda_i u_j \) has the unique decomposition

\[
\Lambda_i u_j = \sum_{k} w_{ijk} u_k + v_{ij}, \quad v_{ij} \in \mathcal{V}, \quad (14.28)
\]

for some set of constants \( w_{ijk} \). To show that the vectors \( v_{ij} \), which number \( m(n - 1) \), are independent, let us suppose

\[
0 = \sum_{i=1}^{n-1} \sum_{j=1}^{m} c_{ij} v_{ij} = \sum_{i=1}^{n-1} \sum_{j=1}^{m} c_{ij} (\Lambda_i u_j - \sum_{k=1}^{m} w_{ijk} u_k). \quad (14.29)
\]
By letting \( \Lambda_n \) act on this equation and taking into account that (14.8) only has the trivial solution \( u = 0 \) we see that
\[
\sum_{i=1}^{n-1} \sum_{j=1}^{m} \sum_{k=1}^{m} c_{ij} w_{ijk} u_k = 0. \tag{14.30}
\]
Then substituting this in (14.29) and letting \( \Lambda_i, i \neq n \), act on (14.29) and again taking into account that (14.8) only has the trivial solution \( u = 0 \) we obtain
\[
\sum_{j=1}^{m} c_{ij} u_j = 0, \tag{14.31}
\]
which shows that all the \( c_{ij} \) must be zero. Therefore let us take the vectors \( v_{ij} \) as our basis for \( \mathcal{V} \).

The identities
\[
\Pi_1 \Lambda_i \Gamma_0 u_j = v_{ij}, \quad \Gamma_0 \Lambda_i \Gamma_0 u_j = \sum_k w_{ijk} u_k, \tag{14.32}
\]
which follow from (14.28) then gives the matrix representations for \( \Pi_1 \Lambda_i \Gamma_0 \) and \( \Gamma_0 \Lambda_i \Gamma_0 \) in these bases, when \( i \neq m \). Using the fact that \( \Lambda_n = I - \sum_{i \neq n} \Lambda_i \) we obtain
\[
\Gamma_0 \Pi_1 = z_n \Gamma_0 + \sum_{i=1}^{n-1} (z_i - z_n) \Gamma_0 \Lambda_i \Gamma_0, \quad \Pi_1 \Gamma_0 \Pi_1 = \sum_{i=1}^{n-1} (z_i - z_n) \Pi_1 \Lambda_i \Gamma_0. \tag{14.33}
\]

Now for \( p \neq n \) (and \( i \neq n \)) (14.28) implies (no sum over \( p \))
\[
\Lambda_p v_{ij} = \sum_k (\delta_{pi} \delta_{kj} - w_{ijk}) \Lambda_p u_k \]
\[
= \sum_k (\delta_{pi} \delta_{kj} - w_{ijk})(v_{pk} + \sum_q w_{pkq} u_q). \tag{14.34}
\]
Thus we deduce
\[
\Gamma_0 \Lambda_p \Pi_1 v_{ij} = \sum_k (\delta_{pi} \delta_{kj} - w_{ijk}) \sum_q w_{pkq} u_q, \]
\[
\Pi_1 \Lambda_p \Pi_1 v_{ij} = \sum_k (\delta_{pi} \delta_{kj} - w_{ijk}) v_{pk}, \tag{14.35}
\]
which gives the matrix representation for the operators \( \Gamma_0 \Lambda_p \Pi_1 \) and \( \Pi_1 \Lambda_p \Pi_1 \) in these bases \( (p \neq n) \), in terms of which we obtain the representation for the operators
\[
\Gamma_0 \Pi_1 = \sum_{p=1}^{n-1} (z_p - z_n) \Gamma_0 \Lambda_p \Pi_1, \quad \Pi_1 \Gamma_0 \Pi_1 = z_n \Pi_1 + \sum_{p=1}^{n-1} (z_p - z_n) \Pi_1 \Lambda_p \Pi_1. \tag{14.36}
\]
Thus all the matrices representing the operators entering (14.27), aside from $Y$, only depend on the parameters $w_{ijk}$ and these parameters can be obtained from the representation in the basis $u_1, u_2, \ldots, u_m$ of $Z$ when the differences $z_i - z_n$, $i = 1, 2, \ldots, n - 1$ are small. To first order in these differences, (14.27), (14.33), and (14.36) imply

$$ Zu_j \approx z_n u_j + \sum_{i=1}^{n-1} (z_i - z_n) \sum_k w_{ijk} u_k. \quad (14.37) $$

Thus knowing this expansion one can recover all the parameters $w_{ijk}$.

The idea to developing the continued fraction is that by a succession of reduction and normalization operations one obtains a series of identities

$$ Z = \Gamma_0 L \Gamma_0 - \Gamma_0 L \Pi_1 (Y + \Pi_1 L \Pi_1)^{-1} \Pi_1 L \Gamma_0, \quad (14.38) $$

$$ Y = M^{(1)} Z^{(1)} K^{(1)}, \quad (14.39) $$

$$ Z^{(1)} = \Gamma_0^{(1)} L^{(1)} \Gamma_0^{(1)} - \Gamma_0^{(1)} L^{(1)} \Pi_1^{(1)} (Y^{(1)} + \Pi_1^{(1)} L^{(1)} \Pi_1^{(1)})^{-1} \Pi_1^{(1)} L^{(1)} \Gamma_0^{(1)}, \quad (14.40) $$

$$ Y^{(1)} = M^{(2)} Z^{(2)} K^{(2)}, \quad (14.41) $$

$$ Z^{(2)} = \Gamma_0^{(2)} L^{(2)} \Gamma_0^{(2)} - \Gamma_0^{(2)} L^{(2)} \Pi_1^{(2)} (Y^{(2)} + \Pi_1^{(2)} L^{(2)} \Pi_1^{(2)})^{-1} \Pi_1^{(2)} L^{(2)} \Gamma_0^{(2)}, \quad (14.42) $$

and so forth, until the dimension of the remaining space goes to zero, or until one (or more) of the assumptions necessary to proceed with the normalization or reduction operation does not hold. By substituting (14.39) in (14.38), then substituting (14.40) in the resulting expression, and subsequently substituting (14.41) in this expression, and so on, one develops the continued fraction expansion. We do not address in this paper how to go ahead with the continued fraction expansion when the assumptions made to proceed with the normalization or reduction operation do not hold.

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