Nonlocal damping consideration for the computer modelling of linear and nonlinear systems vibrations under the stochastic loads

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Abstract. The paper is devoted to the analysis of the nonlocal damping consideration influence on the computer modelling results of linear and nonlinear systems’ vibrations subjected to stochastic stationary loads. The shallow arc dynamic behavior is examined. The problem is solved using the Galerkin method.

1. Introduction
The engineering structures are subjected to various external dynamic loads such as seismic loads, wind loads, transport loads, pedestrian loads, etc. It is important to take the vibration energy dissipation into account for the design process. The precision of calculations is highly correlated to the damping model choice especially for the structures that are made of modern composite materials. Damping models that better represent the properties of the dynamic system allow decreasing the risks for the engineering structures.

Damping in the certain point of the structure with longitudinal coordinate x1, obviously depends not only on local value of motion velocity at this point v(x1), but also on the values of motion velocity in the neighboring points. In the case when it is more distance between the two points the lower influence one of them has on the other [2].

The modern structures along with linear elements often contain geometrically nonlinear elements. Moreover, under external impacts the certain linear elements may change the shape and acquire some nonlinear features. In this research the problem of linear and nonlinear systems forced vibrations modeling with nonlocal damping consideration is solved. Nonlocal damping model is assumed considering the time hysteresis effect that was proposed in paper [2].

The dynamic process of the linear and nonlinear systems is modeled under the stochastic stationary load.

2. Nonlocal damping model for the linear system
The Kelvin-Voigt material model is commonly used in engineering practice to describe the damping process:

\[ \sigma = E\varepsilon + \gamma \dot{\varepsilon}, \]  

where \( \sigma \), \( \varepsilon \) - normal stress and axial strain, \( \dot{\varepsilon} \) – rate of strain change, \( E \) – Young modulus, \( \gamma \) – damping ratio. Dot indicates the time derivative in the paper.

Consider nonlocal damping equation (1) transforms to [2]:
\[ \sigma(x,t) = E \left[ \epsilon(x,t) + \gamma \int_0^t C_v(|x - \theta|) \dot{\epsilon}(\theta,t) d\theta \right]. \] (2)

Here \( C_v(|x - \theta|) \) is the kernel function, which characterize the nonlocal. The \( C_v(|x - \theta|) \) function responds to normalization requirement, which is:

\[ \int_{-\infty}^{\infty} C_v(|x - \theta|) d\theta = 1. \] (3)

Exponential kernel function is used in the paper:

\[ C_v(|x - \theta|) = \frac{\mu}{2} e^{-|x-\theta|}. \] (4)

Here \( \mu \) is the parameter that characterize the influence distance in the nonlocal damping model, \( x \) are the longitudinal coordinate.

As an example of a linear system consider the rod 10 m long with bending stiffness \( 9.82 \cdot 10^5 \) t·m². Both ends of the rod are fixed.

The equilibrium equation for an elementary part of the rod is:

\[ \frac{\partial^2 M(x,t)}{\partial x^2} = m \frac{\partial^2 w(x,t)}{\partial t^2} - q(x,t), \] (5)

where \( w(x,t) \) – rod deflection, \( m \) – distributed mass, \( q(x,t) \) – distributed load.

Considering the plane sections assumption the bending moment expression is:

\[ M(x,t) = -EI \left[ \frac{\partial^3 w(x,t)}{\partial x^2} + \gamma \int_0^t C_v(|x - \theta|) \frac{\partial^3 w(\theta,t)}{\partial \theta^2 \partial t} d\theta \right] \] (6)

Here \( EI \) is rod bending stiffness.

Substitute the second derivative of the moment expression to the left part of the equation (5), and obtain the expression regarding the deflection function \( w(\theta,t) \):

\[ \frac{\partial^2 w(x,t)}{\partial t^2} + \frac{EI}{m} \left[ \frac{\partial^4 w(x,t)}{\partial x^4} + \gamma \int_0^t C_v(|x - \theta|) \frac{\partial^3 w(\theta,t)}{\partial \theta^2 \partial t} d\theta \right] = \frac{q(x,t)}{m}. \] (7)

Solution of this equation should satisfy the boundary conditions for \( x=0 \) and \( x=l \).

Function \( w(x,t) \) is searched as normal modes of arc vibrations:

\[ w(x,t) = \sum_{i=1}^{n} f_i(t)V_i(x). \] (8)

Here \( f_i(t) \) – generalized displacements, \( V_i(x) \) – basis functions. For the rod with fixed ends the basis functions are [8]:

\[ V_i(x) = (shk_l - \sin k lx)(ch_k x - \cos k lx) - (chk_l - \cos k lx)(shk_l x - \sin k lx). \] (9)

\( k_i \) is calculated as the i-root of the characteristic equation \( ch(kx) \cos(kx) = 1 \).

To obtain the generalized displacements \( f_i(\tau) \) the Galerkin method is used. The resulting system of the integral-differential equations is:

\[ \ddot{f}_j(\tau) + \left( \frac{k_j}{\tau} \right)^4 f_j(\tau) + \frac{2\epsilon}{a_j k_j^4 t^4} \sum_{i=1}^{n} \int_0^t V_j^*(z) \int_0^t \frac{1}{2} e^{-|\tau|} \bar{V}_i(y) dy dz f_j(\tau) \]

\[ = \frac{I^4}{k_j^4 a_j} \int_0^t q(y,\tau) V_j(y) dy, \] (10)
where \( k_j = m \omega_j^2 / EI \), \( \omega_j \) - natural frequency, \( a_j = \int_0^l V_j^2(x)dx \), \( z \) and \( y \) - dimensionless coordinates

\[-z = x/l, \ y = \theta/l, \ \tau - \text{dimensionless time} \ \tau = \omega_l t, \ \omega_l^2 = \frac{EI_k^4}{l m}, \ 2 \varepsilon = \gamma \omega_l, \ \omega_l \ - \text{minimum natural frequency}.

The \( f_i(\tau) \) values are obtained by solving the equation system, and arc deflections - using expression (8).

3. Linear system vibration under the stochastic load

3.1. The random stationary load

Let’s consider the influence of nonlocal damping on the rod vibration process under the stochastic distributed load. The load is represented as the random stationary process with the zero mathematical expectation and spectral density:

\[
S(\omega) = \frac{2 \sigma^2 \cdot \delta (\Delta^2 + \theta^2)}{\pi \left[ \omega^2 - \omega^2 - \delta^2 (\Delta^2 + \theta^2)^2 + 4 \delta^2 \cdot \omega^2 \right]}. \tag{11}
\]

Here \( \sigma^2 \) is the variance of random process, \( \delta \) and \( \theta \) parameters, that characterize the correlation scale and latent frequency of the stochastic load.

The method of the canonic expansion [4] is used for modeling the random stationary process. For this purpose the random is presented as:

\[
q(t) = \sum_{k=1}^{n} (U_k \cos \omega_k t + V_k \cos \omega_k t). \tag{12}
\]

Here \( U_k, V_k \) – uncorrelated normally distributed random values with the zero mathematical expectation and equal variances for each pair of random values with equal indices \( k \). To calculate these variances the \( \omega \)-axis section \( 2L \) long is selected. The origin is in the middle of the selected section. According to \( |\omega| > L \), the spectral density could be assumed zero. The whole selected section is divided by the equal parts \( \Delta \omega \) long. Then the variance of the random values \( U_k, V_k \) is calculated by the formula:

\[
D_k = 2S(\omega_k) \cdot \Delta \omega. \tag{13}
\]

The random stationary load is Gaussian and permanent along the rod in each moment. The correlation function for the load was obtained using 400 realizations of the random process is presented in Figure 1. The solid line is for the theoretical correlation function which was calculated by the formula:

\[
K(\tau_1 - \tau_2) = \sigma^2 e^{-\delta |\tau_1 - \tau_2|} \left[ \cos \theta(\tau_1 - \tau_2) + \theta \sin \theta(\tau_1 - \tau_2) \right]. \tag{14}
\]

The dash line is corresponded to the correlation function that was obtained using the numerical modeling data.

3.2. Modelling of linear system random vibrations

The rod dynamics under the load was described above is a random function. Ten realizations of this function are presented in Figure 2. The solid line shows the mathematical expectation of the random process.

The nonlocal damping in the numerical model is determined by the influence distance (\( \mu \) value). The higher \( \mu \), the closer the numerical model to the Kelvin-Voight damping. The results of the modeling for different \( \mu \) values are presented as histograms (Figure 3). For both cases the mathematical expectation is zero and variances are \( 0.0013 \ m^2 \) for \( \mu=0.2 \ l/m \), and \( 0.00056 \ m^2 \) – for \( \mu=1 \ l/m \).
Figure 1. The correlation function for the random stationary load.

Figure 2. Ten realizations of the rod dynamics random process for $\delta l=0.05$ и $\theta l=0.25$.

Figure 3. Rod displacements histograms.

When the model is closer to the Kelvin-Voight damping (higher $\mu$ values) the maximum displacements decrease (Figure 3). It is also obvious from the confident intervals which are presented in Table 1.
4. Nonlinear system vibrations under the stochastic load

The nonlinear system was studied using the shallow arc as an example. The arc has the following parameters: span length 5 m, radius of curvature 10 m, stiffness 37.5 t/m². Both ends are fixed. For such structure the system of equations (10) will transform to:

\[
\frac{N}{A_o} \int_0^1 \frac{1}{l^2} f_j(z) dz = \frac{1}{l^4} \int_0^1 q(y, \tau) V_j(y) dy dz.
\]

(15)

Here \( N \) – longitudinal internal force.

Neglecting the longitudinal vibrations of arc the longitudinal force could be determined as:

\[
N(t, x) = EA \varepsilon^0(x, t),
\]

(16)

where \( A \) – arc cross section area, \( \varepsilon^0(x, t) \) – longitudinal axial strain, which is determined as:

\[
\varepsilon^0(x, t) = \frac{\partial u(t, x)}{\partial x} + \frac{1}{2} \left( \frac{\partial w(t, x)}{\partial x} \right)^2 - \frac{w(t, x)}{R}.
\]

(17)

Where \( u(t, x) \) – arc axis displacement

In order to compare the local and nonlocal model the arc vibrations modeling results are presented as histograms (Figure 4).

![Figure 4. Shallow arc deflection distribution distribution under the stochastic load.](image)

Shallow arc dynamics has specific features. Under the certain loads and geometry the shallow arc deflection leaps. Hence the resulting histogram is bimodal. To analyze the obtained data the histogram was stratified (Figure 5).
Figure 5. Stratification of the shallow arc deflection histogram (a) left part, (b) right part.

The both parts were assumed normal. This assumption was verified by Pearson fitting criterion. For the significance level 0.05 for the left part $\chi^2 = 5.02 < \chi^2_{cr} = 7.81$, and for the right part $\chi^2 = 4.98 < \chi^2_{cr} = 9.49$, hence the both parts could be treated like if they were normally distributed.

The histograms were obtained to compare the results for the local and nonlocal model. The arc deflection amplitude is less when the local model is used (Figure 6).

Figure 6. Histograms of arc deflections for the stochastic stationary load.

The confidence intervals are presented in Table 2.
Table 2. The shallow arc deflection confident intervals.

| Confidence interval | Intervals’ limits, m |
|---------------------|----------------------|
|                     | Nonlocal model       | Local model        |
| 68.27% (from $-\sigma$ to $\sigma$) | from 0.3609 to 0.8901 | from $-0.3656$ to $0.0342$ | from 0.4741 to 0.8083 |
| 95.45% (from $-2\sigma$ to $2\sigma$) | from $-0.5270$ to 1.0928 | from $-0.5312$ to 0.1314 | from 0.3069 to 0.9755 |
| 99.73% (from $-3\sigma$ to $3\sigma$) | from $-0.6931$ to 1.2955 | from $-0.6969$ to 0.2971 | from 0.1398 to 1.1427 |

5. Discussion
The nonlocal damping consideration influence on the results of linear and nonlinear systems computer modeling subjected to stochastic stationary loads is studied in this paper. The computer models of vibration process considering nonlocal damping for the rod and shallow arc are developed. In comparison to local model computer modelling with consideration of nonlocal damping gives an increase of deflection both for linear and nonlinear systems.

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