Unitarily invariant valuations on convex functions

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Continuous, dually epi-translation invariant valuations on the space of finite-valued convex functions on \( \mathbb{C}^n \) that are invariant under the unitary group are investigated. It is shown that elements belonging to the dense subspace of smooth valuations admit a unique integral representation in terms of two families of Monge-Ampère-type operators. In addition, it is proved that homogeneous valuations are uniquely determined by restrictions to subspaces of appropriate dimension and that this information is encoded in the Fourier-Laplace transform of the associated Goodey-Weil distributions. These results are then used to show that a continuous unitarily invariant valuation is uniquely determined by its restriction to a certain finite family of subspaces of \( \mathbb{C}^n \).

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1 Introduction

One of the most important classical results in integral geometry is Hadwiger’s characterization of the space $\text{Val}(\mathbb{R}^n)^{SO(n)}$ of all continuous, rigid motion invariant valuations on the space $\mathcal{K}(\mathbb{R}^n)$ of all nonempty, convex, compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric. Here a map $\mu : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ is called a valuation if

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

for all $K, L \in \mathcal{K}(\mathbb{R}^n)$ such that $K \cup L \in \mathcal{K}(\mathbb{R}^n)$. Hadwiger showed that $\text{Val}(\mathbb{R}^n)^{SO(n)}$ is finite dimensional and a basis of this space is given by the intrinsic volumes $[32]$. This result directly implies many classical integral geometric formulas, including kinematic formulas as well as Crofton and Cauchy-Kubota formulas. More recently, Alesker proved that Hadwiger-type theorems hold not only for rigid motion invariant valuations, but also for smaller subgroups of the affine group:

**Theorem 1.1** (Alesker [1] Theorem 8.1). Let $G \subset SO(n)$ be a compact subgroup. The space $\text{Val}(\mathbb{R}^n)^G$ of all continuous, translation and $G$-invariant valuations on $\mathcal{K}(\mathbb{R}^n)$ is finite dimensional if and only if $G$ operates transitively on the unit sphere.

His results [2] led to rapid advances in the integral geometry of these transitive compact groups $[5, 6, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]$, in particular for complex space forms [17], as well as non-compact groups [0, 11, 12]. Alesker [3] was the first to give explicit bases for $\text{Val}(\mathbb{C}^n)^{U(n)}$. For their study of kinematic formulas for $\text{Val}(\mathbb{C}^n)^{U(n)}$, Bernig and Fu [13] introduced an additional basis, the so-called Hermitian intrinsic volumes, which are characterized by their restriction to lower dimensional subspaces in $\mathbb{C}^n$. Before we state their result, recall that a valuation $\mu$ on $\mathcal{K}(\mathbb{C}^n)$ is called $k$-homogeneous if $\mu(tK) = t^k \mu(K)$ for all $t \geq 0$, $K \in \mathcal{K}(\mathbb{C}^n)$. Let us also denote by $E_{k,q} \in \text{Gr}_k(\mathbb{C}^n)$ the subspace $E_{k,q} = \mathbb{C}^q \times \mathbb{R}^{k-2q}$, where $\text{Gr}_k(\mathbb{C}^n)$ denotes the space of $k$-dimensional real subspaces of $\mathbb{C}^n$.

**Theorem 1.2** (Bernig-Fu [13] Theorem 3.2). For $0 \leq k \leq 2n$, $\max(0,k-n) \leq q \leq \left[ \frac{k}{2} \right]$ there exists a unique $k$-homogeneous valuation $\mu_{k,q} \in \text{Val}(\mathbb{C}^n)^{U(n)}$ such that

$$\mu_{k,q}|_{E_{k,p}} = \delta_{pq} \text{vol}_k,$$

where $\text{vol}_k$ denotes the $k$-dimensional Lebesgue measure on $E_{k,p}$. Moreover, these valuations form a basis for $\text{Val}(\mathbb{C}^n)^{U(n)}$.

Let us add some more context to this description. By a result due to Hadwiger [32], the restriction of any $k$-homogeneous, translation invariant, continuous valuation on $\mathcal{K}(\mathbb{R}^n)$ to a $k$-dimensional subspace $E \subset \mathbb{R}^n$ is a multiple of the $k$-dimensional Lebesgue measure, that is, $\mu|_E = Kl_\mu(E) \cdot \text{vol}_k$ for some $Kl_\mu(E) \in \mathbb{R}$. The function $Kl_\mu : \text{Gr}_k(\mathbb{R}^n) \to \mathbb{R}$ is called the Klain function of $\mu$, and due to a result by Klain [35] it determines $\mu$ uniquely if $\mu$ is an even valuation, that is, if it satisfies $\mu(-K) = \mu(K)$ for all $K \in \mathcal{K}(\mathbb{R}^n)$. This applies in particular to any $k$-homogeneous valuation $\mu \in \text{Val}(\mathbb{C}^n)^{U(n)}$, which is thus uniquely determined by the $U(n)$-invariant function $Kl_\mu$ on $\text{Gr}_k(\mathbb{C}^n)$. By associating to every subspace $E \in \text{Gr}_k(\mathbb{C}^n)$ a quantity called the multiple Kähler angle, Tasaki [18] showed that the orbits of $U(n)$ on $\text{Gr}_k(\mathbb{C}^n)$ are in 1-to-1 correspondence with the $\left[ \frac{\min(k,2n-k)}{2} \right]$-dimensional simplex. The orbits containing the spaces $E_{k,q}$ correspond exactly to the vertices of this simplex under this identification.
thus implies that any \( k \)-homogeneous valuation \( \mu \in \text{Val}(\mathbb{C}^n)^{U(n)} \) is uniquely determined by its restriction to these extremal orbits. The goal of this article is a functional version of this result.

The notion of valuation generalizes to the functional setting as follows. If \( \mathcal{F} \) is a family of (extended) real-valued functions, then we call \( \mu : \mathcal{F} \to \mathbb{R} \) a valuation if

\[
\mu(f) + \mu(h) = \mu(f \vee h) + \mu(f \wedge h)
\]

whenever the pointwise maximum \( f \vee h \) and pointwise minimum \( f \wedge h \) belong to \( \mathcal{F} \). If \( \mathcal{F} \) denotes the family of indicator functions associated to convex bodies, this definition recovers the classical notion of valuations on sets. In recent years, valuations on many different classical spaces have been the focus of intense research, for example Sobolev and \( L^p \) spaces [10, 11, 12, 13, 16, 52, 53], functions of bounded variation [55], Lipschitz functions [23, 29], as well as definable functions [7] and general Banach lattices [49]. Due to their intimate relation to convex bodies, valuations on spaces related to convexity [17, 18, 19, 20, 38] and, more specifically, spaces of convex functions [5, 21, 22, 23, 24, 25, 26, 27, 33, 36, 37, 45] have been one of the most active areas of research in modern valuation theory. Let us introduce some notation. For a finite-dimensional real vector space \( V \), let \( \text{Conv}(V, \mathbb{R}) \) denote the space of finite-valued convex functions on \( V \). We consider this space with the topology of uniform convergence on compact subsets and denote by \( \text{VConv}(V, \mathbb{R}) \) the space of all continuous valuations on \( \text{Conv}(V, \mathbb{R}) \) that are in addition \textit{dually epi-translation invariant}, that is, that satisfy

\[
\mu(f + l) = \mu(f) \quad \text{for all } f \in \text{Conv}(V, \mathbb{R}), \ l : V \to \mathbb{R} \text{ affine.}
\]

This space of functionals is intimately related to continuous translation invariant valuations on convex bodies (see [5, 37]), and this connection has led to the discovery of many structural results for valuations of this type. For example, similar to a classical result by McMullen [41], there exists a homogeneous decomposition as shown by Colesanti, Ludwig and Mussnig [24]: If we denote by \( \text{VConv}_k(V) \) the subspace of \( \text{VConv}(V) \) consisting of \( k \)-homogeneous valuations, that is, all valuations \( \mu \in \text{VConv}(V) \) such that \( \mu(tf) = t^k \mu(f) \) for all \( t \geq 0, \ f \in \text{Conv}(V, \mathbb{R}) \), then

\[
\text{VConv}(V) = \bigoplus_{k=0}^{\dim V} \text{VConv}_k(V).
\]

Note that \( \text{VConv}_0(V) \) is 1-dimensional and given by constant valuations. In a series of articles [22, 25, 26, 27] Colesanti, Ludwig and Mussnig also obtained a version of Hadwiger’s characterization result for the space \( \text{VConv}(\mathbb{R}^n)^{SO(n)} \) of all rotation invariant valuations in \( \text{VConv}(\mathbb{R}^n) \). Their classification result may be summarized as follows: There is precisely one family of invariant valuations for each degree of homogeneity, and any such valuation admits a singular integral representation with respect to the so-called \textit{Hessian measure} of the appropriate degree. Following the analogy with Hadwiger’s classification of all continuous and rigid motion invariant valuations on convex bodies, these valuations are called \textit{functional intrinsic volumes}.

Let us remark that the Hessian measures are related to the real Monge-Ampère operator and assign to any convex function a non-negative measure on \( \mathbb{R}^n \). As such, they play an important role in the study of so-called Hessian equations, a class of fully non-linear partial differential equations introduced by Trudinger and Wang [40, 51]. For the role of Monge-Ampère-type
operators in the construction of valuations on convex functions and convex bodies we refer to [23] and [4, 5].

This is the first in a series of articles on the classification of the space \( \text{VConv}(\mathbb{C}^n)^{U(n)} \) of all \( U(n) \)-invariant valuations in \( \text{VConv}(\mathbb{C}^n) \), that is, all valuations \( \mu \in \text{VConv}(\mathbb{C}^n) \) such that

\[
\mu(f \circ g) = \mu(f) \quad \text{for all} \quad f \in \text{Conv}(\mathbb{C}^n, \mathbb{R}), g \in U(n).
\]

As \( U(1) = \text{SO}(2) \), the case \( n = 1 \) is already covered by the results by Colesanti, Ludwig and Mussnig, so we will assume that \( n \geq 2 \) throughout this article without explicitly stating this restriction in the results. Obviously the homogeneous decomposition is compatible with the action of \( U(n) \), so

\[
\text{VConv}(\mathbb{C}^n)^{U(n)} = \bigoplus_{k=0}^{2n} \text{VConv}_k(\mathbb{C}^n)^{U(n)}
\]

decomposes into its homogeneous components. In this article we are going to work towards a further decomposition of this space, which is inspired by the behavior of the Hermitian intrinsic volumes under restrictions to lower dimensional subspaces.

First, for a linear map \( T : V \to W \) into a finite-dimensional vector space \( W \), we may define the pushforward \( T^* : \text{VConv}(V) \to \text{VConv}(W) \) by assigning to \( \mu \in \text{VConv}(V) \) the valuation \( T^* \mu \in \text{VConv}(W) \) given by

\[
[T^* \mu](f) := \mu(T^* f) \quad \text{for} \quad f \in \text{Conv}(W, \mathbb{R}).
\]

If \( V = \mathbb{C}^n \) and \( E \subset \mathbb{C}^n \) is a subspace, we may in particular consider the pushforward along the orthogonal projection \( \pi_E : \mathbb{C}^n \to E \), which we call the restriction of a valuation to the subspace \( E \). The first result of this article shows that \( U(n) \)-invariant valuations in \( \text{VConv}(\mathbb{C}^n) \) are uniquely determined by the restrictions to the extremal subspaces \( E_{k,q} \).

**Theorem 1.** A valuation \( \mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)} \) satisfies \( \mu \equiv 0 \) if and only if \( \pi_{E_{k,q}} \mu = 0 \) for all \( \max(0,k-n) \leq q \leq \lfloor \frac{k}{2} \rfloor \).

Let us equip \( \text{VConv}(\mathbb{C}^n) \) with the topology of uniform convergence on compact subsets in \( \text{Conv}(\mathbb{C}^n, \mathbb{R}) \) (see [37], Proposition 2.4 for a description of these subsets), and consider for \( \max(0,k-n) \leq q \leq \lfloor \frac{k}{2} \rfloor \) the closed subspace

\[
\text{VConv}_{k,q}(\mathbb{C}^n)^{U(n)} := \{ \mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)} : \pi_{E_{k,p}} \mu = 0 \text{ for all } p \neq q \}.
\]

Following the analogy with valuations on convex bodies, one may consider valuations in these spaces as functional analogs to the Hermitian intrinsic volumes. Note that Theorem 1 implies that the sum

\[
\bigoplus_{q=\max(0,k-n)}^{\lfloor \frac{k}{2} \rfloor} \text{VConv}_{k,q}(\mathbb{C}^n)^{U(n)} \subset \text{VConv}_k(\mathbb{C}^n)^{U(n)}
\]  
(1)

is direct for all \( 0 \leq k \leq 2n \) and coincides with \( \text{VConv}_k(\mathbb{C}^n)^{U(n)} \) for \( k = 0, 1, 2n - 1, 2n \). It is natural to ask whether this inclusion is actually an equality for the intermediate degrees.
as well, and we will address this question in a future article. The main goal of this article is to show that one may obtain Theorem [1] and consequently the decomposition in (1) from a corresponding decomposition for the subspace of smooth valuations, which were introduced by the author in [36] and which form a sequentially dense subspace of all dually epi-translation invariant valuations by the main result of the same article. Here a valuation \( \mu \in \text{VConv}_k(\mathbb{C}^n) \) is called smooth if admits a representation of the form

\[
\mu(f) = D(f)[\tau] \quad \text{for all } f \in \text{Conv}(\mathbb{C}^n, \mathbb{R}),
\]

where \( D(f) \) denotes the differential cycle of \( f \) introduced by Fu in [30], which is an integral current on the cotangent bundle \( T^*\mathbb{C}^n \), and \( \tau \in \Omega^{n-k}_c(\mathbb{C}^n) \otimes \Lambda^k\mathbb{C}^n \subset \Omega^n(T^*\mathbb{C}^n) \) is a differential form with bounded support in the first coordinate of \( T^*\mathbb{C}^n \cong \mathbb{C}^n \times \mathbb{C}^n \). The differential cycle may be considered as the natural extension of the graph of the differential of a convex function, considered as an oriented submanifold of the cotangent bundle \( T^*\mathbb{C}^n \). In particular, \( D(f) \) is given by integration over the graph of \( df \) if \( f \) is a smooth function. We will denote the subspace of smooth valuations by \( \text{VConv}_k(\mathbb{C}^n)^{\text{sm}} \).

To describe smooth unitarily invariant valuations, we introduce two families of invariant differential forms on \( T^*\mathbb{C}^n \) for \( 0 \leq k \leq 2n \) in Section 3.4:

\[
\Theta^n_{k,q} \quad \text{for } \max(0, k - n) \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor, \text{ and}
\]

\[
\Upsilon^n_{k,q} \quad \text{for } \max(1, k - n) \leq q \leq \left\lfloor \frac{k - 1}{2} \right\rfloor.
\]

To avoid unnecessary distinctions, we set \( \Theta^n_{k,q} = 0, \Upsilon^n_{k,q} = 0 \) if \( k \) and \( q \) do not meet these conditions. The next result shows that the decomposition (1) holds for the space of smooth unitarily invariant valuations and provides an integral representation of these functionals in terms of the differential forms \( \Theta^n_{k,q} \) and \( \Upsilon^n_{k,q} \).

**Theorem 2.** For \( 1 \leq k \leq 2n \),

\[
\text{VConv}_k(\mathbb{C}^n)^{U(n)} \cap \text{VConv}_k(\mathbb{C}^n)^{\text{sm}} = \bigoplus_{q=\max(0,k-n)}^{\left\lfloor \frac{k}{2} \right\rfloor} \text{VConv}^k_{k,q}(\mathbb{C}^n)^{U(n)} \cap \text{VConv}_k(\mathbb{C}^n)^{\text{sm}}.
\]

Moreover, for every \( \max(0, k - n) \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor \) and \( \mu \in \text{VConv}^k_{k,q}(\mathbb{C}^n)^{U(n)} \cap \text{VConv}_k(\mathbb{C}^n)^{\text{sm}} \) there exist unique \( \phi_q, \psi_q \in C_c^\infty([0, \infty)) \) such that

\[
\mu(f) = D(f)\left[\phi_q(|z|^2)\Theta^n_{k,q}\right] + D(f)\left[\psi_q(|z|^2)\Upsilon^n_{k,q}\right],
\]

where we set \( \psi_q = 0 \) if \( q = 0 \) or \( q = \left\lfloor \frac{k}{2} \right\rfloor \).

The differential forms \( \Theta^n_{k,q} \) are constant and the measures defined by \( \phi \mapsto D(f)[\phi(z)\Theta^n_{k,q}] \) are related to certain complex Monge-Ampère-type operators. In this sense, this family of differential forms may be considered as a Hermitian analog of the Hessian measures. At the present time, we lack a satisfying geometric interpretation for the valuations constructed from the differential forms \( \Upsilon^n_{k,q} \). We have chosen this form such that all of the valuations \( f \mapsto D(f)[\phi(|z|^2)\Upsilon^n_{k,q}] \) vanish identically on rotation invariant functions independent of the
choice of $\phi \in C_c^\infty([0, \infty))$, although we will not prove this in this article.

Note that Theorem 2 implies that the sum of the spaces $V_{Conv_k,q}(\mathbb{C}^n)^{U(n)}$ is dense in $V_{Conv_k}(\mathbb{C}^n)$. However, it is not sufficient to establish Theorem 1 as it is not clear whether smooth valuations form a dense subspace of the pairwise intersection of two of the spaces $V_{Conv_k,q}(\mathbb{C}^n)^{U(n)}$.

To circumvent this problem, we introduce a tool that will play a role similar to the Klain decomposition of $V_{Conv}(\mathbb{R}^n)$: First note that the homogeneous smooth valuations form a dense subspace of the pairwise intersection of two of the spaces $V_{Conv_k,q}(\mathbb{C}^n)^{U(n)}$.

In particular, we obtain an explicit representation for $\bar{\mu}$ in $\mathbb{R}^n$.

Essentially, $\bar{\mu}$ is a multilinear functional on $\text{Conv}(\mathbb{R}^n, \mathbb{R})$ that uniquely determines $\mu$. In [37] it was shown that the polarization $\bar{\mu}$ lifts to a distribution on $(\mathbb{R}^n)^k$ for any $\mu \in V_{Conv_k}(\mathbb{R}^n)$.

More precisely, there exists a unique symmetric distribution $GW(\mu)$ on $(\mathbb{R}^n)^k$ with compact support such that

$$GW(\mu)[f_1 \otimes \cdots \otimes f_k] = \bar{\mu}(f_1, \ldots, f_k) \quad \text{for } f_1, \ldots, f_k \in \text{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n).$$

This distribution is called the Goodey-Weil distribution of $\mu$ and uniquely determines $\mu$. Thus the same applies to the Fourier-Laplace transform of $GW(\mu)$. As the support of $GW(\mu)$ is compact, its Fourier-Laplace transform is an entire function on $(\mathbb{C}^n)^k$ and given by

$$\mathcal{F}(GW(\mu))[z_1, \ldots, z_k] = GW(\mu)[\exp(i\langle z_1, \cdot \rangle) \otimes \cdots \otimes \exp(i\langle z_k, \cdot \rangle)] \quad \text{for } z_1, \ldots, z_k \in \mathbb{C}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the $\mathbb{C}$-linear extension of the standard scalar product on $\mathbb{R}^n$. Note that for $z_1, \ldots, z_k \in \mathbb{R}^n$ the value of $\mathcal{F}(GW(\mu))[z_1, \ldots, z_k]$ is essentially given by evaluating $\mu$ on functions defined on the space spanned by $z_1, \ldots, z_k$. In particular, $\mathcal{F}(GW(\mu))$ encodes all the information we can obtain by restricting $\mu$ to $k$-dimensional subspaces. As a direct consequence, we obtain that $\mu \in V_{Conv_k}(\mathbb{R}^n)$ is uniquely determined by these restrictions.

**Theorem 3.** If $\mu \in V_{Conv_k}(\mathbb{R}^n)$ satisfies $\pi_E \ast \mu = 0$ for all $k$-dimensional subspaces $E \subset \mathbb{R}^n$, then $\mu \equiv 0$.

Let us remark that a version of this result was also obtained by Colesanti, Ludwig and Mussnig in [27] using a very different approach. As a simple application, we will show that a valuation vanishes if it vanishes on convex polynomial functions, see Corollary 2.4. We also obtain certain restrictions on the support of a valuation, see Section 2.2.

Let us return to the case of unitarily invariant valuations. Theorem 1 is equivalent to showing that $\mu \in V_{Conv_k}(\mathbb{C}^n)^{U(n)}$ vanishes identically if the restriction of the function $\mathcal{F}(GW(\mu))$ to the space $(E_{k,q})^k$ vanishes for all $\max(0, k - n) \leq q \leq \lfloor \frac{k}{2} \rfloor$. We will show the much stronger result that one can explicitly reconstruct $\mathcal{F}(GW(\mu))$ from the restriction of $\mu$ to the spaces $E_{k,q}$, $\max(0, k - n) \leq q \leq \lfloor \frac{k}{2} \rfloor$.

In particular, we obtain an explicit representation for $\mathcal{F}(GW(\mu))$ for $\mu \in V_{Conv_k}(\mathbb{C}^n)^{U(n)}$. In Section 4.1 and Section 4.2 we will introduce a family $P_{k,q}$ of real polynomials on $(\mathbb{C}^n)^k$ as
Theorem 4. For every \( \mu \) of the space spanned by their arguments. The symbols of the differential operators see Proposition 4.5. In particular, they essentially only depend on the multiple Kähler angle of the space spanned by their arguments. The symbols of the differential operators \( D^k \) can similarly be described in terms of the multiple Kähler angle, although the relation is more complicated. We extend both of these families by \( C \)-linearity to \( \mathbb{C} \)-linearity to \( w_1, \ldots, w_k \in \mathbb{C}^n \otimes \mathbb{R} \mathbb{C} \).

Our last main result provides a decomposition of \( F(W(\mu)) \) for arbitrary \( \mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)} \) in terms of the polynomials \( P_{k,q} \) and differential operators \( D^k \) and \( D^k \). Let \( \langle \cdot, \cdot \rangle \) denote the \( \mathbb{C} \)-linear extension of the Hermitian scalar product on \( \mathbb{C}^n \) to \( \mathbb{C}^n \otimes \mathbb{R} \mathbb{C} \).

**Theorem 4.** For every \( \mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)} \) there exist entire functions

\[
\Phi_q(\mu) : \mathbb{C} \rightarrow \mathbb{C} \quad \text{for max}(0, k - n) \leq q \leq \left| \frac{k}{2} \right| , \text{ and} \\
\Psi_q(\mu) : \mathbb{C} \rightarrow \mathbb{C} \quad \text{for max}(1, k - n) \leq q \leq \left| \frac{k - 1}{2} \right| ,
\]

such that the holomorphic functions \( \Phi_q(\mu)[w] := \Phi_q((w, w)), \Psi_q(\mu)[w] := \Psi_q((w, w)) \) on \( \mathbb{C}^n \otimes \mathbb{R} \mathbb{C} \) satisfy

\[
F(W(\mu))[w_1, \ldots, w_k] = \frac{(-1)^k}{k!} \sum_{q = \max(0, k - n)}^k P_{k,q}(w_1, \ldots, w_k) \Phi_q(\mu) \left[ \sum_{j=1}^k w_j \right] \\
+ \frac{(-1)^k}{k!} \sum_{q = \max(1, k - n)}^k \left[ \left( \frac{1}{q} D^k_{\beta,w} - \frac{2}{k - 2q} D^k_{\gamma,w} \right) \Psi_q(\mu) \right] \left[ \sum_{j=1}^k w_j \right].
\]

This decomposition is compatible with the decomposition in the smooth case: Valuations constructed from the differential forms \( \Theta^n \) correspond precisely to the contribution of the terms involving \( P_{k,q} \), while valuations obtained from \( \Upsilon^n \) correspond to the terms involving the differential operators \( D^k \). In this sense, Theorem 4 establishes an abstract version of the decomposition in equation (1).

1.1 Plan of the article

In Section 2 we introduce and discuss the basic properties of the Fourier-Laplace transform of Goodey-Weil distributions and prove Theorem 3. Section 3 provides a characterization of the differential forms used in the classification of smooth unitarily invariant valuations and we show that any smooth valuation admits a representation of the type stated in Theorem 2. The polynomials and differential operators are introduced in Section 4.1 and 4.2 and we use these objects to explicitly calculate the Fourier-Laplace transform of the valuations induced by \( \Theta^n \) and \( \Upsilon^n \) in Section 4.3. We also use this description to show that the representation of smooth valuations in Theorem 2 is unique, completing the proof.

Finally, we prove Theorem 4 in Section 4.4 and apply these results to obtain Theorem 4.
2 The Fourier-Laplace transform of Goodey-Weil distributions

2.1 Basic properties and relation to restrictions to subspaces

In this section we recall some facts about the Goodey-Weil distributions. Let us remark that this concept is based on ideas by Goodey and Weil [31] in the context of translation invariant valuations on convex bodies.

**Theorem 2.1** ([37] Theorem 2). For every \( \mu \in \text{VConv}_k(V) \) there exists a unique distribution \( GW(\mu) \in D'(V^k) \) with compact support which satisfies the following property: If \( f_1, \ldots, f_k \in \text{Conv}(V, \mathbb{R}) \cap C^\infty(V) \), then

\[
GW(\mu)(f_1 \otimes \cdots \otimes f_k) = \bar{\mu}(f_1, \ldots, f_k).
\]

Here, \( \bar{\mu} \) denotes the polarization of \( \mu \) defined in the introduction. The distribution \( GW(\mu) \) is called the Goodey-Weil distribution of \( \mu \in \text{VConv}_k(V) \) and uniquely determines \( \mu \). We thus obtain a well defined and injective map \( GW : \text{VConv}_k(V) \rightarrow D'(V^k) \) into the space of distributions on \( V^k \), called Goodey-Weil embedding. Moreover, \( GW(\mu) \) satisfies the inequality (compare Section 5.1 in [37])

\[
|GW(\mu)(\phi_1 \otimes \cdots \otimes \phi_k)| \leq c_k \|
\mu\|_K \prod_{i=1}^k \|
\phi_i\|_{C^2(V)}
\]

for \( \phi_1, \ldots, \phi_k \in C^\infty_c(V) \), where \( c_k > 0 \) is a constant depending on \( k \) only, \( K \subset \text{Conv}(V, \mathbb{R}) \) is a compact subset and

\[
\|
\mu\|_K := \sup_{f \in K} |\mu(f)|
\]

defines a continuous norm on \( \text{VConv}_k(V) \).

According to [37] Theorem 5.5, the support of the Goodey-Weil distribution of a \( k \)-homogeneous valuation is contained in the diagonal of \( V^k \), which leads to the following notion of support: If \( \mu = \sum_{k=0}^{\dim V} \mu_k \) is the decomposition of \( \mu \) into its homogeneous components, then we set

\[
\text{supp } \mu := \bigcup_{k=1}^{\dim V} \Delta_k^{-1}(\text{supp } GW(\mu_k)) \subset V,
\]

where \( \Delta_k : V \rightarrow V^k \) denotes the diagonal embedding. In particular, \( \text{supp } \mu = \emptyset \) for \( \mu \in \text{VConv}_0(V) \). Note that the support of \( \mu \in \text{VConv}(V) \) is always a compact set. In [37] the following alternative characterization of the support was given.

**Proposition 2.2** ([37] Proposition 6.3). The support of \( \mu \in \text{VConv}(V) \) is minimal (with respect to inclusion) among the closed sets \( A \subset V \) with the following property: If \( f, g \in \text{Conv}(V, \mathbb{R}) \) satisfy \( f = g \) on an open neighborhood of \( A \), then \( \mu(f) = \mu(g) \).

Let us turn to the case \( V = \mathbb{R}^n \) with its standard scalar product. As the Goodey-Weil distribution of \( \mu \in \text{VConv}_k(\mathbb{R}^n) \) is compactly supported by Theorem 2.1 its Fourier-Laplace transform is an entire function on \((\mathbb{C}^n)^k \) and is given by

\[
\mathcal{F}(GW(\mu))[w_1, \ldots, w_k] = GW(\mu)[\exp(i\langle w_1, \cdot \rangle) \otimes \cdots \otimes \exp(i\langle w_k, \cdot \rangle)]
\]
for $w_1, \ldots, w_k \in \mathbb{C}^n$. Here, $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is the $\mathbb{C}$ linear extension of the scalar product on $\mathbb{R}^n$. In particular, $GW(\mu)$ (and thus $\mu$) is uniquely determined by this function. Let us make the following observation:

**Remark 2.3.** As $\mathcal{F}(GW(\mu))$ is an entire function, it is uniquely determined by its restriction to the real subspace $(i\mathbb{R}^n)^k \subset (\mathbb{C}^n)^k$, that is, by

$$\mathcal{F}(GW(\mu))[-y_1 \otimes i, \ldots, -y_k \otimes i] = GW(\mu)[\exp((y_1, \cdot)) \otimes \cdots \otimes \exp((y_k, \cdot))]$$

$$= \frac{1}{k!} \frac{\partial^k}{\partial \lambda_1 \ldots \partial \lambda_k} \bigg|_0 \mu \left( \sum_{i=1}^k \lambda_i \exp((y_i, \cdot)) \right)$$

for $y_1, \ldots, y_k \in \mathbb{R}^n$. Here we have used the characterizing properties of the Goodey-Weil distribution from Theorem 2.1 and the polarization of $\mu$, as well as the fact that $\exp((y, \cdot))$ is a convex function for $y \in \mathbb{R}^n$.

**Proof of Theorem 3.** Assume that $\mu \in VConv_k(\mathbb{R}^n)$ satisfies $\pi_W^* \mu = 0$ for all orthogonal projections $\pi_W : \mathbb{R}^n \to W$ onto a $k$-dimensional subspace $W \in \text{Gr}_k(\mathbb{R}^n)$. By Remark 2.3 $\mathcal{F}(GW(\mu))$ is uniquely determined by

$$\mathcal{F}(GW)[-y_1 \otimes i, \ldots, -y_k \otimes i] = \frac{1}{k!} \frac{\partial^k}{\partial \lambda_1 \ldots \partial \lambda_k} \bigg|_0 \mu \left( \sum_{i=1}^k \lambda_i \exp((y_i, \cdot)) \right)$$

for $(y_1, \ldots, y_k) \in (\mathbb{R}^n)^k$.

Set $W := \text{span}_\mathbb{R}(y_1, \ldots, y_k)$ and consider the function $f \in \text{Conv}(W, \mathbb{R})$ given by

$$f := \sum_{i=1}^k \lambda_i \exp((y_i, \cdot)).$$

Then $\pi_W^* f(x) = \sum_{i=1}^k \lambda_i \exp((y_i, x))$ for $x \in \mathbb{R}^n$ and thus

$$\mu \left( \sum_{i=1}^k \lambda_i \exp((y_i, \cdot)) \right) = \mu(\pi_W^* f) = 0$$

by assumption. As this holds for all $\lambda_i \geq 0$, we deduce

$$\mathcal{F}(GW)[-y_1 \otimes i, \ldots, -y_k \otimes i] = \frac{1}{k!} \frac{\partial^k}{\partial \lambda_1 \ldots \partial \lambda_k} \bigg|_0 \mu \left( \sum_{i=1}^k \lambda_i \exp((y_i, \cdot)) \right) = 0$$

for all $(y_1, \ldots, y_k) \in (\mathbb{R}^n)^k$ and thus $GW(\mu) = 0$. As $GW : VConv_k(\mathbb{R}^n) \to D'(\mathbb{R}^n)^k$ is injective, we obtain $\mu = 0$. □

**Corollary 2.4.** If $\mu \in VConv(\mathbb{R}^n)$ vanishes on all convex polynomials, then $\mu = 0$.

**Proof.** Without loss of generality, we may assume that $\mu$ is $k$-homogeneous, where $0 \leq k \leq n$. For $k = 0$ the statement is obviously true, so we may assume $k > 0$.

It is easy to see that every polynomial on $\mathbb{R}^n$ may be written as a linear combination of convex polynomials. If $\mu \in VConv_k(\mathbb{R}^n)$ vanishes on all convex polynomials, this implies that
GW(µ) vanishes on all functions of the form $P_1 \otimes \cdots \otimes P_k \in C^\infty((\mathbb{R}^n)^k)$, where $P_1, \ldots, P_k$ are polynomials on $\mathbb{R}^n$. Now observe that the sum

$$\exp(\langle y, \cdot \rangle) = \sum_{j=0}^{\infty} \frac{\langle y, \cdot \rangle^j}{j!}$$

converges locally uniformly in the $C^\infty$-topology for all $y \in \mathbb{R}^n$. In particular,

$$\mathcal{F}(GW(\mu))[-y_1 \otimes i, \ldots, -y_k \otimes i] = \sum_{j_1, \ldots, j_k=1}^{\infty} \frac{1}{j_1! \ldots j_k!} GW(\mu)[\langle y_1, \cdot \rangle^{j_1} \otimes \cdots \otimes \langle y_k, \cdot \rangle^{j_k}] = 0$$

for all $y_1, \ldots, y_k \in \mathbb{R}^n$. Thus Remark 2.3 implies $\mu = 0$. \hfill \Box

Note that the restriction of $\mu \in \text{VConv}_k(\mathbb{R}^n)$ to a $k$-dimensional subspace defines a valuation of maximal degree on this space. These functionals all admit integral representations with respect to the real Monge-Ampère operator $\text{MA}$, which assigns to any $f \in \text{Conv}(\mathbb{R}^n, \mathbb{R})$ a non-negative Radon measure $\text{MA}(f)$ on $\mathbb{R}^n$, compare [5]. These measures depend continuously in the weak topology on $f \in \text{Conv}(\mathbb{R}^n, \mathbb{R})$ and are given by $\det(D^2 f(x)) dx$ if $f$ is a smooth function, where $D^2 f$ denotes the Hessian of $f$. If $E \subset \mathbb{R}^n$ is a $k$-dimensional subspace, we denote the real Monge-Ampère operator on $E$ by $\text{MA}_E$.

**Theorem 2.5** (Colesanti-Ludwig-Mussnig [24] Theorem 5). $\mu \in \text{VConv}_n(\mathbb{R}^n)$ if and only if there exists a (necessarily unique) function $\phi \in C_c(\mathbb{R}^n)$ such that

$$\mu(f) = \int_{\mathbb{R}^n} \phi d\text{MA}(f).$$

This has the following implication for the structure of the Fourier-Laplace transform of the Goodey-Weil distributions.

**Lemma 2.6.** Let $\mu \in \text{VConv}_k(\mathbb{R}^n)$. For $E \in \text{Gr}_k(\mathbb{R}^n)$ let $\phi_E \in C_c(E)$ denote the unique function such that $\pi_{E^*}\mu = \int_E \phi_E d\text{MA}_E$. Then for $w_1, \ldots, w_k \in E_C := E \otimes_\mathbb{R} \mathbb{C}$

$$\mathcal{F}(GW(\mu))[w_1, \ldots, w_k] = (-1)^k \frac{k!}{k!} \det(\langle w_i, w_j \rangle)_{i,j=1}^{k} \mathcal{F}_E(\phi_E) \left[ \sum_{i=1}^{k} w_i \right].$$

Here $\mathcal{F}_E$ denotes the Fourier-Laplace transform on $L^1(E)$ and $\langle \cdot, \cdot \rangle$ denotes the $\mathbb{C}$-linear extension of the standard scalar product on $\mathbb{R}^n$.

We will omit the proof, as it is a straightforward calculation. Notice that this implies that $\mathcal{F}(GW(\mu))$ is uniquely determined by its values on $k$-tuples of orthogonal vectors. The following corollary shows that the Fourier-Laplace transform encodes precisely the information obtained from restricting the valuations to subspaces.

**Corollary 2.7.** Let $\mu \in \text{VConv}_k(\mathbb{R}^n)$, $E \in \text{Gr}_k(\mathbb{R}^n)$. Then $\mathcal{F}(GW(\mu))[w_1, \ldots, w_k] = 0$ for all $w_1, \ldots, w_k \in E_C$ if and only if $\pi_{E^*}\mu = 0$.

**Proof.** Remark 2.3 shows that $\pi_{E^*}\mu = 0$ implies that $\mathcal{F}(GW(\mu))[w_1, \ldots, w_k] = 0$ for all $w_1, \ldots, w_k \in E_C$. For the converse statement, let $\phi_E \in C_c(E)$ denote the unique function such that $\pi_{E^*}\mu(f) = \int_E \phi_E d\text{MA}_E(f)$ for all $f \in \text{Conv}(E, \mathbb{R})$.

If $\mathcal{F}(GW(\mu))$ vanishes on $(E_C)^k$, then $\mathcal{F}_E(\phi_E) = 0$ by Lemma 2.6 so $\phi_E = 0$, which implies $\pi_{E^*}\mu = 0$. \hfill \Box
For $A \subset \mathbb{R}^n$ let us denote by $V\text{Conv}_{k,A}(\mathbb{R}^n) \subset V\text{Conv}_k(\mathbb{R}^n)$ the subspace of valuations with support contained in $A$. In addition, consider the space $\mathcal{O}(\mathbb{C}^\times)^k$ of all holomorphic function on $(\mathbb{C}^\times)^k$. This is a Fréchet space with respect to the topology of uniform convergence on compact subsets.

The following observation will be crucial for the proof of Theorem 4.

**Proposition 2.8.** The map

$$\mathcal{F} \circ \text{GW} : V\text{Conv}_{k,A}(\mathbb{R}^n) \to \mathcal{O}(\mathbb{C}^\times)^k$$

is continuous for all $A \subset (\mathbb{R}^n)^k$ compact.

**Proof.** By [2], the Goodey-Weil distributions satisfy

$$|\text{GW}(\mu)[\phi_1 \otimes \cdots \otimes \phi_k]| \leq c_k \|\mu\|_K \prod_{i=1}^k \|\phi_i\|_{C^2(\mathbb{R}^n)}$$

for all $\phi_1, \ldots, \phi_k \in C^\infty_c(\mathbb{R}^n)$ for some $c_k > 0$ depending on $k$ only, where $\| \cdot \|_K$ is a continuous norm on $V\text{Conv}(\mathbb{R}^n)$. Now choose $\phi \in C^\infty_c(\mathbb{R}^n, [0,1])$ with $\phi \equiv 1$ on a neighborhood of $A \subset \mathbb{R}^n$ and $\text{supp} \phi \subset A + \delta B_1(0)$ for some $\delta > 0$. By the Leibnitz rule, there exists a constant $C > 0$ (depending on $\| \phi \|_{C^2(\mathbb{R}^n)}$ only) such that $\|\phi \cdot \psi\|_{C^2(\mathbb{R}^n)} \leq C \|\psi\|_{C^2(A + \delta B_1(0))}$ for all $\psi \in C^\infty(\mathbb{R}^n)$. Thus for $\mu \in V\text{Conv}_{k,A}(\mathbb{R}^n)$ and $\psi_1, \ldots, \psi_k \in C^\infty(\mathbb{R}^n)$,

$$|\text{GW}(\mu)[\psi_1 \otimes \cdots \otimes \psi_k]| = |\text{GW}(\mu)[(\phi \cdot \psi_1) \otimes \cdots \otimes (\phi \cdot \psi_k)]| \leq c_k C^k \|\mu\|_K \prod_{i=1}^k \|\psi_i\|_{C^2(A + \delta B_1(0))}.$$  

The function $\psi_i = \exp(i\langle w_i, \cdot \rangle)$ satisfies

$$\|\psi_i\|_{C^2(A + \delta B_1(0))} \leq 2(1 + |w_i|^2) \exp(h_{A + \delta B_1(0)}(-\text{Im}(w_i))),$$

where $h_{A + \delta B_1(0)}(y) := \sup_{x \in A + \delta B_1(0)} |y, x|$, $y \in \mathbb{R}^n$, denotes the support function of $A + \delta B_1(0)$. Thus

$$|\mathcal{F}(\text{GW}(\mu))[w_1, \ldots, w_k]| \leq c_k 2^k C^k \|\mu\|_K \exp \left( \sum_{i=1}^k h_{A + \delta B_1(0)}(-\text{Im}(w_i)) \right) \prod_{i=1}^k (1 + |w_i|^2).$$

In particular, for any compact $B \subset (\mathbb{C}^\times)^k$ there exists a constant $C(B) > 0$ such that

$$|\mathcal{F}(\text{GW}(\mu))[w_1, \ldots, w_k]| \leq C(B) \|\mu\|_K$$

for all $(w_1, \ldots, w_k) \in B$.

Thus $\mathcal{F} \circ \text{GW} : V\text{Conv}_{k,A}(\mathbb{R}^n) \to \mathcal{O}(\mathbb{C}^\times)^k$ is continuous. \hfill $\square$

### 2.2 Restrictions on the support

This section establishes some restrictions on the support of a $k$-homogeneous valuation, which strengthen [37] Proposition 6.4. Before we state these results, let us first add the following fact on the behavior of the support under the pushforward by a linear map $T : V \to W$ between finite dimensional vector spaces.
Proposition 2.9. Let \( T : V \rightarrow W \) be a linear map and \( \mu \in \text{VConv}(V) \). Then \( \text{supp}(T_\ast \mu) \subset T(\text{supp} \mu) \).

Proof. Let \( f, g \in \text{Conv}(W, \mathbb{R}) \) be two functions with \( f = g \) on a neighborhood \( U \) of \( T(\text{supp} \mu) \). Then \( T^\ast f \) and \( T^\ast g \) coincide on \( T^{-1}(U) \), which is a neighborhood of \( \text{supp} \mu \). Thus \( [T_\ast \mu](f) = \mu(T^\ast f) = \mu(T^\ast g) = [T_\ast \mu](g) \) by Proposition 2.2. As this is true for all \( f \) and \( g \) with this property, Proposition 2.10 implies \( \text{supp}(T_\ast \mu) \subset T(\text{supp} \mu) \).

Proposition 2.10. Let \( 1 \leq k \leq n \) and assume that the support of \( \mu \in \text{VConv}_k(V) \) is contained in a \((k-1)\)-dimensional affine subspace. Then \( \mu = 0 \). In particular, its support is empty.

Proof. Let us assume that \( V = \mathbb{R}^n \) with its standard scalar product. We will start with valuations of degree \( k = n \). If \( \mu \in \text{VConv}_n(\mathbb{R}^n) \), there exists a unique function \( \phi \in C_c(\mathbb{R}^n) \) such that

\[
\mu(f) = \int_{\mathbb{R}^n} \phi(x) \det(D^2 f(x)) \, dx \quad \text{for all } f \in \text{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n)
\]

by Theorem 2.5. Let us assume that the support of \( \mu \) is contained in an affine hyperplane \( H \) of \( \mathbb{R}^n \). Let \( H^\pm \) denote the positive and negative open half spaces with respect to some orientation of \( H \). If \( \psi \in C_c^\infty(\mathbb{R}^n) \) is a function with \( \text{supp} \psi \subset \mathbb{R}^n \setminus H \), the characterization of the support in Proposition 2.2 implies

\[
\mu(f + t\psi) = \mu(f)
\]

for all \( t \in \mathbb{R} \) and \( f \in \text{Conv}(\mathbb{R}^n, \mathbb{R}) \) such that \( f + t\psi \) is convex. We may choose \( f(x) = \frac{1}{2} \mid x \mid^2 \). Then this equation holds for all \( t \in (-\epsilon, \epsilon) \) for some \( \epsilon > 0 \). Thus

\[
0 = \frac{d}{dt} \bigg|_0 \int_{\mathbb{R}^n} \phi(x) \det(Id_n + tD^2 \psi(x)) \, dx = \int_{\mathbb{R}^n} \phi(x) \left. \frac{d}{dt} \right|_0 \det(Id_n + tD^2 \psi(x)) \, dx
\]

\[
= \int_{\mathbb{R}^n} \phi(x) \text{tr}(D^2 \psi(x)) \, dx = \int_{\mathbb{R}^n} \phi(x) \Delta \psi(x) \, dx.
\]

In other words, \( \phi \) satisfies \( \Delta \phi = 0 \) on \( H^\pm \) in the distributional sense. It is a standard fact from regularity theory that any distributional solution of Laplace’s equation is a classical solution. Thus \( \phi \) is harmonic on \( H^\pm \) and thus in particular analytic on \( H^\pm \). However, \( \phi \) is compactly supported, so there exists an open set in \( H^\pm \) such that \( \phi = 0 \) on this set. By the identity theorem, \( \phi = 0 \) on the open sets \( H^\pm \), that is, \( \phi = 0 \) on the dense open subset \( H^+ \cup H^- \). Thus \( \phi = 0 \) by continuity, which implies \( \mu = 0 \).

Now let \( 1 \leq k \leq n - 1 \) be given and let \( \mu \in \text{VConv}_k(\mathbb{R}^n) \) be such that \( \text{supp} \mu \) is contained in a \((k-1)\)-dimensional affine subspace \( H \). If \( W \) is a \( k \)-dimensional subspace, then the image of \( H \) under the orthogonal projection \( \pi : \mathbb{R}^n \rightarrow W \) is an affine subspace of dimension at most \( k - 1 \). By Proposition 2.9 the support of \( \pi_\ast \mu \in \text{VConv}_k(W) \) is thus contained in an affine subspace of dimension at most \( k - 1 \). However, this is a valuation of degree \( k = \text{dim} W \), so the previous argument implies \( \pi_\ast \mu = 0 \). As this is true for all orthogonal projections \( \pi : \mathbb{R}^n \rightarrow W \) onto \( k \)-dimensional subspaces \( W \subset \mathbb{R}^n \), Theorem 3 implies \( \mu = 0 \).

A slight variation of this argument shows the following result.

Corollary 2.11. If \( \mu \in \text{VConv}_n(\mathbb{R}^n) \) is given by \( \mu = \int_{\mathbb{R}^n} \phi \text{d} \mu \) for \( \phi \in C_c(\mathbb{R}^n) \), then \( \text{supp} \Delta \phi \subset \text{supp} \mu \), where we understand \( \Delta \phi \) in the sense of distributions. In particular, \( \partial \text{supp} \phi \subset \text{supp} \mu \subset \text{supp} \phi \subset \text{conv}(\text{supp} \mu) \).
Note that the support of a \( k \)-homogeneous valuation may still be contained in a union of lower dimensional affine subspaces, as shown by the valuation \( \mu \in \text{VConv}_k(V) \) with discrete support given by
\[
\mu(f) = f(x) + f(-x) - 2f(0) \quad \text{for all } f \in \text{Conv}(V, \mathbb{R}),
\]
where \( x \in V \setminus \{0\} \) is an arbitrary point. However, the shape of the connected components of the support is still restricted as exemplified by the following result.

**Corollary 2.12.** For \( k \geq 2 \) the support of \( \mu \in \text{VConv}_k(V) \) is not discrete unless \( \mu = 0 \).

**Proof.** Let \( \mu \in \text{VConv}_k(V) \) be a valuation with discrete support, which is thus a finite set due to the compactness of the support. Consider the projection \( \pi : V \to E \) onto a \( k \)-dimensional subspace. By Proposition 2.9 \( \text{supp}\pi_*\mu \subset \pi(\text{supp}\mu) \), so \( \text{supp}\pi_*\mu \) is a discrete set. If \( \pi_*\mu \neq 0 \), then it is a non-trivial valuation of degree \( k = \dim E \) on \( E \), so its support contains the boundary of an open, non-empty, relatively compact subset by the previous corollary. As \( \dim E = k \geq 2 \), the boundary of such a subset is not discrete, so we obtain a contradiction. Thus \( \pi_*\mu = 0 \). But this holds for all projections \( \pi : V \to E \) onto \( k \)-dimensional subspaces of \( V \), so Theorem 3 implies \( \mu = 0 \). \( \square \)

Note that the example in (3) is 1-homogeneous and supported on three points. This example is minimal in the following sense:

**Corollary 2.13.** If the support of \( \mu \in \text{VConv}(V) \) is contained in a two-point set then \( \text{supp}\mu = \emptyset \) and \( \mu \) is constant.

**Proof.** We may assume that \( \mu \) is \( k \)-homogeneous, \( k > 0 \), supp\(\mu \subset \{x_1, x_2\} \). Then \( x_1 \) and \( x_2 \) belong to a 1-dimensional affine subspace, so for \( k > 1 \) this implies \( \mu = 0 \) by Proposition 2.10.

For \( k = 1 \), we may apply Theorem 3 and assume that \( V = \mathbb{R} \) is 1-dimensional, \( x_1 < x_2 \), and that \( \mu \) is given by \( \mu(f) = \int_{\mathbb{R}} \psi(x)f''(x)dx \) for all \( f \in \text{Conv}(\mathbb{R}, \mathbb{R}) \cap C^2(\mathbb{R}) \) for some \( \psi \in C^\infty_c(\mathbb{R}) \), compare Theorem 2.5. Let \( \phi \in C^\infty_c(\mathbb{R}) \) be a function with \( \text{supp}\phi \subset \mathbb{R} \setminus \{x_1, x_2\} \). As in the proof of Proposition 2.10 this implies
\[
0 = \frac{d}{dt} \bigg|_0 \mu(\cdot|t^2 + t\phi) = \int_{\mathbb{R}} \psi(x)\phi''(x)dx.
\]
Thus \( \psi \) is a distributional solution of \( \psi'' = 0 \) on \( (-\infty, x_1) \cup (x_1, x_2) \cup (x_2, \infty) \). As in the proof of Proposition 2.10 this implies \( \psi = 0 \) on \( (-\infty, x_1) \) and \( (x_2, \infty) \), so \( \psi(x_1) = 0 = \psi(x_2) \) by continuity. But then \( \psi = 0 \) on \( (x_1, x_2) \) as well, so \( \psi = 0 \) and thus \( \mu = 0 \). \( \square \)

### 3 Unitarily invariant differential forms and smooth valuations

The goal of this section is a classification of the relevant differential forms \( \omega \in \Omega^{n-k}_c(\mathbb{C}^n) \otimes \Lambda^k\mathbb{C}^n \) inducing smooth \( U(n) \)-invariant valuations. Starting point is the following observation, which is a special case of [37] Corollary 6.7.

**Proposition 3.1.** Every smooth valuation \( \mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)} \) is given by \( f \mapsto \mu(f) = D(f)[\omega] \) for a \( U(n) \)-invariant differential form \( \omega \in \Omega^{n-k}_c(\mathbb{C}^n) \otimes \Lambda^k\mathbb{C}^n \).
We can thus restrict ourselves to $U(n)$-invariant differential forms. However, we will see that there exist a large number of invariant forms in addition to the differential forms $\Theta_{k,q}^n, \Upsilon_{k,q}^n$ which do not induce any additional valuations. The relations between the valuations induced by these forms can be obtained from the following description of the kernel of the differential cycle. It uses a certain second order differential operator $\bar{D}$, called the symplectic Rumin operator in [36], which was also previously considered in [54]. We will not need its precise definition, only that it vanishes on closed forms as well as multiples of the symplectic form $\omega_s$ (see [36] Proposition 5.13).

**Theorem 3.2 (36 Theorem 2).** $\omega \in \Omega^{n-k}(\mathbb{C}^n) \otimes \Lambda^k \mathbb{C}^n$ satisfies $D(f)[\omega] = 0$ for all $f \in \text{Conv}(\mathbb{C}^n, \mathbb{R})$ if and only if

1. $\bar{D} \omega = 0$,
2. $\int_{\mathbb{C}^n} \omega = 0$, where we consider the zero section $\mathbb{C}^n \hookrightarrow T^*\mathbb{C}^n$ as a submanifold.

Note that for $k > 0$ the second condition is always satisfied, so a given valuation $\mu \in V\text{Conv}_k(\mathbb{C}^n)^{sm}$ is uniquely determined by $\bar{D} \omega$ for some representing differential form $\omega$. If $k = 0$, then the first condition is always satisfied and we only have to consider the second condition.

Finally, let us fix the following convention: $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product on $\mathbb{C}^n$. In particular, $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-linear in its first and conjugate $\mathbb{C}$-linear in its second argument.

### 3.1 The algebra $[\Lambda^*(\mathbb{C}^n \times \mathbb{C}^n)]^{U(n)}$

We start with some results concerning constant invariant forms. Let us equip $T^*\mathbb{C}^n \cong \mathbb{C}^n \times \mathbb{C}^n$ with real coordinates $x + iy$ on the first factor and induced real coordinates $\xi + i\eta$ on the second. Consider the following $U(n)$-invariant differential forms on $\mathbb{C}^n \times \mathbb{C}^n$:

\begin{align*}
\theta_0 &:= \sum_{j=1}^{n} dx_j \wedge dy_j, \\
\theta_1 &:= \sum_{j=1}^{n} dx_j \wedge d\eta_j - dy_j \wedge d\xi_j, \\
\theta_2 &:= \sum_{j=1}^{n} d\xi_j \wedge d\eta_j, \\
\omega_s &:= \sum_{j=1}^{n} dx_j \wedge d\xi_j + dy_j \wedge d\eta_j.
\end{align*}

Note that $\omega_s$ is the symplectic form on $\mathbb{C}^n \times \mathbb{C}^n$. Let us consider the space $[\Lambda^*(\mathbb{C}^n \times \mathbb{C}^n)]^{U(n)}$ of all $U(n)$-invariant forms. Note that this is an algebra with respect to the wedge product.

**Theorem 3.3 (Park 47 Theorem 2.12 and 2.13).** The algebra $[\Lambda^*(\mathbb{C}^n \times \mathbb{C}^n)]^{U(n)}$ is generated by $\theta_0, \theta_1, \theta_2, \omega_s$. Moreover, there exists no polynomial relation in degree less or equal to $n$ between these forms.
In other words, the algebra \([\Lambda^*(\mathbb{C}^n \times \mathbb{C}^n)]^{U(n)}\) is isomorphic to a quotient of the polynomial ring \(\mathbb{R}[X,Y,Z,W]\) by an ideal that does not contain polynomials of degree less or equal to \(n\). We also need the Lefschetz decomposition (see [34] Proposition 1.2.30):

**Theorem 3.4.** Let \((W,\omega_s)\) be a symplectic vector space of dimension \(2n\) and let \(L : \Lambda^*W^* \to \Lambda^*W^*, \tau \mapsto \omega_s \wedge \tau\) be the Lefschetz operator. For \(0 \leq k \leq n\) let \(P^kW := \{\tau \in \Lambda^kW^* : L^{n-k+1}\tau = 0\}\) denote the space of primitive \(k\)-forms on \(W\). Then the following holds:

1. There exists a direct sum decomposition \(\Lambda^kW^* = \bigoplus_{i \geq 0} L^iP^{2i-k}W\).

2. \(L^{n-k} : \Lambda^kW^* \to \Lambda^{2n-k}W^*\) is an isomorphism.

Note that this decomposition is compatible with linear symplectomorphisms. In particular, consider \(W = \mathbb{C}^n \times \mathbb{C}^n\) with its natural symplectic form and the diagonal action of \(U(n)\). Then \(U(n)\) operates by symplectomorphisms. If \(\omega \in [\Lambda^k(\mathbb{C}^n \times \mathbb{C}^n)]^{U(n)}\), this implies that every term in the Lefschetz decomposition of \(\omega\) is \(U(n)\)-invariant. In particular, every factor of the Lefschetz decomposition of \(P(\theta_0,\theta_1,\theta_2,\omega_s)\) for \(P \in \mathbb{R}[X,Y,Z,W]\) is again a polynomial in \(\theta_0,\theta_1,\theta_2,\omega_s\). We are now able to prove the following key result:

**Theorem 3.5.** Let \(P \in \mathbb{R}[X,Y,Z]\) be a homogeneous polynomial of degree \(n\). Then there exists a unique homogeneous polynomial \(Q \in \mathbb{R}[X,Y,Z,W]\) of degree \(n-2\) such that \(P(\theta_0,\theta_1,\theta_2) - \omega_s^2 \wedge Q(\theta_0,\theta_1,\theta_2,\omega_s)\) is primitive.

**Proof.** Let \(P \in \mathbb{R}[X,Y,Z]\) be a homogeneous polynomial of degree \(n\) and \(\tilde{Q} \in \mathbb{R}[X,Y,Z,W]\) the unique homogeneous polynomial of degree \(n-1\) given by the Lefschetz decomposition in Theorem 3.4 such that

\[\omega_s \wedge [P(\theta_0,\theta_1,\theta_2) - \omega_s \wedge \tilde{Q}(\theta_0,\theta_1,\theta_2,\omega_s)] = 0.\]

We thus have to show that \(\tilde{Q}\) is divisible by \(W\). Set

\[
\begin{align*}
\theta_0' &:= \theta_0 - dx_1 \wedge dy_1, \\
\theta_0 &= dx_1 \wedge dy_1, \\
\theta_1' &= \theta_1 - (dx_1 \wedge dy_1 - dy_1 \wedge d\xi_1), \\
\theta_1 &= dx_1 \wedge d\eta_1 - dy_1 \wedge d\xi_1, \\
\theta_2' &= \theta_2 - d\xi_1 \wedge d\eta_1, \\
\theta_2 &= d\xi_1 \wedge d\eta_1, \\
\omega_1 &= \omega_s - dx_1 \wedge \eta_1 \wedge d\xi_1 + dy_1 \wedge d\xi_1, \\
\omega_2 &= \omega_s - dx_1 \wedge \eta_1 \wedge d\xi_1 + dy_1 \wedge d\eta_1.
\end{align*}
\]

Note that \(\theta_0',\theta_1',\theta_2',\omega_2\) may be considered as the generators of \([\Lambda^*(\mathbb{C}^{n-1} \times \mathbb{C}^{n-1})]^{U(n-1)}\). As \(\theta_0 \wedge \tilde{\theta}_2 \wedge \eta_1 = \theta_0^2 = \tilde{\theta}_2^2 = \omega_1 \wedge \eta_1 = 0\), given any polynomial \(R \in \mathbb{R}[X,Y,Z,W]\),

\[
R(\theta_0,\theta_1,\theta_2,\omega_s) = R(\theta_0',\theta_1',\theta_2',\omega_2) + \frac{\partial R}{\partial W}(\theta_0',\theta_1',\theta_2',\omega_2) \wedge \omega_1 + \frac{\partial^2 R}{2 \partial W \partial W}(\theta_0',\theta_1',\theta_2',\omega_2) \wedge \omega_1 \wedge \omega_1
\]

\[
+ \frac{\partial R}{\partial Y}(\theta_0',\theta_1',\theta_2',\omega_2) \wedge \tilde{\theta}_1 + \frac{\partial^2 R}{2 \partial Y \partial Y}(\theta_0',\theta_1',\theta_2',\omega_2) \wedge \tilde{\theta}_1 \wedge \tilde{\theta}_1
\]

\[
+ \frac{\partial R}{\partial Z}(\theta_0',\theta_1',\theta_2',\omega_2) \wedge \tilde{\theta}_2 + \frac{\partial R}{\partial X}(\theta_0',\theta_1',\theta_2',\omega_2) \wedge \tilde{\theta}_0 + \frac{\partial^2 R}{\partial X \partial Z}(\theta_0',\theta_1',\theta_2',\omega_2) \wedge \tilde{\theta}_0 \wedge \tilde{\theta}_2.
\]
Plugging in the coordinate vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial y_1}$, this implies

$$i_{\frac{\partial}{\partial x_1}} i_{\frac{\partial}{\partial x_2}} i_{\frac{\partial}{\partial x_3}} R(\theta_0, \theta_1, \theta_2, \omega) = \left[ -\frac{\partial^2 R}{\partial W \partial W}(\theta_0', \theta_1', \theta_2', \omega) - \frac{\partial^2 R}{\partial Y \partial Y}(\theta_0', \theta_1', \theta_2', \omega) + \frac{\partial^2 R}{\partial X \partial Z}(\theta_0', \theta_1', \theta_2', \omega) \right].$$

We apply this relation to $R(X, Y, Z, W) := W[P(X, Y, Z) − W\tilde{Q}(X, Y, Z, W)]$, which evaluates to zero in $[\Lambda^*(\mathbb{C}^n \times \mathbb{C}^n)]^{U(n)}$ by assumption, and obtain

$$0 = \left[ 2\tilde{Q} + 2W \frac{\partial \tilde{Q}}{\partial W} + W^2 \frac{\partial^2 \tilde{Q}}{\partial W^2} − W \frac{\partial^2 P}{\partial Y^2} + W^2 \frac{\partial^2 \tilde{Q}}{\partial Y^2} + W \frac{\partial^2 P}{\partial X \partial Z} − W^2 \frac{\partial^2 \tilde{Q}}{\partial X \partial Z} \right](\theta_0', \theta_1', \theta_2', \omega_2).$$

But this is a polynomial relation of degree $n − 1$ in $[\Lambda^*(\mathbb{C}^{n−1} \times \mathbb{C}^{n−1})]^{U(n−1)}$, so Theorem 3.3 implies that this relation holds in $\mathbb{R}[X, Y, Z, W]$. Thus $\tilde{Q}$ is divisible by $W$. □

### 3.2 Invariant differential forms

In addition to the forms considered in the previous section, consider the 1-forms

$$\gamma_1 := \sum_{j=1}^{n} x_j dx_j + y_j dy_j, \quad \gamma_2 := \sum_{j=1}^{n} x_j dy_j − y_j dx_j,$$

$$\beta_1 := \sum_{j=1}^{n} x_j d\xi_j + y_j dy_j, \quad \beta_2 := \sum_{j=1}^{n} x_j dy_j − y_j d\xi_j,$$

and set

$$\omega_1 := \gamma_1 \wedge \beta_1 + \gamma_2 \wedge \beta_2,$$

$$\omega_2 := |z|^2 \omega_2 − \omega_1,$$

$$\theta_0' := |z|^2 \theta_0 − \gamma_1 \wedge \gamma_2,$$

$$\theta_1' := |z|^2 \theta_1 − (\gamma_1 \wedge \beta_2 − \gamma_2 \wedge \beta_1),$$

$$\theta_2' := |z|^2 \theta_2 − \beta_1 \wedge \beta_2.$$ (4)

**Proposition 3.6.** Let $\omega \in \bigoplus_{k=0}^{n}(\Omega^{n−k}(\mathbb{C}^n) \otimes \Lambda^k\mathbb{C}^n)^{U(n)}$. For every $z \in \mathbb{C}^n \setminus \{0\}$ there exist unique polynomials $R, R' \in \mathbb{R}[X, Y, Z, W]$ of degree $n − 2$, $R_{\gamma_1 \gamma_2}, \ldots, R_{\beta_1 \beta_2} \in \mathbb{R}[X, Y, Z, W]$ of degree $n − 1$ such that for all $\zeta \in \mathbb{C}^n$

$$\omega|_{(z, \zeta)} = \gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2 + R(\theta_0', \theta_1', \theta_2', \omega_2) + \omega_2^2 \wedge R'(\theta_0', \theta_1', \theta_2', \omega_2)
$$

$$+ \gamma_1 \wedge \gamma_2 \wedge R_{\gamma_1 \gamma_2}(\theta_0', \theta_1', \theta_2', \omega_2) + \gamma_1 \wedge \beta_1 \wedge R_{\gamma_1 \beta_1}(\theta_0', \theta_1', \theta_2', \omega_2)
$$

$$+ \gamma_2 \wedge \beta_2 \wedge R_{\gamma_2 \beta_2}(\theta_0', \theta_1', \theta_2', \omega_2) + \gamma_1 \wedge \beta_1 \wedge R_{\gamma_1 \beta_1}(\theta_0', \theta_1', \theta_2', \omega_2)
$$

$$+ \gamma_2 \wedge \beta_2 \wedge R_{\gamma_2 \beta_2}(\theta_0', \theta_1', \theta_2', \omega_2) + \beta_1 \wedge \beta_2 \wedge R_{\beta_1 \beta_2}(\theta_0', \theta_1', \theta_2', \omega_2).$$

**Proof.** As $w$ is invariant with respect to translations in the second coordinate, we may assume $\zeta = 0$. Using the $U(n)$-invariance, we may further assume that $z$ is a non-trivial real multiple of $e_1$. Then $\omega|_{(z, 0)}$ is invariant under the stabilizer of $e_1$, so

$$\omega|_{(z, 0)} \in [\Lambda^n(\mathbb{C}^n \times \mathbb{C}^n)]^{U(n−1)},$$
where \(U(n-1)\) operates diagonally and leaves the first coordinate of \(\mathbb{C}^n\) invariant. We may decompose \(\mathbb{C}^n = \mathbb{R}z \oplus \mathbb{R}iz \oplus \mathbb{C}^{n-1}\), so setting \(U = \mathbb{R}z, V = \mathbb{R}iz, \) we see that \(\omega|_{(z,0)}\) belongs to the space
\[
\bigoplus_{l_1,k_1,l_2,k_2=0}^1 \Lambda^{l_1}U \otimes \Lambda^{k_1}V \otimes \Lambda^{l_2}U \otimes \Lambda^{k_2}V \otimes [\Lambda^{n-l_1-l_2-k_1-k_2}(\mathbb{C}^{n-1} \times \mathbb{C}^{n-1})]^{U(n-1)}.
\]

The space \(\Lambda^{l_1}U \otimes \Lambda^{k_1}V \otimes \Lambda^{l_2}U \otimes \Lambda^{k_2}V\) is 1-dimensional and spanned by a suitable product of the forms \(\gamma_1, \gamma_2, \beta_1, \beta_2, \) while the space \([\Lambda^{n-l_1-l_2-k_1-k_2}(\mathbb{C}^{n-1} \times \mathbb{C}^{n-1})]^{U(n-1)}\) is spanned by homogeneous polynomials of degree \(n - l_1 - l_2 - k_1 - k_2\) in \(\theta'_0, \theta'_1, \theta'_2, \omega_2\).

Unless \(l_1 = k_1 = l_2 = k_2 = 0, \) this gives us the desired polynomial. Moreover this polynomial is unique due to Theorem 3.3. In the remaining case, we obtain some homogeneous polynomial \(\tilde{R} \in \mathbb{R}[X, Y, Z, W]\) of degree \(n\) such that the corresponding differential form in the decomposition \(\omega\) is given by \(\tilde{R}(\theta'_0, \theta'_1, \theta'_2, \omega_2)\). Note that \(\tilde{R}\) might not be unique, but the differential form \(\tilde{R}(\theta'_0, \theta'_1, \theta'_2, \omega_2)\) is. Theorem 3.4 implies that there exists a polynomial \(R' \in \mathbb{R}[X, Y, Z, W]\) of degree \(n - 2\) such that \(\tilde{R}(\theta'_0, \theta'_1, \theta'_2, \omega_2) = \omega_2^2 \wedge R'(\theta'_0, \theta'_1, \theta'_2, \omega_2)\). Because the square of the Lefschetz operator is injective for the given degree, \(R'(\theta'_0, \theta'_1, \theta'_2, \omega_2)\) is uniquely determined by \(\tilde{R}(\theta'_0, \theta'_1, \theta'_2, \omega_2)\). As this is a relation of degree \(n-2\) in \([\Lambda^{k}(\mathbb{C}^{n-1} \times \mathbb{C}^{n-1})]^{U(n-1)}\), this determines \(R'\) uniquely by Theorem 3.3.

We need two simple and well known results about smooth functions, the proofs of which are included for the convenience of the reader.

**Lemma 3.7.** If \(f: \mathbb{R} \to \mathbb{R}\) is a smooth even function then there exists \(\phi \in C^\infty([0, \infty))\) such that \(f(r) = \phi(r^2)\) for all \(r \in \mathbb{R}\).

**Proof.** We only have to show that \(f(r) := f(\sqrt{r})\) is smooth in \(r = 0\). As \(f\) is even, \(f(2k-1)(0) = 0\) for all \(k \in \mathbb{N}\). Let us show by induction that \(\phi^{(k-1)}\) is continuous and differentiable in \(r = 0\) with \(\phi^{(k)}(0) = \frac{(2k)!}{k^k} f^{(2k)}(0)\). This is trivial for \(k = 0\), as \(f\) is continuous. Assume that the assertion is true for all derivatives up to order \(k - 1\). By L’Hospital’s Theorem,

\[
f^{(2k)}(0) = \lim_{s \to 0} \frac{(2k)!}{s^{2k}} f(s) + f(-s) - \sum_{i=0}^{k-1} \frac{s^{2i}}{(2i)!} f^{(2i)}(0)
\]

\[
= \lim_{s \to 0} \frac{(2k)!}{s^{2k}} \phi(s^2) - \sum_{i=0}^{k-1} \frac{s^{2i}}{(2i)!} \phi^{(i)}(0)
\]

As \(\phi\) is \((k - 1)\)-times continuously differentiable on \((0, \infty)\) by assumption, we may apply L’Hospital’s Theorem to obtain

\[
f^{(2k)}(0) = \lim_{s \to 0} \frac{(2k)!}{s^{2k}} \phi^{(k-1)}(s) - \frac{\phi^{(k-1)}(0)}{k!} = \frac{(2k)!}{k!} \phi^{(k)}(0).
\]

In particular, \(\phi^{(k-1)}\) is continuous. □

**Lemma 3.8.** If \(f \in C^\infty(\mathbb{R})\) satisfies \(f^{(i)}(0) = 0\) for all \(0 \leq i \leq k\) for some \(k \in \mathbb{N}\), then there exists \(\phi \in C^\infty(\mathbb{R})\) such that

\[f(r) = r^{k+1} \phi(r)\]

for all \(r \in \mathbb{R}\).
Proof. By Taylor’s Theorem,
\[ f(r) = \sum_{i=0}^{k} \frac{f^{(i)}(0)}{i!} r^i + \int_{0}^{r} \frac{(r-t)^k}{k!} f^{(k+1)}(t)dt = \int_{0}^{r} \frac{(r-t)^k}{k!} f^{(k+1)}(t)dt. \]
Thus for \( r \neq 0 \)
\[ \frac{f(r)}{r^{k+1}} = \frac{1}{r^{k+1}} \int_{0}^{r} \frac{(r-t)^k}{k!} f^{(k+1)}(t)dt = \frac{1}{r^{k+1}} \int_{0}^{1} \frac{(1-ts)^k}{k!} f^{(k+1)}(rs)rdts \]
so \( \phi(r) := \int_{0}^{1} \frac{(1-ts)^k}{k!} f^{(k+1)}(rs)rdts \) is a smooth function on \( \mathbb{R} \) with \( f(r) = r^{k+1} \phi(r) \) for all \( r \in \mathbb{R} \).

Let us denote the subspace of \( \mathbb{R}[X, Y, Z, W] \) of \( k \)-homogeneous polynomials by \( \mathbb{R}[X, Y, Z, W]_k \).

**Theorem 3.9.** For any \( \omega \in \bigoplus_{k=0}^{n} (\Omega^{n-k}(\mathbb{C}^n) \otimes \Lambda^k \mathbb{C}^n)^{|U(n)|} \) there exist
\[
R_{\alpha} \in C^\infty([0, \infty), \mathbb{R}[X, Y, Z, W]_{n-2}), \quad R_{\gamma_{12}, R_{\gamma_{1}}, R_{\gamma_{12}}, R_{\gamma_{2}}, R_{\gamma_{1}}, R_{\gamma_{1}}, R_{\gamma_{1}}} \in C^\infty([0, \infty), \mathbb{R}[X, Y, Z, W]_{n-1}), \quad R_{\delta} \in C^\infty([0, \infty), \mathbb{R}[X, Y, Z, W]_n),
\]
such that for \((z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^n\),
\[
\omega|_{(z, \zeta)} = \gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2 \wedge R_{\alpha}(\theta_0, \theta_1, \theta_2, \omega_s) [|z|^2] \\
+ \gamma_1 \wedge \gamma_2 \wedge R_{\gamma_{12}}(\theta_0, \theta_1, \theta_2, \omega_s) [|z|^2] + \gamma_1 \wedge \beta_1 \wedge R_{\gamma_{1}}(\theta_0, \theta_1, \theta_2, \omega_s) [|z|^2] \\
+ \gamma_1 \wedge \beta_2 \wedge R_{\gamma_{2}}(\theta_0, \theta_1, \theta_2, \omega_s) [|z|^2] + \gamma_2 \wedge \beta_1 \wedge R_{\gamma_{1}}(\theta_0, \theta_1, \theta_2, \omega_s) [|z|^2] \\
+ \gamma_2 \wedge \beta_2 \wedge R_{\gamma_{2}}(\theta_0, \theta_1, \theta_2, \omega_s) [|z|^2] + \beta_1 \wedge \beta_2 \wedge R_{\gamma_{12}}(\theta_0, \theta_1, \theta_2, \omega_s) [|z|^2] \\
+ R_{\delta}(\theta_0, \theta_1, \theta_2, \omega_s) [|z|^2].
\]

**Proof.** Due to the invariance of \( \omega \), it is enough to show that such a decomposition holds along the line \( \mathbb{R}(e_1, 0) \). First, observe that \( \omega|_{(0,0)} \) is \( U(n) \)-invariant, so there exists a unique polynomial \( P \in \mathbb{R}[X, Y, Z, W] \) of degree \( n \) such that \( \omega|_{(0,0)} = P(\theta_0, \theta_1, \theta_2, \omega_s) \) by Theorem 3.3. By subtracting \( P(\theta_0, \theta_1, \theta_2, \omega_s) \) from \( \omega \), we may thus assume that \( \omega|_{(e_1, 0)} \) vanishes in \( t = 0 \).

By Proposition 3.6 we may write \( \omega|_{(e_1, 0)} \) for \( t \neq 0 \) uniquely as a linear combinations of certain products of \( \gamma_1, \gamma_2, \beta_1, \beta_2 \) and \( R(\theta_0', \theta_1', \theta_2', \omega_2') \) for suitable polynomials \( R \) in \( \mathbb{R}[X, Y, Z, W] \) depending on \( t \in \mathbb{R} \setminus \{0\} \), where \( \theta_0', \theta_1', \theta_2' \) are defined by (4). But along the line \( \mathbb{R}e_1 \),
\[
\gamma_1 = tdx_1, \quad \gamma_2 = tdy_1, \quad \beta_1 = tdx_1, \quad \beta_2 = tdy_1,
\]
while for any homogeneous polynomial \( R \in \mathbb{R}[X, Y, Z, W] \) of degree \( k \)
\[
R(\theta_0', \theta_1', \theta_2', \omega_2')|_{(e_1, 0)} = t^{2k} R \left( \sum_{i=2}^{n} dx_i \wedge dy_i, \sum_{i=2}^{n} dx_i \wedge d\eta_i - dy_i \wedge d\xi_i, \sum_{i=2}^{n} dx_i \wedge d\xi_i, \sum_{i=2}^{n} dx_i \wedge d\eta_i \right).
\]
We may thus consider $\omega|_{(t_{1},0)}$ as a smooth curve in the finite-dimensional vector space spanned by suitable products of the constant forms $dx_{1}, dy_{1}, d\xi_{1}, d\eta_{1}$ and $R$ for $R \in \mathbb{R}[X, Y, W, W]$ of appropriate degree given by

$$
\tilde{R} := R \left( \sum_{i=2}^{n} dx_{i} \wedge dy_{i}, \sum_{i=2}^{n} dx_{i} \wedge d\eta_{i} - dy_{i} \wedge d\xi_{i}, \sum_{i=2}^{n} d\xi_{i} \wedge d\eta_{i}, \sum_{i=2}^{n} dx_{i} \wedge d\xi_{i} + dy_{i} \wedge d\eta_{i} \right).
$$

Note that this corresponds exactly to the evaluation of $R \in \mathbb{R}[X, Y, Z, W]$ in $\Lambda^{*}(\mathbb{C}^{n-1} \times \mathbb{C}^{n-1})$. Let us set $A := \mathbb{R}[X, Y, Z, W]_{n-2} \oplus (\mathbb{R}[X, Y, Z, W]_{n-1})^{6} \oplus \mathbb{R}[X, Y, Z, W]_{n-2}$. We will denote the different components of $R \in A$ by $R = (R_{\alpha}, R_{\gamma_1 \gamma_2}, \ldots, R_{\beta_1 \beta_2}, R_{\delta})$. We obtain a map

$$
A \rightarrow \Lambda^{n}(\mathbb{C}^{n} \times \mathbb{C}^{n})
$$

$$
(R_{\alpha}, R_{\gamma_1 \gamma_2}, \ldots, R_{\beta_1 \beta_2}, R_{\delta}) \mapsto dx_{1} \wedge dy_{1} \wedge d\xi_{1} \wedge d\eta_{1} \wedge \tilde{R}_{\alpha}
+ dx_{1} \wedge dy_{1} \wedge \tilde{R}_{\gamma_1 \gamma_2} + dx_{1} \wedge d\xi_{1} \wedge \tilde{R}_{\gamma_1 \beta_1}
+ dx_{1} \wedge d\eta_{1} \wedge \tilde{R}_{\gamma_2 \beta_2} + dy_{1} \wedge d\xi_{1} \wedge \tilde{R}_{\beta_1 \beta_2}
+ dy_{1} \wedge d\eta_{1} \wedge \tilde{R}_{\gamma_2 \beta_2} + d\xi_{1} \wedge d\eta_{1} \wedge \tilde{R}_{\beta_1 \beta_2}
+ W^{2} \tilde{R}_{\delta}.
$$

Note that this is an isomorphism onto its image: If $R = (R_{\alpha}, R_{\gamma_1 \gamma_2}, \ldots, R_{\beta_1 \beta_2}, R_{\delta}) \in A$ belongs to the kernel of this map, then necessarily $\tilde{R}_{\alpha} = \tilde{R}_{\gamma_1 \gamma_2} = \cdots = \tilde{R}_{\beta_1 \beta_2} = W^{2} \tilde{R}_{\delta} = 0$. But these are relations in $[\Lambda^{*}(\mathbb{C}^{n-1} \times \mathbb{C}^{n-1})]^{U(n-1)}$, so Theorem 3.3 directly implies $R_{\alpha} = R_{\gamma_1 \gamma_2} = \cdots = R_{\beta_1 \beta_2} = 0$. From the Lefschetz decomposition of $\Lambda^{*}(\mathbb{C}^{n-1} \times \mathbb{C}^{n-1})$, we deduce that $W^{2} \tilde{R}_{\delta} = 0$ implies $\tilde{R}_{\delta} = 0$, so Theorem 3.3 shows $R_{\delta} = 0$. Thus $R = 0$, so this is indeed an isomorphism onto its image. Using this isomorphism, we may thus consider $t \mapsto \omega|_{(t_{1},0)}$ as a smooth curve in $A$. Let us denote this curve by $R^{\omega}(t) = (R^{\omega}_{\alpha}(t), R^{\omega}_{\gamma_1 \gamma_2}(t), \ldots, R^{\omega}_{\beta_1 \beta_2}(t), R^{\omega}_{\delta}(t))$. Using that $\omega$ is $U(n)$-invariant, this implies that for all $z \in \mathbb{C}^{n} \setminus \{0\}$

$$
\omega|_{(z,0)} = \gamma_{1} \wedge \gamma_{2} \wedge \beta_{1} \wedge \beta_{2} \wedge |z|^{-4} R^{\omega}_{\alpha}(0, \theta_{1}, \theta_{2}, \omega_{s}) \left[ |z|^{2} \right]
+ \gamma_{1} \wedge \gamma_{2} \wedge |z|^{-2} R^{\omega}_{\gamma_1 \gamma_2}(0, \theta_{1}, \theta_{2}, \omega_{s}) \left[ |z|^{2} \right]
+ \gamma_{1} \wedge \beta_{1} \wedge |z|^{-2} R^{\omega}_{\gamma_{2} \beta_{1}}(0, \theta_{1}, \omega_{s}, \gamma_{2} \wedge \beta_{2}) \left[ |z|^{2} \right]
+ \gamma_{1} \wedge \beta_{2} \wedge |z|^{-2} R^{\omega}_{\beta_{1} \beta_{2}}(0, \theta_{1} + \frac{\gamma_{2} \wedge \beta_{1}}{|z|^{2}}, \theta_{2}, \omega_{s}) \left[ |z|^{2} \right]
+ \gamma_{2} \wedge \beta_{1} \wedge |z|^{-2} R^{\omega}_{\gamma_{1} \beta_{2}}(0, \theta_{1} - \frac{\gamma_{1} \wedge \beta_{2}}{|z|^{2}}, \theta_{2}, \omega_{s}) \left[ |z|^{2} \right]
+ \gamma_{2} \wedge \beta_{2} \wedge |z|^{-2} R^{\omega}_{\gamma_{2} \beta_{2}}(0, \theta_{1}, \omega_{s} - \frac{\gamma_{1} \wedge \beta_{1}}{|z|^{2}}) \left[ |z|^{2} \right]
+ \beta_{1} \wedge \beta_{2} \wedge |z|^{-2} R^{\omega}_{\beta_{1} \beta_{2}}(0, \theta_{1} - \frac{\gamma_{1} \wedge \gamma_{2}}{|z|^{2}}, \theta_{2}, \omega_{s}) \left[ |z|^{2} \right]
+ \left( \omega_{s} - \frac{(\gamma_{1} \wedge \beta_{1} + \gamma_{2} \wedge \beta_{2})}{|z|^{2}} \right)^{2} \left[ |z|^{2} \right]
+ R^{\omega}_{\delta}(0, \theta_{1} - \frac{\gamma_{1} \wedge \gamma_{2}}{|z|^{2}}, \theta_{1} - \frac{\gamma_{1} \wedge \beta_{2} - \gamma_{2} \wedge \beta_{1}}{|z|^{2}}, \theta_{2} - \frac{\beta_{1} \wedge \beta_{2}}{|z|^{2}}, \omega_{s} - \frac{(\gamma_{1} \wedge \beta_{1} + \gamma_{2} \wedge \beta_{2})}{|z|^{2}}) \left[ |z|^{2} \right].
$$

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Collecting all terms containing $\gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2$ and setting
\[
R'[z^2] := \left( R'_0 - \frac{\partial R'_{\gamma_1 \gamma_2}}{\partial Z} + \frac{\partial R'_{\gamma_1 \beta_2}}{\partial Y} - \frac{\partial R'_{\gamma_2 \beta_2}}{\partial Y} - \frac{\partial R'_{\beta_1 \beta_2}}{\partial W} - \frac{\partial^2 (W^2 R'_z)}{\partial W^2} \right) [z^2],
\]
we obtain for $z \neq 0$
\[
\omega_{(z,0)} = \gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2 \wedge |z|^{-2} R' \left( \theta_0, \theta_1, \theta_2, \omega_s \right) [z^2]
+ \gamma_1 \wedge \gamma_2 \wedge |z|^{-2} \left( R'_{\gamma_1 \gamma_2 \gamma_2} - W \frac{\partial R'_z}{\partial X} \right) \left( \theta_0, \theta_1, \theta_2, \omega_s \right) [z^2]
+ \gamma_1 \wedge \beta_1 \wedge |z|^{-2} \left( R'_{\gamma_1 \beta_1 \gamma_2} - W \frac{\partial R'_z}{\partial Y} \right) \left( \theta_0, \theta_1, \theta_2, \omega_s \right) [z^2]
+ \gamma_2 \wedge \beta_1 \wedge |z|^{-2} \left( R'_{\gamma_2 \beta_1 \gamma_2} + W \frac{\partial R'_z}{\partial Y} \right) \left( \theta_0, \theta_1, \theta_2, \omega_s \right) [z^2]
+ \gamma_2 \wedge \beta_2 \wedge |z|^{-2} \left( R'_{\gamma_2 \beta_2 \gamma_2} - W \frac{\partial R'_z}{\partial W} \right) \left( \theta_0, \theta_1, \theta_2, \omega_s \right) [z^2]
+ \beta_1 \wedge \beta_2 \wedge |z|^{-2} \left( R'_{\beta_1 \beta_2 \gamma_2} - W \frac{\partial R'_z}{\partial Z} \right) \left( \theta_0, \theta_1, \theta_2, \omega_s \right) [z^2]
+ \omega^2 \wedge R'_z \left( \theta_0, \theta_1, \theta_2, \omega_s \right) [z^2],
\]
where the polynomial coefficients are smooth functions that vanish in $t = 0$. We may thus apply Lemma 3.7 and Lemma 3.8 to see that almost all of these polynomials may be replaced by $R(\theta_0, \theta_1, \theta_2, \omega_s)|[z^2]$ for some $R \in C^\infty([0, \infty), \mathbb{R}[X,Y,Z,W])$. This is true for all terms except $|z|^{-4} R'(\theta_0, \theta_1, \theta_2, \omega_s) [z^2]$. Let us set $P(t) := R'[t] - \frac{dR'}{dt}(0) t$. Then the derivatives of $P$ vanish up to order 1, so we may write $|z|^{-4} P(\theta_0, \theta_1, \theta_2, \omega_s) [z^2] = \hat{P}(\theta_0, \theta_1, \theta_2, \omega_s) [z^2]$ for some $\hat{P} \in C^\infty([0, \infty), \mathbb{R}[X,Y,Z,W])$ according to Lemma 3.7 and Lemma 3.8.

Subtracting all of these terms, we may thus assume that $\omega$ is a smooth differential form given for $z \neq 0$ by
\[
\omega_{(z,0)} = \gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2 \wedge |z|^{-2} P(\theta_0, \theta_1, \theta_2, \omega_s)
\]
for a polynomial $P' \in \mathbb{R}[X,Y,Z,W]$ homogeneous of degree $n - 2$. However, such a form cannot be smooth unless $P' = 0$. Consider for example the plane $\mathbb{R} e_1 \oplus \mathbb{R} e_2$. Then for $z := x_1 e_1 + y_2 i e_2 \in \mathbb{R} e_1 \oplus \mathbb{R} i e_2$
\[
|z|^{-2} \gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2
=(x_1^2 + y_2^2)^{-1} (x_1 dx_1 + y_2 dy_2) \wedge (x_1 dy_1 - y_2 dx_2) \wedge (x_1 d\xi_1 + y_2 d\eta_2) \wedge (x_1 d\eta_1 - y_2 d\xi_2),
\]
so none of the coefficient functions in front of the products of four of the 1-forms $dx_1, dy_1, d\xi_1, \eta_1$ and $dx_2, dy_2, d\xi_2, \eta_2$ are smooth. Thus the product of $P'(\theta_0, \theta_1, \theta_2, \omega_s)$ with any product of four of these 1-forms vanishes. This holds for any choice of orthonormal coordinates, so in particular $\omega^2 \wedge P'(\theta_0, \theta_1, \theta_2, \omega_s) = 0$. The Lefschetz decomposition implies $P'(\theta_0, \theta_1, \theta_2, \omega_s) = 0$. $\square$
3.3 A pointwise relation

For a 1-form $\eta$ on $T^*\mathbb{C}^n$ let $X_{\eta}$ denote the unique vector field with $i_{X_{\eta}}\omega_s = \eta$. For the 1-forms $\gamma_1, \beta_1$ these vector fields are given by

$$X_{\gamma_1} = \sum_{j=1}^{n} -x_j \frac{\partial}{\partial x_j} - y_j \frac{\partial}{\partial y_j},$$

$$X_{\beta_1} = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}.$$

Lemma 3.10. If $\omega \in \Omega^n(T^*\mathbb{C}^n)$ is primitive, then

$$|z|^2 \omega = \gamma_1 \wedge i_{X_{\beta_1}} \omega - \beta_1 \wedge i_{X_{\gamma_1}} \omega + \omega_s \wedge i_{X_{\beta_1}} i_{X_{\gamma_1}} \omega.$$

Proof. Note that $i_{X_{\gamma_1}} \beta_1 = -|z|^2$, $i_{X_{\beta_1}} \gamma_1 = |z|^2$. Thus

$$i_{X_{\gamma_1}} (\beta_1 \wedge \omega) = -|z|^2 \omega - \beta_1 \wedge i_{X_{\gamma_1}} \omega.$$

On the other hand, $\omega$ is primitive, that is, $\omega_s \wedge \omega = 0$, which implies

$$i_{X_{\gamma_1}} (\beta_1 \wedge \omega) = i_{X_{\gamma_1}} (i_{X_{\beta_1}} \omega_s \wedge \omega) = i_{X_{\gamma_1}} \left( i_{X_{\beta_1}} (\omega_s \wedge \omega) - \omega_s \wedge i_{X_{\beta_1}} \omega \right)$$

$$= -i_{X_{\gamma_1}} \omega_s \wedge i_{X_{\beta_1}} \omega - \omega_s \wedge i_{X_{\gamma_1}} i_{X_{\beta_1}} \omega = -\gamma_1 \wedge i_{X_{\beta_1}} \omega - \omega_s \wedge i_{X_{\gamma_1}} i_{X_{\beta_1}} \omega.$$

Thus,

$$|z|^2 \omega = -i_{X_{\gamma_1}} (\beta_1 \wedge \omega) - \beta_1 \wedge i_{X_{\gamma_1}} \omega$$

$$= \gamma_1 \wedge i_{X_{\beta_1}} \omega + \omega_s \wedge i_{X_{\gamma_1}} i_{X_{\beta_1}} \omega - \beta_1 \wedge i_{X_{\gamma_1}} \omega.$$

Definition 3.11. For $0 \leq k \leq 2n$, $\max(0, k - n) \leq q \leq \lfloor \frac{k}{2} \rfloor$ we set

$$\theta^n_{k,q} := \theta_0^{n-k+q} \wedge \theta_1^{k-2q} \wedge \theta_2^q.$$

To avoid unnecessary distinctions, we set $\theta^n_{k,q} := 0$ if $n, k, q$ do not satisfy these relations.

Corollary 3.12.

$$|z|^2 \theta^n_{k,q} \equiv \gamma_1 \wedge \left((n - k + q) \gamma_2 \wedge \theta^{n-1}_{k,q} + (k - 2q) \beta_2 \wedge \theta^{n-1}_{k-2,q} \right)$$

$$+ \beta_1 \wedge \left((k - 2q) \gamma_2 \wedge \theta^{n-1}_{k-2,q} + q \beta_2 \wedge \theta^{n-1}_{k-2,q} \right) \mod \omega_s.$$

Proof. By Theorem 3.5 there exists a differential form $\xi \in \Lambda^{n-4}(\mathbb{C}^n \times \mathbb{C}^n)^{U(1)}$ such that

$$\omega := \theta^n_{k,q} - \omega_s \wedge \xi$$

is primitive. We can thus apply Lemma 3.10 to obtain

$$|z|^2 \theta^n_{k,q} = |z|^2 \omega + |z|^2 \omega_s \wedge \xi$$

$$\equiv \gamma_1 \wedge i_{X_{\beta_1}} \omega - \beta_1 \wedge i_{X_{\gamma_1}} \omega + \omega_s \wedge i_{X_{\gamma_1}} i_{X_{\beta_1}} \omega + |z|^2 \omega_s \wedge \xi$$

$$\equiv \gamma_1 \wedge i_{X_{\beta_1}} \left( \theta^n_{k,q} - \omega_s \wedge \xi \right) - \beta_1 \wedge i_{X_{\gamma_1}} \left( \theta^n_{k,q} - \omega_s \wedge \xi \right) \mod \omega_s$$

$$\equiv \gamma_1 \wedge i_{X_{\beta_1}} \theta^n_{k,q} - \beta_1 \wedge i_{X_{\gamma_1}} \theta^n_{k,q} \mod \omega_s.$$
Note that
\[ i_{X_{\gamma_1}} \theta_0 = 0, \quad i_{X_{\gamma_1}} \theta_1 = -\gamma_2, \quad i_{X_{\gamma_1}} \theta_2 = -\beta_2, \]
\[ i_{X_{\beta_1}} \theta_0 = \gamma_2, \quad i_{X_{\beta_1}} \theta_1 = \beta_2, \quad i_{X_{\beta_1}} \theta_2 = 0. \]

Thus
\[ i_{X_{\gamma_1}} \theta_{k,q}^n = -(k - 2q)\gamma_2 \wedge \theta_{0}^{n-k+q} \wedge \theta_{1}^{1-k+q-1} \wedge \theta_{2}^{q}\beta_2 \wedge \theta_{0}^{n-k+q} \wedge \theta_{1}^{1-k+q-1} \]
\[ = -(k - 2q)\gamma_2 \wedge \theta_{k-1,q}^{n-1} - q\beta_2 \wedge \theta_{k-2,q-1}^{n-1}, \]
\[ i_{X_{\beta_1}} \theta_{k,q}^n = (n - k + q)\gamma_2 \wedge \theta_{0}^{n-k+q} \wedge \theta_{1}^{1-k+q-1} \wedge \theta_{2}^{q} + (k - 2q)\beta_2 \wedge \theta_{0}^{n-k+q} \wedge \theta_{1}^{1-k+q-1} \]
\[ = (n - k + q)\gamma_2 \wedge \theta_{k,q}^{n-1} + (k - 2q)\beta_2 \wedge \theta_{k-1,q}^{n-1}, \]
so we obtain
\[ |z|^{2} \theta_{k,q}^{n-1} \equiv \gamma_1 \wedge [(n - k + q)\gamma_2 \wedge \theta_{k,q}^{n-1} + (k - 2q)\beta_2 \wedge \theta_{k-1,q}^{n-1}] \]
\[ + \beta_1 \wedge [(k - 2q)\gamma_2 \wedge \theta_{k-1,q}^{n-1} + q\beta_2 \wedge \theta_{k-2,q-1}^{n-1}] \quad \text{mod } \omega_s. \]

\[ \square \]

### 3.4 Representation of smooth valuations

In this section we establish the representation formula for smooth valuations given in Theorem 2. By Theorem 3.2, a valuation \( \mu \in \text{VConv}_{k}(\mathbb{C}^n)^{sm} \) is uniquely determined by the symplectic Rumin differential of a representing form if \( k \geq 1 \). We thus begin by determining some relations between the symplectic Rumin differentials of the invariant forms from the previous section.

**Lemma 3.13.** Let \( R \in \mathbb{R}[X, Y, Z] \) be a homogeneous polynomial of degree \( n-1 \), \( \phi \in C_c^\infty([0, \infty)) \). Define \( \psi \in C_c^\infty([0, \infty)) \) by \( \psi(t) := -\int_t^\infty \phi(s) ds \) for \( t \geq 0 \). Then
\[ \bar{D}(\phi(|z|^2)\gamma_1 \wedge \gamma_2 \wedge R(\theta_0, \theta_1, \theta_2)) = -\bar{D}(\psi(|z|^2)\theta_0 \wedge R(\theta_0, \theta_1, \theta_2)), \]
\[ \bar{D}(\phi(|z|^2)\gamma_1 \wedge \beta_1 \wedge R(\theta_0, \theta_1, \theta_2)) = 0, \]
\[ \bar{D}(\phi(|z|^2)\gamma_1 \wedge \beta_2 \wedge R(\theta_0, \theta_1, \theta_2)) = -\frac{1}{2} \bar{D}(\psi(|z|^2)\theta_1 \wedge R(\theta_0, \theta_1, \theta_2)), \]
\[ \bar{D}(\phi(|z|^2)\gamma_2 \wedge \beta_2 \wedge R(\theta_0, \theta_1, \theta_2)) = 0. \]

**Proof.** If \( \eta \) is any of the forms \( \gamma_2, \beta_1, \beta_2 \), then
\[ \bar{D}(\phi(|z|^2)\gamma_1 \wedge \eta \wedge R(\theta_0, \theta_1, \theta_2)) = \frac{1}{2} \bar{D}(d\psi(|z|^2)\eta \wedge R(\theta_0, \theta_1, \theta_2)) \]
\[ = -\frac{1}{2} \bar{D}(\psi(|z|^2)d\eta \wedge R(\theta_0, \theta_1, \theta_2)), \]
because \( \bar{D} \) vanishes on closed forms. As \( d\gamma_2 = 2\theta_0, d\beta_1 = \omega_s \) and \( d\beta_2 = \theta_1 \), this implies the first three relations using that \( \bar{D} \) vanishes on multiples of \( \omega_s \).

For the last equation, let us show that \( \bar{D}(\phi(|z|^2)\omega_1 \wedge R(\theta_0, \theta_1, \theta_2)) = 0 \). First note that \( \omega_1 \wedge \theta_j' = |z|^2 \omega_1 \wedge \theta_j \) for all \( j = 0, 1, 2 \), where \( \theta_j' \) denotes the forms defined in (4). Let \( Q \in \mathbb{R}[X, Y, Z, W] \)
be the unique homogeneous polynomial of degree \( n - 3 \) such that \( W(R - W^2 Q) = 0 \) in \( \Lambda^*(\mathbb{C}^{n-1} \times \mathbb{C}^{n-1})^{U(n-1)} \), which exists due to Theorem 3.3. As

\[
\omega_1 \wedge \omega_2^2 = \omega_1 \wedge ((z^2 \omega_1 - \omega_1)^2 = \omega_1 \wedge (|z|^4 \omega_1^2 - 2|z|^2 \omega_1 \wedge \omega_1 + \omega_1^1)
\]

is a multiple of \( \omega_s \), we obtain in \( (0, \zeta) \neq (z, \zeta) \in T^*\mathbb{C}^n \)

\[
\omega_1 \wedge R(\theta_0, \theta_1, \theta_2) = |z|^{-2(n-1)} \omega_1 \wedge (R - W^2 Q)(|z|^2 \omega_0, |z|^2 \omega_1, |z|^2 \omega_2)
\]

\[
= |z|^{-2(n-1)} \omega_1 \wedge |z|^2 \omega_1 \wedge Q(|z|^2 \omega_0, |z|^2 \omega_1, |z|^2 \omega_2)
\]

\[
= |z|^{-2(n-1)} \omega_1 \wedge (R - W^2 Q)(|z|^2 \omega_0, |z|^2 \omega_1, |z|^2 \omega_2) \mod \omega_s
\]

\[
= |z|^{-2(n-1)} (\omega_1 + \omega_2) \wedge (R - W^2 Q)(\theta_0', \theta_1', \theta_2')
\]

Here we used that \( \omega_2 \wedge (R - W Q)(\theta_0', \theta_1', \theta_2') = 0 \) and that \( \omega_1 \wedge |z|^2 \theta_j = \omega_1 \wedge \theta_j' \). As \( \bar{D} \) vanishes on multiples of \( \omega_s \), this implies

\[
\bar{D}(\phi(|z|^2) \omega_1 \wedge R(\theta_0, \theta_1, \theta_2)) = 0 \quad \text{for} \quad (z, \zeta) \in T^*\mathbb{C}^n, z \neq 0,
\]

which implies \( \bar{D}(\phi(|z|^2) \omega_1 \wedge R(\theta_0, \theta_1, \theta_2)) = 0 \) by continuity. In particular,

\[
\bar{D}(\phi(|z|^2) \gamma_2 \wedge \beta_2 \wedge R(\theta_0, \theta_1, \theta_2))
\]

\[
= \bar{D}(\phi(|z|^2) \gamma_2 \wedge \beta_2 \wedge R(\theta_0, \theta_1, \theta_2)) + \bar{D}(\phi(|z|^2) \gamma_1 \wedge \beta_1 \wedge R(\theta_0, \theta_1, \theta_2))
\]

\[
= \bar{D}(\phi(|z|^2) \omega_1 \wedge R(\theta_0, \theta_1, \theta_2)) = 0.
\]

\[\square\]

Lemma 3.14. For \( \phi \in C^\infty_c([0, \infty)) \) define \( \psi \in C^\infty_c([0, \infty)) \) by \( \psi(t) := -\int_t^\infty \phi(s) ds \) for \( t \geq 0 \). Then

\[
\bar{D}\left( \left[ \phi(|z|^2)|z|^2 + \frac{2n - k}{2} \psi(|z|^2) \right] \theta^n_{q,q} \right)
\]

\[
= (k - 2q) \bar{D}(\phi(|z|^2) \beta_2 \wedge \gamma_2 \wedge \theta^n_{k-1,q}) + q \bar{D}(\phi(|z|^2) \beta_1 \wedge \beta_2 \wedge \theta^n_{k-2,q-1}).
\]

Proof. As \( \psi' = \phi \), Lemma 3.13 implies

\[
\bar{D}(\psi(|z|^2) \theta^n_{k,q}) = -\bar{D}(\phi(|z|^2) \gamma_1 \wedge \gamma_2 \wedge \theta^n_{k,q}) = -2 \bar{D}(\phi(|z|^2) \gamma_1 \wedge \beta_2 \wedge \theta^n_{k-1,q}).
\]

According to Corollary 3.12

\[
|z|^2 \theta^n_{k,q} \equiv \gamma_1 \wedge [(n - k + q) \gamma_2 \wedge \theta^n_{k,q} + (k - 2q) \beta_2 \wedge \theta^n_{k-1,q}] + \beta_1 \wedge [(k - 2q) \gamma_2 \wedge \theta^n_{k-1,q} + q \beta_2 \wedge \theta^n_{k-2,q-1}] \mod \omega_s.
\]
As $\bar{D}$ vanishes on multiples of $\omega_s$, this implies
\[
\bar{D}(\phi(|z|^2)|z|^2\theta^n_{k,q}) = (n - k + q) \bar{D}(\phi(|z|^2)\gamma_1 \wedge \gamma_2 \wedge \theta^{n-1}_{k-1,q}) + (k - 2q) \bar{D}(\phi(|z|^2)\gamma_1 \wedge \beta_1 \wedge \theta^{n-1}_{k-1,q}) + (k - 2q) \bar{D}(\phi(|z|^2)\beta_1 \wedge \gamma_2 \wedge \theta^{n-1}_{k-1,q}) + q \bar{D}(\phi(|z|^2)\beta_1 \wedge \beta_2 \wedge \theta^{n-1}_{k-2,q-1}).
\]

Rearranging this equation, we obtain the desired result. 

**Lemma 3.15.** Let $R \in \mathbb{R}[X,Y,Z]$ be a homogeneous polynomial of degree $n - 2$ and $\phi \in C_\infty^\infty((0,\infty))$. Then
\[
2\bar{D}(\phi'(|z|^2)\gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2 \wedge R(\theta_0,\theta_1,\theta_2)) = \bar{D}(\phi(|z|^2)[\beta_1 \wedge \gamma_2 \wedge \theta_1 - 2\beta_1 \wedge \beta_2 \wedge \theta_0] \wedge R(\theta_0,\theta_1,\theta_2)).
\]

**Proof.** Using that $\bar{D}$ vanishes on closed forms, we obtain
\[
\bar{D}(\phi'(|z|^2)\gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2 \wedge R(\theta_0,\theta_1,\theta_2)) = \frac{1}{2} \bar{D}(d\phi(|z|^2)\gamma_2 \wedge \beta_1 \wedge \beta_2 \wedge R(\theta_0,\theta_1,\theta_2)) = \frac{1}{2} \bar{D}(d\phi(|z|^2)[2\theta_0 \wedge \beta_1 \wedge \beta_2 - \gamma_2 \wedge \omega_s \wedge \beta_2 + \gamma_2 \wedge \beta_1 \wedge \theta_1] \wedge R(\theta_0,\theta_1,\theta_2)) = 0 - \frac{1}{2} \bar{D}(\phi(|z|^2)[2\theta_0 \wedge \beta_1 \wedge \beta_2 - \gamma_2 \wedge \omega_s \wedge \beta_2 + \gamma_2 \wedge \beta_1 \wedge \theta_1] \wedge R(\theta_0,\theta_1,\theta_2)) = \frac{1}{2} \bar{D}(\phi(|z|^2)\beta_1 \wedge \gamma_2 \wedge \theta_1 - 2\beta_1 \wedge \beta_2 \wedge \theta_0) \wedge R(\theta_0,\theta_1,\theta_2)),
\]
due to the fact that $\bar{D}$ vanishes on multiples of $\omega_s$. 

We can now establish the representation formula in Theorem 2. Let us consider the forms
\[
\Theta^n_{k,q} := c_{n,k,q} \theta^n_{k,q} \quad \text{for} \max(0,k-n) \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor, \quad \text{and} \n\]
\[
\Upsilon^n_{k,q} := c_{n,k,q} \left[\beta_1 \wedge \beta_2 \wedge \theta^{n-1}_{k-2,q-1} - 2\beta_1 \wedge \gamma_2 \wedge \theta^{n-1}_{k-1,q}\right] \quad \text{for} \max(1,k-n) \leq q \leq \left\lfloor \frac{k-1}{2} \right\rfloor, \quad \text{where we set} \quad c_{n,k,q} := \frac{1}{(n-k+q)(q-k-1)!}.
\]

**Theorem 3.16.** Let $1 \leq k \leq 2n$. For every smooth valuation $\mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)}$ there exist $\phi_q \in C_\infty^\infty([0,\infty))$ for $\max(0,k-n) \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor$ and $\psi_q \in C_\infty^\infty([0,\infty))$ for $\max(1,k-n) \leq q \leq \left\lfloor \frac{k-1}{2} \right\rfloor$ such that
\[
\mu(f) = \sum_{q=\max(0,k-n)}^{\left\lfloor \frac{k}{2} \right\rfloor} D(f)[\phi_q(|z|^2)\Theta^n_{k,q}] + \sum_{q=\max(1,k-n)}^{\left\lfloor \frac{k-1}{2} \right\rfloor} D(f)[\psi_q(|z|^2)\Upsilon^n_{k,q}]
\]
for all $f \in \text{Conv}(\mathbb{C}^n, \mathbb{R})$. 

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Proof. Any smooth \( \mu \in VConv_k(\mathbb{C}^n)^{U(n)} \) of degree \( k \geq 1 \) may be represented by a differential form \( \omega \in (\Omega_k^{n-1}(\mathbb{C}^n) \otimes \Lambda^k\mathbb{C}^n)^{U(n)} \) of degree \( 2n \) by Proposition 3.11. By Theorem 3.9 \( \omega \) is given by

\[
\omega = \gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2 \wedge R(\theta_0, \theta_1, \theta_2, \omega_\kappa) \left[ |z|^2 \right] 
+ \gamma_1 \wedge \gamma_2 \wedge R_{\gamma_1\gamma_2}(\theta_0, \theta_1, \theta_2, \omega_\kappa) \left[ |z|^2 \right] 
+ \gamma_1 \wedge \beta_1 \wedge R_{\gamma_1\beta_1}(\theta_0, \theta_1, \theta_2, \omega_\kappa) \left[ |z|^2 \right] 
+ \gamma_2 \wedge \beta_1 \wedge R_{\gamma_2\beta_1}(\theta_0, \theta_1, \theta_2, \omega_\kappa) \left[ |z|^2 \right] 
+ \gamma_2 \wedge \beta_2 \wedge R_{\gamma_2\beta_2}(\theta_0, \theta_1, \theta_2, \omega_\kappa) \left[ |z|^2 \right] 
+ \gamma_1 \wedge \beta_1 \wedge R_{\beta_1\beta_1}(\theta_0, \theta_1, \theta_2, \omega_\kappa) \left[ |z|^2 \right] 
+ \gamma_2 \wedge \beta_2 \wedge R_{\beta_2\beta_2}(\theta_0, \theta_1, \theta_2, \omega_\kappa) \left[ |z|^2 \right] 
+ R'(\theta_0, \theta_1, \theta_2, \omega_\kappa) \left[ |z|^2 \right]
\]

for suitable \( R, R_{\gamma_1\gamma_2}, \ldots, R' \in C^\infty([0, \infty), \mathbb{R}[X, Y, Z, W]) \). If the support of \( \omega \) is contained in \( B_R(0) \times \mathbb{C}^n \subset \mathbb{T}^*\mathbb{C}^n \), we may choose \( \psi \in C_c^\infty([0, \infty), [0, 1]) \) with \( \psi \equiv 1 \) on \([0, R^2] \). By multiplying the equation above with \( \psi(|z|^2) \), we may thus assume that the support of every coefficient of these polynomials is also bounded. As the differential cycle vanishes on multiples of \( \omega_\kappa \), we may further assume that these polynomials do not depend on \( \omega_\kappa \). Then according to Lemma 3.13

\[
\bar{D} \omega = \bar{D} \left( \gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2 \wedge R(\theta_0, \theta_1, \theta_2) \left[ |z|^2 \right] \right) 
+ \bar{D} \left( \gamma_1 \wedge \gamma_2 \wedge R_{\gamma_1\gamma_2}(\theta_0, \theta_1, \theta_2) \left[ |z|^2 \right] \right) + 0 
+ \bar{D} \left( \gamma_1 \wedge \beta_1 \wedge R_{\gamma_1\beta_1}(\theta_0, \theta_1, \theta_2) \left[ |z|^2 \right] \right) + \bar{D} \left( \gamma_2 \wedge \beta_1 \wedge R_{\gamma_2\beta_1}(\theta_0, \theta_1, \theta_2) \left[ |z|^2 \right] \right) 
+ 0 + \bar{D} \left( \beta_1 \wedge \beta_2 \wedge R_{\beta_1\beta_2}(\theta_0, \theta_1, \theta_2) \left[ |z|^2 \right] \right)
+ \bar{D} \left( R'(\theta_0, \theta_1, \theta_2) \left[ |z|^2 \right] \right).
\]

Using Lemma 3.13, we may replace the terms involving \( \gamma_1 \wedge \gamma_2 \) and \( \gamma_1 \wedge \beta_1 \) by a suitable function in \( C_c^\infty([0, \infty), \mathbb{R}[X, Y, Z]) \). Similarly, Lemma 3.13 lets us replace the term involving \( \gamma_1 \wedge \gamma_2 \wedge \beta_1 \wedge \beta_2 \) by multiples of \( \beta_1 \wedge \gamma_2 \) and \( \beta_1 \wedge \beta_2 \). Thus there exist \( R_1, R_2, R_3 \in C_c^\infty([0, \infty), \mathbb{R}[X, Y, Z]) \) such that

\[
\bar{D} \omega = \bar{D}(R_1(\theta_0, \theta_1, \theta_2) \left[ |z|^2 \right]) + \bar{D}(\beta_1 \wedge \gamma_2 \wedge R_2(\theta_0, \theta_1, \theta_2) \left[ |z|^2 \right]) 
+ \bar{D}(\beta_1 \wedge \beta_2 \wedge R_3(\theta_0, \theta_1, \theta_2) \left[ |z|^2 \right]).
\]

The first term can clearly be expressed in terms of the forms \( \Theta_{k,q}^n \), while the last two terms can be written in terms of the forms

\[
(k - 2q)\beta_1 \wedge \gamma_2 \wedge \theta_1^{n-1}_{k-1,q} + q\beta_1 \wedge \beta_2 \wedge \theta_1^{n-1}_{k-2,q-1}, 
\beta_1 \wedge \beta_2 \wedge \theta_1^{n-1}_{k-2,q-1} - 2\beta_1 \wedge \gamma_2 \wedge \theta_1^{n-1}_{k-1,q},
\]

where we may replace the first term by a term involving \( \Theta_{k,q}^n \) due to Lemma 3.14, while the second is a multiple of \( \Upsilon_{k,q}^n \) for \( 1 \leq q < \frac{k}{2} \). In the extremal cases \( q = 0 \) or \( q = \frac{k}{2} \), both of the forms coincide and we may replace them by a term involving \( \Theta_{k,q}^n \) due to Lemma 3.14. We thus find \( \phi_q \in C_c^\infty([0, \infty)) \) for \( \max(0, k-n) \leq q \leq \frac{k}{2} \) and \( \psi_q \in C_c^\infty([0, \infty)) \) for \( \max(1, k-n) \leq q \leq \frac{k-1}{2} \) such that

\[
\bar{D} \omega = \sum_{q = \max(0, k-n)}^{\frac{k}{2}} \bar{D}(\phi_q(|z|^2))\Theta_{k,q}^n \omega + \sum_{q = \max(1, k-n)}^{\frac{k-1}{2}} \bar{D}(\psi_q(|z|^2))\Upsilon_{k,q}^n \omega.
\]
The Kernel Theorem thus implies for all \( f \in \text{Conv}(\mathbb{C}^n, \mathbb{R}) \)
\[
\mu(f) = D(f) [\omega] = \sum_{q = \max(0,k-n)}^{\lfloor \frac{k}{2} \rfloor} D(f)[\phi_q(|z|^2)\theta_{k,q}^n] + \sum_{q = \max(1,k-n)}^{\lfloor \frac{k-1}{2} \rfloor} D(f) [\psi_q(|z|^2)\Upsilon_{k,q}] .
\]
Thus \( \mu \) has the desired representation.

4 The Fourier-Laplace transform for \( U(n) \)-invariant valuations

We begin this section by introducing the polynomials \( P_{k,q} \) and the differential operators \( D_{\gamma,w}^{k,q} \) and \( D_{\beta,w}^{k,q} \). These objects are constructed using the mixed discriminant of certain matrices associated to \( w_1, \ldots, w_k \in \mathbb{C}^n \).

4.1 A family of invariant polynomials

For \( w_1, \ldots, w_k, z \in \mathbb{C}^n \) set \( w := (w_1, \ldots, w_k) \) and consider the real \( k \times k \)-matrices
\[
I_w := (\text{Im}(w_j, w_l))_{j,l=1}^k ,
\]
\[
R_w := (\text{Re}(w_j, w_l))_{j,l=1}^k .
\]

Lemma 4.1. If \( w_1, \ldots, w_k \in \mathbb{C}^n \) are linearly dependent over \( \mathbb{R} \), then the rank of
\[
sI_w + tR_w
\]
is at most \( k-1 \) for all \( t, s \in \mathbb{R} \).

Proof. Without loss of generality assume that \( w_k = \sum_{j=1}^{k-1} a_j w_j \) for \( a_1, \ldots, a_{k-1} \in \mathbb{R} \).
It is sufficient to show that the rank of \( R_w + tI_w \) is smaller than \( k \) for all \( t \in \mathbb{R} \). Note that
\[
(R_w + tI_w)_{jl} = \text{Re}(w_j, w_l) + t \text{Im}(w_j, w_l).
\]
Consider the \( \mathbb{R} \)-bilinear form \( \langle v, w \rangle_t := \text{Re}(v, w) + t \text{Im}(v, w) \). We thus have to consider the rank of the matrix
\[
R_w + tiI_w = \begin{pmatrix}
\langle w_1, w_1 \rangle_t & \ldots & \langle w_1, w_k \rangle_t \\
\ldots & \ldots & \ldots \\
\langle w_k, w_1 \rangle_t & \ldots & \langle w_k, w_k \rangle_t
\end{pmatrix},
\]
which is at most \( k-1 \), as we can subtract \( a_j \) times the \( j \)th row from the last row for \( 1 \leq j \leq k-1 \) to obtain 0 in the \( k \)th row. Thus the rank of this matrix is at most \( k-1 \).

Definition 4.2. We define a (real) polynomial on \((\mathbb{C}^n)^k\) by
\[
P_{k,q}(w_1, \ldots, w_k) := \sum_{j=q}^{\lfloor \frac{k}{2} \rfloor} (-1)^{j-q} \binom{j}{q} \det_k(I_w[2j], R_w[k-2j]).
\]
Here, \( \det_k \) denotes the mixed discriminant on the space of \((k \times k)\)-matrices.
Note that the mixed discriminant \( \det_k(A_1, \ldots, A_k) \) vanishes if the sum of the images of the matrices \( A_1, \ldots, A_k \) does not span \( \mathbb{R}^k \).

**Lemma 4.3.** For every \( E \in \text{Gr}_k(\mathbb{C}^n) \) there exists a constant \( c(E) \in \mathbb{R} \) such that

\[
P_{k,q}(w_1, \ldots, w_k) = c(E) \det_k(\text{Re}(w_j, w_l))_{j,l=1}^k \quad \text{for all } w_1, \ldots, w_k \in E. \tag{6}
\]

**Proof.** Due to Lemma 4.1, both sides of (6) vanish if \( w_1, \ldots, w_k \in E \) are linearly dependent. Considered as polynomials on \( E^k \cong (\mathbb{R}^k)^k \), the two sides thus vanish on the zero set of the determinant. As this is an irreducible polynomial, both sides are divisible by the determinant. Comparing the degrees of homogeneity and using that both sides are symmetric with respect to permutations of the vectors \( w_1, \ldots, w_k \), we see that the quotients define multilinear, anti-symmetric polynomials, that is, they are scalar multiples of the determinant. Thus the two sides differ by a multiplicative constant depending on \( E \in \text{Gr}_k(\mathbb{C}^n) \).  

It follows from basic properties of the mixed determinant that \( P_{k,q} \) is invariant with respect to the diagonal action of \( U(n) \) on \( (\mathbb{C}^n)^k \). We may thus identify \( P_{k,q} \) with a certain \( U(n) \)-invariant function on \( \text{Gr}_k(\mathbb{C}^n) \).

In his study of Poincare-Crofton formulas for submanifolds of complex space forms, Tasaki [48] derived a description of the orbits of the natural operation of \( U(n) \) on \( \text{Gr}_k(\mathbb{C}^n) \). For \( k \leq n \) he showed that the family of \( U(n) \)-orbits is in 1-to-1-correspondence with the \( \lfloor \frac{k}{2} \rfloor \)-dimensional simplex

\[
0 \leq \theta_1 \leq \cdots \leq \theta_{\lfloor \frac{k}{2} \rfloor} \leq \frac{\pi}{2}.
\]

Using this correspondence, the invariant \( \theta(E) := (\theta_1(E), \ldots, \theta_{\lfloor \frac{k}{2} \rfloor}(E)) \) is called the multiple Kähler angle of \( E \in \text{Gr}_k(\mathbb{C}^n) \). For \( k > n \) it is defined by

\[
\theta(E) := (0, \ldots, 0, \theta(E_{k-n})) \quad \text{for } E \in \text{Gr}_k(\mathbb{C}^n), \tag{7}
\]

where we use the convention used in [13] (Tasaki [48] defined the multiple Kähler angle to be \( \theta(E_{k-n}) \) in this case). Tasaki showed in [48] Proposition 3 that there exists a complex orthonormal basis \( e_1, \ldots, e_n \) of \( \mathbb{C}^n \) such that the vectors

\[
e_1, i e_1 \ldots e_{k-n}, i e_{k-n} \quad \text{if } k > n, \text{ and}
\]

\[
e_{2j-1}, \cos \theta_j(E) i e_{2j-1} + \sin(\theta_j(E)) e_{2j} \quad \text{for } \max(0, k-n) + 1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor, \text{ and}
\]

\[
e_k \quad \text{if } k \text{ is odd}
\]

form an orthonormal basis of \( E \) as a Euclidean vector space and we call such a basis a Tasaki basis for \( E \). Note that the first family of vectors corresponds formally to \( \theta_1 = \cdots = \theta_{k-n} = 0 \) for \( k > n \), which is consistent with the convention for the multiple Kähler angle in [7]. A simple calculation shows that the multiple Kähler angle is encoded in the matrix \( I_w \).

**Lemma 4.4.** If \( w_1, \ldots, w_k \) is a Tasaki basis of \( E \in \text{Gr}_k(\mathbb{R}^n) \), then

\[
I_w = \begin{cases}
\text{diag} \left( \begin{pmatrix} 0 & -\cos \theta_j \\ \cos \theta_j & 0 \end{pmatrix} \right), & 1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor, \quad \text{if } k \text{ is even}, \\
\text{diag} \left( \begin{pmatrix} 0 & -\cos \theta_1 \\ \cos \theta_1 & 0 \end{pmatrix} \right), \ldots, \left( \begin{pmatrix} 0 & -\cos \theta_{\lfloor \frac{k}{2} \rfloor} \\ \cos \theta_{\lfloor \frac{k}{2} \rfloor} & 0 \end{pmatrix} \right), & \text{if } k \text{ is odd}.
\end{cases}
\]
Proposition 4.5. For $E \in \text{Gr}_k(\mathbb{C}^n)$ define $\sigma_q(\theta(E)) := \sigma_q(\cos^2 \theta_1(E), \ldots, \cos^2 \theta_{\frac{k}{2}}(E))$, where $\sigma_q$ is the $q$th elementary symmetric polynomial. If $w_1, \ldots, w_k \in E$ then

$$P_{k,q}(w_1, \ldots, w_k) = \det_k(\text{Re}(w_i, w_j))^k_{i,j=1} \sum_{i=q}^{\lfloor \frac{k}{2} \rfloor} (-1)^{i+q} \binom{i}{q} \sigma_i(\theta(E)) = \det_k(\text{Re}(w_i, w_j))^k_{i,j=1} K_{\mu_k,q}(E).$$

Proof. Due to Lemma 4.3 we only have to show that the two sides coincide on a Tasaki basis $w_1, \ldots, w_k$ of $E$. As $R_w = \text{Id}_k$ in this case, the first equation follows from basic properties of the mixed discriminant in combination with Lemma 4.4. We refer to [13] Section 3.4 for the Klain function of the Hermitian intrinsic volumes. 

4.2 Two families of equivariant differential operators

In addition to the matrices considered in the previous section, we associate to $w_1, \ldots, w_k \in \mathbb{C}^n$ and $z \in \mathbb{C}^n$ the matrices

$$Z^I_w(z) := \begin{pmatrix} \text{Re}(w_1, z) \\ \vdots \\ \text{Re}(w_k, z) \end{pmatrix} \cdot \begin{pmatrix} \text{Im}(w_1, z) \\ \vdots \\ \text{Im}(w_k, z) \end{pmatrix}^T - \begin{pmatrix} \text{Im}(w_1, z) \\ \vdots \\ \text{Im}(w_k, z) \end{pmatrix} \cdot \begin{pmatrix} \text{Re}(w_1, z) \\ \vdots \\ \text{Re}(w_k, z) \end{pmatrix}^T,$$

$$Z^R_w(z) := \begin{pmatrix} \text{Re}(w_1, z) \\ \vdots \\ \text{Re}(w_k, z) \end{pmatrix} \cdot \begin{pmatrix} \text{Re}(w_1, z) \\ \vdots \\ \text{Re}(w_k, z) \end{pmatrix}^T.$$

In the calculations below, we will denote the entries of these matrices by $Z^I_{ji}, Z^R_{ji}$, suppressing the dependency on $w_1, \ldots, w_k, z \in \mathbb{C}^n$.

Lemma 4.6. The image of $Z^I_w(z)$ and $Z^R_w(z)$ is contained in the sum of the images of $I_w$ and $R_w$. In particular, the rank of

$$\lambda_1 I_w + \lambda_2 R_w + \lambda_3 Z^I_w(z) + \lambda_4 Z^R_w(z)$$

is at most $k - 1$ for all $\lambda_1, \ldots, \lambda_4 \in \mathbb{R}$ if $w_1, \ldots, w_k \in \mathbb{C}^n$ are linearly dependent over $\mathbb{R}$.

Proof. From the definition one directly obtains that the image of $Z^R_w(z)$ is contained in the image of $Z^I_w(z)$, which is spanned by

$$\begin{pmatrix} \text{Re}(w_1, z) \\ \vdots \\ \text{Re}(w_k, z) \end{pmatrix} \text{ and } \begin{pmatrix} \text{Im}(w_1, z) \\ \vdots \\ \text{Im}(w_k, z) \end{pmatrix}.$$

It is thus sufficient to show that these vectors belong to the sum of the images of $I_w$ and $R_w$. Note that we can write $z = \sum_{j=1}^k a_j w_j + \tilde{z}$ for $a_1, \ldots, a_k \in \mathbb{R}$, where $\tilde{z}$ is perpendicular to $w_1, \ldots, w_k$ with respect to $\text{Re} \langle \cdot, \cdot \rangle$. Thus

$$\begin{pmatrix} \text{Re}(w_1, z) \\ \text{Re}(w_k, z) \end{pmatrix} = \sum_{j=1}^k a_j \begin{pmatrix} \text{Re}(w_1, w_j) \\ \text{Re}(w_k, w_j) \end{pmatrix}.$$
is contained in the image of \( R_w \). Now note that \( \text{Re}(w, iz) = \text{Im}(w, z) \) for any \( w \in \mathbb{C}^n \). Thus applying the same argument to \( iz \) shows that \( (\text{Im}(w_1, z), \ldots, \text{Im}(w_k, z))^T \) belongs to the image of \( R_w \).

Combined with Lemma 4.1 this implies that the dimension of the image of the matrix \( \lambda_1 I_w + \lambda_2 R_w + \lambda_3 Z_w(z) + \lambda_4 Z_{w_R}^R(z) \) is at most \( k - 1 \) if \( w_1, \ldots, w_k \in \mathbb{C}^n \) are linearly dependent over \( \mathbb{R} \). Thus its rank is at most \( k - 1 \). \( \square \)

**Definition 4.7.** For fixed \( w_1, \ldots, w_k \in \mathbb{C}^n \) we define two differential operators \( D_{\beta,w}^{k,q} \) and \( D_{\gamma,w}^{k,q} \) on \( \mathbb{C}^n \) by the following symbols:

\[
D_{\beta,w}^{k,q}[z] := \frac{1}{2} \sum_{j=q}^{\lfloor \frac{k}{2} \rfloor} (-1)^{q+j} \binom{j-1}{q-1} \det(I_w[2(j-1)], Z_w^I(z), R_w[k-2j]),
\]

\[
D_{\gamma,w}^{k,q}[z] := \sum_{j=q}^{\lfloor \frac{k}{2} \rfloor} (-1)^{q+j} \binom{j}{q} \det_k(I_w[2j], Z_w^R(z), R_w[k-2j]) - D_{\beta,w}^{k,q+1}[z].
\]

As in the previous section, these symbols vanish identically if \( w_1, \ldots, w_k \in \mathbb{C}^n \) are linearly dependent. If we fix \( E \in \text{Gr}_k(\mathbb{C}^n) \) and consider the restriction of these symbols to \( E^k \), we may apply the same reasoning as in Lemma 4.3 to see that these restrictions are given by \( \det_k(\text{Re}(w_j, w_i))_{j,i=1}^k P_E(z) \) for a quadratic polynomial \( P_E \) depending on the subspace \( E \) only. To determine the polynomial \( P_E \), it is thus again sufficient to evaluate this expression in a Tasaki basis. We will need the following lemma.

**Lemma 4.8.** If \( w_1, \ldots, w_k \in E \in \text{Gr}_k(\mathbb{C}^n) \) form a Tasaki basis, then

\[
\det(I_w[2q-1], Z_w^I(z), R_w[k-2q]) = \frac{2}{(q-1)!(\lfloor \frac{k}{2} \rfloor - q)!) \sum_{\sigma \in S(\lfloor \frac{k}{2} \rfloor)} \left( \prod_{j=1}^{q-1} \cos \theta_{\sigma(j)}^2 \right) \cos \theta_{\sigma(q)} Z_{2\sigma(q)-1,2\sigma(q)}^I
\]

and

\[
\det_k(I_w[2q], Z_w^R(z), R_w[k-2q-1]) = \frac{1}{q!((\lfloor \frac{k}{2} \rfloor - q - 1)!} \sum_{\sigma \in S(\lfloor \frac{k}{2} \rfloor)} \left( \prod_{j=1}^{q} \cos \theta_{\sigma(j)}^2 \right) \left( Z_{2\sigma(q+1)-1,2\sigma(q+1)}^R + Z_{2\sigma(q+1),2\sigma(q+1)}^R \right)
\]

\[
+ \frac{1}{q!(\lfloor \frac{k}{2} \rfloor - q)!} Z_{k,k}^R \sum_{\sigma \in S(\lfloor \frac{k}{2} \rfloor)} \prod_{j=1}^{q} \cos \theta_{\sigma(j)}^2 \text{ if } k \text{ is odd}
\]

**Proof.** We will only show the first equation for even \( k \), the other calculations are similar. Set

\[
A(s,t) = sI_w + tR_w = \text{diag} \left( \begin{pmatrix} t & -s \cos \theta_j \\ s \cos \theta_j & t \end{pmatrix}, 1 \leq j \leq \lfloor \frac{k}{2} \rfloor \right).
\]

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Then
\[ A(s, t)^{-1} = \text{diag} \left( \frac{1}{t^2 + s^2 \cos \theta_j^2} \begin{pmatrix} t & s \cos \theta_j \\ -s \cos \theta_j & t \end{pmatrix}, 1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor \right), \]
and we obtain
\[
\det_k(I_w[2q], Z_{w}^l(z), R_w[k - 2q - 1]) = \frac{1}{(2q)! (k - 2q - 1)!} \frac{\partial^{2q} \partial^{k-2q-1}}{\partial s^{2q} \partial t^{k-2q-1}} \frac{\partial}{\partial \lambda_1} \det_k(A(s, t) + \lambda Z_{w}^l(z)) = \frac{1}{(2q)! (k - 2q - 1)!} \frac{\partial^{2q} \partial^{k-2q-1}}{\partial s^{2q} \partial t^{k-2q-1}} 0 \det_k(A(s, t)) \text{tr}(A(s, t)^{-1} Z_{w}^l(z)).
\]

As \( Z_{w}^l(z) \) is skew symmetric, this implies
\[
(2q - 1)!(k - 2q)! \det_k(I_w[2q - 1], Z_{w}^l(z), R_w[k - 2q]) = \frac{\partial^{2q-1} \partial^{k-2q}}{\partial s^{2q-1} \partial t^{k-2q}} \left| \prod_{j=1}^{[\frac{k}{2}]} \left( t^2 + s^2 \cos \theta_j^2 \right) \sum_{j=1}^{k} -2I_{2j-1,2j} Z_{2j-1,2j} \right| \frac{\partial}{\partial \lambda_1} \det_k(A(s, t) + \lambda Z_{w}^l(z)) = \frac{2}{(q - 1)! ([\frac{k}{2}] - q)!} \sum_{\sigma \in S_{\frac{k}{2}}} \left( \prod_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \left( t^2 + s^2 \cos \theta_{\sigma(j)}^2 \right) \cos \theta_{\sigma(\left\lfloor \frac{k}{2} \right\rfloor)} Z_{2\sigma(\left\lfloor \frac{k}{2} \right\rfloor) - 1, 2\sigma(\left\lfloor \frac{k}{2} \right\rfloor)} \right).\]

**Corollary 4.9.** For \( w_1, \ldots, w_k \in E_{k,p} = \mathbb{C}^p \times \mathbb{R}^{k-2p}, \)
\[
D_{\beta_{\gamma,w}^{k,q}}[z] = \delta_{pq} \det_k(\text{Re}(w_j, w_l))_{j,l=1}^{k} \sum_{j=1}^{q} |z_j|^2, \\
D_{\gamma_{\beta,w}^{k,q}}[z] = \delta_{pq} \det_k(\text{Re}(w_j, w_l))_{j,l=1}^{k} \sum_{j=q+1}^{k} |\text{Re}(z_j)|^2.
\]

**Proof.** We will again only consider the first equation and leave the second to the reader. It is sufficient to evaluate this expression in a Tasaki basis for \( E_{k,p} \) due to Lemma 4.6. We can thus assume that this basis is given by
\[
w_{2j-1} = e_j, w_{2j} = ie_j \quad \text{for} \quad 1 \leq j \leq p, \\
w_j = e_{p+j} \quad \text{for} \quad k - 2p + 1 \leq j \leq k.
\]
If \( p < q, \) then it is easy to see that the expression vanishes. Let us thus assume that \( p \geq q. \) Then
\[
Z_{2j-1,2j}^l = \begin{cases} 
|z_j|^2 & 1 \leq j \leq p, \\
0 & p < j \leq \left\lfloor \frac{k}{2} \right\rfloor.
\end{cases}
\]

Using Lemma 4.8, we obtain

\[ D_{k,q}^{\gamma_p}[z] = \sum_{j=q}^{p} (-1)^{q-j} \frac{(j-1)}{(q-1)! \left( \frac{1}{2} \right)!} \sum_{\sigma \in S_{\frac{1}{2}}} \prod_{l=1}^{j-1} \cos \theta_{\sigma(l)} \cos \theta_{\sigma(j)} \sum_{l=1}^{p} \frac{1}{(l-1)!} \left( \frac{1}{2} \right)! \]

\[ = \sum_{j=q}^{p} (-1)^{q-j} \frac{(j-1)}{(q-1)!} \sum_{l=1}^{p} \left( p - 1 \right) \left( j - 1 \right) |z_l|^2 = \sum_{j=q}^{p} (-1)^{q-j} \left( p - q \right) \sum_{l=1}^{p} |z_l|^2. \]

The last term vanishes unless \( p = q \), so we obtain the desired formula. \( \square \)

### 4.3 The Fourier-Laplace transform in the smooth case

**Proposition 4.10.** For \( \phi \in C_\infty^\infty(\mathbb{C}^n) \) consider the valuations

\[ \mu_{k,q,\phi}(f) = c_{n,k,q} D^{\phi}(f) \left[ \frac{1}{k} \right], \quad \max(0, k - n) \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor, \]

\[ \mu_{k,q,\phi}^\beta(f) = c_{n,k,q} D^{\phi}(f) \left[ \frac{1}{k} \right] \left[ \beta \right], \quad \max(1, k - n) \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor, \]

\[ \mu_{k,q,\phi}^\gamma(f) = c_{n,k,q} D^{\phi}(f) \left[ \frac{1}{k} \right] \left[ \gamma \right], \quad \max(0, k - n) \leq q \leq \left\lfloor \frac{k - 1}{2} \right\rfloor, \]

where \( c_{n,k,q} = \frac{1}{(n-k+q)!}. \) Then

\( \mathcal{F}(\text{GW}(\mu_{k,q,\phi}))[w_1, \ldots, w_k] = \frac{(-1)^k}{k!} P_{k,q}(w_1, \ldots, w_k) \mathcal{F}(\phi) \left[ \sum_{j=1}^{k} w_j \right], \)

\( \mathcal{F}(\text{GW}(\mu_{k,q,\phi}^\beta))[w_1, \ldots, w_k] = \frac{(-1)^k}{k!} \frac{1}{q} D_{\beta,\phi}^{k,q} \mathcal{F}(\phi) \left[ \sum_{j=1}^{k} w_j \right], \)

\( \mathcal{F}(\text{GW}(\mu_{k,q,\phi}^\gamma))[w_1, \ldots, w_k] = \frac{(-1)^k}{k!} \frac{1}{k - 2q} D_{\gamma,\phi}^{k,q} \mathcal{F}(\phi) \left[ \sum_{j=1}^{k} w_j \right]. \)

**Proof.** We will only prove these formulas for the valuations \( \mu_{k,q,\phi}^\gamma \), the other two cases are similar but simpler. It is enough to evaluate both sides in \( -w_1 \otimes i, \ldots, -w_k \otimes i \) for \( w_1, \ldots, w_k \in \mathbb{C}^n \). Due to Remark 2.3

\( \mathcal{F}(\text{GW}(\mu_{k,q,\phi}^\gamma))[-w_1 \otimes i, \ldots, -w_k \otimes i] = \frac{1}{k!} \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_k} \mu_{k,q,\phi} \left[ \sum_{j=1}^{k} \lambda_j \exp(w_j, i) \right] \)

\( = c_{n,k,q} \frac{1}{k!} \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_k} \int_{\mathbb{C}^n} \phi(z) G_f^\gamma(\beta_1 \wedge \beta_2 \wedge \theta_{k-2,q-1}^{n-1}) \)

for \( f(z) = \sum_{j=1}^{k} \lambda_j \exp Re(w_j, z) \) and the map

\( G_f : \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n \)

\( z \mapsto (z, df(z)). \)
It is convenient to work with complex-valued differential forms. Using the differential operators
\[ \frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}), \]
we have \( dg = \partial g + \bar{\partial} g \) for any \( g \in C^\infty(\mathbb{C}^n) \), where
\[
\partial g = \sum_{j=1}^{n} \frac{\partial g}{\partial z_j} dz_j, \quad \bar{\partial} g = \sum_{j=1}^{n} \frac{\partial g}{\partial \bar{z}_j} d\bar{z}_j
\]
denote the holomorphic and anti-holomorphic part of \( dg \) respectively, and \( dz_j := dx_j + idy_j, d\bar{z}_j = dx_j - idy_j \). A short calculation shows
\[
G_f^\theta_1 = 2i \sum_{j=1}^{k} \lambda_j \exp \Re(w_j, z) \partial \Re(w_j, z) \wedge \bar{\partial} \Re(w_j, z),
\]
\[
G_f^\theta_2 = -\frac{i}{2} \sum_{j=1}^{k} \lambda_j \Im(w_j, w_k) \exp \Re(w_j + w_k, z) d \Re(w_j, z) \wedge d \Re(w_k, z),
\]
\[
G_f^\beta_1 = \sum_{j=1}^{k} \lambda_j \Re(w_j, z) \exp \Re(z, w_j) d \Re(z, w_j),
\]
\[
G_f^\gamma_2 = -\frac{i}{2} (\partial |z|^2 - \bar{\partial} |z|^2).
\]
We will identify \( \theta_0 \) with the corresponding form on \( \mathbb{C}^n \), that is, the natural symplectic form. Thus
\[
\frac{1}{c_{n,k,q}} F(GW(\mu_{k,q}))[-w_1 \otimes i, \ldots, -w_k \otimes i]
\]
\[
= -\frac{i}{2} (2i)^{k-1} 2q \left( -\frac{1}{2} \right)^q \left( \frac{k-1-2q}{k} \right) q! \sum_{\sigma \in S_k} \int_{\mathbb{C}^n} \phi(z) \exp \left( \sum_{j=1}^{k} w_j, z \right) \Theta_0^{n-k+q}
\]
\[
\wedge \prod_{j=1}^{q} \Im(w_{\sigma(2j-1)}, w_{\sigma(2j)}) d \Re(w_{\sigma(2j-1)}, z) \wedge d \Re(w_{\sigma(2j)}, z)
\]
\[
\wedge \partial \Re(w_{\sigma(2q+1)}, z) \wedge \bar{\partial} \Re(w_{\sigma(2q+1)}, z) \ldots \partial \Re(w_{\sigma(k-1)}, z) \wedge \bar{\partial} \Re(w_{\sigma(k-1)}, z)
\]
\[
\wedge \Re(w_{\sigma(k)}, z) d \Re(z, w_{\sigma(k)}) \wedge (\partial |z|^2 - \bar{\partial} |z|^2).
\]
Due to Lemma 2.7 it is enough to choose the vectors \( w_1, \ldots, w_k \) to be scalar multiples of a
Tasaki basis. Set \( \tilde{q} := \left\lfloor \frac{\min(k,2n-k)}{2} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor - \max(0, k - n) \) and consider the vectors
\[
w_{2j-1} := a_{2j-1} e_{2j-1}, \quad w_{2j} = a_{2j} (\cos(\theta_j) i e_{2j-1} + \sin(\theta_j)) e_{2j} \quad 1 \leq j \leq \tilde{q},
\]
\[
w_{2j-1} := a_{2j-1} e_{j+\tilde{q}}, \quad w_{2j} = a_{2j} i e_{j+\tilde{q}}
\]
\[
w_k := a_k e_k \quad k \text{ odd, } \tilde{k} := \left\lfloor \frac{k}{2} \right\rfloor + 1 \quad k \geq n,
\]
for \( a_1, \ldots, a_k \in \mathbb{R} \). These vectors belong to two different types of complex subspaces: For
\( 1 \leq j \leq \tilde{q}, \) \( w_{2j-1} \) and \( w_{2j} \) belong to the complex 2-dimensional subspace spanned by \( e_{2j-1} \) and \( e_{2j} \), while for \( \tilde{q} + 1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor \) they belong to the 1-dimensional subspace spanned by \( e_{j+\tilde{q}} \).
In particular, a permutation \( \sigma \in S_k \) only contributes a non-trivial term if the following conditions are met:

1. For \( 1 \leq j \leq \hat{q} \):
   - \( \{2j - 1, 2j\} \subset \sigma(\{1, \ldots, 2q\}) \), or
   - \( \{2j - 1, 2j\} \subset \sigma(\{2q + 1, \ldots, k - 1\}) \), or
   - \( 2j - 1 = \sigma(k) \) and \( 2j \in \sigma(\{2q + 1, \ldots, k - 1\}) \), or
   - \( 2j = \sigma(k) \) and \( 2j - 1 \in \sigma(\{2q + 1, \ldots, k - 1\}) \).

2. For \( \hat{q} + 1 \leq j \leq \lfloor \frac{k}{2} \rfloor \):
   - \( \{2j - 1, 2j\} \subset \sigma(\{1, \ldots, 2q\}) \).

3. If \( k \) is odd, then either
   - \( k = \sigma(k) \), or
   - \( k \in \sigma(\{2q + 1, \ldots, k - 1\}) \).

We will only consider the case where \( k \) is odd, the other case is similar and simpler. We will split the permutations in \( S_k \) into two families: Either \( \sigma(k) = k \) or \( \sigma(k) \neq k \). In both cases, we can thus reduce the calculation to a sum of permutations of the pairs \( \{2j - 1, 2j\} \) for \( 1 \leq j \leq \lfloor \frac{k}{2} \rfloor \).

Let us consider the different contributions of the pairs \( \{2j - 1, 2j\} \) and the term corresponding to \( w_k \).

**1st set of pairs**: \( 1 \leq j \leq \hat{q} \)

Assume \( \{2j - 1, 2j\} \subset \sigma(\{1, \ldots, k - 2q\}) \). As \( \partial \text{Re}(w_{2j-1}, z) \wedge \bar{\partial} \text{Re}(w_{2j-1}, z) = \frac{|w_{2j-1}|^2}{4} dz_{2j-1} \wedge d\bar{z}_{2j-1} \), this pair contributes

\[
\partial \text{Re}(w_{2j-1}, z) \wedge \bar{\partial} \text{Re}(w_{2j-1}, z) \wedge \partial \text{Re}(w_{2j}, z) \wedge \bar{\partial} \text{Re}(w_{2j}, z) = \frac{a_{2j-1}^2 a_{2j}^2}{16} (1 - \cos^2 \theta_j) dz_{2j-1} \wedge d\bar{z}_{2j-1} dz_{2j} \wedge d\bar{z}_{2j}.
\]

If \( \{2j - 1, 2j\} \subset \sigma(\{k - 2q + 1, \ldots, k - 2\}) \), then we have to consider

\[
\text{Im}(w_{2j-1}, w_{2j}) d \text{Re}(w_{2j-1}, z) \wedge d \text{Re}(w_{2j}, z) = \text{Im}(w_{2j-1}, w_{2j}) a_{2j-1} d \text{Re}(z_{2j-1}) \wedge a_{2j} d(\cos \theta_j \text{Im}(z_{2j-1}) + \sin \theta_j \text{Re}(z_{2j})).
\]

Note that no other term contains multiples of \( dz_{2j-1} \) and \( dz_{2j} \) and their conjugates, except for the symplectic form, but the corresponding terms annihilate this form. Thus this term contributes

\[
a_{2j-1} a_{2j} \cos \theta_j \text{Im}(w_{2j-1}, w_{2j}) \cdot \frac{i}{2} dz_{2j-1} d\bar{z}_{2j-1} = \text{Im}(w_{2j-1}, w_{2j})^2 \cdot \frac{i}{2} dz_{2j-1} d\bar{z}_{2j-1} = a_{2j-1}^2 a_{2j}^2 \cos^2 \theta_j \cdot \frac{i}{2} dz_{2j-1} d\bar{z}_{2j-1}.
\]
For \( \{2j-1, 2j\} = \sigma(\{k-1 \ldots, k\}) \) we obtain with a similar argument that the corresponding terms contribute

\[
\text{Re}(w_{2j-1}, z) \text{Im}(z, w_{2j}) d\text{Re}(z, w_{2j-1}) \wedge d\text{Re}(z, w_{2j})
\]

\[
= \frac{i}{2} \text{Im}(w_{2j-1}, w_{2j}) \text{Re}(w_{2j-1}, z) \text{Im}(w_{2j}, z) d\bar{z}_{2j-1} \wedge d\bar{z}_{2j-1},
\]

\[
\text{Re}(w_{2j}, z) \text{Im}(z, w_{2j-1}) d\text{Re}(z, w_{2j}) \wedge d\text{Re}(z, w_{2j-1})
\]

\[
= -\frac{i}{2} \text{Im}(w_{2j-1}, w_{2j}) \text{Re}(w_{2j}, z) \text{Im}(w_{2j-1}, z) d\bar{z}_{2j-1} \wedge d\bar{z}_{2j-1},
\]

depending on the permutation. Thus this pair contributes

\[
\frac{i}{2} \text{Im}(w_{2j-1}, w_{2j}) (\text{Re}(w_{2j-1}, z) \text{Im}(z, w_{2j}) - \text{Re}(w_{2j}, z) \text{Im}(z, w_{2j-1})) d\bar{z}_{2j-1} \wedge d\bar{z}_{2j-1}
\]

\[
= \frac{i}{2} I_{2j-1, 2j} Z_{2j-1, 2j} d\bar{z}_{2j-1} \wedge d\bar{z}_{2j-1}.
\]

**2nd set of pairs:** \( \tilde{q} + 1 \leq j \leq \left[ \frac{k}{2} \right] \)

If \( \{2j-1, 2j\} = \sigma(\{k-2q+1 \ldots, k-2\}) \), then this pair contributes

\[
\frac{i}{2} I_{2j-1, 2j} Z_{2j-1, 2j} d\bar{z}_{j+\tilde{q}} \wedge d\bar{z}_{j+\tilde{q}},
\]
as in the previous case. If \( \{2j-1, 2j\} \subset \sigma(\{k-2q+1 \ldots, k-2\}) \), then we have to consider the form

\[
\text{Im}(w_{2j-1}, w_{2j}) d\text{Re}(w_{2j-1}, z) \wedge d\text{Re}(w_{2j}, z) = \frac{i}{2} a_{2j-1}^2 a_{2j}^2 \cos^2 \theta_j d\bar{z}_{j+\tilde{q}}.
\]

**3rd case:** \( k \) is odd

If \( \sigma(k) = k \), then the form \( \text{Re}(w_{k}, z) d\text{Re}(z, w_{k}) \wedge (\partial|z|^2 - \bar{\partial}|z|^2) \) contributes

\[
\frac{i}{2} \text{Re}(w_{k}, z)^2 d\bar{z}_{k} \wedge d\bar{z}_{k}.
\]

If \( \sigma(k) \neq k \), then the only other new contribution occurs if \( \{2j-1, 2j\} \subset \sigma(\{2q+1 \ldots, k\}) \) and either \( 2j-1 = \sigma(k) \) or \( \sigma(2j) = k \). In the first case, we obtain the form

\[
\partial \text{Re}(w_{2j}, z) \wedge \bar{\partial} \text{Re}(w_{2j}, z) \wedge \text{Re}(w_{2j-1}, z) d\text{Re}(z, w_{2j-1}) \wedge \gamma_2.
\]

This form contributes

\[
\partial \text{Re}(w_{2j}, z) \wedge \bar{\partial} \text{Re}(w_{2j}, z) \wedge \text{Re}(w_{2j-1}, z) d\text{Re}(z, w_{2j-1}) \wedge \frac{i}{2} \sum_{l=2j-1}^{2j} z_l d\bar{z}_l - \bar{z}_l d\bar{z}_l
\]

\[
= -\frac{i}{8} \text{Re}(w_{2j-1}, z) \text{Re}(w_{2j-1}, z) + \text{Im}(w_{2j-1}, w_{2j}) \text{Im}(w_{2j}, z) d\bar{z}_{2j-1} \wedge d\bar{z}_{2j-1} \wedge d\bar{z}_{2j} \wedge d\bar{z}_{2j}.
\]

Similarly, for \( 2j = \sigma(k) \) the corresponding form contributes

\[
\frac{i}{8} \text{Re}(w_{2j}, z) \text{Re}(w_{2j}, z) + \text{Im}(w_{2j}, w_{2j-1}) \text{Im}(w_{2j-1}, z) d\bar{z}_{2j-1} \wedge d\bar{z}_{2j-1} \wedge d\bar{z}_{2j} \wedge d\bar{z}_{2j}.
\]
In total, this pair thus contributes the sum of these two terms, which is given by

\[
\frac{1}{c_{n,k,q}} \mathcal{F}(\text{GW}(\mu^{\gamma}_{k,q,\phi})[-w_1 \otimes i, \ldots, -w_k \otimes i])
\]

\[
=(2i)^{k-2q-1} \cdot \left( -\frac{1}{2} \right)^q \cdot (\frac{1}{2})^q \cdot \frac{(k-2q-1)(k-2q-2)!}{k!}
\]

\[
\int_{C^n} \phi(z) \exp \left( \sum_{j=1}^{k} w_j, z \right) \theta_0^{n-k+q} \cdot d\bar{z}_1 \wedge d\bar{z}_2 \ldots d\bar{z}_k \wedge d\bar{z}_k
\]

\[
\frac{\prod_{j=1}^{k-3} \theta_0^j}{q!(n-k+q)!} \sum_{\sigma \in S_{\frac{n}{2}}} \prod_{j=1}^{\frac{n}{2}} \cos^2(\theta_{\sigma(j)}) \prod_{j=1}^{\frac{n}{2}} \frac{1}{16} (1 - \cos^2(\theta_{\sigma(j)})) \cdot \frac{a_0^2}{4}
\]

\[
\cdot \frac{i}{8} \left[ Z_{2\sigma}((\frac{1}{2})) - 1, 2\sigma((\frac{1}{2})) + Z_{2\sigma}((\frac{1}{2})) + I_{2\sigma}((\frac{1}{2})) - 1, 2\sigma((\frac{1}{2})) - 1, 2\sigma((\frac{1}{2})) \right]
\]

\[
+ \frac{\prod_{j=1}^{k-1} \theta_0^j}{q!(n-k+q)!} \sum_{\sigma \in S_{\frac{n}{2}}} \prod_{j=1}^{\frac{n}{2}} \cos^2(\theta_{\sigma(j)}) \cdot \left( \frac{\frac{1}{2}}{2} \prod_{j=1}^{\frac{n}{2}} \frac{1}{16} (1 - \cos^2(\theta_{\sigma(j)})) \right) \cdot \frac{i}{2} Z_{k,k}^{R}
\]

By expanding the product involving \(1 - \cos^2(\theta_{\sigma(j)})\) and combining the constants, as well as using that \(\theta_0^{n-k+q} \cdot d\bar{z}_1 \wedge d\bar{z}_2 \ldots d\bar{z}_k \wedge d\bar{z}_k = \frac{(n-k+q)!}{n!} \theta_0^n\), we may use Lemma 10.8 to see that this expression equals

\[
\frac{(k-2q-1)!q!(n-k+q)!}{k!} \int_{C^n} \phi(z) \exp \left( \sum_{j=1}^{k} w_j, z \right) \frac{1}{n!} \theta_0^n
\]

\[
\left[ -\frac{1}{2} \sum_{j=q}^{\frac{n}{2}+1} (-1)^{j-q} \binom{j}{q} \det_k(I[2(j+1) - 1], Z_{w}^I(x), R[k-2(j+1)])
\]

\[
+ \sum_{j=q}^{\frac{n}{2}} (-1)^{j-q} \binom{j}{q} \det_k(I[2j], Z_{w}^I(x), R[k-2j - 1]) \right].
\]

The definitions of \(c_{n,k,q}\) and \(D_{\gamma,w}^{k,q}\) thus imply

\[
\mathcal{F}(\text{GW}(\mu^{\gamma}_{k,q,\phi})\{w_1, \ldots, w_k\} = \frac{(-1)^k}{k!} \cdot \frac{1}{k-2q} D_{\gamma,w}^{k,q} \mathcal{F}(\phi) \left[ \sum_{j=1}^{k} w_j \right],
\]

where we used that \(D_{\gamma,w}^{k,q}\) is homogeneous of degree 2 in each component of \(w = (w_1, \ldots, w_k)\).

\[\square\]

**Corollary 4.11.** The restriction of \(\mu_{k,q,\phi}, \mu_{k,q,\phi}^{\tilde{\gamma}}, \mu_{k,q,\phi}^{\tilde{\gamma}_0}\) to \(E_{k,p}\) vanishes for \(p \neq q\) for all \(\phi \in C_{C}^{\infty}(\mathbb{R}^n)\).

**Proof.** This follows from Proposition 4.10 in combination with Corollary 2.7 using Proposition 4.9 and Corollary 4.9. \(\square\)
4.4 Restrictions to the extremal subspaces $E_{k,q}$ and proof of Theorem 2

If $\mu \in V\text{Con}_k(\mathbb{C}^n)^{U(n)}$, then the valuation $\pi_{E_{k,q}} \ast \mu \in V\text{Con}_k(E_{k,q})$ is invariant under the stabilizer of $E_{k,q}$, that is, it is invariant under $U(q) \times O(k-2q)$. Using Theorem 2, it is easy to see that

$$\pi_{E_{k,q}} \ast \mu(f) = \int_{E_{k,q}} \phi_q \, d\text{MA}_{E_{k,q}}(f)$$

for a $U(q) \times O(k-2q)$-invariant function $\phi_q \in \mathcal{C}(E_{k,q})$. If we denote points in $E_{k,q} \cong \mathbb{C}^q \times \mathbb{R}^{k-2q}$ by $(z, x)$, this implies that $\phi_q(z, x) = \tilde{\phi}(|z|, |x|)$ for some $\tilde{\phi} \in \mathcal{C}_c((0, \infty)^2)$. The goal of this section is the proof of the following result.

**Theorem 4.12.** Let $\mu \in V\text{Con}_k(\mathbb{C}^n)^{U(n)}$. There exist unique functions $a_q(\mu), b_q(\mu) \in \mathcal{C}_c((0, \infty))$ with $a_q(\mu)[0] = b_q(\mu)[0]$ such that for all $f \in \text{Conv}(E_{k,q}, \mathbb{R})$

$$\pi_{E_{k,q}} \ast \mu(f) = \int_{E_{k,q}} \left[ |z|^2 a_q(\mu)[((z, x))] + |x|^2 b_q(\mu)[((z, x))] \right] \, d\text{MA}_{E_{k,q}}(f)[z, x].$$

Here we extend $(z, x) \mapsto |z|^2 a_q(\mu)[((z, x))] + |x|^2 b_q(\mu)[((z, x))]$ by continuity to $(z, x) = 0$.

We will deduce this result from the classification of smooth $U(n)$-invariant valuations. For these valuations, the claim can be deduced from Proposition 2 using the Abel transform: Set $C_b((0, \infty)) := \{ \phi \in C((0, \infty)) : \text{supp } \phi \subset (0, R) \text{ for some } R > 0 \}$. The Abel transform $\mathcal{A} : C_b((0, \infty)) \to C_b((0, \infty))$ is given by

$$\mathcal{A}(\phi)[t] := \int_{-\infty}^{+\infty} \phi(\sqrt{t^2 + s^2}) \, ds \text{ for } t > 0.$$ 

This implies that $\mathcal{A}(\phi)$ is a smooth function if $\phi$ is smooth. If $\psi \in C_b((0, \infty))$ is continuously differentiable, then it is contained in the image of $\mathcal{A}$ and

$$\mathcal{A}^{-1}(\psi)[t] = -\frac{1}{\pi} \int_{t}^{\infty} \frac{\psi'(s)}{\sqrt{s^2 - t^2}} \, dt.$$ 

In particular, $\mathcal{A} : C_b((0, \infty)) \to C_b((0, \infty))$ is injective. The Abel transform is intimately related to the Fourier-Laplace transform of rotation invariant functions: For $\phi \in C_c((0, \infty))$, $w \in E \in \text{Gr}_k(\mathbb{C}^n)$

$$\mathcal{F}(\phi([ \cdot |])[w] = \int_{\mathbb{C}^n} \phi(|z|) \exp(i \langle w, z \rangle) \, dz = \int_{E} \mathcal{A}^{2n-k}(\phi)(|z'|) \exp(i \langle w, z' \rangle) \, dz'.$$

By considering the Fourier-Laplace transforms of the Goodey-Weil distributions of the smooth valuations calculated in Proposition 1.10, we obtain the following formulas for the restriction of smooth valuations to the extremal subspaces $E_{k,q}$.

**Corollary 4.13.** For $\phi \in C_c^\infty((0, \infty))$ set $\tilde{\phi}(t) := \phi(t^2)$. Then

$$\pi_{E_{k,q}} \ast \mu_{k,q, \tilde{\phi}([\cdot |])} = \int_{E_{k,q}} \mathcal{A}^{2n-k}(\tilde{\phi})(|z, x|) \, d\text{MA}_{E_{k,q}},$$

$$\pi_{E_{k,q}} \ast \mu_{k,q, \tilde{\phi}([\cdot |])} = \frac{1}{q} \int_{E_{k,q}} |z|^2 \mathcal{A}^{2n-k}(\tilde{\phi})(|z, x|) \, d\text{MA}_{E_{k,q}},$$

$$\pi_{E_{k,q}} \ast \mu_{k,q, \tilde{\phi}([\cdot |])} = \frac{1}{k - 2q} \int_{E_{k,q}} |x|^2 \mathcal{A}^{2n-k}(\tilde{\phi})(|z, x|) \, d\text{MA}_{E_{k,q}}.$$
Proof. Use the representation of the symbols of operators $D_{\beta,w}^{k,q}, D_{\beta,w}^{k,q}$ in Corollary 4.9 and Proposition 4.10.

In order to deduce Theorem 4.12 from the characterization of smooth valuations, we need a slightly refined version of Theorem 2.5. Let us equip $C_c(\mathbb{R}^n)$ with the following sequential topology: A sequence $(\phi_j)_j$ converges to $\phi \in C_c(\mathbb{R}^n)$ if and only if

1. there exists a compact subset $K \subset \mathbb{R}^n$ with $\text{supp} \phi_j \subset K$ for all $j \in \mathbb{N}$, and
2. $(\phi_j)_j$ converges uniformly to $\phi$.

For $\phi \in C_c(\mathbb{R}^n)$ let us consider $\mu_\phi \in VConv_n(\mathbb{R}^n)$ defined by

$$\mu_\phi(f) = \int_{\mathbb{R}^n} \phi d \text{MA}(f).$$

Recall also that $h_K(y) := \sup_{x \in K} \langle y, x \rangle$ denotes the support function of $K \in \mathcal{K}(\mathbb{R}^n)$ and defines a convex function.

**Proposition 4.14.** The map

$$C_c(\mathbb{R}^n) \rightarrow VConv_n(\mathbb{R}^n)$$

$$\phi \mapsto \mu_\phi$$

is continuous and bijective. Its inverse is given by

$$S : VConv_n(\mathbb{R}^n) \rightarrow C_c(\mathbb{R}^n)$$

$$\mu \mapsto \left[ x \mapsto \frac{1}{\omega_n} \mu(h_{B_1(0)}(\cdot-x)) \right]$$

and is sequentially continuous.

**Proof.** The first map is bijective by Theorem 2.5. Let us show that it is continuous. Let $(\phi_j)_j$ be a sequence in $C_c(\mathbb{R}^n)$ that converges uniformly to $\phi \in C_c(\mathbb{R}^n)$ and such that there exists a compact subset $K \subset \mathbb{R}^n$ with $\text{supp} \phi_j \subset K$ for all $j \in \mathbb{N}$. If $\psi \in C_c(\mathbb{R}^n)$ is a non-negative function with $\psi \equiv 1$ on a neighborhood of $K$, then

$$|\mu_{\phi_j}(f) - \mu_{\phi}(f)| \leq \int_{\mathbb{R}^n} |\phi_j - \phi| d \text{MA}(f) \leq \|\phi_j - \phi\|_{\infty} \int_{\mathbb{R}^n} \psi d \text{MA}(f)$$

$$=\|\phi_j - \phi\|_{\infty} \mu_{\psi}(f).$$

If $A \subset \text{Conv}(\mathbb{R}^n, \mathbb{R})$ is compact, then

$$\sup_{f \in A} |\mu_{\phi_j}(f) - \mu_{\phi}(f)| \leq \|\phi_j - \phi\|_{\infty} \sup_{f \in A} \mu_{\psi}(f),$$

which converges to 0 for $j \rightarrow \infty$ as $\mu_{\psi}$ is continuous and thus bounded on $A$. Thus the map $\phi \mapsto \mu_{\phi}$ is continuous.

Let us turn to the inverse map. It is well known that the Monge-Ampère measure of the support function $h_K$ of $K \in \mathcal{K}(\mathbb{R}^n)$ is given by $\text{MA}(h_K) = \text{vol}(K)\delta_0$, see for example [5].
Lemma 2.4. It is easy to see that the Monge-Ampère operator is equivariant with respect to translations. Thus
\[ \mu_{\phi}(h_K(\cdot - x)) = \int_{\mathbb{R}^n} \phi \text{MA}(h_K(\cdot - x)) = \int_{\mathbb{R}^n} \phi(\cdot + x) \text{MA}(h_K) = \phi(x) \text{vol}(K). \]

Choosing \( K = B_1(0) \), we thus obtain the formula for the inverse.

Let us show that the map \( S : \text{VConv}_n(\mathbb{R}^n) \to C_c(\mathbb{R}^n) \) is sequentially continuous. First, Corollary 2.11 implies that \( \text{supp} S(\mu) \subset \text{conv}(\text{supp} \mu) \). If \((\mu_j)_j \) is a sequence in \( \text{VConv}_n(\mathbb{R}^n) \) converging to \( \mu \in \text{VConv}_n(\mathbb{R}^n) \), then according to \[37\] Proposition 1.2 there exists a compact subset \( K_0 \subset \mathbb{R}^n \) such that \( \text{supp} \mu_j \subset K_0 \) for all \( j \in \mathbb{N} \). We may assume that \( K_0 \) is convex. Thus \( \text{supp} S(\mu_j) \subset \text{conv}(\text{supp} \mu_j) \subset K_0 \) for all \( j \in \mathbb{N} \). It is thus sufficient to show that the sequence \( (S(\mu_j))_j \) converges uniformly to \( \mu \) on compact subsets. As \( h_{B_1(0)} \) is continuous, the map
\[
\mathbb{R}^n \to \text{Conv}(\mathbb{R}^n, \mathbb{R})
\]
\[ x \mapsto h_{B_1(0)}(\cdot - x) \]
is continuous. If \( K \subset \mathbb{R}^n \) is any compact subset, this implies that
\[
\{ h_{B_1(0)}(\cdot - x) : x \in K \} \subset \text{Conv}(\mathbb{R}^n, \mathbb{R})
\]
is compact, so \((\mu_j)_j \) converges uniformly to \( \mu \) on this set. In particular, \( S(\mu_j)[x] = \frac{1}{\omega_n} \mu_j(h_{B_1(0)}(\cdot - x)) \) converges uniformly on \( K \) to \( S(\mu) = \frac{1}{\omega_n} \mu(h_{B_1(0)}(\cdot - x)) \), which shows the claim. \( \Box \)

Proof of Theorem 4.12. As \( S(\pi_{E_{k,q}} \cdot \mu) \) is uniquely determined by \( \mu \in \text{VConv}_k(\mathbb{R}^n) \), it is easy to see that the functions are unique if they exist.

Fix \( \max(0, k - n) \leq q \leq \lfloor \frac{k}{2} \rfloor \). For \( \mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)} \) define
\[
a_q(\mu)[s] := S(\pi_{E_{k,q}} \cdot \mu)(se_1, 0),
b_q(\mu)[t] := S(\pi_{E_{k,q}} \cdot \mu)(0, te_{q+1}),
T(\mu)[z, x] := |z|^2 \frac{a_q(\mu)[(z, x)]}{|z, x|^2} + |x|^2 \frac{b_q(\mu)[(z, x)]}{|z, x|^2}.
\]
Then \( T(\mu) \) extends to a continuous function on \( E_{k,q} \) with compact support such that
\[
|T(\mu)[z, x]| \leq |S(\pi_{E_{k,q}} \cdot \mu)[(z, x)] e_1, 0| + |S(\pi_{E_{k,q}} \cdot \mu)[0, (z, x)] e_{q+1}|.
\]
In particular, \( T : \text{VConv}_k(\mathbb{C}^n)^{U(n)} \to C_c(E_{k,q}) \) is sequentially continuous. We thus have to establish the equation
\[
S(\pi_{E_{k,q}} \cdot \mu)[z, x] = T(\mu)[z, x] \quad \text{for } (z, x) \in E_{k,q}
\]
for all \( \mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)} \), but because both sides are sequentially continuous and smooth valuations are sequentially dense, it is sufficient to only consider smooth \( U(n) \)-invariant valuations. For these valuations the claim follows from Corollary 4.13. \( \Box \)

Note that the proof shows the following slightly finer result.

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Corollary 4.15. For $\mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)}$ the functions $a_q(\mu), b_q(\mu) \in \mathcal{C}_c([0, \infty))$ are given by

$$a_q(\mu)[s] = S(\pi_{E_{k,q}} \mu)[se_1],$$
$$b_q(\mu)[t] = S(\pi_{E_{k,q}} \mu)[te_{q+1}],$$

and the maps $a_q, b_q : \text{VConv}_k(\mathbb{C}^n) \to \mathcal{C}_c([0, \infty))$ are sequentially continuous.

From Proposition 4.10 we obtain $a_q$ and $b_q$ for smooth valuations.

Corollary 4.16. Let $\mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)} \cap \text{VConv}(\mathbb{C}^n)^{sm}$ be given by

$$\mu(f) = \sum_{q=\max(0,k-n)}^{\lfloor \frac{k}{2} \rfloor} D(f) \left[ \phi_q(|z|^2) \Theta_{k,q}^n \right] + \sum_{q=\max(1,k-n)}^{\lfloor \frac{k-1}{2} \rfloor} D(f) \left[ \psi_q(|z|^2) \Upsilon_{k,q}^n \right].$$

Then with $\tilde{\phi}_q(t) := \phi_q(t^2), \tilde{\psi}_q(t) := \psi_q(t^2)$:

$$a_q(\mu)[s] = \begin{cases} A^{2n-k}(\tilde{\phi}_q)[s] + \frac{2}{q} A^{2n-k}(\tilde{\psi}_q)[s] & 0 < q < \frac{k}{2}, \\
A^{2n-k}(\tilde{\phi}_q)[s] & q = 0 \text{ or } q = \frac{k}{2}, \\
A^{2n-k}(\tilde{\psi}_q)[s] & q = 0 \text{ or } q = \frac{k}{2}. \\
\end{cases}$$

$$b_q(\mu)[t] = \begin{cases} A^{2n-k}(\tilde{\phi}_q)[t] - \frac{2q}{k-2q} A^{2n-k}(\tilde{\psi}_q)[t] & 0 < q < \frac{k}{2}, \\
A^{2n-k}(\tilde{\phi}_q)[t] & q = 0 \text{ or } q = \frac{k}{2}. \\
\end{cases}$$

We are now able to complete the proof of Theorem 2.

Proof of Theorem 2. By Theorem 3.16 any valuation $\mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)} \cap \text{VConv}(\mathbb{C}^n)^{sm}$ may be written as

$$\mu(f) = \sum_{q=\max(0,k-n)}^{\lfloor \frac{k}{2} \rfloor} D(f) \left[ \phi_q(|z|^2) \Theta_{k,q}^n \right] + \sum_{q=\max(1,k-n)}^{\lfloor \frac{k-1}{2} \rfloor} D(f) \left[ \psi_q(|z|^2) \Upsilon_{k,q}^n \right]$$

for certain $\phi_q, \psi_q \in \mathcal{C}^\infty_c([0, \infty))$. Due to Corollary 4.11 it is thus sufficient to show that these functions are uniquely determined by $\mu$. This follows from Corollary 4.13. If we restrict $\mu$ to $E_{k,q}$, we obtain

$$\pi_{E_{k,q}} \mu = \int_{E_{k,q}} \left[ A^{2n-k}(\tilde{\phi}_q)[((z,x),x)]^2 \right] + \left( \frac{|z|^2}{q} - \frac{2|x|}{k-2q} \right) A^{2n-k}(\tilde{\psi}_q)[((z,x),x)]^2 \right] dMA_{E_{k,q}}$$

for $\max(0,k-n) \leq q \leq \frac{k}{2}$, where the second term vanishes for $q = 0$ and $q = \frac{k}{2}$. Due to Theorem 2.5 this expression vanishes if and only if $A^{2n-k}(\tilde{\phi}_q) = A^{2n-k}(\tilde{\psi}_q) = 0$. As the Abel transform is injective, this implies $\phi_q = \psi_q = 0$. Thus $\phi_q$ and $\psi_q$ are uniquely determined by $\mu$, which shows that the sum

$$\text{VConv}_k(\mathbb{C}^n)^{U(n)} \cap \text{VConv}_k(\mathbb{C}^n)^{sm} = \bigoplus_{q=\max(0,k-n)}^{\lfloor \frac{k}{2} \rfloor} \text{VConv}_{k,q}(\mathbb{C}^n)^{U(n)} \cap \text{VConv}_k(\mathbb{C}^n)^{sm}.$$
4.5 The Fourier-Laplace transform in the general case

We need the following simple observation for the proof of Theorem 4.

Lemma 4.17. If $\psi \in C_c([0, \infty))$, $k \geq 3$, then $A^{k-1} \left( \frac{\psi}{|\cdot|^2} \right) \in L^1((0, \infty))$.

Proof. This follows from a change to polar coordinates:

$$
\int_0^\infty \left| A^{k-1} \left( \frac{\psi}{|\cdot|^2} \right) \right| ds \leq \int_0^\infty \left[ \int_{|s|^2+x^2} |\psi(\sqrt{s^2+x^2})| dx \right] ds = \int_0^\infty \int_0^\pi \frac{|\psi(r)|}{r^2} r^{k-1} dr d\phi < \infty.
$$

Define an entire function $e \in \mathcal{O}_\mathbb{C}$ on $\mathbb{C}$ by

$$
e(z) = \sum_{n=0}^\infty (-1)^n \frac{z^n}{(2n)!},
$$

that is, $e(z^2) = \cos(z)$. If $g \in L^1(\mathbb{R})$ is even and has compact support, this implies

$$
\int_\mathbb{R} g(x) e(x^2z^2) dx = \int_\mathbb{R} g(x) \exp(ixz) dx = F(g)[z] \quad \text{for } z \in \mathbb{C}.
$$

Let us show the following refined version of Theorem 4. It uses the functions $a_q, b_q : \text{VConv}_k(\mathbb{C}^n) \to C_c([0, \infty))$ defined in the previous section.

Theorem 4.18. For $\mu \in \text{VConv}_k(\mathbb{C}^n)$, define $\Phi_q(\mu), \Psi_q(\mu) \in \mathcal{O}_\mathbb{C}$ by

$$
\Phi_q(\mu)[z] := \frac{1}{k} \int_\mathbb{R} \left| A^{k-1} \left( 2q a_q(\mu) + (k-2q)b_q(\mu) \right) \right| |t||e(t^2 \cdot z)| dt \quad \text{for } 0 \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor,
$$

$$
\Psi_q(\mu)[z] := \frac{q(k-2q)}{k} \int_\mathbb{R} A^{k-1} \left( \frac{a_q(\mu) - b_q(\mu)}{|\cdot|^2} \right) |t||e(t^2 \cdot z)| dt \quad \text{for } 0 < q < \frac{k}{2}.
$$

Consider the holomorphic functions $\tilde{\Phi}_q(\mu)[w] := \Phi_q(\langle w, w \rangle), \tilde{\Psi}_q(\mu)[w] := \Psi_q(\langle w, w \rangle)$ for $w \in \mathbb{C}^n \otimes \mathbb{R} \mathbb{C}$. Then

$$
\mathcal{F}(GW(\mu))[w_1, \ldots, w_k] = (-1)^k \frac{1}{k!} \sum_{q=\max(0,k-n)}^{\left\lfloor \frac{k}{2} \right\rfloor} P_{k,q}(w_1, \ldots, w_k) \tilde{\Phi}_q(\mu) \left[ \sum_{j=1}^k w_j \right] 
$$

$$
+ (-1)^k \frac{1}{k!} \sum_{q=\max(1,k-n)}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left[ \frac{1}{q} D_{\beta,w}^{k,q} - \frac{2}{k-2q} D_{\gamma,w}^{k,q} \right] \tilde{\Psi}_q(\mu) \left[ \sum_{j=1}^k w_j \right],
$$

(8)

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Proof. Note first that $\Psi_q(\mu)$ is well defined due to Lemma 4.17. If $\mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)}$ is smooth and given by

$$
\sum_{q = \max(0, k-n)}^{\left\lfloor \frac{k}{2} \right\rfloor} D(f) \left[ \phi_q(|z|^2) \Theta^n_{k,q} \right] + \sum_{q = \max(1, k-n)}^{\left\lfloor \frac{k-1}{2} \right\rfloor} D(f) \left[ \psi_q(|z|^2) \Upsilon^n_{k,q} \right],
$$

then we may use the relations in Corollary 4.16 to obtain

$$
\Phi_q(\mu)[z^2] = \int_{\mathbb{R}} A^{2n-k-1}(\phi_q(|\cdot|^2))[|t|] e(t^2 \cdot z^2) dt
= \int_{\mathbb{R}} A^{2n-k-1}(\phi_q(|\cdot|^2))[|t|] \exp(i t \cdot z) dt
= \int_{\mathbb{C}^n} \phi_q(|x|^2) \exp(i \langle x, ze_1 \rangle)
$$

and similarly $\Psi_q(\mu)[z^2] = \int_{\mathbb{C}^n} \psi_q(|x|^2) \exp(i \langle x, ze_1 \rangle)$. Thus for $w \in \mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}$:

$$
\Phi_q(\mu)[(w, w)] = F(\phi_q(|\cdot|^2))[w],
\Psi_q(\mu)[(w, w)] = F(\psi_q(|\cdot|^2))[w].
$$

The representation of the Fourier-Laplace transform of $GW(\mu)$ for smooth valuations in Proposition 4.10 thus implies that (8) holds for all smooth valuations. Using Proposition 2.8 and Corollary 4.15, it is easy to see that the maps $\mu \mapsto \Phi_q(\mu)[z]$ and $\mu \mapsto \Psi_q(\mu)[z]$ are sequentially continuous. In particular, both sides of (8) are sequentially continuous as functions of $\mu \in \text{VConv}_k(\mathbb{C}^n)^{U(n)}$, so the claim follows from the fact that smooth $U(n)$-invariant valuations are sequentially dense.

Proof of Theorem 4. If $\mu = 0$, then $\pi_{E_{k,q}} \mu = 0$ for all $\max(0, k-n) \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor$. Conversely, $\pi_{E_{k,q}} \mu = 0$ implies that $a_q(\mu) = b_q(\mu) = 0$, compare Theorem 4.12. Using the definition of the functions $\Phi_q(\mu), \Psi_q(\mu)$ in Theorem 4.18 we see that $\tilde{\Phi}_q(\mu) = \tilde{\Psi}_q(\mu) = 0$. If this holds for all $\max(0, k-n) \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor$, then $F(GW(\mu)) = 0$ by Theorem 4.18 so $\mu = 0$ as $F \circ GW$ is injective.

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