Conservative Discontinuous Cut Finite Element Methods

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Abstract

We develop a conservative cut finite element method for an elliptic coupled bulk-interface problem. The method is based on a discontinuous Galerkin framework where stabilization is added in such a way that we retain conservation on macro elements containing one element with a large intersection with the domain and possibly a number of elements with small intersections. We derive error estimates and present confirming numerical results.

1 Introduction

Simulations in many applications involve approximating solutions to Partial Differential Equations (PDEs) in complex geometries, such geometries may for example be defined by cell membranes, interfaces separating immiscible fluids, or heart valves guarding the exits of the heart cavities. There has been an extensive development and progress of computational methods for efficiently approximating solutions to PDEs in complex geometries, see e.g. \[10, 13, 21, 23, 25\]. In the present work, we develop further one such computational technique, namely the Cut Finite Element Method (CutFEM). In cut finite element methods the domain of interest is embedded into a polygonal domain equipped with a quasi-uniform mesh referred to as the fixed background mesh. An active mesh is defined which consists of all elements in the mesh that intersect the domain of interest. Associated to the active mesh is a finite dimensional function space and a weak formulation with bilinear forms defined from the variational formulation of the PDE. Often consistent stabilization terms are added in the weak form to ensure stability and avoid ill-conditioned linear systems of equations. See \[5\] for an introduction to CutFEM.

In this paper we develop a cut finite element method based on a discontinuous Galerkin (DG) framework with local conservation in mind. We consider a model consisting of two subdomains separated by an interface with convection diffusion equations in the two bulk domains linearly coupled to a convection diffusion equation on the interface. The diffusion operators are in divergence form with variable tensor valued coefficients. For a similar elliptic model problem with constant coefficients an unfitted finite element method based on the continuous Galerkin (CG) framework has been developed and analyzed in \[14\]. In \[9\] a CutFEM based on CG is proposed and analyzed for a coupled bulk-surface diffusion problem considering one bulk domain and later a CutFEM based on a discontinuous Galerkin (DG) framework was proposed in \[22\]. For an early analysis of a finite element method based on fitted meshes for the coupled bulk-interface diffusion problem see \[12\]. In all the work described above stationary interfaces were considered. A space time CutFEM for simulations of coupled bulk-surface convection diffusion equations with moving
interfaces, modelling for example the evolution of soluble surfactant concentrations, was developed in [17]. Cut finite element methods based on DG have also been proposed and analyzed for other problems for surface PDEs in e.g. [8] and bulk PDEs in [15,19].

Compared to DG, the CG framework leads to finite element methods with fewer unknowns. An advantage of DG is its local conservation property [1,3,11]. In this work we develop a CutFEM based on DG that inherits this property which is important in many applications. However, a straightforward application of the ghost penalty stabilization [4] often used in CutFEM to ensure stability and avoid poor conditioning of the linear systems will destroy the local conservation. We propose and analyze a CutFEM based on DG were local conservation properties hold on so called macro elements. We give a criteria for dividing elements in an active mesh into small and large elements. A small element is then connected via a chain of face neighbours to a large element and only on those faces stabilization [4] is applied. In this way, we create macro elements, consisting of a large element and possibly a few small elements. These macro elements behave as standard finite elements and local conservation properties from the DG framework are inherited to these macro elements. In this way we apply stabilization very restrictively but can prove that the macro element stabilization provides the same control as the full stabilization. In [20] we proposed a stabilization for continuous high order CutFEM for PDEs on interfaces where similar macro elements were created, but only as a tool in the proof. We note that the agglomeration techniques for discontinuous piecewise polynomial spaces developed in [19] may be viewed as a strong version of the macro element stabilization we develop here. We also mention approaches for stabilization of continuous finite element spaces based on various extension techniques, see [2,7,26]. Note that all these references are restricted to bulk problems.

The paper is organized as follows. In Section 2 we introduce the model problem and its weak formulation. In Section 3 we formulate the proposed discontinuous cut finite element method and define the macro element stabilization and prove results on the properties provided by the stabilization. In Section 4 we analyze the method and prove optimal order a priori error estimates in the energy and $L^2$ norms, and in Section 5 we show results from numerical experiments that support our theoretical findings.

## 2 The Model Problem

### 2.1 Basic Notation

Let $\Omega \subset \mathbb{R}^d$ be a $d$-dimensional domain, $d = 2$ or $d = 3$, with convex polygonal boundary $\partial \Omega$. Let $\Omega$ be partitioned into two subdomains $\Omega_1$ and $\Omega_2$ by a smooth closed $d - 1$ dimensional internal interface $\Omega_0$. Let $U_\delta(\Omega_0) = \bigcup_{x \in \Omega_0} B_\delta(x)$, where $B_\delta(x)$ is the open ball with radius $\delta > 0$ centred at $x$, be the tubular neighborhood of $\Omega_0$ with thickness $\delta > 0$. We assume that there is $\delta_0 > 0$ such that the closest point mapping $p : U_{\delta_0}(\Omega_0) \to \Omega_0$ is well defined and that $U_{\delta_0}(\Omega_0) \subset \Omega$, and thus the interface does not come arbitrarily close to the external boundary $\partial \Omega$. Assume that $\Omega_2$ is the domain inside $\Omega_0$ and thus $\partial \Omega \subset \partial \Omega_1$. Let $n_i$ denote the exterior unit normal to $\partial \Omega_i$, $i = 1, 2$.

### 2.2 The Problem

Let $\nabla_i v = \nabla v$ be the $\mathbb{R}^d$ gradient for the flat domains $\Omega_i$, $i = 1, 2$, and let $\nabla_0 v = (I - n_0 \otimes n_0) \nabla v$ be the tangential gradient on the curved domain $\Omega_0$, where $n_0$ is a unit
normal to $\Omega_0$, i.e., $n_0 = n_2$ or $n_0 = n_1|_{\Omega_0}$. For $i = 0, 1, 2$, let $\beta_i$ be a smooth tangential vector field on $\Omega_i$ and assume that $\text{div}_i \beta_i = 0$ in $\Omega_i$, where $\text{div}_i \beta_i$ is the divergence for $i = 1, 2$, and the tangential divergence $\text{tr}(\beta_i \otimes \nabla)_{\Omega_i}$, for $i = 0$. We also assume that $n_i \cdot \beta = 0$ on $\partial \Omega_i$ for $i = 1, 2$. For $i = 0, 1, 2$, let $A_i : \Omega_i \to \mathbb{R}^{d \times d}$ be smooth uniformly positive definite matrix fields in the sense that there is a constant $\alpha_{0,i} > 0$ and a function $\alpha_{0,i} \leq \alpha_i(x)$ such that for $x \in \Omega_i$, $\alpha_{0,i}\|\xi\|_{\mathbb{R}^d}^2 \leq \alpha_i(x)\|\xi\|_{\mathbb{R}^d}^2 \leq (A_i \xi, \xi)_{\mathbb{R}^d}$, $\xi \in T_x(\Omega_i)$ (2.1)

where $T_x(\Omega_i)$ is the tangent space at $x$ to $\Omega_i$ (which for $i = 1, 2$, is $\mathbb{R}^d$). Define the following operators

$$[n \cdot A \nabla v]^2 = \sum_{i=1}^2 n_i \cdot A_i \nabla v_i, \quad [\kappa v]_i = \kappa_i v_i - \kappa_{0,i} v_0, \quad i = 1, 2 \quad (2.2)$$

where $\kappa_i \in \mathbb{R}_+, \kappa_{0,i} \in \mathbb{R}_+$, are constant parameters. We shall consider the following stationary convection-diffusion problem

$$-\nabla \cdot (A_i \nabla u_i) + \nabla \cdot (\beta u_i) = f_i \quad \text{in } \Omega_i, \ i = 1, 2 \quad (2.3)$$

$$-\nabla_0 \cdot (A_0 \nabla u_0) + \nabla_0 \cdot (\beta_0 u_0) + [n \cdot A \nabla u] = f_0 \quad \text{on } \Omega_0 \quad (2.4)$$

$$-n_i \cdot A_i u_i = [\kappa v]_i \quad \text{on } \Omega_0 \quad (2.5)$$

$$u_1 = 0 \quad \text{on } \partial \Omega \quad (2.6)$$

For simplicity, we consider the case of homogeneous Dirichlet data on the exterior boundary $\partial \Omega$ since we focus on the coupling at the interface and the extension to more general boundary conditions follows by standard techniques.

The model problem (2.3)-(2.6) is obtained by restricting the general time dependent model describing the evolution of soluble surfactants with an interface, separating two immiscible fluids, moving with normal velocity $n \cdot \beta$, to an equilibrium state, see [14] for a derivation. Note that in [14], the coefficients are scalars but here we allow matrix coefficients. Finally, we remark that if the boundary condition on the outer boundary $\partial \Omega$ is replaced by a Neumann condition the right hand side must satisfy the equilibrium condition $\sum_{i=0}^2 \int_{\Omega_i} f_i = 0$.

### 2.3 Weak Formulation

To derive the weak form we let

$$W = \{ v \in \oplus_{i=0}^2 H^1(\Omega_i) \mid v_1 = 0 \text{ on } \partial \Omega \} \quad (2.7)$$

and note that $v \in W$ takes the form $v = (v_0, v_1, v_2)$ with components $v_i \in H^1(\Omega_i)$. Let for brevity $\widetilde{\kappa}_i = \kappa_i \kappa_{0,i}^{-1}$, multiplying (2.3) by the scaled test functions $\widetilde{\kappa}_i v_i$, and then using Green’s formula, followed by the interface condition (2.5), and the equation on the
interface (2.4), and finally using Green’s formula on the interface we get

\[
\sum_{i=1}^{2} (f_i, \tilde{\kappa}_i v_i)_{\Omega_i} = \sum_{i=1}^{2} - (\nabla \cdot A_i \nabla u_i, \tilde{\kappa}_i v_i)_{\Omega_i} + (\nabla \cdot (\beta u_i), \tilde{\kappa}_i v_i)_{\Omega_i} \tag{2.8}
\]

\[
= \sum_{i=1}^{2} \tilde{\kappa}_i (A_i \nabla u_i, \nabla v_i)_{\Omega_i} - (n_i \cdot A_i \nabla u_i, \tilde{\kappa}_i v_i)_{\partial \Omega_1 \setminus \partial \Omega} \tag{2.9}
\]

\[
+ \sum_{i=1}^{2} \tilde{\kappa}_i \left( \frac{1}{2} (\beta \cdot \nabla u_i, v_i)_{\Omega_i} - \frac{1}{2} (u_i, \beta \cdot \nabla v_i)_{\Omega_i} \right) \tag{2.10}
\]

\[
= \star \tag{2.11}
\]

\[
\star = \sum_{i=1}^{2} \tilde{\kappa}_i a_i(u_i, v_i) + \tilde{\kappa}_i b_i(u_i, v_i) - \left( \frac{1}{2} (\beta \cdot \nabla u_i, \nabla v_i)_{\Omega_i} - \frac{1}{2} (u_i, \beta \cdot \nabla v_i)_{\Omega_i} \right)
\]

Thus, we arrive at the following weak formulation:

**Weak Problem.** Find \( u \in W \) such that

\[
A(u, v) = L(v) \quad \forall v \in W \tag{2.18}
\]

with forms defined, for \( v, w \in W \), by

\[
A(v, w) = \sum_{i=0}^{2} \tilde{\kappa}_i \left( a_i(v_i, w_i) + b_i(v_i, w_i) \right) + \sum_{i=1}^{2} (\kappa_{0,i}^{-1} \beta v_i, \beta w_i)_{\Omega_i} \tag{2.19}
\]

\[
L(v) = \sum_{i=0}^{2} (f_i, \tilde{\kappa}_i v_i)_{\Omega_i} \tag{2.20}
\]

where \( \tilde{\kappa}_0 = 1, \tilde{\kappa}_i = \kappa_{0,i}^{-1} \kappa_i \) for \( i = 1, 2 \), and

\[
a_i(v_i, w_i) = (A_i \nabla v_i, \nabla w_i)_{\Omega_i} \tag{2.21}
\]

\[
b_i(v_i, w_i) = \frac{1}{2} \left( (\beta \cdot \nabla v_i, w_i)_{\Omega_i} - (v_i, \beta \cdot \nabla w_i)_{\Omega_i} \right) \tag{2.22}
\]
Existence and Uniqueness. Introducing the energy norm
\[ \|v\|^2_A = A(v, v), \quad v \in W \] (2.23)
associated with the form \( A \), we have the Poincaré inequality.

Lemma 2.1. There is a constant such that for all \( v \in W \),
\[ \sum_{i=0}^{\iota} \bar{\kappa}_i \|v\|^2_{\Omega_i} \lesssim \|v\|^2_A \] (2.24)

Proof. In order to show that (2.24) holds, we let \( \phi \) be the solution to the problem
\[ -\Delta \phi = \psi \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial \Omega \] (2.25)
where \( \psi \in L^2(\Omega) \). Given \( v \in W \), multiplying \(-\Delta \phi = \psi \) by \( v = \sum_{i=1}^{\iota} \bar{\kappa}_i v_i \chi_i \), where \( \chi_i \) is the characteristic function of \( \Omega_i \), and integrating by parts on the bulk domains \( \Omega_i \), \( i = 1, 2 \), we obtain
\[ \sum_{i=1}^{\iota} (\bar{\kappa}_i v_i, \psi)_{\Omega_i} = \sum_{i=1}^{\iota} (\bar{\kappa}_i v_i, \Delta \phi)_{\Omega_i} \]
(2.26)
\[ = \sum_{i=1}^{\iota} (\bar{\kappa}_i \nabla v_i, \nabla \phi)_{\Omega_i} - (\bar{\kappa}_i v_i, \nabla_n \phi)_{\partial \Omega_i} \]
(2.27)
\[ = \sum_{i=1}^{\iota} (\bar{\kappa}_i \nabla v_i, \nabla \phi)_{\Omega_i} - (\bar{\kappa}_i v_i - v_0, \nabla_n \phi)_{\Omega_0} - (v_0, [\nabla_n \phi]_{\Omega_0}) \]
(2.28)
\[ = \sum_{i=1}^{\iota} (\bar{\kappa}_i \nabla v_i, \nabla \phi)_{\Omega_i} - (\kappa_{0,i}^{-1} \kappa v, \nabla_n \phi)_{\Omega_0} \]
(2.29)
\[ \leq \left( \sum_{i=1}^{\iota} \bar{\kappa}_i \|\nabla v_i\|^2_{\Omega_i} + \kappa_{0,i}^{-1} \|\kappa v\|_{\Omega_0}^2 \right)^{1/2} \left( \sum_{i=1}^{\iota} \bar{\kappa}_i \|\nabla \phi\|^2_{\Omega_i} + \kappa_{0,i}^{-1} \|\nabla_n \phi\|^2_{\Omega_0} \right)^{1/2} \]
(2.30)
\[ \lesssim \|v\|_A \|\psi\|_\Omega \]
(2.31)
where we used a trace inequality and elliptic regularity to obtain
\[ \sum_{i=1}^{\iota} \bar{\kappa}_i \|\nabla \phi\|^2_{\Omega_i} + \kappa_{0,i}^{-1} \|\nabla_n \phi\|^2_{\Omega_0} \lesssim \max_{i \in \{1, 2\}} \left( \sum_{i=1}^{\iota} \bar{\kappa}_i \|\nabla \phi\|^2_{\Omega_i} \right) \lesssim \|\psi\|_\Omega^2 \]
(2.32)
Setting \( \psi = \sum_{i=1}^{\iota} \bar{\kappa}_i v_i \chi_i \) gives
\[ \sum_{i=1}^{\iota} \bar{\kappa}_i \|v_i\|^2_{\Omega_i} \lesssim \|v\|^2_A \]
(2.33)
with hidden constant dependent on \( \max_{i \in \{1, 2\}} \kappa_{0,i}^{-1} \max(1, \kappa_i) \). Finally, using a trace inequality on \( \Omega_2 \) we get
\[ \kappa_{2,0} \|v_0\|_{\Omega_0} \lesssim \|\kappa_{2,0} v_0 - \kappa_2 v_2\|_{\Omega_0} + \kappa_2 \|v_2\|_{\Omega_0} \]
(2.34)
\[ \lesssim \|\kappa_{2,0} v_0 - \kappa_2 v_2\|_{\Omega_0} + \kappa_2 \|v_2\|_{H^1(\Omega_2)} \]
(2.35)
\[ \lesssim \|v\|_A \]
(2.36)
where we used (2.33) to conclude that \( \|v_2\|^2_{H^1(\Omega_2)} \lesssim \|v\|_A \). This completes the proof of the Poincaré inequality (2.24).
Thanks to the Poincaré inequality, the energy norm $\| \cdot \|_A$ is indeed a norm and by definition $A$ is coercive and continuous with respect to $\| \cdot \|_A$ and we may apply the Lax-Milgram lemma to conclude that there exists a unique solution $u \in W$ to the weak problem (2.18).

3 Discontinuous CutFEM

Here we formulate the discontinuous cut finite element method. We begin by introducing some preliminaries including the construction of the mesh and the finite element spaces. Then we formulate the method, define the stabilization forms, and provide some useful technical results for the stabilization forms. We end the section with a derivation of the method and the local conservation property.

3.1 The Mesh and Finite Element Spaces

We introduce the following notation.

- Let $T_h$ be a quasiuniform partition of $\Omega$ into shape regular simplicies with mesh parameter $h \in (0, h_0]$ and let $F_h$ be the set of internal faces in $T_h$.

- Define the active meshes

$$T_{h,i} = \{ T \in T_h : T \cap \Omega_i \neq \emptyset \}, \quad i = 0, 1, 2$$

associated with the subdomains $\Omega_i$ and let $F_{h,i}$ be the set of interior faces in $T_{h,i}$. The corresponding intersections with $\Omega_i$, are defined by

$$K_{h,i} = \{ K = T \cap \Omega_i : T \in T_{h,i} \}, \quad E_{h,i} = \{ K = F \cap \Omega_i : F \in F_{h,i} \}$$

- Let

$$V_h = \{ v \in \bigoplus_{T \in T_h} P_1(T) \mid v = 0 \text{ on } \partial \Omega \}$$

be the space of discontinuous piecewise linear polynomials on $T_h$, which are zero on the external boundary $\partial \Omega$. Let $W_{h,i} = V_h|_{T_{h,i}}$ be the active finite element space associated with $T_{h,i}$ and let

$$W_h = \bigoplus_{i=0}^2 W_{h,i}$$

be the finite element space associated with the full system.

- For a face $F$ in $F_{h,i}$, $i = 0, 1, 2$, shared by neighbouring elements $T_1$ and $T_2$ in $T_{h,i}$ we let $\nu_{i,j}$ be the exterior unit normal vector to $T_j$. For $i = 0$, let $\nu_{0,j}$ be the exterior unit co-normal to $K_j \in K_{h,0}$, i.e. the vector which is tangent to $K_j$ and normal to $\partial K_j$. Define the jump and average operators at the face $F$ for scalar functions by

$$[v] = v_1 - v_2, \quad \langle v \rangle = \theta_{F,1} v_1 + \theta_{F,2} v_2$$

where $v_l = v|_{T_l}$, $l = 1, 2$, and for functions of the form $\nu \cdot v$, with $v$ a vector field, by

$$[\nu \cdot v] = \nu_{i,1} v_{i,1} + \nu_{i,2} v_{i,2}, \quad \langle \nu \cdot v \rangle = \theta_{F,1} \nu_{i,1} \cdot v_1 - \theta_{F,2} \nu_{i,2} \cdot v_2$$

where $\theta_{F,j} \geq 0$ are weights defining a convex combination $\theta_{F,1} + \theta_{F,2} = 1$. The dual average $\langle \cdot \rangle^\ast$ is obtained by switching the weights in the average. Note that if the weights are equal, i.e., $\theta_{F,1} = \theta_{F,2} = 1/2$ we have $\langle w \rangle^\ast = \langle w \rangle$. 

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• In particular, when \( v = \beta \tilde{v} \) for a smooth vector field \( \beta \) and scalar \( \tilde{v} \) we write

\[
[v_i \cdot (\beta \tilde{v})] = v_{i,1} \cdot \beta [\tilde{v}]
\]

(3.7)

where we note that the right hand side is independent of the order of the enumeration of elements \( T_1 \) and \( T_2 \), i.e. setting index 2 to 1 and index 1 to 2, since \( v_{i,2} = -v_{i,1} \) and changing the enumeration corresponds to multiplying \([\tilde{v}]\) by \(-1\).

- We have the following identity

\[
[v_i \cdot v w] = \langle v_i \cdot v \rangle [w] + [v_i \cdot v] \langle w \rangle^*
\]

(3.8)

Here we note that changing the enumeration of \( T_1 \) and \( T_2 \), corresponds to multiplying \( \langle v_i \cdot v \rangle \) and \([ w] \) by \(-1\) and therefore the product \( \langle v_i \cdot v \rangle [w] \) is independent of the enumeration.

### 3.2 The Method

The discontinuous cut finite element method takes the form: find \( u_h = (u_{h,0}, u_{h,1}, u_{h,2}) \in \mathbf{W}_h \) such that

\[
A_h(u_h, v) = L_h(v) \quad \forall v \in \mathbf{W}_h
\]

(3.9)

The forms are defined by

\[
A_h(v, w) = \sum_{i=0}^{2} \tilde{\kappa}_i \left( a_{h,i}(v, w) + b_{h,i}(v, w) + s_{h,i}(v, w) \right) + \sum_{i=1}^{2} (\kappa_{0,i}^{-1}[\kappa v], [\kappa w])_{\Omega_0}
\]

(3.10)

\[
L_h(v) = \sum_{i=0}^{2} (f_i, \tilde{\kappa}_i v_i)_{\Omega_i}
\]

(3.11)

with \( \tilde{\kappa}_0 = 1 \), \( \tilde{\kappa}_i = \kappa_{0,i}^{-1} \kappa_i \) for \( i = 1, 2 \), and the forms \( a_{h,i} \) and \( b_{h,i} \) defined by

\[
a_{h,i}(v_i, w_i) = (A_i \nabla_i v_i, \nabla_i w_i)_{\kappa_{h,i}} - ((v_i \cdot A_i \nabla_i v_i), [w_i])_{\mathcal{E}_{h,i}}
\]

(3.12)

\[
- (|v_i|, (v_i \cdot A_i \nabla_i w_i))_{\mathcal{E}_{h,i}} + (\lambda_{a_i} h^{-1}[v_i], [w_i])_{\mathcal{E}_{h,i}}
\]

(3.13)

\[
b_{h,i}(v_i, w_i) = \frac{1}{2} \left( ((\beta \cdot \nabla_i v_i, w_i)_{\kappa_{h,i}} - (v_i, \beta \cdot \nabla_i w_i)_{\kappa_{h,i}}
\right)
\]

(3.14)

\[
+ \frac{1}{2} \left( ((v_i \cdot \beta)(v_i), [w_i])_{\mathcal{E}_{h,i}} - ((v_i \cdot \beta)[v_i], (w_i))_{\mathcal{E}_{h,i}} \right)
\]

(3.15)

\[
+ (\lambda_{b_i}[v_i], [w_i])_{\mathcal{E}_{h,i}}
\]

(3.16)

where

\[
\lambda_{a_i} = \tau_{a_i} \left\| v_i \right\|_{A_i}^2 = \tau_{a_i} v_i \cdot A_i v_i, \quad \lambda_{b_i} = \tau_{b_i} |v_i \cdot \beta|
\]

(3.17)

with parameters \( \tau_{a_i} > 0 \) sufficiently large to guarantee coercivity, see Lemma 4.4 below, and \( \tau_{b_i} \geq 0 \). The stabilization forms \( s_{h,i} \), for \( i = 0, 1, 2 \), are defined in (3.21), (3.22), and (3.24) below and we derive the method in Section 3.5.
3.3 Definition of the Stabilization Forms

We shall divide the elements in each active mesh $T_{h,i}$ in two types, those with a large intersection with the domain $\Omega_i$ and those with a small intersection. The small elements will be connected through a chain of face neighbours to a large element and together that set of elements form a macro element with a large intersection. The macro elements will essentially behave as standard finite elements in the bulk domains and for the surface domain we will have to add a stabilization in the direction normal to the surface since the partial differential equation at the surface only involves tangential derivatives. We now make this approach precise with the following definitions.

- An element $T$ in $T_{h,i}$ has a large intersection if

$$\gamma_i \leq \frac{|T \cap \Omega_i|}{h_T^{d_i}}$$

where $h_T$ is the diameter of element $T$, $d_i$ is the dimension of the domain $\Omega_i$, and $\gamma_i$ is a positive constant which is independent of the element and the mesh parameter. The elements that are not large are defined to be small. Note that since the mesh is quasiuniform we have $h \sim h_T$ and it follows from (3.18) that

$$h_T^{d_i} \lesssim |T \cap \Omega_i|$$

- Let $\mathcal{M}_{h,i}$ be a macro element partition derived from $T_{h,i}$, such that: each element $T \in T_{h,i}$ belongs to precisely one $T_{h,i,M}$, each macro element $M \in \mathcal{M}_{h,i}$ is a union of elements in $T_{h,i,M}$,

$$M = \bigcup_{T \in T_{h,i,M}} T$$

such that in $T_{h,i,M}$ there is one element $T_M$ with a large intersection and all elements in $T_{h,i,M}$ are connected to $T_M$ via a uniformly bounded number of internal faces in $T_{h,i,M}$.

- Let $\mathcal{F}_{h,i}(M)$ be the set of interior faces in $T_{h,i,M}$. Note that $\mathcal{F}_{h,i}(M)$ is empty when $M$ consist of only one element (with a large intersection). For each $M \in \mathcal{M}_{h,i}$ with $i = 1, 2$, (the bulk domains) we define the stabilization form

$$s_{h,i,M}(v, w) = \sum_{k=0}^{1} \tau_{i,k} h_T^{2k-1} ([\nabla^k v], [\nabla^k w])_{\mathcal{F}_{h,i}(M)}$$

and for $i = 0$, (the surface domain)

$$s_{h,0,M}(v, w) = \sum_{k=0}^{1} \tau_{0,k} h_T^{2k-2} ([\nabla^k v], [\nabla^k w])_{\mathcal{F}_{h,0}(M)} + \tau_{0,2} h_T^2 (\nabla_n v, \nabla_n w)_{\mathcal{M} \cap \Omega_0}$$

Here, $[\nabla^0] = [v]$, $[\nabla^1 v] = [\nabla v]$, and $\tau_{i,k} > 0$ are positive parameters of the form

$$\tau_{i,k} \sim c_{i,k} \| A_i \|_{L^\infty(M)}$$

with $c_{i,k} > 0$. The macro element stabilization control the full jump $[\nabla v]$ across faces internal to the macro element and the normal derivative $\nabla_n v$ at the interface.
• Given the macro element stabilization forms defined by (3.21) and (3.22) we define the global stabilization forms

\[ s_{h,i}(v, w) = \sum_{M \in M_{h,i}} s_{h,i,M}(v, w) \quad (3.24) \]

Remark 3.1. Note that there is no stabilization acting on the faces at the interface between two neighbouring macro elements. In contrast, standard stabilization, which we refer to as full stabilization, includes stabilization terms also on those faces. See Figure 1 for an illustration of a macro element partition and the edges on which stabilization is applied and Figure 4 for a comparison with full stabilization.

Remark 3.2. The scaling, with respect to the mesh parameter \( h \), in the different stabilization terms is chosen in such a way that the forthcoming coercivity estimate in Lemma 4.4, which depend on the technical Lemma 4.3, holds.

We next formulate a basic algorithm for computation of the macro element partition. We seek to generate macro elements that contain as few elements as possible, which corresponds to stabilization on as few faces as possible. Note, however, that the theoretical developments only requires a uniform bound on the number of elements in each macroelement and therefore it is not critical to find the optimal partition into macro elements. The algorithm computes a set

\[ \mathcal{F}_{h,i}^* = \bigcup_{M \in M_{h,i}} \mathcal{F}_{h,i}(M) \subset \mathcal{F}_{h,i} \quad (3.25) \]

containing all the faces where stabilization is applied.

Algorithm 1. Initiate \( \mathcal{F}_{h,i}^* \) as the empty set.

1. Mark each element in \( \mathcal{T}_{h,i} \) as large or small according to criteria (3.18).

2. For every small element that is face connected to one or several large elements, choose one face that connects the small element to a large neighbouring element. Add that face to \( \mathcal{F}_{h,i}^* \) for stabilization and mark the small element as large.

3. Repeat 2 until all elements are marked as large.

Note that the macro element partition and the generated set \( \mathcal{F}_{h,i}^* \) is not unique since there is no preference in the choice of face in Step 2. For the coupled bulk-surface problem we use the algorithm described above to generate \( \mathcal{F}_{h,0}^* \) from the mesh \( \mathcal{T}_{h,0} \). To generate \( \mathcal{F}_{h,i}^* \), for \( i = 1, 2 \) we use the same algorithm but in the choice of face in Step 2, faces that are already in \( \mathcal{F}_{h,0}^* \) are preferred. When the three sets \( \mathcal{F}_{h,i}^* \), \( i = 0, 1, 2 \), have been generated the stabilization forms are given by:

\[ s_{h,0}(v, w) = \sum_{k=0}^{1} \tau_{0,k} h^{2k-2}([\nabla^k v_0], [\nabla^k w_0])_{\mathcal{F}_{h,0}} + \tau_{0,2} h^2 (\nabla n_0 v_0, \nabla n_0 w_0)_{K_h,0} \quad (3.26) \]

\[ s_{h,i}(v, w) = \sum_{k=0}^{1} \tau_{i,k} h^{2k-1}([\nabla^k v_i], [\nabla^k w_i])_{\mathcal{F}_{h,i}}^*, \quad i = 1, 2 \quad (3.27) \]
Figure 1: Illustration of macro elements and the patchwise stabilization. Elements in the active mesh $T_{h,1}$ (left panel) and in $T_{h,2}$ (right panel) are shown. Macro elements consisting of several triangles are marked in purple and their interior edges, the dotted lines, are edges on which stabilization is applied. There are in total 4 edges in $F_{h,1}^*$ and 16 edges in $F_{h,2}^*$.

3.4 Properties of the Macro Element Stabilization

We now present some useful results on the properties provided by the macro element stabilization. A common theme is the observation that the macro element stabilization enables us to control the difference between a general discontinuous piecewise linear function on the active mesh and an approximation which is discontinuous piecewise linear on the macro elements. For the latter function standard finite element estimates holds since the macro elements have a large intersection with the domain.

The first bound shows that we can estimate the difference between a discontinuous piecewise linear function and a suitable linear function on a macro element by the stabilization terms on the macro element. The second bound is the stabilization property that follows essentially from [20], where continuous higher order element were considered and the macro elements appeared as a technical tool in the proofs.

Lemma 3.1. There is a constant such that for all $v \in W_{h,i}$ and all macro elements $M \in \mathcal{M}_{h,i}$, $i = 0, 1, 2$,

$$\inf_{v_M \in P_1(M)} \|\nabla^m (v - v_M)\|_M^2 \lesssim h^{2(1-m)} \sum_{k=0}^{1} h^{2k-1} ||[\nabla^k v]||_{F_{h,i}(M)}^2, \quad m = 0, 1 \quad (3.28)$$

Proof. Let $T_M \in T_{h,i}$ be the element in $M$ that has a large intersection with $\Omega_i$ and let $v_M \in P_1(M)$ be such that $v_M|_{T_M} = v|_{T_M}$. We recall that for two elements $T_1$ and $T_2$ sharing a face $F$ there is a constant such that for each $w = (w_1, w_2) \in P_1(T_1) \oplus P_1(T_2)$,

$$\|\nabla^m w_1\|_{T_1}^2 \lesssim \|\nabla^m w_2\|_{T_2}^2 + h^{2(1-m)} \sum_{k=0}^{1} h^{2k-1} ||[\nabla^k w]||_F^2 \quad (3.29)$$

Setting $w = v - v_M$, noting that $v - v_M = 0$ on $T_M$, and then using (3.29) repeatedly together with the fact that the number of elements in each macro element is uniformly bounded completes the proof. 

$\blacksquare$
Lemma 3.2. There is a constant such that for all \( v \in W_{h,i} \) and all macro elements \( M \in \mathcal{M}_{h,i} \), \( i = 0, 1, 2 \),

\[
\|\nabla^m v\|_M^2 \lesssim \begin{cases} \|\nabla_i^m v\|_{M \cap \Omega_i}^2 + h^{2(1-m)} \left( \sum_{k=0}^{1} h^{2k-1} \|\nabla^k v\|_{\mathcal{F}_{h,i}(M)}^2 \right), & i = 1, 2 \\ h \|\nabla^m v\|_{\Omega_0}^2 + h^{2(1-m)} \left( \sum_{k=0}^{1} h^{2k-1} \|\nabla^k v\|_{\mathcal{F}_{h,0}(M)}^2 \right) + h \|\nabla v\|_{3M \cap \Omega_0}^2 \end{cases}
\]

for \( m = 0, 1 \).

Proof. Let \( T_M \in \mathcal{T}_{h,i} \) be the element in the macro element \( M \) that has a large intersection with \( \Omega_i \), see (3.18). Let \( v_M \in P_1(M) \) be such that \( v_M|_{T_M} = v|_{T_M} \). Then we have using (3.28) in Lemma 3.1

\[
\|\nabla^m v\|_M^2 \lesssim \|\nabla^m v_M\|_{T_M}^2 + \|\nabla^m (v - v_M)\|_{T_M}^2 \lesssim \|\nabla^m v_M\|_{M \cap \Omega_i}^2 + \|\nabla^m (v - v_M)\|_{M \cap \Omega_i}^2, \quad i = 1, 2
\]

To estimate the first term on the right hand side in (3.32) we have the following inverse inequality in the case of bulk domains

\[
\|\nabla^m v_M\|_{T_M}^2 \lesssim \|\nabla^m v_M\|_{T_M}^2 \lesssim \|\nabla^m v_M\|_{M \cap \Omega_i}^2 \lesssim \|\nabla^m v\|_{M \cap \Omega_i}^2, \quad i = 1, 2
\]

For the surface domain, \( i = 0 \), we instead of (3.33) have

\[
\|\nabla^m v_M\|_{M}^2 \lesssim \|\nabla^m v_M\|_{T_M}^2 \lesssim \|\nabla^m v_M\|_{M \cap \Omega_0}^2 + h^{3-2m} \|\nabla^m v_M\|_{M \cap \Omega_0}^2 \lesssim h \|\nabla^m v\|_{M \cap \Omega_0}^2 + h^{2(1-m)} \|\nabla^m v\|_{M \cap \Omega_0}^2
\]

where we recall that \( \nabla_0 \) is the tangential derivative associated with \( \Omega_0 \) and that the codimension for the surface is \( d - d_0 = 1 \). We also note that we need the normal stabilization, the last term on the right hand side of (3.22), at \( \Omega_0 \) to pass from \( T_M \) to the intersection \( T_M \cap \Omega_0 \). Together, (3.32), (3.33), and (3.36) complete the proof. ■

We next show that, on the bulk domains, the macro element stabilization together with terms that are controlled by the weak form in the method, control face stabilization on all faces. Recall that the stabilization is only added inside the macro elements and that the Nitsche penalty term only acts on the intersection between a face and the domain and is therefore significantly weaker compared to stabilization on all faces. To prepare for the estimate we first show a Poincaré type estimate for macro elements in Lemma 3.3.

Lemma 3.3. For the bulk domains \((i = 1, 2)\), there is a constant such that for all \( v \in W_{h,i} \) and all macro elements \( M \in \mathcal{M}_{h,i} \),

\[
\inf_{w_M \in \Phi(M)} \|v - w_M\|_M^2 \lesssim h^2 \left( \|\nabla v\|_{M \cap \Omega_i}^2 + \sum_{k=0}^{1} h^{2k-1} \|\nabla^k v\|_{\mathcal{F}_{h,i}(M)}^2 \right)
\]

(3.37)
**Proof.** Let $T_M \in T_{h,i}$ be the element in $M$ that has a large intersection with $\Omega_i$ and let $v_M \in P_1(M)$ be such that $v_M|_{T_M} = v|_{T_M}$, and $w_M \in P_0(M)$ be such that $w_M|_{T_M}$ is the $L^2$ projection of $v_M|_{T_M} = v|_{T_M}$ onto constants at $T_M$. Then we have

$$
\|v - w_M\|_{T_M}^2 \lesssim \|v - v_M\|_{T_M}^2 + \|v_M - w_M\|_{T_M}^2 \quad (3.38)
$$

$$
\lesssim \|v - v_M\|_{T_M}^2 + h^2 \|\nabla v_M\|_{T_M}^2 \quad (3.39)
$$

$$
\lesssim \|v - v_M\|_{T_M}^2 + h^2 \|\nabla w_M\|_{T_M}^2 + h^2 \|\nabla v\|_{T_M\cap\Omega_i}^2 \quad (3.40)
$$

$$
\lesssim \|v - v_M\|_{T_M}^2 + h^2 \|\nabla v - v_M\|_{T_M\cap\Omega_i}^2 + h^2 \|\nabla v\|_{T_M\cap\Omega_i}^2 \quad (3.41)
$$

$$
\lesssim h^2 \sum_{k=0}^1 h^{2k-1} \|\nabla_k v\|_{T_{h,i}^k(M)}^2 + h^2 \|\nabla v\|_{T_M\cap\Omega_i}^2 \quad (3.42)
$$

where in inequality (3.42) we employed Lemma 3.1 with $m = 0$ and $m = 1$. ■

**Lemma 3.4.** For the bulk domains ($i = 1, 2$), there is a constant such that for all $v \in W_{h,i}$,

$$
\sum_{k=0}^1 h^{2k-1} \|\nabla_k v\|_{T_{h,i}^k}^2 \lesssim \|\nabla v\|_{T_{h,i}}^2 + h^{-1} \|v\|_{E_{h,i}}^2 + \sum_{M \in M_{h,i}, k=0}^1 h^{2k-1} \|\nabla_k v\|_{T_{h,i}^k(M)}^2 \quad (3.43)
$$

**Proof.** Let us start with the estimation of $h^{-1} \|v\|_{T_{h,i}}^2$. We note that the Nitsche penalty term provides control over $h^{-1} \|v\|_{T_{h,i}}^2$, where we recall that $E_{h,i} = T_{h,i} \cap \Omega_i$, and it therefore remains to control $h^{-1} \|v\|_{T_{h,i}}^2$ for faces $F \in F_{h,i}$ which intersect the boundary $\Omega_0$. If $F$ has a large intersection with $\Omega_i$, i.e. there is a constant such that

$$
|F \cap \Omega_i| \gtrsim h^{d-1} \quad (3.44)
$$

and an inverse inequality directly gives

$$
h^{-1} \|v\|_{F}^2 \lesssim h^{-1} \|v\|_{T_{h,i}^k(M)}^2 \quad (3.45)
$$

Next we consider the case when (3.44) does not hold. Let $F$ be such a face, which is shared by two macro elements $M_i$ and $M_j$. Then there is a chain of macro elements $\{M_n\}_{n=1}^N$ connecting $M_i$ to $M_j$ ($M_1 = M_i$ and $M_N = M_j$) of uniformly bounded length $l$, such that two consecutive macro elements $M_n$ and $M_{n+1}$ share a face $F_n$ such that $|F_n \cap \Omega_i| \gtrsim h^{d-1}$. Observing that a macro element must have at least one face on its boundary that has a large intersection with $\Omega_i$, we can then use that face to pass to a neighboring macro element.

Starting with the estimation of $h^{-1} \|v\|_{F}^2$. Letting $v_n = v|M_n$, for $n = 1, \ldots, l$, and keeping in mind that $v_i = v_1$ and $v_j = v_l$, we have for any $w \in \oplus_{M \in M_{h,i}} P_0(M)$, with $w_n = w|M_n \in P_0(M)$ for $n = 1, \ldots, l$,

$$
h^{-1} \|v\|_{F}^2 \lesssim h^{-1} \|w\|_{F}^2 + h^{-1} \|v_1 - w_1\|_{F}^2 + h^{-1} \|v_l - w_l\|_{F}^2 \quad (3.46)
$$

$$
\lesssim h^{-1} \|w\|_{F}^2 + h^{-2} \|v_1 - w_1\|_{M_1}^2 + h^{-2} \|v_l - w_l\|_{M_l}^2 \quad (3.47)
$$

Here we added and subtracted $w_1$ and $w_l$, used the triangle inequality, and an inverse trace inequality on the macro elements. To estimate the first term on the right hand side we add and subtract $w_2, \ldots, w_{l-1}$ and use the triangle inequality.

$$
h^{-1} \|v\|_{F}^2 = h^{-1} \|w_1 - w_l\|_{F}^2 \lesssim \sum_{n=1}^{l-1} h^{-1} \|w_n - w_{n+1}\|_{F}^2 \lesssim \sum_{n=1}^{l-1} h^{-1} \|w_n - w_{n+1}\|_{F_n}^2 \quad (3.48)
$$
Finally, summing over all faces $m$ the elements, collecting the elements into macro elements, and then using the triangle inequality

$$h^{-1} ||[w]||_{F_n}^2 \lesssim h^{-1} ||[v]||_{F_n}^2 + h^{-1} ||w_n - v_n||_{F_n}^2 + h^{-1} ||w_{n+1} - v_{n+1}||_{F_n}^2$$

$$\lesssim h^{-1} ||[v]||_{F_n}^2 + h^{-1} ||w_n - v_n||_{F_n}^2 + h^{-2} ||w_{n+1} - v_{n+1}||_{F_{n+1}}^2$$

Collecting the bounds, taking the infimum over $w \in \bigoplus_{M \in \mathcal{M}_{h,i}} P_1(M)$, and using the Poincaré estimate in Lemma 3.3 on the macro elements, give

$$h^{-1} ||[v]||_{F_{h,i}}^2 \lesssim \sum_{n=1}^{l-1} h^{-1} ||[v]||_{F_n}^2 + \sum_{n=1}^l h^{-2} ||v_n - w_n||_{M_n}^2$$

$$\lesssim \sum_{n=1}^{l-1} h^{-1} ||[v]||_{F_n \cap \Omega_i}^2 + \sum_{n=1}^{l-1} \left( ||\nabla v_n||_{M_n \cap \Omega_i}^2 + \sum_{k=0}^1 h^{2k-1} ||[\nabla^k v_n]||_{F_{h,i}(M)}^2 \right)$$

Finally, summing over all faces $F \in \mathcal{F}_{h,i}$ and using the bounds (3.45) and (3.52) we obtain

$$h^{-1} ||[v]||_{F_{h,i}}^2 \lesssim h^{-1} ||[v]||_{F_{h,i}}^2 + \sum_{M \in \mathcal{M}_{h,i}} ||\nabla v||_{M \cap \Omega_i}^2 + \sum_{k=0}^{1} h^{2k-1} ||[\nabla^k v]||_{F_{h,i}(M)}^2$$

It remains to estimate $h||[\nabla v]||_F^2$ for any $F \in \mathcal{F}_{h,i} \setminus \mathcal{F}_{h,i}^*$, where we recall that $\mathcal{F}_{h,i}^*$ is the set of faces where stabilization is applied. Using an inverse trace inequality on the elements, collecting the elements into macro elements, and then using the stability estimate in Lemma 3.2 with $m = 1$ and $i = 1, 2$, we obtain

$$h||[\nabla v]||_{F_{h,i}}^2 \lesssim ||\nabla v||_{F_{h,i}}^2 = \sum_{M \in \mathcal{M}_{h,i}} ||\nabla v||_{M}^2$$

$$\lesssim \sum_{M \in \mathcal{M}_{h,i}} \left( ||\nabla^m v||_{M \cap \Omega_i}^2 + \sum_{k=0}^{1} h^{2k-1} ||[\nabla^k v]||_{F}^2 \right)$$

Summing (3.53) and (3.55) and using the fact that $\mathcal{F}_{h,i} \cap \Omega_i = \mathcal{E}_{h,i}$ and $\mathcal{T}_{h,i} \cap \Omega_i = \mathcal{K}_{h,i}$ completes the proof.

We end this section by showing a trace inequality for the discontinuous function spaces on the bulk domains, which is of general interest, and will also be used in the derivation of a Poincaré inequality for the discrete spaces. The difficulty is that the functions are discontinuous so we can not directly employ a standard trace inequality and an element wise trace inequality would produce a non optimal factor of $h^{-1}$.

**Lemma 3.5.** There is a constant such that for all $v \in W_{h,i}$, $i = 1, 2$,

$$||v||_{H^1(K_{h,i})} \lesssim ||v||_{H^1(K_{h,i})} + h^{-1} ||[v]||_{F_{h,i}}^2$$

$$\lesssim ||v||_{H^1(K_{h,i})} + h^{-1} ||[v]||_{F_{h,i}}^2 + \sum_{M \in \mathcal{M}_{h,i}} \sum_{k=0}^{1} h^{2k-1} ||[\nabla^k v]||_{F_{h,i}(M)}^2$$

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Figure 2: The face $F$ with a small intersection with $\Omega_1$ (right side of the interface) and a large intersection with $\Omega_2$ (left side of the interface) and the chain of macro elements $\{M_n\}_{n=1}^4$ that enables us to estimate the jump over the face $F$.

**Proof.** Let us consider $i = 1$ and let $O_{h,1} : W_{h,1} \rightarrow W_{h,1} \cap C(\Omega_{h,1})$ be the Oswald interpolant with nodal values given by the average of the nodal values of the discontinuous function $v$, more precisely for a node $x_i$ we define

$$O_{h,1}v|_{x_i} = \frac{1}{|T_{h,1}(x_i)|} \sum_{T \in T_{h,1}(x_i)} v|_T(x_i)$$

(3.58)

where $T_{h}(x_i)$ is the set of elements in $T_{h,1}$ sharing node $x_i$. We then have the well known estimate

$$\|v - O_{h,1}v\|_{T_{h,1}}^2 \lesssim h\|[v]\|_{F_{h,1}}^2$$

(3.59)

see [6]. Adding and subtracting the Oswald interpolant and using the inverse trace inequality

$$\|w\|_{\partial \Omega_0 \cap T} \lesssim h^{-1}\|w\|_T$$

for the difference $v - O_{h,1}v$ and the standard trace inequality

$$\|w\|_{\Omega_0} \lesssim \|w\|_{H^1(\Omega_1)}$$

for $w = O_{h,1}v$ we get

$$\|v\|_{T_{h,1}}^2 \lesssim \|v - O_{h,1}v\|_{T_{h,1}}^2 + \|O_{h,1}v\|_{T_{h,1}}^2$$

(3.60)

$$\lesssim h^{-1}\|v - O_{h,1}v\|_{T_{h,1}(\Omega_0)}^2 + \|O_{h,1}v\|_{T_{h,1}(\Omega_1)}^2$$

(3.61)

$$\lesssim h^{-1}\|v - O_{h,1}v\|_{T_{h,1}(\Omega_0)}^2 + \|O_{h,1}v - v\|_{T_{h,1}(\Omega_1)}^2 + \|v\|_{T_{h,1}(\Omega_0 \cap \Omega_1)}^2$$

(3.62)

$$\lesssim (1 + h^{-1} + h^{-2})\|v - O_{h,1}v\|_{T_{h,1}(\Omega_0)}^2 + \|v\|_{T_{h,1}(\Omega_1)}^2$$

(3.63)

$$\lesssim h^{-1}\|[v]\|_{T_{h,1}}^2 + \|v\|_{T_{h,1}(\Omega_0 \cap \Omega_1)}^2$$

(3.64)

$$\lesssim \|v\|_{T_{h,1}(\Omega_0 \cap \Omega_1)}^2 + h^{-1}\|[v]\|_{T_{h,1}(\Omega_0 \cap \Omega_1)}^2 + \sum_{M \in M_{h,i}} \sum_{k=0}^{1} h^{2k-1}\|\nabla^k v\|_{T_{h,1}(M)}^2$$

(3.65)

for all $h \in (0, h_0]$, and we used Lemma 3.4 in the last inequality.

**3.5 Derivation of the Method and Consistency**

To derive the method we test the exact bulk equations (2.3) by $\tilde{\kappa}_i v_i$ with $v_i \in W_{h,i}$, $i = 1, 2$, and integrate element wise using Green's formula. Using the assumptions on the
velocity field $\beta$ we obtain

$$\sum_{i=1}^{2}(f_i, \tilde{\kappa}_i v_i)_{\Omega_h} = \sum_{i=1}^{2} -\left(\nabla \cdot (A_i \nabla u_i), \tilde{\kappa}_i v_i\right)_{T_{h,i} \cap \Omega} + \left(\nabla \cdot (\beta u_i), \tilde{\kappa}_i v_i\right)_{T_{h,i} \cap \Omega} \quad (3.66)$$

$$= \sum_{i=1}^{2} \left( (A_i \nabla u_i, \nabla \tilde{\kappa}_i v_i)_{K_{h,i}} - (\nu_i \cdot A_i \nabla u_i, \tilde{\kappa}_i v_i)_{\partial K_{h,i}} \right)$$

$$+ \frac{1}{2} \left( (\beta \cdot \nabla u_i, \tilde{\kappa}_i v_i)_{K_{h,i}} - (u_i, \beta \cdot \nabla (\tilde{\kappa}_i v_i))_{K_{h,i}} + ((\nu_i \cdot \beta) u_i, \tilde{\kappa}_i v_i)_{\partial K_{h,i}} \right) \quad (3.67)$$

$$= \sum_{i=1}^{2} \tilde{\kappa}_i \left( (A_i \nabla u_i, \nabla v_i)_{K_{h,i}} - (\langle \nu_i \cdot A_i \nabla u_i, [v_i]\rangle_{\mathcal{E}_{h,i}} \right)$$

$$- \sum_{i=1}^{2} (n_i \cdot A_i \nabla u_i, \tilde{\kappa}_i v_i - v_0)_{\partial \Omega, i \cup \Omega_0} - \sum_{i=1}^{2} (n_i \cdot A_i \nabla u_i, v_0)_{\partial \Omega, i \cup \Omega_0} \quad (3.70)$$

$$+ \frac{1}{2} \sum_{i=1}^{2} \tilde{\kappa}_i \left( (\beta \cdot \nabla u_i, v_i)_{K_{h,i}} - (u_i, \beta \cdot \nabla v_i)_{K_{h,i}} + ((\nu_i \cdot \beta) \langle u_i, [v_i]\rangle_{\mathcal{E}_{h,i}} \right) \quad (3.71)$$

Since the components $v_i$ of $v$ are discontinuous across the internal faces in $T_{h,i}$, the partial integration manufactures certain jump terms. By adding terms that are zero we can make the diffusion dependent parts symmetric and the convection dependent parts skew-symmetric.

$$\sum_{i=1}^{2}(f_i, \tilde{\kappa}_i v_i)_{\Omega_h} = \sum_{i=1}^{2} \tilde{\kappa}_i \left( (A_i \nabla u_i, \nabla v_i)_{K_{h,i}} - (\langle \nu_i \cdot A_i \nabla u_i, [v_i]\rangle_{\mathcal{E}_{h,i}} \right)$$

$$- \langle [u_i], (\nu_i \cdot A_i \nabla v_i)_{\mathcal{E}_{h,i}} \rangle + \tilde{\kappa}_i \left( \lambda_h h^{-1} [u_i], [v_i] \right)_{\mathcal{E}_{h,i}} \quad (3.72)$$

$$+ \frac{1}{2} \sum_{i=1}^{2} \tilde{\kappa}_i \left( (\beta \cdot \nabla u_i, v_i)_{K_{h,i}} - (u_i, \beta \cdot \nabla v_i)_{K_{h,i}} \right)$$

$$+ \left(\langle \nu_i \cdot \beta \rangle \langle u_i \rangle, [v_i] \right)_{\mathcal{E}_{h,i}} - \left( (\nu_i \cdot \beta) [u_i], \langle v_i \rangle\right)_{\mathcal{E}_{h,i}} + \tilde{\kappa}_i \left( \lambda_h [u_i], [v_i] \right)_{\mathcal{E}_{h,i}} \quad (3.75)$$

$$+ \sum_{i=1}^{2} (\kappa_{0,i}^{-1} [\kappa u_i], [\kappa v_i])_{\Omega_0} - \langle [n \cdot A \nabla v], v_0 \rangle_{\Omega_0} \quad (3.76)$$

$$= \sum_{i=1}^{2} \tilde{\kappa}_i a_{h,i}(u_i, v_i) + \tilde{\kappa}_i b_{h,i}(u_i, v_i) + (\kappa_{0,i}^{-1} [\kappa u_i], [\kappa v_i])_{\Omega_0} - \langle [n \cdot A \nabla v], v_0 \rangle_{\Omega_0} \quad (3.77)$$
Next using partial integration on the surface $\Omega_0$ we obtain

\[
\star = -(f_0, v_0)_{\Omega_0} - (\nabla_0 \cdot (A_0 \nabla_0 u_0), v_0)_{\Omega_0} + (\nabla_0 \cdot (\beta u_0), v_0)_{\Omega_0} \tag{3.78}
\]

\[
= -(f_0, v_0)_{\Omega_0} + (A_0 \nabla_0 u_0, \nabla_0 v_0)_{\mathcal{K}_{h,0}} - (\nu_0 \cdot A_0 \nabla_0 u_0, v_0)_{\partial \mathcal{K}_{h,0}} \tag{3.79}
\]

\[
+ \frac{1}{2}((\beta \cdot \nabla_0 u_0, \nu_0)_{\mathcal{K}_{h,0}} - (\nu_0, \beta \cdot \nabla_0 v_0)_{\mathcal{K}_{h,0}} + ((\nu_0 \cdot \beta) u_0, v_0)_{\partial \mathcal{K}_{h,0}}) \tag{3.80}
\]

\[
= -(f_0, v_0)_{\Omega_0} + (A_0 \nabla_0 u_0, \nabla_0 v_0)_{\mathcal{K}_{h,0}} - (\nu_0 \cdot A_0 \nabla_0 u_0, [v_0])_{\mathcal{E}_{h,0}} \tag{3.81}
\]

\[
- \left(\left(u_0, \nu_0 \cdot A_0 \nabla_0 v_0\right)_{\mathcal{E}_{h,0}} + \left(\lambda_0 [u_0], [v_0]\right)_{\mathcal{E}_{h,0}}\right) = 0 \tag{3.82}
\]

\[
+ \frac{1}{2}((\beta \cdot \nabla_0 u_0, \nu_0)_{\mathcal{K}_{h,0}} - (u_0, \beta \cdot \nabla_0 v_0)_{\mathcal{K}_{h,0}}) \tag{3.83}
\]

\[
+ \frac{1}{2}((\nu_0 \cdot \beta) u_0, [v_0])_{\mathcal{E}_{h,0}} - ((\nu_0 \cdot \beta) [u_0], [v_0])_{\mathcal{E}_{h,0}} + \left(\lambda_0 [u_0], [v_0]\right)_{\mathcal{E}_{h,0}} = 0 \tag{3.84}
\]

\[
= -(f_0, v_0)_{\Omega_0} + a_{h,0}(u_0, v_0) + b_{h,0}(u_0, v_0) \tag{3.85}
\]

We finally add the stabilization terms defined by (3.26) and (3.27) to the formulation. The above derivation, together with the fact that the stabilization terms vanish on the exact solution, prove the next lemma, which states that the proposed discontinuous cut finite element method is consistent.

**Lemma 3.6.** The solution $u \in W \cap \bigoplus_{i=0}^{2} H^2(\Omega_i)$ to the convection diffusion problem (2.3) - (2.6) satisfies

\[
A_h(u, v) = L_h(v) \quad \forall v \in W_h \tag{3.86}
\]

### 3.6 The Local Conservation Property

We shall now establish a conservation property for the so called Nitsche flux on the macro elements. Since we do not have any additional stabilization at the interfaces between the macro elements the natural local conservation property inherent in the discontinuous Galerkin formulation is preserved on the macro element level. Consider a macro element $M \in \mathcal{M}_{h,i}$. Testing with the characteristic function $\chi_M$ associated with the macro element $M$, yields

\[
L_h(\chi_M) = A_h(u_h, \chi_M) \tag{3.87}
\]

where

\[
L_h(\chi_M) = (\tilde{\kappa}_i f_i, 1)_{M \cap \Omega_i} \tag{3.88}
\]

and using the fact that $s_{h,i}(u_h, \chi_M) = 0$ since there is no stabilization across the macro element boundaries we get

\[
A_h(u_h, \chi_M) = \sum_{i=0}^{2} a_{h,i}(u_{h,i}, \chi_M) + b_{h,i}(u_{h,i}, \chi_M) + \sum_{i=1}^{2} ([\kappa u_h]_i, [\chi_M]_i)_{M \cap \Omega_0} \tag{3.89}
\]

with

\[
a_{h,i}(u_{h,i}, \chi_M) = -((\nu_i \cdot A_i \nabla_i u_{h,i}), \chi_M)_{\partial M \cap \mathcal{E}_{h,i}^\prime} + \left(\lambda_{a_i} h^{-1}[u_{h,i}], [\chi_M]_{\partial M \cap \mathcal{E}_{h,i}^\prime}\right) \tag{3.90}
\]
and
\[ b_{h,i}(u_{h,i}, \chi_M) = \frac{1}{2} \left( (\beta_i \cdot \nabla_i u_{h,i}, \chi_M)_{\Omega \cap E_{h,i}} - (\omega_{h,i}, \chi_M)_{\Omega \cap E_{h,i}} \right) \quad (3.91) \]
\[ + \frac{1}{2} \left( (\nu_i \cdot \beta_i) (u_{h,i}, \chi_M)_{\partial M \cap E_{h,i}} + (\lambda_{h,i}, \chi_M)_{\partial M \cap E_{h,i}} \right) \quad (3.92) \]
\[ = (\nu_i \cdot \beta_i) (u_{h,i}, \chi_M)_{\partial M \cap E_{h,i}} - \frac{1}{2} \left( (\text{div}_i \beta_i) u_{h,i} \cdot \nu_i, \chi_M \right)_{\partial M \cap E_{h,i}} + (\lambda_{h,i}, \chi_M)_{\partial M \cap E_{h,i}} \quad (3.93) \]
where we used partial integration and relation (3.8). Introducing the discrete normal flux \( \Sigma_n (u_{h,i}) \) such that
\[ \Sigma_n (u_{h,i}) = (\nu_i \cdot A_i \nabla_i u_{h,i}) - (\omega_{h,i}, u_{h,i}) - (\nu_i \cdot \beta_i) (u_{h,i}, 1)_{\partial M \cap \Sigma_{h,i}} \quad (3.94) \]
We get, for each macro element \( M \) in one of the bulk macro meshes \( M_{h,i} \), the conservation law
\[ (\Sigma_n (u_{h,i}), 1)_{\partial M \cap \Omega_i} + ([\kappa u_{h,i}], 1)_{M \cap \Omega_i} + \frac{1}{2} \left( (\text{div}_i \beta_i) u_{h,i}, 1 \right)_{M \cap \Omega_i} + (f_i, 1)_{M \cap \Omega_i} = 0 \quad (3.95) \]
and for \( M \) a macro element in the surface mesh \( M_{h,0} \) we get the same expression except for the sign of the second term, which couples the bulk domains to the interface
\[ (\Sigma_n (u_{h,0}), 1)_{\partial M \cap \Omega_0} - \sum_{i=1}^{2} ([\kappa u_{h,i}], 1)_{M \cap \Omega_0} \]
\[ + \frac{1}{2} \left( (\text{div}_0 \beta_0) u_{h,0}, 1 \right)_{M \cap \Omega_0} + (f_0, 1)_{M \cap \Omega_0} = 0 \quad (3.96) \]

4 Analysis of the Method

4.1 Properties of the Forms

We establish the basic properties of the form \( A_h \) including a discrete Poincaré inequality, coercivity, and continuity. In order to identify the proper local scaling of the different terms in the method we will take the variable coefficients \( A_i \) into account in the proof of coercivity. However, to keep the overall presentation as simple as possible we elsewhere work with the global bounds on parameters \( A_i \). We start by defining the norms
\[ \|v_i\|_{h,i}^2 = \|\nabla_i v_i\|_{A_i, \Sigma_{h,i}}^2 + h^2 \langle v_i, v_i \rangle_{A_i, \Sigma_{h,i}} + h^{-1} \|v_i[v_i]\|_{A_i, \Sigma_{h,i}}^2 + \|v_i\|_{E_{h,i}}^2 \quad (4.1) \]
and
\[ \|v_i\|_{h,i,\star}^2 = \|v_i\|_{h,i}^2 + \|v_i\|_{E_{h,i}}^2 + h \langle v_i, v_i \rangle_{E_{h,i}} \quad (4.2) \]
for \( v_i \in H^s (T_{h,i}) + W_{h,i}, s > 3/2, \) and
\[ \|v\|_{h}^2 = \sum_{i=0}^{s} \|v_i\|_{h,i}^2 + \sum_{i=1}^{2} \kappa_{0,i}^{-1} \|\kappa v_i\|_{E_{h,i}}^2 \quad (4.3) \]
\[ \|v\|_{h,\star}^2 = \sum_{i=0}^{s} \|v_i\|_{h,i}^2 + \sum_{i=1}^{2} \kappa_{0,i}^{-1} \|\kappa v_i\|_{E_{h,i}}^2 \quad (4.4) \]
Here \( \|w\|_{A,\omega}^2 = (Aw, w)_{\omega} \) is the \( A \) weighted \( L^2 \) norm of \( w \) over \( \omega \). We now turn to the discrete Poincaré estimate.
Lemma 4.1. There is a constant such that for all $v \in W_h$,
\[
\sum_{i=0}^{2} h^{-(d-d_i)} \| v_i \|_{K_{h,i}}^2 \lesssim \| v \|_{h}^2
\]  
(4.5)

Proof. The proof is similar to the proof of the continuous Poincaré inequality (2.24) but a bit more complicated since we work with discontinuous piecewise polynomials. First we use Lemma 3.2 with $m = 0$, and the fact $h \leq h_0 \lesssim 1$ to conclude that
\[
\| v_i \|_{M}^2 \lesssim \| v_i \|_{M \cap \Omega_i}^2 \| v_i \|_{s_{h,i}}^2 + \| v_i \|_{s_{h,i}}^2
\]  
(4.6)

for the bulk domains $\Omega_i, i = 1, 2$, and for the surface domain $\Omega_0$,
\[
 h^{-1} \| v_0 \|_{M}^2 \lesssim \| v_0 \|_{M \cap \Omega_0}^2 + \| v \|_{s_{h,0}}^2
\]  
(4.7)

We conclude that
\[
\sum_{i=0}^{2} h^{-(d-d_i)} \| v_i \|_{K_{h,i}}^2 \lesssim \sum_{i=0}^{2} \| v_i \|_{K_{h,i}}^2 + \| v_i \|_{s_{h,i}}^2 \lesssim \| v_0 \|_{K_{h,0}}^2 + \sum_{i=1}^{2} \| v_i \|_{K_{h,i}}^2 + \| v \|_{h}^2
\]  
(4.10)

We now continue with estimates of terms $I$ and $II$.

Term I. Adding and subtracting suitable terms and using the discrete trace inequality in Lemma 3.5 with $i = 2$, we get
\[
\| v_0 \|_{\Omega_0}^2 \lesssim \| v_0 \|_{K_{h,0}}^2 \lesssim \| v_0 \|_{K_{h,0}}^2 + \| v_2 \|_{H(\kappa_{h,2})}^2 + h^{-1} \| [v] \|_{K_{h,i}}^2
\]  
(4.11)

where we used the fact that $\kappa_{0,2}$ and $\kappa_2$ are positive constants.

Term II. Let $\phi$ be the solution to the problem
\[- \Delta \phi = \psi \quad \text{in} \ \Omega, \quad \phi = 0 \quad \text{on} \ \partial \Omega\]  
(4.15)

where $\psi \in L^2(\Omega)$. (Note that this is not an interface problem.) Multiplying $-\Delta \phi = \psi$ by $v = \sum_{i=1}^{2} \kappa_i v_i \chi_i$, where $\chi_i$ is the characteristic function of $\Omega_i$, and integrating by parts
we obtain
\[
\sum_{i=1}^{2} (\tilde{\kappa}_i v_i, \psi)_{\Omega_i} = \sum_{i=1}^{2} \langle \tilde{\kappa}_i v_i, \Delta \phi \rangle_{\Omega_i} = \sum_{i=1}^{2} (\tilde{\kappa}_i v_i, \nabla \phi)_{K_h,i} - (\tilde{\kappa}_i v_i, \nabla n_i \phi)_{\partial K_h,i} - (\tilde{\kappa}_i v_i - n_i \phi)_{\Omega_0} - (v_0, \nabla n_i \phi)_{\Omega_0}
\]
(4.16)
\[
= \sum_{i=1}^{2} (\tilde{\kappa}_i \nabla v_i, \nabla \phi)_{K_h,i} - (\tilde{\kappa}_i v_i, \nabla n_i \phi)_{\partial K_h,i} - (\tilde{\kappa}_i v_i - v_0, \nabla n_i \phi)_{\Omega_0}
\]
(4.17)
\[
\leq \left( \sum_{i=1}^{2} \tilde{\kappa}_i \| \nabla v_i \|^2_{\Omega_i} + \tilde{\kappa}_i h^{-1} \| [v] \|^2_{\Delta h,i} + \kappa_{0,i} h^{-1} \| [v] \|^2_{\Omega_0} \right)^{1/2}
\times \left( \sum_{i=1}^{2} \tilde{\kappa}_i \| \nabla \phi \|^2_{\Omega_i} + \tilde{\kappa}_i h \| \nabla n_i \phi \|^2_{\Delta h,i} + \kappa_{0,i} h \| \nabla n_i \phi \|^2_{\Omega_0} \right)^{1/2}
\]
(4.18)
\[
\leq \| v \|_h \| \psi \|_\Omega
\]
(4.19)
where we used the uniform coercivity of \( A_i \) to pass to the \( A_i \) weighted energy norm. To establish the bound \( \star \leq \| \psi \|_\Omega \) we use a trace inequality on \( \Omega_2 \) followed by elliptic regularity, which holds since \( \Omega \) is a convex polygonal domain, to conclude that
\[
\| \nabla n_i \phi \|_{\Omega_0} \leq \| \nabla \phi \|_{\Omega_0} \leq \| \phi \|_{H^2(\Omega_2)} \leq \| \phi \|_{H^2(\Omega)} \leq \| \psi \|_\Omega
\]
(4.20)
which holds for \( i = 1, 2 \), since the gradient of \( \phi \in H^1(\Omega) \). Then we may apply element wise trace inequalities followed by elliptic regularity to estimate the contributions from the faces
\[
h \| \nabla n_i \phi \|^2_{\Delta h,i} \lesssim \sum_{T \in \mathcal{T}_h,i} h \| \nabla \phi \|^2_T \lesssim \sum_{T \in \mathcal{T}_h,i} \| \nabla \phi \|^2_T + h^2 \| \nabla^2 \phi \|^2_T
\]
(4.21)
\[
\lesssim \| \phi \|^2_{H^2(\mathcal{T}_h,i)} \lesssim \| \phi \|^2_{H^2(\Omega)} \lesssim \| \psi \|^2_\Omega
\]
(4.22)
and finally using the energy stability \( \| \nabla \phi \|_{\Omega_i} \lesssim \| \psi \|_{\Omega} \) to conclude the estimate of \( \star \). We thus arrive at the estimate
\[
\sum_{i=1}^{2} (\tilde{\kappa}_i v_i, \psi)_{\Omega_i} \leq \| v \|_h \| \psi \|_\Omega
\]
(4.23)
Setting \( \psi_i = v_i \) and using the fact that \( \tilde{\kappa}_i > 0 \) we get
\[
II = \sum_{i=1}^{2} \| v_i \|^2_{\Omega_i} \leq \| v \|_h^2
\]
(4.24)
Together the bounds (4.14) and (4.27) of \( I \) and \( II \) complete the proof.

**Lemma 4.2.** There is a constant such that for all \( v \in W_h \),
\[
\| v \|_{h, \star} \lesssim \| v \|_h
\]
(4.25)
followed by the Poincaré estimate in Lemma 4.1, we directly obtain the desired estimate

\[
\|v_i\|^2_{\mathcal{K}_h} \lesssim h^{-(d-d_i)}\|v_i\|^2_{T_h,i} \quad (4.29)
\]

\[
h\|\langle v_i \rangle \|^2_{\mathcal{E}_{h,i}} \lesssim \sum_{T \in \mathcal{T}_{h,i}} h\|v_i\|^2_{\partial T \cap \Omega_i} \lesssim \sum_{T \in \mathcal{T}_{h,i}} h^{-(d-d_i)}\|v_i\|^2_T = h^{-(d-d_i)}\|v_i\|^2_{T_h,i} \quad (4.30)
\]

Proof. Recalling the definition of the norms \((4.1)\)–\((4.4)\), using the inverse estimates

\[
\|v_i\|^2_{\mathcal{K}_h} \lesssim h^{-(d-d_i)}\|v_i\|^2_{T_h,i} \quad (4.29)
\]

\[
h\|\langle v_i \rangle \|^2_{\mathcal{E}_{h,i}} \lesssim \sum_{T \in \mathcal{T}_{h,i}} h\|v_i\|^2_{\partial T \cap \Omega_i} \lesssim \sum_{T \in \mathcal{T}_{h,i}} h^{-(d-d_i)}\|v_i\|^2_T = h^{-(d-d_i)}\|v_i\|^2_{T_h,i} \quad (4.30)
\]

To show coercivity of the form \(A_h\) we will start by proving that the forms \(a_{h,i} + s_{h,i}\) are coercive. Here, as we mentioned above, we will take the variable coefficients \(A_i\) into account in order to identify the proper scalings of the Nitsche penalty and stabilization terms.

**Lemma 4.3.** There is a constant such that for all \(v \in W_{h,i}\) and all macro elements \(M \in \mathcal{M}_{h,i}\), \(i = 0, 1, 2,\)

\[
h\|\nabla_i v\|^2_{A_i,\mathcal{E}_{h,i}} \lesssim c_{A_i,M} \|\nabla_i v\|^2_{A_i,\mathcal{M} \cap \Omega_i} + \|v\|^2_{s_{h,i},M} \quad (4.33)
\]

where the hidden constant is independent of \(A_i\) and \(c_{A_i,M} = \|A_i\|_{L^\infty(M)} \inf_{x \in \mathcal{M}} \alpha_i(x)\) where \(\alpha_i(x)\) is the coercivity constant of \(A_i(x)\), see \((2.1)\).

**Proof.** For \(M \in \mathcal{M}_{h,i}\) let \(\mathcal{F}_{h,i}(\overline{M}) = \{F \in \mathcal{F}_{h,i} : F \subset \overline{M}\}\) be the faces in \(M\) including the faces on the boundary of \(M\) and let \(\mathcal{E}_{h,i}(\overline{M}) = \mathcal{F}_{h,i}(\overline{M}) \cap \Omega_i\) be the intersection of the faces with \(\Omega_i\). Then \(\mathcal{E}_{h,i} = \cup_{M \in \mathcal{M}_{h,i}} \mathcal{E}_{h,i}(\overline{M})\) and we have

\[
h\|\nabla_i v\|^2_{A_i,\mathcal{E}_{h,i}} \lesssim \sum_{M \in \mathcal{M}_{h,i}} h\|A_i\|_{L^\infty(M)} \|\nabla_i v\|^2_{\mathcal{E}_{h,i}(\overline{M})} \quad (4.34)
\]

\[
\lesssim \sum_{M \in \mathcal{M}_{h,i}} \|A_i\|_{L^\infty(M)} \left( h\|\nabla_i (v - v_M)\|^2_{\mathcal{E}_{h,i}(\overline{M})} + h\|\nabla_i v_M\|^2_{\mathcal{E}_{h,i}(\overline{M})} \right) \quad (4.35)
\]

where we added and subtracted \(v_M \in P_1(M)\), such that \(v_M|_{T_M} = v|_{T_M}\), where \(T_M\) is the element with a large intersection with \(\Omega_i\).

**Term I.** We use an inverse inequality to pass from the \(d_i - 1\) dimensional intersections \(E = F \cap \Omega_i\) to the \(d\) dimensional element \(T\), manufacturing a scaling with \(h^{-(d-(d_i-1))}\), and then we apply Lemma \((3.1)\)

\[
h\|\nabla_i (v - v_M)\|^2_{\mathcal{E}_{h,i}(\overline{M})} \lesssim hh^{-(d-(d_i-1))}\|\nabla (v - v_M)\|^2_{M} \quad (4.36)
\]

\[
\lesssim h^{-(d-d_i)} \left( \sum_{k=0}^{1} h^{2k-1}\|\nabla^k v\|^2_{\mathcal{E}_{h,i}(\overline{M})} \right) \quad (4.37)
\]

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Term II. For the bulk domains \((i = 1, 2)\) we use an inverse estimate to pass from the \(d - 1\) dimensional intersection \(E = F \cap \Omega_t\) to the \(d\) dimensional element \(T\), then again using an inverse estimate we pass to the element with a large intersection \(T_M\), then we add and subtract \(v\), use the triangle inequality, and finally Lemma 3.1.

\[
h\|\nabla_i v_M\|_{\mathcal{E}_{h,i}(M)}^2 \lesssim \|\nabla_i v_M\|_{M}^2 \lesssim \|\nabla_i v_M\|_{M}^2
\]

\[
\lesssim \|\nabla_i (v_M - v)\|_{T_M}^2 + \|\nabla_i v\|_{T_M}^2 \lesssim \sum_{k=0}^{1} h^{2k-1} \|\nabla_i v\|_{T_M}^2 + \|\nabla_i v\|_{T_M}^2 \lesssim \|\nabla_i v\|_{T_M}^2
\]

We then have

\[
h\|\nabla_i v\|_{A_i,\mathcal{E}_{h,i}}^2 \lesssim \sum_{M \in \mathcal{M}_{h,i}} \|A_i\|_{L^\infty(M)} \|\nabla_i v\|_{M \cap \Omega_i}^2 + \|A_i\|_{L^\infty(M)} \left( \sum_{k=0}^{1} h^{2k-1} \|\nabla_i v\|_{T_M}^2 \right)
\]

\[
\lesssim \sum_{M \in \mathcal{M}_{h,i}} \|A_i\|_{L^\infty(M)} \|\nabla_i v\|_{A_i, M \cap \Omega_i}^2 + \|v\|_{h,i,M}^2
\]

For the surface domain \((i = 0)\) we need a more refined argument to find the required scaling on the normal gradient stabilization. To that end let \(P_i = I - n_i \otimes n_i\) be the tangent projection associated with the exact domain \(\Omega_i\) and let \(P_M = I - n_M \otimes n_M\) be a constant projection such that

\[
\|P_i - P_M\|_{L^\infty(M)} \lesssim h
\]

Since the normal field is smooth and \(\text{diam}(M) \sim h\) we can construct \(P_M\) using a constant approximation \(n_M\) of \(n\) on the macro element \(M\). Adding and subtracting \(P_M\) and using the triangle inequality we get

\[
h\|\nabla_i v_M\|_{\mathcal{E}_{h,i}(M)}^2 = h\|P_i \nabla v_M\|_{\mathcal{E}_{h,i}(M)}^2
\]

\[
\lesssim h\|P_i - P_M\| \nabla v_M\|_{\mathcal{E}_{h,i}(M)}^2 + h\|P_M \nabla v_M\|_{\mathcal{E}_{h,i}(M)}^2
\]

\[
 = II_1 + II_2
\]

For \(II_1\) we use (4.33) and (3.28) to conclude that

\[
II_1 = h\|P_i - P_M\| \nabla v_M\|_{\mathcal{E}_{h,i}(M)}^2
\]

\[
\lesssim h^3 \|\nabla v_M\|_{\mathcal{E}_{h,i}(M)}^2
\]

\[
\lesssim h\|\nabla v_M\|_{M}^2
\]

\[
\lesssim h\|\nabla v_M\|_{T_M}^2
\]

\[
\lesssim h^2 \|\nabla v_M\|_{T_M}^2
\]

\[
\lesssim h^2 \|\nabla v_M\|_{T_M}^2
\]

\[
\lesssim h^2 \|\nabla (v_M - v)\|_{M \cap \Omega_i}^2 + h^2 \|\nabla v\|_{M \cap \Omega_i}^2
\]

\[
\lesssim h\|\nabla (v_M - v)\|_{M \cap \Omega_i}^2 + h^2 \|\nabla v\|_{M \cap \Omega_i}^2 + h^2 \|\nabla v\|_{M \cap \Omega_i}^2
\]

\[
\lesssim h \sum_{k=0}^{1} h^{2k-1} \|\nabla_i v\|_{T_M}^2 + h^2 \|\nabla_i v\|_{M \cap \Omega_i}^2 + h^2 \|\nabla_i v\|_{M \cap \Omega_i}^2
\]
Next for $II_2$ we pass from edges to the macro element using an inverse inequality, then using the fact that $P_M\nabla v_M$ is constant on $M$ we pass from $M$ to $T_M \cap \Omega_i$, then we add and subtract $P_i$ and employ (4.43),

$$II_2 = h\|P_M\nabla v_M\|^2_{E_{h,i}(M)}$$  \hspace{1cm} (4.55)

$$\lesssim h^{-1}\|P_M\nabla v_M\|^2_M$$  \hspace{1cm} (4.56)

$$\lesssim \|P_M\nabla v_M\|^2_{T_M \cap \Omega_i}$$  \hspace{1cm} (4.57)

$$\lesssim \|P_i\nabla v\|^2_{T_M \cap \Omega_i} + \|(P_M - P_i)\nabla v\|^2_{T_M \cap \Omega_i}$$  \hspace{1cm} (4.58)

$$\lesssim \|\nabla v\|^2_{T_M \cap \Omega_i} + h^2\|\nabla v\|^2_{T_M \cap \Omega_i}$$  \hspace{1cm} (4.59)

$$\lesssim (1 + h^2)\|\nabla v\|^2_{T_M \cap \Omega_i} + h^2\|\nabla n v\|^2_{T_M \cap \Omega_i}$$  \hspace{1cm} (4.60)

$$\lesssim \|\nabla v\|^2_{M \cap \Omega_i} + h^2\|\nabla n v\|^2_{M \cap \Omega_i}$$  \hspace{1cm} (4.61)

Together the bounds (4.37), (4.54), and (4.61) of $I$, $II_1$ and $II_2$ give

$$I + II \lesssim \|\nabla v\|^2_{M \cap \Omega_i} + \sum_{k=0}^1 h^{2k-2}\|\nabla^k v\|^2_{E_{h,i}(M)} + h^2\|\nabla n v\|^2_{M \cap \Omega_i}$$  \hspace{1cm} (4.62)

which inserted into (4.35) give

$$h\|\nabla v\|^2_{A_{h,i,M}} \lesssim \sum_{M \in \mathcal{M}_{h,i}} \|A_i\|_{L^\infty(M)}\|\nabla v\|^2_{M \cap \Omega_i}$$

$$\quad + \|A_i\|_{L^\infty(M)}\left(\frac{1}{\alpha_{i,M}}\sum_{k=0}^1 h^{2k-2}\|\nabla^k v\|^2_{E_{h,i}(M)} + h^2\|\nabla n v\|^2_{M \cap \Omega_i}\right)$$

$$\quad \lesssim \sum_{M \in \mathcal{M}_{h,i}} \|A_i\|_{L^\infty(M)}\alpha_{i,M}^{-1}\|\nabla v\|^2_{A_i,M \cap \Omega_i} + \|v\|^2_{s_{h,i,M}}$$  \hspace{1cm} (4.63)

and thus the proof is complete.  \hspace{1cm} $\blacksquare$

**Lemma 4.4.** If the stabilization parameters $\tau_{a_{h,i}}$ see (3.17), are sufficiently large, then the forms $a_{h,i} + s_{h,i}$ satisfy the coercivity

$$\|\|v\|^2_{h,i} \lesssim a_{h,i}(v,v) + s_{h,i}(v,v) \quad v \in W_{h,i}, \quad i = 0, 1, 2$$  \hspace{1cm} (4.64)

**Proof.** Using the definition of $a_{h,i}$ we get

$$a_{h,i}(v,v) + s_{h,i}(v,v) = \|\nabla v\|^2_{A_i,\mathcal{K}_{h,i}} + \|v\|^2_{s_{h,i}}$$

$$- 2(\langle \nu_i \cdot A_i \nabla v, [v] \rangle_{\mathcal{E}_{h,i}} + \tau_{a_i} h^{-1}\|\nu_i[v]\|^2_{A_i,\mathcal{K}_{h,i}})$$  \hspace{1cm} (4.65)

Here we can estimate the third term on the right hand side using Lemma 4.3 together with the assumption $\sup_{M \in \mathcal{M}_{h,i}} c_{A_i,M} \lesssim 1$ to conclude that

$$2(\langle \nu_i \cdot A_i \nabla v, [v] \rangle_{\mathcal{E}_{h,i}} \lesssim 2h^{1/2}\|\nabla v\|^2_{A_i,\mathcal{K}_{h,i}} h^{-1/2}\|\nu_i[v]\|^2_{A_i,\mathcal{K}_{h,i}}$$

$$\lesssim h\|\nabla v\|^2_{A_i,\mathcal{K}_{h,i}} + h^{-1}\|\nu_i[v]\|^2_{A_i,\mathcal{K}_{h,i}}$$

$$\lesssim \frac{\delta C}{\delta + \frac{1}{\delta} h^{-1}} \left(\|\nabla v\|^2_{A_i,\mathcal{K}_{h,i}} + \|v\|^2_{s_{h,i}}\right) + h^{-1}\|\nu_i[v]\|^2_{A_i,\mathcal{K}_{h,i}}$$  \hspace{1cm} (4.66)
with $\delta > 0$. We then have

$$a_{h,i}(v,v) + s_{h,i}(v,v) \geq (1 - C\delta)(\|\nabla_i v\|_{A_i,\mathcal{E}_{h,i}}^2 + \|v\|_{\mathcal{E}_{h,i}}^2) + h^{-1}(\tau_{a_i} - \delta^{-1})\|\nu_i[v]\|_{A_i,\mathcal{E}_{h,i}}^2$$

(4.72)

where we may take $\delta$ small enough and $\tau_{a_i}$ large enough to obtain

$$a_{h,i}(v,v) + s_{h,i}(v,v) \geq (1 - C\delta)(\|\nabla_i v\|_{A_i,\mathcal{E}_{h,i}}^2 + \|v\|_{\mathcal{E}_{h,i}}^2) + h^{-1}(\tau_{a_i} - \delta^{-1})\|\nu_i[v]\|_{A_i,\mathcal{E}_{h,i}}^2$$

(4.74)

Here we at last used Lemma 4.3 to control $h\|\nabla_i v\|_{A_i,\mathcal{E}_{h,i}}^2$.

Lemma 4.5. The form $A_h$ is continuous

$$A_h(v,w) \lesssim \|v\|_{h,\star}\|w\|_{h,\star} v, w \in \oplus_{i=0}^2 (H^1(\mathcal{T}_{h,i}) + W_{h,i})$$

(4.75)

and if the stabilization parameters $\tau_{a_i}, i = 0, 1, 2$, defined in (3.17), are sufficiently large, then $A_h$ is coercive

$$\|v\|_{h,\star}^2 \lesssim A_h(v,v) \quad v \in W_h$$

(4.76)

Proof. The continuity (4.75) follows directly from the Cauchy-Schwarz inequality and the definition (4.4) of the norm. To show the coercivity (4.76) we first note that by construction $b_{h,i}(v,v) = \lambda_h \|\nu_i[v]\|_{\mathcal{E}_{h,i}}^2$, and therefore

$$A_h(v,v) = \sum_{i=0}^2 \rho_i \left( a_{h,i}(v,v) + s_{h,i}(v,v) \right)$$

(4.77)

$$+ \kappa_i \lambda_h \|\nu_i[v]\|_{\mathcal{E}_{h,i}}^2 + \sum_{i=1}^2 \kappa_{0,i}^{-1} \|\nu_i[v]\|_{\mathcal{E}_{h,i}}^2 \gtrsim \|v\|_{h,\star}^2$$

(4.78)

where we at last used Lemma 4.4. Finally, using Lemma 4.2 we obtain the desired estimate

$$\|v\|_{h,\star}^2 \lesssim \|v\|_{h}^2 \lesssim A_h(v,v)$$

(4.79)

4.2 Interpolation Error Estimates

To define the interpolant we need extensions of functions in $\Omega_i$ to $\mathcal{T}_{h,i}$. For the bulk domains, $\Omega_1$ and $\Omega_2$, the Stein extension theorem, see [24], provides operators $E_i : H^s(\Omega_i) \to H^s(\mathbb{R}^d)$, $i = 1, 2$, $s \geq 0$

such that

$$\|E_i v\|_{H^s(\mathbb{R}^d)} \lesssim \|v_i\|_{H^s(\Omega_i)}, \quad i = 1, 2, \quad s \geq 0$$

(4.81)

For the interface, $\Omega_0$, we construct an extension operator by composition with the closest point mapping $E_0 : H^s(\Omega_0) \ni v \mapsto v \circ p \in H^s(U_{\delta_0}(\Omega_0))$

$$E_0 : H^s(\Omega_0) \ni v \mapsto v \circ p \in H^s(U_{\delta_0}(\Omega_0))$$

(4.82)
where we recall that \( U_{\delta_0}(\Omega_0) \subset \Omega \) is an open tubular neighborhood of \( \Omega_0 \) of thickness \( \delta_0 > 0 \) and \( p : U_{\delta_0}(\Omega_0) \to \Omega_0 \), is the closest point mapping. We then have the stability estimate
\[
\| E_0 v \|_{H^s(U_{\delta_0}(\Omega_0))} \lesssim \delta_0^{1/2} \| v \|_{H^s(\Omega_0)}, \quad s \geq 0
\] (4.83)

Letting \( U_\delta(\Omega_i) = \cup_{x \in \Omega_i} B_\delta(x) \), where \( B_\delta(x) \) is the open ball of diameter \( \delta \) with center \( x \) we may define the extension operator
\[
E : \oplus_{i=0}^2 H^s(\Omega_i) \ni (v_0, v_1, v_2) \mapsto (E_0v_0, E_1v_1, E_2v_2) \in \oplus_{i=0}^2 H^s(U_\delta(\Omega_i))
\] (4.84)

Let \( \pi_{h,i} : L^2(\mathcal{T}_{h,i}) \to W_{h,i} \) be the Clément interpolation operator. For all \( v_i \in H^2(\mathcal{T}_{h,i}) \) and \( T \in \mathcal{T}_{h,i} \) recall the following elementwise trace inequality that holds for elements \( T \in \mathcal{T}_{h,i} \cap \mathcal{T}_{h,i}' \) and for \( i \in \mathcal{T}_{h,i} \)
\[
\| v_i - \pi_{h,i}v_i \|_{H^m(T)} \lesssim h^{s-m} \| v_i \|_{H^s(\mathcal{N}_{h,i}(T))} \quad m \leq s \leq 2, \quad m = 0, 1, 2
\] (4.85)

where \( \mathcal{N}_{h,i}(T) \subset \mathcal{T}_{h,i} \) is the union of the elements in \( \mathcal{T}_{h,i} \) which share a node with \( T \). In particular, we have the stability estimate
\[
\| \pi_{h,i}v_i \|_{H^m(\mathcal{N}_{h,i})} \lesssim \| v_i \|_{H^m(\mathcal{N}_{h,i})}
\] (4.86)

For \( h \in (0, h_0] \) with \( h_0 \) small enough we have \( \mathcal{T}_{h,i} \subset U_{\delta_0}(\Omega_i) \) and we may define interpolation operators by composing the Clément interpolation operator and the continuous extension operators. More precisely
\[
\pi_h : \oplus_{i=0}^2 L^2(\mathcal{T}_{h,i}) \ni (v_0, v_1, v_2) \mapsto (\pi_h E_0v_0, \pi_h E_1v_1, \pi_h E_2v_2) \in W_h
\] (4.87)

Next we show an interpolation estimate in the dG norm.

**Lemma 4.6.** There is a constant such that for all \( v \in \oplus_{i=0}^2 H^2(\Omega_i) \),
\[
\| ||v^e - \pi_h v^e||_{h^\ast,\star}^2 \| \lesssim h^2 \left( \sum_{i=0}^2 \| v_i \|_{H^2(\Omega_i)}^2 \right)
\] (4.88)

**Proof.** Using the notation \( \rho_i = v_i - \pi_{h,i}v_i \) we have
\[
\| \rho_i \|_{h^\ast,\star}^2 = \sum_{i=0}^2 \| \rho_i \|_{h,i}^2 + \sum_{i=1}^2 \| \kappa_{i,i}^{-1} [\kappa \rho_i]_i \|_{h,i}^2
\] (4.89)

with
\[
\| \rho_i \|_{h,i}^2 \lesssim \| \rho_i \|^2_{K_{h,i}} + \| \nabla_i \rho_i \|^2_{K_{h,i}} + h \| \rho_i \|^2_{\mathcal{E}_{h,i}} + h^2 \| \nabla_i \rho_i \|^2_{\mathcal{E}_{h,i}} + h^{-1} \| \rho_i \|^2_{\mathcal{F}_{h,i}} + \| \rho_i \|^2_{n,i}
\] (4.90)

see (4.1)–(4.4). To proceed with the estimates we first recall the following elementwise trace inequality that holds for elements \( T \in \mathcal{T}_{h,i} \) in the bulk domain meshes \( i = 1, 2, \)\n\[
\| w \|_{K}^2 \lesssim h^{-1} \| w \|_{T}^2 + h \| \nabla w \|_{T}^2 \quad v \in H^1(T)
\] (4.92)

where for \( T \subset \Omega_i \) we have \( K = T \) and therefore \( \partial K = \partial T \) and for \( T \) that intersect the interface \( \Omega_0 \) we have \( \partial K = (\partial T \cap \Omega_i) \cup (T \cap \Omega_0) \), see [16] and [18].

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Using (4.92) followed by the interpolation error estimate (4.85) and the stability (4.81) of the extension operator we obtain
\[ \| \rho_i \|_{H_0^1(T)}^2 \lesssim h^{-2} \| \rho_i \|_{T_{h,i}}^2 + \| \nabla \rho_i \|_{T_{h,i}}^2 + h^2 \| \nabla^2 \rho_i \|_{T_{h,i}}^2 \] (4.93)
for \( i = 1, 2 \). Next the interface term is estimated in a similar way
\[ \kappa_0^{-1} \| \kappa \rho_i \|_{T_{h,i}}^2 \lesssim \| \rho_i \|_{T_{h,i}}^2 + \| \rho_i \|_{\Gamma_{h,i}}^2 + h^{-1} \| \rho_i \|_{T_{h,i}(\Omega_0)} + h \| \nabla \rho_i \|_{T_{h,i}(\Omega_0)} \] (4.95)
\[ \lesssim \| \rho_i \|_{\Omega_0}^2 + h^3 \| E_i v_i \|_{H^2(\Gamma_{h,i})} + h^3 \| \nabla E_i v_i \|_{H^2(\Gamma_{h,i})} \lesssim \| \rho_i \|_{\Omega_0}^2 + h \| v_i \|_{H^2(\Omega)} \] (4.96)
where we used the triangle inequality and the trace inequality (4.92).

Finally, for the surface domain \( \Omega_0 \) we proceed in the same way but we instead employ the trace inequality
\[ \| w \|_{\delta K} \lesssim \sum_{l=0}^{2} h^{2l-4} \| \nabla^l w \|_{T}^2 \quad w \in H^2(T) \] (4.97)
see [9], and then the interpolation error estimate (4.85) and the stability of the extension operator (4.83), where we first use the inclusion \( \bar{N}_h(\Gamma_{h,0}) \subset U_\delta(\Omega_0) \) with \( \delta \sim h \),
\[ \| \rho_i \|_{H_0^1(\Omega_0)}^2 \lesssim \sum_{l=0}^{2} h^{2l-3} \| \nabla^l \rho_i \|_{\Gamma_{h,0}}^2 \lesssim \sum_{l=0}^{2} h^{2l-3} h^{2(2-l)} \| \nabla^l v_0 \|_{H^2(\Omega_0)}^2 \] (4.98)
\[ \lesssim h \| \nabla v_0 \|_{H^2(\Omega_0)}^2 \leq h \delta \| \pi v_0 \|_{H^2(\Omega_0)}^2 \lesssim h^2 \| v_0 \|_{H^2(\Omega_0)} \] (4.99)
which completes the proof. \qed

4.3 A Priori Error Estimates

Theorem 4.1. Let \( u \in W \cap \bigoplus_{i=0}^{2} H^2(\Omega_i) \) solve the convection diffusion problem (2.3) - (2.6) and \( u_h \in W_h \) be the finite element solution defined by (3.9). Then, we have
\[ \| u_h - u^c \|_{h,*} \lesssim h \left( \sum_{i=0}^{2} \| u \|_{H^2(\Omega_i)} \right)^{1/2}, \quad \| u_h - u^c \| \lesssim h \left( \sum_{i=0}^{2} \| u \|_{H^2(\Omega_i)} \right)^{1/2} \] (4.100)

Proof. Adding and subtracting the interpolant \( \pi_h u \) and using the triangle inequality we get
\[ \| u - u_h \|_{h,*} \leq \| u - \pi_h u \|_{h,*} + \| \pi_h u - u_h \|_{h,*} \] (4.101)
To estimate the second term we employ coercivity (4.76), the linearity of \( A_h \), the definition (3.9) of the dG method, the consistency (3.86), and finally the continuity (4.75), as follows
\[ \| \pi_h u - u_h \|_{h,*}^2 \lesssim A_h(\pi_h u - u_h, \pi_h u - u_h) \] (4.102)
\[ = A_h(\pi_h u - u, \pi_h u - u_h) + A_h(u - u_h, \pi_h u - u_h) \] (4.103)
\[ = A_h(\pi_h u - u, \pi_h u - u_h) + (A_h(u, \pi_h u - u_h) - L_h(\pi_h u - u_h)) \] (4.104)
\[ = \| \pi_h u - u \|_{h,*}^2 + \| \pi_h u - u_h \|_{h,*}^2 \] (4.105)
and thus
\[ \| \pi_h u - u_h \|_{h,*} \lesssim \| \pi_h u - u \|_{h,*} \] (4.106)
We complete the proof by the interpolation result in Lemma 4.6. The proof of the \( L^2 \) estimate follows in the standard way using a duality argument. \qed
5 Numerical Example

We consider a similar example as in [14]. The computational domain $\Omega$ is $[-1.5, 1.5] \times [-1.5, 1.5]$, the interface $\Omega_0$ is the unit circle, $\beta = (y, -x)$, $A_0 = 1$, $A_1 = 1$, $A_2 = 0.5$, $\kappa_1 = 2$, $\kappa_2 = 0.5$, $\kappa_{0,1} = 1$, $\kappa_{0,2} = 2$ and the source terms $f_i$, $i = 0, 1, 2$ and the boundary data are taken such that the exact solution is

$$u_0(x, y) = 3x^2y - y^3$$  \hspace{1cm} (5.1)

$$u_1(x, y) = e^{1-x^2-y^2}u_0(x, y)$$  \hspace{1cm} (5.2)

$$u_2(x, y) = 2u_1(x, y)$$  \hspace{1cm} (5.3)

Note that $d_1 = d_2 = 2$ and $d_0 = 1$.

We approximate the interface $\Omega_0$ using a cubic spline parametrization, see [27]. The proposed discontinuous CutFEM in Section 3.2 is used with stabilization parameters $\tau_{i,0} = A_i$, $\tau_{i,1} = 0.1A_i$ for $i = 1, 2$, (in equation (3.21)) and $\tau_{0,0} = \tau_{0,1} = A_0$, $\tau_{0,2} = 0.1A_0$ (in equation (3.22)). The resulting linear systems are solved by a direct solver.

The numerical solution on a uniform background mesh with mesh size $h = 0.15$ is shown in Figure 3. We used $\gamma_0 = 0.25$ and $\gamma_1 = \gamma_2 = 0.125$ (see equation (3.18)) in Algorithm 1. This resulted in a macro element partition with 20 edges in $F_{h,0}$, 32 edges in $F_{h,1}$, and 24 edges in $F_{h,2}$. In case of full stabilization we would for this mesh instead apply stabilization on 90 edges when $i = 0$, 138 edges when $i = 1$, and 132 edges when $i = 2$. We illustrate the difference between full stabilization and the macro element stabilization on the active mesh $T_{h,1}$ for a coarser mesh, $h = 0.3$, in Figure 4. We note that in the middle panel when $\gamma_1 = 0.5$ each cut element is marked as a small element in Algorithm 1 and always connected to an element that is entirely inside $\Omega_1$. However, also in this case when we apply stabilization following Algorithm 1 there are fewer edges (46 edges) on which stabilization is applied compared to using full stabilization (76 edges).

For $\gamma_0 = 0.25$ and $\gamma_1 = \gamma_2 = 0.125$ we show $L^2$- and $H^1$-errors in the bulk and the interface concentration for different mesh sizes in Figure 5. We obtain as expected first order convergence in the $H^1$-norm and second order convergence in the $L^2$-norm. We also show the spectral condition number of the scaled stiffness matrix associated with the form 

$$\tilde{A}_h(v, w) = A_h((h^{1/2}v_0, v_1, v_2), (h^{1/2}w_0, w_1, w_2)).$$  \hspace{1cm} (5.4)
Figure 4: Macro elements in the active mesh $T_{h,1}$ are shown in purple and edges on which stabilization is applied are shown with dotted lines for $\gamma_1 = 0.125$ (left panel), $\gamma_1 = 0.5$ (middle panel), and for full stabilization (right panel). The mesh size is $h = 0.3$. Stabilization is applied in total on 12 edges ($\gamma_1 = 0.125$), 46 edges ($\gamma_1 = 0.5$), and 72 edges (full stabilization).

and we observe the expected $O(h^{-2})$ behaviour. We refer to [9] for an estimate of the condition number for a continuous CutFEM approximation of a coupled bulk-surface problem, with the same $h$-scaling as in $\tilde{A}_h$, that may be extended to the current discontinuous Galerkin method. In fact, combining the discrete Poincaré inequality in Lemma 4.1 and Lemma 3.2 we obtain a bound corresponding to Lemma 5.1 in [9] and then the estimate of the condition number follows using the coercivity and continuity of the form $\tilde{A}_h$ as in Theorem 5.1 in [9].

Note that for different mesh sizes the interface is positioned differently relative the background mesh but neither the errors or the condition numbers shown in Figure 5 are affected by how this relative position changes even though much less stabilization is applied than full stabilization. However, in Figure 6 where we compare results with different constants $\gamma_i$ we see that when $\gamma_i$ is too small both the error in the approximation of the interface concentration and the condition number of the resulting linear system are very sensitive to the position of the interface relative the background mesh. Errors both in $L^2$-norm and $H^1$-norm, and the condition number can become very large when enough stabilization is not applied. This is expected since coercivity of the form $A_h$ as well as the discrete Poincaré inequality do not hold in that case. The results in Figure 6 depend on how large the penalty parameters are chosen but here we have chosen the same penalty parameter independent of the parameters $\gamma_i$ in the stabilization terms $s_{h,i}$. Full stabilization give errors that are independent of the interface position relative the background mesh but we see that it also increases the magnitude of the errors. The macro element stabilization with $\gamma_0 = 0.25$ and $\gamma_1 = \gamma_2 = 0.125$ results in linear systems with the same condition number as using full stabilization but solutions with smaller errors. We observe similar behaviour for $H^1$-errors and therefore we only show the errors in $L^2$-norm in Figure 6.

6 Conclusions

We have developed a stabilization method for discontinuous CutFEM approximations of coupled bulk-interface problems, which is localized to macro elements. This macro element stabilization approach leads to convenient proofs of basic stability results and
Figure 5: The error and condition number versus mesh size $h$. Macro element stabilization with $\gamma_0 = 0.25$ and $\gamma_1 = \gamma_2 = 0.125$. Circles represent the error in the bulk and squares represent the error on the interface. Top left panel: The error measured in the $L^2$-norm versus mesh size $h$. The dashed line is $0.8h^2$. Top right panel: The error measured in the $H^1$-norm versus mesh size $h$. The dashed line is $4h$. Bottom: The spectral condition number of the scaled matrix associated with the form (5.4) versus mesh size $h$. The dashed line is $10^4h^{-2}$. 


also preserves the local conservation properties of the discontinuous Galerkin formulation on macro elements. Furthermore, the macro stabilization may be applied to continuous Galerkin methods as well as mass matrices and produces a diagonal block matrix with less coupling compared to standard full stabilization. We consider variable coefficients and diffusion as well as convection. The method enjoys optimal convergence and conditioning properties. Further developments include extension to higher order elements, time dependent domains, and improved implementations including efficient algorithms for computation of nearly optimal macro element partitions on surface domains.

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