On the monodromies and the limit mixed Hodge structures of families of algebraic varieties *

Takahiro SAITO † and Kiyoshi TAKEUCHI ‡

Abstract

We study the monodromies and the limit mixed Hodge structures of families of complete intersection varieties over a punctured disk in the complex plane. For this purpose, we express their motivic nearby fibers in terms of the geometric data of some Newton polyhedra. In particular, the limit mixed Hodge numbers and some part of the Jordan normal forms of the monodromies of such a family will be described very explicitly.

1 Introduction

Families of algebraic varieties are basic objects in algebraic geometry. Here we are interested in the special but fundamental case where such one $Y$ is smooth and defined over the punctured disk $B(0; \varepsilon)^*$ let us consider its fibers $Y_t = \pi^{-1}(t) \subset Y$ $(0 < |t| < \varepsilon)$ and their cohomology groups $H^j(Y_t; \mathbb{C})$ $(j \in \mathbb{Z})$. Then it is our primary interest to know $H^j(Y_t; \mathbb{C})$ themselves and the monodromy operators acting on them. Moreover, in addition to the classical mixed Hodge structure of Deligne, each cohomology group $H^j(Y_t; \mathbb{C})$ is endowed with the limit mixed Hodge structure which encodes some information of the monodromy (see El Zein [10] and Steenbrink-Zucker [37] etc.). However in general, it is very hard to compute the monodromies and the limit mixed Hodge numbers explicitly. Very recently, based on our previous works [11], [22], [23] and some new results in [17], [18], Stapledon [35] succeeded in computing the latter ones for families $Y \subset B(0; \varepsilon)^* \times \mathbb{C}^n$ of schön hypersurfaces in $\mathbb{C}^n$. Here the schönness is a very weak condition which is almost always satisfied. More precisely, in [35] the author expressed the motivic nearby fiber $\psi_t([Y])$ of such $Y \rightarrow B(0; \varepsilon) = \{ t \in \mathbb{C} \mid |t| < \varepsilon \} = B(0; \varepsilon)^* \sqcup \{0\}$ by the function $t = \text{id}_\mathbb{C} : \mathbb{C} \rightarrow \mathbb{C}$ in terms of the tropical variety associated to the defining Laurent polynomial $f(t, x) \in \mathbb{C}(t)[x_1, \ldots, x_n]$ of the hypersurface $Y \subset B(0; \varepsilon)^* \times \mathbb{C}^n$ and obtained a complete description of the limit mixed Hodge numbers of $H^j(Y_t; \mathbb{C})$ $(0 < |t| < \varepsilon)$. His idea is to subdivide $\mathbb{C}^n$ into some algebraic tori $(\mathbb{C}^*)^k$ $(0 \leq k \leq n)$ by the additivity of $\psi_t$ and apply the arguments of Batyrev-Borisov [4] and Borisov-Mavlyutov [2] etc. in

---

*2010 Mathematics Subject Classification: 14E18, 14M25, 32C38, 32S35, 32S40

†Institute of Mathematics, University of Tsukuba, 1-1-1, Tennodai, Tsukuba, Ibaraki, 305-8571, Japan. E-mail: takahiro@math.tsukuba.ac.jp

‡Institute of Mathematics, University of Tsukuba, 1-1-1, Tennodai, Tsukuba, Ibaraki, 305-8571, Japan. E-mail: takemicro@nifty.com
mirror symmetry to each piece \((\mathbb{C}^*)^k\) by using the new special polynomials introduced in Katz-Stapledon [17]. In particular, he effectively used the purity and the generalized Poincaré duality for the intersection cohomology groups of some singular hypersurfaces in the toric compactifications of \((\mathbb{C}^*)^k\) to obtain these remarkable results. Thanks to them, the first author T. Saito obtained some new results on the weight filtrations of the stalks of intersection cohomology complexes. See [32] for the details. However the motivic nearby fiber \(\psi_t([Y])\) used in the paper [35] is the one introduced by Steenbrink [36] very recently with the help of the semi-stable reduction theorem and it is not clear for us if it coincides with the classical (and more standard) one of Denef-Loeser [6], [7] and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to simplify Stapledon’s arguments and extend his results to families of schön complete inter-

fibers and describe them without using the tropical geometry. The aim of this paper is to

in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this sub-
ject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

for us if it coincides with the classical (and more standard) one of Denef-Loeser [6], [7] and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to

and Guibert-Loeser-Merle [14] etc. (see also Raibaut [28] for a nice introduction to this subject). Moreover the arguments in [35] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this paper is to
Theorem 1.1. Assume that the family $Y$ of hypersurfaces in $(\mathbb{C}^*)^n$ is schön. Then we have an equality
\[
\psi_t([Y]) = \sum_{\text{rel.int } F \subset \text{Int } P} [V_F \cap \hat{\mu}] \cdot (1 - \mathbb{L})^{n-\dim F}
\]
in $\mathcal{M}_c'$. We prove this theorem by using only the classical toric geometry and a result of Guibert-Loeser-Merle [14] (see Theorem 2.1). Then in Section 4 we apply it to families $Y \subset B(0;\varepsilon)^n \times \mathbb{C}^n$ of schön hypersurfaces in $\mathbb{C}^n$ and describe some parts of the Jordan normal forms of the monodromies on $H^j(Y_t; \mathbb{C})$ ($0 < |t| < \varepsilon$) explicitly. More precisely, as in [35] we define a finite subset $R_f$ of $\mathbb{C}$ by $\text{UH}_f$ and describe the Jordan normal forms for the eigenvalues $\lambda \notin R_f$. For this purpose, first we prove the following concentration theorem. For $j \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ let
\[
H_j^j(Y_t; \mathbb{C})_\lambda \subset H_j^j(Y_t; \mathbb{C}) \quad \text{(resp. } H_c^j(Y_t; \mathbb{C})_\lambda \subset H_c^j(Y_t; \mathbb{C}))
\]
be the generalized eigenspace of the monodromy automorphism $\Phi_j : H_j^j(Y_t; \mathbb{C}) \sim H_j^j(Y_t; \mathbb{C})$ (resp. $\Phi_j : H_c^j(Y_t; \mathbb{C}) \sim H_c^j(Y_t; \mathbb{C})$) for $0 < |t| < \varepsilon$.

Theorem 1.2. (see Theorem 4.6, Corollary 4.7 and Remark 4.8) Assume that the family $Y$ of hypersurfaces in $\mathbb{C}^n$ is schön. Then for any $\lambda \notin R_f$ and $t \in \mathbb{C}^*$ such that $0 < |t| < 1$ we have isomorphisms
\[
H_j^j(Y_t; \mathbb{C})_\lambda \simeq H_j^j(Y_t; \mathbb{C})_\lambda \quad (j \in \mathbb{Z})
\]
and the concentration
\[
H_j^j(Y_t; \mathbb{C})_\lambda \simeq 0 \quad (j \neq n - 1).
\]

By the proofs of this theorem (see Theorem 4.6) and Sabbah’s one [31, Theorem 13.1] we see also that for $\lambda \notin R_f$ the filtration on the only non-trivial cohomology group $H^{n-1}(Y_t; \mathbb{C})_\lambda$ induced by Deligne’s weight filtration on $H^{n-1}(Y_t; \mathbb{C})$ is concentrated in degree $n - 1$. Since $R_f$ is just a small part of the set of the eigenvalues of the monodromies of $Y$, Theorem 1.2 asserts that the geometric complexity of the family $Y$ is concentrated in the middle dimension $n - 1 = \dim Y_t$. This enables us to describe the Jordan normal forms of the middle-dimensional monodromies
\[
\Phi_{n-1} : H^{n-1}(Y_t; \mathbb{C})_\lambda \sim H^{n-1}(Y_t; \mathbb{C})_\lambda \quad (0 < |t| < \varepsilon)
\]
for the eigenvalues $\lambda \notin R_f$ as follows, in terms of the equivariant limit mixed Hodge polynomials obtained by Stapledon’s results in [35].

Theorem 1.3. Assume that the family $Y$ of hypersurfaces in $\mathbb{C}^n$ is schön. For $\lambda \in \mathbb{C}$ and $m \geq 1$ denote by $J_{\lambda,m}$ the number of the Jordan blocks in the monodromy
\[
\Phi_{n-1} : H^{n-1}(Y_t; \mathbb{C})_\lambda \sim H^{n-1}(Y_t; \mathbb{C})_\lambda \quad (0 < |t| < \varepsilon)
\]
for the eigenvalue $\lambda$ with size $m$. Then for $\lambda \notin R_f$ we have
\[
\sum_{m=0}^{n-1} J_{\lambda,n-m}s^{m+2} = \sum_{F \in \mathcal{S}} s^{\dim F + 1}l^*_\lambda(F, \nu_f|F; 1) \cdot \tilde{I}_p(S, F; s^2)
\]
(for the definitions of the polynomials $l^*_\lambda(F, \nu_f|F; u) \in \mathbb{Z}[u]$ and $\tilde{I}_p(S, F; t) \in \mathbb{Z}[t]$ see Sections 2 and 3).
The proof of Theorem 1.2 relies on Morihiko Saito’s theorem on the primitive decom-
positions of the mixed Hodge modules over nearby cycle preverse sheaves associated to
normal crossing divisors and we apply it to nearby cycle preverse sheaves on some smooth
toric varieties. For the proof, we are also indebted to [35, Theorem 5.7] which is proved
by using [22, Section 2] and some deep results on combinatorics obtained by Stanley [34].
Theorem 1.2 holds true without any assumption on the shape of the Newton polytope

\[ P = p(UH_f) \subset \mathbb{R}^n \]

In particular, we do not require here that \( P \) is convenient as in [35]. The convenience of \( P \) of [35] is stronger than the usual one (see Definition 4.1) and we cannot expect it in general. For the treatment of the non-convenient case, we have to prove the topological concentration

\[ H^j(Y_t; \mathbb{C})_{\lambda} \cong 0 \quad (\lambda \notin R_f, j \neq n - 1) \]

in Theorem 1.2 which does not follow from the results in Danilov-Khovanskii [4] and
Stapledon [35]. Moreover in Section 5 we also extend these results to families of schön complete intersection subvarieties in \( \mathbb{C}^n \) and obtain a formula for the Jordan normal forms of their monodromies. To our surprise, the results that we obtain in this generalized situation are completely parallel to the ones for families of hypersurfaces in \( \mathbb{C}^n \). See Section 5 for the details.

Acknowledgement: We thank Professors Masaharu Ishikawa and Sampei Usui
for their encouragements.

2 Preliminary notions and results

In this section, we introduce some preliminary notions and results which will be used in
this paper.

2.1 Motivic nearby fibers

Throughout this paper we consider only varieties over the field \( \mathbb{C} \) of complex numbers.
From now on we shall introduce the theory of motivic nearby fibers of Denef-Loeser [6], [7]
and Guibert-Loeser-Merle [14] in this special case (see also Raibaut [28]). For a variety
\( S \) denote by \( K_0(\text{Var}_S) \) the Grothendieck ring of varieties over \( S \). Recall that the ring structure is defined by the fiber products over \( S \). Moreover we denote by \( \mathcal{M}_S \) the ring obtained from it by inverting the Lefschetz motive \( L \cong \mathbb{C} \times \mathbb{C} \in K_0(\text{Var}_S) \). If \( S = \text{Spec}(\mathbb{C}) \) we denote \( K_0(\text{Var}_S) \) and \( \mathcal{M}_S \) simply by \( K_0(\text{Var}_{\mathbb{C}}) \) and \( \mathcal{M}_{\mathbb{C}} \) respectively. Note that \( \mathcal{M}_S \) has a natural structure of an \( \mathcal{M}_{\mathbb{C}} \)-module. For \( d \in \mathbb{Z}_{>0} \), let \( \mu_d = \{ \zeta \in \mathbb{C} \mid \zeta^d = 1 \} \cong \mathbb{Z}/d \mathbb{Z} \) be the multiplicative group consisting of \( d \) roots of unity in \( \mathbb{C} \). We denote by \( \hat{\mu} \) the projective limit \( \lim_{\leftarrow d} \mu_d \) of the projective system \( \{ \mu_d \}_{d \geq 1} \) with morphisms \( \mu_{id} \rightarrow \mu_i \), given by \( t \mapsto t^d \).

Then we define the Grothendieck ring \( K_0^\hat{\mu}(\text{Var}_S) \) of varieties over \( S \) with good \( \hat{\mu} \)-actions and its localization \( \mathcal{M}_S^\hat{\mu} \) as in [7, Section 2.4]. Note that \( \mathcal{M}_S^\hat{\mu} \) is naturally a \( \mathcal{M}_{\mathbb{C}} \)-module. Recall also that for a morphism \( \pi : S \rightarrow S' \) of varieties we have a group homomorphism

\[ \pi_1 : \mathcal{M}_S^\hat{\mu} \rightarrow \mathcal{M}_{S'}^\hat{\mu} \] (2.1)
obtained by the composition with \( \pi \). Now let \( Z \) be a smooth variety and \( U \subset Z \) its Zariski open subset such that \( D = Z \setminus U \) is a normal crossing divisor in \( Z \). Moreover let \( f : Z \rightarrow \mathbb{C} \) be a regular function on \( Z \) such that \( f^{-1}(0) \subset Z \) is contained in \( D \). Denote by \( D' \subset D \) the union of irreducible components of \( D \) which are not contained in \( f^{-1}(0) \) and set \( \Omega = Z \setminus D' \). Then we have \( U \subset \Omega \). Let \( E_1, E_2, \ldots, E_k \) be the irreducible components of the normal crossing divisor \( \Omega \cap f^{-1}(0) \) in \( \Omega \subset Z \). For each \( 1 \leq i \leq k \), let \( b_i > 0 \) be the order of the zero of \( f \) along \( E_i \). For a non-empty subset \( I \subset \{1, 2, \ldots, k\} \), let us set
\[
E_I = \bigcap_{i \in I} E_i, \quad E_I^0 = E_I \setminus \bigcup_{i \not\in I} E_i
\]
and \( d_I = \gcd(b_i)_{i \in I} > 0 \). Then, as in \([4, \text{Section 3.3}]\), we can construct an unramified Galois covering \( \tilde{E}_I^0 \rightarrow E_I^0 \) of \( E_I^0 \) as follows. First, for a point \( p \in E_I^0 \) we take an affine open neighborhood \( W \subset \Omega \setminus (\cup_{i \not\in I} E_i) \) of \( p \) on which there exist regular functions \( \xi_i \) \( (i \in I) \) such that \( E_i \cap W = \{ \xi_i = 0 \} \) for any \( i \in I \). Then on \( W \) we have \( f = f_{1,W}(f_{2,W})^{d_I} \), where we set \( f_{1,W} = f \prod_{i \in I} \xi_i^{-b_i} \) and \( f_{2,W} = \prod_{i \in I} \xi_i^{b_i} \). Note that \( f_{1,W} \) is a unit on \( W \) and \( f_{2,W} : W \rightarrow \mathbb{C} \) is a regular function. It is easy to see that \( E_I^0 \) is covered by such affine open subsets \( W \) of \( \Omega \setminus (\cup_{i \not\in I} E_i) \). Then as in \([4, \text{Section 3.3}]\) by gluing the varieties
\[
\tilde{E}_{I,W} = \{(t, z) \in \mathbb{C}^* \times (E_I^0 \cap W) \mid t^{d_I} = (f_{1,W})^{-1}(z)\}
\]
together in an obvious way, we obtain the variety \( \tilde{E}_I^0 \) over \( E_I^0 \). This unramified Galois covering \( \tilde{E}_I^0 \) of \( E_I^0 \) admits a natural \( \mu_{d_I} \)-action defined by assigning the automorphism \( (t, z) \mapsto (\zeta_{d_I} t, z) \) of \( \tilde{E}_I^0 \) to the generator \( \zeta_{d_I} := \exp(2\pi \sqrt{-1}/d_I) \in \mu_{d_I} \). Namely the variety \( \tilde{E}_I^0 \) is endowed with a good \( \mu \)-action in the sense of \([4, \text{Section 2.4}]\) and defines an element \([\tilde{E}_I^0]\) of \( \mathcal{M}_{d_I}^{\mu^{-1}(0)} \). Finally we set
\[
S_{f,U} = \sum_{I \neq \emptyset} (1 - \mathbb{L})^{d_I - 1} \cdot [\tilde{E}_I^0] \in \mathcal{M}_{d_I}^{-\mu^{-1}(0)}. \tag{2.4}
\]
Then we have the following result.

**Theorem 2.1.** (\([4, \text{Theorem 3.9}]\)) Let \( X \) be a variety and \( g : X \rightarrow \mathbb{C} \) a regular function on it. Then there exists a morphism of \( \mathcal{M}_\mathbb{C} \)-modules
\[
\psi_g : \mathcal{M}_X \rightarrow \mathcal{M}_{d_I}^{-\mu^{-1}(0)} \tag{2.5}
\]
such that for any proper morphism \( \pi : Z \rightarrow X \) from a smooth variety \( Z \) and its Zariski open subset \( U \subset Z \) whose complement \( D = Z \setminus U \) is a normal crossing divisor in \( Z \) containing \( (g \circ \pi)^{-1}(0) \) we have the equality
\[
\psi_g([U \rightarrow X]) = (\pi|_{(g \circ \pi)^{-1}(0)})!([S_{g \circ \pi, U}]) \tag{2.6}
\]
in \( \mathcal{M}_{d_I}^{-\mu^{-1}(0)} \).

**Definition 2.2.** In the situation of Theorem 2.1, for a variety \([Y \rightarrow X] \in \mathcal{M}_X \) over \( X \) we call \( \psi_g([Y \rightarrow X]) \in \mathcal{M}_{d_I}^{-\mu^{-1}(0)} \) the motivic nearby fiber of \( Y \) by \( g \) and denote it simply by \( \psi_g([Y]) \).
Following the notations in [7, Sections 3.1.2 and 3.1.3], we denote by $\text{HS}^{\text{mon}}$ the abelian category of Hodge structures with a quasi-unipotent endomorphism. Let 

$$\chi_h : \mathcal{M}^\mu_C \longrightarrow K_0(\text{HS}^{\text{mon}})$$

(2.7)

be the Hodge characteristic morphism defined in [7] which associates to a variety $Z$ with a good $\mu_d$-action the Hodge structure 

$$\chi_h([Z]) = \sum_{j \in \mathbb{Z}} (-1)^j [H^j_c(Z; \mathbb{Q})] \in K_0(\text{HS}^{\text{mon}})$$

(2.8)

with the actions induced by the one $z \mapsto \exp(2\pi \sqrt{-1}/d) z$ ($z \in Z$) on $Z$. We can generalize this construction as follows. For a variety $X$ let $\text{MHM}_X$ be the abelian category of mixed Hodge modules on $X$ (see [15, Section 8.3] etc.) and $K_0(\text{MHM}_X)$ its Grothendieck ring. Then there exists a group homomorphism 

$$H_X : \mathcal{M}_X \longrightarrow K_0(\text{MHM}_X)$$

(2.9)

such that for any morphism $\pi : Z \longrightarrow X$ from a smooth variety $Z$ and the trivial Hodge module $\mathbb{Q}^{H}_Z$ on it we have 

$$H_X([Z \longrightarrow X]) = \sum_{j \in \mathbb{Z}} (-1)^j [H^j R\pi_!(\mathbb{Q}^{H}_Z)].$$

(2.10)

Here the Grothendieck ring $K_0(\text{MHM}_X)$ has a natural $\mathcal{M}_C$-module structure defined by the Hodge realization map $H : \mathcal{M}_C \longrightarrow K_0(\text{MHM}_{\text{Spec}(\mathbb{C})})$ and $H_X$ is moreover $\mathcal{M}_C$-linear. By using the abelian category $\text{MHM}_X^{\text{mon}}$ of mixed Hodge modules on $X$ with a finite order automorphism and its Grothendieck ring $K_0(\text{MHM}_X^{\text{mon}})$ we have also a group homomorphism 

$$H_X^{\text{mon}} : \mathcal{M}_X^{\mu} \longrightarrow K_0(\text{MHM}_X^{\text{mon}})$$

(2.11)

(see [14] and [28] for the details).

**Proposition 2.3.** ([14, Proposition 3.17]) In the situation of Theorem 2.1 there exists a commutative diagram 

$$
\begin{array}{ccc}
\mathcal{M}_X & \xrightarrow{\psi_g} & \mathcal{M}_{g^{-1}(0)} \\
H_X \downarrow & & \downarrow \quad H_{g^{-1}(0)}^{\text{mon}} \\
K_0(\text{MHM}_X) & \xrightarrow{\psi_g} & K_0(\text{MHM}_{g^{-1}(0)}^{\text{mon}})
\end{array}
$$

(2.12)

where $\psi_g$ is induced by the nearby cycles of mixed Hodge modules.

### 2.2 Equivariant Ehrhart theory of Katz and Stapledon

In this section, we introduce some polynomials in the Equivariant Ehrhart theory in Katz-Stapledon [17] and Stapledon [35]. Throughout this paper, we regard the empty set $\emptyset$ as a $(-1)$-dimensional polytope, and as a face of any polytope. Let $P$ be a polytope. If a subset $F \subset P$ is a face of $P$, we write $F \prec P$. For a pair of faces $F \prec F' \prec P$ of $P$, we denote by $[F, F']$ the face poset $\{F'' \prec P \mid F \prec F'' \prec F'\}$, and by $[F, F']^*$ a poset which is equal to $[F, F']$ as a set with the reversed order.
Definition 2.4. Let $B$ be a poset $[F, F']$ or $[F, F']^*$. We define a polynomial $g(B, t)$ of degree $\leq (\dim F' - \dim F)/2$ as follows. If $F = F'$, we set $g(B; t) = 1$. If $F \neq F'$ and $B = [F, F']$ (resp. $B = [F, F']^*$), we define $g(B; t)$ inductively by
\[
\ell^{\dim F' - \dim F} g(B; t^{-1}) = \sum_{F'' \in [F, F']} (t - 1)^{\dim F' - \dim F''} g([F, F'']; t).
\]
(resp. $\ell^{\dim F' - \dim F} g(B; t^{-1}) = \sum_{F'' \in [F, F']^*} (t - 1)^{\dim F' - \dim F''} g([F', F'']; t).$)

In what follows, we assume that $P$ is a lattice polytope in $\mathbb{R}^n$. Let $S$ be a subset of $P \cap \mathbb{Z}^n$ containing the vertices of $P$, and $\omega : S \to \mathbb{Z}$ be a function. We denote by $\text{UH}^\omega$ the convex hull in $\mathbb{R}^n \times \mathbb{R}$ of the set $\{(v, s) \in \mathbb{R}^n \times \mathbb{R} \mid v \in S, s \geq \omega(v)\}$. Then, the set of all the projections of the bounded faces of $\text{UH}^\omega$ to $\mathbb{R}^n$ defines a lattice polyhedral subdivision $\mathcal{S}$ of $P$. Here a lattice polyhedral subdivision $\mathcal{S}$ of a polytope $P$ is a set of some polytopes in $P$ such that the intersection of any two polytopes in $\mathcal{S}$ is a face of both and all vertices of any polytope in $\mathcal{S}$ are in $\mathbb{Z}^n$. Moreover, the set of all the bounded faces of $\text{UH}^\omega$ defines a piecewise $\mathbb{Q}$-affine convex function $\nu : P \to \mathbb{R}$. For a cell $F \in \mathcal{S}$, we denote by $\sigma(F)$ the smallest face of $P$ containing $F$, and $\text{lk}_\mathcal{S}(F)$ the set of all cells of $\mathcal{S}$ containing $F$. We call $\text{lk}_\mathcal{S}(F)$ the link of $F$ in $\mathcal{S}$. Note that $\sigma(\emptyset) = \emptyset$ and $\text{lk}_\mathcal{S}(\emptyset) = \mathcal{S}$.

Definition 2.5. For a cell $F \in \mathcal{S}$, the $h$-polynomial $h(\text{lk}_\mathcal{S}(F); t)$ of the link $\text{lk}_\mathcal{S}(F)$ of $F$ is defined by
\[
\ell^{\dim P - \dim F} h(\text{lk}_\mathcal{S}(F); t^{-1}) = \sum_{F' \in \text{lk}_\mathcal{S}(F)} g([F, F']; t)(t - 1)^{\dim P - \dim F'}.
\]

The local $h$-polynomial $l_\mathcal{S}(\mathcal{S}, F; t)$ of $F$ in $\mathcal{S}$ is defined by
\[
l_\mathcal{S}(\mathcal{S}, F; t) = \sum_{\sigma(F) \prec Q \prec P} (-1)^{\dim P - \dim Q} h(\text{lk}_\mathcal{S}_Q(F); t) \cdot g([Q, P]^*; t).
\]

For $\lambda \in \mathbb{C}$ and $v \in mP \cap \mathbb{Z}^n$ ($m \in \mathbb{Z}_+ := \mathbb{Z}_{\geq 0}$) we set
\[
w_\lambda(v) = \begin{cases} 1 & \left(\exp \left(2\pi \sqrt{-1} \cdot m\nu(\frac{v}{m})\right) = \lambda\right) \\ 0 & \text{(otherwise).} \end{cases}
\]

We define the $\lambda$-weighted Ehrhart polynomial $f_\lambda(P, \nu; m) \in \mathbb{Z}[m]$ of $P$ with respect to $\nu : P \to \mathbb{R}$ by
\[
f_\lambda(P, \nu; m) := \sum_{v \in mP \cap \mathbb{Z}^n} w_\lambda(v).
\]
Then $f_\lambda(P, \nu; m)$ is a polynomial in $m$ with coefficients $\mathbb{Z}$ whose degree is $\leq \dim P$ (see [35]).

Definition 2.6. (35)

(i) We define the $\lambda$-weighted $h^*$-polynomial $h^*_\lambda(P, \nu; u) \in \mathbb{Z}[u]$ by
\[
\sum_{m \geq 0} f_\lambda(P, \nu; m)u^m = \frac{h^*_\lambda(P, \nu; u)}{(1 - u)^{\dim P + 1}}.
\]
If $P$ is the empty polytope, we set $h^*_1(P, \nu; u) = 1$ and $h^*_\lambda(P, \nu; u) = 0$ ($\lambda \neq 1$).
(ii) We define the $\lambda$-local weighted $h^*$-polynomial $l^*_\lambda(P, \nu; u) \in \mathbb{Z}[u]$ by

$$l^*_\lambda(P, \nu; u) = \sum_{Q < P} (-1)^{\dim P - \dim Q} h^*_\lambda(Q, \nu|Q; u) \cdot g([Q, P]^*; u).$$

If $P$ is the empty polytope, we set $l^*_\lambda(P, \nu; u) = 1$ and $l^*_\lambda(P, \nu; u) = 0$ ($\lambda \neq 1$).

**Definition 2.7.** ([35])

(i) We define the $\lambda$-weighted limit mixed $h^*$-polynomial $h^*_\lambda(P, \nu; u) \in \mathbb{Z}[u, v]$ by

$$h^*_\lambda(P, \nu; u, v) := \sum_{F \in S} v^{\dim F + 1} l^*_\lambda(F, \nu|F; uv^{-1}) \cdot h(\text{lk}_S(F); uv).$$

(ii) We define the $\lambda$-local weighted limit mixed $h^*$-polynomial $l^*_\lambda(P, \nu; u) \in \mathbb{Z}[u, v]$ by

$$l^*_\lambda(P, \nu; u, v) := \sum_{F \in S} v^{\dim F + 1} l^*_\lambda(F, \nu|F; uv^{-1}) \cdot l_P(S, F; uv).$$

(iii) We define $\lambda$-weighted refined limit mixed $h^*$-polynomial $h^*_\lambda(P, \nu; u, v, w) \in \mathbb{Z}[u, v, w]$ by

$$h^*_\lambda(P, \nu; u, v, w) := \sum_{Q < P} w^{\dim Q + 1} l^*_\lambda(Q, \nu|Q; u, v) \cdot g([Q, P]; uwv^2).$$

### 3 Monodromies and limit mixed Hodge structures of families of hypersurfaces in algebraic tori $(\mathbb{C}^*)^n$

Let $\mathbb{K} = \mathbb{C}(t)$ be the field of rational functions of $t$ and $f(t, x) = \sum_{a \in \mathbb{Z}_+^n} a_v(t)x^n \in \mathbb{K}[x_1^+; \ldots, x_n^+]$ ($a_v(t) \in \mathbb{K}$) a Laurent polynomial of $x = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{K}$. For $v \in \mathbb{Z}^n$ by the Laurent expansion $a_v(t) = \sum_{j \in \mathbb{Z}} a_{v,j} t^j$ ($a_{v,j} \in \mathbb{C}$) of the rational function $a_v(t)$ we set

$$o(v) := \text{ord}_t a_v(t) = \min\{j \mid a_{v,j} \neq 0\}.$$ 

If $a_v(t) \equiv 0$ we set $o(v) = +\infty$. Then we define an (unbounded) polyhedron $\text{UH}_f$ in $\mathbb{R}^{n+1}$ by

$$\text{UH}_f = \text{Conv} \left( \bigcup_{v \in \mathbb{Z}^n} \{(v, s) \in \mathbb{R}^{n+1} \mid s \geq o(v)\} \right) \subset \mathbb{R}^{n+1},$$

where $\text{Conv}(\cdot)$ stands for the convex hull. Throughout this paper we assume that the dimension of $\text{UH}_f$ is $n + 1$. Then by the projection $p : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ we obtain an $n$-dimensional polytope $P := p(\text{UH}_f) \subset \mathbb{R}^n$. We call it the Newton polytope of $f$. Set $\mathbb{R}_+ := \mathbb{R}_{\geq 0} \subset \mathbb{R}$. Let $\Sigma_0$ be the dual fan of $\text{UH}_f$ in $\mathbb{R}^n \times \mathbb{R}_+^n \subset \mathbb{R}^{n+1}$. We call its subfan $\Sigma_0$ in $\mathbb{R}^n \cong \mathbb{R}^* \times \{0\}$ consisting of cones $\sigma \in \Sigma_0$ contained in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ the recession fan of $\text{UH}_f$. Let $\nu_f : P \to \mathbb{R}$ be the function defining the bottom part of the boundary $\partial \text{UH}_f$ of $\text{UH}_f$ and $S$ the subdivision of $P$ by the lattice polytopes $p(\tilde{F}) \subset \mathbb{R}^n$ ($\tilde{F} \subset \text{UH}_f$). Then for each cell $F$ in $S$ the restriction $\nu_f|_F$ of $\nu_f$ to $F \subset P$ is an affine $\mathbb{Q}$-linear function taking integral values on the vertices of $F$. Let us identify the affine subspace $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$ of $\mathbb{R}^{n+1}$ with $\mathbb{R}^n$ by the projection. Then by using the cones $\sigma \in \Sigma_0$
such that \( \dim \sigma < n + 1 \) we define a polyhedral hypersurface \( \text{Trop}(Y) \) in \( \mathbb{R}^n \times \{1\} \simeq \mathbb{R}^n \) by

\[
\text{Trop}(Y) = \bigcup_{\dim \sigma < n + 1} \left\{ \sigma \cap (\mathbb{R}^n \times \{1\}) \right\} \subset \mathbb{R}^n \times \{1\} \simeq \mathbb{R}^n.
\]

We call it the tropical variety of \( Y \) (see [29] etc.). It has a decomposition

\[
\text{Trop}(Y) = \bigcup_{\dim \sigma < n + 1} \left\{ \text{rel.int}\sigma \cap (\mathbb{R}^n \times \{1\}) \right\}
\]

into the (locally closed) cells \( \text{rel.int}\sigma \cap (\mathbb{R}^n \times \{1\}) \subset \text{Trop}(Y) \). It is clear that there exists a one to one correspondence between the cells \( F \) in \( S \) such that \( \dim F > 0 \) and those in \( \text{Trop}(Y) \). In [35] the author used the cell decomposition of \( \text{Trop}(Y) \) to express the motivic nearby fiber \( \psi_t([Y]) \). However in this paper, we use only the subdivision \( S \) of \( P \). For a cell \( F \) in \( S \) by taking the (unique) compact face \( \bar{F} \prec \text{UH}_f \) of \( \text{UH}_f \) such that \( F = p(\bar{F}) \) we define the initial Laurent polynomial \( \text{Trop}(F)(x) \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm] \) by

\[
\text{Trop}(F)(x) = \sum_{(v,j) \in \bar{F}} a_{v,j} x^v \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm].
\]

By identifying \( \text{Aff}(F) \cap \mathbb{Z}^n \) with \( \mathbb{Z}^{\dim F} \) we may consider \( \text{Trop}(F)(x) \) as a Laurent polynomial on the algebraic torus \( T_F = \text{Spec}(\mathbb{C}[\text{Aff}(F) \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim F} \). We denote by \( V_F \subset T_F \) the hypersurface defined by \( \text{Trop}(F)(x) \) in \( T_F \). By the affine \( \mathbb{Q} \)-linear extension \( \nu_F : \text{Aff}(F) \rightarrow \mathbb{R} \) of \( \nu_F|_F \) to \( \text{Aff}(F) \simeq \mathbb{R}^{\dim F} \) we define an element \( e_F \) of the algebraic torus \( T_F = \text{Spec}(\mathbb{C}[\text{Aff}(F) \cap \mathbb{Z}^n]) \simeq \text{Hom}_{\text{group}}(\text{Aff}(F) \cap \mathbb{Z}^n, \mathbb{C}^*) \) by

\[
e_F(v) = \exp\left(-2\pi \sqrt{-1} \nu_F(v)\right) \quad (v \in \text{Aff}(F) \cap \mathbb{Z}^n).
\]

Then by the multiplication \( \ell_{e_F} : T_F \xrightarrow{\sim} T_F \) by \( e_F \in T_F \) we have \( \ell_{e_F}(V_F) = V_F \). We thus obtain an element \( [V_F \circ \hat{\mu}] \in \mathcal{M}^\mu_{\mathbb{C}_n} \). The hypersurface \( f^{-1}(0) \subset T = \mathbb{C}_t^* \times (\mathbb{C}^*)^n_x \) over a small punctured disk \( B(0; \varepsilon)^* = \{ t \in \mathbb{C} \mid 0 < |t| < \varepsilon \} \) \( 0 < \varepsilon \ll 1 \). By the projection \( \pi : T = \mathbb{C}_t^* \times (\mathbb{C}^*)^n_x \rightarrow \mathbb{C}_t^* \), for \( t \in \mathbb{C} \) such that \( 0 < |t| < \varepsilon \) we set \( Y_t := \pi^{-1}(t) \cap Y \subset \{ t \} \times T_0 \simeq T_0 \simeq (\mathbb{C}^*)^n_x \).

**Definition 3.1.** We say that the family \( Y \) of hypersurfaces \( \{Y_t\}_{0 < |t| < \varepsilon} \) of \( T_0 \simeq (\mathbb{C}^*)^n_x \) is schön if for any cell \( F \) in \( S \) the hypersurface \( V_F \subset T_F \) of \( T_F \) is smooth and reduced.

For the family \( Y \) over the punctured disk, denote by \( \psi_t([Y]) \in \mathcal{M}^\mu_{\mathbb{C}_n} \) its motivic nearby fiber by the function \( t = \text{id}_\mathbb{C} : \mathbb{C} \rightarrow \mathbb{C} \) (see Section [2]). Then the following beautiful result was first obtained by Stapledon [35].

**Theorem 3.2.** Assume that the family \( Y \) is schön. Then we have an equality

\[
\psi_t([Y]) = \sum_{\text{rel.int } F \subset \text{Int } P} [V_F \circ \hat{\mu}] \cdot (1 - \mathbb{L})^{n - \dim F}
\]

in \( \mathcal{M}^\mu_{\mathbb{C}_n} \).
Proof. Let $\Sigma$ be a smooth subdivision of the dual fan $\Sigma_0$ of $UH_f$ and $\Xi$ its subfan in $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}$ consisting of cones $\sigma \in \Sigma$ contained in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. We denote by $\Lambda$ the fan $\{\{0\}, \mathbb{R}_+^1\}$ in $\mathbb{R}^1$ formed by the faces of the closed half line $\mathbb{R}_+^1$ in $\mathbb{R}^1$. Let $X_\Sigma$ (resp. $X_\Xi$) be the toric variety associated to $\Sigma$ (resp. $\Xi$). Recall that the algebraic torus $T = (\mathbb{C}^*)^{n+1}$ acts on $X_\Sigma$. For a cone $\sigma \in \Sigma$ denote by $T_\sigma \simeq (\mathbb{C}^*)^{\dim \sigma}$ the $T$-orbit in $X_\Sigma$ associated to it. Then by the morphism $\Sigma \to \Lambda$ of fans induced by the projection $\mathbb{R}^n \times \mathbb{R}_+^1 \to \mathbb{R}_+^1$ we obtain a morphism

$$\pi_\Sigma : X_\Sigma \to \mathbb{C}$$

of toric varieties. Restricting it to $\mathbb{C}^* \subset \mathbb{C}$, we obtain the projection $\mathbb{C}^* \times X_\Xi \to \mathbb{C}^*$ which extends naturally the previous one $\pi : T = \mathbb{C}^* \times T_0 \to \mathbb{C}^*$. Let $\rho_1, \ldots, \rho_N \in \Sigma$ be the rays i.e. the 1-dimensional cones in $\Sigma$. We may assume that for some $r \leq N$ we have $\rho_i \cap (\mathbb{R}^n \times \{0\}) = \{0\} \iff 1 \leq i \leq r$. For $1 \leq i \leq r$ the $T$-orbit $T_i = T_{\rho_i} \subset X_\Sigma$ associated to $\rho_i$ satisfies the condition $\pi_\Sigma(T_i) = \{0\} \subset \mathbb{C}$. We can easily see that for their closures $\overline{T_i} \subset X_\Sigma$ in $X_\Sigma$ we have

$$\pi_\Sigma^{-1}(\{0\}) = \bigcup_{i=1}^r \overline{T_i}.$$ 

For $1 \leq i \leq N$ let $\alpha_i \in \rho_i \cap (\mathbb{Z}^n_+ \setminus \{0\})$ be the primitive vector on the ray $\rho_i$. We define a non-negative integer $b_i \geq 0$ by $b_i = q(\alpha_i)$, where $q : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^1$ is the projection. Then it is easy to see that for $1 \leq i \leq r$ the order of the zeros of the function $t \circ \pi_\Sigma : X_\Sigma \to \mathbb{C}$ along the toric divisor $\overline{t_i} \subset X_\Sigma$ is equal to $b_i > 0$. For a cone $\sigma \in \Sigma$ such that $\text{rel.int}. \sigma \subset \text{Int}(\mathbb{R}^n \times \mathbb{R}_+^1)$ we denote by $\overline{F_\sigma} : UH_f$ its supporting face in $UH_f$. Then by the schönness of $Y$, the closure $\overline{Y}$ of $Y \subset T \subset X_\Sigma$ in $X_\Sigma$ intersects the $T$-orbit $T_\sigma \simeq (\mathbb{C}^*)^{\dim \sigma} \subset X_\Sigma$ transversally. This in particular implies that $\overline{Y}$ is smooth on a neighborhood of $\pi_\Sigma^{-1}(\{0\})$ in $X_\Sigma$. The resulting smooth hypersurface $W_\sigma := \overline{Y} \cap T_\sigma \subset T_\sigma$ of $T_\sigma$ is defined by the $\overline{F_\sigma}$-part $f_{\overline{F_\sigma}}(t,x)$ of $f(t,x)$. We can also easily see that for any cone $\sigma \in \Xi$ the hypersurface $\overline{Y}$ intersects $T_\sigma$ transversally on a neighborhood of $\pi_\Sigma^{-1}(\{0\})$ in $X_\Sigma$. Moreover the divisor $D = \overline{Y} \setminus Y = \overline{Y} \cap (X_\Sigma \setminus T)$ in $\overline{Y}$ is normal crossing. Let $\pi_{\overline{Y}} = \pi_\Sigma |_{\overline{Y}} : \overline{Y} \to \mathbb{C}$ be the restriction of $\pi_\Sigma$ to $\overline{Y}$. Then by Theorem 2.1 and $t \circ \pi_{\overline{Y}} = \pi_{\overline{Y}}$ we have

$$\psi_t([Y]) = (\pi_{\overline{Y}} |_{\pi_{\overline{Y}}^{-1}(0)})(S_{\pi_{\overline{Y}} Y}).$$

Moreover the morphism $(\pi_{\overline{Y}} |_{\pi_{\overline{Y}}^{-1}(0)})_!$ sends $S_{\pi_{\overline{Y}} Y}$ to its underlying variety over the point $\{0\} \simeq \text{Spec}(\mathbb{C})$, which we still denote by $S_{\pi_{\overline{Y}} Y}$ for short. Define an open subset $\Omega$ of $\overline{Y}$ containing $Y = \overline{Y} \cap T$ by

$$\Omega = \overline{Y} \cap (X_\Sigma \setminus \bigcup_{\sigma \in \Xi \setminus \{0\}} \overline{T_\sigma}) \subset \overline{Y}.$$ 

Let $\Sigma' \subset \Sigma$ be the subset of $\Sigma$ consisting of cones $\sigma$ satisfying the condition $\sigma \cap (\mathbb{R}^n \times \{0\}) = \{0\}$. Then we have

$$\Omega = \overline{Y} \cap \left( \bigcup_{\sigma \in \Sigma'} \overline{T_\sigma} \right)$$

10
Recall that the action of the group \( \hat{\mathcal{G}} \) introduce a new basis of the lattice \( \mathbf{d} \).

Let us set
\[
\rho = \mathbb{R}_+ \alpha_1 \oplus \cdots \oplus \mathbb{R}_+ \alpha_l,
\tau = \mathbb{R}_+ \alpha_1 \oplus \cdots \oplus \mathbb{R}_+ \alpha_{n+1}.
\]

Let us set \( \sigma' = \mathbb{R}_+ \alpha_{l+1} \oplus \cdots \oplus \mathbb{R}_+ \alpha_{n+1} \prec \tau \) and
\[
I = \{1, 2, \ldots, l\}, \quad I' = \{l + 1, l + 2, \ldots, n + 1\}.
\]

Moreover set \( d_I = \gcd\{b_i \mid 1 \leq i \leq l\} \geq 1 \) and let \( d_{I'} \geq 0 \) be the generator of the subgroup \( \mathbb{Z} b_{l+1} + \mathbb{Z} b_{l+2} + \cdots + \mathbb{Z} b_{n+1} \subset \mathbb{Z} \). We may define \( \gcd\{b_i \mid l + 1 \leq i \leq n + 1\} \) to be \( d_{I'} \). Then by the smoothness of the cone \( \tau \) similarly we have \( \gcd(d_I, d_{I'}) = \gcd\{b_i \mid 1 \leq i \leq n + 1\} = 1 \). We shall prove the equality \([3.1]\) only in the case \( d_{I'} \geq 1 \). In the case \( d_{I'} = 0 \) \( (\iff \sigma' \subset \mathbb{R}^n \times \{0\}) \) we have \( d_I = 1 \) and the proof is much easier. Assume that \( d_{I'} \geq 1 \). Let \( \alpha_1^*, \ldots, \alpha_{n+1}^* \in (\mathbb{Z}^{n+1})^* \simeq \mathbb{Z}^{n+1} \) be the dual basis of \( \alpha_1, \ldots, \alpha_{n+1} \in \mathbb{Z}^{n+1} \) and \( \tau^\vee \subset (\mathbb{R}^{n+1})^* \simeq \mathbb{R}^{n+1} \) the dual cone of \( \tau \subset \mathbb{R}^{n+1} \). Then the affine open subset \( \mathbb{C}^{n+1}(\tau)(\simeq \mathbb{C}^{n+1}) \) of \( X_{\Sigma} \) is defined by
\[
\mathbb{C}^{n+1}(\tau) = \text{Spec}(\mathbb{C}[\tau^\vee \cap \mathbb{Z}^{n+1}]) \simeq \text{Spec}(\mathbb{C}[\mathbb{Z}_+ \alpha_1^* + \cdots + \mathbb{Z}_+ \alpha_{n+1}^*])
\]
\[
\simeq \text{Spec}(\mathbb{C}[\xi_1, \ldots, \xi_{n+1}]) \simeq \mathbb{C}^{n+1}_{\xi} \quad \left( \alpha_i^* \leftrightarrow \xi_i \right).
\]

By the coordinates \( \xi = (\xi_1, \ldots, \xi_{n+1}) \) of \( \mathbb{C}^{n+1}(\tau) \subset X_{\Sigma} \) its subset \( T_{\sigma} = \text{Spec}(\mathbb{C}[\text{Aff}(\sigma)^{\perp} \cap \mathbb{Z}^{n+1}]) \simeq (\mathbb{C}^{*})^{n+1-l} \) is explicitly given by
\[
T_{\sigma} = \{ \xi \in \mathbb{C}^{n+1}(\tau) \mid \xi_i = 0 (1 \leq i \leq l), \xi_i \neq 0 (l + 1 \leq i \leq n + 1) \}.
\]

Moreover the restriction of the function \( t \circ \pi_{\Sigma} : X_{\Sigma} \to \mathbb{C} \) to \( \mathbb{C}^{n+1}(\tau) \) is equal to \( \xi_{b_1} \cdots \xi_{n+1}^* \in \mathbb{C}[\xi_1, \ldots, \xi_{n+1}] \) which corresponds to the element \( b_1 \alpha_1^* + \cdots + b_{n+1} \alpha_{n+1}^* \) in the group ring \( \mathbb{C}[(\tau^\vee \cap \mathbb{Z}^{n+1})] \). From this we see that the unramified Galois covering \( \tilde{W}_\sigma \) of \( W_\sigma = \overline{\mathbf{Y}} \cap T_{\sigma} \) is given by
\[
\tilde{W}_\sigma = \{ (t, (\xi_{l+1}, \ldots, \xi_{n+1})) \in \mathbb{C}^* \times T_{\sigma} \mid t^{d_I} = \xi_{l+1}^{-b_{l+1}} \cdots \xi_{n+1}^{-b_{n+1}}, f_{\tilde{W}_\sigma}(\xi_{l+1}, \ldots, \xi_{n+1}) = 0 \}.
\]

Recall that the action of the group \( \hat{\mu} \) on \( \tilde{W}_\sigma \in \mathcal{M}_C^\beta \) is defined by the multiplication of \( \zeta_{d_I} = \exp(2\pi \sqrt{-1}/d_I) \in \mathbb{C} \) to the coordinate \( t \). In order to rewrite \( [\tilde{W}_\sigma] \) we shall introduce a new basis of the lattice \( \mathbb{Z}^{n+1} \). First by our assumption on \( \sigma \), the affine span \( \text{Aff}(\sigma) \simeq \mathbb{R}_l \) of \( \sigma \) intersects the hyperplane \( \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1} \) transversally in \( \mathbb{R}^{n+1} \) and

and \( S_{\Sigma Y} \) is described by some varieties over the normal crossing divisor
\[
\Omega \cap \pi_{\Sigma Y}^{-1}(0) = \overline{\mathbf{Y}} \cap \left( \bigcup_{\sigma \in \Sigma \setminus \{0\}} T_{\sigma} \right) = \Omega \setminus Y.
\]
there exists a basis $\beta_1, \ldots, \beta_l$ of the lattice $\text{Aff}(\sigma) \cap \mathbb{Z}^{n+1} \cong \mathbb{Z}^l$ such that $\beta_1, \ldots, \beta_{l-1} \in \mathbb{Z}^n \times \{0\}$ ( $\iff q(\beta_1) = \cdots = q(\beta_{l-1}) = 0$ and $q(\beta_l) = d_l$. By the condition $\sigma' \not\in \mathbb{R}^n \times \{0\}$ there exists also a basis $\beta_{l+1}, \ldots, \beta_{n+1}$ of the lattice $\text{Aff}(\sigma') \cap \mathbb{Z}^{n+1} \cong \mathbb{Z}^{n+1-l}$ such that $\beta_{l+1}, \ldots, \beta_n \in \mathbb{Z}^n \times \{0\}$ and $q(\beta_{l+1}) = d_{l'}$. By $\mathbb{R}^{n+1} = \text{Aff}(\sigma) \oplus \text{Aff}(\sigma')$ we thus obtain a basis $\beta_1, \ldots, \beta_l, \beta_{l+1}, \ldots, \beta_{n+1}$ of the lattice $\mathbb{Z}^{n+1} = \{\text{Aff}(\sigma) \cap \mathbb{Z}^{n+1}\} \oplus \{\text{Aff}(\sigma') \cap \mathbb{Z}^{n+1}\}$.

For the dual basis $\alpha_1^*, \ldots, \alpha_{n+1}^*$ we have the decomposition

$$ w_1 + w_2 = b_1 \alpha_1^* + \cdots + b_{n+1} \alpha_{n+1}^* = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, $$

where we set $w_1 = b_1 \alpha_1^* + \cdots + b_l \alpha_l^* \subset \text{Aff}(\sigma')^\perp \cap \mathbb{Z}^{n+1} \cong \mathbb{Z}^l$ and $w_2 = b_{l+1} \alpha_{l+1}^* + \cdots + b_{n+1} \alpha_{n+1}^* \subset \text{Aff}(\sigma)^\perp \cap \mathbb{Z}^{n+1} \cong \mathbb{Z}^{n+1-l}$. Moreover by the construction of $\beta_1, \ldots, \beta_{n+1}$, for the dual basis $\beta_1^*, \ldots, \beta_{n+1}^*$ of it we have also

$$ d_l \beta_l^* + d_{l'} \beta_{n+1}^* = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \quad (3.2) $$

We thus obtain $w_1 = d_l \beta_l^*, w_2 = d_{l'} \beta_{n+1}^*$. By the condition $\sigma' \not\in \mathbb{R}^n \times \{0\}$ we have $w_2 \neq 0$. It follows also from our assumption on $\sigma$ that the restriction of the projection $p: \mathbb{R}^{n+1} \to \mathbb{R}^n$ to $\text{Aff}(\sigma) \cong \mathbb{R}^{n+1-l}$ is injective. Hence we get $p(w_2) \neq 0$ and the non-vanishing $p(w_1) \neq 0$ follows from $p(w_1) + p(w_2) = 0$. Since the two vectors $p(w_1)$ and $p(w_2)$ in $\mathbb{Z}^n \subset \mathbb{R}^n$ are divisible by $d_l$ and $d_{l'}$ respectively and $\gcd(d_l, d_{l'}) = 1$, they are divisible also by $d_l d_{l'}$. Let us show that the lattice vector

$$ \frac{1}{d_l d_{l'}} p(w_2) = \frac{1}{d_l} p(\beta_{n+1}^*), $$

thus obtained is primitive. Suppose that it is divisible again by an integer $d \geq 2$. Then we have

$$ 0 = \langle \beta_1, w_2 \rangle = \langle p(\beta_1), p(w_2) \rangle + q(\beta_1) \cdot q(w_2) = \langle p(\beta_1), p(w_2) \rangle + d_l d_{l'} q(\beta_{n+1}^*). $$

This implies that

$$ q(\beta_{n+1}^*) = -\left\langle \frac{1}{d_l d_{l'}} p(w_2), p(\beta_1) \right\rangle $$

is divisible by $d$. So $\beta_{n+1}^*$ is also divisible by $d \geq 2$, which contradicts the fact that $\beta_{n+1}^*$ is primitive. It follows also from

$$ 1 = \langle \beta_i, \beta_i^* \rangle = \langle p(\beta_i), p(\beta_i^*) \rangle \quad (l + 1 \leq i \leq n) $$

that the vectors $p(\beta_i^*) \in \mathbb{Z}^n \subset \mathbb{R}^n$ ($l + 1 \leq i \leq n$) are primitive. Note that $\beta_{l+1}^*, \ldots, \beta_{n+1}^*$ form a basis of the lattice $\text{Aff}(\sigma)^\perp \cap \mathbb{Z}^{n+1} \cong \mathbb{Z}^{n+1-l}$. For their projections by $p: \mathbb{R}^{n+1} \to \mathbb{R}^n$ we have the following result.
Proposition 3.3. The vectors \( p(\beta_{l+1}^*), \ldots, p(\beta_n^*), \frac{1}{d_I} p(\beta_{n+1}^*) \in p(\text{Aff}(\sigma)^{\perp}) \cap \mathbb{Z}^n \cong p(\text{Aff}(\widetilde{F}_\sigma)) \cap \mathbb{Z}^n = \text{Aff}(F_\sigma) \cap \mathbb{Z}^n \) form a basis of the lattice \( p(\text{Aff}(\sigma)^{\perp}) \cap \mathbb{Z}^n \cong \mathbb{Z}^{n+l}. \)

Proof. We have seen that \( p(\beta_{l+1}^*), \ldots, p(\beta_n^*), \frac{1}{d_I} p(\beta_{n+1}^*) \) are primitive. First let us show that they are linearly independent over \( \mathbb{R} \). Suppose that we have

\[
\lambda_{l+1} p(\beta_{l+1}^*) + \cdots + \lambda_n p(\beta_n^*) + \lambda_{n+1} \frac{p(\beta_{n+1}^*)}{d_I} = 0
\]

for some \( \lambda_i \in \mathbb{R} \) \((l + 1 \leq i \leq n + 1)\). Then by taking the pairings with \( \beta_i \) \((l + 1 \leq i \leq n)\) we obtain \( \lambda_i = 0 \) \((l + 1 \leq i \leq n)\) and hence \( \lambda_{n+1} = 0 \). Next we show that they generate the lattice \( p(\text{Aff}(\sigma)^{\perp}) \cap \mathbb{Z}^n \) over \( \mathbb{Z} \). For this we use the following result.

Lemma 3.4. The vectors \( p(\beta_{l+1}^*), \ldots, p(\beta_n^*) \) form a basis of the lattice \( \{ \mathbb{R}p(\beta_{l+1}^*) \oplus \cdots \oplus \mathbb{R}p(\beta_n^*) \} \cap \mathbb{Z}^n \).

Proof. Suppose that they do not generate the lattice over \( \mathbb{Z} \). Then there exist integers \( d \geq 2 \) and \( \lambda_{l+1}, \ldots, \lambda_n \in \mathbb{Z} \) such that

\[
\frac{1}{d} \left\{ \lambda_{l+1} p(\beta_{l+1}^*) + \cdots + \lambda_n p(\beta_n^*) \right\} \in \mathbb{Z}^n
\]

and \( \frac{\lambda_n}{d} \in \mathbb{Q} \setminus \mathbb{Z} \) for some \( l + 1 \leq i_0 \leq n \). By taking the pairing with \( p(\beta_{i_0}) \) we obtain \( \frac{\lambda_n}{d} \in \mathbb{Z} \), which is a contradiction. \( \Box \)

Let us continue the proof of Proposition 3.3. By Lemma 3.4, if the vectors \( p(\beta_{l+1}^*), \ldots, p(\beta_n^*), \frac{1}{d_I} p(\beta_{n+1}^*) \) do not generate the lattice \( p(\text{Aff}(\sigma)^{\perp}) \cap \mathbb{Z}^n \), then there exist integers \( d \geq 2 \) and \( \lambda_{l+1}, \ldots, \lambda_n \in \mathbb{Z} \) such that

\[
\frac{1}{d} \left\{ \frac{1}{d_I} p(\beta_{n+1}^*) - \lambda_{l+1} p(\beta_{l+1}^*) - \cdots - \lambda_n p(\beta_n^*) \right\} \in \mathbb{Z}^n.
\]

Set \( \gamma^* = \beta_{n+1}^* - \lambda_{l+1} d_I \beta_{l+1}^* - \cdots - \lambda_n d_I \beta_n^* \). Then we obtain a new basis \( \beta_{l+1}^*, \ldots, \beta_n^*, \gamma^* \) of the lattice \( \text{Aff}(\sigma)^{\perp} \cap \mathbb{Z}^{n+1} \cong \mathbb{Z}^{n+1-l} \). By taking the pairing with \( \beta_i \in \text{Aff}(\sigma) \cap \mathbb{Z}^{n+1} \) we get

\[
0 = \langle \beta_i, \gamma^* \rangle = \langle p(\beta_i), p(\gamma^*) \rangle + d_I \cdot q(\gamma^*).
\]

Since the lattice vector \( p(\gamma^*) \in \mathbb{Z}^n \) is divisible by \( dd_I \), the integer \( q(\gamma^*) \in \mathbb{Z} \) and hence \( \gamma^* \in \mathbb{Z}^{n+1} \) itself is so. This contradicts the fact that \( \gamma^* \) is primitive. \( \Box \)

Now we return to the proof of Theorem 3.2. By the new basis \( \beta_{l+1}^*, \ldots, \beta_{n+1}^* \) of the lattice \( \text{Aff}(\sigma)^{\perp} \cap \mathbb{Z}^{n+1} \cong \mathbb{Z}^{n+1-l} \) we have an isomorphism

\[
T_\sigma = \text{Spec}(\mathbb{C}[\text{Aff}(\sigma)^{\perp} \cap \mathbb{Z}^{n+1}]) \cong \text{Spec}(\mathbb{C}[\mathbb{Z}\beta_{l+1}^* + \cdots + \mathbb{Z}\beta_{n+1}^*])
\]

\[
\cong \text{Spec}(\mathbb{C}[z_{l+1}, \ldots, z_{n+1}]) \quad (\beta_i^* \longleftrightarrow z_i).
\]

By the new coordinates \( z = (z_{l+1}, \ldots, z_{n+1}) \) of \( T_\sigma \cong (\mathbb{C}^*)^{n+1-l} \) we have

\[
\widetilde{W}_\sigma = \{(t, (z_{l+1}, \ldots, z_{n+1})) \in \mathbb{C}^* \times T_\sigma \mid t^{d_I} = z_{n+1}^{-d_I}, f_{\widetilde{F}_\sigma}(z) = 0\}.
\]

13
On the other hand, by taking the projection \( q : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^1 \) of the both sides of the decomposition (3.2), we obtain
\[
d_I \cdot q(\beta^*_s) + d_{I'} \cdot q(\beta^*_{n+1}) = 1. \tag{3.3}
\]
Note that for the basis \( p(\beta^*_1), \ldots, p(\beta^*_n), \frac{1}{d_I}p(\beta^*_{n+1}) \) of the lattice \( \text{Aff}(F_\sigma) \cap \mathbb{Z}^n \) constructed in Proposition 3.3, we have \( \nu_f(p(\beta^*_i)), \nu_f(p(\beta^*_{n+1})) \in \mathbb{Z} \) and \( \nu_f(\frac{1}{d_I}p(\beta^*_{n+1})) \equiv \frac{q(\beta^*_{n+1})}{d_I} \mod \mathbb{Z} \). By Proposition 3.3, we obtain an isomorphism
\[
V_{F_\sigma} \simeq \{(s, (z_{l+1}, \ldots, z_{n+1})) \in \mathbb{C}^* \times T_\sigma \mid s^{d_I} = z_{n+1}, \ f_{F_\sigma}(z) = 0\}.
\]
Moreover the action of the group \( \hat{\mu} \) on \( [V_{F_\sigma} \circ \hat{\mu}] \in \mathcal{M}_C^\alpha \) corresponds to the multiplication of \( \exp(-2\pi \sqrt{-1}q(\beta^*_{n+1})/d_I) \) to the coordinate \( s \). By the equality (3.3), we have
\[
\exp \left( -\frac{2\pi \sqrt{-1}q(\beta^*_{n+1})}{d_I} \right)^{-d_{I'}} = \exp \left( \frac{2\pi \sqrt{-1}}{d_I} \right) = \zeta_{d_I}. \tag{3.4}
\]
Furthermore by \( \gcd(d_I, d_{I'}) = 1 \) the morphism \( V_{F_\sigma} \rightarrow \tilde{W}_\sigma \) defined by \( (s, z) \mapsto (s^{-d_{I'}}, z) \) is an isomorphism. It is compatible with the actions of \( \hat{\mu} \) on the both sides by the equality (3.4). We thus obtained the required isomorphism \( [\tilde{W}_\sigma] \simeq [V_{F_\sigma} \circ \hat{\mu}] \) in \( \mathcal{M}_C^\alpha \). If \( \dim \sigma + \dim F_\sigma < n + 1 \), similarly we can prove an isomorphism
\[
\tilde{W}_\sigma \simeq V_{F_\sigma} \times (\mathbb{C}^*)^{n+1-\dim \sigma-\dim F_\sigma},
\]
but the action of \( \hat{\mu} \) on the second factor \( (\mathbb{C}^*)^{n+1-\dim \sigma-\dim F_\sigma} \) of the right hand side might be non-trivial. Nevertheless by [35, Example 2.2] (which follows essentially from the definition of \( \mathcal{M}_C^\alpha \)) we obtain an isomorphism
\[
[\tilde{W}_\sigma] \simeq [V_{F_\sigma} \circ \hat{\mu}] \cdot (\mathbb{L} - 1)^{n+1-\dim \sigma-\dim F_\sigma}
\]
in \( \mathcal{M}_C^\alpha \). Then by Theorem 2.1, we have
\[
\psi_l([Y]) = \sum_{\sigma \in \Sigma \setminus \{0\}} [\tilde{W}_\sigma] \cdot (1 - \mathbb{L})^{\dim \sigma - 1} = \sum_{\sigma \in \Sigma \setminus \{0\}} (-1)^{n+1-\dim \sigma-\dim F_\sigma} [V_{F_\sigma} \circ \hat{\mu}] \cdot (1 - \mathbb{L})^{n-\dim F_\sigma}.
\]
For a cell \( F \in S \) denote by \( \tilde{F} < \text{UH}_f \) the unique compact face of \( \text{UH}_f \) such that \( p(\tilde{F}) = F \) and let \( F^\circ \in \Sigma_0 \) be the cone which corresponds to it in the dual fan \( \Sigma_0 \). Then we can easily show
\[
\sum_{\sigma \in \Sigma \setminus \{0\}} (-1)^{n+1-\dim \sigma-\dim F_\sigma} [V_{F_\sigma} \circ \hat{\mu}] \cdot (1 - \mathbb{L})^{n-\dim F_\sigma} = \begin{cases} [V_F \circ \hat{\mu}] \cdot (1 - \mathbb{L})^{n-\dim F} & (\text{rel.int } F \subset \text{Int } P) \\ 0 & \text{(otherwise)} \end{cases}
\]
\[14\]
(cf. the proof of Matsui-Takeuchi [22, Theorem 5.7 and Proposition 5.5]). We thus obtain the desired formula

$$\psi_t(Y) = \sum_{\text{rel. int } F \subset \text{Int } P} [V_F \cap \hat{\mu}] \cdot (1 - L)^{n - \dim F}.$$ 

This completes the proof. \(\square\)

**Remark 3.5.** As is clear from the above proof of Theorem 3.2, it can be immediately generalized to any schön family of subvarieties of \(T_0 \simeq (\mathbb{C}^*)^n\) as in [35, Theorem 3.2 and Corollary 5.3]. However in this paper, we do not use such a generalization.

By the proof of the above theorem we obtain the following result.

**Lemma 3.6.** Assume that the family \(Y\) is schön. Then there exists small \(\varepsilon > 0\) such that the hypersurface \(Y_t = Y \cap \pi^{-1}(t) \subset T_0 = (\mathbb{C}^*)^n\) is Newton non-degenerate (see [20]) for any \(t \in \mathbb{C}^*\) satisfying the condition \(0 < |t| < \varepsilon\).

Recall that for a constructible sheaf \(F \in D^b_c(\mathbb{C})\) on \(\mathbb{C}\) its nearby cycle sheaf \(\psi_t(F) \in D^b_c(\{0\})\) by the function \(t\) has a direct sum decomposition

$$\psi_t(F) = \bigoplus_{\lambda \in \mathbb{C}} \psi_{t,\lambda}(F)$$

with respect to the generalized eigenspaces \(\psi_{t,\lambda}(F) \in D^b_c(\{0\})\) for eigenvalues \(\lambda \in \mathbb{C}\) (see Dimca [8] etc.). Let \(j : B(0; \varepsilon)^* \hookrightarrow B(0; \varepsilon) = B(0; \varepsilon)^* \cup \{0\}\) be the inclusion map. By the rotation on the punctured disk we obtain the monodromy automorphisms

$$\Phi_j : H^j_c(Y_t; \mathbb{C}) \xrightarrow{\sim} H^j_c(Y_t; \mathbb{C}) \quad (j \in \mathbb{Z})$$

for \(0 < |t| < \varepsilon\). For \(\lambda \in \mathbb{C}\) let

$$H^j_c(Y_t; \mathbb{C})_\lambda \subset H^j_c(Y_t; \mathbb{C})$$

be the generalized eigenspace of \(\Phi_j\) for the eigenvalue \(\lambda\). Then we have an isomorphism

$$H^j\psi_{t,\lambda}(j_t R\pi_! \mathbb{C}_Y) \simeq H^j_c(Y_t; \mathbb{C})_\lambda$$

for any \(j \in \mathbb{Z}\) and \(\lambda \in \mathbb{C}\).

**Proposition 3.7.** Assume that the family \(Y\) is schön. Then for \(t \in \mathbb{C}^*\) such that \(0 < |t| \ll 1\) we have

$$H^j_c(Y_t; \mathbb{C}) \simeq 0 \quad (j < n - 1)$$

and the Gysin map

$$H^j_c(Y_t; \mathbb{C}) \longrightarrow H^{j+2}(T_0; \mathbb{C})$$

associated to the inclusion map \(Y_t \hookrightarrow T_0\) is an isomorphism (resp. surjective) for \(j > n - 1\) (resp. \(j = n - 1\)). Moreover the monodromy \(\Phi_j : H^j_c(Y_t; \mathbb{C}) \xrightarrow{\sim} H^j_c(Y_t; \mathbb{C})\) is identity for any \(j > n - 1\). In particular, for any \(\lambda \neq 1\) and \(t \in \mathbb{C}^*\) such that \(0 < |t| \ll 1\) we have the concentration

$$H^j_c(Y_t; \mathbb{C})_\lambda \simeq 0 \quad (j \neq n - 1).$$
Proof. Since $Y_t \subset T_0$ is smooth and affine, the first assertion follows from the (generalized) Poincaré duality theorem

$$H^j_c(Y_t; \mathbb{C}) \simeq |H^{2n-2-j}(Y_t; \mathbb{C})|^* \quad (j \in \mathbb{Z}).$$

Moreover by the above lemma, the second assertion follows from the weak Lefschetz theorem (see Danilov-Khovanskii [4] Proposition 3.9) for the Newton non-degenerate hypersurface $Y_t \subset T_0 = (\mathbb{C}^*)^n$. Note also that the morphism

$$\psi_t(j_t^* R\pi_t|_{\mathbb{C}_T}) \longrightarrow \psi_t(j_t^* R\pi_t|_{\mathbb{C}_Y})$$

induced by the one $\mathbb{C}_T \rightarrow \mathbb{C}_Y$ is compatible with the monodromy automorphisms on $\psi_t(j_t^* R\pi_t|_{\mathbb{C}_T})$ and $\psi_t(j_t^* R\pi_t|_{\mathbb{C}_Y})$. Then the remaining assertion follows immediately from the previous ones.

Now let $X$ be a general variety and for some $\varepsilon > 0$ consider a family $Y \subset B(0; \varepsilon)^* \times X$ of subvarieties of $X$ over the punctured disk $B(0; \varepsilon)^* \subset \mathbb{C}$. Let $\pi : B(0; \varepsilon)^* \times X \to B(0; \varepsilon)^*$ be the projection and for $t \in \mathbb{C}$ such that $0 < |t| < \varepsilon$ set $Y_t = Y \cap \pi^{-1}(t) \subset X$. Then we obtain the monodromy automorphisms

$$\Phi_j : H^j_c(Y_t; \mathbb{C}) \simto H^j_c(Y_t; \mathbb{C}) \quad (j \in \mathbb{Z})$$

for $0 < |t| < \varepsilon$. For $\lambda \in \mathbb{C}$ let

$$H^j_c(Y_t; \mathbb{C})_\lambda \subset H^j_c(Y_t; \mathbb{C})$$

be the generalized eigenspace of $\Phi_j$ for the eigenvalue $\lambda$. Then we have an isomorphism

$$H^j_c\psi_{t,\lambda}(j_t^* R\pi_t|_{\mathbb{C}_Y}) \simeq H^j_c(Y_t; \mathbb{C})_\lambda$$

for any $j \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$. Considering the mixed Hodge module over the left hand side $H^j_c\psi_{t,\lambda}(j_t^* R\pi_t|_{\mathbb{C}_Y})$ we can endow the right hand side $H^j_c(Y_t; \mathbb{C})_\lambda$ with a mixed Hodge structure which coincides with the classical limit mixed Hodge structure (see El Zein [10] and Steenbrink-Zucker [37] etc.). The weight filtration $M_{\bullet}$ on it is the “relative” monodromy filtration with respect to its Deligne’s weight filtration $W_{\bullet}$ in the following sense. Let $\Phi_j^*$ be the unipotent part of the monodromy $\Phi_j$ and set $N = \log \Phi_j^* : H^j_c(Y_t; \mathbb{C}) \to H^j_c(Y_t; \mathbb{C})$. Then for any $r \in \mathbb{Z}$ the filtration $M(r)_{\bullet}$ on the graded piece $V_r = G^W_r H^j_c(Y_t; \mathbb{C})$ induced by $M_{\bullet}$ and the morphism $N(r) : V_r \to V_r$ induced by $N$ give rise to isomorphisms

$$N(r)^k : G^M(r+k)_{r+k} V_r \simto G^M(r)_r V_r \quad (k \geq 0).$$

Namely the filtration $M(r)_{\bullet}$ on $V_r$ is the monodromy filtration of the automorphism $G^W_r(\Phi_j) : V_r \simto V_r$ centered at $r$. Deligne proved that such a filtration $M_{\bullet}$ on $H^j_c(Y_t; \mathbb{C})$ is unique (if it exists). For $\lambda \in \mathbb{C}$ and $p, q, r \in \mathbb{Z}$ let $h^{p,q}(G^W_r H^j_c(Y_t; \mathbb{C})_\lambda) \geq 0$ be the dimension of the $(p, q)$-part of the above limit mixed Hodge structure on the graded piece $G^W_r H^j_c(Y_t; \mathbb{C})_\lambda$ defined by the weight filtration $M(r)_{\bullet}$.

**Definition 3.8.** (Stapledon [35]) For $\lambda \in \mathbb{C}$ we define the equivariant refined limit mixed Hodge polynomial (resp. the equivariant limit mixed Hodge polynomial) $E_{\lambda}(Y_t; u, v, w) \in \mathbb{Z}[u, v, w]$ (resp. $E_{\lambda}(Y_t; u, v, w) \in \mathbb{Z}[u, v]$) for the eigenvalue $\lambda \in \mathbb{C}$ by

$$E_{\lambda}(Y_t; u, v, w) = \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}(G^W_r H^j_c(Y_t; \mathbb{C})_\lambda) \, u^p v^q w^r,$$

$$E_{\lambda}(Y_t; u, v) = \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}(H^j_c(Y_t; \mathbb{C})_\lambda) \, u^p v^q.$$
By this definition, obviously we have $E_{\lambda}(Y_i; u, v, 1) = E_{\lambda}(Y_i; u, v)$ for any $\lambda \in \mathbb{C}$.

**Lemma 3.9.** Let $Z \subset Y \subset B(0; \varepsilon)^* \times X$ be a subfamily of $Y$. Then for any $\lambda \in \mathbb{C}$ we have

$$E_{\lambda}(Y_i; u, v, w) = E_{\lambda}(Z_i; u, v, w) + E_{\lambda}((Y \setminus Z)_i; u, v, w) \quad (0 < |t| < \varepsilon).$$

**Proof.** There exists a long exact sequence

$$\cdots \to H^j R\pi_{\gamma}Q_Y^H \to H^j R\pi_{\gamma}Q_Y^H \to H^j R\pi_{\gamma}Q_Z^H \to H^{j+1} R\pi_{\gamma}Q_Y^H \to \cdots$$

of mixed Hodge modules. For any $r \in \mathbb{Z}$ by taking the $r$-th graded piece $Gr^r_F (\cdot)$ of each term in it, we obtain again a long exact sequence. Then the assertion follows by applying the (exact) nearby cycle functor $\psi_t(\cdot)$ of mixed Hodge modules to them.

Now let us return to the family $Y \subset B(0; \varepsilon)^* \times (\mathbb{C}^*)^n$ of hypersurfaces of $T_0 = (\mathbb{C}^*)^n$ over the punctured disk $B(0; \varepsilon)^*$. Then by Proposition 2.3 and Theorem 3.2 we obtain the following corollary.

**Corollary 3.10.** Assume that the family $Y \subset B(0; \varepsilon)^* \times (\mathbb{C}^*)^n$ is schön. Then we have

$$E_{\lambda}(Y_i; u, v) = \sum_{\text{rel. int } F \subset \text{Int } F} E_{\lambda}(V_F \otimes \hat{\mu}; u, v) \cdot (1 - uv)^{n - \text{dim } F},$$

for any $\lambda \in \mathbb{C}$. Here the equivariant mixed Hodge polynomials $E_{\lambda}(V_F \otimes \hat{\mu}; u, v) \in \mathbb{Z}[u, v]$ are defined by Deligne’s mixed Hodge structure of the variety $V_F$ and the semisimple action on its cohomology groups as

$$E_{\lambda}(V_F \otimes \hat{\mu}; u, v) = \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}(H^j_{\lambda}(V_F; \mathbb{C})) u^p v^q.$$

The following fundamental result was obtained by Stapledon in [35]. For $\lambda \in \mathbb{C}$ set

$$\varepsilon(\lambda) = \begin{cases} 1 & (\lambda = 1) \\ 0 & (\lambda \neq 1) \end{cases}$$

and recall that we have

$$h^*_{\lambda}(P, \nu_F; u, v, w) = \sum_{Q < P} w^{\text{dim } Q + 1} h^*_{\lambda}(Q, \nu_F|Q; u, v) \cdot g([Q, P]; uvw^2). \quad (3.5)$$

**Theorem 3.11.** ([35, Theorem 5.7]) Assume that the family $Y$ is schön. Then for any $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ and $\lambda \in \mathbb{C}$ we have

$$uvw^2 E_{\lambda}(Y_i; u, v, w) = \varepsilon(\lambda) \cdot (uvw^2 - 1)^n + (-1)^{n-1} h^*_{\lambda}(P, \nu_F; u, v, w). \quad (3.6)$$

In [17] and [35] Katz and Stapledon proved this theorem from Corollary 3.10 by using [22, Section 2] and some deep results on combinatorics obtained by Stanley [34]. For a cell $F \in \mathcal{S}$ by using the affine linear extension $\nu_F: \text{Aff}(F) \simeq \mathbb{R}^{\text{dim } F} \to \mathbb{R}$ of $\nu|_F: F \to \mathbb{R}$ we set

$$m_F = [\nu_F(\text{Aff}(F) \cap \mathbb{Z}^n) : \nu_F(\text{Aff}(F) \cap \mathbb{Z}^n) \cap \mathbb{Z}] > 0.$$

17
Then we define a finite subset \( R_f \subset \mathbb{C} \) by
\[
R_f = \bigcup_{F \subset \partial P} \{ \lambda \in \mathbb{C} \mid \lambda^{m_F} = 1 \} \subset \mathbb{C}.
\]
Note that we have \( 1 \in R_f \). Then by Theorem 3.11 and Proposition 3.7 we immediately obtain the following result. Note that by the definition of the integers \( m_F \) the condition \( \lambda \notin R_f \) implies the vanishing \( l^*_t(Q, \nu_f|Q; u, v) = 0 \) for any proper face \( Q \neq P \) of \( P \).

**Theorem 3.12.** Assume that the family \( Y \) is schön. Then for any \( \lambda \notin R_f \) and \( t \in \mathbb{C}^* \) such that \( 0 < |t| \ll 1 \) the equivariant mixed Hodge polynomial \( E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w] \) for the eigenvalue \( \lambda \) is concentrated in degree \( n - 1 \) in the variable \( w \) and given by
\[
E_\lambda(Y_t; u, v, w) = (-1)^{n-1}w^{n-1} \sum_{p,q} h^{p,q}(H^{n-1}_c(Y_t; \mathbb{C}) \lambda) u^p v^q
\]
\[
= (-1)^{n-1}w^{n-1} \sum_{P \in \mathcal{S}} \sum_{t \geq 0} \sum_{k \geq 0} h^{p,q}(H^{n-1}_c(Y_t; \mathbb{C}) \lambda) t^k \cdot l_t(P, F; u, v). \]

In particular, by setting \( u = v = s \) and \( w = 1 \) we have
\[
E_\lambda(Y_t; s, s) = (-1)^{n-1} \sum_{k \geq 0} \left( \sum_{p+q=k} h^{p,q}(H^{n-1}_c(Y_t; \mathbb{C}) \lambda) \right) s^k
\]
\[
= (-1)^{n-1} \frac{1}{s^2} \sum_{F \in \mathcal{S}} \sum_{t \geq 0} s^{\dim F + 1} \sum_{k \geq 0} h^{p,q}(H^{n-1}_c(Y_t; \mathbb{C}) \lambda) t^k \cdot l_t(P, F; s^2).
\]

At the end of this section, by using mixed Hodge modules we will give a geometric proof to this concentration in degree \( n - 1 \). We also obtain the following corollary. Note that for \( \lambda \notin R_f \) by Theorem 3.12 and the construction of the weight filtration of the limit mixed Hodge structure of \( H^{n-1}_c(Y_t; \mathbb{C}) \) the filtration on \( H^{n-1}_c(Y_t; \mathbb{C}) \lambda \) induced by it is equal to the monodromy filtration for the monodromy \( \Phi_{n-1} : H^{n-1}_c(Y_t; \mathbb{C}) \lambda \rightarrow H^{n-1}_c(Y_t; \mathbb{C}) \lambda \) centered at \( n - 1 \).

**Corollary 3.13.** Assume that the family \( Y \subset B(0; \varepsilon)^* \times (\mathbb{C}^*)^n \) is schön. Then for any \( \lambda \notin R_f \) and \( t \in \mathbb{C} \) such that \( 0 < |t| \ll 1 \) we have the symmetry
\[
\sum_{p+q=n-1+k} h^{p,q}(H^{n-1}_c(Y_t; \mathbb{C}) \lambda) = \sum_{p+q=n-1-k} h^{p,q}(H^{n-1}_c(Y_t; \mathbb{C}) \lambda)
\]
for any \( k \geq 0 \).

By Theorem 3.12 and Corollary 3.13 for any \( \lambda \notin R_f \) the Jordan normal form of the middle-dimensional monodromy
\[
\Phi_{n-1} : H^{n-1}_c(Y_t; \mathbb{C}) \lambda \sim H^{n-1}_c(Y_t; \mathbb{C}) \lambda
\]
on \( H^{n-1}_c(Y_t; \mathbb{C}) \lambda \) can be recovered from the polynomial \( E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w] \) as follows. By Theorem 3.12 for the polynomial \( E_\lambda(Y_t; u, v) = E_\lambda(Y_t; u, v, 1) \in \mathbb{Z}[u, v] \) we have
\[
E_\lambda(Y_t; u, v, w) = E_\lambda(Y_t; u, v) \cdot w^{n-1}.
\]
Moreover the polynomial 
\[ \tilde{E}_\lambda(Y_t; s) := (-1)^{n-1}E_\lambda(Y_t; s, s) \in \mathbb{Z}[s] \] has only non-negative coefficients and the symmetry centered at \( n - 1 \). By the Lefschetz decomposition of 
\[ H^{n-1}_c(Y_t; \mathbb{C})_\lambda \] there exist non-negative integers \( q_{\lambda,i} \geq 0 \) (\( 0 \leq i \leq n - 1 \)) such that

\[ \tilde{E}_\lambda(Y_t; s) = q_{\lambda,0}(1 + s^2 + \cdots + s^{2n-4} + s^{2n-2}) + q_{\lambda,1}(s + s^3 + \cdots + s^{2n-3}) + q_{\lambda,2}(s^2 + \cdots + s^{2n-4}) + \cdots + q_{\lambda,n-1}s^{n-1}. \]

For \( \lambda \in \mathbb{C} \) and \( m \geq 1 \) denote by \( J_{\lambda,m} \) the number of the Jordan blocks in the monodromy automorphism

\[ \Phi_{n-1} : H^{n-1}_c(Y_t; \mathbb{C}) \xrightarrow{\sim} H^{n-1}_c(Y_t; \mathbb{C}) \quad (0 < |t| \ll 1) \]

for the eigenvalue \( \lambda \) with size \( m \).

**Proposition 3.14.** Assume that the family \( Y \) is schön. Then for any \( \lambda \notin R_f \) we have

\[ J_{\lambda,m} = q_{\lambda,n-m} \quad (1 \leq m \leq n). \]

Recall that for a cell \( F \in S \) the local \( h \)-polynomial \( l_P(S, F; t) \in \mathbb{Z}[t] \) has non-negative coefficients and the symmetry

\[ l_P(S, F; t) = t^{n-\dim F}l_P(S, F; t^{-1}) \]

(see [35, Remark 4.9]). Moreover it is unimodal. Hence there exist non-negative integers \( l_{F,i} \) (\( 0 \leq i \leq \lfloor \frac{n-\dim F}{2} \rfloor \)) such that

\[ l_P(S, F; t) = l_{F,0}(1 + t + t^2 + \cdots + t^{n-\dim F}) + l_{F,1}(t + t^2 + \cdots + t^{n-\dim F-1}) + l_{F,2}(t^2 + \cdots + t^{n-\dim F-2}) + \cdots. \]

We set

\[ \tilde{l}_P(S, F; t) = \sum_{i=0}^{\lfloor \frac{n-\dim F}{2} \rfloor} l_{F,i}t^i. \]

Then by Theorem 3.12 and Proposition 3.14 we obtain the following result.

**Theorem 3.15.** Assume that the family \( Y \) is schön. Then for \( \lambda \notin R_f \) we have

\[ \sum_{m=0}^{n-1} J_{\lambda,n-m} s^{m+2} = \sum_{F \in S} s^{\dim F+1}l_\lambda^*(F, \nu_f|F; 1) \cdot \tilde{l}_P(S, F; s^2). \]

In particular, we have

\[ J_{\lambda,n} = \sum_{F \in S, \dim F=1} l_\lambda^*(F, \nu_f|F; 1) \cdot l_{F,0}. \]
The multiplicities of the eigenvalues \( \lambda \neq 1 \) in the middle-dimensional monodromy \( \Phi_{n-1} \) are described more simply as follows.

**Theorem 3.16.** Assume that the family \( Y \) is schön. Then for \( \lambda \neq 1 \) the multiplicity of the factor \( t - \lambda \) in the characteristic polynomial of the monodromy
\[
\Phi_{n-1} : H^{n-1}_c(Y; \mathbb{C}) \to H^{n-1}_c(Y; \mathbb{C}) \quad (0 < |t| \ll 1)
\]
is equal to that in
\[
\prod_{\text{rel.int } F \subset \text{Int } F, \dim F = n} (t^{m_F} - 1)^{\text{Vol}_n(F)},
\]
where \( \text{Vol}_n(F) \in \mathbb{Z}_{>0} \) is the normalized volume i.e. the \( n! \) times usual volume \( \text{Vol}(F) \) of \( F \) with respect to the lattice \( \text{Aff}(F) \cap \mathbb{Z}^n \cong \mathbb{Z}^n \) in \( \text{Aff}(F) \cong \mathbb{R}^n \).

**Proof.** By Proposition \[\text{3.7}\] and the proof of Theorem \[\text{3.2}\] the assertion can be proved by calculating monodromy zeta functions as in \[\text{[21]}\]. We can obtain it also just by taking the Euler characteristics of the both sides of the equality in Theorem \[\text{3.2}\].

Now let us give a geometric proof to the concentration in Theorem \[\text{3.12}\].

**Theorem 3.17.** Assume that the family \( Y \) is schön. Then for any \( \lambda \notin R_f \) the morphism
\[
\psi_{\lambda, \lambda}(j_Y R_{\pi_1} \mathbb{C}_Y) \to \psi_{\lambda, \lambda}(j_Y R_{\pi_2} \mathbb{C}_Y)
\]
induced by the one \( R_{\pi_1} \mathbb{C}_Y \to R_{\pi_2} \mathbb{C}_Y \) is an isomorphism.

**Proof.** The proof is similar to that of \[\text{[38, Theorem 4.5]}\]. We shall use the notations \( \Sigma, \Xi, X_\Sigma, \pi_\Sigma : X_\Sigma \to \mathbb{C}, \overline{Y}, \pi_T : \overline{Y} \to \mathbb{C} \) etc. in the proof of Theorem \[\text{3.2}\]. By the schönness of \( Y \) the hypersurface \( \overline{Y} \subset X_\Sigma \) intersects \( T \)-orbits in \( \pi_\Sigma^{-1}(\{0\}) \) transversally. Recall that for any cone \( \sigma \in \Xi \) the hypersurface \( \overline{Y} \) intersects the \( T \)-orbit \( T_\sigma \) associated to it transversally on a neighborhood of \( \pi_\Sigma^{-1}(\{0\}) \) in \( X_\Sigma \). Recall also that the divisor \( D = \overline{Y} \setminus Y = \overline{Y} \cap (X_\Sigma \setminus T) \) in \( \overline{Y} \) is normal crossing there. Let \( i_D : D \hookrightarrow \overline{Y} \) and \( j_Y : Y \hookrightarrow \overline{Y} \) be the inclusion maps. Then there exists a commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{j_Y} & \overline{Y} \\
\downarrow_{\pi_Y = \pi|_Y} & & \downarrow_{\pi_T} \\
\mathbb{C}^* & \xrightarrow{i_D} & \mathbb{C}.
\end{array}
\]
Since \( \pi_T : \overline{Y} \to \mathbb{C} \) is proper, we have isomorphisms
\[
\left\{
\begin{array}{ll}
\psi_l(j_Y R_{\pi_1} \mathbb{C}_Y) \simeq \psi_l(R(\pi_T)_* (j_Y)_! \mathbb{C}_Y), \\
\psi_l(j_Y R_{\pi_2} \mathbb{C}_Y) \simeq \psi_l(R(j_Y)_! R(\pi_Y)_* \mathbb{C}_Y) \simeq \psi_l(R(\pi_T)_* (R(j_Y)_! \mathbb{C}_Y)).
\end{array}
\right.
\]
Therefore, by applying the functor \( \psi_l \circ R(\pi_T)_* : \mathbf{D}^b(\overline{Y}) \to \mathbf{D}^b(\{0\}) \) to the distinguished triangle
\[
(j_Y)_! \mathbb{C}_Y \to R(j_Y)_* \mathbb{C}_Y \to (i_D)_* (i_D)^{-1} R(j_Y)_* \mathbb{C}_Y \to 1
\]
we obtain the new one
\[
\psi_l(j_Y R_{\pi_1} \mathbb{C}_Y) \to \psi_l(j_Y R_{\pi_2} \mathbb{C}_Y) \to \psi_l(R(\pi_T)_* (i_D)_* (i_D)^{-1} R(j_Y)_* \mathbb{C}_Y) \to 1.
\]
This implies that for the proof of the theorem it suffices to prove the vanishing
\[ \psi_{l,\lambda}(R(\pi_Y)_*(i_D)_*(i_D)^{-1}R(j_Y)_*C_Y) \simeq 0 \]
for any \( \lambda \notin R_f \). Since \( \pi_Y : \overline{Y} \to \mathbb{C} \) is proper, by [8] Proposition 4.2.11 and [16] Exercise VIII.15 for any \( \lambda \in \mathbb{C} \) we have an isomorphism
\[ \psi_{l,\lambda}(R(\pi_Y)_*(i_D)_*(i_D)^{-1}R(j_Y)_*C_Y) \simeq R\Gamma(\pi_Y^{-1}D_{\{0\}} ; \psi_{\pi_Y,\lambda}( (i_D)_*(i_D)^{-1}R(j_Y)_*C_Y))). \]
Now let us set
\[ D' = D \setminus \pi_Y^{-1}(\{0\}) = \bigcup_{\sigma \in \Xi} (\overline{Y} \cap \overline{T_\sigma}) \subset D \]
and \( Y' = \overline{Y} \setminus D' \supset Y \). Then \( D' \) is a normal crossing divisor of \( \overline{Y} \) on a neighborhood of \( \pi_Y^{-1}(\{0\}) \) and for the inclusion maps \( i_{D'} : D' \hookrightarrow \overline{Y} \) and \( j_{Y'} : Y' \hookrightarrow \overline{Y} \) there exists an isomorphism
\[ \psi_{\pi_Y,\lambda}( (i_D)_*(i_D)^{-1}R(j_Y)_*C_Y) \simeq \psi_{\pi_Y,\lambda}( (i_{D'})_*(i_{D'})^{-1}R(j_{Y'})_*C_{Y'}). \]
Note that for the natural stratification of the normal crossing divisor \( D' \subset \overline{Y} \) by the strata
\[ D'_\sigma := \overline{Y} \cap (\overline{T_\sigma} \setminus \bigcup_{\tau \in \Xi \setminus \{0\}} \overline{T_\tau}) \subset D' \ (\sigma \in \Xi \setminus \{0\}) \]
the cohomology sheaves \( H^j(i_{D'})^{-1}R(j_{Y'})_*C_{Y'} \ (j \in \mathbb{Z}) \) are constructible. Moreover their restrictions to each stratum \( D'_\sigma \subset D' \) are constant. Then by cutting the support of the complex \( (i_{D'})^{-1}R(j_{Y'})_*C_{Y'} \in \mathbf{D}^b_c(D') \) by the stratification and truncating each of the resulting complexes, it suffices to prove the vanishing
\[ R\Gamma(\pi_Y^{-1}(\{0\}) ; \psi_{\pi_Y,\lambda}(C_{\overline{Y} \cap \overline{T_\sigma}})) \simeq 0. \]
for any \( \lambda \notin R_f \) and any cone \( \sigma \in \Xi \setminus \{0\} \). Fixing such \( \lambda \) and \( \sigma \) we shall prove the vanishing from now. Set \( \pi_\sigma = \pi_Y|_{\overline{Y} \cap \overline{T_\sigma}} : \overline{Y} \cap \overline{T_\sigma} \to \mathbb{C} \) and \( F_\sigma = C_{\overline{Y} \cap \overline{T_\sigma}} \). Then it is enough to show the vanishing
\[ R\Gamma(\pi_\sigma^{-1}(\{0\}) ; \psi_{\pi_\sigma,\lambda}(F_\sigma)) \simeq 0. \]
According to the general theory of mixed Hodge modules (for the details see e.g. Dimca-Saito [9] Section 1.4 and Denef-Loeser [6] etc.), each graded piece of the nearby cycle perverse sheaf \( \psi_{\pi_\sigma,\lambda}(F_\sigma)[n - \dim \sigma - 1] \in \mathbf{D}^b_c(\pi_\sigma^{-1}(\{0\})) \) with respect to its weight filtration has a primitive decomposition into some minimal extension perverse sheaves of \( L_\tau[n - \dim \tau] \in \mathbf{D}^b_c(\overline{Y} \cap T_\tau) \), where \( \tau \) is a cone in \( \Sigma \setminus \Xi \) such that \( \sigma \prec \tau \) and \( L_\tau \) is a rank one local system on the \( (n - \dim \tau) \)-dimensional smooth hypersurface \( \overline{Y} \cap T_\tau \subset T_\tau \simeq (\mathbb{C}^*)^{n - \dim \tau + 1} \). Moreover the cones \( \tau \) appearing in this decomposition should satisfy the following condition. Let \( \tau \) be a cone in \( \Sigma \setminus \Xi \) such that \( \sigma \prec \tau \) and let \( \alpha_1, \ldots, \alpha_k \in \tau \cap (\mathbb{Z}^{n+1} \setminus \{0\}) \) be the primitive vectors on the edges of \( \tau \) not contained in the hyperplane \( \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1} \). Set \( b_i = q(\alpha_i) > 0 \ (1 \leq i \leq k) \) and \( d_\tau = \gcd\{b_i \mid 1 \leq i \leq k\} > 0 \). Then, for any \( 1 \leq i \leq k \) the order of the zero of the function \( \pi_\sigma \) along the divisor of \( \overline{Y} \cap \overline{T_\sigma} \) associated to the ray containing \( \alpha_i \) is equal to \( b_i > 0 \). Moreover for the cone \( \tau \) to appear in the decomposition it should satisfy the condition \( \lambda^{d_\tau} = 1 \). In such a case, the rank one local system \( L_\tau \) on \( \overline{Y} \cap T_\tau \) has the following condition. Let \( \rho \in \Sigma \setminus \Xi \) be a ray
such that \( \tau \cap \rho = \{0\} \) and \( \tau(\rho) := \tau + \rho \) is a cone in \( \Sigma \). Let \( \beta \in \rho \cap (\mathbb{Z}^{n+1} \setminus \{0\}) \) be the primitive vector on it. Then for any such \( \rho \) the monodromy of the local system \( L_\tau \) around the divisor \( \overline{Y} \cap \overline{T_{\tau(\rho)}} \subset \overline{Y} \cap \overline{T_\tau} \) is given by the multiplication by the complex number \( \lambda^{-q(\beta)} \in \mathbb{C} \). By cutting the supports of the minimal extension perverse sheaves by the toric stratifications of \( \overline{T_\tau} \), for the proof of the theorem it suffices to prove the vanishing

\[
R \Gamma_c(\overline{Y} \cap \overline{T_\tau}; L_\tau) \cong 0
\]

for any cone \( \tau \) in \( \Sigma \setminus \Xi \) such that \( \sigma \prec \tau \) and \( \lambda^{d_\tau} = 1 \). For a cell \( F \) in \( \mathcal{S} \) let \( \tilde{F} \prec UH_f \) be the unique compact face of \( UH_f \) such that \( F = p(\tilde{F}) \) and \( F^0 \in \Sigma_0 \) the cone which corresponds to it in the dual fan \( \Sigma_0 \). Then for the cone \( \tau \) there exists a unique cell \( F \in \mathcal{S} \) such that \( F \subset \partial P \) and \( \text{rel} \text{.int}(\tau) \subset \text{rel} \text{.int}(F^0) \). Let \( \tau' \) be a cone in \( \Sigma \setminus \Xi \) such that \( \tau' \subset F^0 \), \( \dim \tau' = \dim F^0 \) and \( \tau \prec \tau' \). Then by the smoothness of the cone \( \tau' \) we can easily show the equality \( m_F = d_{\tau'} \). By our assumption \( \lambda \notin R_f \) we thus obtain \( \lambda^{d_{\tau'}} \neq 1 \). This implies that there exists a ray \( \rho \) of \( \tau' \) such that \( \tau \cap \rho = \{0\} \) and the primitive vector \( \beta \in \rho \cap (\mathbb{Z}^{n+1} \setminus \{0\}) \) on it satisfies the condition \( \lambda^{-q(\beta)} \neq 1 \). Moreover, since the supporting faces of \( \tau \) and \( \tau' \) in \( UH_f \) coincide and are equal to \( \tilde{F} \prec UH_f \), we have a product decomposition

\[
\overline{Y} \cap \overline{T_\tau} \cong Z \times (\mathbb{C}^*)^k
\]

for some variety \( Z \) and \( k > 0 \) such that the equation of the divisor \( \overline{T_{\tau(\rho)}} \subset \overline{T_\tau} \) corresponds to a coordinate of the torus \( (\mathbb{C}^*)^k \). Now the desired vanishing follows from the Künneth formula. This completes the proof. \( \square \)

By Theorem 3.17 we can reprove the concentration in Theorem 3.12 as follows.

**Corollary 3.18.** Assume that the family \( Y \) is schön. Then for any \( \lambda \notin R_f \) and \( t \in \mathbb{C}^* \) such that \( 0 < |t| \ll 1 \) we have the concentration

\[
H^j_c(Y_t; \mathbb{C})_{\lambda} \simeq 0 \quad (j \neq n-1).
\]

Moreover the equivariant mixed Hodge polynomial \( E_\lambda(Y_t; u,v,w) \in \mathbb{Z}[u,v,w] \) for the eigenvalue \( \lambda \) is concentrated in degree \( n-1 \) in the variable \( w \).

**Proof.** Assume that \( \lambda \notin R_f \). Then the first assertion is already shown in Proposition 3.7. However we shall give a new proof to it by using Theorem 3.17. By Theorem 3.17 for \( t \in \mathbb{C} \) such that \( 0 < |t| \ll 1 \) there exist isomorphisms

\[
H^j_c(Y_t; \mathbb{C})_{\lambda} \cong H^j(Y_t; \mathbb{C})_{\lambda} \quad (j \in \mathbb{Z}).
\]

Since the \((n-1)\)-dimensional variety \( Y_t \) is affine and smooth, by the (generalized) Poincaré duality the left (resp. right) hand side vanishes for \( j < n-1 \) (resp. \( j > n-1 \)). We thus obtain the concentration

\[
H^j_c(Y_t; \mathbb{C})_{\lambda} \simeq 0 \quad (j \neq n-1).
\]

By applying the proof of Sabbah [31, Theorem 13.1] to the above isomorphisms we see that the only non-trivial cohomology group \( H^{n-1}_c(Y_t; \mathbb{C})_{\lambda} \) has a pure weight \( n-1 \). \( \square \)
4 Monodromies and limit mixed Hodge structures of families of hypersurfaces in $\mathbb{C}^n$

Let $f(t,x) = \sum_{v \in \mathbb{Z}^n_+} a_v(t)x^v \in \mathbb{K}[x_1,\ldots,x_n]$ be a polynomial of $x = (x_1,\ldots,x_n)$ over the field $\mathbb{K} = \mathbb{C}(t)$ of rational functions of $t$. Then as in Section 2 we can define an (unbounded) polyhedron $UH_f$ associated to it in $\mathbb{R}^n_+ \times \mathbb{R}^3$ and its projection $P = p(UH_f) \subset \mathbb{R}^n_+$. We call $P$ the Newton polytope of $f \in \mathbb{K}[x_1,\ldots,x_n]$. Throughout this section we assume that $\dim P = n$.

**Definition 4.1.** (Stapledon [35]) We say that $P$ is convenient if for any coordinate subspace $H$ of $\mathbb{R}^n$ we have $\dim P \cap H = \dim H$.

By this definition, for a convenient polytope $P$ we have $0 \in P$. Let $\Sigma_0$ be the dual fan of $UH_f$ in $\mathbb{R}^n_+ \times \mathbb{R}^1_+ \subset \mathbb{R}^{n+1}$ and $\nu_f : P \to \mathbb{R}$ the function defining the bottom part of the boundary $\partial UH_f$ of $UH_f$. Moreover by the subdivision $\mathcal{S}$ of $P$ into the lattice polytopes $p(F)$ ($\widetilde{F} \subset UH_f$) we define polynomials $I^j_f(x) \in \mathbb{C}[x_1,\ldots,x_n]$ and elements $[V_F \cap \mu] \in \mathcal{M}_C^\mu$ for cells $F \in \mathcal{S}$ as in Section 3. In this situation, the hypersurface $f^{-1}(0) \subset X = C^*_t \times C^\mu_n$ defines a family $Y$ of hypersurfaces of $X_0 = C^\mu_n$ over a small punctured disk $B(0;\varepsilon)^* (0 < \varepsilon \ll 1)$. By the projection $\pi : C^*_t \times C^\mu_n \to C^*_t$ for $t \in \mathbb{C}$ such that $0 < |t| < \varepsilon$ we set $Y_t := \pi^{-1}(t) \cap Y \subset \{t\} \times X_0 \cong X_0 = C^\mu_n$. We define also the schönness of the family as in Definition 3.1. Then by the proof of the weak Lefschetz theorem (see Danilov-Khovanskii [4] Proposition 3.9) we can prove the following proposition.

**Proposition 4.2.** Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön and $P$ is convenient. Then for $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ we have

$$H^j_c(Y_t; \mathbb{C}) \simeq 0 \quad (j < n - 1)$$

and the Gysin map

$$H^j_c(Y_t; \mathbb{C}) \to H^{j+2}_c(X_0; \mathbb{C})$$

associated to the inclusion map $Y_t \hookrightarrow X_0$ is an isomorphism (resp. surjective) for $j > n - 1$ (resp. $j = n - 1$). Moreover the monodromy $\Phi_j : H^j_c(Y_t; \mathbb{C}) \to H^j_c(Y_t; \mathbb{C})$ is identity for any $j > n - 1$.

We set $P_\infty := \partial P \cap \mathbb{R}^n_+ \subset \partial P$.

**Definition 4.3.** (Stapledon [35]) We say that the polynomial $f(t,x) = \sum_{v \in \mathbb{Z}^n_+} a_v(t)x^v \in \mathbb{K}[x_1,\ldots,x_n]$ satisfies the condition (S) if the function $\nu_f : P \to \mathbb{R}$ is constant on $P_\infty \subset \partial P$.

The following result was proved in Stapledon [35] by calculating the related equivariant mixed Hodge polynomials $E_\lambda(Y_t;u,v,w) \in \mathbb{Z}[u,v,w]$ very precisely.

**Theorem 4.4.** (Stapledon [35] Corollary 6.3) Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön, $P$ is convenient and $f(t,x) \in \mathbb{K}[x_1,\ldots,x_n]$ satisfies the condition (S). Then for any $\lambda \neq 1$ the equivariant refined limit mixed Hodge polynomial $E_\lambda(Y_t;u,v,w) \in \mathbb{Z}[u,v,w]$ for the eigenvalue $\lambda$ is concentrated in degree $n - 1$ in the variable $w$. 

23
We can drop the condition (S) and the one on $P$ in Theorem 4.4 as follows. We define a finite subset $R_f \subset \mathbb{C}$ by

$$R_f = \bigcup_{F \subset P_\infty} \{ \lambda \in \mathbb{C} \mid \lambda^{mf} = 1 \} \subset \mathbb{C}.$$ 

Then we have the following result.

**Theorem 4.5.** Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön. Then for $\lambda \notin R_f$ and $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ the equivariant refined limit mixed Hodge polynomial $E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w]$ for the eigenvalue $\lambda$ is concentrated in degree $n - 1$ in the variable $w$ and given by

$$E_\lambda(Y_t; u, v, w) = (-1)^{n-1}w^{n-1}\sum_{p,q} h^{p,q}(H^{n-1}_\lambda(Y_t; \mathbb{C})u^pv^q = \sum_{p,q} h^{p,q}(P, \nu_f|u, v)$$

$$= (-1)^{n-1}w^{n-1}\sum_{P \in S} v^{\dim F + 1}l^*_\lambda(F, \nu_f|F; uv^{-1}) \cdot l_P(S, F; uv).$$

**Proof.** For a possibly empty subset $I \subset \{1, \ldots, n\}$, we define a subset $T^I$ of $X_0 = \mathbb{C}^n$ by

$$T^I := \{(x_1, \ldots, x_n) \in X_0 \mid x_i = 0 \ (i \notin I), x_i \neq 0 \ (i \in I) \} \approx (\mathbb{C}^*)^{|I|}.$$ 

Then we have a decomposition $X_0 = \mathbb{C}^n = \bigcup_{I \subset \{1, \ldots, n\}} T^I$ of $X_0 = \mathbb{C}^n$. We also define a polynomial $f_t \in \mathbb{K}[(x_i)_{i \in I}]$ by substituting $0$ into the variable $x_i \ (i \notin I)$ of $f$, a family of hypersurfaces $Y^I$ of $T^I$ by $Y^I := f_t^{-1}(0) \subset B^* \times T^I$ and a polytope $P^I$ in $\mathbb{R}^I = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0 \ (i \notin I) \} \subset \mathbb{R}^n$ by $P^I := P \cap \mathbb{R}^I$. Then by Lemma 3.9 we have

$$E_\lambda(Y_t; u, v, w) = \sum_{I \subset \{1, \ldots, n\}} E_\lambda(Y^I_t; u, v, w)$$

for $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$. We shall say that a face $Q \prec P$ of $P$ is relevant if $Q \subset P$. If $Q \prec P$ is relevant, then for any face $\sigma$ of the first quadrant $\mathbb{R}^d_+$ containing $Q$ the face $\sigma \cap P \prec P$ of $P$ is also relevant. Moreover there exist a possibly empty subset $I \subset \{1, \ldots, n\}$ such that $Q = P^I$ and $\dim P^I = |I|$. We denote by $S$ the set consisting of possibly empty subsets $I \subset \{1, \ldots, n\}$ such that $P^I$ are relevant. Then by Theorem 3.11 for $\lambda \notin R_f$ we have

$$uvw^2E_\lambda(Y_t; u, v, w) = \sum_{I \subset S} (-1)^{|I|-1}h^*_\lambda(P^I, \nu_f|P^I; u, v, w).$$

Moreover for each relevant face $P^I \prec P$ of $P$ by Definition 2.7 we have

$$h^*_\lambda(P^I, \nu_f|P^I; u, v, w) = \sum_{Q \prec P^I} w^{\dim Q + 1}l^*_\lambda(Q, \nu_f|Q; u, v) \cdot g([Q, P^I]; uvw^2). \quad (4.1)$$

If $Q \prec P^I$ is not a relevant face of $P$, then $Q \subset P_\infty$ and for $\lambda \notin R_f$ we have $l^*_\lambda(Q, \nu_f|Q; u, v) = 0$. Note also that $l^*_\lambda(\emptyset, \nu_f|\emptyset; u, v) = 0$ for $\lambda \neq 1$. Moreover for any
I, I' ∈ S such that I' ⊂ I we have $g([P', P]; uvw^2) = 1$. Hence for each fixed I' ∈ S we have
\[
\sum_{I : I' \subset I} (-1)^{|I| - 1} g([P', P]; uvw^2) = \sum_{I : I' \subset I} (-1)^{|I| - 1} = \begin{cases} (-1)^{n-1} & (I' = \{1, \ldots, n\}) \\ 0 & \text{(otherwise)}. \end{cases}
\]

We thus obtain
\[
uvw^2 E_\lambda(Y; u, v, w) = \sum_{I \in S} \left( -1 \right)^{|I| - 1} \sum_{I' \in S : I' \subset I} w^{|I'| + 1} \sum_{\lambda \in \Sigma' \cap I} \left( P', \nu_f |_{\lambda'} \right) \cdot g([P', P]; uvw^2) = \sum_{I \in S} w^{|I'| + 1} \lambda \left( P', \nu_f \right) \cdot \left( \sum_{I : I' \subset I} (-1)^{|I| - 1} g([P', P]; uvw^2) \right) = (-1)^{n-1} w^{n+1} \lambda \left( P, \nu_f \right).
\]

We shall say that a face $\sigma < \mathbb{R}^n_+$ of the first quadrant $\mathbb{R}^n_+$ is relevant if the condition $(P \setminus P_\infty) \cap \sigma \neq \emptyset$ is satisfied. It is easy to see that if $\sigma < \mathbb{R}^n_+$ is relevant then we have $\dim(P \cap \sigma) = \dim \sigma$. Let $\Sigma_1$ be the fan in $\mathbb{R}^n$ consisting of all the faces of $\mathbb{R}^n_+$ and regard it as the dual fan of the first quadrant $\mathbb{R}^n_+$. Denote by $\Sigma_1 \subset \Sigma_1$ its subset consisting of the dual cones of the relevant faces of $\mathbb{R}^n_+$. Then we can easily see that $\Sigma_1$ is a subfan of $\Sigma_1$. Denote by $\Omega_0$ the toric variety associated to $\Sigma_1$. Then $\Omega_0$ is an open subset of $X_0 = \mathbb{C}^n$ and $X_0 \setminus \Omega_0$ is a closed subset in it. Moreover for the action of $T_0 = (\mathbb{C}^*)^n$ on $X_0 = \mathbb{C}^n$ it is a union of some $T_0$-orbits. Set $Y^0 = Y \cap (\mathbb{C}^* \times \Omega_0) \subset \mathbb{C}^* \times \Omega_0$ and let $\pi^0 : \mathbb{C}^* \times \Omega_0 \to \mathbb{C}^*$ be the projection.

**Theorem 4.6.** Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön. Then for any $\lambda \notin R_f$ the morphism
\[
\psi_{t, \lambda}(j_! R\pi_! \mathcal{C}_{Y^0}) \to \psi_{t, \lambda}(j_! R\pi_! \mathcal{C}_Y)
\]
induced by the one $\mathcal{C}_{Y^0} \to \mathcal{C}_Y$ is an isomorphism. Moreover for such $\lambda$ the morphism
\[
\psi_{t, \lambda}(j_! R(\pi^0)_! \mathcal{C}_{Y^0}) \to \psi_{t, \lambda}(j_! R(\pi^0)_* \mathcal{C}_{Y^0})
\]
induced by the one $R(\pi^0)_! \mathcal{C}_{Y^0} \to R(\pi^0)_* \mathcal{C}_{Y^0}$ is an isomorphism.

**Proof.** The proof is similar to that of Theorem 3.17. By decomposing the normal crossing divisor $X_0 \setminus \Omega_0$ into tori and applying Proposition 3.7 and Theorem 3.12 to each of them, for $\lambda \notin R_f$ we obtain the vanishing
\[
\psi_{t, \lambda}(j_! R\pi_! \mathcal{C}_{Y^0}) \simeq 0
\]
from which the first assertion follows. Let $\Xi_0$ be the subfan of the dual fan $\Sigma_0$ in $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}$ consisting of the cones $\sigma \in \Sigma_0$ contained in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^n+1$. Then $\Xi_0$ is the dual fan of the $n$-dimensional polytope $P \subset \mathbb{R}^n$. Moreover by the definition of $\Sigma_0$ we can easily see that $\Sigma_0$ is a subfan of $\Xi_0$. By this property we can construct a smooth subdivision $\Sigma$ of $\Sigma_0$ such that $\Sigma_0 \subset \Sigma$. Then the toric variety $X_\Sigma$ associated to $\Sigma$ is a smooth variety containing $\mathbb{C}^* \times \Omega_0$ and the second assertion can be proved as in the proof of Theorem 3.17. \[\square\]
By Theorem 4.6 we can drop the condition on $P$ in Proposition 4.2 as follows. For $\lambda \in \mathbb{C}$ let
\[ H^j(Y_\lambda; \mathbb{C}) \subset H^j(Y_\lambda; \mathbb{C}) \]
be the generalized eigenspace of $\Phi$ for the eigenvalue $\lambda$. Similarly for $Y^\circ \subset Y$ we define linear subspaces
\[ H^j(Y_\lambda^\circ; \mathbb{C}) \subset H^j(Y_\lambda^\circ; \mathbb{C}) \quad (\lambda \in \mathbb{C}). \]
Then by Theorem 4.6 for any $\lambda \notin R_f$ there exist isomorphisms
\[ H^j_c(Y_\lambda^\circ; \mathbb{C}) \simeq H^j_c(Y_\lambda; \mathbb{C}) \quad (j \in \mathbb{Z}). \]

**Corollary 4.7.** Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön. Then for any $\lambda \notin R_f$ and $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ we have the concentration
\[ H^j_c(Y_\lambda; \mathbb{C}) \simeq 0 \quad (j \neq n - 1) \]
and the filtration on the only non-trivial cohomology group $H^{n-1}_c(Y_\lambda; \mathbb{C})$ induced by Deligne’s weight filtration on $H^{n-1}_c(Y_\lambda; \mathbb{C})$ is concentrated in degree $n - 1$.

**Proof.** For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ let
\[ H^j(Y_\lambda^\circ; \mathbb{C}) \subset H^j(Y_\lambda^\circ; \mathbb{C}) \]
be the generalized eigenspace of the monodromy $H^j(Y_\lambda^\circ; \mathbb{C}) \sim H^j(Y_\lambda^\circ; \mathbb{C})$. Then by Theorem 4.6 for any $\lambda \notin R_f$ and $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ we have isomorphisms
\[ H^j_c(Y_\lambda^\circ; \mathbb{C}) \simeq H^j_c(Y_\lambda; \mathbb{C}) \quad (j \in \mathbb{Z}). \]

By the proof of Sabbah [31, Theorem 13.1] this implies that for any $j \in \mathbb{Z}$ the filtration on $H^j(Y_\lambda^\circ; \mathbb{C})$ induced by Deligne’s weight filtration of $H^j_c(Y_\lambda^\circ; \mathbb{C})$ is concentrated in degree $j$. Then the assertion follows immediately from Theorem 4.5.

**Remark 4.8.** By the proofs of Theorems 3.17 and 4.6 if the family $Y$ is schön we can also show that for any $\lambda \notin R_f$ and $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ there exist isomorphisms
\[ H^j(Y_\lambda; \mathbb{C}) \simeq H^j(Y_\lambda^\circ; \mathbb{C}) \quad (j \in \mathbb{Z}). \]
Since the proof is similar, we omit the details.

By Corollary 4.7 we can prove the following formula for the multiplicities of the eigenvalues $\lambda \notin R_f$ in the monodromy $\Phi_{n-1}$ by calculating monodromy zeta functions as in [21]. For a cell $F \in S$ denote by $Q_F < P$ the unique face of $P$ such that $\text{rel.int} F \subset \text{rel.int} Q_F$.

**Theorem 4.9.** Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön. Then for $\lambda \notin R_f$ the multiplicity of the factor $t - \lambda$ in the characteristic polynomial of the monodromy
\[ \Phi_{n-1} : H^{n-1}_c(Y_\lambda; \mathbb{C}) \sim H^{n-1}_c(Y_\lambda; \mathbb{C}) \quad (0 < |t| \ll 1) \]
is equal to that in
\[ \prod_{F \notin P_\infty, \dim F = \dim Q_F} (t^{m_F} - 1)(-1)^{n-\dim F} \text{Vol}_2(\tilde{F}), \]
where $\text{Vol}_2(\tilde{F}) \in \mathbb{Z}_{>0}$ is the normalized volume of $\tilde{F}$ with respect to the lattice $\text{Aff}(\tilde{F}) \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^{\dim F}$ in $\text{Aff}(\tilde{F}) \simeq \mathbb{R}^{\dim F}$.
Moreover by Theorems 3.15 and 4.7 and Corollary 4.4 we can easily obtain results similar to the ones in Corollary 3.13, Proposition 3.14 and Theorem 3.15. In particular as in Theorem 3.15 by Corollary 4.7 and Theorem 4.5 we can describe the numbers $J_{\lambda,m}$ of the Jordan blocks in the middle-dimensional monodromy

$$
\Phi_{n-1} : H^{n-1}(Y; \mathbb{C}) \to H^{n-1}(Y; \mathbb{C}) \quad (0 < |t| \ll 1)
$$

for the eigenvalues $\lambda \notin R_f$ with size $m \geq 0$ in terms of $U_H f$. See Theorem 1.3

5 Monodromies and limit mixed Hodge structures of families of complete intersection varieties

In this section, we extend our previous results to families complete intersection subvarieties in $(\mathbb{C}^*)^n$ or $\mathbb{C}^n$. Throughout this section, for $1 \leq k \leq n$ let $f_i(t, x)$ ($1 \leq i \leq k$) be Laurent polynomials $f_i(t, x) = \sum_{v \in \mathbb{Z}^n} a_{i,v}(t) x^v \in \mathbb{K}[x_1, \ldots, x_n]$ or polynomials $f_i(t, x) = \sum_{v \in \mathbb{Z}^n} a_{i,v}(t) x^v \in \mathbb{K}[x_1, \ldots, x_n]$ over the field $\mathbb{K} = \mathbb{C}(t)$. Then the subvariety $f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ in $T = \mathbb{C}_t^* \times (\mathbb{C}^*)^n$ or $X = \mathbb{C}_t^* \times \mathbb{C}_x^n$ defines a family $Y$ of subvarieties of $T_0 = (\mathbb{C}^*)^n$ or $X_0 = \mathbb{C}^n$ over a small punctured disk $B(0; \varepsilon)^* \subset \mathbb{C}$ ($0 < \varepsilon \ll 1$).

We shall describe its monodromy and limit mixed Hodge structure. As in Section 3 we define $U_H f_i \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}_x$ and their second projections $P_i = p(U_H f_i) \subset \mathbb{R}^n$. Set $f = (f_1, \ldots, f_k)$ and let

$$U_H := U_H f_1 + \cdots + U_H f_k \subset \mathbb{R}^{n+1}_x$$

be the Minkowski sum of $U_H f_1, \ldots, U_H f_k$. Set $P = p(U_H f) = P_1 + \cdots + P_k \subset \mathbb{R}^n$. Throughout this section we assume that $\dim P = n$. By using $U_H f$, we define a function $\nu f : P \to \mathbb{R}$, a subdivision $\mathcal{S}$ of $P$ into lattice polytopes and a closed subset $P_\infty \subset P$ as in Sections 3 and 4. Moreover for each cell $F \in \mathcal{S}$ and $1 \leq i \leq k$ the initial Laurent polynomial $f_i^T(x) \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ of $f_i$ with respect to $F$ is defined.

Definition 5.1. (Stapledon [35]) We say that the family $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ of subvarieties of $T_0 = (\mathbb{C}^*)^n$ or $X_0 = \mathbb{C}^n$ is schön if for any $J \subset \{1, \ldots, k\}$ and any cell $F \in \mathcal{S}$ the subvariety $V_F = \bigcap_{j \in J} \{ f_j^T = 0 \} \subset T_F$ of $T_F \simeq (\mathbb{C}^*)^{\dim F}$ is a non-degenerate complete intersection (see [25]).

It follows easily from the proof of Theorems 3.11 and 4.17 that if the family $Y$ in $T_0 = (\mathbb{C}^*)^n$ is schön its generic fiber $Y_t = Y \cap \pi^{-1}(t) \subset T_0$ ($0 < |t| \ll 1$) is a smooth complete intersection. Moreover by Danilov-Khovanskii [4, Theorem 6.4] we obtain the following results. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ let

$$H^j(Y_t; \mathbb{C}) \subset H^j(Y_t; \mathbb{C})$$

be the generalized eigenspace of the monodromy automorphism

$$\Phi_j : H^j(Y_t; \mathbb{C}) \to H^j(Y_t; \mathbb{C})$$

for the eigenvalue $\lambda$. 

27
Proposition 5.2. Let $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ be a family of subvarieties in $T_0 = (\mathbb{C}^*)^n$ (resp. $X_0 = \mathbb{C}^n$). Assume that $Y$ is schön and $\dim P_i = n$ (resp. $P_i$ is convenient) for any $1 \leq i \leq k$. Then for $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ we have
\[ H^j_c(Y; \mathbb{C}) \simeq 0 \quad (j < n - k) \]
and the Gysin map
\[ H^j_c(Y; \mathbb{C}) \longrightarrow H^{j+2k}_c(T_0; \mathbb{C}) \]  
(resp. $H^j_c(Y; \mathbb{C}) \longrightarrow H^{j+2k}_c(X_0; \mathbb{C})$)  
(5.1)
(5.2)
associated to the inclusion map $Y_i \hookrightarrow T_0$ (resp. $Y_i \hookrightarrow X_0$) is an isomorphism for $j > n - k$ and surjective for $j = n - k$. Moreover the monodromy $\Phi_j: H^j_c(Y; \mathbb{C}) \rightarrow H^j_c(Y; \mathbb{C})$ is identity for $j > n - k$. In particular, for any $\lambda \neq 1$ and $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ we have the concentration
\[ H^j_c(Y; \mathbb{C})_{\lambda} \simeq 0 \quad (j \neq n - k). \]

Defining a finite subset $R_f \subset \mathbb{C}$ by using $\partial P, P_\infty \subset P$ as in Sections 3 and 4 we obtain the following results. In the case where $Y$ is family of subvarieties in $X_0 = \mathbb{C}^n$ we define $\Omega_0 \subset \mathbb{C}^n, Y^0 \subset Y$ and the projection $\pi^0: \mathbb{C}^* \times \Omega_0 \rightarrow \mathbb{C}^*$ as in Section 4.

Theorem 5.3. Let $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ be a family of subvarieties in $T_0 = (\mathbb{C}^*)^n$. Assume that $Y$ is schön. Then for any $\lambda \notin R_f$ the morphism
\[ \psi_{t,\lambda}(j_!R\pi_!\mathbb{C}_Y) \longrightarrow \psi_{t,\lambda}(j_!R\pi_!\mathbb{C}_Y) \]
induced by the one $R\pi_!\mathbb{C}_Y \rightarrow R\pi_*\mathbb{C}_Y$ is an isomorphism.

Corollary 5.4. In the situation of Theorem 5.3, for any $\lambda \notin R_f$ and $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ we have the concentration
\[ H^j_c(Y; \mathbb{C})_{\lambda} \simeq 0 \quad (j \neq n - k) \]
and the filtration on the only non-trivial cohomology group $H^{n-k}_c(Y; \mathbb{C})_{\lambda}$ induced by Deligne’s weight filtration on $H^{n-k}_c(Y; \mathbb{C})$ is concentrated in degree $n - k$.

Theorem 5.5. Let $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ be a family of subvarieties in $X_0 = \mathbb{C}^n$. Assume that $Y$ is schön. Then for any $\lambda \notin R_f$ the morphism
\[ \psi_{t,\lambda}(j_!R\pi_!\mathbb{C}_{Y^0}) \longrightarrow \psi_{t,\lambda}(j_!R\pi_!\mathbb{C}_{Y^0}) \]
induced by the one $\mathbb{C}_{Y^0} \rightarrow \mathbb{C}_Y$ is an isomorphism. Moreover for such $\lambda$ the morphism
\[ \psi_{t,\lambda}(j_!R(\pi^0)_!\mathbb{C}_{Y^0}) \longrightarrow \psi_{t,\lambda}(j_!R(\pi^0)_!\mathbb{C}_{Y^0}) \]
induced by the one $R(\pi^0)_!\mathbb{C}_{Y^0} \rightarrow R(\pi^0)_!\mathbb{C}_{Y^0}$ is an isomorphism.

We have also a generalization of Remark 1.8. By Corollary 5.4 and Bernstein-Khovanski-Kushnirenko’s theorem, in the case where $Y$ is family of subvarieties in $T_0 = (\mathbb{C}^*)^n$ we obtain the following formula for the multiplicities of the eigenvalues $\lambda \notin R_f$ in the middle-dimensional monodromy $\Phi_{n-k}: H^{n-k}_c(Y; \mathbb{C}) \rightarrow H^{n-k}_c(Y; \mathbb{C})$ by calculating monodromy zeta functions as in [21].

28
**Definition 5.6.** Let $\Delta_1, \ldots, \Delta_n$ be lattice polytopes in $\mathbb{R}^n$. Then we define their normalized $(n$-dimensional) mixed volume $\Vol_{Z}(\Delta_1, \ldots, \Delta_n) \in \mathbb{Z}$ by the formula

$$\Vol_{Z}(\Delta_1, \ldots, \Delta_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \sum_{I \subseteq \{1, \ldots, n\}} \Vol_{Z} \left( \sum_{i \in I} \Delta_i \right)$$

(5.3)

where $\Vol_{Z}(\cdot) = n! \Vol(\cdot) \in \mathbb{Z}$ is the normalized $(n$-dimensional) volume with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Let $\Sigma_0$ be the dual fan of $\UH_j$ in $\mathbb{R}^{n+1}$. For a cell $F$ in $\mathcal{S}$ let $\tilde{F} \prec \UH_j$ be the unique compact face of $\UH_j$ such that $F = \pi(\tilde{F})$ and $F^\circ \in \Sigma_0$ the cone which corresponds to it in the dual fan $\Sigma_0$. Then for the supporting faces $\tilde{F}_i \prec \UH_j$ of $F^\circ$ in $\UH_j$ ($1 \leq i \leq k$) we have $\tilde{F}_1 + \cdots + \tilde{F}_k = \tilde{F}$.

**Theorem 5.7.** Assume that the family $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ of subvarieties in $T_0 = (\mathbb{C}^*)^n$ is schön. Then for $\lambda \notin R_\ell$ the multiplicity of the factor $t - \lambda$ in the characteristic polynomial of the middle-dimensional monodromy

$$\Phi_{n-k}: H_{e}^{n-k}(Y_t; \mathbb{C}) \xrightarrow{\sim} H_{e}^{n-k}(Y_t; \mathbb{C}) \quad (0 < |t| \ll 1)$$

is equal to that in

$$\prod_{\text{rel.int} F \subset \int P, \dim F = n} (t^{m_F} - 1)^{K_F},$$

where we set

$$K_F = \sum_{m_1, \ldots, m_k \geq 1 \atop m_1 + \cdots + m_k = \dim F} \Vol_{Z}(\tilde{F}_1, \ldots, \tilde{F}_1, \ldots, \tilde{F}_k, \ldots, \tilde{F}_k)$$

by using the normalized mixed volumes with respect to the lattice $\text{Aff}(\tilde{F}) \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^{\dim F}$ in $\text{Aff}(\tilde{F}) \simeq \mathbb{R}^{\dim F}$.

For each subset $J \subset \{1, \ldots, k\}$, set $\mathbb{R}^J := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_j = 0 (j \notin J)\} \simeq \mathbb{R}^{|J|}$ and

$$U_J := \text{Conv}\left( \bigcup_{j \in J} \{e_j\} \times \UH_j \right) \subset \mathbb{R}^J \times \mathbb{R}^{n+1},$$

$$P_J := \text{Conv}\left( \bigcup_{j \in J} \{e_j\} \times P_j \right) \subset \mathbb{R}^J \times \mathbb{R}^n,$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^J$ is the standard vector. Obviously, the image of $U_J$ by the projection $p_J: \mathbb{R}^J \times \mathbb{R}^n \times \mathbb{R}_s \to \mathbb{R}^J \times \mathbb{R}^n$ is $P_J$. We write $\tilde{p}$, $\tilde{U}$ and $\tilde{P}$ for $p_{\{1, \ldots, k\}}$, $U_{\{1, \ldots, k\}}$ and $P_{\{1, \ldots, k\}}$, respectively. Let $\tilde{\nu}: \tilde{P} \to \mathbb{R}$ be the function defining the bottom part of the boundary $\partial \tilde{U}$ of $\tilde{U}$ and $\tilde{\mathcal{S}}$ the subdivision of $\tilde{P}$ by the lattice polytopes $\tilde{p}(\tilde{F}) \subset \tilde{P}$ ($\tilde{F} \prec \tilde{U}$). By the assumption that the dimension of $P$ is $n$, we have $\dim \tilde{U} = n+k$ and $\dim \tilde{P} = n+k-1$. We obtain the following generalization of Theorem 3.11 to families of complete intersection subvarieties of $T_0 = (\mathbb{C}^*)^n$. Recall that for $\lambda \in \mathbb{C}$ we set

$$\varepsilon(\lambda) = \begin{cases} 1 & (\lambda = 1) \\ 0 & (\lambda \neq 1). \end{cases}$$

29
Theorem 5.8. Let \( Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) \) be a family of subvarieties of \( T_0 = (\mathbb{C}^*)^n \). Assume that \( Y \) is schön. Then for any \( t \in \mathbb{C}^* \) such that \( 0 < |t| \ll 1 \) and \( \lambda \in \mathbb{C} \) the equivariant refined limit mixed Hodge polynomial \( E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w] \) for the eigenvalue \( \lambda \) is given by

\[
(uvw^2)^k E_\lambda(Y_t; u, v, w) = \varepsilon(\lambda) \cdot (uvw^2 - 1)^n + \sum_{\emptyset \neq J \subseteq \{1, \ldots, k\}} \alpha J \dim P_J - 1 (uvw^2 - 1)^n |J| - 1 - \dim P_J \cdot h_\lambda^k(P_J; \nu|_{P_J}; u, v, w).
\]

Proof. We prove the assertion only in the case where \( \lambda \neq 1 \). The proofs for the other cases are similar. We use the Cayley trick of Danilov-Khovanskii [4] in its refined form of [11]. For sufficiently small \( \varepsilon > 0 \) we set \( B^* = B(0, \varepsilon)^* \subset \mathbb{C} \). Then we have \( Y \subset B^* \times T_0 \). Moreover we set

\[
\Omega := \{(t, (x_1, \ldots, x_n), [\alpha_1 : \cdots : \alpha_k]) \in B^* \times T_0 \times \mathbb{P}^{k-1} | \sum_{i=1}^{k} \alpha_i f_i(t, x) \neq 0 \}.
\]

Then there exists a projection \( \Omega \rightarrow (B^* \times T_0) \setminus Y \) which is a locally trivial fibration with fiber \( \mathbb{C}^{k-1} \). Hence it follows from Lemma [3.9] that for \( t \in B^* \) and any \( \lambda \neq 1 \) we have

\[
E_\lambda(\Omega_t; u, v, w) = (uvw^2)^{k-1} E_\lambda(\{(t) \times T_0 \} \setminus Y_t; u, v, w)
\]

\[
= (uvw^2)^{k-1} (E_\lambda(\{(t) \times T_0; u, v, w) - E_\lambda(Y_t; u, v, w))
\]

\[
= - (uvw^2)^{k-1} E_\lambda(Y_t; u, v, w).
\]

For each non-empty subset \( J \subset \{1, \ldots, k\} \) we define a subset \( T_J \simeq (\mathbb{C}^*)^{|J|-1} \) of \( \mathbb{P}^{k-1} \) by

\[
T_J := \{[\alpha_1 : \cdots : \alpha_k] \in \mathbb{P}^{k-1} | \alpha_j \neq 0 (j \in J), \alpha_j = 0 (j \notin J)\} \simeq (\mathbb{C}^*)^{|J|-1}.
\]

Moreover we set

\[
\Omega_J := \Omega \cap (B^* \times T_0 \times T_J),
\]

\[
Y_J := (B^* \times T_0 \times T_J) \setminus \Omega_J
\]

\[
= \{(t, (x_1, \ldots, x_n), [\alpha_1 : \cdots : \alpha_k]) \in B^* \times T_0 \times T_J | \sum_{j \in J} \alpha_j f_j(t, x) = 0 \}.
\]

Then for \( t \in B^* \) and \( \lambda \neq 1 \) we have a decomposition \( \Omega_t = \bigsqcup_{J \neq \emptyset} \Omega_{J,t} \) and hence

\[
E_\lambda(\Omega_t; u, v, w) = \sum_{J \neq \emptyset} E_\lambda(\Omega_{J,t}; u, v, w)
\]

\[
= \sum_{J \neq \emptyset} (E_\lambda(\{t\} \times T_0 \times T_J; u, v, w) - E_\lambda(Y_{J,t}; u, v, w))
\]

\[
= - \sum_{J \neq \emptyset} E_\lambda(Y_{J,t}; u, v, w).
\]

We thus obtain the equality

\[
(uvw^2)^{k-1} E_\lambda(Y_t; u, v, w) = \sum_{J \neq \emptyset} E_\lambda(Y_{J,t}; u, v, w).
\]
It is easy to check that $Y_J$ is schön. Note that $U_J$ is $\mathrm{UH}_{\sum_{j \in J} a_j f_j}$. Therefore applying Theorem \ref{thm:intersection_subvarieties} to the families $Y_J \subset B^* \times T_0 \times T_J$ we obtain

$$(uvw^2) E_\lambda(Y_J; u, v, w) = (-1)^{\dim P_J - 1}(uvw^2 - 1)^{n + |J| - 1 - \dim P_J} \cdot h_\lambda(P_J, \widetilde{v}|_{T_J}; u, v, w).$$

Now the assertion follows immediately. \hfill $\square$

**Corollary 5.9.** In the situation of Theorem \ref{thm:corollary_5.9} assume also that $\dim P_i = n$ for any $1 \leq i \leq k$. Then for any $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ and $\lambda \in \mathbb{C}$ we have

$$(uvw^2)^k E_\lambda(Y_i; u, v, w) = \varepsilon(\lambda) \cdot (uvw^2 - 1)^n + \sum_{\emptyset \neq J \subset \{1, \ldots, k\}} (-1)^{n + |J|} h_\lambda(P_J, \widetilde{v}|_{T_J}; u, v, w).$$

From now on, we consider only families $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ of subvarieties of $X_0 = \mathbb{C}^n$. The corresponding results for families of subvarieties of $T_0 = (\mathbb{C}^*)^n$ can be obtained similarly. For a cell $F \in \tilde{S}$ let $\widetilde{v}_F : \text{Aff}(F) \simeq \mathbb{R}^{\dim F} \to \mathbb{R}$ be the (affine) linear extension of $v|_F : F \to \mathbb{R}$ and set

$$\widetilde{m}_F = [\widetilde{v}_F(\text{Aff}(F) \cap \mathbb{Z}^{n+k}) : \widetilde{v}_F(\text{Aff}(F) \cap \mathbb{Z}^{n+k}) \cap \mathbb{Z}] > 0.$$ 

Let $\Delta$ be the convex hull of the points $e_1, \ldots, e_k$ in $\mathbb{R}^k$. Then $\Delta$ is a $(k - 1)$-dimensional lattice simplex and we have $\tilde{P} \subset \Delta \times \mathbb{R}^n_+$. Now let us set

$$\tilde{P}_\infty := \{\text{Int}(\Delta) \times \text{Int}(\mathbb{R}_+^n)\} \cap \partial \tilde{P} \subset \partial \tilde{P}.$$ 

Then we define a finite subset $\tilde{R}_f \subset \mathbb{C}$ by

$$\tilde{R}_f = \bigcup_{F \subset \tilde{P}_\infty} \{\lambda \in \mathbb{C} \mid \lambda \widetilde{m}_F = 1\} \subset \mathbb{C}.$$ 

**Lemma 5.10.** We have $R_f = \tilde{R}_f$.

**Proof.** We shall say that a face $Q \prec \tilde{P}$ is a side face of $\tilde{P}$ if its image by the projection $r : \Delta \times \mathbb{R}^n_+ \to \Delta$ is equal to $\Delta$. Note also that the inverse image of the barycenter of $\Delta$ by the map $r|_\widetilde{P} : \tilde{P} \to \Delta$ is similar to the Minkowski sum $P = P_1 + \cdots + P_k$. Hence there exists a natural bijection between the set of the side faces of $\tilde{P}$ and that of the faces of $P$. Moreover for any cell $F \in \tilde{S}$ in $\tilde{P}_\infty$ there exist another cell $F' \in \tilde{S}$ in $\tilde{P}_\infty$ and a side face $Q \prec \tilde{P}$ of $\tilde{P}$ such that $F \prec F'$ and rel.int$F' \subset$ rel.int$Q$. Then we have $\widetilde{m}_F|_{\widetilde{m}_F}$. This implies that for the definition of $\tilde{R}_f$ it suffices to consider only cells $F \in \tilde{S}$ in $\tilde{P}_\infty$ whose relative interiors are contained in those of side faces of $\tilde{P}$. In fact, there exists also a natural bijection between the set of such cells $F \in \tilde{S}$ and that of the cells $G \in S$ in $P_\infty$. For the cell $F \in \tilde{S}$ in $\tilde{P}_\infty$ let $F_{\text{red}} \in S$ be the corresponding cell in $P_\infty$. Then it is easy to show that $\widetilde{m}_F = m_{F_{\text{red}}}$. We thus obtain the equality $R_f = \tilde{R}_f$. \hfill $\square$

Now we have the following generalization of Theorem \ref{thm:relation} to families of complete intersection subvarieties of $\mathbb{C}^n$. 

31
Theorem 5.11. Let \( Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) \) be a family of subvarieties of \( X_0 = \mathbb{C}^n \). Assume that \( Y \) is schön. Then for any \( \lambda \notin R_f = \widetilde{R_f} \) and \( t \in \mathbb{C}^* \) such that \( 0 < |t| \ll 1 \) the equivariant refined limit mixed Hodge polynomial \( E_\lambda(Y_i; u, v, w) \in \mathbb{Z}[u, v, w] \) for the eigenvalue \( \lambda \) is concentrated in degree \( n - k \) in the variable \( w \) and given by

\[
E_\lambda(Y_i; u, v, w) = (-1)^{n-k} w^{n-k} \sum_{p,q} h^{p,q}(H^{n-k}_c(Y_i; \mathbb{C})_{\lambda}) u^p v^q
\]

\[
= (-1)^{n-k} \frac{w^{n-k}}{u^k v^k} \prod_{\lambda} (\tilde{P}, \tilde{\nu}; u, v)
\]

\[
= (-1)^{n-k} \frac{w^{n-k}}{u^k v^k} \sum_{F \in \mathcal{S}} v^{\dim F + 1} l_\lambda^*(F, \tilde{\nu}|_F; uv^{-1}) \cdot l_{\tilde{P}}(\mathcal{S}, F; uv).
\]

In particular, by setting \( u = v = s \) and \( w = 1 \) we have

\[
E_\lambda(Y_i; s, s) = (-1)^{n-k} \sum_{m \geq 0} \left( \sum_{p+q=m} h^{p,q}(H^{n-k}_c(Y_i; \mathbb{C})_{\lambda}) \right) s^m
\]

\[
= (-1)^{n-k} \frac{1}{s^{2k}} \sum_{F \in \mathcal{S}} s^{\dim F + 1} l_\lambda^*(F, \tilde{\nu}|_F; 1) \cdot l_{\tilde{P}}(\mathcal{S}, F; s^2).
\]

Proof. For a possibly empty subset \( I \subseteq \{1, \ldots, n\} \), we define a subset \( I^f \) of \( X_0 = \mathbb{C}^n \) by

\[
I^f := \{(x_1, \ldots, x_n) \in X_0 \mid x_i = 0 \ (i \notin I), x_i \neq 0 \ (i \in I) \} \simeq (\mathbb{C}^*)^{|I|}.
\]

Then we have a decomposition \( X_0 = \mathbb{C}^n = \bigcup_{I \subseteq \{1, \ldots, n\}} I^f \) of \( X_0 = \mathbb{C}^n \). We also define polynomials \( f_j \in \mathbb{K}[(x_i)_{i \in I}] \) by substituting 0 into the variable \( x_i \ (i \notin I) \) of \( f_j \), a family of subvarieties \( Y^f \) of \( I^f \) by \( Y^f := f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) \subset B^* \times I^f \) and polytopes \( P_i^f \) in \( \mathbb{R}^I \) by \( P_i^f := P_j \cap \mathbb{R}^I \). We set \( P^f = P_1^f + \cdots + P_k^f = P \cap \mathbb{R}^I \). Then by Lemma 3.9 we have

\[
E_\lambda(Y_i; u, v, w) = \sum_{I \subseteq \{1, \ldots, n\}} E_\lambda(Y_i^f; u, v, w)
\]

for \( t \in \mathbb{C}^* \) such that \( 0 < |t| \ll 1 \). For each non-empty subset \( J \subseteq \{1, \ldots, k\} \) we define a polytope \( P_j^f \) in \( \mathbb{R}^J \times I^f \) by \( P_j^f := \text{Conv}(\bigcup_{j \in J} \{e_j\} \times P_j^f) \). We shall say that a face \( Q \prec \tilde{P} \) of \( \tilde{P} \) is relevant if \( Q \not\subset \tilde{P} \). If \( Q \prec \tilde{P} \) is relevant, then for any face \( \sigma \) of the polyhedron \( \Delta \times \mathbb{R}_+^I \) containing \( Q \) the face \( \sigma \cap \tilde{P} \prec \tilde{P} \) of \( \tilde{P} \) is also relevant. Moreover there exist a possibly empty subset \( I \subseteq \{1, \ldots, n\} \) and a non-empty one \( J \subseteq \{1, \ldots, k\} \) such that \( Q = P_j^f \) and \( \dim P_j^f = |I| + |J| - 1 \). For each \( I \subseteq \{1, \ldots, n\} \) denote by \( S^I \) the set consisting of non-empty subsets \( J \subseteq \{1, \ldots, k\} \) such that \( P_j^f \) is relevant. Then by Theorem 5.8 for \( \lambda \notin R_f = \widetilde{R_f} \) we have

\[
(uvw^2)^k E_\lambda(Y_i^f; u, v, w) = \sum_{J \in S^I} (-1)^{|I|+|J|} l_\lambda^*(P_j^f, \tilde{\nu}|_{P_j^f}; u, v, w).
\]

Moreover for each relevant face \( P_j^f \prec \tilde{P} \) of \( \tilde{P} \) by Definition 2.7 we have

\[
h_\lambda^*(P_j^f, \tilde{\nu}|_{P_j^f}; u, v, w) = \sum_{Q \prec P_j^f} w^{\dim Q + 1} l_\lambda^*(Q, \tilde{\nu}|_Q; u, v) \cdot g([Q, P_j^f]; uvw^2).
\]
Assume that the family $Y$ the Jordan blocks in the monodromy automorphism $\Phi$ each other. Eventually, we obtain the desired formula for the eigenvalue $\lambda$.

\[ (uvw^2)^k E_\lambda(Y; u, v, w) = (-1)^{n+k} w^{n+k} l_\lambda^*(\tilde{P}; \tilde{\nu}, u, v). \]

As in Corollary 4.7, by Theorems 5.5 and 5.11 we obtain the following result.

**Corollary 5.12.** In the situation of Theorem 5.11, for any $\lambda \notin R_f$ and $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ we have the concentration

\[ H^j_\lambda(Y; \mathbb{C}) \simeq 0 \quad (j \neq n-k) \]

and the filtration on the only non-trivial cohomology group $H^{n-k}_c(Y; \mathbb{C})_\lambda$ induced by Deligne’s weight filtration on $H^{n-k}_c(Y; \mathbb{C})$ is concentrated in degree $n-k$.

By Theorem 5.11 and Corollary 5.12, for any $\lambda \notin R_f$ we can describe the Jordan normal form of the middle-dimensional monodromy

\[ \Phi_{n-k} : H^{n-k}_c(Y; \mathbb{C})_\lambda \xrightarrow{\sim} H^{n-k}_c(Y; \mathbb{C})_\lambda \]

as in Theorem 3.15. Recall that the dimension of $\tilde{P}$ is $n+k-1$, and hence for a cell $F \in \tilde{S}$ the local $h$-polynomial $l_{\tilde{P}}(\tilde{S}, F; t) \in \mathbb{Z}[t]$ has non-negative coefficients and the symmetry

\[ l_{\tilde{P}}(\tilde{S}, F; t) = t^{n+k-1-\dim F} l_{\tilde{P}}(\tilde{S}, F; t^{-1}). \]

Moreover it is unimodal. Hence there exist non-negative integers $l_{F,i}$ ($0 \leq i \leq \lfloor \frac{n+k-1-\dim F}{2} \rfloor$) such that

\[ l_{\tilde{P}}(\tilde{S}, F; t) = l_{F,0}(1 + t + t^2 + \ldots + t^{n+k-1-\dim F}) + l_{F,1}(t + t^2 + \ldots + t^{n+k-1-\dim F - 1}) + l_{F,2}(t^2 + \ldots + t^{n+k-1-\dim F - 2}) + \ldots. \]

We set

\[ \tilde{l}_{\tilde{P}}(\tilde{S}, F; t) = \sum_{i=0}^{\lfloor \frac{n+k-1-\dim F}{2} \rfloor} l_{F,i} t^i. \]

**Theorem 5.13.** Let $Y = f^{-1}_1(0) \cap \cdots \cap f^{-1}_k(0)$ be a family of subvarieties of $X_0 = \mathbb{C}^n$. Assume that the family $Y$ is schön. For $\lambda \in \mathbb{C}$ and $m \geq 1$ denote by $J_{\lambda,m}$ the number of the Jordan blocks in the monodromy automorphism

\[ \Phi_{n-k} : H^{n-k}_c(Y; \mathbb{C}) \xrightarrow{\sim} H^{n-k}_c(Y; \mathbb{C}) \quad (0 < |t| \ll 1) \]

for the eigenvalue $\lambda$ with size $m$. Then for $\lambda \notin R_f$ we have

\[ \sum_{m=0}^{n-k} J_{\lambda,n-k+1-m} s^{m+2k} = \sum_{F \in \tilde{S}} s^{\dim F + 1} l_\lambda^*(F, \tilde{\nu}|_F; 1) \cdot \tilde{l}_{\tilde{P}}(\tilde{S}, F; s^2). \]
The multiplicities of the eigenvalues $\lambda \notin R_f$ in the monodromy $\Phi_{n-k}$ are described more simply as follows. For a cell $F \in S$ let $Q_F \prec P$ be the unique face of $P$ such that $\text{rel.int} F \subset \text{rel.int} Q_F$.

**Theorem 5.14.** In the situation of Theorem 5.13, for $\lambda \notin R_f$ the multiplicity of the factor $t - \lambda$ in the characteristic polynomial of the middle-dimensional monodromy

$$
\Phi_{n-k} : H^{n-k}_c(Y_t; \mathbb{C}) \sim H^{n-k}_c(Y_t; \mathbb{C}) \quad (0 < |t| \ll 1)
$$

is equal to that in

$$
\prod_{F \not\subset P_\infty, \dim F = \dim Q_F} (t^{m_F} - 1)^{(-1)^{n-\dim F}K_F},
$$

where we define the integers $K_F$ as in Theorem 5.7.

**References**

[1] Batyrev, V. and Borisov, L. “Mirror duality and string-theoretic Hodge numbers”, *Invent. Math.*, 126 (1996): 183-203.

[2] Borisov, L. and Mavlyutov, A. “String cohomology of Calabi-Yau hypersurfaces in mirror symmetry”, *Adv. in Math.*, 180 (2003): 335-390.

[3] Broughton, S. A. “Milnor numbers and the topology of polynomial hypersurfaces”, *Invent. Math.*, 92 (1988): 217-241.

[4] Danilov, V. I. and Khovanskii, A. G. “Newton polyhedra and an algorithm for computing Hodge-Deligne numbers”, *Math. Ussr Izvestiya*, 29 (1987): 279-298.

[5] Denef, J. and Loeser, F. “Weights of exponential sums, intersection cohomology, and Newton polyhedra”, *Invent. Math.*, 106 (1991): 275-294.

[6] Denef, J. and Loeser, F. “Motivic Igusa zeta functions”, *J. Alg. Geom.*, 7 (1998): 505-537.

[7] Denef, J. and Loeser, F. “Geometry on arc spaces of algebraic varieties”, *Progr. Math.*, 201 (2001): 327-348.

[8] Dimca, A. *Sheaves in topology*, Universitext, Springer-Verlag, Berlin, 2004.

[9] Dimca, A. and Saito, M. “Some consequences of perversity of vanishing cycles”, *Ann. Inst. Fourier*, 54 (2004): 1769-1792.

[10] El Zein, F. “Théorie de Hodge des cycles évanescent”, *Ann. Sci. École Norm. Sup.*, 19 (1986): 107-184.

[11] Esterov, A. and Takeuchi, K. “Motivic Milnor fibers over complete intersection varieties and their virtual Betti numbers”, *Int. Math. Res. Not.*, Vol. 2012, No. 15 (2012): 3567-3613.

[12] Fulton, W. *Introduction to toric varieties*, Princeton University Press, 1993.
[13] García López, R. and Némethi, A. “Hodge numbers attached to a polynomial map”, *Ann. Inst. Fourier*, 49 (1999): 1547-1579.

[14] Guibert, G., Loeser, F. and Merle, M. “Iterated vanishing cycles, convolution, and a motivic analogue of a conjecture of Steenbrink”, *Duke Math. J.*, 132 (2006): 409-457.

[15] Hotta, R., Takeuchi, K. and Tanisaki, T. *D-modules, perverse sheaves, and representation theory*, Birkhäuser Boston, 2008.

[16] Kashiwara, M. and Schapira, P. *Sheaves on manifolds*, Springer-Verlag, 1990.

[17] Katz, E. and Stapledon, A. “Local \(h\)-polynomials, invariants of subdivisions, and mixed Ehrhart theory”, *Adv. in Math.*, 286 (2016): 181-239.

[18] Katz, E. and Stapledon, A. “Tropical geometry, the motivic nearby fiber and limit mixed Hodge numbers of hypersurfaces”, *Res. Math. Sci.*, 3:10 (2016): 36pp.

[19] Kouchnirenko, A. G. “Polyédres de Newton et nombres de Milnor”, *Invent. Math.*, 32 (1976): 1-31.

[20] Libgober, A. and Sperber, S. “On the zeta function of monodromy of a polynomial map”, *Compositio Math.*, 95 (1995): 287-307.

[21] Matsui, Y. and Takeuchi, K. “Monodromy zeta functions at infinity, Newton polyhedra and constructible sheaves”, *Mathematische Zeitschrift*, 268 (2011): 409-439.

[22] Matsui, Y. and Takeuchi, K. “Monodromy at infinity of polynomial maps and Newton polyhedra, with Appendix by C. Sabbah”, *Int. Math. Res. Not.*, Vol. 2013, No. 8 (2013): 1691-1746.

[23] Matsui, Y. and Takeuchi, K. “Motivic Milnor fibers and Jordan normal forms of Milnor monodromies”, *Publ. Res. Inst. Math. Sci.*, 50 (2014): 207-226.

[24] Matsui, Y. and Takeuchi, K. “On the sizes of the Jordan blocks of monodromies at infinity”, *Hokkaido Math. Journal*, 44 (2015): 1-14.

[25] Oda, T. *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties*, Springer-Verlag, 1988.

[26] Oka, M. *Non-degenerate complete intersection singularity*, Hermann, Paris (1997).

[27] Raibaut, M. “Fibre de Milnor motivique à l’infini”, *C. R. Acad. Sci. Paris Sér. I Math.*, 348 (2010): 419-422.

[28] Raibaut, M. “Singularités à l’infini et intégration motivique”, *Bull. Soc. Math. France*, 140 (2012): 51-100.

[29] Richter-Gebert, J., Sturmfels, B. and Theobald, T. “First steps in tropical geometry”, Idempotent mathematics and mathematical physics, *Contemp. Math.*, 377 (2005): 289-317.
[30] Sabbah, C. “Monodromy at infinity and Fourier transform”, *Publ. Res. Inst. Math. Sci.*, 33 (1997): 643-685.

[31] Sabbah, C. “Hypergeometric periods for a tame polynomial”, *Port. Math.*, 63 (2006): 173-226.

[32] Saito, T. “On the mixed Hodge structures of the intersection cohomology stalks of complex hypersurfaces”, [arXiv:1602.02976](https://arxiv.org/abs/1602.02976).

[33] Siersma, D. and Tibăr, M. “Singularities at infinity and their vanishing cycles”, *Duke Math. J.*, 80 (1995): 771-783.

[34] Stanley, R. “Subdivisions and local h-vectors”, *J. Amer. Math. Soc.*, 5, no.4 (1992), 805-851.

[35] Stapledon, A. “Formulas for monodromy”, [arXiv:1405.5355](https://arxiv.org/abs/1405.5355).

[36] Steenbrink, J. “Motivic Milnor fibre for nondegenerate function germs on toric singularities”, [arXiv:1310.6914](https://arxiv.org/abs/1310.6914).

[37] Steenbrink, J. H. M. and Zucker, S. “Variation of mixed Hodge structure I”, *Invent. Math.*, 80 (1985): 489-542.

[38] Takeuchi, K. and Tibăr, M. “Monodromies at infinity of non-tame polynomials”, *Bull. Soc. Math. France*, 144, no.3 (2016), 477-506.

[39] Tibăr, M. “Topology at infinity of polynomial mappings and Thom regularity condition”, *Compositio Math.*, 111, no.1 (1998), 89-109.

[40] Varchenko, A. N. “Zeta-function of monodromy and Newton’s diagram”, *Invent. Math.*, 37 (1976): 253-262.