Algorithm for determination the type of bilateral motions of a carrier with a mobile load along a horizontal plane

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Abstract. The motion of a mechanical system consisting of a carrier and a load is considered. The carrier, located all the time on a horizontal plane, can move translationally along a rectilinear trajectory. The carrier has a rectilinear channel through which the load can move. The load is considered further on as a material point. The load can move in the channel according to a predetermined motion law. The channel axis is located in a vertical plane passing through the trajectory of the carrier. The Coulomb dry friction model is applied for simulation the forces of resistance to the motion of the carrier from the side of the underlying plane. In the conditions of the carrier motion along horizontal plane without detachment, the carrier motion differential equations are a system of three linear second-order differential equations. The influence of the system parameters on the motion of the carrier from the rest state is studied. In the considered mechanical system (carrier+load) only two types of motion are possible: regular motions in which the relative moments of time being the switching times from one of the three differential equations to the other are unchanged, and the irregular motions when such switching times are different in each of the half-periods of the load motion. Definitive expressions allowing determine the type of carrier motion for the studied systems are derived. An algorithm for classifying two-sided carrier motions for a predetermined load motion law in a channel is presented based on the values of the defining expressions. Examples of specific systems are considered and the results of computational experiments are presented.

1. Carrier motion differential equations

The motion of a mechanical system consisting of a carrier and a load is considered [1–5]. The carrier, located all the time on a horizontal plane, moves translationally along a rectilinear trajectory. The carrier has a rectilinear channel through which the load can move. The channel axis is located in a vertical plane passing through the trajectory of the carrier.

Let the law of motion of the load in the channel be given in the form \( x_2(t) = \ell \cdot \sin(\omega t) \), where \( \ell = \text{const} \), \( \omega = \text{const} \), and the forces of resistance of the medium to the motion of the carrier are modeled by forces of the Coulomb friction type, then the carrier motion differential equations (CMDE), according to [1, 2], are

\[
\ddot{x} = \beta \cdot (\cos \varphi + f \cdot \sin \varphi) \cdot \sin (\omega t) - \gamma \quad \text{for} \quad \dot{x} > 0; \tag{1}
\]

\[
\ddot{x} = \beta \cdot (\cos \varphi - f \cdot \sin \varphi) \cdot \sin (\omega t) + \gamma \quad \text{for} \quad \dot{x} < 0; \tag{2}
\]
\[ \dot{x} = 0 \quad \text{for} \quad \dot{x} = 0, \tag{3} \]

where \( x \) is the carrier coordinate; \( \varphi \) is channel setting angle; \( \beta = \ell \cdot \omega^2 \cdot \frac{m}{m + M} \); \( \gamma = g \cdot f \); \( M \) is mass of the carrier; \( m \) is weight of load; \( f \) is coefficient of sliding friction in motion (equal to the coefficient of sliding friction at rest) for a pair of materials “carrier–underlying horizontal plane”. It is assumed that the inequality [3]

\[ \beta \cdot |\sin \varphi| \leq g \tag{4} \]

is satisfied. Inequality (4) guarantees the uninterrupted motion of the carrier from the horizontal plane. If \( \beta \geq g \), then from (4) there follows a restriction on the channel setting angle

\[ |\varphi| \leq \arcsin \left( \frac{g}{\beta} \right). \tag{5} \]

2. Conditions for the motion of a carrier from a state of rest

A necessary and sufficient condition for the CMDE (1) to take place in the real dynamics of the carrier is the inequality

\[ \gamma < \beta \cdot (\cos \varphi + f \cdot \sin \varphi), \tag{6} \]

and for the realization of the CMDE (2) in real dynamics there is an inequality

\[ \gamma < \beta \cdot (\cos \varphi - f \cdot \sin \varphi). \tag{7} \]

The replacement \( z = \tan \frac{\varphi}{2} \) leads these inequalities to the following form, respectively

\[ (\beta + \gamma) \cdot z^2 - 2\beta f \cdot z - (\beta - \gamma) < 0, \tag{8} \]

\[ (\beta + \gamma) \cdot z^2 + 2\beta f \cdot z - (\beta - \gamma) < 0. \tag{9} \]

Inequality (6) (or (7)) is satisfied, if the channel setting angle is such that \( \tan \frac{\varphi}{2} \in (z_1; z_2) \), where \( z_1 \) and \( z_2 \) are the real roots of the square trinomial corresponding to (8) (or (9)). These roots are real when

\[ \beta \geq \frac{\gamma}{\sqrt{1 + f^2}}. \tag{10} \]

Therefore, if (10) holds, and \( \varphi \in \Phi_0 \equiv (\varphi_1; \varphi_2) \), where

\[ \varphi_1 = 2 \arctan \frac{\beta \cdot f - \sqrt{\beta^2(1 + f^2) - \gamma^2}}{\beta + \gamma}, \quad \varphi_2 = 2 \arctan \frac{\beta \cdot f + \sqrt{\beta^2(1 + f^2) - \gamma^2}}{\beta + \gamma}, \]

then the carrier can move from a state of rest in the positive direction of the axis \( Ox \). If (10) is satisfied, and \( \varphi \in \Phi_0 \equiv (\varphi_1; \varphi_2) \), where \( \varphi_1 = -\varphi_2 \), \( \varphi_2 = -\varphi_1 \), then the carrier can move from the state of rest in the negative direction of the axis \( Ox \). Simultaneous fulfillment of both conditions (6) and (7) (i.e. (8) and (9)) means that the intersection of intervals \( \Phi_0 \) and \( -\Phi_0 \) is not empty, i.e. \( \varphi_1 < 0 \), which leads to the requirement

\[ \beta > \gamma. \tag{11} \]

Therefore, if (11) takes place, and the channel setting angle is such that \( \varphi \in \Phi_0 \equiv (\varphi_1; \varphi_2) \), then the carrier can move from the state of rest both in the positive and negative directions of the axis \( Ox \).
If the parameter $\beta$ is such that

$$\beta = \frac{\gamma}{\sqrt{1 + f^2}},$$

(12)

then $\varphi_1 = \varphi_2 = \varphi_* = 2 \arctg \frac{\sqrt{1 + f^2} - 1}{f}$, $\varphi_1 = \varphi_2 = -\varphi_*$. 

If the parameter $\beta$ is such that

$$\beta = \gamma,$$

(13)

then $\varphi_1 = \varphi_2 = 0$, $\varphi_1 = \varphi_* = 2 \arctg f$, $\varphi_1 = -\varphi_*$. 

Thus, it is determined that if the parameter of the investigated system is such that

$$\gamma \sqrt{1 + f^2} < \beta < \gamma,$$

(14)

then in the real dynamics of the carrier it is possible either only one-way motion in the positive direction of the axis $Ox$ when the channel setting angle $\varphi \in \Phi_1 \equiv (\varphi_1; \varphi_2)$, 

(15)

where $\varphi_1 > 0$, $\varphi_2 > 0$, or only one-way motion in the negative direction of the axis $Ox$ when the installation angle $\varphi \in \Phi_2 \equiv (\varphi_2; \varphi_1)$, 

(16)

where $\varphi_1 = -\varphi_2$, $\varphi_2 = -\varphi_1$. 

If the parameter $\beta > \gamma$, i.e. (11) is satisfied, then one-way carrier motion from the state of rest only in the positive direction is possible when $\varphi \in \Phi_2+ \equiv (\varphi_2; \varphi_1)$, and the one-way carrier motion in the negative direction of the axis $Ox$ when the channel setting angle $\varphi \in \Phi_2- \equiv (\varphi_1; \varphi_2)$.

In a further analysis of the dynamics of the carrier, lets study its possible bilateral motions from a state of rest with a positive value of the channel setting angle, i.e. select from the set $\Phi_0 \equiv (\varphi_1; \varphi_2)$ a subset $\Phi_{0+} \equiv [0; \varphi_{\text{max}}]$, where

$$\varphi_{\text{max}} = \begin{cases} 
2 \arctg \frac{-\beta \cdot f + \sqrt{\beta^2 (1 + f^2) - \gamma^2}}{\beta + \gamma} & \text{for } \beta \leq \tilde{\beta}; \\
\arcsin \left( \frac{g}{\tilde{\beta}} \right) & \text{for } \beta \geq \tilde{\beta}.
\end{cases}$$

(17)

Here $\tilde{\beta}$ is the root of equation

$$2 \arctg \frac{-\beta \cdot f + \sqrt{\beta^2 (1 + f^2) - \gamma^2}}{\beta + \gamma} = \arcsin \left( \frac{g}{\tilde{\beta}} \right).$$

(18)

Note. In the considered case one-way carrier motions are possible only in the positive direction of the axis $Ox$ when $\beta < \tilde{\beta}$, and the case of unilateral motions of the carrier for $\beta \geq \tilde{\beta}$ is impossible in principle.
3. Algorithm for determination the type of possible bilateral carrier motions from the state of rest for $\Phi_{0+}$.

Let $\varphi > 0$, $\varphi \in \Phi_{0+}$ and $\beta \sin \varphi < g$. Let’s compute

$$
\int_{\tau}^{T+\tau} \left[ \beta \cdot (\cos \varphi + f \cdot \sin \varphi) \cdot \sin (\omega t) - \gamma \right] \cdot dt = 
$$

$$
= \frac{\cos \varphi + f \sin \varphi}{\omega} \left\{ \sqrt{\beta^2 - \gamma_+^2} + \sqrt{\beta^2 - \gamma_-^2} - \gamma_+ \cdot \left[ \pi + \arcsin \left( \frac{\gamma_+}{\beta} \right) - \arcsin \left( \frac{\gamma_+}{\beta} \right) \right] \right\} 
- \int_{T+\tau}^{T+\tau} \left[ \beta \cdot (\cos \varphi - f \cdot \sin \varphi) \cdot \sin (\omega t) + \gamma \right] \cdot dt =
$$

$$
= \frac{\cos \varphi - f \sin \varphi}{\omega} \left\{ \sqrt{\beta^2 - \gamma_-^2} + \sqrt{\beta^2 - \gamma_+^2} - \gamma_- \cdot \left[ \pi + \arcsin \left( \frac{\gamma_-}{\beta} \right) - \arcsin \left( \frac{\gamma_-}{\beta} \right) \right] \right\},
$$

where $\tau_+ = \frac{1}{\omega} \arcsin \left( \frac{\gamma_+}{\beta} \right)$, $\tau_- = \frac{1}{\omega} \arcsin \left( \frac{\gamma_-}{\beta} \right)$, $\gamma_+ = \frac{\gamma}{\cos \varphi + f \sin \varphi}$, $\gamma_- = \frac{\gamma}{\cos \varphi - f \sin \varphi}$,

$T = \frac{2\pi}{\omega}$, and take as the first and the second defining expressions

$$
I_1 = \sqrt{\beta^2 - \gamma_+^2} + \sqrt{\beta^2 - \gamma_-^2} - \gamma_+ \cdot \left[ \pi + \arcsin \left( \frac{\gamma_+}{\beta} \right) - \arcsin \left( \frac{\gamma_+}{\beta} \right) \right]; 
$$

$$
I_2 = \sqrt{\beta^2 - \gamma_-^2} + \sqrt{\beta^2 - \gamma_+^2} - \gamma_- \cdot \left[ \pi + \arcsin \left( \frac{\gamma_-}{\beta} \right) - \arcsin \left( \frac{\gamma_-}{\beta} \right) \right].
$$

From the system of equations

$$
\begin{aligned}
\int_{\tau}^{T+\tau} \left[ \beta \cdot (\cos \varphi + f \cdot \sin \varphi) \cdot \sin (\omega t) - \gamma \right] \cdot dt &= 0; \\
\int_{T+\tau}^{T+\tau} \left[ \beta \cdot (\cos \varphi - f \cdot \sin \varphi) \cdot \sin (\omega t) + \gamma \right] \cdot dt &= 0,
\end{aligned}
$$

which reduces to the form

$$
\begin{aligned}
\beta \cdot \cos (\omega \theta) + \sqrt{\beta^2 - \gamma_+^2} - \gamma_+ \cdot \left( \pi + \omega \theta - \omega \tau_+ \right) &= 0; \\
\beta \cdot \cos (\omega \theta) + \sqrt{\beta^2 - \gamma_-^2} - \gamma_- \cdot \left( \pi + \omega \tau_- - \omega \theta \right) &= 0,
\end{aligned}
$$

where $\theta > \tau_-$, we find the third defining expression

$$
I_3 = \sqrt{\beta^2 - \gamma_+^2} + \beta \cdot \cos \left( f \cdot \pi \cdot \text{tg} \varphi + \arcsin \left( \frac{\gamma_+}{\beta} \right) \right) - \frac{\gamma \pi}{\cos \varphi}.
$$

The algorithm for determination the type of CM for the considered cases has the form (here DAC means that “dynamics analysis is completed“):
**Step 1.** Compute the value $I_1$.
**Step 1.1.** If $I_1 < 0$, then the CM has a type $R2$ and DAC.
**Step 1.2.** If $I_1 = 0$, then the CM has a type $R3$ and DAC.
**Step 1.3.** If $I_1 > 0$, then go to Step 2.

**Step 2.** Compute the value $I_2$.
**Step 2.1.** If $I_2 < 0$, then the CM has a type $R3$ and DAC.
**Step 2.2.** If $I_2 = 0$, then the CM has a type $R3$ and DAC.
**Step 2.3.** If $I_2 > 0$, then go to Step 3.

**Step 3.** Compute the value $I_3$.
**Step 3.1.** If $I_3 < 0$, then the CM has a type $R3$ and DAC.
**Step 3.2.** If $I_3 = 0$, then the CM has a type $R5$ and DAC.
**Step 3.3.** If $I_3 > 0$, then the CM has a type $NR$ and DAC.

Here, the CM of type $R2$, $R3$ and $R5$ [1, 2] are realized in carrier dynamics with the following regular sequences of alternation of CMDE (1)-(3):

- For $R2$: $(3); (1), (3), (2), (3); (1), (3), (2), (3); (1), (3), (2), (3); \ldots$
- For $R3$: $(3); (1), (2), (3); (1), (2), (3); (1), (2), (3); \ldots$
- For $R5$: $(3); (1), (2); (1), (2); (1), (2); \ldots$

Let $\varphi = 0$, then $\gamma_+ = \gamma_- = \gamma = g f$, and defining expressions (19), (20) and (21) turn into

$$I_1 = I_2 = I_3 = I = 2\sqrt{\beta^2 - \gamma^2} - \gamma \pi.$$  

The algorithm for determination the type of CM for $\varphi = 0$ is:

**Step 1.** Compute the value $I$.
**Step 1.1.** If $I < 0$, then the CM has a type $R2$ and DAC.
**Step 1.2.** If $I = 0$, then the CM has a type $R5$ and DAC.
**Step 1.3.** If $I > 0$, then the CM has a type $NR$ and DAC.

4. Results of computational experiments

The dependences $x(t)$, $\dot{x}(t)$ and phase portraits of CM are presented for $\varphi=0.5000$, $f=0.2000$, $\omega=4.0000$ [1/s], $g=9.8100$ [m/s²], $\pi=3.1416$ and $\beta \in \{\beta_{1,1}=3.0000, \beta_{1,2}=4.0254, \beta_{2,1}=4.2000, \beta_{2,2}=4.4136, \beta_{3,1}=4.4400, \beta_{3,2}=4.4494, \beta_{3,3}=7.0000\}$ [m/s²].

- If $\beta = \beta_{1,1}$, then $I_1 < 0$ and the CM realizes according the $R2$ type (figures 1, 3, 5).
- If $\beta = \beta_{1,2}$, then $I_1 = 0$ and the CM realizes according the $R3$ type (figures 2, 4, 6).

**Figure 1.** Dependence $x(t)$ for $\beta = \beta_{1,1}$.

**Figure 2.** Dependence $x(t)$ for $\beta = \beta_{1,2}$.  

| 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | t  |
|---|---|---|---|---|---|---|---|
| 0 | 0.05 | 0.1 | 0.15 | 0 | 0.5 | 1 | x  |

| 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | t  |
|---|---|---|---|---|---|---|---|
| 0 | 0.05 | 0.1 | 0.15 | 0 | 0.5 | 1 | x  |

If $\beta = \beta_{2.1}$, then $I_1 > 0$, $I_2 < 0$ and the CM realizes according the $R3$ type (figures 7, 9, 11).

If $\beta = \beta_{2.2}$, then $I_1 > 0$, $I_2 = 0$ and the CM realizes according the $R3$ type.

If $\beta = \beta_{3.1}$, then $I_1, I_2 > 0$, $I_3 < 0$ and the CM realizes according the $R3$ type (figures 8, 10, 12).
If $\beta = \beta_{3.2}$, then $I_1, I_2 > 0, I_3 = 0$ and the CM realizes according the $R5$ type (figures 13,15,17).

If $\beta = \beta_{3.3}$, then $I_1, I_2, I_3 > 0$ and the CM realizes according the $NR$ type (figures 14,16,18).

**Figure 9.** Dependence $\dot{x}(t)$ for $\beta = \beta_{2.1}$.

**Figure 10.** Dependence $\dot{x}(t)$ for $\beta = \beta_{3.1}$.

**Figure 11.** Phase portrait for $\beta = \beta_{2.1}$.

**Figure 12.** Phase portrait for $\beta = \beta_{3.1}$.

**Figure 13.** Dependence $x(t)$ for $\beta = \beta_{3.2}$.

**Figure 14.** Dependence $x(t)$ for $\beta = \beta_{3.3}$.

**Figure 15.** Dependence $x(t)$ for $\beta = \beta_{3.3}$.
Figure 15. Dependence $\dot{x}(t)$ for $\beta = \beta_{3.2}$.

Figure 16. Dependence $\dot{x}(t)$ for $\beta = \beta_{3.3}$.

Figure 17. Phase portrait for $\beta = \beta_{3.2}$.

Figure 18. Phase portrait for $\beta = \beta_{3.3}$.

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