Semi-discrete and fully discrete HDG methods for Burgers’ equation

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Abstract

This paper proposes semi-discrete and fully discrete hybridizable discontinuous Galerkin (HDG) methods for the Burgers’ equation in two and three dimensions. In the spatial discretization, we use piecewise polynomials of degrees \( k (k \geq 1), k - 1 \) and \( l (l = k - 1; k) \) to approximate the scalar function, flux variable and the interface trace of scalar function, respectively. In the full discretization method, we apply a backward Euler scheme for the temporal discretization. Optimal a priori error estimates are derived. Numerical experiments are presented to support the theoretical results.

Key Words: Burgers’ equation, HDG method, semi-discrete scheme, fully discrete scheme, error estimate

1 Introduction

Let \( \Omega \subset \mathbb{R}^d (d = 2, 3) \) be a polyhedral domain with boundary \( \partial\Omega \), and let \( T > 0 \) be a given final time. We consider the following Burgers’ equation \([7, 8]\):

\[
\begin{align*}
    u_t - \nu \Delta u + b(u) \cdot \nabla u &= f & \text{in } \Omega \times [0, T), \\
    u &= 0 & \text{on } \partial\Omega \times [0, T), \\
    u(\cdot, 0) &= u_0 & \text{in } \Omega,
\end{align*}
\]

where \( u(x, t) \) is the unknown scalar function with initial value \( u_0(x) \), \( b(u) = [u, u]^T \) when \( d = 2 \) and \( b(u) = [u, u, u]^T \) when \( d = 3 \), \( \nu > 0 \) is the coefficient of viscosity, and \( f(x, t) \) is the prescribed force.

The Burgers’ equation, which can be viewed as a simplified model of Navier-Stokes equation, is a nonlinear partial differential equation that simulates the propagation and reflection of shock waves. It is widely used in many physical fields such as fluid mechanics, nonlinear acoustics, and gas dynamics. In recent decades, there have developed many finite element (FE) methods for Burgers’ equation, such as conforming methods \([1, 4, 9, 10, 16, 25, 29]\), B-spline methods \([3, 26, 37]\), least-squares methods \([23, 35, 39]\), mixed methods \([12, 19, 20, 30, 33]\), discontinuous Galerkin (DG) methods \([5, 32, 40]\), and weak Galerkin methods \([14, 21]\).

The hybridizable discontinuous Galerkin (HDG) framework, presented in \([27]\) for second order elliptic problems, provides a unifying strategy for hybridization of finite element methods. By the local elimination of the unknowns defined in the interior of elements, the HDG method leads to a system where the unknowns are only the globally coupled degrees of freedom describing the introduced Lagrange multiplier. In \([28]\), an implicit high-order
The HDG method was presented for nonlinear convection–diffusion equations. The HDG method used piecewise polynomials of degrees \( k(k \geq 0) \) for the approximations of the scalar variable, corresponding flux and trace of the scalar variable. A numerical example for the two dimensional Burgers’ equation with the third-order backward difference formula in time discretization was presented. We refer the readers to [13, 15, 17, 24, 31] for some developments and applications of the HDG method.

In this paper, we consider an HDG discretization of the Burgers’ equation in two and three dimensions. In the spatial discretization, the flux variable \( q \), the scalar variable \( u \) and its trace are approximated respectively by piecewise polynomials of degrees \( k \), \( k \) and \( l(l=k,k-1) \) with \( k \geq 1 \). In the fully discrete scheme, a backward Euler scheme is adopted for the temporal derivative.

The rest of the paper is arranged as follows. In section 2, we introduce some notations and the weak problem. Section 3 presents the semi-discrete HDG scheme, prove the existence and uniqueness of the solution and carries out the error analysis. Section 4 discusses the fully discrete HDG scheme, including the stability, the existence and uniqueness of the solution and carries out the error analysis. Finally, we provides some numerical examples to verify the theoretical results in section 5.

2 Notation and weak problem

For any bounded domain \( D \subset \mathbb{R}^d(s = d, d-1) \) and integer \( m \geq 0 \), let \( H^m(D) \) and \( H_0^m(D) \) denote the usual \( m \)th-order Sobolev spaces on \( D \), and \( \| \cdot \|_{m,D}, \| \cdot \|_{m,D} \) denote the norm and semi-norm on these spaces, respectively. We use \( \langle \cdot, \cdot \rangle_{m,D} \) to denote the inner product of \( H^m(D) \), with \( (\cdot, \cdot)_D := \langle \cdot, \cdot \rangle_{0,D} \). When \( D = \Omega \), we set \( \| \cdot \|_m := \| \cdot \|_{m,\Omega}, \| \cdot \|_m := \| \cdot \|_{m,\Omega} \) and \( (\cdot, \cdot) := (\cdot, \cdot)_\Omega \). In particular, when \( D \subset \mathbb{R}^{d-1} \), we use \( (\cdot, \cdot)_D \) to replace \( (\cdot, \cdot)_D \). We denote by \( P_m(D) \) the set of all polynomials on \( D \) with degree at most \( m \).

Let \( \mathcal{T}_h = \bigcup\{K\} \), consisting of arbitrary open polygons/polyhedrons, be a partition of the domain \( \Omega \). For any \( K \in \mathcal{T}_h \), let \( h_K \) be the infimum of the diameters of circles (or spheres) containing \( K \) and denote by \( h := \max_{K \in \mathcal{T}_h} h_K \) the mesh size. We assume that \( \mathcal{T}_h \) is shape-regularin the sense that the following two assumptions hold (cf. [11]):

(M1) There exists a positive constant \( \theta \), such that the following holds: for each element \( K \in \mathcal{T}_h \), there exists a point \( M_K \in K \) such that \( K \) is star-shaped with respect to every point in the circle (or sphere) of center \( M_K \) and radius \( \theta h_K \).

(M2) There exists a positive constant \( l \), such that for every element \( K \in \mathcal{T}_h \), the distance between any two vertexes is no less than \( l h_K \).

We denote by \( \varepsilon_h \) the set of all faces in the mesh, and by \( \varepsilon_h^0 \) and \( \varepsilon_h^\partial \) the sets of interior edges/faces and boundary edges/faces, respectively. For any \( e \in \varepsilon_h \), we denote by \( h_e \) the diameter of \( e \). We also introduce the following mesh-dependent inner products and norms:

\[
\langle u, v \rangle_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle u, v \rangle_K, \quad \| u \|_{0,\mathcal{T}_h}^2 := \sum_{K \in \mathcal{T}_h} \| u \|_{0,K}^2, \quad \| u \|_{0,\partial \mathcal{T}_h}^2 := \sum_{K \in \mathcal{T}_h} \| u \|_{0,\partial K}^2, \quad \langle u, v \rangle_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle u, v \rangle_{\partial K}.
\]

For convenience, we use the notation \( a \leq b \) to denote that there exists a generic positive constant \( C \), independent of the spatial and temporal mesh parameters, \( h \) and \( \Delta t \), such that \( a \leq Cb \).

To give the weak problem for the model (1.1a)-(1.1c), we need to introduce the bilinear form \( A(\cdot, \cdot) \) and trilinear form \( B(\cdot, \cdot, \cdot) \):

\[
A(u, v) := \nu (\nabla u, \nabla v), \quad \forall u, v \in H_0^1(\Omega),
\]
Since
\[ (b(u) \cdot \nabla u, v) = \frac{1}{3} \langle \nabla \cdot b(u), uv \rangle + \frac{2}{3} (b(u), v \nabla u) \]
\[ = \frac{1}{3} (b(u), \nabla (uv)) + \frac{2}{3} (b(u), v \nabla u) \]
\[ = -\frac{1}{3} (b(u), u \nabla v) + \frac{1}{3} (b(u), v \nabla u) \quad \forall v \in H^1_0(\Omega), \]
it is easy to see that
\[ B(u, v, w) = -B(u, w, v) \quad \forall u, v, w \in H^1_0(\Omega). \quad (2.1) \]

With the above notations, the weak form of (1.1a)-(1.1c) is given as follows: find \( u(t) \in H^1_0(\Omega) \) such that
\[
\begin{cases}
(\dot{u}, v) + A(u, v) + B(u, u, v) = (f, v), \quad \forall v \in H^1_0(\Omega), \ t \in (0, T] \\
u(0) = u_0, \quad x \in \Omega.
\end{cases}
\]
(2.2)

From [36, Section III. Theorem 3.1], it holds the following wellposedness result for the weak problem (2.2).

**Lemma 2.1.** In the case of \( d = 2 \), given \( f \in L^\infty(0, T; L^2(\Omega)) \) and \( u_0(x) \in L^2(\Omega) \), the problem (2.2) admits a unique solution \( u \) satisfying
\[ u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)). \]
Furthermore, if \( u_0(x) \in H^1_0(\Omega) \), then
\[ u \in C([0, T]; H^1_0(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)). \]

### 3 Semi-discrete HDG method

#### 3.1 Semi-discrete scheme

Introduce a new variable \( q = -\nabla u \), then we rewrite (1.1a)-(1.1c) as
\[
\begin{cases}
q + \nabla u = 0, \quad \text{in} \quad \Omega \times (0, T], \\
\dot{u} + \nu \nabla \cdot q + b(u) \cdot \nabla u = f, \quad \text{in} \quad \Omega \times (0, T], \\
u(\cdot, 0) = u_0, \quad \text{on} \quad \partial \Omega \times [0, T], \\
\end{cases}
\]
(3.1)

For any integer \( k \geq 1 \) and \( l = k - 1, k \), we introduce the following finite element spaces:
\[
\begin{align*}
Q_h &= \{ v_h \in [L^2(\Omega)]^d : v_h|_K \in [P_{k-1}(K)]^d, \forall K \in T_h \}, \\
v_h &= \{ v_h \in L^2(\Omega) : v_h|_K \in P_k(K), \forall K \in T_h \}, \\
\hat{v}_h &= \{ \hat{v}_h \in L^2(\varepsilon_h) : \hat{v}_h|_E \in P_l(E), \forall E \in \varepsilon_h, \hat{v}_h|_{\varepsilon_h} = 0 \}.
\end{align*}
\]

Let us introduce the standard \( L^2 \) projections \( \Pi^0_{k-1} : [L^2(\Omega)]^d \rightarrow Q_h \), \( \Pi^0_k : L^2(\Omega) \rightarrow v_h \), and \( \Pi^0_l : L^2(\varepsilon_h) \rightarrow \hat{v}_h \), which satisfy
\[
\begin{align*}
\langle \Pi^0_{k-1} q, r \rangle_K &= (q, r)_K, \quad \forall r \in [P_{k-1}(K)]^d, \\
\langle \Pi^0_k u, w \rangle_K &= (u, w)_K, \quad \forall w \in P_k(K), \\
\langle \Pi^0_l u, \mu \rangle_e &= (u, \mu)_e, \quad \forall \mu \in P_l(e).
\end{align*}
\]
Then the semi-discrete HDG scheme for (3.1) reads as follows: find \((q_h, u_h, \tilde{u}_h) \in Q_h \times V_h \times \hat{V}_h\) such that

\[
\begin{cases}
A_h(q_h, r) - C_h(u_h, \tilde{u}_h; r) = 0, \\
(u_{h,t}, w)_{\mathcal{T}_h} + \nu C_h(w; q_h) + S_h(u_h, \tilde{u}_h; w, \mu) + B_h(u_h, \tilde{u}_h; w, \mu) = (f, w)_{\mathcal{T}_h}, \\
u u_h(0) = \Pi_k^0 u_0,
\end{cases}
\]

for all \((r, w, \mu) \in Q_h \times V_h \times \hat{V}_h\). Here

\[
A_h(q_h, r) := (q_h, r)_{\mathcal{T}_h}, \\
C_h(u_h, \tilde{u}_h; r) := (u_h, \nabla \cdot r)_{\mathcal{T}_h} - \langle \tilde{u}_h, r \cdot n \rangle_{\partial \mathcal{T}_h}, \\
S_h(u_h, \tilde{u}_h; w, \mu) := \nu \langle \tau (\Pi_k^0 u_h - \tilde{u}_h), \Pi_k^0 w - \mu \rangle_{\partial \mathcal{T}_h}, \\
B_h(v_h; u_h, \tilde{u}_h; w, \mu) := -\frac{1}{3} \langle b(v_h) u_h, \nabla w \rangle_{\mathcal{T}_h} + \frac{1}{3} \langle b(v_h) \cdot \nabla u_h, w \rangle_{\mathcal{T}_h} - \frac{1}{3} \langle b(v_h) \cdot n u_h, \mu \rangle_{\partial \mathcal{T}_h} + \frac{1}{3} \langle b(v_h) \cdot n \tilde{u}_h, w \rangle_{\partial \mathcal{T}_h},
\]

and \(\tau |_{\partial K} = h_K^{-1}\).

We introduce an operator \(K_h : V_h \times \hat{V}_h \rightarrow Q_h\) defined by

\[
(K_h(u_h, \mu), r)_{\mathcal{T}_h} := -(w, \nabla_h \cdot r)_{\mathcal{T}_h} + \langle \mu, r \cdot n \rangle_{\partial \mathcal{T}_h}, \forall r \in Q_h.
\]

It is easy to see that \(K_h\) is well defined. From (3.2) we can immediately get

\[
q_h = -K_h(u_h, \tilde{u}_h).
\]

Hence, the system (3.2) can be reduced to the following one: find \((q_h, u_h, \tilde{u}_h) \in Q_h \times V_h \times \hat{V}_h\) such that, for all \((w, \mu) \in V_h \times \hat{V}_h\),

\[
\begin{cases}
(q_h + K_h(u_h, \tilde{u}_h), r)_{\mathcal{T}_h} = 0, \\
(u_{h,t}, w)_{\mathcal{T}_h} + \nu (K_h(u_h, \tilde{u}_h), K_h(u_h, \mu))_{\mathcal{T}_h} + S_h(u_h, \tilde{u}_h; w, \mu) + B_h(u_h, \tilde{u}_h; w, \mu) = (f, w)_{\mathcal{T}_h}, \\
u u_h(0) = \Pi_k^0 u_0.
\end{cases}
\]

**Remark 3.1.** We note that the degree of approximation polynomials in \(Q_h\) can be chosen as \(k\). Such a choice makes no difference in the subsequent analysis, and then leads to the same convergence rates as the degree \(k - 1\).

### 3.2 Basic results

We introduce the following semi-norm on \(V_h \times \hat{V}_h\): For \(\forall (u_h, \tilde{u}_h) \in V_h \times \hat{V}_h\),

\[
\| (u_h, \tilde{u}_h) \|^2 := \| (K_h(u_h, \tilde{u}_h)) \|^2_{\mathcal{T}_h} + \| \tau \frac{1}{2} (\Pi_k^0 u_h - \tilde{u}_h) \|^2_{\partial \mathcal{T}_h}.
\]

We can show that \(\| (\cdot, \cdot) \|\) is a norm. In fact, if \(\| (u_h, \tilde{u}_h) \| = 0\), then \(\nabla u_h = 0\) and \(\Pi_k^0 u_h |_{\partial K} = u_h |_{\partial K} = \tilde{u}_h |_{\partial K}\), which means that \(u_h\) is piecewise constant with respect to \(\mathcal{T}_h\) and \(u_h = \tilde{u}_h\) on \(\varepsilon_K\). Since \(\tilde{u}_h = 0\) on \(\partial \Omega\), then \(u_h = \tilde{u}_h = 0\).

By using the trace theorem, the inverse inequality, and scaling arguments, we can easily get the following lemma.

**Lemma 3.1.** [34] For all \(K \in \mathcal{T}_h, \omega \in W^{1, \tilde{q}}(K)\), and \(1 \leq \tilde{q} \leq \infty\), we have

\[
\| \omega \|_{0, \tilde{q}, \partial K} \lesssim h_K^{\frac{1}{\tilde{q}}} \| \omega \|_{0, \tilde{q}, K} + h_K^{1 - \frac{1}{\tilde{q}}} |\omega|_{1, \tilde{q}, K}.
\]

In particular, for all \(\omega \in P_h(K)\),

\[
\| \omega \|_{0, \tilde{q}, \partial K} \lesssim h_K^{\frac{1}{\tilde{q}}} \| \omega \|_{0, \tilde{q}, K}.
\]
Lemma 3.2. For any \((w, \mu) \in V_h \times \hat{V}_h\) we have
\[
\|\nabla_h w\|_{0,T_h} + \tau^{\frac{1}{2}} (w - \mu)\|_{0,T_h} \lesssim \|(w, \mu)\|.
\] (3.5)

Proof. For any \((w, \mu) \in V_h \times \hat{V}_h\), by the definition of \(K_h\), it holds
\[
\langle K_h(w, \mu), r \rangle_{T_h} = -\langle w, \nabla_h \cdot r \rangle_{T_h} + \langle \mu, r \cdot n \rangle_{\partial T_h} = \langle \nabla_h w, r \rangle_{T_h} + \langle \mu - w, r \cdot n \rangle_{\partial T_h},
\]
for all \(r \in Q_h\). By taking \(r = \nabla_h w\), the estimate (3.5) follows from the H"older’s inequality and the inverse inequality.

Lemma 3.3. [34] The embedding relationship
\[
W^{1,2}(\Omega) \hookrightarrow W^{0,\tilde{q}}(\Omega)
\]
holds for \(\tilde{q}\) satisfying \(2 \leq \tilde{q} < \infty\) when \(d = 2\), \(2 \leq \tilde{q} \leq 6\) when \(d = 3\).

Lemma 3.4. [22] There exists an interpolation operator, called Oswald interpolation, \(I_h : V_h \rightarrow V_h \cap H^1_0(\Omega)\), such that, for any \(w \in V_h\),
\[
\sum_{K \in T_h} \|w - I_h w\|_{0,K}^2 \lesssim \sum_{c \in e_h} h_c \|[w]\|_{0,c,e}^2.
\] (3.6)
\[
\sum_{K \in T_h} |w - I_h w|_{1,K}^2 \lesssim \sum_{c \in e_h} h_c^{-1} \|[w]\|_{0,c,e}^2.
\] (3.7)

Lemma 3.5. It holds
\[
\|w\|_{0,\tilde{q},T_h} \lesssim \|(w, \mu)\|, \quad \forall (w, \mu) \in V_h \times \hat{V}_h,
\] (3.8)
where \(2 \leq \tilde{q} < \infty\) when \(d = 2\) and \(2 \leq \tilde{q} \leq 6\) when \(d = 3\).

Proof. For all \((w, \mu) \in V_h \times \hat{V}_h\), from Lemma 3.3 and the Poincaré inequality, we have
\[
\|I_h w\|_{0,\tilde{q},T_h} \lesssim \|I_h w\|_{1,2,T_h} \lesssim \|
abla I_h w\|_{0,T_h}.
\]
From (3.7) and Lemma 3.2 it follows
\[
\|
abla I_h w\|_{0,T_h} \lesssim \|
abla h w\|_{0,T_h} + \left(\sum_{c \in e_h} h_c^{-1} \|[w]\|_{0,c,e}^2\right)^{\frac{1}{2}}
\lesssim \|(w, \mu)\| + \left(\sum_{c \in e_h} h_c^{-1} \|[w - \mu]\|_{0,c,e}^2\right)^{\frac{1}{2}}
\lesssim \|(w, \mu)\|.
\] (3.9)

Using Lemma 3.3, the inverse inequality and the properties of the projection-mean operator \(([34]) P_h : V_h \rightarrow W^{1,2}(\Omega) \cap W^{0,\tilde{q}}(\Omega),\) we obtain
\[
\|w - I_h w\|_{0,\tilde{q},T_h} \lesssim \|w - P_h w\|_{0,\tilde{q},T_h} + \|P_h w - I_h w\|_{0,\tilde{q},T_h}
\lesssim h^{1 - \frac{\tilde{q}}{2} + \frac{1}{2}} \|
abla h w\|_{0,T_h} + \|P_h w - I_h w\|_{1,2,T_h}
\lesssim \|
abla h w\|_{0,T_h} + \|w - P_h w\|_{1,2,T_h} + \|w - I_h w\|_{1,2,T_h}
\lesssim \|
abla h w\|_{0,T_h} + \|\nabla_h (w - I_h w)\|_{0,T_h}
\lesssim \|(w, \mu)\|,
\]
which, together with (3.9), yields the desired estimate (3.8). 
\[
\square
\]
3.3 Well-posedness of the semi-discrete HDG scheme

First, we have the following boundedness results for $\mathcal{B}_h$.

**Lemma 3.6.** For any $(v_h, \tilde{u}_h), (u_h, \hat{u}_h), (w, \mu) \in V_h \times \hat{V}_h$, it holds

\[
|\mathcal{B}_h (v_h; u_h, \tilde{u}_h; w, \mu)\| \lesssim \|\nabla u_h\|_{0,2,K} \|w\|_{0,6,K} + \sum_{K \in T_h} \|v_h\|_{0,3,K} \|u_h\|_{0,6,K} \|\nabla w\|_{0,2,K},
\]

(3.10)

\[
|\mathcal{B}_h (v_h; u_h, \tilde{u}_h; w, \mu)\| \lesssim \|v_h\|_{0,3,T_h} \cdot ||(u_h, \tilde{u}_h)|| \cdot ||(w, \mu)||.
\]

(3.11)

**Proof.** For any $(v_h, \tilde{u}_h), (u_h, \hat{u}_h), (w, \mu) \in V_h \times \hat{V}_h$, by the definition of $\mathcal{B}_h$, we have

\[
3\mathcal{B}_h (v_h; u_h, \tilde{u}_h; w, \mu) = \left[ (b(v_h) \cdot \nabla u_h, w)_{\Omega_T} - (b(v_h) \cdot u_h, \nabla w)_{\Omega_T} \right]
\]

\[
+ \left[ (b(v_h) \cdot n \tilde{u}_h, w)_{\partial \Omega_T} - (b(v_h) \cdot n u_h, \mu)_{\partial \Omega_T} \right]
\]

(3.12)

Using the Hölder inequality and Lemma 3.5, we have

\[
|R_1| \lesssim \sum_{K \in T_h} \|v_h\|_{0,3,K} \|\nabla u_h\|_{0,2,K} \|w\|_{0,6,K} + \sum_{K \in T_h} \|v_h\|_{0,3,K} \|u_h\|_{0,6,K} \|\nabla w\|_{0,2,K},
\]

(3.13)

From triangle inequality we have

\[
|R_2| \leq \|b(v_h) \cdot n (u_h - \tilde{u}_h), w\|_{\partial \Omega_T} + \|b(v_h) \cdot n u_h, w - \mu\|_{\partial \Omega_T}
\]

(3.14)

By the Hölder inequality, Lemma 3.1 and Lemma 3.5, we obtain

\[
|T_1| \lesssim \sum_{K \in T_h} \|v_h\|_{0,3,K} \|u_h - \tilde{u}_h\|_{0,2,K} \|w\|_{0,6,K},
\]

(3.15)

Similarly, we have

\[
|T_2| \lesssim \sum_{K \in T_h} \|v_h\|_{0,3,K} \|u_h\|_{0,6,K} \|w - \mu\|_{0,2,K}
\]

(3.16)

As a result, the desired inequality (3.11) follows from (3.12)-(3.16). Since

\[
\|v_h\|_{0,3,T_h} \lesssim \|(v_h, \tilde{u}_h)\|
\]

by Lemma 3.5, the inequality (3.11) indicates (3.10).

We also have the following stability result.
Theorem 3.1. For the numerical solution $(q_h,u_h,\hat{u}_h) \in Q_h \times V_h \times \tilde{V}_h$ to the scheme (3.4) with initial setting $u_h(0)$, it holds
\[
\|u_h(t)\|^2_{0,T} + \nu \int_0^t \|u_h(t)\|^2 dt \leq \|u_h(0)\|^2_{0,T} + \frac{1}{\nu} \int_0^t \|f(\tau)\|^2_{0,T} d\tau,
\] (3.17)
i.e., the numerical solution is stable with respect to initial approximate value and source term.

Proof. Taking $(w,\mu) = (u_h, \hat{u}_h)$ in (3.4), we get
\[(u_h(t),u_h(t)) + \nu\|u_h(t)\|^2 = (f,u_h(t)),\]
then, using the Young’s inequality and Lemma 3.5, we know that
\[(f,u_h(t)) \leq \frac{C}{2\nu} \|f\|^2_{0,T} + \frac{\nu}{2} \|u_h(t)\|^2,
\]
where $C$ is a positive constant independent of $h$. Thus
\[
\frac{1}{2} \frac{d}{dt} \|u_h\|^2_{0,T} + \frac{\nu}{2} \|u_h(t)\|^2 \leq \frac{C}{2\nu} \|f\|^2_{0,T},
\]
Integrating the above inequality with respect to $t$ yields the desired result (3.17).

We are now in a position to show the global existence and uniqueness of the semi-discrete solution by the standard theory of ordinary differential equations.

Theorem 3.2. If $f(\cdot,t)$ is continuous with respect to $t$, then the problem (3.4) admits a unique solution $(q_h,u_h(t),\hat{u}_h(t))$ for any $t \in [0,T]$.

Proof. Let $\varphi,\phi$ and $\psi$ be the bases of $V_h|K, \tilde{V}_h|K$ and $[P_{k-1}(K)]^d$, respectively, with
\[
\varphi = (\varphi_1, \ldots, \varphi_m), \quad \phi = (\phi_1, \ldots, \phi_n), \quad \psi = (\psi_1, \ldots, \psi_p).
\]
Denote $u_h(t)|_K := \varphi U(t), \tilde{u}_h(t)|_K := \phi \tilde{U}(t)$ and $-q_h(t)|_K := \psi Q(t)$ with
\[U(t) = (U_1(t), \ldots, U_m(t))^T, \quad \tilde{U}(t) = (\tilde{U}_1(t), \ldots, \tilde{U}_n(t))^T, \quad Q(t) = (Q_1(t), \ldots, Q_p(t))^T.
\]
Set
\[
\begin{align*}
M_1 &= \sum_{K \in T_h} \int_K \psi^T \varphi dx, & M_2 &= \sum_{K \in T_h} \int_K (\nabla \cdot \psi)^T \varphi dx, & M_3 &= \sum_{K \in T_h} \int_{\partial K} (\varphi n)^T \varphi ds, \\
M_4 &= \sum_{K \in T_h} \int_K \varphi^T \varphi dx, & M_5 &= \sum_{K \in T_h} \int_K \tau \varphi^T \varphi ds, & M_6 &= \sum_{K \in T_h} \int_{\partial K} \tau \varphi^T \varphi ds, \\
M_7 &= \sum_{K \in T_h} \int_{\partial K} \tau \phi^T \varphi ds, & M_8(\tilde{U}) &= \sum_{K \in T_h} \int_{\partial K} \frac{1}{3} (\nabla \varphi)^T b(\varphi \tilde{U}) \varphi dx, & M_9(\tilde{U}) &= \sum_{K \in T_h} \int_{\partial K} \frac{1}{3} (b(\varphi \tilde{U}) n) \varphi^T \varphi ds, \\
F(t) &= \sum_{K \in T_h} \int_K \psi^T f(t) dx.
\end{align*}
\]
Here we denote that $\tilde{Q}(t)|_K = Q(t), \tilde{U}(t)|_K = U(t)$ and $\tilde{u}_h(t)|_K = \tilde{u}_h(t)$. Thus, the system (3.4) can be written as
\[
\begin{cases}
M_1 \dot{Q}(t) + M_2 \dot{U}(t) - M_3 \ddot{U}(t) = 0, \\
M_4 \frac{d\tilde{U}(t)}{dt} - \nu M_5 \dot{Q}(t) + (\nu M_6 - M_8(\tilde{U}(t)) + M_9(\tilde{U}(t))) \dot{U}(t) + (M_9(\tilde{U}(t)) - \nu M_6) \ddot{U}(t) = F(t), \\
\nu M_5 \dot{Q}(t) - (\nu M_6 + M_7(\tilde{U}(t))) \dot{U}(t) + \nu M_7 \ddot{U}(t) = 0.
\end{cases}
\] (3.18)
Since $M_1, M_4$ and $M_7$ are symmetric positive defined, we can eliminate $\tilde{Q}(t)$ and $\tilde{U}(t)$ in (3.18) to get
\[ M_4 \frac{d\tilde{U}(t)}{dt} + M(\tilde{U}(t))\tilde{U}(t) = F(t), \tag{3.19} \]
where
\[ M(\tilde{U}(t)) = \nu M_2 M_1^{-1} M_2 + \nu M_5 - M_8(\tilde{U}) + M_8(\tilde{U}) - (\nu M_2 M_1^{-1} M_3 - M_9(\tilde{U}) + \nu M_6) M^* \]
and
\[ M^* = (M_7 + M_9 M_1^{-1} M_3)^{-1}(M_7 M_1^{-1} M_2 + M_9^2 + \frac{1}{\nu} M_8(\tilde{U})). \]

According to the stability result (3.17), we know that $M_8(\tilde{U}(t))$ and $M_9(\tilde{U}(t))$ are bounded in $[0, T]$, which implies that $M(\tilde{U}(t))\tilde{U}(t)$ is globally Lipschitz continuous with respect to $\tilde{U}(t)$. In addition, $M_4$ is symmetric positive defined, by the standard ODE theory [38], there exist a unique solution to (3.19) on $[0, T]$, which means the existence and uniqueness of $(q_6(t), u_6(t), \tilde{w}_6(t))$ in (3.4) on the interval $[0, T]$. \hfill \Box

### 3.4 A priori error estimation

This section is devoted to the error estimation of the HDG scheme (3.4). For the $L^2$ projections $(\Pi_{k-1}^0 q(t), \Pi_k^0 u(t), \Pi_0^1 u(t))$, we have the following standard estimates [6]:

\[
\|q - \Pi_{k-1}^0 q\|_{0, \tau_h} \leq C h^k \|q\|_{k, \Omega}, \quad \forall q \in [H^k(\Omega)]^d,
\]
\[
\|u - \Pi_k^0 u\|_{0, \tau_h} \leq C h^{k+1} \|u\|_{k+1, \Omega}, \quad \forall u \in H^{k+1}(\Omega),
\]
\[
\|u - \Pi_k^0 u\|_{0, \partial \tau_h} \leq C h^{k+\frac{1}{2}} \|u\|_{k+1, \Omega}, \quad \forall u \in H^{k+1}(\Omega),
\]
\[
\|w\|_{0, \partial \tau_h} \leq \frac{C}{\nu} h^\frac{1}{2}, \quad \forall w \in V_h.
\]

We also need the Gronwall inequality:

**Lemma 3.7.** Suppose $\rho(t) \geq 0$ satisfies
\[
\rho(t) \leq \alpha + \int_0^t \beta(s) \rho(s) ds,
\]
with $\alpha, \beta(s) \geq 0$. Then it follows
\[
\rho(t) \leq \alpha \exp(\int_0^t \beta(s) ds).
\]

**Lemma 3.8.** For any $u \in H_0^1(\Omega)$, $(w, \mu) \in V_h \times \hat{V}_h$, it holds
\[
B_h(\Pi_k^0 u(t); \Pi_k^0 u(t), \Pi_0^1 u(t); w, \mu) = (b(u) \cdot \nabla u, w) + E_N(u; u, w, \mu), \tag{3.20}
\]
where
\[
E_N(u; u, w, \mu) := \frac{1}{3} \left( (b(u - \Pi_k^0 u) \cdot \nabla u, w)_{\tau_h} + \frac{1}{3} (b(\Pi_k^0 u)(u - \Pi_k^0 u), \nabla w)_{\tau_h} \right.
\]
\[
- \frac{1}{3} (b(u - \Pi_k^0 u) \cdot \nabla \Pi_k^0 u, w)_{\tau_h} - \frac{1}{3} (b(u) \cdot (\nabla u - \nabla \Pi_k^0 u), w)_{\tau_h}
\]
\[
+ \frac{1}{3} (b(u - \Pi_k^0 u) \cdot \mu \Pi_k^0 u)_{\partial \tau_h} + \frac{1}{3} (b(u) \cdot \mu (u - \Pi_k^0 u))_{\partial \tau_h}
\]
\[
- \frac{1}{3} (b(u - \Pi_k^0 u) \cdot w \Pi_k^0 u)_{\partial \tau_h} - \frac{1}{3} (b(u) \cdot w (u - \Pi_0^1 u)_{\partial \tau_h}
\].
Proof. For any \( w \in V_h \), by \( \langle b(u) \cdot n, u \mu \rangle_{\partial \Omega_h} = 0 \) we have

\[
(b(u) \cdot \nabla u, w)_h = \frac{1}{2} \langle \nabla b(u), uw \rangle_h + \frac{1}{3} (b(u), w \nabla u)_h
\]

\[
= \frac{1}{3} (b(u), w \nabla u)_{\partial \Omega_h} - \frac{1}{3} (b(u), w \nabla u)_h + \frac{2}{3} (b(u), w \nabla u)_h
\]

and

\[
\langle \Pi_k u(t), \Pi_k u(t) \rangle_{\partial \Omega_h} \quad \Rightarrow \quad \frac{1}{3} (b(u), w \nabla u)_{\partial \Omega_h} - \frac{1}{3} (b(u), w \nabla u)_h + \frac{3}{3} (b(u), w \nabla u)_h,
\]

which imply

\[
- \frac{1}{3} (b(u), w \nabla u)_{\partial \Omega_h} + \frac{1}{3} (b(u), w \nabla u)_h
\]

and

\[
\langle \Pi_k u(t), \Pi_k u(t) \rangle_{\partial \Omega_h} \quad \Rightarrow \quad - \frac{1}{3} (b(u), w \nabla u)_{\partial \Omega_h} + \frac{1}{3} (b(u), w \nabla u)_h
\]

Combining the above four equalities gives (3.20).

**Lemma 3.9.** For \( u \in H^{k+1}(\Omega) \), it holds

\[
|E_N (u; u, w, \mu)| \lesssim h^k||u||_{k+1}||u||_2||(w, \mu)||, \quad \forall (w, \mu) \in V_h \times \tilde{V}_h.
\]  

**Proof.** From the Hölder inequality, the sobolev inequality, and the projection properties, we have the following estimates:

\[
|(b(u - \Pi_k u)u, \nabla w)_h| \lesssim \sum_{K \in \Omega_h} ||u - \Pi_k u||_{0,2,K} ||u||_{0,\infty,K} ||\nabla w||_{0,2,K}
\]

\[
\lesssim h^{k+1}||u||_{k+1}||u||_2||\nabla w||_{0,\Omega_h}
\]

\[
\lesssim h^{k+1}||u||_{k+1}||u||_2||(w, \mu)||,
\]

\[
|(b(\Pi_k u)(u - \Pi_k u), \nabla w)_h| \leq C \sum_{K \in \Omega_h} ||\Pi_k u||_{0,\infty,K} ||u - \Pi_k u||_{0,2,K} ||\nabla w||_{0,2,K}
\]

\[
\lesssim \sum_{K \in \Omega_h} ||u||_{0,\infty,h} h^{k+1,K} ||u||_{k+1,K} ||\nabla w||_{0,2,K}
\]

\[
\lesssim h^{k+1}||u||_{k+1}||u||_2||\nabla w||_{0,\Omega_h}
\]

\[
\lesssim h^{k+1}||u||_{k+1}||u||_2||(w, \mu)||,
\]

\[
|(b(u - \Pi_k u) \cdot \nabla \Pi_k u, w)_h| \lesssim \sum_{K \in \Omega_h} ||u - \Pi_k u||_{0,3,K} ||\nabla \Pi_k u||_{0,2,K} ||w||_{0,6,K}
\]

\[
\lesssim \sum_{K \in \Omega_h} h^{k+1-\frac{1}{2}} ||u||_{k+1,K} ||u||_{1,2,K} ||w||_{0,6,K}
\]

\[
\lesssim h^{k+1-\frac{1}{2}} ||u||_{k+1}||u||_2||w||_{0,6,h},
\]

\[
\lesssim h^{k+1-\frac{1}{2}} ||u||_{k+1}||u||_2||(w, \mu)||,
\]

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\begin{align*}
|\langle b(u) \cdot (\nabla u - \nabla \Pi_h^o u), w \rangle_{\partial T_h}| & \lesssim \sum_{K \in T_h} |u|_{0,\infty,K} |\nabla u - \nabla \Pi_h^o u|_{0,2,K} \|w\|_{0,2,K} \\
& \lesssim \sum_{K \in T_h} |u|_{0,\infty,K} h^k |u|_{k+1,K} \|w\|_{0,2,K} \\
& \lesssim h^k \|u\|_{k+1} \|u\|_2 \|w\|_{0,T_h} \\
& \lesssim h^k \|u\|_{k+1} \|u\|_2 \|w,\mu\|_2,
\end{align*}

As a result, the desired result follows from the definition of $E_N(u; w, \mu)$. 

\textbf{Lemma 3.10.} Let $(u, q)$ be the solution to the problem (3.1), then for any $(w, \mu) \in V_h \times \hat{V}_h$ it holds

\begin{align}
\Pi_h^{k-1} q = -K_h (\Pi_h^o u, \Pi_h^o u),
\end{align}

\begin{align}
\{(\Pi_h^o u)_t, w \rangle_{\partial T_h} + \nu (K_h (\Pi_h^o u, \Pi_h^o u), K_h (w, \mu))_{\partial T_h} + S_h (\Pi_h^o u, \Pi_h^o u; w, \mu) + B_h (\Pi_h^o u; \Pi_h^o u, \Pi_h^o u; w, \mu) = ((\Pi_h^o u - u)_t, w \rangle_{\partial T_h} + f, w) + E_L (u, w, \mu) + E_N (u, w, \mu),
\end{align}
where

\[ E_L(u,w,\mu) := -\nu \langle (\nabla u - \Pi^o_{k-1} \nabla u) \cdot n, w - \mu \rangle_{\partial \Omega} + \nu \langle \tau (\Pi^o_k u - u), \Pi^o_k w - \mu \rangle_{\partial \Omega}. \]

Proof. By the definition of \( K_h \), we get

\[
((\Pi^o_k u), t), w)_{\Omega} + \nu (K_h (\Pi^o_k u, \Pi^o_k u), K_h (w, \mu))_{\Omega} + S_h (\Pi^o_k u, \Pi^o_k u; w, \mu) + B_h (\Pi^o_k u; \Pi^o_k u, \Pi^o_k u; w, \mu) \\
= ((\Pi^o_k u), t), w)_{\Omega} + \nu (\nabla \cdot \Pi^o_{k-1} q, w)_{\Omega} - \nu (\nabla \cdot (\Pi^o_{k-1} q) \cdot n, w - \mu)_{\partial \Omega} + \nu \langle \tau (\Pi^o_k u - \Pi^o_k u), \Pi^o_k w - \mu \rangle_{\partial \Omega} \\
+ \sum_{i=0}^4 R_i.
\]

From the Green’s formula, the properties of the projections \( \Pi^o_{k-1}, \Pi^o_k \) and \( \Pi^3_k \), it follows

\[ R_1 = (\Pi^o_k u - u), t), w)_{\Omega} + (u, w), \]
\[ R_2 = \nu (\nabla \cdot (\Pi^o_{k-1} q), w)_{\Omega} - \nu (\Pi^o_{k-1} q \cdot n, \mu)_{\partial \Omega} \\
= -\nu (\nabla \cdot (\Pi^o_k u - \Pi^o_k u), w)_{\partial \Omega} + \nu \langle (q - \Pi^o_k q), n, w - \mu \rangle_{\partial \Omega} \\
+ \nu \langle (q - \Pi^o_k q) \cdot n, w \rangle_{\partial \Omega} - \nu \langle (q - \Pi^o_k q) \cdot n, \mu \rangle_{\partial \Omega} \\
= \nu \langle \tau (\Pi^o_k u - \Pi^o_k u), \Pi^o_k w - \mu \rangle_{\partial \Omega} + \nu \langle \tau (\Pi^o_k u - \Pi^o_k u), \Pi^o_k w - \mu \rangle_{\partial \Omega}, \]

and, by (3.20) we have

\[ R_4 = (b(u) \cdot \nabla u, w) + E_N (u, w, \mu). \]

Finally, combining the above equations implies the desired relation (3.22).

Lemma 3.11. For \( u \in H^{k+1}(\Omega) \), it holds

\[ |E_L(u,w,\mu)| \lesssim \nu h^k \| u \|_{k+1} \| (w, \mu) \|, \quad \forall (w, \mu) \in V_h \times \hat{V}_h. \]  

(3.23)

Proof. The desired conclusion follows from

\[
|\langle (\nabla u - \Pi^o_{k-1} \nabla u) \cdot n, w - \mu \rangle_{\partial \Omega}| \lesssim \sum_{K \in \mathcal{T}_h} |\nabla u - \Pi^o_{k-1} \nabla u|_{0,2,\partial K} h^{\frac{1}{2}} (h^{-\frac{1}{2}} |w - \mu|_{0,2,\partial K}) \\
\lesssim h^k \| \nabla u \|_{k+1} \| (w, \mu) \|, \\
|\langle \tau (\Pi^o_k u - u), \Pi^o_k w - \mu \rangle_{\partial \Omega}| \lesssim \sum_{K \in \mathcal{T}_h} \tau^\frac{1}{2} |\Pi^o_k u - u|_{0,2,\partial K} \tau^\frac{1}{2} |\Pi^o_k w - \mu|_{0,2,\partial K} \\
\lesssim h^k \| u \|_{k+1} \| (w, \mu) \|. \]

Set

\[
\xi^k = \Pi^o_{k-1} q - q_h, \quad \xi^k = \Pi^o_k u - u_h, \quad \xi^k = \Pi^o_k u - \tilde{u}_h, \quad \eta^k = q - \Pi^o_k q, \quad \eta^k = u - \Pi^o_k u, \quad \eta^k = u - \Pi^o_k u.
\]

By substracting (3.4) from (3.22), we can obtain the following error equations.
Lemma 3.12. Let \((u, q)\) be the solution to the problem (3.1) and \((q_h, u_h, \tilde{u}_h) \in Q_h \times V_h \times \tilde{V}_h\) be the solution of (3.4). Then, for any \((w, \mu) \in V_h \times \tilde{V}_h\), we have

\[
\begin{align*}
\xi_h^n + K_h(\xi_h^n, \xi_h^n) = 0, \\
\left( (\xi_h^n)_t, w \right)_{\Gamma_h} + \nu \left( K_h(\xi_h^n, \xi_h^n), K_h(w, \mu) \right)_{\Gamma_h} + S_h(\xi_h^n, \xi_h^n; w, \mu) \\
+ B_h(\Pi_h^n u; \Pi_h^n u, \Pi_h^n u; w, \mu) - B_h(u_h; u_h, \tilde{u}_h; w, \mu) \\
- (\Pi_h^n u - u, t)_{\Gamma_h} - E_L(u, w, \mu) - E_N(u, w, \mu) = 0.
\end{align*}
\] (3.24)

Lemma 3.13. Assume that \(u \in \mathcal{L}^2(\Omega)\), then we have

\[
B_h(\xi_h^n; \Pi_h^n u, \Pi_h^n u; \xi_h^n, \xi_h^n) \lesssim \left\| \xi_h^n \right\|_{0, \Gamma_h} \left\| u \right\|_{2, \Gamma_h} \left\| (\xi_h^n, \xi_h^n) \right\|, \forall (\xi_h^n, \xi_h^n) \in V_h \times \tilde{V}_h.
\] (3.25)

Proof. From the definition of \(B_h\), we have

\[
3B_h(\xi_h^n; \Pi_h^n u, \Pi_h^n u; \xi_h^n, \xi_h^n) = -(b(\xi_h^n) \Pi_h^n u, \nabla \xi_h^n)_{\Gamma_h} + (b(\xi_h^n) \cdot \nabla \Pi_h^n u, \xi_h^n)_{\partial \Omega_h} \\
= -(b(\xi_h^n) \Pi_h^n u, \nabla \xi_h^n)_{\Gamma_h} + (b(\xi_h^n) \cdot \nabla \Pi_h^n u, \xi_h^n)_{\partial \Omega_h} \\
+ (b(\xi_h^n) \cdot \nabla \Pi_h^n u, \xi_h^n - \xi_h^n)_{\partial \Omega_h} - (b(\xi_h^n) \cdot \nabla \Pi_h^n u - \Pi_h^n u, \xi_h^n)_{\partial \Omega_h} \\
= \sum_{i=1}^4 R_i.
\]

Using the Hölder inequality, Lemmas 3.1, 3.3 and 3.5, we can obtain

\[
R_1 = -(b(\xi_h^n) \Pi_h^n u, \nabla \xi_h^n)_{\Gamma_h} \lesssim \left\| \xi_h^n \right\|_{0, \Gamma_h} \| \Pi_h^n u \|_{0, \Gamma_h} \left\| \nabla \xi_h^n \right\|_{0, \Gamma_h} \lesssim \left\| \xi_h^n \right\|_{0, \Gamma_h} \left\| u \right\|_{2, \Gamma_h} \left\| (\xi_h^n, \xi_h^n) \right\|,
\]

\[
R_2 = (b(\xi_h^n) \cdot \nabla \Pi_h^n u, \xi_h^n)_{\Gamma_h} \lesssim \left\| \xi_h^n \right\|_{0, \Gamma_h} \left\| \nabla \Pi_h^n u \right\|_{0, \Gamma_h} \left\| \xi_h^n \right\|_{0, \Gamma_h} \lesssim \left\| \xi_h^n \right\|_{0, \Gamma_h} \left\| u \right\|_{2, \Gamma_h} \left\| (\xi_h^n, \xi_h^n) \right\|,
\]

\[
R_3 = (b(\xi_h^n) \cdot \nabla \Pi_h^n u, \xi_h^n - \xi_h^n)_{\partial \Omega_h} \lesssim h^{-\frac{1}{2}} \left\| \xi_h^n \right\|_{0, \Gamma_h} \left\| \Pi_h^n u \right\|_{0, \Gamma_h} \left\| \xi_h^n - \xi_h^n \right\|_{0, \Gamma_h} \lesssim \left\| \xi_h^n \right\|_{0, \Gamma_h} \left\| u \right\|_{2, \Gamma_h} \left\| (\xi_h^n, \xi_h^n) \right\|,
\]

\[
R_4 = -(b(\xi_h^n) \cdot \nabla \Pi_h^n u - \Pi_h^n u, \xi_h^n)_{\partial \Omega_h} \\
\leq h^{-\frac{1}{2}} \left\| \xi_h^n \right\|_{0, \Gamma_h} \left\| \Pi_h^n u - \Pi_h^n u \right\|_{0, \Gamma_h} h^{-\frac{1}{2}} \left\| \xi_h^n \right\|_{0, \Gamma_h} \\
\leq h^{-\frac{1}{2}} \left\| \xi_h^n \right\|_{0, \Gamma_h} \left\| \Pi_h^n u - u \right\|_{0, \Gamma_h} \left\| u - \Pi_h^n u \right\|_{0, \Gamma_h} h^{-\frac{1}{2}} \left\| \xi_h^n \right\|_{0, \Gamma_h} \\
\lesssim \left\| \xi_h^n \right\|_{0, \Gamma_h} \left\| u \right\|_{2, \Gamma_h} \left\| (\xi_h^n, \xi_h^n) \right\|.
\]

As a result, the estimate (3.25) holds.

Lemma 3.14. Let \((u, q)\) be the solution to the problem (3.1) with \(u \in \mathcal{L}^1(0, T; \mathcal{L}^{k+1}(\Omega))\) and \(u_t \in \mathcal{L}^2(0, T; \mathcal{L}^{k+1}(\Omega))\), and let \((q_h, u_h, \tilde{u}_h) \in Q_h \times V_h \times \tilde{V}_h\) be the solution of (3.4). Then we have

\[
\left\| \xi_h^n(t) \right\|_{0, \Gamma_h}^2 + \nu \int_0^t \left\| (\xi_h^n(\tau), \xi_h^n(\tau)) \right\|^2 d\tau \lesssim h^{2k} (\left\| u(t) \right\|_{k+1} + \left\| u_t(t) \right\|_{k+1}),
\] (3.26)

Proof. Taking \((w, \mu) = (\xi_h^n, \xi_h^n)\) in (3.24) and using the antisymmetry of \(B_h\), together with
Cauchy-Schwarz inequality, Lemmas 3.9, 3.11, 3.13 and the Young’s inequality, we have

\[
\frac{1}{2} \frac{d}{dt} \|\zeta_h^n\|^2_{0,T_n} + \nu \|\langle \zeta_h^n, \xi_h^n \rangle\|^2 \leq \langle \Pi_h^n u - u, \zeta_h^n \rangle_{T_n} + E_L(u, \zeta_h^n, \xi_h^n) + E_N(u; u, \zeta_h^n, \xi_h^n) + B_h(\zeta_h^n; \Pi_h^n u, \Pi_h^n u; \zeta_h^n, \xi_h^n)
\]

\[
\lesssim h^{k+1} \|u_t\|_{k+1} \|\zeta_h^n\|_{0,T_n} + C h^k (\|u\|_{k+1} \|u_t\| + \nu \|u\|_{k+1}) \|\zeta_h^n, \xi_h^n\|
\]

\[
+ C \|\zeta_h^n, \xi_h^n\|_{0,T_n} \|u_t\| \|\zeta_h^n, \xi_h^n\| \lesssim \frac{C h^{2k+2}}{2} \|u_t\|_{k+1}^2 + \frac{1}{2} \|\zeta_h^n\|_{0,T_n}^2 + C \frac{h^{2k}}{\nu} (\|u\|_{k+1} \|u_t\| + \nu \|u_t\|_{k+1})^2 + \frac{\nu}{4} \|\zeta_h^n, \xi_h^n\|_{0,T_n}^2.
\]

Here \(C\) is a positive constant independent of \(h\). Integrating the above inequality with respect to \(t\) yields

\[
\|\zeta_h^n(t)\|_{0,T_n}^2 + \nu \int_0^t \|\zeta_h^n, \xi_h^n\|_{0,T_n}^2 \, d\tau \leq \|\zeta_h^n(0)\|_{0,T_n}^2 + C \int_0^t \|\zeta_h^n, \xi_h^n\|_{0,T_n} \, d\tau + C h^{2k},
\]

which, together with the Gronwall’s inequality, gives the desired result (3.26). \(\square\)

From Lemma 3.14 and the triangle inequality, we easily get the following error estimates for the semi-discrete scheme.

**Theorem 3.3.** Let \((q(t), u(t))\) and \((q_h(t), u_h(t))\) be the solutions to the problem (3.1) and (3.4), respectively. Suppose that \(u \in H^1(0,T; H^{k+1}(\Omega)) \) and \(u_t \in L^2(0,T; H^k(\Omega))\), then it holds

\[
\|u(t) - u_h(t)\|_{0,T_n} \lesssim h^k (\|u(t)\|_{k+1} + \|u_t(t)\|_{k+1}), \quad t \in [0,T]
\]

and

\[
\left( \int_0^T \|\mathbf{q} - q_h(\tau)\|_{0,T_n}^2 \, d\tau \right)^{\frac{1}{2}} \lesssim h^k \left( \int_0^T (\|u(\tau)\|_{k+1}^2 + \|u_t(\tau)\|_{k+1}^2) \, d\tau \right)^{\frac{1}{2}}.
\]

## 4 Fully discrete HDG method

### 4.1 Backward Euler fully discrete scheme

Given a positive integer \(N\), let \(0 = t_0 < t_1 < \ldots < t_N = T\) be a uniform division of time domain \([0,T]\) with the time step \(\Delta t := \frac{T}{N}\). We refer to \(q_h^n, u_h^n, \bar{u}_h^n\) as the approximation of \(q_h(t_n), u_h(t_n), \bar{u}_h(t_n)\) respectively at the discrete time \(t_n = n \Delta t\) for \(n = 1, 2, \ldots, N\). By replacing the time derivative \((u_h)_t\) at time \(t_n\) by the backward difference quotient

\[
\partial_t u_h^n = \frac{u_h^n - u_h^{n-1}}{\Delta t}
\]

in (3.4), the linearized backward Euler HDG scheme is given as follows: for each \(1 \leq n \leq N\), find \((q_h^n, u_h^n, \bar{u}_h^n) \in Q_h \times V_h \times \hat{V}_h\) such that

\[
\begin{cases}
(\partial_t u_h^n, w)_{T_n} + \nu (K_h(u_h^n, \bar{u}_h^n), K_h(w, \mu))_{T_n} + S_h(u_h^n, \bar{u}_h^n; w, \mu) \\
\quad + B_h(u_h^n; u_h^n, \bar{u}_h^n; w, \mu) = (f^n, w)_{T_n},
\end{cases}
\]

for all \((w, \mu) \in V_h \times \hat{V}_h\).
Theorem 4.1. For the fully discrete scheme (4.1), we have the following stability result: for any $1 \leq n \leq N$,

$$\|u^n_h\|_{0,T_N}^2 + \sum_{i=1}^n \|u^n_h - u^{i-1}_h\|_{0,T_n}^2 + \nu \Delta t \sum_{i=1}^n \|((u^n_h, \hat{u}^n_h))\|^2 \lesssim \|u^0_h\|_{0,T_n}^2 + \frac{\Delta t}{\nu} \sum_{i=1}^n \|f^i\|_{0,T_n}^2. \quad (4.2)$$

Proof. Taking $(w, \mu) = (u^n_i, \hat{u}^n_i)$ in (4.1), we have

$$\frac{1}{\Delta t}((u^n_i - u^{i-1}_h, u^n_i)_{T_n} + \nu \|(u^n_i, \hat{u}^n_i)\|^2 = (f^i, u^n_h)_{T_n},$$

using the Cauchy-Schwarz inequality, the Young’s inequality and Lemma 3.5, we get

$$\frac{1}{\Delta t}((u^n_h)^2_{0,T_n} - \|u^{i-1}_h\|_{0,T_n}^2 + \|u^n_h - u^{i-1}_h\|_{0,T_n}^2) + \nu \|(u^n_i, \hat{u}^n_i)\|^2 \leq \frac{C}{2\nu} \|f^i\|_{0,T_n}^2 + \frac{\nu}{2} \|(u^n_i, \hat{u}^n_i)\|^2.$$

Summing up the above inequality from $i = 1$ to $i = n$ leads to the desired result.

The following theorem shows a result the existence and uniqueness of the fully discrete solution.

Theorem 4.2. Given $u^{n-1}_h$, the fully discrete scheme (4.1) admits a unique solution $(q^n_R, u^n_h, \hat{u}^n_h)$ for $1 \leq n \leq N$.

Proof. As (4.1) is a linear square system, we know that uniqueness is equivalent to existence. So we only need to show that if $(q^n_{1h}, u^n_{1h}, \hat{u}^n_{1h})$ and $(q^n_{2h}, u^n_{2h}, \hat{u}^n_{2h})$ are two solutions of (4.1), then $(q^n_{1h}, u^n_{1h}, \hat{u}^n_{1h}) = (q^n_{2h}, u^n_{2h}, \hat{u}^n_{2h})$.

In fact, we have

$$\frac{u^n_{1h} - u^{n-1}_h}{\Delta t}, w)_{T_n} + \nu (K_h(u^n_{1h}, \hat{u}^n_{1h}), K_h(w, \mu))_{T_n} + S_h(u^n_{1h}, \hat{u}^n_{1h}; w, \mu) + \mathcal{B}_h(u^{n-1}_h; u^n_{1h}, \hat{u}^n_{1h}; w, \mu) = (f^n, w)_{T_n}, \quad (4.3)$$

$$\frac{u^n_{2h} - u^{n-1}_h}{\Delta t}, w)_{T_n} + \nu (K_h(u^n_{2h}, \hat{u}^n_{2h}), K_h(w, \mu))_{T_n} + S_h(u^n_{2h}, \hat{u}^n_{2h}; w, \mu) + \mathcal{B}_h(u^{n-1}_h; u^n_{2h}, \hat{u}^n_{2h}; w, \mu) = (f^n, w)_{T_n}. \quad (4.4)$$

Subtracting (4.4) from (4.3), taking $(w, \mu) = (u^n_{1h} - u^n_{2h}, \hat{u}^n_{1h} - \hat{u}^n_{2h}) = (\eta^n, \bar{\eta}^n)$ and using the antisymmetry of $\mathcal{B}_h$, we get

$$\frac{1}{\Delta t}((\eta^n, \bar{\eta}^n) + \nu \|\eta^n, \bar{\eta}^n\|^2 = 0,$$

which means that $\eta^n = \bar{\eta}^n = 0$. This completes the proof.

4.2 A priori error estimation

We first recall the discrete version of the Gronwall inequality in a slightly more general form used in [18].

Lemma 4.1. Let $\rho, G$ and $a_j, b_j, c_j, \gamma_j$, for integers $j \geq 0$, be nonnegative numbers such that

$$a_n + \rho \sum_{j=0}^n b_j \leq \rho \sum_{j=0}^n \gamma_j a_j + \rho \sum_{j=0}^n c_j + G, \quad \forall n \geq 0.$$

Suppose that $\rho \gamma_j < 1$ for all $j$, and set $\sigma_j = (1 - \rho \gamma_j)^{-1}$, then

$$a_n + \rho \sum_{j=0}^n b_j \leq \exp(\rho \sum_{j=0}^n \sigma_j \gamma_j) (\rho \sum_{j=0}^n c_j + G), \quad \forall n \geq 0.$$
By following the same line as in the proof of Lemma 3.10, we can derive the following lemma.

**Lemma 4.2.** Let \((u, q)\) be the solution to the problem (3.1), then for any \((w, \mu)\) \(\in \mathcal{V}_h \times \hat{\mathcal{V}}_h\), it holds

\[
\begin{align*}
\Pi_{k-1}^n q(t_n) &= -\mathcal{K}_h(\Pi_{k}^n u(t_n), \Pi_{k}^n u(t_n)), \\
(\partial_t \Pi_{k}^n u(t_n), w)_{\mathcal{T}_h} + \nu (\mathcal{K}_h(\Pi_{k}^n u(t_n), \Pi_{k}^n u(t_n)), \mathcal{K}_h(w, \mu))_{\mathcal{T}_h} \\
&\quad + S_h(\Pi_{k}^n u(t_n), \Pi_{k}^n u(t_n); w, \mu) + \mathcal{B}_h(\Pi_{k}^n u(t_n); \Pi_{k}^n u(t_n), \Pi_{k}^n u(t_n); w, \mu) \\
&\quad = (\partial_t u(t_n) - u(t_n), w)_{\mathcal{T}_h} + (f(t_n), w) \\
&\quad + E_L(u(t_n), w, \mu) + E_N(u(t_n); u(t_n), w, \mu).
\end{align*}
\]  
(4.5)

**Theorem 4.3.** Let \((q(t), u(t))\) and \((q_h^n, u_h^n)\) be the solutions to the problem (3.1) and (4.1), respectively. Suppose \(u \in L^\infty(0, T; H^{k+1}(\Omega))\), \(u_t \in L^\infty(0, T; H^{k+1}(\Omega))\) and \(u_{tt} \in L^2(0, T; H^{k+1}(\Omega))\). Then for any \(1 \leq n \leq N\), it holds the error estimate

\[
\|u(t_n) - u_h^n\|_{0,T_h}^2 + \Delta t \sum_{j=1}^n \|q(t_j) - q_h^n\|_{0,T_h}^2 \\
\lesssim h^{2k}(\sum_{j=1}^n \|u(t_j)\|_{2k+1}^2 + \int_{t_0}^{t_n} \|u(t)\|_{2k+1}^2 ds + \Delta t^2(\int_{t_0}^{t_n} \|u(t)\|_{2k+1}^2 ds + \max_{x \in \Omega, t \in [0, t_n]} |u(t)|^2)).
\]  
(4.6)

**Proof.** Subtracting (4.1) from (4.5), for any time \(t_j\) we have

\[
\begin{align*}
(\partial_t \xi_i^n(t_j), w)_{\mathcal{T}_h} + \nu (\mathcal{K}_h(\xi_i^n(t_j), \xi_i^n(t_j)), \mathcal{K}_h(w, \mu))_{\mathcal{T}_h} + S_h(\xi_i^n(t_j), \xi_i^n(t_j); w, \mu) \\
&\quad - \mathcal{B}_h(u_h^n - u^n, h^n; w, \mu) - \mathcal{B}_h(u_h^n - u^n, h^n; w, \mu) \\
&\quad + (\partial_t \Pi_{k}^n u(t_j) - u(t_j), w)_{\mathcal{T}_h} + E_L(u(t_j), w, \mu) + E_N(u(t_j); u(t_j), w, \mu).
\end{align*}
\]  
(4.7)

Taking \((w, \mu) = (\xi_i^n(t_j), \xi_i^n(t_j))\) in (4.7), we get

\[
\begin{align*}
\frac{1}{\Delta t} (\xi_i^n(t_j) - \xi_i^n(t_{j-1}), \xi_i^n(t_j))_{\mathcal{T}_h} + \nu \|((\xi_i^n(t_j), \xi_i^n(t_j))\|_2^2 \\
= (\partial_t \Pi_{k}^n u(t_j) - u(t_j), \xi_i^n(t_j))_{\mathcal{T}_h} + E_L(u(t_j), \xi_i^n(t_j), \xi_i^n(t_j)) \\
&\quad + E_N(u(t_j); u(t_j), \xi_i^n(t_j), \xi_i^n(t_j)) - \mathcal{B}_h(\xi_i^n(t_j); \Pi_{k}^n u(t_j), \Pi_{k}^n u(t_j); \xi_i^n(t_j), \xi_i^n(t_j)) \\
&\quad - \mathcal{B}_h(u_h^n - u_h^{n-1}; \Pi_{k}^n u(t_j), \Pi_{k}^n u(t_j); \xi_i^n(t_j), \xi_i^n(t_j)) \\
:= Q_1^n + Q_2^n + Q_3^n + Q_4^n + Q_5^n.
\end{align*}
\]  
(4.8)

From the Cauchy-Schwarz inequality, it follows

\[
Q_1^n = (\partial_t \Pi_{k}^n u(t_j) - u(t_j), \xi_i^n(t_j))_{\mathcal{T}_h} \\
= (\partial_t \Pi_{k}^n u(t_j) - \partial_t u(t_j), \xi_i^n(t_j))_{\mathcal{T}_h} + (\partial_t u(t_j) - u(t_j), \xi_i^n(t_j))_{\mathcal{T}_h} \\
\leq (\|\partial_t \Pi_{k}^n u(t_j) - \partial_t u(t_j)\|_{0,T_h} + \|\partial_t u(t_j) - u(t_j)\|_{0,T_h}) \|\xi_i^n(t_j)\|_{0,T_h}.
\]

Then using the property of projection yields

\[
\|\partial_t \Pi_{k}^n u(t_j) - \partial_t u(t_j)\|_{0,T_h} = \|\Pi_{k}^n \partial_t u(t_j) - \partial_t u(t_j)\|_{0,T_h} = \frac{1}{\Delta t} \int_{t_{j-1}}^{t_j} |\Pi_{k}^n u_t - u_t| ds \\
\lesssim \frac{h^{k+1}}{\Delta t} \int_{t_{j-1}}^{t_j} |u_t|_{k+1} ds \\
\lesssim \frac{h^{k+1}}{\sqrt{\Delta t}} (\int_{t_{j-1}}^{t_j} |u_t|_{k+1}^2 ds)^{\frac{1}{2}}.
\]
Similarly, we can get
\[
\|\partial_t u(t_j) - u_t(t_j)\|_{0,\tau_n} \leq \frac{1}{\Delta t} \left( \int_{t_{j-1}}^{t_j} (s - t_{j-1}) \|u_{tt}\|_0 ds \right) \leq \frac{1}{\Delta t} \left( \int_{t_{j-1}}^{t_j} (s - t_{j-1})^2 ds \right)^{\frac{1}{2}} \left( \int_{t_{j-1}}^{t_j} \|u_{tt}\|^2 ds \right)^{\frac{1}{2}} \leq \sqrt{\Delta t} \left( \int_{t_{j-1}}^{t_j} \|u_{tt}\|^2 ds \right)^{\frac{1}{2}}.
\]

So
\[
Q_j^1 \lesssim \frac{h^{k+1}}{\sqrt{\Delta t}} \left( \int_{t_{j-1}}^{t_j} |u_t|_{k+1} ds \right)^{\frac{1}{2}} + \sqrt{\Delta t} \left( \int_{t_{j-1}}^{t_j} \|u_{tt}\|^2 ds \right)^{\frac{1}{2}} \|\xi_h(t_j)\|_{0,\tau_n}.
\]

By Lemmas 3.9, 3.11 and 3.13, we have
\[
Q_2^1 \lesssim h^k \|u(t_j)\|_{k+1} \|\xi_h(t_j), \xi_h^u(t_j)\|, \\
Q_3^1 \lesssim h^k \|u(t_j)\|_{k+1} \|u(t_j)\|_2 \|\xi_h(t_j), \xi_h^u(t_j)\|, \\
Q_4^1 \lesssim \|\xi_h^u(t_j)\|_{0,\tau_n} \|u(t_j)\|_2 \|\xi_h(t_j), \xi_h^u(t_j)\|, \\
Q_5^1 \lesssim \|u_h^j - u_h^{j-1}\|_{0,\tau_n} \|u(t_j)\|_2 \|\xi_h(t_j), \xi_h^u(t_j)\|.
\]

From the triangle inequality and the property of projection, it follows
\[
\|u_h^j - u_h^{j-1}\|_{0,\tau_n} = \|u_h^j - \Pi_h^u u(t_j) + \Pi_h^u u(t_j) - u_h^{j-1} + \Pi_h^u u(t_j) - u(t_{j-1})\|_{0,\tau_n} \\
\lesssim \|\xi_h^u(t_j) - \xi_h^u(t_{j-1})\|_{0,\tau_n} + \|\Pi_h^u (u(t_j) - u(t_{j-1})) - (u(t_j) - u(t_{j-1}))\|_{0,\tau_n} \\
+ \|u(t_j) - u(t_{j-1})\|_{0,\tau_n} \\
\lesssim \|\xi_h^u(t_j) - \xi_h^u(t_{j-1})\|_{0,\tau_n} + h^{k+1} \|u(t_j) - u(t_{j-1})\|_{k+1} + \Delta t \max_{x \in \Omega, t \in [t_{j-1}, t_j]} |u_t|.
\]

Thus,
\[
Q_j^1 \lesssim \left( \|\xi_h^u(t_j) - \xi_h^u(t_{j-1})\|_{0,\tau_n} + h^{k+1} \|u(t_j) - u(t_{j-1})\|_{k+1} \right) + \Delta t \max_{x \in \Omega, t \in [t_{j-1}, t_j]} |u_t| \|u_{tt}\|^2 \|\xi_h(t_j), \xi_h^u(t_j)\|.
\]

Substituting the estimates of \(Q_m\) \((m = 1, 2, \cdots, 5)\) into (4.8), summing up the obtained inequality from \(j = 1\) to \(j = n\), and noticing that
\[
\frac{1}{\Delta t} \left( \xi_h^u(t_j) - \xi_h^u(t_{j-1}) \right) \|\tau_n\| = \frac{1}{2\Delta t} \left( \|\xi_h^u(t_j)\|^2_{0,\tau_n} - \|\xi_h^u(t_{j-1})\|^2_{0,\tau_n} + \|\xi_h^u(t_j) - \xi_h^u(t_{j-1})\|^2_{0,\tau_n} \right),
\]
we get
\[ \|\xi_h^*(t_n)\|_{\mathcal{T}_h}^2 + \sum_{j=1}^{n} \|\xi_h^*(t_j) - \xi_h^*(t_{j-1})\|_{\mathcal{T}_h}^2 + 2\nu \Delta t \sum_{j=1}^{n} \|\langle \xi_h^*(t_j), \xi_h^*(t_j) \rangle\|^2 \]
\[ \lesssim \Delta t \sum_{j=1}^{n} \left( \frac{h^{k+1}}{\Delta t} \left( \int_{t_{j-1}}^{t_j} |u_t|^2 \, ds \right) \right)^2 + \sqrt{\Delta t} \left( \int_{t_{j-1}}^{t_j} \|u_{tt}\|^2 \, ds \right) \left\|\xi_h^*(t_j)\right\|_{0,\mathcal{T}_h} \]
\[ + \Delta t \sum_{j=1}^{n} (h^k \|u(t_j)\|_{k+1} + \|u(t_{j-1})\|_{k+1}) + \Delta t \max_{x \in \Omega, t \in [0,t_n]} |u_t| + \|\xi_h^*(t_j)\|_{0,\mathcal{T}_h} \]
\[ + \|\xi_h^*(t_j) - \xi_h^*(t_{j-1})\|_{0,\mathcal{T}_h} \|\xi_h^*(t_j), \xi_h^*(t_j)\| \]
\[ \lesssim h^{2k+2} \int_0^{t_n} |u_t|^2 \, ds + \Delta t^2 \int_0^{t_n} \|u_{tt}\|^2 \, ds + \Delta t \sum_{j=1}^{n} \|\xi_h^*(t_j)\|_{0,\mathcal{T}_h}^2 \]
\[ + h^{2k} \sum_{j=0}^{n} \|u(t_j)\|^2_{k+1} + \Delta t^2 \max_{x \in \Omega, t \in [0,t_n]} |u_t|^2 + \Delta t \sum_{j=1}^{n} \|\xi_h^*(t_j)\|_{0,\mathcal{T}_h}^2 + \Delta t \sum_{j=1}^{n} \|\xi_h^*(t_j) - \xi_h^*(t_{j-1})\|_{0,\mathcal{T}_h}^2 \]
\[ + \|\xi_h^*(t_j), \xi_h^*(t_j)\| \]
\[ \|\xi_h^*(t_j)\|_{0,\mathcal{T}_h} \]
\[ \lesssim (4.9) \]

which, together with Lemma 4.1 and the triangle inequality, indicates the desired result. \( \Box \)

**Remark 4.1.** Due to the use of backward Euler scheme for the temporal discretization, the fully discretization (4.4) is only of first order temporal accuracy. In fact, we can also apply other higher order implicit time-stepping schemes such as the diagonally implicit Runge–Kutta (DIRK) methods.

Consider the following two-stage and third-order DIRK(2,3) formulas \[2\] written in the form of Butcher’s table for time integration:

\[
\begin{array}{ccc}
0 & 1 & 0 \\
1 & 2 & 2 \\
\end{array}
\]

where

\[
\begin{array}{ccc}
a_{11} & a_{12} & c_1 \\
b_{1} & b_2 & c_2 \\
\end{array}
\]

We apply the DIRK(2,3) method to the semi-discrete HDG scheme (3.4). To simplify notation we write

- \( t_{n,i} \) for \( t^n + c_i \Delta t \),

- \( y_h^{n,i} \) for \( (q_h^{n,i}, u_h^{n,i}) \), and

- \( y_h^{n,i} \) for \( (q_h^{n,i}, u_h^{n,i}) = (q_h(t_{n,i}), u_h(t_{n,i})) \).

The numerical solution \( y_h^{n+1} = (q_h^{n+1}, u_h^{n+1}) \) at time level \( n + 1 \) given by the DIRK(2,3) method is computed as follows:

\[
y_h^{n+1} = y_h^n + \Delta t (b_1 f_{h,1} + b_2 f_{h,2}), \quad (4.10)
\]

where

\[
f_{h,1} = \frac{y_h^{n+1} - y_h^n}{a_{11} \Delta t}, \quad f_{h,2} = \frac{y_h^{n+2} - y_h^n}{a_{22} \Delta t} - \frac{a_{21}}{a_{22}} f_{h,1}. \quad (4.11)
\]

The intermediate states \( y_h^{n,i} = (q_h^{n,i}, u_h^{n,i}), i = 1, 2 \), with the Ossen iteration, are determined as follows: given \( u_h^{n,i,0} \), find \( (q_h^{n,i}, u_h^{n,i,0}, \xi_h^{n,i}) = (q_h^{n,i}, u_h^{n,i,0}, \xi_h^{n,i}) \in Q_h \times V_h \times \hat{V}_h \) such
that for \( r = 1, 2, \ldots \),

\[
\begin{cases}
q_h^{n,i,r} + \mathcal{K}_h(u_h^{n,i,r}, \hat{u}_h^{n,i,r}) = 0, \\
\left( \frac{1}{a_{ii}} \Delta t \right) u_h^{n,i,r}(w)_{\tau_h} + \nu(\mathcal{K}_h(u_h^{n,i,r}, \hat{u}_h^{n,i,r}), \mathcal{K}_h(w, \mu))_{\tau_h} \\
+ \mathcal{S}_h(u_h^{n,i,r}, \hat{u}_h^{n,i,r}; w, \mu) + \mathcal{B}_h(u_h^{n,i,r-1}; u_h^{n,i,r}, \hat{u}_h^{n,i,r}; w, \mu) = (f(t^n, i), w)_{\tau_h} + (\hat{z}_h^{n,i}, w)_{\tau_h}, \\
u_0^i = \Pi_b u_0,
\end{cases}
\]  

(4.12)

and the terms \( z_h^{n,i} \), \( i = 1, 2 \), on the right-hand side of (4.12) are given by

\[
z_h^{n,1} = \frac{u_h^n}{a_{11} \Delta t}, \quad z_h^{n,2} = \frac{u_h^n}{a_{22} \Delta t} + \frac{a_{21}}{a_{11} \Delta t} \cdot \frac{u_h^{n-1}}{a_{11} \Delta t} - z_h^{n,1}.
\]

We note that the resulting system (4.12) at each \( (i) \)-th stage of the DIRK(2,3) method is very similar to the backward–Euler system (4.1). And we will give some numerical experiments in next section to show the efficiency of DIRK(2,3) fully discrete scheme (4.10).

5 Numerical experiments

In this section, we present some numerical results to demonstrate the performance of our proposed fully discrete HDG schemes for solving the Burgers’ equation.

We consider two cases of the HDG methods with \( k \geq 1 \):

\[
\begin{align*}
\text{HDG - I : } & l = k, \\
\text{HDG - II : } & l = k - 1.
\end{align*}
\]  

(5.1)

Example 5.1. This example is to test the accuracy of the back Euler fully discrete scheme (4.1). Take \( \Omega = [0, 1] \times [0, 1], T = 1, \) \( \nu = 1, 0.01 \). The exact solution to the problem (1.1a)-(1.1c) is given by

\[
u = e^{-t} x(x - 1) y(y - 1), \quad \text{in } \Omega \times [0, T],
\]

Then the force term and the boundary condition can be derived explicitly.

We use \( M \times M \) uniform triangular meshes (c.f. Figure 1) for the spatial discretization and, to verify the spatial accuracy, take the time step as \( \Delta t = h^2 / 2 \) (i.e. \( N = M^2 \)) for \( k = 1 \) and \( \Delta t = \sqrt{2} h^3 / 4 \) for \( k = 2 \), respectively.

Some numerical results of the relative errors for the approximations of \( u \) and \( q \) at the final time with \( k = 1, 2 \) are shown in Tables 1-4. We can see that the scheme (4.1) yields \( (k + 1) \)-th and \( (k) \)-th spatial convergence orders of \( \| u(T) - u_h(T) \|_0 \) and \( \| q(T) - q_h(T) \|_0 \), respectively. The results of \( \| u(T) - u_h(T) \|_0 \) are conformable to Theorems 4.3.
Figure 1: The domain : $4 \times 4$ (left) and $8 \times 8$ (right) mesh

Table 1: History of convergence for Example 5.1 with $\nu = 1, k = 1$

(a) Method: HDG-I ($l = 1$)

| mesh  | $\|u(T) - u_h(T)\|_{0}$ | error | order | $\|q(T) - q_h(T)\|_{0}$ | error | order |
|-------|--------------------------|--------|--------|--------------------------|--------|--------|
| $4 \times 4$ | $2.1597e-01$ | - | - | $2.9311e-01$ | - | - |
| $8 \times 8$ | $4.1132e-02$ | 2.00 | 0.98 | $1.4864e-01$ | 2.00 | 1.00 |
| $16 \times 16$ | $1.3543e-02$ | 2.00 | 0.99 | $3.7322e-02$ | 2.00 | 1.00 |
| $32 \times 32$ | $6.3865e-03$ | 2.00 | 1.00 | $1.8665e-02$ | 2.00 | 1.00 |

(b) Method: HDG-II ($l = 0$)

| mesh  | $\|u(T) - u_h(T)\|_{0}$ | error | order | $\|q(T) - q_h(T)\|_{0}$ | error | order |
|-------|--------------------------|--------|--------|--------------------------|--------|--------|
| $4 \times 4$ | $2.4145e-01$ | - | - | $3.1279e-01$ | - | - |
| $8 \times 8$ | $6.0180e-02$ | 2.00 | 0.98 | $1.5806e-01$ | 2.00 | 1.00 |
| $16 \times 16$ | $1.5038e-02$ | 2.00 | 1.00 | $7.9255e-02$ | 2.00 | 1.00 |
| $32 \times 32$ | $3.7593e-03$ | 2.00 | 1.00 | $3.9656e-02$ | 2.00 | 1.00 |
| $64 \times 64$ | $9.3980e-04$ | 2.00 | 1.00 | $1.9832e-02$ | 2.00 | 1.00 |

Table 2: History of convergence for Example 5.1 with $\nu = 0.01, k = 1$

(a) Method: HDG-II ($l = 1$)

| mesh  | $\|u(T) - u_h(T)\|_{0}$ | error | order | $\|q(T) - q_h(T)\|_{0}$ | error | order |
|-------|--------------------------|--------|--------|--------------------------|--------|--------|
| $4 \times 4$ | $1.1207e-01$ | - | - | $2.9056e-01$ | - | - |
| $8 \times 8$ | $3.3444e-02$ | 1.74 | 0.96 | $1.4978e-01$ | 1.97 | 1.00 |
| $16 \times 16$ | $8.5650e-03$ | 1.97 | 1.00 | $7.4578e-02$ | 2.00 | 1.00 |
| $32 \times 32$ | $2.1460e-03$ | 2.00 | 1.00 | $3.7359e-02$ | 2.00 | 1.00 |
| $64 \times 64$ | $5.3674e-04$ | 2.00 | 1.00 | $1.8669e-02$ | 2.00 | 1.00 |

(b) Method: HDG-II ($l = 0$)

| mesh  | $\|u(T) - u_h(T)\|_{0}$ | error | order | $\|q(T) - q_h(T)\|_{0}$ | error | order |
|-------|--------------------------|--------|--------|--------------------------|--------|--------|
| $4 \times 4$ | $1.3157e-01$ | - | - | $3.0275e-01$ | - | - |
| $8 \times 8$ | $4.1890e-02$ | 1.65 | 0.89 | $1.6356e-01$ | 1.65 | 1.00 |
| $16 \times 16$ | $8.5650e-03$ | 1.97 | 1.00 | $7.4578e-02$ | 2.00 | 1.00 |
| $32 \times 32$ | $2.5616e-03$ | 2.00 | 1.00 | $3.9656e-02$ | 2.00 | 1.00 |
| $64 \times 64$ | $6.3806e-04$ | 2.00 | 1.00 | $1.9936e-02$ | 2.00 | 1.00 |
Table 3: History of convergence for Example 5.1 with $\nu = 1, k = 2$

(a) Method: HDG-II($l = 2$)

| mesh     | $\|u(T) - u_h(T)\|_0$ | $\|q(T) - q_h(T)\|_0$ |
|----------|------------------------|------------------------|
|          | error                  | order                  | error                  | order                  |
| 4 $\times$ 4 | 1.3706e-02              | –                      | 4.1763e-02              | –                      |
| 8 $\times$ 8 | 1.5937e-03              | 3.10                   | 1.0597e-02              | 1.98                   |
| 16 $\times$ 16 | 1.9284e-04              | 3.05                   | 2.6642e-03              | 1.99                   |
| 32 $\times$ 32 | 2.3736e-05              | 3.02                   | 6.6756e-04              | 2.00                   |
| 64 $\times$ 64 | 2.9445e-06              | 3.01                   | 1.6705e-04              | 2.00                   |

(b) Method: HDG-II($l = 1$)

| mesh     | $\|u(T) - u_h(T)\|_0$ | $\|q(T) - q_h(T)\|_0$ |
|----------|------------------------|------------------------|
|          | error                  | order                  | error                  | order                  |
| 4 $\times$ 4 | 1.4714e-02              | –                      | 3.3600e-02              | –                      |
| 8 $\times$ 8 | 1.7267e-03              | 3.09                   | 1.0915e-02              | 1.98                   |
| 16 $\times$ 16 | 2.1016e-04              | 3.04                   | 2.7449e-03              | 1.99                   |
| 32 $\times$ 32 | 2.5953e-05              | 3.02                   | 6.8868e-04              | 2.00                   |
| 64 $\times$ 64 | 3.2566e-06              | 3.01                   | 1.7228e-04              | 2.00                   |

Table 4: History of convergence for Example 5.1 with $\nu = 0.01, k = 2$

(a) Method: HDG-II($l = 2$)

| mesh     | $\|u(T) - u_h(T)\|_0$ | $\|q(T) - q_h(T)\|_0$ |
|----------|------------------------|------------------------|
|          | error                  | order                  | error                  | order                  |
| 4 $\times$ 4 | 9.4294e-02              | –                      | 1.0218e-01              | –                      |
| 8 $\times$ 8 | 1.2210e-02              | 2.95                   | 1.6179e-02              | 2.66                   |
| 16 $\times$ 16 | 1.5272e-03              | 3.00                   | 3.0797e-03              | 2.39                   |
| 32 $\times$ 32 | 1.9068e-04              | 3.00                   | 6.9507e-04              | 2.15                   |
| 64 $\times$ 64 | 2.3817e-05              | 3.00                   | 1.6880e-04              | 2.04                   |

(b) Method: HDG-II($l = 1$)

| mesh     | $\|u(T) - u_h(T)\|_0$ | $\|q(T) - q_h(T)\|_0$ |
|----------|------------------------|------------------------|
|          | error                  | order                  | error                  | order                  |
| 4 $\times$ 4 | 9.4922e-02              | –                      | 1.0298e-01              | –                      |
| 8 $\times$ 8 | 1.2250e-02              | 2.95                   | 1.6577e-02              | 2.64                   |
| 16 $\times$ 16 | 1.5310e-03              | 3.00                   | 3.1770e-03              | 2.38                   |
| 32 $\times$ 32 | 1.9114e-04              | 3.00                   | 7.2041e-04              | 2.14                   |
| 64 $\times$ 64 | 2.3876e-05              | 3.00                   | 1.7523e-04              | 2.04                   |

Example 5.2. This example is to test the accuracy of the DIRK(2,3) scheme $(1.10)$. Take $\Omega = [0,1] \times [0,1], T = 1$ and $\nu = 0.1$. The exact solution to the problem $(1.1a)$-$(1.1c)$ is given by

$$u = (e^t - 1)xy \tanh\left(\frac{1-x}{\nu}\right) \tanh\left(\frac{1-y}{\nu}\right), \quad in \ \Omega \times [0,T].$$

We use $M \times M$ uniform triangular spatial meshes (c.f. Figure 1) for the computation.

To test the temporal accuracy, we take $k = 3$ and use a very fine spatial mesh with $N = 256$. Numerical results of the errors at the final time $T$ Table 5 show that the temporal convergence rate of the scheme is close to third order.

To verify the spatial accuracy, we adopt a small time step $\Delta t = 0.005$ so that the overall error is governed by the spatial error. Numerical results in Tables 6 and 7 for $k = 1$ and $k = 2$ show that scheme gives $(k+1)$-th and $(k)$-th spatial convergence orders of $\|u(T) - u_h(T)\|_0$ and $\|q(T) - q_h(T)\|_0$, respectively.
Example 5.3. This is a three-dimensional example to test the accuracy of the DIRK(2,3) scheme (4.10). We take $\Omega = [0,1] \times [0,1] \times [0,1]$, $T = 1$ and $\nu = 1$. The exact solution to the problem (1.1a)-(1.1c) is of the form

$$u = e^{-t}(1-x)y(1-y)z(1-z), \quad \text{in } \Omega \times [0,T].$$

We use $N \times N \times N$ uniform triangular spatial meshes (c.f. Figure 2) for the computation.
To test the spatial accuracy, we take a small time step $\Delta t = 0.005$. Numerical results are given in Table 8 and 9 for $k = 1$ and $k = 2$, respectively, which show that the scheme (4.10) yields $(k+1)$-th and $k$-th spatial convergence orders of $\|u(T) - u_h(T)\|_0$ and $\|q(T) - q_h(T)\|_0$, respectively.

### Table 8: History of convergence with $k = 1$: Example 5.3

(a) Method: HDG-I($l = 1$)

| mesh | $\|u(T) - u_h(T)\|_0$ | error | order | $\|q(T) - q_h(T)\|_0$ | error | order |
|------|----------------------|-------|-------|----------------------|-------|-------|
| $2 \times 2 \times 2$ | $6.9815e-01$ | - | - | $5.6066e-01$ | - | - |
| $4 \times 4 \times 4$ | $1.7672e-01$ | 1.98 | - | $3.0285e-01$ | 0.89 | - |
| $8 \times 8 \times 8$ | $4.4207e-02$ | 2.00 | - | $1.5438e-01$ | 0.97 | - |
| $16 \times 16 \times 16$ | $1.0350e-02$ | 2.00 | - | $7.7566e-02$ | 0.99 | - |
| $32 \times 32 \times 32$ | $2.5875e-03$ | 2.00 | - | $3.8830e-02$ | 1.00 | - |

(b) Method: HDG-II($l = 0$)

| mesh | $\|u(T) - u_h(T)\|_0$ | error | order | $\|q(T) - q_h(T)\|_0$ | error | order |
|------|----------------------|-------|-------|----------------------|-------|-------|
| $2 \times 2 \times 2$ | $8.8064e-01$ | - | - | $6.1036e-01$ | - | - |
| $4 \times 4 \times 4$ | $2.0917e-01$ | 2.07 | - | $3.1971e-01$ | 0.93 | - |
| $8 \times 8 \times 8$ | $5.1528e-02$ | 2.02 | - | $1.6186e-01$ | 0.98 | - |
| $16 \times 16 \times 16$ | $1.2835e-02$ | 2.00 | - | $8.1193e-02$ | 0.99 | - |
| $32 \times 32 \times 32$ | $3.2055e-03$ | 2.00 | - | $4.0630e-02$ | 1.00 | - |

### Table 9: History of convergence with $k = 2$: Example 5.3

(a) Method: HDG-I($l = 2$)

| mesh | $\|u(T) - u_h(T)\|_0$ | error | order | $\|q(T) - q_h(T)\|_0$ | error | order |
|------|----------------------|-------|-------|----------------------|-------|-------|
| $2 \times 2 \times 2$ | $1.3095e-01$ | - | - | $1.9906e-01$ | - | - |
| $4 \times 4 \times 4$ | $1.4825e-02$ | 3.14 | - | $5.3052e-02$ | 1.85 | - |
| $8 \times 8 \times 8$ | $1.7531e-03$ | 3.08 | - | $1.3659e-02$ | 1.96 | - |
| $16 \times 16 \times 16$ | $2.1463e-04$ | 3.03 | - | $3.4459e-03$ | 1.99 | - |

(b) Method: HDG-II($l = 1$)

| mesh | $\|u(T) - u_h(T)\|_0$ | error | order | $\|q(T) - q_h(T)\|_0$ | error | order |
|------|----------------------|-------|-------|----------------------|-------|-------|
| $2 \times 2 \times 2$ | $1.4706e-01$ | - | - | $1.9582e-01$ | - | - |
| $4 \times 4 \times 4$ | $1.6034e-02$ | 3.20 | - | $5.3962e-02$ | 1.86 | - |
| $8 \times 8 \times 8$ | $1.8868e-03$ | 3.09 | - | $1.3871e-02$ | 1.96 | - |
| $16 \times 16 \times 16$ | $2.3104e-04$ | 3.03 | - | $3.4979e-03$ | 1.99 | - |
| $32 \times 32 \times 32$ | $3.2681e-05$ | 3.01 | - | $8.7713e-04$ | 2.00 | - |

### 6 Conclusion

In this paper, we have developed a class of semi-discrete and fully discrete HDG methods for the Burgers’ equation in two and three dimensions. The existence and uniqueness of the semi-discrete solution and error estimation for the semi-discrete and fully discrete schemes have been derived. Finally, numerical experiments have verified the theoretical results.
Figure 2: The domain $2 \times 2 \times 2$ (left) and $4 \times 4 \times 4$ (right) mesh

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