Tail sums of Wishart and GUE eigenvalues beyond the bulk edge.

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Abstract
Consider the classical Gaussian unitary ensemble of size \( N \) and the real Wishart ensemble \( W_N(n, I) \). In the limits as \( N \to \infty \) and \( N/n \to \gamma > 0 \), the expected number of eigenvalues that exit the upper bulk edge is less than one, 0.031 and 0.170 respectively, the latter number being independent of \( \gamma \). These statements are consequences of quantitative bounds on tail sums of eigenvalues outside the bulk which are established here for applications in high dimensional covariance matrix estimation.

1 Introduction
This paper develops some tail sum bounds on eigenvalues outside the bulk that are needed for results on estimation of covariance matrices in the spiked model, Donoho et al. [2017]. This application is described briefly in Section 4. It depends on properties of the eigenvalues of real white Wishart matrices, distributed as \( W_N(n, I) \), which are the main focus of this note.

Specifically, suppose that \( A \sim W_N(n, I) \), and that \( \lambda_1 \geq \ldots \geq \lambda_N \) are eigenvalues of the sample covariance matrix \( n^{-1}A \). In the limit \( N/n \to \gamma > 0 \), it is well known that the empirical distribution of \( \{\lambda_i\} \) converges to the Marcenko-Pastur law (see e.g. Pastur and Shcherbina [2011], Corollary 7.2.5), which is supported on an interval \( I_\gamma \) — augmented with 0 if \( \gamma > 1 \) — having upper endpoint \( \lambda(\gamma) = (1 + \sqrt{\gamma})^2 \). We focus on the eigenvalues \( \lambda_i \) that exit this “bulk” interval \( I_\gamma \) on the upper side. In statistical application, such exiting eigenvalues might be mistaken for “signal” and so it is useful to have some bounds on what can happen under the null hypothesis of no signal. Section 3 studies the mean value behavior of quantities such as

\[
T_N = \sum_{i=1}^{N} (\lambda_i - \lambda(\gamma))^q, \quad q \geq 0
\]

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which for $q = 0$ reduces to the number $T^0_N$ of exiting eigenvalues.

It is well known that the largest eigenvalue $\lambda_1 \overset{\text{as}}{\to} \lambda(\gamma)$ [Geman, 1980], and that closed intervals outside the bulk support contain no eigenvalues for $N$ large with probability one [Bai and Silverstein, 1998]. However these and even large deviation results for $\lambda_1$ [Majumdar and Vergassola, 2009] and $T^0_N$ [Majumdar and Vivo, 2012] seem not to directly yield the information on $E(T_N)$ that we need. Marino et al. [2014] looked at the variance of $T^0_N$ using methods related to those of this note. Recently, Chiani [2017] has studied the probability that all eigenvalues of Gaussian, Wishart and double Wishart random matrices lie within the bulk, and derived universal limiting values of 0.6921 and 0.9397 in the real and complex cases respectively.

In summary, the motivation for this note is high-dimensional covariance estimation, but there are noteworthy byproducts: the asymptotic values of $E(T^0_N)$ are perhaps surprising small, and numerically for the Gaussian Unitary Ensemble (GUE), it is found that the chance of even two exiting eigenvalues is very small, of order $10^{-6}$.

2 The Gaussian Unitary Ensemble Case (GUE)

We begin with GUE to illustrate the methods in the simplest setting, and to note an error in the literature. Recall that the Gaussian Unitary ensemble $\text{GUE}(N)$ is the Gaussian probability measure on the space of $N \times N$ Hermitian matrices with density proportional to $\exp\{-\frac{1}{2} N \text{tr} A^2\}$.

**Theorem 1.** Let $\lambda_1, \ldots, \lambda_N$ be eigenvalues of an $N$-by-$N$ matrix from the GUE. Denote by $\lambda_+ = 2$ the upper edge of the Wigner semicircle, namely, the asymptotic density of the eigenvalues. For $q \geq 0$, let

$$T_N = \sum_{i=1}^{N} (\lambda_i - \lambda_+)^q. \quad (1)$$

Then, with a constant $c_q$ specified at (3) below,

$$E(T_N) = c_q N^{-2q/3}(1 + o(1)).$$

In particular, for $q = 0$ and $T_N = \#\{i : \lambda_i > \lambda_+\}$,

$$E(T_N) \to c_0 = \frac{1}{6\sqrt{3\pi}} \approx 0.030629. \quad (2)$$

**Proof.** We use the so-called one-point function and bounds due to Tracy and Widom [1994, 1996]. To adapt to their notation, let $(y_i)_1^N$ be the eigenvalues of GUE with joint density proportional to $\exp(-\sum_1^N y_i^2)\Delta^2(y)$, where $\Delta(y)$ is the usual Vandermonde. In this scaling the eigenvalue bulk concentrates as the semi-circle on $[-\mu_N, \mu_N]$ with $\mu_N = \sqrt{2N}$.

We have $y_i = \sqrt{N/2 \lambda_i}$ and $\mu_N = \sqrt{N/2 \lambda_+}$, for $\lambda_+ = 2$, so that

$$T_N = \sum_{i=1}^{N} (\lambda_i - \lambda_+)^q = \left(\frac{2}{N}\right)^{q/2} \sum_{i=1}^{N} (y_i - \mu_N)^q.$$

2
From the determinantal structure of GUE, the marginal density of a single (unordered) eigenvalue \( y_i \) is given by the one-point function

\[
N^{-1} S_N(y, y) = N^{-1} \sum_{k=0}^{N-1} \phi_k^2(y),
\]

where \( \phi_k(y) \) are the (Hermite) functions obtained by orthonormalizing \( y^k e^{-y^2/2} \). Thus

\[
E(T_N) = \left( \frac{2}{N} \right)^{q/2} \int_{\mu_N}^\infty (y - \mu_N)^q S_N(y, y) dy.
\]

Now introduce the TW scaling

\[
y = \mu_N + \tau_N x, \quad \tau_N = \frac{1}{\sqrt{2N^{1/6}}},
\]

and let \( \text{Ai} \) denote the Airy function. Tracy and Widom [1996, p 745-6] show that

\[
S_{\tau_N}(x, x) = \tau_N S_N(\mu_N + \tau_N x, \mu_N + \tau_N x) \rightarrow K_A(x, x) = \int_0^\infty \text{Ai}^2(x + z) dz,
\]

with the convergence being dominated: \( S_{\tau_N}(x, x) \leq M^2 e^{-2x} \). Consequently,

\[
E(T_N) = \left( \frac{2\tau_N^2}{N} \right)^{q/2} \int_0^\infty x^q S_{\tau_N}(x, x) dx \sim N^{-2q/3} \int_0^\infty x^q K_A(x, x) dx.
\]

In particular, \( E(T_N) = O(N^{-2q/3}) \), and if \( q = 0 \), then \( E(T_N) \) converges to a positive constant.

Integration by parts and Olver et al. [2010, 9.11.15] yield

\[
c_q = \int_0^\infty x^q K_A(x, x) dx = \int_0^\infty x^q \int_x^\infty \text{Ai}^2(z) dz dx = \frac{1}{q + 1} \int_0^\infty x^{q+1} \text{Ai}^2(x) dx = \frac{2\Gamma(q + 1)}{\sqrt{\pi} 12^{(2q+9)/6} \Gamma((2q + 9)/6)}.
\]

(3)

For \( q = 0 \) the constant becomes \( c_0 = 1/(6\sqrt{3}\pi) \).

Remarks. 1. Ullah [1983] states, in our notation, that the expected number of eigenvalues above the bulk edge, \( E(T_N) \sim 0.25N^{-1/2} \). This claim cannot be correct: a counterexample uses the limiting law \( F_2 \) for \( y(1) = \max_i y_i \) of Tracy and Widom [1994]:

\[
E(T_N) \geq \Pr(y(1) > \sqrt{2N}) \rightarrow 1 - F_2(0) = 0.030627.
\]

(4)

We evaluated numerically in Mathematica the formulas (U3), (U6) and (U7) for \( p = (2/N)E(T_N) \) given in Ullah [1983]. While numerical results from intermediate formula (U3)
Table 1: For GUE($N$), the probabilities $p_N(k)$ of exactly $k$ eigenvalues exceeding the upper bulk edge $\sqrt{2N}$, along with the expected number $E(T_N)$, to be compared with limiting value (2).

| $N$  | $p_N(1)$       | $p_N(2)$       | $p_N(3)$       | $E(T_N)$  |
|------|----------------|----------------|----------------|-----------|
| 10   | $2.868 \cdot 10^{-2}$ | $1.36 \cdot 10^{-6}$ | $6.9 \cdot 10^{-14}$ | 0.028681  |
| 25   | $2.955 \cdot 10^{-2}$ | $1.70 \cdot 10^{-6}$ | $1.4 \cdot 10^{-13}$ | 0.029551  |
| 50   | $2.994 \cdot 10^{-2}$ | $1.88 \cdot 10^{-6}$ | $1.9 \cdot 10^{-13}$ | 0.029944  |
| 100  | $3.019 \cdot 10^{-2}$ | $2.00 \cdot 10^{-6}$ | $2.3 \cdot 10^{-13}$ | 0.030195  |
| 250  | $3.039 \cdot 10^{-2}$ | $2.09 \cdot 10^{-6}$ | $2.6 \cdot 10^{-13}$ | 0.030392  |
| 500  | $3.048 \cdot 10^{-2}$ | $2.14 \cdot 10^{-6}$ | $2.8 \cdot 10^{-13}$ | 0.030480  |

are consistent with our (2), neither those from (U6) nor those from the final result (U7) are consistent with (U3), or indeed with each other!

2. The striking closeness of the right side of (4) to (2) led us to use the Matlab toolbox of Bornemann [2010] to evaluate numerically

$$p_N(k) = \text{Pr}(\text{exactly } k \text{ of } \{y_i\} > \sqrt{2N}) = E_2^{(n)}(k, J)$$

with $J = (\sqrt{2N}, \infty)$, in the notation of Bornemann [2010]. The results, in Table 1 confirm that the probability of 2 or more eigenvalues exiting the bulk is very small, of order $10^{-6}$, for all $N$. This is also suggested by the plots of the densities of $y(1), y(2), \ldots$ in the scaling limit in Figure 4 of Bornemann [2010], which itself extends Figure 2 of Tracy and Widom [1994].

3 The real Wishart case

Suppose $\lambda_i$ are eigenvalues of $n^{-1}XX^\top$ for $X$ a $N \times n$ matrix with i.i.d. $N(0,1)$ entries. Assume that $\gamma_N = N/n \to \gamma \in (0,1]$. Set $\lambda(\gamma) = (1 + \sqrt{\gamma})^2$.

We recall the scaling for the Tracy-Widom law from the largest eigenvalue $\lambda_1$:

$$\lambda_1 = \lambda_2(\gamma_N) + N^{-2/3} \tau(\gamma_N) W_N$$

where $W_N$ converges in distribution to $W \sim TW_1$ and $\tau(\gamma) = \sqrt{\gamma(\sqrt{\gamma} + 1)^{1/3}}$.

**Theorem 2.** (a) Suppose $\eta(\lambda, c) \geq 0$ is jointly continuous in $\lambda$ and $c$, and satisfies

$$\eta(\lambda, c) = 1 \quad \text{for } \lambda \leq \lambda(c)$$

$$\eta(\lambda, c) \leq M\lambda \quad \text{for some } M \text{ and all } \lambda.$$  

Suppose also that $c_N - \gamma_N = O(N^{-2/3})$. Then for $q > 0$,

$$E\left(\sum_{i=1}^{N} [\eta(\lambda_i, c_N) - 1]^q\right) \to 0. \quad (5)$$
(b) Suppose $c_N - \gamma_N \sim s(\gamma)N^{-2/3}$, where $\sigma(\gamma) = \tau(\gamma)/\lambda'(\gamma) = \gamma(1 + \sqrt{\gamma})^{1/3}$. Then

$$
E\left(\sum_{i=1}^{N} (\lambda_i - \lambda(c_N))^q_+\right) \sim \tau^q(\gamma) N^{-2q/3} \int_0^\infty (x - s)^q_+ K_1(x, x) dx.
$$

(6)

where $K_1$ is defined at (9) below.

(c) In particular, let $N_n = \#\{i : \lambda_i \geq \lambda(c_N)\}$ and suppose that $c_N - \gamma_N = o(N^{-2/3})$. Then

$$
EN_n \to c_0 = \int_0^\infty K_1(x, x) dx \approx 0.17.
$$

Remarks.

1. Part (b) represents a sharpening of (5) that is relevant when $\eta(\lambda) = \eta(\lambda, \gamma)$ is Hölder continuous in $\lambda$ near the bulk edge $\lambda(\gamma)$,

$$
\eta(\lambda) - \eta(\lambda(\gamma)) \sim (\lambda - \lambda(\gamma))^{q_+}.
$$

The example $q = 1/2$ occurs commonly for optimal shrinkage rules $\eta^*(\lambda)$ in Donoho et al. [2017].

2. Section 4 explains why we allow $c_N$ to differ from $\gamma_N$.

Proof. Define

$$
T_N = \sum_{i=1}^{N} F(\lambda_i, c_N), \
F(\lambda, c) = \begin{cases} 
[\eta(\lambda, c) - 1]^q & (a) \\
[\lambda - \lambda(c)]^q_+ & (b).
\end{cases}
$$

We adapt the discussion here to the notation used in Tracy and Widom [1998] and Johnstone [2001]. Let $(y_i)_1^N = n\lambda_i$ be the eigenvalues of $W_N(n, I)$ with joint density function $P_N(y_1, \ldots, y_N)$ with explicit form given, for example, in [Johnstone, 2001, eq. (4.1)]. We obtain

$$
E(T_N) = \int_0^\infty F(y/n, c_N) R_1(y) dy,
$$

where $R_1(y_1) = N \int_{(y,\infty)^{N-1}} P_N(y_1, \ldots, y_N) dy_2 \cdots dy_N$ is the one-point (correlation) function. It follows from Tracy and Widom [1998, p814-16] that

$$
R_1(y) = T_1(y) = \frac{1}{2} \text{tr} \ K_N(y, y)
$$

(7)

where $K_N(x, y)$ is the $2 \times 2$ matrix kernel associated with $P_N$, see e.g. Tracy and Widom [1998, eq. (3.1)]. It follows from Widom [1999] that

$$
\frac{1}{2} \text{tr} \ K_N(y, y) = S(y, y) + \psi(y)(\epsilon \phi)(y) = S_1(y, y),
$$

(8)

where the functions $S(y, y')$, $\psi(y)$ and $\phi(y)$ are defined in terms of orthonormalized Laguerre polynomials in Widom [1999] and studied further in Johnstone [2001]. The function $\epsilon(x) = \frac{1}{2} \text{sgn} x$ and the operator $\epsilon$ denotes convolution with the kernel $\epsilon(x - y)$.

\[\square\]
For convergence, introduce the Tracy-Widom scaling
\[ y = \mu_N + \sigma_N x, \]
where we set \( N_h = N + \frac{1}{2} \) and \( n_h = n + \frac{1}{2} \) and define
\[ \mu_N = (\sqrt{N_h} + \sqrt{n_h})^2, \quad \sigma_N = c(N_h/n_h)^{1/3}, \]
where \( c(\gamma) = (1 + \sqrt{\gamma})^{1/3}(1 + 1/\sqrt{\gamma}) = (1 + \sqrt{\gamma})^{1/3}\lambda(\gamma) \)
We now rescale the scalar-valued function (8):
\[ S_{1\tau}(x, x) = \sigma_N S_1(\mu_N + \sigma_N x, \mu_N + \sigma_N x). \]
We can rewrite our target \( E(T_N) \) using (7), (8) and this rescaling in the form
\[ E(T_N) = \int_{\delta_N}^{\infty} F(\ell_N(x), c_N) S_{1\tau}(x, x) dx, \]
where \( \ell_N(x) = (\mu_N + \sigma_N x)/n, \delta_N = (n\lambda(c_N) - \mu_N)/\sigma_N \) and we used the fact that \( F(\lambda, c) = 0 \) for \( \lambda \leq \lambda(c) \).

It follows from [Johnstone, 2001, eq. (3.9)] that
\[ S_{1\tau}(x, x) = 2 \int_0^{\infty} \phi_{\tau}(x + u)\psi_{\tau}(x + u)du + \psi_{\tau}(x) \left[c_{\phi} - \int_x^{\infty} \phi_{\tau}(u)du\right]. \]
It is shown in equations (3.7), 3.8 and Sec. 5 of that paper that
\[ \phi_{\tau}(x), \psi_{\tau}(x) \to \frac{1}{\sqrt{2}}\text{Ai}(x) \]
and, uniformly in \( N \) and in intervals of \( x \) that are bounded below, that
\[ \phi_{\tau}(x), \psi_{\tau}(x) = O(e^{-x}). \]
Along with \( c_{\phi} \to 1/\sqrt{2} \) (cf. App. A7 of same paper), this shows that
\[ S_{1\tau}(x, x) \to K_1(x, x) = \int_0^{\infty} \text{Ai}^2(x + z)dz + \frac{1}{2}\text{Ai}(x) \left[1 - \int_x^{\infty} \text{Ai}(z)dz\right] > 0 \quad (9) \]
with the convergence being dominated
\[ S_{1\tau}(x, x) \leq M'e^{-2x} + M'e^{-x}. \quad (10) \]
Before completing the argument for (a) – (c), we note it is easily checked that
\[ n^{-1}\mu_N = \lambda(\gamma_N) + O(N^{-1}), \quad (11) \]
so that
\[ \delta_N = \frac{n}{\sigma_N} [\lambda(c_N) - \lambda(\gamma_N)] + O(N^{-1/3}). \]
If \( c_N - \gamma_N = \theta_N N^{-2/3} \) for \( \theta_N = O(1) \) then
\[ \delta_N \sim \frac{n}{\sigma_N} N^{-2/3}\theta_N \lambda'(\gamma) \sim \theta_N/\sigma(\gamma), \]
since we have
\[ N^{2/3} \sigma_N/n \sim \sigma(\gamma)X(\gamma) = \tau(\gamma). \]  

(12)

In case (a), then, \( \delta_N \geq -A \) for some \( A \). We then have \( \ell_N(x) \to \lambda(\gamma) \) for all \( x \geq -A \), and so from joint continuity
\[ \eta(\ell_N(x), c_N) \to \eta(\lambda(\gamma), \gamma) = 1, \]
and hence for all \( x \geq -A \),
\[ F(\ell_N(x), c_N) = [\eta(\ell_N(x), c_N) - 1]^q \to 0 \]

(13)

The convergence is dominated since the assumption \( \eta(\lambda, c) \leq M \lambda \) implies that \( |F(\ell_N(x), c_N)| \leq C(1 + |x|^q) \). Hence the convergence (13) along with (10) and the dominated convergence theorem implies (5).

For case (b),
\[ N^{2q/3} \mathbb{E}(T_N) = \int_{\delta_N}^{\infty} \left[ N^{2/3}(\ell_N(x) - \lambda(c_N)) \right]^q S_1 \tau(x, x) dx. \]

Observe that
\[ N^{2/3}(\lambda(\gamma_N) - \lambda(c_N)) \sim N^{2/3}X(\gamma)(\gamma_N - c_N) \sim -s \tau(\gamma), \]
and so from (11) and (12), we have
\[ N^{2/3}(\ell_N(x) - \lambda(c_N)) = O(N^{-1/3}) + N^{2/3}(\lambda(\gamma_N) - \lambda(c_N)) + N^{2/3}n^{-1} \sigma_N x \sim \tau(\gamma)(x - s). \]

(14)

In addition, from (14), we have
\[ N^{2/3}|\ell_N(x) - \lambda(c_N)| \leq M(1 + |x|), \]
so that the convergence is dominated and (6) is proven.

For case (c), we have only to evaluate
\[ c_0 = \int_0^{\infty} K_1(x, x) dx = \int_0^{\infty} K_A(x, x) dx + \frac{1}{4} \int_0^{\infty} G'(x) dx = I_1 + I_2, \]
where \( I_1 \) was evaluated in the previous section and \( G(x) = [1 - \int_x^{\infty} \text{Ai}(z) dz]^2 \). Since \( \int_0^{\infty} \text{Ai}(z) dz = 1/3 \), from Olver et al. [2010, 9.10.11], we obtain
\[ 4I_2 = G(\infty) - G(0) = 1 - (2/3)^2 = 5/9, \]
with the result
\[ c_0 = \frac{1}{6 \sqrt{3} \pi} + \frac{5}{36} \approx 0.031 + 0.139 = 0.16952. \]
4 Application to covariance estimation

We indicate how Theorem 2 is applied to covariance estimation in the spiked model studied in [Donoho et al. 2017]. Consider a sequence of statistical problems indexed by dimension $p$ and sample size $n$. In the $n$th problem $\mathbf{X} \sim \mathcal{N}_p(0, \Sigma)$ where $p = p_n$ satisfies $p_n/n \to \gamma \in (0, 1]$ and the population covariance matrix $\Sigma = \Sigma_p$ has fixed ordered eigenvalues $\ell_1 \geq \ldots \geq \ell_r > 1$ for all $n$, and then $\ell_{r+1} = \ldots = \ell_{p_n} = 1$.

Suppose that the sample covariance matrix $\hat{\mathbf{S}} = \hat{\mathbf{S}}_{n,p}$ has eigenvalues $\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_p$ and corresponding eigenvectors $v_1, \ldots, v_p$. Consider shrinkage estimators of the form

$$\hat{\Sigma}_{\eta} = \sum_{j=1}^{p} \eta(\hat{\lambda}_j, c_p) v_j v_j^\top,$$

where $\eta(\lambda, c)$ is a continuous bulk shrinker, that is, satisfies the conditions (a) of Theorem 2. Without loss of generality, as explained in the reference cited, we may also assume that $\lambda \to \eta(\lambda, c)$ is non-decreasing. In the spiked model, the typical choice for $c_p$ in practice would be to set $c_p = p/n$, and we adopt this choice below.

It is useful to analyse an “oracle” or “rank-aware” variant of (15) which takes advantage of the assumed structure of $\Sigma_p$, especially the fixed rank $r$ of $\Sigma_p - I$:

$$\hat{\Sigma}_{\eta,r} = \sum_{j=1}^{r} \eta(\hat{\lambda}_j, c_p) v_j v_j^\top + \sum_{j=r+1}^{p} v_j v_j^\top.$$

The error in estimation of $\Sigma$ using $\hat{\Sigma}$ is measured by a loss function $L_p(\Sigma, \hat{\Sigma})$. One seeks conditions under which the losses $L_p(\Sigma, \hat{\Sigma}_{\eta})$ and $L_p(\Sigma, \hat{\Sigma}_{\eta,r})$ are asymptotically equivalent. They consider a large class of loss functions which satisfy a Lipschitz condition which implies that, for some $q$,

$$|L_p(\Sigma, \hat{\Sigma}_{\eta}) - L_p(\Sigma, \hat{\Sigma}_{\eta,r})| \leq C(\ell_1, \eta(\hat{\lambda}_1)) \sum_{j=r+1}^{p} [\eta(\hat{\lambda}_j, c_p) - 1]^q.$$

Suppose now that $\Pi : \mathbb{R}^p \to \mathbb{R}^{p-r}$ is a projection on the span of the $p-r$ unit eigenvectors of $\Sigma$. Let $X = \Pi \mathbf{X}$ and let $\lambda_1 \geq \ldots \geq \lambda_{p-r}$ denote the eigenvalues of $n^{-1}XX^\top$. By the Cauchy interlacing Theorem (e.g. [Bhatia 1997, p. 59]), we have

$$\hat{\lambda}_j \leq \lambda_{j-r} \quad \text{for } r + 1 \leq j \leq p,$$

where the $(\lambda_i)_{i=1}^{p-r}$ are the eigenvalues of a white Wishart matrix $W_{p-r}(n, I)$. From the monotonicity of $\eta$,

$$\sum_{j=r+1}^{p} [\eta(\hat{\lambda}_j, c_p) - 1]^q \leq \sum_{i=1}^{p-r} [\eta(\lambda_i, c_p) - 1]^q.$$

Now apply part (a) of Theorem 2 with the identifications

$$N \leftarrow p-r, \quad c_N \leftarrow c_p.$$
Clearly $\gamma_N = N/n \to \gamma$ and
\[
c_N - \gamma_N = \frac{N + r}{n} - \frac{N}{n} = O(N^{-2/3}),
\]
since $r$ is fixed. We conclude that the right side of (17) and hence $|L_p(\Sigma, \hat{\Sigma}_\eta) - L_p(\Sigma, \hat{\Sigma}_{\eta,r})|$ converge to 0 in $L_1$ and in probability.

Part (c) of Theorem 2 helps to give an example where the losses $L_p(\Sigma, \hat{\Sigma}_\eta)$ and $L_p(\Sigma, \hat{\Sigma}_{\eta,r})$ are not asymptotically equivalent. Indeed, let $L_p(\Sigma, \hat{\Sigma}_\eta) = \|\hat{\Sigma}_\eta^{-1} - \Sigma^{-1}\|$, with $\|\cdot\|$ denoting matrix operator norm. Here the optimal shrinkage rule $\eta = \eta^*(\lambda, c)$ is discontinuous at the upper bulk edge $\lambda(c) = (1 + \sqrt{c})^2$:
\[
\eta^*(\lambda, c) = 1 \quad \text{for } \lambda \leq \lambda(c)
\]
\[
\eta^*(\lambda, c) \to 1 + \sqrt{c} \quad \text{for } \lambda \downarrow \lambda(c).
\]
Proposition 3 of [Donoho et al., 2017] shows that
\[
\|\hat{\Sigma}_\eta^{-1} - \Sigma^{-1}\| - \|\hat{\Sigma}_{\eta,r}^{-1} - \Sigma^{-1}\| \overset{D}{\to} W,
\]
where $W$ has a two point distribution $(1 - \pi)\delta_0 + \pi\delta_w$ with non-zero probability $\pi = \Pr(TW_1 > 0)$ at location $w = f(\ell_+) - f(\ell_r)$, where $\ell_+ = 1 + \sqrt{c}$ and the function
\[
f(\ell) = \left[ \frac{c(\ell - 1)}{\ell(\ell - 1 + \gamma)} \right]^{1/2}
\]
is strictly decreasing for $\ell \geq \ell_+$.

Part (c) of Theorem 2 along with interlacing inequality (10), is used in the proof to establish that $N_n = \#\{i \geq r + 1 : \lambda_{in} > \lambda_+(c_n)\}$, the number of noise eigenvalues exiting the bulk, is bounded in probability.

5 Final Remarks

It is apparent that the same methods will show that the value of $c_0$ for the Gaussian Orthogonal Ensemble will be the same as for the real Wishart (Laguerre Orthogonal Ensemble), and similarly that the value of $c_0$ for the white complex Wishart (Laguerre Unitary Ensemble) will agree with that for GUE.

Some natural questions are left for further work. First, the evaluation of $c_0$ for values of $\beta$ other than 1 and 2, and secondly universality, i.e. that the limiting constants do not require the assumption of Gaussian matrix entries.

Finally, this article appears in a special issue dedicated to the memory of Peter Hall. Hall’s many contributions to high dimensional data have been reviewed by [Samworth, 2016]. However, it seems that Peter did not publish specifically on problems connected with the application of random matrix theory to statistics — the exception that proves the rule of his extraordinary breadth and depth of interests. Nevertheless the present author’s work on this specific topic, as well as on many others, has been notably advanced by Peter’s support — academic, collegial and financial – in promoting research visits to Australia and contact with specialists there in random matrix theory, particularly at the University of Melbourne, Peter’s academic home since 2006.
References

Z. D. Bai and Jack W. Silverstein. No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. *Annals of Probability*, 26(1):316–345, 1998. ISSN 0091-1798.

Rajendra Bhatia. *Matrix Analysis*, volume 169 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. ISBN 0-387-94846-5.

F. Bornemann. On the numerical evaluation of distributions in random matrix theory: a review. *Markov Processes and Related Fields*, 16(4):803–866, 2010. ISSN 1024-2953. arXiv:0904.1581.

M. Chiani. On the probability that all eigenvalues of Gaussian, Wishart, and double Wishart random matrices lie within an interval. *IEEE Transactions on Information Theory*, 63(7):4521–4531, 2017.

DLMF. NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.9 of 2014-08-29, 2014. Online companion to [Olver et al.](https://dlmf.nist.gov/).

David Donoho, Matan Gavish, and Iain M. Johnstone. Optimal shrinkage of eigenvalues in the spiked covariance model. arxiv:1311.0851v3; in press, *Annals of Statistics*, 2017.

Stuart Geman. A limit theorem for the norm of random matrices. *Annals of Probability*, 8:252–261, 1980.

Iain M. Johnstone. On the distribution of the largest eigenvalue in principal components analysis. *Annals of Statistics*, 29:295–327, 2001.

Satya N. Majumdar and Massimo Vergassola. Large deviations of the maximum eigenvalue for Wishart and Gaussian random matrices. *Physical Review Letters*, 102:060601, Feb 2009.

Satya N. Majumdar and Pierpaolo Vivo. Number of relevant directions in principal component analysis and Wishart random matrices. *Physical Review Letters*, 108:200601, May 2012.

Ricardo Marino, Satya N. Majumdar, Grégory Schehr, and Pierpaolo Vivo. Phase transitions and edge scaling of number variance in Gaussian random matrices. *Physical Review Letters*, 112:254101, Jun 2014.

F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010. Print companion to [DLMF](https://dlmf.nist.gov/).

Leonid Pastur and Mariya Shcherbina. *Eigenvalue Distribution of Large Random Matrices*, volume 171 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2011. ISBN 978-0-8218-5285-9.

Richard J. Samworth. Peter Hall’s work on high-dimensional data and classification. *Annals of Statistics*, 44(5):1888–1895, 2016. ISSN 0090-5364.
Craig A. Tracy and Harold Widom. Level-spacing distributions and the Airy kernel. *Communications in Mathematical Physics*, 159:151–174, 1994.

Craig A. Tracy and Harold Widom. On orthogonal and symplectic matrix ensembles. *Communications in Mathematical Physics*, 177:727–754, 1996.

Craig A. Tracy and Harold Widom. Correlation functions, cluster functions, and spacing distributions for random matrices. *Journal of Statistical Physics*, 92:809–835, 1998.

N Ullah. Number of energy levels outside Wigner’s semicircle. *Journal of Physics A: Mathematical and General*, 16(18):L767, 1983.

H. Widom. On the relation between orthogonal, symplectic and unitary ensembles. *Journal of Statistical Physics*, 94:347–363, 1999.