Retraction

Retraction: Global Stability Analysis of Dengue Transmission Model with Awareness, Vector Control and Time Delays (J. Phys.: Conf. Ser. 1899 012104)

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Global Stability Analysis of Dengue Transmission Model with Awareness, Vector Control and Time Delays

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Abstract. In this paper, a mathematical model for a single-strain dengue virus transmission, incorporating vector control, disease awareness of susceptible humans, and both the latent delays for human and mosquitoes, is proposed and studied. The global stability properties of disease-free equilibrium and endemic equilibrium are completely established through Lyapunov functionals and LaSalle’s invariance principle. The global dynamics of the equilibrium points are characterized by the value of basic reproductive number $R_0$. If $R_0 < 1$, then the disease-free equilibrium is globally asymptotically stable. If $R_0 > 1$, then the disease-free equilibrium is unstable, and the endemic equilibrium exists which is globally asymptotically stable. Lastly, this paper presents numerical simulations and possible recommendations for future works.

1. Introduction
Dengue is a mosquito-borne viral infection caused by the dengue virus (DENV) that has four serotypes (DEN1, DEN2, DEN3 and DEN4) which belong to the genus Flavivirus, family Flaviviridae. Dengue causes a severe flu-like illness which may sometimes lead to a more complicated condition called severe dengue. Severe dengue (formerly known as dengue hemorrhagic fever) was first recognized in the 1950s during dengue epidemics in the Philippines and Thailand. This has become a leading cause of hospitalization and deaths among children and adults in the Asian and Latin American countries [15].

Dengue viruses are transmitted to humans through the bite of infective females $Aedes aegypti$ and $Aedes albopictus$ mosquitoes. Once humans are infected with the virus, they can transmit the infection to susceptible mosquitoes after the first symptoms appear (after the incubation period of 4-10 days) [15]. When the virus enters the mosquitoes’ blood, the virus will require an additional 8-12 days incubation before it can then be transmitted to another human. The mosquito remains infected for the remainder of its life, which might be days or a few weeks [2]. A human who recovers from a dengue infection of one serotype becomes immune to that particular serotype. However, subsequent infection from the other serotypes increases the risk of developing severe dengue [15].

There are already some vaccines under development, including the Dengvaxia which has been recommended for use in countries with high number of dengue cases [15]. However, until the reliability of the use of this vaccine is fully established, the most effective way to control the spread of dengue in the community is by taking preventive measures. These can be done through reduction of mosquito population and personal protection awareness against mosquito bites [2]. Mosquito reduction includes the elimination of mosquito breeding sites and the use of insecticide to kill mosquitoes. On the other
hand, personal protection awareness includes avoiding mosquito bites (by using mosquito net when sleeping, window and door screens, long-sleeved clothes and long pants, mosquito repellent, etc.).

Mathematical models can provide a useful tool in analyzing the spread of dengue virus in a community. Several compartmental models for dengue have been introduced and studied (see [5], [7], [8]). In 1998, Esteva and Vargas [5] introduced a model for dengue virus transmission by using ordinary differential equations. This model assumed that when a susceptible mosquito consumes a blood meal from an infectious human, that mosquito immediately becomes infectious. They further assumed that when an infectious mosquito bites a susceptible human, that bitten human becomes infectious instantaneously. In reality, it takes some time before these infected humans and mosquitoes become infectious [15]. These time periods are called latent periods. In 2008, Wei, Li and Martcheva [14] considered an SIR-SI vector-borne disease model incorporating a latent delay for the infected vectors. In 2012, Lashari, Hattaf and Zaman [11] studied how the latent delay for infected humans affects the dynamics of the SIR-SI vector-borne disease model. In 2017, Guan et al. [8] introduced a model for dengue fever transmission with latent time delays for both humans and mosquitoes, and found out that these delays resulted to the occurrence of Hopf bifurcation (see [10, p. 58]). This model is described by the following system of differential equations

\[
S_H'(t) = \mu_H N_H(t) - \mu_H S_H(t) = \lambda_H I_V(t) S_H(t) \tag{1}
\]

\[
I_H'(t) = \lambda_H I_V(t - \tau_1) S_H(t) - (\mu_H + \delta) I_H(t) \tag{2}
\]

\[
R_H'(t) = \delta I_H(t) - \mu_H R_H(t) \tag{3}
\]

\[
S_V'(t) = A_2 - \lambda_V I_H(t) S_V(t) - \mu_V S_V(t) \tag{4}
\]

\[
I_V'(t) = \lambda_V I_H(t - \tau_2) S_V(t) - \mu_V I_V(t), \tag{5}
\]

where \( S_H, I_H, R_H, S_V \) and \( I_V \) represent the number of susceptible humans, infected humans, recovered humans, susceptible mosquitoes and infected mosquitoes, respectively. The parameters \( \mu_H, \mu_V \) and \( \delta \) are all constants which represent the death rate of humans, death rate of mosquitoes and recovery rate of humans from the disease. Here, \( N_H(t) \) represents the total human population at time \( t \). It is also assumed that humans and mosquitoes in the infected compartments at time \( t \) are not all capable of transmitting the virus. Hence, the latent delays \( \tau_1 \) and \( \tau_2 \) for mosquitoes and humans, respectively, are included in the model. In (1), the term \( \lambda_H I_V(t) S_H(t) \) represents the rate at which susceptible humans get infected by the dengue virus at time \( t \), where \( \lambda_H \) is the effective contact rate sufficient to transmit the virus from an infected mosquito to a susceptible human. Similarly, in (4), the term \( \lambda_V I_H(t) S_V(t) \) represents the rate at which susceptible mosquitoes get infected by the dengue virus at time \( t \), where \( \lambda_V \) is the effective contact rate sufficient to transmit the virus from an infected human to a susceptible mosquito. However, not all mosquitoes and humans in the infected compartment at time \( t \) are infectious. This poses a problem with the previous assumptions that the infected mosquitoes and humans are not all capable of transmitting the virus.

In this paper, we make improvements to the dengue model in [8] so that the above problem is addressed. Moreover, human protection awareness and vector control are introduced in the model to study their effects on the spread of dengue virus in the community.

The rest of the paper is organized as follows: in Section 2, we present the formulation of the mathematical model and discuss the properties of the solutions. We determine the equilibrium points of the model and its basic reproductive number \( R_0 \) in Section 3. In Section 4, we study the global stability of the equilibrium points when \( R_0 < 1 \) and \( R_0 > 1 \). In Section 5, we perform numerical simulations to illustrate the theoretical results. Finally, we present some concluding remarks and suggestions for future work in Section 6.

2. The Mathematical Model

The total human population size at time \( t \), denoted by \( N_H(t) \), is partitioned into three compartments of individuals who are susceptible \( S_H(t) \), infected \( I_H(t) \) and removed \( R_H(t) \). On the other hand, the
total mosquito population size at time $t$, denoted by $N_V(t)$, is partitioned to only two compartments of mosquitoes that are susceptible $S_V(t)$ and infected $I_V(t)$.

Assume that all recruitment of human and mosquito population goes only in the susceptible compartments which happen at constant rates $A_1$ and $A_2$, respectively. Denote the natural death of humans and mosquitoes by $\mu_H$ and $\mu_V$, respectively. Assume also that an infected human dies at a rate $\varepsilon$ due to the severity and complications of the disease. Furthermore, susceptible and infected mosquitoes die due to vector control at a rate $c$. Susceptible humans who practice self-protection (awareness) against the mosquito bites move to the removed compartment at a rate $a$. Moreover, an infected human recovers from the disease at a rate $\delta$ and goes to the removed compartment. Only one strain of dengue virus is considered in this study, hence, the recovered individuals will acquire permanent immunity from that particular serotype. On the other hand, once a mosquito becomes infectious, such mosquito is capable of transmitting virus for the rest of its life.

We now introduce the time delays $\tau_1$ and $\tau_2$ in the model describing the incubation times of the humans and mosquitoes to become infectious, respectively. Observe that at time $t$, the infectious mosquitoes are those who have been infected already by the virus $\tau_1$ time units ago, provided that they are still alive after that $\tau_1$ period. Hence, the incidence term for infected humans is given by $e^{-(\mu_V+c)\tau_1} \lambda_H I_V(t - \tau_1) S_H(t)$, where $e^{-(\mu_V+c)\tau_1}$ represents the probability that an infected mosquito remains alive after a period of $\tau_1$ units. Similarly, at time $t$, the infectious humans are those who have been infected already by the virus $\tau_2$ time units ago, provided that they are still alive after that $\tau_2$ period. This suggests that the incidence term of infected vectors is $e^{-(\mu_H+\delta+\varepsilon)\tau_2} \lambda_V I_H(t - \tau_2) S_V(t)$, where $e^{-(\mu_H+\delta+\varepsilon)\tau_2}$ represents the probability that an infected human remains alive and infected after a period of $\tau_2$ units.

Figure 1: Progression of Dengue Infection for the Human and Mosquito Population

The above diagram of dengue virus transmission is described by the following system of differential
equations:

\[
S'_H(t) = A_1 - \mu_H S_H(t) - e^{-(\mu_V+c)\tau_1} \lambda_H I_V(t - \tau_1) S_H(t) - aS_H(t) \tag{6}
\]

\[
I'_H(t) = e^{-(\mu_V+c)\tau_1} \lambda_H I_V(t - \tau_1) S_H(t) - (\mu_H + \epsilon + \delta) I_H(t) \tag{7}
\]

\[
R'_H(t) = aS_H(t) + \delta I_H(t) - \mu_H R_H(t) \tag{8}
\]

\[
S'_V(t) = A_2 - e^{-(\mu_H+\delta+\epsilon)\tau_2} \lambda_H I_V(t - \tau_2) S_V(t) - \mu_V S_V(t) - cS_V(t) \tag{9}
\]

\[
I'_V(t) = e^{-(\mu_H+\delta+\epsilon)\tau_2} \lambda_H I_V(t - \tau_2) S_V(t) - \mu_V I_V(t) - cI_V(t) \tag{10}
\]

where \(N_H(t) = S_H(t) + I_H(t) + R_H(t), N_V(t) = S_V(t) + I_V(t), S_H, I_H, R_H, S_V, I_V, \) and \(I_V \) are continuous real-valued functions on \( \mathbb{R} \). All parameters are positive constants. Throughout this paper, the system of equations (6)-(10) shall be referred to as system \((M)\).

For \(\tau = \max \{\tau_1, \tau_2\} > 0\), let \(C := C\left([-\tau, 0], \mathbb{R}^5\right)\) be the Banach space of continuous functions from \([-\tau, 0]\) into \(\mathbb{R}^5\) with norm \(|\phi|_\ast = \sup_{-\tau \leq \theta \leq 0} ||\phi(\theta)||\) for each \(\phi \in C\).

For any given continuous function \(X = (S_H, I_H, R_H, S_V, I_V) : [-\tau, \zeta) \rightarrow \mathbb{R}^5\) with \(\zeta > 0\), define \(X_t \in C\) for each \(t \geq 0\) by

\[X_t(\theta) = X(t + \theta) = (S_H(t + \theta), I_H(t + \theta), R_H(t + \theta), S_V(t + \theta), I_V(t + \theta))\]

for all \(\theta \in [-\tau, 0]\).

Denote by \(\mathbb{R}_{0+}\) the set of nonnegative real numbers. For system \((M)\), we will only consider solutions \(X = (S_H, I_H, R_H, S_V, I_V)\) on \([-\tau, +\infty)\) with initial condition \(\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)\) from the set

\[\Phi = \{\phi \in C([-\tau, 0], \mathbb{R}^5) : \phi_1(0) > 0, \phi_4(0) > 0\} .\]

That is, for all \(\theta \in [-\tau, 0]\), we have

\[S_H(\theta) = \phi_1(\theta), \quad I_H(\theta) = \phi_2(\theta), \quad R_H(\theta) = \phi_3(\theta), \quad S_V(\theta) = \phi_4(\theta), \quad I_V(\theta) = \phi_5(\theta).\]

The following theorems can be established by using the properties of retarded functional differential equations (see [4, 10]).

**Theorem 1.** For each \(\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in \Phi\), the solution to system \((M)\) with initial condition \(\phi\) exists and is unique.

**Theorem 2.** Every solution to system \((M)\) with initial condition \(\phi \in \Phi\) is nonnegative and bounded on \([0, +\infty)\).

Hereafter, we shall denote by \(\Omega\) the space of all solutions to system \((M)\) with initial condition \(\phi \in \Phi\).

### 3. The Basic Reproductive Number and Equilibrium Points

System \((M)\) always has the disease-free equilibrium

\[E^0 = (S_{H0}, I_{H0}, R_{H0}, S_{V0}, I_{V0}) = \left( \frac{A_1}{\mu_H + a}, 0, \frac{aA_1}{(\mu_H + a)\mu_V + c}, \frac{A_2}{(\mu_H + a)(\mu_V + c)^2 + (\mu_H + c + \delta)}, 0 \right)\]

in \(\Omega\). Define

\[R_0 = \frac{A_1A_2e^{-(\mu_V+c)\tau_1}e^{-(\mu_H+\delta+c)\tau_2}A_HA_V}{(\mu_H + a)(\mu_V + c)^2(\mu_H + c + \delta)} .\]

This quantity is called the **basic reproductive number**, which represents the average number of secondary infections that a single infectious human can produce in a totally susceptible population.
of humans and mosquitoes. To see this, note that there are $\frac{A_2}{\mu V + c}$ susceptible mosquitoes, and $e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_V \frac{A_2}{\mu V + c}$ will be infected per unit time and shall be infected for a period of $\frac{1}{\mu V + c}$ time units. This suggests that the total number of infected vectors from one infectious human is given by $e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_V \frac{A_2}{\mu V + c} \cdot \frac{1}{\mu V + c}$. The fact that there are $\frac{A_1}{\mu H + a}$ susceptible humans, these infectious mosquitoes will bite and transmit the virus to $e^{-(\mu_V + c)\tau_1} \lambda_H$ susceptibles per unit time. Note that such human individuals shall be infected for a period of $\frac{1}{\mu H + \epsilon + \delta}$ so that the total number of secondary human infections is given by

$$\frac{A_1 A_2 e^{-(\mu_V + c)\tau_1} e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_H \lambda_V}{(\mu_H + a)(\mu V + c)^2(\mu_H + \epsilon + \delta)}.$$  

It is important to note that the dynamics of system (M) is characterized by $R_0$. When $R_0 > 1$, the model has another equilibrium point, the endemic equilibrium $E^* = (S_{H^*}, I_{H^*}, R_{H^*}, S_{V^*}, I_{V^*})$, where

$$S_{H^*} = \frac{\mu V + c}{\lambda_V} \left[ A_1 e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_V + (\mu V + c)(\mu_H + \epsilon + \delta) \right],$$

$$I_{H^*} = \frac{A_1 A_2 e^{-(\mu_V + c)\tau_1} e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_H \lambda_V - (\mu_H + a)(\mu V + c)\lambda_V}{e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_V (\mu_H + \epsilon + \delta)((\mu_H + a)(\mu V + c) + A_2 \lambda_H e^{-(\mu_V + c)\tau_1})},$$

$$R_{H^*} = \frac{\delta I_{H^*} + a S_{H^*}}{\mu_H},$$

$$S_{V^*} = \frac{(\mu_H + \epsilon + \delta) \left[ A_2 e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_H + (\mu_H + a)(\mu V + c) \right]}{e^{-(\mu_V + c)\tau_1} \lambda_H \left[ A_1 e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_V + (\mu H + \epsilon + \delta)(\mu V + c) \right]},$$

$$I_{V^*} = \frac{A_1 A_2 e^{-(\mu_V + c)\tau_1} e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_H \lambda_V - (\mu_H + a)(\mu V + c)^2(\mu_H + \epsilon + \delta) A_2 \lambda_H e^{-(\mu_V + c)\tau_1}}{e^{-(\mu_V + c)\tau_1} \lambda_H (\mu V + c) \left[ A_1 e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_V + (\mu H + \epsilon + \delta) \right]}.$$  

Because $R_0$ is dependent on the values of the model parameters, health authorities may introduce intervention strategies that can influence the values of certain parameters yielding $R_0 < 1$. In such case, the ideal state where the disease is eventually eliminated from the community may be achieved.

In the following remarks, we look at how the level of human awareness $a$ and the level of effective vector control $c$ affect the value of $R_0$.

**Remark 3.** For fixed parameters $A_1$, $A_2$, $\mu V$, $\mu H$, $\delta$, $\epsilon$, $\lambda H$, $\lambda V$, $\tau_1$ and $\tau_2$, $R_0$ is a decreasing function of $a$.

This is because

$$\frac{\partial R_0}{\partial a} = -\frac{A_1 A_2 e^{-(\mu_V + c)\tau_1} e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_H \lambda_V}{(\mu_H + a)^2(\mu V + c)^2(\mu_H + \epsilon + \delta)} < 0.$$  

This implies that increasing one’s knowledge on the methods of preventing mosquito bites and practicing these would eventually lead to a decrease in the rate of spread of dengue virus in the community.

**Remark 4.** For fixed parameters $A_1$, $A_2$, $\mu V$, $\mu H$, $\delta$, $\epsilon$, $\lambda H$, $\lambda V$, $\tau_1$ and $\tau_2$, $R_0$ is a decreasing function of $c$.

Again, observe that

$$\frac{\partial R_0}{\partial c} = -\frac{A_1 A_2 \tau_1 e^{-(\mu_V + c)\tau_1} e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_H \lambda_V}{(\mu_H + a)(\mu V + c)^2(\mu_H + \epsilon + \delta)} - \frac{2 A_1 A_2 e^{-(\mu_V + c)\tau_1} e^{-(\mu_H + \delta + \epsilon)\tau_2} \lambda_H \lambda_V}{(\mu_H + a)(\mu V + c)^2(\mu_H + \epsilon + \delta)} < 0$$
which justifies the above remark. Hence, practicing vector control and extending its level of effectiveness are potential factors that may contribute to slowing down the spread of dengue virus in the community.

4. Global Stability Analysis

In this section, we will study the global stability of the equilibrium points through Lyapunov functionals and the LaSalle’s Invariance Principle [10]. Because the variable \( R_H \) is not explicit in equations (6), (7), (9), and (10), we may first consider the following subsystem \((M_R)\) of system \((M)\):

\[
S_H'(t) = A_1 - e^{-(\mu_V + \epsilon + \delta)}H(t - \tau_1)S_H(t) - (\mu_H + a)S_H(t)
\]

\[
I_H'(t) = e^{-(\mu_V + \epsilon + \delta)}H(t - \tau_1)S_H(t) - (\mu_H + \epsilon + \delta)I_H(t)
\]

\[
S_V'(t) = A_2 - e^{-(\mu_H + \epsilon + \delta)}H(t - \tau_2)S_V(t) - (\mu_V + c)S_V(t)
\]

\[
I_V'(t) = e^{-(\mu_H + \epsilon + \delta)}H(t - \tau_2)S_V(t) - (\mu_V + c)I_V(t)
\]

Certainly, system \((M_R)\) always has the disease-free equilibrium \( E_0 = (S_{H_0}, 0, S_{V_0}, 0) \), where \( S_{H_0} = \frac{A_1}{\mu_H + a} \) and \( S_{V_0} = \frac{A_2}{\mu_V + c} \). Also, if \( R_0 > 1 \), then system \((M_R)\) has a unique endemic equilibrium \( E_* = (S_{H_*}, I_{H_*}, S_{V_*}, I_{V_*}) \), where \( S_{H_*}, I_{H_*}, S_{V_*} \) and \( I_{V_*} \) are given in (11), (12), (14), and (15), respectively.

Denote by \( \mathbb{R}_{0+} \) the set of nonnegative real numbers. For system \((M_R)\), we will consider solutions \( \tilde{X} = (S_H, I_H, S_V, I_V) \) on \([-\tau, +\infty)\) with initial condition \( \phi = (\phi_1, \phi_2, \phi_4, \phi_5) \) from the set

\[
\Phi_R = \{ \phi = (\phi_1, \phi_2, \phi_4, \phi_5) \in C([-\tau, 0], \mathbb{R}^4_+) : \phi_1(\theta) > 0, \phi_4(\theta) > 0 \}
\]

Furthermore, let \( \Omega_R \) be the space of all solutions to system \((M_R)\) with initial condition \( \phi \in \Phi_R \).

Observe also that for any \( \phi = (\phi_1, \phi_2, \phi_4, \phi_5) \in \Phi_R \), there exists a unique solution to system \((M_R)\) with initial condition \( \phi \), which is nonnegative and bounded on \([0, +\infty)\).

**Theorem 5.** If \( R_0 < 1 \), then the disease-free equilibrium \((S_{H_0}, 0, S_{V_0}, 0)\) of system \((M_R)\) is globally asymptotically stable in \( \Omega_R \).

**Proof.** Define the functional \( V : \Phi_R \to \mathbb{R} \) by

\[
V(\gamma) = V_1(\gamma) + V_2(\gamma) + V_3(\gamma) + V_4(\gamma) + V_5(\gamma) + V_6(\gamma)
\]

where

\[
V_1(\gamma) = \frac{e^{-(\mu_H + \epsilon + \delta)}H(t - \tau_1) - \gamma_1(0) - S_{H_0}}{\gamma_1(0) - S_{H_0} \ln \frac{\gamma_1(0)}{S_{H_0}}}
\]

\[
V_2(\gamma) = \frac{e^{-(\mu_H + \epsilon + \delta)}H(t - \tau_2) - \gamma_2(0) - S_{V_0}}{\gamma_2(0) - S_{V_0} \ln \frac{\gamma_2(0)}{S_{V_0}}}
\]

\[
V_3(\gamma) = \frac{e^{-(\mu_H + \epsilon + \delta)}H(t - \tau_2) - \gamma_3(0) - S_{V_0}}{\gamma_3(0) - S_{V_0} \ln \frac{\gamma_3(0)}{S_{V_0}}}
\]

\[
V_4(\gamma) = \frac{e^{-(\mu_H + \epsilon + \delta)}H(t - \tau_2) - \gamma_4(0) - S_{V_0}}{\gamma_4(0) - S_{V_0} \ln \frac{\gamma_4(0)}{S_{V_0}}}
\]

\[
V_5(\gamma) = \frac{e^{-(\mu_H + \epsilon + \delta)}H(t - \tau_2) - \gamma_5(0) - S_{V_0}}{\gamma_5(0) - S_{V_0} \ln \frac{\gamma_5(0)}{S_{V_0}}}
\]

\[
V_6(\gamma) = \frac{e^{-(\mu_H + \epsilon + \delta)}H(t - \tau_2) - \gamma_6(0) - S_{V_0}}{\gamma_6(0) - S_{V_0} \ln \frac{\gamma_6(0)}{S_{V_0}}}
\]
Certainly, $V$ is continuous and nonnegative on $\Phi_R$ if $R_0 < 1$. Furthermore, $V(\gamma) = 0$ if and only if $\gamma = E_{0b}$, where $E_{0b} = (S_{H0}, I_{H0}, S_{V0}, I_{V0}) \in \Phi_R$, the function identically equal to $E_0 = (S_{H0}, 0, S_{V0}, 0)$.

Let $X = (S_H, I_H, S_V, I_V)$ be a solution to system $(M_R)$ with initial condition $\phi = (\phi_1, \phi_2, \phi_4, \phi_5) \in \Phi_R$. Then $X_t \in \Phi_R$ for $t \geq 0$, with $X_0 = \phi$, and $V(X_t) = V_1(X_t) + V_2(X_t) + V_3(X_t) + V_4(X_t) + V_5(X_t) + V_6(X_t)$, where

\[
V_1(X_t) = \frac{e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V A_2}{(\mu_H + \epsilon + \delta)(\mu_V + c)} \left[ S_H(t) - S_{H0} - S_{H0} \ln \frac{S_H(t)}{S_{H0}} \right],
\]

\[
V_2(X_t) = \left[ S_V(t) - S_{V0} \ln \frac{S_V(t)}{S_{V0}} \right],
\]

\[
V_3(X_t) = \frac{e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V A_2}{(\mu_H + \epsilon + \delta)(\mu_V + c)} I_H(t) + I_V(t),
\]

\[
V_4(X_t) = \frac{e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V A_2}{(\mu_H + \epsilon + \delta)(\mu_V + c)} \int_{t-\tau_2}^{t} I_H(\theta)d\theta,
\]

\[
V_5(X_t) = \frac{e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V A_2}{(\mu_H + \epsilon + \delta)(\mu_V + c)} \left( \int_{t-\tau_1}^{t} I_V(\theta)d\theta \right)^2,
\]

\[
V_6(X_t) = \left( \int_{t-\tau}^{t} \frac{S_V(\theta)}{S_H(\theta)} d\theta \right)^2 + (\mu_V + c) (1 - R_0) \int_{t-\tau}^{t} I_V(\theta)d\theta.
\]

We now get the derivative of $V$ along the solutions of system $(M_R)$ with initial condition $\phi \in \Phi_R$. Considered as a function of $t \in [0, +\infty)$, we have

\[
\frac{d}{dt} V_1(X_t) = \frac{e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V A_2}{(\mu_H + \epsilon + \delta)(\mu_V + c)} \left( 1 - \frac{S_{H0}}{S_H(t)} \right) \times \left[ A_1 - (\mu_H + a) S_H(t) - e^{-(\mu_V + c)\tau_1} \lambda_H I_V(t - \tau_1) S_H(t) \right].
\]

By substituting $(\mu_H + a) S_H(t)$ to $A_1$ in (21), we get

\[
\frac{d}{dt} V_1(X_t) = -\frac{e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V A_2 (\mu_H + a) (S_H(t) - S_{H0})^2}{(\mu_H + \epsilon + \delta)(\mu_V + c) S_H(t)}.
\]

Similarly, we have

\[
\frac{d}{dt} V_2(X_t) = -\left( \mu_V + c \right) \left( \frac{S_V(t) - S_{V0}}{S_V(t)} \right)^2 - e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V I_H(t - \tau_2) S_V(t),
\]

\[
+ e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V I_H(t - \tau_2) S_{V0}.
\]
Now, if \( V = 0 \), then

\[
\frac{d}{dt} V(t) = -\frac{e^{-(\mu_H+\epsilon+\delta)\tau_2} \lambda v A_2 (\mu_H + \mu)(S_H(t) - S_H0)^2}{(\mu_H + \epsilon + \delta)(\mu_H + \epsilon)} < 0
\]

Thus, for \( t \geq 0 \), \( V(\bar{X}_t) = \frac{d}{dt} V(\bar{X}_t) \leq 0 \) if \( R_0 < 1 \).

Note that in (22), \( V'(\bar{X}_t) = 0 \) if \( S_H(t) = S_H(t-\tau) = S_H0, S_V(t) = S_V(t-\tau) = S_V0, I_H(t-\tau) = 0 \) and \( I_V(t-\tau) = 0 \). For simplicity, let

\[ k_1 = \frac{e^{-(\mu_H+\epsilon+\delta)\tau_2} \lambda v A_2 (\mu_H + \mu)}{(\mu_H + \epsilon + \delta)(\mu_H + \epsilon)} \quad \text{and} \quad k_2 = (\mu_H + \epsilon + \delta). \]

Now, if \( V'(\bar{X}_t) = 0 \), then

\[
-k_1 (S_H(t-\tau) - S_H0)^2 - k_2 (S_V(t-\tau) - S_V0)^2 = k_2 (1 - R_0) I_V(t - \tau).
\]

Observe that the left-hand side expression of (23) is always nonpositive while its right-hand side expression is always nonnegative when \( R_0 < 1 \). This means that (23) only holds when

\[
k_1 (S_H(t-\tau) - S_H0)^2 + k_2 (S_V(t-\tau) - S_V0)^2 = 0 \quad \text{and} \quad k_2 (1 - R_0) I_H(t - \tau) = 0.
\]

Hence, (24) and (25) imply that \( S_H(t-\tau) = S_H0, S_V(t-\tau) = S_V0, I_H(t-\tau) = 0 \), and \( I_V(t - \tau) = 0 \). Therefore, for all \( t \geq 0 \), \( V'(\bar{X}_t) = 0 \) if and only if \( S_H(t) = S_H(t-\tau) = S_H0, S_V(t) = S_V(t-\tau) = S_V0, I_H(t-\tau) = 0 \) and \( I_V(t - \tau) = 0 \).

By LaSalle’s Invariance Principle [10], \( \bar{X}_t \rightarrow (S_{H0}, I_{H0}, S_{V0}, I_{V0}) \) as \( t \rightarrow +\infty \). Accordingly, \( \bar{X}(t) \) converges to \((S_{H0}, 0, S_{V0}, 0)\) as \( t \rightarrow +\infty \), that is, the disease-free equilibrium \((S_{H0}, 0, S_{V0}, 0)\) is globally asymptotically stable in \( \Omega_R \).

**Corollary 6.** If \( R_0 < 1 \), then the disease-free equilibrium \((S_{H0}, 0, R_{H0}, S_{V0}, 0)\) of system (M) is globally asymptotically stable in \( \Omega \).
**Proof.** Let \((S_H(t), I_H(t), R_H(t), S_V(t), I_V(t))\) be a solution to system \((M)\) with initial condition \(\phi \in \Phi\). Note that the solution for \(R_H(t)\) of system \(M\) is given by,

\[
R_H(t) = \int_{-\tau}^{t} e^{\mu_H(t-\theta)}(\delta I_H(\theta) + aS_H(\theta))d\theta.
\]

Since \(R_0 < 1\), by Theorem 5, we have

\[
\lim_{t \to +\infty} S_H(t) = S_{H_0}, \quad \lim_{t \to +\infty} I_H(t) = 0, \quad \lim_{t \to +\infty} S_V(t) = S_{V_0}, \quad \text{and} \quad \lim_{t \to +\infty} I_V(t) = 0.
\]

Hence, \(\lim_{t \to +\infty} R_H(t) = \lim_{t \to +\infty} \frac{\delta I_H(t) + aS_H(t)}{\mu_H} = R_{H_0}\). Accordingly, the disease-free equilibrium \((S_{H_0}, 0, R_{H_0}, S_{V_0}, 0)\) of system \((M)\) is globally asymptotically stable in \(\Omega\). \(\square\)

**Theorem 7.** If \(R_0 > 1\), then the disease-free equilibrium \((S_{H_0}, 0, S_{V_0}, 0)\) of system \((M)\) is unstable.

**Proof.** By linearizing system \((M)\) at the disease-free equilibrium \((S_{H_0}, 0, S_{V_0}, 0)\), we obtain the matrix \(J((S_{H_0}, 0, S_{V_0}), -\lambda)\) as

\[
\begin{bmatrix}
-\mu_H - a - \lambda & 0 & 0 & -s_2 \\
0 & -(\mu_H + c + \delta) - \lambda & 0 & s_2 \\
0 & -s_1 \left( \frac{A_2}{\mu_V + \epsilon} \right) & -(\mu_H + c) - \lambda & 0 \\
0 & s_1 \left( \frac{A_2}{\mu_V + \epsilon} \right) & 0 & -(\mu_V + c) - \lambda
\end{bmatrix}
\]

where \(J((S_{H_0}, 0, S_{V_0}))\) is the Jacobian matrix at \((S_{H_0}, 0, S_{V_0}, 0)\), \(\lambda\) is the eigenvalue, \(I\) is the \(4 \times 4\) identity matrix, \(s_1 = e^{-(\mu_H + c + \delta)T} - \lambda, V\), and \(s_2 = e^{-(\mu_V + c + \lambda)}\). Therefore, the characteristic equation at the disease-free equilibrium \((S_{H_0}, 0, S_{V_0})\) is given by

\[
0 = \det (J((S_{H_0}, 0, S_{V_0}, 0)) - \lambda I)
\]

\[
= -\frac{1}{(\mu_V + c)(\mu_H + a)}(c + 2\lambda + \mu_V)(a + 2\lambda + \mu_H)
\]

\[
\times \left[ -4(\mu_V + c)(\mu_H + a)\lambda^2 - 2(\mu_V + c)(\mu_H + a)(c + \delta + \epsilon + \mu_H + \mu_V)\lambda
\right.
\]

\[
+ A_1 A_2 e^{-T(\lambda + \mu_V + c)} e^{-T(\mu_H + \mu_V + \delta + \epsilon + \lambda)} \lambda H_{H}(\mu_V + c)^2 (\delta + \epsilon + \mu_H + \mu_H + a).
\]

We show that (26) has a positive root so that \((S_{H_0}, 0, S_{V_0}, 0)\) is unstable. Clearly, \(\lambda = -\frac{\mu_V + c}{2}\) and \(\lambda = -\frac{\mu_H + a}{2}\) are roots of (26). The other roots are determined by the equation

\[
4(\mu_V + c)(\mu_H + a)\lambda^2 + 2(\mu_V + c)(\mu_H + a)(c + \delta + \epsilon + \mu_H + \mu_V)\lambda
\]

\[
+ A_1 A_2 e^{-T(\lambda + \mu_V + c)} e^{-T(\mu_H + \mu_V + \delta + \epsilon + \lambda)} \lambda H_{H}(\mu_V + c)^2 (\delta + \epsilon + \mu_H + \mu_H + a).
\]

Denote the left hand-side expression of (27) as \(P(\lambda)\) and the right hand-side as \(Q(\lambda)\). Suppose that \(\lambda\) is real. Observe that \(P(0) = 0\) and \(\lim_{\lambda \to +\infty} P(\lambda) = +\infty\). Note also that \(Q(\lambda)\) is a decreasing function of \(\lambda\) and \(Q(0) = [\mu_V + c]^2 (\delta + \epsilon + \mu_H + \mu_H + a)] [R_0 - 1] > 0\). Since \(R_0 > 1\). Hence, the functions \(P\) and \(Q\) must intersect for some \(\lambda_0 > 0\), which establishes the existence of a positive root of (26). \(\square\)
Corollary 8. If \( R_0 > 1 \), then the disease-free equilibrium \((S_{H_0}, 0, R_{H_0}, S_{V_0}, 0)\) of system (M) is unstable in \( \Omega \).

Denote by \( \Phi_+ \) and \( \Phi_{R+} \) the sets of initial conditions for system (M) and system \((M_R)\), respectively, such that for \( \phi \in \Phi \) and \( \psi \in \Phi_R \), we have \( \phi(\theta) > 0 \) and \( \psi(\theta) > 0 \). Also, let \( \Omega_+ \) and \( \Omega_{R+} \) be the spaces of all positive solutions to system (M) and system \((M_R)\) with initial conditions in \( \Phi_+ \) and \( \Phi_{R+} \), respectively.

Theorem 9. If \( R_0 > 1 \), then the endemic equilibrium \((S_{H*}, I_{H*}, S_{V*}, I_{V*})\) of system \((M_R)\) is globally asymptotically stable in \( \Omega_{R+} \).

Proof. Define the functional \( V : \Phi_{R+} \to \mathbb{R} \) by

\[
V(\gamma) = V_1(\gamma) + V_2(\gamma) + V_3(\gamma) + V_4(\gamma) + V_5(\gamma) + V_6(\gamma) + V_7(\gamma) + V_8(\gamma)
\]

where

\[
V_1(\gamma) = e^{-(\mu_H+\epsilon+\delta)T}S_{V*}I_{H*} \left[ \frac{\gamma_1(0)}{I_{H*}} - S_{H*} - S_{H*} \ln \frac{\gamma_1(0)}{I_{H*}} \right],
\]

\[
V_2(\gamma) = e^{-(\mu_H+\epsilon+\delta)T}S_{V*}I_{H*} \left[ \frac{\gamma_2(0)}{I_{H*}} - I_{H*} - I_{H*} \ln \frac{\gamma_2(0)}{I_{H*}} \right],
\]

\[
V_3(\gamma) = \gamma_4(0) - S_{V*} - S_{V*} \ln \frac{\gamma_4(0)}{S_{V*}},
\]

\[
V_4(\gamma) = \gamma_5(0) - I_{V*} - I_{V*} \ln \frac{\gamma_5(0)}{I_{V*}},
\]

\[
V_5(\gamma) = e^{-(\mu_H+\epsilon+\delta)T}S_{V*}I_{H*} \int_0^T \left[ \frac{\gamma_5(-\theta)}{I_{V*}} - 1 - \ln \frac{\gamma_5(-\theta)}{I_{V*}} \right] d\theta,
\]

\[
V_6(\gamma) = e^{-(\mu_H+\epsilon+\delta)T}S_{V*}I_{H*} \int_0^T \left[ \frac{\gamma_2(-\theta)}{I_{H*}} - 1 - \ln \frac{\gamma_2(-\theta)}{I_{H*}} \right] d\theta,
\]

\[
V_7(\gamma) = e^{-(\mu_H+\epsilon+\delta)T}S_{V*}I_{H*} \int_0^T \left[ \frac{\gamma_1(\theta)}{S_{H*}} - 2 + \frac{S_{H*}}{\gamma_1(\theta)} \right] d\theta,
\]

\[
V_8(\gamma) = (\mu_V + \epsilon)S_{V*} \int_{-T}^T \left[ \frac{\gamma_3(\theta)}{S_{V*}} - 2 + \frac{S_{V*}}{\gamma_3(\theta)} \right] d\theta.
\]

Certainly, \( V \) is continuous and nonnegative on \( \Phi_{R+} \). Also, \( V(\gamma) = 0 \) if and only if \( \gamma = E_{ss} \), where \( E_{ss} = (S_{H*}, I_{H*}, S_{V*}, I_{V*}) \in \Phi_{R+} \), the function identically equal to \( E_* = (S_{H*}, I_{H*}, S_{V*}, I_{V*}) \).

Let \( X = (S_H, I_H, S_V, I_V) \) be a solution identically equal to \( E_* = (S_{H*}, I_{H*}, S_{V*}, I_{V*}) \). Then \( X_1 = X_2 = 0 \), with \( X_0 = \phi \), and

\[
V(X_t) = V_1(X_t) + V_2(X_t) + V_3(X_t) + V_4(X_t) + V_5(X_t) + V_6(X_t) + V_7(X_t) + V_8(X_t)
\]
where

\[
V_1(\tilde{X}_t) = \frac{e^{-(\mu_H + c)\tau_2} \lambda_V S_V (\mu_H + c)}{e^{-(\mu_H + c)\tau_1} \lambda_H I_{V*} S_{H*}} \left[ S_H(t) - S_{H*} - S_{H*} \ln \left( \frac{S_H(t)}{S_{H*}} \right) \right],
\]

\[
V_2(\tilde{X}_t) = \frac{e^{-(\mu_H + c)\tau_2} \lambda_V S_V (\mu_H + c)}{e^{-(\mu_H + c)\tau_1} \lambda_H I_{V*} S_{H*}} \left[ I_H(t) - I_{H*} - I_{H*} \ln \left( \frac{I_H(t)}{I_{H*}} \right) \right],
\]

\[
V_3(\tilde{X}_t) = S_V(t) - S_{V*} - S_{V*} \ln \frac{S_V(t)}{S_{V*}},
\]

\[
V_4(\tilde{X}_t) = I_V(t) - I_{V*} - I_{V*} \ln \frac{I_V(t)}{I_{V*}},
\]

\[
V_5(\tilde{X}_t) = e^{-(\mu_H + c)\tau_2} \lambda_V S_V I_{H*} \int_0^{\tau_1} \left[ \frac{I_V(t - \theta)}{I_{V*}} - 1 - \ln \frac{I_V(t - \theta)}{I_{V*}} \right] d\theta,
\]

\[
V_6(\tilde{X}_t) = e^{-(\mu_H + c)\tau_2} \lambda_V S_V I_{H*} \int_0^{\tau_2} \left[ \frac{I_H(t - \theta)}{I_{H*}} - 1 - \ln \frac{I_H(t - \theta)}{I_{H*}} \right] d\theta,
\]

\[
V_7(\tilde{X}_t) = e^{-(\mu_H + c)\tau_2} \lambda_V S_V I_{H*} (\mu_H + a) \int_{t - \tau}^t \left[ \frac{S_H(\theta)}{S_{H*}} - 2 + \frac{S_{H*}}{S_H(\theta)} \right] d\theta,
\]

\[
V_8(\tilde{X}_t) = (\mu_V + c) S_{V*} \int_{t - \tau}^t \left[ \frac{S_V(\theta)}{S_{V*}} - 2 + \frac{S_{V*}}{S_V(\theta)} \right] d\theta.
\]

We now get the derivative of \( V \) along the solutions of system \((M_R)\) with initial condition \( \phi \in \Phi_{R+} \).

Considered as a function of \( t \in [0, +\infty) \), we have

\[
\frac{d}{dt} V_1(\tilde{X}_t) = e^{-(\mu_H + c)\tau_2} \lambda_V S_V I_{H*} \left( 1 - \frac{S_{H*}}{S_H(t)} \right)
\times \left[ A_1 - (\mu_H + a) S_H(t) - e^{-(\mu_H + c)\tau_1} \lambda_H I_V(t - \tau_1) S_H(t) \right].
\]

Note that \( A_1 = e^{-(\mu_H + c)\tau_1} \lambda_H I_V S_{H*} + (\mu_H + a) S_{H*} \). Consequently, we have

\[
\frac{d}{dt} V_1(\tilde{X}_t) = e^{-(\mu_H + c)\tau_2} \lambda_V S_V I_{H*} N_1 - e^{-(\mu_H + c)\tau_2} \lambda_V S_V I_{H*} N_2
\]

where

\[
N_1 = 1 - \frac{I_V(t - \tau_1) S_H(t)}{I_{V*} S_{H*}} + \frac{\mu_H + a}{e^{-(\mu_H + c)\tau_1} \lambda_H I_{V*} S_{H*}} - \frac{(\mu_H + a) S_H(t)}{e^{-(\mu_H + c)\tau_1} \lambda_H I_{V*} S_{H*}}
\]

and

\[
N_2 = \frac{S_{H*}}{S_H(t)} - \frac{I_V(t - \tau_1)}{I_{V*}} + \frac{(\mu_H + a) S_{H*}}{e^{-(\mu_H + c)\tau_1} \lambda_H I_{V*} S_{H*}} - \frac{\mu_H + a}{e^{-(\mu_H + c)\tau_1} \lambda_H I_{V*}}.
\]

Also,

\[
\frac{d}{dt} V_2(\tilde{X}_t) = e^{-(\mu_H + c)\tau_2} \lambda_V S_V I_{H*} \left[ \frac{I_V(t - \tau_1) S_H(t)}{I_{V*} S_{H*}} - \frac{I_H(t)}{I_{H*}} - \frac{I_H(t) I_V(t - \tau_1) S_H(t)}{I_{H*} I_V(t) I_{V*} S_{H*}} + 1 \right],
\]

\[
\frac{d}{dt} V_3(\tilde{X}_t) = e^{-(\mu_H + c)\tau_2} \lambda_V \left[ I_{H*} S_{V*} - I_H(t - \tau_2) S_V(t) \right] + (\mu_V + c) \left[ S_{V*} - S_V(t) \right]
- e^{-(\mu_H + c)\tau_2} \lambda_V \left[ I_{H*} S_{V*} - I_H(t - \tau_2) S_V(t) \right] - (\mu_V + c) \left[ S_{V*} S_{V*} - S_V(t) \right].
\]
\[
\frac{d}{dt} V_4(X_t) = e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V I_H(t-\tau_2)S_V(t) - (\mu_V+c)I_V(t) + e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V I_{H*}S_{V*}
\]
\[
- \frac{I_{V*}}{I_V(t)}e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V I_H(t-\tau_2)S_V(t).
\]
\[
\frac{d}{dt} V_5(X_t) = e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V S_V I_{H*} \frac{I_V(t)}{I_V(t)} - \frac{I_V(t-\tau_1)}{I_V(t)} + \ln \frac{I_V(t-\tau_1)}{I_V(t)}.
\]
\[
\frac{d}{dt} V_6(X_t) = e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V S_V I_{H*} \frac{I_H(t)}{I_H(t)} - \frac{I_H(t-\tau_2)}{I_H(t)} + \ln \frac{I_H(t-\tau_2)}{I_H(t)}.
\]
\[
\frac{d}{dt} V_7(X_t) = \left(\frac{\mu_V+c}{\lambda_H}I_{V*}\right) - \frac{e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V S_V I_{H*}\left(\mu_H+a\right)}{e^{-(\mu_V+\epsilon+\delta)\tau_1}\lambda_H I_{V*}} \times \left[\frac{S_H(t)}{S_{H*}} - 2 + \frac{S_{H*}}{S_H(t)} - \left(\frac{S_{H*}}{S_H(t)} - 2 + \frac{S_{H*}}{S_H(t-\tau)}\right)\right].
\]
\[
\frac{d}{dt} V_8(X_t) = (\mu_V+c)S_{V*} \left[2 + \frac{S_{V*}}{S_V(t)} - \left(\frac{S_V(t-\tau)}{S_V(t)} - 2 + \frac{S_{V*}}{S_V(t-\tau)}\right)\right].
\]
Accordingly, we have
\[
\frac{d}{dt} V(X_t) = e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V S_V I_{H*} \left[4 - \frac{S_{H*}}{S_H(t)} - \frac{I_V(t-\tau_1)S_H(t)I_{H*}}{I_V(t)S_{V*}I_{H*}} - \frac{S_{H*}}{S_H(t)} - \frac{S_{H*}}{S_H(t)}\right]
\]
\[
+ (\mu_V+c)S_{V*} \left[2 - \frac{S_{V*}}{S_V(t)} - \frac{S_{V*}}{S_V(t)}\right]
\]
\[
+ \left[\ln \frac{I_H(t-\tau_2)}{I_H(t)} + \ln \frac{I_V(t-\tau_1)}{I_V(t)}\right] e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V S_V I_{H*}
\]
\[
+ e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V S_V I_{H*}\left(\mu_H+a\right) \left[2 - \frac{S_{H*}}{S_H(t)} - \frac{S_{H*}}{S_H(t)}\right]
\]
\[
- e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V S_V I_{H*}\frac{S_{V*}}{S_V(t)}
\]
\[
+ e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V S_V I_{H*}\frac{I_V(t)}{I_V(t)} - (\mu_V+c)I_V(t)
\]
\[
+ e^{-(\mu_H+\epsilon+\delta)\tau_2}\lambda_V S_V I_{H*}\left(\mu_H+a\right)
\]
\[
\times \left[\frac{S_H(t)}{S_{H*}} - 2 + \frac{S_{H*}}{S_H(t)} - \left(\frac{S_{H*}}{S_H(t)} - 2 + \frac{S_{H*}}{S_H(t-\tau)}\right)\right]
\]
\[
+ (\mu_V+c)S_{V*} \left[2 + \frac{S_{V*}}{S_V(t)} - \left(\frac{S_V(t-\tau)}{S_V(t)} - 2 + \frac{S_{V*}}{S_V(t-\tau)}\right)\right].
\]

Note that
\[
\frac{I_H(t-\tau_2)I_V(t-\tau_1)}{I_H(t)} = \left(\frac{S_{V*}}{S_V(t)}\right) \left(\frac{I_{V*}I_H(t-\tau_2)S_V(t)}{I_V(t)S_{V*}I_{H*}}\right) \left(\frac{S_{H*}}{S_H(t)}\right) \left(\frac{I_V(t-\tau_1)S_H(t)I_{H*}}{I_V(t)S_{V*}I_{H*}}\right).
\]
Let \( r_1 = e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V S_V I_{H*} \). Then
\[
\frac{d}{dt} V(\bar{X}_t) = r_1 \left[ \left( 1 - \frac{S_{H*}}{S_H(t)} + \ln \left( \frac{S_{H*}}{S_H(t)} \right) \right) + \left( 1 - \frac{S_{V*}}{S_V(t)} + \ln \left( \frac{S_{V*}}{S_V(t)} \right) \right) \right]
\]
\[
+ r_1 \left[ 1 - \frac{I_V(t - \tau_1)S_H(t)I_{H*}}{I_V S_H(t)} + \ln \frac{I_V(t - \tau_1)S_H(t)I_{H*}}{I_V S_H(t)} \right]
\]
\[
+ r_1 \left[ 1 - \frac{I_V(t)S_V(t)I_{H}(t - \tau_2)}{I_V(t)S_V(t)I_{H}(t - \tau_2)} + \ln \frac{I_V(t)S_V(t)I_{H}(t - \tau_2)}{I_V(t)S_V(t)I_{H}(t - \tau_2)} \right]
\]
\[
+ \frac{e^{-(\mu_H + \epsilon + \delta)\tau_2} \lambda_V S_V I_{H*}(\mu_H + \alpha)}{e^{-(\mu_V + \epsilon)\tau_1} \lambda_H I_{V*}} \left[ 2 - \frac{S_{V*}}{S_V(t - \tau)} + \frac{S_{V*}}{S_V(t - \tau)} \right]
\]
\[
+ (\mu_V + \epsilon) S_V \left[ 2 - \frac{S_{V*}}{S_V(t - \tau)} + \frac{S_{V*}}{S_V(t - \tau)} \right]
\]
\[
+ \frac{\lambda_H}{\lambda_V S_V(t)} \left[ 2 - \frac{S_{H*}}{S_H(t - \tau)} + \frac{S_{H*}}{S_H(t - \tau)} \right]
\]
\[
+ \frac{\lambda_V}{\lambda_H S_V(t)} \left[ 2 - \frac{S_{H*}}{S_H(t - \tau)} + \frac{S_{H*}}{S_H(t - \tau)} \right]
\]
Thus, for \( t \geq 0 \), \( V'(\bar{X}_t) = \frac{d}{dt} V(\bar{X}_t) \leq 0 \) if \( R_0 > 1 \).

Therefore, for all \( t \geq 0 \), \( V'(\bar{X}_t) = 0 \) if and only if \( S_{H}(t) = S_{H}(t - \tau) = S_{H*}, S_{V}(t) = S_{V}(t - \tau) = S_{V*}, I_{H}(t - \tau) = I_{H*} \) and \( I_{V}(t - \tau) = I_{V*} \).

Following a similar argument as in the proof of Theorem 5, we conclude that the endemic equilibrium \((S_{H*}, I_{H*}, S_{V*}, I_{V*})\) is globally asymptotically stable in \( \Omega_{R+} \).

**Corollary 10.** If \( R_0 > 1 \), then the endemic equilibrium \((S_{H*}, I_{H*}, R_{H*}, S_{V*}, I_{V*})\) of system \((M)\) is globally asymptotically stable in \( \Omega_{R+} \).

**Proof.** The proof is similar to Corollary 6.

### 5. Numerical Simulations

To illustrate the theoretical results in Section 4, we perform numerical simulations using MATLAB DDE23. We only take constant functions as our initial conditions.

| Parameter   | Description                              | Value        | Unit       | Taken from |
|-------------|------------------------------------------|--------------|------------|------------|
| \( \mu_H \) | Natural death rate for humans            | 0.00000397   | day\(^{-1}\) | Assumed    |
| \( \mu_V \) | Natural death rate for mosquitoes        | 0.0714       | day\(^{-1}\) | [7]        |
| \( \epsilon \) | Dengue disease death rate                | 0.00035      | day\(^{-1}\) | Assumed    |
| \( \delta \) | Human recovery rate                      | 0.143        | day\(^{-1}\) | Assumed    |
| \( \lambda_H \) | Effective contact rate (mosquito to human) | (0.0146, 0.2241) | day\(^{-1}\) | [12]       |
| \( \lambda_V \) | Effective contact rate (human to mosquito) | (0.1299, 0.6821) | day\(^{-1}\) | [12]       |
| \( A_1 \) | Recruitment rate for humans              | Variable     | day\(^{-1}\) | Assumed    |
| \( A_2 \) | Recruitment rate for mosquitoes          | Variable     | day\(^{-1}\) | Assumed    |
| \( a \) | Degree of protection awareness           | [0, 1]       | –          | Assumed    |
| \( c \) | Degree of vector control                 | [0, 1]       | –          | Assumed    |
| \( \tau_1 \) | Latency period for infected human        | [1, 11]      | days       | [6]        |
| \( \tau_2 \) | Latency period for infected mosquito     | [1, 16]      | days       | [6]        |

**Simulation 1.** Consider the parameter values in Table 1 with \( A_1 = 200, A_2 = 200, a = 0.31, c = 0.42, \)
\( \lambda_H = 0.0016, \lambda_V = 0.1311, \tau_1 = 11 \) and \( \tau_2 = 13 \). Hence, \( R_0 = 0.544 \) and the disease-free equilibrium is \((645, 0, 407, 0)\). Also, we take the following initial conditions:
(a) \( (\phi_1, \phi_2, \phi_4, \phi_5) = (850, 500, 550, 250) \);
(b) \( (\phi_1, \phi_2, \phi_4, \phi_5) = (250, 150, 275, 125) \);
(c) \( (\phi_1, \phi_2, \phi_4, \phi_5) = (1000, 250, 750, 575) \); and
(d) \( (\phi_1, \phi_2, \phi_4, \phi_5) = (1250, 570, 850, 300) \).

Figure 2 shows that for different initial conditions, the trajectories of the solutions converge to 
\( (645, 0, 407, 0) \). This agrees with our result that the disease-free equilibrium is globally asymptotically
stable when \( R_0 < 1 \).

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**Simulation 2.** Consider the same parameter values as in Simulation 1 except that we decrease the values
of \( A_1 = A_2 = 200 \) to \( A_1 = A_2 = 100 \), \( \tau_1 = 11 \) to \( \tau_1 = 9 \), \( a = 0.31 \) to \( a = 0.27 \) and \( c = 0.42 \)
to \( c = 0.29 \). Hence, \( R_0 = 2.487 \) and the disease-free equilibrium is \( (370, 0, 277, 0) \). By choosing 100
different initial conditions \( (\phi_1, \phi_2, \phi_4, \phi_5) \), we see in Figure 3 that the trajectories of the solutions do not
converge to \( (370, 0, 277, 0) \). This agrees with our result that the disease-free equilibrium is unstable when
\( R_0 > 1 \).
Figure 3: (Simulation 2) When $R_0 > 1$, the disease-free equilibrium is unstable and the endemic equilibrium exists which is globally asymptotically stable.

Remark 11. In Simulation 2, we obtained $R_0 = 2.487$ and the endemic equilibrium $(S_{H*}, I_{H*}, S_{V*}, I_{V*}) = (357.1, 24.8, 115.3, 116.3)$ now exists. Observe that each of the solution curves of $S_H, I_H, S_V,$ and $I_V$ converges to $S_{H*}, I_{H*}, S_{V*},$ and $I_{V*}$, respectively, as shown in Figure 3. This agrees with our result that the endemic equilibrium is globally asymptotically stable whenever $R_0 > 1$.

Simulation 3. Take $A_1 = A_2 = 100, \tau_1 = 9, \tau_2 = 5, a = 0.15$ and $c = 0.2$. Hence, $R_0 = 9.270661$ and the endemic equilibrium is $(500, 174, 9, 360)$. Take the following initial conditions:
(a) $(\phi_1, \phi_2, \phi_4, \phi_5) = (350, 186, 300, 190)$;
(b) $(\phi_1, \phi_2, \phi_4, \phi_5) = (250, 150, 273, 125)$;
(c) $(\phi_1, \phi_2, \phi_4, \phi_5) = (1000, 250, 750, 575)$; and
(d) $(\phi_1, \phi_2, \phi_4, \phi_5) = (1250, 570, 850, 300)$.

Figure 4 shows that for the above initial conditions, the trajectories of the solutions converge to $(500, 174, 9, 360)$. This is again consistent with our result that the endemic equilibrium is globally asymptotically stable if $R_0 > 1$. 

Retracted
Figure 4: (Simulation 3) The endemic equilibrium is globally asymptotically stable if $R_0 > 1$.

**Simulation 4.** Consider the parameter values in Table 1 with $A_1 = 200$, $A_2 = 200$, $c = 0.42$, $\lambda_H = 0.0016$, $\lambda_V = 0.1311$, $\tau_1 = 11$, $\tau_2 = 13$, and take the initial condition $(\phi_1, \phi_2, \phi_4, \phi_5) = (850, 500, 550, 250)$. We vary the values of $a$ and obtain the following (see Figure 5):

(a) When $a = 0.05$, we have $R_0 = 3.37$ and the endemic equilibrium $(3842, 54, 126, 281)$ is globally asymptotically stable;

(b) When $a = 0.15$, we have $R_0 = 1.12$ and the endemic equilibrium $(1330, 3, 363, 44)$ is still globally asymptotically stable;

(c) When $a = 0.25$, we have $R_0 = 0.67$ and the disease-free equilibrium $(800, 0, 407, 0)$ is now globally asymptotically stable; and

(d) When $a = 0.35$, we have $R_0 = 0.48$ and the disease-free equilibrium $(571, 0, 407, 0)$ is also globally asymptotically stable.
Figure 5: (Simulation 4) The effect of varying the values of $a$ to the stability of equilibrium points.

The results in Simulation 4 coincide with Remark 3, that is, the basic reproductive number $R_0$ is a decreasing function of the level of self-protection awareness $a$. This result is useful in identifying ways in order to decrease the spread of dengue virus in the community.

We now illustrate how the level of vector control $c$ affects the stability of the equilibrium points.

**Simulation 5.** Consider the parameter values in Table 1 with $A_1 = 200$, $A_2 = 200$, $a = 0.27$, $\lambda_H = 0.0016$, $\lambda_V = 0.1311$, $\tau_1 = 11$, $\tau_2 = 13$, and take the initial condition $(\phi_1, \phi_2, \phi_4, \phi_5) = (850, 500, 550, 250)$. We vary the values of $c$ and obtain the following (see Figure 6):

(a) When $c = 0.05$, we have $R_0 = 599.63$ and the endemic equilibrium $(208, 1002, 10, 1638)$ is globally asymptotically stable;

(b) When $c = 0.25$, we have $R_0 = 9.47$ and the endemic equilibrium $(676, 121, 72, 550)$ is still globally asymptotically stable;

(c) When $c = 0.40$, we have $R_0 = 0.84$ and the disease-free equilibrium $(740, 0, 424, 0)$ is now globally asymptotically stable; and

(d) When $c = 0.45$, we have $R_0 = 0.39$ and the disease-free equilibrium $(740, 0, 384, 0)$ is also globally asymptotically stable.
Figure 6: (Simulation 5) The effect of varying the values of $c$ to the stability of equilibrium points.

Simulation 5 illustrates that $R_0$ is also a decreasing function of the level of vector control $c$, as stated in Remark 4. This implies that utilizing some known vector controls and extending their level of effectiveness are potential factors that may contribute in controlling the spread of virus in the community.

6. Concluding Remarks and Future Works

In this paper, we have incorporated the incubation times of dengue virus for human and mosquito in an SIR-SI human-mosquito dengue transmission model. The human population is completely described by the interaction of susceptible, infected and recovered individuals, and the mosquito population is subdivided into the classes of susceptible and infected. We described the properties of the solutions in Section 2. In Section 3, we defined the basic reproductive number $R_0$ and calculated the equilibrium points of the model. In Section 4, we established the global stability of the equilibrium points through Lyapunov functionals and LaSalle's Invariance Principle. In particular, if $R_0 < 1$, then all solutions converge to the disease-free equilibrium which implies that the disease will eventually be eliminated from the community. If $R_0 > 1$, then the disease-free equilibrium is unstable and the endemic equilibrium exists. In this case, the endemic equilibrium is globally asymptotically stable, implying that the disease will stay in the community. Numerical simulations to illustrate the theoretical results were presented in Section 5.

Contrary to the work in [8], this paper shows that the presence of latent delays $\tau_1$ and $\tau_2$ does not lead to the occurrence of Hopf bifurcation. Moreover, this study confirms that increasing the level of human self-protection awareness helps in controlling the spread of the dengue virus in the community. Furthermore, the use of extensive vector control methods can minimize the extent of dispersion of the virus in the community.

In 2019, Chunqing Wu and Patricia J. Y. Wong [16] introduced a similar model for dengue transmission in the paper entitled *Dengue transmission: mathematical model with discrete time delays*...
and estimation of the reproduction number. Vector controls and human awareness are not considered in the model. Moreover, the mosquito population is assumed to be constant. In contrast to our paper, they have only established the local stability of the equilibrium points. They have additional results on the estimation of reproductive number when the population is partially susceptible to the disease. A parallel study on such estimation can be done in our proposed model. We leave this for future work.

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