On the Analog Strong CP Problem in the $CP^{N-1}$ Models

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Addressed is the question of whether a natural mechanism exists to resolve the strong CP problem. The analogous issue for the two-dimensional $CP^{N-1}$ models is analyzed using computer simulations.

1 Introduction

Non-perturbative effects can introduce CP violation into the strong interactions. Since no such violations of this discrete symmetry have been observed, an avoidance mechanism must be operative. Although many proposals have been suggested to resolve this issue, the strong CP problem remains one of the outstanding low-energy mysteries of the standard model of particle physics.

A four-dimensional Yang-Mills theory has instantons. They represent tunnelling between different classical $n$-vacua, where $n$ is an integer characterizing a topological winding number. The true quantum vacua are believed to be linear combinations of the $n$-vacua weighted by $\exp(in\theta)$. Alternatively, one can add to the standard Yang-Mills action the term $\theta Q$, where $Q = g^2/(32\pi^2) \int d^4xF^{a\mu}_{\mu\nu}F^{\mu\nu}_a(x)$ is the total instanton number. The resulting theory explicitly breaks parity, time reversal invariance and CP unless $\theta$ equals $0$ or $\pi$. When $\theta$ equals $\pi$, CP is believed to be spontaneously broken.

When quarks are included in the analysis, the question of CP violation is more complicated. An overall phase in the quark mass matrix $\mathcal{M}$ can be

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eliminated through an axial $U_A(1)$ transformation, but this produces an extra contribution to $\theta$ of $\text{Arg det} M$. The relevant parameter is thus $\theta_{\text{eff}} = \theta + \text{Arg det} M$. In QCD, strong CP violation arises unless $\theta_{\text{eff}} = 0$. From measurements on the neutron’s electric dipole moment, a bound of $\theta_{\text{eff}} < 10^{-9}$ is obtained. The strong CP problem can be stated as follows. Why is $\theta_{\text{eff}} < 10^{-9}$ when $\theta_{\text{eff}}$ is a parameter of order unity? In other words, $\theta$ must be tuned to cancel the contribution to $\text{Arg det} M$ to one part in a billion. If this is adjusted by hand at the tree level, loop effects ruin the cancellation. Various proposals have been made to resolve this problem, of which the most elegant is the Pecci-Quinn mechanism. It automatically sets $\theta_{\text{eff}}$ to zero.

A number of years ago, I raised the question of whether the strong CP problem in pure Yang-Mills theories could be solved naturally. One should realize that it is strong-coupling dynamics that is most relevant to the issue of $\theta$ dependence. Why is this so? Long-distance physics always determines vacuum structure. For example, local fluctuations in magnetic moments do not determine whether an iron bar magnetizes. A microscopic region may look disordered but the overall system can be ordered, or vice-versa. Since $\theta$-vacua are at the heart of the CP problem, the existence or non-existence of long-distance correlations is crucial. Due to our current limited understanding of long-distance physics and strong-coupling dynamics in Yang-Mills theory, perhaps some subtle effect concerning the strong CP problem has been overlooked.

There exists a toy model in which the strong CP problem is dynamically resolved. It is the $(2+1)$-dimensional Georgi-Glashow model that was analyzed by Polyakov. It exhibits a natural relaxation mechanism by producing its own axion-like field to eliminate the $\theta$ dependence.

The model has instantons: They are the ’t Hooft-Polyakov monopoles. Polyakov showed that these monopoles lead to confinement and produce a mass for the unbroken vector boson. One can repeat Polyakov’s analysis in the presence of a $\theta$ term. To the action, one adds $\theta Q$, where $Q$ is the net monopole number. The dilute gas approximation gives

$$Z_\theta = \frac{1}{N} \int D\phi \exp \left[ -\frac{1}{2} \int dx \partial^\mu \phi \partial_\mu \phi(x) + 2\lambda \cos (\phi(x)/\sqrt{\mu} + \theta) \right]$$

for the partition function, where $\lambda$ and $\mu$ are parameters. Here $\phi$, a field that is dynamically generated, reproduces the Coulombic interactions between monopoles. Since the shift $\phi \rightarrow \phi - \sqrt{\mu} \theta$, eliminates the $\theta$ dependence in $Z_\theta$, the $(2+1)$-dimensional analog of the strong CP problem is resolved.

There is a physical reason why the $\theta$ dependence goes away. Three-dimensional Coulombic gas systems are overall neutral. If $Q = 0$, then the
term $\theta \times$ (monopole number) has no effect when added to the Lagrangian.

But why is the three-dimensional Coulombic gas system neutral? The answer is due to the attractive $1/r$ potential between monopoles and antimonopoles. The force between the monopoles and the anti-monopole is sufficiently strong so as to render the gas neutral.

Once one realizes this, it is easy to establish criteria for when the $\theta$ dependence in a theory naturally goes away. Let $V(x)$ be the anti-instanton–instanton interaction potential in a $d$-dimensional system computed in the absence of any other instantons and let $R$ be the distance between the instanton and the anti-instanton.

If $V(R) \sim 1/R^n$ with $n < d$ then there is no $\theta$ dependence.

\begin{equation}
\text{If } V(R) \sim 1/R^n \text{ with } n < d \text{ then there is no } \theta \text{ dependence.} \tag{2}
\end{equation}

For four-dimensional Yang-Mills theories, there is no strong CP problem if the instanton is attracted to the anti-instanton through a potential stronger that $1/R^4$ for $R$ large. Because of our ignorance of the long-distance behavior of Yang-Mills theories, it is not known whether the criterion in Eq. (2) is satisfied.

\section{Computer Results of Schierholz et al for $CP^{N-1}$ Models}

During the past few years, G. Schierholz and co-workers have performed some intriguing Monte Carlo studies for the two-dimensional $CP^{N-1}$ models in the presence of a $\theta$ term. These models have several features in common with $d = 4$ Yang-Mills theories: They have confinement and instantons.

The free-energy difference per unit volume $f(\theta)$ is determined from

\begin{equation}
\exp(-V f(\theta)) = Z(\theta)/Z(0) ,
\end{equation}

where $V$ is the volume of the system and $Z(\theta)$ is the partition function with the two-dimensional $\theta$ term of $\theta/(2\pi) \int d^2 x F_{01}$.

As a function of $\theta$, the free-energy obtained from Monte Carlo simulations exhibited a flattening behavior that is reproduced well by

\begin{equation}
f = \begin{cases} 
a(\beta) \theta^2 & \theta < \theta_c \\
c(\beta) & \theta > \theta_c \end{cases},
\end{equation}

In other words, $f$ rises quadratically with $\theta$, turns over and then displays no dependence on $\theta$ beyond a critical value $\theta_c$. For two-dimensional confining systems, the string-tension $\sigma$ for a particle of charge $e$ can be computed from

\begin{equation}
\sigma(e, \theta) = f(\theta + 2\pi e) - f(\theta) .
\end{equation}
Thus the behavior in Eq. (4) implies that confinement is lost for $\theta > \theta_c$ for sufficiently small charges $e$. Furthermore, the Monte Carlo simulations suggested that $\theta_c(\beta)$ goes to zero as the continuum limit is taken. Taken together, the above statements imply that a confining continuum $CP^{N-1}$ theory would have to have $\theta$ adjusted to zero. The conclusion is that the requirement of confinement necessitates a solution of the two-dimensional analog strong CP problem in the $CP^{N-1}$ models.

By the way, a flattening behavior in $f(\theta)$ was also seen for the $d = 4$ $SU(2)$ lattice gauge theory in subsequent simulations performed by G. Schierholz. In $d = 4$, however, this does not imply the lost of confinement so that the interpretation of this result is not so clear.

Inspired by Schierholz’s work, Jan Plefka and I decided to perform our own studies. We were particularly interested in whether the “deconfinement effect” could be related to a natural relaxation method.

3 General Discussion of Lattice Measurements of the Free-Energy for a System with a $\theta$ Term

It is not easy to simulate a system with a $\theta$ term. This is because one usually uses the $\theta = 0$ vacuum to analyze the $\theta \neq 0$ vacua. If “barrier penetration” effects are strong, poor results can ensue.

The free-energy difference $f(\theta)$ can be computed from

$$\exp(-V f(\theta)) = \sum_Q P(Q) \exp(i\theta Q).$$  \hspace{1cm} (6)

Here, $P(Q)$ is the probability that the system has topological charge $Q$. Let $N_{MC}(Q)$ be the number of times that Monte Carlo configurations with topological charge $Q$ are generated. Then

$$P_{MC}(Q) = \frac{N_{MC}(Q)}{\sum_{Q'} N_{MC}(Q')}$$  \hspace{1cm} (7)

provides a Monte Carlo estimate $P_{MC}$ of $P(Q)$. Using Eq. (3), a Monte Carlo value $f_{MC}(\theta)$ of $f(\theta)$ can be obtained from

$$\exp(-V f_{MC}(\theta)) = \sum_Q P_{MC}(Q) \exp(i\theta Q).$$  \hspace{1cm} (8)

This method of computing $f(\theta)$ can lead to an anomalous flattening effect. Suppose there is a dominant statistical error in the $Q = 0$ sector and that the
probability of having a $Q = 0$ configuration is over estimated: $P_{MC}(0) > P(0)$. Then one can show that

$$f_{MC} \approx \begin{cases} f(\theta) & \theta < \theta_b \\ f(\theta_b) & \theta > \theta_b \end{cases}, \quad (9)$$

where $\delta P(0) = P_{MC}(0) - P(0)$ and

$$f(\theta_b) \approx \frac{1}{V} |\log |\delta P(0)|| \quad . \quad (10)$$

Here are the key points (see reference [14]):

1. When the volume is sufficiently big, a limiting $\theta_b$ arises. For $\theta > \theta_b$, $f(\theta)$ cannot be reliably measured.

2. If a flattening behavior in $f(\theta)$ is seen, one should be suspicious of the result. One needs to check whether $|\delta P(0)|$ is bigger than it should be.

3. If a large-$V$ simulation shows a flattening effect for $f(\theta)$, but a smaller-$V$ simulation does not, then one should probably trust the smaller-$V$ result.

In refs.[11,12], no flattening effect was seen in simulations with small volumes. Point (3) says that one should trust the smaller volume studies. If true, the $CP^{N-1}$ models do not undergo a deconfining phase transition.

4 Monte Carlo Results for the Exactly Solvable $d = 2$ $U(1)$ Gauge Theory

One way to decide the question of anomalous flattening behavior is to study an exactly solvable system. The $d = 2$ $U(1)$ gauge theory is such a case. The action is given by

$$S_{U(1)} = \beta \sum_p (U_p + U_p^*) \quad , \quad (11)$$

where $U_p$ is the product of the $U(1)$ link phases around a plaquette $p$ and $\beta$ is the inverse coupling.

The lattice analog of the continuum $\theta$-term action $i\theta/2\pi \int d^2 x F_{01}$ is

$$S_{\theta \text{ term}} = \frac{\theta}{2\pi} \sum_p \log (U_p) \quad . \quad (12)$$

The exact analytic result for the partition function governed by the sum of the action in Eqs. (11) and (12) has been computed in ref.[15].

When Jan Plefka and I performed Monte Carlo simulations of this system, we sometimes found a flattening behavior in the free energy that disagreed with exact analytic results [14]. Figure 1 illustrates this.
Figure 1: The Free Energy of the U(1) Gauge Model Versus $\theta$. The solid line is the exact result.

Noteworthy features are:

1. A barrier $\theta_b$ arises.
2. A short calculation verifies that $\theta_b$ is the value expected from Eq. (10).
3. The flattening effect is due to $P_{MC}(0) > P(0)$.

It turns out that $\theta_b$ moves to smaller values as $V$ increases. For a $7 \times 7$ lattice, the free energy agreed with exact analytic results for all $f(\theta)$.

5 Monte Carlo Results for a $CP^{N-1}$ Model

We also decided to simulate the $CP^3$ model for a lattice action involving an auxiliary $U(1)$ gauge field. The advantage of using this latticization is that analytic strong coupling series are available. The action is given by

$$ S = \beta N \sum_{x,\Delta} \left( z_x^* \cdot z_{x+\Delta} U (x, x + \Delta) + z_x \cdot z_{x+\Delta}^* U^* (x, x + \Delta) \right) $$

(13)
where the complex scalar fields \( z_i^x \) satisfy \( \sum_{i=1}^{N} z_i^x z_i^{* x} = 1 \) and \( N = 4 \) for the \( CP^3 \) model. Here, the sum is over unit shifts \( \Delta \) in the \( d \) positive directions, where \( d = 2 \). The field \( U(x, x + \Delta) \) is a phase associated with the link between \( x \) and \( x + \Delta \); it is the \( U(1) \) auxiliary gauge field. Eq. (13) and \( S_\theta \) of Eq. (12) are added to obtain the full action. This lattice action differs from the one used in refs.[11,12] but is equally good.

When we performed Monte Carlo simulations, again, spurious flattening behavior in the free energy sometimes occurred. Figure 2 is an example.

6 Conclusion

Here are the main points:

(1) Subtleties arise when simulating systems with a \( \theta \) term. The basic difficulty is that one is trying to simulate one vacuum (a \( \theta \neq 0 \) vacuum) using another vacuum (\( \theta = 0 \) vacuum). For good Monte Carlo results, barrier
penetration needs to be suppressed. When the volume of the system is large, the barrier penetration becomes sufficiently severe that reliable results cannot be obtained, particularly for $\theta$ near $\pi$.

(2) Because current simulation methods tend to have the biggest error in the measurement for the $Q = 0$ probability, Monte Carlo results can generate an anomalous flattening behavior in the free-energy for $\theta$ large.

(3) Reliable results for the free-energy appear to be obtainable throughout the entire $\theta$ region as long as the volume is keep sufficiently small.

(4) The ideas and results of this presentation suggest that the $CP^{N-1}$ model with a $\theta$ term does not undergo a deconfining phase transition.

(5) In short, it is unlikely that the solution of the strong CP problem as suggested by previous numerical studies in the $CP^3$ model actually works.

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