INVERTIBILITY OF RANDOM MATRICES: NORM OF THE INVERSE

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Abstract. Let $A$ be an $n \times n$ matrix, whose entries are independent copies of a centered random variable satisfying the subgaussian tail estimate. We prove that the operator norm of $A^{-1}$ does not exceed $Cn^{3/2}$ with probability close to 1.

1. Introduction.

Let $A$ be an $n \times n$ matrix, whose entries are independent identically distributed random variables. The spectral properties of such matrices, in particular invertibility, have been extensively studied (see, e.g. the survey [DS]). While $A$ is almost surely invertible whenever its entries are absolutely continuous, the case of discrete entries is highly non-trivial. Even in the case, when the entries of $A$ are independent random variables taking values $\pm 1$ with probability $1/2$, the precise order of probability that $A$ is degenerate is unknown. Komlós [K1, K2] proved that this probability is $o(1)$ as $n \to \infty$. This result was improved by Kahn, Komlós ans Szemerédi [KKS], who showed that this probability is bounded above by $\theta^n$ for some absolute constant $\theta < 1$. The value of $\theta$ has been recently improved in a series of papers by Tao and Vu [TV1, TV2] to $\theta = 3/4 + o(1)$ (the conjectured value is $\theta = 1/2 + o(1)$).

However, these papers do not address the quantitative characterization of invertibility, namely the norm of the inverse matrix, considered as an operator from $\mathbb{R}^n$ to $\mathbb{R}^n$. Random matrices are one of the standard tools in geometric functional analysis. They are used, in particular, to estimate the Banach–Mazur distance between finite-dimensional Banach spaces and to construct sections of convex bodies possessing certain properties. In all these questions the distortion $\|A\| \cdot \|A^{-1}\|$ plays the crucial role. Since the norm of $A$ is usually highly concentrated, the distortion is determined by the norm of $A^{-1}$. The estimate of the norm of $A^{-1}$ is known only in the case when $A$ is a matrix with independent $N(0, 1)$ Gaussian entries. In this case Szarek [Sz2] proved that $\|A^{-1}\| \leq c\sqrt{n}$ with probability close to 1 (see also [Sz1] where

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the spectral properties of a Gaussian matrix are applied to an important question from geometry of Banach spaces). For other random matrices, including a random $\pm 1$ matrix, even a polynomial bound was unknown. Proving such polynomial estimate is the main aim of this paper.

More results are known about rectangular random matrices. Let $\Gamma$ be an $N \times n$ matrix, whose entries are independent random variables. If $N > n$, then such matrix can be considered as a linear operator $\Gamma : \mathbb{R}^n \to Y$, where $Y = \Gamma \mathbb{R}^n$. If we consider a family $\Gamma_n$ of such matrices with $n/N = \alpha$ for a fixed constant $\alpha > 1$, then the norms of $(\Gamma_n|_Y)^{-1}$ converge a.s. to $(1 - \sqrt{\alpha})^{-1}n^{-1/2}$, provided that the fourth moments of the entries are uniformly bounded [BY]. The random matrices for which $n/N = 1 - o(1)$ are considered in [LPRT]. If the entries of such matrix satisfy certain moment conditions and $n/N > 1 - c/\log n$, then $\|\Gamma|_Y^{-1}\| \leq C(n/N) \cdot n^{-1/2}$ with probability exponentially close to 1.

The proof of the last result is based on the $\varepsilon$-net argument. To describe it we have to introduce some notation. For $p \geq 1$ let $B^n_p$ denote the unit ball of the Banach space $\ell^n_p$. Let $E \subset \mathbb{R}^n$ and let $B \subset \mathbb{R}^n$ be a convex symmetric body. Let $\varepsilon > 0$. We say that a set $F \subset \mathbb{R}^n$ is an $\varepsilon$-net for $E$ with respect to $B$ if

$$E \subset \bigcup_{x \in F} (x + \varepsilon B).$$

The smallest cardinality of an $\varepsilon$-net will be denoted by $N(E, B, \varepsilon)$. For a point $x \in \mathbb{R}^n$, $\|x\|$ stands for the standard Euclidean norm, and for a linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$, $\|T\|$ denotes the operator norm of $T : \ell^n_2 \to \ell^m_2$.

Let $E \subset S^{n-1}$ be a set such that for any fixed $x \in E$ there is a good bound for the probability that $\|\Gamma x\|$ is small. We shall call such bound the small ball probability estimate. If $N(E, B^2_n, \varepsilon)$ is small, this bound implies that with high probability $\|\Gamma x\|$ is large for all $x$ from an $\varepsilon$-net for $E$. Then the approximation is used to derive that in this case $\|\Gamma x\|$ is large for all $x \in E$. Finally, the sphere $S^{n-1}$ is partitioned in two sets for which the above method works. This argument is possible because the small ball probability is controlled by a function of $N$, while the size of an $\varepsilon$-net depends on $n < N$.

The case of a square random matrix is more delicate. Indeed, in this case the small ball probability estimate is too weak to produce a non-trivial estimate for the probability that $\|\Gamma x\|$ is large for all points of an $\varepsilon$-net. To overcome this difficulty, we use the $\varepsilon$-net argument for one part of the sphere and work with conditional probability on the other part. Also, we will need more elaborate small ball probability
estimates, than those employed in [LPRT]. To obtain such estimates we use the method of Halász, which lies in the foundation of the arguments of [KKS], [TV1], [TV2].

Let \( P(\Omega) \) denote the probability of the event \( \Omega \), and let \( E\xi \) denote the expectation of the random variable \( \xi \). A random variable \( \beta \) is called subgaussian if for any \( t > 0 \)
\[
P( |\beta| > t ) \leq C \exp(-ct^2).
\]

The class of subgaussian variables includes many natural types of random variables, in particular, normal and bounded ones. It is well-known that the tail decay condition (1.1) is equivalent to the moment condition \( (E|\beta|^p)^{1/p} \leq C\sqrt{p} \) for all \( p \geq 1 \).

The letters \( c, C, C' \) etc. denote unimportant absolute constants, whose value may change from line to line. Besides these constants, the paper contains many absolute constants which are used throughout the proof. For reader's convenience we use a standard notation for such important absolute constants. Namely, if a constant appears in the formulation of Lemma or Theorem x.y, we denote it \( C_{x,y} \) or \( c_{x,y} \).

The main result of this paper is the following polynomial bound for the norm of \( A^{-1} \).

**Theorem 1.1.** Let \( \beta \) be a centered subgaussian random variable of variance 1. Let \( A \) be an \( n \times n \) matrix whose entries are independent copies of \( \beta \). Then for any \( \varepsilon > c_{1.1}/\sqrt{n} \)
\[
P \left( \exists x \in \mathbb{R}^n \mid \|Ax\| < \frac{\varepsilon}{C_{1.1}n^{3/2}} \right) < \varepsilon
\]
if \( n \) is large enough.

More precisely, we prove that the probability above is bounded by \( \varepsilon/2 + 4 \exp(-cn) \) for all \( n \in \mathbb{N} \).

The inequality of Theorem 1.1 means that \( \|A^{-1}\| \leq C_{1.1} \cdot n^{3/2}/\varepsilon \) with probability greater than \( 1 - \varepsilon \). Equivalently, the smallest singular number of \( A \) is at least \( \varepsilon/(C_{1.1}n^{3/2}) \).

An important feature of Theorem 1.1 is its universality. Namely, the probability estimate holds for all subgaussian random variables, regardless of their nature. Moreover, the only place, where we use the assumption that \( \beta \) is subgaussian, is Lemma 2.3 below.

### 2. Preliminary results.

Assume that \( l \) balls are randomly placed in \( k \) urns. Let \( V \in \{1, \ldots, k\}^l \) be a random vector whose \( i \)-th coordinate is the number of balls contained in the \( i \)-th urn. The distribution of \( V \), called random allocation,
has been extensively studied, and many deep results are available (see [KSC]). We need only a simple combinatorial lemma.

**Lemma 2.1.** Let \( k \leq l \) and let \( X(1), \ldots, X(l) \) be i.i.d. random variables uniformly distributed on the set \( \{1, \ldots, k\} \). Let \( \eta < 1/2 \). Then with probability greater than \( 1 - \eta \) there exists a set \( J \subset \{1, \ldots, l\} \) containing at least \( l/2 \) elements such that

\[
\sum_{i=1}^{k} |\{j \in J \mid X(j) = i\}|^2 \leq C(\eta) \frac{l^2}{k}.
\]

**Remark 2.2.** The proof yields \( C(\eta) = \eta^{-16} \). This estimate is by no means exact.

**Proof.** Let \( X = (X(1), \ldots, X(l)) \). For \( i = 1, \ldots, k \) denote

\[
P_i(X) = |\{j \mid X(j) = i\}|.
\]

Let \( 2 < \alpha < k/2 \) be a number to be chosen later. Denote

\[
I(X) = \{i \mid P_i(X) \geq \alpha \frac{l}{k}\}.
\]

For any \( X \) we have \( \sum_{i=1}^{k} P_i(X) = l \), so \( |I(X)| \leq k/\alpha \). Set \( J(X) = \{j \mid X(j) \in I(X)\} \). Assume that \( |J(X)| \leq l/2 \). Then for the set \( J'(X) = \{1, \ldots, l\} \setminus J(X) \) we have \( |J'(X)| \geq l/2 \) and

\[
\sum_{i=1}^{k} |\{j \in J'(X) \mid X(j) = i\}|^2 \leq \sum_{i \in I(X)} P_i^2(X) \leq k \cdot \left( \alpha \frac{l}{k} \right)^2 = \alpha^2 \frac{l^2}{k}.
\]

Now we have to estimate the probability that \( |\{J(X)\}| \geq l/2 \). To this end we estimate the probability that \( J(X) = J \) and \( I(X) = I \) for fixed subsets \( J \subset \{1, \ldots, l\} \) and \( I \subset \{1, \ldots, k\} \) and sum over all relevant choices of \( J \) and \( I \). We have

\[
\mathbb{P}(|J(X)| \geq l/2) \leq \sum_{|J| \geq l/2} \sum_{|I| \leq k/\alpha} \mathbb{P}(J(X) = J, I(X) = I) \leq \sum_{|J| \geq l/2} \sum_{|I| \leq k/\alpha} \mathbb{P}(X(j) \in I \text{ for all } j \in J) \leq 2^l (k/\alpha) \cdot \binom{k}{\alpha} (1/\alpha)^{l/2} \leq k \cdot (e\alpha)^{k/\alpha} \cdot (4/\alpha)^{l/2},
\]

since the random variables \( X(1), \ldots, X(l) \) are independent. If \( k \leq l \) and \( \alpha > 100 \), the last expression does not exceed \( \alpha^{-l/8} \). To complete the proof, set \( \alpha = \eta^{-8} \). If \( \eta > (2/k)^{1/8} \), then the assumption \( \alpha < k/2 \)
is satisfied. Otherwise, we can set $C(\eta) = \alpha^2 > (k/2)^2$, for which the inequality (2.1) becomes trivial.

□

The following result is a standard large deviation estimate (see e.g. [DS] or [LPRT], where a more general result is proved).

**Lemma 2.3.** Let $A = (a_{i,j})$ be an $n \times n$ matrix whose entries are i.i.d subgaussian random variables. Then

$$\mathbb{P} \left( \|A : B^n_2 \to B^n_2\| \geq C_{2.3} \sqrt{n} \right) \leq \exp(-n).$$

We will also need the volumetric estimate of the covering numbers $N(K, D, t)$ (see e.g. [P]). Denote by $|K|$ the volume of $K \subset \mathbb{R}^n$.

**Lemma 2.4.** Let $t > 0$ and let $K, D \subset \mathbb{R}^n$ be convex symmetric bodies. If $tD \subset K$, then

$$N(K, D, t) \leq \frac{3^n |K|}{|tD|}.$$

3. **Halász Type Lemma.**

Let $\xi_1, \ldots, \xi_n$ be independent centered random variables. To obtain the small ball probability estimates below, we have to bound the probability that $\sum_{j=1}^n \xi_j$ is concentrated in a small interval. One standard method of obtaining such bounds is based on Berry-Esseen Theorem (see, e.g. [LPRT]). However, this method has certain limitations. In particular, if $\xi_j = t_j \varepsilon_j$, where $t_j \in [1, 2]$ and $\varepsilon_j$ are $\pm 1$ random variables, then Berry-Esseen Theorem does not “feel” the distribution of the coefficients $t_j$, and thus does not yield bounds better than $c/\sqrt{n}$ for the small ball probability. To obtain better bounds we use the approach developed by Halász [Ha1, Ha2].

**Lemma 3.1.** Let $c > 0$, $0 < \Delta < a/(2\pi)$ and let $\xi_1, \ldots, \xi_n$ be independent random variables such that $\mathbb{E}\xi_i = 0$, $\mathbb{P}(\xi_i > a) \geq c$ and $\mathbb{P}(\xi_i < -a) \geq c$. For $y \in \mathbb{R}$ set

$$S_\Delta(y) = \sum_{j=1}^n \mathbb{P}(\xi_j - \xi'_j \in [y - \pi\Delta, y + \pi\Delta]),$$

where $\xi'_j$ is an independent copy of $\xi_j$. Then for any $v \in \mathbb{R}$

$$\mathbb{P} \left( \left| \sum_{j=1}^n \xi_j - v \right| < \Delta \right) \leq \frac{C}{n^{5/2}\Delta} \int_{3a/2}^{\infty} S^2_\Delta(y) \, dy + ce^{-c'n}.$$
Proof. For \( t \in \mathbb{R} \) define
\[
\varphi_k(t) = \mathbb{E} \exp(i\xi_k t)
\]
and set
\[
\varphi(t) = \mathbb{E} \exp \left( it \sum_{k=1}^{n} \xi_k \right) = \prod_{k=1}^{n} \varphi_k(t).
\]
Then by a Lemma of Esséen [E], for any \( v \in \mathbb{R} \)
\[
Q = \mathbb{P} \left( \sum_{j=1}^{n} \xi_j - v < \Delta \right) \leq c \int_{[-\pi/2, \pi/2]} |\varphi(t/\Delta)| \, dt.
\]
Let \( \xi_k' \) be an independent copy of \( \xi_k \) and let \( \nu_k = \xi_k - \xi_k' \). Then
\[
\mathbb{P}(|\nu_k| > 2a) \geq 2c^2 = \bar{c}.
\]
We have
\[
|\varphi_k(t)|^2 = \mathbb{E} \cos \nu_k t
\]
and
\[
|\varphi(t)| \leq \left( \prod_{k=1}^{n} \exp \left( -1 + |\varphi_k(t)|^2 \right) \right)^{1/2} = \exp \left( -\frac{1}{2} \sum_{k=1}^{n} (1 - |\varphi_k(t)|^2) \right).
\]
Define a new random variable \( \tau_k \) by conditioning on \( |\nu_k| > 2a \). For a Borel set \( A \subset \mathbb{R} \) put
\[
\mathbb{P}(\tau_k \in A) = \frac{\mathbb{P}(\nu_k \in A \setminus [-2a, 2a])}{\mathbb{P}(|\nu_k| > 2a)}.
\]
Then by (3.1),
\[
1 - |\varphi_k(t)|^2 \geq \mathbb{E}(1 - \cos \tau_k t) \cdot \mathbb{P}(|\nu_k| > 2a) \geq \bar{c} \cdot \mathbb{E}(1 - \cos \tau_k t),
\]
so
\[
|\varphi(t)| \leq \exp(-\bar{c} f(t)),
\]
where
\[
f(t) = \mathbb{E} \sum_{k=1}^{n} (1 - \cos \tau_k t).
\]
Let \( T(m, r) = \{ t \mid f(t/\Delta) \leq m, \, |t| \leq r \} \) and let
\[
M = \max_{|t| \leq \pi/2} f(t/\Delta).
\]
To estimate $M$ from below, notice that

$$M = \max_{|t| \leq \pi/2} f(t/\Delta) \geq \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \mathbb{E} \sum_{k=1}^{n} (1 - \cos(\tau_k/\Delta)t) \, dt$$

$$= \mathbb{E} \sum_{k=1}^{n} \left(1 - \frac{2}{\pi} \cdot \frac{\sin(\tau_k/\Delta)\pi/2}{\tau_k/\Delta}\right) \geq cn,$$

since $|\tau_k|/\Delta > 2a/\Delta > 4\pi$.

To estimate the measure of $T(m, \pi/2)$ we use the argument of [Ha1]. For reader’s convenience we present a complete proof.

**Lemma 3.2.** Let $0 < m < M/4$. Then

$$|T(m, \pi/2)| \leq c \sqrt{m/M} \cdot |T(M/4, \pi)|.$$

**Proof.** Let $l = \sqrt{M/4m}$. Taking the integer part if necessary, we may assume that $l$ is an integer. For $k \in \mathbb{N}$ set

$$S_k = \{ \sum_{j=1}^{k} t_j \mid t_j \in T(m, \pi/2) \}.$$

Note that $S_1 = T(m, \pi/2)$. Since

$$1 - \cos \alpha = 2 \sin^2(\alpha/2)$$

and

$$\sin^2 \left( \sum_{j=1}^{k} \alpha_j \right) \leq \left( \sum_{j=1}^{k} |\sin \alpha_j| \right)^2 \leq k \sum_{j=1}^{k} \sin^2 \alpha_j,$$

we conclude that $S_k \subset T(k^2m, k\pi/2)$. For $k \leq l$ we have $k^2m < M$, so $(-\pi/2, \pi/2) \setminus T(k^2m, k\pi/2) \neq \emptyset$. For a Borel set $A$ denote $\mu(A) = |A \cap [-\pi, \pi]|$. Now we shall prove by induction that for all $k \leq l$

$$\mu(S_k) \geq (k/2) \cdot \mu(S_1).$$

Obviously, $\mu(S_2) = |S_2| \geq 2 \cdot |S_1|$, so this inequality holds for $k = 2$. Assume that $\mu(S_{k-1}) \geq (k-1)/2 \cdot \mu(S_1)$. Note that the sets $S_k$ are closed. Let $v \in (-\pi/2, \pi/2)$ be a boundary point of $S_k$. Such point exists since $(-\pi/2, \pi/2) \setminus S_k \neq \emptyset$. Let $\{v_j\}_{j=1}^{\infty}$ be a sequence of points in $(-\pi/2, \pi/2) \setminus S_k$ converging to $v$. Then $(v_j - S_1) \cap S_{k-1} = \emptyset$, so by continuity we have

$$\mu((v - S_1) \cap S_{k-1}) = 0.$$

Since the set $S_1$ is symmetric, this implies

$$\mu((v + S_1) \cup S_{k-1}) = \mu(v + S_1) + \mu(S_{k-1}).$$
Both sets in the right hand side are contained in $S_{k+1}$ (to see it for $S_{k-1}$ note that $0 \in S_2$). Since $v + S_1 \subset [-\pi, \pi]$, the induction hypothesis implies

$$\mu(S_{k+1}) \geq \mu(v + S_1) + \mu(S_{k-1}) \geq \mu(S_1) + \frac{k - 1}{2} \cdot \mu(S_1) = \frac{k + 1}{2} \cdot \mu(S_1).$$

Finally, since $S_l \cap [-\pi, \pi] \subset T(l^2m, \pi)$, we get

$$|T(l^2m, \pi)| \geq \frac{l}{2} \cdot |T(m, \pi/2)|.$$

\[\square\]

We continue to prove Lemma 3.1. Since

$$Q \leq C \int_{[-\pi/2, \pi/2]} |\varphi(t/\Delta)| dt \leq C \int_{[-\pi/2, \pi/2]} \exp(-c' f(t/\Delta)) dt \leq \tilde{C} \int_0^n |T(m, \pi/2)| e^{-c'm} dm,$$

Lemma 3.2 implies

(3.2) $Q \leq \frac{C'}{\sqrt{M}} \cdot |T(M/4, \pi)| + ce^{-C'M/16} \leq \frac{C'}{\sqrt{M}} \cdot |T(M/4, \pi)| + ce^{-c'n}$.

Here for $m > M/4$ we used a trivial estimate $|T(m, \pi/2)| \leq \pi$.

To complete the proof we have to estimate the measure of $T = T(M/4, \pi)$ from above. For any $t \in T$ we have

$$g(t) = \sum_{k=1}^n E \cos(\tau_k t/\Delta) \geq n - M/4 \geq n/2.$$

Let $w(x) = (1 - |x|/\pi) \cdot \chi_{[-\pi, \pi]}(x)$ and put $W = \hat{w}$. Then $W \geq 0$ and $W(t) \geq c$ for $|t| \leq \pi$. Hence by Parseval’s equality,

$$|T| \leq \left(\frac{n}{2}\right)^{-2} \int_T |g(t)|^2 dt \leq C \left(\frac{n}{2}\right)^{-2} \int_\mathbb{R} W^2(t) |g(t)|^2 dt = \frac{C}{n^2} \int_\mathbb{R} \left| \frac{\sum_{k=1}^n w(\tau_k/\Delta - y)}{n} \right|^2 dy.$$

Since $w \leq \chi_{[-\pi, \pi]}$, the last expression does not exceed

$$\frac{C}{n^2} \int_\mathbb{R} \left( \sum_{k=1}^n \mathbb{P} \left( \frac{\tau_k}{\Delta} \in [y - \pi, y + \pi] \right) \right)^2 dy \leq \frac{C}{n^2\Delta} \int_\mathbb{R} \left( \sum_{k=1}^n \mathbb{P} \left( \tau_k \in [z - \pi\Delta, z + \pi\Delta] \right) \right)^2 dz.$$
Since \( \tau_k \notin [-2a, 2a] \) and \( \pi \Delta < a/2 \), we can integrate only over \( \mathbb{R} \setminus [-3a/2, 3a/2] \).

Substituting this estimate into (3.2), we get

\[
Q \leq \frac{C}{n^{5/2} \Delta} \int_{\mathbb{R} \setminus [-3a/2, 3a/2]} \left( \sum_{k=1}^{n} P(\tau_k \in [z - \pi \Delta, z + \pi \Delta]) \right)^2 dz + ce^{-c'n}.
\]

To finish the proof, recall that the variables \( \tau_k \) are symmetric. This allows to change the integration set in the previous inequality to \( (3a/2, \infty) \).

Moreover, if \( z \in (3a/2, \infty) \), then

\[
P(\tau_k \in [z - \pi \Delta, z + \pi \Delta]) \leq \frac{1}{c} \cdot P(\nu_k \in [z - \pi \Delta, z + \pi \Delta]),
\]

so the random variables \( \tau_k \) can be replaced by \( \nu_k = \xi_k - \xi'_k \).

**Remark 3.3.** A more delicate analysis shows that the term \( ce^{-c'n} \) in the formulation of Lemma 3.1 can always be eliminated. However, we shall not prove it since this term does not affect the results below.

We shall apply Lemma 3.1 to weighted copies of the same random variable. To formulate the result we have to introduce a new notion.

**Definition 3.4.** Let \( x \in \mathbb{R}^m \). For \( \Delta > 0 \) define the \( \Delta \)-profile of the vector \( x \) as a sequence \( \{P_k(x, \Delta)\}_{k=1}^{\infty} \) such that

\[
P_k(x, \Delta) = |\{j \mid |x_j| \in (k\Delta, (k+1)\Delta)\}|.
\]

**Theorem 3.5.** Let \( \beta \) be a random variable such that \( \mathbb{E} \beta = 0 \) and \( \mathbb{P}(\beta > c) \geq c' \cdot \mathbb{P}(\beta < -c) \geq c' > 0 \). Let \( \beta_1 \ldots \beta_m \) be independent copies of \( \beta \). Let \( \Delta > 0 \) and let \( (x_1 \ldots x_m) \in \mathbb{R}^m \) be a vector such a \( < |x_j| < \sqrt{\frac{m}{\pi}} a \) for some \( a > 0 \). Then for any \( \Delta < a/(2\pi) \) and for any \( v \in \mathbb{R} \)

\[
\mathbb{P}\left( \left| \sum_{j=1}^{m} \beta_j x_j - v \right| < \Delta \right) \leq \frac{C}{m^{5/2}} \sum_{k=1}^{\infty} P_k^2(x, \Delta).
\]

**Proof.** We shall apply Lemma 3.1 to the random variables \( \xi_j = x_j \beta_j \).

Let \( \mathcal{M}(\mathbb{R}) \) be the set of all probability measures on \( \mathbb{R} \). Consider the function \( F : \mathcal{M}(\mathbb{R}) \to \mathbb{R}_+ \) defined by

\[
F(\mu) = \int_{3a/2}^{\infty} \tilde{S}^2_{\Delta}(y) \, dy,
\]

where

\[
\tilde{S}_{\Delta}(y) = \sum_{j=1}^{m} \mu\left( \frac{1}{x_j} \cdot [y - \pi \Delta, y + \pi \Delta] \right).
\]
Since $F$ is a convex function on $\mathcal{M}(\mathbb{R})$, it attains the maximal value at an extreme point of this set, i.e. at some delta-measure $\delta_t$, $t \in \mathbb{R}$. Note that in this case

$$\tilde{S}_\Delta(y) = \{ j \mid t|x_j| \in [y-\pi\Delta, y+\pi\Delta] \} = \sum_{j=1}^m \chi(t|x_j| - y),$$

where $\chi = \chi_{[-\pi\Delta, \pi\Delta]}$ is the indicator function of $[-\pi\Delta, \pi\Delta]$. For $t < \frac{1}{2C}$ we have $t|x_j| < a/2$, so $\tilde{S}_\Delta(y) = 0$ for any $y \geq 3a/2$, and thus $F(\delta_t) = 0$. If $t \geq \frac{1}{2C}$, then

$$F(\delta_t) = \sum_{j=1}^m \sum_{l=1}^m \int_{3a/2}^{\infty} \chi(t|x_j| - y)\chi(t|x_l| - y) \, dy \leq 2\pi\Delta|\{(j, l) \mid |t|x_j| - |x_l|| \leq \pi\Delta\}| = g(t).$$

Since the function $g$ is decreasing,

$$F(\delta_t) \leq g\left(\frac{1}{2C}\right) \leq 2\pi\Delta \sum_{l=1}^{\infty} \{|\{j \mid |x_j| - l\Delta\} \leq 2\pi\Delta \cdot C\}|^2 \leq \tilde{C}\Delta \sum_{k=1}^{\infty} |\{j \mid |x_j| \in (k\Delta, (k+1)\Delta)\}|^2.$$

The last inequality holds since we can cover each interval $[l\Delta-2\pi\Delta C, l\Delta+2\pi\Delta C]$ by at most $2\pi C + 2$ intervals $(k\Delta, (k+1)\Delta]$.

Let $\mu$ be the distribution of the random variable $\beta - \beta'$, where $\beta'$ is an independent copy of $\beta$. Applying Lemma 3.1 to the random variables $\xi_j = x_j \cdot \beta_j$, we have

$$\mathbb{P}\left( \left| \sum_{j=1}^m \beta_j x_j - v \right| < \Delta \right) \leq \frac{C}{m^{5/2}\Delta} \mathbb{F}(\mu) + ce^{-c'\Delta} \leq \frac{C'}{m^{5/2}} \sum_{k=1}^{\infty} |\{j \mid |x_j| \in (k\Delta, (k+1)\Delta)\}|^2 + ce^{-c'\Delta}.$$
4. Small ball probability estimates.

Let $G$ be an $n \times n$ Gaussian matrix. If $x \in S^{n-1}$ is any unit vector, then $y = Gx$ is the standard Gaussian vector in $\mathbb{R}^n$. Hence for any $t > 0$ we have $\mathbb{P}(|y_j| < t) \leq t \cdot \sqrt{2/\pi}$ for any coordinate. Moreover,

$$\mathbb{P} \left( \|y\| \leq t \cdot \sqrt{n} \right) \leq (2\pi)^{-n/2} \text{vol}(t\sqrt{n}B_n^2) \leq (Ct)^n.$$

We would like to have the same small ball probability estimates for the random vector $y = Ax$. However, it is easy to see that it is impossible to achieve such estimate for all directions $x \in S^{n-1}$. Indeed, if $A$ is a random $\pm 1$ matrix and $x = (1/\sqrt{2}, 1/\sqrt{2}, 0 \ldots, 0)$, then $\mathbb{P}(y_j = 0) = 1/2$ and $\mathbb{P}(y = 0) = 2^{-n}$. Analyzing this example, we see that the reason that the small ball estimate fails is the concentration of the Euclidean norm of $x$ on a few coordinates. If the vector $x$ is “spread”, we can expect a more regular behavior of the small ball probability.

Although we cannot prove the Gaussian type estimates for all directions and all $t > 0$, it is possible to obtain such estimates for most directions provided that $t$ is sufficiently large ($t > t_0$). Moreover, the more we assume about the regularity of distribution of the coordinates of $x$, the smaller value of $t_0$ we can take. This general statement is illustrated by the series of results below.

The first result is valid for any direction. The following Lemma is a particular case of [LPRT], Proposition 3.4.

**Lemma 4.1.** Let $A$ be an $n \times n$ matrix with i.i.d. subgaussian entries. Then for every $x \in S^{n-1}$

$$\mathbb{P} (\|Ax\| \leq C_{4.1} \sqrt{n}) \leq \exp(-c_{4.1} n).$$

The example considered at the beginning of this section shows that this estimate cannot be improved for a general random matrix.

If we assume that all coordinates of the vector $x$ are comparable, then we have the following Lemma, which is a particular case of Proposition 3.4 [LPRTV2] (see also Proposition 3.2 [LPRT]).

**Lemma 4.2.** Let $\beta$ be a random variable such that $\mathbb{E}\beta = 0$, $\mathbb{E}\beta^2 = 1$ and let $\beta_1, \ldots, \beta_m$ be independent copies of $\beta$. Let $0 < r < R$ and let $x_1, \ldots, x_m \in \mathbb{R}$ be such that $r/\sqrt{m} \leq |x_j| \leq R/\sqrt{m}$ for any $j$. Then for any $t \geq r/\sqrt{m}$ and for any $v \in \mathbb{R}$

$$\mathbb{P} \left( \left| \sum_{j=1}^m \beta_j x_j - v \right| < t \right) \leq C_{4.2}.t.$$

Here $C_{4.2}$ and $C_{4.2}$ depend only on $r$ and $R$. 
Proof. The proof is based on Berry-Esseen theorem (cf., e.g., [St], Section 2.1).

Theorem 4.3. Let $\{\zeta_j\}_{j=1}^m$ be a sequence of independent random variables with expectation 0 and finite third moments, and let $A^2 := \sum_{j=1}^m E|\zeta_j|^2$. Then for every $\tau \in \mathbb{R}$ one has

$$|P\left(\sum_{j=1}^m \zeta_j < \tau A\right) - P\left(g < \tau\right)| \leq \frac{(c/A^3)}{m} \sum_{j=1}^m E|\zeta_j|^3,$$

where $g$ is a Gaussian random variable with $N(0, 1)$ distribution and $c \geq 1$ is a universal constant.

Let $\zeta_j = \beta_j x_j$. Then $A^2 := \sum_{j=1}^m E\zeta_j^2 = \|x\|^2 \geq r^2$. Since the random variables $\beta_j$ are copies of a subgaussian random variable $\beta$, $E|\beta|^3 \leq C$ for some absolute constant $C$. Hence, $E \sum_{j=1}^m |\zeta_j|^3 \leq C \sum_{j=1}^m |x_j|^3 \leq C'/\sqrt{m}$. By Theorem 4.3 we get

$$P\left(\left|\sum_{j=1}^m \beta_j x_j - v\right| < t\right) \leq P\left(\frac{v - t}{c} \leq g < \frac{v + t}{c}\right) + \frac{c'}{\sqrt{m}}$$

$$\leq C'' t + \frac{c'}{\sqrt{m}} \leq 2C'' t,$$

provided $t \geq \frac{c''}{c' \sqrt{m}}$. \qed

If $x = (1/\sqrt{m}, \ldots, 1/\sqrt{m})$, then

$$P\left(\left|\sum_{j=1}^m \beta_j x_j\right| = 0\right) \geq C/\sqrt{m}.$$

This shows that the bound $t \geq \frac{c''}{c' \sqrt{m}}$ in Lemma 4.2 is necessary.

The proofs of Lemma 4.1 and Lemma 4.2 are based on Paley–Zygmund inequality and Berry–Esseen Theorem respectively. To obtain the linear decay of small ball probability for $t \leq \frac{c''}{c' \sqrt{m}}$, we use the third technique, namely Halász method. However, since the formulation of the result requires several technical assumptions on the vector $x$, we postpone it to Section 6, where these assumptions appear.

To translate the small ball probability estimate for a single coordinate to a similar estimate for the norm we use the Laplace transform technique, developed in [LPRT]. The following Lemma improves the argument used in the proof of Theorem 3.1 [LPRT].

Lemma 4.4. Let $\Delta > 0$ and let $Y$ be a random variable such that for any $v \in \mathbb{R}$ and for any $t \geq \Delta$, $P(|Y-v| > t) \leq L t$. Let $y = (Y_1, \ldots, Y_n)$
be a random vector, whose coordinates are independent copies of $Y$. Then for any $z \in \mathbb{R}^n$
\[ P \left( \|y - z\| \leq \Delta \sqrt{n} \right) \leq (C_{14} L \Delta)^n. \]

**Proof.** We have
\[
P \left( \|y - z\| \leq \Delta \sqrt{n} \right) = P \left( \sum_{i=1}^{n} (Y_i - z_i)^2 \leq \Delta n \right) = P \left( n - \frac{1}{\Delta} \sum_{i=1}^{n} (Y_i - z_i)^2 \geq 0 \right)
\[
\leq E \exp \left( n - \frac{1}{\Delta} \sum_{i=1}^{n} (Y_i - z_i)^2 \right) = e^n \cdot \prod_{i=1}^{n} E \exp\left(-\frac{1}{\Delta} (Y_i - z_i)^2\right).
\]

To estimate the last expectation we use Lemma 6.1.
\[
E \exp\left(-\frac{1}{\Delta} (Y_i - z_i)^2\right) = \int_{0}^{1} P \left( \exp\left(-\frac{1}{\Delta} (Y_i - z_i)^2\right) > s \right) ds
\]
\[
= \int_{0}^{1} 2ue^{-u^2/2} \mathbb{P}(|Y_i - z_i| < \Delta u) du
\]
\[
\leq \int_{0}^{1} ue^{-u^2/2} L \Delta du + \int_{1}^{\infty} ue^{-u^2/2} L \Delta u du
\]
\[
\leq \tilde{C} L \Delta.
\]
Substituting this into the previous inequality, we get
\[ P \left( \|y - z\| \leq \Delta \sqrt{n} \right) \leq (e \cdot \tilde{C} L \Delta)^n. \]

\[ \square \]

5. **Partition of the sphere.**

To apply the small ball probability estimates proved in the previous section we have to decompose the sphere into different regions depending on the distribution of the coordinates of a point. We start by decomposing the sphere $S^{n-1}$ in two parts following [LPRT, LPRTV1, LPRTV2]. We shall define two sets: $V_P$ – the set of vectors, whose Euclidean norm is concentrated on a few coordinates, and $V_S$ – the set of vectors whose coordinates are evenly spread. Let $r < 1 < R$ be
the numbers to be chosen later. Given \( x = (x_1, \ldots, x_n) \in S^{n-1} \), set \( \sigma(x) = \{ i \mid |x_i| \leq R/\sqrt{n} \} \). Let \( P_I \) be the coordinate projection on the set \( I \subset \{1, \ldots, n\} \).

Set \( V_P = \{ x \in S^{n-1} \mid \|P_{\sigma(x)}x\| < r \} \)
\( V_S = \{ x \in S^{n-1} \mid \|P_{\sigma(x)}x\| \geq r \} \).

First we shall show that with high probability \( \|Ax\| \geq C\sqrt{n} \) for any \( x \in V_P \).

For a single vector \( x \in \mathbb{R}^n \) this probability was estimated in Lemma 4.1. We shall combine this estimate with an \( \varepsilon \)-net argument.

**Lemma 5.1.** For any \( r < 1/2 \)

\[
\log N(V_P, B_{2}^{n}, 2r) \leq \frac{n}{R} \cdot \log \left( \frac{3R}{r} \right).
\]

**Proof.** If \( x \in B_{2}^{n} \), then \(|\{1, \ldots, n\} \setminus \sigma(x)| \leq n/R\). Hence, the set \( V_P \) is contained in the sum of two sets: \( rB_{2}^{n} \) and \( W_P = \{ x \in B_{2}^{n} \mid |\text{supp}(x)| \leq n/R^2 \} \).

Since \( W_P \) is contained in the union of unit balls in all coordinate subspaces of dimension \( l = n/R \), Lemma 2.4 implies

\[
N(W_P, B_{2}^{n}, r) \leq \binom{n}{l} \cdot N(B_{2}^{l}, B_{2}^{l}, r) \leq \binom{n}{l} \cdot \left( \frac{3}{r} \right)^l.
\]

Finally,

\[
\log N(V_P, B_{2}^{n}, 2r) \leq \log N(W_P, B_{2}^{n}, r) \leq l \cdot \log \left( \frac{3n}{lr} \right) \leq \frac{n}{R} \cdot \log \left( \frac{3R}{r} \right).
\]

\[\square\]

Recall that \( C_{4.1} < C_{2.3} \). Set \( r = C_{4.1}/2C_{2.3} \) and choose the number \( R > 1 \) so that

\[
\frac{1}{R} \cdot \log \left( \frac{3R}{r} \right) < \frac{C_{4.1}}{2}.\]

For these parameters we prove that the norm of \( Ax \) is bounded below for all \( x \in V_P \) with high probability.

**Lemma 5.2.**

\[
\mathbb{P} \left( \exists x \in V_P \mid \|Ax\| \leq C_{4.1}\sqrt{n}/2 \right) \leq 2 \exp(-C_{4.1}n).
\]

**Proof.** By Lemma 5.1 the set \( V_P \) contains a \( (C_{4.1}/2C_{2.3}) \)-net \( N \) in the \( \ell_2 \)-metric of cardinality at most \( \exp(C_{4.1}/2) \). Let

\[
\Omega_0 = \{ \omega \mid \|A\| > C_{2.3}\sqrt{n} \}
\]
and let

$$
\Omega_P = \{ \omega \mid \exists x \in \mathcal{N} \| A(\omega)x \| \leq C_{\text{4.1}} \sqrt{n} \}.
$$

Then Lemma 4.1 implies

$$
\mathbb{P}(\Omega_0) + \mathbb{P}(\Omega_P) \leq \exp(-n) + \exp(-c_{\text{4.1}} n) \leq 2 \exp(-c_{\text{4.1}} n).
$$

Let $\omega \notin \Omega_P$. Pick any $x \in V_P$. There exists $y \in \mathcal{N}$ such that

$$
\|Ax\| \geq \|Ay\| - \|A(x - y)\| \geq C_{\text{4.1}} \sqrt{n} - \|A : B^n_2 \rightarrow B^n_2\| : \|x - y\|_2 \geq C_{\text{4.1}} \sqrt{n}.
$$

\[\square\]

For $x = (x_1, \ldots, x_n) \in V_S$ denote

$$
(5.1) \quad J(x) = \left\{ j \mid \frac{r}{2\sqrt{n}} \leq |x_j| \leq \frac{R}{\sqrt{n}} \right\}.
$$

Note that

$$
\sum_{j \in J(x)} x_j^2 \geq \sum_{j \in \sigma(x)} x_j^2 - \frac{r^2}{2} \geq \frac{r^2}{2},
$$

so

$$
|J(x)| \geq (r^2/2R^2) \cdot n =: m.
$$

Let $0 < \Delta < r/2\sqrt{n}$ be a number to be chosen later. We shall cover the interval $[\frac{r}{2\sqrt{n}}, \frac{R}{\sqrt{n}}]$ by

$$
k = \left\lceil \frac{R - r/2}{\sqrt{n}\Delta} \right\rceil
$$

consecutive intervals $(j\Delta, (j+1)\Delta]$, where $j = k_0, (k_0+1), \ldots, (k_0+k)$, and $k_0$ is the largest number such that $k_0\Delta < r/2\sqrt{n}$. Then we shall decompose the set $V_S$ in two subsets: one containing the points whose coordinates are concentrated in a few such intervals, and the other containing points with evenly spread coordinates. This will be done using the $\Delta$-profile, defined in 3.4. Note that if $m$ coordinates of the vector $x$ are evenly spread among $k$ intervals, then

$$
\sum_{i=1}^{\infty} P_i^2(x, \Delta) \sim \frac{m^2}{k} \sim m^{5/2}\Delta.
$$

This observation leads to the following
Definition 5.3. Let $\Delta > 0$ and let $Q > 1$. We say that a vector $x \in V_S$ has a $(\Delta, Q)$-regular profile if there exists a set $J \subset J(x)$ such that $|J| \geq m/2$ and
\[
\sum_{i=1}^{\infty} P_i^2(x|J, \Delta) \leq Q m^{5/2} \Delta =: C_{5.3} Q \cdot m^2 / k.
\]
Here $x|J \in \mathbb{R}^n$ is a vector with coordinates $x|J(j) = x(j) \cdot \chi_J(j)$.

If such set $J$ does not exist, we call $x$ a vector of $(\Delta, Q)$-singular profile.

Note that $\sum_{i=1}^{\infty} P_i^2(x|J, \Delta) \geq m/2$. Hence, if $\Delta < m^{-3/2}/2$, then every vector in $V_S$ will be a vector of a $(\Delta, Q)$-singular profile.

Vectors of regular and singular profile will be treated differently. Namely, in Section 6 we prove that vectors of regular profile satisfy the small ball probability estimate of the type $Ct$ for $t \geq \Delta$. This allows to use conditioning to estimate the probability that $\|Ax\|$ is small for some vector $x$ of regular profile. In Section 7 we prove that the set of vectors of singular profile admits a small $\epsilon$-net. This fact combined with Lemma 5.2 allows to estimate the probability that there exists a vector $x$ of singular profile such that $\|Ax\|$ is small using the standard $\epsilon$-net argument.

6. Vectors of a regular profile.

To estimate the small ball probability for a vector of a regular profile we apply Theorem 5.5.

Lemma 6.1. Let $\Delta \leq \frac{r}{4 \pi \sqrt{n}}$. Let $x \in V_S$ be a vector of $(\Delta, Q)$-regular profile. Then for any $t \geq \Delta$
\[
\mathbb{P} \left( \left| \sum_{j=1}^{n} \beta_j x_j - v \right| < t \right) \leq C_{6.1} Q \cdot t.
\]

Proof. Let $J \subset \{1, \ldots, n\}$, $|J| \geq m/2$ be the set from Definition 5.3. Denote by $\mathbb{E}_J$ the expectation with respect to the random variables $\beta_j$, where $j \in J^c = \{1, \ldots, n\} \setminus J$. Then
\[
\mathbb{P} \left( \left| \sum_{j=1}^{n} \beta_j x_j - v \right| < t \right) = \mathbb{E}_{J^c} \mathbb{P} \left( \left| \sum_{j \in J} \beta_j x_j - (v + \sum_{j \in J^c} \beta_j x_j) \right| < t \left| \beta_j, \, j \in J^c \right) \right)
\]

Hence, it is enough to estimate the conditional probability.
Recall that $\beta$ is a centered subgaussian random variable of variance 1. It is well-known that such variable satisfies $\mathbb{P}(\beta > c) \geq c'$, $\mathbb{P}(\beta < -c) \geq c'$ for some absolute constants $c, c'$. Moreover, a simple Paley–Zygmund type argument shows that this estimates hold if we assume only that $\mathbb{E}\beta = 0$ and the second and the fourth moment of $\beta$ are comparable. Hence, for $t = \Delta$ the Lemma follows from Theorem 3.5, where we set $a = r/\sqrt{n}$, $C_{3.5} = R/r$.

To prove the Lemma for other values of $t$, assume first that $t = \Delta s = 2^s \Delta < \frac{r}{4\pi \sqrt{n}}$ for some $s \in \mathbb{N}$. Consider the $\Delta s$-profile of $x|_J$:

$$P_l(x|_J, \Delta s) = |\{ j \in J \mid |x_j| \in (l\Delta, (l+1)\Delta]\}|.$$

Notice that each interval $(l\Delta, (l+1)\Delta]$ is a union of $2^s$ intervals $(i\Delta, (i+1)\Delta]$. Hence

$$\sum_{l=1}^{\infty} P_l^2(x|_J, \Delta s) \leq 2^s \sum_{l=1}^{\infty} P_l^2(x|_J, \Delta) \leq 2^s Q m^{5/2} \Delta = Q m^{5/2} t.$$

Applying Theorem 3.5 with $\Delta$ replaced by $\Delta s$ and $v' = v + \sum_{j \in J^c} \beta_j x_j$, we obtain

$$\mathbb{P} \left( \left| \sum_{j \in J} \beta_j x_j - (v + \sum_{j \in J^c} \beta_j x_j) \right| < t \mid \beta_j, j \in J^c \right) \leq C_{3.5} Q t.$$

For $2^s \Delta < t < 2^{s+1} \Delta$ the result follows from the previous inequality applied for $t = 2^s \Delta$. If $t \geq \frac{c_{4.2}}{\sqrt{n}}$, Lema 4.2 implies

$$\mathbb{P} \left( \left| \sum_{j \in J} \beta_j x_j - (v + \sum_{j \in J^c} \beta_j x_j) \right| < t \mid \beta_j, j \in J^c \right) \leq C_{4.2} \leq C_{4.2} Q t.$$

Finally, if $\frac{r}{4\pi \sqrt{n}} < t < \frac{\sqrt{2} \pi R}{r \sqrt{n}}$, the previous inequality applied to $t_0 = \frac{\sqrt{2} \pi R}{r \sqrt{n}}$ implies

$$\mathbb{P} \left( \left| \sum_{j \in J} \beta_j x_j - (v + \sum_{j \in J^c} \beta_j x_j) \right| < t \mid \beta_j, j \in J^c \right) \leq C_{4.2} Q t_0 \leq C Q t,$$

where $C = C_{4.2} \frac{\sqrt{2} \pi R}{r} \frac{4\pi}{r}$.

Now we estimate the probability that $\|A(\omega)x\|$ is small for some vector of a regular profile.

**Theorem 6.2.** Let $\Delta > 0$ and let $U$ be the set of vectors of $(\Delta, Q)$-regular profile. Then

$$\mathbb{P} \left( \exists x \in U \mid \|Ax\| \leq \frac{\Delta}{2\sqrt{n}} \right) \leq C_{6.1} Q \Delta n.$$
Proof. Set

\[ s = \frac{\Delta}{2\sqrt{n}}. \]

Let \( \Omega \) be the event described in Theorem 6.2. Denote the rows of \( A \) by \( a_1, \ldots, a_n \). Note that since \( \|A^{-1}\| = \|(A^{-1})^T\| \), for any \( \omega \in \Omega \) there exists a vector \( u = (u_1, \ldots, u_n) \in S^{n-1} \) such that

\[ u_1a_1 + \ldots + u_na_n = z, \]

where \( \|z\| < s \). Then \( \Omega = \cup_{k=1}^n \Omega_k \), where \( \Omega_k \) is the event \( |u_k| \geq 1/\sqrt{n} \). Since the events \( \Omega_k \) have the same probability, it is enough to estimate \( P(\Omega_n) \).

To this end we condition on the first \( n-1 \) rows of the matrix \( A = A(\omega) \):

\[ P(\Omega_n) = E_{a_1,\ldots,a_{n-1}} P(\Omega_n \mid a_1, \ldots, a_{n-1}). \]

Here \( E_{a_1,\ldots,a_{n-1}} \) is the expectation with respect to the first \( n-1 \) rows of the matrix \( A \). Take any vector \( y \in U \) such that

\[ \sum_{j=1}^{n-1} (a_j, y)^2 < s^2. \]

If such vector does not exist, then \( \|Ay\| \geq s \) for all \( y \in U \), and so \( \omega \notin \Omega \). Note that the vector \( y \) can be chosen using only \( a_1, \ldots, a_{n-1} \).

We have

\[ a_n = \frac{1}{u_n} (u_1a_1 + \ldots + u_{n-1}a_{n-1} - z), \]

so for \( \omega \in \Omega_n \)

\[ |(a_n, y)| = \frac{1}{|u_n|} \left| \sum_{j=1}^{n-1} u_j (a_j, y) - \langle z, y \rangle \right| \leq \sqrt{n} \left( \left( \sum_{j=1}^{n-1} u_j^2 \right)^{1/2} \left( \sum_{j=1}^{n-1} (a_j, y)^2 \right)^{1/2} + \|z\| \right) \leq 2\sqrt{n} \cdot s = \Delta. \]

The row \( a_n \) is independent of \( a_1, \ldots, a_{n-1} \). Hence, Lemma 6.1 implies

\[ P(\Omega_n) \leq P (|\langle a_n, y \rangle| \leq \Delta \mid a_1, \ldots, a_{n-1}) = P \left( \left| \sum_{j=1}^n \beta_{n,j} y_j \right| \leq \Delta \mid a_1, \ldots, a_{n-1} \right) \leq C_{6.1} Q \Delta, \]

and so \( P(\Omega) \leq C_{6.1} Q \Delta n \). \( \square \)
7. Vectors of a singular profile.

We prove first that the set of vectors of singular profile admits a small ∆-net in the ℓ∞-metric.

Lemma 7.1. Let $C_{\Delta} n^{-3/2} \leq \Delta \leq n^{-1/2}$, where $C_{\Delta} = \frac{2R^3}{\Delta}$ and let $W_S$ be the set of vectors of $(\Delta, Q)$-singular profile. Let $\eta > 0$ be such that $C(\eta) < C_{\Delta}$, where $C(\eta)$ is the function defined in Lemma 2.1. Then there exists a ∆-net $N$ in $W_S$ in ℓ∞-metric such that

$$|N| \leq \left( \frac{C_{\Delta} \eta}{\Delta \sqrt{n} \eta} \right)^n.$$ 

Remark 7.2. Lemma 2.4 implies that there exists a ∆-net for $S_{n-1}$ in the ℓ∞-metric with less than $\left( \frac{C \Delta}{\sqrt{n}} \right)^n$ points. Thus, considering only vectors of a singular profile, we gain the factor $\eta^C_n \Delta^n$ in the estimate of the size of a ∆-net.

Proof. Let $J \subset \{1, \ldots, n\}$ and denote $J' = \{1, \ldots, n\} \setminus J$. Let $W_J \subset W_S$ be the set of all vectors $x$ of a $(\Delta, Q)$-singular profile for which $J(x) = J$. We shall construct ∆-nets in each $W_J$ separately. To this end we shall use Lemma 2.1 to construct a ∆-net for the set $P_J W_J$, where $P_J$ is the coordinate projection on $\mathbb{R}^J$. Then the product of this ∆-net and a ∆-net for the ball $B_{2J'}^2$ will form a ∆-net for the whole $W_J$. 

Assume that $J = \{1, \ldots, l\}$, where $l \geq m$. Let $I_1, \ldots, I_k$ be consecutive subintervals $(i \Delta, (i + 1) \Delta]$, $i = k_0, \ldots, k_0 + k$, covering the interval $[\frac{r}{2\sqrt{n}}, \frac{R}{\sqrt{n}}]$, which appear in the definition of profile. Recall that $k = \left\lceil \frac{R - r/2}{\sqrt{n} \Delta} \right\rceil$.

The restriction on ∆ implies that $k \leq m$. Let $d_i$ be the center of the interval $I_i$. Set

$$M_J = \{ x \in \mathbb{R}^J \mid |x_j| \in \{d_1, \ldots, d_k\} \text{ for } j \in J \}.$$ 

Then $|M_J| = (2k)^l$. Let $\mathcal{N}_J$ be the set of all $x \in M_J$ for which there exists a vector $y \in W_J$ such that $-\Delta/2 < y_j - x_j \leq \Delta/2$ for all $j \in J$. The set $\mathcal{N}_J$ forms a ∆-net for $W_J$ in the ℓ∞ metric. To estimate its cardinality we use the probabilistic method.

Let $X(1), \ldots, X(l)$ be independent random variables uniformly distributed on the set $\{1, \ldots, k\}$. Let $N \subset \{1, \ldots, k\}^l$ be the set of all $l$-tuples $(v(1), \ldots, v(l))$ such that $|x_j| = d_{v(j)}$, $j = 1, \ldots, l$ for some
\[ x = (x_1, \ldots, x_l) \in \mathcal{N}_J. \] Since both \( \mathcal{M}_J \) and \( \mathcal{N}_J \) are invariant under changes of signs of the coordinates,

\[
P((X(1), \ldots, X(l)) \in N) = \frac{|\mathcal{N}_J|}{|\mathcal{M}_J|}.
\]

Let \( (X(1), \ldots, X(l)) \in N \) and let \( x \in \mathbb{R}^l \) be such that \( x_j = d_{X(j)} \). Let \( y \in W_J \) be a vector such that \(-\Delta/2 < y_j - x_j \leq \Delta/2\) for all \( j \in J \). Then for any \( j \in J \), \( y_j \in I_i \) implies that \( X(j) = i \). Let \( E \subset J \) be any set containing at least \( m/2 \) elements. Then

\[
\sum_{i=1}^{\infty} P_i^2(y|E, \Delta) = \sum_{i=1}^{\infty} |\{j \in E \mid X(j) = i\}|^2.
\]

Since \( y \) is a vector of a singular profile, this implies

\[
\sum_{i=1}^{k} |\{j \in E \mid X(j) = i\}|^2 \geq Qm^{5/2}\Delta = C_{350} \cdot Q \frac{m^2}{k} > C(\eta) \cdot \frac{m^2}{k}.
\]

Now Lemma 2.1 implies that \( \mathbb{P}(N) \leq \eta^l \), so

\[
|\mathcal{N}_J| \leq (2k\eta)^l = \left( \frac{R - 2r}{\Delta \sqrt{n} \eta} \right)^l.
\]

To estimate the cardinality of the \( \Delta \)-net for the whole \( W_J \) we use Lemma 2.4. Since \( \Delta \leq 1/\sqrt{|J|} \), \( \Delta B^J_\infty \subset B^J_2 \), so

\[
N(P_J W_J, B^J_\infty, \Delta) \leq N(B^J_2, B^J_\infty, \Delta) \leq 3^{n-l} \frac{|B^J_2|}{|\Delta B^J_\infty|} \leq \left( \frac{c}{\Delta \sqrt{n} - l} \right)^{n-l}.
\]

Since the function \( f(t) = (a/t)^l \) is increasing for \( 0 < t < a/e \), the right-hand side of the previous inequality is bounded by \( (c/\Delta \sqrt{n})^n \). Hence,

\[
N(W_J, B^J_\infty, \Delta) \leq N(P_J W_J, B^J_\infty, \Delta) \cdot N(P_J W_J, B^J_\infty, \Delta) \leq \left( \frac{c}{\Delta \sqrt{n}} \eta^{l/n} \right)^n.
\]

Finally, set

\[
\mathcal{N} = \bigcup_{|J| \geq m} \mathcal{N}_J.
\]

Then

\[
|\mathcal{N}| \leq \sum_{l=m}^{n} \sum_{|J|=l} |\mathcal{N}_J| \leq 2^n \left( \frac{c'}{\Delta \sqrt{n} \eta^{m/n}} \right)^n.
\]

Thus, Lemma 7.1 holds with \( c_{7.1} = m/n = \frac{r^2}{2R^2} \). \( \square \)
Now we are ready to show that \( \|Ax\| \geq c \) for all vectors of a \((\Delta, Q)\)-singular profile with probability exponentially close to 1.

**Theorem 7.3.** There exists an absolute constant \( Q_0 \) with the following property. Let \( \Delta \geq C_{7.3}^{-3/2} \), where \( C_{7.3} = \max(C_{4.2}, C_{7.1}) \). Denote by \( \Omega_\Delta \) the event that there exists a vector \( x \in V_S \) of \((\Delta, Q_0)\)-singular profile such that \( \|Ax\| \leq \frac{\Delta}{2}n \). Then

\[
\mathbb{P}(\Omega_\Delta) \leq 3 \exp(-n).
\]

**Proof.** We consider two cases. First, we assume that \( \Delta \geq \Delta_1 = \frac{10}{C_{7.1}}/n \). In this case we estimate the small ball probability using Lemma 4.2 and the size of the \( \varepsilon \)-net using Lemma 7.1. Note that only the second estimate uses the profile of the vectors. Then we conclude the proof with the standard approximation argument.

The case \( \Delta \leq \Delta_1 \) is more involved. From Case 1 we know that there exists \( Q_1 \) such that all vectors of \((\Delta_1, Q_1)\)-singular profile satisfy \( \|Ax\| \geq \frac{\Delta}{2}n \) with probability at least \( 1 - e^{-n} \). Hence, it is enough to consider only vectors whose profile is regular on the scale \( \Delta_1 \) and singular on the scale \( \Delta \). For these vectors we use the regular profile in Lemma 6.1 to estimate the small ball probability and singular profile in Lemma 7.1 to estimate the size of the \( \varepsilon \)-net. The same approximation argument finishes the proof.

**Case 1.** Assume first that \( \Delta \geq \Delta_1 = \frac{10}{C_{7.1}}/n \). Let \( Q_1 > 1 \) be a number to be chosen later. Let \( \mathcal{M} \) be the smallest \( \frac{\Delta}{2} \)-net in the set of the vectors of \((\Delta, Q_1)\)-singular profile in \( \ell_\infty \) metric.

Let \( x \in V_S \) and let \( J = J(x) \) defined in (5.1). Denote \( J^c = \{1, \ldots, n\} \setminus J \). Then Lemma 11.2 implies

\[
\mathbb{P} \left( \left| \sum_{j=1}^n \beta_j x_j \right| \leq t \right) \leq \mathbb{E}_{J^c} \mathbb{P} \left( \left| \sum_{j \in J} \beta_j x_j + \sum_{j \in J^c} \beta_j x_j \right| \leq t \mid \beta_j, j \in J^c \right) \leq C_{4.3} \quad \text{for all } t \geq \frac{4.3}{\sqrt{n}}.
\]

Since \( \Delta \sqrt{n} \geq \frac{4.3}{\sqrt{n}} \), by Lemma 4.4 we have

\[
\mathbb{P} \left( \|Ax\| \leq \Delta n \right) \leq (C_{4.3} \Delta \sqrt{n})^n
\]

and so,

\[
(7.1) \quad \mathbb{P} \left( \exists x \in \mathcal{M} \mid \|Ax\| \leq \Delta n \right) \leq |\mathcal{M}|(C_{4.3} \Delta \sqrt{n})^n.
\]

We shall show that \( Q_1 \) can be chosen so that the last quantity will be less than \( (2e)^{-n} \). Recall that by Lemma 7.1 there exists a \( \Delta \)-net \( N \)
for the set of vectors of \((\Delta, Q_1)\)-singular profile satisfying

\[|\mathcal{N}| \leq \left( \frac{C_{\text{me}}}{\Delta \sqrt{n}} \eta^{n/2} \right)^n,\]

provided

(7.2) \hspace{1cm} C(\eta) < C_{\text{me}} Q_1.

Covering each cube of size \(\Delta\) with the center in \(\mathcal{N}\) by the cubes of size \(\frac{\Delta}{2C_{\text{me}}}\) and using Lemma 2.4, we obtain

\[|\mathcal{M}| \leq |\mathcal{N}| \cdot N(\Delta B^n_\infty, \Delta B^n_\infty) \leq \left( \frac{6C_{\text{me}} \cdot C_{\text{me}} Q_1 \eta^{n/2}}{\Delta \sqrt{n}} \right)^n.\]

Substitution of this estimate into (7.1) yields

\[\mathbb{P}(\exists x \in \mathcal{N} \mid \|Ax\| \leq \Delta n) \leq \left( \frac{6C_{\text{me}} \cdot C_{\text{me}} Q_1 \eta^{n/2}}{\Delta \sqrt{n}} \right)^n \cdot (C_{\text{me}} \Delta \sqrt{n})^n \leq (C' \eta^{n/2})^n.\]

Now choose \(\eta\) so that \(C' \eta^{n/2} < 1/e\) and choose \(Q_1\) satisfying (7.2). With this choice the probability above is smaller than \(e^{-n}\). Combining this estimate with Lemma 2.3, we have that \(\|A\| \leq C_{\text{me}} \Delta \sqrt{n}\) and \(\|Ax\| \geq \Delta n\) for all \(x \in \mathcal{N}\) with probability at least \(1 - 2e^{-n}\).

Let \(y \in V_S\) be a vector of \((\Delta, Q_1)\)-singular profile. Choose \(x \in \mathcal{N}\) such that \(\|x - y\|_\infty \leq \frac{\Delta}{2C_{\text{me}}}\). Then \(\|x - y\| \leq \frac{\Delta \sqrt{n}}{2C_{\text{me}}}\) and

\[\|Ay\| \geq \|Ax\| - \|A(x - y)\| \geq \Delta n - \|A\| \|x - y\| \geq \frac{\Delta}{2} n.\]

**Case 2** Assume that \(C_{\text{me}} n^{-3/2} \leq \Delta \leq \Delta_1 = c_{\text{me}} n^{1/4}/n\). Let \(\Omega_1\) be the event that \(\|Ax\| < \frac{\Delta}{2} n = c_{\text{me}} n^{1/4}/2\) for some vector of \((\Delta_1, Q_1)\)-singular profile. We proved in Case 1 that

(7.3) \hspace{1cm} \mathbb{P}(\Omega_1) < 2e^{-n}.

Let \(Q_2 > 1\) be a number to be chosen later and let \(W\) be the set of all vectors of \((\Delta_1, Q_1)\)-singular and \((\Delta, Q_2)\)-regular profile. By Lemma 6.1 any vector \(x \in W\) satisfies

\[\mathbb{P} \left( \left| \sum_{j=1}^{n} \beta_j x_j \right| \leq t \right) \leq C_{\text{me}} Q_1 t \]

for all \(t \geq \Delta_1\).

Now we can finish the proof as in Case 1. Since \(\Delta \sqrt{n} \geq \Delta_1\), Lemma 4.4 implies

\[\mathbb{P}(\|Ax\| \leq \Delta n) \leq (C' \Delta \sqrt{n})^n\]

for any \(x \in W\). Here \(C' = C_{\text{me}} \cdot C_{\text{me}} Q_1\).
Let $\mathcal{N}$ be the smallest $\frac{\Delta}{\Delta \sqrt{n}}$-net in $W$ in $\ell_\infty$ metric. Note that $\Delta \geq C_{\ref{lem:net}}^{-1/2} \geq C_{\ref{lem:net}}^{-3/2}$. Arguing as in the Case 1, we show that
\[ |N| \leq \left( \frac{6C_{\ref{lem:net}} C_{\ref{lem:net}}}{\Delta \sqrt{n} \eta} \right)^n \]
for any $\eta$ satisfying
\[ C(\eta) < C_{\ref{lem:net}} Q_2. \]
Hence,
\[ \mathbb{P}(\exists x \in \mathcal{N} | \|Ax\| \leq \Delta n) \leq |N|(C\Delta \sqrt{n})^n \leq (C" \eta \Delta)^n. \]
Choose $\eta$ so that the last quantity is less than $e^{-n}$ and choose $Q_2$ so that (7.4) holds. Then the approximation argument used in Case 1 shows that the inequality
\[ \|Ay\| \geq \frac{\Delta}{2} n \]
holds for any $y \in W$ with probability greater than $1 - e^{-n}$. Combining it with (7.3), we complete the proof of Case 2. Finally, we unite two cases setting $Q_0 = \max(Q_1, Q_2)$. 

8. Proof of Theorem 1.1

To prove Theorem 1.1 we combine the probability estimates of the previous sections. Let $\varepsilon > \frac{1}{C_{\ref{lem:net}} \sqrt{n}}$, where the constant $q_{\ref{lem:net}}$ will be chosen later. Define the exceptional sets:
\[ \Omega_0 = \{ \omega | \|A\| > C_{\ref{lem:net}} \sqrt{n} \}, \]
\[ \Omega_\psi = \{ \omega | \exists x \in V_{\psi} | \|Ax\| < C_{\ref{lem:net}} \sqrt{n} \}. \]
Then Lemma 2.3 and Lemma 5.2 imply
\[ \mathbb{P}(\Omega_0) + \mathbb{P}(\Omega_\psi) \leq 3 \exp(-a_{\ref{lem:net}}). \]

Let $Q_0$ be the number defined in Theorem 7.3. Set
\[ \Delta = \frac{\varepsilon}{2C_{\ref{lem:net}} Q_0 \cdot n}. \]
The assumption on $\varepsilon$ implies $\Delta \geq C_{\ref{lem:net}}^{-1/2} n^{-3/2}$ if we set $q_{\ref{lem:net}} = 2C_{\ref{lem:net}} Q_0$. Denote by $W_\Delta$ the set of vectors of $(\Delta, Q_0)$-singular profile and by $W_\Delta$ the set of vectors of $(\Delta, Q_0)$-regular profile. Set
\[ \Omega_S = \{ \omega | \exists x \in W_\Delta | \|Ax\| < \frac{\Delta}{2} n = \frac{1}{4C_{\ref{lem:net}} Q_0} \varepsilon \}, \]
\[ \Omega_R = \{ \omega | \exists x \in W_\Delta | \|Ax\| < \frac{\Delta}{2 \sqrt{n}} = \frac{1}{4C_{\ref{lem:net}} Q_0} \varepsilon \cdot n^{-3/2} \}. \]
By Theorem 7.3, \(P(\Omega_S) \leq 3e^{-n}\), and by Theorem 6.2, \(P(\Omega_R) \leq \varepsilon/2\).

Since \(S^{n-1} = V_F \cup W_S \cup W_R\), we conclude that

\[
P(\omega \mid \exists x \in S^{n-1} \parallel Ax \parallel < 1/2) \leq \frac{\varepsilon \cdot n^{-3/2}}{2C_{Q_0}} \leq \varepsilon/2 + 4 \exp(-c_4 n) < \varepsilon
\]

for large \(n\).

\(\square\)

**Remark 8.1.** The proof shows that the set of vectors of a regular profile is critical. On the other sets the norm of \(Ax\) is much greater with probability exponentially close to 1.

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