Several questions and hypotheses concerning the limit polynomials for Chacon\(_{(3)}\) transformation

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We study the weak closure \(\mathcal{L} = WCl(\{\hat{T}^k\})\) of powers of non-singular Chacon transformation with 2-cuts. It is still an open question does \(\mathcal{L}\) contain any Markov operator except an orthogonal projector to the constants \(\Theta\) and some polynomials \(P(\hat{T})\)? In this paper we calculate a particular set of limit polynomials

\[
P_m(\hat{T}) = \lim_{n \to \infty} \hat{T}^{-m h_n}, \quad m \in \mathbb{Z},
\]

where \(h_n = (3^n - 1)/2\) are the sequence of heights of towers in a standard rank one representation of the Chacon map. We show that for any \(d \geq 2\) the family of limit polynomials contains infinitely many polynomials of degree \(d\). We also formulate hypotheses and open questions concerning the sequence \(P_m\) and the entire set \(\mathcal{L}\).

I. INTRODUCTION

Chacon transformation in terms of symbolic dynamics can be defined as a substitution system over the finite alphabet \(\mathbb{A} = \{0, 1\}\) via a pair of substitution rules

\[
0 \mapsto 0010, \quad 1 \mapsto 1.
\]

Starting with an initial word \(w_0 = 0\) and applying the substitution transform we construct the sequence \(w_n\),

\[
\begin{align*}
0 &= w_0, \\
0010 &= w_1, \\
0010001010010001000101001010010001010010 & = w_2, \\
0010001010010001000101001010010001010010 & = w_3, \\
& \quad \ldots
\end{align*}
\]

and then define an infinite word \(w_\infty\) such that each \(w_n\) is a prefix of \(w_\infty\). Further, considering the closure \(X\) of all shifts of \(w_\infty\) in the space \(\mathbb{A}^\infty\) endowed with the Tikhonov topology we come to a topological dynamical system \((S, X, B)\), where \(B\) is the \(\sigma\)-algebra of Borel sets and \(T\) is the shift transformation,

\[
T: \ldots, x_0, x_1, \ldots \mapsto \ldots, x_1, x_2, \ldots, x_j, \ldots
\]

Let us consider a natural invariant measure \(\mu\) on the measurable space \((X, B)\) defined as follows. For a finite word \(w\) let \(\mu([w])\) be the empirical probability of observing \(w\) in \(w_\infty\), where \([w]\) is the set encoded by \(w\):

\[
[w] := \{x \in X : x_0 = w(0), \ldots, x_{\ell-1} = w(\ell-1)\},
\]

\(\ell = |w|\) is the length of \(w\) and \(w(j)\) denotes the letter at position \(j\) in \(w\).

**Definition 1.** The map \(T\) considered as a measure-preserving invertible transformation of the probability space \((X, B, \mu)\) is called non-singular Chacon transformation with 2-cuts or Chacon\(_{(3)}\) transformation (see \cite{Cha69, Fri70}).

Transformation \(T\) has an interesting combination of ergodic properties. It is known to be weakly mixing and power weakly mixing \cite{Dan04}, but not strongly mixing \cite{Cha69}. It has trivial centralizer \cite{JL78} and minimal self-joinings \cite{JRS80}. It is also known that the spectral measure \(\sigma\) of Chacon transformation \(T\) is singular and its convolutions satisfy the following condition of pairwise singularity \cite{PR}.

\[
\sigma \perp \sigma \ast \sigma, \quad \sigma \ast \sigma \perp \sigma \ast \sigma \ast \sigma, \quad \ldots
\]

\(\sigma^k \perp \sigma^{*\ell}\) for any \(k \neq \ell\).

The study of convolutions of the spectral type measure \(\sigma\) goes back to the Kolmogorov’s question concerning the hypothetic group property of spectrum: *is it true that \(\sigma \ast \sigma \ll \sigma\)? This property holds for the discrete part of spectrum, but it is false for the singular component. Moreover, now we know many examples of ergodic transformations \(T\) such that \(\sigma \ast \sigma \perp \sigma\) (see \cite{Ose69, Ste87, Goo99, dJL92}).

For a survey of problems in modern spectral theory of dynamical systems the reader can refer to \cite{Lem09} and \cite{KT07}.

**Definition 2.** We say that a map \(T\) is mixing if

\[
\mu(T^k A \cap B) \to \mu(A) \mu(b) \quad \text{as} \quad k \to \infty,
\]

for any measurable sets \(A\) and \(B\), and we call \(T\) weakly mixing if the convergence holds for a subsequence \(k_j\).

Both mixing and weak mixing properties can be described in spectral terms.

**Definition 3.** Let \(\hat{T}\) be the unitary Koopman operator, associated with \(T\) and acting in the separable Hilbert space \(H = L^2(X, \mu)\) by the following rule

\[
\hat{T}: f(x) \mapsto f(Tx).
\]
A sequence of bounded linear operators $A_j : H \to H$ in a Hilbert space $H$ converges weakly to $A$ if for any $f, g \in H$

$$\langle A_j f, g \rangle \to \langle Af, g \rangle, \quad j \to \infty.$$ 

Let $\Theta$ denote the orto-projector to constants,

$$(\Theta f)(x) \equiv \int_X f(z) \, d\mu(z).$$

A transformation $T$ is weakly mixing if and only if

$$T^{k_j} \to \Theta$$

for some subsequence $k_j$. It means that $\Theta$ is in the weak closure $\mathcal{L} = \text{WCl}\{\hat{T}^k\}$ of powers $\hat{T}^k$.

II. LIMIT POLYNOMIALS

In our investigation \[PR\] to prove the pairwise singularity of the convolutions $\sigma^* k$ we used the following observation.

**Lemma 4.** In the weak close of powers $\mathcal{L} = \text{WCl}\{\hat{T}^k\}$ for Chacon transformation $T$ one can find an infinite family of non-trivial square polynomials

$$Q_m(\hat{T}) = \frac{(3^s - 1)I + 2(3^s + 1)^s\hat{T} + (3^s - 1))\hat{T}^2}{4 \cdot 3^s},$$

for $m = 3^s + 1$ and, moreover,

$$Q_m(\hat{T}) = \lim_{n \to \infty} \hat{T}^m h_n - h_s,$$

where $h_s = (3^s - 1)/2$ and $I$ is the identity operator.

In order to understand this phenomenon let us consider a simpler case.

**Lemma 5.** There exists a sequence $k_j \to \infty$ such that

$$\hat{T}^{k_j} \to I + \hat{T}.$$

**Proof.** Another way to define Chacon transformation as a measure-preserving transformation is to use the concept of rank one transformation.

**Definition 6.** Let $T$ be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Then $T$ is called **rank one transformation** if there exists a sequence of Rokhlin tower partitions

$$\xi_j = \{B_j, TB_j, T^2 B_j, \ldots, T^{h_n - 1} B_j, E_j\}$$

of the phase space such that $\mu(E_j) \to 0$ and for any measurable set $A$ one can fins $\xi_j$-measurable sets $A_j$ approximating $A$: $\mu(A_j \Delta A) \to 0$ as $j \to \infty$.

In fact, Chacon transformation is rank one and can be constructed using so-called cutting-and-stacking construction.

**Construction 7.** We start with a unit segment $[0, 1]$ interpreted as a Rokhlin tower $U_0$ of height $h_0 = 1$. Then we cut this segment twice, in three equal parts

$$L_{1,0} = [0, 1/3], \quad L_{1,1} = [1/3, 2/3], \quad L_{1,2} = [2/3, 1],$$

and add one additional “level”, a segment $S_1$ of length $1/3$ which is drawn above the middle part $[1/3, 2/3]$ (see fig. 1),

$$S_1$$

Next we stack all these segments in the natural order: $L_{1,0} L_{1,1} S_1 L_{1,2}$ and we get the next Rokhlin tower $U_1$ of height $h_1 = 4$ (see fig. 2). In other words, we assume that

$$L_{1,0} \xrightarrow{T} L_{1,1} \xrightarrow{T} S_1 \xrightarrow{T} L_{1,2},$$

and $T$ will be defined on $L_{1,2}$ on the next steps of the construction. We repeat the same procedure with the new tower: we cut it in three equal columns, put one additional level to the top of the middle column and stack together (fig. 3).

At each step of the construction we have a Rokhlin tower $U_n$ of height $h_n = (3^n - 1)/2$. It can be easily checked that this sequence serves as an approximating
sequence of Rokhlin towers in the definition of rank one transformation.

Note that if we draw all the additional level above the corresponding subcolumns without restacking the tower $U_n$ at the step $n$ we come to the following representation of the Chacon map (see fig. 4).

Construction 8. Let us consider the compact group of 3-adic integers $\Gamma = \mathbb{Z}_{(3)}$. We associate $\Gamma$ with the set of one-sided 3-adic sequences

$$y = (y_1, y_2, \ldots, y_k, \ldots), \quad y_k \in \{0, 1, 2\}.$$
and

\[ f(T^{hn}x) = f(T^{-1}x), \quad \text{if} \quad x \in U_n^{(1)} \]

for all points \( x \in U_n \) except the first level \( B_n \) of the tower \( U_n \) (observe that \( \mu(B_n) \to 0 \)). Thus,

\[ \hat{T}^{hn} \to \frac{1 + \hat{T}^{-1}}{2}, \]

since \( \mu(U_n^{(0)}) = \mu(U_n^{r(0)}) \), and applying conjugation we complete the proof.

Analyzing the effects used in the proof we see that lemma \[ 5 \] can be easily extended in the following way. Given \( m \in \mathbb{N} \) let us consider the sum

\[ \phi_0^{(m)} = \phi_0(y) + \phi_0(Sy) + \cdots + \phi_0(S^{m-1}y) \]

and define the corresponding distribution \( \rho_m \) of the values of \( \phi_0^{(m)} \). Actually \( \rho_m \) is the measure on \( \mathbb{Z} \) with a finite support which is the image by \( \phi_0^{(m)} \) of the Haar probability measure on \( \Gamma \).

**Lemma 9.** For any \( m \in \mathbb{N} \) the sequence \( \hat{T}^{-mh_n} \) converges weakly to a polynomial depending on \( \hat{T} \), and

\[ P_m(\hat{T}) := \lim_{n \to \infty} \hat{T}^{-mh_n} = \int_{\mathbb{Z}} \hat{T}^k \, d\rho_m(k). \]

The scheme of the proof can be found in [PR], and the idea can be explained as follows. Passing the tower \( U_n \) \( m \) times we count (in addition to \( mh_n \)) the values of the cocycle \( \phi(y) \) at the points

\[ \phi_0(y), \quad \phi_0(Sy), \quad \cdots, \quad \phi_0(S^{m-1}y). \]

Let us consider several first polynomials \( P_n(\hat{T}) \):

\[ P_1(\hat{T}) = \frac{1}{2}(1 + \hat{T}) \]
\[ P_2(\hat{T}) = \frac{1}{6}(1 + 4\hat{T} + \hat{T}^2) \]
\[ P_3(\hat{T}) = \frac{1}{2}(1 + \hat{T} + \hat{T}^2) \]
\[ P_4(\hat{T}) = \frac{1}{9}(2\hat{T} + 5\hat{T}^2 + 2\hat{T}^3) \]
\[ P_5(\hat{T}) = \frac{1}{18}(\hat{T} + 8\hat{T}^2 + 8\hat{T}^3 + \hat{T}^4) \]

Since the weak closure \( \overline{\text{WCL}}(\{\hat{T}^s\}) \) is invariant under multiplication by \( \hat{T}^s \) for any \( s \in \mathbb{Z} \) we can reduce the polynomials \( P_m(\hat{T}) \) by the smallest power \( l_m \) of \( \hat{T} \) in \( P_m(\hat{T}) \). Set

\[ \tilde{P}_m(z) = z^{-l_m} \cdot P_m(z). \]

Let us represent \( \tilde{P}_m(z) \) in the form

\[ \tilde{P}_m(z) = a_{m,0} + a_{m,1}z + \cdots + a_{m,d(m)}z^{d(m)}, \]

where \( d(m) = \deg \tilde{P}_m(z) \).

**Lemma 10.** The coefficients \( a_{m,j} \in \mathbb{Q} \) satisfy the following Markov property

\[ \sum_{j=0}^{d(m)} a_{m,j} = 1, \quad \text{and} \quad a_{m,j} \geq 0. \]

In table 1 of the Appendix we list the first 365 polynomials \( P_m(z) \).

Let us discuss several remarks explaining the structure of this table. First, for simplicity of calculations we apply the transform \( y \to y + 1 \) to the base of the tower \( U_n \) and consider the following cocycle \( \phi(y) \) instead of \( \phi_0(y) \) (see fig. 6),

\[ \phi(y) = \begin{cases} 0, & \text{if } y = 000\ldots01* \\ 1, & \text{if } y = 000\ldots02* \end{cases} \]

The function \( \phi(y) \) is more convenient for calculation of the iterates \( \phi(S^k y) \).

**Lemma 11.** For any power \( 3^\ell \) of three we have

\[ \phi^{(3^\ell)}(y) = \begin{cases} 0, & \text{if } y = *^\ell(0)1* \\ 1, & \text{if } y = *^\ell(0)1* \end{cases} \]

where the notation \( (0) \) is used for any sequence of zeros (including empty sequence), and \( *^\ell \) denotes an arbitrary word of length \( \ell \). An equivalent way how we can state this property of the cocycle is to say that

\[ \phi^{(3^\ell)}(y) = \phi(Ay), \]

where \( A \) is the non-invertible left shift:

\[ A(y_1y_2\cdots y_k\cdots) = y_2y_3\cdots y_{k+1}\cdots \]
It follows immediately from this lemma that polynomials \( \tilde{P}_m(z) \) repeat after multiplication by 3,

\[
P_{3m}(z) = P_m(z).
\]

**Theorem 12.** For any \( d \in \mathbb{N} \) the family \( \{ \tilde{P}_m \} \) contains infinitely many polynomials of degree \( d \).

**Proof.** The theorem is based on the following observation. If we consider functions \( \phi(m)(y) \) and \( \phi(S^k y) \) as random variables defined on \( \Gamma \), then \( \phi(m)(y) \) and \( \phi(y) \) are almost independent. Thus, extending the proof of Lemma 4, we see that configurations

\[
m(\ell_1, \ldots, \ell_{d-1}) = 10^{\ell_1}10^{\ell_2}100 \ldots 0^{\ell_{d-1}}13
\]
generates for sufficiently big \( \ell_j \) polynomials \( P(\ell_j)(z) \) of degree \( d \) such that

\[
\lim_{\ell_j \to \infty} P(\ell_j)(T) = \frac{1}{2^d}(1 + T)^d.
\]

To illustrate the construction used in the proof let us consider configuration

\[
m(\ell_1, \ell_2) = 10^{\ell_1}10^{\ell_2}13 = 100 \ldots 01000 \ldots 013
\]

Set

\[
p^{-[i,j]} = p^{-i} + p^{-i-1} + \ldots + p^{-j},
\]

and notice that \( 3^{-[1,\infty]} = 1/2 \).

**Lemma 13.** \( \mathcal{P}_m(\ell_1, \ell_2) \) is a self-reciprocal polynomial,

\[
\mathcal{P}_m(\ell_1, \ell_2) = \gamma z^3 + (1/2 - \gamma)z^2 + (1/2 - \gamma)z + \gamma,
\]

where

\[
\gamma = 3^{-[1,\ell_1]}3^{-[1,\ell_2]} + 3^{-[1,\ell_1]}3^{-(\ell_2+1)} + 3^{-(\ell_1+1)}3^{-[1,\ell_2]}.
\]

**Proof.** The proof of this lemma is very close to that of lemma 4.

In the next section we formulate a set of hypotheses concerning the properties of the limit polynomials \( \tilde{P}_m(z) \).

In hypothesis 4, we conjecture that all polynomials \( \tilde{P}_m(z) \) are self-reciprocal, i.e. they have coefficients symmetric under the transform \( k \mapsto d - 1, \ d = \deg \tilde{P}_m \),

\[
\tilde{P}_m(z) = \sum_{j=0}^{d(m)} a_{m,j}z^j, \quad a_{m,j} = a_{m,d(m)-j}.
\]

In other words, the sequence \( a_{m,j} \) is symmetric.

**Lemma 14.** If hypothesis 4 holds then \( d(m) \in 2\mathbb{Z} + 1 \) implies that \((-1)\) is a root of \( \tilde{P}_m \). In particular, any polynomial \( \tilde{P}_m \) of odd degree is factorized,

\[
\mathcal{P}_m(z) = (z + 1)R_m(z).
\]

The proof is a simple calculation. Nevertheless, we can ask a question: is it the only way to factorize \( \tilde{P}_m \)?

**Theorem 15.** The family of limit polynomials \( \mathcal{P}_m(z) \) contains infinitely many cubic polynomials for which \( R_m(z) = (z + 1)^{-1}\tilde{P}_m(z) \) are irreducible.

**Proof.** Indeed, consider cubic polynomials given by configurations \( 10^{\ell_1}10^{\ell_2}1 \) with \( \ell_1 = \ell_2 \) (see table 2 of the Appendix). With a simplified notation \( \ell = \ell_1 \) we have

\[
\mathcal{P}_m(\ell,\ell)(z) = (3a^2 + 2a)(z^3 + 1) + (3^{2\ell + 1} - 3a^2 - 2a)(z^2 + z),
\]

where

\[
3^{-[1,\ell]} = 1 \ldots 1 = \frac{a}{3^\ell}, \quad \gcd(a, 3) = 1.
\]

Next, let us apply the transform \( z = -1 + w \) to \( \mathcal{P}_m(\ell,\ell) \). We get a new polynomial

\[
P^*(w) = (3a^2 + 2a)w^3 + (3^{2\ell + 1} - 8a - 12a^2)(w - 1).
\]

What are the common divisors of

\[
X = 3a^2 + 2a \quad \text{and} \quad Y = 3^{2\ell + 1} - 8a - 12a^2?
\]

We have

\[
Y + 4X = 3^{2\ell + 1}
\]

and, at the same time,

\[
X = a(3a + 2)
\]

is factorized in two numbers, both are relatively prime to 3. Thus, taking any prime divisor of \( X \) and applying Eisenstein’s criterion we see that \( P^*(w) \) is irreducible over \( \mathbb{Q} \).

**Lemma 16** (Eisenstein’s criterion). Consider a polynomial \( P \in \mathbb{Q}[z] \),

\[
P(z) = a_nz^n + \cdots + a_1z + a_0,
\]

and suppose that there exists a prime number \( p \) such that

\[
p \not| a_n, \quad p^2 \not| a_0, \quad p \mid a_j \ \text{for} \ j = 0, 1, \ldots, n - 1.
\]

Then \( P(z) \) is irreducible over \( \mathbb{Q} \).

It is interesting to remark that the quadratic polynomials given in lemma 4 are factorized over \( \mathbb{Q} \), thus, to see that there exists irreducible polynomial \( \tilde{P}_m(z) \) we have to consider a particular example:

\[
\tilde{P}_2(z) = \frac{1}{6}(z^2 + 4z + 1).
\]

Letting \( z = -1 + w \) we get a polynomial

\[
P^*(w) = z^2 + 2z - 2.
\]

We can apply Eisenstein’s criterion to \( P^* \), since 2 divides all the coefficient except the coefficient in \( z^2 \), and 4 do not divide 2.
III. QUESTIONS AND HYPOTHESES

Hypothesis 1. The limit polynomials \( \bar{P}_m(z) \) are self-reiprocal, that is

\[
\bar{P}_m(z) = \sum_{k=0}^{d(m)} a_k z^k, \quad a_k = a_{d(m)-k}.
\]

Corollary. If hypothesis 1 is true then \(-1\) is a root of a polynomial \( \bar{P}_m(z) \), whenever \( d(m) \in 2\mathbb{Z} + 1 \).

We have to mention that most questions below presume or at least require hypothesis 1 for a particular \( m \).

Definition 17. Consider two configurations of the same length

\[ c_1 c_2 \ldots c_N \quad \text{and} \quad c'_1 c'_2 \ldots c'_N, \]

with \( c_j, c'_j \in \{0, 1, 2\} \). We say that \( m' \) is conjugate to \( m \) and write \( m' = m^* \) if \( c'_j = c_{N+1-j} \).

Hypothesis 2. The polynomials \( \bar{P}_m(z) \) and \( \bar{P}_{m^*}(z) \) coincide for any pair of conjugate configurations \( m \) and \( m^* \).

It can be observed from table 1 that some polynomials coincide even for non-conjugate configurations, for example, for \( m = 10 = 101_3 \) and \( m' = 26 = 222_3 \).

Question 3. Which pairs of polynomials \( \bar{P}_m(z) \) and \( \bar{P}_{m^*}(z) \) coincide?

Let \( |m|_3 \) be the length of the 3-adic expansion of \( m \) if \( 3 \nmid m \), and \( |m|_3 = |3^{-1}m|_3 \) otherwise.

Hypothesis 4. \( P^x_m(z) = 2 \cdot 3^{|m|_3} \cdot \bar{P}_m(z) \) is a polynomial with integer coefficients,

\[ P^x_m = b_m, 0 + b_m, 1 z + \cdots + b_m, d(m) z^d. \]

The greatest common divisor of \( b_m, j \) is 1 or 2.

This is a well-known fact that the set of all weak limits of powers \( \mathcal{L} \) is a semigroup. Thus, it is a natural question: can we get a polynomial \( P_m(z) \) as a product of two different elements of \( \mathcal{L} \)?

Hypothesis 5. The polynomial \( \bar{P}_m(z) \) has two or more factors which are not \((z+1)\) if and only if (see table 3)

\[ |m|_3 \in 2\mathbb{Z} \quad \text{and} \quad m = m^*. \]

For example, the polynomial

\[ \bar{P}_{68}(z) = \frac{1}{81}(3 + 5z + z^2)(1 + 5z + 3z^2) \]

corresponds to a symmetric configuration \( 68 = 2112_3 \).

In particular, if hypothesis 5 is true then the roots \( r_j \) of a polynomial \( P^x_m(z) \) starting with \( z^d + \ldots \) are algebraic integers.

Hypothesis 6. All roots of any \( \bar{P}_m(z) \) are real numbers (Lee–Yang property).

Remark 18. It follows directly from hypothesis 1 as well as the definition of the polynomials \( \bar{P}_m(z) \) that the roots of \( \bar{P}_m(z) \) must be negative, and they appear in pairs: \( r \) and \( r^{-1} \).

Now, if we assume hypotheses 1 and 6 then applying to our polynomials the transformation

\[ z = \kappa_1(z) = i \frac{z - 1}{z + 1} \]

mapping \( \mathbb{R} \) to the unit circle in the complex plane, we can define the dual polynomials

\[ Q_m(w) = \bar{P}_m(\kappa_1(z)). \]

Hypothesis 7. The polynomials \( Q_m(w) \) are self-reciprocal polynomials having all roots \( \lambda_j \) on the unit circle and in the right-half plane:

\[ |\lambda_j| = 1, \quad \text{Re} \lambda_j > 0. \]
Let us consider, for example, the polynomials

\[ P_{122}^d(z) = z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1, \]
\[ P_{124}^d(z) = z^6 + 23z^5 + 119z^4 + 200z^3 + 119z^2 + 23 + 1, \]
\[ P_{130}^d(z) = z^6 + 22z^5 + 120z^4 + 200z^3 + 120z^2 + 22z + 1, \]

and, in particular,

produced to sums of \( d\phi \)

These configurations generate sums of \( d\phi \)

corresponding to the configuration

\[ 122 = 11112_3, \quad 124 = 11121_3, \quad 130 = 11211_3. \]

The dual polynomials \( Q_m(w) \) are

\[ Q_{122}(w) = -\frac{2i}{486} \left( 35w^6 - 117w^5 + 209w^4 - 250w^3 + 209w^2 - 117w + 35 \right), \]
\[ Q_{124}(w) = -\frac{i}{486} \left( 77w^6 - 232w^5 + 415w^4 - 496w^3 + 415w^2 - 232w + 77 \right), \]
\[ Q_{130}(w) = -\frac{2i}{486} \left( 39w^6 - 117w^5 + 205w^4 - 250w^3 + 205w^2 - 117w + 39 \right), \]

and the root of these polynomials are shown on fig. \( 8 \).

**Question 8.** What is the asymptotic behaviour of the distributions \( \rho_m \)?

**Remark.** In the proof of theorem [12] we consider, for a given degree \( d \), a set of polynomials corresponding to configurations

\[ m = 1 \ 0^{\ell_1} 10^{\ell_2} 1 \ldots 0^{\ell_{d-1}} 1_3, \]

where ones are separated by long sequences of zeroes. These configurations generate sums \( d(m) \) which are reduced to sums of \( d \) almost independent random variables, and, in particular,

\[ \tilde{P}_m(z) \to \frac{1}{2^d} (1 + z)^d, \quad \ell_j \to \infty. \]

Thus, it is easy to see that the corresponding distributions \( \rho_m \) converge to the binomial distribution.

**Question 9.** Is it true that the distributions \( \rho_m \), centered and scaled, converge to the normal distribution as \( d(m) \to \infty \) independently on the structure of \( \tilde{P}_m(z) \)? (see fig. 9)

**Hypothesis 10.** The first polynomials \( \tilde{P}_m(z) \) of degree \( d \) is observed at

\[ m = \frac{3^{d-1} + 1}{2}. \]

**Hypothesis 11.** Consider a subsequence of polynomials described in hypothesis [10] if \( m \) is given by formula [1] and \( m \in 2\mathbb{Z} \) then the corresponding polynomials \( \tilde{P}_m^d(z) \) are irreducible monic self-reciprocal polynomials. If \( m \) is odd then the same is true for \( (z+1)^{-1} \tilde{P}_m^d(z) \). The roots \( r_j \) of \( \tilde{P}_m^d(z) \) are algebraic integers, moreover, \( r_j \in \mathbb{R} \).

**Remark 19.** In particular, if hypothesis [10] is true then the following estimate holds

\[ d(m) \leq 1 + \log_2(2m - 1). \]

**Question 12.** How \( d(m) \) depends on \( m \)?

**Question 13.** Is it true that no one polynomial \( \tilde{P}_m(z) \) divides another polynomials in this sequence, and any \( \tilde{P}_m(z) \) is never a product of different polynomials \( \tilde{P}_m^d(z) \) of smaller degree?

**Question 14.** Is it true that any operator \( \tilde{P}_m(T) \) is not a product of different operators \( A_j \in \mathcal{L} \) in the weak closure of powers of Chacon transformation \( T \)?

**Question 15.** Can we find \( \tilde{P}_m(z) \) which is an isolated point in the semigroup generated by all \( \{ P_m \} \), and can we find \( \tilde{P}_m(T) \) which is an isolated point in \( \mathcal{L} \)?

**Question 16.** Is it true that the set \( \mathcal{L} \) does not contain operators given by series

\[ \sum_{j \in \mathbb{Z}} a_j T^j, \]

where infinitely many \( a_j \neq 0 \)? Is it possible to find among elements \( V \in \mathcal{L} \) operators of the form

\[ V = \kappa \Theta + \sum_{j} a_j T^j, \quad \kappa \neq 0? \]

The following well-known question still has no answer as well.

**Question 17.** Is Chacon(\( \kappa \)) transformation is \( \kappa \)-mixing, which means that there exists \( V \in \mathcal{L} \) such that

\[ V = \kappa \Theta + V_2, \quad \kappa \neq 0? \]

**Hypothesis 18.** There exists \( \varepsilon_0 > 0 \) such that among the polynomials \( P_m(T) \) and in the set \( \mathcal{L} \) there is no polynomials

\[ \sum_{j=0}^{d} a_j T^j, \quad a_j > 0, \]
having the property
\[ \left| \frac{a_{j+1}}{a_j} - 1 \right| < \varepsilon_0. \]

**Question 19.** Can we find \( \tilde{P}_m(z) \) which is an isolated point in the semigroup generated by all \( \{\tilde{P}_m^j\} \), and can we find \( \tilde{P}_m(T) \) which is an isolated point in \( \mathcal{L} \)?

**Question 20.** How can we describe the entire set \( \mathcal{L} \) for Chacon\(_{(3)}\) transformation?

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### V. APPENDIX: THE LIMIT POLYNOMIALS

**Table 1. First 122 limit polynomials \( \tilde{P}_m(z) \)**

The columns of this table indicate: the number \( m \), 3-adic expansion of \( m \) (configuration), and the polynomial \( \tilde{P}_m(z) \). We mark by * the indexes corresponding to configurations 111 \ldots 123. We skip symmetrical configurations like 1123 \sim 2113 following Hypothesis 2 which is true in this interval.

| Index \( m \) | Configuration | Polynomial \( \tilde{P}_m(z) \) |
|---------------|--------------|--------------------------------|
| 1*            | 13           | \( \tilde{P}_1(z) = \tilde{P}_3(z) = \tilde{P}_5(z) = \cdots = \frac{1}{2}(1 + z) \) |
| 2*            | 23           | \( \tilde{P}_2(z) = \tilde{P}_6(z) = \cdots = \frac{1}{6}(1 + 4z + z^2) \) |
| 4             | 113          | \( \tilde{P}_4(z) = \frac{1}{9}(2z^2 + 5z + 2) \) |
| 5*            | 123          | \( \tilde{P}_5(z) = \frac{1}{18}(z^3 + 8z^2 + 8z + 1) \) |
| 8             | 223          | \( \tilde{P}_8(z) = \frac{1}{9}(2z^2 + 5z + 2) \) |
| 10            | 1013         | \( \tilde{P}_{10}(z) = \frac{1}{54}(13z^2 + 28z + 13) \) |
| 11            | 1023         | \( \tilde{P}_{11}(z) = \frac{1}{54}(4z^3 + 23z^2 + 23z + 4) \) |
| 13            | 1113         | \( \tilde{P}_{13}(z) = \frac{1}{54}(5z^3 + 22z^2 + 22z + 5) \) |
| 14*           | 1123         | \( \tilde{P}_{14}(z) = \frac{1}{54}(z^4 + 13z^3 + 26z^2 + 13z + 1) \) |
| 16            | 1213         | \( \tilde{P}_{16}(z) = \frac{1}{54}(z^4 + 12z^3 + 28z^2 + 12z + 1) \) |
| Index \(m\) | Configuration | Polynomial \(\tilde{P}_m(z)\) |
|-------------|---------------|-----------------------------|
| 17          | 122_3         | \(\tilde{P}_{17}(z) = \frac{1}{54}(4z^3 + 23z^2 + 23z + 4)\) |
| 20          | 202_3         | \(\tilde{P}_{20}(z) = \frac{1}{54}(z^4 + 12z^3 + 28z^2 + 12z + 1)\) |
| 23          | 212_3         | \(\tilde{P}_{23}(z) = \frac{1}{54}(5z^3 + 22z^2 + 22z + 5)\) |
| 26          | 222_3         | \(\tilde{P}_{26}(z) = \frac{1}{54}(13z^2 + 28z + 13)\) |
| 28          | 1001_3        | \(\tilde{P}_{28}(z) = \frac{1}{81}(20z^2 + 41z + 20)\) |
| 29          | 1002_3        | \(\tilde{P}_{29}(z) = \frac{1}{162}(13z^3 + 68z^2 + 68z + 13)\) |
| 31          | 1011_3        | \(\tilde{P}_{31}(z) = \frac{1}{162}(17z^3 + 64z^2 + 64z + 17)\) |
| 32          | 1012_3        | \(\tilde{P}_{32}(z) = \frac{1}{81}(2z^4 + 20z^3 + 37z^2 + 20z + 2)\) |
| 34          | 1021_3        | \(\tilde{P}_{34}(z) = \frac{1}{162}(4z^4 + 39z^3 + 76z^2 + 39z + 4)\) |
| 35          | 1022_3        | \(\tilde{P}_{35}(z) = \frac{1}{162}(16z^3 + 65z^2 + 65z + 16)\) |
| 38          | 1102_3        | \(\tilde{P}_{38}(z) = \frac{1}{162}(5z^4 + 39z^3 + 74z^2 + 39z + 5)\) |
| 40          | 1111_3        | \(\tilde{P}_{40}(z) = \frac{1}{81}(3z^4 + 20z^3 + 35z^2 + 20z + 3)\) |
| 41*         | 1112_3        | \(\tilde{P}_{41}(z) = \frac{1}{162}(z^5 + 19z^4 + 61z^3 + 61z^2 + 19z + 1)\) |
| 43          | 1121_3        | \(\tilde{P}_{43}(z) = \frac{1}{162}(z^5 + 17z^4 + 63z^3 + 63z^2 + 17z + 1)\) |
| 44          | 1122_3        | \(\tilde{P}_{44}(z) = \frac{1}{81}(2z^4 + 20z^3 + 37z^2 + 20z + 2)\) |
| 47          | 1202_3        | \(\tilde{P}_{47}(z) = \frac{1}{162}(z^5 + 16z^4 + 64z^3 + 64z^2 + 16z + 1)\) |
| 50          | 1212_3        | \(\tilde{P}_{50}(z) = \frac{1}{162}(5z^4 + 39z^3 + 74z^2 + 39z + 5)\) |
| 52          | 1221_3        | \(\tilde{P}_{52}(z) = \frac{1}{81}(2z^4 + 18z^3 + 41z^2 + 18z + 2)\) |
| 53          | 1222_3        | \(\tilde{P}_{53}(z) = \frac{1}{162}(13z^3 + 68z^2 + 68z + 13)\) |
| Index $m$ | Configuration | Polynomial $\tilde{P}_m(z)$ |
|-----------|--------------|----------------------------|
| 56        | 2002₃       | $\tilde{P}_{56}(z) = \frac{1}{81}(2z^4 + 18z^3 + 41z^2 + 18z + 2)$ |
| 59        | 2012₃       | $\tilde{P}_{59}(z) = \frac{1}{162}(z^5 + 17z^4 + 63z^3 + 63z^2 + 17z + 1)$ |
| 62        | 2022₃       | $\tilde{P}_{62}(z) = \frac{1}{162}(4z^4 + 39z^3 + 76z^2 + 39z + 4)$ |
| 68        | 2112₃       | $\tilde{P}_{68}(z) = \frac{1}{81}(3z^4 + 20z^3 + 35z^2 + 20z + 3)$ |
| 71        | 2122₃       | $\tilde{P}_{71}(z) = \frac{1}{162}(17z^3 + 64z^2 + 64z + 17)$ |
| 80        | 2222₃       | $\tilde{P}_{80}(z) = \frac{1}{81}(20z^2 + 41z + 20)$ |
| 82        | 10001₃      | $\tilde{P}_{82}(z) = \frac{1}{486}(121z^2 + 244z + 121)$ |
| 83        | 10002₃      | $\tilde{P}_{83}(z) = \frac{1}{486}(40z^3 + 203z^2 + 203z + 40)$ |
| 85        | 10011₃      | $\tilde{P}_{85}(z) = \frac{1}{486}(53z^3 + 190z^2 + 190z + 53)$ |
| 86        | 10012₃      | $\tilde{P}_{86}(z) = \frac{1}{486}(13z^4 + 121z^3 + 218z^2 + 121z + 13)$ |
| 88        | 10021₃      | $\tilde{P}_{88}(z) = \frac{1}{486}(13z^4 + 120z^3 + 220z^2 + 120z + 13)$ |
| 89        | 10022₃      | $\tilde{P}_{89}(z) = \frac{1}{486}(52z^3 + 191z^2 + 191z + 52)$ |
| 91        | 10101₃      | $\tilde{P}_{91}(z) = \frac{1}{486}(56z^3 + 187z^2 + 187z + 56)$ |
| 92        | 10102₃      | $\tilde{P}_{92}(z) = \frac{1}{486}(17z^4 + 120z^3 + 212z^2 + 120z + 17)$ |
| 94        | 10111₃      | $\tilde{P}_{94}(z) = \frac{1}{486}(21z^4 + 121z^3 + 202z^2 + 121z + 21)$ |
| 95        | 10112₃      | $\tilde{P}_{95}(z) = \frac{1}{486}(4z^5 + 61z^4 + 178z^3 + 178z^2 + 61z + 4)$ |
| 97        | 10121₃      | $\tilde{P}_{97}(z) = \frac{1}{486}(4z^5 + 56z^4 + 183z^3 + 183z^2 + 56z + 4)$ |
| 98        | 10122₃      | $\tilde{P}_{98}(z) = \frac{1}{486}(16z^4 + 121z^3 + 212z^2 + 121z + 16)$ |
| 100       | 10201₃      | $\tilde{P}_{100}(z) = \frac{1}{243}(8z^4 + 60z^3 + 107z^2 + 60z + 8)$ |
| Index $m$ | Configuration | Polynomial $\tilde{P}_m(z)$ |
|-----------|---------------|-----------------------------|
| 101       | $10202_3$     | $\tilde{P}_{101}(z) = \frac{1}{486}(4z^5 + 55z^4 + 184z^3 + 184z^2 + 55z + 4)$ |
| 103       | $10211_3$     | $\tilde{P}_{103}(z) = \frac{1}{486}(4z^5 + 59z^4 + 180z^3 + 180z^2 + 59z + 4)$ |
| 104       | $10212_3$     | $\tilde{P}_{104}(z) = \frac{1}{243}(10z^4 + 60z^3 + 103z^2 + 60z + 10)$ |
| 106       | $10221_3$     | $\tilde{P}_{106}(z) = \frac{1}{486}(16z^4 + 117z^3 + 220z^2 + 117z + 16)$ |
| 107       | $10222_3$     | $\tilde{P}_{107}(z) = \frac{1}{486}(52z^3 + 191z^2 + 191z + 52)$ |
| 110       | $11002_3$     | $\tilde{P}_{110}(z) = \frac{1}{486}(17z^4 + 117z^3 + 218z^2 + 117z + 17)$ |
| 112       | $11011_3$     | $\tilde{P}_{112}(z) = \frac{1}{243}(11z^4 + 60z^3 + 101z^2 + 60z + 11)$ |
| 113       | $11012_3$     | $\tilde{P}_{113}(z) = \frac{1}{486}(5z^5 + 61z^4 + 177z^3 + 177z^2 + 61z + 5)$ |
| 115       | $11021_3$     | $\tilde{P}_{115}(z) = \frac{1}{486}(5z^5 + 59z^4 + 179z^3 + 179z^2 + 59z + 5)$ |
| 116       | $11022_3$     | $\tilde{P}_{116}(z) = \frac{1}{243}(10z^4 + 60z^3 + 103z^2 + 60z + 10)$ |
| 119       | $11102_3$     | $\tilde{P}_{119}(z) = \frac{1}{486}(6z^5 + 61z^4 + 176z^3 + 176z^2 + 61z + 6)$ |
| 121       | $11111_3$     | $\tilde{P}_{121}(z) = \frac{1}{486}(7z^5 + 65z^4 + 171z^3 + 171z^2 + 65z + 7)$ |
| 122$^*$   | $11112_3$     | $\tilde{P}_{122}(z) = \frac{1}{486}(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1)$ |

**Table 2.** Several remarkable limit polynomials $\tilde{P}_m(z)$ for $m \leq 1094$. 

| Index $m$ | Configuration | Polynomial $\tilde{P}_m(z)$ |
|-----------|---------------|-----------------------------|
| 1         | $1_3$         | $\tilde{P}_1(z) = \tilde{P}_3(z) = \tilde{P}_5(z) = \cdots = \frac{1}{2}(1 + z)$ |
| 2$^*$     | $2_3$         | $\tilde{P}_2(z) = \tilde{P}_6(z) = \cdots = \frac{1}{6}(1 + 4z + z^2)$ |

*First occurrence of degree $d$*
| Index $m$ | Configuration | Polynomial $\tilde{P}_m(z)$ |
|----------|---------------|----------------------------|
| 5*   | 12s           | $\tilde{P}_5(z) = \frac{1}{18}(z^3 + 8z^2 + 8z + 1)$ |
| 14*  | 112s          | $\tilde{P}_{14}(z) = \frac{1}{54}(z^4 + 13z^3 + 26z^2 + 13z + 1)$ |
| 41*  | 1112s         | $\tilde{P}_{41}(z) = \frac{1}{162}(z^5 + 19z^4 + 61z^3 + 61z^2 + 19z + 1)$ |
| 122* | 11112s        | $\tilde{P}_{122}(z) = \frac{1}{486}(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1)$ |
| 365* | 111112s       | $\tilde{P}_{365}(z) = \frac{1}{1458}(z^7 + 34z^6 + 211z^5 + 483z^4 + 483z^3 + 211z^2 + 34z + 1)$ |
| 1094*| 1111112s      | $\tilde{P}_{1094}(z) = \frac{1}{4374}(z^8 + 43z^7 + 343z^6 + 1050z^5 + 1500z^4 + 1050z^3 + 343z^2 + 43z + 1)$ |

**Similar configurations**

| Index $m$ | Configuration | Polynomial $\tilde{P}_m(z)$ |
|----------|---------------|----------------------------|
| 122     | 11112s        | $\tilde{P}_{122}(z) = \frac{1}{486}(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1)$ |
| 124     | 11121s        | $\tilde{P}_{124}(z) = \frac{1}{486}(z^6 + 23z^5 + 119z^4 + 200z^3 + 119z^2 + 23z + 1)$ |
| 130     | 11211s        | $\tilde{P}_{130}(z) = \frac{1}{486}(z^6 + 22z^5 + 120z^4 + 200z^3 + 120z^2 + 22z + 1)$ |
| 148     | 12111s        | $\tilde{P}_{148}(z) = \frac{1}{486}(z^6 + 23z^5 + 119z^4 + 200z^3 + 119z^2 + 23z + 1)$ |
| 202     | 21111s        | $\tilde{P}_{202}(z) = \frac{1}{486}(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1)$ |

**Irreducible up to a root $(-1)$ cubic polynomials**

| Index $m$ | Configuration | Polynomial $\tilde{P}_m(z)$ |
|----------|---------------|----------------------------|
| 91       | 10101s        | $\tilde{P}_{91}(z) = \frac{1}{486}(56z^3 + 187z^2 + 187z + 56)$ |
| 253      | 100101s       | $\tilde{P}_{253}(z) = \frac{1}{1458}(173z^3 + 556z^2 + 556z + 173)$ |
| 739      | 1000101s      | $\tilde{P}_{739}(z) = \frac{1}{4374}(524z^3 + 1663z^2 + 1663z + 524)$ |
| 757      | 1001001s      | $\tilde{P}_{757}(z) = \frac{1}{4374}(533z^3 + 1654z^2 + 1654z + 533)$ |
### Table 3. Non-irreducible polynomials $\tilde{P}_m(z)$ for $m \leq 122$ (up to a root $-1$).

| Index $m$ | Configuration | Factorization of $\tilde{P}_m(z)$ |
|-----------|---------------|-----------------------------------|
| 4         | 113           | $\tilde{P}_4(z) = \frac{1}{9}(2+z)(1+2z)$ |
| 8         | 223           | $\tilde{P}_8(z) = \frac{1}{9}(2+z)(1+2z)$ |
| 28        | 10013         | $\tilde{P}_{28}(z) = \frac{1}{81}(5+4z)(4+5z)$ |
| 40        | 11113         | $\tilde{P}_{40}(z) = \frac{1}{81}(3+5z+z^2)(1+5z+3z^2)$ |
| 52        | 12213         | $\tilde{P}_{52}(z) = \frac{1}{81}(2+6z+z^2)(1+6z+2z^2)$ |
| 56        | 20023         | $\tilde{P}_{56}(z) = \frac{1}{81}(2+6z+z^2)(1+6z+2z^2)$ |
| 68        | 21123         | $\tilde{P}_{68}(z) = \frac{1}{81}(3+5z+z^2)(1+5z+3z^2)$ |
| 80        | 22223         | $\tilde{P}_{80}(z) = \frac{1}{9}(5+4z)(4+5z)$ |
| 244       | 1111113       | $\tilde{P}_{244}(z) = \frac{1}{729}(14+13z)(13+14z)$ |

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