An analytical approximation of the growth function in Friedmann-Lemaître universes

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ABSTRACT

We present an analytical approximation formula for the growth function in a spatially flat cosmology without a cosmological constant. Our approximate formula is written simply in terms of a rational function. The single rational function applies for all, open, closed and flat universes. Our results involve no elliptic functions, and have very small relative error of less than 0.2 per cent over the range of the scale factor 0 ≤ a ≤ 1, and less than 0.4 per cent over the range 0.2 ≤ Ω_m ≤ 4 for a cosmology without a cosmological constant.

Key words: cosmology: theory – large scale structure of the universe

1 INTRODUCTION

Heath [1977] has shown that the growth function in a dust cosmology can be written as

\[ D_1(a) \propto H(a) \int_0^a \frac{da'}{(a'H(a'))^3}, \]

\[ D_2 \propto H(a), \]

\[ H(a) = \sqrt{\Omega_m a^{-3} + (1 - \Omega_m - \Omega_\Lambda)a^{-2} + \Omega_\Lambda}, \]

where a is the scale factor of the universe, normalised to unity at present, Ω_m and Ω_Λ are the density parameters of dust matter and a cosmological constant, respectively.

Although a compact expression using the incomplete beta function has been shown by Bildhauer et al. [1992], so far, no analytic solution of \( D_1(a) \) has been presented for \( \Omega_\Lambda \neq 0 \). Here we restrict ourselves to the case \( \Omega_m + \Omega_\Lambda = 1 \), and present an approximate formula in a simple algebraic form.

Following Eisenstein [1997], we adopt a normalization for \( D_1(a) \) as

\[ D_1(a) = \frac{5\Omega_m}{2} H(a) \int_0^a \frac{da'}{(a'H(a'))^3}. \]

Then, in a flat cosmology \( \Omega_m + \Omega_\Lambda = 1 \), our formula is

\[ D_1(a) = a\sqrt{1 + x} \left( 1 + 1.175x + 0.3064x^2 + 0.005355x^3 \right), \]

\[ 1 + 1.857x + 0.021x^2 + 0.1530x^3, \]

where,

\[ x = \frac{1 - \Omega_m a^3}{\Omega_m}. \]

Note that our approximate formula is exact when \( \Omega_m = 1 \).

2 APPROXIMATION

We define

\[ F(a) = \frac{5}{2}\Omega_m^2 \int_0^a \frac{da'}{(a'(\Omega_m a^{-3} + 1 - \Omega_m))^3}. \]

The power series expansion of \( F(a) \) around \( a = 0 \) yields

\[ F(a) = a^{\hat{2}} \left( 1 + \frac{15}{22} x + \frac{75}{136} x^2 - \frac{175}{368} x^3 + \cdots \right), \]

where \( x \) is defined in Eq. (6). After expanding \( F(a) \) up to \( O(x^4) \), we can obtain the Padé approximant to the following order:

\[ F(a) = a^{\hat{2}} \left( 1 + b_1 x + b_2 x^2 + b_3 x^3 \right), \]

\[ 1 + c_1 x + c_2 x^2 + c_3 x^3, \]

where the numerical constants are determined as follows:

\[ b_1 = \frac{619226202351}{527102715964}, \]

\[ b_2 = \frac{1247831282519}{40730664415400}, \]

\[ b_3 = \frac{232758215919527}{43467765064114880}, \]

\[ c_1 = \frac{88964947071}{47918428724}, \]

\[ c_2 = \frac{88964947071}{47918428724}, \]

\[ c_3 = \frac{88964947071}{47918428724}. \]
Table 1. The maximal relative error $\Delta E$ (per cent) for the normalised growth function $D(a)$ by our formula Eq. (5) for $\Omega_m + \Omega_\Lambda = 1$

| $\Omega_m$ | 0.2 | 0.3 | 0.4 | 0.5 | 1 |
|-----------|-----|-----|-----|-----|---|
| maximal $\Delta E$ | 0.20 | 0.03 | < 0.01 | < 0.01 | 0 |

Table 2. A comparison of the relative error of Eq. (13) and our $D_1(1)$ from Eq. (9).

| $\Omega_m$ | 0.2 | 0.3 | 0.5 | 0.9 |
|-----------|-----|-----|-----|-----|
| $\Delta E$ of Eq. (13) | 0.54 | 0.134 | 0.057 | 0.019 |
| $\Delta E$ of Eq. (9) | 0.19 | < 0.01 | < 0.01 | < 0.01 |

$c_2 = \frac{244568735707}{2395921436200}$
$c_3 = \frac{17010766061223}{11170754639680}$

We have checked the Taylor expansion and the Padé approximant using Maxima. It is convenient to use a normalised growth function

$$D(a) \equiv \frac{D_1(a)}{D_1(1)},$$

so that $D(1) = 1$.

In order to check the accuracy of our formula, we calculate the following relative error

$$\Delta E = \left| \frac{D_{\text{app}} - D_{\text{num}}}{D_{\text{num}}} \right| \times 100 \text{ (percent)},$$

where $D_{\text{app}}$ and $D_{\text{num}}$ represent the values of normalised growth functions calculated by using approximate formula and numerical method, respectively.

Table 1 shows the maximal relative error $\Delta E$ for $1/1000 \leq a \leq 1$ in our method. It is apparent that our approximate formula has sufficiently small uncertainties over the wide range of parameters $0.2 \leq \Omega_m \leq 1$ and $1/1000 \leq a \leq 1$.

The approximation of Carroll et al. (1992), which was adopted from Lahav et al. (1993), is

$$D^C_1 = \frac{5\Omega_m}{2(1 + \Omega_m) + \left(1 + \frac{\Omega_\Lambda}{\Omega_m}\right)\left(1 + \frac{1 - \Omega_\Lambda}{\Omega_m}\right)},$$

for $\Omega_m + \Omega_\Lambda = 1$.

A comparison of the relative error $\Delta E$ at $a = 1$ for both methods by Carroll et al. (1992) and us, is shown in Table 2 for $\Omega_m + \Omega_\Lambda = 1$. Our formula has generally smaller relative error in the range $0.2 \leq \Omega_m < 1$.

3 ZERO COSMOLOGICAL CONSTANT CASE

In the $\Omega_\Lambda = 0$ case, the analytical form for $D_1(a)$ is widely studied (e.g., Weinberg 1972, Heath 1977, Bildhauer et al. 1992) also have shown the growth function as a function of the scale factor $a$ in various Friedmann-Lemaître universes.

Table 3. The maximal relative error $\Delta E$ for the normalised growth function $D(a)$ by our formula Eq. (19) for $\Omega_\Lambda = 0$

| $\Omega_m$ | 0.2 | 0.5 | 1 | 2.0 | 4.0 |
|-----------|-----|-----|---|-----|-----|
| maximal $\Delta E$ | 0.34 | 0.02 | 0 | 0.07 | 0.36 |

in a slightly different context. Here, we show a simple and efficient approximate formula, directly as a function of $a$. In a dust cosmology with zero cosmological constant, $\Omega_\Lambda = 0$, our formula is

$$D_1(a) = a\sqrt{1 + y}\left(1 + 1.13y + 0.1247y^2 - 0.003893y^3\right),$$

where,

$$y = \frac{1 - \Omega_m}{\Omega_m}a.$$

It is straightforward to derive Eq. (19) from the Padé approximant, in the same way as shown in the previous section. Table 3 shows the maximal relative error of the our formula for the normalised growth function $D(a)$ in the range $0.2 \leq \Omega_m \leq 4$ and $1/1000 \leq a \leq 1$.

4 CONCLUSION

We have presented a simple approximation formula for the growth function in a spatially flat cosmology with dust and a cosmological constant. Our formula is written in terms of a rational function, and widely applicable over the range $1/1000 \leq a \leq 1$ with sufficiently small relative error of less than 0.2 per cent.

We have also shown the approximate formula in a dust cosmology without a cosmological constant in terms of a rational function. The single rational function applies for all, open, closed and flat universes with relative error of less than 0.4 per cent over the range $1/1000 \leq a \leq 1$ and $0.2 \leq \Omega_m \leq 4$.

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