TRANSFINITE DIGRAPHS

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Abstract — Transfinite graphs have been defined and examined in a variety of prior works, but transfinite digraphs had not as yet been investigated. The present work embarks upon such a task. As with the ordinals, transfinite digraphs appear in a hierarchy of ranks indexed by the countable ordinals. The digraphs of rank 0 are the conventional digraphs. Those of rank 1 are constructed by defining certain extremities of 0-ranked digraphs, and then partitioning those extremities to obtain vertices of rank 1. Then, digraphs of rank 0 are connected together at those vertices of rank 1 to obtain a digraph of rank 1. This process can be continued through the natural-number ranks. However, to achieve a digraph whose rank is the first infinite ordinal $\omega$ (i.e., the first limit ordinal), a special kind of transfinite digraph, which we call a digraph with an "arrow rank" must first be constructed in a way different from those of natural-number rank. Then, digraphs of still higher ranks can be constructed in a way similar to that for the natural-number ranked digraphs. However, just before each limit-ordinal rank, a digraph of arrow rank must be set up.

Key Words: Transfinite digraphs, ranks of digraphs, pristine digraphs.

1 Introduction

This is the second of a series of three works in which digraphs are generalized in nonstandard and transfinite ways. The first work [3] defined and examined nonstandard digraphs. The present work does the same for transfinite digraphs. The third work will discuss digraphs that are both transfinite and nonstandard.

Transfinite digraphs appear in a hierarchy of ranks indexed by the ordinals $\nu = 0, 1, 2, \ldots, \omega, \omega + 1, \ldots$ along with a special "arrow rank" $\vec{\omega}$ that appears after all the natural-number ranks but precedes the first infinite-ordinal rank $\omega$. Other "arrow ranks" appear subsequently just before the countable limit ordinal ranks. A transfinite digraph of rank $\nu$ is
called a $\nu$-digraph. A 0-digraph is a conventional digraph. Our definitions and analysis proceed through all the natural-number ranks of digraphs recursively. However, before proceeding on to $\omega$-digraphs, we have to first introduce the $\vec{\omega}$-digraphs through a quite different construction. Having done so, we can then proceed recursively through the countably infinite ordinal ranks $\nu = \omega, \omega + 1, \omega + 2, \ldots$, but then must consider the arrow-ranked $(\omega + \vec{\omega})$-digraph before proceeding through the $\nu$-digraphs where $\nu = \omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots$, and so forth. Since discussions of the $\nu$-digraphs where $\nu > \omega$ are so similar to those for $\nu \leq \omega$, we merely mention them in the last section.

Our notations and symbols herein are the same as those used in [3].

2 Pristine transfinite digraphs

In general, the higher-rank vertices of transfinite digraphs can "embrace" vertices of lower ranks; that is, a vertex of high rank can contain a vertex of lower rank, which in turn can contain a vertex of still lower rank, and so forth through a sequence of vertices of decreasing ranks. This leads to some complications in the analysis of such digraphs. Some simplicity can be achieved by prohibiting such embraced vertices. We shall do so in this discussion of transfinite digraphs and will call the so-restricted digraphs pristine digraph. Every vertex of a pristine transfinite digraph of, say, rank $\rho$ will contain only "intips" and "outtips" of rank $\rho - 1$ but no vertex of rank less than $\rho$, as will be explicated below.

Actually, this prohibition is no restriction at all because any digraph can be converted into a pristine digraph by "extracting" the embraced vertices. The process is the same as that for transfinite graphs and is described in [1, Section 1.4]; the only difference arising for transfinite digraphs is that branches in the "extraction paths" are replaced by pairs of oppositely directed, parallel arcs. The resulting pristine digraphs are simpler to analyze but have an expanded structure due to the introduction of the extraction paths.

3 1-digraphs

As was indicated in the Introduction, 0-digraphs are simply the standard digraphs discussed in [3, Section 2]. On the other hand, "transfinite digraphs" appear in a hierarchy of ranks
of transfiniteness, with the 0-digraphs having the initial—and not transfinite—rank in that hierarchy. The "1-digraphs" comprise the first truly transfinite rank and are the subject of this section.

It turns out that basing the construction of transfinite digraphs of ranks 2 and higher on "directed walks," rather than on "directed paths," leads to a more general and less complicated structure. The reason for this is virtually the same as that encountered when constructing undirected transfinite graphs—namely, path-connectedness of ranks 1 and higher is not in general transitive as a binary relation among transfinite vertices, whereas walk-connectedness is transitive in this regard. This matter is discussed in [2, Chapter 5] and will not be repeated here. On the other hand, the construction of digraphs of 1 can be based either on conventional paths or on conventional walks (i.e., either on paths or walks of rank 0); the two methods are entirely equivalent. To be consistent with our subsequent constructions of digraphs of ranks 2 or higher, we will use walk-connectedness rather than the equivalent path-connectedness when first constructing transfinite digraphs of rank 1.

Assume we have at hand a (standard) 0-digraph $D^0 = \{A, V^0\}$. The vertices in $V^0$ are now called 0-vertices. Moreover, the intips and outtips of an arc are assigned the rank of $-1$ and are called $(-1)$-intips and $(-1)$-outtips, denoted by $s^{-1}$ and $t^{-1}$, respectively. Also, both kinds of tips are called $(-1)$-ditips.

We start by defining a nontrivial directed walk of rank 0, also called a (nontrivial) 0-diwalk. A 0-diwalk $W^0$ is an alternating sequence of 0-vertices $v_m^0$ and arcs $a_m$, with that sequence having at least two 0-vertices:

$$W^0 = \langle \ldots, v_{m-1}^0, a_{m-1}, v_m^0, a_m, v_{m+1}^0, \ldots \rangle$$

where $a_m = \langle s_m^{-1}, t_m^{-1} \rangle$ with $s_m^{-1} \in v_m^0$ and $t_m^{-1} \in v_{m+1}^0$, where the indices $m$ traverse a set of consecutive integers, and wherein the 0-vertices and arcs may repeat.\(^1\) For the last property in particular it may happen that $v_m^0 = v_k^0$ for some $m \neq k$ and $a_m = a_k$ again for some $m \neq k$.\(^2\) Each arc $a_m$ is directed from $v_m^0$ toward $v_{m+1}^0$. For each $m$, we say that $v_m^0$ and $a_m$ are incident inward and that $v_{m+1}^0$ and $a_m$ are incident outward. Thus, the direction of

\(^1\) Compare this definition with that of a finite dipath given by [3, Equation (7.1)] in symbolic notation, wherein the 0-vertices and arcs do not repeat and the sequence is finite.

\(^2\) It may happen that $k = m + 1$ because self-loops are permitted.
incidence is taken from the perspective of the arcs—not of the vertices. Moreover, we say that $W^0$ is directed in the same direction as that of its arcs (i.e., from left to right in (1).

A trivial 0-diwalk is one having just one vertex and no arc.

We require that, if the sequence (1) terminates on either side, it terminates at a 0-vertex. $W^0$ is called two-ended or finite if it terminates on both sides; $W^0$ is called one-ended if it terminates on only one side; $W^0$ is called endless if it terminates on neither side.

We can define a more general kind of walk wherein the arc directions need not conform, that is, $a_m$ could be either from $v^0_m$ to $v^0_{m+1}$ (because $s^{-1}_m \in v^0_m$ and $t^{-1}_m \in v^0_{m+1}$) or from $v^0_{m+1}$ to $v^0_m$ (because $s^{-1}_m \in v^0_{m+1}$ and $t^{-1}_m \in v^0_m$). In this more general case, we refer to (1) as a 0-semiwalk—not as a 0-diwalk. A 0-semiwalk is a walk in the "underlying graph" of $D^0$.

$W^0$ is called extended inward (resp. extended outward) if there exists an index $m_0$ such that all entries within (1) exist and are distinct for $m < m_0$ (resp. $m > m_0$), that is, if those parts of 1 are one-ended 0-dipaths.

A 0-intip $s^0$ (resp. a 0-outtip $t^0$) is a maximal set of extended inward (resp. extended outward) 0-diwalks that are pairwise identical for all $m < m_0$ (resp. $m > m_0$); $m_0$ will depend upon the choice of the pair of 0-diwalks. Any such 0-diwalk in $s^0$ (resp. in $t^0$) will be called a representative of $s^0$ (resp. of $t^0$).

Now, consider the set $T^0$ of all 0-intips and 0-outtips of the 0-digraph $D^0 = \{A, V^0\}$. We assume that $T^0$ is not empty. Partition $T^0$ arbitrarily to get

$$T^0 = \cup_{i \in I^0} T^0_i.$$  \hspace{1cm} (2)

Here, $I^0$ is the index set of the partition. At this point we change notation by setting $v^1_i = T^0_i$ and call each $v^1_i$ a 1-vertex.\footnote{We may think of the $v^1_i$ as the connections between the extremities of $D^0$.} Thus, each $v^1_i$ is a set of 0-intips and/or 0-outtips in the chosen partition of $T^0$. Also, we set $V^1 = \{v^1_i; i \in I^0\}$; thus, $V^1$ is the resulting set of 1-vertices.

Then, the 1-digraph $D^1$ is defined as the triple:

$$D^1 = \{A, V^0, V^1\}$$  \hspace{1cm} (3)
The "underlying 1-graph" $G^1$ of $D^1$ is defined as follows.\(^4\) First, remove the directions of the arcs. Thus, each arc $a = \langle s^{-1}, t^{-1} \rangle$ becomes a branch $b = \langle t_1^{-1}, t_2^{-1} \rangle$. In the event that there exist two oppositely directed arcs incident to the same two vertices, the resulting two branches are connected in parallel, but we allow parallel branches (as well as parallel arcs). $B$ will denote the set of the resulting branches. Also, each vertex of $D^0$ now becomes a 0-node, and $V^0$ is replaced by a set $X^0$ of 0-nodes. Next, each 0-intip $s^0$ or 0-outtip $t^0$ becomes a 0-tip. Then, the same partitioning as before of the set $T^0$ of 0-ditips to get the 1-vertices is now applied to the set of all 0-tips to get a set of 1-nodes $x^1_i$, with $x^1_i$ being the $i$th set of the partition. $X^1$ denotes the set of these 1-nodes. Then, the underlying 1-graph $G^1$ of $D^1$ is

$$G^1 = \{B, X^0, X^1\} \quad (4)$$

Returning to digraphs, here is some more terminology we shall use. A one-ended or endless 0-diwalk $W^0$, as given by (1), is said to traverse a 0-intip $s^0$ (resp. a 0-outtip $t^0$) if there exists an index $m_0$ such that $W^0$ is identical to a representative of $s^0$ for all $m < m_0$ (resp. a representative of $t^0$ for all $m > m_0$).

Let $u^1$ (resp. $v^1$) be a 1-vertex containing $s^0$ (resp. $t^0$), and let $W^0$ traverse $s^0$ (resp. $t^0$). We then say that $u^1$ and $W^0$ are incident inward (resp. $v^1$ and $W^0$ are incident outward).

At this point, we need to define a (nontrivial) 1-diwalk $W^1$.\(^5\) This is an alternating sequence of 1-vertices and 0-diwalks:

$$W^1 = \langle \ldots, v^1_{m-1}, W^0_{m-1}, v^1_m, W^0_m, v^1_{m+1}, \ldots \rangle \quad (5)$$

where the indices $m$ traverse a set of consecutive integers and the following condition holds for each $m$: Each $W^0_m$ is a 0-diwalk reaching $v^1_m$ and $v^1_{m+1}$ (thus, $v^1_m$ and $W^0_m$ are incident inward, and $v^1_{m+1}$ and $W^0_m$ are incident outward). The elements of (5) may repeat. The direction of $W^1$ is the same as the direction of its 0-walks; that is, from left to right in (5). We allow a 1-diwalk to terminate on either side. For instance, the sequence (5) may

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\(^4\)We use the same definitions and notations for graphs as those explicated in [1] (or in [2] when embraced nodes are disallowed).

\(^5\)Now, a 1-diwalk and a 1-dipath are not in general equivalent.
terminate on the left and/or on the right at a 1-vertex. Another way is for it to terminate at a 0-vertex. For example, the left-most 0-diwalk, say, $W^0_m$ in (5) may be a one-ended 0-diwalk starting at a 0-vertex $v^0_m$ and extending outward to reach $v^1_{m+1}$; we can represent this case by writing (5) as

$$W^1 = \langle v^0_m, W^0_m, v^1_{m+1}, W^0_{m+1}, v^1_{m+2}, \ldots \rangle$$

Alternatively, we may have

$$W^1 = \langle \ldots, v^1_{m-2}, W^0_{m-2}, v^1_{m-1}, W^0_{m-1}, v^0_m \rangle$$

where now $W^0_{m-1}$ terminates on the right at the 0-vertex $v^0_m$ and is incident inward at $v^1_{m-1}$. If $W^1$ terminates on both the left and the right, we call it a finite diwalk. Finally, as a special case, we may have that the 1-diwalk is in fact a 0-diwalk of the form

$$W^1 = \langle v^0_m, W^0_m, v^0_{m+1} \rangle$$

where now $W^0_m$ is a finite 0-diwalk terminating at $v^0_m$ and $v^0_{m+1}$.

A trivial 1-diwalk has just one element, a 1-vertex or a 0-vertex.

As an example of a nontrivial 1-digraph consider Figure 1 (located at the end of this report). It consists of a single one-ended 1-diwalk:

$$W^1 = \langle v^1_1, W^0_1, v^1_2, W^0_2, v^1_3, \ldots \rangle$$

The first 0-diwalk $W^0_1$ in $W^1$ is incident inward at $v^1_1$, proceeds downward to pass through the 0-vertex $v^0_0$, then upward to become incident outward at the 1-vertex $v^1_2$. The second 0-diwalk $W^0_2$ proceeds in the same way, being incident inward at $v^1_2$, going through $v^0_2$, and then becoming incident outward at $v^1_3$. This continues infinitely to traverse all of $W^1$.

The definition of a 1-semiwalk is obtained from the definition of a 1-diwalk (5) by relaxing the restrictions on the directions of the 0-diwalks $W^0_m$ and of the arcs in $W^0_m$. In other words, a 1-semiwalk in the 1-digraph $D^1$ corresponds to a 1-walk in the underlying 1-graph $G^1$ of $D^1$. Again, as a special case, a 1-semiwalk may in fact be a 0-semiwalk corresponding to (6).
Two vertices $u$ and $v$ of ranks 0 or 1 are said to be strongly 1-diwalk-connected (resp. unilaterally 1-diwalk-connected, resp. weakly 1-semiwalk-connected)\(^6\) if there exists a 1-diwalk from $u$ to $v$ and another 1-diwalk from $v$ to $u$ (resp. a 1-diwalk from $u$ to $v$ or from $v$ to $u$ but not necessarily both, resp. a 1-semiwalk terminating at $u$ and $v$).

A 1-strong component is a maximal set of vertices such that every two of those vertices are strongly 1-diwalk-connected; in addition, as a special case, a single vertex may comprise by itself a 1-strong component connected to itself by a trivial 1-diwalk. Fact: Every vertex lies in exactly one 1-strong component.

A 1-unilateral component is a maximal set of vertices such that every two of them are unilaterally 1-diwalk-connected. (Now, a single vertex by itself cannot be a 1-unilateral component.) Fact: Every vertex lies in at least one 1-unilateral component.

A 1-weak component is a maximal set of vertices such that every two of them are weakly 1-semiwalk-connected. (Here again, a single vertex by itself cannot be a 1-weak component.) Fact: Every vertex lies in exactly one 1-weak component.

Let us note again that a 1-diwalk may in fact be a 0-diwalk. Thus, a 1-strong component may actually be a 0-strong component. Similarly, a 1-lateral component or a 1-weak component may be of 0 rank.\(^7\)

4 2-digraphs

Just as the definition of 1-vertices is based on partitions of set of 0-intips and 0-outtips, the definition of ”2-vertices” uses partitions of sets of ”1-intips” and ”1-outtips”. The definitions of the latter entities require the idea of ”extended” 1-walks.

A 1-walk $W^1$ (see (5)) in the 1-digraph $D^1$ is called extended inward (resp. extended outward) if there exists an index $m_0$ such that all elements of (5) exist and are distinct for $m < m_0$ (resp. $m > m_0$). A 1-tip (resp. 1-outtip) of $D^1$ is a maximal set of extended inward (resp. extended outward) 1-diwalks that are pairwise identical for all $m < m_0$ (resp. $m > m_0$), where the index $m_0$ depends upon the choice of the pair of 1-diwalks. Let $T^1$ be the set of all 1-intips and 1-outtips. We assume that $T^1$ is not empty. Partition $T^1$

\(^6\)The dw" herein is an abbreviation diwalk”, and the ”sw” stands for ”semiwalk”.
\(^7\)See [3, Section 8] with regard to these components of 0 rank.
arbitrarily to get $T^1 = \bigcup_{i \in I} T^1_i$. Here, $I^1$ is the index set of the partition. Again, we change notation by setting $v_i^2 = T^1_i$. We call each $v_i^2$ a 2-vertex. Also, $V^2$ denotes the set of all 2-vertices. Then, the 2-digraph $D^2$ is defined as the quadruple:

$$D^2 = \{A, V^0, V^1, V^2\}$$ (7)

At this point, we could continue this discussion of 2-digraphs much as we did for 1-digraphs after they were defined. For instance, the "underlying 2-graph" $G^2$ of $D^2$ could be obtained by removing the directions of the arcs to obtain a set $B$ of branches and in turn getting sets $X_i$ ($i = 0, 1, 2$) of $i$-nodes, all of which would lead to

$$G^2 = \{B, X^0, X^1, X^2\}.$$ 

Furthermore, we could construct in turn "2-walks", "2-intips", and "2-outtips", a set $V^3$ of "3-vertices", and finally the "3-digraph":

$$D^3 = \{A, V^0, V^1, V^2, V^3\}.$$ 

Moreover, still more generally we could proceed recursively to define "$\mu$-digraphs" for all natural numbers $\mu \in \mathbb{N}$. This we do in the next section.

5 $\mu$-digraphs

We turn now to the construction of a transfinite $\mu$-digraph for the natural number $\mu$ by means of recursion. So far, we have done so for $\mu = 0, 1, 2$. We now assume, that for each natural number $\rho = 0, 1, \ldots, \mu - 1$, the $\rho$-digraphs

$$D^\rho = \{A, V^0, \ldots, V^\rho\}$$ (8)

have been defined along with their associated structures such as $\rho$-vertices and $\rho$-diwalks $W^\rho$. In particular, we have

$$W^\rho = \langle \ldots, t_{m-1}^\rho, W_{m-1}^\rho, v_m^\rho, W_{m-1}^\rho, v_{m+1}^\rho, \ldots \rangle$$ (9)

where the indices $m$ traverse a consecutive set of integers. Here, for $\rho = 0$, $W_{m-1}^\rho$ is replaced by an arc $a_m = \langle s_m^{-1}, t_m^{-1} \rangle$, and $v_m^0$ is a 0-vertex, as in (1). For $\rho = 1, \ldots, \mu - 1$, all the
\((\rho - 1)\)-diwalks \(W_{\rho m}^{\rho - 1}\) are incident inward at \(v_{\rho m}^\rho\) and incident outward at \(v_{\rho m+1}^\rho\), and thus the direction of \(W^\rho\) is from left to right in (9). Because \(W^\rho\) is a diwalk, the elements in (9) may repeat. The sequence \(W^\rho\) may terminate on either side at an \(\alpha\)-vertex \((0 \leq \alpha \leq \rho)\).

All the ideas in the preceding paragraph and some others that we have yet to discuss in this section have been defined for \(\rho = 0\) and \(\rho = 1\) in Sections 3 and 4. For \(\rho = 2, \ldots, \mu\), they will be precisely specified when we complete this cycle of our recursive development from \(\rho = \mu - 1\) to \(\rho = \mu\).

The \((\mu - 1)\)-diwalk \(W^{\mu - 1}\) is call extended inward (resp. extended outward) if there exists an index \(m_0\) such that all elements within (9) exist and are distinct for \(m < m_0\) (resp. \(m > m_0\)). A \((\mu - 1)\)-intip \(s^{\mu - 1}\) (resp. a \((\mu - 1)\)-outtip \(t^{\mu - 1}\)) is a maximal set of extended inward (resp. extended outward) \((\mu - 1)\)-diwalks that are pairwise identical for all \(m < m_0\) (resp. \(m > m_0\)); \(m_0\) will depend upon the choice of the pair of \((\mu - 1)\)-diwalks. Any such \((\mu - 1)\)-diwalk in \(s^{\mu - 1}\) (resp. \(t^{\mu - 1}\)) will be called a representative of \(s^{\mu - 1}\) (resp. \(t^{\mu - 1}\)). We may on occasion refer to a 1-intip or a 1-outtip as a 1-ditip.

Let \(T^{\mu - 1}\) be the set of all \((\mu - 1)\)-intips and \((\mu - 1)\)-outtips of the \((\mu - 1)\)-digraph \(D^{\mu - 1}\). Here too, we assume that \(T^{\mu - 1}\) is not empty. We partition \(T^{\mu - 1}\) arbitrarily to get \(T^{\mu - 1} = \bigcup_{i \in I^{\mu - 1}} T_i^{\mu - 1}\), where \(I^{\mu - 1}\) is the index set of the partition. Once again, we change notation by setting \(v_i^\mu = T_i^{\mu - 1}\) and call each \(v_i^\mu\) a \(\mu\)-vertex. Thus, each \(v_i^\mu\) is a set of \((\mu - 1)\)-intips and/or \((\mu - 1)\)-outtips. The set \(V^\mu = \{v_i^\mu : i \in I^{\mu - 1}\}\) denotes the set of \(\mu\)-vertices.

All this yields the definition of a \(\mu\)-digraph, given by (8) when \(\rho\) is replaced by \(\mu\). In particular, for \(\rho = \mu - 1\), this cycle of our recursion has yielded one more higher rank \(\mu\) of the transfinite digraphs, namely,

\[
D^\mu = \{A, V^0, \ldots, V^\mu\}. \tag{10}
\]

Upon repeating our prior definitions and constructions but this time for the rank \(\mu\), we obtain the (nontrivial) \(\mu\)-diwalk:

\[
W^\mu = \langle \cdots, v_{\mu m - 1}^\mu, W_{\mu m - 1}^{\mu - 1}, v_{\mu m}^\mu, W_{\mu m}^{\mu - 1}, v_{\mu m + 1}^\mu, \ldots \rangle. \tag{11}
\]

Here, each \(W_{\mu m}^{\mu - 1}\) is an endless \((\mu - 1)\)-diwalk that is incident inward at \(v_{\mu m}^\mu\) (resp. incident
outward at \( v_{m+1}^\mu \); in other words, \( W_{m}^{\mu-1} \) reaches \( v_{m}^\mu \) (resp. \( v_{m+1}^\mu \)) along a representative of a \((\mu - 1)\)-intip \( s_{m}^{\mu-1} \) in \( v_{m}^\mu \) (resp. along a representative of a \((\mu - 1)\)-outtip \( t_{m}^{\mu-1} \) in \( v_{m+1}^\mu \)). As before, the direction of \( W_\mu \) conforms with the directions of its \( W_{m}^{\mu-1} \), that is, from left to right in (11).

Here, too, the \( \mu \)-diwalk may terminate on the left and/or on the right. That is, \( W_\mu \) may be of the form:

\[
W_\mu = \langle v_{m}^\alpha, W_{m}^{\mu-1}, v_{m+1}^\mu, W_{m+1}^{\mu-1}, v_{m+2}^{\mu}, \ldots \rangle
\]

where \( 0 \leq \alpha \leq \mu \) and \( W_{m}^{\mu-1} \) is incident inward at \( v_{m}^\alpha \) along an \((\alpha - 1)\)-intip. Similarly, we may also have

\[
W_\mu = \langle \ldots, v_{n-1}^\mu, W_{n-1}^{\mu-1}, v_{n}^\mu, W_{n}^{\mu-1}, v_{n+1}^{\beta} \rangle
\]

where \( 0 \leq \beta \leq \mu \) and \( W_{n}^{\mu-1} \) is incident outward at \( v_{n+1}^\beta \) along a \((\beta - 1)\)-outtip. Also possible as a special case is the three-element diwalk:

\[
W_\mu = \langle v_{1}^\alpha, W_{1}^\gamma, v_{2}^\beta \rangle
\]

where \( \gamma \geq \max(\alpha - 1, \beta - 1) \). Thus, diwalks of ranks lower than \( \mu \) are taken to be special cases of \( \mu \)-diwalks.

A trivial \( \mu \)-diwalk has just one element, an \( \alpha \)-vertex, where \( 0 \leq \alpha \leq \mu \).

The underlying \( \mu \)-graph \( G^\mu \) of \( D^\mu \) is obtained by removing the directions of all the arcs and thereby the directions of all the dipaths. Thus, each intip and outip becomes simply an (undirected) tip, each vertex becomes a node, and we get a \( \mu \)-graph as defined in [1, Section 2.2].

A \( \mu \)-semiwalk in \( D^\mu \) is defined as is a \( \mu \)-diwalk (11) except that now directions need not conform. That is, a tracing along a \( \mu \)-semiwalk may encounter arcs in opposite directions. This possible nonconformity in directions also holds for all the semiwalks of ranks less than \( \mu \) within \( D^\mu \). To state this another way, a \( \mu \)-semiwalk in \( D^\mu \) corresponds to a \( \mu \)-walk in the underlying \( \mu \)-graph \( G^\mu \) of \( D^\mu \). As was true with diwalks, a semiwalk of rank less than \( \mu \) is taken to be a special case of a \( \mu \)-semiwalk.

Two vertices \( u \) and \( v \) of ranks \( \alpha \) and \( \beta \) \((0 \leq \alpha, \beta \leq \mu)\) are said to be strongly \( \mu \)-dwconnected (resp. unilaterally \( \mu \)-dwconnected, resp. weakly \( \mu \)-swconnected) if there exists
a $\mu$-diwalk from $u$ to $v$ and another $\mu$-diwalk from $v$ to $u$ (resp. a $\mu$-diwalk from $u$ to $v$ or from $v$ to $u$ but not necessarily both, resp. a $\mu$-semiwalk terminating at $u$ and $v$). Here, too, the stated $\mu$-diwalk or $\mu$-semiwalk may in fact have a rank less than $\mu$.

A $\mu$-strong component is a maximal set of vertices such that every two of them are strongly $\mu$-dwconnected. As a special case, a single vertex may be comprise such a maximal set by itself. Fact: Every vertex lies in exactly on $\mu$-strong component. Note: Since a $\mu$-diwalk may in fact be a diwalk of lower rank, a $\mu$-strong component may actually be a $\gamma$-strong component where $0 \leq \gamma < \mu$.

A $\mu$-unilateral component is a maximal set of vertices such that every two of them are unilaterally $\mu$-dwconnected. The special case of a single vertex comprising a $\mu$-unilateral component does not arise now. Fact: Every vertex lies in at least one $\mu$-unilateral component. Note: Again a $\mu$-unilateral component may in fact be of lower rank.

A $\mu$-weak component is a maximal set of vertices such that every two of then are weakly $\mu$-swconnected. Fact: Every vertex lies in exactly one $\mu$-weak component. Note: A $\mu$-weak component may be of lower rank as well.

6 $\vec{\omega}$-digraphs

We now assume that our process for constructing $\mu$-digraphs can be continued for ever-increasing ranks $\mu$ through all the natural numbers. This yields a digraph $D^{\vec{\omega}}$ that has vertices for every natural-number ranks $\mu$:

$$D^{\vec{\omega}} = \{A, V^0, V^1, V^2, \ldots\}$$

(12)

$D^{\vec{\omega}}$ will be called an $\vec{\omega}$-digraph, where $\vec{\omega}$ is viewed as a rank that is larger than any natural number but precedes the first transfinite ordinal $\omega$ [2, pages 2 and 3]. $\vec{\omega}$ is the first of the "arrow ranks".

We define in $D^{\vec{\omega}}$ two kinds of one-ended diwalks; they are substantially different from the one-ended diwalks we have so far discussed and may or may not exist in $D^{\vec{\omega}}$. Assuming that they do, one kind is the one-ended $\vec{\omega}$-outdiwalk:

$$W_\omega^{\vec{\omega}} = \langle v_0^\mu, W_0^\mu, v_1^{\mu+1}, W_1^{\mu+1}, v_2^{\mu+2}, W_2^{\mu+2}, \ldots \rangle$$

(13)
where, for each natural number $k = 0, 1, 2, \ldots$, $v_{k}^{\mu+k}$ is a $(\mu + k)$-vertex, and $W_{k}^{\mu+k}$ is a one-ended, extended outward $(\mu + k)$-diwalk that has $v_{k}^{\mu+k}$ as its terminal vertex, is directed rightward, and reaches $v_{k+1}^{\mu+k+1}$ through a $(\mu + k)$-outtip, that is, $W_{k}^{\mu+k}$ is a representative of that $(\mu + k)$-outtip. Thus, $W_{\mu}^{3}$ is also directed toward the right and extends infinitely rightward.

In contrast to (13), we also have a one-ended $\vec{\omega}$-indiwalk:

$$W_{i}^{\vec{\omega}} = \langle \ldots, W_{-3}^{\mu+2}, v_{-2}^{\mu+2}, W_{-2}^{\mu+1}, v_{-1}^{\mu+1}, W_{-1}^{\mu}, v_{0}^{\mu} \rangle$$

In this case, for $k = \ldots, -3, -2, -1$, $W_{k}^{\mu-k-1}$ is a one-ended, extended inward $(\mu - k - 1)$-diwalk that has $v_{k+1}^{\mu-k-1}$ as its terminal vertex, is directed rightward, and reaches $v_{k}^{\mu-k}$ through a $(\mu - k - 1)$-intip, that is, $W_{k}^{\mu-k-1}$ is a representative of that $(\mu - k - 1)$-intip. Again, $W_{\mu}^{3}$ is directed rightward but now extends infinitely toward the left.

An endless $\vec{\omega}$-diwalk $W_{\mu}^{3}$ is obtained by joining these two walks at $v_{0}^{\mu}$.

$W_{\mu}^{3}, W_{\mu}^{i}$, and $W_{\mu}^{3}$ will themselves be called extended if their terms are all distinct except for possibly a finite number of terms.

An example of an $\vec{\omega}$-digraph that has no extended $\vec{\omega}$-diwalk consists of an infinite set of $\rho$-diwalks with $\rho = 0, 1, 2, \ldots$, with only one $\rho$-diwalk for each $\rho$, which share only one vertex, a 0-vertex, where they all meet.

Of course, and example of an $\vec{\omega}$-digraph having an extended $\vec{\omega}$-diwalk is any one of $W_{\mu}^{3}, W_{\mu}^{i}$, or $W_{\mu}^{3}$.

An $\vec{\omega}$-outtip is a maximal set of one-ended extended $\vec{\omega}$-outdiwalks that are pairwise eventually identical; by this we mean that the two one-ended $\vec{\omega}$ diwalks (as given by (13)) are identical except for a finite number of terms. Similarly, an $\vec{\omega}$-intip is a maximal set on one-ended, extended $\vec{\omega}$-indiwalks that are pairwise eventually identical (i.e., identical except for a finite number of terms in any two sequences like (14)). Any diwalk in an $\vec{\omega}$-outtip is called a representative of that $\vec{\omega}$-outtip, and similarly for a diwalk in an $\vec{\omega}$-intip. We will refer to both the $\vec{\omega}$-outtips and $\vec{\omega}$-intips simply as $\vec{\omega}$-ditips. We will use these $\vec{\omega}$-ditips to construct $\omega$-vertices in the next section.
7 \( \omega \)-digraphs

The next higher rank for transfinite digraphs is the rank \( \omega \), where \( \omega \) is the first infinite ordinal \([2, \text{Section 1.2}]\). Assuming now that we have at hand an \( \varnothing \)-digraph \( D^{\varnothing} \) having at least one \( \varnothing \)-ditip, let \( T^{\varnothing} \) be the set of all its \( \varnothing \)-ditips. Partition \( T^{\varnothing} \) arbitrarily to get \( T^{\varnothing} = \bigcup_{i \in I^{\varnothing}} T^{\varnothing}_i \), where \( I^{\varnothing} \) is the index set of the partition. Here, too, we change notation by setting \( v^{\varnothing}_i = T^{\varnothing}_i \) and call \( v^{\varnothing}_i \) an \( \omega \)-vertex. Thus, \( v^{\varnothing}_i \) is a set of \( \varnothing \)-ditips. The set \( V^{\varnothing} = \{v^{\varnothing}_i : i \in I^{\varnothing}\} \) denotes the set of all the \( \omega \)-vertices.

We can now state the definition of an \( \omega \)-digraph as follows:

\[
D^{\omega} = \{A, V^0, V^1, \ldots, V^{\omega}\}
\] (15)

where the ellipses indicate that vertex sets for all the natural numbers are included.

Then, we can define an \( \omega \)-diwalk \( W^{\omega} \) in \( D^{\omega} \) as the sequence:

\[
W^{\omega} = \langle \ldots, v^{\omega}_{m-1}, W^{\varnothing}_{m-1}, v^{\omega}_m, W^{\varnothing}_m, v^{\omega}_{m+1}, \ldots \rangle
\] (16)

just as we did for a \( \mu \)-diwalk in (11) except that \( v^{\omega}_m \) here replaces \( v^\mu_m \) there and \( W^{\varnothing}_m \) here replaces \( W^{\mu}_{m-1} \) there. In particular, if \( v^\mu_m \) and \( v^{\omega}_{m-1} \) both exist in (16), then \( W^{\varnothing}_m \) is an endless \( \varnothing \)-diwalk that reaches \( v^{\omega}_m \) through an \( \varnothing \)-intip and reaches \( v^{\omega}_{m+1} \) through an \( \varnothing \)-outtip.

In fact, all of the discussion of Section 5 can be transferred to this section so long as the natural-number rank \( \mu - 1 \) is replaces by the arrow rank \( \varnothing \) and \( \mu \) is replaced by \( \omega \). Because of this, we will not pursue this discussion of \( \omega \)-digraphs any further.

8 Transfinite digraphs of still higher order

With an \( \omega \)-digraph \( D^{\omega} \) in hand, we can define \( \omega \)-intips and \( \omega \)-outtips and, assuming these exist, can then define \( (\omega + 1) \)-vertices, and finally \( (\omega + 1) \)-digraphs:

\[
D^{\omega+1} = \{A, V^0, V^1, \ldots, V^{\omega}, V^{\omega+1}\}
\]

Proceeding recursively, we can construct, for \( k = 1, 2, \ldots, (\omega + k) \)-intips and \( (\omega + k) \)-outtips, and then \( (\omega + k + 1) \)-vertices and \( (\omega + k + 1) \)-digraphs:

\[
D^{\omega+k+1} = \{A, V^0, V^1, \ldots, V^{\omega}, V^{\omega+1}, \ldots, V^{\omega+k+1}\}
\]
After all the \((\omega + m)\)-ranks \((m = 0, 1, 2, \ldots)\) have been so traversed, we come to the next arrow rank \(\omega + \omega\), whose digraphs \(D^{\omega + \omega}\) are constructed in the same way as that explicated in Section 6 with the proviso that \(\omega\) is replaced by \(\omega + \omega\). Next in line are the \(\omega \cdot 2\)-digraphs whose construction mimics that of Section 7. We can then proceed onward through the ranks \(\omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots\), and indeed through still larger ordinal ranks.

Can we proceed through all the countable ordinals? The answer would be "yes" if we could develop a completely general recursive construction for an arbitrary countable ordinal rank and also for an arbitrary arrow rank preceding a countable limit-ordinal rank. But, this has yet to be done.

References

[1] A.H. Zemanian, Pristine Transfinite Graphs and Permissive Electrical Networks, Birkhauser, Boston, 2001.

[2] A.H. Zemanian, Graphs and Networks: Transfinite and Nonstandard, Birkhauser, Boston, 2004.

[3] A.H. Zemanian, Nonstandard Digraphs, ECE Technical Report 1, Department of Electrical Engineering, University at Stony Brook, Stony Brook, New York, April 15, 2009.