UNRAMIFIED COHOMOLOGY OF CLASSIFYING VARIETIES
FOR EXCEPTIONAL SIMPLY CONNECTED GROUPS

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Abstract. Let $BG$ be a classifying variety for an exceptional simple simply connected algebraic group $G$. We compute the degree 3 unramified Galois cohomology of $BG$ with values in $(\mathbb{Q}/\mathbb{Z})'(2)$ over an arbitrary field $F$. Combined with a paper by Merkurjev, this completes the computation of these cohomology groups for $G$ semisimple simply connected over all fields.

These computations provide another family of examples of simple simply connected groups $G$ such that $BG$ is not stably rational.

Let $G$ be an algebraic group over a field $F$ with an embedding $\rho: G \hookrightarrow SL_n$ over $F$. We call the quotient variety $SL_n/G$ a classifying space of $G$ and denote it by $BG$.

The goal of this paper is to compute the unramified cohomology of $BG$, defined as follows. Write $H^d(F)$ for the Galois cohomology group $H^d(\text{Gal}(F), (\mathbb{Q}/\mathbb{Z})'(d-1))$, where $(\mathbb{Q}/\mathbb{Z})'(d-1) = \varinjlim_n \mu_n^{\otimes (d-1)}$ for $n$ not divisible by the characteristic of $F$.

For each $d \geq 2$, define $H^d_{nr}(BG/F)$ (or simply $H^d_{nr}(BG)$) to be the intersection of the kernels of the residue homomorphisms

$$\partial_v: H^d(F(BG)) \rightarrow H^{d-1}(F(v))$$

as $v$ ranges over the discrete valuations of $F(BG)$ over $F$. The natural homomorphism $H^d_{nr}(BG) \rightarrow H^d_{nr}(BG/F)$ is split by evaluation at the distinguished point of $BG$; this gives a direct sum decomposition of $H^d_{nr}(BG)$, and we denote the complement of $H^d_{nr}(BG)$ by $H^d_{nr}(BG)_{\text{norm}}$. This group depends only on $G$ and $F$, and not on $\rho$ or $n$ [M02, 2.3].

This paper completes the computation of $H^3_{nr}(BG/F)_{\text{norm}}$ for $G$ semisimple simply connected and $F$ arbitrary. The computation is quickly reduced to the case where $G$ is simple simply connected [M02, §4]. In [M02], $H^3_{nr}(BG)_{\text{norm}}$ was computed for $G$ simple and classical. We compute it for the remaining cases, where $G$ is exceptional, that is, where $G$ is of type $G_2$, $3D_4$, $6D_4$, $F_4$, $E_6$, $E_7$, or $E_8$.

Main Theorem 0.2. Let $G$ be a simple simply connected exceptional algebraic group defined over a field $F$. Then

$$H^3_{nr}(BG)_{\text{norm}} = \begin{cases} \mathbb{Z}/2 & \text{if } \text{char } F \neq 2, G \text{ is of type } 3D_4, \text{ and } G \text{ has a nontrivial Tits algebra}, \\ 0 & \text{otherwise}. \end{cases}$$

(See §5.3 for an explanation of the characteristic $\neq 2$ hypothesis.) The general motivation for studying $H^3_{nr}(X)$ is that it can sometimes detect if $X$ is not stably rational; see [C] pp. 35–39. It was an open question whether $BG$ is

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stably rational for $G$ semisimple simply connected. The first counterexamples were provided in [M02], where Merkurjev exhibited groups $G$ of type $2A_n$, $2D_3$, and $1D_4$ with $H^3_{nr}(BG)_{norm} \neq 0$, hence with $BG$ not stably rational. The results here give another class of such $G$’s; see Corollary 7.2.

Our basic tool is that one can compute $H^3_{nr}(BG)_{norm}$ by inspecting the ramification of the Rost invariant of $G$; see [M02] or Lemma 5.2. Many questions are settled by hopping along the chain of inclusions

$$G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$$

of split groups; see §§6 and 5.5.

The most interesting part of the proof is where we show that the mod 4 portion of the Rost invariant is ramified for groups of type $2E_6$. We prove (in Theorem 3.1) that every isotropic trialitarian group embeds in a group of type $2E_6$ with trivial Tits algebras. This settles the question, since the mod 4 portion of the Rost invariant for groups of type $4D_4$ is easily shown to be ramified (Lemma 6.3). The proof of Theorem 3.1 uses Galois descent and interpretations of exceptional groups as acting on nonassociative algebras.

**Remark 0.4.** Computations of $H^d_{nr}(X/F)$ in the literature for $X$ a smooth variety (e.g., a classifying variety) typically assume that $F$ is algebraically closed. The examples of nonrational classifying varieties $BG$ provided here and in [M02] require that $F$ is not algebraically closed (e.g., $F = \mathbb{Q}$).

1. Vocabulary

An (affine) algebraic group is **simple** if it is $\neq 1$, is connected, and has no noncentral connected normal subgroups over an algebraic closure. (These groups are often called “absolutely almost simple”.) Simple groups are classified in, e.g., [KMRT Ch. VI]. We say that a group is of type $T_n$ if it is simple with root system of type $T_n$ and of type $tT_n$ if additionally the absolute Galois group of $F$ acts as a group of automorphisms of order $t$ on the Dynkin diagram.

Let $V$ be a finite-dimensional irreducible representation of an algebraic group $G$ over $F$. The $F$-algebra $\text{End}_G(V)$ is a skew field by Schur’s Lemma, and it is finite dimensional over $F$; it is called a **Tits algebra** for $G$. If it is a (commutative) field, we say that it is **trivial**.

The **Dynkin index** $n_G$ of a simple simply connected algebraic group $G$ is a natural number which depends only upon the type of $G$ and the (Schur) indices of its Tits algebras. The value of $n_G$ can be found in [M03] App. B, for example:

| type of $G$ | $3.5D_4$, all Tits alg’s trivial | $3.5D_4$, some Tits alg’s nontrivial | $F_4$ | $1E_6$ | $2E_6$ | $E_7$ | $E_8$ |
|-------------|---------------------------------|---------------------------------|-------|-------|-------|-------|-------|
| $n_G$       | 2                               | 6                               | 12    | 6     | 6     | 12    | 12    | 60    |

We have functors $H^1(\ast, G)$ and $H^3(\ast)$ which take a field extension of $F$ and give a pointed set and abelian group, respectively. A **degree 3 invariant** $\text{Inv}^3(G)$ of $G$ is a morphism of functors

$$H^1(\ast, G) \longrightarrow H^3(\ast)$$

which takes base points to base points. Such invariants are often called “normalized”. We write $\text{Inv}^3(G)$ for the abelian group of such invariants. (Clearly, this definition makes sense for every algebraic group $G$ over $F$.)
Let \( p = \text{char } F \) if the characteristic is prime and \( p = 1 \) otherwise. Write \( n_G = p^k n_G' \), where \( n_G' \) is a natural number prime to \( p \). The group \( \text{Inv}_3(G) \) is cyclic of order \( n_G' \). It has a canonical generator \( r_G \) which we call the Rost invariant of \( G \). (This is the prime-to-\( p \) part of what is called the Rost invariant in [M03]. I do not know how to define a residue map for the \( p \)-primary part.)

For \( \alpha : H \to G \) a map between simple simply connected algebraic groups over \( F \), there is a positive integer \( n_\alpha \) called the Rost multiplier or “Dynkin index” of \( \alpha \); see [M03, \S7]. It has the properties: \( n_H \) divides \( n_\alpha n_G \) and for every extension \( E \) of \( F \), the composition

\[
H^1(E, H) \xrightarrow{\alpha} H^1(E, G) \xrightarrow{r_G} H^3(E)
\]

is \( n_\alpha r_H \).

The standard inclusions in [L3] all have Rost multiplier 1. This observation goes back to [Dyn, p. 192]. Excepting \( G_2 \), this can be seen as follows: The inclusion of \( D_4 \) into \( E_8 \) arises from the natural inclusion of root systems. It is easy to check from the definition that the Rost multiplier is 1. Since multipliers are nonnegative integers and the Rost multiplier of a composition is the product of the Rost multipliers [M03, 7.9], the intervening Rost multipliers are all 1. An alternative, explicit argument for the inclusion \( F_4 \subset E_6 \) can be found in [Ga01, 2.4].

All inclusions of groups of type \( G_2, D_4, F_4, E_n \) occurring in this paper are conjugates of the standard inclusions of split groups over an algebraic closure of the base field. Since Rost multipliers are unchanged by field extensions [M03, 7.9.4], the multipliers for these inclusions are all 1.

2. \( A_2 \subset D_4 \)

2.1. In this section, we assume that \( F \) contains a primitive cube root of unity and hence has characteristic \( \neq 3 \). Let \( L \) be a cubic Galois extension of \( F \); by Kummer theory it is obtained by adjoining a cube root of some element \( \lambda \in F^* \). We write \((\lambda)\) for the corresponding class in \( F^*/F^{*3} = H^1(F, \mu_3) \), where \( \mu_3 \) is the algebraic group of cube roots of unity.

The short exact sequence \( 1 \to \mu_3 \to SL_3 \to PGL_3 \to 1 \) induces a connecting homomorphism \( \delta : H^3(F, PGL_3) \to H^3(F, \mu_3) \).

**Lemma 2.2.** Continue the hypotheses of 2.1. Let \( G \) be the quasi-split simply connected group of type \( ^3D_4 \) associated with the extension \( L/F \). Then \( G \) contains a subgroup isomorphic to \( PGL_3 \) such that for every extension \( E \) of \( F \) the diagram

\[
\begin{array}{ccc}
H^1(E, PGL_3) & \xrightarrow{x \mapsto \delta(x)(\lambda)} & H^3(E, \mu_3^2) \\
\downarrow & & \downarrow \\
H^1(E, G) & \xrightarrow{r_G} & H^3(E)
\end{array}
\]

commutes up to sign, where the arrow on the right comes from the natural map \( \mu_3^2 \to (\mathbb{Q}/\mathbb{Z})/(2) \).

**Proof.** We have maps

\[
PGL_3 \times \mu_3 \longrightarrow \text{Spin}_8 \times \mathbb{Z}/3 \longrightarrow F_4,
\]

where \( F_4 \) denotes the split algebraic group of that type and we identify \( \mu_3 \) with \( \mathbb{Z}/3 \) using the primitive cube root of unity in \( F \). The first map comes from the fact that \( PGL_3 = \text{Aut}(M_3(F)) \) preserves the subspace of trace zero elements in \( M_3(F) \); see
The second map comes from the Springer decomposition of Albert algebras; see [KMRT, 38.7]. The group $\mathbb{Z}/3$ acts on $\text{Spin}_8$ in a manner which cyclically permutes the vector and half-spin representations and fixes $\text{PGL}_3$ elementwise. The map $\text{Spin}_8 \to F_4$ has Rost multiplier 1.

When we twist the groups in (3.3) by $(\lambda)$ and restrict to connected components, we obtain the sequence

$$\text{PGL}_3 \xrightarrow{\epsilon} G \xrightarrow{\sigma} F_4.$$

The map $H^1(\sigma_t)$ sends a class $[A] \in H^1(E, \text{PGL}_3)$ of a central simple $E$-algebra of degree 3 to the class of the first Tits construction $[J(A, \lambda)] \in H^1(F, F_4)$; see [KMRT, 39.9]. The composition $r_{F_4} \circ H^1(\sigma_t)$ is, up to sign, the composition of $\delta$ with the cup product $\cdot \cup$ by $[\text{KMRT}]$ p. 537. Since $\sigma$ has Rost multiplier 1, $r_{F_4} \circ H^1(\sigma) = r_G$, and the lemma is proved.

3. $3D_4, 6D_4 \subset 2E_6$

In this section, we assume that $F$ has characteristic $\neq 2$. A simple algebraic group is said to be trialitarian if it is of type $3D_4$ or $6D_4$. It is well known that every quasi-split simply connected trialitarian group is a subgroup of the split $F_4$, hence of every simply connected quasi-split $E_6$. In this section, we prove:

**Theorem 3.1** (char $F \neq 2$). Let $T$ be a trialitarian simply connected group over $F$ which is $F$-isotropic but not $F$-quasi-split. Let $K$ be a quadratic extension of $F$ such that $T$ is $K$-quasi-split. Then there exists a simply connected group $G$ of type $2E_6$ over $F$ such that

1. all of $G$’s Tits algebras are trivial;
2. $G$ is of type $1E_6$ over $K$; and
3. $T$ is a subgroup of $G$ with Rost multiplier 1.

Given a $T$ as in the first sentence of Theorem 3.1, such a $K$ always exists by [Ga98, 0.1]. We postpone the proof of the theorem until the end of this section.

3.2. Let $\mathcal{C}$ be the split Cayley algebra over $F$ with canonical involution (also known as conjugation) $\pi_\mathcal{C}$. Fix the basis $u_1, \ldots, u_8$ of $\mathcal{C}$ as in [Ga98] and [Ga01] so that the bilinear norm form $n(x, y) = x\pi_\mathcal{C}(y) + y\pi_\mathcal{C}(x)$ is given by $n(u_i, u_j) = \delta_{(i+j),9}$ (Kronecker delta). Write $\sigma$ for the involution on $GL(\mathcal{C})$ which is adjoint for $n$.

Let $R$ denote the subgroup of $GL(\mathcal{C})^{\times 2}$ consisting of so-called related triples of proper similitudes of $n$; see [Ga98] §1 or [KMRT] §35 for a definition. This group is reductive with center of rank 2; its derived subgroup consists of triples $t = (t_0, t_1, t_2)$ with $t_i \in SO(n)$ for all $i$ and is isomorphic to $\text{Spin}_8$ [KMRT] 35.7.

The group $S_3 = \langle r, \pi \mid r^3 = \pi^2 = 1, \pi r = r^2 \pi \rangle$ acts on $R$ via

$$r_t = (t_1, t_2, t_0) \quad \text{and} \quad \pi (t_0, t_1, t_2) = (\pi t_0 \pi t_1 \pi t_2, \pi t_1 \pi t_2, \pi t_2 \pi t_1).$$

Define $R \rtimes S_3$ to be the Cartesian product $R \times S_3$ with multiplication

$$(t, \alpha) \cdot (t', \beta) = (t \cdot ^t \alpha t', \alpha \beta).$$

The split Albert algebra $J$ has as an underlying vector space the matrices in $M_3(\mathcal{C})$ fixed by the conjugate transpose. With that in mind, we may write a general element of $J$ as in (3.3) below where $\varepsilon_i \in F$, $c_i \in \mathcal{C}$, and the entries given as $\cdot$ are forced by symmetry. The algebra $J$ has a canonically determined norm form; write $\text{Inv}(J)$ for the group of norm isometries.
There is an injection $g: R \times S_3 \hookrightarrow \text{Inv}(J)$ defined by

$$g_L\left(\begin{array}{ccc}
\varepsilon_0 & c_2 & \varepsilon_1 \\
c_1 & \varepsilon_0 & \varepsilon_2 \\
0 & 0 & 1
\end{array}\right) = \left(\begin{array}{ccc}
\mu(t_0)^{-1} & t_2(c_2) & t_0(c_2) \\
\mu(t_1)^{-1} & t_0(c_0) & t_1(c_0) \\
\mu(t_2)^{-1} & \varepsilon_1 & \varepsilon_2
\end{array}\right),$$

$$g_r\left(\begin{array}{ccc}
\varepsilon_0 & c_2 & \varepsilon_1 \\
c_1 & \varepsilon_0 & \varepsilon_2 \\
0 & 0 & 1
\end{array}\right) = \left(\begin{array}{ccc}
\varepsilon_1 & c_0 & \varepsilon_2 \\
0 & 0 & 1 \\
c_2 & \varepsilon_0 & 1
\end{array}\right),$$

and $g_\pi\left(\begin{array}{ccc}
\varepsilon_0 & c_2 & \varepsilon_1 \\
c_1 & \varepsilon_0 & \varepsilon_2 \\
0 & 0 & 1
\end{array}\right) = \left(\begin{array}{ccc}
\varepsilon_0 & \pi \varepsilon c_1 & \varepsilon_2 \\
\pi \varepsilon c_2 & \varepsilon_0 & 1
\end{array}\right).$

Here $\mu(t_i)$ denotes the factor by which $t_i$ scales the norm $n$. That is, $n(t_ix, t_iy) = \mu(t_i)n(x, y)$ for all $x, y \in \mathbb{C}$.

### 3.3. Construction of a quasi-split $^{2}E_6$. The algebra $J$ is also endowed with a nondegenerate symmetric bilinear form $s$ defined by

$$s(x, y) = \text{Tr}_J(xy) = \sum_{i=0}^{2} [\varepsilon_i \mu_i + n(c_i, d_i)]$$

for

$$x = \left(\begin{array}{ccc}
\varepsilon_0 & c_2 & \varepsilon_1 \\
c_1 & \varepsilon_0 & \varepsilon_2 \\
0 & 0 & 1
\end{array}\right), \quad y = \left(\begin{array}{ccc}
v_0 & d_2 & \varepsilon_1 \\
v_1 & 0 & \varepsilon_0 \\
d_1 & d_2 & 1
\end{array}\right).$$

For each $f \in GL(J)$, there is a unique $f^\dagger \in GL(J)$ such that $s(f(x), f^\dagger(y)) = s(x, y)$ for all $x, y \in J$.

The map $f \mapsto f^\dagger$ restricts to automorphisms of $\text{Inv}(J)$ and $R \times S_3$ defined over $F$. We have

$$r^\dagger = r, \quad \pi^\dagger = \pi, \quad \text{and} \quad t^\dagger = (\sigma(t_0)^{-1}, \sigma(t_1)^{-1}, \sigma(t_2)^{-1}).$$

Let $i$ be a generator for $\text{Gal}(K/F)$. We define the groups $E_6^K$ and $H$ to be the groups $\text{Inv}(J)$ and $R \times S_3$ with twisted $i$-actions. For $f$ a $K$-point, we set $\tilde{f} = if^\dagger t$ where the action on the left is the new action and juxtaposition denotes the usual action. The group $E_6^K$ is quasi-split of type $^{2}E_6$.

We consider $E_6^K$ as a closed subgroup of $GL(V)$ as follows. Let $V$ be the $F$-subspace of $(J \otimes K) \oplus (J \otimes K)$ fixed by the $i$-semilinear automorphism

$$(j \otimes k, j' \otimes k') \mapsto (j' \otimes \iota(k'), j \otimes \iota(k)).$$

The homomorphism

$$\text{Inv}(J) \to GL(J \otimes J) \quad \text{given by} \quad f \mapsto f \oplus f^\dagger$$

defines an injective $F$-homomorphism $E_6^K \to GL(V)$. It is the usual 54-dimensional representation of $E_6^K$; it is irreducible over $F$.

### 3.5. Proof of Theorem 3.1

Since $T$ is isotropic and not quasi-split, it has a Tits algebra which is a non-split quaternion algebra $Q$ over a cubic extension $L$ of $F$ [Ga08 0.1]. Put $K = F(\sqrt{b})$. Since $T$ is $K$-quasi-split, $Q$ is split by $L(\sqrt{b})$, hence $Q$ is of the form $(a, b)_L$ for some $a \in L^\times$ such that $N_{L/F}(a) = 1$ [KMRT 43.9]. Since $Q$ is not split over $L$, it is not split over the normal closure $L^c$ of $L/F$ [Ga08 3.2]. In particular, $L^c$ does not contain a square root of $b$, so $P = K \otimes_F L^c$ is a quadratic field extension of $L^c$.

To simplify our argument, we assume that $L$ is not Galois over $F$, so $\text{Gal}(L^c/F)$ is isomorphic to $S_3$. (This is the case that will be used in the rest of the paper. The other case — where $L$ is Galois over $F$ — is only easier.) Then, the group $\text{Gal}(P/F)$ is isomorphic to $S_3 \times \mu_2 \cong \mathbb{Z}/6 \times \mu_2$, where the factor of $\mathbb{Z}/6$ corresponds to the subgroup $\text{Gal}(P/\Delta)$ for $\Delta$ the unique quadratic extension of $F$ in $L^c$. We fix
generators $\zeta := (r^{-1}, -1) \in S_3 \times \mu_2$ (which generates the copy of $\mathbb{Z}/6$) and $\tau = (\pi, 1)$ (which generates the copy of $\mu_2$ in $\mathbb{Z}/6 \times \mu_2$ corresponding to $\text{Gal}(P/L(\sqrt{6}))$).

We construct the group $G$ by descent as follows. We identify $E_6^K$ — and hence $H \times S_3$ — with a closed subgroup of $GL(V)$ as in (3.3). We call an additive homomorphism $f : V \otimes P \to V \otimes P$ \emph{\textbackslash{}semilinear} if there is some $\varpi \in \text{Gal}(P/F)$ such that

$$f(pv) = \varpi(p)f(v) \quad \text{for all } p \in P \text{ and } v \in V \otimes P.$$ 

Let $\widetilde{GL}(V)$ denote the (abstract) group of such maps $f$. We define a group homomorphism $\phi : \text{Gal}(P/F) \to \widetilde{GL}(V)$ such that $\phi(\varpi)$ is \textbackslash{}semilinear for all $\varpi$ and $T(F)$ and $G(F)$ are the subgroups of Spin$_8(P)$ and $E_6^K(P)$ commuting with $\phi(\varpi)$ for all $\varpi$.

Define $\mathfrak{t} = (t_0, t_1, t_2) \in GL(3)$ by setting $t_i = m_i P$ for $m_i = \text{diag}(1, \rho^i(a), -\rho^i(a), \rho^{i+2}(a)^{-1}, \rho^{i+1}(a)^{-1}, -1, 1, \rho^i(a))$ with $\rho := \zeta^2$, and $P$ the matrix permuting the basis vectors as $(12)(36)(45)(78)$, for the basis of $\mathfrak{c}$ fixed in (3.2) above. Since $N_{L/F}(a) = 1$, $\mathfrak{t}$ is a related triple by [Ga01, 1.5(3), 1.6, 1.8]. Set

\begin{equation}
\phi(\zeta) = t_\mathfrak{t} \zeta \quad \text{and} \quad \phi(\tau) = \pi \tau.
\end{equation}

We have

\begin{equation}
\zeta \mathfrak{t}^{-1} = (\sigma(\zeta(t_0)), \sigma(\zeta(t_1)), \sigma(\zeta(t_2)))^{-1} = (\sigma(t_0), \sigma(t_2), \sigma(t_1))^{-1} = r^{-1} \mathfrak{t}^{-1} r,
\end{equation}

since $\sigma(t_i) = t_i$ for all $i$. Hence

\begin{equation}
\phi(\zeta)^2 = r^2 \zeta^2, \quad \phi(\zeta)^3 = \mathfrak{t}^3, \quad \text{and} \quad \phi(\zeta)^6 = \text{Id}_{V \otimes P}.
\end{equation}

Since $\pi$ and $\tau$ commute, we have

\begin{equation}
\phi(\tau)^2 = \text{Id}_{V \otimes P}.
\end{equation}

Since

\begin{equation}
\phi(\zeta)^5 = \phi(\zeta)^2 \cdot \phi(\zeta)^3 = \mathfrak{t}^2 \zeta^5,
\end{equation}

and $\pi \tau$ and $\mathfrak{t}$ commute, it is easy to verify that

\begin{equation}
\phi(\tau) \phi(\zeta) = \phi(\zeta)^5 \phi(\tau).
\end{equation}

Equations (3.7)–(3.9) give that (3.6) defines a homomorphism $\phi : \text{Gal}(P/F) \to \widetilde{GL}(V)$. Then the set map $z : \text{Gal}(P/F) \to H \times S_3$ defined by $z_\varpi := \phi(\varpi) \varpi^{-1}$ is in fact a 1-cocycle. (This correspondence between groups of semilinear transformations and 1-cocycles is well-explained in [Jac, §3] for $z$.) Set $G$ to be the twisted group $(E_6^K)_z$; it automatically satisfies (2). Since $z$ takes values in the simply connected group $E_6^K$, (1) holds.

Since the values of $z$ normalize the subgroup Spin$_8$ of $E_6^K$, the twisted group $(\text{Spin}_8)_z$ is a subgroup of $G$. The inclusion has Rost multiplier 1 since the inclusion Spin$_8 \subset E_6^K$ over an algebraic closure arises from the natural inclusion of root systems $D_4 \subset E_6$. The restriction of $z$ to Spin$_8$ is the descent given in [Ga98, 4.7] to construct $T$, i.e., $(\text{Spin}_8)_z$ is isomorphic to $T$, hence (3).

\begin{remark}
3.10. The isotropic group $G$ occurring in Theorem 3.4 is typically not quasi-split, even over $L$. This can be seen by examining the mod 2 portion of the Rost invariant for $(z) \in H^1(P/L, E_6^K)$, which is typically nontrivial by [Ga01, 6.7].
\end{remark}
4. A CONSTRUCTION

The purpose of this section is to construct a suitable extension of $F$ over which we may apply Theorem 3.4.

**Proposition 4.1.** Let $F$ be a field of characteristic $\neq 2$, and let $K$ be a quadratic extension of $F$. There is a regular extension $E$ of $F$ and a group $T$ of type $^4D_4$ over $E$ such that $T$ is $E$-isotropic and $(E \otimes F)K$-quasi-split, but not $E$-quasi-split.

Presumably one could prove Proposition 4.1 by applying [M99] to produce a group $T_0$ of type $^4D_4$ whose Tits algebras have index 8 and then extending scalars to function fields of transfers of generalized Severi-Brauer varieties so that the Tits algebras of $T_0$ have index 2 and are split by a quadratic extension of the base field. Then by [Ga98] there is a group $T$ as in Proposition 4.1 with the same Tits algebras as in $T_0$. We give a low-tech argument here.

**Lemma 4.2** (char $F \neq 2$). For $p, q \in F^*$, the ring $L = F(t)[x]/(x^3 + px + qt)$ is a separable cubic field extension of $F(t)$ which is regular over $F$ and not Galois over $F(t)$. There is a prolongation of the $t$-adic valuation on $F(t)$ to $L$ which is unramified with residue degree 1 and with respect to which $x$ has value 1.

**Proof.** If $L$ is not a field, then there is some $a \in F(t)$ such that $a^3 + pa + qt = 0$. Since $a$ is integral over the unique factorization domain $F[t]$, it belongs to $F[t]$, so it makes sense to speak of the degree of $a$. In particular, at least two of the terms $a^3, pa, qt$ must have the same degree, which is also the maximum of the degrees. This implies that $a$ cannot have positive degree. But then $qt$, with degree 1, is the unique term of maximal degree, which is a contradiction.

Since $p, q$ are in $F^*$, the discriminant $-4p^3 - 27q^2t^2$ of $L$ is not 0, hence $L$ is separable over $F$. An argument similar to the one in the preceding paragraph shows that the discriminant is not a square in $F(t)$: Any square root $b \in F(t)$ of the discriminant would belong to $F[t]$ and have degree 1. Then the coefficient of $t$ in $b^2$ would be nonzero. Thus $L$ is not Galois over $F(t)$. If $\ell \in L \setminus F(t)$ is algebraic over $F$, then $L$ is generated by $\ell$ as an $F(t)$-algebra and the discriminant of the extension $L/F(t)$ comes from $F^*/F(t)^2$, which is a contradiction. Hence $F$ is algebraically closed in $L$ and $L$ is regular over $F$.

Hensel’s Lemma gives that $x^3 + px + qt$ has a linear factor of the form $x - \pi$ in $F((t))[x]$, where $\pi$ has $t$-adic value 1. The map $x + \pi$ gives an isomorphism of $L$ with the subfield $F(t)/(\pi)$ of $F((t))$, and the $t$-adic valuation obviously extends to $L$ so that $x$ has value 1. Since $F((t))$ is the completion of $F(t)$ with respect to the $t$-adic valuation and hence is unramified with residue degree 1, the claims about ramification and residue degree of our prolongation to $L$ follow.

**Lemma 4.3** (char $F \neq 2$). Let $p, b \in F^*$ be such that the quaternion algebra $(p, b)_F$ is nonsplit. Let $L$ be as in Lemma 4.2. Then the quaternion algebra $(x, b)_L$ is nonsplit and is not isomorphic to $(-qt, b)_L$.

**Proof.** Since $N_{L/F(t)}(x) = -qt$, the corestriction of $(x, b)_L$ down to $F(t)$ is Brauer-equivalent to $(-qt, b)_F(t)$. This algebra is split if and only if the quadratic form $\langle 1, -b, qt \rangle$ is isotropic over $F(t)$. Over the completion $F((t))$, this form has residue forms $\langle 1, -b \rangle$ and $\langle q \rangle$. Since the algebra $(p, b)_F$ is nonsplit, the first form is anisotropic, hence $\langle 1, -b, qt \rangle$ is anisotropic over $F((t))$ by Springer’s Theorem. Thus $(-qt, b)_F(t)$ is nonsplit, and hence so is $(x, b)_L$. 


For the sake of contradiction, suppose that \((x, b)_L\) is isomorphic to \((-qt, b)_L\), i.e., the algebra \((-xqt, b)_L\) is split. Since 
\[-x(qt) = -x(-x^3 - px) = x^4 + px^2 \equiv x^2 + p \pmod{L^{1/2}},\]
the algebra \((x^2 + p, b)_L\) is split.

Let \(\tilde{L}\) be a completion of \(L\) with respect to the prolongation of the \(t\)-adic valuation on \(F(t)\) given by Lemma 4.2. The norm of \((x^2 + p, b)_L\) is the form \(\langle 1, -(x^2 + p), -b, b(x^2 + p) \rangle \) over \(L\). Since \(x\) has value 1, over \(\tilde{L}\) this form has one residue form \((1, -p, -b, bp)\) over the residue field \(F\). This is the norm of the algebra \((p, b)_F\), which is anisotropic because the algebra is nonsplit. By Springer’s Theorem, the norm of \((x^2 + p, b)_L\) is anisotropic over \(\tilde{L}\), hence the algebra is not \(L\)-split, which contradicts our assumption that \((x, b)_L\) is isomorphic to \((-qt, b)_L\). \(\square\)

4.4. Proof of Proposition 4.5. Write \(K = F(\sqrt{b})\). Let \(F_0 := F(p, t)\) for \(p, t\) indeterminates. Set \(L_0 := F_0[x]/(x^3 + px + t)\) as in Lemma 4.3. Set \(E\) to be the function field of the Severi-Brauer variety of the quaternion algebra \((t, b)_{F_0}\). Since \(E\) is regular over \(F_0\), \(L := L_0 \otimes_{F_0} E\) is a field which is cubic and not Galois over \(E\); it is the function field of the Severi-Brauer variety of \((t, b)_{L_0}\).

Since \(b\) is not a square in \(E\), the quaternion algebra \((p, b)_{F_0}\) is not split. By Lemma 4.3 \((x, b)_{L_0}\) is nonsplit and is not isomorphic to \((-t, b)_{L_0}\), hence \((x, b)_L \cong (x, b)_{L_0} \otimes_{L_0} L\) is not split by a well-known theorem of Amitsur. The corestriction \(\text{cor}_{L/E}(x, b)_L\) is \((-t, b)_E\), which is split. Thus there is a simply connected \(E\)-isotropic — but not \(E\)-quasi-split — trialitarian group \(T\) over \(E\) with nontrivial Tits algebra \((x, b)_L\) [Ga98, 4.7]. It is of type \(\text{D}_4\) since \(L\) is not Galois over \(E\). It is \(E(\sqrt{b})\)-quasi-split since \((x, b)\) is split over \(L(\sqrt{b})\) [Ga98, 5.6]. \(\square\)

5. Ramification

Let \(G\) be an algebraic group over \(F\). We say that an invariant \(a \in \text{Inv}^3(G)\) is unramified if the composition
\[H^1(E((t)), G) \xrightarrow{a} H^3(E((t))) \xrightarrow{\partial} H^2(E)\]
is 0 for every field extension \(E\) of \(F\). Otherwise we say that \(a\) is ramified. The following example is typical:

Example 5.1. Let \(F\) be a field with a primitive cube root of unity, and let \(L = F(\lambda^{1/3})\) be a cubic Galois extension of \(F\). We claim that the invariant \(a \in \text{Inv}^3(PGL_3)\) given by \(a(x) = \delta(x) \cup (\lambda)\) is ramified.

The set \(H^1(F, PGL_3)\) classifies degree 3 cyclic central simple algebras \((C, d)\) for \(C\) a cubic Galois extension of \(F\) and \(d \in F^*\). Let \([C] \in H^1(F, \mathbb{Z}/3)\) denote the class corresponding to \(C\). Then \(\delta(C, d) \cup (\lambda) = \pm[C] \cup (d) \cup (\lambda)\) in \(H^3(F, \mu_{2\mathbb{Z}}^2)\). Taking \(E = F(u)\) for \(u\) an indeterminate and \(C = E((t))(t^{1/3})\) a cubic Galois extension of \(E((t))\), we have
\[\partial[\delta(C, u) \cup (\lambda)] = \pm(u) \cup (\lambda) \in H^2(E)\]
This is nonzero in \(H^2(E)\) since \(u\) is not a norm from the extension \(E(\lambda^{1/3})/E\). Hence \(a\) is ramified, as claimed. \(\square\)

For the rest of the section, we assume that \(G\) is simple and simply connected. We write \(\text{Inv}^3_{\text{nr}}(G)\) for the subset of unramified invariants in \(\text{Inv}^3(G)\). It is a subgroup since \(\partial\) is a group homomorphism.
Lemma 5.2 ([M02]). \( H^3_{\text{nr}}(BG)_{\text{norm}} \cong \text{Inv}^3_{\text{nr}}(G) \). □

In particular, \( H^3_{\text{nr}}(BG)_{\text{norm}} \) is necessarily finite; see [11]

5.3. If \( F \) has prime characteristic \( p \), then multiplication by \( p \) is an isomorphism of \( (\mathbb{Q}/\mathbb{Z})'(2) \). Hence \( H^3(F), \text{Inv}^3(G) \), and — by Lemma 5.2 — \( H^3_{\text{nr}}(BG/F)_{\text{norm}} \) have no nontrivial \( p \)-torsion.

This explains the hypothesis “\( \text{char} \ F \neq 2 \)” in the Main Theorem: a simply connected group \( G \) of type \( ^3D_4 \) with nontrivial Tits algebras “should” have \( H^3_{\text{nr}}(BG)_{\text{norm}} \) equal to \( \mathbb{Z}/2 \), but this is impossible in characteristic 2.

Strongly Inner Lemma 5.4. Let \( G \) be a simple simply connected group over \( F \), and fix \( z \in Z^1(F,G) \). The canonical identification \( \text{Inv}^3(G) = \text{Inv}^3(G_z) \) defined by \( r_G \mapsto r_{G_z} \) restricts to an identification \( \text{Inv}^3_{\text{nr}}(G) = \text{Inv}^3_{\text{nr}}(G_z) \).

Groups \( G_z \) obtained by twisting \( G \) by a 1-cocycle \( z \) as in the lemma are called strongly inner forms of \( G \). Such groups have the same type and Tits algebras as \( G \), hence also the same Dynkin index as \( G \). In particular, \( n'_{G_z} = n'_{G} \).

Proof. Let \( E \) be an extension of \( F \), and consider the diagram

\[
\begin{array}{ccc}
H^1(E((t)), G) & \xrightarrow{mr_G} & H^3(E((t))) \\
\tau_z \cong & -mr_G(z) & \downarrow \quad \downarrow \\
H^1(E((t)), G_z) & \xrightarrow{mr_{G_z}} & H^3(E((t))) \\
\end{array}
\]

where \( \tau_z \) is the twisting isomorphism and \( m \) is an integer. The left box commutes by [G00] p. 76, Lem. 7 or [MPT], 1.7. The right box commutes because \( \partial \) is a group homomorphism and \( \partial(r_{G}(z)) = 0 \). Hence \( mr_G \) is ramified if and only if \( mr_{G_z} \) is. □

5.5. Functoriality (homomorphisms). Let \( \alpha : H \to G \) be a morphism of algebraic groups. Then \( \alpha \) induces natural homomorphisms

\[ \alpha^* : \text{Inv}^3(G) \to \text{Inv}^3(H) \quad \text{and} \quad \alpha_{\text{nr}}^* : \text{Inv}^3_{\text{nr}}(G) \to \text{Inv}^3_{\text{nr}}(H). \]

Now suppose that \( H \) and \( G \) are simple simply connected. If the Rost multiplier of \( \alpha \) is 1, then \( n_H \) divides \( n_G \), hence \( n'_{G}/n'_{H} \) divides \( n'_{G}/n'_{H} \). (See [11] for definitions.) Also, \( \alpha^* \) is a surjection with kernel of order \( n'_{G}/n'_{H} \). Then we have: If \( \text{Inv}^3_{\text{nr}}(H) \) is trivial, then \( \text{Inv}^3_{\text{nr}}(G) \) is \( (n_G/n_H)\)-torsion.

5.6. Functoriality (scalar extension). Let \( K \) be an extension field of \( F \), and write \( G_K \) for \( G \times_F K \). The restriction homomorphism

\[ \text{res}_{K/F} : \text{Inv}^3(G/F) \to \text{Inv}^3(G/K) \]

is the natural surjection \( \mathbb{Z}/n'_{G} \to \mathbb{Z}/n'_{G_K} \); its kernel is the \( (n'_G/n'_{G_K})\)-torsion in \( \text{Inv}^3(G/F) \). It restricts to a homomorphism

\[ (\text{res}_{K/F})_{\text{nr}} : \text{Inv}^3_{\text{nr}}(G/F) \to \text{Inv}^3_{\text{nr}}(G/K). \]

The kernel of this map is killed by \( n'_{G}/n'_{G_K} \), hence by \( n_G/n_G \). We have: If \( \text{Inv}^3_{\text{nr}}(G/K) \) is trivial, then \( \text{Inv}^3_{\text{nr}}(G/F) \) is \( (n_G/n_{G_K})\)-torsion.
6. The case where $G$ has trivial Tits algebras

In this section, we prove:

**Proposition 6.1.** Let $G$ be a simple simply connected exceptional algebraic group. If $G$ has only trivial Tits algebras, then $\text{Inv}_{nr}^3(G) = 0$.

That is, the Main Theorem holds for groups with only trivial Tits algebras by Lemma 5.2. In proving the proposition, we may assume that $G$ is quasi-split by the Strongly Inner Lemma 5.4.

The proposition still holds if the hypothesis “exceptional” is dropped; the classical groups are treated in [M02].

**6.2. Type $3D_4$.** Let $L$ be a separable cubic extension of $F$, such that the quasi-split group $G$ is of type $3D_4$ over $L$. Since $G$ is quasi-split, $n_{G_L} = 2$ and $\text{Inv}_{nr}^3(G/L) = 0$ by [M02] 8.5. Hence $\text{Inv}_{nr}^3(G/F) = 3$-torsion by 3.6. By 5.3 we may assume that char $F \neq 3$.

Let $F'$ be the extension obtained from $F$ by adjoining (if not already in $F$) a primitive cube root of unity. Then $G$ is still of type $3D_4$ over $F'$ and the invariants $2r_G$ and $4r_G$ are ramified over $F'$ by Lemma 2.2 and Example 6.1. Since $2r_G$ and $4r_G$ are the only nontrivial 3-torsion elements of $\text{Inv}^3(G/F)$, we have shown that $\text{Inv}_{nr}^3(G/F) = 0$.

**Lemma 6.3.** Let $G$ be simply connected of type $6D_4$ (no restriction on the Tits algebras). Then $\text{Inv}_{nr}^3(G) = 0$.

**Proof.** Let $\Delta$ be the unique quadratic extension of the base field $F$ over which $G$ is of type $3D_4$. Let $K$ be a generic quasi-splitting field for $G$ over $\Delta$ as in [KR]. Then $G$ is quasi-split of type $3D_4$ over $K$, hence $n_{G_K} = 6$ and $\text{Inv}_{nr}^3(G/K) = 0$ by the $3D_4$ case (6.2). We have that $\text{Inv}_{nr}^3(G/F)$ is 2-torsion by 3.6.

Let $L$ be a cubic extension of $F$ over which $G$ is of type $2D_4$. Then $n_{G_L} = 2$ or 4 and $\text{Inv}_{nr}^3(G/L) = 0$ by [M02] 8.5. Since $n_G = 6$ or 12 (as $n_{G_L} = 2$ or 4), $\text{Inv}_{nr}^3(G/F) = 3$-torsion.

Combining the two previous paragraphs, we find that $\text{Inv}_{nr}^3(G/F) = 0$. 

**6.4. Type $G_2$.** Here $n_G = 2$, so we may assume that char $F \neq 2$. The Rost invariant for the split group of type $G_2$ is given explicitly in [KMRT] p. 441. It is the Elman-Lam invariant for 3-Pfister quadratic forms, which is clearly ramified.

**6.5. Type $F_4$.** The split $G_2$ is contained in our split group $G$ of type $F_4$ with Rost multiplier 1. (See the end of [14].) We omit this sort of observation below.) Since $n_{F_4} = 6$ and $n_{G_2} = 2$, the group $\text{Inv}_{nr}^3(G)$ is 3-torsion by 6.4 and the $G_2$ case (6.4). In characteristic $\neq 3$, the mod 3 part of the Rost invariant is described in [PR] 3.2, and it is clearly ramified. So $\text{Inv}_{nr}^3(G) = 0$.

**6.6. Type $^{1}E_6$.** The split group $G$ of type $E_6$ contains a subgroup which is split of type $F_4$ ([Sp] 14.20, 14.24), and we have $n_{F_4} = n_G = 6$. Hence $\text{Inv}_{nr}^3(G) = 0$ by [5.3] and the $F_4$ case (6.5).

**6.7. Type $^2E_6$.** The group $G$ is split by a quadratic extension and $n_G = 12$. By 5.6 and the $^1E_6$ case (6.6), $\text{Inv}_{nr}^3(G)$ is 2-torsion, hence we may assume that char $F \neq 2$.

Let $T$ and $E$ be as in Proposition 4.1. By Theorem 5.1 $T$ is contained in a strongly inner form $G'$ of $G$ over $E$. By Lemma 6.3 $\text{Inv}_{nr}^3(T/E) = 0$. Since $T$ has...
a nontrivial Tits algebra over $E$, the Dynkin index $n_T$ is 12. Since $n_{E_f}$ is also 12, we have $\text{Inv}^{3}_{\text{nr}}(G'/E) = 0$. By the Strongly Inner Lemma, $\text{Inv}^{3}_{\text{nr}}(G/E) = 0$. Since $G$ is of type $^2E_6$ over $F$ and $E$, we have $n_G = n_{GE}$, hence $\text{Inv}^{3}_{\text{nr}}(G/F) = 0$ by §5.6.

6.8. Type $E_7$. The natural inclusion of root systems gives a split simply connected subgroup of type $E_6$ inside the split group $G$ of type $E_7$. Since $n_G = 12$ and $n_{E_6} = 6$, $\text{Inv}^{3}_{\text{nr}}(G)$ is 2-torsion by §5.5 and the $^1E_6$ case (§6.6). Hence we may assume that $\text{char } F \neq 2$.

Set $F' = F(x)$ and $K = F'(\sqrt{2})$. There is a quasi-split simply connected group $E_6^K$ over $F$ of type $^2E_6$ associated with the extension $K/F'$; it injects into $G_{F'}$ with Rost multiplier 1 [Ga01] §3. Since $n_{E_6^K}$ and $n_{G_{F'}}$ are both 12, $\text{Inv}^{3}_{\text{nr}}(G/F') = 0$ by §5.5 and the $^2E_6$ case (§6.6). Since $n_G = 12$, we have $\text{Inv}^{3}_{\text{nr}}(G/F) = 0$.

6.9. Type $E_8$. As in the previous cases, we may assume that our group $G$ of type $E_8$ is actually split. The natural inclusion of root systems gives an embedding of a split simply connected group of type $E_7$ in $G$, so $\text{Inv}^{3}_{\text{nr}}(G)$ is 5-torsion by §5.5 and the $E_7$ case (§6.8). In particular, we may assume that $\text{char } F \neq 5$ by §6.3.

Let $F''$ be the extension obtained from $F$ by adjoining two indeterminates and (if necessary) a primitive 5th root of unity. There is an $F''$-central division algebra $D$ of dimension $5^2$, namely the symbol algebra determined by the two indeterminates. There are obvious copies of $SL_5$ in $G$ arising from inclusions of sets of simple roots, and these inclusions have Rost multiplier 1. Arguing in a manner similar to [Go02] §1, one can twist such an inclusion to find a strongly inner form $G'$ of $G$ over $F''$ and an injection $SL_5(D) \hookrightarrow G'$. Now $n_{SL_5(D)} = 5$ and $\text{Inv}^{3}_{\text{nr}}(SL_5(D)/F') = 0$ (as can be seen from the explicit formula for the Rost invariant in [Mo03] p. 138), hence $\text{Inv}^{3}_{\text{nr}}(G'/F')$ is 12-torsion by §5.5. By the Strongly Inner Lemma, $\text{Inv}^{3}_{\text{nr}}(G'/F')$ is 12-torsion. Since the Dynkin index of $G$ is 60 over $F$ and $F'$, $\text{Inv}^{3}_{\text{nr}}(G/F)$ is 12-torsion by §5.6.

Combining the two preceding paragraphs gives that $\text{Inv}^{3}_{\text{nr}}(G)$ is 0. This completes the proof of Proposition 6.1.

7. Proof of the Main Theorem

Let $G$ be as in the Main Theorem 7.2. If $G$ has only trivial Tits algebras (e.g., $G$ is of type $G_2$, $F_4$, or $E_8$), then the Main Theorem holds for $G$ by Proposition 6.1 and Lemma 6.2.

If $G$ is of type $E_6$ or $E_7$, then we pick a generic quasi-splitting field $K$ of $G$ over $F$. We have $n_G = n_G$ and $\text{Inv}^{3}_{\text{nr}}(G/K) = 0$ (by Proposition 5.1), hence $\text{Inv}^{3}_{\text{nr}}(G/F) = 0$ by §5.6.

If $G$ is of type $^3D_4$, the Main Theorem holds by Lemma 6.3. The remaining case is where $G$ is of type $^3D_4$ with non trivial Tits algebras. We have $n_G = 12$ and as in the proof of Lemma 6.3 $\text{Inv}^{3}_{\text{nr}}(G)$ is 2-torsion. Hence we may assume that $\text{char } F \neq 2$.

The only nontrivial 2-torsion element of $\text{Inv}^{3}(G)$ is $6r_G$, so we will complete the proof of the Main Theorem if we show that $6r_G$ is unramified. That is, if we show that for every extension $E$ of $F$, the composition

$$H^1(E((t)), G) \xrightarrow{6r_{GE((t))}} H^3(E((t))) \xrightarrow{\partial} H^2(E)$$

is 0.
If $G$ is of type $1D_4$ over $E$, then it is of type $1D_4$ over $E((t))$ and $n_{G,E((t))}$ is 2 or 4. Hence $6r_{G,E((t))} = 2r_{G,E((t))}$ and the composition (7.1) is 0 by [M02, 8.2].

Otherwise, $G$ is of type $3D_4$ over $E$. That is, if $L$ is a cubic Galois extension of $F$ over which $G$ is of type $1D_4$, the tensor product $L' := L \otimes_F E$ is a cubic field extension of $E$. We have a diagram

\[
\begin{array}{ccccc}
H^1(E((t)), G) & \rightarrow & H^3(E((t))) & \rightarrow & H^2(E) \\
\downarrow \text{res} & & \downarrow \text{res} & & \downarrow \text{res}_{L'/E} \\
H^1(L'((t)), G) & \rightarrow & H^3(L'((t))) & \rightarrow & H^2(L').
\end{array}
\]

The left box commutes because the Rost invariant is compatible with restriction, and the right box commutes because the extension $E((t)) \subset L'((t))$ is unramified, hence the whole diagram commutes. Fix a class $\alpha$ in $H^1(E((t)), G)$. Since $G_{L'((t))}$ is of type $1D_4$, the composition of the two bottom arrows is 0 by the preceding paragraph, and the image of $\alpha$ in $H^2(L')$ is 0. Let $\beta \in H^2(E)$ be the image of $\alpha$; it is 2-torsion because $6r_G$ is 2-torsion. Hence

\[
\beta = 3\beta = \text{cor}_{L'/E} \text{res}_{L'/E}(\beta),
\]

which is 0 by the commutativity of the diagram. This shows that the composition (7.1) — which is the top row of the diagram — is 0 in this case.

Thus $\text{Im}_{\text{nr}}^3(G) = \mathbb{Z}/2$ for $G$ of type $3D_4$ with nontrivial Tits algebras when $\text{char} F \neq 2$. This completes the proof of the Main Theorem 0.2.

**Corollary 7.2.** Let $G$ be a simply connected group of type $3D_4$ or $6D_4$ over a field $F$ of characteristic $\neq 2$. If $G$ has a nontrivial Tits algebra, then $BG$ is not stably rational.

**Proof.** If $G$ is of type $3D_4$, this is a direct consequence of the Main Theorem.

If $G$ is of type $6D_4$, let $K$ be the unique quadratic extension of $F$ over which $G$ is of type $3D_4$. It follows from [Ga98, 3.2] that $G$ has nontrivial Tits algebras when $G$ is considered as a $K$-group. Therefore $BG$ is not stably rational as a $K$-variety, hence not as an $F$-variety.

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