A multiple-scales asymptotic approach for dynamic response analysis of piezoelectric laminated composites

Gopal Kondagunta¹ and Manoj Kumar Jain²

¹ Assistant Professor, Department of Aerospace Engineering, IIT Madras, Chennai-600036, India.
² Edison Engineer, GE Aviation, Bangalore - 560066, India.

Email: gopal@ae.iitm.ac.in

Abstract. A multiple-scales asymptotic theory is formulated for predicting the coupled electro-
elastic vibration characteristics of simply supported laminated piezoelectric plates composed of
orthotropic layers. The equations of motion and charge equations are solved within the
framework of three-dimensional piezoelectricity. The inhomogeneity is considered to be in the
thickness direction. The equations are expressed in a non-dimensional form and the
introduction of multiple scales in the formulation leads to a uniform expansion of the field
variables in even powers of a plate thickness parameter. Two different laminate configurations
are analyzed for the free vibration and electro-elastic response. The results obtained are
compared with available exact solutions and numerical solutions from finite element analysis.

1. Introduction
Piezoelectric materials are used in combination with composites to build structures with integrated
sensing and actuation abilities. The design of such multifunctional structures requires accurate and
efficient predictive modelling tools along with experimental results. Several analytical and numerical
procedures have been developed over the years to model the static and dynamic response of these
piezoelectric structures. Accurate analytical models are always desirable because they give a
functional relation between the input and response and enable a quick and efficient analysis of the
change in response with varying input parameters.

Tiersten [1] made the first attempt to study the governing equations, fundamental behaviour, and
obtain exact and approximation solution methodologies for single-ply piezoelectric plates. Heylinger
and Saravanos [2] developed exact solutions for the free vibrations of simply supported, laminated,
rectangular plates composed of orthotropic piezoelectric layers. Classical plate theories and higher
order shear deformation theories for laminated and thick plates have been used for static and dynamic
analysis of composite laminates with piezoelectric layers [3, 4]. Benjeddou [5] describes various finite
element analysis based numerical approaches used to model adaptive structural elements. Tarn and
Wang [6, 7] proposed asymptotic theory based formulations, based on 3-D elasticity for stress analysis
of laminated plates subject to thermo-mechanical loading and for dynamic response analysis of
laminated composites. One of the attractive features of this formulation is that there is no apriori
assumption for the field variables and their actual variation is obtained as part of the solution. For
laminated plates, the inter-laminar traction conditions and displacement continuity between layers are
inherently satisfied by the theory.
In this work, a multiple-scales based asymptotic expansion formulation is used to model the free vibrations of laminated piezoelectric plates. The formulation has been applied to two different configurations with piezoelectric layers bonded on the top and bottom surfaces assuming linear piezoelectric behaviour.

2. Formulation of the asymptotic method

In this work, the theory is developed for orthotropic, inhomogeneous plates. Consider a piezoelectric plate of uniform thickness $2h$, having orthotropic properties. A Cartesian coordinate system is selected such that the plane $x_3 = 0$ coincides with mid plane of the plate and the $x_3$ axis is positive downward. Appropriate boundary constraints are prescribed along the edges of the plate. On the lateral surface $x_3 = -h$, the transverse load $q(x_1, x_2, x_3, t)$ and potential $\phi(x_1, x_2, x_3, t)$ are prescribed. On the surface $x_3 = h$, the plate is free from external load. The three-dimensional electro-elastic governing equations expressed in the chosen axes of the plate are given by the equations

$$\sigma_i = c_{ij}\varepsilon_j - e_{ik}E_k, \quad D_k = e_{ki}\varepsilon_i + \eta_{ik}E_i$$

$$E_k = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right)$$

(1)

Here $\sigma_i, \varepsilon_i$ are the six components of the stress and strain tensor, $u_i$ are the three displacement components, $c_{ij}$ are the nine orthotropic elastic constants, $e_{ik}$ are the five piezoelectric coupling coefficients, $E_k$ are the electric field components in orthogonal directions, $\eta_{ik}$ the three dielectric constants and $D_k$ the three electric displacements. The material is inhomogeneous in the thickness direction. Laminated plates can be considered to be piecewise inhomogeneous.

The equations of motion, charge and electric field in terms of the potential are given by

$$\sigma_{ij} = \rho \ddot{u}_i$$

$$D_{ij} = 0, \quad E_k = -\Phi_k$$

(2)

The stresses and electric displacements are expressed in terms of $u_1, u_2, u_3$ and then rearranged, such that, for all the field variables, the derivative with respect to the thickness coordinate is expressed in terms of the derivatives with respect to the in-plane co-ordinates, along with inertia terms. These equations are then non-dimensionalized with respect to a suitable parameter chosen with close study on behaviour and relative magnitudes of field variables, as given below

$$x = x_1 / L, \quad y = x_2 / L, \quad z = x_3 / h; \quad u = u_1 / h, \quad v = u_2 / h, \quad w = u_3 / L$$

$$\sigma_{xx} = \sigma_{11} / Q, \quad \sigma_{xy} = \sigma_{12} / Q, \quad \sigma_{yx} = \sigma_{21} / Q; \quad \sigma_{zz} = \sigma_{13} / \hat{Q}, \quad \sigma_{yz} = \sigma_{23} / \hat{Q}, \quad \sigma_{zy} = \sigma_{32} / \hat{Q};$$

$$\sigma_{xz} = \sigma_{31} / \hat{Q}, \quad \sigma_{zx} = \sigma_{13} / \hat{Q}; \quad D_x = D_1 / \hat{Q}, \quad D_y = D_2 / \hat{Q}, \quad D_z = D_3 / \Phi, \quad \Phi = \phi e / Q L \varepsilon^2$$

(3)

$\hat{Q} = h / L$ is a plate thickness parameter, $L$ is a typical in-plane length dimension and $Q, e$ are the reference elastic constant and piezoelectric modulus. In this work we take $Q = c_{33}$ and $e = e_{33}$.

Each field variables is then expanded as an asymptotic series in terms of the plate thickness parameter as given in equation (4).

$$f(x, y, z, \tau_k, \hat{Q}) = f_{(0)}(x, y, z, \tau_k) + \hat{Q} f_{(1)}(x, y, z, \tau_k) + \hat{Q}^2 f_{(2)}(x, y, z, \tau_k) + ...$$

(4)
The expansions are substituted in the non-dimensional equations and suitable boundary conditions are applied. Comparing coefficients of the same power of the parameters, different orders of equations are obtained. The equations corresponding to $\epsilon^0$ are known as first order solutions. The associated dimensionless traction, displacement and electrical displacement boundary conditions are:

$$
\begin{bmatrix}
\sigma_{x_0} & \sigma_{y_0} \\
\sigma_{y_0} & \sigma_{z_0}
\end{bmatrix} = [0 \ 0] \text{ on } z = \pm 1
$$

$\dot{\sigma}^0 : \sigma_{z_0} = -\dot{q}, \ \Phi_{(0)} = \dot{\phi} \text{ on } z = -1 \text{ and } \sigma_{z_0} = 0, \ \Phi_{(0)} = 0 \text{ on } z = 1$

$$
\begin{bmatrix}
n_0 & 0 & n_2 \\
0 & n_2 & n_1
\end{bmatrix}
\begin{bmatrix}
\sigma_{x_0} \\
\sigma_{y_0} \\
\sigma_{z_0}
\end{bmatrix} = \begin{bmatrix}
\dot{\hat{p}}_1 \\
\dot{\hat{p}}_2
\end{bmatrix} \text{ and }
\begin{bmatrix}
n_1 & n_2
\end{bmatrix}
\begin{bmatrix}
\sigma_{x_0} \\
\sigma_{y_0} \\
\sigma_{z_0}
\end{bmatrix} = \dot{\hat{p}}_3 \text{ on } \Gamma_\sigma
$$

$$u_{(0)} = u^0, \ v_{(0)} = v^0, \ w_{(0)} = w^0, \ D_{z_{(0)}} = D_z^0 \text{ on } \Gamma_u$$

Similar equations are derived for higher order approximations corresponding to $\dot{\sigma}^{2k}$. The boundary conditions corresponding to $\dot{\sigma}^{2k}$ are given by:

$$
\begin{bmatrix}
\sigma_{x_k} & \sigma_{y_k} \\
\sigma_{y_k} & \sigma_{z_k}
\end{bmatrix} = [0 \ 0] \text{ on } z = \pm 1
$$

$\dot{\sigma}^0 : \sigma_{z_k} = 0, \ \Phi_{(k)} = 0 \text{ on } z = -1 \text{ and } \sigma_{z_k} = 0, \ \Phi_{(0)} = 0 \text{ on } z = 1$

$$
\begin{bmatrix}
n_0 & 0 & n_2 \\
0 & n_2 & n_1
\end{bmatrix}
\begin{bmatrix}
\sigma_{x_k} \\
\sigma_{y_k} \\
\sigma_{z_k}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix} \text{ and }
\begin{bmatrix}
n_1 & n_2
\end{bmatrix}
\begin{bmatrix}
\sigma_{x_k} \\
\sigma_{y_k} \\
\sigma_{z_k}
\end{bmatrix} = 0 \text{ on } \Gamma_\sigma
$$

$$u_{(k)} = 0, \ v_{(k)} = 0, \ w_{(k)} = 0, \ D_{z_{(k)}} = 0 \text{ on } \Gamma_u$$

The direct use of the classical small parameter perturbation technique for the asymptotic expansion formulation can result in secular terms which grow unbounded with time. This is because in each higher order approximation, terms from the lower order approximations occur as “forcing” functions. If multiple time scales are not used, the formulation will end up with the higher order solutions having secular terms that are not small after some time. Consequently the asymptotic expansion would break down. The multiple time scales approach used in this formulation ensures that the expansion would be uniform and valid regardless of the time span and ensures that the higher order terms are free of the secular terms. The use of multiple time scales is introduced as given in equation (7).

$$
\tau_k = \frac{\dot{\sigma}^{2k}}{l} \sqrt{\frac{c_s}{\rho_0}}, \ k = 0, 1, 2, 3... \ \rho_0 = \text{reference mass density}
$$

In equation (7), $\dot{\sigma}$ is the small plate parameter. When $k = 0$ correspondingly $\tau_0$ represents a fast scale. Likewise for higher values of $k$, $\tau_k$ represents a slower scale, $\tau_1$ represents an even slower scale and so on. The equations for each order are integrated successively with respect to $z$, and along with the traction and electric potential boundary conditions, give the required solution. The first order equations expressed in terms of the three displacements are as follows:
\[ A_1 u_{0,xx} + A_{66} u_{0,yy} + (A_{12} + A_{66}) v_{0,xy} - B_{11} w_{0,xxx} - (B_{66} + B_{12} + B_{66}) w_{0,xy} + G_{46} D_{a_{,r}} = I_{10} \frac{\partial^2 u_0}{\partial \tau_0^2} - I_{11} \frac{\partial^2 w_{0,x}}{\partial \tau_0^2} \]

\[ A_{22} v_{0,xy} + A_{66} v_{0,yy} + (A_{12} + A_{66}) u_{0,xy} - B_{22} w_{0,xyy} - (B_{66} + B_{12} + B_{66}) w_{0,xy} + G_{56} D_{2_{,v}} = I_{10} \frac{\partial^2 v_0}{\partial \tau_0^2} - I_{11} \frac{\partial^2 w_{0,y}}{\partial \tau_0^2} \]

\[ D_{11} w_{0,xxxx} + 2(D_{12} + 2D_{66}) w_{0,xyy} + D_{22} w_{0,yyyy} - B_{11} u_{0,xxx} - (B_{12} + 2B_{66}) u_{0,xy} - B_{22} v_{0,xyy} - (B_{12} + 2B_{66}) v_{0,xy} \]

\[-F_{66} D_{z_{,x}} - F_{46} D_{z_{,r}} = \dot{q} - I_{20} \frac{\partial^2 w_0}{\partial \tau_0^2} + I_{12} \frac{\partial^2}{\partial \tau_0^2} (w_{0,xx} + w_{0,yy}) - I_{11} \frac{\partial^2}{\partial \tau_0^2} (u_{0,r} + v_{0,y}) \]

\[ I_{10} = \int_{-1}^{1} \rho_1 dz, \quad I_{11} = \int_{-1}^{1} \rho_2 dz, \quad A_g = \int_{-1}^{1} \bar{Q}_g dz, \quad B_g = \int_{-1}^{1} \bar{Q}_g z dz, \quad G_{46} = \int_{-1}^{1} P_{46} dz, \quad G_{56} = \int_{-1}^{1} P_{56} dz, \]

\[ P_{46} \text{ and } P_{56} \text{ are functions of the electroelastic properties} \]

(8)

3. Model Configuration

3.1 Three-ply symmetric laminate

First, a three-ply symmetric laminate with an orthotropic PVDF layer bonded to transversely isotropic PZT layers on top and bottom is considered such that there is a mismatch in electric and elastic constants. Table 1 gives the electro-elastic properties of the materials and figure 1(a) shows the schematic of the configuration.

Table 1. Electro-elastic constants of the materials (from [2], \( \eta_0 = 8.854 \times 10^{-12} Fm^{-1} \))

| Property | PZT-4 | PVDF | Orthotropic Laminate |
|----------|-------|------|----------------------|
| \( c_{11} \) (GPa) | 139.021 | 238.003 | 134.855 |
| \( c_{12} \) | 77.847 | 3.976 | 5.156 |
| \( c_{13} \) | 74.327 | 2.195 | 5.156 |
| \( c_{22} \) | 139.021 | 23.6 | 14.352 |
| \( c_{23} \) | 74.327 | 1.922 | 7.132 |
| \( c_{33} \) | 115.449 | 10.671 | 14.351 |
| \( c_{44} \) | 25.6 | 2.15 | 3.606 |
| \( c_{55} \) | 25.6 | 4.4 | 5.654 |
| \( c_{66} \) | 30.6 | 6.43 | 5.654 |
| \( e_{31} \) (C/m²) | -5.2 | -0.13 | 0 |
| \( e_{32} \) | -5.2 | -0.14 | 0 |
| \( e_{33} \) | 15.08 | -0.28 | 0 |
| \( e_{15} \) | 12.72 | -0.01 | 0 |
| \( e_{24} \) | 12.72 | -0.01 | 0 |
| \( \eta_1 / \eta_0 \) | 1475 | 12.5 | 3.5 |
| \( \eta_{12} / \eta_0 \) | 1475 | 11.98 | 3.0 |
| \( \eta_{13} / \eta_0 \) | 1300 | 11.98 | 3.0 |
Two aspect ratios of 10 and 50 are considered to represent a thick and a thin plate configuration. This is a problem for which the exact solution was given in [2] and hence the same was chosen for validating the proposed approach. The plate thickness is taken to be 0.01 m. For ease of calculations, the same unit density is assumed for both materials. The plate is assumed to be a square plate.

---

3.2 Hybrid Laminate with cross-ply laminate

After the validation study, the approach was used to analyse a hybrid cross-ply laminate. The hybrid laminate is a 3 layered symmetric cross-ply laminate bonded with PZT-4 patches at the top and bottom [PZT/0°/90°/0°/PZT] to give a symmetric stacking sequence as shown in figure 1(b). This is a commonly used for extension mode configuration for piezoelectric sensor and actuator applications. Six different thickness ratios were modelled for calculating the natural frequency and other electroelastic field variables. The problem was also modelled using 3-D finite element analysis in ABAQUS® and the numerical and analytical results were compared. The in-plane dimensions are assumed equal.

4. Solution Methodology

The governing equations and the boundary conditions in the proposed model get further simplified for symmetric cross-ply laminates. The cross-coupling stiffness terms and a few other terms become zero for this configuration. The square plate is simply supported on all four edges. For free vibration analysis the external loads (the transverse load and the electric potential) are taken to be zero. The solution for the first order approximation can be taken to be of the form:

\[
\begin{align*}
  u_0 &= U_0 \cos(\alpha x) \sin(\beta y) \cos(\omega_{mn} \tau_0 - \delta_{mn}) \\
  v_0 &= V_0 \sin(\alpha x) \cos(\beta y) \cos(\omega_{mn} \tau_0 - \delta_{mn}) \\
  w_0 &= W_0 \sin(\alpha x) \sin(\beta y) \cos(\omega_{mn} \tau_0 - \delta_{mn}) \\
  D_{z_0} &= d_{z_0} \sin(\alpha x) \cos(\beta y) \cos(\omega_{mn} \tau_0 - \delta_{mn})
\end{align*}
\]

\[\alpha = m\pi, \beta = n\pi \text{ and } \omega_{mn} \text{ are the circular frequencies of the motion. } U_0, V_0, W_0, d_{z_0} \text{ are the amplitudes. The phase angles } \delta_{mn} \text{ are independent of } \tau_0 \text{ and are as yet undetermined functions of time scales } \tau_1, \tau_2, \text{ etc.} \]

The assumed displacement field is substituted into the governing equations to obtain the required eigenvalue problem to solve for the natural frequencies of the plate. The in-plane and flexural motions...
are uncoupled and can be solved separately for the natural frequencies. The first order approximation for the natural flexural vibration frequencies is given in terms of the plate stiffness and modes as

\[ \omega_{mn} = \left\{ \left[ D_{11} \alpha^4 + 2(D_{12} + 2D_{66}) \alpha^2 \beta^2 + D_{22} \beta^4 / \left[ I_{20} + I_{12}(\alpha^2 + \beta^2) \right] \right] \right\}^{1/2} \]  

(10)

Similarly, the higher order approximations of the frequencies can be obtained as corrections to the first order approximations. For the first order approximation, the amplitudes are normalized according to \( U_0^2 + V_0^2 + W_0^2 = 1 \). Once the mid-plane displacements are known, the thickness distribution can be obtained through integration of the relations (for higher order approximations) and subsequently the stresses and the electric potential can be calculated.

5. Results and Discussion

5.1 Three-ply symmetric laminate

The natural frequencies of the three-ply plate were obtained from the analysis along with the field variation of the displacements, stresses, electric potential and electric displacement. The analysis was performed up to four orders of approximation to obtain accurate variation of these field variables through the thickness of the plate for the two thickness ratios. Natural frequencies for the plate were also obtained from 3-D finite element analysis in ABAQUS® for both the thickness ratios and compared with the analytical results. The fundamental frequencies for various orders are given in Table 2. For the thin plate, the results are compared with the exact 3-D elasticity results of Heylinger and Saravanos [2].

Table 2. Non-dimensional fundamental frequency for 3-ply laminate with dissimilar materials

| \( \frac{a}{2h} \) | \( \varepsilon^0 \) | \( \varepsilon^2 \) | \( \varepsilon^4 \) | \( \varepsilon^6 \) | Heylinger et.al. [2] | FEA Result |
|-------|-------|-------|-------|-------|----------------|------------|
| 50    | 17.732| 17.548| 17.551| 17.551| 17.551         | 17.551     |
| 10    | 17.593| 13.181| 14.920| 14.184| -              | 14.412     |

Figure 2. Variation of the displacements and electric potential through plate thickness
The variation of the displacements and the electric potential through the thickness are shown in figure 2 for the different orders of approximation. The electric potential shows considerable non-linear variation through the thickness even for the thin plate while the non-linearity in the displacements is pronounced in the thick plate. Figure 3 shows the variation of the non-dimensional transverse shear stresses through the normalized thickness coordinate. The variation of the non-dimensional normal stress and the electric displacement through thickness is shown in figure 4. For the transverse shear and normal stresses, significant difference is observed in the prediction of the higher order and lower order approximations. It is clear that the higher order approximations give a more accurate estimate of the transverse shear and normal stresses while the lower order approximations over predict these values. As expected, the electric displacement shows a sharp variation in the outer piezo-ceramic layers of the laminate while remaining almost constant within the composite laminate.

**Figure 3.** Variation of non-dimensional transverse shear stresses through thickness for $a/2h=10$

$z = x_3/h, \ \sigma_{xz} = \sigma_{13}a^2/h^2c_{33}, \ \sigma_{yz} = \sigma_{23}a^2/h^2c_{33}$

**Figure 4.** Variation of non-dimensional normal stresses and electric displacement for $a/2h=10$

$z = x_3/h, \ \sigma_{zz} = \sigma_{33}a^3/h^3c_{33}, \ D_z = D_3a/he_{33}$
5.2 Hybrid composite laminate

Six different thickness ratios from 1/100 to 1/10 were analysed. Table 3 shows the comparison between fundamental frequencies for different thickness ratios. Exact solution is available only for one thickness ratio. The finite element results are obtained for all the six thickness ratios. Rapid convergence is observed even for thick plates. For the thick plate, in-plane displacement variations can be differentiated and show pronounced non-linearity. The flexural displacement also varies through the thickness. Figure 5 shows the variation of the non-dimensional electric potential along the normalized thickness coordinate for a/2h=10 for different orders of approximation. The core material has no coupled electromechanical properties, but the potential varies through the thickness for the higher order approximations. This behaviour is more visible in thick plates and diminishes for relatively thin plates.

Table 3. Non-dimensionalized fundamental frequency for different values of a/2h

| a/2h | ε₀   | ε²       | ε⁴       | ε⁶       | Exact Solution ([2]) | FEA solution |
|------|------|----------|----------|----------|-----------------------|--------------|
| 100  | 12.893 | 12.962   | 12.962   | 12.962   | -                     | 12.962       |
| 50   | 12.979 | 12.899   | 12.899   | 12.899   | 12.899                | 12.899       |
| 25   | 12.967 | 12.644   | 12.657   | 12.656   | -                     | 12.656       |
| 20   | 12.957 | 12.455   | 12.486   | 12.484   | -                     | 12.484       |
| 12.5 | 12.916 | 11.651   | 11.845   | 11.812   | -                     | 11.817       |
| 10   | 12.873 | 10.931   | 11.387   | 11.270   | -                     | 11.290       |

Figure 5. Variation of the non-dimensional electric displacement and potential for a/2h = 10

\[ z = x_1/h, \quad \Phi = \phi e/QL_3 \]

Figure 6 shows the plot of the transverse shear stress variation while the variation of the transverse normal stress and electric displacement through the plate thickness is shown in figure 7 for a/2h = 10. The stresses converge for higher order approximations even for the thick plate. The predictions of transverse shear stress and transverse normal shows significant improvement using the higher order approximations.
Figure 6. Variation of the non-dimensional transverse shear stresses through thickness for $a/2h = 10$

$$z = x/h, \quad \sigma_{xz} = \sigma_{23} a^2 / h^2 c_{33}, \quad \sigma_{yz} = \sigma_{33} a^2 / h^2 c_{33}$$

Figure 7. Variation of non-dimensional stresses and electrical displacement through thickness.

$$z = x/h, \quad \sigma_{zz} = \sigma_{33} a^3 / h^3 c_{33}, \quad D_z = D_3 a / h e_{33}$$

6. Summary
The multiple-scales based asymptotic expansion method is shown to be an efficient and accurate analytical method for the analysis of free vibration and electro-elastic response of piezo-laminate composites. The method considers the full 3-D electro-elastic governing equations without any apriori assumptions on variation of the field variables through the plate thickness and the multiple orders of approximations obtain the thickness variation of field variables such as the transverse displacement, stresses and electric potential with increasing accuracy. The effect of dissimilar electromechanical properties on the electric potential and the effect of thickness on the plate response were effectively modelled by the approach. While the first order approximation for the asymptotic solution is
essentially the same as the classical laminate theory, the higher order approximations fully include the effect of shear deformation and rotary inertia. Fast convergence was noticed even for thick plates for the natural frequencies and the transverse stresses.

References
[1] Tiersten H F Linear Piezoelectric Plate Vibrations, Plenum, New York, 1969
[2] Heylinger P and Saravanos D A 1995 Exact free-vibration analysis of laminated plates with embedded piezoelectric layers, *Acoustical Society of America* 98 (3), 1547-57
[3] Duan W H, Quek S T and Wang Q 2005 Free vibration analysis of piezoelectric coupled thin and thick annular plate *J. Sound and Vibration* 281 119-39
[4] Zhou Y L 2001 Modelling a piezoelectric composite laminate using a third order plate theory *The International Society of Offshore and Polar Engineers* 4 131-39
[5] Benjeddou A 2000 Advances in piezoelectric finite element modeling of adaptive structural elements: a survey *Computers and Structures* 76 347-63
[6] Tarn J Q and Wang Y M 1993 Asymptotic thermoelastic analysis of anisotropic inhomogeneous and laminated plates *J. of Thermal Stresses* 18 35-58
[7] Tarn J Q and Wang Y M 1994 An asymptotic theory for dynamic response of anisotropic inhomogeneous and laminated plates *Int. J of Sol and Struc* 31 231-46