Digital quantum simulation of boson systems with Jordan-Wigner transformation: Generating Hong-Ou-Mandel dip in quantum computers

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This work proposes an efficient digital quantum simulation protocol for the linear scattering process of bosons, which provides a simple extension to partially distinguishable boson cases. Our protocol is achieved by combining the boson-fermion correspondence relation and fermion to qubit encoding (F2QE) protocols. As a proof of concept, we designed a quantum circuits for simulating the Hong-Ou-Mandel scattering process with a variation of particle distinguishability. The circuits were verified with the classical and quantum simulations using the IBM Quantum and IonQ cloud services.

I. INTRODUCTION

Quantum simulation imitates an evolution of one quantum system with another artificially organized quantum system, i.e., quantum simulator [1]. Digital quantum simulators can encode with qubits an arbitrary quantum system comprising various particles, such as spins, fermions, and bosons, either exactly or approximately, depending on the particle nature. Qubits can be realized with several physical systems, such as trapped ions [2, 3], nuclear magnetic resonance [4, 5], superconducting circuits [6, 7], quantum dots [8], and photons [9]. Therefore, we can simulate any quantum system using digital quantum simulators without qubit encoding protocols regardless of the physical nature of the simulator. Recent studies [10, 11] indicate that achieving quantum supremacy using digital simulations and its application to many-particle systems may become a realistic target in the foreseeable future.

Among various many-particle quantum systems, bosonic systems are considered to have the significant benefit from digital quantum simulations. Knill, Laflamme and Milburn (KLM) showed that the postselected linear optics is capable of universal quantum computing [12]. Also, boson sampling proposed by Aaronson and Arkhipov [13] is a strong candidate for demonstrating the computational superiority of quantum devices. The boson sampling problem is believed to belong to classically hard sampling problems.

Inspired by the computational power of noninteracting bosonic systems, several boson to qubit encoding (B2QE) protocols have been proposed to simulate bosonic problems with digital quantum computers [14–19]. The majority of studies discretize bosonic creation and annihilation operators directly using unary or binary qubit representations of the Fock states as qubit encoding protocols. The required resources, such as the numbers of qubits and gates, vary according to the encoding protocols. Ref. [17] compared the resource efficiency among encoding protocols. It is noteworthy that Ref. [18] presents a method for the digital quantum simulation of linear and nonlinear optical elements.

In this paper, we propose an alternative many-boson digital simulation method by combining the boson-fermion correspondence analyzed by Shchesnovich [20] and fermion to qubit encoding (F2QE) protocols [21, 22]. Specifically, our protocol transforms bosonic states into fermionic states with internal degrees of freedom, which are then transformed to qubit states via a F2QE protocol (JW transformation). With our simulation model, quantum circuits with M bundles of N qubits can simulate the number-conserving scattering process of N bosons in M modes. Our protocol is summarized in Fig. 1. The most significant advantage of our protocol is that it can simulate non-ideal partially distinguishable bosons, i.e., bosons with internal degrees of freedom, using a direct extension of qubit numbers.

As a proof of concept, we generate the Hong-Ou-Mandel (HOM) dip [23] with our protocol. The

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multi-fermions with an internal degree of freedom. number-conserving bosonic systems with entangled
tool, we express the simulate many-boson systems with qubits. Our pro-
discusses its possible future extensions.
Finally, section IV concludes our present work and
partial distinguishability with an eight-qubit-circuit. HOM dip experiment. We simulate the two-photon
simulation. In section III, we apply our model to the
with the JW transformation for the digital bosonic
protocol, we show how to combine this transformation
explain our B2QE protocol to

In this section, we explain our B2QE protocol to
simulate many-boson systems with qubits. Our pro-
tool, the JW transformation [21].

A. Effective bosonic states of multi-fermions

We first explain how a specific form of entan-
gled multi-fermions can effectively behave as multi-
bosons. In the second quantization language, the
bosonic creation and annihilation operators $\hat{a}_i^\dagger$ and
$\hat{a}_i$ ($i = 1, \ldots, M$) obey the following commutation relations:

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \quad (1)$$

while the fermionic operators $\hat{b}_i^\dagger$ and $\hat{b}_i$ obey the anti-
commutation relations:

$$\{\hat{b}_i, \hat{b}_j^\dagger\} = \delta_{ij}, \quad \{\hat{b}_i, \hat{b}_j\} = \{\hat{b}_i^\dagger, \hat{b}_j^\dagger\} = 0, \quad (2)$$

where $\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$. The above relations
satisfy the Pauli exclusion principle for fermions, which prohibits the superposition of two fermions
in the same state. Indeed, we see that $\hat{b}_i^\dagger\hat{b}_j^\dagger|\text{vac}\rangle =
-\hat{b}_j^\dagger\hat{b}_i^\dagger|\text{vac}\rangle = 0$ by Eq. (2). On the other hand, if
the fermions have internal degrees of freedom, such as spin, fermions mode with different internal states
can occupy the same spatial mode. By denoting a $K$-dimensional internal degree of freedom as $\mu$ ($\mu = 0, \ldots, K-1$), a fermionic operator with internal
degrees of freedom $\mu$ is defined as $\hat{b}_i^\mu$ and $\hat{b}_i^\mu$. The anticommutation relations for the operators are as
follows:

$$\{\hat{b}_i^\mu, \hat{b}_j^\nu\} = \delta_{ij}\delta^{\mu\nu}, \quad \{\hat{b}_i^\mu, \hat{b}_j^\nu\} = \{\hat{b}_i^\mu, \hat{b}_j^\nu\} = 0. \quad (3)$$

In such a case, the fermions can condensate in the same spatial mode up to $K$. We aim to employ this
feature of multi-fermionic states for mimicking the Bose-Einstein condensation (BEC) with the cutoff
$K$. Fig. 2 explains the concept of fermionic condensation.

On the other hand, for the fermionic condensation
to operate like the BEC, we must properly con-
sider the fundamental differences between bosons and fermions, i.e., the exchange symmetry and an-
tisymmetry indicated in Eqs. (1) and (2). Shchesnovich [20] showed that the interchangeability of entanglement and exchange symmetry can render entangled multi-fermions symmetric under the ex-
change of spatial modes. Here, we introduce the ef-
fective bosonic state of multi-fermions with the con-
densation limit $K$ in the second quantization lan-
guage, which offers a more refined explanation than
of the first quantization language used in Ref. [20].

II. DIGITIZING BOSONIC SYSTEMS

In this section, we explain our B2QE protocol to
simulate many-boson systems with qubits. Our pro-
tool, which offers a more refined explanation than
of the second quantization language used in Ref. [20].
Let us consider an $N$-fermionic state,
\[ \hat{b}_{i_1}^{[\mu_1]} \hat{b}_{i_2}^{[\mu_2]} \cdots \hat{b}_{i_N}^{[\mu_N]} |\text{vac}\rangle, \] (4)
where $i_1, i_2, \ldots, i_N$ are the internal indices of fermions.

Since the antisymmetrical entanglement of the fermions is essential for effective bosonic states to behave like bosons, the exchange symmetry of the state must be preserved under evolutions. In other words, if we want to simulate the bosonic scattering process with fermions, the transformation operators of fermions must preserve the antisymmetrical entanglement. We observe that some transformation operators satisfy this restriction. We first consider a bosonic operator $T$ of the following form:
\[ T = \exp \left[ it \sum_{j,k=1}^{M} \Phi_{jk} \hat{a}^\dagger_j \hat{a}_k \right], \] (10)
where $t$ is the evolution time and $\Phi_{jk} \in \mathbb{C}$. Note that Eq. (11) is the effective Hamiltonian of linear transformations with a fixed $t$. We note that
\[ \sum_{jk} \Phi_{jk} \hat{a}^\dagger_j \hat{a}_k \] (11)
behaves as the Hamiltonian of the given system by setting $\Phi_{jk} = \Phi_{kj}^\ast$. Then, the transformation of $\hat{a}^\dagger_i$ under $T$ is given by
\[ T \hat{a}^\dagger_i T^\dagger = \sum_j \exp(it \Phi_{ij}^\ast) \hat{a}^\dagger_j \]
\[ \equiv \sum_j u_{ij} \hat{a}^\dagger_j, \] (12)
where $\sum_j u_{ij} u_{kj}^\ast = \delta_{ik}$. In the fermionic system, the corresponding operator $T_f$ is expressed as follows:
\[ T_f = \exp(it \sum_{\mu} \sum_{j,k} \Phi_{jk} \hat{b}_{i_1}^{[\mu]} \hat{b}_{i_2}^{[\mu]}), \] (13)
which gives
\[ T_f b^\mu_i T_f = \sum_j u_{ij} b^\mu_j. \]  
(14)

Then, the state Eq. (5) evolves via \( T_f \) as follows:
\[ |\Psi\rangle_f = \frac{1}{\sqrt{N!}} \sum_{j_1,\cdots,j_N} u_{k_1}^j \cdots u_{j_N}^k b^{\mu_1}_j \cdots b^{\mu_N}_k |\text{vac}\rangle = \frac{1}{\sqrt{N!}} \sum_{j_1,\cdots,j_N} u_{j_1}^{k_1} \cdots u_{j_N}^{k_N} b^{\mu_1}_{j_1} \cdots b^{\mu_N}_{j_N} |\text{vac}\rangle. \]  
(15)

The second line of the above equation shows that the transformed state is a linear combination of effective multi-boson states, which itself is an effective multi-boson state. In a more general form, we see that any number-conserving Hamiltonian looks like \( H = \sum_j \Phi_j \hat{a}_j \hat{a}_j^\dagger + c.c. \).

Finally, we check whether the measurement of the state Eq. (5) that evolves with Eq. (10) is effectively bosonic. Suppose first that we postselect terms without bunching, irrespective of what the internal states of the particles are. Without loss of generality, we can assume the boson number distribution vector as follows:
\[ \vec{n} = (1,1,\cdots,1,0,\cdots,0). \]  
(16)

Then, the scattering probability is given with a projector \( E = \sum_{\mu_1,\cdots,\mu_N} (b^\mu_j)^* \cdots b^\mu_N |\text{vac}\rangle \langle \text{vac}| b^\mu_N \cdots b^\mu_1 \) as follows:
\[ P = \text{Tr}(E \rho_f) = \sum_{\mu_1,\cdots,\mu_N} \langle \text{vac}| b^\mu_N \cdots b^\mu_1 |\Psi\rangle \langle \Psi| b^\mu_1 \cdots b^\mu_N |\text{vac}\rangle. \]  
(17)

Using the relation,
\[ \hat{b}^{\mu_1}_j \cdots \hat{b}^{\mu_N}_{j_N} b^{\nu_1}_{k_1} \cdots b^{\nu_N}_{k_N} |\text{vac}\rangle = \delta^{\{k_1,\cdots,k_N\}}_{\{\mu_1,\cdots,\nu_N\}} |\text{vac}\rangle, \]  
(18)

we have
\[ P \sim |\text{perm}(u)|^2, \]  
(19)

where \( u \) is an \( N \times N \) matrix whose entries are \( u_{ij} \) and \( \text{perm}(u) \) denotes the permanent of \( u \), as expected for a bosonic systems with \( T \). If the postselected states permit bunching, the probability becomes proportional to the permanent of the submatrix of \( u \) as expected.

B. Simulating multi-boson systems with qubits

Since a fermionic state of the form indicated in Eq. (5) can simulate a linear scattering of bosons, we conclude that digital quantum computers can also simulate the same system using the JW transformation. Before explaining how we actually organize quantum circuits and algorithms for such a simulation, we first review the JW transformation, which maps fermions to qubits.

In the JW transformation, qubit states \(|0\rangle\) and \(|1\rangle\) correspond to the empty and occupied states of fermions for a given mode, i.e., the following isomorphism should hold:
\[ N \text{ qubit state } |\vec{n}\rangle = |n_1,\cdots,n_N\rangle \quad (n_j = 0,1) \]
\[ \cong N \text{ fermionic state } (\hat{b}^\dagger)^{n_1} \cdots (\hat{b}^\dagger)^{n_N} |\text{vac}\rangle. \]  
(20)

The left and right hand side denotes an \( N \)-qubit state and an \( N \)-fermionic state, respectively, and \( \cong \) represents that the two sides are in a correspondence relationship with each other. For this relationship to hold, there must be operators acting on the \( N \)-qubit system that play the roles of creation and annihilation operators. Indeed, we can construct such operators by combining the Pauli operators \( X_j, Y_j \) and \( Z_j \) (\( j = 1,\cdots,L \)), i.e., \( \hat{b}^\dagger_j (X,Y,Z) \cong \hat{b}^\dagger_j \) and \( \hat{b}_j (X,Y,Z) \cong \hat{b}_j \).

We can see that \(|\vec{n}\rangle\) and \(\hat{b}_j^\dagger (X,Y,Z)\) must satisfy the following conditions:

- If \( n_j = 0 \), then \( \hat{b}_j |\vec{n}\rangle = 0 \)
- If \( n_j = 1 \), then \( \hat{b}_j |\vec{n}\rangle = (-1)^{s_{j\hat{b}}} |\vec{n}\rangle \)

where \( s_{j\hat{b}} \equiv \sum_{k=1}^L n_k \). Note that \( (-1)^{s_{j\hat{b}}} \) comes from the anticommutation property of the creation-annihilation operators.

It can easily be verified that
\[ \hat{b}_j (X,Y,Z) \equiv (\otimes_{k=1}^L Z_k) \otimes \sigma_{-j}^x ; \]
\[ \hat{b}_j^\dagger (X,Y,Z) \equiv (\otimes_{k=1}^L Z_k) \otimes \sigma_{-j}^+ \]  
(21)

(\( \sigma^+ \equiv |0\rangle\langle 1| \) and \( \sigma^- \equiv |1\rangle\langle 0| \)) satisfy the above conditions. One can also check that Eq. (21) satisfies the anticommutation relations, i.e., \( \{\hat{b}_j,\hat{b}_k\} = \delta_{jk} \) and \( \{\hat{b}^\dagger_j,\hat{b}^\dagger_k\} = \{\hat{b}_j,\hat{b}_k\} = 0 \). The state transformation of Eq. (20) and operator transformations in Eq. (21) define the JW transformation for the digital simulation of fermionic systems.
as follows:

\[ \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} sgn(\sigma) |\chi_{\sigma(1)},\chi_{\sigma(2)},\ldots,\chi_{\sigma(N)},\chi_0,\ldots,\chi_0\rangle. \]

(24)

On the other hand, if all the bosons are in the same mode, e.g., the first mode, the state can be written as follows:

\[ \frac{1}{\sqrt{N!}} \hat{b}_1^{[1]} \cdots \hat{b}_1^{[N]} |\text{vac}\rangle = \hat{b}_1^{[1]} \cdots \hat{b}_1^{[N]} |\text{vac}\rangle \]

\[ \cong |(1,1,\ldots,1),\chi_0,\ldots,\chi_0\rangle. \]

(25)

For the case with \( N = 2 \) and \( M = 3 \), Eq. (24) takes the following form:

\[ \frac{1}{\sqrt{2}} \left( |\chi_1,\chi_2,\chi_0\rangle - |\chi_2,\chi_1,\chi_0\rangle \right) \]

\[ = \frac{1}{\sqrt{2}} (|10,01,00\rangle - |01,10,00\rangle), \]

(26)

which corresponds to the bosonic state \( \hat{a}_1^{[1]} \hat{a}_1^{[2]} |\text{vac}\rangle \), while Eq. (25) becomes \( |11,00,00\rangle \), which corresponds to the bosonic state \( \frac{1}{\sqrt{2}} (\hat{a}_1^{[1]} \hat{a}_1^{[2]} |\text{vac}\rangle \). 

Since Eqs. (22) and (23) represent a mapping from bosonic systems to qubits, we can digitally simulate multi-boson systems with the following process:

1. Preparation of the initial state: We first need to prepare the initial states of the form shown in Eq. (7), which can be achieved by adopting one of the known antisymmetrization algorithms, e.g., those in Refs. 30, 31. On the other hand, we can find optimal algorithms for the states with small \( N \) case-by-case.

2. Evolution: The unitary operations can be executed by substituting Eq. (23) into the Hamiltonian operator of Eq. (11).

3. Measurement: While the order of the excited states is unimportant, the number of excited states in each bundle is crucial because it determines the distributions of boson numbers. For example, if \( N = 3 \), \((100), (010), \) and \((001)\), in all cases a mode has one particle with different internal state. Nevertheless, we only
In the fermion system, set the HOM effect for $N$ fermions using the JW transformation, which produces the following transformation operator in Eq. (24), we prepare the following initial state $|\Psi\rangle$: needs four qubits here. Ideal bosons simply with a direct extension of qubit generalization shows our protocol can simulate non-ideal bosons with a two-dimensional internal degree of freedom. This generalization shows our protocol can simulate non-ideal bosons simply with a direct extension of qubit numbers.

### III. APPLICATION: HONG-OU-MANDEL DIP

In this section, we use our protocol to simulate the HOM effect for $N = 2$. We first simulate ideal photon case (with no internal degree of freedom), which is then generalized to non-ideal photons with a two-dimensional internal degree of freedom. This generalization shows our protocol can simulate non-ideal bosons with a two-dimensional internal degree of freedom. This generalization shows our protocol can simulate non-ideal bosons with a two-dimensional internal degree of freedom. This generalization shows our protocol can simulate non-ideal bosons with a two-dimensional internal degree of freedom.

#### A. HOM experiment with ideal photons

Since two qubits can represent a bosonic mode with a maximal photon number of two, our protocol needs four qubits here.

**Preparation.**— Using the notations given before Eq. (24), we prepare the following initial state $|\Psi\rangle_i$:

$$|\Psi\rangle_i = \frac{1}{\sqrt{2}}(|\chi_1, \chi_2⟩ - |\chi_2, \chi_1⟩) = \frac{1}{\sqrt{2}}(|10, 01⟩ - |01, 10⟩).$$ (27)

**Evolution.**— For the case of HOM scattering, we set

$$t = \frac{\pi}{4}, \quad \Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$ (28)

which produces the following transformation operator $T_H$:

$$T_H = \exp[\frac{i\pi}{4}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)].$$ (29)

In the fermion system, $T_H$ is given as follows:

$$T_H = \exp[\frac{i\pi}{4}(\hat{b}_1^{1\dagger} \hat{b}_2^\dagger + \hat{b}_2^{1\dagger} \hat{b}_1^\dagger + \hat{b}_1^{2\dagger} \hat{b}_2^\dagger + \hat{b}_2^{2\dagger} \hat{b}_1^\dagger)]$$

$$= \exp[i\frac{\pi}{4}(b_1^{1\dagger}b_2^\dagger + b_2^{1\dagger}b_1^\dagger)] \exp[i\frac{\pi}{4}(b_1^{2\dagger}b_2^\dagger + b_2^{2\dagger}b_1^\dagger)].$$ (30)

Using the JW transformation, we obtain

$$b_1^{1\dagger}b_2^\dagger + b_2^{1\dagger}b_1^\dagger = \frac{1}{2}(X \otimes Z \otimes X + Y \otimes Z \otimes Y) \otimes \mathbb{I},$$

$$b_1^{2\dagger}b_2^\dagger + b_2^{2\dagger}b_1^\dagger = \frac{1}{2} \mathbb{I} \otimes (X \otimes Z \otimes X + Y \otimes Z \otimes Y)$$ (31)

in the qubit system. Since $X \otimes Z \otimes X$ and $Y \otimes Z \otimes Y$ commute, $T_H$ can be further decomposed as:

$$T_H = \exp[i\frac{\pi}{8}(X \otimes Z \otimes X \otimes \mathbb{I})] \exp[i\frac{\pi}{8}(Y \otimes Z \otimes Y \otimes \mathbb{I})]$$

$$\times \exp[i\frac{\pi}{8}(\mathbb{I} \otimes X \otimes Z \otimes X)] \exp[i\frac{\pi}{8}(\mathbb{I} \otimes Y \otimes Z \otimes Y)].$$ (32)

Note that we have not used the Trotter decomposition, because all the terms in the exponential terms commute with each other. This is true for the general linear optical transformations.

**Measurement.**— The final state transformed by Eq. (29) is given by

$$|\Psi\rangle_f = \frac{i}{\sqrt{2}}(|11, 00⟩ + |00, 11⟩).$$ (33)

The interpretation of the above state is that two bosons always bunch, i.e., the HOM effect occurs.

The full circuit for the HOM digital simulation is shown in Fig. 4. We used IBM Quantum and IonQ for the digital quantum simulation (the results are shown in Fig. 5).

#### B. HOM dip

We will now simulate the HOM dip (see, e.g., for a pedagogic review) with a two-dimensional internal degree of freedom that creates distinguishability. By denoting the internal state of bosons as $|\beta⟩$, the creation and annihilation operators are written as $\hat{a}_i^{\dagger}$ and $\hat{a}_i$ with $[\hat{a}_i^{\dagger}, \hat{a}_j] = \delta_{ij} \delta_{sr}$. Then, an $N$-boson state $\hat{a}_i^{\dagger} \hat{a}_i^{\dagger} \hat{a}_i^{\dagger} \hat{a}_i^{\dagger} |\chi⟩$ ($i_\alpha \in \{1, \ldots, N\}$, $s_\beta \in \{0, 1\}$ for $\alpha, \beta \in \{1, \ldots, N\}$) can effectively be expressed as a fermionic state as follows:

$$\frac{1}{\sqrt{N!}} \hat{b}_i^{[\mu_1]_{s_1}} \hat{b}_i^{[\mu_2]_{s_2}} \cdots \hat{b}_i^{[\mu_N]_{s_N}} |\chi⟩.$$ (34)

Therefore, the general initial state for the HOM dip with two photons can be written as follows:

$$|\Psi\rangle_i = \hat{a}_i^{\dagger} \hat{a}_i^{\dagger} |\chi⟩ \overset{\frac{1}{\sqrt{2}}}{\equiv} \frac{1}{\sqrt{2}} \hat{b}_i^{[1]_{s_1}} \hat{b}_i^{[1]_{s_2}} |\chi⟩,$$ (35)

where $|s⟩$ and $|r⟩$ are the general internal states of the form $|0⟩ + |1⟩$ ($\zeta, \xi \in \mathbb{C}$ and $|\zeta|^2 + |\xi|^2 = 1$). To simulate this type of HOM dip, we need eight qubits, which are displayed in Fig. 6. Each qubit corresponds to the particle states $(i, \mu, s)$ as indicated in the figure.
Preparation. — Without loss of generality, we can assume the internal state of the photons as $|s\rangle = |0\rangle$ and $|r\rangle = \zeta |0\rangle + \xi |1\rangle$. Therefore, the initial state for partially distinguishable photons can be described as follows:

$$|\Psi\rangle_i = \frac{1}{\sqrt{2}} \hat{b}_{10}^{\dagger} \hat{b}_{20}^{\dagger} |\text{vac}\rangle$$

$$= \frac{1}{\sqrt{2}} \left( \zeta b_{10}^{\dagger} b_{20}^{\dagger} + \xi b_{10}^{\dagger} b_{21}^{\dagger} \right) |\text{vac}\rangle$$

$$= \frac{1}{\sqrt{2}} \left( \zeta (|1000, 0010\rangle - |0010, 1000\rangle) + \xi (|1000, 0001\rangle - |0010, 0100\rangle) \right). \quad (36)$$

We can prepare this state by first creating

$$\frac{1}{\sqrt{2}} (|1000, 0010\rangle - |0010, 1000\rangle), \quad (37)$$

and then applying the following gates:

between $(2, 0, 1)$ and $(2, 1, 1)$ and between $(2, 0, 2)$ and $(2, 1, 2)$. The above gates can be represented in a matrix form as follows:

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{i\gamma} \cos \left( \frac{\theta}{2} \right) & -e^{i\phi} \sin \left( \frac{\theta}{2} \right) & 0 \\
0 & e^{-i\phi} \sin \left( \frac{\theta}{2} \right) & e^{-i\gamma} \cos \left( \frac{\theta}{2} \right) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (38)$$

where $\zeta$, and $\xi$ are given with $(\gamma, \phi, \theta)$ by

$$\zeta = e^{i\gamma} \cos \left( \frac{\theta}{2} \right), \quad \xi = -e^{i\phi} \sin \left( \frac{\theta}{2} \right). \quad (39)$$

For two indistinguishable bosons (ideal photons), i.e., $\zeta = 1$, and the initial state is as follows:

$$|\Psi\rangle_i^{\text{ind}} = \frac{1}{\sqrt{2}} \hat{b}_{10}^{\dagger} \hat{b}_{20}^{\dagger} |\text{vac}\rangle$$

$$= \frac{1}{\sqrt{2}} (|1000, 0010\rangle - |0010, 1000\rangle). \quad (40)$$

On the other hand, if two bosons are fully distinguishable, i.e., $\xi = 1$, the initial state can be given

FIG. 4: Full circuit for HOM experiment. As seen in figures in the second line, $T_H$ given by Eq. (32) is further decomposed into one- or two-qubit gates. We set, for example, $R_{XZX}(\Theta) = \exp[i\Theta(X \otimes Z \otimes X)]$, where the index indicates the operator in the exponent.

FIG. 5: Quantum simulations of HOM effect with (left) IBM Quantum and (right) IonQ.
without loss of generality as follows:

\[
|\Psi\rangle_{\text{dis}} = \frac{1}{\sqrt{2}} \hat{b}_{10}^{\dagger} \hat{b}_{21}^{\dagger} |\text{vac}\rangle
= \frac{1}{\sqrt{2}} (|1000, 0000\rangle - |0010, 0100\rangle). \tag{41}
\]

**Evolution.** — The evolution operator with distinguishability is simply obtained by generalizing Eq. (30) as follows:

\[
T_H
= \exp\left[\frac{i \pi}{4} \sum_{s,\mu} (\hat{b}_{1s}^{\mu} \hat{b}_{2s}^{\mu} + \hat{b}_{1s}^{\dagger \mu} \hat{b}_{2s}^{\dagger \mu})\right]
= \exp\left[\frac{i \pi}{4} (\hat{b}_{10}^{\dagger 11} \hat{b}_{20}^{12} + \hat{b}_{12}^{11} \hat{b}_{21}^{12})\right] \exp\left[\frac{i \pi}{4} (\hat{b}_{11}^{11} \hat{b}_{21}^{12} + \hat{b}_{12}^{11} \hat{b}_{21}^{12})\right]
\times \exp\left[\frac{i \pi}{4} (\hat{b}_{10}^{12} \hat{b}_{20}^{12} + \hat{b}_{12}^{12} \hat{b}_{20}^{12})\right] \exp\left[\frac{i \pi}{4} (\hat{b}_{12}^{12} \hat{b}_{21}^{12} + \hat{b}_{12}^{12} \hat{b}_{21}^{12})\right]. \tag{42}
\]

**Measurement.** — When the bosons are indistinguishable, the final state \(|\Psi\rangle_{\text{f}}^{\text{ind}}\) is as follows:

\[
|\Psi\rangle_{\text{f}}^{\text{ind}} = \frac{i}{\sqrt{2}} (\hat{b}_{10}^{11} \hat{b}_{10}^{11} + \hat{b}_{20}^{12} \hat{b}_{20}^{12}) |\text{vac}\rangle
= \frac{i}{\sqrt{2}} (|1010, 0000\rangle + |0000, 1010\rangle), \tag{43}
\]
i.e., two particles are always in the same mode and the coincidence probability becomes zero.

When they are distinguishable, the final state \(|\Psi\rangle_{\text{f}}^{\text{dis}}\) is given as follows:

\[
|\Psi\rangle_{\text{f}}^{\text{dis}}
= \frac{1}{\sqrt{2}} \left(i \hat{b}_{10}^{11} \hat{b}_{10}^{11} + \hat{b}_{20}^{12} \hat{b}_{20}^{12}\right) |\text{vac}\rangle
+ \frac{\xi}{\sqrt{2}} \left(i (|1001, 0000\rangle - |0110, 0000\rangle + |1000, 0001\rangle - |0010, 0100\rangle - (|0001, 1000\rangle - |0100, 0010\rangle)\right), \tag{44}
\]

which means that each particle can arrive at each of the two detectors with probability 0.5.

In general, the final state with an arbitrary distinguishability \(|r\rangle = \zeta |0\rangle + \xi |1\rangle\) is given by

\[
|\Psi\rangle_{\text{f}} = \frac{i \zeta}{\sqrt{2}} (|1010, 0000\rangle + |0000, 1010\rangle)
+ \frac{\xi}{\sqrt{2}} \left( i (|1001, 0000\rangle - |0110, 0000\rangle) + (|1000, 0001\rangle - |0010, 0100\rangle) - (|0001, 1000\rangle - |0100, 0010\rangle) + i (|0000, 1001\rangle - |0000, 0110\rangle) \right). \tag{45}
\]

We can predict that the coincidence probability for the photons varies from 0 (fully indistinguishable) to 0.5 (fully distinguishable). We have demonstrated a classical simulation of the HOM dip experiment. Fig. 7 reveals a clear pattern of the HOM dip.
IV. CONCLUSIONS

We have proposed an efficient method for the digital simulation of linear-optical networks by using the property that suitably entangled fermions can effectively behave like bosons. Unlike the existing B2QE protocols, our approach provides a simple and intuitive extension of an ideal bosonic system to a non-ideal one by introducing additional internal degrees of freedom. We successfully executed a digital simulation of the HOM dip for partially distinguishable photons with our method using the IBM Quantum and IonQ cloud services.

The obvious extension of our B2QE approach would be the non-number-conserving bosonic system simulations, such as Gaussian boson sampling [33] and molecular simulations [34]. However, confining the infinite bosonic Hilbert space to the finite qubit Hilbert space will intrinsically generate errors for the non-number-conserving bosonic problems. In future work, we will attempt to optimize the required resources and errors induced by the confinement. We also intend to develop another efficient quantum algorithm for computing the matrix permanent based on our B2QE protocol. With the help of the new B2QE protocol, we envisage developing efficient qubit-based quantum algorithms for bosonic systems, e.g., the boson sampling with nonideal photons, the Bose-Hubbard model, and the spin-boson model.

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