On the compatibility of Lorentz metrics with linear connections on 4-dimensional manifolds

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Abstract. This paper considers 4-dimensional manifolds upon which there is a Lorentz metric $h$ and a symmetric connection $\Gamma$ and which are originally assumed unrelated. It then derives sufficient conditions on $h$ and $\Gamma$ (expressed through the curvature tensor of $\Gamma$) for $\Gamma$ to be the Levi-Civita connection of some (local) Lorentz metric $g$ and calculates the relationship between $g$ and $h$. Some examples are provided which help to assess the strength of the sufficient conditions derived.

Submitted to: Class. Quantum Grav.

PACS numbers: 04.20.-q, 04.20.Cv, 02.40.Ky
1. Introduction

If $M$ is a connected $n$-dimensional manifold admitting a smooth metric $g$ of arbitrary signature, associated Levi-Civita connection $\Gamma$ and corresponding curvature tensor $R$ then, if $\nabla$ denotes the covariant derivative from $\Gamma$, one has $\nabla g = 0$ and, in components in any coordinate system,

$$g_{ae} R^e_{bcd} + g_{be} R^e_{acd} = 0, \quad g_{ae} R^e_{bcd,f} + g_{be} R^e_{acd,f} = 0, \quad \ldots$$

(1)

where a semicolon denotes the covariant derivative $\nabla$ in component form. This has led to the following question (or variants of it) in the literature. Suppose a connected $n$-dimensional manifold $M$ admits a smooth symmetric linear connection $\Gamma$ with corresponding curvature tensor $R$ and suppose, in addition, that $M$ admits a global smooth metric $h$ of arbitrary signature such that, on $M$

$$h_{ae} R^e_{bcd} = 0$$

$$h_{ae} R^e_{bcd,f} = 0$$

(3.1)

for some integer $k$. Under which conditions is $\Gamma$ a metric (or locally metric) connection, that is, under which conditions does there exist a global metric $g$ on $M$, of arbitrary signature, whose Levi-Civita connection is $\Gamma$ (or, given $m \in M$, does there exist an open neighbourhood $U$ of $m$ and metric $g$ of arbitrary signature on $U$ whose Levi-Civita connection is the restriction to $U$ of $\Gamma$) and, if so, how are $g$ and $h$ related?

If one is somehow able to find the holonomy group of $\Gamma$ then the problem is partly solved because then $\Gamma$ is a metric connection if and only if for some $m \in M$ there is a non-degenerate quadratic form on the tangent space $T_m M$ to $M$ at $m$, of signature $(p, q)$ for non-negative integers $p, q$ with $p + q = n$, which is preserved by the holonomy group associated with $\Gamma$. If such is the case, $\Gamma$ is compatible with a metric $g$ on $M$ of signature $(p, q)$, that is $\nabla g = 0$, and the holonomy group of $\Gamma$ is a Lie-subgroup of $O(p, q)$ [1, 2]. In the case $n = 4$ and if $\Gamma$ is fixed and given to be the Levi-Civita connection of a Lorentz metric on $M$, the holonomy group of $\Gamma$ can be used to find all metrics on $M$ compatible with $\Gamma$ and these may (depending on the holonomy group) have any of the signatures $(1, 3)$, $(0, 4)$ or $(2, 2)$ [3]. Remaining in the $n = 4$ case and with $R$ fixed and assumed to arise from a Levi-Civita connection $\Gamma$ compatible with a Lorentz metric on $M$, it is known that this metric is generically determined up to a constant conformal factor and hence, generically, $\Gamma$ is uniquely determined [4, 5, 6, 7]. It is also known that on a large class of 4-dimensional manifolds a symmetric linear connection $\Gamma$ satisfying (2) for some metric $h$ on $M$ is necessarily locally metric (in the sense that each $m \in M$ admits an open neighbourhood $U$ such that the restriction of $\Gamma$ to $U$ is metric). However, the procedure involves the investigation of a $36 \times 36$ matrix and is geometrically obscure [8]. In this paper, the question raised at the beginning of this section for the case $n = 4$ and with $h$ of Lorentz signature will be considered, that is, if $M$ admits a symmetric linear
connection $\Gamma$, associated curvature $R$ and metric tensor $h$ of Lorentz signature (and not assumed related to $\Gamma$ or $R$ in any way) such that (2) and (3.1)-(3.k) hold for some $k$, is $\Gamma$ a metric (or locally metric) connection on $M$ and, if so, how is $h$ related to the metric or metrics compatible with $\Gamma$?

A standard notation will be used, with round and square brackets denoting the usual symmetrisation and and skew-symmetrisation of indices, respectively, and a comma denotes a partial derivative. A skew-symmetric tensor $F$ of type $(0,2)$ or $(2,0)$ at $m$ is called a bivector. If $F(\neq 0)$ is such a bivector, the rank of any of its (component) matrices is either two or four. In the former case, one may write (e.g. in the $(2,0)$ case) $F^{ab} = 2r^{[a}s^{b]}$ for $r, s \in T_mM$ (or alternatively, $F = r \wedge s$) and $F$ is called simple, with the 2-dimensional subspace (2-space) of $T_mM$ spanned by $r, s$ referred to as the blade of $F$. In the latter case, $F$ is called non-simple.

The metric $h(m)$ converts $T_mM$ into a Lorentz inner product space and thus it makes sense to refer to vectors in $T_mM$ and covectors in the cotangent space $T^*_mM$ to $M$ at $m$ (using $h(m)$ to give a unique isomorphism $T_mM \leftrightarrow T^*_mM$, that is, to raise and lower tensor indices) as being timelike, spacelike, null or orthogonal, using the signature $(-, +, +, +)$. The same applies to 1-dimensional subspaces (directions) and 2- and 3-dimensional subspaces of $T_mM$ or $T^*_mM$. A simple bivector at $m$ is then called timelike (respectively, spacelike or null) if its blade at $m$ is a timelike (respectively a spacelike or null) 2-space at $m$. A non-simple bivector $F$ at $m$ may, by a standard argument ([9], see also [1]), be shown to uniquely determine an orthogonal pair of 2-spaces at $m$, one spacelike and one timelike, and which are referred to as the canonical pair of blades of $F$. A tetrad $(l, n, x, y)$ of members of $T_mM$ is called a null tetrad at $m$ if the only non-vanishing inner products between its members at $m$ are $h(l, n) = h(x, x) = h(y, y) = 1$. Thus $l$ and $n$ are null.

It is also remarked that the tensor, with local components $R_{abcd} \equiv h_{ae}R^e_{bcd}$ satisfies $R_{abcd} = -R_{bacd} = -R_{abdc}$ and $R_{abcd} + R_{adbc} + R_{acdb} = 0$. It then follows after some index juggling (see, e.g. [10]), that $R_{abcd} = R_{cdab}$.

2. Preliminary Results

Let $M$ be a 4-dimensional smooth connected Hausdorff manifold admitting a smooth symmetric linear connection $\Gamma$ with associated curvature tensor $R$ and a global smooth Lorentz metric $h$ such that (2) holds. No relation between $h$ and $\Gamma$ is assumed other than (2). It will be convenient to describe a simple algebraic classification of $R$ (relative to $h$). This classification is easily described geometrically and is a pointwise classification. It has been described before in a more specific context (see, e.g. [4] [11]) but since its use here is slightly different, it will be briefly described.

Define a linear map $f$ from the 6-dimensional vector space of type $(2,0)$ bivectors at $m$ into the vector space of type $(1,1)$ tensors at $m$ by $f : F^{ab} \rightarrow R_{abcd}^aF^{cd}$. The condition (2) shows that if a tensor $T$ is in the range of $f$ then

$$h_{ae}T^e_b + h_{be}T^e_a = 0 \quad (\Rightarrow T_{ab} = -T_{ba}, \ T_{ab} = h_{ae}T^e_b) \quad (4)$$
and so $T$ can be regarded as a member of the matrix representation of the Lie algebra of the pseudo-orthogonal (Lorentz) group of $h(m)$. Using $f$ one can divide the curvature tensor $R(m)$ into five classes.

**Class A** This is the most general curvature class and the curvature will be said to be of (curvature) class A at $m \in M$ if it is not in any of the classes B, C, D or O below.

Class B The curvature tensor is said to be of (curvature) class B at $m \in M$ if the range of $f$ is 2-dimensional and consists of all linear combinations of type $(1,1)$ tensors $F$ and $G$ where $F_{a b}^a = x^a y^b - y^a x_b$ and $G_{a b}^a = l^a n_b - n^a l_b$ with $l, n, x, y$ a null tetrad at $m$. The curvature tensor at $m$ can then be written as

$$R_{abcd} \equiv h_{ac} R_{bcd}^e = \frac{\alpha}{2} F_{a b} F_{c d} - \frac{\beta}{2} G_{a b} G_{c d},$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0 \neq \beta$, the symmetrised cross term in $F$ and $G$ vanishes because $R_{a[bcd]} = 0$. This class can be further split in classes $B_1$ and $B_2$ where $B_1$ (respectively, $B_2$) applies if $\alpha \neq \beta$ (respectively, $\alpha = \beta$).

Class C The curvature tensor is said to be of (curvature) class C at $m \in M$ if the range of $f$ is 2- or 3-dimensional and if there exists $0 \neq k \in T_m M$ such that each of the type $(1,1)$ tensors in the range of $f$ contains $k$ in its kernel (i.e. each of their matrix representations $F$ satisfies $F_{a b}^a k = 0$).

Class D The curvature tensor is said to be of (curvature) class D at $m \in M$ if the range of $f$ is 1-dimensional. It follows that the curvature components satisfy

$$R_{abcd} = \lambda F_{a[b} F_{cd]}$$

at $m$ ($0 \neq \lambda \in \mathbb{R}$) for some bivector $F$ at $m$ which then satisfies $F_{a[b} F_{cd]}$ and is thus simple. It also follows that there exist two independent members $r, s \in T_m M$ such that $F_{a b} r^b = F_{a b} s^b = 0$ and hence that $r$ and $s$ lie in the kernel of each tensor in the range of $f$.

Class O The curvature tensor is said to be of (curvature) class O at $m \in M$ if it vanishes at $m$.

The following remarks can be checked in a straightforward manner [4].

(i) For the classes A and B there does **not** exist $0 \neq k \in T_m M$ such that $F_{a b}^a k^b = 0$ for every $F$ in the range of $f$.

(ii) For class A, the range of $f$ has dimension at least two and if this dimension is four or more the class is necessarily $A$.

(iii) The vector $k$ in the definition of class C is unique up to a scaling.

(iv) For the classes A and B there does **not** exist $0 \neq k \in T_m M$ such that $R_{a b c d}^a k^d = 0$, whereas this equation has exactly one independent solution for class C and two for class D.

(v) The five classes A, B, C, D and O are mutually exclusive and exhaustive for the curvature tensor at $m$. If the curvature class is the same at each $m \in M$ then $M$ will be said to be of that class.
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Denoting by the same symbols $A$, $B$ (or $B_1$ and $B_2$), $C$, $D$ and $O$ those subsets of $M$ consisting of precisely those points where the curvature tensor has that class, one may decompose $M$, disjointly as $M = A \cup B_1 \cup B_2 \cup C \cup D \cup O$ and then, disjointly, as

$$M = A \cup \text{int}B_1 \cup \text{int}B_2 \cup \text{int}C \cup \text{int}D \cup \text{int}O \cup Z$$

(6)

where $\text{int}$ denotes the manifold topology interior operator and $Z$ is the (necessarily closed) subset of $M$ defined by the disjointness. The idea is to show that $Z$ has empty interior, $\text{int}Z = \emptyset$. A similar decomposition was considered in [4] [11] [12], but in these works $B$ was not subdivided into $B_1$ and $B_2$ and so the ”leftovers” subset $Z$ may differ from that in (6) since $\text{int}B_1 \cup \text{int}B_2$ may be a proper subset of $\text{int}B$. To show that $\text{int}Z = \emptyset$ in (6) it is first noted that $A$ is open in $M$ (i.e. $\text{int}A = A$) and that $A \cup B$ is also open in $M$ [4]. So let $U \subseteq Z$ be open in $M$. Then by the disjointness of the decomposition (6) $U \cap A = \emptyset$ and so $U \cap (A \cup B) = U \cap B = W$ and $W$ is open in $M$. The disjointness of (6) also shows that, if $W \neq \emptyset$, $W$ must intersect each of $B_1$ and $B_2$ non-trivially, so let $m \in W \cap B_1$ and consider the linear map $f'$ from the vector space of type $(2,0)$ bivectors at $m$ into itself given by $f' : F^{ab} \rightarrow R^{ab}_{\ cd}F^{cd}$. The characteristic polynomial of this map has, from (5), three distinct solutions $\alpha$, $\beta$ and $0$ at $m$ and since the first two of these are simple roots they give rise to smooth functions $\tilde{\alpha}$ and $\tilde{\beta}$ defined on some neighbourhood $V$ of $m$ (and with $\tilde{\alpha}(m) = \alpha$ and $\tilde{\beta}(m) = \beta$) which are solutions of the characteristic polynomial of $f'$ and distinct from each other and from zero in $V$ [13]. It follows that $m \in V \cap W \subseteq B_1$ with $V \cap W$ open and disjoint from $B_2$ (since at points of $B_2$ the characteristic polynomial of $f'$ has only two distinct solutions $\alpha (= \beta)$ and 0). Thus $V \cap W \subseteq \text{int}B_1$ and hence $U \cap \text{int}B_1 \neq \emptyset$, contradicting the disjointness of (6). It follows that $W$ and hence $U \cap B$ are empty. From here the argument follows [4] to get $U = \emptyset$ and hence $\text{int}Z = \emptyset$. The following is thus established.

**Theorem 1** In the disjoint decomposition (6), $Z$ is closed, $\text{int}Z = \emptyset$ and so $M \setminus Z$ is an open dense subset of $M$.

It is remarked that the reason for decomposing $B$ as $B_1 \cup B_2$ is that the above argument leading to the local smooth functions $\tilde{\alpha}$ and $\tilde{\beta}$ and which extend $\alpha$ and $\beta$ does not apply if $\alpha = \beta$ at $m$ unless $m \in \text{int}B_2$, (i.e. the problem lies with the points in $B_2 \setminus \text{int}B_2$).

It is also remarked that in the open subset $\text{int}C$ of $M$ the non-zero tangent vectors $k$ introduced in the definition of this type give rise to a smooth integrable 1-dimensional distribution on $\text{int}C$. To see this one recalls that the curvature tensor satisfies $R^{\alpha}_{\ bcd}k^d = 0$ at each $m \in \text{int}C$ and that $k$ is unique (up to a scaling) in satisfying this equation. The smoothness of the distribution defined follows from the smoothness of the curvature in the following way [14] [4]. Let $m \in \text{int}C$ and let $0 \neq k \in T_mM$ satisfy the above equation. Choose a coordinate domain $U \subseteq \text{int}C$ about $m$ in which $k^4 = 1$ at $m$. The conditions $R^{\alpha}_{\ bcd}k^d = 0$ on $U$ then reduce to a system of equations of the form $\sum_{\alpha=1}^{3} E_{\alpha p}k^\alpha = E_p$ for functions $E_{\alpha p}$ and $E_p : U \rightarrow \mathbb{R}$ and $p = 1,2,3$. Since the linear system so formed has rank equal to three at $m$ (since there the $k^\alpha$ are uniquely determined) then by taking $k^4 = 1$ on $U$ one obtains a linear system for the $k^\alpha$ of rank
three over $U$. Cramer’s rule then reveals smooth solutions for the components of $k^a$ and hence a local nowhere zero vector field $k$ satisfying $R^a_{bcd}k^d = 0$.

In the open subset $\text{int}D$ it can be shown that for $m \in \text{int}D$ there exists an open coordinate neighbourhood $U \subseteq \text{int}D$ of $m$ on which the curvature tensor satisfies $R_{abcd} = \lambda F_{ab}F_{cd}$ where $F$ is a smooth bivector and $\lambda$ a smooth real-valued function on $U$. A similar argument to the previous one shows that $U$ may be chosen so that there are smooth vector fields $r$ and $s$ on $U$ such that at each $m' \in U$, $r(m')$ and $s(m')$ are independent and satisfy $R^a_{bcd}r^d = R^a_{bcd}s^d = 0$ [13].

If $m \in \text{int}B_1$ the curvature tensor takes the form (5) over some coordinate neighbourhood $U \subseteq \text{int}B_1$ of $m$ with $\alpha, \beta$ smooth nowhere zero, nowhere equal real-valued functions and $F$ and $G$ smooth bivectors on $U$. Similar remarks apply if $m \in \text{int}B_2$ with $\alpha = \beta$ on $U \subseteq \text{int}B_2$. For the latter two ($B_1$ and $B_2$) cases, a null tetrad $l, n, x, y$ of smooth vector fields exists on $U$ satisfying (5) on $U$.

3. The Main Results

Let $M$, $\Gamma$, $R$ and $h$ be as described at the beginning of section 2 and suppose that (2) holds. If (3.1) to (3.9) also hold then it follows that on $M$

$$h_{ae;f_1} R^e_{bcd} + h_{be;f_1} R^e_{acd} = 0$$

(7.1)

$$\vdots$$

$$h_{ae;f_1 \ldots f_k} R^e_{bcd} + h_{be;f_1 \ldots f_k} R^e_{acd} = 0$$

(7.9)

and conversely the set of equations (2) and (7.1) to (7.9) imply the set (2) and (3.1) to (3.k). Thus the conditions (2) and (3.1) to (3.9) are equivalent to the conditions (2) and (7.1) to (7.9). The aim is to use this equivalence to show when the original conditions (2) and (3.k) for a particular $k$ imply the existence of a local or global Lorentz metric $g$ on $M$ compatible with $\Gamma$ and to display the relationship between $g$, $h$ and the geometry of $R$ as expressed through its curvature type. The method to be used employs a theorem in [15] and involves the following idea. Let $m \in M$ and $T$ be a type $(0, a)$ tensor at $m$ with $a \geq 2$ and with $T$ symmetric in its first two indices (so $T$ has components $T_{abc \ldots d}$ and $T_{abc \ldots d} = T_{bac \ldots d}$. Suppose also that at $m$ (cf. (2))

$$T_{aeg \ldots h} R^e_{bcd} + T_{beg \ldots h} R^e_{acd} = 0$$

(8)

Then for any tensor $S$ of type $(a - 2, 0)$, the tensor $h'_{ab} = T_{abc \ldots d}S^{c \ldots d}$ is symmetric and satisfies (2). This means that for any $F$ in the range of the map $f$ at $m$ described earlier

$$h'_{ae} F^e_b + h'_{be} F^e_a = 0$$

(9)

Each such $F$ in the range of $f$ imposes strong algebraic constraints on $h'$, these constraints being conveniently written down for each of the curvature classes for $R$ at $m$ in terms of the original Lorentz metric $h$ and the features of the particular curvature class. They are, in the notation of section 2 [15]
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| Class A | $h'_{ab} = \alpha h_{ab}$ \quad (10a) |
| Class B | $h'_{ab} = \alpha h_{ab} + 2\beta l(a n_b) = (\alpha + \beta)h_{ab} - \beta(x_a x_b + y_a y_b)$ \quad (10b) |
| Class C | $h'_{ab} = \alpha h_{ab} + \beta k_a k_b$ \quad (10c) |
| Class D | $h'_{ab} = \alpha h_{ab} + \beta r_a r_b + \gamma s_a s_b + 2\delta r(a s_b)$ \quad (10d) |

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and the completeness relation $h_{ab} = 2l(a n_b) + x_a x_b + y_a y_b$ was used in (10b). These expressions allow, depending on the curvature class, expressions for the covariant derivatives of $h$ to be written out reasonably conveniently since, from (2) and (7.1) to (7.k), they satisfy the conditions asked for in (8). Hence, for any tensor $S$ of the appropriate type, each tensor $h'_{ab} = h_{abc...d}S^{c...d}$ (for the appropriate number of derivatives) satisfies (9) for each such $F$ and hence the appropriate equation in the set (10a) to (10d). From the arbitrariness of $S$ one obtains the desired expressions for these covariant derivatives. For example, if the curvature class at $m$ is $C$ then, at $m$, and if (7.1) and (7.2) are assumed

$$h_{abc} = h_{ab}\alpha_c + k_a k_b\beta_c, \quad h_{abcd} = h_{ab}\alpha_{cd} + k_a k_b\beta_{cd}$$

for covectors $\alpha_c$ and $\beta_c$ and tensors $\alpha_{cd}$ and $\beta_{cd}$ at $m$.

These ideas can now be applied to the open subsets $A$, int$B_1$, int$B_2$, int$C$ and int$D$ in the decomposition (6).

**Theorem 2** Let $M$ be a smooth 4-dimensional connected manifold admitting a smooth symmetric linear connection $\Gamma$ and associated curvature $R$ and also admitting a smooth Lorentz metric $h$ such that (2) holds. Then

(i) if, on $A$, (3.1) also holds, $\Gamma$ is compatible with a local Lorentz metric on $A$,

(ii) if, on int$B_1$ (respectively int$B_2$), (3.1) also holds, $\Gamma$ is compatible with a local Lorentz metric on int$B_1$ (respectively int$B_2$),

(iii) if, on int$C$, (3.1) and (3.2) also hold, $\Gamma$ is compatible with a local Lorentz metric on an open dense subset of int$C$,

(iv) if, on int$D$, (3.1), (3.2) and (3.3) also hold, $\Gamma$ is compatible with a local Lorentz metric on an open dense subset of int$D$.

**Proof**

(i) The remarks preceding the theorem (see (10a)) together with the imposition of (3.1) or, equivalently, (7.1) show that $h_{abc} = h_{ab}w_c$ for some 1-form $w$ on the open subset $A$ which is easily seen to be smooth since $h$ and $\Gamma$ are. Now use (2) and the Ricci identity on $h$ to get

$$h_{abc}c_{[cd]} = 0 \quad (\Rightarrow h_{ab}w_{[cd]} = 0)$$

Thus $w_{[cd]} = 0$ and so $w_a$ is locally a gradient. Hence, for each $m \in A$, there is an open neighbourhood $W(\subseteq A)$ of $m$ on which $w_a = w_c$ for some smooth function $w : W \to \mathbb{R}$. Then on $W$, $g_{ab} = e^{-w}h_{ab}$ satisfies $g_{abc} = 0$ and is a local Lorentz metric for $\Gamma$. Further, if $g'$ is any other local metric defined on some neighbourhood $W'$ of $m$ and compatible with $\Gamma$ then $g'$ satisfies (2) on $W'$ and hence, on $W \cap W'$, $g' = \phi g$ for some positive smooth function $\phi : W \cap W' \to \mathbb{R}$ (see (10a)). From this and the result $g'_{abc} = 0$ it
follows that $g'$ is a constant multiple of $g$ on $W \cap W'$.

(ii) The remarks preceding the theorem (see (10b)) together with the imposition of (3.1) and (7.1) show that for each $m \in \text{int}B_1$ (respectively, $m \in \text{int}B_2$) there is a coordinate neighbourhood $U \subseteq \text{int}B_1$ (respectively, $U \subseteq \text{int}B_2$) such that on $U$

\[ h_{abc} = h_{ab}w_c + 2l(a)n_b)\lambda_c \]  

(13)

where $l$ and $n$ are the smooth null members of the null tetrad $l, n, x, y$ on $U$ established in section 2 and $w$ and $\lambda$ are (necessarily) smooth 1-forms on $U$. Since the contravariant components of $h$ satisfy $h^{ac}h_{cb} = \delta^a_b$, the result $(h^{ac}h_{cb})_c = 0$ implies that

\[ h^{ab} = -h^{ab}w_c - 2l(a)n_b)\lambda_c \]  

(14)

Thus $h^{ab}$ contracts with $l^a_n$ on $U$. It easily checked that the nowhere zero symmetric tensor $h$.

Now $U$ can be chosen so that, in addition to the above results, the curvature tensor satisfies (5) on $U$ with $\alpha, \beta$ smooth nowhere zero functions and $F$ and $G$ as given in (5) in terms of the null tetrad on $U$. The Bianchi identities between $R$ and $\Gamma$ are $R^a_{bcde} = 0$. One now employs a technique first used in [16] (where $h$ was, in fact, a metric compatible with $\Gamma$). First, one writes out the Bianchi identity using (5) and contracts with $l_{a}x^{b}y^{d}x^{c}$. This operation eliminates all terms except one and gives, on $U$

\[ \beta G^a_{b;c}G_{cd}l_{a}x^{b}y^{d}x^{c} = 0 \]  

(15)

Similar contractions with $l_{a}x^{b}y^{d}x^{e}$ and $l_{a}x^{b}y^{d}x^{c}y^{d}x^{e}$ and the result $l^a y_{a;b} = -y^a l_a$ which follows from applying (14) to the expansion of $(h^{ab}l_{a}y_{b})_c = 0$ lead to

\[ l_{a;b}x^a y^b = l_{a;b}y^a x^b = l_{a;b}y^a n^b = 0 \]  

(16)

Observations of the symmetry in these contractions then give

\[ l_{a;b}x^a y^b = l_{a;b}y^a x^b = l_{a;b}x^a l^b = l_{a;b}x^a n^b = 0 \]  

(17)

From these results it easily follows that $l_{a;b}x^a = l_{a;b}y^a = 0$. Further, one has $(h_{ab}l^a b)_c = 0$ and so, using (13), one has $l^a y^a = 0$ and hence $l_{a;b}l^a = 0$. It follows that $l_{a;b} = l_{a;p}$ for some smooth 1-form $p$ on $U$ and then, from (14), that $l^a_{;b} = l^a(p_b - w_b - \lambda_b)$. Thus $l^a$ and $\lambda_a$ are recurrent on $U$. Similar arguments show that $n^a$ and $\lambda_a$ are recurrent on $U$ and, since $(l^a n_a)_b = 0$, that $n_{a;b} = n_a(\lambda_b + w_b - p_b)$ and $n^a_{;b} = -n^a p_b$. It is then easily checked that the nowhere zero symmetric tensor $T_{a;b} = 2l(a)n_b$ on $U$ is also recurrent in the sense that $T_{a;bc} = T_{ab}r_c$ where $r$ is the 1-form $r_a = \lambda_a + w_a$ on $W$. When the Ricci identity is applied to $T_{ab}$ and use is made of (5) one gets

\[ T_{a;[cd]} = 2n_c l(a)R^c_{b)d} + 2l_c n(a)R^c_{b)d} = 0 \]  

(18)

and so $r_{[a;b]} = 0$. Thus one may assume, by shrinking $W$ if necessary, that, on $W$, $r_a = r_c$ for a smooth function $r : W \to \mathbb{R}$. It follows that, on $W$, the tensor $t_{ab} = e^{-r} T_{ab}$ is symmetric, nowhere zero and covariantly constant, $t_{ab;c} = 0$. When (13) is rewritten as $h_{abc} = h_{ab}w_c + t_{ab}c$ for some smooth 1-form $\gamma_a$ on $W$ and the Ricci identity is applied to $h$ (using (2)) one finds

\[ h_{ab}w_{c;d} + t_{ab}(\gamma_{[c;d]} - \gamma_{c}w_{d}) = 0 \]  

(19)
At each point of $W$ the matrices $h_{ab}$ and $t_{ab}$ have (different) ranks four and two, respectively, and so the two quantities on the left hand side of (19) must vanish separately. The first of these equations leads, as before, to the fact that $w_a$ is locally a gradient and the second gives $\gamma_{[c,d]} - \gamma_{[e]w_d} = 0$. Thus by again shrinking $W$ if necessary one has $w_a = w_a$ on $W$ and then the second condition can be rewritten as

\[(e^{-w}\gamma_a)_b - (e^{-w}\gamma_b)_a = 0\]  

(20)

This means, by again reducing $W$ if necessary, that there exists a smooth function $\delta : W \to \mathbb{R}$ such that $\gamma_a = e^w\delta_a$. Now consider a tensor $g$ on $W$ of the form

\[g_{ab} = \phi h_{ab} + \epsilon t_{ab}\]  

(21)

for smooth functions $\phi, \epsilon : W \to \mathbb{R}$ with $\phi$ nowhere zero. Then, by again considering rank on the indices $a, b$ one sees that the condition that $g$ is covariantly constant, $g_{ab}^{;c} = 0$, is equivalent to the two differential equations

\[\phi_a + \phi w_a = 0 \quad \epsilon_a + \phi e^w\delta_a = 0\]  

(22)

which are to be regarded as equations to find $\phi$ and $\epsilon$ with $w$ and $\delta$ given. The general solutions are $\phi = De^{-w}$ and $\epsilon = C - D\delta$ for $C, D \in \mathbb{R}$ and so if $g_{ab} = De^{-w}h_{ab} + (C - D\delta)t_{ab}$, then $g_{abc} = 0$. Further, it is clear that $C$ and $D$ may be chosen so that $g$ is non-degenerate at $m$ and hence on a (possibly reduced) open neighbourhood $W$ of $m$. It is also clear that the original null tetrad $l, n, x, y$ (with respect to $h$) is, after a possible smooth rescaling of these tetrad members, a null tetrad with respect to $g$ on $W$ and so $g$ is a Lorentz metric on $W$ compatible with $\Gamma$. This construction of $g$ and (10b) show that any Lorentz metric compatible with $\Gamma$ is of this general form on some open neighbourhood of $m$.

(iii) The assumption that (2), (3.1) and (3.2) hold on int$C$ means that (7.1) and (7.2) also hold on int$C$ and hence that for $m \in$ int$C$, there is an open neighbourhood $U$ of $m$ and a smooth nowhere zero vector field $k$ on $U$ such that

\[h_{abc} = h_{ab}w_c + k_a k_b \lambda_c\]  

(23)

\[h_{abcd} = h_{ab}X_{cd} + k_a k_b Y_{cd}\]  

(24)

for necessarily smooth 1-forms $w$ and $\lambda$ and tensors $X$ and $Y$. If (23) is covariantly differentiated and equated with (24) and the resulting equation contracted at each $m \in U$ with $t^a t^b$, where $t \in T_m \mathbb{M}$ is such that $h_{ab} t^a t^b \neq 0$, $h_{ab} k^a t^b = 0$, one finds that on $U$

\[X_{ab} = w_a w_b + w_{a;b}\]  

(25)

When this information is replaced in the equation from which it came, one finds, after cancellation, that

\[k_a(k_b Y_{cd} - k_b \lambda_{cd} - k_b w_c \lambda_d - k_b d \lambda_c) = k_{a;d} k_b \lambda_c\]  

(26)

The value of $\lambda_a$ in (23) depends, of course on the choice of $k$ and $U$ there being the freedom to replace $k$ by $\mu k$ for some nowhere zero smooth function $\mu : U \to \mathbb{R}$. However,
the condition that \( \lambda(m) = 0 \) is independent of the choice of \( k \). So let \( C_1 \) (respectively \( C_2 \)) be the subset of points of \( \text{int} C \) at which this condition holds (respectively, does not hold). Then \( C_2 \) is open in \( \text{int} C \) and hence in \( M \) and one has a disjoint decomposition of \( \text{int} C \) in the form

\[
\text{int} C = C_1 \cup C_2 = \text{int} C_1 \cup C_2 \cup Z_1
\]

where \( \text{int} C_1 \) is the interior of \( C_1 \) in either the topology of \( M \) or the subspace topology of \( \text{int} C \) (these being equal). The subset \( Z_1 \) is defined by the disjointness of the decomposition (27) and hence, since \( Z_1 \subseteq C_1, \ Z_1 \) has empty interior in either \( M \) or \( \text{int} C \). Thus \( \text{int} C_1 \cup C_2 \) is open in \( M \) and open and dense in \( \text{int} C \). It follows from (26) that \( k \) is recurrent on \( C_2, k_{ab} = k_a p_b \), where \( p \) is now a smooth 1-form on \( C_2 \).

On \( \text{int} C_1, \lambda = 0 \) and so (23) gives \( h_{abc} = h_{ab} w_c \). A similar argument to one earlier then shows that each \( m \in \text{int} C_1 \) admits an open neighbourhood \( W \) on which \( w_a = w_{,a} \) and on which \( g_{ab} = e^{-w} h_{ab} \) is then a Lorentz metric compatible with \( \Gamma \).

On \( C_2 \), the recurrence of \( k \) together with the properties of class \( C \) and the Ricci identity give

\[
k_{a:b;c} = 0 \quad \Rightarrow k_a p_{[b;c]} = 0 \Rightarrow p_{[a;b]} = 0
\]

and so \( p_a \) is locally a gradient. Thus each \( m \in C_2 \) admits a neighbourhood \( W \) on which \( p_a = p_{,a} \) for some smooth function \( p : W \to \mathbb{R} \). Thus on \( W \) the 1-form \( k'_a = e^{-p} k_a \) is covariantly constant, \( k'_{a;b} = 0 \). Now return to (23), which can be assumed to hold on \( W \) with \( k_a \) replaced by \( k'_a \) (and the prime on \( k' \) dropped), construct \( h_{ab;cd} \) and skew symmetrise over the indices \( c \) and \( d \) (using \( h_{ab;[cd]} = 0 \) from (2)) to get

\[
h_{ab} w_{[c;d]} + k_a k_b (\lambda_{[c;d]} - \lambda_{[c} w_{d]}) = 0
\]

Now one proceeds in a similar manner to that in part \((ii)\) to get the relations

\[
w_{[a;b]} = 0 \quad \lambda_{[a;b]} - \lambda_{[a} w_{b]} = 0
\]

From this one has on \( W \) (by reducing \( W \) if necessary) a function \( w : W \to \mathbb{R} \) such that \( w_a = w_{,a} \) and a relation like (20) with \( \gamma_a \) replaced by \( \lambda_a \) and hence a function \( \delta : W \to \mathbb{R} \) such that \( \lambda_a = e^w \delta_a \). The potential metrics on \( W \) are thus of the form

\[
g_{ab} = \phi h_{ab} + \epsilon k_a k_b
\]

for necessarily smooth functions \( \phi, \epsilon : W \to \mathbb{R} \) and with \( \phi \) positive which, from \( g_{abc} = 0 \), are easily seen to satisfy the same conditions as \( \phi \) and \( \epsilon \) do in (22). Thus the general solution is \( \phi = De^{-w} \) and \( \epsilon = C - D \delta \) for \( C, D \in \mathbb{R} \) chosen to preserve Lorentz signature on a (possibly reduced) open subset of \( W \) (and the signature is automatically Lorentz at those points of \( W \) where \( k \) is null with respect to \( h \)). Thus \( g \) is a Lorentz metric on \( W \) compatible with \( \Gamma \).

\((iv)\) For this part one assumes that (2) together with (3.1), (3.2) and (3.3) hold. This is equivalent to assuming (2) together with (7.1), (7.2) and (7.3). Thus if \( m \in \text{int} D \) there are smooth covector fields \( r \) and \( s \) defined on some open neighbourhood \( U \) of \( m \) such
that \( r(m') \) and \( s(m') \) are independent members of \( T^*_m M \) at each \( m' \in U \) (section 2) and such that, on \( U \),

\[
\begin{align*}
    h_{abc} &= h_{ab}w_c + r_ar_b\beta_c + s_as_b\gamma_c + 2r(as)b\delta_c \\
    h_{abcd} &= h_{ab}X_{cd} + r_ar_bY_{cd} + s_as_bZ_{cd} + 2r(as)bW_{cd} \\
    h_{abcde} &= h_{ab}X_{cde} + r_ar_bY_{cde} + s_as_bZ_{cde} + 2r(as)bW_{cde}
\end{align*}
\]

where \( w_a, \beta_a, \ldots W_{abc} \) are smooth tensors of the appropriate type on \( U \). Now, by shrinking \( U \) if necessary, one may assume that, in addition to the smooth covector fields \( r \) and \( s \) on \( U \), there are smooth covector fields \( e \) and \( f \) on \( U \) such that \( r(m'), s(m'), e(m') \) and \( f(m') \) are independent members of \( T^*_m M \) at each \( m' \in U \). Then one defines a positive definite metric \( \gamma \) on \( U \) by \( \gamma_{ab} = r_ar_b + s_as_b + e_ae_b + f_af_b \) (and where \( \gamma^{ab} \) temporarily denotes the inverse of \( \gamma_{ab} \) and not indices raised with metric \( h \)). Then define smooth vector fields \( R, S, E \) and \( F \) on \( U \) by \( R^a = \gamma^{ab}r_b, \ldots, F^a = \gamma^{ab}f_b \) and smooth covector fields \( p, q, p' \) and \( q' \) on \( U \) by \( p_a = R^bp_{ba}, q_a = S^b_{ba}, p'_a = R^b_s_{b}a \) and \( q'_a = S^b_s_{b}a \). Finally define two type \((0,2)\) tensor fields \( u \) and \( u' \) on \( U \) by

\[
\begin{align*}
    r_{a;b} &= r_ap_b + s_aq_b + u_{ab} \\
    s_{a;b} &= r_ap_b + s_aq_b' + u'_{ab}
\end{align*}
\]

Thus \( R, S, E \) and \( F \) form a \( \gamma \)-orthogonal tetrad at each point of \( U \) and a contraction of each of (35) and (36) with \( R^a \) and with \( S^a \) gives

\[
R^au_{ab} = R^au'_{ab} = S^au_{ab} = S^au'_{ab} = 0
\]

Now take the covariant derivative of (32) using (35) and (36) and equate to (33) [It is noted at this point that the above construction of the vector fields \( R, S, E \) and \( F \) may be achieved whilst, at the same time, ensuring that \( E \) is nowhere null on \( U \) with respect to \( h \) (that is, \( h_{ab}E^aE^b \) is nowhere zero on \( U \))]. A contraction of the resulting equation with \( E^aE^b \) gives

\[
X_{ab} = w_{a;b} + w_aw_b
\]

whilst successive contractions of this same equation with \( R^aR^b, S^aS^b \) and \( R^aS^b \), using (37) and taking into account (38) give

\[
\begin{align*}
    Y_{ab} &= w_a\beta_b + 2\beta_ap_b + 2\delta_ap'_b + \beta_{a;b} \\
    Z_{ab} &= w_a\gamma_b + 2\gamma_aq'_b + 2\delta_ao_b + \gamma_{a;b} \\
    W_{ab} &= w_\delta_{a}b + \beta_aq_b + \gamma_ap'_b + \delta_ao_b + \delta_{a;b}
\end{align*}
\]

A back substitution of (39) into the equation from which they arose and using (35) and (36) then gives after a long but straightforward calculation

\[
r_{(a}u_{b)d}\beta_c + s_{(a}u_{b)d}\delta_c + r_{(a}u'_{b)d}\delta_c + s_{(a}u'_{b)d}\gamma_c = 0
\]

Finally, contractions of (40) with \( R^a \) and \( S^a \) using (37) give, respectively, at \( m \)

\[
\begin{align*}
    u_{bd}\beta_c + u'_{bd}\delta_c &= 0 \\
    u'_{bd}\gamma_c + u_{bd}\delta_c &= 0
\end{align*}
\]
Now given \( m \in U \), as above, the covector fields \( r \) and \( s \) are determined up to changes \( r \to r' \) and \( s \to s' \) where

\[
r' = \rho r + \sigma s \quad s' = \rho' r + \sigma' s
\]

where \( \rho, \sigma, \rho' \) and \( \sigma' \) are smooth functions : \( U \to \mathbb{R} \) (with \( U \) possibly reduced) such that \( r'(m') \) and \( s'(m') \) are independent at each \( m' \in U \). The covector field \( w \) in (32) is clearly independent of the choice of \( r \) and \( s \) but the covector fields \( \beta, \gamma \) and \( \delta \) in (32) are not. However, the condition that \( \beta(m) = \gamma(m) = \delta(m) = 0 \) is independent of the choice of \( r \) and \( s \). Let \( D_1 \) (respectively, \( D_2 \)) be the subset if \( \text{int} D \) of those points where this condition is satisfied (respectively, not satisfied). Then \( \text{int} D_1 \) and \( D_2 \) are open in \( \text{int} D \) and hence in \( M \). Thus \( \text{int} D \) admits the disjoint decomposition \( \text{int} D = \text{int} D_1 \cup \text{int} D_2 \cup Z_2 \) where \( Z_2 \) has empty interior in \( \text{int} D \) and in \( M \). Next, let \( m \in D_2 \). The tensors \( u \) and \( u' \) at \( m \) depend on the choice of \( r \) and \( s \) from (35), (36) and (42). However, it is straightforward to check that the condition that \( u(m) = u'(m) = 0 \) is independent of the choice of \( r \) and \( s \). So let \( D_3 \) (respectively, \( D_4 \)) be the subset of \( D_2 \) of those points where this condition is satisfied (respectively, not satisfied). Then \( \text{int} D_3 \) and \( D_4 \) are open subsets of \( D_2 \) and hence of \( M \) and one has a disjoint decomposition of \( D_2 \) given by \( D_2 = \text{int} D_3 \cup \text{int} D_4 \cup Z_3 \) where \( Z_3 \) has empty interior in \( M \). Thus one has a disjoint decomposition of \( \text{int} D \) given by

\[
\text{int} D = \text{int} D_1 \cup \text{int} D_3 \cup \text{int} D_4 \cup Z_4
\]

where \( Z_4 = Z_2 \cup Z_3 \). Now \( Z_3 \subseteq D_2 \) and \( Z_2 \cap D_2 = \emptyset \) and so if \( V(\neq \emptyset) \) is open in \( M \) with \( V \subseteq Z_4 \) then, since \( Z_2 \) and \( Z_3 \) have empty interiors in \( M \), \( V \) cannot be contained in either \( Z_2 \) or \( Z_3 \) and hence \( V \cap Z_2 \neq \emptyset \neq V \cap Z_3 \). But since \( D_2 \) is open, \( V \cap D_2 \) is open and a subset of \( Z_3 \) and is hence empty. Now \( V \cap Z_3 \subseteq V \cap D_2 \) and so one achieves the contradiction that \( V \cap Z_3 = \emptyset \). It follows that \( V = \emptyset \) and so \( Z_4 \) has empty interior in \( M \).

Let \( m \in \text{int} D_1 \) so that, from (32), \( h_{ab,c} = h_{ab} w_c \) on some open neighbourhood \( U \) of \( m \). Then, by reducing \( U \) if necessary, one has, as before (see (12)), a smooth function \( w : U \to \mathbb{R} \) such that \( w_a = w_{,a} \) and then \( e^{-w} h_{ab} \) is a Lorentz metric on \( U \) compatible with \( \Gamma \).

Let \( m \in \text{int} D_3 \) so that, whichever covector fields \( r \) and \( s \) are chosen, (35) and (36) hold on some open neighbourhood \( U \) of \( m \) with \( u \) and \( u' \) zero on \( U \). Then it follows from these equations and the definition of the curvature class \( D \) that

\[
\begin{align*}
    r_a R^a_{bcd} &= 0, & r_a R^a_{bcd;f_1...f_n} &= 0 \quad (n = 1, 2, \ldots) \\
    s_a R^a_{bcd} &= 0, & s_a R^a_{bcd;f_1...f_n} &= 0 \quad (n = 1, 2, \ldots)
\end{align*}
\]

Thus the infinitesimal holonomy group of \( \Gamma \) is 1-dimensional at each \( m' \in U \) and, given \( U \) is chosen connected and simply-connected (as it always can be), is isomorphic (as a Lie group) to the holonomy group of \( \Gamma \) (restricted to \( U \)) and with holonomy algebra represented by the matrices in the range of the map \( f \) described in section 2. Since (2) holds this holonomy group is a subgroup of the orthogonal group of \( h \) and it follows that
a metric on $U$ compatible with $\Gamma$ is now assured [2] (see, also [1]). To actually construct such a metric one notes that the members of $T_n\mathfrak{M}$ associated with $r(m)$ and $s(m)$ under $h$ annihilate each matrix in the above representation of the holonomy algebra. Hence, by exponentiation, they give rise to holonomy invariant (i.e. covariantly constant) vector fields on $U$ which span the same distribution on $U$ as do the vector fields associated with $r$ and $s$ under $h$ (see, e.g. [1]). Hence one may now suppose, using the freedom (42), that $r$ and $s$ above are covariantly constant. In this case, covariantly differentiating (32) and using the Ricci identity for $h$ and (2), one finds

\[ w_{[ab]} = 0, \quad \beta_{[ab]} = \beta_{[ab]} w, \quad \gamma_{[ab]} = \gamma_{[ab]} w, \quad \delta_{[ab]} = \delta_{[ab]} w \]  

(45)

and so since $U$ is simply connected, $w_a = w_a$ with $w : U \to \mathbb{R}$. Then (45) shows that $e^{-w}\beta$, $e^{-w}\gamma$ and $e^{-w}\delta$ are closed 1-forms on $U$ and so there exist $\mu, \nu, \lambda : U \to \mathbb{R}$ such that $\beta_a = e^w \mu_a$, $\gamma_a = e^w \nu_a$, and $\delta_a = e^w \lambda_a$. Now construct the tensor $g$ on $U$ by

\[ g_{ab} = e^{-w} h_{ab} + (c_1 - \mu) r_a r_b + (c_2 - \nu) s_a s_b + 2(c_3 - \lambda) r(a) s_b \]  

(46)

where $c_1, c_2, c_3 \in \mathbb{R}$. It is easily checked that $g$ is covariantly constant and, by appropriate choice of $c_1, c_2$ and $c_3$ (e.g. $c_1(m) = \mu(m)$, $c_2(m) = \nu(m)$ and $c_3(m) = \lambda(m)$), $g$ can be made to have Lorentz signature in some neighbourhood of $m$.

Now let $m \in D_4$ so that at least one of $\beta(m)$, $\gamma(m)$ and $\delta(m)$ is not zero. Then, by using the freedom permitted by (42), one may arrange that $\beta$, $\gamma$ and $\delta$ are each non-zero at $m$ and hence in some open neighbourhood $U \subseteq D_4$ of $m$. It then follows from (41) and the definition of the region $D_4$ that $u$ and $u'$ are nowhere zero and proportional on $U$ so that, on $U$, $u' = \kappa u$ with $\kappa : U \to \mathbb{R}$ nowhere zero. Now one can again use the freedom in (42) by replacing $r$ by $r' = \kappa r - s$ (with $s$ unchanged) to achieve (35) and (36) with $u$ zero and $u'$ nowhere zero on $U$. It then follows from (41) that $\gamma$ and $\delta$ vanish on $U$ (and hence that $\beta$ is nowhere zero on $U$). Then (32) is

\[ h_{abc} = h_{ab} w_c + r_a r_b \beta_c \]  

(47)

where, for convenience, the prime on $r$ is omitted. From (47) and (35) (with $u = 0$) one then finds

\[ h_{abcd} = h_{ab}(w_{cd} + w_c w_d) + r_a r_b (\beta_{cd} + w_c \beta_d + 2 \beta c p d) + 2 \beta_{(a} s_{b)} \beta_{c} q d \]  

(48)

At this point the extra condition (3.3) (or, equivalently (7.3) or (34)) is introduced. Thus one takes one more covariant derivative of (48) and equates it to the right hand side of (34) to obtain on equation on $U$ of the form

\[ h_{ab} A_{cde} + r_a r_b B_{cde} + s_a s_b C_{cde} + 2 r_{(a} s_{b)} D_{b c d} + 2 r_{(a} u_{b)} e \beta_{c} q d = 0 \]  

(49)

where the tensors $A, B, C$ and $D$ can be calculated but whose exact form is not needed. A contraction with $E^a E^b$ (cf. the proof of (38)) gives $A = 0$ on $U$ whilst contractions with $R^a R^b$, $S^a S^b$ and $R^a S^b$ reveal that $B = C = D = 0$ on $U$. Thus, on $U$, (49) becomes

\[ r_a u_{b e} \beta_{c} q d + r_b u_{a e} \beta_{c} q d = 0 \]  

(50)

A contraction of (50) with $R^a$ and the use of (37) then reveals that $q_a = 0$ on $U$ (since $u'$ and $\beta$ are nowhere zero on $U$). Thus from (35), $r_{a b} = r_{a} p_b$ and then the Ricci identity
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on \( r \) reveals that \( p_{[a;b]} = 0 \) and so, by shrinking \( U \) if necessary, one has \( p_a = p_a \) for some function \( p : U \to \mathbb{R} \). Then one can replace \( r \) in the above by \( e^{-pr} \) where \( (e^{-pr_a})_b = 0 \). This latter (covariantly constant) vector field will still be labelled \( r \). Then (47) and (2) show that \( w_{[a;b]} = 0 \) and (similar to case (iii)) \( \beta_{[a,b]} - \beta_{[a} w_{b]} = 0 \). Thus, again by reducing \( U \) if necessary one has functions \( w, \theta : U \to \mathbb{R} \) such that \( w_a = w_a \) and \( \beta_a = e^w \theta_a \). Then tensor \( g \) defined on \( U \) by

\[
g_{ab} = e^{-w} h_{ab} - (\theta + C) r_a r_b \tag{51}\]

is now easily checked to be covariantly constant, where the arbitrary constant \( C \) can be adjusted to ensure that \( g \) has Lorentz signature in some open neighbourhood of \( m \).

In summary, the manifold \( M \) has been disjointly decomposed as \( M = M' \cup Z' \) where \( M' \) is the open subset of \( M \) given by

\[
M' = A \cup \text{int} B_1 \cup \text{int} B_2 \cup \text{int} C_1 \cup C_2 \cup \text{int} D_1 \cup \text{int} D_3 \cup D_4 \cup \text{int} O \tag{52}
\]

each point of which (under the appropriate conditions in theorem 2) admits an open neighbourhood and a metric on that neighbourhood compatible with \( \Gamma \) (this being clearly true with no further conditions on the region \( \text{int} O \)) and where \( Z' = Z \cup Z_1 \cup Z_4 \). Now each of \( Z, Z_1 \) and \( Z_4 \) has empty interior in \( M \) and \( Z \) is closed in \( M \). Also \( Z_1 \subseteq \text{int} C \) and \( Z_4 \subseteq \text{int} D \) and so (since, from (6), \( Z \) is disjoint from \( \text{int} C \) and \( \text{int} D \)) \( Z, Z_1 \) and \( Z_4 \) are mutually disjoint. It now follows that \( Z' \) is a closed subset of \( M \) with empty interior and hence that \( M' \) is open and dense in \( M \). To see this one first notes that \( Z' \) is closed by definition and that from the preliminary decompositions (6) and (27), \( Z \cup Z_1 \) is closed. Now let \( \emptyset \neq V \subseteq Z' \) with \( V \) open. Then \( V' \equiv V \cap (M \backslash Z) \) is open and \( V' \cap Z = \emptyset \). Then \( V'' \equiv V' \cap (M \backslash (Z \cup Z_1)) \) is open and \( V'' \subseteq Z_4 \). Thus \( V'' = \emptyset \) and hence \( V' \subseteq Z \cup Z_1 \). Thus \( V' \subseteq Z_1 \) and so \( V' = \emptyset \). This means that \( V \subseteq Z \) which is a contradiction. Thus \( \text{int} Z' = \emptyset \) and \( M' \) is open and dense in \( M \). This completes the proof of theorem 2.

The next two theorems are consequences of theorem 2.

**Theorem 3** Under the conditions of theorem 2, if \( \Gamma \) is Ricci flat then \( D = \emptyset \) and equations (2), (3.1) and (3.2) on \( M \) are sufficient to ensure a local Lorentz metric compatible with \( \Gamma \) on the open subset \( M' \) of \( M \).

**Proof**

If the curvature class at \( m \in M \) is \( D \) then the curvature tensor takes the form \( R_{abcd} (= h_{ae} R_{bcd}^e) \) for some simple bivector \( F \) at \( m \) and \( 0 \neq \lambda \in \mathbb{R} \) (section 2). The Ricci flat condition \( R_{bcd}^c = 0 \) then gives \( F_{bc} F_{cd} = 0 \). Since \( F \) is simple one may write \( F = p \wedge q \) for \( p, q \in T^* m M \) and with \( p \) and \( q \) (h-)orthogonal. The Ricci flat condition then shows that \( p \) and \( q \) are each (h-)null, contradicting the Lorentz signature of \( h \). Thus \( D = \emptyset \) and the result now follows from theorem 2.

**Theorem 4**

(i) If \( m \in M' \), with \( M' \) given by (52) and if \( h \) satisfies (2) and those equations in (3) appropriate to the region of \( m' \) in which \( m' \) is located, and as given in theorem 2, in that region, then \( h \) satisfies (3.k) for all \( k \) at each point of that region.
(ii) If the infinitesimal holonomy algebra associated with $\Gamma$ has constant dimension on $M$ and $M$ is simply connected and if the conditions appropriate for each of the regions as specified in theorem 2 hold in these regions, $M$ admits a global Lorentz metric which is compatible with $\Gamma$.

**Proof**

(i) The proof of this follows rather quickly by using the various relationships obtained in the proof of theorem 2 between $h$ and the constructed local metric $g$. Thus either $h$ and $g$ were conformally related or linked by various recurrent tensors $t$ or recurrent 1-forms $k$. The result now follows from (1).

(ii) The constancy of the dimension of the infinitesimal holonomy algebra means that the infinitesimal holonomy groups at each $m \in M$ are isomorphic to each other and to the restricted holonomy group of $M$ and hence (since $M$ is simply connected) to the holonomy group of $M$ \[I\]. Then part (i) shows that $h$ satisfies (3.1), for all $k$, on $M'$ and hence on $M$. Thus the infinitesimal holonomy algebra at each $m \in M$ is a subalgebra of the orthogonal algebra of $h$ (since each contribution to the infinitesimal holonomy algebra satisfies (4)). It follows that the holonomy group of $M$ is isomorphic to a subgroup of the orthogonal group of $h$ (that is, its members preserve the quadratic form $h$ at each $m \in M$ [2] - see, also, [1] and section 1). The result in (ii) now follows. (In fact the original conditions (2) and (3.1)-(3.1) imposed may be regarded as forcing a certain subspace of the infinitesimal holonomy algebra to lie within the orthogonal algebra of $h$.) \(\bullet\)

4. Remarks and Examples

In this paper the tensor $h$ was assumed to be of Lorentz signature. However it is clear that if $h$ is assumed positive-definite, the curvature types $A$, $B$, $C$, $D$ and $O$ still make sense after an obvious change for type $B$ where the range of $f$ would be 2-dimensional and spanned, in an obvious notation, by $x^a y_b - y^a x_b$ and $z^a w_b - w^a z_b$ where $x, y, z, w$ is an orthonormal tetrad at the point in question. Also, even when $h$ is taken as having Lorentz signature, it has been pointed out that metrics of both Lorentz and positive-definite signature have sometimes been constructed and which are compatible with $\Gamma$ (cf. [4]).

The results of theorem 2 give sufficient conditions in terms of equations (2) and (3.1)-(3.1) for each curvature class for a connection to be metric. The work of Edgar shows that for a fairly general class of cases the condition (2) is sufficient, but the class to which this applies is somewhat obscure [8]. Here some examples will be given which show that condition (2) is not a sufficient condition for the existence of a local metric for any of the curvature classes $A$, $B$, $C$ or $D$.

**Example 1.** This example is taken from [17]. Let $(R^4, \eta)$ denote Minkowski space and let $U$ be the open submanifold $t > 0$ of $R^4$. Define $\phi : U \to \mathbb{R}$ by $\phi(x, y, z, t) = \log t$
so that $\phi_{a;b} = -\phi_a \phi_b$ where $\phi_a = \phi_{,a}$. Define a metric $g$ on $U$ by $g = e^{-2\phi} \eta$ and let $\Gamma$ and $R$ be the associated Levi-Civita connection and curvature tensor, respectively. Then in the coordinates on $U$ inherited from $R^4$

\begin{align}
\Gamma^a_{bc} &= -(\delta^a_b \phi_c + \delta^a_c \phi_b - \phi^a g_{bc}) \\
R^a_{bcd} &= \phi^e \phi_e (\delta^a_d g_{bc} - \delta^a_c g_{bd})
\end{align}

(53)  (54)

where $\phi^a = g^{ab} \phi_b$. Then define a 1-form $\psi$ on $U$ by $\psi_a = (1 - e^{\phi})^{-1} \phi_a$ and a new symmetric connection $\Gamma'$ on $U$ by the components (cf. [8])

\begin{align}
\Gamma'^a_{bc} &= \Gamma^a_{bc} + \delta^a_b \psi_c + \delta^a_c \psi_b
\end{align}

(55)

This has the curvature tensor $R'$ with components

\begin{align}
R'^a_{bcd} &= \phi^e \phi_e \left(1 - (1 - e^{\phi})^{-1}\right) (\delta^a_d g_{bc} - \delta^a_c g_{bd})
\end{align}

(56)

It is easy to check that $R'$ and $g$ satisfy (2) and that the map $f$ associated with $R'$ (section 2) has rank equal to 6 everywhere on $U$ and so the curvature class is everywhere $A$. However, using $R'$ and $\Gamma'$ one can check after a short calculation that (3.1) fails. In fact $\Gamma'$ is not a local metric connection because if it was compatible with a metric $g'$ on some open subset $V$ of $U$ then from (10a) one has without loss of generality $g' = e^{\alpha} g$ for some smooth function $\alpha : V \to R$. But $g$ has zero covariant derivative with respect to $\Gamma$ and so the condition that $g'$ has zero covariant derivative with respect to $\Gamma'$ gives, using (55)

\begin{align}
\alpha_{,c} g_{ab} - 2g_{ab} \psi_c - g_{ac} \psi_b - g_{bc} \psi_a = 0
\end{align}

(57)

On contracting (57) with $g^{ab}$ one finds $2\alpha_{,c} = 5\psi_c$ and a back substitution gives $g_{ab} \psi_c = 2g_{ac} \psi_b + 2g_{bc} \psi_a$. A final contraction at any $p \in V$ with $X^c$ for any $X \in T_p V$ and a consideration of rank reveals that $X^a \psi_a = 0$ for all such $X$ and hence the contradiction that $\psi \equiv 0$ on $V$. Thus although $\Gamma'$ yields a curvature tensor of class $A$ satisfying (2), (3.1) fails and $\Gamma'$ is not locally metric.

**Example 2.** This example is quite general and can be applied to each of the curvature classes $A, B, C$ and $D$. Let $(M, g)$ be a space-time which admits a nowhere zero recurrent null vector field $l$ with the property that no rescaling of $l$ is covariantly constant over some non-empty open subset of $M$. This means that at some $p \in M$, $R^a_{bcd} l^d \neq 0$, for otherwise, one has $l_a ; b = l^{,a} l_b$ for some covector field $p$ on $M$ which from the Ricci identity satisfies $p_{[a;b]} = 0$ on $M$. Thus for any $p \in M$ there is an open neighbourhood $U$ of $p$ and a function $\phi : U \to R$ such that $p_a = \phi_{,a}$ and then $e^{-\phi} l$ is a covariantly constant rescaling of $l$ over $U$. Thus there exists $p \in M$ and, by continuity, some open neighbourhood $V$ of $p$ on which $R^a_{bcd} l^d$ is nowhere zero. The recurrence condition and the Ricci identity on $l$ in fact show that $R^a_{bcd} l^d = F^a_{bc} l^e$ for some tensor $F$ on $V$ which is hence nowhere zero on $V$. Restricting attention to $V$ with the original metric $g$ and associated Levi-Civita connection $\Gamma$ it is noticed that $l_a l^a_{[bc]} = 0$ and hence that $l$ is hypersurface orthogonal on $V$. Thus, shrinking $V$, if necessary, one may assume $l$ is scaled to a gradient vector field $l'$ on $V$ such that $l'_{,a} = \psi l'_a l'_b$ for some function $\psi : V \to R$. Now define a new symmetric
connection \( \Gamma' \) on \( V \) by the components \( \Gamma'^{a}_{bc} = \Gamma^{a}_{bc} + \Lambda^{a}_{b} \Lambda_{c}^{\nu} \). It is easily calculated that the curvature tensor \( R' \) on \( V \) arising from \( \Gamma' \) is identical to the original one (this can be quickly checked by first noting that if \( P^{a}_{bc} = \Gamma'^{a}_{bc} - \Gamma^{a}_{bc} = \Lambda^{a}_{b} \Lambda_{c}^{\nu} \) then \( P^{a}_{bcd} - P^{a}_{bdc} = 0 \) and \( P^{a}_{bc} P^{e}_{de} = 0 \) and then using a convenient formula in \([19, 20]\)). Thus in any chart on \( V \),

\[
R'^{a}_{bcd} = R^{a}_{bcd} \tag{58}
\]

Using a vertical stroke to denote a \( \Gamma' \) covariant derivative one thus has on \( V \), using another convenient result in \( [19, 20]\),

\[
R'^{a}_{bcd|e} = R^{a}_{bcd|e}
\]

\[
= R^{a}_{bcd|e} + \tilde{R}^{a}_{bcd} f^{a}_{ef} - R^{a}_{fde} P^{f}_{be} - R^{a}_{bfe} P^{f}_{de} - R^{a}_{ecf} P^{f}_{de}
\]

\[
= R^{a}_{bcd|e} - 2 \Lambda^{a}_{b} \Lambda^{\nu}_{c} \tilde{F}^{d} \tag{59}
\]

It now easily follows that for \( R'^{a}_{bcd}, \tilde{R}^{a}_{bcd|e} \) and \( g, (2) \) holds and \( (3.1) \) fails on \( V \).

Several cases of these results can now be considered which cover each of the curvature classes \( A, B, C \) and \( D \) and for none of which \( \Gamma' \) is locally metric.

Consider first the situation where the original \( g \) is a vacuum metric of the Goldberg-Kerr type \([21, 22]\). Then the associated space-time may be taken as \( V \) and admitting a recurrent null vector field \( l \) as described above. The Petrov type is \( III \) and so the curvature tensor (being equal to the Weyl tensor because of the vacuum condition) has an associated map \( f \) of rank 4 (see, e.g. \([4]\)). The curvature class is thus \( A \). If \( \Gamma' \) defined as above was compatible with a metric \( g' \) on some open subset \( W \) of \( V \) then \( \Gamma \) and \( \Gamma' \) are each metric connections on \( W \) with the same curvature tensor. It then follows from \([5, 4]\) that, on \( W \), \( g' = cg \) (\( 0 \neq c \in \mathbb{R} \)) and hence the contradiction that \( \Gamma' = \Gamma \) on \( W \).

Now consider the situation when \( V \) is a space-time of curvature class \( B \) and hence of holonomy type \( R_{\gamma} \) \([1, 23]\) and thus the metric \( g \) is of the Bertolli-Robinson type (see, e.g. \([24]\)). Further, if \( V \) is chosen to be simply connected (as it always can) then two independent recurrent null vector fields \( l \) and \( n \) are admitted by \( V \) with each having all of the properties that \( l \) had in the previous paragraph and satisfying \( l^{a} n_{a} = 1 \), \( l_{a;b} = l_{a} p_{b} \) and \( n_{a;b} = -n_{a} p_{b} \) \([4, 16, 23]\). Thus the tensor \( h \) on \( V \) defined by \( h_{ab} = 2 l^{a} n_{b} \) is covariantly constant on \( V \), \( h_{abc} = 0 \). Now define a symmetric connection \( \Gamma' \) by the components \( \Gamma'^{a}_{bc} = \Gamma^{a}_{bc} + \Lambda^{a}_{b} \Lambda_{c}^{\nu} \) and suppose that \( \Gamma' \) is compatible with a metric \( g' \) on some open subset \( W \) of \( V \). Then \( \Gamma \) and \( \Gamma' \) give the same curvature tensor as before and it follows from \([4, 15, 16]\) (see \( (10b) \)) that \( g' = \phi g + \psi h \) for functions \( \phi, \psi : W \rightarrow \mathbb{R} \). Imposing the condition \( g_{abc} = 0 = g'_{abc} \), recalling that \( h_{abc} = 0 \) and taking \( P^{a}_{bc} = l^{a} l_{b}^{\nu} l_{c}^{\rho} \) as before gives

\[
0 = g'_{abc} = \phi_{c} g_{ab} + \phi g_{abc} + \psi_{c} h_{ab} + \psi h_{abc}
\]

\[
= \phi_{c} g_{ab} + \phi (g_{abc} - g_{ae} P^{e}_{bc} - g_{eb} P^{e}_{ac})
\]

\[
+ \psi_{c} h_{ab} + \psi (h_{abc} - h_{ae} P^{e}_{bc} - h_{eb} P^{e}_{ac})
\]

\[
\Rightarrow (\phi + \psi)_{c} g_{ab} = 2 (\phi + \psi) l_{a}^{\nu} l_{b}^{\nu} l_{c}^{\rho} \tag{60}
\]
A rank argument shows that \( \phi = -\psi \) and hence that \( g' = \phi(g - h) \). But then \( g_{ab}b^b = g_{ab}h^b = 0 \) and so \( g' \) is not non-degenerate. This contradiction completes the proof.

The final two cases may be dealt with briefly. First, let \( (M, g) \) be a space-time of holonomy type \( R_6 \) \([11, 16, 23]\) - and for existence and examples see \([25]\). Then, restricting to an open subset \( U \) of \( M \), if necessary, one may assume that, on \( U \), there are vector fields \( y \) and \( l \) with \( y \) spacelike and covariantly constant and \( l \) null and recurrent. Thus \( y_{a;b} = 0 \) and \( l_{a;b} = l_a p_b \) with \( p \) a nowhere zero 1-form on \( U \) and one may assume \( U \) is chosen so that the map \( f \) (section 2) has everywhere rank 2. Then, on \( U \), \( R^a_{bcd} y^d = 0 \) (that is, the curvature class is \( C \)) and \( R^a_{bcd} l^d \) is nowhere zero. On scaling \( l \) to \( l' \) as in the previous examples and constructing the connection \( \Gamma' \) as before, one again achieves (58) and (59) so that \( g, \Gamma \) and \( R' \) satisfy (2) but not (3.1). Because of (58) it follows from (10c) \([4, 15, 5]\) that if \( g' \) is a metric compatible with \( \Gamma' \), one must have \( g'_{ab} = \phi g_{ab} + \psi a y b \) for functions \( \phi, \psi : U \to \mathbb{R} \). Now the condition \( g'_{ab;c} = 0 \) yields, by an argument which is very similar to (60), the consequence that \( \phi \equiv 0 \) on \( U \) and so that such a tensor \( g' \) is not non-degenerate. Hence \( \Gamma' \) is not locally a metric connection.

Finally for a space-time of holonomy type \( R_2 \) \([4, 16, 23, 25]\) one may arrange an open subset \( U \) of it to admit vector fields \( l, n, x \) and \( y \) with all inner products between them vanishing on \( U \) except \( l^a n_a = x^a x_a = y^a y_a = 1 \). Also, on \( U \), \( l \) and \( n \) are (null and) recurrent and \( x \) and \( y \) are covariantly constant. The set \( U \) may be chosen so that the map \( f \) has rank 1 everywhere on \( U \) (and hence, the curvature class is \( D \)). One then proceeds as above to construct \( \Gamma' \) (using \( l \) or \( n \)) so that \( g, \Gamma \) and \( R' \) satisfy (2) but not (3.1) and notes that, from (58), any metric \( g' \) compatible with \( \Gamma' \) satisfies \([4, 15, 5]\) \( g'_{ab} = \phi g_{ab} + \alpha x_a x_b + \beta y_a y_b + 2 \gamma x(a y b) \) for functions \( \phi, \alpha, \beta, \gamma : U \to \mathbb{R} \). Another argument, very similar to (60) then shows that \( \phi \equiv 0 \) on \( U \) and thus \( g' \) is not non-degenerate. Hence \( \Gamma' \) is not locally a metric connection.

As a final remark, it is noted how the tensor \( h \) and any local metric compatible with the connection \( \Gamma \) are related geometrically. For the case relevant to theorem 2(i), that is, class \( A \), any local metric is just a conformal rescaling of \( h \). For class \( B \), any local metric is a (functional) combination of \( h \) and the local covariantly constant tensor \( t \), where the latter tensor’s eigenspaces (with respect to \( h \)) are, at each relevant point \( m \), identical to the two 2-dimensional subspaces of \( T_m M \) into which \( T_m M \) is decomposed by the curvature tensor for this class. For class \( C \), any local metric is either a rescaling of \( h \) or a (functional) combination of \( h \) and the covariantly constant tensor \( k_a k_b \) where \( k \) is a local covariantly constant vector field spanning the unique direction picked out at each relevant point by a class \( C \) curvature tensor. For class \( D \), any local metric \( g \) is either a rescaling of \( h \) or a (functional) combination of \( h \) and appropriate products of either one or two local covariantly constant vector fields and where these latter vector fields lie in the 2-dimensional subspace determined at each appropriate point by this curvature class. In addition, it is noted that if \( g \) and \( g' \) are two local metrics compatible with \( \Gamma \) in some neighbourhood \( U \), the latter being contained in one of the regions \( A, \) int\(B_1\),
int\(B_2\) etc. considered above, then \(g\) and \(g'\), having the same Levi-Civita connection, are either conformally related with a constant conformal factor or are jointly related to the appropriate covariantly constant tensor(s) mentioned above by simple (constant coefficient) combinations and which reflect the local holonomy group [4, 3].

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