Penetrative Brinkman ferroconvection via internal heating in high porosity anisotropic porous layer: influence of boundaries

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1. Introduction

Ferrofluids (FFs) are synthesized by suspending single domain ferromagnetic nanoparticles stabilized in various nonmagnetic carrier fluids, which exhibit both magnetic and fluid properties [1, 2]. These fluids are now termed as magnetic nanofluids and the study of such fluids has been a subject of intensive investigations over decades due to their potential applications in magnetically heat controlled thermostypons for technological purposes [3, 4]. Thermal convection in a horizontal FF layer in the existence of magnetic field, called ferro-thermal-convection (FTC), has been studied extensively both theoretically and experimentally over the years to understand the heat transfer mechanisms and the details are amply documented in the review article by Nkurikiyimfura et al. [5].

The study of FFs through porous media has also captivated the attention of researchers because of their potential utility in subsurface environmental engineering wherein externally applied magnetic field are used to control and direct the FFs flow. Rosensweig et al. [6] experimentally demonstrated the FFs penetration in the Hele-Shaw cell, while Zahn et al. [7] studied the FTC in a porous medium in the existence of oblique magnetic field. Moridis et al. [8] carried out experimental and theoretical studies on FFs for guiding and detecting fluids in the subsurface using geophysical methods. Oldenburg et al. [9] developed capabilities of simulation for both miscible and immiscible of FF-saturated porous media. Borglin et al. [10] described simple laboratory-type investigations on penetrative convection in a FF-saturated porous medium.

To understand the heat transfer characteristics in a FF-saturated porous medium, FTC in a porous medium has also been studied extensively (Sekar et al. [11], Shivakumara et al. [12, 13], Nanjundappa et al. [14, 15, 16]). Besides, the implications of additional effects such as magnetic field dependent viscosity (Nanjundappa et al. [17, 18], Shivakumara et al. [19, 20]), rotation (Shivakumara et al. [21], Nanjundappa et al. [22]), internal heating (Nanjundappa et al. [23, 24]), throughflow (Nanjundappa et al. [25]), non-uniform basic temperature gradients (Nanjundappa et al. [26]), local thermal nonequilibrium (Shivakumara et al. [27, 28, 29, 30]), Cattaneo heat flux law (Shivakumara et al. [31, 32]) have also been investigated on FTC in porous media.

In many engineering applications the porous matrix used turns out to be anisotropic both mechanically and thermally. Shin and Inagaki [33] reported the improvement of heat transfer by the inclusion of high thermal conductivity porous medium. But the...
suppression of natural convection by the porous medium becomes a matter of concern in spite of the enhanced thermal conductivity. In this context, Shiina and Hishida [34] studied the convection in an anisotropic ordinary viscous fluid-saturated porous layer and compared their findings with experimental results. Nonetheless, such a study is missing for FF-saturating a porous medium despite the study finds importance in heat transfer problems encountered in many engineering applications involving FFs.

The novelty of the current paper is, therefore, to investigate FTC in a high porosity anisotropic porous medium saturated by a FF subjected to uniform internal heating. Since Darcy’s law fails to give satisfactory results for high porosity porous medium, Brinkman-extended Darcy model is used to illustrate the FF flow, which takes care of inertia and boundary effects. In the investigation of the problem, different boundary conditions considered are evaluated. In this context, Shiina and Hishida [34] studied the convection in an anisotropic ordinary viscous fluid-saturated porous layer and compared their findings with experimental results. Nonetheless, such a study is missing for FF-saturating a porous medium despite the study finds importance in heat transfer problems encountered in many engineering applications involving FFs.

As far as the temperature is concerned, the boundaries are kept at different uniform temperatures. The study undertaken is more general and the results obtained for the stability eigenvalue problem under the limiting cases are compared with those published earlier and found good agreement.

To achieve the said objectives, this work is considered as follows: In Section 2, the problem is formulated mathematically and the steady linearized disturbance equations as well as different boundary conditions considered are given. In Section 3, the numerical method of solving the stability eigenvalue problem is briefly outlined. Section 4 contains results and discussions and finally some significant conclusions are given in the last Section.

2. Mathematical formulation

Figure 1 shows an infinite layer of horizontal FF-saturated high porosity anisotropic porous medium bounded between \( z = 0 \) and \( z = d \) are maintained at the constant temperatures \( T_0 \) and \( T_1 (= T_0 - \Delta T) (\Delta T > 0) \), respectively. In addition, the porous medium is uniformly heated volumetrically.

The FTC is studied under the following assumptions:

- The porous medium is in local thermal equilibrium.
- Anisotropy is considered in the effective thermal diffusivity and the permeability.
- The permeability ratio in horizontal direction to the vertical direction is fixed at 0.5, while the effect of effective thermal diffusivities ratio is allowed to vary arbitrarily.
- The principal axes of thermal diffusivity tensor are coinciding with the coordinate axes and a horizontal isotropy is considered.
- The incompressible FF is electrically non-conducting and the Oberbeck-Boussinesq approximation holds.
- The fluid properties such as viscosity, porosity and thermal conductivity are taken as constants.
- The viscous dissipation and magnetocaloric effects are absent.

The equations governing in the appropriate context as (Rosensweig [1]; Shiina and Hishida [34]):

\[
\nabla \cdot \vec{u} = 0 \tag{1}
\]

Momentum conservation
\[
\left[ \frac{\partial}{\partial t} + (\nabla \cdot \vec{v}) \right] \vec{v} = -\frac{1}{\rho_0} \nabla p + \left[ 1 - a_1(T - T_0) \right] \vec{g} + \frac{\mu}{\rho_0} \nabla^2 \vec{v} - \frac{\mu}{\rho_0} \nabla \cdot \vec{v} + \frac{\mu_0}{\rho_0} (\vec{M} \cdot \nabla) \vec{H} \tag{2}
\]

Energy conservation
\[
A \frac{\partial T}{\partial t} + \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + Q \tag{3}
\]

Maxwell equations
\[
\nabla \cdot \vec{B} = 0 \tag{4}
\]

\[
\nabla \times \vec{H} = 0 \text{ or } \vec{H} = \nabla \phi \tag{5}
\]

\[
\vec{B} = \mu_0 (\vec{M} + \vec{H}) \tag{6}
\]

where
\[
\vec{M} = [M_x + \chi(H - H_0) - K_z(T - T_0)] \vec{H} / \vec{H} \tag{7}
\]

The following basic state is taken into account
\[
[\vec{V}, \rho, \rho, T, \vec{M}, \vec{H}] = \left[ 0, p_0, p_0, T_0, \bar{M}_s, \bar{H}_s \right] (z). \tag{8}
\]

Using Eq. (8), Eqs. (1), (2), (3), (4), (5), and (6) reduce to
\[
-p_0 \vec{g} - \nabla p_0 + \mu_0 (\bar{M}_s \cdot \nabla) \bar{H}_s = 0 \tag{9}
\]

\[
T_0 - T_0 = \left( \frac{Qd}{2\kappa_x} - \beta \right) z - \frac{Q^2}{2\kappa_x} \tag{10}
\]

\[
H_s(z) = H_0 + \frac{K_z}{1 + \chi} (T_0 - T_0) \tag{11}
\]

\[
M_s(z) = M_0 - \frac{K_z}{1 + \chi} (T_0 - T_0) \tag{12}
\]

The basic state is perturbed to study its stability in the form
\[
[\vec{V}, \rho, \rho, T, \vec{M}, \vec{H}] = \left[ 0, p_0, p_0, T_b, \bar{M}_s, \bar{H}_s \right] (z) + [\vec{V}', \rho', \rho', T', \vec{M}', \vec{H}'] (x, y, z, t). \tag{13}
\]

Substitution of Eq. (13) into Eqs. (1), (2), (3), (4), (5), (6), and (7), using Eqs. (9), (10), (11), and (12) and linearizing by ignoring the product of primed quantities leads to the following stability equations
\[
\left[ \rho_0 \frac{\partial}{\partial t} + \mu \nabla^2 \right] \nabla^2 w = \mu_0 K_z \frac{\partial}{\partial z} \left( \nabla^2 \phi \right) \frac{d T_b}{d z} - \left[ \rho_0 \theta g - \mu_0 K_z^2 \frac{1}{1 + \chi} \frac{d T_b}{d z} \right]
\]

\[
\nabla^2 T = \frac{\mu}{K_x} \nabla^2 w - \frac{\mu}{K_x} \frac{\partial^2 w}{\partial z^2} \tag{14}
\]

\[
A \frac{\partial T}{\partial t} + \frac{\partial T}{\partial t} = k_v \nabla^2 T + k_v \frac{\partial^2 T}{\partial z^2} + Q \tag{15}
\]

\[
(1 - \frac{M_0}{H_0}) \nabla^2 \phi + (1 + \chi) \frac{\partial^2 \phi}{\partial z^2} - K_v \frac{\partial T}{\partial z} = 0 \tag{16}
\]

where \( \frac{\partial T}{\partial z} = \frac{\partial T}{\partial z} - \frac{\partial T}{\partial z} = k_y = k_h \text{ and } K_x = K_y = K_h. \)
We seek the solution of Eqs. (14), (15), and (16) using the method of normal modes of the form (noting that the principle of exchange of stability holds)

$$\{w, T, \Phi\}(x, y, z) = \{W, \Theta, \Phi\}(z) \exp(i n x + i k y)$$

where $a_1$ and $a_2$ are the wave numbers in the $x$ and $y$-directions, respectively. Substituting Eq. (17) into Eqs. (14), (15), and (16) and considering the permeability in the $z$-direction is twice the permeability in the horizontal direction (Shiina and Hishida [34]), we get

$$\rho_0 C_1 a_1^2 \frac{d^2 W}{dz^2} + \mu_0 K_1 a_1^2 \frac{dT_b}{dz} - \frac{K_2}{1 + \chi} = 0$$

where $\chi = \frac{a_1^2}{a_2^2}$.

Eqs. (18), (19), and (20) are then non-dimensionalized using the following transformations

$$\bar{z} = \frac{z}{a_1}, \bar{w} = \frac{w}{a_1}, \bar{\Phi} = \frac{\Phi}{a_1}, \bar{\Theta} = \frac{\Theta}{a_1^2},$$

$$\bar{m} = \frac{m}{a_1}, \bar{M} = \frac{M}{a_1}, \bar{N} = \frac{N}{a_1}, \bar{D} = \frac{D}{a_1},$$

Eqs. (18), (19), and (20) after using Eq. (21) become

$$\left[\left(D^2 - \eta^2\right) \Theta \right] = [N(1 - 2z) - 1]1(D\Theta - \Theta) + \Theta R \Theta$$

$$\left(D^3 - \Phi M_1\right) \Phi - \Theta = 0.$$  \hspace{1cm} (24)

The following boundary combinations are considered:

Case (i): Lower and upper rigid surfaces are ferromagnetic (R-F), and isothermal

$$W = D W = 0, \quad \Theta = 0, \quad \Phi = 0 \quad \text{at} \quad z = 0, 1$$

Case (ii): Lower and upper free surfaces are ferromagnetic (F-F), and isothermal

$$W = D W = 0, \quad \Theta = 0, \quad D \Phi = 0 \quad \text{at} \quad z = 0, 1$$

Case (iii): Lower and upper rigid surfaces are paramagnetic (R-P), and isothermal

$$W = D W = 0, \quad \Theta = 0 \quad \text{at} \quad z = 0, 1$$

$$D \Phi = \begin{cases} \frac{a}{1+z} \Phi & \text{at} \quad z = 0 \\ \frac{a}{1+z} \Phi & \text{at} \quad z = 1 \end{cases}$$  \hspace{1cm} (28)

3. Numerical solution

Eqs. (22), (23), and (24) subject to any choice of boundary conditions cases form an eigenvalue problem and solved numerically by the Galerkin technique. We expand the unknown variables in power series of the form

$$\{W, \Theta, \Phi\}(z) = \sum_{n=1}^{n} \{A_n W_n(z), B_n \Theta_n(z), C_n \Phi_n(z)\}$$  \hspace{1cm} (29)

where $A_n, B_n, C_n$ (m = 1, 2, ..., n) are constants and basis functions $W_n, \Theta_n, \Phi_n$ are the trial functions chosen usually satisfying (25)-(28).

(i) R-F boundaries

$$W_n = (\zeta^{n+1} - 2\zeta^{n+1} + \zeta^n)T_m^*, \quad \Theta_n = (\zeta^{n+1} - \zeta^n)T_m^*, \quad \Phi_n = (\zeta^{n+1} - \zeta^n)T_m^*.$$  \hspace{1cm} (30)

(ii) F-F boundaries

$$W_n = (\zeta^{n+1} - 2\zeta^{n+1} + \zeta^n)T_m^*, \quad \Theta_n = (\zeta^{n+1} - \zeta^n)T_m^*, \quad \Phi_n = (\zeta^{n+1} - \zeta^n)T_m^*.$$  \hspace{1cm} (31)

(iii) R-P boundaries

$$W_n = (\zeta^{n+1} - 2\zeta^{n+1} + \zeta^n)T_m^*, \quad \Theta_n = (\zeta^{n+1} - \zeta^n)T_m^*, \Phi_n = (\zeta^{n+1} - \zeta^n)T_m^*.$$  \hspace{1cm} (32)

where $T_m^*$ (m = 0, 1, 2, ...) are Chebyshev polynomials of the second kind.

Substituting Eq. (29) into Eqs. (22), (23), and (24), multiplying the resulting Eqs. (22), (23), and (24), respectively, by $W_n(z), \Theta_n(z)$ and $\Phi_n(z)$, carrying out the integration by parts between $z = 0$ and $z = 1$, and using Eqs. (25), (26), (27), and (28), we obtain the following linear system of homogenous algebraic equations:

$$\begin{bmatrix} C_{av} & M_{av} & F_{av} \\ H_{av} & B_{av} & 0 \\ G_{av} & J_{av} & C_{av} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = 0.$$  \hspace{1cm} (33)

where

$$C_{av} = \begin{bmatrix} \int_0^1 [D^2 W_0 D^2 W_1 + 2a^2 D W_0 D W_1 + a^2 W_0 W_1] dz \\ + \int_0^1 [D a_1^2 D W_0 W_1 + a^2 W_0 W_1] dz \end{bmatrix}$$

$$M_{av} = - a^2 D a_1^2 \begin{bmatrix} \int_0^1 [R_0 R_1 M_0 N_1(1 - 2z) - 1] W_0 W_1 dz \end{bmatrix}$$

$$F_{av} = - a^2 R_1 M_1 D a_1^2 \begin{bmatrix} \int_0^1 [N_0(1 - 2z) - 1] D \Phi_0 W_1 dz \end{bmatrix}$$

$$G_{av} = \begin{bmatrix} \int_0^1 [N_0(1 - 2z) - 1] \Theta_0 W_1 dz \end{bmatrix}$$

$$H_{av} = \begin{bmatrix} \int_0^1 [D \Theta_0 D \Theta_1 + a^2 \eta \Theta_0 \Theta_1] dz \end{bmatrix}$$

$$J_{av} = \begin{bmatrix} \int_0^1 \Phi_0 a_1 D \Phi_0 dz \end{bmatrix}$$

$$\begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = \begin{bmatrix} a \end{bmatrix} [\Phi_0(1) \Phi_0(1) - \Phi_0(0) \Phi_0(0)] + \int_0^1 [D \Phi_0 D \Phi + a^2 M_1 \Phi_0 \Phi_1] dz.$$  \hspace{1cm} (34)
functions into Eq. (35) for each case of boundary conditions leads to a characteristic equation of the form

\[ R a_n - \chi - \frac{F_0.01}{C_{12}} = 0. \]  

(34)

The eigenvalues are extracted from Eq. (34). Substitution of trial functions into Eq. (35) for each case of boundary conditions leads to a characteristic equation of the form

\[ f(R_c, R_s, Da, \eta, M_1, \eta, a) = 0. \]  

(35)

From Eq. (35), the critical value of \( R_c(R_s) \) or \( R_{ac}(R_{ac}) \) is computed with respect to 'u' (wave number) for fixed governing physical parameters \( Da, M_1, \eta, \) and \( N_s \). A symbolic algebraic package of Microsoft Mathematica 12.0 (see [35]) is used for this purpose.

Table 1. Process of convergence of results for different orders of approximations for different \( Da \) and \( \chi \) when \( M_1 = M_3 = 1, \lambda = \eta = 1 \) and \( N_s = 2 \)

| Boundaries | \( Da \) | \( m = n - 1 \) | \( m = n - 3 \) | \( m = n - 5 \) | \( m = n - 7 \) | \( m = n - 8 \) |
|------------|--------|----------------|----------------|----------------|----------------|----------------|
|            | \( a_c \) | \( a_c \) | \( a_c \) | \( a_c \) | \( a_c \) | \( a_c \) |
| R-F        | 0.01   | 2599.56       | 3.535          | 2355.389       | 3.854          | 2319.357       |
|            | 0.1    | 1051.344      | 3.2003         | 1004.235       | 3.395          | 1015.141       |
|            | 1      | 892.715       | 3.1259         | 861.5872       | 3.301          | 872.9511       |
|            | 10     | 876.761       | 3.1174         | 847.1565       | 3.291          | 858.4873       |
|            | 100    | 875.165       | 3.1166         | 845.7116       | 3.289          | 857.0381       |
| F-F        | 0.01   | 1850.495      | 3.378          | 2156.847       | 4.061          | 2159.087       |
|            | 0.1    | 606.3761      | 3.243          | 769.464        | 3.312          | 770.086        |
|            | 1      | 473.3096      | 3.189          | 615.070        | 3.118          | 615.196        |
|            | 10     | 459.7059      | 3.167          | 599.147        | 3.094          | 599.2614       |
|            | 100    | 458.3428      | 3.189          | 597.548        | 3.092          | 597.6618       |
| R-P        | \( \chi = 0 \) | 0.01 | 2287.48       | 3.700          | 2422.24        | 3.961          |
|            | \( \chi = 100 \) | 0.1  | 1181.75       | 3.335          | 1037.45        | 3.488          |
|            | \( \chi = 100 \) | 1    | 1006.34       | 3.256          | 891.162        | 3.392          |
|            | \( \chi = 100 \) | 10   | 988.690       | 3.247          | 876.362        | 3.381          |
|            | \( \chi = 100 \) | 100  | 986.924       | 3.246          | 874.880        | 3.380          |
| R-P        | \( \chi = 0 \) | 0.01 | 2265.09       | 4.018          | 2638.23        | 4.268          |
|            | \( \chi = 100 \) | 0.1  | 949.940       | 3.592          | 1147.30        | 3.645          |
|            | \( \chi = 100 \) | 1    | 813.603       | 3.501          | 988.760        | 3.540          |
|            | \( \chi = 100 \) | 10   | 799.858       | 3.490          | 792.699        | 3.528          |
|            | \( \chi = 100 \) | 100  | 798.462       | 3.489          | 971.091        | 3.527          |

Table 2. Comparison \( R_{ac} \) and \( a_c \) for different \( \eta \) and \( Da \) for \( M_1 = M_3 = 0 \) and \( N_s = 1 \)

| \( \eta \) | \( Da \) | \( R_{ac} \) | \( a_c \) | \( R_{ac} \) | \( a_c \) |
|------------|--------|--------------|----------|--------------|----------|
| 1          | 100    | 1708.28      | 3.1163   | 1708.11      | 3.1164   |
|            | 10     | 1711.39      | 3.1172   | 1711.21      | 3.1173   |
|            | 1      | 1742.40      | 3.1263   | 1742.22      | 3.1263   |
|            | 0.1    | 2049.34      | 3.2949   | 2049.12      | 3.2950   |
|            | 0.01   | 4907.55      | 3.5322   | 4966.93      | 3.5323   |
|            | 0.001  | 32214.40     | 3.7642   | 32213.8      | 3.7640   |
| 0.01       | 0.001  | 295631.40    | 3.7641   | 295630.90    | 3.7641   |
| 0.01       | 100    | 831.37       | 4.2326   | 831.31       | 4.2325   |
|            | 10     | 832.60       | 4.2344   | 832.54       | 4.2344   |
|            | 1      | 844.91       | 4.2523   | 844.85       | 4.2523   |
|            | 0.1    | 965.49       | 4.4126   | 965.42       | 4.4126   |
|            | 0.01   | 2055.78      | 5.2219   | 2055.63      | 5.2219   |
|            | 0.001  | 11746.39     | 6.3295   | 11747.9      | 6.3299   |
| 0.01       | 0.001  | 706.60       | 4.6932   | 706.55       | 4.6932   |
|            | 10     | 707.55       | 4.6958   | 707.506      | 4.6958   |
|            | 1      | 717.02       | 4.7201   | 716.976      | 4.7200   |
|            | 0.1    | 809.05       | 4.9412   | 809.00       | 4.9411   |
|            | 0.01   | 1601.71      | 6.2069   | 1601.61      | 6.2068   |
|            | 0.001  | 7942.14      | 9.0575   | 7942.12      | 9.0579   |
| 0.001      | 100    | 692.53       | 4.7651   | 692.49       | 4.7650   |
|            | 10     | 693.45       | 4.7676   | 693.407      | 4.7676   |
|            | 1      | 702.58       | 4.7932   | 702.54       | 4.7932   |
|            | 0.1    | 791.21       | 5.0267   | 791.16       | 5.0267   |
|            | 0.01   | 1545.94      | 6.4065   | 1545.86      | 6.4065   |
|            | 0.001  | 7335.96      | 10.1541  | 7336.64      | 10.154   |
Figure 2. Variation of (a) $R_{tc}$ and (b) $a_c$ against $\eta$ for different $Da$ when $M_1 = N_1 = 2$ and $M_3 = 1$.
4. Results and discussion

The penetrative FTC instilling due to volumetric heating in a FF-saturated high porosity anisotropic porous layer has been investigated. The process of convergence of results for representative values of governing parameters are shown in Table 1 and note that eight terms ($m = n = 8$) in the Galerkin expansion are needed to achieve the convergence, in general. In Table 2, the results obtained for R–F surfaces when $M_3 = 1$ and $N_s = 2$ are presented.

![Figure 3. Variation of (a) $R_{tc}$ and (b) $a_c$ against $Da$ for different $M_1$ when $\eta = M_3 = 1$ and $N_s = 2$](image-url)
and $Ns = 0$ are compared with Shiina and Hishida [34] for different values of $\eta$ and $Da$, and the results are in excellent agreement.

The results obtained covering a wide range of governing parameters for R–P boundaries with $\chi \neq 0$ (solid curves) and $\chi = 0$ (dashed curves), R–F boundaries (dotted curves) and F–F boundaries (dashed-dotted curves) are exhibited in Figures 2, 3, 4, 5, 6, and 7. Figure 2(a) shows the critical Rayleigh number $R_{tc}$ plotted against thermal anisotropy parameter $\eta$ for $M_3 = 1$, $Da = 10$ and $Ns = 2$. The effect of increasing $\eta$ is to increase $R_{tc}$ indicating its effect is to postpone the onset of FTC. This may be attributed to the fact that, as $\eta$ decreases, a heated fluid parcel carries a lesser amount of heat in the horizontal direction. Further, R–P surfaces with large magnetic susceptibility $(1 + \chi = 10^2)$

![Figure 4](image-url)

Figure 4. Variation of (a) $R_{tc}$ and (b) $a_c$ against $M_1$ for different $\eta$ for $M_3 = 1$, $Da = 10$ and $Ns = 2$. 
show more stabilizing effect against FTC when compared to the low magnetic susceptibility ($\chi = 0$) case, while F–F surfaces show more destabilizing influence on the same. Thus the impact of R–F surfaces on the stability of the system lies between those of R–P and F–F surfaces. That is, the R–P surfaces have sturdy effect in stabilizing the thermally unstable FF-saturated anisotropic porous layer than R–F and F–F surfaces because the rigid boundaries with an increase in the magnetic induction suppress the disturbances more effectively. These results are in accordance with those of Gotoh and Yamada [36]. The critical wave number $a_c$ decreases with increasing $\eta$ and $Da$ indicating their effect is to broaden the convection cell size. Further, the values of $a_c$ for F–F surfaces are the least as compared to R–F surfaces and highest for R–P surfaces.

The influence of $M_1$ (magnetic number) and $Da$ (Darcy number) on $R_{tc}$ is demonstrated in Figure 3 (a) for $M_3 = 1$, $\eta = 1$ and $Ns = 2$. Increasing $Da$ and $M_1$ is to decrease $R_{tc}$ representing their effect is to hasten the onset of FTC. For a fixed porous layer thickness, increase in $Da$ amounts to increase in the permeability of the porous medium which in turn speed up the flow of FF in porous media and hence lower $R_{tc}$ values are required for the onset of FTC. Moreover, the convection cell size becomes widened and contracted with increasing $Da$ and $M_1$, respectively.

Figure 5. Variation of (a) $R_{tc}$ and (b) $a_c$ against $M_3$ for different $Da$ for $M_1 = Ns = 2$ and $\eta = 1$.  

The critical wave number $a_c$ decreases with increasing $\eta$ and $Da$ indicating their effect is to broaden the convection cell size. Further, the values of $a_c$ for F–F surfaces are the least as compared to R–F surfaces and highest for R–P surfaces.
Figure 4(a) illustrates the variation of $R_{tc}$ as a function of $M_1$ for different $\eta$ values when $M_3 = 1$, $Da = 10$ and $Ns = 2$. The values of $R_{tc}$ decrease quite rapidly at first then slowly and finally the curves of different $\eta$ remain invariant as $M_1$ increases and this is due to the saturation of destabilizing magnetic force. The critical wave numbers increases with $M_1$, reach maximum, and thereafter decrease with further increase in $M_1$ (Figure 4b). With increasing value of $M_1$, the curves of $a_c$ for two values of $\eta$ coalesce for different bounding surfaces.

The variation of $M_3$ (nonlinearity of fluid magnetization parameter) on $R_{tc}$ is exhibited in Figure 5(a) for $M_1 = 2$, $\eta = 1$ and $Ns = 2$. The figure demonstrates that increasing $M_3$ has a destabilizing effect on the system but it is only marginal. Thus, the system becomes unstable with a smaller
temperature gradient as $M_3$ increases. From the Figure 5 (b), we note that an increase in $M_3$ is to decrease $\alpha$ and thus to increase the convection cell size. Figure 6(a) represents $R_{tc}$ against the internal heat source parameter $N_s$ for different $\eta$ when $M_1 = 2$, $Da = 10$ and $M_3 = 1$. It is seen that $R_{tc}$ decreases monotonically with increasing $N_s$ and thus hastens the onset of FTC. This is because; increasing $N_s$ amounts to increase in the energy supply to the system. Figure 6(b) shows the variation of $\alpha$ and note that it increases as $N_s$ increases and thus its effect is to decrease the convection cell size.

Figure 7 shows the locus of $R_{tc}$ versus $R_{nc}$ for different $N_s$ and bounding surfaces when $M_1 = 2$, $M_3 = 1$, $\eta = 1$ and $Da = 10$. It is seen that there is a strong coupling between $R_{nc}$ and $R_{mc}$, and an increase in $R_{nc}$ has a destabilizing effect on the onset of FTC. Thus, the gravitational force ($R_{nc}$) becomes negligible when the magnetic force ($R_{mc}$) is predominant and vice versa. Moreover, the stability curves are slightly convex. The instability sets in at higher $R_{nc}$ representing the system is more stable when magnetic forces alone are present. In all these figures, the results for F–F, R–F and R–P boundaries with $\chi = 0$ and $\chi >> 1$ are presented. It is observed that

$$(R_{nc} \text{ or } R_{mc})_{\chi = 0} < (R_{nc} \text{ or } R_{mc})_{\chi = \chi_1} < (R_{nc} \text{ or } R_{mc})_{\chi = \chi_2} < (R_{nc} \text{ or } R_{mc})_{\chi = \chi_3}.$$  

Thus we note that the system is more stabilizing against FTC if the surfaces are rigid with high magnetic susceptibility ($\chi >> 1$) and least stable if the surfaces are F–F.

5. Conclusions

The onset of penetrative FTC in a FF-saturated high porosity anisotropic porous layer is investigated by uniformly heating the porous medium internally. The Galerkin method is applied to solve the stability eigenvalue problem numerically. The results of the present investigations can be summarized as follows:

- The effect of increasing thermal anisotropy ($\eta$) and internal heat source ($N_s$) is to delay and hasten the onset of FTC in an anisotropic porous medium, respectively. Thus, the porous medium anisotropy can be used effectively to control (suppress or augment) FTC.
- The critical stability parameters satisfy the following inequality suggesting different boundaries significantly influence the instability characteristics of FTC. That is,

$$R_{mc} < R_{nc} < (R_{nc} \text{ or } R_{mc})_{\chi = \chi_1} < (R_{nc} \text{ or } R_{mc})_{\chi = \chi_2} < (R_{nc} \text{ or } R_{mc})_{\chi = \chi_3}.$$  

- The increase in magnetic susceptibility $\chi$ is to delay the penetrative FTC.
- Increase in nonlinearity of fluid magnetization and the strength of magnetic parameters tend to destabilize the system.
- The magnetic ($R_m$) and the gravitational ($R_g$) forces are complementary with each other and the buoyancy force has more stabilizing effect on the system compared to magnetic force. i.e. $R_{nc} < R_{mc}$.
- The effect of increasing $\eta$, $Da$ as well as decrease in $M_1$, $M_3$ and $Ns$ to increase the critical wave number and hence their effect is to contract the convection cells size.

Declarations

Author contribution statement

C. E. Nanjundappa: Conceived and designed the analysis; Analyzed and interpreted the data; Wrote the paper.
Savitha Y. L. & I.S. Shivakumara: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.
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