ASYMPTOTIC REGULARITY CONDITIONS FOR THE STRONG CONVERGENCE TOWARDS WEAK LIMIT SETS AND WEAK ATTRACTIONS OF THE 3D NAVIER-STOKES EQUATIONS

RICARDO M. S. ROSA

Abstract. The asymptotic behavior of solutions of the three-dimensional Navier-Stokes equations is considered on bounded smooth domains with no-slip boundary conditions or on periodic domains. Asymptotic regularity conditions are presented to ensure that the convergence of a Leray-Hopf weak solution to its weak \( \omega \)-limit set – weak in the sense of the weak topology of the space \( H \) of square-integrable divergence-free velocity fields – are achieved also in the strong topology of \( H \). In particular, if a weak \( \omega \)-limit set is bounded in the space \( V \) of velocity fields with square-integrable vorticity then the attraction to the set holds also in the strong topology of \( H \). Corresponding results for the strong convergence towards the weak global attractor of Foias and Temam are also presented.

1. INTRODUCTION

The notions of limit sets and attractors (whether local or global) permeate the theory of dynamical systems both in finite and in infinite dimensions. In the case of infinite dimensions, the existence of such sets, in particular that of the global attractor, is a major issue (the global attractor is the minimal set for the inclusion relation which uniformly attracts all bounded sets of initial conditions – the global attractor contains all locally attracting sets and \( \omega \)-limit sets). Existence results have been obtained for a number of nonlinear partial differential equations modeling various phenomena.

In this note we address the celebrated system associated with the Navier-Stokes equations for an incompressible fluid filling a region in a three-dimensional space (3D NSE for short). Due in particular to the lack of a result on the global well-posedness for the 3D NSE the notion of attractor in this case is not settled, and the study of the asymptotic behavior of this system is a major challenge.

In [7] Foias and Temam introduced a notion of weak global attractor (see the definition in [6,1]), which is loosely speaking a global attractor for the weak topology of the natural phase space of square-integrable divergence-free vector fields (see also the related notion of trajectory attractor \([2,4,13]\)). A notion of weak limit set (limit set for the weak topology, see definition [5,3])) can be similarly considered. The study of attractors and limit sets for
the strong topology is more delicate due to the lack of global regularity and uniqueness of the solutions.

Our aim in this note is to consider weak limit sets and the weak global attractor of Foias and Temam and present a condition for convergence in the strong topology of the phase space. It is an asymptotic regularity condition. More precisely, we prove that if the weak \( \omega \)-limit set (respectively weak global attractor) is made of points through which pass solutions satisfying the energy equation (with equality, not just inequality, see (2.5)), then it is a strong \( \omega \)-limit set (resp. global attractor for the strong topology). A corollary of this results with a simpler condition for strong convergence is that the weak \( \omega \)-limit set (resp. weak global attractor) be bounded in the space of divergence-free velocity fields with finite enstrophy (i.e. square-integrable vorticity). This includes weakly attracting fixed points, weak limit cycles, weakly attracting quasi-periodic orbits, etc, which turn out to be strongly attracting provided they have bounded enstrophy. It also includes hyperbolic objects, in which case the weakly attracting invariant manifolds turn out to be also strongly attracting provided they have bounded enstrophy.

The proof of this result is based on an idea devised by Ball (see [1, 3, 8, 14, 10]), exploiting energy-type equations to prove asymptotic compactness of the trajectories. Let us consider limit sets for simplicity. We start with a trajectory \( u = u(t) \). This trajectory is said to be asymptotically compact (in a given space) if given any time sequence \( t_n \to \infty \), there exists a convergent subsequence for \( \{u(t_n)\} \). This asymptotic compactness implies the existence of the corresponding \( \omega \)-limit set. A similar result holds for global attractors [12, 9, 18, 16].

The idea of the energy-equation method to obtain the asymptotic compactness can be broken down into two steps: i) weak compactness of the sequence \( \{u(t_n)\} \), and ii) norm convergence \( |u(t_{n_j})| \to |v_0| \) of a weakly convergent subsequence \( u(t_{n_j}) \to v_0 \), as \( j \to \infty \). In uniformly convex spaces (such as Hilbert spaces), weak plus norm convergences implies strong convergence, hence asymptotic compactness in the strong topology. In practice, the first step follows from classical a priori estimates obtained from energy-type inequalities, while the second one, as developed in [1, 3], follows from energy-type equations (the equality is important!).

At this point one may be skeptical about the condition of an energy equation for the 3D NSE since the (Leray-Hopf) weak solutions are known to satisfy only an energy inequality. The crucial point that we present here is based on the simple observation that in fact the energy equation need only be satisfied by the limit solution associated with \( v_0 \). Hence, asymptotic regularity is all what is needed for the strong convergence of a Leray-Hopf weak solution towards its limit set. The precise results are given in Lemma 4.1, Proposition 4.1, and Theorems 4.1 to 4.4 below.

It should be clear that this idea may be adapted to yield similar results for other differential equations in which uniqueness and lack of regularity are troublesome, such as wave equations with critical nonlinearities.
2. Preliminaries

We recall now some classical results which can be found for instance in [11, 13, 5, 17]. We consider the three-dimensional Navier-Stokes equations with either periodic or no-slip boundary conditions. In the periodic case, we consider the whole space $\mathbb{R}^3$, and the flow is assumed periodic with period $L_i$ in each direction $x_i$. We define $\Omega = \prod_{i=1}^3 (0, L_i)$ and assume that the average flow on $\Omega$ vanishes, i.e.

$$\int_{\Omega} u(x) \, dx = 0.$$ 

Here, $u = (u_1, u_2, u_3)$ denotes the velocity vector, and $x = (x_1, x_2, x_3)$, the space-variable.

In the no-slip case, the flow is considered in a bounded domain $\Omega$ of $\mathbb{R}^3$, with smooth boundary $\partial \Omega$, and the no-slip boundary condition on $\partial \Omega$ is assumed, i.e. $u = 0$ on $\partial \Omega$.

In either periodic or no-slip case, we obtain a functional equation formulation for the time-dependent velocity field $u = u(t)$ in a suitable space $H$:

$$\frac{du}{dt} + \nu Au + B(u, u) = f. \quad (2.1)$$

We consider the test spaces $V = \{ u \in C_c^\infty(\Omega); \nabla \cdot u = 0, \ u(x) \text{ is periodic with period } L_i \text{ in each direction } x_i \}$, in the periodic case, and

$$V = \{ u \in C_c^\infty(\Omega)^3; \nabla \cdot u = 0 \},$$

in the no-slip case, and define $H$ as the completion of $V$ under the $L^2(\Omega)^3$ norm (where $C_c^\infty(\Omega)$ denotes the space of infinitely-differentiable real-valued functions with compact support on $\Omega$). We also consider the space $V$ defined as the completion of $V$ under the $H^1(\Omega)^3$ norm.

We identify $H$ with its dual and consider the dual space $V'$, so that $V \subset H \subset V'$. We denote by $H_w$ the space $H$ endowed with its weak topology.

We consider the inner products in $H$ and $V$ given respectively by

$$\langle u, v \rangle = \int_{\Omega} u(x) \cdot v(x) \, dx, \quad \langle u, v \rangle = \int_{\Omega} \sum_{i=1,3} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} \, dx,$$

and the associated norms $|u| = (u, u)^{1/2}$, $\|u\| = (\langle u, u \rangle)^{1/2}$.

We denote by $P_{LH}$ the (Leray-Helmholtz) orthogonal projector in $L^2(\Omega)^3$ onto the subspace $H$. In (2.1), $A$ is the Stokes operator $Au = -P_{LH} \Delta u; B(u, v) = P_{LH}(u \cdot \nabla)v$ is a bilinear term corresponding to the inertial term; $f$ represents the mass density of volume forces applied to the fluid, and we assume that $f \in H$; and $\nu > 0$ is the kinematic viscosity. The Stokes operator is a positive self-adjoint operator on $H$, and we denote by $\lambda_1 > 0$ its first eigenvalue.

A Leray-Hopf weak solution on an open time interval $I = (t_0, t_1)$, $-\infty \leq t_0 < t_1 \leq \infty$, is defined as a function $u = u(t)$ on $(t_0, t_1)$ with values in $H$ and satisfying the following properties:

(i) $u \in L^\infty(t_0, t_1; H) \cap L^2_{loc}(t_0, t_1; V);$
(ii) $\partial u/\partial t \in L^{4/3}_{\text{loc}}(t_0, t_1; V')$;
(iii) $u \in C(I; H^1_w)$ (i.e. weakly continuous);
(iv) $u$ satisfies the functional equation (2.1) almost everywhere on $I = (t_0, t_1)$;
(v) $u$ satisfies the following energy inequality in the distribution sense on $I = (t_0, t_1)$:
\[
\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu \|u(t)\|^2 \leq (f, u(t)). \tag{2.2}
\]

A *Leray-Hopf weak solution* on an interval of the form $[t_0, t_1)$ is defined as a Leray-Hopf weak solution on $(t_0, t_1)$ which is continuous at $t = t_0$, i.e.

(vi) $u(t) \to u(t_0)$ in $H$, as $t \to t_0^+$.

From now on, for notational simplicity, a weak solution will always mean a Leray-Hopf weak solution. For a weak solution on an arbitrary interval $I$, it follows that
\[
|u(t)|^2 \leq |u(t')|^2 e^{-\nu \lambda_1 (t-t')} + \frac{1}{\nu^2 \lambda_1^2} |f|^2 \left(1 - e^{-\nu \lambda_1 (t-t')}\right), \tag{2.3}
\]
for all $t$ in $I$ and almost all $t'$ in $I$ with $t' < t$. The allowed times $t'$ are the Lebesgue points of the function $t \mapsto |u(t)|$. In the case of a weak solution on an interval of the form $[t_0, t_1)$, the point $t_0$ is a point of continuity of $t \mapsto |u(t)|$, hence a Lebesgue point, so that the estimate above is also valid for the initial time $t' = t_0$.

Another classical estimate obtained from the energy inequality is
\[
|u(t)|^2 + \nu \int_{t'}^t \|u(s)\|^2 \, ds \leq |u(t')|^2 + \frac{1}{\nu \lambda_1} |f|^2 (t - t'), \tag{2.4}
\]
for all $t$ in $I$ and almost all $t'$ in $I$ with $t' < t$, with the set of allowed times $t'$ consisting again as the Lebesgue points of the function $t \mapsto |u(t)|$.

A *strong solution* on an arbitrary interval $I$ is defined as a weak solution on $I$ satisfying
(vii) $u \in C(I; V)$.

Any strong solution satisfies the energy equation
\[
\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu \|u(t)\|^2 = (f, u(t)) \tag{2.5}
\]
in the distribution sense on its interval of definition.

It is well established that given any initial time $t_0$ and any initial condition $u_0$ in $H$, there exists at least one global weak solution on $[t_0, \infty)$ satisfying $u(t_0) = u_0$. It is also known that if $u_0$ belongs to $V$, then there exists a local strong solution, defined on some interval $[t_0, t_1)$, $t_1 > t_0$, with $u(t_0) = u_0$. Finally, any strong solution is unique on its interval of definition. The uniqueness in this case is with respect to the larger class of all weak solution, i.e. any weak solution with $u(t_0) = u_0$ must agree with the strong solution on the interval of definition of the latter.
3. Weak limit sets and the weak global attractor

The weak global attractor, as defined in [7], is the set

\[ A_w = \begin{cases} 
    v_0 \in H; & \text{there exists at least one global weak solution } v = v(t) \\
    \text{defined for all } t \in \mathbb{R} \text{ which is uniformly bounded in } H, \\
    \text{i.e. } \sup_{t \in \mathbb{R}} |v(t)| < \infty, \text{ and such that } v(0) = v_0 
\end{cases} \]  \tag{3.1}

Due to the energy estimate (2.3) and the uniform bound on the global solutions in the definition of \( A_w \) it follows that \( A_w \) is a bounded set in \( H \):

\[ |v_0| \leq \frac{1}{\nu \lambda_1} |f|, \quad \forall v_0 \in A_w. \]

It is proved in [7] that \( A_w \) is weakly compact in \( H \) and that it attracts all weak solutions in the following sense: If \( u = u(t) \) is a weak solution on \([t_0, \infty)\) for some \( t_0 \in \mathbb{R} \), then for any neighborhood \( O \) of \( A_w \) in the weak topology of \( H \), there exists a time \( T \geq t_0 \) such that \( u(t) \in O \) for all \( t \geq T \). Since \( H \) is separable the weak topology is metrizable on bounded sets and the convergence above can be rewritten in terms of this metric. Finally, a certain invariance property holds, namely if \( v_0 \) belongs to \( A_w \) and \( v = v(t), t \in \mathbb{R} \), is a uniformly bounded global weak solution through \( v_0 \) then \( v(t) \in A_w \) for all \( t \in \mathbb{R} \).

Besides the pointwise attraction (attraction of individual weak solutions) of the weak global attractor, one can show that the attraction is in fact uniform with respect to uniformly bounded sets of initial condition (see [8]). More precisely, given \( t_0 \in \mathbb{R} \) and \( R > 0 \), then for every neighborhood \( O \) of \( A_w \) in the weak topology of \( H \), there exists a time \( T \geq t_0 \) such that \( u(t) \in O \) for all \( t \geq T \) and for every weak solution \( u \) on \([t_0, \infty)\) with \( \sup_{t \geq t_0} |u(t)| \leq R \). Since \( A_w \) is bounded in \( H \) and the weak topology of \( H \) is metrizable on bounded subsets this uniform attraction in the weak topology can be rewritten in terms of an associated metric.

These properties define \( A_w \) and justify its definition as the weak global attractor. They can also be used to characterize \( A_w \) in a more classical way:

\[ A_w = \begin{cases} 
    v_0 \in H; & \text{there exist global weak solutions } u_n = u_n(t), n \in \mathbb{N}, \text{ defined} \\
    \text{for all } t \geq 0, \text{ with } \sup_{t \in \mathbb{R}} |u_n(t)| \leq |f|/\nu \lambda_1, \text{ and a time} \\
    \text{sequence } \{t_n\}_n, t_n \geq 0, t_n \to \infty, \text{ such that } u_n(t_n) \rightharpoonup v_0 
\end{cases} \]  \tag{3.2}

Now, given an arbitrary weak solution \( u = u(t) \) on an interval of the form \((t_0, \infty)\) or \([t_0, \infty)\), for some \( t_0 \in \mathbb{R} \), we define its weak \( \omega \)-limit set by

\[ \omega_w(u) = \{v_0 \in H; \exists \{t_n\}_n, t_n > t_0, t_n \to \infty, u(t_n) \rightharpoonup v_0\}, \]  \tag{3.3}

where \( \rightharpoonup \) denotes the weak convergence in \( H \). This set is always nonempty since \( \{u(t)\}_{t > t_0} \) is bounded in \( H \) (thanks to (2.3)), hence weakly precompact. Since the weak topology is metrizable on bounded subsets of \( H \), the classical characterization \( \omega_w(u) = \cap_{t \geq 0} \overline{\bigcup_{t \geq s} \{u(t)\}_w} \) holds, where \( \overline{\cdot} \) denotes the closure in the weak topology. Hence, \( \omega_w(u) \) is weakly compact. By classical dynamical system arguments one can also show that \( \omega_w(u) \) attracts \( u \) in the sense that for any weakly open set \( O \) containing \( \omega_w(u) \), there exists a time \( T > t_0 \) such that \( u(t) \in O \) for all \( t \geq T \).
As for the invariance property, it is possible to show that for every \( v_0 \) in \( \omega_w(u) \), there exists a global weak solution \( v = v(t) \), \( t \in \mathbb{R} \), with \( v(0) = v_0 \) and \( v(t) \in \omega_w(u) \) for all \( t \in \mathbb{R} \). This is achieved by passing to the limit the solutions \( u(t_n + \cdot) \) over time intervals \([-T, T]\), for arbitrarily large times \( T \). In fact, classical a priori estimates (derived from (2.3) and (2.4)) yield that \( \{u(t_n + \cdot)\}_{n} \) is bounded on \( L^\infty(-T, T; H) \cap L^2(-T, T; V) \) and that \( \{\partial u/\partial t(t_n + \cdot)\}_{n} \) is bounded on \( L^{4/3}(-T, T; V') \), which imply precompactness in \( L^2(-T, T; H) \) and \( C([-T, T], H_w) \). A diagonalization process guarantees convergence on any bounded set to a global weak solution defined on all \( \mathbb{R} \). However, due to the possible lack of uniqueness one cannot guarantee the invariance for every solution passing through \( v_0 \).

With the above invariance property in mind, given \( v_0 \) in \( \omega_w(u) \) we defined \( \mathcal{G}_u(v_0) \) as the set of all global weak solutions \( v = v(t) \), \( t \in \mathbb{R} \), with \( v(0) = v_0 \), and such that there exists a sequence \( \{t_n\}_n \), \( t_n \geq t_0 \), \( t_n \to \infty \), with the property that \( u(t_n + \cdot) \) converges to \( v \) in \( C([-T, T], H_w) \), for all \( T > 0 \). Note that this implies that \( \mathcal{G}_u(v_0) \subseteq \omega_w(u) \), for all \( v_0 \) in \( \omega_w(u) \).

Finally, by classical a priori estimates (derived from (2.3) and (2.4)) and Aubin’s compactness theorem the convergence of \( u(t_n + \cdot) \) to \( v \) holds weakly-star in \( L^\infty(-T, T; H) \), weakly in \( L^2(-T, T; V) \), and strongly in \( L^2(-T, T; H) \). Then, the convergence required in the definition of \( \mathcal{G}_u(v_0) \) guarantees the uniqueness of the limit and the convergence without passing to further subsequences.

A similar argument for the weak global attractor yields that for every \( v_0 \) in \( A_w \) and every sequences \( \{u_n\}_n \) and \( \{t_n\}_n \) as in the characterization (3.2), with \( u_n(t_n) \to v_0 \), there exists subsequences \( \{u_{n_j}\}_j \) and \( \{t_{n_j}\}_j \) such that \( u_{n_j}(t_{n_j} + \cdot) \) converges weakly-star in \( L^\infty(-T, T; H) \), weakly in \( L^2(-T, T; V) \), strongly in \( L^2(-T, T; H) \) and strongly in \( C([-T, T], H_w) \), for all \( T > 0 \), to a global weak solution \( v = v(t) \), with \( v(t) \in A_w \), for all \( t \in \mathbb{R} \), and with \( v(0) = v_0 \).

4. Asymptotic regularity conditions for the strong convergence towards weak limit sets and the weak global attractor

As mentioned in the introduction the required regularity condition is that the limit solutions satisfy the energy equation exactly. More precisely, we have the following result.

**Lemma 4.1.** Let \( u \) be a weak solution defined on some interval of the form \((t_0, \infty)\) or \([t_0, \infty)\), for some \( t_0 \in \mathbb{R} \). Let \( \nu > 0 \) and \( \{t_n\}_n \) be such that \( t_n > t_0 \), \( t_n \to \infty \), and \( u(t_n) \to v_0 \) weakly in \( H \). If there exists a global weak solution \( v = v(t), t \in \mathbb{R} \), such that \( u(t_n + \cdot) \) converges to \( v \) in \( C([-T, T], H_w) \), for all \( T > 0 \), and which satisfies the energy inequality (2.2), then \( u(t_n) \) converges strongly in \( H \) to \( v_0 \).

**Proof.** In what follows, given \( T > 0 \) we restrict ourselves to \( n \) such that \( t_n - T > t_0 \).

The weak solution satisfies the energy inequality (2.2). By adding and subtracting the term \( \nu \lambda_1 |u(t)|^2 / 2 \), we arrive at the inequality

\[
\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \frac{\nu \lambda_1}{2} |u(t)|^2 + \nu |u(t)|^2 \leq (f, u(t)),
\]  

(4.1)
where
\[ [u]^2 = \|u\|^2 - \frac{\lambda_1}{2}|u|^2, \quad (4.2) \]
with \([\cdot]\) being a norm in \(V\) equivalent to \(\|\cdot\|\). By “multiplying” (4.1) by appropriate nonnegative test functions approximating \(\theta\) in the distribution formulation of the equation, where \(\theta_m(t)\) is the continuous piecewise linear function joining linearly the values \(\theta_m(-T) = 0, \theta_m(-T + 1/m) = 1, \theta_m(-1/m) = 1, \theta_m(0) = 0\), and let \(m \to \infty\) one finds at the limit that
\[ |u(t_n)|^2 \leq |u(t_n - T)|^2 e^{-\nu \lambda_1 T} - 2 \int_T^0 e^{\nu \lambda_1 s} \{\nu|u(t_n + s)|^2 - (f, u(t_n + s))\} \, ds, \]
for almost all \(T\) such that \(t_n - T > t_0\) (more precisely, it holds for all Lebesgue points \(T\) of the function \(T \to |u(t_n - T)|\)).

Passing to the limit as \(n\) goes to infinity and using that \(u(t_n + \cdot)\) converges to \(v\) weakly in \(L^2(-T, T; V)\) we find
\[ \limsup_{n \to \infty} |u(t_n)|^2 \leq \limsup_{n \to \infty} |u(t_n - T)|^2 e^{-\nu \lambda_1 T} - 2 \int_T^0 e^{\nu \lambda_1 s} \{\nu|v(s)|^2 - (f, v(s))\} \, ds. \]
Note that we have used that \([\cdot]\) is an equivalent norm for \(V\) so that
\[ \int_T^0 e^{\nu \lambda_1 s}|v(s)|^2 \, ds \leq \liminf_{n \to \infty} \int_T^0 e^{\nu \lambda_1 s}|u(t_n + s)|^2 \, ds. \]

As for the limit solution \(v\) it satisfies the energy equation, so that
\[ |v(0)|^2 = |v(-T)|^2 e^{-\nu \lambda_1 T} - 2 \int_T^0 e^{\nu \lambda_1 s} (\nu|v(s)|^2 - (f, v(s))) \, ds. \]
By subtracting this equality from the inequality for the limsup and using that both \(v(-T)\) and \(u(t_n - T)\) are bounded in \(H\) independently of \(T\) and \(n\) we find
\[ \limsup_{n \to \infty} |u(t_n)|^2 - |v(0)|^2 \leq 2Re^{-\nu \lambda_1 T}, \]
for some bound \(R > 0\). The previous passage should make clear why we need the equality in the energy equation only for the limit solution. Now, since \(T\) is arbitrary (except for a set of measure zero), we may let \(T \to \infty\) to find that
\[ \limsup_{n \to \infty} |u(t_n)|^2 - |v(0)|^2 \leq 0. \]
Since \(v(0) = v_0\), we obtain
\[ \limsup_{n \to \infty} |u(t_n)| \leq |v_0|. \]
On the other hand, since \(u(t_n)\) converges weakly in \(H\) to \(v_0\) we have \(|v_0| \leq \liminf_{n \to \infty} |u(t_n)|\). Thus \(\lim_{n \to \infty} |u(t_n)| \leq |v_0|\), which together with the weak convergence implies the strong convergence \(u(t_n) \to v_0\) in \(H\). This concludes the proof. \qed
Proposition 4.1. Let $u$ be a weak solution defined on some interval of the form $(t_0, \infty)$ or $[t_0, \infty)$, for some $t_0 \in \mathbb{R}$. If for every $v_0$ in $\omega_w(u)$ all the global weak solutions in $G_u(v_0)$ satisfy the energy equation \[(2.3)\] then $\omega_w(u)$ attracts $u$ in the strong topology of $H$.

Proof. If this were not true we would find a time sequence $\{t_n\}_n$, with $t_n > t_0$, $t_n \to \infty$, and such that $\{u(t_n)\}_n$ does not have any subsequence converging strongly in $H$. Now, since $\{u(t_n)\}_n$ is bounded (thanks to \[(2.3)\]) it has a weakly convergent subsequence $\{u(t_{n_j})\}_{n_j}$, with a weak limit $v_0$ belonging to $\omega_w(u)$. Since any solution $v$ in $G_u(v_0)$ satisfies the condition of Lemma 4.1 it follows that $\{u(t_{n_j})\}_{n_j}$ converges strongly to $v_0$, which is a contradiction. Thus, $\omega_w(u)$ attracts $u$ in the strong topology of $H$. \qed

We now define the set

$$V_{\text{reg}} = \{v_0 \in V; \exists \delta > 0 \text{ and } \exists \text{ a strong solution } v \in C((-\delta, \delta), V) \text{ with } v(0) = v_0\}.$$ 

Theorem 4.1. Let $u$ be a weak solution defined on some interval of the form $(t_0, \infty)$ or $[t_0, \infty)$, for some $t_0 \in \mathbb{R}$. If $\omega_w(u) \subset V_{\text{reg}}$, then $\omega_w(u)$ attracts $u$ in the strong topology of $H$.

Proof. In view of Proposition 4.1 it suffices to show that any weak solution in $G_u(v_0)$ satisfies the energy equation for all $v_0$ in $\omega_w(u)$. Note first that $G_u(v_0) \subset V_{\text{reg}}$ since it is included in $\omega_w(u)$, which is assumed to be included in $V_{\text{reg}}$. So any weak solution $v$ in $G_u(v_0)$ is included in $V_{\text{reg}}$, i.e. $v(t) \in V_{\text{reg}}$ for all $t \in \mathbb{R}$. By the uniqueness of strong solutions among the larger class of weak solutions it follows that $v$ is a global strong solution and belongs to $C(\mathbb{R}, V)$. Therefore, $v$ satisfies the energy equation, and this completes the proof. \qed

Theorem 4.2. Let $u$ be a weak solution defined on some interval of the form $(t_0, \infty)$ or $[t_0, \infty)$, for some $t_0 \in \mathbb{R}$. If $\omega_w(u)$ is a bounded subset of $V$, then $\omega_w(u)$ attracts $u$ in the strong topology of $H$.

Proof. Following the idea in the proof of Theorem 4.1 any global weak solution $v$ in $G_u(v_0)$, $v_0 \in \omega_w(u)$, is uniformly bounded in $V$, hence it is a global strong solution belonging to $C(\mathbb{R}, V)$, which is sufficient to apply Proposition 4.1 and conclude the proof. \qed

We now present a result concerning the weak global attractor. This result is not a direct consequence of the previous ones simply because of the uniform attraction required (reflecting the fact that the weak global attractor may be larger than the union of weak $\omega$-limit sets).

Theorem 4.3. If $A_w \subset V_{\text{reg}}$ or $A_w$ is a bounded subset of $V$, then $A_w$ attracts every weak solution in the strong topology of $H$, and this attraction is uniform with respect to uniformly bounded sets of weak solutions. More precisely, given $t_0 \in \mathbb{R}$ and $R > 0$, then for every $\varepsilon > 0$, there exists a time $T \geq t_0$ such that \[\text{dist}_H(u(t), A_w) \overset{\text{def}}{=} \sup_{v_0 \in A_w} |u(t) - v_0| < \varepsilon, \text{ for every weak solution } u \text{ on } [t_0, \infty) \text{ with } \sup_{t \geq t_0} |u(t)| \leq R.\]

Proof. Suppose the result is not true. Then there exists $t_0 \in \mathbb{R}$, $R > 0$, $\varepsilon > 0$, a sequence $u_n$ of weak solutions on $[t_0, \infty)$ with $\sup_{t \geq t_0} |u_n(t)| \leq R$, and a time sequence $\{t_n\}_n$, $t_n \geq t_0$, $t_n \to \infty$, such that \[|u_n(t_n) - v_0| \geq \varepsilon, \text{ for all } n \text{ and all } v_0 \in A_w.\] (4.3)
Consider the sequence $v_n(t) = u_n(t_n + t)$, defined for $t \geq t_0 - t_n$. By the assumption of uniform estimate on $u_n$ and by classical a priori estimates for the 3D NSE (derived from (2.3) and (2.4)) the sequence $\{v_n\}_n$ is bounded in $L^\infty(-T,T;H) \cap L^2(-T,T;V)$, and $\{\partial v_n/\partial t\}_n$ is bounded in $L^{4/3}(-T,T;V')$, which imply precompactness in $L^2(-T,T;H)$ and $C([-T,T],H_w)$. By passing to the limit as in the classical theory of existence of weak solutions of the 3D NSE and by using a diagonalization process we find a subsequence converging on any bounded interval $[-T,T]$, $T > 0$, to a global weak solution $v = v(t)$, $t \geq 0$. In particular, $u_{n_j}(t_{n_j})$ converges weakly to some element $v_0 = v(0)$ in $H$.

At the limit, we retain a uniform bound for $v$, $\sup_{t \geq t_0} |v(t)| \leq R$, so that $v(t)$ belongs to $A_w$ for all $t \in \mathbb{R}$, and in particular $v_0 \in A_w$. Since $A_w$ either belongs to $V_{\text{reg}}$ or is a bounded subset of $V$ it follows (see the proofs of Theorems 1.1 and 1.2) that $v$ is a global strong solution, $v \in C(\mathbb{R},V)$, hence $v$ satisfies the energy equation on $\mathbb{R}$.

Now we proceed as in the proof of Lemma 4.1. The only difference being that instead of working with $u(t_n + \cdot)$ we work with $u_n(t_n + \cdot)$ (we still have the inequalities holding for almost all $T$, as the countable union of zero-measure sets is still of zero measure). Then, by repeating the energy-equation argument with $u_n(t_n + \cdot)$ we find that $\limsup_{n \to \infty} |u_{n_j}(t_{n_j})| \leq |v_0|$, so that in fact $u_{n_j}(t_{n_j})$ converges strongly to $v_0$. But this contradicts 1.3 since $v_0$ belongs to $A_w$. This completes the proof. □

Our last result concerns just part of the weak global attractor. It does not reduce to a statement about weak $\omega$-limit sets included in the weak global attractor since it allows for asymptotic limits of sequences of weak solutions instead of a single solution. It applies, for instance, to regular (in the sense of being a strong solution) connecting orbits which are not necessarily $\omega$-limit sets.

**Theorem 4.4.** Suppose $v = v(t)$, $t \in \mathbb{R}$, is a global strong solution uniformly bounded in $H$ (in other words, a global weak solution included in $A_w \cap V_{\text{reg}}$) and set $v_0 = v(0)$. Let $u_n = u_n(t)$, $t \geq 0$, and $\{t_n\}_n$ be as in the characterization (3.2), with $u_n(t_n) \to v_0$. Then, $u_n(t_n) \to v_0$ strongly in $H$.

**Proof.** By assumption, $v = v(t)$ is a strong solution, hence it satisfies the energy equation (2.5). Then, as in the last paragraph of the proof of Theorem 1.3 we apply the energy-equation method to the sequence of weak solutions $\{u_n(t_n + \cdot)\}_n$ to show that $|u_n(t_n)| \to |v_0|$. Then, we conclude that $u(t_n) \to v_0$ strongly in $H$. (At first one may need to pass to further subsequences, but since the weak limit $u_n(t_n) \to v_0$ exists and hence is unique, the strong converge in $H$ must hold for the whole sequence.) □

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(Ricardo M. S. Rosa) DEPARTAMENTO DE MATEMÁTICA APLICADA, INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, CAIXA POSTAL 68530 ILHA DO FUNDÃO, RIO DE JANEIRO RJ 21945-970, BRAZIL

E-mail address: rrosa@ufrj.br