CONVERGENCE OF THE $J$-FLOW
ON KÄHLER SURFACES

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1. Introduction

In [Do], Donaldson described how a number of geometric situations fit
into a general framework of diffeomorphism groups and moment maps. In
the Kähler setting, he used this framework to define a natural parabolic flow,
as follows. Suppose that $(M, \omega)$ is a compact Kähler manifold of dimension
$n$ and let $\chi_0$ be another Kähler form on $M$, in a different Kähler class.
Consider the infinite-dimensional manifold $M$ of diffeomorphisms $f : M \to M$, homotopic to the identity. $M$ carries a natural symplectic form $\Omega$ defined
by
$$\Omega_f(v, w) = \int_M \omega(v, w) \frac{\chi_0^n}{n!},$$
for sections $v, w$ of $f^*(TM)$. The group $G$ of exact $\chi_0$-symplectomorphisms
of $M$ acts on $M$ by composition on the right, preserving $\Omega$. We can identify
the Lie algebra of $G$ with the space of functions on $M$ of integral zero with
respect to the volume form induced by $\chi_0$. A moment map $\mu : M \to \text{Lie}(G)^*$
for the group action is given by
$$\mu(f) = \frac{\int_M f^*(\omega) \wedge \chi_0^{n-1}}{\chi_0^n} - \frac{\int_M \omega \wedge \chi_0^{n-1}}{\int_M \chi_0^n},$$
where we are using the $L^2$ inner product to identify $\text{Lie}(G)$ with its dual. It
is natural to look for solutions of
$$\mu(f) = 0 \pmod{G}. \quad (1.1)$$
These points form the symplectic quotient. Under certain conditions, one
would hope that the gradient flow $f_t$ of the function $\|\mu\|^2$ on $M$ would
converge to give a solution of (1.1). The gradient flow can be rewritten as a
flow of Kähler forms $(f_t^*)^{-1}(\chi_0)$ on $M$. This defines a parabolic flow on the
space of Kähler potentials and is the object of study of this paper.

At around the same time, Chen [C1] independently discovered the same
flow as the gradient flow of his $J$-functional. He later called it the $J$-flow.
He showed in [C1] that the $J$-functional is related to the Mabuchi K-energy $[Ma]$, which plays a key role in the study of Kähler geometry and stability in the sense of geometric invariant theory (see [Y2], [T2], [T3] and [PS] for example).

Explicitly, the $J$-flow is defined as follows. Let $c$ be the constant given by
\[ c = \frac{\int_M \omega \wedge \chi_0^{n-1}}{\int_M \chi_0^n}, \]
and let $\mathcal{H}$ be the space of Kähler potentials
\[ \mathcal{H} = \{ \phi \in C^\infty(M) \mid \chi_\phi = \chi_0 + \sqrt{-1} \partial \bar{\partial} \phi > 0 \}. \]

The $J$-flow is the flow on $\mathcal{H}$ given by
\[ \frac{\partial \phi_t}{\partial t} = c - \frac{\omega \wedge \chi_{\phi_t}^{n-1}}{\chi_{\phi_t}^n}. \]

A critical point of the $J$-flow gives a Kähler metric $\chi$ satisfying
\[ \omega \wedge \chi^{n-1} = c \chi^n. \]

Donaldson [Do] asked whether one can find a solution to (1.3) in the class $[\chi_0]$ under certain assumptions. He noted that a necessary condition is that $[nc \chi_0 - \omega]$ be a Kähler class, and conjectured that this condition be sufficient. Chen [C1] confirmed this conjecture in the case $n = 2$, without using the $J$-flow, by observing that (1.3) reduces to a Monge-Ampère equation which can be solved by the well-known result of Yau [Y1]. The conjecture is still open for $n > 2$.

Chen [C1] shows that Donaldson’s conjecture would imply a result on the lower bound of the Mabuchi K-energy for compact Kähler manifolds $M$ with negative first Chern class. Namely, if $-\omega \in c_1(M)$ with $\omega > 0$, then for Kähler classes $[\chi_0]$ satisfying
\[ nc[\chi_0] - [\omega] > 0, \]
the Mabuchi K-energy would have a lower bound in the class $[\chi_0]$.

Solutions of the $J$-flow exist for a short time by general theory, since the flow is parabolic. In [C2], Chen showed that the flow always exists for all time for any smooth initial data. He also showed that if the bisectional curvature of $\omega$ is non-negative then the $J$-flow converges to a critical metric.
In general, the behaviour of the flow is not known. In this paper, we deal with the case \( n = 2 \) with no curvature restrictions. Our main result is as follows.

**Main Theorem** Suppose that \((M, \omega)\) has dimension \( n = 2 \) and that

\[
nc\chi_0 - \omega > 0.
\]

Then the \( J \)-flow \((1.2)\) converges in \( C^\infty \) to a smooth critical metric.

The outline of the paper is as follows. In section 2 we state some preliminary facts about the flow and introduce notation. In section 3, the maximum principle is used to derive an estimate on the second derivatives of \( \phi \) in terms of \( \phi \) itself. In section 4, a \( C^0 \) estimate for \( \phi \) is given. The argument uses the second order estimate, a Moser iteration argument applied to the exponential of \(-\phi\) and the result of Tian \([T1]\) (see also \([TY]\)) on the existence of constants \( \alpha > 0 \) and \( C \) such that

\[
\int_M e^{-\alpha\phi} \frac{\chi_0^n}{n!} \leq C,
\]

for all \( \phi \) in \( \mathcal{H} \) with \( \sup_M \phi = 0 \). In section 5, the proof of the main theorem is completed.

2. Preliminaries and notation

From now on, assume that \( \omega \) has been scaled so that \( c = 1/n \). We will work in local coordinates, and write

\[
\omega = \frac{\sqrt{-1}}{2} g_{i\overline{j}} dz^i \wedge dz^\overline{j}, \quad \chi_0 = \frac{\sqrt{-1}}{2} \chi_{0i\overline{j}} dz^i \wedge dz^\overline{j},
\]

and

\[
\chi = \frac{\sqrt{-1}}{2} \chi_{i\overline{j}} dz^i \wedge dz^\overline{j} = \frac{\sqrt{-1}}{2} (\chi_{0i\overline{j}} + \partial_i \partial_\overline{j} \phi) dz^i \wedge dz^\overline{j},
\]

where \( \chi = \chi_\phi \) (suppressing the \( t \)-subscript.) The operators \( \Lambda_\omega \) and \( \Lambda_\chi \) act on \((1,1)\) forms \( \alpha = \frac{\sqrt{-1}}{2} \alpha_{i\overline{j}} dz^i \wedge dz^\overline{j} \) by

\[
\Lambda_\omega \alpha = g^{i\overline{j}} \alpha_{i\overline{j}}, \quad \text{and} \quad \Lambda_\chi \alpha = \chi^{i\overline{j}} \alpha_{i\overline{j}}.
\]

The \( J \)-flow \((1.2)\) can be written

\[
\frac{\partial \phi}{\partial t} = \frac{1}{n} (1 - \Lambda_\chi \omega), \quad \phi|_{t=0} = 0.
\]

(2.1)
Differentiating with respect to $t$ gives
\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial t} \right) = \tilde{\Delta} \left( \frac{\partial \phi}{\partial t} \right),
\] (2.2)
where the operator $\tilde{\Delta}$ acts on functions $f$ by
\[
\tilde{\Delta} f = \frac{1}{n} \chi^k \chi^\sigma g_{ij} \partial_i \partial_j f.
\]
For convenience, write
\[
h^k_l = \chi^j \chi^\sigma g_{ij}.
\]
The tensor $h^k_l$ is positive definite and its inverse defines a Hermitian metric on $M$. The operator $\tilde{\Delta}$ is, up to a constant factor, the Laplacian associated to this Hermitian metric.

By the maximum principle for parabolic equations, (2.2) implies that
\[
\inf_M (\Lambda \chi_0 \omega) \leq \Lambda \chi \omega \leq \sup_M (\Lambda \chi_0 \omega),
\] (2.3)
which gives a lower bound for $\chi$,
\[
\chi \geq \frac{1}{\sup_M (\Lambda \chi_0 \omega)} \omega.
\] (2.4)

The $J$-functional \[ C1 \] is defined by
\[
J_{\omega, \chi_0} (\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \omega \wedge \chi_t^{n-1} \frac{1}{(n-1)!} dt,
\]
where $\{ \phi_t \}$ is a path in $\mathcal{H}$ between 0 and $\phi$. The functional is independent of the choice of path. We will need the following formula for the functional in the case $n = 2$. Taking the path $\phi_t = t \phi$, we see that
\[
J_{\omega, \chi_0} (\phi) = \frac{1}{2} \int_M \phi \omega \wedge (\chi_0 + \chi).
\] (2.5)

Chen also makes use of the $I$-functional, \[ M1 \]
\[
I_{\omega, \chi_0} (\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \chi_t^n \frac{1}{n!} dt.
\]
This is a well-known functional in Kähler geometry (see \[ M1 \]). Notice that $I(\phi) = 0$ along the flow. For $n = 2$, this functional is given by
\[
I_{\omega, \chi_0} (\phi) = \frac{1}{6} \int_M \phi (\chi_0^2 + \chi \wedge \chi_0 + \chi^2).
\] (2.6)
In the course of the paper, $C_0, C_1, \ldots$ will denote constants depending only on the initial data $\omega$ and $\chi_0$. Curvature expressions such as $R_{ijkl}$ will always refer to the metric $g_{\overline{\sigma}}$.

3. Second order estimate

We use the maximum principle to obtain an estimate on the second derivative of $\phi$ in terms of $\phi$. We choose to calculate the evolution of $(\log \Lambda_\omega \chi - A \phi)$ for some constant $A$ (compare to [Y1], [Au] or [Si] for the analogous estimate for the well-known Monge-Ampère equation, and [Ca] for the Kähler-Ricci flow.)

**Theorem 3.1** Suppose that $(M, \omega)$ has dimension $n = 2$ and that

$$\chi_0 - \omega > 0. \quad (3.1)$$

Let $\phi = \phi_t$ be a solution of the $J$-flow (2.1) on $[0, \infty)$. Then there exist constants $A > 0$ and $C > 0$ depending only on the initial data such that for any time $t \geq 0$, $\chi = \chi_{\phi_t}$ satisfies

$$\Lambda_\omega \chi \leq Ce^{A(\phi - \inf_{M \times [0, t]} \phi)}. \quad (3.2)$$

**Proof** We will calculate

$$(\tilde{\Delta} - \frac{\partial}{\partial t})(\log(\Lambda_\omega \chi) - A \phi).$$

Using normal coordinates for $\omega$, first calculate

$$\tilde{\Delta}(\Lambda_\omega \chi) = \frac{1}{n} h^{kl} \partial_k \partial_l (g^{\overline{\sigma}} \chi_{\overline{\sigma}})$$

$$= \frac{1}{n} h^{kl} R_{kl} \chi_{\overline{\sigma}} + \frac{1}{n} h^{kl} g^{\overline{\sigma}} \partial_k \partial_{\overline{\sigma}} \chi.$$ 

And

$$\frac{\partial}{\partial t}(\Lambda_\omega \chi) = \frac{\partial}{\partial t}(g^{\overline{\sigma}} \partial_i \partial_{\overline{\sigma}} \phi)$$

$$= -\frac{1}{n} g^{\overline{\sigma}} \partial_i \partial_{\overline{\sigma}} (\chi^{\overline{\sigma}} g_{kl})$$

$$= \frac{1}{n} (g^{\overline{\sigma}} \partial_i (\chi^{\overline{\sigma}} \partial_{\overline{\sigma}} \chi_{\overline{\sigma}}) g_{kl} + g^{\overline{\sigma}} \chi^{\overline{\sigma}} R_{kl})$$

$$= \frac{1}{n} (g^{\overline{\sigma}} \partial_i \chi_{\overline{\sigma}} \partial_{\overline{\sigma}} - g^{\overline{\sigma}} h^{\overline{\tau}} \chi^{\overline{\sigma}} \partial_i \chi_{\overline{\tau}} \partial_{\overline{\sigma}} \chi_{\overline{\tau}}$$

$$- g^{\overline{\sigma}} h^{\overline{\tau}} \chi^{\overline{\sigma}} \partial_i \chi_{\overline{\tau}} \partial_{\overline{\sigma}} + \chi^{k\overline{l}} R_{k\overline{l}}).$$
Now
\[ \tilde{\Delta} \log(\Lambda \omega \chi) = \frac{\tilde{\Delta}(\Lambda \omega \chi)}{\Lambda \omega \chi} - \frac{|\tilde{\nabla}(\Lambda \omega \chi)|^2}{(\Lambda \omega \chi)^2}, \]
where
\[ |\tilde{\nabla}(\Lambda \omega \chi)|^2 = \frac{1}{n} h^{ij} \partial_k (\Lambda \omega \chi) \partial_k (\Lambda \omega \chi). \]

Note that by the Kähler property of \( \chi \), we have
\[ \partial_i \partial_j \chi_{kl} = \partial_k \partial_l \chi_{ij}. \]

Then
\[ (\tilde{\Delta} - \frac{\partial}{\partial t}) \log(\Lambda \omega \chi) = \frac{1}{n \Lambda \omega \chi} (h^{ij} R_{k \overline{j}} \chi_{k \overline{j}} - n \frac{|\tilde{\nabla}(\Lambda \omega \chi)|^2}{\Lambda \omega \chi} + g^{ij} h^{i \overline{m}} \chi^{j \overline{r}} \partial_i \chi_{r} \partial_j \chi_{\overline{m}} + g^{j \overline{r}} h^{i \overline{m}} \chi^{j \overline{r}} \partial_i \chi_{r} \partial_j \chi_{\overline{m}} - \chi^{k \overline{r}} R_{k \overline{r}}). \]

We need the following lemma to deal with the second term on the right hand side.

**Lemma 3.2**
\[ n|\tilde{\nabla}(\Lambda \omega \chi)|^2 \leq (\Lambda \omega \chi) g^{ij} h^{i \overline{m}} \chi^{j \overline{r}} \partial_i \chi_{r} \partial_j \chi_{\overline{m}}. \]

**Proof** Using normal coordinates for \( \omega \) in which \( \chi \) is diagonal, and making use of the Cauchy-Schwartz inequality, we obtain
\[ n|\tilde{\nabla}(\Lambda \omega \chi)|^2 = \sum_{i,j,k} \chi^{k \overline{k}} \chi^{j \overline{j}} \partial_k \chi_{\overline{r}} \partial_k \chi_{\overline{r}} \]
\[ \leq \sum_i \left( \sum_k (\chi^{k \overline{k}})^2 |\partial_k \chi_{\overline{r}}|^2 \right)^{1/2} \left( \sum_k (\chi^{k \overline{k}})^2 |\partial_k \chi_{\overline{r}}|^2 \right)^{1/2} \]
\[ = \left( \sum_i \left( \sum_k (\chi^{k \overline{k}})^2 |\partial_k \chi_{\overline{r}}|^2 \right)^{1/2} \right)^2 \]
\[ = \left( \sum_i \sqrt{\chi_{\overline{r}}} \left( \sum_k (\chi^{k \overline{k}})^2 \chi^{j \overline{r}} |\partial_k \chi_{\overline{r}}|^2 \right)^{1/2} \right)^2 \]
\[ \leq \sum_i \chi_{\overline{r}} \sum_{i,k} (\chi^{k \overline{k}})^2 \chi^{j \overline{r}} |\partial_k \chi_{\overline{r}}|^2 \]
\[
(\Lambda \omega \chi) \sum_{i,k} (\chi^{k\overline{q}})^2 \chi^{\overline{p}} \partial_k \chi_{i\overline{r}} \partial_k \chi_{i\overline{r}} \\
= (\Lambda \omega \chi) \sum_{i,k} (\chi^{k\overline{q}})^2 \chi^{\overline{p}} \partial_k \chi_{i\overline{r}} \partial_k \chi_{i\overline{r}} \\
\leq (\Lambda \omega \chi) \sum_{i,j,k} (\chi^{k\overline{q}})^2 \chi^{\overline{p}} \partial_k \chi_{i\overline{r}} \partial_k \chi_{i\overline{r}} \\
= (\Lambda \omega \chi) g^{j\overline{q}} h^{k\overline{q}} \chi^{\overline{p}} \partial_k \chi_{i\overline{r}} \partial_k \chi_{i\overline{r}}.
\]

Let \( C_0 \) be a constant satisfying
\[
R_{k\overline{l}} \geq -C_0 g_{k\overline{l}} g^{j\overline{q}}.
\]

Then,
\[
(\tilde{\Delta} - \partial_t) \log(\Lambda \omega \chi) \geq \frac{1}{n \Lambda \omega \chi} (-C_0 h^{k\overline{q}} g_{k\overline{l}} g^{j\overline{q}} \chi_{i\overline{r}} - \chi^{k\overline{q}} R_{k\overline{l}}) \\
= \frac{1}{n} (-C_0 h^{k\overline{q}} g_{k\overline{l}} - \frac{1}{\Lambda \omega \chi} \chi^{k\overline{q}} R_{k\overline{l}}).
\]

Now calculate
\[
(\tilde{\Delta} - \partial_t) \phi = \frac{1}{n} (h^{k\overline{q}} \partial_k \partial_l \phi + \chi^{j\overline{q}} g^{j\overline{q}} - 1) \\
= \frac{1}{n} (\chi^{k\overline{q}} \chi^{j\overline{q}} g_{j\overline{q}} \chi_{k\overline{l}} - h^{k\overline{q}} \chi_{i\overline{r}} + \chi^{j\overline{q}} g^{j\overline{q}} - 1) \\
= \frac{1}{n} (2 \chi^{j\overline{q}} g^{j\overline{q}} - h^{k\overline{q}} \chi_{i\overline{r}} - 1).
\]

At this point we must choose our value of \( \Lambda \). From our assumption (3.1), we can choose \( 0 < \epsilon < 1/3 \) to be sufficiently small so that
\[
\chi_0 \geq (1 + 3\epsilon) \omega. \tag{3.3}
\]

Let \( A \) be given by
\[
A = \frac{C_0}{\epsilon}.
\]

Fix a time \( t > 0 \). There is a point \((x_0, t_0)\) in \( M \times [0, t] \) at which the maximum of \((\log(\Lambda \omega \chi) - A\phi)\) is achieved. We may assume that \( t_0 > 0 \). At this point, we have
\[
0 \geq (\tilde{\Delta} - \partial_t)(\log(\Lambda \omega \chi) - A\phi)
\]
\[
\begin{align*}
&\geq \frac{1}{n} (-C_0 h^{\gamma_l} g_{k\ell} - \frac{1}{\Lambda_{\omega}(\chi)} \chi^{k\ell} R_{k\ell} - 2 \lambda_1^{\gamma_l} g_{i\ell} + A h^{\gamma_l} \chi_{0k\ell} + A) \\
&\geq \frac{1}{n} (-C_0 h^{\gamma_l} g_{k\ell} - \frac{1}{\Lambda_{\omega}(\chi)} \chi^{k\ell} R_{k\ell} - 2 \lambda_1^{\gamma_l} g_{i\ell} + (1 - \epsilon) A h^{\gamma_l} \chi_{0k\ell} \\
&\quad + \epsilon A h^{\gamma_l} g_{k\ell} + A) \\
&= \frac{1}{n} (-\frac{1}{\Lambda_{\omega}(\chi)} \chi^{k\ell} R_{k\ell} - 2 \lambda_1^{\gamma_l} g_{i\ell} + (1 - \epsilon) A h^{\gamma_l} \chi_{0k\ell} + A).
\end{align*}
\]

From the lower bound (2.4) on \(\chi_{k\ell}\), the term \(\chi^{k\ell} R_{k\ell}\) is bounded above and hence at \((x_0, t_0)\), we have
\[
1 + (1 - \epsilon) h^{\gamma_l} \chi_{0k\ell} - 2 \lambda_1^{\gamma_l} g_{i\ell} \leq \frac{C_1}{\Lambda_{\omega}(\chi)}.
\]

From (3.3), we get
\[
1 + (1 + \epsilon) h^{\gamma_l} g_{k\ell} - 2 \lambda_1^{\gamma_l} g_{i\ell} \leq \frac{C_1}{\Lambda_{\omega}(\chi)}.
\]

We will compute in normal coordinates at \(x_0\) for \(\omega\) in which \(\chi\) is diagonal and has eigenvalues \(\lambda_1, \lambda_2\). From (2.4), \(\lambda_1\) and \(\lambda_2\) are bounded below by a positive constant. We want to show that they are also bounded above. First, observe that for \(n = 2\),
\[
1 + (1 - \epsilon) \frac{1}{\Lambda_{\omega}(\chi)} = \frac{\det \chi}{\det(\omega)(\Lambda_{\omega}(\chi))},
\]
and by (2.3), this is bounded along the flow.

Multiplying (3.3) by \((\det \chi) / \det(\omega)\) gives,
\[
\lambda_1 \lambda_2 + (1 + \epsilon) \left( \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2} \right) - 2(\lambda_1 + \lambda_2) \leq C_2.
\]

From (2.3), we may suppose that one of the eigenvalues, say \(\lambda_2\), is bounded from above. Rewrite the inequality as
\[
\lambda_1 (\lambda_2 + (1 + \epsilon) \frac{1}{\lambda_2} - 2) + (1 + \epsilon) \frac{\lambda_2}{\lambda_1} - 2 \lambda_2 \leq C_2.
\]

Then, since the function \(f : (0, \infty) \to \mathbb{R}\) defined by
\[
f(x) = x + (1 + \epsilon) \frac{1}{x} - 2,
\]

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is bounded below by a small positive constant depending on $\epsilon$, we see that $\lambda_1$ must also be bounded above. Hence at the point $(x_0, t_0)$, there exists $C$ depending only on the initial data such that

$$\Lambda_\omega \chi \leq C.$$

Then, on $M \times [0, t]$,

$$\log(\Lambda_\omega \chi) - A\phi \leq \log C - A \inf_{M \times [0, t]} \phi.$$

Exponentiating gives

$$\Lambda_\omega \chi \leq C e^{A(\phi - \inf_{M \times [0, t]} \phi)},$$

completing the proof of the theorem.

4. Zero order estimate

We prove an estimate on the $C^0$ norm of $\phi$ using a Moser iteration method applied to the exponential of the solution rather than a power of the solution (compare to [Y1]) and the estimate of Theorem 3.1.

**Theorem 4.1** Suppose that $(M, \omega)$ has dimension $n = 2$ and that

$$\chi_0 - \omega > 0.$$

Let $\phi_t$ be a solution of the $J$-flow (2.7) on $[0, \infty)$. Then there exists a constant $\tilde{C}$ depending only on the initial data such that

$$\|\phi_t\|_{C^0(M)} \leq \tilde{C}.$$

**Proof** Suppose first that $\inf_M \phi_t$ is bounded from below uniformly in time. We will show that this implies the above estimate. Since the functional $J_{\omega, \chi_0}$ decreases along the flow, there exists a constant $C_0$ such that

$$\int_M \phi_t \omega \wedge (\chi_0 + \chi_{\phi_t}) \leq C_0,$$

using (2.5). Let $C_1$ be a positive constant satisfying

$$\omega^2 \leq C_1 \omega \wedge \chi_0.$$
Then

\[
\int_M \phi_t \omega^2 = \int_M (\phi_t - \inf_M \phi_t) \omega^2 + \int_M \inf_M \phi_t \omega^2
\]

\[
\leq C_1 \int_M (\phi_t - \inf_M \phi_t) \omega \wedge \chi + \inf_M \phi_t \left( \int_M \omega^2 - C_1 \int_M \omega \wedge \chi \right)
\]

\[
= C_1 C_0 - C_1 \int_M (\phi_t - \inf_M \phi_t) \omega \wedge \chi
\]

\[
+ \inf_M \phi_t \left( \int_M \omega^2 - 2C_1 \int_M \omega \wedge \chi \right)
\]

\[
\leq C_1 C_0 + \inf_M \phi_t \left( \int_M \omega^2 - 2C_1 \int_M \omega \wedge \chi \right).
\]

This gives an upper bound for \( \int_M \phi_t \omega^2 \) depending on the lower bound for \( \inf_M \phi_t \). Since \( \Delta \omega \phi_t > -\Lambda \omega \chi_0 \) along the flow, it follows from the existence of a lower bound on the Green’s function of \( \omega \) that \( \sup_M \phi_t \) is bounded from above, giving us the required estimate.

Now suppose that no such lower bound for \( \inf_M \phi_t \) exists. Then we can assume that there is a sequence of times \( t_i \to \infty \) such that

(i) \( \inf_M \phi_{t_i} = \inf_{t \in [0, t_i]} \inf_M \phi_t \)

(ii) \( \inf_M \phi_{t_i} \to -\infty \).

We will seek a contradiction. For a fixed \( i \), write

\[
\psi_{t_i} = \phi_{t_i} - \sup_M \phi_{t_i}.
\]

Notice that \( \sup_M \phi_{t_i} \) is bounded from below by zero from (2.6) and the fact that \( I(\phi_t) = 0 \). Hence

\[
\|\psi_{t_i}\|_{C^0} \to \infty.
\]

The following proposition is the key result of this section.

**Proposition 4.2** Let \( M \) be a compact complex surface with two Kähler metrics \( \chi_0 \) and \( \omega \). Suppose that \( \psi \in C^\infty(M) \) satisfies the conditions

\[
\chi_\psi = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi > 0, \quad \sup_M \psi = 0,
\]

and

\[
\Lambda \omega \chi_\psi \leq C e^{A(\psi - \inf_M \psi)}.
\]
Then there exists a constant $C'$ depending only on $M$, $\omega$, $\chi_0$ and the constants $A$ and $C$ such that

$$\|\psi\|_{C^0} \leq C'.$$

We apply this proposition to $\psi = \psi_{t_i}$ and obtain a contradiction since

$$\Lambda_\omega \chi_{\psi_{t_i}} = \Lambda_\omega \chi_{\phi_{t_i}} \\
\leq Ce^{A(\phi_{t_i} - \inf_{t \in [0,t_i]} \inf_M \phi_t)} \\
= Ce^{A(\psi_{t_i} - \inf_M \psi_{t_i})},$$

where we have used Theorem 3.1 and condition (i) above. It remains to prove the proposition.

**Proof of Proposition 4.2** Let $\delta$ be a small positive constant, to be determined later. Set $B = A/(1 - \delta)$ and let $u = e^{-B\psi}$.

Now, for $\beta = n/(n - 1) = 2$, the Sobolev inequality for functions $f$ on $(M, \omega)$ is

$$\|f\|_{L^\beta}^2 \leq C_2(\|\nabla f\|_2^2 + \|f\|_2^2),$$

for $C_2$ depending on $\omega$. We will apply this to $u^{p/2}$ for $p \geq 1$. This gives

$$\left(\int_M e^{-Bp\beta\psi} \omega^2 \right)^{1/\beta} \leq C_2 \left(\int_M \|\nabla e^{-Bp\psi/2}\|_2^2 \omega^2 + \int_M e^{-Bp\psi} \omega^2 \right). \quad (4.1)$$

Now calculate

$$\int_M \|\nabla e^{-Bp\psi/2}\|_2^2 \omega^2 = \int_M \nabla e^{-Bp\psi/2} \wedge \bar{\nabla} e^{-Bp\psi/2} \wedge \omega = B^2 p^2 e^{-Bp\psi} \nabla \psi \wedge \bar{\nabla} \psi \wedge \omega = -B^2 p \int_M \nabla (e^{-Bp\psi}) \wedge \bar{\nabla} \psi \wedge \omega = B^2 p \int_M e^{-Bp\psi} \sqrt{1 - \beta} \psi \wedge \omega = B^2 p \int_M e^{-Bp\psi} (\chi_\psi - \chi_0) \wedge \omega \leq CBp \int_M e^{-Bp\psi} \omega^2 \leq CBp \int_M e^{-Bp\psi} e^{A(\inf_M \psi)} \omega^2 \leq CBp e^{-A\inf_M \psi} \int_M e^{-(p-(1-\delta))Bp\psi} \omega^2,$$
where we have used the estimate
\[ \Lambda \omega \chi \psi \leq C e^{A(\psi - \inf_M \psi)}. \]

Then in (4.1),
\[ \left( \int_M u^{p\beta} \frac{\omega^2}{2} \right)^{1/\beta} \leq C_3 p e^{-A \inf_M \psi} \int_M u^{p-(1-\delta)} \frac{\omega^2}{2}. \]

Raising to the power \( 1/p \) and writing \( \gamma = 1 - \delta \) gives
\[ \|u\|_{p\beta} \leq C_3^{1/p} p^{1/p} e^{-(A/p) \inf_M \psi \|u\|_{p-\gamma}/p}. \]

Take the logarithm of both sides to get
\[ \log \|u\|_{p\beta} \leq \frac{1}{p} \log C_3 + \frac{1}{p} \log p + \frac{1}{p} \sup_M (-A\psi) + \frac{(p-\gamma)}{p} \log \|u\|_{p-\gamma}. \]

We now apply the iteration. First, replace \( p \) with \( p\beta + \gamma \) to get
\[ \log \|u\|_{p\beta + \gamma} \leq \frac{1 + \beta}{p\beta + \gamma} \log C_3 + \frac{1 + \beta}{p\beta + \gamma} (\beta \log p + \log(p\beta + \gamma)) \]
\[ + \frac{1 + \beta}{p\beta + \gamma} \sup_M (-A\psi) + \frac{\beta(p-\gamma)}{p\beta + \gamma} \log \|u\|_{p-\gamma}. \]

Repeat this procedure, replacing \( p \) with \( p\beta + \gamma \) to obtain for any positive integer \( k \),
\[ \log \|u\|_{p\beta^k + 1 + \gamma(\beta + \beta^2 + \ldots + \beta^k)} \leq \frac{1 + \beta + \beta^2 + \ldots + \beta^k}{p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^k-1)} \log C_3 \]
\[ + \frac{1 + \beta + \beta^2 + \ldots + \beta^k}{p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^k-1)} (\beta^k \log p + \beta^{k-1} \log(p\beta + \gamma) + \ldots \]
\[ + \log(p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^k-1)) \]
\[ + \frac{1 + \beta + \beta^2 + \ldots + \beta^k}{p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^k-1)} \sup_M (-A\psi) \]
\[ + \frac{\beta^k(p-\gamma)}{p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^k-1)} \log \|u\|_{p-\gamma}. \]

(4.2)

Now set \( p = 1 + \delta \). Then, since \( \beta = 2 \) we have
\[ p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^k-1) = 1 + \beta + \beta^2 + \ldots + \beta^k + \delta. \]
Notice that the second term on the right hand side of (4.2) is bounded by
\[
\log p + \frac{1}{\beta} \log \beta^2 + \ldots + \frac{1}{\beta^k} \log (\beta^{k+1}) \leq \log p + \log \beta \left( \sum_{i=1}^{k} \frac{i+1}{\beta^i} \right) 
\leq C_4.
\]
Then
\[
\log \|u\|_{p\beta^{k+1+\gamma(\beta+\beta^2+\ldots+\beta^k)}} \leq \log C_3 + C_4 + \sup_{M} (-A\psi) + 2\delta \max \left( \log \|u\|_{2\delta}, 0 \right).
\]
Using the fact that \( A = (1 - \delta)B \) and \(-B\psi = \log u\), and letting \( k \) tend to infinity,
\[
\log \|u\|_{C_0} \leq C_5 + 2 \max \left( \log \|u\|_{2\delta}, 0 \right).
\]
Hence we get the following inequality for \( \psi \),
\[
\|\psi\|_{C_0} \leq C_6 + C_7 \max \left( \log \left( \int_{M} e^{-\alpha \Phi \frac{n}{\sqrt{2}}} \frac{\omega^2}{2} \right)^{1/2\delta} \right) \quad (4.3)
\]
We can now finish the estimate. First, define
\[
P(M, \chi_0) = \{ \Phi \in C^2(M) \mid \chi_0 + \frac{\sqrt{-1}}{2} \partial \overline{\partial} \Phi \geq 0, \sup_{M} \Phi = 0 \}.
\]
Then Proposition 2.1 of [11] (see section 4.4, [Ho]) states that there exist constants \( \alpha > 0 \) and \( C_8 \) depending only on \((M, \chi_0)\) such that
\[
\int_{M} e^{-\alpha \Phi \frac{n}{\sqrt{2}}} \frac{\omega^2}{2} \leq C_8 \quad \text{for all } \Phi \in P(M, \chi_0).
\]
Define \( \delta \) to be
\[
\delta = \min \left\{ \frac{\alpha}{4A}, \frac{1}{2} \right\} > 0.
\]
Then the required estimate follows from (4.3), since \( \psi \) belongs to \( P(M, \chi_0) \).
5. Convergence of the flow

In this section we complete the proof of the main theorem. We assume, using the result of [C2], that a solution \( \phi = \phi_t \) for the J-flow exists for all time. From Theorem 3.1 and Theorem 4.1 we have uniform estimates on \( \phi \) and the derivatives \( \partial_i \partial_j \phi \), using the fact that

\[
\chi_{ij} = \chi_{0ij} + \partial_i \partial_j \phi > 0.
\]

Since the operator

\[
\frac{1}{n} (1 - \Lambda \chi \omega),
\]

is concave in the \( \chi_{ij} \), it is well known that, by the work of Evans [E1, E2] and Krylov [Kr] (see also [Tr]), one can deduce a uniform Hölder estimate on the second derivatives \( \partial_i \partial_j \phi \). By differentiating the equation (2.1) and applying standard Schauder estimates for parabolic equations (see [LSU] for example), one can obtain uniform estimates on all of the derivatives of \( \phi \). It then follows that there is a sequence of times \( t_j \to \infty \) such that \( \phi_t \) converges in \( C^\infty \) to some smooth function \( \phi_\infty \). In order to show that we have convergence without having to pass to a subsequence, we will use a modification of the argument in [Ca].

Notice that \( \partial \phi / \partial t \) satisfies the heat equation

\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial t} \right) = \tilde{\Delta} \left( \frac{\partial \phi}{\partial t} \right).
\]

Since we have uniform bounds for \( \chi_{ij} \) from above and away from zero, and bounds on \( \frac{\partial}{\partial t} \chi_{ij} \) and all the covariant derivatives of \( \chi_{ij} \) and \( \frac{\partial}{\partial t} \chi_{ij} \), it follows from the Harnack inequality of Li and Yau [LY] and the argument in [Ca] that there exist positive constants \( C_0 \) and \( \eta \), which are independent of \( t \), such that

\[
\sup_M \left( \frac{\partial \phi}{\partial t} \right) - \inf_M \left( \frac{\partial \phi}{\partial t} \right) \leq C_0 e^{-\eta t}.
\]

Since

\[
\int_M \frac{\partial \phi}{\partial t} \chi^2 = 0,
\]

\( \partial \phi / \partial t \) must take on the value zero somewhere on \( M \) for each \( t \), and so

\[
\left| \frac{\partial \phi}{\partial t} \right| \leq C_0 e^{-\eta t}.
\]
Hence for any $0 < s < s'$, and any $x \in M$,
\[
|\phi(x, s') - \phi(x, s)| = \left| \int_s^{s'} \frac{\partial \phi}{\partial t}(x, t) dt \right|
\leq \int_s^{s'} \left| \frac{\partial \phi}{\partial t}(x, t) \right| dt
\leq C_0 \int_s^{s'} e^{-\eta t} dt
= C_0 \frac{1}{\eta} (e^{-\eta s} - e^{-\eta s'}),
\]
which tends to zero as $s$ and $s'$ tend to infinity. Hence $\phi_t$ converges in the $C_0$ norm to $\phi_\infty$. It must converge also in the $C^\infty$ topology, since otherwise there would exist an integer $N$, an $\epsilon > 0$ and a sequence $t_j \to \infty$ with
\[
\|\phi_{t_j} - \phi_\infty\|_{C^N} \geq \epsilon.
\]
Since $\phi$ is bounded in all the $C^k$ norms, one could pass to a subsequence of the $\phi_{t_j}$ which would converge to some $\phi'_\infty \neq \phi_\infty$, giving the contradiction. This completes the proof.

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