High-speed quantum transducer with a single-photon emitter in an atomically thin resonator

Xingyu Gao,∗1,2 Zhang-qi Yin,†3 and Tongcang Li1,4,5,6,†

1Department of Physics and Astronomy, Purdue University, West Lafayette, IN 47907, USA
2School of the Gifted Young, University of Science and Technology of China, Hefei, Anhui, 230026, China
3Center for Quantum Information, Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing 100084, China
4School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907, USA
5Purdue Quantum Center, Purdue University, West Lafayette, IN 47907, USA
6Birck Nanotechnology Center, Purdue University, West Lafayette, IN 47907, USA
(Dated: December 27, 2017)

Quantum transducers can transfer quantum information between different systems. Microwave-optical photon conversion is important for future quantum networks to interconnect remote superconducting quantum computers with optical fibers. Here we propose a high-speed quantum transducer based on a single-photon emitter in an atomically thin membrane resonator that can couple single microwave photons to single optical photons. The 2D resonator will be a freestanding van der Waals heterostructure (may consist of graphene, hexagonal boron nitride, or other 2D materials) that hosts a quantum emitter. The mechanical vibration (phonon) of the 2D resonator interacts with optical photons by shifting the optical transition frequency of the single-photon emitter with strain or the Stark effect. The mechanical vibration couples to microwave photons by shifting the resonant frequency of a LC circuit that includes the membrane. Thanks to the small mass of the 2D resonator, both the single-photon optomechanical coupling strength and the electromechanical coupling strength can be about 100 MHz. This provides a way for high-speed quantum state transfer between a microwave photon, a phonon, and an optical photon.

I. INTRODUCTION

In the past few years, optomechanical and electromechanical systems have gained remarkable attentions for achieving coherent quantum control [1]. These hybrid devices are leading candidates for transferring quantum information between different forms, such as photonic, phononic, electronic, and spin states [2][12]. In particular, the opto-electro-mechanical coupling of single microwave (or radio-frequency) photons to single optical photons is attractive for future quantum technologies [4][6]. One potential application will be to use optical photons to coherently interconnect remote superconducting quantum computers that use microwave photons. Converting classical microwaves to optical lights has been achieved with metal-coated Si3N4 (or SiN) membrane resonators (thickness ∼ 100 nm) [5][6] embedded in LC circuits, and nanobeam piezo-optomechanical crystal cavities (thickness ∼ 200 nm) [4][7].

Converting a single microwave photon to a single optical photon has been a challenging task. It is difficult to reach the strong coupling regime with single photons, which requires the single-photon optomechanical and electromechanical coupling strengths to be larger than the optical decay rate, the mechanical decay rate, and the microwave decay rate. In 2016, a state-of-the-art experiment reported a remarkable single-photon optomechanical coupling strength of 1.1 MHz in a nanobeam optomechanical crystal cavity [7]. While it is much larger than the 30 kHz optomechanical coupling strength in a previous experiment [4], the 1.1 MHz optomechanical coupling strength is still much smaller than the optical decay rate (5.2 GHz) of the nanobeam optomechanical crystal cavity in that experiment [7].

In this paper, we propose a quantum transducer that couples a single microwave photon to a single optical photon with a quantum emitter in a suspended 2D membrane (Fig.1). The 2D membrane is a freestanding van der Waals heterostructure. It may consist of graphene, hexagonal boron nitride (h-BN), transition metal dichalcogenide (TMDC), or other 2D materials. The graphene will be a part of a capacitor in a LC circuit that couples the mechanical vibration (phonon) of the 2D membrane to microwave photons in the circuit [13][14]. The h-BN/TMDC layers host a single-photon emitter [15][20] that couples its mechanical vibration to single optical photons with strain or the Stark effect [21][23]. The single-photon emitter replaces the role of the optical cavity in the former opto-electro-mechanical systems [4][9]. Here the mechanical vibration of the 2D resonator couples to the electron orbital state of the single-photon emitter, instead of its electron spin state [24][25]. Thus it does not require a magnetic field gradient. We propose to apply a constant voltage to the graphene electrode to increase the strain of the 2D membrane and the charge in the LC circuit to achieve the single-photon strong coupling regime. The mechanical vibration fre-
FIG. 1. (Color online) Schematic of a high-speed quantum transducer with a single-photon emitter in a 2D resonator. (a) A doubly capped 2D membrane hosting a quantum emitter is suspended to from a mechanical resonator. The device couples to an optical system and an electric microwave resonator by the mechanical vibration of the 2D membrane. |e⟩ and |g⟩ are electron orbital states of the single-photon emitter. The decay from |e⟩ to |g⟩ can emit a visible or infrared photon with frequency \( \omega(x) \). (b) An ultrathin 2D membrane contains 3 layers. A top graphene monolayer is connected to the electric circuit. An intermediate h-BN insulating monolayer separates the graphene from the bottom h-BN or TMDC (e.g., WSe_2) monolayer which contains a single-photon emitter. (c) A 2D membrane contains 2 layers. The bottom graphene monolayer is connected to the electric circuit. The top h-BN or TMDC monolayer contains a single photon emitter. The graphene has a hole to avoid direct contact of the single-photon emitter with graphene. (d) The 2D membrane capacitor \( C_m(x) \) is a part of a LC resonator that contains additional capacitors \( C_0, C_1 \) and an inductor \( L_1 \). A constant bias voltage \( V_{dc} \) is applied to the membrane to tune the resonant frequency and the coupling strength. A very large inductor \( L_2 \gg L_1 \) and a large capacitor \( C_1 \gg C_0 + C_m(x) \) are used to isolate the constant bias voltage \( V_{dc} \) and the GHz microwave signal in the LC resonator.

The proposed quantum transducer with a single-photon emitter in a 2D resonator (Fig. 1) has several advantages. At low temperatures, the intrinsic linewidth of the zero-phonon line (ZPL) of a single-photon emitter can be much smaller than the linewidth of a nanoscale optical cavity (typically a few GHz) [7]. Linewidths down to 50 MHz has been experimentally observed for quantum emitters in h-BN at 5K [31]. Thanks to the small mass of a 2D resonator, both single-photon optomechanical coupling strength and single-photon electromechanical coupling strength of this system can be about 100 MHz under suitable conditions, which can be larger than the optical decay rate (\( \sim 50 \) MHz), mechanical decay rate (\( \sim 100 \) kHz), and the microwave decay rate (\( \sim 100 \) kHz) of this system [13, 31–33]. Thus the system can reach the strong coupling regime with single photons. Ideally, the quantum transducer can be put together with a superconducting quantum processor in a dilution refrigerator below 50 mK to operate [29, 33–35]. Because of the high vibration frequency (> 1 GHz), the relevant mechanical mode of a nanoscale 2D resonator will be automatically prepared in the quantum ground state in a dilution refrigerator. Thus no optical cooling is required to reach the quantum regime.

Our proposal is based on recent progresses in atomically thin mechanical resonators (e.g., a suspended graphene membrane) [33], and single-photon emitters in TMDC monolayers [13–15] and h-BN [19, 20]. Suspended 2D membranes have outstanding mechanical properties such as ultrahigh Young’s modulus and the ability to sustain remarkable strain without breaking [36–37]. These mechanical systems can be precisely engineered with high quality factors [34, 35, 38, 39]. A graphene mechanical resonator with a quality factor of \( 10^5 \) has been capacitively coupled to a superconducting microwave cavity [39]. Their extremely small mass and out-of-plane stiffness give them a large zero-point vibration amplitude for strong coupling. h-BN has a bandgap of about 6 eV, and can host single photon emitters deep inside its bandgap, which is stable even at room temperature [19, 20]. These quantum emitters exhibit narrow ZPL distributed over a large range from 550 nm to 800 nm [31]. Monolayer tungsten diselenide (WSe_2) has a direct bandgap of about 1.9 eV, and can host localized single photon emitters at low temperature [15, 18]. Optical properties and energy-levels of single-photon emitters in 2D materials can be modulated by applying strains or external fields [20, 25, 40]. For example, bright and photostable single-photon emitters embedded in hexagonal boron nitride shows great spectra tunability under strain control [20]. Localized emitters in a monolayer WSe_2 are demonstrated to have efficient spectra tunability by the Stark effect [40]. It is remarkable that a 5-layer van der Waals heterostructure consisting graphene, h-BN, WSe_2, h-BN, and graphene has been assembled to measure the Stark effect [40]. These emitters are stable during material transfer. These properties make it possible to realize strong optomechanical and electromechanical couplings with single-photon emitters in a suspended 2D resonator. Our proposal only requires a 3-layer or 2-layer freestanding van der Waals heterostructure (Fig. 1). Similar heterostructure 2D resonators have been demonstrated at room temperature already [41]. Their quality factor will be much higher at low temperatures.

In the following parts of this paper, we will first describe the model in section [41] and discuss the mechan-
ical vibration feature in section III. The electromechanical coupling and optomechanical coupling based on the strain and the Stark effect will be calculated in section IV and V. We show that this hybrid system can reach strong coupling regime under proper experimental conditions. In section VI, we simulate a scheme of high speed quantum state transfer from a microwave photon to an optical photon, which demonstrates that a high fidelity can be achieved through optical readout. In the last section, we briefly summarize the results of the paper.

II. MODEL

As show in Fig.1, we consider a quantum transducer with a freestanding van der Waals heterostructure consisting of graphene and hexagonal boron nitride (and/or a TMDC monolayer) that hosts a quantum emitter. The electron orbital state of a single photon emitter embedded in this membrane couples to the mechanical bending mode $\hat{b}$. Therefore, the spontaneous emission spectrum (e.g., around 600 nm) of the single photon emitter is dependent on the mechanical mode $\hat{b}$. The mechanical oscillator also forms a capacitor with the bottom electrode, which couples the mechanical mode $\hat{b}$ to the electric microwave mode $\hat{\epsilon}$.

An example of the 2D membrane is shown in Fig.1(b). In this example, a top graphene monolayer is connected to the electric circuit and forms a capacitor with the bottom electrode (Fig.1(a)). An intermediate h-BN insulating monolayer separates the graphene from the bottom h-BN or TMDC (e.g., WSe$_2$, MoS$_2$) monolayer which contains a single-photon emitter. At low temperature ($T < 50$ mK), the quality factor of the mechanical resonator will be very high. The electronic excited state $|\epsilon\rangle$ of the single-photon emitter in this 2D membrane will change the electric field. Thus the electronic orbital state $|\epsilon\rangle$ of the quantum emitter can be coupled to the mechanical bending mode $\hat{b}$ by the lattice strain or the Stark effect. Finally, the photon emission from the spontaneous decay of the electron orbital state $|\epsilon\rangle$ couples to the mechanical mode $\hat{b}$.

Another possible membrane containing 2 atomic layers is shown in Fig.1(c). The top layer hosts a single photon emitter to couple the optical photon and mechanical phonon through strain effect. The bottom layer will be used to apply electric force onto the membrane to tune its strain and resonant frequency, and couple its mechanical motion to a microwave photon in a LC circuit. A hole is fabricated in the bottom graphene monolayer to minimize the absorption of photons by graphene.

A driving laser at the red sideband of the emitter’s optical transition $|g\rangle \leftrightarrow |\epsilon\rangle$ will be applied to induce resonant coupling between the electron orbital state and the mechanical mode $\hat{b}$, and read out the quantum state optically. The frequency of the driving laser will be $\omega_L = \omega_0 - \omega_m$, where $\omega_0$ is the optical resonant frequency of the single photon emitter when the 2D resonator is at its equilibrium position, and $\omega_m$ is the vibration frequency of the 2D membrane. As will be discussed further in Sec. VI, the optical driving field is typically assumed to only a few times of the single photon field, which is extremely weak.

The suspended membrane also forms a capacitor which couples the mechanical vibration to microwave photons at the frequency $\omega_{LC}$. As shown in Fig.1(d), the 2D membrane capacitor $C_m(x)$ is a part of a LC resonator that contains additional capacitors $C_0$, $C_1$ and an inductor $L_1$. Since $C_m(x)$ depends on the position of the 2D membrane, the mechanical mode $\hat{b}$ couples to the microwave mode $\hat{\epsilon}$. A constant bias voltage $V_{dc}$ is applied to the membrane to tune the coupling strength. A very large inductor $L_2 \gg L_1$ and a large capacitor $C_1 \gg (C_0 + C_m(x))$ are used to isolate the constant bias voltage $V_{dc}$ and the high-frequency microwave signal in the LC resonator. The large inductor $L_2$ prevents the GHz microwave photon from leaking to the $V_{dc}$ connector. The capacitor $C_1$ prevents a DC current passing the $L_1$ inductor. The resonant frequency of the LC circuit is $\omega_{LC} \approx 1 / \sqrt{L_1(C_m(x) + C_0)}$. We assume that the coupling rates are much less than the mode spacing of these resonators so that only one mode is relevant for each degree of freedom.

The Hamiltonian of the system takes the form

$$H = \hbar \omega(x) |\epsilon\rangle \langle \epsilon| + \frac{p^2}{2m} + \frac{m \omega_m^2 x^2}{2} + \frac{\phi^2}{2L} + \frac{q^2}{2C(x)} - qV_{dc}$$

where $\phi$ and $q$ correspond to the flux in the inductor and the charge on the capacitors, respectively. For the excited state $|\epsilon\rangle$ of the single photon emitter, the electron-phonon interaction leads to the energy shift of the ZPL, which results in the dependence of the frequency $\omega(x)$ of the ZPL on the displacement of the membrane $x$. Here we introduce the phonon creation(annihilation) operator $b\dagger(b)$ and microwave photon creation(annihilation) operator $\epsilon\dagger(\epsilon)$, with

$$x = x_{zp}(b + b\dagger), \quad p = im \epsilon \omega_m x_{zp}(b - b\dagger), \quad q = q_{zp}\epsilon(c + c\dagger), \quad \phi = iL \omega_{LC} q_{zp}\epsilon(c - c\dagger)$$

where $x_{zp} = \sqrt{\hbar/(2m \epsilon \omega_m)}$, $q_{zp} = \sqrt{\hbar/(2L \omega_{LC})}$ are zero-point fluctuations of the mechanical mode and the microwave mode, respectively. Then the Hamiltonian
can be given by $H = H_0 + H_I$, where (see details in Appendix A)

$$H_0 = h\omega_0 |e\rangle \langle e| + h\omega_m b^\dagger b + h\omega_{LC} c^\dagger c,$$  

(2)

$$H_I = hg_{om} (b^\dagger b) |e\rangle \langle e| + g_{em} (b^\dagger b) (c + c^\dagger).$$  

(3)

Here $\omega_0$ is the frequency of the optical transition at equilibrium position, $g_{om}$ and $g_{em}$ are optomechanical and electromechanical coupling rates, respectively. The optical, mechanical and electrical damping rates are $\kappa$ ($\sim 2\pi\cdot50$ MHz), $\Gamma_m$ ($\sim 2\pi\cdot100$ kHz) and $\Gamma_{LC}$ ($\sim 2\pi\cdot100$ kHz), respectively [33, 32, 33, 39]. To enter the strong coupling regime, $g_{om}$ and $g_{em}$ need to exceed these three damping rates [33]. For some applications, only large cooperativities [44, 45], $g_{om} \equiv g_{om}^2/(\kappa \Gamma_m) > 1$ and $c_{em} \equiv g_{em}/(\Gamma_m \Gamma_{LC}) > 1$, are required.

III. MECHANICAL VIBRATION

The mechanical resonator (Fig. 1) that we discuss here is based on doubly clamped ultrathin 2D materials like graphene, h-BN or WSe$_2$. These materials have ultrahigh Young’s modulus ($\sim 1000$ GPa) and can sustain strains up to 25% without breaking. Experiments have examined the relation between the frequency of the fundamental mechanical bending mode and the dimensions of the 2D membrane [33, 46]. For mechanical resonators under tension $T$, the fundamental flexural mode frequency $\omega_m$ is given by [33, 46]:

$$\frac{\omega_m}{2\pi} = \left( \frac{A^2 Y h^2}{\rho l^4} + 0.57 A^2 T}{\rho^2 w h} \right)^{1/2},$$  

(4)

where $w$, $l$ and $h$ are the width, length and thickness of the structure, respectively. $A$ is the clamping coefficient, which is 1.03 for doubly clamped membranes [36]. $Y$ is the Young’s modulus and $T$ is the pre-tension of the suspended 2D materials. The typical initial tension of a doubly clamped suspended graphene has been measured in Ref. [36] to be about 0.01 N/m. The frequency of the fundamental flexural mode as a function of the thickness of the 2D membrane is shown in Fig. 2(a). The length and width of the membrane are assumed to be $l = 110$ nm and $w = 1$ $\mu$m, respectively. The thickness of a real 2D membrane is discrete, corresponds to certain points in this plot. A transition from flexible membrane to stiff plate mechanical behaviors can be observed in this figure. For very thin layers, the frequency in Eq. (4) is dominated by the second term which is determined by the pre-tension $T$. For a thick flake, the tensile strain induced by vibration is more important than the pre-tension. So the first term in Eq. (4) will dominate.

In the following discussions, the typical dimensions of the mechanical membrane are assumed to be $(l, w, h) = (110$ nm, 1 $\mu$m, 1.1 nm). The frequency of the fundamental flexural mode when there is no bias voltage is calculated to be about 2.07 GHz, and the zero-point fluctuation amplitude is about 0.14 pm. The initial distance between the membrane and the capacitor chip is important in order to achieve high coupling rate at small DC bias voltages. Here we assume the initial distance to be $d = 10$ nm. A bias voltage $V_{dc}$ can be tuned to achieve optimum coupling rates.

The elastic properties of freestanding 2D materials have been studied in several experiments. The relationship between the force and the deformation $\delta$ at the center of the doubly clamped structure is [33, 49]

$$F = \left[ \frac{30.78wh^3}{l^3} Y + \frac{12.32}{l} T \right] \delta + \frac{8whY}{3L^3} \delta^3.$$

(5)

Because the membrane is very thin, a bias voltage will induce an electric field that can cause a dramatic deflection of the membrane. For a doubly clamped suspended membrane with a parallel bottom electrode that form a capacitor, the electrostatic force as a function of the bias voltage is given by [33, 50]

$$F = \frac{\varepsilon_0 wlv^2_{dc}}{2(d - x)^2}.$$

(6)

The equilibrium position can be derived from Eq. (5) and Eq. (6). The deformation will increase the tension $T$ and change the mechanical resonance frequency as showed in Fig. 2(b). In this system, we can tune the mechanical resonance frequency by adjusting the bias voltage to match the frequency of a superconducting qubit. Typically, the frequency of a superconducting circuit is around 5 GHz [29, 90]. A 3.3 V bias voltage will cause a $x = 2.4$ nm displacement and can shift the mechanical vibration frequency from 2.07 GHz to 5 GHz to match the frequency of a superconducting qubit. 

FIG. 2. (Color online) (a) Thickness dependence of the resonance frequency of a 2D membrane without a bias voltage (solid blue line). Young’s modulus $Y = 1000$ GPa, pre-tension $T = 10$ nN [33, 46]. The length and width of the membrane are assumed to be $l = 110$ nm and $w = 1$ $\mu$m, respectively. The thickness of a real 2D membrane is discrete, corresponds to certain points in this plot. A transition from flexible membrane (black dotted line) to stiff plate (red dashed line) mechanical behaviors can be observed in this figure. (b) Voltage dependence of the resonance frequency of a 2D membrane. The thickness is taken as 1.1 nm, corresponding to 3 layers. A bias voltage leads to a displacement of the membrane which increases the tension $T$. Other parameters are the same as those in (a).
IV. ELECTROMECHANICAL COUPLING

As shown in Fig. 1, the 2D membrane and the bottom electrode forms a capacitor $C_m(x)$. Its capacitance depends on the separation $(d-x)$ between the 2D membrane and the bottom electrode. The vibration of the 2D membrane changes the value of $C_m(x)$, and thus couples to the microwave photon in the LC circuit. With another paralleled tuning capacitor $C_0$, the total capacitance is $C(x) \equiv C_0 + C_m(x)$. Here we assume $C_1 \gg C_0 + C_m(x)$. So the effect of $C_1$ on the high frequency microwave signal can be neglected.

As discussed in the former section, a bias voltage $V_{dc}$ will be applied to the capacitor to tune the mechanical vibration frequency $\omega_m$ of the 2D membrane to match the LC circuit’s resonance frequency $\omega_{LC}$. The bias voltage will charge the capacitors $\bar{q} = V_{dc}C(x)$. In this case, the electromechanical coupling between the microwave photon and the mechanical phonon can be described by the Hamiltonian

$$H_{em} = \hbar g_{em}(b + b^\dagger)(c + c^\dagger)$$

where $g_{em} = G \bar{q} \omega_{zpf} \mu_{dc}$ and $G = \frac{1}{2} \left( \frac{1}{C}\right)_{x=x}$. Thus the electromechanical coupling strength depends on the parameter $G$, which is proportional to the bias voltage $V_{dc}$ and $C_m(x)/C(x)$.

The strong electromechanical coupling works when $\omega_m = \omega_{LC}$. Assuming the inductor is $L_1 = 1\mu H$ \cite{39} and the quality factor is $Q_{LC} = 50000$, the total capacitor should be $C = 1.3 \text{ fF}$ to have a resonant frequency of 5 GHz. The electric damping rate will be

$$\Gamma_{LC} = \omega_{LC}/Q_{LC} = 2\pi \cdot 100 \text{ kHz}.$$

The relationship between the electromechanical coupling rate ($g_{em}/2\pi$) and the bias voltage is displayed in Fig. 3.

V. OPTOMECHANICAL COUPLING

In cavity optomechanics, the mechanical vibration of a mirror couples to photons by changing the length and the resonant frequency of a cavity \cite{1}. In our case, the mechanical vibration of a membrane will change the lattice constant of the membrane, and thus shifts the frequency $\omega(x)$ of the optical transition $|g\rangle \leftrightarrow |e\rangle$ of a single photon emitter in the 2D membrane. So the mechanical phonon of the 2D membrane can be coupled to optical photons by the strain effect. The zero phonon line (ZPL) of a single photon emitter in a 2D membrane is also sensitive to electric fields due to the Stark effect. In our proposed device (Fig. 1(c)), the vibration of the 2D membrane will change the electric field $V_{dc}/(d-x)$ between the membrane and the bottom electrode, and thus can couple to optical transitions of the single photon emitter by the Stark effect. The coupling between a mechanical vibration mode of the 2D membrane and the quantum emitter’s electronic state can be described by a Hamiltonian

$$H_{om} = \hbar g_{om}(b + b^\dagger) |e\rangle \langle e|.$$

Here $g_{om} = x_{zpf} \omega_{zpf} / dx$ is the optomechanical coupling strength. We call it optomechanical coupling because a red sideband photon at frequency $\omega_L = \omega_m - \omega_{zpf}$ is involved in this process, although it is not explicit in this Hamiltonian in the form $H_{om}$. The electronic transition between $|g\rangle$ and $|e\rangle$ will happen by absorbing a phonon at frequency $\omega_m$ and a photon at $\omega_L$ together.

A. Strain-Mediated Optomechanical Coupling

In this part, we consider a strain along the length direction caused by the bending of the doubly clamped membrane which has $A_1$ symmetry and preserves the $C_{3v}$ symmetry of a defect center. An $A_1$-symmetric strain preserves the degeneracy of $E_g$ and $E_u$ orbital states and uniformly shifts the energy of both states. Thus under a $A_1$-symmetric strain, the ZPL will have a frequency shift without splitting. Experiments have investigated the spectral shift of single photon emitters in h-BN under different applied strain fields \cite{20}. The studied emitters...
The strain mediated coupling strength is thousand times larger than the typical decay rate of $2\pi \times 53$ MHz \cite{19,20,31}. To achieve strong coupling by the strain effect, we can use a red-detuned laser to drive the system more than 20 times stronger than a single-photon field. As will be discussed in the following section, the Stark effect will be larger.

B. The Stark Effect-Induced Optomechanical Coupling

With a bias voltage $V_{dc}$, there will be an electric field at the location of the single photon emitter for a three-layer 2D membrane as shown in Fig. 1(b). The Stark effect of localized emitters in a monolayer 2D material has been studied recently. For an emitter in a WSe$_2$ monolayer, a spectral shift up to $21$meV by a $4 \times 10^8$ V/m electric field was observed \cite{40}. In our proposed system, the ZPF of the 2D membrane induces an electric field variation $\Delta E_{zp} = x_{zp}\partial E/\partial x = x_{zp}V_{dc}/d^2$, which can be quite high ($\sim 10$ kV/m). The coupling rate between the optical transition and the mechanical vibration induced by the Stark effect is given by

$$g_{om2} = x_{zp}\frac{\partial \omega}{\partial x} = \Delta E_{zp}\frac{\partial \omega}{\partial E}.$$ \hspace{1cm} (10)

The Stark effect-induced coupling rate $g_{om2}$ is much larger than the strain induced coupling rate $g_{om1}$. The Stark effect-induced coupling rate $g_{om2}$ is plotted as a function of the displacement and bias voltage in Fig. 5. The optomechanical coupling rate is larger than 50 MHz when the bias voltage exceeds 2.5 V, and thus the system can reach the strong optomechanical coupling regime. Assuming the decay rate the the excited state $|e\rangle$ is 53 MHz and the decay rate of the mechanical oscillator is 100kHz, the optomechanical cooperativity is $c_{om2} = g_{om2}^2/(\Gamma_m \kappa) \sim 500$ when the bias voltage is 2.5V.

![FIG. 4. (Color online) The strain-induced optomechanical coupling strength ($g_{om1}/2\pi$) between an electron orbital states of the single-photon emitter and the mechanical vibration of a 2D membrane as a function of (a) the displacement and (b) the bias voltage. Parameters are the same as in Fig. 3](image)

The optomechanical cooperativity due to the strain effect is $c_{om1} = g_{om1}^2/(\Gamma_m \kappa) \sim 1.8$.

However, the single-photon optomechanical coupling strength $g_{om1}$ is still much smaller than the typical decay rate of the excited state of a single-photon emitter in a 2D membrane. The typical lifetime of the excited electronic state in h-BN is 3 ns, corresponding to the atomic decay rate of $2\pi \times 53$ MHz \cite{19,20,31}. To achieve strong coupling by the strain effect, we can use a red-detuned laser to drive the system more than 20 times stronger than a single-photon field. As will be discussed in the following section, the Stark effect will be larger.

![FIG. 5. (Color online) The Stark effect-induced optomechanical coupling strength ($g_{om2}/2\pi$) between an electron orbital states of the single-photon emitter and the mechanical vibration of a 2D membrane as a function of (a) the displacement and (b) the bias voltage. Parameters of the 2D membrane are the same as in Fig. 3](image)
VI. HIGH SPEED QUANTUM STATE TRANSFER

Most of optomechanical devices are hitherto in the limit of the weak single-photon optomechanical coupling regime, \( g_{om} \ll \kappa \). In that case, a strong driving field is required to enhance the effective coupling strength, which results in a large average photon number and a low signal-to-noise ratio. Our proposed system uses a single-photon emitter in a 2D membrane to replace the optical cavity to mediate the coupling between an optical photon and a mechanical phonon. Thanks to the small mass of a 2D membrane and the relatively narrow ZPL of a single-photon emitter at low temperature, it is much easier to reach the strong coupling regime. The single-photon coupling rate can be about 100 MHz, making it possible to realize high speed quantum state transfer under a weak laser driving field.

Here we consider the quantum state transfer from a microwave photon in the LC circuit to an optical single-photon pulse output. We assume that the system works at low temperature (<50 mK). So the electron orbital state (\( \psi_e \)) of the single photon emitter and the mechanical vibration mode (\( \psi_m \)) of the 2D membrane are initialized at ground states, i.e., \( |\psi_e(t = 0)\rangle = |g\rangle \) and \( |\psi_m(t = 0)\rangle = |0\rangle \). Then the LC resonator is prepared in state \( |\psi_{LC}(t = 0)\rangle = |1\rangle \). The process aims to transfer the quantum state from \( |\psi_{e}(t = 0)\rangle \otimes |\psi_{m}(t = 0)\rangle \) to \( |\psi_{LC}(t = 0)\rangle = |g\rangle |01\rangle \) to \( |e\rangle |00\rangle \). As spontaneous emission couples electron orbital state to the free-space continuum optical modes, once the electron state evolves into excited state \( |e\rangle \), an optical photon may be spontaneously emitted from the transition \( |e\rangle \) to \( |g\rangle \) and the system will output a single-photon pulse. However, to simplify the calculation, as the strong coupling conditions are fulfilled, we can first derive the effective Hamiltonian by ignoring the decay and then take it back into consideration during the second part. Here a laser driving field is required. Its effect can be described by a Hamiltonian

\[
H_d = \hbar \frac{\Omega}{2} (e^{-i \omega_L t} |e\rangle \langle g| + e^{i \omega_L t} |g\rangle \langle e|),
\]

(11)

where \( \omega_L \) is the frequency of the driving laser and \( \Omega \) is the Rabi frequency. Applying the Schrieffer-Wolff transformation \( U = \exp\left[ \frac{g_{om}}{\omega_m} (b^\dagger - b) |e\rangle \langle e| \right] \)

(12)
to the total Hamiltonian yields

\[
\tilde{H} = \hbar \omega_0 |e\rangle \langle e| + \hbar \omega_m b^\dagger b + \hbar \omega_{LC} c^\dagger c + \hbar \frac{g_{om}}{\omega_m} |e\rangle \langle e|
+ \hbar \frac{\Omega}{2} (e^{-i \omega_L t} e^{\frac{g_{om}}{\omega_m} (b^\dagger - b)} |e\rangle \langle g| + e^{i \omega_L t} e^{\frac{g_{om}}{\omega_m} (b - b^\dagger)} |g\rangle \langle e|).
\]

(13)

The Hamiltonian in the rotating-wave frame is \( H = H_0 + H_1 \), where

\[
H_0 = -\hbar \Delta |e\rangle \langle e| + \frac{\hbar \omega_m}{2} b^\dagger b
+ \hbar \omega_{LC} c^\dagger c,
\]

(14)

\[
H_1 = \hbar \frac{\Omega}{2} (e^{\frac{g_{om}}{\omega_m} (b^\dagger - b)} |e\rangle \langle g| + e^{\frac{g_{om}}{\omega_m} (b - b^\dagger)} |g\rangle \langle e|)
+ h_{gem} (b + b^\dagger) (c + c^\dagger) + \hbar \frac{g_{om} g_{em}}{\omega_m} (c^\dagger c + c c^\dagger) \langle e| \langle e|.
\]

(15)

Here \( \Delta = \omega_L - \omega_0 - \frac{\omega_m}{2} \) is the detuning between the driving laser and the optical resonance frequency. Transforming the Hamiltonian to the interaction picture and simplifying it by using the rotating wave approximation, we obtain the effective interaction Hamiltonian

\[
\tilde{H}_I = \frac{\hbar \Omega}{2} (e^{-i \Delta t} e^{\frac{g_{om}}{\omega_m} (b^\dagger e^{i \omega_m t} - be^{-i \omega_m t})} |e\rangle \langle g| + e^{i \Delta t} e^{\frac{g_{om}}{\omega_m} (be^{-i \omega_m t} - b^\dagger e^{i \omega_m t})} |g\rangle \langle e|)
+ h_{gem} (b + b^\dagger) c^\dagger c + \hbar \frac{g_{om} g_{em}}{\omega_m} (c^\dagger e^{-i \omega_{LC}} + ce^{i \omega_{LC}}) \langle e| \langle e|.
\]

(16)

We assume that the laser driving field is tuned to the red phonon sideband of the \( \bar{g} \) transition (\( \Delta \approx -\omega_m = -\omega_{LC} \)). The last term in Eq. (16) oscillates rapidly in the interaction picture. So its average influence on the energy level of atom excited state \( |e\rangle \) can be ignored. Then, expanding \( \tilde{H}_I \) in Eq. (16) by \( g/\omega_m \) and keeping only the near resonant terms, we can approximate the interaction Hamiltonian as

\[
\tilde{H}_I = \frac{\hbar \Omega}{2} \frac{g_{om}}{\omega_m} (be^{-i(\Delta + \omega_m) t} |e\rangle \langle g| + b^\dagger e^{i(\Delta + \omega_m) t} |g\rangle \langle e|)
+ h_{gem} (b c^\dagger + b^\dagger c)
+ \hbar g_{om} (b \sigma_{eg} + b^\dagger \sigma_{ge}) + h_{gem} (b c^\dagger + b^\dagger c),
\]

(17)

where \( \tilde{g}_{om} = \frac{\Omega}{2} \frac{g_{om}}{\omega_m} \) is the effective coupling strength between the electronic state of the single photon emitter and the mechanical vibration mode. Without loss of generality, we further assume \( \tilde{g}_{om} = g_{em} = g_e \) for simplicity of notation. If the system is initialized into the state \( |\psi_e\rangle |\psi_m\rangle |\psi_{LC}\rangle = |g\rangle |01\rangle \), the quantum state will evolve within the subspace \( |g\rangle |01\rangle, |g\rangle |10\rangle \) and \( |e\rangle |00\rangle \). This Hamiltonian has three eigenstates which are \( |\psi_1\rangle = \frac{1}{\sqrt{2}}(|g\rangle |01\rangle - \sqrt{2} |g\rangle |10\rangle + |e\rangle |00\rangle \), \( |\psi_2\rangle = \frac{1}{\sqrt{2}}(|g\rangle |01\rangle + \sqrt{2} |g\rangle |10\rangle + |e\rangle |00\rangle \) and \( |\psi_3\rangle = \sqrt{2} (|g\rangle |01\rangle - |e\rangle |00\rangle ) \). Then we get the quantum state of this system

\[
|\psi(t)\rangle = e^{-i \sqrt{2} g_{et} t} |\psi_1\rangle + e^{i \sqrt{2} g_{et} t} |\psi_2\rangle + |\psi_3\rangle
\]

(18)

\[
= \frac{1}{2} (1 + \cos \sqrt{2} g_{et} t) |g\rangle |01\rangle
- \frac{1}{2} (1 - \cos \sqrt{2} g_{et} t) |e\rangle |00\rangle
- \frac{i}{2} \sin \sqrt{2} g_{et} t |g\rangle |10\rangle.
\]

(19)
In the above simplified discussion, we neglect the coupling of the electronic transition \(|g⟩ \leftrightarrow |e⟩\) to the optical photon output and other decay channels. Now we consider these decay channels. This system couples to three decay channels: the optical channel with decay rate \(\kappa\), the mechanical channel with damping rate \(\Gamma_m\) and the electric channel with damping rate \(\Gamma_LC\). Because of stimulated emission, the effective damping rate for each decay channel depends on the thermal occupation \(n_i\) of the environment. The increase of stimulated emission will lead to the acceleration of dephase in this system, whose rate is proportional to the number of thermal quanta. So the effective decay rate takes the form \(\langle n_i + 1 \rangle \kappa\) \[9\], where the number of thermal quanta in each channel is given by \(n_i = 1/(e^{\hbar \omega_i/k_B T} - 1)\). \(T\) is the temperature of the environment. At \(T = 50\) mK, \(n_0 \approx 0\), \(n_m = 0.008\) and \(n_{LC} = 0.008\) when \(\omega_m = \omega_{LC} = 2\pi \times 5\) GHz.

The photon output from the system will be at the resonant frequency \(\omega_0\), which has a frequency shift \(\omega_n\) from the frequency of the driving laser \(\omega_L\). This photon output is the result of the microwave-optical photon conversion. This means that the optical decay channel contains the information of the quantum state transfer, rather than leading to the loss in the process. The main channels leading to quantum decoherence are the mechanical dissipation and electric circuit dissipation, which are quite small since \(\Gamma_m\) and \(\Gamma_{LC}\) are much smaller than \(\kappa\).

To qualify the coupling between the transition \(|g⟩ \leftrightarrow |e⟩\) and single photon output through the optical decay channel, we introduce \(a\) to denote the one-dimensional free-space photon modes which couples to the atomic transition with coupling strength \(\sqrt{\kappa/2\pi}\). Using the method in Ref. \[52\], the whole conditional Hamiltonian \[53\ 54\], including the mechanical decay and electric circuit decay, can be written in the following form in the rotating frame:

\[
H_I = \hbar g_a(b|e⟩⟨g| + b^\dagger|g⟩⟨e|) + \hbar g_c(b_c^\dagger + b^\dagger c) + i\hbar \sqrt{\kappa/2\pi} \int_{-\omega_a}^{\omega_a} d\omega (|e⟩⟨g|a(\omega) - |g⟩⟨e|a^\dagger(\omega)) + \int_{-\omega_a}^{\omega_a} d\omega [\omega a^\dagger(\omega)a(\omega)] - i\hbar \Gamma_m b^\dagger b - i\hbar \Gamma_{LC} c^\dagger c.
\]

(20)

To solve the dynamics governed by the Hamiltonian (20), we can expand the state \(|\psi⟩\) of the whole system into the following superposition:

\[
|\psi⟩ = c_1|e⟩|00⟩ + c_2|g⟩|10⟩ + c_3|g⟩|01⟩ \otimes |\text{vac}\rangle + |g⟩|00⟩ \otimes |\phi⟩,
\]

(21)

where \(|\text{vac}\rangle\) denotes the vacuum state of the free-space photon mode \(a\), and

\[
|\phi⟩ = \int_{-\omega_a}^{\omega_a} c_\omega a^\dagger(\omega)|\text{vac}\rangle d\omega
\]

(22)

represents the state (not normalized) of the single-photon output pulse. The coefficients \(c_1, c_2, c_3\) and \(c_\omega\) are time dependent. At time \(t = 0\), we have \(c_1 = 0, c_2 = 0, c_3 = 1\) and \(c_\omega = 0\).

After applying a red sideband driving laser at \(\omega_L\), these coefficients changes with \(t\). We need to compute the time evolution of these coefficients by substituting \(|\psi⟩\) into the Schrödinger equation \(i\hbar \partial_t |\psi⟩ = H|\psi⟩\). For numerical representation of the Hamiltonian (20), we discretize the free-space field \(a(\omega)\) by introducing a finite but small frequency interval \(\delta \omega\) between two adjacent modes. Then, we have about \(N = 2\omega_a/\delta \omega\) free-space modes in total. The \(j\)-th mode is denoted by \(a_j\), whose frequency detuning from the central frequency is given by \(\omega_j = (j - N/2)\delta \omega\). Here \(\delta \omega\) is much smaller than the inverse of the total evolution time \(T\). The integral bandwidth \(\omega_a\) is much larger than the optical decay rate \(\kappa\), but is much smaller than \(\omega_n\) to guarantee that there will be no change of the physical result by discretization. We rewrite the single-photon state as

\[
|\phi⟩ = \sum_{j=1}^{N} c_{\omega_j} a_j^\dagger|\text{vac}\rangle.
\]

(23)

Then we can obtain the following set of equations for coefficients \(c_1, c_2, c_3\) and \(c_{\omega_j}\):

\[
c_1 = -ig_c c_2 + k' \sum_{j=1}^{N} c_{\omega_j},
\]

(24)

\[
c_2 = -ig_c c_1 - ig_c c_3 - \frac{\Gamma_m}{2} c_2,
\]

(25)

\[
c_3 = -ig_c c_2 - \frac{\Gamma_{LC}}{2} c_3,
\]

(26)

\[
c_{\omega_j} = -k' c_1 - i(j - N/2)\delta \omega c_{\omega_j},
\]

(27)

where the effective optical decay rate \(\kappa' = \sqrt{\kappa \omega_a/2\pi}\). We numerically integrating Eqs. (24)-(27) to obtain the solutions.

FIG. 6. (Color online) Time dependence of the possibility \(P_n\) of evolution without mechanical or electronic loss when the coupling rates are (a) \(g_c/2\pi = 5\) MHz and (b) \(g_c/2\pi = 50\) MHz. Other parameters are \(\kappa/2\pi=50\) MHz, \(\Gamma_m/2\pi=100\) kHz, \(\Gamma_{LC}/2\pi=100\) kHz, and temperature \(T = 50\) mK. In the numerical simulations, the intervals and numbers of the free-space modes are taken as (a) \(\delta \omega/2\pi = 0.25\) MHz, \(N=2000\), and (b) \(\delta \omega/2\pi = 1\) MHz, \(N=500\). The results in the figure are independent of the exact values of \(\delta \omega\) and \(N\) when \(\delta \omega\) is sufficiently small and \(N\) is sufficiently large.
In this system, the property we concern most is the fidelity of converting a microwave photon in the LC circuit to an optical single-photon pulse output. If the total quantum state is a pure state, the whole process of the quantum state transfer from electric circuit to an optical photon output is reversible, which means that the information in the electric circuit can be totally transferred to optical photons. However, the Hamiltonian \( \hat{H} \) is not Hermitian because of the mechanical and electrical circuit decay terms \(-i\hbar \frac{\kappa}{2 \pi} b \hat{b} \) and \(-i \hbar \frac{\Gamma_m}{2 \pi} c \hat{c} \). Because of these two decay terms, the initial signal may be lost due to thermal dissipation by the 2D resonator or the LC circuit, which may lead to no optical photon output. To quantify the influence of these leakages, we introduce the possibility of quantum state evolution without leakage through the mechanical or LC decay channel:

\[
P_n = |c_1(t)|^2 + |c_2(t)|^2 + |c_3(t)|^2 + \sum_{j=1}^{N} |c_{\omega_j}(t)|^2,
\]

which is the normalization coefficient of the total quantum state at time \( t \). Fig. 6 displays the calculated \( P_n \) for two different coupling strengths \( g_c/2\pi = 5 \) MHz and \( g_c/2\pi = 50 \) MHz. Other parameters are \( \kappa/2\pi = 50 \) kHz, \( \Gamma_m/2\pi = 100 \) kHz, \( \Gamma_{LC}/2\pi = 100 \) kHz, and temperature \( T = 50 \) mK. The possibility \( P_n \) for quantum state evolution without mechanical or electric circuit loss is larger than 99% when the coupling rate \( g \) matches the atomic decay rate \( \kappa \) as shown in Fig. 6(b). When the coupling rate is small, the efficiency of the state transfer is lower. When \( g_c/2\pi = 5 \) MHz, about 10% of information is lost due to mechanical and electronic circuit dissipation (Fig. 6(a)).

The single-photon output through the optical decay channel is our signal, while all other dissipation channels contribute to the loss. Here we quantify the state-transfer fidelity by \( F_{\text{trans}} \), which is the normalization coefficient of the single-photon pulse state at the time \( t \)

\[
F_{\text{trans}} = \sum_{j=1}^{N} |c_{\omega_j}(t)|^2.
\]

In Fig. 7 we show the fidelity as a function of the evolution time \( t \) with different coupling rates. The fidelity increases with evolving time, and eventually reaches its maximum when the fidelity \( F_{\text{trans}} \) equals to the possibility \( P_n \). It is found that when the effective coupling rate largely surpasses other decay rates, \( G \sim \kappa \gg \Gamma_m, \Gamma_{LC} \), the fidelity is over 99% as shown in Fig. 7(b)-(d). The dependence of \( P_n \) and \( F_{\text{trans}} \) on the system temperature and the decay rate \( \kappa \) is provided in Fig. 8. It shows that the fidelity can maintain a high value (> 95%) within a large scale of the temperature and optical decay rate (the coupling rate is assumed to be the same as the optical decay rate in Fig. 8). It is worthwhile to notice that the fidelity maintains > 0.95 when the system works at 1K temperature. This is because our proposed system can do high-speed laser cooling of the mechanical resonator and the LC circuit. 

![Figure 7](image-url)  
**Fig. 7.** (Color online) Time evolution of the fidelity \( F_{\text{trans}} \) of the state transfer from a microwave photon in the LC circuit to a free-space optical photon output for different coupling rates (a) \( g_c/2\pi = 5 \) MHz, (b) \( g_c/2\pi = 20 \) MHz, (c) \( g_c/2\pi = 50 \) MHz, and (d) \( g_c/2\pi = 200 \) MHz. The maximum fidelities are (a) \( F_{\text{trans}} = 0.905 \), (b) \( F_{\text{trans}} = 0.990 \), (c) \( F_{\text{trans}} = 0.995 \) and (d) \( F_{\text{trans}} = 0.996 \). In Fig. (a) where \( g_c/2\pi = 5 \) MHz, it takes about 800 ns to reach the maximum fidelity \( F = 0.905 \). In the other three conditions, they take (b) 45 ns, (c) 33 ns and (d) 30 ns to reach a fidelity \( F_{\text{trans}} > 0.95 \). Other parameters are taken as temperature \( T = 50 \) mK, \( \kappa/2\pi = 50 \) MHz, \( \Gamma_m/2\pi = 100 \) kHz, \( \Gamma_{LC}/2\pi = 100 \) kHz.

![Figure 8](image-url)  
**Fig. 8.** (Color online) (a) Temperature dependence of the fidelity \( F_{\text{trans}} \) (blue solid curve) of transferring a microwave photon to an optical photon pulse output, and the possibility \( P_n \) of evolution without loss through mechanical and LC channels (red dashed curve). The evolution time is taken as 50 ns. Other parameters are taken as \( g_c/2\pi = 50 \) MHz, \( \kappa/2\pi = 50 \) MHz, \( \Gamma_m/2\pi = 100 \) kHz, \( \Gamma_{LC}/2\pi = 100 \) kHz. (b) Dependence of the fidelity \( F_{\text{trans}} \) (blue solid curve) and the possibility \( P_n \) (red circles) on the atomic decay rate \( \kappa \) at 50 mK. Here the possibility \( P_n \) equals to the fidelity \( F_{\text{trans}} \), meaning that the fidelity has reached its maximum value. We assume that the coupling rate \( g \) is always tuned to match the decay rate \( \kappa \). Other parameters are the same as in (a).
The graphene will absorb 2.1(b) photons emitted by the single-photon emitter needs most photons can be collected by a high numerical aperture (Fig. 1(a)) will reflect photons emitted toward it. So optical photon efficiently. The bottom metallic electrode need to minimize additional losses to detect the generated photon in free space. As in other optical systems, we will ing a microwave photon in a LC circuit to an optical state transfer fidelity. As shown in Fig.9, the required evolving time decreases significantly as the decay rate increases.

The speed of quantum state transfer is also important. A low transferring speed will make the whole process slow and lead to a poor fidelity. So we also calculate the dependence of the required evolution time to achieve a high state transfer fidelity $F_{\text{trans}} > 0.95$ on the decay rate $\kappa$ (we assume that the coupling rate is tuned to match the decay rate). As shown in Fig.9, the required evolving time decreases significantly as the decay rate increases.

The above calculation considers the fidelity of converting a microwave photon in a LC circuit to an optical photon in free space. As in other optical systems, we will need to minimize additional losses to detect the generated optical photon efficiently. The bottom metallic electrode (Fig. 1(a)) will reflect photons emitted toward it. So most photons can be collected by a high numerical aperture (NA) objective lens above the 2D resonator. In Fig. 1(b), photons emitted by the single-photon emitter needs to pass through the graphene electrode to be detected. The graphene will absorb 2.3% or more of the light [55], which will reduce the overall fidelity. A doped semiconducting monolayer may replace the graphene electrode to minimize light absorption. To avoid light absorption by graphene, we may use the structure as shown in Fig. 1(c). The device can be integrated with an optical cavity to further improve the efficiency of photon collection [56].

VII. CONCLUSION

In this paper, we propose a scheme to realize high speed quantum state transfer between a microwave photon in a LC circuit and an optical photon in free space with a single photon emitter in an atomically thin 2D resonator. The 2D mechanical resonator couples an electric microwave photon to atomic ZPL with a coupling strength larger than 100 MHz. The conversion to a single photon pulse output can be realized by applying a weak driving laser at the red sideband of the atomic transition of the single-photon emitter. High speed quantum state transfer between a single microwave photon and a single optical photon with a fidelity larger than 0.95 can be realized. When the environmental temperature is relatively high ($\sim 1$ K), the red-detuned driving laser can cool and initialize the mechanical mode and the LC circuit to their quantum ground states, and maintain the high fidelity of the quantum state transfer of the transducer.

ACKNOWLEDGMENTS

T. L. acknowledges the support from NSF under Grant No. 1555035-PHY and the Tellabs Foundation. Z.-Q. Y. is supported by the National Natural Science Foundation of China Grant 61771278, 61435007 and 11574176.

Appendix A: Derivation of Hamiltonian

The Hamiltonian of the system is [13, 14, 21]

$$H = \hbar \omega(x) |e \rangle \langle e| + \frac{m \Omega^2 x^2}{2} + \frac{p^2}{2m} + \frac{q^2}{2C(x)} + \frac{\phi^2}{2L} - q V_{dc}. \quad (A1)$$

The corresponding Langevin equations are

$$\dot{x} = \frac{p}{m}, \quad (A2a)$$
$$\dot{p} = - \frac{\hbar}{2} \partial_x \left( \frac{1}{C(x)} \right) - \hbar \partial_x \omega(x) |e \rangle \langle e| - \Gamma_m p - F_{th}, \quad (A2b)$$
$$\dot{q} = \frac{\phi}{L}, \quad (A2c)$$
$$\dot{\phi} = - \frac{q}{C(x)} - \Gamma_{LC} \phi + V_{dc} + V_{th}. \quad (A2d)$$

where $\Gamma_m$ and $\Gamma_{LC}$ refer to the dissipation rates of mechanical membrane and the LC circuit. Here $F_{th}$ is the thermal noise force and $V_{th}$ is the thermal noise voltage. The equilibrium state is then characterized by

$$\ddot{\phi} = \ddot{p} = 0, \quad (A3a)$$
$$m \omega_m^2 \ddot{x} = - \frac{\hbar}{2} \partial_x \left( \frac{1}{C(x)} \right) |_{x=\bar{x}} - \hbar \partial_x \omega(x) |_{x=\bar{x}} |e \rangle \langle e| \quad (A3b)$$
$$\simeq - \frac{\hbar}{2} \partial_x \left( \frac{1}{C(x)} \right) |_{x=\bar{x}} \quad (A3c)$$
$$\ddot{q} = V_{dc} C(\bar{x}).$$
We rewrite the Hamiltonian using \( x = \bar{x} + \delta x \), \( q = \bar{q} + \delta q \), and obtain
\[
H = \hbar \omega_0 |\psi\rangle \langle \psi| e + \frac{\phi^2}{2L} + \frac{p^2}{2m} + \frac{m \omega_m^2 (\delta x^2 + 2 \bar{x} \delta x + \delta x^2)}{2} + \frac{(\bar{q}^2 + 2 \delta q \bar{q} + \delta q^2)}{2C(x)} - (\bar{q} + \delta q) V_{dc}
\]
\[
= \hbar \omega_0 |\psi\rangle \langle \psi| e + \frac{\phi^2}{2L} + \frac{p^2}{2m} + \frac{m \omega_m^2 \bar{x} \delta x}{2C(\bar{x})} + \frac{\bar{q}^2}{2C(\bar{x})} - \delta q V_{dc}
\]
Eventually, by ignoring the constant term \( H_{\text{const}} \), the Hamiltonian takes the form \( H = H_0 + H_I \), where
\[
H_0 = \hbar \omega_0 |\psi\rangle \langle \psi| e + \hbar \omega_m b^\dagger b + \hbar \omega_L c^\dagger c,
\]
\[
H_I = \hbar g_{om} (b + b^\dagger) |\psi\rangle \langle \psi| e + \hbar g_{em} (b + b^\dagger) (c + c^\dagger).
\]
Here \( g_{om} = x_{zp} f (\partial \omega(x)/\partial x) \) and \( g_{em} = \bar{q} \frac{\partial}{\partial x} (1/\bar{C}(\bar{x})) x_{zp} f q_{zp} \) are the optomechanical and electromechanical coupling strength, respectively.
[27] Y. Ma, Z.-q. Yin, P. Huang, W. L. Yang, and J. Du, Phys. Rev. A 94, 053836 (2016).
[28] Y. Ma, T. M. Hoang, M. Gong, T. Li, and Z.-q. Yin, Phys. Rev. A 96, 023827 (2017).
[29] J. Kelly et al., Nature 519, 66 (2015).
[30] Y. Zheng et al., Phys. Rev. Lett. 118, 210504 (2017).
[31] A. Dietrich, M. Bürk, E. Steiger, L. Antoniuk, T. Tran, M. Nguyen, I. Aharonovich, F. Jelezko, and A. Kubanek, arXiv:1712.06938 (2017).
[32] D. Lee, K. W. Lee, J. V. Cady, P. Ovartchaiyapong, and A. C. B. Jayich, Journal of Optics 19, 033001 (2017).
[33] A. Castellanos-Gomez, V. Singh, H. S. van der Zant, and G. A. Steele, Annalen der Physik 527, 27 (2015).
[34] A. Eichler, J. Moser, J. Chaste, M. Zdrojek, I. Wilson-Rae, and A. Bachtold, Nature nanotechnology 6, 339 (2011).
[35] X. Song, M. Oksanen, J. Li, P. Hakonen, and M. A. Sillanpää, Phys. Rev. Lett. 113, 027404 (2014).
[36] C. Lee, X. Wei, J. W. Kysar, and J. Hone, Science 321, 385 (2008).
[37] C. Wong, M. Annamalai, Z. Wang, and M. Palanipan, Journal of Micromechanics and Microengineering 20, 115029 (2010).
[38] N. Morell, A. Reserbat-Plantey, I. Tsioutsios, K. G. Schadler, F. Dubin, F. H. Koppens, and A. Bachtold, Nano letters 16, 5102 (2016).
[39] P. Weber, J. Guttinger, I. Tsioutsios, D. E. Chang, and A. Bachtold, Nano letters 14, 2854 (2014).
[40] C. Chakraborty, K. M. Goodfellow, S. Dhara, A. Yoshimura, V. Meunier, and A. N. Vamivakas, Nano Letters 17, 2253 (2017).
[41] F. Ye, J. Lee, and P. X.-L. Feng, Nanoscale 9, 18208 (2017).
[42] A. Reserbat-Plantey, K. G. Schädler, L. Gaudreau, G. Navickaité, J. Güttinger, D. Chang, C. Toninelli, A. Bachtold, and F. H. Koppens, Nature communications 7 (2016).
[43] D. Vitali, S. Gigan, A. Ferreira, H. Böhm, P. Tombesi, A. Guerreiro, V. Vedral, A. Zeilinger, and M. Aspelmeyer, Physical review letters 98, 030405 (2007).
[44] S. Gröblacher, K. Hammerer, M. R. Vanner, and M. Aspelmeyer, Nature 460, 724 (2009).
[45] A. Clerk, F. Marquardt, and K. Jacobs, New Journal of Physics 10, 095010 (2008).
[46] J. S. Bunch, A. M. Van Der Zande, S. S. Verbridge, J. W. Frank, D. M. Tanenbaum, J. M. Parpia, H. G. Craighead, and P. L. McEuen, Science 315, 490 (2007).
[47] A. Neto and K. Novoselov, Materials Express 1, 10 (2011).
[48] Q. H. Wang, K. Kalantar-Zadeh, A. Kis, J. N. Coleman, and M. S. Strano, Nature nanotechnology 7, 699 (2012).
[49] I. Frank, D. M. Tanenbaum, A. M. van der Zande, and P. L. McEuen, Journal of Vacuum Science & Technology B: Microelectronics and Nanometer Structures Processing, Measurement, and Phenomena 25, 2558 (2007).
[50] C. Wong, M. Annamalai, Z. Wang, and M. Palanipan, Journal of Micromechanics and Microengineering 20, 115029 (2010).
[51] J. Teufel, T. Donner, D. Li, J. Harlow, M. Allman, K. Cicak, A. Sirois, J. D. Whittaker, K. Lehner, and R. W. Simmonds, Nature 475, 359 (2011).
[52] L.-M. Duan, A. Kuzmich, and H. Kimble, Physical Review A 67, 032305 (2003).
[53] M. B. Plenio and P. L. Knight, Rev. Mod. Phys. 70, 101 (1998).
[54] Y. Huang, Z.-q. Yin, and W. L. Yang, Phys. Rev. A 94, 022302 (2016).
[55] R. R. Nair, P. Blake, A. N. Grigorenko, K. S. Novoselov, T. J. Booth, T. Stauber, N. M. R. Peres, and A. K. Geim, Science 320, 1308 (2008).
[56] H. Kelkar, D. Wang, D. Martín-Cano, B. Hofmann, S. Christiansen, S. Göttinger, and V. Sandoghdar, Phys. Rev. Applied 4, 054010 (2015).