Extensions of maps to $M(\mathbb{Z}_m, 1)$

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Abstract
We show that a Moore space $M(\mathbb{Z}_m, 1)$ is an absolute extensor for finite dimensional metrizable spaces of cohomological dimension $\dim_{\mathbb{Z}_m} \leq 1$.

Keywords: Cohomological Dimension, Extension Theory

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1 Introduction

All spaces are assumed to be metrizable. A map means a continuous function and a compactum means a compact metrizable space. By cohomology we always mean the Čech cohomology. Let $G$ be an abelian group. The cohomological dimension $\dim_G X$ of a space $X$ with respect to the coefficient group $G$ does not exceed $n$, $\dim_G X \leq n$ if $H^{n+1}(X, A; G) = 0$ for every closed $A \subset X$. Note that this condition implies that $H^{n+k}(X, A; G) = 0$ for all $k \geq 1$ [7], [2]. Thus, $\dim_G X =$ the smallest integer $n \geq 0$ satisfying $\dim_G X \leq n$ (provided it exists), and $\dim_G X = \infty$ if such an integer does not exist.

Cohomological dimension is characterized by the following basic property: $\dim_G X \leq n$ if and only for every closed $A \subset X$ and a map $f : A \longrightarrow K(G, n)$, $f$ continuously extends over $X$ where $K(G, n)$ is the Eilenberg-MacLane complex of type $(G, n)$ (we assume that $K(G, 0) = G$ with discrete topology and $K(G, \infty)$ is a singleton). This extension characterization of Cohomological Dimension gives a rise to Extension Theory (more general than Cohomological Dimension Theory) and the notion of Extension Dimension. The extension dimension of a space $X$ is said to be dominated by a CW-complex $K$, written $e\text{-dim} X \leq K$, if every map $f : A \longrightarrow K$ from a closed subset $A$ of $X$ continuously extends over $X$. Thus $\dim_G X \leq n$ is equivalent to $e\text{-dim} X \leq K(G, n)$ and $\dim X \leq n$ is equivalent to $e\text{-dim} X \leq S^n$. The property $e\text{-dim} X \leq K$ is also denoted by $X \tau K$ and it is also referred to as $K$ being an absolute extensor of $X$.

The following theorem shows a close connection between extension and cohomological dimensions.
Theorem 1.1 (Dranishnikov Extension Theorem) Let $K$ be a CW-complex and $X$ a metrizable space. Denote by $H_*(K)$ the reduced integral homology of $K$. Then

(i) $\dim H_n(K)X \leq n$ for every $n \geq 0$ if $e\dim X \leq K$;
(ii) $e\dim X \leq K$ if $K$ is simply connected, $X$ is finite dimensional and $\dim H_n(K)X \leq n$ for every $n \geq 0$.

Theorem 1.1 was proved in [1] for the compact case and extended in [3] to the metrizable case.

Let $G$ be an abelian group. We always assume that a Moore space $M(G,n)$ of type $(G,n)$ is an $(n-1)$-connected CW-complex whose reduced integral homology is concentrated in dimension $n$ and equals $G$. Theorem 1.1 implies that for a finite dimensional metrizable space $X$ and $n > 1$, $\dim G X \leq n$ if and only if $e\dim X \leq M(G,n)$. The main open problem for $n = 1$ is:

Problem 1.2 Let $G$ be an abelian group and let $M(G,1)$ be a Moore space whose fundamental group is abelian. Is $M(G,1)$ an absolute extensor for finite dimensional metrizable spaces of $\dim G \leq 1$?

This problem was affirmatively answered in [6] for $M(\mathbb{Z}_2,1) = \mathbb{R}P^2$. In this paper we extend the result of [6] to Moore spaces $M(\mathbb{Z}_m,1)$. In this particular case we choose a specific model and by $M(\mathbb{Z}_m,1)$ we mean the space obtained by attaching a disk to a circle by an $m$-fold covering map of the disk boundary. Our main result is:

Theorem 1.3 The Moore space $M(\mathbb{Z}_m,1)$ is an absolute extensor for finite dimensional metrizable spaces of cohomological dimension mod $m$ at most 1.

The case of metrizable spaces of $\dim \leq 3$ in Theorem 1.3 was independently obtained by A. Nagórkó by generalizing the approach of [5] to 3-dimensional lens spaces.

2 Preliminaries

In this section we present a few general notations and facts that will be used later.

For a CW-complex $L$ we denote by $L^{[k]}$ the $k$-skeleton of $L$.

Let $A$ and $B$ be compact spaces and $A' \subset A$ are $B' \subset B$ closed subsets. We denote by $\frac{A \times B}{A' \times B'}$ the quotient space of $A \times B$ by the partition consisting of the singletons of $(A \times B) \setminus (A' \times B')$ and the sets $\{a\} \times B'$, $a \in A'$. Clearly, the spaces $\frac{A' \times B}{A' \times B'}$ and $\frac{A \times B'}{A \times B''}$ can be considered as subspaces of the space $\frac{A \times B}{A' \times B'}$. In a similar way we define for closed subsets $A' \subset A'' \subset A$ and $B' \subset B'' \subset B$ the space

$$\frac{A \times B}{A'' \times B' \cup A' \times B''}$$

as the quotient space of $A \times B$ by the partition consisting of the sets $\{a\} \times B'$ for $a \in A'' \setminus A'$, the sets $\{a\} \times B''$ for $a \in A'$ and the singletons not contained in the sets listed before.
**Proposition 2.1** Let $f, g : M(\mathbb{Z}_m, 1) \to M(\mathbb{Z}_m, 1)$ be maps inducing the zero-homomorphism of the fundamental group. Then $f \circ g$ is null-homotopic.

**Proof.** Note that the universal cover $\tilde{M}$ of $M(\mathbb{Z}_m, 1)$ is homotopy equivalent to a bouquet of 2-spheres and the maps $f$ and $g$ lift to $\tilde{M}$. Thus the map $f \circ g$ factors through maps $\tilde{M} \to M(\mathbb{Z}_m, 1) \to M$ whose composition induces the zero-homomorphism of $H_2(M)$ since $H_2(M(\mathbb{Z}_m, 1)) = 0$. Hence the composition $\tilde{M} \to M(\mathbb{Z}_m, 1) \to M$ is a null-homotopic map resulting in $f \circ g$ being null-homotopic. ■

**Proposition 2.2** Let $T$ be a $G$-bundle over $M(\mathbb{Z}_m, 1)$ and $M_0$ a singleton in $M(\mathbb{Z}_m, 1)$. If the structure group $G$ of the bundle is arcwise connected then $T$ is trivial over $M(\mathbb{Z}_m, 1) \setminus M_0$.

**Proof.** Take a sufficiently fine triangulation of $M(\mathbb{Z}_m, 1)$ and observe that there is a retraction $r : M(\mathbb{Z}_m, 1) \setminus M_0 \to L$ to a 1-dimensional subcomplex $L$ of $M(\mathbb{Z}_m, 1)$ such that $r$ can decomposes into the composition of retractions that move the points of $M(\mathbb{Z}_m, 1)$ only inside small sets over which $T$ is trivial. Then $r$ induces a bundle map from $T$ over $M(\mathbb{Z}_m, 1) \setminus M_0$ to $T$ over $L$. Note that every $G$-bundle over a one-dimensional simplicial complex is trivial if $G$ is arcwise connected. Thus $T$ over $L$ is trivial and hence $T$ over $M(\mathbb{Z}_m, 1) \setminus M_0$ is trivial as well. ■

The following two propositions are simple exercises left to the reader.

**Proposition 2.3** Let $T$ be a ball bundle over a metrizable space $L$, $T_0$ the fiber of $T$ over a point in $L$ and $U$ a neighborhood of $T_0$ in $T$. Then $T/T_0$ embeds into $T$ so that $T \setminus (T/T_0) \subset U$ and the projection of $T/T_0$ to $L$ coincides with the projection of $T$ to $L$ restricted to $T/T_0$.

**Proposition 2.4** Let $X$ be a metrizable space, $g : K \to L$ a map of a CW-complex $K$ to a simplicial complex $L$ (with the CW topology) such that for every simplex $\Delta$ of $L$ we have that $g^{-1}(\Delta)$ is a subcomplex of $K$ and $\text{e-dim}X \leq g^{-1}(\Delta)$. Then a map $f : F \to K$ from a closed subset $F$ of $X$ extends over $X$ if $g \circ f : F \to L$ extends over $X$.

We will also need

**Proposition 2.5** (ii) Let $K$, $L$ and $M$ be finite CW-complexes, $L_0$ a singleton in $L$, $X$ a metrizable space and $F$ closed subset of $X$ such that $L$ is connected, $L$ admits a simplicial structure and $\text{e-dim}X \leq \Sigma K$.

(i) If a map $f : F \to \frac{L \times K}{L_0 \times K}$ followed by the projection of $\frac{L \times K}{L_0 \times K}$ to $L$ extends over $X$ then $f$ extends over $X$ as well.

(ii) If $f : F \to L \times K$ and $g : L \times K \to M$ are maps such that $f$ followed by the projection of $L \times K$ to $L$ extends over $X$ and $g$ is null-homotopic on $L_0 \times K$ then $f$ followed by $g$ extends over $X$ as well.
For the reader’s convenience let us outline the proof of Proposition 2.4. Fix a triangulation of $L$ for which $L_0$ is a vertex. Observe that the projection of $L \times K_{L_0 \times K}$ to $L$ factors up to homotopy through the space $L^{[0]}_0 \times K$ where $L^{[0]}$ is the 0-skeleton of $L$ with respect to the triangulation of $L$. Also observe that for every $n$-simplex $\Delta$ the space $\Delta \times K$ is homotopy equivalent to the wedge of $n$ copies of $\Sigma K$ and hence $\text{e-dim} X \leq \frac{\Delta \times K}{\Delta^{[0]}_0 \times K}$. Then, by Proposition 2.4, we have that $f$ in (i) followed by the projection of $L \times K$ to $L^{[0]}_0 \times K$ extends over $X$. Hence $f$ extends over $X$ as well and (i) is proved. Note that in (ii) the map $g \times f$ factors up to homotopy through $L^{[0]}_0 \times K$ and hence (ii) follows from (i).

3 Lens spaces

By $\mathbb{R}^n$, $\mathbb{B}^n$, $\mathbb{S}^n$ we denote the $n$-dimensional Euclidean space, the unit ball in $\mathbb{R}^n$, and the unit sphere in $\mathbb{R}^{n+1}$ respectively. A topological $n$-sphere is denoted by $S^n$ with $S^0$ being a singleton. We usually assume that that $\mathbb{R}^m \subset \mathbb{R}^k$ if $m \leq k$. Thus we will use the subscript $\bot$ to write $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^n_\bot$ for $n = m + k$ in order to emphasize that $\mathbb{R}^m$ and $\mathbb{R}^n_\bot$ are not subspaces of each other.

Recall that for a CW-complex $L$ we denote by $L^{[k]}$ the $k$-skeleton of $L$. For a covering space $\tilde{L}$ of $L$ we will consider $\tilde{L}$ with the CW-structure induced by the CW-structure of $L$ and hence we have that the $k$-skeleton $\tilde{L}^{[k]}$ of $\tilde{L}$ is the preimage of $L^{[k]}$ under the covering map.

In the proof of Theorem 1.3 we will use the infinite dimensional lens space $L_m$ as a model of $K(Z_m, 1)$. Let us remind the construction of $L_m$. Decompose $\mathbb{R}^{2n}$ into the direct sum of $n$ two-dimensional coordinate planes $\mathbb{R}^2$ and consider the orthogonal transformation $\theta$ of $\mathbb{R}^{2n}$ induced by rotating counterclockwise each $\mathbb{R}^2$ in the decomposition of $\mathbb{R}^{2n}$ by the angle $2\pi/n$. Thus $\mathbb{Z}_m = \{\theta, \theta^2, \ldots, \theta^n\}$ acts on $\mathbb{R}^{2n}$ by orientation preserving orthogonal transformations and $\mathbb{Z}_m$ acts freely on the unit sphere $\mathbb{S}^{2n-1}$ of $\mathbb{R}^{2n}$. We will refer to $\theta$ as the generating transformation of $\mathbb{Z}_m$. Denote $L_m^{[2n-1]} = \mathbb{S}^{2n-1}/\mathbb{Z}_m$. Representing $\mathbb{R}^{2n+2}$ as $\mathbb{R}^{2n+2} = \mathbb{R}^n \oplus \mathbb{R}^2_\bot$, we can regard $\mathbb{S}^{2n-1}$ as a subset of $\mathbb{S}^{2n+1}$ and $L_m^{[2n-1]}$ as a subset of $L_m^{[2n+1]}$. The infinite dimensional lens space $L_m$ is defined as $L_m = \text{dirlim} L_m^{[2n-1]}$. Clearly $\mathbb{Z}_m$ freely acts on $\mathbb{S}^\infty = \text{dirlim} \mathbb{S}^{2n-1}$ and $L_m = \mathbb{S}^\infty/\mathbb{Z}_m$. Thus we have that $L_m = K(\mathbb{Z}_m, 1)$ since $\mathbb{S}^\infty$ is contractible.

The CW-structure of $L_m$ is defined so that $L_m$ has only one cell in each dimension, see 4. The CW-structure of $L_m$ agrees with our previous notation since $L_m^{[2n-1]}$ is indeed the $(2n - 1)$-skeleton of $L_m$. Set $L_m = \mathbb{S}^\infty$ and $L_m^{[2n-1]} = \mathbb{S}^{2n-1}$. The preimage $\tilde{L}_m^{[2n]}$ of $L_m^{[2n]}$ under the projection of $\mathbb{S}^{2n+1}$ to $L_m$ can be described as follows. Represent $\mathbb{R}^{2n+2}$ as $\mathbb{R}^{2n+2} = \mathbb{R}^{2n} \oplus \mathbb{R}^2_\bot$ of orthogonal coordinate subspaces invariant under the action of $\mathbb{Z}_m$ on $\mathbb{R}^{2n+2}$ and let $\mathbb{S}^{2n-1}$ and $\mathbb{S}_1^1$ be the unit sphere and the unit circle in $\mathbb{R}^{2n}$ and $\mathbb{R}^2_\bot$ respectively. Take a point $a \in \mathbb{S}_1^1$ and consider the $(2n + 1)$-dimensional linear subspace of $\mathbb{R}^{2n+2}$ containing $\mathbb{R}^{2n}$ and the point $a$, and in this subspace consider the unit sphere $\mathbb{S}_a^{2n}$. Then $\mathbb{S}^{2n-1}$ divides $\mathbb{S}_a^{2n}$ into two hemispheres and denote by $C_a$ the
hemisphere containing the point \( a \). It is clear that \( gC_a = C_{ga} \) for \( g \in \mathbb{Z}_m \). Fix any orbit \( A \) of \( \mathbb{Z}_m \) in \( S^1 \). The space \( \tilde{L}_m^{[2n]} \) is defined as the union of \( \tilde{L}_m^{[2n]} = S^{2n-1} \) with the \((2n)\)-dimensional hemispheres \( C_a, a \in A \) which are defined to be the \((2n)\)-cells of \( \tilde{L}_m^{[2n]} \). Clearly \( \tilde{L}_m^{[2n]} \) is invariant under the action of \( \mathbb{Z}_m \) on \( \mathbb{R}^{2n+2} \). Then \( L_m^{[2n]} \) is defined as the orbit space \( L_m^{[2n]} = \tilde{L}_m^{[2n]} / \mathbb{Z}_m \) and it is obvious that \( L_m^{[2n]} \) is obtained from \( L_m^{[2n-1]} \) by attaching one \((2n)\)-cell.

We will call the models of \( L_m^{[2n-1]} \) and \( L_m^{[2n]} \) described above the covering models. Note that \( \tilde{L}_m^{[k]} \) is the universal cover of \( L_m^{[k]} \) for \( k > 1 \) and \( \tilde{L}_m^{[1]} = S^1 \) is an \( m \)-fold cover of \( L_m^{[1]} = S^1 \). The 0-skeleton \( L_m^{[0]} \) of \( L_m \) is a singleton in \( L_m^{[1]} \). Note that \( L_m^{[2]} = M(\mathbb{Z}_m, 1) \).

The space \( L_m^{[2n]} \) can be also described in the following way. Consider the unit ball \( \mathbb{B}^{2n} \) in \( \mathbb{R}^{2n} \). Then \( L_m^{[2n]} \) is the quotient space of \( \mathbb{B}^{2n} \) under the action of \( \mathbb{Z}_m \) on \( \partial \mathbb{B}^{2n} = S^{2n-1} \). By this we mean the quotient space whose equivalence classes are the orbits of the points in \( S^{2n-1} \) and the singletons in \( \mathbb{B}^{2n} \setminus S^{2n-1} \). We will refer to such representation of \( L_m^{[2n]} \) as the ball model of \( L_m^{[2n]} \).

The space \( L_m^{[2n+1]} \) also admits a similar description. Represent \( \mathbb{R}^{2n+1} = \mathbb{R}^{2n} \oplus \mathbb{R} \), consider the unit sphere \( S^{2n-1} \) in \( \mathbb{R}^{2n} \) and the action of \( \mathbb{Z}_m \) on \( \mathbb{R}^{2n} \). Consider the unit ball \( \mathbb{B}^{2n+1} \) in \( \mathbb{R}^{2n+1} \) and define an equivalence relation on \( \mathbb{B}^{2n+1} \) with the equivalence classes to be: the orbits of the action of \( \mathbb{Z}_m \) on \( S^{2n-1} \), the singletons of \( \mathbb{B}^{2n+1} \setminus \partial \mathbb{B}^{2n+1} \) and the sets \( \{(x, t), (\theta x, -t)\} \) where \( \theta \) is the generating transformation of \( \mathbb{Z}_m \) and \( (x, t) \in \mathbb{R}^{2n+1} = \mathbb{R}^{2n} \oplus \mathbb{R} \) such that \( (x, t) \in \partial \mathbb{B}^{2n+1} \) and \( t < 0 \). Then \( L_m^{[2n+1]} \) is the quotient space of \( \mathbb{B}^{2n+1} \) under this equivalence relation. Similarly we refer to such representation of \( L_m^{[2n+1]} \) as the ball model of \( L_m^{[2n+1]} \).

## 4 Extensions of maps to Lens spaces

In this section we prove two auxiliary propositions. By a Moore space \( M(\mathbb{Z}_m, k) \) we mean a space obtained by attaching a \((k + 1)\)-ball to a \( k \)-sphere \( S^k \) by a map degree \( m \) from the ball boundary to \( S^k \) and we denote the \( k \)-sphere \( S^k \) in \( M(\mathbb{Z}_m, k) \) by \( \partial M(\mathbb{Z}_m, k) \).

**Proposition 4.1** Let \( \psi : S^1 \times S^{2n-1} \longrightarrow L_m^{[2n]}, n \geq 1 \) be a map such that \( \psi \) restricted to \( S^1 \times S^0 \) generates the fundamental group of \( \tilde{L}_m^{[2n]} \) and \( \psi \) restricted to \( S^0 \times S^{2n-1} \) is null-homotopic. Then \( \psi \) considered as a map from \( S^1 \times \partial M(\mathbb{Z}_m, 2n - 1) \) extends over \( S^1 \times M(\mathbb{Z}_m, 2n - 1) \).

**Proof.** Replacing \( \psi \) by a homotopic map assume that \( \psi \) factors through \( \frac{S^1 \times S^{2n-1}}{S^0 \times S^{2n-1}} \). Represent \( S^1 \) as the quotient space of \( I = [0, 1] \) under the projection sending the end-points \( \partial I \) of \( I \) to \( S^0 \) and consider the induced projection from the \((2n)\)-sphere \( \frac{I \times S^{2n-1}}{\partial I \times S^{2n-1}} \) to \( \frac{S^1 \times S^{2n-1}}{S^0 \times S^{2n-1}} \).

Then this projection followed by the map induced by \( \psi \) from \( \frac{S^1 \times S^{2n-1}}{S^0 \times S^{2n-1}} \) to \( L_m^{[2n]} \) lifts to a map \( \psi_l : \frac{I \times S^{2n-1}}{\partial I \times S^{2n-1}} \longrightarrow \tilde{L}_m^{[2n]} \) to the universal cover \( \tilde{L}_m^{[2n]} \) of \( L_m^{[2n]} \). Denote by \( g \in \mathbb{Z}_m \) the element of the fundamental group \( \mathbb{Z}_m \) of \( \tilde{L}_m^{[2n]} \) represented by the map \( \psi \) restricted to
$S^1 \times S^0$ with $S^0 \times S^0$ and $\psi(S^0 \times S^0)$ being the base points in $S^1 \times S^0$ and $L_m^{[2n]}$ respectively and recall that $g$ is a generator of $\mathbb{Z}_m$.

Represent $M(\mathbb{Z}_m, 2n - 1)$ as the quotient space of a $(2n)$-ball $B$ under the projection from $B$ to $M(\mathbb{Z}_m, 2n - 1)$ sending $\partial B = S^{2n-1}$ to $\partial M(\mathbb{Z}_m, 2n - 1) = S^{2n-1}$ by a map of degree $m$. Consider the induced projection from $\frac{I \times B}{\partial I \times \partial B}$ to $\frac{I \times M(\mathbb{Z}_m, 2n-1)}{\partial I \times M(\mathbb{Z}_m, 2n-1)}$ and denote by $\psi_B : \frac{I \times \partial B}{\partial I \times \partial B} \to \tilde{L}_m^{[2n]}$ this projection restricted to $\frac{I \times \partial B}{\partial I \times \partial B}$ and followed by the map $\psi$. Then the problem of extending $\psi$ reduces to the problem of extending $\psi_B$ to a map $\psi_B' : \frac{I \times \partial B}{\partial I \times \partial B} \to \tilde{L}_m^{[2n]}$ so that for every $x \in B/\partial B$ and $(0, x), (1, x) \in \partial I \times (B/\partial B) = \frac{\partial I \times B}{\partial I \times \partial B}$ we have that $\psi_B'(1, x) = g(\psi_B'(0, x))$ with the element $g$ of the fundamental group of $L_m^{[2n]}$ being considered as acting on $\tilde{L}_m^{[2n]}$.

Note that $S^2_{\#} = \frac{I \times \partial B}{\partial I \times \partial B}$ is a $(2n)$-sphere and $\psi_{\#} = \psi_B|S^2_{\#} : S^2_{\#} \to \tilde{L}_m^{[2n]}$ factors through a map of degree $m$ from $S^2_{\#}$ to $S^2_m = \frac{I \times \partial M(\mathbb{Z}_m, 2n-1)}{\partial I \times \partial M(\mathbb{Z}_m, 2n-1)}$. Also note that under the projection of $I \times B$ to $\frac{I \times \partial B}{\partial I \times \partial B}$ the $(2n)$-sphere $S^2_{\#} = \partial(I \times B)$ goes to the space $\frac{I \times B}{\partial I \times \partial B}$ which is the union of the spheres $S^2_0 = \{0\} \times (B/\partial B)$, $S^2_1 = \{1\} \times (B/\partial B)$ and $S^2_{\#}$ so that $S^2_0$ and $S^2_1$ are disjoint and each of them intersects $S^2_{\#}$ at only one point. And finally note that $\tilde{L}_m^{[2n]}$ is homotopy equivalent to a bouquet of $(2n)$-spheres. Then the problem of extending $\psi_B$ to $\psi_B'$ boils down to constructing a map $\psi_0 : S^2_0 \to \tilde{L}_m^{[2n]}$ so that $(\psi_0)_*(\alpha) - g_*((\psi_0)_*(\alpha)) + (\psi_{\#})_*((\psi_0)_*(\alpha)) = 0$ in the homology group $H_{2n}(\tilde{L}_m^{[2n]})$ with $\alpha$ and $\beta$ being the generators of $H_{2n}(S^2_0)$ and $H_{2n}(S^2_{\#})$ determined by the orientations of $S^2_0$ and $S^2_{\#}$ induced by an orientation of $S^2_{\#}$.

Recall that $\gamma = (\psi_0)_*(\beta)$ is divisible by $m$ since $\psi_0$ factors through a map of degree $m$ of a $(2n)$-sphere and $g$ comes from an orientation preserving orthogonal transformation of the $(2n+1)$-sphere $\tilde{L}_m^{[2n+1]}$. Consider the cellular homology of $\tilde{L}_m^{[2n]}$, fix an oriented $(2n)$-cell $C$ of $\tilde{L}_m^{[2n]}$ and index the $(2n)$-cells $C_1, C_2, \ldots, C_m$ of $\tilde{L}_m^{[2n]}$ so that $C_i = g^i(C), 1 \leq i \leq m$. Let $\gamma_1 C_1 + \gamma_2 C_2 + \cdots + \gamma_m C_m, \gamma_1 + \cdots + \gamma_m = 0, \gamma_i \in \mathbb{Z}$, be the cycle representing $\gamma$ and $y_1 C_1 + \cdots + y_m C_m, y_1 + \cdots + y_m = 0, y_i \in \mathbb{Z}$, the cycle representing $y = (\psi_0)_*(\alpha)$. Then $g_* (y)$ is represented by the cycle $y_m C_1 + y_1 C_2 + \cdots + y_{m-1} C_m$ and we arrive at the system of linear equations over $\mathbb{Z}$:

\[
\begin{cases}
\gamma_1 + \cdots + \gamma_m = 0 \\
y_1 + \cdots + y_m = 0 \\
y_1 - y_m + \gamma_1 = 0 \\
y_2 - y_1 + \gamma_2 = 0 \\
\cdots \\
y_m - y_{m-1} + \gamma_m = 0.
\end{cases}
\]
Representing \( y_m = -y_1 - y_2 - \cdots - y_{m-1} \) get

\[
\begin{align*}
2y_1 + y_2 + \cdots + y_{m-1} + \gamma_1 &= 0 \\
y_2 - y_1 + \gamma_2 &= 0 \\
\vdots \\
y_{m-1} - y_{m-2} + \gamma_{m-1} &= 0.
\end{align*}
\]

Eliminating \( y_1, \ldots, y_{m-2} \) from the first equation get

\[
y_{m-1} = -\frac{1}{m} (\gamma_1 + 2\gamma_2 + 3\gamma_3 + \cdots + (m-1)\gamma_{m-1})
\]

and find \( y_{m-2}, y_{m-3}, \ldots, y_1 \) from the remaining equations. Recall that \( \gamma \) is divisible by \( m \) and hence \( \gamma_1, \ldots, \gamma_m \) are divisible by \( m \) as well. Thus we conclude that the system is solvable over \( \mathbb{Z} \). Set \( \psi_0 \) to be a map with \( (\psi_0)_*(\alpha) = y \) and the proposition follows.  

**Proposition 4.2** Let \( \psi : \frac{L_m[2] \times S^{2n-1}}{L_m[0] \times S^{2n-1}} \longrightarrow \frac{L_m[2n+1]}{L_m[0] \times M(Z_m, 2n-1)} \), \( n \geq 1 \), be a map. Then \( \psi \) considered as a map from \( \frac{L_m[2] \times \partial M(Z_m, 2n-1)}{L_m[0] \times \partial M(Z_m, 2n-1)} \) extends over \( \frac{L_m[2] \times M(Z_m, 2n-1)}{L_m[0] \times M(Z_m, 2n-1)} \).

**Proof.** Note that \( L_m[1] \subset L_m[2] = \frac{L_m[2] \times S^0}{L_m[0] \times S^0} \subset \frac{L_m[2] \times S^{2n-1}}{L_m[0] \times S^{2n-1}} \), denote by \( g \in \mathbb{Z}_m \) the element of the fundamental group of \( L_m[2n+1] \) represented by \( \psi \) restricted to the circle \( L_m[1] \) and consider \( g \) as an orthogonal transformation acting on the universal cover \( S^{2n+1} / L_m[2n+1] \) of \( L_m[2n+1] \).

Let the projection \( p_I : [0, 1] \longrightarrow L_m[1] \) send the end points \( \partial I \) of \( I \) to \( L_m[0] \). This projection induces a projection of the \((2n)\)-sphere \( \frac{L_m[2] \times S^{2n-1}}{L_m[0] \times S^{2n-1}} \) to \( \frac{L_m[1] \times S^{2n-1}}{L_m[0] \times S^{2n-1}} \) and this projection followed by \( \psi \) lifts to a map \( \psi_I : \frac{L_m[2] \times S^{2n-1}}{L_m[0] \times S^{2n-1}} \longrightarrow \frac{L_m[2n+1]}{L_m[0] \times S^{2n-1}} = S^{2n+1} \). Then \( \psi_I \) factors up to homotopy relative to \( \frac{L_m[2] \times S^{2n-1}}{L_m[0] \times S^{2n-1}} = \partial I \) through the space \( \frac{L_m[1] \times S^{2n-1}}{L_m[0] \times S^{2n-1}} = I \). It implies that \( \psi \) factors up to homotopy through the space \( \frac{L_m[2] \times S^{2n-1}}{L_m[0] \times S^{2n-1}} \). Thus replacing \( \psi \) by a map from the last space we may assume that \( \psi : \frac{L_m[2] \times S^{2n-1}}{L_m[0] \times S^{2n-1}} \longrightarrow \frac{L_m[2n+1]}{L_m[0] \times M} \) and look for an extension of \( \psi \) over the space

\[
\frac{L_m[2] \times M}{L_m[0] \times M \cup L_m[1] \times \partial M}
\]

where we shorten \( M(Z_m, 2n-1) \) and \( \partial M(Z_m, 2n-1) \) to \( M \) and \( \partial M \) respectively.

Represent \( L_m[2] \) as the quotient space of a disk \( D \) under the projection \( p_D : D \longrightarrow L_m[2] \) which sends \( \partial D = S^1 \) to \( L_m[1] = S^1 \) by an \( m \)-fold map and denote \( D^0 = p^{-1}(L_m[0]) \).

Denote

\[
K = \frac{D \times M}{D^0 \times M \cup \partial D \times \partial M}
\]

\[
\partial D K = \frac{\partial D \times M}{D^0 \times M \cup \partial D \times \partial M} \subset K
\]
The projection from $\partial D K = \frac{D \times S^{2n-1}}{\partial D \times S^{2n-1}} = S^{2n+1}_2$ induced by $p_D$ and followed by $\psi$ lifts to a map $\psi_M : \partial M \to L^{[2n+1]}_m = S^{2n+1}$. Consider a rotation of $\partial D$ by the angle $2\pi/m$ under which the map $p_D$ restricted to $\partial D$ is invariant. Then this rotation induces a rotation (homeomorphism) $\omega$ of the space $\partial D K$. Thus the problem of extending $\psi$ reduces to the problem of extending $\psi_M$ to a map $\psi'_M : K \to \tilde{L}^{[2n+1]}_m$ so that $\psi'_M(\omega(x)) = g(\psi_M(x))$ for $x \in \partial D K$.

Note that $\partial D K$ is the union of $(2n+1)$-spheres $S^{2n+1}_1, \ldots, S^{2n+1}_m$ intersecting each other at points of $D^0$ (we consider $D$ as a natural subset of $K$). Also note that $\partial M K \cap \partial D K = \partial D$ and $\partial M K = S^{2n+1}_m$ is a $(2n+1)$-sphere intersecting the spheres $S^{2n+1}_i, 1 \leq i \leq m$, at points of $\partial D$. Clearly $\partial D K$ is invariant under $\omega$ and and the spheres $S^{2n+1}_i$ can be indexed so that $S^{2n+1}_i = \omega^i(S^{2n+1}_m)$.

Consider a projection from a $(2n)$-ball $B$ to $M$ sending $\partial B$ to $\partial M$ by a map of degree $m$. This projection induces a projection $p : D \times B \to K$ from the $(2n+2)$-ball $D \times B$ to $K$ under which the $(2n+1)$-sphere $\partial(D \times B)$ goes to $\partial M K \cup \partial D K$ so that the sphere $S^{2n+1}_i$ is covered $m$-times and each of the spheres $S^{2n+1}_i, 1 \leq i \leq m$, is covered only once. Recall that $S^{2n+1}_i = \omega^i(S^{2n+1}_m), \psi_M(\omega(x)) = g(\psi_M(x))$ and $g$ is an orientation preserving orthogonal transformation of $S^{2n+1}_i = \tilde{L}^{[2n+1]}_m$. Consider the spheres $S^{2n+1}_i, S^{2n+1}_1, \ldots, S^{2n+1}_m$ with the orientation induced by an orientation of the sphere $\partial(D \times B)$ and define a map $\psi_m : S^{2n+1}_m \to S^{2n+1}_i = \tilde{L}^{[2n+1]}_m$ so that deg $\psi_m = -\deg \psi_M|\tilde{S}^{2n+1}_m$ and $\psi_m$ extends $\psi_M$ restricted to $S^{2n+1}_m$. Now define $\psi_i = g^i \circ \psi_m \circ \omega^{-i} : S^{2n+1}_i \to \tilde{L}^{[2n+1]}_m$. Thus we have extended $\psi_M$ over $\partial M K \cup \partial D K$ so that that the map $p$ restricted to $\partial(D \times B)$ and followed by this extension is of degree 0 and hence extends to a map from $D \times B$ to $\tilde{L}^{[2n+1]}_m$. Clearly the last extension induces a map $\psi'_M : K \to \tilde{L}^{[2n+1]}_m$ with the required properties and the proposition follows. □

5 Pushing maps off the $(2n+1)$-skeleton of $L_m$

In this section we will prove

Proposition 5.1 Let $X$ be a metrizable space with dim$_Z X \leq 2n - 1, n \geq 2$, and let $f : X \to L^{[2n+1]}_m$ be a map. Then there is a map $f' : X \to L^{[2n]}_m$ which coincides with $f$ on $f^{-1}(L^{[2n-1]}_m)$.

The proof of Proposition 5.1 is based on a modification of $L^{[2n+1]}_m$. This modification is defined for $n \geq 1$ and will be referred to as the basic modification of $L^{[2n+1]}_m$. We describe this modification in such a way and using such notations that it can be used in Section 6 for constructing a similar modification of $L^{[2n+2]}_m$.

Consider the covering model of $L^{[2n+1]}_m$. Let $\mathbb{R}^{2n+2} = \mathbb{R}^{2n} \oplus \mathbb{R}^2$ and let $S^{2n+1}, S^{2n-1}$ and $S^3_1$ be the unit spheres and the unit circle in $\mathbb{R}^{2n+2}, \mathbb{R}^{2n}$ and $\mathbb{R}^2$, respectively. Fix
a sufficiently small $\epsilon > 0$ and take a closed $\epsilon$-neighborhood $E_S^1$ of $S^1_\perp$ in $S^{2n+1}$, such that $E_S^1$ does not intersect $S^{2n-1}$. Clearly $E_S^1$ is invariant under the action of $Z_m$ on $S^{2n+1}$ and $E_S^1$ can be considered as a trivial $(2n)$-ball bundle over $S^1_\perp$, with respect to the group $SO(2n)$ of orientation preserving orthogonal transformations of a $(2n)$-ball. The bundle $E_S^1$ over $S^1_\perp$ can be visualized as follows. Take a point $a \in S^1_\perp$ and consider the unit sphere $S^2_{a \perp}$ in the linear $(2n)$-dimensional subspace of $\mathbb{R}^{2n+2}$ containing $S^{2n-1}$ and $a$. Then the closed $\epsilon$-neighborhood of $a$ in $S^2_{a \perp}$ will be the $(2n)$-ball over $a$ in the bundle $E_S^1$. The sphere $S^{2n-1}$ divides $S^{2n}$ into two hemispheres and for the hemisphere $C_a$ containing the point $a$ we consider the natural deformation retraction of $C_a \setminus \{a\}$ to $S^{2n-1}$ along the shortest arcs in $S^2_{a \perp}$ connecting $a$ with the points of $S^{2n-1}$. Then this retraction induces the corresponding deformation retraction $r_S^1 : S^{2n+1} \setminus S^1_\perp \longrightarrow S^{2n-1}$ which commutes with the transformations of $Z_m$. Note that from this description of the bundle it can be seen that the transformations in $Z_m$ induce bundle maps of $E^1_S$. Let $p : S^{2n+1} \longrightarrow L^{[2n+1]}_m = S^{2n+1} / Z_m$ be the projection. Denote $S^1_\perp = p(S^1_\perp) = S^1_\perp / Z_m$ and $E^1 = p(E^1_S)$. Then $E^1$ is a trivial $(2n)$-ball bundle over $S^1_\perp$ (since $E^1_S$ is a bundle with respect to the orientation preserving orthogonal transformations) and $r^1_S$ induces the deformation retraction $r^1 : L^{[2n+1]}_m \setminus S^1_\perp \longrightarrow L^{[2n-1]}_m$. Represent $E^1 = S^1 \times B$ where $B$ is an $(2n)$-ball and denote $\partial E^1 = S^1 \times \partial B = S^1 \times S^{2n-1}$. By $S^0$ we denote a singleton in a sphere $S^k$. Note that, since $r^1$ is a deformation retraction, $r^1$ sends the circle $S^1 \times S^0 \subset \partial E^1$ to a circle in $L^{[2n-1]}_m$ homotopic to $S^1 \times S^0$ in $L^{[2n+1]}_m$. On the other hand $S^1 \times S^0$ homotopic to the circle $S^1_\perp$ which represents a generator of the fundamental group of $L^{[2n]}_m$ and hence represents a generator of the fundamental group of $L^{[2n]}_m$ as well. Also note that $S^0 \times S^{2n-1} \subset \partial E^1$ is contractible in the ball $E^1 \cap L^{[2n]}_m$. Thus $r^1$ restricted to $S^1 \times S^0$ and $S^0 \times S^{2n-1}$ and followed by the inclusion of $L^{[2n-1]}_m$ into $L^{[2n]}_m$ represent a generator of the fundamental group of $L^{[2n]}_m$ and a null-homotopic map to $L^{[2n]}_m$ respectively.

By the basic surgery of $L^{[2n+1]}_m$ we mean replacing $E = E^1 = S^1 \times B$ with $E_M = S^1 \times M(\mathbb{Z}_m, 2n-1)$ such that $\partial E = S^1 \times \partial B$ is identified with $\partial E_M = S^1 \times \partial M(\mathbb{Z}_m, 2n-1)$ through an identification of $\partial M(\mathbb{Z}_m, 2n-1) = S^{2n-1}$ with $\partial B = S^{2n-1}$. The basic modification $M$ of $L^{[2n+1]}_m$ is the space obtained from $L^{[2n+1]}_m$ by the basic surgery of $L^{[2n+1]}_m$. Clearly $L^{[2n-1]}_m$ remains untouched in $M$.

The basic surgery of $L^{[2n+1]}_m$ can be even easier described in the ball model of $L^{[2n+1]}_m$. In this model the set $E^1$ is represented by the closed $\epsilon$-neighborhood of $\mathbb{R}_\perp \cap \mathbb{B}^{2n+1}$ in $\mathbb{B}^{2n+1}$ and the retraction $r^1$ is represented by the natural retraction from $\mathbb{B}^{2n+1} \setminus \mathbb{R}$ to $S^{2n-1} = \mathbb{R}^{2n} \cap \partial \mathbb{B}^{2n+1}$ which sends $(x, t) \in \mathbb{B}^{2n+1}$ with $\|x\| > 0$ to the point $(x, \|x\|, 0)$ in $S^{2n-1}$. We described in detail the basic surgery of $L^{[2n-1]}_m$ in the covering model because, as we mentioned before, this description will be used in Section 6.

**Proposition 5.2** The identity map of $L^{[2n-1]}_m$, $n \geq 1$, extends to a map $\alpha : M \longrightarrow L^{[2n]}_m$ from the basic modification $M$ of $L^{[2n+1]}_m$ to $L^{[2n]}_m$ so that $\alpha$ restricted to $S^0 \times S^{2n-1} \subset S^1 \times S^{2n-1} = \partial E_M$ is null-homotopic where $S^0$ is a singleton in $S^1$. 

9
**Proof.** Recall that \( \partial E = S^1 \times S^{2n-1} \) and \( r^1 \) restricted to \( S^1 \times S^0 \) and \( S^0 \times S^{2n-1} \) and followed by the inclusion of \( L_m^{[2n-1]} \) into \( L_m^{[2n]} \) represent a generator of the fundamental group of \( L_m^{[2n]} \) and a null-homotopic map to \( L_m^{[2n]} \) respectively. Then, By Proposition 5.1 the map \( r^1 \) restricted to \( \partial E = \partial E_M \) extends over \( E_M \) as a map to \( L_m^{[2n]} \) and this extension together with \( r^1 \) restricted to \( L_m^{[2n+1]} \setminus (E \setminus \partial E) \) provides the map required in the proposition. ■

**Proof of Proposition 5.1** Consider the basic modification \( M \) of \( L_m^{[2n+1]} \). By Theorem 1 e-dim \( X \leq M(\mathbb{Z}_m, 2n - 1) \). Recall that \( \partial E = \partial E_M \subset E_M = S^1 \times M(\mathbb{Z}_m, 2n - 1) \). Then if \( f \) restricted to \( f^{-1}(\partial E) \) and followed by the projection of \( E_M = S^1 \times M(\mathbb{Z}_m, 2n - 1) \) to \( M(\mathbb{Z}_m, 2n - 1) \) extends over \( f^{-1}(E) \) as a map to \( M(\mathbb{Z}_m, 2n - 1) \) and hence \( f \) restricted to \( f^{-1}(\partial E) \) extends over \( f^{-1}(E) \) as a map to \( E_M \). The last extension together with \( f \) provides a map \( f_M : X \rightarrow M \) which coincides with \( f \) on \( f^{-1}(L_m^{[2n-1]}) \). By Proposition 6.2, take a map \( \alpha : M \rightarrow L_m^{[2n]} \) which extends the identity map of \( L_m^{[2n-1]} \). Then \( f' = \alpha \circ f_M : X \rightarrow L_m^{[2n]} \) is the map required in the proposition. ■

6 Pushing maps off the \((2n + 2)\)-skeleton of \( L_m \)

In this section we will prove

**Proposition 6.1** Let \( X \) be a metrizable space with \( \text{dim}_{\mathbb{Z}_m} X \leq 2n, n \geq 1 \), and let \( f : X \rightarrow L_m^{[2n+2]} \) be a map. Then there is a map \( f' : X \rightarrow L_m^{[2n+1]} \) which coincides with \( f \) on \( f^{-1}(L_m^{[2n]}) \).

The proof of Proposition 6.1 is based on a modification of \( L_m^{[2n+2]} \). This modification is defined for \( n \geq 1 \) and will be referred to as the basic modification of \( L_m^{[2n+2]} \). Consider the ball model of \( L_m^{[2n+2]} \) with the projection \( p : \mathbb{B}^{2n+2} \rightarrow L_m^{[2n+2]} \) where \( \mathbb{B}^{2n+2} \) is the unit ball in \( \mathbb{R}^{2n+2} = \mathbb{R}^n \oplus \mathbb{R}^2 \). Denote by \( S^{2n+1}, S^1 \) and \( \mathbb{B}^2 \) the unit sphere, the unit circle and the unit ball in \( \mathbb{R}^{2n+2} \) and \( \mathbb{R}^2 \) respectively. Also denote \( L_m^{[2n+1]} = p(S^{2n+1}) \) and \( (L_m^2)_{\perp} = p(\mathbb{B}^2) \), and let \( L_m^{[2n]} \subset L_m^{[2n+1]} \) be the \((2n)\)-skeleton of \( L_m^{[2n+1]} \) constructed as described in Section 2.

Consider the construction of the basic modification of \( L_m^{[2n+1]} \) as described in Section 5. Extend the neighborhood \( E^1_0 \) of \( S^1 \) in \( S^{2n+1} \) to a neighborhood \( E^3_0 \) of \( \mathbb{B}^2 \) in \( \mathbb{B}^{2n+2} \) and extend the \((2n)\)-ball \( SO(2n)\)-bundle structure of \( E^1_0 \) over \( S^1 \) to a \((2n)\)-ball \( SO(2n)\)-bundle structure of \( E^3_0 \) over \( \mathbb{B}^2 \) so that the transformations of \( \mathbb{Z}_m \) on \( \mathbb{R}^{2n+2} \) induce bundle maps of \( E^3_0 \). Then the neighborhood \( E^2 = p(E^3_0) \) of \( (L_m^2)_{\perp} = p(\mathbb{B}^2) \) in \( L_m^{[2n+2]} \) will have the induced \((2n)\)-ball \( SO(2n)\)-bundle structure over \((L_m^2)_{\perp}\). The retraction \( r^1_S : S^{2n+1} \setminus S^1 \rightarrow S^{2n-1} \) can be extended to a retraction \( r^2_S : \mathbb{B}^{2n+2} \setminus \mathbb{B}^2 \rightarrow \mathbb{S}^{2n-1} \) which induces the retraction \( r^2 : L_m^{[2n+2]} \setminus (L_m^2)_{\perp} \rightarrow L_m^{[2n-1]} \).

Note that \((L_m^0)_{\perp} \cap (L_m^2)_{\perp}\) is a singleton lying in \((L_m^1)_{\perp} = p(S^1) = S^1\). Also note that the pair \(((L_m^2)_{\perp}, (L_m^1)_{\perp})\) is homeomorphic to the pair \((L_m^2, L_m^1)\). In order to simplify
Proposition 2.3 embed the space \( E \) in the notation, from now we will write \( L_m^0, L_m^1, L_m^2 \) instead of \( (L_m^0)_\perp, (L_m^1)_\perp, (L_m^2)_\perp \) keeping in mind that any skeleton whose dimension does not depend on \( n \) should be interpreted as having the subscript \( \perp \).

Let \( E^0 = E^2 \cap L_m^{2n} \) be the fiber of the bundle \( E^2 \) over the point \( L_m^0 \). Denote by \( \partial E^0 \) the boundary of the ball \( E^0 \) and by \( \partial E^2 \) the induced \( S^{2n-1} \)-bundle formed by the boundaries of the fibers of \( E^2 \) which are \((2n)\)-balls. Consider the retraction \( r^2 \) as a map to \( L_m^{2n-1} \) followed by the inclusion into \( L_m^{2n} \). Note that \( r^2 \) extends \( r^1 \) and recall that \( r^1 \) is a deformation retraction on \( L_m^{2n} \setminus L_m^1 \). Then, by Proposition 2.5, \( r^2 \) can be homotoped into a map \( r^2 : L_m^{2n+2} \setminus L_m^2 \to L_m^{2n} \) which does not move the points of \( L_m^{2n} \setminus L_m^2 = L_m^{2n} \setminus L_m^1 \). Thus we can define the map \( r^2_0 : (L_m^{2n+2} \setminus (E^2 \setminus \partial E^2)) \cup E^0 \to L_m^{2n} \) which coincides with \( r^2 \) on \( L_m^{2n+2} \setminus (E^2 \setminus \partial E^2) \) and does not move the points of \( E^0 \). Take a neighborhood \( U \) of \( E^0 \) in \( E \) and extend \( r^2_0 \) to a map \( r^2_U : (L_m^{2n+2} \setminus (E^2 \setminus \partial E^2)) \cup U \to L_m^{2n} \). Consider separately the quotient space \( E = E^2/\partial E^0 \) and consider \( \partial E = \partial E^2/\partial E^0 \) as a subspace of \( E \). By Proposition 2.3, embed the space \( E \) into \( L_m^{2n+2} \) so that \( E \subset E^2 \), \( E \cap L_m^{2n} = L_m^0 = \) the singleton \( E^0 \) in \( E \) and \( E^2 \setminus E \subset U \). Thus we have that \( r^2_U \) is defined on \( L_m^{2n+2} \setminus (E \setminus \partial E) \), \( L_m^{2n} \subset L_m^{2n+2} \setminus (E \setminus \partial E) \) and hence \( r^2_U \) acts on \( L_m^{2n+2} \setminus (E \setminus \partial E) \) as a retraction to \( L_m^{2n} \).

The basic modification of \( L_m^{2n+2}, n \geq 1 \), is defined as follows. By Proposition 2.2 represent \( \partial E = \partial E^2/\partial E^0 \) as
\[
\partial E = \frac{L_m^2 \times S^{2n-1}}{L_m^0 \times S^{2n-1}}.
\]
Denote
\[
E_M = \frac{L_m^2 \times M(Z_m, 2n-1)}{L_m^0 \times M(Z_m, 2n-1)}, \quad \partial E_M = \frac{L_m^2 \times \partial M(Z_m, 2n-1)}{L_m^0 \times \partial M(Z_m, 2n-1)}
\]
and consider \( \partial E_M \) as a subset of \( E_M \). By the basic surgery of \( L_m^{2n+2} \) we mean replacing \( E \) with \( E_M \) such that \( \partial E_M \) is identified with \( \partial E \) through an identification of \( \partial M(Z_m, 2n-1) \) with \( S^{2n-1} \). By the basic modification \( M \) of \( L_m^{2n+2} \) we mean the space obtained from \( L_m^{2n+2} \) by the basic surgery of \( L_m^{2n+2} \). Clearly \( L_m^{2n} \) remains untouched in \( M \).

**Proposition 6.2** The identity map of \( L_m^{2n}, n \geq 1 \), extends to a map from the basic modification \( M \) of \( L_m^{2n+2} \) to \( L_m^{2n+1} \).

**Proof.** By Proposition 4.2 the map \( r^2_U \) restricted to \( \partial E = \partial E_M \) extends over \( E_M \) and this extension together with \( r^2_U \) restricted to \( L_m^{2n+2} \setminus (E \setminus \partial E) \) provides the map required in the proposition. ■

**Proof of Proposition 6.1.** Consider the basic modification \( M \) of \( L_m^{2n+2} \). By Theorem 11 e-dim \( X \leq \Sigma M(Z_m, 2n-1) \). Recall that \( \partial E = \partial E_M \subset E_M = \frac{L_m^2 \times M(Z_m, 2n-1)}{L_m^0 \times M(Z_m, 2n-1)} \).

Then, by Proposition 2.5 \( f \) restricted to \( f^{-1}(\partial E) \) extends over \( f^{-1}(E) \) as a map to \( E_M \) and this extension together with \( f \) provides a map \( f_M : X \to M \) which coincides with
Proposition 7.3. Let \( f \) be a map \( f : M \to L_{m}^{[2n]} \). By Proposition 6.2, take a map \( \alpha : M \to L_{m}^{[2n+1]} \) which extends the identity map of \( L_{m}^{[2n]} \). Then \( f' = \alpha \circ f : X \to L^{[2n+1]} \) is the map required in the proposition. ■

7 Pushing maps off the 3-skeleton of \( L_{m} \)

The goal of this section is to prove Theorem 1.3. Clearly Propositions 5.1 and 6.1 imply

**Theorem 7.1** Let \( X \) be a finite dimensional metrizable space with \( \dim_{\mathbb{Z}} X \leq 2 \) and let \( f : X \to L_{m}^{[n]}, n \geq 3, \) be a map. Then there is a map \( f' : X \to L_{m}^{[3]} \) which coincides with \( f \) on \( f^{-1}(L_{m}^{[2]}) \).

An easy corollary of Theorem 7.1 is

**Corollary 7.2** Let \( X \) be a finite dimensional metrizable space with \( \dim_{\mathbb{Z}} X \leq 1 \) and \( f_{F} : F \to L_{m}^{[2]} \) a map from a closed subset \( F \) of \( X \). Then \( f_{F} \) extends to a map \( f : X \to L_{m}^{[3]} \).

**Proof.** Since \( L_{m} = K(\mathbb{Z}, 1) \) we have \( e\dim X \leq L_{m} \). Then \( f_{F} \) extends to a map \( f : X \to L_{m} \). Since \( X \) is finite dimensional we can assume that there is \( n \) such that \( f(X) \subset L_{m}^{[n]} \). Then, by Theorem 7.1 one can replace \( f \) by a map to \( L_{m}^{[3]} \) which coincides with \( f_{F} \) on \( F \) and the corollary follows. ■

Thus the only missing part of proving Theorem 1.3 is to push maps from \( L_{m}^{[3]} \) to \( L_{m}^{[2]} \). We will do this in two steps. The first one is

**Proposition 7.3** Let \( X \) be a finite dimensional metrizable space with \( \dim_{\mathbb{Z}} X \leq 1 \) and let \( f : X \to L_{m}^{[3]} \) be a map. Then there is a map \( f' : X \to L_{m}^{[2]} \) which coincides with \( f \) on \( f^{-1}(L_{m}^{[1]}) \).

**Proof.** Consider the basic modification \( M \) of \( L_{m}^{[3]} \). Recall that \( M \) is obtained from \( L_{m}^{[3]} \) by the basic surgery which replaces \( E = S^{1} \times B \subset L_{m}^{[3]} \) with \( E_{M} = S^{1} \times M(\mathbb{Z}, 1) = S^{1} \times L_{m}^{[2]} \) by identifying \( \partial E = S^{1} \times \partial B = S^{1} \times S^{1} \) with \( \partial E_{M} = S^{1} \times L_{m}^{[1]} = S^{1} \times S^{1} \). Also recall that \( L_{m}^{[1]} \) remains untouched in \( M \) and does not meet \( E_{M} \).

Enlarge \( M \) to the space \( M^{+} \) by enlarging \( E_{M} = S^{1} \times L_{m}^{[2]} \) to \( E_{M}^{+} = S^{1} \times L_{m}^{[3]} \). Apply again Corollary 7.2 to the projection of \( S^{1} \times L_{m}^{[3]} \) to \( L_{m}^{[3]} \) to extend the map \( f \) restricted to \( f^{-1}(E) \) over \( f^{-1}(E) \) as a map to \( S^{1} \times L_{m}^{[3]} \) and this way to get a map \( f^{+} : X \to M^{+} \) which differs from \( f \) only on \( f^{-1}(E) \).

The space \( M^{++} \) is obtained from \( M^{+} \) by replacing \( E_{M}^{+} = S^{1} \times L_{m}^{[3]} \) with \( E_{M}^{++} = S^{1} \times M \) by identifying \( L_{m}^{[1]} \) in \( L_{m}^{[3]} \) with \( L_{m}^{[1]} \) in \( M \). Thus \( M^{++} \) differs from \( M^{+} \) on the set \( E_{M}^{++} = S^{1} \times S^{1} \times L_{m}^{[2]} \subset M^{++} \).

By Proposition 6.2 the identity map of \( L_{m}^{[1]} \) extends to a map \( \alpha : M \to L_{m}^{[2]} \) which is null-homotopic on \( S^{0} \times L_{m}^{[1]} \subset E_{M} \) where \( S^{0} \) is a singleton in \( S^{1} \). Then we get that the map
\[id \times \alpha : E_M^{++} = S^1 \times M \rightarrow S^1 \times L_m^3 \subset E_M^{++} \text{ induces } \beta : M^{++} \rightarrow M^+ \text{ so that } \beta(M^{++}) \subset M \subset M^+.\]

Consider the map \( \gamma = \alpha \circ \beta : M^{++} \rightarrow L_m^2 \) and note that \( \gamma \) restricted to \( S^0 \times S^0 \times L_m^2 \subset E_M^{++} \) is the composition of the maps \( S^0 \times S^0 \times L_m^2 \rightarrow S^0 \times L_m^2 \rightarrow L_m^2 \) each of them acting as \( \alpha \) restricted to \( S^0 \times L_m^2 \subset E_M \). Hence, by Proposition 2.4 \( \gamma \) restricted to \( S^0 \times S^0 \times L_m^2 \) is null-homotopic. Then, by Proposition 2.5 \( f^+ \) restricted to \( (f^+)^{-1}(\partial E_M^{++}) \) and followed by \( \gamma \) for \( \partial E_M^{++} = S^1 \times S^1 \times L_m^1 \subset E_M^{++} \) extends over \( (f^+)^{-1}(E_M^{++}) \) and this extension provides a map \( f' : X \rightarrow L_m^2 \) that coincides with \( f \) on \( f^{-1}(L_m^1) \). The proposition is proved. \[\blacksquare\]

**Proposition 7.4** Let \( X \) be a finite dimensional metrizable space with \( \dim_{\mathbb{Z}} X \leq 1 \) and let \( f : X \rightarrow L_m^3 \) be a map. Then there is a map \( f' : X \rightarrow L_m^2 \) which coincides with \( f \) on \( f^{-1}(L_m^2) \).

**Proof.** Consider the ball model of \( L_m^3 \) and let \( p : \mathbb{B}^3 \rightarrow L_m^2 \). In the decomposition \( \mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}_z \) we will refer to \( \mathbb{R}^2 \) and \( \mathbb{R}_z \) as the \( xy \)-coordinate plane and the \( z \)-axis respectively. By a rotation of \( \mathbb{B}^3 \) we mean an orthogonal rotation around the \( z \)-axis. Clearly a rotation \( \phi \) of \( \mathbb{B}^3 \) induces a homeomorphism \( \phi_L \) of \( L_m^3 \) which will be called the induced rotation of \( L_m^3 \). Note that the rotations of \( \mathbb{B}^3 \) and \( L_m^2 \) commute with the projection \( p \). By this we mean that \( p \circ \phi = \phi_L \circ p \). Also note that \( L_m^2 \) is invariant under the induced rotations of \( L_m^3 \).

Take a disk \( B \) of radius 1/3 lying in the \( xz \)-coordinate plane and centered at the point \((1/2, 0, 0) \in \mathbb{R}^3 \). Denote by \( E = S^1 \times B \) the solid torus obtained by rotating \( B \) around the \( z \)-axis and denote by \( \partial E = S^1 \times \partial B = S^1 \times S^1 \subset E \) the boundary of \( E \). Clearly \( E \) can be considered as subsets of \( L_m^2 \).

Let \( \mathcal{I} = \mathbb{B}^3 \cap \mathbb{R}_z \) be the \([-1, 1]\)-interval of the \( z \)-axis and denote \( \partial \mathcal{I} = \{-1, 1\} \subset \mathbb{R}_z \) the end points of \( \mathcal{I} \). Consider an obvious retraction \( r_\mathcal{B} : \mathbb{B}^3 \setminus (E \setminus \partial E) \rightarrow \mathcal{I} \cup \partial \mathbb{B}^3 = \mathcal{I} \cup S^2 \) such that \( r_\mathcal{B} \) commutes with the rotations of \( \mathbb{B}^3 \) and consider the map \( \gamma : \mathcal{I} \cup \partial \mathbb{B}^3 \rightarrow L_m^2 \) such that \( \gamma \) coincides on \( S^2 \) with \( p \) and \( \gamma \) sends \( \mathcal{I} \) to the point \( p(\partial \mathcal{I}) \). Note that \( r_\mathcal{B} \) followed by \( \gamma \) induces the retraction \( r : L_m^3 \setminus (E \setminus \partial E) \rightarrow L_m^3 \) such that \( r \) commutes with the rotations of \( L_m^3 \).

Consider the surgery of \( L_m^3 \) which replaces \( E = S^1 \times B \) with \( E_M = S^1 \times L_m^2 \) by identifying the boundary \( \partial B = S^1 \) of \( B \) with the 1-skeleton \( L_m^1 = S^1 \times L_m^2 \). Denote by \( M \) the space obtained from \( L_m^3 \) by this surgery. Clearly the 2-skeleton \( L_m^2 \) of \( L_m^3 \) remains untouched in \( M \). Note that this surgery and this modification are different from the basic surgery and the basic modification of \( L_m^3 \) considered before.

Observe that any map from \( L_m^1 \) to \( L_m^2 \) extends over \( L_m^2 \). Fix a singleton \( S^0 \) in \( S^1 \) and extend \( r \) restricted to \( S^0 \times L_m^2 \) to a map \( r^0 : S^0 \times L_m^2 \rightarrow L_m^2 \). Then, since \( r \) commutes with the rotations of \( L_m^3 \), \( r^0 \) can be extended by the rotations of \( L_m^3 \) to a map \( r^1 : E_M = S^1 \times L_m^2 \rightarrow L_m^2 \) so that \( r^1 \) extends \( r \) restricted to \( \partial E \). Thus \( r^1 \) together with \( r \) provide a retraction \( \beta : M \rightarrow L_m^2 \) which extends the identity map of \( L_m^2 \).

Now denote \( F = f^{-1}(\partial E) \) and consider the restriction \( f|F : F \rightarrow \partial E = S^1 \times S^1 = S^1 \times L_m^1 \). Then, by Corollary 7.2 and Proposition 7.3 \( f|F \) followed by the projection
to $L^{[1]}_m$ extends over $f^{-1}(E)$ as a map to $L^{[2]}_m$ and this extension provides an extension $f_E : f^{-1}(E) \to E = S^1 \times L^{[2]}_m$ of $f|F$ over $f^{-1}(E)$. Thus we get a map $f_M : X \to M$ which coincides with $f$ on $f^{-1}(L^{[2]}_m)$. Set $f' = \beta \circ f_M : X \to L^{[2]}_m$ and the proposition follows. ■

**Proof of Theorem 1.3.** Theorem 1.3 follows from Corollary 7.2 and Proposition 7.4. ■

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