Contact resolutions of projectivised nilpotent orbit closures

Baohua Fu

July 16, 2018

Abstract

The projectivised nilpotent orbit closure $\mathbb{P}(O)$ carries a natural contact structure on its smooth part, which is induced by a line bundle $L$ on $\mathbb{P}(O)$. A resolution $\pi : X \to \mathbb{P}(O)$ is called contact if $\pi^*L$ is a contact line bundle on $X$. It turns out that contact resolutions, crepant resolutions and minimal models of $\mathbb{P}(O)$ are all the same. In this note, we determine when $\mathbb{P}(O)$ admits a contact resolution, and in the case of existence, we study the birational geometry among different contact resolutions.

1 Introduction

Recall that a nilpotent orbit $O$ in a semi-simple complex Lie algebra $\mathfrak{g}$ enjoys the following properties:

(i) it is $C^*$-invariant, where $C^*$ acts on $\mathfrak{g}$ by linear scalars;
(ii) it carries the Kirillov-Kostant-Souriau symplectic 2-form $\omega$;
(iii) $\lambda^*\omega = \lambda \omega$ for any $\lambda \in C^*$.

One deduces from (iii) that this symplectic structure on $O$ gives a contact structure on the projectivisation $\mathbb{P}(O)$, which is induced by the line bundle $L := \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{\mathbb{P}(O)}$. When $\mathfrak{g}$ is simple, the variety $\mathbb{P}(O) \subset \mathbb{P}(\mathfrak{g})$ is closed if and only if $O$ is the minimal nilpotent orbit $O_{\text{min}}$ (see for example Prop. 2.6 [Be]). In this case, $\mathbb{P}(O_{\text{min}})$ is a Fano contact manifold. It is generally believed that these are the only examples of such varieties ([Be], [Le]). A positive answer to this would imply that every compact quaternion-Kähler manifold...
with positive scalar curvature is homothetic to a Wolf space (Theorem 3.2 [LeSa]).

If we take the closure $\overline{\mathbb{P}(\mathcal{O})} = \mathbb{P}(\mathcal{O})$, then it is in general singular. We say that a resolution $\pi : X \to \mathbb{P}(\mathcal{O})$ is contact if $\pi^*L$ is a contact line bundle on $X$. It follows that $X$ is a projective contact manifold. Such varieties have drawn much attention recently (see for example [Pe] and the references therein).

The first aim of this note is to find all contact resolutions that $\mathbb{P}(\mathcal{O})$ can have. More precisely we prove that (Theorem 4.5) if the normalization $\tilde{\mathbb{P}(\mathcal{O})}$ of $\mathbb{P}(\mathcal{O})$ is not smooth, then the resolution $X$ is isomorphic to $\mathbb{P}(T^*(G/P))$ for some parabolic sub-group $P$ in the adjoint group $G$ of $\mathfrak{g}$ and $\pi$ is the natural resolution. The proof relies on the main result in [KPSW] and that in [Fu1]. A classification (Corollary 4.6) of $\mathcal{O}$ such that $\mathbb{P}(\mathcal{O})$ admits a contact resolution can be derived immediately, with the help of [Be].

Once we have settled the problem of existence of a contact resolution, we turn to study the birational geometry among different contact resolutions in the last section, where (Theorem 5.2) the chamber structure of the movable cone of a contact resolution is given, based on the main result in [Na]. This gives another way to prove the aforesaid result under the condition that $\mathcal{O}$ admits a symplectic resolution, since minimal models, contact resolutions and crepant resolutions of $\mathbb{P}(\mathcal{O})$ are the same objects (Proposition 3.3).

Acknowledgements: The author would like to express his gratitude to IHES for its hospitality. During my visit there, the discussions with F. Q. Fang and S. S. Roan are the original impetus for this work. I am grateful to O. Biquard for discussions on quaternion-Kähler geometry. I want to thank M. Brion for remarks to a first version of this note, especially for the proof of Proposition 5.1, which ensures the validity of Theorem 5.2 for all simple Lie algebras. I would like to thank the referees for their critics on a previous version.

2 Singularities in $\mathbb{P}(\widetilde{\mathcal{O}})$

Let $\mathfrak{g}$ be a simple complex Lie algebra and $\mathcal{O}$ a nilpotent orbit in $\mathfrak{g}$. The normalization of the closure $\overline{\mathcal{O}}$ will be denoted by $\tilde{\mathcal{O}}$. The scalar $\mathbb{C}^*$-action on $\overline{\mathcal{O}}$ lifts to $\tilde{\mathcal{O}}$. There is only one $\mathbb{C}^*$-fixed point on $\tilde{\mathcal{O}}$, say $o$. We denote by $\mathbb{P}(\tilde{\mathcal{O}})$ the geometric quotient of $\tilde{\mathcal{O}} \setminus \{o\}$ by the $\mathbb{C}^*$-action. Similarly we denote
by \( \mathbb{P}(\tilde{O}) \) the geometric quotient \( \tilde{O} \setminus \{0\} / \mathbb{C}^* \). Note that \( \mathbb{P}(\tilde{O}) \) is nothing but the normalization of \( \mathbb{P}(\tilde{O}) \).

Recall that a contact structure on a smooth variety \( X \) is a corank 1 sub-bundle \( F \subset TX \) such that the O'Neil tensor \( F \times F \to L := TX/F \) is everywhere non-degenerate. In this case, \( L \) is called a contact line bundle on \( X \) and we have \( K_X \simeq L^{-1} \), where \( n = (\dim X - 1)/2 \). If we regard the natural map \( TX \to L \) as a section \( \theta \in H^0(X, \Omega^1_X(L)) \) (called a contact form), then the non-degenerateness is equivalent to the condition that \( \theta \wedge (d\theta)^n \) is nowhere vanishing when considered locally as an element in \( H^0(X, \Omega^{2n+1}_{\mathbb{C}}(L^n+1)) = H^0(X, \mathcal{O}_X) \).

For a point \( v \in O \), the tangent space \( T_vO \) is naturally isomorphic to \([v, g] = \text{Im}(g \cdot ad_v)\). The map \([v, x] \mapsto \kappa(v, x)\) defines a one-form \( \theta' \) on \( O \), where \( \kappa \) is the Killing form of \( g \). Then \( \omega := d\theta' \) is the Kirillov-Kostant-Souriau symplectic form on \( O \). Notice that \( \lambda^* \theta' = \lambda \theta' \) for every \( \lambda \in \mathbb{C}^* \), so it defines an element \( \theta \in H^0(\mathbb{P}(O), \Omega^1_{\mathbb{P}(O)}(L)) \), where \( L \) is the pull-back of \( \mathcal{O}_{\mathbb{P}(g)}(1) \) to \( \mathbb{P}(O) \). This is in fact a contact form, i.e. \( \theta \wedge (d\theta)^\wedge n \) is everywhere non-zero, where \( n = (\dim O - 2)/2 \). Since the codimension of the complement of \( \mathbb{P}(O) \) in \( \mathbb{P}(\tilde{O}) \) is at least 2, \( \theta \) extends to a contact form on the smooth part of \( \mathbb{P}(\tilde{O}) \).

**Remark 2.1.** Let \( G \) be the adjoint group of \( g \). Then the contact structure on \( \mathbb{P}(O) \) is \( G \)-invariant, which is precisely the contact structure on \( \mathbb{P}(\tilde{O}) \) when \( \mathbb{P}(O) \) is viewed as a twistor space of a quaternion-Kähler manifold ([Sw]).

**Proposition 2.1.** The projective variety \( \mathbb{P}(\tilde{O}) \) is projectively normal with only rational Gorenstein singularities.

**Proof.** By abusing the notations, we denote also by \( L \) the pull-back of \( L \) to the normalization \( \tilde{O} \), which is a line bundle. Note that the complement of \( \mathbb{P}(O) \) in \( \mathbb{P}(\tilde{O}) \) has codimension at least 2, so \( K_{\mathbb{P}(\tilde{O})} = L^{-n-1} \) is locally free, which implies that \( \mathbb{P}(\tilde{O}) \) is Gorenstein. Notice that \( \tilde{O} \setminus \{0\} \) has rational singularities by results of Panyushev and Hinich (see [Pa]), so its quotient by the \( \mathbb{C}^* \) action \( \mathbb{P}(\tilde{O}) \) has only rational Gorenstein singularities.

The following proposition is easily deduced from Proposition 5.2 in [Be], which plays an important role to our classification result (Corollary 4.6).

**Proposition 2.2.** Let \( g \) be a simple Lie algebra and \( O \subset g \) a non-zero nilpotent orbit. Then \( \mathbb{P}(\tilde{O}) \) is smooth if and only if either \( O \) is the minimal nilpotent orbit or \( g \) is of type \( G_2 \) and \( O \) is the nilpotent orbit of dimension 8.
Singularities in $\mathbb{P}(\tilde{O})$ are examples of the so-called contact singularities in [CF]. Projectivised nilpotent orbits have already been studied, for example, in [Be] (for relation with Fano contact manifolds), [Ko] (for relation with harmonic maps) and [Sw] (from the twistor aspect). Their closures have also been studied, for example in [Po] (for the self-duality), which give examples of non-smooth, self-dual projective varieties.

3 Minimal models

For a proper morphism between normal varieties $f : X \to W$, we denote by $N_1(f)$ the vector space (over $\mathbb{R}$) generated by reduced irreducible curves contained in fibers of $f$ modulo numerical equivalence. Let $N_1(f)$ be the group $\text{Pic}(X) \otimes \mathbb{R}$ modulo numerical equivalence (w. r. t. $N_1(f)$), then we have a perfect pairing $N_1(f) \times N_1(f) \to \mathbb{R}$.

If $f$ is a resolution, then $X$ is called a minimal model of $W$ if $K_X$ is $f$-nef.

**Proposition 3.1.** Let $W$ be a projective normal variety with only canonical singularities and $f : X \to W$ a resolution. Then $f$ is crepant if and only if $X$ is a minimal model of $W$.

**Proof.** If $f$ is crepant, then $K_X = f^*K_W$, which gives $K_X \cdot [C] = 0$ for every $f$-exceptional curve $C$, so $X$ is a minimal model of $W$.

Suppose $K_X$ is $f$-nef, then so is $K_X - f^*K_W = \sum a_i E_i$, where $E_i$ are exceptional divisors of $f$. By the negativity lemma (see Lemma 13-1-4 [Ma]), $a_i \leq 0$ for all $i$. On the other hand, $W$ has only canonical singularities, so $a_i \geq 0$, which gives $a_i = 0$ for all $i$, thus $f$ is crepant.

**Corollary 3.2.** Let $W$ be a projective normal variety with only terminal singularities and $f : X \to W$ a resolution. Then the following statements are equivalent:

(i) $f$ is crepant;
(ii) $X$ is a minimal model of $W$;
(iii) $f$ is small, i.e. $\text{codim}(\text{Exc}(f)) \geq 2$.

By the previous section, there is a contact structure on $\mathbb{P}(O)$, induced by the line bundle $L$ on $\mathbb{P}(\tilde{O})$. A contact resolution of $\mathbb{P}(\tilde{O})$ is a resolution $\pi : X \to \mathbb{P}(\tilde{O})$ such that $\pi^*L$ is a contact line bundle on $X$. 
Proposition 3.3. Let $\pi : X \to \mathbb{P}(\mathcal{O})$ be a resolution, then the following statements are equivalent:

(i) $\pi$ is crepant;
(ii) $K_X$ is $\pi$-nef;
(iii) $\pi$ is a contact resolution.

Proof. The equivalence between (i) and (ii) follows from Prop. 2.1 and Prop. 3.1. The implication (iii) to (i) is clear from the definitions. Now suppose that $\pi$ is crepant, then $K_X \simeq \pi^*(L^{-(n+1)}) \simeq (\pi^*L)^{-(n+1)}$. Let $\tilde{X}$ be the fiber product $X \times_{\mathbb{P}(\mathcal{O})} (\mathcal{O} \setminus \{o\})$ and $h : \tilde{X} \to \mathcal{O} \setminus \{o\}$ the natural projection. Then $h$ is a resolution of singularities and $h^*\omega$ extends to a 2-form $\tilde{\omega}$ on $\tilde{X}$ since $\mathcal{O} \setminus \{o\}$ has only symplectic singularities, where $\omega$ is the symplectic form on the smooth part of $\mathcal{O}$. $\tilde{X}$ inherits a $\mathbb{C}^*$-action from that on $\mathcal{O}$. Contracting $\tilde{\omega}$ with the vector field generating the $\mathbb{C}^*$-action, we obtain an element $\tilde{\theta} \in H^0(X, \Omega_X \otimes \pi^*L)$. Now it is clear that $\tilde{\theta}$ gives the contact form on $X$ extending $\theta$.

4 Contact resolutions

Let $f : Z \to \mathbb{P}(\mathcal{O})$ be a resolution and let $\hat{Z}$ be the fiber product $Z \times_{\mathbb{P}(\mathcal{O})} W_0$ and $\hat{f} : \hat{Z} \to W_0$ the natural projection, where $W_0 = \mathcal{O} \setminus \{0\}$. Recall that $L$ is the restriction of $\mathcal{O}_{\mathbb{P}(\mathcal{O})}(1)$ to $\mathbb{P}(\mathcal{O})$.

Lemma 4.1. $\hat{Z}$ is isomorphic to the complement of the zero section in the total space of the line bundle $(f^*L)^*$ and $\hat{f}$ is a resolution of singularities.

Proof. This follows from that $W_0$ is naturally isomorphic to the complement of the zero section in $L^*$ and the fiber product $Z \times_{\mathbb{P}(\mathcal{O})} L^*$ is isomorphic to $f^*(L^*) \simeq (f^*L)^*$.

Proposition 4.2. The map $f$ is a contact resolution if and only if $\hat{f}$ is a symplectic resolution.

Proof. Let $\omega$ be the Kostant-Kirillov-Souriau symplectic form on $\mathcal{O}$, then $(\hat{f})^*\omega$ extends to $\tilde{\omega} \in H^0(\hat{Z}, \Omega_{\hat{Z}}^2)$. $\hat{Z}$ admits a $\mathbb{C}^*$-action induced from the one on $W_0$ and for this action, one has $\lambda^*\tilde{\omega} = \lambda \tilde{\omega}$ for all $\lambda \in \mathbb{C}^*$. By contracting $\tilde{\omega}$ with the vector field generating the $\mathbb{C}^*$-action, we obtain a 1-form $\theta'$ on $\hat{Z}$ satisfying $\lambda^*\theta' = \lambda \theta'$, this gives an element $\theta$ in $H^0(\hat{Z}, \Omega_{\hat{Z}}(f^*L))$. Then $\theta$ is a contact form if and only if $\tilde{\omega}$ is a symplectic form.
From now on, we let $\mathcal{O}$ be a nilpotent orbit such that $\mathbb{P}(\overline{\mathcal{O}})$ is singular.

**Proposition 4.3.** Let $\overline{\pi} : X \to \mathbb{P}(\overline{\mathcal{O}})$ be a contact resolution and $\overline{L} = \overline{\pi}^*(L)$ the contact line bundle on $X$. Then $(X, \overline{L})$ is isomorphic to $(\mathbb{P}(T^*Y), \mathcal{O}_{\mathbb{P}(T^*Y)}(1))$ for some smooth projective variety $Y$.

**Proof.** Note that $K_X \simeq \overline{L} - n - 1$, where $n = (\text{dim} \mathcal{O})/2 - 1$. For a curve $C$ in $X$, we have $K_X \cdot C = -(n + 1)L \cdot \overline{\pi}_*[C]$, thus $K_X$ is not nef. By [KPSW], $X$ is either a Fano contact manifold or $(X, \overline{L})$ is isomorphic to $(\mathbb{P}(T^*Y), \mathcal{O}_{\mathbb{P}(T^*Y)}(1))$ for some smooth projective variety $Y$.

The map $\overline{\pi}$ factorizes through the normalization, so we obtain a birational map $\nu : X \to \mathbb{P}(\overline{\mathcal{O}})$. By assumption, $\mathbb{P}(\overline{\mathcal{O}})$ is singular. Zariski’s main theorem then implies that there exists a curve $C$ contained in a fiber of $\nu$. Now $K_X \cdot C = 0$, thus $-K_X$ is not ample, which shows that $X$ is not Fano. $\square$

Let us denote by $\pi_0 : \tilde{X} \to W_0$ the symplectic resolution provided by Proposition 4.2. By lemma 4.1, $\tilde{X}$ is isomorphic to $T^*Y \setminus Y$.

**Lemma 4.4.** $\pi_0$ extends to a morphism $\pi : T^*Y \to \overline{\mathcal{O}}$.

**Proof.** Note that $\pi_0$ lifts to a morphism $\tilde{X} \to \tilde{W}_0$, where $\tilde{W}_0$ is the normalization of $W_0$, which gives a homomorphism $H^0(\tilde{W}_0, \mathcal{O}_{\tilde{W}_0}) \to H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$. Notice that $H^0(\tilde{W}_0, \mathcal{O}_{\tilde{W}_0}) = H^0(\overline{\mathcal{O}}, \mathcal{O}_{\overline{\mathcal{O}}})$ and $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^0(T^*Y, \mathcal{O}_{T^*Y})$. On the other hand, we have a natural morphism $T^*Y \to \text{Spec}(H^0(T^*Y, \mathcal{O}_{T^*Y}))$, which composed with the map $\text{Spec}(H^0(T^*Y, \mathcal{O}_{T^*Y})) \simeq \text{Spec}(H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})) \to \text{Spec}(H^0(\tilde{W}_0, \mathcal{O}_{\tilde{W}_0})) \simeq \text{Spec}(H^0(\overline{\mathcal{O}}, \mathcal{O}_{\overline{\mathcal{O}}})) = \overline{\mathcal{O}} \to \overline{\mathcal{O}}$ gives $\pi$. $\square$

Notice that $\pi$ is a symplectic resolution of $\overline{\mathcal{O}}$, thus the main theorem in [Fu1] implies that $\pi$ is isomorphic to the moment map of the $G$-action on $T^*G/P$ for some parabolic subgroup $P$ in $G$. So we obtain

**Theorem 4.5.** Let $\mathcal{O}$ be a nilpotent orbit in a semi-simple Lie algebra $\mathfrak{g}$ such that $\mathbb{P}(\overline{\mathcal{O}})$ is singular. Suppose that we have a contact resolution $\pi : Z \to \mathbb{P}(\overline{\mathcal{O}})$, then $Z \simeq \mathbb{P}(T^*(G/P))$ for some parabolic subgroup $P$ in the adjoint group $G$ of $\mathfrak{g}$ and the morphism $\pi$ is the natural one.

Now Proposition 2.2 implies the following

**Corollary 4.6.** Suppose $\mathfrak{g}$ is simple. The projectivised nilpotent orbit closure $\mathbb{P}(\overline{\mathcal{O}})$ admits a contact resolution if and only if either
(i) \( \mathcal{O} \) is the minimal nilpotent orbit, or
(ii) \( \mathfrak{g} \) is of type \( G_2 \) and \( \mathcal{O} \) is of dimension 8, or
(iii) \( \mathcal{O} \) admits a symplectic resolution.

The classification of nilpotent orbits satisfying case (iii) has been done in [Fu1] and [Fu2]. For example, every projectivised nilpotent orbit closure in \( \mathfrak{sl}_n \) admits a contact resolution, which is given by the projectivisation of cotangent bundles of some flag varieties.

Recall that the twistor space of a compact quaternion-Kähler manifold is a contact Fano manifold ([Sa]). One may wonder if a contact resolution of \( \mathbb{P}(\mathcal{O}) \) could be the twistor space of a quaternion-Kähler manifold. Unfortunately, the answer to this is in general no, as shown by the following:

**Proposition 4.7.** Let \( G \) be a simple complex Lie group with Lie algebra \( \mathfrak{g} \) and \( P \) a parabolic sub-group of \( G \). Then \( \mathbb{P}(T^*(G/P)) \) is a twistor space of a quaternion-Kähler manifold if and only if \( G/P \simeq \mathbb{P}^n \) for some \( n \).

**Proof.** Recall that the image of the moment map \( T^*(G/P) \to \mathfrak{g} \) is a nilpotent orbit closure \( \mathcal{O} \), which gives a generically finite morphism \( \pi : \mathbb{P}(T^*(G/P)) \to \mathbb{P}(\mathcal{O}) \). There are two cases:

(i) there is a fiber of positive dimension, then as proved in Proposition 4.3, \( \mathbb{P}(T^*(G/P)) \) is not Fano.

(ii) every fiber of \( \pi \) is zero-dimensional, then \( \pi \) is a finite \( G \)-equivariant surjective morphism. If \( \mathbb{P}(T^*(G/P)) \) is Fano, then by Proposition 6.3 [Be], either \( \pi = id \) and \( \mathcal{O} = \mathcal{O}_{\text{min}} \) or \( \pi \) is one of the \( G \)-covering in the list of Brylinski-Kostant (see table 6.2 [Be]). In both cases, one has that \( \mathbb{P}(T^*(G/P)) \) is isomorphic to \( \mathbb{P}(\mathcal{O}_{\text{min}}') \) for some minimal nilpotent orbit \( \mathcal{O}_{\text{min}}' \subset \mathfrak{g}' \), which is possible only if \( G/P \) is isomorphic to \( \mathbb{P}^n \) for some \( n \).

Now suppose that \( \mathbb{P}(T^*G/P) \) is a twistor space of a quaternion-Kähler manifold \( M \). Then the scalar curvature of \( M \) would be positive, which implies ([Sa]) that \( \mathbb{P}(T^*G/P) \) is Fano, so \( G/P \) is isomorphic to \( \mathbb{P}^n \) for some \( n \). □

As pointed out by Prof. A. Swann, this proposition follows also from [LeSa], where it is shown that a contact Fano variety with \( b_2 \geq 2 \) is isomorphic to \( \mathbb{P}(T^*\mathbb{P}^n) \) for some \( n \).

## 5 Birational geometry

Let \( \mathfrak{g} \) be a simple complex Lie algebra and \( \mathcal{O} \) a non-zero nilpotent orbit in \( \mathfrak{g} \). We now try to understand the birational geometry between different contact
resolutions of $\mathbb{P}(\mathcal{O})$. We can assume that $\mathcal{O}$ is not the minimal nilpotent orbit, since $\mathbb{P}(\mathcal{O}_{\text{min}})$ is smooth.

Suppose that $\mathcal{O}$ admits a symplectic resolution, then by [Fu1], it is given by the natural map $\pi : X := T^*(G/P) \to \mathcal{O}$ for some parabolic sub-group $P$ in $G$. Let us denote by $\pi_0$ the restriction of $\pi$ to $X_0 := T^*(G/P) \setminus (G/P)$, then $\pi_0$ is a symplectic resolution of $W_0 := \mathcal{O} \setminus \{0\}$.

I’m indebted to M. Brion for the proof of the following proposition, which allows us to remove the restriction that $\mathfrak{g}$ is of classical type in an earlier version of this note.

**Proposition 5.1.** We have $N_1(\pi_0) = N_1(\pi)$ and $N^1(\pi_0) = N^1(\pi)$.

**Proof.** Consider the natural projections: $X_0 \xrightarrow{p_0} G/P \xleftarrow{\mathfrak{g}} X$, then $\text{Pic}(X_0) \otimes \mathbb{R}$ is identified with $\text{Pic}(G/P) \otimes \mathbb{R} = N^1(G/P)$ via $p_0^*$. Notice that for a complete curve $C$ on $X_0$ and a divisor $D \in \text{Pic}(G/P)$, we have $C \cdot p_0^* D = (p_0)_* (C) \cdot D$. Thus we need to show that images of complete curves in $X_0$ under $(p_0)_*$ generate $H_2(G/P, \mathbb{R}) = N_1(G/P)$.

Let $I$ be the set of simple roots of $G$ which are not roots of the Levi subgroup of $P$, i.e. $I$ is the set of marked roots in the marked Dynkin diagram of $\mathfrak{p} = \text{Lie}(P)$. A basis of $H_2(G/P, \mathbb{R})$ is given by Schubert curves $C_\alpha := P_\alpha / B$, where $\alpha \in I$ and $P_\alpha$ is the corresponding minimal parabolic subgroup containing the Borel subgroup $B$. We need to lift every $C_\alpha$ to a curve in $X_0$. There are two cases:

(i) $I$ consists of a single simple root $\alpha$, then $b_2(G/P) = 1$. Since $\mathcal{O}$ is supposed to be non-minimal, and the 8-dimensional nilpotent orbit closure in $G_2$ has no symplectic resolution (Proposition 3.21 [Fu1]), by Proposition 2.2 we can assume that $\mathcal{O} \setminus \{0\}$ is not smooth. By Zariski’s main theorem, there exists a fiber of $\pi_0$ which has positive dimension. Take an irreducible curve $C$ containing in this fiber, then $(p_0)_* C$ is non-zero in $H_2(G/P, \mathbb{R}) \simeq \mathbb{R}$, which generates (over $\mathbb{R}$) $N_1(G/P)$.

(ii) $I$ contains at least two simple roots. To lift $C_\alpha$, we take a simple root $\beta \in I$ different to $\alpha$, then $\mathfrak{g}_\beta$ generates a $G_\alpha$-submodule $M$ of $\mathfrak{g}$ contained in $\mathfrak{n}$, where $G_\alpha$ is the simple subgroup of $G$ associated with the simple root $\alpha$ and $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$. Then in $T^*(G/P) \simeq G \times^P \mathfrak{n}$, there is the closed subvariety $P_\alpha \times^B M \simeq G_\alpha \times^B_\alpha M$ which is mapped to $G_\alpha M = M$ with fibers $G_\alpha / B_\alpha \simeq P_\alpha / B$, where $B_\alpha = B \cap G_\alpha$. Now any fiber of this map lifts $C_\alpha$.

Let $\tilde{\pi} : \mathbb{P}(X) \to \mathbb{P}(\mathcal{O})$ be the induced map, which is a contact resolution.
The contact structure on $\mathbb{P}(X)$ is given by the line bundle $\tilde{L} = O_p(1)$, where $p : \mathbb{P}(X) \to G/P$ is the natural map. We have $Pic(\mathbb{P}(X)) \simeq Pic(G/P) \oplus \mathbb{Z}[\tilde{L}]$. Notice that $\tilde{L} = \pi^*L$, so for any $\pi$-exceptional curve $C$, one has $C \cdot \tilde{L} = C \cdot \pi^*L = 0$, so $\tilde{L}$ is zero in $N^1(\pi)$. This provides the identifications $N^1(\pi) = N^1(\pi_0) = N^1(\pi)$ and $N_1(\pi) = N_1(\pi_0) = N_1(\pi)$.

Recall that the cone $NE(\pi) = NE(G/P)$ is generated by Schubert curves in $G/P$ over $\mathbb{R} \geq 0$. As shown in the proof of Proposition 4.1, these Schubert curves are images of curves in the fibers of $\pi_0$, thus $NE(\pi_0) = NE(\pi)$. Since $NE(\pi_0) = NE(\tilde{\pi})$, we obtain $NE(\tilde{\pi}) = NE(\pi)$. By Kleiman’s criterion, $\text{Amp}(\pi_0) = \text{Amp}(\pi) = \text{Amp}(\tilde{\pi})$. By [Na] Theorem 4.1 (ii), this is a simplicial polyhedral cone.

Let $g : X_0 \to \mathbb{P}(X)$ and $h : W_0 \to \mathbb{P}(\mathcal{O})$ be the natural projections, then $\tilde{\pi}_0g = h\pi_0$. Take a $\pi_0$-movable divisor $p_0^*D$, then $(\pi_0)_*p_0^*D = h^*\tilde{\pi}_0\tilde{p}^*D \neq 0$, which gives that $\tilde{\pi}_0\pi^*D \neq 0$. Notice that $\pi_0^*(\pi_0)_*p_0^*D = g^*\pi^*\tilde{\pi}_0\tilde{p}^*D$ and $p_0^*D = g^*\tilde{p}^*D$, so the cokernel of $\tilde{\pi}_0\pi^*D \to \tilde{p}^*D$ has support of codimension $\geq 2$. In conclusion $\tilde{p}^*D$ is $\pi$-movable and vice versa. So we obtain $\text{Mov}(\pi_0) = \text{Mov}(\pi) = \text{Mov}(\tilde{\pi})$.

To remember the parabolic subgroup $P$, from now on, we will write $\pi_P$ instead of $\pi$ (similarly for $\pi_0, \tilde{\pi}$). For two parabolic subgroups $Q, Q'$ in $G$, we write $Q \sim Q'$ (called equivalent) if $T^*(G/Q)$ and $T^*(G/Q')$ give both symplectic resolutions of a same nilpotent orbit closure. In [Na], Namikawa found a way to describe all parabolic subgroups which are equivalent to a given one. Furthermore the chamber structure of $\text{Mov}(\pi_P)$ has been described in loc. cit. Theorem 4.1. By our precedent discussions $\text{Mov}(\pi_0) = \text{Mov}(\pi) = \text{Mov}(\tilde{\pi})$, thus we obtain the chamber structure of $\text{Mov}(\pi)$, namely:

**Theorem 5.2.** Let $\mathcal{O}$ be a non-minimal nilpotent orbit in a simple complex Lie algebra $g$ whose closure admits a symplectic resolution, say $T^*(G/P)$, where $G$ is the adjoint group of $g$. Let $\tilde{\pi}_P : \mathbb{P}(T^*(G/P)) \to \mathbb{P}(\mathcal{O})$ be the associated contact resolution. Then $\text{Mov}(\pi_P) = \cup_{Q \sim P} \text{Amp}(\tilde{\pi}_Q)$.

By Mori theory (see for example [Ma], Theorem 12-2-7), this implies that every minimal model of $\mathbb{P}(\mathcal{O})$ is of the form $\mathbb{P}(T^*(G/Q))$ for some parabolic subgroup $Q \subset G$ such that $P \sim Q$. Now by Proposition 3.3, this gives another proof of Theorem 4.5 in the case where $\mathcal{O}$ admits a symplectic resolution.

Similarly, as a by-product of our argument, we obtain the description of the movable cone of a symplectic resolution of $W_0$, which shows by Mori
theory that every symplectic resolution of $\mathcal{O} \setminus \{0\}$ is the restriction of a Springer map, thus

**Corollary 5.3.** Let $\mathfrak{g}$ be a simple Lie algebra and $\mathcal{O} \subset \mathfrak{g}$ a nilpotent orbit. Suppose that $\mathcal{O}$ admits a symplectic resolution, then every symplectic resolution of $\mathcal{O} \setminus \{0\}$ can be extended to a symplectic resolution of $\mathcal{O}$.

**Remark 5.1.** The condition that $\mathcal{O}$ admits a symplectic resolution cannot be removed, due to the following two examples:

(i). if $\mathfrak{g}$ is not of type $A$, then $\mathcal{O}_{\text{min}}$ admits no symplectic resolution ([Fu1]), however $\mathcal{O}_{\text{min}} \setminus \{0\}$ is smooth, so trivially it admits a symplectic resolution;

(ii). if $\mathfrak{g}$ is of type $G_2$ and $\mathcal{O}$ is the 8-dimensional nilpotent orbit, then $W_0 := \mathcal{O} \setminus \{0\}$ is not smooth, and its normalization map $\mu : \tilde{W}_0 \to W_0$ is a symplectic resolution which does not extends to $\mathcal{O}$, since $\mathcal{O}$ is not a Richardson nilpotent orbit (Prop. 3.21 [Fu1]). Here we used the result in [LeSm] and [Kr] which says that $\tilde{W}_0$ is is fact the minimal nilpotent orbit in $\text{so}_7$, thus it is smooth and symplectic.

Are these two examples the only exceptions?

**References**

[Be] Beauville, A.: *Fano contact manifolds and nilpotent orbits*, Comment. Math. Helv **73** (1998), 566–583

[CF] Campana, F.; Flenner, H.: *Contact singularities*, Manuscripta Math. **108** (2002), no. 4, 529–541

[Fu1] Fu, B.: *Symplectic resolutions for nilpotent orbits*, Invent. Math. **151** (2003), 167-186

[Fu2] Fu, B.: *Extremal contractions, stratified Mukai flops and Springer maps*, math.AG/0605431, to appear in Adv. Math.

[KPSW] Kebekus, S.; Peternell, T.; Sommese, A. J.; Wiśniewski, J. A.: *Projective contact manifolds*, Invent. Math. **142** (2000), no. 1, 1–15

[Ko] Kobak, P. Z.: *Twistors, nilpotent orbits and harmonic maps*, in *Harmonic maps and integrable systems*, 295–319, Aspects Math., E23, Vieweg, Braunschweig, 1994
REFERENCES

[Kr] Kraft, H.: *Closures of conjugacy classes in* $G_2$, J. Algebra **126** (1989), no. 2, 454–465

[Le1] LeBrun, C.: *Fano manifolds, contact structures, and quaternionic geometry*, Int. J. of Math. **6** (1995), 419–437

[LeSa] LeBrun, C.; Salamon, S.: *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. **118** (1994), no. 1, 109–132

[LeSm] Levasseur, T.; Smith, S. P.: *Primitive ideals and nilpotent orbits in type* $G_2$, J. Algebra **114** (1988), no. 1, 81–105

[Ma] Matsuki, K.: *Introduction to the Mori program*, Universitext. Springer-Verlag, New York, 2002

[Na] Namikawa, Y.: *Birational geometry of symplectic resolutions of nilpotent orbits II*, math.AG/0408274

[Pa] Panyushev, D. I.: *Rationality of singularities and the Gorenstein property of nilpotent orbits*, Funct. Anal. Appl. 25 (1991), no. 3, 225–226 (1992)

[Pe] Peternell, T.: *Contact structures, rational curves and Mori theory*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 509–518

[Po] Popov, V. L.: *Self-dual algebraic varieties and nilpotent orbits*, in Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000), 509–533, Tata Inst. Fund. Res. Stud. Math., 16, Bombay, 2002

[Sa] Salamon, S.: *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), no. 1, 143–171

[Sw] Swann, A.: *Homogeneous twistor spaces and nilpotent orbits*, Math. Ann. **313** (1999), no. 1, 161–188

Laboratoire J. Leray (Mathématiques)
Faculté des sciences
2, Rue de la Houssinière, BP 92208
F-44322 Nantes Cedex 03 - France
fu@math.univ-nantes.fr