Modified Lorenz-Malkus water wheel model: dry friction versus chaos

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Abstract. In this paper, we propose a model of the water wheel under dry friction (the modified Lorenz-Malkus system). Dynamical features of this system are investigated and analyzed depending on the dry friction parameter. Particularly, we present the analysis of stationary points, Lyapunov exponents, bifurcation diagrams, and stochastic properties of the considered system. Based on the results of numerical simulations we showed that the dry friction may serve an effective mechanism to control the chaotic dynamics.

1. Introduction
The history of Lorenz system started in 1963 when E. Lorenz published his famous work titled “Deterministic Nonperiodic Flow” [1]. In this article he showed, that simple system of first-order ordinary differential equations with quadratic nonlinearities may demonstrate a quite complex behavior. In our days this article can be considered as a basis for quite interesting and important area of the nonlinear dynamics - the theory of deterministic chaos.

The system investigated by E. Lorenz had the following form:

\[
\begin{align*}
\dot{x} &= \sigma(y-x) \\
\dot{y} &= x(r-z) - y, \\
\dot{z} &= xy - bz
\end{align*}
\]

where variables have following physical meaning: \(x\) is a parameter proportional to the rate of convection, \(y\) is a parameter proportional to the temperature gradient between ascending and descending streams, \(z\) is a parameter characterizing the vertical temperature variation. Parameters \(\sigma\), \(r\) and \(b\) correspond to the Prandtl number, Rayleigh number, and certain physical dimensions of the convection layer, respectively.

One of the features discovered by E. Lorenz during numerical experiments was the fact that solutions to the system (1) demonstrate the strong sensitivity to a small change in initial conditions. Using the standard iteration process he showed that initially closed solutions...
dive with exponential speed (Figure 1). Nowadays, such a dependence of solutions on the initial condition is one of the most important signs of the non-linear dynamic systems with chaotic behavior.

It should be noted that the chaotic behavior of different physical models was observed in a large number of works (see, e.g. [2, 3, 4, 5, 6, 7, 8, 9, 10] and related references). Some real-life physical systems can be modeled by the Lorenz system (1), for example: lasers [2], convection in tubes [3], a dissipative oscillator with inertial excitation [3], chemical reactions [4], and other.

In [5] the fractional Lorenz system (where time derivatives replaced by fractional derivatives) was investigated. Particularly, the adaptive and non-adaptive synchronization between two Lorenz systems was analyzed and discussed in this paper. Also, the authors suggested a control strategy which reduces the number of control parameters (the proposed strategy is based on a number of restrictions). In [6] the Lorenz system being considered as the foundation of the atmospheric disturbance model. In this work, an analytic expression for the dynamical characteristic of the atmospheric disturbance was obtained. In [7] the novel method for generating pseudo-random numbers was suggested. This method is based on the generalized Lorenz system (with some additional terms). It is showed that the proposed method may find an application in the field of cryptology security.

In this work, we propose the extended Lorenz system modified by the inclusion of the dry friction in the first equation. Moreover, such phenomena may serve as the control mechanism which reduces the chaotic behavior in the considered system. In other words, we propose the novel method for control of the chaotic dynamics based on the fundamental physical principles.

2. Water wheel model under dry friction

One of the simple physical system described by the Lorenz system (1) is the so-called Lorenz-Malkus water wheel. This model firstly was proposed and experimentally examined by American scientist W. Malkus in 1970s.

Let us briefly describe this system. The water wheel model has a set of cups on the rim. Each cup has a hole at the bottom where water can leak from. Also, the input flow pours on the cups from above (Figure 2). The rate of this flow strongly defines the behavior of the water wheel. However, scientists who deal with this system (e.g. [8, 9, 10]) do not take into account the dry friction which nevertheless, always takes place in the real-life technical systems.

To derive equations describing the water wheel model under dry friction let we do the following assumptions: cups are continuously distributed on the rim; the input water flow continuously
Figure 2. Water wheel model with different rates of water flow. Left panel - low rate and the water wheel remains motionless; middle panel moderate rate and the water wheel moves with a steady speed; right panel big rate and the water wheel moves chaotically.

enters the top cup; cups can not overflow; water wheel is placed in the vertical plane relative to the Earth surface; cup leakages are proportional to the water mass in a cup. The equation describing the dynamics of the water mass in the water wheel is:

$$\frac{dM}{dt} = Q - \lambda M, \quad (2)$$

where $Q$ is the input water mass rate, $\lambda$ is the cup leakage parameter, $M$ is the total mass of the water. The solution to this equation can be obtained in the explicit form:

$$M(t) = \frac{Q}{\lambda} (1 - e^{-\lambda t}) \quad (3)$$

Obviously that the asymptotic limit ($t \to \infty$) of the right side of the equation (3) is $Q/\lambda$. It is well-known that the radius vector of the center of mass for a system has the following form:

$$\vec{r} = \sum_i M_i \vec{r}_i \quad (4)$$

where $\vec{r} = (z, y)$. Now let we choose an arbitrary moment of time $t_0$. At this moment the top cup is completely filled, but others are empty. For the time $t > t_0$ the water wheel starts to rotate, and the input flow begins filling other cups which were initially empty. It seems suitable to split the mass of the water into two $M_a$ and $M_b$ parts for the moments $t < t_0$ and $t > t_0$, respectively (note, these masses depend on the time). Let us note that this split is not necessary, however, it helps us to derive equations for the center of mass of the water in the simplest way. Thus, coordinates of the center of mass are:

$$y = \frac{M_a(t)y_a + M_b(t)y_b}{M_a(t) + M_b(t)}, \quad (5)$$

$$z = \frac{M_a(t)z_a + M_b(t)z_b}{M_a(t) + M_b(t)}, \quad (6)$$

and corresponding derivatives are:

$$\frac{dy}{dt} = \frac{1}{M} \left( M_a \frac{dy_a}{dt} + M_b \frac{dy_b}{dt} + \frac{dM_a}{dt} y_a + \frac{dM_b}{dt} y_b \right), \quad (7)$$
\[
\frac{dz}{dt} = \frac{1}{M} \left( M_a \frac{dz_a}{dt} + M_b \frac{dz_b}{dt} + \frac{dM_a}{dt} z_a + \frac{dM_b}{dt} z_b \right).
\] (8)

The velocity of the center of mass \( M_a \) and its two components can be written as \( v = \omega r \) and corresponding derivatives are:

\[
\frac{dy_i}{dt} = v \cos(\phi) = \omega r \cos(\phi) = \omega z_i
\] (9)

\[
\frac{dz_i}{dt} = -v \sin(\phi) = -\omega r \sin(\phi) = -\omega y_i
\] (10)

where \( r = \sqrt{y_i^2 + z_i^2} \) and \( \phi = \arctan(z/y) \). At \( t = t_0 \) for the mass \( M_a \) we have the following relations:

\[
\frac{dM_a}{dt} = -\lambda M, \quad \frac{dy_a}{dt} = \omega z, \quad \frac{dz_a}{dt} = -\omega y.
\] (11)

and for \( M_b \):

\[
M_b = 0, \quad \frac{dM_b}{dt} = \lambda M, \quad \frac{dy_b}{dt} = 0, \quad \frac{dz_b}{dt} = 0, \quad z_b = R, \quad y_b = 0.
\] (12)

Next, we substitute equations (11)-(12) into (7)-(8). Some simple transformations lead us to the following system of two ordinary differential equations describing the evolution of \( y \) and \( z \):

\[
\frac{dy}{dt} = \omega z - \lambda y,
\] (13)

\[
\frac{dz}{dt} = -\omega y + \lambda(R - z).
\] (14)

It is known, that the full moment of inertia for the hoop (with respect to the symmetry axis) has a constant value and is presented as:

\[
I = MR^2.
\] (15)

Following the second Newton’s law, the rate of change of the angular momentum equals to the torque which is the sum of \( N_g \) (gravity component), \( N_d \) (dry friction component), \( N_v \) (viscous friction component), \( N_w \) (component corresponding to bringing the incoming water flow up to the speed of the cup into which it falls):

\[
I \frac{d\omega}{dt} = N_g + N_w + N_v + N_d.
\] (16)

Expressions for each of these components have the following form:

\[
N_g = Mg y,
\] (17)

\[
N_v = -\alpha \omega,
\] (18)

\[
N_d = -m \text{sign}(\omega),
\] (19)

\[
N_w = -\omega \lambda I_w,
\] (20)

where \( g \) is the gravitational acceleration, \( \alpha \) is a viscous friction coefficient, \( m \) is the coefficient of the dry friction, \( I_w \) is the falling water’s moment of inertia. Then, the differential equation of (16) can be written as:

\[
\frac{d\omega}{dt} = \frac{Mg}{I} y - \left( \frac{\alpha}{I} + \frac{I_w}{I} \right) \omega - \frac{m}{I} \text{sign}(\omega),
\] (21)
Turning to dimensionless parameters and variables, we obtain the following system:

\[
\begin{aligned}
\dot{x} &= \sigma(y - x) - M_{df}\text{sign}(x) \\
\dot{y} &= x(r - z) - y \\
\dot{z} &= xy - bz \\
\end{aligned}
\]  

(22)

where $M_{df}$ is a coefficient of the dry friction moment. In the frame of this model, the variables have the following meaning: $x$ is the angular velocity of the water wheel, $y$ and $z$ are coordinates of the center of mass of the water.

3. Stationary points investigation

It is well-known that the stationary points play an important role in the analysis of dynamical systems described by differential equations. Thus, we start the analysis of the modified Lorenz system from the identification of stationary points. Following the standard procedure, we have to suppose right parts of equations equal to zero. As a result, we obtain a set of algebraic equations. A solution to these equations determines the set of stationary points. For the modified Lorenz-Malkus system (with parameters $\sigma = 10$, $r = 28$, $b = 8/3$, $M_{df} = 3$) this solution give us the following stationary points:

$(0, 0, 0)$, $(8.64, 8.34, 27)$, $(-8.33, -8.63, 26.96)$, $(-0.011, -0.31, 0.0013)$, $(0.011, 0.31, 0.0013)$.

Stationary points were studied following the first Lyapunov method. The numerical analysis demonstrates that the point $(0, 0, 0)$ became a stable node in contrast to the classical Lorenz system where it is of a saddle-node type. Two focuses have similar coordinates in both modified and classical system. Additionally, our numerical results for the system (22) exhibit the existence of two “new” saddle points which do not exists in the classical system. These points arise during the bifurcation of saddle point $(0, 0, 0)$ from the saddle-node point to two saddle-node points and one stable node point. However, these stationary points can not be shown in explicit form because of enormous expressions they described by. As it can be seen from Figure 3 phase portraits are of dissimilar look as compared to the classical Lorenz system. Firstly, at any moment of the dry friction, trajectories are going to stable zero point. It should also be pointed out, that the modified Lorenz-Malkus system (22) behaves chaotically during some finite time interval. This phenomenon is due to the additional nonlinear term in an already strongly nonlinear system. Moreover, we can state that the dry friction in the considered system may serve an effective mechanism for control of the chaotic dynamics.

![Figure 3. Phase portraits of the modified Lorenz-Malkus system for different values of parameter $M_{df}$. Left panel - $M_{df} = 0.5$, middle panel - $M_{df} = 2$, right panel - $M_{df} = 3$.](image-url)
4. Lyapunov exponents
Instability of dynamical systems closely related to the existence of chaotic regimes of these systems. To identify the system stability the well-known method of Lyapunov exponents are usually applied. To calculate these exponents, authors frequently use a standard orthogonalization procedure (Gram-Schmidt procedure) together with the well-known Wolf’s algorithm [11]. Analysis of Lyapunov exponents for the modified Lorenz-Malkus system (22) with parameters $\sigma = 10$, $r = 28$, $b = 8/3$, $M_{df} = 3$ exhibits that the system goes to the stable fixed point during some finite time interval. Figure 4 reflects the dynamics of the Lyapunov exponents spectrum (as functions of the time). As we can see from these figures at the initial period of time (up to 150 units of model time) the system exhibits chaotic behavior, but, after that, the highest Lyapunov exponent becomes negative, so the dynamics from chaotically turns to regular.

![Figure 4](image)

**Figure 4.** Left panel - Lyapunov exponents versus time. Right panel - corresponding phase portrait of the modified Lorenz-Malkus system. Modelling parameters are $\sigma = 10$, $r = 28$, $b = 8/3$, $M_{df} = 3$ and initial conditions are $(1, 1, 1)$.

5. Bifurcation diagrams
Another important method which is focused on the analysis of the dynamical system behavior is the bifurcation diagram method. In this work, we construct a set of bifurcation diagrams for the modified Lorenz-Malkus system (22). These bifurcation diagrams were constructed by the following receipt: when the trajectory of a solution finally achieved the zero point, we stepped back to another initial point and repeated the calculation process, after that the obtained set of points was glued. Resulting bifurcation diagrams are presented in Figure 5. Let us note, that the transition to chaotic behavior is coming a bit slowly for the modified Lorenz-Malkus system than the classical system (see Figure 5 (a), (c)).

6. Stochastic features
In this section, we consider stochastic features of the modified Lorenz-Malkus system. Numerical results show that the duration of chaotic behavior exhibits a stochastic nature. To construct the corresponding histograms we generated 10000 sets of initial conditions from the cube in phase space (cube dimensions are: edges of the cube are 20 and the center is placed at origin). Further, we analyzed the behavior of the solution’s trajectory and the duration of this behavior was stored. Obviously, the histogram form depends on the moment of dry friction. We did not identify a distribution function for each quantity of the moment of dry friction. However,
we established that for some partial quantities of the moment of dry friction the distribution function can be approximated by the Gamma distribution.

\[ f(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta} \left( \frac{x}{\beta} \right)^{\alpha-1} e^{-x/\beta} \]

As it can be seen from the right panel of Figure 6 the histogram for the modified Lorenz-Malkus system with dry friction moment \( M_{df} = 10 \) is well approximated by the Gamma distribution (6) with parameters \( \alpha = 4.9, \beta = 1.98 \).

7. Conclusions
In this work, the model of water wheel under dry friction (the modified Lorenz-Malkus system) was investigated. We presented step by step derivation of the system of equations which describes the dynamics of this mechanical system. It turned out, that point \((0,0,0)\) is a stable fixed point for the considered system. It is noted that the system exhibits the chaotic behavior during the initial evolution, however after a finite time interval it becomes regular (analysis of Lyapunov exponents confirmed this result, namely the highest Lyapunov exponent remains positive during a finite time interval). Also, stationary points of modified Lorenz-Malkus system were investigated. Fundamental differences between the classical and modified Lorenz-Malkus
Figure 6. Histograms of duration of the chaotic behavior for the modified Lorenz-Malkus system at different values of the moment of dry friction. Left panel - $M_{df} = 3$; right panel - $M_{df} = 10$. Red curve represents the Gamma distribution (6) with parameters $\alpha = 4.9$, $\beta = 1.98$.

system was established and analyzed. Particularly, we demonstrated that dry friction can be considered as an effective mechanism for control of the chaotic dynamics. We showed also, that for some partial values of the moment of dry friction the duration of chaotic behavior in this system demonstrate stochastic properties and can be described by the Gamma distribution.

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