Submodular Function Maximization in Parallel via the Multilinear Relaxation*

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Abstract

Balkanski and Singer [5] recently initiated the study of adaptivity (or parallelism) for constrained submodular function maximization, and studied the setting of a cardinality constraint. Very recent improvements for this problem by Balkanski, Rubinstein, and Singer [6] and Ene and Nguyen [21] resulted in a near-optimal \((1 - 1/e - \varepsilon)\)-approximation in \(O(\log n/\varepsilon^2)\) rounds of adaptivity. Partly motivated by the goal of extending these results to more general constraints, we describe parallel algorithms for approximately maximizing the multilinear relaxation of a monotone submodular function subject to packing constraints. Formally our problem is to maximize \(F(x)\) over \(x \in [0, 1]^n\) subject to \(Ax \leq 1\) where \(F\) is the multilinear relaxation of a monotone submodular function. Our algorithm achieves a near-optimal \((1 - 1/e - \varepsilon)\)-approximation in \(O(\log^2 m \log n/\varepsilon^4)\) rounds where \(n\) is the cardinality of the ground set and \(m\) is the number of packing constraints. For many constraints of interest, the resulting fractional solution can be rounded via known randomized rounding schemes that are oblivious to the specific submodular function. We thus derive randomized algorithms with poly-logarithmic adaptivity for a number of constraints including partition and laminar matroids, matchings, knapsack constraints, and their intersections.

Our algorithm takes a continuous viewpoint and combines several ideas ranging from the continuous greedy algorithm of \([35, 13]\), its adaptation to the MWU framework for packing constraints \([11]\), and parallel algorithms for packing LPs \([27, 37]\). For the basic setting of cardinality constraints, this viewpoint gives rise to an alternative, simple to understand algorithm that matches recent results \([6, 21]\). Our algorithm to solve the multilinear relaxation is deterministic if it is given access to a value oracle for the multilinear extension and its gradient; this is possible in some interesting cases such as the coverage function of an explicitly given set system.

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1 Introduction

A real-valued set function \( f : 2^\mathcal{N} \rightarrow \mathbb{R} \) over a finite ground set \( \mathcal{N} \) is submodular iff

\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \text{ for all } A, B \subseteq \mathcal{N}.
\] (1)

Submodular set functions play a significant role in classical combinatorial optimization. More recently, due to theoretical developments and a plethora of applications ranging from algorithmic game theory, machine learning, and information retrieval & analysis, their study has seen a resurgence of interest. In this paper we are interested in constrained submodular function maximization. Given a non-negative submodular set function \( f : 2^\mathcal{N} \rightarrow \mathbb{R}_+ \) over a finite ground set \( \mathcal{N} \) the goal is to find \( \max_{S \in \mathcal{I}} f(S) \) where \( \mathcal{I} \) is down-closed family of sets that captures some packing constraint of interest. The canonical problem here is the cardinality constrained problem \( \max f(S) \) where \( |S| \leq k \). Among many other applications, this problem captures NP-Hard problems including the Maximum \( k \)-Cover problem which can not be approximated to better than a \((1 - 1/e - \epsilon)\)-factor for any \( \epsilon > 0 \) unless \( P = NP \) \[22\]. The cardinality constrained problem has been well-studied from the 70’s with an optimal \((1 - 1/e)\)-approximation established via a simple greedy algorithm when \( f \) is monotone \[33\]. There has been extensive theoretical work in the last decade on approximation algorithms for submodular function maximization. Several new algorithmic ideas were developed to obtain improved approximation ratios for various constraints, and to handle non-monotone functions. One of these new ingredients is the multilinear relaxation approach \[13\] that brought powerful continuous optimization techniques to submodular function maximization. We refer the reader to a recent survey \[12\] for some pointers to the new developments on greedy and continuous methods, and to \[11\] on local search methods.

Recent applications of submodular function maximization to large data sets, and technological trends, have motivated new directions of research. These include the study of faster algorithms in the sequential model of computation \[3, 31, 19, 32, 21\], algorithms in distributed setting \[30, 26, 29, 8, 9\], and algorithms in the streaming setting \[4, 14, 18\]. Barbosa et al. \[9\] developed a general technique to obtain a constant round algorithm in the MapReduce model of computation that gets arbitrarily close to the approximation achievable in the sequential setting. The MapReduce model captures the distributed nature of data but allows for a polynomial amount of sequential work on each machine. In some very recent work Balkanski and Singer \[5\] suggested the study of adaptivity requirements for submodular function maximization which is closer in spirit to the traditional parallel computation model such as the PRAM. To a first order approximation the question is the following. Assuming that the submodular function \( f \) can be evaluated efficiently in parallel, how fast can constrained submodular function maximization be done in parallel? To avoid low-level considerations of the precise model of parallel computation, one can focus on the number of adaptive rounds needed to solve the constrained optimization problem; this corresponds to the depth in parallel computation. The formal definition of the notion of adaptivity from \[5\] is the following. An algorithm with oracle access to a submodular function \( f : \mathcal{N} \rightarrow \mathbb{R} \) is \( r \)-adaptive for an integer \( r \) if for \( i \in [r] \), every query \( q \) to \( f \) in round \( i \) depends only on the answers to queries in rounds 1 to \((i-1)\) (and is independent of all other queries in rounds \( i \) and greater). We believe that the definition is intuitive and use other terms such as depth, rounds and iterations depending on the context.

Balkanski and Singer \[5\] considered the basic cardinality constrained problem and showed that in the value oracle model (where one assumes black box access to \( f \)), one needs \( \Omega(\log n / \log \log n) \) rounds of adaptivity for a constant factor approximation. They also developed a randomized algorithm with an approximation ratio of \( 1/3 - \epsilon \). In recent work, Balkanski et al. \[6\] and Ene and Nguyen \[21\] described randomized algorithms that achieved a near-optimal approximation ratio of...
(1 − 1/e − ε) with O(log n/ε²) adaptivity. The algorithm of Ene and Nguyen \cite{EneNg19} uses Ō(n poly(1/ε)) function calls, while the algorithm of Balkanski et al. \cite{BalkBBL19} uses Ō(nk² poly(1/ε)) function calls.

We refer the reader to \cite{BalkBBL19} for extensive justification for the study of adaptivity of submodular function maximization. We believe that the close connections to parallel algorithms is already a theoretically compelling motivation. For instance, specific problems such as Set Cover and Maximum k-Cover have been well-studied in the PRAM model (see \cite{FeCo66} and references therein). Our goals here are twofold. First, can we obtain parallel algorithms for other and more general classes of constraints than the cardinality constraint? Second, is there a unified framework that cleanly isolates the techniques and ideas that lead to parallelization for submodular maximization problems?

Our Contribution: We address our goals by considering the following general problem. Given a monotone submodular function \( f : 2^N \rightarrow \mathbb{R}_+ \) maximize \( f \) subject to a set of explicitly given packing constraints in the form \( Ax \leq 1 \), \( x \in \{0, 1\}^n \); here \( n = |N| \), and \( A \in [0, 1]^{m \times n} \) is a non-negative matrix. Packing constraints in this form capture many constraints of interest including cardinality, partition and laminar matroids, matchings in graphs and hypergraphs, independent sets in graphs, multiple knapsack constraints, and their intersections to name a few. To solve these in a unified fashion we consider the problem of solving in parallel the following multilinear relaxation:

\[
\text{maximize } F(x) \text{ s.t } Ax \leq 1 \text{ and } x \in [0, 1]^N. \quad \text{(Pack-ML)}
\]

Here \( F : [0, 1]^N \rightarrow \mathbb{R}_+ \) is the multilinear extension of \( f \) \cite{FeCo66}, a continuous extension of \( f \) defined formally in Section 2. We mention that solving a packing LP of the form

\[
\text{maximize } \langle c, x \rangle \text{ s.t } Ax \leq 1 \text{ and } x \in [0, 1]^n \quad \text{(Pack-LP)}
\]

with \( c \geq 0 \) is a special case of our problem.

The multilinear relaxation is used primarily for the sake of discrete optimization. For this reason we make the following convenient assumption: for every element \( j \) of the ground set \( N \), the singleton element \( \{j\} \) satisfies the packing constraints, that is, \( Ae_j \leq 1 \). Any element which does not satisfy the assumption can be removed from consideration. We make this assumption for the rest of the paper as it helps the exposition and avoids uninteresting technicalities.

Our main result is the following theorem.

\textbf{Theorem 1.1.} There is a parallel/adaptive algorithm that solves the multilinear relaxation of a monotone submodular function subject to \( m \) packing constraints with the following properties. For a given parameter \( \epsilon > 0 \):

- It outputs a \((1 - 1/e - \epsilon)\)-approximation to the multilinear relaxation.
- It runs in \( O\left(\frac{1}{\epsilon^4 \log^2 m \log n}\right) \) adaptive rounds.
- The algorithm is deterministic if given value oracle access to \( F \) and its gradient \( F' \). The total number of oracle queries to \( F \) and \( F' \) is \( O(n \text{ poly}(\log n/\epsilon)) \).
- If only given access to a value oracle for \( f \) the algorithm is randomized and outputs a \((1 - 1/e - \epsilon)\)-approximate feasible solution with high probability, and deterministically finishes in the prescribed number of rounds. The total number of oracle accesses to \( f \) is \( O(n^2 \text{ poly}(\log n/\epsilon)) \).

\footnote{1}{We use \( \tilde{O}(\cdot) \) notation to suppress poly-logarithmic factors.}
Our algorithm solves the continuous relaxation and outputs a fractional solution $x$. To obtain an integer solution we need to round $x$. Several powerful and general rounding strategies have been developed over the years including pipage rounding, swap rounding, and contention resolution schemes \cite{13,7,17,23,23,12}. These establish constant factor integrality gaps for the multilinear relaxation for many constraints of interest. In particular, for cardinality constraints and more generally matroid constraints there is no loss from rounding the multilinear relaxation. Thus solving the multilinear relaxation in \textbf{Theorem 1.1} already gives an estimate of the value of the integer optimum solution. One interesting aspect of several of these rounding algorithms is the following: with randomization, they can be made oblivious to the objective function $f$ (especially for monotone submodular functions). Thus one can convert the fractional solution into an integer solution without any additional rounds of adaptivity. Of course, in a fine-grained parallel model of computation such as the PRAM, it is important to consider the parallel complexity of the rounding algorithms. This will depend on the constraint family and is outside the scope of this paper. We mention that the case of partition matroids is relatively straightforward and one can derive a randomized parallel algorithm with an approximation ratio of $(1 - 1/e - \epsilon)$ with poly-logarithmic depth.

For the case of cardinality constraint we are able to derive a more oracle-efficient algorithm with similar parameters as the ones in \cite{6,21}. The efficient version is presented as a discretization of the continuous algorithm, and we believe it provides a different perspective from previous work\cite{3}. The algorithm can be extended to a single knapsack constraint while maintaining a depth of $O(\log n/\epsilon^2)$.

\textbf{Remark 1.2.} The adaptivity of our algorithm has a dependence of $1/\epsilon^4$ on the error parameter $\epsilon$. We note that even for the case of pure packing LPs, until recently, this was the best dependence known for parallel algorithms \cite{37}. Recent improvements lead to a $1/\epsilon^2$ dependence for pure packing LPs \cite{28,1}. It is an interesting future research direction to see if these improvements can be extended to the multilinear relaxation.

\textbf{Remark 1.3.} Our parallel algorithm for the multilinear relaxation relies only on “monotone concavity” of the multilinear extension (as defined in Section 2). Thus our parallel algorithm also applies to yield a $(1 - 1/e - \epsilon)$-approximation for maximizing any monotone concave function subjecting to packing constraints. Even for non-decreasing concave functions, which can be optimized almost exactly in the sequential setting, it is not clear that they can be solved efficiently and near optimally in the parallel setting when in the oracle model with black box access to the the function and its gradient.

\subsection{1.1 Technical overview}

We build upon several ingredients that have been developed in the past. These include the continuous greedy algorithm for approximating the multilinear relaxation \cite{35,13} and its adaptation to the multiplicative weight update method for packing constraints \cite{19}. The parallelization is inspired by fast parallel approximation schemes for positive LPs pioneered by Luby and Nisan \cite{27} and subsequently developed by Young \cite{37}. Here we briefly sketch the high-level ideas which are in some sense not that complex.

We will first consider the setting of a single constraint ($m = 1$), which corresponds to a knapsack constraint of the form $\langle a, x \rangle \leq 1$. For linear objective functions $f(x) = \langle c, x \rangle$, we know that the optimal solution is obtained by greedily sorting the coordinates in decreasing order of $c_j/a_j$ and choosing each coordinate in turn to its fullest extent of the upper bound 1 until the budget of one unit is exhausted (the last job may be fractionally chosen). One way to parallelize the greedy algorithm\footnote{Balkanski and Singer \cite{5, Section D} describe very briefly a connection between their $1/3$-approximation algorithm and the multilinear relaxation but not many details are provided.}
(and taking a continuous view point) while losing only a \((1 - \epsilon)\)-factor is the following. We bucket the ratios \(c_j/a_j\) into a logarithmic number of classes by some appropriate discretization. Starting with the highest ratio class, instead of choosing only one coordinate, we choose all coordinates in the same bucket and increase them simultaneously in parallel until the budget is met or all coordinates reach their upper bound. If the budget remains we move on to the next bucket. It is not hard to see that this leads to a parallel algorithm with poly-logarithmic depth; the approximation loss is essentially due to bucketing.

Consider now the nonlinear case, \([\text{Pack-ML}]\) under a knapsack constraint. In the sequential setting, the continuous greedy algorithm \([35, 13]\) is essentially the following greedy algorithm presented as a continuous process over time. At any time \(t\), if \(x(t)\) is the current solution, we increase only \(x_j\) for the best “bang-for-buck” coordinate \(j = \arg \max_h F'_h(x)/a_h\); here \(F'_h(x)\) is the gradient of \(F\) with respect to \(h\). In the special case of the cardinality constraint, this is the coordinate with the largest partial derivative. Multilinearity of \(F\) implies that we should increase the same coordinate \(j\) until it reaches its upper bound. A natural strategy to parallelize this greedy approach is to bucket the ratios of the coordinates (by some appropriate discretization) and simultaneously increase all coordinates in the best ratio bucket. This won’t quite work because \(F\) is non-linear and the gradient values decrease as \(x\) increases. Here is a simple but key idea. Let \(\lambda\) be the current highest ratio and let us call any coordinate \(j\) in the highest bucket a good coordinate. Suppose we increase all good coordinates by some \(\delta\) until the average ratio of the good coordinates falls, after the increase, to \((1 - \epsilon)\lambda\). During the step we have a good rate of progress, but the step size \(\delta\) may be very small. However, one can argue that after the step, the number of good coordinates for current gradient level falls by an \(\epsilon\) fraction. Hence we cannot make too many such steps this bucket empties, and have made “dual” progress in terms of decreasing the \(\ell_\infty\)-norm of the gradient. This simple scheme suffices to recover a polylogarithmic depth algorithm for the knapsack constraint. With some additional tricks we can convert the algorithm into a randomized discrete algorithm that recovers the parameters of \([6, 21]\) for the cardinality constraint. We note that viewing the problem from a continuous point of view allows for a clean and deterministic algorithm (assuming value oracles for \(F\) and its gradient \(F'\)).

The more technical aspect of our work is when \(m > 1\); that is, when there are several constraints. Here we rely on a Lagrangean relaxation approach based on the multiplicative weight update (MWU) method for positive LPs, which has a long history in theoretical computer science \([2]\). The MWU approach maintains non-negative weights \(w_1, w_2, \ldots, w_m\) on the constraints and solves a sequence of Lagrangean relaxations of the original problem while updating the weights. Each relaxed problem is obtained by collapsing the \(m\) constraints into a single constraint \(\langle w, Ax \rangle \leq \langle w, 1 \rangle\) obtained by taking a weighted linear combination of the original constraints. Note that this single constraint is basically a knapsack constraint. However, the weights are updated after each step and hence the knapsack constraint evolves dynamically. Nevertheless, the basic idea of updating many variables with the same effective ratio that we outlined for the single knapsack constraint can be generalized. One critical feature is that the weights increase monotonically. In the sequential setting, \([13]\) developed a framework for \([\text{Pack-ML}]\) that allowed a clean combination of two aspects: (a) an analysis of the continuous greedy algorithm for proving a \((1 - 1/e)\)-approximation for the multilinear relaxation and (b) the analysis of the step size and weight updates in MWU which allows one to argue that the final solution (approximately) satisfies the constraints. We borrow the essence of this framework, but in order to parallelize the algorithm we need both the dual gradient-decreasing viewpoint discussed above and another idea from previous work on parallel algorithms for positive LPs \([27, 37]\). Recall

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\(^3\)This tension is also central to the recent works \([3, 8, 21]\). We believe that it is easier to understand it in the continuous setting where one can treat the matter deterministically.
that in the setting of a single knapsack constraint, when we update multiple variables, there are two bottlenecks for the step size: the total budget and the change in gradient. In the MWU setting, the step size is further controlled by weight update considerations. Accordingly, the step size update rule is constrained such that if we are increasing along the $j$ coordinate with a current value of $x_j$, then the updated value is at most $(1 + \epsilon^2 / \log m)x_j$. This limit is conservative enough to ensure the weights do not grow too fast, but can only limit the step size a small number of times before the geometrically increasing coordinates exceed a certain upper bound.

**Organization:** The rest of the paper is organized as follows. **Section 2** describes relevant background on submodular functions and the multilinear extension. In **Section 3**, we first describe and analyze an algorithm for the multilinear relaxation when we have a single cardinality constraint. This gives an algorithm with $O(\log n/\epsilon^2)$ depth assuming oracle access to the multilinear extension $F$ and its derivative $F'$, which in turn can be implemented via (many more) oracle calls to $f$ without increasing the adaptivity. We describe and analyze our algorithm for general packing constraints in **Section 3**. In **Appendix A**, we analyze a randomized discretization of the continuous algorithm for cardinality constraints with a better oracle complexity w.r.t $f$. In **Appendix B**, we describe and analyze $O(\log n/\epsilon^2)$-adaptive algorithms for maximizing a monotone submodular function subject to a single knapsack constraint.

Note that **Section 3** is largely included to develop some intuition ahead of the more complicated constraints in **Section 4** but none of the formal observations in **Section 3** are invoked explicitly in **Section 4**. Moreover, the bounds obtained in **Section 3** for the cardinality constraint are already known [6, 21]. The reader primarily interested in the main result regarding general packing constraints may prefer to skip ahead to **Section 4**.

## 2 Submodular set functions and the Multilinear relaxation

In this section we provide some relevant background and notation that we use in the rest of the paper. Let $f : 2^N \to \mathbb{R}$ assign real values to subsets of $N$. $f$ is nonnegative if $f(S) \geq 0$ for all $S \subseteq N$. $f$ is monotone if $S \subseteq T$ implies $f(S) \leq f(T)$. $f$ is normalized if $f(\emptyset) = 0$.

We have already seen one definition of submodularity in (1). Another useful (and equivalent) definition is via marginal values. For a real-valued set function $f : 2^N \to \mathbb{R}$, the marginal value of a set $U$ with respect to a set $S$ is defined as $f(S \cup U) - f(S)$, which we abbreviate by $f_S(U)$. If $U$ is a singleton $\{i\}$ we write $f_S(i)$ instead of $f_S(\{i\})$. We also use the notation $S + i$ and $S + i + j$ as short hand for $S \cup \{i\}$ and $S \cup \{i, j\}$. A set function $f$ is submodular iff it satisfies the following property modeling decreasing marginal returns:

$$f_S(i) \geq f_T(i) \text{ for all } S \subset T \subseteq N \text{ and } i \not\in T.$$  

The following seemingly restricted form of this property also suffices: $f_S(i) \geq f_{S+j}(i) \forall S, i, j \not\in S$ and we will see a continuous analogue of this latter property subsequently. In this paper we restrict attention to normalized, nonnegative and monotone submodular set functions.

### 2.1 Multilinear extension and relaxation

In this section, we outline basic properties of a continuous extension of submodular functions to the fractional values in $[0,1]^N$ called the multilinear extension [13].

**Notation 2.1.** For two vectors $x, y$, let $x \lor y$ be the coordinatewise maximum of $x$ and $y$, and let $x \land y$ denote the coordinatewise minimum, and let $x \setminus y = x - x \land y$. We identify an element $j$ with
the coordinate vector $e_j$, and a set of elements $S \subseteq \mathcal{N}$ with the sum of coordinate vectors, $\sum_{j \in S} e_j$.

In particular, for a vector $x \in [0, 1]^\mathcal{N}$ and a set of coordinates $S$, $x \land S$ is the vector obtained from $x$ by setting all coefficients not indexed by $S$ to 0, and $x \setminus S = x - x \land S$ is the vector obtained from $x$ setting all coordinates indexed by $S$ to 0.

**Definition 2.2.** Given a set function $f : 2^\mathcal{N} \to \mathbb{R}$, the multilinear extension of $f$, denoted $F$, extends $f$ to the product space $[0, 1]^\mathcal{N}$ by interpreting each point $x \in [0, 1]^\mathcal{N}$ as an independent sample $S \subseteq \mathcal{N}$ with sampling probabilities given by $x$, and taking the expectation of $f(S)$. Equivalently,

$$F(x) = \sum_{S \subseteq \mathcal{N}} \left( \prod_{i \in S} x_i \prod_{i \not\in S} (1 - x_i) \right).$$

We extend $F$ to the cone $\mathbb{R}_{\geq 0}$ by truncation: $F(x) = F(x \land \mathbb{1})$. where $x \land \mathbb{1}$ takes the coordinatewise minimum of $x$ and the all-ones vector $\mathbb{1}$.

We also write $F_i(x) = F(x \lor y) - F(y)$ which generalizes the definition of marginal values to the continuous setting. We let $F'(x)$ denote the gradient of $F$ at $x$ and $F''(x)$ denote the Hessian of $F$ at $x$. $F'_i(x)$ denotes the partial derivative of $F$ with respect to $i$, and $F''_{ij}(x)$ denotes the second order partial derivative with respect to $i$ and $j$. The following lemma captures several submodularity properties of $F$ that it inherits from $f$. The properties are paraphrased from [35, 13] and can be derived from the algebraic formula for $F$ and submodularity of $f$.

**Lemma 2.3.** Let $F$ be the multilinear extension of a set function $f$, and $x \in [0, 1]^\mathcal{N}$.

1. (Multilinearity) For any $i \in \mathcal{N}$, $F(x) = F(x \setminus i) + x_i F_{x \setminus i}(i)$. In particular, $F(x)$ is linear in $x_i$.

2. (Monotonicity) For any $i \in \mathcal{N}$, $F'_i(x) = F_{x \setminus i}(i)$. In particular, if $f$ is monotone, then $F'$ is nonnegative, and $F$ is monotone (that is, $F(y) \geq F(x)$ if $y \geq x$).

3. For any $i \neq j \in \mathcal{N}$, for $y = x \setminus \{i, j\}$, we have

$$F''_{ij}(x) = F(y \lor \{i, j\}) - F(y \lor i) - F(y \lor j) + F(y).$$

If $f$ is submodular, then $F''_{ij}(x) \leq 0$.

4. (Monotone concavity) For any $d \geq 0$, the function $\lambda \mapsto F(x + \lambda d)$ is concave in $\lambda$ (whenever $F(x + \lambda d)$ is defined).

**Multilinear relaxation:** The multilinear extension $F$ of a submodular function $f$ has many uses, but a primary motivation is to extend the relax-and-round framework of approximation algorithms for linear functions to submodular function maximization. Given a discrete optimization problem of the form $\max_{S \subseteq \mathcal{I}} f(S)$ we relax it to the continuous optimization problem $\max_{x \in P_\mathcal{I}} F(x)$ where $P_\mathcal{I}$ is a polyhedral or convex relaxation for the feasible solutions of constraint family $\mathcal{I}$. The problem $\max_{x \in P_\mathcal{I}} F(x)$ is referred to as the multilinear relaxation. It is useful to assume that linear optimization over $P_\mathcal{I}$ is feasible in polynomial time in which case it is referred to as solvable. The multilinear relaxation is not exactly solvable even for the simple cardinality constraint polytope $\{x \in [0, 1]^n : \sum_i x_i \leq k\}$. The continuous greedy algorithm [35] gives an optimal $(1 - 1/e)$ approximation for solvable polytopes when $f$ is monotone. Our focus in this paper is the restricted setting of explicit packing constraints.
Preprocessing: Recall that we made an assumption that for all \( j \in \mathcal{N}, Ae_j \leq \mathbb{1} \). With this assumption in place we can do some useful preprocessing of the given instance. First, we can get lower and upper bounds on OPT, the optimum solution value for the relaxation. We have \( \text{OPT} \geq \ell = \max_j f(j) \) and \( \text{OPT} \leq \sum_j f(j) = u \leq n\ell \). Since we are aiming for a \((1 - 1/\epsilon - \epsilon)\)-approximation we can assume that for all \( j \), \( f(j) \geq \frac{\epsilon}{n}\ell \); any element which does not satisfy this assumption can be discarded and the total loss is at most \( \epsilon \text{OPT} \). Further, we can see, via subadditivity of \( f \) and \( F \) that \( F\left(\frac{\epsilon}{n}\mathbb{1}\right) \leq \frac{\epsilon}{n} \sum_j f(j) \leq \epsilon \text{OPT} \). We can also assume that \( A_{i,j} = 0 \) or \( A_{i,j} \geq \epsilon/n \) for all \( i, j \); if \( A_{i,j} < \epsilon/n \) we can round it down to 0. Let \( A' \) be the modified matrix. If \( A'x \leq \mathbb{1} \) then we have that \( Ax \leq (1 + \epsilon)\mathbb{1} \). Therefore \( A(1 - O(\epsilon))x \leq \mathbb{1} \). From monotone concavity we also see that \( F((1 - O(\epsilon))x) \geq (1 - O(\epsilon))F(x) \). Thus, solving with respect to \( A' \) does not lose more than a \((1 - O(\epsilon)) \) multiplicative factor when compared to solving with \( A \).

Evaluating \( F \) and \( F' \): The formula for \( F(x) \) gives a natural random sampling algorithm to evaluate \( F(x) \) in expectation. Often we need to evaluate \( F(x) \) and \( F'(x) \) to high accuracy. This issue has been addressed in prior work via standard Chernoff type concentration inequalities when \( f \) is non-negative.

Lemma 2.4 \((14)\). Suppose \( F'(i) \in [0, M'] \). Then with \( r = O\left(\frac{1}{\epsilon^2}p \log d\right) \) parallel evaluations of \( f \) one can find an estimate \( Z \) of \( F'(i) \) such that \( \mathbb{P}\left[|Z - F'(i)| \geq \epsilon F'(i) + \frac{\epsilon}{p} M'\right] \leq \frac{1}{d^3} \). Similarly, if \( F(x) \in [0, M] \), then with \( r = O(\frac{1}{\epsilon^2}p \log d) \) parallel evaluations of \( f \), one can find an estimate \( Z \) of \( F(x) \) such that \( \mathbb{P}\left[|Z - F(x)| \geq \epsilon F(x) + \frac{\epsilon}{p} M\right] \leq \frac{1}{d^3} \).

Choosing \( d = n \) and \( p = n \) we can estimate \( F'(i) \) and \( F(x) \) to within a \((1 \pm \epsilon)\) multiplicative error, and an additive error of \( \frac{\epsilon}{n} M' \) and \( \frac{\epsilon}{n} M \) respectively. Via the preprocessing that we already discussed, we can assume that \( M \leq n \text{OPT} \) and \( M' \leq \text{OPT} \). For any \( x \) such that \( F(x) \geq \frac{\epsilon}{n} \text{OPT} \) we can set \( p = n^2/\epsilon \) to obtain a \((1 + \epsilon)\)-relative approximation. Similarly if \( F'(i) \geq \frac{\epsilon}{n} \text{OPT} \) we can obtain a \((1 + \epsilon)\)-relative approximation by setting \( p = n/\epsilon \).

In some cases an explicit and simple formula for \( F \) exists from which one can evaluate it deterministically and efficiently. A prominent example is the coverage function of a set system. Let \( f \) be defined via a set system on \( n \) sets \( A_1, A_2, \ldots, A_n \) over a universe \( \mathcal{U} \) of size \( r \) as follows. For \( S \subseteq [n] \) we let \( f(S) = \bigcup_{i \in S} A_i \), the total number of elements covered by the sets in \( S \). It is then easy to see that

\[ F(x) = \sum_{e \in \mathcal{U}} \left(1 - \prod_{i \in A_e} (1 - x_i)\right). \]

Thus, given an explicit representation of the set system, \( F(x) \) and \( F'(x) \) can be evaluated efficiently and deterministically.\(^4\)

Throughout the paper we assume that \( \epsilon > 0 \) is sufficiently small. We also assume that \( \epsilon > \text{poly}(1/n) \), since otherwise sequential algorithms already achieve \( 1/\epsilon \)-adaptivity.

\(^4\)We ignore the numerical issues involved in the computation. One can approximate the quantities of interest with a small additive and multiplicative error via standard tricks.
parallel-greedy($f, N, k, \epsilon$)

1. $x \leftarrow 0$
2. $\lambda \leftarrow \text{OPT} // \text{or any upper bound for OPT}$
3. while $\langle x, 1 \rangle \leq k$ and $\lambda \geq e^{-1} \text{OPT}$
   A. let $S = \{ j \in N : F'_j(x) \geq \frac{(1 - \epsilon)\lambda}{k} \}$
   B. while $S$ is not empty and $\langle x, 1 \rangle \leq k$
      i. chose $\delta$ maximal s.t.
         a. $F(x + \delta S) \geq (1 - \epsilon)^2 \lambda |S|$
         b. $\langle x + \delta S, 1 \rangle \leq k$
      ii. $x \leftarrow x + \delta S$
      iii. update $S$
   C. $\lambda \leftarrow (1 - \epsilon)\lambda$
4. return $x$

**Figure 1:** A parallel implementation of the continuous-greedy algorithm specialized to the cardinality polytope.

### 3 Parallel maximization with a cardinality constraint

We first consider the canonical setting of maximizing the multilinear extension of a submodular function subject to a cardinality constraint specified by an integer $k$. The mathematical formulation is below.

$$\text{maximize } F(x) \text{ over } x \in \mathbb{R}_{\geq 0}^N \text{ s.t. } \langle 1, x \rangle \leq k.$$ 

This problem was already considered and solved to satisfaction by Balkanski et al. [6] and Ene and Nguyen [21]. The approach given here is different (and simple enough), and is based on the continuous-greedy algorithm of Călinescu et al. [13], specialized to the cardinality constraint polytope. Establishing this connection lays the foundation for general constraints in Section 4. That said, there is no formal dependence between Section 4 and this section. As the bounds presented in this section have been obtained in previous work [6, 21], the reader primarily interested in new results may want to skip ahead to Section 4.

We propose the algorithm parallel-greedy, given in Figure 2. It is a straightforward parallelization of the original continuous-greedy algorithm due to Călinescu et al. [13], specialized to the cardinality polytope. continuous-greedy is an iterative and monotonic algorithm that, in each iteration, computes the gradient $F'(x)$ and finds the point $v$ in the constraint polytope that maximizes $\langle F'(x), v \rangle$. In the case of the cardinality polytope, $v$ is $e_j$ for the coordinate $j = \arg\max_h F'_h(x)$ with the largest gradient. continuous-greedy then adds $\delta e_j$ to $x$ for a fixed and conservative step size $\delta > 0$. The new algorithm parallel-greedy makes two changes to this algorithm. First, rather than increase $x$ along the single best coordinate, we identify all “good” coordinates with gradient values nearly as large as the best coordinate, and increase along all of these coordinates uniformly.
Second, rather than increase $x$ along these coordinates by a fixed increment, we choose $\delta$ dynamically. In particular, we greedily choose $\delta$ as large as possible such that, after updating $x$ and thereby decreasing the gradient coordinatewise, the set of good coordinates is still nearly as good on average.

The dynamic choice of $\delta$ accounts for the fact that increasing multiple coordinates simultaneously can affect their gradients. The importance of greedily choosing the step size is to geometrically decrease the number of good coordinates. It is shown below (in Lemma 3.3) that, when the many good coordinates are no longer nearly-good on average, a substantial fraction of these coordinates are no longer good. When there are no nearly-good coordinates remaining, the threshold for “good” decreases. The threshold can decrease only so much before we can conclude that the current solution $x$ cannot be improved substantially and obtains the desired approximation ratio. Thus parallel-greedy takes a primal-dual approach equally concerned with maximizing the objective as driving down the gradient.

We first assume oracle access to values $F(x)$ and gradients $F'(x)$. The algorithm and analysis immediately extends to approximate oracles that return relative approximation to these quantities. Such oracles do exist (and are readily parallelizable) for many real submodular functions of interest. Given oracle access to $f$, one can implement sufficiently accurate oracles to $F(x)$ and $F'(x)$ without increasing the depth but with many more oracle calls to $f$. In Section 3.3, we present a randomized discretization of parallel-greedy that improves the oracle complexity w/r/t $f$. Note that the algorithms in [6, 21] call $f$ directly and do not assume oracle access to $F$ or $F'$.

3.1 Approximation ratio

We first analyze the approximation ratio of the output solution $\hat{x}$. The main observation is that $\lambda$ is an upper bound on the gap $\text{OPT} - F(x)$.

**Lemma 3.1.** At any point, we have $\lambda \geq \text{OPT} - F(x)$.

**Proof.** The claim holds initially. Whenever $x$ is increased, $\text{OPT} - F(x)$ decreases since $F(x)$ is monotone, and hence the claim continues to hold. Whenever $\lambda$ is about to be decreased in (2.B.1.a), we have $S$ empty (or the algorithm terminates since $\langle x, \mathbb{1} \rangle = k$) with respect to the current value of $\lambda$. Thus, if $z$ is an optimal solution then we have

$$
\text{OPT} - F(x) \leq F_x(z) \leq \langle F'(x), z \rangle \leq (1 - \epsilon)\frac{\lambda}{k}\langle z, \mathbb{1} \rangle \leq (1 - \epsilon)\lambda
$$

by (1) monotonicity of $F$, (2) monotonic concavity of $F$, (3) emptiness of $S$ w/r/t $\lambda$, and (4) the fact that $\langle z, \mathbb{1} \rangle \leq k$.

The connection between $\lambda$ and $\text{OPT} - F(x)$ allows us to reinterpret (2.B.1.a) as saying that we are closing the objective gap at a good rate in proportion to the increase in the (fractional) cardinality of $x$. This is the basic invariant in standard analyses of the greedy algorithm that implies that greedy achieves a (near) $(1 - e^{-1})$-approximation, as follows.

**Lemma 3.2.** The output $\hat{x}$ satisfies $F(\hat{x}) \geq (1 - O(\epsilon))(1 - e^{-1})\text{OPT}$.

**Proof.** Let $t = \sum_i x_i$ be the total sum of the coordinates. From the preceding lemma and the choice of $\delta$ in the algorithm, we have $F_S(x + \delta S) \geq (1 - \epsilon)^2 \frac{\delta |S|}{k} (\text{OPT} - F(x))$ in (2.B.1.a), hence

$$
\frac{dF(x)}{dt} \geq \frac{(1 - \epsilon)^2}{k} (\text{OPT} - F(x)),
$$
hence
\[ F(x) \geq \left( 1 - \exp\left(-\frac{(1 - \epsilon)^2 t}{k}\right) \right) \OPT. \]

In particular, if \( t = \langle 1, x \rangle = k \) at the end of the algorithm, we have
\[ F(x) \geq (1 - O(\epsilon))(1 - e^{-1}) \OPT. \]

If \( \lambda \leq e^{-1} \OPT \), then \( F(x) \geq (1 - e^{-1}) \OPT \). In either case, the output \( \hat{x} \) satisfies \( F(\hat{x}) \geq (1 - O(\epsilon))(1 - e^{-1}) \OPT \).

### 3.2 Iteration count

We now analyze the iteration count of parallel-greedy. The key observation lies in line (2.B.1.a). If \( \delta \) is determined by line (2.B.1.a), then the margin of taking \( S \) uniformly has dropped significantly. In this case, as the next lemma shows, a significant fraction of the coordinates in \( S \) must have had their marginal returns decrease enough to force them to drop out of \( S \). The iteration can then be charged to the geometric decrease in \( |S| \).

**Lemma 3.3.** If \( F_x(x + \delta S) = (1 - \epsilon)^2 \frac{\delta |S|}{k} \), then the step (3.B.1) decreases \( |S| \) by at least a \((1 - \epsilon)\)-multiplicative factor. This implies that, for fixed \( \lambda \), the loop at (3.B) iterates at most \( O\left( \frac{\log n}{\epsilon^2} \right) \) times, and at most \( O\left( \frac{\log n}{\epsilon^2} \right) \) times total. That is, each step in greedy iterates at most \( O\left( \frac{\log n}{\epsilon^2} \right) \) times.

**Proof.** Let \( x' \) and \( S' \) denote the values of \( x \) and \( S \) before updating, and let \( x'' \) and \( S'' \) denote the values of \( x \) and \( S \) after. We want to show that \( |S''| \leq (1 - \epsilon)|S'| \). We have
\[
(1 - \epsilon)^2 \frac{\delta |S'|}{k} \overset{(1)}{=} F_x(x'') \overset{(2)}{\geq} \langle F'(x''), \delta S'' \rangle \overset{(3)}{\geq} \langle F'(x''), \delta S'' \rangle \geq (1 - \epsilon)\lambda \frac{\delta |S''|}{k}.
\]

by (1) choice of \( \delta \), (2) monotonicity, and (3) definition of \( S'' \). Dividing both sides by \((1 - \epsilon)\lambda\), we have \( |S''| \leq (1 - \epsilon)|S'| \).

One implementation detail is finding \( \delta \) in the inner loop. We can assume that \( \text{poly}(\epsilon/n) \leq \delta \leq k \) (since below \( \text{poly}(\epsilon/n) \), the gradient \( F'(x) \) does not change substantially). It is easy to see that a (say) \((1 + \epsilon/2)\)-multiplicative approximation of the exact value of \( \delta \) suffices. (A more detailed discussion of approximating \( \delta \) is in the more subtle setting of generic packing constraints is given later in Section 4.4.) Hence we can try all \( O(\log(n)/\epsilon) \) powers of \((1 + \epsilon)\) between \( \text{poly}(\epsilon/n) \) and 1 to find a sufficiently good approximation of \( \delta \). A second implementation detail regards to initial value of \( \lambda \) for upper bounding \( \OPT \). Standard tricks allow us to obtain a constant factor without increasing the depth; see the related discussing w/r/t general packing constraints in Section 4.4.

**Remark 3.4.** One might try to extend parallel-greedy to non-monotone (but still nonnegative) submodular functions. Here the difficulty is not so much in adapting Lemma 3.1 as similar technicalities have been overcome in the sequential setting. The difficulty lies in Lemma 3.3 specifically, we invoke monotonicity in step (2) of the derivation. The basic packing argument is that if the gradient is bad on average, and all the coordinates are nonnegative, then many of the coordinates should be pretty bad. If we remove monotonicity, then the gradient may be bad just because a few coordinates become negative.
randomized-parallel-greedy($f, N, k, \epsilon$)

1. $Q \leftarrow \emptyset$, $t \leftarrow 0$, $\lambda \leftarrow \text{OPT}$ // or any $\lambda \geq \text{OPT}$
2. while $t \leq (1 - 2\epsilon)k$ and $\lambda \geq e^{-1}\text{OPT}$
   A. let $S = \left\{ j \in N : f_Q(j) \geq \frac{(1 - \epsilon)\lambda}{k} \right\}$
   B. while $S$ is not empty and $t \leq (1 - 2\epsilon)k$
      i. chose $\delta$ maximal s.t.
         a. $F_Q(Q + \delta S) \geq (1 - \epsilon)^2\lambda \delta |S| / k$
         b. $t + \delta |S| \leq (1 - 2\epsilon)k$
      ii. sample $R \sim \delta S$
      iii. $Q \leftarrow Q \cup R$, $t \leftarrow t + \delta |S|$
      iv. update $S$
   C. $\lambda \leftarrow (1 - \epsilon)\lambda$
3. return $x$

**Figure 2**: A randomized, combinatorial variant of the previous parallel-greedy algorithm for cardinality constraints.

### 3.3 Oracle complexity w/r/t $f$

The preceding algorithm and analysis were presented under the assumption that gradients of the multilinear extension $F$ were easy to compute (at least, approximately). This assumption holds for many applications of interest. In this section, we consider a model where we only have oracle access to the underlying set function $f$.

We first note that $F(x)$ and $F'(x)$ can still be approximated (to sufficient accuracy) by taking the average of $f(Q)$ for many random samples $Q \sim x$. To obtain $(1 \pm \epsilon)$-accuracy with high probability for either $F(x)$ or a single coordinate of $F'(x)$, one requires about $O\left(\frac{n \log n}{\epsilon^2}\right)$ samples, each of which may be computed in parallel (see Lemma 2.4). Thus parallel-greedy still has $\tilde{O}\left(\frac{1}{\epsilon^2}\right)$ depth in this model. However, the total number of queries to $f$ increases to $\tilde{O}\left(n^2 \text{poly}(1/\epsilon)\right)$, because computing an entire gradient to assemble $Q$ in line (3.A) requires $\tilde{O}\left(\frac{n^2}{\epsilon^2}\right)$ queries to $f$.

To reduce the oracle complexity w/r/t $f$, we propose the alternative algorithm parallel-greedy in Figure 2, which is guided by the previous parallel-greedy algorithm, but maintains a discrete set $Q \subseteq N$ rather than a fractional solution $x$. The primary difference is in steps (2.B.ii) and (2.B.iii), where rather than add the fractional solution $\delta S$ to our solution, we first sample a set $R \sim \delta S$ (where each coordinate in $S$ is drawn independently with probability $\delta$), and then we add $R$ to the running solution. The primary benefit to this rounding step is that computing the gradient $F'(x)$ is replaced by computing the margins $f_Q$, which requires only a constant number of oracle calls per element.

We defer the analysis of randomized-parallel-greedy to Appendix A. At a high level, one
can see that the key points to the analysis of parallel-greedy now hold in expectation. Further techniques from randomized analysis adapt the essential invariants from parallel-greedy to the additional randomization to obtain the following bounds.

Theorem 3.5. Let \( \epsilon > 0 \) be given, let \( f : 2^N \rightarrow \mathbb{R}_{\geq 0} \) be a normalized, monotone submodular function in the oracle model, and let \( k \in \mathbb{N} \). Then with high probability, randomized-parallel-greedy computes a \((1 - \epsilon)(1 - e^{-1})\) multiplicative approximation to the maximum value set of cardinality \( k \) with \( O\left(\log \frac{n}{\epsilon^2}\right) \) expected adaptivity and \( \tilde{O}\left(\frac{n}{\epsilon^4}\right) \) expected oracle calls to \( f \).

4 Parallel maximization with packing constraints

We now consider the general setting of maximizing the multilinear relaxation in the setting of explicit packing constraints in the form below.

\[
\text{maximize } F(x) \text{ over } x \in [0,1]^N \text{ s.t. } Ax \leq 1.
\]

We refer the reader to some preprocessing steps outlined in Section 2. In Figure 3, we give a parallel algorithm that combines the many-coordinate update and greedy step size of parallel-greedy with multiplicative weight update techniques that navigates the packing constraints. The high-level MWU framework follows the one from [19].

We briefly explain the algorithm. The framework from [19] has a notion of time, maintained in the variable \( t \), that goes from 0 to 1. The algorithm maintains non-negative weights \( w_i \) for each constraint \( i \) that reflect how tight is each constraint. In the sequential setting, the algorithm in [19] combines continuous-greedy and MWU as follows. In each iteration, given the current vector \( x \), it finds a solution to the following linear optimization problem with a single non-trival constraint obtained via a weighted linear combination of the \( m \) packing constraints:

\[
\text{max } \langle F'(x), y \rangle \text{ s.t. } \langle w, Ax \rangle \leq \langle w, 1 \rangle \text{ and } y \geq 0.
\]

The optimum solution to this relaxation is a single coordinate solution \( \gamma e_j \) where \( j \) maximizes the ratio \( \frac{F'(x)}{(w, Ae_h)} \). The algorithm then updates \( x \) by adding \( \delta e_j \) for some appropriately small step size \( \delta \) and then updates the weights. The weights are maintained according to the MWU rule and (approximately) satisfy the invariant that \( w_i = \exp(\eta(Ax_i)) \) for \( \eta = \Theta(\log m/\epsilon) \).

The parallel version differs from the sequential version as follows. When solving the Lagrangean relaxation it considers all good coordinates (the set \( S \)) whose ratios are close to \( \lambda = \max_h \frac{F_h'(x)}{(w, Ae_h)} \) and simultaneously updates them. The step size has to be adjusted to account for this, and the adjusted step size is a primary difference from the algorithm in [19]. The sequential algorithm takes a greedy step for the sake of obtaining width independence. In the parallel setting, two different considerations come in to play. First, the simultaneous update to many coordinates means that the step size needs to be small enough such that the gradient does not change too much, but that it does change sufficiently so that we can use an averaging argument to limit the number of iterations. Second, if the gradient is not the bottleneck, then the bottleneck comes from limiting the change in \( x \) to ensure the weights do not grow too rapidly. In this case, we ensure that each coordinate \( j \in S \) increases by at least \((1 + \epsilon^2/\log m)\) multiplicative factor, which can only happen a limited number of times due to the starting value of \( x \).

We organize the formal analysis into four parts. The first part, Section 4.1, concerns the packing constraints, and shows that the output \( \hat{x} \) satisfies \( A\hat{x} \leq (1 + O(\epsilon))1 \). The second part, Section 4.2...
parallel-mwu-greedy

1. for each $j \in [n]$
   
   A. choose $x_j$ maximal in $[0, 1]$ s.t. $A_{ij} x_j \leq \frac{\varepsilon}{n}$ for all $i \in [m]$

2. $t \leftarrow 0$, $\lambda \leftarrow \text{OPT}$ // or any upper bound for OPT

3. define $w = w(x)$ by $w_i = \exp\left(\frac{\log m}{\varepsilon} (Ax)_i\right)$ for $i \in [m]$

4. while $t < 1$ and $\lambda \geq e^{-1} \text{OPT}$
   
   A. $W \leftarrow \langle w, 1 \rangle$
   
   B. let $S = \left\{ j \in \mathcal{N} : \frac{F_j'(x)}{\langle w, Ae_j \rangle} \geq (1 - \varepsilon)^3 \frac{\lambda}{W}, F_j'(x) \geq \frac{\varepsilon (1 - \varepsilon) \lambda}{n} \right\}$

   C. while $S$ is not empty, $(1 - \varepsilon)\langle w, 1 \rangle \leq W$, and $t \leq 1$
      
      i. $\gamma \leftarrow \frac{W}{\langle w, A(x \wedge S) \rangle}$
      
      ii. choose $\delta > 0$ large as possible s.t. for $x' = x + \delta \gamma (x \wedge S)$
           
           a. $F(x') \geq (1 - \varepsilon)^4 \delta \lambda$.
           
           b. $\gamma \delta \leq \frac{\varepsilon^2}{4 \log m}$
           
           c. $t + \delta \leq 1$.
      
      iii. $x \leftarrow x + \delta \gamma (x \wedge S)$ // update $x$

      iv. $t \leftarrow t + \delta$ // update time

      v. update $w$ and $S$
   
   D. if $(1 - \varepsilon)\langle w, 1 \rangle \leq W$ and $S$ is empty
      
      i. $\lambda \leftarrow (1 - \varepsilon) \lambda$.

5. return $x$

Figure 3: A parallel implementation of the MWU/continuous-greedy algorithm of Chekuri et al. [19].

concerns the approximation ratio, and shows that the output $\hat{x}$ has an approximation factor of $F(\hat{x}) \geq \left( (1 - e^{-1}) - O(\varepsilon) \right) \text{OPT}$. The third part, Section 4.3, analyzes the number of iterations and shows that each step in Figure 3 is executed at most $\tilde{O}\left(\frac{1}{\varepsilon^4}\right)$ times. The last part, Section 4.4, addresses the total number of oracle calls. The lemmas in these parts together prove Theorem 1.1.

We first note the monotonicity of the various variables at play.

Observation 4.1. Over the course of the algorithm, $x$ is increasing, $w$ is increasing, $t$ is increasing, $F(x)$ is increasing, $W$ is increasing, $F'(x)$ is decreasing, and $\lambda$ is decreasing. Within the loop at 4.C. *, $S$ is decreasing.
4.1 Feasibility of the packing constraints

We first show that the algorithm satisfies the packing constraints to within a \((1 + O(\epsilon))\)-factor. The first fact shows that the weights grow at a controlled rate as \(x\) increases. The basic proof idea, which appears in Young [37], combines the fact that we increase (some coordinates) of \(x\) by a small geometric factor, and the fact that \(x\) is recursively near-feasible. This implies that the increase in load of any constraint is by at most a small additive factor, hence the weights (which exponentiate the loads) increase by at most a small geometric factor.

**Lemma 4.2.** At the beginning of each iteration of step 4.C.iii, if \(Ax \leq 21\), then

\[
w(x + \delta \gamma(x \land S)) \leq (1 + \epsilon)\delta(w, \mathbb{1}).
\]

**Proof.** For each constraint \(i \in [m]\), we have

\[
w_i(x + \delta \gamma(x \land S)) = w_i(\delta \gamma(x \land S))w_i(x) \leq w_i \left( \frac{\epsilon^2}{4 \log m} (x \land S) \right)w_i(x)
\]

\[
= e^{\frac{\epsilon^2}{4 \log m}}(A(x \land S))_i w_i(x) \leq e^{\epsilon/2}w_i(x) \leq (1 + \epsilon)w_i(x)
\]

by (1) choice of \(\delta\) per step 4.C.ii.b, (2) \((A(x \land S))_i \leq 2\), and (3) upper bounding the Taylor expansion of \(e^{\epsilon/2}\).

**Lemma 4.3.** At the beginning of each iteration of step 4.C.iii, if \(Ax \leq 21\), then

\[
\langle w(x + \delta \gamma(x \land S)), \mathbb{1} \rangle \leq \delta(1 + \epsilon)\log m \frac{\epsilon}{\epsilon} \langle w, \mathbb{1} \rangle.
\]

**Proof.** The following is a standard proof from the MWU framework, where the important invariant is preserved by choice of \(\delta\) w.r.t. step 4.C.ii.b. Let \(x' = x + \delta \gamma(x \land S)\). Define \(\omega(\tau) = w(x + \tau \gamma(x \land S))\), where we recall that

\[
w_i(x + \tau \gamma(x \land S)) = w_i \exp \left( \frac{\tau \gamma \log m}{\epsilon} (A(x \land S)) \right).
\]

We have

\[
w(x') - w(x) = \omega(\tau) - \omega(0) = \int_0^\delta \frac{d}{d\tau} \omega(\tau) d\tau = \gamma \frac{\log m}{\epsilon} \int_0^\delta \langle \omega(\tau), (A(x \land S)) \rangle d\tau
\]

\[
\leq (1 + \epsilon)\gamma \frac{\log m}{\epsilon} \int_0^\delta \langle w, A(x \land S) \rangle d\tau \leq (1 + \epsilon)\frac{\log m}{\epsilon} \int_0^\delta \langle w, \mathbb{1} \rangle d\tau
\]

by (1) monotonicity of \(\omega\) and Lemma 4.8 and (2) choice of \(\gamma\).

**Lemma 4.4.** The output of the algorithm \(\hat{x}\) satisfies \(A\hat{x} \leq (1 + 3\epsilon)\mathbb{1}\).

**Proof.** We prove a slightly stronger claim; namely, that at each time \(t\), one has \(Ax \leq ((1 + \epsilon)t + 2\epsilon)\mathbb{1}\).

Consider Lemma 4.3 so long as \(Ax \leq 2\mathbb{1}\), by interpolating (the upper bound on) \(\langle w, \mathbb{1} \rangle\) as a continuous function of \(t\), we have

\[
\frac{d}{dt} \langle w, \mathbb{1} \rangle \leq (1 + \epsilon)\frac{\log m}{\epsilon} \langle w, \mathbb{1} \rangle.
\]
Initially, when \( t = 0 \), we have \( Ax \leq \epsilon \mathbb{1} \) by choice of \( x \). Thus, \( w_i \leq e^{\log m} = m \) for each \( i \in [m] \), and in sum we have \( \langle w, \mathbb{1} \rangle \leq m^2 \). Since \( w_i(x) = \exp \left( \frac{\log m}{\epsilon} (Ax)_i \right) \leq \langle w, \mathbb{1} \rangle \leq m^2 \), this also implies that \( (Ax)_i \leq 2e \leq 2 \) initially.

Solving the differential inequality (5) with initial value \( m^2 \), we have

\[
\langle w, \mathbb{1} \rangle \leq m^2 \exp \left( (1 + \epsilon) \frac{\log m}{\epsilon} t \right) = \exp \left( \frac{\log m}{\epsilon} ((1 + \epsilon)t + 2\epsilon) \right).
\]

for all \( t \in [0,1] \) as long as \( Ax \leq 2 \). In particular, since \( w_i = \exp \left( \frac{\log m}{\epsilon} (Ax)_i \right) \leq \langle w, \mathbb{1} \rangle \) for each \( i \), we have

\[
Ax \leq ((1 + \epsilon)t + 2\epsilon) \mathbb{1} \leq (1 + 3\epsilon) \mathbb{1} \leq 21.
\]

By induction on \( t \), we have \( Ax \leq (1 + 3\epsilon) \mathbb{1} \) for all \( t \in [0,1] \). \( \square \)

### 4.2 Approximation ratio

We now analyze the approximation ratio of the output solution \( \hat{x} \). The main observation, similar to Lemma 3.1 for parallel-greedy, is that \( \lambda \) is an upper bound on the gap \( \text{OPT} - F(x) \).

**Lemma 4.5.** At all times, \( \lambda \geq \text{OPT} - F(x) \).

**Proof.** The claim holds initially with \( \lambda \geq \text{OPT} \) and \( F(x) \geq 0 \). Whenever \( x \) is increased and \( \lambda \) is unchanged, \( F(x) \) increases due to monotonicity of \( F \), hence the claim continuous to hold. Whenever \( \lambda \) is about to be decreased in \((4.D.i)\), we have \( S \) empty with respect to the current value of \( \lambda \). Thus, letting \( z \) be an optimal solution, we have

\[
\text{OPT} - F(x) \leq \frac{F(z)}{W} + (1 - \epsilon) \lambda \frac{\langle w, Az \rangle}{W} + \epsilon(1 - \epsilon)\lambda
\]

by (1) monotonicity of \( F \), (2) nonnegative concavity, (3) \( S = \emptyset \), (4) \( Az \leq \mathbb{1} \), and (5) \( (1 - \epsilon)\langle w, \mathbb{1} \rangle \leq W \). Thus, after replacing \( \lambda \) with \( (1 - \epsilon)\lambda \), we still have \( \lambda \geq \text{OPT} - F(x) \). \( \square \)

**Lemma 4.6.** The output \( \hat{x} \) of the algorithm satisfies \( F(\hat{x}) \geq (1 - O(\epsilon))(1 - e^{-1}) \text{OPT} \).

**Proof.** From the preceding lemma and line \((4.C.i.a)\) of the algorithm, we have the following. Suppose \( x \) changes to \( x' \) with step size \( \delta \). We have

\[
F(x') - F(x) \geq (1 - \epsilon)^4 \delta \lambda \geq (1 - \epsilon)^4 \delta (\text{OPT} - F(x)),
\]

and \( t \) increases by \( \delta \). Therefore, \( F(x) \), as a function of \( t \), increases at a rate of \( F(x) \geq \left(1 - e^{-1} - O(\epsilon) t\right) \text{OPT} \). In particular, since the algorithm terminates with either \( \lambda \leq e^{-1} \text{OPT} \) or \( t \geq 1 \), the output \( \hat{x} \) satisfies \( F(\hat{x}) \geq (1 - O(\epsilon))(1 - e^{-1}) \text{OPT} \). \( \square \)
4.3 Iteration count

In this section, we analyze the total number of iterations in parallel-mwu. parallel-mwu consists of two nested loops: an outer loop (4.1), where $W$ and $\lambda$ are adjusted, and an inner loop (4.3), which increases $x$ uniformly along “good” coordinates $w/r/t$ fixed values of $W$ and $\lambda$. We first analyze the number of iterations of the outer loop.

**Lemma 4.7.** In each iteration of the outer loop (4.1) except for the last, either $\lambda$ decreases by a $(1-\epsilon)$-multiplicative factor, or $\langle w, 1 \rangle$ increases by a $\frac{1}{1-\epsilon}$-multiplicative factor. This implies that, since $\lambda$ ranges from OPT to $e^{-1}$OPT, and $\langle w, 1 \rangle$ ranges from $m$ to $m^{O(1/\epsilon)}$, there are at most $O\left(\frac{\log m}{\epsilon^2}\right)$ iterations of the outer loop.

**Proof.** The inner loop (4.3) terminates when either $S = \emptyset$, $(1-\epsilon)\langle w, 1 \rangle = W$, or $t = 1$. If $S = \emptyset$, then $\lambda$ decreases in line (4.D.i). If $(1-\epsilon)\langle w, 1 \rangle = W$, then since $W$ was the value of $\langle w, 1 \rangle$ at the beginning of the iteration, we have that $\langle w, 1 \rangle$ increased by a $\frac{1}{1-\epsilon}$-multiplicative factor. If $t = 1$, then this is the last iteration of (4.1).

We now analyze the number of iterations of the inner loop (4.1) for each fixed iteration of the outer loop (4.1). Each iteration of the inner loop, except for possibly the last, fixes $\delta$ based on either (4.C.ii.a) or (4.C.ii.b). We first bound the number of times $\delta$ can be chosen based on (4.C.ii.b). The key idea to the analysis (due to [37]) is that one can only geometrically increase the coordinates in $S$ a small number of times before violating the upper bounds on the coordinates implied by $Ax \leq (1+O(\epsilon))I$.

**Lemma 4.8.** In each iteration of the outer loop (4.1), $\delta$ is determined by (4.C.ii.b) at most $O\left(\frac{\log n \log m}{\epsilon^2}\right)$ times.

**Proof.** If $\delta$ is determined by (4.C.ii.b) more than $O\left(\frac{\log n \log m}{\epsilon^2}\right)$ times, consider the coordinate $j$ that survives in $S$ throughout all of these many iterations. Such a coordinate exists because the set $S$ is decreasing throughout the inner loop. The initial value of $x_j$ sets $A_{ij}x_j \geq \epsilon/n$ for some $i \in [m]$, and by Lemma 4.3 $A_{ij}x_j \leq (Ax)_i$ cannot exceed $1+O(\epsilon)$. Each iteration where $\delta$ is determined by (4.C.ii.b) increases $x_j$ (hence $A_{ij}x_j$) by a $\left(1 + \Omega\left(\frac{\epsilon^2}{\log m}\right)\right)$-multiplicative factor. Applying this multiplicative increase more than $O\left(\frac{\log n \log m}{\epsilon^2}\right)$ times would violate the upper bound on $A_{ij}x_j$.

We now analyze the number of times $\delta$ can be chosen based on (4.C.ii.a). The following lemma is analogous to Lemma 3.3 but the analysis is more subtle because of (a) the general complexity added by the weights and (b) the fact that the underlying potential function is not monotone.

**Lemma 4.9.** For a fixed iteration of the outer loop (4.1), $\delta$ is determined by (4.C.ii.a) at most $O\left(\frac{\log n}{\epsilon}\right)$ times.

**Proof.** The overall proof is based on the potential function $\langle F'(x), x \land S \rangle$, which is always in the range $\text{poly}\left(\frac{\epsilon}{n}\right)OPT \leq \langle F'(x), x \land S \rangle \leq \text{poly}(n)OPT$ for nonempty $S$. An important distinction
from the potential function of Lemma 3.3 is that $\langle F'(x), x \wedge S \rangle$ is not monotone: $F'(x)$ and $S$ are both decreasing, but $x$ is increasing. On one hand, the total growth by $x$ is bounded above by an $O(n/\epsilon)$-multiplicative factor coordinatewise by our initial choice of $x_j$, as discussed in Lemma 4.8. On the other hand, we claim that whenever $\delta$ is determined by (4.C.ii.a), $\langle F'(x), x \wedge S \rangle$ decreases by a $(1 - \Omega(\epsilon))$-multiplicative factor. We prove the claim below, but suppose for the moment that this claim holds. Then we have a poly$(n/\epsilon)$-multiplicative range for $\langle F'(x), x \wedge S \rangle$ with non-empty $S$, and that the total increase of $\langle F'(x), x \wedge S \rangle$ is bounded above (via the bound on the growth of $x$)

by a $O\left(\frac{n}{\epsilon}\right)$-multiplicative factor. It follows that $\delta$ is determined by (4.C.ii.a) at most $O\left(\frac{\log n}{\epsilon}\right)$

times until $\langle F'(x), x \wedge S \rangle$ falls below the lower bound poly$(\epsilon/n)$, and $S$ is empty.

Now we prove the claim. Let $x' = x + \delta \gamma(x \wedge S)$, $w' = w(x')$, and $S'$ denote the values of $x$, $w$ and $S$ after the update in step (4.C.v). We want to show that $\langle F'(x'), x' \wedge S \rangle \leq (1 - \epsilon \lambda)\langle F'(x), x \wedge S \rangle$

for some constant $c > 0$. We have

\[
\delta \gamma \langle F'(x'), x' \wedge S' \rangle \leq \left(1 + \frac{\epsilon^2}{2 \log m}\right) \delta \gamma \langle F'(x'), x \wedge S \rangle
\]

by (1) $x' \leq \left(1 + \frac{\epsilon^2}{2 \log m}\right) x$ by choice of $\delta \gamma$, (2) nonnegativity of $F'$ and $S' \subseteq S$, (3) monotonic concavity of $F$, (4) choice of $\delta$, (5) choice of $\gamma$, and (6) definition of $S$. Dividing both sides by $\delta \gamma$ gives the inequality we seek.

\[
\text{Lemma 4.10. Each step of parallel-mwu has at most } O\left(\frac{\log^2 m \log n}{\epsilon^4}\right) \text{ iterations.}
\]

\[
\text{Proof. The preceding lemmas show that we have } O\left(\frac{\log m}{\epsilon^2}\right) \text{ iterations of the outer loop and }\]

\[
O\left(\frac{\log m \log n}{\epsilon^2}\right) \text{ iterations of the inner loop per iteration of the outer loop.}
\]

\[
4.4 \text{ Number of oracle calls and additional implementation details}
\]

In this section, we briefly account for the total work and number of oracle calls of the algorithm. The bottlenecks are step (4.C.ii), where we search for a value of $\delta$ satisfying certain constraints, and (4.C.v), where one updates $w$ and $S$.

\[
\text{Estimating OPT: The algorithm requires a value } \lambda \text{ that is an upper bound on OPT. Preprocessing allows us to choose } \lambda = \sum_j f(j) \text{ and we have OPT} \leq \lambda \leq n \text{ OPT. It is also useful to}
\]
have an estimate that is within a constant factor. This can be done (a standard idea) by running
the algorithm in parallel with $O(\log n)$ geometrically increasing values of $\lambda$ and picking the best
solution from the parallel runs.

**Step size:** We first note that the greedy step size $\delta$ does not have to be computed exactly, and
that a $(1 + \epsilon)$-multiplicative factor approximation suffices. Indeed, suppose the algorithm is at step
(4.C.ii). Let $\delta$ be the exact maximum value satisfying the conditions (4.C.ii.a), and let $\tilde{\delta}$ be
a value such that $\delta \leq \tilde{\delta} \leq \min\left\{(1 + \epsilon)\delta, \frac{2^\epsilon}{\log m}\right\}$. We want to show that $\tilde{\delta}$ approximately satisfies
(4.C.ii.a). Indeed, we have

$$F_x\left(x + \tilde{\delta}(x \wedge S)\right) \geq F_x(x + \delta(x \wedge S)) \geq (1 + \epsilon)^4\delta \lambda \geq \frac{(1 - \epsilon)^4}{1 + \epsilon} \delta \lambda.$$ by (1) monotonicity, (2) choice of $\delta$, and (3) choice of $\tilde{\delta}$. The inequality (4.C.ii.a) is invoked
only in Lemma 4.6. It is easy to see that the slightly weaker inequality with $\delta$ is enough to prove
Lemma 4.6 with only a change in the hidden constants. The other point where (4.C.ii.a) is invoked in the proof is when $\delta$ is determined by (4.C.ii.a). Here we need only observe that increasing $x$ further along $x \wedge S$ with a larger step size $\delta > \delta$ only decreases the coordinate values $F(x)$ and thereby $\langle F'(x), S \rangle$.

The second claim is that one needs only guess $O(\log(n)/\epsilon)$ values of $\delta$. Indeed, we know that
$\gamma \delta \leq \epsilon^2 / \log m$. On the other hand, if $\gamma \delta \leq \text{poly}(\epsilon/n)$, then it is easy to show that (4.C.ii.a) is still satisfied. Thus one only needs to check $\tilde{\delta}$ for $O(\log(n)/\epsilon)$ powers of $(1 + \epsilon)$ between between
poly $(\epsilon/n)$ and $\epsilon^2 / \log m$.

**Oracle calls to $F$ and $F'$:** Now, to execute step (4.C.ii), we must evaluate $F(x')$ for $O(\log(n)/\epsilon)$ different possible choices of $\delta$. To execute step (4.C.iii), we need $\tilde{O}(N)$ work to update
$w$, and then obtain a partial derivative $F'_j(x)$ for each coordinate $j \in [n]$. Since the depth of either
step is $O\left(\frac{\log^2 m \log n}{\epsilon^2}\right)$, we see that the algorithm requires $\tilde{O}(N/\epsilon^2)$ total work (where $N$ is
the total number of nonzeros in $A$), $O\left(\frac{\log^2 m \log^2 n}{\epsilon^2}\right)$ calls to evaluate $F(x)$, and $O\left(\frac{n \log^2 m \log^2 n}{\epsilon^2}\right)$
to individual coordinates of $F'(x)$. It is not hard to see that if we have a $(1 + \epsilon)$ multiplicative
approximation for these quantities then the whole analysis still goes through.

**Oracle calls to $f$:** If we assume only oracle access to the underlying submodular function $f$, then
we need to estimate $F(x)$ and $F'_j(x)$ based on random sampling. Each sample constitutes a query
to $f$ and the samples are done in parallel followed by aggregation. We note that the starting value
of $x$ in the algorithm satisfies the property that $F(x) \geq \frac{\epsilon}{n} \text{OPT}$. Moreover, we can assume at any
point that $F'_j(x) \geq \frac{\epsilon}{n} \text{OPT}$; for if it is smaller than even taking all such coordinates to 1 would
contribute at most $\epsilon \text{OPT}$. Following the discussion after Lemma 2.4 we see that one can obtain
an estimate for $F'_j(x)$ that is, with high probability, a $(1 + \epsilon)$ multiplicative approximation using
$\tilde{O}(n/\epsilon^3)$ samples. Similarly with $\tilde{O}(n/\epsilon^3)$ samples one can get a $(1 + \epsilon)$-approximation for $F(x)$
with high probability. From the preceding analysis the algorithm makes $\tilde{O}(n/\epsilon^3)$ calls to individual
coordinates of $F'$ and $\tilde{O}(1/\epsilon^2)$ calls to $F$. Thus the total number of oracle calls to $f$ is $\tilde{O}(n^2/\epsilon^5)$.
Since the correctness probability of each estimate is inversely polynomial in $n$, we can take a union
bound over the $\tilde{O}(n/\epsilon^2)$ estimates that the algorithm requires.
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A Analysis of randomized-parallel-greedy

In this section, we analyze randomized-parallel-greedy and obtain the bounds of Theorem 3.5. As mentioned when introducing the algorithm earlier, the basic idea is that randomized-parallel-greedy preserves the most important invariants of parallel-greedy in expectation.

The variable $t$ is used to track the expected cardinality of $|Q|$. One can simplify the algorithm by replacing the variable $t$ with $|Q|$ wherever it appears, and the analysis would generally still hold. (In this case, a variable similar to $t$ could be introduced in the analysis.) The inclusion of $t$ makes the analysis more straightforward, as we can use $t$ to track progress throughout the algorithm.

Intuitively, the key points to the analysis of parallel-greedy now hold in expectation. The only source of randomization is in (2.B.ii), where we sample a set $R \sim \delta Q$ to add to $S$. Let us
assume that \( k \geq C \frac{\log n}{\epsilon^2} \) for some constant \( C > 1 \), since otherwise one can simply run the sequential greedy algorithm and obtain the desired depth. Then the cardinality of the computed solution \( Q \) is randomized, but is concentrated at \( t \leq (1-\epsilon)k \) because each independent coin toss adjusts the cardinality by at most 1, and \( t \) is at least \( (1-\epsilon)C \frac{\log n}{\epsilon^2} \).

**Lemma A.1.** With high probability, we have \( |S| \leq (1+\epsilon)t \leq k \) throughout the algorithm.

**Proof.** We assume that \( k \geq C \frac{\log n}{\epsilon^2} \) for a large constant \( c > 0 \), since otherwise one can simply run the sequential greedy algorithm instead. In this case we also have \( t \geq C \frac{\log n}{\epsilon^2} \) throughout the algorithm. As discussed above, \( t \) tracks the expected cardinality of \( Q \), summing the expected increase \( |R| = \delta|S| \) over each iteration of (2.B.11). On the other hand, each random coin toss from sampling \( R \sim \delta S \) affects the cardinality of \( Q \) by at most 1. Since the expected cardinality of \( Q \) is \( t \geq C \frac{\log n}{\epsilon^2} \), this is at most a \( \frac{\epsilon^2}{C \log n} \)-fraction of the expected total. It follows from (online extensions of) the multiplicative Chernoff inequality that \( |Q| \leq (1+\epsilon)t \) all throughout the algorithm.

To make this argument formal, for \( \ell \in \mathbb{N} \) and \( j \in \mathcal{N} \) let \( X_{\ell,j} \in \{0,1\} \) indicate whether or not \( j \) is sampled by \( R \) in the \( \ell \)th iteration of (2.B.11). Each \( X_{\ell,j} \) depends on \( X_{\ell',j'} \) for previous iterations \( \ell' < \ell \) and \( j \in [n] \), but once these outcomes are fixed, each \( X_{\ell,j} \) is independent of the other indicator variables in the \( \ell \)th iteration. Let \( Y_{\ell,j} = \mathbb{E}[X_{\ell,j} \mid X_{\ell',j'} \text{ for } \ell' < \ell, j \in j'] \) by the expected value of \( X_{\ell,j} \) going in to the \( \ell \)th round. If we let \( S_\ell \) and \( \delta_\ell \) denote the values of the set \( S \) and step size \( \delta \) during the \( \ell \)th iteration of (2.B.11), then we have \( Y_{\ell,j} = \delta \) if \( j \in S_\ell \), and 0 otherwise. In particular, we have \( \sum_{\ell \in \mathbb{N}} \sum_{j \in [n]} Y_{\ell,j} = \sum_{\ell \in \mathbb{N}} \delta_\ell |S_\ell| = t \leq (1-2\epsilon)k \) deterministically. By online extensions of the Chernoff inequality [Lemma C.1], we have

\[
P \left( \sum_{\ell,e} X_{\ell,e} \geq k \right) \leq P \left( (1+\epsilon) \sum_{\ell,e} Y_{\ell,e} + \epsilon k \right) \leq P \left( (1+\epsilon) \sum_{\ell,e} Y_{\ell,e} + O \left( \frac{\log n}{\epsilon} \right) \right) \leq \text{poly}(1/n),
\]

as desired. \( \Box \)

**Lemma A.2.** At each time \( t \), we have \( \mathbb{E}[f(Q)] \geq (1 - O(\epsilon)) \left( 1 - e^{-t/k} \right) \text{OPT} \). In particular, since the algorithm exits with either \( t = (1 - 2\epsilon)k \) or \( \text{OPT} - f(\widehat{Q}) \leq \epsilon^{-1} \text{OPT} \), the final set \( \widehat{Q} \) satisfies \( \mathbb{E}[f(\widehat{Q})] \geq (1 - O(\epsilon))(1 - \epsilon^{-1}) \text{OPT} \).

**Proof.** Given \( Q \), we have

\[
\mathbb{E}[f_Q(R) \mid Q] \overset{(1)}{=} F_Q(Q + \delta S) \geq (1 - \epsilon)^2 \frac{|S|}{k} \text{OPT} - f(Q)
\]

by (1) definition of the multilinear extension and (2) [Lemma 3.1] Taking expectations over \( Q \)

\[
\mathbb{E} \left[ \frac{df(Q)}{dt} \right] \geq \mathbb{E} \left[ \frac{(1 - \epsilon)^2}{k} \text{OPT} - f(Q) \right] = \frac{(1 - \epsilon)^2}{k} \text{OPT} - \mathbb{E}[f(Q)].
\]

By Fubini’s theorem, one can interchange the expectation and the derivative, which gives

\[
\frac{d}{dt} \mathbb{E}[f(Q)] = \mathbb{E} \left[ \frac{df(Q)}{dt} \right] \geq \frac{(1 - \epsilon)^2}{k} \text{OPT} - \mathbb{E}[f(Q)].
\]
Solving the differential inequality, we have
\[ E[f(Q)] \geq \left(1 - e^{-(1-\epsilon)^2 t/k}\right) \text{OPT} \geq (1 - O(\epsilon))(1 - e^{-t/k}) \text{OPT}, \]
as desired.

\[\text{Lemma A.3.} \quad \text{If } F_Q(Q + \delta S) = (1 - \epsilon^2 \lambda \frac{\delta |S|}{k}, \text{ then } |S| \text{ decreases by a } (1 - \epsilon)\text{-multiplicative factor in expectation. This implies that, for fixed } \lambda, \text{ the loop at (2.B.\*) iterates at most } O\left(\frac{\log n}{\epsilon^2}\right) \text{ times in expectation (via Lemma C.2). In total, the loop at (2.B.\*) iterates at most } O\left(\frac{\log n}{\epsilon^2}\right) \text{ times in expectation.} \]

\[\text{Proof.} \quad \text{We have}\]
\[(1 - \epsilon)^2 \lambda \frac{\delta |S|}{k} \overset{(1)}{=} F_Q(Q + \delta S) \overset{(2)}{=} \delta \langle F'(Q + \delta S), S \rangle \]
\[\overset{(3)}{=} \delta \sum_{j \in S} E[f_{Q\cup R}(j)] \overset{(4)}{=} \delta E \left[ \sum_{j \in S} f_{Q\cup R}(j) \right] \]
\[\overset{(5)}{=} \delta E \left[ \sum_{j \in S'} f_{Q\cup R}(j) \right] \overset{(6)}{=} \delta E \left[ |S'| \frac{(1 - \epsilon)\lambda}{k} \right] = (1 - \epsilon)\lambda \frac{\delta}{k} E [|S'|] \]

by (1) choice of \(\delta\), (2) monotonic concavity, (3) definition of \(F'\), (4) linearity of expectation, (5) monotonicity, and (6) definition of \(S'\). Dividing both sides by \(\epsilon \lambda \delta / k\), we have \(E [|S'|] \leq (1 - \epsilon) |S|\) in expectation.

We now prove Theorem 3.5

\[\text{Proof.} \quad \text{Let } \mu = E[f(Q)] \geq (1 - O(\epsilon))(1 - \epsilon^{-1}) \text{OPT}, \text{ per Lemma A.2. We first show that } f(Q) \geq \mu \text{ with high probability. Let } p = P[f(S) \geq (1 - \epsilon)\mu]. \text{ We first observe that}\]
\[P[f(S) \geq \text{OPT}] \overset{(1)}{=} P[|S| \geq k] \overset{(2)}{=} \frac{1}{\text{poly}(n)}\]

because (1) \(f(Q) \leq \text{OPT}\) whenever \(|Q| \leq k\) and (2) by the concentration bound Lemma A.1. We have
\[
\mu = E[f(Q)] \overset{(3)}{=} p E[f(Q) \mid f(Q) \leq (1 - \epsilon)\mu] + \left(1 - p - \frac{1}{\text{poly}(n)}\right) E[f(Q) \mid (1 - \epsilon)\mu \leq f(Q) \leq \mu] + \frac{1}{\text{poly}(n)} E[f(Q) \mid f(Q) \geq \mu] \overset{(4)}{\leq} p(1 - \epsilon)\mu + \left(1 - p - \frac{1}{\text{poly}(n)}\right)\mu + \frac{1}{\text{poly}(n)} n\mu \overset{(4)}{\leq} -\epsilon p \mu + \mu + \frac{\mu n}{\text{poly}(n)}, \]

by (3) conditional expectations and (4) bounding \(f(Q)\) by \((1 - \epsilon)\mu, \mu, \text{ and } n\mu\) respectively. Dividing both sides by \(\mu\) and rearranging, we have
\[p \leq \frac{n}{\epsilon \text{poly}(n)} = \frac{1}{\text{poly}(n)} \]

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parallel-greedy-knapsack($f, \mathcal{N}, a$)

1. $x \leftarrow \left\{ j : a_j \leq \frac{\epsilon}{n} \right\}, \lambda \leftarrow \text{OPT} \; // \; \text{or any upper bound for OPT}$

2. while $\langle a, x \rangle \leq 1$ and $\lambda \geq e^{-1} \text{OPT}$
   A. let $S = \{ j \in \mathcal{N} : F_j(x) \geq (1 - \epsilon) \lambda a_j \}$
   B. while $S$ is not empty and $\langle a, x \rangle \leq 1$
      i. if $a_j \geq \Omega \left( \frac{\epsilon^2}{\log n} \right)$ for some $j \in S$
         a. if $\langle a, x \rangle + a_j \leq 1$
             1. $x_j \leftarrow 1$
         b. else return $x$
      ii. else
         a. choose $\delta$ maximal s.t.
            1. $F_x(x + \delta S) \geq (1 - \epsilon)^2 \lambda \delta \langle a, S \rangle$
            2. $\langle a, x + \delta S \rangle \leq 1$
         b. $x \leftarrow x + \delta S$
   C. $\lambda \leftarrow (1 - \epsilon) \lambda$

3. return $x$

Figure 4: A parallel implementation of the continuous-greedy algorithm specialized to the knapsack polytope.

for a slightly smaller polynomial $\text{poly}(n)$, as desired.

It remains to account for the depth and oracle calls, and work. The depth was proven in Lemma A.3. The expected number of oracle calls is bounded by multiplying the expected depth by $\tilde{O}(n/\epsilon^2)$, which is the number of random samples needed to estimate $F(x)$.

B Knapsack constraints

In this section, we consider the parallel continuous greedy algorithm a single knapsack packing constraint, an intermediate setting in between the cardinality constraint and general packing constraints. Formally, we consider the following problem:

$$\text{maximize } F(x) \text{ over } x \geq 1 \text{ s.t. } \langle a, x \rangle \leq 1,$$

where $a : \mathcal{N} \rightarrow [0, 1]$ is a positive cost vector. Here we have normalized the costs so that the size of the knapsack is 1. We let $\|a\|_\infty = \max_j a_j$ be the maximum cost of any item. We first present algorithms that obtain approximation factor that depend on $a$; the dependency can then be removed by partial enumeration (without increasing the depth, but increasing the total amount of work).

As with the cardinality constraint, we first consider a model where we have oracle access to the multilinear extension $F$ and its derivatives $F'$. We present an algorithm that is called
parallel-greedy-knapsack and given in Figure 4. It is very similar to parallel-greedy, and can be interpreted as a parallel extension of the continuous-greedy algorithm of Călinescu et al. [13] specialized to the knapsack polytope. The primary differences from parallel-greedy are as follows. First, we simply take any coordinate with cost at most \( \epsilon/n \). This only uses an \( \epsilon \)-fraction of the budget, and the fact that all remaining coordinates have cost at least \( \epsilon/n \) will be useful in the analysis. The second difference is probably the most significant difference, and is as follows. When gathering the set of “good” coordinates \( S \), rather than comparing the partial derivative \( F'_j(x) \) of each coordinate to a fixed threshold, we compare the “bang-for-buck ratio” \( F'_j(x)/a_j \) of the partial derivative to the cost to the threshold. Third, when adding coordinates to our solution, we take special exception for items whose costs are at least \( \Omega(\epsilon^2/\log n) \)-fraction of the budget. When a good coordinate \( j \) has such a high cost or partial derivative, we directly set \( x_j = 1 \) rather than take a fractional amount. Maintaining the invariant \( x_j \in \{0, 1\} \) for all coordinates \( j \) with \( a_j \geq \epsilon^2/\log n \) is convenient for applying the Chernoff inequality should one want to round \( x \) to a discrete solution later.

The final bounds and proof are similar to that of the cardinality constraints in Section 3, and many of the differences are analogous to the differences between cardinality and knapsack constraints in the well-known sequential setting. Consequently we restrict ourselves to brief sketches of proofs, highlighting the main differences from the proofs of Section 3.

**Lemma B.1.** At any point, we have \( \lambda \geq \text{OPT} - F(x) \). This implies the following.

1. In either step (2.B.i.a.1) or (2.B.i.b), we have

\[
\frac{dF(x)}{d(a,x)} \geq (1 - \epsilon)^2(\text{OPT} - F(x)),
\]

hence

\[
F(x) \geq (1 - O(\epsilon))(1 - e^{-\langle a, x \rangle}) \text{OPT}
\]

at any point.

2. If \( \lambda \leq e^{-1} \text{OPT} \), then \( F(x) \geq (1 - e^{-1}) \text{OPT} \).

Thus, if the algorithm terminates with \( (a, x) = 1 \) or \( \lambda \leq e^{-1} \text{OPT} \), then we have \( F(x) \geq (1 - O(\epsilon))(1 - e^{-1}) \text{OPT} \). Otherwise, the algorithm terminates in step (2.B.i.b), in which case there is an item \( j \in N \) such that \( F(x) \geq (1 - O(\epsilon))(1 - e^{-1-a_j}) \text{OPT} \) and \( F(x+j) \geq (1 - O(\epsilon))(1 - e^{-1}) \text{OPT} \).

**Proof sketch.** The proof is the same as Lemma 3.1 with the only change being that we now have

\[
\langle F'(x), z \rangle \leq (1 - \epsilon)\lambda \langle z, a \rangle \leq (1 - \epsilon)\lambda
\]

by (1) emptiness of \( S \) and (2) \( \langle z, a \rangle \leq k \). We should note that, at the beginning of each iteration of step (2.B.i.a.1), we have \( F(x+e_j) = F(x) + F'_j(x) \) by multilinearity of \( F \), which gives the differential inequality [6] when increasing \( x \) in step (2.B.i.a.1). 

**Lemma B.2.** parallel-greedy-knapsack enters the if clause (2.B.1.*) at most \( O\left(\frac{\log n}{\epsilon^2}\right) \) times over the course of the algorithm.

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Proof. Each time we enter the if clause (except possibly the last), we increase $t$ by at least $\Omega\left(\frac{e^2}{\log n}\right)$. But we always have $t \leq 1$.

Lemma B.3. If $F(x + \delta S) = (1 - e)^2 \lambda \delta (a, S)$, then the step $(2.B.iii.b)$ decreases $(a, S)$ by at least a $(1 - e)$-multiplicative factor. Since $(a, S)$ ranges from at most $n$ to at least $\epsilon/n$ (unless $S$ is empty), this implies that, for fixed $\lambda$, the steps $(2.B.iii)$ iterate at most $O(\log n/\epsilon)$ times, and at most $O\left(\frac{\log n}{e^2}\right)$ times total.

Proof sketch. The proof is the same as in Lemma 3.3 except the modified definition of $S$ changes the endpoints of equation (2) to $(1 - e)^2 \lambda \delta (a, S')$ and $(1 - e) \lambda \delta (a, S'')$, respectively.

We conclude with a theorem summarizing the analysis. We note that the upper bound on the depth comes from the combination of Lemma B.2 and Lemma B.3. The oracle complexity and total work follow from essentially the same analysis as the cardinality setting.

Theorem B.4. parallel-greedy-knapsack computes a vector $x$ with the following properties.

- $a < (a, x) \leq 1$
- $b \; F(x) \geq (1 - O(\epsilon))(1 - e^{-1\|a\|_\infty})\,\text{OPT}$, where $\|a\|_\infty = \max_j a_j$ is the maximum cost of any item.
- $c \; \text{If } F(x) < (1 - O(\epsilon))(1 - e^{-1})\,\text{OPT}$, then there is an item $j \in N$ such that $f(x + j) \geq (1 - O(\epsilon))(1 - e^{-1})\,\text{OPT}$.
- $d \; \text{If } a_j \geq \frac{ce^2}{\log n}$ (for any desired constant $c > 0$), then $x_j \in \{0, 1\}$.

parallel-greedy-knapsack has depth $O\left(\frac{\log n}{e^2}\right)$, uses $O\left(\frac{n \log n}{e^2}\right)$ oracle calls to $F(x)$ and coordinates of $F'(x)$, and does total work $\tilde{O}\left(\frac{n}{e^2}\right)$.

B.1 Oracle complexity w/r/t $f$

Lemma B.5. Let $\hat{Q}$ be the final set $Q$ output by randomized-parallel-greedy-knapsack. With high probability, $(1 - \epsilon)t \leq \langle a, x \rangle \leq (1 + \epsilon)t \leq 1$.

Proof sketch. We use $t$ to track the expected size of $\langle a, Q \rangle$. Every item with cost at least $ce^2/\log n$, for some small constant $c > 0$, is treated deterministically. Conversely, every randomized decision contributes at most $ce^2/\log n$ to $\langle a, Q \rangle$. By online extensions of the Chernoff inequality (Lemma C.1) applied similarly to the proof of Lemma A.1, the total cost is at most $(1 + \epsilon)t \leq 1$ with probability $\geq 1 - 1/poly(n)$.

Lemma B.6. At each time $t$, $E[f(Q)] \geq (1 - e^2)(1 - O(\epsilon))(1 - e^{-t/k})$. In particular, since the algorithm exits with either $t = (1 - 2\epsilon)k$ or $\text{OPT} - f(S) \leq e^{-1}\,\text{OPT}$, the final set $\hat{Q}$ satisfies

$$E[f(\hat{Q})] \geq (1 - O(\epsilon))(1 - e^{-1})\,\text{OPT}.$$ 

Proof sketch. The proof essentially the same as Lemma A.2, except Lemma 3.1 is replaced by Lemma B.1.
randomized-parallel-greedy-knapsack($f, a, \epsilon$)

1. $Q \leftarrow \emptyset$, $t \leftarrow 0$, $\lambda \leftarrow \text{OPT}$ // or any upper bound for OPT
2. while $t \leq 1$ and $\lambda \geq e^{-1} \text{OPT}$
   A. let $S = \{ j \in N : f_Q(j) \geq (1 - \epsilon)\lambda a_j \}$
   B. while $S$ is not empty and $t \leq 1 - \epsilon$
      i. if $a_j \geq \Omega \left( \frac{\epsilon^2}{\log n} \right)$ or $F_Q(j) \geq \Omega \left( \frac{\epsilon^2 \text{OPT}}{\log n} \right)$ for some $j \in S$
         a. if $t + a_j \leq 1 - \epsilon$
            1. $Q \leftarrow Q + j$, $t \leftarrow t + a_j$
         b. else return $Q$
      ii. else
         a. choose $\delta > 0$ maximal s.t.
            1. $F_Q(Q + \delta S) \geq (1 - \epsilon)^2 \delta \langle a, S \rangle$
            2. $t + \delta \langle a, S \rangle \leq 1 - \epsilon$
         b. sample $R \sim \delta S$
         c. $Q \leftarrow Q \cup R$, $t \leftarrow t + \delta \langle a, S \rangle$
      iii. $\lambda \leftarrow (1 - \epsilon)\lambda$
3. return $Q$

Lemma B.7. If $F'(Q + \delta S)Q = (1 - \epsilon)^2 \delta \langle a, S \rangle$

Proof sketch. The same proof as Lemma A.3 goes through, except the modified definition of $S$
changes the first term of the derivation [5] is replaced with $(1 - \epsilon)^2 \lambda \delta \langle a, S \rangle$ and the last two terms
of [5] are replaced by

$$
\cdots \geq \delta \mathbb{E}[(1 - \epsilon)\lambda \langle a, S' \rangle] \geq (1 - \epsilon)\lambda \delta \mathbb{E}[(\langle a, S' \rangle)].
$$

We note that the objective value is tightly concentrated because any item with margin $\geq \Omega \left( \frac{\epsilon^2 \text{OPT}}{\log n} \right)$ is decided deterministically.

Theorem B.8. randomized-parallel-greedy-knapsack returns a randomized set $Q$ such that

(a) $\langle a, Q \rangle \leq 1$ with high probability.

(b) $E[f(Q)] \geq (1 - O(\epsilon))(1 - e^{-1 \cdot \|a\|_\infty}) \text{OPT}$ and $f(Q) \geq (1 - \epsilon) E[f(Q)]$ with high probability, where $\|a\|_\infty = \max_j a_j$ is the maximum cost of any item.

(c) $E\left[ \max_{j \in [n]} f(Q + j) \right] \geq (1 - O(\epsilon))(1 - e^{-1}) \text{OPT}$, and $\max_{j \in [n]} f(Q + j) \geq E\left[ \max_{j \in [n]} f(Q + j) \right]$ with high probability.

randomized-parallel-greedy-knapsack has depth $O\left( \frac{\log n}{\epsilon^2} \right)$, and uses $\tilde{O}\left( \frac{n}{\epsilon^2} \right)$ oracle calls to $f$. 

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B.2 Partial enumeration

Above we derived low-depth algorithms for knapsack constraints with an approximation factor that degrades with the maximum cost of an item. This suffices for many real applications, where large costs are exceptional. For theoretical purposes, it is preferable to obtain approximation ratios independent of the cost of any large item, which may be as much as 1 (for which the corresponding approximation bound is vacuous). In the sequential setting, the well-known technique of “partial enumeration” removes the dependence on the maximum cost and obtains the same approximation ratio as the cardinality constraint [34]. In partial enumeration, one initializes the solution \((x \text{ or } Q)\) with different combinations of a constant number of initial elements (3 suffices), hoping to guess the largest margin items in the optimal solution. It is easy to see that partial enumeration extends here as well, and obviously can be done in parallel without increasing the depth.

**Theorem B.9.** In \(O\left(\frac{\log n}{\epsilon^2}\right)\) depth, one can compute an \((1 - O(\epsilon))(1 - e^{-1})\)-multiplicative approximation to maximizing a normalized monotone submodular function subject to a knapsack constraint.

Note that although partial enumeration does not increase the depth, it does increase the total number of oracle queries and work by a \(O(n^3)\)-multiplicative factor. Ene and Nguyen [20] recently obtained an alternative to partial enumeration that increases the total work and number of oracle queries by a \(O(\exp(\text{poly}(1/\epsilon)))\)-multiplicative factor instead, which is preferable for modest values of \(\epsilon\). The techniques may extend here, but the details are tedious and beyond the scope of this paper.

C Concentration bounds

C.1 Online Chernoff inequalities

We employ the following online extension of multiplicative Chernoff inequalities, previously used in [36, 15].

**Lemma C.1.** Let \(X_1, \ldots, X_n, Y_1, \ldots, Y_n \in [0, 1]\) be random variables and let \(\epsilon > 0\) be sufficiently small. If \(E[X_i \mid X_1, \ldots, X_{i-1}, Y_1, \ldots, Y_i] \leq Y_i\) for \(i \in [n]\), then for any \(\delta > 0\),

\[
P\left(\sum_{i=1}^{n} X_i \geq (1 + \epsilon) \sum_{i=1}^{n} Y_i + \delta\right) \leq (1 + \epsilon)^{-\delta}.
\]

C.2 Concentration bounds for decay processes

**Lemma C.2.** Let \(X_1, X_2, \cdots \in \mathbb{N}\) where \(X_1 = n\), and for \(i \geq 2\), \(E[X_i \mid X_1, \ldots, X_{i-1}] \leq \max\{(1 - \epsilon)X_{i-1}, 1\}\). Then there exists a constant \(c > 0\) such that, for \(k \in \mathbb{N}\)

\[
P[X_k > 1] \leq \exp\left(-c \frac{k\epsilon}{\log n}\right).
\]

**Proof.** Let \(Y_i = \log(X_{i-1}) - \log(X_i)\). Then \(0 \leq Y_i \leq \log n\), and \(\sum Y_i \leq \log n\). For each \(i\), if \(X_{i-1} = 1\) (and the process has essentially halted), we have \(Y_i = 0\). Otherwise, we have

\[
E[Y_i \mid X_1, \ldots, X_{i-1}] = \log(X_{i-1}) - E[\log(X_i)] \overset{(1)}{\geq} \log(X_{i-1}) - \log(E[X_i]) = \log\left(\frac{1}{1 - \epsilon}\right) \geq c\epsilon
\]
for some constant $c > 0$. The claim now follows from applying the following lemma to the variables $\{Y_i/c\}$ with $K = \frac{\log n}{c\epsilon}$.

Lemma C.3. Let $Y_1, Y_2, \ldots \geq 0$ and $K > 0$ such that

1. For any $\ell \in \mathbb{N}$, $\sum_{i=1}^{\ell} Y_i \leq K$.

2. For any $\ell \in \mathbb{N}$, if $\sum_{i=1}^{\ell} Y_i < K$, then $E[Y_{\ell+1} \mid Y_1, \ldots, Y_\ell] \geq 1$.

3. For any $\ell \in \mathbb{N}$, if $\sum_{i=1}^{\ell} Y_i = K$, the $Y_{\ell+1} = 0$.

Then for $i \in \mathbb{N}$, $P \left( \sum_{j=1}^{[iK]} Y_j < K \right) \leq e^{-ci}$, for some absolute constant $c > 0$.

Proof. Define $Z_1, Z_2, \ldots \geq 0$ by

\[
Z_i = \begin{cases} 
Y_i & \text{if } \sum_{j=1}^{i-1} Y_j < K, \\
1 & \text{if } \sum_{j=1}^{i-1} Y_j = K.
\end{cases}
\]

For each $i$, $\sum_{j=1}^{i} Y_j = K \iff \sum_{j=1}^{i} Z_j \geq K$. Moreover, we have $E[Z_i \mid Z_1, \ldots, Z_{i-1}] \geq 1$ for all $i$ and any values of $Z_1, \ldots, Z_{i-1}$.

For $L = \lceil K \rceil$, we divide the $Z_i$'s into groups of $L$. For $i \in \mathbb{N}$, let $W_i = \sum_{j=1}^{L} Z_{(i-1)L+j}$. For each $i$, we have $0 \leq W_i \leq 2K - 1$ unconditionally, and $E[W_i \mid W_1, \ldots, W_{i-1}] \geq K$ for any values of $W_1, \ldots, W_{i-1}$.

We now have

\[
P \left( \sum_{j=1}^{L-1} Y_j < K \right) = P \left( \sum_{j=1}^{L-1} Z_j < K \right) = P \left( \sum_{j=1}^{i} W_j < K \right) \leq P \left( \sum_{j=1}^{i} W_j < 2K \right) \leq e^{-ci}
\]

by (1) Chernoff inequalities, for some constant $c > 0$. ■