Tracy-Widom fluctuations for perturbations of the log-gamma polymer in intermediate disorder

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Abstract

In 1+1 dimensions, the free-energy fluctuations of the directed polymer is conjecturally in the Tracy-Widom universality class at all finite temperatures. Seppäläinen’s log-gamma polymer was proven to have Tracy-Widom fluctuations in a restricted temperature range by Borodin et al. [10]. We remove this restriction, and extend this result into the intermediate disorder regime, where the inverse temperature is scaled to zero along with the length of the polymer. Using a perturbation argument, we show that any polymer that matches a certain number of moments with the log-gamma polymer also has Tracy-Widom fluctuations in intermediate disorder.

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1 Introduction

In 1999-2000 Baik et al. [6] and Johansson [14] proved that the asymptotic fluctuations of the energy in certain point-to-point last passage problems were governed by the same Tracy-Widom law which arises in the large $N$ limit of the top eigenvalue of an $N \times N$ matrix from the Gaussian Unitary Ensemble (GUE). It was then conjectured that this holds for very general distributions, and furthermore that it extends to the positive temperature case of directed polymers in a random environment in $1+1$ dimensions. Here, the free energy takes the form

$$F(\beta, N) = \log \sum_{x} \exp \left( \beta \sum_{i=1}^{N} \xi_{x(i)} \right)$$  \hspace{1cm} (1)

where the paths $x$ have fixed starting and end point, and the $\xi_{i,j}$ are independent identically distributed random variables, collectively referred to as the disorder.

To date, the only progress that has been made on the positive temperature conjecture is: 1) It has been verified for the special exactly solvable log-gamma case [18], in certain range of the parameter values [10]; 2) It has been shown to hold under certain double scaling regimes a) for long thin rectangles [4], and b) in the intermediate disorder limit [2].

In $1+1$ dimensions, the directed random polymers are in the strong disorder regime for all values of inverse temperature $\beta > 0$. The intermediate disorder regime means to take $\beta \to 0$ with the length of the polymer to probe the transition: The more slowly $\beta$ is taken to 0, the closer one is to the pure asymptotics. The special case where $\beta_{N} = O(N^{-1/4})$ was studied in detail in [2]. It probes the regime governed by the Kardar-Parisi-Zhang (KPZ) equation, crossing over between the Gaussian (Edwards-Wilkinson) regime $\beta_{N} \ll O(N^{-1/4})$, and the Tracy-Widom regime $\beta_{N} \gg O(N^{-1/4})$.

In this article we develop a perturbation argument (Theorem 2.4) which shows to some extent the universality in the last regime $1 \gg \beta_{N}$ with $\beta_{N} \gg O(N^{-1/4})$. If one has two disorder distributions whose moments up to a certain order are sufficiently close, then there is a $\beta_{N}$ in that regime such that the appropriately rescaled free energy fluctuations are the same asymptotically.

In principle, one would like to use this to prove some universality of the Tracy-Widom law in intermediate disorder for directed polymers free energies of the form (1). However, the only case in which the Tracy-Widom law is known, the log-gamma polymer, is not even really of the form (1). The log-gamma distributions form a two parameter family, parameters which can be thought of as mean and variance. The mean is a trivial parameter in the directed polymer (1), and one should really think of $\beta$ as controlling the variance. However, the variance parameter of the log-gamma does not appear multiplicatively. Hence, although we can use the perturbation result to compare nearby models to the solvable log-gamma model, the statement (Corollary 2.5) is not as simple as it would be if there were a solvable model of the form (1).

Finally, the intermediate disorder regime of the log-gamma polymer, $\beta_{N} \to 0$, turns out to be outside of the range of the best available result [10], which requires $\beta \geq \beta^{*} > 0$. Most of the present article is devoted to removing this restriction, caused by the form of the contours employed in the exact formula for the log-gamma polymer. We start with an exact formula using nicer contours, and we thank Ivan Corwin for suggesting using the formula from [9] to obtain it.

In this way, we obtain the Tracy-Widom GUE law for the point-to-point log-gamma polymer for all parameter values and appropriate “nearby” distributions.
2 Perturbation theorem for directed polymer free energies in the intermediate disorder regime

The random disorder field is given by variables $\xi_{i,j}(\beta)$, $i,j \in \{1,2,\ldots\}$ which are independent for each $\beta > 0$. We consider up/right directed lattice paths $x$ from $(1,1)$ to $(N,N)$. The energy of such a path is given by

$$H_\beta(x) = -\sum_{(i,j) \in x} \xi_{i,j}(\beta).$$

The partition function is given by

$$Z_N^\beta = \sum_x e^{-H_\beta(x)}.$$  \hfill (2)

Typically one would have $\xi_{i,j}(\beta) = \beta \xi_{i,j}$ but we will want to consider as one of our main examples the log-gamma polymer, where the parameter enters in a somewhat more complicated way. The limiting free energy is given by the (see Theorem 2.4 in Seppäläinen [18]),

$$F(\beta) = \lim_{N \to \infty} \frac{1}{N} \log Z_N^{(N,N)}.$$  

The scaled and centered free energy fluctuations are given by

$$h_N := \log Z_N^\beta - N F(\beta) \sigma(\beta) N^{-1/3}.$$ \hfill (3)

In general

$$\sigma(\beta) \approx C \beta^{4/3} \quad \text{as} \quad \beta \searrow 0,$$ \hfill (4)

with a constant $C$ depending only on the distribution of the weights $\xi$.

We will be primarily interested in the intermediate disorder regime, in which $\beta$ goes to zero as $N \to \infty$, but $\lim_{N \to \infty} \sigma(\beta) N^{1/3} > 0$. In particular, if

$$\lim_{N \to \infty} \sigma(\beta) N^{1/3} = \infty,$$ \hfill (5)

we expect the fluctuations to be Tracy-Widom. If $\lim_{N \to \infty} \sigma(\beta) N^{1/3} \in (0,\infty)$ we get a KPZ crossover distribution. For example, in Alberts et al. [2] the case $\beta = CN^{-1/4}$ was studied. If $\lim_{N \to \infty} \sigma(\beta) N^{1/3} = 0$ the fluctuations are Gaussian, as can be seen by doing a chaos expansion in $\beta$ and checking that only the leading term, linear in the noise, survives.

In the case (5) the limiting fluctuations are supposed to have the Tracy-Widom law in wide generality, but the only case where there are any results is the special log-gamma polymer. Here $e^{-\xi(\beta)}$ have the Gamma distribution, or $e^{\xi(\beta)}$ have the inverse Gamma distribution which is supported on $x > 0$, with density

$$P(e^{\xi(\beta)} \in dx) = \frac{1}{\Gamma(\theta)} x^{-\theta-1} e^{-1/x} dx$$  \hfill (6)

where

$$\theta = \beta^{-2} + 1.$$ \hfill (7)
The moments of the inverse Gamma distribution are given by
\[ E[e^{k \xi}] = \frac{\Gamma(\theta - k)}{\Gamma(\theta)} \]
and the log moments are given by
\[ E[\xi^k] = -\Psi^{(k)}(\theta), \]
where \( \Psi \) is the digamma function
\[ \Psi(\gamma) = \frac{\Gamma'(\theta)}{\Gamma(\theta)}. \] (8)

Hence, subtracting the expectation, \( E[\xi] = \Psi(\theta) \approx 2|\log \beta| \) as \( \beta \downarrow 0 \), which we can do without changing the problem, we have
\[ \text{Var}(\xi) \approx \beta^2, \quad \beta \downarrow 0 \]

mimicking the way in which the inverse temperature \( \beta \) would enter the standard polymer \( \xi(\beta) = \beta \xi \).

For the log-gamma distribution (6),
\[ F(\beta) = -2\Psi(\theta/2), \quad \sigma(\beta) = (-\Psi''(\theta/2))^{1/3}. \] (9)

Our first theorem concerns the fluctuations of the log-gamma model

**Theorem 2.1.** Let \( -\xi_i,j(\beta) \) have the log-gamma distribution (6) and \( \beta_N \to \beta \in [0, \infty) \) with \( \sigma(\beta)N^{1/3} \to \infty \). Then
\[ \lim_{N \to \infty} P(h_N < r) = F_{\text{GUE}}(r) \]
where \( h_N \) is the scaled-centered log partition function in (3), and \( F_{\text{GUE}} \) is the GUE Tracy-Widom distribution.

This was proved for \( \beta \geq \beta^* > 0 \) in Borodin et al. [10]. Our result removes this restriction.

Our next result extends the \( \beta_N \downarrow 0 \) part of this result to “nearby” distributions.

**Definition 2.2** (Moment matching condition). Two parametrized families of weights \( \xi = \xi(\beta) \) and \( \tilde{\xi} = \tilde{\xi}(\beta) \) are said to match moments up to order \( k \) if for some \( C < \infty \) and for all sufficiently small \( \beta \),
\[ |E[\xi^n] - E[\tilde{\xi}^n]| \leq C\beta^k \quad n = 1, \ldots, k - 1, \]
and
\[ |E[\xi^k]|, |E[\tilde{\xi}^k]| \leq C\beta^k. \] (10)

Denote by \( C^k(\mathbb{R}) \) the space of functions on \( \mathbb{R} \) whose derivatives up to order \( k \) are all uniformly bounded on all of \( \mathbb{R} \).

**Lemma 2.3.** Suppose two families of weights match moments up to order \( k \) (as in Definition 2.2) and let \( \varphi \in C^k(\mathbb{R}) \). Then there is a \( C < \infty \) such that,
\[ |E[\varphi(h_N)] - E[\varphi(h_N)]| \leq C\frac{N^{2-\frac{1}{3}}\beta^k}{\sigma(\beta)}. \] (11)
Lemma 2.3 is proved in Section 3, and the following perturbation follows as a consequence of the fact that weak convergence is equivalent to convergence of expectations of functions in $C^k(\mathbb{R})$ functions (see, for example, Billingsley [7]).

**Theorem 2.4** (Perturbation theorem). Suppose $\xi(\beta)$ and $\tilde{\xi}(\beta)$ match moments up to order $k$ (as in Definition 2.2) and $\beta_N \searrow 0$ with

$$\lim_{N \to \infty} \frac{N^{2-\frac{1}{3}\beta_N^2}}{\sigma(\beta_N)} = 0.$$  \hfill (12)

Then

$$\lim_{N \to \infty} P(h_N \leq r) = \lim_{N \to \infty} P(\tilde{h}_N \leq r).$$

**Corollary 2.5.** Suppose $\xi(\beta)$ and $\tilde{\xi}(\beta)$ match moments up to order $k$ with $-\tilde{\xi}(\beta)$ log-gamma as above and (12) holds. Then

$$\lim_{N \to \infty} P(h_N \leq r) = F_{\text{GUE}}(r).$$

For example, if $\beta_N = N^{-\alpha}, \alpha \in (0, 1/4]$, we need to match

$$k > \frac{5}{3\alpha} + \frac{4}{3}$$

moments with a log-gamma to prove the Tracy-Widom law. When $\alpha = 1/4$ one knows that 6 moments suffice to get the crossover law [13], so the result is slightly suboptimal. Note that it gets worse as $\alpha$ decreases, whereas the truth is supposed to be that 5 moments suffice when $\alpha = 0$ [8].

**Remark 1.** The number of moments required can be reduced by 1 using truncation; see Chatterjee [12], for example. However, this isn’t optimal either, and hence we choose not to write down the additional details.

### 3 Proof of the perturbation theorem by Lindeberg replacement

The Lindeberg proof of the Central Limit theorem is a now standard argument for proving universality [15, 16]. Recently, perturbation ideas were used to prove universality of the free energy $N^{-1} \log Z_N(\beta)$ in the Sherrington-Kirkpatrick (SK) and other spin glasses by Carmona and Hu [11]. Their idea was to compare the log partition functions with Gaussian weights and general weights using interpolation and Taylor expansion. Carmona and Hu’s third moment condition for universality was improved using truncation to a second moment condition by Chatterjee [12]¹. Chatterjee also applied the Lindeberg technique to several other examples, including Wigner random-matrices to prove the universality of the semi-circle law. Similar ideas were used by Auffinger and Chen [5] to show that the *Gibbs measure* on configurations was also universal. The extension to the polymer

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¹A similar truncation idea also appears in Trotter’s presentation of Lindeberg’s proof of the CLT
model is not difficult, and Auffinger [3] showed that the Gibbs measure is universal in intermediate disorder. Using the conjectured relationship KPZ fluctuation relationship, it follows that the free energy is universal. However, the universality theorems are without much content currently because the limit of the Gaussian polymers (or any other standard polymer) has not been determined for any weight distribution. Since we need more terms in the Taylor expansion to prove Lemma 2.3, we reproduce the standard argument for completeness.

For a fixed vertex \( x = (i, j) \), define
\[
Z(y) = Z_{x^c} + Z_x e^y,  
\]
where \( Z_{x^c} \) represents the sum in (2) over paths that do not pass through \( x \), and \( Z_x \) is the sum over paths that do pass through \( x \), but do not include weight at \( x \). Let \( h(y) = N^{-1/3} \sigma^{-1} (\log Z(y) - NF) \). \( Z(y) \) and \( h(y) \) do indeed depend on all the other weights for \( z \neq x \), but the dependence is suppressed in the notation because we want to isolate the effect of replacing \( \xi_x \) by \( \tilde{\xi}_x \). For a fixed smooth function \( \varphi \) we will show that
\[
\left| \mathbb{E}[\varphi(h(\xi_x))] - \mathbb{E}[\varphi(h(\tilde{\xi}_x))] \right| \leq C(\sigma N^{-1/3})^{-1} \beta^k.  
\]
where the expectation is over the entire disorder. We obtain (11) by replacing \( \xi_x \) by \( \tilde{\xi}_x \) \( N^2 \) times for each \( x \in \{1, \ldots, N\}^2 \).

Fix all the other weights in the disorder, and write Taylor’s theorem for \( \varphi(h(y)) \):
\[
\varphi(h(\xi_x)) = \sum_{j=0}^{k-1} \frac{\partial_j^k \varphi(h)(0)}{j!} \xi_x^j + \frac{\partial_j^k \varphi(h)(\zeta)}{k!} \zeta^k,  
\]
with \( \zeta \) between 0 and \( \xi_x \). Taking expectation and using the independence of \( \{\xi_z\}_{z \in \mathbb{R}^2} \), we get
\[
\mathbb{E}[\varphi(h(\xi_x))] = \sum_{j=0}^{k-1} \frac{a_j}{j!} \mathbb{E}[\xi_x^j] + \frac{a_k}{k!} \mathbb{E}[\zeta^k],  
\]
where \( a_j = \mathbb{E}[\partial_j^j \varphi(h)(0)], j = 1, \ldots, k-1 \) and \( a_k = \mathbb{E}[\partial_k^k \varphi(h)(\zeta)] \). One has an analogous expression for \( \mathbb{E}[\varphi(h(\tilde{\xi}_x))] \), but note that in fact \( a_j = \tilde{a}_j \) for \( j = 1, \ldots, k-1 \) since all the other weights are the same in the two expressions at each stage of the switching. Hence, from the moment matching condition (10),
\[
\left| \mathbb{E}[\varphi(h(\xi_x))] - \mathbb{E}[\varphi(h(\tilde{\xi}_x))] \right| \leq \left( \sum_{j<k} |a_j| + |\tilde{a}_k| \right) C \beta^k,  
\]
To control the error term, we will show that for any \( k \geq 1 \), and all \( y \in \mathbb{R} \),
\[
|\partial_k^k \varphi(h(y))| \leq C_{k, \varphi}(\sigma N^{1/3})^{-1},  
\]
where \( C_{k, \varphi} \) is a constant dependent only on \( \varphi \), \( k \) and the constant from the moment matching condition. This estimate (17), the Taylor expansion (15) and the moment matching condition in Definition 2.2 together imply (14).

To prove (17), we expand the derivative of a composition (à la Faa di Bruno)
\[
\partial_k^k \varphi(h) = \sum_{\sum m_s = k} C_{m_1 \cdots m_k} \partial^{\sum m_s} \varphi \prod_{s=1}^{k} (\partial^s h)^{m_s},  
\]
where the $C_{m_1 \ldots m_k}$ are multinomial coefficients, and $m_s \geq 0$ for $s = 1, \ldots, k$. Since $\varphi$ is smooth with bounded derivatives up to order $k$, we only need to control $\partial^r h(0)$ for $r \geq 1$. Computing derivatives in (13),
\[
\frac{\partial_y \log Z(y)}{Z_x e^{y}} = p(y),
\]
\[
\frac{\partial_y^i \log Z(y)}{Z_x e^{y}} = P_i(p(y)), \quad i > 1
\]
where $P_i$ is the polynomial given by the recurrence
\[
P_{i+1}(p) = P'_i(p)p(1 - p).
\]
The recurrence is just the chain rule, and $P'_i(p) = P_i(p)(1 - P_i(p))$. Since $0 \leq P_i(p) \leq 1$ for all $y \in \mathbb{R}$, we can bound each of the polynomials $P_i$ by constants for $i = 1, \ldots, k$. Putting the last few observations together, we get (17) for $k \geq 1$.

4 Tracy-Widom fluctuations for the log-gamma polymer

4.1 Fredholm determinant formula

Theorem 4.1. For $N \geq 9$, let $Z^N_\beta$ be the partition function of the log-gamma polymer with $\theta = \beta^{-2} + 1$. Then for $\text{Re} u > 0$,
\[
\mathbb{E}[e^{-u Z^N_\beta}] = \det(I + K^N_u)_{L^2(C_\varphi)}
\]
where
\[
K^N_u(v, v') = \frac{1}{2\pi i} \int_{z_{\text{crit}} - \delta}^{z_{\text{crit}} + \delta} \frac{1}{\sin(\pi(w - v))} \left( \frac{\Gamma(v)}{\Gamma(w)} \frac{\Gamma(\theta - w)}{\Gamma(\theta - v)} \right)^N \frac{u^{w - v}}{w - v'} \, dw + \sum_{i=1}^{q(v)} \text{Res}_{u,i}(v, v')
\]
where $0 < \delta < \frac{z_{\text{crit}}}{2}$, $q(v) = \lfloor z_{\text{crit}} + \delta - \text{Re}(v) \rfloor$,
\[
\text{Res}_{u,i}(v, v') = \left( \frac{\Gamma(v)}{\Gamma(v + i)} \frac{\Gamma(\theta - v - i)}{\Gamma(\theta - v)} \right)^N \frac{u^i}{v + i - v'},
\]
and
\[
z_{\text{crit}} = \theta/2.
\]
The contours are defined as follows: For any $\varphi \in (0, \pi/4]$, the $C_\varphi$ contour is given by $\{z_{\text{crit}} + e^{i(\pi + \varphi)}y\}_{y \in \mathbb{R}^+} \cup \{z_{\text{crit}} + e^{i(\pi - \varphi)}y\}_{y \in \mathbb{R}^+}$ oriented so as to have increasing imaginary part. The $\ell_x$ contour is a vertical straight-line with real part $x$ (see Figure 1). They're both oriented to have increasing imaginary part.

Remark 2. Theorem 4.1 is proved by setting $\tau = 0$ in Theorem 2.1 in Borodin et al. [9]. This requires a new estimate, and this is done in Section A.

We'll see in the next section that critical point of the integrand of $K^N_u$ in (19) is at $z_{\text{crit}}$. The contours $\ell_{z_{\text{crit}} + \delta}$ and $C_\varphi$ are located at the critical point (as $\delta \to 0$).
Asymptotics of the fluctuation field

We are interested in the asymptotic probability distribution of (3). The trick is to rewrite the left hand side of (18) as

$$ E[\exp\{-e^{\sigma N^{1/3} (h_N - r)}\}] $$

by taking

$$ u = e^{-NF - r\sigma N^{1/3}}. $$

As $N \to \infty$, by (5) $\sigma N^{1/3} \to \infty$, and (21) becomes $\lim_{N \to \infty} P(h_N < r)$. Now we consider the same limit of the right hand side of (18). We start with a formal critical point analysis of the kernel which can be rewritten as

$$ K_N(v, v') = -\frac{1}{2\pi i} \int_{C_{\text{crit}}} \frac{\pi}{\sin(\pi(w - v))} e^{N[G(v) - G(w)] + r\sigma N^{1/3}(v - w)} dw \frac{d\tilde{w}}{w - v'} $$

where we’ve ignored the residues, dropped the subscript $u$ in the kernel, and let

$$ G(z) = \log \Gamma(z) - \log \Gamma(\theta - z) + F(\beta)z. $$

We have $G'(z) = \Psi(z) + \Psi(\theta - z) + F$. From (9), it follows that the critical point, i.e. $G'(z_{\text{crit}}) = 0$ is at $z_{\text{crit}} = \theta/2$, and $G''$ vanishes there as well. Therefore the exponent is cubic near the critical point and it is natural to define

$$ \tilde{v} = \sigma N^{1/3}(v - z_{\text{crit}}), \quad \tilde{w} = \sigma N^{1/3}(w - z_{\text{crit}}), $$

to get, using (9), $K_N(\tilde{v}, \tilde{v}') \to K_{\text{Ai}}(\tilde{v}, \tilde{v}')$, where the Airy kernel is defined as

$$ K_{\text{Ai}}(\tilde{v}, \tilde{v}') := \frac{1}{2\pi i} \int \exp\left\{-\frac{1}{3} \tilde{v}^3 + r\tilde{v}\right\} \frac{d\tilde{w}}{\exp\left\{\frac{1}{3} \tilde{w}^3 + r\tilde{w}\right\} (\tilde{v} - \tilde{w})(\tilde{v} - \tilde{v}')}.$$
The Airy kernel acts on the contour $e^{-2\pi i/3}\mathbb{R}_+ \cup e^{2\pi i/3}\mathbb{R}_+$ (oriented to have increasing imaginary part) and the integral in $\tilde{w}$ is on the (likewise oriented) contour $\{e^{-\pi i/3}\mathbb{R}_+ + \delta\} \cup \{e^{\pi i/3}\mathbb{R}_+ + \delta\}$ for any horizontal shift $\delta > 0$. The determinant of the right hand side of (25) is $F_{\text{GUE}}(r)$ (see, for example, [10]).

4.3 Estimates along the contours

We prove that $\det(1 + K^N) \to \det(1 + K_{\text{Ai}})$ rigorously in this section. Recall the kernel (19)

$$K^N(v, v') = \frac{1}{2\pi i} \int_{\ell_{z_{\text{crit}}+\delta(\sigma N)^{-1/3}}} I(v, v', w - v)dw + \sum_{i=1}^{q(v)} \text{Res}_i(v, v')$$

where $I(v, v', w - v)$ is the integrand in (23) and $u$ is in (22). We drop the subscripts on $K^N, I$ and $\text{Res}_i$ in this section indicate that we’ve set $u$ as in (22). The little extra displacement of the $\ell_{z_{\text{crit}}+\delta(\sigma N)^{-1/3}}$ is a necessary technicality that we will address in due course. Therefore we will henceforth simply write $\ell_{z_{\text{crit}}}$ as a shorthand.

We will show in Lemma 4.7 that

$$|K^N(v, v')| \leq f(v, N)$$

for $(v, v')$ on the $C_{\pi/4}$ contour. $f(v, N)$ is integrable over $C_{\pi/4}$, and depends favorably on $N$. Then, from the Hadamard inequality for determinants, we get for $m > 1$,

$$|\det(K^N(v_i, v_j))|_{1 \leq i, j \leq m} \leq m^{m/2} \prod_{i=1}^{m} f(v_i, N).$$

It follows that the Fredholm expansion of the determinant,

$$\det(I + K^N)_{L^2(C_{\pi/4})} = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{C_{\pi/4}} dv_1 \cdots \int_{C_{\pi/4}} dv_m \det(K^N(v_i, v_j))_{1 \leq i, j \leq m},$$

is absolutely integrable and summable uniformly in $N$. Thus, we can take the $N \to \infty$ limit inside the series and integrals and replace $K^N$ by its pointwise limit, the Airy kernel. This is similar to what was done in Borodin et al. [10], but now the constants in (27) must depend favorably on $N$ and $\theta$. The rigorous estimates are shown in Lemma 4.7.

Recall the function $G(z)$ defined in (24)

$$G(z) = \log \Gamma(z) - \log \Gamma(\theta - z) - 2\Psi(z_{\text{crit}})z.$$  (29)

The bound in (27) will follow from an analysis of this function along the contours $C_{\pi/4}$ and $\ell_{z_{\text{crit}}}$, and an estimate on the residues $\text{Res}_i(v, v')$. The analysis will be performed in several steps.

1. We first show that the Taylor approximation is effective in the region $|z - z_{\text{crit}}| \leq c(\sigma N^{1/3})^{-1}$. Although $G$ is analytic, one must be careful because the derivatives of $G$ are a function of $\theta$, which is allowed to go to infinity with $N$.

Using the Taylor expansion, we may also arrange for an estimate of the form (recall $z_{\text{crit}} = \theta/2$)

$$\text{Re}(G(z) - G(z_{\text{crit}})) \leq -c|z - z_{\text{crit}}|^{3} , \quad |z - z_{\text{crit}}| \leq \frac{z_{\text{crit}}}{2},$$

(30)
where $c > 0$ can be explicitly chosen. This is needed to show that the pointwise limit of $K^N$ is $K_{Ai}$.

2. Next we show that the real part of $G$ decreases sufficiently rapidly away from the critical point on the $C_{\pi/4}$ contour. The upper and lower halves of the $C_{\pi/4}$ contour are parametrized as

$$z(r) = z_{\text{crit}} + r\hat{e}$$

where $\hat{e} = -1 \pm i$. We show in Lemma 4.4 that the derivative of $G$ satisfies

$$\frac{d}{dr} \text{Re}(G(z(r)) - G(z_{\text{crit}})) \leq -\frac{2r^2}{(1 + z_{\text{crit}} + 2r)^2}.$$  

This captures the cubic behavior of $G$ near the critical point, and the linear decay for large $r$.

3. On the $\ell_{z_{\text{crit}}}$ contour, we use the parametrization

$$w(r) = z_{\text{crit}} + r\hat{e} + \delta(\sigma N^{1/3})^{-1},$$

where $\hat{e} = \pm i$. We show that $\text{Re}(G(w(r)) - G(z_{\text{crit}}))$ is non-decreasing in $r$. Since $\ell_{z_{\text{crit}}}$ is not the steepest descent contour, we can’t show that the derivative of $\text{Re}(G)$ is strictly positive as in (32). However, this is sufficient for our purposes since we use Taylor expansion close to critical point to get a better estimate.

4. Finally, we estimate the contribution of the residues to the bound in (27). This is shown in Lemma 4.6. Using Lemma 4.6 and steps 1-3, we show that the $m^{th}$ term of Fredholm series in (28) can be uniformly bounded in $N$. The bound in Lemma 4.6 also shows that the residues vanish in the limit (5).

Using steps 1-4, it’s easy to see that the pointwise limit of the first integral in (26) is the Airy kernel. We first consider the integral term, and split it as

$$\int_{r < c(\sigma N^{1/3})^{-1}} I(v, v', w(r) - v) dw(r) + \int_{r \geq c(\sigma N^{1/3})^{-1}} I(v, v', w(r) - v) dw(r)$$

where $w(r)$ is parametrization in (33) of the $\ell_{z_{\text{crit}}}$ contour. We take $N \to \infty$ followed by $c \to \infty$ in the first integral. Before we do so, since the integrand is analytic in a tiny region of size $c(\sigma N^{1/3})^{-1}$, we modify the contours so that they’re locally aligned with the Airy contours. Then, by the argument in Section 4.2, the first integral goes to $K_{Ai}$ in the rescaled variables $(\tilde{v}, \tilde{v}')$. To show that the second integral goes to 0, we use $|\sin(\pi(s))|^{-1} \leq e^{-\pi|\text{Im}(s)|}$, and the following bound implied by Prop. 4.1 and Lemma 4.5: there exist constants $C$ and $r_0 > 0$ such that

$$G(w(r)) - G(z_{\text{crit}}) \geq Cz_{\text{crit}}^{-2}r^3 \quad r \leq r_0,$$

$$G(w(r)) \geq G(w(r_0)) \quad r > r_0.$$ 

Since $\sigma N^{1/3} = O\left((z_{\text{crit}}N)^{1/3}\right) \to \infty$ as $N \to \infty$, for each fixed $c$ the second integral converges to 0. From Lemma 4.6, it follows that residues converge pointwise to 0 too. The pointwise convergence of $K^N(v, v')$ to $K_{Ai}$ and the estimate on $K^N$ in Lemma 4.7 shows that the Fredholm determinants converge. This proves Theorem 2.5.
Step 1. Taylor expansion near the critical point. We’ve already seen in Section 4.2 that the first two derivatives of $G$ vanish near the critical point. If the third and fourth derivatives of $G$ were well-behaved, the Taylor expansion for $G(z)$ is an effective approximation when $z$ is close to the critical point:

$$G(z) - G(z_{\text{crit}}) = \frac{G^{(3)}(z_{\text{crit}})}{3!}(z - z_{\text{crit}})^3 + \frac{G^{(4)}(\xi)}{4!}(z - z_{\text{crit}})^4,$$

for $\xi \in \{y : |y - z_{\text{crit}}| < |z - z_{\text{crit}}|\}$. Since $G$ is an analytic function, it’s clear that $G^{(3)}$ and $G^{(4)}$ are well-behaved for fixed $z_{\text{crit}}$. However, we allow $z_{\text{crit}} \to \infty$; Prop. 4.2 shows roughly that $G^{(4)}(z) \approx C z_{\text{crit}}$ for a constant $C > 0$, when $|z(r) - z_{\text{crit}}| \leq z_{\text{crit}}/2$ and $z_{\text{crit}} \to \infty$.

Proposition 4.2. When $|z - z_{\text{crit}}| \leq z_{\text{crit}}/2$,

$$\frac{2}{(2 + z_{\text{crit}})^2} \leq -G^{(3)}(z_{\text{crit}}) \leq \frac{4}{z_{\text{crit}}^3},$$

$$|G^{(4)}(z)| \leq \frac{96}{z_{\text{crit}}^4} + \frac{32}{z_{\text{crit}}^3}.$$  \hspace{1cm} (34)

Proof of Prop. 4.2. The Digamma function can be written as (6.3.16 in [1])

$$\Psi(z) = -\gamma_{EM} + \sum_{n=0}^{\infty} \left( \frac{1}{n + 1} - \frac{1}{n + z} \right),$$

where $\gamma_{EM}$ is the Euler-Mascheroni constant. Note that the series is absolutely convergent when $z$ is bounded away from the nonpositive integers. Differentiating (29) twice we obtain

$$-G^{(3)}(z_{\text{crit}}) = -2\Psi^{(2)}(z_{\text{crit}}) = 4 \sum_{n=0}^{\infty} \frac{1}{(n + z_{\text{crit}})^3} = \frac{4}{z_{\text{crit}}^3} + 2 \sum_{n=1}^{\infty} \frac{1}{(n + z_{\text{crit}})^3}. $$

Estimating the sum by an integral, we get

$$4 \int_{1}^{\infty} \frac{1}{(x + z_{\text{crit}})^3} dx \leq -G^{(3)}(z_{\text{crit}}) - \frac{4}{z_{\text{crit}}^3} \leq 4 \int_{0}^{\infty} \frac{1}{(x + z_{\text{crit}})^3} dx$$

which proves (34). We estimate $G^{(4)}(z)$ similarly: To apply the integral test as before, we first show that $|x + z|$ is increasing with $x \in \mathbb{R}^+$. It’s clear that if $|z - z_{\text{crit}}| \leq z_{\text{crit}}/2$, then $z$ has positive real part and consequently, so does $x + z$ for all $x > 0$. It follows that $|x + z|$ increases with $x$. Then, using (36) and $|x + z| \geq 2^{-1}(x + z_{\text{crit}})$,

$$\left| \Psi^{(3)}(z) \right| \leq \sum_{n=0}^{\infty} \frac{6}{|n + z|^4} \leq \frac{96}{|z_{\text{crit}}|^4} + \int_{0}^{\infty} \frac{96}{|x + z_{\text{crit}}|^4} dx,$$

which proves (35).

$\square$
Step 2. Decay of $G$ along the $C_{\pi/4}$ contour. In the following lemma, we first compute the derivative of $G$ along a general contour. We will use this computation repeatedly to estimate $G$ along the $C_{\pi/4}$ and $\ell_{\text{crit}}$ contours (Lemma 4.5), and to estimate the residues in Lemma 4.6.

**Lemma 4.3.** Let $z(r) = z_{\text{crit}} + v(r)$ be a contour. Then, the derivative of $\text{Re}(G)$ (29) is

$$
\frac{d}{dr} \text{Re}(G(z(r))) = 2 \sum_{n=0}^{\infty} -\text{Re}(v'(r)v^2(n + z_{\text{crit}})^2 + \text{Re}(v'(r))|v(r)|^4)
\frac{(n + z_{\text{crit}})(n + z_{\text{crit}})^2 - v^2)^2}{(n + z_{\text{crit}})(n + z_{\text{crit}})^2 - v^2)^2}. 
$$

**Proof.** From (29),

$$
\frac{d}{dr} G(z(r)) = z'(r) (\Psi(z_{\text{crit}} + v(r)) - \Psi(z_{\text{crit}})) + z'(r) (\Psi(z_{\text{crit}} - v(r)) - \Psi(z_{\text{crit}})).
$$

Using (36),

$$
\frac{d}{dr} G(z(r)) = z'(r) \sum_{n=0}^{\infty} \frac{2}{n + z_{\text{crit}}} - \frac{1}{n + z(r)} - \frac{1}{n + \theta - z(r)}
$$

$$
= 2 \sum_{n=0}^{\infty} -v'(r)v^2(n + z_{\text{crit}})^2 + v'(r)|v(r)|^4)
\frac{(n + z_{\text{crit}})(n + z_{\text{crit}})^2 - v^2)^2}{(n + z_{\text{crit}})(n + z_{\text{crit}})^2 - v^2)^2}. 
$$

**Lemma 4.4.** $\text{Re}(G)$ in (29) satisfies the following derivative bound:

$$
\frac{d}{dr} \text{Re}(G(z(r))) \leq -\frac{2r^2}{1 + z_{\text{crit}} + 2r^2}. 
$$

where $z(r)$ is the parametrization of $C_{\pi/4}$ given in (31).

**Proof.** We parametrize the upper-half of the $C_{\pi/4}$ contour as in Lemma 4.3 with $v(r) = r\hat{e}$ where $\hat{e} = -1 + i$. Then,

$$
\frac{d}{dr} \text{Re}(G(z(r))) = 2 \sum_{n=0}^{\infty} \frac{-2r^2(n + z_{\text{crit}})^2 - 4r^4}{(n + z_{\text{crit}})(n + z_{\text{crit}})^2 - 2r^2)^2}
$$

$$
\leq -4 \sum_{n=0}^{\infty} \frac{r^2}{(n + z_{\text{crit}} + 2r)^3} \leq -4 \int_{1}^{\infty} \frac{r^2}{(x + z_{\text{crit}} + 2r)^3} dx
$$

$$
= -2 \frac{r^2}{(1 + z_{\text{crit}} + 2r)^2}. 
$$

This captures the behavior of $G$ along the steepest-descent contours rather well: Cubic near the critical point, and then linear decay for $r \geq C z_{\text{crit}}$. $G$ behaves symmetrically in the lower half plane, and hence satisfies the same estimates on the lower half of the $C_{\pi/4}$ contour.

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Step 3. Decay along the $\ell_{z_{\text{crit}}}$ contour.

Lemma 4.5. Re$(G)$ in (29) increases away from the critical point along the $\ell_{z_{\text{crit}}}$ contour.

Proof. The $\ell_{z_{\text{crit}}}$ contour doesn’t quite start off at the critical point, but at a distance $\delta_N = \delta(\sigma N^{1/3})^{-1}$ away from $z_{\text{crit}}$. Let $w(r)$ be the parametrization of $\ell_{z_{\text{crit}}}$ in (33). Using Lemma 4.3 and $v(r) = ri + \delta_N$

$$\frac{d}{dr} \text{Re}(G(w(r))) = 2 \sum_{n=0}^{\infty} - \text{Re}(i(\delta_N - r^2 + 2\delta_N ri)(n + z_{\text{crit}})^2 + \text{Re}(i)|v|^4) \geq 0.$$

\hfill \square

Step 4. Triviality of the residues

Lemma 4.6. There exist constants $c_1, C > 0$ independent of $N$ and $z_{\text{crit}}$ such that the residues in (26) satisfy for $\ell = 1, \ldots, |\text{Im}(v)|$

$$\log |\text{Res}_\ell(v, v')| \leq \begin{cases} -c_1 N\ell |\text{Im}(v)| \frac{1}{z_{\text{crit}}} & 1 \leq |\text{Im}(v)| \leq Cz_{\text{crit}} \\ -c_1 N\ell \log \left(1 + \frac{|\text{Im}(v)|}{z_{\text{crit}}} \right) & |\text{Im}(v)| > Cz_{\text{crit}} \end{cases}$$

when $v, v' \in C_{\pi/4}$.

Lemma 4.6 helps show the estimate on the kernel in (27) that’s used in the Hadamard bound; it also shows that the residues go to 0 as $\sigma N^{1/3} \to \infty$.

Proof. From (20), we can write the residues in the form

$$|\text{Res}_\ell(v, v')| = \left| \frac{\Gamma(v) \Gamma(\theta - v - s)}{\Gamma(v + s) \Gamma(\theta - v)} \right|^N e^{2\Psi(z_{\text{crit}})N\ell - r\ell\sigma N^{1/3}} \frac{v + \ell - v'}{u + \ell - u'},$$

$$\leq e^{N(G(v) - G(v + \ell) - r\ell\sigma N^{1/3})}.$$ (37)

We estimate $G(v) - G(v + \ell)$ using Lemma 4.3. Letting $v(r) = k\hat{e} + r$ in the parametrization of the contour in Lemma 4.3 let’s us interpolate between $G(v)$ and $G(v + \ell)$. Then,

$$\frac{d}{dr} \text{Re}(G(z(r)))$$

$$= 2 \sum_{n=0}^{\infty} \frac{r(2k - r)(n + z_{\text{crit}})^2 + ((r - k)^2 + k^2)^2}{(n + z_{\text{crit}})(n + z_{\text{crit}} + v^2)(n + z_{\text{crit}} - v)^2}$$

$$\geq 2 \sum_{n=0}^{\infty} \frac{r(2k - r)(n + z_{\text{crit}})^2 + ((r - k)^2 + k^2)^2}{(n + z_{\text{crit}})(n + z_{\text{crit}} + 2k - r)^4}$$

$$\geq 2 \int_{1 + \max(k, z_{\text{crit}})}^{\infty} \frac{r(2k - r)x}{(x + 2k - r)^4} \, dx + 2 \int_{1 + z_{\text{crit}}}^{\infty} \frac{(r - k)^2 + k^2)^2}{x(x + 2k - r)^4} \, dx := I_1 + I_2.$$
To get the limits of integration in $I_1$, notice that $f(x) = x/(x + 2k - r)^4$ is decreasing for $x \geq \max(k, \z_{\text{crit}})$. This lets us use the integral test to estimate the sum.

Our bounds on the integrals $I_j, j = 1, 2$ will ensure the following:

1. $\Re(G(v+\ell) - G(v)) \geq C\z_{\text{crit}}^{-2}$. Then, the exponent in (37) will contain a negative term of order at least $N\z_{\text{crit}}^{-2}$ which grows to infinity by (5), and thus overwhelms $\sigma N^{1/3} = \mathcal{O} \left( (\z_{\text{crit}}^{-2} N)^{1/3} \right)$.

2. $G(v+1) - G(v) = \mathcal{O}(\log(1 + \z_{\text{crit}}^{-1}|\Im(v)|))$. This is so that the residue is integrable in $|\Im(v)| = k$ as $k \to \infty$.

Explicitly computing $I_1$, we get

$$I_1 = \frac{r(2k-r)(3(1 + \max(k, \z_{\text{crit}})) + 2k-r)}{3(1 + \max(k, \z_{\text{crit}}) + 2k-r)^3} \geq \frac{rk}{3(1 + \max(k, \z_{\text{crit}}) + 2k)^3},$$

for $r \leq k$. This is sufficient for requirement 1. For the second integral,

$$I_2 \geq k^4 \left( \frac{1}{(2k-r)^4} \log \left( 1 + \frac{2k-r}{1 + \z_{\text{crit}}} \right) - \frac{11}{(2k-r)^3(1 + \z_{\text{crit}} + (2k-r))} \right),$$

where we’ve used

$$\int_{a}^{\infty} \frac{dx}{x(x+c)^4} = -\frac{6a^2 + 15ac + 11c^2}{6c^4(a+c)^3} + \frac{1}{c^4} \log \left( 1 + \frac{c}{a} \right).$$

Here, for $k \geq C\z_{\text{crit}}$ where $C$ is some constant, the first term in $I_2$ dominates the second for all $r \leq k$.

Therefore, integrating over $r$, we get for some constants $c_1, C$,

$$G(v + \ell) - G(v) \geq \begin{cases} 
    c_1 \ell^2 \frac{|\Im(v)|}{\z_{\text{crit}}} & |\Im(v)| \leq C\z_{\text{crit}} \\
    c_1 \ell \log \left( 1 + \frac{|\Im(v)|}{\z_{\text{crit}}} \right) & |\Im(v)| > C\z_{\text{crit}}
\end{cases}$$

Finally, we prove the inequality used in the Hadamard bound in (27).

**Lemma 4.7.** There exist constants $c_1, c_2, C > 0$ that are independent of $N$ and $\z_{\text{crit}}$ such that for all $N$ large enough,

$$|K^N(v, v')| \leq \begin{cases} 
    c_1 \exp \left( -c_2 N \z_{\text{crit}}^{-2} |\Im(v)|^3 \right) & |\Im(v)| < 1 \\
    c_1 \exp \left( -c_2 N \z_{\text{crit}}^{-2} |\Im(v)| \right) & 1 \leq |\Im(v)| \leq C\z_{\text{crit}} \\
    c_1 \left( 1 + \frac{|\Im(v)|}{\z_{\text{crit}}} \right)^{-c_2 N} & |\Im(v)| > C\z_{\text{crit}}
\end{cases}$$

\[ \square \]
on the contour $C_{\pi/4}$. Consequently the $m$th term of the Fredholm series for $K^N$ in (28) satisfies

$$\frac{1}{m!} \int_{C_{\pi/4}} dv_1 \cdots \int_{C_{\pi/4}} dv_m \det(K^N(v_i,v_j))_{1 \leq i,j \leq m} \leq \frac{C}{m^{(m-1)/2}}.$$  \hfill (38)

**Proof.** From the estimates in Lemma 4.4 and Lemma 4.5, it follows that the integral in (26) has two regimes of behavior: for constant $c_1,c_2,C > 0$

$$\int_{\ell_{\text{crit}}} I(v,v',\omega)dw \leq \begin{cases} 
  c_1 \exp \left( -c_2 Nz_{\text{crit}}^{-2} \right) |\text{Im}(v)|^3 & |\text{Im}(v)| \leq Cz_{\text{crit}} \\
  c_1 \exp \left( -c_2 N |\text{Im}(v)| \right) & |\text{Im}(v)| > Cz_{\text{crit}}
\end{cases}$$

The $\ell_{\text{crit}}$ contour is a small distance $\delta(\sigma N^{1/3})^{-1}$ away from $z_{\text{crit}}$. Hence, when $|\text{Im}(v)| < 1$, for $N$ large enough, there are no residues. We estimate the contribution of the residues to $K^N$ when $|\text{Im}(v)| > 1$. There are about $|\text{Im}(v)|$ many residues. For $N$ large enough, and $1 \leq |\text{Im}(v)| \leq Cz_{\text{crit}}$, Lemma 4.6 implies

$$\sum_{\ell=1}^{[|\text{Im}(v)|]} \text{Res}_\ell (v,v') \leq \sum_{\ell=1}^{[|\text{Im}(v)|]} c_1 \exp \left( -c_1 N |\text{Im}(v)| \right) \leq c_1 \exp \left( -c_2 N \right),$$

since $Nz_{\text{crit}}^{-2} \to \infty$. The constant $c_1$ was allowed to change from inequality to inequality. When $|\text{Im}(v)| > Cz_{\text{crit}}$,

$$\sum_{\ell=1}^{[|\text{Im}(v)|]} \text{Res}_\ell (v,v') \leq \sum_{\ell=1}^{[|\text{Im}(v)|]} c_1 \left( 1 + \frac{|\text{Im}(v)|}{z_{\text{crit}}} \right)^{-c_2 N} \leq c_1 \left( 1 + |\text{Im}(v)|/z_{\text{crit}} \right)^{-c_2 N},$$

where again $c_1$ was allowed to change between inequalities.

Then, in (28),

$$\int_{C_{\pi/4}} f(v,N)dv \leq \left( \int_0^1 e^{-c_2 N z_{\text{crit}}^{-2} x} dx + \int_{C_{\text{crit}}} e^{-c_2 N z_{\text{crit}}^{-2} x} dx + \int_{C_{\text{crit}}} \left( 1 + \frac{x}{z_{\text{crit}}} \right)^{-c_2 N} \right) \leq \left( \frac{1}{Nz_{\text{crit}}^{-2}} + e^{-c_2 N z_{\text{crit}}^{-2}} + \frac{(1+C)^{-c_2 N}}{Nz_{\text{crit}}^{-2}} \right).$$

Since $Nz_{\text{crit}}^{-2} \to \infty$, it’s clear that this bound behaves favorably as $N \to \infty$. This shows (38) and implies that the Fredholm series can be uniformly truncated for all $N$.

Plugging these estimates into (28), shows that for $m$ large enough,
A  Fredholm determinant as a limit of the formula of Borodin-Corwin-Ferrari-Veto

A.1  The BCFV theorem

Borodin et al. [9] consider a mixed polymer that includes Seppäläinen’s log-gamma polymer [18] and the O’Connell-Yor semi-discrete polymer [17]. For \( N \geq 1 \), the up-right paths \( x \) consist of a discrete portion \( x^d \) adjoined to a semi-discrete portion \( x^{sd} \). The discrete portion is an up-right nearest-neighbor path on \( \mathbb{Z}^2 \) that goes from \((-N,1)\) to \((-1,n)\) for some \( 1 \leq n \leq N \). For \( 0 \leq s_n < \cdots < s_{N-1} \leq \tau \), the semi-discrete path consists of horizontal segments on \((s_i, s_{i+1})\) for \( i = n, \ldots, N - 2 \) and a final interval \((s_{N-1}, \tau)\) connected by vertical jumps of size 1 at each \( s_i \). For \( 1 \leq m, n \leq N \) let \( \xi_{-m,n} \) be independent log-gamma random variables with parameter \( \theta \), and for all \( 1 \leq n \leq N \) let \( B_n \) be independent Brownian motions. Then, the paths have energy

\[
H_\beta(x) = - \sum_{(i,j) \in \mathbb{Z}^d} \xi_{i,j} + B_n(s_n) + (B_{n+1}(s_{n+1}) - B_{n+1}(s_n)) + \ldots + (B_N(\tau) - B_N(s_{N-1})).
\]

(39)

The partition function is given by

\[
Z^N(\tau) = \sum_{i=1}^N \sum_{x^d((-N,1),(-1,i))} \int_{x^d((0,i))} \int_{(\tau,N)^{\mathbb{E}}} e^{-H_\beta(x)} dx^{sd}
\]

where \( dx^{sd} \) represents the Lebesgue measure on the simplex \( 0 < s_n < s_{n+1} < \cdots < s_{N-1} \leq \tau \).

Remark 3. When \( \tau = 0 \), the polymer is simply the standard log-gamma polymer. There is no semi-discrete part.

Theorem A.1 (Borodin et al. [10], Theorem 2.1). Fix \( N \geq 9 \) and \( \tau > 0 \). For all \( u \in \mathbb{C} \) with positive real part

\[
E\left[e^{-uZ^N(\tau)}\right] = \det(1 + K_{u,\tau}^N)_{L^2(C_{\varphi})}
\]

where

\[
K_{u,\tau}^N(v,v') = \frac{1}{2\pi i} \int_{\mathcal{D}_v} ds \frac{1}{\sin(\pi s)} \frac{1}{\Gamma(s+v)} \prod_{n=1}^N \frac{1}{\Gamma(s+v)} \frac{1}{\Gamma(\theta - v - s)} \frac{1}{\Gamma(\theta - v)} e^{(sv+z^2/2)u^s}.
\]

The \( C_{\varphi} \) contour is defined in Theorem 4.1. The \( \mathcal{D}_v \) contour depends on \( v \) and parameters \( R \) and \( d \). For every \( v \in C_{\varphi} \) we choose \( R = -\text{Re}(v) + 3\theta/4 \), \( d > 0 \), and let \( \mathcal{D}_v \) consist of straight lines from \( R - i\infty \) to \( R - id \) to \( 1/2 - id \) to \( 1/2 + id \) to \( R + id \) to \( R + i\infty \). The parameter \( d \) must be taken small enough so that \( v + \mathcal{D}_v \) does not intersect \( C_{\varphi} \). Both contours are oriented to have increasing imaginary part.

Remark 4. We write \( \mathcal{D}_v \) contour as a union of the “sausage” \( \mathcal{D}_v,\square \) and the vertical contour \( \ell_{-\text{Re}(v)+R} \) defined in Theorem 4.1. Closing the \( \mathcal{D}_v \) contour picks up residues that we will estimate separately. For each \( v \), there is some wiggle room in the \( R \) parameter that allows the vertical contour \( \ell_{-\text{Re}(v)+R} \) to avoid the singularity of the sine function in \( I_{u,\tau}(v,v',s) \).
A.2 Proof of Theorem 4.1

In this section, we obtain Theorem 4.1 by letting \( \tau \searrow 0 \) in Theorem A.1. We may do so if we can truncate the series that defines the Fredholm determinant of \( Z^N(\tau) \) uniformly in \( \tau \). This is done by proving an estimate on \( K^N_{u,\tau}(v,v') \) that depends favorably with \( \tau \), and then using Hadamard’s inequality in the usual way (see Section 4.3). The constants in the following propositions may depend on the angle of the contour \( \phi \).

Proposition A.2 (BCFV kernel estimate). When \( |\text{Im}(v)| > \max((2e^{\tau\theta/2}|u|^{1/2N}, c_3\theta)) \), for some \( c_1, c_2, c_3 > 0 \), the kernel in (A.1) satisfies the bound

\[
|K^N_{u,\tau}(v,v')| \leq 2e^{\tau\theta/2}|u|e^{-c_1\tau|\text{Im}(v)|} + e^{C(\theta,N,\tau)}e^{-c_2N|\text{Im}(v)||\log|v|-c_3\theta|},
\]

where \( C(\theta,N,\tau) \) is some explicit constant identified in (53).

Proposition A.2 is proved by estimating \( I_u \) on the closed rectangular contour \( \mathcal{D}_v,\square \), and the vertical line \( \ell_R \).

Proposition A.3 (Integral over the \( \mathcal{D}_v,\square \) contour). When \( |\text{Im}(v)| > (2e^{\tau\theta/2}|u|)^{1/2} \),

\[
\left| \int_{\mathcal{D}_v,\square} I_{u,\tau}(v,v',s) \right| \leq \frac{e^{\tau\theta/2}|u|}{|\text{Im}(v)|^{2N}}e^{-c\tau|\text{Im}(v)|},
\]

where \( c > 0 \) is some constant.

Proposition A.4 (Integral over the \( \ell_R \) contour). For some constants \( c_1, c_2, c_3 > 0 \), \( |\text{Im}(v)| \geq c_3\theta \)

\[
\int_{\ell_R} I_{u,\tau}(v,v',s) \, ds \leq e^{C(\theta,N,\tau)}e^{-c_1N|\text{Im}(v)||\log|v|-c_2\theta|}
\]

where \( C(\theta,N,\tau) \) is the constant in Prop. A.2.

Before proving the propositions, we complete the proof of Theorem 4.1:

1. The contour \( \ell_R \) must be moved so that it has real part \( -\text{Re}(v) + z_{\text{crit}} + \delta \), for arbitrarily small \( \delta \). This is done by using the bound on \( I_{u,\tau}(v,v',s) \) in (53). This implies that we can truncate the vertical contour at large \( |\text{Im}(s)| \), and use Cauchy’s theorem to move over the vertical contour.

2. We have to set \( \tau = 0 \) in \( K^N_{u,\tau} \) without changing its Fredholm determinant. Using the Hadamard inequality argument in Section 4.3 and the bounds in Prop. A.3 and Prop. A.4, we may truncate the Fredholm series and take a pointwise limit \( \tau \to 0 \) of the kernel without changing its Fredholm determinant. This proves Theorem 4.1.
**Proof of Proposition A.3.** The integral over $D_{v, \Box}$ simply collects residues from the poles of $\sin^{-1}(\pi s)$ (up to signs). Then,

$$
\int_{D_{v, \Box}} I_{u, \tau}(v, v', s) = \sum_{q(v)} \left( \frac{\Gamma(v)\Gamma(\theta - v - i)}{\Gamma(v + i)\Gamma(\theta - v)} \right)^N u^i e^{\tau(\Re(v) + i^2/2)} |v + i - v'| \left( \sum_{i=1}^{q(v)} \text{Res}_{u,i}(v, v') \right),
$$

where $q(v) \leq R$ is the number of zeros of the sine caught inside the sausage. Since $v, v' \in C_\varphi$, we have

$$
\Re(v) = \frac{\theta}{2} - \cot(\varphi)|\Im(v)|
$$

where $0 < \delta \leq \frac{\theta}{4}$. Then, we may estimate $q(v)$:

$$
q(v) \leq R = -\Re(v) + \frac{\theta}{2} + \delta = \cot(\varphi)|\Im(v)| + \delta.
$$

For our bound, the number of residues doesn’t matter, and the contribution of the first residue dominates. The ratio of gamma functions in the residues become $\frac{\Gamma(v)}{\Gamma(v + i)} = \prod_{j=0}^{\frac{1}{2}}(v + j)^{-1}$ and $\Gamma(\theta - v - i)/\Gamma(\theta - v) = \prod_{j=1}^{\frac{1}{2}}(\theta - v - j)^{-1}$. It’s clear that $|v + i| \geq |\Im(v)|$ and $|\theta - v - i| \geq |\Im(v)|$. The $|v - v' + i|^{-1}$ term can be bounded above by a constant. Therefore for $|\Im(v)| > (e^{\theta/2}|u|/2)\frac{1}{\pi \varphi}$,

$$
\sum_{i=1}^{q(v)} \text{Res}_{u,i}(v, v') \leq \sum_{i=1}^{q(v)} \frac{1}{|\Im(v)|^{2N}} |u|^{i + \tau \theta} e^{-\tau i(\cot(\varphi)|\Im(v)| - i/2)}
\leq \sum_{i=1}^{q(v)} \left( \frac{e^{\theta/2}|u|}{|\Im(v)|^{2N}} \right)^i e^{-\tau c |\Im(v)|},
\leq 2 \frac{e^{\theta/2}|u|}{|\Im(v)|^{2N}} e^{-\tau c |\Im(v)|},
$$

where $c$ is a $\varphi$-dependent constant that comes from bounding the $i(\cot(\varphi)|\Im(v)| - i/2)$ term on the interval $1 \leq i \leq \cot(\varphi)|\Im(v)| + \delta$. We choose the constant $c_3$ in Prop. A.4 to ensure that $\cot(\varphi)|\Im(v)| \geq \cot(\varphi)c_3\theta > \theta/4 \geq \delta$. \hfill \Box

**Proof of Prop. A.4.** We will focus first on estimating the product of Gamma functions in $I_{u, \tau}(v, v', s)$. For $s \in \ell - \Re(v) + R$,

$$
\Re(s) = \delta + \cot(\varphi)|\Im(v)|.
$$

Stirling’s formula holds whenever $\arg(z)$ remains bounded away from $\pm \pi$ (see Abramowitz and Stegun [1]),

$$
\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + O \left( \frac{1}{|z|} \right),
$$

and

$$
\Re(\log \Gamma(z)) = -\Im(z) \arg(z) + \Re(z) (\log |z| - 1) - \frac{1}{2} \log |z| + O \left( \frac{1}{|z|} \right).
$$
This gives

\[ \log \left( \frac{\Gamma(v)}{\Gamma(v+s)} \right) + \log \left( \frac{\Gamma(\theta - v - s)}{\Gamma(\theta - v)} \right) = \]

\[ - \text{Im}(v) \arg(v) + \text{Im}(\theta - v) \arg(\theta - v) \tag{43} \]

\[ + \text{Re}(v)(\log |v| - 1) - \text{Re}(\theta - v)(\log |\theta - v| - 1) \tag{44} \]

\[ + \text{Im}(v + s) \arg(v + s) - \text{Im}(\theta - v - s) \arg(\theta - v - s) \tag{45} \]

\[ - \text{Re}(v + s)(\log |v + s| - 1) + \text{Re}(\theta - v - s)(\log |\theta - v - s| - 1) \tag{46} \]

\[ - \frac{1}{2} \log \left| \frac{v(\theta - v - s)}{(v + s)(\theta - v)} \right| + \mathcal{O}(\theta^{-1}) \tag{47} \]

since \(|v|, |\theta - v|, |\theta - v - s|, |v + s| \geq c\theta\) for some constant \(c > 0\).

Both terms in (44) have the correct sign and dominate the terms in (43). This will give us the exponential decay in \(v\) that we need. In particular,

\[ \text{Re}(v)(\log |v| - 1) - \text{Re}(\theta - v)(\log |\theta - v| - 1) = \frac{\theta}{2} \log \frac{|v|}{|\theta - v|} - \cot(\varphi) \text{Im}(v)(\log(|v|\theta - v) - 2). \]

Since the ratio inside the logarithm is \(\mathcal{O}(1)\) for all \(v \in \mathcal{C}_\varphi\) and some \(\varphi\)-dependent constant \(c\),

\[ (44) \leq \mathcal{O}(\theta) - c|v|\log(|v|\theta - v). \tag{48} \]

Since \(\text{Im}(\theta - v - s) = -\text{Im}(v + s), (45)\) becomes \(\text{Im}(v + s)(\arg(v + s) + \arg(\theta - v - s))\). From (40) and (42), we get \(\text{Re}(v + s) \geq \text{Re}(\theta - (v + s))\), and clearly \(\text{Im}(v + s) \geq -\text{Im}(\theta - (v + s))\). This implies that \(\text{Im}(v + s)\) and \(\arg(v + s) + \arg(\theta - v - s)\) have opposite signs, and hence

\[ (45) \leq \text{Im}(v + s)(\arg(v + s) + \arg(\theta - v - s)) \leq 0. \tag{49} \]

For some constant \(c > 0\),

\[ (46) = -\frac{\theta}{2} \log \frac{|v + s|}{|\theta - v - s|} - \delta \log |v + s||\theta - v - s| \]

\[ \leq -\frac{\theta}{2} c - \delta \log \left( \frac{\theta}{2} - \delta \right) \left( \frac{\theta}{2} + \delta \right) \tag{50} \]

\[ = \mathcal{O}(\theta) + \mathcal{O}(\log(\theta)). \]

Equation 47 should be split up into two terms: the first term is \(- \log |v/(\theta - v)|\) that is order \(\log \theta\) for small \(v\), and \(\mathcal{O}(1)\) for \(|v| \geq c\theta\). The second term \(- \log |(\theta - v - s)/(v + s)|\) behaves similarly, and we get

\[ (47) = \mathcal{O}(1) + \mathcal{O}(\log(\theta)). \tag{51} \]

From (40) and (42), we get \(|v + s - v'|^{-1} \leq \frac{2}{\theta} \). Finally, to analyze \(\exp(\tau(sv + s^2/2))\), we look at the
real part of $su + s^2/2$:

$$\operatorname{Re}(su + s^2/2)$$

$$= \operatorname{Re}(s) \operatorname{Re}(v) - \operatorname{Im}(s) \operatorname{Im}(v) + \frac{\operatorname{Re}(s)^2 - \operatorname{Im}(s)^2}{2}$$

$$= \operatorname{Re}(s) \operatorname{Re}(v) + \frac{\operatorname{Re}(s)^2}{2} + \frac{\operatorname{Im}(v)^2}{2} - \frac{(\operatorname{Im}(s) + \operatorname{Im}(v))^2}{2}$$

$$= (\delta + \cot(\varphi)|\operatorname{Im}(v)|)(\frac{\theta}{2} - \cot(\varphi)|\operatorname{Im}(v)|) + \frac{(\delta + \cot(\varphi)|\operatorname{Im}(v)|)^2}{2} + \frac{\operatorname{Im}(v)^2}{2} - \frac{(\operatorname{Im}(s) + \operatorname{Im}(v))^2}{2}$$

$$= \frac{\theta \delta + \delta^2}{2} + \frac{\cot(\varphi)|\operatorname{Im}(v)|}{2} + (1 - \cot(\varphi)^2)\frac{\operatorname{Im}(v)^2}{2} - \frac{(\operatorname{Im}(s) + \operatorname{Im}(v))^2}{2}$$

$$\leq C\theta + \cot(\varphi)\frac{\theta}{2} |\operatorname{Im}(v)|,$$  \hspace{1cm} (52)

using $\cot(\varphi) \geq 1$.

Putting (48), (49), (50), (51), (52) together with $\frac{1}{|\sin(\pi s)|} \leq Ce^{-\pi|\operatorname{Im}(s)|}$, we get

$$I_{\alpha,\tau}(v, v', s) \leq e^{C(\theta, N, \tau)} e^{-N|\operatorname{Im}(v)|(|\log |v| - c \cot(\varphi)\theta| - \pi|\operatorname{Im}(s)|)}$$  \hspace{1cm} (53)

where

$$C(\theta, N, \tau) = O(N(\theta + |\log \theta| + \theta^{-1} + 1) + \tau \theta).$$

Integrating this over $s$ completes the proof. \hfill \Box

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