Use of a modified Markov models for parallel reliability systems that are subject to maintenance

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Abstract. In this paper, we shall describe the reliability function \( R(t) \) as probability that the system will not malfunction within a given interval of execution times for maintained systems. Although, we shall be concerned with developing the reliability functions and predicted time to equipment breakdown (MTTF), Time between failures on average (MTBF) and the overall repair time (MTTR) from concepts of Markov model and some of transformations of Laplace to obtain the reliability and availability.

1. Introduction

The Markov approach can be used to model the random behaviour of systems that differ in time and space discretely or continuously [1]. A stochastic process is a type of random variation that can be discrete or continuous. Although the simple Markov approach cannot be used to model all stochastic processes, there are techniques for modelling certain additional stochastic processes using extensions of this basic framework. In this search, we'll go over these additional strategies [2]. The system's behaviour must be characterized by a lack of memory for the simple Markov method to function, which ensures that the system's future states are independent of all previous states except the one immediately preceding them [3]. As a result, a system's potential random action is solely determined by where it is right now, not by where it has been in the past or how it got there. In addition, for the solution to function, the process must be stationary, also known as homogeneous [4]. This means that the system's behaviour must be consistent over time, regardless of the time span under consideration, i.e., the likelihood of moving from one state to another must be the same (stationary) at all periods in the past and future. The Markov method is applicable to such systems because of these two factors, loss of memory and being stationary [5, 6].

2. Some important preliminaries

Definition (2.1) Stochastic Processes

Is a family of \( X = \{X(t): t \in T\} \) random variables, where \( t \) normally denotes time. That is, at all times in the collection, \( t, T \), the concept of a random number \( X(t) \) is observed: \( \{X(t): t \in T\} \) is a discrete time process, whether the set \( T \) is finite or countable. In fact, this usually means \( T = \{0, 1, 2, 3...\} \). A discrete-time process is then \( \{X(0), X(1), X(2), X(3), ...\} \): a random number correlated with 0,1,2,3 per time.

Definition (2.2) Markov process

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Is probability of transformation from any state \( i \) to any state \( j \) is the probability of any Markov model identified by the set of probabilities \( P_{ij} \). One of the most significant features of the Markov model is that the probability of transformation \( p_{ij} \) depends only on states \( i \) and \( j \) and is totally independent of all previous states except the last one.

**Definition (2.4) Failure rates** [7, 8]

The failure rates are the probability of failure happening in the interval \([t+\Delta t]\) per unit time, because a failure did not occur before \( t \). That is the rate of failure at which failures have occurred in \([t+\Delta t]\) failures have occurred in \([t\Delta t]\) failures have occurred in \([t\Delta t]\)

\[
\text{Failure rate} = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{R(T) \Delta t}
\]

(2.1)

**Definition (2.5) Mean time to failure (MTTF)** [9]

MTTF is based on the value to be expected the lifetime before a malfunction happens. Suppose that a system's reliability role is given by \( R(t) \) It is possible to calculate the MTTF as

\[
\text{MTTF} = \int_0^\infty t f(t) dt = \int_0^\infty R(t) dt
\]

(2.2)

**Definition (2.6) The transition probability matrix** [10]

It is possible to arrange the transition probabilities as a transition probability matrix

\[
P = (P_{ij})
\]

\[
P = \begin{bmatrix}
P_{0,0} & P_{0,1} & \cdots \\
P_{1,0} & P_{1,1} & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\]

The row \( i \) includes the probabilities of change from state and other states

**Definition (2.7) Time between failures on average (MTBF)** [11, 12]

The Mean Time Between Failure (MTBF) is a term that is commonly used in maintenance system reliability work. It is known as an item's (average) or (expected) lifetime, \( \mu \) or \( E(t) \) is an equivalent notation:

\[
\text{MTBF} = \int_0^\infty t f(t) dt,
\]

(2.3)

or

\[
\text{MTBF} = \int_0^\infty [1 - F(t)] dt
\]

(2.4)

MTBF is most generally expressed in terms of a reliability function.

\[
\text{MTBF} = \int_0^\infty R(t) dt
\]

\[
= \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t R(T)}
\]

(2.5)

**Definition (2.8) The overall repair time (MTTR)** [13]
Often known as the mean downtime or mean time to repair. The MTTR (mean time to restore) is the approximate value of the random variable repair time, not failure time, and it is measured as follows:

$$MTTR = \int_0^\infty g(t) \, t \, dt$$

(2.6)

MTTR = $1/\mu$ is obtained from the above equation because the distribution has a repair time density of $g(t) = \mu e^{-\mu t}$. When the log normal density function $g(t)$ is applied to the repair time $T$, and the density function is

$$g(t) = \frac{1}{\sqrt{2\pi \sigma^2 t}} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}}$$

(2.7)

Then it can be shown that

$$MTTR = me^{\sigma^2}$$

(2.8)

The log normal distribution's median is denoted by $m$.

**Definition (2.9) Availability $A(t)$** [14]

is a measure that enables a system to restore itself when it fails; instead, the availability of a system is specified as the possibility that it will succeed at time $t$. In terms of math,

$$Availability = \frac{MTTF}{MTTR + MTTF}$$

**Definition (2.13) Maintainability** [14, 15]

When maintenance is done according to required processes and services, the possibility of restoring a failed device to stated conditions within a given period of time is known as maintainability.

**Definition (2.10) Parallel System** [14, 16]

If $n$ devices are linked in parallel, such that the device works as long as at least one element works (signal from 1 to O).

$$R_p = (1 - q_0)^n - q_0^n$$

(2.9)

![Figure (1.1) an-n-stage parallel system](image)
In solving differential equations, the Laplace transform is important. The transformation of a time function \( f(t) \) by Laplace is defined by the integral

\[
L[f(t)] = f^*(s) = \int_0^\infty e^{-st} f(t) dt, \text{ Re}(s) > 0 \tag{2.10}
\]

Where \( L \) represents the Laplace transform.

**Definition (2.12) Inverse Laplace transform (ILT) [15, 17]**

The \( \text{ILT} \) of \( F(S) \) denoted \( \mathcal{L}^{-1} [F(s)] \), is the function \( f \) defined on \([0, \infty)\) which has the fewest number of discontinuities and satisfies

\[
L[f(x)] = F(s) \tag{2.11}
\]

**Illustrative Example (1) [17]**

Find \( \mathcal{L}^{-1} \left[ \frac{-1 + 7s}{(1 + s)(2 + s)(-3 + s)} \right] \)

Solution:
We compose the articulation in the structure

\[
\frac{-1 + 7s}{(1 + s)(2 + s)(-3 + s)} = \frac{A}{1 + s} + \frac{B}{2 + s} + \frac{C}{-3 + s}
\]

Solving for the constants yields: \( A = 2, B = -3, \text{ and } C = 1 \).

Thus, we get

\[
\mathcal{L}^{-1} \left[ \frac{-1 + 7s}{(1 + s)(2 + s)(-3 + s)} \right] = \mathcal{L}^{-1} \left[ \frac{2}{1 + s} \right] + \mathcal{L}^{-1} \left[ \frac{3}{2 + s} \right] + \mathcal{L}^{-1} \left[ \frac{-1}{-3 + s} \right] = 2e^{-x} + 3xe^{x} - e^{3x}
\]

**Example (2) [17]**

Solve the following differential equations

\[
\frac{dp_0}{dt} = -\lambda p_0(t) \tag{2.12}
\]

\[
\frac{dp_1}{dt} = \lambda p_0(t) - \lambda p_1(t) \tag{2.13}
\]

Solution:
Taking the Laplace change of eqns. (2.12) and (2.13) we have

\[
sp_0^*(s) - p_0(0) = -\lambda p_0^*(s)
\]

\[
sp_1^*(s) - p_1(0) = \lambda p_0^*(s) - \lambda p_1^*(s)
\]

Here

\[
p_0(0) = 1 \quad \text{and} \quad p_1(0) = 0
\]

Thus

\[
sp_0^*(s) = -\lambda p_0^*(s) + 1 \tag{2.14}
\]

\[
sp_1^*(s) = \lambda p_0^*(s) - \lambda p_1^*(s) \tag{2.15}
\]

Solving eqns. (2.14) and (2.15) simultaneously, we obtain
On reversal, we get the time dependent functions

\[ p_0(t) = e^{-\lambda t} \]
\[ p_1(t) = (\lambda t)e^{-\lambda t} \]

These are the terms of the Poisson distribution \[\text{[13]}\] for \( n = 0 \) and \( n = 1 \)

**Example (3)**

It’s clear that the reliability of parallel system with three components is

\[ R_S = 1 - (1 - R_1)(1 - R_2)(1 - R_3) \]

\[ R_S = 1 - \prod_{i=1}^{3}(1 - R_i) \]

(2.16)

Or

**Ri-exponential** we get

\[ R_S = 1 - (1 - e^{-\lambda t})^3 \]

(2.17)

By utilizing Binomial Theorem, we get

\[ (1 - e^{-\lambda t})^3 = 1 - 3e^{-\lambda t} + 3e^{-2\lambda t} - e^{-3\lambda t} \]

(2.18)

To figure the MTTF, we should utilize the equation (2.16)

\[ \text{MTTF} = \int_0^\infty (1 - (1 - e^{-\lambda t})^n)dt \]

(2.19)

If \( n = 3 \) we have

\[ \text{MTTF} = \int_0^\infty (1 - (1 - e^{-\lambda t})^3) dt \]
\[ = \int_0^\infty (3e^{-\lambda t} - 3e^{-2\lambda t} + e^{-3\lambda t})dt = \frac{7}{6\lambda} \]

**Definition (2.14) Argotic Process** \[\text{[18]}\]

It has no absorption states, allowing \( X(t) \) to travel about forever, as in the operation of a ground power plant, where failure only causes a temporary disruption.

**Illustrative Example**

If we had a maintainable system consisting of three equipment connected in parallel and had the same failure rate, the equipment would operate simultaneously. The state space is 0, 1, 2 and 3 representing the number of equipment failed. The system is failed (Argotic state) when it researches state (3).

If a device fails to work correctly, it is usually repaired to detect and address the error. By making a small change or removing a part, the machine is brought back to working order \[\text{[19, 20]}\].

\[
\begin{array}{cccc}
0 & 1-3\lambda & 3\lambda & 0 & 0 \\
1 & \mu & 1-(2\lambda + \mu) & 2\lambda & 0 \\
2 & 0 & 2\mu & 1-(\lambda + 2\mu) & \lambda \\
3 & 0 & 0 & 3\mu & (1-3\mu) \\
\end{array}
\]

Therefore if the limiting state probability vector is \([p_0 \ p_1 \ p_2 \ p_3]\)
Which in explicit form gives

\[(1 - 3\lambda)p_0 + \mu p_1 = p_0\]
\[3\lambda p_0 + (1 - 2\lambda - \mu)p_1 + 2\mu p_2 = p_1\]
\[2\lambda p_1 + (1 - \lambda - 2\mu)p_2 + 3\mu p_3 = p_2\]
\[\lambda p_2 + (1 - 3\mu)p_3 = p_3\]

Rearranging gives

\[-3\lambda p_0 + \mu p_1 = 0\]
\[3\lambda p_0 - (2\lambda + \mu)p_1 + 2\mu p_2 = 0\]
\[2\lambda p_1 - (\lambda + 2\mu)p_2 + 3\mu p_3 = 0\]
\[\lambda p_2 - 3\mu p_3 = 0\]
\[p_0 + p_1 + p_2 + p_3 = 1\]

The limiting state probabilities can be obtained by using straightforward method or matrix techniques and are

\[p_0 = \frac{\mu^3}{(\lambda + \mu)^3}\]
\[p_1 = \frac{3\mu^2\lambda}{(\lambda + \mu)^3}\]
\[p_2 = \frac{3\mu\lambda^2}{(\lambda + \mu)^3}\]
\[p_3 = \frac{\lambda^3}{(\lambda + \mu)^3}\]

In the case of three identical components connected in parallel, state 3 also becomes an upstate giving

Availability,

\[A = p_0 + p_1 + p_2\]
\[A = \frac{\mu^2 + 3\mu^2\lambda + 3\mu\lambda^2}{(\lambda + \mu)^3}\]

Unavailability,

\[U = p_3\]

Conclusions

To analyze reliability models that are subject to maintenance, a parallel-linked system was used. This study gives us a clear idea of how long it would take for a system to break down and how long it would take to maintain it and this is depending on argotic process. As a result, these values led to understand the behavior of the availability and reliability of the given system.
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