C1,α ISOMETRIC EXTENSIONS

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Abstract. In this paper we consider the Cauchy problem for isometric immersions. More precisely, given a smooth isometric immersion of a codimension one submanifold we construct C1,α isometric extensions for any α < 1/(n(n+1)+1) via the method of convex integration.

1. Introduction

The problem of extending isometric embeddings is considered in this paper. More precisely, let Σ be a codimension one submanifold of a smooth n-dimensional Riemannian manifold (M, g) and let f : Σ → Rq be a smooth isometric immersion, q ≥ n + 1. The problem is to extend f to be an isometric immersion u : M → Rq. This problem was first considered by Jacobowitz in [20] when the target dimension is q = n∗ = 1/2n(n + 1) in the high-regularity setting. In particular, Jacobowitz showed that this extension problem can viewed as a Cauchy problem, and showed that local extensions can be obtained under certain geometric conditions (i) by a variant of the Cauchy-Kowalevskaja Theorem in the analytic setting and (ii) by an adaptation of Nash’s implicit function theorem in the Ck setting with k ≥ 17. The geometric sufficient condition of Jacobowitz (see below for precise statement) can be seen as stating that the image f(Σ) has to be “more curved” than Σ, and it was shown in [20] that this condition is almost necessary – indeed, if f(Σ) is geodesic but Σ is not, no isometric extension can exist.

Viewing the extension problem as an initial-value problem has also played an important role in progress concerning Schläfli’s conjecture (see [14]) on the local embedding of surfaces in R3. For extending from one point where the Gaussian curvature is changing signs in a 2-dimensional manifold, sufficiently smooth embeddings into R3 have been constructed in a small neighbourhood of such point in [16, 23] by imposing some non-degeneracy conditions on the vanishing of the curvature. Similarly, for the isometric extension from a small curve across which Gaussian curvature vanishes at some order, sufficiently smooth local or semi-global isometric embeddings into R3 also exist in some neighbourhood of such curve under some conditions, see [11, 17, 21].

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The isometric extension problem can also be considered in the low-regularity and low codimension setting. For the classical global isometric immersion problem of \( u : \mathcal{M} \rightarrow \mathbb{R}^{n+1} \) the celebrated Nash-Kuiper Theorem \([22, 24]\) provides solutions of class \( u \in C^1 \). More precisely, any smooth strictly short immersion \( \bar{u} : \mathcal{M} \rightarrow \mathbb{R}^{n+1} \) may be uniformly approximated by \( C^1 \) isometric immersions – this high flexibility is a paradigm example of Gromov’s h-principle \([13]\). We recall that being isometric amounts to, in local coordinates, to the following system of first order partial differential equations:

\[
\frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} = g_{ij},
\]

where \( g = \sum_{i,j=1}^n g_{ij} dx^i dx^j \) is the metric in local coordinates, whereas \( u \) is strictly short if the metric tensor \( g_{ij} - \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} \) is a positive definite tensor. The Nash-Kuiper Theorem has been extended to \( C^{1,\alpha} \) in \([1]\) and subsequently in \([6]\) for \( \alpha < (2n_\ast + 1)^{-1} \) in the local case and for \( \alpha < (2(n+2)n_\ast + 1)^{-1} \) in the global case, where \( n_\ast = \frac{1}{2}n(n+1) \). Further progress in the local 2-dimensional case was achieved in \([9]\), for \( \alpha < 1/5 \). Let us mention that the interest in \( C^{1,\alpha} \) immersions is two-fold. On the one hand there is an interesting dichotomy between the high flexibility provided by the Nash-Kuiper Theorem for \( C^{1,\alpha} \) with small \( \alpha \) and extensions of classical rigidity theorems \([2, 6]\) for \( \alpha \) close to 1. On the other hand there is an unexpected parallel between the theory \( C^{1,\alpha} \) isometric immersions and the theory of \( C^\alpha \) weak solutions of the incompressible Euler equations and more general classes of hydrodynamic equations — this connection eventually leads to the resolution of Onsager’s conjecture in 3-dimensional turbulence, see \([3–5, 10, 19]\). The Cauchy problem for the Euler equations has been considered in \([7, 8]\).

In this paper, we aim to explore more in this parallel direction, that is, study the local extension problem in low codimension and low regularity. In \([18]\) Hungerbühler and Wasem first considered this problem and showed that in the \( C^1 \) category the analogue of the Nash-Kuiper statement is valid for one-sided isometric extensions. In this paper we consider \( C^{1,\alpha} \) isometric extensions from adapted short immersions (see definition below). In particular, we also construct a one-sided adapted short immersions from short immersion, and then a \( C^{1,\alpha} \) isometric extension.

1.1. Main results and ideas. In order to keep the presentation simple, we focus on the case of a single chart. That is, \( \Omega \subset \mathbb{R}^n \) is an open bounded set equipped with a smooth metric \( g = (g_{ij}) \). In the coordinates \( x_1, \ldots, x_n \in \Omega \) we denote the derivative of a map \( u : \Omega \rightarrow \mathbb{R}^{n+1} \) by \( \nabla u = (\partial_j u_i)_{i,j} \). Then, a differentiable map \( u \) is isometric if \( g = \nabla u^T \nabla u \) and strictly short if \( g - \nabla u^T \nabla u > 0 \), i.e. the \( n \times n \) matrix \( (g_{ij} - \partial_i u \cdot \partial_j u)_{ij} \) is positive definite for all \( x \). As usual, we call \( u \) an immersion if \( \nabla u(x) \) has rank \( n \) for every \( x \), and an embedding if in addition \( u \) is 1-1.

In analogy with adapted subsolutions introduced in \([8]\), we define adapted short immersions as follows.
**Definition 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set equipped with a smooth metric $g = (g_{ij})$. We call a map $u : \overline{\Omega} \to \mathbb{R}^{n+1}$ a $C^{1,\alpha}$-adapted short immersion (embedding) with constants $M, r > 0$, if $u \in C^{1,\alpha}(\overline{\Omega})$ is a short immersion (embedding) in $\overline{\Omega}$ with
\[
g - \nabla u^T \nabla u = \rho^2(Id + G)
\]
with a nonnegative function $\rho \in C(\overline{\Omega})$ and symmetric tensor $G \in C(\overline{\Omega}; \mathbb{R}^{n \times n})$ satisfying
\[
|G(x)| \leq r \quad \text{for all } x \in \overline{\Omega}
\]
and the following additional estimates hold: $u \in C^2(\overline{\Omega} \setminus \{\rho = 0\})$ and $\rho, G \in C^1(\overline{\Omega} \setminus \{\rho = 0\})$ with
\[
|\nabla^2 u(x)| \leq M\rho(x)^{-2},
\]
\[
|\nabla \rho(x)| \leq M\rho(x)^{-1},
\]
\[
|\nabla G(x)| \leq M\rho(x)^{-3},
\]
for any $x \in \overline{\Omega} \setminus \{\rho = 0\}$.

One of our main results is to show how the existence of a $C^{1,\alpha}$ extension can be reduced to the existence of an $C^{1,\alpha}$ adapted short immersion.

**Theorem 1.1.** [Isometric approximation] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set equipped with a smooth metric $g = (g_{ij})$ and let $\Sigma \subset \overline{\Omega}$ be a $C^2$ codimension one submanifold. Let $u : \overline{\Omega} \to \mathbb{R}^{n+1}$ be a $C^{1,\infty}$ adapted short immersion for some $\alpha_0 \geq \frac{1}{n(n+1)+1}$ with $\nabla u^T \nabla u = g$ on $\Sigma$. Then, for any $\epsilon > 0$ and any $\alpha < \frac{1}{n(n+1)+1}$, there exists a $C^{1,\alpha}$ isometric immersion $\bar{u} : \overline{\Omega} \to \mathbb{R}^{n+1}$ such that $\|\bar{u} - u\|_{C^0(\Omega)} < \epsilon$.

The basic method for obtaining $C^{1,\alpha}$ isometric immersions has been developed in [3]. Indeed, the bound $\alpha < \frac{1}{n(n+1)+1} = \frac{1}{2n+1}$ on the Hölder exponent agrees with the bound obtained in [6]. However, we need to substantially modify the proof in order to make sure $\bar{u} = u$ on $\Sigma$. This requires us to modify the mollification argument (see Section 3), which is designed to handle the problem with loss of derivatives. Then we adapt the strategy introduced in [8] of decomposing the domain $\overline{\Omega} \setminus \Sigma$ in level sets of the metric error (see Section 5).

In turn, one needs to consider criteria under which such an adapted short extension exists. In this paper we give a sufficient criterion, following the approach in [18]. To recall the general setting, let $(\mathcal{M}, g)$ be a smooth Riemannian manifold with a smooth codimension 1 submanifold $\Sigma$ and an isometric immersion $f : \Sigma \to \mathbb{R}^{n+1}$. For any point $p \in \Sigma$ we denote, as usual, by $T_p\Sigma$ the tangent vector space of $\Sigma \subset \mathcal{M}$, and $L : T_p\Sigma \times T_p\Sigma \to \mathbb{R}^1$ and $\hat{L} : T_p\Sigma \times T_p\Sigma \to N_{f(p)}$ are the second fundamental forms of $\Sigma \subset \mathcal{M}$ and $f(\Sigma) \subset \mathbb{R}^{n+1}$ respectively, with $N_{f(p)}$ denoting the normal space to $f(\Sigma)$ at $f(p)$. Here $\langle \cdot, \cdot \rangle$ denotes Euclidean scalar product.

In [20] Jacobowitz showed that a sufficient condition for the existence of a smooth isometric extension of $f : \Sigma \to \mathbb{R}^{n+1}$ is that there exists a vectorfield $\mu : \Sigma \to \mathbb{R}^{n+1}$

with

\begin{align}
(i) & \quad \mu(p) \in N_{f(p)}, \\
(ii) & \quad |\mu(p)| < 1, \\
(iii) & \quad \langle \mu(p), \bar{L}(\cdot, \cdot) \rangle = L(\cdot, \cdot) \quad \text{on } T_p \Sigma 
\end{align}

for any \( p \in \Sigma \). Moreover, a simple example in [20] shows that condition (1.5) is optimal in the sense that in general one cannot replace (ii) with \( |\mu(p)| \leq 1 \). On the other hand, in [18] Hungerbühler-Wasem showed that a sufficient condition for the existence of a \( C^1 \) one-sided isometric extension is that there exists a vectorfield \( \mu : \Sigma \rightarrow \mathbb{R}^{n+1} \) with

\begin{align}
(i) & \quad \mu(p) \in N_{f(p)}, \\
(ii) & \quad |\mu(p)| = 1, \\
(iii) & \quad \langle \mu(p), \bar{L}(\cdot, \cdot) \rangle - L(\cdot, \cdot) \quad \text{is positive definite on } T_p \Sigma 
\end{align}

for any \( p \in \Sigma \). We recall that a one-sided extension is defined in [18] as the extension of \( f \) to a one-sided neighbourhood of \( \Sigma \). The latter can be defined as follows. Let \( B \) be a ball centered at zero in \( \mathbb{R}^n \), which is a local chart of \( M \) for an open set containing \( \Sigma \) and let \( B_0 = B \cap (\mathbb{R}^{n-1} \times \{0\}) \), then \( B \cap (\mathbb{R}^{n-1} \times \mathbb{R}_{\leq 0}) \) and \( B \cap (\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}) \) are both called one-sided neighbourhoods of \( B_0 \). A one-sided neighbourhood \( \Omega \) of a point in \( \Sigma \) is the image of a one-sided neighbourhood of \( B_0 \) under the inverse of such local chart.

Our second main result is the existence of an adapted short extension under the condition (1.6).

**Theorem 1.2. [Adapted short extension]** Let \( f : \Sigma \rightarrow \mathbb{R}^{n+1} \) be a smooth isometric immersion, where \( \Sigma \) is a codimension one submanifold of an \( n \)-dimensional Riemannian manifold \((M, g)\). If there exists a vectorfield \( \mu \) satisfying condition (1.6) at any point \( p \in \Sigma \), then there exists a one-sided neighbourhood \( \Omega \) of \( \Sigma \) at \( p \), such that for any \( \alpha_0 < 1/3 \) there exists a \( C^{1,\alpha_0} \) adapted short immersion \( u : (\Omega, g) \rightarrow \mathbb{R}^{n+1} \) with \( \nabla u^T \nabla u = g \) and \( u = f \) on \( \Sigma \).

Obviously, combining Theorem 1.2 with Theorem 1.1 one can easily obtain

**Theorem 1.3. [Isometric extension]** Assume \( f : \Sigma \rightarrow \mathbb{R}^{n+1} \) is a smooth isometric immersion, where \( \Sigma \) is a codimension one submanifold of an \( n \)-dimensional Riemannian manifold \((M, g)\). If there exists a vectorfield \( \mu \) satisfying condition (1.6) at any point \( p \in \Sigma \), then there exists a one-sided neighbourhood \( \Omega \) of \( \Sigma \) at \( p \), such that for any \( \alpha < \frac{1}{n(n+1)+1} \) there exists a \( C^{1,\alpha} \) isometric immersion \( u : (\Omega, g) \rightarrow \mathbb{R}^{n+1} \) with \( u = f \) on \( \Sigma \).

We expect that in the case \( n = 2 \) the techniques of [9] can be adapted to construct \( C^{1,\alpha} \) isometric extensions for any \( \alpha < 1/5 \) (rather than just \( \alpha < 1/7 \)).
2. Preliminaries

In this section we introduce some notation, function spaces and basic lemmas. For a function \( f : \Omega \rightarrow \mathbb{R}^q \) and for a multi-index \( \beta \) the Hölder norms are defined as follows:

\[
\|f\|_0 = \sup_{\Omega} f, \quad \|f\|_m = \sum_{j=0}^{m} \max_{|\beta| = j} \|\partial^\beta f\|_0,
\]

and

\[
[f]_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, \quad [f]_{m+\alpha} = \max_{|\beta| = m} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x - y|^\alpha}, \quad 0 < \alpha \leq 1.
\]

Then the Hölder norms are given as

\[
\|f\|_{m+\alpha} = \|f\|_m + [f]_{m+\alpha}.
\]

We recall the standard interpolation inequality

\[
[f]_r \leq C \|f\|_{0}^{1-r} [f]_{s}^{r}
\]

for \( s > r \geq 0 \) and the approximation of Hölder functions by smooth functions:

**Lemma 2.1.** For any \( r, s \geq 0 \), and \( 0 < \alpha \leq 1 \), we have

\[
[f * \varphi_l]_{r+s} \leq C l^{-s} [f]_r,
\]

\[
\|f - f * \varphi_l\|_r \leq C l^{1-r} [f]_1, \quad \text{if } 0 \leq r \leq 1,
\]

\[
\|(fg) * \varphi_l - (f * \varphi_l)(g * \varphi_l)\|_r \leq C l^{2\alpha-r} \|f\|_\alpha \|g\|_\alpha,
\]

with constant \( C \) depending only on \( s, r, \alpha, \varphi \).

In the above \( \varphi_l \) is a standard mollifying kernel at length-scale \( l > 0 \). Other properties about Hölder norm can be found in references such as [6,9,10].

For an \( n \times n \) matrix \( P \) we will use the operator norm as defined by

\[
|P| := \sup_{\xi \in S^{n-1}} |P\xi|.
\]

The \( n \times n \) identity matrix is denoted by \( \text{Id} \). We recall the following lemmas. The first one is the decomposition of a metric into a sum of primitive metrics [24], where we refer to the form used in [6,9].

**Lemma 2.2.** For any \( n \geq 2 \) there exists a geometric constant \( r_0 > 0 \), vectors \( \nu_1, \ldots, \nu_n \in S^{n-1} \) and smooth functions \( a_k \in C^\infty(B_{r_0}(\text{Id})) \) such that, for any positive definite matrix \( P \in \mathbb{R}^{n \times n} \) with

\[
|P - \text{Id}| \leq r_0,
\]

the identity

\[
P = \sum_{k=1}^{n} a_k^2(P) \nu_k \otimes \nu_k
\]

holds.
Here we recall that $n_\ast = \frac{1}{2}n(n+1)$. Of course we may assume without loss of generality that $r_0 < 1$. For future reference we fix some radii $0 < r_1 < r_2 < r_0$ such that
\[ r_1 \leq \frac{r_2}{5} \leq \frac{1}{25}r_0, \] \hspace{1cm} (2.2)
with $r_0$ from Lemma 2.2.

Finally, we recall the following lemma, which describes the profile of the building block of convex integration (the corrugation) for the isometric immersion problem, from [6], see also [18].

**Lemma 2.3.** There exist some $\delta_\ast > 0$ and a pair of functions $(\Gamma_1, \Gamma_2) \in C^\infty([0, \delta_\ast] \times \mathbb{R}, \mathbb{R}^2)$ such that $(\Gamma_1, \Gamma_2)(s,t) = (\Gamma_1, \Gamma_2)(s,t+2\pi)$ and
\[
(1+\partial_t \Gamma_1)^2 + (\partial_t \Gamma_2)^2 = 1 + s^2; \\
\|\partial_s \partial_t \Gamma_1(s,t)\|_0 + \|\partial_t (\Gamma_1, \Gamma_2)(s,t)\|_0 \leq C(k)s, \text{ for } k \geq 0.
\]

3. Main Iteration Propositions

In this section, we state and prove our main iteration propositions which form the building block of the convex integration iteration. We start by recalling how to “add” a primitive metric - this is called a “step” in the terminology of [24], see also Proposition 2 in [6].

**Proposition 3.1. [Step]** Let $u \in C^2(\Omega, \mathbb{R}^{n+1})$ be an immersion, $\nu \in S^{n-1}$ and $a \in C^2(\Omega)$. Assume that
\[
\frac{1}{\gamma}Id \leq \nabla u^T \nabla u \leq \gamma Id \quad \text{in } \Omega 
\] \hspace{1cm} (3.1)
\[
\|u\|_2 \leq M\delta^{1/2}\theta, 
\] \hspace{1cm} (3.2)
\[
\|a\|_0 \leq \left(\frac{1}{2}\gamma \varepsilon\right)^{1/2}, \quad \|a\|_1 \leq M\varepsilon^{1/2}\theta, \quad \|a\|_2 \leq M\varepsilon^{1/2}\theta\bar{\theta}, 
\] \hspace{1cm} (3.3)
for some $M, \gamma \geq 1, \varepsilon \leq \delta \leq 1$ and $\theta \leq \bar{\theta}$. There exists a constant $c_0 = c_0(M, \gamma)$ such that, for any
\[
\lambda \geq c_0 \frac{\delta^{1/2}}{\varepsilon^{1/2} \bar{\theta}}, 
\] \hspace{1cm} (3.4)
there exists an immersion $v \in C^2(\Omega, \mathbb{R}^{n+1})$ such that
\[
\frac{1}{2\gamma}Id \leq \nabla v^T \nabla v \leq 2\gamma Id \quad \text{in } \Omega 
\] \hspace{1cm} (3.5)
\[
v = u \quad \text{on } \Omega \setminus \text{supp } a, 
\] \hspace{1cm} (3.6)
\[
\|v - u\|_j \leq M\varepsilon^{1/2}\lambda^{j-1}, \quad j = 0, 1, 2 
\] \hspace{1cm} (3.7)
\[
\|v\|_2 \leq M\varepsilon^{1/2}\lambda, 
\] \hspace{1cm} (3.8)
\[
\|\nabla v^T \nabla v - (\nabla u^T \nabla u + a^2 \nu \otimes \nu)\|_j \leq C(M, \gamma)\varepsilon^{1/2}\delta^{1/2}\theta\lambda^{j-1}, \quad j = 0, 1. 
\] \hspace{1cm} (3.9)
Here $M$ is a constant depending only on $\gamma$. 


Proof. Let us fix $\lambda \geq c_0 \frac{\delta^{1/2}}{a} \geq \bar{\theta}$, where $c_0$ will be chosen later during the proof. For the moment it suffices to assume that $c_0 \geq 1$, so that, in particular, in the sequel we may assume the inequalities

$$\theta \leq \frac{\delta^{1/2}}{a} \lambda \leq \lambda. \quad (3.10)$$

We start by regularizing $u$ on length-scale $\lambda^{-1}$ to get a smooth immersion $\tilde{u}$ satisfying

$$\|\tilde{u} - u\|_1 \leq C(M)\delta^{1/2} \lambda^{-1}, \quad \|\tilde{u}\|_2 \leq C(M)\lambda^{1/2}, \quad \|\tilde{u}\|_3 \leq C(M)\lambda. \quad (3.11)$$

Observe that

$$\nabla \tilde{u}^T \nabla \tilde{u} = \nabla u^T \nabla u - (\nabla u - \nabla \tilde{u})^T \nabla u - \nabla \tilde{u}^T (\nabla u - \nabla \tilde{u}),$$

and hence

$$\frac{1}{2\gamma} \text{Id} \leq \nabla \tilde{u}^T \nabla \tilde{u} \leq 2\gamma \text{Id}, \quad (3.12)$$

provided $\delta^{1/2} \lambda^{-1} \leq c_2^{-1}$ for some $c_2 = c_2(M, \gamma)$. Choosing $c_0 \geq c_2$ in the inequality (3.4) constraining $\lambda$ will ensure this. Then it follows that $\nabla \tilde{u}^T \nabla \tilde{u}$ is invertible, and hence we can set

$$\tilde{\xi} = \nabla \tilde{u}(\nabla \tilde{u}^T \nabla \tilde{u})^{-1} \nu, \quad \xi = \frac{\tilde{\xi}}{|	ilde{\xi}|^2},$$

$$\tilde{\zeta} = *(\partial_1 \tilde{u} \wedge \partial_2 \tilde{u} \wedge \cdots \wedge \partial_n \tilde{u}), \quad \zeta = \frac{\tilde{\zeta}}{|	ilde{\xi}||\tilde{\zeta}|},$$

$$\tilde{a} = |	ilde{\xi}| a,$$

where $*$ denotes the Hodge star operator. Observe that we then have, by construction,

$$\nabla \tilde{u}^T \xi = \frac{1}{|	ilde{\xi}|^2} \nu, \quad \nabla \tilde{u}^T \zeta = 0. \quad (3.13)$$

It follows from (3.11)-(3.12) that

$$\|(\xi, \zeta)\|_0 \leq C(\gamma), \quad (3.14)$$

$$\|(\xi, \zeta)\|_1 \leq C(\gamma, M)\delta^{1/2},$$

$$\|(\xi, \zeta)\|_2 \leq C(\gamma, M)\lambda.$$
where in the final inequality we have used that $\tilde{\theta} \leq \lambda$. We set

$$v = u + \frac{1}{\lambda}(\Gamma_1(\tilde{a}, \lambda x \cdot \nu)\xi + \Gamma_2(\tilde{a}, \lambda x \cdot \nu)\zeta).$$

Using Lemma 2.3 we see that $v = u$ outside $\text{supp } a$, so that (3.6) holds.

Next, note that

$$\|v - u\|_j \leq \frac{1}{\lambda}(\|\Gamma_1\|_j \|\xi\|_j + \|\Gamma_2\|_j \|\zeta\|_j)
\leq \frac{C}{\lambda}(\|\Gamma_1\|_j \|\xi\|_0 + \|\Gamma_1\|_0 \|\xi\|_j + \|\Gamma_2\|_j \|\zeta\|_0 + \|\Gamma_2\|_0 \|\zeta\|_j)$$

for $j = 0, 1, 2$. Therefore we need to estimate $\|\Gamma_i\|_j$ for $i = 1, 2$ and $j = 0, 1, 2$, where we refer, with a slight abuse of notation, to the $C^j$-norms in $x \in \Omega$ of the composition $x \mapsto \Gamma_i(\tilde{a}(x), \lambda x \cdot \nu)$. Using Lemma 2.3 we deduce for $i = 1, 2$

$$\|\partial_i \Gamma_1\|_0 + \|\partial_i \Gamma_1\|_0 + \|\partial_i^2 \Gamma_1\|_0 \leq C\|\tilde{a}\|_0 \leq C(\gamma)\varepsilon^{1/2},$$

$$\|\Gamma_1\|_1 \leq \|\partial_i \Gamma_1\|_0 \lambda + \|\partial_i \Gamma_1\|_0 \|\nabla \tilde{a}\|_0
\leq C(\gamma)\varepsilon^{1/2} \lambda + C(M, \gamma)\varepsilon^{1/2} \theta$$

$$\leq C(\gamma)\varepsilon^{1/2} \lambda,$$

$$\|\partial_i \Gamma_1\|_1 \leq \|\partial_i^2 \Gamma_1\|_0 \lambda + \|\partial_i \partial_i \Gamma_1\|_0 \|\nabla \tilde{a}\|_0
\leq C(\gamma)\varepsilon^{1/2} \lambda,$$

where we have used that $\lambda \geq C(M, \gamma)\theta$ - this can be ensured by an appropriate choice of $c_0$. Similarly, we also have

$$\|\partial_2 \Gamma_1\|_0 \leq C\|\tilde{a}\|_0 \leq C(\gamma)\varepsilon^{1/2},$$

$$\|\partial_2 \Gamma_2\|_0 \leq C;$$

$$\|\partial_2 \Gamma_1\|_1 \leq \|\partial_2 \partial_2 \Gamma_1\|_0 \lambda + \|\partial_2^2 \Gamma_1\|_0 \|\nabla \tilde{a}\|_0 \leq C(\gamma)\varepsilon^{1/2} \lambda,$$

$$\|\partial_2 \Gamma_2\|_1 \leq \|\partial_2^3 \Gamma_2\|_0 \lambda + \|\partial_2^2 \Gamma_2\|_0 \|\nabla \tilde{a}\|_0 \leq C(\gamma)\lambda.$$
Consequently, we derive
\[
\|v - u\|_0 \leq C(\gamma)\varepsilon^{1/2}\lambda^{-1},
\]
\[
\|v - u\|_1 \leq C(\gamma)\varepsilon^{1/2} + C(M, \gamma)\varepsilon^{1/2}\delta^{1/2}\theta \lambda^{-1}
\]
\[
\leq C(\gamma)\varepsilon^{1/2},
\]
\[
\|v - u\|_2 \leq C(\gamma)\varepsilon^{1/2}\lambda + C(M, \gamma)\varepsilon^{1/2}\delta^{1/2}\theta
\]
\[
\leq C(\gamma)\varepsilon^{1/2}\lambda.
\]
Summarizing, we arrive at (3.7), and since \(\varepsilon^{1/2}\lambda \geq M\delta^{1/2}\theta\), also at (3.8).

Next, we derive estimates on the metric error. We calculate:
\[
\nabla v = \nabla u + (\partial_t \Gamma_1 \xi \otimes \nu + \partial_t \Gamma_2 \zeta \otimes \nu) + \frac{1}{\lambda}(\Gamma_1 \nabla \xi + \Gamma_2 \nabla \zeta)
\]
\[
+ \frac{1}{\lambda}(\partial_s \Gamma_1 \xi \otimes \nabla \tilde{a} + \partial_s \Gamma_2 \zeta \otimes \nabla \tilde{a})
\]
\[
= \nabla u + A + E_1 + E_2,
\]
where we have set
\[
A = \partial_t \Gamma_1 \xi \otimes \nu + \partial_t \Gamma_2 \zeta \otimes \nu, \quad E_1 = \frac{1}{\lambda}(\Gamma_1 \nabla \xi + \Gamma_2 \nabla \zeta)
\]
and \(E_2 = E_2^{(1)} + E_2^{(2)}\) with
\[
E_2^{(1)} = \frac{1}{\lambda}\partial_s \Gamma_1 \xi \otimes \nabla \tilde{a}, \quad E_2^{(2)} = \frac{1}{\lambda}\partial_s \Gamma_2 \zeta \otimes \nabla \tilde{a}.
\]
Using (3.13) and Lemma 2.3 we have
\[
\nabla \tilde{u}^T A + A^T \nabla \tilde{u} + A^T A = a^2 \nu \otimes \nu
\]
and
\[
\nabla \tilde{u}^T E_2^{(2)} = 0.
\]
Therefore we may write, using the notation \(\text{sym}(B) = (B + B^T)/2\),
\[
\nabla v^T \nabla v - (\nabla u^T \nabla u + a^2 \nu \otimes \nu) = \text{sym} \left[(\nabla u - \nabla \tilde{u})^T (A + E_2^{(2)})\right]
\]
\[
+ \text{sym} \left[A^T E_2^{(2)}\right] + \text{sym} \left[(\nabla u + A)^T (E_1 + E_2^{(1)})\right] + (E_1 + E_2)^T (E_1 + E_2).
\]
(3.19)
Using the estimates (3.14), (3.15), (3.17) and (3.18) we obtain
\[
\|A\|_0 \leq C(\gamma)\varepsilon^{1/2},
\]
\[
\|E_1\|_0 \leq C(M, \gamma)\lambda^{-1}\varepsilon^{1/2}\delta^{1/2}\theta,
\]
\[
\|E_2^{(1)}\|_0 \leq C(M, \gamma)\lambda^{-1}\varepsilon\theta,
\]
\[
\|E_2^{(2)}\|_0 \leq C(M, \gamma)\lambda^{-1}\varepsilon^{1/2}\theta.
\]
Using that \(\varepsilon \leq \delta\) and \(\lambda \geq \theta\), we deduce
\[
\|\nabla v^T \nabla v - (\nabla u^T \nabla u + a^2 \nu \otimes \nu)\|_0 \leq C(M, \gamma)\varepsilon^{1/2}\delta^{1/2}\lambda^{-1}\theta.
\]
(3.20)
Similarly, using the Leibniz-rule we obtain
\[ \|A\|_1 \leq C(\|\partial_1 \Gamma_1\|_1 \|\xi\|_0 + \|\partial_1 \Gamma_2\|_1 \|\xi\|_0 + \|\partial_1 \Gamma_1\|_0 \|\xi\|_1 + \|\partial_1 \Gamma_2\|_0 \|\xi\|_1) \]
\[ \leq C(\gamma, M)(\varepsilon^{1/2} \lambda + \varepsilon^{1/2} \delta^{1/2} \theta) \]
\[ \leq C(\gamma, M) \varepsilon^{1/2} \lambda, \]
\[ \|E_1\|_1 \leq \frac{C}{\lambda}(\|\Gamma_1\|_1 \|\nabla \xi\|_0 + \|\Gamma_1\|_0 \|\nabla \xi\|_1 + \|\Gamma_2\|_1 \|\nabla \zeta\|_0 + \|\Gamma_2\|_0 \|\nabla \zeta\|_1) \]
\[ \leq C(\gamma, M) \varepsilon^{1/2} \delta^{1/2} \theta, \]
\[ \|E_2^{(1)}\|_1 \leq \frac{C}{\lambda}(\|\partial_s \Gamma_1\|_1 \|\xi\|_0 \|\nabla \tilde{a}\|_0 + \|\partial_s \Gamma_1\|_0 \|\xi\|_1 \|\nabla \tilde{a}\|_0 + \ldots + \|\partial_s \Gamma_1\|_0 \|\xi\|_0 \|\nabla \tilde{a}\|_1) \]
\[ \leq \frac{C(\gamma, M)}{\lambda}(\varepsilon \lambda \theta + \varepsilon \delta^{1/2} \tilde{\theta}) \]
\[ \leq C(\gamma, M) \varepsilon \theta, \]
\[ \|E_2^{(2)}\|_1 \leq \frac{C}{\lambda}(\|\partial_s \Gamma_2\|_1 \|\xi\|_1 \|\nabla \tilde{a}\|_0 + \|\partial_s \Gamma_2\|_0 \|\xi\|_1 \|\nabla \tilde{a}\|_0 + \ldots + \|\partial_s \Gamma_2\|_0 \|\xi\|_0 \|\nabla \tilde{a}\|_1) \]
\[ \leq \frac{C(\gamma, M)}{\lambda}(\varepsilon^{1/2} \lambda \theta + \varepsilon^{1/2} \delta^{1/2} \tilde{\theta}) \]
\[ \leq C(\gamma, M) \varepsilon^{1/2} \theta. \]

Differentiating (3.19), collecting terms and using the inequalities (3.10) we deduce
\[ \|\nabla v^T \nabla v - (\nabla u^T \nabla u + a^2 \nu \otimes \nu)\|_1 \leq C(M, \gamma) \varepsilon^{1/2} \delta^{1/2} \theta. \]

This concludes the verification of (3.9).

Finally, we verify that \( v \) is an immersion. From (3.20) it follows that
\[ \|\nabla v^T \nabla v - (\nabla u^T \nabla u + a^2 \nu \otimes \nu)\|_0 \leq \frac{1}{2 \gamma}, \]
provided we choose \( c_0 \geq 2 \gamma C(M, \gamma) \). Using (3.3) and \( \varepsilon \leq 1 \) we observe
\[ 0 \leq a^2 \nu \otimes \nu \leq \frac{\gamma}{2} \Id, \]
so that from (3.11) we readily deduce (3.5). This concludes the proof. \( \square \)

Next we show how to utilize Proposition 3.1 to “add” a term of the form
\[ \sum_{k=1}^{N} a_k^2 \nu_k \otimes \nu_k \]
to the metric. This corresponds to a “stage” in the terminology of [24], compare also with Proposition 4 in [6].
Proposition 3.2. [Stage] Let $u \in C^2(\Omega, \mathbb{R}^{n+1})$ be an immersion, $\nu_k \in S^{n-1}$ and $a_k \in C^2(\Omega)$ for $k = 1, \ldots, N$. Assume that

$$\frac{1}{\gamma} \text{Id} \leq \nabla u^T \nabla u \leq \gamma \text{Id} \quad \text{in } \Omega \quad (3.21)$$

$$\| u \|_2 \leq M \delta^{1/2} \theta, \quad (3.22)$$

$$\| a_k \|_0 \leq \left( \frac{1}{2} \gamma \varepsilon \right)^{1/2}, \quad \| a_k \|_1 \leq M \varepsilon^{1/2} \theta, \quad \| a_k \|_2 \leq M \varepsilon^{1/2} \theta \tilde{\theta}, \quad (3.23)$$

for some $M, \gamma \geq 1, \varepsilon \leq \delta \leq 1$ and $\theta \leq \tilde{\theta}$. Then there exists a constant $c_1 = c_1(M, \gamma)$ such that for any $K \geq c_1 \tilde{\theta} \theta^{-1}$ there exists an immersion $v \in C^2(\Omega, \mathbb{R}^{n+1})$ such that

$$v = u \text{ on } \Omega \setminus \bigcup_k \text{supp } a_k, \quad (3.24)$$

$$\| v - u \|_j \leq M \varepsilon^{1/2} (\varepsilon^{-1/2} \delta^{1/2} \theta K)^{j-1} \quad j = 0, 1, \quad (3.25)$$

$$\| v \|_2 \leq M \delta^{1/2} \theta K^N. \quad (3.26)$$

Furthermore, there exists $E \in C^1(\Omega, \mathbb{R}^{n \times n})$ such that

$$\nabla v^T \nabla v = \nabla u^T \nabla u + \sum_{k=1}^N a_k^2 \nu_k \otimes \nu_k + E \quad \text{in } \Omega \quad (3.28)$$

with

$$\| E \|_0 \leq C(M, \gamma) \varepsilon K^{-1},$$

$$\| E \|_1 \leq C(M, \gamma) \varepsilon^{1/2} \delta^{1/2} \theta K^{N-1}. \quad (3.27)$$

The constant $M$ depends only on $\gamma$ and $N$.

Proof. Set $u_0 = u, \gamma_1 = \gamma$ and construct $u_1$ by applying Proposition 3.1 to $u_0$ and $a_1$ with constants

$$\varepsilon_1 = \varepsilon, \quad \delta_1 = \delta, \quad \lambda_1 = \theta K \frac{\delta^{1/2}}{\varepsilon^{1/2}}, \quad \theta_1 = \theta, \quad \tilde{\theta}_1 = \tilde{\theta}. \quad (3.29)$$

Then conditions (3.1)-(3.3) are satisfied, and we may in addition choose $c_1$ so that (3.1) is fulfilled for $K \geq c_1 \tilde{\theta} \theta^{-1}$. We obtain $u_1$ with the properties

$$\| u_1 - u_0 \|_j \leq M_1 \varepsilon^{1/2} \lambda_1^{j-1}, \quad j = 0, 1, 2,$$

$$\| u_1 \|_2 \leq M_1 \varepsilon^{1/2} \lambda_1, \quad (3.28)$$

$$\| \nabla u_1^T \nabla u_1 - (\nabla u_1^T \nabla u_1 + a_1^2 \nu_1 \otimes \nu_1) \|_j \leq C(M, \gamma_1) \varepsilon^{1/2} \delta^{1/2} \theta_1 \lambda_1^{j-1}, \quad j = 0, 1,$$

with $M_1$ depending only on $\gamma$. Note that this time $u_1$ and $a_2$ satisfy once again the assumptions (3.1)-(3.3) of Proposition 3.1 with

$$\varepsilon_2 = \delta_2 = \varepsilon, \quad \gamma_2 = 2 \gamma_1, \quad \theta_2 = \tilde{\theta}_2 = \lambda_1,$$
and hence we may again apply Proposition 3.1 with \( \lambda_2 = K\lambda_1 \). It is easy to verify that condition (3.4) is then valid. Thus, in this way we may successively apply Proposition 3.1 to obtain \( u_2, \ldots, u_N \) with constants

\[
\varepsilon_k = \delta_k = \varepsilon, \quad \lambda_k = \lambda_1 K^{k-1}, \quad \theta_k = \bar{\theta}_k = \lambda_{k-1}, \quad \gamma_k = 2\gamma_{k-1}.
\]

We conclude for \( k = 2, \ldots, N \)

\[
\|u_k - u_{k-1}\|_j \leq M \varepsilon^{1/2} \lambda_k^{j-1}, j = 0, 1, 2,
\]

\[
\|u_k\|_2 \leq M \varepsilon^{1/2} \lambda_k,
\]

\[
\|\nabla \! u_k^T \nabla u_k - (\nabla \! u_{k-1}^T \nabla u_{k-1} + a_k^2 \nu_k \otimes \nu_k)\|_j \leq C(M, \gamma_k) \varepsilon \lambda_{k-1} \lambda_k^{j-1}, j = 0, 1.
\]

We obtain an immersion \( v = u_N \in C^2(\Omega, \mathbb{R}^{n+1}) \) satisfying, for \( j = 0, 1, 2 \),

\[
\|v - u\|_j \leq \sum_{k=1}^N \|u_k - u_{k-1}\|_j \leq M \sum_{k=1}^N \varepsilon^{1/2} \lambda_k^{j-1}.
\]

Hence (3.25) easily follows, with a possibly larger constant \( M \). Moreover, we further deduce

\[
\|v\|_2 \leq \|u\|_2 + \|v - u\|_2 \leq M \delta^{1/2} \theta + M \varepsilon^{1/2} \sum_{k=1}^N \lambda_k
\]

\[
\leq M \delta^{1/2} \theta + C M \delta^{1/2} \theta K^N,
\]

so that (3.26) follows.

Set

\[
E = \nabla \! u^T \nabla v - \left( \nabla \! u^T \nabla u + \sum_{k=1}^N a_k^2 \nu_k \otimes \nu_k \right)
\]

\[
= \sum_{k=1}^N \left[ \nabla \! u_k^T \nabla u_k - (\nabla \! u_{k-1}^T \nabla u_{k-1} + a_k^2 \nu_k \otimes \nu_k) \right].
\]

Combining with (3.28) and (3.29) we obtain

\[
\|E\|_0 \leq C(M, \gamma) \varepsilon^{1/2} \delta^{1/2} \theta \lambda_1^{-1} + \sum_{k=2}^N \varepsilon \lambda_{k-1} \lambda_k^{-1}
\]

\[
\leq C(M, \gamma) \varepsilon K^{-1},
\]

\[
\|E\|_1 \leq C(M, \gamma) \varepsilon^{1/2} \delta^{1/2} \theta + \sum_{k=2}^N \varepsilon \lambda_{k-1}
\]

\[
\leq C(M, \gamma) \varepsilon^{1/2} \delta^{1/2} \theta K^{N-1},
\]

which are the desired estimates of the metric error (3.27). \( \square \)
As a corollary, we can add a term of the form $\rho^2(\text{Id} + G)$ with $|G| \leq r_0$ (c.f. Lemma 2.2) to the metric.

**Corollary 3.1.** Let $u \in C^2(\Omega, \mathbb{R}^{n+1})$ be an immersion such that
\[
\frac{1}{\gamma} \text{Id} \leq \nabla u^T \nabla u \leq \gamma \text{Id}, \quad \text{in } \Omega
\]
\[
\|u\|_2 \leq M \delta^{1/2} \theta,
\]
and let $\rho \in C^1(\Omega)$, $G \in C^1(\Omega, \mathbb{R}^{n \times n})$ such that
\[
\|\rho\|_0 \leq \left(\frac{1}{2} \gamma \varepsilon\right)^{1/2}, \quad \|\rho\|_1 \leq M \varepsilon^{1/2} \theta,
\]
\[
\|G\|_0 \leq r_0, \quad \|G\|_1 \leq M \theta
\]
with $\varepsilon \leq \delta \leq 1$ and $M, \gamma \geq 1$. Then there exists a constant $K_0 = K_0(M, \gamma)$ such that for any $K \geq K_0$ there exists an immersion $v \in C^2(\Omega, \mathbb{R}^{n+1})$ such that
\[
v = u \text{ on } \Omega \setminus \text{supp } \rho,
\]
\[
\|v - u\|_j \leq M \varepsilon^{1/2} (\varepsilon^{-1/2} \delta^{1/2} \theta K)^{j-1} \quad j = 0, 1,
\]
\[
\|v\|_2 \leq M \delta^{1/2} \theta K^n - 1.
\]
Furthermore, there exists $E \in C^1(\Omega, \mathbb{R}^{n \times n})$ such that
\[
\nabla v^T \nabla v = \nabla u^T \nabla u + \rho^2(\text{Id} + G) + E \quad \text{in } \Omega
\]
with
\[
\|E\|_0 \leq C(M, \gamma) \varepsilon K^{-1},
\]
\[
\|E\|_1 \leq C(M, \gamma) \varepsilon^{1/2} \delta^{1/2} \theta K^n - 1.
\]

The constant $M$ depends only on $\gamma$.

**Proof.** Let us fix $K \geq K_0$ with $K_0 = K_0(M, \gamma)$ as in Proposition 3.2. We start by regularizing $\rho$ and $G$ at length-scale $\ell = \tilde{\theta}^{-1}$, with
\[
\tilde{\theta} = \frac{K \theta}{K_0} \geq \theta.
\]
We obtain
\[
\|\tilde{\rho}\|_0 \leq \left(\frac{1}{2} \gamma \varepsilon\right)^{1/2}, \quad \|\tilde{\rho} - \rho\|_0 \leq C(M) \varepsilon^{1/2} \theta \ell, \quad \|\tilde{\rho}\|_j \leq C(M) \varepsilon^{1/2} \theta \ell^{1-j},
\]
\[
\|\tilde{G}\|_0 \leq r_0, \quad \|\tilde{G} - G\|_0 \leq C(M) \theta \ell, \quad \|\tilde{G}\|_j \leq C(M) \theta \ell^{1-j}
\]
for $j = 1, 2$. Let $h = \rho^2(\text{Id} + G)$ and $\tilde{h} = \tilde{\rho}^2(\text{Id} + \tilde{G})$. Then
\[
\|\tilde{h} - h\|_0 \leq \|\tilde{\rho}^2 - \rho^2\|_0 + \|\tilde{\rho}^2 \tilde{G} - \rho^2 G\|_0
\]
\[
\leq C(M, \gamma) \varepsilon \theta \ell \leq K_0 C(M, \gamma) \varepsilon K^{-1},
\]
\[
\|\tilde{h} - h\|_1 \leq C(M, \gamma) \varepsilon \theta \leq C(M, \gamma) \varepsilon^{1/2} \delta^{1/2} \theta.
\]
From Lemma 2.2 we obtain, with \( n_\ast = \frac{n(n+1)}{2} \),

\[
\text{Id} + \tilde{G} = \sum_{k=1}^{n_\ast} \tilde{a}_k^2 \nu_k \otimes \nu_k,
\]

where

\[
\|\tilde{a}_k\|_1 \leq C(M)\theta, \quad \|\tilde{a}_k\|_2 \leq C(M)\theta \tilde{\theta}.
\]

Define \( a_k = \tilde{\rho}\tilde{a}_k \), so that

\[
\tilde{h} = \sum_{k=1}^{n_\ast} a_k^2 \nu_k \otimes \nu_k,
\]

and \( a_k \) satisfies

\[
\|a_k\|_1 \leq C(M)\varepsilon^{1/2}\theta, \quad \|a_k\|_2 \leq C(M)\varepsilon^{1/2}\theta \tilde{\theta}.
\]

Furthermore, by taking the trace of (3.35) and using that \( r_0 \leq 1 \) we deduce

\[
\|a_k\|_0 \leq \|\text{tr} \tilde{h}\|_0^{1/2} \leq \sqrt{2n}\tilde{\rho}\|_0 \leq (n\gamma\varepsilon)^{1/2}.
\]

Then, by replacing \( \gamma \) with \( 2n\gamma \) we observe that the conditions of Proposition 3.2 are satisfied with \( N = n_\ast \). Combining the conclusions (3.24)-(3.27) with (3.34) we deduce the desired estimates (3.30)-(3.33).

\[\Box\]

4. **Proof of Theorem 1.2**

In this section, we will prove Theorem 1.2 through Taylor expansion and applying one stage. As is well-known, for any point in \( \Sigma \), there exists a neighbourhood in \( \mathcal{M} \) admitting a geodesic coordinate system \((x_1, \ldots, x_n)\) such that \( \Sigma = \{x_n = 0\} \) and the metric is of the form

\[
g = \sum_{i,j=1}^{n-1} g_{ij} dx^i dx^j + (dx^n)^2.
\]

In the sequel we write \( x = (x', x_n) \), so that in particular the given isometric immersion \( f : \Sigma \to \mathbb{R}^{n+1} \) is locally a function \( f = f(x') \). For simplicity and without loss of generality we may assume that, in these coordinates, the neighbourhood is given by \( \{x : |x'| < 1 \text{ and } |x_n| < 1\} \). The later construction is split into two steps.

**Step 1. Initial short extension.** This step is same as [18]. We make the ansatz

\[
u(x) = f(x') + f_1(x')x_n + f_2(x')x_n^2 + f_3(x')x_n^3
\]

with \( f_m(x') \), \( m = 1, 2, 3 \) to be fixed. An easy calculation gives, for any \( i \leq n-1 \)

\[
\partial_i u(x) = \partial_i f(x') + \partial_i f_1(x')x_n + \partial_i f_2(x')x_n^2 + \partial_i f_3(x')x_n^3,
\]

\[
\partial_n u(x) = f_1(x') + 2f_2(x')x_n + 3f_3(x')x_n^2,
\]
and hence, using that by assumption $f|\Sigma \to \mathbb{R}^{n+1}$ is isometric,
\[
\begin{align*}
\partial_i u \cdot \partial_j u &= g_{ij}(x', 0) + [\partial_i f_1 \cdot \partial_j f + \partial_j f_1 \cdot \partial_i f] x_n \\
&\quad + [\partial_i f_2 \cdot \partial_j f + \partial_j f_2 \cdot \partial_i f + 2\partial_i f_1 \cdot \partial_j f_1] x_n^2 + o(x_n^2) \\
\partial_t u \cdot \partial_n u &= f_1 \cdot \partial_t f + (2f_2 \cdot \partial_t f + f_1 \cdot \partial_t f_1) x_n \\
&\quad + [f_1 \cdot \partial_t f_2 + 2\partial_t f_1 \cdot f_2 + 3f_1 \cdot \partial_t f] x_n^2 + o(x_n^2),
\end{align*}
\]
\[
\partial_n u \cdot \partial_n u = f_1 \cdot f_1 + 4f_1 \cdot f_2 x_n + (6f_1 \cdot f_3 + 4f_2 \cdot f_2) x_n^2 + o(x_n^2).
\]
Comparing the Taylor expansion of $g$
\[
g(x) = \begin{pmatrix} g_{ij}(x', 0) + \partial_n g_{ij}(x', 0) x_n + \frac{1}{2} \partial_n^2 g_{ij}(x', 0) x_n^2 + o(x_n^2) & 0 \\ 0 & 1 \end{pmatrix},
\]
with the matrix $\nabla u^T \nabla u$, our aim is to choose $f_1$, $f_2$ and $f_3$ in such a way as to ensure that
\[
g - \nabla u^T \nabla u = x_n P_1(x') + o(x_n) \text{ for } x_n > 0,
\]
where $P_1$ are uniformly positive definite tensors in $\Sigma$. Thus we set
\[
f_1 = \mu, \quad f_2 = -\mu, \quad f_3 = 0,
\]
where $\mu$ is the vectorfield satisfying $L_0$, so that $\mu = \mu(x')$ satisfies
\[
\mu \cdot \partial_i f(x') = 0, \quad |\mu| = 1, \quad \mu \cdot \partial_n^2 f(x') - L_{ij}(x') > 0, \quad (4.1)
\]
where the last inequality is understood in the sense of positive definite quadratic forms. Since in our geodesic coordinate system $L_{ij}(x') = -\frac{1}{2} \partial_n g_{ij}(x', 0)$, our choice of $f_1, f_2, f_3$ implies
\[
g - \nabla u^T \nabla u = x_n \begin{pmatrix} 2f_1(x') \cdot \partial_n^2 f(x') + \partial_n g_{ij}(x', 0) & 0 \\ 0 & 4 \end{pmatrix} + O(x_n^2) \quad (4.2)
\]
For $d_0 < 1$ sufficiently small we see that $L_0$ is positive definite for all $0 < x_n < d_0$. Therefore
\[
\Omega = \{ x : |x'| < 1 \text{ and } 0 < x_n < d_0 \}
\]
is a one-sided neighbourhood of $\Sigma$ in which $u$ is strictly short and extends $f|\Sigma$.

Step 2. Adapted short extension. We shall utilize one stage of adding primitive metric errors to construct an adapted short immersion. First of all, the immersion $u : \Omega \to \mathbb{R}^{n+1}$ satisfies $u \in C^2(\Omega)$ with
\[
\frac{1}{\gamma} \mathrm{Id} \leq \nabla u^T \nabla u \leq \gamma \mathrm{Id},
\]
\[
\|u\|_{C^2(\Omega)} \leq M
\]
for some $\gamma, M > 1$. Let
\[
\rho^2 = \frac{1}{n} \text{tr}(g - \nabla u^T \nabla u).
\]

From the construction in Step 1, in particular the expression (4.2), we deduce that there exists a constant $C \geq 1$ so that for all $x \in \Omega$

$$\frac{1}{C}x_n^{1/2} \leq \rho(x) \leq Cx_n^{1/2}, \quad |\nabla \rho(x)| \leq Cx_n^{-1/2}, \quad |\nabla^2 \rho(x)| \leq Cx_n^{-3/2}. \quad (4.3)$$

Furthermore, there exists $\tau > 0$ such that

$$g - \nabla u^T \nabla u \geq 2\tau \rho^2 \text{Id}.$$

In particular, using Lemma 1 from [24] (see also Lemma 1.9 in [26]), we obtain the decomposition

$$g - \nabla u^T \nabla u - \tau \rho^2 \text{Id} = \sum_{k=1}^N b_k^2 \rho \chi_k \otimes \chi_k,$$

for some $\rho \in S^{n-1}, b_k \in C^\infty(\Omega)$ and some integer $N$, with estimates of the form

$$\|b_k\|_{C^j(\Omega)} \leq C(M) \quad (4.4)$$

for $j = 0, 1, 2$. By setting $b_k = \tilde{b}_k \rho$ we then deduce

$$g - \nabla u^T \nabla u - \tau \rho^2 \text{Id} = \sum_{k=1}^N \chi_k b_k^2 \omega_k \otimes \omega_k,$$

with estimates, for $j = 0, 1, 2$ and $k = 1, \ldots, N$,

$$|\nabla^j b_k(x)| \leq C(M)x_n^{1/2-j} \quad \text{for } x \in \Omega. \quad (4.5)$$

Next, we define a Whitney-decomposition of the domain $\Omega$ as follows: Set $d_q = 2^{-q}d_0$ for $q = 1, 2, \ldots$ and define

$$\Omega_q = \{x : |x'| < 1 \text{ and } x_n \in (d_{q+1}, d_q-1)\}.$$ 

Moreover, let $\{\chi_q\}_q$ be a partition of unity on $\Omega$ subordinate to the decomposition $\Omega = \bigcup_{q=1}^\infty \Omega_q$, with the following standard properties:

(a) $\text{supp} \chi_q \subset \Omega_q$, in particular $\text{supp} \chi_q \cap \text{supp} \chi_{q+2} = \emptyset$;

(b) $\sum_{q=0}^\infty \chi_q^2 = 1$ in $\Omega$;

(c) For any $q$ and $j = 0, 1, 2$ we have $\|\chi_q\|_{C^j(\Omega_q)} \leq Cd_q^{-j}$. Thus

$$g - \nabla u^T \nabla u - \tau \rho^2 \text{Id} = \sum_{k=1}^N \sum_{q \text{ odd}} (\chi_q b_k)^2 \omega_k \otimes \omega_k + \sum_{k=1}^N \sum_{q \text{ even}} (\chi_q b_k)^2 \omega_k \otimes \omega_k. \quad (4.6)$$

We apply Proposition 3.2 with the first sum on the right hand side of (4.6). From property (c) and (4.5) we deduce

$$\|\chi_q b_k\|_{C^j(\Omega_q)} \leq C(M)d_q^{1/2-j},$$

so that the assumptions (3.21)-(3.23) hold in each $\Omega_q$ with parameters

$$\delta = 1, \quad \varepsilon = d_q, \quad \theta = \tilde{\theta} = d_q^{-1}.$$
Observe that, using property (a), we may “add” each metric term \( \sum_{k=1}^{N} (\chi_q b_k)^2 \varpi_k \otimes \varpi_k \) with \( q \) odd in parallel. Proposition 3.2 leads, for any \( K \geq K_0(M, \gamma) \), to an immersion \( w \in C^2(\Omega, \mathbb{R}^{n+1}) \) such that for all \( q \in \mathbb{N} \)

\[
\|w - u\|_{C^0(\Omega_q)} \leq \frac{M d_q^2}{K}
\]

\[
\|w - u\|_{C^1(\Omega_q)} \leq \frac{M d_q^{1/2}}{K}
\]

\[
\|w\|_{C^2(\Omega_q)} \leq \frac{M d_q^{-1} K^N}{K}.
\]

Moreover

\[
\nabla w^T \nabla w = \nabla u^T \nabla u + \sum_{k=1}^{N} \sum_{q \text{ odd}} (\chi_q b_k)^2 \varpi_k \otimes \varpi_k + \mathcal{E}_{\text{odd}}
\]

with

\[
\|\mathcal{E}_{\text{odd}}\|_{C^0(\Omega_q)} \leq C(M, \gamma) d_q \frac{1}{K}
\]

\[
\|\mathcal{E}_{\text{odd}}\|_{C^1(\Omega_q)} \leq C(M, \gamma) d_q^{-1/2} K^{N-1}.
\]

Then, \( w \) again satisfies the assumptions (3.21)-(3.23) in each \( \Omega_q \) with parameters

\[
\delta = 1, \quad \varepsilon = d_q, \quad \theta = \bar{\theta} = d_q^{-1} K^N.
\]

Therefore, applying Proposition 3.2 once more (with the same \( K \)), with the second term in (4.6) (the even \( q \)'s) leads to an immersion \( v \in C^2(\Omega, \mathbb{R}^{n+1}) \) with

\[
\|v - w\|_{C^0(\Omega_q)} \leq \frac{M d_q^2}{K^{N+1}}
\]

\[
\|v - w\|_{C^1(\Omega_q)} \leq \frac{M d_q}{K^{1/2}}
\]

and

\[
\|v\|_{C^2(\Omega_q)} \leq \frac{M d_q^{-1} K^{2N}}{K}. \tag{4.7}
\]

Moreover

\[
\nabla v^T \nabla v = \nabla u^T \nabla u + \sum_{k=1}^{N} \sum_{q \in \mathbb{N}} (\chi_q b_k)^2 \varpi_k \otimes \varpi_k + \mathcal{E}_{\text{odd}} + \mathcal{E}_{\text{even}} \tag{4.8}
\]

\[
= \nabla u^T \nabla u + \sum_{k=1}^{N} b_k^2 \varpi_k \otimes \varpi_k + \mathcal{E}
\]

with \( \mathcal{E} = \mathcal{E}_{\text{odd}} + \mathcal{E}_{\text{even}} \) and

\[
\|\mathcal{E}_{\text{even}}\|_{C^0(\Omega_q)} \leq C(M, \gamma) d_q \frac{1}{K}
\]

\[
\|\mathcal{E}_{\text{even}}\|_{C^1(\Omega_q)} \leq C(M, \gamma) d_q^{-1/2} K^{2N-1}.
\]
Putting things together we deduce for every \( q \in \mathbb{N} \)
\[
\| v - u \|_{C^0(\Omega_q)} \leq \frac{M d_q^2}{K} \frac{1}{K},
\]
\[
\| v - u \|_{C^1(\Omega_q)} \leq M d_q^{1/2},
\]
\[
\| E \|_{C^0(\Omega_q)} \leq C(M, \gamma) d_q^1 \frac{1}{K},
\]
\[
\| E \|_{C^1(\Omega_q)} \leq C(M, \gamma) d_q^{-1/2} K^{2N-1}.
\]

Now we are in a position to show \( v \) is our desired adapted short immersion. First of all, observe that for \( x \in \Omega_q \) we have \( x_n \sim d_q \sim \rho^2(x) \). Therefore from (4.3) and (4.7), we get (1.3) and (1.2). Besides,
\[
|E(x)| \leq C(M, \gamma) K^{-1} \rho^2(x),
\]
\[
|\nabla E(x)| \leq C(M, \gamma) K^{2N-1} \rho^{-1}(x).
\]

Let
\[
G(x) = - \frac{E(x)}{\tau \rho^2(x)},
\]
so that
\[
g - \nabla v^T \nabla v = \tau \rho^2 \text{Id} - E = \tau \rho^2 (\text{Id} + G),
\]
and, using (4.3)
\[
|G(x)| \leq C(M, \gamma) (\tau K)^{-1}, \quad |\nabla G(x)| \leq C(M, \gamma) \rho^{-3}(x).
\]

In particular, by choosing \( K \) sufficiently large, we can ensure that (5.4) is satisfied. Finally, observe that for any \( \alpha_0 < 1/3 \) and any \( x \in \Omega_q \)
\[
\| v - u \|_{C^{1,\alpha_0}(\Omega_q)} \leq \| v - u \|_{C^1(\Omega_q)}^{1-\alpha_0} \| v - u \|_{C^{2}(\Omega_q)}^{\alpha_0} \leq C(M, \gamma, K) d_q^{(1-3\alpha_0)/2}
\]
is bounded independently of \( q \). Consequently \( v \in C^{1,\alpha_0}(\Omega) \). This concludes the proof.

5. Proof of Theorem 1.1

In this section, we will show that any adapted short immersion can be approximated by isometric immersions with the aid of Corollary 3.1 and some ideas in [8]. The proof is divided into three steps.

**Step 1. Parameter definition.** Since \( v \) is an adapted short immersion, we can write
\[
g - \nabla v^T \nabla v = \rho_0^2 (\text{Id} + G_0).
\]

From the definition of adapted short immersion, one has
\[
\rho_0 \geq 0, \quad |G_0| \leq r_1.
\]

Let
\[
\varepsilon_0 = \max \{ \max_{x \in \Omega} \rho_0^2(x), 1 \}, \quad 0 < a < \frac{1}{2}.
\]
and define two sequences of constants \( \{\varepsilon_q\}, \{\theta_q\} \) as
\[
\varepsilon_q = \varepsilon_0 A^{-2a^q}, \quad \theta_q = A^{(n^*+a)q+3a}
\] (5.3)
with large \( A > 1 \) to be fixed during the proof. For future reference we define the sets
\[
\Omega_j^{(q)} = \left\{ x \in \Omega : \rho_q(x) > \frac{9}{8} \varepsilon_{j+1}^{1/2} \right\}
\]
for any \( j, q = 0, 1, 2, \cdots \). Here \( \rho_q(x), q = 1, 2, \cdots \) will be defined in Step 2. Then it is easy to see that \( \Omega_j^{(q)} \subset \Omega_{j+1}^{(q)} \) and
\[
\bigcup_j \Omega_j^{(q)} = \{ x \in \Omega : \rho_q(x) > 0 \}.
\]
In particular, when \( q = 0 \), \( \Omega_j^{(0)} = \{ x \in \Omega : \rho_0(x) > \frac{9}{8} \varepsilon_j^{1/2} \} \). Using Definition 1.1 it is not difficult to verify that, whenever \( x \in \Omega_j^{(0)} \) for some \( j \geq 0 \), we have
\[
|\nabla^2 v(x)| \leq M \varepsilon_j^{1/2} \theta_j, \quad |\nabla \rho_0(x)| \leq M \varepsilon_j^{1/2} \theta_j, \quad |\nabla G_0(x)| \leq M \theta_j. \quad (5.4)
\]
In fact, if \( \rho_0(x) > \frac{9}{8} \varepsilon_j^{1/2} \) for some \( j \), from (1.2) and using that \( \varepsilon_0 \geq 1 \) and \( n^* \geq 1 \geq 2a \), we obtain
\[
|\nabla^2 v(x)| \leq M \rho_0(x)^{-2} \leq MA^{2aj+2a} = (MA^{-n^*+2aj-a}) \varepsilon_0^{1/2} A^{(n^*+a)j+3a} A^{-aj} \leq M \varepsilon_j^{1/2} \theta_j.
\]
Similarly, from (1.3) we have
\[
|\nabla \rho_0(x)| \leq M \rho_0(x)^{-1} \leq MA^{aj+a} = (MA^{aj-n^*+a}) \varepsilon_0^{1/2} A^{-a(j+1)} A^{(n^*+a)j+3a} \leq M \varepsilon_j^{1/2} \theta_j,
\]
and from (1.4)
\[
|\nabla G_0(x)| \leq M \rho_0(x)^{-2} \leq MA^{3aj+3a} = (MA^{2aj-n^*}) A^{(n^*+a)j+3a} \leq M \theta_j.
\]
Furthermore we may assume without loss of generality that \( M \geq \overline{M} \), where \( \overline{M} \) is the constant in Corollary 3.1.

**Step 2. Inductive construction.** We now use Corollary 3.1 to construct a sequence of smooth adapted short immersions \( \{v_q\} \), and corresponding \( \{\rho_q\}, \{G_q\} \) such that in \( \Omega \)
\[
g - \nabla v_q^T \nabla v_q = \rho_q^2 (\text{Id} + G_q), \quad (5.5)
\]
and the following statements hold:

1) \( q \) For any \( x \in \Omega \),
\[
|G_q(x)| \leq r_2 \quad \text{and} \quad 0 \leq \rho_q(x) \leq 4 \varepsilon_q^{1/2};
\]

2) \( q \) If \( \rho_q(x) \leq 2 \varepsilon_{q+1}^{1/2} \), then \( |G_q(x)| \leq r_1; \)
(3) If $x \in \Omega^{(q)}_j$, for some $j \geq q$, then
\[ |\nabla^2 v_q(x)| \leq M \varepsilon_j^{1/2} \theta_j, \quad |\nabla \rho_q(x)| \leq M \varepsilon_j^{1/2} \theta_j, \quad |\nabla G_q(x)| \leq M \theta_j. \]

In addition, if $x \in \overline{\Omega} \setminus \Omega^{(q)}_q$, $v_q(x) = v_{q-1}(x)$ and for any $x \in \overline{\Omega}$,
\[ \|v_q - v_{q-1}\|_j \leq M \varepsilon_j^{1/2} \theta_j^{-1}, j = 0, 1. \quad (5.6) \]

Set $v_0 = v$. From (5.1), (5.3) and (5.4), we see that conditions $(1)_0 - (3)_0$ are satisfied by $v_0$. Next, suppose $v_q, \rho_q, G_q$ have been defined satisfying relation (5.3) and conditions $(1)_q - (3)_q$ hold. Let $\chi(s) \in C^\infty(0, \infty)$ be a smooth cut-off function satisfying
\[ \chi(s) = \begin{cases} 1 & s \geq 2, \\ 0 & s \leq \frac{7}{4}. \end{cases} \]
and set
\[ \phi_q(x) = \chi\left(\frac{\rho_q(x)}{\varepsilon_{q+1}}\right), \quad \psi_q(x) = \chi\left(\frac{4\rho_q(x)}{3\varepsilon_{q+1}}\right). \]

Observe that $\text{supp } \phi_q \subset \text{supp } \psi_q \subset \Omega^{(q)}_q$, so that, using (3)_q
\[ \|\nabla \phi_q(x)\|_0 \leq CM \theta_q, \quad \|\nabla \psi_q(x)\|_0 \leq CM \theta_j. \quad (5.7) \]
Hereafter $C$ denotes geometric constant. Define
\[ h_q := (g - \nabla v_q^T \nabla v_q - \varepsilon_{q+1} \text{Id}) \phi_q^2 = \tilde{\rho}_q^2 \text{Id} + \tilde{G}_q, \quad (5.8) \]
with
\[ \tilde{\rho}_q = \phi_q \sqrt{\rho_q^2 - \varepsilon_{q+1}}, \quad \tilde{G}_q = \frac{\psi_q \rho_q^2}{\rho_q^2 - \varepsilon_{q+1}} G_q. \]

Observe that the second equality in (5.8) holds because $\psi_q = 1$ on $\text{supp } \phi_q$ and that, furthermore, $\tilde{\rho}_q$ is well-defined because $\rho_q \geq \frac{3}{2} \varepsilon_{q+1}$ on $\text{supp } \phi_q$. Consequently $\tilde{\rho}_q \in C^1(\Omega), \tilde{G}_q \in C^1(\Omega; \mathbb{R}^{n \times n})$. Next we derive estimates on $\tilde{\rho}_q$ and $\tilde{G}_q$.

**Estimates for $\tilde{\rho}_q$:** First of all, $0 \leq \tilde{\rho}_q \leq \rho_q \leq 4 \varepsilon_{q+1}^{1/2}$. Moreover, thanks to the fact that $\rho_q(x) \geq \frac{3}{2} \varepsilon_{q+1}$ on $\text{supp } \phi_q$, and $\frac{3}{2} \varepsilon_{q+1}^{1/2} \leq \rho_q(x) \leq 2 \varepsilon_{q+1}^{1/2}$ on $\text{supp } \nabla \phi_q$, using (1)_q and (3)_q we have
\[ \|\nabla \tilde{\rho}_q\|_0 \leq \|\phi_q \nabla \sqrt{\rho_q^2 - \varepsilon_{q+1}}\|_0 + \|\sqrt{\rho_q^2 - \varepsilon_{q+1}} \nabla \phi_q\|_0 \]
\[ \leq CM \varepsilon_{q+1}^{1/2} \theta_q. \]

**Estimates for $\tilde{G}_q$:** Note that
\[ \tilde{G}_q = \psi_q G_q + \frac{\varepsilon_{q+1} \psi_q}{\rho_q^2 - \varepsilon_{q+1}} G_q. \]
Therefore, using (1) and the fact that $\rho_q(x) \geq \frac{9}{8}\varepsilon_{q+1}^{1/2}$ on $\text{supp} \psi_q \subset \Omega^{(q)}_q$,

$$
\|\tilde{G}_q\|_0 \leq \|G_q\| + \left\| \frac{\varepsilon_{q+1}\psi_q}{\rho^2_q - \varepsilon_{q+1}} G_q \right\|_0 \leq \frac{81}{17}\|G_q\|_0 \leq 5r_2 \leq r_0.
$$

Moreover,

$$
\|\nabla \tilde{G}_q\|_0 \leq \left\| \frac{\psi_q\varepsilon_{q+1}G_q}{(\rho^2_q - \varepsilon_{q+1})^2} \nabla \rho^2_q \right\|_0 + \left\| \frac{\rho^2_q\psi_q}{\rho^2_q - \varepsilon_{q+1}} \nabla G_q \right\|_0 + \left\| \frac{\rho^2_q\nabla \psi_q}{\rho^2_q - \varepsilon_{q+1}} G_q \right\|_0 \\
\leq CM\theta_q.
$$

Combining (3) with the above estimates of $\tilde{\rho}_q, \tilde{G}_q$, we deduce that $\tilde{\rho}_q, \tilde{G}_q, v_q$ satisfy all the assumptions of Corollary 3.1 with constants

$$
\varepsilon = \delta = \varepsilon_q, \quad \theta = \theta_q.
$$

Therefore, applying Corollary 3.1 yields $v_{q+1}$ and $\mathcal{E}$, satisfying the following estimates

$$
\|v_{q+1} - v_q\|_j \leq M\varepsilon_q^{1/2}\theta_q^{-1}A^{j-1}, j = 0, 1; \quad (5.9)
$$
$$
\|v_{q+1}\|_2 \leq M\varepsilon_q^{1/2}A^{n_\ast}; \quad (5.10)
$$
$$
\|\mathcal{E}\|_0 \leq C(M)\varepsilon_qA^{-1}; \quad (5.11)
$$
$$
\|\mathcal{E}\|_1 \leq C(M)\varepsilon_q\theta_qA^{n_\ast-1}; \quad (5.12)
$$

for some constant $C(M)$, along with the identity

$$
\mathcal{E} = g - \nabla v^T_{q+1} \nabla v_{q+1} - [(1 - \phi^2_q)(g - \nabla v^T_q \nabla q) + \phi^2_q\varepsilon_{q+1}\text{Id}].
$$

Set

$$
\rho^2_{q+1} = \rho^2_q(1 - \phi^2_q) + \varepsilon_{q+1}\phi^2_q; \quad G_{q+1} = \frac{\rho^2_qG_q}{\rho^2_{q+1}} (1 - \phi^2_q) + \frac{\mathcal{E}}{\rho^2_{q+1}};
$$

then

$$
g - \nabla v^T_{q+1} \nabla v_{q+1} = \rho^2_{q+1}(\text{Id} + G_{q+1}).
$$

Thus (5.5) holds. We then give two remarks. The first one is that $\mathcal{E}$ and $\phi_q$ share the same support which can be seen from the construction in Corollary 3.1. Thus $G_{q+1}$ is well defined. The other one is that if $x \in \bar{\Omega} \setminus \Omega^{(q)}_q$, $\rho_q \leq \frac{9}{8}\varepsilon_{q+1}^{1/2}$, then $\phi_q = 0$ and

$$
(v_{q+1}, \rho_{q+1}, G_{q+1}) = (v_q, \rho_q, G_q).
$$

Using (5.3) and (5.9), we gain (5.6). It remains to verify $1_{q+1} - 3_{q+1}$.

Verification of $1_{q+1}$: First, on $\text{supp} (1 - \phi^2_q)$, one gets $\rho_q(x) \leq 2\varepsilon_{q+1}^{1/2}$, thus

$$
\rho^2_{q+1} \geq \rho^2_q(1 - \phi^2_q) + \frac{1}{4}\rho^2_q\phi^2_q \geq \frac{1}{4}\rho^2_q.
$$
and $|G_q| \leq r_1$ from (2). So

$$\left| \frac{\rho_{q+1}^2 G_q}{\rho_{q+1}^2} (1 - \phi_q^2) \right| \leq 4r_1. \quad (5.13)$$

Secondly, from the formula of $\rho_{q+1}$ and the fact that $\rho_q(x) \geq \frac{3}{2} \varepsilon_{q+1}$ when $x \in \text{supp} \phi_q$, one has $\rho_{q+1}^2 \geq \varepsilon_{q+1}$ on $\text{supp} \phi_q$. Besides, we gain from (5.11), on $\text{supp} \phi_q$,

$$\|\mathcal{E}\|_0 \leq \omega \varepsilon_{q+1},$$

for some small $\omega$ to be fixed, provided taking $A$ larger. Thus on $\text{supp} \phi_q$, we have

$$\left| \frac{\mathcal{E}}{\rho_{q+1}^2} \right| \leq \omega. \quad (5.14)$$

Finally, from (5.13) and (5.14), we gain

$$|G_{q+1}| \leq 4r_1 + \omega \leq r_2,$$

provided $\omega < r_2 - 4r_1$.

Moreover, we can derive the estimate of $\|\rho_{q+1}\|_0$ as follows.

$$\rho_{q+1}^2 \leq 4\varepsilon_{q+1} (1 - \phi_q^2) + \varepsilon_{q+1} \phi_q^2 \leq 4\varepsilon_{q+1},$$

which leads to $\|\rho_{q+1}\|_0 \leq 4\varepsilon_{q+1}^{1/2}$, where we again use the fact that $\rho_q \leq 2\varepsilon_{q+1}^{1/2}$ when $x \in \text{supp} (1 - \phi_q^2)$.

**Verification of (2)$_{q+1}$**: If $\rho_{q+1}(x) \leq 2\varepsilon_{q+2}^{1/2} < \frac{7}{4}\varepsilon_{q+1}^{1/2}$, so $\phi_q(x) = 0$ at such point, and then $v_{q+1} = v_q$, which helps us get (2)$_{q+1}$ directly from (2).

**Verification of (3)$_{q+1}$**: First of all, observe that $\rho_{q+1}(x) \geq \varepsilon_{q+1} > \frac{a}{8}\varepsilon_{q+2}^{1/2}$ when $x \in \text{supp} \phi_q$, whereas we recall that

$$(v_{q+1}, \rho_{q+1}, G_{q+1}) = (v_q, \rho_q, G_q) \text{ when } x \not\in \text{supp} \phi_q,$$

thus we only need to show the case in which $j = q + 1$ and $x \in \text{supp} \phi_q$.

For $|\nabla^2 v_{q+1}|$, using (5.2), (5.3) and (5.10), one has

$$\|v_{q+1}\|_2 \leq M\varepsilon_{q+1}^{1/2} \theta_{q+1},$$

after taking $A$ larger. For $|\nabla \rho_{q+1}|$, we can calculate

$$\|\nabla \rho_{q+1}\|_0 \leq \|2\rho_q \nabla \rho_q (1 - \phi_q^2)\|_0 + \|2\phi_q (\varepsilon_{q+1} - \rho_q^2) \nabla \phi_q\|_0 \leq CM\varepsilon_{q+1} \theta_{q+1},$$

which leads to

$$|\nabla \rho_{q+1}| \leq \frac{\|\nabla \rho_{q+1}\|_0}{2\rho_{q+1}} \leq CM\varepsilon_{q+1}^{1/2} \theta_{q+1} \leq M\varepsilon_{q+2}^{1/2} \theta_{q+1},$$

where we have used $\rho_{q+1}(x) \geq \varepsilon_{q+1}^{1/2}$ when $x \in \text{supp} \phi_q$, and taken $A$ larger.
For $|\nabla G_{q+1}|$, using (5.2) and (5.12), one is able to obtain
\[
\|\nabla G_{q+1}\|_0 \leq \frac{1}{\min \rho_{q+1}^2} \left( \|G_q(1 - \phi_q) \nabla \rho_q^2\|_0 + \|\rho_q^2(1 - \phi_q) \nabla G_q\|_0 + \|\rho_q^2 G_q \nabla \phi_q^2\|_0 \right) \\
+ \frac{\|\mathcal{E}\|_1}{\min \rho_{q+1}^2} + \frac{\|G_q^2\|_0}{\min \rho_{q+1}^4} (\|\rho_q^2(1 - \phi_q)\|_0 + \|\mathcal{E}\|_0) \\
\leq CM\theta_q + \frac{C(M)\varepsilon_q \theta_q A^{n_*-1}}{\varepsilon_{q+1}} \leq M\theta_{q+1},
\]
where we have utilized the fact that $\rho_{q+1}^2 \geq \varepsilon_{q+1}$ when $E \neq 0$.

**Step 3. Convergence and conclusion.** Finally, we concentrate on the convergence of $\{v_q\}$. From (3) and $v_q = v_{q-1}$ when $x \in \overline{\Omega} \setminus \Omega_q^{(q)}$, we gain
\[
\|v_q - v_{q-1}\|_2 \leq \|v_q\|_2 + \|v_{q-1}\|_2 \leq 2M\varepsilon_q^{1/2}\theta_q.
\]
Interpolation $\|v_q - v_{q-1}\|_1 \leq M\varepsilon_q$ and (5.15) gives
\[
\|v_q - v_{q-1}\|_{1+\alpha} \leq 2M^2\varepsilon_q^{1/2}\theta_q^\alpha \leq K(M, \varepsilon_0) A^{((n_*+\alpha)\alpha-a)q},
\]
to make which convergent when $q$ goes to infinity, we should take
\[
\alpha < \frac{a}{n_* + a} \to \frac{1}{2n_* + 1}, \text{ as } a \to \frac{1}{2}.
\]
Then
\[
\sum_{q>m} \|v_q - v_{q-1}\|_{1+\alpha} \to 0, \text{ as } m \to \infty,
\]
provided taking $\alpha \in [0, \frac{1}{2n_* + 1})$. Obviously such $\alpha \leq \alpha_0$, thus the sequence of adapted short immersions $\{v_q\}$ is convergent in $C^{1,\alpha}(\overline{\Omega})$. Let $\bar{v}$ be the limit, then
\[
\|\bar{v} - v_0\|_0 \leq \sum_{q=1}^{\infty} \|v_q - v_{q-1}\|_0 \leq \sum_{q=1}^{\infty} M\varepsilon_q^{1/2}\theta_q^{-1} \leq 2M\varepsilon_0^{1/2} A^{-n_*-2a}.
\]
Hence, upon taking $A$ larger, one can get
\[
\|\bar{v} - v\|_0 = \|\bar{v} - v_0\|_0 \leq \varepsilon.
\]
Furthermore,
\[
\|g - \nabla \bar{v}^T \nabla \bar{v}\|_0 \leq \lim_{q \to \infty} (1 + r_0) n\rho_q^2 \leq \lim_{q \to \infty} 32n\varepsilon_q = 0,
\]
which means that $\bar{v}$ is isometric in $\overline{\Omega}$. Therefore, we get our desired isometric immersion.

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