On three-dimensional topological field theories constructed from $D^\omega(G')$ for finite groups

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ABSTRACT

We investigate the 3d lattice topological field theories defined by Chung, Fukuma and Shapere. We concentrate on the model defined by taking a deformation $D^\omega(G)$ of the quantum double of a finite commutative group $G$ as the underlying Hopf algebra. It is suggested that Chung-Fukuma-Shapere partition function is related to that of Dijkgraaf-Witten by $Z_{\text{CFS}} = |Z_{\text{DW}}|^2$ when $G = \mathbb{Z}_{2^{N+1}}$. For $G = \mathbb{Z}_{2^N}$, such a relation does not hold.

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1 Introduction

Three-dimensional topological field theories have been attracting interests of mathematicians and physicists. There are several ways of constructing 3d topological field theories \[1-5\]. It is important to study a relation among them in view of the ultimate goal of classifying all topological field theories. Recently Chung, Fukuma and Shapere defined a class of 3d lattice topological field theories \[6\]. They are in one-to-one correspondence with ‘nice’ Hopf algebras satisfying a few conditions. It is interesting that the class of Hopf algebras is not identical with the class of ribbon Hopf algebras, on which Chern-Simons type theories are based \[7\]. So we are interested in establishing a relation between Chern-Simons theory and Chung-Fukuma-Shapere (CFS) theory.

In this note, we would like to investigate relations between CFS theory for finite groups and Chern-Simons theory for finite gauge groups (Dijkgraaf-Witten theory \[2\]). We define CFS theory for finite groups by taking \( D^\omega(G) \), a deformation of the quantum double of a finite commutative group \( G \), as the underlying Hopf algebra. It is worth noting that, in every known construction of 3d topological field theories, models based on finite groups have rich structure and are suitable for explicit calculations \[2,5,8\]. In the present case, the theory becomes completely rigorous and is free from divergences because the Hopf algebra is finite-dimensional.

We find that CFS partition function is related to Dijkgraaf-Witten (DW) partition function by

\[
Z_{CFS}^{D^\omega(G)}(M) = |Z_{DW}^{D^\omega(G,\omega)}(M)|^2
\]

if one of the following conditions is met: (i) \( \omega \equiv 1 \). (ii) \( G = \mathbb{Z}_N \) with odd \( N \) and the 3-manifold \( M \) can be constructed from the 3-sphere, lens spaces, or 2-manifold \( \times S^1 \) via the connected sum. We also find that some CFS theories constructed from \( D^\omega(\mathbb{Z}_{2N}) \) do not fall into the category of DW theories for finite cyclic groups.

2 Chung-Fukuma-Shapere theory

Let us recall the definition of CFS theory. We pick a semi-simple Hopf algebra \( (A; m, u, \Delta, \epsilon, S) \) over \( \mathbb{C} \). Symbols \( m, u, \Delta, \epsilon \) and \( S \) denote multiplication, unit, comultiplication, counit
and antipode, respectively. Using a basis \( \{ \phi_x | x \in X \} \) of \( A \), we write these operations as

\[
m(\phi_x \otimes \phi_y) = C_{xy}^z \phi_z, \tag{2}
\]
\[
u(1) = u^x \phi_x, \tag{3}
\]
\[
\Delta(\phi_x) = \Delta_y^z \phi_y \otimes \phi_z, \tag{4}
\]
\[
\epsilon(\phi_x) = \epsilon_x, \tag{5}
\]
\[
S(\phi_x) = S_y^x \phi_y, \tag{6}
\]

where \( C_{xy}^z, u^x, \epsilon_x, 1 \in \mathbb{C} \). Summation over the repeated indices is assumed hereafter. We define the metric \( g_{xy} \) and the cometric \( h^{xy} \) by

\[
g_{xy} \equiv C_{xv}^u C_{uy}^v, \quad h^{xy} \equiv \Delta_u^v \Delta_v^y. \tag{7}
\]

Since the Hopf algebra \( A \) is semi-simple, the existence of the inverse \( g^{xy} \) of \( g_{xy} \) is guaranteed.

Furthermore we impose two conditions on the Hopf algebra \( A \):

\[
h^{xy} \text{ has the inverse } h_{xy}, \tag{8}
\]
\[
h^{xz} g_{zy} = \Lambda S_x^y \quad (\Lambda \equiv \epsilon^x u_x = |X|), \tag{9}
\]

where \( |X| \) denotes the order of the set \( X \). In virtue of (8), we raise and lower the indices \( x, y, z, \ldots \) by \( g_{xy} \) and \( g^{xy} \) for \( C_{xy}^z \) and \( u^x \), and \( h^{xy} \) and \( h_{xy} \) for \( \Delta_x^{yz} \) and \( \epsilon_x \). Imposing the strong constraint (9) amounts to requiring that applying the direction changing operator (defined below) twice is equivalent to the identity operation.

One can calculate the partition function \( Z_A^{\text{CFS}}(M) \) for a 3-manifold \( M \) following the prescription below.

1. Choose an arbitrary lattice \( L \) of \( M \). A lattice \( L \) means a composition of polygonal faces which are glued together along edges. We assume that every edge in \( L \) is a boundary of at least three polygons. A simplicial decomposition and its dual are the lattices. The number of \( i \)-cells in \( L \) is denoted by \( N_i \).

2. Decompose \( L \) into the set of polygonal faces \( F = \{ f \} \) and that of hinges \( H = \{ h \} \) as depicted in fig. 1. We pick an orientation of each face \( f \) and put arrows on an edge according to it. The edges of an \( n \)-gon \( f \) is numbered \( i = 1, \ldots, n (= n_f) \) in this order.
3. Assign

\[ C_{x_{f[1]}x_{f[2]} \cdots x_{f[n]}} \equiv C_{a_1 x_{f[1]} a_2 x_{f[2]} a_3} \times \cdots \times C_{a_{n-1} x_{f[n-1]} a_n} C_{a_n x_{f[n]}} a_1 \in \mathbb{C} \]  \hspace{1cm} (10)

to each \( n \)-gonal face \( f \). Symbol \( f[i] \) stands for one of the integers \( \{1, \ldots, \sum_{f \in F} n_f\} \) and \( f[i] \neq f'[i'] \) if \( f \neq f' \) or \( i \neq i' \). The index \( x_i \) runs over \( X \). One can imagine the variable \( x_{f[i]} \) lives on the \( i \)-th edge of \( f \).

4. The assignment of an arrow to each edge of faces induces those to \( n \)-hinges \( \{h\} \). The \( n \) arrows on the edges of a hinge \( h \) are not always in the same direction.

If all arrows in the \( n \)-hinge \( h \) are in the same direction, we number the edges \( i = 1, \ldots, n(=n_h) \) in the clockwise order around the arrows. Let the symbol \( h[j] = f[k] \) if and only if the \( j \)-th edge of \( h \) and the \( k \)-th edge of \( f \) is glued. We associate

\[ \Delta_{x_{h[1]}x_{h[2]} \cdots x_{h[n]}} \equiv \Delta_{a_1} x_{h[1]} a_2 \Delta_{a_2} x_{h[2]} a_3 \times \cdots \times \Delta_{a_{n-1}} x_{h[n-1]} a_n \Delta_{a_n} x_{h[n]} a_1 \in \mathbb{C} \]  \hspace{1cm} (11)

to \( h \).

If the directions of arrows in \( h \) are not the same as the rest, we change the direction of the arrows so as to make directions of all the arrows match by multiplying an additional factor (the direction changing operators) \( S_{x,x'} \in \mathbb{C} \) (See fig. 2) for each hinge.

5. We have defined the weight \( C_{x_{f[1]} \cdots x_{f[n]}} \) for each face \( f \) and \( \Delta_{x_{h[1]} \cdots x_{h[j]} \cdots x_{h[n]}} \prod_{j \in R_h} S_{x_{h[j]}, x_{h[j]}}^{x'_{h[j]}} \) for each hinge \( h \). The set \( R_h \) corresponds to the set of all direction changing edges.
Figure 2: direction changing operator

of a hinge \( h \). The partition function is defined by contracting indices:

\[
Z_A^{\text{CFS}}(M) = \mathcal{N} \prod_{f \in F} C_{x_f[1] \cdots x_f[n_f]} \prod_{h \in H} \left[ \Delta_{x_h[1] \cdots x_h[n_h]} \prod_{j \in B_h} S_{x_h[j] x'_h[j]} \right],
\]

(12)

where \( \mathcal{N} \) is a normalization factor.

In ref. [6] it is shown that, with an appropriate normalization, \( Z_A^{\text{CFS}}(M) \) is a topological invariant.

3 Finite-dimensional Hopf Algebras \( A \)

As the authors of ref. [6] pointed out, the partition function (12) can suffer from the divergence in the normalization factor. Therefore the definition (12) sometimes stays at the formal level. On the other hand, if we consider a finite-dimensional Hopf algebra \( A \), the definition is completely rigorous. We will study the case hereafter.

We find that the partition function (12) becomes topological invariant if we choose the correct normalization

\[
\mathcal{N} = \Lambda^{-N_3 - N_1}
\]

(13)

for any finite-dimensional Hopf algebra \( A \).\footnote{A normalization \( \mathcal{N} = \Lambda^{-N_3} \) is employed for \( C[G] \) in ref. [6]. There is, however, no contradiction. We use \( \sum_{x \in X} \) to sum over indices instead of the Haar measure \( \int_G dg \) in ref. [6]. So an additional factor \( \Lambda^{-N_1} \) appears here.}
We claim that the partition function for the sphere depends only on the dimension of the Hopf algebra $A$:

$$Z_A^{\text{CFS}}(S^3) = \Lambda^{-1} = |X|^{-1}. \quad (14)$$

This is shown as follows. Take, for instance, a lattice consisting of three triangular faces and three 3-hinges each of which pastes the three faces ($N_0 = N_1 = N_2 = N_3 = 3$). The partition function is $\Lambda^{-6}C_{rst}C_{uvw}C_{xyz}\Delta^{ruv}_{r}\Delta^{syw}_{s}\Delta^{wz}. \quad (14)$ This expression can be reduced to eq. (14) as a consequence of the axioms of Hopf algebras and eqs. (8) and (9).

4 $A = \mathbb{C}[G]$ for a finite group $G$

Let $G$ be a finite group with the unit element $e$. The group algebra $\mathbb{C}[G] = \bigoplus_{x \in G} \mathbb{C}\phi_x$, where $\phi_x$ is a formal basis, has a natural Hopf algebra structure. This finite-dimensional Hopf algebra satisfies the conditions (8) and (9) and therefore induces a topological field theory [6]. The partition function becomes

$$Z_{\mathbb{C}[G]}(M) = |G|^{-N_0} \prod_{f \in F} |G|^{N_0} \delta_{x_f[1]}x_f[2]\cdots x_f[|f|]e \times \prod_{h \in H} \left[ \delta_{x_f[1]}x_{h[1]}x_{h[2]}\delta_{x_{h[1]}}x_{h[3]}\cdots \delta_{x_{h[1]}}x_{h[|h|]}\cdots \delta_{x_{h[1]}}x_{h[|k|]}\prod_{j \in R_h} \delta_{x_jx_{j'}e} \right]$$

$$= |G|^{-N_0} \sum_{\{g([PQ])\} \mid \{PQ\} \in H} \prod_{f \in F} \delta_{g(\partial f),e}. \quad (15)$$

In the second line, $g([PQ])$ running over the group $G$ is the link variable on the hinge $[PQ]$ between two vertices $P$ and $Q$. Note that $g(\partial f) = \prod_{[PQ] \in f} g([PQ])$ and $g([PQ]) = g([QP])^{-1}$. We have used that $\sum_{i=0}^{3}(-1)^i N_i = 0$ for closed 3-manifolds. The summation in eq. (15) counts the number of flat gauge field configurations (not up to gauge transformation) in the lattice gauge theoretic picture. The factor $|G|^{-N_0+1}$ cancels the gauge volume of the local gauge transformations defined on vertices and we have

$$Z_{\mathbb{C}[G]}^{\text{CFS}}(M) = |G|^{-1} \left| \text{hom}(\pi_1(M), G) \right|. \quad (16)$$

The partition function (16) is sensitive only of the fundamental group of $M$. It is not surprising that eq. (15) catches the information of the fundamental group of $M$. CFS theory depends only on the 2-skeleton $L_2$ of the lattice but $\pi_1(L_2)$ is isomorphic to $\pi_1(M)$. 

5
We note that (16) is not proportional to the number of flat gauge fields up to gauge transformation since we do not introduce the equivalence relation \( \exists g \in G \text{ s.t. } \rho(\cdot) \sim g\rho(\cdot)g^{-1} \) for \( \rho \in \text{hom}(\pi_1(M), G) \).

The partition function (16) is exactly the same form as that of Dijkgraaf-Witten theory for the trivial 3-cocycle \( \omega \equiv 1 \) [2]. A 3-cocycle is a map \( \omega : G \times G \times G \rightarrow U(1) \) which satisfies the condition

\[
\omega(g, x, y)\omega(gx, y, z)^{-1}\omega(g, xy, z)^{-1}\omega(x, y, z) = 1, \\
\omega(e, y, z) = \omega(x, e, z) = \omega(x, y, e) = 1
\]

for any \( g, x, y, z \in G \). \( \omega \equiv 1 \) is a 3-cocycle. In ref. [2], it is argued that the Chern-Simons theories with a finite gauge group (DW theories) are labeled by its 3-cocycles.

The partition function of DW theory is

\[
Z_{DW(G,\omega)} = \sum_{f \in \text{hom}(\pi_1(M), G)} W(f; \omega),
\]

where \( W(f; \omega) \) is the weight satisfying \( W(f; \omega) \equiv 1 \) for \( \omega \equiv 1 \).

5 \quad A = D^\omega(G) \text{ for a finite commutative group } G

In DW theories, the \( \omega \neq 1 \) theories are more interesting than the \( \omega \equiv 1 \) one. The former theories can distinguish distinct 3-manifolds with identical fundamental groups. Therefore it is natural to ask whether one can obtain DW theory with non-trivial \( \omega \) from CFS theory by a choice of Hopf algebra \( A \).

In the following, we investigate the theory defined by \( A = D^\omega(G) \). \( D^\omega(G) \) is a quasi-Hopf algebra [1] introduced in ref. [10]. This choice seems promising since DW theory from a cocycle \( \omega \) is suggested to be equivalent to the Altschuler-Coste theory which uses the regular representation of \( D^\omega(G) \) [5].

Of course, the quasi-Hopf algebra \( D^\omega(G) \) is not always a Hopf algebra. If \( G \) is commutative, however, it can be verified that \( D^\omega(G) \) becomes a Hopf algebra and satisfies the conditions (8) and (9). From now on, we restrict ourselves to the case of commutative group \( G \). Let us recall the definition of \( D^\omega(G) \) for a commutative finite group \( G \). \( D^\omega(G) \) is spanned by the formal basis \( \{ \phi(g, x) | g, x \in G \} \) as a \( C \)-module. Hopf algebra structure

\footnote{The base \( \phi(g, x) \) is usually written as \( \frac{1}{2} \).}
of $D^\omega(G)$ is
\begin{align*}
C_{(g,x)(h,y)}^{(k,z)} &= \delta_{g,h} \delta_{g,k} \delta_{xy,z} \theta_g(x, y), \\
u^{(g,x)} &= \delta_{x,e}, \\
\Delta_{(g,x)}^{(h,y)(k,z)} &= \delta_{x,y} \delta_{x,z} \delta_{g,hk} \theta_x(h, k), \\
\epsilon_{(g,x)} &= \delta_{g,e}, \\
S^{(h,y)}_{(g,x)} &= \delta_{gh,e} \delta_{xy,e} \theta_h(y^{-1}, y)^{-1} \theta_{y^{-1}}(h, h^{-1})^{-1},
\end{align*}
(20)
where $\theta_g(x, y) \equiv \omega(x, y) \omega(x, y, g) \omega(x, y)^{-1}$. It is known that $\theta_g(x, y) = \theta_{g^{-1}}(x, y)$ and that there exists functions $c_g(x)$ on $G$ labeled by $g \in G$ such that
\begin{equation}
\theta_g(x, y) = c_g(x)c_g(y)c_{g(xy)}^{-1}.
\end{equation}
(21)
It follows that
\begin{equation}
\theta_g(x, y) = \theta_{g^{-1}}(x, y), \quad \theta_g(x, x^{-1})\theta_h(x, x^{-1}) = \theta_{gh}(x, x^{-1}).
\end{equation}
(22)
Eqs.(20) imply
\begin{align*}
C_{(g_1,h_1)(g_2,h_2)\cdots(g_k,h_k)} &= |G|^{-1} \delta_{g_1,g_2} \delta_{g_1,g_3} \cdots \delta_{g_1,g_k} \delta_{h_1\cdots h_k,e} \times \\
&\times \prod_{j=1}^{k-1} \theta_{g_j} \left( \prod_{\ell=1}^{j} h_{\ell}, h_{j+1}^{-1} \right) \quad \text{(for } k \geq 3), \\
\Delta^{(h_1,g_1)(h_2,g_2)\cdots(h_k,g_k)} &= C_{(g_1,h_1)(g_2,h_2)\cdots(g_k,h_k)}.
\end{align*}
(23)
(24)
We find that $Z_{D^\omega(G)}^{\text{CFS}} = |Z_{D^\omega(G)}^{\text{DW}}|^2$ for $\omega \equiv 1$ and any commutative finite group $G$. In fact, $D^\omega(G)$ reduces to the quantum double $D(G)$ and we have
\begin{align*}
C_{(g,x)(h,y)}^{(k,z)} &= \tilde{C}_{ghk} \tilde{\Delta}_{xyz}, \\
\Delta^{(g,x)(h,y)}_{(k,z)} &= \tilde{\Delta}_{ghk} \tilde{C}_{xyz}, \\
S^{(h,y)}_{(g,x)} &= \delta_{gh,e} \delta_{xy,e}.
\end{align*}
(25)
The quantities with tilde are those for $A = \mathbb{C}[G]$. Though $D[G]$ is not the tensor product of $\mathbb{C}[G]$'s, the theory decouples into two sectors and we have
\begin{equation}
Z_{D(G)}^{\text{CFS}}(M) = |Z_{\mathbb{C}[G]}^{\text{CFS}}(M)|^2.
\end{equation}
(26)
In view of eq.(25), one can imagine these two sectors live on the lattice and on the dual lattice, respectively.
For $\omega \neq 1$ theories, the decoupling does not occur. However, the partition function $Z_{D^\omega(G)}(\Sigma_g \times S^1)$, where $\Sigma_g$ is the closed surface with genus $g$, is the square of that for $C[G]$:

$$Z_{D^\omega(G)}(\Sigma_g \times S^1) = |G|^{4g}. \quad (27)$$

We can prove this by picking a lattice and using $(18),(21), (22),(25)$, etc. Another result we obtain is that each CFS theory has the factorization property

$$Z_{D^\omega(G)}(M_1)Z_{D^\omega(G)}(M_2) = Z_{D^\omega(G)}(M)Z_{D^\omega(G)}(S^3) \quad (M = M_1 \# M_2). \quad (28)$$

We can prove it by calculating the weight factor coming from neighborhoods of the $S^2$ boundaries.

Let us explain the case of lens spaces $L(p, q)$ ($p \geq 3$) in detail. We take the lattice depicted in fig. 3. We obtain the partition function

![Figure 3: A lattice for $L(p, q)$.

Figure 3: A lattice for $L(p, q)$. $N_0 = 2, N_1 = p + 1, N_2 = 2p, N_3 = p + 1$. All faces are triangular. Edges of two shaded triangles are pair-wisely identified: $BX_iX_{i+1} \sim AX_{i+q}X_{i+q+1}$. As a consequence, $A \sim B, X_i \sim X_j$.

$$Z_{D^\omega(G)}(L(p, q)) = |G|^{-2p-2} \prod_{i=0}^{p-1} C_{a_i,b_i,c_i} C_{x_1,y_1,z_1} \Delta^{x_0a_0}\Delta^{x_0a_0x_1a_1}\Delta^{x_1a_1x_2a_2}\Delta^{x_2a_2x_3a_3}\cdots \Delta^{x_{p-1}a_{p-1}}$$
\[ p \prod_{j=0}^{p-1} \Delta b_j c_{j+1} z_{j+1} + q y_j s^b_j y_j s^y_j \]  \\ &= |G|^{-2} \sum_{g, h \in G} \delta_{p, e} \delta_{q, e} \prod_{i=0}^{p-1} \theta_g (h^i, h) \theta_h (g^i, g). \tag{30} \]

Suffices of \( x, y, z, a, b, c \) are understood by \( \mod p \). This expression turns out to be true also for \( L(0, 1) = S^2 \times S^1, L(1, 1) = S^3, L(2, 1) = \mathbb{RP}^3 \).

Let us specialize to the case \( G = Z_N \). \( Z_N \) has \( N \) different 3-cocycles \( \omega^\ell (\ell = 0, 1, \ldots, N - 1) \). Explicitly,

\[ \omega^\ell_N (x, y, z) = \exp \left( \frac{2\pi \sqrt{-1}}{N^2} z (x + y - x + y) \right) \tag{31} \]

\[ = \begin{cases} \exp \left( \frac{2\pi \sqrt{-1}}{N} \bar{z} \right) & \text{if } \bar{x} + \bar{y} \geq N \\ 1 & \text{otherwise} \end{cases} \tag{32} \]

Here, \( \bar{x} \in \{0, 1, \ldots, N - 1\} \) is the representative of \( x \in Z/NZ \approx Z_N \): \( \bar{x} \equiv x \mod N \). In this case, the partition function \( (30) \) is always a positive integer independent of \( q \):

\[ \frac{Z_{D^{\omega}}^{\text{CFS}}(Z_N)(L(p, q))}{Z_{D^{\omega}}^{\text{CFS}}(Z_N)(S^3)} = \sum_{r, s=0}^{m-1} \exp \left( \frac{2p\ell rs}{m^2} \cdot 2\pi \sqrt{-1} \right) \tag{33} \]

\[ = m \times \left| \{ s \in \{0, \ldots, m - 1\} | 2p\ell s \equiv 0 \mod m^2 \} \right|, \tag{34} \]

where \( m = (N, p) \) is the greatest common divisor of \( N \) and \( p \). We have used \( \overline{pz} = 0 \iff \exists r \in \{0, 1, \ldots, m - 1\} \) s.t. \( \overline{z} = r N/m \).

### 6 Discussions

DW theory has been studied elaborately \cite{2, 12, 14}. On manifolds \( \Sigma_g \times S^1 \) and \( S^3 \), the DW partition function for a commutative group \( G \) takes the form

\[ Z_{(G, \omega)}^{\text{DW}}(S^3) = |G|^{-2}, \quad Z_{(G, \omega)}^{\text{DW}}(\Sigma_g \times S^1) = |G|^{2g}. \tag{35} \]

For \( G = Z_N \), the partition function on a lens space \( L(p, q) \) has an expression

\[ \frac{Z_{(Z_N, \omega)}^{\text{DW}}(L(p, q))}{Z_{(Z_N, \omega)}^{\text{DW}}(S^3)} = \sum_{r=0}^{m-1} \exp \left( \frac{pnr^2}{m^2} \cdot 2\pi \sqrt{-1} \right), \tag{36} \]

where \( m = (N, p) \) and \( n \in \{0, \ldots, p - 1\}, nq \equiv 1 \mod p \).
We find that for every $\ell$ and odd $N$,

$$Z_{D^\omega}(Z_N)(M) = |Z_{DW}(Z_N)^\omega(M)|^2$$

(37)

holds for $M = S^3$, $\Sigma_g \times S^1$ and $L(p, q)$. The relation (37) is preserved under the connected sum provided that the both theories have factorization property. Therefore the relation (37) holds for a wide variety of manifolds and it is suggested that $Z_{D^\omega}(Z_N)$ and $\left|Z_{DW}(Z_N)^\omega\right|^2$ for odd $N$ are equivalent as topological invariants.

Eq.(37) can be shown as follows. The proof for $M = S^3$, $\Sigma_g \times S^1$ and $L(p, q)$ is obvious. For lens spaces, we have

$$|Z_{DW}(Z_N, \omega\ell)(L(p,q))|^2 = N^{-2} \sum_{r, s = 0}^{m-1} \exp\left(\frac{p\ell n(r^2 - s^2)}{m^2} \cdot 2\pi \sqrt{-1}\right)$$

(38)

$$= N^{-2} \sum_{0 \leq \alpha \leq 2(m-1),\ -(m-1) \leq \beta \leq (m-1),\ \alpha, \beta \in Z, \alpha \equiv \beta \mod 2} \exp\left(\frac{p\ell n\alpha\beta}{m^2} \cdot 2\pi \sqrt{-1}\right)$$

(39)

$$= N^{-2} \sum_{\alpha, \beta = 0}^{m-1} \exp\left(\frac{p\ell n\alpha\beta}{m^2} \cdot 2\pi \sqrt{-1}\right).$$

(40)

In the second line, we have set $\alpha = r + s$, $\beta = r - s$. In the third line, we have used the fact that the summand is invariant under the shift of $\alpha$ or $\beta$ by $m$. The last expression agrees with (33) since $\{n\alpha \mod m|0 \leq \alpha \leq m-1\} = \{2\alpha \mod m|0 \leq \alpha \leq m-1\}$ due to the equality $(n, m) = (2, m) = 1$.

In contrast with the case of odd integer $N$, eq.(37) is not always true for even $N$. To see this, we consider the theory for which $N$ and $\ell$ are both odd. Then it can be verified that $Z_{DW}^{\omega\ell}(Z_N)(L(2,1)) = 0$ from eq.(36). On the other hand, $Z_{D^\omega}(Z_N)(L(p,q))$ is always a positive integer. Therefore $Z_{CFS}$ and $\left|Z_{DW}\right|^2$ cannot be equivalent as topological invariants.

We sometimes used the term ‘topological field theory’ for CFS theory above. It meant, in fact, no more than a set of weights giving rise to a topological invariant for closed manifolds. It is not known whether each CFS theory has an underlying functor in Atiyah’s axioms [13]. We expect that CFS theory for a finite-dimensional Hopf algebra $A$ with the normalization (13) has the underlying functor. This is true for $A = C[G]$ because it is equivalent to DW theory for $\omega \equiv 1$ and each DW theory has the functor [2]. If eq.(37) is valid for arbitrary manifolds, the theory for $A = D^\omega(Z_{2N+1})$ has an underlying functor,
too. It is equivalent to DW theory for $G = \mathbb{Z}_N \times \mathbb{Z}_N$, $\omega((g_1, g_2), (h_1, h_2), (k_1, k_2)) = \omega^\ell(g_1, h_1, k_1) \times \omega^{N-\ell}(g_2, h_2, k_2)$, since

$$|Z_{(\mathbb{Z}_N; \omega^\ell)}^{\text{DW}}(M)|^2 = Z_{(\mathbb{Z}_N; \omega^\ell)}^{\text{DW}}(M) \times Z_{(\mathbb{Z}_N; \omega^{N-\ell})}^{\text{DW}}(M) = Z_{(\mathbb{Z}_N \times \mathbb{Z}_N; \omega^\ell \times \omega^{N-\ell})}^{\text{DW}}(M). \quad (41)$$

For other theories with $A = D^\omega(G)$, it is also likely that the functors exist. Recall that the value of the partition function $Z(\Sigma_g \times S^1)$ is equal to the dimension of the Hilbert space $\mathcal{H}_{\Sigma_g}$ associated with $\Sigma_g$ in any topological field theories. In CFS theory, the partition function $Z_{(\mathbb{Z}_N)}^{\text{CFS}}(\Sigma_g \times S^1)$ is a positive integer as should be if the functor exists. The existence of the functor explains the factorization property (28) since $\dim \mathcal{H}_{S^2} = Z_{(\mathbb{Z}_N \times \mathbb{Z}_N; \omega^\ell \times \omega^{N-\ell})}^{\text{CFS}}(M) = 1$.

If CFS theory for $A = D^\omega(G)$ has an underlying functor, it is ‘irreducible’ and cannot be a direct sum of other topological field theories since $\dim \mathcal{H}_{S^2} = 1$. We note that, in the case, some CFS theories for $D^\omega(\mathbb{Z}_N)$ cannot be a tensor product of a number of DW theories for finite cyclic groups. Let us assume that CFS theory for $A = D^\omega(\mathbb{Z}_4)$ is a tensor product of DW theories. Due to eqs. (27) and (35), it has to be a product of $\mathbb{Z}_4$ or $\mathbb{Z}_2$ DW theories. But it can be checked that no combinations of these theories reproduces $Z_{D^\omega(\mathbb{Z}_4)}^{\text{CFS}}(L(4, 1))$.

In DW theory, some pairs of homotopy inequivalent lens spaces with an identical fundamental group can be distinguished (e.g. $L(5, 1)$ and $L(5, 2)$). The reversal of the orientation of a manifold amounts to complex conjugation of the partition function: $Z_{D^\omega(G)}^{\text{DW}}(M^*) = Z_{D^\omega(G)}^{\text{DW}}(M)^*$ (e.g. $L(3, 1) = L(3, 2)^*$). Because of the absolute square in (37), $Z_{D^\omega(\mathbb{Z}_{2N+1})}^{\text{CFS}}$ is insensitive of these. It is desirable to have a Hopf algebra $A$ for which $Z_{D^\omega(\mathbb{Z}_{N+1})}^{\text{CFS}} = Z_{D^\omega(\mathbb{Z}_N; \omega)}^{\text{DW}}$. However, we suppose $Z_{A}^{\text{CFS}}$ for any Hopf algebra $A$ is insensitive of the orientation because CFS construction does not refer to the orientation of manifolds.

So far we cannot relate DW theory for non-commutative finite groups to CFS theory. We comment that, for a non-commutative finite group $G$, $D(G)$ is a Hopf algebra which induces a topological field theory via CFS construction. It will be interesting to investigate such models. The decoupling (23) does not occur in the models in general. However, there seems to be no room for twisting since $D^\omega(G)$ with $\omega \neq 1$ is not always a Hopf algebra.

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