Hamiltonian Superfield Formalism
with $N$ Supercharges

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Abstract

An action principle that applies uniformly to any number $N$ of supercharges is proposed. We perform the reduction to the $N = 0$ partition function by integrating out superpartner fields. As a new feature for theories of extended supersymmetry, the canonical Pfaffian measure factor is a result of a Gaussian integration over a superpartner. This is mediated through an explicit choice of direction $n^a$ in the $\theta$-space, which the physical sector does not depend on. Also, we re-interpret the metric $g^{ab}$ in the Susy algebra $[D^a, D^b] \sim g^{ab} \partial_\theta$ as a symplectic structure on the fermionic $\theta$-space. This leads to a superfield formulation with a general covariant $\theta$-space sector.

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1 Introduction: Review of $N = 1$

A few years ago, we developed a $N = 1$ superfield formulation of Hamiltonian field theories [1, 2], where all the fields $z_0^A$ are replaced by superfields

$$ z^A = z_0^A + \theta z_1^A \quad (1.1) $$

The basic idea is that $\theta$-translations should encode the BRST symmetry, so that the superpartners

$$ z_1^A = \{\Omega(z_0), z_0^A\}_{PB} \quad (1.2) $$

are the corresponding BRST-transformed fields. Furthermore, the supersymmetry should be implemented as a square root of time translations

$$ D^2 = \frac{\partial}{\partial t} \quad (1.3) $$

As the superderivative is a combination of $\theta$ and $t$ derivatives,

$$ D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial t} \quad (1.4) $$

one should introduce a matching supercharge, which is a combination of the BRST charge and the Hamiltonian

$$ Q = \Omega - \theta H \quad (1.5) $$

It is natural to postulate the following “superequations” of motion

$$ DF = \{Q, F\}_{PB} + D_{\text{expl}}F \quad (1.6) $$

for any quantity $F = F(z(t, \theta); t, \theta)$, where “expl” denotes explicit differentiation. Applying the equations of motion twice on a field $z^A$, we get the equations for time evolution

$$ \dot{z}^A = -\{H, z^A\}_{PB} \quad (1.7) $$

if we let the Hamiltonian be

$$ -H = \frac{1}{2} \{Q, Q\}_{PB} + D_{\text{expl}}Q \quad (1.8) $$

Note that $H$ is expressed entirely in terms of the supercharge $Q$. Similarly, eq. (1.5) can be thought of as a definition of $\Omega$ in terms of $Q$. At this point, the definition (1.8) appears to be a very restrictive choice of Hamiltonian, but this is not so. If we for instance consider $\Omega$ to be nilpotent and without explicit $\theta$-dependence (as would normally be the case), then any BRST-invariant Hamiltonian can be written in the form (1.8)! For further details, including the necessary integrability conditions, we refer to the Ref. [1, 2]. Lagrangian and related superfield formulations have been discussed in Ref. [1, 3] and Ref. [4], respectively.

*For comparison, the signs of $Q$ and $\Omega$ are the opposite of the conventions used in Ref. [1, 2].
1.1 Path Integral

The $N = 1$ operator and path integral formalism were worked out in Ref. [1, 2]. Here we review the $N = 1$ path integral construction, as it serves as an important prototype for further developments. Assume in the following that the supercharge $Q$ has no explicit time dependence. The action reads

$$S = \int dt \, d\theta \left[ z^A \bar{\omega}_{AB}(z) \, Dz^B(-1)^{\epsilon_B} + Q(z) \right]$$

(1.9)

$$= \int dt \left[ z_0^A \bar{\omega}_{AB}(z_0) \, z_0^B \frac{1}{2} \bar{z}_1^A \omega_{AB}(z_0) \bar{z}_1^B(-1)^{\epsilon_B} + (z_1^A \partial_A + \partial_{\theta}^{\text{expl}})Q(z_0) \right],$$

(1.10)

where we in the last expression have written the action out in components. (For details concerning the symplectic 2-tensor $\omega_{AB}$, see Section 3.1 and eq. (3.51).) Note that the superpartners $z_1^A$ only appear up to the Gaussian order in the action (1.10). Hence it is possible to integrate them out of the path integral. This reduces in a very direct way the $N = 1$ path integral to the original $N = 0$ partition function:

$$Z = \int [dz_0][dz_1] \exp \left[ \frac{i}{\hbar} S \right] = \int [dz_0] \, \text{Pf}(\omega_{AB}(z_0)) \, \exp \left[ \frac{i}{\hbar} \int dt \left( z_0^A \bar{\omega}_{AB}(z_0) \, z_0^B - H(z_0) \right) \right].$$

(1.11)

Remarkably the Gaussian integration has just the right impact on both the classical and the quantum part of the partition function:

1. Completing the square generates a shift-term in the action, which restores the Hamiltonian to the form (1.8).

2. The Gaussian integration produces the canonical Pfaffian measure factor of the the original $N = 0$ formalism. The factor is needed (at the naive and purely formal level) to maintain a reparametrization invariant path integral.

The simplicity of the above reduction (1.11) from $N = 1$ to $N = 0$ suggests that it should serve as a cornerstone for further theoretical developments of the superfield formalism.

1.2 The Plan of the Paper

The purpose of the paper is twofold:

1. We would like to generalize to higher supersymmetries [5], first and foremost to the $N = 2$ case (see Section 2 below). Such a situation arises in theories with both a BRST and an anti-BRST symmetry. The Susy algebra and the equations of motion are easy to generalize [6]. The main hurdle is the formulation of an action principle that gives rise to a correct path integral. As a minimum requirement, any proposals for extended supersymmetry should reduce to the original $N = 0$ partition function via the above $N = 1$ path integral (1.11). Previous works [6, 7] do not meet this test. For instance, the path integral of Gozzi et al. [7] is confined to the classical trajectories. In their approach, the rigid $N = 2$ geometry has completely ironed out quantum fluctuations. The proposal of Ref. [6] has correct quantum behavior in the original $z_0$-sector, but their superpartner $z_1$ is constrained, in contrast to the superpartner $z_1$ in the $N = 1$ action (1.10). Furthermore, the quantum measure factor of Ref. [6] is generated with the help of a vielbein $h_A^B$. We shall here give a new proposal that remedies this with the caveat, that we have explicitly selected a $\theta$-direction $n^a$. However, the path integral does not depend on $n^a$.

2. For $N > 2$ an additional challenge arises, as flat coordinate systems for the $\theta$-space are no longer protected (by the requirement of no external fermionic constants). Hence, it is of interest to develop a general covariant theory for the $\theta$-space (see Section 3).
2 \( N = 2 \) Revisited

The \( N = 2 \) case is physically motivated by Hamiltonian \( Sp(2) \)-symmetric theories [8]. Such theories are endowed with a BRST and an anti-BRST charge \( \Omega^a \), \( a = 1, 2 \), of ghost number ±1, respectively. The theories are invariant under rotations \( \Omega^a = \Lambda^a_b \Omega^b \) with \( 2 \times 2 \) matrices \( \Lambda^a_b \in Sp(2) \cong SL(2,\mathbb{R}) \). The idea is now to implement geometrically the two fermionic BRST/anti-BRST symmetries by introducing two fermionic parameters \( \theta_a \), \( a = 1, 2 \). The \( N = 2 \) superfield

\[
z^A = z^A_0 + \theta_a z^{aA} + \theta^2 z^A_3 ,
\]

\[
\theta^a := \frac{1}{2} \epsilon^a{}_{b} \theta_a \theta_b = \theta_1 \theta_2 ,
\]

has four component fields, \( z^A_0 \), \( z^A_1 \), \( z^A_2 \) and \( z^A_3 \). Of these four fields, the three superpartners are BRST/anti-BRST transformed fields,

\[
z^{aA} = \{ \Omega^a(z_0), z^A_0 \}_{PB} , \quad a = 1, 2 ,
\]

\[
z^A_3 = \frac{1}{2} \epsilon_{ab} \left( \{ \Omega^a(z_0), \{ \Omega^b(z_0), z^A_0 \}_{PB} \}_{PB} + \{ \partial^a \Omega^b(z_0), z^A_0 \}_{PB} \right) .
\]

The three “BRST/anti-BRST transformation laws” are implemented via a suitable choice of equations of motion (see Section 2.6).

2.1 The Metric \( g^{ab} \)

We can now move around along two linearly independent directions in the fermionic \( \theta \)-plane. Therefore, we need to introduce two superderivatives \( D^a \), \( a = 1, 2 \), such that the Susy algebra inevitably acquires a symmetric \( 2 \times 2 \) metric\(^1\) \( g^{ab} \):

\[
[D^a, D^b] \equiv D^{(a} D^{b)} = g^{ab} \frac{\partial}{\partial t} \quad , \quad g^{ab} = g^{ba} .
\]

The appearance of a symmetric metric \( g^{ab} \) is perhaps the single most important new feature for the \( N = 2 \) case, so let us investigate it in further detail. It has effectively two degrees of freedom, as an overall normalization just rescales the time variable \( t \). It transforms as a tensor \( g^{ad} = \partial^a \theta'_b \ g^{bc} \partial^d \theta'_c \) under reparametrization of the fermionic coordinates \( \theta_a \rightarrow \theta'_a \). As we ignore a less attractive possibility\(^2\) of introducing fermionic constants, the most general reparametrization is of the form \( \theta'_a = \theta_a \Lambda^a_b \), where \( \Lambda^a_b \in GL(2,\mathbb{R}) \) is a constant bosonic matrix with \( \det(\Lambda^a_b) \neq 0 \), i.e. the geometry is completely rigid. Arguments along similar lines combined with the fact that \( g^{ab} \) should be cohomologically closed (see Section 3.2), show that the metric \( g^{ab} \) does not depend on \( \theta_a \).

The origin of the metric tensor \( g^{ab} \) can be related to a real \( 2 \times 2 \) matrix \( G^a_b \) for the ghost number operator \( G \):

\[
\{ G^a , \Omega^b \}_{PB} = G^a_b \ \Omega^b .
\]

This matrix \( G^a_b \) also has two degrees of freedom, because its determinant and trace are fixed from the onset:

\[
\det(G^a_b) = \det(\sigma_3) = -1 , \quad \text{tr}(G^a_b) = \text{tr}(\sigma_3) = 0 ,
\]

where \( \sigma_3 \) is the 3rd Pauli matrix. One may identify

\[
\sqrt{-g} g^{ac} \equiv G^a_b \epsilon^{bc} , \quad g := \det(g_{ab}) < 0 ,
\]

\(^1\)For comparison, the reference Ref. [6] has a metric \( g^{ab} = \frac{1}{2} g^{ab} \).

\(^2\)This possibility is covered in complete generality in Section 3. However, there is no physical motivation to introduce external fermionic parameters in a theory.
where there is included a determinant factor $\sqrt{-g}$ on the lhs. to absorb an inessential overall normalization. The symmetry of $g^{ab}$ is a consequence of $G^{a}_{b}$ being traceless. As the metric $g^{ab}$ should be real, it acquires an indefinite (1,1) signature (see Section 3.2). An indefinite metric does not pose a fundamental challenge to our construction, but it is however a technical nuisance, and for simplicity, we assume from now on that the metric $g^{ab}$ is positive definite.

2.2 Equations of Motion

In general, we use the metric $g^{ab}$ to raise and lower $Sp(2)$ indices. For instance, $\theta^a := g^{ab}\theta_b$. As the superderivatives are realized as

$$D^a = \frac{\partial}{\partial \theta^a} + \frac{1}{2} \theta^a \frac{\partial}{\partial t},$$

one should introduce matching supercharges

$$Q^a = \Omega^a - \frac{1}{2} \theta^a H,$$

and impose the following equations of motion

$$D^a F = \{Q^a, F\}_{PB} + D^a_{\text{expl}} F.$$

Applying the equations of motion twice on a field $z^A$, and then symmetrizing, we get the equations for time evolution

$$\dot{z}^A = -\{H, z^A\}_{PB},$$

if we let the Hamiltonian be

$$-H = \frac{1}{2} g^{ab} \left(\{Q^a, Q^b\}_{PB} + D^a_{\text{expl}} Q^b \right).$$

2.3 Two Sectors: Tilde and Check

It turns out that the equations of motion eq. (2.10) in their present formulation are not directly applicable for action and path integral building. Instead, one can give an equivalent, more tractable formulation [6]. First, introduce a Hodge-dual

$$\tilde{\theta}^a := \epsilon^{ab} \sqrt{g} \theta_b$$

(2.13)

to $\theta_a$. (See the Appendices for further details.) Now build two bosonic supercharges

$$\tilde{Q} := \tilde{\theta}_a Q^a = \tilde{\theta}_a \Omega^a + \delta^2(\theta) H$$

(2.14)

$$\tilde{Q} := \theta_a Q^a = \theta_a \Omega^a,$$

(2.15)

and two bosonic superderivatives

$$\tilde{D} := \tilde{\theta}_a D^a = \tilde{\theta}_a \partial^a - \delta^2(\theta) \partial_t$$

(2.16)

$$\tilde{D} := \theta_a D^a = \theta_a \partial^a.$$

(2.17)

\textit{\textsuperscript{5}}The supercharges $Q^a$ could in principle have both explicit $t$ and explicit $\theta$ dependence. However, whenever we work in a path integral formalism, we assume that the $Q^a$ contain no explicit time dependence.

\textit{\textsuperscript{6}}$\tilde{D}$ and $\tilde{Q}$ were denoted $D$ and $Q$, respectively, in Ref. [6].
One may show (see Section 3.8-3.10) that the equations of motion eq. (2.10) are equivalent to the following set of equations:

\[
\begin{align*}
\tilde{D} F &= \{\tilde{Q}, F\}_{PB} + \tilde{D}_{\text{expl}} F \\
\hat{D} F &= \{\hat{Q}, F\}_{PB} + \hat{D}_{\text{expl}} F,
\end{align*}
\]

(2.18) (2.19)

provided the pertinent integrability conditions are satisfied.

### 2.4 New Action

Our new action proposal consists of three parts:

\[
S = \tilde{S}[z] + \hat{S}[z, w] + S_n[w, \pi],
\]

(2.20)

\[
\tilde{S}[z] = -\int dt\sqrt{g} \, d^2\theta \left[ z^A \tilde{\omega}_{AB}(z) \tilde{D} z^B + \tilde{Q}(z) \right],
\]

(2.21)

\[
\hat{S}[z, w] = \int dt\sqrt{g} \, d^2\theta \, w^A \left[ \omega_{AB}(z) \tilde{D} z^B + \partial_A \hat{Q}(z) \right],
\]

(2.22)

\[
S_n[w, \pi] = \int dt\sqrt{g} \, d^2\theta \, w^A(\theta) \pi_A(n^a\theta_a).
\]

(2.23)

Here \(z^A\) and \(w^A\) are \(N=2\) superfields, while \(\pi_A\) is a \(N=1\) auxiliary superfield. This yields \(2^2+2^2+2^1 = 10\) components for each \(A = 1, \ldots, 2M\). All three superfields carry the same Grassmann-parity \(\epsilon_A\).

Alternatively, the superfield \(w^A\) may be viewed as a "collective field" or "shift field",

\[
\hat{S}[z, w] = \int dt\sqrt{g} \, d^2\theta \left[ w^A \omega_{AB}(z) \tilde{D} z^B + \hat{Q}(z+w) - \hat{Q}(z) + O(w^2) \right].
\]

(2.24)

The main new ingredient is provided by a gauge-fixing real unit-vector \(n^a\), which satisfies

\[
n^a g_{ab} n^b = 1.
\]

(2.25)

It represents an explicit choice of direction \(\theta_{\|} := n^a\theta_a\) in the 2-dimensional \(\theta\)-plane.

We claim that the variation of the action yields the equations of motion

\[
\begin{align*}
\tilde{D} z^A &= \{\tilde{Q}, z^A\}_{PB}, \\
\hat{D} z^A &= \{\hat{Q}, z^A\}_{PB}, \\
w^A &= 0, \\
\pi_A &= 0.
\end{align*}
\]

(2.26) (2.27) (2.28) (2.29)

Parts of this are proved below. It follows immediately that the variation of the tilde part \(\tilde{S}\) wrt. the superfield \(z^A\) produces the tilde equation of motion (2.26), but there could potentially be "reaction force" contributions from the second sector \(\hat{S}\). This is prohibited, as we will see, because the reaction terms vanishes, mediated by a marvelous compatibility between the two sectors. In particular, we will see that the full \(w^A\) superfield are annihilated on-shell: \(w^A \cong 0\). Similarly, variation of the second part of the action \(\hat{S}\) wrt. the superfield \(w^A\) produces the second equation (2.27). Again possible interfering contributions from the third sector \(S_n\) are avoided, although the actual details in this case are less important, as we are mostly interested in the \(z\)-sector.
The action reads in components

\[ \hat{S}[z] = \int dt \left[ z_0^A \omega_{AB}(z_0) \dot{z}_0^B + \frac{1}{2} z^aA g_{ab} \omega_{AB}(z_0) z^{bB} (-1)^{\epsilon_B} 
+ g_{ab} (\epsilon^aA \partial_A + \partial^a_{\text{expl}}) Q^b(z_0) \right], \]

\[ \hat{S}[z, w] = \int dt \sqrt{g} \left[ \epsilon_{ab} w^{aA}[\omega_{AB}(z_0) z^{bB} (-1)^{\epsilon_B} + \partial_A Q^b(z_0)] 
+ 2w_0^A \omega_{AB}(z_0) z_0^B - (-1)^{\epsilon_a} \epsilon_{ab} z^aA z^{bB} \partial_B \omega_{AC}(z_0) w_0^C 
+ \epsilon_{ab} (\partial_A Q^b(z_0)) \partial_B w_0^B \right], \]

\[ S_n[w, \pi] = \int dt \left[ \sqrt{g} w_0^A \pi_A^0 - \pi_A^1 \tilde{n}_a w^{aA} \right], \]

where we have introduced an orthogonal unit-vector

\[ \tilde{n}_a := \sqrt{g} \epsilon_{ab} n^b. \]  

Together the pair \( n^a \) and \( \tilde{n}^a \) form an orthonormal basis satisfying a completeness relation

\[ n^a n_b + \tilde{n}^a \tilde{n}_b = \delta^0_b. \]

Any quantity \( F^a = \theta^a, Q^a, z^aA, w^{aA}, \ldots \), carrying \( Sp(2) \) indices has projections

\[ F^\parallel = n_a F^a, \quad F^\perp = \tilde{n}_a F^a, \]

onto \( n^a \) and \( \tilde{n}^a \), respectively. They decompose as

\[ F^a = n^a F^\parallel + \tilde{n}^a F^\perp. \]

Alternatively, one may think of the above decomposition as a change of coordinates \( \theta_a \rightarrow \theta'_b = \theta_a \Lambda^a_b \), where

\[ \theta'_1 \equiv \theta^\parallel, \quad \theta'_2 \equiv \theta^\perp, \quad \Lambda^a_b = n^a \delta^1_b + \tilde{n}^a \delta^2_b. \]

The unit-vector \( \tilde{n} \) becomes parallel to the new 1'-axis, and the metric becomes diagonal in the new coordinates:

\[ n'_b = n_a \partial^a \theta'_b = \delta^1_b, \quad g'_{ad} = \partial^b \theta'_a g_{bc} \partial^c \theta'_d = \delta_{ad}. \]

If we expand the kinetic part of the action \( S \) to the quadratic order, the 10 component fields pair off, as indicated in the following the diagram

\[ z_0 \quad z_\parallel \quad z_\perp \quad z_3 
\uparrow \quad \uparrow \quad \uparrow 
\downarrow \quad \downarrow \quad \downarrow 
w_3 \quad w_\parallel \quad w_\perp \quad w_0 
\uparrow \quad \uparrow \quad \uparrow 
\pi_0 \quad \pi_\parallel \quad \pi_\perp \]

In other words, the arrows indicate non-zero, off-diagonal entries of the kinetic action Hessian.

### 2.5 Integration over \( \pi^A \)

It is clear from the third part of the action (2.32) (and in accordance with the above diagram), that the integration over the \( N = 1 \) superfield \( \pi_A(\theta) = \pi_A^0 + \theta^\parallel \pi_A^1 \) annihilates two components \( w_3^A \cong 0 \) and \( w_\perp^A \cong 0 \). Hence the \( N = 2 \) superfield

\[ w^A \cong w^A_0 + \theta^\parallel w^A_\parallel \]

(2.40)
reduces to a $N=1$ superfield. The integrations over the superpartners $\pi_A^0$ and $\pi_A^1$ produce no contribution to the measure. To recapitulate, the field content is now a $N=2$ superfield $z^A$ and a $N=1$ auxiliary superfield $w_0^A + \theta^A = w^A$, yielding $2^2 + 2^1 = 6$ components for each $A = 1, \ldots, 2M$. The partially reduced action reads

$$S \cong \tilde{S}[z] + \bar{S}[z, w_0, w^A], \quad \text{(2.41)}$$

$$\tilde{S}[z] = \int dt \left[ \frac{1}{2} \dot{z}_0^A \omega_{AB}(z_0) z_0^B \right] + \frac{1}{2} \dot{z}_0^A \omega_{AB}(z_0) z_0^B (-1)^{c_B} + \frac{1}{2} \dot{z}_0^A \omega_{AB}(z_0) z_0^B (-1)^{c_B} \right] \right), \quad \text{(2.42)}$$

$$\bar{S}[z, w_0, w^A] = - \int dt \left[ w^A \omega_{AB}(z_0) z_0^B (-1)^{c_B} + \partial_A Q^B(z_0) \right] + 2 \sqrt{\pi} w_0^A \omega_{AB}(z_0) \dot{z}_0^B + (-1)^{c_A}(z_0^A z_0^B - z_0^A z_0^B) \partial_A \omega_{AC}(z_0) w_0^C + \left[ (z_0^A \partial_A + \partial_A^\text{expl}) Q^B(z_0) - (z_0^A \partial_A + \partial_A^\text{expl}) Q^B(z_0) \right] \partial_B w_0^B \right]. \quad \text{(2.43)}$$

### 2.6 Classical Equations

Now let us vary the action. The $z_3^A$ field only appears at one place in the action and to the linear order. Hence, the variation wrt. $z_3^A$ annihilates $w_0^A \cong 0$:

$$\delta z_3^A: \quad w_0^A \cong 0. \quad \text{(2.44)}$$

Variation wrt. $z^A$ and $z^{-A}$ yield their own equations,

$$\delta z^A: \quad z^A \cong \{ Q^B(z_0), z_0^A \}_P + \mathcal{O}(w_0), \quad \text{(2.45)}$$

$$\delta z^{-A}: \quad z^{-A} \cong \{ Q^B(z_0), z_0^A \}_P + w^A + \mathcal{O}(w_0), \quad \text{(2.46)}$$

respectively, with the notable appearance of a “reaction force” $w^A$ in the equation for $z^{-A}$. On the other hand the $w^A$ field only appears linearly in the action. The variation wrt. $w^A$ enforces the correct equation for $z^{-A}$

$$\delta w^A: \quad z^{-A} \cong \{ Q^B(z_0), z_0^A \}_P + \mathcal{O}(w_0). \quad \text{(2.47)}$$

Note the difference between the $z^A$ and the $z^{-A}$ sector: The $z^A$ appears Gaussian and free of reaction forces, while the $z^{-A}$ is constrained. By comparing the two eqs. (2.46) and (2.47), we conclude that the reaction force vanishes on-shell:

$$w^A \cong \mathcal{O}(w_0). \quad \text{(2.48)}$$

The vanishing of the reaction force is a result of a remarkable balance between the tilde part $\tilde{S}$ and the check part $\bar{S}$ of the action. Finally, variation wrt. $w_0^A$ produces an equation for $z_3^A$. After substitution of appearances of $z^A$, $z^{-A}$ and $w_0^A$ with their respective equations of motion, the $z_3$-equation takes the form:

$$z_3^A \cong \frac{1}{2} \epsilon_{ab} \left( \{ Q^a(z_0), Q^b(z_0), z_0^A \}_P + \{ \partial^a_{\text{expl}} Q^b(z_0), z_0^A \}_P \right). \quad \text{(2.49)}$$

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The preservation of measure factors is a very important **generic** property of a superfield integration, as the component fields automatically carry opposite Grassmann statistics. This trivial fact will be used so many times in the following, that we will not always bother to mention it explicitly.
2.7 Reduction to $N = 0$

Of the six remaining component fields, let us integrate out $z_A^3$, $w_0^A$, $w_A^\parallel$ and $z_A^\perp$, leaving $z_0^A$ and $z_A^\parallel$ (See Diagram 2.39). This produces no measure factor contributions. The action becomes the $N = 1$ action (1.10)

$$S \cong \tilde{S}[z_0, z^\parallel] + \int dt \left[ \frac{1}{2} \{Q^\perp(z_0), Q^\perp(z_0)\}_{PB} + \partial_{\text{expl}}^\perp Q^\perp(z_0) \right],$$

(2.50)

shifted with an additional contribution to the Hamiltonian from the perpendicular sector. An integration over the remaining superpartner $z_A^\parallel$ reproduces the $N=0$ path integral with the Hamiltonian (2.12).

Note that (2.50) is not a manifest $N = 2$ to $N = 1$ reduction, because of the $N = 0$ shift-term. Integrating out the anti-BRST symmetry in a $Sp(2)$ theory will encode half of the Hamiltonian in a $N = 1$ action and half of the Hamiltonian will be deposited outside in a $N = 0$ term. This is the inevitable consequence of a BRST and an anti-BRST symmetry, which share the same Hamiltonian, or equivalently, that there are two $\theta$’s but only one $t$. The upshot is that the $N = 2$ theory can only be compared with the fully reduced $N=0$ theory, and not with intermediate stages.

3 $N$ Supercharges

We now generalize the construction to arbitrary number $N$ of supercharges and provide further details, that we previously skipped or glozed over. The corresponding $\theta$-space is a $N$-dimensional manifold with global, real, Grassmann-odd coordinates $\theta_1, \ldots, \theta_N$.

3.1 Symplectic $z$-space

The superfield $z^A$, $A = 1, \ldots, 2M$, has Grassmann-parity $\epsilon_A$ and takes value in a field-theoretic phase space. This phase space is endowed with a closed, non-degenerated, symplectic, Grassmann-even 2-form

$$\omega = \frac{1}{2} dz^A \omega_{AB} dz^B, \quad \omega_{AB} = (-1)^{(\epsilon_A+1)(\epsilon_B+1)} \omega_{BA}.$$ (3.1)

The closeness relation $d\omega = 0$ reads in components

$$\sum_{\text{cycl. } A,B,C} (-1)^{(\epsilon_A+1)\epsilon_C} \partial_A \omega_{BC} = 0, \quad \partial_A \equiv \frac{\partial}{\partial z^A}.$$ (3.2)

The superfields can be expanded in $2^N$ component fields

$$z^A(t, \theta) = z_0^A(t) + \theta_a z^a A(t) + \theta_a \theta_b z^{ab} A(t) + \ldots + (\theta_1 \ldots \theta_N) z^{1 \ldots N} A(t).$$ (3.3)

3.2 Symplectic $\theta$-space

We now postulate that as a first principle the $\theta$-space should be symplectic, i.e. endowed with a closed, non-degenerated, symplectic, Grassmann-even 2-form

$$g = \frac{1}{2} d\theta_a g^{ab} d\theta_b,$$ (3.4)
with a reality condition \((g^{ab})^* = g^{ab}\). We will sometimes refer to the \(\theta\)-space as a fermionic world volume. The closeness relation \(dg = 0\) reads in components
\[
\sum_{\text{cycl. } a,b,c} \partial^a g^{bc} = 0 , \quad \partial^a \equiv \frac{\partial}{\partial \theta_a} .
\tag{3.5}
\]
The full potential and naturalness of this definition will become completely clear after the construction of the superderivative in Section 3.4. We begin by discussing some of its immediate consequences. First of all, the symplectic 2-tensor turns out to be symmetric,
\[
g^{ab} = g^{ba} ,
\tag{3.6}
\]
because of the Grassmann nature of the \(\theta\)-parameters. This remarkable virtue of supermathematics, i.e. that a fermionic symplectic manifold can superficially “appear” Riemannian, is one of the main points that we want to state. From our experience with the \(N = 2\) case, we already know that a symmetric metric \(g^{ab}\) is precisely what is needed in the Susy algebra construction (2.4). The metric
\[
g^{ab}(\theta) = g^{ab}_0 + \theta_c g^{c,ab}_1 + \theta_d g^{cd,ab}_2 + \ldots
\tag{3.7}
\]
may depend on the \(\theta_a\)’s but not on the time variable \(t\), as this would generate unwanted terms in the Susy algebra, see eq. (3.16) below.

Despite the Riemannian “look”, it is crucial that the \(\theta\)-space metric is of symplectic rather than of Riemannian origin, so that powerful symplectic techniques apply:

1. First of all, a global Poincaré Lemma is at our disposal, i.e. there exists a global, Grassmann-even 1-form, known as a symplectic potential,
\[
\frac{1}{2} \vartheta = \frac{1}{2} \vartheta^a d\theta_a , \quad (\vartheta^a)^* = \vartheta^a .
\tag{3.8}
\]
such that
\[
g = \frac{1}{2} d\vartheta , \quad g^{ab} = \frac{1}{2} (\vartheta^a \vartheta^b + \vartheta^b \vartheta^a) .
\tag{3.9}
\]
It is proved in Appendix B that for any choice of potential, there exist fermionic constants \(\theta_0^a\) such that
\[
\vartheta_a := g_{ab} \vartheta^b = (\theta_a - \theta_0^a) + \mathcal{O}((\theta - \theta_0)^2) .
\tag{3.10}
\]
The fermionic constants \(\theta_0^a\) are theoretically needed for translational invariance in \(\theta\)-space. They may in practice be set to zero. (A reader only interested in the flat case, may identify \(\vartheta_a\) and \(\theta_a\). This is why we choose the unconventional factor \(\frac{1}{2}\) in eqs. (3.8) and (3.9).)

2. Secondly, we have a global version of the Darboux theorem, i.e. there exists global coordinates such that \(g^{ab}\) is constant. Moreover, the real metric \(g^{ab}\) may be chosen to be a diagonal matrix with diagonal entries equal to \(\pm 1\). In fact, the metric has an invariant \((p, q)\) signature, \(p + q = N\). This is in accordance with a theorem known in the mathematical Literature as the Sylvester Law of Inertia. For simplicity, we assume in the following that the metric \(g^{ab}\) is positive definite.

3. Thirdly, the Liouville theorem, “Hamiltonian flows are divergenceless”, will play an important role later on in our construction. The canonical world volume measure factor can be written \(\sqrt{g}\), where \(g = \det(g_{ab})\). The Liouville theorem may conveniently be recasted as
\[
\Delta := \frac{1}{\sqrt{g}} \vartheta^a \sqrt{g} g_{ab} \vartheta^b = 0 .
\tag{3.11}
\]
Superficially, this appears as the vanishing of a Laplace-Beltrami operator.
3.3 Poisson Bracket

The inverse symplectic matrix $\omega^{AB}$ is a Poisson bi-vector, which gives rise to a Poisson bracket in the exterior algebra $\mathcal{A}$ of world volume forms

$$\{F, G\}_{PB} = F_A \omega^{AB} \partial_B G = -(-1)^{\epsilon(F)\epsilon(G)+p(F)p(G)}\{G, F\}_{PB}, \quad F, G \in \mathcal{A}, \quad (3.12)$$

where $p$ denotes the world volume form-degree and $\epsilon$ denotes the Grassmann-parity. (The form-basis $d\theta_a$ are passive spectators to the Poisson bracket.) The Jacobi identity reads

$$\sum_{\text{cycl. } F, G, H} (-1)^{\epsilon(F)\epsilon(H)+p(F)p(H)}\{\{F, G\}_{PB}, H\}_{PB} = 0. \quad (3.13)$$

Also note that the Poisson bi-vector $\omega^{AB} = \omega^{AB}(z(t, \theta))$ does not have explicit $t$ and $\theta$ dependence.

3.4 Susy Algebra

We use the symplectic potential $\vartheta^a$ to define the superderivative

$$D^a = \partial^a + \frac{1}{N} \vartheta^a \partial_t, \quad \partial_t \equiv \frac{\partial}{\partial t}. \quad (3.14)$$

The exterior superderivative becomes

$$D := d\theta_a D^a = d - \frac{1}{N} \vartheta \partial_t, \quad (3.15)$$

while the Susy algebra itself reads

$$D^2 = -\frac{2}{N} g \partial_t, \quad [D^a, D^b] \equiv D^{\{a} D^{b\}} = \frac{2}{N} g^{ab} \partial_t, \quad (3.16)$$

in 2-form notation and in component notation, respectively. Note in particular that the exterior superderivative is not nilpotent, but acts as a square root of time translations. The metric $g^{ab}$ reappears in the Susy algebra, because of its dual role as a symplectic field strength (cf. eq. (3.9)). This is the real reason the $\theta$-space is promoted to be a symplectic manifold.

3.5 Equations of Motion

Superevolution of a quantity $F = F(z(t, \theta); t, \theta)$ is governed by $N$ supercharges $Q^a = Q^a(z(t, \theta); t, \theta)$, $a = 1, \ldots, N$, which can be neatly packed into a Grassmann-even supercharge 1-form

$$Q := d\theta_a Q^a. \quad (3.17)$$

At the classical level, the equations of motion for a world volume scalar $F$ is

$$DF = \{Q, F\}_{PB} + D_{\text{expl}} F, \quad (3.18)$$

or in components

$$D^a F = \{Q^a, F\}_{PB} + D^\text{expl}_a F, \quad (3.19)$$

where the subscript “expl” as usual denotes explicit differentiation.
As an aside, if $F$ in eq. (3.19) is not a scalar, but if $F$ transforms covariantly as a tensor under world volume reparametrizations, one should replace $\partial^a$ with a covariant derivative $\nabla_a$. Hence the covariant superderivative reads

$$D^a = \nabla^a + \frac{1}{N} g^a \partial_t.$$

(3.20)

This of course depends on the specific choice of connection. Unfortunately, in contrast to Riemannian manifolds, which can always be endowed with a Levi-Civita connection, symplectic spaces do not have a canonical choice of connection. So in other words, in order to give a covariant and consistent recipe for the equations of motion (3.19) for a non-scalar tensor $F$, we should assume that the $\theta$-space comes equipped with a connection. For technical reasons, we should assume that the connection is torsion free, and other conditions to be discussed elsewhere, as it would be out of scope here.

We now indicate 3 different derivations of the Susy algebra of the $Q^a$'s:

1. First of all, let us derive the Susy algebra, while only referring to manifest scalar objects. It is here that the form notation comes in extra handy, as the exterior superderivative $D$ has the nice property, that it takes a world volume scalar to a world volume scalar. Hence we may apply (3.18) repeatedly on itself, without having to rely on a specific connection. (Note in particular that this is not so for the superderivative $D^a F$, which is not a world volume scalar.) Applying the equations of motion (3.18) successively, and using the Jacobi identity, we get

$$-\frac{2}{N} g \partial_t F = D^2 F = \left\{ \frac{1}{2} \{Q, Q\}_{PB} + D_{expl}\{Q, F\}\right\}_{PB} - \frac{2}{N} g \partial_t^{expl} F.$$

(3.21)

Written out in components

$$\frac{2}{N} g^{ab} \partial_t F = [D^a, D^b] F = \left\{ \{Q^a, Q^b\}_{PB} + D_{expl}^{(a}Q^{b)}{,} F\right\}_{PB} + \frac{2}{N} g^{ab} \partial_t^{expl} F.$$

(3.22)

2. Secondly, if one do not like forms, to justify the component expression (3.22) directly in components, one merely has to check that (3.22) transforms covariantly under change of coordinates.

3. Thirdly, one can derive eq. (3.22) in components by applying the superderivative $D^a$ twice and then symmetrize, with the implicit understanding that the $D^a$ in eq. (3.22) stands for the covariant superderivative (3.20). However, if the connection is torsion free, it is easy to see that the Christoffel symbols drops out of the symmetrized combination in eq. (3.22), so one may replace $D^a$ with ordinary superderivatives (3.14), in agreement with the two previous derivations.

### 3.6 Hamiltonian

Contracting on both sides of eq. (3.22) with the inverse metric $g_{ab}$, we get the equation for time evolution

$$\partial_t F = \left\{ H, F\right\}_{PB} + \partial_t^{expl} F,$$

(3.23)

where we define the Hamiltonian as

$$-H := \frac{1}{2} g_{ab} \left\{ \{Q^a, Q^b\}_{PB} + D_{expl}^{(a}Q^{b)}{\} \right.$$

(3.24)

Note that $H$ depends explicitly on the metric $g_{ab}$.
3.7 Integrability

Similarly, the $\theta$-evolution is governed by the following equations of motion

$$\partial^a F = \{\Omega^a, F\}_{PB} + \partial^a_{\text{expl}} F , \quad dF = \{\Omega, F\}_{PB} + d_{\text{expl}} F ,$$

(3.25)

with generators

$$\Omega^a := Q^a + \frac{1}{N} \vartheta^a H , \quad \Omega := d\vartheta_a \Omega^a = Q - \frac{1}{N} \vartheta H .$$

(3.26)

The integrability conditions $[\partial_t, d] F = 0$ and $d^2 F = 0$ for eqs. (3.23) and (3.25) are zero-curvature equations

$$\{\Omega, H\}_{PB} + d_{\text{expl}} H + \partial^a_{\text{expl}} \Omega = 0$$

(3.27)

$$\frac{1}{2} \{\Omega, \Omega\}_{PB} + d_{\text{expl}} \Omega = 0 .$$

(3.28)

3.8 Two Sectors: Tilde and Check

In any dimension $N$ there are two 1-forms readily at our disposition. They are the symplectic potential $\vartheta^a$ and its Hodge-like dual

$$\tilde{\vartheta}^a := D^a \left( \frac{\vartheta^N}{\sqrt{g}} \right) = \partial^a \delta^N (\theta - \theta^0) ,$$

(3.29)

respectively. (See the Appendices for further details.) As usual, the metric $g_{ab}$ raises and lowers indices. Hence we may construct two types of scalar supercharges

$$\tilde{Q} := \tilde{\vartheta}_a Q^a = \tilde{\vartheta}_a \Omega^a = (-1)^N \delta^N (\theta - \theta^0) H$$

(3.30)

$$\tilde{Q} := \vartheta_a Q^a = \vartheta_a \Omega^a .$$

(3.31)

and two types of superderivatives$^{**}$

$$\tilde{D} := \tilde{\vartheta}_a D^a = \tilde{\vartheta}_a \vartheta^a - (-1)^N \delta^N (\theta - \theta^0) \partial_t$$

(3.32)

$$\tilde{D} := \vartheta_a D^a = \vartheta_a \vartheta^a .$$

(3.33)

In the last equality in both eqs. (3.31) and (3.33) we used the fact that $\vartheta_a g_{ab} \vartheta^b = 0$. The equations of motion eq. (3.18) for $D^a$ imply the corresponding equation of motion for $\tilde{D}$ and $\tilde{D}$:

$$\tilde{D} F = \{\tilde{Q}, F\}_{PB} + \tilde{D}_{\text{expl}} F$$

(3.34)

$$\tilde{D} F = \{\tilde{Q}, F\}_{PB} + \tilde{D}_{\text{expl}} F .$$

(3.35)

We show in the next two Sections 3.9-3.10, that the opposite is also true, i.e. that the two eqs. (3.34) and (3.35) imply the eq. (3.18).

$^{**}$\(\tilde{D}\) and \(\tilde{Q}\) were denoted \(D\) and \(Q\), respectively, in Ref. [6].
3.9 Tilde Sector

First multiply the tilde equation (3.34) with $\bar{\theta}^a$ (or the check equation (3.35) with $\bar{\theta}^a$):

$$
\delta^N(\theta - \theta^0) \ D^a \ F = \delta^N(\theta - \theta^0) \ \{Q^a, F \}_P + \delta^N(\theta - \theta^0) \ \hat{D}_a \ expl \ F ,
$$  
(3.36)

leading to

$$
dF(\theta^0) = \{\Omega(\theta^0), F(\theta^0) \}_P + d_{expl} F(\theta^0) .
$$  
(3.37)

Our aim is now to derive a differentiated version

$$
0 = d^2 F = d\{\Omega(\theta^0), F(\theta^0) \}_P + d_{expl} F(\theta^0) .
$$  
(3.38)

One may not proceed by direct differentiation, because eq. (3.25) is only known for $\theta_a = \theta_a^0$. Instead substitute $F$ with $\{\Omega, F \}_P + d_{expl} F$ in the above eq. (3.37):

$$
d\{\Omega(\theta^0), F(\theta^0) \}_P + d_{expl} F(\theta^0) = \{\Omega(\theta^0), \{\Omega(\theta^0), F(\theta^0) \}_P \}_P + \{\Omega(\theta^0), d_{expl} F(\theta^0) \}_P + d_{expl} \{\Omega(\theta^0), F(\theta^0) \}_P = 0 ,
$$  
(3.39)

where the Jacobi identity and a integrability condition (3.28) were applied in the last equality.

Next Berezin integrate the lhs. of the tilde equation (3.34). After integrating by part and using the Liouville theorem (3.11), we get

$$
\int \sqrt{g} \ d^N \theta \ \hat{D} \ F = - (-1)^N \ \hat{\partial}_t F(\theta^0) .
$$  
(3.40)

Similarly, after use of eqs. (3.37) and (3.38), the rhs. of the tilde equation (3.34) becomes

$$
\int \sqrt{g} \ d^N \theta \ \{\hat{Q}, F \} + \hat{D}_{expl} F \ = \ (-1)^N \ \{H(\theta^0), F(\theta^0) \} - (-1)^N \ \hat{\partial}_{expl}^t F(\theta^0) .
$$  
(3.41)

Hence we have derive the equations of motion (3.25) and (3.23) (and thereby the superequation (3.18)) for $\theta_a = \theta_a^0$.

3.10 Check Sector

Imagine that we are given initial conditions $z^A(t_0, \theta_0) = z_{00}^A$. It is shown above that one can determine uniquely $z^A(t, \theta_a)$ for arbitrary times $t$ along the line $\theta_a = \theta_a^0$. We would like to extend the solution to arbitrary $\theta_a \neq \theta_a^0$, at some arbitrary but fixed time $t$. The key to solve the check equation $\hat{D} z^A = \{\hat{Q}, z^A \}_P$, is to keep track of powers of $\theta_a - \theta_a^0$. Let $F[n]$ be the part of a quantity $F$ that contains $n$ powers of $\theta_a - \theta_a^0$. We expand the relevant quantities accordingly:

$$
z^A = \sum_{n=0} \ z^A_{[n]} , \quad z^A_{[0]} := z^A(t, \theta_0) ,
$$  
(3.42)

$$
\{\hat{Q}, z^A \}_P = \sum_{n=1} \ \{\hat{Q}, z^A \}_{P[n]} ,
$$  
(3.43)

$$
\hat{D} = \sum_{n=0} \ \hat{D}_{[n]} .
$$  
(3.44)

All sums truncate because the $\theta$’s are Grassmann-odd. The leading contribution to $\hat{D}$ is the conformal operator

$$
\hat{D}_{[0]} = (\theta_a - \theta_a^0) \hat{\partial}^a .
$$  
(3.45)
The operator $\tilde{D}_{[0]}$ preserves and counts the powers of $\theta_a - \theta_0^0$. Applying this on the check equation, we get
\[
nz^A_{[n]} = \tilde{D}_{[0]}z^A_{[n]} = (\tilde{Q}, z^A)^{[n]}_{PB} - \sum_{k=1}^{n} \tilde{D}_{[k]}z^A_{[n-k]} .
\] (3.46)

It is easy to solve for $z^A_{[0]}$. We can do it successively, for increasing $n > 0$, because eq. (3.46) is of a triangular form. Recall namely that $\tilde{Q}$ contains explicitly one power of $\theta_a - \theta_0^0$. Therefore, for a given $n > 0$, the rhs. can only depend on previous components $z^A_{[0]}, \ldots, z^A_{[n-1]}$.

This shows that there is enough information in the tilde and check sector eqs. (3.34) and (3.35) combined to construct a unique solution $z^A(t, \theta)$ for arbitrary $t$ and $\theta$. Will this solution $z$ satisfy the $D^a$ equations of motion eq. (3.18) as well? Yes, because a solution $z'$ to eq. (3.18) (with the same initial condition) is trivially also a solution to eqs. (3.34) and (3.35). By uniqueness $z = z'$.

3.11 Action

Our action is a natural generalization of the $N=2$ action (2.20),
\[
S = \tilde{S}[z] + \check{S}[z, w] + S_a[w, \pi] ,
\] (3.47)
\[
\tilde{S}[z] = -(-1)^N \int dt \sqrt{g} d^N \theta \left[ z^A \check{\omega}_{AB}(z) \check{D}z^B (-1)^{N_{AB}} + \check{Q}(z) \right] ,
\] (3.48)
\[
\check{S}[z, w] = \int dt \sqrt{g} d^N \theta \, w^A \left[ \omega_{AB}(z) \check{D}z^B + \partial_A \check{Q}(z) \right] ,
\] (3.49)
\[
S_a[w, \pi] = \int dt \sqrt{g} d^N \theta \, w^A(\theta) \, \pi_A(n^a \theta_a) .
\] (3.50)

Here $z^A$ and $w^A$ are superfields, while $\pi_A$ is a $N=1$ auxiliary superfield. Of these, $z^A$ and $\pi_A$ carry the same Grassmann-parity $\epsilon_A$, while $w^A$ carries Grassmann-parity $\epsilon_A + N$. The real unit-vector $n^a = n^a(\theta)$ represents an explicit choice of direction in the $N$-dimensional $\theta$-space. We will show how the unit-vector $n^a$ points out a single superpartner $z^A := n_a z^a A$, which appears to the Gaussian order in the action, while all the remaining $N-1$ orthogonal $z^a A$ superpartners are constrained. The $\check{\omega}_{AB}$ is defined as
\[
\check{\omega}_{AB} := \left( z^C \partial_C + 2 \right)^{-1} \omega_{AB} = \int_0^1 d\alpha \, \omega_{AB}(\alpha z) .
\] (3.51)

We claim that the variation of the action yields the equations of motion
\[
\check{D}z^A = \{ \check{Q}, z^A \}^A_{PB} ,
\] (3.52)
\[
\check{D}z^A = \{ \tilde{Q}, z^A \}^A_{PB} ,
\] (3.53)
\[
w^A = 0 ,
\] (3.54)
\[
\pi_A = 0 .
\] (3.55)

We prove parts of this below, in particular that the full superfield $w^A$ vanishes on-shell: $w^A \equiv 0$. As a corollary to this, we immediately deduce that the tilde part $\check{S}$ alone is responsible for the equation of motion for the superfield $z^A$. On the other hand, it is straightforward to perform a manifest superfield variation of $\check{S}$. This is to a large extent insensitive to the number of superpartners, and the resulting equation is eq. (3.52).

It is proved in Appendix B that the (lowered) symplectic potential $\theta_a$ has a unique fermionic zero $\theta_a^0$. We assume from now on that $\theta_a^0$ is shifted to $\theta_a^0 = 0$, either by redefining the $\vartheta_a$ or the $\theta_a$. 

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The first part of the action $\tilde{S}$ turns out to consist of terms proportional to either the delta function $\delta^N(\theta)$ itself or its first derivatives. Now integrate the latter type of terms by part. After use of the Liouville Theorem $\partial^a(\sqrt{g}g_{ab}) = 0$, we arrive at a component expression

$$\tilde{S}[z] = \int dt \left[ z_0^A \bar{\omega}_{AB}(z_0) \dot{z}^B_0 + \frac{1}{2} z^{aA}_0 g_{ab} \omega_{AB}(z_0) z^{bB} (-1)^{e_B} + g^0_{ab} (z^{aA}_0 \partial_A + \partial^{a}_{\text{expl}}) Q^b(z_0) \right].$$ (3.56)

It is hard to provide useful component expressions for the second part $\tilde{S}$. Instead, we will see below how an inductive approach may yield manageable expressions (cf. Section 3.13). The third part reads in components

$$S_n[w, \pi] = \int dt \sqrt{g_0} \left[ w^A_{2N-1} \pi^0_A - (-1)^{(N+1)/2} \pi^1_A n_0^{a_1,...,a_N} w^{a_2,...,a_N, A} \right],$$ (3.57)

if $g = g_0$ is constant. In the non-constant case, there will be additional terms proportional to derivatives of the metric.

We should mention that it is always possible to go to new $\theta'_a$-coordinates, such that the unit-vector $\bar{\eta}$ becomes parallel to the new 1'-axis, and the metric becomes the unit matrix (cf. eq. (2.38)). However, here we continue to work with as general coordinates as possible.

### 3.12 Case $N = 1$

In the $N=1$ case, there are only two discrete choices for the unit-vector $n^1 = \pm \sqrt{g^{11}}$, corresponding to the two points on a zero-sphere $S^0$. The second and third part of the action read in components

$$\tilde{S}[z, w] = -\int dt \sqrt{g_0} w^A_0 \left[ \omega_{AB}(z_0) z^{1B} (-1)^{e_B} + \partial_A Q^1(z_0) \right],$$ (3.58)

$$S_n[w, \pi] = \int dt \left[ \sqrt{g_0} w^A_1 \pi^0_A \pm \pi^1_A w^0_A \right],$$ (3.59)

respectively. A closeness relation $\partial^1 g^{11} = 0$ was used to derive the component expression for the third part $S_n$. An $N=1$ superfield integration over $\pi_a$ annihilates $w^A$ completely, so that the action reduces to merely the $z$-sector

$$S \cong \tilde{S}[z].$$ (3.60)

If we go to flat coordinates $g^{11} = 1$, we arrive at the usual $N=1$ action (1.9) with a Gaussian superpartner $z^{1A}$.

### 3.13 Reduction from $N$ to $N - 1$

Since we have already covered the $N=1$ case we may assume $N > 1$. The reductive step consists of integrating out a $\theta$-direction in the $N$-dimensional $\theta$-space. The direction may be chosen arbitrarily, as long as it is orthogonal to the given $n^a$ direction. By performing a change of coordinates $\theta_a \rightarrow \theta'_b$, if necessary, one may without loss of generality assume that the direction, that is integrated out, is $\theta_1$, and that $n^a$ and $g^{ab}$ are of the block form

$$n^a = \begin{bmatrix} 0 \\ n^a \end{bmatrix}, \quad g^{ab} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g^{ab} \end{bmatrix}. \quad (3.61)$$
A prime denotes quantities associated with the $N-1$ remaining directions. For instance, $	heta' := (\theta_2, \theta_3, \ldots, \theta_N)$. One may take the symplectic potential and its dual to be of the form

$$
\vartheta^a = \begin{bmatrix} \theta_1 \\ \varrho^a \end{bmatrix}, \\
\bar{\vartheta}^a = \partial^a \delta^{N}(\theta) = \begin{bmatrix} \delta^{N-1}(\theta') \\ -\theta_1 \bar{\vartheta}^a \end{bmatrix},
$$

(3.62)

respectively, so that the tilde and check superderivatives and supercharges decompose as

$$
\check{D} = \vartheta_a \partial^a = D' + \theta_1 \partial^1,
$$

(3.63)

$$
\check{D} = \hat{\vartheta}_a \partial^a = (-1)^N \delta^{N}(\theta) \partial_t = \delta^{N-1}(\theta') \partial^1 - \theta_1 \check{D}',
$$

(3.64)

$$
\check{Q} = \check{\vartheta}_a Q^a = \check{Q}' + \theta_1 Q^1,
$$

(3.65)

$$
\hat{Q} = \hat{\vartheta}_a Q^a = \delta^{N-1}(\theta') Q^1 - \theta_1 \hat{Q}',
$$

(3.66)

respectively. The superfields themselves decompose as

$$
z^A(\theta) = z'^A(\theta') + \theta_1 z^A(\theta') \quad \longrightarrow \quad z'^A(\theta'),
$$

$$
w^A(\theta) = \check{w}^A(\theta') + \theta_1 w^A(\theta') \quad \longrightarrow \quad w'^A(\theta'),
$$

$$
\pi_A(n^a \theta_a) = \pi_A(n'^a \theta'_a).
$$

(3.67)

In detail, to go from $N$ to $N-1$ number of $\theta'$s in the path integral, the hat superfields $\check{z}^A(\theta')$ and $\check{w}^A(\theta')$ should be integrated out, while the superfields $z'^A$, $w'^A$ and $\pi_A$ should be kept. Note that the Grassmann parities of the $w^A$ and $w'^A$ fields are opposite. We do not touch the $\pi_A$ field, as $n^1 = 0$. (Recall that the $\theta$-direction, that is integrated out, is orthogonal to $n^a$.) Therefore, the third part $S_n$ is unaltered by the reduction

$$
S_n[w, \pi] - S'_n[w', \pi] = 0.
$$

(3.68)

The change in the first action part is

$$
\check{S}[z] - \check{S}'[z'] = \int dt \left[ \frac{1}{2} z^A \omega_{AB}(z_0) z^B (-1)^{e_B} + (z^A \partial_A + \partial^1_{\text{expl}}) Q^1(z_0) \right].
$$

(3.69)

This follows either from the component expression (3.56), or one may give a manifest $N-1$ superfield derivation by applying the decomposition formulas (3.64) and (3.66) in the superfield action (3.48). The change in the second action part becomes

$$
\check{S}'[z, w] - \check{S}'[z', w'] = (-1)^N \int dt \sqrt{\theta'} \, d^{N-1} \theta' \check{w}^A \left[ \omega_{AB}(z') (\check{D}' + 1) \check{z}^B (-1)^{e_B} \right. \\
\left. + (-1)^r \check{z}^A \partial_A \omega_{AB}(z') \check{D}' \check{z}^B + \partial_A[\check{z}^B \partial_B + \partial^1_{\text{expl}}] Q'(z') + Q^1(z') \right].
$$

(3.70)

Now expand everything in powers $n$ of $\theta'$,

$$
z'^A = \sum_{n=0} z'^A_{[n]}, \quad z^A = \sum_{n=0} z^A_{[n]}, \quad \check{w}^A = \sum_{n=0} \check{w}^A_{[n]}, \quad \check{D}' = \sum_{n=0} \check{D}'_{[n]}, \ldots
$$

(3.71)

We now make the following

Claim: For all $r = 0, \ldots, N-1$, the variation of $S - S'$ wrt. $\check{z}_{[N-1-r]}$ leads to the equations of motion $\check{w}_{[r]} \approx 0$.

Induction proof in $r$: We argue successively, for increasing $r = 0, \ldots, N-1$. According to the induction assumption we may discard terms proportional to previous components $\check{w}^A_{[0]}, \ldots, \check{w}^A_{[r-1]}$. Moreover, we
only have to keep terms that contain \( \hat{z}^B_{[N-1-r]} \) explicitly. Coincidentally, this leave no room for \( \theta' \) appearances that are not already accounted for inside either \( \hat{w}_{[r]} \) or inside \( \hat{z}^B_{[N-1-r]} \).

**Case \( r < N-1 \):** The relevant parts boil down to only one term

\[
\dot{S}[z, w] - \dot{S}'[z', w'] \sim (-1)^N \int dt \sqrt{g_0} \ d^{N-1} \theta' \ \hat{w}^A_{[r]} \omega_{AB}(z_0) (N - r) \hat{z}^B_{[N-1-r]}(-1)^{rB} .
\]  

(3.72)

Variation wrt. \( \hat{z}^B_{[N-1-r]} \) yields the claim \( \hat{w}^A_{[r]} \equiv 0 \).

The last step \( r = N-1 \): We would like to vary wrt. \( \hat{z}^A_{[0]} \equiv z^1A \). Up to now we have showed that the full \( N-1 \) superfield \( \hat{w}^A \) reduces on-shell to a single top-component,

\[
\hat{w}^A \equiv \theta_2 \theta_3 \ldots \theta_N w^{23...N,A}.
\]

(3.73)

Inserted into the second part of the action, we get

\[
\dot{S}[z, w] - \dot{S}'[z', w'] \equiv (-1)^N \int dt \sqrt{g_0} \ w^{23...N,A} \left[ \omega_{AB}(z_0) z^B (-1)^{rB} + \partial_A Q^1(z_0) \right] .
\]

(3.74)

We should not forget contributions from the tilde sector (3.69), which also depend on \( \hat{z}^A_{[0]} \equiv z^1A \). Variation wrt. \( w^{23...N,A} \) enforces the correct equations of motion

\[
\delta w^{23...N,A} : \quad z^1B \equiv \{Q^1(z_0), z^0B\}_PB ,
\]

(3.75)

while the variation wrt. \( z^1B \) yields the same equation

\[
\delta z^1B : \quad z^1A \equiv \{Q^1(z_0), z^0A\}_PB - (-1)^N \sqrt{g_0} w^{23...N,A} ,
\]

(3.76)

with an additional reaction force term. We conclude that the top-component

\[
w^{23...N,A} \equiv 0
\]

(3.77)

of the superfield \( \hat{w}^A \) must vanishes on-shell as well. Completing the square yields the following shift in the action

\[
S - S' \equiv \int dt \left[ \frac{1}{2} \{Q^1(z_0), Q^1(z_0)\}_PB + \partial_{expl}^1 Q^1(z_0) \right] .
\]

(3.78)

It is now clear how the complete reduction down to \( N=0 \) proceeds. The outcome is the usual \( N=0 \) partition function

\[
\mathcal{Z} = \int[dz] \exp \left[ \frac{i}{\hbar} \dot{S} \right] = \int[dz_0] \mathrm{Pf}(\omega_{AB}(z_0)) \ \exp \left[ \frac{i}{\hbar} \int dt \left( z^A_0 \omega_{AB}(z_0) z^B_0 - H(z_0) \right) \right] .
\]

(3.79)

From each step of the reduction, there is a classical action shift (3.78). The shifts accumulate in the Hamiltonian of the form (3.24). Finally, in the very last reduction step from \( N = 1 \) to \( N = 0 \), all references to the choice of unit-vector \( n^a \) disappear, and the correct Pfaffian measure factor is reinstated.

### 3.14 An Example: \( N = 3 \)

The fact that the new action principle (3.47) is formulated for arbitrary \( N \), makes it is applicable to numerous situations. For instance, from the perspective of Hamiltonian \( Sp(2) \) theories, it opens up
a tantalizing possibility to have a manifest 2-dimensional $\theta$-plane inside a $N = 3$ theory, consisting of three supercharges $Q^a = \Omega^a + \frac{1}{3} \vartheta^a H$, $a = 1, 2, 3$, with an explicitly broken 3rd direction $n^a = \delta^a_3$, and with a corresponding dummy charge $\Omega^3 = 0$. Let us assume the coordinates are flat $\vartheta^a = \theta_a$ for simplicity. From our analysis, we know that the construction will implement all the correct equations of motion. If we use the 3-dimensional Levi-Civita symbol to raise and lower indices on superpartners, i.e. $z^A_a := \frac{1}{2} \epsilon_{abc} z^b c A$, we may indicate how the $2^3 + 2^3 + 2^1 = 18$ component fields pair off inside the kinetic part of the action to the quadratic order:

$$
\begin{array}{cccccccc}
z_0 & z^1 & z^2 & z^3 & z_3 & z_2 & z_1 & z_7 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
w_7 & w_1 & w_2 & w_3 & w^3 & w^2 & w^1 & w_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\pi^0 & \pi^1
\end{array}
$$

(3.80)

The $N = 2$ superpartners $z^1, z^2$ and $z_3$ will be constrained to their equations of motion, respectively, while the Gaussian integration over $z^3$ will produce the Pfaffian measure factor.

4 Discussions

To recapitulate, we have recasted the equations of motion (3.18) for superevolution into an equivalent set of two equations, a tilde eq. (3.34) and a check eq. (3.35). The tilde and check equation of motion lead directly to a corresponding action and path integral prescription. The third and last ingredient is a gauge-fixing term $S_n$, which explicitly single out a direction $n_a$ in the $\theta$-space without affecting the physical sector. One may speculate that the above rearrangement is necessary to disentangle the $t$ and the $\theta$ derivatives, and thereby unravel the $z_0$ sector from its many superpartners. The caveat is of course that all manipulations should be executed in a supersymmetric manner.

We saw that a comparison of a supersymmetric theory to some of its partially reduced cousins is tricky, because non-supersymmetric shift-terms are generated in the action (cf. Section 2.7). However, the reduction scheme is very successful in making contact to the fully reduced $N = 0$ theory. One may imagine that the “democracy” among the $N \theta$-directions in the Susy algebra, in which they share one and the same $t$ coordinate, makes it hard to partially integrate out $\theta$-directions without affecting the common $t$ parameter.

In conclusion, we have formulated a simple action principle for a superfield formulation of Hamiltonian field theories with $N$ supercharges. This action gives rise to a viable path integral of superfields, which may easily be reduced to the original sector.

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A Superconventions

The Berezin integration is defined as
\[
\int d^N \theta := \partial^N \ldots \partial^1 = \frac{1}{N!} \epsilon_{a_1 \ldots a_N} \partial^{a_1} \ldots \partial^{a_N},
\] (A.1)
where we have the following convention for the Levi-Civita symbol
\[
\epsilon_{N,1} := +1, \quad \epsilon_{1,\ldots,N} = (-1)^{\frac{N}{2}}.
\] (A.2)

An invariant integration measure is
\[
\sqrt{g} d^N \theta, \quad g := \text{Ber}(g^{ab}) = \det(g_{ab}).
\] (A.3)

The invariant N-dimensional Dirac delta function is correspondingly
\[
\delta^N(\theta) := \frac{\theta^N}{\sqrt{g}} = \frac{\theta^N}{\sqrt{g_0}},
\] (A.4)
where
\[
\theta^N := \theta_1 \ldots \theta_N = \frac{\epsilon_{a_1 \ldots a_N}}{N!} \theta_{a_1} \ldots \theta_{a_N}, \quad \epsilon^{1,\ldots,N} := +1.
\] (A.5)

We may define a 1-form
\[
\tilde{\theta}^a := \partial^a \delta^N(\theta) = \frac{\epsilon_{a_1 \ldots a_N}}{\sqrt{g_0} (N-1)!} \theta_{a_2} \theta_{a_3} \ldots \theta_{a_N}.
\] (A.6)

Fierz’ relations:
\[
\theta_{a_1} \ldots \theta_{a_N} = (-1)^{\frac{N}{2}} \epsilon_{a_1,\ldots,a_N} \theta^N,
\theta_a \tilde{\theta}^b = \delta_a^b \delta^N(\theta).
\] (A.7)

B The Symplectic Potential

In general, \(\partial^a\) has a \(\theta\)-expansion
\[
\partial^a(\theta) = \partial^a_0 + g_{0}^{ab} \theta_b + O(\theta^2),
\] (B.1)
where \(\partial^a_0\) are fermionic constants. It is easy to see that the equation \(\partial^a(\theta) = 0\) has precisely one solution, which we will denote
\[
\theta^a_0 = - g_{0}^{ab} \theta_b^0 + O(\theta^2).
\] (B.2)
(For instance, rewrite eq. (B.1) as a fixed point equation \(\theta_a = -g_{0}^{ab} \theta_b^0 + O(\theta^2)\), and eliminate all \(\theta\)-appearances recursively on the rhs. This process terminates after finite many steps because of the Grassmann nature.) Hence we may reorganize \(\partial^a\) as a polynomial in \((\theta_a - \theta^a_0)\) with no constant term:
\[
\partial^a(\theta) = g_{0}^{ab} (\theta_b - \theta^0_b) + O((\theta - \theta^0)^2),
\] (B.3)
Therefore, the symplectic field strength \(g^{ab}(\theta) = g_{0}^{ab} + O(\theta - \theta^0)\) is a function of \((\theta_a - \theta^a_0)\) as well. We conclude in particular that
\[
g^{ab}(\theta^0) = g_{0}^{ab} \equiv g^{ab}(0),
\] (B.4)
which can be traced to the fact that $g^{ab}$ does not depend on the fermionic constants $\vartheta^a_0$. Moreover,

$$\vartheta^a := g_{ab} \vartheta^b = (\theta^a - \theta^a_0) + \mathcal{O}((\theta - \theta^0)^2). \quad (B.5)$$

A $N$-fold product of the $\vartheta^a$'s leads to a shifted delta function

$$\vartheta^N = \vartheta^N_0 = \delta^N(\theta - \theta^0), \quad \vartheta^N := \vartheta_1 \ldots \vartheta_N. \quad (B.6)$$

We may define a 1-form

$$\tilde{\vartheta}^{a_1} := \vartheta^{a_1} \left(\frac{\vartheta^N}{\sqrt{g}}\right) = \frac{\epsilon^{a_1 \ldots a_N}}{\sqrt{g_0(N-1)!}} (\theta_{a_2} - \theta_{a_2}^0) (\theta_{a_3} - \theta_{a_3}^0) \ldots (\theta_{a_N} - \theta_{a_N}^0). \quad (B.7)$$

Fierz’ relation:

$$\vartheta^a \tilde{\vartheta}^b = \delta^b_a \delta^N(\theta - \theta^0). \quad (B.8)$$

One may also give explicit formulas for $\vartheta^a$ in terms of $g^{ab}$ by using homotopy operators. A convenient choice of potential, satisfying additionally $\vartheta^a \theta^a = 0$ and $\vartheta^a |_{\theta = 0} = 0$, is

$$\frac{1}{2} g^{ab} = \theta^a \bar{g}^{ab}, \quad \bar{g}^{ab} := (\theta^a \partial^a + 2)^{-1} g^{ab} = \int_0^1 \alpha \, d\alpha \, g^{ab}(\alpha \theta). \quad (B.9)$$

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