A note on the Uniformed Patroller Game

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Abstract

Patrolling Games were introduced by Alpern, Morton and Papadaki (2011) to model the adversarial problem where a mobile Patroller can only thwart an attack at some location by visiting it during the attack period, which has a prescribed integer duration $m$. In the present note we modify the problem by allowing the Attacker to go to his planned attack location early and observe the presence or absence of the Patroller (who wears a uniform), there. The Attacker can then choose to delay attacking for some number of periods $d$ after the Patroller leaves. This extra information for the Attacker can reduce thwarted attacks by as much as a factor of 4 in specific models.

Keywords: two-person game; constant-sum game; patrolling game; star network; uniformed patroller; attack duration; delay.

1 Introduction

Patrolling games were introduced by Alpern, Morton and Papadaki (2011) to model the adversarial problem where a mobile Patroller can only thwart an attack at some location by visiting (or inspecting) there during the attack period. The attack period has a prescribed duration that is an integer number $m$ of consecutive periods. In this paper we introduce a modification of the original model, whereby the Attacker can arrive at his intended attack location early and observe the presence or absence of the Patroller there. We say that the Patroller is wearing a uniform in order to stress that she is observable, but only at short range, when being at the location of the Attacker. This property gives the Attacker an additional choice variable, namely the number of periods $d$ to delay the start of his attack after the Patroller leaves the location chosen for the attack. If the Patroller comes back before these $d$ periods, the Attacker must restart his count. Thus the attack begins in the first period in which the Patroller has been recorded to be away for $d$ consecutive periods.

To illustrate the waiting parameter $d$, suppose that the Patroller’s presence at the attack node $i$ is indicated by a 1, and her absence is indicated by a 0. Suppose further that the Attacker waits until the Patroller arrives at the attack location, and after that her presence-absence sequence that he records is given by 11010100\ldots. If for example $d = 2$, then his decision as to whether or not to attack in each subsequent period is illustrated in Table \ref{table:Example}. If for example the attack difficulty is $m = 3$, then the attack will be successful only if the Patroller’s sequence continues with two more 0’s, that is, 1101010000\ldots. Note that the attack can begin in the same period that the Patroller’s absence has been observed. Thus, we consider $m \geq 2$ as otherwise the Attacker could win simply by attacking as soon as the Patroller is not present.

In our model the Patroller has a base (not considered a ‘location’) from which she can reach any of the $n$ locations she has to defend in a single period. In principle, the Patroller could visit a location in consecutive periods and her base could also be attacked, but we show easily that such a behavior is dominated. Thus after visiting (inspecting) a location, the Patroller returns

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
Period & Patroller\textsuperscript{\textasteriskcentered} & Attacker\textsuperscript{\textasteriskcentered} & Attacker\textsuperscript{\textasteriskcentered} & Attacker\textsuperscript{\textasteriskcentered} & Attacker\textsuperscript{\textasteriskcentered} & Attacker\textsuperscript{\textasteriskcentered} & Attacker\textsuperscript{\textasteriskcentered} \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
4 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{tabular}
\caption{Example of the waiting parameter $d$.}
\end{table}

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to her base. One might think that the Patroller should always visit some location immediately after returning to her base, but in fact she might want to stay at her base for consecutive periods to produce additional uncertainty for the Attacker. We determine the optimal probability $r$ in which to remain at her base for an additional period, as a function of the number of locations $n$ and the attack duration $m$. The Patroller knows her current location, but we assume she does not remember her previous sequence of locations (i.e. she is Markovian). For the Attacker, we determine his optimal delay $d$. We call this game the *The Uniformed Patroller Game* $U = U(n, m)$.

After solving the game, it is easy to compare its value (the optimal probability that an attack is thwarted) to the corresponding easily tractable version where the Patroller wears no uniform. We find, for example, that for short attack duration $m = 2$ and for a large number of locations $n$, wearing a uniform reduces the interception probability for an attack by a factor of 4. Hence, we feel that the common occurrence of uniformed Patrollers is likely explained not by their ability to intercept attacks, but to the effect on uniforms revealing the presence of a Patroller and thereby deterring attacks from taking place at all. In our model, as in previous patrolling models, the Attacker will never be deterred from carrying out an attack. Deterrence could be modeled in future work. When there are multiple Patrollers with prescribed paths (a single direction around a circle) but varying start times, it has been found in a thought provoking paper by Lin (2019) that, unlike our result, wearing a uniform does not hurt the Patroller(s). In this sense our paper and that by Lin (2019) show that the effect of wearing a uniform depends greatly on whether fixed or unpredictable patrolling paths are used. This is where the Markovian nature of the paths in our model is particularly important.

Patrolling problems have been studied for long, see e.g. Morse and Kimball (1951), but only from the Patroller’s viewpoint. A game theoretic approach, modelling an adversarial Attacker who wants to infiltrate or attack a network at a node of his choice, has only recently been introduced by Alpern, Morton and Papadaki (2011). The techniques developed there were also applied to the class of line networks by Papadaki et al. (2016), with the interpretation of patrolling a border. A continuous version of patrolling games on a network has recently been given by Garrec (2019). Other research of a similar reasoning includes Lin et al. (2013) for random attack times, Lin, Atkinson and Glazebrook (2014) for imperfect detection, Hochbaum, Lyu and Ordóñez (2014) on security routing games, and Basilico, De Nittis and Gatti (2017) for uncertain alarm signals. See also Baykal-Gürsoy et al. (2014) on infrastructure security games. Earlier work on patrolling a channel/border with different paradigms, includes Washburn (1982, 2010), Szechtman et al. (2008), Zoroa N, Fernández-Sáez and Zoroa P (2012), and Collins et al. (2013). The case where the Patroller is restricted to periodic walks has recently been studied by Alpern, Lidbetter and Papadaki (2019). The related problem of ambush is studied by Baston and Kikuta (2004, 2009), while an artificial intelligence approach to patrolling is given by Basilico, Gatti and Amigoni (2012). Applications to airport security and counter terrorism, are given respectively by Pita et al. (2008), and Fokkink and Lindelauf (2013). Our problem is also akin to the problem faced by a waiter in a restaurant who must return to each table with a certain frequency depending on the number of diners at the table. An approach to the problem discussed here when the locations to be defended have a graph theoretic structure is given in Alpern and Katsikas (2019), where the Patroller can only move along edges.
2 Analysis

We view the Uniformed Patroller game as being played on a star $S_n$ consisting of a center $C$ (the Patroller’s base) connected to $n$ end nodes (representing the $n$ locations the Patroller has to defend). We restrict to a Markovian Patroller, such that she reflects from the $n$ end nodes with probability $s$, from the center $C$ goes to each end with probability $p$, and remains at the center with probability $r$. This setting simplifies the game by introducing a single parameter family of patrolling strategies.

![Figure 1: The Star $S_n$.](image)

We begin by assuming that the attack takes place at an end node, which we denote by $E$, and then we show that the Patroller should reflect from the ends ($s = 1$). Note that since we have taken $m \geq 2$, reflecting from the ends will further imply that the Attacker should never attack at the center because in that case the Patroller will never be away from it for two consecutive periods.

2.1 Attack duration $m = 2$

In this case an attack at location $E$ cannot be intercepted if it starts when the Patroller is at another end node $e \neq E$, but only if she is at the center. Thus, an attack at location $E$ will be intercepted with probability $q \cdot p$, where $q$ is the probability that the Patroller is at $C$ at the beginning of the attack, which in turn implies that the Attacker should choose the waiting time $d$ to minimize $q$.

We wish to calculate how the probability $q$ that the Patroller is at $C$ changes over time, as she continues to be away from the Attacker’s chosen attack location $E$. Suppose that at some period $t$ the Patroller is not at $E$, but she is either at $C$ with probability $q$, or at one of the remaining $n-1$ locations other than $E$ with probability $1 - q$. Then, in the following period $t+1$ the Patroller will be at $C$ with probability $q \cdot r + (1 - q) \cdot s$, and at $E$ with probability $q \cdot p + (1 - q) \cdot 0$.

Hence, conditional on the Patroller not being at location $E$ at period $t$, the probability that she is at her base $C$ at period $t + 1$ is given by

$$f(q, s) = \frac{q \cdot r + (1 - q) \cdot s}{1 - p \cdot q}. \quad (1)$$

Fraction $1$ is increasing in $s$, and $f(q, s)$ is maximized for $s = 1$ at

$$f(q) = f(q, 1) = \frac{q \cdot r + (1 - q)}{1 - p \cdot q} = \frac{1 - n \cdot p \cdot q}{1 - p \cdot q}. \quad (2)$$
Additionally, since from equation (2) we find that
\[ f'(q) = -\frac{p(n-1)}{(1-pq)^2} < 0, \]
then \( f(q) \) is decreasing and hence minimized for \( q = 1 \), with \( f(1) = \hat{q} \), where
\[ \hat{q} = \frac{(1 - n \cdot p)}{(1 - p)}. \quad (3) \]

This is easily explained. We have shown that it is optimal for the Patroller to reflect at the ends \((s = 1)\) so \( f(q) \) is simply the probability that the Patroller is at the center, given that she was at the center with probability \( q \) last period (and has not appeared at the Attacker’s end). Since we have also shown that \( f(q) \) is decreasing in \( q \), and since \( q \) is a probability, then \( f(q) \) is minimized when \( q \) is the maximum probability of being at the center, namely \( q = 1 \). This occurs in the first period that the Patroller has left the Attacker’s end. So the minimum probability \( \hat{q} \) of the Patroller being at the center occurs two period after she has left the Attacker’s end. Hence the optimal delay is given by \( d = 2 \), regardless of the Patroller’s choice of \( p \), and is a dominant strategy.

The attack \((e, 2)\) will be intercepted if the Patroller is at \( C \) in its first period and she goes to \( E \) in its second period, that is, with probability
\[ a_2(n, p) = \hat{q} \cdot p = b(p) = \frac{(1 - n \cdot p) \cdot p}{1 - p}. \quad (4) \]

For a number \( n \) of candidate attack locations, the Patroller chooses the value of \( p \) that maximizes the interception probability \((4)\). Figure 2 shows the variation of \((4)\) with \( p \), for \( n = 2, \ldots, 8 \), where \( p \in [0, 1/n] \) is the probability that the Patroller moves to each of the \( n \) locations from her base \( C \).

Figure 2: The interception probability \( a_2(n, p) \), for \( n = 2 \) (blue), \ldots, 8 (yellow).

For the \( n = 2 \) arc star (which is equivalent to a line network with three nodes), the Patroller should move from the center towards each end with probability about 0.3, and remain at the center with probability about 0.4. For the \( n = 3 \) arc star, the Patroller should move towards each of the three end nodes with probability about 0.2, and remain at the center with probability about 0.4, etc..

We can be more precise about these Patroller’s optimal strategies. In particular, the optimal value \( \hat{p} = \hat{p}(n) \) for \( p \) depends on \( n \) and can be found by solving the first order equation
\[ a_2'(p) = \frac{(n \cdot p^2 - 2 \cdot n \cdot p + 1)}{(1 - p)^3}. \]

This can be simply written in the form
\[ n \cdot p^2 - 2 \cdot n \cdot p + 1 = 0, \]
Equations (5), (6) show respectively that the optimal probability $\hat{p}$ is asymptotic to $1/(2\cdot n)$, while the optimal probability $\hat{r}$ goes to $1/2$. The value $V$ of the game is given by

$$V = a(n, \hat{p}) = (2\cdot n - 1) - 2\cdot \sqrt{n \cdot (n - 1)}.$$  

We can pull together the above results into the following proposition:

**Proposition 1.** Consider the Uniformed Patroller game $U(n, 2)$. The optimal Patroller’s strategy is to reflect off the end nodes and to remain at the center with probability given by (6). The optimal Attacker’s strategy is to locate at a random end node and begin the attack in the second period that the Patroller is away ($d = 2$). The interception probability (value of the game) is then given by (7).

### 2.2 Attack duration $m$ odd

Suppose now that $m = 2j + 1$ is odd. Say the Patroller adopts a random walk on the star $S_n$ (always goes to a random adjacent node, never remains at the same node). In any $2j$ consecutive periods, she visits $j$ ends counting multiplicity, randomly chosen. The probability that a particular end $e$ is visited among these $j$ is given by $1 - \left(\frac{n-1}{n}\right)^j$ and so this is a lower bound on the interception probability. Similarly, if the Attacker begins his attack at any time the Patroller is away from his chosen location, then the number $k$ of end nodes the Patroller visits during the rest of the attack satisfies $k \leq j$, and since they are chosen randomly (independently) the probability that the Attacker’s node is among them is given by $1 - \left(\frac{n-1}{n}\right)^k \leq 1 - \left(\frac{n-1}{n}\right)^j$. Note that only the random walk always gives $k = j$. Hence, we have established the following result:

**Proposition 2.** Consider the Uniformed Patroller game $U(n, m)$ where the attack duration $m$ is odd. The unique optimal Patroller’s strategy is the random walk. The optimal Attacker’s strategy is to attack at any end with an arbitrary delay. The value of the game is then given by

$$V = 1 - \left(\frac{n-1}{n}\right)^{(m-1)/2}.$$  

### 2.3 Attack duration $m = 4$

Here, if the attack starts when the Patroller is at the center, then she gets two chances to intercept it (i.e. to find the correct end). However, if the attack starts when the Patroller is at an end, then she gets only one chance. In the first case, the Patroller can intercept the attack with any of the sequences

$$CE_-, CCE_-, CCCE, CeCE,$$

where $e$ is any end other than the attack node $E$, while in the second case, with any of the sequences
It follows that the attack will be intercepted with overall probability

\[ a_4(n, p, s, q) = q \cdot \left( 1 + r + r^2 + (1 - p - r) \right) \cdot p + (1 - q) \cdot \left( s + s \cdot r + (1 - s) \cdot s \right) \cdot p \]

\[ = -p \cdot (r \cdot s - r^2 - s^2 + 2 \cdot s + p - 2) \cdot q + p \cdot (2 \cdot s + r \cdot s - s^2), \]  \hspace{1cm} (9)

where \( q \) is the conditional probability that the Patroller is at the center \( C \) at the beginning of the attack given that she is not at the attack node \( E \). Note that the coefficient of \( q \) in \( (9) \) is given by the product of \( p \) with the expression

\[ (-r \cdot s + r^2 + s^2 - 2 \cdot s - p + 2) = 2 + (s^2 - 2 \cdot s) - r \cdot s + r^2 - p \]

\[ \geq 2 - 1 - rs + r^2 - p \geq 1 - (r + p) + r^2 \geq 1 - (r + p) \geq 0, \]

since \( r + p \leq r + n \cdot p = 1, \) and \( s \leq 1. \) It follows that for fixed \( n, p, \) the interception probability \( \hat{q} \) is increasing in \( q. \) By the same reasoning we have used for \( m = 2, \) it further follows that the Attacker should choose to wait for \( d = 2 \) to attain \( q = \hat{q}, \) and the minimum interception probability

\[ a_4 = \frac{p}{1 - p} \left( (p + r - 1) \cdot s + (2 - r - p \cdot r - r^2 - 2 \cdot p) \cdot s + (2 \cdot r - p \cdot r + r^3) \right) \]. \hspace{1cm} (10)

To check that \( (11) \) is increasing in \( s, \) notice that the derivative with respect to \( s \) in the bracketed quadratic above is given by

\[ 2 \cdot (p + r - 1) \cdot s + (2 - r - p \cdot r - r^2 - 2 \cdot p) \geq 2 \cdot (p + r - 1) \cdot 1 + (2 - r - p \cdot r - r^2 - 2 \cdot p), \]

since \( r + p < 1, \) which is equivalent to \( r \cdot (1 - p - r) \geq 0, \) since both factors are non-negative.

Consequently, it turns out that the Patroller maximizes the interception probability for fixed \( p \) by choosing \( s = 1 \) (reflecting at the ends). Taking \( s = 1 \) in \( (10) \), gives

\[ a_4(n, p) = \frac{p}{1 - p} \cdot \left( 2 \cdot r - p - 2 \cdot p \cdot r - r^2 + r^3 + 1 \right) \]

\[ = -n^3 \cdot p^3 + 2 \cdot n^2 \cdot p^3 + 2 \cdot n \cdot p^3 - 3 \cdot n \cdot p^2 - 3 \cdot p^2 + 3 \cdot p \]

\[ = \frac{n^3 \cdot p^3 + 2 \cdot n^2 \cdot p^3 - 3 \cdot n \cdot p^2 - 3 \cdot p^2 + 3 \cdot p}{1 - p}. \hspace{1cm} (11) \]

In Figure 3 we plot the variation of the interception probability \( a_4(n, p) \) with \( p, \) for different number of nodes, while in Table 2 we give the maxima of \( a_4(n, \hat{p}) \) and the optimal \( \hat{r} = 1 - n \cdot \hat{p}. \)

| \( n \) | \( \hat{r} \) | \( \hat{p} \) | \( a_4(n, \hat{p}) \) |
|-------|------|------|---------------|
| 2     | 0.1778 | 0.4111 | 0.5391 |
| 3     | 0.1885 | 0.2705 | 0.3618 |
| 4     | 0.1924 | 0.2019 | 0.2720 |
| 5     | 0.1945 | 0.1611 | 0.2179 |
| 6     | 0.1960 | 0.1340 | 0.1817 |
| 7     | 0.1971 | 0.1147 | 0.1559 |
| 8     | 0.1976 | 0.1003 | 0.1364 |
| 9     | 0.1981 | 0.0891 | 0.1213 |

Figure 3: \( a_4(n, p) \) for \( n = 2 \) (blue), \ldots, 9 (black).

Table 2: \( a_4(n, \hat{p}) \) for \( n = 2, \ldots, 9. \)
We can also get an exact algebraic expression for the value \( p = \hat{p}(n) \) which maximizes the interception probability \( \left( \frac{p}{1-p} \right)^1 \), by differentiating \( \left( \frac{p}{1-p} \right)^1 \) with respect to \( p \) and setting the numerator of the resulting fraction \( f(p)/(1-p)^2 \) equal to 0, where \( f(p) \) is the fourth degree polynomial

\[
3 - (6 + 6 \cdot n) \cdot p + (3 + 9 \cdot n + 6 \cot n^2) \cdot p^2 + (-4 \cdot n - 4 \cdot n^2 + 4 \cdot n^3) \cdot p^3 + (3 \cdot n^3)^4.
\]

The real root of this polynomial which is a probability, is given by

\[
\hat{p}(n) = \frac{2 \cdot n^2 + G \sqrt{2} - 3 \cdot n^2 \sqrt{\frac{C}{B \cdot n^3}} + \frac{E}{3 \cdot n^2 \cdot D} + \frac{4 \cdot \sqrt{2} \cdot F}{D} - \frac{D}{8 \cdot n^3} + 2 \cdot n + 2}{6 \cdot n^2},
\]

where

\[
A = 8 \cdot n^6 - 18 \cdot n^5 + 6 \cdot n^4 + 9 \cdot n^3 - 3 \cdot n, \quad B = (n - 1)^6 \cdot n^3 \cdot (32 \cdot n^3 + 24 \cdot n^2 - 3 \cdot n - 4),
\]

\[
C = 8 \cdot (n^2 + n + 1)^2 - 12 \cdot n \cdot (2 \cdot n^2 + 3 \cdot n + 1), \quad F = (4 \cdot n^4 + 2 \cdot n^3 + 6 \cdot n^2 + 11 \cdot n + 4) \cdot (n - 1)^2,
\]

\[
D = \sqrt[3]{A + 2 \cdot \sqrt{B - 3 \cdot n + 1}}, \quad E = (4 \cdot n^2 - 1) \cdot (n - 1)^2, \quad G = \sqrt{C - \frac{6 \cdot n \cdot E}{D}} + 6 \cdot n \cdot D.
\]

### 2.4 Asymptotic analysis for \( m = 4 \)

We now consider the optimal play for \( m = 4 \) when \( n \) is large. Since \( p \) goes to 0, it is more transparent to work with \( r \). We start by writing \( \left( \frac{p}{1-p} \right)^1 \) in terms of \( r \), using that \( p = (1-r)/n \), and calling it

\[
\pi(n, r) = \frac{-1 + r \cdot (-1 + 2 \cdot r^2 + n \cdot (1 + 2 \cdot r - r^2 + r^3))}{n \cdot (-1 + n + r)},
\]

so that,

\[
n \cdot \pi(n, r) \to \text{poly}(r) = r^4 - 2 \cdot r^3 + 3 \cdot r^2 - r - 1, \quad \text{as} \quad n \to \infty.
\]

The first order condition on \( r \) is given by the cubic \( 4 \cdot r^3 - 6 \cdot r^2 + 6 \cdot r - 1 = 0 \), with solution in \([0, 1]\) of

\[
\hat{r}_\infty = \frac{1}{2} \left(1 - (1 + \sqrt{2})^{-1/3} + (-1 + \sqrt{2})^{1/3} \right) \approx 0.20196.
\]

We also have that

\[
a \equiv \text{poly}(\hat{r}_\infty) = -\frac{3 \cdot \left(5 - 4 \cdot \sqrt{2} - 7 \cdot (-1 + \sqrt{2})^{4/3} + (-1 + \sqrt{2})^{2/3} \cdot (-1 + 2 \cdot \sqrt{2})\right)}{16 \cdot (-1 + \sqrt{2})^{4/3}} \approx 1.0944,
\]

which implies that

\[
\pi(n, \hat{r}_\infty) \to a/n \approx 1.0944/n.
\]

To sum up, we show that for large \( n \) the Patroller should stay at the center about 20\% of the time, which ensures him an interception probability of about 1.09/n. The optimal delay remains 2.
2.5 Giving the Patroller some memory

Our original model gives the Patroller no memory. Here we allow her when at the center, to remember if she came there from another end (state \(eC\)) or remained there from the previous period (state \(CC\)). State \(e\) denotes her presence at an end other than the Attacker’s node \(E\). To simplify the analysis, we assume that when at an end she always reflects back to the center, as found in Section 2.1. From the center she now has two probabilities, denoted \(\alpha\) and \(\beta\), that she moves to a random end: the former (\(\alpha\)) from state \(CC\) and the latter (\(\beta\)) from state \(eC\). If the Patroller follows such a strategy \((\alpha, \beta)\), then conditional on not going to node \(E\), her transition probabilities are given by the matrix

\[
\begin{pmatrix}
1 - n_a & 0 & (n - 1) a \\
1 - n_b & 0 & (n - 1) b \\
0 & 1 & 0
\end{pmatrix}.
\]

For simplicity, we will focus on the case when \(n = 3\) and \(m = 2\), where the corresponding transition matrix will be denoted by \(A\). The Attacker knows that the Patroller’s distribution over the three states, conditional on the fact that she has not been seen at the chosen attack node \(E\), is given by the three tuple (with respective probabilities of being at states \(CC\), \(eC\) and \(e\))

\[
x^\kappa = (0, 1, 0) \cdot A^\kappa,
\]

after \(\kappa\) periods away from \(E\). When the Patroller leaves \(E\) she is at the center, and more specifically she is at the center having just arrived there from an end, so in state \(eC\). That is, we have that \(x^1 = (0, 1, 0)\). If the attack takes place with delay \(d\), and \(x^d = (x_1^d, x_2^d, x_3^d)\), it will be intercepted if and only if the Patroller moves to \(E\) in the next period, which has probability

\[
P(\alpha, \beta, d) = \alpha \cdot x_1^d + \beta \cdot x_2^d = x^d \cdot (\alpha, \beta, 0).
\]

Thus, the Patroller’s objective is to choose \(\alpha\) and \(\beta\) (both in the interval \([0, 1/3]\)) to maximize the minimum of \(13\) over \(d\) (chosen by the Attacker). We have that

\[
V = \max_{\alpha, \beta \in [0, 1/3]} \min_{1 \leq d \leq D} x^d \cdot (\alpha, \beta, 0).
\]

Note that \(x^d\) is also a function of \(\alpha\) and \(\beta\), though this is not indicated in our notation.

We solve (14) by first numerical and then exact algebraic methods. For our numerical work we take \(D = 10\). Unlike the similar problem in Section 2.1, it turns out that there is not a unique delay \(d\) which the Attacker can adopt without knowing \(\alpha\) and \(\beta\) (here it is either \(d = 2\) or \(d = 3\)). If the Attacker moves first, then he will have to use a mixture of \(d = 2\) and \(3\), but even then he cannot reduce the interception probability \(P\) to the maximum value \(V\). Numerical work fixes \(\alpha\) and \(\beta\), generates the sequence of interception probabilities \(x^d \cdot (\alpha, \beta, 0)\) in (14), and selects the delay \(d\) which minimizes the entry. By varying \(\alpha\) and \(\beta\) in a grid of values, we find that the optimal \(\alpha\) is near 0.3, the optimal \(\beta\) is near 0.2, and the minimizing delay \(d\) is either 2 or 3. To illustrate these ideas we give the sequences \(x^d \cdot (\alpha, \beta, 0)\) for the two strategy pairs \(\alpha = 0.30, \beta = 0.21\) and \(\alpha = 0.30, \beta = 0.22\)

\[
0.21, 0.141, 0.132^*, 0.162, 0.139, 0.149, 0.148, 0.145, 0.148, 0.147, \text{ for } (\alpha, \beta) = (0.30, 0.21),
\]

\[
0.22, 0.131^*, 0.143, 0.159, 0.142, 0.152, 0.148, 0.148, 0.149, 0.148, \text{ for } (\alpha, \beta) = (0.30, 0.22).
\]

(15)
If the Patroller adopts \((0.30, 0.21)\), then the optimal Attacker’s response is \(d = 3\) with interception probability 0.132, while if the Patroller adopts \((0.30, 0.22)\), then the optimal response is \(d = 2\) with interception probability 0.131. Thus we have a numerical function \(h(\alpha, \beta)\), which is calculated as \(h(0.30, 0.21) = 0.132\) and \(h(0.20, 0.22) = 0.131\). By carrying out such numerical work over \(\alpha\) and \(\beta\) and taking the maximum, a good approximation to the maximizing values can be found.

To get more accurate results we adopt algebraic methods, based on our numerical results for \(d\). As functions of \(\alpha, \beta\), the first three terms of the \(x^\kappa\) distribution sequence [12] are given respectively by

\[
(0, 1, 0), \quad (0, 1, 0) \cdot A = \left( \frac{1 - 3\beta}{1 - \beta}, 0, \frac{2\beta}{1 - \beta} \right), \quad \text{and}
\]

\[
\left( \frac{1 - 3\beta}{1 - \beta}, 0, \frac{2\beta}{1 - \beta} \right) A = \left( \frac{(1 - 3\alpha)(1 - 3\beta)}{1 - \alpha(1 - \beta)}, \frac{4\beta}{1 - \beta}, \frac{2\alpha(1 - 3\beta)}{(1 - \alpha)(1 - \beta)} \right).
\]

Then the first terms of the sequence \(x^\kappa \cdot (\alpha, \beta, 0)\) of interception probabilities are

\[
(0, 1, 0) \cdot (\alpha, \beta, 0) = \beta,
\]

\[
\left( \frac{1 - 3\beta}{1 - \beta}, 0, \frac{2\beta}{1 - \beta} \right) \cdot (\alpha, \beta, 0) = \frac{\alpha(1 - 3\beta)}{1 - \beta} = u_2(\alpha, \beta),
\]

\[
\left( \frac{(1 - 3\alpha)(1 - 3\beta)}{1 - \alpha(1 - \beta)}, \frac{4\beta}{1 - \beta}, \frac{2\alpha(1 - 3\beta)}{(1 - \alpha)(1 - \beta)} \right) \cdot A = \frac{\alpha(1 - 3\alpha)(1 - 3\beta)}{1 - \alpha(1 - \beta)} + \frac{2\beta^2}{1 - \beta} = u_3(\alpha, \beta),
\]

where \(u_2(\alpha, \beta)\) and \(u_3(\alpha, \beta)\) are the payoffs if the Attacker chooses \(d = 2\) or \(d = 3\) while the Patroller chooses \(\alpha\) and \(\beta\) as described above. Thus, given our knowledge from numerical work that the optimal waiting time \(d\) is always 2 or 3, we can rewrite the value problem [14] as

\[
V = \max_{\alpha, \beta \in [0, 1]} \min(u_2(\alpha, \beta), u_3(\alpha, \beta)).
\]

Solving the equation \(u_2(\alpha, \beta) = u_3(\alpha, \beta)\), we get the optimal \(\beta\) as a function of \(\alpha\)

\[
\hat{\beta}(\alpha) = -3\alpha^2 + \sqrt{4\alpha^2 - 4\alpha^3 + 9\alpha^4}
\]

\[
2(1 - \alpha)
\]

and it follows that the value of the game is given by \(\max_{\alpha} u(\alpha)\), where \(u(\alpha)\) is given by

\[
u(\alpha) = u_2(\alpha, \hat{\beta}(\alpha)) = u_3(\alpha, \hat{\beta}(\alpha)) = \frac{\alpha(2 - 2\alpha + 9\alpha^2 - 3\alpha\sqrt{4 - 4\alpha + 9\alpha^2})}{2 - 2\alpha + 3\alpha^2 - \alpha\sqrt{4 - 4\alpha + 9\alpha^2}}.
\]

The maximum of [16] on \([0, 1]\) cannot be expressed simply, but it can be approximated at \(\tilde{\alpha} \approx 0.305\), with \(\hat{\beta} = \hat{\beta}(\tilde{\alpha}) \approx 0.217\) and \(V = u(\tilde{\alpha}) \approx 0.136\). Recall that the corresponding interception probability when the Patroller has no memory was shown in Section [2.1] to be \(5 - 2\sqrt{6} \approx 0.101\). Thus a Patroller with limited memory intercepts the attack about 13% of the time compared with 10% for a Patroller with no memory, which is about a 34% increase. It is also useful to compare the optimal probabilities of staying still when at the center. For the Markovian Patroller this probability is 0.449 (\(\hat{r}\) of [10]). For the Patroller with limited memory, it is \(1 - 3\tilde{\alpha} \approx 0.0855\) if he was previously again at the center, and \(1 - 3\hat{\beta} \approx 0.349\) if he has just come there from an end.

Hence, we have the following result:
**Proposition 3.** Consider that the Patroller, when at the center, knows whether she has just arrived or was already there the previous period. Let $\alpha$ and $\beta$ denote her choice of probability to move to an end from the center, respectively, in the two cases. The Patroller maximizes the interception probability at 0.136 by choosing $\alpha$ and $\beta$ with respective probabilities approximately 0.305 and 0.136. The interception probability increases approximately 36% over what it was for a memoryless Patroller. At the optimal Patroller’s values of $\alpha$ and $\beta$, both delays $d = 2$ and $d = 3$ are optimal for the Attacker.

### 2.6 The Cost of Wearing a Uniform

In the above analysis, the Attacker uses his information about the presence and absence of the Patroller at his location in optimizing the timing of his attack. So it would seem intuitive that forcing the Patroller to wear a uniform reduces the optimal probability of intercepting the attack. However, in a distant but related multiple Patroller game of Lin (2019) it is shown that wearing a uniform does not affect this probability (value of the game). So motivated by that paper we decided to compare the two values (with and without a uniform) in our model. We find in all cases that there is indeed a cost of wearing a uniform, but that this cost varies with the parameters of our game. In Lin’s model, a Patroller who leaves the Attacker’s position will never return, so the Attacker is only vulnerable to the next Patroller, and this may in part be why there is no cost of wearing a uniform in that model.

Consider the original patrolling game where the Patroller is non-uniformed and Markovian. Assuming the attack duration is $m = 2$, in any interval of length equal to 2 she visits only one end, so the interception probability is given by $W = 1/n$. Recall that the corresponding interception probability $V$ for the Uniformed Patroller game is given by \(7\). For large $n$, taking the ratio $V/W$ (uniformed over non-uniformed) and applying twice L’Hôpital’s rule when indeterminate forms occur, we get

\[
\lim_{n \to \infty} \frac{V}{W} = \lim_{n \to \infty} \frac{\frac{2 \cdot n - 1}{n} - 2 \cdot \sqrt{n \cdot (n - 1)}}{1/n} = \lim_{n \to \infty} \frac{n \cdot (2 - 1/n - 2 \cdot \sqrt{1 - 1/n})}{n \cdot 1/n^2} = \lim_{n \to \infty} \frac{1/n^2 - (1 - 1/n)^{-1/2} \cdot 1/n^2}{-2 \cdot 1/n^3} = \lim_{n \to \infty} \frac{1/n^2 \cdot (1 - 1/n)^{-1/2}}{1/n^2 \cdot (-2 \cdot 1/n)} = \frac{1/2 \cdot (1 - 1/n)^{-3/2} \cdot 1/n^2}{2 \cdot 1/n^2} = \frac{1}{4}.
\]

**Proposition 4.** For large $n$ and attack duration $m = 2$, wearing a uniform reduces the interception probability (value) by a factor of 4.

Similar results can be obtained if we consider alternative values of $m$. For example, recall that when $m$ is odd the interception probability $V$ for the Uniformed Patroller game on the star $S_n$ is given by \(8\). Without a uniform the Patroller starts equiprobably a random walk at $t = 1$ either an an end or at the center, while the Attacker also equiprobably initiates his attack either at an odd or at an even period. That is, the interception probability in the case when $m = 2 \cdot j + 1$ is given by

\[
W = \frac{1}{2} \cdot \left(1 - \left(\frac{n-1}{n}\right)^j\right) + \frac{1}{2} \cdot \left(1 - \left(\frac{n-1}{n}\right)^{j+1}\right).
\]

We look again at the ratio of the two interception probabilities. It is easy to show that as $n$ goes to infinity $V/W$ converges to $2j/(2j + 1) = (m - 1)/m$ (the proof is similar to the one for $m = 2$).
Proposition 5. When the attack duration \( m \) is odd, the relative loss in interception probability of wearing a uniform for large \( n \) is the reciprocal of the attack duration. That is,

\[
\lim_{n \to \infty} \frac{W - V}{W} = \frac{1}{m}.
\]

3 Conclusion

This paper examined the aspects of patrolling games that change when the Attacker can detect the presence of the Patroller at his location. We call such problems Uniformed Patroller Games, as the wearing of a uniform is a metaphor for this type of detection. The Patroller’s problem does not change much from the earlier (non-uniformed) version, but the Attacker must now decide how long to wait after the Patroller has left the location where he waits to carry out his attack. We determine solutions to this game when the Patroller must defend an arbitrary number of locations and the attack lasts an arbitrary number of periods. In this model the game without a uniform is easy to solve so that we can determine the reduction in the probability of intercepting an attack when wearing a uniform. This reduction turns out to be sensitive to the parameters involved. These results contrast with the lack of any reduction in interception probability in the continuous time model of [Lin (2019)], with multiple agents. Since wearing a uniform make intercepting attacks more difficult, we plan to investigate the role the uniform plays in deterring attacks. This should add to the current lively debate on the importance of uniformed police ‘on the beat’ as opposed to undercover agents.

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