Berry’s phase in view of quantum estimation theory, and its intrinsic relation with the complex structure

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Abstract

In this paper, it is pointed out that the Berry’s phase is a good index of degree of no-commutativity of the quantum statistical model. Intrinsic relations between the ‘complex structure’ of the Hilbert space and Berry’s phase is also discussed.

Keywords: Berry’s phase quantum estimation theory, attainable Cramer-Rao type bound, complex structure, antiunitary operator

1 Introduction

Berry’s phase, convinced by many experiments, is naturally interpreted as a curvature of natural connection introduced on the line bundle over the space of pure states [1][3][20][21].

In this paper, it is shown that the Berry’s phase is a good index of non-commutativity of the quantum statistical model, and that Berry’s phase has intrinsic relation with the ‘complex structure’ of the Hilbert space.

The paper is organized as follows. Section 2 is review of quantum estimation theory and Berry’s phase. In section 3, sections 4-6, relations between Berry’s phase and quantum estimation theory are discussed. In section 4 and sections 5-sec:timeresversal, relations between Berry’s phase and the ‘complex structure’ is studied.

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2 Preliminaries

2.1 Quantum measurement theory, the unbiased estimator

We denote by $\mathcal{P}_1(H)$ the space of density operators of pure states in a separable Hilbert space $H$. $\mathcal{P}_1(H)$ is often simply denoted by $\mathcal{P}$, $\mathcal{P}_1$.

Let $\Omega$ be a space of all possible outcomes of an experiment, and $\sigma(\Omega)$ be a $\sigma$-field in $\Omega$. When the density operator of the system is $\rho$, the probability that the data $\omega \in \Omega$ lies in $B \in \sigma(\Omega)$ writes

$$\Pr\{\omega \in B|\rho\} = \text{tr} \rho M(B),$$

by use of the map $M$ from $\sigma(\Omega)$ to nonnegative Hermitian operator which satisfies

$$M(\phi) = O, M(\Omega) = I,$$

$$M\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} M(B_i) \quad (B_i \cap B_j = \phi, i \neq j),$$

so that (1) define a probability measure (see Ref. [9], p.53 and Ref. [10], p.50). We call the map $M$ the measurement, because there always exist an physical experiment corresponds to the map $M$ which satisfies (2) [22][17].

The quantum estimation theory deals with identification of the density operator of the given physical system from the data obtained by the experiment. For simplicity, we usually assume that the density operator is a member of a model, or a manifold of $\mathcal{M} = \{\rho(\theta) | \theta \in \Theta \subset \mathbb{R}^m\} \subset \mathcal{P}$, and that the finite dimensional parameter $\theta$ is to be estimated statistically. For example, $\mathcal{M}$ is the set of spin states with given wave function part and unknown spin part. In this paper, we restrict ourselves to the pure state model case, where any member of the model is pure state,

$$\rho(\theta) = \pi(|\phi(\theta)\rangle)$$

$$\equiv |\phi(\theta)\rangle \langle \phi(\theta)|.$$

To estimate the parameter, we perform the experiment and obtain the data $\omega$ by which we calculate an estimator $\hat{\theta}$ by estimator $\hat{\theta}(\omega)$. A pair
(Ω, σ(Ω), M, ˆθ) of an estimator ˆθ(*), a measurement M, a space Ω of data, a σ-field σ(Ω) is also called an estimator. The expectation of f(ω) with respect to the probability distribution P is denoted by Eθ[f(ω)|M].

The estimator (Ω, σ(Ω), M, ˆθ) is said to be unbiased if

$$E_θ[\hat{\theta}(\omega)|M] = \theta$$

(3)

holds for all θ ∈ Θ. If (3) and

$$\partial_i E_θ[\theta^i(\omega)|M] = \delta^j_i \quad (i, j = 1, ..., m),$$

where ∂i stands for ∂/∂θi, hold at a particular θ, (Ω, σ(Ω), M, ˆθ) is called locally unbiased at θ.

As a measure of the error of the locally unbiased estimator, we use the weighed sum of covariance matrix,

$$\sum_{i=1}^{m} g_i [V_θ[\hat{\theta}(\omega)|M]]_{ii},$$

(4)

where Vθ[θ(ω)|M] is the covariance matrix of the random variable θ(ω) which obeys the probability distribution P, and g; (i = 1, ..., m) are non-negative real numbers, which corresponds to the ‘cost’ caused by the wrong estimation of the true value of the i-th component of the parameter. More generally,

$$\text{Tr}GV_θ[\hat{\theta}(\omega)|M],$$

(5)

where G is a real symmetric nonnegative matrix, is also used as a measure of the error, and its infimum

$$\text{CR}(θ, G, M) = \inf\{\text{Tr}GV_θ[\hat{\theta}(\omega)|M] | (Ω, σ(Ω), M, ˆθ) is locally unbiased at θ\}$$

(6)

is called the attainable Cramer-Rao (CR) type bound, in analogy with the Cramer-Rao inequality in the classical estimation theory (throughout the paper, the term ‘classical estimation theory’ means the estimation theory of the family of probability distributions). The real symmetric nonnegative matrix G is called weight matrix. For CR(θ, G, M), abbreviated notations such as CR(θ, G) CR(G) are often used.
2.2 SLD CR inequality, the attainable CR type bound

We have the following *SLD CR inequality*, which is proved for the exact state model by Helstrom \[8,9\], and is proved for the pure state model by Fujiwara and Nagaoka \[6\]:

\[ CR(\theta, G) \geq \text{Tr} G \left( J^S(\theta) \right)^{-1}. \tag{7} \]

Here \( J^S(\theta) \), called *SLD Fisher information matrix*, is defined by

\[ J^S(\theta) \equiv [\text{Re}\langle l_i(\theta)|l_j(\theta)\rangle]. \tag{8} \]

where \(|l_i(\theta)\rangle (i = 1, ..., m)\) are defined below.

The horizontal lift \(|l_X(\theta)\rangle\) of a tangent vector \(X \in T_{\rho(\theta)}(M)\) to \(|\phi(\theta)\rangle\), is an element of \(\mathcal{H}\) which satisfies

\[ X \rho(\theta) = \pi_* (|l_X\rangle) \equiv \frac{1}{2}(|l_X\rangle\langle \phi(\theta)| + |\phi(\theta)\rangle\langle l_X|), \tag{9} \]

and

\[ \langle l_X|\phi(\theta)\rangle = 0. \tag{10} \]

Here, \(X\) in the left hand side \[(9)\) of is to be understood as a differential operator. \(|l_i(\theta)\rangle\) is defined to be a horizontal lift of \(\partial_i \in T_{\rho(\theta)}(M)\).

The uniqueness of the horizontal lift is proved easily. As for the existence, given a manifold \(\mathcal{N} = \{|\phi(\theta)\rangle, \theta \in \Theta\}\) in \(\mathcal{H}\) such that

\[ \mathcal{M} = \pi(\mathcal{N}) \equiv \{\rho(\theta) | \rho(\theta) = \pi(\theta), \theta \in \Theta\}, \tag{11} \]

the horizontal lift explicitly writes

\[ |l_X(\theta)\rangle = 2X|\phi(\theta)\rangle - 2\rho(\theta)X|\phi(\theta)\rangle. \tag{12} \]

The inequality \[(7)\) is of special interest, because of the following theorems.

**Theorem 1** *(Fujiwara and Nagaoka \[6\]*) *If the model is one dimensional, the equality in \[(7)\) establishes.*
Theorem 2 If the weight matrix is in the form of
\[ G = \text{diag}(0, \ldots, 0, g_i, 0, \ldots, 0), \] (13)
the equality in (7) establishes.

However, the bound is not always attainable\[13\].

Theorem 3 The SLD CR inequality is attainable for a strictly positive definite weight matrix, iff
\[ \text{Im}\langle l_i(\theta)|l_j(\theta)\rangle = 0 \] (14)
for any \(i, j\).

The model is said to be locally quasi-classical at \(\theta\) iff (14) holds true at \(\theta\).

In general,
\[
\inf \left\{ \sum_{i=1}^{m} g_i[V_\theta[M]]_{ii} \mid M \text{ is locally unbiased at } \theta \right\}
\geq \sum_{i=1}^{m} \inf \left\{ g_i[V_\theta[M]]_{ii} \mid M \text{ is locally unbiased at } \theta \right\}
= \sum_{i} g_i \left[ J^{S-1}(\theta) \right]_{ii},
\]
holds true, where the last equality comes from theorem 2, and if the model is not quasi-classical and the weight matrix is strictly positive, the equality in first inequality does not establish. The gap between the both sides of the inequality can be considered to be caused by the non-commutativity of quantum theory.

2.3 Berry’s phase

In this section, we review the geometrical theory of Berry’s phase.

Let us denote by \(\tilde{\mathcal{H}}\) the totality of state vectors, or member of \(\mathcal{H}\) with unit length. Because the two state vectors correspond to the same state iff they differ with each other only in their phase factor, it is natural to consider
\( \hat{\mathcal{H}} \) as a principal fiber bundle with the base space \( P_1 \) and the structure group \( U(1) \).

A horizontal lift \( \hat{C} = \{ |\phi(t)\rangle | 0 \leq t \leq 1 \} \) of the curve \( C = \{ \rho(t) | 0 \leq t \leq 1 \} \) in \( P_1 \) is defined to be a curve in \( \hat{\mathcal{H}} \) which satisfies \( \rho(t) = \pi(|\phi(t)\rangle) \) and

\[
|l_{d/dt}(t)\rangle = 2 \frac{d}{dt} |\phi(t)\rangle.
\]  

(15)

The connection introduced by this horizontal lift is called Pancharatnam connection. Then, the Berry’s phase \( \gamma(C) \) for the curve \( C = \{ \rho(t) | 0 \leq t \leq 1 \} \) is defined by

\[
|\phi(1)\rangle = e^{i\gamma(C)} |\tilde{\phi}(1)\rangle,
\]

(16)

where \( |\tilde{\phi}(1)\rangle \) satisfies \( \pi(|\tilde{\phi}(1)\rangle) = \rho(1) \) and

\[
\text{Im}\langle\phi(0)|\tilde{\phi}(1)\rangle = 0.
\]

(17)

The model is said to be parallel iff Berry’s phase for any curve in the model vanishes.

The Berry’s phase for the infinitesimal loop

\[
(\theta^1, \ldots, \theta^i, \ldots, \theta^j, \ldots, \theta^m) \leftarrow (\theta^1, \ldots, \theta^i + d\theta^i, \ldots, \theta^j + d\theta^j, \ldots, \theta^m) \downarrow \uparrow \theta = (\theta^1, \ldots, \theta^i, \ldots, \theta^j, \ldots, \theta^m) \rightarrow (\theta^1, \ldots, \theta^i + d\theta^i, \ldots, \theta^j, \ldots, \theta^m)
\]

(18)

is calculated up to the second order of \( d\theta \) as

\[
\frac{1}{2} \tilde{J}_{ij} d\theta^i d\theta^j + o(d\theta)^2,
\]

where \( \tilde{J}_{ij} \) is equal to \( \text{Im}\langle l_i|l_j\rangle \).

Mathematically,

\[
\frac{1}{2} \sum_{i,j} \tilde{J}_{ij} d\theta^i d\theta^j
\]

(19)

is an representation of the curvature form of Pancharatnam connection.
3 SLD and Fubini-Study metric

It should be noted that the SLD FIsher information matrix is deeply concerned with this fiber bundle structure. Given a curve \( C = \{ \rho(t) \} \) in \( \mathcal{P}_1 \), let us consider the minimization

\[
\min \left\{ \langle \dot{\psi}(t) | \dot{\psi}(t) \rangle \mid \pi(\psi(t)) = \rho(t) \right\},
\]

which corresponds to ‘the shortest distance’ between infinitesimally distant fibers \( \pi^{-1}(\rho(t)) \) and \( \pi^{-1}(\rho(t + dt)) \). It is pointed out that the minimum is achieved by the horizontal lift \( \hat{C} \), and

For any curve \( \hat{C}' = \{ |\phi(t)| \mid 0 \leq t \leq 1 \} \) such that \( C = \pi(\hat{C}') \), we can prove the inequality

\[
V_t[M \mid \hat{t}] \geq \frac{1}{4(\dot{\phi}'(t)|\dot{\phi}'(t))},
\]

almost in the same way as the SLD CR inequality, where \( (\Omega, \sigma(\Omega), M, \hat{t}) \) is a locally unbiased estimator of the parameter \( t \), and dot “\( \cdot \)” stands for the differentiation with respect to \( t \). Most strict inequality of this type is obtained by the minimization of the denominator of the right hand side of the inequality. It is already known that the minimum is given by horizontal lift \( \hat{C} \). Given this fact, it is easily understood that the minimum is equal to the SLD Fisher information matrix.

Therefore, SLD Fisher information \( J^S_t(t) \) of the parameter \( t \) is proportional to ‘the minimum distance’ between infinitesimally distant two fibers \( \pi^{-1}(\rho(t)) \) and \( \pi^{-1}(\rho(t + dt)) \), and the minimization corresponds to the search of the best possible bound.

Remember that the inverse of the Fisher information matrix is the attainable lower bound of the covariance matrix of the unbiased estimator when the model is 1-dimensional. This fact implies that if SLD Fisher information is smaller, it is harder to distinguish \( \rho(t) \) and \( \rho(t + dt) \). We can also say that the closer two fibers \( \pi^{-1}(\rho(t)) \) and \( \pi^{-1}(\rho(t + dt)) \) are, the harder it is to distinguish \( \rho(t) \) from \( \rho(t + dt) \).
In the following, we define inner product \( \langle \ast, \ast \rangle_\theta \) in \( T_\theta(M) \) by
\[
\langle \partial_i, \partial_j \rangle_\theta = \left[ J^S(\theta) \right]_{ij},
\]
(22)
because this metric (SLD metric, hereafter) seems to be estimation-theoretically and geometrically natural. Notice the unique existence of the horizontal lift assures that the equality (22) certainly defines a metric.

4 D-transform

We define a linear transform \( D_\theta \) in \( T_\theta(M) \) by
\[
\langle \partial_i, D_\theta(\partial_j) \rangle_\theta = \tilde{J}_{ij}(\theta).
\]
(23)

\( D_\theta \) is called D-transform. The non-zero eigenvalues of \( D_\theta \) are denoted by \( \pm i \beta_j(\theta) \), where \( \beta_j(\theta) \) is positive real number, and \( j \) runs from 1 to the half of the rank of D-transform, and \( \beta_j(\theta)s \) are sorted so that \( \beta_1 \geq \beta_2 \geq ... \).

When \( \dim M = 2 \),
\[
\beta_1(\theta) = \frac{\tilde{J}_{12}(\theta)}{\sqrt{\det J^S(\theta)}},
\]
(24)
left hand side of which is Berry’s along the curve which encloses unit area, where the unit of area is naturally induced by the SLD metric.

It is worthy of remarking that the \( D_\theta \) is a manifestation of the natural complex structure of the Hilbert space \( \mathcal{H} \). Actually, D-transform is obtained by the following procedure.

First, multiply the imaginary unit \( i \) to \( |l_X\rangle \). Since \( i|l_X\rangle \) is not a horizontal lift of any element of \( \mathcal{T}_\rho(M) \) generally, we project \( i|l_X\rangle \) to \( \text{span}_\mathbb{R}\{\{|l_1\rangle, ..., |l_m\rangle\} \) with respect to the metric \( \text{Re}\langle\ast|\ast\rangle \). \( DX \in \mathcal{T}_\rho(M) \) is defined to be a tangent vector whose horizontal lift is identical to the product of the projection.

Defining D-transform in this way, the curvature form of Pancharatnam connection is defined by the equation (23). Therefore, Berry’s phase is a manifestation of the complex structure of the Hilbert space.
5 Berry’s phase and local non-commutativity

In this section, it is pointed out that Pancharatnam curvature is a good index of non-commutativity of two components of the parameter. As is already mentioned, the difference between \( \text{CR}(G) \) and \( \text{Tr}GJS^{-1} \) is a manifestation of the non-commutativity of the quantum mechanics. The author conjectures the difference increases as \( \beta_j \)s increase, because of the following reasons.

First, in the 2-dimensional pure state model, we have the following theorem [13].

**Theorem 4** Let \( \mathcal{M} \) and \( \mathcal{M}' \) be a 2-dimensional pure state model. Then, if \( \beta_1(\theta', \mathcal{M}') \geq \beta_1(\theta, \mathcal{M}) \) and \( JS(\theta', \mathcal{M}') = JS(\theta, \mathcal{M}) \), then for any weight matrix \( G \),

\[
\text{CR}(G, \theta', \mathcal{M}') \geq \text{CR}(G, \theta, \mathcal{M}).
\]  

Second, by virtue of theorem [3], if Pancharatnam curvature vanishes, the gap between both sides of the inequality (??) vanishes.

Third, as for general multi-dimensional pure state models, we have [13],

\[
\text{CR} \left( \theta, JS(\theta), \mathcal{M} \right) = \left\{ \text{TrRe} \sqrt{I_m + iJS^{-1/2}(\theta)J(\theta)JS^{-1/2}(\theta)} \right\}^{-2}
\]
\[ \sum_j \frac{4}{1 + \sqrt{1 - |\beta_j(\theta)|^2}}. \] (26)

Notice that the transform of the parameter,

\[ \theta^i \rightarrow \theta'^i = a_i \theta^i, \]

and the change of the weight,

\[ g_i \rightarrow g'_i = a_i^2 g_i, \]

effects the attainable CR type bound in the exactly the same manner. Therefore, if every component are to be equally weighed in the sum, the weight matrix must be the one which ‘normalize’ the ‘length’ of each component. When the model is parameterized so that \( J^S \) is diagonal, the weight matrix \( J^S \) gives one of good normalizations, because \( [J^{S-1}]_{ii} \) is minimum variance of locally unbiased estimator of the submodel of \( M \) such that all the components of the parameter other than \( \theta^1 \) are fixed to some constant.

However, in the general case, the estimation theoretical meaning of \( \text{CR}(J^S) \) is hard to verify. Still, this value is geometrical in the sense that it remains invariant under any transform of the coordinate in the model \( M \). Considering the non-commutativity of the model should be invariant under the coordinate transform, \( \text{CR}(J^S) \) can be one of good measures of non-commutativity.

6 Berry’s phase and global non-commutativity

Even if the model is locally quasi-classical at any \( \theta \in \Theta \), the best locally unbiased estimator \((\Omega, \sigma(\Omega), M, \hat{\theta})\) is dependent on true value of the parameter \( \theta \). A locally quasi-classical model is said to be quasi-classical when the measurement \( M \) in the best pair \((\Omega, \sigma(\Omega), M, \hat{\theta})\) is independent of true value of the parameter, for, in this case, using the globally best measurement \( M \), the adaptation of the calculation of the estimate from the data, which is the problem of the classical estimation theory, is only left to be done to find the globally best pair \((\Omega, \sigma(\Omega), M, \hat{\theta})\).
In the faithful model case, or the case where the model is consisted of the strictly positive density operator, it is pointed out the model quasi-classical iff Uhlmann’s RPF (relative phase factor) vanishes for any curve in the model Keiji:1997a. For Uhlmann’s RPF is nothing but a generalization of Berry’s phase to the non-pure model, it is of interest to study if the analogical fact holds true for the pure state model.

For simplicity, we say that the manifold $\mathcal{N}$ in $\mathcal{P}_1$ is a horizontal lift of the model $\mathcal{M}$ if

$$\pi(\mathcal{N}) = \mathcal{M}, \quad \forall |\phi(\theta)\rangle \in \mathcal{N}, 2\frac{\partial}{\partial \theta} |\phi(\theta)\rangle = |l_i(\theta)\rangle,$$

holds true. The horizontal lift $\mathcal{N}$ exists iff $\mathcal{M}$ is locally quasi-classical at any point. The model is said to be quasi-parallel iff Berry’s for any curve is 0 or $\pi$.

**Theorem 5** If the model $\mathcal{M}$ is quasi-parallel, that model is quasi-classical.

**Proof** First, apply Schmidt’s orthonormalization to the horizontal lift $\mathcal{N}$ of $\mathcal{M}$, to obtain the orthonormal basis $B = \{|e_i\rangle | i = 1, 2, ...\}$ such that $\mathcal{N}$ is a subset of the real span of $B$. We immerse Hilbert space $\mathcal{H}$ into $L^2(\mathbb{R}, \mathbb{C})$ as

$$|\phi(\theta)\rangle = \sum_i a_i |e_i\rangle \mapsto \sum_i a_i(\theta) \psi_i(x),$$

where $\{ \psi_i(x) | i = 1, 2, ...\}$ is an orthonormal basis in $L^2(\mathbb{R}, \mathbb{C})$. Then, letting $E(dx)$ be a projection valued measure which is obtained as a spectral decomposition of the position operator, the pair $(\mathbb{R}, \sigma(\mathbb{R}), E, \hat{\theta})$ is one of the best estimators. This assertion is easily proved by calculating the Fisher information matrix of the family,

$$\left\{ p(x, \theta) = \sum_i (a_i(\theta))^2 |\psi_i(x)|^2 \right\}$$

(28)
of probability distributions.

**Example** Let $\psi(x)$ is a wave function, or an element of $L^2(\mathbb{R}^d, \mathbb{C})$, then, the model

$$
\mathcal{M}_x(\psi(x)) = \pi(N_x)
$$

$$
N_x(\psi(x)) = \left\{ |\phi(\theta)\rangle \mid |\phi(\theta)\rangle = \psi(x-\theta), \theta \in \mathbb{R}^d \right\},
$$

is said to be a (d-dimensional) position shifted model, and the (d-dimensional) momentum shifted model $\mathcal{M}_p(\psi(x))$ and the (d-dimensional) position-momentum shifted model $\mathcal{M}_{x,p}(\psi(x))$ are defined almost in the same way.

If $\psi(x)$ takes real value only and is symmetric about orgin (for example, the eigenstates of the harmonic oscillator), then the position shifted model $\mathcal{M}_x(\psi(x))$ and the momentum shifted model $\mathcal{M}_p(\psi(x))$ are quasi-parallel, and, therefore, are quasi-classical.

**Example** Denote by $J_z$ the z-component of the spin operator $J$, by $|m\rangle (m = -S, -S + 1, \ldots, S)$ the $m$-th eigenstate of $J_z$, where $S$ is the total spin, and define the model $\mathcal{M}_{J_z}(|\psi\rangle)$ by

$$
\mathcal{M}_{J_z}(|\psi\rangle) = \pi(N_{J_z}(|\psi\rangle))
$$

$$
N_{J_z}(|\psi\rangle) = \left\{ |\phi(\theta)\rangle \mid e^{-i\frac{\hbar}{\sqrt{2}} J_z |\phi(\theta)\rangle, \theta \in \mathbb{R} \right\}.
$$

If and only if $|\psi\rangle$ satisfies

$$
|\langle m|\psi\rangle| = |\langle -m|\psi\rangle| \quad (m = -S, -S + 1, \ldots, S),
$$

the model $\mathcal{M}_{J_z}(|\psi\rangle)$ is quasi-parallel and quasi-classical.

**Example** The Riemannian Geodesic with respect to the metric tensor $J^S(\theta)$ is quasi-parallel and quasi-classical.

The converse of the theorem is, however, not true, because the following counter-examples exist.

**Example** We consider the model $\mathcal{M}_x$ which is defined by

$$
\mathcal{M}_x = \pi(N_x)
$$

$$
N_x = \left\{ |\phi(\theta)\rangle \mid |\phi(\theta)\rangle = c\text{const.} \times \left( x - \theta \right)^2 e^{-(x-\theta)^2 + ig(x-\theta), \theta \in \mathbb{R} \right\},
$$
where \( g \) the function such that

\[
g(x) = \begin{cases} 
0 & (x \geq 0), \\
\alpha & (x < 0). 
\end{cases}
\]

Then, as easily checked, \( \mathcal{N}_x \) is a horizontal lift of the model \( \mathcal{M}_x \), and \( \langle \phi(\theta) | \phi(\theta') \rangle \) is not real unless \( \alpha = n\pi \) \((n = 0, 1, \ldots)\). However, SLD CR bound is uniformly attained by the measurement obtained by the spectral decomposition \( E(dx) = |x\rangle\langle x| dx \) of the position operator, where \( |x_0\rangle = \delta(x-x_0) \). as is checked by comparing SLD Fisher information of the model \( \mathcal{M}_x \) and the classical Fisher information of the probability distribution family

\[
\left\{ p(x|\theta) \left| p(x|\theta) = |\langle \phi(\theta) | x \rangle|^2, \ \theta \in \mathbb{R} \right. \right\}.
\]

Note that \( |\phi(\theta)\rangle \) is an eigenstate of the Hamiltonian

\[
H(\theta) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\hbar^2}{m} \left( 2(x-\theta)^2 + \frac{1}{(x-\theta)^2} \right),
\]

whose potential has two wells with infinite height of wall between them.

**Example** Let \( \mathcal{H} \) be \( L^2([0, 2\pi], \mathbb{C}) \), and define a one parameter model \( \mathcal{M} \) such that,

\[
\mathcal{M} = \pi(\mathcal{N}) \quad \mathcal{N} = \left\{ |\phi(\theta)\rangle \left| |\phi(\theta)\rangle = \text{const.} \times (2 - \cos \omega) e^{i\alpha(f(\omega-\theta)+\theta)}, \ (0 \leq \omega, \theta < 2\pi) \right. \right\},
\]

where \( \alpha \) is a real number and \( f \) the function defined by

\[
f(\omega-\theta) = \begin{cases} 
\omega - \theta & (\omega - \theta \geq 0), \\
\omega + 2\pi - \theta & (\omega - \theta < 0). 
\end{cases}
\]

Physically, (29) is an eigenstate of the Hamiltonian \( H \) such that,

\[
H(\theta) = -\frac{\hbar^2}{2m} \left( \frac{d}{d\omega} - i\alpha \right)^2 + \frac{A - B \cos(\omega - \theta)}{2 - \cos(\omega - \theta)},
\]

which characterize the dynamics of an electron confined to the one-dimensional ring which encircles magnetic flux \( \Phi = 2\pi ac/e \), where \( m \) is the mass of the
electron, $-e$ the charge of the electron, $c$ the velocity of light. $A$ and $B$ are the appropriately chosen constant.

It is easily checked that $\mathcal{N}$ is a horizontal lift of the model $\mathcal{M}$, and that the model $\mathcal{M}$ is not parallel unless $\alpha = n\pi$ ($n = 0, 1, ...$). However, consider the projection valued measure $E_\omega$ such that

$$E_\omega(d\omega) = |\omega\rangle\langle\omega|d\omega,$$

where $|\omega_0\rangle = \delta(\omega - \omega_0)$. Then, it is easily checked that the classical Fisher information of the probability distribution family

$$\left\{ p(\omega|\theta) \left| p(\omega|\theta) = |\langle\phi(\theta)|\omega\rangle|^2, \ 0 \leq \omega, \theta < 2\pi \right. \right\}$$

is equal to the SLD Fisher information of $\mathcal{M}$.

\section{The antiunitary operator and Time reversal symmetry}

\subsection{The antiunitary operator}

As is pointed out in the section, Berry’s phase seems to have some intrinsic relation with the ‘complex structure’. In this section, we study this point using the antiunitary operator.

The transformation $A$

$$|\tilde{a}\rangle = A|a\rangle, \ |\tilde{b}\rangle = A|b\rangle$$

is said to be antiunitary iff

$$\langle\tilde{a}|\tilde{b}\rangle = \overline{\langle a|b\rangle},$$

$$A(\alpha|a\rangle + \beta|b\rangle) = \overline{\alpha}A|a\rangle + \overline{\beta}A|b\rangle,$$

where $\overline{z}$ means complex conjugate of $z$ (see Ref. [18] p.266).

\textbf{Theorem 6} The model is quasi-parallel iff the horizontal lift of the model is invariant by some antiunitary operator.
Proof. Suppose that any member of the manifold $\mathcal{N} = \{|\phi\rangle\}$ in $\tilde{\mathcal{H}}$ is invariant by the antiunitary operator $A$, and let $|\tilde{\phi}\rangle = A|\phi\rangle, |\tilde{\phi}'\rangle = A|\phi'\rangle$. Then, we have

$$\langle \phi | \phi' \rangle = \langle \tilde{\phi}' | \tilde{\phi} \rangle = \langle \phi' | \phi \rangle \in \mathbb{R}.$$  

Conversely, if $\langle \phi | \phi' \rangle$ is real for any $|\phi\rangle, |\phi'\rangle \in \mathcal{N}$, by Schmidt’s orthonormalization, we can obtain the orthonormal basis $\mathcal{B} = \{|i\rangle | i = 1, 2, ..., d\}$ such that $\mathcal{N}$ is subset of the real span of $\mathcal{B}$, which means any member of $\mathcal{N}$ is invariant by the antiunitary operator $K_{\mathcal{B}}$, which is defined by,

$$K_{\mathcal{B}} \sum_i \alpha_i |i\rangle = \sum_i \overline{\alpha_i} |i\rangle.$$  

$\Box$

7.2 Time reversal symmetry

As an example of the antiunitary operator, we discuss time reversal operator (see Ref. [8], pp. 266-282). The time reversal operator $T$ is an antiunitary operator in $L^2(\mathbb{R}^3, \mathbb{C})$ which transforms the wave function $\psi(x) \in L^2(\mathbb{R}^3, \mathbb{C})$ as:

$$T\psi(x) = \overline{\psi(x)} = K_{\{|x\rangle\}} \psi(x).$$

The term ‘time reversal’ came from the fact that if $\psi(x, t)$ is a solution of the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V \right) \psi,$$

then $\overline{\psi(x, -t)}$ is also its solution.

The operator $T$ is sometimes called motion reversal operator, since it transforms the momentum eigenstate $e^{i\mathbf{p} \cdot \mathbf{x}/\hbar}$ corresponding to eigenvalue $\mathbf{p}$ to the eigenstate $e^{-i\mathbf{p} \cdot \mathbf{x}/\hbar}$ corresponding to eigenvalue $-\mathbf{p}$.

Define the position shifted model by

$$\mathcal{M}_x = \{\rho(\theta) | \rho(\theta) = \pi(\psi(x - x_0)), x_0 \in \mathbb{R}^3\},$$

$$\mathcal{M}_x = \{\rho(\theta) | \rho(\theta) = \pi(\psi(x - x_0)), x_0 \in \mathbb{R}^3\},$$

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and suppose that any member of the horizontal lift $\mathcal{N}_x$ of the model $\mathcal{M}_x$ has time reversal symmetry. Then, since time reversal operator $T$ is antiunitary, the model $\mathcal{M}_x$ is quasi-classical in the wider sense. The spectral decomposition of the position operator gives optimal measurement.

Now, we discuss the generalization of time reversal operator. In this paper, the antiunitary transform

$$T_\alpha : e^{i p \cdot x / \hbar} \rightarrow e^{i \alpha(p)} e^{-i p \cdot x / \hbar}$$

is also called motion reversal operator, or time reversal operator.

If the wave function $\psi(x)$ is invariant by the time reversal operator $T_\alpha$,

$$\int_{\mathbb{R}^3} \psi(x - x_0) \overline{\psi(x - x'_0)} \, dx \in \mathbb{R} \tag{30}$$

holds true for any $x_0, x'_0$, which means that the position shifted model $\mathcal{M}_x$ is quasi-parallel and, therefore, is quasi-classical.

Conversely, if (30) holds true, Fourier transform of (30) leads to

$$|\Psi(p)|^2 = |\Psi(-p)|^2,$$

where

$$\Psi(p) = \frac{1}{\sqrt{2\pi}} \int \psi(x) e^{-i p \cdot x / \hbar} \, dx.$$ 

Therefore, the wave function $\psi(x)$ is transformed to itself by the time reversal operator $T_\alpha$ such that

$$T_\alpha : e^{i p \cdot x / \hbar} \rightarrow e^{i(\beta(p) + \beta(-p))} e^{-i p \cdot x / \hbar},$$

where

$$e^{i \beta(p)} = \frac{\Psi(p)}{|\Psi(p)|}.$$ 

**Theorem 7** A position shifted model is quasi-parallel if and only if there exists the time reversal operator which transforms the wave function $\psi(x)$ to itself.
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