Full Fermion-Boson Vertex Function Derived in terms of the
Ward-Takahashi Relations in Abelian Gauge Theory

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Abstract

I present an approach to derive the full fermion-boson vertex function in four-
dimensional Abelian gauge theory in terms of a set of normal (longitudinal)
and transverse Ward-Takahashi relations for the fermion-boson and axial-
vector vertices in momentum space in the case of massless fermion. Such a
derived fermion-boson vertex function should be satisfied both perturbatively
and non-perturbatively. I show that, by an explicit computation, such a
derived full fermion-boson vertex function to one-loop order leads to the same
result as one obtained in perturbation theory.

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I. INTRODUCTION

It is understood that the interactions determine the structure and properties of any theory, and the basic interactions are described by the basic vertices. On the other hand, gauge symmetry imposes powerful constraints on the basic vertex functions of gauge theories, leading to the exact relations among Green’s functions—referred to as the Ward-Takahashi (WT) relations[1]. They play an important role in providing the consistency conditions in perturbation theory as well as in the nonperturbative study of gauge theories through the use of the Dyson-Schwinger equations (DSEs)[2,3]. But the normal WT identity for the vertex specifies only its longitudinal part, leaving the transverse part undetermined. The transverse part of the vertex has long been known to play the crucial role in ensuring multiplicative renormalizability of the propagator and even in determining the propagator [2,4,5]. Therefore, in past years much effort has been devoted to constructing the transverse part of the vertex based on an ansatz guided by perturbative constraints[2,4,5]. However, such a constructed vertex is not unique since it is not fixed by the symmetry of the system. The latter provides the key point in determining the transverse part of the vertex: Like the longitudinal part, the transverse part of the vertex should be determined also by the WT-type constraint relations called the transverse WT relations[6,7].

In this work, I present an approach to derive the full fermion-boson vertex function in four-dimensional Abelian gauge theory in terms of a set of normal and transverse WT relations for the fermion-boson and axial-vector vertices in momentum space in the case of massless fermion without any ansatz. Such a derived fermion-boson vertex function should be satisfied both perturbatively and non-perturbatively because it is determined completely from the symmetry relations. This approach was proposed but not completed yet by present author in Refs.[8,9]. This paper presents a complete description for this approach and the complete results for these transverse WT relations and the full fermion-boson vertex function, as well as a proof that such a derived full fermion-boson vertex function to one-loop order leads to the same result as one obtained in perturbation theory.
II. TRANSVERSE WARD-Takahashi Relations for the Vector and the Axial-Vector Vertex Functions

At first, let me write the complete expressions of the normal and transverse WT relations for the fermion-boson and axial-vector vertices in momentum space. The normal WT identity for the fermion-boson vertex (i.e., vector vertex) \( \Gamma^\mu_V \) in momentum space is well-known:

\[
q_\mu \Gamma^\mu_V(p_1, p_2) = S^{-1}_F(p_1) - S^{-1}_F(p_2),
\]

where \( q = p_1 - p_2 \), and \( S_F \) is the full fermion propagator. This WT identity in coordinate space is related to the divergence of the time-ordered products of the three-point Green function involving the vector current operator[1,10], while the transverse WT relation for the vector vertex is related to the curl of the time-ordered products of the three-point function involving the vector current operator[7]:

\[
\partial^\mu_x \left\langle 0 \left| T j^\nu(x) \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle - \partial^\nu_x \left\langle 0 \left| T j^\mu(x) \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle \\
= i\sigma^{\mu\nu} \left\langle 0 \left| T \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle \delta^4(x_1 - x) + i \left\langle 0 \left| T \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle \sigma^{\mu\nu} \delta^4(x_2 - x) \\
+ 2m \left\langle 0 \left| T \bar{\psi}(x) \sigma^{\mu\nu} \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle \\
+ \lim_{x' \to x} i(\partial^x_\lambda - \partial^x_\lambda') \varepsilon^{\lambda\mu\rho} \left\langle 0 \left| T \bar{\psi}(x') \gamma_\rho \gamma_\sigma U_P(x', x) \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle,
\]

where \( j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x) \), and \( \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \). The Wilson line \( U_P(x', x) = P \exp(-ig \int^x_{x'} dy^\rho A_\rho(y)) \) is introduced in order that the operator be locally gauge invariant, where \( A_\mu \) are the gauge fields. In the QED case, \( g = e \) and \( A_\rho \) are the photon fields.

Eq.(2) shows that the transverse part of the vector vertex is related to the tensor and axial-vector vertices. Therefore, to obtain complete constraint on the vector vertex, the WT relations for the axial-vector and tensor vertices are required to build as well. In the case of massless fermion, \( m = 0 \), which will be considered in the following discussions, the tensor vertex contribution disappears and so it is only needed to consider the normal and transverse WT relations for the axial-vector vertex.
The normal WT identity for the axial-vector vertex function $\Gamma_A^\mu$ in momentum space is known as[10]:

$$q_\mu \Gamma_A^\mu(p_1, p_2) = S_F^{-1}(p_1) \gamma_5 + \gamma_5 S_F^{-1}(p_2) + \frac{ig^2}{16\pi^2} F(p_1, p_2),$$  

(3)

where $F(p_1, p_2)$ denotes the contribution of the axial anomaly[11] in momentum space[9,10].

The transverse WT relation for the axial-vector vertex function can be derived with the similar procedure as that for obtaining Eq.(2), and the result is[8,9]

$$\partial_x^\mu \left\langle 0 \left| T j_5^\mu(x) \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle - \partial_x^\nu \left\langle 0 \left| T j_5^\nu(x) \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle = i \sigma^{\mu\nu} \gamma_5 \left\langle 0 \left| T \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle \delta^4(x_1 - x) - i \left\langle 0 \left| T \psi(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle \sigma^{\mu\nu} \gamma_5 \delta^4(x_2 - x)$$

$$+ \lim_{x'\to x} \left( \partial_{x'}^\nu - \partial_x^\nu \right) \varepsilon^{\lambda\mu\nu\rho} \left\langle 0 \left| T \bar{\psi}(x') \gamma_\rho U_P(x', x) \psi(x) \bar{\psi}(x_1) \bar{\psi}(x_2) \right| 0 \right\rangle,$$

(4)

where $j_5^\mu(x) = \bar{\psi}(x) \gamma^\mu \gamma_5 \psi(x)$. There is no transverse axial anomaly[12].

The transverse WT relations for the vector and the axial-vector vertex functions can be written in more clear and elegant form in momentum space by computing the Fourier transformations of Eqs.(2) and (4), which give

$$iq^\mu \Gamma_V^\mu(p_1, p_2) - iq^\nu \Gamma_V^\nu(p_1, p_2)$$

$$= S_F^{-1}(p_1) \sigma^{\mu\nu} + \sigma^{\mu\nu} S_F^{-1}(p_2)$$

$$+(p_{1\lambda} + p_{2\lambda}) \varepsilon^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2) - \int \frac{d^4k}{(2\pi)^4} 2k_\chi \varepsilon^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2; k),$$

(5)

and

$$iq^\mu \Gamma_A^\mu(p_1, p_2) - iq^\nu \Gamma_A^\nu(p_1, p_2)$$

$$= S_F^{-1}(p_1) \sigma^{\mu\nu} \gamma_5 - \sigma^{\mu\nu} \gamma_5 S_F^{-1}(p_2)$$

$$+(p_{1\lambda} + p_{2\lambda}) \varepsilon^{\lambda\mu\nu\rho} \Gamma_{V\rho}(p_1, p_2) - \int \frac{d^4k}{(2\pi)^4} 2k_\chi \varepsilon^{\lambda\mu\nu\rho} \Gamma_{V\rho}(p_1, p_2; k),$$

(6)

where the integral-terms involve $\Gamma_{A\rho}(p_1, p_2; k)$ and $\Gamma_{V\rho}(p_1, p_2; k)$, respectively, with the internal momentum $k$ of the gauge boson appearing in the Wilson line. $\Gamma_{A\rho}(p_1, p_2; k)$ and $\Gamma_{V\rho}(p_1, p_2; k)$ are defined respectively by
\begin{align*}
\int d^4 x d^4 x' d^4 x_1 d^4 x_2 e^{i[p_1 \cdot x_1 - p_2 \cdot x_2 + (p_2 - k) \cdot x - (p_1 - k) \cdot x']}(0|T \bar{\psi}(x')\gamma_\rho \gamma_5 U_P(x', x)\psi(x)\psi(x_1)\bar{\psi}(x_2)|0) \\
= (2\pi)^4 \delta^4(p_1 - p_2 - q)iS_F(p_1)\Gamma_A(\rho, p_1, p_2; k)iS_F(p_2), \\
(7)
\end{align*}

\begin{align*}
\int d^4 x d^4 x' d^4 x_1 d^4 x_2 e^{i[p_1 \cdot x_1 - p_2 \cdot x_2 + (p_2 - k) \cdot x - (p_1 - k) \cdot x']}(0|T \bar{\psi}(x')\gamma_\rho U_P(x', x)\psi(x)\psi(x_1)\bar{\psi}(x_2)|0) \\
= (2\pi)^4 \delta^4(p_1 - p_2 - q)iS_F(p_1)\Gamma_V(\rho, p_1, p_2; k)iS_F(p_2), \\
(8)
\end{align*}

where \( q = (p_1 - k) - (p_2 - k) \). Eqs.(7) and (8) show that \( \Gamma_A(\rho, p_1, p_2; k) \) and \( \Gamma_V(\rho, p_1, p_2; k) \) are the non-local axial-vector and vector vertex functions, respectively, which and hence the integral-terms are the four-point-like functions. These integral-terms, with relations (7) and (8), are essential for the transverse WT relations (5) and (6) to be satisfied both perturbatively and non-perturbatively, which were missing in Refs.[7,8]. Indeed, as shown by Ref.[13,14], the integral-term in Eq.[5] is crucial to prove the transverse WT relation (5) being satisfied to one-loop order in perturbation theory.

Eqs.(5) and (6) show that the transverse parts of the vector and the axial-vector vertex functions are coupled each other. It implies that the transverse parts of the vector and axial-vector vertex functions are not independent of each other in four-dimensional space-time.

Now there are the normal WT identities (1) and (3), which impose the constraints on longitudinal parts of the vector and the axial-vector vertices, respectively, and the transverse WT relations (5) and (6), which impose the constraints on transverse parts of these vertices. In the case of zero fermion mass, Eqs.(1), (3), (5) and (6) form formally a complete set of WT relations for the vector and the axial-vector vertices. Then the full vector and axial-vector vertex functions can be derived in terms of this set of WT relations.

*** FULL FERMION-BOSON VERTEX FUNCTION ***

Now let me derive the full fermion-boson vertex (vector vertex) function \( \Gamma_V^\mu \) by consistently solving this set of WT relations for the vector and the axial-vector vertex functions. To do this, multiplying both sides of Eqs.(5) and (6) by \( iq_\nu \), and then moving the terms proportional to \( q_\nu \Gamma_V^\nu \) and \( q_\nu \Gamma_A^\nu \) into the right-hand side of the equations, I thus have
\[ q^2 \Gamma_V^{\mu}(p_1, p_2) = q^\mu [q_\nu \Gamma_V(p_1, p_2) + i S_f^{-1}(p_1) q_\nu \sigma^{\mu \nu} + i q_\nu \sigma^{\mu \nu} S_f^{-1}(p_2) \]
\[ + i(p_{1 \lambda} + p_{2 \lambda}) q_\nu \varepsilon^{\lambda \mu \nu \rho} \Gamma_{A \rho}(p_1, p_2) - i q_\nu C_A^{\mu \nu}, \quad (9) \]

\[ q^2 \Gamma_A^{\mu}(p_1, p_2) = q^\mu [q_\nu \Gamma_A^{\nu}(p_1, p_2) + i S_f^{-1}(p_1) q_\nu \sigma^{\mu \nu} \gamma_5 - i q_\nu \sigma^{\mu \nu} \gamma_5 S_f^{-1}(p_2) \]
\[ + i(p_{1 \lambda} + p_{2 \lambda}) q_\nu \varepsilon^{\lambda \mu \nu \rho} \Gamma_{V \rho}(p_1, p_2) - i q_\nu C_V^{\mu \nu}, \quad (10) \]

where

\[ C_A^{\mu \nu} = \int \frac{d^4k}{(2\pi)^4} 2k_\lambda \varepsilon^{\lambda \mu \nu \rho} \Gamma_{A \rho}(p_1, p_2; k), \quad (11) \]

\[ C_V^{\mu \nu} = \int \frac{d^4k}{(2\pi)^4} 2k_\lambda \varepsilon^{\lambda \mu \nu \rho} \Gamma_{V \rho}(p_1, p_2; k). \quad (12) \]

Substituting Eq.(10) into Eq.(9) and using Eqs.(1) and (3), after lengthy computations, I obtain the full fermion-boson vertex function as follows:

\[ \Gamma_V^{\mu}(p_1, p_2) = \Gamma_{V(L)}^{\mu}(p_1, p_2) + \Gamma_{V(T)}^{\mu}(p_1, p_2) \quad (13) \]

with

\[ \Gamma_{V(L)}^{\mu}(p_1, p_2) = q^{-2} q^\mu [S_f^{-1}(p_2) - S_f^{-1}(p_1)] \quad (14) \]

and

\[ \Gamma_{V(T)}^{\mu}(p_1, p_2) = [q^2 + (p_1 + p_2)^2 - ((p_1 + p_2) \cdot q) q^{-2}]^{-1} \]
\[ \times \{ i S_f^{-1}(p_1) \sigma^{\mu \nu} q_\nu + i \sigma^{\mu \nu} q_\nu S_f^{-1}(p_2) \]
\[ + i[S_f^{-1}(p_1) \sigma^{\mu \lambda} - \sigma^{\mu \lambda} S_f^{-1}(p_2)](p_{1 \lambda} + p_{2 \lambda}) \]
\[ + i[S_f^{-1}(p_1) \sigma^{\lambda \nu} - \sigma^{\lambda \nu} S_f^{-1}(p_2)] q_\nu(p_{1 \lambda} + p_{2 \lambda}) q^\mu q^{-2} \]
\[ - i[S_f^{-1}(p_1) \sigma^{\mu \nu} - \sigma^{\mu \nu} S_f^{-1}(p_2)] q_\nu(p_1 + p_2) \cdot q q^{-2} \]
\[ + i[S_f^{-1}(p_1) \sigma^{\lambda \nu} + \sigma^{\lambda \nu} S_f^{-1}(p_2)] q_\nu(p_{1 \lambda} + p_{2 \lambda}) [p_1^\mu + p_2^\mu - q^\mu(p_1 + p_2) \cdot qq^{-2}] q^{-2} \]
\[ - i q_\nu C_A^{\mu \nu} + q_\nu q_\rho q^{-2}(p_{1 \lambda} + p_{2 \lambda}) \varepsilon^{\lambda \mu \nu \rho} C_V^{\rho \alpha} \]
\[ - i q_\nu(p_{1 \lambda} + p_{2 \lambda}) [p_1^\mu + p_2^\mu - q^\mu(p_1 + p_2) \cdot qq^{-2}] q^{-2} C_A^{\lambda \nu} \}. \quad (15) \]
Here the longitudinal part of the vertex, given by Eq.(14), is a natural result of the normal WT relation (1), while the transverse part of the vertex, given by Eq.(15), is derived from the transverse WT relations for the vector and axial-vector vertices, Eqs.(5) and (6). Consequently, the longitudinal vertex (14) as well as the transverse vertex (15) and then the full vertex function (13) should be satisfied both perturbatively and nonperturbatively. In next section I will show that, by an explicit computation, the fermion-boson vertex given by Eqs.(13)-(15) is satisfied indeed to one-loop order in perturbation theory.

IV. FULL FERMION-BOSON VERTEX FUNCTION TO ONE-LOOP ORDER

The full fermion-boson vertex (13) consists of the longitudinal part $\Gamma_{V(L)}^\mu$ and the transverse part $\Gamma_{V(T)}^\mu$, where $\Gamma_{V(T)}^\mu$ involves two parts of contributions from the full fermion propagator and the four-point-like functions, which may be denoted as $\Gamma_{V(T)}^{\mu(I)}$ and $\Gamma_{V(T)}^{\mu(II)}$, respectively. To prove $\Gamma_{V}^\mu$ given by Eqs.(13)-(15) being satisfied to one-loop order, it is required to calculate the contributions from the four-point-like functions. In perturbation theory, the four-point-like functions (7) and (8) and then the integral-terms (11) and (12) can be calculated in the interaction representation order by order. At one-loop order the integral-term (11) can be written from Eq.(7) straightforwardly:

$$C_{A}^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} 2k_\lambda \epsilon^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2; k)$$

$$= g^2 \int \frac{d^4k}{(2\pi)^4} 2k_\lambda \epsilon^{\lambda\mu\nu\rho} \gamma^\alpha \frac{1}{\not{p}_1 - \not{k} - m} \gamma^\rho \gamma_5 \frac{1}{\not{p}_2 - \not{k} - m} \gamma^\beta \frac{-i}{k^2} [g_{\alpha\beta} + (\xi - 1) \frac{k_\alpha k_\beta}{k^2}]$$

$$+ g^2 \int \frac{d^4k}{(2\pi)^4} 2\epsilon^{\alpha\mu\nu\rho} \gamma^\beta \frac{1}{\not{p}_1 - \not{k} - m} \gamma^\rho \gamma_5 + \gamma^\rho \gamma_5 \frac{1}{\not{p}_2 - \not{k} - m} \gamma^\beta \frac{-i}{k^2} [g_{\alpha\beta} + (\xi - 1) \frac{k_\alpha k_\beta}{k^2}],$$

(16)

where $\not{k} = \gamma_\mu k^\mu$, and $\xi$ is the covariant gauge parameter. The last two terms in the right-hand side of Eq.(16) are the one-loop self-energy contributions accompanying the vertex correction. Replacing $\gamma^\rho \gamma_5$ by $\gamma^\rho$ in Eq.(16), then the integral-term (12) at one-loop order, $C_{V}^{\mu\nu}$, can be written.

The integral-term at one-loop order given by Eq.(16) was computed in Refs.[13,14] where the transverse WT relation (5) was proved to be satisfied to one-loop order. The result is
\[ C_A^{\mu\nu} = -\Sigma(p_1)\sigma^{\mu\nu} - \sigma^{\mu\nu}\Sigma(p_2) - Q_V^{\mu\nu}, \]  

where \( \Sigma(p_i) \) ( \( i = 1, 2 \)) is the one-loop fermion self-energy and

\[
Q_V^{\mu\nu} = -\frac{i\alpha}{4\pi^3}\{\gamma\gamma_1^\dagger(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2^\dagger)\gamma^\alpha J^{(0)}(1) - \gamma_\alpha y_1^\dagger(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2^\dagger)\gamma^\lambda
+ \gamma^\lambda(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2^\dagger)\gamma^\alpha J^{(0)}(1) + \gamma_\alpha y_1^\dagger(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2^\dagger)\gamma^\gamma\gamma J^{(2)}_{\lambda\eta}
+ (\xi - 1)(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2)K^{(0)} - (p_1^2\sigma^{\mu\nu}\gamma^\lambda + p_2^2\sigma^{\mu\nu}\gamma^\lambda + y_1^\dagger\sigma^{\mu\nu}y_2^\dagger\gamma^\lambda
+ \gamma^\lambda(y_1\sigma^{\mu\nu}x_2^\dagger)J^{(1)} + \gamma^\lambda(p_1^2\sigma^{\mu\nu}y_2^\dagger + p_2^2\sigma^{\mu\nu}y_2^\dagger)\gamma^\eta I^{(2)}_{\lambda\eta})\}.
\]

Here \( \alpha = g^2/4\pi \), \( J^{(0)}, J^{(1)}_{\lambda\eta}, K^{(0)} \) and \( I^{(2)}_{\lambda\eta} \) are some integrals:

\[
J^{(0)} = \int_M d^4k \frac{1}{k^2[(p_1 - k)^2 + i\varepsilon][(p_2 - k)^2 + i\varepsilon]},
\]

\[
K^{(0)} = \int_M d^4k \frac{1}{[(p_1 - k)^2 + i\varepsilon][(p_2 - k)^2 + i\varepsilon]},
\]

\[
I^{(2)}_{\lambda\eta} = \int_M d^4k \frac{k_\lambda k_\eta}{k^4[(p_1 - k)^2 + i\varepsilon][(p_2 - k)^2 + i\varepsilon]}.
\]

Replacing 1 by \( k_\lambda \) and \( k_\lambda k_\eta \) in the numerator of Eq.(19) gives the expressions of \( J^{(1)}_{\lambda\eta} \) and \( J^{(2)}_{\lambda\eta} \), respectively. These integrals can be carried out in the cutoff regularization scheme or in the dimensional regularization scheme[4,5].

The integral-term (12) at one-loop order can be computed similarly, which gives

\[
C_V^{\mu\nu} = -\Sigma(p_1)\sigma^{\mu\nu}\gamma_5 + \sigma^{\mu\nu}\gamma_5\Sigma(p_2) - Q_A^{\mu\nu},
\]

where

\[
Q_A^{\mu\nu} = -\frac{i\alpha}{4\pi^3}\{\gamma\gamma_1^\dagger(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2^\dagger)\gamma^\alpha J^{(0)}(1) - \gamma_\alpha y_1^\dagger(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2^\dagger)\gamma^\lambda
+ \gamma^\lambda(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2^\dagger)\gamma^\alpha J^{(0)}(1) + \gamma_\alpha y_1^\dagger(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2^\dagger)\gamma^\gamma\gamma J^{(2)}_{\lambda\eta}
+ (\xi - 1)(y_1\sigma^{\mu\nu} + \sigma^{\mu\nu}y_2)K^{(0)} - (p_1^2\sigma^{\mu\nu}\gamma^\lambda + p_2^2\sigma^{\mu\nu}\gamma^\lambda + y_1^\dagger\sigma^{\mu\nu}y_2^\dagger\gamma^\lambda
- \gamma^\lambda(y_1\sigma^{\mu\nu}y_2^\dagger)J^{(1)} + \gamma^\lambda(p_1^2\sigma^{\mu\nu}y_2^\dagger + p_2^2\sigma^{\mu\nu}y_2^\dagger)\gamma^\eta I^{(2)}_{\lambda\eta})\}.
\]
Now substituting the fermion propagator to one-loop order $S_F^{-1}(p_i) = \gamma_i - \Sigma(p_i), \quad i = 1,2$, together with Eqs.(17)-(18) and Eqs.(22)-(23) into Eqs.(13)-(15), after some algebraic calculations, I obtain

$$\Gamma^\mu_V = \Gamma^\mu_{V(L)} + \Gamma^\mu_{V(T)} = \gamma^\mu + \Lambda^\mu_V,$$

(24)

where

$$\Lambda^\mu_V(p_1,p_2) = -\frac{i\alpha}{4\pi^3} \{ \gamma^\alpha\gamma^\mu\gamma^\lambda\gamma^\delta J^{(0)} + (\gamma^\alpha\gamma^\mu\gamma^\lambda\gamma^\delta J^{(1)} + \gamma^\lambda\gamma^\mu\gamma^\delta J^{(2)} + \gamma^\lambda\gamma^\mu\gamma^\delta J^{(2)} \}, (25)$$

which is the familiar expression of one-loop vector vertex function in perturbation theory [4,5].

This shows that the full fermion-boson vertex function given by Eqs.(13)-(15) to one-loop order leads to the same result as one obtained in perturbation theory. Since the four-point-like functions, $\Gamma_{\rho\rho}(p_1,p_2; k)$ and $\Gamma_{V\rho}(p_1,p_2; k)$, can be calculated order by order in perturbation theory, thus one may demonstrate that the full fermion-boson vertex function derived from symmetry relations should be satisfied order by order in perturbation theory.

**V. CONCLUSION AND REMARK**

In this paper, I have derived the full fermion-boson vertex function in four-dimensional Abelian gauge theory in terms of a set of normal and transverse WT relations for the vector and axial-vector vertices in momentum space in the case of massless fermion. The longitudinal part of the vertex is specified by the normal WT relation in terms of the fermion propagator, while the transverse part of the vertex is determined by the transverse WT relations for the vector and the axial-vector vertices. Such a derived full fermion-boson vertex function, in principle, should be satisfied both perturbatively and nonperturbatively because it is determined by symmetry relations which are satisfied perturbatively and nonperturbatively. I have shown that, by an explicit computation, such a full fermion-boson vertex is
satisfied indeed to one-loop order in perturbation theory. By the parallel procedure, the full axial-vector vertex function can be derived also. Thus this provides an approach to derive the basic vertex functions from the symmetry relations.

Notice that the transverse part of the vertex, $\Gamma_{V(T)}^\mu$, derived this way separates naturally two parts: $\Gamma_{V(T)}^{\mu\,(I)}$ expressed in terms of the full fermion propagator, and $\Gamma_{V(T)}^{\mu\,(II)}$ related to the four-point-like functions. Neglecting $\Gamma_{V(T)}^{\mu\,(II)}$ corresponds to the cutoff of the four-point-like functions. In this case, it is easy to check that $\Gamma_{V(L)}^\mu(p_1, p_2) + \Gamma_{V(T)}^{\mu\,(I)}(p_1, p_2) \to \gamma^\mu$ when the fermion propagator is taken as bare one. It shows that $\Gamma_{V(T)}^{\mu\,(I)}$ gives the leading contribution to the transverse part of the vertex. The contribution related to the four-point-like functions, $\Gamma_{V(T)}^{\mu\,(II)}$, needs to be studied further.

Additionally, I checked the transverse WT relation for the fermion-boson vertex to one-loop order in d-dimensions, with $d = 4 + \epsilon$. The result shows that this relation in d-dimensions has the same form as one, Eq.(5) with Eq.(16), given in 4-dimensions and so there is no need for an additional piece $\sim (d - 4)$ to include for this relation to hold in 4-dimensions. Thus it can be checked that the full fermion-boson vertex function expressed by Eqs.(13) to (15) holds exactly in 4-dimensions. Recently, Ref.[15] made a good comment on the potential of the transverse WT relation to determine the full fermion-boson vertex and then checked the transverse WT relation to one loop order in d-dimensions. However, the authors of Ref.[15] separated out a so-called additional piece $\sim (d - 4)$ from the integral-term by defining a modifying integral-term, which in fact does not change the original formula of the transverse WT relation to one-loop order. Thus it needs not have introduced such an additional piece. The detailed discussions will be given in a separate paper[16].

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