The geodesic rule for higher codimensional global defects

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Abstract

We generalize the geodesic rule to the case of formation of higher codimensional global defects. Relying on energetic arguments, we argue that, for such defects, the geometric structures of interest are the totally geodesic submanifolds. On the other hand, stochastic arguments lead to a diffusion equation approach, from which the geodesic rule is deduced. It turns out that the most appropriate geometric structure that one should consider is the convex hull of the values of the order parameter on the causal volumes whose collision gives rise to the defect. We explain why these two approaches lead to similar results when calculating the density of global defects by using a theorem of Cheeger and Gromoll. We present a computation of the probability of formation of strings/vortices in the case of a system, such as nematic liquid crystals, whose vacuum is $\mathbb{R}P^2$.

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1. Introduction

Topological defects [1] are static solutions of the Euler-Lagrange equations of a classical field theory, whose non-triviality is guaranteed by the homotopy type of the vacuum. The stability of a topological defect can be ascribed to the infinite amount of energy required to reduce it to the trivial configuration, whose energy is zero [1]. This highlights one of the reasons for the methodological importance of topological defects in field theories: they provide a glimpse of non-perturbative aspects of the vacuum of a model, a vacuum for which very few things are generically known.

The geodesic rule [2]-[11] is a way that allows us, in principle, to make predictions about the density of such topological defects. There seems to be a significant body of experimental [12]-[16] and numerical results [17],[18] that makes us be reasonably confident about the validity of the geodesic rule for the case of global defects. The formulation and the validity of this rule for local defects, however, is a much more contentious issue, both experimentally and numerically [19],[20]. For this reason so we will confine our analysis to just global defects in the present work. It is worth noticing that the geodesic rule is not the only known way for producing topological defects. Another way is through order parameter oscillations [21],[22],[23] whose contributions are dominant is when there is no significant dissipation during bubble collisions.

To be more concrete, we will assume that the phase transition after which a topological defect is formed is of first order and proceeds by bubble nucleation [1],[10],[11]. Then the geodesic rule states that, when two such bubbles collide, the order parameter (spacetime scalar) $\phi$ smoothly interpolates between its values in the bubbles $\phi_1$, $\phi_2$ following the shortest path (geodesic) on the vacuum manifold. There are two ways to justify the validity of geodesic rule: The first relies on minimizing the energy density of the model under consideration in the spontaneously broken phase [4],[5]. The second relies on the stochastic nature of the bubble collisions in the broken phase and the Markovian character of the bubble effective dynamics during their expansions and their collisions [24],[25]. In this paper we extend the arguments of both of these approaches to the case of formation of higher co-dimensional defects (monopoles, textures etc). We justify the reason why these two approaches lead to the same estimate for the density of global defects by using a theorem of Cheeger and Gromoll [26]. We use an integral geometric argument to determine the probability of formation of global strings/vortices
formed in a system whose vacuum is $\mathbb{R}P^2$ as was also done in an earlier work.

2. THE ENERGETIC APPROACH AND TOTALLY GEODESIC SUBMANIFOLDS

A classical, quantum or thermal field theory describing the phase transition that gives rise to topological defects, has as basic kinematic variables the order parameter(s) $\phi$, which, for simplicity, we will assume to be spacetime scalar fields in this work. Let the spacetime, on which the model is defined, be $\mathbb{R}^k$, endowed with the usual flat metric, for concreteness. The scalar fields $\phi$ are sections of a vector bundle with base $\mathbb{R}^k$ and typical fiber an, irreducible most of the times, representation of a compact, semisimple Lie group $G$, which incorporates the internal symmetries of the system. The vacuum then consists of all such maps $\phi$ that minimize an appropriate energy functional. In most cases we choose to focus our attention on the image of the space-time points under $\phi$, rather than on the set of such maps themselves, as the basic kinematic variables. Such a choice is largely unimportant however, especially within the domain of perturbation theory around a smooth $\phi$, like the ones used in the Ginzburg-Landau description of phase transitions, which we will be working with in what follows.

We assume that the vacuum of the model is an $m$-dimensional Riemannian manifold $\mathcal{M}$, endowed with a metric $\tilde{g}$. In most cases of interest, as was mentioned above, $\mathcal{M}$ is a semisimple, compact Lie group $G$, endowed with its Cartan-Killing metric $\tilde{g} = \text{Tr}(u^{-1}du)^2$, $u \in G$. Then the Cartan-Killing metric is non-degenerate and positive definite, so $(G, \tilde{g})$ is, indeed, a Riemannian manifold. The vacuum $\mathcal{M}$ is also very frequently a homogeneous space $G/H$, endowed with the induced metric from $\tilde{g}$. The energy functional of the model can be written as

$$E[\phi] = \int_{\mathbb{R}^k} \left\{ \frac{1}{2}(\partial_\alpha \phi)(\partial^\alpha \phi) + V(\phi) \right\}$$

where $\alpha = 1, 2, \ldots, k$ and $V(\phi)$ denotes the potential energy density of the model. The possible existence of a mass term $\frac{m^2}{2}\phi^2$ is assumed to be incorporated inside $V(\phi)$. Local energy extremization, under an infinitesimal energy variation $\delta E$ gives

$$\delta E[\phi] = \int_{\mathbb{R}^k} \left\{ (\partial^\alpha \phi)\delta(\partial_\alpha \phi) + \delta V(\phi) \right\}$$

All the points of the stable vacuum have the same potential energy density, so we can set $\delta V(\phi) = 0$ on $\mathcal{M}$. Then the vanishing of the first variation of the energy reduces
to

$$\partial_{\alpha} \delta \phi = 0$$  \hspace{1cm} (3)

This implies that the energy between two points of the vacuum $\phi_1$ and $\phi_2$ will be extremized when $\delta \phi$ is constant [4],[5]. This constant can be made equal to zero by an affine reparametrization. Such a result can be heuristically interpreted, as stating that the scalar field must follow a path of “stationary length”, with respect to perturbations, when it is arc-length parametrized, namely it must follow a geodesic, with endpoints $\phi_1$ and $\phi_2$. Such a geodesic is locally length minimizing [26],[27], as is also noted right after eq. (28) below. Generally, the variation $\delta$ is in the space of all maps $\phi$ that are sufficiently smooth for our purposes. However, since we have decided to focus on the image of points of $\mathbb{R}^k$ under $\phi$, rather than on $\phi$ themselves, the variations $\delta$ effectively reduce to variations on $\mathcal{M}$.

One can extend, still heuristically, this argument to the case of higher co-dimensional defects. For concreteness, let us consider the case of monopoles, which are produced by the collision of four causally disconnected bubbles with scalar field values $\phi_i, \ i = 1, \ldots, 4$. According to the geodesic rule, $\phi$ will interpolate between each pair of bubbles, along the geodesic(s) $\phi_{ij}, i, j = 1, \ldots, 4, \ i \neq j$ that join the values of $\phi$ in these bubbles, on $\mathcal{M}$. Consider a 2-simplex in $\mathcal{M}$ with vertices $\phi_i$ and edges $\phi_{ij}$. By analogy with the geodesic rule, and as we will prove in what follows, we should require the faces of this simplex to be least-area manifolds having as boundary the union of the geodesic segments $\phi_{ij}$. We can then determine the homotopy properties of the 2-simplex, and decide, in particular, whether it corresponds to a trivial element of the second homotopy group $\pi_2(\mathcal{M})$. If it does, then a monopole will not be formed, on topological grounds, from this bubble collision. Otherwise, a monopole formation is possible. By enumerating the number of simplices with trivial and non-trivial images in $\pi_2(\mathcal{M})$ with vertices uniformly distributed on $\mathcal{M}$, we can provide an estimate of the number of monopoles that will be formed in this phase transition. One can inductively extend this construction to the case of higher co-dimensional defects.

We want to derive a local analytic condition that, if possible, is satisfied by the heuristic multi-dimensional extension of the geodesic rule described in the previous paragraph. Such an extension applies to the formation of $n$-codimensional defects. The homotopy group guaranteeing the stability of such defects, when non-trivial, is $\pi_{k-n-1}$. 

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As an example of this homotopic classification, when \( k = 3 \) and \( n = 0 \) we get the condition for the stability of monopoles expressed by the non-trivial \( \pi_2 \). Similarly, we get conditions for the stability of strings/vortices, textures and other defects. Here \( n \) is the dimension of an appropriate sub-manifold \( \mathcal{N} \) properly embedded in \( \mathcal{M} \) as \( l : \mathcal{N} \to \mathcal{M} \). Although \( l \) is a proper embedding, on physical grounds, i.e. no self-intersections of \( \mathcal{N} \) should be allowed, it turns out that such a condition is not too overly restrictive. Indeed, the following computations can be appropriately generalized when \( l \) are immersions with finite self-intersections, resulting in integral \( k \)-varifolds for instance, without any major geometric modifications [27]. We allow for \( \mathcal{N} \) to have a piecewise smooth boundary \( \partial\mathcal{N} \). This boundary is composed of simplices iteratively generated by the values of \( \phi_i, i = 1, \ldots, n+2 \) in the colliding bubbles, the geodesics \( \phi_{ij}, i, j = 1, \ldots, n+2, i \neq j \) joining them, and their higher dimensional counterparts. Let \( g = l_\ast \tilde{g} \) indicate the induced metric on \( \mathcal{N} \) and \( x \in \mathcal{N} \). We choose an orthonormal basis \( e_1, \ldots, e_n, \nu_{n+1}, \ldots, \nu_m \) of \( T_x\mathcal{M} \), with \( e_i, i = 1, \ldots, n \) orthonormal in \( T_x\mathcal{N} \) and \( \nu_j, j = n+1, \ldots, m \) orthonormal in the fiber \( N_x\mathcal{N} \). Then any \( X \in T_x\mathcal{M} \) can be decomposed as \( X = X^\top + X^\bot \) where \( X^\top \in T_x\mathcal{N} \) and \( X^\bot \in N_x\mathcal{N} \). Let \( \tilde{\nabla} \) and \( \nabla \) denote the Levi-Civita connections on \( \mathcal{M} \) and \( \mathcal{N} \), respectively. Then, for \( X, Y \in T\mathcal{N} \) [28],

\[
\nabla_X Y = (\tilde{\nabla}_X Y)^\top
\]

Consider a smooth isotopy ("smooth deformation") of \( \mathcal{N} \) which keeps \( \partial\mathcal{N} \) fixed. Such an isotopy is a map

\[
F(x, t) : \mathcal{N} \times (-\epsilon, \epsilon) \to \mathcal{M}
\]

with \( \epsilon > 0 \), which a smooth enough diffeomorphism \( F(\cdot, t) : \mathcal{N} \to \mathcal{N} \), for each \( t \in (-\epsilon, \epsilon) \). The fact that such a diffeomorphism exists is guaranteed by the implicit function theorem. The vector field generating \( F(x, t) \) is

\[
F_t = \left. \frac{\partial F(x, t)}{\partial t} \right|_{t=0}
\]

and let the induced metric on \( F(\mathcal{N}, t) \) be denoted by \( g_t \). Obviously \( g_0 = g \). According the area formula [29],

\[
Vol(F(\mathcal{N}, t)) = \int_{\mathcal{N}} J(x, t) \, d^n x
\]
where \( J(x, t) \) denotes the Jacobian associated with the action of \( F_t \). This Jacobian can also be expressed in terms of the metric as \([30]\)

\[
J(x, t) = \sqrt{\det g_t}
\]

which means that

\[
\frac{d}{dt} \text{Vol}(F(N, t) \bigg|_{t=0}) = \int_N \left( \frac{d}{dt} \sqrt{\det g_t} \bigg|_{t=0} \right) \sqrt{\det g_0} \, d^n x
\]

Since \( g_t \) is a positive-definite symmetric matrix, it satisfies the identity

\[
\det g_t = \exp(tr \ln g_t)
\]

where \( tr \) denotes the trace of \( g_t \) and we get

\[
\frac{d}{dt} \sqrt{\det g_t} \bigg|_{t=0} = \frac{1}{2} \sqrt{\det g_0} \, tr \left( g_0^{-1} \frac{dg_t}{dt} \bigg|_{t=0} \right)
\]

which implies, taking into account that \( \{e_i\}, \ i = 1, \ldots, n \) is an orthonormal basis, that

\[
\frac{d}{dt} \sqrt{\det g_t} \bigg|_{t=0} = \frac{1}{2} tr \left( \frac{dg_t}{dt} \bigg|_{t=0} \right) = \frac{1}{2} \sum_{i=1}^n \frac{d}{dt} g_t(e_i, e_i) \bigg|_{t=0}
\]

where we have used that the operations of \( tr \) and differentiation of \( g_t \) commute with each other. Because \( \vec{\nabla} \) is a Levi-Civita connection, we have

\[
\frac{d}{dt} g_t(e_i, e_i) = 2\tilde{g} (\vec{\nabla}_{F_t} e_i, e_i)
\]

and because it is torsion-free, namely

\[
\vec{\nabla}_{F_t} e_i - \vec{\nabla}_{e_i} F_t = [F_t, e_i] = 0
\]

we find

\[
\frac{d}{dt} g_t(e_i, e_i) \bigg|_{t=0} = \tilde{g}(\vec{\nabla}_{e_i} F_t, e_i)
\]

which, upon using Leibniz’s rule with the Levi-Civita condition once more, gives

\[
\vec{\nabla}_{e_i} [\tilde{g}(F_t, e_i)] = \tilde{g}(\vec{\nabla}_{e_i} F_t, e_i) + \tilde{g}(F_t, \vec{\nabla}_{e_i} e_i)
\]

resulting in

\[
\frac{d}{dt} \sqrt{\det g_t} \bigg|_{t=0} = \sum_{i=1}^n \tilde{g}(\vec{\nabla}_{e_i} [\tilde{g}(F_t, e_i)] - \tilde{g}(F_t, \vec{\nabla}_{e_i} e_i)
\]
Because $\tilde{g}(F_t, e_i)$ is a scalar field and by the definition of divergence [27],[30]

$$\text{div}_\mathcal{N}X = \sum_{i=1}^{n} g(\nabla_{e_i}X, e_i), \quad \forall X \in T_x\mathcal{N} \quad (18)$$

we find that

$$\frac{d}{dt} \sqrt{\det g_t} \bigg|_{t=0} = \text{div}_\mathcal{N}F_t^\top - \sum_{i=1}^{n} \tilde{g}(F_t, \tilde{\nabla}_{e_i} e_i) \quad (19)$$

Combining Stokes’ theorem and the initial assumption that $F_t = 0$ on $\partial\mathcal{N}$, we conclude that

$$\int_{\mathcal{N}} \text{div}_\mathcal{N}F_t^\top \sqrt{\det g_0} \ d^n x = 0 \quad (20)$$

The second term on the right hand side of (19) can be re-expressed, due to linearity, as

$$\sum_{i=1}^{n} \tilde{g}(F_t, \tilde{\nabla}_{e_i} e_i) = \sum_{i=1}^{n} \tilde{g}(F_t, (\tilde{\nabla}_{e_i} e_i)^\perp) + \sum_{i=1}^{n} \tilde{g}(F_t, (\tilde{\nabla}_{e_i} e_i)^\top) \quad (21)$$

We observe that the second term reduces to

$$\sum_{i=1}^{n} \tilde{g}(F_t, (\tilde{\nabla}_{e_i} e_i)^\top) = \sum_{i=1}^{n} \tilde{g}(F_t, \nabla_{e_i} e_i) \quad (22)$$

By choosing a normal coordinate system around $x \in \mathcal{N}$, we see that $\nabla_{e_i} e_i = 0$ and, since this is a vector relation, it is true in all coordinate systems. Therefore

$$\sum_{i=1}^{n} \tilde{g}(F_t, (\tilde{\nabla}_{e_i} e_i)^\top) = 0 \quad (23)$$

The second fundamental form at $x \in \mathcal{N}$ is a vector in $N_x\mathcal{N}$ given by

$$S(X,Y) = (\tilde{\nabla}_XY)^\perp, \quad \forall X,Y \in T_x\mathcal{N} \quad (24)$$

We have used the sign convention of [28] in this definition. The trace of the second fundamental form is called the mean curvature vector $H$, so,

$$H = \frac{1}{n} \sum_{i=1}^{n} S(e_i, e_i) \quad (25)$$

Then, the first term of (21) becomes

$$\sum_{i=1}^{n} \tilde{g}(F_t, (\tilde{\nabla}_{e_i} e_i)^\perp) = n \tilde{g}(F_t, H) \quad (26)$$
so, (9) eventually gives

\[
\frac{d}{dt} \text{Vol}(F(\mathcal{N}, t)) \bigg|_{t=0} = \int_{\mathcal{N}} n \tilde{g}(F_t, H) \sqrt{\text{det} g} \, d^n x
\]  

(27)

Since the isotopy \( F(x, t) \) is arbitrary, we conclude from (27) that the multidimensional generalization of the geodesic rule that we are seeking, amounts to determining all \( \mathcal{N} \) for which

\[ H = 0, \quad \forall \ x \in \mathcal{N} \]  

(28)

Submanifolds for which \( H = 0 \) are called minimal [31],[32]. Obviously, geodesics are minimal 1-dimensional submanifolds of \( \mathcal{M} \). The word “minimal” may appear to be a misnomer, since all that we have determined are the extrema of the volume functional, which are not necessarily minima. This is indeed true globally. To determine the volume minimizing minimal submanifolds, one would have to use the second variation formula [28],[29]. Locally, however, a minimal manifold without singularities is volume minimizing [27]. To be more concrete, assume that \( x \) is not on the boundary \( \partial \mathcal{N} \). It turns out, then, that there is a small enough open set \( U \subset \mathcal{N} \), with \( x \in U \) such that

\[
\text{Vol}(\mathcal{N} \cap U) \leq \text{Vol}(F(\mathcal{N} \cap U, t))
\]  

(29)

when \( \mathcal{N} \) is minimal. It is worth pointing out that although an appropriately modified version of (27) is true for manifolds with singularities, as mentioned above, (29) does not hold in such a case [27]. An obvious class of embeddings \( l \) satisfying (27) are the ones for which \( S = 0 \), instead of the weaker \( H = 0 \). Such submanifolds \( \mathcal{N} \) are called totally geodesic [28], because they have the important property that a geodesic \( \gamma(s) \) of \( \mathcal{M} \) starting at \( x \in \mathcal{N} \) with an initial direction \( \frac{d\gamma(0)}{ds} \in T_x \mathcal{N} \) always stays in \( \mathcal{N} \). The order parameter of the broken phase in the colliding bubbles and the minimal geodesics joining them should be entirely inside \( \mathcal{N} \). In any other case, there is at least a part of a geodesic segment which is in the complement of \( \mathcal{N} \), thus failing to extremize the energy (1). Such cases are ruled out by having requiring the extremization (2). This argument demonstrates that we must choose as physically relevant only the subset of minimal submanifolds that are totally geodesic. These are the higher dimensional generalizations of the geodesic rule that we were seeking. Obviously the geodesics themselves fulfill this requirement.

3. Stochasticity, convexity and equivalence of results
In previous work [24],[25] one of us presented an argument leading to the geodesic rule on stochastic grounds. The argument relied, in an essential way, on the existence of two widely distinct time scales, one relatively short for the bubble collisions $\tau_C$, and a much longer one for the bubble coalescence $\tau_M$. With the additional assumptions of the Markovian character of the bubble dynamics and the almost isotropy of the vacuum $\mathcal{M}$, a diffusion equation governing the evolution of the order parameter inside the bubbles in the low-temperature phase was derived. The short “time” asymptotics of the corresponding solution gave rise to the geodesic rule [24],[25].

There is no obvious a priori way to generalize this result and determine the higher dimensional analogues of the geodesics, call them $Q$, in the stochastic approach. However we can proceed inductively as follows: Let’s consider the value of the order parameter $\phi_i, i = 1, 2, 3$ inside three bubbles that collide to potentially give rise to a string-like defect. Assume that these three values are within the injectivity radii of each other so that the geodesic rule as argued in [24] holds. Let $\phi_{ij}, i, j = 1, 2, 3, i \neq j$ denote the minimal geodesics joining the bubbles whose order parameter is $\phi_i$ and $\phi_j$. Due to the proximity of $\phi_i$ and $\phi_j$ there is just one such segment (minimal geodesic) for each pair $ij, i \neq j$. Let $p \in \phi_{ij}$ and $q \in \phi_{ik}$ with $i \neq k$. Then, the unique geodesic $\phi_{pq}$ must be contained in the set $Q$, due to continuity. Continuity also demands all pairs of interior points of $Q$ be connected by segments. All the points of such segments must belong to $Q$. In Euclidean space we would call a set $Q$ satisfying such a requirement “convex”. We must be careful, however, since there are several inequivalent concepts of convexity in the Riemannian context, all of which coincide in the Euclidean case [23],[25],[27]. To state the problem: what we are seeking is a set $Q \subset \mathcal{M}$, such that for any two $p, q \in Q$ there exists a segment $\phi_{pq} \subset Q$ such that $\phi_{pq}$ is the unique segment connecting $p$ and $q$ in $\mathcal{M}$. The set $Q$ satisfying this condition is called “convex” even in the Riemannian setting [26],[28],[30]. The geometric structure $Q$ we wanted to determine, therefore, is the convex hull of $\phi_i, i = 1, 2, 3$ in $\mathcal{M}$. We see that this result can be inductively generalized for all higher dimensional topological defects, without any further difficulty.

The class of classical vacua $\mathcal{M}$ usually employed satisfies one further condition. As was pointed out above, $\mathcal{M}$ is very frequently a Lie group $(G, \tilde{g})$, where $\tilde{g}$ is the Cartan-Killing metric or one of its homogeneous spaces $G/H$ with the induced metric. For such cases more can be stated. Indeed, let $W, Z, V \in TG$ be left-
invariant vector fields on $G$ and $[W, Z]$ be their commutator. If $|W \wedge Z| = \tilde{g}(W, W)\tilde{g}(Z, Z) - [\tilde{g}(W, Z)]^2$ and the Riemann tensor is

$$R(W, Z)V = \nabla_W \nabla_Z V - \nabla_Z \nabla_W V - \nabla_{[W, Z]} V$$

(30)

it turns out that the sectional curvature

$$K(W, Z) = \frac{\tilde{g}(R(W, Z)Z, W)}{|W \wedge Z|}$$

can be computed to be [28]

$$K(W, Z) = \frac{\tilde{g}([W, Z], [W, Z])}{4|W \wedge Z|}$$

(31)

Therefore any Lie group $(G, \tilde{g})$ has non-negative sectional curvature. The same is true for its homogeneous spaces $G/H$ with their induced metrics. To see that, it suffices to notice that the principal fibration $H \to G \xrightarrow{\pi} G/H$, with $\pi$ being the natural projection, is a Riemannian submersion. Then, a formula [33] gives

$$K(\pi_*W, \pi_*Z) = K(W, Z) + \frac{3}{4} \tilde{g}([W, Z]^\perp, [W, Z]^\perp)$$

(32)

where $W, Z$ are in addition assumed to be orthonormal and, in accordance with the notation of Section 2, $W^\perp$ denotes the normal component (fiber tangential) of $W \in TG$. Therefore we are interested in the convexity properties of subsets $Q$ of vacua $\mathcal{M}$ which are compact manifolds of non-negative sectional curvature. Consequently the Ricci curvature along a particular direction of $T\mathcal{M}$, which is the average of the sectional curvatures along all two-planes containing this direction [28],[30], is strictly positive. The most important point for the stochastic approach isn’t so much that the Ricci curvature on $\mathcal{M}$ is positive, but that it has a lower bound. Such a lower bound guarantees that the diffusion equation that is used to establish the geodesic rule on $\mathcal{M}$ has several “reasonable”, from a physical viewpoint, properties such as the total conservation of heat (stochastic completeness), uniqueness of the associated heat kernel etc [34]. Such properties can be expected to be true on general physical grounds even if the underlying smooth structure of $\mathcal{M}$ were to be abandoned, a fact toward which the quantum/thermal cases seem to be pointing, as will be explained in the next Section.

It is of some interest to generically ascertain that such convex $Q$, as predicted by the geodesic rule, exist for physically relevant vacua $\mathcal{M}$. If this were not true, then
topological defects would not form even when they should have, despite the homotopic arguments and the experimental and numerical evidence to the contrary. Then the stochastic approach, as we have developed it, would be invalid. As a crude first attempt, one can at least try to guarantee the existence of convex subsets of $\mathcal{M}$ by estimating their maximum “size”. An upper bound for the linear dimensions of such a convex subset is provided by the convexity radius $\text{Conv}(\mathcal{M})$. It is defined for $x \in \mathcal{M}$ by

$$\text{Conv}_x = \sup \{ \rho : B_x(r) \text{ is convex for all } r < \rho \}$$  \hspace{1cm} (33)

where $B_x(r)$ stands for the solid ball centered at $x$ of radius $r$. Then it can be proved that [35]

$$\text{Conv}(\mathcal{M}) \leq \frac{\text{inj}(\mathcal{M})}{2}$$  \hspace{1cm} (34)

where $\text{inj}(\mathcal{M})$ indicates the injectivity radius, which is the minimum distance around any $x \in \mathcal{M}$ for which the exponential map is a diffeomorphism. There are various upper and lower bounds for the injectivity radius of a manifold, depending on conditions that the curvature, volume and other geometric characteristics of the manifold satisfy, bounds that we omit since are not explicitly needed in the sequel [36]. It is encouraging that the convexity radius is not identically equal to zero, although this may happen for some $x \in \mathcal{M}$. It would be useful to be able to find a lower bound estimate for the convexity radius. Such an estimate, however, is more difficult to obtain [36], but careful examination shows that this is not as serious a problem as it might appear at a first glance. Indeed, the existence of convex $\mathcal{Q}$ is guaranteed and explicitly provided by the interior construction of Cheeger and Gromoll [26] that leads to the proof of their structure theorem mentioned in the next paragraph.

As we have seen, the generalizations of the geodesic rule resulting from the energetic and the stochastic approaches are embedded totally geodesic submanifolds and convex subsets of $\mathcal{M}$, respectively. The question that naturally arises is whether these two classes of objects give rise to the same prediction about the density of topological defects. We expect that they should, since they describe the same physical phenomenon from two different viewpoints. This expectation turns out to be correct, as proved by Cheeger and Gromoll in the fundamental [26]. In that work, it was demonstrated that if $\mathcal{Q}$ is a closed connected convex subset of a Riemannian manifold $\mathcal{M}$, then $\mathcal{Q}$ has the structure of an embedded $n$-dimensional submanifold of $\mathcal{M}$ with a smooth totally geodesic interior and possibly non-smooth boundary $\partial \mathcal{Q}$. Comparing the properties
of $\mathcal{N}$ in Section 2 and of $\mathcal{Q}$ of the present Section, we see that the Cheeger-Gromoll theorem implies that for each $\mathcal{Q}$ there is exactly one admissible $\mathcal{N}$. Therefore, the predictions about the density of topological defects derived by the energetic and by the stochastic arguments coincide, as they should. The proof of the Cheeger-Gromoll theorem is attained in several steps and it is quite geometric. The methods employed as well as the constructions leading to the result, although highly transparent, appear to have only limited utility for physical purposes, so we skip them and refer to the original [26] for further details.

4. Further comments and a sample calculation

In this Section we are making three mutually loosely connected comments, related to the topics discussed above. First, we comment on the time scale in which the geodesic rule was established. As mentioned in Section 3, the stochastic derivation of the geodesic rule relied on the assumption that $\tau_M \gg \tau_C$ [24],[25]. In the case $\tau_M \sim \tau_C$, the strong mixing of the order parameters of the colliding bubbles suppresses the number of topological defects that can be formed. This can occur, for instance, when there is a high rate of bubble nucleation. To enforce this suppression explicitly, we can introduce as a regulator a “mass” parameter $M$ in the diffusion equation that established the geodesic rule. Then the diffusion equation would become

$$\frac{\partial \phi}{\partial t} = (D + M^2)\nabla^2 \phi$$

where $D$ is an effective diffusion constant characterizing the dynamics of $\phi$ in the broken symmetry phase carried by the colliding bubbles. The exact form of such a regulating mass term $M$ should, ideally, be derived from the dynamics of bubble collisions. However it is sufficient for our purposes to know that $M \sim 0$ for $t \sim \tau_C$ and that $M \to \infty$ for $t \sim \tau_M$. This behavior can easily be enforced by the “time”-dependence

$$M = M_o t^\sigma$$

where $M_o > 0$, $\sigma > 0$ are constants. The effect of the addition of $M$ in to the diffusion equation can be clearly seen in the, simple and instructive, case of $\mathcal{M} = \mathbb{R}^m$. Then the “massive” heat kernel $K_M(\phi_1, \phi_2; t)$ gets expressed in terms of the “massless” heat kernel $K_0(\phi_1, \phi_2; t)$ by

$$K_M(\phi_1, \phi_2; t) = K_0(\phi_1, \phi_2; t) e^{-tM^2D}$$
Combining (36) and (37), we observe that the presence of $M$ suppresses super-exponentially the contributions of the large $t$ to the heat kernel, which is the desired result. The basic role of (37) rests in the fact that the “massive” heat kernel $K_M(\phi_1, \phi_2; t)_M$ for any generic vacuum manifold $\mathcal{M}$ is expressed as an asymptotic expansion of $K_M(\phi_1, \phi_2; t)_{\mathbb{R}^k}$ in terms of $t$ through [34]

$$K_M(\phi_1, \phi_2; t)_M = K_M(\phi_1, \phi_2; t)_{\mathbb{R}^k} \left\{ 1 + t a_2(\phi_1, \phi_2) + t^2 a_4(\phi_1, \phi_2) + \ldots \right\}$$  \hspace{1cm} (38)

Here $a_2, a_4, \ldots$ are polynomials of the Riemann tensor of $\mathcal{M}$, its contractions, its covariant derivatives, and subsequent combinations of them [37]. We see that, due to (38) the super-exponential decay of the large $t$, contributions still persist for any $\mathcal{M}$ as in the case of $\mathbb{R}^m$. The effect of suppressing the large $t$ contribution through $M$ can also be seen at the level of the scalar propagator on $\mathcal{M}$ which is [37]

$$Q^{-1}(\phi_1, \phi_2)_M = \int_0^\infty K_M(\phi_1, \phi_2; t)_M \, dt$$  \hspace{1cm} (39)

which when combined with (37) gives

$$Q^{-1}(\phi_1, \phi_2)_M = \int_0^\infty K_0(\phi_1, \phi_2; t)_M \, e^{-t M^2 \mathcal{D}} \, dt$$  \hspace{1cm} (40)

Second, we have to notice that everything we have mentioned so far applies for classical or zero-temperature field theories. But it is far more realistic, and thus desirable, for this formulation to be applicable when quantum or thermal contributions are taken into account. Although these two classes of contributions are quite distinct physically, they are handled by very similar methods, within the regime of perturbation theory. For this reason from now on, we will refer only to quantum corrections. Quantum corrections can be taken into account by replacing the potential energy density $V$ by the effective potential $V_{\text{eff}}$ [38] in (1). To illustrate our points we rely, for simplicity, on one of the most studied models, the $\phi^4$ model, whose classical Lagrangian is, for $\alpha = 1, 2, \ldots, k$

$$L(\phi) = \frac{1}{2} (\partial^\alpha \phi)(\partial_\alpha \phi) + \frac{1}{2} \tilde{m}^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$  \hspace{1cm} (41)

Its quantum effective potential $V_{\text{eff}}$ has a loop expansion given by

$$V_{\text{eff}} = V_d + \hbar V_1 + \hbar^2 V_2 + \ldots$$  \hspace{1cm} (42)

where $V_d = \frac{\tilde{m}^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4$. The one-loop corrections to $V_{\text{eff}}$ receive contributions from the following two momentum integrals, for $\epsilon > 0$

$$\frac{\lambda \phi^2}{4!} \int \frac{d^k p}{(2\pi)^k} \frac{i}{p^2 - \tilde{m}^2 + i\epsilon} \quad \text{and} \quad \frac{i}{4} \left( \frac{\lambda \phi^2}{2} \right)^2 \int \frac{d^k p}{(2\pi)^k} \frac{1}{(p^2 - \tilde{m}^2 + i\epsilon)^2}$$  \hspace{1cm} (43)
for the propagator and the four-vertex diagram, respectively [38]. These integrals show explicitly how the dimension $k$ of spacetime $\mathbb{R}^k$ enters the effective potential. Since the vacuum minimizes $V_{\text{eff}}$, the quantum corrections, unlike the classical potential, bring an explicit dependence of $V_{\text{eff}}$ on $k$. This behavior is also in accordance with the experimental data: the density of topological defects is observed to depend on the spacetime dimension [1]. In the physically important case $k = 4$ if we carry out explicitly these integrals we find that [38]

$$V_{\text{eff}} = \frac{\tilde{m}^2 \phi^2}{2} + \frac{\lambda \phi^4}{4!} + \frac{1}{(8\pi)^2} \left\{ \left( \frac{\lambda \phi^2}{2} + \tilde{m}^2 \right)^2 \ln \left( 1 + \frac{\lambda \phi^2}{2\tilde{m}^2} \right) - \frac{\lambda \phi^2}{2} \left( \frac{3\lambda \phi^2}{4} + \tilde{m}^2 \right) \right\}$$

(44)

We observe that this expression has a divergence for $\tilde{m} = 0$. This is due to the ultraviolet subtractions in the one loop contribution, which were set up to enforce the condition $\frac{d^4V_{\text{eff}}}{d\phi^4}|_{\phi=0} = 0$. In case we want to extrapolate the calculation of $V_{\text{eff}}$ to $\tilde{m} = 0$, we choose a new renormalization scale $\phi = \tilde{M}$ and a renormalized coupling constant $\lambda_{\tilde{M}}$ at that scale and we obtain the familiar Coleman-Weinberg expression [39]

$$V_{\text{eff}} = \lambda_{\tilde{M}} \frac{\phi^4}{4!} + \frac{\lambda_{\tilde{M}} \phi^4}{(16\pi)^2} \left( \ln \frac{\phi^2}{\tilde{M}^2} - \frac{25}{6} \right) + \ldots$$

(45)

We observe that $V_{\text{eff}}$ has a singularity when $\ln \frac{\phi^2}{\tilde{M}^2} = \frac{25}{6}$. It is very typical for $V_{\text{eff}}$ to have such singularities. Upon minimization of $V_{\text{eff}}$ to determine the vacuum, such singularities will persist, will become more numerous and get more complicated as the order of the loop expansion increases [38]. As a result, the vacuum $\mathcal{M}$ will inherit such singularities, so when the quantum corrections are taken into account we can no longer assume $\mathcal{M}$ to be a manifold. Although the exact geometric role of such singularities is unknown, it is clear that the underlying differentiable structure of $\mathcal{M}$ loses its smoothness on them. Since higher orders of perturbation theory keep bringing more and more such singularities in the determination of $\mathcal{M}$, we wonder whether it makes any sense to speak about a smooth structure of $\mathcal{M}$ at all! One way out of such a difficulty would be to excise, by hand, all the singularities of $\mathcal{M}$. This is an iterative and ad hoc process, however, so it is less than satisfying, theoretically. Another approach, would be to enlarge the structure that $\mathcal{M}$ can be allowed to have, from that of a manifold to that of a metric space. The advantage of this enlargement is that “mild” singularities are already incorporated in the formalism of metric spaces [27],[29]. Moreover, there is no a priori requirement of smoothness of such a $\mathcal{M}$. In this context, it appears that the metric and measure structures of
which now, unlike the Riemannian case are disjoint, will keep existing with appropriate modifications, despite the presence of singularities. Therefore, when quantum corrections are taken into account we may have to abandon the Riemannian structure of $\mathcal{M}$ in favor of a metric-measure space structure [40],[42]. In such a case one can still define a diffusion equation like the one that gave rise to the geodesic rule, even without any smoothness assumptions [41],[42]. We find it very unlikely that in such a case the vacuum will maintain its positive sectional curvature (now defined in the sense of Alexandrov-Toponogov) [42], as in the classical/non-thermal case. However, we expect that relative volume increments (as defined through a Hausdorff measure) of $\mathcal{M}$ would have an upper bound, a fact which, in the Riemannian case, corresponds to a lower bound on the Ricci curvature [30]. Lower Ricci curvature bounds are necessary for the stochastic completeness of the diffusion equation in the Riemannian case [34] and this may still turn out to be true for the metric-measure spaces of interest [42].

Last, we provide a sample of an analytic computation regarding the probability of formation of vortices (string-like defects) in a system whose vacuum is $\mathbb{R}P^2$. An example of such a system is furnished by nematic liquid crystals. This calculation can also be found in [43]. We view $\mathbb{R}P^2$ as the unit sphere $S^2$ with opposite points identified. Such a vacuum can arise as a result of the symmetry breaking pattern $SO(3) \to SO(2) \times \mathbb{Z}_2$, for instance. To begin with, it is possible to have vortex formation in this case, since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$. To compute this probability, we have to assume that a vortex is formed from the collision of three bubbles, the order parameters of which $\phi_i$, $i = 1, 2, 3$ are independent random variables uniformly distributed, with respect to the Lebesgue measure, namely the area, on $\mathbb{R}P^2$. Without loss of generality, we can take $\phi_1$ located at the north pole of $S^2$, which is the double covering of $\mathbb{R}P^2$.

The probability we are seeking places the second point $\phi_2$ at an angular distance of at least $\pi/2$ from $\phi_1$, since we are interested in a vortex formation. This probability is given by the ratio of the area of the spherical cap around the north pole ($\phi_1$) of angular radius $\theta$ over the area of the northern hemisphere. We consider the northern hemisphere since a value of $\phi_2$ inside it does not give rise to a vortex, according to the geodesic rule and what follows. Such an area ratio equals $\frac{2\theta}{2\pi} = \frac{\theta}{\pi}$, for $0 \leq \theta < \pi$. Then $\phi_3$ can be located uniformly anywhere in the hemisphere whose equator is the great circle joining $\phi_1$ with $\phi_2$ since $\phi_3 \in \mathbb{R}P^2$ instead of $S^2$. By using spherical
coordinates, the probability of formation of a vortex is given by

\[
\int_{0}^{\pi} \frac{\theta}{\pi} \sin \theta \, d\theta = \frac{1}{\pi} \tag{46}
\]

This type of integral geometric computation is possible analytically only in very limited cases, in which the geometry is so simple as to allow a straightforward enough parametrization of \( M \) and subsequent explicit integrations. In all other cases, such as almost all computations of the density of higher codimensional defects, a numerical estimate seems to be the only feasible way to reach any results.

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