Regularity of edge ideal of a graph

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Abstract
In this paper, we introduce some reduction processes on graphs which preserve the regularity of related edge ideals. As a consequence, an alternative proof for the theorem of R. Fröberg on linearity of resolution of edge ideal of graphs is given.

1 Introduction and Preliminaries

Throughout this paper, we assume that $G$ is a simple finite graph on vertex set $[n] = \{1, \ldots, n\}$. A graph $G$ is called chordal, if every induced cyclic subgraph of $G$ has length 3. A vertex $v$ of a graph $G$ is simplicial, if the neighborhood of $v$ in $G$ is a complete subgraph. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field $K$ with standard grading. The edge ideal of $G$ is defined by

$$I(G) = (x_ix_j : \{i, j\} \text{ is an edge in } G) \subset S.$$

Let $I \neq 0$ be a homogeneous ideal of $S$ and $\mathbb{N}$ be the set of non-negative integers. For every $i \in \mathbb{N}$, one defines:

$$t_i^S(I) = \max\{j : \beta_{i,j}^S(I) \neq 0\}$$

where $\beta_{i,j}^S(I)$ is the $i, j$-th graded Betti number of $I$ as an $S$-module. The Castelnuovo–Mumford regularity of $I$ is given by:

$$\text{reg } (I) = \sup\{t_i^S(I) - i : i \in \mathbb{Z}\}.$$

We say that the ideal $I$ has a $d$-linear resolution, if $I$ is generated by homogeneous polynomials of degree $d$ and $\beta_{i,j}^S(I) = 0$, for all $j \neq i + d$ and $i \geq 0$. For
an ideal which has a $d$-linear resolution, the Castelnuovo–Mumford regularity would be $d$.

Recently, several mathematicians have studied the regularity of edge ideals of graphs. Kummini in [7] has computed the Castelnuovo–Mumford regularity of Cohen–Macaulay bipartite graphs and Van Tuyl in [14] has generalized it for sequentially Cohen–Macaulay bipartite graphs. In [9] the regularity was computed for very well-covered graphs, in [10], some bounds were obtained for the regularity of edge ideals of vertex decomposable and shellable graphs and in [15], the Castelnuovo–Mumford regularity was calculated for edge ideals of several other classes of graphs. Also [12] has studied the topology of the lcm-lattice of edge ideals and derived upper bounds on the Castelnuovo–Mumford regularity of the ideals.

The Alexander dual of a square-free monomial ideals, plays an essential role in combinatorics and commutative algebra. For a square-free monomial ideal $I = (M_1, \ldots, M_q) \subset S = K[x_1, \ldots, x_n]$, the Alexander dual of $I$, denoted by $I^\vee$, is defined to be:

\[ I^\vee = P_{M_1} \cap \cdots \cap P_{M_q} \]

where, $P_{M_i}$ is prime ideal generated by $\{x_j : x_j | M_i\}$.

We begin with a well-known result of Eagon and Reiner and its generalization by Terai concerning the relation of the regularity of a square-free monomial ideal and the Cohen-Macaulayness of its Alexander dual. For a complete discussion of this fact, one can refer to [13].

**Theorem 1.1** (Eagon-Reiner theorem [2, Theorem 3]). Let $I$ be a square-free monomial ideal in $S = K[x_1, \ldots, x_n]$. The ideal $I$ has a $q$-linear resolution if and only if $S/I^\vee$ is Cohen-Macaulay of dimension $n - q$.

**Theorem 1.2** ([13, Theorem 2.1]). Let $I$ be a square-free monomial ideal in $S = K[x_1, \ldots, x_n]$ with $\dim S/I \leq n - 2$. Then,

\[ \dim \frac{S}{I^\vee} - \depth \frac{S}{I^\vee} = \reg (I) - \indeg (I). \]

Here, $\indeg (I)$ indicates the initial degree of $I$. That is, the minimal degree of a minimal generator of $I$.

The following lemma was proved in [11].

**Lemma 1.3.** Let $I, I_1$ and $T$ be ideals in a commutative Noetherian local ring $(R, \mathfrak{m})$ such that, $I = I_1 + T$ and

\[ r := \depth \frac{R}{I_1 \cap T} \leq \depth \frac{R}{T}. \]
Then, for all $i < r - 1$ one has:

$$H^i_m \left( \frac{R}{I_1} \right) \cong H^i_m \left( \frac{R}{I} \right).$$

**Remark 1.4.** Let $I, J$ be square-free monomial ideals generated by elements of degree $d \geq 2$ in $S = K[x_1, \ldots, x_n]$. By Theorem 1.2, we have

$$\text{reg} (I) = n - \text{depth} \frac{S}{I^\vee}, \quad \text{reg} (J) = n - \text{depth} \frac{S}{J^\vee}.$$ 

Therefore, $\text{reg} (I) = \text{reg} (J)$ if and only if $\text{depth} S/I^\vee = \text{depth} S/J^\vee$.

For a graph $G$, let $\bar{G}$ denotes the complement of graph $G$. That is, $V(\bar{G}) = V(G)$ and

$$E(\bar{G}) = \{ \{i, j\} : \{i, j\} \notin E(G) \}.$$ 

Frequently in this paper, we take a graph $G$ and we let $I = I(\bar{G})$ be the edge ideal of graph $\bar{G}$. The following proposition was proved in [6, Proposition 4.1.1].

**Proposition 1.5.** If $H$ is an induced subgraph of $G$ on a subset of the vertices of $G$, then:

$$\beta^S_{i,j} \left( I(\bar{H}) \right) \leq \beta^S_{i,j} \left( I(\bar{G}) \right)$$

for all $i, j$.

**Corollary 1.6.** Let $G$ be a graph and $H$ an induced subgraph of $G$. If $I(\bar{H})$ does not have linear resolution, then the ideal $I(\bar{G})$ does not have linear resolution.

## 2 Reduction processes on graphs

In this section we introduce some reduction processes on vertices and edges of a graph which preserve the regularity of the edge ideal of the complement of the graph.

In the following, for convenience we use this notation:

$$x = x_1, \ldots, x_n, \quad z = z_1, \ldots, z_r, \quad y = y_1, \ldots, y_m.$$ 

Also for a subset $F \subset [n]$, we set $x_F = \prod_{i \in F} x_i$ and $P_F = (x_i : i \in F).$
Lemma 2.1. Let $S = K[x, y]$ be the polynomial ring and $I$ be an ideal in $K[y]$. Then,

$$\text{depth } \frac{S}{(x_1 \cdots x_n)IS} = \text{depth } \frac{SI}{IS}.$$ 

Lemma 2.2. Let $I \neq 0$ be square-free monomial ideal in $K[x, z]$ and $J$ be the ideal

$$J = I + (x_i y_j : 1 \leq i \leq n, 1 \leq j \leq m) \subset S := K[x, y, z].$$

Then, we have the followings:

(i) $J^\vee = I^\vee \cap (x_{[n]} y_{[m]}).$

(ii) If $z_i z_j \notin I$ for all $1 \leq i < j \leq r$, then $\text{reg } (I) = \text{reg } (J)$.

Proof. (i) This is an easy computation.

(ii) By Remark 1.4, it is enough to show that, $\text{depth } S/I^\vee = \text{depth } S/J^\vee$. We know that $I^\vee$ is intersection of prime ideals $P_F$, such that:

$$|F| = 2, \quad G(P_F) \subset \{x, z\} \quad \text{and} \quad x_F \in I.$$

Since $z_i z_j \notin I$, for all $1 \leq i < j \leq r$, it follows that $P \not\in \{z\}$, for all $P \in \text{Ass } (I)$. Hence $x_{[n]} \in P$, for all $P \in \text{Ass } (I)$. This means that, $x_{[n]} \in I^\vee$. Now, by part (i) of this theorem, we have:

$$J^\vee = I^\vee \cap (x_{[n]} y_{[m]}) = (x_{[n]}) + (y_{[m]} I^\vee).$$

(1)

Clearly, $(x_{[n]}) \cap ((y_{[m]} I^\vee)) = (x_{[n]} y_{[m]}).$ Hence by Lemma 1.3, we have:

$$H_i^m \left( \frac{S}{J^\vee} \right) \cong H_i^m \left( \frac{S}{(y_{[m]} I^\vee)} \right), \quad \text{for all } i < (m + n + r) - 2.$$ (2)

Since,

$$\dim \frac{S}{J^\vee} = (m + n + r) - 2 = \dim \frac{S}{I^\vee},$$

from (2) and Lemma 2.2 we conclude that, $\text{depth } S/I^\vee = \text{depth } S/J^\vee$.  

Theorem 2.3. Let $G_1$ and $G_2$ be graphs on two vertex sets $V_1$ and $V_2$ respectively, such that $V_1 \cap V_2 = \{z\}$ and $\{z_i, z_j\} \in E(G_1) \cap E(G_2)$, for all $1 \leq i < j \leq r$. Let

$$I_1 = I(G_1) \subset K[x, z],$$

$$I_2 = I(G_2) \subset K[y, z],$$

$$I = I(G_1 \cup G_2) \subset S = K[x, y, z].$$

be corresponding non-zero circuit ideals. Then,
(i) \( \text{depth } \frac{S}{I} = \min\{\text{depth } \frac{S}{I_1}, \text{depth } \frac{S}{I_2}\} \).

(ii) \( \text{reg } (I) = \max\{\text{reg } (I_1), \text{reg } (I_2)\} \).

(iii) The ideal \( I \) has a 2-linear resolution if and only if both of \( I_1 \) and \( I_2 \) have a 2-linear resolution.

Proof. (i) We know that:
\[
I = I_1 + I_2 + (x_i y_j : 1 \leq i \leq n, 1 \leq j \leq m).
\]

Let,
\[
J_1 = I_1 + (x_i y_j : 1 \leq i \leq n, 1 \leq j \leq m),
J_2 = I_2 + (x_i y_j : 1 \leq i \leq n, 1 \leq j \leq m).
\]

Then, \( I^\vee = J_1^\vee \cap J_2^\vee \) and by Lemma 2.2(ii), we have:
\[
J_1^\vee + J_2^\vee = (x_{[n]}, y_{[m]}).
\]

From Mayer-Vietoris long exact sequence ([4, Proposition 5.1.8.]), we have the long exact sequence:
\[
\cdots \rightarrow H^{i-1}_m \left( \frac{S}{(x_{[n]}, y_{[m]})} \right) \rightarrow H^i_m \left( \frac{S}{I^\vee} \right) \rightarrow H^i_m \left( \frac{S}{J_1^\vee} \right) \oplus H^i_m \left( \frac{S}{J_2^\vee} \right) \rightarrow \left( \frac{S}{(x_{[n]}, y_{[m]})} \right) \rightarrow \cdots.
\]

Hence, for all \( i < (m + n + r) - 2 \), we have:
\[
H^i_m \left( \frac{S}{J_1^\vee} \right) \oplus H^i_m \left( \frac{S}{J_2^\vee} \right) \cong H^i_m \left( \frac{S}{J^\vee} \right).
\]
This implies that,

\[ \text{depth } \frac{S}{I'} = \min \{ \text{depth } \frac{S}{J'}, \text{depth } \frac{S}{J''} \}. \tag{3} \]

By Lemma 2.2(ii) and Remark 1.4, we have:

\[ \text{depth } \frac{S}{I'} = \text{depth } \frac{S}{J'}, \quad \text{for } i = 1, 2. \]

Hence, (i) follows from (3) and the above equality.

(ii) This is an easy consequence of (i) and Remark 1.4.

(iii) This is a direct consequence of (ii). \qed

**Lemma 2.4.** Let \( G \) be a graph on vertex set \([n]\) such that, \( \{1, 2\} \in E(G) \) and

\[ \{\{1, i\}, \{2, i\}\} \notin E(G), \quad \text{for all } i > 2. \tag{4} \]

Let \( I = I(G) \subset S = K[x_1, \ldots, x_n] \) be the circuit ideal of \( G \). Then,

(i) \( \text{depth } \frac{S}{I'} + (x_1, x_2) \geq \text{depth } \frac{S}{I'} - 1. \)

(ii) \( \text{depth } \frac{S}{I' \cap (x_1, x_2)} \geq \text{depth } \frac{S}{I'}. \)

**Proof.** Let \( t := \text{depth } S/I' \leq \dim S/I' = n - 2. \)

(i) One can easily check that, condition (4) is equivalent to say that:

for all \( r > 2 \), there exists \( F \in E(G) \) such that, \( P_F \subset (x_1, x_2, x_r) \).

Therefore,

\[ I' = \bigcap_{F \in E(G)} P_F = \left( \bigcap_{F \in E(G)} P_F \right) \cap (x_1, x_2, x_3) \cap \cdots \cap (x_1, x_2, x_n) \]

\[ = \left( \bigcap_{F \in E(G)} P_F \right) \cap (x_1, x_2, x_3 \cdots x_n) \]

\[ = I' \cap (x_1, x_2, x_3 \cdots x_n). \]

Clearly, \( x_3 \cdots x_n \in I' \). Thus, from Mayer–Vietoris long exact sequence,

\[ \cdots \to H^{i-1}_m \left( \frac{S}{I'} \right) \oplus H^{i-1}_m \left( \frac{S}{(x_1, x_2, x_3 \cdots x_n)} \right) \to H^{i-1}_m \left( \frac{S}{I' + (x_1, x_2)} \right) \to H^i_m \left( \frac{S}{I'} \right) \to \cdots. \]

we have:

\[ H^{i-1}_m \left( \frac{S}{I' + (x_1, x_2)} \right) = 0, \quad \text{for all } i < t \leq n - 2. \tag{5} \]
This proves (i).

(ii) From Mayer–Vietoris long exact sequence

\[ \cdots \to H_{m-1}^i \left( \frac{S}{I^\vee + (x_1, x_2)} \right) \to H_m^i \left( \frac{S}{I^\vee \cap (x_1, x_2)} \right) \to H_m^i \left( \frac{S}{I^\vee} \right) \oplus H_m^i \left( \frac{S}{(x_1, x_2)} \right) \to \cdots. \]

and (5), we have:

\[ H_m^i \left( \frac{S}{I^\vee \cap (x_1, x_2)} \right) = 0, \quad \text{for all } i < t \leq n - 2, \]

which completes the proof of (ii).

\[ \square \]

**Theorem 2.5.** Let \( G \) be a graph on vertex set \([n]\) such that, \( \{1, 2\} \in E(G) \) and \( \{1, i\}, \{2, i\} \notin E(G) \), for all \( i > 2 \). Let,

\[ G_1 = (G \setminus \{1, 2\}) \cup \{\{0, 1\}, \{0, 2\}\} \]

be a graph on \( \{0\} \cup [n] \) and \( I = I(\bar{G}), J = I(\bar{G}_1) \) be circuit ideals in \( S = K[x_0, x_1, \ldots, x_n] \). Then,

\[ \text{reg}(I) = \text{reg}(J). \]

**Proof.** By Remark 1.4, it is enough to show that, depth \( S/I^\vee = \text{depth} S/J^\vee \).

Let \( G_1 = G \setminus \{1, 2\} \) and \( I_1 = I(\bar{G}_1) \). Clearly, \( I_1^\vee = (x_1, x_2) \cap I^\vee \) and

\[ J^\vee = \left( \bigcap_{i=3}^{n}(x_0, x_i) \right) \cap I_1^\vee = (x_0, x_3 \cdots x_n) \cap I_1^\vee. \]

Moreover, our assumption implies that for all \( i > 2 \), there exists \( F \in E(\bar{G}) \) such that, \( P_F \subset (x_1, x_2, x_i) \). Therefore,

\[ I_1^\vee + (x_0, x_3 \cdots x_n) = (x_0, x_3 \cdots x_n, I_1^\vee) \]

\[ = (x_0) + \left( x_3 \cdots x_n, \left[ (x_1, x_2) \cap \left( \bigcap_{F \in \bar{G}} P_F \right) \right] \right) \]

\[ = (x_0) + \left( (x_1, x_2, x_3) \cap \cdots \cap (x_1, x_2, x_n) \cap \left( \bigcap_{F \in \bar{G}} P_F \right) \right) \]

\[ = (x_0) + \left( \bigcap_{F \in \bar{G}} P_F \right) = (x_0, I^\vee). \]
Now, consider Mayer-Vietoris long exact sequence
\[
\cdots \to H_{m-1}^i \left( \frac{S}{I_1^{\vee}} \right) \oplus H_{m-1}^i \left( \frac{S}{(x_0, x_3, \ldots, x_n)} \right) \to H_{m-1}^i \left( \frac{S}{(x_0, I^{\vee})} \right) \to
\]
\[
H_{m}^i \left( \frac{S}{I_1^{\vee}} \right) \to H_{m}^i \left( \frac{S}{J^{\vee}} \right) \oplus H_{m}^i \left( \frac{S}{(x_0, x_3, \ldots, x_n)} \right) \to \cdots.
\]

Let \( t := \text{depth} \frac{S}{I_1^{\vee}} \leq \dim \frac{S}{I_1^{\vee}} = (n + 1) - 2. \) Consider two cases:

**Case 1.** \( t = (n + 1) - 2. \)
In this case, using Lemma 2.4(ii), we have:
\[
(n + 1) - 2 = \text{dim} \frac{S}{I_1^{\vee}} = \text{depth} \frac{S}{I_1^{\vee}} \leq \text{depth} \frac{S}{I_1^{\vee}} \leq \text{depth} \frac{S}{I_1^{\vee}} = (n + 1) - 2.
\]
This means that, depth \( S/I_1^{\vee} = (n + 1) - 2. \) Hence, by (7), we have:
\[
H_{m-1}^i \left( \frac{S}{(x_0, I^{\vee})} \right) \cong H_{m}^i \left( \frac{S}{J^{\vee}} \right), \quad \text{for all } i < (n + 1) - 2
\]
which implies that depth \( \frac{S}{J^{\vee}} = (n + 1) - 2 = t. \)

**Case 2.** \( t < (n + 1) - 2. \)
Since depth \( S/I_1^{\vee} \geq t, \) by Lemma 2.4(ii) and the exact sequence (7), \( H_{m}^i \left( \frac{S}{J^{\vee}} \right) = 0, \) for all \( i < t \) and we get the exact sequence
\[
0 \longrightarrow H_{m-1}^i \left( \frac{S}{(x_0, I^{\vee})} \right) \longrightarrow H_{m}^i \left( \frac{S}{J^{\vee}} \right).
\]
This implies that, \( H_{m}^i \left( \frac{S}{J^{\vee}} \right) \neq 0. \) Therefore, depth \( S/J^{\vee} = t. \)

Let \( G \) be a graph without any cycle of length 3 and \( G_1 \) a subdivision of \( G, \) that is, \( G_1 \) is obtained by adding some vertices on edges of \( G; \) then Theorem 2.5 implies that \( \text{reg} \left( I(\bar{G}) \right) = \text{reg} \left( I(\bar{G}_1) \right). \) As an application of the last reduction process, we state the following.

**Corollary 2.6.** Let \( C \) be a cycle of length \( n > 3 \) and \( I = I(\bar{C}) \subset S = K[x_1, \ldots, x_n] \) be the circuit ideal of \( C. \) Then,

(i) \( \text{reg} \left( I \right) = 3; \) in particular \( I \) does not have linear resolution.

(ii) If \( G \) is not chordal graph, then the ideal \( I(\bar{G}) \) does not have linear resolution.
Proof. (i) Let $E(C) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. We use induction on $n$. For $n = 4$ an easy computation shows that, the minimal free resolution of $I(C)$ is:

$$0 \rightarrow S(-4) \rightarrow S^2(-2) \rightarrow I,$$

which is not linear. Assume that $n > 4$ and the theorem holds for cycles of length $n - 1$. For a cycle $C$ of length $n$, let $C'$ be the graph $(C \setminus 1) \cup \{1, 3\}$. Then $C'$ is a cycle of length $n - 1$ and by induction hypothesis, $\text{reg}(I(C')) = 3$. Using Theorem 2.5, we have $\text{reg}(I(C)) = \text{reg}(I(C')) = 3$.

(ii) If $G$ is not chordal, then $G$ contains an induced cycle $C_n$ with $n > 3$.

Now, from (i) and Corollary 1.6 we conclude that the ideal $I(G)$ does not have linear resolution.

Now, we state another reduction which is removing a simplicial vertex in a graph.

**Theorem 2.7.** Let $G$ be a graph on $[n]$ and $v$ be a simplicial vertex of $G$. Let $G_1 = G \setminus v$ and $I = I(G), J = I(G_1)$ be the corresponding non-zero circuit ideals in $S = K[x_1, \ldots, x_n]$. Then,

$$\text{reg} (I) = \text{reg} (J).$$

**Proof.** By Remark 1.4, it is enough to show that, $\text{depth} S/I^v = \text{depth} S/J^v$. Without loss of generality, we may assume that, $N(v) = \{1, \ldots, v - 1\}$ and $J \subset K[x_1, \ldots, \hat{x}_v, \ldots, x_n]$. Therefore, we have:

$$I = J + (x_v x_i : v < i \leq n).$$

Moreover, since $v$ is a simplicial vertex, we conclude that, $x_{v+1} \cdots x_n \in J^v$. Hence we have:

$$I^v = J^v \cap \left( \bigcap_{i=v+1}^n (x_v, x_i) \right) = J^v \cap (x_v, x_{v+1} \cdots x_n) = ((x_v) \cap J^v) + (x_{v+1} \cdots x_n).$$

Clearly, $((x_v) \cap J^v) \cap (x_{v+1} \cdots x_n) = (x_v \cdots x_n)$. Hence by Lemma 1.3,

$$H^i_m \left( \frac{S}{I^v} \right) \cong H^i_m \left( \frac{S}{(x_v) \cap J^v} \right), \quad \text{for all } i < n - 2.$$ 

Since $\dim S/I^v = n - 2$, the above isomorphism and Lemma 2.1 implies that, $\text{depth} S/I^v = \text{depth} S/J^v$. \qed
Remark. Let $G$ be a non-complete graph, $v$ be a simplicial vertex of $G$ and $G_1 = G \setminus v$. If $G_1$ is a complete graph, then the ideal $I = I(\bar{G}) = (x_v x_i : \{v, i\} \in E(\bar{G}))$ is a non-zero ideal and

$$I^v = (x_v, \prod_{\{v, i\} \in E(\bar{G})} x_i).$$

In particular, $I^v$ is Cohen-Macaulay and the ideal $I$ has a 2-linear resolution (Theorem 1.1).

If $G_1$ is not a complete graph, then Theorem 2.7 implies that $reg I(\bar{G}) = reg I(\bar{G}_1)$.

The following nice characterization of chordal graphs and Theorem 2.7, enable us to prove that the ideal $I(\bar{G})$ has a linear resolution, whenever $G$ is a chordal graph.

**Theorem 2.8** ([8], essentially [1]). A graph $G$ is chordal if and only if every induced subgraph of $G$ has a simplicial vertex.

**Corollary 2.9.** If $G$ is a non-complete chordal graph, then the ideal $I = I(\bar{G})$ has a 2-linear resolution over any field $K$.

**Proof.** Let $G$ be a non-complete chordal graph. By Theorem 2.8, $G$ has simplicial vertex $v$. If $G_1 = G \setminus v$, then $G_1$ is again chordal graph. Now, the induction and Theorem 2.7 together with the remark after Theorem 2.7, yield the conclusion.

By Corollaries 2.6(ii) and 2.9 we have the following result which was first proved by Fröberg in [3].

**Corollary 2.10.** A graph $G$ is chordal if and only if $I(\bar{G})$ has a linear resolution.

The class of chordal graphs are contained in the class of decomposable graphs (c.f. [4, Lemma 9.2.1]). Using our reduction processes, we can find the regularity of decomposable graphs in terms of its indecomposable components.

**Definition 2.11** (Decomposable Graph). Let $G$ be a graph on vertex set $[n]$. We say that $G$ is decomposable, if there exists proper subsets $P$ and $Q$ of $[n]$ with $P \cup Q = [n]$ such that,

(a) $\{i, j\} \in E(G)$, for all $i, j \in P \cap Q$, $i \neq j$. 

10
Remark 2.12 (Regularity of Decomposable Graphs). Let $G$ be a decomposable graph and $P, Q$ be proper subsets of $V(G) = [n]$ which satisfies in the mentioned conditions.

- If both of $G_P$ and $G_Q$ are complete graphs, then:

\[ I(\bar{G}) = (x_iy_j : i \in P \setminus Q, j \in Q \setminus P). \]

Hence,

\[ I(\bar{G})^\vee = \left( \prod_{i \in P \setminus Q} x_i, \prod_{i \in Q \setminus P} y_i \right) \]

which is Cohen-Macaulay of dimension $n - 2$. Thus, $\text{reg} \ I(\bar{G}) = 2$, by Theorem 1.1.

- If $G_P$ is complete graph but $G_Q$ is not complete graph, then all $v \in P \setminus Q$ are simplicial vertex. Hence by Theorem 2.7, $\text{reg} \ I(\bar{G}) = \text{reg} \ I(\bar{G}_Q)$.

If $|P| = 1$, we conclude that $\text{reg} \ I(\bar{G}) = \text{reg} \ I(\bar{G}_Q)$. Otherwise, the graph $G' = G \setminus v$ is again decomposable with the components $P' = P \setminus v$ and $Q$. Note that, $G'_{P'}$ is again a complete graph. Going on this argument, we conclude that, $\text{reg} \ I(\bar{G}) = \text{reg} \ I(\bar{G}_Q)$.

- If non of $G_P$ and $G_Q$ are complete graphs, then Theorem 2.3 implies that, $\text{reg} \ I(\bar{G}) = \max\{\text{reg} \ I(\bar{G}_P), \text{reg} \ I(\bar{G}_Q)\}$.

Remark 2.13. Let $G$ be a (indecomposable) graph. After our reduction processes (Theorems 2.5 and 2.7), finally we get a graph $G'$ with $\text{reg} \ I(\bar{G}) = \text{reg} \ I(\bar{G}')$ and $G'$ has neither a simplicial vertex nor a subdivision. If at least one of the connected components of $G'$ has cycle of length greater that 3, then $I(\bar{G})$ does not have a 2-linear resolution (Corollary 2.6(ii)).

But, sometimes we are not able to do more reduction on a graph. For example, if $G$ is the Peterson graph or the following Hamiltonian graph, then we cannot apply our reduction process to further simplify $G$. 

11
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