Co-clustering of Nonsmooth Graphons

David Choi

July 24, 2015

Abstract

Performance bounds are given for exploratory co-clustering/blockmodeling of bipartite graph data, where we assume the rows and columns of the data matrix are samples from an arbitrary population. This is equivalent to assuming that the data is generated from a nonsmooth graphon. It is shown that co-clusters found by any method can be extended to the row and column populations, or equivalently that the estimated blockmodel approximates a blocked version of the generative graphon, with estimation error bounded by $O_P(n^{-1/2})$. Analogous performance bounds are also given for degree-corrected blockmodels and random dot product graphs, with error rates depending on the dimensionality of the latent variable space.

1 Introduction

In the statistical analysis of network data, blockmodeling (or community detection) and its variants are a popular class of methods that have been tried in many applications, such as modeling of communication patterns [7], academic citations [17], protein networks [1], online behavior [21, 30], and ecological networks [15].

In order to develop a theoretical understanding, many recent papers have established consistency properties for the blockmodel. In these papers, the observed network is assumed to be generated using a set of latent variables that assign the vertices into groups (the “communities”), and the inferential goal is to recover the correct group membership from the observed data. Various conditions have been established under which recovery is possible [5, 6] and also computationally tractable [10, 11, 20, 24, 28]. Additionally, conditions are also known under which no algorithm can correctly recover the group memberships [13, 23].

The existence of a true group membership is central to these results. In particular, they assume a generative model in which all members of the same group are statistically identical. This implies that the group memberships explain the entirety of the network structure. In practice, we might not expect this assumption to even approximately hold, and the objective of finding “true communities” could be difficult to define precisely, so that a more reasonable goal might be to discover group labels which partially explain structure that is evident in the data. Comparatively little work has been done to understand blockmodeling from this viewpoint.
To address this gap, we consider the problem of blockmodeling under model misspecification. We assume that the data is generated not by a blockmodel, but by a much larger nonparametric class known as a graphon. This is equivalent to assuming that the vertices are sampled from an underlying population, in which no two members are identical and the notion of a true community partition need not exist. In this setting, blockmodeling might be better understood not as a generative model, but rather as an exploratory method for finding high-level structure: by dividing the vertices into groups, we divide the network into subgraphs that can exhibit varying levels of connectivity. This is analogous to the usage of histograms to find high and low density regions in a nonparametric distribution. Just as a histogram replicates the binned version of its underlying distribution without restrictive assumptions, we will show that the blockmodel replicates structure in the underlying population when the observed network is generated from an arbitrary graphon.

Our results are restricted to the case of bipartite graph data. Such data arises naturally in many settings, such as customer-product networks where connections may represent purchases, reviews, or some other interaction between people and products.

The organization of the paper is as follows. Related work is discussed in Section 2. In Section 3, we define the blockmodeling problem for bipartite data generated from a graphon, and present a result showing that the blockmodel can detect structure in the underlying population. In Section 4, we discuss extensions of the blockmodel, such as mixed-membership, and give a result regarding the behavior of the excess risk in such models. Section 5 contains a sketch and proof for the main theorem. Auxiliary results and extensions are proven in the Appendix.

2 Related Works

The papers [2, 14, 19, 25], and [12] are most similar to the present work, in that they consider the problem of approximating a graphon by a blockmodel. The papers [2, 14, 19] and [25] consider both bipartite and non-bipartite graph data, and require the generative graphon to satisfy a smoothness condition, with [14] establishing a minimax error rate and [19] extending the results to a class of sparse graphon models. In a similar vein, [29] shows consistent and computationally efficient estimation assuming a type of low rank generative model. While smoothness and rank assumptions are natural for many non-parametric regression problems, it seems difficult to judge whether they are appropriate for network data and if they are indeed necessary for good performance.

In [12] and in this present paper, which consider only bipartite graphs, the emphasis is on exploratory analysis. Hence no assumptions are placed on the generative graphon. Unlike the works which assume smoothness or low rank structure, the object of inference is not the generative model itself, but rather a blocked version of it (this is defined precisely in Section 3). This is reminiscent of some results for confidence intervals in nonparametric regression, in which the interval is centered not on the generative function or density itself, but rather on a smoothed or histogram-ed version [31, Sec 5.7 and Thm 6.20]. The present paper can be viewed as a substantial improvement over [12]; for example, Theorem 1 improves the rates of convergence from $O_P(n^{-1/4})$ to $O_P(n^{-1/2})$, and also applies to computationally efficient...
estimates.

3 Co-clustering of nonsmooth graphons

In this section, we give a formulation for co-clustering (or co-blockmodeling) in which the rows and columns of the data matrix are samples from row and column populations, and correspond to the vertices of a bipartite graph. We then present an approximation result which implies that any co-clustering of the rows and columns of the data matrix can be extended to the populations. Roughly speaking, this means that if a co-clustering “reveals structure” in the data matrix, then similar structure will also exist at the population level.

3.1 Problem Formulation

Data generating process Let \( A \in \{0, 1\}^{m \times n} \) denote a binary \( m \times n \) matrix representing the observed data. For example, \( A_{ij} \) could denote whether person \( i \) rated movie \( j \) favorably, or whether gene \( i \) was expressed under condition \( j \).

We assume that \( A \) is generated by the following model, in which each row and column of \( A \) is associated with a latent variable that is sampled from a population:

**Definition 1 (Bipartite Graphon).** Given \( m \) and \( n \), let \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_n \) denote i.i.d. uniform \([0, 1]\) latent variables

\[
x_1, \ldots, x_m \overset{iid}{\sim} \text{Unif}[0, 1], \quad \text{and} \quad y_1, \ldots, y_n \overset{iid}{\sim} \text{Unif}[0, 1].
\]

Let \( \omega : [0, 1]^2 \mapsto [0, 1] \) specify the distribution of \( A \in \{0, 1\}^{m \times n} \), conditioned the latent variables \( \{x_i\}_{i=1}^m \) and \( \{y_j\}_{j=1}^n \),

\[
A_{ij} \sim \text{Bernoulli} (\omega(x_i, y_j)), \quad i \in [m], j \in [n]
\]

where the Bernoulli random variables are independent.

We will require that \( \omega \) be measurable and bounded between 0 and 1, but may otherwise be arbitrarily non-smooth. We will use \( \mathcal{X} = [0, 1] \) and \( \mathcal{Y} = [0, 1] \) to denote the populations from which \( \{x_i\} \) and \( \{y_j\} \) are sampled.

Co-clustering In co-clustering, the rows and columns of a data matrix \( A \) are simultaneously clustered to reveal submatrices of \( A \) that have similar values. When \( A \) is binary valued, this is also called blockmodeling (or co-blockmodeling).

Our notation for co-clustering is the following. Let \( K \) denote the number of clusters. Let \( S \in [K]^m \) denote a vector identifying the cluster labels corresponding to the \( m \) rows of \( A \), e.g., \( S_i = k \) means that the \( i \)th row is assigned to cluster \( k \). Similarly, let \( T \in [K]^n \) identify the cluster labels corresponding to the \( n \) rows of \( A \). Given \( (S, T) \), let \( \Phi_A(S, T) \in [0, 1]^{K \times K} \) denote the normalized sums for the submatrices of \( A \) induced by \( S \) and \( T \):

\[
[\Phi_A(S, T)]_{st} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} 1(S_i = s, T_j = t), \quad s, t \in [K].
\]
Let $\pi_S \in [0, 1]^K$ and $\pi_T \in [0, 1]^K$ denote the fraction of rows or columns in each cluster:

$$
\pi_S(s) = \frac{1}{m} \sum_{i=1}^{m} 1(S_i = s) \quad \text{and} \quad \pi_T(t) = \frac{1}{n} \sum_{j=1}^{n} 1(T_j = t).
$$

Let the average value of the $(s, t)$th submatrix be denoted by $\hat{\theta}_{st}$, given by

$$
\hat{\theta}_{st} = \frac{[\Phi_A(S, T)]_{st}}{\pi_S(s)\pi_T(t)}.
$$

Generally, $S$ and $T$ are chosen heuristically to make the entries of $\hat{\theta}$ far from the overall average of $A$. A common approach is to perform k-means clustering of the spectral coordinates for each row and column of $A$ [26]. Heterogeneous values of $\hat{\theta}$ can be interpreted as revealing subgroups of the rows and columns in $A$.

**Population co-blockmodel** Given a co-clustering $(S, T)$ of the rows and columns of $A$, we will consider whether similar subgroups also exist in the unobserved populations $X$ and $Y$. Let $\sigma : X \mapsto [K]$ and $\tau : Y \mapsto [K]$ denote mappings that co-cluster the row and column populations $X$ and $Y$. Let $\Phi_\omega(\sigma, \tau) \in [0, 1]^{K \times K}$ denote the integral of $\omega$ within the induced co-clusters, or the blocked version of $\omega$:

$$
[\Phi_\omega(\sigma, \tau)]_{st} = \int_{X \times Y} \omega(x, y) \, 1(\sigma(x) = s, \tau(y) = t) \, dx \, dy, \quad s, t \in [K].
$$

Let $\Phi_\omega(S, \tau) \in [0, 1]^{K \times K}$ denote the integral of $\omega$ within the induced co-clusters, over $\{x_1, \ldots, x_n\} \times Y$:

$$
[\Phi_\omega(S, \tau)]_{st} = \frac{1}{m} \sum_{i=1}^{m} \int_{Y} \omega(x_i, y) \, 1(S_i = s, \tau(y) = t) \, dy.
$$

Let $\pi(\sigma)$ and $\pi(\tau)$ denote the fraction of the population in each cluster:

$$
\pi_\sigma(s) = \int_X 1(\sigma(x) = s) \, dx \quad \text{and} \quad \pi_\tau(t) = \int_Y 1(\tau(y) = t) \, dy.
$$

Theorem 1 will show that for each clustering $S, T$, there exists $\sigma : X \mapsto [K]$ and $\tau : Y \mapsto [K]$ which cluster the populations $X$ and $Y$ such that $\Phi_A(S, T) \approx \Phi_\omega(S, \tau)$ and $\Phi_A(S, T) \approx \Phi_\omega(\sigma, \tau)$, as well as $\pi_S \approx \pi_\sigma$ and $\pi_T \approx \pi_\tau$, implying that subgroups found by co-clustering $A$ are indicative of similar structure in the populations $X$ and $Y$.

### 3.2 Approximation Result for Co-clustering

Theorem 1 states that for each $(S, T) \in [K]^m \times [K]^n$, there exists population co-clusters $\sigma_S : X \mapsto [K]$ and $\tau_T : Y \mapsto [K]$ such that $\Phi_A(S, T) \approx \Phi_\omega(S, \tau_T) \approx \Phi_\omega(\sigma_S, \tau_T)$, and also $\pi_S \approx \pi_\sigma_S$ and $\pi_T \approx \pi_\tau_T$. 


Theorem 1. Let $A \in \{0, 1\}^{m \times n}$ be generated by some $\omega$ according to Definition 1, with fixed ratio $m/n$. Let $(S, T)$ denote vectors in $[K]^m$ and $[K]^n$ respectively, with $K \leq n^{1/2}$.

1. For each $T \in [K]^n$, there exists $\tau_T : \mathcal{Y} \mapsto [K]$ such that

$$\max_{S,T \in [K]^m \times [K]^n} \| \Phi_A(S, T) - \Phi_\omega(S, \tau_T) \| + \| \pi_T - \pi_{\tau_T} \| = O_P \left( \sqrt{\frac{K^2 \log n}{n}} \right)$$  \hspace{1cm} (1)$$

2. For each $S \in [K]^m$, there exists $\sigma_S : \mathcal{X} \mapsto [K]$, such that

$$\sup_{S,\tau \in [K]^m \times [K]^n} \| \Phi_\omega(S, \tau) - \Phi_\omega(\sigma_S, \tau) \| + \| \pi_S - \pi_{\sigma_S} \| = O_P \left( \sqrt{\frac{K^2 \log m}{m}} \right)$$  \hspace{1cm} (2)$$

3. Combining (1) and (2) yields

$$\max_{S,T \in [K]^m \times [K]^n} \| \Phi_\omega(\sigma_S, \tau_T) - \Phi_A(S, T) \| + \| \pi_T - \pi_{\tau_T} \| + \| \pi_S - \pi_{\sigma_S} \| = O_P \left( \sqrt{\frac{K^2 \log n}{n}} \right).$$  \hspace{1cm} (3)$$

Remarks for Theorem 1 To give context to Theorem 1, suppose that $A \in \{0, 1\}^{m \times n}$ represents product-customer interactions, where $A_{ij} = 1$ indicates that product $i$ was purchased (or viewed, reviewed, etc.) by customer $j$. We assume $A$ is generated by Definition 1, meaning that the products and customers are samples from populations. This could be literally true if $A$ is sampled from a larger data set, or the populations might only be conceptual, perhaps representing future products and potential customers.

Suppose that we have discovered cluster labels $S \in [K]^m$ and $T \in [K]^n$ producing a density matrix $\hat{\theta}$ with heterogeneous values. These clusters can be interpreted as product categories and customer subgroups, with heterogeneity in $\hat{\theta}$ indicating that each customer subgroup may prefer certain product categories over others. We are interested in the following question: will this pattern generalize to the populations $\mathcal{X}$ and $\mathcal{Y}$? Or is it descriptive, holding only for the particular customers and products that are in the data matrix $A$?

An answer is given by Theorem 1. Specifically, (1) and (3) show different senses in which the co-clustering $(S, T)$ may generalize to the underlying populations. (1) implies that the customer population $\mathcal{Y}$ will be similar to the $n$ observed customers in the data, regarding their purchases of the $m$ observed products when aggregated by product category. (3) implies a similar result, but for their purchases of the entire population $\mathcal{X}$ of products aggregated by product category, as opposed only to the $m$ observed products in the data.

Since Theorem 1 holds for all $(S, T)$, it applies regardless of the algorithm that is used to choose the co-blockmodel. It also applies to nested or hierarchical clusters. If (1) or (3) holds at the lowest level of hierarchy with $K$ classes, then it also holds for the aggregated values at higher levels as well, albeit with the error term increased by a factor which is at most $K$. 

5
Theorem 1 controls the behavior of $\Phi_A, \pi_S$, and $\pi_T$, instead of the density matrix $\hat{\theta}$ which may be of interest. However, since $\hat{\theta}$ is derived from the previous quantities, it follows that Theorem 1 also implies control of $\hat{\theta}$ for all co-clusters involving $\gg m^{1/2}$ rows or $\gg n^{1/2}$ columns.

All constants hidden by the $O_P(\cdot)$ notation in Theorem 1 are universal, in that they do not depend on $\omega$ (but do depend on the ratio $m/n$).

4 Application of Theorem 1 to Bipartite Graph Models

In many existing models for bipartite graphs, the rows and columns of the adjacency matrix $A \in \{0, 1\}^{m \times n}$ are associated with latent variables that are not in $X$ and $Y$, but in other spaces $S$ and $T$ instead. In this section, we give examples of such models and discuss their estimation by minimizing empirical squared error. We define the population risk as the difference between the estimated and actual models, under a transformation mapping $X$ to $S$ and $Y$ to $T$. Theorem 2 shows that the empirical error surface converges uniformly to the population risk. The theorem does not assume a correctly specified model, but rather that the data is generated by an arbitrary $\omega$ following Definition 1.

4.1 Examples of Bipartite Graph Models

We consider models in which the rows and columns of $A$ are associated with latent variables that take values in spaces other than $X$ and $Y$. To describe these models, we will use $S = (S_1, \ldots, S_m)$ and $T = (T_1, \ldots, T_n)$ to denote the row and column latent variables, and $S$ and $T$ to denote their allowable values. Let $\Theta$ denote a parameter space. Given $\theta \in \Theta$, let $\omega_\theta : S \times T \mapsto [0, 1]$ determine the distribution of $A$ conditioned on $(S, T)$, so that the entries $\{A_{ij}\}$ are conditionally independent Bernoulli variables, with $P(A_{ij} = 1 | S, T) = \omega_\theta(S_i, T_j)$.

1. **Stochastic co-blockmodel with $K$ classes:** Let $S = T = [K]$ and $\Theta = [0, 1]^{K \times K}$. For $\theta \in \Theta$, let $\omega_\theta$ be given by

$$
\omega_\theta(s, t) = \theta_{st}, \quad s, t \in S \times T
$$

where $s \in S$ and $t \in T$ are row and column co-cluster labels.

2. **Degree-corrected co-blockmodel [18, 32]:** Let $S = T = [K] \times [0, 1)$ and $\Theta = [0, 1]^{K \times K}$. Given $u, v \in [K]$ and $b, d \in [0, 1)$, let $s = (u, b)$ and $t = (v, d)$. Let $\omega_\theta$ be given by

$$
\omega_\theta(s, t) = bd \theta_{uv}, \quad s, t \in S \times T.
$$

In this model, $u, v \in [K]$ are co-cluster labels, and $b, d \in [0, 1)$ are degree parameters, allowing for degree heterogeneity within co-clusters.

3. **Random Dot Product [16, 29]:** Let $S = T = \{c \in [0, 1]^d : \|c\| \leq 1\}$. Let $\omega$ be given by

$$
\omega(s, t) = s^T t, \quad s, t \in S \times T.
$$
4. Dot Product + Blockmodel: Models 1-3 are instances of a somewhat more general model. Let \( D = \{ c \in [0, 1] : \| c \| \leq 1 \} \). Let \( S = T = [K] \times D \) and \( \Theta = [0, 1]^{K \times K} \). Given \( u, v \in [K] \) and \( b, d \in D \), let \( s = (u, b) \) and \( t = (v, d) \). Let \( \omega_\theta \) be given by
\[
\omega_\theta(s, t) = b^T d \theta_{uv}.
\] (4)

4.2 Empirical and Population Risk

Given a data matrix \( A \in \{0, 1\}^{m \times n} \), and a model specification \((S, T, \Theta)\), one method for estimating \((S, T, \theta) \in S^m \times T^n \times \Theta \) is to minimize the empirical squared error \( R_A \), given by
\[
R_A(S, T; \theta) = \frac{1}{nm} \sum_{i=1}^{m} \sum_{j=1}^{n} (A_{ij} - \omega_\theta(S_i, T_j))^2.
\]

Generally, the global minimum of \( R_A \) will be intractable to compute, so a local minimum is used for the estimate instead.

If a model \((S, T, \theta)\) is found by minimizing or exploring the empirical risk surface \( R_A \), does it approximate the generative \( \omega \)? We will define the population risk in two different ways:

1. Approximation of \( \omega \) by \( \omega_\theta \): Let \( \sigma \) and \( \tau \) denote mappings \( \mathcal{X} \mapsto S \) and \( \mathcal{Y} \mapsto T \), and let \( R_\omega \) be given by
\[
R_\omega(\sigma, \tau; \theta) = \int_{\mathcal{X} \times \mathcal{Y}} [\omega(x, y) - \omega_\theta(\sigma(x), \tau(y))]^2 \, dx \, dy,
\]
denoting the error between the mapping \((x, y) \mapsto \omega_\theta(\sigma(x), \tau(y), \theta)\) and the generative \( \omega \). If there exists \( \theta \) such that \( R_\omega(\sigma, \tau; \theta) \) is low for some \( \sigma : \mathcal{X} \mapsto S \) and \( \tau : \mathcal{Y} \mapsto T \), then \( \omega_\theta \) (or more precisely, its transformation \((x, y) \mapsto \omega_\theta(\sigma(x), \tau(y))\)) can be considered a good approximation to \( \omega \).

2. Approximation of \( \sigma^* = \arg \min_\sigma R_\omega(\sigma, \tau, \theta) \) by \( S \): Overloading notation, let \( R_\omega(S, \tau, \theta) \) denote
\[
R_\omega(S, \tau; \theta) = \frac{1}{m} \sum_{i=1}^{m} \int_{\mathcal{Y}} [\omega(x_i, y) - \omega_\theta(S_i, \tau(y))]^2 \, dy.
\]

To motivate this quantity, consider that given \((\tau, \theta)\), the optimal partition \( \sigma^* : [0, 1] \mapsto [K] \) is the greedy assignment for each \( x \in [0, 1] \):
\[
\sigma^*(x) = \arg \min_{s \in [K]} \int_{0,1} [\omega(x_i, y) - \omega_\theta(s, \tau(y))]^2 \, dy.
\]

If there exists \((S, \theta)\) such that \( R_\omega(S, \tau; \theta) \) is low for some choice of \( \tau \), then \( S \) can be considered a good approximation to the corresponding \( \{\sigma^*(x_i)\}_{i=1}^{m} \).

Theorem 2 will imply that for models of the form (4), minimizing \( R_A \) is asymptotically a reasonable proxy for minimizing \( R_\omega \) (by both metrics described above), with rates of convergence depending on the covering numbers of \( S \) and \( T \).
4.3 Convergence of the Empirical Risk Function

Theorem 2 gives uniform bounds between $R_A$ and $R_\omega$ for models of form (4). Specifically, for each choice of $(S, T) \in S^m \times T^n$, there exists transformations $\sigma_S : \mathcal{X} \mapsto \mathcal{S}$ and $\tau_T : \mathcal{Y} \mapsto \mathcal{T}$ such that $R_A(S, T; \theta) \approx R_\omega(\sigma_S, \tau_T; \theta) \approx R_\omega(S, \tau_T; \theta)$, up to an additive constant and with uniform convergence rates depending on $d$ and $K$. As a result, minimization of $R_A(S, T; \theta)$ is a reasonable proxy for minimizing $R_\omega$, by either measure defined in Section 4.2.

In addition, the mappings $\sigma_S$ and $\tau_T$ will resemble $S$ and $T$, in that they will induce similar distributions over the latent variables. To quantify this, we define the following quantities. Given $S \in [K]^m \times D^m$, we will let $S = (U, B)$, where $U \in [K]^m$ and $B \in D^m$, and similarly let $T = (V, D)$ where $V \in [K]^n$ and $D \in D^n$. Likewise, given $\sigma : \mathcal{X} \mapsto [K] \times D$, we will let $\sigma = (\mu, \beta)$, where $\mu : \mathcal{X} \mapsto [K]$ and $\beta : \mathcal{X} \mapsto D$, and similarly let $\tau = (\nu, \delta)$ where $\nu : \mathcal{Y} \mapsto [K]$ and $\delta : \mathcal{Y} \mapsto D$. Let $\Psi_S, \Psi_T, \Psi_\sigma$, and $\Psi_\tau$ denote the CDFs of the values given by $S, T, \sigma$ and $\tau$, which are functions $[K] \times [0, 1]^d \mapsto [0, 1]$ equaling:

\[
\Psi_S(k, c) = \frac{1}{m} \sum_{i=1}^m 1\{U_i \leq k, B_i \leq c\} \quad \Psi_T(k, c) = \frac{1}{n} \sum_{j=1}^n 1\{V_i \leq k, D_i \leq c\}
\]

\[
\Psi_\sigma(k, c) = \int_\mathcal{X} 1\{\mu(x) \leq k, \beta(x) \leq c\} \, dx \quad \Psi_\tau(k, c) = \int_\mathcal{Y} 1\{\nu(y) \leq k, \delta(y) \leq c\} \, dy,
\]

where inequalities of the form $c \leq c'$ for $c, c' \in [0, 1]^d$ are satisfied if they hold entrywise.

**Theorem 2.** Let $A \in \{0, 1\}^{m \times n}$, with fixed ratio $m/n$, be generated by some $\omega$ according to Definition 1. Let $(S, T, \Theta)$ denote a model of the form (4).

1. For each $T \in T^n$, there exists $\tau_T : \mathcal{Y} \mapsto \mathcal{T}$ such that

\[
\max_{S, T, \theta \in S^m \times T^n \times \Theta} |R_A(S, T; \theta) - R_\omega(S, \tau_T; \theta) - C_1| + \frac{\|\Psi_T - \Psi_\tau\|_F^2}{Kd} \leq O_P \left( d^{1/2} \left( \frac{K^2 \log n}{\sqrt{n}} \right)^{1+\theta} \right),
\]

where $C_1 \in \mathbb{R}$ is constant in $(S, T, \theta)$.

2. For each $S \in S^m$, there exists $\sigma_S : \mathcal{X} \mapsto \mathcal{S}$ such that

\[
\sup_{S, \tau, \theta \in S^m \times T^n \times \Theta} |R_\omega(S, \tau; \theta) - R_\omega(\sigma_S, \tau; \theta) - C_2| + \frac{\|\Psi_S - \Psi_\sigma\|_F^2}{Kd} \leq O_P \left( d^{1/2} \left( \frac{K^2 \log n}{\sqrt{n}} \right)^{1+\theta} \right),
\]

where $C_2 \in \mathbb{R}$ is constant in $(S, \tau, \theta)$.
3. Combining (5) and (6) yields
\[
\begin{align*}
\max_{S,T,\theta \in S^m \times T^n \times \Theta} & |R_\omega(\sigma_S, \tau_T; \theta) - R_A(S, T; \theta) - C_1 - C_2| + \frac{||\Psi_S - \Psi_{\sigma_S}||^2}{Kd} \\
& + \frac{||\Psi_T - \Psi_{\tau_T}||^2}{Kd} = O_P \left( d^{1/2} \left( \frac{K^2 \log n}{\sqrt{n}} \right)^{1+\epsilon} \right).
\end{align*}
\]

Remarks for Theorem 2  Theorem 2 states that any assignment $S$ and $T$ of latent variables to the rows and columns can be extended to the populations, such that the population exhibits a similar distribution of values in $S$ and $T,$ and the population risk as a function of $\theta$ is close to the empirical risk.

The theorem may also be viewed as an oracle inequality, in that for any fixed $S$ and $T,$ minimizing $\theta \mapsto R_A(S, T; \theta)$ is approximately equivalent to minimizing $\theta \mapsto R_\omega(\sigma_S, \tau_T, \theta),$ as if the model $\omega$ were known. This implies that the best parametric approximation to $\omega$ can be learned, for any choice of $\sigma_S$ and $\tau_T.$ However, it is not known whether the mappings $S \mapsto \sigma_S$ and $T \mapsto \tau_T$ are approximately onto; if not, minimization of $R_A$ over $(S, T, \theta)$ is a reasonable proxy for minimization of $R_\omega$ over $(\sigma, \tau, \theta),$ but only over a subset of the possible mappings $\sigma : X \mapsto S$ and $\tau : Y \mapsto T.$

The convergence of $\Psi_S$ to $\Psi_{\sigma_S}$ is established in Euclidean norm. This implies pointwise convergence at every continuity point of $\Psi_{\sigma_S},$ thus implying weak convergence and also convergence in Wasserstein distance.

The proof is contained in Appendix B. It is similar to that of Theorem 1, but requires substantially more notation due to the additional parameters. Essentially, the proof approximates the model of (4) by a blockmodel, and then applies Theorem 1 to bound the difference between $R_A$ and $R_\omega.$

5  Proof of Theorem 1

We present a sketch of the proof for Theorem 1, which defines the most important quantities. We then present helper lemmas and give the proof of the theorem.

5.1  Proof Sketch

Let $W \in [0, 1]^{m \times n}$ denote the expectation of $A,$ conditioned on the latent variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$:
\[
W_{ij} = \omega(x_i, y_j), \quad i \in [m], j \in [m],
\]
and let $\Phi_W(S, T)$ denote the conditional expectation of $\Phi_A(S, T)$:
\[
[\Phi_W(S, T)]_{st} = \frac{1}{nm} \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} 1\{S_i = s, T_j = t\}.
\]
Given co-cluster labels $S \in [K]^m$ and $T \in [K]^n$, let $1_{S=s} \in \{0, 1\}^m$ and $1_{T=t} \in \{0, 1\}^n$ denote the indicator variables

$$1_{S=s}(i) = \begin{cases} 1 & \text{if } S_i = s \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad 1_{T=t}(j) = \begin{cases} 1 & \text{if } T_j = t \\ 0 & \text{otherwise} \end{cases}. $$

Let $g_{T=t} \in [0, 1]^m$ denote the vector $n^{-1}W_{1_{T=t}}$, or

$$g_{T=t}(i) = \frac{1}{n} \sum_{j=1}^{n} W_{ij} 1\{T_j = t\}. $$

It can be seen that the entries of $\Phi_W(S, T)$ can be written as

$$[\Phi_W(S, T)]_{st} = \frac{1}{m} \langle 1_{S=s}, g_{T=t} \rangle, $$

where $\langle \cdot, \cdot \rangle$ denotes inner product. Similarly, the entries of $\Phi_\omega(S, \tau)$ can be written as

$$[\Phi_\omega(S, \tau)]_{st} = \frac{1}{m} \langle 1_{S=s}, g_{\tau=t} \rangle, $$

where $g_{\tau=t} \in [0, 1]^m$ is the vector

$$g_{\tau=t}(i) = \int_{\mathcal{Y}} \omega(x_i, y) 1\{\tau(y) = t\} \, dy, \quad i \in [m]. $$

The proof of Theorem 1 will require three main steps:

**S1:** In Lemma 1, a concentration inequality will be used to show that $\Phi_A(S, T) \approx \Phi_W(S, T)$ uniformly over all possible values of $(S, T)$.

**S2:** For each $T \in [K]^n$, we will show there exists $\tau : \mathcal{Y} \mapsto [K]$ such that $g_{T=t} \approx g_{\tau=t}$ for $t \in [K]$. By (7) and (8), this will imply that $\Phi_W(S, T) \approx \Phi_\omega(S, \tau)$ uniformly for all $S \in [K]^m$. The mapping $\tau$ will also satisfy $\pi_T \approx \pi_\tau$ as well, so that $T$ and $\tau$ have similar class frequencies.

**S3:** Analogous to S2, we will show that for each $S \in [K]^m$, there exists $\sigma : \mathcal{X} \mapsto [K]$ such that $\Phi_\omega(S, \tau) \approx \Phi_\omega(\sigma S, \tau)$ uniformly over $\tau$, and also that $\pi_S \approx \pi_{\sigma S}$.

Steps S1 and S2 correspond to (1) in Theorem 1, while step S3 corresponds to (2).

Let $G_T$ and $G_\tau$ denote the stacked vectors in $\mathbb{R}^{mK+K}$ given by

$$G_T = \left( \frac{g_{T=1}}{\sqrt{m}}, \ldots, \frac{g_{T=K}}{\sqrt{m}}, \pi_T \right) \quad \text{and} \quad G_\tau = \left( \frac{g_{\tau=1}}{\sqrt{m}}, \ldots, \frac{g_{\tau=K}}{\sqrt{m}}, \pi_\tau \right), $$

and let $G_n$ and $G$ denote the set of all possible values for $G_T$ and $G_\tau$:

$$G_n = \{G_T : T \in [K]^n\} \quad \text{and} \quad G = \{G_\tau : \tau \in \mathcal{Y} \mapsto [K]\}. $$
Step S2 is established by showing that the sets $G_n$ and $G$ converge in Hausdorff distance. This will require the following facts. The Hausdorff distance (in Euclidean norm) between two sets $B_1$ and $B_2$ is defined as

$$d_{\text{Haus}}(B_1, B_2) = \max \left\{ \sup_{B_1 \in B_1} \inf_{B_2 \in B_2} \| B_1 - B_2 \|, \sup_{B_2 \in B_2} \inf_{B_1 \in B_1} \| B_1 - B_2 \| \right\}.$$  

Given a Hilbert space $\mathbb{H}$ and a set $B \subset \mathbb{H}$, let $\Gamma_B(H) = \sup_{B \in B} \langle H, B \rangle$ denote the support function of $B$, defined as

$$\Gamma_B(H) = \sup_{B \in B} \langle H, B \rangle.$$  

It is known that the convex hull $\text{conv}(B)$ equals the intersection of its supporting hyperplanes:

$$\text{conv}(B) = \{ x \in \mathbb{H} : \langle x, H \rangle \leq \Gamma_B(H) \text{ for all } H \in \mathbb{H} \},$$  

and that the Hausdorff distance between $\text{conv}(B_1)$ and $\text{conv}(B_2)$ is given by [27, Thm 1.8.11], [3, Cor 7.59]

$$d_{\text{Haus}}(\text{conv}(B_1), \text{conv}(B_2)) = \sup_{H : \|H\| = 1} |\Gamma_{B_1}(H) - \Gamma_{B_2}(H)|.$$  

To establish S2, Lemma 2 will show that

$$\sup_{H : \|H\| = 1} |\Gamma_{G_n}(H) - \Gamma_G(H)| = O_P(K(\log n)n^{-1/2}),$$  

and Lemma 3 will show that

$$d_{\text{Haus}}(\text{conv}(G_n), G) = 0.$$  

By (9) and (10), $\text{conv}(G_n)$ and $\text{conv}(G)$ converge in Hausdorff distance, which by (11) implies that $\text{conv}(G_n)$ and $G$ converge in Hausdorff distance. This implies that for each $G_T \in G_n$, there exists $G_T \in G$ such that $\max_T \|G_T - G_T\| \to 0$. This will establish S2, since $G_T \approx G_T$ implies by (7) and (8) that $\Phi_W(S, T) \approx \Phi_w(S, \tau)$ uniformly over $S \in [K]^m$, and it also implies that $\pi_T \approx \pi_T$ as well.

The proof of S3 will be similar to S2. It can be seen that $\Phi_w(S, \tau)$ and $\Phi_w(\sigma, \tau)$ can be written as

$$[\Phi_w(S, \tau)]_{st} = \langle f_{S=s}, 1_{\tau=t} \rangle \quad \text{and} \quad [\Phi_w(\sigma, \tau)]_{st} = \langle f_{\sigma=s}, 1_{\tau=t} \rangle,$$

where the functions $f_{S=s}, 1_{\tau=t},$ and $f_{\sigma=s}$ are given by

$$1_{\tau=t}(y) = \begin{cases} 1 & \text{if } \tau(y) = t \\ 0 & \text{otherwise} \end{cases}$$

$$f_{S=s}(y) = \frac{1}{m} \sum_{i=1}^m \omega(x_i, y) 1\{S_i = s\}$$

$$f_{\sigma=s}(y) = \int_X \omega(x, y) 1\{\sigma(x) = s\} \, dx.$$
Analogous to S2, we will define sets $F_S$ and $F_\sigma$ given by

$$F_S = (f_{S=1}, \ldots, f_{S=K}, \pi_S) \quad \text{and} \quad F_\sigma = (f_{\sigma=1}, \ldots, f_{\sigma=K}, \pi_\sigma),$$

whose possible values are given by

$$F_n = \{F_S : S \in [K]^m\} \quad \text{and} \quad \mathcal{F} = \{F_\sigma : \sigma \in \mathcal{X} \mapsto [K]\}.$$

Lemma 2 will show that the support functions $\Gamma_{\mathcal{F}_n}$ and $\Gamma_{\mathcal{F}}$ converge, and Lemma 3 will show that $d_{\text{Haus}}(\text{conv}(\mathcal{F}), \mathcal{F}) = 0$. Using (12), this will establish S3 by arguments that are analogous to those used to prove S2.

5.2 Intermediate Results for Proof of Theorem 1

Lemmas 1 - 3 will be used to prove Theorem 1, and are proven in Section 5.4.

Lemma 1 states that $\Phi_A \approx \Phi_W$ for all $(S, T)$.

**Lemma 1.** Under the conditions of Theorem 1,

$$\max_{S, T} \|\Phi_A(S, T) - \Phi_W(S, T)\|^2 = O_P((\log K)n^{-1}) .$$

(13)

Lemma 2 states that the support functions of $\mathcal{G}$ and $\mathcal{G}_n$ and of $\mathcal{F}$ and $\mathcal{F}_n$ converge.

**Lemma 2.** Under the conditions of Theorem 1,

$$\sup_{\|H\|=1} |\Gamma_{\mathcal{G}_n}(H) - \Gamma_{\mathcal{G}}(H)| \leq O_P(K(\log n)n^{-1/2})$$

(14)

$$\sup_{\|H\|=1} |\Gamma_{\mathcal{F}_m}(H) - \Gamma_{\mathcal{F}}(H)| \leq O_P(K(\log m)m^{-1/2}),$$

(15)

which implies

$$d_{\text{Haus}}(\text{conv}(\mathcal{G}_n), \text{conv}(\mathcal{G})) \leq O_P(K(\log n)n^{-1/2})$$

$$d_{\text{Haus}}(\text{conv}(\mathcal{F}_m), \text{conv}(\mathcal{F})) \leq O_P(K(\log m)m^{-1/2}).$$

Lemma 3 states that the sets $\mathcal{F}$ and $\mathcal{G}$ are essentially convex.

**Lemma 3.** It holds that

$$d_{\text{Haus}}(\text{conv}(\mathcal{G}), \mathcal{G}) = 0$$

(16)

$$d_{\text{Haus}}(\text{conv}(\mathcal{F}), \mathcal{F}) = 0.$$  

(17)
5.3 Proof of Theorem 1

Proof of Theorem 1. We bound \( \|\Phi_W(S, T) - \Phi_\omega(S, \tau)\|^2 \) uniformly over \( S \), as follows:

\[
\|\Phi_W(S, T) - \Phi_\omega(S, \tau)\|^2 = \sum_{s=1}^{K} \sum_{t=1}^{K} (\|\Phi_W(S, T)\|_{st} - \|\Phi_\omega(S, \tau)\|_{st})^2
\]

\[
= \sum_{s=1}^{K} \sum_{t=1}^{K} \frac{1}{m^2} \langle 1_{S=s}, 1_{T=t} \rangle \langle 1_{S=s}, 1_{T=t} \rangle
\]

\[
\leq \sum_{s=1}^{K} \sum_{t=1}^{K} \left( \frac{1}{m} \|1_{S=s}\|^2 \right) \left( \frac{1}{m} \|1_{T=t} - 1_{T=t}\|^2 \right)
\]

\[
\leq \left( \sum_{s=1}^{K} \left( \frac{1}{m} \|1_{S=s}\|^2 \right) \right) \left( \sum_{t=1}^{K} \left( \frac{1}{m} \|1_{T=t} - 1_{T=t}\|^2 \right) \right)
\]

(18)

where (18) holds because \( m^{-1} \sum_{s=1}^{K} \|1_{S=s}\|^2 = 1 \).

By Lemma 2 and Lemma 3, it holds that \( d_{\text{Haus}}(\text{conv}(G_n), G) = O_P(K(\log n)n^{-1/2}) \). Given \( T \), let \( \tau \equiv \tau_T \) denote the minimizer of \( \|G_T - G_\tau\| = \langle G_T - G_\tau, G_T - G_\tau \rangle \). It follows that

\[
\max_T \|G_T - G_\tau\|^2 = \max_T \sum_{t=1}^{K} \frac{1}{m} \|1_{T=t} - 1_{T=t}\|^2 + \|\pi_T - \pi_\tau\|^2
\]

\[
= O_P \left( \frac{K^2 \log n}{n} \right)
\]

(19)

Combining (13), (19), and (18) yields

\[
\max_{S, T} \|\Phi_A(S, T) - \Phi_\omega(S, \tau_T)\|^2 + \|\pi_T - \pi_\tau\|^2 = O_P \left( \frac{K^2 \log n}{n} \right),
\]

establishing (1).

The proof of (2) proceeds in similar fashion. The quantity \( \|\Phi_\omega(S, \tau) - \Phi_\omega(\sigma, \tau)\|^2 \) may be bounded uniformly over \( \tau \):

\[
\|\Phi_\omega(S, \tau) - \Phi_\omega(\sigma, \tau)\|^2 = \sum_{s=1}^{K} \sum_{t=1}^{K} (\|\Phi_\omega(S, \tau)\|_{st} - \|\Phi_\omega(\sigma, \tau)\|_{st})^2
\]

\[
= \sum_{s=1}^{K} \sum_{t=1}^{K} \langle f_{S=s} - f_{\sigma=s}, 1_{T=t} \rangle^2
\]

\[
\leq \left( \sum_{s=1}^{K} \|f_{S=s} - f_{\sigma=s}\|^2 \right),
\]

(20)
where all steps parallel the derivation of (18). It follows from Lemma 2 and 3 that $d_{	ext{Haus}}(\text{conv}(\mathcal{F}_m), \mathcal{F}) = O_P(K(\log m)m^{-1/2})$. Given $S$, let $\sigma \equiv \sigma_S$ denote the minimizer of $\|F_S - F_\sigma\|$, so that

$$\max_S \sum_{t=1}^K \|f_{S=t} - f_{\sigma=t}\|^2 + \|\pi_S - \pi_\sigma\|^2 = O_P \left( \frac{K^2 \log m}{m} \right).$$

(21)

Combining (21) and (20) yields

$$\max_{S,\tau} \|\Phi_\omega(S, \tau) - \Phi_\omega(\sigma_S, \tau)\|^2 + \|\pi_S - \pi_\sigma\|^2 = O_P \left( \frac{K^2 \log m}{m^2} \right),$$

establishing (2) and completing the proof.

5.4 Proof of Lemmas 1 – 3

The proof of Lemma 2 will rely on Lemma 4, which is a very slight modification of Lemma 4.3 in [4]. Lemma 4 is proven in the Appendix.

**Lemma 4.** Let $\mathbb{H}$ denote a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $g : \mathcal{Y} \mapsto \mathbb{H}$, and let $y_1, \ldots, y_n \in \mathcal{Y}$ be i.i.d. Let $L_n : \mathbb{H}^K \mapsto \mathbb{R}$ be defined as

$$L_n(H) = \frac{1}{n} \sum_{j=1}^n \max_{k \in [K]} \langle h_k, g(y_j) \rangle, \quad H = (h_1, \ldots, h_K) \in \mathbb{H}^K.$$  

(22)

Let $\mathcal{H} = \{ H \in \mathbb{H}^K : \|h_k\| \leq 1, t \in [K] \}$. It holds that

$$\mathbb{E} \sup_{H \in \mathcal{H}} |L_n(H) - \mathbb{E}L_n(H)| \leq 2K \left( \frac{\mathbb{E}\|g(y)\|^2}{n} \right)^{1/2}.$$  

To prove Lemma 3, we will require a theorem for finite dimensional convex hulls:

**Theorem 3.** [27, Thm 1.1.4] If $B \subset \mathbb{R}^d$ and $x \in \text{conv}(B)$, there exists $B_1, \ldots, B_{d+1}$ such that $x \in \text{conv}\{B_1, \ldots, B_{d+1}\}$.

Additionally, we will also require some results on Hilbert-Schmidt integral operators. A kernel function $\omega : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ is Hilbert-Schmidt if it satisfies

$$\int_{\mathcal{X} \times \mathcal{Y}} |\omega(x, y)|^2 dxdy < \infty.$$  

It can be seen that $\omega$ defined by Definition 1 is Hilbert-Schmidt. Let $\Omega$ denote the integral operator induced by $\omega$, given by

$$(\Omega f)(x) = \int_{\mathcal{Y}} \omega(x, y)f(y)dy.$$
It is known that a Hilbert-Schmidt operator $\Omega$ is a limit (in operator norm) of a sequence of finite rank operators, so that its kernel $\omega$ has singular value decomposition given by

$$\omega(x, y) = \sum_{q=1}^{\infty} \lambda_q u_q(x)v_q(y),$$

where $\{u_q\}_{q=1}^{\infty}$ and $\{v_q\}_{q=1}^{\infty}$ are sets of orthonormal functions mapping $X \mapsto \mathbb{R}$ and $Y \mapsto \mathbb{R}$, and $\lambda_1, \lambda_2, \ldots$ are scalars decreasing in magnitude and satisfying $\sum_{q=1}^{\infty} \lambda_q^2 < \infty$.

**Proof of Lemma 1.** Given $(S, T)$, let $\Delta \in [-1, 1]^{K \times K}$ denote the quantity

$$\Delta_{st} = \frac{1}{nm} \sum_{i=1}^{m} \sum_{j=1}^{n} (A_{ij} - W_{ij}) 1(S_i = s, T_j = t).$$

It holds that $\mathbb{E}[\Delta|W] = 0$, and by Hoeffding’s inequality,

$$\mathbb{P}(|\Delta_{st}| \geq \epsilon|W) \leq 2e^{-2nm\epsilon^2}, \quad s, t \in [K].$$

Conditioned on $W$, each entry of $\Delta$ is independent of the others. Given $\delta \in [-1, 1]^{K \times K}$, it follows that

$$\mathbb{P}(\Delta = \delta|W) = \prod_{s=1}^{K} \prod_{t=1}^{K} \mathbb{P}(\Delta_{st} = \delta_{st}|W) \leq 2^{K^2} \exp\left(-2nm \sum_{s=1}^{K} \sum_{t=1}^{K} \delta_{st}^2\right).$$

Let $B$ denote the set

$$B = \left\{ \delta \in [-1, 1]^{K \times K} : \sum_{s,t} \delta_{st}^2 \geq \epsilon, \delta \in \text{supp}(\Delta) \right\}.$$

The cardinality of $B$ is smaller than the support of $\Delta$, which is less than $(nm)^{K^2}$ when conditioned on $W$. It follows by a union bound over $B$ that

$$\mathbb{P}(\Delta \in B|W) \leq 2|B|e^{-2nm\epsilon} \leq 2(nm)^{K^2}e^{-2nm\epsilon}.$$

It can be seen that $\|\Phi_A(S, T) - \Phi_W(S, T)\|^2 = \sum_{s,t} \Delta_{st}^2$, implying that $\Delta \in B$ is equivalent to the event that $\|\Phi_A(S, T) - \Phi_W(S, T)\|^2 \geq \epsilon$. A union bound over all $S, T$ implies that

$$\mathbb{P}\left(\max_{S,T} \|\Phi_A(S, T) - \Phi_W(S, T)\|^2 \geq \epsilon \right) \leq 2K^{n+m}(nm)^{K^2}e^{-2nm\epsilon}.$$

Letting $\epsilon = C(1 + n/m)(\log K)n^{-1}$ for some $C$ proves the lemma. \qed
Proof of Lemma 2. Let \( g_y \in [0, 1]^m \) denote the column of \( W \) induced by \( y \in \mathcal{Y} \), and let \( f_x \in [0, 1]^n \) denote the row of \( \omega \) corresponding to \( x \in \mathcal{X} \):
\[
g_y(i) = \omega(x, y), \quad i \in [m] \quad \text{and} \quad f_x(y) = \omega(x, y), \quad y \in \mathcal{Y}.
\]
Algebraic manipulation shows that \( g_{T=t}, g_{\tau=t}, f_{S=s}, \) and \( f_{\sigma=s} \) can be written as
\[
g_{T=t} = \frac{1}{m} \sum_{j=1}^{m} g_y(T_j = t) \quad g_{\tau=t} = \int_{\mathcal{Y}} g_y(\tau(y) = t) \, dy
\]
\[
f_{S=s} = \frac{1}{m} \sum_{i=1}^{m} f_x(S_i = s) \quad f_{\sigma=s} = \int_{\mathcal{X}} f_x(\sigma(x) = s) \, dx.
\]
Given \( H = (h_1, \ldots, h_K, \pi_H) \), it follows that the inner products \( \langle H, G_T \rangle, \langle H, G_\tau \rangle, \langle H, F_S \rangle, \) and \( \langle H, F_\sigma \rangle \) equal
\[
\langle H, G_T \rangle = \frac{1}{n} \sum_{j=1}^{n} \left[ \left\langle h_{T_j}, \frac{g_y}{\sqrt{m}} \right\rangle + \pi_H(T_j) \right], \quad \langle H, G_\tau \rangle = \int_{\mathcal{Y}} \left\langle h_{\tau(y)}, \frac{g_y}{\sqrt{m}} \right\rangle + \pi_H(\tau(y)) \, dy
\]
\[
\langle H, F_S \rangle = \frac{1}{m} \sum_{i=1}^{m} \left[ \left\langle h_{S_i}, f_x \right\rangle + \pi_H(S_i) \right], \quad \langle H, F_\sigma \rangle = \int_{\mathcal{X}} \left\langle h_{\sigma(x)}, f_x \right\rangle + \pi_H(\sigma(x)) \, dx,
\]
and hence that the support functions equal
\[
\Gamma_{g_n}(H) = \frac{1}{n} \sum_{j=1}^{n} \max_{k \in [K]} \left[ h_k, \frac{g_y}{\sqrt{m}} \right] + \pi_H(k), \quad \Gamma_g(H) = \int_{\mathcal{Y}} \max_{k \in [K]} \left[ h_k, \frac{g_y}{\sqrt{m}} \right] + \pi_H(k) \, dy
\]
\[
\Gamma_{f_n}(H) = \frac{1}{m} \sum_{i=1}^{m} \max_{k \in [K]} \left[ h_k, f_x \right] + \pi_H(k), \quad \Gamma_f(H) = \int_{\mathcal{X}} \max_{k \in [K]} \left[ h_k, f_x \right] + \pi_H(k) \, dx,
\]
which implies that \( \mathbb{E} \Gamma_{g_n}(H) = \Gamma_g(H) \) and \( \mathbb{E} \Gamma_{f_n}(H) = \Gamma_f(H) \).

To show (14), we observe that \( \Gamma_{g_n} \) can be rewritten as
\[
\Gamma_{g_n}(H) = \frac{1}{n} \sum_{j=1}^{n} \max_{k \in [K]} \left[ \left[ \frac{h_k}{\pi_H(k)} \right], \left[ \frac{m^{-1/2}g_y}{1} \right] \right],
\]
which matches (22) so that Lemma 4 can be applied. Applying Lemma 4 results in
\[
\mathbb{E} \sup_{\|H\|=1} \left| \Gamma_{g_n}(H) - \Gamma_g(H) \right| \leq \frac{4K}{\sqrt{n}}, \tag{23}
\]
where we have used \( \{ H : \| H \| = 1 \} \subset \mathcal{H} \) and \( \left\| \left[ \frac{m^{-1/2}g_y}{1} \right] \right\|^2 \leq 2 \).

Let \( Z(y_1, \ldots, y_n) = \sup_{\|H\|=1} \left| \Gamma_{g_n}(H) - \Gamma_g(H) \right| \). For \( \ell \in [n] \), changing \( y_\ell \) to \( y'_\ell \) changes \( Z \) by at most \( 4/n \). Applying McDiarmid’s inequality yields
\[
\mathbb{P} \left( |Z - \mathbb{E}Z| \geq \epsilon \right) \leq 2e^{-2\epsilon^2 n/8}.
\]
Letting $\epsilon = n^{-1/2} \log n$ implies that $Z - \mathbb{E}Z = O_P(n^{-1/2} \log n)$, which combined with (23) implies (14).

To show (15), we observe that
\[
\Gamma_{\mathcal{F}_m}(H) = \frac{1}{m} \sum_{i=1}^{m} \max_{k \in [K]} \left\langle \left[ h_k \right], \left[ \frac{f_{x_i}}{1} \right] \right\rangle,
\]
so that Lemma 4 and McDiarmid’s inequality can be used analogously to the proof of (14).

We divide the proof of Lemma 3 into two sub-lemmas, one showing (16) and the other showing (17). This is because the proof of (17) will require additional work, due to the fact that the elements of $\mathcal{F}$ are infinite dimensional.

**Lemma 5.** For each $G^* \in \text{conv}(G)$, there exists $G_1, G_2, \ldots \in G$ such that $\lim_{\ell \to \infty} \|G^* - G_\ell\| = 0$.

**Lemma 6.** For each $F^* \in \text{conv}(F)$, there exists $F_1, F_2, \ldots \in F$ such that $\lim_{\ell \to \infty} \|F^* - F_\ell\| = 0$.

**Proof of Lemma 5.** Recall the definition of $g_y \in [0,1]^m$ as defined in the proof of Lemma 4:
\[
g_y(i) = \omega(x_i, y), \quad i \in [m],
\]
and that $g_{\tau=t}$ can be written as
\[
g_{\tau=t} = \int_{\mathcal{Y}} g_y 1\{\tau(y) = t\} \, dy.
\]

We note the following properties of $\{g_y : y \in \mathcal{Y}\}$:

- **P1:** Each $G^* \in \text{conv}(G)$ is a finite convex combination of elements in $G$. This holds by Theorem 3, since $G$ is a subset of $[0,1]^{mK} + K$, a finite dimensional space.

- **P2:** For all $\epsilon$, there exists a finite set $B$ that is an $\epsilon$-cover of $\{g_y : y \in \mathcal{Y}\}$ in Euclidean norm. This holds because $\{g_y : y \in \mathcal{Y}\}$ is a subset of the unit cube $[0,1]^m$.

By P1, each $G^* \in \text{conv}(G)$ can be written as a finite convex combination of elements in $G$, so that for some integer $N > 0$ there exists $G_{\tau_1}, \ldots, G_{\tau_N} \in G$ such that
\[
G^* = \sum_{i=1}^{N} \eta_i G_{\tau_i},
\]
where $\eta$ is in the $N$-dimensional unit simplex. It follows that for some $\mu : \mathcal{Y} \mapsto [0,1]^K$ satisfying $\sum_k \mu_k(y) = 1$ for all $y$, $G^* \equiv (g^*_1, \ldots, g^*_K, \pi^*_G)$ satisfies
\[
g^*_k = \int_{\mathcal{Y}} g_y \mu_k(y) \, dy \quad \text{and} \quad \pi^*_G(k) = \int_{\mathcal{Y}} \mu_k(y) \, dy, \quad k \in [K].
\]
We now construct \( \tau : \mathcal{X} \mapsto [K] \) inducing \( G_\tau \in \mathcal{G} \) which approximates \( G^* \in \text{conv}(\mathcal{G}) \). By P2, let \( \mathcal{B} \) denote an \( \varepsilon \)-cover of \( \{g_y : y \in \mathcal{Y}\} \), and enumerate its elements as \( b_1, \ldots, b_{|\mathcal{B}|} \). For each \( y \in \mathcal{Y} \), let \( \ell : \mathcal{Y} \mapsto [|\mathcal{B}|] \) assign \( y \) to its closest member in \( \mathcal{B} \), so that \( \|g_y - b_{\ell(y)}\| \leq \varepsilon \). For \( i = 1, \ldots, |\mathcal{B}| \), let \( \mathcal{Y}_i \) denote the set \( \{y : \ell(y) = i\} \). Arbitrarily divide each region \( \mathcal{Y}_i \) into \( K \) disjoint sub-regions \( \mathcal{Y}_{i1}, \ldots, \mathcal{Y}_{iK} \) such that \( \cup_k \mathcal{Y}_{ik} = \mathcal{Y}_i \), where the measure of each sub-region is given by
\[
\int_{\mathcal{Y}_{ik}} 1 \, dy = \int_{\mathcal{Y}_i} \mu_k(y) \, dy, \quad k \in [K].
\] (24)

Let \( \tau : \mathcal{Y} \mapsto [K] \) assign each region \( \mathcal{Y}_{ik} \) to \( k \), so that
\[
\tau(y) = k \quad \text{for all} \quad y \in \mathcal{Y}_{ik}, \ i = 1, \ldots, |\mathcal{B}|.
\]

By (24), it holds that \( \pi_\tau = \pi^*_G \), and also that
\[
g_{\tau=k} - g^*_k = \int_{\mathcal{Y}} g_y \left[ \{\tau(y) = k\} - \mu_k(y) \right] \, dy
\]
\[
= \int_{\mathcal{Y}} \left[ b_{\ell(y)} + g_y - b_{\ell(y)} \right] \left[ \{\tau(y) = k\} - \mu_k(y) \right] \, dy
\]
\[
= \sum_{i=1}^{|\mathcal{B}|} b_i \left[ \int_{\mathcal{Y}_{ik}} 1 \, dy - \int_{\mathcal{Y}_i} \mu_k(y) \, dy \right] + \int_{\mathcal{Y}} (g_y - b_{\ell(y)}) \left[ \{\tau(y) = k\} - \mu_k(y) \right] \, dy
\]
\[
= 0 + \int_{\mathcal{Y}} (g_y - b_{\ell(y)}) \left[ \{\tau(y) = k\} - \mu_k(y) \right] \, dy,
\]
which implies that
\[
\|g_{\tau=k} - g^*_k\| \leq \left\| \int_{\mathcal{Y}} (g_y - b_{\ell(y)}) \{\tau(y) = k\} \, dy \right\| + \left\| \int_{\mathcal{Y}} (g_y - b_{\ell(y)}) \mu_k(y) \, dy \right\|
\]
\[
\leq 2 \int_{\mathcal{Y}} \|g_y - b_{\ell(y)}\| \, dy
\]
\[
\leq 2\varepsilon.
\]
It follows that \( \|G_\tau - G^*\|^2 = \sum_{k=1}^K m^{-1} \|g_{\tau=k} - g^*_k\|^2 + \|\pi_\tau - \pi^*_G\|^2 \leq 4K\varepsilon^2 m^{-1} \), and hence that \( \lim_{\varepsilon \to 0} \|G_\tau - G^*\| = 0 \), proving the lemma.

\[\square\]

**Proof of Lemma 6.** Recall the definition of \( f_\varepsilon : \mathcal{Y} \mapsto [0, 1] \) as defined in the proof of Lemma 4:
\[
f_\varepsilon(y) = \omega(x, y),
\]
and that \( f_{\sigma=s} \) can be written as
\[
f_{\sigma=s} = \int_{\mathcal{X}} f_\varepsilon \{\sigma(x) = s\} \, dx.
\]

18
Because \( \{ f_x : x \in \mathcal{X} \} \) is not finite dimensional, the arguments of Lemma 5 do not directly apply. To circumvent this, we will approximate the space \( \mathcal{F} \) by a finite dimensional \( \hat{\mathcal{F}} \), such that the convex hulls \( \text{conv}(\mathcal{F}) \) and \( \text{conv}(\hat{\mathcal{F}}) \) converge.

For \( Q = 1, 2, \ldots \), let \( \omega_Q \) be the best rank-Q approximation to \( \omega \),

\[
\omega_Q(x, y) = \sum_{q=1}^{Q} \lambda_q u_q(x) v_q(y).
\]

Given \( D > 0 \), let \( \hat{u}_q \) denote a truncation of \( u_q \), defined as

\[
\hat{u}_q^D(x) = \begin{cases} 
D & \text{if } u_q(x) \geq D \\
u_q(x) & \text{if } -D \leq u_q(x) \leq D \\
-D & \text{if } u_q(x) \leq -D,
\end{cases}
\]

and let \( \hat{\omega} : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R} \) be defined as

\[
\hat{\omega}(x, y) = \sum_{q=1}^{Q} \lambda_q \hat{u}_q(x) v_q(y).
\]

Let \( \hat{f}_x : \mathcal{Y} \mapsto \mathbb{R} \) and \( \hat{f}_{\sigma=s} \) be defined as

\[
\hat{f}_x(y) = \hat{\omega}(x, y) \quad \text{and} \quad \hat{f}_{\sigma=s} = \int_{\mathcal{X}} \hat{f}_x \mathbf{1}\{ \sigma(x) = s \} \, dx.
\]

Let \( \hat{F}_\sigma \) and \( \hat{\mathcal{F}} \) be defined as

\[
\hat{F}_\sigma = (\hat{f}_{\sigma=1}, \ldots, \hat{f}_{\sigma=K}, \pi_{\sigma}) \quad \text{and} \quad \hat{\mathcal{F}} = \{ \hat{F}_\sigma : \sigma \in [K]^{\mathcal{X}} \}.
\]

We bound the difference \( \| \hat{f}_x - f_x \|^2 \):

\[
\| \hat{f}_x - f_x \|^2 = \sum_{q=1}^{Q} \lambda_q^2 (\hat{u}_q(x) - u_q(x))^2 + \sum_{q=Q+1}^{\infty} \lambda_q^2 u_q(x)^2,
\]

where we used the fact \( f_x = \sum_{q=1}^{\infty} \lambda_q u_q(x) v_q \), and that the functions \( \{ v_q \} \) are orthonormal. It follows that

\[
\int_{\mathcal{X}} \| \hat{f}_x - f_x \|^2 \, dx = \sum_{q=1}^{Q} \lambda_q^2 \int_{\mathcal{X}} (\hat{u}_q(x) - u_q(x))^2 \, dx + \sum_{q=Q+1}^{\infty} \lambda_q^2 \int_{\mathcal{X}} u_q(x)^2 \, dx
\]

\[
= \sum_{q=1}^{Q} \lambda_q^2 \int_{\mathcal{X}} (\hat{u}_q(x) - u_q(x))^2 \, dx + \sum_{q=Q+1}^{\infty} \lambda_q^2
\]

\[
\leq \sum_{q=1}^{Q} \lambda_q^2 \int_{x : |u_q(x)| \geq D} u_q(x)^2 \, dx + \sum_{q=Q+1}^{\infty} \lambda_q^2.
\]
from whence it can be seen that
\[
\lim_{\min(Q,D) \to \infty} \int_{\mathcal{X}} \|\hat{f}_x - f_x\|^2 \, dx = 0.
\]

We use this result to bound \(\|\hat{f}_{\sigma=s} - f_{\sigma=s}\|\):
\[
\max_{s,\sigma} \|\hat{f}_{\sigma=s} - f_{\sigma=s}\|^2 = \max_{s,\sigma} \left\| \int_{\mathcal{X}} (\hat{f}_x - f_x) 1_{\sigma=s}(x) \, dx \right\|^2 \\
\leq \int_{\mathcal{X}} \|\hat{f}_x - f_x\|^2 \, dx \\
\to 0 \text{ as } \min(Q, D) \to \infty.
\]

Since \(\|\hat{F}_\sigma - F_\sigma\|^2 = \sum_{k=1}^K \|\hat{f}_{\sigma=k} - f_{\sigma=k}\|^2 + \|\pi_\sigma - \pi_\sigma\|^2\), it follows that for any \(\epsilon > 0\), there exists \((Q, D)\) inducing \(\hat{F} = \{\hat{F}_\sigma : \sigma \in [K]^X\}\) such that
\[
\sup_{\sigma} \|\hat{F}_\sigma - F_\sigma\| \leq \epsilon,
\]
so that the support functions of \(\mathcal{F}\) and \(\hat{\mathcal{F}}\) can be bounded by
\[
\sup_{H, \|H\|=1} |\Gamma_{\mathcal{F}}(H) - \Gamma_{\hat{\mathcal{F}}}(H)| \leq \max_{\|H\|=1,\sigma} \left| \langle H, F_\sigma - \hat{F}_\sigma \rangle \right| \\
\leq \max_{\sigma} \|F_\sigma - \hat{F}_\sigma\| \\
\leq \epsilon,
\]
implying that
\[
d_{\text{Haus}}(\text{conv}(\mathcal{F}), \text{conv}(\hat{\mathcal{F}})) \leq \epsilon,
\]
which in turn implies that for any \(F^* \in \text{conv}(\mathcal{F})\), there exists \(\hat{F}^* \in \text{conv}(\hat{\mathcal{F}})\) such that \(\|F^* - \hat{F}^*\| \leq \epsilon\).

For any choice of \((Q, D)\), we observe that properties P1 and P2 as described in Lemma 5 for \(\mathcal{G}\) also hold for \(\hat{\mathcal{F}}\):

P1: Each \(\hat{F} \in \text{conv}(\hat{\mathcal{F}})\) is a finite convex combination of elements in \(\hat{\mathcal{F}}\). This holds because each \(\hat{f}_x\) can be written as
\[
\hat{f}_x = \sum_{q=1}^Q \lambda_q \hat{\mu}_q(x) v_q,
\]
showing that \(\{\hat{f}_x : x \in \mathcal{X}\}\) is a finite dimensional subspace of \(\mathcal{Y} \mapsto \mathbb{R}\), and hence \(\hat{\mathcal{F}}\) is as well, allowing Theorem 3 to be applied.

P2: For all \(\epsilon\), there exists a finite \(\epsilon\)-cover of \(\{\hat{f}_x : x \in \mathcal{X}\}\) in Euclidean norm. This holds because the set \(\{\hat{u}(x) : x \in \mathcal{X}\}\) is a subset of the hypercube \([-D, D]^Q\).
As a result, the same arguments used to prove Lemma 5 also apply to \( \tilde{\mathcal{F}} \), implying that for each \( \tilde{\mathcal{F}} \in \text{conv}(\mathcal{F}) \), there exists for any \( \epsilon > 0 \) a mapping \( \sigma : \mathcal{X} \mapsto [K] \) such that
\[
\| \tilde{\mathcal{F}}_\sigma - \tilde{\mathcal{F}} \| \leq 4K\epsilon^2.
\] (27)

It thus follows that for any \( \epsilon > 0 \) and \( F^* \in \text{conv}(\mathcal{F}) \), there exists \( \tilde{F}^* \in \text{conv}(\tilde{\mathcal{F}}) \) and \( \sigma : \mathcal{X} \mapsto [K] \) such that
\[
\| F^* - F_\sigma \| \leq \| F^* - \tilde{F}^* \| + \| \tilde{F}^* - \tilde{F}_\sigma \| + \| \tilde{F}_\sigma - F_\sigma \|
\]
\[
\leq 2\epsilon + 4\epsilon^2 K.
\]

As a result, it follows that there exists \( F_1, F_2, \ldots \in \mathcal{F} \) such that \( \lim_{i \to \infty} \| F^* - F_i \| = 0. \)

**Proof of Lemma 3.** Lemma 3 follows immediately from Lemmas 5 and 6, which establish (16) and (17) respectively.

\[ \square \]

### A Proof of Lemma 4

To prove Lemma 4, we will use a result from [4], which we state and prove here:

**Lemma 7.** [4, Lemma 4.3] Let \( \mathbb{H} \) denote a Hilbert space, and let \( g : \mathcal{Y} \mapsto \mathbb{H} \). Let \( y_1, \ldots, y_n \in \mathcal{Y} \) be i.i.d, and let \( L_n : \mathbb{H}^K \mapsto \mathbb{R} \) be defined as follows:

\[
L_n(H) = \frac{1}{n} \sum_{j=1}^{n} \max_{t \in [K]} \langle h_t, g(y_j) \rangle, \quad H = (h_1, \ldots, h_K) \in \mathbb{H}^K
\]

Let \( B = \{ H \in \mathbb{H}^K : \| h_k \| \leq 1, k \in [K] \} \). Then the following three statements hold:

\[
\mathbb{E} \sup_{H \in B} L_n(H) - \mathbb{E} L_n(H) \leq 2\mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \max_{t \in [K]} \langle h_t, g(y_j) \rangle, \quad (28)
\]

where \( \epsilon_1, \ldots, \epsilon_j \overset{iid}{\sim} \pm 1 \) w.p. 1/2,

\[
\mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \max_{t \in [K]} \langle h_t, g(y_j) \rangle \leq 2K \mathbb{E} \sup_{\| h \| = 1} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \langle h, g(y_j) \rangle, \quad (29)
\]

and

\[
\mathbb{E} \sup_{\| h \| = 1} \frac{1}{n} \sum_{j=1}^{n} \epsilon_i \langle h, g(y_j) \rangle \leq \left( \frac{\mathbb{E} \| g(y) \|^2}{n} \right)^{1/2}. \quad (30)
\]
Proof of Lemma 7. (28) is a standard symmetrization argument [9]. Letting \(y_1', \ldots, y_j'\) denote i.i.d Uniform \([0, 1]\) random variables, and \(\epsilon_1, \ldots, \epsilon_n \overset{iid}{\sim} \pm 1\) w.p. 1/2, it holds that

\[
\mathbb{E} \sup_{H \in B} L_n(H) - \mathbb{E} L_n(H) \leq \mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \max_{t \in [K]} \langle h_t, g(y_j) \rangle - \max_{t \in [K]} \langle h_t, g(y_j') \rangle
\]

\[
= \mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \left( \max_{t \in [K]} \langle h_t, g(y_j) \rangle - \max_{t \in [K]} \langle h_t, g(y_j') \rangle \right)
\]

\[
\leq \mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \max_{t \in [K]} \langle h_t, g(y_j) \rangle + \mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \max_{t \in [K]} \langle h_t, g(y_j') \rangle
\]

\[
= 2\mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \max_{t \in [K]} \langle h_t, g(y_j) \rangle.
\]

To show (29), let \(\mathcal{R}(\mathcal{F})\) denote the (non-absolute valued) Rademacher complexity of a function class \(\mathcal{F}\):

\[
\mathcal{R}(\mathcal{F}) = \mathbb{E} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j f(y_j).
\]

The following contraction principles for Rademacher complexity hold: [4, 22]

1. \(\mathcal{R}(|\mathcal{F}|) \leq \mathcal{R}(\mathcal{F})\), where \(|\mathcal{F}| = \{|f| : f \in \mathcal{F}\}\). [8, Thm 11.6]

2. \(\mathcal{R}(\mathcal{F}_1 \oplus \mathcal{F}_2) \leq \mathcal{R}(\mathcal{F}_1) + \mathcal{R}(\mathcal{F}_2)\), where \(\mathcal{F}_1 \oplus \mathcal{F}_2 = \{f_1 + f_2 : (f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2\}\).

For \(K = 2\), (29) follows from the following steps,

\[
\mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \max_{t \in [2]} \langle h_t, g(y_j) \rangle = \frac{1}{2} \mathbb{E} \left\{ \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \left[ \langle h_1, g(y_j) \rangle + \langle h_2, g(y_j) \rangle \right. \right.
\]

\[
\left. \left. + \left| \langle h_1, g(y_j) \rangle - \langle h_2, g(y_j) \rangle \right| \right] \right\}
\]

\[
= \mathbb{E} \left\{ \sup_{\|h_1\|_1 = 1} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \langle h_1, g(y_j) \rangle + \sup_{\|h_2\|_1 = 1} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \langle h_2, g(y_j) \rangle \right\}
\]

\[
= K \mathbb{E} \sup_{\|h\|_1 = 1} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \langle h, g(y_j) \rangle,
\]

which holds by \(\max(a, b) = (a+b+|a-b|)/2\) and the contraction principles. The induction rule for general \(K\) is straightforward, using the fact that \(\max(a_1, \ldots, a_K) = \max(\max(a_1, \ldots, a_{K-1}), a_K)\).
To show (30), observe that

$$\mathbb{E} \sup_{\|h\|=1} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \langle h, g(y_j) \rangle = \mathbb{E} \sup_{\|h\|=1} \left( \frac{1}{n} \sum_{j=1}^{n} \epsilon_j g(y_j) \right)$$

$$= \mathbb{E} \left\| \frac{1}{n} \sum_{j=1}^{n} \epsilon_j g(y_j) \right\|$$

$$\leq \left( \mathbb{E} \left\| \frac{1}{n} \sum_{j=1}^{n} \epsilon_j g(y_j) \right\|^2 \right)^{1/2}$$

$$= \left( \frac{1}{n} \mathbb{E} \| g(y_1) \|^2 \right)^{1/2}.$$

Proof of Lemma 4. (28) - (30) imply that

$$\mathbb{E} \sup_{H \in B} L_n(H) - \mathbb{E} L_n(H) \leq K \left( \frac{\mathbb{E} \| g(y) \|^2}{n} \right)^{1/2}. \quad (31)$$

It also holds that

$$\mathbb{E} \inf_{H \in B} L_n(H) - \mathbb{E} L_n(H) \geq 2 \mathbb{E} \inf_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \max_{t \in [K]} \langle h_t, g(y_j) \rangle$$

$$= -2 \mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} (-\epsilon_j) \max_{t \in [K]} \langle h_t, g(y_j) \rangle$$

$$= -2 \mathbb{E} \sup_{H \in B} \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \max_{t \in [K]} \langle h_t, g(y_j) \rangle$$

$$\geq -2 K \left( \frac{\mathbb{E} \| g(y) \|^2}{n} \right)^{1/2}, \quad (32)$$

where the first inequality holds by a symmetrization analogous to (28); the second by algebraic manipulation; the third because $\epsilon_1, \ldots, \epsilon_n$ are $\pm 1$ with probability $1/2$; the fourth by (29) and (30).

Combining (31) and (32) proves the lemma. \qed

B Proof of Theorem 2

Preliminaries

Let $\mathcal{D} = \{c \in [0, 1]^d : \|c\| \leq 1\}$, $\mathcal{S} = [K] \times \mathcal{D}$, $\mathcal{T} = [K] \times \mathcal{D}$, and $\Theta = [0, 1]^{K \times K}$. Let $\bar{\mathcal{D}}$ denote the smallest $\epsilon$-cover in 2-norm of $\mathcal{D}$. Let $\bar{\mathcal{S}} = [K] \times \bar{\mathcal{D}}$ and let $\bar{\mathcal{T}} = [K] \times \bar{\mathcal{D}}$. Let $K = |\bar{\mathcal{S}}| = |\bar{\mathcal{T}}| \leq K(\sqrt{de^{-1}})^d$. 

23
As described in Section 4.3, recall that we may write \( S, T, \sigma \) and \( \tau \) as \( S = (U, B), T = (V, D), \sigma = (\mu, \beta), \) and \( \tau = (\nu, \delta) \). Given \( S = (U, B) \), let \( \tilde{S} \) denote its closest approximation in \( \mathbb{S}^m \). This means that \( \tilde{S} = (U, \tilde{B}) \), with \( \tilde{B} \in \mathbb{D}^m \) satisfying \( \tilde{B}_i = \arg\min_{c \in \mathbb{D}} \| B_i - c \| \) for \( i \in [m] \). Similarly, given \( T = (V, D) \) or \( \tau = (\nu, \delta) \), let \( \tilde{T} = (V, \tilde{D}) \) or \( \tilde{\tau} = (\nu, \delta) \) be defined analogously.

Let \( Z \in [0, 1]^{m \times n} \) be defined by

\[
Z_{ij} = \tilde{B}_i^T \tilde{D}_j W_{ij},
\]

and let \( \Phi_Z(U, V) \) be defined by

\[
[\Phi_Z(U, V)]_{uv} = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n Z_{ij} 1\{U_i = u, V_j = v\}.
\]

Let \( \Phi_\xi(U, \nu) \) and \( \Phi_\xi(\mu, \nu) \) denote population versions of \( \Phi_Z \), defined by

\[
[\Phi_\xi(U, \nu)]_{uv} = \frac{1}{m} \sum_{i=1}^m \int_Y \tilde{B}_i^T \delta(y) \omega(x_i, y) 1\{U_i = u, \nu(y) = v\} \, dy,
\]

\[
[\Phi_\xi(\mu, \nu)]_{uv} = \int_X \tilde{\beta}(x)^T \tilde{\delta}(y) \omega(x, y) 1\{\mu(x) = u, \nu(y) = v\} \, dx \, dy.
\]

Let \( \pi^B_{U=k}, \pi^D_{V=k}, \pi^\beta_{\mu=k}, \) and \( \pi^\delta_{\nu=k} \) be defined for \( k \in [K] \) as

\[
\pi^B_{U=k} = \frac{1}{m} \sum_{i=1}^m \tilde{B}_i \tilde{B}_i^T 1\{U_i = k\} \quad \pi^D_{V=k} = \frac{1}{n} \sum_{j=1}^n \tilde{D}_j \tilde{D}_j^T 1\{V_j = k\}
\]

\[
\pi^\beta_{\mu=k} = \int_X \tilde{\beta}(x)^T \tilde{\beta}(x) 1\{\mu(x) = k\} \, dx \quad \pi^\delta_{\nu=k} = \int_Y \tilde{\delta}(y)^T \tilde{\delta}(y) 1\{\nu(y) = k\} \, dy.
\]

We observe that \( \sum_{k=1}^K \| \pi^B_{U=k} \|_F \leq 1 \), since by triangle inequality,

\[
\sum_{k=1}^K \| \pi^B_{U=k} \|_F \leq \frac{1}{m} \sum_{i=1}^m \| \tilde{B}_i \tilde{B}_i^T \|_F \leq 1,
\]

where we have used \( \| \tilde{B}_i \| \leq 1 \) for all \( \tilde{B}_i \in \mathbb{D} \).

Recall the definitions for \( g_y \in [0, 1]^m \) and \( f_x : \mathcal{Y} \mapsto [0, 1] \):

\[
g_y(i) = \omega(x_i, y) \quad \text{and} \quad f_x(y) = \omega(x, y).
\]

Define for \( k \in [K] \) the matrices \( g^B_{V=k} \) and \( g^\delta_{\nu=k} \) in \([0, 1]^{m \times d}\), and the functions \( f^B_{U=k} \) and \( f^\delta_{\mu=k} \) mapping \( \mathcal{Y} \mapsto \mathcal{D} \):

\[
g^B_{V=k} = \frac{1}{n} \sum_{j=1}^n g_{y_j} \tilde{D}_j^T 1\{V_j = k\} \quad g^\delta_{\nu=k} = \int_\mathcal{Y} g_y \tilde{\delta}(y)^T 1\{\nu(y) = k\} \, dy
\]

\[
f^B_{U=k} = \frac{1}{m} \sum_{i=1}^m f_{x_i} \tilde{B}_i 1\{U_i = k\} \quad f^\delta_{\mu=k} = \int_X f_x \tilde{\beta}(x) 1\{\mu(x) = k\} \, dx.
\]
Define the matrix $1^B_{U=u} \in [0, 1]^{m \times d}$ and function $1^\delta_{v=v} : \mathcal{Y} \mapsto \mathcal{D}$

$$1^B_{U=u}(i, j) = \begin{cases} \bar{B}_i(j) & \text{if } U_i = u \\ 0 & \text{otherwise} \end{cases} \quad 1^\delta_{v=v}(y) = \begin{cases} \bar{\delta}(y) & \text{if } v(y) = v \\ 0 & \text{otherwise} \end{cases}.$$

We observe that $m^{-1} \sum_{k=1}^K \|1^B_{U=k}\|^2 \leq 1$ since $\|\bar{B}_i\|^2 \leq 1$ for all $\bar{B}_i \in \mathcal{D}$. Analogous to $G_T, G_r, F_S, F_\sigma$ as defined in Section 5.1, let $G_V^D, G^\delta_{\nu}, F_U^\delta$, and $F^\beta_{\mu}$ be defined by:

$$G_V^D = \left( \frac{g_{V=1}}{\sqrt{m}}, \ldots, \frac{g_{V=K}}{\sqrt{m}}, \frac{\pi_{V=1}}{\sqrt{m}}, \ldots, \frac{\pi_{V=K}}{\sqrt{m}}, \frac{\Psi_{\bar{\nu}}}{Kd} \right) \quad G^\delta_{\nu} = \left( \frac{g_{\nu=1}}{\sqrt{m}}, \ldots, \frac{g_{\nu=K}}{\sqrt{m}}, \frac{\pi_{\nu=1}}{\sqrt{m}}, \ldots, \frac{\pi_{\nu=K}}{\sqrt{m}}, \frac{\Psi_{\bar{\nu}}}{Kd} \right)$$

$$F_U^B = \left( f_{U=1}, \ldots, f_{U=K}, \frac{\pi_{U=1}}{\sqrt{m}}, \ldots, \frac{\pi_{U=K}}{\sqrt{m}}, \frac{\Psi_{\bar{\nu}}}{Kd} \right) \quad F^\beta_{\mu} = \left( f_{\mu=1}, \ldots, f_{\mu=K}, \frac{\pi_{\mu=1}}{\sqrt{m}}, \ldots, \frac{\pi_{\mu=K}}{\sqrt{m}}, \frac{\Psi_{\bar{\nu}}}{Kd} \right).$$

Define the sets $\bar{\mathcal{F}}_m, \bar{\mathcal{G}}_n, \bar{\mathcal{F}},$ and $\bar{\mathcal{G}}$ by

$$\bar{\mathcal{F}}_m = \{ F^B_{U} : \bar{S} = (U, \bar{B}) \in \bar{S}^m \} \quad \bar{\mathcal{F}} = \{ F^\beta_{\mu} : \bar{\sigma} = (\mu, \bar{\beta}) \in \mathcal{X} \mapsto \bar{S} \}$$

$$\bar{\mathcal{G}}_n = \{ G^D_V : \bar{T} = (V, \bar{D}) \in \bar{T}^n \} \quad \bar{\mathcal{G}} = \{ G^\delta_{\nu} : \bar{\tau} = (\nu, \bar{\delta}) \in \mathcal{Y} \mapsto \bar{T} \}.$$

### B.1 Intermediate Results for Proof of Theorem 2

Lemmas 8 and 9 are analogs to Lemmas 2 and 3.

**Lemma 8.** Under the conditions of Theorem 2,

$$\sup_{\|H\|=1} |\Gamma_{\bar{\mathcal{G}}_n}(H) - \Gamma_{\bar{\mathcal{G}}}(H)| \leq O_P(\bar{K}(\log n)n^{-1/2}) \quad (33)$$

$$\sup_{\|H\|=1} |\Gamma_{\bar{\mathcal{F}}_m}(H) - \Gamma_{\bar{\mathcal{F}}}(H)| \leq O_P(\bar{K}(\log m)m^{-1/2}), \quad (34)$$

which implies

$$d_{\text{Haus}}(\text{conv}(\bar{\mathcal{G}}_n), \text{conv}(\bar{\mathcal{G}})) \leq O_P(\bar{K}(\log n)n^{-1/2})$$

$$d_{\text{Haus}}(\text{conv}(\bar{\mathcal{F}}_m), \text{conv}(\bar{\mathcal{F}})) \leq O_P(\bar{K}(\log m)m^{-1/2}).$$

**Lemma 9.** It holds that

$$d_{\text{Haus}}(\text{conv}(\bar{\mathcal{G}}), \bar{\mathcal{G}}) = 0 \quad (35)$$

$$d_{\text{Haus}}(\text{conv}(\bar{\mathcal{F}}), \bar{\mathcal{F}}) = 0. \quad (36)$$

Lemmas 10 - 12 bound various error terms that appear in the proof of Theorem 2. They bound on the approximation error that arises when substituting $(\bar{S}, \bar{T})$, and also the differences $|R_A(\bar{S}, \bar{T}; \theta) - R_W(\bar{S}, \bar{T}; \theta)|, |R_W(\bar{S}, \bar{T}; \theta) - R_\omega(\bar{S}, \bar{T}; \theta)|$ and $|R_\omega(\bar{S}, \bar{T}; \theta) - R_\omega(\bar{\sigma}, \bar{\tau}; \theta)|$.  

25
Lemma 10. It holds that
\[ |R_A(S, T; \theta) - R_A(S, \bar{T}; \theta)| \leq 12\varepsilon \] (37)
\[ |R_\omega(S, \tau; \theta) - R_\omega(S, \bar{\tau}; \theta)| \leq 12\varepsilon \]
and that
\[ \|\Psi_S - \Psi_S\|^2 \leq Kd\varepsilon \]
\[ \|\Psi_{\bar{\tau}} - \Psi_{\bar{\tau}}\|^2 \leq Kd\varepsilon \] (38)
\[ \|\Psi_T - \Psi_T\|^2 \leq Kd\varepsilon \]
\[ \|\Psi_{\bar{\tau}} - \Psi_{\bar{\tau}}\|^2 \leq Kd\varepsilon. \]

Lemma 11. If $\bar{K} \leq n^{1/2}$, it holds that
\[ |R_A(S, \bar{T}; \theta) - R_W(S, \bar{T}; \theta) - C_1| \leq 2\bar{K}O_P(\bar{K}(\log n)(n^{-1})) \] (39)

Lemma 12. Given $\bar{T} = (V, \bar{D}) \in \mathcal{T}^n$, let $\bar{\tau} = (\nu, \bar{\delta}) \in \mathcal{Y} \mapsto \bar{T}$ minimize $\|G_{\nu}^\beta - G_{\bar{\nu}}^\beta\|$. It holds that
\[ |R_W(S, \bar{T}; \theta) - R_\omega(S, \bar{\tau}; \theta) - C_2| \leq O_P(\bar{K}(\log n)n^{-1/2}). \] (40)

Given $S = (U, \bar{B}) \in \tilde{S}^n$, let $\bar{\sigma} = (\mu, \bar{\beta}) \in \mathcal{X} \mapsto \tilde{S}$ minimize $\|F_{\nu}^\beta - F_{\bar{\nu}}^\beta\|$. It holds that
\[ |R_\omega(S, \bar{\tau}; \theta) - R_\omega(\bar{\sigma}, \bar{\tau}; \theta) - C_3| \leq O_P(\bar{K}(\log m)m^{-1/2}). \] (41)

B.2 Proof of Theorem 2

Proof of Theorem 2. Given $\bar{T} = (V, \bar{D})$, let $\bar{\tau} = (\nu, \bar{\delta})$ minimize $\|G_{\nu}^\beta - G_{\bar{\nu}}^\beta\|$, which by Lemmas 8 and 9 is bounded by $O_P(\bar{K}(\log n)n^{-1/2})$. Using this fact and (38), the quantity $\|\Psi_T - \Psi_\tau\|^2$ can be bounded by
\[ \|\Psi_T - \Psi_\tau\|^2 \leq 2\|\Psi_T - \Psi_T\|^2 + 2\|\Psi_T - \Psi_\tau\|^2 \]
\[ \leq 2\|\Psi_T - \Psi_\tau\|^2 + 2\|G_{\nu}^\beta - G_{\bar{\nu}}^\beta\|^2 \]
\[ \leq 2Kd\varepsilon + O_P(\bar{K}(\log n)(n^{-1})). \] (42)

Using (37), (39), (40), (41), and (42), it holds for $\bar{K} \leq n^{1/2}$ that
\[ |R_A(S, T; \theta) - R_\omega(S, \bar{\tau}; \theta) - C_1 - C_2| + \frac{\|\Psi_T - \Psi_\tau\|^2}{Kd} \leq |R_A(S, T; \theta) - R_A(S, \bar{T}; \theta)| \]
\[ + |R_A(S, \bar{T}; \theta) - R_W(S, \bar{T}; \theta) - C_1| \]
\[ + |R_W(S, \bar{T}; \theta) - R_\omega(S, \bar{\tau}; \theta) - C_2| \]
\[ + |R_\omega(S, \bar{\tau}; \theta) - R_\omega(S, \bar{\tau}; \theta)| \]
\[ + \frac{\|\Psi_T - \Psi_\tau\|^2}{Kd} \]
\[ \leq 26\varepsilon + O_P\left(\bar{K}^2\log(n)\right) + O_P\left(\bar{K}\bar{K}\log(n)\right). \] (43)
Using $\bar{K} \leq K(d^{1/2} \epsilon^{-1})^d$ and letting $\epsilon = \left( \frac{K^{2d/2} \log n}{n^{1/2}} \right)^{1/d}$ yields that $\bar{K} \leq n^{1/2}$ eventually, so that substituting into (43) yields

$$|R_A(S, T; \theta) - R_\omega(S, \bar{\tau}; \theta) - C_1 - C_2| + \frac{\|\Psi_T - \Psi_T\|^2}{Kd} \leq O_P \left( d^{1/2} \left( \frac{K^2 \log n}{n^{1/2}} \right)^{1/d} \right),$$

proving (5).

Similarly, it holds that

$$|R_\omega(S, \tau; \theta) - R_\omega(\bar{\sigma}, \tau; \theta) - C_3| + \frac{\|\Psi_S - \Psi_\sigma\|^2}{Kd} \leq |R_\omega(S, \tau; \theta) - R_\omega(\bar{S}, \bar{\tau}; \theta)|$$

$$+ |R_\omega(\bar{S}, \bar{\tau}; \theta) - R_\omega(\bar{\sigma}, \bar{\tau}; \theta) - C_3|$$

$$+ |R_\omega(\bar{\sigma}, \bar{\tau}; \theta) - R_\omega(\bar{\sigma}, \tau; \theta)|$$

$$+ \frac{\|\Psi_S - \Psi_\sigma\|}{Kd}$$

$$\leq 26 \epsilon + O_P \left( K^{1/2} \log(m) \right) + O_P \left( K \bar{K} \log(m) \right),$$

and letting $\epsilon = \left( \frac{K^{2d/2} \log m}{m^{1/2}} \right)^{1/d}$ proves (6). \(\square\)

### B.3 Proof of Lemmas 8 - 12

**Proof of Lemma 8.** Let $H = (h_1, \ldots, h_K, \pi_1, \ldots, \pi_v, \Psi_H)$, where $h_k \in \mathbb{R}^{m \times d}$, $\pi_k \in \mathbb{R}^{d \times d}$, and $\Psi_H : [K] \times \mathcal{D} \mapsto [0, 1]$. Given $(v, d) \in [K] \times \mathcal{D}$, let $1_{v,d} : [K] \times \mathcal{D} \mapsto [0, 1]$ denote the indicator function

$$1_{v,d}(k, c) = 1\{v \leq k, d \leq c\}.$$

Given $G_V^D \in \mathcal{G}_n$ and $G_V^\delta \in \bar{G}$, the inner products $\langle H, G_V^D \rangle$ and $\langle H, G_V^\delta \rangle$ equal

$$\langle H, G_V^D \rangle = \sum_{k=1}^K \left\langle h_k, \frac{g_D^k}{\sqrt{m}} \right\rangle + \sum_{k=1}^K \left\langle \pi_k, \frac{\pi^D_k}{\sqrt{m}} \right\rangle + \frac{1}{Kd} \langle \Psi_H, \Psi_T \rangle$$

$$= \frac{1}{n} \sum_{j=1}^n \left[ \left\langle h_{V_j}, \frac{g_{D_j}}{\sqrt{m}} \right\rangle + \left\langle \pi_{V_j}, \bar{D}_j \bar{D}_j^T \right\rangle + \frac{1}{Kd} \langle \Psi_H, 1_{V_j,D_j} \rangle \right]$$

$$\langle H, G_V^\delta \rangle = \sum_{k=1}^K \left\langle h_k, \frac{g_\delta^k}{\sqrt{m}} \right\rangle + \sum_{k=1}^K \left\langle \pi_k, \frac{\pi^\delta_k}{\sqrt{m}} \right\rangle + \frac{1}{Kd} \langle \Psi_H, \Psi_T \rangle$$

$$= \int_\mathcal{Y} \left\langle h_{\nu(y)}, \frac{g_\delta(\nu(y))^T}{\sqrt{m}} \right\rangle + \left\langle \pi_{\nu(y)}, \bar{\delta}(y) \bar{\delta}(y)^T \right\rangle + \frac{1}{Kd} \langle \Psi_H, 1_{\nu(y),\bar{\delta}(y)} \rangle dy.$$
It follows that the support functions $\Gamma_{g_n}$ and $\Gamma_{\tilde{g}}$ equal

$$
\Gamma_{g_n}(H) = \frac{1}{n} \sum_{j=1}^{n} \max_{v \in [K], c \in D} \left[ \left\langle h_v, \frac{g_y c^T}{\sqrt{m}} \right\rangle \right.
$$

$$
+ \left\langle \pi_v, cc^T \right\rangle + \frac{1}{Kd} \left\langle \Psi_H, 1_v c \right\rangle 
$$

$$
= \frac{1}{n} \sum_{j=1}^{n} \max_{(v,c) \in T} \left[ \left[ \begin{array}{c} h_v c \\ \pi_v c c^T \end{array} \right] \right.
$$

$$
\left[ \begin{array}{c} \frac{1}{Kd} \end{array} \right] \left[ \begin{array}{c} \pi_v c c^T \\ \Psi_H \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]
$$

$$
= \frac{1}{n} \max_{v \in (K) \in C} \int \left[ \left[ \begin{array}{c} h_v c \\ \pi_v c c^T \end{array} \right] \right.
$$

$$
\left[ \begin{array}{c} \frac{1}{Kd} \end{array} \right] \left[ \begin{array}{c} \pi_v c c^T \\ \Psi_H \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] dy.
$$

Given $t = (v, c) \in \tilde{T}$, let $h'_t = \left[ \begin{array}{c} h_v c \\ \pi_v c c^T \end{array} \right]$. Since $\|c\| \leq 1$ and $\|(Kd)^{-1}v,c\| \leq 1/d$, it follows that $\|h'_t\|_F^2 \leq \|h_v\|_F^2 + \|\pi_v\|_F^2 + \|\Psi_H\|_F^2 / d^2$, where $\| \cdot \|_F$ denotes Frobenius norm, so that if $\|H\| \leq 1$, then $\|h'_t\| \leq 1$ for all $t \in \tilde{T}$. As a result, the proof of Lemma 2 can be copied here: Lemma 4 implies that

$$
\mathbb{E} \sup_{\|H\|=1} |\Gamma_{\tilde{g}}(H) - \Gamma_{g_n}(H)| \leq \frac{6K}{\sqrt{n}},
$$

and McDiarmid’s inequality applied to $Z = \sup_{\|H\|=1} |\Gamma_{\tilde{g}}(H) - \Gamma_{g_n}(H)|$ implies that $Z - \mathbb{E}Z = O_P(n^{-1/2} \log n)$.

The proof for $\sup_{\|H\|=1} |\Gamma_{F_m}(H) - \Gamma_{\tilde{F}}(H)|$ follow parallel arguments. \qed

**Proof of Lemma 9.** Enumerate the members of $\tilde{T}$ as $1, \ldots, \tilde{K}$. Given $(u, c) \in \tilde{T}$, let $t(u, c)$ denote its corresponding index in $1, \ldots, \tilde{K}$. Given $\tilde{T} = (V, \tilde{D}) \in \tilde{T}$, recall the definition of

$$
G_{\tilde{T}} = \left( \frac{g_{\tilde{T}=1}}{\sqrt{m}}, \ldots, \frac{g_{\tilde{T}=\tilde{K}}}{\sqrt{m}}, \pi_{\tilde{T}} \right),
$$

a vector in $\mathbb{R}^{m\tilde{K} + \tilde{K}}$. It can be seen that

$$
G_{\tilde{D}} = \left( \frac{g_{\tilde{V}=1}}{\sqrt{m}}, \ldots, \frac{g_{\tilde{V}=\tilde{K}}}{\sqrt{m}}, \pi_{\tilde{V}=1}, \ldots, \pi_{\tilde{V}=\tilde{K}}, \frac{\Psi_{\tilde{T}}}{Kd} \right)
$$

28
is a linear transformation of $G_T$, given by
\[
g^D_{V=k} = \sum_{c \in \mathcal{D}} g^{\mathcal{T} = t(k,c)} c^T, \quad k \in [K] \\
\pi^D_{V=k} = \sum_{c \in \mathcal{D}} \pi^{\mathcal{T} = t(k,c)} c c^T, \quad k \in [K] \\
\Psi_T = \sum_{k=1}^{K} \sum_{c \in \mathcal{D}} \pi^{t(k,c)} 1_{k,c}.
\]

By Lemma 3, it holds that $\mathcal{G} = \{ G_T : \bar{T} \in \mathcal{T} \}$ is convex. Since $\mathcal{G} = \{ G^D_T : \bar{T} = (V, \bar{D}) \in \mathcal{T} \}$ is related to $\mathcal{G}$ by a linear transform, and as linear transformations preserve convexity, it follows that $\mathcal{G}$ is also convex. By parallel arguments, it also follows that $\mathcal{F}$ is a linear transformation of $\mathcal{F}$ and hence convex as well. \[\square\]

**Proof of Lemma 10.** If $\| B_i - \bar{B}_i \| \leq \epsilon$ for all $i \in [m]$ and $\| D_i - \bar{D}_i \| \leq \epsilon$ for all $j \in [n]$, then
\[
|R_A(S, T; \theta) - R_A(\bar{S}, \bar{T}; \theta)| \leq \left| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (A_{ij} - B^T_i D_j \theta_{U_i V_j})^2 - (A_{ij} - \bar{B}^T_i \bar{D}_j \theta_{U_i V_j})^2 \right| \\
\leq 12 \epsilon,
\]
where we use the fact that $\| B_i \|, \| D_j \|, \theta_{uv}$, and $A_{ij}$ are all between 0 and 1. By similar arguments, it also holds that
\[
|R_\omega(S, \tau; \theta) - R_\omega(\bar{S}, \bar{\tau}; \theta)| \leq 12 \epsilon \\
|R_\omega(\sigma, \tau; \theta) - R_\omega(\sigma, \bar{\tau}; \theta)| \leq 12 \epsilon.
\]

We also show that $\| \Psi_S - \Psi_\bar{S} \|^2 \leq K d \epsilon$ by
\[
\| \Psi_S - \Psi_\bar{S} \|^2 = \sum_{k=1}^{K} \int_{[0,1]^d} \left[ \frac{1}{mn} \sum_{i=1}^{m} \left( \{ U_i \leq k, \beta_i \leq c \} - \{ U_i \leq k, \bar{\beta}_i \leq c \} \right) \right]^2 dc \\
\leq \sum_{k=1}^{K} \int_{[0,1]^d} \frac{1}{mn} \sum_{i=1}^{m} \left[ \{ U_i \leq k, \beta_i \leq c \} - \{ U_i \leq k, \bar{\beta}_i \leq c \} \right]^2 dc \\
= \frac{1}{m} \sum_{i=1}^{m} \sum_{k=1}^{K} \int_{[0,1]^d} \left| \{ U_i \leq k, \beta_i \leq c \} - \{ U_i \leq k, \bar{\beta}_i \leq c \} \right| dc \\
\leq K d \epsilon,
\]
where the first inequality holds by Jensen’s inequality, and the second inequality holds because $\| \beta_i - \bar{\beta}_i \| \leq \epsilon$ and the integral is over $[0,1]^d$. The quantities $\| \Psi_T - \Psi_\bar{T} \|^2, \| \Psi_\sigma - \Psi_\bar{\sigma} \|^2$, etc., are bounded similarly. \[\square\]
Proof of Lemma 11. Given $\theta \in [0, 1)^{K\times K}$, let $\bar{\theta} \in [0, 1)^{K\times K}$ be given by
$$\bar{\theta}_{st} = \bar{b}^T d \theta_{uw} \quad \text{for all } s = (u, \bar{b}) \in \bar{S}, t = (v, d) \in \bar{T}.$$ 

For $\bar{S} = (U, \bar{B}) \in \bar{S}^m$ and $\bar{T} = (V, \bar{D}) \in \bar{T}^n$,
$$R_A(\bar{S}, \bar{T}; \theta) - R_W(\bar{S}, \bar{T}; \theta) = C_1 - 2 \sum_{s \in \bar{S}} \sum_{t \in \bar{T}} ( [\Phi_A(\bar{S}, \bar{T})]_{st} - [\Phi_W(\bar{S}, \bar{T})]_{st} ) \bar{\theta}_{st},$$
(44)
where $C_1$ is constant in $(\bar{S}, \bar{T}, \theta)$. This implies
$$|R_A(\bar{S}, \bar{T}; \theta) - R_W(\bar{S}, \bar{T}; \theta) - C_1| \leq 2\| \Phi_A(\bar{S}, \bar{T}) - \Phi_W(\bar{S}, \bar{T}) \|_1$$
$$\leq 2K(\| \Phi_A(\bar{S}, \bar{T}) - \Phi_W(\bar{S}, \bar{T}) \|_2$$
$$\leq 2KO_P(K(\log n)n^{-1})$$
where the inequalities follow by (44), the equivalence of norms, and Lemma 1, which requires $K \leq n^{1/2}$. \hfill \Box

Proof of Lemma 12. It holds that
$$R_W(\bar{S}, \bar{T}; \theta) - R_{\omega}(\bar{S}, \bar{\tau}; \theta) = C_2 - 2 \sum_{u=1}^{K} \sum_{v=1}^{K} ( [\Phi_Z(U, V)]_{uv} - [\Phi_\zeta(U, V)]_{uv} ) \theta_{uv}$$
$$+ \sum_{u=1}^{K} \sum_{v=1}^{K} \left( \langle \bar{\pi}^b_{U=1}, \bar{\pi}^b_{V=1} \rangle - \langle \bar{\pi}^b_{U=1}, \bar{\pi}^b_{V=1} \rangle \right) \theta_{uv}^2,$$
where $C_2$ is constant in $\bar{S}, \bar{T}, \theta$ and $\bar{\tau}$. This implies
$$|R_W(\bar{S}, \bar{T}; \theta) - R_{\omega}(\bar{S}, \bar{\tau}; \theta) - C_2| \leq \| \Phi_Z(U, V) - \Phi_\zeta(U, \nu) \|_1$$
$$+ \left( \sum_{u=1}^{K} \| \bar{\pi}^b_{U=1} \| \right) \left( \sum_{v=1}^{K} \| \bar{\pi}^b_{V=1} - \bar{\pi}^b_{V=1} \| \right)$$
$$\leq K\| \Phi_Z(U, V) - \Phi_\zeta(U, \nu) \| + \sqrt{K} \left( \sum_{v=1}^{K} \| \bar{\pi}^b_{V=1} - \bar{\pi}^b_{V=1} \|^2 \right)^{1/2},$$
(45)
where the final inequality uses the fact that $\sum_{u=1}^{K} \| \bar{\pi}^b_{U=1} \| \leq 1$.

It can be seen that the entries of $\Phi_Z(U, V)$ and $\Phi_\zeta(U, \nu)$ equal the inner products
$$[\Phi_Z(U, V)]_{uv} = \frac{1}{m} \langle 1_{U=1}^\delta, g_{V=1}^\delta \rangle$$
$$[\Phi_\zeta(U, \nu)]_{uv} = \frac{1}{m} \langle 1_{U=1}^\delta, g_{V=1}^\delta \rangle,$$
which implies
$$\| \Phi_Z(U, V) - \Phi_\zeta(U, \nu) \|^2 \leq \left( \sum_{u=1}^{K} \frac{1}{m} \| 1_{U=1}^\delta \|^2 \right) \left( \sum_{v=1}^{K} \frac{1}{m} \| g_{V=1}^\delta - g_{V=1}^\delta \|^2 \right)$$
$$\leq \sum_{v=1}^{K} \frac{1}{m} \| g_{V=1}^\delta - g_{V=1}^\delta \|^2,$$
(46)
Given $G^D_U \in \tilde{G}_n$, let $C^3_U \in \tilde{G}$ minimize $\|G^D_U - C^3_U\|$. Using (45), (46) and Lemma 8 implies

$$|R_W(\tilde{S}, \tilde{T}; \theta) - R_\omega(\tilde{S}, \tilde{\tau}; \theta) - C_2| \leq \sqrt{2K}\|G^D_U - C^3_U\| \leq O_P(KK(\log n)n^{-1/2}).$$

Similarly, it holds that

$$R_\omega(\tilde{S}, \tilde{\tau}; \theta) - R_\omega(\tilde{\sigma}, \tilde{\tau}; \theta) = C_3 - 2\sum_{u=1}^{K} \sum_{v=1}^{K} ([\Phi_\zeta(U, \nu)]_{uv} - [\Phi_\zeta(\mu, \nu)]_{uv}) \theta_{uv}$$

$$+ \sum_{u=1}^{K} \sum_{v=1}^{K} \left(\langle \bar{\pi}^B_{U=u}, \bar{\pi}^\delta_{\nu=v} \rangle - \langle \bar{\pi}^{\beta}_{\mu=u}, \bar{\pi}^{\delta}_{\nu=v} \rangle \right) \theta_{uv}^2,$$

where $C_3$ is constant in $\tilde{S}, \tilde{\tau}, \tilde{\sigma}$ and $\theta$. This implies

$$|R_\omega(\tilde{S}, \tilde{\tau}; \theta) - R_\omega(\tilde{\sigma}, \tilde{\tau}; \theta) - C_3| \leq 2\|\Phi_\zeta(U, \nu) - \Phi_\zeta(\mu, \nu)\|_1$$

$$+ \left(\sum_{u=1}^{K} \|\pi^B_{U=u} - \pi^{\beta}_{\mu=u}\|\right) \left(\sum_{v=1}^{K} \|\pi^{\delta}_{\nu=v}\|\right)$$

$$\leq 2K\|\Phi_\zeta(U, \nu) - \Phi_\zeta(\mu, \nu)\| + \sqrt{K} \left(\sum_{u=1}^{K} \|\pi^B_{U=u} - \pi^{\beta}_{\mu=u}\|^2\right)^{1/2},$$

(47)

where we have used $\sum_v \|\pi^{\delta}_{\nu=v}\| \leq 1$.

The entries of $\Phi_\zeta(U, \nu)$, and $\Phi_\zeta(\mu, \nu)$ equal the inner products

$$[\Phi_\zeta(U, \nu)]_{uv} = \langle f^B_{U=u}, 1^\delta_{\nu=v} \rangle \quad \text{and} \quad [\Phi_\zeta(\mu, \nu)]_{uv} = \langle f^{\beta}_{\mu=u}, 1^\delta_{\nu=v} \rangle,$$

which implies

$$\|\Phi_\zeta(U, \nu) - \Phi_\zeta(\mu, \nu)\|^2 \leq \left(\sum_{u=1}^{K} \frac{1}{m} \|f^B_{U=u} - f^{\beta}_{\mu=u}\|^2\right) \left(\sum_{v=1}^{K} \frac{1}{m} \|1^\delta_{\nu=v}\|^2\right)$$

$$\leq \sum_{u=1}^{K} \frac{1}{m} \|f^B_{U=u} - f^{\beta}_{\mu=u}\|^2. \quad (48)$$

Given $F^B_U \in \overline{F}_m$, let $F^{\beta}_\mu \in \overline{F}$ minimize $\|F^B_U - F^{\beta}_\mu\|$. It follows from (47), (48), and Lemma 8 that

$$|R_\omega(\tilde{S}, \tilde{\tau}; \theta) - R_\omega(\tilde{\sigma}, \tilde{\tau}; \theta) - C_3| \leq \sqrt{2K}\|F^B_U - F^{\beta}_\mu\| \leq O_P(KK(\log m)m^{-1/2}).$$

□
References

[1] Edoardo M Airoldi, David M Blei, Stephen E Fienberg, and Eric P Xing. Mixed membership stochastic blockmodels. In Advances in Neural Information Processing Systems, pages 33–40, 2009.

[2] Edoardo M Airoldi, Thiago B Costa, and Stanley H Chan. Stochastic blockmodel approximation of a graphon: Theory and consistent estimation. In Advances in Neural Information Processing Systems, pages 692–700, 2013.

[3] Charalambos D Aliprantis and Kim C Border. Infinite Dimensional Analysis: A Hitchhiker’s Guide. Berlin: Springer-Verlag, 2006.

[4] Gérard Biau, Luc Devroye, and Gábor Lugosi. On the performance of clustering in hilbert spaces. Information Theory, IEEE Transactions on, 54(2):781–790, 2008.

[5] Peter J Bickel and Aiyou Chen. A nonparametric view of network models and newman–girvan and other modularities. Proceedings of the National Academy of Sciences, 106(50):21068–21073, 2009.

[6] Peter J Bickel, Aiyou Chen, Elizaveta Levina, et al. The method of moments and degree distributions for network models. The Annals of Statistics, 39(5):2280–2301, 2011.

[7] Vincent D Blondel, Jean-Loup Guillaume, Renaud Lambiotte, and Etienne Lefebvre. Fast unfolding of communities in large networks. Journal of Statistical Mechanics: Theory and Experiment, 2008(10):P10008, 2008.

[8] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. Concentration inequalities: A nonasymptotic theory of independence. Oxford University Press, 2013.

[9] Olivier Bousquet, Stéphane Boucheron, and Gábor Lugosi. Introduction to statistical learning theory. In Advanced Lectures on Machine Learning, pages 169–207. Springer, 2004.

[10] T Tony Cai, Xiaodong Li, et al. Robust and computationally feasible community detection in the presence of arbitrary outlier nodes. The Annals of Statistics, 43(3):1027–1059, 2015.

[11] Aiyou Chen, Arash A Amini, Elizaveta Levina, and Peter J Bickel. Fitting community models to large sparse networks. Ann. Stat., 41(arXiv: 1207.2340):2097–2122, 2012.

[12] David Choi and Patrick J Wolfe. Co-clustering separately exchangeable network data. The Annals of Statistics, 42(1):29–63, 2014.

[13] Aurelien Decelle, Florent Krzakala, Cristopher Moore, and Lenka Zdeborová. Asymptotic analysis of the stochastic block model for modular networks and its algorithmic applications. Physical Review E, 84(6):066106, 2011.
[14] Chao Gao, Yu Lu, and Harrison H Zhou. Rate-optimal graphon estimation. *arXiv preprint arXiv:1410.5837*, 2014.

[15] Michelle Girvan and Mark EJ Newman. Community structure in social and biological networks. *Proceedings of the national academy of sciences*, 99(12):7821–7826, 2002.

[16] Peter D Hoff, Adrian E Raftery, and Mark S Handcock. Latent space approaches to social network analysis. *Journal of the american Statistical association*, 97(460):1090–1098, 2002.

[17] Pengsheng Ji and Jiashun Jin. Coauthorship and citation networks for statisticians. *arXiv preprint arXiv:1410.2840*, 2014.

[18] Brian Karrer and Mark EJ Newman. Stochastic blockmodels and community structure in networks. *Physical Review E*, 83(1):016107, 2011.

[19] Olga Klopp, Alexandre B Tsybakov, and Nicolas Verzelen. Oracle inequalities for network models and sparse graphon estimation. *arXiv preprint arXiv:1507.04118*, 2015.

[20] Florent Krzakala, Cristopher Moore, Elchanan Mossel, Joe Neeman, Allan Sly, Lenka Zdeborová, and Pan Zhang. Spectral redemption in clustering sparse networks. *Proceedings of the National Academy of Sciences*, 110(52):20935–20940, 2013.

[21] Pierre Latouche, Etienne Birmelé, and Christophe Ambroise. Overlapping stochastic block models with application to the french political blogosphere. *The Annals of Applied Statistics*, pages 309–336, 2011.

[22] Michel Ledoux and Michel Talagrand. *Probability in Banach Spaces: isoperimetry and processes*, volume 23. Springer Science & Business Media, 2013.

[23] Elchanan Mossel, Joe Neeman, and Allan Sly. A proof of the block model threshold conjecture. *arXiv preprint arXiv:1311.4115*, 2013.

[24] MEJ Newman. Spectral community detection in sparse networks. *arXiv preprint arXiv:1308.6494*, 2013.

[25] Sofia C Olhede and Patrick J Wolfe. Network histograms and universality of blockmodel approximation. *Proceedings of the National Academy of Sciences*, 111(41):14722–14727, 2014.

[26] Karl Rohe, Tai Qin, and Bin Yu. Co-clustering for directed graphs: the stochastic co-blockmodel and spectral algorithm di-sim. *arXiv preprint arXiv:1204.2296*, 2012.

[27] Rolf Schneider. *Convex bodies: the Brunn–Minkowski theory*. Cambridge University Press, 2013.
[28] Daniel L Sussman, Minh Tang, Donniell E Fishkind, and Carey E Priebe. A consistent adjacency spectral embedding for stochastic blockmodel graphs. *Journal of the American Statistical Association*, 107(499):1119–1128, 2012.

[29] Daniel L Sussman, Minh Tang, and Carey E Priebe. Universally consistent latent position estimation and vertex classification for random dot product graphs. *arXiv preprint arXiv:1207.6745*, 2012.

[30] Amanda L Traud, Eric D Kelsic, Peter J Mucha, and Mason A Porter. Comparing community structure to characteristics in online collegiate social networks. *SIAM review*, 53(3):526–543, 2011.

[31] Larry Wasserman. *All of nonparametric statistics*. Springer Science & Business Media, 2006.

[32] Yunpeng Zhao, Elizaveta Levina, Ji Zhu, et al. Consistency of community detection in networks under degree-corrected stochastic block models. *The Annals of Statistics*, 40(4):2266–2292, 2012.