Dedicated to the memory of Andrei Aleksandrovich Gonchar and Herbert Stahl

M. Riesz-Schur-type inequalities for entire functions of exponential type

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Abstract. We prove a general M. Riesz-Schur-type inequality for entire functions of exponential type. If \( f \) and \( Q \) are two functions of exponential types \( \sigma > 0 \) and \( \tau > 0 \), respectively, and if \( Q \) is real-valued and the real zeros of \( Q \), not counting multiplicities, are bounded away from each other, then
\[
|f(x)| \leq (\sigma + \tau)(A_{\sigma+\tau}(Q))^{-1/2} \|Qf\|_{C(\mathbb{R})}, \quad x \in \mathbb{R},
\]
where
\[
A_{\sigma}(Q) \overset{\text{def}}{=} \inf_{x \in \mathbb{R}} (|Q'(x)|^2 + s^2|Q(x)|^2).
\]
We apply this inequality to the weights \( Q(x) \overset{\text{def}}{=} \sin(\tau x) \) and \( Q(x) \overset{\text{def}}{=} x \) and describe the extremal functions in the corresponding inequalities.

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§ 1. Introduction

For \( n \in \mathbb{N} \), let \( \mathbb{P}_n \) and \( \mathbb{T}_n \) denote the sets of algebraic and trigonometric polynomials of degree at most \( n \) with complex coefficients, respectively. Next, given \( \gamma > 0 \), let \( B_\gamma \) be the collection of entire functions of exponential type \( \gamma \). In other words, \( f \in B_\gamma \) if and only if
\[
\limsup_{|z| \to \infty} |f(z)| \exp(-\gamma + \varepsilon)|z| < \infty
\]
for every \( \varepsilon > 0 \). Finally, let \( C(\Omega) \) be the space of all continuous complex-valued functions \( f \) on \( \Omega \subset \mathbb{R} \) for which\(^1\)
\[
\|f\|_{C(\Omega)} \overset{\text{def}}{=} \sup_{t \in \Omega} |f(t)| < \infty.
\]
The M. Riesz-Schur inequality\(^2\)
\[
\|P\|_{C([-1,1])} \leq (n + 1)\|\sqrt{1 - t^2}P(t)\|_{C([-1,1])}, \quad P \in \mathbb{P}_n, \quad (1.1)
\]

\(^1\)With apologies for the repulsive \( \|f(t)\|_{C(\Omega)} \) notation that we occasionally use in this paper.
\(^2\)Hitherto, this used to be called the Schur Inequality; see, for instance, [1] and [2] for reasons why one should add Marcel Riesz's name to it.

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is well known and plays an essential role in approximation theory. In particular, the simplest proof of Markov's inequality is based on applying (1.1) to the Bernstein Inequality. We have no doubts that every person reading this paper has seen this done on countless occasions, so let us give just one such reference here, namely, M. Riesz’s [3], §3, Satz IV, p.359, from 1914, which is the granddaddy of them all.

The following M. Riesz-Schur-type

$$\|T\|_{C((-\pi,\pi))} \leq (n+1)\|\sin t \ T(t)\|_{C((-\pi,\pi))}, \quad T \in \mathbb{T}_n,$$  

(1.2)

and Schur-type

$$\|P\|_{C([-1,1])} \leq (n+1)\|tP(t)\|_{C([-1,1])}, \quad P \in \mathbb{P}_n,$$  

(1.3)

inequalities are less known. In fact, polling top experts, the third of us found not a single person who was aware of (1.2) despite it looking so deceptively similar to (1.1). Although for even trigonometric polynomials (1.2) is indeed equivalent to (1.1), we do not know if any simple trick could be used to extend it from even to all trigonometric polynomials. We found (1.2) hiding as [4], Lemma 15.1.3, p.567, with no reference to any original source, and, therefore, we must assume that [4] is the one and only place where the proof is presented and there is a good probability that the proof belongs to Q.I. Rahman.3 Inequality (1.3) was proved in a 1919 paper by Schur: see [5], (29), p.285. As a matter of fact, Schur proved a little more. Namely, for odd \(n\) the factor \(n+1\) in (1.3) can be replaced by \(n\).

In this note we discuss various versions of inequalities (1.2) and (1.3) for entire functions of exponential type. They are special cases of the following general M. Riesz-Schur-type inequality.

**Theorem 1.1.** Let \(\tau \geq 0\) and \(\sigma > 0\). Let \(Q \in B_{\tau}\) be a function that is real-valued on \(\mathbb{R}\) and let \(Z_Q \overset{\text{def}}{=} \{t_n\}\) denote the real zeros of \(Q\), not counting multiplicities. Assume that if \(\text{card} \ Z_Q > 1\), then \(\inf_{k \neq \ell} |t_k - t_\ell| > 0\). Let

$$A_s(Q) \overset{\text{def}}{=} \inf_{t \in \mathbb{R}}([Q'(t)]^2 + s^2[Q(t)]^2), \quad s \geq 0.$$  

(1.4)

Then the following two statements hold true.

(a) If \(f \in B_{\sigma}\), then we have

$$|f(x)| \leq (\sigma + \tau)(A_{\sigma + \tau}(Q))^{-1/2}\|Qf\|_{C(\mathbb{R})}, \quad x \in \mathbb{R}.$$  

(1.5)

(b) If equality holds in (1.5) for a point \(x = x_0 \in \mathbb{R}\) and for a function \(f \in B_{\sigma}\) that is real-valued on \(\mathbb{R}\), with \(f \not\equiv 0\), then either

$$|(Qf)(x_0)| = \|Qf\|_{C(\mathbb{R})},$$

or there exist two real constants \(S\) and \(C\) such that \(|S| + |C| > 0\) and

$$Q(x)f(x) \equiv S\sin((\sigma + \tau)x) + C\cos((\sigma + \tau)x), \quad x \in \mathbb{R}.$$  

(1.6)

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3Qazi Ibadur Rahman died on July 21, 2013, and, thus, the source of (1.2) may never be found.
Remark 1.2. Of course, if $\|Qf\|_{C(\mathbb{R})} = \infty$ in (1.5), then Theorem 1.1, (a) is self-evident and has no intrinsic value. The latter holds if, say, $f$ is a non-constant function of exponential type 0.

Remark 1.3. Our proof of Theorem 1.1 is based on the Duffin-Schaeffer inequality for entire functions of exponential type: see [6], Theorem II, p.555. The proof of Theorem 1.1, (a) is similar to the proof of Lemma 15.1.3 (p.567) in the book of Rahman and Schmeisser [4].

Remark 1.4. If equality holds in (1.5) for a point $x = x_0 \in \mathbb{R}$ and for a function $f \in B_\sigma$ that is real-valued on $\mathbb{R}$, then $f'(x_0) = 0$ so that if $f \not\equiv 0$ and $Q'(x_0) \neq 0$, then

$$|(Qf)(x_0)| < \|Qf\|_{C(\mathbb{R})}.$$  

Remark 1.5. Theorem 1.1, (b) shows that equality holds in (1.5) only in rather exceptional cases since $f$ in (1.6) must be an entire function (of order 1 and type at most $\sigma$) so that the zeros of $Q$, counting multiplicities, must be zeros of the right-hand side as well.

In the following two corollaries we apply Theorem 1.1 with the two M.Riesz-Schur weights that appear in (1.2) and (1.3).

**Corollary 1.6.** Let $\tau > 0$ and $\sigma > 0$. If $f \in B_\sigma$, then

$$|f(x)| \leq \left(\frac{\sigma}{\tau} + 1\right)\|\sin(\tau t)f(t)\|_{C(\mathbb{R})}, \quad x \in \mathbb{R}. \quad (1.7)$$

Moreover, equality holds in (1.7) for a point $x = x_0 \in \mathbb{R}$ such that $\cos(\tau x_0) \neq 0$ and for a function $f \in B_\sigma$ that is real-valued on $\mathbb{R}$ with $f \not\equiv 0$ if and only if $\sigma/\tau \in \mathbb{N}$ and

$$f(x) \equiv S \frac{\sin((\sigma + \tau)x)}{\sin(\tau x)}$$

for some real $S \neq 0$.

**Corollary 1.7.** Let $\sigma > 0$. If $f \in B_\sigma$, then

$$|f(x)| \leq \sigma\|tf(t)\|_{C(\mathbb{R})}, \quad x \in \mathbb{R}. \quad (1.8)$$

Moreover, equality holds in (1.8) for a point $x = x_0 \in \mathbb{R}$ and for a function $f \in B_\sigma$ that is real-valued on $\mathbb{R}$ with $f \not\equiv 0$ if and only if $f(x) \equiv S\sin(\sigma x)/x$ for some real $S \neq 0$.

Remark 1.8. Inequality (1.2) is a special case of (1.7) in Corollary 1.6, and, since (1.1) is a special case of (1.2), inequality (1.1) also follows from (1.7). It remains to be seen whether (1.3) can be derived either directly from Corollary 1.7 or from Theorem 1.1, (a).

First, we discuss some lemmas in §2, and then we give the proofs of these results in §3.

Throughout §§2 and 3 we will assume that $\tau \geq 0$, $\sigma > 0$, and that the function $Q \in B_\tau$ is real-valued on $\mathbb{R}$. When necessary, we will refer to the conditions

$$A_\theta(Q) > 0, \quad \text{where} \quad A_s(Q) \overset{\text{def}}{=} \inf_{t \in \mathbb{R}}([Q'(t)]^2 + s^2[Q(t)]^2), \quad s \geq \theta, \quad (1.9)$$

where $\theta > \tau$ is fixed, and

$$d > 0, \quad \text{where } d \overset{\text{def}}{=} \begin{cases} \inf_{k \neq \ell} |t_k - t_\ell|, & \text{card } Z_Q \geq 2, \\ 1 & \text{otherwise}, \end{cases}$$

(1.10)

where $Z_Q \overset{\text{def}}{=} \{t_n\}$ denote the real zeros of $Q$, not counting multiplicities.

Note that $A_s(Q) > 0$ for some particular value of $s > 0$ if and only if it is positive for every $s > 0$.

In what follows, the Lebesgue measure of a (Lebesgue) measurable set $E \subset \mathbb{R}$ will be denoted by $m(E)$.

§ 2. Some lemmas

We will discuss properties of $Q$ and of other entire functions of exponential type that are needed for the proof of Theorem 1.1. We start with two simple properties.

Lemma 2.1. (a) $A_s(Q)$ defined by (1.4) is a continuous nondecreasing function of $s \in [0, \infty)$.

(b) If $Q$ satisfies condition (1.9), then

$$\limsup_{x \to \infty} |Q(x)| > 0.$$  

Proof. (a) Clearly, $A_s(Q)$ is a nondecreasing function of $s \in [0, \infty)$. Let $c \in [0, \infty)$ be a fixed number. Then, for each $\varepsilon > 0$, there exists $u_0 = u_0(\varepsilon, c) \in \mathbb{R}$ such that

$$A_c(Q) \geq [Q'(u_0)]^2 + c^2(Q(u_0))^2 - \varepsilon,$$

and, therefore,

$$A_c(Q) \leq A_s(Q) \leq [Q'(u_0)]^2 + s^2(Q(u_0))^2 \leq A_c(Q) + (s^2 - c^2)[Q(u_0)]^2 + \varepsilon, \quad s \in (c, \infty),$$

from which

$$A_c(Q) \leq \lim_{s \to c^+} A_s(Q) \leq A_c(Q) + \varepsilon,$$

and, letting here $\varepsilon \to 0^+$, this proves right continuity of $A_s(Q)$ at $c$.

The left-continuity at $c > 0$ follows from the inequalities

$$A_s(Q) \leq A_c(Q) \leq \left(\frac{c}{s}\right)^2 A_s(Q), \quad s \in (0, c).$$

(b) Fix $s \geq \theta$. If $\limsup_{x \to \infty} |Q(x)| = 0$, that is, $\lim_{x \to \infty} Q(x) = 0$, then, for $x > 0$ and by the mean value theorem,

$$|x| \inf_{t \in [x, \infty)} |Q'(t)| \leq |Q(2x) - Q(x)|,$$

so that, dividing both sides by $|x|$ and letting $x \to \infty$ we get

$$\liminf_{x \to \infty} |Q'(x)| \leq \lim_{x \to \infty} \frac{|Q(2x) - Q(x)|}{|x|} = 0.$$

\footnote{Actually, $Q(x) = o(|x|)$ as $x \to \infty$ is sufficient here.}
and then
\[
\liminf_{x \to \infty} \left( [Q'(x)]^2 + s^2 [Q(x)]^2 \right) \leq \liminf_{x \to \infty} [Q'(x)]^2 + s^2 \lim_{x \to \infty} [Q(x)]^2 = 0,
\]
which contradicts (1.9). If \( \limsup_{x \to -\infty} |Q(x)| = 0 \), then we apply the previous part to \( Q(-x) \), which also satisfies the conditions. The proof is complete.

Next, we discuss some metric properties of \( Q \) and of other entire functions of exponential type.

**Definition 2.2.** Let \( 0 < \delta \leq L < \infty \). A measurable set \( E \subseteq \mathbb{R} \) is called \((L, \delta)\)-dense if for every interval \( \Delta \subset \mathbb{R} \) with \( m(\Delta) = L \) we have \( m(E \cap \Delta) \geq \delta \).

**Lemma 2.3.** Let \( 0 < \delta \leq L < \infty \) and \( \sigma > 0 \). If the measurable set \( E \subseteq \mathbb{R} \) is \((L, \delta)\)-dense, then there is a constant \( C_1 > 0 \), such that for every \( f \in B_\sigma \) the Remez-type inequality
\[
\|f\|_{C(\mathbb{R})} \leq C_1 \|f\|_{C(E)}
\] holds.

Lemma 2.3 is a special case of Katsnelson’s Theorem 3 in [7], where he proved it for polysubharmonic functions on \( \mathbb{C}^n \).

**Lemma 2.4.** Let \( \tau \geq 0 \) and \( \theta \geq \tau \). If \( Q \in B_{\tau} \) satisfies conditions (1.9) and (1.10), then there exists \( \alpha^* > 0 \) such that the set
\[
E^* \overset{\text{def}}{=} \{ x \in \mathbb{R} : |Q(x)| \geq \alpha^* \}
\]
is \((d/2, d/4)\)-dense, where \( d \) is defined in (1.10).

**Proof.** Let \( s \geq \theta \); see (1.9). It follows immediately from (1.9) that there exists a constant \( C_2 > 0 \) such that
\[
[Q'(x)]^2 + s^2 [Q(x)]^2 \geq C_2^2, \quad x \in \mathbb{R}.
\]
Next, given \( \alpha > 0 \), the open set \( E_\alpha \overset{\text{def}}{=} \{ x \in \mathbb{R} : |Q(x)| < \alpha \} \) can be represented as \( E_\alpha = \bigcup_{I \in \mathscr{A}_\alpha} I \), where \( \mathscr{A}_\alpha \) is a family of pairwise disjoint open intervals \( I \). Then (2.2) shows that for \( \alpha \in (0, C_2/s) \) the set \( E_\alpha \) does not contain zeros of \( Q' \). Therefore, \( Q \) is strictly monotone on each interval \( I \in \mathscr{A}_\alpha \).

If \( \alpha \in (0, C_2/s) \) and \( I \in \mathscr{A}_\alpha \) is a bounded interval then \( |Q| \) takes the same value at the endpoints of \( I \) so that \( I \) contains precisely one zero of \( Q \). If \( \alpha \in (0, C_2/s) \) and \( I \in \mathscr{A}_\alpha \) is an unbounded interval, say \((a, \infty)\), then, \( Q \) being monotone on \( I \), the finite or infinite \( \Gamma \overset{\text{def}}{=} \lim_{x \to \infty} Q(x) \) exists and, by Lemma 2.1, (b), \( \Gamma \neq 0 \). Hence, if \( \beta > 0 \) is sufficiently small, say, \( 0 < \beta < \alpha_1 \leq C_2/s \), then \( I \not\subseteq E_\beta \) so that \( I \not\in \mathscr{A}_\beta \). In other words if \( 0 < \alpha < \alpha_1 \leq C_2/s \), then \( \mathscr{A}_\alpha \) has no unbounded components.

Summarizing the above, there exists \( \alpha_1 \in (0, C_2/s) \) such that, for all \( \alpha \in (0, \alpha_1] \), each interval \( I \in \mathscr{A}_\alpha \) is bounded and it contains precisely one zero \( t_I \) of \( Q \). In particular, if \( Q \) has no real zeros, then \( E_\alpha = \varnothing \) for \( \alpha \in (0, \alpha_1] \).

\[\text{[5]}\text{Viktor Katsnelson informed us that B.Ja. Levin knew of this result, at least for} n = 1, \text{and lectured about it way before 1971 although he published it in the form of a Remez-type inequality only much later.}\]
It remains to estimate the length of each interval $I \in \mathscr{A}_\alpha$. Let $\alpha_2 \in (0, C_2/s]$ be the unique positive solution of the equation $\alpha/\sqrt{C_2^2 - \alpha^2s^2} = d/8$. Let $\alpha^* \overset{\text{def}}{=} \min\{\alpha_1, \alpha_2\}$. Taking account of (2.2) and using the mean value theorem, we see that for each $x \in I$ there exists $\xi \in I$ such that

$$|x - t_I| = \frac{|Q(x)|}{|Q'(\xi)|} \leq \frac{\alpha}{\sqrt{C_2^2 - \alpha^2s^2}} \leq \frac{d}{8}$$

whenever $0 < \alpha \leq \alpha^*$. Hence $m(I) \leq d/4$ for $\alpha \in (0, \alpha^*]$. Therefore, due to condition (1.10), for every closed interval $\Delta$ of length $d/2$, we have $m(\Delta \cap E_{\alpha^*}) \leq d/4$. This shows that $E^* \overset{\text{def}}{=} \mathbb{R} \setminus E_{\alpha^*}$ is $(d/2, d/4)$-dense. The lemma is proved.

Now we are in a position to prove the following Schur-type inequality.

**Lemma 2.5.** Let $\tau \geq 0$ and $\theta \geq \tau$. Let $Q \in \mathbb{B}_\tau$ satisfy conditions (1.9) and (1.10). Let $\sigma > 0$. Then there is a constant $C_3 > 0$ such that for every $f \in \mathbb{B}_\sigma$ we have

$$\|f\|_{C(\mathbb{R})} \leq C_3 \|Qf\|_{C(\mathbb{R})}. \quad (2.3)$$

**Proof.** Let the number $\alpha^* > 0$ and the $(d/2, d/4)$-dense set $E^* \subset \mathbb{R}$ be chosen with the help of Lemma 2.4. Then

$$\|f\|_{C(E^*)} \leq \frac{1}{\alpha^*} \|Qf\|_{C(\mathbb{R})} \quad (2.4)$$

so that (2.3) follows from Lemma 2.3.

§ 3. Proofs of the main results

**Proof of Theorem 1.1, (a).** Let $\tau$, $\sigma$ and $Q$ satisfy the conditions of Theorem 1.1, and let $f \in \mathbb{B}_\sigma$. Assume $A_{\sigma+\tau}(Q) > 0$, $f \neq 0$ and $\|Qf\|_{C(\mathbb{R})} < \infty$ since otherwise all claims are self-evident.

The proof will consist of three short steps. Using Lemma 2.5 with $\theta = \sigma + \tau$, we obtain $\|f\|_{C(\mathbb{R})} < \infty$ and this will be used in all three steps.

**Step 1.** First, we assume that $f$ is real-valued on $\mathbb{R}$ and that there exists $x^* \in \mathbb{R}$ such that $|f(x^*)| = \|f\|_{C(\mathbb{R})}$. Since $Qf \in \mathbb{B}_{\sigma+\tau}$, we can apply the Duffin-Schaeffer inequality to $Qf$ to get

$$[(Qf)'(x)]^2 + (\sigma + \tau)^2[(Qf)(x)]^2 \leq (\sigma + \tau)^2 \|Qf\|_{C(\mathbb{R})}^2, \quad x \in \mathbb{R}; \quad (3.1)$$

see [6], Theorem II. Since $f'(x^*) = 0$, we can set $x = x^*$ in (3.1) to arrive at

$$\|f\|_{C(\mathbb{R})}^2 = f^2(x^*) \leq \frac{(\sigma + \tau)^2 \|Qf\|_{C(\mathbb{R})}^2}{[Q'(x^*)]^2 + (\sigma + \tau)^2 [Q(x^*)]^2},$$

which implies (1.5) immediately.

**Step 2.** Second, we assume that $f$ is real-valued on $\mathbb{R}$, but $|f|$ does not necessarily take its maximum value on $\mathbb{R}$. Given $\varepsilon > 0$, pick $x_\varepsilon \in \mathbb{R}$ such that $\|f\|_{C(\mathbb{R})} \leq |f(x_\varepsilon)| + \varepsilon$. Next, observe that, for fixed $\delta > 0$, the function $F_\delta$ defined by

$$F_\delta(x) \overset{\text{def}}{=} f(x) \frac{\sin(\delta(x - x_\varepsilon))}{\delta(x - x_\varepsilon)}, \quad x \in \mathbb{R},$$

is real-valued on $\mathbb{R}$. Since

Given $\varepsilon > 0$, pick $x_\varepsilon \in \mathbb{R}$ such that $\|f\|_{C(\mathbb{R})} \leq |f(x_\varepsilon)| + \varepsilon$. Next, observe that, for fixed $\delta > 0$, the function $F_\delta$ defined by

$$F_\delta(x) \overset{\text{def}}{=} f(x) \frac{\sin(\delta(x - x_\varepsilon))}{\delta(x - x_\varepsilon)}, \quad x \in \mathbb{R},$$

is real-valued on $\mathbb{R}$. Since

$$\|f\|_{C(\mathbb{R})} \leq |f(x_\varepsilon)| + \varepsilon.$$
belongs to $B_{\sigma+\delta}$, is real-valued on $\mathbb{R}$, $|F_\delta(x)| \leq |f(x)|$ for $x \in \mathbb{R}$, and, since $F_\delta(x)$ goes to 0 as $x \to \pm\infty$, it takes its maximal value at some point in $\mathbb{R}$ so that, as proved in Step 1, we can use (1.5) to obtain

$$\|f\|_{C(\mathbb{R})} - \varepsilon \leq |f(x_\varepsilon)| = |F_\delta(x_\varepsilon)| \leq \|F_\delta\|_{C(\mathbb{R})} \leq (\sigma + \tau + \delta)(A_{\sigma+\tau+\delta}(Q))^{-1/2}\|QF_\delta\|_{C(\mathbb{R})} \leq (\sigma + \tau + \delta)(A_{\sigma+\tau+\delta}(Q))^{-1/2}\|Qf\|_{C(\mathbb{R})}.$$  \\

First, letting here $\varepsilon \to 0+$, we get

$$\|f\|_{C(\mathbb{R})} \leq (\sigma + \tau + \delta)(A_{\sigma+\tau+\delta}(Q))^{-1/2}\|Qf\|_{C(\mathbb{R})},$$

and then, letting $\delta \to 0+$ and using continuity of $A_s(Q)$ as proved in Lemma 2.1, (a), we obtain (1.5) for $f$ as well.

Step 3. Third, we assume that $f$ is not necessarily real-valued on $\mathbb{R}$. Then $f$ can be written as $f = f_1 + if_2$, where both $f_1$ and $f_2$ given by

$$f_1(z) \overset{\text{def}}{=} \frac{1}{2}(f(z) + f(\overline{z})), \quad f_2(z) \overset{\text{def}}{=} \frac{1}{2i}(f(z) - f(\overline{z})), $$

are real-valued on $\mathbb{R}$ and belong to $B_{\sigma}$.

Given $\varepsilon > 0$, pick $x_\varepsilon \in \mathbb{R}$ such that

$$f(x_\varepsilon) \neq 0 \quad \text{and} \quad \|f\|_{C(\mathbb{R})} \leq |f(x_\varepsilon)| + \varepsilon. $$

Let $\eta \in \mathbb{R}$ be defined by $\exp(i\eta) = f(x_\varepsilon)/|f(x_\varepsilon)|$, and let the function $G$ be defined by

$$G \overset{\text{def}}{=} \cos \eta \ f_1 + \sin \eta \ f_2. $$

Then $G$ is real-valued on $\mathbb{R}$ and satisfies the relations $|G(x)| \leq |f(x)|$ for $x \in \mathbb{R}$ and $|G(x_\varepsilon)| = |f(x_\varepsilon)|$. Applying (1.5) to $G$, we obtain

$$\|f\|_{C(\mathbb{R})} - \varepsilon \leq |f(x_\varepsilon)| = |G(x_\varepsilon)| \leq \|G\|_{C(\mathbb{R})} \leq (\sigma + \tau)(A_{\sigma+\tau}(Q))^{-1/2}\|QG\|_{C(\mathbb{R})} \leq (\sigma + \tau)(A_{\sigma+\tau}(Q))^{-1/2}\|Qf\|_{C(\mathbb{R})};$$

and, letting $\varepsilon \to 0+$, we again arrive at (1.5) for $f$ as well.

Thus, Theorem 1.1, (a) has been fully proved.

Proof of Theorem 1.1, (b). Let $\tau$, $\sigma$, and $Q$ satisfy the conditions of Theorem 1.1. If there exist $x = x_0 \in \mathbb{R}$ and $f \in B_{\sigma}$ that is real-valued on $\mathbb{R}$ such that equality holds in (1.5), then $\|Qf\|_{C(\mathbb{R})} < \infty$ and $|f(x_0)| = \|f\|_{C(\mathbb{R})} < \infty$, so that $f'(x_0) = 0$. Hence, again using (1.5) and the definition of $A_s(Q)$ in (1.4),

$$[(Qf)'(x_0)]^2 + (\sigma + \tau)^2[(Qf)(x_0)]^2 = (|Q'(x_0)|)^2 + (\sigma + \tau)^2[Q(x_0)]^2 \leq \frac{(\sigma + \tau)^2\|Qf\|^2_{C(\mathbb{R})}}{A_{\sigma+\tau}(Q)} \|Qf\|^2_{C(\mathbb{R})} \geq (\sigma + \tau)^2\|Qf\|^2_{C(\mathbb{R})}.$$  \\

Summarizing,

$$[(Qf)'(x_0)]^2 + (\sigma + \tau)^2[(Qf)(x_0)]^2 \geq (\sigma + \tau)^2\|Qf\|^2_{C(\mathbb{R})},$$
and, by the Duffin-Schaeffer inequality,
\[
[(Qf)'(x_0)]^2 + (\sigma + \tau)^2[(Qf)(x_0)]^2 \leq (\sigma + \tau)^2\|Qf\|_{C(R)}^2
\]
(see (3.1)), so that
\[
[(Qf)'(x_0)]^2 + (\sigma + \tau)^2[(Qf)(x_0)]^2 = (\sigma + \tau)^2\|Qf\|_{C(R)}^2.
\]
If \(|(Qf)(x_0)| < \|(Qf)\|_{C(R)}\), then this equality is possible if and only if \(Qf\) is of the form
\[
(Qf)(x) = S \sin((\sigma + \tau)x) + C \cos((\sigma + \tau)x)
\]
with some real constants \(S\) and \(C\) (see [6], Theorem II), and, since \(f \not\equiv 0\), we have \(|S| + |C| > 0\).

Thus, Theorem 1.1, (b) has been fully proved as well.

**Proof of Corollaries 1.6 and 1.7.** The weights \(Q(x) \equiv \sin(\tau x) \in B_\tau\) and \(Q(x) \equiv x \in B_0\) both satisfy conditions (1.9) and (1.10) of Theorem 1.1, and in both cases the corresponding \(A_s(Q)\) is easy to compute. Therefore, inequalities (1.7) and (1.8) follow immediately from (1.5).

Next we describe all extremal functions in Corollaries 1.6 and 1.7.

Let \(Q(x) \equiv \sin(\tau x)\) with \(\tau > 0\). If \(\sigma/\tau \in \mathbb{N}\), then
\[
f(x) \equiv S \frac{\sin((\sigma + \tau)x)}{\sin(\tau x)}
\]
is an extremal function for (1.7) in \(x = 0\). Let us assume that \(f \in B_\sigma\) with \(f \not\equiv 0\) is an extremal function for (1.7) at some \(x = x_0 \in \mathbb{R}\) such that \(\cos(\tau x_0) \neq 0\). Then, by Theorem 1.1, (b) and Remark 1.4,
\[
\sin(\tau x)f(x) \equiv S \sin((\sigma + \tau)x) + C \cos((\sigma + \tau)x), \quad x \in \mathbb{R},
\]
where \(S\) and \(C\) are real, and \(|S| + |C| > 0\). Clearly, \(C\) must be 0 since, otherwise, \(f\) is not even continuous at 0. Furthermore, \(\sin((\sigma + \tau)x)/\sin(\tau x)\) belongs to \(B_\sigma\) if and only if the zeros of the denominator are also zeros of the numerator. Then the numerator must vanish at \(\pi/\tau\) which is the first positive zero of the denominator, so that \(\sigma\) must be an integer multiple of \(\tau\). This proves Corollary 1.6.

Next let \(Q(x) \equiv x\). Then
\[
f(x) \equiv S \frac{\sin(\sigma x)}{x}
\]
is an extremal function for (1.8) at \(x = 0\). Moreover, if \(f \in B_\sigma\) with \(f \not\equiv 0\) is an extremal function for (1.8) at some \(x = x_0 \in \mathbb{R}\), then since \(Q' \equiv 1 \not\equiv 0\), by Theorem 1.1, (b) and Remark 1.4,
\[
x f(x) \equiv S \sin(\sigma x) + C \cos(\sigma x), \quad x \in \mathbb{R},
\]
where \(S\) and \(C\) are real, and \(|S| + |C| > 0\). As before, \(C\) must be 0. Thus, Corollary 1.7 has been proved as well.

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Addendum. Just before this English version went to print, Sergey Tikhonov informed us that (1.2) can be found in S. N. Bernstein’s Russian language book “Extremal Properties of Polynomials”; see Corollary III on page 167. We use this opportunity to point out regarding Bernstein’s claim the following. First, it is not true as stated because, for instance, \( s_n(\theta) \equiv 1 - \cos \theta \) is a counterexample. Second, it becomes true if one additionally assumes that \( s_n(\pi) = 0 \) as well, in which case one can factor out \( \sin \theta \) from \( s_n \) so that \( s_n(\theta) / \sin \theta \) is itself a trigonometric polynomial. Third, there is a typo in the inequality (62) of Corollary III; namely, \( M_n \) should be replaced by \( M \times n \). Fourth, an arbitrary constant multiplier is missing from Bernstein’s extremal function \( \sin(n\theta) \). After all these corrections, the properly formulated version of Bernstein’s statement is precisely our (1.2). We also mention that the proof of (1.2) in [4] is identical to Bernstein’s proof. Whether or not Q. I. Rahman was familiar with Bernstein’s book remains a mystery.

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