THE COMBINATORICS OF HOPPING PARTICLES AND POSITIVITY IN MARKOV CHAINS

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Abstract. The asymmetric simple exclusion process (ASEP) is a model for translation in protein synthesis and traffic flow; it can be defined as a Markov chain describing particles hopping on a one-dimensional lattice. In this article I give an overview of some of the connections of the stationary distribution of the ASEP to combinatorics (tableaux and multiline queues) and special functions (Askey-Wilson polynomials, Macdonald polynomials, and Schubert polynomials). I also make some general observations about positivity in Markov chains.

1. Introduction

The goal of this article is to illustrate some of the elegant connections between combinatorics and probability that arise when one studies Markov chains. We will focus in particular on several variations of the asymmetric simple exclusion process, illustrating combinatorial formulas for its stationary distribution and connections to special functions. The last section of this article makes some general observations about positivity in Markov chains, in the context of the Markov Chain Tree Theorem.

The asymmetric simple exclusion process (or ASEP) is a model for particles hopping on a one-dimensional lattice (e.g. a line or a ring) such that each site contains at most one particle. The ASEP was introduced independently in biology by Macdonald–Gibbs–Pipkin [MGP68] and in mathematics by Spitzer [Spi70] around 1970, see also [Lig85]. It exhibits boundary-induced phase transitions, and has been cited as a model for translation in protein synthesis, sequence alignment, the nuclear pore complex, and traffic flow.

The ASEP has remarkable connections to a number of topics, including the XXZ model [San94], vertex models [BP18, BW18], the Tracy-Widom distribution [Joh00, TW09], and the KPZ equation [BG97, CST18, CS18, CK21]. The ASEP is often viewed as a prototypical example of a random growth model from the so-called KPZ universality class in (1 + 1)-dimensions, see [KPZ86, Cor12, Qua12]. However, in this article we restrict our attention to the ASEP’s relations to combinatorics (including staircase tableaux and multiline queues), as well as to special functions (including Askey-Wilson polynomials, Macdonald polynomials, and Schubert polynomials). Much of this article is based on joint works with Sylvie Corteel [CW07b, CW07a, CW10, CW11], as well as Olya Mandelshtam [CMW22], and Donghun Kim [KW21].

2. The open boundary ASEP

In the ASEP with open boundaries (see Figure 1), we have a one-dimensional lattice of $n$ sites such that each site is either empty or occupied by a particle. At most one particle may occupy a given site. We can describe it informally as follows. During each infinitesimal time interval $dt$, each particle at a site $1 \leq i \leq n-1$ has a probability $dt$ of jumping to the next site on its right, provided it is empty, and each particle at a site $2 \leq i \leq n$ has a probability $qdt$ of jumping to the next site.
on its left, provided it is empty. Furthermore, a particle is added at site \( i = 1 \) with probability \( \alpha dt \) if site 1 is empty and a particle is removed from site \( n \) with probability \( \beta dt \) if this site is occupied.

More formally, we define this model as a discrete-time Markov chain.

![Figure 1. The (three-parameter) open boundary ASEP.](image)

**Definition 2.1.** Let \( \alpha, \beta, \) and \( q \) be constants between 0 and 1. Let \( B_n \) be the set of all \( 2^n \) words of length \( n \) in \( \{\circ, \bullet\} \). The open boundary ASEP is the Markov chain on \( B_n \) with transition probabilities:

- If \( \tau = A\bullet\circ B \) and \( \sigma = A\circ\bullet B \) (where \( A \) and \( B \) are words in \( \{\circ, \bullet\} \)), then we have that \( \Pr(\tau \rightarrow \sigma) = \frac{1}{n+1} \) and \( \Pr(\sigma \rightarrow \tau) = \frac{q}{n+1} \) (particle hops right or left).
- If \( \tau = \circ B \) and \( \sigma = \bullet B \) then \( \Pr(\tau \rightarrow \sigma) = \frac{\alpha}{n+1} \) (particle enters the lattice from left).
- If \( \tau = B\bullet \) and \( \sigma = B\circ \) then \( \Pr(\tau \rightarrow \sigma) = \frac{\beta}{n+1} \) (particle exits the lattice to the right).
- Otherwise \( \Pr(\tau \rightarrow \sigma) = 0 \) for \( \sigma \neq \tau \) and \( \Pr(\tau \rightarrow \tau) = 1 - \sum_{\sigma \neq \tau} \Pr(\tau \rightarrow \sigma) \).

In the long time limit, the system reaches a steady state where all the probabilities \( \pi(\tau) \) of finding the system in configuration \( \tau \) are stationary, i.e. satisfy \( \frac{d}{dt} \pi(\tau) = 0 \). Moreover, the stationary distribution is unique. We can compute it by solving for the left eigenvector of the transition matrix with eigenvalue 1, or equivalently, by solving the **global balance** equations: for all states \( \tau \in B_n \), we have

\[
\pi(\tau) \sum_{\sigma \neq \tau} \Pr(\tau \rightarrow \sigma) = \sum_{\sigma \neq \tau} \pi(\sigma) \Pr(\sigma \rightarrow \tau),
\]

where both sums are over all states \( \sigma \neq \tau \).

The steady state probabilities are rational expressions in \( \alpha, \beta \) and \( q \). For convenience, we clear denominators, obtaining “unnormalized probabilities” \( \Psi(\tau) \) which are equal to the \( \pi(\tau) \) up to a constant: that is, \( \pi(\tau) = \frac{\Psi(\tau)}{Z_n} \), where \( Z_n = Z_n(\alpha, \beta, q) \) is the **partition function** \( \sum_{\tau \in B_n} \Psi(\tau) \).

![Figure 2. The state diagram of the open-boundary ASEP on a lattice of 2 sites.](image)

**Example 2.2.** Figure 2 shows the state diagram of the open-boundary ASEP when \( n = 2 \), and Table 1 gives the corresponding unnormalized probabilities. Therefore we have \( \pi(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \pi(\bullet\circ) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}, \pi(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \) and \( \pi(\circ\circ) = \frac{\beta^2}{Z_2} \), where \( Z_2 = \alpha^2 + \alpha\beta(\alpha + \beta + q) + \alpha\beta + \beta^2 \).
For \( n = 3 \), if we again write each probability \( \pi(\tau) = \frac{\Psi(\tau)}{Z_3} \), we find that \( Z_3(\alpha, \beta, q) \) is a polynomial which is \textit{manifestly positive} – that it, it has only positive coefficients. Also, \( Z_3 \) has 24 terms (counted with multiplicity): \( Z_3(1,1,1) = 24 \). Computing more examples quickly leads to the conjecture that the partition function \( Z_n = Z_n(\alpha, \beta, q) \) is a (manifestly) positive polynomial with \((n - 1)!\) terms.

In algebraic combinatorics, if a quantity of interest is known or believed to be a positive integer or a polynomial with positive coefficients, one seeks an interpretation of this quantity as counting some combinatorial objects. For example, one seeks to express such a polynomial as a generating function for certain tableaux or graphs or permutations, etc. A prototypical example is the Schur polynomial \( s_\lambda(x_1, \ldots, x_n) \) [Sta99]: there are several formulas for it, including the bialternant formula and the Jacobi-Trudi formula, but both of these involve determinants and neither makes it obvious that the Schur polynomial has positive coefficients. However, one can express the Schur polynomial as the generating function for semistandard tableaux of shape \( \lambda \), and this formula makes manifest the positivity of coefficients [Sta99].

Given the above, and our observations on the positivity of the partition function \( Z_n(\alpha, \beta, q) \), the natural question is: can we express each probability as a (manifestly positive) sum over some set of combinatorial objects? We will explain how to answer this question using (a special case of) the staircase tableaux of [CW11].

### 2.1. The open-boundary ASEP and \( \alpha\beta \)-staircase tableaux.

In what follows, we will depict Young diagrams in Russian notation (with the corner at the bottom).

**Definition 2.3.** An \( \alpha\beta \)-staircase tableau \( T \) of size \( n \) is a Young diagram of shape \((n, n-1, \ldots, 2, 1)\) (drawn in Russian notation) such that each box is either empty or contains an \( \alpha \) or \( \beta \), such that:

1. no box in the top row is empty
2. each box southeast of a \( \beta \) and in the same diagonal as that \( \beta \) is empty.
3. each box southwest of an \( \alpha \) and in the same diagonal as that \( \alpha \) is empty.

See Figure 3. It is an exercise to verify that there are \((n + 1)!\) \( \alpha\beta \)-staircase tableaux of size \( n \).

![Figure 3](image-url)

**Figure 3.** At left: an \( \alpha\beta \)-staircase tableau \( T \) of type \((\circ \bullet \circ \bullet \bullet \circ \bullet \bullet)\). At right: \( T \) with a \( q \) in each unrestricted box. We have \( \text{wt}(T) = \alpha^5\beta^4q^2 \).
**Definition 2.4.** Some boxes in a tableau are forced to be empty because of conditions (2) or (3) above; we refer to all other empty boxes as *unrestricted*. (The unrestricted boxes are those whose nearest neighbors on the diagonals to the northwest and northeast, respectively, are an α and β.)

After placing a q in each unrestricted box, we define the *weight* \( wt(T) \) of \( T \) to be \( \alpha^i \beta^j q^k \) where \( i, j \) and \( k \) are the numbers of \( \alpha \)'s, \( \beta \)'s, and \( q \)'s in \( T \).

The *type* of \( T \) is the word obtained by reading the letters in the top row of \( T \) and replacing each \( \alpha \) by \( \bullet \) and \( \beta \) by \( \circ \), see Figure 3.

The following result [CW07b, CW07a, CW11] gives a combinatorial formula for the steady state probabilities of the ASEP. (Note that [CW07b, CW07a] used bijection with staircase tableaux, and are closely connected to the positive Grassmannian.) The \( q = 0 \) case was previously studied by Duchi–Schaeffer in [DS05].

**Theorem 2.5.** Consider the ASEP with open boundaries on a lattice of \( n \) sites. Let \( \tau = (\tau_1, \ldots, \tau_n) \in \{\bullet, \circ\}^n \) be a state. Then the unnormalized steady state probability \( \Psi(\tau) \) is equal to \( \sum_T wt(T) \), where the sum is over the \( \alpha \beta \)-staircase tableaux of type \( \tau \).

Equivalently, if we let \( T_n \) be the set of \( \alpha \beta \)-staircase tableaux of size \( n \), and \( Z_n := \sum_{T \in T_n} wt(T) \) be the weight generating function for these tableaux, then the steady state probability \( \pi(\tau) \) is \( \frac{\sum_T wt(T)}{Z_n} \), where the sum is over the \( \alpha \beta \)-staircase tableaux of type \( \tau \).

In the case \( n = 2 \), there are six tableaux of size 2, shown in Figure 4 and arranged by type. Computing the weights of the tableaux of the various types reproduces the results from Example 2.2.

![Figure 4. The six \( \alpha \beta \)-staircase tableau \( T \) of size 2.](image)

**2.2. Lumpings of Markov chains.** Given a combinatorial formula such as Theorem 2.5, how can we prove it? One option is to realize the ASEP as a lumping (or projection) of a Markov chain on tableaux [CW07a]. (See also [DS05] for the case \( q = 0 \).) Recall that we have a surjection \( f : T_n \to B_n \), which maps an \( \alpha \beta \)-staircase tableau to its *type*. We’d like to construct a Markov chain on tableaux whose projection via \( f \) recovers the ASEP. If we can do so, and moreover show that the steady state probability \( \pi(T) \) is proportional to \( wt(T) \), then we will have proved Theorem 2.5.

**Definition 2.6.** Let \( \{X_t\} \) be a Markov chain on state space \( \Omega_X \) with transition matrix \( P \), and let \( f : \Omega_X \to \Omega_Y \) be a surjective map. Suppose there is an \( |\Omega_Y| \times |\Omega_Y| \) matrix \( Q \) such that for all \( y_0, y_1 \in \Omega_Y \), if \( f(x_0) = y_0 \), then

\[
\sum_{x : f(x) = y_1} P(x_0, x) = Q(y_0, y_1).
\]

Then \( \{f(X_t)\} \) is a Markov chain on state space \( \Omega_Y \) with transition matrix \( Q \). We say that \( \{f(X_t)\} \) is a *strong lumping* of \( \{X_t\} \) and \( \{X_t\} \) is a *strong lift* of \( \{f(X_t)\} \).

Suppose \( \pi \) is a stationary distribution for \( \{X_t\} \), and let \( \pi_f \) be the measure on \( \Omega_Y \) defined by \( \pi_f(y) = \sum_{x : f(x) = y} \pi(x) \). Then \( \pi_f \) is a stationary distribution for \( \{f(X_t)\} \).

See [KS60, Pan19] for a thorough discussion of lumping.
Figure 5. Transitions in the Markov chain on tableaux.

The ASEP can be lifted to a Markov chain on $\alpha\beta$-staircase tableaux [CW07a], see Figure 5. In each diagram, the grey boxes represent boxes that must be empty. Note that the remaining empty boxes on the left and right side of a $\rightarrow$ are in bijection with each other; they must be filled the same way. The lifted chain has the particularly nice property that the left hand side of (1) always has at most one nonzero term.

If we identify $\alpha$’s and $\beta$’s inside the tableaux with particles and holes, then the chain on tableaux reveals a circulation of particles and holes in the second row of the tableaux; this is similar to a phenomenon observed in [DS05].

2.3. The Matrix Ansatz. One can also prove Theorem 2.5 using the following Matrix Ansatz, first introduced by Derrida, Evans, Hakim and Pasquier [DEHP93].

**Theorem 2.7** (Derrida-Evans-Hakim-Pasquier). Consider the ASEP with open boundaries on a lattice of $n$ sites. Suppose that $D$ and $E$ are matrices, $|V\rangle$ is a column vector, $\langle W |$ is a row vector, and $c$ is a constant, such that:

\begin{align*}
(2) & \quad DE - qED = c(D + E) \\
(3) & \quad \beta D |V\rangle = c |V\rangle \\
(4) & \quad \alpha \langle W |E = c \langle W | \\

If we identify $\tau = (\tau_1, \ldots, \tau_n) \in \{0, 1\}^n$ with a state (by mapping 1 and 0 to $\bullet$ and $\circ$, respectively), then the steady state probability $\pi(\tau)$ is equal to

$$
\pi(\tau) = \frac{\langle W |(\prod_{i=1}^n (\tau_i D + (1 - \tau_i E)))|V\rangle}{\langle W |(D + E)^n|V\rangle}.
$$

For example, the steady state probability of state $\circ \bullet \circ \bullet \circ \bullet$ is $\frac{\langle W |EDDED|V\rangle}{\langle W |(D+E)^2|V\rangle}$.

\[\text{[DEHP93] stated this result with } c = 1, \text{ but we will use a representation with } c = \alpha \beta \text{ in order to prove Theorem 2.5.}\]
We note that Theorem 2.7 does not imply that a solution $D, E, |V\rangle, \langle W|$ exists nor that it is unique. Indeed there are multiple solutions, which in general involve infinite-dimensional matrices.

To prove Theorem 2.5 using the Matrix Ansatz, we let $D_1 = (d_{ij})$ be the (infinite) upper-triangular matrix with rows and columns indexed by $\mathbb{Z}^+$, defined by $d_{i,i+1} = \alpha$ and $d_{ij} = 0$ for $j \neq i + 1$. Let $E_1 = (e_{ij})$ be the (infinite) lower-triangular matrix defined by $e_{ij} = 0$ for $j > i$ and

$$
e_{ij} = \beta^{i-j+1}(q^{j-1}(i-1)/j - 1) + \alpha \sum_{r=0}^{j-2} \binom{i-j+r}{r} q^r \text{ for } j \leq i.$$

That is,

$$D_1 = \begin{pmatrix} 0 & \alpha & 0 & 0 & \ldots \\ 0 & 0 & \alpha & 0 & \ldots \\ 0 & 0 & 0 & \alpha & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad E_1 = \begin{pmatrix} \beta & 0 & 0 & \ldots \\ \beta^2 & \beta(\alpha + q) & 0 & \ldots \\ \beta^3 & \beta^2(\alpha + 2q) & \beta(\alpha + \alpha q + q^2) & \ldots \\ \beta^4 & \beta^3(\alpha + 3q) & \beta^2(\alpha + 2\alpha q + 3q^2) & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We also define the (infinite) row and column vectors $\langle W_1 | = (1, 0, 0, \ldots)$ and $| V_1 \rangle = (1, 1, 1, \ldots)^t$. Then one can check that $D_1, E_1, \langle W_1 |, | V_1 \rangle$ satisfy (2), (3), and (4), with $c = \alpha \beta$. One can also show $D_1$ and $E_1$ are transfer matrices whose products enumerate $\alpha \beta$-staircase tableaux. For example, \(\langle W_1 | E_1 D_1 E_1 D_1 E_1 D_1 \rangle | V_1 \rangle\) enumerates the staircase tableaux of type \(\bullet \circ \bullet \bullet \bullet \bullet \). Now Theorem 2.7 implies Theorem 2.5.

### 2.4. Generalization to the five-parameter open boundary ASEP

More generally, we would like to understand a generalized ASEP in which particles can both enter and exit the lattice at the left (at rates $\alpha, \gamma$), and exit and enter the lattice at the right (at rates $\beta, \delta$). There is a version of the Matrix Ansatz for this setting [DEHP93], as well as suitable tableaux filled with $\alpha, \beta, \gamma$ and $\delta$’s (which we will simply call staircase tableaux) [CW11].

![Figure 6. The (five-parameter) open boundary ASEP.](image)

**Definition 2.8.** A **staircase tableau** $T$ of size $n$ is a Young diagram of shape $(n, n-1, \ldots, 2, 1)$ such that each box is either empty or contains an $\alpha$, $\beta$, $\gamma$, or $\delta$ such that:

1. no box in the top row is empty
2. each box southeast of a $\beta$ or $\delta$ and in the same diagonal as that $\beta$ or $\delta$ is empty.
3. each box southwest of an $\alpha$ or $\gamma$ and in the same diagonal as that $\alpha$ or $\gamma$ is empty.

See Figure 7 for an example. It is an exercise to verify that there are exactly $4^n n$! staircase tableaux of size $n$.

**Definition 2.9.** We call an empty box of a staircase tableau $T$ **distinguished** if either:

- its nearest neighbor on the diagonal to the northwest is a $\delta$, or
- its nearest neighbor on the diagonal to the northwest is an $\alpha$ or $\gamma$, and its nearest neighbor on the diagonal to the northeast is a $\beta$ or $\gamma$. 
After placing a $q$ in each distinguished box, we define the weight $\text{wt}(T)$ of $T$ to be the product of all letters in the boxes of $T$.

The type of $T$ is the word obtained by reading the letters in the top row of $T$ and replacing each $\alpha$ or $\delta$ by $\bullet$, and each $\beta$ or $\gamma$ by $\circ$, see Figure 7.

The following result from [CW10, CW11] subsumes Theorem 2.5. It can be proved using a suitable generalization of the Matrix Ansatz.

**Theorem 2.10.** Consider the ASEP with open boundaries on a lattice of $n$ sites as in Figure 6. Let $\tau \in \{\bullet, \circ\}^n$ be a state. Then the unnormalized steady state probability $\Psi(\tau)$ is equal to $\sum_T \text{wt}(T)$, where the sum is over the staircase tableaux of type $\tau$.

Remarkably, there is another solution to the Matrix Ansatz, found earlier by Uchiyama, Sasamoto, and Wadati [USW04], which makes use of orthogonal polynomials. More specifically, one can find a solution where $D$ and $E$ are tridiagonal matrices, such that the rows of $D + E$ encode the three-term recurrence relation characterizing the Askey-Wilson polynomials; these are a family of orthogonal polynomials $p_n(x; a, b, c, d|q)$ which are at the top of the hierarchy of classical one-variable orthogonal polynomials (including the others as special or limiting cases) [AW85].

The connection of Askey-Wilson polynomials with the ASEP via [USW04] leads to applications on both sides. On the one hand, it facilitates the computation of physical quantities in the ASEP such as the phase diagram [USW04]; it also leads to a relation between the ASEP and the Askey-Wilson stochastic process [BW017]. On the other hand, this connection has applications to the combinatorics of Askey-Wilson moments. Since the 1980’s there has been a great deal of work on the combinatorics of classical orthogonal polynomials (e.g. Hermite, Charlier, Laguerre) [Vie85, ISV87, CKS16]; the connection of staircase tableaux to ASEP, and of ASEP to Askey-Wilson polynomials, led to the first combinatorial formula for moments of Askey-Wilson polynomials [CW11, CSSW12].

Even more generally, one can study a version of the ASEP with open boundaries in which there are different species of particles. This version is closely connected [Can17, CW18, CGdGW16] to Koornwinder polynomials [Koo92], a family of multivariate orthogonal polynomials which generalize Askey-Wilson polynomials.

3. **The (multispecies) ASEP on a ring**

It is also natural to consider the ASEP on a lattice of sites arranged in a ring, of which some sites are occupied by a particle. Each particle in the system can jump to the next site either clockwise or counterclockwise, provided that this site is empty. In this model, the resulting stationary distribution is always the uniform distribution. This motivates considering a multiespecies generalization of the ASEP, in which particles come with different weights, which in turn influence the hopping rates.
3.1. The multispecies ASEP, multiline queues, and Macdonald polynomials. In the multispecies ASEP (mASEP) on a ring, two neighboring particles exchange places at rates 1 or \( t \), depending on whether the heavier particle is clockwise or counterclockwise from the lighter one.

**Definition 3.1.** Let \( t \) be a constant such that \( 0 \leq t \leq 1 \), and let \( \lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \) be a partition. Let \( B_n(\lambda) \) be the set of all words of length \( n \) obtained by permuting the parts of \( \lambda \). The multispecies ASEP on a ring is the Markov chain on \( B_n(\lambda) \) with transition probabilities:

- If \( \mu = (\mu_1, \ldots, \mu_n) \) and \( \nu \) are in \( B_n(\lambda) \), and \( \nu \) is obtained from \( \mu \) by swapping \( \mu_i \) and \( \mu_{i+1} \) for some \( i \) (indices considered modulo \( n \)), then \( \Pr(\mu \to \nu) = \frac{t}{n} \) if \( \mu_i > \mu_{i+1} \) and \( \Pr(\mu \to \nu) = \frac{1}{n} \) if \( \mu_i < \mu_{i+1} \).
- Otherwise \( \Pr(\mu \to \nu) = 0 \) for \( \nu \neq \mu \) and \( \Pr(\mu \to \mu) = 1 - \sum_{\nu \neq \mu} \Pr(\mu \to \nu) \).

We think of the parts of \( \lambda \) as representing various types of particles of different weights.

As before, one would like to find an expression for each steady state probability as a manifestly positive sum over some set of combinatorial objects. One may give such a formula in terms of Ferrari-Martin’s *multiline queues* shown in Figure 8, see [Mar20, CMW22].

One fascinating aspect of the multispecies ASEP on a ring is its close relation [CdGW15] to *Macdonald polynomials* \( P_\lambda(x_1, \ldots, x_n; q, t) \) [Mac95], a remarkable family of polynomials that generalize Schur polynomials, Hall-Littlewood polynomials, and Jack polynomials. The next result follows from [CdGW15] and [CMW22].

**Theorem 3.2.** Let \( \mu \in B_n(\lambda) \) be a state of the mASEP on a ring. Then the steady state probability \( \pi(\mu) \) is

\[
\pi(\mu) = \frac{\Psi(\mu)}{Z_\lambda},
\]

where \( \Psi(\mu) \) is obtained from a permuted basement Macdonald polynomial and \( Z_\lambda \) is obtained from the Macdonald polynomial \( P_\lambda \) by specializing \( q = 1 \) and \( x_1 = x_2 = \cdots = x_n = 1 \).

The following table shows the probabilities of the mASEP when \( \lambda = (4, 3, 2, 1) \). Note that because of the circular symmetry in the mASEP, e.g. \( \pi(1, 2, 3, 4) = \pi(2, 3, 4, 1) = \pi(3, 4, 1, 2) = \pi(4, 1, 2, 3) \), it suffices to list the probabilities for the states \( w \) with \( w_1 = 1 \).

In light of Theorem 3.2 and the connection to multiline queues, it is natural to ask if one can give a formula for Macdonald polynomials in terms of multiline queues. This is indeed possible, see [CMW22] for details.

We remark that there is a family of Macdonald polynomials associated to any affine root system; the “ordinary” Macdonald polynomials discussed in this section are those of type \( \tilde{A} \). It is interesting that they are related to particles hopping on a ring (which resembles an affine A Dynkin diagram).
Meanwhile, the Koornwinder polynomials from the previous section are the Macdonald polynomials attached to the non-reduced affine root system of type $\tilde{C}_n^\vee$. It is interesting that they are related to particles hopping on a line with open boundaries (which resembles a Dynkin diagram of type $\tilde{C}_n^\vee$).

We note that there are other connections between probability and Macdonald polynomials, including Macdonald processes [BC14], and a Markov chain on partitions whose eigenfunctions are coefficients of Macdonald polynomials [DR12]. There is also a variation of the exclusion process called the multispecies zero range process, whose stationary distribution is related to modified Macdonald polynomials [AMM20].

### 3.2. The inhomogeneous TASEP, multiline queues, and Schubert polynomials

Another multispecies generalization of the exclusion process on a ring is the inhomogeneous totally asymmetric exclusion process (TASEP). In this model, two adjacent particles with weights $i$ and $j$ with $i < j$ can swap places only if the heavier one is clockwise of the lighter one, and in this case, they exchange places at rate $x_i - y_j$, see Figure 9.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{inhomogeneous_TASEP.png}
\caption{The inhomogeneous multispecies TASEP on a ring, with $\lambda = (6, 5, 4, 3, 2, 1)$.}
\end{figure}

**Definition 3.3.** Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be constants such that $0 < x_i - y_j \leq 1$ for all $i, j$, and let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be a partition. Let $B_n(\lambda)$ be the set of all words of length $n$ obtained by permuting the parts of $\lambda$. The inhomogeneous TASEP on a ring is the Markov chain on $B_n(\lambda)$ with transition probabilities:

- If $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu$ are in $B_n(\lambda)$, and $\nu$ is obtained from $\mu$ by swapping $\mu_i$ and $\mu_{i+1}$ for some $i$ (indices considered mod $n$), then $\Pr(\mu \to \nu) = \frac{x_{\mu_i} - y_{\mu_i}}{\sum_{\nu \neq \mu} \Pr(\mu \to \nu)}$ if $\mu_i < \mu_{i+1}$.
- Otherwise $\Pr(\mu \to \nu) = 0$ for $\nu \neq \mu$ and $\Pr(\mu \to \mu) = 1 - \sum_{\nu \neq \mu} \Pr(\mu \to \nu)$.

When $y_i = 0$ for all $i$, there is a formula for the stationary distribution of the inhomogeneous TASEP in terms of multiline queues; this can be proved using a version of the Matrix Ansatz [AM13].
Recall that the mASEP on a ring is closely connected to Macdonald polynomials. Curiously, when $y_1 = 0$ the inhomogeneous TASEP on a ring is related to Schubert polynomials, a family of polynomials which give polynomial representatives for the Schubert classes in the cohomology ring of the complete flag variety. For example, many (unnormalized) steady state probabilities are equal to products of Schubert polynomials [Can16, KW21], and all of them are conjecturally positive sums of Schubert polynomials [LW12].

Given $w = (w_1, \ldots, w_n)$ a permutation in the symmetric group $S_n$ and $p = (p_1, \ldots, p_m) \in S_m$ with $m < n$, we say that $w$ contains $p$ if $w$ has a subsequence of length $m$ whose letters are in the same relative order as those of $p$. For example, the permutation $(3, 2, 6, 5, 1, 4)$ contains the pattern $(2, 4, 1, 3)$ because its letters $3, 6, 1, 4$ have the same relative order as those of $(2, 4, 1, 3)$. If $w$ does not contain $p$ we say that $w$ avoids $p$. We say that $w \in S_n$ is evil-avoiding if $w$ avoids the patterns $(2, 4, 1, 3), (4, 1, 3, 2), (4, 2, 1, 3)$ and $(3, 2, 1, 4)$.

We have the following result, see [KW21] for details.

**Theorem 3.4.** Let $\lambda = (n, n - 1, \ldots, 1)$ so that the inhomogeneous TASEP can be viewed as a Markov chain on the $n!$ permutations of the set $\{1, 2, \ldots, n\}$. Let $w \in S_n$ be a permutation with $w_1 = 1$ which is evil-avoiding, and let $k$ be the number of descents of $w^{-1}$. Then the steady state probability $\pi(w)$ equals

$$\pi(w) = \frac{\Psi(w)}{Z_n},$$

where $\Psi(w)$ is a monomial in $x_1, \ldots, x_{n-1}$ times a product of $k$ Schubert polynomials, and $Z_n = \prod_{i=1}^{\lambda(n)} h_{n-i}(x_1, x_2, \ldots, x_{i-1}, x_i)$ with $h_i$ the complete homogeneous symmetric polynomial.

The following table shows the probabilities of the inhomogeneous TASEP when $\lambda = (4, 3, 2, 1)$.

| State $w$ | Unnormalized probability $\Psi(w)$ |
|-----------|-----------------------------------|
| 1234      | $x_1^2 x_2$                       |
| 1243      | $x_1^2 (x_1 x_2 + x_1 x_3 + x_2 x_3) = x_1^2 S_{1342}$ |
| 1324      | $x_1 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3) = x_1 S_{1432}$ |
| 1342      | $x_1 x_2 (x_1^2 + x_1 x_2 + x_2^2) = x_1 x_2 S_{1423}$ |
| 1423      | $x_1^2 x_2 (x_1 + x_2 + x_3) = x_1^2 x_2 S_{1243}$ |
| 1432      | $(x_1^2 + x_1 x_2 + x_2^2) (x_1 x_2 + x_1 x_3 + x_2 x_3) = S_{1423} S_{1342}$ |

Table 3. Probabilities for the inhomogeneous TASEP when $\lambda = (4, 3, 2, 1)$.

For general $y_i$, there is a version of Theorem 3.4 involving double Schubert polynomials [KW21]. Very often, beautiful combinatorial properties go hand-in-hand with integrability of a model. While this topic goes beyond the scope of this article, the reader can learn about integrability and the exclusion process from [Can16, CRV14], or more generally about integrable probability from [BG16].

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2We call these permutations evil-avoiding because if one replaces $i$ by 1, $e$ by 2, $l$ by 3, and $u$ by 4, then evil and its anagrams ule, vile and leiv become the four patterns $2413, 4132, 4213$ and $3214$. (Leiv is a name of Norwegian origin meaning “heir.”)
4. Positivity in Markov chains

The reader may at this point wonder how general is the phenomenon of positivity in Markov chains? That is, how often can one express the steady state probabilities of a Markov chain in terms of polynomials with all coefficients positive (ideally as a sum over combinatorial objects)?

In some sense, the answer to this question is all the time: the Markov Chain Tree Theorem gives a formula for the stationary distribution of a finite-state irreducible Markov chain as a positive sum indexed by rooted trees of the state diagram. However, the number of terms of this formula grows fast very quickly! (By Cayley’s formula, the complete graph on $n$ vertices has $n^{n-2}$ spanning trees.) Moreover, for many Markov chains, there is a common factor which can be removed from the above formula for the stationary distribution, resulting in a more compact formula. Sometimes the more compact formula involves polynomials with negative coefficients.

Let $G$ be the state diagram of a finite-state irreducible Markov chain whose set of states is $V$. That is, $G$ is a weighted directed graph with vertices $V$, with an edge $e$ from $i$ and $j$ weighted $Pr(e) := Pr(i,j)$ whenever the probability $Pr(i,j)$ of going from state $i$ to $j$ is positive. We call a connected subgraph $T$ a spanning tree rooted at $r$ if $T$ includes every vertex of $V$, $T$ has no cycle, and all edges of $T$ point towards the root $r$. (Irreducibility of the Markov chain implies that for each vertex $r$, there is a spanning tree rooted at $r$.) Given a spanning tree $T$, we define its weight as $wt(T) := \prod_{e \in T} Pr(e)$.

**Theorem 4.1 (Markov Chain Tree Theorem).** The stationary distribution of a finite-state irreducible Markov chain is proportional to the measure that assigns the state $\tau$ the “unnormalized probability”

$$\Psi(\tau) := \sum_{\text{root}(T)=\tau} wt(T).$$

That is, the steady state probability $\pi(\tau)$ equals $\pi(\tau) = \frac{\Psi(\tau)}{Z}$, where $Z = \sum_{\tau} \Psi(\tau)$.

Theorem 4.1 first appeared in [Hil66] and was proved for general Markov chains in [LR83]. It has by now many proofs, one of which involves lifting the Markov chain to a chain on the trees themselves; the result then follows from Kirchhoff’s Matrix Tree Theorem. See [AT90], [LP16], [PT18], and references therein.

**Example 4.2.** Consider the Markov chain with five states 1, ..., 5, whose transition matrix is as follows:

$$
\begin{pmatrix}
\frac{2-q}{3} & 0 & \frac{1}{3} & \frac{q}{3} & 0 \\
0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\
\frac{q}{3} & \frac{1-q}{3} & 0 & 0 & \frac{1}{3} \\
\frac{1}{3} & \frac{q}{3} & 0 & \frac{2-2q}{3} & \frac{q}{3} \\
0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2-q}{3}
\end{pmatrix}
$$

(5)

The state diagram is shown in Figure 10. (We have omitted the factors of $\frac{1}{3}$ from each transition probability as they do not affect the eigenvector of the transition matrix). We also omitted the loops at each state.

If one applies Theorem 4.1, one finds e.g. that there are six spanning trees rooted at state 1, with weights $q^3, q^3, q^2, q, 1,$ and $1$. Adding up these contributions gives $\Psi(1) = 2q^3 + q^2 + q + 2$. Computing the spanning trees rooted at the other states gives rise to the unnormalized probabilities $\Psi(\tau)$ for the stationary distribution shown in Table 4.

Note that the unnormalized probabilities from Table 4 share a common factor of $(q+1)$. Dividing by this common factor gives the (more compact) unnormalized probabilities $\Psi(\tau)$ shown in Table 5.
Figure 10. The state diagram from Example 4.2, plus a spanning tree rooted at 1 with weight $q^3$.

| State $\tau$ | Unnormalized probability $\Psi(\tau)$ |
|--------------|--------------------------------------|
| 1            | $2q^3 + q^2 + q + 2$                  |
| 2            | $q^4 + 3q^3 + 4q^2 + 3q + 1$         |
| 3            | $2q^3 + 2q^2 + q + 1$                |
| 4            | $q^3 + q^2 + 2q + 2$                 |
| 5            | $2q^3 + 4q^2 + 4q + 2$               |

Table 4. Unnormalized probabilities for the Markov chain from Example 4.2 as given by the Markov Chain Tree Theorem.

| State $\tau$ | Unnormalized probability $\Psi(\tau)$ |
|--------------|--------------------------------------|
| 1            | $2q^2 - q + 2$                       |
| 2            | $q^2 + 2q^2 + 2q + 1$                |
| 3            | $2q^2 + 1$                           |
| 4            | $q^2 + 2$                            |
| 5            | $2q^2 + 2q + 2$                      |

Table 5. Unnormalized probabilities from Table 4 after dividing by the joint common factor $(q + 1)$.

We see that when we write the stationary distribution in “lowest terms,” we obtain a vector of polynomials which do not have only nonnegative coefficients.

This example motivates the following definitions.

**Definition 4.3.** Consider a measure $(\Psi_1, \ldots, \Psi_n)$ on the set $\{1, 2, \ldots, n\}$ in which each component $\Psi_i(q_1, \ldots, q_N)$ is a polynomial in $\mathbb{Z}[q_1, \ldots, q_N]$.

We say the formula $(\Psi_1, \ldots, \Psi_n)$ is **manifestly positive** if all coefficients of $\Psi_i$ are positive for all $i$. And we say $(\Psi_1, \ldots, \Psi_n)$ is **compact** if there is no polynomial $\phi(q_1, \ldots, q_N) \neq 1$ which divides all the $\Psi_i$.

Theorem 4.1 shows that every finite-state Markov chain has a manifestly positive formula for the stationary distribution. Meanwhile, Example 4.2 shows that in general this formula is not compact, and that there are Markov chains whose compact formula for the stationary distribution is not manifestly positive.

In light of Theorem 4.1, it is interesting to revisit e.g. the stationary distribution of the open boundary ASEP with parameters $\alpha$, $\beta$, and $q$. One can use Theorem 2.5 to express the components $\Psi_{tab}(\tau)$ of the stationary measure as a sum over the tableaux of type $\tau$. On the other hand, one

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3We don’t require that $\sum_i \Psi_i = 1$; to obtain a probability distribution we can just divide each term by $Z := \sum_i \Psi_i$. 

---
can use Theorem 4.1 to express the components $\Psi_{\text{tree}}(\tau)$ of the stationary measure as a sum over spanning trees rooted at $\tau$ of the state diagram. Both $\Psi_{\text{tab}}(\tau)$ and $\Psi_{\text{tree}}(\tau)$ are polynomials in $\alpha, \beta, q$ with positive coefficients; however, the former is compact, and has many fewer terms than the latter. Because the stationary measure is unique (up to an overall scalar), for each $n$ there is a polynomial $Q_n(\alpha, \beta, q)$ such that $\frac{\Psi_{\text{tree}}(\tau)}{\Psi_{\text{tab}}(\tau)} = Q_n(\alpha, \beta, q)$ for all $\tau \in \{0, 1\}^n$. The number of terms in $Q_n$ appears in Table 6.

| $n$ | $Q_n(1, 1, 1)$ |
|-----|----------------|
| 2   | 1              |
| 3   | 4              |
| 4   | 840            |
| 5   | 2285015040     |
| 6   | 11335132600511975880768000 |

Table 6. The ratio of the numbers of terms between the Markov Chain Tree theorem formula and the staircase tableaux formula for the stationary distribution of the (three-parameter) open boundary ASEP on a lattice of $n$ sites.

It would be interesting to reprove e.g. Theorem 2.5 using the Markov Chain Tree Theorem. We note that the analysis of the ASEP and its variants would be easier if these Markov chains were reversible; in general they are not (except for special cases of the parameters). Nevertheless there has been progress on the mixing time of the ASEP, see [GNS21] and references therein.

Besides the ASEP, there are other interesting Markov chains arising in statistical mechanics whose stationary distributions admit manifestly positive formulas as sums over combinatorial objects (which are often compact). These include the Razumov-Stroganov correspondence [DF04, dGR04, CS14], the Tsetlin library [Tse63, Hen72], and many other models of interacting particles [AM10, AN21], see also [Ayy22].

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