Uniqueness of positive bound states to Schrodinger systems with critical exponents

Congming Li ∗ Li Ma†

Abstract

We prove the uniqueness for the positive solutions of the following elliptic systems:

\[
\begin{align*}
-\Delta(u(x)) &= u(x)^\alpha v(x)^\beta \\
-\Delta(v(x)) &= u(x)^\beta v(x)^\alpha
\end{align*}
\]

Here \( x \in \mathbb{R}^n, n \geq 3, \) and \( 1 \leq \alpha, \beta \leq \frac{n+2}{n-2} \) with \( \alpha + \beta = \frac{n+2}{n-2} \). In the special case when \( n = 3 \) and \( \alpha = 2, \beta = 3 \), the systems come from the stationary Schrodinger system with critical exponents for Bose-Einstein condensate. As a key step, we prove the radial symmetry of the positive solutions to the elliptic system above with critical exponents.

Keyword: Moving plane, positive solutions, radial symmetric, uniqueness
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1 Introduction

In this paper, we consider the uniqueness of positive solutions to the following stationary Schrodinger system:

\[
\begin{align*}
-\Delta (u(x)) &= u(x)^\alpha v(x)^\beta \\
-\Delta (v(x)) &= u(x)^\beta v(x)^\alpha
\end{align*}
\]  

(1)

Here \(x \in \mathbb{R}^n, n \geq 3, \) and \(1 \leq \alpha, \beta \leq \frac{n+2}{n-2} \) with \(\alpha + \beta = \frac{n+2}{n-2} \). In the special case when \(n = 3\) and \(\alpha = 2, \beta = 3\), the systems come from the stationary Schrodinger system with critical exponents for Bose-Einstein condensate ([11],[15],[16], and [18]). In the earlier works [15],[16], and [18], people pay more attention to the elliptic system (1) with subcritical exponents. Very interestingly, Chen and Li have proved that the best constant in weighted Hardy-Littlewood-Sobolev inequality can be achieved by explicit radially symmetric functions (see [4] and [13]). As a consequence of their work, the uniqueness of positive solutions to the corresponding elliptic system (it is (1) in the case when \(\alpha = 0\) and \(\beta = \frac{n+2}{n-2}\)) has been settled down. However, when \(0 < \alpha, \beta \), the uniqueness of smooth positive solutions to the stationary Schrodinger system (1) is an open question. Generally speaking, there are very few result even for the uniqueness of positive solutions to the ordinary differential systems. The aim of this paper is to prove the radial symmetry and uniqueness of positive solutions to (1) with critical exponents and \(1 \leq \alpha < \beta \leq \frac{n+2}{n-2}\).

As one can expect, just like in the work M.W. Weinstein [21] in the scalar case with the sub-critical exponent, that there is a closed relationship between the stationary Schrodinger system with critical exponent with the Hardy-Littlewood-Sobolev inequality. As we show below, this is true. Since we shall use Hardy-Littlewood-Sobolev inequality to prove radial symmetry of our solutions, let’s do an excursion about recent progress of Lieb’s conjecture. Let us begin by recalling the well-known Hardy-Littlewood-Sobolev inequalities. Let \(0 < \lambda < n, 1 < s, r < \infty, \) and \(\|f\|_p\) be the \(L_p(\mathbb{R}^n)\) norm of the function \(f\). We shall write by \(\|f\|_{p,\Omega}\) the \(L_p\) norm of the function on the domain \(\Omega\). Then, classical Hardy-Littlewood-Sobolev inequality (HLS) states that:

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} dx dy \leq C_{s,\lambda,n}\|f\|_r\|g\|_s
\]

(2)

for any \(f \in L^r(\mathbb{R}^n), g \in L^s(\mathbb{R}^n), \) and for \(\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2\). Hardy and Littlewood also introduced the double weighted inequality, which was later generalized by Stein and Weiss in [19] in the following form:

\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\alpha_0}|x-y|^\lambda|y|^{\beta_0}} dx dy \right| \leq C_{\alpha_0,\beta_0,s,\lambda,n}\|f\|_r\|g\|_s
\]

(3)

where \(\alpha_0 + \beta_0 \geq 0, \)

\[
1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha_0}{n} < 1 - \frac{1}{r}, \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha_0 + \beta_0}{n} = 2.
\]

(4)
The best constant in the weighted inequality (3) can be obtained by maximizing the functional

$$J(f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\alpha_0} |x-y|^{\lambda}|y|^{\beta_0}} \, dx \, dy$$  \hspace{1cm} (5)$$

under the constraints $\|f\|_r = \|g\|_s = 1$. Then the corresponding Euler-Lagrange equations are the system of integral equations:

$$\begin{cases} 
\lambda_1 r f(x)^{r-1} = \frac{1}{|x|^{\alpha_0}} \int_{\mathbb{R}^n} \frac{g(y)}{|y|^{\gamma_0}|x-y|^\lambda} \, dy \\
\lambda_2 s g(x)^{s-1} = \frac{1}{|x|^{\beta_0}} \int_{\mathbb{R}^n} \frac{f(y)}{|y|^{\delta_0}|x-y|^\lambda} \, dy 
\end{cases}$$  \hspace{1cm} (6)$$

where $f, g \geq 0$, $x \in \mathbb{R}^n$, and $\lambda_1 r = \lambda_2 s = J(f, g)$.

Let $u = c_1 f^{r-1}$, $v = c_2 g^{s-1}$, $p = \frac{1}{r-1}$, $q = \frac{1}{s-1}$, $pq \neq 1$, and for a proper choice of constants $c_1$ and $c_2$, system (6) becomes

$$\begin{cases} 
u(x) = \frac{1}{|x|^p} \int_{\mathbb{R}^n} \frac{u(y)^p}{|y|^q|x-y|^\lambda} \, dy \\
v(x) = \frac{1}{|x|^q} \int_{\mathbb{R}^n} \frac{v(y)^q}{|y|^p|x-y|^\lambda} \, dy 
\end{cases}$$  \hspace{1cm} (7)$$

where $u, v \geq 0$, $0 < p, q < \infty$, $0 < \lambda < n$, $\alpha_0 = \lambda < \frac{1}{p+1} < \frac{\lambda+\alpha_0}{n}$, and $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha_0+\beta_0}{n}$.

Note that in the special case where $\alpha_0 = 0$ and $\beta_0 = 0$, system (7) reduces to the following system:

$$\begin{cases} 
u(x) = \int_{\mathbb{R}^n} \frac{u(y)^p}{|y|^q|x-y|^\lambda} \, dy \\
v(x) = \int_{\mathbb{R}^n} \frac{v(y)^q}{|y|^p|x-y|^\lambda} \, dy 
\end{cases}$$  \hspace{1cm} (8)$$

with

$$\frac{1}{q+1} + \frac{1}{p+1} = \frac{\lambda}{n}. \hspace{1cm} (9)$$

It is well-known that this integral system is closely related to the system of partial differential equations

$$\begin{cases} (-\Delta)^{\gamma/2} u = v^p, \quad u > 0, \text{ in } \mathbb{R}^n, \\
(-\Delta)^{\gamma/2} v = u^q, \quad v > 0, \text{ in } \mathbb{R}^n, 
\end{cases}$$  \hspace{1cm} (10)$$

where $\gamma = n - \lambda$.

When $p = q = \frac{n+\gamma}{n-\gamma}$, and $u(x) = v(x)$, system (8) becomes the single equation:

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+\gamma}{n-\gamma}}}{|x-y|^{\frac{n-\gamma}{n}}} \, dy, \quad u > 0, \text{ in } \mathbb{R}^n. \hspace{1cm} (11)$$

The corresponding PDE is the well-known family of semi-linear equations

$$(-\Delta)^{\gamma/2} u = u^{(n+\gamma)/(n-\gamma)}, \quad u > 0, \text{ in } \mathbb{R}^n$$  \hspace{1cm} (12)$$

In particular, when $n \geq 3$, and $\gamma = 2$, (12) becomes
\[ -\Delta u = u^{(n+2)/(n-2)}, \quad u > 0, \quad \text{in } R^n. \] (13)

The classification of the solutions of (13) has provided an important ingredient in the study of the well-known Yamabe problem and the prescribing scalar curvature problem. Equation (13) were studied by Gidas, Ni, and Nirenberg [8], Caffarelli, Gidas, and Spruck [1], Chen and Li [2], and Li [12]. They classified all the solutions. Recently, Wei and Xu [20] generalized this result to the solutions of more general equation (12) with \( \gamma \) being any even numbers between 0 and \( n \).

Although the systems for other real values of \( \alpha, \beta \) between 0 and \( n \) are of interest to people, we shall only concentrate in this paper to the system (1) with critical exponents when \( 1 \leq \alpha, \beta \leq \frac{n+2}{n-2} \) and \( \alpha + \beta = \frac{n+2}{n-2} \).

Our main results are

**Theorem 1**. Any \( L_{\frac{n^2}{n-2}}(R^n) \times L_{\frac{n^2}{n-2}}(R^n) \) positive solution pair \( (u, v) \) to the system (1) with critical exponents are radial symmetric functions.

and

**Theorem 2**. Assume that \( 1 \leq \alpha < \beta \leq \frac{n+2}{n-2} \). Then any \( L_{\frac{n^2}{n-2}}(R^n) \times L_{\frac{n^2}{n-2}}(R^n) \) radial symmetric solution pair \( (u, v) \) to the system (1) with critical exponents are unique such that \( u = v \).

We point out that when \( u = v \), the elliptic system (1) reduces to the elliptic equation with critical exponent (13). Then \( u = v \) is in a special family of functions:

\[
\phi_{x_0,t}(x) = c\left(\frac{t}{t^2 + |x - x_0|^2}\right)^{(n-2)/2} \quad (14)
\]

where \( t > 0, x_0 \in R^n \), with some positive constants \( c \) such that each \( \phi_{x_0,t}(x) \) solves (13). This family of functions are important in the study of (1).

Our results are motivated from the previous work [6], where Chen, Li, and Ou considered more general system (8) and established the symmetry and monotonicity of the solutions. In [3], Chen and Li also obtained a regularity result of the solutions to (8). To establish the symmetry of the solution to (8), Chen, Li, and Ou [5] [6] [7] introduced a new idea, an integral form of the method of moving planes. It is entirely different from the traditional method used for partial differential equations. Instead of relying on maximum principles, certain integral norms were estimated. The new method is a very powerful tool in studying qualitative properties of other integral equations and systems. In fact, following Chen, Li, and Ou’s work, Jin and Li [9] studied the symmetry of the solutions to the more general system (7).

Chen and Ma [17] discussed the Liouville type theorem for the positive solutions to the elliptic system (10).

In this paper, we first prove the radial symmetry of the solutions to (1) with critical exponents. It is obvious that the radial symmetry of the solutions reduces (1) to a system
of ODEs, which has special solution pair \((\phi_{o,t}(x), \phi_{o,t}(x))\). To prove the uniqueness, we prove that \(u(0) = v(0)\). Then by the uniqueness of the initial value problem for ODE, we conclude that \(u = v = \phi_{o,t}\). This is the key observation in establishing the uniqueness of positive solutions for (1) with critical exponents.

Theorems 1 and 2 will be proved in the next two sections.

2 Proof of the Radial symmetry

We use the moving plane method introduced by Chen-Li-Ou in [5]. We shall use the Hardy-Littlewood-Sobolev inequality:

\[
|Tf|_p \leq |f|_{\frac{np}{n-2p}},
\]

where \(C(n, p)\) is a uniform positive constant and

\[
Tf(x) = \int_{\mathbb{R}^n} |x - y|^{2-n} f(y) dy.
\]

**The Proof of Theorem 1.** For each \(\lambda \in \mathbb{R}\), we denote by

\[H_\lambda = \{x \in \mathbb{R}^n; x_1 < \lambda\}.
\]

For each \(x = (x_1, x') \in \mathbb{R}^n\), we let

\[x_\lambda = (2\lambda - x_1, x')\]

be the reflection point of \(x\) with respect to the hyperplane \(\partial H_\lambda\). We let \(e_1 = (1, 0, ...0)\).

We define

\[u_\lambda(x) = u(x_\lambda), \quad B^u_\lambda = \{x \in H_\lambda; u_\lambda(x) > u(x)\},\]

and

\[v_\lambda(x) = v(x_\lambda), \quad B^v_\lambda = \{x \in H_\lambda; v_\lambda(x) > v(x)\}.
\]

To do the moving plane method, we need the following formula, which is obtained by a change of variables.

\[u(x) = \int_{H_\lambda} \frac{u^\alpha v^\beta(y)}{|x - y|^{n-2}} dy + \int_{H_\lambda} \frac{u^\alpha v^\beta(y)}{|x_\lambda - y|^{n-2}} dy,
\]

and

\[u_\lambda(x) = \int_{H_\lambda} \frac{u^\alpha v^\beta(y)}{|x_\lambda - y|^{n-2}} dy + \int_{H_\lambda} \frac{u^\alpha v^\beta(y)}{|x - y|^{n-2}} dy.
\]

Then we have

\[u_\lambda(x) - u(x) = \int_{H_\lambda} (u^\alpha v^\beta_\lambda - u^\alpha v^\beta)(y)\left(\frac{1}{|x - y|^{n-2}} - \frac{1}{|x_\lambda - y|^{n-2}}\right) dy.
\]
Note that for $x \in H_{\lambda}$, we have

$$\frac{1}{|x - y|^{n-2}} > \frac{1}{|x \lambda - y|^{n-2}}.$$ 

Then for $x \in B_{\lambda}^v$, we have

$$\left\{ \begin{align*}
0 & \leq u_{\lambda}(x) - u(x) \\
& \leq \alpha \int_{B_{\lambda}^\alpha} \frac{u_{\lambda}^{\alpha - 1}v_{\lambda}^{\beta - 1}(u_{\lambda} - u)}{|x - y|^{n-2}} \, dy + \beta \int_{B_{\lambda}^\beta} \frac{u_{\lambda}^{\alpha - 1}v_{\lambda}^{\beta - 1}(v_{\lambda} - v)}{|x - y|^{n-2}} \, dy \\
\text{: = } I + II
\end{align*} \right. \quad (17)$$

Let $p = \frac{2n}{n-2}$. Using the Hardy-Littlewood-Sobolev inequality (15) we can bound the first term $I$ in (17) by

$$\left\{ \begin{align*}
|I|_p & \leq C(n, p)|u_{\lambda}^{\alpha - 1}v_{\lambda}^{\beta - 1}(u_{\lambda} - u)|_{\frac{2n}{n+2}} \\
& \leq C(n, p)|u_{\lambda}^{\alpha - 1}|v_{\lambda}^{\beta - 1}|u_{\lambda} - u|_p
\end{align*} \right. \quad (18)$$

Here the integrations are over the set $B_{\lambda}^\alpha$.

Using again the Hardy-Littlewood-Sobolev inequality (15) we can bound the first term $II$ in (17) by

$$\left\{ \begin{align*}
|II|_p & \leq C(n, p)|u_{\lambda}^{\alpha - 1}v_{\lambda}^{\beta - 1}(v_{\lambda} - v)|_{\frac{2n}{n+2}} \\
& \leq C(n, p)|u_{\lambda}^{\alpha - 1}|v_{\lambda}^{\beta - 1}|v_{\lambda} - v|_p
\end{align*} \right. \quad (19)$$

Here the integrations are over the domain $B_{\lambda}^\beta$. Hence, we have

$$|u_{\lambda} - u|_{p, B_{\lambda}^v} \leq C(n, p)(|u_{\lambda}|_{p, B_{\lambda}^v}^{\alpha - 1}|v_{\lambda}|_{p, B_{\lambda}^v}^{\beta - 1}|u_{\lambda} - u|_{p, B_{\lambda}^v} + |u_{\lambda}|_{p, B_{\lambda}^v}^{\alpha - 1}|v_{\lambda}|_{p, B_{\lambda}^v}^{\beta - 1}|v_{\lambda} - v|_{p, B_{\lambda}^v}) \quad (20)$$

Similarly, we have for following formulae for $v$ and $v_{\lambda}$.

$$v(x) = \int_{H_{\lambda}} \frac{u^{\alpha}v^{\beta}(y)}{|x - y|^{n-2}} \, dy + \int_{H_{\lambda}} \frac{v_{\lambda}^{\alpha}v_{\lambda}^{\beta}(y)}{|x - y|^{n-2}} \, dy,$$

and

$$v_{\lambda}(x) = \int_{H_{\lambda}} \frac{u^{\alpha}v^{\beta}(y)}{|x_{\lambda} - y|^{n-2}} \, dy + \int_{H_{\lambda}} \frac{v_{\lambda}^{\alpha}u_{\lambda}^{\beta}(y)}{|x_{\lambda} - y|^{n-2}} \, dy.$$  

Then we have the following estimate

$$|v_{\lambda} - v|_p \leq C(n, p)(|v_{\lambda}|_{p, B_{\lambda}^v}^{\alpha - 1}|u_{\lambda}|_{p, B_{\lambda}^v}^{\beta - 1}|u_{\lambda} - v|_{p, B_{\lambda}^v} + |v_{\lambda}|_{p, B_{\lambda}^v}^{\alpha - 1}|u_{\lambda}|_{p, B_{\lambda}^v}^{\beta - 1}|u_{\lambda} - u|_{p, B_{\lambda}^v}) \quad (21)$$

After these preparations, we can use the moving plane method as developed in [5] to prove the radial symmetry of the solutions.

At first, let’s start the plane from the infinity. Indeed, for $\lambda >> 1$ large enough, we know that the quantities

$$|v_{\lambda}|_{p, B_{\lambda}^v}, |u_{\lambda}|_{p, B_{\lambda}^v}, |v_{\lambda}|_{p, B_{\lambda}^v}$$
and

$$|u_\lambda|_{p,B^v_\lambda}$$

all are small, which give us that

$$|u_\lambda - u|_{p,B^u_\lambda} \leq \frac{1}{2}|v_\lambda - v|_{p,B^v_\lambda}$$

and

$$|v_\lambda - v|_{p,B^v_\lambda} \leq \frac{1}{2}|u_\lambda - u|_{p,B^u_\lambda}.$$

These imply that $$|u_\lambda - u|_{p,B^u_\lambda} = 0$$ and $$|v_\lambda - v|_{p,B^v_\lambda} = 0$$. These say that $$B^u_\lambda = \phi$$ and $$B^v_\lambda = \phi$$.

Next we define

$$\lambda_0 = \{\lambda \in \mathbb{R}; B^u_\lambda = \phi \text{ for all } \lambda' \geq \lambda\}.$$ 

Then it follows from the fact that $$u(x) \to 0$$ as $$|x| \to \infty$$ and $$u(x) > 0$$ in $$\mathbb{R}^n$$ that $$\lambda < +\infty$$. By the definition of $$\lambda_0$$, we have $$u_{\lambda_0}(x) \leq u(x)$$ for $$x \in H_{\lambda_0}$$. Using the expression (16), we see that $$u_{\lambda_0}(x) < u(x)$$ for $$x \in H_{\lambda_0}$$. This implies that $$|2\lambda e_1 - B^u_\lambda| \to 0$$ as $$\lambda \to \lambda_0$$. It is now standard to know (see [5]) that $$u_{\lambda_0} = u$$, which then gives us $$v_{\lambda_0} = v$$.

Since $$x_1$$ can be any directions, we conclude that $$u$$ and $$v$$ are radial symmetric about some point $$x_o$$.

### 3 Proof of the Uniqueness

In some sense, the proof of Theorem 2 is just at hand by using the integral expression of the solution pair $$(u, v)$$.

**Proof of Theorem 2.** Let $$(u, v) \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \times L^{\frac{2n}{n-2}}(\mathbb{R}^n)$$ be a pair of solutions to system (1). By Theorem 1, we know that $$u$$ and $$v$$ are radial symmetric about the some point $$x_0$$. We may say, $$x_0 = 0$$.

Recall that we have assumed that $$1 \leq \alpha < \beta < \frac{n+2}{n-2}$$. Since, $$u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$$ and $$v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$$. Using the same method in [5], we have that $$u \in C^2(\mathbb{R}^n)$$ and $$v \in C^2(\mathbb{R}^n)$$ with

$$u(x) \to 0, v(x) \to 0,$$

as $$|x| \to \infty$$.

Since our solution $$u$$ is radially symmetric, hence we can write, in polar coordinates, the first equation in (1) as

$$(r^{n-1}u'(r))' = -r^{n-1}u(r)^\alpha v(r)^\beta,$$

where $$r = |x|$$.

Integrating both sides from 0 to $$r$$ yields

$$r^{n-1}u'(r) = -\int_0^r s^{n-1}u^\alpha v^\beta(s)ds.$$
It follows by another integration that
\[ u(r) = u(0) - \int_0^r \frac{1}{\tau^{n-1}} \int_0^\tau s^{n-1} u^\alpha v^\beta dsd\tau. \] (22)

Similarly, for \( v(r) \), we have
\[ v(r) = v(0) - \int_0^r \frac{1}{\tau^{n-1}} \int_0^\tau s^{n-1} u^\alpha v^\beta dsd\tau. \] (23)

As we mentioned in the introduction, we need only to show that \( u(0) = v(0) \). Otherwise, suppose
\[ u(0) < v(0), \] (24)
then by continuity, for all small \( r > 0 \),
\[ u(r) < v(r). \] (25)

In other word, there exists an \( R > 0 \), such that
\[ u(r) < v(r), \, \forall r \in (0, R). \] (26)

Let \( R_0 \) be the supreme value of \( R \), such that (26) holds. Then \( R_0 \leq \infty \) and \( u(R_0) = v(R_0) \), where we have used the fact that \( u(+\infty) = v(+\infty) = 0 \). By the definition of \( R_0 \) and \( \alpha < \beta \), we have that
\[ u(r)^\alpha v(r)^\beta > v(r)^\alpha u(r)^\beta, \, \forall r \in (0, R_0). \] (27)

Then we have from (22) and (23) that
\[ 0 > u(0) - v(0) = \int_0^{R_0} \frac{1}{\tau^{n-1}} \int_0^\tau s^{n-1}(u^\alpha v^\beta - u^\beta v^\alpha)(s) dsd\tau > 0. \]

This is impossible.

Similarly, one can show that \( u(0) > v(0) \) is impossible. Therefore, we must have
\[ u(0) = v(0). \]

Finally, by the standard ODE theory, we arrive at
\[ u(r) \equiv v(r). \]

Hence, our elliptic system (1) has been reduced to the elliptic equation with critical exponent (13). By now, it is standard to know that our solutions pair \( u \) and \( v \) are of the form (14). This completes the proof of Theorem 2.
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