Convergence of two obstructions for projective modules

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Abstract: Let $X = \text{Spec}(A)$ denote a regular affine scheme, over a field $k$, with $1/2 \in k$ and $\text{dim} X = d$. Let $P$ denote a projective $A$-module of rank $n \geq 2$. Let $\pi_0(\mathcal{LO}(P))$ denote the (Nori) Homotopy Obstruction set \cite{M2, MM2}, and $\tilde{CH}^n(X, \Lambda^n P)$ denote the Chow Witt group, of Barge and Morel \cite{BM}. In this article, we define a natural (set theoretic) map

$$\Theta_P : \pi_0(\mathcal{LO}(P)) \longrightarrow \tilde{CH}^n(X, \Lambda^n P)$$

The main results are published in the book \cite{M23}, and this article remains unpublished here in arxiv.

1 Introduction

This is an update for the readers. The main results in this article are included in my recently published Book \cite{M23}. This is a landmark paper on Obstruction theory of projective modules. Since the results are already available in the book \cite{M23} it will not make sense to publish this paper again. As usual, the paper remains here in Arxiv, for reference. Only problem left in this theory is the Agreement question \cite{LZ}. It has been established that

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the Nori Homotopy obstruction detects splitting $[\text{MM2}]$, under suitable conditions. Question remains whether Chow-Witt obstruction of Barge-Morel would also do the same. I would leave this problem for the Chow-Witt cohort to resolve.

In this article we establish a natural (set theoretic) map from (Nori) Homotopy Obstruction set $[\text{M2} \text{MM2}]$, to the Chow Witt group obstructions $[\text{BM}]$, for projective modules to split off a free direct summand. We avoid repeating the extensive background comments given in $[\text{MM1} \text{MM2}]$, on this set of problems on obstructions for projective modules. To facilitate further discussions, in this introduction, let $X = \text{Spec} \, (A)$ denote a regular affine scheme, over a field $k$, with $1/2 \in k$ and $\dim X = d$. Let $P$ denote a projective $A$-module, with $\text{rank} (P) = n$. The episode began, when (around 1989) through some verbal communications M. V. Nori posed the following Homotopy question.

**Question 1.1** (Homotopy Question). Suppose $X = \text{Spec} \, (A)$ is a smooth affine variety, with $\dim X = d$. Let $P$ be a projective $A$-module of rank $n$ and $f_0 : P \to I$ be a surjective homomorphism, onto an ideal $I$ of $A$. Assume $Y = V(I)$ is smooth with $\dim Y = d - n$. Also suppose $Z = V(J) \subseteq \text{Spec} \, (A[T]) = X \times A^1$ is a smooth subscheme, such that $Z$ intersects $X \times 0$ transversally in $Y \times 0$. Now, suppose that $\varphi : P[T] \to J$ is a surjective map such that $\varphi|_{T=0} = f_0 \otimes A^1$. Then the question is, whether there is a surjective map $F : P[T] \to J$ such that (i) $F|_{T=0} = f_0$ and (ii) $F|_Z = \varphi$. Assume $2n \geq d + 3$.

Refer to $[\text{MM2} \text{Mu} \text{M2}]$ and others for slightly varying versions of this question (1.1). The Homotopy question (1.1) was settled affirmatively by Bharwadekar-Keshari $[\text{BK}]$, when $A$ is essentially smooth over an infinite perfect field $k$, and $2n \geq d + 3$. The proof used the corresponding result in the local case in $[\text{MV}]$, along with results in $[\text{M2}]$.

**Euler class groups:** Subsequent to (1.1), Nori also communicated (1992) a definition of Euler class groups to house obstructions for splitting, in the case $d = n$ (see $[\text{MR}]$). Given integers $0 \leq n \leq d$ and invertible modules $\mathcal{L}$, Nori’s definition was expanded ($[\text{BS1} \text{BS2} \text{MY}]$) to define Euler class groups $E^n(A, \mathcal{L})$. These were defined ideal theoretically, using generators
and relations. However, only when \( n = d \), the Euler class groups worked well enough. For projective \( A \)-modules \( P \) of rank \( d \), an Euler class \( e(P) \in E^d(A, \Lambda^d P) \) was defined. It was conjectured and proved [BS2, BS3] that
\[
e(P) = 0 \iff P \cong Q \oplus A.
\]
(1)

For projective \( A \)-modules \( P \), with \( \text{rank}(P) = n < d \), attempts to define Euler classes \( e(P) \) failed.

This set of ideas of Nori (along with other ongoing activities, e.g. [BS1, BS2, BS3, Mu, M2]) received an added significance in 2000, with the introduction of Chow Witt groups, by Barge and Morel [BM, Mo, F1], to house such a possible obstruction. (Chow Witt groups are also known as Oriented Chow groups.) For integers \( 0 \leq n \leq d \), and line bundles \( \mathcal{L} \) on \( X \), groups \( \tilde{CH}^n(X, \mathcal{L}) \), to be called the Chow Witt groups, were introduced. Further, with \( \text{rank}(P) = n \), an obstruction class \( \varepsilon_{CW}(P) \in \tilde{CH}^n(X, \Lambda^n P) \), to be called the Chow Witt obstruction, was defined. As in the case of Euler class groups (1), when \( n = d \), it was conjectured [BM] and proved that [Mo, Theorem 8.14]
\[
\varepsilon_{CW}(P) = 0 \iff P \cong Q \oplus A
\]
(2)
when \( A \) is smooth over a perfect field \( k \). By then, it started appearing not so promising, that Euler class groups \( E^n(A, \Lambda^n P) \) may be able to house such obstructions, for splitting, when \( n = \text{rank}(P) \leq d - 1 \). It remains an open question, whether Chow Witt obstruction \( \varepsilon_{CW}(P) \in \tilde{CH}^n(X, \Lambda^n P) \), would detect splitting, under certain reasonable conditions, when \( n = \text{rank}(P) \leq d - 1 \).

While it was not explicitly articulated, it is obvious that there is an ingrained homotopy relation in the Homotopy question (1.1). For this reason, a local \( P \)-orientation is defined to be a pair \((I, \omega)\), where \( I \subseteq A \) is an ideal and \( \omega : P \to I^2 \) is a surjective map. Let \( \mathcal{LO}(P) \) denote the set of all local \( P \)-orientations. By substituting \( T = 0, 1 \), one obtains two maps
\[
\mathcal{LO}(P) \xrightarrow{T=0} \mathcal{LO}(P[T]) \xrightarrow{T=1} \mathcal{LO}(P)
\]
This induces chain equivalence relations \( \sim \) on \( \mathcal{LO}(P) \). The homotopy obstruction set \( \pi_0(\mathcal{LO}(P)) \) was defined to be the set of all equivalence classes. Further, an obstruction class \( \varepsilon_H(P) \in \pi_0(\mathcal{LO}(P)) \) is defined. It was also established [MM2] that
\[
\varepsilon_H(P) \text{ is neutral} \iff P \cong Q \oplus A
\]
(3)
when $2n \geq d + 3$ and $A$ is essentially smooth over a perfect field. This is clearly an improvement over the results [1, 2], which work only when the rank $n = d$. The possibility to use the homotopy relations ingrained in (1.1) to construct a house $\pi_0(\mathcal{L}O(P))$ for obstructions $\varepsilon_H(P)$ was considered only recently [MM2]. However, $\pi_0(\mathcal{L}O(P))$ is an invariant of $P$ itself, while Chow Witt groups $\widetilde{CH}^n(X, \mathcal{L})$ are invariants of $X$ (and of the determinant). Further, as invariants, Chow Witt groups $\widetilde{CH}^n(X, \Lambda^n P)$ are very similar to Chow groups $CH^n(X)$ [Fu]. We summarize the some of the above:

1. The homotopy obstruction $\varepsilon_H(P) \in \pi_0(\mathcal{L}O(P))$ is well defined and it detects splitting [3],

2. The Chow Witt obstruction $\varepsilon_{CW}(P) \in \widetilde{CH}^n(X, \Lambda^n P)$ is well defined. Only when $n = d$, it is known to detect splitting [2].

3. The Euler class $e(P) \in E^d(A, \Lambda^d P)$ is defined only when $n = d$ and it detects splitting in this case. For $n \leq d - 1$, it is unlikely that an obstruction (Euler) class $e(P)$ in $E^n(A, \Lambda^n P)$ would be definable. Further, in [MM2], for any projective module $P$, an Euler class group $E(P)$ is defined. This group is generated by the set $\mathcal{L}O^n(P)$ of all $P$ orientations $(I, \omega)$ with $\text{height}(I) = n$; modulo the global orientations. Further, $E^n(A, \mathcal{L}) = E(\mathcal{L} \oplus A^{n-1})$. The Euler class groups $E(P)$ works very well when $n = d$ [MM1]. While the definition of $E(P)$ is very natural, it fails decisively when $n \leq d - 1$. Under some stringent conditions, there is a natural map $E(P) \rightarrow \pi_0(\mathcal{L}O(P))$ [MM2].

At this stage it is fairly transparent that Homotopy obstructions of Nori [M2, MM2], and the Chow Witt obstructions of Barge-Morel [BM] must converge, some way. This is precisely what we respond to, in this article, by establishing a natural map (set theoretic)

$$\Theta_P : \pi_0(\mathcal{L}O(P)) \rightarrow \widetilde{CH}^n(X, \Lambda^n P) \quad \text{with} \quad \Theta_P(\varepsilon_H(P)) = \varepsilon_{CW}(P^*). \quad (4)$$

We establish this for all projective modules $P$, with $2 \leq n = \text{rank}(P) \leq d$. While it was established in [MM2] that $\pi_0(\mathcal{L}O(P))$ has a additive structure, when $2n \geq d + 2$, there is no such well defined structure outside this range of $n$. Therefore, the map $\Theta_P$, defined above, would be a set theoretic map only, in general, and respects additivity when $2n \geq d + 2$. Further, recall that the
obstruction class \( \varepsilon_H(P) \in \pi_0(\mathcal{L}O(P)) \) is defined, for all \( n := \text{rank}(P) \). The obstruction class \( \varepsilon_H(P) \) detects splitting properties of \( P \), under the conditions stated above (3). With this result in mind, the following natural question emerges.

**Question 1.2** (Agreement Question). Whether the map \( \Theta_P \) is injective?

If and when the answer to (1.2) is affirmative, Chow Witt obstructions \( \varepsilon_{CW}(P) \) would detect splitting, under the same hypotheses above (3).

This set of problems was referred to as the "Homotopy Program" by this author (e.g. [MS]). More precise outline of the program was given in [MM2]. The possibility of the existence of a map \( \Theta_P \), as above, was mentioned as a part of the program [MM2 Part 2, pp. 173], which we accomplish fully in this article.

We briefly comment on the place of \( \mathbb{A}^1 \)-homotopy in this program [MM2, Part 2, pp. 173]. We denote the \( \mathbb{A}^1 \)-homotopy category by \( \mathcal{H}(k) \). Perhaps, this stems from variety of spheres \( S^{p,q} \) considered in \( \mathcal{H}(k) \) [MVV, pp.111], and

\[
Q_{2n} = 
\text{Spec} \left( k[x_1, \ldots, x_n, y_1, \ldots, y_n, z] \middle/ \sum_{i=1}^n x_i y_i + z(z-1) \right) \cong (\mathbb{P}^1)^n \cong S^{2n,n} \text{ being one of them} \]  

[Mo, Rem. 6.42]. [ADF, Thm. 2.2.5]. Further, \( Q_{2n} \) is \( \mathbb{A}^1 \)-naive [AF, Thm. 1.1.1], in the sense

\[
\pi_{\mathbb{A}^1}(X, Q_{2n}) \xrightarrow{\sim} Mor_{\mathcal{H}(k)}(X, Q_{2n}) \quad \forall \text{ smooth affine schemes } X = \text{Spec} (A),
\]

where, following the Suslin-Voevodsky construction, \( \pi_{\mathbb{A}^1}(X, Q_{2n}) \) denotes the usual set defined by homotopy (see [Mo, pp. 199]). Other notations were used in [MM1, MM2, MM3, for \( \pi_{\mathbb{A}^1}(X, Q_{2n}) \). Combining this with results in [MM2, Lem. 2.9, Thm. 7.3], when \( 2n \geq d + 2 \), we have maps and isomorphisms

\[
E^n(A, A) = E(A^n) \longrightarrow \pi_0(\mathcal{L}O(A^n)) \cong \pi_{\mathbb{A}^1}(X, Q_{2n}) \xrightarrow{\sim} Mor_{\mathcal{H}(k)}(X, Q_{2n})
\]

(6)

The second arrow is also an isomorphism, when \( A \) is essentially smooth, \( k \) is an infinite perfect field, and \( 2n \geq d + 3 \) [MM1, Thm. 1.4]. Combining with (4), we obtain a natural map

\[
Mor_{\mathcal{H}(k)}(X, Q_{2n}) \longrightarrow \widetilde{CH}^n(X, A) \quad \text{if } 2n \geq d + 3.
\]

(7)
This map (7) was also obtained in [AF] Thm. 1, when \(2n \geq d + 2\), which is a particular case of (4), with \(P = A^n\). Given these isomorphisms (6), the \(A^1\)-homotopy invariant \(\text{Mor}_{\mathcal{H}(k)}(X, Q_{2n})\) does not seem to be of any additional help, from our perspective of obstruction theory. However, as indicated in [MM2, Appendix. A], the obstructions \(\pi_0(\mathcal{L}O(P))\) should have a \(A^1\)-homotopy interpretation, which would be of interest, by its own rights. Such an interpretation may also throw some light on the question (1.2).

As is [BM], our work is fully reliant on the methods of classical (commutative) algebra, and is independent of the methods of \(A^1\)-homotopy.

We comment on the organization of the article. In section 2, we provide some background on Homotopy obstruction, mainly from [MM2, MM1]. In section 3, we associate a symmetric form \(\Phi(I, \omega)\) to certain representatives \((I, \omega) \in \pi_0(\mathcal{L}O(P))\). In section 4, we provide some preliminaries on Chow Witt groups. In section 5, we establish the map \(\Theta_P\).

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2 Preliminaries on Homotopy Obstructions

Throughout this article \(A\) will denote a noetherian commutative ring, with \(\dim A = d\), and \(A[T]\) will denote the polynomial ring in one variable \(T\). We assume \(1/2 \in A\) and \(\text{Spec}(A)\) is connected. For an \(A\)-module \(M\), denote \(M[T] := M \otimes A[T]\). Likewise, for a homomorphism \(f : M \rightarrow N\) of \(A\)-modules, \(f[T] := f \otimes A[T]\). Our main results would assume \(A\) is a regular ring, containing a field \(k\), with \(1/2 \in k\). Further, \(P\) will denote a projective \(A\)-module with \(\text{rank}(P) = n\), and \(2 \leq n \leq d\). Denote \(|P| := \Lambda^n P\), the determinant of \(P\).

For a such a projective \(A\)-module \(P\), as above, the set \(\pi_0(\mathcal{L}O(P))\) of equivalence classes of homotopy obstructions was defined in [MM2]. We recall some of the essential elements of the definition of, and alternative descriptions of \(\pi_0(\mathcal{L}O(P))\) from [MM1].

Definition 2.1. Let \(A\) be a noetherian commutative ring, with \(\dim A = d\) and \(P\) be a projective \(A\)-module with \(\text{rank}(P) = n\). By a local \(P\)-
orientation, we mean a pair \((I, \omega)\) where \(I\) is an ideal of \(A\) and \(\omega : P \to \mathbb{F}\) is a surjective homomorphism. We will use the same notation \(\omega\) for the map \(\frac{P}{IT} \to \frac{I}{IT}\), induced by \(\omega\). A local local \(P\)-orientation will simply be referred to as a local orientation, when \(P\) is understood. Denote

\[
\begin{align*}
\mathcal{L}O(P) &= \{(I, \omega) : (I, \omega) \text{ is a local } P \text{ orientation}\} \\
\tilde{\mathcal{L}}O(P) &= \{(I, \omega) \in \mathcal{L}O(P) : \text{height}(I) \geq n\} \\
\mathcal{L}O^n(P) &= \{(I, \omega) \in \mathcal{L}O(P) : \text{height}(I) = n\} \\
\text{Note } \tilde{\mathcal{L}}O(P) &= \mathcal{L}O^n(P) \cup \{(A, 0)\}
\end{align*}
\]

(8)

For \((I_0, \omega_0), (I_1, \omega_1) \in \mathcal{L}O(P)\), we write \((I_0, \omega_0) \sim_H (I_1, \omega_1)\), if there is an \((I, \omega) \in \mathcal{L}O(P[T])\) such that \((I(0), \omega(0)) = (I_0, \omega_0)\) and \((I(1), \omega(1)) = (I_1, \omega_1)\). In this case, we say \((I_0, \omega_0)\) is homotopic to \((I_1, \omega_1)\). The relation \(\sim_H\) generates a chain equivalence relation \(\sim\) on \(\mathcal{L}O(P)\), which we call the chain homotopy relation. The (Nori) Homotopy obstruction set \(\pi_0(\mathcal{L}O(P)) := \tilde{\mathcal{L}}O(P) / \sim\) is defined to be the set of equivalence classes of elements in \(\mathcal{L}O(P)\).

So, we have a push forward diagram:

\[
\begin{CD}
\mathcal{L}O(P[T]) @>T=0>> \mathcal{L}O(P) \\
@V T=1 VV @VV \text{in } \text{Sets.} V \\
\mathcal{L}O(P) @>>> \pi_0(\mathcal{L}O(P))
\end{CD}
\]

(9)

Similarly, the homotopy relation \(\sim_H\) on \(\tilde{\mathcal{L}}O(P)\) leads to a chain equivalence relation \(\sim\) on \(\tilde{\mathcal{L}}O(P)\). Note, due to moving lemma arguments (basic element theory), it would not make any difference, whether \(\sim_H\) is defined by using \(P[T]\)-orientations \((I, \omega)\) in \(\tilde{\mathcal{L}}O(P[T])\) or in \(\mathcal{L}O(P[T])\). Define

\[
\pi_0(\tilde{\mathcal{L}}O(P)) := \tilde{\mathcal{L}}O(P)/\sim.
\]

Note, a push forward diagram, as in in (9), to define \(\pi_0(\tilde{\mathcal{L}}O(P))\), would not be possible, because the substitution \(T = 0, T = 1\) do not behave too well, in this case. However, there is natural map

\[
\varphi : \pi_0(\tilde{\mathcal{L}}O(P)) \to \pi_0(\mathcal{L}O(P))
\]
The following is from [MM2].

**Proposition 2.2.** Let $A$ and $P$ be as in (2.1). Then, the map $\varphi$ is surjective. Assume further that $A$ is a regular ring containing a field $k$, with $1/2 \in k$. Then $\sim$ is an equivalence relation on $\tilde{LO}(P)$. Moreover, $\varphi$ is a bijection.

**Remark 2.3.** With notations as in (2.1) the following are some useful observations.

1. Suppose $(I_0, \omega_0), (I_1, \omega_1) \in \tilde{LO}(P)$ and $(I_0, \omega_0) \sim (I_1, \omega_1)$. By definition there is homotopy $H(T) = (I, \omega) \in LO(P[T])$ such that $H(0) = (I_0, \omega_0)$ and $H(1) = (I_1, \omega_1)$. By moving Lemma arguments, similar to [MM2, Lemma 4.5], we can assume that $H(T) \in \tilde{LO}(P[T])$.

2. If $A$ is Cohen Macaulay, then $\tilde{LO}(P)$ is in bijection with the set

\[
\big\{ (I, \omega) \in LO^n(P) : \omega : \frac{P}{IP} \sim \frac{I}{I^2} \text{ is an isomorphism} \big\} \cup \{(A, 0)\}
\]

Moreover, for $(I, \omega) \in LO^n(P)$, $I$ is a local complete intersection ideal.

We recall the definition of the neutral element and the obstruction class.

**Definition 2.4.** Use the notations as above (2.1). The neutral element $e_1 \in \pi_0(LO(P))$ is defined to be the image of $(A, 0)$. The Nori Homotopy obstruction $\varepsilon_H(P) \in \pi_0(LO(P))$ is defined to be the image of $(0, 0)$.

Now assume $A$ is a Cohen Macaulay ring and $(I, \omega) \in \tilde{LO}(P)$. Suppose $\omega$ lifts to a surjective map $f : P \rightarrow I$. It follows

\[
\varepsilon_H(P) = \text{image}(I, \omega) \in \pi_0(LO(P))
\]

3 **The symmetric form**

The essence of the arguments in the this section, can be traced back to the following theorem of Altman and Kleiman [AK, Theorem 4.5].
**Theorem 3.1.** Suppose $A$ is a commutative noetherian ring and $I$ is a locally complete intersection ideal, with $\text{height}(I) = n$. Suppose $L$ is an invertible $A$-module. Then, there is a natural isomorphism

$$\chi : \text{Hom} \left( \Lambda^n I, \frac{L}{IL} \right) \xrightarrow{\sim} \text{Ext}^n \left( A_I, L \right)$$

For convenience, recall other notations:

$$\text{Hom} \left( \Lambda^n I, \frac{L}{IL} \right) =: \left| \frac{L}{IL} \right|^{-1}$$

For our purpose, the following formulations would be helpful.

**Lemma 3.2.** Let $A$ be a noetherian commutative ring and $P$ be a projective $A$-module with $\text{rank}(P) = n$. Let $I$ be a locally complete intersection ideal with $\text{height}(I) = n$. Suppose $\omega : \frac{P}{IP} \xrightarrow{\sim} \frac{I}{I^2}$ is an isomorphism, and let $|\omega| : \left| \frac{P}{IP} \right| \xrightarrow{\sim} \left| \frac{I}{I^2} \right|$ denote the determinant of $\omega$, where $\left| \frac{P}{IP} \right| := \Lambda^n I_{IP}$, $\left| \frac{I}{I^2} \right| := \Lambda^n I_{I^2}$ denote the determinants.

Let $f : P \twoheadrightarrow I$ be a surjective lift of $\omega$, and consider its Koszul complex. Then, for any finitely generated $A$-module $M$, the sequence

$$\text{Hom} \left( \Lambda^{n-1} P, M \right) \xrightarrow{d_n} \text{Hom} \left( \Lambda^n P, M \right) \xrightarrow{\varphi} \text{Hom} \left( \Lambda^n IP, M \right) \longrightarrow 0 \quad (10)$$

is exact. Consequently, the following maps are isomorphisms:

$$\text{Hom} \left( \left| \frac{P}{IP} \right|, \frac{M}{IM} \right) \xrightarrow{\chi(\omega)} \text{Ext}^n \left( \frac{A}{I}, M \right)$$

where

$$\chi(\omega) = \iota(f) \sim \varphi$$

$$\iota(f)$$ is the identification

$$\left| \omega \right|$$ sends $\lambda \mapsto \lambda |\omega|$

$$\chi(\omega) = \iota(f) \left| \omega \right|$$

Further, the map $\chi(\omega)$ depends only on $|\omega|$, and is independent of the lift $f$ of $\omega$. 
Proof. Note there is a natural surjective map $\varphi$, by reduction modulo $I$. Further, the composition is zero. We check the exactness of (10) locally. So, we can assume $P = \oplus_{i=1}^n A e_i$ is free, and let $f(e_i) = x_i$. Suppose $\varphi(\lambda) = 0$ for some $\lambda : A e_1 \wedge \cdots \wedge e_n \to M$. So,

$$\lambda(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^n x_i m_i \quad \text{for some } m_i \in M$$

Define

$$\tilde{\lambda} : \Lambda^{n-1} P \to M \quad \text{by} \quad \tilde{\lambda}(e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_n) = (-1)^{i-1} m_i$$

Then, $d_n^*(\tilde{\lambda}) = \lambda$. This establishes that that sequence (10) is exact.

The map $\varphi$ is independent of the lift $f$. So, by definition $\chi(\omega)$ depends only on $|\omega|$. The proof is complete. 

Remark 3.3. Recall, as always, $\text{Ext}^n(A, M)$ in (3.2) is defined only up to isomorphism and the natural identification $\iota$ in (11) needs to be understood with care. Suppose $A, I, P, M$ be as in (3.2), and let $f, g : P \to I$ be two surjective maps, (without any reference to $\omega$). Then, corresponding to Koszul complexes of $f, g$ are related as follows

$$
\begin{array}{cccccccc}
0 & \to & \Lambda^n P & \xrightarrow{d_n(f)} & \Lambda^{n-1} P & \to & \cdots & \to & P & \xrightarrow{f} & A & \xrightarrow{\iota} & 0 \\
\downarrow \det(\Delta) & & \downarrow \det(\Delta) & & \downarrow & & \downarrow & & \downarrow & & 1 & & 1 \\
0 & \to & \Lambda^n P & \xrightarrow{d_n(g)} & \Lambda^{n-1} P & \to & \cdots & \to & P & \xrightarrow{g} & A & \xrightarrow{\iota} & 0
\end{array}
$$

By (10), we have two representations $\text{Hom}(\Lambda^n P \frac{P}{IP} \frac{M}{TM}, M) \xrightarrow{\varphi} \text{Ext}^n(A, M)$, related, as follows

$$
\begin{array}{cccccccc}
\text{Hom}(\Lambda^n P, M) & \xrightarrow{d(f)_n} & \text{Hom}(\Lambda^n P, M) & \xrightarrow{\varphi} & \text{Hom}(\Lambda^n P \frac{P}{IP} \frac{M}{TM}, M) & \xrightarrow{\det(\Delta)} & 0 \\
\downarrow & & \downarrow \det(\Delta) & & \downarrow \det(\Delta) & & 0 \\
\text{Hom}(\Lambda^n P, M) & \xrightarrow{d(g)_n} & \text{Hom}(\Lambda^n P, M) & \xrightarrow{\varphi} & \text{Hom}(\Lambda^n P \frac{P}{IP} \frac{M}{TM}, M) & \xrightarrow{\det(\Delta)} & 0
\end{array}
$$

10
where $\det(\Delta) = \Delta \mod I$ is an unit. One can summarize, that the diagram

$$
\begin{array}{ccc}
\text{Hom} \left( \left| \frac{P}{IP} \right| , \frac{M}{IM} \right) & \xrightarrow{\iota(f)} & \text{Ext}^n \left( \frac{A}{I}, M \right) \\
\text{Hom} \left( \left| \frac{I}{IP} \right| , \frac{M}{IM} \right) & & \\
\end{array}
$$

commutes.

The following is a further relaxed version of (3.2), which suits our purpose.

**Corollary 3.4.** Let $A$ be a noetherian commutative ring and $P$ be a projective $A$-module with $\text{rank}(P) = n$. Let $I$ be a locally complete intersection ideal with $\text{height}(I) = n$, and $\omega : \frac{P}{IP} \sim \frac{I}{IP}$ be an isomorphism. Let $f : P \rightarrow IJ$ be a surjective lift of $\omega$, where $J$ is an ideal with $I + J = A$.

Then, for any finitely generates $A$-module $M$, the following maps are isomorphisms:

$$
\begin{array}{ccc}
\text{Hom} \left( \left| \frac{P}{IP} \right| , \frac{M}{IM} \right) & \xrightarrow{\iota(f)} & \text{Ext}^n \left( \frac{A}{I}, M \right) \\
\text{Hom} \left( \left| \frac{I}{IP} \right| , \frac{M}{IM} \right) & & \\
\end{array}
$$

where $\iota(f)$ is the identification $\tilde{\omega}$ sends $\lambda \mapsto \lambda |\omega|$

$$
\chi(\omega) = \iota(f) |\omega|
$$

Further, the map $\chi(\omega)$ depends only on $|\omega|$, and is independent of the lift $f$ of $\omega$, or the ideal $J$.

**Proof.** Let $s + t = A$ with $s \in I$ and $t \in J$. Then, the natural map $\text{Ext}^n \left( \frac{A}{I}, M \right) \xrightarrow{\sim} \text{Ext}^n \left( \frac{A}{T}, M_t \right)$ is isomorphism. Further, $f_t : P_t \rightarrow I_t$ is a surjective map. It follows from (3.2), the sequence

$$
\begin{array}{c}
\text{Hom} \left( \Lambda^{n-1} P_t, M_t \right) \xrightarrow{d^*_n} \text{Hom} \left( \Lambda^n P_t, M_t \right) \xrightarrow{\varphi} \text{Hom} \left( \Lambda^n \frac{P}{IP}, \frac{M}{IM} \right) \longrightarrow 0
\end{array}
$$

is exact. The rest follows as in (3.2). The proof is complete.

In the rest of this section, we further elaborate all these (12), to associate symmetric isomorphism $(I, \omega) \mapsto \Phi(I, \omega)$. 

11
Suppose \( X = \text{Spec}(A) \) is Cohen Macaulay scheme, with \( \dim X = d \). Recall, for integers \( 0 \leq n \leq d \), in [M1], the subcategory \( CM^n(X) \subseteq \text{Coh}(X) \), was defined to be the full subcategory of objects

\[
CM^n(X) = \{ F \in \text{Coh}(X) : \text{grade}(F) = \text{proj} \dim(F) = n \} \quad (14)
\]

where \( \text{proj} \dim(F) \) denotes the locally free dimension of \( F \). Given an invertible sheaf \( \mathcal{L} \), on \( X \), the association \( F \mapsto F^\vee := \mathcal{E}xt^n(F, \mathcal{L}) \) is a duality in \( CM^n(X) \). This endows \( CM^n(X) \) with a structure of an exact category with duality, to be denoted by \( CM^n(X, \mathcal{L}) \). The following is a key definition for our purpose.

**Definition 3.5.** Now assume \( A \) is Cohen Macaulay ring, with \( \dim A = d \). Let \( P \) be a projective \( A \)-module with \( \text{rank}(P) = n \leq d \). With the notations as in section 2, let \( (I, \omega) \in L\mathcal{O}^n(P) \). As before, \( |\omega| : \left| \frac{P}{TP} \right| \to \left| \frac{I}{TP} \right| \) denote the determinant map. With \( M = |P| \), the diagram (12) reduces to:

\[
\begin{array}{ccc}
\text{End} \left( \left| \frac{P}{TP} \right| \right) & \xrightarrow{(f)} & \text{Ext}^n \left( \frac{A}{I}, |P| \right) \\
\uparrow{\omega} & & \downarrow{\chi(\omega)} \\
\text{Hom} \left( \left| \frac{I}{TP} \right|, \left| \frac{P}{TP} \right| \right) & \sim & \text{End} \left( \left| \frac{I}{TP} \right| \right)
\end{array}
\quad (15)
\]

Therefore,

\[
\chi(\omega)(|\omega|^{-1}) = \iota(f)(1\left| \frac{P}{TP} \right|) \in \text{Ext}^n \left( \frac{A}{I}, |P| \right)
\]

is a free generator.

Define the isomorphism

\[
f_I: \frac{A}{I} \sim \text{Ext}^n \left( \frac{A}{I}, |P| \right) \quad \text{by} \quad f_I(\overline{a}) := a\iota(f)(1\left| \frac{P}{TP} \right|) = \chi(\omega)(|\omega|^{-1}) \quad \forall a \in A
\]

(16)

1. In other word, with identification \( \text{Ext}^n \left( \frac{A}{I}, |P| \right) = \text{End} \left( \left| \frac{P}{TP} \right| \right) \), the isomorphism \( f_I: \frac{A}{I} \sim \text{End} \left( \left| \frac{P}{TP} \right| \right) \), sends \( 1 \mapsto 1\left| \frac{P}{TP} \right| \).
2. Since \( \text{Ext}^r \left( \frac{A}{I}, |P| \right) = 0, \forall r \neq n \), it follows \( f_I \) is a symmetric isomorphism in \( CM^n(X, |P|) \).
3. We let $\Phi(I, \omega) := \left( \frac{A_I}{f}, f \right)$ denote the symmetric space in $\text{CM}^n(X, |P|)$. Further, for $(A, 0) \in \mathcal{L}O(P)$, let $\Phi(A, 0) := 0$ be the trivial symmetric space in $\text{CM}^n(X, |P|)$.

4. Let $\textbf{M} (\text{CM}^n(X)) := \textbf{M} (\text{CM}^n(X, |P|))$ denote the monoid of the isometry classes of symmetric spaces in $\text{CM}^n(X, |P|)$. We will continue to use the same notation $\Phi(I, \omega)$ for the symmetric space in $\text{CM}^n(X, |P|)$, and its isometry class in $\textbf{M} (\text{CM}^n(X, |P|))$. Thus, the association $(I, \omega) \mapsto \Phi(I, \omega)$, defines a set theoretic map:

$$\Phi : \tilde{\mathcal{L}}O(P) \longrightarrow \textbf{M} (\text{CM}^n(X, |P|))$$

(17)

For future correspondences with the derived categories, we further analyze this symmetric space $\Phi(I, \omega)$.

**Remark 3.6.** Let $X = \text{Spec}(A)$ be an affine Cohen Macaulay scheme with $\dim X = d$. Let $P$ be a projective $A$-module with $\text{rank}(P) = n$. For any $A$-module $M$, write $M^* = \text{Hom}(A, |P|)$. Throughout, on the category $\mathcal{P}(X)$ of finitely generated projective $A$-modules, we work with the duality $Q \mapsto Q^*$. Further, duality we consider in other associated categories (e.g. $\text{CM}^n(X)$), would be induced by this duality on $\mathcal{P}(X)$.

Let $(I, \omega) \in \mathcal{L}O^n(P)$. We have $\frac{A}{I} \in \text{CM}^n(X)$ and $\omega : \frac{P}{fP} \sim \frac{I}{fI}$ is an isomorphism. Consider a surjective lift $f$ of $\omega$, as in the diagram

$$\begin{array}{ccc}
P & \xrightarrow{f} & I \cap J \\
\downarrow & & \downarrow \\
\frac{P}{fP} & \sim & \frac{I}{fI}
\end{array}$$

with $I + J = A$, $\text{height}(J) \geq n$

Let $\omega_J : \frac{P}{fP} \sim \frac{J}{fJ}$, and $\omega_{IJ} : \frac{P}{fP} \sim \frac{IJ}{fIJ}$ denote the isomorphism induced by $f$. Note $\omega_{IJ} = \omega \oplus \omega_J$. There is a natural map from the Koszul complex
of $f$ to its dual, as follows:

$$
0 \xrightarrow{\phi_n} \Lambda^n P \xrightarrow{d_n} \cdots \xrightarrow{d_1 = f} \Lambda^1 P \xrightarrow{\phi_1} A \xrightarrow{d_0} \frac{A}{I_f} \xrightarrow{\phi_0} 0
$$

$$
0 \xrightarrow{\phi_n} \Lambda^0 P^* \xrightarrow{\phi_1} (\Lambda^{n-1} P)^* \xrightarrow{(d^*)_1} \text{End} (|P|) \xrightarrow{\text{End} \left( \frac{|P|}{|P|_I} \right)} 0
$$

(18)

Of course, $|P| = \Lambda^0 P^* = \Lambda^n P$. The rectangle at degree $r, r - 1$ is given by

$$
\Lambda^r P \xrightarrow{d_r} \Lambda^{r-1} P
$$

$$
\Lambda^{n-r} P^* \xrightarrow{(d^*)_r} \Lambda^{n-r+1} P^*
$$

(19)

We further explain the set up:

1. $d_r(p_1 \wedge \cdots \wedge p_r) = \sum_{i=1}^r (-1)^{i-1} f(p_i) p_1 \wedge \cdots \wedge p_{i-1} \wedge p_{i+1} \wedge \cdots \wedge p_r$.

2. We follow the sign conventions in [S1], [S2, Rem. 13], while we use homology complexes. So, $(d^*)_r = (-1)^{r-1} d_{n-r+1}^*$ and the double dual identification is given by $(-1)^{r(n-r)} : \Lambda^r P \sim \Lambda^{n-r} P^*$.

3. The maps $\varphi_r : \Lambda^r P \rightarrow (\Lambda^{n-r} P)^*$ are obtained by the perfect duality

$$
\left\{ \begin{array}{l}
\Lambda^r P \otimes \Lambda^{n-r} P \rightarrow |P| & \text{sending} & \text{[H] Ex.5.16(b), pp.127]}
\end{array} \right.
$$

$$(p_1 \wedge \cdots \wedge p_r) \otimes (p_{r+1} \wedge \cdots \wedge p_n) \mapsto p_1 \wedge \cdots \wedge p_r \wedge p_{r+1} \wedge \cdots \wedge p_n$$

4. The isomorphisms $\varphi_0$ and $\varphi_{IJ}$ are given by the natural multiplication maps.

5. We split $\varphi_{IJ} = \varphi_I \oplus \varphi_J : \frac{A}{I_f} \oplus \frac{A}{I_J} \sim \text{End} \left( \frac{|P|}{|P|_I} \right) \oplus \text{End} \left( \frac{|P|}{|P|_J} \right)$. Combining with the natural identifications, we have $f_{IJ} = f_I \oplus f_J$, as in the
4 Preliminaries on Chow Witt groups

We recall two different formulations of the Gersten Witt complex from \[BW\]. Let \(X\) be a regular scheme over a field \(k\), with \(1/2 \in k\) with \(\dim X = d\). Let \(L\) be an invertible sheaf on \(X\) and \(0 \leq n \leq d\) be an integer. We denote \(X^{(n)} := \{x \in X : \text{co}\dim(x) = n\}\). For \(x \in X\), let \(X_x := \text{Spec}(\mathcal{O}_{X,x})\) and let \(D^b_{fl}(\mathcal{Y}(X_x, \mathcal{L}_x))\) denote the bounded derived category of complexes, of projective \(X_x\)-modules, with finite length homologies, and duality induced by \(P \mapsto \text{Hom}(P, \mathcal{L}_x)\). Further, let \(W^n(D^b_{fl}(\mathcal{Y}(X_x, \mathcal{L}_x)))\) denote the \(n\)-shifted Witt group. For convenience, with \(X\) and \(\mathcal{L}\) being understood as above, we introduce the following two notations

\[
\begin{align*}
C^n(D^b_{fl}, W) &= \bigoplus_{x \in X^{(n)}} W^n(D^b_{fl}(\mathcal{Y}(X_x, \mathcal{L}_x))) \\
C^n(C^{\mathcal{M}}, W) &= \bigoplus_{x \in X^{(n)}} W(C^{\mathcal{M}}(X_x, \mathcal{L}_x))
\end{align*}
\]

(21)

For \(x \in X^{(n)}\), \(W(C^{\mathcal{M}}(X_x, \mathcal{L}_x))\) denotes the Witt group of the exact category \(C^{\mathcal{M}}(X_x)\) with duality \(M \mapsto \text{Ext}^n(M, \mathcal{L}_x)\). Note, under the standing regularity hypotheses, \(C^{\mathcal{M}}(X_x)\) turns out to be the category of finite length \(\mathcal{O}_{X,x}\)-modules. For brevity of notations, we would often ignore to include \(\mathcal{L}\) or \(\mathcal{L}_x\) in the notations.

From the work of Balmer and Walter \[B\ [BW]\] on Witt groups of triangulated categories, we have the following two isomorphic descriptions of the
Gersten Witt complex, on $X$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & W(X, \mathcal{L}) & \longrightarrow & C^0(\mathbb{D}_{fl}^b, W) & \longrightarrow & \cdots & \longrightarrow & C^m(\mathbb{D}_{fl}^b, W) & \overset{\partial^0_W}{\longrightarrow} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \vdots & & \downarrow & & \downarrow \\
0 & \longrightarrow & W(X, \mathcal{L}) & \longrightarrow & C^0(\mathbb{CM}_l, W) & \longrightarrow & \cdots & \longrightarrow & C^m(\mathbb{CM}_l, W) & \overset{d^0_W}{\longrightarrow} & \cdots
\end{array}
$$

(22)

terminating at the $\bigoplus_{x \in X^{(0)}}$-term, and where $W(X, \mathcal{L})$ denotes the Witt group of the category of $\mathcal{P}(X)$ with duality $P \mapsto \text{Hom}(P, \mathcal{L})$. We would be using these notations in this diagram (22), in particular, the notations $\partial^0_W, d^0_W$ for the differentials.

We proceed to recall the definition of the Chow Witt group $\widetilde{CH}^n(X, \mathcal{L})$ of codimension $n$ (oriented) cycles, due to Barge and Morel [BM]. However, [FL] provides the most comprehensive foundation available on Chow Witt groups. Before we give the definition, we need to set up some notations, for typographical reasons. For $x \in X$, denote the residue field, at $x$, by $\kappa(x)$, and $u(\kappa(x)) = K_1(\kappa(x))$ denotes the group of units in $\kappa(x)$. Further, for $x \in X^{(k)}$, denote

$$
\begin{cases}
I(X_x) \subseteq W(C^{\mathbb{M}_k}(X_x), \mathcal{L}_x), & \text{the fundamental ideal} \\
I^n(X_x) = I(X_x)^n, & \forall n \geq 1 \\
I^n(X_x) = W(C^{\mathbb{M}_k}(X_x), \mathcal{L}_x), & \forall n \leq 0
\end{cases}
$$

So, $\forall x \in X^{(k)}$, by [L, Cor. 1.6], [L, Cor. 2.3], we have

$$
\frac{I^0(X_x)}{I^1(X_x)} = \frac{\mathbb{Z}}{2\mathbb{Z}}, \quad \frac{u(\kappa(x))}{u(\kappa(x))^2} \sim \frac{I^1(X_x)}{I^2(X_x)}
$$

(23)

We ignore the coordinate associated to the duality. Fix an integer $0 \leq n \leq d$. We would define $\widetilde{CH}^n(X, \mathcal{L})$, as cohomology of a certain complex. With that in mind, denote the following, which would be degree $n - 1, n, n + 1$ terms of variety of complexes:

$$
\begin{align*}
\mathcal{K}^{n-1}_1(X) &= \bigoplus_{x \in X^{(n-1)}} u(\kappa(x)) = \bigoplus_{x \in X^{(n-1)}} K_1(\kappa(x)) \\
\mathcal{K}^n_0(X) &= \bigoplus_{x \in X^{(n)}} K_0(\kappa(x)) =: \bigoplus_{x \in X^{(n)}} \mathbb{Z}[x] \\
\mathcal{T}^{n-1}_1(X) &= \bigoplus_{x \in X^{(n-1)}} I^1(X_x), \quad \mathcal{T}^{n-1}_2(X) = \bigoplus_{x \in X^{(n-1)}} I^2(X_x) \\
\mathcal{T}^n_0(X) &= \bigoplus_{x \in X^{(n)}} I^n_0(X_x), \quad \mathcal{T}^n_0(X) = \bigoplus_{x \in X^{(n)}} I^n_1(X_x) \\
\mathcal{T}^{n+1}_1(X) &= \bigoplus_{x \in X^{(n+1)}} I^{-1}(X_x) = \bigoplus_{x \in X^{(n+1)}} W(X_x)
\end{align*}
$$

(24)
Thus, by (23), we have isomorphisms

\[
\begin{aligned}
K_{n-1}(X) &\cong \bigoplus_{x \in X} \frac{\mathcal{O}(x)}{\mathcal{O}(x)^2} \rightarrow \bigoplus_{x \in X} \mathcal{O}(x) \rightarrow T_{n-1}(X) \\
K_0(X) &\cong \bigoplus_{x \in X} \mathbb{Z} \rightarrow \bigoplus_{x \in X} \mathbb{Z} \rightarrow T_0(X)
\end{aligned}
\]

and

\[
\begin{aligned}
K_{n-1}(X) &\cong \bigoplus_{x \in X} \mathcal{O}(x) \rightarrow \bigoplus_{x \in X} \mathcal{O}(x) \rightarrow T_{n-1}(X) \\
K_0(X) &\cong \bigoplus_{x \in X} \mathbb{Z} \rightarrow \bigoplus_{x \in X} \mathbb{Z} \rightarrow T_0(X)
\end{aligned}
\]

Now consider the diagram:

\[
\begin{aligned}
G_1^{n-1}(X) \xrightarrow{\zeta_1} &\rightarrow I_1^{n-1}(X) \\
G_0^{n}(X) \xrightarrow{d_0^n} &\rightarrow I_0^n(X) \\
K_1^{n-1}(X) \xrightarrow{d_K^{n-1}} &\rightarrow K_0^n(X) \xrightarrow{d_0^n} \rightarrow I_0^n(X) \\
K_0^n(X) \xrightarrow{d_0^n} &\rightarrow K_0^n(X) \xrightarrow{d_0^n} \rightarrow I_0^n(X) \xrightarrow{d_0^n} \rightarrow I_0^{n+1}(X)
\end{aligned}
\]

The differential \(d_0^n\) at the lower left corner is the usual order map. The upper right complex is, the degree \(n-1,n,n+1\) portion, of a properly constructed subcomplex of the Gersten Witt complex (22). The upper left diagonal of the diagram is obtained by cartesian product, of the lower left and upper right complex. We define three groups:

\[
\begin{aligned}
CH^n(X) &= \text{co ker}(d_K^{n-1}), \\
C^n(C,L) &= \frac{\text{ker}(d_0^n)}{\text{image}(d_0^n)}, \\
\hat{C}H^n(X,L) &= \frac{\text{ker}(d_0^n)}{\text{image}(d_0^n)}
\end{aligned}
\]

(26)

The first one is the Chow group of codimension \(n\)-cycles [Fu], and the last one \(\hat{C}H^n(X,L)\) is to be called the **Chow Witt groups**, (of co dimension \(n\) oriented cycles).
Remark 4.1. The following are some information on the diagram (25):

1. In [BM], for integers $0 \leq n \leq d$, a complex $G^n_{\bullet}$ of length $d$ was defined. (Since we fixed $n$, diagram (25) shows only one coordinate, in the superscript $G_1^{n-1}(X) \to G_0^n(X) \to \cdots$, indicating the degree. The additional subscript is also helpful.) The upper left diagonal of (25), is the degree $n-1, n, n+1$ portion, of $G^n_{\bullet}$. The complex $G^n_{\bullet}$ was defined to be the cartesian product of the Milnor $K$-theory complex $[Mi]$, at the lower left corner (which terminates as shown in diagram (25)), and a subcomplex $I^{\bullet}_{n-1}(X)$ of the Gersten Witt complex (22).

   However, our interest remains limited to the degree $n-1, n, n+1$ portion of $G^n_{\bullet}$, in which case the complexes are fairly transparent and elementary. This is similar to the fact that, the Chow group $CH^n(X)$ is defined by the tail end of the Milnor $K$-theory complex on the lower left corner.

2. There are natural decompositions $G^n_0(X, \mathcal{L}) \cong \bigoplus_{x \in X^{(n)}} G^n_0(X_x)$, and $G_1^{n-1}(X, \mathcal{L}) \cong \bigoplus_{x \in X^{(n-1)}} G_1^{n-1}(X_x)$

3. There are other descriptions of the complex $G^n_{\bullet}(X, \mathcal{L})$, of the Chow Witt groups, as follows:

   (a) Given a field $F$, with $1/2 \in F$, analogous to Milnor $K$-groups $K_r(F)$, there are groups $K_r^{MW}(F)$, known as Milnor Witt groups [Mo]. Using the Milnor Witt groups $K_r^{MW}(\kappa(x))$, one can construct Gersten complexes $C^n_{\bullet}(X, \mathcal{L}, K^{MW})$, exactly as in the lower right corner of the diagram (25). One can prove that $G^n_{\bullet}(X, \mathcal{L}) \cong C^n_{\bullet}(X, K^{MW}, \mathcal{L})$. See [Mo], [F2, Theorem 1.2, pp. 6] for further details.

   (b) Chow-Witt group $\widetilde{CH}^n(X, \mathcal{L})$ can also be defined as the homology of the Gersten complexes of the Grothendieck Witt groups [FS, S1, M1].
We remark on variety of descriptions of Witt groups of regular local rings.

**Remark 4.2.** Consider a regular local ring \((R, m)\), with \(\dim R = d\) and \(1/2 \in R\). Let \(k = R/m\). Given these, we can associate a number of Witt groups, as follows:

1. Most fundamental is the Witt group \(W(k)\) of quadratic forms \([L]\).
2. The Witt group \(W(\mathcal{V}(k), *)\) of the category \(\mathcal{V}(k)\), of finite dimensional \(k\)-vector spaces, with duality \(V \mapsto Hom(V, k)\).
3. The Witt group \(W(CM^d(R), Ext^d)\) of the category \(CM^d(R)\), of finite length \(R\)-modules, with duality \(M \mapsto Ext^d(M, A)\).
4. The Witt group \(W(D^b(CM^d(R)), Ext^d)\) of the bounded derived category \(D^b(CM^d(R))\), of the category \(CM^d(R)\), of finite length \(R\)-modules, with duality induced by \(M \mapsto Ext^d(M, A)\).
5. The Witt group \(W^d(D_{fl}^d(\mathcal{P}(R)), Hom(\cdot, A))\) of the bounded derived category \(D_{fl}^d(\mathcal{P}(R))\), of complexes finite rank free \(R\)-modules, with finite length homology, and duality induced by \(P \mapsto Hom(P, A)\). This was mentioned above \([21]\). (We would not consider skew dualities.)

We tried to be consistent with the notations in \([BW]\). Readers are referred to \([BW]\) for further details on derived Witt groups. It turns out that all these Witt groups are (naturally) isomorphic \([B, BW, QSS, L]\). The isomorphism of the two descriptions of Gersten Witt complexes \([22]\), mentioned above, is a consequence of this fact. However, some of the isomorphisms would depend on some choices to be made. While the first one \(W(k)\) would be elementary \([L]\), the last one \(W^d(D_{fl}^d(\mathcal{P}(R)), Hom(\cdot, A))\) may also be nicer to work with. This is because the duality \(P \mapsto Hom(P, A)\) is much more tangible than, dualities associated to "Ext".

We restate \([F1, Lemma 10.3.4]\), as follows.
Lemma 4.3. Let $X = \text{Spec}(A)$ be a regular scheme, $1/2 \in A$, and $\mathcal{L}$ be an invertible sheaf on $X$. Consider the duality induced by $P \mapsto \text{Hom}(P, \mathcal{L})$ on the bounded derived category $\mathbf{D}^b(\mathcal{P}(X))$ of the category $\mathcal{P}(X)$ of projective $A$-modules of finite rank. For $x := \varphi \in X$ denote $X_x := \text{Spec}(A_{\varphi})$. Let

$$\partial^n_W : \bigoplus_{x \in X^{(n)}} W^n (\mathcal{D}^b_f (X_x, \mathcal{L}_x)) \to \bigoplus_{x \in X^{(n+1)}} W^{n+1} (\mathcal{D}^b_f (X_y, \mathcal{L}_y))$$

(27)

denote the differential of the Gersten Witt complex (which is a map between Witt groups of two quotient categories). Let $f_1, \ldots, f_n \in A$ be a regular sequence and $K(f_1, \ldots, f_n)$ be the Koszul complex. Let $\varphi : K(f_1, \ldots, f_n) \simto K(f_1, \ldots, f_n)^*$ be the corresponding symmetric form. Let $t \in A$ be a non zero divisor (or isomorphism) on $\frac{A}{(f_1, \ldots, f_n)}$. Then,

$$\partial^n_W : [(K(f_1, \ldots, f_n), t\varphi)] = [(K(f_1, \ldots, f_n; t), \varphi \wedge t)]$$

(28)

where $\varphi \wedge t : K(f_1, \ldots, f_n; t) \simto K(f_1, \ldots, f_n; t)^*$ denotes the symmetric form. Evidently, $[(\cdot, t\varphi)]$ and $[(\cdot, \varphi \wedge t)]$ denote the respective elements in the Witt groups of the quotient categories.

Proof. Consider the multiplication map $t : K(f_1, \ldots, f_n) \to K(f_1, \ldots, f_n)$. The cone of this map is the Koszul complex $K(f_1, \ldots, f_n; t)$. Now, the lemma follows from the following Lemma 4.4.

The following is a more general formulation of Lemma 4.3.

Lemma 4.4. Let $X = \text{Spec}(A)$ be a regular scheme, $1/2 \in A$, and $\dim X = d$. Let $\mathcal{L}$ be an invertible sheaf on $X$. Consider duality induced by $P \mapsto \text{Hom}(P, \mathcal{L})$ on $\mathbf{D}^b(\mathcal{P}(X))$, and on the other associated categories. Fix an integer $n$. Let $\partial^n_W$ and $d^n_W$ denote the differentials in the Gersten Witt complex (22). Let $\mathcal{P}^n(\mathcal{P}(X)) \subseteq \mathbf{D}^b(\mathcal{P}(X))$ denote the subcategory of objects $Q_\bullet$, with $\text{codim}(H_i(Q_\bullet)) \geq n$, $\forall i$. In $\mathcal{P}^n(\mathcal{P}(X))$, consider a complex $P_\bullet$, as follows:

$$0 \to P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to 0$$

with $H_i(P_\bullet) = 0 \forall i \neq 0$, and $M := H_0(P_\bullet)$. Then $M \in \text{CM}^n(X)$. 20
Let \( \varphi : P_\bullet \simto P_\bullet^* \) be a symmetric (quasi) isomorphism. Then, \( \varphi \) induces a symmetric isomorphism \( \overline{\varphi} : M \simto M'^\vee \) in \( \text{CM}^n(X, \mathcal{L}) \) where \( M'^\vee := \text{Ext}^n(M, \mathcal{L}) \).

Let \( t \in A \) be a non zero divisor in \( M \). The map
\[
\begin{array}{ccccccccc}
P_\bullet & - & - & - & - & - & - & - & - & 0 \\
\varphi_t & & \downarrow{\varphi_n t} & & \downarrow{\varphi_{n-1} t} & & \downarrow{\varphi_1 t} & & \downarrow{\varphi_0 t} \\
P_\bullet^* & - & - & - & - & - & - & - & - & 0 \\
\end{array}
\]
is a symmetric morphism in \( \mathcal{D}^n(\mathcal{P}(X)) \), which is a lift of its image in \( \mathcal{D}^n(\mathcal{P}(X)) \). By [B] Definition 5.16, [BW] Thm 2.1, 3.1, we have
\[
\partial_{W}^n [(P_\bullet, \varphi_t)] = [(\text{Cone}(P_\bullet, \varphi_t), \psi)]
\]
where \( \psi : \text{Cone}(P_\bullet, \varphi_t) \to \text{Cone}(P_\bullet, \varphi_t)^* \) denotes the induced duality map.

By (TR3) axiom (see [B]), we have a map of the triangles:
\[
\begin{array}{ccccccccc}
P_\bullet & \xrightarrow{t} & P_\bullet & \xrightarrow{\varphi} & \text{Cone}(t) & \xrightarrow{\text{TP}_\bullet} \\
\downarrow{t} & & \downarrow{\varphi} & & \downarrow{\text{TP}_\bullet} & & \downarrow{\text{TP}_\bullet} \\
P_\bullet^* & \xrightarrow{t\varphi} & P_\bullet^* & \xrightarrow{\text{Cone}(t\varphi)} & \text{TP}_\bullet & & \downarrow{\text{TP}_\bullet} \\
\end{array}
\]

Since \( \varphi \) is a quasi isomorphism, \( \text{T} : \text{Cone}(t) \cong \text{Cone}(t\varphi) \) is also a quasi isomorphism. So, \( \varphi \wedge t := \text{T}^* \varphi \text{T} : \text{Cone}(t) \simto \text{Cone}(t)^* \) is a symmetric isomorphism in \( \mathcal{D}^n(\mathcal{P}(X)) \). Consequently, \( \text{T} : (\text{Cone}(t), \varphi \wedge t) \simto (\text{Cone}(\varphi t), \psi) \) is an isometry in \( \mathcal{D}^{n+1}(\mathcal{P}(X)) \). In fact, at degree \( r \):
\[
\begin{align*}
\text{Cone}(t)_r &= P_{r-1} \oplus P_r \\
\text{Cone}(t)^*_r &= P_{n-r+1}^* \oplus P_{n-r}^* \\
\text{Cone}(\varphi t)_r &= P_{r-1} \oplus P_r \\
\text{Cone}(\varphi t)^*_r &= P_{n-r+1}^* \oplus P_{n-r}^*
\end{align*}
\]

One checks, \( \varphi \wedge t : \text{Cone}(t) \to (\text{Cone}(t))^* \) has a natural description, similar to the natural dualities in Koszul complexes. (We remark that cones are very similar to Koszul complexes. For this reason the degree \( r \) term of the cone can be viewed as \( (\text{Cone}(P_\bullet, t))_r = P_r \oplus P_{r-1} = P_r \oplus P_{r-1} \wedge Ae_0 \).) Therefore,
\[
\partial_{W}^n [(P_\bullet, t\varphi)] = [(\text{Cone}(P_\bullet, t\varphi), \psi)] = [((\text{Cone}(P_\bullet, t), \varphi \wedge t)] \quad (29)
\]
Further, since $t$ is a non zero divisor on $M = H_0(P_\bullet)$, the sequence
\[
\begin{array}{ccccccc}
0 & \longrightarrow & H_0(P_\bullet) & \stackrel{t}{\longrightarrow} & H_0(P_\bullet) & \longrightarrow & H_0(\text{Cone}(P_\bullet, t)) & \longrightarrow & 0
\end{array}
\]
and $H_i(\text{Cone}(P_\bullet, t)) = 0 \quad \forall \ i \neq 0$

We obtain the commutative diagram
\[
\begin{array}{cccccccc}
0 & \longrightarrow & M & \stackrel{t}{\longrightarrow} & M & \longrightarrow & \frac{M}{tM} & \longrightarrow & 0 \\
\psi & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Ext}^n(M, \mathcal{L}) & \stackrel{t}{\longrightarrow} & \text{Ext}^n(M, \mathcal{L}) & \longrightarrow & \text{Ext}^{n+1}(\frac{M}{tM}, \mathcal{L}) & \longrightarrow & 0
\end{array}
\]
(30)

where $\overline{\psi}$ is the induced symmetric space in $\widehat{CM}^{n+1}(X, \mathcal{L})$. Consequently,
\[
d^n_W [(M, t\overline{\varphi})] = \left[ \left( \frac{M}{tM}, \overline{\psi} \right) \right]
\]
(31)

**Proof.** As elaborated! \[\Box\]

### 5 The convergence of two obstructions

In this final section, we prove our main result by establishing a natural map $\Theta_P : \pi_0(\mathcal{LO}(P)) \longrightarrow \widehat{CH}^n(X)$.

**Definition 5.1.** Let $X = \text{Spec} (A)$ be a Cohen-Macaulay affine scheme, with $\text{dim} X = d$ and $1/2 \in A$. Let $P$ be a projective $A$-module with $\text{rank}(P) = n$. We adapt all the notations in above (section 4) with $\mathcal{L} = \Lambda^n P = |P|$, the determinant. Refer to the diagram \([25]\) of definition of Chow Witt groups $\widehat{CH}^n(X, |P|)$. For $(I, \omega) \in \widehat{\text{LO}}(P)$, consider the symmetric isomorphism $\Phi(I, \omega) = (\underline{\frac{f}{t}}, f_I)$ in $\text{CM}^n(X, |P|)$, as defined in Definition 3.5, item \([3]\), and equation \([17]\). Recall, that the same notation $\Phi(I, \omega)$ was used to denote the symmetric form and its isometry class in the monoid $\text{MCM}^n(X, |P|)$. This defines an elements
\[
\begin{align*}
&\Phi(I, \omega)_I \in \mathcal{I}^n_0(X) = \bigoplus_{x \in X^{(n)}} \mathcal{I}^0(X_x) \\
&\Phi(I, \omega)_K \in \mathcal{K}^n_0(X) = \bigoplus_{x \in X^{(n)}} K_0(X_x)
\end{align*}
\]
They patch to define an element in \( \Phi(I, \omega)_G \in G^n_0(X, |P|) \). Note \( \Phi(I, \omega)_I \) comes from a symmetric space in \( CM^n(X) \), or equivalently from \( \mathcal{P}(X) \). Consequently, \( d^n_I(\Phi(I, \omega)_I) = 0 \), which follows from construction of the Gersten Witt complex \( [22, \text{BW}] \). Hence \( d^n_G(\Phi(I, \omega)_G) = 0 \). Therefore, \( \Phi(I, \omega)_G \) represents an element in \( \widetilde{CH}^n(X, |P|) \). Define, a set theoretic map

\[
\Omega : \mathcal{L}O(P) \longrightarrow \widetilde{CH}^n(X, |P|) \quad \text{by} \quad \Omega(I, \omega) := \Phi(I, \omega)_G \in \widetilde{CH}^n(X, |P|) \quad (32)
\]

The following commutative diagram would be helpful,

\[
\begin{array}{cccccc}
\mathcal{L}O(P) & \xrightarrow{\Phi} & \mathcal{M}(CM^n(X, |P|)) & \xrightarrow{\iota_n} & \ker(d^n_G) & \xrightarrow{q} \\
\downarrow{\Omega} & & \downarrow{q} & & & \\
\widetilde{CH}^n(X, |P|) & & & & & \\
\end{array}
\quad (33)
\]

where \( \iota_n \) is a the map obtained by point wise localization map.

**Remark 5.2.** Chow Witt groups \( \widetilde{CH}^n(X, |P|) \) can also be defined, equivalently, as the cohomology of the Gersten complex of the Grothedieck Witt groups. So, there is a direct GW-way to look at all these [M1, Remark 4.14(1), 4.5(1)].

We state the following on homotopy invariance of Chow Witt groups.

**Proposition 5.3.** Let \( X \) be a regular scheme over a field \( k \), with \( 1/2 \in k \), 
\( d = \dim X \). Fix an integer \( 0 \leq n \leq d \) and an invertible sheaf \( \mathcal{L} \) on \( X \). All the dualities considered are induced by the duality \( Q \mapsto Hom(Q, \mathcal{L}) \) on \( \mathcal{P}(X) \). Let \( p : X \times \mathbb{A}^1 \longrightarrow X \) be the projection map. Then, \( p \) induces an isomorphism

\[
p^* : \widetilde{CH}^n(X, \mathcal{L}) \sim \longrightarrow \widetilde{CH}^n(X \times \mathbb{A}^1, p^* \mathcal{L})
\]

of the Chow Witt groups.

**Proof.** See [F1, Cor. 11.3.3].

The following is the main theorem in this article.
**Theorem 5.4.** Suppose $A$ is a regular ring, containing a field $k$ with $1/2 \in k$, and $\dim A = d \geq 2$. Let $P$ be a projective $A$-module of rank $n$. Then, the map $\Omega$ in (32) factors through a set theoretic map $\Theta_P$, as in the commutative diagram:

$$
\begin{array}{ccc}
\tilde{\mathcal{L}}O(P) & \xrightarrow{\beta} & \pi_0(\mathcal{L}O(P)) \\
\downarrow{\Omega} & & \downarrow{\Theta_P} \\
\tilde{C}H^n(X, |P|) & \xrightarrow{\pi} & C(n, X, |P|)
\end{array}
$$

(34)

**Proof.** By (2.2), $\pi_0(\mathcal{L}O(P)) = \pi_0(\tilde{\mathcal{L}}O(P))$. Let $(I_0, \omega_0), (I_1, \omega_1) \in \tilde{\mathcal{L}}O(P)$ be such that $\beta((I_0, \omega_0)) = \beta((I_1, \omega_1)) \in \pi_0(\tilde{\mathcal{L}}O(P))$. By (2.3), there is $(I, \omega) \in \mathcal{L}O^n(P[T])$ such that $(I, \omega)|_{T=0} = (I_0, \omega_0)$ and $(I, \omega)|_{T=1} = (I_1, \omega_1)$. The projection map $p : X \times A^1 \longrightarrow X$ induces a pull back map

$$
p^* : \tilde{C}H^n(X, |P|) \sim \tilde{C}H^n(X \times A^1, p^* |P|)
$$

which is an isomorphism (5.3) or [F2, Theorem 2.15]. Consider the commutative diagram (while we duplicate notations $\Omega$, $\Phi$ etc.)

$$
\begin{array}{ccc}
\tilde{\mathcal{L}}O(P) & \xrightarrow{\Phi} & M(CM^n(X, |P|)) \\
\downarrow{\tilde{\mathcal{L}}O(P[T])} & & \downarrow{\Phi} \\
\tilde{\mathcal{L}}O(P[T]) & \xrightarrow{\Phi} & M(CM^n(X \times A^1, p^* |P|))
\end{array}
$$

We are required to prove that $\Omega(I_0, \omega_0) = \Omega(I_1, \omega_1)$. Since $p^*$ (the last vertical arrow) is an isomorphism, it is enough to prove that

$$
p^* (\Omega(I_i, \omega_i)) = \Omega(I, \omega) \quad \text{for} \quad i = 0, 1
$$

(36)

Since $T$ and $T - 1$ are interchangeable, it would suffice to establish the case $i = 0$. This is established below, in Lemma [5.5] The proof is complete. 

24
5.1 The retraction $T=0$

We establish the equation (36), in this subsection.

**Lemma 5.5.** Let $X = \text{Spec}(A)$ be a regular affine scheme, over $\text{Spec}(k)$, where $k$ is a field with $1/2 \in k$ and $\dim X = d$. Let $P$ be a projective $A$-module with $\text{rank}(P) = n$, with $2 \leq n \leq d$. Let $(I, \omega) \in \mathcal{LO}^n(P[T])$ be such that $(I_0, \omega_0) := (I, \omega)|_{T=0} \in \widetilde{\mathcal{O}}(P)$. Then, $p^*(\Omega(I_0, \omega_0)) = \Omega(I, \omega)$, with notations as in (35).

**Proof.** Let $f_0 : P \to I_0 \cap J_0$ be a lift of $\omega_0 : P \to \mathcal{J}_0^0$ such that $\text{height}(J_0) \geq n$, and $I_0 + J_0 = A$. By [M2, Lem. 2.2], there is a lift $h(T) : P[T] \to I$ of $\omega$ such that $h(0) = f_0$. Now, $I = (h(P[T]), s)$, with $s \in I^2$. Consider the set, $V = \{ \varphi \in \text{Spec}(A[T]) : \text{height}(\varphi) \leq n - 1, sT \not\in \varphi \}$. Now, $(h, sT) \in P[T]^* \oplus A[T]$ is basic on $V$. By basic element theory [M3], we have $f := h + sTg$ is basic on $V$, for some $g \in P[T]^*$. It follows, (1) $f(0) = f_0$ and (2) $f$ is a lift of $\omega$ and (3) $\text{height}(f(P[T])) \geq n$. So, we can write $f(P[T]) = I \cap J$, with $\text{height}(J) \geq n$ and $I + J = A[T]$. We fix such a lift $f$ of $\omega$. Also, let $\omega_J : P[T] \to \mathcal{J}_T$, $\omega_{IJ} : P[T] \to \mathcal{J}_{IJ}$, be induced by $f$. Then, $(J, \omega_J), (IJ, \omega_{IJ}) \in \mathcal{LO}(P[T])$.

Likewise, denote $(J_0, \omega_{j_0}), (I_0J_0, \omega_{j_0j_0}) \in \widetilde{\mathcal{O}}(P)$. The Koszul complex of $f$ and $f_0$ leads to the symmetric spaces, which may be considered in the $\mathcal{CM}^n(-)$-categories or in the derived categories:

\[
\begin{array}{ccccccc}
0 & \to & \Lambda^0 P[T]^* & \to & \cdots & \to & \Lambda^{n-1} P[T]^* & \to & \Lambda^n P[T]^* & \to & \text{Ext}^k \left( \frac{A[T]}{(I \cap J)}, \mathcal{L}[T] \right) & \to & 0 \\
\phi & & \phi & & \phi & & & & \phi & & & & & & \\
0 & \to & \Lambda^n P[T] & \to & \cdots & \to & \Lambda^1 P[T] & \xrightarrow{f_0} & \Lambda^0 P[T] & \to & \text{Ext}^k \left( \frac{A[T]}{(I \cap J)}, \mathcal{L}[T] \right) & \to & 0 \\
\phi(0) & & \phi(0) & & \phi(0) & & & & \phi(0) & & & & & & \\
0 & \to & \Lambda^0 P^* & \to & \cdots & \to & \Lambda^{n-1} P^* & \to & \Lambda^n P^* & \to & \text{Ext}^k \left( \frac{A}{(I(0) \cap J(0))}, \mathcal{L} \right) & \to & 0
\end{array}
\]

Denote $X[T] := X \times \mathbb{A}^1 = \text{Spec}(A[T])$. We proceed to use Lemma 4.4 [30].
Denote the symmetric spaces in $\mathcal{D}^{n+1}(\mathcal{P}(X[T]))$, as follows

\[
\begin{align*}
    C(T\varphi) := (\text{Cone}(T), \varphi \wedge T) &\cong \text{Cone}(T\varphi) \quad \text{cone of } T\varphi : \Lambda^*P[T] \to \Lambda^*P[T] \\
    C(T\varphi(0)[T]) := (\text{Cone}(T), \varphi(0)[T] \wedge T) &\cong \text{Cone}(T\varphi(0)[T]) : \Lambda^*P[T] \to \Lambda^*P[T]
\end{align*}
\]

We claim that these symmetric spaces $C(T\varphi)$, $C(T\varphi(0)[T])$ are isometric. Write $f(T) = f_0 + T f_1 + \cdots + T^m f_m$, with $f_i \in \text{Hom}(P, A)$. Consider the map

\[
\Delta = \begin{pmatrix}
1_{A[T]} & \frac{\partial f_0}{T} \\
0 & 1_{P[T]}
\end{pmatrix} : A[T] \oplus P[T] \xrightarrow{\sim} A[T] \oplus P[T]
\]

Then,\[
\begin{pmatrix}
-T \\
f
\end{pmatrix}
\begin{pmatrix}
1_{A[T]} & \frac{\partial f_0}{T} \\
0 & 1_{P[T]}
\end{pmatrix}
\begin{pmatrix}
0 \\
f_0[T]
\end{pmatrix} = \begin{pmatrix}
-T \\
f_0[T]
\end{pmatrix}
\]

Therefore, the diagram

\[
\begin{array}{ccc}
A[T] \oplus P[T] & \xrightarrow{(-T, f_0[T])} & A[T] \\
\Delta \downarrow & & \downarrow 1 \\
A[T] \oplus P[T] & \xrightarrow{(-T, f)} & A[T]
\end{array}
\]

\[
\begin{array}{ccc}
A[T] & \xrightarrow{\Delta} & A[T] \\
0 & & 0
\end{array}
\]

commutes.

We write $Q = A[T] \oplus P[T]$ and $Q_r = \Lambda^r Q$. Then, the cone $C(T\varphi)$ is the Koszul complex of the map $(-T, f) : Q \to A[T]$, and $C(T\varphi(0)[T])$ is the Koszul complex of the map $(-T, f(0)[T]) : Q \to A[T]$.

Write

\[
\begin{align*}
M = \frac{\Lambda[T]}{(J, J)} &= \frac{B}{(T f_0, f_0[T])} \\
M^\vee &= \text{Ext}^{n+1}(M, [P[T]])
\end{align*}
\]

Consider the following composition of maps of Koszul complexes, and duality:

\[
\begin{array}{cccccccccccc}
0 & \xrightarrow{\Lambda^{n+1} Q} & \cdots & \xrightarrow{\Lambda^2 Q} & \xrightarrow{\Lambda Q} & \xrightarrow{Q} & \xrightarrow{(-T, f_0[T])} & A[T] & \xrightarrow{\Delta} & M & \xrightarrow{\text{can}} & 0 \\
0 & \xrightarrow{\Lambda^{n+1} Q} & \cdots & \xrightarrow{\Lambda^2 Q} & \xrightarrow{\Lambda Q} & \xrightarrow{Q} & \xrightarrow{(-T, f)} & A[T] & \xrightarrow{\text{can}} & M & \xrightarrow{\text{can}} & 0 \\
0 & \xrightarrow{A[T]} & \cdots & \xrightarrow{\Lambda^{n-1} Q} & \xrightarrow{\Lambda^n Q} & \xrightarrow{\Lambda^n Q} & \xrightarrow{\Lambda^n Q} & \xrightarrow{\Lambda^n Q} & \xrightarrow{\Lambda^n Q} & \xrightarrow{M^\vee} & 0 \\
0 & \xrightarrow{A[T]} & \cdots & \xrightarrow{\Lambda^{n-1} Q} & \xrightarrow{\Lambda^n Q} & \xrightarrow{\Lambda^n Q} & \xrightarrow{\text{can}} & \xrightarrow{\text{can}} & M^\vee & 0
\end{array}
\]
Since $\det(\Delta) = 1$, it follows from (38), the composition is the canonical map $\Lambda \cdot Q \overset{\sim}{\to} \Lambda \cdot Q^*$. Therefore, $\Lambda \cdot \Delta : C(T \varphi(0)[T]) \overset{\sim}{\to} C(T \varphi)$ is an isometry, in $\mathcal{P}^{n+1}(\mathcal{P}(X))$. At degree zero, the compositions agree. It follows that from this that

$$f_I \oplus f_J = f_{I_0}[T] \oplus f_{J_0}[T] : M \overset{\sim}{\to} M'$$

So, $\forall \ y \in X[T]^{(n+1)}$, the images agree:

$$\left( \frac{A[T]}{I}, f_I \right) = \left( \frac{A[T]}{I_0}, f_{I_0}[T] \right) \in W\left( C \mathcal{M}^{n+1}(X[T], |P| [T]_y) \right)$$

(37)

Up to isometry, we have

$$\begin{cases}
\Phi(I, \omega) = \left( \frac{A[T]}{I}, f_I \right) \in \mathcal{M} \left( C \mathcal{M}^n(X[T]) \right), \\
\Phi(I_0, \omega_0) = \left( \frac{A}{I_0}, f_{I_0} \right) \in \mathcal{M} \left( C \mathcal{M}^n(X) \right) \\
p^*\Phi(I, \omega_0) = \left( \frac{A[T]}{I_0[T]}, f_{I_0}[T] \right) \in \mathcal{M} \left( C \mathcal{M}^n(X[T]) \right)
\end{cases}$$

We obtain $\Phi(I, \omega)_I \in \mathcal{I}^n_0(X[T])$ and $\Phi(I_0, \omega_0)_I \in \mathcal{I}^n_0(X)$, by point wise localization. Now,

$$\begin{cases}
I \cdot \Phi(I, \omega) := \left( \frac{A[T]}{I}, T f_I \right) \\
T \cdot p^*\Phi(I, \omega) := \left( \frac{A[T]}{I_0[T]}, T f_{I_0}[T] \right)
\end{cases}$$

represent symmetric forms in $C \mathcal{M}^n(X[T])$. We use the notation $\langle I \rangle \cdot \Phi(I, \omega)_I$ and $\langle T \rangle \cdot p^*\Phi(I_0, \omega_0)_I$ to denote the corresponding elements in $\mathcal{I}^n_0(X[T])$. Refer to the notations in [22 24 25]. By Lemma 4.4 we have

$$\begin{cases}
\partial_W^p (\Lambda \cdot P[T], T \varphi) = \left[ ((\Lambda \cdot P[T], Cone(T)), \varphi \wedge T) \right]. \\
\text{And, equivalently,} \\
d_W^p (T \cdot (\Phi(I, \omega)_I) \perp \Phi(J, \omega_J)_J) = \sum_{x \in X[T]^{(n+1)}} \left[ \left( \frac{A[T]}{(I\cup J)[T]}, \psi \right)_x \right]
\end{cases}$$

where $\psi : \frac{A[T]}{(I\cup J)[T]} \overset{\sim}{\to} Ext^{n+1} \left( \frac{A[T]}{(I\cup J)[T]}, p^* \mathcal{L} \right)$ is the symmetric isomorphism induced by $\varphi \wedge T$ (see 30 31). Since $I + J = A[T]$ and $I_0 + J_0 = A$, it follows

$$\begin{cases}
d_W^p (\langle T \rangle \cdot \Phi(I, \omega)_I) = \sum_{x \in X[T]^{(n+1)}} \left[ \left( \frac{A[T]}{(I\cup J)[T]}, \psi \right)_x \right] \\
d_W^p ((\langle T \rangle - \langle 1 \rangle) \cdot \Phi(I, \omega)_I)
\end{cases}$$

(38)
in $\mathcal{I}_{0}^{n+1}(X[T], p^{∗}L)$. Similarly,

\[
\begin{cases}
  d_{n}^{W}(\langle T \rangle \cdot p^{∗}\Phi(I_{0}, \omega_{0})_{T}) = \sum_{x \in X[T]}\left[\left(\frac{A[T]}{I_{0}, T}, \psi_{0}\right)_{x}\right] \\
  = d_{n}^{p}(\langle T \rangle - \langle 1 \rangle \cdot p^{∗}\Phi(I_{0}, \omega_{0})_{T})
\end{cases}
\]

(39)

where $\psi_{0}$ is induced by $\varphi(0)[T] \wedge T$ (see 30, 31). It follows from (37) that

\[
d_{n}^{p}(\langle (T) - (1) \rangle \cdot \Phi(I_{0}, \omega_{0})_{T}) = d_{n}^{p}(\langle (T) - (1) \rangle \cdot p^{∗}\Phi(I_{0}, \omega_{0})_{T})
\]

(40)

in $\mathcal{I}_{0}^{n+1}(X[T], |P[T]|)$. Similarly, we have

\[
d_{n}^{K}(T \cdot \Phi(I, \omega)_{K}) = d_{n}^{p}(T \cdot p^{∗}\Phi(I(0), \omega)_{K}) \in \mathcal{X}_{0}^{n+1}(X[T])
\]

Patching these two identities, it follows

\[
d_{n}^{G}(T \cdot \Phi(I, \omega)_{G}) = d_{n}^{p}(T \cdot p^{∗}\Phi(I(0), \omega)_{G}) \in G_{0}^{n+1}(X[T], |P[T]|)
\]

(41)

Consider the homotopy invariance,

\[
p^{∗} : \widetilde{CH}^{n}(X, |P|) \xrightarrow{\sim} \widetilde{CH}^{n}(X \times \mathbb{A}^{1}, |P[T]|)
\]

It follows from (41), and the definition of the retraction, that

\[
(p^{∗})^{-1}\Omega(I, \omega) = (p^{∗})^{-1}(p^{∗}\Omega(I_{0}, \omega_{0})) = \Omega(I_{0}, \omega_{0})
\]

The proof is complete. $\blacksquare$

**Corollary 5.6.** With the notations as above (5.4), we have

\[
\Theta_{P}(\varepsilon_{H}(P)) = \varepsilon_{CW}(P) \in \widetilde{CH}^{n}(X)
\]

where $\varepsilon_{H}(P) \in \pi_{0}\left(\mathcal{L}O(P)\right)$ is the homotopy obstruction defined in (2.4), and $\varepsilon_{CW}(P)$ denotes the oriented Chern class, as defined in [BM, F1].

The following lemma is standard that we used above.

**Lemma 5.7.** Suppose $A$ is a commutative noetherian ring and $Q$ is a projective $A$-module with $\text{rank}(Q) = n$ and let $L := \Lambda^{n}Q = |Q|$. For $A$-modules $M$, we denote $M^{∗} := Hom(M, L)$. Let $0 \leq r \leq n$ and $\text{can} : \Lambda^{r}Q \rightarrow \Lambda^{n-r}Q^{∗}$
be the isomorphism induced by the perfect duality $\Lambda^r Q \otimes \Lambda^{n-r} Q \to \mathcal{L}$. Let $\Delta : Q \to Q$ be an isomorphism, with $\det(\Delta) = 1$. Then, the diagram

\[
\begin{array}{ccc}
\Lambda^r Q & \xrightarrow{\Lambda^r \Delta} & \Lambda^r Q \\
\downarrow{\text{can}} & & \downarrow{\text{can}} \\
\Lambda^{n-r} Q^* & \xrightarrow{\Lambda^{n-r} \Delta^*} & \Lambda^{n-r} Q^*
\end{array}
\]

commutes.

**Proof.** First, we prove this for $r = 1$. Let $q \in Q$. Let $\text{can}\Delta(q) \in \Lambda^{n-1} Q^*$ is given by $\text{can}\Delta(q)(q_2 \wedge \cdots \wedge q_n) = \Delta(q) \wedge q_2 \wedge \cdots \wedge q_n$. Then, $\lambda_q := \Lambda^{n-r} \Delta^* \text{can}\Delta(q)$ is given by the commutative diagram

\[
\begin{array}{ccc}
\Lambda^{n-1} Q & \xrightarrow{\Lambda^{n-1} \Delta} & \Lambda^{n-1} Q \\
\downarrow{\lambda_q} & & \downarrow{\text{can}\Delta(q)} \\
\Lambda^n Q & & \Lambda^n Q
\end{array}
\]

So,

$$\lambda_q(q_2 \wedge \cdots \wedge q_n) = \Delta(q) \wedge \Lambda^{n-1} \Delta(q_2 \wedge \cdots \wedge q_n) = \Lambda^n(\Delta)(q \wedge q_2 \wedge \cdots \wedge q_n).$$

Since $\det \Delta = 1$, it follows that, the upper right composition (42), sends $q \mapsto (q_2 \wedge \cdots \wedge q_n \mapsto q \wedge q_2 \wedge \cdots \wedge q_n)$, which is the map $\text{can} : Q \to \Lambda^{n-1} Q^*$.

For the general case, for $q_1, \ldots, q_r \in Q$, let $ev_{q_1 q_2 \ldots q_r} \in \Lambda^{n-r} Q^*$, denote the map $q_{r+1} \wedge \cdots \wedge q_n \mapsto q_1 \wedge \cdots \wedge q_{r+1} \wedge \cdots \wedge q_n$. So, $\text{can}\Lambda^r \Delta(q) = ev_{\Delta(q_1) \Delta(q_2) \ldots \Delta(q_r)}$. So, $\Lambda^{n-r} \text{can}\Lambda^r \Delta(q)$ is given by the commutative diagram

\[
\begin{array}{ccc}
\Lambda^{n-r} Q & \xrightarrow{\Lambda^{n-r} \Delta} & \Lambda^{n-r} Q \\
\downarrow{\text{ev}_{\Delta(q_1) \Delta(q_2) \ldots \Delta(q_r)}} & & \downarrow{\Lambda^n Q} \\
\Lambda^n Q & & \Lambda^n Q
\end{array}
\]

which sends

$$q_{r+1} \wedge \cdots \wedge q_n \mapsto \Delta(q_1) \wedge \cdots \wedge \Delta(q_r) \wedge \Delta(q_{r+1}) \wedge \cdots \wedge \Delta(q_n) = \Lambda^n(\Delta(q_1) \wedge \cdots \wedge q_r \wedge q_{r+1} \wedge \cdots \wedge q_n)$$

This is precisely the map $\text{can} : \Lambda^r Q \to \Lambda^{n-r} Q^*$. The proof is complete. ■
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