GLOBAL REGULARITY OF CRITICAL SCHÖDINGER MAPS: SUBTHRESHOLD DISPERSED ENERGY

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ABSTRACT. We consider the energy-critical Schrödinger map initial value problem with smooth initial data from $\mathbb{R}^2$ into the sphere $S^2$. Given sufficiently energy-dispersed data with subthreshold energy, we prove that the system admits a unique global smooth solution. This improves earlier analogous conditional results [18]. The key behind this improvement lies in exploiting estimates on the commutator of the Schrödinger map and harmonic map heat flows.

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1. INTRODUCTION

We consider the Schrödinger map initial value problem

$$\begin{cases}
\partial_t \phi &= \phi \times \Delta \phi \\
\phi(x,0) &= \phi_0(x)
\end{cases}$$

(1.1)

with $\phi_0 : \mathbb{R}^d \to S^2 \to \mathbb{R}^3$. This system formally enjoys conservation of energy

$$E(\phi(t)) := \frac{1}{2} \int_{\mathbb{R}^d} |\partial_x \phi(t)|^2 dx$$

(1.2)

and mass

$$M(\phi(t)) := \int_{\mathbb{R}^d} |\phi(t) - Q|^2 dx$$

where $Q \in S^2$ is fixed. When $d = 2$, both (1.1) and (1.2) are invariant with respect to the scaling

$$\phi(x, t) \to \phi(\lambda x, \lambda^2 t), \quad \lambda > 0$$

(1.3)

The author was supported by NSF grant DMS-1103877.
and so when \( d = 2 \) the system (1.1) is called energy-critical. This is the setting we consider in this article.

For the physical significance of (1.1), see [6, 16, 17, 13]. The system may also be interpreted as a geometric analogue of the free linear Schrödinger equation (see [18, Introduction]). For more general analogues of (1.1), e.g., for Kähler targets other than \( S^2 \), see [7, 14, 15]. See also [11, 12, 3] for connections with other spin systems.

The local theory of (1.1) is developed in [21, 6, 7, 14]. Global wellposedness given data with small critical Sobolev norm (in all dimensions \( d \geq 2 \)) is shown in [4]. A conditional extension of this result that requires assuming an \( L^4 \) spacetime bound on the solution and smallness of a critical Besov norm of the data appears in [18]. Recently it has been shown [1] that global existence and uniqueness as well as scattering hold given 1-equivariant data with energy less than \( 4\pi \). In the radial setting (which excludes harmonic maps), [8] establishes global wellposedness at any energy level. In this article we pursue an extension of our work [18], removing the \( L^4 \) boundedness condition. To state our main results, we first introduce some Sobolev spaces.

For \( \sigma \in [0, \infty) \), let \( H^\sigma = H^\sigma (\mathbb{R}^2) \) denote the usual Sobolev space of complex-valued functions on \( \mathbb{R}^2 \). For any \( Q \in S^2 \), set

\[
H^\sigma_Q := \{ f : \mathbb{R}^2 \to \mathbb{R}^3 \text{ such that } |f(x)| \equiv 1 \text{ a.e. and } f - Q \in H^\sigma \}
\]

This is a metric space with induced distance \( d^\sigma_Q(f, g) = \| f - g \|_{H^\sigma} \). For \( f \in H^\sigma_Q \) we set \( \| f \|_{H^\sigma_Q} = d^\sigma_Q(f, Q) \) for short. Define also

\[
H^\infty := \bigcap_{\sigma \in \mathbb{Z}_+} H^\sigma \quad \text{and} \quad H^\infty_Q := \bigcap_{\sigma \in \mathbb{Z}_+} H^\sigma_Q
\]

Throughout \( \mathbb{Z}_+ = \{0, 1, 2 \ldots \} \) denotes the nonnegative integers.

These definitions may be naturally extended to any spacetime slab \( \mathbb{R}^2 \times [-T, T], T \in (0, \infty) \). For any \( \sigma, \rho \in \mathbb{Z}_+ \), let \( H^{\sigma, \rho}(T) \) denote the Sobolev space of complex-valued functions on \( \mathbb{R}^2 \times [-T, T] \) with the norm

\[
\| f \|_{H^{\sigma, \rho}(T)} := \sup_{t \in (-T, T)} \sum_{\rho' = 0}^\rho \| \partial_t^{\rho'} f(\cdot, t) \|_{H^\sigma}
\]

and for \( Q \in S^2 \) endow

\[
H^{\sigma, \rho}_Q := \{ f : \mathbb{R}^2 \times [-T, T] \to \mathbb{R}^3 \text{ such that } |f(x, t)| \equiv 1 \text{ a.e. and } f - Q \in H^{\sigma, \rho}(T) \}
\]

with the metric induced by the \( H^{\sigma, \rho}(T) \) norm. Also, define the spaces

\[
H^{\infty, \infty}(T) = \bigcap_{\sigma, \rho \in \mathbb{Z}_+} H^{\sigma, \rho}(T) \quad \text{and} \quad H^{\infty, \infty}_Q(T) = \bigcap_{\sigma, \rho \in \mathbb{Z}_+} H^{\sigma, \rho}_Q(T)
\]

For \( f \in H^{\infty} \) and \( \sigma \geq 0 \) we define the homogeneous Sobolev norms as

\[
\| f \|_{\dot{H}^\sigma} = \| \hat{f}(\xi) \cdot |\xi|^\sigma \|_{L^2}
\]
Here $\hat{f}$ stands for the Fourier transform in the spatial variables only. We similarly define $\tilde{H}_Q^\sigma$ by first translating $f$ by $Q$.

We now state our main global result.

**Theorem 1.1** (Global regularity). Fix $Q \in S^2$. Then there exists $\varepsilon_0 > 0$ such that for all $\phi_0 \in H_Q$ with $E_0 := E(\phi_0) \leq E_{\text{crit}}$ and

$$\sup_{k \in \mathbb{Z}} \| P_k \partial_x \phi_0 \|_{L^2_x} \leq \varepsilon_0,$$

equation (1.1) admits a unique global solution $\phi \in C(\mathbb{R} \to H_Q^\infty)$.

The subthreshold assumption on the initial data is important only insofar as it is used to establish certain bounds on the gauge. It may be possible to establish such bounds without making use of the subthreshold assumption, though we do not pursue this here.

**Theorem 1.2** (Uniform bounds). Fix $Q \in S^2$. Let $\sigma_1 \geq 1$. Then there exists $\tilde{\varepsilon}_0(\sigma_1) \in (0, \varepsilon_0]$ such that for all $\phi_0 \in H_Q$ with $E_0 := E(\phi_0) < E_{\text{crit}}$ and

$$\sup_{k \in \mathbb{Z}} \| P_k \partial_x \phi_0 \|_{L^2_x} \leq \tilde{\varepsilon}_0,$$

the global solution $\phi$ constructed in Theorem 1.1 satisfies the uniform bounds

$$\sup_{t \in \mathbb{R}} \| \phi(t) - Q \|_{H^\sigma} \lesssim \| \phi_0 - Q \|_{H^\sigma}, \quad 1 \leq \sigma \leq \sigma_1$$

To prove Theorems 1.1 and 1.2, we first establish estimates at fixed dyadic scales. In doing so we make use of an important technical tool, frequency envelopes in particular, defined as follows.

**Definition 1.3** (Frequency envelopes). Let $\delta > 0$ be fixed. A positive sequence $\{a_k\}_{k \in \mathbb{Z}}$ is a frequency envelope if it belongs to $\ell^2$ and is slowly varying:

$$a_k \leq a_j 2^{[k-j]}, \quad j, k \in \mathbb{Z}$$ (1.4)

A frequency envelope $\{a_k\}_{k \in \mathbb{Z}}$ is $\varepsilon$-energy dispersed if it satisfies the additional condition

$$\sup_{k \in \mathbb{Z}} a_k \leq \varepsilon$$

Note in particular that frequency envelopes satisfy the following summation rules:

$$\sum_{k' \leq k} 2^{pk'} a_{k'} \lesssim (p - \delta)^{-1} 2^{pk} a_k \quad p > \delta$$ (1.5)

$$\sum_{k' \geq k} 2^{-pk'} a_{k'} \lesssim (p - \delta)^{-1} 2^{-pk} a_k \quad p > \delta$$ (1.6)
In practice we work with $p$ bounded away from $\delta$, e.g., $p > 2\delta$ suffices, and iterate these inequalities only $O(1)$ times. Therefore in applications we drop the factors $(p - \delta)^{-1}$ appearing in (1.3) and (1.6).

Given initial data $\phi_0 \in H^\infty_Q$, define for all $\sigma \geq 0$ and $k \in \mathbb{Z}$

$$c_k(\sigma) := \sup_{k' \in \mathbb{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'} \|P_{k'} \partial_x \phi_0\|_{L^2_x}$$

and set $c_k := c_k(0)$ for short. For $\sigma \in [0, \sigma_1]$ it then holds that

$$\|\partial_x \phi_0\|_{H^\sigma_x}^2 \sim \sum_{k \in \mathbb{Z}} c_k^2(\sigma) \quad \text{and} \quad \|P_k \partial_x \phi_0\|_{L^2_x} \leq c_k(\sigma) 2^{-\sigma k}$$

Similarly, for $\phi \in H^\infty_{Q,T}$, define for all $\sigma \geq 0$ and $k \in \mathbb{Z}$

$$\gamma_k(\sigma) := \sup_{k' \in \mathbb{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'} \|P_{k'} \phi\|_{L^\infty_t L^2_x}$$

and set $\gamma_k := \gamma_k(1)$.

For technical reasons we construct a solution $\phi$ on a time interval $(-2^{2K}, 2^{2K})$ for some given $K \in \mathbb{Z}_+$ and then proceed to prove bounds that are uniform in $K$. We assume $1 \ll K \in \mathbb{Z}_+$ is chosen and hereafter fixed. By the local theory, we may assume that we have a solution $\phi \in C([-T, T] \to H^\infty_Q)$ of (1.1) on the time interval $[-T, T]$ for some $T \in (0, 2K)$. In order to extend $\phi$ to a solution on all of $(-2^{2K}, 2^{2K})$ with uniform bounds (uniform in $T, K$) it suffices to prove uniform a priori estimates on

$$\sup_{t \in [-T, T]} \|\phi(t)\|_{H^\sigma_Q}$$

for, say, $\sigma$ in the interval $[1, \sigma_1]$, with $\sigma_1 \gg 1$ chosen sufficiently large.

The first step in our approach, carried out in [2], is to lift the Schrödinger map system (1.1) to the tangent bundle and view it with respect to the caloric gauge. The lift of (1.1) becomes a system of coupled magnetic nonlinear Schrödinger equations of the form

$$(i \partial_t + \Delta) \psi_m = B_m + V_m, \quad m = 1, 2$$

with initial data $\psi_1(0), \psi_2(0)$. Here $B_m$ and $V_m$ respectively denote the magnetic and electric potentials (see (2.8) and (2.9) for definitions). At the level of the tangent bundle the goal is to prove a priori bounds on $\|\psi_m\|_{L^\infty_t H^\sigma_Q}$, which we establish by proving stronger frequency-localized estimates. The proof of Theorem [1.1] is then completed by transferring bounds on the derivative fields $\psi_m$ back to bounds on the map $\phi$.

Energy spaces are not sufficient for controlling $P_k \psi_m$, and so we combine in addition to these a host of Strichartz, local smoothing, and maximal function type spaces into one space $G_k(T)$ (see [33] for precise definitions). As our goal will be to express control of the $G_k(T)$ norms of $P_k \psi_m$ in terms of the energy
of the frequency localizations of the initial data, we introduce the following
frequency envelopes. Let $\sigma_1 \in \mathbb{Z}_+$ be positive. For $\sigma \in [0, \sigma_1 - 1]$, set

$$b_k(\sigma) := \sup_{k' \in \mathbb{Z}} 2^{\sigma k' - 2^{-\delta} k - k'} \| P_{k'} \psi_x \|_{L^2_k(T)}$$

(1.11)

By (2.25) these envelopes are finite and in $L^2$. We abbreviate $b_k(0)$ by setting $b_k := b_k(0)$.

We now state the key result for solutions of the gauge field equation (1.10) (with the caloric gauge as the gauge choice).

**Theorem 1.4.** Assume $T \in (0, 2^{2K}]$ and $Q \in S^2$. Choose $\sigma_1 \in \mathbb{Z}_+$ positive. Let $\varepsilon_0 > 0$ and let $\phi \in H^\infty \times (T)$ be a solution of the Schrödinger map system (1.7) whose initial data $\phi_0$ has energy $E_0 := E(\phi_0) < E_{\text{crit}}$ and satisfies the energy dispersion condition

$$\sup_{k \in \mathbb{Z}} c_k \leq \varepsilon_0$$

(1.12)

Suppose that the bootstrap hypothesis

$$b_k \leq \varepsilon_0^{-1/10} c_k$$

(1.13)

is satisfied. Then, for $\varepsilon_0$ sufficiently small,

$$b_k(\sigma) \lesssim c_k(\sigma)$$

(1.14)

holds for all $\sigma \in [0, \sigma_1 - 1]$ and $k \in \mathbb{Z}$.

We use a continuity argument to prove Theorem 1.4. For $T' \in (0, T)$, let

$$\Psi(T') = \sup_{k \in \mathbb{Z}} c_k^{-1} \| P_k \psi_m(s = 0) \|_{L^2_k(T')}$$

Then $\Psi : (0, T] \to [0, \infty)$ is well-defined, increasing, continuous, and satisfies

$$\lim_{T' \to 0} \Psi(T') \lesssim 1$$

The critical implication to establish is

$$\Psi(T') \leq \varepsilon_1^{-1/10} \implies \Psi(T') \lesssim 1$$

which in particular follows from

$$b_k \lesssim c_k$$

(1.15)

We must also similarly establish

$$b_k(\sigma) \lesssim c_k(\sigma)$$

(1.16)

for $\sigma \in (0, \sigma_1 - 1]$ in order to bring under control the higher-order Sobolev norms. The bounds (1.16), however, will be seen to follow as an easy consequence of the proof of (1.15), and therefore throughout this article the emphasis will be upon proving (1.15).
Corollary 1.5. Given the conditions of Theorem 1.4
\[ \|P_k |\partial_x| \partial_m \phi| \|_{L_t^\infty L_x^2 ((-T,T) \times \mathbb{R}^2)} \lesssim c_k (\sigma) \] (1.17)
holds for all \( \sigma \in [0, \sigma_1 - 1] \).

In view of the local theory and (1.8), this corollary implies Theorems 1.1 and 1.2.

One strategy for proving (1.15), i.e., \( b_k \lesssim c_k \), is to show that the nonlinearity is perturbative. Along these lines, we project (2.7) to frequencies \( \sim 2^k \) and apply the linear estimate of Proposition 3.6:
\[ \|P_k \psi_m\|_{G_k(T)} \lesssim \|P_k \psi_m(0)\|_{L_x^2} + \|P_k N_m\|_{N_k(T)} \] (1.18)
If the nonlinearity is perturbative in the sense that \( \|P_k N_m\|_{N_k(T)} \lesssim \varepsilon b_k \), then (1.18) implies \( b_k \lesssim c_k + \varepsilon b_k \), which establishes (1.15). This is the general approach of [4] in the case of small critical norm. In the dispersed setting, the entirety of \( P_k N_m \) does not obey such a bound; it fails to be perturbative in this sense. Nevertheless, several pieces of \( N_m \) are perturbative, and showing this plays a crucial role in enabling us to gain control over the remaining nonperturbative part of the nonlinearity, as in [18]. The key there was a certain bilinear Strichartz estimate adapted to a magnetic NLS. In this article we are able to establish stronger bounds on the electric potential \( V_m \), which then puts us into a position to apply the machinery of [18] but without the conditional \( L^4 \) condition required there. Hence the most important technical contribution of this article lies in \( \S 5 \). This improvement also leads to some slight technical simplifications over [18] in establishing (1.16) for \( \sigma > 0 \). We remark that it would be interesting to extend the results of [18] in a direction other than the one we take in this article; in particular, one expects global wellposedness under the assumption of an \( L^4 \)-type bound even if the data is not energy-dispersed.

2. Gauge selection

We begin with some constructions valid for any smooth function \( \phi : \mathbb{R}^2 \times (-T,T) \to \mathbb{S}^2 \). Space and time derivatives of \( \phi \) are denoted by \( \partial_\alpha \phi(x,t) \), where \( \alpha = 1, 2, 3 \) ranges over the spatial variables \( x_1, x_2 \) and time \( t \) with \( \partial_3 = \partial_t \).

Select a smooth orthonormal frame \( (v(x,t), w(x,t)) \) for \( T_\phi(x,t) \mathbb{S}^2 \), i.e., smooth functions \( v, w : \mathbb{R}^2 \times (-T,T) \to T_\phi(x,t) \mathbb{S}^2 \) such that for each point \( (x,t) \) in the domain the vectors \( v(x,t), w(x,t) \) form an orthonormal basis of \( T_\phi(x,t) \mathbb{S}^2 \). As a matter of convention we assume that \( v \) and \( w \) are chosen so that \( v \times w = \phi \).

We define the so-called derivative fields \( \psi_\alpha \) by writing \( \partial_\alpha \phi \) with respect to \( (v, w) \) and identifying \( \mathbb{R}^2 \) with \( \mathbb{C} \):
\[ \psi_\alpha := v \cdot \partial_\alpha \phi + iw \cdot \partial_\alpha \phi \] (2.1)
In other words, $\psi_\alpha$ is determined by
\[ \partial_\alpha \phi = v \text{Re}(\psi_\alpha) + w \text{Im}(\psi_\alpha) \tag{2.2} \]
Through this identification, the Levi-Civita connection on $S^2$ pulls back to a covariant derivative on $\mathbb{C}$, denoted by
\[ D_\alpha := \partial_\alpha + i A_\alpha \]
where the real-valued connection coefficients $A_\alpha$ are determined via
\[ A_\alpha = w \cdot \partial_\alpha v \tag{2.3} \]
Due to the fact that the Levi-Civita connection on $S^2$ is torsion-free, the derivative fields satisfy the compatibility relations
\[ D_\beta \psi_\alpha = D_\alpha \psi_\beta \tag{2.4} \]
A straightforward calculation (which uses the fact that the sphere has constant curvature) shows
\[ \partial_\beta A_\alpha - \partial_\alpha A_\beta = \text{Im}(\psi_\beta \overline{\psi_\alpha}) =: q_{\beta\alpha} \]
so that the curvature of the connection is given by
\[ [D_\beta, D_\alpha] := D_\beta D_\alpha - D_\alpha D_\beta = i q_{\beta\alpha} \tag{2.5} \]
Suppose $\phi$ is a smooth solution of the Schrödinger map system \((1.1)\). This system lifts to
\[ \psi_t = i D_\ell \psi_\ell \tag{2.6} \]
because
\[ \phi \times \Delta \phi = J(\phi)(\phi^* \nabla_j \partial_j \phi \]
where $J(\phi)$ denotes the complex structure $\phi \times$ and $(\phi^* \nabla_j$ the pullback of the Levi-Civita connection $\nabla$ on the sphere. Here we are using the convention that repeated Roman indices are summed over the spatial variables. In fact, throughout we consistently use Roman indices only for spatial variables and Greek indices for variables that are either spatial or temporal.

Using \((2.4)\) and \((2.5)\) in \((2.6)\) yields
\[ D_t \psi_m = i D_\ell D_\ell \psi_m + q_{\ell m} \psi_\ell \]
which is equivalent to the nonlinear Schrödinger equation
\[ (i \partial_t + \Delta) \psi_m = \mathcal{N}_m \tag{2.7} \]
with nonlinearity $\mathcal{N}_m$ defined by the formula
\[ \mathcal{N}_m := -i A_\ell \partial_\ell \psi_m - i \partial_\ell (A_\ell \psi_m) + (A_\ell + A^2_\ell) \psi_m - i \psi_\ell \text{Im}(\overline{\psi_\ell \psi_m}) \]
We split this nonlinearity as a sum $\mathcal{N}_m = B_m + V_m$ with $B_m$ and $V_m$ defined by
\[ B_m := -i A_\ell \partial_\ell \psi_m - i A_\ell \overline{\partial_\ell \psi_m} \tag{2.8} \]
and
\[ V_m := (A_\ell + A^2_\ell) \psi_m - i \psi_\ell \text{Im}(\overline{\psi_\ell \psi_m}) \tag{2.9} \]
We call $B_m$ the magnetic part of the potential and $V_m$ the electric part. The differentiated Schrödinger map system then takes the form

$$
\begin{align*}
(i\partial_t + \Delta)\psi_m &= B_m + V_m \\
D_\alpha\psi_\beta &= D_\beta\psi_\alpha \\
\partial_\alpha A_\beta - \partial_\beta A_\alpha &= \text{Im}(\psi_\alpha\psi_\beta)
\end{align*}
$$

A solution $\psi_m$ of (2.10) cannot be determined uniquely without first choosing an orthonormal frame $(v, w)$. Changing a given choice of orthonormal frame induces a gauge transformation that modifies the derivative fields and connection coefficients according to

$$
\psi_m \to e^{-i\theta}\psi_m, \quad A_m \to A_m + \partial_m \theta
$$

The system (2.10) is invariant with respect to such gauge transformations.

Gauges were first used to study (1.1) in [6]. The caloric gauge, which is the gauge we select, was introduced by Tao in [23] in the setting of wave maps into hyperbolic space. In the series of unpublished papers [24, 25, 26, 27, 28], Tao used this gauge in establishing global regularity of wave maps into hyperbolic space. In his unpublished note [22], Tao also suggested the caloric gauge as an alternative to the Coulomb gauge in studying Schrödinger maps. The caloric gauge was first used in the Schrödinger maps problem by Bejenaru, Ionescu, Kenig, and Tataru in [4] to establish global well-posedness in the setting of initial data with sufficiently small critical norm. We recommend [23, 25, 22, 4] for background on the caloric gauge and for helpful heuristics.

**Theorem 2.1** (The caloric gauge). Let $T \in (0, \infty)$, $Q \in S^2$, and let $\phi(x, t) \in H^\infty_H(T)$ be such that $\sup_{t \in (-T, T)} E(\phi(t)) < E_{\text{crit}}$. Then there exists a unique smooth extension $\phi(s, x, t) \in C(0, \infty) \to H_{Q}^\infty(T)$ solving the covariant heat equation

$$
\partial_s \phi = \Delta \phi + \phi \cdot |\partial_\rho \phi|^2
$$

and with $\phi(0, x, t) = \phi(x, t)$. Moreover, for any given choice of a (constant) orthonormal basis $(v_{\infty}, w_{\infty})$ of $T_0S^2$, there exist smooth functions $v, w : [0, \infty) \times \mathbb{R}^2 \times (-T, T) \to S^2$ such that at each point $(s, x, t)$, the set $\{v, w, \phi\}$ naturally forms an orthonormal basis for $\mathbb{R}^3$, the gauge condition

$$
w \cdot \partial_s v \equiv 0
$$

is satisfied, and

$$
|\partial_\rho f(s)| \lesssim_\rho (s)^{-(\rho+1)/2}
$$

for each $f \in \{\phi - Q, v - v_\infty, w - w_\infty\}$, multiindex $\rho$, and $s \geq 0$.

We now record a couple of the technical bounds from [19], which will be useful later on in controlling terms along the heat flow.
Theorem 2.2. The following bounds hold:
\[
\int_0^\infty \sup_{x \in \mathbb{R}^2} |\psi_x(s, x)|^2 ds \lesssim E_0 1 
\]
(2.14)
\[
\|\mathbf{A}_x(s)\|_{L_x^2(\mathbb{R}^2)} \lesssim E_0 1 
\]
(2.15)

Proof. For (2.14), see [19, §4]. For (2.15), see [19, §§7, 7.1]. □

From now on we assume that our Schrödinger map \( \varphi = \varphi(x,t) \) has been extended via harmonic map heat flow to a map \( \tilde{\varphi}(s, x, t) \), \( s \geq 0 \). From here on out we also call the extension \( \varphi \).

Let \( \partial_0 = \partial_s \) and define for all \((s, x, t) \in [0, \infty) \times \mathbb{R}^2 \times (-T, T)\) the various gauge components
\[
\psi_\alpha := v \cdot \partial_\alpha \varphi + iw \cdot \partial_\alpha \varphi \quad A_\alpha := w \cdot \partial_\alpha v \quad D_\alpha := \partial_\alpha + A_\alpha \quad q_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha
\]
The parallel transport condition \( w \cdot \partial_s v \equiv 0 \) is equivalently expressed in terms of the connection coefficients as
\[
A_s \equiv 0 
\]
(2.16)

Expressed in terms of the gauge, the heat flow (2.11) lifts to
\[
\psi_s = D_\ell \psi_\ell 
\]
(2.17)

Using (2.4) and (2.5), we may take the \( D_m \) covariant derivative of (2.17) and rewrite it as
\[
\partial_s \psi_m = D_\ell D_\ell \psi_m + i \text{Im}(\psi_m \overline{\psi_\ell}) \psi_\ell
\]
or equivalently
\[
(\partial_s - \Delta) \psi_m = i A_\ell \partial_\ell \psi_\alpha + i \partial_\ell (A_\ell \psi_m) - A_\ell^2 \psi_m + i \psi_\ell \text{Im}(\overline{\psi_\ell} \psi_m) 
\]
(2.18)

More generally, taking the \( D_\alpha \) covariant derivative, we obtain
\[
(\partial_s - \Delta) \psi_\alpha = U_\alpha 
\]
(2.19)

with
\[
U_\alpha := i A_\ell \partial_\ell \psi_\alpha + i \partial_\ell (A_\ell \psi_\alpha) - A_\ell^2 \psi_\alpha + i \psi_\ell \text{Im}(\overline{\psi_\ell} \psi_\alpha) 
\]
(2.20)

From (2.5) and (2.16) it follows that
\[
\partial_s A_\alpha = \text{Im}(\overline{\psi_\alpha} \psi_\alpha)
\]
and integrating back from \( s = \infty \) (justified using (2.13)) yields
\[
A_\alpha(s) = - \int_s^\infty \text{Im}(\overline{\psi_\alpha} \psi_\alpha)(s') ds'
\]
(2.21)

At \( s = 0 \), \( \varphi \) satisfies both (1.1) and (2.11), or equivalently, \( \psi_\ell(s = 0) = i \psi_s(s = 0) \). While for \( s > 0 \) it continues to be the case that \( \psi_\ell = D_\ell \psi_\ell \) by construction, we no longer necessarily have \( \psi_\ell(s) = i D_\ell(s) \psi_\ell(s) \), i.e.,
\( \phi(s, x, t) \) is not necessarily a Schrödinger map at fixed \( s > 0 \). We recall from [LS] §2 the following description of their commutator \( \Psi = \psi_t - i\psi_s \).

**Lemma 2.3** (Flows do not commute). Set \( \Psi := \psi_t - i\psi_s \). Then

\[
\partial_s \Psi = D_t D_t \Psi + i\text{Im}(\psi_t \bar{\psi}_t)\psi_t - \text{Im}(\psi_s \bar{\psi}_t)\psi_t = D_t D_t \Psi + i\text{Im}(\Psi \bar{\psi}_t)\psi_t + i\text{Im}(i\psi_s \bar{\psi}_t)\psi_t - \text{Im}(\psi_s \bar{\psi}_t)\psi_t
\]

We now record some frequency-localized energy estimates which find application in controlling the paralinearized nonlinearity.

**Theorem 2.4** (Frequency-localized energy estimates for heat flow). Let \( f \in \{\phi, v, w\} \). Then for \( \sigma \in [1, \sigma_1] \) the bound

\[
\|P_k f(s)\|_{L^\infty_t L^2_x} \lesssim 2^{-\sigma k} \gamma_k(\sigma)(1 + s2^{2k})^{-20}\tag{2.22}
\]

holds, and, for any \( \sigma, \rho \in \mathbb{Z}_+ \), it holds that

\[
\sup_{k \in \mathbb{Z}} \sup_{s \in [0, \infty)} (1 + s)^{\sigma/2} 2^{\sigma k} \|P_k \partial_t^\rho f(s)\|_{L^\infty_t L^2_x} < \infty \tag{2.23}
\]

**Corollary 2.5** (Frequency-localized energy estimates for the caloric gauge). For \( \sigma \in [0, \sigma_1 - 1] \) it holds that

\[
\|P_k \psi_x(x)\|_{L^\infty_t L^2_x} + \|P_k A_m(s)\|_{L^\infty_t L^2_x} \lesssim 2^k 2^{-\sigma k} \gamma_k(\sigma)(1 + s2^{2k})^{-20}\tag{2.24}
\]

Moreover, for any \( \sigma \in \mathbb{Z}_+ \),

\[
\sup_{k \in \mathbb{Z}} \sup_{s \in [0, \infty)} (1 + s)^{\sigma/2} 2^{\sigma k} \left( \|P_k (\partial_t^\rho \psi_x(s))\|_{L^\infty_t L^2_x} + \|P_k (\partial_t^\rho A_m(s))\|_{L^\infty_t L^2_x} \right) < \infty \tag{2.25}
\]

and

\[
\sup_{k \in \mathbb{Z}} \sup_{s \in [0, \infty)} (1 + s)^{\sigma/2} 2^{\sigma k} \left( \|P_k (\partial_t^\rho \psi_t(s))\|_{L^\infty_t L^2_x} + \|P_k (\partial_t^\rho A_t(s))\|_{L^\infty_t L^2_x} \right) < \infty \tag{2.26}
\]

Full proofs of Theorem 2.4 and its corollary may be found in [LS]. These are extensions to the energy-dispersed setting of analogous small-energy bounds from [4].

### 3. Function spaces

#### 3.1. Definitions.

**Definition 3.1** (Littlewood-Paley multipliers). Let \( \eta_0 : \mathbb{R} \to [0, 1] \) be a smooth even function vanishing outside the interval \([-8/5, 8/5]\) and equal to 1 on \([-5/4, 5/4]\). For \( j \in \mathbb{Z} \), set

\[
\chi_j(\cdot) = \eta_0(\cdot/2^j) - \eta_0(\cdot/2^{j-1}), \quad \chi_{\leq j}(\cdot) = \eta_0(\cdot/2^j)
\]
Let $P_k$ denote the operator on $L^\infty(\mathbb{R}^2)$ defined by the Fourier multiplier $\xi \mapsto \chi_k(|\xi|)$. For any interval $I \subset \mathbb{R}$, let $\chi_I$ be the Fourier multiplier defined by $\chi_I = \sum_{j \in I \cap \mathbb{Z}} \chi_j$ and let $P_I$ denote its corresponding operator on $L^\infty(\mathbb{R}^2)$. We denote $P(-\infty,k]$ by $P_{\leq k}$ for short. For $\theta \in S^1$ and $k \in \mathbb{Z}$, we define the operators $P_{k,\theta}$ by the Fourier multipliers $\xi \mapsto \chi_k(\xi \cdot \theta)$.

Some frequency interactions in the nonlinearity of (2.7) can be controlled using the following Strichartz estimate:

Lemma 3.2 (Strichartz estimate). Let $f \in L^2_x(\mathbb{R}^2)$ and $k \in \mathbb{Z}$. Then the Strichartz estimate
\[
\|e^{it\Delta} f\|_{L^4_t L^2_x} \lesssim \|f\|_{L^2_x}
\]
holds, as does the maximal function bound
\[
\|e^{it\Delta} P_k f\|_{L^4_t L^\infty_x} \lesssim 2^{k/2} \|f\|_{L^2_x}
\]

The first bound is the original Strichartz estimate [20] and the second follows from scaling.

For a unit length $\theta \in S^1$, we denote by $H_\theta$ its orthogonal complement in $\mathbb{R}^2$ with the induced Lebesgue measure. Define the lateral spaces $L^{p,q}_\theta$ as those consisting of all measurable $f$ for which the norm
\[
\|h\|_{L^{p,q}_\theta} := \left( \int_{\mathbb{R}} \left( \int_{H_\theta \times \mathbb{R}} |h(x_1 \theta + x_2, t)|^q dx_2 dt \right)^{p/q} dx_1 \right)^{1/p}
\]
is finite. We make the usual modifications when $p = \infty$ or $q = \infty$. For proofs of the following lateral Strichartz estimates, see [4, §3, §7].

Lemma 3.3 (Lateral Strichartz estimates). Let $f \in L^2_x(\mathbb{R}^2)$, $k \in \mathbb{Z}$, and $\theta \in S^1$. Let $2 < p \leq \infty$, $2 \leq q \leq \infty$ and $1/p + 1/q = 1/2$. Then
\[
\|e^{it\Delta} P_{k,\theta} f\|_{L^{p,q}_\theta} \lesssim 2^{k(2/p-1/2)} \|f\|_{L^2_x}, \quad p \geq q
\]
\[
\|e^{it\Delta} P_{k,\theta} f\|_{L^{p,q}_\theta} \lesssim 2^{k(2/p-1/2)} \|f\|_{L^2_x}, \quad p \leq q
\]

In the Schrödinger map setting, local smoothing spaces were first used in [9] and subsequently in [10, 2, 5, 4].

Lemma 3.4 (Local smoothing [9, 10]). Let $f \in L^2_x(\mathbb{R}^2)$, $k \in \mathbb{Z}$, and $\theta \in S^1$. Then
\[
\|e^{it\Delta} P_{k,\theta} f\|_{L^{\infty,2}_\theta} \lesssim 2^{-k/2} \|f\|_{L^2_x}
\]
For $f \in L^2_x(\mathbb{R}^d)$, the maximal function space bound
\[
\|e^{it\Delta} P_{k} f\|_{L^{\infty,2}_\theta} \lesssim 2^{k(d-1)/2} \|f\|_{L^2_x}
\]
holds in dimension $d \geq 3$. 
In $d = 2$, the maximal function bound fails due to a logarithmic divergence. In order to overcome this, we exploit Galilean invariance as in [4] (the idea goes back to [29] in the setting of wave maps).

For $p, q \in [1, \infty], \theta \in S^1, \lambda \in \mathbb{R}$, define $L^{p,q}_{\theta,\lambda}$ using the norm

$$
\|f\|_{L^{p,q}_{\theta,\lambda}} := \|T_{\lambda\theta}(f)\|_{L^{p,q}_{\theta,\lambda}} = \left( \int_{\mathbb{R}} \left( \int_{H_0 \times \mathbb{R}} |f((x_1 + t\lambda)\theta + x_2, t)|^q dx_2 dt \right)^{p/q} dx_1 \right)^{1/p}
$$

where $T_w$ denotes the Galilean transformation

$$
T_w(f)(x, t) := e^{-ix \cdot w/2} e^{-it|w|^2/4} f(x + tw, t)
$$

With $W \subset \mathbb{R}$ finite we define the spaces $L^{p,q}_{\theta,W}$ by

$$
L^{p,q}_{\theta,W} := \sum_{\lambda \in W} L^{p,q}_{\theta,\lambda}, \quad \|f\|_{L^{p,q}_{\theta,W}} := \inf_{f = \sum_{\lambda \in W} f_{\lambda}} \sum_{\lambda \in W} \|f_{\lambda}\|_{L^{p,q}_{\theta,\lambda}}
$$

For $k \in \mathbb{Z}, K \in \mathbb{Z}_+$, set

$$
W_k := \{\lambda \in [-2^k, 2^k]: \lambda \in \mathbb{Z} \}
$$

In our application we shall work on a finite time interval $[-2^K, 2^K]$ in order to ensure that the $W_k$ are finite. This still suffices for proving global results so long as our effective bounds are proved with constants independent of $T, K$. As discussed in [4 §3], restricting $T$ to a finite time interval avoids introducing additional technicalities.

Lemma 3.5 (Local smoothing/maximal function estimates). Let $f \in L^2_x(\mathbb{R}^2)$, $k \in \mathbb{Z}$, and $\theta \in S^1$. Then

$$
\|e^{it\Delta} P_{k,\theta} f\|_{L^{\infty}_{\theta,2}} \lesssim 2^{-k/2} \|f\|_{L^2}, \quad |\lambda| \leq 2^{k-40}
$$

and moreover, if $T \in (0, 2^{2K}]$, then

$$
\|1_{[-T,T]}(t)e^{it\Delta} P_k f\|_{L^{2,\infty}_{\theta,W_{k+40}}} \lesssim 2^{k/2} \|f\|_{L^2}
$$

Proof. The first bound follows from Lemma 3.4 via a Galilean boost. The second is more involved and proven in [4 §7].

We now introduce the main function spaces, which follow the modifications in [18] of spaces introduced in [4].

Let $T > 0$. For $k \in \mathbb{Z}$, let $I_k = \{\xi \in \mathbb{R}^2: |\xi| \in [2^{k-1}, 2^{k+1}]\}$. Let

$$
L^2_k(T) := \{f \in L^2(\mathbb{R}^2 \times [-T, T]) : \text{supp } \hat{f}(\xi, t) \subset I_k \times [-T, T]\}$$
For \( f \in L^2(\mathbb{R}^2 \times [-T, T]) \), let
\[
\|f\|_{F^k_0(T)} := \|f\|_{L^\infty_T L^2_x} + \|f\|_{L^3 T L^{\infty}_x} + 2^{-k/2} \|f\|_{L^4_T L^\infty_x} + 2^{-k/6} \sup_{\theta \in S^1} \|f\|_{L^3_\theta L^6_x} \sup_{\theta \in S^1} \|f\|_{L^{\infty,2}_\theta W^{k+40}_x}.
\]
Define \( F_k(T) \), \( G_k(T) \), \( N_k(T) \) as the normed spaces of functions in \( L^2_k(T) \) for which the corresponding norms are finite:
\[
\|f\|_{F_k(T)} := \inf_{J, m_1, \ldots, m_J \in \mathbb{Z}^+} \inf_{f = f_{m_1} + \cdots + f_{m_j}} \sum_{j=1}^J 2^{m_j} \|f_{m_j}\|_{F^0_k(T)}
\]
\[
\|f\|_{G_k(T)} := \|f\|_{F^0_k(T)} + 2^{k/6} \sup_{|j-k| \leq 20} \sup_{\theta \in S^1} \|P_{j, \theta} f\|_{L^{6,3}_\theta}
\]
\[
\|f\|_{N_k(T)} := \inf_{f = f_1 + f_2 + f_3} \|f_1\|_{L^4_{\hat{\theta}_1}} + 2^{k/6} \|f_2\|_{L^{3,2/5}_\hat{\theta}_1} + 2^{k/6} \|f_3\|_{L^6_{\hat{\theta}_2}} + 2^{-k/6} \|f_4\|_{L^{6/5,3/2}_\theta} + 2^{-k/6} \|f_5\|_{L^{6/5,3/2}_\theta} + 2^{-k/6} \sup_{\theta \in S^1} \|f_6\|_{L^{1,2}_\theta W^{k+40}_x},
\]
where \((\hat{\theta}_1, \hat{\theta}_2)\) denotes the canonical basis in \( \mathbb{R}^2 \).

These spaces are related via the following linear estimate, which is proved in [4].

**Proposition 3.6 (Main linear estimate).** Assume \( K \in \mathbb{Z}^+ \), \( T \in (0, 2^K] \) and \( k \in \mathbb{Z} \). Then for each \( u_0 \in L^2 \) that is frequency-localized to \( I_k \) and for any \( h \in N_k(T) \), the solution \( u \) of
\[
(i\partial_t + \Delta_x)u = h, \quad u(0) = u_0
\]
satisfies
\[
\|u\|_{G_k(T)} \lesssim \|u(0)\|_{L^2_x} + \|h\|_{N_k(T)}
\]

The spaces \( G_k(T) \) are used to hold projections \( P_k \psi_m \) of the derivative fields \( \psi_m \) satisfying (2.7). The main components of \( G_k(T) \) are the local smoothing/maximal function spaces \( L^\infty_{0, \lambda}, L^{2,\infty}_{0, W^{k+40}_x} \), and the angular Strichartz spaces. The local smoothing and maximal function space components play an essential role in recovering the derivative loss that arises from the magnetic nonlinearity.

The spaces \( N_k(T) \) hold frequency projections of the nonlinearities in (2.7). Here the main spaces are the inhomogeneous local smoothing spaces \( L^{1,2}_{0, \phi W^{k+40}_x} \) and the Strichartz spaces, both chosen to match those of \( G_k(T) \).
The spaces $G_k(T)$ clearly embed in $F_k(T)$. Two key properties enjoyed only by the larger spaces $F_k(T)$ are

$$\|f\|_{F_k(T)} \approx \|f\|_{F_{k+1}(T)}$$

for $k \in \mathbb{Z}$ and $f \in F_k(T) \cap F_{k+1}(T)$, and

$$\|P_k(uv)\|_{F_k(T)} \lesssim \|u\|_{F_{k'}(T)} \|v\|_{L^\infty_{t,x}}$$

for $k, k' \in \mathbb{Z}$, $|k - k'| \leq 20$, $u \in F_{k'}(T)$, $v \in L^\infty(\mathbb{R}^2 \times [-T, T])$. Both of these properties follow readily from the definitions.

In order to bound the nonlinearity of \[2.7\] in $N_k(T)$, it is important to gain regularity from the parabolic heat-time smoothing effect. The desired frequency-localized bounds do not (at least not readily) propagate in heat-time in the spaces $G_k(T)$, whereas these bounds do propagate with decay in the larger spaces $F_k(T)$. Note that since the $F_k(T)$ norm is translation invariant, it holds that

$$\|e^{s\Delta}h\|_{F_k(T)} \lesssim (1 + s2^{2k})^{-20}\|h\|_{F_k(T)}, \quad s \geq 0$$

for $h \in F_k(T)$. In certain bilinear estimates we do not need the full strength of the spaces $F_k(T)$ and instead can use the bound

$$\|f\|_{F_k(T)} \lesssim \|f\|_{L^2_2L^\infty_{t,x}} + \|f\|_{L^4_{t,x}}$$

(3.1)

which follows from

$$\|f\|_{L^2_{\theta,\infty}W_{k+m,j}} \leq \|f\|_{L^2_{\theta,\infty}} \lesssim 2^{k/2}\|f\|_{L^2_2L^\infty_{t,x}}$$

3.2. Bilinear estimates. For proofs of Lemmas \[3.7\], \[3.8\], and \[3.9\] see [4, §3].

**Lemma 3.7** (Bilinear estimates on $N_k(T)$). For $k, k_1, k_3 \in \mathbb{Z}$, $h \in L^2_{t,x}$, $f \in F_{k_1}(T)$, and $g \in G_{k_3}(T)$, we have the following inequalities under the given restrictions on $k_1, k_3$.

$$|k_1 - k| \leq 80: \quad \|P_k(hf)\|_{N_k(T)} \lesssim \|h\|_{L^2_{t,x}} \|f\|_{F_{k_1}(T)}$$

(3.2)

$$k_1 \leq k - 80: \quad \|P_k(hf)\|_{N_k(T)} \lesssim 2^{-|k-k_1|/6}\|h\|_{L^2_{t,x}} \|f\|_{F_{k_1}(T)}$$

(3.3)

$$k \leq k_3 - 80: \quad \|P_k(hg)\|_{N_k(T)} \lesssim 2^{-|k-k_3|/6}\|h\|_{L^2_{t,x}} \|g\|_{G_{k_3}(T)}$$

(3.4)

**Lemma 3.8** (Bilinear estimates on $L^2_{t,x}$). For $k_1, k_2, k_3 \in \mathbb{Z}$, $f_1 \in F_{k_1}(T)$, $f_2 \in F_{k_2}(T)$, and $g \in G_{k_3}(T)$, we have

$$\|f_1 \cdot f_2\|_{L^2_{t,x}} \lesssim \|f_1\|_{F_{k_3}(T)} \|f_2\|_{F_{k_2}(T)}$$

(3.5)

$$k_1 \leq k_3: \quad \|f \cdot g\|_{L^2_{t,x}} \lesssim 2^{-|k_1-k_3|/6}\|f\|_{F_{k_1}(T)} \|g\|_{G_{k_3}(T)}$$

(3.6)
We also have the following stronger estimates, which rely upon the local smoothing and maximal function spaces.

**Lemma 3.9** (Bilinear estimates using local smoothing/maximal function bounds). For $k, k_1, k_2 \in \mathbb{Z}$, $h \in L^2_{t,x}$, $f \in F_{k_1}(T)$, $g \in G_{k_2}(T)$, we have the following inequalities under the given restrictions on $k_1, k_2$.

\[
\begin{align*}
  k_1 &\leq k - 80 : \quad \| P_k(hf) \|_{N_k(T)} \lesssim 2^{-|k-k_1|/2} \| h \|_{L^2_{t,x}} \| f \|_{F_{k_1}(T)} \\
  k_1 &\leq k_2 : \quad \| f \cdot g \|_{L^2_{t,x}} \lesssim 2^{-|k_1-k_2|/2} \| f \|_{F_{k_1}(T)} \| g \|_{G_{k_2}(T)}
\end{align*}
\]  

(3.7)  

(3.8)

### 3.3. Trilinear Estimates and Summation.

We combine the bilinear estimates to establish some trilinear estimates. As we do not control local smoothing norms along the heat flow, we will oftentimes be able to put only one term in a $G_k$ space. Nonetheless, such estimates still exhibit good off-diagonal decay.

Define the sets $Z_1(k), Z_2(k), Z_3(k) \subset \mathbb{Z}^3$ as follows:

\[
Z_1(k) := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1, k_2 \leq k - 40 \text{ and } |k_3 - k| \leq 4\}
\]

(3.9)

\[
Z_2(k) := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_3 \leq k_1 - 40 \text{ and } |k_2 - k_1| \leq 45\}
\]

\[
Z_3(k) := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_3 \leq k \text{ and } |k - \max\{k_1, k_2\}| \leq 40 \text{ or } k_3 > k \text{ and } |k_3 - \max\{k_1, k_2\}| \leq 40\}
\]

In our main trilinear estimate, we avoid using local smoothing / maximal function spaces. The following is proven in [18] Lemma 3.10.

**Lemma 3.10** (Main trilinear estimate). Let $C_{k,k_1,k_2,k_3}$ denote the best constant $C$ in the estimate

\[
\| P_k (P_{k_1} f_1 P_{k_2} f_2 P_{k_3} g) \|_{N_k(T)} \lesssim C \| P_{k_1} f_1 \|_{F_{k_1}(T)} \| P_{k_2} f_2 \|_{F_{k_2}(T)} \| P_{k_3} g \|_{G_{k_3}(T)}
\]  

(3.10)

Then $C_{k,k_1,k_2,k_3}$ satisfies the bounds

\[
C_{k,k_1,k_2,k_3} \lesssim \begin{cases} 
  2^{-|k_1+k_2+k_3|/6} & (k_1, k_2, k_3) \in Z_1(k) \\
  2^{-k_3/6} & (k_1, k_2, k_3) \in Z_2(k) \\
  2^{-|\Delta k|/6} & (k_1, k_2, k_3) \in Z_3(k) \\
  0 & (k_1, k_2, k_3) \in \mathbb{Z}^3 \setminus \{Z_1(k) \cup Z_2(k) \cup Z_3(k)\}
\end{cases}
\]

where $\Delta k = \max\{k_1, k_1, k_2, k_3\} - \min\{k_1, k_1, k_2, k_3\} \geq 0$.

The following two corollaries are [18] Corollary 3.11 and [18] Corollary 3.12, respectively.

**Corollary 3.11.** Let $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ be $\delta$-frequency envelopes. Let $C_{k,k_1,k_2,k_3}$ be as in Lemma 3.10. Then

\[
\sum_{(k_1,k_2,k_3) \in \mathbb{Z}^3 \setminus Z_2(k)} C_{k,k_1,k_2,k_3} a_{k_1} b_{k_2} c_{k_3} \lesssim a_k b_k c_k
\]
Corollary 3.12. Let \( \{a_k\}, \{b_k\} \) be \( \delta \)-frequency envelopes. Let \( C_{k,k_1,k_2,k_3} \) be as in Lemma 3.10. Then
\[
\sum_{(k_1,k_2,k_3) \in Z_2(k) \cup Z_3(k)} 2^{\max\{k_3\}-\max\{k_1,k_2\}} C_{k,k_1,k_2,k_3} a_k b_k c_k \lesssim a_k b_k c_k
\]

If we take advantage of the local smoothing/maximal function spaces, then we can obtain the following improvement ([18, Lemma 3.13]).

Lemma 3.13 (Main trilinear estimate improvement over \( Z_1 \)). The best constant \( C_{k,k_1,k_2,k_3} \) in (3.10) satisfies the improved estimate
\[
C_{k,k_1,k_2,k_3} \lesssim 2^{-\lfloor (k_1+k_2)/2 \rfloor}
\]
when \( \{k_1,k_2,k_3\} \in Z_1(k) \).

There are certain situations, such as when bounding the cubic-type term, where we can place each term in a \( G_k \) space, in which case we get better estimates.

Lemma 3.14 (Improved trilinear estimate). Let \( C_{k,k_1,k_2,k_3} \) denote the best constant \( C \) in the estimate
\[
\|P_k (P_{k_1}g_1 P_{k_2}g_2 P_{k_3}g_3)\|_{N_k(T)} \lesssim C \|P_{k_1}g_1\|_{G_{k_1}(T)} \|P_{k_2}g_2\|_{G_{k_2}(T)} \|P_{k_3}g_3\|_{G_{k_3}(T)}
\]
Then \( C_{k,k_1,k_2,k_3} \) satisfies the bounds
\[
C_{k,k_1,k_2,k_3} \lesssim \begin{cases} 
2^{-\lfloor (k_1+k_2)/2 \rfloor} & (k_1,k_2,k_3) \in Z_1(k) \\
2^{-\lfloor k-k_1 \rfloor/6} 2^{-\lfloor k-k_2 \rfloor/6} & (k_1,k_2,k_3) \in Z_2(k) \\
2^{-\Delta k/6} & (k_1,k_2,k_3) \in Z_3(k) \\
0 & (k_1,k_2,k_3) \in Z^3 \setminus \{Z_1(k) \cup Z_2(k) \cup Z_3(k)\}
\end{cases}
\]
where \( \Delta k = \max\{k,k_1,k_2,k_3\} - \min\{k,k_1,k_2,k_3\} \geq 0 \).

Proof. We seek an improvement over Lemma 3.10 only on the set \( Z_2(k) \). Here we apply (3.4) so that
\[
\|P_k (P_{k_1}g_1 P_{k_2}g_2 P_{k_3}g_3)\|_{N_k(T)} \lesssim 2^{-\lfloor k-k_1 \rfloor/6} \|P_{k_1}g_1\|_{L_{t,x}^2(T)} \|P_{k_2}g_2\|_{L_{t,x}^2(T)} \|P_{k_3}g_3\|_{G_{k_3}(T)}
\]
We conclude with (3.6). □

Corollary 3.15. Let \( \{a_k\}, \{b_k\}, \{c_k\} \) be \( \delta \)-frequency envelopes. Let \( C_{k,k_1,k_2,k_3} \) be as in Lemma 3.14. Then
\[
\sum_{(k_1,k_2,k_3) \in \mathbb{Z}^3} C_{k,k_1,k_2,k_3} \lesssim a_k b_k c_k
\]
Lemma 4.1. In view of Corollary (3.11) we need only establish the bound on $Z_2(k)$. We have

$$
\sum_{\ell = 1 \atop \ell \neq b, c} b_\ell \leq \sum_{\ell = 1 \atop \ell \neq b, c} b_\ell ^2
$$

To finish, sum on $k_3$, then on $k_1$. \qed

4. Bounds along the heat flow

We recall from [4, 18] some estimates in the $F_k$ spaces that propagate along the heat flow. We assume throughout that $\phi$ has subthreshold energy and that $\varepsilon > 0$ is a very small number such that $b_k \leq \varepsilon$ and $\varepsilon^{1/2} \sum_j b_j^2 \ll 1$.

Lemma 4.1. Let $k \in \mathbb{Z}$, $s \geq 0$. Let $F \in \{ A_\ell^2, \partial_t A_\ell, f g : \ell = 1, 2; f, g \in \{ \psi_m, \psi_m : m = 1, 2 \} \}$. Then

$$
\| P_k \psi_m(s) \|_{F_k(T)} \lesssim (1 + s 2^{2k})^{-3} 2^{-\sigma k} b_k(\sigma)
$$

(4.1)

$$
\| P_k (A_\ell \psi_m(s)) \|_{F_k(T)} \lesssim \varepsilon (1 + s 2^{2k})^{-2} (s 2^{2k})^{-3/8} 2^{-\sigma k} b_k(\sigma)
$$

(4.2)

$$
\| P_k F(s) \|_{F_k(T)} \lesssim \varepsilon^{1/2} (1 + s 2^{2k})^{-2} (s 2^{2k})^{-5/8} 2^{-\sigma k} b_k(\sigma)
$$

(4.3)

$$
\| P_k \int_0^s e^{(s-s')t} U_m(s') ds' \|_{F_k(T)} \lesssim (1 + s 2^{2k})^{-4} 2^{-\sigma k} b_k(\sigma)
$$

(4.4)

$$
\| P_k A_m(0) \|_{F_k(T)} \lesssim \sum_p b_p^2
$$

(4.5)

These $F_k$ estimates can then be used to establish various $L^4$ bounds along the heat flow. In particular, we have [18 Lemma 7.17]:

Lemma 4.2. The following bounds hold

$$
\| P_k \psi_t(s) \|_{L^4_{t,x}} + \| P_k \psi_s(s) \|_{L^4_{t,x}} \lesssim (1 + s 2^{2k})^{-2} 2^k (1 + \sum_j b_j^2) b_k
$$

(4.6)

$$
\| \int_0^s e^{(s-r)\Delta} P_k U_\alpha(r) dr \|_{L^4_{t,x}} \lesssim \varepsilon (1 + s 2^{2k})^{-2} 2^k (1 + \sum_j b_j^2) b_k
$$

(4.7)

where $\alpha \in \{ 0, 3 \}$.

Taking advantage of the fact that the heat and Schrödinger flows share the same initial data at $s = 0$ for all $t$, we may use the above bounds to control the commutator of these flows with an $\varepsilon$ gain.
Corollary 4.3. Let $\Psi = \psi_t - i\psi_s$. Then
$$\|P_k \Psi\|_{L^4_t L^4_x} \lesssim \varepsilon 2^k (1 + \sum_j b_j^2) b_k$$

Proof. From (2.19) and (2.20) we have
$$(\partial_s - \Delta) \Psi = U_t - iU_s$$
As $\Psi(s = 0) = 0$, Duhamel implies
$$\Psi(s) = \int_0^s e^{(s-r)\Delta} (U_t - iU_s)(r)dr$$
The conclusion follows from (4.7). \hfill \Box

This next lemma is a special case of [18, Lemma 7.18]:

Lemma 4.4. It holds that
$$\|P_k A_m(0)\|_{L^4_t L^4_x} \lesssim b_k^2$$

5. The electric potential

In this section we prove that $V_m$ (defined by 2.9) is perturbative in the sense that $\|P_k V_m\|_{N_k(T)} \lesssim \varepsilon b_k$. Throughout this section, $\varepsilon > 0$ is a very small number such that $b_k \leq \varepsilon$ and $\varepsilon^{1/2} \sum_j b_j^2 \ll 1$. In application, it will be set equal to a suitable power of the energy dispersion parameter $\varepsilon_0$.

Proposition 5.1. The term $V_m = (A_t + A^2_x)\psi_m - i\psi_t \text{Im}(\bar{\psi}_l \psi_m)$ is perturbative.

We prove this in several steps, starting with the cubic term.

The cubic term $-i\psi_t \text{Im}(\bar{\psi}_l \psi_m)$.

To control the cubic term we use off-diagonal decay and take advantage of the fact that any $\psi$ may be placed in a $G_k$ space. Hence Lemma 3.14 applies and from Corollary 3.15 we conclude
$$\|P_k (\psi_t \text{Im}(\bar{\psi}_l \psi_m))\|_{N_k(T)} \lesssim \varepsilon b_k$$

The term $A^2_x \psi$.

We conclude as a corollary of Lemma 3.14 that
$$\|A^2_x(0)\|_{L^2_t L^2_x} \lesssim \|A_x(0)\|_{L^4_t L^4_x}^2 \lesssim \sum_{k \in \mathbb{Z}} \|P_k A_x(0)\|_{L^4_t L^4_x}^2 \lesssim \sup_{j \in \mathbb{Z}} b_j^2 \cdot \sum_{k \in \mathbb{Z}} b_k^2$$

We next show how to use this $L^2$ bound to control $A^2_x \psi_m$ in $N_k$ spaces.
Lemma 5.2. Let \( f \in L^2_{t,x} \). Then
\[
\| P_k (f \psi_m) \|_{N_k(T)} \lesssim \| f \|_{L^2_{t,x}} b_k
\] (5.2)

Proof. Begin with the following Littlewood-Paley decomposition of \( P_k (f \psi_x) \):
\[
P_k (f \psi_x) = P_k(P_{<k-80} f P_{k-5< \cdot < k+5} \psi_x) + \sum_{|k_1-k| \leq 4} P_k (P_{k_1} f P_{k_2} \psi_x) + \sum_{|k_1-k_2| \leq 90, k_1, k_2 > k-80} P_k (P_{k_1} f P_{k_2} \psi_x)
\]
The first term is controlled using (3.2):
\[
\| P_k(P_{<k-80} f P_{k-5< \cdot < k+5} \psi_x) \|_{N_k(T)} \leq \| P_k(P_{<k-80} f P_{k-5< \cdot < k+5} \psi_x) \|_{L^2_{t,x}^2}
\]
\[
\leq \| P_{<k-80} f \|_{L^2_{t,x}^2} \| P_{k-5< \cdot < k+5} \psi_x \|_{G_k(T)}
\]
To control the second term we apply (3.3):
\[
\| P_k (P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim 2^{-|k_2-k|/6} \| P_{k_1} f \|_{L^2_{t,x}^2} \| P_{k_2} \psi_x \|_{G_{k_2}(T)}
\]
To control the high-high interaction, apply (3.4):
\[
\| P_k (P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim 2^{-|k_2-k_1|/6} \| P_{k_1} f \|_{L^2_{t,x}^2} \| P_{k_2} \psi_x \|_{G_{k_2}(T)}
\]
Therefore
\[
\sum_{|k_1-k_2| \leq 90, k_1, k_2 > k-80} \| P_k (P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim \sum_{|k_1-k_2| \leq 90, k_1, k_2 > k-80} 2^{-|k_2-k|/6} \| P_{k_1} f \|_{L^2_{t,x}^2} b_{k_2}
\]
and so by Cauchy-Schwarz
\[
\sum_{|k_1-k_2| \leq 90, k_1, k_2 > k-80} \| P_k (P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim b_k \left( \sum_{k_1 \geq k-80} \| P_{k_1} f \|_{L^2_{t,x}^2}^2 \right)^{1/2}
\]
Upon interchanging the \( L^2_{t,x} \) and \( \ell^2 \) norms, we conclude from the standard square function estimate that
\[
\sum_{|k_1-k_2| \leq 90, k_1, k_2 > k-80} \| P_k (P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim \| f \|_{L^2_{t,x}} b_k
\]

Together (5.1) and Lemma 5.2 imply
\[
\| P_k (A^2_x \psi_m) \|_{N_k(T)} \lesssim \epsilon b_k
\] (5.3)

The leading term \( A_t \psi \).

This term requires more effort to bound, as its behavior blends that of the cubic term and \( A^2_x \psi \). The main difficulty arises from the fact that we do not
control $\psi_t(s)$ in $F_k$ spaces for positive heat flow times $s > 0$. While at $s = 0$ we do indeed have $\psi_t = iD_j \psi_j$ (because at $s = 0$ it holds that $\psi_t = i\psi_s$) as a consequence of the fact that $\phi$ is a Schrödinger map, along the heat flow we do not have such an explicit representation of $\psi_t$ and instead must access it through the commutator of the heat and Schrödinger flows. Thus our first step is to represent $\psi_t$ as $i\psi_s + \Psi$. It may be tempting to try to place $\Psi$ in the $F_k$ spaces, as we do have such bounds for $\psi_s$. The $F_k$ bounds for $\psi_s$, however, are obtained as a consequence of the formula $\psi_s = D_j \psi_j$ and the bounds on $\psi_x$ and $A_x$; therefore a different approach would be required to bound $\Psi$ in $F_k$. The bounds on $\Psi(s), s > 0$ that can be readily obtained are those in $L^4$. Owing to the fact that $\Psi(0) = 0$, it turns out that under the assumption of energy dispersion, these bounds come with an $\varepsilon$ gain.

Representing $A_t$ as

$$A_t(s) = - \int_0^\infty \text{Im}(i\bar{\psi}_s \psi_s)(s')ds' - \int_0^\infty \text{Im}(\bar{\Psi} \psi_s)(s')ds'$$

we show that the $L^2$ norm of the second integral is small, enabling us to treat its total contribution to $A_t \psi$ using Lemma 5.2.

Now $\psi_s = D_j \psi_j$ and as already mentioned does enjoy bounds in the $F_k$-spaces. The effect of the integration is cancel the derivative $s$ that appear, and so in principle the first integral in (5.4) is on par with the cubic term $\psi_t \text{Im}(\bar{\psi}_t \psi_m)$. However, any one of the terms in the cubic term may be placed in a $G_k$ space, whereas, for $s' > 0$, we cannot place $\psi_s(s')$ in $G_k$ spaces. For this reason we further decompose $\psi_s$, representing it as $\psi_s = \partial_t \psi + iA_t \psi_t$. Rewriting the first integral in (5.4) as

$$- \int_0^\infty |\psi_s|^2(s')ds'$$

we see that it suffices (by Young’s inequality) to just consider the contributions from $|\partial_t \psi|^2$ and from $|A_t \psi_t|^2$. For the latter term, we pull out $A_t$ in $L^2$ (which is largest at $s = 0$), and control $\int_0^\infty |\psi_t|^2 \lesssim E_0$ using heat flow bounds; hence this term has a contribution like that of $A^2 x \psi$. For the $|\partial_t \psi|^2$ term, we expand $\psi(s)$ using the Duhamel formula as the sum of its linear evolution and nonlinear evolution. While we do not propagate bounds in the $G_k$-spaces along the nonlinear heat flow, such bounds do in fact propagate along the linear flow; the contribution from the three linear terms taken together therefore is comparable to that of the cubic term. The nonlinear evolution terms must be dealt with more delicately, as here we must resort to placing these in $F_k$-spaces, resulting in worse off-diagonal gains. The upshot, however, is that these terms come with an energy-dispersion gain that is enough to offset the consequences of inferior decay. We turn to the details.
Lemma 5.3. It holds that
\[ \| \int_0^\infty (\overline{\Psi} \cdot D_\ell \psi_\ell)(s) \, ds \|_{L^2_{t,x}} \lesssim \varepsilon (1 + \sum_j b_j^2)^2 \sum_k b_k^2 \]

Proof. We first bound \((\overline{\Psi} \cdot D_\ell \psi_\ell)(s)\) in \(L^2\). Define
\[
\mu_k(s) := \sup_{k' \in \mathbb{Z}} 2^{-|j|k-k'} \| P_k \overline{\Psi}(s) \|_{L^4_{t,x}} \quad \text{and} \quad \nu_k(s) := \sup_{k' \in \mathbb{Z}} 2^{-|j|k-k'} \| P_k (D_\ell \psi_\ell)(s) \|_{L^4_{t,x}}
\]
Then
\[ \| (\overline{\Psi} \cdot D_\ell \psi_\ell)(s) \|_{L^2_{t,x}} \lesssim \sum_k \mu_k(s) \sum_{j \leq k} \nu_j(s) + \sum_k \nu_k(s) \sum_{j \leq k} \mu_j(s) \quad (5.5) \]
From Corollary 4.3 and from (4.6) it follows that
\[ \mu_k(s) \lesssim \varepsilon (1 + s^{2k})^{-2} 2^k (1 + \sum_p b_p^2) b_k \quad (5.6) \]
and
\[ \nu_k(s) \lesssim (1 + s^{2k})^{-2} 2^k (1 + \sum_p b_p^2) b_k \quad (5.7) \]
Therefore
\[ \int_0^\infty \| (\overline{\Psi} \cdot D_\ell \psi_\ell)(s) \|_{L^2_{t,x}}^2 \, ds \lesssim \int_0^\infty \sum_k \mu_k(s) \sum_{j \leq k} \nu_j(s) \, ds \]
\[ \lesssim \varepsilon (1 + \sum_p b_p^2)^2 \sum_k 2^k b_k \sum_{j \leq k} 2^j b_j \times \]
\[ \int_0^\infty (1 + s^{2j})^{-2} (1 + s^{2k})^{-2} \, ds \]
Then
\[ \sum_{j \leq k} 2^j b_j \int_0^\infty (1 + s^{2j})^{-2} (1 + s^{2k})^{-2} \, ds \lesssim \int_0^\infty (1 + s^{2k})^{-2} \, ds \sum_{j \leq k} 2^j b_j \]
and
\[ \int_0^\infty (1 + s^{2k})^{-2} \, ds \lesssim 2^{-2k} \]
so that
\[ \sum_k 2^k b_k \sum_{j \leq k} 2^j b_j \int_0^\infty (1 + s^{2j})^{-2} (1 + s^{2k})^{-2} \, ds \lesssim \sum_k b_k^2 \]
\[ \square \]

Lemma 5.4. It holds that
\[ \| \int_0^\infty |A_\ell \psi_\ell|^2(s) ds \|_{L^2_{t,x}} \lesssim \sup_j b_j^2 \sum_k b_k^2 \]
Proof. Start by taking a Littlewood-Paley decomposition of $|A_\ell|^2$:

$$\int_0^\infty |A_\ell \psi_\ell|^2(s)ds = \int_0^\infty \sum_{j,k} P_j A_\ell(s) \cdot P_k \overline{A_\ell(s)} \cdot |\psi_x|^2(s)ds$$

We bound this expression in absolute value by

$$\sum_{j,k} \sup_{s \geq 0} |P_j A_x(s)| \cdot \sup_{s' \geq 0} |P_k A_x(s')| \cdot \int_0^\infty |\psi_x|^2(r)dr$$

Next we take the $L^2_{t,x}$ norm, which we control by placing the integral in $L^\infty_{t,x}$ and the summation in $L^2_{t,x}$. We control the summation by

$$\sum_{j,k} \| \sup_{s \geq 0} |P_j A_x(s)| \cdot \sup_{s' \geq 0} |P_k A_x(s')| \|_{L^2_{t,x}} \lesssim \sum_j \| \sup_{s \geq 0} |P_j A_x(s)| \|^2_{L^2_{t,x}}$$

Now

$$\sup_{s \geq 0} |P_j A_m(s)| \lesssim \sup_{s \geq 0} \int_s^\infty |P_j (\overline{\psi_m(s')} D_\ell \psi_\ell(s'))|ds' \leq \int_0^\infty |P_j (\overline{\psi_m(s')} D_\ell \psi_\ell(s'))|ds'$$

and in view of the proof of Lemma 4.4, the $L^2$ norm of the right hand side is bounded by $b_j^2$.

Thus it remains to show

$$\| \int_0^\infty |\psi_\ell|^2(s)ds \|_{L^\infty_{t,x}} \lesssim 1$$

For fixed $s, x, t$, however, $\psi_\ell$ is simply the representation of $\partial_\ell \phi$ with respect to an orthonormal basis of $T\phi(s,x,t)S^2$. Therefore $|\psi_\ell| = |\partial_\ell \phi|$ and so we may invoke (the uniform in time) estimate (2.14).

□

Lemma 5.5. It holds that

$$P_k \left( \int_0^\infty |\partial_\ell \psi_\ell|^2(s')ds' \psi_m(0) \right) \lesssim \varepsilon b_k \tag{5.8}$$

Proof. We write

$$\psi_\ell(s) = e^{s\Delta} \psi_\ell(0) + \int_0^s e^{(s-r)\Delta} U_\ell(r)dr$$

and expand (5.8) accordingly. Additionally, we perform a Littlewood-Paley decomposition of each term. The trilinear term

$$P_k \left( \sum_{k_1,k_2,k_3} \int_0^\infty e^{s\Delta} P_{k_1} \partial_\ell \psi_\ell(0) \cdot e^{s\Delta} P_{k_2} \partial_\ell \psi_\ell(0) \cdot |\psi_x|^2(s)ds \cdot P_{k_3} \psi_m(0) \right)$$
we bound in $N_k(T)$ by

$$
\sum_{(k_1,k_2,k_3) \in Z_1(k) \cup Z_2(k) \cup Z_3(k)} C_{k,k_1,k_2,k_3} 2^{k_1} \|P_{k_1} \psi_x(0)\|_{G_{k_1}(T)} 2^{k_2} \|P_{k_2} \psi_x(0)\|_{G_{k_2}(T)} \times \\
\times \|P_{k_3} \psi_m(0)\|_{G_{k_3}(T)} \int_0^\infty (1 + s^{2k_1})^{-20} (1 + s^{2k_2})^{-20} ds
$$

where $C_{k,k_1,k_2,k_3}$ is as in Lemma 3.14 and $Z_1, Z_2, Z_3$ are given by (3.9). As

$$
2^{k_1}2^{k_2} \int_0^\infty (1 + s^{2k_1})^{-20} (1 + s^{2k_2})^{-20} ds \lesssim 1
$$

it suffices to control

$$
\sum_{(k_1,k_2,k_3) \in Z_1(k) \cup Z_2(k) \cup Z_3(k)} C_{k,k_1,k_2,k_3} \|P_{k_1} \psi_x(0)\|_{G_{k_1}(T)} \times \\
\times \|P_{k_2} \psi_x(0)\|_{G_{k_2}(T)} \|P_{k_3} \psi_m(0)\|_{G_{k_3}(T)}
$$

(5.9)

We invoke Corollary 3.15 to bound (5.9) by $b_k^2$, which suffices for this term in view of the energy dispersion assumption.

Next we must consider products involving either one or two terms. The arguments are similar to those of the case already considered. Here, however, we only place $P_{k_3} \psi_m(0)$ in a $G_k$ space; the remaining two terms we control in $F_k$ spaces using either (4.1) or (4.4). Hence in our $N_k(T)$ bound we use $C_{k,k_1,k_2,k_3}$ as in Lemma 3.10 rather than Lemma 3.14 and to sum we must use Corollary 3.11 instead of Corollary 3.15. Corollary 3.11 however, only supplies the bound over $Z^3 \setminus Z_2(k)$. On $Z_2$ the sum is controlled by

$$
\varepsilon \sum_{(k_1,k_2,k_3) \in Z_2(k)} 2^{-|k-k_3|/6} b_{k_1} b_{k_2} b_{k_3} \lesssim \varepsilon \sum_{k,k_3 \leq k_1-40} 2^{-|k-k_3|/6} b_{k_1} b_{k_1} b_{k_3}
$$

$$
\lesssim \varepsilon b_k \sum_{k,k_3 \leq k_1-40} 2^{-|k-k_3|/6} 2^{\delta |k-k_3|} b_{k_1}^2
$$

$$
\lesssim \varepsilon b_k \sum_{k_1 \geq k+40} b_{k_1}^2
$$

6. The Magnetic Potential

We begin by introducing a paradifferential decomposition of the magnetic nonlinearity, splitting it into two pieces. This decomposition depends upon a frequency parameter $k \in \mathbb{Z}$, which we suppress in the notation; this frequency $k$ is the output frequency. The decomposition also depends upon a universal frequency gap parameter $\varpi \in \mathbb{Z}_+$ that need only be taken sufficiently large (see [18, §5] for discussion).
Define
\[
A_{m, \text{lo\&lo}}(s) := - \sum_{k_1, k_2 \leq k - w} \int_s^\infty \text{Im}(P_{k_1} \overline{\psi_m} P_{k_2} \psi_s)(s') ds'
\]
\[
A_{m, \text{hi\&hi}}(s) := - \sum_{\max\{k_1, k_2\} > k - w} \int_s^\infty \text{Im}(P_{k_1} \overline{\psi_m} P_{k_2} \psi_s)(s') ds'
\]
so that \(A_m = A_{m, \text{lo\&lo}} + A_{m, \text{hi\&hi}}\). Similarly define
\[
B_{m, \text{lo\&lo}} := -i \sum_{k_3} (\partial_\ell (A_{\ell, \text{lo\&lo}} P_{k_3} \psi_m) + A_{\ell, \text{lo\&lo}} \partial_\ell P_{k_3} \psi_m)
\]
\[
B_{m, \text{hi\&hi}} := -i \sum_{k_3} (\partial_\ell (A_{\ell, \text{hi\&hi}} P_{k_3} \psi_m) + A_{\ell, \text{hi\&hi}} \partial_\ell P_{k_3} \psi_m)
\]
so that \(B_m = B_{m, \text{lo\&lo}} + B_{m, \text{hi\&hi}}\).

Our goal is to control \(P_k B_m\) in \(N_k(T)\). We consider first \(P_k B_{m, \text{hi\&hi}}\), performing a Littlewood-Paley decomposition. In order for frequencies \(k_1, k_2, k_3\) to have an output in this expression at a frequency \(k\), we must have \((k_1, k_2, k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)\), where \(Z_0(k) := Z_1(k) \cap \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1, k_2 > k - \varpi\}\) and \(Z_2, Z_3\) are given by (3.9). We apply Lemma 3.10 to bound \(P_k B_{m, \text{hi\&hi}}\) in \(N_k(T)\) by
\[
\sum_{(k_1, k_2, k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)} \int_0^\infty 2^{\max\{k_1, k_2, k_3\}} |C_{k_1, k_2, k_3} P_{k_1} \psi_x(s)| F_{k_1} \times
\]
\[
\times \|P_{k_3} (D_\ell \psi_x(s))\| F_{k_3} \|P_{k_3} \psi_m(0)\| G_{k_3} ds
\]
which, thanks to (4.1) and (4.2), is controlled by
\[
\sum_{(k_1, k_2, k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)} 2^{\max\{k_1, k_2, k_3\}} |C_{k_1, k_2, k_3} b_1 b_2 b_3| \times
\]
\[
\times \int_0^\infty (1 + s^2 k_1)^{-4} 2^{k_2} (s^2 2^{k_2})^{-3/8} (1 + s^2 2^{k_2})^{-2} ds
\]
As
\[
\int_0^\infty (1 + s^2 k_1)^{-4} 2^{k_2} (s^2 2^{k_2})^{-3/8} (1 + s^2 2^{k_2})^{-2} ds \lesssim 2^{-\max\{k_1, k_2\}}
\]
we reduce to
\[
\sum_{(k_1, k_2, k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)} 2^{\max\{k_1, k_2, k_3\} - \max\{k_1, k_2\}} |C_{k_1, k_2, k_3} b_1 b_2 b_3|
\]
To estimate \(P_k B_{m, \text{hi\&hi}}\) on \(Z_2 \cup Z_3\), we apply Corollary 3.12 and use the energy dispersion hypothesis. As for \(Z_0(k)\), we note that its cardinality \(|Z_0(k)|\) satisfies \(|Z_0(k)| \lesssim \varpi\) independently of \(k\). Hence for fixed \(\varpi\) summing over this set is harmless given sufficient energy dispersion.
The last term to consider is $P_k B_{m,lo\wedge lo}$. We cannot control this term directly in $N_k$, but instead can get a handle on it via the following bilinear estimate from [18 §5].

**Theorem 6.1.** Let $\sigma \in [0, \sigma_1 - 1]$, $s \geq 0$, and $2^{-k} \ll 1$. Suppose that $V_m$ and $B_{m,hi\vee hi}$ are perturbative and that this can be established using only mixed $L^p$ spaces with all $p$ lying in a compact subset of $(1, \infty)$, e.g., the local smoothing and maximal function spaces are excluded. Then

$$2^{-j} (1 + s 2^j)^8 \| P_j \psi(s) \cdot P_k \psi_m(0) \|^2_{L^2_{t,x}} \lesssim 2^{-2\sigma k} c_j^2 (\sigma) + \varepsilon^2 b_j^2 b_k^2 (\sigma) \quad (6.2)$$

We show how to apply this theorem in the next section.

**7. Proofs of the main theorems**

We have in place all of the estimates needed to prove (1.15) and hence Theorem 1.4.

Using the main linear estimate of Proposition 3.6 and the decomposition introduced in §6 we write

$$\| P_k \psi_m \|_{G_k(T)} \lesssim \| P_k \psi_m(0) \|_{L^2_T} + \| P_k V_m \|_{N_k(T)} + \| P_k B_{m,lo\wedge lo} \|_{N_k(T)} \quad (7.1)$$

It was shown in the two preceding sections that $P_k V_m$ and $P_k B_{m,hi\vee hi}$ are perturbative in the sense that

$$\| P_k V_m \|_{N_k(T)} + \| P_k B_{m,hi\vee hi} \|_{N_k(T)} \lesssim \varepsilon b_k$$

where $\varepsilon > 0$ is assumed to satisfy $b_k \leq \varepsilon$ and $\varepsilon^{1/2} \sum_j b_j^2 \ll 1$. In view of (1.12) and the bootstrap hypothesis (1.13), we set $\varepsilon := \varepsilon_0^{9/10}$. Clearly this satisfies $b_k \leq \varepsilon$. Moreover,

$$\varepsilon^{1/2} \sum_j b_j^2 \leq \varepsilon_0^{3/10} \sum_j c_j^2 \lesssim E_0 \varepsilon_0^{3/10}$$

To handle $P_k B_{m,lo\wedge lo}$, we first rewrite it as

$$P_k B_{m,lo\wedge lo} = -i \partial \ell (A_{\ell,lo\wedge lo} P_k \psi_m) + R$$

where $R$ is a perturbative remainder (see [18 §5] for details). Therefore

$$\| P_k \psi_m \|_{G_k(T)} \lesssim c_k + \varepsilon b_k + \| \partial \ell (A_{\ell,lo\wedge lo} P_k \psi_m) \|_{N_k(T)} \quad (7.2)$$

Thus it remains to control $-i \partial \ell (A_{\ell,lo\wedge lo} P_k \psi_m)$, which we expand as

$$-i P_k \partial \ell \sum_{k_1, k_2 \leq k - \infty \atop |k_3 - k| \leq 4} \int_0^\infty \text{Im} (P_{k_1} \psi P_{k_2} \psi)(s') P_{k_3} \psi_m(0) ds' \quad (7.3)$$
and whose $N_k(T)$ norm we denote by $N_{lo}$. The key now is to apply Theorem 6.1 to $P_k \psi_t(s')$ and $P_k \psi_m(0)$, after first placing all of (7.3) in $N_k(T)$ using (3.7). We obtain

$$N_{lo} \lesssim 2^k \sum_{k_1, k_2 \leq k-\omega} 2^{-|k-k_1|/2} 2^{-\max\{k_1, k_2\} b_{k_2} (c_{k_1} c_{k_3} + \varepsilon b_{k_1} b_{k_3})}$$

$$\lesssim 2^k \sum_{k_1, k_2 \leq k-\omega} 2^{(k_1+k_2)/2} 2^{-\max\{k_1, k_2\} b_{k_2} (c_{k_1} c_k + \varepsilon b_{k_1} b_k)}$$

Without loss of generality we restrict the sum to $k_1 \leq k_2$:

$$\sum_{k_1 \leq k_2 \leq k-\omega} 2^{(k_1-k_2)/2} b_{k_2} (c_{k_1} c_k + \varepsilon b_{k_1} b_k)$$

Using the frequency envelope property to sum off the diagonal, we reduce to

$$N_{lo} \lesssim \sum_{j \leq k-\omega} (b_j c_j c_k + \varepsilon b_j^2 b_k)$$

Combining this with (7.2) and the fact that $R$ is perturbative, we obtain

$$b_k \lesssim c_k + \varepsilon b_k + \sum_{j \leq k-\omega} (b_j c_j c_k + \varepsilon b_j^2 b_k)$$

which, in view of our choice of $\varepsilon$, reduces to

$$b_k \lesssim c_k + c_k \sum_{j \leq k-\omega} b_j c_j$$

Squaring and applying Cauchy-Schwarz yields

$$b_k^2 \lesssim (1 + \sum_{j \leq k-\omega} b_j^2) c_k^2$$

(7.4)

Setting

$$B_k := 1 + \sum_{j < k} b_j^2$$

in (7.3) leads to

$$B_{k+1} \leq B_k (1 + C c_k^2)$$

with $C > 0$ independent of $k$. Therefore

$$B_{k+m} \leq B_k \prod_{\ell=1}^m (1 + C c_{k+\ell}^2) \leq B_k \exp(C \sum_{\ell=1}^m c_{k+\ell}^2) \lesssim_{E_0} B_k$$

Since $B_k \to 1$ as $k \to -\infty$, we conclude

$$B_k \lesssim_{E_0} 1$$

uniformly in $k$, so that, in particular,

$$\sum_{j \in \mathbb{Z}} b_j^2 \lesssim 1$$

(7.5)

which, joined with (7.4), implies (1.15).
The proof of (1.16) is almost an immediate consequence. A bit of care must be exercised, though. For instance, we will have summations of terms such as $C_{k_1,k_2,k_3}b_{j_1}b_{j_2}b_{j_3}b_jj_3^2 - \sigma j_3$ to control (see for instance Corollary 3.11), where $\{j_1,j_2,j_3\}$ is some permutation of $\{k_1,k_2,k_3\}$. Clearly such a term does not sum over $j_3 \ll C$. The way out, however, is straightforward.

Recall that we need only sum over $Z_1(k), Z_2(k), Z_3(k)$, defined in (3.9). If we encounter a sum over $Z_1(k)$, bound the $k_3$ term with $b_{k_3}j_{3}^2 - \sigma k_3$ and the remaining two terms with $b_{k_1}j_{1}^2$ and $b_{k_2}j_{2}^2$. Over $Z_2(k)$, always bound the $k_1$ term by $b_{k_1}j_{1}^2 - \sigma k_1 \leq b_{k_1}j_{1}^2 - \sigma k_1$. Similarly, on $Z_3(k)$ we also bound the $k_1$ term by $b_{k_1}j_{2}^2 - \sigma k_1$. Such a strategy suffices for controlling the perturbative terms.

We obtain

$$b_{k}(\sigma) \lesssim c_{k}(\sigma) + \varepsilon b_{k}(\sigma) + \sum_{j \leq k-\omega} (b_{j}c_{j}j_{k}^2 + \varepsilon b_{j}^2 b_{k}(\sigma))$$

which is enough to prove (1.16) in view of (7.5).

Equipped with Theorem 1.4 we conclude Corollary 1.5 as in [18, §4.6].

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