GEOMETRIC STRUCTURES ON FIELDS.

0. Introduction.

Let $X$ be a connected differentiable manifold, $G$ a Lie group which acts transitively on $X$, and which action verifies the unique extension property (u.e.p): this means that two elements of $G$ which coincide on an open set of $X$, coincide on $X$. A geometric structure on $X$, or an $(X,G)$–structure on $X$, is a differentiable manifold $M$ endowed with an atlas $(\phi_i : U_i \to X)_{i \in I}$, where $\phi_i$ is a diffeomorphism onto its image such that:

$$\phi_i \circ \phi_j^{-1}|_{U_i \cap U_j} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$$

is the restriction of an element $g_{ij}$ of $G$. The family $g_{ij}$ satisfies the Chasles relation:

$$g_{ij}g_{jk} = g_{ik}.$$ 

The $(X,G)$ structure of $M$ pulls back to its universal cover $\hat{M}$. This last structure is defined by a local diffeomorphism:

$$D_M : \hat{M} \longrightarrow X$$

called the developing map, which gives rise to a representation

$$h_M : \pi_1(M) \longrightarrow G$$

called the holonomy representation of the $(X,G)$ structure of $M$.

The developing map can also be constructed as follows: we consider the sheaf of local $(X,G)$ transformations from $M$ to $X$, we denote by $F$ its Etale space, we have a map:

$$F \longrightarrow X$$

$$[f]_x \longrightarrow f(x),$$

where $x$ is an element of $M$ and $[f]_x$ an element of the fiber of $x$, this application is well defined since $G$ satisfied the unique extension property. Its restriction to a connected component of $F$ is the developing map up to a cover.

Examples of such structures are:

$n$–dimensional affine manifolds where $X$ is $\mathbb{R}^n$ and $G$ is $Aff(\mathbb{R}^n)$, the group of affine transformations of $\mathbb{R}^n$. 

n–dimensional projective manifolds where $X$ is the projective space $PIR^n$, and $G$ is the group of projective transformations $PGl(n, IR)$, etc...

For more information about those structures, see...

We remark that to develop a theory of $(X, G)$ manifolds, we need essentially 1–homotopy theory. The 1–homotopy of toposes is well understood. The goal of this paper is to generalize this theory to toposes to give applications to algebraic geometric and fields theory.

1. The fundamental group of a topos.

The theory of the fundamental group of a topos is well understood, it is proposed as an exercise in 4. p. 321 exercise 2.7.5.

Definition 1.1.
Let $C$ be a category, a sieve on $X$ is a subclass $R$ of the class $Ob(C)$ of objects of $C$ such that for every map $m : Y' \to Y$ such that $Y$ is in $R$, $Y'$ is also in $R$. Let $f : C' \to C$ be functor, we denote by $R^f$ the pull back of $R$ to $C'$, it is the sieve of $C'$ which elements are objects which image by $f$ is in $R$.
We denote by $C_Y$ subcategory of objects over $Y$.

Definition 1.2.
A topology on $C$ is an application which assigns to each object $S$ a non empty subclass $J(S)$ of the class of sieves over $S$, such that
For every map $f : T \to S$, and every sieve $R$ of $J(S)$, $R^f$ is a sieve of $J(T)$, (here $f$ is considered as a morphism between the categories $C_T$ and $C_S$.
For every object $S$ of $C$, every element $R$ of $J(S)$, and every element $R'$ of $C_X$, $R'$ is an element of $J(S)$ if for every object $f : T \to S$ of $R$, $R^f$ is an object of $J(T)$.
The elements of $J(S)$ are called the refinements of $S$.

Definition. 1.3.
A presheaf of sets on $C$, is a contravariant functor from $C$ to the category of sets.
A sheaf of sets on $C$, is a presheaf of sets on $C$ such that for every refinement $R$ of $S$, the map
$$F(S) \to F_R$$
is bijective, where $F_R$ is the presheaf defined on $R$ by $F_R(f) = F(T)$, where $f : T \to S$ is a map of $R$.

Example.
We consider the category $e$ whose the class of object has one element $x$, and such that the set of morphisms of $x$ is the singleton $\{Id_x\}$. A sheaf on this category, is a contravariant functor which sends $x$ to a set.
Let $C$ be a category whose set of objects is not empty, there exists a projection functor $e_C : C \to e$, which assigns $e$ to each object of $C$. 

2
Definition 1.4.
A constant sheaf, is a sheaf which factor through \( e \): that is, such that there exists a sheaf \( F_e \) of \( e \) such that \( F = F_e \circ e_C \).

We will say that the sheaf \( F \) is locally constant, if and only if there is a covering family \( (X_i)_{i \in I} \), such that the restriction of \( F \) to the category over \( X_i \) is a constant sheaf.

In the sequel, we will suppose that the category \( C \) is a topos, this means that \( C \) is equivalent to the category of sheaves defined on a standard \( U \)-sieve for a given universe \( U \), or that one of the following properties is satisfied:

(i) Endowed with its canonical topology, \( C \) is a \( U \)-sieve on which every sheaf is representable.
(ii) Endowed with its canonical topology, \( C \) is a \( U \)-sieve such that:
- The projective limits exist on \( C \).
- Summand indexed by an element of \( U \) exist, are disjoint, an universal
- Equivalence relations are effective and universal on \( C \).

Definition 1.5.
A morphism \( f : X \to Y \) of toposes is a functor \( f^{-1} : Y \to X \) such that: the presheaf \( f_*(F)(Y) = F(f^{-1}(Y)) \) is a sheaf.

The left adjoint functor \( f^* \) of \( f_* \) commute with finite projective limits.

Definition 1.6.
Let \( Z \) be the final object of a topos we will say that \( C \) is connected if we cannot find a covering family \( \{S_1, S_2\} \) of \( C \) such that \( S_1 \times_Z S_2 \) represents the empty object.

Definitions 1.7.
- A topological covering family \( (X_i)_{i \in I} \) of \( C \) is connected if for every \( X_i \), the topos \( C_{X_i} \) is connected.
- A topos is locally connected if and only if for every covering family \( (X_i)_{i \in I} \), there is a connected covering family \( (Y_j)_{j \in J} \) such that for each \( j \), there is a \( k(j) \) such that \( Y_j \) is a subobject of \( X_{k(j)} \).

Definition 1.8.
Let \( (X_i)_{i \in I} \) be a connected covering family of a topos, a path of this family denoted by \( (i_1, ..., i_n) \), will be a family of object \( (X_1, ..., X_n) \) such that \( X_i \times_Z X_{i+1} \) is different from the initial object, where \( Z \) is the final object.

Proposition 1.9.
A locally connected topos \( C \) is connected if and only if there is a topological covering family \( (X_i)_{i \in I} \) such that for each object \( X \) of \( C \), and for each \( i \) of \( I \), there is a path \( (i_1 = i, ..., i_n) \) such that \( X_n \times_Z X \) is not the initial object.

Proof.
Let \( C \) be a locally connected and connected topos. Suppose that there is an object \( X_i \) of the family \( (X_i)_{i \in I} \) and an object \( X \) of \( C \), such that for every path
We will call $C(X_i)$ the connected component of $X_i$. A locally connected topos is the direct summand of its connected components.

We endow the set of paths associated to the topological covering family $(X_i)_{i \in I}$ of the topos $C$ with the following relation:

Two paths $x$ and $y$ will be said equivalent if and only if there is a sequence of paths $z_1, ..., z_n$, and a $k \leq n$ such that $z_k = (i_1, ..., i_{k+1} = i_i, ..., i_m)$ and $z_{k+1} = (i_1, ..., i_{l+1}, i_{l+2}, ..., i_m)$, with $x = z_1$ and $y = z_n$.

We denote by $\text{Path}((X_i)_{i \in I})$, the set of equivalence classes of paths of $(X_i)_{i \in I}$.

In each equivalence class of a path $x$, we can find a representant $\bar{x} = (j_1, ..., j_k)$, such that $j_r \neq j_{r+1}$.

Let $x = (i_1, ..., i_n)$ and $y = (j_1, ..., j_m)$ be two paths representing the elements $\bar{x}$ and $\bar{y}$ of $\text{Path}((X_i)_{i \in I})$ such that $i_n = j_1$. We associate to $\bar{x}$ and $\bar{y}$ the element $x * y$, which has $(i_1, ..., i_n, j_2, ..., j_m)$ as a representant.

We can now define the groupoid $\text{Gr}((X_i)_{i \in I})$, whose objects are the elements of $I$. A morphism between $i$ and $j$ is the class of a path $(i_1, ..., i_n)$ such that $i_1 = i$ and $i_n = j$.

The inverse of the path $(i_1, ..., i_n)$ is $(i_n, ..., i_1)$.

The set of morphisms represented by the elements $(i_1 = i, ..., i_n = i)$ is a group denoted $\text{Aut}(i)$.

Consider now a locally constant sheaf $F$ defined on $C$, $(X_i)_{i \in I}$ a locally constant connected topological covering family, such that the restriction of $F$ to $X_i$ is a constant sheaf. Such a family will be called a trivializing family.

Let $X_i$ and $Y$, be two objects of $C$ such that the restriction of $F$ to $X_i$ and $Y$ is constant. Recall $Z$ is the final object of $C$. We suppose that $X \times_Z Y$ is not the initial object. The projections $p_x : X \times_Z Y \to X$ and $p_y : X \times_Z Y \to Y$ gives rise to the isomorphisms of sets: $g_x : F(X) \to F(X \times_Z Y)$ and $g_y : F(Y) \to F(X \times_Z Y)$. We deduce the isomorphism $g_x^{-1} g_y : F(Y) \to F(X)$.

We can apply this to the path $(i_1, ..., i_n)$, we deduce for each $i_k$ an isomorphism $g_{i_k i_{k+1}} : F(X_{i_{k+1}}) \to F(X_{i_k})$, and a morphism $g_{i_1 i_n} : g_{i_1 i_2} ... g_{i_{n-1} i_n}$ : $F(X_{i_n}) \to F(X_{i_1})$.

It results a representation

$$\text{hol}_F : \text{Aut}(i) \to \text{Aut}(F(X_i))$$

where $\text{Aut}(F(X_i))$ is the group of automorphisms of the sets $F(X_i)$. We set $F_S(X_i) = \text{hol}_F(\text{Aut}(i))$. 

4
Proposition 1.10.
Let $C$ be a locally connected and connected topos, and $F$ a sheaf on $C$, for each objects $X_i$ of $X_j$ of the connected trivializing family $(X_i)_{i \in I}$, the groups $\text{hol}_F(X_i)$ and $\text{hol}_F(X_j)$ are isomorphic. Moreover the class of isomorphism of $\text{hol}_F(X_i)$ does not depend of the trivializing family.

Proof.
Let $X_i$ and $X_j$ be two elements of $(X_i)_{i \in I}$, there exists a path $(i_1, ..., i_n)$ between $X_i$ and $X_j$. This path induces an isomorphism between $\text{hol}_F(X_i)$ and $\text{hol}_F(X_j)$.

Now, we show that the isomorphism class of $\text{hol}_F(X_i)$ does not depend of the chosen trivializing family. Let $(Y_j)_{j \in J}$ be another trivializing family. There exists another trivializing family $(Z_k)_{k \in K}$, such that each $Z_k$ is a sub object of an object $X_{i(k)}$ of $(X_i)_{i \in I}$ and $Y_{j(k)}$ of $(Y_j)_{j \in J}$.

It suffices to show that $\text{hol}_F(X_i)$ and $\text{hol}_F(Z_k)$ are isomorphic for each element $Z_k$ of $(Z_k)_{k \in K}$.

Let $(k_1 = k, ..., k_n = k)$ be a path of $(Z_k)_{k \in K}$, for each $k_i$, we choose an element $X_{k_i}$ of $(X_i)_{i \in I}$ such that $X_{k_i}$ is a sub object of $X_i$, thus we obtain a morphism of group from $\text{hol}_F((Z_k)_{k \in K})$ to $\text{hol}_F((X_i)_{i \in I})$.

Define now its inverse. Let $(i_1 = i, ..., i_n = i)$ be a path of $(X_i)_{i \in I}$, for every $X_i$, there exists an object $Z_{k_i}$ of the family $(Z_k)_{k \in K}$ which is a subobject of such that $Z_{k_i} \times X_{k_{i+1}}$ is different from the initial object. There exists a path between a component of $Z_{k_i} \times X_{k_{i+1}}$ and $Z_{k_{i+1}}$ since $X_{k_{i+1}}$ is connected. We thus obtain a path of $(Z_k)_{k \in K}$ we define the inverse of the previous morphism.

We will denote by $(G_i)_{i \in I}$, the family of groups such that for each $i$ there exists a sheaf $F_i$ on $C$, such that $G_i$ is the holonomy of $F_i$.

Given two sheaves $F_i$ and $F_j$, a morphism of sheaves $f_{ij} : F_i \rightarrow F_j$ induces a morphism between the group $G_i$ and $G_j$. In this case, we will say that $G_i \leq G_j$ if the morphism $f_{ij}$ is a surjective morphism.

We thus define a projective system of groups $(G_i)_{i \in I}$ with projective completion is the projective fundamental group of the topos $C$. We denote it by $\text{pro}\text{fund}_1(C)$.

For a locally connected topos, we can define the profoundamental group of each of its connected component.

Definition 1.11.
We will say that a locally connected, and connected topos $C$ is simply connected if and only if each locally constant sheaf on $C$ is a constant sheaf, otherwise, this means that the profoundamental group of $C$ is trivial.

We will say that $C$ is a locally simply connected topos if for each topological covering family $(X_i)_{i \in I}$ of $C$, there exists a sub family $(Y_j)_{j \in J}$ such that $Y_j$ is simply connected.

Remark.
For a locally connected and locally simply connected topos $C$, the profoundamental group of $C$ is a called the fundamental group which has the following description:
Let \((X_i)_{i \in I}\) be a topological covering family of \(C\), such that for each \(i\), \(X_i\) is connected and simply connected.

We denote by \(G\) the intersection of the kernel of all representation \(Aut(i) \to hol_P(X_i)\), the fundamental group is the quotient of \(Aut(i)\) by \(G\).

2. Geometric structures in topos theory.

The main goal of this part is to extend the notion of geometric structures to the notion of topos.

**Definition 2.1.**

Let \(C\) be a topos, and \(G\) a subgroup of automorphisms of \(C\), we will say that \(G\) satisfies the unique extension property (uep) if two elements of \(G\) which coincide on the category over an object of \(C\), agree on \(C\).

In the sequel, topoi considered, will be locally connected topoi, and we will assume that they have a finite number of connected components.

**Definition 2.2.**

Let \(C\) be a topos, and \(G\) a group of automorphisms of \(C\) whose elements verify the unique extension property. We will say that a topos \(D\) is a \((C,G)\) topos if and only if there exists a topological covering family \((X_i)_{i \in I}\) of \(D\) such that for each \(X_i\), there exists a local isomorphism

\[\phi_i : X_i \to C\]

this means that there exists an object \(Y_i\) of \(C\), such that \(\phi_i\) factor through an isomorphism \(\phi'_i : X_i \to Y_i\).

Suppose that \(X_i \times_Z X_j\) is different from the initial object, (recall that \(Z\) is the final object) then there exists and element \(g_{ij}\) of \(G\) such that

\[g_{ij}(\phi_j)_{X_i \times Z X_j} = \phi_{X_i \times X_j}\]

where \((\phi_i)_{X_i \times Z X_j}\) (resp. \((\phi_j)_{X_i \times Z X_j}\)) is the restriction of \(\phi_i\) (resp. \(\phi_j\)) to the category above \(X_i \times_Z X_j\).

We have

\[g_{jk}(\phi_k)_{X_i \times Z X_j \times Z X_k} = (\phi_j)_{X_i \times Z X_j \times Z X_k},\]

\[g_{ij}(\phi_j)_{X_i \times Z X_j \times Z X_k} = (\phi_i)_{X_i \times Z X_j \times Z X_k},\]

We deduce that

\[g_{ij}g_{jk}(\phi_k)_{X_i \times Z X_j \times Z X_k} = (\phi_i)_{X_i \times Z X_j \times Z X_k}.\]

The unique extension property implies that

\[(1) \quad g_{ij}g_{jk} = g_{ik}\]
The relation (1) allows us to define a locally constant sheaf $F$ on $D$, such that the restriction of $F$ to the category above $X_i$ is the constant sheaf $G$, for each object $X$ of $D$, $F(X)$ is the kernel of

$$\prod F(X_i) \rightarrow \prod F(X_i \times_X X_j).$$

The transitions morphisms of $F$ are induced by the elements $g_{ij}$ of $G$.

Let $D_0$ be a connected component of $D$, we deduce from the universal property of $\text{pro}\pi_1(D_0)$, a representation:

$$\text{hol}_{D_0,C,G} : \text{pro}\pi_1(D_0) \rightarrow G$$

given by the projection

$$\text{pro}\pi_1(D_0) \rightarrow \text{hol}_F(X_i).$$

This representation is called the holonomy representation of the $(C,G)$ structure of $D$.

The representations deduced from $X_i$ and $X_j$ are conjugated, so the conjugacy class of the holonomy representation does not depend of $X_i$.

**Definition 2.3.**

Let $D$ and $D'$ be two $(C,G)$ topoi, we will say that a local isomorphism $f : D \rightarrow D'$ is a $(C,G)$ morphism if and only if

$$\psi_{i(j)} f|_{X_i} = k_{ij} \phi_i,$$

where $k_{ij}$ is an element of $G$, $((X_i)_{i \in I}, \phi_i)$ and $((Y_j)_{j \in J}, \psi_j)$ are two topological covering families which define respectively the $(C,G)$ structures of $D$ and $D'$, and $f|_{X_i}$ is the restriction of $f$ to the category above $X_i$. Moreover we suppose that $f|_{X_i}$ factors through an sub object $Y_{i(j)}$ of $D'$. We thus endowed the class of $(C,G)$ structures to a structure of category.

Since the existence of a universal cover of the topos $D$ is not sure, we replace it by the sheaf $H_S$ of local $(C,G)$ morphisms from $D$ to $C$, in order to define a developing map.

To $H_S$ we can associate the category whose objects are couples $(X,s)$ where $X$ is an object of $D$ and $s$ is an element of $H_S(X)$. A map between two objects $(X,s)$ and $(Y,s')$ is a morphism $f : X \rightarrow Y$ such that $H_S(f)(s') = s$. This category $H$ is a topos.

We can define the morphism:

$$\text{Dev} : H \rightarrow C$$

defined on the category above $(X,s)$ by $s$.

The morphism is called the developing map of the $(C,G)$ structure of $D$.

We consider now $C$ (resp. $C'$) a topos $C$, and a group $G$ (resp. $G'$) of automorphisms of $C$ (resp. $C'$) whose elements verify the unique extension
property. Moreover, we suppose that we have a local isomorphism of topoi \( \phi : C \to C' \), and a morphism of groups \( \Phi : G \to G' \) such that for each element \( g \) of \( G \), we have \( \phi \circ g = \Phi(g) \circ \phi \).

To each \((C,G)\) topos defined by the covering family \((X_i)_{i \in I}\), we can associate the \((C',G')\) topos defined by the family \((\phi_i \circ \phi)\).

We thus define a functor \( \phi^* \) from the category of \((C,G)\) topoi to the category of \((C',G')\) topoi.

3. The space of \((C,G)\) structure.

We consider a topos \( D \) endowed with a \((C,G)\) structure. Let \((X_i)_{i \in I}\) be the covering family which defines this structure. Let consider the topos \( S((X_i)_{i \in I}) \) whose final object is obtained by gluing the product of topoi \( X_i \times C \) by the relation

\[
\tilde{g}_{ij} : U_i \times Z \to U_j \times C
\]

induced by the transitions functions \( g_{ij} \). \( S((X_i)_{i \in I}) \) is called the structural topos of the \((C,G)\) structure of \( D \).

The family of projections \( p_i : U_i \times C \to D \) define a projection \( p : S((X_i)_{i \in I}) \to D \), since the transition functions induce the identity on the first factor.

Definition 1.3.
A morphism \( s : D \to S((X_i)_{i \in I}) \) such that \( p \circ s = Id \) will be called a transverse section of the topos \( S((X_i)_{i \in I}) \).

A section \( s : D \to S((X_i)_{i \in I}) \) is transverse if and only if \( p_i \circ s_i \) is the identity, where \( s_i \) is the restriction of \( s \) to the category over \( X_i \).

A \((C,G)\) topos on \( D \) is defined by a transverse section \( s_0 \). Conversely, we have the fact:

Proposition 3.2.
A transverse section to the topos \( S((X_i)_{i \in I}) \) defines a \((C,G)\) structure on \( D \).

The space of \((C,G)\) structures on the topos \( D \) can also be describe as the set of the following triples:

A \((C,G)\) structure on the topos \( D \), defined by the trivializing family \((X_i)_{i \in I}\),
- we fixe a chart \((X_{i_0}, \phi_{i_0})\)
- an isomorphism of topoi \( D \to D' \).

We denote \( S(D,C,G) \) the set of \((C,G)\) structures of \( D \).
We have an application:

\[
S(D,C,G) \to Hol(pro\pi_1(D),G)
\]

which associates to each element of \( S(D,C,G) \) its holonomy representation.

4. Applications to algebraic geometry.

We consider a scheme \( S \) define on a field \( k \), \( G \) a group of automorphisms of \( S \) which verifies the unique extension property. We can suppose for instance
that $S$ is irreducible, and $G$ a subgroup of isomorphisms of $S$. It acts also on the Etale sieve $Et_S$ of $S$. We can define a notion of $(S, G)$ schemes in the category of schemes defined over $k$.

An $(S, G)$ etale scheme is a scheme $T$ such that its Etale sieve $Et_T$ is endowed with and $(S, G)$ structure. This means that there exists an etale covering $(U_i)_{i \in I}$ of $T$, etale morphisms

$$f_i : Et_{U_i} \longrightarrow Et_S$$

such that there exists a morphism $g_{ij}$ of $G$ such that:

$$f_i|_{U_i \times_T U_j} = g_{ij}f_j|_{U_i \times_T U_j}.$$

Let $\hat{T} \rightarrow T$ be an etale covering. We denote by $\pi_{\hat{T}}$ the group of Deck transformations of this covering map. We can lift the $(S, G)$ structure on $\hat{T}$, by setting $\hat{f}_i : U_i \times_{\hat{T}} \hat{T} \rightarrow S = f_i \circ p_{U_i}$, where $p_{U_i}$ is the first projection $U_i \times_T T \rightarrow U_i$.

Suppose that the holonomy group of the $(S, G)$ structure of $T$ is finite, its define a sheaf $F$ with finite fiber over $T$. We deduce from the theory of fundamental group that there exists a finite cover $\hat{T}$ such that the pulls back of $F$ on $\hat{T}$ is trivial. We called $\hat{T}$ a holonomy cover of the $(S, G)$ structure of $T$.

Consider an $(S, G)$ structure defined on the scheme $T$ with finite holonomy group $h(T)$, we denote $\hat{T}$ the finite covering space corresponding to $h(T)$. For a fixed $\hat{T}$, the set of deformations of the $(S, G)$ structures is the set of fibered spaces $D(T, \hat{T}, S, G)$ which are quotient defined by the relations:

$$\hat{T} \times S \longrightarrow \hat{T} \times S$$

$$(x, y) \rightarrow (x, hol(T)(\gamma)(y))$$

where $hol(T)$ is the holonomy representation of an $(S, G)$ structure of $T$.

For such a bundle, we have an horizontal foliation $\mathcal{F}$ which is the pull forward of the foliation of $\hat{T} \times S$ whose leaves are the subschemes $\hat{T} \times y$ where is a point of $S$. 

\[9\]
A section $s$ transverse to this foliation defines the $(S, G)$ structure of $T$, since it lifts to a section $\hat{s} : \hat{T} \to \hat{T} \times S$.

If $T'$ is another finite covering space of $T$ such that there exists an etale morphism $\hat{T} \to T'$, we have a map $D(T, T', S, G) \to D(T, \hat{T}, S, G)$. The inductive limit of the space $D(T, \hat{T}, S, G)$ is the classifying space of the $(S, G)$ structure of $T$. We denote it by $D(T, S, G)$.

Suppose now that the schemes are defined on $\mathcal{C}$, we endowed the group $G$ with the compact-open topology induced by the analytic topology of $S$. We can now study the deformation space of the representation $\rho : \text{hol}(T) \to G$.

**Proposition 4.1.**

Let $(\rho_t)_{t \in I}$ be a deformation of the representation $\rho = \rho_0$ of the holonomy of an $(S, G)$ structure of $T$ where $I$ is an open interval, there exists an open interval $J$ which contains $0$ such that for every $t$ in $I$, the groups $\rho_0(\text{hol}(T))$ and $\rho_t(\text{hol}(T))$ are isomorphic.

**Proof.**

Let $g_1, ..., g_n$ be the elements of $\text{hol}(T)$. Consider the set $X$ of elements of $S$ such that the cardinal of the family $(g_1(x), ..., g_n(x))$ is $n$. The set $X$ is a non empty set. There exists open sets $U_1, ..., U_n$ disjoint each other containing respectively $g_1(x), ..., g_n(x)$ for the induced analytic topology, an open interval $J$ contained in $I$, such that $\rho_t(g_i(x))$ is an element of $V_i$ for $i = 1, ..., n$, where $V_i$ is an open set contained in $U_i$, moreover if $g_ig_k(x)$ is an element of $V_i$, then $\rho_t(g_ig_k(x))$ is also an element of $V_i$ for $t$ in $J$. We deduce that the map

$$\rho_t(\text{hol}(T)) \to \rho(\text{hol}(T))$$

$$\rho_t(g_i) \longrightarrow g_i$$

is an isomorphism.

**Theorem 4.2.**

Let $S$ and $T$ be two $\mathcal{C}$-schemes, compact for the analytic topology, then the image of the map $D(T, \hat{T}, S, G) \to \text{Hom}(\text{hol}(T), G)$ is open.

**Proof.**

Under the conditions of the theorem, for each deformation $(\rho_t)_{t \in I}$ of the holonomy representation of $T$, there exists an interval $J$ contained in $I$ such that the groups $\rho(\text{hol}(T))$ and $\rho_t(\text{hol}(T))$ are isomorphic. The flat bundle $D(T, T, S, G)$ and the flat bundle induced by the representation $\rho_t$ for $t \in J$ are isomorphic. We identified them. We denote by $\mathcal{F}_t$ the horizontal foliation of the flat bundle induced by $\rho_t$, $t \in J$. For $t$ small, this foliation remains transverse to the vertical foliation. It implies that $\rho_t$ is the holonomy of a $(S, G)$ structure of $T$.

**The Beyli theorem.**

Let $\mathbb{P}$ be the projective line defined over the rational numbers, it has been shown by Beyli that finite covers of $C_0 = \mathbb{P} - \{0, 1, \infty\}$ are algebraic curves
defined by equations with algebraic coefficients. If we denote by \( G \) the group of automorphisms of \( C_0 \), we can restate this result as follows:

**Theorem 4.3.**
An algebraic curve has a complete \((C_0, G)\) structure which holonomy is finite if and only if it has a finite cover which is an algebraic curve defined by algebraic coefficient.

The interest of the set of finite covers of \( C_0 \) is emphasized by the fact that the Galois group \( \text{Gal}(\mathbf{Q}/\mathbf{Q}) \) acts naturally on it by acting on coefficients of algebraic polynomials. Since the structure of \( \text{Gal}(\mathbf{Q}/\mathbf{Q}) \) is not well-known, the previous action is a tool which allow to study this group.

We will define a natural action of \( \text{Gal}(\mathbf{Q}/\mathbf{Q}) \) on the set of \((C_0, G)\) structures. This will give another tool to study this group.

Let \( \mathbf{C}_0 \) be the \( \mathbf{Q} \)-projective line without 3-points, and \( G_{\mathbf{C}_0} \) a subgroup of its automorphisms group. Consider a curve \( C \) defined over \( \mathbf{C}_0 \) endowed with an \((\mathbf{C}_0, G_{\mathbf{C}_0})\) structure. This structure is defined by an etale covering family \((U_i \rightarrow C)\) of \( C \), and morphisms \( f_i : U_i \rightarrow V_i \), where \( V_i \rightarrow C_{\mathbf{C}_0} \) is an etale morphism. The group \( \text{Gal}(\mathbf{C}/\mathbf{C}) \) acts naturally on the space of curves defined over \( \mathbf{C}_0 \), one of its element \( \sigma \), sends a curve \( C \) defined by a polynomial which coefficients are algebraic, to the curve \( C \sigma \) defined by \( P \sigma \). For an element \( \sigma \) of \( \text{Gal}(\mathbf{Q}/\mathbf{Q}) \), we will denote by \( C \sigma \) the image of \( C \) by \( \sigma \). The action of \( \sigma \) induces a map \( f_i \sigma : U_i \sigma \rightarrow V_i \sigma \), we define an \((\mathbf{C}_0, G_{\mathbf{C}_0})\) structure on \( C \). Remark that we can have \( C \sigma = C \), but the \((\mathbf{C}_0, G_{\mathbf{C}_0})\) are not fixed by \( \sigma \).

The action of the Galois group \( \text{Gl}(\mathbf{Q}/\mathbf{Q}) \) on the fundamental groupoid of \( C_0 \).

We can define the topology \( S_{C_0} \) generated by the family of finite covers maps \( C \rightarrow C_0 \). We have seen that a finite cover \( C \) of \( C_0 \) is an algebraic curve, define by polynomial which coefficients are algebraic. Let \( \sigma \) be an element of \( \text{Gl}(\mathbf{Q}/\mathbf{Q}) \), \( \sigma \) acts on finite covers of \( C_0 \). Consider a path \((C_1, ..., C_n)\) which represents an element of the groupoid \( \text{Gr}(C_0) \) where the index family is the set of finite cover of \( C_0 \), we define the action of \( \sigma \) on \( \text{Gr}(C_0) \) by

\[
\sigma(C_1, ..., C_n) = (C_1 \sigma, ..., C_n \sigma)
\]

For each locally constant sheaf \( F \) on \( S_{C_0} \), we can define the sheaf \( F \sigma \) defined by \( F \sigma(C) = F(C \sigma) \). The action of \( \sigma \) on \( \text{Gr}(C_0) \) induces an automorphism \( \bar{\sigma} \) on the holonomy groupoid \( G_F \) of \( F \). This action is compatible with morphisms between locally constant sheaves. We thus deduce an action of \( \sigma \) on the fundamental groupoid of \( S_{C_0} \). If \( C \) is defined by polynomials with rational coefficients, this action induces an action of \( \text{Gal}(\mathbf{Q}/\mathbf{Q}) \) on the fundamental group of \( S_{C_0} \) in \( C \).

**The case of field theory.**

We will now apply the previous study to field Theory. Let \( k \) be a field, \( E \) an extension of \( k \). We denote by \( G \) the Galois group of this extension. The unique
extension property is satisfied by the Galois group $G(E : k)$ acting on $\text{Spec}(E)$, endowed with its Zariski topology, since the unique non empty set of $\text{Spec}(E)$, is $\text{Spec}(E)$.

Recall that an etale cover of $\text{Spec}(E)$, is the spectrum of a finite product of separable extensions $(k_i)_{i \in I}$ of $E$. The Galois group of the etale cover is the product of the Galois groups of the extensions $E \rightarrow k_i$.

We want to classify finite products of finite separable extension of $k$ that we call $k$–separable algebras, which are endowed with $(E,G)$ structures. Let $F = k_1 \times \ldots \times k_n$ be such a $k$–separable algebra, an etale covering of $\text{Spec}(F)$ is given by $\text{Spec}(F')$, where $F'$ is a finite product of separable extensions of $k$ which is an extension of $F$. Let $k_S$, be the separable closure of $k$. The family of separable extensions $(H_i = (k_i^1 \times \ldots \times k_i^n)_{i \in I}$ of $F$, is a covering family if and only if the $k_S$ is the inductive limit of $(k_i^1)$. This is equivalent to saying that $\text{Gal}(k_S : F)$ is the projective limit of the family $(\text{Gal}(k_S : k_i^1))$.

Thus the $(S,G)$ structure on $\text{Spec}(F)$ is defined by a family $(F_i)_{i \in I}$ of finite product of separable extensions of $k$ such that $\text{Gal}(k_S : k_i^1)$ is the projective limit of the family $\text{Gal}(F_S : k_i^1)$, and such that for each $i \in I$, there exists a morphism $D_i' : \text{Et}_{F_i} \rightarrow \text{Et}_E$.

The Chasles relation $g_{ij}D_{ij}'_{\text{Et}(\text{Spec}(F_i) \times \text{Spec}(F_j) = D_{ij}'_{\text{Et}(\text{Spec}(F_i) \times \text{Spec}(F_j)^\prime}$, a morphism $g_{ij}$ of $G$ which satisfies $g_{ij}D_{ij}'_{\text{Et}(\text{Spec}(F_i) \times \text{Spec}(F_j)} = D_{ij}'_{\text{Et}(\text{Spec}(F_i) \times \text{Spec}(F_j)^\prime}$.

We will consider only morphisms induced by an etale map $D_i' : \text{Spec}(F_i) \rightarrow \text{Spec}(E_i)$. Such a morphism is induced by an separable extension $D_i : E_i \rightarrow F_i$.

Thus the family $(g_{ij})$ define a flat sheaf $S(E,G)$ on the etale site of $\text{Spec}(F)$, with holonomy representation

$$\text{hol} : \text{Gal}(F_S : F) \rightarrow \text{Gal}(E : k)$$

**Proposition.**

Endow a field $F$ with an $(S,G)$ structure defined by the family $(F_i)_{i \in I}$, suppose that each $F$–algebra $F_i$ is a field, then there exists a separable extension $F_{S,G}$ of $F$, such that every field $F_i$ is a subfield of $F_{S,G}$, we denote by $i$ the canonical map $F_i \rightarrow F_{S,G}$, there exists a map $D : E \rightarrow F_{S,G}$, and a representation $\text{hol} : \text{Gal}(F_{S,G} : F) \rightarrow \text{Gal}(E : k)$. We have $i \circ D = D_i$. The fields $F_{S,G}$ is universal in the following sense: If $K$, is a field such that there exist a map $D_K : E \rightarrow K$, a map $i_K : F_i \rightarrow K$ such that $i_K \circ D_K = D_i$, then there exist a map $f : F_{S,G} \rightarrow K$, such that $D_K = f \circ D$.

**Proof.**

The field $F_{S,G}$ is the Artin cover which trivializes the bundle defined by the holonomy of the $(E,G)$ structure of $F$.

**Remark.**
An \((E,G)\) structures allow us to construct representations of galois groups of separable extensions of \(F\) to \(k\)–vector spaces.

**Definition.**
Let \(F\) (resp. \(F')\), be a \(k\)–separable algebra endowed with the respective \((E,G)\) structure defined by the etale covering \((\text{Spec}(F_i))_{i \in I}\) (resp. \((\text{Spec}(F'_i))_{i' \in I'}\), and the family of maps \(f_i : \text{Spec}(F_i) \to \text{Spec}(E_i)\), (resp. \(f'_i : \text{Spec}(F'_i) \to \text{Spec}(E'_i)\)).

An \((E,G)\) morphism between \(F\) and \(F'\) is an etale morphism \(h : \text{Spec}(F) \to \text{Spec}(F')\) induced by a family of maps \(h_i : \text{Spec}(F_i) \to \text{Spec}(F'_i)\), such that there exists a map \(g_{ij}\) of \(G\) with the property: 
\[
f'_j \circ h_i \circ u_i = g_{ij} \circ f'_j \circ h_j \circ u_j,
\]
where \(u_i\) is the canonical map \(\text{Spec}(F_i) \times \text{Spec}(F_j) \to \text{Spec}(F_i)\), \(h_i\) is the lift of \(h\) to \(\text{Spec}(F_i)\), and \(g_{ij}\) is an element of \(G\).

We denote by \(\text{Aut}(F,E,G)\) the group of \((E,G)\) maps of \(F\).

**The complete structures.**
Recall that an \((E,G)\) structure on the field \(F\) is said to be complete if the developing map is a covering map.

**Proposition.**
Endow a field \(F\) with an \((E,G)\) structure, suppose that the \((E,G)\) structure is defined by fields \((F_i)_{i \in I}\), the structure is complete if and only if \(E_{S,G}\) is a separable extension of \(E\). The developing map is an isomorphism if and only if and \(E\) is isomorphic to the separable of \(F\).

**Proof.**
The \((E,G)\) structure of \(F\) is defined by a family of maps \(E \to E_i \to F_i\) where \(E_i\) and \(F_i\) are respectively separable extensions of \(E\) and \(F\). This structure is complete if and only if \(\lim(F_i) = \text{the separable closure of } F\) is a separable extension of \(E\). This implies that the developing map is an isomorphism if and only if \(F\) is separately closed and \(E\) is isomorphic to \(F\).

On the other hand we have the following:

**Proposition.**
Suppose that \(F\) is a separable extension of the field \(E\), then \(F\) is endowed with a complete \((E,G)\) structure.

Now suppose that the holonomy group \(\text{hol}(Gal(F_k : F))\) is finite and the structure is complete, we have seen that this implies that \(E\) is a separable extension of \(F\). If the characteristic of \(k\) is zero, and \(G = Gal(E \mid k)\), every subfield \(F\) of \(E\), such that \(E\) is a finite extension of \(F\) endows \(E\) with with a complete \((F,G_F)\) structure because every extension of \(k\) is separable.

We can restate the fundamental theorem of the Galois theory as follows:

**Proposition.**
Let \(E\) be a Galois extension of \(k\) with galois group \(G\), then the complete \((F,G_F)\) structures on \(E\) which holonomies groups are subgroup of \(G\), is one to one with the subgroups of \(G(E : k)\)?
Operations on \((E,G)\) structures.

The change of basis.

Let \(E \rightarrow E'\) a morphism of \(k\)-fields, and \(G\) and \(G'\) be the Galois groups of \(E\) and \(E'\) over \(k\). Suppose moreover that \(E'\) is separable over \(E\), then to any \((E',G')\) defined on the field \(F\) by the family of \(F\)-algebras \((F_i^t)_{t \in I}\), and the family of maps \(f : \phi_i : E_i \rightarrow F_i^t\), we can assign an \((E,G)\) structure defined by the same data, since in this case separable extensions of \(E'\) are also separable extensions of \(E\).

We thus obtain a functor

\[
f^* : S(E',G') \rightarrow S(E,G).
\]

Conversely, consider an \((E,G)\) structure defined on the field \(F\), by the family of \(F\)-separable algebras \((F_i)_{t \in I}\), and the family of \(E\)-separable algebras \((E_i)_{t \in I}\), the maps \(\phi : E_i \rightarrow F_i\), gives rise to maps \(\phi'_i : E_i \rightarrow F_i^t\), remark that the fields \(E_i E'\) and \(F_i E'\) are respectively separable extensions of \(E'\), and \(FE'\).

Let \(g_{ij} : E_i \otimes E_j \rightarrow E_i \otimes E_j\) be the coordinates change of the \((E,G)\) structure on \(F\). The coordinates change of the \((E',G')\) structure on \(FE'\) are defined by \(h_{ij} : E_i E' \otimes E_j E' = (E_i \otimes E_j)E' \rightarrow E_i E' \otimes E_j E' = (E_i \otimes E_j)E'\), where \(h_{ij} = g_{ij}1_{d_{E'}}\).

we obtain an \((E',G')\) structure on \(FE'\), we have just define a functor

\[
f_* : S(E,G) \rightarrow S(E',G').
\]

**Proposition.**

Let \(s\) be an \((E',G')\) structure defined on the field \(F\), the \(S(E,G)\) structure \(f_* f^*(s)\) is isomorphic to \(s\).

**Proof.**

This result from the trivial fact that for each extension \(E' \rightarrow F'\), the field \(E'F'\) is canonical isomorphic to \(F'\).

**Proposition.**

The functor \(f_*\) is the left adjoint of the functor \(f^*\), that is we have an isomorphism between \(\text{Hom}_{(E',G')}(f_*(s), t)\) and \(\text{Hom}_{(E,G)}(s, f^*(t))\) where \(s\) and \(t\) are respectively an \((E,G)\) and an \((E',G')\) structure.

**Proof.**

The forgetful functor \(h'\) from the category \(\text{Ext}(E')\) of extensions of \(E'\) to the category \(\text{Ext}(E)\) of extensions of \(E\) is right adjoint to the functor \(h\) from \(\text{Ext}(E)\) to \(\text{Ext}(E')\) which assigns to \(F\), the field \(FE'\). This implies the proposition.

Let \(F\) and \(F'\) be two \(k\)-algebras, endowed with the respective \((E,G)\) structures \(A\) and \(B\) defined respectively by the families of separable extensions \((F_i)_{t \in I}\) and \((F_j^t)_{t \in J}\) of \(F\) and \(F'\). The \(k\)-algebras \(E_i \otimes E_j\) and \(F_i \otimes F_j\) are respectively separable extensions of \(E\) and \(F \otimes F'\). The maps \(D_i : E_i \rightarrow F_i\),
and $D_j : E_j \rightarrow F_j$, define a map $D_{i \otimes j} : E_i \otimes E_j \rightarrow F_i \otimes F_j$, thus it defines an $(E, G)$ structure denoted $A \otimes B$.

We have just endowed the category of $(E, G)$ $k-$algebras with a tensor product.

We will denote by $I$ be the $(E, G)$ structure defined on $E$ by the identity of the etale site of $E$.

Suppose now that the field $k$ is $\mathbb{Q}$, and $E$ is a product of finite extensions of $\mathbb{Q}$. Let $F$, be a $\mathbb{Q}-$algebra endowed with an $(E, G)$ structure, defined by the respective covering families $(F_i)_{i \in I}$, and $(E_i)_{i \in I}$ of $F$ and $E$. Let $\sigma$ be an element of $Gal(\mathbb{Q}/\mathbb{Q})$. We will denote by $L^\sigma$ the image of a $\mathbb{Q}-$algebra $L$ by $\sigma$. The $(E, G)$ structure of $F$ is defined by a family of maps $f_i : E_i \rightarrow F_i$, the action of $\sigma$ on $\mathbb{Q}$, induces an action $f_i^{\sigma} : E_i^{\sigma} \rightarrow F_i^{\sigma}$. This action defined and $(E^\sigma, G^\sigma)$ structure on $F^\sigma$, where $G^\sigma = \sigma G \sigma^{-1}$. Remark that we can have $F^\sigma = F$ but the $(E, G)$ of $F$ are not fixed by $\sigma$.

**Applications.**

Let $E$ be an algebraic extension of a field $k$, and $G$ be a subgroup of the Galois group of $E$. We want first to answer the following question, given an algebraic extension $F$ of $k$, is there an $(E, G)$ structure on $F$. We will find a non commutative two cocycle with take $H^2(Et_F, L)$ which is the obstruction to solving this problem, where $L$ is a sheaf on $Et_F$.

Recall that, given an $(E, G)$ structure on the field $F$, is defined by maps $E_i \rightarrow F_i$ where $E_i$ and $F_i$ are respectively finite products of separable extensions of $E$ and $F$.

We can remark that the map $E_i \rightarrow F_i$ gives rise to an $(E, G)$ structure over $Spec(F_i)$, defined as follow: the subfamily of etale covers $F_k$ of $F$ which are etale cover of $F_i$ is a covering family of $F_i$.

Now suppose that the characteristic of the field is zero, for every extension (separable) of $F$, there exists always an extension (separable) $H$ which contains $F$ and $E$. Thus we can define an $(E, G)$ structure on $H$ which assigns to every etale morphism $H \rightarrow K$, the map $E \rightarrow H \rightarrow K$. The category $C(H, E, G)$ of $(E, G)$ structures over $H$ is not empty.

we can define a sheaf of categories on $Et_F$ defines as follow:

To every etale covering space $H$ of $F$, we assigns the category $C(E, H, G)$.

This sheaf of categories is a stack. If the group of automorphisms of each element $H$ of $C(E, F, G)$ is isomorph to $L(H)$, where $L$ is a sheaf on $Et_F$, then this stack is a gerbe. The obstruction of the existence of an $(E, G)$ structure is then given by a 2–Cech cocycle.

Now, we will give a procedure to build examples of $(E, G)$ structures.

Consider an algebraic extension $E$ over $k$, with Galois group $G$. Let $E_S$ be a separable closure of $E$. Suppose that we have a representation $h : Gal(E_S : E) \rightarrow G$, and consider an element $c$ of $H^1(Gal(E_S : E), G)$ for this representation. With this cocycle, we can built a flat $L G-$bundle on $Et_E$. Now consider a trivialization of this bundle. It is defined by a family $(E_i)_{i \in I}$ of separable
extensions of $E$. We define the $(E,G)$ structure by etale morphism $E_i \to E_i$. 
The coordinates change will be given by the coordinates change of the bundle $L$. Obviously the holonomy of this $(E,G)$ structure is $h$.

Proposition.

If $c = 0$, then the $(E,G)$ structure that we have just defined is trivial.

Bibliography.

1. Deligne, P. Le groupe fondamental de la droite projective moins trois points. Math. Sci. Res. Inst. Publ, 16.
2. Giraud, J. Cohomologie non abélienne. Springer 1971.
3. Goldman, W. Geometric structures on affine manifolds and varieties of representations. Cont. Math, 74.
4. S.G.A.1 Séminaire dirigé par A.Grothendieck, Revêtement étale et groupe fondamental. Springer.
5. S.G.A.4 Séminaire dirigé par M, Artin, A. Grothendieck et J-L Verdier. Théorie des topos et cohomologie étale des schémas. Spinger 1972
6. Micali, A. Corps de fonctions algebriques, notas de matematicas. Instituto Matematica Nacional de Pesquisas. 1963