Twisted quadratic foldings of root systems
and liftings of Schubert classes

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Abstract

Given a finite crystallographic root system $\Phi$ whose Dynkin diagram has a non-trivial
automorphism, it yields a new root system $\Phi_\tau$ by a so-called classical folding. On the other
hand, Lusztig’s folding (1983) folds the root system of type $E_8$ to $H_4$ starting from an
automorphism of the root lattice of type $E_8$. The notion of a twisted quadratic folding
of a root system was introduced by Lanini–Zainoulline (2018) to describ e both the classical
foldings and Lusztig’s folding on the same footing. The structure algebra $Z(G)$ of the moment
graph $G$ associated with a finite root system and its reflection group $W$ is an algebra over
a certain polynomial ring $S$, whose underlying module is free with a distinguished basis
$\{\sigma^w | w \in W\}$ called combinatorial Schubert classes. By Lanini–Zainoulline (2018), a
twisted quadratic folding $\Phi \hookrightarrow \Phi_\tau$ induces an embedding of the respective Coxeter groups
$\varepsilon: W_\tau \hookrightarrow W$ and a ring homomorphism $\varepsilon^*: Z(G) \rightarrow Z(G_\tau)$ between the corresponding
structure algebras. This paper studies the $\varepsilon^*$-preimage of Schubert classes and in particular
provides a combinatorial criterion for a Schubert class $\sigma^\tau(\mathbf{u})$ of $Z(G_\tau)$ to admit a Schubert
class $\sigma^w(\mathbf{w})$ of $Z(G)$ such that the relation $\varepsilon^*(\sigma^w(\mathbf{w})) = c \cdot \sigma^\tau(\mathbf{u})$ holds for some nonzero scalar $c$.

1 Introduction

This paper builds on the connection of two major mathematical constructions, foldings of finite
root systems and the structure algebras of moment graphs associated with finite root systems.
These two were recently connected in the work of Lanini–Zainoulline [1].

A folding of a root system is a procedure long known in the literature to exploit the symmetry
of a given root system and yield a smaller root system of a different type (e.g. Steinberg [2]).
There are four families of classical foldings of finite crystallographic root systems: foldings from
type $A_{2n-1}$ to $C_n (n \geq 2)$, from type $D_{n+1}$ to $B_n (n \geq 3)$, from type $E_6$ to $F_4$, and from type
$D_4$ to $G_2$ (cf. Springer [3, Section 10]), in which each folding starts with an automorphism
of the Dynkin diagram of the given root system. The other well-known folding is the Lusztig
folding from type $E_8$ to $H_4$ discovered by Lusztig in [4, Section 3.9 (b)], in which the folding
begins with an automorphism of the root lattice of type $E_8$ not induced by an automorphism
of the Dynkin diagram (cf. Moody–Patera [5]). Recently, Lanini–Zainoulline [1] introduced the
notion of a twisted quadratic folding of a root system which explains the construction of both
classical quadratic foldings (those listed above except the one from type $D_4$ to $G_2$) and the
Lusztig folding on the same footing via the notion of a folded representation of a root system.
Let $K$ be a unital subring of $\mathbb{R}$, $L$ be a free quadratic $K$-algebra given by $L = K[x]/(p(x))$ for
some monic separable $K$-polynomial $p(x) = x^2 - c_1 x + c_2$, and $\tau$ be one of the two distinct roots
of $p(x)$ in $L$. A free $K$-module $U$ obtained from some free $L$-module $U$ by restriction of scalars
is equipped with a $K$-linear map $T$ given by multiplication by $\tau$ on $U$. A folded representation
realizes a finite crystallographic root system $\Phi$ as a subset of $U$ so that its root lattice is stable
under $\mathcal{T}$ and its simple roots admit a certain partition with respect to $\mathcal{T}$ ([1] Definition 3.1]). Then the $\tau$-twisted folding of $\Phi$ is the procedure to pass from $\Phi$ to a new finite root system $\Phi_\tau$ under the projection onto the $\tau$-eigenspace of $\mathcal{T}$. In this framework, the classical foldings are obtained by taking $p(x) = x^2 - 1$, and the Lusztig folding is obtained by taking $p(x) = x^2 - x - 1$ ([1] Examples 4.3–4.5]).

Our second objects of study are the notions of a structure algebra and its Schubert classes, stem from the study of equivariant cohomology rings of flag varieties and its generalization to arbitrary finite Coxeter groups. Let $G$ be a connected semisimple linear algebraic group over $\mathbb{C}$ with a maximal torus $T$, a Borel subgroup $B$ containing $T$, and the Weyl group $W$. Let $\Lambda$ be the character group of $T$ and $S := \text{Sym}_\mathbb{Z} \Lambda$ be the symmetric algebra. Then the structure algebra associated with $W$ is given by

$$Z(\mathcal{G}) = \{ \xi \in \bigoplus_{v \in W} S \mid \xi(v) - \xi(v') \text{ is divisible by } \alpha \in \Phi^+ \text{ if } v <_\alpha v' \},$$

where $\xi(v)$ denotes the $v$-coordinate of $\xi$ for each $v \in W$, $<_\alpha$ denotes the Bruhat cover relation on $W$ and $\mathcal{G}$ is a labelled graph called a moment graph (Fiebig [6] Sections 3), in which this case coincides with the Bruhat graph of $W$. The structure algebra $Z(\mathcal{G})$ is an $S$-algebra under the coordinate-wise addition and multiplication, and by a famous result of Goresky–Kottwitz–MacPherson [7, Theorem 7.2], the $T$-equivariant cohomology ring $H^*_T(G/B, \mathbb{Z})$ is isomorphic to $Z(\mathcal{G})$ as $S$-algebras via the so-called localization map (see also [8, Theorem 8.11]). The notions of a moment graph and its structure algebra can be extended to any finite real reflection groups by an appropriate choice of $S$.

It is well-known that the ordinary cohomology ring $H^*(G/B, \mathbb{R})$ has two descriptions, one via the additive basis called Schubert classes ([9] and the other via the so-called coinvariant algebra $S/\Delta^W$ ([10]), where $\Delta^W$ is the ideal generated by $W$-invariant polynomials with no constant terms. In their independent work, Bernstein–Gelfand–Gelfand [11] and Demazure [12] constructed polynomials in the coinvariant algebra which correspond to the Schubert classes of $H^*(G/B)$ under Borel’s isomorphism. Hiller [13] proved that the polynomials constructed in [11] form an additive basis of the coinvariant algebra for any finite real reflection group $W$. Kaji [14] extended Hiller’s result to the equivariant setting, constructing additive basis elements in the double coinvariant algebra $S \otimes_{S^W} S$ for any finite real reflection group $W$, which in case $W$ is crystallographic correspond to the Schubert classes of $H^*_T(G/B)$ under the Borel-type isomorphism from $H^*_T(G/B)$ to $S \otimes_{S^W} S$ ([14] Proposition 4.1]). He further showed an isomorphism from the double coinvariant algebra to the structure algebra $Z(\mathcal{G})$ ([14] Corollary 4.10]), thereby giving an explicit formula ([1] for the images $\sigma^w(v) (w \in W)$ of his double Schubert polynomials under this isomorphism: for $w, v \in W$ and $v = s_1 \cdots s_\ell$ any reduced expression,

$$\sigma^w(v) = \sum \prod_{p=1}^{l(w)} r_{i_p}(\alpha_{i_p}),$$

where the sum runs over all $1 \leq i_1 < \cdots < i_{l(w)} \leq \ell$ such that $s_{i_1} \cdots s_{i_{l(w)}} = w$, $\alpha_j$ is a simple root corresponding to $s_j$, and $r_{i_p}(\alpha_{i_p}) = s_1 s_2 \cdots s_{i_p - 1}(\alpha_{i_p})$.

Let $\Phi \leadsto \Phi_\tau$ be a twisted quadratic folding of a finite crystallographic root system. Let $W$ and $W_\tau$ be the finite reflection groups of $\Phi$ and $\Phi_\tau$ respectively. The folding $\Phi \leadsto \Phi_\tau$ induces an embedding $\varepsilon : W_\tau \hookrightarrow W$ of Coxeter groups (cf. Lusztig [1] Corollary 3.3]). If we denote the moment graph associated with the original root system $\Phi$ by $\mathcal{G}$ and the one associated with the $\tau$-folded root system $\Phi_\tau$ by $\mathcal{G}_\tau$, the main result of Lanini–Zainoulline [1] Theorem 6.2] states that there is a ring homomorphism $\varepsilon^* : Z(\mathcal{G}) \to Z(\mathcal{G}_\tau)$ induced from the embedding $\varepsilon$. This paper studies the Schubert calculus of the folding map $\varepsilon^*$. We take ([1] as definition of a Schubert class of $Z(\mathcal{G})$ (Definition 4.1), and investigate how the Schubert classes of the original structure algebra $Z(\mathcal{G})$ relate to those of the folded structure algebra $Z(\mathcal{G}_\tau)$ under $\varepsilon^*$. As a first
step to understand the $\varepsilon^*$-preimages of the Schubert classes of $Z(G_\tau)$, we find an answer to the following question.

**Question 1.1.** Which Schubert class $\sigma^\tau_w \in Z(G_\tau)$ admits a Schubert class $\sigma^w \in Z(G)$ satisfying the following condition?

$$\varepsilon^*(\sigma^w) = c \cdot \sigma^\tau_w \text{ for some } c \in L^\times. \quad (2)$$

**Definition 1.2.** We say that a Schubert class $\sigma^\tau_w$ is liftable or admits a lifting if there exists some $\sigma^w \in Z(G)$ satisfying (2). Such a Schubert class $\sigma^w$ is called a lifting of $\sigma^\tau_w$, and we say that $\sigma^\tau_w$ lifts to $\sigma^w$.

Our interesting discovery is that the condition for a given Schubert class $\sigma^\tau_w$ of $Z(G_\tau)$ to admit a lifting in $Z(G)$ involves only the existence of an element $w \in W$ satisfying certain combinatorial conditions in relation to $w \in W_\tau$. More precisely, we introduce a map $\varphi : S \to S_\tau$ on the Coxeter generating sets of $W$ and $W_\tau$ (Section 6), and define the folding subset $W^F$ of $W$ in terms of how $\varphi$ behaves on the reduced expressions of each element $w \in W$ (Definition 1.2). Our main result is the following.

**Theorem 1.3** (Corollary 1.5 Proposition 1.10). Let $u \in W_\tau$ and $w \in W$. Let $w = s_{i_1} \cdots s_{i_\ell}$ be any $W$-reduced expression. Then the Schubert class $\sigma^w \in Z(G)$ is a lifting of $\sigma^\tau_w \in Z(G_\tau)$ if and only if $w \in W^F$ and $u = \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$. Moreover, the coefficient $c$ in (2) equals $\tau^m$ for some unique nonnegative integer $m$, where $\tau$ is the parameter appearing in the definition of the underlying twisted quadratic folding.

If every Schubert class of $Z(G_\tau)$ admits a lifting, then we say that $\varepsilon^*$ has the lifting property. In general, $\varepsilon^*$ does not have this property. Even in the smallest examples of a twisted quadratic folding from type $A_3$ to $C_2$ or from type $A_4$ to $H_2$, the Schubert class of $Z(G_\tau)$ corresponding to the longest element $w_0$ of $W_\tau$ is not liftable (cf. Example 1.12). Our criterion for nonliftable (Corollary 1.5) suggests that as the size of the folded root system increases, there will be more and more nonliftable Schubert classes in $Z(G_\tau)$. However, by passing to the parabolic setting, in some cases, it is possible to choose a parabolic subgroup $W_P \subseteq W$ so that the folding map $\varepsilon^* : Z(G^P) \to Z(G\tau^P)$ does have the lifting property (Example 1.13). As an application of Theorem 1.3 in Section 9 we provide a list of parabolic subgroups $W_P$ such that the folding map $\varepsilon^* : Z(G^P) \to Z(G\tau^P)$ has the lifting property (Proposition 9.1).

The paper is organized as follows. In Section 2 we recall the notions of a root datum and a root system and some facts about finite Coxeter groups essential to the later discussion. In Sections 3–4 we introduce the notions of a parabolic moment graph and its structure algebra, then discuss a distinguished basis of the structure algebra called (combinatorial) Schubert classes. In Section 5 we briefly summarize the Lanini-Zainoulline construction of a twisted quadratic folding of a root system and introduce our main object of study, the induced folding map $\varepsilon^* : Z(G^P) \to Z(G\tau^P)$ between the structure algebras. Sections 6–7 investigate the liftings of Schubert classes under the folding map $\varepsilon^*$. In Section 6 we define a combinatorial folding map $\varphi : S \to S_\tau$ and prove its key properties. In Section 7 we prove Theorem 1.3 and describe some examples. Section 8 discusses how to detect nonliftable Schubert classes and provides a useful criterion (Corollary 8.5). Finally, Section 9 applies the results of Sections 7 and 8 to each known twisted quadratic folding to provide a complete list of parabolic subsets $P \subset S$ such that the folding map $\varepsilon^* : Z(G^P) \to Z(G\tau^P)$ has the lifting property.

## 2 Coxeter groups and root systems

This section sets the notation and collects some facts about root data, finite root systems, and finite Coxeter groups that will be frequently used throughout the text.
2.1 Root data and root systems

This paper works with two approaches to a finite root system, that is, the notion of a root datum following [15, Exposé XXI] and that of a finite root system following [16, Section 1.2].

We denote a root datum by $\Phi \rightarrow \Lambda^\vee$, where the convention is as follows: 1) $\Lambda$ is a free $\mathbb{Z}$-module of finite rank, 2) $\Phi \subset \Lambda$ is a nonempty finite subset, 3) $\Lambda^\vee$ is the dual $\mathbb{Z}$-module, 4) $\alpha^\vee$ denotes the image of $\alpha \in \Phi$, 5) $s_\alpha(x) = x - \alpha^\vee(x)\alpha$ for $x \in \Lambda$, 6) $W(\Phi) := \{s_\alpha \mid \alpha \in \Phi\} \leq \text{GL}_\mathbb{Z}(\Lambda)$ is the Weyl group of $\Phi$. 7) $\Delta$ denotes a simple system, 8) $\Lambda_r := \text{span}_\mathbb{Z}(\Phi)$ is the root lattice. We assume that a root datum is reduced, that is, if $\Phi \cap c\Phi \neq \emptyset$ for some $c \in \mathbb{Z}$, then $c = \pm 1$.

**Definition 2.1.** Let $\Phi \rightarrow \Lambda^\vee$ be a root datum and $K \subset \mathbb{R}$ be a (unital) subring. Let $U$ be a free $K$-module containing $\Lambda \otimes K$ as a $K$-submodule. Let $(-,-)$ denote a dot product on $U$. If $\alpha^\vee(x) = \frac{2(x,\alpha)}{(\alpha,\alpha)}$ for all $\alpha \in \Phi$ and $x \in \Lambda$, we call the inclusion $\Phi \subset \Lambda \rightarrow \Lambda \otimes K \subset U$ a geometric $K$-representation of the root datum and denote it by $(\Phi, U)_K$. By extending the formula of $s_\alpha (\alpha \in \Phi)$ over $U$, we obtain $s_\alpha \in \text{GL}_K(U)$. We define the Weyl group of $(\Phi, U)_K$ to be the subgroup $\langle s_\alpha \mid \alpha \in \Phi \rangle$ of $\text{GL}_K(U)$.

Let $\mathbb{E}$ be a Euclidean space with a dot product $(-,-)$. We denote a root system by $\Phi$ and use the following convention: 1) $\Phi \subset \mathbb{E}\setminus\{0\}$ is a nonempty finite subset, 2) $s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$ for $x \in \mathbb{E}$, 3) $W(\Phi) := \{s_\alpha \mid \alpha \in \Phi\} \leq \text{GL}_\mathbb{E}(\mathbb{E})$ is the Weyl group of $\Phi$. We assume that a root system is reduced, that is, if $\Phi \cap c\Phi \neq \emptyset$ for some $c \in \mathbb{R}$, then $c = \pm 1$. The minimal coefficient ring of $\Phi$ is the minimal subring $K_{\text{min}} \subset \mathbb{R}$ containing all the values $\frac{2(x,\alpha)}{(\alpha,\alpha)}$ where $\alpha_i, \alpha_j \in \Delta$. If $\Phi$ is crystallographic, $K_{\text{min}} = \mathbb{Z}$.

**Definition 2.2.** Let $\Phi$ be a root system with the minimal coefficient ring $K_{\text{min}}$ and $K \subset \mathbb{R}$ be any subring containing $K_{\text{min}}$. Let $U$ be a free $K$-module equipped with a dot product $(-,-)$. Then $\Phi$ admits a realization $\Phi \subset U$ satisfying the defining conditions of a root system. The realization $\Phi \subset U$ is called a $K$-realization of the root system. We define the Weyl group of $\Phi \subset U$ to be the subgroup $\langle s_\alpha \mid \alpha \in \Phi \rangle$ of $\text{GL}_K(U)$.

The two distinct notions as defined in Definitions 2.1 and 2.2 will be used in the construction of a twisted quadratic folding in Section 5, where the input of a folding is a root datum and the output is a not necessarily crystallographic finite root system.

2.2 Coxeter groups

Let $(W,S)$ be a finite Coxeter system. We use the following notation: 1) $m(s,t) \in \mathbb{Z}_{\geq 2}$ is the order of $st$ in $W$ for $s,t \in S$, 2) $l : W \rightarrow \mathbb{Z}_{\geq 0}$ is the length function, 3) $(W, \leq)$ is the Bruhat poset, 4) $F(S)$ is the free monoid generated by $S$, 5) $[s,t]_m$ is the word $sts\cdots \in F(S)$ of length $m$, 6) $\mathcal{R}(w) \subset F(S)$ is the set of all reduced words representing $w \in W$, 7) $\mathcal{R}(W) := \bigcup_{w \in W} \mathcal{R}(w)$, 8) $W_P := \{s \mid s \in P\} \leq W$ for a (possibly empty) subset $P \subset S$, 9) $W^P := \{w \in W \mid l(ws) > l(w)\}$ for all $s \in P$, 10) $w$ is the unique element of $wW_P \cap W^P$ for $w \in W$.

The following list of well-known results will be frequently used in the main discussion. The reader is referred to [17] and [16] for the details.

**Lemma 2.3** ([16]). (a) Let $w \in W$, $\alpha \in \Phi^\pm$. Then $l(ws_\alpha) > l(w)$ if and only if $w(\alpha) \in \Phi^\pm$. Equivalently, $l(s_\alpha w) > l(w)$ if and only if $w^{-1}(\alpha) \in \Phi^\pm$.

(b) Let $w, w' \in W$. Let $w = s_1\cdots s_k$ be a reduced expression. Then $w' \leq w$ if and only if $w'$ admits a reduced expression which is a subexpression of $s_1\cdots s_k$.

We identify $W$ with a finite real reflection group acting on a finite dimensional Euclidean space $U$ via the standard geometric representation. Let $\theta \in U$ be a dominant vector with respect to a
simple system $\Delta$ whose stabilizer subgroup is $W_P$. The $W$-orbit $W\theta$ is equipped with a partial order given by taking the transitive closure of the following binary relations for $x \in U$:

$$x < s_\alpha(x) \text{ for all } \alpha \in \Phi^+ \text{ with } (x, \alpha) > 0.$$  

Lemma 2.4 ([18] Proposition 1.1). The orbit map $W \to W\theta : w \mapsto w(\theta)$ restricts to a poset isomorphism $(W^P, \leq) \simeq (W\theta, \leq)$.

3 Moment graphs and structure algebras

In this paper, we work with a class of moment graphs called parabolic moment graphs (cf. [19, Example 4.3]). Let $\Delta_P \subseteq \Delta$ be a (possibly empty) subset. The Bruhat poset $(W, \leq)$ restricts to the subposet $(W^P, \leq)$. The following data defines a parabolic moment graph $G^P$ associated with the pair $(\Phi, \Phi_P)$.

(PM1) The vertex set is the poset $(W^P, \leq)$,

(PM2) The edge set is $E := \{w \to s_\alpha w \mid w \in W^P, \alpha \in \Phi^+, w < s_\alpha w\}$,

(PM3) The labelling function is $L(w \to s_\alpha w) = \alpha$ for each $\alpha \in \Phi^+$.

Note that $w < s_\alpha w$ if and only if $w < s_\alpha w$ and $w \neq s_\alpha w$ (use Lemma 2.4). A proof for the well-definedness of the labelling function $L$ can be found in [19, Example 4.3] for example. We will denote the parabolic moment graph associated with $(\Phi, \emptyset)$ by $G$ instead of $G^\emptyset$ to simplify notation. It coincides with the Bruhat graph as defined in [17].

Example 3.1 (Type $A_2$). Let $\Phi$ be the root system of type $A_2$ with simple roots denoted by $\{\alpha_1, \alpha_2\}$. Let $s_i$ denote the simple reflection associated with $\alpha_i$. Let $\Delta_P = \{\alpha_1\}$. Below are the moment graphs associated with $(\Phi, \emptyset)$ (left) and $(\Phi, \Phi_P)$ (right).

Let $G^P = ((W^P, \leq), E, L)$ be the moment graph associated with $(\Phi, \Phi_P)$ defined above. Let $\Phi \subseteq U$ be a $K$-realization of the root system for some subring $K \subseteq \mathbb{R}$ containing $K_{\text{min}}$ (Definition 2.2). Set $S := \text{Sym}_K U$, the symmetric algebra of $U$ over $K$.

Definition 3.2 ([6, Section 3.2]). The structure algebra of $G^P$ is given by

$$Z(G^P) := \{(z_w)_{w \in W^P} \in \bigoplus_{w \in W^P} S \mid z_w - z_{w'} \in L(w \to w')S \text{ for all } w \to w' \in E\}.$$

By direct computation, one can verify that $Z(G^P)$ is an $S$-algebra under the coordinate-wise addition, multiplication, and scalar multiplication.

Remark 3.3. (a) An element $\xi \in Z(G^P)$ can be viewed either as an $|W^P|$-tuple of $S$-elements or as a function $W^P \to S$. Using this second perspective, we will often denote the $w$-coordinate of $\xi$ by $\xi(w)$.
(b) Notice that the polynomial algebra $S$ depends on the $K$-realization of the root system $\Phi \subset U$. When the $K$-realization $\Phi \subset U$ is coming from a geometric $K$-representation $(\Phi, U)_K$ of a root datum, there is a canonical choice for $S$ as follows. Let $\Phi \hookrightarrow A^\vee$ be a root datum with a simple system $\Delta$ and a (possibly empty) subset $\Delta_P \subset \Delta$. Let $K \subset \mathbb{R}$ be a (unital) subring. Let $(\Phi, U)_K$ be a geometric $K$-representation of the root datum. By definition of $(\Phi, U)_K$, $U$ contains $U_A := \Lambda \otimes_{\mathbb{Z}} K$ as a free $K$-submodule. We take $S = \text{Sym}_K U_A$ in Definition 3.2 to obtain the structure algebra $Z(G^P)$.

(c) Each label $\alpha \in \Phi^+$ of the moment graph $G^P$ has degree one as an element of $S$.

**Example 3.4** (Type $A_2$). Let $K = \mathbb{Z}$. Let $\Phi \subset U$ be a $K$-realization of the root system of type $A_2$ such that $U = \text{span}_K \Phi$ with a simple system $\Delta = \{\alpha_1, \alpha_2\}$ and a subset $\Delta_P = \{\alpha_1\}$. Then $S = \mathbb{Z}[\alpha_1, \alpha_2]$ and

$$Z(G^P) = \left\{ (z_e, z_{s_2}, z_{s_1 s_2}) \in S \oplus S \oplus S \mid z_e - z_{s_2} \in \alpha_2 S, \right.$$

$$z_{s_2} - z_{s_1 s_2} \in \alpha_1 S, z_e - z_{s_1 s_2} \in (\alpha_1 + \alpha_2) S \right\}.$$

### 4 Combinatorial Schubert classes

In this section, we define the notion of *combinatorial Schubert classes* of a structure algebra $Z(G^P)$ and state their key properties mostly without proofs. The reader is referred to [20 Appendix D] and [14] for the details.

**Definition 4.1** (cf. [20 Appendix D.3], [14 Corollary 4.11]). The combinatorial Schubert class corresponding to $w \in W$ is a function $\sigma^{(w)} : W \to S$ given by the following formula: let $x \in W$ and let $x = s_1 s_2 \cdots s_r$ be a (not necessarily reduced) expression and we denote the simple root corresponding to $s_i$ by $\alpha_i$ for each $1 \leq i \leq r$. Then we define if $w \not\leq x$, $\sigma^{(w)}(x) = 0$; and if $w \leq x$,

$$\sigma^{(w)}(x) = \sum_{i} \prod_{p=1}^{\ell} s_{1 s_2 \cdots s_{i_p-1} (\alpha_{i_p})}$$

(3)

where $\ell = l(w)$, $i = (i_1, \ldots, i_\ell) \in \mathbb{Z}^\ell$ such that $1 \leq i_1 < i_2 < \cdots < i_\ell \leq r$ and $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. By convention, we set $\sigma^{(e)}(x) = 1$ for all $x \in W$.

It is well-known that the formula (3) does not depend on the choice of the expression $x = s_{1 s_2 \cdots s_r}$ (cf. [21] Section 4) for when $W$ is crystallographic.

From Definition 4.1 it is not clear if the Schubert class $\sigma^{(w)}$ belongs to $Z(G)$ for each $w \in W$. However, this is indeed the case ([20 Lemma D.4]), and for $w \in W^P$, the restriction $\sigma^{(w)}|_{W^P}$ belongs to $Z(G^P)$ (Lemma 4.6 (b)). Moreover, the set of all Schubert classes $\{\sigma^{(w)} \mid w \in W^P\}$ forms an $S$-module basis for $Z(G^P)$ (Lemma 4.7).

**Example 4.2** (Type $A_2$). Below are the Schubert classes of the structure algebra $Z(G)$ where $G$ is of type $A_2$. From top to bottom, left to right, the diagrams depict the Schubert classes $\sigma^{(e)}$, $\sigma^{(s_1)}$ and $\sigma^{(s_2)}$, $\sigma^{(s_2 s_1)}$ and $\sigma^{(s_1 s_2)}$; and finally $\sigma^{(s_1 s_2 s_1)}$.

![Diagram](image-url)
Here are the basic properties of a Schubert class $\sigma^{(w)}$ stated without a proof.

**Lemma 4.3** ([20] Appendix D.1]). Let $w \in W$.

(a) For each $x \in W$, $\sigma^{(w)}(x)$ is homogeneous of degree $l(w)$.

(b) $\sigma^{(w)}(w) = \prod_{\alpha \in \Phi^+, \ell(w) = \ell} \alpha \in \Phi^- \alpha$.

(c) If $w \leq x$, then $\sigma^{(w)}(x) \neq 0$.

**Remark 4.4.** By (a), we may call the common homogeneous degree of $\sigma^{(w)}(x) (x \in W)$ the degree of $\sigma^{(w)}$, denoted by $\deg \sigma^{(w)}$. By Lemmas 4.3 (b) and 2.3 (a), we have

$$\deg \sigma^{(w)} = l(w) = |\{\alpha \in \Phi^+ \mid l(s_{w_\alpha} w < l(w))\}| = |\{w' \rightarrow w \in E \mid w' \in WP\}|.$$

Lemmas 4.3, 4.6 establish that for $w \in WP$, the restriction $\sigma^{(w)}|_{WP}$ belongs to $Z(G_P)$.

**Lemma 4.5** ([20] Appendix D.2]). Let $x \in W$ and let $\alpha$ be a simple root.

$$\sigma^{(w)}(xs_\alpha) = \begin{cases} \sigma^{(w)}(x), & \text{if } l(ws_\alpha) > l(w) \\ \sigma^{(w)}(x) + \sigma^{(w)}(s_\alpha)(x), & \text{if } l(ws_\alpha) < l(w) \end{cases} \quad (4)$$

**Lemma 4.6.** (a) Let $w \in WP$, $x \in W$ and $y \in WP$. Then $\sigma^{(w)}(xy) = \sigma^{(w)}(x)$.

(b) Let $w, x \in WP$ and $\alpha \in \Phi^+$. Then $\sigma^{(w)}(x) - \sigma^{(w)}(s_\alpha x) \in \alpha S$.

**Proof.** (a) Let $y = s_1s_2 \cdots s_\ell$ be a reduced expression and denote the simple reflection associated with $s_i$ by $\alpha_i$ for each $1 \leq i \leq \ell$. Note that each $\alpha_i$ belongs to $\Delta_P$. Since $w \in WP$, we have $l(ws_i) > l(w)$ for each $1 \leq i \leq \ell$. By applying (4) repeatedly, we obtain

$$\sigma^{(w)}(x) = \sigma^{(w)}(xs_1) = \sigma^{(w)}(xs_1s_2) = \cdots = \sigma^{(w)}(xs_1s_2 \cdots s_\ell) = \sigma^{(w)}(xy).$$

(b) We have $s_\alpha x = s_\alpha xy$ for some $y \in WP$. Applying (a), we obtain

$$\sigma^{(w)}(x) - \sigma^{(w)}(s_\alpha x) = \sigma^{(w)}(y^{-1}) - \sigma^{(w)}(s_\alpha xy^{-1}) \in \alpha S.$$
When we speak of a Schubert class of \( Z(G^P) \) corresponding to \( w \in W^P \), we mean the restriction \( \sigma(w)|_{W^P} \). For the notational convenience, we denote \( \sigma(w)|_{W^P} \) also by \( \sigma(w) \).

**Lemma 4.7** ([20], Appendix D.5). The set \( \{\sigma(w) \mid w \in W^P\} \) is an \( S \)-basis of \( Z(G^P) \).

The following characterization of a Schubert class will be the key in Section 7.

**Lemma 4.8** (cf. [14], Proposition 4.9). Let \( w \in W^P \) and \( \xi := (x_{w'})_{w' \in W^P} \in Z(G^P) \). The following are equivalent:

(a) \( \xi \) is homogeneous of degree \( l(w) \) such that \( x_w \neq 0 \) and \( x_{w'} = 0 \) for all \( w' \not\geq w \),

(b) \( \xi = c \cdot \sigma(w) \) for some nonzero constant \( c \in K \).

**Proof.** We give a proof that (a) implies (b) (the other direction is clear). Let \( w = e \). If \( \xi \) is homogeneous of degree \( l(e) = 0 \) with \( c := x_e \neq 0 \), then \( x_{w'} = c \) for all \( w' \in W^P \) by the edge relations of \( Z(G^P) \). Thus \( \xi = c \cdot \sigma(e) \). Let \( w > e \). Since \( x_{w'} = 0 \) for all \( w' < w \), for \( x_w(\neq 0) \) to satisfy all the relations \( x_{w'} - x_w \in L(w' \to w)S \), it must be divisible by the product \( \prod_{w' < w} L(w' \to w) \) (since each label is a prime element of \( S \)). Notice that \( \deg(\prod_{w' < w} L(w' \to w)) = \sum_{w' < w} \deg(L(w' \to w)) = l(w) \) (Remarks 3.3 (c) and 4.4). Since \( x_w \) also has degree \( l(w) \), we obtain \( x_w = c \prod_{w' < w} L(w' \to w) \) for some nonzero \( c \in K \). Take the difference \( \xi - c \cdot \sigma(w) := (y_{w'})_{w' \in W^P} \). Observe that \( y_w = 0 \). We also have \( y_{w'} = 0 \) for all \( w' \not\geq w \) by the assumption on \( \xi \) as well as Proposition 4.3 (c). We show that \( y_{w'} = 0 \) for all \( w' > w \) as well. Let \( v \in W^P \) such that \( v > w \) and \( l(v) = l(w) + 1 \). Then for all \( w' < v \), we obtain \( y_{w'} = 0 \) thanks to \( l(w') \leq l(w) \), \( y_w = 0 \) and \( y_{w'} = 0 \) for all \( w'' \not\geq w \). Hence for \( y_v \) to satisfy all the relations \( y_{w'} - y_v \in L(w' \to v)S \), it must be divisible by the product \( \prod_{w' < v} L(w' \to v) \), which has degree \( l(v) \) by the same argument made for \( x_v \). Since \( \deg(\xi - c \cdot \sigma(w)) = l(w) < l(v) \), \( y_v = 0 \) follows. We repeat the argument to obtain \( y_{w'} = 0 \) for all \( w' > v \) and \( l(w') = l(v) + 1 \). We repeat this process until we finally obtain \( y_{w''} = 0 \) for the longest element \( w'' \) of \( W^P \). Thus we obtain \( y_{w'} = 0 \) for all \( w' > w \) and complete the proof. \( \square \)

## 5 Twisted quadratic foldings and induced folding map

In this section, we recall the notion of a twisted quadratic folding of a finite crystallographic root system introduced by Lanini–Zainoulline [1].

Let \( K \subset \mathbb{R} \) be a (unital) subring and let \( L \) be a free quadratic \( K \)-algebra. The term **quadratic** indicates that the underlying free \( K \)-module is of rank 2. Then \( L = K[x]/(p(x)) \), where \( p(x) = x^2 - c_1 x + c_2 \) is some monic polynomial over \( K \), and \( (p(x)) \) denotes the ideal of the polynomial ring \( K[x] \) generated by \( p(x) \). We assume that \( L \) is separable over \( K \). Equivalently, the discriminant \( D = c_1^2 - 4c_2 \) is invertible in \( K \) (cf. [22], Appendix C)]. The trivial quadratic \( K \)-algebra \( K \times K \) can be obtained by taking \( p(x) = x^2 - 1 \). We will refer to this case as split case. We will refer to the case where \( L \) is not isomorphic to \( K \times K \) as non-split case. When \( L \) is non-split, we further assume \( D > 0 \) so that \( L \) can be identified with a subring of \( \mathbb{R} \). It is known that \( p(x) \) splits over \( L \). We denote the two distinct roots of \( p(x) \) in \( L \) by \( \tau \) and \( \sigma \), so that \( p(x) = (x - \tau)(x - \sigma) \) over \( L \). In the split case, we choose \( \tau = [1, -1] \) and \( \sigma = [-1, 1] \) in the basis \( \{[1, 0], [0, 1]\} \) of \( K \times K \). In the non-split case, we will assume \( \tau > \sigma \). In what follows, we fix a \( K \)-basis of \( L \) as \( \{1, \tau\} \).

### 5.1 Construction

The central step in the construction of a twisted quadratic folding is to define a folded \( K \)-representation of a given root datum \( \Phi \mapsto \Lambda^\vee \) (Definition 5.2). We first recall two key notions involved in the definition, namely, the \( \tau \)-form and the \( \tau \)-operator.
Let $\mathcal{U}$ be a free $L$-module of rank $n$ equipped with a basis and the dot product $(-,-)$. Let $U$ denote the free $K$-module of rank $2n$ obtained from $\mathcal{U}$ by restriction of scalars from $L$ to $K$. For $u = (x_i, y_i)_{i=1}^n \in U$ ($x_i, y_i \in K$) with respect to the $\{1,\tau\}$-basis, we write $u_L := (x_i + y_i\tau)_{i=1}^n \in \mathcal{U}$. Let $\text{pr}_\tau : L \to K$ be the $K$-linear projection given by $\text{pr}_\tau(x + y\tau) = x$ where $x, y \in K$. Then the $\tau$-form on $U$ is a symmetric $K$-bilinear form $(-,-)$ given by

$$(u, u')_\tau = \text{pr}_\tau(u_L \cdot u'_L) \quad (5)$$

for $u, u' \in U$. The $\tau$-operator on $U$ is the $K$-linear automorphism $T : U \to U$ defined by the property $(T u)_L = \tau \cdot u_L$, where on the right-hand side, the element $u_L \in \mathcal{U}$ is multiplied by the scalar $\tau$. The $\tau$-operator $T$ is adjoint with respect to the $\tau$-form ([1] Lemma 2.8). A nonzero element $u \in U$ is called $\tau$-rational if $(u_L \cdot u_L) \in K$.

**Lemma 5.1 ([1] Lemma 2.10).** Let $u \in U$ be $\tau$-rational. Then the following hold:

(a) $(u, Tu)_\tau = 0$,

(b) $(Tu, Tu)_\tau = -c_2(u, u)_\tau = -c_2(u_L \cdot u_L),

(c) If $(u_L \cdot u_L) \neq 0$, then $u \neq Tu$.

**Definition 5.2** (Folded representation, [1] Definition 3.1). Let $\Phi \mapsto \Lambda^\vee$ be a root datum with a simple system $\Delta \subset \Phi$ specified. Let $U$ be a free $K$-module of some even rank $2n$ equipped with a basis and the dot product $(-,-)$. A geometric $K$-representation $(\Phi, U)_K$ of the root datum $\Phi \mapsto \Lambda^\vee$ (Definition 2.1) is called a folded $K$-representation if the following conditions are satisfied:

(F1) There exists a free quadratic separable $K$-algebra $L$ and a free $L$-module $\mathcal{U}$ of rank $n$ such that $U$ is obtained from $\mathcal{U}$ by restriction of scalars to $K$. [Then fix the two roots $\tau, \sigma \in L$, the $\tau$-operator $T$, and the $\tau$-form $(-,-)_\tau$ on $U$.]

(F2) The $\tau$-form $(-,-)_\tau$ coincides with the dot product $(-,-)$ on $U$.

(F3) $T(\Lambda) = \Lambda$ and $T(\Lambda_\tau) = \Lambda_\tau$.

(F4) The simple system $\Delta$ admits a partition $\Delta = \Delta_T \sqcup \Delta_{\text{rat}} \sqcup T(\Delta_{\text{rat}})$, where $\Delta_T := \{ \alpha \in \Delta \mid T\alpha = \alpha \}$ ($T$-invariant simple roots) and $\Delta_{\text{rat}}$ consists of simple roots that are $\tau$-rational.

Here are some remarks. First, (F2) is equivalent to $c_2 = -1$ ([1] Lemma 2.3). In particular, the defining equation $\tau(\tau - c_1) = 1$ of $\tau$ shows that $\tau$ is invertible in $L$. Second, in the split case ($c_1 = 0$), all simple roots that are not fixed under $T$ are $\tau$-rational. Hence we divide these simple roots into $\Delta_{\text{rat}} \sqcup T(\Delta_{\text{rat}})$ by making a choice. Finally, since $T$ is an automorphism of $U$, $\Delta_{\text{rat}} \cap T(\Delta_{\text{rat}}) = \emptyset$ implies $|\Delta_{\text{rat}}| = |T(\Delta_{\text{rat}})|$. In the non-split case, it is known that $\Delta_T = \emptyset$.

We now consider the eigenspaces of $T$. In $U_L := U \otimes_K L$, the extended $\tau$-operator $T_L$ admits two eigenspaces $U_L^{(\tau)}$ and $U_L^{(\sigma)}$ corresponding to eigenvalues $\tau$ and $\sigma$ respectively. Moreover, the decomposition $U_L = U_L^{(\tau)} \oplus U_L^{(\sigma)}$ is orthogonal with respect to $(-,-)_\tau$ on $U_L$ ([1] Lemma 2.9). We compose the canonical inclusion $U \hookrightarrow U_L$ with the projection $U_L \to U_L^{(\tau)}$ to obtain the map $\pi_\tau : U \to U_L^{(\tau)}$. For $u \in U$, we have the formula:

$$\pi_\tau(u) = \frac{1}{\tau - \sigma}(T - \sigma I)(u),$$

and $\pi_\tau$ is known to be bijective ([1] after Lemma 2.9)]. Given $u, u' \in U$, we also have

$$(\pi_\tau(u), \pi_\tau(u'))_\tau = \frac{\sigma^2 - c_2}{\tau - \sigma}(u_L \cdot u'_L). \quad (6)$$

We set $\Delta_\tau := \pi_\tau(\Delta_T \sqcup \Delta_{\text{rat}})$ and for each $\alpha \in \Delta \sqcup \Delta_{\text{rat}}$, we write $\overline{\alpha} := \pi_\tau(\alpha)$. Given $\overline{\alpha} \in \Delta_\tau$, the following $L$-linear map $R_{\overline{\alpha}} \in \text{Aut}_L(U_L^{(\tau)})$ is known to behave as a reflection with respect to
\(\pi\): if \(\alpha \in \Delta^T\), define \(R_{\pi}\) to be the reflection \(s_\alpha \in \text{Aut}_K(U)\) extended to \(U_L\), then restricted to \(U_L^{(r)}\); if \(\alpha \in \Delta_{\text{rat}}\), define \(R_{\pi}\) to be the composite \(s_\alpha s_{\tau_\alpha} \in \text{Aut}_K(U)\) extended to \(U_L\), then restricted to \(U_L^{(r)}\). This procedure is well-defined and the map \(R_{\pi}\) commutes with \(T_L|_{U_L^{(r)}}\). We define \(W_\tau := \langle R_{\pi} \mid \alpha \in \Delta^T \sqcup \Delta_{\text{rat}} \rangle\) and \(\Phi_\tau := \{ u(\pi) \mid \pi \in \Delta_\tau, u \in W_\tau \}\). Then \(W_\tau\) is a finite \(L\)-reflection group with root system \(\Phi_\tau\). The set \(\Delta_\tau\) is a simple system of \(\Phi_\tau\). The passage from \((\Phi, U)_K\) to \(\Phi_\tau \subset U_L^{(r)}\) is called a twisted quadratic folding. It is noted that \(\Phi_\tau\) is finite but not necessarily crystallographic. Finally, the following assignment of generators extends to an embedding of Coxeter groups:

\[
e: W_\tau \hookrightarrow W: R_{\pi} \mapsto s_\alpha \\
e: W_\tau \hookrightarrow W: s_\alpha s_{\tau_\alpha} \quad \text{if } \alpha \in \Delta^T, \\
e: W_\tau \hookrightarrow W: s_\alpha s_{\tau_\alpha} \quad \text{if } \alpha \in \Delta_{\text{rat}}.
\]

All of this is established in \cite{1} Sections 3–4. By direct computation, we may check

**Lemma 5.3.** \(u(\pi_\tau(x)) = \pi_\tau(e(u)(x))\) for all \(u \in W_\tau\), \(x \in U\).

Given a (possibly empty) subset \(\Delta_P \subset \Delta\) such that \(T(\text{span}_K \Delta_P) = \text{span}_K \Delta_P\), we can conduct the procedure we have described so far for the subsystem \(\Phi_P \subset U\) and its reflection group \(W_P = \langle s_\alpha \mid \alpha \in \Delta_P \rangle\), and obtain the folded root system \((\Phi_P)_\tau \subset U_L^{(r)}\) and associated reflection group \((W_P)_\tau\). By \cite{1} Lemma 5.2 and Proposition 2.4, we have

**Lemma 5.4.** The embedding \(e: W_\tau \hookrightarrow W\) restricts to \(W_\tau^P \hookrightarrow W_P\), where \(W_\tau^P\) denotes the set of minimal length representatives of the coset space \(W_\tau/(W_P)_\tau\).

### 5.2 Induced folding map

Let \(\Phi \simeq \Phi_\tau\) be a twisted quadratic folding and \(\Phi_P \simeq (\Phi_P)_\tau\) be its subfolding as discussed above. We keep the notation of Section 5.1. Let \(G_P\) be the moment graph associated with the pair \((\Phi_\tau, (\Phi_P)_\tau)\). Recall that by Definition 2.4 \(U\) contains \(U_\Lambda := \Lambda \otimes \mathbb{Z} K\) as a free \(K\)-submodule. Let \(S_\tau := \text{Sym}_L \pi_\tau(U_\Lambda)\). We replace \(G_P\) by \(G_P^\tau\) and \(S_\tau\) by \(S_\tau\) in Definition 3.2 to obtain the folded structure algebra \(Z(G_P^\tau)\). The map \(\pi_\tau|_{U_\Lambda}: U_\Lambda \rightarrow \pi_\tau(U_\Lambda)\) naturally extends to the ring homomorphism \(\pi_\tau: S \rightarrow S_\tau\) (recall that \(S = \text{Sym}_K U_\Lambda\)).

**Proposition 5.5** (\cite{1} Theorem 6.2). There is an induced ring homomorphism \(\varepsilon^*: Z(G_P^\tau) \rightarrow Z(G_P^\tau)\), called the folding map, given by

\[
\varepsilon^*(\xi)(u) = \pi_\tau(\xi(e(u)))
\]

for all \(\xi \in Z(G_P^\tau)\) and \(u \in W_P\). Moreover, it commutes with the \(S\)-action on \(Z(G_P^\tau)\), that is, \(\varepsilon^*(f \cdot \xi) = \pi_\tau(f) \cdot (\varepsilon^*(\xi))\) for \(\xi \in Z(G_P^\tau)\) and \(f \in S\).

**Remark 5.6.** Since the map \(\pi_\tau: U \rightarrow U_L^{(r)}\) is bijective, in particular, its restriction \(\pi_\tau|_{U_\Lambda}: U_\Lambda \rightarrow \pi_\tau(U_\Lambda)\) is injective, and so is the induced map \(\pi_\tau: S \rightarrow S_\tau\).

### 5.3 Local analysis of a folding

The following result about the local behaviour of the map \(\pi_\tau|_\Delta: \Delta \rightarrow \Delta_\tau\) is used to establish combinatorial properties of the folding map \(\varepsilon^*\) in Section 5 (Lemma 5.7).

**Lemma 5.7.** Let \(\Phi \simeq \Phi_\tau\) be a twisted quadratic folding of \(\Phi\) simply-laced. Let \(\alpha, \beta \in \Delta^T \sqcup \Delta_{\text{rat}}\) such that \(m := m(R_{\Pi}, R_{\beta}) \geq 3\). The possible configuration of the full subgraph \((\pi_\tau|_\Delta)^{-1}(\{\alpha, \beta\})\) of the Coxeter diagram of \(W\) is one of the following up to symmetry of \(\alpha\) and \(\beta\):

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Proof. We may assume, without loss of generality, $(\alpha, \beta, T) = (p + a, c, d)$ for some $a, b \in K$. Hence at least one of the above three values must be zero. Since $\Phi$ is taken to be simply connected, we have $(\alpha, T) = (p + a, c, d)$. Hence if $(\alpha, \beta) = (1 + 2, 0)$, then $(\alpha, \beta) = (1 + 2, 0)$, and $(\alpha, \beta) = (1 + 2, 0)$. Hence either $a \neq 0$ or $b \neq 0$ (or both).

We know $(\alpha, T\alpha) = 0$, $(\beta, T\beta) = 0$ (Lemma 5.1 (a)), and $(\alpha, T\beta) = (T\alpha, \beta)$ (the adjoint property of $T$). We compute the remaining combinations below:

$$(\alpha, \beta) = pr_\tau(\alpha_L \cdot \beta_L), \quad (\tau, \tau) = pr_\tau(\alpha_L \cdot \beta_L).$$

Since $m \geq 3$, we have $(\tau, \tau) \neq 0$, and hence $\alpha_L \cdot \beta_L \neq 0$. Hence either $a \neq 0$ or $b \neq 0$ (or both).

We know $(\alpha, T\alpha) = 0$, $(\beta, T\beta) = 0$ (Lemma 5.1 (a)), and $(\alpha, T\beta) = (T\alpha, \beta)$ (the adjoint property of $T$). We compute the remaining combinations below:

$$(\alpha, \beta) = \frac{\pi}{2}(a + b\tau), \quad (\alpha, \beta) = \frac{\pi}{2}(a + b\tau).$$

Since the original root system $\Phi$ is finite crystallographic, its Coxeter diagram contains no circuit. Hence at least one of the above three values must be zero. Since $\Phi$ is taken to be simply laced, $a \neq 0$ (resp. $b \neq 0$) implies $a = (\alpha, \beta) = -(\frac{1}{2})$ (resp. $b = (\alpha, T\beta) = -(\frac{1}{2})$). We also have

$$\alpha_L \cdot \alpha_L = \begin{cases} 1 + \tau & \text{if } \alpha \in \Delta_T, \\ 1 & \text{if } \alpha \in \Delta_{rat}. \end{cases}$$

For $\alpha \in \Delta_{rat}$, this is a consequence of Lemma 5.1 (b). For $\alpha \in \Delta_T$, write $\alpha_L \cdot \alpha_L = c + d\tau$ for some $c, d \in K$. Then combine $c = (\alpha, \alpha) = (\alpha, T\alpha) = d$ and $(\alpha, \alpha) = 1$.

We denote the angle between $\alpha$ and $\beta$ by $\theta$.

**Case 1:** $c_1 = 0$ (split case). We have $(\alpha, \beta) = (T\alpha, T\beta) = a$. Suppose $\alpha = T\alpha$. Then we also have $b = (T\alpha, \beta) = (\alpha, \beta) = a$, and hence $a \neq 0$ and $b \neq 0$. We have

$$(\alpha, \beta) = \frac{\pi}{2}(a + b\tau), \quad (\alpha, \alpha) = \frac{\pi}{2}(1 + \tau),$$

$$(\beta, \beta) = \frac{\pi}{2}(1 + \tau)$$

if $\beta \in \Delta_T$, and

$$(\beta, \beta) = \frac{\pi}{2}$$

if $\beta \in \Delta_{rat}$.

Hence if $\beta \in \Delta_T$, we obtain $-\cos \theta = -\frac{1}{2}$, yielding the diagram (A). If $\beta \in \Delta_{rat}$, we obtain $-\cos \theta = -\frac{1}{\sqrt{2}}$, yielding the diagram (B).

Now suppose $\alpha, \beta \notin \Delta_T$. If $a = 0$, then $b \neq 0$. We have

$$(\alpha, \beta) = \frac{\pi}{2}(a + b\tau), \quad (\alpha, \alpha) = (\beta, \beta) = \frac{\pi}{2}.$$ 

Hence $-\cos \theta = -\frac{1}{2}$, yielding the diagram (C). If $b = 0$, then $a \neq 0$. We have

$$(\alpha, \beta) = \frac{\pi}{2}(a + b\tau), \quad (\alpha, \alpha) = (\beta, \beta) = \frac{\pi}{2}.$$ 

Hence $-\cos \theta = -\frac{1}{2}$, yielding the diagram (D).
Hence \( -\cos \theta = -\frac{1}{\tau} \), yielding the diagram (C).

**Case 2:** \( c_1 \neq 0 \) (non-split case). If \( a = 0 \), then \( b \neq 0 \). The equality \( (T_\alpha, T_\beta)_\tau = -\frac{1}{2}c_1 = -\frac{1}{2} \) enforces \( c_1 = 1 \), and \( \tau = \frac{1+\sqrt{5}}{2} \) (the golden ratio). We have

\[
(\alpha, \beta)_\tau = \frac{\tau^2 + 1}{2}(-\frac{1}{2} \tau), \quad (\alpha, \alpha)_\tau = (\beta, \beta)_\tau = \frac{\tau^2 + 1}{2}.
\]

Hence \( -\cos \theta = -\frac{1}{\sqrt{5}} \), yielding the diagram (E). If \( b = 0 \), then \( a \neq 0 \). We have

\[
(\alpha, \beta)_\tau = \frac{\tau^2 + 1}{2}(-\frac{1}{2} \tau), \quad (\alpha, \alpha)_\tau = (\beta, \beta)_\tau = \frac{\tau^2 + 1}{2}.
\]

Hence \( -\cos \theta = -\frac{1}{\tau} \), yielding the diagram (C). If \( a + c_1 b = 0 \), then both \( a \neq 0 \) and \( b \neq 0 \). The equality \( -\frac{1}{2} - \frac{1}{2}c_1 = 0 \) enforces \( c_1 = -1 \), and \( \tau = \frac{1+\sqrt{5}}{2} \). We have

\[
(\alpha, \beta)_\tau = \frac{\tau^2 + 1}{2}(-\frac{1}{2} \tau), \quad (\alpha, \alpha)_\tau = (\beta, \beta)_\tau = \frac{\tau^2 + 1}{2}.
\]

Hence \( -\cos \theta = -\frac{1}{\sqrt{5}} \), yielding the diagram (F).

### 5.4 Folding diagrams

This subsection collects the Coxeter diagrams of all known examples of twisted quadratic foldings (cf. [1] Examples 4.3–4.5). The number at each vertex is the index of each simple reflection that the vertex represents. In the Coxeter diagram associated with the original root system (drawn first in each folding), the simple reflections represented by black dots correspond to the elements of \( \Delta \), those represented by white dots correspond to the elements of \( \Delta_{rat} \), and those represented by double white dots correspond to the elements of \( \Delta^T \) respectively.

![Coxeter diagrams](image)

### 6 Combinatorial folding map

Let \( \Phi \leadsto \Phi_\tau \) be a twisted quadratic folding with \( \Phi \) simply laced. Let \( (W, S) \) denote the Coxeter system associated with the \( K \)-reflection group \( W \) of \( \Phi \) where \( S = \{ s_\alpha \mid \alpha \in \Delta \} \), and let \( (W_\tau, S_\tau) \) denote the Coxeter system associated with the \( L \)-reflection group \( W_\tau \) where \( S_\tau = \{ R_\beta \mid \beta \in \Delta_\tau \} \). We will use \( l_\tau(u) \) to denote the length of \( u \in W_\tau \).

We study liftings of Schubert classes (Definition [1.2]) by treating the folding \( \Phi \leadsto \Phi_\tau \) on the level of the Coxeter generating sets \( S \) and \( S_\tau \). We define a set-theoretic map \( \varphi : S \to S_\tau \) by

\[
\varphi : S \to S_\tau : s_\alpha, s_{\tau \alpha} \mapsto R_\beta
\]

for each \( \alpha \in \Delta^T \cup \Delta_{rat} \). Note that \( \varphi \) is surjective due to the surjectivity of \( \pi_\tau|_\Delta : \Delta \to \Delta_\tau \)
Definition 6.1. Let $R \in S_\tau$ be such that $\varphi^{-1}(R)$ has exactly two elements. If $s$ is one of them, the other one is denoted by $s^*$ and called the opposite simple reflection of $s$.

The following is the rephrasing of Lemma 6.6 in terms of $\varphi$.

**Lemma 6.2.** Let $R, R' \in S_\tau$ such that $m(R, R') \geq 3$. The possible configuration of the full subgraph $\varphi^{-1}([R, R'])$ of the Coxeter diagram of $(W, S)$ is one of the following, up to symmetry of $R$ and $R'$.

- (A) $s \cdot t \cdot s \cdot t$ (B) $s \cdot t \cdot s \cdot t$ (C) $s \cdot t \cdot s^* \cdot t^*$ (D) $s \cdot t \cdot s^* \cdot t^*$

For a pair $(R, R')$ of simple reflections in $W_\tau$ such that $m(R, R') = 2$, we have the following.

**Lemma 6.3.** Let $R, R' \in S_\tau$ such that $m(R, R') = 2$. Let $s \in \varphi^{-1}(R)$ and $t \in \varphi^{-1}(R')$. Then $m(s, t) = 2$.

**Proof.** Let us write $R = R_\tau$ and $R' = R_\tau'$ for some $\alpha, \beta \in \Delta^T \sqcup \Delta_{\text{rat}}$. The condition $m(R, R') = 2$ is equivalent to $(\alpha, \beta)_\tau = 0$. By formula (6), $(\alpha L \cdot \beta L) = 0$. Hence $(\alpha, \beta)_\tau = pr_\tau(\alpha L \cdot \beta L) = 0$, $(\alpha, T \beta)_\tau = (T \alpha, \beta)_\tau = pr_\tau(\tau \alpha L \cdot \beta L) = 0$, and $(T \alpha, T \beta)_\tau = pr_\tau(\tau \alpha L \cdot \tau \beta L) = 0$. Thus for all possible choices of $s \in \varphi^{-1}(R_\tau)$ and $t \in \varphi^{-1}(R_\tau')$, we have $m(s, t) = 2$.

The map $\varphi$ has the following property.

**Lemma 6.4.** Let $s_{i_1} \cdots s_{i_\ell}$ be a reduced expression in $W_\tau$ and let $R_{j_1} \cdots R_{j_k}$ be a reduced expression in $W_\tau$. Suppose $\varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}})$ for all $1 \leq p \leq \ell - 1$. Then $s_{i_1} \cdots s_{i_\ell}$ is a subexpression of $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ if and only if $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ is a subexpression of $R_{j_1} \cdots R_{j_k}$.

**Proof.** Let $R_j \in S_\tau$. Then $s_i \in S$ is a subexpression of $\varphi(R_j)$ if and only if $\varphi(s_{i_p}) = R_j$. This follows from the surjectivity of $\varphi$ and the property $\varphi(s_i) = s_i$ (if $\alpha_i \in \Delta^T$) or $s_is_i^*$ (if $\alpha_i \in \Delta_{\text{rat}} \sqcup T(\Delta_{\text{rat}})$). Let $s_{i_1} \cdots s_{i_\ell}$ be a subexpression of $\varphi(R_{j_1}) \cdots \varphi(R_{j_k})$. Then for each $1 \leq p \leq \ell$, $s_{i_p}$ is a subexpression of $\varphi(R_{j_p})$ for some $1 \leq q(p) \leq k$ such that $q(p) \leq q(p+1)$ and we have $\varphi(s_{i_p}) = R_{j_q(p)}$. Now the condition $\varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}})$ for all $1 \leq p \leq \ell - 1$ guarantees that $q(p) \neq q(p+1)$ for each $p$. Therefore, $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ is a subexpression of $R_{j_1} \cdots R_{j_k}$. Conversely, let $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ be a subexpression of $R_{j_1} \cdots R_{j_k}$. Then for each $1 \leq p \leq \ell$, $\varphi(s_{i_p}) = R_{j_q(p)}$ for some $1 \leq q(1) < \cdots < q(\ell) \leq k$. Each $s_{i_p}$ is a subexpression of $\varphi(R_{j_q(p)})$. Therefore, $s_{i_1} \cdots s_{i_\ell}$ is a subexpression of $\varphi(R_{j_1}) \cdots \varphi(R_{j_k})$.

In order to yield further properties of the map $\varphi : S_\tau \rightarrow S_\tau$ and the group embedding $\varphi : W_\tau \hookrightarrow W$, we introduce the notion of a folding branch and a collapsing part.

**Definition 6.5.** Let $\Gamma$ and $\Gamma_\tau$ denote the Coxeter diagram of $W$ and $W_\tau$ respectively. A folding branch of $\Gamma$ with respect to $\varphi : S_\tau \rightarrow S_\tau$ is a full subgraph $\Gamma'$ of $\Gamma$ obtained by choosing one vertex from $\varphi^{-1}(R)$ for each $R \in \Gamma_\tau$ such that $m(s, s') = 3$ for all $s, s' \in \Gamma'$ with $m(\varphi(s), \varphi(s')) \geq 3$. A collapsing part of $\Gamma$ with respect to $\varphi$ is the full subgraph $\varphi^{-1}([R, R'])$ of $\Gamma$ where $R, R' \in S_\tau$ and $m(R, R') \geq 4$.

A folding branch exists by Lemma 6.2 since for every pair $R, R' \in \Gamma_\tau$ with $m(R, R') \geq 3$, there exists a pair $s, s' \in \Gamma$ such that $\varphi(s) = R$, $\varphi(s') = R'$ and $m(s, s') = 3$. 

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Example 6.6 (Type $E_8$ to $H_4$). We describe folding branches and a collapsing part for the folding from type $E_8$ to $H_4$ shown in Section 5.3. We have $S = \{s_1, \cdots, s_8\}$ and $S_t = \{R_1, \cdots, R_4\}$. The map $\varphi : S \to S_t$ sends $s_1, s_7$ to $R_1$, $s_2, s_6$ to $R_2$, $s_3, s_5$ to $R_3$, and $s_4, s_8$ to $R_4$. The Coxeter diagram of the original graph admits three folding branches: $\Gamma_1 = \{s_1, s_2, s_3, s_4\}$, $\Gamma_2 = \{s_7, s_6, s_5, s_4\}$, and $\Gamma_3 = \{s_7, s_6, s_5, s_8\}$. It has a collapsing part $\Gamma' = \{s_3, s_4, s_5, s_8\}$.

Definition 6.5 allows us to prove the following key property of the embedding $\varepsilon : W_r \hookrightarrow W$.

**Proposition 6.7.** If $R_{j_1} \cdots R_{j_q}$ is a reduced expression in $W_\tau$, then $\varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_q})$ is a reduced expression in $W$.

**Proof.** We break down the map $\varepsilon$ into two steps. First, for each $1 \leq p \leq \ell$, let us choose $s_{i_p} \in \varphi^{-1}(R_{j_p})$ so that $s_{i_1}, \cdots, s_{i_\ell}$ all belong to a single folding branch. This is possible, since $\varphi$ maps each folding branch of the Coxeter diagram $\Gamma$ of $W$ onto the Coxeter diagram $\Gamma_\tau$ of $W_\tau$. By Lemmas 6.2 and 6.3 for those $1 \leq p \leq \ell - 1$ such that $m(R_{j_p}, R_{j_p+1}) = 2$ or $3$, we have $m(s_{i_p}, s_{i_p+1}) = m(R_{j_p}, R_{j_p+1})$. Hence the only situation where the word $s_{i_1} \cdots s_{i_\ell}$ is not reduced is when $R_{j_1} \cdots R_{j_q}$ contains a consecutive subword of the form $[RR']_m R(R')$ where $m(R, R') \geq 4$, that is, equal to either 4 or 5 by Lemma 6.2.

Next, for each $1 \leq p \leq \ell$, if $|\varphi^{-1}(R_{j_p})| = 2$, then insert $s_{i_p}^*$ on the right of $s_{i_p}$. When $|\varphi^{-1}(R_{j_p})| = 1$, we adopt the convention $s_{i_p}^* = s_{i_p}$. By Lemma 6.2, $s_{i_1}, \cdots, s_{i_\ell}$ also belong to a single folding branch (different from the first one if $|\varphi^{-1}(R_{j_p})| = 2$ for at least one $p$). In particular, the observation in the previous paragraph applies to the word $s_{i_1}^* \cdots s_{i_\ell}^*$ as well. We also notice that if $s_{i_p}^* \neq s_{i_p}$ and $s_{i_p}^*$ does not belong to a collapsing part of $\Gamma$, then it commutes with all $s_{i_1}, \cdots, s_{i_{\ell}}$. This is because by Lemma 6.2 each $s \in \Gamma$ not belonging to a collapsing part is disconnected from any folding branch not containing $s$.

Based on these observations, in order to determine if the $W$-word $\varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_q})$ is reduced or not, it suffices to look at the subwords consisting of elements of a collapsing part of $\Gamma$. More precisely, for $R, R' \in S_t$ with $m(R, R') \in \{4, 5\}$, we will study the word of the form $[\varepsilon(R) \varepsilon(R')]_m$ where $4 \leq m \leq m(R, R')$, and show that it is reduced.

First, consider the case $m(R, R') = 4$. The full subgraph $\varphi^{-1}(\{R, R'\})$ is depicted in Lemma 6.2 (C). We have

$$\varepsilon(R)\varepsilon(R')\varepsilon(R)\varepsilon(R') = ss^*tssts^* \sim s^*ststs^*$$

where $\sim$ denotes the braid relation on $F(S)$. Notice that in the second word to last, the second $s^*$ (boxed) is playing the role of a separator between $sts$ and $t$, preventing the occurrence of $sts$. In the last word, the second $s$ (boxed) is separating $s^*sts$ and $t$, preventing the occurrence of $s^*sts^*$. One also checks (by direct manipulation of the word) that $sts$ and $ts^*ts^*$ cannot occur. Thus the word $ss^*sts^*$ is reduced. Next, consider the case $m(R, R') = 5$. The full subgraph $\varphi^{-1}(\{R, R'\})$ is depicted in Lemma 6.2 (D). We have

$$\varepsilon(R)\varepsilon(R')\varepsilon(R)\varepsilon(R') = ss^*tt^*ss^*tt^* \sim s^*tt^*sts^*$$

Observe that in the word on the right, the second $s^*$ (boxed) is separating $sts$ and $t$, preventing the occurrence of $sts$. By looking at all other braid equivalent words, one checks (by a similar separation phenomenon) that $sts$, $s^*ts^*$, $ts^*ts^*$, $s^*t^*t^*$, or $t^*s^*t^*s^*$ cannot occur either. Thus the word $ss^*tt^*ss^*tt^*$ is reduced.

Finally, $\varepsilon(R)\varepsilon(R')\varepsilon(R)\varepsilon(R') = ss^*tt^*ss^*tt^*ss^* \sim s^*tt^*sts^*$. Observe that in the word on the right, once again $s^*$ (boxed) is separating $sts$ and $t$ preventing the occurrence of either $sts$ or $ststs$. By looking at all other braid equivalent words, one checks (by a similar separation phenomenon) that $sts$, $s^*ts^*$, $ts^*ts^*$, $s^*t^*t^*$, $t^*s^*t^*$, and $t^*s^*t^*t^*$ cannot occur either. Thus the word $ss^*tt^*ss^*tt^*$ is reduced. \qed
Corollary 6.8. If $w' \leq u$ in $W_\tau$, then $\varepsilon(w') \leq \varepsilon(u)$ in $W$.

Recall the notation $\mathcal{R}(W)$ from Section 22; Let $F(S_\tau)$ be the free monoid generated on the Coxeter generating set $S_\tau$. The map $\varphi : S \to S_\tau$ induces a new map $\hat{\varphi} : \mathcal{R}(W) \to F(S_\tau)$ as follows. Let $i := s_{i_1} \cdots s_{i_k} \in \mathcal{R}(W)$. Then apply $\varphi$ to each $s_{i_p}$, and whenever $\varphi(s_{i_p}) = \varphi(s_{i_{p+1}})$ occurs, replace the word $\varphi(s_{i_p})\varphi(s_{i_{p+1}})$ by $\varphi(s_{i_p})$. We introduce one notation to facilitate our discussion.

Definition 6.9. For $w \in W$, we define $\mathcal{R}_\varphi(w)$ to be the subset of $\mathcal{R}(w)$ consisting of reduced words $i$ such that the number of adjacency of opposite simple reflections appearing in $i$ is maximal among all the elements of $\mathcal{R}(w)$. We define

$$\mathcal{R}_\varphi(W) := \bigcup_{w \in W} \mathcal{R}_\varphi(w).$$

For example, in the folding from type $A_3$ to $C_2$, consider $w = s_2s_1s_2s_3 \in W$. We have

$$\mathcal{R}(w) = \{s_2s_1s_2s_3, s_1s_2[s_3s_1], s_1s_2s_3s_1\}, \quad \mathcal{R}_\varphi(w) = \{s_1s_2[s_3s_1], s_1s_2s_3s_1\},$$

where the adjacent opposite simple reflections are square-bracketed. The maximal number of adjacency of opposite simple reflections is one.

The following statement about $\hat{\varphi}$ is the foundation for the proof of Proposition 6.10.

Proposition 6.10. Let $w \in W$ and let $i = s_{i_1} \cdots s_{i_k} \in \mathcal{R}_\varphi(w)$. Then $\hat{\varphi}(i) \in \mathcal{R}(W_\tau)$.

Proof. First, we prove the following claim:

Claim 6.11. Let $w$ and $i$ be as in the statement. Let $j := \hat{\varphi}(i) = R_{j_1} \cdots R_{j_k}$ where $1 \leq k \leq \ell$ and each $R_{j_q}$ denotes a simple reflection in $W_\tau$. Then for $j' := R_{j_1} \cdots R_{j_{k-1}}$, there exists $w' \in W$ and $i' \in \mathcal{R}_\varphi(w')$ and $\hat{\varphi}(i') = j'$.

There are $1 = a_1 < \cdots < a_k \leq \ell$ such that $\varphi(s_{i_{a_p}}) = R_{j_p}$ and $\varphi(s_{i_{a_p}}) \neq \varphi(s_{i_{a_p+1}})$ for each $1 \leq p \leq k$. Take $w' = s_{i_1}s_{i_2} \cdots s_{i_{a_k-1}}$. The word $i' := s_{i_1}s_{i_2} \cdots s_{i_{a_k-1}}$ belongs to $\mathcal{R}(w')$ since it is a consecutive subword of the reduced word $i$. Moreover, since $i'$ is obtained from $i \in \mathcal{R}_\varphi(w)$ by removing either $s_{i_{a_k}}$, or the word $s_{i_{a_k}}^* s_{i_{a_k}}$ from the right end, $i' \in \mathcal{R}_\varphi(w')$ must be the case (otherwise, $i \notin \mathcal{R}_\varphi(w)$). This proves the claim.

Since by construction we have $R_{j_q} \neq R_{j_{q+1}}$ for all $1 \leq q \leq k-1$, in order for the sequence $R_{j_1} \cdots R_{j_k}$ not to be reduced, the only possibility is that it contains a pattern $[RR']_m$ where $R, R' \in S_\tau$ and $m > m(R, R')$. We argue that this situation cannot occur.

Let us consider all possible patterns of the $\varphi$-preimage of each pair of simple reflections $R, R' \in S_\tau$ such that $m(R, R') \geq 3$. There are four distinct patterns as in Lemma 6.2. In (A), there is nothing to prove. In (B), let us list all words $i \in \mathcal{R}_\varphi(W)$ such that $\hat{\varphi}(i) = RR'R$. Up to symmetry between $s$ and $s^*$ (resp. $t$ and $t^*$), they are:

$$[ss^*][tt^*][ss^*], \quad [ss^*][tt^*]s, \quad s[tt^*][ss^*], \quad s[tt^*]s, \quad s[tt^*]^*s, \quad sts.$$

where a pair of opposite simple reflections are enclosed by brackets. Direct computation shows that adding $t$ or $t^*$ to the right of each of the above reduced word will result in either a non-reduced word or a reduced word not in $\mathcal{R}_\varphi(W)$. We give two examples to describe these points:

$$[ss^*][tt^*][ss^*]t \sim stst^*s^*s^*, \quad stst^* \sim s[tt^*]s.$$  

The first shows a case where the word is not reduced (see the underlined part), and the second shows a case where the word is reduced but not in $\mathcal{R}_\varphi(W)$.

In (C), let us list all words $i \in \mathcal{R}_\varphi(W)$ such that $\hat{\varphi}(i) = RR'R$. We give two examples to describe these points:

$$[ss^*[t][ss^*]t, \quad st[ss^*]t, \quad s^*t[ss^*]t.$$  

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Direct computation shows that adding $s$ or $s^*$ to the right end of each of the above word results in either a non-reduced word or a reduced word not in $\mathcal{R}_\varphi(W)$:

\[
\begin{align*}
[ss^*]t[ss^*]tts & \sim ss^* tsts^*tst & st[ss^*]tsts^*t & \sim ststs^*t \\
[ss^*]t[ss^*]tts & \sim s^* tsts^*tst & s^t[ss^*]tsts^*t & \sim s^* tsts^*tst \\
st[ss^*]tts & \sim [ss^*]t[ss^*]t & s^t[ss^*]t[ss^*]t & \sim [ss^*]t[ss^*]t
\end{align*}
\]

The underlined parts indicate that the word is not reduced. The last line of each column shows examples where the word is reduced but not in $\mathcal{R}_\varphi(W)$.

In (D), let us list all reduced words $i \in \mathcal{R}_\varphi(W)$ such that $\varphi(i) = RR'R'R'R$.

\[
\begin{align*}
[sst^*][tt^*][ss^*][tt^*][ss^*] & \quad [ss^*][tt^*][ss^*][tt^*][ss^*] & \quad [ss^*][tt^*][tt^*][ss^*][tt^*] & \quad [ss^*][tt^*][tt^*][ss^*][tt^*] \\
s[tt^*][ss^*][tt^*][ss^*] & \quad s[tt^*][ss^*][tt^*][ss^*] & \quad s[tt^*][ss^*][tt^*][ss^*] & \quad s[tt^*][ss^*][tt^*][ss^*] \\
s^*tt^*[ss^*][tt^*][ss^*] & \quad s^*tt^*[ss^*][tt^*][ss^*] & \quad s^*tt^*[ss^*][tt^*][ss^*] & \quad s^*tt^*[ss^*][tt^*][ss^*] \\
[ss^*][tt^*][tt^*][ss^*][tt^*] & \quad [ss^*][tt^*][tt^*][ss^*][tt^*] & \quad [ss^*][tt^*][tt^*][ss^*][tt^*] & \quad [ss^*][tt^*][tt^*][ss^*][tt^*] \\
&s[tt^*][tt^*][ss^*] & \quad s^*tt^*[ss^*][tt^*] & \quad [ss^*][ss^*][tt^*][ss^*] & \quad [ss^*][ss^*][tt^*][ss^*] \\
&s^*tt^*[tt^*][ss^*] & \quad [ss^*][ss^*][tt^*][ss^*] & \quad [ss^*][ss^*][tt^*][ss^*] & \quad [ss^*][ss^*][tt^*][ss^*] \\
&s[tt^*][ss^*][ss^*] & \quad st[ss^*][tt^*][ss^*] & \quad st[ss^*][tt^*][ss^*] & \quad st[ss^*][tt^*][ss^*]
\end{align*}
\]

Direct computation shows that adding $t$ or $t^*$ to the right end of each of the above words results in either a non-reduced word, or a reduced word not in $\mathcal{R}_\varphi(W)$. Given below are two examples to illustrate these points. The first is an example of a non-reduced word (see the underlined part). The second is an example of a reduced word not in $\mathcal{R}_\varphi(W)$ (compare with the last line).

\[
\begin{align*}
[ss^*][tt^*][ss^*][tt^*][ss^*]t & \sim ss^* tt^* s^* ststs^* s^* & s[tt^*][ss^*][tt^*][ss^*]t & \sim t^* ststs^* tt^* ss^* t \\
ss^*tt^*ts^*tst^*s^*t & \sim ss^*tt^*ts^*tst^*s^*t & t^*ststs^*tt^*ss^*t & \sim [tt^*][ss^*][tt^*][ss^*][tt^*] \\
ss^*tt^*ts^*tst^*s^*sts^* & \sim ss^*tt^*ts^*tst^*s^*sts^* & [tt^*][ss^*][tt^*][ss^*][tt^*] & \sim st^*ss^*tt^*ts^*sts^*t
\end{align*}
\]

Finally, by Claim 6.11, we may discard the possibility of the appearance of the word $[RR']_m$ in $R_j, \ldots, R_k$ where $R, R' \in S_{\tau}$ for all $m > m(R, R')$. This completes the proof.

\section{Liftings of Schubert classes}

Let $u \in W_{\tau}$ with $l_{\tau}(u) = \ell$ so that $\deg \sigma^{(u)}_{\tau} = \ell$. Motivated by Lemma 4.8 in order to find a lifting of $\sigma^{(u)}_{\tau}$, we look for an element $w \in W_{\tau}$ with the following properties:

\begin{enumerate}
\item[(C1)] $l(w) = \ell,$
\item[(C2)] $w \leq \varepsilon(u),$
\item[(C3)] $w \nleq \varepsilon(u')$ for all $u' \in W_{\tau}$ such that $u \nleq u'$.
\end{enumerate}

\begin{proposition}
Let $u \in W_{\tau}$ with $l_{\tau}(u) = \ell$ and $w \in W_{\tau}$. Then $\varepsilon^*(\sigma^{(w)}_{\tau}) = c \cdot \sigma^{(w)}_{\tau}$ for some nonzero $c \in L$ if and only if $w$ satisfies the above conditions (C1)–(C3).
\end{proposition}

\begin{proof}
($\Rightarrow$) Suppose $\varepsilon^*(\sigma^{(w)}_{\tau}) = c \cdot \sigma^{(w)}_{\tau}$ for some $c \in L$. Then we have $l(w) = \deg \sigma^{(w)}_{\tau} = \deg \varepsilon^*(\sigma^{(w)}_{\tau}) = \deg \sigma^{(u)}_{\tau} = l_{\tau}(u) = \ell$. Thus (C1) is verified. Since $0 \neq c \cdot \sigma^{(u)}_{\tau}(u) = \varepsilon^*(\sigma^{(w)}_{\tau})(u) = \pi_{\tau}(\sigma^{(w)}_{\tau}(\varepsilon(u)))$, we have $\sigma^{(w)}_{\tau}(\varepsilon(u)) = 0$. By Definition 4.8, $w \leq \varepsilon(u)$ follows, verifying (C2).

Finally, let $u' \in W_{\tau}$ with $u \nleq u'$. We have $0 = c \cdot \sigma^{(u')}_{\tau}(u') = \varepsilon^*(\sigma^{(w)}_{\tau})(u') = \pi_{\tau}(\sigma^{(w)}_{\tau}(\varepsilon(u')))$. Since $\pi_{\tau}$ is injective (Remark 5.9), we obtain $\sigma^{(w)}_{\tau}(\varepsilon(u')) = 0$. Now Lemma 4.3(c) gives us $w \nleq \varepsilon(u')$, verifying (C3).
\end{proof}
Let \( w \) satisfy (C1)–(C3). We show that \( \varepsilon^*(\sigma^*(w)) \) is homogeneous of degree \( \ell \) with properties \( \varepsilon^*(\sigma^*(w))(u) \neq 0 \) and \( \varepsilon^*(\sigma^*(w))(u') = 0 \) for all \( u' \neq u \), then apply Lemma 1.8. First, we have \( \deg \varepsilon^*(\sigma^*(w)) = \deg \sigma^*(w) = l(w) = \ell \) by (C1). Next, we have \( \varepsilon^*(\sigma^*(w))(u) = \pi_\tau(\sigma^*(w)(\varepsilon(u))) \). By (C2), we have \( \varepsilon(u) \geq w \), hence \( \sigma^*(w)(\varepsilon(u)) \neq 0 \) by Lemma 4.3 (c). Since \( \pi_\tau \) is injective, we obtain \( \varepsilon^*(\sigma^*(w))(u) \neq 0 \). Finally, let \( u' \neq u \). By (C3), we have \( \varepsilon(u') \notin w \) which implies \( \sigma^*(w)(\varepsilon(u')) = 0 \) by Definition 4.1. Thus \( \varepsilon^*(\sigma^*(w))(u') = \pi_\tau(\sigma^*(w)(\varepsilon(u')))) = 0. \)

Proposition 7.1 states the condition for a Schubert class \( \sigma^*(w) \) of \( Z(G_P^*) \) to admit a lifting \( \sigma^*(w) \) in \( Z(G_P^*) \) in terms of the relationship between the group elements \( u \in W_P^* \) and \( w \in P \). This provides the first answer to Question 1.1. However, verifying (C3) is not easy even for the smallest examples of foldings since it requires the full description of the moment graphs involved. We aim to restate the conditions (C1)–(C3) using the combinatorics of Coxeter groups.

**Definition 7.2.** The folding subset of \( W \), denoted by \( W^F \), is a subset of \( W \) consisting of the identity element \( e \in W \) and elements \( w \in W \) satisfying the following.

1. **(FS1)** If \( s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w) \), then \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \in \mathcal{R}(W_\tau) \).
2. **(FS2)** If \( s_{i_1} \cdots s_{i_\ell}, s_{j_1} \cdots s_{j_\ell} \in \mathcal{R}(w) \), then \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}), \varphi(s_{j_1}) \cdots \varphi(s_{j_\ell}) \in \mathcal{R}(u) \) for some common \( u \in W_\tau \).

**Remark 7.3.** (a) Let \( w \in W^F \) and \( s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w) \). For any \( 1 \leq p < q \leq \ell \), the consecutive subexpression \( w' = s_{i_p} \cdots s_{i_q} \) satisfies (FS1) and (FS2). Hence \( w' \in W^F \).

(b) Let us recall the map \( \tilde{\varphi} : \mathcal{R}(W) \to F(S_\tau) \) defined before Proposition 6.10. By (FS1), the map \( \varphi : \mathcal{R}(W) \to F(S_\tau) \) restricted to \( \mathcal{U}_{w \in W^F} \mathcal{R}(w) \) is the same as applying \( \varphi : S \to S_\tau \) to each simple reflection in \( s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w) \).

(c) Let \( w \in W \) with a unique reduced expression \( s_{i_1} \cdots s_{i_\ell} \). Then \( w \in W^F \) by the following argument. By (b) and Proposition 6.10, \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \in \mathcal{R}(W_\tau) \), verifying (FS1). Since \( \mathcal{R}(w) \) is a singleton, (FS2) trivially holds.

By definition of the folding subset \( W^F \), the map \( \varphi : S \to S_\tau \) extends to

\[
\overline{\varphi} : W^F \to W_\tau : w = e \mapsto e
w = s_{i_1} \cdots s_{i_\ell} \mapsto \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})
\]

where \( w = s_{i_1} \cdots s_{i_\ell} \) is any reduced expression. The condition (FS2) guarantees that \( \overline{\varphi} \) is well-defined, and (FS1) guarantees that it preserves the length, that is, \( l_\tau(\overline{\varphi}(w)) = l(w) \) for \( w \in W^F \).

Theorem 7.7 will restate the lifting conditions (C1)–(C3) solely in terms of \( \overline{\varphi} \).

**Proposition 7.4.** Let \( u \in W_\tau \) with \( l_\tau(u) = \ell \). Suppose \( w \in W \) satisfies (C1), (C2) and \( w \in W^F \). Then \( w \) satisfies (C3).

**Proof.** Let \( w \in W^F \) and let \( u = R_{j_1} \cdots R_{j_\ell} \) be a reduced expression. By applying the group homomorphism \( \varepsilon \), we obtain

\[
\varepsilon(u) = \varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_\ell}),
\]

which is a reduced expression in \( W \) by Proposition 6.7. Since \( w \leq \varepsilon(u) \), by Lemma 2.3 (b), \( w \) admits some reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \) which is a subexpression of \( \varepsilon \). Since \( w \in W^F \), we have \( \varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}}) \) for all \( 1 \leq p \leq \ell - 1 \). By Lemma 6.4, the sequence \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \) is a subexpression of \( u = R_{j_1} \cdots R_{j_\ell} \). Since the numbers of simple reflections in the two expressions are the same, we obtain \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = R_{j_1} \cdots R_{j_\ell} = u. \)
Let \( u' \in W_\tau \) such that \( w \leq \varepsilon(u') \). We aim to show \( u \leq u' \). Let \( u' = R_{j_1}^{\prime} \cdots R_{j_k}^{\prime} \) be a reduced expression. Then

\[
\varepsilon(u') = \varepsilon(R_{j_1}^{\prime}) \cdots \varepsilon(R_{j_k}^{\prime})
\]

(9)
is a reduced expression in \( W \). Since \( w \leq \varepsilon(u') \), by Lemma 2.3 (b), \( w \) admits some reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \) which is a subexpression of \( u' \). Since \( w \in W^F \), we have \( \varphi(s_{i_\ell}) \neq \varphi(s_{i_{\ell+1}}) \) for all \( 1 \leq p \leq \ell - 1 \). By Lemma 6.4, it follows that \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \) is a subexpression of \( u' = R_{j_1}^{\prime} \cdots R_{j_k}^{\prime} \). Since \( w \in W^F \), \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_k}) = u \). By Lemma 2.3 (b), \( u \leq u' \) follows.

**Proposition 7.5.** Let \( u \in W_\tau \) with \( l_\tau(u) = \ell \). Suppose \( w \in W \) satisfies the conditions (C1)–(C3). Then \( w \in W^F \).

**Proof.** Our aim is to establish the defining properties (FS1), (FS2) of \( W^F \) for \( w \). First, we prove the following claim, which is apparently weaker than (FS1):

**Claim 7.6.** Every reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \) satisfies \( \varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}}) \) for all \( 1 \leq p \leq \ell - 1 \).

By the definition of the set \( R_\varphi(w) \) (see before Proposition 6.10), it suffices to prove the statement for the elements of \( R_\varphi(w) \). Let \( s_{i_1} \cdots s_{i_\ell} \in R_\varphi(w) \) and recall the map \( \hat{\varphi} : R(W) \to F(S_\tau) \). By Proposition 6.10, \( \hat{\varphi}(s_{i_1} \cdots s_{i_\ell}) \) is \( W_\tau \)-reduced. Write the \( W_\tau \)-reduced word so obtained as \( R_{j_1}^{\prime} \cdots R_{j_k}^{\prime} \) for some \( k \leq \ell \), and let \( u' := R_{j_1}^{\prime} \cdots R_{j_k}^{\prime} \in W_\tau \). Suppose \( \varphi(s_{i_\ell}) = \varphi(s_{i_{\ell+1}}) \) for some \( 1 \leq p \leq \ell - 1 \). Then \( k < \ell \), that is, \( l_\tau(u') < l_\tau(u) \). Since \( w = s_{i_1} \cdots s_{i_\ell} \) is a subexpression of \( \varepsilon(u') = \varepsilon(R_{j_1}^{\prime}) \cdots \varepsilon(R_{j_k}^{\prime}) \) by construction, by Lemma 2.3 (b), we obtain \( w \leq \varepsilon(u') \) where \( u \neq u' \), a contradiction to (C3). Hence \( \varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}}) \) for all \( 1 \leq p \leq \ell - 1 \).

We now proceed to prove (FS1). Knowing that every element of \( R(w) \) has no adjacency of pairs of opposite simple reflections has two consequences. First, Proposition 6.10 applies to each \( s_{i_1} \cdots s_{i_\ell} \in R(w) \). Second, the map \( \hat{\varphi} \) coincides with the operation of applying \( \varphi \) to each \( s_{i_p} \). Thus, for each \( s_{i_1} \cdots s_{i_\ell} \in R(w) \), the expression \( \hat{\varphi}(s_{i_1} \cdots s_{i_\ell}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \) is \( W_\tau \)-reduced by Proposition 6.10. Finally, we prove (FS2). Let \( u = R_{j_1}^{\prime} \cdots R_{j_k}^{\prime} \) be a reduced expression. Since \( w \leq \varepsilon(u) \), by Lemma 2.3 (b), \( w \) has a reduced expression \( s_{i_1} \cdots s_{i_\ell} \) which is a subexpression of \( \varepsilon(R_{j_1}^{\prime}) \cdots \varepsilon(R_{j_k}^{\prime}) \). Since \( \varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}}) \) holds for all \( 1 \leq p \leq \ell - 1 \), by Lemma 6.4, \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \) is a subexpression of \( R_{j_1}^{\prime} \cdots R_{j_k}^{\prime} \). Since the numbers of simple reflections involved in the two expressions are the same, we obtain \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = R_{j_1}^{\prime} \cdots R_{j_k}^{\prime} \). Suppose that there is \( s_{i_1}^{\prime} \cdots s_{i_\ell}^{\prime} \in R(w) \) with \( \varphi(s_{i_1}^{\prime}) \cdots \varphi(s_{i_\ell}) \in \varepsilon(u') \) for some \( u' \neq u \in W_\tau \). By construction, \( s_{i_1}^{\prime} \cdots s_{i_\ell}^{\prime} \) is a subexpression of \( \varepsilon(\varphi(s_{i_1}^{\prime})) \cdots \varepsilon(\varphi(s_{i_\ell}^{\prime})) \). Since \( u' = \varphi(s_{i_1}^{\prime}) \cdots \varphi(s_{i_\ell}^{\prime}) \) is \( W_\tau \)-reduced by (FS1), \( \varepsilon(u') = \varepsilon(\varphi(s_{i_1}^{\prime})) \cdots \varepsilon(\varphi(s_{i_\ell}^{\prime})) \) is a \( W \)-reduced expression (Proposition 6.7). By Lemma 2.3 (b), we obtain \( u = s_{i_1} \cdots s_{i_\ell} \leq \varepsilon(u') \) where \( l_\tau(u') < l_\tau(u) \) and \( u' \neq u \). This is a contradiction to (C3).

**Theorem 7.7.** Let \( u \in W_\tau \) with \( l_\tau(u) = \ell \) and \( w \in W \). Then \( w \) satisfies (C1)–(C3) if and only if \( w \in W^F \) and \( u = \overline{\tau}(w) \).

**Proof.** Suppose \( w \) satisfies (C1)–(C3). Then by Proposition 2.3, \( w \in W^F \). Let \( u = R_{j_1} \cdots R_{j_k} \) be a reduced expression. Since \( l(w) = \ell \) and \( w \leq \varepsilon(u) \), by Lemma 2.3 (b), \( w \) has a reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \) which is a subexpression of \( \varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_k}) \). Since we have \( \varphi(s_{i_1}) \neq \varphi(s_{i_{\ell+1}}) \) for all \( 1 \leq p \leq \ell - 1 \) thanks to \( w \in W^F \), by Lemma 6.4, it follows that \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \) is a subexpression of \( R_{j_1} \cdots R_{j_k} \). Since the numbers of simple reflections in the two expressions are the same, we obtain \( \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = R_{j_1} \cdots R_{j_k} \), that is, \( \overline{\tau}(w) = u \).

Conversely, suppose \( w \in W^F \) and \( u = \overline{\tau}(w) \). Since \( \overline{\tau} \) is length-preserving, \( l(w) = \ell \) must hold, establishing (C1). Next, \( u = \overline{\tau}(w) \) implies that for some \( W \)-reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \), we have \( u = \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \) which is \( W_\tau \)-reduced. Then \( s_{i_1} \cdots s_{i_\ell} \) is a subexpression
of $\varepsilon(\varphi(s_{i_1})) \cdots \varepsilon(\varphi(s_{i_\ell}))$ by construction. Hence by Lemma 2.3(b), $w \leq \varepsilon(u)$, establishing (C2).

Finally, by Proposition 7.3, $w \in W^F$ implies (C3).

By Proposition 7.1 and Theorem 7.7 we obtain the following lifting criterion.

**Corollary 7.8.** Let $u \in W^P_r$. Then the Schubert class $\sigma_r^{(u)}$ admits a lifting to $Z(G^P_r)$ if and only if $u \in \overline{P}(W^F \cap W^P_r)$.

**Corollary 7.9.** Let $u \in W^P_r$ with $l_r(u) = \ell$. If there exists $w \in W^P$ with a unique reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ and if $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = u$, then $\sigma_r^{(u)}$ lifts to $\sigma_r^{(w)}$.

**Proof.** By Remark 7.3(c), $w$ belongs to $W^F$, and we have $\overline{P}(w) = u$. Hence by Corollary 7.8, $\sigma_r^{(u)}$ admits a lifting, namely $\sigma_r^{(w)}$.

When a Schubert class $\sigma_r^{(u)} \in Z(G^P_r)$ lifts to $\sigma_r^{(w)} \in Z(G^P_r)$, we explicitly know the nonzero constant $c \in \mathbb{L}$ appearing in the Definition 12.

**Proposition 7.10.** Let $\sigma_r^{(w)} \in Z(G^P_r)$ be a lifting of $\sigma_r^{(u)} \in Z(G^P_r)$. Then $\varepsilon^*(\sigma_r^{(w)}) = \tau^m \cdot \sigma_r^{(u)}$ for some unique $m \in \mathbb{Z}_{\geq 0}$.

**Proof.** We have $\varepsilon^*(\sigma_r^{(w)}) = c \cdot \sigma_r^{(u)}$ for some nonzero $c \in \mathbb{L}$. In particular, by looking at the coordinate at $\varepsilon(u)$, this implies $\pi_r(\varepsilon(\sigma_r^{(w)}(\varepsilon(u)))) = c \cdot \sigma_r^{(u)}(u)$. Let $w = s_{i_1} \cdots s_{i_\ell}$ be a $W$-reduced expression, and write $\varphi(s_{i_1}) = R_i$. Then $u = \overline{P}(w) = R_1 \cdots R_\ell$ is a $W$-reduced expression (since $w \in W^F$ by Theorem 7.7). Apply $\varepsilon$ to this expression and write the $W$-reduced expression so obtained as $\varepsilon(u) = s'_{i_1} \cdots s'_{i_\ell}$ for appropriate $L$. By construction, this expression contains $w = s_1 \cdots s_\ell$ as a subexpression. Denote this subexpression by $s'_{i_1} \cdots s'_{i_\ell}$ where $1 \leq i_1 < \cdots < i_\ell \leq \ell$. We have $s'_{i_p} = s_p$ and $\varphi(s'_{i_p}) = \varphi(s_p) = R_p$ for each $1 \leq p \leq \ell$. Let $\beta_p$ denote the simple root corresponding to $s_p$, and let $\beta_p$ denote the simple root corresponding to $R_p$. By Definition 4.1 we compute

$$\sigma_r^{(w)}(\varepsilon(u)) = \prod_{p=1}^\ell r_p(\alpha_p),$$

where $r_p(\alpha_p) = s'_{i_1}s'_{i_2} \cdots s'_{i_{p-1}}(\alpha_p)$. By Lemma 5.3,

$$\pi_r(r_p(\alpha_p)) = \pi_r(\varepsilon(R_1 \cdots R_{p-1})(\alpha_p)) = \begin{cases} R_1 \cdots R_{p-1}(\beta_p) & \text{if } \alpha_p \in \Delta^T \sqcup \Delta_{\text{rat}}, \\ R_1 \cdots R_{p-1}(\tau \beta_p) & \text{if } \alpha_p \in \mathcal{T}(\Delta_{\text{rat}}). \end{cases}$$

Hence we obtain

$$\pi_r(\sigma_r^{(w)}(\varepsilon(u))) = \tau^m \prod_{p=1}^\ell R_1 \cdots R_{p-1}(\beta_p) = \tau^m \cdot \sigma_r^{(u)}(u)$$

where $m$ is the number of elements $\alpha_p \in \Delta_{\text{rat}}$ appearing in (10). Thus $c = \tau^m$.

**Definition 7.11.** Let $u \in W^P_r$. We say that $u$ is liftable or admits a lifting if there exists an element $w \in W^P$ such that $w \in W^F$ and $\overline{P}(w) = u$. Any such element $w \in W^P$ is called a lifting of $u$ and we say that $u$ lifts to $w$.

Using this terminology, Corollary 7.8 says that a Schubert class $\sigma_r^{(u)} \in Z(G^P_r)$ lifts to $\sigma_r^{(w)} \in Z(G^P_r)$ if and only if $u$ lifts to $w$.

**Example 7.12 (Type $A_4$ to $H_2$).** Let $\Phi \to \Phi_r$ be the folding of the root system of type $A_4$ to $H_2$. See the folding diagram in Section 3.4. The folded Coxeter group $W_r$ is the dihedral group $D_{10}$. For each $u \in W_r$ with $0 \leq l_r(u) \leq 4$, Table 1 shows a complete list of liftings $w \in W^F$. In this case, every lifting $w \in W^F$ has a unique reduced expression. It will be shown in Lemma 8.6 (VI)(a) that the longest element $u = R_1 R_2 R_1 R_2 R_1 = R_2 R_1 R_2 R_1 R_2$ is not liftable.
Table 1: Liftings of $W_\tau$-elements: type $A_4$ to type $H_2$

| $l_\tau(u)$ | $u \in W_\tau$ | Liftings $w \in W^F \cap W^P$ |
|-----------|----------------|----------------------------------|
| 4         | $R_1 R_2 R_1 R_2$ | $s_1 s_2 s_3 s_4$ |
| 3         | $R_1 R_2 R_2$ | $s_1 s_2 s_3 s_4$ |
| 2         | $R_1 R_2$ | $s_1 s_2 s_3 s_4$ |
| 1         | $R_1$ | $s_1$ |
| 0         | $e$ | $e$ |

Table 2: Liftings of $W_\tau$-elements: type $D_6/A_4$ to type $H_3/H_2$

| $l_\tau(u)$ | $u \in W_\tau$ | Liftings $w \in W^F \cap W^P$ |
|-----------|----------------|----------------------------------|
| 10        | $R_1 R_2 R_3 R_2 R_1 R_2 R_3 R_2 R_1$ | $s_1 s_2 s_3 s_4 s_5 s_6 s_8 s_4 s_3 s_2 s_1$ |
| 9         | $R_2 R_3 R_2 R_1 R_3 R_2 R_3 R_2 R_1$ | $s_2 s_3 s_4 s_5 s_6 s_8 s_3 s_2 s_1$ |
| 8         | $R_3 R_2 R_1 R_3 R_2 R_3 R_2 R_1$ | $s_3 s_4 s_5 s_6 s_8 s_3 s_2 s_1$ |
| 7         | $R_2 R_3 R_2 R_3 R_2 R_1 R_2 R_3 R_1$ | $s_4 s_5 s_6 s_8 s_3 s_2 s_1$ |
| 6         | $R_1 R_3 R_2 R_3 R_2 R_1$ | $s_5 s_6 s_8 s_3 s_2 s_1$ |
| 5         | $R_3 R_2 R_3 R_2 R_1$ | $s_6 s_3 s_2 s_1$ |
| 4         | $R_2 R_3 R_2 R_1$ | $s_7 s_8 s_3 s_2 s_1$ |
| 3         | $R_3 R_2 R_1$ | $s_8 s_3 s_2 s_1$ |
| 2         | $R_2 R_1$ | $s_9 s_8 s_3 s_2 s_1$ |
| 1         | $R_1$ | $s_1, s_3$ |
| 0         | $e$ | $e$ |

Example 7.13 (Type $D_6/A_4$ to $H_3/H_2$). Let $\Phi \rightarrow \Phi_\tau$ be the folding of the root system of type $D_6$ to $H_3$. See the folding diagram in Section 5.4. We take the parabolic subgroup $W_P := \langle s_2, s_3, s_4, s_6 \rangle$, which is of type $A_4$. We have $(W_P)_\tau = \langle R_2, R_3 \rangle$, and it forms a parabolic subgroup of $W_\tau$ of type $H_2$. For each $u \in W_\tau^F$, Table 2 shows a complete list of liftings $w \in W^F \cap W^P$. In Table 2 observe first that each $w = s_{i_1} \cdots s_{i_p}$ (reduced expression) satisfies $\varphi(s_{i_p}) = R_{i_p}$ for each $1 \leq p \leq \ell$, where $u = R_{i_1} \cdots R_{i_\ell}$ is the prescribed reduced expression of $u$. For those $w \in W^P$ with $0 \leq l(w) \leq 5$, $w$ has a unique reduced expression. Hence by Corollary 7.9, $\sigma_w^{(u)}$ lifts to $\sigma^{(w)}$. For those $w \in W^P$ with $6 \leq l(w) \leq 10$, observe that each $w$ has precisely two distinct reduced expressions by commuting $s_5$ and $s_6$, and that the two reduced expressions obtained by applying $\varphi$ to each $s_{i_p}(1 \leq p \leq l(w))$ give rise to the same element $u \in W_\tau$. Hence $w \in W^F \cap W^P$, and by Corollary 7.8, $\sigma_w^{(u)}$ lifts to $\sigma^{(w)}$ for $w \in W$ given in Table 2.

8 Nonliftable elements

In this section, we discuss a way to detect nonliftable elements of $W_\tau$.

Lemma 8.1. Let $u \in W_\tau$ with $l_\tau(u) = \ell$ and let $w \in W^F$ be a lifting of $u$. Let $R_{i_1}, \cdots R_{i_\ell} \in R(u)$ and suppose that there exists $s_{i_1} \cdots s_{i_p} \in R(u)$ such that $\varphi(s_{i_p}) = R_{i_p}$ for each $1 \leq p \leq \ell$. Then for any $1 \leq p \leq q \leq \ell$, the consecutive subexpression $u' := R_{i_p} R_{i_{p+1}} \cdots R_{i_q}$ lifts to $w' := s_{i_p} s_{i_{p+1}} \cdots s_{i_q}$.
Proof. Since \( w' = s_{i_p} \cdots s_{i_t} \) is a consecutive subexpression of \( w = s_{i_1} \cdots s_{i_t} \in W^F \), \( w' \) also belongs to \( W^F \) (Remark 7.3 (a)). By construction, \( \varphi(w') = u' \). Hence \( u' \) is a lifting of \( u' \). \( \square \)

Lemma 8.2. Let \( R, R' \in S_r \).

(a) If \( m(R, R') = 4 \), then \( u = [R, R']_4 = RR'R'R' \) is not liftable.

(b) If \( m(R, R') = 5 \), then \( u = [R, R']_5 = RR'R'R'R \) is not liftable.

Remark 8.3. If \( m(R, R') = 3 \), then \( u = RR'R \) is always liftable (by the existence of a folding branch, see after Definition 6.5).

Proof. (a) The set \( \mathcal{R}(u) \) consists of precisely two elements: \( RR'R \) and \( R'R'R'R. \) By Lemma 6.2 in the Coxeter diagram, the configuration of the full subgraph \( \varphi^{-1}([R, R']) \) is (C) (up to symmetry of \( R \) and \( R' \)). Suppose, for the sake of contradiction, \( w \in W^F \) is a lifting of \( u \).

Case 1: \( RR'R'R \in \widehat{\varphi}(\mathcal{R}(w)) \). There exists \( \mathbf{i} := s_{i_1} \cdots s_{i_t} \in \mathcal{R}(w) \) such that
\[
\widehat{\varphi}(\mathbf{i}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_t}) = RR'R'R.
\]

By Lemma 8.1, \( w = s_{i_1}s_{i_2}s_{i_3} \) is a lifting of \( u' = RR'R'. \) Since \( w' \in W^F \), the only possibilities are \( s_{i_1}s_{i_2}s_{i_3} = sts^* \) or \( s^*ts. \) On the other hand, \( \varphi(s_{i_1}) = R' \) forces \( s_{i_3} = t. \) However, both \( w = sts^t \) and \( w = ts^st \) violate (FS2) of Definition 7.2.

Case 2: \( R'R'R'R \in \widehat{\varphi}(\mathcal{R}(w)) \). There exists \( \mathbf{i} := s_{i_1} \cdots s_{i_t} \in \mathcal{R}(w) \) such that
\[
\widehat{\varphi}(\mathbf{i}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_t}) = R'R'R'R.
\]

By Lemma 8.1, \( w = s_{i_1}s_{i_2}s_{i_3}s_{i_4} \) is a lifting of \( u' = RR'R'. \) Since \( w' \in W^F \), the only possibilities are \( s_{i_1}s_{i_2}s_{i_3} = sts^* \) or \( s^*ts. \) On the other hand, \( \varphi(s_{i_1}) = R \) forces \( s_{i_2} = s \) or \( s^*. \) However, \( w = sts^t \) and \( w = ts^st \) both violate (FS2) of Definition 7.2.

(b) The set \( \mathcal{R}(u) \) consists of precisely two elements: \( RR'R'R \) and \( R'R'R'R'R. \) By Lemma 6.2 in the Coxeter diagram, the configuration of the full subgraph \( \varphi^{-1}([R, R']) \) is (D) (up to symmetry of \( R \) and \( R' \)). Suppose, for the sake of contradiction, \( w \in W^F \) is a lifting of \( u \).

Case 1: \( RR'R'R \in \widehat{\varphi}(\mathcal{R}(w)) \). There exists \( \mathbf{i} := s_{i_1} \cdots s_{i_t} \in \mathcal{R}(w) \) such that
\[
\widehat{\varphi}(\mathbf{i}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_t}) = RR'R'R.
\]

By Lemma 8.1, \( w = s_{i_1}s_{i_2}s_{i_3}s_{i_4} \) is a lifting of \( u' = RR'R'R. \) Since \( w' \in W^F \), the only possibility is \( s_{i_1}s_{i_2}s_{i_3}s_{i_4} = sts^*t. \) On the other hand, \( \varphi(s_{i_1}) = R \) forces \( s_{i_2} = s \) or \( s^*. \) However, \( w = sts^ts \) and \( w = tsts^t \) both violate (FS2) of Definition 7.2.

Case 2: \( R'R'R'R \in \widehat{\varphi}(\mathcal{R}(w)) \). We may argue in exactly the same way as in Case 1. \( \square \)

Proposition 8.4. Let \( u \in W_r \) with \( l_r(u) = \ell \) and let \( w \in W^F \) be a lifting of \( u \). Then for every \( R_{j_1} \cdots R_{j_\ell} \in \mathcal{R}(w) \), there exists \( s_{i_1} \cdots s_{i_t} \in \mathcal{R}(w) \) such that \( \varphi(s_{i_p}) = R_{j_p} \) for each \( 1 \leq p \leq \ell \).

Proof. We first remark that the last part of the claim is equivalent to \( \varphi(s_{i_1} \cdots s_{i_t}) = R_{j_1} \cdots R_{j_\ell} \) (Remark 7.3 (b)). Let \( \mathbf{i} \in \mathcal{R}(w) \). Assume \( \varphi(\mathcal{R}(w)) \subseteq \mathcal{R}(u) \) and let \( j \in \mathcal{R}(u) \setminus \widehat{\varphi}(\mathcal{R}(w)) \). Since both \( \widehat{\varphi}(\mathbf{i}) \) and \( j \) belong to \( \mathcal{R}(u) \), there exists a finite sequence of \( W_r \)-braid moves that transforms \( \widehat{\varphi}(\mathbf{i}) \) into \( j \), and the corresponding sequence \( \widehat{\varphi}(\mathbf{i}) = j_0 \rightarrow j_1 \rightarrow \cdots \rightarrow j_\ell = j \) of elements of \( \mathcal{R}(u) \). Since \( \widehat{\varphi}(\mathbf{i}) \in \widehat{\varphi}(\mathcal{R}(w)) \) and \( j \notin \widehat{\varphi}(\mathcal{R}(w)) \), there exists an integer \( 0 \leq k \leq r-1 \) such that \( j_k \in \widehat{\varphi}(\mathcal{R}(w)) \) and \( j_{k+1} \notin \widehat{\varphi}(\mathcal{R}(w)) \). Let us write \( j_k = \widehat{\varphi}(y') \) for some \( y' = s_{i'_1} \cdots s_{i'_t} \in \mathcal{R}(w) \), and set \( R_{j'_k} := \varphi(s_{i'_q}) \) for each \( 1 \leq q \leq \ell \). We will argue that in the \( W_r \)-braid move \( \{R_{j'_k}R_{j'_{k+1}}\}_m = [R_{j_{k+1}}R_{j'_k}]_m \) that transforms \( j_k \) into \( j_{k+1} \), \( m \) cannot be equal to 2 or 3. After that, we show that \( m \geq 4 \) is also not possible, yielding a contradiction to our hypothesis.

The case of \( m = 2 \) is discarded as follows. By Lemma 6.3, \( m = 2 \) would imply that any \( s \in \varphi^{-1}(R_{j'_p}) \) and \( t \in \varphi^{-1}(R_{j'_{p+1}}) \) commute. Hence, there is a corresponding \( W_r \)-braid move
Lemma 8.1. If $m \not\in \mathcal{S}_2$, $\mathcal{S}_2$, then

Proposition 8.6. (I)(a), (II)(a), (III)(a): Proof for these three cases follows the same argument as one

Proposition 8.6. (IV)\ 

Similarly, $u' = [RR]'_4$ is not liftable. Since $j_k \in \hat{\varphi}(\mathcal{R}(w))$, this is a contradiction to Lemma 8.1. If $m = 5$, by Lemma 8.2(b), $u'' = [RR]'_5$ is not liftable. Since $j_k \in \hat{\varphi}(\mathcal{R}(w))$, this is a contradiction to Lemma 8.1. By Lemma 6.2 the cases $m \geq 6$ do not exist. This completes the proof.

By a direct application of Lemma 8.1 and Proposition 8.4, we obtain

Corollary 8.5. Let $u \in \mathcal{W}_r$ with $l_r(u) = \ell$. If $u$ admits a $\mathcal{W}_r$-reduced expression $u = R_{j_1} \cdots R_{j_q}$ which has a consecutive subexpression $u' = R_{j_p} R_{j_{p+1}} \cdots R_{j_q}$ $(1 \leq p \leq q \leq \ell)$ such that $u'$ is not liftable, then $u$ is not liftable.

Proposition 8.6. The following elements of $\mathcal{W}_r$ are not liftable:

(1) Type $A_{2n-1}$ to $C_n$: (a) $R_n R_{n-1} R_n$, (b) $R_{n-1} R_{n-2} R_n R_{n-1} R_{n-1}$, (c) $R_{n-1} R_n R_{n-1} R_n$.

(2) Type $D_{n+1}$ to $B_n$: (a) $R_{n-1} R_n R_{n-1}$, (b) $R_n R_{n-1} R_{n-1}$.

(3) Type $E_6$ to $F_4$: (a) $R_2 R_3 R_2$, (b) $R_3 R_4 R_3 R_2 R_3$.

(4) Type $E_8$ to $H_4$: (a) $R_3 R_2 R_3 R_4 R_3$, (b) $R_3 R_4 R_3 R_4 R_3$.

(5) Type $D_6$ to $H_4$: (a) $R_2 R_1 R_2 R_3 R_2$, (b) $R_2 R_3 R_2 R_3 R_2$.

(6) Type $A_4$ to $H_2$: (a) $R_1 R_2 R_1 R_2 R_3$.

Proof. (I)(a), (II)(a), (III)(a): Proof for these three cases follows the same argument as one can see from the folding diagrams in Section 5.3. We present here a proof for (I)(a). Set $u = R_n R_{n-1} R_n$. If $u$ is liftable, then there exists some $w = s_{i_1} s_{i_2} s_{i_3} \in W^F$ such that $\varphi(s_{i_1}) = R_n$, $\varphi(s_{i_2}) = R_{n-1}$ and $\varphi(s_{i_3}) = R_n$ (Proposition 8.4). There are precisely two candidates: $w_1 = s_n s_n s_{n+1}$ and $w_2 = s_n s_{n+1} s_n$. Observe that $s_n s_{n-1} s_n = s_{n-1} s_n s_{n-1}$ in $W$, but

\[ \varphi(s_n) \varphi(s_{n+1}) \varphi(s_n) = R_n R_{n-1} R_n \neq R_{n-1} R_n R_{n-1} = \varphi(s_{n-1}) \varphi(s_n) \varphi(s_{n-1}). \]

Hence $w_1 \not\in W^F$. Similarly, $w_2 \not\in W^F$.

(I)(b) and (III)(a): Proof for these two cases follows the same argument. We present here a proof for (I)(b). Set $u = R_{n-1} R_{n-2} R_{n-1} R_{n-1} R_n$. If $u$ is liftable, then by Proposition 8.4 there exists some $w = s_{i_1} \cdots s_{i_5} \in W^F$ such that $\varphi(s_{i_p}) = R_{j_p}$ for each $1 \leq p \leq 5$. By Lemma 8.3, $s_{i_1} s_{i_2}$ is a lifting of $u_1 = R_{n-1} R_{n-2}$ and $s_{i_3} s_{i_4} s_{i_5}$ is a lifting of $u_2 = R_{n-1} R_{n-1} R_{n-1}$. The element $u_1$ has two liftings $w_1 = s_{n-1} s_n$ and $w_1' = s_{n+1} s_n$, and $u_2$ has also two liftings $w_2 = s_n s_{n+1} s_{n+1}$ and $w_2' = s_{n+1} s_{n+1} s_{n+1}$.

Thus there are four candidates for $w$:

\[
\begin{align*}
    \text{v}_1 &= \text{w}_1 \text{w}_2' = s_{n-1} s_n - 2 s_n s_{n+1} s_{n+1} = s_{n-1} s_n - 2 s_n s_{n+1} s_{n+1}, \\
    \text{v}_2 &= \text{w}_1' \text{w}_2' = s_{n-1} s_n - 2 s_n s_{n+1} s_{n+1}, \\
    \text{v}_3 &= \text{w}_1' \text{w}_2 = s_n s_{n+1} s_{n+1} s_{n+1}, \\
    \text{v}_4 &= \text{w}_1' \text{w}_2' = s_{n+1} s_{n+1} s_{n+1} s_{n+1} = s_{n+1} s_{n+1} s_{n+1} s_{n+1}.
\end{align*}
\]
Notice that a reduced expression of $v_1$ and $v_2$ contains a consecutive subexpression $s_{n-2}s_{n+1}$ and observe that $s_{n-2}s_{n+1} = s_{n+1}s_{n-2}$ in $W$ but $R_{n-2}R_{n-1} \neq R_{n-1}R_{n-2}$ in $W_r$. Hence $v_1, v_2 \notin W^F$. A similar argument shows $v_3, v_4 \notin W^F$ (look at the subexpression $s_{n+2}s_{n-1}$). Hence $u$ is not liftable.

(I)(c), (II)(b), (III)(c) follow from Lemma 8.2(a).

(IV)(a) and (V)(a): Proof for these two cases follows the same argument as one can see from the folding diagrams in Section 5.4. We present here a proof for (IV)(a). Set $u = R_3R_2R_3R_1R_3$. If $u$ is liftable, then by Proposition 8.4 there exists some $w = s_{i_1} \cdots s_{i_5} \in W^F$ such that $\varphi(s_{i_p}) = R_{i_p}$ for each $1 \leq p \leq 5$. By Lemma 8.6, $s_{i_1}s_{i_2}$ is a lifting of $u_1 = R_3R_2$ and $s_{i_1}s_{i_4}s_{i_5}$ is a lifting of $u_2 = R_3R_4R_3$. The element $u_1$ has two liftings $w_1 = s_3s_2$ and $w'_1 = s_5s_6$, and $u_2$ also has two liftings $w_2 = s_3s_4s_5$ and $w'_2 = s_5s_4s_3$. Thus there are four candidates for $w$:

\[
\begin{align*}
    v_1 &= w_1w_2 = s_3s_2s_3s_4s_5 = s_2s_3s_4s_2s_5, \\
    v_2 &= w_1w'_2 = s_3s_2s_5s_4s_3, \\
    v_3 &= w'_1w_2 = s_5s_6s_3s_4s_5, \\
    v_4 &= w'_1w'_2 = s_5s_6s_4s_5s_3 = s_6s_5s_4s_6s_3.
\end{align*}
\]

Notice that reduced expressions of $v_1$ and $v_2$ contain a consecutive subexpression $s_2s_5$ and observe that $s_2s_5 = s_5s_2$ in $W$ but $R_2R_3 \neq R_3R_2$ in $W_r$. Hence $v_1, v_2 \notin W^F$. A similar argument shows $v_3, v_4 \notin W^F$ (look at the subexpression $s_6s_3$). Hence $u$ is not liftable.

(IV)(b), (V)(b) and (VI)(a) follow from Lemma 8.2(b).

Using Corollary 8.5 and Proposition 8.6 we can detect less trivial nonliftable elements of $W_r$ as in the following example.

**Example 8.7** (Type $E_8/D_5$ to $H_4/H_3$). Let $\Phi \hookrightarrow \Phi_r$ be a folding of the root system of type $E_8$ to $H_4$. See the folding diagram in Section 5.4. We take the parabolic subgroup $W_P := \langle s_2, s_3, s_4, s_5, s_6, s_8 \rangle$, which is of type $D_6$. We have $(W_P)_r = \langle R_2, R_3, R_4 \rangle$, which is a parabolic subgroup of $W_r$ of type $H_3$. Below is an example of a nonliftable element of $W^P_r$ (of the smallest length).

\[ u = R_3R_2R_3R_4R_3R_1R_2R_1R_3R_4R_3R_2R_1 \in W^P_r. \]

This reduced expression contains a nonliftable consecutive subexpression $u' = R_3R_2R_3R_4R_3$ (Lemma 8.6 (IV)(a)). Hence by Corollary 8.5 it follows that $u$ is not liftable.

9 Lifting property of the folding map

We say that the folding map $\varepsilon^*: Z(G^P) \to Z(G^P_r)$ has the lifting property if the Schubert class $\sigma^{(u)}_r$ admits a lifting for every $u \in W^P$. In this section, for each of the twisted quadratic foldings listed in Section 5.4 we apply the results established in Sections 7 and 8 to obtain a complete list of parabolic subsets $P \subset S$ (recall that $S$ denotes a Coxeter generating set of the original reflection group $W$) such that $\varepsilon^*$ has the lifting property. We use $P_r \subset S_r$ to denote a parabolic subset for the folded Coxeter group $W_r$. It is recalled that by construction of a folding of a subsystem $\Phi_P$, the simple system $\Delta_P$ corresponding to $P$ must satisfy $T(\text{span}_K \Delta_P) = \text{span}_K \Delta_P$ (see before Proposition 5.4). We use the following notational convention: if $S = \{s_1, \ldots, s_n\}$ and $I$ is a (possibly empty) subset of $\{1, \ldots, n\}$, then we write $P_I$ to denote the subset $\{s_i | i \notin I\} \subset S$.

**Proposition 9.1.** The folding map $\varepsilon^*: Z(G^P) \to Z(G^P_r)$ has the lifting property if and only if $P \subset S$ is one of the following.

(a) Type $A_{2n-1}$ to $C_n$: $P_{1,2n-1} \subset P$ (equivalently, $P = P_{1,2n-1}$ or $S$).

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Proof. (a) \(\Rightarrow\) Suppose \(P_{1,2n-1} \not\subset P\), or equivalently, \((P_r)_1 \not\subset P_r\). We will find an element \(u \in W^P_r\) that is not liftable. There exists some \(2 \leq i \leq n\) such that \(R_i \not\subset P_r\). If \(R_n \not\subset P_r\), then \(R_n R_{n-1} R_n \in W^P_r\), which is not liftable (Lemma \(8.6\) (I)(a)). If \(R_{n-1} \not\subset P_r\), then \(R_{n-1} R_{n-2} R_{n-1} R_n R_{n-1} \in W^P_r\), which is not liftable (Lemma \(8.6\) (I)(b)). For \(2 \leq i \leq n - 2\), take \(u = R_{n-1} R_n R_{n-2} R_{n-1} R_n R_{n-1} (R_{n-3} R_{n-2}) \cdots (R_{i+1} R_i) \in W^P_r\). Since \(u\) contains a nonliftable consecutive subexpression \(u' = R_{n-1} R_{n-2} R_{n-1} R_n R_{n-1}\), by Corollary \(8.5\) \(u\) is not liftable. Thus, \(\varepsilon^*\) does not have the lifting property.

\((=)\) For \(P = S\), the statement is clear. Let \(P = P_{1,2n-1}\), or equivalently, \(P_r = (P_r)_1\). An arbitrary element \(u \in W^P_r\) with \(l_r(u) \geq 1\) has a reduced expression either

\[
u = \begin{cases} R_j R_{j-1} \cdots R_1 & \text{for some } 1 \leq j \leq n, \text{ or}, \\ R_k R_{k+1} \cdots R_{n-2} R_{n-1} R_n R_{n-1} R_{n-2} \cdots R_2 R_1 & \text{for some } 1 \leq k \leq n - 1. \end{cases}
\]

In the former case, \(u\) has the following two liftings in \(W^P\):

\[
w_1 = s_j s_{j-1} \cdots s_2 s_1, \\
w_2 = s_{2n-j} s_{2n-j+1} \cdots s_{2n-2} s_{2n-1}.
\]

Observe that these are unique reduced expressions for \(w_1\) and \(w_2\) respectively, hence belong to \(W^F\) (Remark \(7.3\) (c)). In the latter case, \(u\) has the following two liftings in \(W^P\):

\[
w_3 = s_k s_{k+1} \cdots s_{n-2} s_{n-1} s_n s_{n+1} s_{n+2} \cdots s_{2n-2} s_{2n-1}, \\
w_4 = s_{2n-k} s_{2n-k-1} \cdots s_n s_{n+1} s_{n+2} \cdots s_{2n-1}.
\]

Again, these are unique reduced expressions for \(w_3\) and \(w_4\) respectively, hence belong to \(W^F\). Thus \(\varepsilon^*\) has the lifting property.

(b) \((\Rightarrow)\) Suppose \(P_{n,n+1} \not\subset P\), or equivalently, \((P_r)_n \not\subset P_r\). Then there exists \(1 \leq i \leq n - 1\) such that \(R_i \not\subset P_r\). Take \(u = R_{n-1} R_n R_{n-1} R_n \cdots R_i \in W^P_r\). Since \(u\) contains a nonliftable consecutive subexpression \(u' = R_{n-1} R_n R_{n-1}\) (Lemma \(8.6\) (II)(a)), by Corollary \(8.5\) \(u\) is not liftable. Hence \(\varepsilon^*\) does not have the lifting property.

\((=)\) For \(P = S\), the statement is clear. Let \(P = P_{n,n+1}\), or equivalently, \(P_r = (P_r)_n\). First, we establish that each element of \(W^P_r\) has a reduced expression of the following form:

\[\rightarrow R_1, \rightarrow R_2, \ldots, \rightarrow R_j, \rightarrow R_{n-2}, \rightarrow R_{n-1}, R_n\]

where each block \(B_j\) is either empty (\(\emptyset\)) or a word of the form \(R_k R_{k-1} \cdots R_j\) for some \(j \leq k \leq n\) (the arrow within each block \(B_j\) indicates that the subscripts of the simple reflections are decreasing), satisfying the following conditions: if \(B_j \neq \emptyset\), then \(B_j' \neq \emptyset\) for all \(j' > j\); if the block \(B_j\) starts with \(R_n\), then every block \(B_j'\) with \(j' > j\) also starts with \(R_n\); if the block \(B_j\) does not start with \(R_n\), then \(0 \leq |B_1| \leq |B_2| \leq \cdots \leq |B_j|\), where each \(|B_i|\) denotes the size of the block \(B_i\). By construction, the above word is reduced, and every reduced word that is braid equivalent to it ends with \(R_n\), hence the group element represented by the above reduced word
belongs to $W^P$. It remains to show that the reduced expressions of the above form exhaust all the elements of $W^P$. We will use a counting argument.

We denote $b_i := |B_i|$ for each $1 \leq i \leq n$. Let $j$ be the smallest number such that block $B_j$ begins with $R_n$. Then we have

$$0 \leq b_1 \leq b_2 \leq \cdots \leq b_{j-1} \leq b_j = n - (j - 1).$$

(11)

We first take care of two extreme cases, namely, $j = 1, n + 1$. When $j = 1$, every block begins with $R_n$, yielding $(R_n R_{n-1} \cdots R_1)(R_n R_{n-1} \cdots R_2) \cdots (R_n R_{n-1}) R_n$. When $j = n + 1$, no block begins with $R_n$, yielding the empty word corresponding to $e \in W_\tau$. Now suppose $2 \leq j \leq n$. We count the $(j - 1)$-tuple $(b_1, \ldots, b_{j-1})$ satisfying (11).

For $x \in \mathbb{Q}$, we denote the largest integer no greater than $x$ by $\lfloor x \rfloor$. We need to distribute the numbers $\{0, 1, 2, \ldots, n - (j - 1)\}$ among $b_1, b_2, \ldots, b_{j-1}$ in the weakly ascending order. Let $0 \leq k \leq \min\{j - 2, n - (j - 1)\}$. We place $k$ separation walls between $j - 1$ objects, and in the resulting $k + 1$ separated sections, we distribute $k + 1$ numbers in strictly ascending order from left to right. The number of choices for such distribution is $(j - 2) \cdot \binom{n - (j - 1) + 1}{k}$. By varying $1 \leq j \leq n + 1$, the total number of choices is found to be

$$F(n) := 2 + \sum_{j=2}^{\lfloor \frac{n+3}{2} \rfloor} \sum_{k=0}^{j-2} \binom{n - (j - 1) + 1}{k} + \sum_{j=\frac{n+3}{2}+1}^{n} \sum_{k=0}^{n - (j - 1)} \binom{j - 2}{k} \cdot \binom{n - (j - 1) + 1}{k}.$$

On the other hand, we have $|W^P| = |W_\tau| |A_{n-1}| = 2^n n! = 2^n$.

Claim 9.2. We have $F(n) = 2^n$ for all $n \geq 2$. (Proof postponed at the end.)

Let $\mu = B_1 B_2 \cdots B_n$ be a $W_\tau$-reduced word of the form described above, and let $u \in W^P$ be the element represented by $\mu$. We build a $W_\tau$-reduced word $\lambda$ of the following form:

$$\lambda = \begin{array}{ccccccc}
A_1 & \rightarrow & s_1 & \rightarrow & s_2 & \cdots & \rightarrow & s_j & \cdots & \rightarrow & s_{n-2} & \rightarrow & s_{n-1} & \rightarrow & s_n & \text{or} & s_{n+1}
\end{array}$$

where each block $A_j$ is obtained from block $B_j$ as follows: if $B_j = \emptyset$, then $A_j = \emptyset$; if $B_j = R_k R_{k-1} \cdots R_j$, then $A_j = s_k s_{k-1} \cdots s_j$ where $s = s_k$ if $k \neq n$, and $s = s_n$ or $s_{n+1}$ if $k = n$ such that if $A_j$ begins with $s_n$ (resp. $s_{n+1}$), then $A_{j+1}$ begins with $s_{n+1}$ (resp. $s_n$).

Claim 9.3. The element $w \in W^P$ represented by $\lambda$ above belongs to $W^F$.

We verify the conditions (FS1) and (FS2) of Definition 7.2. By definition of $\lambda$, no element of $R(u)$ contains the consecutive subword $s_n s_{n+1}$ or $s_{n+1} s_n$. Hence Proposition 6.10 applies to $\lambda$, and we obtain that $\tilde{\varphi}(\lambda)$ is $W_\tau$-reduced. Moreover, since $s_n$ and $s_{n+1}$ are not adjacent in $\lambda$, $\tilde{\varphi}(\lambda)$ coincides with applying $\varphi$ to each simple reflection in $\lambda$. Thus (FS1) is verified.

Next, we observe from the Coxeter diagrams in Section 5.4 that among the $W$-braid moves $[st]_{m(s,t)} = [ts]_{m(s,t)}$, the only ones for which the corresponding $W_\tau$-moves $[\varphi(s) \varphi(t)]_{m(s,t)} = [\varphi(t) \varphi(s)]_{m(s,t)}$ are false are the following:

$$s_n s_{n-1} s_n = s_{n-1} s_n s_{n-1}, \quad s_{n+1} s_n s_{n+1} = s_{n-1} s_n s_{n+1},$$

namely, the long braid moves within the collapsing part. By definition of $w$ via $\lambda$, we can directly check that no element of $R(w)$ contains the consecutive subword $s_n s_{n-1} s_n$, $s_{n-1} s_n s_{n-1}$, $s_{n+1} s_n s_{n+1}$ or $s_{n-1} s_{n+1} s_{n-1}$. Hence, for every $W$-braid move $[st]_{m(s,t)} = [ts]_{m(s,t)}$ within $R(u)$, there is a corresponding $W_\tau$-braid move $[\varphi(s) \varphi(t)]_{m(s,t)} = [\varphi(t) \varphi(s)]_{m(s,t)}$. Thus $\tilde{\varphi}(R(u)) \subset R(u)$. This verifies (FS2). Hence $w \in W^F$ and we have $\tilde{\varphi}(w) = u$ as desired.
Proof of Claim 9.2: Set
\[
    f(n, j) := \begin{cases}
        \sum_{k=0}^{j-1} \binom{j-1}{k} \left( \binom{n-j+1}{k+1} \right) & \text{if } 1 \leq j \leq \left\lceil \frac{n+1}{2} \right\rceil, \\
        \sum_{k=0}^{n-j} \binom{j-1}{k} \left( \binom{n-j+1}{k+1} \right) & \text{if } \left\lceil \frac{n+1}{2} \right\rceil + 1 \leq j \leq n - 1.
    \end{cases}
\]

We aim to show \( f(n, j) = \binom{n}{j} \) for all \( 1 \leq j \leq n - 1 \). Once we have this, \( F(n) = \sum_{k=0}^{n} \binom{n}{j} = 2^n \) will follow. We prove by induction on \( n \geq 2 \). If \( n = 2 \), then \( j = 1 \) and we have \( f(2, 1) = 2 = \binom{2}{1} \).

Now assume \( f(n, j) = \binom{n}{j} \) for \( n \geq 2 \). Let \( 1 \leq j \leq \left\lfloor \frac{n+1}{2} \right\rfloor \) (the same argument will apply to the case \( \left\lceil \frac{n+1}{2} \right\rceil + 1 \leq j \leq n - 1 \)). It suffices to show \( f(n+1, j) - f(n, j) = (\binom{n}{j-1}) \). We have
\[
f(n+1, j) - f(n, j) = \sum_{k=0}^{j-1} \binom{j-1}{k} \left( \binom{n-j+1}{k+1} \right) = \sum_{k=0}^{j-1} \binom{j-1}{k} \left( \binom{n-j}{k} \right) + \sum_{k=1}^{j-1} \binom{j-1}{k} \left( \binom{n-j}{k-1} \right).
\]

On the other hand,
\[
    (\binom{n}{j-1}) = \frac{j-1}{n-j+1} \binom{n}{j} = \sum_{k=0}^{j-1} \binom{j}{k+1} \left( \binom{n-j}{k} \right) = \sum_{k=0}^{j-1} \binom{j-1}{k} \left( \binom{n-j}{k} \right) + \sum_{k=1}^{j-1} \binom{j-1}{k+1} \left( \binom{n-j}{k} \right).
\]

Thus \( f(n+1, j) - f(n, j) = (\binom{n}{j-1}) \) holds and we finish the proof.

(c) Suppose \( P \neq S \), or equivalently, \( P_r \neq S_r \). If \( R_2 \notin P_r \), then \( R_2R_3R_2 \in W^P_r \), which is not liftable (Lemma 8.6 (III)(a)). If \( R_4 \notin P_r \), then \( R_3R_4R_3R_1 \in W^P_r \), which is not liftable (Corollary 8.5). If \( R_3 \notin P_r \), then \( R_2R_3R_3 \in W^P_r \), which is not liftable (Proposition 8.6 (III)(b)). Finally, if \( R_4 \notin P_r \), then \( R_2R_3R_3R_4 \in W^P_r \), which is not liftable (Corollary 8.5). Thus \( \varepsilon^* \) does not have the lifting property. The other direction is clear.

(d) Suppose \( P \neq S \), or equivalently, \( P_r \neq S_r \). If \( R_3 \notin P_r \), then \( R_2R_3R_4 R_3 \in W^P_r \), which is not liftable (Example 8.7). Similarly, if \( R_3 \notin P_r \), then \( R_2R_3R_4 R_3 \in W^P_r \), which is not liftable (Example 8.7). Thus \( \varepsilon^* \) does not have the lifting property. The other direction is clear.

(e) (\Rightarrow) Suppose \( P_{1.5} \notin P \), or equivalently, \( (P_r)_1 \notin P_r \). If \( R_2 \notin P_r \), then \( R_2R_1R_2R_3 \in W^P_r \), which is not liftable (Proposition 8.6 (V)(a)). If \( R_3 \notin P_r \), then \( R_2R_1R_3R_2 \in W^P_r \), which is not liftable (Corollary 8.5). Thus \( \varepsilon^* \) does not have the lifting property.

(e) (\Leftarrow) This direction is proved in Example 7.12 for \( P = P_{1.5} \) and it is trivial for \( P = S \).

(f) By Example 7.12 and Proposition 8.6 (VI)(a), \( u = R_1R_2R_1R_2R_1 \) is the only nonliftable element of \( W_r \). We have \( u \notin W^P_r \) if and only if \( P_r \neq \emptyset \). Hence it follows that \( \varepsilon^* \) has the lifting property if and only if \( P \neq \emptyset \), i.e., \( P = P_{2.4}, P_{1.3} \) or \( S \).

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