A high-robustness and low-cost model for cascading failures

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Abstract – We study numerically the cascading failure problem by using artificially created scale-free networks and the real network structure of the power grid. The capacity for a vertex is assigned as a monotonically increasing function of the load (or the betweenness centrality). Through the use of a simple functional form with two free parameters, it is revealed that it is indeed possible to make networks more robust at a lower cost. We suggest that our method to prevent cascade by protecting less vertices is particularly important for the design of more robust real-world networks to cascading failures.

The network robustness has been one of the most central topics in the complex network research [1]. In scale-free networks, the existence of hub vertices with high degrees has been shown to yield fragility to intentional attacks, while at the same time the network becomes robust to random failures due to the heterogeneous degree distribution [2–5]. On the other hand, for the description of dynamic processes on top of networks, it has been suggested that the information flow across the network is one of the key issues, which can be captured well by the betweenness centrality or the load [6].

Cascading failures can happen in many infrastructure networks, including the electrical power grid, Internet, road systems, and so on. At each vertex of the power grid, the electric power is either produced or transferred to other vertices, and it is possible that from some reasons a vertex is overloaded beyond the given capacity, which is the maximum electric power the vertex can handle. The breakdown of the heavily loaded single vertex will cause the redistribution of loads over the remaining vertices, which can trigger breakdowns of newly overloaded vertices. This process will go on until all the loads of the remaining vertices are below their capacities. For some real networks, the breakdown of a single vertex is sufficient to collapse the entire system, which is exactly what happened on August 14, 2003 when an initial minor disturbance in Ohio triggered the largest blackout in the history of United States in which millions of people suffered without electricity for as long as 15 hours [7].

A number of aspects of cascading failures in complex networks have been discussed in the literature [8–16], including the model for describing cascade phenomena [8], the control and defense strategy against cascading failures [9,10], the analytical calculation of capacity parameter [11], and the modelling of the real-world data [12]. In a recent paper [16], the cascade process in scale-free networks with community structure has been investigated, and it has been found that a smaller modularity is easier to trigger cascade, which implies the importance of the modularity and community structure in cascading failures. In the research of the cascading failures, the following two issues are closely related to each other and of significant interests: One is how to improve the network robustness to cascading failures, and the other particularly important issue is how to design manmade networks with a lower cost. In most circumstances, a high robustness and a low cost are difficult to achieve simultaneously. For example, while a network with more edges is more robust to failures, in practice, the number of edges is often limited by the cost to construct them. In brevity, it costs much to build a robust network. Very recently, Schäfer et al. proposed a new proactive measure to increase the robustness of heterogeneous loaded networks to cascades. By defining the load-dependent weights, the network turns to be more homogeneous and the total load is decreased, which means the investment cost is also reduced [15]. In the present letter, for simplicity, we try to find a possible way of protecting networks based on the flow along shortest-hop path, first proposed by Motter-Lai [8]. Through the use of our improved capacity model, we
numerically examine the cascades in scale-free networks and the electrical power grid network. Since for heterogeneously loaded networks, overload avalanches can be triggered by the failure of only one of the most loaded vertices, the following results are all based on the removal of one vertex with the highest load. Our results suggest that networks can indeed be made more robust while spending less.

We first construct the Barabási-Albert (BA) scale-free network [17] of the size $N = 5000$ with the average degree $(k) \approx 4$ to study the cascading failures. The BA network is characterized by the degree distribution $p(k) \sim k^{-\gamma}$ with the degree exponent $\gamma = 3$, and it has been shown that the load distribution also exhibits the power law behavior [6], which means that there exist a few vertices with very large loads.

The betweenness centrality for each vertex, defined as the total number of shortest paths passing through it, is used as the measure of the load and computed by using the efficient algorithm [18]. The capacity $c_v$ for the vertex $v$ is assigned as

$$c_v = \lambda(l_v)l_v,$$

where $l_v$ is the initial load without failed vertices. Although it should be possible to find, via a kind of the variational approach, the optimal functional form of $\lambda(l_v)$ which gives rise to the lower cost and the higher robustness (see below for the definitions of the two) we in this work simplify $\lambda(l_v)$ as shown in fig. 1:

$$\lambda(l_v) = 1 + \alpha \Theta(l_v/l_{\text{max}} - \beta),$$

where $\Theta(x) = 0(1)$ for $x < 0(>0)$ is the Heaviside step function, $l_{\text{max}} = \text{max}_v l_v$, and we use $\alpha \in [0, \infty)$ and $\beta \in [0, 1]$ as two control parameters in the model. In ref. [8] a constant $\lambda$ has been used (see fig. 1 for comparison), which corresponds to the limiting case of $\beta = 0$ with the identification $\lambda = 1 + \alpha$ in our model.

At the initial time $t = 0$, the vertex with the highest load is removed from the network, and then new loads for all other vertices are recomputed\(^1\). We then check the failure condition $c_v < l_v(t)$ for each vertex, and remove all overloaded vertices to get the network at $t + 1$. The above process continues until all existing vertices fulfill the condition $c_v > l_v(t)$, and the size of the giant component $N^*$ at the final stage is measured. The relative size of the cascading failures is conveniently captured by the ratio [8]

$$g = \frac{N^*}{N},$$

which we call the robustness from now on. For networks of homogeneous load distributions, the cascade does not happen and $g \approx 1$ has been observed [8]. Also for networks of scale-free load distributions, one can have $g \approx 1$ if randomly chosen vertices, instead of vertices with high loads, are destroyed at the initial stage [8].

In general, one can split, at least conceptually, the total cost for the networks into two different types: On the one hand, there should be the initial construction cost to build a network structure, which may include, e.g., the cost for the power transmission lines in power grids, and the cost proportional to the length of road in road networks. Another type of the cost is required to make the given network functioning, which can be an increasing function of the amount of flow and can be named as the running cost. For example, we need to spend more to have bigger memory sizes and faster network card and so on for the computer server which delivers more data packets. In the present letter, we assume that the network structure is given (accordingly the construction cost is fixed), and focus only on the running cost which should be spent in addition to the initial construction cost.

Without consideration of the cost to protect vertices, the cascading failure can be made never to happen by assigning extremely high values to capacities. However, in practice, the capacity is severely limited by cost. We expect the cost to protect the vertex $v$ should be an increasing function of $c_v$, and for convenience define the cost $e$ as

$$e = \frac{\sum_{v=1}^{N} (\lambda(l_v) - 1)}{N}.$$

It is to be noted that for a given value of $\alpha$, the original Motter-Lai (ML) capacity model in ref. [8] has always a higher value of the cost than our model (see fig. 1). Although $e = 0$ at $\beta = 1$, it should not be interpreted as a costfree situation; we have defined $e$ only as a relative measure in comparison to the case of $\lambda(l) = 1$ for all vertices. For a given network structure, the key quantities to be measured are $g(\alpha, \beta)$ and $e(\alpha, \beta)$, and we aim to

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\(^1\)In real situations of failures, the initial breakdown can happen at any vertex in the network. However, the eventual scale of damages must be greater when a heavily loaded vertex is broken, and accordingly we in this work restrict ourselves to the worst case when the vertex with the highest load is initially broken.
increase $g$ and decrease $e$, which will eventually provide us a way to achieve the high robustness and the low cost at the same time.

In fig. 2(a), we report the robustness $g$ for the BA network of the size $N = 5000$ with the average degree $(k) \approx 4$ as a function of $\beta$ at $\alpha = 0.10, 0.15, 0.20, 0.25, 0.30$, and 1.0 (from bottom to top). As $\beta$ increases further beyond the region in fig. 2(a), the robustness $g$ is found to decrease toward zero (not shown here), which is as expected since the larger $\beta$ makes vertices with larger loads less protected (see fig. 1). We also skip in fig. 2 small values of $\beta$ below approximately 0.001: If $\beta < l_{\text{min}}/l_{\text{max}}$, with the minimum load $l_{\text{min}}$, all vertices are given $\lambda(l) = 1 + \alpha$, equivalent to the ML model corresponding to $\beta = 0$. It is shown in fig. 2(a) that for $\alpha \lesssim 0.30$, $g$ first increases and then decreases as $\beta$ is increased, exhibiting a well-developed maximum $g_{\text{max}}$ at $\beta = \beta^{*}$. This is a particularly interesting observation since the network becomes more robust (larger $g$) protecting less vertices (larger $\beta$). In more detail, the curve for $\alpha = 0.20$ in fig. 2(a) shows the maximum $g_{\text{max}} \approx 0.62$ (at $\beta^{*} \approx 0.00133$), which is about 3.5 times bigger than $g \approx 0.175$ (at $\beta = 0$). In other words, the network can be made much more robust by assigning smaller capacities to vertices with less loads. For larger values of $\alpha$, on the other hand, it is found that $g_{\text{max}}$ occurs at $\beta = 0$, which indicates that the above finding, i.e., possibility of making a network more robust by protecting less vertices, does not hold, as exemplified by the curve for $\alpha = 1$ in fig. 2(a).

The above observation is closely related with ref. [9], where it has been found that in order to reduce the size of cascades (or to have a larger $g$), some of less loaded vertices should be removed just after the initial attack. In reality, however, we believe that the direct application of this strategy of intentional breakdowns is not easy, for cascading failures usually propagate across the whole network very soon just after the initial breakdown. In contrast, we propose in this work a way to make the network better prepared to breakdowns, by protecting less vertices.

In order to look at the cost benefit of protecting less vertices in a more careful way, we plot in fig. 2(b) the cost $e$ in eq. (4) vs. $\beta$ at various values of $\alpha$. As is expected from fig. 1, the cost $e$ is shown to be a monotonically decreasing (increasing) function of $\beta$ ($\alpha$) at fixed $\alpha$ ($\beta$). Take again the case with $\alpha = 0.20$ as an example with $e(\beta^{*}) \approx 0.153$ and $e(\beta = 0) = 0.2$: It is then concluded that for $\alpha = 0.2$ one can make the network $3.5 \approx 0.62/0.175$ times more robust while spending only $76.5\% \approx 0.153/0.2$ of the original cost. In fig. 2(c), we use the same data as in fig. 2(a) and (b), and show the relation between the robustness and the cost for $\alpha = 0.10, 0.15, 0.20$ from bottom to top. For comparison, the values $(g, e)$ for $\beta = 0$, corresponding to the ML model, are also displayed as symbols at the end of curves. It is clearly shown that for a given $\alpha$, one can achieve the higher robustness and the lower cost by tuning $\beta$ toward the right-most point on each curve. We can also use fig. 2(c) to choose the most efficient way to get a given robustness $g$: For example, suppose that $g = 0.6$ is the required robustness. The vertical line for $g = 0.6$ crosses several different curves, and one can choose the crossing point which has the lowest cost.

We next study the cascading failures in the real network structure of the North American power grid of the size $N = 4941$ [19]. Although the electrical power grid network is a very homogeneous network in terms of the degree distribution, the load distribution, in a sharp contrast, shows a strong heterogeneity as shown in fig. 3. In other words, the degree distribution is more like an exponential one, while the load distribution is similar to the power.
Fig. 3: The cumulative load distribution of power grid network $P(l)$ in log-log scale. The inset shows the cumulative degree distribution $P(k)$ of the power grid in linear-log scale.

law form. The broad load distribution can be one of the reasons for the fragility of the power grid to cascading failures [8].

We then apply, the same method as we used above, to the power grid, and obtain $g$ and $e$ as functions of $\beta$ for given values of $\alpha$. Figure 4 for the cascading failures of the power grid is in parallel to fig. 2 for the BA network: fig. 4(a) for $g$ vs. $\beta$, (b) for $e$ vs. $\beta$, and (c) for $e$ vs. $g$. There are some quantitative differences between curves for the power grid and the BA network. However, qualitatively speaking, both networks are shown to exhibit the following common features: i) For a given $\alpha$, the robustness has a maximum $g_{\text{max}}$ at $\beta = \beta^*$, ii) $e$ is a monotonically decreasing function of $\beta$ at a given $\alpha$, and iii) there exists a lob-like structure in the $g$-$e$ plane, which indicates that one can make the network exhibit a higher robustness and a lower cost at the same time than the corresponding values for the ML model. It is worth mentioning that the power grid in fig. 4 can be made to show the higher $g$ and the lower $e$ than the ML model in a broader region of $\alpha$: Even at $\alpha = 1$, the power grid can have much better robustness and much lower cost in comparison to the ML model. Specifically, at $\alpha = 1.0$ the ML model has $g \approx 0.40$ and $e \approx 1.0$ while our model can yield $g \approx 0.73$ and $e \approx 0.26$ (at $\beta \approx 0.00583$) (see fig. 4(c)), which occurs when only 26% of vertices are given the higher capacity $\lambda(l) = 2$, and the other remaining 74% of vertices have the lower capacity $\lambda(l) = 1$. In other words, by assigning lower capacities to 74% of vertices, the network becomes much more robust.

In reality, it is also interesting to observe the effect of noise on the dynamical process. In ref. [20], when noise is introduced into the nonlinear dynamical system, it has been shown that noise changes the singularity at a special time to a statistical time distribution and shows various interesting behaviors. In the present work, we are interested in how the presence of noise influences the final cascading failure behavior within our scheme. Here, we introduce effects of noise as an erroneous assignment of the capacity function. In detail, at a given error probability $\epsilon$, the vertex $v$ is assigned the capacity $c_v'$ instead of its correct $c_v$:

$$c_v' = c_v(1 + r), \quad (5)$$

where $r$ is the uniform random variable with zero mean ($r \in [-1, 1]$). We believe that this erroneous behavior is plausible in reality, since the perfect knowledge for the true value of the load for each vertex may not be available, which may cause an erroneous assignment of the capacity on a vertex. In the limiting case of $\epsilon = 0$, we recover our error-free results presented above. In fig. 5, we report the results at $\alpha = 0.2$ for the robustness $g$ for the BA network...
as a function of $\beta$ for different error probability $\varepsilon$ (see fig. 2(a) for comparison). It is seen that for small $\varepsilon$, the overall behavior is qualitatively the same as in fig. 2(a), i.e., the existence of a well-developed robustness peak and gradual decrease as $\beta$ is increased. The peak height of the robustness is found to decrease as $\varepsilon$ is increased, indicating the negative effect of the noise. An interesting observation in fig. 5 is that as $\varepsilon$ becomes larger there exists a region of $\beta$ in which the robustness is actually higher than the error-free case of $\varepsilon = 0$.

In summary, we have suggested a new capacity model to cascading failures, by improving the existing ML capacity model in ref. [8]. The main idea in our model is the same as in existing studies: In a highly heterogeneous network with a broad load distribution, vertices with large loads should be more protected by assigning large capacities. Different from other studies in which the capacity is assigned in proportion to the load, i.e., $c = \lambda l$, we generalize the model so that the proportionality constant $\lambda$ is now changed to an increasing function $\lambda(l)$ of $l$. In more detail, we use the Heaviside step function for $\lambda(l)$ characterized by two parameters, the step height $\alpha$, and the step position $\beta$. By applying this capacity model to the artificial BA network as well as the real network of the power grid, we have clearly shown that it is indeed possible to make the network more robust, while at the same time the cost to assign capacities is drastically reduced. We believe that our suggested model to assign capacities to vertices should be practically useful in designing infrastructure networks in an economic point of view. As a final remark, it needs to be pointed out that the model proposed in this work should be considered as only the first step to find the optimal functional form of $\lambda(l)$.

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