Heavy Quark Symmetry and the Skyrme Model

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Abstract

We present a consistent way of describing heavy baryons containing a heavy quark as bound states of an SU(2) soliton and heavy mesons. The resulting mass formula reveals the heavy quark symmetry explicitly. By extending the model to the orbitally excited states, we establish the generic structure of the heavy baryon spectrum. As anticipated from the heavy quark spin symmetry, the $c$-factor denoting the hyperfine splitting constant vanishes and the baryons with the same angular momentum of light degrees of freedom form degenerate doublets. This approach is also applied to the pentaquark exotic baryons, where the conventional $c$-factor plays no more a role of the hyperfine constant. After diagonalizing the Hamiltonian of order $N_c^{-1}$, we get the degenerate doublets, which implies the vanishing of genuine hyperfine splitting.

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I. INTRODUCTION

Hadrons containing a single heavy quark \(Q\) with its mass \(m_Q\) much greater than a typical scale of strong interactions \(\Lambda_{\text{QCD}}\) can be viewed as a freely propagating point-like color source dressed by light degrees of freedom such as light quarks and gluons. Besides the chiral symmetry for the light quark system, such a system reveals an additional symmetry, so-called the heavy quark spin-flavor symmetry [1–3] as the heavy quark mass goes to infinity. In this limit, the heavy quark spin decouples to the rest of the strongly interacting light quark system, since their coupling is a relativistic effect of order \(1/m_Q\). Furthermore, since the heavy quark can hardly change its velocity due to the strong interactions via soft gluons, the dynamics of the system is independent of its mass and, therefore, its flavor. As a consequence of the heavy quark spin symmetry, the hadrons come in degenerate doublets [4] with total spin (regardless of the number of light quarks)

\[ j_\pm = j_\ell \pm \frac{1}{2}, \tag{1.1} \]

(unless \(j_\ell = 0\)) which are formed by combining the spin of the heavy quark with the total angular momentum of the light degrees of freedom \(j_\ell\). In other words, the total angular momentum of the light degrees of freedom, \(\vec{J}_\ell = \vec{J} - \vec{S}_Q\), is conserved and the corresponding quantum number \(j_\ell\) can classify the hadrons with a single heavy quark.

In the Skyrme model à la Callan and Klebanov (CK) [5], heavy baryons can be described by bound states of a soliton of the \(SU(2)\) chiral Lagrangian and the heavy meson containing the heavy quark. This picture which was originally suggested for the strange hyperons has been shown to work successfully in describing the static properties of the heavy baryons with charm(c)- or bottom(b)-quark [6–8]. It is also extended to the study of the exotic pentaquark-baryons (\(P\)-baryons, in short) as the bound states of soliton and antiflavored heavy mesons [9]. (Here, by “antiflavored” heavy mesons, we denote the \(\bar{Q}q\) heavy mesons carrying opposite heavy flavor, \(i.e., C = -1\) or \(B = +1\). They are the antiparticles of normal heavy mesons consist of \(Q\bar{q}\), which we will call as heavy mesons.) However, one should be careful in applying the bound state approach to the heavy baryons, especially in case of \(P\)-baryons, due to following respects:

Firstly, one should be more careful on the interpretation and the calculation of the so-called hyperfine constant: When quantized, the bound system of the soliton and heavy mesons with energy \(\omega_B\) and grand spin quantum number \(k\) (the corresponding operator being defined as \(\vec{K} = \vec{I}_h + \vec{J}_h\); \(\vec{I}_h\): isospin of heavy mesons, \(\vec{J}_h\): spin of heavy mesons) can be identified with the heavy baryon of isospin \(i\) and spin \(j\). The masses of such heavy baryons in the conventional approach [8] turns out to be

\[ m_{(i,j)} = M_{\text{sol}} + \omega_B + \frac{1}{2\mathcal{Z}} \left\{cj(j + 1) + (1 - c)i(i + 1) + c(c - 1)k(k + 1) \right\}, \tag{1.2} \]

where \(M_{\text{sol}}\) and \(\mathcal{Z}\) are the soliton mass and its moment of inertia with respect to the collective isospin rotation, respectively. The spin of the heavy baryon \(j\) takes a priori one of the values \(|i - k|, \cdots, i + k\) and \(c\), “hyperfine constant”, is a constant defined through

\[ \langle k, k'_3|\vec{\Theta}|k, k_3\rangle \equiv -c(k, k'_3|\vec{K}|k, k_3), \tag{1.3} \]
on the analogy of the Lande’s g-factor in atomic physics. Here, $\vec{\Theta}$ is the meson field operator induced by the collective rotation, which forms the first rank tensor in the space of the grand spin eigenstates $|k, k_3\rangle$. The $c$ yields the hyperfine splittings between the heavy baryon masses and it has been known to play the role of an order parameter for the heavy quark symmetry; it is required to vanish in the heavy meson mass limit so that the heavy baryon masses do not depend on the total spin. This requirement for the $c$, however, has been over-emphasized. In fact, the heavy quark symmetry does not necessarily require such an entire independence of mass on the total spin, as will be discussed in this work.

To heavy baryons carrying a heavy quark one may extend straightforwardly the applicability of Eq. (1.2). The heavy quark symmetry is manifested by vanishing $c$-factors of bound heavy meson states, provided a subtle point is corrected [10]. In the strangeness sector the vector mesons $K^*$ are eliminated out in favor of the pseudoscalar meson $K$, in analogy to the $\rho$ mesons for the light quark system. [11] This approximation is valid only when the vector meson masses are sufficiently larger than those of the corresponding pseudoscalar mesons. As the heavy quark mass in both mesons increases, the heavy vector meson and the heavy pseudoscalar meson should be treated on the same footing. Defects in taking the conventional bound state approach of CK to heavy baryons have been pointed out and correct heavy baryon mass spectra with the explicit heavy quark symmetry have been obtained by Jenkins et al. [12] and other groups [13–16] in the infinitely heavy meson mass limit. In Refs. [12,13], a different but equivalent quantization scheme is adopted: the soliton is first quantized to nucleons and $\Delta$’s and then the heavy mesons are bound to them to form a heavy baryon, while in conventional bound state approach [4] the whole soliton-heavy meson bound system is quantized by using the collective coordinates. The mass predictions [14,13] based on Eq. (1.2), however, give slightly different results from those of Ref. [12], i.e., constant shifts of heavy baryon masses to the amount of $3/8I$. This difference comes from the approximate treatment on the last term of Eq. (1.2). We will show that the two predictions are equivalent by treating the term in proper way.

We have met a completely different situation in the study of penta-quark heavy baryons which carry one heavy anti-quark. [17] The $c$-factors associated with the bound states of the antiflavored heavy mesons do not vanish. In some specific cases, one gets involved with a serious problem such as negative $c$-factors, with which the mass formula (1.2) would yield lower mass for the baryon with higher spin. However, this flaw can be cured by the following observation. There appear multiply degenerate heavy meson bound states, which prevent us from using the mass formula (1.2) in the present form. The $c$-factor defined through Eq. (1.3) cannot play the role of the hyperfine constant. Therefore, the hyperfine splitting being of order $1/N_c$ should be obtained, due to the appearance of the off-diagonal terms, by diagonalizing the Hamiltonian matrix with respect to the soliton-antiflavored heavy meson bound states degenerate up to order $N_c^0$. We shall see that the mixing of the states at the $N_c^{-1}$ order cannot be neglected in anti-flavor case. The resulting $P$-baryon masses do respect the heavy quark symmetry and the resulting hyperfine constants vanish.

The main purpose of this paper is to clarify the things associated with the hyperfine constant in the conventional bound state approach and develop a consistent bound state approach to be applied not only to the normal heavy baryons but also to negative parity heavy baryons and exotic baryons carrying an antiflavor. It also supplies some of the details
left out in our previous paper \cite{17} where we investigated the pentaquark exotic baryons. We will work in an extreme limit where both the soliton and heavy mesons are infinitely heavy and sit on the same point in space. This approximation enables us to get useful informations without getting involved with any complicated numerical calculations.

This paper is organized as follows. In Sec. II, we briefly introduce our working Lagrangian density. The positive and negative parity eigenstates of the heavy mesons under the static potentials provided by the soliton configuration are found in Sec. II. Section IV is to discuss the collective coordinate quantization procedure for the bound system of a single heavy meson to the soliton. In Sec. V we describe the heavy baryons as the bound heavy mesons to soliton. We also derive a mass formula for the heavy baryons, which is more appropriate to appreciate the heavy quark symmetry than Eq.(1.2). The realization of heavy quark symmetry in heavy baryon spectrum is also discussed. We study the pentaquark exotic baryons by considering the bound states of the “anti-flavored” heavy mesons in Sec. VI. We shall show that there exist degenerate doublets as given in Eq.(1.1) in pentaquark states. A few concluding remarks are given in Sec. VII and explicit formulas are provided in Appendices A and B.

II. HEAVY MESON EFFECTIVE LAGRANGIAN

We start with describing briefly the effective Lagrangian for the heavy mesons interacting with Goldstone bosons, which respects both heavy quark symmetry and chiral SU(2)_L × SU(2)_R symmetry. (See Refs. \cite{18,19} for details.)

Consider heavy mesons containing a heavy quark Q and a light antiquark ¯q. Here, the light antiquark in a heavy meson is assumed to form a point-like object with the heavy quark, endowing it with appropriate color, flavor, angular momentum and parity. Let Φ and Φ* be the field operators that annihilate j^z=0^- and 1^- heavy mesons with C = +1 or B = −1. These fields Φ and Φ* form an SU(2) antidoublets: for example, when the heavy quark constituent is the c-quark,

\[
\Phi = (D^0, D^+) \quad \left( \Phi^\dagger = (\bar{D}^0, D^-)^T \right),
\]

\[
\Phi^* = (D^{*0}, D^{*+}) \quad \left( \Phi^{*\dagger} = (\bar{D}^{*0}, D^{*-})^T \right).
\]

The traditional Lagrangian for the free fields is

\[
\mathcal{L}_{\text{free}} = \partial_\mu \Phi \partial^\mu \Phi^\dagger - m_\Phi^2 \Phi \Phi^\dagger - \frac{1}{2} \Phi^{\*\mu\nu} \Phi^*_{\mu\nu} + m_{\Phi^*}^2 \Phi^{*\mu} \Phi^*_{\mu\dagger},
\]

where \( \Phi^{*\mu\nu} \equiv \partial_\mu \Phi^*_{\nu} - \partial_\nu \Phi^*_{\mu} \) is the field strength tensor of the heavy vector meson fields and \( m_\Phi \) and \( m_{\Phi^*} \) are the masses of heavy pseudoscalar and vector mesons, respectively.

In the limit of infinite heavy quark mass, the heavy quark symmetry implies that the dynamics of the heavy mesons depends trivially on their spin and mass. Such trivial dependence can be eliminated by introducing a redefined 4 × 4 matrix field \( H(x) \) as \cite{20}.

\footnote{We will work with two light flavors. For the generalization to three flavors, see Ref. \cite{21}.}
instead of the traditional heavy meson fields, \( \Phi \) and \( \Phi^\ast \). Here, we use the conventional Dirac \( \gamma \)-matrices and \( v / \) denotes \( v^\mu \gamma^\mu \). The fields \( \Phi_v \) and \( \Phi^\ast_v \), respectively, represent the heavy pseudoscalar field and heavy vector fields in the moving frame with a four velocity \( v^\mu \). They are related to the \( \Phi \) and \( \Phi^\ast \) as \[ \Phi = e^{-iv \cdot x m} \frac{1}{\sqrt{2m}} \Phi_v, \]
\[ \Phi^\ast = e^{-iv \cdot x m} \frac{1}{\sqrt{2m}} \Phi^\ast_v. \] (2.3a)

Under the heavy quark spin rotation, \( H \) transforms
\[ H \rightarrow SH, \] (2.3b)
with \( S \in SU(2)_v \) (the heavy quark spin symmetry group boosted by the velocity \( v \)). In the heavy meson rest frame, \( i.e. \), \( v^\mu = (1, \vec{0}) \), \( S \) can be written explicitly as
\[ H \rightarrow e^{i \frac{1}{2} \vec{\sigma} \cdot \vec{\sigma}} H, \] (2.3c)
where \( \sigma_i = \frac{1}{2} \epsilon_{ijk} [\gamma_j, \gamma_k] \), the Dirac spin matrices. To leading order in the heavy meson mass, the free Lagrangian density describing the heavy mesons propagating with a four-velocity \( v^\mu \) is nothing but that for the freely propagating heavy quark \[ L_{HQS}^{\text{free}} = -iv \mu \text{Tr}(\partial^\mu H \bar{H}). \] (2.4)

For later convenience, we have introduced \( \bar{H} = \gamma_0 H^\dagger \gamma_0 \), which transforms under the heavy quark spin rotation as \( \bar{H} \rightarrow S \bar{H} S^{-1} \). One may easily check that it comes as the leading order term in \( m_\Phi (= m_{\Phi^\ast}) \) by substituting Eq. (2.3a) into \( L_{\text{free}} \).

On the other hand, the dynamics of the light quark system is governed by the \( SU(2)_L \times SU(2)_R \) chiral symmetry, which is realized in a nonlinear way via a \( 2 \times 2 \) unitary matrix
\[ \Sigma = \exp \left( \frac{i}{f_\pi} \left( \begin{array}{c} \pi^0 \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- \pi^0 \end{array} \right) \right), \] (2.5)
with the triplet of Goldstone bosons \( (\pi^+, \pi^0 \text{ and } \pi^-) \) and the pion decay constant \( f_\pi (=93 \text{ MeV}) \). Under the \( SU(2)_L \times SU(2)_R \) transformation, \( \Sigma \) transforms as
\[ \Sigma \rightarrow L \Sigma R^\dagger, \] (2.5a)
with global transformations \( L \in SU(2)_L \) and \( R \in SU(2)_R \). In terms of \( \Sigma \), the interactions among the Goldstone bosons are described by the Lagrangian density
\[ \mathcal{L}_M = \frac{f^2}{4} \text{Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + \cdots, \] (2.6)
where terms with more derivatives are abbreviated by the ellipsis.

Construction of a chirally invariant Lagrangian for the couplings of the heavy mesons to the Goldstone bosons can be done by assigning \( H \) a proper transformation rule under the
chiral transformation. There may be a considerable freedom. A standard one is to introduce a redefined matrix

$$\xi = \sqrt{\Sigma}, \quad (2.7)$$

which transforms under the $SU(2)_L \times SU(2)_R$ as

$$\xi \rightarrow L\xi U^\dagger = U\xi R^\dagger. \quad (2.8)$$

Here, $U$ is a special unitary matrix depending on $L$, $R$ and the Goldstone fields. From $\xi$ one can construct a vector field $V_\mu$ and an axial vector field $A_\mu$

$$V_\mu = \frac{i}{2}(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger),$$
$$A_\mu = \frac{i}{2}(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger), \quad (2.9)$$

which have simple chiral transformation properties:

$$V_\mu \rightarrow UV_\mu U^\dagger + U\partial_\mu U^\dagger,$$
$$A_\mu \rightarrow UA_\mu U^\dagger. \quad (2.10)$$

Let the light quark doublet $q$ transforms as

$$q = (u, d)^T \rightarrow Uq, \quad (2.11)$$

so that the heavy meson field $H(x)$ transforms as

$$H \rightarrow HU^\dagger. \quad (2.12)$$

This choice defines a simple transformation rule of $H$ under the parity operation:

$$\xi(\vec{r}, t) \rightarrow \xi^\dagger(-\vec{r}, t), \quad \text{and} \quad H(\vec{r}, t) \rightarrow \gamma^0 H(-\vec{r}, t)\gamma^0. \quad (2.13)$$

In terms of the vector field $V_\mu$, a covariant derivative can be constructed as

$$D_\mu H(x) = H(\vec{r} + V_\mu), \quad (2.14)$$

which transforms under a chiral transformation as $D_\mu H \rightarrow (D_\mu H)U^\dagger$.

Now, it is easy to write down a “heavy-quark-symmetric” and “chirally-invariant” Lagrangian for the couplings of the heavy meson fields to the Goldstone bosons. To the leading order in the derivatives on the Goldstone boson fields, it reads

$$\mathcal{L}_{HQS} = \mathcal{L}_M - iv_\mu \text{Tr}(D^\mu H\bar{H}) + g \text{Tr}(H\gamma^\mu\gamma_5 A_\mu \bar{H}), \quad (2.15)$$

with a universal coupling constant $g$ for the $\Phi\Phi^*\pi$ and $\Phi^*\Phi^*\pi$ interactions. The nonrelativistic quark model provides a naive estimation $[19]$ for the value of $g$ as

$$g = -\frac{3}{4}. \quad (2.16)$$

On the other hand, the Lagrangian leads to the decay widths

$$\Gamma(\Phi^*_{\pm} \rightarrow \pi^+\Phi_{-\frac{1}{2}}) = 2\Gamma(\Phi^*_{\pm} \rightarrow \pi^0\Phi_{\frac{3}{2}}) = \frac{1}{12\pi} \frac{g^2}{f_\pi} |\vec{p}_\pi|^3, \quad (2.17)$$

where the subscript $\pm \frac{1}{2}$ of $\Phi$ and $\Phi^*$ denotes the third component of their isospin. In case of $Q=c$, the $c$-quark, the experimental upper limit $[23]$ of 131 keV on the $D^*$ width implies that $|g|^2 \lesssim 0.5$ when combined with the $D^{*+} \rightarrow D^+\pi^0$ and $D^{*+} \rightarrow D^0\pi^+$ branching ratios $[20]$. In this work, however, we will take the pion decay constant $f_\pi$ and the heavy meson coupling constant $g$ as adjustable parameters.
III. SOLITON-HEAVY MESON BOUND STATE

With a suitable stabilizing term, the nonlinear Lagrangian $\mathcal{L}_M$ supports a classical soliton solution \[ \Sigma_0(\vec{r}) = \exp(i\vec{\tau} \cdot \vec{r} F(r)), \] (3.1) with the wavefunction $F(r)$ satisfying the boundary conditions

\[ F(0) = \pi \quad \text{and} \quad F(r) \xrightarrow{r \to \infty} 0. \] (3.1a)

The soliton solution carries a winding number identified with the baryon number

\[ \text{(B. No.)} = -\frac{1}{24\pi^2} \int d^3r \varepsilon^{ijk} \text{Tr}(\Sigma_0^\dagger \partial_i \Sigma_0 \Sigma_0^\dagger \partial_j \Sigma_0 \Sigma_0^\dagger \partial_k \Sigma_0) = 1, \] (3.1b)

and a finite mass

\[ M_{\text{sol}} = 4\pi \int_0^\infty r^2 dr \left[ F'' + 2\frac{\sin^2 F}{r^2} \right] + \cdots, \] (3.1c)

with $F' = dF/dr$ and the ellipsis denoting the contributions from the soliton-stabilizing terms.

Our main interest is the bound heavy-meson states (if any) due to the static potentials provided by the baryon-number-one soliton configuration (3.1) sitting at the origin. Note that we are working with the infinitely heavy soliton. Explicitly, the potentials are in the form of

\[ V^\mu = (V^0, \vec{V}) = (0, iv(r)\hat{r} \times \vec{r}), \]
\[ A^\mu = (A^0, \vec{A}) = (0, \frac{1}{2}(a_1(r)\vec{r} + a_2(r)\hat{r} \vec{r} \cdot \hat{r})), \] (3.2)

with

\[ v(r) = \frac{\sin^2 (F/2)}{r}, \quad a_1(r) = \frac{\sin F}{r} \quad \text{and} \quad a_2(r) = \frac{F' - \sin F}{r}. \] (3.2a)

In the rest frame $v^\mu = (1, \vec{0})$, the equation of motion for the eigenmodes $H_n(\vec{r})e^{-i\varepsilon_n t}$ of the $H$-field can be read off as

\[ \varepsilon_n H_n(\vec{r}) = gH_n(\vec{r})\vec{A} \cdot \vec{\sigma}, \] (3.3)

where we have used that $H(x)\vec{\gamma} \gamma_5 = -H(x)\vec{\sigma}$. Here $\varepsilon_n$ is the eigenenergy and $n$ denotes a set of quantum numbers which classify the eigenmodes. The “hedghog” configuration (3.1) correlates the isospin and the angular momentum, while the heavy-quark symmetry implies the heavy quark spin decoupling from the set. Thus, the equation of motion is

\[ \text{The explicit form of the stabilizing terms is not essential for our discussions. Later we will adopt the Skyrme term as a stabilizing term for numerical results. For a review of the Skyrme model we refer to Ref. 23.} \]
invariant under the parity operation, the heavy quark spin rotation, and the simultaneous rotations in the ensemble of spaces: isospin space, “light-quark spin” space, and ordinary space. Let $\vec{L}$, $\vec{S}_\ell$, $\vec{S}_Q$, and $\vec{I}_h$ be the orbital angular momentum, light quark spin, heavy quark spin, and isospin operators of the heavy mesons, respectively, and let $Y_{\ell m}(\hat{r})$, $\ell(\pm \frac{1}{2})$, $|\pm \frac{1}{2}\rangle_Q$, and $\tilde{\phi}_{\pm \frac{1}{2}}$ be corresponding eigenstates, respectively. (See Appendix A for their explicit representations.) The simultaneous rotations mentioned above are generated by the “light-quark grand spin” operators defined as

$$\vec{K}_{\ell} = \vec{L} + \vec{S}_\ell + \vec{I}_h.$$  

(3.4)

By the subscript $\ell$, we distinguish $\vec{K}_{\ell}$ from the traditional grand spin operators ($\vec{K} = \vec{L} + \vec{S} + \vec{I}_h$ with $\vec{S} = \vec{S}_\ell + \vec{S}_Q$ being the spin operators of the heavy mesons) used in the bound state approach in the Skyrme model. Then, the eigenmodes of the heavy meson can be classified by the third component of heavy quark spin with the Clebsch-Gordan coefficients ($\ell$ from the traditional grand spin operators ($\vec{K} = \vec{L} + \vec{S} + \vec{I}_h$ with $\vec{S} = \vec{S}_\ell + \vec{S}_Q$ being the spin operators of the heavy mesons) used in the bound state approach in the Skyrme model. Then, the eigenmodes of the heavy meson can be classified by the third component of heavy quark spin $s_Q$, the grand spin $(k_\ell, k_3)$ and the parity $\pi$, so that $n = \{k_\ell, k_3, \pi, s_Q\}$.

The situation is very similar to obtaining the eigenmodes of the confined quarks in the chiral bag model [29]. We start with the construction of the eigenfunctions of the grand spin and the heavy quark spin by taking direct products of the four eigenstates, $Y_{\ell m\ell}, \tilde{\phi}_{\pm \frac{1}{2}}$, $\ell(\pm \frac{1}{2})$, and $|\pm \frac{1}{2}\rangle_Q$, which yields four $\mathcal{K}^{(i)}_{k_hk_3s_Q}$, $i=1,2,3,4$:

$$\mathcal{K}^{(i)}_{k_hk_3s_Q} = \sum_{m_\ell,m_t}(\ell_1,m_\ell,\frac{1}{2},m_t|\ell_i,m_\ell + m_t)(\ell_i,m_\ell + m_t,\frac{1}{2},m_s|k_\ell,k_3)Y_{\ell m\ell}(\hat{r})\tilde{\phi}_{m_\ell}(m_s||s_Q),$$  

(3.5)

with the Clebsch-Gordan coefficients $(\ell_1,m_1,\ell_2,m_2|\ell,m)$. Here, for a later convenience, we first combine the angular momentum and the isospin to form $\vec{X}(= \vec{L} + \vec{I}_h)$ and then combine the light quark spin with it. Although the heavy quark spin does not involve in the combination of the light-quark grand spin, we have included them in the definition of $\mathcal{K}^{(i)}_{k_hk_3s_Q}$ in order to shorten the expressions. The explicit forms of $\mathcal{K}^{(i)}_{k_hk_3s_Q}$ are given in Appendix B.

In terms of these $\mathcal{K}^{(i)}_{k_hk_3s_Q}$, the heavy meson wavefunction can be written as

$$H_n(\vec{r}) = \sum_{i=1,2} h^{(i)}_{k_\ell}(r)\mathcal{K}^{(i)}_{k_hk_3s_Q}, \text{ for } \pi = -(-1)^{k_\ell} \text{ states},$$  

$$H_n(\vec{r}) = \sum_{i=3,4} h^{(i)}_{k_\ell}(r)\mathcal{K}^{(i)}_{k_hk_3s_Q}, \text{ for } \pi = +(-1)^{k_\ell} \text{ states},$$  

(3.6)

with radial functions $h^{(i)}_{k_\ell}(r)$. Note that the electric modes $(i=1,2$ states) and the magnetic modes $(i=3,4$ states) are decoupled due to their different parities.

Since the heavy mesons and the soliton are assumed infinitely heavy, their kinetic effects can be neglected and the heavy mesons are expected just to sit at the center of the soliton where the potentials have the lowest value. That is, in the heavy mass limit, all the radial functions $h^{(i)}_{k_\ell}(r)$ can be approximated as

$$h^{(i)}_{k_\ell}(r) = \alpha_i f(r),$$  

(3.7)
with a constant $\alpha_i$ and a function $f(r)$ which is strongly-peaked at the origin and normalized as $\int_0^\infty r^2 dr |f(r)|^2 = 1$. The problem is to find the eigenfunction of the equation

$$\varepsilon \mathcal{K}_{k^* k_3 s_Q} = \frac{1}{2} g F'(0) \mathcal{K}_{k^* k_3 s_Q} \{ (\vec{\tau} \cdot \bar{\vec{r}})[\vec{\sigma} \cdot \bar{\vec{r}}](\vec{\tau} \cdot \bar{\vec{r}}) \}, \quad (3.8)$$

as a linear combination of $\mathcal{K}_{k^* k_3 s_Q}^{(i)}$, i.e., $\mathcal{K}_{k^* k_3 s_Q} = \sum_i \alpha_i \mathcal{K}_{k^* k_3 s_Q}^{(i)}$ (i.e., $\mathcal{K}_{k^* k_3 s_Q}^{(i)} = \mathcal{K}_{k^* k_3 s_Q}$). In obtaining Eq. (3.8), we have used that $F(r) = \pi + F'(0) r + O(r^2)$ near the origin so that $a_1(r) \sim -F'(0) + O(r^2)$, and $a_2(r) \sim 2F'(0) + O(r^2)$ and the identity $(2\vec{\tau} \cdot \vec{\pi} - \vec{\sigma} \cdot \vec{\tau}) = (\vec{\tau} \cdot \vec{\pi})(\vec{\sigma} \cdot \vec{\tau})(\vec{\tau} \cdot \vec{\pi})$.

The expansion coefficients $\alpha_i$ are obtained by solving the secular equation

$$\sum_j \mathcal{M}_{ij} \alpha_j = -\varepsilon \alpha_i, \quad (i, j = 1, 2, 3, 4) \quad (3.9)$$

where the matrix elements $\mathcal{M}_{ij}$ (i, j = 1, 2, 3, 4) are defined as

$$\mathcal{M}_{ij} = \frac{1}{2} g F'(0) \int d\Omega \text{Tr} \left\{ \mathcal{K}_{k^* k_3 s_Q}^{(i)} (\bar{\vec{r}} \cdot \vec{\tau}) [(\vec{\sigma} \cdot \bar{\vec{r}})] (\vec{\tau} \cdot \bar{\vec{r}}) \mathcal{K}_{k^* k_3 s_Q}^{(j)} \right\}, \quad (3.10)$$

with $\mathcal{K}_{k^* k_3 s_Q}^{(j)} = \gamma^0 \mathcal{K}_{k^* k_3 s_Q}^{(j)\dagger} \gamma^0$. The minus sign in the right hand side of Eq. (3.9) comes from the fact that the basis states $\mathcal{K}_{k^* k_3 s_Q}^{(i)}$ are normalized as (B4).

Explicit matrix elements are presented in Appendix B, according to which $\mathcal{K}_{k^* k_3 s_Q}^{(1,2)} (\bar{\vec{r}})$ are already the eigenstates of Eq. (3.8) with the degenerate eigenenergy $\frac{1}{2} g F'(0)$ and $\mathcal{M}$ should be diagonalized for $i, j = 3, 4$. The diagonalization of $\mathcal{M}_{ij}$ (i, j = 3, 4) leads to two eigenstates:

$$\mathcal{K}_{k^* k_3 s_Q}^{(+)} = \sqrt{\frac{k_l}{2k_l+1}} \mathcal{K}_{k^* k_3 s_Q}^{(3)} + \sqrt{\frac{k_l+1}{2k_l+1}} \mathcal{K}_{k^* k_3 s_Q}^{(4)}, \quad (\varepsilon = +\frac{1}{2} g F'(0))$$

$$\mathcal{K}_{k^* k_3 s_Q}^{(-)} = \sqrt{\frac{k_l}{2k_l+1}} \mathcal{K}_{k^* k_3 s_Q}^{(3)} - \sqrt{\frac{k_l+1}{2k_l+1}} \mathcal{K}_{k^* k_3 s_Q}^{(4)}, \quad (\varepsilon = -\frac{3}{2} g F'(0)) \quad (3.11)$$

Thus, for each set of different quantum numbers $\{ k_l(\neq 0), k_3, s_Q \}$, if we have one state with the eigenenergy $\varepsilon = -\frac{3}{2} g F'(0)$ and three degenerate states with $\varepsilon = -\frac{1}{2} g F'(0)$. Since $g < 0$ and $F'(0) < 0$ (in case of baryon-number-one soliton solution), we have one soliton-heavy meson bound state with a binding energy $\frac{3}{2} g F'(0)$. The positive energy states imply three (one for $k_l = 0$) soliton-antiflavored heavy meson bound states with a binding energy $\frac{1}{2} g F'(0)$.

[17] (See Sec. VI) In Fig. 1 given are energy levels of the bound states of heavy mesons and antiflavoured ones. Shaded areas denote the continuum states and triply degenerate states are represented as thick lines.

In case of a typical soliton solution stabilized by the Skyrme term [21], we have $F'(0) \sim -2e f_\pi$ with $e$ being the Skyrme parameter. When the parameters are fixed as $f_\pi=64.5$ MeV

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3In case of $k_l=0$, we have two eigenstates;

$$\mathcal{K}_{000}^{(1)} (\bar{\vec{r}}) : \varepsilon = +\frac{1}{2} g F'(0),$$

$$\mathcal{K}_{000}^{(3)} (\bar{\vec{r}}) : \varepsilon = -\frac{3}{2} g F'(0).$$
and \( e = 5.45 \) for the soliton to fit well the nucleon and Delta masses \([30]\), \( F'(0) \) amounts to \( \sim -0.70 \) GeV which implies that the binding energies of the soliton-heavy meson and soliton-antiflavored heavy meson bound states are \( \frac{3}{2}gF'(0) \sim 0.79 \) GeV and \( \frac{1}{2}gF'(0) \sim 0.26 \) GeV, respectively. Compared with those of Refs. \([30]\), one can see that the binding energies are reduced by a factor half and more. It should be emphasized further that in Refs. \([30]\) the binding energy increases as the heavy meson mass increases and also that our results are obtained with infinite heavy meson mass.

Each state can be combined with the heavy quark spin to produce doubly-degenerate grand spin eigenstates with \( k_\pm = k_\ell \pm 1/2 \) (provided \( k_\ell \neq 0 \), for which we have only a grand spin \( k = 1/2 \) state). These degeneracies are the direct consequence of the heavy quark symmetry and a special attention should be paid to the quantization procedure, although the degeneracy in \( k_\ell \) (and in principal quantum numbers which are related with the radial excitations) may be an artifact from the approximation (3.7) on the radial function \( h^{(i)}_{k_\ell}(r) \).

In general, when the heavy meson’s kinetic term is taken into account, the radial function feels the centrifugal potential \( \ell_{\text{eff}}(\ell_{\text{eff}} + 1)/r^2 \) near the origin so that it behaves as \( h^{(i)}_{k_\ell} \sim r^{\ell_{\text{eff}}} \). Here, \( \ell_{\text{eff}} \) is the “effective” angular momentum \([3]\), which is related to \( \ell \) as

\[
\ell_{\text{eff}} = \begin{cases} 
\ell + 1, & \text{if } \lambda = \ell + \frac{1}{2}, \\
\ell - 1, & \text{if } \lambda = \ell - \frac{1}{2}.
\end{cases}
\]

Due to the vector potential \( \vec{V} \) \((\sim i(\hat{\mathbf{r}} \times \hat{\mathbf{r}})/r\), near the origin) the singular structure of \( \vec{D}^2 = (\vec{\nabla} - \vec{V})^2 \) is altered from \( \ell(\ell + 1)/r^2 \) (that of \( \vec{\nabla}^2 \)) to \( \ell_{\text{eff}}(\ell_{\text{eff}} + 1)/r^2 \). Thus, only those states with \( \ell_{\text{eff}} = 0 \) can have strongly peaked radial function and the degeneracies will be broken in such a way that the states with higher \( \ell_{\text{eff}} \) have the higher energy. For the positive parity states, \( \ell_{\text{eff}} = 0 \) can be achieved only when \( \ell = 1 \). In Table I listed are a few heavy meson eigenstates which involve \( \ell = 0, 1 \) angular basis. From now on, we will restrict our considerations to these states. Note that the \( \ell_{\text{eff}} \) of the wavefunctions \( K_{1ms_Q}^{(1)}, K_{2ms_Q}^{(4)} \) (included in \( K_{2ms_Q}^{(\pm)} \)), and \( K_{2ms_Q}^{(3)} \) (included in \( K_{2ms_Q}^{(\pm)} \)) is 2 and that the \( K_{1ms_Q}^{(4)} \) (included in \( K_{1ms_Q}^{(\pm)} \)) is 1. In case of the finite heavy meson mass, these states will have higher eigenenergies and will become even unbound.

In terms of the eigenmodes, we can expand the heavy meson field operator as

\[
H(x) = \sum_n H_n(\vec{r})e^{-i\epsilon_n t}a_n,
\]
with the heavy meson annihilation operators \( a_n \). Note that we don’t need to include the term for the antiparticles. The Fock states on which the quark field operators act are obtained as

\[
|n_1, n_2, \cdots \rangle = a_{n_1}^{\dagger}a_{n_2}^{\dagger} \cdots |\text{vac}\rangle,
\]
where \(|\text{vac}\rangle\) is the vacuum of the heavy meson fields. Hereafter, we will denote the Fock states of a single heavy meson occupying the corresponding state as in Table I. (To simplify the notations, unless necessary, we will not specify such trivial quantum numbers as the third component of the grand spin, the heavy quark spin and the parity; \( k_3, s_Q \) and \( \pi \).)
What we have obtained so far is the soliton-heavy meson (or antiflavored heavy meson) bound state which carries a baryon number and a heavy flavor (or anti-heavy flavor). To endow the states with correct quantum numbers such as spin and isospin, we have to go to next order in $1/N_c$, while remaining in the same order in $m_Q$, namely $O(m_Q^0 N_c^{-1})$. This can be done by quantizing the zero modes associated with the invariance under simultaneous $SU(2)$ rotation of the soliton configuration together with the heavy meson fields:

$$\xi_0 \rightarrow C\xi_0 C^\dagger, \quad \text{and} \quad H \rightarrow HC^\dagger,$$

with an arbitrary constant $SU(2)$ matrix $C$ and $\xi_0^2 \equiv U_0$. The rotation becomes dynamical by giving time dependence to the $SU(2)$ collective variables as

$$\xi(\vec{r}, t) = C(t)\xi_0(\vec{r})C^\dagger(t), \quad \text{and} \quad H(\vec{r}, t) = H_{bf}(\vec{r}, t)C^\dagger(t),$$

and then the quantization is done by elevating the collective variables to the corresponding quantum mechanical operators. In Eq.(4.2), $H_{bf}$ refers to the heavy meson field in the isospin-co-moving frame, while $H(\vec{r}, t)$ refers to that in the laboratory frame. Substitution of Eq.(4.2) into Eq.(2.15) leads us to the Lagrangian (in the reference frame where the heavy meson is at rest in space but rotating in isospin space)

$$L^{\text{rot}} = -M_{\text{sol}} + \int d^3r \left\{ -i \text{Tr}(\partial_0 H_{bf}\bar{H}_{bf}) + g \text{Tr}(H_{bf}\vec{A}\cdot\vec{\sigma}\bar{H}_{bf}) \right\} + \frac{i}{2}\mathcal{I}\omega^2 - \frac{1}{2}\int d^3r \left\{ \text{Tr}(H_{bf}\frac{1}{2}((\xi^\dagger\vec{\tau}\cdot\vec{\omega}\xi + \xi\vec{\tau}\cdot\vec{\omega}\xi^\dagger) \bar{H}_{bf}) \right\},$$

where we have kept terms up to $O(m_Q^0 N_c^{-1})$. The “angular velocity”, $\vec{\omega}$, of the collective rotation is defined by

$$C^\dagger\partial_0 C \equiv \frac{1}{2}i\vec{r}\cdot\vec{\omega},$$

and $\mathcal{I}$ is the moment of inertia of the soliton configuration with respect to the rotation

$$\mathcal{I} = \frac{8\pi}{3} f_\pi^2 \int_0^\infty r^2 dr (\sin^2 F + \cdots),$$

where the contributions from the soliton-stabilizing-Lagrangian are abbreviated simply by ellipsis.

Given the Lagrangian (4.3) that describes dynamics up to order $O(m_Q^0 N_c^{-1})$, one has equation of motion

$$i\partial_0 H_{bf} = H_{bf} \left[ g\vec{A}\cdot\vec{\sigma} - \frac{1}{4}(\xi^\dagger\vec{\tau}\cdot\vec{\omega}\xi + \xi\vec{\tau}\cdot\vec{\omega}\xi^\dagger) \right]$$

consistent to that order. The last “Coriolis” term in the equation of motion couples the fast and slow degrees of freedom. Although the heavy mesons are infinitely heavy, their angular momentum and isospin are associated with the light constituents. Thus, we may take those light degrees of freedom of the heavy meson fields as the fast ones and the collective rotations as the slow ones. Note that the scale of the eigenenergies $|\varepsilon_n|$ of the heavy mesons is much greater than that of the rotational velocity; $|\varepsilon_n| \gg |\omega|$. 

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Generally accepted procedure of handling these different scales is as follows. We first solve the equation of motion for fast degrees of freedom with slow degrees of freedom “frozen”. In this way, we get “snap-shot” pictures of the fast motion. Next we solve the equation of motion for slow degrees of freedom taking into account the “relic” of the fast motion that has been “integrated out”, in a manner completely analogous to the incorporation of Berry phases. It is also analogous to the “strong-coupling limit” of the particle-rotor model in nuclear physics, where the coupling between the rotating “core” and the particle is much stronger than the perturbation of the single-particle motion by Coriolis interaction. Here, the roles of the particle and the rotor are played by the bound heavy-mesons and the rotating soliton configuration. Thus, we may take the assumption that the bound heavy mesons rotate together with the soliton core in the unchanged eigen-modes. It enables us to expand the $H_{bf}(x)$ in terms of the classical eigenmodes obtained in Sec. III as Eq. (3.13):

$$H_{bf}(x) = \sum_n H_n(\vec{r}) e^{-i\varepsilon_n t} a_n.$$  

(4.5)

Taking Legendre transformation of the Lagrangian, we obtain the Hamiltonian as

$$H = \int d^3r \left\{ \delta L^\text{rot}_{bf,\alpha\beta} \delta \dot{H}_{bf,\alpha\beta} + \frac{\delta L^\text{rot}_{bf,\alpha}}{\delta \omega_\alpha} \omega_\alpha - L^\text{rot} \right\}$$

$$= M_{sol} - g \int d^3r \text{Tr}(H_{bf} \vec{A} \cdot \vec{\sigma} H_{bf}) + \frac{1}{2I}(\vec{R} - \vec{\Theta}(\infty))^2,$$

(4.6)

where the rotor spin $\vec{R}$ is the canonical momenta conjugate to the collective variables $C(t)$:

$$R_\alpha \equiv \frac{\delta L^\text{rot}_{bf,\alpha}}{\delta \omega_\alpha} = \mathcal{I} \omega_\alpha + \Theta_\alpha(\infty),$$

(4.6a)

with $\vec{\Theta}(\infty)$ defined as

$$\vec{\Theta}(\infty) \equiv -\frac{1}{2} \int d^3r \text{Tr}[H_{bf} \frac{1}{2}(\xi^\dagger \vec{\tau} \xi + \xi \vec{\tau} \xi^\dagger) \vec{A}_{bf}],$$

(4.6b)

whose expectation value with respect to the state $|n\rangle$ is the Berry phase associated with the collective rotation. Note that it is nothing but the isospin of the heavy mesons modulo the sign. (See Eq. (A13b) below)

With the collective variable introduced in Eq. (1.2), the isospin of the fields $U(x)$ and $H(x)$ is entirely shifted to $C(t)$. To see this, consider the isospin rotation

$$\Sigma \rightarrow \mathcal{A} \Sigma \mathcal{A}^\dagger, \quad H \rightarrow H \mathcal{A}^\dagger,$$

(4.7)

with $\mathcal{A} \in SU(2)_V$, under which the collective variables and fields in body-fixed frame transform as

$$C(t) \rightarrow \mathcal{A} C(t), \quad H_{bf}(x) \rightarrow H_{bf}(x).$$

(4.8)

Note that the $H$-field becomes isospin blind in the (isospin) co-moving frame. The conventional Noether construction leads the isospin of the system as
\[ I_a = \frac{1}{2} \text{Tr}(\tau_a C \tau_b C^\dagger) (I \omega_b + \Theta_b(\infty)) = D_{ab}(C) R_b, \] (4.9)

with \( D_{ab}(C) \) being the \( SU(2) \) adjoint representation associated with the collective variables \( C(t) \).

Under a spatial rotation (together with the spin rotation in case of the heavy meson), with the help of the \( K \)-symmetry, the fields transform as

\[
\begin{align*}
\Sigma(\vec{r}, t) &\rightarrow \Sigma(\vec{r}', t) = C(t) B^\dagger \Sigma_0(\vec{r}) B C^\dagger(t), \\
H(\vec{r}, t) &\rightarrow e^{i \frac{1}{2} \vec{a} \cdot \vec{\sigma}} H(\vec{r}', t) e^{-i \frac{1}{2} \vec{a} \cdot \vec{\sigma}} = (e^{i \frac{1}{2} \vec{a} \cdot \vec{\sigma}} H_{bf}(\vec{r}', t) e^{-i \frac{1}{2} \vec{a} \cdot (\vec{\sigma} + \vec{r})}) B^\dagger C(t),
\end{align*}
\] (4.10)

with \( \vec{r}' = \exp(i \vec{a} \cdot \vec{L}) \vec{r} \) and \( B = \exp(i \frac{1}{2} \vec{a} \cdot \vec{r}) \in SU(2) \). This means that the spatial rotation is equivalent to the transformation of the collective variables and \( H \)-fields in the body fixed frame as

\[
\begin{align*}
C(t) &\rightarrow C(t) B^\dagger, \\
H_{bf}(\vec{r}', t) &\rightarrow e^{i \frac{1}{2} \vec{a} \cdot \vec{\sigma}} H_{bf}(\vec{r}', t) e^{-i \frac{1}{2} \vec{a} \cdot (\vec{\sigma} + \vec{r})}.
\end{align*}
\] (4.11)

Therefore, we get the fact that the spin of the \( H_{bf}(x) \) is the grand spin; that is, the isospin of the \( H \)-field is transmuted into the part of the spin in the isospin co-moving frame. Remember that the \( H_{bf}(x) \) becomes isospin blind in that frame. Applying the Noether theorem to the Lagrangian (4.3), we obtain the spin of the system explicitly as

\[ \vec{J} = \vec{R} + \vec{K}_{bf}, \] (4.12)

with the grand spin of the heavy meson fields in the isospin co-moving frame

\[ \vec{K}_{bf} = - \int d^3 r \text{Tr}\{ (\vec{L} H_{bf} + [\frac{1}{2} \vec{\sigma}, H_{bf}] + H_{bf}(-\frac{1}{2} \vec{r})) \vec{H}_{bf} \}. \] (4.13)

Finally, the heavy-quark spin symmetry of the Lagrangian under the transformation

\[ H(x) \rightarrow e^{i \frac{1}{2} \vec{a} \cdot \vec{\sigma}} H(x) = (e^{i \frac{1}{2} \vec{a} \cdot \vec{\sigma}} H_{bf}(x)) C(t), \] (4.14)

has nothing to do with the collective rotations. The heavy-quark spin operator remains unchanged in the isospin co-moving frame:

\[ \vec{S}_Q = - \int d^3 r \text{Tr}(\vec{\sigma} \vec{H} \vec{H}) = - \int d^3 r \text{Tr}(\frac{1}{2} \vec{\sigma} H_{bf} \vec{H}_{bf}). \] (4.15)

Because of this heavy-quark spin decoupling, it is convenient to proceed with the spin operators \( \vec{J}_\ell \) for the light degrees of freedom in the soliton-heavy meson bound system defined as

\[ \vec{J}_\ell = \vec{J} - \vec{S}_Q = \vec{R} + \vec{K}_{bf}. \] (4.16)

Upon canonical quantization, the collective variables become the quantum mechanical operators; the isospin (\( \vec{I} \)), the spin (\( \vec{J}_\ell \)) and the spin of the rotor (\( \vec{R} \)) discussed so far become the corresponding operators \( \hat{I}_a, \hat{J}_{\ell,a} \) and \( \hat{R}_a \), respectively. We distinguish those operators
associated with the collective coordinate quantization by using a tilde on them. Let the
eigenstates of the rotor-spin operator \( \tilde{R}_a \) be denoted by \( |i; m_1, m_2 \rangle \) (\( m_1, m_2 = -i, -i + 1, \cdots, i \)):

\[
\begin{align*}
\tilde{R}_z |i; m_1, m_2 \rangle &= m_2 |i; m_1, m_2 \rangle, \\
\tilde{I}_z |i; m_1, m_2 \rangle &= m_1 |i; m_1, m_2 \rangle.
\end{align*}
\] (4.17)

These states are represented explicitly by the Wigner D-functions

\[
\sqrt{2i + 1} D_{m_1, m_2}^{(i)}(C).
\] (4.18)

At this point, it should be mentioned that we are following exactly the same quantization
procedure of CK [3]. We quantize the whole soliton-heavy meson bound system to obtain
the heavy baryon states. In Refs. [12,13], only the soliton is quantized to nucleons and \( \Delta \)'s
by using the collective coordinate quantization. Then the heavy mesons with good isospin
and spin are bound to form a heavy baryon. The corresponding isospin and spin operator
of the heavy baryon system are different from Eqs. (4.9) and (4.12), respectively. However,
both approaches lead to the same final results for the heavy baryon spectrum.

V. HEAVY BARYONS

The eigenstates \( |i, i_3; j_\ell, j_{\ell,3}; s_Q \rangle \) of the operators \( \tilde{I}_a \) and \( \tilde{J}_{\ell,a} \) with their corresponding
quantum numbers \( i, i_3 \) (isospin) and \( j_\ell, j_{\ell,3} \) (spin of the light degrees of freedom) are given
by the linear combinations of the direct product of the the rotor-spin eigenstates \( |i; m_1, m_2 \rangle \)
and the single-particle Fock state \( |n \rangle \):

\[
|i, i_3; j_\ell, j_{\ell,3}; s_Q \rangle_a = \sum_m (i, j_\ell,3 - m, k_{j_\ell}^a, m|j_\ell, j_{\ell,3}|i, i_3, j_\ell,3 - m) |k_{j_\ell}, m, s_Q \rangle_a. \] (5.1)

One may combine further the heavy quark spin and the spin of the light degrees of freedom
to construct the states with a good total spin, which is, however, not necessary in our
discussion. Remember that, in the infinite heavy quark mass limit, \( (j_\ell, j_{\ell,3}) \) themselves are
good quantum numbers to label the heavy hadrons. For a given set of \( (i, j_\ell) \), there can be
more than one state depending on which Fock state \( |n \rangle \) is involved in the combination (5.1).
We will distinguish them by using a sequential number, \( a(=1,2,\cdots) \), in \( |i, j_\ell \rangle \). Here again,
to shorten the expressions, we will not specify the quantum numbers \( i_3, j_3 \) and \( s_Q \) unless
necessary.

In Table II, we list a few \( |i, j_\ell^p \rangle \) states for low-lying heavy baryons. Here, we consider the
\( |i, j_\ell^p \rangle \) states made of the integer rotor-spin-states so that they describe the heavy baryons
of a half-integer spin \( j = j_\ell \pm \frac{1}{2} \). In the Table, we do not present the states such as \( |0, 1^+ \rangle \),
\( |1, 0^+ \rangle \) and \( |0, 0^- \rangle \), which cannot form the bound states. The incorporation of the \( 1/N_c \)
order corrections due to collective rotations cannot turn them into bound systems. We also
exclude two other possible \( |1, 1^- \rangle \) states from the list, since they come from the heavy-meson states \( |2 \rangle_{1,2} \) of \( \ell = 2 \) and thus they do not mix up with the state \( |1, 1^- \rangle \) shown in Table II
by the collective rotation.
The physical heavy baryons of our concern appear as the eigenstates of the Hamiltonian $\tilde{H}$. We will not try to find the exact eigenstates of the Hamiltonian. Remind that we have kept the terms only up to $O(m_Q^0 N_c^{-1})$ in the collective coordinate quantization procedure. We take the last term in the Hamiltonian

$$\tilde{H}_{rot} \equiv \frac{1}{2\ell} (\vec{R} - \vec{\Theta}(\infty))^2,$$  

(5.2)

as a perturbation of order $O(m_Q^0 N_c^{-1})$ and we will consistently search for the approximate eigenstates to that order.

Except for the case of $|1, 1^{+}\rangle$, we have only one bound state for a given $(i, j^{\pi})$. To the first order, the eigenstate of $\tilde{H}$ is approximated by the unperturbed one $|i, j^{\pi}\rangle_1$ and the mass correction to the corresponding baryon is obtained by taking the expectation value of the $\tilde{H}_{rot}$ with respect to it:

$$M_{(i, j^{\pi})}^{rot} = \frac{1}{2\ell} \langle \langle i, j^{\pi}_{\ell}|(\vec{R} - \vec{\Theta}(\infty))^2|i, j^{\pi}_{\ell}\rangle \rangle.$$  

(5.3)

Now, our problem is reduced to evaluating the expectation values of the operators $-2\vec{R} \cdot \vec{\Theta}(\infty)$ and $\vec{\Theta}^2(\infty)$. Using Eq.(5.2), we easily obtain the expectation value of the operator $-2\vec{R} \cdot \vec{\Theta}(\infty)$ as

$$a \langle \langle i, j^{\pi}_{\ell}| - 2\vec{R} \cdot \vec{\Theta}(\infty)|i, j^{\pi}_{\ell}\rangle \rangle_a = c_{aa} \{j_{\ell}(j_{\ell} + 1) - i(i + 1) - k_{\ell}(k_{\ell} + 1)\},$$  

(5.4)

with $c_{aa}$ being the $c$-value associated with the single-particle Fock state $|k_{\ell}\rangle_a$ participating in the construction of the state $|i, j^{\pi}_{\ell}\rangle$. As for the expectation value of the operator $\vec{\Theta}^2(\infty)$, in many CK-based models [5] it has been approximated as

$$\langle \langle i, j^{\pi}_{\ell}|\vec{\Theta}^2(\infty)|i, j^{\pi}_{\ell}\rangle \rangle_a \approx |a\langle k_{\ell}|\vec{\Theta}(\infty)|k_{\ell}\rangle_a|^2 = c_{aa}^2 k_{\ell}(k_{\ell} + 1).$$  

(5.5)

In the heavy meson mass limit, it can be exactly evaluated as

$$a\langle k_{\ell}, k_3|\vec{\Theta}^2(\infty)|k_{\ell}, k_3\rangle_a = \sum_{k'_{\ell}, k'_3, b} |a\langle k_{\ell}, k_3|\vec{\Theta}(\infty)|k'_{\ell}, k'_3\rangle_b|^2,$$  

(5.6)

where the summation runs over the complete set of intermediate Fock states $|k'_{\ell}, k'_3\rangle_b$. Including all the Fock states that have non-vanishing expectation value $\langle n|\vec{\Theta}(\infty)|m\rangle$ with the help of the approximation (3.4, 5) on the radial functions, we obtain

$$a\langle k_{\ell}|\vec{\Theta}^2(\infty)|k_{\ell}\rangle_a = \frac{3}{4}.$$  

(5.7)

One may obtain the same result by using the fact that $\vec{\Theta}(\infty)$ defined as in Eq.(1.6b) is nothing but the isospin operator $I_h$ (modulo opposite sign) of the heavy meson field.

Thus, for the heavy baryons with quantum numbers $(i, j^{\pi})$ which allow only one bound state, we obtain the mass formula as

$$m_{i, j^{\pi}} = M_{sol} + \omega_B + \frac{3}{8\ell} + \frac{1}{2\ell} \{c j_{\ell}(j_{\ell} + 1) + (1 - c)i(i + 1) - c k_{\ell}(k_{\ell} + 1)\},$$  

(5.8)
where $c$ is an abbreviation for $c_{aa}$ and $\omega_B \equiv m_\Phi - \frac{3}{2}g'F'(0)$. In order to compare it with Eq.\,(1.2), we have included the weight averaged heavy-meson mass $m_\Phi(\equiv \frac{1}{3}(3m_{q*} + m_\Phi))$. Eq.\,(5.8) is in a quite different form from Eq.\,(1.2) and satisfies the heavy-quark symmetry regardless of the $c$-value. However, all the $c$-values associated with the heavy-meson bound states $|k_\ell\rangle_-$ vanish identically; viz.,

$$c_{--} = \frac{k_\ell + 1}{2k_\ell + 1}c_{33} - 2\sqrt{\frac{k_\ell(k_\ell + 1)}{2k_\ell + 1}}c_{34} + \frac{k_\ell}{2k_\ell + 1}c_{44} = 0,$$

as given in Appendix B. Consequently, both mass formulas (1.2) and (5.8) yield the same heavy baryon masses apart from a constant shift coming from the way of evaluating the expectation value of $\vec{\Theta}^2(\infty)$ as in (5.5) and (5.7). It should be emphasized that such a coincidence owes entirely to the vanishing $c$-values and the presence of the unique bound state. When one has non-vanishing $c$-value as we shall see in Sec. VI, Eq.\,(1.2) cannot be applied anymore.

If we have multiple degenerate bound states, $|i,j^\pi\rangle_a (a = 1, 2, \cdots)$, the situation becomes a little bit complicated. The mass corrections and the corresponding eigenstate are obtained by diagonalizing the energy matrix $\mathcal{E}$ whose matrix element is defined by

$$\mathcal{E}_{ab} = a\langle i, j^\pi \mid \hat{H}_{rot} \mid i, j^\pi \rangle_b. \quad (a, b = 1, 2, \cdots)$$

For the $(i = 1, j^\pi = 1^+)$ heavy baryons, we have doubly degenerate bound states; $|1,1\rangle_1$ and $|1,1\rangle_2$. Since they are, respectively, made of $|k_\ell = 0\rangle_3$ and $|k_\ell = 2\rangle_-$, the first rank tensor $\Theta(\infty)$ cannot lead to nonvanishing energy matrix between the two states. Thus, each state can be separately the eigenstate with their degenerate mass given by the mass formula (5.8).

Because of vanishing $c$-value, we can write the mass formula for the heavy baryon simply as

$$m_{(i,j^\pi)} = M_{sol} + \omega_B + \frac{1}{2I}(i(i + 1) + \frac{3}{4}).$$

In Fig. 2, the resulting heavy baryon spectrum is presented schematically. It is interesting to observe the degeneracy in mass of the heavy baryons with positive and negative parities. Since we are working upon an assumption on the radial functions with ignoring any effects of the kinetic term, we are not at a position to conclude whether such a parity doubling has any physical importance or it is just an artifact of the approximation.

Explicitly, we have the masses of $(0,0^+)$ and $(1,1^+)$ heavy baryons as

$$m_{(0,0^+)} = M_{sol} + \omega_B + 3/8I,$$

$$m_{(1,1^+)} = M_{sol} + \omega_B + 11/8I,$$

$$m_{(1,1^+)} = M_{sol} + \omega_B + 11/8I,$$

$\text{The exact degeneracy is however an artifact of the approximation of using the same radial function for all the state. When the heavy mesons are allowed to move, } |1,1\rangle_2 \text{ will have higher energy.}$
which correspond to the mass of $\Lambda_Q$ and degenerate mass of $\Sigma_Q$ and $\Sigma_Q^*$, respectively and are consistent with the results of Ref. [12]. Eq. (5.12) yields an interesting model-independent relation for the mass difference of $\Sigma_Q (\Sigma_Q^*)$ and $\Lambda_Q$:

$$m_{\Sigma_Q \Sigma_Q^*} - m_{\Lambda_Q} = \frac{2}{3}(m_\Delta - m_N) \approx 0.20 \text{ GeV},$$  

(5.13)

where we have used the fact that the $SU(2)$ collective quantization of the bare soliton leads to the the nucleon and delta masses as

$$m_N = M_{\text{sol}} + \frac{3}{8I}, \quad \text{and} \quad m_\Delta = M_{\text{sol}} + \frac{15}{8I}.$$  

(5.14)

In the mass formula for the heavy baryons, we have three parameters, $M_{\text{sol}}, 1/I$ and $gF'(0)$. For a naive prediction on the heavy baryon masses, we fit them to produce experimental values of $m_N(-939 \text{ MeV})$, $m_\Delta(-1232 \text{ MeV})$ and $m_{\Lambda_c}(=2285 \text{ MeV})$, which leads to

$$M_{\text{sol}} = 866 \text{ MeV}, \quad 1/I = 195 \text{ MeV} \quad \text{and} \quad gF'(0) = 419 \text{ MeV}.$$  

(5.15)

Combined with the slope of the soliton wavefunction $F'(0) \sim -690 \text{ MeV}$ (in case of the Skyrme term-stabilized soliton solution), Eq.(5.15) implies $g$-value as $g \approx -0.61$ which is comparable to that of the non-relativistic quark model ($-0.75$) and the experimental estimation via the $D^*$-decay ($|g|^2 \lesssim 0.5$). This set of parameters yields a prediction on the the $\Lambda_b$ mass and the average mass of the $\Sigma_c - \Sigma_c^*$ multiplets, $m_{\Sigma_c} (\equiv \frac{1}{3}(2m_{\Sigma_c} + m_{\Sigma_c^*}))$ as

$$m_{\Lambda_b} = M_{\text{sol}} + m_B - \frac{3}{2}gF'(0) + 3/8I = 5623 \text{ MeV},$$

$$m_{\Sigma_c} = M_{\text{sol}} + m_D - \frac{3}{2}gF'(0) + 11/8I = 2483 \text{ MeV},$$  

(5.16)

which are comparable with the experimental value of the $\Lambda_b$ mass (5641 MeV) and $\Sigma_c$ mass (2453 MeV) [33]. Recent experimental data for the $\Sigma_c^*$ mass of 2530 MeV [34] (although it needs the verifications of other groups) gives 2504 MeV for the experimental value of $m_{\Sigma_c}$, which is not far from our estimation. In our approach, the $\Sigma_c$ and $\Sigma_c^*$ are degenerate in mass. To get the splitting between them, one needs to include $1/m_Q$ corrections [10].

With $1/I \sim 200 \text{ MeV}$, we can estimate the discrepancy in the masses of the heavy baryons given by the mass formulas, (1.2) and (5.8), as $3/8I \sim 60 \text{ MeV}$. It amounts 10% of the binding energy $-\frac{3}{2}gF'(0) \sim 630 \text{ MeV}$ and 30% of the rotational energy $\sim 1/I \sim 200 \text{ MeV}$. One should not say that the discrepancy is at most 3% for the case of the charmed baryons by comparing it with the whole heavy baryon masses. Although we are working with the heavy baryons, our scheme is valid only in the low energy region below $\Lambda_{QCD}$.

So far, we have considered the first order perturbations in the masses of the bound $|i, j^\pi\rangle$ states. To estimate naively the effects of the other states, we take into account the positive energy states into our procedure. It leads to a mass correction to the heavy baryon described by $|i, j^\pi\rangle_a$ as

$$\Delta m_{(i,j^\pi)} = \sum_b |\langle i,j^\pi|H_{\text{rot}}|i,j^\pi\rangle_b|^2 \varepsilon_a - \varepsilon_b + \cdots,$$  

(5.17)
where the summation runs over the unbound states \(|i, j_\ell^\pi\rangle\). They are, thus, at most of second order in \(1/N_c\), which is out of our concern. Here, to check a consistency, we evaluate the leading order correction to \((i = 1, j_\ell^\pi = 0^-)\) heavy baryon state. We have

\[
\Delta m_{(1,0^-)} = -\frac{1}{2} \frac{1}{2gF'(0)} \frac{1}{18I^2} \sim -2.6 \text{ MeV},
\]

which is negligibly small compared to the first order corrections, \(11/8I \sim 270 \text{ MeV}\). The coupling of \(1/N_c\) order due to collective rotations cannot compete with the energy difference \(2gF'(0)\) of order \(N_c^0\).

VI. PENTAQUARK EXOTIC BARYONS

In the limit of infinite heavy quark mass, the quark model predicts stable pentaquark\((P)\) exotic baryons whose quark contents are \(\bar{Q}q^4\). With the exact \(SU(3)_F\) symmetry assumed for the light quarks, it was shown that a strange anti-charmed baryon \(P_{\bar{c}s} (\bar{c}sq_0^3, q_0 = u, d)\) is stable against the decays into \(\Lambda D\) or \(ND_s\) with binding energy about 150 MeV. The binding energy becomes down to \(\sim 85\) MeV if included a realistic \(SU(3)_F\) symmetry breaking [37] and it becomes even unbound when the motions of the heavy constituent are taken into account [38–40]. In the Skyrme model, as the heavy meson masses increases, there appears bound state(s) for the antiflavored heavy mesons to the soliton, which reveals a possibility for the stable nonstrange \(P\)-baryon(s) [9]. It is interesting to note that in quark model such a nonstrange anticharmed baryon \((\bar{c}q_0^4)\) cannot have sufficient symmetry to yield a hyperfine binding.

Our approach can be easily switched to the one for the soliton-antifavored heavy meson bound system by considering the negative energy solutions with the four velocity \(v_\mu\) in the equations replaced by \(-v_\mu\). (One may develop an effective Lagrangian proper for the antifavored heavy mesons. See Ref. [17].) Now, for each \(k_\ell(\neq 0)\), we have three degenerate bound states of the antifavored heavy mesons, \(|k_\ell\rangle_{1,2,+,}\), with the binding energy \(\frac{1}{2}gF'(0)\). Compared with that for the heavy mesons, the binding energy is reduced by a factor 3. With \(gF'(0) = 419\) MeV as given by Eq. (5.15), it amounts to \(\sim 210\) MeV and is comparable to \(1/I\).

In Table III listed are the bound \(|i, j_\ell^\pi\rangle\) states proper for the discussions of the \(P\)-baryons. As for the states with quantum numbers \((i = 0, j_\ell^\pi = 0^-), (1, 0^-)\) and \((0, 1^-)\), we have only one bound state. To first order in \(1/N_c\), the masses of such \(P\)-baryons are given by the same mass formula as Eq. (5.8):

\[
m_{(i,j_\ell^\pi)}^P = M_{sol} + \omega_B' + \frac{1}{2I} \{cji(j_\ell + 1) + (1 - c)i(i + 1) - c\ell(k_\ell + 1) + \frac{3}{4}\},
\]

with \(\omega_B' = \pi_0 - \frac{1}{2}gF'(0)\). The \(c\)-factors associated with \(|0\rangle_1\) and \(|1\rangle_+\) states are obtained as 0 and \(-1/4\), respectively. [41] The negative \(c\)-value is remarkable. However, these states are associated with the zero rotor-spin state and such nonvanishing \(c\)-values do not play any important role in their masses which are simply obtained as

\[
m_{(0,0^-)} = m_{(0,1^-)} = M_{sol} + \omega_B' + \frac{3}{8I}.
\]
For the case of $(1, 0^-)$, the mass formula (6.1) gives
\[ m_{(1,0^-)} = M_{\text{sol}} + \omega_B' + \frac{15}{8T} = M_N + \bar{m}_\Phi - \frac{1}{2}gF'(0) + \frac{3}{2T}. \]  
(6.3)

Since $\frac{1}{2}gF'(0) \sim 210 \text{ MeV}$ and $1/T \sim 195 \text{ MeV}$, the rotational energy blows up the state above the decay threshold of the nucleon-heavy meson bound system, $M_{\text{th}}(\equiv M_N + \bar{m}_\Phi)$.

In cases of the $(0, 1^+)\text{ and } (1, 0^+)$, we have two degenerate states $|0, 1^+\rangle_{1,2}$ and $|1, 0^+\rangle_{1,2}$, respectively. Since the states come from the single-particle Fock states of the same $k_\ell$, the energy matrices can be expressed in a form of Eq.(6.1) with the constant $c$ replaced by the $2 \times 2$ matrix associated with $k_\ell = 1$:
\[ c = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}. \]  
(6.4)

Explicitly, we have
\[ E_{(0,1^+)} = M_{\text{sol}} + \omega_B' + \frac{3}{8T}, \quad \text{and} \quad E_{(1,0^+)} = M_{\text{sol}} + \omega_B' + \frac{11}{8T} - \frac{2c}{T}. \]  
(6.5)

Here again, the energy matrix $E_{(0,1^+)}$ is independent of $c$ because the states are made of the zero rotor-spin state. Anyway, both are diagonal so that $|0, 1^+\rangle_{1,2}$ and $|1, 0^+\rangle_{1,2}$ are the eigenstates of the Hamiltonian with the masses
\[ m_{(0,1^+),1,2} = M_{\text{sol}} + \omega_B' + \frac{3}{8T} = M_N + \bar{m}_\Phi - \frac{1}{2}gF'(0), \]  
(6.6)

and
\[ m_{(1,0^+),1} = M_{\text{sol}} + \omega_B' + \frac{7}{8T} = M_N + \bar{m}_\Phi - \frac{1}{2}gF'(0) + \frac{1}{2T}, \]
\[ m_{(1,0^+),2} = M_{\text{sol}} + \omega_B' + \frac{19}{8T} = M_N + \bar{m}_\Phi - \frac{1}{2}gF'(0) + \frac{2}{T}. \]  
(6.7)

The state $|1, 0\rangle_2$ lies above the decay threshold.

For the $(i = 1, j_\ell = 1^+)$ states, we have three possible combinations; $|1, 1^+\rangle_{1,2,3}$. As far as the first two states $|1, 1^+\rangle_{1,2}$ are concerned, the energy matrix reads simply
\[ E_{(1,1^+)} = M_{\text{sol}} + \omega_B' + \frac{11}{8T} + \frac{c}{T}, \]  
(6.8)

with the same $2 \times 2$ matrix $c$ as in Eq.(6.4). It leads us to the mass eigenvalues
\[ m_{(1,1^+),1} = M_{\text{sol}} + \omega_B' + \frac{9}{8T}, \]
\[ m_{(1,1^+),2} = M_{\text{sol}} + \omega_B' + \frac{15}{8T}. \]  
(6.9)

With including the third state $|1, 1^+\rangle_3$, the full $3 \times 3$ energy matrix is obtained as
\[ E_{(1,1^+)} = M_{\text{sol}} + \omega_B' + \frac{11}{8T} + \frac{1}{4T} \begin{pmatrix} -1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ \sqrt{3} & 0 & 1 \end{pmatrix}. \]  
(6.10)

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The coupling between $|1, 1^+\rangle_1$ and $|1, 1^+\rangle_3$ modifies $m_{(1,1^+)}$ as

$$m_{(1,1^+)} = M_{\text{sol}} + \omega'_B + \frac{7}{8\mathcal{L}}, \quad (6.11)$$

with $|1, 1^+\rangle_3 = \sqrt{3} |1, 1^+\rangle_1 - \frac{1}{2} |1, 1^+\rangle_3$ and adds doubly degenerate unbound states of mass $M_{\text{sol}} + \omega'_B + 15/8\mathcal{L}$. Note that in obtaining those off-diagonal matrix elements one cannot use Eq.(B9) and they cannot be written in a form of Eq.(6.1) anymore.

As for the $|1, 1^-\rangle$ states, we have four possible combinations. They are made of the heavy-meson bound state with different $k_\ell(=0,1,2)$. By using Eq.(B7), we obtain the energy matrix as

$$\mathcal{E}_{(1,-)} = M_{\text{sol}} + \omega'_B + \frac{11}{8\mathcal{L}} + \frac{1}{12\mathcal{L}} \begin{pmatrix} 0 & 4\sqrt{3} & 0 & 0 & 4\sqrt{3} & 3 & 0 & \sqrt{15} \\ 4\sqrt{3} & 3 & 0 & \sqrt{15} & 0 & 0 & -6 & 0 \\ 0 & 0 & -6 & 0 & 0 & \sqrt{15} & 0 & 9 \end{pmatrix}. \quad (6.12)$$

The couplings between the states are somewhat strong. It yields four mass eigenenergies

$$m_{(1,1^-)} = M_{\text{sol}} + \omega'_B + \frac{7}{8\mathcal{L}},$$
$$m_{(1,1^-)}^{1+} = M_{\text{sol}} + \omega'_B + \frac{15}{8\mathcal{L}},$$
$$m_{(1,1^-)}^{2+} = M_{\text{sol}} + \omega'_B + \frac{19}{8\mathcal{L}},$$

where the states with $m_{(1,1^-)}$ are doubly degenerate and expected to be bound.

Our remarkably simple results on the $P$-baryon masses are summarized in Table IV and Fig. 3. To provide a rough scale, in Fig. 3, the heavy baryon spectrum is presented in the left hand side. Here again, the degeneracy of the $P$-baryon states in the parity is apparent. In Table IV, we also give a rough prediction obtained by using the values for the parameters given by Eq.(5.15). However, all the states listed in Table IV do not seem to survive under the finite heavy meson mass corrections. Recently, we have reported that such finite mass corrections reduce the binding energy by an amount from 25% (in case of bottomed baryons) to 35% (in case of the charmed baryons) of their infinite mass limit, $\frac{3}{2}gF'(0)$ \textcircled{1}. Note that 35% of $\frac{3}{2}gF'(0)$ is comparable to the binding energy $\frac{1}{2}gF'(0)$ for the soliton-antiflavored heavy mesons. The finite mass corrections will be more crucial for the heavy meson eigenstates with $\ell_{\text{eff}} \neq 0$. All the degeneracies in the heavy meson eigenstates will be broken. Then, the couplings between the states as represented by the energy matrices $\mathcal{E}_{(1,0^+)}$, $\mathcal{E}_{(1,-)}$ and $\mathcal{E}_{(1,1^+)}$ will do less important roles. However, it does not mean that one can apply the mass formula Eq.(1.2) for the $P$-baryons in the present form.

In order to show it, here, we compare our results with what would have been obtained by straightforwardly extending the bound state approach of CK as in Ref. \textcircled{1}. As a nontrivial example, we consider the $P$-baryons with $i = 1$ and $j^\pi = \frac{1}{2}^+, \frac{3}{2}^+ (j^\pi_{\ell} = 0^+, 1^+)$, which are obtained by combining the rotor-spin state with $i = 1$ to the heavy meson bound states with $k^\pi_{\ell} = 1^+$. To simplify the process, we will include only the $|1\rangle_1$ state into the consideration, in which case the $P$-baryon masses are simply given by the diagonal elements of the energy matrices $\mathcal{E}_{(1,0^+)}$ or $\mathcal{E}_{(1,1^+)}$ associated with $|1, 0^+\rangle_1$ or $|1, 1^+\rangle_1$.
\(m_{(1, 1/2^+)} = M_{\text{sol}} + \omega_B' + (11 - 16c_1)/8\mathcal{L},\) 
\(m_{(1, 3/2^+)} = m_{(1, 1/2^+)} = M_{\text{sol}} + \omega_B' + (11 - 8c_1)/8\mathcal{L},\) 
(6.14)

with \(c_1 = \frac{1}{3};\) the \(c\)-value associated with the state \(|1\rangle_1\). In the traditional bound state approach with sufficiently heavy meson masses, \(|1\rangle_2\) will appear as nearly doubly degenerate states, say, \(|k = \frac{1}{2}\rangle\) and \(|k = \frac{3}{2}\rangle\), which become to completely degenerate in the infinitely heavy mass limit. When combined with \(i=1\) rotor spin state, the former yields \(P\)-baryons of \(j^\pi = \frac{1}{2}^+\), \(\frac{3}{2}^+\) with masses

\(m_{(1, 1/2^+)} = M_{\text{sol}} + \omega_B' + (11 - 8c_{1/2^+})/8\mathcal{L},\)
\(m_{(1, 3/2^+)} = M_{\text{sol}} + \omega_B' + (11 + 4c_{3/2^+})/8\mathcal{L},\)
(6.15)

and the latter yields

\(m_{(1, 1/2^+)} = M_{\text{sol}} + \omega_B' + (11 - 20c_{3/2^+})/8\mathcal{L},\)
\(m_{(1, 3/2^+)} = M_{\text{sol}} + \omega_B' + (11 - 8c_{5/2^+})/8\mathcal{L}.\)
(6.16)

Here, \(c_{1/2^+}\) and \(c_{3/2^+}\) are the \(c\)-values associated with the \(|k = \frac{1}{2}\rangle\) and \(|k = \frac{3}{2}\rangle\) states. These \(c\)-values are obtained as

\[c_{1/2^+} = \frac{4}{3}c_1 \quad \text{and} \quad c_{3/2^+} = \frac{2}{3}c_1.\]
(6.17)

Note that the \(P\)-baryon masses given by Eqs.(6.17) are quantitatively different from those of Table IV and furthermore that they violate the heavy quark symmetry; that is, there is no signal for any degenerate pairs of \((1, 1/2^+)\) and \((1, 3/2^+)\).

Then, what goes wrong in this straightforward extension? In the quantization procedure for obtaining Eq.(6.12), only a single bound state is involved. Thus, to obtain the hyperfine energy of order \(1/N_c\), it is enough to take the expectation value of \(\hat{H}_{\text{rot}}\) with respect to the unperturbed soliton-heavy meson bound state of order \(N_c^0\). However, in the heavy meson mass limit, all the heavy meson bound states come in degenerate doublets with grand spin \(k = k_\ell \pm 1/2\) (unless \(k_\ell = 0\)) and consequently we have degenerate states up to order \(m_{00}^0N_c^0\); for example, \(|i = 1, j = \frac{1}{2}\rangle\) coming from \(k = \frac{1}{2}\) bound state and \(|i = 1, j = \frac{3}{2}\rangle\) from \(k = \frac{3}{2}\) bound state. In evaluating the hyperfine energy of next order due to collective rotation, as is well-known in standard quantum mechanics, we have to diagonalize the energy matrices for the degenerate basis. As for our sample case, we obtain the energy matrices as

\[\mathcal{E}_{(1, 1/2^+)} = M_{\text{sol}} + \omega_B' + \frac{11}{8\mathcal{L}} + \frac{c_1}{6\mathcal{L}} \begin{pmatrix} -8 & 2\sqrt{2} \\ 2\sqrt{2} & -10 \end{pmatrix},\]
\[\mathcal{E}_{(1, 3/2^+)} = M_{\text{sol}} + \omega_B' + \frac{11}{8\mathcal{L}} + \frac{c_1}{6\mathcal{L}} \begin{pmatrix} 4 & -2\sqrt{5} \\ -2\sqrt{5} & -4 \end{pmatrix},\]
(6.18)

which yield the \(P\)-baryon masses consistent with Eq.(6.14) as
$$m^{(1,\frac{1}{2}^+)} = M_{\text{sol}} + \omega_B + (11 - 16c_1)/8I,$$
$$m^{(1,\frac{1}{2}^+)} = M_{\text{sol}} + \omega_B' + (11 - 8c_1)/8I,$$
$$m^{(1,\frac{3}{2}^+)} = M_{\text{sol}} + \omega_B + (11 - 8c_1)/8I,$$
$$m^{(1,\frac{3}{2}^+)} = M_{\text{sol}} + \omega_B' + (11 + 8c_1)/8I.$$  \hfill (6.19)

It shows that the inclusion of the degenerate states into the quantization procedure is, thus, essential in restoring the heavy quark symmetry in the bound state approach. If we work with finite but sufficiently heavy meson masses, we have only approximate degenerate states of the grand spin \( k = k_t \pm \frac{1}{2} \). Although the mass corrections due to the nondegenerate states are at most of order \( 1/\sqrt{N_c} \), because of the small energy discrepancy of order \( 1/m_Q \) in the denominator, their couplings can compete with diagonal terms of order \( m_Q^0 N^{-1} \) and should be taken into account properly.

\section*{VII. SUMMARY AND CONCLUSION}

In this paper, we have discussed the heavy quark symmetry in describing heavy baryons containing a single heavy quark or antiquark as bound states of the \( SU(2) \) soliton and heavy mesons. We have developed a consistent bound state approach so that the heavy quark symmetry is realized explicitly in the heavy baryon spectrum in the infinitely heavy mass limit. The resulting mass formula reads

$$m^{(0,j^\pi)} = M_{\text{sol}} + \omega_B + \frac{3}{8I}, \quad (j^\pi = \frac{1}{2}, \frac{3}{2}^-)$$
$$m^{(1,j^\pi)} = M_{\text{sol}} + \omega_B + \frac{11}{8I}, \quad (j^\pi = \frac{1}{2}, \frac{3}{2}^\pm)$$

for the heavy baryons and

$$m^{(0,j^\pi)} = M_{\text{sol}} + \omega_B' + \frac{3}{8I}, \quad (j^\pi = \frac{1}{2}, \frac{3}{2}^\pm)$$
$$m^{(1,j^\pi)} = M_{\text{sol}} + \omega_B' + \frac{7}{8I}, \quad (j^\pi = \frac{1}{2}, \frac{3}{2}^\pm)$$

for the \( P \)-baryons. All the masses are consistent with the heavy quark symmetry and the genuine hyperfine splittings vanish. As for the heavy baryons, one can still apply the mass formula (1.2) to obtain their masses and the \( c \)-factor defined by Eq.(1.3) plays the role of the hyperfine constant. For the \( P \)-baryons, the degenerate states up to order \( m_Q^0 N^{-1} \) requires diagonalizing the Hamiltonian with respect to the degenerate basis to obtain the hyperfine energy of next order in \( 1/N_c \). In this case, the nonvanishing \( c \)-factors defined by Eq.(1.3) should not be treated as the hyperfine constant. In restoring the heavy quark symmetry in the \( P \)-baryon spectrum, the couplings of the states made of the degenerate heavy meson bound states with the grand spins \( k = k_t \pm 1/2 \) are shown to play an important role.

Except the ground state with \( i = 0 \) and \( j^\pi = \frac{1}{2}^+ \), the occurrence of the parity doublets in the spectrum (i.e., two states with the same angular momentum but opposite parity occurring
at the same mass) is interesting. However, we are not in the position to conclude whether such a parity doubling has any physical importance as discussed in the context of chiral symmetry \[42\] and Regge-pole theory \[43\] or just an artifact from our approximation on the radial functions. To extract more decisive conclusion, we should work with finite heavy mesons incorporating the kinetic terms \[44\] and see whether the parity doubling occurs in the heavy quark limit. Iachello \[45\] has reported that a similar parity doubling in the excited baryon spectra is occurred in the baglike models and stringlike models and it was analyzed as a consequence of the geometric structure of baryons.

We have corrected a mistake committed in applying the traditional bound state approach of evaluating the expectation value of the operator \(\vec{\Theta}^2\) to the heavy baryons. It has been approximated simply by the square of the expectation value of \(\vec{\Theta}\) as

\[
\langle \langle i, j\ell | \vec{\Theta}^2 | i, j\ell \rangle \rangle \approx c^2 k\ell (k\ell + 1). \tag{7.3}
\]

In our approach, it can be exactly obtained as

\[
\langle \langle i, j\ell | \vec{\Theta}^2 | i, j\ell \rangle \rangle = \frac{3}{4}. \tag{7.4}
\]

In case of heavy baryons with \(c = 0\), the correction is an overall shift of the heavy baryon masses by an amount \(3/8\mathcal{I} \sim 60\) MeV.

Furthermore, the simple structure of the model enables us to illustrate explicitly how the spin-grandspin transmutation occurs in the bound state approach. In the isospin co-moving frame, the total spin of the soliton-heavy meson bound system can be obtained as

\[
\vec{J} = \vec{R} + \vec{K}_{bf}, \tag{7.5}
\]

with the rotor spin \(\vec{R}\) and the grand spin \(\vec{K}_{bf}\) of the heavy meson fields in the isospin co-moving frame. The latter plays the role of the heavy-meson spin; that is, the isospin of the heavy mesons is transmuted to the part of their spin.

We have worked with infinitely heavy mesons (and soliton). It provides a useful instruction to the bound state approaches with finite-mass heavy mesons so that it has a correct heavy quark limit; that is, one has to include the nearly doubly degenerate heavy meson bound states of grand spin \(k = k\ell \pm 1/2\) into the quantization procedure.

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APPENDIX A: SPIN AND ISO(ISPIN OPERATORS AND THEIR Eigenstates

The invariance of the Lagrangian density \[2.2\] under the infinitesimal Lorentz transformation
\[ x^\mu \to x'^\mu = x^\mu + \epsilon^\mu_{\nu} x^\nu, \text{ with } \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \]

\[ \Phi \to \Phi'(x') = \Phi(x), \]

\[ \Phi^*_{\alpha}(x) \to \Phi^*_{\alpha'}(x') = \frac{1}{2} \epsilon^{\mu\nu} (S_{\mu\nu})_{\alpha\beta} \Phi^*_\beta(x), \text{ with } (S^{\mu\nu})_{\alpha\beta} = g^{\mu}_{\alpha} g^{\nu}_{\beta} - g^{\mu}_{\beta} g^{\nu}_{\alpha} \]

defines conserved angular momentum operators

\[ J^i = \frac{1}{2} \epsilon^{ijk} \int d^3r M_{0jk}, \]

\[ M^{0jk} = \left( x^j P^{0k} - x^k P^{0j} \right) + \left( \Pi^{*m} (S^{kj})_{mn} \Phi^* n + \Phi^* m (S^{kj})_{mn} \Pi^* n \right) \]

where \( P^{\mu\nu} \) is the canonical energy-momentum tensor and \( \Pi^{*m} (\equiv \partial L_{\text{free}} / \partial \dot{\Phi}^* n) \) is the momentum conjugate to the field \( \Phi^* n \). Here, the indices run from 1 to 3. The first part corresponds to the orbital angular momentum, \( \vec{L} \), and the second part correspond to the spin angular momentum, \( \vec{S} \). In the heavy meson mass limit \( (m_{\Phi, \Phi^*} \to \infty) \), substitution of Eq.(2.3a) with \( v = (1, \vec{0}) \) leads to the spin operator of the heavy meson fields as

\[ \vec{S} = i \int d^3r \bar{\Phi}^* v \times \Phi^* v \]

as the leading order term in the meson masses. From now on, we will work in the rest frame of the heavy mesons. In terms of the \( 4 \times 4 \) matrix field \( H(x) \), it can be simply rewritten as

\[ \vec{S} = -\int d^3r \text{Tr}([\frac{1}{2} \bar{\sigma}, H] \bar{H}), \]

which implies that the corresponding quantum mechanical spin operators in the \( 4 \times 4 \) matrix representation is the Dirac spin matrices acting on the wavefunction \( H_n(x) \) as

\[ \vec{S} \{ H_n \} \equiv [\frac{1}{2} \bar{\sigma}, H_n]. \]

The minus sign in Eq.(A4) is due to the normalization convention of the field \( H(x) \) and \( \bar{H}(x) \). Note that the meson number operator is given by

\[ N = -\int d^3r \text{Tr}(H \bar{H}). \]

One can easily find the eigenstates of the spin operators as

\[ (s=0; s_3=0) = [\frac{1}{\sqrt{2}} (1 + \gamma^0)] \gamma_5, \]

\[ (s=1; s_3=0) = [\frac{1}{\sqrt{2}} (1 + \gamma^0)] \gamma^3, \]

\[ (s=1; s_3=\pm1) = [\frac{1}{\sqrt{2}} (1 + \gamma^0)][\frac{\mp 1}{\sqrt{2}} (\gamma^1 \pm i \gamma^2)], \]

which are normalized as

\[ \text{Tr}\{(s; s_3) (s'; s'_3)\} = -\delta_{s's} \delta_{s_3 s'_3}. \]

Furthermore, the invariance of the Lagrangian (2.4) under the heavy-quark spin rotation (2.3b) leads to conserved heavy-quark spin operators \( \vec{S}_Q \) as
\[ \vec{S}_Q = - \int d^3 r \ Tr(\frac{1}{2} \sigma H \bar{H}), \]
that is, the action of the heavy-quark spin operators on the wavefunction is the multiplication of the Dirac spin matrices to its left-hand-side:
\[ \vec{S}_Q \{ H_n \} = \frac{1}{2} \sigma H_n. \]

Since the heavy-quark spin decouples in the heavy meson mass limit, it is convenient to introduce the “light-quark” spin operators for the light degrees of freedom in the heavy mesons:
\[ \vec{S}_\ell \equiv \vec{S} - \vec{S}_Q \]
\[ \vec{S}_\ell \{ H_n \} = H_n \left( -\frac{1}{2} \vec{\sigma} \right). \]

The eigenstates of \( \vec{S}_Q \) and \( \vec{S}_\ell \) can be explicitly obtained as
\[ \ell(\pm \frac{1}{2}|| \pm \frac{1}{2})_Q = (s=1; s_3 = \pm 1), \]
\[ \ell(\pm 1|| 0) = \sqrt{2} \{ (s=1; s_3 = 0) + (s=0; s_3 = 0) \}, \]
\[ \ell(-\frac{1}{2}|| -\frac{1}{2})_Q = s=1; s_3 = 1 \].

On the other hand, the isospin operators associated with the invariance of the Lagrangian (2.13) under the isovector transformations (2.8) and (2.12) (with \( L = R = U \)) are
\[ \vec{I} = \vec{I}_M + \vec{I}_h, \]
with \( \vec{I}_M \) the isospin operator of the Goldstone boson fields
\[ \vec{I}_M = i \int d^3 r \frac{f^2}{2} Tr \{ \frac{1}{2} \vec{\tau} (\Sigma^\dagger \partial_0 \Sigma + \Sigma \partial_0 \Sigma^\dagger) \} + \cdots, \]
and \( \vec{I}_h \) those of the heavy meson fields interacting with Goldstone pions
\[ \vec{I}_h = \frac{1}{\sqrt{2}} \int d^3 r \ Tr \{ H \left( -\frac{1}{2} \vec{\tau} \right) \bar{H} \}. \]

Especially, the isospin operator of the free heavy meson fields is
\[ \vec{I}_h = - \int d^3 r \ Tr \{ H \left( -\frac{1}{2} \vec{\tau} \right) \bar{H} \}. \]

Thus, the quantum mechanical isospin operator for the wavefunction is the \( 2 \times 2 \) Pauli matrices \( (-\frac{1}{2} \vec{\tau}) \) acting on the right-hand-side of the anti-isodoublet \( H_n(x) \):
\[ \vec{I}_h \{ H_n(x) \} = H_n(x) \left( -\frac{1}{2} \vec{\tau} \right). \]

Explicitly, their eigenstates can be written as
\[ \tilde{\phi}_{+\frac{1}{2}} = (0, -1), \quad \text{and} \quad \tilde{\phi}_{-\frac{1}{2}} = (1, 0). \]
APPENDIX B: $K_\ell$-BASIS

Here, we present the explicit expressions of the angular part of the wavefunctions, which are the eigenstates of $K_\ell^2$, $K_\ell z$ and $S_Q^2$, $S_{Q,z}$. They can be obtained by linear combinations of direct products of four angular momentum bases; $Y_{\ell m}(\hat{r})$ (the eigenstate of $L^2$, $L_z$), $\phi_{\pm 1/2}$ (the eigenstate of $I^2_h$, $I_h z$, $\ell(\pm 1/2)$ (the eigenstate of $S^2_\ell$, $S_{\ell z}$), and $|\pm 1/2\rangle_Q$ (the eigenstate of $S_Q^2$, $S_{Q,z}$). For a given quantum number, $k_\ell, k_3$, we have four states $K_{k_\ell k_3 s_Q}^{(i)}$, $i=1,2,3,4$ according to the numbers of the different combination of $I_h$ and $S_\ell$.

We first combine the spherical harmonics $Y_{\ell m}(\hat{r})$, and the isospin basis $\phi_{\pm 1/2}$ to obtain $Y_{\lambda \lambda_3}^{(\pm)}(\hat{r})$, the eigenstates of $\Lambda^2$ and $\lambda_3$ ($\Lambda \equiv \tilde{L} + \tilde{I}_h$):

\[ Y_{\lambda \lambda_3}^{(\pm)}(\hat{r}) = \frac{\lambda + \lambda_3}{2\lambda} Y_{\ell \lambda_3 + \frac{1}{2}}(\hat{r}) \phi_+ + \frac{\lambda - \lambda_3}{2\lambda} Y_{\ell \lambda_3 - \frac{1}{2}}(\hat{r}) \phi_-, \quad \lambda = \ell + \frac{1}{2} \]

\[ Y_{\lambda \lambda_3}^{(-)} = -\sqrt{\frac{\lambda - \lambda_3 + 1}{2(\lambda + 1)}} Y_{\ell \lambda_3 - \frac{1}{2}}(\hat{r}) \phi_+ + \sqrt{\frac{\lambda + \lambda_3 + 1}{2(\lambda + 1)}} Y_{\ell \lambda_3 + \frac{1}{2}}(\hat{r}) \phi_-, \quad \lambda = \ell - \frac{1}{2} \]  

(B1)

It provides a convenient basis in evaluating the expectation values of the operator including $(\vec{\tau} \cdot \hat{r})$, since $Y^{(\pm)}$ satisfy a useful identity

\[ Y_{\lambda \lambda_3}^{(\pm)}((\vec{\tau} \cdot \hat{r})) = Y_{\lambda \lambda_3}^{(\mp)} \]  

(B2)

Next, the eigenstates $K_{k_\ell k_3 s_Q}^{(i)}$ is obtained by combining $Y^{(\pm)}$ and $\ell(\pm 1/2 || \pm 1/2)_Q$:

(i) $i=1$; $\lambda = \ell + \frac{1}{2}$, $k_\ell = \lambda - \frac{1}{2} = \ell$

\[ K_{k_\ell k_3 s_Q}^{(1)} = -\sqrt{\frac{k_\ell - k_3 + 1}{2(k_\ell + 1)}} Y_{\lambda k_3 - \frac{1}{2}}(\hat{r}) \phi_+ + \sqrt{\frac{k_\ell + k_3 + 1}{2(k_\ell + 1)}} Y_{\lambda k_3 + \frac{1}{2}}(\hat{r}) \phi_-, \]  

(ii) $i=2$; $\lambda = \ell - \frac{1}{2}$, $k_\ell = \lambda + \frac{1}{2} = \ell$

\[ K_{k_\ell k_3 s_Q}^{(2)} = +\sqrt{\frac{k_\ell + k_3 - 1}{2k_\ell}} Y_{\lambda k_3 + \frac{1}{2}}(\hat{r}) \phi_+ + \sqrt{\frac{k_\ell - k_3}{2k_\ell}} Y_{\lambda k_3 - \frac{1}{2}}(\hat{r}) \phi_-, \]  

(iii) $i=3$; $\lambda = \ell - \frac{1}{2}$, $k_\ell = \lambda - \frac{1}{2} = \ell$

\[ K_{k_\ell k_3 s_Q}^{(3)} = -\sqrt{\frac{k_\ell - k_3 + 1}{2(k_\ell + 1)}} Y_{\lambda k_3 + \frac{1}{2}}(\hat{r}) \phi_+ + \sqrt{\frac{k_\ell + k_3 + 1}{2(k_\ell + 1)}} Y_{\lambda k_3 - \frac{1}{2}}(\hat{r}) \phi_-, \]  

(iv) $i=4$; $\lambda = \ell + \frac{1}{2}$, $k_\ell = \lambda + \frac{1}{2} = \ell + 1$

\[ K_{k_\ell k_3 s_Q}^{(4)} = +\sqrt{\frac{k_\ell + k_3 - 1}{2k_\ell}} Y_{\lambda k_3 + \frac{1}{2}}(\hat{r}) \phi_+ + \sqrt{\frac{k_\ell - k_3}{2k_\ell}} Y_{\lambda k_3 - \frac{1}{2}}(\hat{r}) \phi_-, \]  

(B3)

They are normalized as

\[ \int d\Omega \text{Tr}(K_{k_\ell k_3 s_Q}^{(i)} K_{k_\ell' k_3' s_Q'}^{(j)}) = -\delta_{ii'}\delta_{k_\ell k_\ell'}\delta_{k_3 k_3'}\delta_{s_Q s_Q'}. \]  

(B4)

One can easily check that $K_{k_\ell k_3 s_Q}^{(1)}$ and $K_{k_\ell k_3 s_Q}^{(2)}$ have the parity $\pi = -(-1)^{k_\ell}$ and the other two have the parity $\pi = (-1)^{k_\ell}$ and they are related to each other as
\[ K^{(i)}_{k_\ell k_3 s_Q}(\vec{q}, \vec{r}) = K^{(i+2)}_{k_\ell k_3 s_Q}. \quad (i=1,2) \] (B5)

These \( K_\ell \)-bases evaluate the matrix elements \( \mathcal{M}_{ij} \) defined by Eq.(3.10) as

\[
\mathcal{M} = -\frac{1}{2} g F'(0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(for } i,j=1,2) \label{B6}
\]
\[
\mathcal{M} = \frac{1}{2} g F'(0) \begin{pmatrix} 2k_\ell + 3 & -4 \sqrt{k_\ell(k_\ell + 1)} \\ -4\sqrt{k_\ell(k_\ell + 1)} & 2k_\ell - 1 \end{pmatrix} \quad \text{(for } i,j=3,4),
\]

independent of \( k_3 \) and the heavy quark spin \( s_Q \).

In Sec. V, we work with the expectation values of the operator \( \vec{\Theta}(\infty) \) with respect to the single-particle Fock states \( |n\rangle \) for which we need to evaluate the expectation values of \( \vec{\Theta} \) with respect to the \( K_\ell \)-basis. Wigner-Eckart theorem enables us to express them as

\[ j\langle k'_3 k'_3 | \vec{\Theta}(\infty) | k_\ell k_3 \rangle_i \equiv -\int d\Omega \ Tr\{K^{(i)}_{k_\ell k_3 s_Q} \vec{r} \cdot \hat{r} \left( \frac{1}{2} \hat{r} \right) \vec{q} \cdot \hat{q} \vec{K}^{(j)}_{k'_3 k'_3 s_Q} \} = \frac{(k'_3 k_3 1 q | k'_3 k'_3)}{(2k'_3 + 1)} j(k'_3 || \vec{\Theta} || k_\ell)_i, \] (B7)

with the “reduced matrix elements”:

\[
\begin{align*}
1(k_\ell||\vec{\Theta}||k_\ell)_1 &= \frac{1}{2} \frac{\sqrt{k_\ell(2k_\ell + 1)}}{k_\ell + 1}, \\
2(k_\ell||\vec{\Theta}||k_\ell)_2 &= -\frac{1}{2} \frac{(k_\ell + 1)(2k_\ell + 1)}{k_\ell}, \\
3(k_\ell||\vec{\Theta}||k_\ell)_3 &= -\frac{1}{2} \frac{(2k_\ell + 3)\sqrt{k_\ell}}{k_\ell + 1}, \\
3(k_\ell||\vec{\Theta}||k_\ell)_4 &= -\frac{\sqrt{2k_\ell + 1}}{2}, \label{B8}
\end{align*}
\]
\[
\begin{align*}
4(k_\ell||\vec{\Theta}||k_\ell)_4 &= \frac{1}{2} \frac{(2k_\ell - 1)\sqrt{k_\ell + 1}}{k_\ell(2k_\ell + 1)}, \\
1(k_\ell - 1||\vec{\Theta}||k_\ell)_3 &= -\frac{(k_\ell + 1)(2k_\ell - 1)}{2k_\ell + 1}, \\
1(k_\ell - 1||\vec{\Theta}||k_\ell)_4 &= -\frac{1}{2} \frac{(2k_\ell - 1)}{k_\ell(2k_\ell + 1)}, \\
2(k_\ell + 1||\vec{\Theta}||k_\ell)_3 &= \frac{1}{2} \frac{(2k_\ell + 3)}{(k_\ell + 1)(2k_\ell + 1)}, \\
2(k_\ell + 1||\vec{\Theta}||k_\ell)_4 &= -\frac{\sqrt{k_\ell(2k_\ell + 3)}}{2k_\ell + 1},
\end{align*}
\]

and others are zero. As far as the single-particle Fock states of the same \( k_\ell \) are concerned, we can rewrite Eq.(B7) in a more convenient form as
\[ a \langle k_\ell, m' | \tilde{\Theta}(\infty) | k_\ell, m \rangle_b = -c_{ab} \langle k_\ell, m' | \tilde{K}_\ell | k_\ell, m \rangle, \]  

(B9)

where \((k_\ell, m' | \tilde{K}_\ell | k_\ell, m)\) denotes the expectation value of the operator \(\tilde{K}_\ell\) with respect to its eigenstates. The multiplication coefficients analogous to the Lande’s \(g\)-factor can be read off from Eq. (B7) as

\[ \begin{align*}
  c_{33} &= + \frac{2k_\ell + 3}{2(k_\ell + 1)(2k_\ell + 1)}, \\
  c_{44} &= - \frac{2k_\ell - 1}{2k_\ell(2k_\ell + 1)}, \\
  c_{34} &= + \frac{1}{\sqrt{k_\ell(k_\ell + 1)(2k_\ell + 1)}}, \\
  c_{11} &= - \frac{1}{2(k_\ell + 1)}, \\
  c_{22} &= + \frac{1}{2k_\ell},
\end{align*} \]  

(B9a)

and others zero. With respect to the states \(|k_\ell\rangle_{\pm}\), we have

\[ \begin{align*}
  c_{--} &= 0, \\
  c_{++} &= + \frac{1}{2k_\ell(k_\ell + 1)}, \\
  c_{-+} &= + \frac{1}{2\sqrt{k_\ell(k_\ell + 1)}}.
\end{align*} \]  

(B9b)
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FIGURES

FIG. 1. Bound states of (a) heavy mesons and (b) antiflavored heavy mesons.

FIG. 2. Heavy baryon spectrum.

FIG. 3. P-baryon spectrum.
TABLES

TABLE I. Heavy-meson eigenstates with $\ell = 0$ and 1.

| $\{k, k_3, \pi, s_Q\}$ | $\varepsilon$ | eigenfunct. | $\ell_{\text{eff}}$ | $k$ | $|n\rangle$ |
|------------------------|--------------|-------------|-----------------|-----|---------|
| $\{0, 0, -, s_Q\}$    | $+\frac{1}{2}gF'(0)$ | $f(r)\mathcal{K}_{00s_Q}^{(1)}$ | 1   | $\frac{1}{2}$ | $|0\rangle$ |
| $\{1, m, -, s_Q\}$    | $-\frac{3}{2}gF'(0)$ | $f(r)\mathcal{K}_{1ms_Q}^{(-)}$ | 1   | $\frac{1}{2}, \frac{3}{2}$ | $|1\rangle_-$ |
| $\{1, m, -, s_Q\}$    | $+\frac{1}{2}gF'(0)$ | $f(r)\mathcal{K}_{1ms_Q}^{(+)}$ | 1   | $\frac{1}{2}, \frac{3}{2}$ | $|1\rangle_+$ |
| $\{0, 0, +, s_Q\}$    | $-\frac{3}{2}gF'(0)$ | $f(r)\mathcal{K}_{00s_Q}^{(3)}$ | 0   | $\frac{1}{2}$ | $|0\rangle_3$ |
| $\{1, m, +, s_Q\}$    | $+\frac{1}{2}gF'(0)$ | $f(r)\mathcal{K}_{1ms_Q}^{(1)}$ | 2   | $\frac{1}{2}, \frac{3}{2}$ | $|1\rangle_1$ |
| $\{1, m, +, s_Q\}$    | $+\frac{1}{2}gF'(0)$ | $f(r)\mathcal{K}_{1ms_Q}^{(-)}$ | 0   | $\frac{1}{2}, \frac{3}{2}$ | $|1\rangle_2$ |
| $\{2, m, +, s_Q\}$    | $-\frac{3}{2}gF'(0)$ | $f(r)\mathcal{K}_{2ms_Q}^{(-)}$ | 2   | $\frac{3}{2}, \frac{5}{2}$ | $|2\rangle_-$ |
| $\{2, m, +, s_Q\}$    | $+\frac{1}{2}gF'(0)$ | $f(r)\mathcal{K}_{2ms_Q}^{(+)}$ | 2   | $\frac{3}{2}, \frac{5}{2}$ | $|2\rangle_+$ |

TABLE II. $|i, j^\ell_i\rangle$ states for heavy baryons.

| $i$ | $j^\ell_i$ | $|n\rangle$ | $|i, j^\ell_i\rangle_{a}$ | $\varepsilon$ | $j$ |
|-----|------------|-------------|----------------|--------------|-----|
| 0   | 0$^+$      | $|0\rangle$ | $|0, 0^+\rangle$ | $-\frac{3}{2}gF'(0)$ | $\frac{1}{2}$ | $\Lambda_Q$ |
| 1   | 1$^+$      | $|0\rangle_3$ | $|1, 1^+\rangle_1$ | $-\frac{3}{2}gF'(0)$ | $\frac{1}{2}, \frac{3}{2}$ | $\Sigma_Q, \Sigma_Q^*$ |
|     | 1$^+$      | $|2\rangle_-$ | $|1, 1^+\rangle_2$ | $-\frac{3}{2}gF'(0)$ | $\frac{1}{2}, \frac{3}{2}$ | $\Sigma_Q, \Sigma_Q^*$ |
| 0   | 1$^-$      | $|1\rangle_-$ | $|0, 1^-\rangle$ | $-\frac{3}{2}gF'(0)$ | $\frac{1}{2}, \frac{3}{2}$ | $\Sigma_Q, \Sigma_Q^*$ |
| 1   | 0$^-$      | $|1\rangle_-$ | $|1, 0^-\rangle$ | $-\frac{3}{2}gF'(0)$ | $\frac{1}{2}$ | $\Sigma_Q, \Sigma_Q^*$ |
| 1   | 1$^-$      | $|1\rangle_-$ | $|1, 1^-\rangle$ | $-\frac{3}{2}gF'(0)$ | $\frac{1}{2}, \frac{3}{2}$ | $\Sigma_Q, \Sigma_Q^*$ |
TABLE III. $|i, j_{\ell}^{\pi})\rangle$ states for the $P$-baryons.

| $i$ | $j_{\ell}^{\pi}$ | $|n\rangle$ | $|i, j_{\ell}^{\pi})\rangle_i$ | $\varepsilon$ | $j$ |
|-----|-----------------|-------------|----------------------------|-------------|-----|
| 0   | 0$^-$           | $|0\rangle_1$ | $|0, 0^-(0)\rangle$ | $-\frac{1}{2}gF'(0)$ | $\frac{1}{2}$ |
| 0   | 1$^-$           | $|1\rangle_+$ | $|0, 1^-(0)\rangle$ | $-\frac{1}{2}gF'(0)$ | $\frac{1}{2}, \frac{3}{2}$ |
| 1   | 0$^-$           | $|1\rangle_+$ | $|0, 1^-(0)\rangle$ | $-\frac{1}{2}gF'(0)$ | $\frac{1}{2}$ |
| 1   | 1$^-$           | $|0\rangle_1$ | $|1, 1^-(0)\rangle_1$ |             |     |
|     | $|1\rangle_2$ | $|1, 1^-(0)\rangle_2$ |             | $-\frac{1}{2}gF'(0)$ | $\frac{1}{2}, \frac{3}{2}$ |
| 0   | 1$^+$           | $|1\rangle_1$ | $|0, 1^+(0)\rangle_1$ |             |     |
|     | $|1\rangle_2$ | $|0, 1^+(0)\rangle_2$ |             |             |     |
| 1   | 0$^+$           | $|1\rangle_1$ | $|1, 0^+(0)\rangle_1$ |             |     |
|     | $|1\rangle_2$ | $|1, 0^+(0)\rangle_2$ |             |             |     |
| 1   | 1$^+$           | $|1\rangle_1$ | $|1, 1^+(0)\rangle_1$ |             |     |
|     | $|1\rangle_2$ | $|1, 1^+(0)\rangle_2$ |             |             |     |

TABLE IV. Positive and Negative Parity $P$-baryon ($P$) masses (in MeV).

| $i$ | $j_{\ell}^{\pi}$ | $j^{\pi}$ | Mass Formula | $m_{P_c}$ | $m_{P_b}$ | b.e.$^*$ |
|-----|-----------------|-----------|--------------|-----------|-----------|--------|
| 0   | 0$^-$           | $\frac{1}{2}^-$ | $M_{solt} + \omega'_B + 3/8I$ | 2704 | 6042 | 210 |
| 0   | 1$^-$           | $\frac{1}{2}^-, \frac{3}{2}^-$ | $M_{solt} + \omega'_B + 3/8I$ | 2704 | 6042 | 210 |
| 1   | 1$^-$           | $\frac{1}{2}^-, \frac{3}{2}^-$ | $M_{solt} + \omega'_B + 7/8I$ | 2802 | 6140 | 112 |
| 0   | 1$^+$           | $\frac{1}{2}^+, \frac{3}{2}^+$ | $M_{solt} + \omega'_B + 3/8I$ | 2704 | 6042 | 210 |
| 1   | 0$^+$           | $\frac{1}{2}^+$ | $M_{solt} + \omega'_B + 7/8I$ | 2802 | 6140 | 112 |
| 1   | 1$^+$           | $\frac{1}{2}^+, \frac{3}{2}^+$ | $M_{solt} + \omega'_B + 7/8I$ | 2802 | 6140 | 112 |

* Binding energy below $M_{th}$. 