LOCAL GEOMETRIC INVARIANTS OF INTEGRABLE EVOLUTION EQUATIONS

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ABSTRACT. The integrable hierarchy of commuting vector fields for the localized induction equation of 3D hydrodynamics, and its associated recursion operator, are used to generate families of integrable evolution equations which preserve local geometric invariants of the evolving curve or swept-out surface.

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Introduction. The concept of the soliton and the inverse scattering transform approach to solving certain non-linear equations were introduced twenty-five years ago in two landmark papers by Zabusky-Kruskal and Gardner-Greene-Kruskal-Miura [1,2]. Since then, soliton theory, or the theory of integrable systems, has had enormous impact on applied mathematics and mathematical physics. Water wave theory [3], nonlinear optics [4], field theory [5], and relativity [6] are but a few of the areas which have been influenced by these ideas. An important aspect of the study of soliton equations is that a single example can be of interest in a wide variety of contexts; this universality may well be related to the fact that such equations frequently have an underlying geometric meaning. For instance, the sine-Gordon equation, which first appeared in differential geometry [7], is also used as a model for dislocation of crystals [8], for field theory [9], and nonlinear optics (self-induced transparency) [10]. In the same vein, several authors [11], [12], [13],[14],[15], [16], have developed the connection between the Serret-Frenet equations and other elements of classical differential geometry and various well-known integrable models, including the non-linear Schrödinger equation, and the modified Korteweg-de Vries equation.

Continuing this geometric theme, we describe in this paper certain families of integrable equations which govern the motion of curves in the plane, on the sphere, and in three-dimensional space. We focus our attention on evolution equations which preserve distinguished local invariants of the evolving space curve or associated swept-out surface.
Our point of departure is the localized induction equation (LIE): $\gamma_t = \gamma_s \times \gamma_{ss} = \kappa B$. LIE is an idealized local model of the evolution of the centerline of a thin vortex tube in a three dimensional inviscid incompressible fluid. Here, the subscripts $t$ and $s$ denote partial differentiation with respect to time $t$ and arclength $s$ for the evolving curve $\gamma(s, t)$ in $R^3$, which has curvature $\kappa$, torsion $\tau$, and Frenet frame $\{T, N, B\}$; the multiplication sign denotes cross product. For derivation and history, we refer to [17],[18].

More accurate, non-local models have been considered; we refer the reader to the work of Moore-Saffman [19], Klein-Majda [20] and the references mentioned therein. LIE has a second interpretation as a “potential” form for the classical one-dimensional continuous Heisenberg ferromagnet [21].

Hasimoto [22] uncovered the relation of LIE to soliton theory by showing that it induces an evolution on the complex curvature $\psi = \kappa e^{i \int^s \tau du}$ governed by the nonlinear Schrödinger equation (NLS): $\psi_t = i(\psi_{ss} + \frac{1}{2} |\psi|^2 \psi)$. NLS is a well known example of an equation with soliton solutions. Since Hasimoto’s discovery, the structure of LIE has been more fully spelled out in the context of infinite dimensional Hamiltonian systems. In fact the Hamiltonian for LIE is just the arclength functional on curves (as shown by Marsden-Weinstein [23], who also described the relevant Poisson structure). Further, associated to LIE is an infinite sequence of commuting Hamiltonian vector fields, with Hamiltonians which can be expressed as global geometric invariants of the curves. All of these equations are of the form $\gamma_t = W = aT + bN + cB$, where $W$ is a geometric vector field, which means the coefficients $a, b, c$ are functions of $\kappa, \tau, \kappa' = \kappa_s, \tau' = \tau_s$, and
higher derivatives with respect to $s$. We list the first few terms of the \textit{localized induction hierarchy} (LIH) \cite{24} of commuting vector fields, as well as their associated Hamiltonians (the vector field $X_0$ is exceptional):

$$
X_0 = -T,
$$

$$
X_1 = \kappa B, \quad I_1 = \int_\gamma ds,
$$

$$
X_2 = \frac{\kappa^2}{2}T + \kappa'N + \kappa\tau B, \quad I_2 = \int_\gamma -\tau \ ds,
$$

$$
X_3 = \kappa^2\tau T + (2\kappa'\tau + \kappa\tau')N + (\kappa\tau^2 - \kappa'' - \frac{1}{2}\kappa^3)B,
$$

$$
I_3 = \int_\gamma \frac{1}{2}\kappa^2 \ ds,
$$

$$
X_4 = (-\kappa\kappa'' + \frac{1}{2}(\kappa')^2 + \frac{3}{2}\kappa^2\tau^2 - \frac{3}{8}\kappa^4)T
$$

$$
+ (-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \frac{3}{2}\kappa^2\kappa')N
$$

$$
+ (\kappa\tau^3 - 3(\kappa'\tau)' - \frac{3}{2}\kappa^3\tau - \kappa\tau'')B,
$$

$$
I_4 = \int_\gamma \frac{1}{2}\kappa^2\tau \ ds,
$$

$$
X_n = \mathcal{R}(X_{n-1}), \quad n > 0.
$$

As the last line suggests, LIH is generated by successively applying a recursion operator starting with $X_0$. For background material on recursion operators in the theory of integrable systems, see \cite{25}; the recursion operator for LIH was introduced in \cite{24}, and is defined below.

The vector field $X_2$ also arises in a related fluid mechanical context, via a refined version of LIE. Fukumoto-Miyazaki \cite{26} derived a “localized induction equation” for thin vortex tubes, which allows for axial velocity; their equation be expressed in terms of the
vector fields of LIH via $\gamma_t = (cX_2 + X_1)$, $c$ some parameter. Lamb [11] had previously considered the (essentially equivalent) evolution equation $\gamma_t = X_2 - 3\tau_0 X_1 + 3\tau_0^2 X_0$, which has the following special property: starting with an initial curve $\gamma(s, 0)$ having constant torsion $\tau_0$, the evolution preserves the condition $\tau = \text{constant} = \tau_0$, while the curvature function $\kappa(s, t)$ satisfies the modified Korteweg-deVries equation $\kappa_t = \kappa_{sss} + \frac{3}{2} \kappa^2 \kappa_s$, another well known soliton equation. Lamb’s result can be interpreted as saying that there exist distinguished solutions to the Fukumoto-Miyazaki equation which can be completely described by tracking just one evolving functional parameter (curvature) rather than the usual two (curvature and torsion). The special case $\tau_0 = 0$ corresponding to an evolution of planar curves has been discussed in several recent papers [15],[16],[27]; in fact, in [28], this planar evolution equation is shown to be a ”localized induction approximation” associated with vortex patches in ideal two-dimensional fluids. Part of our aim here is to demonstrate that Lamb’s example is but one of a whole family of evolution equations preserving local geometric invariants, that such examples are geometrically significant, and that they are intimately related to the recursion operator $\mathcal{R}$.

The Recursion Operator $\mathcal{R}$ and Variation formulas. Formulas for the variations of geometric invariants of a curve $\gamma(s, t)$ evolving by a vector field $\gamma_t = W$ have been derived by various authors [16],[24],[29],[30]. Much of the present paper depends on the remarkably simple expressions for these variations in terms of the recursion operator $\mathcal{R}$ in case $W$ is locally arclength preserving (LAP), i.e., in case $<W_s, T> = 0$. In addition to curvature and torsion, we consider natural curvatures $u$ and $v$ related to $\kappa$ and $\tau$ via
\[ u + iv = \psi = \kappa e^{i \int^s \tau(u) du} \]. The corresponding natural frame \( \{ T, U, V \} \) is related to the standard Frenet frame by \( U + iV = (N + iB)e^{i \int^s \tau(u) du} \) and satisfies the natural Frenet equations \( T_s = uU + vV, U_s = -uT, V_s = -vT \). We begin our list of formulas with the definitions of \( \mathcal{R} \) and a parameterization operator \( \mathcal{P} \), which takes an arbitrary vector field \( X = aT + bN + cB = fT + gU + hV \) to an LAP vector field:

\[
\begin{align*}
(a) \quad \mathcal{P}(X) &= \int^s (\kappa b) \, ds \, T + bN + cB \\
&= \int^s (gu + hv) \, ds \, T + gU + hV,
\end{align*}
\]

\[
(b) \quad \mathcal{R}(W) = -\mathcal{P}(T \times W'),
\]

\[
(c) \quad W(\kappa) = < -\mathcal{R}^2(W), N >,
\]

\[
(d) \quad W(\tau) = < -\mathcal{R}^2(W), B/\kappa >',
\]

\[
(e) \quad W(u) = < -\mathcal{R}^2(W), U >,
\]

\[
W(v) = < -\mathcal{R}^2(W), V >.
\]

Note that at each time \( t \), \( u + iv \) is only determined up to multiplication by a complex unit; the formulas for \( W(u) \) and \( W(v) \) should therefore read “for some choice of natural curvatures \( u(s, t) \) and \( v(s, t) \)”. Also, the appropriate choice of antiderivative in the definition of \( \mathcal{P} \) depends upon the class of curves under consideration. In particular, in the asymptotically linear case, it is the “antisymmetrized” antiderivative operator \( \int^s f \, ds = F(s) - \frac{1}{2}(F(\infty) + F(-\infty)), F' = f \), which gives the terms of LIH listed above.
The formulas stated above were derived in [24] and [31], where similar formulas were also derived for the evolution of the frame \( \{T, U, V\} \).

**Curve invariants.** We now present the integrable hierarchies \( X_n, Y_n, Z_n, \Omega_n, \omega_n, \sigma_n, \Sigma_n \), of vector fields preserving geometric invariants of the evolving space curve \( \gamma \). As noted below, the \( X_n \) and \( Y_n \) hierarchies have been discussed previously in the literature (we include them for completeness); the other hierarchies are new.

(i) *Locally arclength preserving:* For \( n \geq 0 \), the LIE hierarchy \( X_n \) is locally arclength preserving. The \( X_n \) satisfy the recursion relation \( X_{n+1} = \mathcal{R}(X_n) \), starting with \( X_0 \). If \( W = X_n \), then the induced evolution on complex curvature \( \psi_t = W(\psi) \) is the corresponding element of the NLS hierarchy; in particular, \( X_1 = \kappa B \) induces the the NLS equation (details appear in [24]). Obviously, any linear combination of the \( X_n \) will also be locally arclength preserving. Our hierarchies will mainly be of this form.

(ii) *Planarity preserving:* For \( n \geq 0 \), the even flows in LIH \( Y_n = X_{2n} \) are planarity preserving (the evolution of a planar curve stays planar). The \( Y \)–recursion operator is just \( \mathcal{R}^2(W) = -\mathcal{P}(W_{ss}) \) restricted to planar vector fields along planar curves. The starting vector field in the sequence is \( Y_0 = X_0 \). If \( W = Y_n \) then the induced evolution on curvature \( \kappa_t = W(\kappa) \) is the corresponding element of the (mKdV) hierarchy; in particular, \( Y_1 \) induces the (mKdV) equation itself [15],[16],[27]. Observe that planarity preserving is equivalent to preserving the condition \( \tau = 0 \).

(iii) *Constant torsion preserving:* For \( n \geq 0 \), the vector fields

\[
Z_n = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-\tau_0)^k X_{2n-k}
\]
preserve the constant torsion condition $\tau = \tau_0$. Each $Z_n$ contains a “phantom” or formal vector field component $X_{-1}$; by definition, $\mathcal{R}(X_{-1}) = X_0$. $X_{-1}$ has trivial geometric action; it is introduced for algebraic purposes, to facilitate the recursion formulas for the $Z_n$. The $Z$-recursion operator is $(\mathcal{R} - \tau_0)^2$; the starting vector field in the sequence is $Z_0 = X_0 - \tau_0 X_{-1}$. We remark that along constant torsion curves, the operator $\mathcal{R} - \tau_0$, thought of as an integro-differential operator on vector fields $W = aT + bN + cB$, has coefficients independent of $\tau$. If $W = Z_n$ then the induced evolution on curvature $\kappa_t = W(\kappa)$ is the corresponding element of the (mKdV) hierarchy; in particular $Z_1$ induces the (mKdV) equation, and we recover the result of Lamb [11]. When $\tau_0 = 0$, we are in case (ii).

A closely related sequence of vector fields is given by the “planar-like” vector fields

$$\Omega_n = \sum_{k=0}^{2n-1} \binom{2n-1}{k} (-\tau_0)^k X_{2n-k},$$

so called because they have no binormal component and their coefficients are independent of $\tau$; in fact, the $\Omega_n$ look identical to the $Y_n$, the only difference being that they are defined along space curves rather than planar curves. We have $\Omega_{n+1} = \mathcal{R}^2 Z_n$. Again, the recursion operator is $(\mathcal{R} - \tau_0)^2$ and the starting vector field in the sequence is $\Omega_1 = X_2 - \tau_0 X_1 = \frac{\kappa^2}{2} T + \kappa' N$. The form of $\Omega_n$, together with formulas (c) and (d), imply that the $Z_n$ are constant torsion preserving and induce the (mKdV) hierarchy on the curvature $\kappa$.

(iv) Constant natural curvature and sphericity preserving: Note that if a natural curvature is constant ($v = v_0$ along a curve $\gamma$) then the natural Frenet equations imply
c = γ + (1/v)V is a constant vector, so γ lies on a sphere of radius 1/v centered at c. (Compare this with the standard sphericity condition $\kappa^{-2} + (\kappa')^2 \tau^{-2} \kappa^{-4} = \text{constant},$ given in most differential geometry books, e.g., [32]). Conversely, if γ lies on a sphere of radius r, then a natural frame {T, U, V} along γ is readily obtained by letting U be normal to T and tangent to the sphere (and so $V = T \times U$ is normal to the sphere); then the corresponding natural curvature $v$ is constant, $v = 1/r$, and $u$ is the geodesic curvature of γ, regarded as a spherical curve. We define a constant $v$-preserving analogue of the planar hierarchy $Y_n$: $\omega_1 = \frac{u^2}{2} T + u_s U,$ etc., are obtained by simply replacing $\kappa$ by $u$ and $N$ by $U$ in the expressions for the corresponding $Y_n$. From this description, it is clear that the $\omega_n$ define evolutions on spherical curves; further, they are generated by an intrinsic recursion operator, $S(W) = -\mathcal{P}(\nabla^2_T(W))$ – here $\nabla$ is covariant differentiation on the sphere – since $\nabla_T(T) = u U$, and $\nabla_T(U) = -u T$ look just like the planar Frenet equations. In terms of $\mathcal{R}$, we have $S(W) = (\mathcal{R}^2 + v^2)(W)$; specifically, if we consider the class of asymptotically geodesic curves – $u$ decays rapidly to zero – the antiderivative operators in both formulas for $S$ are the “antisymmetrized” ones, mentioned above.

We note that, just as Euclidean elastic curves give solitons for LIE evolving by screw motion, “spherical elastic curves” [33] simply rotate when evolving via $\gamma_t = \omega_1$. For a general spherical curve, one might expect $u$ to evolve according to mKdV, by analogy with the planar case. This is not quite true; however, the related spherical vector fields defined by $\sigma_1 = (\frac{u^2}{2} - v^2) T + u_s U$, $\sigma_{n+1} = S \sigma_n$ induce exactly the mKdV hierarchy (and
this links our work with that of Chern-Tenenblat [34] on integrable foliations of surfaces with constant Gauss curvature). The two families are related via $\omega_{n+1} = R^2(\sigma_n)$.

Yet another hierarchy of spherical evolution equations is obtained by simply restricting the even vector fields $X_{2n}$ to curves with $v = v_0$. It is worth remarking that both the $\sigma_n$ and $\omega_n$ can be expressed in terms of the restricted $X_{2n}$:

$$\omega_n = \sum_{k=0}^{n} (-1)^k v_0^{2k} \left( \frac{1}{2} - n \right)_k X_{2n-2k},$$

$$\sigma_n = \sum_{k=0}^{n} (-1)^k v_0^{2k} \left( -\frac{1}{2} - n \right)_k X_{2n-2k}$$

Here, $(a)_k = (a)(a+1)(a+2)\ldots(a+k-1)$ is the standard Pochhammer symbol.

(v) Constant curvature preserving: The interest of the spherical evolution equations discussed above is enhanced by the general connection between spherical curves and space curves, via the tangent indicatrix construction. To be brief, we mention only the following special case: a unit speed curve $\gamma$ on the unit sphere, with geodesic curvature $u$, is the derivative of a unit-speed space curve $\Gamma$, having constant curvature $\kappa = 1$ and torsion $\tau = u$. It follows that we may regard the spherical equations of (iv) as evolution equations on space curves of constant curvature $\kappa = 1$; moreover, in the case of the $\sigma_n$ hierarchy, the torsion of $\Gamma$ evolves according to mKdV. We note that it is also possible to translate the $\sigma_n$ into explicit (non-local) vector fields on the space curve level; e.g., $\sigma_1$ corresponds to $\Sigma_1 = \int^s \left( \frac{\tau^2}{2} - 1 \right) N + \tau_s B \right) ds$, etc.

Surface invariants. Given an evolving space curve, one can also consider the geometric invariants of the surface being swept out by the curve. For example, in case (iv), the curves are sweeping out a sphere, a surface of constant positive Gauss curvature.
We now discuss means of generating constant negative Gaussian curvature (CNGC) surfaces. These surfaces have long been a subject of interest in differential geometry, from the early work of Bäcklund [35], to the recent work of Sym [36] and Melko-Sterling [37], who have approached their study using some of the tools of modern soliton theory. Here, we indicate a complementary avenue of investigation: we discuss evolution equations on curves which sweep out CNGC surfaces.

We begin by observing that the variation formulas \( c, d, e \) clearly simplify in the case that \( W \) is an eigenvector for \( R \). Specifically, if \( \gamma \) is a curve with constant torsion \( \tau = \tau_0 \), one easily verifies that the vector field \( W = \cos(\theta)T - \sin(\theta)N \) satisfies \( R(W) = \tau_0 W \), where \( \theta = \int^{s} \kappa(u) du \). Note that as a consequence of our variation formulas, the evolution equation \( \gamma_t = W \) induces an evolution on curvature \( \kappa_t = -<N, R^2(W)> = -<N, \tau_0^2 W> = \tau_0^2 \sin(\theta) \), so \( \theta \) satisfies the sine-Gordon equation

\[
\theta_{st} = -G \sin(\theta), \quad \text{where } G = -\tau_0^2.
\]

This well-known soliton equation has long been associated with CNGC surfaces. Let us substantiate the claim that the swept out surface \( M \) is indeed CNGC, with curvature \( G \). Observe, via our variation formulas, that \( W(\tau) = 0 \). Thus constant torsion is preserved during the evolution. Next, the normal \( \nu \) to the surface is just the binormal \( B \) of our evolving curve; hence the second fundamental form of the surface, applied to the tangent \( T \) of our curve \( \gamma \), is just \( <-d\nu(T), T> = <-B_s, T> = <-\tau N, T> = 0 \). Thus \( T \) is an asymptotic direction of the surface \( M \). By the Beltrami-Enneper theorem [38], the Gauss
curvature of the surface $M$ along $\gamma$ is the negative of the square of the torsion of $\gamma$; this is just $G$. This dynamical prescription for generating CNGC surfaces, although closely related to standard discussions of the sine-Gordon equation [11],[16], seems to be new (it was recently and independently derived by Segur and McLachlan [39] in their study of evolution equations associated with surface motion in $R^3$).

Let’s consider this evolution equation $\gamma_t = W$ when the initial curve is a Hasimoto filament [22], a curve $\gamma$ with curvature $\kappa = 2a \text{sech}(as)$ and torsion $\tau = \tau_0$. One easily checks (with appropriate choice of antiderivative $\theta = \int \kappa ds$) that $\theta_{ss} = a^2 \sin(\theta)$. Thus $W = \cos(\theta)T - \sin(\theta)N = -a^{-2}(\frac{\kappa^2}{2} - a^2)T + \kappa'N$. Consequently, modulo scaling and slippage along the curve, $W = \Omega_1$, one of our previous planar-like vector fields.

Along $\gamma$, we have $\Omega_1 = X_2 - \tau_0 X_1$. Both $X_2$ and $X_1$ are Killing fields (= infinitesimal isometries) along $\gamma$ [29]; hence the curvature and torsion functions remain unchanged (modulo translation) as the curve evolves. The comments of the previous paragraph therefore apply for all time; and the result is that the Hasimoto filament, evolving via the equation $\gamma_t = \Omega_1$, sweeps out a CNGC surface. We contrast this result with an observation of Melko-Sterling: under the evolution $\gamma_t = X_1$, the Hasimoto filament sweeps out a surface $M$ which is a formal boundary point of the set of CNGC surfaces, although $M$ itself is not CNGC. As our simple example suggests, there is a connection between the hierarchies of vector fields described in this paper, and the geometry of CNGC surfaces – the vector fields $\Omega_n$ play a central role. Details will appear in a subsequent publication.
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