Gravitational Collapse of Self-similar Perfect fluid with Scalar function

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The self-similar spherically symmetric perfect fluid space-time with scalar function incorporating linear equation of state is studied. The investigation of gravitational collapse conditions on the space-time determines that scalar function has an imaginary form $\phi(X) = i \alpha, \varphi(X)$ being a non-zero real valued function of self-similar variable $X$. The gravitational collapse of cloud ends into the formation of black hole or naked singularity according to the choice of physically reasonable initial data relative to the linear equation of state. PACS Numbers: 04.20.Dw, 04.70.Bw

I. INTRODUCTION

The gravitational collapse of a massless scalar field is of immense interest in both cloud collapse situations as well as cosmological scenarios. In cosmology, the evolution of a scalar field has great importance because it supplements understanding for fundamental matter fields towards the transition from matter dominated regime of the Universe to dark energy domination. Linde suggested that scalar fields are important in view of the inflationary scenarios that govern the early universe dynamics because such a field can act as an ‘effective’ cosmological constant in driving the inflation.

The scalar fields are fundamental fields that satisfy the Klein-Gordon (KG) equation and therefore an important member in understanding the realms of basic particles in cosmology and processes in cloud collapse. Christodoulou established the global existence and uniqueness of the solutions of Einstein-scalar field equations. Further the author studied a sufficient condition for the formation of a trapped surface in the evolution of a physically reasonable initial data set.

Choptuik, from a numerical study of spherically symmetric collapse of massless scalar field presented black hole threshold in the space of initial data for classes of massless scalar fields. Scalar field collapse and cosmic censorship hypothesis (CCH), proposed by Penrose, is numerically studied by Goswami et. al. in FLRW model, showing equal chance existence of black hole (BH) and a locally naked singularity (NS) in the end state of collapse according to reasonable initial data. Some other examples of spherical scalar fields collapse are already appeared in literature. Goncalves and Moss studied scalar field and concluded that the formation of black hole depends on the size and mass of the initial field configuration and the mass of the scalar field.

Studies of gravitational collapse of cloud gained its popularity since its onset by Oppenheimer and Snyder in analysing the spherical collapse of a homogeneous dust cloud that led to general concept of trapped surfaces and black hole. The marginally bound self-similar Tolman-Bondi dust solutions, analysed by Eardley and Smarr describe the occurrence of strong NS at the centre of the collapsing cloud. This presents a strong counter example to CCH. In the study of self-similar collapsing Tolman-Bondi space-times, J.P.S. Lemos shown that degree of the in-homogeneity of the collapsing matter as a function of the binding energy is necessary to form a NS. From homogeneous to in-homogeneous dust, to radiation and then to perfect fluid spherical collapse studies are found in the literature.

The studies of self-similar perfect fluid collapse models are important as the pressure gradients might prevent the occurrence of a NS or could reduce the strength of the singularity. Ori and Piran investigated the collapse of an adiabatic perfect fluid numerically, and showed that a NS forms at the centre, and such appearance is a common feature when equation of state is linear. Joshi analyzed gravitational collapse of a perfect fluid obeying an adiabatic linear equation of state in a general spherically symmetric self-similar space-time. The end state of such collapse terminates into the formation of strong curvature naked singularity at $(t = 0, r = 0)$, and non-spacelike rays emanate from their can be observed by global asymptomatic observer. The gravitational collapse of self-similar spherical models analyzed by Harada et. al. for convergence of field equations to a self-similar solution, and classification of types of self-similarity assumption.

Recently, Banerjee and Chakrabarti analyzed spherically symmetric collapsing scalar field model with a dissipative fluid which includes a heat flux and they showed that formation of a BH or the occurrence of a NS depend on the initial collapsing profile. Pereira and Chan determined self-similar solutions of a collapsing perfect fluid with a massless scalar field with kinematic self-similarity of the first kind in 2+1 dimensions, and some of their solutions represent gravitational collapse.

The gravitational collapse of a massless scalar field may give insights into phenomena of end state collapse and the CCH, other than the early universe and cosmological considerations. The studies undertaken earlier of gravitational collapse of perfect fluid models coupled with...
The self-similar spherical cloud with perfect fluid in terms of (i) viability of the scalar function $\phi$ in the evolution of collapse with $\phi(X)$ is explored in this section. Section IV is devoted to express the sufficient condition for existence of radial null geodesics emerging from the singularity. In section V, existence of both BH/NS is shown subject to appropriate choice of initial data of collapsing cloud. The visibility and strength of NS is examined in section VI. The discussion and conclusions are specified in section VII.

II. SELF-SIMILAR PERFECT FLUID SPACE-TIME

The scheme of work is as follows: In section II, we describe self-similar perfect fluid space-time with scalar function $\phi$ and minimum requirement of weak energy condition on the space-time. The validity of cloud collapse is established through matching of interior space-time with outgoing Vaidya solution in section III, and nature of $\phi(X)$ is explored in this section. Section IV is devoted to express the sufficient condition for existence of radial null geodesics emerging from the singularity. In section V, existence of both BH/NS is shown subject to appropriate choice of initial data of collapsing cloud. The visibility and strength of NS is examined in section VI. The discussion and conclusions are specified in section VII.

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where $\nu$, $\psi$ and $S$ are functions of the self-similarity variable $X = t/r$, and

$$d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2$$

(2)

is the metric on a 2-sphere. The metric describing the space-time is self-similar if it admits a radial coordinate $r$ and an orthogonal time coordinate $t$ such that, we have $g_{tt}(dr, dr) = g_{tt}(t, r), g_{rr}(dt, dr) = g_{rr}(t, r)$ for all $d > 0$. Thereby along the integral curves of the Killing vector field $\xi^\mu$ all points are similar. The self-similar parameter $X$ is a dimensionless combination of space and time coordinates. The stress-energy tensor for a perfect fluid:

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}$$

(3)

where $u_a = \delta^t_a e^t$ is the 4-dimensional velocity. The energy momentum tensor with respect to Klein-Gordon (KG) scalar function $\phi(t, r)$ is described by

$$T^{\mu\nu}(\phi)(t, r) = \phi^{\mu\nu} \phi^{\nu} - \frac{1}{2} g^{\mu\nu} \phi^{\nu} \phi^{\nu}$$

(4)

The pressure $P$ and energy density $\rho$ in the self-similar case can be put in the form:

$$P = \frac{p(X)}{r^2}, \quad \rho = \frac{\eta(X)}{r^2}$$

(5)

The Klein-Gordon equation for the motion of free, massless particles (photon, gluon, graviton) is expressed by

$$\Box \phi = -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + g^{\mu\nu} T^\sigma_{\mu\nu} \partial_\sigma \phi = 0$$

(6)

and our study shall deal with motion of photons only. The Einstein field equations in this set up take the form

$$G^\nu_\mu = \frac{8\pi G}{c^4} \left[ T^\nu_\mu + T^\nu_\mu(\phi) \right].$$

(7)

The self-similarity of the space-time implies existence of constants of motion along $dX = 0$, which in turn allows the reduction of Einstein field equations to a set of ordinary differential equations and using the units that fix the speed of light and the gravitational constant via $8\pi G = c = 1$, we have obtained:

$$-\frac{1}{S^2} + \frac{2e^{-2\psi}}{S} \left[ X^2 S^2 - X^2 \dot{S} \psi + XS \dot{\psi} + \frac{(S - X S)^2}{2S} + \frac{\dot{\phi}^2 X^2 S}{4} \right] - \frac{2e^{-2\nu}}{S} \left( \dot{S} \dot{\psi} + \frac{S^2}{4S} - \frac{\dot{\phi}^2 S}{4} \right) = -\eta(X)$$

(8)

$$\ddot{S} - \ddot{\dot{\psi}} - \ddot{\dot{\phi}} \psi + \frac{S \dot{\psi}}{X} = \frac{\dot{\phi}^2 S}{2}$$

(9)

$$-\frac{1}{S^2} + \frac{2e^{-2\psi}}{S} \left[ -S X \nu + X^2 \dot{S} \nu + \frac{(S - X S)^2}{2S} - \frac{\dot{\phi}^2 X^2 S}{4} \right] - \frac{2e^{-2\nu}}{S} \left( \dot{S} + \frac{S^2}{2S} - \nu S + \frac{\dot{\phi}^2 S}{4} \right) = p$$

(10)
\[ e^{-2\psi} \left[ X^2 \ddot{S} + SX^2 \ddot{\nu} + SX \dot{\nu} + SX \dot{\psi} + X^2 \dot{\nu} \dot{S} - SX^2 \dot{\nu} \dot{\psi} - X^2 \ddot{\psi} \dot{S} + \frac{\dot{\phi}^2 X^2 S}{2} \right] \\
+ e^{-2\nu} \left[ \ddot{\phi} + \ddot{\psi} - \ddot{\psi} - S \ddot{\nu} - S \ddot{\psi} - \dot{S} \ddot{\phi} - \frac{\ddot{\phi}^2 S}{2} \right] = p \]  

(11)

The KG equation in terms of \( X = t/r \) takes the form

\[ e^{-2\psi} \left[ \ddot{\phi} + \ddot{\psi} - \ddot{\psi} + \frac{2\ddot{\psi} S}{S} \right] + e^{-2\psi} X^2 \left[ \ddot{\phi} + \ddot{\psi} - \dot{\nu} \ddot{\psi} - \frac{2\ddot{\psi} S}{S} \right] = 0. \]  

(12)

where an overdot denotes the derivative with respect to \( X \). We assume that the collapsing fluid obeys a linear equation of state (EoS)

\[ p(X) = a \eta(X) \]  

(13)

where \( 0 \leq a \leq 1 \) is a constant. The conservation equation is

\[ [T^{\mu\nu} + T^{\mu\nu}(\phi)]_{;\nu} = 0, \]  

(14)

and from it follows:

\[ e^{-2\psi} X^2 \left[ \ddot{\phi}^2 - \ddot{\phi} \ddot{\psi} - \ddot{\psi}^2 - \frac{2S \ddot{\phi} S}{S} \right] + \ddot{\eta} + (\eta + p) \left[ \ddot{\psi} + \frac{2S \ddot{\psi} S}{S} \right] + e^{-2\nu} \left[ \ddot{\psi}^2 + \ddot{\phi} \ddot{\psi} - \ddot{\phi}^2 + \frac{2S \ddot{\phi} S}{S} \right] = 0 \]  

(15)

\[ e^{-2\psi} X^2 \left[ \ddot{\phi} \ddot{\phi} + \ddot{\phi} \ddot{\psi} + \ddot{\psi} \ddot{\phi} + \ddot{\psi} \ddot{\psi} + \frac{2S \ddot{\phi} S}{S} \right] + \ddot{\nu} (\eta + p) + \ddot{\nu} + \frac{2p}{X} \]  

(16)

These conservation equations together lead to

\[ \ddot{\eta} + (\eta + p) \left[ \ddot{\psi} + \frac{2S \ddot{\psi} S}{S} \right] + \left[ \ddot{\nu} (\eta + p) + \ddot{\nu} + \frac{2p}{X} \right] = 0 \]  

(17)

On using (13), above equation has the solution in the form

\[ e^{\psi + \nu} = C_2 \eta^{-1} S^{-2} X^{-2a/1+a} \]  

(18)

where \( C_2 \) is an integration constant. Elimination of \( \ddot{S} \) from equations (8) and (10) lead to

\[ \left( \frac{\ddot{S}}{S} \right)^2 V + \frac{\ddot{S}}{S} \left( \ddot{V} + 4X e^{2\nu} \right) + e^{2\psi + 2\nu} \left( - \eta - e^{-2\psi} \right) + \frac{1}{S^2} - \frac{3\ddot{\phi}^2 X^2 e^{-2\psi}}{2} - \frac{\ddot{\phi}^2 e^{-2\psi}}{2} = 0 \]  

(19)

and

\[ \ddot{V} = X e^{2\nu} (H - 2) + 2X \dot{\phi}^2 (V + 2X^2 e^{2\nu}). \]  

(20)

The KG equation (12) can be put in the form

\[ \ddot{\phi} (U^2 - 1) + \ddot{\psi} (\ddot{\psi} - \ddot{\nu}) (U^2 + 1) + \frac{2S \ddot{\psi} S}{S} (U^2 - 1) = 0 \]  

(21)

where the quantities \( U, H \) and \( V \) are defined by

\[ U = \frac{e^{-2\psi} e^{2\psi}}{X^2}, \quad H = (\eta + p) e^{2\psi}, \quad V(X) = e^{2\psi} - X^2 e^{2\nu} = X^2 e^{2\psi} (U^2 - 1). \]  

(22)

Also \( H \) can be written as \( H = r^2 e^{2\psi} (T_1 - T_0) \). The matter satisfy weak energy condition if and only if \( T_{ij} \chi^i \chi^j \geq 0 \) for all non-space-like vector \( \chi^i \). Thus for matter satisfying weak energy condition \( p \geq 0 \), \( (\rho + p) \geq 0 \), it follows that \( H(X) \geq 0 \) for all \( X \). The physical quantities energy density and pressure given in equation (5) diverge as \( t \to 0, r \to 0 \). The Ricciscalar \( R = [\rho(X) - 3P(X)] - e^{-2\psi} X^2 (U^2 - 1) \ddot{\phi}^2/r^2 \) in these limit diverges as well, so the point \( t = 0, r = 0 \) is a genuine singularity of the space-time.

III. NATURE OF \( \phi(X) \) ON THE MATCHING SURFACE

We consider a spherical surface with its motion described by a time-like four space \( \Sigma \), which divides space-time into interior and exterior manifolds. We shall first cut the space-time along time-like hypersurface, and then join the internal part with the outgoing Vaidya solution. The procedure adopted here is similar to (33) and suitably modified for the present work. The metric on the whole space-time can be written in the form

\[ ds^2 = \left\{ \begin{array}{ll}
-e^{2\nu} dt^2 + e^{2\nu} dr^2 + r^2 S^2 d\Omega^2, & r \leq \rho, \\
-(1 - \frac{2m(r)}{r}) dt^2 - 2d\phi^2 - r^2 d\Omega^2, & r \geq \rho.
\end{array} \right. \]  

(23)

The metric on the hypersurface \( r = \rho \) is given by

\[ ds^2 = -dr^2 + R^2(r) d\Omega^2 \]  

(24)

For the junction conditions we consider the approach given in (33) for our case. Hence we expect

\[ (ds^2)_{\Sigma} = (ds^2)_{+\Sigma} = (ds^2)_{-\Sigma} \]  

(25)
The second junction condition is obtained by requiring the continuity of the extrinsic curvature of $\Sigma$ across the boundary. This yields

$$K_{ij}^- = K_{ij}^+$$  \hspace{1cm} (26)

where $K_{ij}^\pm$ are extrinsic curvatures to $\Sigma$, given by

$$K_{ij} = -n_\alpha^\pm \frac{\partial^2 x^\alpha}{\partial \xi^i \partial \xi^j} - n_\alpha^n \Gamma^\beta_\gamma \frac{\partial x^\beta}{\partial \xi_i} \frac{\partial x^\gamma}{\partial \xi^j}$$

where $\Gamma^\beta_\gamma$ are Christoffel symbols, $n_\alpha^\pm$ the unit normal vectors to $\Sigma$, $x^\alpha$ are the coordinates of the interior and exterior space-time and $\xi^i$ are the coordinates that defines $\Sigma$. From the junction condition (25) we obtain

$$\frac{dt}{d\tau} = \frac{1}{e^{\nu(r,\Sigma, t)}}$$

where subscripts $r$ and $t$ denote partial derivative with respect to $r$ and $t$ respectively. The unit normal to the inner and outer $\Sigma$ are given by

$$n_\alpha^- = (0, e^{\psi(r_S, t)}, 0, 0),$$

$$n_\alpha^+ = \left(1 - \frac{2m(v)}{r}, \frac{2d\bar{r}}{dv}, \frac{2m(v)}{r} \right) \frac{1 - \frac{2m(v)}{r}}{\bar{r}}.$$

From equations (26), (32) and (34) we have

$$\left( \frac{dv}{d\tau} \right) - \frac{2m(v)}{r} \frac{d\bar{r}}{dv} = \left[ e^{-\psi} r S(S + r S_r) \right]_\Sigma \tag{37}$$

Substituting equations (28), (29) and (37) into (36) we obtain

$$\left( \frac{dv}{d\tau} \right) = \left[ \frac{S + r S_r}{e^{\nu}} + \frac{r S_t}{e^{\psi}} \right]_{\Sigma}^{-1} \tag{39}$$

Differentiating (39) with respect to $\tau$ and using equation (37), we can rewrite (38) as

$$-\left( \frac{\nu_r}{e^{\psi}} \right)_\Sigma = \left\{ -\frac{r}{e^{\nu}} S_{tr} + \nu_2 (S + r S_r) + r \nu_2 S_t - \frac{r S_{tt}}{e^{\psi}} + \frac{e^\nu}{r S} \left( (S + r S_r)^2 - 1 \right) \right\} \tag{40}$$

Next we translate the above equation in terms of $X = t/r$, which is given by

$$\frac{1}{S^2} + 2e^{-2\nu} - \frac{2e^{-2\nu}}{S} \left[ -S \dot{X} \ddot{X} + X^2 \dot{X} \ddot{S} + \frac{(S - X \dot{S})^2}{2S} \right]$$

Comparing (41) with (9) and (10), we can finally write

$$[p(X)]_\Sigma = -\left[ \frac{\dot{\phi}^2 \left( e^{-2\nu} X^2 + e^{-2\nu} + X e^{(\psi + \nu)} \right)}{2} \right] \tag{42}$$

It is evident from above that at the surface $X = X_\Sigma$, pressure $p(X_\Sigma) = 0$ only if $\dot{\phi}^2(X_\Sigma) = 0$. The simplest choice is that $\phi(X) = \text{const}$ for all $X$. The general selection of $\phi(X)$ be such that it must be satisfying the KG equation (21) and that physical attributes of the quantities like density, pressure and scale factor appearing in the field equations be maintained on the collapsing cloud. So, to satisfy weak energy condition in general, study of field equations and equation (42) suggest that the term $\dot{\phi}^2$ need to be negative in sign, and therefore function $\phi(X)$ should be an imaginary function satisfying $\dot{\phi}^2(X_\Sigma) = 0$. Herein, we are not aiming to find such a function describing wave equation for massless particles but instead make efforts to study end state of gravitational collapse keeping the form of function $\phi(X)$ arbitrary. We suggest $\phi(X)$ should have the following possible form:

$$\phi(X) = i \alpha_0 \varphi(X),$$

where $\varphi(X)$ is a non-zero real valued function, $\alpha_0$ is a positive constant and $i$ is an imaginary number. In the beginning, we have started with scalar function $\phi(t, r)$, and it is related to $\phi(X)$ by the expression $d\phi(t, r) = \frac{2}{S^2} \left( e^{-2\nu} X^2 + e^{-2\nu} + X e^{(\psi + \nu)} \right)$.
\( \phi, \mu \partial r + \phi, \nu \partial r = \phi (X) dX \), therefore for a given \( \phi (X) \), by integration \( \phi (t, r) \) can be found.

Now the important question is whether such function \( \phi (X) \) would exist satisfying the KG equation \( (12) \). To understand the nature of scalar function, attempts are made in \[37\]. We understand that the nature of \( \phi (X) \) expressed here is needed to explore gravitational interaction of motion of massless particles in the collapsing cloud scenario, so the function \( \phi (X) \), now in terms of function \( \varphi (X) \) shall carry out properties of KG equation. Further, for the above \( \phi (X) \), we find that \( \mathcal{L}_\xi \phi (X) = 0 \), such \( \phi (X) \) befits well with the constants of motion of the self-similarity, though the space-time symmetries will not get mixed up with the internal symmetries due to scalar field, which they should, as per the exposition on gravitational collapse here is needed to explore gravitational interaction of motion of massless particles in the collapsing cloud scenario, so the function \( \phi (X) \) is an imaginary function such that \( \dot{\psi} (t) = 1 + 2 \eta (S^2) \) and \( \dot{\psi} (2) = \frac{S^2}{2} \). The Misner-Sharp mass \( 2M(t, r) \) of the system describing the amount of matter enclosed within the surface \( r < r_s \), is expressed in terms of \( X = t/r \),

\[
\frac{2M}{R} = 1 + S^2 e^{-2\psi} \left[ \frac{(U^2 - 1)X^2 S^2}{2^2} + \frac{2X S^2}{S} - 1 \right] \tag{44}
\]

where \( R = r S \) is the physical radius of the cloud.

\[
\frac{\partial M}{\partial r} = \frac{(S - \dot{S} X)S^2 \eta}{2} + \frac{2 \dot{\psi} X^2 e^{-2\psi} S^2}{4} \left[ (1 + U^2) - \dot{S} (1 + 3U^2) \right] \tag{45}
\]

In general, it is expected that \( M(t, r) \) to be an increasing function of \( r \), i.e. \( \partial M/\partial r \geq 0 \) in the region \( r < r_s \). For collapse consideration, \( \dot{R} < 0 \), so does \( \dot{S} < 0 \) and that \( \phi \) is an imaginary function such that \( \dot{\psi} < 0 \), therefore in the above equation second term is negative. Hence, the scalar field presumptuously does not support increasing nature of mass function unless the density function of mass is finite then dominates the second scalar field involving term. Considering the above, the discussion on trapped regions will be carried out in section V.

IV. THE NULL GEODESICS

In this section, we study the sufficient condition for existence of radial null geodesics emerging from the singularity and for this objective, we express herein the technique developed by Dwivedi and Joshi for our perusal.\[39\] A self-similar space-time is characterized by the existence of a homothetic Killing vector \( \xi^\mu \) which is given by the Lie derivative:

\[
\xi^\mu = r \frac{\partial}{\partial r} + t \frac{\partial}{\partial t} \text{ and } \mathcal{L}_\xi g_{\mu \nu} = \xi_{\mu ;\nu} + \xi_{\nu ;\mu} = 2g_{\mu \nu} \tag{46}
\]

where \( \mathcal{L} \) denotes the Lie derivative. Let \( K^\mu = dx^\mu /dk \) be the tangent vector to the null geodesics, where \( k \) is an affine parameter. Then

\[
g_{\mu \nu} K^\mu K^\nu = 0. \tag{47}
\]

It follows that along null geodesics, we have

\[
\xi^\alpha K_\alpha = \mathcal{C} \tag{48}
\]

where \( \mathcal{C} \) is a constant. From the above algebraic equation and the null condition \( (47) \), we obtain equations

\[
re^{2\psi} K^r - te^{2\nu} K^t = \mathcal{C}, \quad e^{2\psi} (K^r)^2 - e^{2\nu} (K^t)^2 = 0. \tag{49}
\]

The above equations yield the following exact expressions for \( K^t \) and \( K^r \):

\[
K^t = \frac{\mathcal{C} [X \pm e^{2\psi} Q]}{r [e^{2\psi} - e^{2\nu} X^2]}, \quad K^r = \frac{\mathcal{C} [1 \pm X e^{2\nu} Q]}{r [e^{2\psi} - e^{2\nu} X^2]} \tag{50}
\]

where \( Q = \sqrt{e^{-2\nu} - e^{-2\psi}} > 0 \). Equations in \( (50) \), determine the radial null geodesic equation

\[
\frac{dt}{dr} = \frac{X \pm e^{2\nu} Q}{1 \pm X e^{2\nu} Q} \tag{51}
\]

Analyzing above equation, we note that point \( (t = 0, r = 0) \) is a singular point of this differential equation. At this point, a curvature singularity forms at the origin \( (t = 0, r = 0) \), where the energy density diverges. The divergence of the density in this singularity results also in a divergence of curvature scalars there. The nature of the singularity (a NS or a BH) can be revealed by the existence of radial null geodesics emanating from the singularity. The singularity is at least locally naked if there exist such geodesics, and if no such geodesics exist it is a black hole. If the singularity is naked, then there exists a real and positive value of \( X_0 \) as a solution to the following algebraic equation\[10\]

\[
X_0 = \lim_{t \to 0, \ r \to 0} X = \lim_{t \to 0, \ r \to 0} \frac{t}{r} = \lim_{t \to 0, \ r \to 0} \frac{dt}{dr} \tag{52}
\]

Using equation \( (51) \) and L'Hôpital's rule, we can have

\[
V(X_0)Q(X_0) = 0 \tag{53}
\]

Since \( Q > 0 \), this implies that

\[
V(X_0) = 0 \tag{54}
\]

This is an all important equation that governs the behavior of the tangent near the singular points. The central shell focusing singularity is at least locally naked if equation \( (54) \) admits one or more positive roots. The values of the roots give the tangents of the escaping geodesics near the singularity. The smallest value of \( X_0 \), say \( X_0^* \), corresponds to the earliest ray escaping from the singularity which, is called the Cauchy horizon of the space-time and there is no solution in the region \( X < X_0^* \). Thus in the absence of a positive root to equation \( (54) \), the central singularity is not naked because there is no outgoing future directed null geodesics emanating from the singularity.\[23\]
V. EXISTENCE OF NS/BH PHASES

In this section, we analyze evolution of field equations in terms of its parameter $X$ to determine the nature of singularity in the neighbourhood of $(t = 0, r = 0)$. To discuss BH/NS self-similar solutions, we consider the function $U^2 = e^{-2\nu}e^{2\psi}/X^2$ introduced by Cahill and Taub (1970)\(^{[14]}\). $U$ measures the velocity of the $X = \text{const.}$ hypersurfaces relative to the fluid. The induced metric on such a hypersurface is

$$ds^2 = -e^{2\nu}(1 - U^2)dt^2 + r^2S^2d\Omega^2.$$  

So if $U > 1$, all displacements on the surface are spacelike and an observer cannot remain on it. If $U < 1$, timelike displacements are possible on the surface and an observer can remain in such a region. If $U = 1$ and $e^{2\nu}$ is finite, the surface contains a null vector and is either an event horizon or a particle horizon. However, if $e^{2\nu}$ becomes infinite at $U = 1$, one is required to calculate the limit of $e^{2\nu}(1 - U^2)$ as $U \to 1$, in order to determine the nature of the hypersurface.\(^{[14,15]}\)

Now, we solely consider our study on the hypersurface $X = X_0 = \text{const.}$ when $U = 1$. Herein $U^2 = 1$ implies $e^{2\nu} = e^{2\nu}X^2$. The KG equation \(^{[21]}\) on such surface in particular takes the simple form

$$2\phi(\psi - \dot{\nu}) = 0.$$  

(55)

This leads to $\dot{\phi} = 0$ or $(\psi - \dot{\nu}) = 0$ on the surface $X = X_0$.

**Case (i):** When $(\psi - \dot{\nu}) \neq 0$ and $\dot{\phi} = 0$, so the function $\phi(X)$ can be a constant for all $X$ or be a non-zero function satisfying $\phi(X_0) = 0$ and therefore the system of self-similar field equations reduces to usual set of equations not having any effect of scalar function. The results of some of these investigations is mentioned in the introduction and are discussed in detail in\(^{[13]}\) and references therein.

**Case (ii):** When $(\psi - \dot{\nu}) = 0$ on the surface $X = X_0$, the function $\dot{\phi}(X)$, and so $\dot{\phi}(X)$ need not be zero on this surface. So, we are interested in determining various properties of the evolution of field equations, keeping the scalar function $\phi$ arbitrary such that the end state collapse lead to formation of BH or NS in the space-time. When this condition is subjected to field equation \(^{[11]}\) on this surface,

$$p = 2\nu Xe^{-2\psi}.$$  

(56)

On the use of equation \(^{[13]}\) and $\dot{\psi} = \dot{\nu}$, the integration of equation \(^{[17]}\) yields

$$e^{2\nu} = \frac{\gamma}{S^2}X^2\frac{a}{b}$$  

(57)

where $b = 1 + a$ and $\gamma$ is a positive constant of integration. Further solving equations \(^{[56]}\) and \(^{[57]}\),

$$2\dot{\nu} = a \gamma S^{-2}X^{(1-a)/b}.$$  

(58)

In this environment, equations \(^{[19]}\) and \(^{[20]}\) take the form

$$2X\dot{S} = S + \frac{\gamma X^{2/\nu} - \gamma X^{2/b}}{\eta S^2} = 1 + 2\dot{\phi}^2 X^2$$  

(59)

and on using equation \(^{[59]}\), it becomes

$$2 \frac{2M}{R} = 1 + S^2e^{-2\psi} \left[\frac{2X\dot{S}}{S} - 1\right]$$  

(60)

and on further use of equation \(^{[60]}\), the boundary of the trapped region $2M = R$ is expressed as

$$2 \frac{2M}{R} - 1 = \frac{\gamma e^{-2\psi}X^{2/b}}{2} \left[1 - a - \frac{2}{\eta S^2}\right].$$  

(62)

The space-time region where the mass function $2M < R$ is not trapped, while $2M > R$ describes a trapped region. From above equation the Schwarzschild radius $R_s = 2M$ is obtained only when $\left[(1 - a) - 2/(\eta S^2)\right] = 0$ and further this term decides about the sign of $(2M/R - 1)$, thereby indicating formation of trapped/un-trapped regions for various values of $a$. In the dust case at the surface $U = 1$, energy density obtained from conservation equation \(^{[57]}\) is $\eta = \gamma / S^2$ and for obtaining the Schwarzschild radius, it is $2/S^2$. The formation of event horizon and BH region in all the situations of EoS parameter $a$ is schematically shown in fig-01.

It is specified in section 4 that existence of real positive roots for $V(X) = 0$ provides a sufficient condition for the occurrence of NS at the origin. To realize this in terms of physical parameters of self-similar field equations, we analyze these equations on the surface $U = 1$ as follows: We define a function $y = X^{\beta}$ where $\beta = 1/(1 + a)$. On applying these transformations, equations \(^{[59]}\) and \(^{[60]}\) take the form

$$2\beta S\dot{y} - \frac{\gamma}{S^2}y^2 + \frac{\gamma}{\eta S^2}y^2 = 1 + 2\beta^2(\dot{\phi})^2y^2$$  

(64)

$$- \frac{b\gamma}{2S^2}y^2 = 2\beta^2(\dot{\phi})^2y^2.$$  

(65)
where dash denotes differentiation with respect to y. Eliminating $(\phi')^2$ from the equations (64) and (65) to obtain:

$$2\beta \frac{S'}{S} y - \frac{(1-a)\gamma}{2S^2} y^2 + \frac{\gamma}{\eta S^4} y^2 = 1. \quad (66)$$

Already we know, the scale invariant quantity U represents velocity of the fluid relative to the hypersurface $X = X_0$. The case, where $U = 1$, is of special interest to us as this corresponds to a possible situation that the space-time allows a naked singularity at the center i.e. null geodesics terminate at the singularity in the past. Thus we are interested in the values of different parameters for the solution, which take into account the case $U = 1$ for some real $X > 0$. We analyze the above differential equation near the point $y = y_0 = y(X_0)$ with the condition that $U(y_0) = 1$ and that $\eta(y_0) = \eta_0$. Let us write $S(X)$ that satisfy differential equation for $S$ in equation (66) such that

$$S(y) = s_0 + s_0 \sum_{k=1}^{\infty} s_k (y - y_0)^k \quad (67)$$

On using equation (67), equation (66) takes the form

$$(2 - w_0) \gamma y^2 + s_1 w_1 y - w_2 = 0 \quad (68)$$

where

$$w_0 = (1-a)\eta_0 s_0^2, \quad w_1 = 4\beta \eta \eta_0 s_0^4, \quad w_2 = 2 \eta \eta_0 s_0^4$$

are positive constants and $S(y_0) = s_0$, $S'(y_0) = s_0 s_1$. This algebraic equation ultimately decide the final fate of the collapse. In general the existence of real positive roots of above quadratic equation will put a limitation on the physical parameters $\eta_0$, $s_0$ and $s_1$. Thus the existence of real positive roots of $V(X) = 0$ (and, hence existence of naked singularity) is characterized by the values of physical parameters $\eta_0 > 0$, $s_0 > 0$ and $s_1 < 0$ for the collapsing cloud. These parameters form initial data set of positive central density $\eta_0 > 0$, scale factor component $s_0 > 0$ (through physical radius $R_0 = r s_0 > 0$) and condition of gravitational collapse $s_1 < 0$ when the information regarding profiles of energy density and physical radius of a collapsing cloud are given. The roots of the equation (68) are

$$y = \frac{1}{2(2-w_0)\gamma} \left[ -w_1 s_1 \pm \sqrt{\Delta} \right]$$

where $\Delta = w_1^2 s_1^2 + 4 w_2 (2 - w_0) \gamma$. \quad (69)

When $(2 - w_0) > 0$, there is a complete certainty of existence of one positive real root and one negative real root to the above equation. So, it is important to test the condition on $(2 - w_0)$. In the dust case $a = 0$, $\eta = \gamma/S^2$, so $(2 - w_0) = (2 - \gamma) > 0$ for appropriate choice of arbitrary constant $\gamma$ and in the stiff fluid case $a = 1$, $(2 - w_0) = 2 > 0$. For other values of $0 < a < 1$, it is observed graphically that $E_o = \eta_0 s_0^2$ need to be sufficiently small to satisfy the requisite positive sign of $(2 - w_0)$ for various values of $a$. The reasonable initial data satisfying $(2 - w_0) > 0$ is shown in figure 2. This establishes existence of positive real root of $V(X) = 0$ and existence of at least locally NS is guaranteed. On the other hand, when $(2 - w_0) < 0$ there are situations of having only negative real and imaginary roots.

FIG. 1. The requisite condition $[(1-a) - 2/(\eta S^2)] = 0$ for obtaining the Schwarzschild radius $R_o = 2M$ is plotted as a curve where $E_o = \eta_0 S_0^2$. The trapped region relative to wide availability of initial data indicates that, it is well past the Schwarzschild radius forming BH for various values of EoS parameter $a$.  

FIG. 2. The condition $(2 - w_0) > 0$ substantiate the existence of real positive root in equation (69) for occurrence of at least locally NS. The said condition is fulfilled for wide choice of physically reasonable initial data as shown in the shaded region for various values of EoS parameter $a$ and $E_o = \eta_0 S_0^2$. 


VI. VISIBILITY AND STRENGTH OF NS

The visibility of a naked singularity can be considered as physically significant, if light rays can escape from singularity to far away observers for a finite period of time, and such singularity is termed to be a globally NS. The detailed Mathematical description to study this can be found in[15]. The singularities are visible for a finite period of time, if infinity of integral curves escape from singularity. So for this, the equation of geodesics in the form \( r = r(X) \) needs to be analyzed. Now \( dX/dr = 1/r \, dt/dr \, - \, X/r \), using equation (50) in \( dX/dr \), and integrating it in \((X, r)\) plane, we have

\[
 r = C \exp \left[ \int \frac{1 + X e^{2\nu}}{V(X) Q(X)} \, dX \right], \tag{70}
\]

where integration constant \( C \) labels different geodesics. Earlier, we have specified that for a singularity to be naked, \( V(X) = 0 \) must have at least one real positive root. Let \( X = X_0 \) be a real positive root of \( V(X) = 0 \). Near the singularity, in the limit \( t \to 0, r \to 0 \), equation (20) decomposes into \( V(X) = (X - X_0) X_0^\alpha e^{2\nu(X_0)}[H_0 - 2] \) and using this \( V(X) \) in equation (70) and integrating it, it yields

\[
 r = C (X - X_0)^{2/[H_0 - 2]} \tag{71}
\]

where \( H_0 = H(X_0) = [\gamma \, b X_0^{3/2} /S(X_0)^2] > 0 \). So, we observe that when \( H_0 > 2 \), an infinity of integral curves will meet the singularity in the past with tangent \( X = X_0 \), with different curves being characterized by different values of \( C \). Hence an infinity of integral curves with tangent \( X = X_0 \) would escape from singularity, forming at least locally NS. The condition \( H_0 > 2 \) is now replaced by \( X_0^{3/2} > 2S(X_0)^2/(\gamma b) \). Thus above equation represents null geodesics escaping from singularity, reaching to at least local observer with tangent \( X = X_0 \) for all values of EoS parameter \( \alpha \). Also when \( 0 < H_0 < 2 \), it can be shown that in \((t, r)\) plane, singularity is a NS[44].

The locally naked singularities discussed here are not visible to an asymptotic observer. Therefore, now, we briefly describe when such null geodesics can reach far away observer. Let \( X_1 \) and \( X_2 \) (with \( X_1 > X_2 \)) be two consecutive roots of the equation \( V(X) = 0 \) i.e. solutions of \( U^2 - 1 = 0 \) and which are given by \( X_i = [e^{2\nu(S^2\eta/\gamma)]^{1/2} > 0 \) at \( X = X_0 \). For dust case \( X_i = e^\nu > 0 \). Equation (70) is rearranged as follows:

\[
 r = C \exp \left[ \int \frac{1 + X e^{2\nu}}{X - X_1} \frac{dX}{h(X)} \right] - \int \frac{1 + X e^{2\nu}}{X - X_2} \frac{dX}{h(X)} \]

where \( h(X) = \frac{(X_1 - X_2)(X) Q(X)}{(X_1 - X_1)(X - X_2)} \). \tag{72}

The function \( h(X) \) does not change sign in the interval between \( X_1 \) and \( X_2 \) and has same sign at \( X_1 \) and \( X_2 \). Therefore all integral curves can meet singularity in the past with tangent \( X = X_1 \) and same trajectory will reach infinity \( r = \infty \) at \( X = X_2 \), and vice-versa. Thus the singularity will be globally naked and an infinity of curves would emanate from singularity to reach a distant observer.

Next, we study the strength of naked singularity. A space-time singularity is gravitationally strong if volume elements get crushed to zero dimensions at the singularity, and weak otherwise. Further that a space-time does not admit an extension through a singularity if it is a strong curvature singularity in the sense of Tipler[43]. So, to analyze the strength of the singularity in stronger sense, we examine the behaviour of the scalar \( \Psi = R^{ab}K_aK_b \), where \( R_{ab} \) is the Ricci tensor, along null geodesics that terminate at the NS in the past. Clarke and Królak has given a necessary and sufficient condition criterion for a singularity to be strong that for at least one non-spacelike geodesic with affine parameter \( k \), \( \lim_{k \to 0} k^2 \Psi \) is positive[15]. So we have[15]

\[
 \lim_{k \to 0} k^2 \Psi = \lim_{k \to 0} k^2 R_{ab}K^aK^b = \frac{4H_0}{(H_0 + 2)^2} > 0. \tag{73}
\]

This condition is always true for every real positive root \( X_0 \) of \( V(X) = 0 \). Thus along radial null geodesics strong curvature condition is satisfied for every such \( X_0 \). Since \( H_0 > 0 \) is also a necessary condition for holding weak energy condition on the surface \( X = X_0 \). Therefore, it follows that singularities are gravitationally strong if the weak energy condition is satisfied or in other words to say if \( X_0 > 0 \).

VII. DISCUSSION AND CONCLUSIONS

The revealed imaginary nature of the scalar function, \( \phi(X) = i \alpha_0 \, \varphi(X) \), on applying it to the equation (21), we observe that, this equation is totally written in terms of \( \varphi(X) \). So, the solution \( \varphi(X) \) of KG equation will describe the wave nature of free, massless particles and its interaction with the gravity is understood from the analysis of field equations. Further the field equations involves \( \partial^2 \) only, and therefore evolution of field equations with scalar field is related to the real valued function \( \varphi(X) \). Hence the wave motion of the massless particles be taken as an imaginary part of \( \phi(X) \) in the \((t, r)\) plane.

For the conditions, say, first \( \nu = 0 \), and second \( \psi = \dot{\psi} \) equation (12) yields respectively:

\[
 \dot{\phi} = i \alpha_0 S^2 \exp \left[ - \int \frac{(e^{2\nu} + X^2)}{(e^{2\nu} - X^2)} \psi dX \right], \quad \text{and} \quad \dot{\phi} = i \alpha_0 S^2. \tag{74}
\]

Herein, we have six equations comprising KG equation and EoS and there are six unknowns \( \nu, \psi, S, \rho, p \) and \( \phi \), therefore system of equations is closed and there is no arbitrary choice of function available here. So if we choose function, say \( S \) then we expect unique exact solution.
to the above system of equations. Hence it is a little tough job to find such solution. On availability of such functions, integration of above equation will give us wave equation expressing its interaction with gravitation field.

In Minkowski flat space-time, the wave form is \( \varphi(X) = (a_0X + a_1) \), a plane wave front with no gravitational interaction.

Conclusions:
To represent the gravitational collapse of a dense cloud, we have matched the solution to the Vaidya outgoing pure radiation solution and through that we have obtained the condition that scalar field should satisfy \( \dot{\phi}^2(X_\Sigma) = 0 \) on the matching surface \( \Sigma \).

This study has revealed that to maintain weak energy condition, the scalar function needs to be an imaginary function \( \varphi(X) = i a_0 \varphi(X) \), and that its real valued function \( \varphi(X) \) only takes active part in the evolution of the system of field equations.

We have shown that the singularities are gravitationally strong if the positivity of weak energy condition is maintained. The space-time admits globally strong curvature naked singularities provided the algebraic equation \( V(X) = 0 \) has real positive roots and appropriate physically reasonable initial data of energy density and physical radius be supplied.

At the same time black hole formation is as well a regular feature of the end state of collapse subject to physically reasonable initial data. The occurrence of both NS and BH depend upon from what initial data of energy density and physical radius profiles, the collapse begins.

In our study, from equation (21), we have derived the condition \( \dot{\psi} = \nu \) on the surface \( U = 1 \). It is possible that the scale factor \( S \), in the third term of the equation (21), can annihilate the term \( (U^2 - 1) \), but then \( S \) needs to appear in terms of metric functions \( \nu \) and \( \psi \) which is a difficult proposition, such study can incorporate a general approach to the above analysis.

Our analysis deals with radial null geodesics only, so separate studies can be taken up to include timelike geodesics. The mechanism constructed here can help develop models of negative repulsive pressure considering \( \phi^2 > 0 \), therefore the dark energy models can be studied.

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