On the Open Question of The Tracy-Widom Distribution of 
$\beta$-Ensemble With $\beta = 6$

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Abstract: We determine completely the Tracy-Widom distribution for Dysons $\beta$-ensemble with $\beta = 6$. The problem of the Tracy-Widom distribution of $\beta$-ensemble for general $\beta > 0$ has been reduced to find out a bounded solution of the Bloemendal-Virág equation with a specified boundary. Rumanov proposed a Lax pair approach to solve the Bloemendal-Virág equation for even integer $\beta$. He also specially studied the $\beta = 6$ case with his approach and found a second order nonlinear ordinary differential equation (ODE) for the logarithmic derivative of the Tracy-Widom distribution for $\beta = 6$. Grava et al. continued to study $\beta = 6$ and found Rumanov’s Lax pair is gauge equivalent to that of Painlevé II in this case. They started with Rumanov’s basic idea and came down to two auxiliary functions $\alpha(t)$ and $q_2(t)$, which satisfy a coupled first-order ODE. The open question by Grava et al. asks whether a global smooth solution of the ODE with boundary condition $\alpha(x) = 0$ and $q_2(x) = -1$ exists. By studying the linear equation that is associated with $q_2$ and $\alpha$, we give a positive answer to the open question. Moreover, we find that the solutions of the ODE with $\alpha(x) = 0$ and $q_2(x) = -1$ are parameterized by $c_1$ and $c_2$. Not all $c_1$ and $c_2$ give global smooth solutions. But if $(c_1, c_2) \in R_{smooth}$, where $R_{smooth}$ is a large region containing $(0, 0)$, they do give. We prove the constructed solution is a bounded solution of the Bloemendal-Virág equation with the required boundary condition if and only if $(c_1, c_2) = (0, 0)$.

1 Introduction

In the one dimensional case, the interaction energy of two point charges is

$$k_e \ln \left(\frac{r_0}{|x_A - x_B|}\right),$$

where $k_e$ is the electric force constant, $r_0$ is the distance that the interaction energy is 0, $x_A$ and $x_B$ are the positions of the two point charges. Dyson’s Coulomb gas model is $N$ particles with like charges, i.e., $k_e > 0$, in an external field $V = V(x)$. By the canonical ensemble, the probability that the first particle is in $[x_1, x_1 + dx_1]$, $\cdots$, and that $N$-th particle is in $[x_N, x_N + dx_N]$, is

$$p(x_1, x_2, \cdots, x_N)dx_1dx_2\cdots dx_N = \frac{1}{Z_N} e^{-\frac{k_e}{k_B T} \sum_{1 \leq i < j \leq N} k_e \ln \left(\frac{r_0}{|x_i - x_j|}\right) + \sum_{j=1}^{N} V(x_j)} dx_1dx_2\cdots dx_N \tag{1.1}$$

where $k_B$ is the Boltzmann constant, $T$ is the temperature, and $Z_N$ is the normalization constant. Here we assume $V$ is Gaussian, i.e.,

$$V(x) = \frac{1}{2} k_e x^2.$$

Let

$$\beta = \frac{k_e}{k_B T}, \quad \lambda = \sqrt{\frac{k_B}{k_e} T}.$$

Then the particle distribution becomes

$$\tilde{p}(\lambda_1, \lambda_2, \cdots, \lambda_N)d\lambda_1d\lambda_2\cdots d\lambda_N = \frac{1}{Z_N} \left(\prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \right) e^{-\frac{\beta}{2} \sum_{j=1}^{N} \lambda^2_j} d\lambda_1d\lambda_2\cdots d\lambda_N. \tag{1.2}$$
A system of random variables $\lambda_1, \lambda_2, \ldots, \lambda_N$ with distribution \([1,2]\) is called the $\beta$-ensemble. The $\beta$-ensemble, $\beta = 1, 2, 4$, describes the joint density of eigenvalues of the three classical matrix models, i.e., the Gaussian orthogonal ensemble(GOE), Gaussian unitary ensemble(GUE) and Gaussian symplectic ensemble(GSE), respectively. For general $\beta > 0$, the $\beta$-ensemble can be realised as the joint density of eigenvalues of the spiked $\beta$-Hermite matrix ensemble \([2]\), $\beta$-ensemble for general $\beta$ also has other physical applications, for example, it can be mapped to a chiral Liouville theory with central charge \([8]\). Also, in some sense, the harmonic potential \([14]\) is not so serious a limitation since the universality of the applications, for example, it can be mapped to a chiral Liouville theory with central charge \([8]\). Also, spiked $\sigma$-ensemble \((1.4)\) with infinite terms \((1.3)\) can not determine the particle number in $\beta$-ensemble, respectively. For general $\beta$-ensemble, \([2]\) proved the constant term for $\beta$-ensemble had been proved by Bourgade et al. \([4]\). The interesting case is the thermodynamic limit $N \to \infty$. Almost all particles distribute in $[-\sqrt{2N}, \sqrt{2N}]$ obeying the Wigner semicircle law with an approximate density $\sigma(\lambda) = \pi^{-1/2} N \sqrt{2N - \lambda^2}$ \([15]\), i.e., the particle number in $[\lambda, \lambda + d\lambda]$ is about $\sigma(\lambda)d\lambda$. But few particles may lie outside $[-\sqrt{2N}, \sqrt{2N}]$. It is proved that near the edge a proper scaling limit is the soft edge probability distribution \([16]\)

$$F(\beta) = \lim_{N \to \infty} E_{\beta N}^{\text{soft}} \left( 0; \left( \sqrt{2N} + \frac{t}{\sqrt{2N}^{1/6}}, \infty \right) \right),$$

where

$$E_{\beta N}^{\text{soft}}(0; (t, \infty)) = \int_{\lambda_N = -\infty}^t \cdots \int_{\lambda_1 = -\infty}^t \bar{\rho}(\lambda_1, \ldots, \lambda_N) d\lambda_1 \cdots d\lambda_N.$$  

$F(\beta)$ is called the Tracy-Widom distribution.

The explicit expressions for $F(\beta)$ for $\beta = 1, 2, 4$ are classical \([19, 20, 16]\). They represent $\beta$-ensemble, respectively. For general $\beta$-ensemble, \([2]\) obtained and proved the constant term for $\beta$-ensemble had been proved by Bourgade et al. \([4]\). The expansions of $F(\beta)$ at $t = -\infty$ are of special interests. In \([19\) and \([20\), Tracy and Widom obtained and proved $F_2(\beta)$ for $\beta = 1, 2, 4$ without the constant term. They also conjectured the values of the constant term $c_0$ for $\beta = 1, 2, 4$. By the Deift-Zhou nonlinear steepest descent method \([6]\), Deift et al. \([5]\) proved the constant term for $\beta = 2$ and Baik et al. \([11]\) proved the constant terms for $\beta = 1, 2, 4$. Finally Borot et al. \([3]\) derived an amazing asymptotic expression of $F(\beta)$ for general $\beta > 0$ at $t = -\infty$ by the loop-equation technique. Their asymptotic expression is

$$F(\beta) = \exp \left( \frac{\beta}{24} |t|^3 + \frac{\sqrt{2}}{3} \left( \frac{\beta}{2} - 1 \right) |t|^2 + \frac{1}{8} \left( \frac{\beta}{2} + \frac{3}{2} \right) \ln |t| + c_\beta + O(|t|^{-\frac{7}{2}}) \right),$$

where the constant term $c_\beta$ is

$$c_\beta = \frac{\gamma_E}{6 \beta} + \left( \frac{17}{8} - \frac{25}{24} \frac{\beta}{2} + \frac{2}{\beta} \right) \ln 2 - \frac{1}{2} \frac{\ln(\frac{\beta}{2})}{2} - \frac{\ln(2\pi)}{4} + \frac{\beta}{2} \left( \frac{1}{12} - \zeta'(-1) \right) + \int_0^\infty \frac{1}{e^{\beta t/2} - 1} \left( \frac{t}{e^t - 1} - 1 + \frac{t}{2} - \frac{t^2}{12} \right) dt.$$  

Here $\gamma_E$ denotes the Euler’s constant and $\zeta$ refers to the Riemann zeta function. Note the prime $'$ will always be used to denote derivative.

But the asymptotics \((1.3)\) is only valid at $t = -\infty$ and can not be continued to finite $t$. In fact, even with infinite terms \((1.3)\) can not determine $F(\beta)$, see Theorem \((1.2)\) below. So we still need the explicit expression for $F(\beta)$ beyond $\beta = 1, 2, 4$. Following the pioneering work of Dumitriu and Edelman \([7]\), Bloemendal and Virág \([2]\) finally found out a representation of $F(\beta)$ in terms of the solution of a linear partial differential equation(PDE). They represent $F(\beta)$ by the limit of $F(\beta; x; t)$

$$F_{\beta}(t) = \lim_{x \to \infty} F(\beta; x; t),$$

where $F(\beta; x; t)$ is a special solution of the linear PDE

$$\frac{\partial F}{\partial t} + \frac{2}{\beta} \frac{\partial^2 F}{\partial x^2} + (t - x^2) \frac{\partial F}{\partial x} = 0.$$  

More precisely, they proved the following theorem.
Theorem 1.1. \[ PDE \ (1.6) \] with boundary
\begin{equation}
(1.7)
\begin{cases}
F(\beta; x, t) \xrightarrow{\beta \to \infty, \ t \to \infty} 1 \\
F(\beta; x, t) \xrightarrow{\beta \to \infty, \ t \text{ fixed}} 0
\end{cases}
\end{equation}
has a unique bounded smooth solution. \( F_{\beta}(t) \) is represented by the solution through (1.4).

So the remaining problem is to find a bounded solution of the Bloemenand-Virág equation (1.6) with the boundary condition (1.7). In [17], Rumanov proposed a Lax representation of (1.6) for even integer \( \beta \). Let
\[
\Psi_x = \hat{L}\Psi, \quad \Psi_t = \hat{B}\Psi
\]
be Rumanov’s Lax pair, where \( \hat{L} \) and \( \hat{B} \) are 2 \times 2 matrices. \( \Psi \) can denote both a 2 \times 2 non-singular matrix or a 2 \times 1 column vector. Here we assume it is a column vector
\begin{equation}
(1.8)
\Psi(x, t) = \begin{pmatrix} \mathcal{F}(x, t) \\ \mathcal{G}(x, t) \end{pmatrix}.
\end{equation}
The key of Rumanov’s scheme is to let \( \mathcal{F} \) satisfy the rescaled Bloemenand-Virág equation
\begin{equation}
(1.9)
\frac{\beta}{2} \frac{\partial \mathcal{F}}{\partial t} + \frac{\partial^2 \mathcal{F}}{\partial x^2} + (t - x^2) \frac{\partial \mathcal{F}}{\partial x} = 0.
\end{equation}
Combining some other considerations, Rumanov concluded
\begin{equation}
(1.10)
F(\beta; x, t) = \mathcal{F} \left( \frac{\beta}{2} \right)^{\hat{x}} x, \left( \frac{\beta}{2} \right)^{\hat{t}} t \right).
\end{equation}
In [18], Rumanov studied the \( \beta = 6 \) case and expressed \( F_{\beta}(t) \) by an auxiliary function \( \eta(t) \) and the solution of Painlevé II
\begin{equation}
(1.11)
u''(t) = tu(t) + 2u(t)^3,
\end{equation}
which he deduced as the Hastings-McLeod solution [12]. The auxiliary function \( \eta(t) \) satisfies a second order ODE that can be linearized.

Grava, Its, Kapaev and Mezzadri [11] found Rumanov’s Lax pair for \( \beta = 6 \) is gauge equivalent to the Lax pair of Painlevé II. Their gauge transformation is of form
\begin{equation}
(1.12)
\Psi(x, t) = e^{\frac{\hat{x}}{\hat{t}} \kappa(t) \left( \begin{array}{cc} \frac{1+q_2(t)}{2} - \alpha(t) & -1 \\ \frac{1-q_2(t)}{2} & 0 \end{array} \right) \psi(t)^{\sigma_3} \psi_0(x, t),}
\end{equation}
where \( \kappa(t) \) and \( \psi(t) \) are scalar functions, and \( \psi_0(x, t) \) is the 2 \times 1 column vector of the wave function of Painlevé II, and \( \sigma_3 \) is the Pauli matrix \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). They also suggested \( \psi = -\frac{u}{\sqrt{\beta}} \), where \( u \) is the Hastings-McLeod solution of Painlevé II. Then they showed \( q_2(t) \) and \( \alpha(t) \) satisfy the ODE
\begin{equation}
(1.13)
q'_2(t) = 2 \left( \frac{2}{3} \right) q_2 + \left( 1 + q_2 \right) \left( \frac{1}{3} - q_2 \right),
\end{equation}
\begin{equation}
(1.14)
\alpha'(t) = \alpha \left( \frac{2}{3} \alpha + \frac{u' - q_2}{u} \right) - \frac{t}{6} \left( 1 + q_2 \right) - u^2 \left( 1 + q_2 \right) - \frac{u^2}{3} (3 + q_2).
\end{equation}
Moreover, they proved that
\begin{equation}
(1.15)
q_2(t) \xrightarrow{t \to \infty} -1 + o(1), \quad \alpha(t) \xrightarrow{t \to \infty} o(1).
\end{equation}
It is straightforward to verify that
\begin{equation}
(1.16)
q_2(t) = \frac{1}{\sqrt{2}} (-t)^{-3/2} + \frac{21}{8} (-t)^{-3} + 1707 \frac{\sqrt{2}}{64} (-t)^{-9/2} + 49123 \frac{\sqrt{2}}{256} (-t)^{-5} + \cdots,
\end{equation}
\begin{equation}
(1.17)
\alpha(t) = \frac{1}{\sqrt{2}} (-t)^{1/2} - \frac{1}{8} (-t)^{-1} - 37 \frac{\sqrt{2}}{64} (-t)^{-5/2} - 373 \frac{\sqrt{2}}{256} (-t)^{-4} + \cdots
\end{equation}
is an asymptotic solution of Equation (1.13)- (1.14) at \( t = -\infty \).

The open question in [11] contains two parts:
(1) Prove the system (1.13)-(1.14) with (1.15) has a smooth solution on $(-\infty, \infty)$.

(2) Assume (1) succeeds. Prove the solution in (1) has expansions (1.16)-(1.17) at $t = -\infty$.

In this paper we will show that there are a 2-parameter family of solutions of (1.13)-(1.14) that satisfy $q_2 \to -1$ and $\alpha \to 0$. More precisely, at $t = \infty$ these solutions have asymptotics

\begin{align}
q_2(t) &= -1 + Ai(t) \left[ \frac{1}{2} \left( \frac{t}{3^{1/3}} \right) \int_{-\infty}^{t} A_i(s) Ai \left( \frac{s}{3^{1/3}} \right) \, ds - A_i \left( \frac{t}{3^{1/3}} \right) \right] + o \left( e^{-\frac{4t^{1/3}}{3}} \right), \\
\alpha(t) &= -\frac{3^{1/3}}{2} Ai(t) \left[ \frac{1}{2} \left( \frac{t}{3^{1/3}} \right) \int_{-\infty}^{t} A_i(s) Bi \left( \frac{s}{3^{1/3}} \right) \, ds - B_i \left( \frac{t}{3^{1/3}} \right) \right] + o \left( e^{-\frac{4t^{1/3}}{3}} \right),
\end{align}

where $\tilde{M}_2(t) = o \left( Ai(t) \right)$ and $\tilde{N}_2(t) = o \left( A_i(t) \right)$. The detailed expressions of $\tilde{M}_2(t)$ and $\tilde{N}_2(t)$ will be given in Section 5. If $c_1 \neq 0$ or $c_2 \neq 0$, the leading terms in the asymptotics (1.18) and (1.19) are obvious. If $c_1 = 0$ and $c_2 = 0$, the leading terms for the asymptotics of $q_2 + 1$ and $\alpha$ are

\begin{align}
q_2 + 1 &= \frac{4\pi}{3^{1/3}} A_i(t) \left[ B_i \left( \frac{t}{3^{1/3}} \right) \int_{-\infty}^{t} A_i(s) Ai \left( \frac{s}{3^{1/3}} \right) \, ds - A_i \left( \frac{t}{3^{1/3}} \right) \right] + o \left( e^{-\frac{4t^{1/3}}{3}} \right), \\
\alpha &= \frac{2\pi}{3} A_i(t) \left[ A_i' \left( \frac{t}{3^{1/3}} \right) \int_{-\infty}^{t} A_i(s) Bi \left( \frac{s}{3^{1/3}} \right) \, ds - B_i' \left( \frac{t}{3^{1/3}} \right) \right] + o \left( e^{-\frac{4t^{1/3}}{3}} \right).
\end{align}

Not all solutions with asymptotics of (1.18) and (1.19) at $t = \infty$ can be smoothly evolved to $t = -\infty$. It may develop to singularity at $t = t_0$, which depends on $c_1$ and $c_2$. Our first main result of this paper is the following.

**Theorem 1.2.** There is a region $R_{\text{smooth}}$ that is the neighbourhood of the positive $c_2$-axis including the origin $(0,0)$ in the $(c_1, c_2)$-plane, such that if $(c_1, c_2)$ is in the region $R_{\text{smooth}}$ then the solution defined by the asymptotics of (1.18) at $t = \infty$ is smooth for $t \in (-\infty, \infty)$ and has asymptotics of (1.16)-(1.17) at $t = -\infty$.

In fact, the region $R_{\text{smooth}}$ is very large. The numerical results for $R_{\text{smooth}}$ are shown in Figure 1.

![Figure 1](image_url)

Figure 1. $R_{\text{smooth}}$ and $R_{\text{singular}}$. $R_{\text{smooth}}$ is the light green region. If $(c_1, c_2)$ belongs to $R_{\text{smooth}}$, the solution defined at $t = \infty$ by this $(c_1, c_2)$ is smooth on $(-\infty, \infty)$. Else if $(c_1, c_2)$ belongs to $R_{\text{singular}}$ (the light yellow region), the corresponding solution must have singularity at some finite $t = t_0$. The red curve is the boundary between $R_{\text{smooth}}$ and $R_{\text{singular}}$. $P_c$ is a special point on the boundary curve: the boundary curve becomes straight on the right of $P_c$.

Theorem 1.2 gives positive answers to the open questions (1) and (2) of [11]. But the non-uniqueness of $q_2 = q_2(c_1, c_2; t)$ causes the non-uniqueness of $F_0(t)$. In fact, by formula (1.17)

\begin{equation}
F_0 \left( \frac{t}{3^{1/3}} \right) = \frac{q_2 - 1}{2q_2} \exp \left( - \frac{1}{3} \int_{-\infty}^{t} \omega(s) \, ds + \frac{2}{3} \int_{-\infty}^{t} \frac{u'(s)}{u(s)} \frac{1 + q_2(s)}{q_2(s)} \, ds \right),
\end{equation}

where $\omega(s)$ is the function defined in (1.21).
where \( \omega(s) = u(s)^4 + su(s)^2 - u'(s)^2 \), we can verify that \( F_6(t) \) is indeed dependent on \( c_1 \) and \( c_2 \). So we have to determine the values of \( c_1 \) and \( c_2 \) to guarantee there is only a unique \( F_6(t) \). To determine \( c_1 \) and \( c_2 \), we rely on Theorem 1.1. Grava et al. \([11]\) have formulated \( F(\beta = 6; x, t) \) from \( q_2 \) and \( \alpha \) as

\[
\begin{align*}
(1.23) \quad & F \left( \beta = 6; \frac{t}{3^{2/3}} \right) = \kappa u^2 \left[ \frac{1}{2} + \frac{q_2}{2} x - \alpha \right] Y_{12}^{(6)}(x, t) + Y_{22}^{(6)}(x, t), \quad x \geq 0, \\
(1.24) \quad & F \left( \beta = 6; \frac{t}{3^{2/3}} \right) = -\kappa u^2 e^{-\frac{3}{2}x-t} \left[ \frac{1}{2} + \frac{q_2}{2} x - \alpha \right] Y_{11}^{(3)}(x, t) + Y_{21}^{(3)}(x, t), \quad x \leq 0,
\end{align*}
\]

where \( Y^{(3)} \) and \( Y^{(6)} \) are \( 2 \times 2 \) matrices of the wave function of the Painlevé II. By \([1.23]-[1.24]\), \( F(\beta = 6; x, t) \) contains parameters \( c_1 \) and \( c_2 \). By \([1.13]-[1.14]\) and \([1.23]-[1.24]\), we can prove \( c_1 = 0 \) is enough to guarantee the boundary condition \([1.7]\). But if \( c_2 \neq 0 \), \( F(\beta = 6; \frac{t}{3^{2/3}}, \frac{t}{3^{2/3}}) \) will grow exponentially near the line \( x = -\sqrt{t} \) for \( t \to \infty \).

The second main result of this paper is the following:

**Theorem 1.3.** If and only if \( c_1 = c_2 = 0 \), the resulting \( F(\beta = 6; x, t) \) is bounded at the boundary \( x^2 + t^2 = \infty \).

Now all requirements of Theorem 1.1 are satisfied: \( F(\beta = 6; x, t) \) given by \([1.23]-[1.24]\) satisfies the Bloemendal-Virág equation \([1.6]\); \( c_1 = 0 \) guarantees it satisfies the boundary condition \([1.7]\); Theorem 1.3 guarantees it is a bounded solution. So \( F(\beta = 6; x, t) \) is indeed given by \([1.23]\) and \([1.24]\) with \( c_1 = c_2 = 0 \). We also note that \( F_6(t) \) is given by \([1.22]\) with the hidden parameters \( c_1 = c_2 = 0 \).

## 2 Derivation of the ODEs of \( q_2, \alpha \) and \( \kappa \)

The Flaschka-Newell Lax pair of Painlevé II is \([9]\)

\[
\begin{align*}
(2.1) \quad & \frac{d\psi_0}{dx} = \hat{L}_0 \psi_0, \\
(2.2) \quad & \frac{d\psi_0}{dt} = \hat{B}_0 \psi_0
\end{align*}
\]

where

\[
\begin{align*}
(2.3) \quad & \psi_0 = \left( \begin{array}{c} F_0(x, t) \\ G_0(x, t) \end{array} \right), \\
(2.4) \quad & \hat{L}_0 = \frac{x^2}{2} \sigma_3 + x \left( \begin{array}{cc} 0 & u(t) \\ u(t) & 0 \end{array} \right) + \left( \begin{array}{cc} -\frac{\kappa}{2} - u(t)^2 & -u'(t) \\ u'(t) & \frac{\kappa}{2} + u(t)^2 \end{array} \right), \\
(2.5) \quad & \hat{B}_0 = -\frac{x}{2} \sigma_3 - \left( \begin{array}{cc} 0 & u(t) \\ u(t) & 0 \end{array} \right).
\end{align*}
\]

By \([1.8]\) and \([1.12]\), we get

\[
(2.6) \quad \mathcal{F}(x, t) = \frac{1}{2} e^{\frac{x^2}{2} - \frac{t}{2}} (q_2(t)x + x - 2\alpha(t)\kappa(t))(\nabla_0(x, t) - e^{\frac{x^2}{2} - \frac{t}{2}} \kappa(t) G_0(x, t).
\]

Grava et. al suggested

\[
(2.7) \quad \psi = -\frac{i}{\sqrt{u}}
\]

Substituting \([2.3]\) and \([2.7]\) into \([1.9]\) with \( \beta = 6 \), we immediately obtain \([1.13]-[1.14]\) and

\[
(2.8) \quad \frac{\kappa'}{\kappa} = -\frac{2}{3} \alpha - \frac{1}{3} (t + u^2)u' + \frac{u'}{6u} (2q_2 - 1) + \frac{1}{3} (u')^2.
\]

By requiring \( \mathcal{F}(x, t) \xrightarrow{x \to -\infty, t \to -\infty} 1 \), Grava et. al \([11]\) proved \([1.15]\) and

\[
(2.9) \quad \kappa u^2 \xrightarrow{t \to -\infty} 1.
\]

Equation \([2.8]\) with boundary condition \([2.9]\) determine \( \kappa \) completely if a smooth solution of \( (\alpha, q_2) \) has been obtained under the boundary condition \([1.15]\), more precisely, the asymptotics of \([1.15]-[1.19] \).
The expansion of $\kappa(t)$ at $t = -\infty$ can also be obtained. By (2.8), (1.16)–(1.17) and the asymptotics of Hastings-McLeod solution $u$, the asymptotics of $\frac{\kappa'(t)}{\kappa(t)}$ at $t = -\infty$ is obtained as

$$\frac{\kappa'(t)}{\kappa(t)} = \frac{t^2}{12} - \frac{\sqrt{2}}{3} (-t)^{3/2} + \frac{5}{24} (-t)^{-1} + \frac{7}{32\sqrt{2}} (-t)^{-3/2} + \cdots.$$

Therefore

$$(2.10) \quad \kappa(t) = C_\kappa \times e^{-\frac{t^2}{12} + \frac{\sqrt{2}}{3} (-t)^{3/2} - \frac{5}{24} (-t)^{-1} + \frac{7}{32\sqrt{2}} (-t)^{-3/2} + \cdots}.$$

Assuming (1.3), we get

$$(2.11) \quad \ln C_\kappa = c_{\beta = 6} - \frac{\ln 3}{36} + \frac{5}{4} \ln 2.$$

For $\beta = 6$, Borot et al. [3] was able to simplify (2.11) to

$$(2.12) \quad c_{\beta = 6} = \frac{97}{72} \ln 2 - \frac{7}{36} \ln 3 - \frac{\ln(2\pi)}{6} + \frac{\ln \Gamma(\frac{1}{3})}{3} + \frac{\zeta'(-1)}{3}.$$

So we have

$$(2.13) \quad \ln C_\kappa = -\frac{7}{72} \ln 2 - \frac{2}{9} \ln 3 - \frac{\ln(2\pi)}{6} + \frac{\ln \Gamma(\frac{1}{3})}{3} + \frac{\zeta'(-1)}{3}.$$

The value of $\ln C_\kappa$ can be obtained by numerical experiments similar to the ones in Section 7 with $c_1 = c_2 = 0$. Our numerical experiments give

$$(2.14) \quad \ln C_\kappa = -0.3445050500286934815501994065702518 \cdots.$$

In fact, $\ln C_\kappa$ from our numerical experiments coincides with (2.13) for more than 100 digits, which gives a numerical verification of (1.3) and (1.4) for $\beta = 6$.

Altogether, the algorithm is as following. First give the ansatz for $F(x,t)$ as (2.6); then by (1.9) obtain the ODEs for the unknowns; next by the boundary condition for $F(x,t)$ get all boundary conditions for the unknowns, which should determine all unknowns uniquely; at last prove the obtained $F(x,t)$ satisfies all the requirements for it.

### 3 Asymptotics of $q_2$ and $\alpha$ at $t = -\infty$

In this section we will show by linearization analysis that the asymptotics (1.16)–(1.17) are not the asymptotics of a specific solution of (1.11)–(1.14), but of a general solution of (1.11)–(1.14). More detailed analysis of these asymptotics will be given in Section 6.

Suppose $(q_{20}(t), \alpha_{0}(t))$ is a smooth solution of (1.11)–(1.14) with asymptotics of (1.11)–(1.17). Let $(q_2(t), \alpha(t))$ be a solution of (1.11)–(1.14) near $(q_{20}(t), \alpha_{0}(t))$. Then $(q_2(t), \alpha(t))$ can be expressed as

$$q_2(t) = q_{20}(t) + \epsilon \mathcal{Q}(t),$$

$$\alpha(t) = \alpha_0(t) + \epsilon \mathcal{A}(t),$$

where $\epsilon \to 0$ is infinitesimal.

So $(\mathcal{Q}, \mathcal{A})$ satisfies the ODE

$$\mathcal{Q}'(t) = \frac{2}{3} \mathcal{Q}_0 + \frac{u' - 2q_{20}}{u^3} \mathcal{Q} + \frac{2}{3} q_{20} \mathcal{A},$$

$$\mathcal{A}'(t) = -\frac{1}{3} \left( \frac{u'}{u} \mathcal{Q}_0 + \frac{t}{2} + u^2 \right) \mathcal{Q} + \frac{4}{3} \mathcal{Q}_0 + \frac{u' - 2q_{20}}{u^3} \mathcal{A}.$$

At $t = -\infty$, the expansions of $u$, $\frac{u'}{u}$, $\alpha_0$ and $q_{20}$ are known

$$u = \sqrt{-\frac{t}{2}} + \cdots, \quad \frac{u'}{u} = \frac{1}{2t} + \cdots, \quad q_{20} = \frac{1}{\sqrt{2}} (-t)^{-\frac{3}{2}} + \cdots, \quad \alpha_0 = \sqrt{-\frac{t}{2}} + \cdots.$$
Therefore, \( \mathcal{Q} \) and \( \mathcal{A} \) satisfy

\[
\mathcal{Q}'(t) = \left( \frac{\sqrt{-2t}}{3} + \cdots \right) \mathcal{Q} + \left( \frac{\sqrt{t}}{3} (-t)^{-\frac{1}{2}} + \cdots \right) \mathcal{A},
\]

\[
\mathcal{A}'(t) = \left( \frac{1}{6} (-2t)^{-\frac{1}{2}} + \cdots \right) \mathcal{A} + \left( \frac{2\sqrt{-2t}}{3} + \cdots \right) \mathcal{A}.
\]

Now it is clear that the solution \( (q_2, \alpha) \) is exponentially close to the solution \( (q_{20}, \alpha_0) \) in an order of \( e^{-\frac{2\sqrt{t}}{3}(-t)^{1/2}} \).

So we reach the following result.

**Theorem 3.1.** If a solution \( (q_{20}(t), \alpha_0(t)) \) is nonsingular on \((-\infty, t_0] \) and has asymptotics [1.10]-[1.17], then the general solutions that are close to \( (q_{20}(t), \alpha_0(t)) \) are all non-singular on \((-\infty, t_0] \) and also have asymptotics [1.10]-[1.17] at \( t = -\infty \).

### 4 The linear variables and the integral equations

At \( t = \infty \), it is convenient to work with \( \tilde{q}_2 \) and \( \alpha \)

\[
\tilde{q}_2 = 1 + q_2.
\]

Also it is helpful to remember

\[
u(t) = \text{Ai}(t) + \left( \frac{1}{32\pi^{3/2}} t^{-\frac{1}{2}} + \cdots \right) e^{-\frac{2}{3}t^{3/2}} + \cdots.
\]

The ODEs for \( \tilde{q}_2 \) and \( \alpha \) are

\[
\tilde{q}_2'(t) = \frac{2}{3} (\tilde{q}_2 - 1) + \frac{u'(3 - \tilde{q}_2 - \tilde{q}_2'),}{3},
\]

\[
\alpha'(t) = \left( \frac{2}{3} \alpha + \frac{u'(3 - \tilde{q}_2 - \tilde{q}_2)}{3} \right) \alpha - \frac{t}{6} \tilde{q}_2 - \frac{2 + \tilde{q}_2}{3} \alpha^2.
\]

Equations [4.2]-[4.3] are linearized by

\[
\tilde{q}_2(t) = \frac{\phi_1(t)}{\phi_0(t)} u(t), \quad \alpha(t) = \frac{\phi_2(t)}{\phi_0(t)} u(t),
\]

where \( \phi_1, \phi_2 \) and \( \phi_0 \) satisfy

\[
\phi_1'(t) = -\frac{2}{3} \phi_2(t),
\]

\[
\phi_2'(t) = -\frac{2}{3} u(t)\phi_0(t) - \frac{1}{6}(t + 2u(t)^2)\phi_1(t),
\]

\[
\phi_0'(t) = \frac{1}{3} u'(t)\phi_1(t) - \frac{2}{3} u(t)\phi_2(t).
\]

To analyze [4.5]-[4.7], we need the following estimations.

**Proposition 4.1.** There exists \( t_0^P \), such that for \( t \geq t_0^P \):

- \( |\text{Bi}(t)| < e^{\frac{2}{3}t^{3/2}} \) and \( |\text{Bi}'(t)| < e^{\frac{2}{3}t^{3/2}} \sqrt{t} \);
- \( |\text{Ai}(t)| < e^{-\frac{2}{3}t^{3/2}} \) and \( |\text{Ai}'(t)| < e^{-\frac{2}{3}t^{3/2}} \sqrt{t} \);
- \( |u(t)| < e^{-\frac{2}{3}t^{3/2}} \) and \( |u'(t)| < e^{-\frac{2}{3}t^{3/2}} \sqrt{t} \).

The proof is routine. Thus we omit it.

**Remark 4.2.** For example, we can take \( t_0^P = 1 \). By numerical results, \( t_0^P = 1 \) satisfies all requirements for \( t_0^P \) in Proposition 4.1. We assume \( t_0^P \) is sufficiently large. Also, the actual value of \( t_0^P \) is never needed in the following proofs.
4.1 The three independent solutions defined near $t = \infty$

For convenience, denote

$$g_1(t) = Bi\left(\frac{t}{3^{2/3}}\right), \quad g_2(t) = Ai\left(\frac{t}{3^{2/3}}\right), \quad g_3(t) = 0.$$  

We will show, for $k = 1, 2, 0$, the following integral equations

$$\Phi_{1k}(t) = g_k(t) + \frac{2\pi}{3\sqrt{3}} \int_{-\infty}^{t} Bi\left(\frac{s}{3^{2/3}}\right) \left( -\frac{2}{3} u(s) \Phi_{0k}(s) - \frac{1}{3} u(s)^2 \Phi_{1k}(s) \right) ds$$

(4.8)

$$\Phi_{2k}(t) = -\frac{3}{2} g_k'(t) - \pi \int_{-\infty}^{t} Bi\left(\frac{s}{3^{2/3}}\right) \left( -\frac{2}{3} u(s) \Phi_{0k}(s) - \frac{1}{3} u(s)^2 \Phi_{1k}(s) \right) ds$$

(4.9)

$$\Phi_{0k}(t) = \delta_k^0 + \int_{-\infty}^{t} \left( \frac{1}{3} u'(s) \Phi_{1k}(s) - \frac{2}{3} u(s) \Phi_{2k}(s) \right) ds.$$  

(4.10)

define three independent solutions for (1.5)-(1.7), i.e., $(\phi_1(t), \phi_2(t), \phi_0(t)) = (\Phi_{1k}(t), \Phi_{2k}(t), \Phi_{0k}(t))$, $k = 1, 2, 0$, solves (1.5)-(1.7).  

For $k = 1, 2, 0$, define

$$\Phi_{1k}^{(0)}(t) = (1 - \delta_k^0) g_k(t),$$

(4.11)

$$\Phi_{2k}^{(0)}(t) = \frac{3}{2} (1 - \delta_k^0) g_k'(t),$$

(4.12)

$$\Phi_{0k}^{(0)}(t) = \delta_k^0 + \int_{-\infty}^{t} \left( \frac{1}{3} u'(s) \Phi_{1k}^{(0)}(s) - \frac{2}{3} u(s) \Phi_{2k}^{(0)}(s) \right) ds,$$  

(4.13)

where $\delta_k^0 = \left\{ \begin{array}{ll}
1, & k = 0 \\
0, & k \neq 0.
\end{array} \right.$

For $j > 0$, define

$$\Phi_{1k}^{(j+1)}(t) = g_k(t) + \frac{2\pi}{3\sqrt{3}} \int_{-\infty}^{t} Bi\left(\frac{s}{3^{2/3}}\right) \left( -\frac{2}{3} u(s) \Phi_{0k}^{(j)}(s) - \frac{1}{3} u(s)^2 \Phi_{1k}^{(j)}(s) \right) ds$$

(4.14)

$$\Phi_{2k}^{(j+1)}(t) = -\frac{3}{2} g_k'(t) - \pi \int_{-\infty}^{t} Bi\left(\frac{s}{3^{2/3}}\right) \left( -\frac{2}{3} u(s) \Phi_{0k}^{(j)}(s) - \frac{1}{3} u(s)^2 \Phi_{1k}^{(j)}(s) \right) ds$$

(4.15)

$$\Phi_{0k}^{(j+1)}(t) = \delta_k^0 + \int_{-\infty}^{t} \left( \frac{1}{3} u'(s) \Phi_{1k}^{(j+1)}(s) - \frac{2}{3} u(s) \Phi_{2k}^{(j+1)}(s) \right) ds.$$  

(4.16)

Proposition 4.3. For $t \geq t_0^p$,

$$|\Phi_{1k}^{(j+1)}(t) - \Phi_{1k}^{(j)}(t)| < C_{1k}^{(j+1)} e^{-(\frac{\sqrt{3} + k}{2} + \frac{3}{4})^{3/2}}, \quad |\Phi_{2k}^{(j+1)}(t) - \Phi_{2k}^{(j)}(t)| < C_{2k}^{(j+1)} t^{\frac{3}{2}} e^{-(\frac{3}{2} + \frac{3}{4} k + \frac{3}{4})^{3/2}},$$

$$|\Phi_{0k}^{(j+1)}(t) - \Phi_{0k}^{(j)}(t)| < C_{0k}^{(j+1)} e^{-(\frac{\sqrt{3} + k}{2} + \frac{3}{4})^{3/2}},$$

where

$$C_{11}^{(j)} = 2C_{21}^{(j)} = \pi^j 3^{\frac{j-1}{2}} \prod_{i=1}^{j} \frac{6i - 1}{(3i - 1)(3i - 2)}, \quad C_{01}^{(j)} = \pi^j 3^{\frac{j-1}{2}} \prod_{i=1}^{j} \frac{6i - 1}{(3i - 1)(3i + 1)},$$

$$C_{12}^{(j)} = 2C_{22}^{(j)} = \pi^j 3^{\frac{j-1}{2}} \prod_{i=1}^{j} \frac{6i + 1}{3i(3i - 1)}, \quad C_{02}^{(j)} = \frac{1}{2} \pi^j 3^{\frac{j-1}{2}} \prod_{i=1}^{j} \frac{6i + 1}{3i(3i + 2)},$$

$$C_{10}^{(j)} = 2C_{20}^{(j)} = 3\pi^j 3^{\frac{j-1}{2}} \prod_{i=1}^{j} \frac{2i - 1}{3i - 2}, \quad C_{00}^{(j)} = \pi^j 3^{\frac{j-1}{2}} \prod_{i=1}^{j} \frac{2i - 1}{3i - 2}.$$
Theorem 4.4. \((\Phi^{(j)}_{1k}, \Phi^{(j)}_{2k}, \Phi^{(j)}_{0k})\), \(k = 1, 2, 0\), defined by (4.11) - (4.16) converge to the solutions of (4.5) - (4.7) for \(t \geq t_0^p\).

5 Asymptotics of \(q_2\) and \(\alpha\) at \(t = \infty\)

At \(t = \infty\), it is straightforward to verify

(5.1) \(\Phi_{11}(t) = \text{Bi}(\frac{t}{3^{2/3}}) + O(\text{e}^{-\frac{4\pi}{3}t^{3/2}}),\)

(5.2) \(\Phi_{21}(t) = -\frac{3^{1/3}}{2} \text{Bi}'(\frac{t}{3^{2/3}}) + O(\sqrt{t} e^{-\frac{4\pi}{3}t^{3/2}}),\)

(5.3) \(\Phi_{01}(t) = \int_{\infty}^{t} \left(\frac{1}{3} \text{Ai}'(s) \text{Bi}(\frac{s}{3^{2/3}}) + \frac{1}{3^{1/3}} \text{Ai}(s) \text{Bi}'(\frac{s}{3^{2/3}})\right) ds + O(\text{e}^{-\frac{4\pi}{3}t^{3/2}}),\)

(5.4) \(\Phi_{12}(t) = \text{Ai}(\frac{t}{3^{2/3}}) + O(\text{e}^{-\frac{4\pi}{3}t^{3/2}}),\)

(5.5) \(\Phi_{22}(t) = -\frac{3^{1/3}}{2} \text{Ai}'(\frac{t}{3^{2/3}}) + O(\sqrt{t} e^{-\frac{4\pi}{3}t^{3/2}}),\)

(5.6) \(\Phi_{02}(t) = \int_{\infty}^{t} \left(\frac{1}{3} \text{Ai}'(s) \text{Ai}(\frac{s}{3^{2/3}}) + \frac{1}{3} \text{Ai}(s) \text{Ai}'(\frac{s}{3^{2/3}})\right) ds + O(\text{e}^{-\frac{4\pi}{3}t^{3/2}}),\)

(5.7) \(\Phi_{10}(t) = -\frac{4\pi}{3^{2/3}} \left(\text{Ai}(\frac{t}{3^{2/3}}) \int_{\infty}^{t} \text{Bi}(\frac{s}{3^{2/3}}) \text{Ai}(s) ds - \text{Bi}(\frac{t}{3^{2/3}}) \int_{\infty}^{t} \text{Ai}(\frac{s}{3^{2/3}}) \text{Ai}(s) ds\right) + O(\text{e}^{-2t^{3/2}}),\)

(5.8) \(\Phi_{20}(t) = \frac{2\pi}{3} \left(\text{Ai}(\frac{t}{3^{2/3}}) \int_{\infty}^{t} \text{Bi}(\frac{s}{3^{2/3}}) \text{Ai}(s) ds - \text{Bi}(\frac{t}{3^{2/3}}) \int_{\infty}^{t} \text{Ai}(\frac{s}{3^{2/3}}) \text{Ai}(s) ds\right) + O(\sqrt{t} e^{-2t^{3/2}}),\)

(5.9) \(\Phi_{00}(t) = 1 + \int_{\infty}^{t} \left(\frac{1}{3} \text{Ai}'(s) \Phi_{10}^{(1)}(s) - \frac{2}{3} \text{Ai}(s) \Phi_{20}^{(1)}(s)\right) ds + O(\text{e}^{-\frac{4\pi}{3}t^{3/2}}),\)

where \(\Phi_{10}^{(1)}\) and \(\Phi_{20}^{(1)}\) are defined by dropping the error terms of (5.7) and (5.8) respectively.

Lemma 5.1. The asymptotics of a solution of (4.12) - (4.15) at \(t = \infty\) must belong to one of the following three classes.

Class A: \(\tilde{q}_2(t) \xrightarrow{t \to \infty} 0\) and \(\alpha(t) \xrightarrow{t \to \infty} 0\).

(5.10) \(\tilde{q}_2(t) = \left(c_1 \text{Bi}(\frac{t}{3^{2/3}}) + c_2 \text{Ai}(\frac{t}{3^{2/3}}) + c_1 \tilde{M}_2(t) e^{-\frac{4\pi}{3}t^{3/2}}\right) \text{Ai}(t) + o(\text{e}^{-\frac{4\pi}{3}t^{3/2}}),\)

(5.11) \(\alpha(t) = -\frac{3^{1/3}}{2} \left(c_1 \text{Bi}'(\frac{t}{3^{2/3}}) + c_2 \text{Ai}'(\frac{t}{3^{2/3}}) + c_1 \tilde{N}_2(t) e^{-\frac{4\pi}{3}t^{3/2}}\right) \text{Ai}(t) + o(\text{e}^{-\frac{4\pi}{3}t^{3/2}}),\)

where

\(\tilde{M}_2(t) = -\text{Bi}(\frac{t}{3^{2/3}}) \int_{\infty}^{t} \left(\frac{1}{3} \text{Ai}'(s) \text{Bi}(\frac{s}{3^{2/3}}) + \frac{1}{3} \text{Ai}(s) \text{Bi}'(\frac{s}{3^{2/3}})\right) ds,\)

\(\tilde{N}_2(t) = -\text{Bi}'(\frac{t}{3^{2/3}}) \int_{\infty}^{t} \left(\frac{1}{3} \text{Ai}'(s) \text{Bi}(\frac{s}{3^{2/3}}) + \frac{1}{3} \text{Ai}(s) \text{Bi}'(\frac{s}{3^{2/3}})\right) ds.\)

Class B: \(\tilde{q}_2(t) \xrightarrow{t \to \infty} \infty\) and \(\alpha(t) \xrightarrow{t \to \infty} -\infty.\)

(5.13) \(\tilde{q}(t) = \left[\text{Bi}(\frac{t}{3^{2/3}}) + c_2 \left(\text{Ai}(\frac{t}{3^{2/3}}) - \text{Bi}(\frac{t}{3^{2/3}}) \Phi_{02}^{(0)}(t)\right) \Phi_{01}^{(0)}(t)\right] \text{Ai}(t) + o(\text{e}^{-\frac{4\pi}{3}t^{3/2}}),\)

(5.14) \(\alpha(t) = -\frac{3^{1/3}}{2} \left[\text{Bi}'(\frac{t}{3^{2/3}}) + c_2 \left(\text{Ai}'(\frac{t}{3^{2/3}}) - \text{Bi}'(\frac{t}{3^{2/3}}) \Phi_{02}^{(0)}(t)\right) \Phi_{01}^{(0)}(t)\right] \text{Ai}(t) + o(\text{e}^{-\frac{4\pi}{3}t^{3/2}}).\)

Class C: \(\tilde{q}(t) \xrightarrow{t \to \infty} 2\) and \(\alpha(t) \xrightarrow{t \to \infty} \sqrt{7}.\)
Proof. The general solution of (4.5)-(4.7) is \((\phi_1, \phi_2, \phi_0) = (c_1 \Phi_{11} + c_2 \Phi_{12} + c_3 \Phi_{10}, c_1 \Phi_{21} + c_2 \Phi_{22} + c_3 \Phi_{20}, c_1 \Phi_{01} + c_2 \Phi_{02} + c_3 \Phi_{00})\). So \(\bar{q}_2 = c_1 \Phi_{11} + c_2 \Phi_{12} + c_3 \Phi_{10}\) and \(\alpha = c_1 \Phi_{21} + c_2 \Phi_{22} + c_3 \Phi_{20}\). If \(c_0 \neq 0\), \(c_0\) can be taken as 1. By (5.1)-(5.9), we have \(\bar{q}_2 = (c_1 \Phi_{11} + c_2 \Phi_{12} + o(e^{-\frac{t^2}{2}}))(1 - c_1 \Phi_{01} + o(e^{-\frac{t^2}{2}}))u = (c_1 \Phi_{11} + c_2 \Phi_{12} - c_1^2 \Phi_{11} \Phi_{01} + o(e^{-\frac{t^2}{2}}))u\). Also considering \(u(t) = A(t) + o(e^{-2t^2})\) and (5.1)-(5.9), we finally get (5.10). We can verify directly \(\bar{M}_2(t) = o(A(t))\). Similarly, (5.11) is obtained. Therefore, the Class A describes the asymptotics of \(\bar{q}_2(t)\) and \(\alpha(t)\) at \(t = \infty\) for \(c_0 \neq 0\). If \(c_0 = 0\) and \(c_1 \neq 0\), \(c_1\) can be taken as 1. Then we can prove the asymptotics belong to Class B in this case. At last, if \(c_0 = c_1 = 0\), \(c_2\) can be taken as 1. Class C describes the asymptotics of this case. \(\square\)

**Proposition 5.2.** A solution of (1.13)-(1.14), which has property (1.15), must have asymptotics (1.18)-(1.21).

Proof. Let \((\phi_1, \phi_2, \phi_0)\) and \((c_1, c_2, c_0)\) be the ones defined in the proof of Lemma 5.1. By (1.13) and \(\bar{q}_2 = q_2 + 1\), we know \(\bar{q}_2(t) \xrightarrow{t \to \infty} 0\) and \(\alpha(t) \xrightarrow{t \to \infty} 0\). This is the Class A case. The only remaining problem is to verify (1.20)-(1.21) for \(c_1 = c_2 = 0\). It is straightforward to verify this by (5.7)-(5.9). \(\square\)

**Remark 5.3.** The error terms of (1.20)-(1.21) are not optimal. In fact, the error terms can be shown to be \(O(t^{-\frac{1}{2}}e^{-\frac{1}{4}t^2})\) and \(O(t^{-\frac{1}{2}}e^{-\frac{1}{8}t^2})\) respectively by a tedious calculation from (4.11)-(4.16).

### 6 Proof of Theorem 1.2

**Proposition 6.1.** There exists a minimal \(k_0 > 0\) so as to \(t + k_0 u(t)^2 \geq 0\) for all \(t\).

Proof. It is obvious \(k_0 > 0\). Let \(f(t) = t + 2u(t)^2\). \(f(t)\) has minimum since \(f(0) > 0\), \(f(-\infty) = 0\) and \(f(t) < 0\) for large negative enough \(t\). Since \(u(t) \neq 0\) for \(t < \infty\), there exists \(k\) such that \(t + 2u(t)^2 + ku(t)^2 \geq 0\) for all \(t\). Obviously, such \(k\) has minimum. \(\square\)

**Remark 6.2.** By the preceding proof, we know \(k_0 > 2\). To calculate \(k_0\) numerically, we use \(k_0 = -\min(\phi_1)\), by which \(k_0\) is calculated up to more than 100 digits. Though \(k_0\) is so accurately known, its “closed form” is still unknown. \(k_0 \approx 2.1228589561253469\) is achieved at \(t \approx -1.18811911480738777\). In the following, \(k_0 < \frac{10}{\pi}\) is needed. The proof of \(k_0 < \frac{10}{\pi}\) is somewhat technical and digressed, so we put it in Appendix [A].

A visualized estimation of \(k_0\) is given by Figure 2.

![Figure 2](image_url)

Figure 2. Estimate \(k_0\) by the graph of \(\phi_1\). The green curve is the plot of \(\phi_1\) and the horizontal red line is \(y = -2.2\). By the graph, it is obvious \(\min(\phi_1) > -2.2\), i.e., \(k_0 < 2.2\).

**Lemma 6.3.** If a solution of (4.5)-(4.7) has properties \(\phi_1(t_0) > 0\), \(\phi_2(t_0) > 0\) and \(\phi_0(t_0) > 0\) and \(\frac{d}{dt} \phi_0(t_0) - \frac{d^2}{dt^2} u(t_0) \phi_1(t_0) > 0\), then

1. \(\phi_1(t)\), \(\phi_2(t)\), \(\phi_0(t)\) and \(\frac{d}{dt} \phi_0(t) - \frac{d^2}{dt^2} u(t) \phi_1(t)\) are all monotonic decreasing on \((-\infty, t_0]\):
2. \(\phi_1(t) \xrightarrow{t \to \infty} \infty\), \(\phi_2(t) \xrightarrow{t \to \infty} \infty\) and \(\phi_0(t) \xrightarrow{t \to \infty} \infty\).
3. \(\lim_{t \to \infty} \frac{\phi_1(t)}{\phi_0(t)} = 1\), \(\lim_{t \to \infty} \frac{\phi_2(t)}{\phi_0(t)} = 1\).
Correspondingly, near \( t = -\infty \),

\[
q(t) = o(1), \quad \alpha(t) = \sqrt{-\frac{t}{2}} + o((-t)^{\frac{1}{2}}).
\]

Proof. (1) \( \phi'_1(t_0) = -\frac{2}{3} \phi_2(t_0) < 0 \).
\( \phi'_2(t_0) = -\frac{2}{3} u(t_0) \phi_0(t_0) - \frac{k}{6} (t_0 + k_0 u(t_0)^2) \phi_1(t_0) + \frac{1}{6} (k_0 - 2) u(t_0)^2 \phi_1(t_0) < 0 \).
\( \phi'_0(t_0) = \frac{2}{3} u(t_0) \phi_1(t_0) - \frac{2}{3} u(t_0) \phi_2(t_0) < 0 \).
\( \frac{d}{dt} \left( \frac{2}{3} \phi_0(t_0) - \frac{k_0 - 2}{6} u(t) \phi_1(t) \right) = \frac{2}{3} \left( \frac{1}{3} u'(t) \phi_1(t_0) - \frac{2}{3} u(t) \phi_2(t_0) \right) - \frac{k_0 - 2}{6} \phi_1(t_0) \phi_1(t) - \frac{k_0 - 2}{6} u(t_0) \left( \frac{2}{3} \phi_2(t_0) \right) \)
\( = \frac{10 - 3k_0}{18} u'(t) \phi_0(t_0) - \frac{6 - k_0}{9} u(t) \phi_2(t_0) \).

By Remark [6.2], \( k_0 < \frac{4k}{3} \). Therefore, \( \frac{d}{dt} \left( \frac{2}{3} \phi_0(t_0) - \frac{k_0 - 2}{6} u(t_0) \phi_1(t_0) \right) < 0 \). So we have \( \phi_1(t_0) - \epsilon > \phi_1(t_0) > 0, \phi_2(t_0) > 0, \phi_0(t_0) - \epsilon > \phi_3(t_0) > 0 \) and \( \frac{d}{dt} \phi_0(t) - \frac{k_0 - 2}{6} u(t) \phi_1(t) |_{t=t_0} > \frac{2}{3} \phi_0(t) - \frac{k_0 - 2}{6} u(t) \phi_1(t) |_{t=t_0} > 0 \). This process can be repeated endlessly. So the first statement of the lemma is proved.

(2) By the preceding proof, \( \phi_1(t) > 0, \phi_2(t) > 0, \phi'_1(t) = -\frac{2}{3} \phi_2(t) < 0 \) and \( \phi'_2(t) = -\frac{1}{6} (t + k_0 u(t)^2) \phi_1(t) \leq 0 \). So both \( \phi_1(t) \) and \( \phi_2(t) \) grow exponentially to infinity as \( t \to -\infty \). By \( \phi'_0(t) = \frac{2}{3} u'(t) \phi_1(t_0) - \frac{2}{3} u(t) \phi_2(t_0), \phi_0(t) \to \infty \) as \( t \to -\infty \) is got.

(3) Let \( x_1 = \lim_{t \to -\infty} \frac{\phi_1(x)}{\phi_0(x)} \) and \( x_2 = \lim_{t \to -\infty} \frac{\phi_2(x)}{\phi_0(x)} \). We apply L’Hospital’s rule to obtain the values of \( x_1 \) and \( x_2 \). It is legal since \( \phi_1(t) u(t) \to \infty, \phi_2(t) \to \infty, \phi_0(t) \to \infty \) and \( \phi_0(t) < 0 \) for \( t < t_0 \). Then by L’Hospital’s rule and \((6.5)-(6.7)\), we obtain the algebraic equations for \( x_1 \) and \( x_2 \)

\[
x_1 = \frac{-\frac{2}{3} \phi_0(t) + \frac{u'(t)}{u(t)} x_1}{\frac{1}{3} u'(t) x_1 - \frac{2}{3} u(t) x_2},
\]
\[
x_2 = \frac{-\frac{1}{6} \phi_0(t)^2 + x_1 - \frac{2}{3} u(t)}{\frac{1}{3} u'(t) x_1 - \frac{2}{3} u(t) x_2}.
\]

Note \( t \) in \((6.3)-(6.4)\) should be understood as \( t \to -\infty \). The algebraic equations \((6.3)-(6.4)\) for \( x_1 \) and \( x_2 \) have 3 set of solutions. Considering

\[
u(t) = \sqrt{-\frac{t}{2}} \left( 1 - \frac{1}{8} (-t)^{-3} - \frac{73}{128} (-t)^{-6} - \frac{10657}{1024} (-t)^{-9} + \cdots \right),
\]

we can write out explicitly the 3 set of solutions as following.

Set A: \( x_1 = 1 + \frac{1}{\sqrt{2}} (-t)^{-\frac{1}{2}} + \cdots, \ x_2 = 1 - \frac{1}{4\sqrt{2}} (-t)^{-\frac{1}{2}} + \cdots \).

Set B: \( x_1 = -8(-t)^3 + \frac{57}{2} + \cdots, \ x_2 = 2\sqrt{2}(-t)^{-\frac{1}{2}} - \frac{39}{4\sqrt{2}} (-t)^{-\frac{1}{2}} + \cdots \).

Set C: \( x_1 = 1 - \frac{1}{\sqrt{2}} (-t)^{-\frac{1}{2}} + \cdots, \ x_2 = -1 - \frac{1}{\sqrt{2}} (-t)^{-\frac{1}{2}} + \cdots \).

The solutions of Set 2 and Set 3 are contradictory with the fact that \( x_1 > 0 \) and \( x_2 > 0 \). So we get \((6.1)\). Considering \( g_2(t) = \widetilde{q}_2(t) - 1 \) and \( u(t) \sim \sqrt{-\frac{t}{2}}, \ (6.2) \) is immediately obtained. \( \square \)

By Proposition 6.6, we will see \((6.1)\) is the general case.

**Proposition 6.4.** For \( i = 1, 2, 0 \) and \( j = 2, 0, \Phi_{ij} \) are all positive and monotonic decreasing. Furthermore, all of them approach to positive infinity as \( t \to -\infty \).
Proposition 6.5. Proof. 
Both \( \Phi_{12}(t) \xrightarrow{t \to \infty} 0 \) and \( \Phi_{22}(t) \xrightarrow{t \to \infty} 0 \) are obvious. By \( \Phi_{02}(t) \xrightarrow{t \to \infty} \frac{4^{16}}{8 \pi} e^{-\frac{t^{3/2}}{2}} t^{-\frac{3}{4}} \), we get \( \Phi_{02}(t) \xrightarrow{t \to \infty} 0 \). Further, \( \frac{2}{3} \Phi_{02}(t) - \frac{k_0}{6} u(t) \Phi_{12}(t) = \frac{4k_0}{8 \pi} e^{-\frac{t^{3/2}}{2}} (t^{-\frac{3}{4}} + O(t^{-2})) \). Thus \( \frac{2}{3} \Phi_{02}(t) - \frac{k_0}{6} u(t) \Phi_{12}(t) \xrightarrow{t \to \infty} 0 \). By Lemma 6.3 we proved the proposition for \( j = 2 \).

From
\[
- \frac{4 \pi}{3^{3/4}} \left( \text{Ai}(\frac{t}{3^{3/4}}) \text{Bi}(\frac{s}{3^{3/4}}) u(s) ds - \text{Bi}(\frac{t}{3^{3/4}}) \text{Ai}(\frac{s}{3^{3/4}}) u(s) ds \right) = e^{-\frac{t^{3/2}}{2}} \left( \frac{3}{4 \sqrt{\pi}} t^{-\frac{3}{4}} + O(t^{-\frac{3}{2}}) \right)
\]
and
\[
\frac{2}{3} \left( \text{Ai}(\frac{t}{3^{3/4}}) \text{Bi}(\frac{s}{3^{3/4}}) u(s) ds - \text{Bi}(\frac{t}{3^{3/4}}) \text{Ai}(\frac{s}{3^{3/4}}) u(s) ds \right) = e^{-\frac{t^{3/2}}{2}} \left( \frac{3}{8 \sqrt{\pi}} t^{-\frac{3}{4}} + O(t^{-\frac{3}{2}}) \right)
\]
\( \Phi_{10} \xrightarrow{t \to \infty} 0 \) and \( \Phi_{20} \xrightarrow{t \to \infty} 0 \) are obtained. Obviously, \( \Phi_{00}(t) \xrightarrow{t \to \infty} 1 > 0 \). Also, \( \frac{2}{3} \Phi_{02}(t) - \frac{k_0}{6} u(t) \Phi_{10}(t) \xrightarrow{t \to \infty} \frac{2}{3} > 0 \). By Lemma 6.3 the proposition is also true for \( j = 0 \).

Proposition 6.5. For any fixed finite real \( t_0 \), if \( c_2 \geq 0 \), \( c_0 > 0 \) and \( c_1 \) is sufficiently small, then \( c_1 \Phi_{11} + c_2 \Phi_{12} + c_0 \Phi_{10}, c_1 \Phi_{21} + c_2 \Phi_{22} + c_0 \Phi_{20} \) and \( c_1 \Phi_{01} + c_2 \Phi_{02} + c_0 \Phi_{00} \) are all monotonic decreasing and positive on \( (-\infty, t_0] \).

Proof. Let \( \phi_1(t) = c_1 \Phi_{11} + c_2 \Phi_{12} + c_0 \Phi_{10}, \phi_2(t) = c_1 \Phi_{21} + c_2 \Phi_{22} + c_0 \Phi_{20} \) and \( \phi_0(t) = c_1 \Phi_{01} + c_2 \Phi_{02} + c_0 \Phi_{00} \). It is obvious there exists \( \delta_1 > 0 \) such that \( \phi_1(t) > 0, \phi_2(t) > 0, \phi_0(t) > 0 \) and \( \frac{k_0}{6} \phi_0(t) - \frac{2}{3} \phi_2(t) \phi_0(t) > 0 \) for any \( |c_1| < \delta_1 \), since they are all greater than 0 for \( c_1 = 0 \). By Lemma 6.3 \( \phi_1(t) \), \( \phi_2(t) \) and \( \phi_0(t) \) are positive and monotonic decreasing for \( t \leq t_0 \). By Lemma 6.3, there exists \( \delta_2 > 0 \) such that \( \phi_0(t) > 0 \) for all \( t \in [t_0, \infty) \) for \( |c_1| < \delta_2 \), since \( c_2 \Phi_{02}(t) + c_0 \Phi_{00}(t) > 0 \) for all \( t \geq t_0 \). Let \( \delta = \min(\delta_1, \delta_2) \). Then, if \( |c_1| < \delta \), \( \phi_1(t) \), \( \phi_2(t) \) and \( \phi_0(t) \) have all the desired properties.

After changing of variables \( s = \sqrt{-t} \), \( \tilde{\phi}_1(s) = \phi_1(t), \tilde{\phi}_2(s) = \phi_2(t), \tilde{\phi}_0(s) = \phi_0(t) \), we can see the ODE system for \( \tilde{\phi}_1(s), \tilde{\phi}_2(s) \) and \( \tilde{\phi}_0(s) \) satisfies all the requirements of Theorem 12.3 of [21]. After changing the variables back, we get the following result.

Proposition 6.6. At \( t = -\infty \), \( \phi_1(t) \) and \( \phi_2(t) \) and \( \phi_0(t) \) have asymptotics
\begin{align}
\phi_1(t) & \sim k_P \times \varphi_1(t) + k_D \times \varphi_1(t) + k_N \times \varphi_1(t), \\
\phi_2(t) & \sim k_P \times \varphi_2(t) + k_D \times \varphi_2(t) + k_N \times \varphi_2(t), \\
\phi_0(t) & \sim k_P \times \varphi_0(t) + k_D \times \varphi_0(t) + k_N \times \varphi_0(t),
\end{align}
where
\begin{align}
\varphi_1(t) & = \left( \sqrt{\pi}(-t)^{-\frac{3}{4}} + \frac{55}{48}(-t)^{-\frac{5}{4}} + \frac{9107}{1536\sqrt{2}}(-t)^{-\frac{9}{4}} + \cdots \right) (-t)^{\frac{3}{4}} e^{-\frac{2\pi^2}{3}(-t)^{3/2}}, \\
\varphi_2(t) & = \left( 1 - \frac{5}{48\sqrt{2}}(-t)^{-\frac{3}{4}} - \frac{1013}{3072}(-t)^{-\frac{5}{4}} - \frac{2547101}{1327104\sqrt{2}}(-t)^{-\frac{9}{4}} + \cdots \right) (-t)^{\frac{3}{4}} e^{-\frac{2\pi^2}{3}(-t)^{3/2}}, \\
\varphi_0(t) & = \left( 1 + \frac{7}{48\sqrt{2}}(-t)^{-\frac{3}{4}} + \frac{145}{1024}(-t)^{-\frac{5}{4}} + \frac{1496311}{1327104\sqrt{2}}(-t)^{-\frac{9}{4}} + \cdots \right) (-t)^{\frac{3}{4}} e^{-\frac{2\pi^2}{3}(-t)^{3/2}}, \\
\varphi_1(t) & = \left( 1 + \frac{67}{72}(-t)^{-3} + \frac{551671}{10368}(-t)^{-6} + \frac{22894539769}{2239488}(-t)^{-9} + \cdots \right) (-t)^{-\frac{3}{4}}, \\
\varphi_2(t) & = \left( -\frac{1}{4}(-t)^{-1} + \frac{1009}{18\sqrt{2}}(-t)^{-4} + \frac{20411827}{41472}(-t)^{-7} + \cdots \right) (-t)^{-\frac{1}{4}}, \\
\varphi_0(t) & = \left( \frac{1}{\sqrt{2}}(-t)^{-\frac{9}{4}} + \frac{1009}{18\sqrt{2}}(-t)^{-\frac{5}{4}} + \frac{6873355}{648\sqrt{2}}(-t)^{-\frac{9}{4}} + \cdots \right) (-t)^{-\frac{1}{4}}, \\
\varphi_1(t) & = \left( -\sqrt{2}(-t)^{-\frac{3}{4}} + \frac{55}{48}(-t)^{-\frac{5}{4}} - \frac{9107}{1536\sqrt{2}}(-t)^{-\frac{9}{4}} + \cdots \right) (-t)^{\frac{3}{4}} e^{-\frac{2\pi^2}{3}(-t)^{3/2}}, \\
\varphi_2(t) & = \left( 1 + \frac{5}{48\sqrt{2}}(-t)^{-\frac{3}{4}} - \frac{1013}{3072}(-t)^{-\frac{5}{4}} + \frac{2547101}{1327104\sqrt{2}}(-t)^{-\frac{9}{4}} + \cdots \right) (-t)^{\frac{3}{4}} e^{-\frac{2\pi^2}{3}(-t)^{3/2}}, \\
\varphi_0(t) & = \left( 1 - \frac{1}{4}(-t)^{-1} - \frac{1273}{288}(-t)^{-4} - \frac{20411827}{41472}(-t)^{-7} + \cdots \right) (-t)^{-\frac{1}{4}}, \\
\end{align}
Remark 6.7. Proposition 6.6 gives a straightforward explanation for the 3 sets of solutions appearing in the proof of Lemma 6.3. If $k_P \neq 0$, the limits are given by the Set A. If $k_P = 0$ and $k_O \neq 0$, the limits are given by the Set B. Else if $k_P = k_O = 0$, the limits are given by the Set C. There is no other possibility for the limits. However, the understanding of Proposition 6.6 is subtle: for example, if in case the best approximation (obtained by optimal truncation) of $u(t)$ by its asymptotic series has an error more than the order of $e^{-\frac{\alpha}{k}(-t)^{\frac{3}{2}}}$, the lower order terms in (6.6)-(6.8) lost their meaning for REAL $t$. Fortunately, the error order of the best approximation of $u(t)$ by its asymptotic series is $e^{-\frac{\alpha}{k}(-t)^{\frac{3}{2}}}$. So all terms in (6.6)-(6.8) are contributing.

So we have constructed two sets of solutions for (6.3)-(6.7): at $t = \infty$, we have $(\Phi_{1,i}, \Phi_{2,i}, \Phi_{0,i})$, $i = 1, 2, 0$; and at $t = -\infty$, we have $(\varphi_{1,i}, \varphi_{2,i}, \varphi_{0,i})$, $i = P, O, N$. Therefore, they only differ by a constant matrix

\[
\begin{pmatrix}
\Phi_{11}(t) & \Phi_{12}(t) & \Phi_{10}(t) \\
\Phi_{21}(t) & \Phi_{22}(t) & \Phi_{20}(t) \\
\Phi_{01}(t) & \Phi_{02}(t) & \Phi_{00}(t)
\end{pmatrix} = \begin{pmatrix}
\varphi_{1P}(t) & \varphi_{1O}(t) & \varphi_{1N}(t) \\
\varphi_{2P}(t) & \varphi_{2O}(t) & \varphi_{2N}(t) \\
\varphi_{0P}(t) & \varphi_{0O}(t) & \varphi_{0N}(t)
\end{pmatrix}
\begin{pmatrix}
k_P & k_P & k_P \\
k_O & k_O & k_O \\
k_N & k_N & k_N
\end{pmatrix}.
\]

\(\Phi_{ij}(t) \xrightarrow{t \to \infty} \infty\) for $i = 2, 0$ and $j = 1, 2, 0$ mean $k_P > 0$ and $k_P > 0$. In fact, their approximate values are $k_P \approx 0.1678571$ and $k_P \approx 0.6235798$. More accurate values of them are given in Section 7 where they are determined up to more than 100 digits.

Now we are able to prove Theorem 1.2

**Proof.** Let $c_2 > 0$ and $c_1$ be sufficiently small. Define $\phi_1(t) = c_1\Phi_{11} + c_2\Phi_{12} + \Phi_{01}$, $\phi_2(t) = c_1\Phi_{21} + c_2\Phi_{22} + \Phi_{02}$, and $\phi_0(t) = c_1\Phi_{10} + c_2\Phi_{20} + \Phi_{00}$. Then $\phi_1(t)$, $\phi_2(t)$ and $\phi_0(t)$ satisfy (6.3)-(6.7). Next define $\bar{q}_2(t) = \frac{\varphi_{1O}(t)}{\varphi_{0O}(t)} u(t)$ and $\alpha(t) = \frac{\varphi_{1N}(t)}{\varphi_{0N}(t)} u(t)$. By (1.3), $\bar{q}(t)$ and $\alpha(t)$ satisfy (1.2)-(1.4). By Proposition 6.5, $\bar{q}_2(t)$ and $\alpha(t)$ are smooth on $(-\infty, \infty)$. By Proposition 5.2 and 6.6, $\bar{q}_2(t)$ and $\alpha(t)$ have desired asymptotics at $t = \infty$ and $t = -\infty$.

7 Numerical experiments about Figure 1

In this section, we give the details to generate Figure 1. A few important data, such as the numerical values of the connection data, are also given, as well as some interesting observations from the numerical experiments.

7.1 Description of the procedure

By Section 6, we know the singularities of $\bar{q}_2(t)$ and $\alpha(t)$ are completely determined by the zeroes of $\phi_0(t) = \Phi_{01}(t) + c_1\Phi_{01}(t) + c_2\Phi_{02}(t)$. So our first step is to obtain the numerical solutions of $\Phi_{ij}(t)$, $i, j = 1, 2, 0$, for $t \in [t, N, T]$. Since $\phi_0(t) \xrightarrow{t \to \infty} 1 > 0$, we must require $\phi_0(-\infty) \geq 0$ in order that $\bar{q}_2(t)$ and $\alpha(t)$ have no zeroes for $t \in (-\infty, \infty)$. Therefore, our second step is to compute the matrix elements $k_P$, $k_P$ and $k_P$, which will reflect the main behaviors of the solution near $t = -\infty$. For moderate $t$, we use the numerical solutions to resolve if $\phi_0(t)$ has zeroes, which constitutes our last step. More precisely, we determine the boundary between $R_{smooth}$ and $R_{singular}$ by seeking the minimal $c_2$ such that $\Phi_{01}(t) + c_1\Phi_{01}(t) + c_2\Phi_{02}(t) \geq 0$ for all $t \in (-\infty, \infty)$ for given $c_1$. Since $\Phi_{02}(t) > 0$, the problem is simplified to find the minimum of $\Phi_{01}(t) + c_1\Phi_{01}(t) + c_2\Phi_{02}(t)$ for given $c_1$, i.e., $c_2 = \min_{\forall t \in (-\infty, \infty)} \left(\Phi_{01}(t) + c_1\Phi_{01}(t) + c_2\Phi_{02}(t)\right)$.

Obviously, we have to do numerical integration of ODEs. Currently, the most precise ODE integrator, such as Taylor [13] or high-order Runge-Kutta, can integrate an ODE numerically with precision up to 1000 digits. For convenience, we use the build-in ‘NDSolve’ of Mathematica to do the numerical integration for (1.2)-(1.4). The default option of ‘NDSolve’ is inappropriate to do high-precision numerical integration. By explicitly giving the ‘Method’ option of ‘NDSolve’, we can force it to use the Gauss-Legendre Runge-Kutta method, which is suitable for the high-precision purpose. To save running time, we manage to let the typical precision be of order $10^{-120}$ [5]. The stages of the Runge-Kutta method are set according to the precision goal of the numerical integration. As a rule, we always let the stages greater than 100, i.e, the order of the numerical scheme is always more than 200. The step-sizes $h$ are chosen as $0.01 \leq h \leq 0.05$. By rough but careful estimations for each case, we guarantee that the errors

---

1It does not mean the final error or final relative error is less than $10^{-120}$. It just mean, the relative error is less than $10^{-120}$ for every step.
generated by the numeric scheme itself are always negligible, comparing to the errors that exist on the boundaries and are propagated by the ODE system.

7.2 Determine $T_P$

$T_P$ is determined by two key factors: the truncation orders of $\Phi_{ij}$ at $t = \infty$ and the precision goal of the numerical integration. We use (5.1)-(5.9) as the truncation of $\Phi_{ij}$ since the higher order truncation will involve multiple integrals, which is difficult to get satisfactory high-precision results.

We demand the error of $\Phi_{ij}(t)$ at $t = 0$ is of order $10^{-120}$. For the solution $\Phi_{11}(t)$, we can show their errors at $t = 0$ are of order $e^{-\frac{3}{2}t^{5/2}}$. Solving $e^{-\frac{3}{2}t^{5/2}} = 10^{-120}$, we get $t_P = 45.888$. For convenience, we set $t_P = 46$, at which the relative errors are of order $e^{-\frac{1}{3}16^{3/2}} \approx 2.19 \times 10^{-181}$. So we set the precision goal of the numeric scheme as $10^{-182}$ in computing $\Phi_{11}(t)$. By a similar way, we could show it is appropriate to set $t_P = 36$ and the precision goal as $10^{-120}$ in computing $\Phi_{12}(t)$. In computing $\Phi_{10}(t)$, we also use $t_P = 36$ and the precision $10^{-120}$.

7.3 Determine $T_N$

By (6.1)-(6.8), each $k_O$ term contributes to a portion of order $e^{-\frac{3}{2}t^{3/2}}$. By solving $e^{-\frac{3}{2}t^{3/2}} = 10^{-120}$, we get $t = -91.7761$. This means the $k_O$ terms can be neglected when $t < -91.7761$. For convenience, we set $T_N = -92$.

7.4 The numerical solution of $u(t)$

For computation efficiency, the numerical solution of $u(t)$ is first obtained independently on $[t_M, t_H]$. We demand the max error of $u(t)$ is of order $10^{-120}$. Since the best approximation of $u(t)$ by (6.5) has an error of order $e^{-(t)^{3/2}}$, $t_M$ is obtained by solving $e^{-(t)^{3/2}} = 10^{-120}$, i.e., $t_M \approx -42.42$. For safety, we set $t_M = -44$. So, for $T_N \leq t < t_M$, we use the asymptotic expansion (6.6) up to the $(-t)^{-41/2}$ term to compute $u(t)$. For $t_M \leq t \leq t_H$, $u(t)$ is obtained by the high-precision numerical integration of (1.11).

Let

$$\tilde{u}(t) = u(t) + \epsilon \mathcal{U}(t),$$

where $\epsilon$ is an infinitesimal, and $\tilde{u}$ also satisfies (1.11). Then $\mathcal{U}(t)$ satisfies

$$\mathcal{U}''(t) = (t + 6u(t)^2) \mathcal{U}(t).$$

As $u(t) \xrightarrow{t \to -\infty} \sqrt{\frac{t}{2}} + \cdots$, $\mathcal{U}(t)$ is of order $e^{\frac{3}{2}(-2t)^{3/2}}$. In the numerical experiments, $\epsilon \mathcal{U}(t)$ is understood as the error. So $\epsilon \sim 3 \times 10^{-240}$ in order that at $t = -44$ the error is of order $10^{-120}$. Then the error of $u(t)$ at $t = 0$ should be of order $10^{-240}$. The computational error of $u(t)$ behaves like $\epsilon u Ai(t)$ for $t > 0$. So, the relative error at $t = t_H$ should also be of order $10^{-240}$. For safety, we manage the relative error at $t = t_H$ to be of order $10^{-250}$.

The value of $t_H$ is related to how $u(t)$ is approximated near $t = \infty$. We take

$$u(t) \approx Ai(t) - 2\pi Ai(t) \int_0^t Bi(s)Ai(s)^3ds + 2\pi Bi(t) \int_0^t Ai(s)^4ds$$

as the approximation of $u(t)$. The error order of the approximation (7.1) is about $e^{-\frac{3}{2}t^{3/2}}$. So the relative error is of order $e^{-\frac{3}{2}t^{3/2}}$. By solving $e^{-\frac{3}{2}t^{3/2}} = 10^{-250}$, $t_H \approx 35.985$ is obtained. For convenience, we set $t_H = 36$. We use 250 digits in computing the numerical solution of $u(t)$.

7.5 Transformations to avoid small step size

Fixing the step size, the Runge-Kutta method will be generally more accurate to integrate a slowly-varying system. To see the crux, let us consider the approximation of $e^{-t^{3/2}}$ by polynomials. It is easy to see that the relative error of the approximation on interval [100, 100.01] is almost the same as the one on interval [1, 1.1] when using the same degree of approximation polynomials. This means smaller step size is needed for large $t$ if the system increases or decreases too fast. To avoid the small step size for large $|t|$, we use Table 1 to transform the fast variables to slow ones.
Table 1: Transformations used to transform the fast variables to the slow ones

| fast variables | $t < -1$ | $-1 \leq t \leq 1$ | $t > 1$ |
|---------------|----------|------------------|---------|
| $u$           | $u(t)$   | $u(t)$           | $u(t) = u(t)e^{\frac{2\sqrt{2}}{t}}$ |
| $\Phi_{11}$, $i = 1, 2, 0.$ | $\Phi_{11}(t) = \Phi_{11}(t)e^{-\frac{2\sqrt{2}}{t}}(-t)^{-3/2}$ | $\Phi_{11}(t)$ | $\Phi_{11}(t) = \Phi_{11}(t)e^{-\frac{2\sqrt{2}}{t}}(-t)^{-3/2}$ |
| $\Phi_{12}$, $i = 1, 2, 0.$ | $\Phi_{12}(t) = \Phi_{12}(t)e^{-\frac{2\sqrt{2}}{t}}(-t)^{-3/2}$ | $\Phi_{12}(t)$ | $\Phi_{12}(t) = \Phi_{12}(t)e^{-\frac{2\sqrt{2}}{t}}(-t)^{-3/2}$ |
| $\Phi_{10}$, $i = 1, 2, 0.$ | $\Phi_{10}(t) = \Phi_{10}(t)e^{-\frac{2\sqrt{2}}{t}}(-t)^{-3/2}$ | $\Phi_{10}(t)$ | $\Phi_{10}(t) = \Phi_{10}(t)e^{-\frac{2\sqrt{2}}{t}}(-t)^{-3/2}$ |

7.6 Numerical results

The main numerical results are displayed in Figure 1.

7.6.1 The values of $k_{P1}$, $k_{P2}$ and $k_{P0}$

In principle, $k_{P_i}$, $i = 1, 2, 0$, can be computed by any of $\lim_{t \to -\infty} \frac{\Phi_{0i}(t)}{\varphi_{0P}(t)}$, $\lim_{t \to -\infty} \frac{\Phi_{0i}(t)}{\varphi_{0P}(t)}$ or $\lim_{t \to \infty} \frac{\Phi_{0i}(t)}{\varphi_{0P}(t)}$. In our numerical experiments, we use

\[
(7.2) \quad k_{P_i} = \frac{\Phi_{0i}(t_N)}{\varphi_{0P}(t_N)},
\]

which is a little more accurate than the other two choices. In (7.2), $\Phi_{0i}(t_N)$ are obtained directly from the numerical integration of ODEs of $\Phi_{ji}$, while $\varphi_{0P}(t_N)$ is calculated by its asymptotic expansion (6.11), where $\varphi_{0P}(t)$ is computed up to the term $\epsilon_{354}^0 \times (-t)^{354}(-t)^{-3/2}$. It is not surprising that $\epsilon_{354}^0$ is very large since $\epsilon_{354}^0 \times (-t)^{354}$ contributes about $7.74545 \times 10^{-126}$ at $t = -92$. So $k_{P_i}$ is determined with an approximate precision of $10^{-120}$. The final numerical results of $k_{P_i}$ are

\[
(7.3) \quad k_{P1} = -0.0969123435570255523226380385083332 \ldots,
\]
\[
(7.4) \quad k_{P2} = 0.16785710292133850132168687360301197 \ldots,
\]
\[
(7.5) \quad k_{P0} = 0.6235798169501424223251084362366955 \ldots.
\]

7.6.2 $\Phi_{ij}(t)$ near $t = 0$

By (6.1)-(6.11), (6.1)-(6.11) and (7.3)-(7.5), the main behaviors of $\Phi_{ij}(t)$ at $t = \pm \infty$ have been described. We demonstrate their behaviors on the “transition zone” by Figures 3, 4 and 5.
Figure 3. Plots of $\Phi_{11}$ (red), $\Phi_{21}$ (green) and $\Phi_{01}$ (blue). At any $t$, $\Phi_{11}(t)$ is always the largest. $\Phi_{01}(t)$ and $\Phi_{21}(t)$ intersect at $t \approx -4.3166745$.

Figure 4. Plots of $\Phi_{12}$ (red), $\Phi_{22}$ (green) and $\Phi_{02}$ (blue). All of them are positive. There are 4 intersections in the figure: $t_1 \approx 3.30090866$, $t_2 \approx -1.7223227$, $t_3 \approx -3.5443904$, $t_4 \approx -3.72852126$. For $t > t_1$, $\Phi_{22}(t) > \Phi_{12}(t) > \Phi_{02}(t)$. For $t_1 > t > t_2$, $\Phi_{12}(t) > \Phi_{22}(t) > \Phi_{02}(t)$. For $t_2 > t > t_3$, $\Phi_{12}(t) > \Phi_{02}(t) > \Phi_{22}(t)$. For $t_3 > t > t_4$, $\Phi_{02}(t) > \Phi_{12}(t) > \Phi_{22}(t)$. For $t < t_4$, $\Phi_{02}(t) > \Phi_{22}(t) > \Phi_{12}(t)$.

Figure 5. Plots of $\Phi_{10}$ (red), $\Phi_{20}$ (green) and $\Phi_{00}$ (blue). All of them are positive. For all $t$, $\Phi_{00}(t) > \Phi_{20}(t) > \Phi_{10}(t)$.

7.6.3 The critical point $P_c$

In Section 7.2.1 we have explained $c_2 = -\min_{t \in (-\infty, x)} \left( \frac{\Phi_{00}(t)}{\Phi_{22}(t)} + c_1 \frac{\Phi_{01}(t)}{\Phi_{22}(t)} \right)$ on the boundary between $R_{smooth}$ and $R_{singular}$. Given $c_1$, let the minimum is achieved at $t = t_z$. The numerical results show the $t_z$ is unique for any given $c_1$. So, on the boundary curve, $t_z = t_z(c_1)$. It is obvious that both $t_z$ and $c_2$ must approach to $\infty$ when $c_1 \to -\infty$. As $c_1$ increases gradually to $P_c$, $t_z$ decreases and finally approaches to $-\infty$ as displayed by Figure 6.

Figure 6. Plots of $c_2$ (red) and $t_z$ (green). The $c_2$ curve, which is the boundary between $R_{smooth}$ and $R_{singular}$, is smooth. Though it looks very like a straight line, it is indeed a curve. The $t_z$ curve has apparently a singularity near $c_1 = -\frac{1}{2} \frac{t}{s_0} \approx 3.217236287$. 

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On the right of $P_c$, the minimum is always achieved at $t = -\infty$, i.e., $t_2 = -\infty$. So we have

\begin{equation}
(7.6) \quad c_2 = -\left(\frac{k_{P_0}}{k_{P_2}} + c_1 \frac{k_{P_1}}{k_{P_2}}\right) = \frac{k_{P_0}}{k_{P_2}} - \frac{k_{P_1}}{k_{P_2}} c_1,
\end{equation}

which is the straight line right of $P_c$ in Figure 1. For the critical point $P_c$, the interesting observation from the numerical experiment is $c_1 = -\frac{1}{\pi} \frac{2}{k_{P_1}}$. Then, from (7.6), $c_2 = -\frac{1}{\pi} \frac{k_{P_0}}{k_{P_2}}$ at $P_c$.

### 7.6.4 The values of $k_{O1}$, $k_{O2}$, $k_{O0}$, $k_{N1}$, $k_{N2}$ and $k_{N0}$

Integrating $u(t)$ and $\Phi_{ij}(t)$ numerically along the path O-A-B in Figure 7, we have obtained the values of $k_{P1}$, $k_{P2}$ and $k_{P0}$ with about 120 digits of precision. But $k_{O1}$, $k_{O2}$, $k_{O0}$, $k_{N1}$, $k_{N2}$ and $k_{N0}$ can not be obtained in this way. To calculate them, we have to extend our numerical integration from the real line to the complex plane of $t$ as displayed by Figure 7.

![Figure 7: Paths used to integrate $\Phi_{ij}(t)$](image)

\[ \angle BOD = \frac{\pi}{10} \quad \text{and} \quad \angle BOE = \frac{\pi}{4} \arccos\left(\frac{135 \ln 10}{184 \sqrt{10}}\right) \approx 0.879372. \]

The boundary between the light yellow region and the light green one is chosen by solving $e^{\frac{11}{20} (92)\frac{23}{2} \cos\left(\frac{17}{2} + \frac{\pi}{4}\right)} = 10^{60}$, i.e., $\pi - \theta \approx 0.699535$. For simplicity, we choose $\pi - \theta = \frac{\pi}{10}$. It is easy to show that $\Phi_{ij}$ lost their precision when they are integrated numerically along the ray $O-D$ starting from B. So we integrate them numerically along the ray O-D, by which $\Phi_{ij}$ can be guaranteed to have about 120 digits of precision. $k_{O1}$ obtained by this way can be shown to have about 60 digits of precision, which is almost the best that we can expect for the computation of $k_{O1}$ when $u(0)$ is computed with about 240 digits of precision.

To compute $k_{N1}$, we use $k_{N1} = \frac{\Phi_{ij}(t) - k_{P1} \times \Phi_{ij}(t)}{\sqrt{4t}}$, where $t$ is chosen as the point E. The argument of $E$ is chosen by solving $e^{\frac{11}{20} (92)\frac{23}{2} \cos\left(\frac{17}{2} + \frac{\pi}{4}\right)} = 10^{30}$. At the first sight, one may want to evaluate $k_{N1}$ from $\Phi_{ij}$ on the dotted line. But (10) has proved

\begin{equation}
(7.7) \quad u(t) = \sqrt{-\frac{t}{2}} \left(1 + O((-t)^{-3/2})\right) + \frac{i}{2 \sqrt{\pi}} (-t)^{-\frac{1}{4}} e^{-\frac{23}{2} \pi (-t)^{3/2}} \left(1 + O(t^{-\frac{1}{2}})\right),
\end{equation}

for $\frac{23}{4} \leq \arg(t) < \frac{23}{4}$. So we should not use the expansions (6.15) along the light yellow region near $\theta = \frac{23}{4}$. Considering the exponential term of (7.7), we can show $k_{N1}$ is best calculated near $E$. Also, it can be shown that $k_{N1}$ calculated in this way have about 30 digits of precision.
The final numerical values of $k_{ij}$, $i = O, N$, $j = 1, 2, 0$, are

$$(7.8) \quad k_{01}^+ = (k_{01}^-)^* = 0.47478765355557080096 \cdots + i \times 0.0937292652940652656 \cdots,$$

$$(7.9) \quad k_{02}^+ = (k_{02}^-)^* = 0.27411877958821979669 \cdots - i \times 0.158262551185190266698 \cdots,$$

$$(7.10) \quad k_{00}^+ = (k_{02}^-)^* = -1.018336045084649924885 \cdots - i \times 0.58793658975512151298 \cdots,$$

$$(7.11) \quad k_{N1}^+ = (k_{N1}^-)^* = -0.19583328674156168848 \cdots + i \times 0.048456171778512776161 \cdots,$$

$$(7.12) \quad k_{N2}^+ = (k_{N2}^-)^* = 0.048456171778512776161 \cdots + i \times 0.083928551460669295066 \cdots,$$

$$(7.13) \quad k_{N0}^+ = (k_{N0}^-)^* = -0.36002397503083963185 \cdots.$$

From our numerical results, we observe that $k_{N1}^+ = \left(\frac{2n\pi}{3} - \frac{1}{2}i\right) k_{P1}$, $k_{N2}^+ = \left(\frac{2n\pi}{3} + \frac{1}{2}i\right) k_{P2}$ and $k_{N0}^+ = \frac{2n\pi}{3} k_{P0}$ with the errors less than $10^{-4}$, which are consistent with the estimated precision of the numerical $k_{N1}^+$, $k_{N2}^+$ and $k_{N0}^+$. Also it is observed that $\left(\frac{k_{P1}}{k_{P2}}\right)^*$ $= \frac{k_{N0}^+}{k_{P0}^+}$ with more than 60 digits of precision.

### 7.6.5 The solution corresponding to $P_c$

Let us consider the solutions of (1.5)-(1.7) described by Figure 1. We note that the solution corresponding to $P_c$ in Figure 1 has a special property. For simplicity, we scale the solution as

$$\Phi_c = (\Phi_{1c}, \Phi_{2c}, \Phi_{0c})$$

$$= \frac{2}{k_{P0}} (\Phi_{10}, \Phi_{20}, \Phi_{00}) - \frac{1}{k_{P1}} (\Phi_{11}, \Phi_{21}, \Phi_{01}) - \frac{1}{k_{P2}} (\Phi_{12}, \Phi_{22}, \Phi_{02}).$$

By the numerical connection data (7.8)-(7.13), it is easy to verify (within the tolerance of precision) that $\Phi_c(t) \xrightarrow{\cdots} -2\sqrt{3}\left(\varphi_{1N}(t), \varphi_{2N}(t), \varphi_{0N}(t)\right)$. So this special solution decreases exponentially to zero as $t \to -\infty$. We also note the other bounded solutions at $t = -\infty$, which are spanned by $\Phi_c$ and $\frac{1}{k_{P1}} (\Phi_{11}, \Phi_{21}, \Phi_{01}) - \frac{1}{k_{P2}} (\Phi_{12}, \Phi_{22}, \Phi_{02})$, decrease algebraically to 0.

### 8 The wave function of Painlevé II

The Lax pair of Painlevé II is

$$\frac{d\Psi_0}{dx} = \hat{L}_0 \Psi_0,$$

$$\frac{d\Psi_0}{dt} = \hat{B}_0 \Psi_0,$$

where $\hat{L}_0$ and $\hat{B}_0$ are defined by (2.4) and (2.5). Unlike in Section 2 where $\psi_0$ is vector, here, $\Psi_0$ is a $2 \times 2$ matrix.

Define the six regions in the complex $x$-plane as

$$\Omega_j = \left\{ x \left| \frac{\pi}{2} + \frac{j-2}{3} \pi \prec \arg x \prec \frac{\pi}{2} + \frac{j}{3} \pi \right. \right\}, \quad j = 1, 2, \cdots, 6.$$

Equation (8.1) has 6 canonical solutions $\Psi_0^{(j)}(x)$ defined in the regions $\Omega_j$, $j = 1, \cdots, 6$,

$$\Psi_0^{(j)}(x) \xrightarrow{x \to \infty} \left( I + \frac{m_1}{x} + \cdots \right) e^{\left(\frac{\pi}{3} - \frac{\pi}{3}i\right) x}, \quad \frac{\pi}{2} + \frac{j-2}{3} \pi \prec \arg x \prec \frac{\pi}{2} + \frac{j}{3} \pi.$$

For convenience, we denote $\Omega_7 = \Omega_1$ and $\Psi_0^{(j)} = \Psi_0^{(1)}$. If $\Psi_0$ is known, then $u(t)$ can be recovered by $u = (m_1)_{21} = -(m_1)_{12}$.

The sector $\Omega_j$ overlaps with $\Omega_{j+1}$. In the crossover region,

$$\Psi_0^{(j+1)} = \Psi_0^{(j)} S_0^{(j)}.$$

\footnote{Just as the asymptotic series hints, the numerical results show the decrease looks very like $(-t)^{-\frac{7}{2}}$.}
For the case that \( u(t) \) is the Hastings-McLeod solution,

\[
S_0^{(1)} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad S_0^{(2)} = I_{2 \times 2}, \quad S_0^{(3)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\
S_0^{(4)} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad S_0^{(5)} = I_{2 \times 2}, \quad S_0^{(6)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Define \( Y^{(j)}(x,t) \) by

\[
Y^{(j)}(x,t) = \psi^{(j)}_0(x,t)e^{-\left(\frac{t^2}{4} - \frac{x^2}{4}\right)}\sigma_3. \tag{8.4}
\]

For convenience, we call both \( \psi^{(j)}_0 \) and \( Y^{(j)} \) as the wave functions of Painlevé II. By (8.3), \( Y^{(j)} \) satisfy the Riemann-Hilbert problem illustrated by Figure 8.

![Figure 8. The original Riemann-Hilbert Problem.](image)

To prove Theorem 1.3, a detailed analysis for the case \( t \to \infty \) is needed. So we deform the original Riemann-Hilbert problem to Figure 9.

![Figure 9. The final Riemann-Hilbert Problem.](image)

By solving the Riemann-Hilbert problem of \( Y \), one gets the following result.
Lemma 8.1. \( Y^{(6)} \) and \( Y^{(3)} \) have the following asymptotics:

(A) For \( x \to \infty \) and fixed \( t \), \( Y^{(6)}(x, t) \) has expansion \( Y^{(6)}(x, t) = I_{2 \times 2} + \frac{m_1(t)}{x} + \ldots \).

(B) For \( x \to -\infty \) and fixed \( t \), \( Y^{(3)}(x, t) \) has expansion \( Y^{(3)}(x, t) = I_{2 \times 2} + \frac{m_1(t)}{x} + \ldots \).

(C) \[ \lim_{x \to \infty, t \to x, t \neq \pm \sqrt{1}} \frac{Y^{(6)}(x, t)}{x \to \infty, t \to x, t \neq \pm \sqrt{1}} = I_{2 \times 2} \quad \text{and} \quad \lim_{x \to -\infty, t \to x, t \neq \pm \sqrt{1}} \frac{Y^{(3)}(x, t)}{x \to -\infty, t \to x, t \neq \pm \sqrt{1}} = I_{2 \times 2}. \]

(D) For \( t \to \infty \) and \( 0 \leq x < \sqrt{1} - t^{-\frac{1}{4}} \), \( Y^{(6)}(t) \to \begin{pmatrix} 1 & -e^{t^{-\frac{1}{4}} - 2t} \\ 0 & 1 \end{pmatrix} \).

(E) For \( t \to \infty \) and \( -\sqrt{1} + t^{-\frac{1}{4}} < x \leq 0 \), \( Y^{(3)}(t) \to \begin{pmatrix} 1 & 0 \\ -e^{xt^{-\frac{1}{4}}} & 1 \end{pmatrix} \).

In both cases (A) and (B), \( m_1(t) = \begin{pmatrix} (u')^2 - u^4 - tu^2 & -u \\ u & -(u')^2 + u^4 + tu^2 \end{pmatrix} \).

Lemma 8.1 is already known, see for example [10].

Remark 8.2. Because of (8.10), at \( t = -\infty \), \( \kappa(t) \) is 'smaller' than other quantities in the formulae. It is unnecessary to estimate \( Y^{(3)} \) and \( Y^{(6)} \) so accurately at \( t = -\infty \).

Lemma 8.1 fulfils parts of our purpose to prove (13). In fact we still need more detailed behaviour of \( Y^{(3)} \) on \( x = k\sqrt{7} \). For completeness, we also give the results for \( Y^{(6)} \).

Before we study the asymptotics of \( Y^{(6)}(x, t) \) and \( Y^{(3)}(x, t) \) along \( x = k\sqrt{7} \), let us first write down the ODEs for them, which our study will rely on.

By (8.3), \( Y^{(j)} \) satisfies

\[ \frac{dY^{(j)}}{dx} = \hat{L}_0 Y^{(j)} + \left( \frac{t - x^2}{2} \right) Y^{(j)} \sigma_3, \]

\[ \frac{dY^{(j)}}{dt} = \hat{B}_0 Y^{(j)} + \frac{x}{2} Y^{(j)} \sigma_3. \]

The detailed formulae are

\[
\begin{align*}
\frac{d}{dx} \begin{pmatrix} Y_{11}^{(j)} \\ Y_{21}^{(j)} \end{pmatrix} &= \begin{pmatrix} -u(t)^2 & xu(t) - u'(t) \\ xu(t) + u'(t) & t - x^2 + u(t)^2 \end{pmatrix} \begin{pmatrix} Y_{11}^{(j)} \\ Y_{21}^{(j)} \end{pmatrix}, \\
\frac{d}{dt} \begin{pmatrix} Y_{11}^{(j)} \\ Y_{21}^{(j)} \end{pmatrix} &= \begin{pmatrix} 0 & -u(t) \\ -u(t) & x \end{pmatrix} \begin{pmatrix} Y_{11}^{(j)} \\ Y_{21}^{(j)} \end{pmatrix}, \\
\frac{d}{dx} \begin{pmatrix} Y_{12}^{(j)} \\ Y_{22}^{(j)} \end{pmatrix} &= \begin{pmatrix} -t + x^2 - u(t)^2 & xu(t) - u'(t) \\ xu(t) + u'(t) & u(t)^2 \end{pmatrix} \begin{pmatrix} Y_{12}^{(j)} \\ Y_{22}^{(j)} \end{pmatrix}, \\
\frac{d}{dt} \begin{pmatrix} Y_{12}^{(j)} \\ Y_{22}^{(j)} \end{pmatrix} &= \begin{pmatrix} -x & -u(t) \\ -u(t) & 0 \end{pmatrix} \begin{pmatrix} Y_{12}^{(j)} \\ Y_{22}^{(j)} \end{pmatrix}.
\end{align*}
\]

Along the line \( x = k\sqrt{7} \), by

\[ \frac{dY^{(j)}(x, t)}{dt} \bigg|_{x = k\sqrt{7}} = \frac{k}{2\sqrt{7}} \frac{dY^{(j)}(x, t)}{dx} \bigg|_{x = k\sqrt{7}}, \quad j = 3, 6, \]

we get

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} Y_{11}^{(j)} \\ Y_{21}^{(j)} \end{pmatrix} &= \begin{pmatrix} \frac{-ku(t)^2}{2\sqrt{7}} & \frac{k^2 - 2u(t) - \frac{k^2 - 2u(t)}{2\sqrt{7}}}{2\sqrt{7}} \\ \frac{-ku(t)^2}{2\sqrt{7}} + \frac{k^2 - 2u(t)}{2\sqrt{7}} & \frac{k^2 - 2u(t) - \frac{k^2 - 2u(t)}{2\sqrt{7}}}{2\sqrt{7}} \end{pmatrix} \begin{pmatrix} Y_{11}^{(j)} \\ Y_{21}^{(j)} \end{pmatrix}, \\
\frac{d}{dt} \begin{pmatrix} Y_{12}^{(j)} \\ Y_{22}^{(j)} \end{pmatrix} &= \begin{pmatrix} \frac{k^2 - 2u(t)^2}{2\sqrt{7}} & \frac{k^2 - 2u(t) - \frac{k^2 - 2u(t)}{2\sqrt{7}}}{2\sqrt{7}} \\ \frac{k^2 - 2u(t)^2}{2\sqrt{7}} + \frac{k^2 - 2u(t)}{2\sqrt{7}} & \frac{k^2 - 2u(t) - \frac{k^2 - 2u(t)}{2\sqrt{7}}}{2\sqrt{7}} \end{pmatrix} \begin{pmatrix} Y_{12}^{(j)} \\ Y_{22}^{(j)} \end{pmatrix}.
\end{align*}
\]

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8.1 The asymptotics of $Y^{(6)}(x, t)$ for $x \to \infty$ and $t \to \infty$ along $x = k\sqrt{t}$

In this case $k > 0$. First let us assume $k \neq 1$. By Lemma 8.1 we know $Y_{11}^{(6)} \to 1$ and $Y_{22}^{(6)} \to 1$. Therefore, the approximate differential equations for $Y_{21}^{(6)}$ and $Y_{12}^{(6)}$ along $x = k\sqrt{t}$ are

\begin{align}
\frac{d}{dt} Y_{21}^{(6)} &= \left(3 - \frac{k^2}{2}\right)k\sqrt{t}Y_{21}^{(6)} + \frac{k^2 - 2}{2}\text{Ai}(t) + \frac{k}{2\sqrt{t}}\text{Ai}'(t), \\
\frac{d}{dt} Y_{12}^{(6)} &= \left(\frac{k^2 - 3}{2}\right)k\sqrt{t}Y_{12}^{(6)} + \frac{k^2 - 2}{2}\text{Ai}(t) - \frac{k}{2\sqrt{t}}\text{Ai}'(t),
\end{align}

while the approximate differential equations for $Y_{21}^{(6)}$ and $Y_{12}^{(6)}$ for fixed but large $t$ are

\begin{align}
\frac{d}{dx} Y_{21}^{(6)} &= (t - x^2)Y_{21}^{(6)} + x\text{Ai}(t) + \text{Ai}'(t), \\
\frac{d}{dx} Y_{12}^{(6)} &= (-t + x^2)Y_{12}^{(6)} + x\text{Ai}(t) - \text{Ai}'(t).
\end{align}

8.1.1 $Y_{21}^{(6)}$

Case $0 < k < 2$:

In this case, the solution of (8.11) is

\begin{equation}
Y_{21}^{(6)} = \Upsilon_1(k)e^{\frac{k^2}{2}(3-k^2)x^3/2} + e^{\frac{k^2}{2}(k^2-3)x^3/2}\int_{-\infty}^{t} e^{\frac{k^2}{2}(k^2-3)s^3/2} \left(\frac{k^2 - 2}{2}\text{Ai}(s) + \frac{k}{2\sqrt{s}}\text{Ai}'(s)\right) ds.
\end{equation}

The solution of (8.13) is

\begin{equation}
Y_{21}^{(6)} = C_1(t)e^{-\frac{t}{2}x^3} + e^{-\frac{t}{2}x^3}\int_{-\infty}^{\sqrt{t}} e^{-ts+\frac{3}{2}s^3} \left(s\text{Ai}(t) + \text{Ai}'(t)\right) ds.
\end{equation}

(8.15) and (8.16) must coincide at $x = k\sqrt{t}$. Therefore, we have

\begin{equation}
C_1(t) - \Upsilon_1(k) = \int_{-\infty}^{t} e^{\frac{k^2}{2}(k^2-3)x^3/2} \left(\frac{k^2 - 2}{2}\text{Ai}(s) + \frac{k}{2\sqrt{s}}\text{Ai}'(s)\right) ds \\
- \int_{-\infty}^{\sqrt{t}} e^{-ts+\frac{3}{2}s^3} \left(s\text{Ai}(t) + \text{Ai}'(t)\right) ds.
\end{equation}

Note that the right-side of (8.17) is a solution of $\int_{-\infty}^{t} e^{\frac{k^2}{2}(k^2-3)x^3/2} \left(\frac{k^2 - 2}{2}\text{Ai}(s) + \frac{k}{2\sqrt{s}}\text{Ai}'(s)\right) ds = 0$. So (8.17) determines $C_1(t)$ and $\Upsilon_1(k)$ up to a constant. But we know $\Upsilon_1(k) = 0$ for $k \in (0, \sqrt{3})$. So we get

\begin{equation}
C_1(t) = \int_{-\infty}^{t} e^{-\frac{3}{2}s^3/2} \left(-\frac{1}{2}\text{Ai}(s) + \frac{1}{2\sqrt{s}}\text{Ai}'(s)\right) ds.
\end{equation}

Taking the $t \to \infty$ limit of (8.17) and considering (8.18), we obtain

\begin{equation}
\Upsilon_1(k) = 0, \quad k \in (0, 2).
\end{equation}

Therefore, the final result for $0 < k < 2$ is

\begin{equation}
Y_{21}^{(6)} \approx e^{\frac{k^2}{2}(3-k^2)x^3/2}\int_{-\infty}^{t} e^{\frac{k^2}{2}(k^2-3)s^3/2} \left(\frac{k^2 - 2}{2}\text{Ai}(s) + \frac{k}{2\sqrt{s}}\text{Ai}'(s)\right) ds.
\end{equation}

Case $k = 2$:

Similar to the case $0 < k < 2$, we can also derive

\begin{equation}
\Upsilon_1(2) = 0.
\end{equation}

Thus

\begin{equation}
Y_{21}^{(6)} \approx e^{-\frac{3}{2}t^{3/2}} \int_{-\infty}^{t} e^{\frac{3}{2}s^{3/2}} \left(\text{Ai}(s) + \frac{1}{\sqrt{s}}\text{Ai}'(s)\right) ds.
\end{equation}

By the result of Riemann-Hilbert problem, $Y_{21}^{(6)} \to 0$ and $Y_{22}^{(6)} \to 1$ are still true for $k = 1$. 

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Case $k > 2$:
In this case, the solution of (8.11) is

$$Y_{21}^{(6)}(t) = \Upsilon_1(k) e^{\frac{t}{\sqrt{7}}} \int_{t_A}^{t} e^{\frac{t}{\sqrt{7}}} \left( \frac{k^2 - 2}{2} \tilde{A}i(s) + \frac{k}{2\sqrt{7}} \tilde{A}i'(s) \right) ds. \quad (8.20)$$

Clearly, the first term can be neglected. Thus for $k > 2$,

$$Y_{21}^{(6)}(t) \approx e^{\frac{t}{\sqrt{7}}} \int_{t_A}^{t} e^{\frac{t}{\sqrt{7}}} \left( \frac{k^2 - 2}{2} \tilde{A}i(s) + \frac{k}{2\sqrt{7}} \tilde{A}i'(s) \right) ds. \quad (8.21)$$

Note $t_A$ is a fixed arbitrary real number.

**Remark 8.3.** As $k \to \infty$, (8.21) is consistent with $Y_{21}^{(6)}(t) \xrightarrow{t \to \infty} \frac{u(t)}{x}$. 

**8.1.2 $Y_{12}^{(6)}$**

For all $k > 0$, (8.12) has the solution

$$Y_{12}^{(6)}(t) = \Upsilon_2(k) e^{\frac{t}{\sqrt{7}}} \int_{t_A}^{t} e^{\frac{t}{\sqrt{7}}} \left( \frac{k^2 - 2}{2} \tilde{A}i(s) - \frac{k}{2\sqrt{7}} \tilde{A}i'(s) \right) ds. \quad (8.22)$$

Since (8.22) has to approach 0 as $t \to \infty$, we have $\Upsilon_2(k) = 0$ for $k \geq \sqrt{3}$. The solution of (8.11) is

$$Y_{12}^{(6)}(t) = C_2(t) e^{-tx + \frac{t^3}{3}} + e^{-tx + \frac{t^3}{3}} \int_{t_A}^{t} e^{tx - \frac{t^3}{3}} (s \tilde{A}i(s) - \tilde{A}i'(s)) ds. \quad (8.23)$$

By the consistence of (8.22) and (8.23) for $x = k \sqrt{7}$, we get

$$C_2(t) - \Upsilon_2(k) = \int_{t_A}^{t} e^{\frac{k^2 - 2}{2} \tilde{A}i(s) - \frac{k}{2\sqrt{7}} \tilde{A}i'(s)} ds - \int_{\sqrt{7}}^{k\sqrt{7}} e^{\frac{t^3}{3}} (s \tilde{A}i(s) - \tilde{A}i'(s)) ds. \quad (8.24)$$

Unlike the case of $Y_{21}^{(6)}$, we can not recklessly take the $k \to 1$ limit of (8.24).

Let us fix $k$, $k > 1$. Consider the asymptotics of (8.24) as $t \to \infty$. The first term of the right-side of (8.24) can be neglected since it is exponentially small for $t \to \infty$. So we have

$$C_2(t) - \Upsilon_2(k) \approx - \int_{\sqrt{7}}^{k\sqrt{7}} e^{\frac{t^3}{3}} (\tilde{A}i(t) s - \tilde{A}i'(t)) ds$$

$$= - \int_{0}^{k-1} e^{\frac{t^3}{3}} (1 - \frac{1}{3} \frac{t^3}{3} + \ldots) e^{\frac{t^3}{3}} (\tilde{A}i(t) \sqrt{1+r} - \tilde{A}i'(t)) \sqrt{r} dr$$

$$\approx - \int_{0}^{k-1} e^{\frac{t^3}{3}} (1 - \frac{1}{3} \frac{t^3}{3} + \ldots) e^{\frac{t^3}{3}} (\tilde{A}i(t) \sqrt{1+r} - \tilde{A}i'(t)) \sqrt{r} dr$$

$$= - \frac{1}{2} - \frac{1}{12 \sqrt{\pi}} t^{-\frac{3}{2}} + \frac{35}{1728 \sqrt{\pi}} t^{-\frac{5}{2}} + \ldots. \quad (8.25)$$

By the condition that $\Upsilon_2(k) = 0$ for $k \geq \sqrt{3}$, we get

$$C_2(t) = \frac{1}{2} - \frac{1}{12 \sqrt{\pi}} t^{-\frac{3}{2}} + \frac{35}{1728 \sqrt{\pi}} t^{-\frac{5}{2}} + \ldots. \quad (8.26)$$

Similarly, for $0 < k < 1$, we have

$$C_2(t) - \Upsilon_2(k) = \frac{1}{2} - \frac{1}{12 \sqrt{\pi}} t^{-\frac{3}{2}} + \frac{35}{1728 \sqrt{\pi}} t^{-\frac{5}{2}} + \ldots. \quad (8.27)$$
Therefore,

\[ \Upsilon_2(k) = -1, \quad 0 < k < 1. \]

Thus,

\begin{equation}
\Upsilon_2(k) = \begin{cases} 0, & k > 1, \\ -1, & 0 < k < 1. \end{cases}
\end{equation}

One should not try to get the expression of \( C_2(t) \) by setting \( k = 1 \) in (8.24), since \( \Upsilon_2(1) \) is not defined. We claim that

\begin{equation}
C_2(t) = -\frac{1}{2} - \frac{1}{2} \int_{-\infty}^{t} e^{\frac{t}{2} + 3/2} \left( \text{Ai}(s) + \frac{1}{\sqrt{s}} \text{Ai}'(s) \right) ds.
\end{equation}

**Proof.** Let \( k_P = 1 + \epsilon - \frac{c^2}{6} + \frac{5}{12} \epsilon^3 + \cdots \) and \( k_N = 1 - \epsilon - \frac{c^2}{6} - \frac{5}{12} \epsilon^3 + \cdots \) be the two roots of \( \frac{1}{3}(3 - k^2)k = \frac{2}{3} - \epsilon^2 \), \( \epsilon > 0 \). Then

\[ C_2(t) + \frac{1}{2} \left( \frac{t}{2} + 3/2 \right) e^{\frac{t}{2} + 3/2} \left( \text{Ai}(s) + \frac{1}{\sqrt{s}} \text{Ai}'(s) \right) ds = 0. \]

Since \( 4\text{Ai}(s) + \frac{\text{Ai}'(s)}{\sqrt{s}} = e^{-\frac{1}{2} s^{3/2}} \left( \frac{3}{2 \sqrt{s}} s^{-\frac{1}{2}} - \frac{9}{4 \sqrt{s}} s^{-\frac{3}{2}} + \cdots \right) \), we only need to show \( \lim_{\epsilon \to 0} \epsilon^2 \int_{-\infty}^{t} e^{-\epsilon^2 s^{3/2}} s^{-\frac{1}{2}} ds = 0 \). But \( \epsilon^2 \int_{-\infty}^{t} e^{-\epsilon^2 s^{3/2}} s^{-\frac{1}{2}} ds = \frac{2}{3} \epsilon \sqrt{t} \left( \text{Erf}(\epsilon t) - 1 \right) \). Therefore, we set \( \epsilon = \epsilon(t) \) smaller than \( t^{-n} \) for any \( n > \frac{5}{4} \), for example, \( \epsilon(t) = e^{-t} \). Then the terms \( \int_{-\infty}^{t} e^{-\epsilon^2 s^{3/2}} s^{-\frac{1}{2}} ds \) and \( \int_{-\infty}^{t} \frac{1}{\sqrt{s}} \text{Ai}(s) + \frac{\text{Ai}'(s)}{\sqrt{s}} ds \) can all be neglected. So (8.29) is obtained. \( \square \)

**Remark 8.4.** For \( k \to \infty \), (8.22) with (8.28) is consistent with \( Y_{12}^{(6)} \approx -\frac{\sqrt{t}}{x} \).

Altogether, (8.22) with (8.28) is convenient for estimating \( Y_{12}^{(6)} \) on \( x = k\sqrt{t}, k \neq 1 \), and (8.22) with (8.29) is proper for estimating \( Y_{12}^{(6)} \) near \( x = \sqrt{t} \).

### 8.2 The asymptotics of \( Y^{(3)}(x, t) \) for \( x \to -\infty \) and \( t \to \infty \) along \( x = k\sqrt{t} \)

The behaviour of \( Y^{(3)} \) are similar to \( Y^{(6)} \) presented in Section 8.1.

In this case, \( k < 0 \). \( x = -\sqrt{t} \) is the dividing line. By Lemma 8.1, we know \( Y_{11}^{(3)} \to 1 \) and \( Y_{22}^{(3)} \to 1 \). Therefore, the approximate differential equations for \( Y_{21}^{(3)} \) and \( Y_{12}^{(3)} \) along \( x = k\sqrt{t} \) are

\begin{align}
\frac{d}{dt} Y_{21}^{(3)} &= \frac{(3 - k^2)k}{2} \sqrt{t} Y_{21}^{(3)} + k^2 - \frac{2}{2} \text{Ai}(t) + \frac{k}{2\sqrt{t}} \text{Ai}'(t), \\
\frac{d}{dt} Y_{12}^{(3)} &= \frac{(k^2 - 3)k}{2} \sqrt{t} Y_{12}^{(3)} + k^2 - \frac{2}{2} \text{Ai}(t) - \frac{k}{2\sqrt{t}} \text{Ai}'(t).
\end{align}

The approximate differential equations for \( Y_{21}^{(3)} \) and \( Y_{12}^{(3)} \) for fixed but large \( t \) are

\begin{align}
\frac{d}{dx} Y_{21}^{(3)} &= (t - x^2) Y_{21}^{(3)} + x \text{Ai}(t) + \text{Ai}'(t), \\
\frac{d}{dx} Y_{12}^{(3)} &= (-t + x^2) Y_{12}^{(3)} + x \text{Ai}(t) - \text{Ai}'(t).
\end{align}

#### 8.2.1 \( Y_{12}^{(3)} \)

The behaviour of \( Y_{12}^{(3)} \) is similar to \( Y_{21}^{(3)} \).

**Case** \(-2 < k < 0\):
In this case, the solution of (8.31) is

\[ Y_{21}^{(3)} = \Upsilon_3(k) e^{\frac{k}{2} (k^2 - 3)^{3/2}} + e^{\frac{k}{2} (k^2 - 3)^{3/2}} \int_{-\infty}^t e^{\frac{k}{2} (3-k^2)^{3/2}} \left( \frac{k^2 - 2}{2} \text{Ai}(s) - \frac{k}{2\sqrt{s}} \text{Ai}'(s) \right) ds. \]

The solution of (8.33) is

\[ Y_{12}^{(3)} = C_3(t) e^{-tx + \frac{k}{2} s^{3/2}} + e^{-tx + \frac{k}{2} s^{3/2}} \int_{-\infty}^s e^{ts - \frac{k}{2} s^{3/2}} (s\text{Ai}(t) - \text{Ai}'(t)) ds. \]

So we have

\[ C_3(t) - \Upsilon_3(k) = \int_{-\infty}^t e^{\frac{k}{2} (3-k^2)^{3/2}} \left( \frac{k^2 - 2}{2} \text{Ai}(s) - \frac{k}{2\sqrt{s}} \text{Ai}'(s) \right) ds \]

\[ - \int_{-\infty}^s e^{ts - \frac{k}{2} s^{3/2}} (s\text{Ai}(t) - \text{Ai}'(t)) ds. \]

By \( \Upsilon_3(k) = 0 \) for \( k \in [-\sqrt{3}, 0] \), we get

\[ C_3(t) = \int_{-\infty}^t e^{-\frac{k}{2} s^{3/2}} \left( -\frac{1}{2} \text{Ai}(s) + \frac{1}{2\sqrt{s}} \text{Ai}'(s) \right) ds. \]

Taking the \( t \to \infty \) limit of (8.36) and considering (8.37), we obtain

\[ \Upsilon_3(k) = 0, \quad k \in (-2, 0). \]

Therefore, the final result for \(-2 < k < 0\) is

\[ Y_{12}^{(3)} \approx e^{\frac{k}{2} (k^2 - 3)^{3/2}} \int_{-\infty}^t e^{\frac{k}{2} (3-k^2)^{3/2}} \left( \frac{k^2 - 2}{2} \text{Ai}(s) - \frac{k}{2\sqrt{s}} \text{Ai}'(s) \right) ds. \]

Case \( k = -2 \):

\[ Y_{12}^{(3)} \approx e^{-\frac{2}{3} s^{3/2}} \int_{-\infty}^t e^{\frac{2}{3} s^{3/2}} \left( \text{Ai}(s) + \frac{1}{\sqrt{s}} \text{Ai}'(s) \right) ds. \]

Case \( k < -2 \):

In this case, the solution of (8.31) is

\[ Y_{12}^{(3)} = \Upsilon_4(k) e^{\frac{k}{2} (k^2 - 3)^{3/2}} + e^{\frac{k}{2} (k^2 - 3)^{3/2}} \int_{-\infty}^t e^{\frac{k}{2} (3-k^2)^{3/2}} \left( \frac{k^2 - 2}{2} \text{Ai}(s) - \frac{k}{2\sqrt{s}} \text{Ai}'(s) \right) ds. \]

Thus, for \( k < -2 \), \( Y_{12}^{(3)} \) is approximated by

\[ Y_{12}^{(3)} \approx e^{\frac{k}{2} (k^2 - 3)^{3/2}} \int_{-\infty}^t e^{\frac{k}{2} (3-k^2)^{3/2}} \left( \frac{k^2 - 2}{2} \text{Ai}(s) - \frac{k}{2\sqrt{s}} \text{Ai}'(s) \right) ds. \]

**8.2.2 \( Y_{21}^{(3)} \)**

For all \( k < 0 \), (8.30) has the solution

\[ Y_{21}^{(3)} = \Upsilon_4(k) e^{\frac{k}{2} (3-k^2)^{3/2}} + e^{\frac{k}{2} (3-k^2)^{3/2}} \int_{-\infty}^t e^{\frac{k}{2} (k^2 - 3)^{3/2}} \left( \frac{k^2 - 2}{2} \text{Ai}(s) + \frac{k}{2\sqrt{s}} \text{Ai}'(s) \right) ds. \]

The solution of (8.32) is

\[ Y_{21}^{(3)} = C_4(t) e^{tx - \frac{k}{2} s^{3/2}} + e^{tx - \frac{k}{2} s^{3/2}} \int_{-\infty}^s e^{ts - \frac{k}{2} s^{3/2}} (s\text{Ai}(t) + \text{Ai}'(t)) ds. \]
By the consistence of (8.41) and (8.42) for $x = k\sqrt{t}$, we get

$$C_4(t) - \Upsilon_4(k) = \int_{-\infty}^{t} e^{\frac{k}{2}(k^2 - 3)s^{3/2}} \left( \frac{k^2}{2} - 2 \text{Ai}(s) + \frac{k}{2\sqrt{s}} \text{Ai}'(s) \right) ds$$

(8.43)

$$- \int_{-\sqrt{t}}^{-k\sqrt{t}} e^{-ts + s^{3/2}} (s\text{Ai}(t) + \text{Ai}'(t)) ds.$$

Let us fix $k$, $k < -1$. Near $t = \infty$, the first term of the right-side of (8.43) can be neglected since it is exponentially small. Therefore, we obtain

$$C_4(t) - \Upsilon_4(k) \approx -\int_{-\sqrt{t}}^{-k\sqrt{t}} e^{-ts + s^{3/2}} (\text{Ai}(t)s + \text{Ai}'(t)) ds$$

$$= -\int_{0}^{k+1} e^{\frac{2}{3}r^{3/2} - \frac{1}{3}r^{3/2} + \frac{1}{4}t^{3/2}r^{3}} \left( \text{Ai}(t)\sqrt{t}(-1 + r) + \text{Ai}'(t) \right) \sqrt{t} dr$$

$$\approx -\int_{0}^{\infty} e^{-r^{3/2}r^{3}} \left( 1 + \frac{1}{3}r^{3/2}r^{3} + \ldots \right) e^{\frac{2}{3}r^{3/2}} \left( \text{Ai}(t)\sqrt{t}(r - 1) + \text{Ai}'(t) \right) \sqrt{t} dr$$

(8.44)

$$= -\frac{1}{2} - \frac{1}{12\sqrt{\pi}} t^{-\frac{3}{2}} + \frac{35}{1728\sqrt{\pi}} t^{-\frac{3}{2}} + \ldots.$$

By the condition that $\Upsilon_4(k) = 0$ for $k \leq -\sqrt{3}$, we get

$$\Upsilon_4(k) = 0, \quad k < -1,$$

(8.45)

$$C_4(t) = -\frac{1}{2} - \frac{1}{12\sqrt{\pi}} t^{-\frac{3}{2}} + \frac{35}{1728\sqrt{\pi}} t^{-\frac{3}{2}} + \ldots.$$

Similarly, for $-1 < k < 0$, we have

(8.46)

$$C_4(t) - \Upsilon_4(k) = -\frac{1}{2} - \frac{1}{12\sqrt{\pi}} t^{-\frac{3}{2}} + \frac{35}{1728\sqrt{\pi}} t^{-\frac{3}{2}} + \ldots.$$

Therefore,

$$\Upsilon_4(k) = -1, \quad -1 < k < 0.$$

Thus,

(8.47)

$$\Upsilon_4(k) = \begin{cases} 0, & k < -1, \\ -1, & -1 < k < 0. \end{cases}$$

We claim that

(8.48)

$$C_4(t) = -\frac{1}{2} - \frac{1}{12\sqrt{\pi}} t^{-\frac{3}{2}} \left( \text{Ai}(s) + \frac{1}{\sqrt{s}} \text{Ai}'(s) \right) ds.$$

The proof is similar to the case in Section 8.3, and thus we omit it.

9 Proof of Theorem 1.3

By (1.10), we only need to prove that $F(x, t)$ satisfies the Bloemendal-Virág boundary (1.7) and that $F(x, t)$ is bounded at $x^2 + t^2 = \infty$. 

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Formula (9.1) is proper for $F(x, t) \rightarrow 0$. By considering $F(x, t)$ on the boundary $AB$, [11] proved

$$
\begin{pmatrix}
F_0(x, t) \\
G_0(x, t)
\end{pmatrix} = i \begin{pmatrix}
\Psi^{(6)}_{012}(x, t) \\
\Psi^{(6)}_{022}(x, t)
\end{pmatrix},
$$

where $F_0$ and $G_0$ are defined by (2.3), and $\Psi^{(6)}_0$ is a canonical wave solution of Painlevé II.

By (2.9), we have

$$
\dot{F}(x, t) = -\kappa u^2 \left( x - \alpha \right) Y_1^{(6)}(x, t) + Y_2^{(6)}(x, t).
$$

Formula (9.1) is proper for $x \geq 0$.

The expression of $\dot{F}(x, t)$ for $x \leq 0$ has also been given by [11]

$$
\dot{F}(x, t) = -\kappa u^2 \left( x - \alpha \right) Y_1^{(3)}(x, t) + Y_2^{(3)}(x, t).
$$

Note $\alpha = 1$ has been applied to the expression in [11]. By (8.3) and (8.4), (9.1) and (9.2) coincide on \(x = 0\).

### 9.1 Boundedness of $F(x, t)$ to $c_1 = 0$ and $c_2 = 0$

Let us investigate $F(x, t)$ at the boundary of $CD$, i.e., $x \to -\infty$ and $t \to \infty$ simultaneously.

It is convenient to study the case along $x = k\sqrt{t}$, $k \in (-\sqrt{3}, 0)$. In this case, $e^{\frac{x^3}{3} - xt}$ is very large:

$$
e^{\frac{x^3}{3} - xt} = e^{\frac{1}{3}k(k^2 - 3)t^{3/2}}.
$$

The largest case is $k = -1$, i.e.,

$$
e^{\frac{x^3}{3} - xt} = e^{\frac{4}{3}t^{3/2}}.
$$

By (8.3), we know $-\kappa u^2 e^{\frac{x^3}{3} - xt} Y_3^{(3)}(x, t)$ only contributes a finite term $-\Psi_4(k)$. Thus we can temporarily neglect it.

Also we know for $t > 0$, $Y_3^{(3)}(x, t) \to 1$. Then by (8.3), we have

$$
limit_{x = k\sqrt{t}, k \in (-\sqrt{3}, 0), t \to \infty} F(x, t) = -\Psi_4(k) + \lim_{k \in (-\sqrt{3}, 0), t \to \infty} -e^{\frac{1}{3}k(k^2 - 3)t^{3/2}} u(t)^{-1} \left( k\sqrt{t} + \frac{q_2(t)}{2} - \alpha(t) \right).
$$

Since $\frac{1}{3}k(k^2 - 3)$ varies in $(0, \frac{4}{3})$ for $k \in (-\sqrt{3}, 0)$ and $u(t)^{-1}$ has order of $e^{-\frac{t^{3/2}}{3}}$, we have to require $k\sqrt{t} + \frac{q_2(t)}{2} - \alpha(t)$ has order of $e^{-\frac{t^{3/2}}{3}}$. By (1.19) and (1.19), we find it is only possible for $c_1 = c_2 = 0$. 

Figure 10. Diagram of the boundary $x^2 + t^2 = \infty$. $AB$: $x \to \infty$ and $t \to \infty$; $BC$: fixed $x$ and $t \to \infty$; $CD$: $x \to -\infty$ and $t \to \infty$; $DE$: $x \to -\infty$ and fixed $t$; $EF$: $x \to -\infty$ and $t \to -\infty$; $FG$: fixed $x$ and $t \to -\infty$; $GH$: $x \to \infty$ and $t \to -\infty$; $HA$: $x \to \infty$ and fixed $t$.
9.2 \( c_1 = c_2 = 0 \) to boundedness of \( F(\beta = 6; x, t) \) at \( x^2 + t^2 = \infty \)

9.2.1 On \( HA \)

In this case, \( t \) is fixed, \( x \to \infty \),
\[
    F(x, t) = \kappa u^\frac{1}{2} \left[ u^{-1} \left( \frac{1 + q_2}{2} x - \alpha \right) Y_{12}^{(6)}(x, t) + Y_{22}^{(6)}(x, t) \right].
\]

Recall \( Y_{12}^{(6)} \to \frac{u(t)}{\beta} \) and \( Y_{22}^{(6)} \to 1 \) in this case, we get
\[
    \lim_{x \to \infty} F(x, t) = -\kappa(t) u(t)^{\frac{1}{2}} q_2(t) + \frac{1}{2} \kappa(t) u(t)^{\frac{1}{2}} (1 - q_2(t)).
\]

It is straightforward to verify that (9.3) is the same as (1.22).

9.2.2 On \( AB \)

\[
    F(x, t) = \kappa u^\frac{1}{2} \left[ u^{-1} \left( \frac{1 + q_2}{2} x - \alpha \right) Y_{12}^{(6)}(x, t) + Y_{22}^{(6)}(x, t) \right].
\]

By \( Y_{22}^{(6)}(x, t) \to 1 \) on \( AB \), we know
\[
    \lim_{x \to \infty} \kappa(t) u(t)^{\frac{1}{2}} Y_{22}^{(6)}(x, t) = 1.
\]

By \( Y_{12}^{(6)}(x, t) \to 0 \) on \( AB \), we have
\[
    \lim_{x \to \infty} \kappa(t) u(t)^{-\frac{1}{2}} \alpha(t) Y_{12}^{(6)}(x, t) = 0.
\]

Then we will show
\[
    \lim_{x \to \infty} \kappa(t) u(t)^{-\frac{1}{2}} \frac{q_2(t)}{2} + \frac{1}{2} x Y_{12}^{(6)}(x, t) = 0.
\]

To prove (9.6), we divide the problem into 2 cases:
(1) \( x \geq t \); (2) \( x \leq t \). In the case (1), by the Riemann-Hilbert problem of \( Y \), it is easy to show \( Y_{12}^{(6)} = o(1) \times x^{-1} \), and thus (9.6) is true. In the case (2),
\[
    \lim_{x \to \infty} \left| \kappa(t) u(t)^{-\frac{1}{2}} \frac{q_2(t)}{2} + \frac{1}{2} x Y_{12}^{(6)}(x, t) \right| \leq \lim_{x \to \infty} \left| \kappa(t) u(t)^{-\frac{1}{2}} \frac{q_2(t)}{2} + \frac{1}{2} x \right| \times Y_{12}^{(6)}(x, t).
\]

Considering \( Y_{12}^{(6)} \to 0 \) for \( x > 0 \), we know (9.6) is also true in this case. Thus (9.6) is proved.

Gathering (9.4), (9.5) and (9.6), we get
\[
    \lim_{x \to \infty} F(x, t) = 1.
\]

Remark 9.1. By Lemma 8.1 (8.22) and Remark 8.4, we know \( c_1 = 0 \) is enough to guarantee \( \lim_{x \to \infty} F(x, t) = 1 \) on \( AB \).

9.2.3 On \( BC \)

Case \( x \geq 0 \).
\[
    F(x, t) = \kappa u^\frac{1}{2} \left[ u^{-1} \left( \frac{1 + q_2}{2} x - \alpha \right) Y_{12}^{(6)}(x, t) + Y_{22}^{(6)}(x, t) \right].
\]

In this case, \( x \) is finite and \( t \) is positive infinite. So, \( Y_{12}^{(6)}(x, t) \leq 1 \) and \( Y_{22}^{(6)}(x, t) \to 1 \). Therefore,
\[
    \lim_{t \to \infty} F(x, t) = \lim_{t \to \infty} \kappa u^\frac{1}{2} \left[ u^{-1} \left( \frac{1 + q_2}{2} x - \alpha \right) Y_{12}^{(6)}(x, t) + Y_{22}^{(6)}(x, t) \right] \]
\[
    = 1, \quad x \geq 0.
\]

The division is at liberty. For example, for given \( \epsilon > 0 \), any division of \( x \geq t^{\frac{1}{2}+\epsilon} \) and \( x \leq t^{\frac{1}{2}+\epsilon} \) works.
Case $x \leq 0$.

\[
F(x, t) = -\kappa(t)u(t)\frac{d}{dt}e^{\frac{q^2 t}{2} - xt} \left[ u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y_{11}^{(3)}(x, t) + Y_{21}^{(3)}(x, t) \right].
\]

By Lemma 8.1, we know $Y_{11}^{(3)}(x, t) \approx 1$ and $Y_{21}^{(3)}(x, t) \approx -e^{-\frac{t}{2} - \frac{q^2 t}{4}}$. So we get

\[
\lim_{t \to \infty} F(x, t) = \lim_{t \to \infty} -\kappa(t)u(t)\frac{d}{dt}e^{\frac{q^2 t}{2} - xt} \left[ u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y_{11}^{(3)}(x, t) + Y_{21}^{(3)}(x, t) \right] = 1, \quad x \leq 0.
\]

9.2.4 On $CD$

\[
F(x, t) = -\kappa(t)u(t)\frac{d}{dt}e^{\frac{q^2 t}{2} - xt} \left[ u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y_{11}^{(3)}(x, t) + Y_{21}^{(3)}(x, t) \right].
\]

This is the most complicated case. The result is

\[
\lim_{x \to -\infty, t \to \infty} F(x, t) = \begin{cases} 
0, & \frac{x+\sqrt{2}t}{t} \to -\infty, \\
1, & \frac{x+\sqrt{2}t}{t} \to \infty, \\
e(0, 1), & \text{near } x = -\sqrt{t}.
\end{cases}
\]

The corresponding proof is given in Appendix [C].

9.2.5 On $DE$

\[
F(x, t) = -\kappa(t)u(t)\frac{d}{dt}e^{\frac{q^2 t}{2} - xt} \left[ u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y_{11}^{(3)}(x, t) + Y_{21}^{(3)}(x, t) \right].
\]

Since $t$ is finite, $Y_{11}^{(3)}(x, t) \to 1$ and $Y_{21}^{(3)}(x, t) \to 0$, we obtain

\[
\lim_{x \to -\infty} F(x, t) = 0.
\]

9.2.6 On $EF$

\[
F(x, t) = -\kappa(t)u(t)\frac{d}{dt}e^{\frac{q^2 t}{2} - xt} \left[ u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y_{11}^{(3)}(x, t) + Y_{21}^{(3)}(x, t) \right].
\]

Let us first evaluate $Y_{11}^{(3)}(x, t)$ and $Y_{21}^{(3)}(x, t)$ along the curve $x = -\sqrt{2\sqrt{A_1 + t}}$ for $t \in [-A_1, 0]$ with $A_1 \geq 1$.

Along the curve,

\[
\frac{d}{dt} \begin{pmatrix} Y_{11}^{(3)}(t) \\ Y_{21}^{(3)}(t) \end{pmatrix} = \frac{1}{\sqrt{2\sqrt{A_1 + t}}} \begin{pmatrix} u(t)^2 & u(t) \\ -u(t)^2 & -u(t)^2 \end{pmatrix} \begin{pmatrix} Y_{11}^{(3)}(t) \\ Y_{21}^{(3)}(t) \end{pmatrix}.
\]

By (B.6), we get

\[
\ln \left( Y_{11}^{(3)}(0)^2 + Y_{21}^{(3)}(0)^2 \right) \geq \ln \left( Y_{11}^{(3)}(t)^2 + Y_{21}^{(3)}(t)^2 \right) + \frac{1}{2} \int_t^0 \frac{-s - [s + 2u(s)^2]}{\sqrt{2\sqrt{A_1 + s}}^2} ds.
\]

But $Y_{11}^{(3)}(0) = Y_{11}^{(3)}(x = -\sqrt{2\sqrt{A_1 + t}}, t = 0) \approx 1$ and $Y_{21}^{(3)}(0) = Y_{21}^{(3)}(x = -\sqrt{2\sqrt{A_1 + t}}, t = 0) \approx 0$. So

\[
Y_{11}^{(3)}(t)^2 + Y_{21}^{(3)}(t)^2 < \left( e^{-\frac{t}{2} + \frac{s+2u(s)^2}{2\sqrt{A_1 + s}}} \right)^2 ds.
\]

Since $s + 2u(s)^2 \approx 0$ for large negative $s$, we assume $e^{-\frac{t}{2} + \frac{s+2u(s)^2}{2\sqrt{A_1 + s}}} < 1$. Thus $Y_{11}^{(3)}(t)^2 + Y_{21}^{(3)}(t)^2 < 1$ for large negative $t$. Also considering (1.10), (1.11), (2.10) and (6.1), we immediately obtain

\[
\lim_{x \to -\infty, t \to -\infty} F(x, t) = 0.
\]
9.2.7 On \(FG\)

Case \(x \leq 0\).
\[
\mathcal{F}(x, t) = -\kappa(t)u(t) \frac{d^2}{dt^2} \left[ u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y^{(3)}_{11}(x, t) + Y^{(3)}_{21}(x, t) \right].
\]
\[
dY^{(3)}_{11}(x, t) = -u(t)Y^{(3)}_{21}(x, t),
\]
\[
dY^{(3)}_{21}(x, t) = xY^{(3)}_{21}(x, t) - u(t)Y^{(3)}_{11}(x, t).
\]

At \(t = -\infty\), by (6.5), we know
\[
Y^{(3)}_{11}(x, t) = C_1(x) \times \left(1 + \frac{x}{2\sqrt{2(-t)^{1/2}}} + \frac{x^2}{16t} - \frac{8\sqrt{2} + 9\sqrt{2}x^3}{192(-t)^{3/2}} + \ldots \right) e^{4\frac{t}{x^2 - 2t)^{3/2}}} + \ldots,
\]
\[
Y^{(3)}_{21}(x, t) = C_1(x) \times \left(1 - \frac{x}{2\sqrt{2(-t)^{1/2}}} + \frac{x^2}{16t} - \frac{8\sqrt{2} + 9\sqrt{2}x^3}{192(-t)^{3/2}} + \ldots \right) e^{4\frac{t}{x^2 - 2t)^{3/2}}} + \ldots.
\]

Also considering (1.10), (1.11), (2.5) and (2.10), we finally get
\[
\lim_{t \to -\infty} \mathcal{F}(x, t) = 0, \quad x \leq 0,
\]
which, actually, has been proved in Section 9.2.6.

Case \(x \geq 0\).
\[
\mathcal{F}(x, t) = \kappa u(t) \frac{d^2}{dt^2} \left[ u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y^{(6)}_{12}(x, t) + Y^{(6)}_{22}(x, t) \right].
\]
\[
dY^{(6)}_{12}(x, t) = -xY^{(6)}_{12}(x, t) - u(t)Y^{(6)}_{22}(x, t),
\]
\[
dY^{(6)}_{22}(x, t) = -u(t)Y^{(6)}_{12}(x, t).
\]

Similar to the case \(x < 0\), we get
\[
Y^{(6)}_{12}(x, t) = C_2(x) \times \left(1 + \frac{x}{2\sqrt{2(-t)^{1/2}}} + \frac{x^2}{16t} - \frac{8\sqrt{2} + 9\sqrt{2}x^3}{192(-t)^{3/2}} + \ldots \right) e^{4\frac{t}{x^2 - 2t)^{3/2}}} + \ldots,
\]
\[
Y^{(6)}_{22}(x, t) = C_2(x) \times \left(1 - \frac{x}{2\sqrt{2(-t)^{1/2}}} + \frac{x^2}{16t} - \frac{8\sqrt{2} + 9\sqrt{2}x^3}{192(-t)^{3/2}} + \ldots \right) e^{4\frac{t}{x^2 - 2t)^{3/2}}} + \ldots.
\]

Therefore, we also have
\[
\lim_{t \to -\infty} \mathcal{F}(x, t) = 0, \quad x \geq 0,
\]
which can also be inferred from Section 9.2.8.

9.2.8 On \(GH\)

\[
\mathcal{F}(x, t) = \kappa(t) \left( u(t)^{-1} \left( \frac{q_2(t) + 1}{2} x - \alpha(t) \right) Y^{(6)}_{12}(x, t) + u(t)^{1/2}Y^{(6)}_{22}(x, t) \right).
\]

Along the curve \(x = \sqrt{\frac{x}{A_2}} + t\), \(Y^{(6)}_{12}(x, t)\) and \(Y^{(6)}_{22}(x, t)\) satisfy
\[
\frac{d}{dt} \left( \begin{array}{c} Y^{(6)}_{12}(t) \\ Y^{(6)}_{22}(t) \end{array} \right) = \frac{1}{\sqrt{2\sqrt{A_2} + t}} \left( \begin{array}{cc} \frac{1}{2}(t + 2u(t)^2) & -u(t)^{1/2} \\ u(t)^{1/2} & \frac{1}{2}(t + 2u(t)^2) \end{array} \right) \left( \begin{array}{c} Y^{(6)}_{12}(t) \\ Y^{(6)}_{22}(t) \end{array} \right).
\]

By (3.5) and (3.6),
\[
\ln \sqrt{Y^{(6)}_{12}(0)^2 + Y^{(6)}_{22}(0)^2} \leq \ln \sqrt{Y^{(6)}_{12}(t)^2 + Y^{(6)}_{22}(t)^2} \leq 1 \int_{-\infty}^{0} \frac{|s + 2u(s)^2|}{\sqrt{2\sqrt{A_2} + s}} ds,
\]
\[
\ln \sqrt{Y^{(6)}_{12}(0)^2 + Y^{(6)}_{22}(0)^2} \geq \ln \sqrt{Y^{(6)}_{12}(t)^2 + Y^{(6)}_{22}(t)^2} \geq 1 \int_{-\infty}^{0} \frac{|s + 2u(s)^2|}{\sqrt{2\sqrt{A_2} + s}} ds.
\]
Considering (6.5), for a large $A_2$, we have
\[ \int_0^t \frac{\left| s + 2u(s) \right|^2}{\sqrt{2} \sqrt{A_2}} ds \approx 0. \]
Also we know $Y_{12}^{(6)}(0) = Y_{12}^{(6)}(x = \sqrt{2} \sqrt{A_2}, t = 0) \approx 0$ and $Y_{22}^{(6)}(0) = Y_{22}^{(6)}(x = \sqrt{2} \sqrt{A_2}, t = 0) \approx 1$.
Therefore,
\[ (9.9) \quad Y_{12}^{(6)}(t)^2 + Y_{22}^{(6)}(t)^2 \approx 1 \]
on the curve $x = \sqrt{2} \sqrt{A_2} + t$, $t \in [-A_2, 0]$. By (1.16)-(1.17), (2.10), (6.5) and (9.9), we obtain
\[ \lim_{x \to x, t \to -\infty} F(x, t) = \lim_{x \to x, t \to -\infty} \kappa(t) \left( u(t) \frac{q(t) + 1}{2} x - \alpha(t) \right) Y_{12}^{(6)}(x, t) + u(t) Y_{22}^{(6)}(x, t) \]
(9.10)
\[ \frac{1}{2} \kappa(t) u(t) \frac{q(t) + 1}{2} Y_{12}^{(6)}(x, t). \]
By (2.10) and (6.5), if $-t \gg (18 \ln A_2)^{1/2}$, we have $F(t) \sim 0$.
Now let us prove when $-t < 2 \times (18 \ln A_2)^{1/2}$, $F(t) \to 0$ as $t \to -\infty$.
By the mean value theorem,
\[ Y_{12}^{(6)}(t) = Y_{12}^{(6)}(0) + \left( Y_{12}^{(6)} \right)'(\xi) t \]
\[ = Y_{12}^{(6)}(2\sqrt{A_2}, 0) + \frac{1}{\sqrt{2} \sqrt{A_2}} \left( -\frac{1}{2} \left( \xi + 2u(\xi)^2 \right) Y_{12}^{(6)}(\xi) - u'(\xi) Y_{22}^{(6)}(\xi) \right) t, \]
where $t < \xi < 0$.
From
\[ Y_{12}^{(6)}(2\sqrt{A_2}, 0) \approx \frac{-u(0)}{\sqrt{2} \sqrt{A_2}} \frac{\sqrt{2} \sqrt{A_2} + t}{\sqrt{2} \sqrt{A_2} + \xi} < 1, \]
we know
\[ \left| \kappa(t) u(t) - \frac{1}{2} Y_{12}^{(6)}(x(t)) \right| < \left| \kappa(t) u(t) \right| \left| \left( u(0) \right| + \left| \frac{1}{2} \left( \xi + 2u(\xi)^2 \right) Y_{12}^{(6)}(\xi) + u'(\xi) Y_{22}^{(6)}(\xi) \right| t \right| \]
By $t < \xi < 0$, (2.10), (6.5) and (9.9), we have
\[ \lim_{t \to -\infty} \left| \kappa(t) u(t) - \frac{1}{2} Y_{12}^{(6)}(x(t)) \right| = 0. \]
Thus
\[ \kappa(t) u(t) - \frac{1}{2} Y_{12}^{(6)}(x(t)) \approx 0, \]
when $t > -2 \times (18 \ln A_2)^{1/2}$ but large negative enough.
Altogether, we have
\[ \lim_{x \to x, t \to -\infty} F(x, t) = 0. \]

**Appendix A** \quad $k_0 < \frac{10}{3}$

From $k_0 = -\min \left( \frac{u}{\sqrt{2}} \right)$, we know
\[ (A.1) \quad k_0 = 2 \cdot \left[ \min \left( \frac{u}{\sqrt{2}} \right) \right]^{-2}, \quad t \in (-\infty, 0). \]

Following the original arguments of [12], we give a lower bound for the local minimum of $\frac{u}{\sqrt{2}}$ for large negative $t$.

**Proposition A.1.** If there is a local minimum of $\frac{u}{\sqrt{2}}$ for $t < -\frac{11}{8}$, it must be greater than $\sqrt{\frac{1203}{1331}}$. 

Proof. Let \( u(t) = \sqrt{\frac{t}{2}} z(t) \). Obviously, \( z(t) > 0 \). Then \( z \) satisfies 
\[
\frac{z''(t)}{z'(t)} + \frac{z'(t)}{t} = \left( \frac{1}{\sqrt{t}} - \frac{t}{\sqrt{t}} \left( z(t)^2 - 1 \right) \right) z(t). 
\]
At a local minimum, we have \( u'(t) = 0 \) and \( u''(t) > 0 \). Then, we have 
\[
\frac{1}{\sqrt{t}} - t \left( z(t)^2 - 1 \right) > 0, \text{ i.e., } z(t) > \sqrt{1 + \frac{1}{4t}}. \text{ Since } t < -\frac{11}{8}, \text{ the local minimum is greater than } \frac{1203}{1331}. \]
Note that Proposition A.1 does not mean \( \frac{u}{\sqrt{t}} > \frac{1203}{1331} \) for \( t \in (-\infty, -\frac{11}{8}) \) since \( \frac{u}{\sqrt{t}} \) may be smaller near the boundary \( t = -\frac{11}{8} \). But if we could also prove \( \frac{u}{\sqrt{t}} > \frac{1203}{1331} \) for \( t \in [-\frac{11}{8}, 0) \), then we can still conclude \( \frac{u}{\sqrt{t}} > \frac{1203}{1331} \) for \( t \in (-\infty, 0) \). The next proposition fulfills this aim.

**Proposition A.2.** For \( t \in [-\frac{11}{8}, 0) \), \( \frac{u}{\sqrt{t}} > \frac{1203}{1331} \).

Proof. Huang et. al. [13] proved

\begin{equation}
(A.2) \quad \frac{|u(0) - 98}{264} < 11 \times 10^{-4}, \quad \left| u'(0) + \frac{153}{518} \right| < 12 \times 10^{-4}.
\end{equation}

They also defined the approximate solution as

\begin{equation}
(A.3) \quad y_b(t) = \frac{t^{15}}{13206825} + \frac{t^{14}}{717099} + \frac{t^{13}}{81755} + \frac{t^{12}}{15201} + \frac{t^{11}}{47200} + \frac{13t^{10}}{24088} + \frac{39t^9}{53333} + \frac{18t^8}{61523} - \frac{17t^7}{20578} - \frac{93t^6}{35396} - \frac{224t^5}{36015} - \frac{360t^4}{36911} + \frac{203t^3}{10806} + \frac{335302t^2}{688889} + \frac{153t}{518} + \frac{98}{267}.
\end{equation}

and the remainder term as

\begin{equation}
(A.4) \quad R_4(t) = y(t) - ty(t) - 2y(t)^3.
\end{equation}

We can verify

\begin{equation}
(A.5) \quad |R_4(t)| < 2 \times 10^{-3}, \quad t \in [-\frac{11}{8}, 0].
\end{equation}

Let \( \delta_4(t) = u(t) - y_b(t) \). It is easy to show

\begin{equation}
(A.6) \quad \delta'_4(t) = 6(y_b(t)^2 + t)\delta_4(t) + 6y_b(t)\delta_4(t)^2 + 2\delta_4(t)^3 - R_4(t).
\end{equation}

Next, we will show \( \delta_4(t) \) is sufficiently small for \( t \in [-\frac{11}{8}, 0] \).

We can verify \( \frac{4}{8} < 6y_b(t)^2 + t < \frac{11}{8} \) and \( \frac{4}{10} < 6y_b(t)^2 < \frac{49}{10} \) for \( t \in [-\frac{11}{8}, 0] \). Therefore, we have \( \delta_4(t) \geq \delta_b(t) \) in the interval, where \( \delta_b(t) \) is defined by

\begin{equation}
(A.7) \quad \delta''_b(t) = \frac{13}{5} \delta_b(t) - \frac{49}{10} \delta_b(t)^2 + 2\delta_b(t)^3 - \frac{1}{500}, \quad \delta_b(0) = \frac{11}{10000}, \quad \delta_b'(0) = \frac{3}{2500}.
\end{equation}

So we have

\begin{equation}
|\delta_b(t)| < 120 \times 10^{-4} = \frac{3}{2500} \text{ for } t \in [-\frac{11}{8}, 0].
\end{equation}

When \( t \in [-\frac{11}{8}, 0] \), we can show \( \frac{u(t)}{\sqrt{t}} > \sqrt{\frac{1203}{1331}} \). Therefore, for \( t \in [-\frac{11}{8}, 0] \), we have

\begin{equation}
\min \left( \frac{u(t)}{\sqrt{t}} \right) = \min \left( \frac{y_b(t) + \delta_b(t)}{\sqrt{t}} \right) > \min \left( \frac{y_b(t) + \delta_b(t)}{\sqrt{t}} \right) > \min \left( \frac{y_b(t) - \frac{3}{2500}}{\sqrt{t}} \right) > \sqrt{\frac{1203}{1331}}.
\end{equation}

Combining Proposition A.1 and A.2 we obtain \( \min \left( \frac{u(t)}{\sqrt{t}} \right) > \sqrt{\frac{1203}{1331}} \). By (A.1), \( k_0 < \frac{2662}{1203} \) is obtained.

\footnote{Since \( R_4(t) \) is a polynomial, Sturm’s theorem applies. The following several cases of verification can also be done in this way.}
Appendix B The growth rate estimate for the solution of a second order linear ODE

For $a > 0$, $c > 0$, $b^2 - 4ac < 0$ and $V = (x, y)^T$, let us define
\[ |V| = \sqrt{ax^2 + bxy + cy^2}, \]
where $|V|$ can be understood as the length of vector $V$. Denote
\[ M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \]
then
\[ r^2 = \frac{|MV|^2}{|V|^2} \]
has both a minimal value $\mathcal{M}_1 = r_1^2$ and a maximal value $\mathcal{M}_2 = r_2^2$. These two extreme values satisfy
\[(B.1) \quad \Delta \mathcal{M}^2 + 2\delta \mathcal{M} + \Delta D^2 = 0,\]
where
\[ \Delta = b^2 - 4ac, \]
\[ D = \det M, \]
\[ \delta = -b^2(m_{11}m_{22} + m_{12}m_{21}) - 2b(am_{12} - cm_{21})(m_{11} - m_{22}) + 2ac^2m_{12}^2 + 2c^2m_{21}^2 + 2ac(m_{11}^2 + m_{22}^2). \]
Assume the linear ODE is of form
\[(B.2) \quad \frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f_{11}(t) & f_{12}(t) \\ f_{21}(t) & f_{22}(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \]
We know
\[ \lim_{N \to \infty} \begin{pmatrix} x(t_N) \\ y(t_N) \end{pmatrix} = \begin{pmatrix} 1 + hf_{11}(t_0 + (N - 1)h) \\ hf_{21}(t_0 + (N - 1)h) \end{pmatrix} \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix}, \]
\[ \ldots \]
\[ \begin{pmatrix} 1 + hf_{11}(t_0 + h) \\ hf_{21}(t_0 + h) \end{pmatrix} \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix}, \]
where $h = \frac{t - t_0}{N}$. Then by \[(B.1)\] for $t_1 > t_0$, we can prove
\[(B.3) \quad \ln |(x(t_1), y(t_1))^T| \leq \ln |(x(t_0), y(t_0))^T| + \int_{t_0}^{t_1} \left( \frac{1}{2}(f_{11}(s) + f_{22}(s)) + \sqrt{\frac{H(s)}{\sqrt{-\Delta}} \right) ds, \]
\[(B.4) \quad \ln |(x(t_1), y(t_1))^T| \geq \ln |(x(t_0), y(t_0))^T| + \int_{t_0}^{t_1} \left( \frac{1}{2}(f_{11}(s) + f_{22}(s)) - \sqrt{\frac{H(s)}{\sqrt{-\Delta}} \right) ds, \]
where
\[ H(s) = ac(f_{11}(s) - f_{22}(s))^2 - b(a(f_{12}(s) - cf_{21}(s))(f_{11}(s) - f_{22}(s)) + (a(f_{12}(s) + cf_{21}(s))^2 - b^2f_{12}(s)f_{21}(s). \]
If $a = c = 1$, $b = 0$, for $t_1 > t_0$, \[(B.3)\] and \[(B.4)\] are reduced to
\[(B.5) \quad \ln |(x(t_1), y(t_1))^T| \leq \frac{1}{2} \int_{t_0}^{t_1} \left( f_{11}(s) + f_{22}(s) + \sqrt{(f_{11}(s) - f_{22}(s))^2 + (f_{12}(s) + f_{21}(s))^2} \right) ds + \ln |(x(t_0), y(t_0))^T|, \]
\[(B.6) \quad \ln |(x(t_1), y(t_1))^T| \geq \frac{1}{2} \int_{t_0}^{t_1} \left( f_{11}(s) + f_{22}(s) - \sqrt{(f_{11}(s) - f_{22}(s))^2 + (f_{12}(s) + f_{21}(s))^2} \right) ds + \ln |(x(t_0), y(t_0))^T|. \]
Appendix C \( \mathcal{F}(x, t) \) on \( CD \)

\[
\mathcal{F}(x, t) = -\kappa(t)u(t)^{\frac{\pi}{4}}e^{\frac{\pi^2}{4}xt} \left[ u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y_{11}^{(3)}(x, t) + Y_{21}^{(3)}(x, t) \right].
\]

We divide the region \( CD \) into two parts: \( x \leq -\sqrt[3]{3}t \) and \( x \geq -\sqrt[3]{3}t. \)

C.1 Case \( x \leq -\sqrt[3]{3}t. \)

In this case, \( Y_{21}^{(3)}(x, t) \to 0 \) and \( e^{\frac{\pi^2}{4}xt} \leq 1, \) so

\[
\lim_{x \to -\sqrt[3]{3}t, t \to \infty} -\kappa(t)u(t)^{\frac{\pi}{4}}e^{\frac{\pi^2}{4}xt}Y_{11}^{(3)}(x, t) = 0.
\]

By \( Y_{11}^{(3)}(x, t) \to 1, \) we know

\[
\lim_{x \to -\sqrt[3]{3}t, t \to \infty} \kappa(t)u(t)^{\frac{\pi}{4}}e^{\frac{\pi^2}{4}xt}u(t)^{-1}\alpha(t)Y_{11}^{(3)}(x, t) = 0.
\]

Thus

\[
\lim_{x \to -\sqrt[3]{3}t, t \to \infty} \mathcal{F}(x, t) = \lim_{x \to -\sqrt[3]{3}t, t \to \infty} -\kappa(t)u(t)^{\frac{\pi}{4}}e^{\frac{\pi^2}{4}xt}u(t)^{-1}q_2(t) + \frac{1}{2}Y_{11}^{(3)}(x, t)
\]

\[
= \lim_{x \to -\sqrt[3]{3}t, t \to \infty} xe^{\frac{\pi^2}{4}xt}u(t)^{-1}q_2(t) + \frac{1}{2}.
\]

(C.1) can be proved to be 0 by dividing the region \( x \leq -\sqrt[3]{3}t \) into two parts, for example \( x \leq -2\sqrt[3]{3}t \) and \( -2\sqrt[3]{3}t \leq x \leq -\sqrt[3]{3}t. \) In both parts, \( xe^{\frac{\pi^2}{4}xt}u(t)^{-1} \to 0 \) is obvious.

Therefore, we have

\[
\lim_{x \to -\sqrt[3]{3}t, t \to \infty} \mathcal{F}(x, t) = 0.
\]

C.2 Case \( x \geq -\sqrt[3]{3}t. \)

First we show

\[
\lim_{x \to -\infty, t \to \infty, x \geq -\sqrt[3]{3}t} -\kappa(t)u(t)^{\frac{\pi}{4}}e^{\frac{\pi^2}{4}xt}u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y_{11}^{(3)}(x, t) = 0.
\]

In fact, by (1.20) and (1.21), we obtain their expansions at \( t = \infty \) as

\[
q_2(t) + 1 = e^{-\frac{t}{4\sqrt{3}t}} \left( \frac{1}{18\pi} t^{-\frac{3}{2}} - \frac{59}{192\pi} t^{-3} + \ldots \right),
\]
\[
\alpha(t) = e^{-\frac{t}{4\sqrt{3}t}} \left( \frac{3}{16\pi} t^{-1} - \frac{29}{128\pi} t^{-\frac{3}{2}} + \ldots \right).
\]

Also we have

\[
u(t)^{-1} \approx \frac{1}{\lambda(t)} = e^{\frac{t}{4\sqrt{3}t}} \left( 2 \sqrt{3}t^{-\frac{3}{2}} + \frac{5}{24} \sqrt{3}t^{-\frac{5}{2}} + \ldots \right).
\]

Therefore,

\[
\lim_{x \to -\infty, t \to \infty, x \geq -\sqrt[3]{3}t} -\kappa(t)u(t)^{\frac{\pi}{4}}e^{\frac{\pi^2}{4}xt}u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right) Y_{11}^{(3)}(x, t)
\]
\[
= \lim_{x \to -\infty, t \to \infty, x \geq -\sqrt[3]{3}t} -e^{\frac{\pi^2}{4}xt}u(t)^{-1} \left( \frac{1 + q_2(t)}{2} x - \alpha(t) \right)
\]
\[
= \lim_{x \to -\infty, t \to \infty, x \geq -\sqrt[3]{3}t} -e^{\frac{\pi^2}{4}xt}e^{\frac{t}{4\sqrt{3}t}} \left( \frac{1}{4\sqrt{3}t} x^{-\frac{3}{2}} - \frac{3}{8\sqrt{3}t} x^{-\frac{5}{2}} \right).
\]
But \( x^3 - xt - \frac{3}{4}t^2 \leq 0 \) in this case and also \( |x| \leq \sqrt{3t} \). So we get (C.2).

By (8.41), we have

\[
(C.5) \quad \lim_{x \to kV1, k < 0, k \neq -1, t \to \infty} -\kappa(t)u(t)\frac{x^3}{3}e^{\frac{x^3}{3t}}y^{(3)}_{21}(x, t) = -\Upsilon_4(k).
\]

Near \( x = -\sqrt{t} \), by (8.42), we get

\[
(C.6) \quad -\kappa(t)u(t)\frac{x^3}{3}e^{\frac{x^3}{3t}}y^{(3)}_{21}(x, t) \approx \frac{1}{2} + \frac{1}{2} \int_{x}^{t} e^{\frac{s^3}{3t}} \left( \frac{\text{Ai}(s) + \frac{1}{\sqrt{s}}\text{Ai}'(s)}{\sqrt{s}} \right) \, ds - \int_{-\sqrt{t}}^{x} e^{-ts + \frac{3}{3t}s^3} \left( s\text{Ai}(t) + \text{Ai}'(t) \right) \, ds
\]

\[
(C.7) \quad \approx \frac{1}{2} - \int_{0}^{T+1} e^{\frac{3}{3t}s^3 - \frac{3}{3t}s^3 + \frac{3}{3t}s^3} \left( \text{Ai}(t)\sqrt{t}(r - 1) + \text{Ai}'(t) \right) \, \sqrt{t} \, dr.
\]

By (C.6), we see \(-\kappa(t)u(t)\frac{x^3}{3}e^{\frac{x^3}{3t}}y^{(3)}_{21}(x, t)\) is monotonic increasing with \( x \). So it must lie in \((0, 1)\). (C.7) is convenient for estimating its value.

**C.2.1 Case** \( k\sqrt{t} < x \leq 0, k > -1 \).

(C.6) is valid for a large positive \( t \). So does (C.3). Along the line that \( t \) is fixed, by (8.35) we get

\[
(C.8) \quad \frac{d}{dx} \left( -e^{\frac{x^3}{3t}}y^{(3)}_{21}(x, t) \right) = -e^{-tx + \frac{3}{3t}x^3} \left( x\text{Ai}(t) + \text{Ai}'(t) \right).
\]

As

\[
\frac{\text{Ai}'(t)}{\text{Ai}(t)} = -\sqrt{t} - \frac{1}{4t} + \frac{t}{32}t^{-2} + \ldots,
\]

(C.8) never vanishes in the region. Thus \(-e^{\frac{x^3}{3t}}y^{(3)}_{21}(x, t)\) is monotonic increasing in the region. But it is known

\[-e^{\frac{x^3}{3t}}y^{(3)}_{21}(x, t) |_{x = k\sqrt{t}} \approx 1, \quad k > -1,
\]

and

\[-e^{\frac{x^3}{3t}}y^{(3)}_{21}(x, t) |_{x = 0} \approx F(x, t) |_{x = 0} \approx 1.
\]

We must conclude

\[
\lim_{k\sqrt{t} < x < 0, k > -1, t \to \infty} F(x, t) = 1.
\]

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