Characterization of stress concentration in two dimensions*

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Abstract

We consider a boundary value problem of the anti-plane elasticity in a domain containing an inclusion which is nearly touching to the domain’s boundary. We assume that the domain and the inclusion are disks. By using the boundary integral formulation for the interface problem and adopting the bipolar coordinates, we derive the asymptotic formulas which explicitly describe the gradient blow-up of the solution as the distance between the inclusion and the domain’s boundary tends to zero. We also consider the boundary value problem for the Lamé system in a circular domain containing a circular hole. We show that the stress tensor blows up under the uniform boundary traction, as the distance tends to zero. Additionally, we provide a Fourier series solution in bipolar coordinates for the Lamé system in the whole plane with an inclusion of core-shell geometry.

Key words. Stress concentration, Asymptotic analysis, Boundary value problem, Bipolar coordinates, Anti-plane elasticity, Lamé system

Contents

1 Introduction 2

2 Series solution for the interface problem in bipolar coordinates 3
  2.1 Layer potential formulation for the interface problem 3
  2.2 Series solution in bipolar coordinates 5

3 Asymptotic analysis for electric field concentration 6
  3.1 Decomposition of $u$ into singular part and regular part 6
  3.2 Expanding singular stress in series by separation of variables 9
  3.3 Expressing singular stress in terms of Lerch transcendent functions 10
  3.4 Deriving pointwise rate of gradient blow-up 11

4 Image line charge formula 12
  4.1 Approximation by image line charge 12

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1 Introduction

Placing high-contrast elastic materials close together increases the risk of stress accumulation. When the stress intensity exceeds a threshold, it causes fracture, for example, in composite materials. On the other hand, the electric field concentrated in a small region has many practical applications. Describing the concentrated stress by asymptotic analyses on the electricity and linear elasticity problems is the main scope of this paper.

Unfortunately, when the stress concentration takes place, existing numerical solvers demand a high cost. As for the electric field concentration, such a difficulty has been overcome by finding explicit formulas that approximate the solutions [17, 8, 7]. Recently, an analogous study [1] of anti-plane elasticity fully characterized the blow-up behavior of stress in core-shell structure by image line charges.

Another approach using variational principles has been partially successful in linear elasticity. The upper bound of the increasing rate of stress was obtained by Bao et al, as the distance $\varepsilon$ between two convex inclusions gets smaller. The rate turned out to be $O(\varepsilon^{-1/2})$ in the planar case [6], and $O(|\varepsilon \log \varepsilon|^{-1})$ in three dimensions [3], respectively. Bao et al [4] also estimated the stress concentration in all dimensions with the core-shell type geometry of elastic materials. However, so far, there has been no research about finding asymptotic formula for accumulated stress in the small region between two circular inhomogeneities of different sizes.

Throughout this article, we consider the two-dimensional position space separated by two circles, one containing the other inside near the boundary. In each connected component, the material is assumed to be isotropic and homogeneous. Specifically, the main purpose of this article is to characterize the gradient blow-up in the following boundary value problem of electricity:

$$
\begin{align*}
\Delta u &= 0 \quad \text{in} \ B_e \setminus \partial B_i, \\
\frac{\partial u}{\partial \nu} &= g \quad \text{on} \ \partial B_e, \\
\frac{\partial u}{\partial \nu} &= k \frac{\partial u}{\partial \nu} \quad \text{on} \ \partial B_i, \\
|u| &= u_- \quad \text{on} \ \partial B_i,
\end{align*}
$$

where the overall domain $B_e$ is a disk containing the other disk $B_i$ and the smooth function $g$ is arbitrarily given as a flux outward. The solution $u$ determines the electric potential in $B_e$. 

5 Series solution for the interface problem in Lamé system

5.1 Problem of linear elasticity in core-shell type geometry

5.2 Method of separation of variables for bi-harmonic equation

5.3 Solution representation for transmission problem

6 Conclusion

A Proof of Theorem 3.5

B Proof of Corollary 3.8

C Proof of Lemma 5.1
when the inclusion $B_i$ has conductivity $k$ while $B_e \setminus B_i$ has conductivity 1. The problem differs from that of Kim et al [1] in that the incident field is described by the flux $g$ instead of the far-field behavior in the free-space problem. A similar but partial approach to the problem of linear elasticity in the free-space problem is given as well.

More explicitly, let $B_i$ and $B_e$ be the two open disks of radius $r_i$ and $r_e$ centered at $(c_i, 0)$ and $(c_e, 0)$ respectively, where $c_i$ and $c_e$ are defined by

$$c_i = \frac{r_e^2 - r_i^2 - (r_e - r_i - \varepsilon)^2}{2(r_e - r_i - \varepsilon)}$$

and

$$c_e = c_i + r_e - r_i - \varepsilon$$

so that $B_e$ contains $B_i$ and the minimum width of $B_e \setminus B_i$ is $\varepsilon$. The asymptotic formula of the solution to (1.1) is obtained by a modified application of the analysis in [1]. As $\varepsilon$ tends to zero and $k \gg 1$, the resulting electric potential $u$ in the shell $B_e \setminus B_i$ turns out to have bounded tangential flux, whereas the singular behavior of its normal flux coincides with that of a combination of the single and double layer potentials generated by virtual line charges:

$$u^*(z) = \frac{4\pi r_ir_e}{r_e - r_i} \left[ C_x \int_{[\alpha,c_i]} \ln|z - s| \varphi_+(s) ds + C_y \int_{[\alpha,c_i]} \Im\{z - s\} \psi_+(s) ds \right].$$

For details, refer to Theorem 4.2. On the other hand, we provide a method of deriving the Fourier coefficients of the airy stress function in the problem of elasticity.

The remainder is organized as follows. In Section 2, the solution $u$ is decomposed into two parts, only one of which contributing to the stress concentration. Then in Section 3 and 4, we derive the asymptotic formulas that explicitly show how each condition of the problem is related to the gradient blow-up. Finally, in Section 5, we provide the method to compute the solution of the problem of linear elasticity using separation of variables in response to arbitrarily given far incident field of stress.

2 Series solution for the interface problem in bipolar coordinates

2.1 Layer potential formulation for the interface problem

Let $S_{\partial B_i}$ and $S_{\partial B_e}$ be the single-layer potentials for Laplace equation acting on the functions in $L^2_0(\partial B_i)$ and $L^2_0(\partial B_e)$ respectively. Then a candidate for the solution to (1.1) is

$$u = u_e \equiv S_{\partial B_i}[g_i] + S_{\partial B_e}[g_e] \quad \text{in } B_e$$

for some smooth mean-zero functions $g_i$ and $g_e$. It turns out from the uniform convergence of series representations for $g_i$ and $g_e$ in Lemma 2.2 that this is indeed the case for any choice of the conductivity $0 < k \neq 1$.

By plugging (2.1) into the boundary conditions for normal derivatives and applying jump formula, we conclude that the problem (1.1) is equivalent to

$$\begin{bmatrix} \lambda & \frac{-\partial_{\nu_e}}{2} S_{\partial B_e} \end{bmatrix} \begin{bmatrix} g_i \\ g_e \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

in $L^2_0(B_i) \times L^2_0(B_e)$, (2.2)
where \( \lambda = \frac{k+1}{2(k-1)} \). The resulting recursive relations for the densities \( g_i \) and \( g_e \) give the series representations for the densities in terms of \( g \). To simplify the summands of the series and prove their convergence, we introduce Lemma 2.1. For ease of notation in Lemma 2.1, let \( R_{\Gamma_i}(z) := c + r^2 \frac{(z-c)}{|z-c|^2} \), which is the reflection about circles \( \Gamma_i \) of radius \( r \) and center \( c \). In accordance with the conventions, we put \( R_{\Gamma_1, \Gamma_2}[f](z) \equiv f(R_{\Gamma_2} \circ R_{\Gamma_1}(z)) \).

**Lemma 2.1** ([17]). Let \( D \) be a disk centered at \( c \) and \( \nu \) be the outward unit normal vector on \( \partial D \). If \( v \) is harmonic in \( D \) and continuous on \( \partial D \),

\[
S_{\partial D} \left[ \frac{\partial v}{\partial \nu} \right]^{-}(x) = \begin{cases} 
-\frac{1}{2} v(x) + \frac{v(c)}{2} & \text{for } x \in D \\
-\frac{1}{2} R_{\partial D}[v](x) + \frac{v(c)}{2} & \text{for } x \in \mathbb{R}^2 \setminus \overline{D}.
\end{cases}
\]

If \( v \) is harmonic in \( \mathbb{R}^2 \setminus \overline{D} \), continuous on \( \partial D \), and \( \lim_{|x| \to \infty} v(x) = 0 \), then

\[
S_{\partial D} \left[ \frac{\partial v}{\partial \nu} \right]^{+}(x) = \begin{cases} 
\frac{1}{2} R_{\partial D}[v](x) & \text{for } x \in D \\
\frac{1}{2} v(x) & \text{for } x \in \mathbb{R}^2 \setminus \overline{D}.
\end{cases}
\]

The following argumentation solves the system of equations (2.2). First, note that

\[
g_i = \frac{1}{\lambda} \partial S_{\partial B_e}[g_i] = \frac{1}{\lambda} \partial S_{\partial B_e} \left[ \frac{2}{\lambda} \partial S_{\partial B_i} \partial S_{\partial B_e} [g_i] - 2g \right],
\]

which gives the recursion formula for \( g_i \). Thereby the density function \( g_i \) can be represented by a series formally as follows:

\[
g_i = -\frac{2}{\lambda} \partial S_{\partial B_e} \left[ g \right] + \frac{2}{\lambda} \partial S_{\partial B_e} \partial S_{\partial B_i} \left[ g_i \right] \\
= -\frac{2}{\lambda} \partial S_{\partial B_e} \left[ g \right] - \left( \frac{2}{\lambda} \right)^2 \partial S_{\partial B_e} \partial S_{\partial B_i} \partial S_{\partial B_e} \left[ g_i \right] + \left( \frac{2}{\lambda} \right)^3 \partial S_{\partial B_e} \partial S_{\partial B_i} \partial S_{\partial B_e} \partial S_{\partial B_i} \left[ g_i \right] \\
= \ldots \\
= -\sum_{n=0}^{\infty} \left( \frac{2}{\lambda} \right)^{n+1} \partial S_{\partial B_e} \left( \partial S_{\partial B_i} \partial S_{\partial B_e} \right)^n \left[ g \right],
\]

where \( R_i \) and \( R_e \) are the shortcuts of \( R_{\partial B_i} \) and \( R_{\partial B_e} \). Then, by Lemma 2.1

\[
g_i = -\frac{2}{\lambda} \sum_{n=0}^{\infty} \left( -\frac{1}{2\lambda} \right)^n \frac{\partial}{\partial v_i} (R_e R_i)^n S_{\partial B_e}[g]. \tag{2.3}
\]

The series representation for \( g_e \) is a direct consequence of (2.3). Indeed, since

\[
g_e = 2 \frac{\partial S_{\partial B_e}}{\partial v_e} [g_i] - 2g = -2g - \frac{4}{\lambda} \sum_{n=0}^{\infty} \left( -\frac{1}{2\lambda} \right)^n \frac{\partial S_{\partial B_i}}{\partial v_i} \frac{\partial}{\partial v_i} (R_e R_i)^n S_{\partial B_e}[g],
\]

the application of Lemma 2.1 once again yields

\[
ge_e = -2g + \frac{2}{\lambda} \sum_{n=0}^{\infty} \left( -\frac{1}{2\lambda} \right)^n \frac{\partial}{\partial v_e} R_i (R_e R_i)^n S_{\partial B_e}[g]. \tag{2.4}
\]
2.2 Series solution in bipolar coordinates

The formal expressions (2.3) and (2.4) satisfy the system of equations (2.2) if the uniform convergence of the series is assumed. To prove the convergence by the actual computation, we introduce the orthogonal coordinate system, where each point in the complex plane corresponds to the pair \((\zeta, \theta)\) via
\[
e^{\zeta + i\theta} = \frac{\alpha + z}{\alpha - z}, \quad -\infty < \zeta < \infty, \quad -\pi < \theta \leq \pi.
\]
Such \((\zeta, \theta)\) are called bipolar coordinates with foci \(\pm \alpha\).

A direct computation in [1] shows that \(\alpha\) is decided uniquely to satisfy
\[
\partial B_e = \{ z(\zeta, \theta) : \zeta = \zeta_i \} \quad \text{and} \quad \partial B_i = \{ z(\zeta, \theta) : \zeta = \zeta_i \}, \quad \text{where} \quad \zeta_i > \zeta_e > 0,
\]
which enables the expression of the densities \(g_i\) and \(g_e\) as functions of \(\theta\). The boundaries can be visualized as in Figure 2.1.

Figure 2.1: The shaded regions are separated by the inner circle \(\partial B_i\) and are contained in the outer circle \(\partial B_e\). Here, the geometric parameters are determined to be \(r_e = 5\), \(r_i = 2\) and \(\varepsilon = 1\).

Converting the coordinate system into \((\zeta, \theta)\), the validity of (2.3) and (2.4) is proved in the next lemma. For ease of notation, let \(h(\zeta, \theta) := (\cosh \zeta + \cos \theta)/\alpha\) so that the scale factors in both directions become
\[
\frac{\partial z}{\partial \zeta} = \frac{\partial z}{\partial \theta} = \frac{1}{h(\zeta, \theta)}.
\]

Lemma 2.2. Let \(G_\varepsilon := S_{\partial B_\varepsilon}[g]\) for each fixed \(\varepsilon\). The density functions \(g_i\) and \(g_e\) can be expressed in terms of the bipolar coordinate system as
\[
\begin{align*}
\left\{ \begin{array}{l}
g_i(\theta) = -2g \sum_{n=0}^{\infty} \left( -\frac{1}{2\lambda} \right)^n \frac{h(\zeta_i, \theta)}{h(2n(\zeta_i - \zeta_e) + \zeta_i, \theta)} \frac{\partial G_e}{\partial \nu}(2n(\zeta_i - \zeta_e) + \zeta_i, \theta), \\
g_e(\theta) = -2g - 2 \sum_{n=0}^{\infty} \left( -\frac{1}{2\lambda} \right)^n \frac{h(\zeta_e, \theta)}{h(2n(\zeta_i - \zeta_e) + 2\zeta_i - \zeta_e, \theta)} \frac{\partial G_e}{\partial \nu}(2n(\zeta_i - \zeta_e) + 2\zeta_i - \zeta_e, \theta).
\end{array} \right.
\end{align*}
\]

Proof. From the change of coordinates by
\[
\frac{\partial}{\partial \nu} \bigg|_{\zeta = \zeta_0} = -\text{sgn}(\zeta_0)h(\zeta_0, \theta) \frac{\partial}{\partial \zeta} \bigg|_{\zeta = \zeta_0}
\]
the series representations (2.5) are formally obtained. The uniform absolute convergence follows from conditions (i) through (iii):
(i) the inequality $\left| \frac{1}{2k} \right| < 1$ for any choice of $0 < k \neq 1$,

(ii) the monotonicity $h(\zeta_1, \theta) \geq h(\zeta_2, \theta)$ for any $\zeta_1 \geq \zeta_2$, and

(iii) the uniform boundedness of $\| \nabla G_\varepsilon \|_{L^\infty(K)}$, where $K \subset B_i$ is any compact neighborhood of $\alpha$, which corresponds to the point $\zeta = \infty$ in bipolar coordinates.

Note that this proves the uniform convergence of the series (2.5) for each fixed $0 < \varepsilon < r_i$. □

3 Asymptotic analysis for electric field concentration

3.1 Decomposition of $u$ into singular part and regular part

It can be inferred from the newly derived series (2.5) that the function $G_\varepsilon \equiv S_{\partial B_i}[g]$ is a counterpart of the far-field behavior $H$ in the free space problem solved in [1]. However, there are some crucial differences between them in nature. First of all, the function $G_\varepsilon$ may not be extended to a harmonic function on the whole complex plane depending on the choice of $g$, whereas $H$ was assumed in [1] to be an entire function. Moreover, the harmonic function $G_\varepsilon$ depends on the minimum width $\varepsilon$ of the shell-shaped domain $B_\varepsilon \setminus B_i$, unlike $H$. If the two obstacles are overcome, it is possible to characterize the gradient blow-up by reproducing the results from the free space problem.

For this purpose, consider the new domain $B_0 := \{ z : |z - r_e| < r_e \}$ and the $\varepsilon$-independent function $G_0 := S_{\partial B_0}[g]$. Observe that

$$G_0(z) = G_\varepsilon(z + c_\varepsilon - r_e) \quad \forall z \in \mathbb{C}, \quad \text{where} \quad \limsup_{\varepsilon \to 0^+} \frac{c_\varepsilon - r_e}{\varepsilon} < \infty.$$ 

In other words, the graph of $G_0$ is obtained by translating the graph of $G_\varepsilon$ by the scale of $O(\varepsilon)$.

In addition, define the new densities

$$
g_i^0(\theta) = -2 \sum_{n=0}^\infty \left( \frac{1}{2\lambda} \right)^n \frac{h(\zeta_i, \theta)}{h(2n(\zeta_i - \zeta_e) + \zeta_i, \theta)} \frac{\partial G_0}{\partial \nu} (2n(\zeta_i - \zeta_e) + \zeta_i, \theta),$$

$$g_e^0(\theta) = -2g - 2 \sum_{n=0}^\infty \left( -\frac{1}{2\lambda} \right)^n \frac{h(\zeta_e, \theta)}{h(2n(\zeta_i - \zeta_e) + 2\zeta_i - \zeta_e, \theta)} \frac{\partial G_0}{\partial \nu} (2n(\zeta_i - \zeta_e) + 2\zeta_i - \zeta_e, \theta).$$

Remark that

$$2n(\zeta_i - \zeta_e) + 2\zeta_i - \zeta_e \sim \left( \frac{2n + 2}{r_i} - \frac{2n + 1}{r_e} \right) \sqrt{\frac{2r_ir_e}{r_e - r_i}} \sqrt{\varepsilon} \quad \text{as} \; \varepsilon \to 0^+$$

and

$$2n(\zeta_i - \zeta_e) + \zeta_i \sim \left( \frac{2n + 1}{r_i} - \frac{2n}{r_e} \right) \sqrt{\frac{2r_ir_e}{r_e - r_i}} \sqrt{\varepsilon} \quad \text{as} \; \varepsilon \to 0^+,$$

which implies that the circles

$$\zeta = 2n(\zeta_i - \zeta_e) + 2\zeta_i - \zeta_e \quad \text{and} \quad \zeta = 2n(\zeta_i - \zeta_e) + \zeta_i \quad \text{for} \; n \geq 0.$$
are applied to $z$ so that the summands of $g_i^0$ and $g_e^0$ are well defined. The convergence of the defining series of $g_i^0$ and $g_e^0$ is verified in the similar way as the proof of Lemma 2.2.

Finally, we link our boundary value problem to the results in [1] via the next lemma:

Lemma 3.1. Let $u_0 = S_{\partial B_i}[g_i^0] + S_{\partial B_e}[g_e^0]$. Then there exists some $\varepsilon > 0$ such that

$$\|\nabla (u_\varepsilon - u_0)\|_{L^\infty(B_\varepsilon)} \leq C_0 \quad \forall \varepsilon \in (0, \varepsilon_0)$$

for some constant number $C_0$ independent of $\varepsilon$.

Proof. Observe that

$$u_\varepsilon - u_0 = S_{\partial B_i}[g_i - g_i^0] + S_{\partial B_e}[g_e - g_e^0].$$

For the first part, it follows from Lemma 2.1 that for some constant numbers $C_{ii}$ and $C_{ie}$,

$$S_{\partial B_i}[g_i - g_i^0] = \begin{cases} C_{ii} + \frac{1}{\lambda} \sum_{n=0}^{\infty} \left( -\frac{1}{2\lambda} \right)^n (R_i R_i)^n (G_\varepsilon - G_0) & \text{in } B_i, \\ C_{ie} + \frac{1}{\lambda} \sum_{n=0}^{\infty} \left( -\frac{1}{2\lambda} \right)^n R_i (R_i R_i)^n (G_\varepsilon - G_0) & \text{in } B_e \setminus B_i. \end{cases}$$

Similarly, the second part is simplified by Lemma 2.1 as

$$S_{\partial B_e}[g_e - g_e^0] = C_e + \frac{1}{\lambda} \sum_{n=0}^{\infty} \left( -\frac{1}{2\lambda} \right)^n (R_i R_i)^{n+1} (G_\varepsilon - G_0) \quad \text{in } B_e.$$  

Remark that in the bipolar coordinate system, the action of reflections are expressed as

$$(R_i \circ R_i)^n (\zeta, \theta) = (2n(\zeta_i - \zeta_e) + \zeta, \theta) \equiv (\zeta_{+n}, \theta)$$

and

$$(R_i \circ R_e)^n \circ R_i (\zeta, \theta) = (2n(\zeta_i - \zeta_e) + 2\zeta_i - \zeta, \theta) \equiv (\zeta_{-n}, \theta).$$

Hence, $\zeta$ values of the points $z \in B_e$ get larger when any series of reflections among

$$(R_i \circ R_i)^n, \quad n = 0, 1, \ldots$$

are applied to $z$, whereas $\theta$ remains to be the same. Similarly, $\zeta$ values for those points $z \in B_e \setminus B_i$ get larger when the series of reflections among

$$(R_i \circ R_e)^n \circ R_i, \quad n = 0, 1, \ldots$$

are applied. Therefore, using the notation $G_0(\zeta, \theta) = G_\varepsilon(\tilde{\zeta}, \tilde{\theta})$ of which $(\tilde{\zeta}, \tilde{\theta})$ is in fact obtained from translating $(\zeta, \theta)$ by $c_e - r_e$,

$$|\nabla (R_i R_i)^n (G_\varepsilon - G_0)| = \left| \frac{h(\zeta, \theta)}{h(\zeta_{+n}, \theta)} \right| |\nabla (G_\varepsilon - G_0)(\zeta_{+n}, \theta)| \leq |\nabla G_\varepsilon(\zeta_{+n}, \theta) - \nabla G_\varepsilon(\tilde{\zeta}_{+n}, \tilde{\theta})|,$$

$$|\nabla R_i (R_i R_i)^n (G_\varepsilon - G_0)| = \left| \frac{h(\zeta, \theta)}{h(\zeta_{-n}, \theta)} \right| |\nabla (G_\varepsilon - G_0)(\zeta_{-n}, \theta)| \leq |\nabla G_\varepsilon(\zeta_{-n}, \theta) - \nabla G_\varepsilon(\tilde{\zeta}_{-n}, \tilde{\theta})|,$$
where the inequalities are valid for all $\theta$. The only thing left for the proof is to show that the right hand sides are bounded by some $C^*$ independent of $n$ and $\varepsilon$, thereby arriving at the conclusion

$$\|\nabla(u_\varepsilon - u_0)\|_{L^\infty(B_r)} \leq \frac{2}{|\lambda|} \sum_{n=0}^{\infty} \left( \frac{1}{2|\lambda|} \right)^n C^* = \frac{4C^*}{2|\lambda| - 1} \equiv C_0.$$  

**Derivation of the uniform bound $C^*$**: The existence of $C^*$ mainly follows from $c_\varepsilon - r_\varepsilon \simeq \varepsilon$. Let $x_0 := c_\varepsilon - r_\varepsilon$. Since $\zeta_{\pm, n} > \zeta_1$, it suffices to find $C^*$ independent of $\varepsilon$ such that

$$\|\nabla(G_\varepsilon - G_0)\|_{L^\infty(B_i)} = \frac{1}{4\pi} \sup_{z \in B_i} \left| \int_{\partial B_r} \frac{x_0 g(w)}{(z - w)(z - w + x_0)} dw \right| \leq C^*.$$  

Observe that changing variables by $w = c_\varepsilon - r_\varepsilon e^{-it}$ gives

$$\int_{\partial B_r} \frac{x_0 g(w)}{(z - w)(z - w + x_0)} dw \equiv \int_{-\pi}^{\pi} \frac{x_0 r_\varepsilon g(t) dt}{(z - c_\varepsilon + r_\varepsilon e^{it})(z - r_\varepsilon + r_\varepsilon e^{it})}. $$

In addition, the smoothness of $g$ implies $|g(t) - g(0)| \leq C|t|$ for all $t$. Therefore,

$$\left| \int_{-\pi}^{\pi} \frac{x_0 r_\varepsilon g(t) dt}{(z - c_\varepsilon + r_\varepsilon e^{it})(z - r_\varepsilon + r_\varepsilon e^{it})} \right| \leq \left| \int_{-\pi}^{\pi} \frac{x_0 r_\varepsilon |t| dt}{|z - c_\varepsilon + r_\varepsilon e^{it}||z - r_\varepsilon + r_\varepsilon e^{it}|} \right| \equiv I_1 + I_2.$$

The integral $I_1$ can be readily computed to be constantly $0$, for example, by the residue formula of complex function theory. Moreover,

$$I_2 \leq \int_{\sqrt{x_0} < |t| < \pi} \frac{x_0 r_\varepsilon |t| dt}{(C\sqrt{x_0})(C\sqrt{x_0})} + \int_{|t| < \sqrt{x_0}} \frac{x_0 r_\varepsilon |t| dt}{(C\varepsilon)(C\varepsilon)} \equiv C^*,$$

so $x_0 \simeq \varepsilon$ implies $I_2 \leq C^*$.

**Theorem 3.2.** Suppose that $\varepsilon$ is sufficiently small.

(a) For conductivities $k$ in compact intervals $[k_1, k_2] \subset (0, \infty)$, $\|\nabla u_\varepsilon\|_{L^\infty(B_r)}$ is bounded uniformly in both of $\varepsilon$ and $k$.

(b) If $\nabla G_0(z) = O(|z|)$ as $z \to 0$, then $\|\nabla u_\varepsilon\|_{L^\infty(B_r)}$ is bounded uniformly in $\varepsilon$ and $k$.

**Proof.** Part (a) is a direct consequence of Lemma 3.1 and [1]. For part (b) note that

$$\sum_{n=0}^{\infty} \frac{h(\zeta_{\pm, n}, \theta)}{h(\zeta_{\pm, n}, \theta)} |\nabla(G_\varepsilon - G_0)(\zeta_{\pm, n}, \theta)|, \sum_{n=0}^{\infty} \frac{h(\zeta_{\pm, n}, \theta)}{h(\zeta_{\pm, n}, \theta)} |\nabla(G_\varepsilon - G_0)(\zeta_{\pm, n}, \theta)| \leq C$$

can be shown by dividing the sums into $n > 1/\sqrt{\varepsilon}$ and $n < 1/\sqrt{\varepsilon}$, where $C$ is independent of $k$ and $\varepsilon$. Following the procedure of the proof of Lemma 3.6 in [1] using maximum principle, the uniform boundedness over $k$ is obtained. \qed
Remark from Equation (1.1) that $u$ depends linearly on $g$, which also depends linearly on $G_0$. Therefore, the second statement of Theorem 3.2 implies that only the linear term of $G_0$ contributes to the singular behavior of $\nabla u$. Set

$$H_1(z) = z \cdot \lim_{w \in B_0 \atop w \to 0} \nabla G_0(w) \quad \text{and} \quad H_2 = G_\varepsilon - H_1.$$ 

**Corollary 3.3.** Let $g$ be any mean zero smooth function on $\partial B_\varepsilon$. Decompose the solution $u$ to the Neumann-transmission problem (1.1) into $u = u_1 + u_2$, where $u_1$ and $u_2$ are defined by the densities described by Equations (2.5), where $G_\varepsilon$ is replaced by $H_1$ and $H_2$, respectively. Then, $\|\nabla u_2\|_{L^\infty(B_\varepsilon)}$ is bounded uniformly in $\varepsilon$.

### 3.2 Expanding singular stress in series by separation of variables

Separating variables in bipolar coordinate system, any harmonic function $f$ in $B_\varepsilon \setminus B_\iota$ or $B_\iota$ are readily proved to have the series representation

$$f(\zeta, \theta) = a_0 + b_0 \zeta + \sum_{n=1}^\infty \left[ (a_n e^{n\zeta} + b_n e^{-n\zeta}) \cos n\theta + (c_n e^{n\zeta} + d_n e^{-n\zeta}) \sin n\theta \right].$$

For instance, for $\zeta > \zeta_\iota > 0$,

$$x = \alpha \left[ 1 + 2 \sum_{n=1}^\infty (-1)^n e^{-n\zeta} \cos n\theta \right] \quad \text{and} \quad y = -2\alpha \sum_{n=1}^\infty (-1)^n e^{-n\zeta} \sin n\theta.$$

In other words, the complex number $z = x + iy$ indicating the point $z = (x, y)$ is expressed as

$$z = x + iy = \alpha + 2\alpha \sum_{n=1}^\infty (-1)^n e^{-n(\zeta + i\theta)}.$$

**Lemma 3.4.** The singular part $u_1$ in the decomposition of Corollary 3.3 can be expressed as

$$u_1(z) = (-2z + U(z)) \cdot \lim_{w \in B_0 \atop w \to 0} \nabla G_0(w)$$

for the complex valued function $U$ defined as

$$U(z(\zeta, \theta)) = C + \begin{cases} 
\sum_{n=1}^\infty \left( A_n e^{n(\zeta-i\theta-2\zeta_\iota)} + B_n e^{n(\zeta-i\theta)} \right) \quad \text{for } \zeta < \zeta < \zeta_i, \\
\sum_{n=1}^\infty \left( A_n e^{n(-\zeta-i\theta)} + B_n e^{n(-\zeta-i\theta)} \right) \quad \text{for } \zeta > \zeta_i,
\end{cases}$$

where

$$A_n = \frac{4\alpha(-1)^n}{2\lambda - e^{-2n(\zeta-i\zeta_\iota)}}, \quad B_n = \frac{-4\alpha(-1)^n e^{-2n(\zeta-i\zeta_\iota)}}{2\lambda - e^{-2n(\zeta-i\zeta_\iota)}},$$

and $C$ is any constant number.
Proof. It suffices to show that \( v(z) := -2z + U(z) \) solves the problem

\[
\begin{cases}
\Delta v = 0 & \text{in } B_e \setminus \partial B_i, \\
\frac{\partial v}{\partial \nu} = S_{\partial B_e}^{-1}(z - c_e)|_{z \in B_e} & \text{on } \partial B_e, \\
\frac{\partial v}{\partial \nu} = k \frac{\partial v}{\partial \nu} & \text{on } \partial B_i, \\
v_+ = v_- & \text{on } \partial B_i.
\end{cases}
\]

The continuity and harmonicity of \( v \) directly follow from the uniform convergence of \( U \) on every compact subset of \( B_e \). The transmission condition on \( \partial B_i \) reads

\[
2\lambda A_n + B_n = 4\alpha (-1)^n,
\]

and the Neumann boundary condition on \( \partial B_e \) reads

\[
A_n e^{-2\alpha(z_i - c_e)} + B_n = 0,
\]

which are satisfied by the definitions of \( A_n \) and \( B_n \). \( \Box \)

### 3.3 Expressing singular stress in terms of Lerch transcendent functions

Up to the previous subsection, the solution \( u \) to the boundary value problem posed in (1.1) is expressed in uniformly convergent series. In the rest of Section 3, the singular behavior of \( \nabla u \) is revealed by an asymptotic analysis using the integral representation of \( u \).

**Definition 1.** Define Lerch transcendent function \( L \) by

\[
L(z; \beta) = -\int_0^\infty \frac{ze^{-(\beta+1)t}}{1 + ze^{-t}} dt \quad \text{for } z \in \mathbb{C}, \ |z| < 1, \ \beta > 0.
\]

In addition, define another function \( P \) by

\[
P(z; \beta) = -z \frac{\partial}{\partial z} L(z; \beta) = \int_0^\infty \frac{ze^{-(\beta+1)t}}{(1 + ze^{-t})^2} dt.
\]

Ignoring the bounded terms of \( \nabla u \), Theorem 3.5 expresses \( u \) in terms of the Lerch transcendent functions. Since its proof is analogous to that of [1], the details of the proof are provided in the appendix at the end of this article. For simplicity in notation, denote the linear factors of \( G_0 \) by

\[
\hat{x} \cdot \left\{ \lim_{w \to 0} \nabla G_0(w) \right\} = C_x \quad \text{and} \quad \hat{y} \cdot \left\{ \lim_{w \to 0} \nabla G_0(w) \right\} = C_y.
\]

**Theorem 3.5.** The solution \( u = u_\epsilon \) to the Neumann-transmission problem (1.1) is represented by

\[
u(z) = \begin{cases} r_1(z) & \text{when } 0 < k < 1, \\
\frac{2r_i r_e}{r_e - r_i} [C_x \Re\{q(z; \beta, \tau)\} + C_y \Im\{q(z; \beta, \tau)\}] + r_2(z) & \text{when } k > 1,
\end{cases}
\]
where \( \tau = 1/(2\lambda) \), \( \beta = \frac{-\ln \tau}{2\xi - 2\xi_e} \), \( \| \nabla r_j \|_{L^\infty(B_k)} \) are bounded uniformly in \((\varepsilon, k)\), and the singular function \( q \) is defined as

\[
q(z; \beta, \tau) = \begin{cases} 
\tau \left[ L\left( e^{-2(\zeta - \zeta_1) - i\theta}; \beta \right) - L\left( e^{-2(\zeta_1 - 2\xi_e) - i\theta}; \beta \right) \right] & \text{for } \zeta_e < \zeta < \zeta_i, \\
\tau \left[ L\left( e^{-2(\zeta - \zeta_0) - i\theta}; \beta \right) - L\left( e^{-2(\zeta_0 - 2\xi_e) - i\theta}; \beta \right) \right] & \text{for } \zeta > \zeta_i.
\end{cases}
\]

### 3.4 Deriving pointwise rate of gradient blow-up

In order to simplify the representation in Theorem 3.5 further, we introduce some properties of the function \( P \), which directly follow from integration by parts:

\[
|P(e^{-s + i\theta}; \beta)| \leq \frac{1}{2\beta(s + \cos \theta)} \quad \forall s > 0
\]
and

\[
|P(e^{-s_2 + i\theta}; \beta) - P(e^{-s_1 + i\theta}; \beta)| \leq \frac{s_2 - s_1}{2\beta(s_1 + \cos \theta)} \quad \forall s_2 > s_1 > 0.
\]

Equation (3.4) is used to derive the upper bound of stress concentration, and Equation (3.5) simplifies the image charge formula via the next lemma:

**Lemma 3.6.** Fix any positive real number \( \delta \) and choose any \( \zeta^{(1)}, \zeta^{(2)} = O(\sqrt{\varepsilon}) \) satisfying the inequalities \( \zeta^{(1)} \geq \zeta(1 + \delta) \) and \( \zeta^{(2)} \leq \zeta(1 - \delta) \). Then for any \( \zeta \in [\zeta_e, \zeta_i] \),

\[
P\left( e^{-\zeta(1) - i\theta}; \beta \right) - P\left( e^{-\zeta(2) - i\theta}; \beta \right) = O\left( \frac{1}{h(\zeta, \theta)} \right) \quad \text{as } \varepsilon \to 0.
\]

**Theorem 3.7.** For each fixed \( k > 0 \), \( \| \nabla u \|_{L^\infty(B_k)} \) is bounded uniformly in \( \varepsilon \). The gradient blow-up occurs only if \( \tau \to 1 \), i.e., \( k \gg 1 \), only in \( B_k \setminus B_i \). In particular, for \( \zeta_e < \zeta < \zeta_i \),

\[
\nabla u(z(\zeta, \theta)) + O(1)
\]

\[
= -\frac{\tau}{\sqrt{\varepsilon}} \sqrt{\frac{2r \tau e}{(r_0 - r_i)}} \left( \cos \zeta + \cos \theta \right) \left[ C_x R \left\{ 2P \left( e^{-2(\zeta - \zeta_0) - i\theta}; \beta \right) \right\} + C_y \left\{ 2P \left( e^{-2(\zeta - \zeta_1) - i\theta}; \beta \right) \right\} \right] \\
= -\frac{\tau}{\sqrt{\varepsilon}} \sqrt{\frac{2r \tau e}{(r_0 - r_i)}} \left( \cos \zeta + \cos \theta \right) \left[ C_x R \left\{ 2P \left( e^{-\zeta(1) - i\theta}; \beta \right) \right\} + C_y \left\{ 2P \left( e^{-\zeta(2) - i\theta}; \beta \right) \right\} \right].
\]

**Proof.** We may suppose \( k > 1 \). It follows from Lemma 3.6 applied to Equations (A.1) and (A.2) that

\[
\nabla q \cdot \dot{\zeta} = -h(\zeta, \theta) \begin{cases} 
2P\left( e^{-2(\zeta - \zeta_0) - i\theta}; \beta \right) + O\left( \frac{1}{h(\zeta, \theta)} \right) & \text{for } \zeta_e < \zeta < \zeta_i, \\
O\left( \frac{1}{h(\zeta, \theta)} \right) & \text{for } \zeta > \zeta_i,
\end{cases}
\]
and

\[
\nabla q \cdot \dot{\theta} = O(1) \quad \text{for } \zeta > \zeta_e.
\]
Moreover, Equation (3.4) shows
\[
|2h(\zeta, \theta)P(e^{-(2\zeta_i - \zeta)^{-i\theta}}; \beta)| \leq \frac{1}{\alpha \beta} \frac{h(\zeta, \theta)}{h(2\zeta_i - \zeta, \theta)} \leq \frac{C}{|\ln \tau|}
\]
for \( \zeta_e < \zeta < \zeta_i \), which implies that the gradient blow-up doesn’t occur unless \( k \gg 1 \). \( \square \)

From asymptotic analysis on the integral representation in Theorem 3.7, the optimal estimate for the pointwise divergence of \( \nabla u \) follows. For more details, refer to the appendix.

**Corollary 3.8.** For sufficiently large \( k \gg 1 \),
\[
\frac{C_1}{\sqrt{\frac{r_i r_e}{2(r_e - r_i)} k + 1}} \leq \|\nabla u_\varepsilon\|_{L^\infty(B_e \setminus B_i)} \leq \frac{C_2}{\sqrt{\frac{r_i r_e}{2(r_e - r_i)} k + 1}} + \varepsilon
\]
if one of the following holds:
(i) \( C_x \neq 0 \) and \( C_y = 0 \)
(ii) \( C_x = 0, C_y \neq 0 \) and \( \beta \) is bounded as \( \varepsilon \to 0 \).

## 4 Image line charge formula

In this section, the singular behavior of \( \nabla u_\varepsilon \) is described solely by some type of virtual charges of which densities are supported on two line segments, namely, \([-c_i, -\alpha]\) and \([\alpha, c_i]\).

### 4.1 Approximation by image line charge

Since most of the procedure is analogous to \[\Pi\], we will skip the proof and just state the results.

**Lemma 4.1.** Define the line charge densities \( \varphi_\pm \) and \( \psi_\pm \) as
\[
\begin{align*}
\varphi_+(s) &= 2\alpha \beta e^{2\beta \zeta_i} \left( \frac{s - \alpha}{s + \alpha} \right)^{\beta - 1} \left( \frac{s - \alpha}{s + \alpha} \right)^{\beta}, \\
\varphi_-(s) &= 2\alpha \beta e^{2\beta \zeta_i} \left( \frac{s + \alpha}{s - \alpha} \right)^{\beta - 1} \left( \frac{s + \alpha}{s - \alpha} \right)^{\beta}, \\
\psi_+(s) &= e^{2\beta \zeta_i} \left( \frac{s - \alpha}{s + \alpha} \right)^{\beta} \\
\psi_-(s) &= -e^{2\beta \zeta_i} \left( \frac{s + \alpha}{s - \alpha} \right)^{\beta}
\end{align*}
\]
for \( \alpha < s < c_i \), \( -c_i < s < \alpha \).

Then for \( \zeta_e < \zeta < \zeta_i \),
\[
L(e^{-(2\zeta_i - \zeta)^{-i\theta}}; \beta) = \int_{[\alpha, c_i]} \ln |z - s| \varphi_+(s) ds + i \int_{[\alpha, c_i]} \frac{\mathfrak{R}\{z - s\}}{|z - s|^2} \psi_+(s) ds + r_+(z),
\]
\[
L(e^{-(\zeta + 2\zeta_i)^{-i\theta}}; \beta) = \int_{[\alpha, c_i]} \ln |z - s| \varphi_-(s) ds - i \int_{[\alpha, c_i]} \frac{\mathfrak{R}\{z - s\}}{|z - s|^2} \psi_-(s) ds + r_-(z),
\]
where \( |\nabla r_\pm(z)| \leq 1/r_i \).
Theorem 4.2. For \( k \gg 1 \), the solution \( u = u_\varepsilon \) in \( B_\varepsilon \setminus B_i \) shows normal gradient blow-up

\[
u(z) = \frac{4\pi r_1 e}{r_\varepsilon - r_i} \left[ C_x \int_{[\alpha, c_i]} \ln |z - s| \varphi_+(s) ds + C_y \int_{[\alpha, c_i]} \frac{3(z - s)}{|z - s|^2} \psi_+(s) ds \right] + r_1(z),
\]

\( \zeta_\varepsilon < \zeta < \zeta_i \), where \( |\partial r_1 / \partial \nu| \) is bounded uniformly in \( \varepsilon \) and \( z \). The support of the image charge density can be changed to be outside of \( B_\varepsilon \) in the similar way as

\[
u(z) = \frac{4\pi r_1 e}{r_\varepsilon - r_i} \left[ -C_x \int_{[-c, -\alpha]} \ln |z - s| \varphi_-(s) ds + C_y \int_{[-c, -\alpha]} \frac{3(z - s)}{|z - s|^2} \psi_-(s) ds \right] + r_2(z),
\]

\( \zeta_\varepsilon < \zeta < \zeta_i \), where \( |\partial r_2 / \partial \nu| \) is bounded uniformly in \( \varepsilon \) and \( z \).

Note that any convex combination of the two expressions can also represent the singular term.

Proof. It follows directly from applying Lemma 4.1 to Theorem 3.5. \( \square \)

5 Series solution for the interface problem in Lamé system

5.1 Problem of linear elasticity in core-shell type geometry

The aforementioned method of approximation for the conductivity problem can be applied to the linear elasticity problem. In the theory of elasticity, the singular behavior of the stress tensor \( \sigma \) is the main concern. As a tensor, \( \sigma \) defines a linear transformation from \( S^1 \) into \( \mathbb{R}^2 \) at each point of the domain occupied by an elastic material. A stress vector \( \sigma(\hat{n}) \) is defined for each location occupied by an elastic body and each direction \( \hat{n} \in S^1 \).

When a coordinate system is specified, the stress \( \sigma \) can be represented as a matrix-valued function. Denote the matrix representation of \( \sigma \) in the Cartesian coordinate system by \([\sigma]_c = \left[ \begin{array}{cc} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{array} \right] \), where \( \sigma_{xy} = \sigma_{yx} \). It is assumed that the matrix components are governed by the differential equation of linear elasticity as

\[
\nabla \cdot \sigma = \begin{bmatrix} \frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \sigma_{xy} \\ \frac{\partial}{\partial x} \sigma_{yx} + \frac{\partial}{\partial y} \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \nabla^2 (\sigma_{xx} + \sigma_{yy}) = \Delta (\sigma_{xx} + \sigma_{yy}) = 0, \quad (5.1)
\]

where the second equation means there’s no body force applied. Physically, the resulting vectors \( \sigma(\hat{n}) \) stand for the internal forces applied on the infinitesimal hyperplanes perpendicular to \( \hat{n} \).

As in the conductivity problem, the elastic body under consideration is a bounded domain which contains another type of elastic material inside. The pair \((\lambda, \mu)\) of constant numbers, named after Lamé, determines the property of elastic materials and the boundary condition over the boundary of two different types of materials. The general solution of (5.1) is to be found by separating variables.

Let \( \Omega \subset \mathbb{C} \) be a bounded domain where an isotropic homogeneous elastic body lies. We assume that \( \Omega \) is conformally equivalent to an annulus, and the conformal mapping is further assumed to be slightly extended to include \( \partial \Omega \) in its domain. More precisely, it is assumed that there exist some \( \zeta_2 > \zeta_1 > 0, \delta > 0 \), and a conformal mapping \( F \) from the annulus

\[
\{ z \in \mathbb{C} : \, \zeta_1 - \delta < |z| < \zeta_2 + \delta \}
\]
onto an open set $U$ that contains $\Omega$, where $F$ satisfies $F(\{z \in \mathbb{C} : \ ζ_1 < |z| < ζ_2\}) = \Omega$.

Note that the domain $\Omega$ can be parametrized by the orthogonal transformation $U(z) = F(\varepsilon z)$ from $(ζ_1, ζ_2) \times \mathbb{R}$ bijectively onto $\Omega$. Remark that the transformation $U$ is smooth up to the boundary, where

$$\partial \Omega = \{U(z) \equiv U(ζ + iθ) : \mathbb{R}\{z\} = ζ_1 \text{ or } \mathbb{R}\{z\} = ζ_2\}.$$  

The new coordinate system $(ζ, θ)$ defined on $\Omega$ by the transformation $U$ turns out to be useful in solving boundary value problems in core-shell geometry of relatively general shape, which is dealt with in Subsection \textit{5.2}.

### 5.2 Method of separation of variables for bi-harmonic equation

Instead of solving Equation (5.1) directly for the vector-valued function $σ$, Airy's formulation will be introduced to reduce the problem into finding a scalar-valued function. If there's no body force applied to a two-dimensional elastic body, W.J.Ibbetson [23] showed that there exists a scalar function $χ$ satisfying both of

$$σ \equiv \begin{bmatrix} σ_{xx} & σ_{xy} \\ σ_{yx} & σ_{yy} \end{bmatrix} = \begin{bmatrix} \frac{∂^2}{∂y^2}χ & \frac{∂^2}{∂x∂y}χ \\ −\frac{∂^2}{∂y∂x}χ & \frac{∂^2}{∂x^2}χ \end{bmatrix} \quad \text{and} \quad \nabla^4χ \equiv ΔΔχ = 0. \quad (5.2)$$

Remark that the conditions in (5.2) directly imply those in (5.1).

Let $Ω$ be the domain parametrized by $U$ as in Subsection \textit{5.1}. To simplify the arguments, consider the four times continuously differentiable functions satisfying Equations (5.2). The following lemma justifies the method of separating variables to express general stress distributions that correspond to the Airy stress functions. The assumption of $C^{(4)}$ regularity can be generalized into Sobolev spaces by slightly modifying the converging sense and proof.

\textbf{Lemma 5.1.} Suppose that

$$(\nabla^4χ ) \circ U(ζ, θ) = \left( \sum_{j+k\leq 4} U_{j,k}(ζ) \frac{∂^{j+k}}{∂ζ^j∂θ^k} \right) (F(ζ, θ)(χ \circ U)(ζ, θ))$$

for some $U_{j,k}$ smooth functions defined on $[ζ_1, ζ_2]$ and $F$ is a smooth function defined on $\overline{U^{-1}(Ω)}$. Then there exists some constant coefficients $C^{(m)}_n$ and $S^{(m)}_n$ defined for $m = 1, 2, 3, 4$ and $n = 0, 1, 2, \cdots$ that satisfies

$$F(ζ, θ)(χ \circ U)(ζ, θ) = \sum_{n=0}^{∞} \sum_{m=1}^{4} C^{(m)}_nf_{m,n}(ζ) \cos(nθ) + \sum_{n=1}^{∞} \sum_{m=1}^{4} S^{(m)}_ng_{m,n}(ζ) \sin(nθ), \quad (5.3)$$

where the convergence is pointwise and uniform on $Ω$. Here $\{f_{m,n}\}_{m=1}^{4}$ and $\{g_{m,n}\}_{m=1}^{4}$ respectively are families of four linearly independent smooth functions for each $n$ and their determination depends only on the coordinate transformation $U$.

\textbf{Lemma 5.2.} Let $F(\nu) = \alpha \cdot \frac{2ν}{w+1}$ so that $e^{ζ+iθ} = \frac{α+e^{ζ+iθ}}{α-F(\varepsilon + w)}$, which defines the bipolar coordinate system mentioned before. Then

$$(χ \circ U)(ζ + iθ) = \frac{α}{\cosh ζ + \cos θ} \left[ \sum_{n=0}^{∞} f_{n}(ζ) \cos(nθ) + \sum_{n=1}^{∞} g_{n}(ζ) \sin(nθ) \right], \quad (5.4)$$
The functions \( g_n \) has the same form as \( f_n \) with different coefficients \( A'_n \) through \( B'_n \). 

**Proof.** Recall that \( h(\zeta, \theta) = (\cosh \zeta + \cos \theta)/\alpha \) is the scale factor in the bipolar coordinate system. We have to solve the bi-harmonic equation \( \nabla^4 \chi = 0 \), which is equivalent in the bipolar coordinate system to solving

\[
\left( \frac{\partial^4}{\partial \zeta^4} + 2 \frac{\partial^4}{\partial \zeta^2 \theta^2} + \frac{\partial^4}{\partial \zeta^4} - 2 \frac{\partial^2}{\partial \zeta^2} + 2 \frac{\partial^2}{\partial \theta^2} + 1 \right) [h(\zeta, \theta) \chi(U(\zeta + i\theta))] = 0.
\] (5.6)

Expanding \( \chi \circ U \) into a Fourier series in the form of Equation (5.4) gives the differential equations in terms of the Fourier coefficients as

\[
\left( \frac{\partial^4}{\partial \zeta^4} - 2(n^2 + 1) \frac{\partial^2}{\partial \zeta^2} + n^4 - 2n^2 + 1 \right) f_n(\zeta) = 0 \quad \text{for } n \geq 0
\]

and

\[
\left( \frac{\partial^4}{\partial \zeta^4} - 2(n^2 + 1) \frac{\partial^2}{\partial \zeta^2} + n^4 - 2n^2 + 1 \right) g_n(\zeta) = 0 \quad \text{for } n \geq 1,
\]

of which general solutions are expressed as Equations (5.5). See the proof of Lemma 5.1 in Appendix C for the discussion of more general type of coordinate systems, including the polar coordinate system. \( \square \)

**Example 1.** Let \( \sigma \) be the isotropic homogeneous linear stress function satisfying Neumann boundary conditions

\[
\begin{aligned}
\sigma(\mathbf{n}) &= 0 \quad \text{on } \partial B_i, \\
\sigma(\mathbf{n}) &= \mathbf{n} \quad \text{on } \partial B_e.
\end{aligned}
\] (5.7)

Then, as \( \varepsilon \to 0 \), \( \| \sigma \|_{L^\infty(B_e \setminus B_i)} \) diverges.

**Proof.** The solution \( \sigma \) is obtained from the boundary conditions applied to \( \chi \) in its Fourier series (5.4). For notational simplicity, let

\[
\delta = 2 \sinh(\zeta_i - \zeta_e)(1 - \cosh(\zeta_i + \zeta_e) \cosh(\zeta_i - \zeta_e)).
\]

Then the coefficients for the given boundary tractions are

\[
A_1 = \frac{\alpha \sinh(\zeta_e + \zeta_i) - \zeta_i}{\delta}, \quad B_1 = -\frac{\alpha \sinh(\zeta_e - \zeta_i) + \cosh(\zeta_e - \zeta_i) \sinh(2\zeta_i)}{\delta}, \quad C_1 = -\frac{\alpha \cosh(\zeta_e + \zeta_i)}{\delta},
\]

\[
B_0 = \frac{-2\alpha \cosh(\zeta_e - \zeta_i)}{\delta}, \quad \text{and} \quad A_{n+1} = B_{n+1} = C_{n+1} = D_{n+1} = A'_n = B'_n = C'_n = D'_n = 0 \text{ for all } n \geq 1.
\]

Here \( \delta \sim C \varepsilon^{3/2}, \alpha \sim C \varepsilon^{1/2}, \zeta_i \sim C \varepsilon^{1/2}, \text{ and } \zeta_e \sim C \varepsilon^{1/2} \text{ as } \varepsilon \to 0 \). Therefore, \( \| \sigma \|_{C^\infty} \geq \frac{1}{C \varepsilon^{1/2}} \) as \( \varepsilon \to 0 \). For detailed information about the stress components, see [14]. \( \square \)
5.3 Solution representation for transmission problem

Let the disks \( B_i \) and \( B_e \) define the core-shell geometry as in the conductivity problem. Consider the transmission problem in \( \Omega = \mathbb{R}^2 \setminus B_i \) given by

\[
\begin{aligned}
\nabla^4 \chi &= 0 \quad \text{in } \Omega \setminus \partial B_e \\
\sigma[\chi] \hat{n}_+ &= \sigma[\chi] \hat{n}_- \quad \text{on } \partial B_e \\
u[\chi]|_+ &= u[\chi]|_- \quad \text{on } \partial B_e \\
\sigma[\chi] \hat{n}_+ &= 0 \quad \text{on } \partial B_i,
\end{aligned}
\]

where \( u \) is the displacement vector defined up to rigid-body displacements and the strain \( \mathcal{E} \) depends on the Lamé parameters:

\[
\mathcal{E} \equiv \frac{1}{2}(\nabla u + \nabla u^T) = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{xy} \\ \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\lambda + 2\mu}{4\mu(\lambda + \mu)} & 0 & \frac{\lambda - 2\mu}{4\mu(\lambda + \mu)} \\ 0 & 1 & 0 \\ \frac{-\lambda}{4\mu(\lambda + \mu)} & 0 & \frac{\lambda + 2\mu}{4\mu(\lambda + \mu)} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yy} \end{bmatrix}.
\]

Here the second line of Equation (5.8) follows from the continuity of normal traction, and the fourth line from assuming that \( B_i \) is a cavity. Note that the solution \( \chi \) of this problem must be symmetric with respect to the \( x \)-axis, so the coefficients \( g_n(\zeta) \) of sines in Equation 5.4 vanish.

For the next lemma, let \( J \) be the operator defined by

\[
J[s] := h \left( \frac{\partial^2 s}{\partial \zeta^2} + \frac{\partial^2 s}{\partial \theta^2} \right) - 2 \left( \frac{\sinh \zeta \partial s}{\alpha \zeta} - \frac{\sin \theta \partial s}{\alpha \theta} \right) + \left( \frac{\cos \theta - \cos \zeta}{\alpha} + \frac{2 \sin^2 \theta + 2 \sinh^2 \zeta}{\alpha} \right) \frac{1}{h} \cdot s.
\]

Lemma 5.3. Let \( s := h(\zeta, \theta) \chi \). The boundary conditions of the transmission problem (5.8) is equivalent to the following:

1. The continuity of \( s \) and \(-\sin \theta)s + (\cosh \zeta + \cos \theta)\frac{\partial s}{\partial \zeta} \) on \( \partial B_e \),

2. the continuity of

\[
\frac{2\mu + \lambda}{4\mu(\mu + \lambda)} \left\{ \left( h(\zeta, \theta) \frac{\partial^2 s}{\partial \zeta^2} - \frac{\partial}{\partial \zeta} \left( \frac{\sinh \zeta}{\alpha} s \right) + \frac{\cosh \zeta}{\alpha h(\zeta, \theta)} s \right) \right. \\
\left. - \frac{\lambda}{2\mu + \lambda} \left( h(\zeta, \theta) \frac{\partial^2 s}{\partial \theta^2} + \frac{\partial}{\partial \theta} \left( \frac{\sin \theta}{\alpha} s \right) - \frac{\cos \theta}{\alpha h(\zeta, \theta)} s \right) - \frac{1}{r_e} \left( \frac{\partial s}{\partial \zeta} - \frac{\sinh \zeta}{\alpha h(\zeta, \theta)} s \right) \right\}
\]

on \( \partial B_e \),

3. the continuity of

\[
\frac{2\mu + \lambda}{4\mu(\mu + \lambda)} \left[ - \frac{2\sinh \zeta}{\alpha} J[s] - h(\zeta, \theta) \frac{\partial J}{\partial \zeta}[s] \\
+ \frac{2(\mu + \lambda)}{2\mu + \lambda} \left\{ - h \frac{\partial s}{\partial \theta} \frac{\partial^2 s}{\partial \zeta^2} - h^2 \frac{\partial^3 s}{\partial \zeta \partial \theta^2} + h \frac{\partial}{\partial \theta} \left( \frac{\partial s}{\partial \zeta} \right) \right. \\
+ \left( \frac{\partial h}{\partial \theta} \right) \frac{\partial}{\partial \zeta} \left( \frac{1}{h} \frac{\partial h}{\partial \theta} \right) s + h^2 \frac{\partial^2 s}{\partial \theta^2 \partial \zeta} \left( \frac{1}{h} \frac{\partial h}{\partial \theta} \right) \left\} \right.
\]
on $\partial B_e$,

4. the equation

$$\left\{ (\cosh \zeta + \cos \theta) \frac{\partial^2}{\partial \zeta^2} - \sinh \zeta \frac{\partial}{\partial \zeta} - \sin \theta \frac{\partial}{\partial \theta} + \cos \theta \right\} s = 0$$

on $\partial B_i$, and

5. the equation

$$\left\{ (\cosh \zeta + \cos \theta) \frac{\partial^2}{\partial \theta^2} - \sinh \zeta \frac{\partial}{\partial \zeta} - \sin \theta \frac{\partial}{\partial \theta} + \cosh \zeta \right\} s = 0$$

on $\partial B_i$.

**Proof.** On the interface $\partial B_e$, the continuity of displacement and normal traction are induced as boundary conditions. The continuity of normal traction is shown in [21] to be equivalent to the continuity of $\chi$ and $\frac{\partial \chi}{\partial n}$ over $\partial B_e$. In addition, the continuity of displacement is shown in [21] to be equivalent to the continuity of

$$\frac{2\mu + \lambda}{4\mu(\mu + \lambda)} \left\{ \frac{\partial^2 \chi}{\partial n^2} - \frac{\lambda}{2\mu + \lambda} \left( \frac{\partial^2 \chi}{\partial s^2} + \frac{1}{r_e} \frac{\partial \chi}{\partial n} \right) \right\}$$

and

$$\frac{2\mu + \lambda}{4\mu(\mu + \lambda)} \left\{ \frac{\partial}{\partial n} (\nabla^2 u) + \frac{2(\mu + \lambda)}{2\mu + \lambda} \frac{\partial}{\partial s} \left( \frac{\partial^2 \chi}{\partial n \partial s} \right) \right\}$$

over $\partial B_e$. In the core-shell type geometry the signs of the normal and tangential derivatives are chosen as

$$\frac{\partial}{\partial n} = -h(\zeta, \theta) \frac{\partial}{\partial \zeta} \quad \text{and} \quad \frac{\partial}{\partial s} = h(\zeta, \theta) \frac{\partial}{\partial \theta}.$$

The last two conditions follow from zero-traction condition on $\partial B_i$. \hfill \square

**Proposition 5.4.** The solution $\chi$ to the transmission problem (5.8) is represented by

$$\chi(\zeta, \theta) = \begin{cases} \frac{1}{h(\zeta, \theta)} \left[ A_0 \cosh \zeta + C_0 \sinh \zeta + \zeta (B_0 \cosh \zeta + D_0 \sinh \zeta) \right. \\ + \left. \left\{ A_1 \cosh(2\zeta) + B_1 + C_1 \sinh(2\zeta) + D_1 \zeta \right\} \cos \theta \right. \\ + \sum_{n=2}^{\infty} \left\{ A_n \cosh((n+1)\zeta) + B_n \cosh((n-1)\zeta) \\ + C_n \sinh((n+1)\zeta) + D_n \sinh((n-1)\zeta) \right\} \cos(n\theta) \right] \text{ in } B_e \setminus \overline{B_i}; \\
\frac{1}{h(\zeta, \theta)} \left[ P_0 e^\zeta + Q_0 \zeta e^\zeta + \sum_{n=1}^{\infty} e^{n\zeta} (P_n e^\zeta + Q_n e^{-\zeta}) \cos(n\theta) \right] \text{ in } \mathbb{R}^2 \setminus \overline{B_e} \end{cases}$$

(5.9)

where the coefficients are determined by the boundary conditions in Lemma 5.3, which gives a system of infinitely many linear equations in terms of the coefficients of at most finite number.

**Proof.** The coefficients vanished from Lemma 5.2 prevents the solution from generating pole at the foci of bipolar coordinate system. \hfill \square
6 Conclusion

In this article, the interface problems of electricity and elasticity in the presence of an inhomogeneity near the boundary were solved by asymptotic analysis. The main part concerning the electricity was the characterization of the electric field concentration by extracting the linear factors of the single-layer potential $S_{B_0}(g)$, where $g$ is arbitrarily given as an exterior Neumann boundary data. As for the Lamé system, the method to reduce the problem into a solvable linear system was presented. It is left as a future work to characterize the elastic stress concentration by estimating the behavior of the stress as $\varepsilon$ tends to zero.

A Proof of Theorem 3.5

Proof. The proof is analogous to that of Lemma 5.1 in [7], where the following lemma plays a significant role:

Lemma A.1 ([17]). Let $\max\{0,a\} < a_0$ and $0 < \tau < 1$. Then for any $\theta \in (-\pi, \pi]$,

$$a_0 \sum_{m=1}^{\infty} \left( \tau^{m-1} \frac{e^{-ma_0+a+i\theta}}{(1+e^{-ma_0+a+i\theta})^2} \right) - P \left( e^{-(a_0-a)+i\theta}, -\frac{\ln\tau}{a_0} \right) \leq \frac{8a_0}{\cosh(a_0-a)+\cos\theta}$$

and

$$a_0 \sum_{m=1}^{\infty} \left( \tau^{m-1} \frac{e^{-ma_0+a+i\theta}}{(1+e^{-ma_0+a+i\theta})^2} \right) \leq \frac{8a_0}{\cosh(a_0-a)+\cos\theta}.$$

Note that Lemma A.1 guarantees that a certain type of series can be approximated by the derivative of Lerch transcendent functions. The rest of this section is devoted to applying the lemma to the expression of $u_1$ derived in Lemma 3.4. By differentiating Equation (3.2), one obtains

$$\frac{\partial U}{\partial \zeta} = \begin{cases} \sum_{n=1}^{\infty} n \left( A_n e^{n(-\zeta+i\theta-2\zeta_i)} - B_n e^{n(-\zeta-i\theta)} \right) & \text{for } \zeta_e < \zeta < \zeta_i, \\ -\sum_{n=1}^{\infty} n \left( A_n e^{n(-\zeta-i\theta)} + B_n e^{n(-\zeta-i\theta)} \right) & \text{for } \zeta > \zeta_i, \end{cases}$$

and

$$\frac{\partial U}{\partial \theta} = \begin{cases} \sum_{n=1}^{\infty} (-ni) \left( A_n e^{n(-\zeta-i\theta-2\zeta_i)} + B_n e^{n(-\zeta-i\theta)} \right) & \text{for } \zeta_e < \zeta < \zeta_i, \\ \sum_{n=1}^{\infty} (-ni) \left( A_n e^{n(-\zeta-i\theta)} + B_n e^{n(-\zeta-i\theta)} \right) & \text{for } \zeta > \zeta_i. \end{cases}$$

Observe from geometric series expansion that

$$B_n = -4\alpha(-1)^n \sum_{m=1}^{\infty} \tau^m e^{mn(-2\zeta_i+2\zeta_e)} \quad \text{and} \quad A_n = -B_n e^{-n(-2\zeta_i+2\zeta_e)}.$$

Finally, note that for all $|z| < 1$, $z \in \mathbb{C}$ we have

$$\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n.$$
which enables the correspondence of $\nabla U$ with $P$ by Lemma A.1. Namely, we have $\nabla U = O(1)$ uniformly in $\varepsilon$ when $-1 < \tau < 0$, i.e., $0 < k < 1$. On the other hand, when $k > 1$ so that $0 < \tau < 1$,

$$\frac{\partial U}{\partial \zeta} = O(1) + \begin{cases} \frac{4\alpha}{2} \left[ -P\left(e^{-2(2\zeta-\zeta_i)\zeta}; \beta\right) - P\left(e^{-2(2\zeta_i-\zeta)\zeta_i}; \beta\right) \right] \text{ for } \zeta < \zeta < \zeta_i, \\ \frac{4\alpha}{2} \left[ P\left(e^{-2(2\zeta-\zeta_i)\zeta_i}; \beta\right) - P\left(e^{-2(2\zeta_i-\zeta)\zeta_i}; \beta\right) \right] \text{ for } \zeta > \zeta_i, \end{cases}$$

and

$$\frac{\partial U}{\partial \theta} = O(1) + \begin{cases} \frac{4\alpha}{2} \left[ iP\left(e^{-2(2\zeta-\zeta_i)\zeta_i}; \beta\right) - iP\left(e^{-2(2\zeta_i-\zeta)\zeta_i}; \beta\right) \right] \text{ for } \zeta < \zeta < \zeta_i, \\ \frac{4\alpha}{2} \left[ iP\left(e^{-2(2\zeta-\zeta_i)\zeta_i}; \beta\right) - iP\left(e^{-2(2\zeta_i-\zeta)\zeta_i}; \beta\right) \right] \text{ for } \zeta > \zeta_i. \end{cases}$$

(A.1)

Further approximation using

$$\frac{4\alpha}{2} \frac{2r\tau_i}{r - \tau_i} + O(\sqrt{\varepsilon}) \text{ as } \varepsilon \to 0$$

shows that

$$\nabla U(z) = \frac{2r\tau_i}{r - \tau_i} \nabla q(z; \beta, \tau) + O(1), \quad z \in B_\varepsilon,$$

which completes the proof. □

## B Proof of Corollary 3.8

**Proof.** Observe that integrating by parts gives

$$P\left(e^{-2(2\zeta-2\zeta_i)\zeta_i}; \beta\right) = \int_0^\infty e^{-\beta t} \frac{1 + \cosh(2\zeta_i - 2\zeta_i + \zeta + t) \cos \theta - i \sinh(2\zeta_i - 2\zeta_i + \zeta + t) \sin \theta}{2(\cosh(2\zeta_i - 2\zeta_i + \zeta + t) + \cos \theta)^2} dt.$$

In particular, for $\theta = \pi/2$ and $\zeta < \zeta < \zeta_i$,

$$P\left(e^{-2(2\zeta-2\zeta_i)\zeta_i}; \beta\right) = \int_0^\infty e^{-\beta t} \frac{1 - i \sinh(2\zeta_i - 2\zeta_i + \zeta + t)}{2 \cosh^2(2\zeta_i - 2\zeta_i + \zeta + t)} dt.$$

Hence it follows that

$$\frac{C_1}{(\beta + 1)(\beta + 3)} \leq 3 \left\{ P\left(e^{-2(2\zeta-2\zeta_i)\zeta_i}; \beta\right) \right\} \leq \frac{C_2}{(\beta + 1)(\beta + 3)}$$

and

$$\frac{C_1}{\beta + 1} \leq \Re \left\{ P\left(e^{-2(2\zeta-2\zeta_i)\zeta_i}; \beta\right) \right\} \leq \frac{C_2}{\beta + 1}.$$

Finally, the estimation of $\beta$ as $k \gg 1$ completes the proof:

$$\beta = \frac{1}{2\zeta_i - 2\zeta_i} (1 - \tau + O((1 - \tau)^2)) = \frac{1}{\zeta_i - \zeta_i} \left( \frac{1}{k + 1} + O \left( \frac{1}{(k + 1)^2} \right) \right) \sim \sqrt{\frac{r_r \tau_i}{2\varepsilon (r_e - r_i) k + 1}}.$$
C Proof of Lemma 5.1

Proof. Fix any $C^4$ bi-harmonic function $\chi$ and $\zeta_0 \in [\zeta_1, \zeta_2]$. Expand the function of $\theta$ as

$$F(\zeta_0, \theta)(\chi \circ U)(\zeta_0, \theta) = \sum_{n=0}^{\infty} f_n \cos(n\theta) + \sum_{n=1}^{\infty} g_n \sin(n\theta). \tag{C.1}$$

For simplicity, let $\chi_0(\zeta, \theta) = F(\zeta, \theta)(\chi \circ U)(\zeta, \theta)$. Here $f_n$ and $g_n$ are functions of $\zeta_0$:

$$g_n = g_n(\zeta_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} \chi_0(\zeta_0, \theta) \sin(n\theta) d\theta \quad \text{for } n \geq 1,$$

$$f_n = f_n(\zeta_0) = \begin{cases} \frac{1}{\pi} \int_{-\pi}^{\pi} \chi_0(\zeta_0, \theta) \cos(n\theta) d\theta & \text{for } n \geq 1, \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_0(\zeta_0, \theta) \cos(n\theta) d\theta & \text{for } n = 0. \end{cases}$$

Integrating by parts four times using that $\chi_0 \in C^4(U^{-1}(\Omega))$,

$$|g_n(\zeta_0)| = \left| \frac{1}{\pi n^4} \int_{-\pi}^{\pi} \frac{\partial^4}{\partial \theta^4} \chi_0(\zeta_0, \theta) \sin(n\theta) d\theta \right| \leq \frac{2}{n^4} \sup_{\Omega} \left| \frac{\partial^4}{\partial \theta^4} \chi_0 \right|,$$

which shows that $|g_n(\zeta_0)| \leq C/n^4$, where $C$ does not depend on $\zeta_0$. Similarly, $|f_n(\zeta_0)| \leq C/n^4$. Hence the series on the right hand side of (C.1) converges absolutely and uniformly on $\Omega$. Moreover, Leibniz’s rule and the fact that $\sum_{n=1}^{\infty} \frac{1}{n^s} < \infty$ for $s = 2, 3, 4$ shows that term-by-term differentiation up to second order of the right hand side of (C.1) converges uniformly, which guarantees that the uniform limit becomes the same differentiation of $\chi_0 \circ U$. Finally, to find the generating functions $f_{m,n}$ and $g_{m,n}$ of $f_n$ and $g_n$, apply $\nabla_{x,y}^4 \equiv \sum_{j+k \leq 4} U_{j,k}(\zeta_0) \frac{\partial^{j+k}}{\partial \zeta_j \partial \zeta_k} F(\zeta_0, \theta)$ on both sides of (C.1). For each $n = 0, \pm 1, \pm 2, \cdots$, it follows from Leibniz’s rule and integration by parts that the $n$-th complex Fourier coefficient of $\nabla_{x,y} \chi_0$ is

$$0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \nabla_{x,y} \chi_0(\zeta_0, \theta) e^{-i\theta} d\theta = \frac{1}{2\pi} \sum_{j+k \leq 4} (-i)^k U_{j,k}(\zeta_0) \frac{\partial^j}{\partial \zeta_j} \int_{-\pi}^{\pi} \chi_0(\zeta_0, \theta) e^{-i\theta} d\theta, \tag{C.2}$$

where

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_0(\zeta_0, \theta) e^{-i\theta} d\theta = \begin{cases} f_0(\zeta_0) & \text{if } n = 0, \\ \frac{1}{2} (f_n(\zeta_0) - ig_n(\zeta_0)) & \text{if } n > 0, \\ \frac{1}{2} (f_{-n}(\zeta_0) + ig_{-n}(\zeta_0)) & \text{if } n < 0. \end{cases}$$

Here we used the complex Fourier basis $\{ e^{i n \theta} \}_{n=-\infty}^{\infty}$ to make use of the invariance of the basis under differentiation. The conditions given by (C.2) defines a system of ordinary differential equations

$$\sum_{j=0}^{4} U_{j,0}(\zeta_0) \frac{d^j}{d\zeta^j} f_0(\zeta_0) = 0 \quad \text{for } n = 0,$$

$$\sum_{j+k \leq 4} (-i)^k U_{j,k}(\zeta_0) \frac{d^j}{d\zeta^j} \left[ (-1)^k \frac{f_n(\zeta_0) + ig_n(\zeta_0)}{2} \pm \frac{f_n(\zeta_0) - ig_n(\zeta_0)}{2} \right] = 0 \quad \text{for } n \geq 1.$$
Note that for \( n \geq 1 \) the system of two ordinary differential equations for \( f_n \) and \( g_n \) is symmetric, by which we mean that, as placeholders, \( f_n \) and \( g_n \) can be interchanged. In addition, it is a linear differential equation of fourth order. Hence, adding and subtracting the symmetric forms, we get two ODEs to be solve only for \( f_n + g_n \) and \( f_n - g_n \). Therefore, both of \( f_n \) and \( g_n \) are some linear combinations of the same basis set \( \{ f_{m,n} \equiv g_{m,n} \}_{m=1}^4 \) for each \( n \geq 1 \), which completes the proof. \( \square \)

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