Expert-Supervised Reinforcement Learning for Offline Policy Learning and Evaluation

Aaron Sonabend W
Harvard University
asonabend@g.harvard.edu

Junwei Lu
Harvard University
junweilu@hsph.harvard.edu

Leo A. Celi
MIT
lceli@mit.edu

Tianxi Cai
Harvard University
tcai@hsph.harvard.edu

Peter Szolovits
MIT
psz@mit.edu

Abstract

Offline Reinforcement Learning (RL) is a promising approach for learning optimal policies in environments where direct exploration is expensive or unfeasible. However, the adoption of such policies in practice is often challenging, as they are hard to interpret within the application context, and lack measures of uncertainty for the learned policy value and its decisions. To overcome these issues, we propose an Expert-Supervised RL (ESRL) framework which uses uncertainty quantification for offline policy learning. In particular, we have three contributions: 1) the method can learn safe and optimal policies through hypothesis testing, 2) ESRL allows for different levels of risk aversion within the application context, and finally, 3) we propose a way to interpret ESRL’s policy at every state through posterior distributions, and use this framework to compute off-policy value function posteriors. We provide theoretical guarantees for our estimators and regret bounds consistent with Posterior Sampling for RL (PSRL) that account for any risk aversion threshold. We further propose an offline version of PSRL as a special case of ESRL.

1 Introduction

With increasing success in reinforcement learning (RL), there is broad interest in applying these methods to real-world settings. This has brought exciting progress in offline RL and Off-Policy Policy Evaluation (OPPE). These methods allow one to leverage observed data sets collected by expert exploration of environments where, due to costs or ethical reasons, direct exploration is not feasible. Sample-efficiency, reliability, and ease of interpretation are characteristics that offline RL methods must have in order to be used for real-world applications with high risks, where a tendency is exhibited towards sampling bias. In particular, there is a need for policies that shed light into the decision-making at all states and actions, and account for the uncertainty inherent in the environment and in the data collection process. In healthcare data for example, there is a common bias that arises: drugs are mostly prescribed only to sick patients; and so naive methods can lead agents to consider them harmful. Actions need to be limited to policies which are similar to the expert behavior and sample size should be taken into account for decision-making [1,2].

To address these deficits we propose an Expert-Supervised RL (ESRL) approach for offline learning based on Bayesian RL. This method yields safe and optimal policies as it learns when to adopt the expert’s behavior and when to pursue alternative actions. Risk aversion might vary across applications as errors may entail a greater cost to human life or health, leading to variation in tolerance for the target policy to deviate from expert behavior. ESRL can accommodate different risk aversion levels. We provide theoretical guarantees in the form of a regret bound for ESRL, independent of the risk.
aversion level. Finally, we propose a way to interpret ESRL’s policy at every state through posterior distributions, and use this framework to compute off-policy value function posteriors for any policy.

While training a policy, ESRL considers the reliability of the observed data to assess whether there is substantial benefit and certainty in deviating from the behavior policy, an important task in a context of limited data. This is embedded in the method by learning a policy that chooses between the optimal action or the behavior policy based on statistical hypothesis testing. The posteriors are used to test the hypothesis that the optimal action is indeed better than the one from the behavior policy. Therefore, ESRL is robust to the quality of the behavior policy used to generate the data.

To understand the intuition for why hypothesis testing works for offline policy learning, we discuss an example. Consider a medical setting where we are interested in the best policy to treat a complex disease over time. We first assume there is a standardized treatment guideline that works well and that most physicians adopt it to treat their patients. The observed data will have very little exploration of the whole environment—in this case, meaning little use of alternative treatments. However, the state-action pairs observed will be near optimal. For any fixed state, those actions not recommended by the treatment guidelines will be rare in the data set and the posterior distributions will be dominated by the uninformative wide priors. The posteriors for the value associated with the optimal actions will incorporate more information from the data as they are commonly observed. Thus, testing for the null hypothesis that an alternative action is better than the treatment guideline will likely yield a failure to reject the null, and the agent will conclude the physician’s action is best. Unless the alternative is substantially better for a given state, the learned policy will not deviate from the expert’s behavior when there is a clear standard of care.

On the other hand, if there is no treatment guideline or consensus among physicians, different doctors will try different strategies and state-action pairs will be more uniformly observed in the data. At any fixed state, some relatively good actions may have narrower posterior distributions associated with their value. Testing for the null hypothesis that a fixed action is better than what the majority of physicians chose is more likely to reject the null and point towards an alternative action in this case, as variance will be smaller across the sampled actions. Deviation from the (noisy) behavior policy will occur more frequently. Therefore, whether there is a clear care guideline or not, the method will have learned a suitable policy.

A central point in Bayesian RL is that the posterior provides not just the expected value for each action, but also higher moments. We leverage this to produce interpretable policies which can be understood and analyzed within the context of the application. We illustrate this with posterior distributions and credible intervals (CI). We further propose a way to produce posterior distributions for OPPE with consistent and unbiased estimates.

**Handling Uncertainty.** To the best of our knowledge, there is no work that has incorporated hypothesis testing directly into the policy training process. However, accounting for the uncertainty in policy estimation is a successful idea which has been widely explored in other works. Methods range from confidence interval estimation using bootstrap, to model ensembles for guiding online exploration [3, 4, 5]. We adopt the Bayesian framework, which has been proven successful in online RL [6, 7], as it provides a natural way to formalize uncertainty in finite samples. Bayesian model free methods such as temporal difference (TD) learning provide provably efficient ways to explore the dynamics of the MDP [8, 9, 10]. Gaussian Process TD can be also used to provide posterior distributions with mean value and CI for every state-action pair [11]. Although efficient for online exploration, TD methods require large data in high dimensional settings, which can be a challenge in complex offline applications such as healthcare. ESRL is model-based which makes it sample efficient [12]. Within model-based methods, the Bayesian framework allows for natural incorporation of uncertainty measures. Posterior sampling RL proposed by Strens efficiently explores the environment by using a single MDP sample per episode [13]. ESRL fits within this line of methods, which are theoretically guaranteed to be efficient in terms of finite time regret bounds [14, 15].

**Hypothesis Testing for Offline RL.** Naively applying model-based RL to offline, high dimensional tasks can degrade its performance, as the agent can be led to unexplored states where it fails to learn reliable policies. A common strategy is to regularize the learned policy towards the behavior policy whether directly in the state space or in the action space [16, 17, 18]. However, there are cases where the data logging policy is a noisy representation of the expert behavior, and regularization will lead to sub-optimal actions. ESRL can detect these cases through hypothesis testing [19] to check whether
We are interested in finding a policy which improves upon \(\pi\). Directly regularizing the target policy to the behavior might restrict the agent from finding optimal actions, especially when \(\pi\) has a high random component \(\epsilon\), or \(\pi^0\) is not close to optimal. Thus we want to know when to use \(\mu\) versus \(\pi\). This motivates the use of posterior distributions to quantify how well each state has been explored and consistent, and are equipped with uncertainty measures.

**2 Problem Set-up**

We are interested in learning policies that can be used in real-world applications. To develop the framework we will use the clinical example discussed in Section 1. Consider a finite horizon MDP defined by the following tuple: \(<S, A, R^M, P^M, \tau, P_0>\), where \(S\) is the state-space, \(A\) is the action space, \(M\) is the model over all rewards and state transition probabilities with prior \(f(\cdot)\), \(R^M(s, a) : S \times A \rightarrow [0, 1]\) is the reward distribution for fixed state-action pair \((s, a)\) under model \(M\), with mean \(\bar{R}^M(s, a)\). \(P^M_\alpha(s'|s)\) is the probability distribution function for transitioning to state \(s'\) from state-action pair \((s, a)\) under model \(M\), \(\tau \in \mathbb{N}\) is the fixed episode length, and \(P_0\) is the initial state distribution. The true MDP model \(M^*\) has distribution \(f\).

The behavior policy function is a noisy version of a deterministic policy. Going back to the clinical example there is generally a consensus of what the correct treatment is for a disease, but the data will be generated by different physicians who might adhere to the consensus to varying degrees. Thus, we model the standard of care as a deterministic policy function to the behavior might restrict the agent from finding optimal actions, especially when \(\pi\) has a high random component \(\epsilon\), or \(\pi^0\) is not close to optimal. Thus we want to know when to use \(\mu\) versus \(\pi\). This motivates the use of posterior distributions to quantify how well each state has been explored and consistent, and are equipped with uncertainty measures.

**3 Expert-Supervised Reinforcement Learning**

We are interested in finding a policy which improves upon \(\pi\). Directly regularizing the target policy to the behavior might restrict the agent from finding optimal actions, especially when \(\pi\) has a high random component \(\epsilon\), or \(\pi^0\) is not close to optimal. Thus we want to know when to use \(\mu\) versus \(\pi\). This motivates the use of posterior distributions to quantify how well each state has been explored and consistent, and are equipped with uncertainty measures.

We will denote a policy function by \(\mu : S \times \{1, \ldots, \tau\} \rightarrow A\). The associated value function for \(\mu\), model \(M\) is \(V_{\mu,t}^M(s) = \mathbb{E}_{s_{t+1}}\left[\sum_{j=t}^\tau \bar{R}^M(s_j, a_j)|s_t = s\right]\), and the action-value function is \(Q_{\mu,t}^M(s, a) = \bar{R}^M(s, a) + \sum_{s' \in S} P^M_\mu(s'|s) V_{\mu,t+1}^M(s')\).

At any fixed \((s, t)\), \(\mu(s, t) \equiv \arg \max_a Q_{\mu,t}^M(s, a)\), note that we allow \(\mu\) to vary. This will be useful to compare the quality of the behavior policy with \(\pi\). The target policy actions. We consider both the expected values of each action \(Q_{\mu,t}^M(s, \pi(s, t))\) versus \(Q_{\mu,t}^M(s, \mu(s, t))\), and their second moments for any fixed \(\mu\). In particular, posterior distributions of \(Q_{\mu,t}^M(s, a)\), \(a \in A\) are used to test if the value for \(\mu(s, t)\) is significantly better than \(\pi\). This makes the learning process robust to the quality of the behavior policy. Next we formalize this arguments by a sampling scheme, define the ESRL policy, and state its theoretical properties.
Sampling $Q$ functions. The distribution over the MDP model $f(\cdot | D_T)$ implicitly defines a posterior distribution for any $Q$ function: $Q_{\mu, \pi}(s, a) \sim f_Q(s, a, t, D_T)$. As the true MDP model $M^*$ is stochastic, we want to approximate the conditional mean $Q$ value: $E [Q_{m, \pi}^M(s, a) | t, s, a, D_T]$. We do this by sampling $K$ MDP models $M_k$, compute $Q^{(k)}_{m, \pi}(s, a), k = 1, \ldots, K$ and use $Q_{m, \pi}(s, a) \equiv \frac{1}{K} \sum_{k=1}^{K} Q^{(k)}_{m, \pi}(s, a)$.

Lemma 3.1. $Q_{m, \pi}(s, a)$ is consistent and unbiased for $Q_{m, \pi}^M(s, a)$:

$$E \left[ \hat{Q}_{m, \pi}(s, a) | t, s, a, D_T \right] = E \left[ Q_{m, \pi}^M(s, a) | t, s, a, D_T \right],$$

$$\hat{Q}_{m, \pi}(s, a) - E \left[ Q_{m, \pi}^M(s, a) | t, s, a, D_T \right] = O_P \left( K^{-\frac{1}{2}} \right), \forall (t, s, a).$$

Lemma 3.1 establishes desirable properties for our $Q$ function estimation. Choosing $K = 1$ yields an immediate result: every $Q^{(k)}_{m, \pi}(s, a)$ from model $M_k$ is unbiased.

The stochasticity of $M^*$ and $\pi$ suggests the mean $Q$ values for $\pi$ and $\mu$ are not enough to make a decision for whether it is beneficial to deviate from $\pi$. Next we discuss how to directly incorporate this uncertainty assessment into the policy training through Bayesian hypothesis testing.

ESRL Policy Learning Through Hypothesis Testing. For a fixed $\alpha$-level, denote the ESRL policy by $\mu^\alpha$, we next describe the steps to learn this policy. By iterating backwards as in dynamic programming, assume we know $\mu^\alpha(s, j) \forall s \in S, j \in \{ t + 1, \ldots, \tau \}$, and we have $V_{\mu^\alpha, \tau+1}^M(s) = 0, \forall s \in S$. To compute $\mu^\alpha(s, t)$, we start with samples $\{Q^{(k)}_{m, \pi}(s, a)\}_{k=1}^{K}$ to test the null hypothesis that the behavior-chosen action is better than $\mu(s, t) = \arg \max_a Q_{\mu^\alpha}(s, a, t)$. We write the null hypothesis as $H_0: Q_{\mu^\alpha}(s, \mu(s, t)) \leq Q_{\mu^\alpha}(s, \pi(s, t))$. That is, the probability that the value for using action $\mu(s, t)$ is less than the value under action $\pi(s, t)$, and in both cases the agent proceeds with ESRL policy $\mu^\alpha$ onward. If the learned policy does not yield a significantly better value estimate, then we fail to reject the null and proceed to use the behavior policy’s action.

For MDP model $M$, we have

$$P(H_0 | t, s, D_T) = P \left( Q_{\mu^\alpha, \pi}^M(s, \mu(s, t)) \leq Q_{\mu^\alpha, \pi}^M(s, \pi(s, t)) | t, s, D_T \right),$$

(1)

and the ESRL policy at $(s, t)$ is then

$$\mu^\alpha(s, t) = \begin{cases} \mu(s, t) & \text{if } P(H_0 | t, s, D_T) < \alpha, \\ \pi(s, t) & \text{else.} \end{cases}$$

We estimate the null probability in (1), under true MDP $M^*$ and optimal policy $\mu^*$ with samples $\{Q^{(k)}_{t}(s, a)\}_{k=1}^{K}$ as $\hat{P}(H_0 | t, s, D_T) = \frac{1}{K} \sum_{k=1}^{K} I \left( Q^{(k)}_{t}(s, \mu_k(s, t)) \leq Q^{(k)}_{t}(s, \pi(s, t)) \right)$.

Lemma 3.2. Let $P^*(H_0 | t, s, D_T)$ be the null probability under true MDP $M^*$ with policy $\mu^*$,

$$\hat{P}(H_0 | t, s, D_T) - P^*(H_0 | t, s, D_T) = O_P \left( K^{-\frac{1}{2}} \right).$$

Lemma 3.2 guarantees that we can construct a consistent policy $\mu^\alpha$ by sampling from the MDP posterior. There are two factors that come into play in (1): the difference in mean $Q$ values, and the second moments. If $Q_{\mu^\alpha, \pi}^M(s, \mu(s, t))$ is much higher than $Q_{\mu^\alpha, \pi}^M(s, \pi(s, t))$, but there are very few samples in $D_T$ for $(s, \mu(s, t))$, the wide posterior will translate into a high $P(H_0 | t, s, D_T)$ leading ESRL to adopt $\pi(s, t)$. To choose $\mu(s, t)$ there needs to be both a substantial benefit for this new action and a high certainty of such gain. How averse the user is to deviating from $\pi$ is controlled by parameter $\alpha$. A small risk averse $\alpha$ will allow $\mu^\alpha$ to deviate from $\pi$ only with high certainty. When $\alpha = 1$, Algorithm 1 boils down to an offline version of PSRL after $T$ episodes, which uses majority voting for a robust policy.

Algorithm 1 collects these ideas in order to learn an ESRL policy $\mu^\alpha$. It begins by sampling $K$ MDP models from the posterior distribution and splitting the samples into two disjoint sets $\tilde{I}_1, \tilde{I}_2$: we use $\tilde{I}_1$ to draw the final policy based on majority voting and $\tilde{I}_1$ for hypothesis testing. Disjoint sets ensure independence and keep theoretical guarantees under the next Assumption.
We now illustrate how the ESRL framework can be used to construct efficient point estimates of the value function, and their posterior distributions. Hypothesis testing can also be used to assess whether the difference in value of two policies is statistically significant (i.e. $\mu^\alpha$ vs. $\pi$).

To compute the estimated value of a given policy $\hat{\mu}$, we sample $K$ models from the posterior and navigate $M_k$ using $\hat{\mu}$. This yields samples $V^{(k)}_{\hat{\mu},1} \sim f_V(\cdot|\hat{\mu}, D_T)$. We estimate $\mathbb{E} \left[ V^{M^\ast}_{\hat{\mu},1}(s) - V^{M^\ast}_{\pi,1}(s) \right]$ with $V_{\hat{\mu}} = \frac{1}{K} \sum_{k=1}^{K} V^{(k)}_{\hat{\mu},1}$. Note that we average over the initial states as well, as we are interested to know the marginal value of the policy. A conditional value of the policy function $V^{M^\ast}_{\hat{\mu},1}(s)$ can also be computed simply by starting all samples at a fixed state.

**Assumption 3.3.** Let $\mathbb{P}^\ast(H_0|s,t,D_T)$ be defined as in (1) for the true $M^\ast$. The chosen risk-averse parameter $\alpha \in [0,1]$ satisfies $\mathbb{P}^\ast(H_0|s,t,D_T) \neq \alpha \forall (s,t) \in S \times \{1, \ldots, T\}$.

As $\alpha$ is set by the user, Assumption 3.3 is easily satisfied as long as $\alpha$ is chosen carefully. Let $V^{M^\ast,1}_{\mu^\alpha,1}(s)$ be the value under the true MDP $M^\ast$ and let $\mu^\alpha$ be an ESRL policy which uses the null hypotheses in (1) defined under $M^\ast$. Then we can define the expected regret for $\mu_k$ defined by $k^{MV}$ obtained from Algorithm 1 as $\mathbb{E} \left[ \text{Regret}(T)|D_T \right] = \mathbb{E} \left[ \sum_{s \in S} P_0(s) \left( V^{M^\ast}_{\mu^\alpha,1}(s) - V^{M^\ast}_{\mu_k,1}(s) \right) \right] | D_T$.

**Theorem 3.4 (Regret Bound for ESRL).** For any $\alpha \in [0,1]$ which satisfies Assumption 3.3, Algorithm 1 using $D_T$ will yield

$$\mathbb{E} \left[ \text{Regret}(T) \right] = \mathcal{O} \left( \tau S \sqrt{AT \log(SAT)} \right).$$

ESRL is suitable to a large class of models, as this regret bound does not require any specific form of $f$. If we further assume a Dirichlet prior over the transitions, the rate can be improved. This bound is also novel as it is true for any level of risk aversion $\alpha$, Algorithm 1 universally converges to the oracle. This makes ESRL flexible for a wide range of applications. Next we consider how to discern whether ESRL, or any other fixed policy, is an improvement on the behavior policy.

### 4 Off-Policy Policy Evaluation and Uncertainty Estimation

We now illustrate how the ESRL framework can be used to construct efficient point estimates of the value function, and their posterior distributions. Hypothesis testing can also be used to assess whether the difference in value of two policies is statistically significant (i.e. $\mu^\alpha$ vs. $\pi$).

```python
Algorithm 1: Expert-Supervised RL
Sample $M_k \sim f(\cdot|D_T)$ $k = 1, \ldots, K$, set $I_1 = \{1, \ldots, \left\lfloor \frac{K}{2} \right\rfloor \}$, $I_2 = \{ \left\lfloor \frac{K}{2} \right\rfloor + 1, \ldots, K \}$; Set $V^{(k)}_{\pi}(s) \leftarrow 0 \forall s \in S, k = 1, \ldots, K$; Compute behavior distribution $\pi(a|s,t)$ from $D_T$, set $\pi(s,t) = \arg \max_a \pi(a|s,t)$;

for $t = \tau, \ldots, 1$ do
  for $s \in S$ do
    for $k = 1, \ldots, K$ do
      $\mu_k(s,t) \leftarrow \arg \max_a Q_t^{(k)}(s,a)$;
      $\hat{\mu}(s,t) \leftarrow$ maj. vote$\{\mu_k(s,t), k \in I_1\}$;
      Compute $\hat{P}(H_0|s,t,D_T) = \frac{1}{|I_2|} \sum_{k \in I_2} I \left( Q_t^{(k)}(s,\mu_k(s,t)) - Q_t^{(k)}(s,\pi(s,t)) < 0 \right)$;
      for $k = 1, \ldots, K$ do
        $\text{Set } \mu_k^\alpha(s,t) \leftarrow I \left( \hat{P}(H_0|D_T) < \alpha \right) \mu_k(s,t) + I \left( \hat{P}(H_0|D_T) \geq \alpha \right) \pi(s,t)$;
        $\hat{V}_t^{(k)}(s) \leftarrow Q_t^{(k)}(s,\mu_k^\alpha(s,t))$;
      end
      $\hat{\mu}^\alpha(s,t) \leftarrow$ maj. vote$\{\mu_k^\alpha(s,t), k \in I_1\}$;
      $\mathcal{M}^\alpha(s,t) \leftarrow \{k|k \in I_1, \mu_k^\alpha(s,t) = \hat{\mu}^\alpha(s,t)\}$;
    end
  end
end

Define majority voting set: $\mathcal{M}^\alpha = \cap_{(s,t)} \mathcal{M}^\alpha(s,t)$;
if $\exists k \in \mathcal{M}^\alpha$ then choose $k \in \mathcal{M}^\alpha$ at random, set $k^{MV} \leftarrow k$;
else Set $k^{MV}$ to most common $k \in \mathcal{M}^\alpha(s,t)$ $\forall (s,t)$;```

---

**Algorithm 1:** Expert-Supervised RL

Sample $M_k \sim f(\cdot|D_T)$ $k = 1, \ldots, K$, set $I_1 = \{1, \ldots, \left\lfloor \frac{K}{2} \right\rfloor \}$, $I_2 = \{ \left\lfloor \frac{K}{2} \right\rfloor + 1, \ldots, K \}$; Set $V^{(k)}_{\pi}(s) \leftarrow 0 \forall s \in S, k = 1, \ldots, K$; Compute behavior distribution $\pi(a|s,t)$ from $D_T$, set $\pi(s,t) = \arg \max_a \pi(a|s,t)$;

for $t = \tau, \ldots, 1$ do
  for $s \in S$ do
    for $k = 1, \ldots, K$ do
      $\mu_k(s,t) \leftarrow \arg \max_a Q_t^{(k)}(s,a)$;
      $\hat{\mu}(s,t) \leftarrow$ maj. vote$\{\mu_k(s,t), k \in I_1\}$;
      Compute $\hat{P}(H_0|s,t,D_T) = \frac{1}{|I_2|} \sum_{k \in I_2} I \left( Q_t^{(k)}(s,\mu_k(s,t)) - Q_t^{(k)}(s,\pi(s,t)) < 0 \right)$;
      for $k = 1, \ldots, K$ do
        $\text{Set } \mu_k^\alpha(s,t) \leftarrow I \left( \hat{P}(H_0|D_T) < \alpha \right) \mu_k(s,t) + I \left( \hat{P}(H_0|D_T) \geq \alpha \right) \pi(s,t)$;
        $\hat{V}_t^{(k)}(s) \leftarrow Q_t^{(k)}(s,\mu_k^\alpha(s,t))$;
      end
      $\hat{\mu}^\alpha(s,t) \leftarrow$ maj. vote$\{\mu_k^\alpha(s,t), k \in I_1\}$;
      $\mathcal{M}^\alpha(s,t) \leftarrow \{k|k \in I_1, \mu_k^\alpha(s,t) = \hat{\mu}^\alpha(s,t)\}$;
    end
  end
end

Define majority voting set: $\mathcal{M}^\alpha = \cap_{(s,t)} \mathcal{M}^\alpha(s,t)$;
if $\exists k \in \mathcal{M}^\alpha$ then choose $k \in \mathcal{M}^\alpha$ at random, set $k^{MV} \leftarrow k$;
else Set $k^{MV}$ to most common $k \in \mathcal{M}^\alpha(s,t)$ $\forall (s,t)$;
We perform several analyses to assess ESRL policy learning, sensitivity to the risk aversion parameter. We use samples $V$ to assess the risk aversion parameter $\alpha$. Theorem 4.1. Let $\tilde{\mu} : S \times \{1, \ldots, \tau\} \mapsto A$ be a pre-specified policy.

\[
E \left[ \tilde{\nabla}_t(s) | D_T, \tilde{\mu} \right] = E \left[ V_{\tilde{\mu}}(s) | D_T, \tilde{\mu} \right] , \tilde{\nabla}_t - E \left[ V_{\tilde{\mu}}(s) | D_T, \tilde{\mu} \right] = O_p \left( K^{-\frac{1}{2}} \right).
\]

Theorem 4.1 ensures that we are indeed estimating the quantity of interest. It establishes that $\tilde{\nabla}_t$ is consistent and unbiased for $\sum_{s \in S} P_0(s)V_{\tilde{\mu}}(s)$. Point estimates and uncertainty are not sufficient by themselves to evaluate the quality of a policy. For example in an application such as healthcare, there might be policies for which the second best action (treatment) is not significantly different in terms of value, but has less associated secondary risks. Including a secondary risk directly into the method might force us to make strong modeling assumptions. Therefore, testing whether such policies yield a significant difference in value is important as one might devise a policy that always chooses the safest action (e.g. clinical terms) and if this yields an equivalent value then it is preferable.

**Policy-level hypothesis testing.** Define the value function null hypothesis for two fixed policies $\tilde{\mu}_1, \tilde{\mu}_2$ as the event in which policy $\tilde{\mu}_1$ has a higher expected value than $\tilde{\mu}_2$ conditional on $D_T$: $H_0 : \mathbb{E}_{s \sim P_0, M^*} [V_{\tilde{\mu}_1}(s) | D_T, \tilde{\mu}_1] > \mathbb{E}_{s \sim P_0, M^*} [V_{\tilde{\mu}_2}(s) | D_T, \tilde{\mu}_2]$. The probability of the null under the true model $M^*$ is

\[
\mathbb{P}(H_0 | D_T) = \mathbb{P} \left( V_{\tilde{\mu}_1} > V_{\tilde{\mu}_2} | D_T, \tilde{\mu}_1, \tilde{\mu}_2 \right) = \sum_{s \in S} P_0(s) \mathbb{P} \left( V_{\tilde{\mu}_1} > V_{\tilde{\mu}_2} | s, D_T, \tilde{\mu}_1, \tilde{\mu}_2 \right).
\]

We use samples $V^{(k)}$, $\ell = 1, 2$ to estimate the probability of the null with $\tilde{\mathbb{P}}_\mu (H_0 | D_T) = \frac{1}{K} \sum_{k=1}^K I \left( V^{(k)}_{\tilde{\mu}_1}(s) > V^{(k)}_{\tilde{\mu}_2}(s) \right)$. Consistency of this estimator is shown in the supplementary material.

## 5 Experiments and Application

We perform several analyses to assess ESRL policy learning, sensitivity to the risk aversion parameter $\alpha$, value function estimation, and finally illustrate how we can interpret the posteriors within the context of the application. We use the Riverswim environment [23], and a Sepsis data set built from MIMIC-III data [24]. As offline methods that regularize learned policies are designed for continuous state space, we compare ESRL to Value iteration (VI), tabular Q-learning (QL) [20], and Deep Q networks (DQN) [25] with a 2-layer neural network with ReLU activation functions. For Riverswim we use 2-128 unit layers, for Sepsis we use 128, 256 unit layers respectively [26]. For ESRL, we use conjugate Dirichlet/multinomial, and normal-gamma/normal for the prior and likelihood of the transition and reward functions respectively.

### 5.1 Riverswim

The Riverswim environment [23] requires deep exploration for achieving high rewards. There are 6 states and two actions: swim right or left. Only swimming left is always successful. There are only two ways to obtain rewards: swimming left while in the far left state will yield a small reward (5/1000) w.p. 1, swimming right in the far right state will yield a reward of 1 w.p. 0.6. The episode lasts 20 time points. We train policy $\pi^0$ using PSRL [14] for 10,000 episodes, we then generate $D_T$ set with $\pi$, varying both size $T$ and noise $\epsilon$. The offline trained policies are then tested on the environment for 10,000 episodes. This process is repeated 50 times.

**Policy Learning.** We first assess ESRL on Riverswim. The training set sample size $T$ is kept low to make it hard to completely learn the dynamics of the environment. We train an offline policy using ESRL with different risk aversion parameters ($\alpha = 0.01, 0.05, 0.1$), and offline PSRL (ESRL with $\alpha = 1$). Figure 6(a) shows mean reward for $T = 200$ episodes while varying $\epsilon$. ESRL proves to be robust to the behavior policy quality. This is expected as when $\epsilon$ is low the environment is not fully explored. This yields high variance in the $Q$ posteriors, which leads ESRL to reject the null more often and favor the behavior policy. For low quality data generating policies there is greater exploration of the environment, which yields narrower posterior distributions for the $Q$ function posteriors, leading ESRL to reject the null when it is indeed beneficial to do so. When behavior policy is almost deterministic, the smaller risk aversion parameter $\alpha$ seems to yield good results as ESRL almost always imitates the behavior policy. Value iteration and offline PSRL do well when there is
enough exploration in $D_T$ ($\epsilon > 0.01$). Overall $Q$-learning methods lack enough data to learn a good policy. Figure 1(b) compares methods on an almost constant behavior policy ($\epsilon = 0.05$), so there is little exploration in $D_T$. ESRL is robust as wide posteriors lead it to deviate from $\pi$. OPSRL, VI, QL, and DQN generally fail likely to lack of exploration in $D_T$. For value function estimation ESRL is sample efficient and unbiased. Figure 3 shows Mean Squared Error (MSE) for value estimation of an ESRL policy using $D_T$ while varying $T$. In small sample data sets ESRL performs substantially better as it uses the model to overcome rarely visited states in $D_T$.

**Hypothesis testing and interpretability with $Q$ function posterior distributions.** We illustrate interpretability of the ESRL method in Riverswim as it is a simple, intuitive setting. Figure 2 shows 3 $Q$ function posterior distributions $f_Q(\cdot|s, t, D_T)$, each for a fixed state-time pair $(s, t)$. Display (a) shows $Q$ functions for the far left state and an advanced time point $t = 17$. There is high certainty (no overlap in posteriors) that swimming left will yield a higher reward, as left is successful w.p. 1. $Q_{17}(0, \text{left})$ has a wider posterior as this $(s, a)$ is not common in $D_T$. Display (b) is the most interesting, it sheds light into the utility of uncertainty measures. A naive RL method that only considers mean values, would choose the optimal action according to $\mu$: swimming left. However, there is high uncertainty associated with such a choice. In fact, we know that the optimal strategy in Riverswim is $\pi(2, 2) = \text{right}$, hypothesis testing will fail to reject the null and use the behavior action which will lead to a higher expected reward. Display (c) shows $Q$ posteriors for the state furthest to the right, at $t = 5$. Choosing right will be successful with high certainty: narrow $Q_{5}(5, \text{left})$ posterior. Swimming left will still yield a relatively high reward as in the next time point the agent will proceed with the optimal policy (choosing right). As there is no overlap in (a) and (c), the best choice is clear as would be reflected with a hypothesis test.

**5.2 Sepsis.**

We further test ESRL on a Sepsis data set built from MIMIC-III [24]. Sepsis is a state of infection where the immune system gets overwhelmed and its response can cause tissue damage, organ failure, and death. Deciding treatments and medication dosage is a dynamic and highly challenging task for the clinicians. We consider an action space representing dosage of intravenous fluids for hypovolemia (IV fluids) and vasopressors to counteract vasodilation. The action space $\mathcal{A}$ is size 25: a $5 \times 5$
matrix over discretized dose of vasopressors and IV fluids. The state space is composed of 1,000 clusters estimated using K-means on a 46-feature space which contains measures of the patient’s physiological state. We used negative SOFA score as a reward \[26\], we transform it to be between 0 and 1. The data set used has 12,991 episodes of 10 time steps- measurements every 4-hour interval. 80\% of episodes are used for training and 20\% for testing.

Figure 4: (a) & (b) show posterior distributions of \(Q\) functions at fixed \((s,t)\). Display (c) shows posteriors \(\hat{V}\) for policies: \(\pi\) and \(\mu_{\alpha}\) for \(\alpha = 0.01, 0.05, 0.1\), VI, QL, and DQN for \(K = 500\).

Figure 4 (a) & (b) show posterior distributions for two different \((s,t)\) pairs in the Sepsis data set hand-picked to illustrate interpretability. For simplicity we restrict to show the best action: \(\mu(s,t)\), physician’s action \(\pi(s,t)\), and three other low dose actions. Display (a) shows posterior distributions over a state rarely observed in \(D_T\), hence the \(Q\) functions which have relatively high standard errors. The expected cumulative inverse SOFA value for this state seems to be relatively stable no matter what action is taken. The \(Q\) posteriors for \(\mu\) and \(\pi\) are practically overlapping so there’s no reason to deviate from \(\pi\), this is encoded into \(\mu_{\alpha}\) through hypothesis testing. Interpertability is useful in these cases as a physician might see no difference in actions will yield similar SOFA and choose an action with a lower risk of side effects. Display (b) on the other hand shows a common state in \(D_T\): the low standard errors allow the policy to deviate from \(\pi\) at any \(\alpha\) level. Within this state, actions \(\pi\) and \(\mu\) are usually selected so the posteriors for their \(Q\) functions are narrow, as opposed to those for \(\alpha = 0, 3\). These actions are not prevalent in \(D_T\) as they seem to be sub-optimal, so they are less often chosen by doctors and seen in \(D_T\).

Figure 4 (c) shows the posterior distribution of the Sepsis value function for different policies. There seems to be a bi-modal distribution: it is easier to control the SOFA scores for patients in the set of states shown in the right mode of the distribution. Physicians know how to do this well as shown by the posterior value function for \(\pi\); and ESRL picks up on this. The other clusters of states in the left mode seem to be much harder to control. We can appreciate how deviating from the physician’s policy is strikingly damaging to the expected value on the test set for VI, QL and DQN. The \(D_T\) is probably not enough to generalize to the test set due to the high dimensional state and action spaces. ESRL through hypothesis testing captures this and hardly deviates from the behavior policy.

6 Conclusion

We propose an Expert-Supervised RL (ESRL) approach for offline learning based on Bayesian RL. This framework can learn safe policies from observed data sets. It accounts for uncertainty in the MDP and data logging process to assess when it is safe and beneficial to deviate from the behavior policy. ESRL allows for different levels of risk aversion, which are chosen within the application...
context. We show a $\mathcal{O}(S\sqrt{AT})$ Bayesian regret bound that is independent of the risk aversion level. The ESRL framework can be used to obtain interpretable posterior distributions for the $Q$ functions and for OPPE. These posteriors are flexible to account for any possible policy function and are amenable to interpretation within the context of the application. We believe ESRL is a step towards bridging the gap between RL research and real-world applications.

Broader Impact

We believe ESRL is a tool that can help bring RL closer to real-world applications. In particular this will be useful in the clinical setting to find optimal dynamic treatment regimes for complex diseases, or at least assist in treatment decision making. This is because ESRL’s framework lends itself to be questioned by users (physicians) and sheds light into potential biases introduced by the data sampling mechanism used to generate the observed data set. Additionally, using hypothesis testing and accommodating different levels of risk aversion makes the method sensible to offline settings and different real-world applications. It is important when using ESRL and any RL method, to question the validity of the policy’s decisions, the quality of the data, and the method that was used to derive these.

References

[1] Omer Gottesman, Fredrik Johansson, Matthieu Komorowski, Aldo Faisal, David Sontag, Finale Doshi-Velez, and Leo Anthony Celi. Guidelines for reinforcement learning in healthcare. *Nature medicine*, 25(1):16, 2019.

[2] Omer Gottesman, Fredrik D. Johansson, Joshua Meier, Jack Dent, Donghun Lee, Srivatsan Srinivasan, Linying Zhang, Yi Ding, David Wihl, Xuefeng Peng, Jiayu Yao, Isaac Lage, Christopher Mosch, Li-Wei H. Lehman, Matthieu Komorowski, Aldo Faisal, Leo Anthony Celi, David A. Sontag, and Finale Doshi-Velez. Evaluating reinforcement learning algorithms in observational health settings. *CoRR*, abs/1805.12298, 2018.

[3] Leslie Pack Kaelbling. *Learning in Embedded Systems*. A Bradford Book Ser. 1993.

[4] Martha White and Adam White. Interval estimation for reinforcement-learning algorithms in continuous-state domains. In J. D. Lafferty, C. K. I. Williams, J. Shawe-Taylor, R. S. Zemel, and A. Culotta, editors, *Advances in Neural Information Processing Systems 23*, pages 2433–2441. Curran Associates, Inc., 2010.

[5] Thanard Kurutach, Ignasi Clavera, Yan Duan, Aviv Tamar, and Pieter Abbeel. Model-ensemble trust-region policy optimization. 2018.

[6] Mohammad Ghavamzadeh, Shie Mannor, Joelle Pineau, and Aviv Tamar. Bayesian reinforcement learning: A survey. *Foundations and Trends® in Machine Learning*, 8(5-6):359–483, 2015.

[7] Brendan O’Donoghue, Ian Osband, Rémi Munos, and Volodymyr Mnih. The uncertainty bellman equation and exploration. *CoRR*, abs/1709.05380, 2017.

[8] Richard Dearden, Nir Friedman, and Stuart J. Russell. Bayesian q-learning. In *AAAI/IAAI*, 1998.

[9] John Asmuth, Lihong Li, Michael L. Littman, Ali Nouri, and David Wingate. A bayesian sampling approach to exploration in reinforcement learning. 2012.

[10] Alberto Maria Metelli, Amarildo Likmeta, and Marcello Restelli. Propagating uncertainty in reinforcement learning via wasserstein barycenters. In H. Wallach, H. Larochelle, A. Beygelzimer, F. Bach, F. Fox, and R. Garnett, editors, *Advances in Neural Information Processing Systems 32*, pages 4333–4345. Curran Associates, Inc., 2019.

[11] Yaakov Engel, Shie Mannor, and Ron Meir. Reinforcement learning with gaussian processes. In *Proceedings of the 22nd international conference on machine learning*, volume 119 of *ICML ’05*, pages 201–208. ACM, 2005.
[12] Marc Deisenroth and Carl Rasmussen. Pilco: A model-based and data-efficient approach to policy search. pages 465–472, 01 2011.

[13] Malcolm Strens. A bayesian framework for reinforcement learning. In In Proceedings of the Seventeenth International Conference on Machine Learning, pages 943–950. ICML, 2000.

[14] Ian Osband, Daniel Russo, and Benjamin Van Roy. (more) efficient reinforcement learning via posterior sampling. 2013.

[15] Ian Osband and Benjamin Van Roy. Why is posterior sampling better than optimism for reinforcement learning? 2016.

[16] Aviral Kumar, Justin Fu, George Tucker, and Sergey Levine. Stabilizing off-policy q-learning via bootstrapping error reduction. CoRR, abs/1906.00949, 2019.

[17] Rahul Kidambi, Aravind Rajeswaran, Praneeth Netrapalli, and Thorsten Joachims. Morel : Model-based offline reinforcement learning, 2020.

[18] Yifan Wu, George Tucker, and Ofir Nachum. Behavior regularized offline reinforcement learning, 2019.

[19] Quentin F. Gronau, Alexander Ly, and Eric-Jan Wagenmakers. Informed bayesian t-tests. The American Statistician, 74(2):137–143, 2020.

[20] Richard S. Sutton and Andrew G. Barto. Reinforcement Learning: An Introduction. The MIT Press Cambridge, Massachusetts London, England, 2017.

[21] Nan Jiang and Lihong Li. Doubly robust off-policy value evaluation for reinforcement learning. arXiv.org, 2016.

[22] Philip S. Thomas and Emma Brunskill. Data-efficient off-policy policy evaluation for reinforcement learning. 2016.

[23] Alexander L Strehl and Michael L Littman. An analysis of model-based interval estimation for markov decision processes. Journal of Computer and System Sciences, 74(8):1309–1331, 2008.

[24] L. Shen L.-W. H. Lehman M. Feng M. Ghassemi B. Moody P. Szolovits L. Anthony Celi A. E. W. Johnson, T. J. Pollard and R. G. Mark. MIMIC-III. A freely accessible critical care database. Scientific Data, 4(160035), 2016.

[25] Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Andrei A. Rusu, Joel Veness, Marc G. Bellemare, Alex Graves, Martin Riedmiller, Andreas K. Fidjeland, Georg Ostrovski, Stig Petersen, Charles Beattie, Amir Sadik, Ioannis Antonoglou, Helen King, Dharshan Kumaran, Daan Wierstra, Shane Legg, and Demis Hassabis. Human-level control through deep reinforcement learning. Nature, 518(7540):529, 2015.

[26] Aniruddh Raghu, Matthieu Komorowski, Leo Anthony Celi, Peter Szolovits, and Marzyeh Ghassemi. Continuous state-space models for optimal sepsis treatment - a deep reinforcement learning approach. 2017.
Appendix A  Off-Policy Policy Evaluation and Uncertainty Estimation

In this Section, we follow the lines of Section 4 in the main text with more discussion. We show an Algorithm that collects the ideas presently discussed and an additional Lemma regarding the convergence of the null probability estimator.

We leverage \( f(\cdot | \mathbf{D}_T) \) to estimate the value function for any policy, obtain CI and use hypothesis testing for whether there is a meaningful difference in two policy functions (i.e. \( \mu^a \) vs. \( \pi \)). Recall, we compute the estimated value of a given policy \( \tilde{\mu} \), by sampling \( K \) models from the posterior and navigating \( M_k \) using \( \tilde{\mu} \) to obtain \( V^{M_k}_\tilde{\mu} \sim f_V(\cdot | \tilde{\mu}, \mathbf{D}_T) \). We estimate \( \mathbb{E}[V^{M^*}_\mu | \mathbf{D}_T] \) with \( \hat{V}_\mu = \frac{1}{K} \sum_k V^{(k)}_{\tilde{\mu}, 1} \). This process is shown in Algorithm 2.

Algorithm 2: Value function estimation

\[
\begin{align*}
&\text{for } k = 1, \ldots, K \text{ do} \\
&\quad \text{Set } V^{(k)}_0 \leftarrow 0; \\
&\quad \text{Sample } M_k \sim f(\cdot | \mathbf{D}_T), \ k = 1, \ldots, K; \\
&\quad \text{Sample } s \sim P^0_{M_k}; \\
&\quad \text{for } t = 1, \ldots, \tau \text{ do} \\
&\quad\quad \alpha \leftarrow \tilde{\mu}(s, t); \\
&\quad\quad V^{(k)}_t \leftarrow V^{(k)}_{t-1} + R_{M_k}(s, a); \\
&\quad\quad \text{Sample } s' \sim P_{M_k}(s'|s); \\
&\quad\quad \text{Set } s \leftarrow s'; \\
&\quad \text{end} \\
&\quad \text{Set } V^{(k)}_{\tilde{\mu}, 1} \leftarrow V^{(k)}_{\tau}; \\
&\text{end}
\end{align*}
\]

Note that we average over the initial states as well, as we are interested to know the marginal value of the policy. A conditional value of the policy function \( V^{M^*}_{\tilde{\mu}, 1}(s) \) can also be computed simply by starting all samples at a fixed state. Analogous to Section 3, we use samples \( \\{V^{(k)}_s\}_{k=1}^K \) to define a \((1 - \alpha)\) CI using the \( \alpha \) and \( 1 - \alpha \) quantiles. Note that for policies which are very different from the behavior policy, the posterior distribution will have wider CIs due to the wide distribution shift. This signals that there is not enough information in \( \mathbf{D}_T \) for the rarely visited state-action pairs \((s, a)\). This happens with OPPE importance sampling estimators as well [2]. As opposed to only considering point estimators of the value function, these CI help to assess whether the estimated value is likely to be accurate or if the estimate is unreliable given the information in \( D_T \). Importance sampling based estimators reflect this large distribution shift in high variance estimators.

Policy-level hypothesis testing. We use Algorithm 2 to assess whether there is a statistically significant difference in value from two different policies. Define the value function null hypothesis for two fixed policies \( \mu_1, \mu_2 \) as the event in which policy \( \tilde{\mu}_1 \) has a higher expected value than \( \tilde{\mu}_2 \) conditional on \( \mathbf{D}_T \): \( H_0 : \mathbb{E}_{s \sim P_0, M^*}[V_{\tilde{\mu}_1}(s)|\mathbf{D}_T, \tilde{\mu}_1] > \mathbb{E}_{s \sim P_0, M^*}[V_{\tilde{\mu}_2}(s)|\mathbf{D}_T, \tilde{\mu}_2] \). The probability of the null under the true model \( M^* \) is

\[
\mathbb{P}_\mu (H_0|\mathbf{D}_T) = \mathbb{P} \left( V^{M^*}_{\tilde{\mu}_1}(s) > V^{M^*}_{\tilde{\mu}_2}(s) | \mathbf{D}_T, \tilde{\mu}_1, \tilde{\mu}_2 \right) = \sum_{s \in S} P_0(s) \mathbb{P} \left( V_{\tilde{\mu}_1}(s) > V_{\tilde{\mu}_2}(s) | s, \mathbf{D}_T, \tilde{\mu}_1, \tilde{\mu}_2 \right).
\]

We use the following estimator from samples generated from Algorithm 2:

\[
\hat{\mathbb{P}}_\mu (H_0|\mathbf{D}_T) = \frac{1}{K} \sum_{k=1}^K I \left( V^{M_k}_{\tilde{\mu}_1}(s) - V^{M_k}_{\tilde{\mu}_2}(s) > 0 \right).
\]  

(2)

Lemma A.1. Let \( \mu_1, \mu_2 : S \times \{1, \ldots, \tau\} \rightarrow [0, 1] \) be two pre-specified policy functions, and let \( \hat{\mathbb{P}}_\mu (H_0|\mathbf{D}_T) \) be defined as in (2)

\[
\hat{\mathbb{P}}_\mu (H_0|\mathbf{D}_T) - \mathbb{P}_\mu (H_0|\mathbf{D}_T) = O_p \left( K^{-\frac{1}{2}} \right),
\]

Lemma [A.1] ensures consistency of the probability of the null-hypothesis for the value function testing.
Appendix B  Supporting Lemma

**Lemma B.1.** (Lemma 1 in [14]) If $f$ is the distribution of $M^*$ then, for any $\sigma(D_T)$–measurable function $g$, and model $M_k \sim f(\cdot|D_T)$:
\[
E[g(M^*)|D_T] = E[g(M_k)|D_T].
\]

Appendix C  Proof of results in main body

### C.1 Theorem 3.4

In this Subsection we develop the necessary definitions and lemmas, and eventually go on to prove Theorem 3.4. To simplify notation let $\mathbb{P}^*(H_0) \equiv \mathbb{P}^*(H_0|s, t, D_T)$ and $\mathbb{P}(H_0) \equiv \mathbb{P}(H_0|s, t, D_T)$. Given the behavior policy as defined in Algorithm 1 and the optimal policy under the true MDP $\mu^*$, we can write the ESRL policy obtained from Algorithm 1, and it’s equivalent version under $M_k$ as:
\[
\mu^k(s, t) = I\left(\hat{P}(H_0) < \alpha\right) \mu_k(s, t) + I\left(\hat{P}(H_0) \geq \alpha\right) \pi(s, t),
\]
\[
\mu^*(s, t) = I\left(\mathbb{P}^*(H_0) < \alpha\right) \mu^*(s, t) + I\left(\mathbb{P}^*(H_0) \geq \alpha\right) \pi(s, t),
\]
next define the ESRL policy which uses the true null probabilities and $\mu_k$ as:
\[
\mu^k(s, t) = I\left(\mathbb{P}^*(H_0) < \alpha\right) \mu_k(s, t) + I\left(\mathbb{P}^*(H_0) \geq \alpha\right) \pi(s, t).
\]
finally let
\[
\Delta_t^\mu = \sum_{s \in S} P_0(s) \left(V_{\mu^k,1}^{M^*}(s) - V_{\mu_k,1}^{M^*}(s)\right)
\]
\[
\Delta_t^* = \sum_{s \in S} P_0(s) \left(V_{\mu^k,1}^{M^*}(s) - V_{\mu_k,1}^{M^*}(s)\right).
\]

Consider function $g : M \mapsto V_{M^*}^{\mu^k,1}$. $g$ is $\sigma(D_T)$ measurable for a fixed $\alpha \in [0, 1]$ as $\pi(s, t)$, $\mathbb{P}^*(H_0)$ are fixed $\forall(s, t) \in S \times \{1, \ldots, T\}$, thus, by Lemma B.1 for any $M_k \sim f(\cdot|D_T)$
\[
E\left[V_{\mu^k,1}^{M}(s)|D_T\right] = E\left[V_{\mu^k,1}^{M^*}(s)|D_T\right],
\]
now using iterated expectations we get
\[
E\left[V_{\mu^k,1}^{M}(s)\right] = E\left[V_{\mu^k,1}^{M^*}(s)\right].
\]
We use this to re-express the expected regret for episode $i$ under model $k$ computed with Algorithm 1 as
\[
E[\Delta_i] = E\left[\sum_{s \in S} P_0(s) \left(V_{\mu^k,1}^{M^*}(s) - V_{\mu_k,1}^{M^*}(s)\right)\right]
\]
\[
= \sum_{s \in S} P_0(s) \left(E\left[V_{\mu^k,1}^{M^*}(s)\right] - E\left[V_{\mu_k,1}^{M^*}(s)\right]\right)
\]
\[
= \sum_{s \in S} P_0(s) \left(E\left[V_{\mu^k,1}^{M^*}(s)\right] - E\left[V_{\mu_k,1}^{M^*}(s)\right]\right)
\]
\[
= E[\Delta_t^*] + E[\Delta_t^\mu],
\]
where the last step follows from adding and subtracting $E\left[V_{\mu^k,1}^{M^*}(s)\right]$.

We first consider $E[\Delta_t^*]$, we use a strategy similar to [14], but do not make an iid assumption for within-episode observations. Define the following Bellman operator $T_{M^*}^{\mu^k}$ for any MDP $M$, policy $\mu_\alpha$, and value function $V$ to be
\[
T_{M^*}^{\mu^k}V(s) = \bar{R}_M(s, \mu^k(s, t)) + \sum_{s' \in S} P_{\mu^k}(s, t)(s'|s)V(s'),
\]
this lets us write $V_{\mu^k}^{M^*,t}(s) = T_{M^*}^{\mu^k}V_{\mu^k}^{M^*,t+1}(s)$.

The next Lemma will let us express term $E\left[\Delta_t^*|M^*, M_k\right]$ in terms of the Bellman operator.


Lemma C.1. If $f$ is the distribution of $M^*$, then

$$
E \left[ \Delta_i^* \bigg| M^*, M_k \right] = E \left[ \sum_{j=1}^T \left( T_M^{M_k} - T_M^{M^*} \right) \cdot \left( V_M^{M_k} - V_M^{M^*} \right) \bigg| M^*, M_k \right].
$$

We now define a confidence set for the reward and transition estimated probabilities.

**Lemma C.2.** Let $\mathcal{I}$ denote the set of index $i, j$ for episodes in $D_T = \{(s_i, a_i, r_i, \ldots, s_{i\tau}, a_{i\tau}, r_{i\tau})\}_{i=1}^T$, that is: $\mathcal{I} = \{(i, j) | i \in \{1, \ldots, T\}, j \in \{1, \ldots, \tau\}\}$. Further let $N_T(s, a)$ be the number of times $(s, a)$ was sampled in $D_T$: $N_T(s, a) = \sum_{i,j \in \mathcal{I}} I(S_{ij} = s, A_{ij} = a)$, let $P_a(\cdot | s)$ and $R(s, a)$ be non-parametric estimators for the distribution of transitions and rewards observed after sampling $T$ episodes:

$$
P_a(s' | s) = \frac{\sum_{i,j \in \mathcal{I}} I(s_{i,j+1} = s') I(s_{ij} = s, a_{ij} = a)}{N_T(s, a)}, \quad R(s, a) = \frac{\sum_{i,j \in \mathcal{I}} I(s_{ij} = s, a_{ij} = a) r_{ij}}{N_T(s, a)}.
$$

Define the confidence set:

$$
\mathcal{M}_T \equiv \left\{ M : \| P_a(\cdot | s) - P^M(\cdot | s) \|_1 \leq \beta_T(s, a), \| R(s, a) - R^M(s, a) \|_1 \leq \beta_T(s, a) \forall (s, a) \right\},
$$

where $\beta_T(s, a) \equiv \sqrt{\frac{\lambda S T \log(2 SAT)}{\max\{1, N_T(s, a)\} T}}$, then $P(M^* \notin \mathcal{M}_T) < \frac{1}{T}$.

**Proof of Theorem 3.4.** We start by summing $\Delta_i^*$ over all episodes:

$$
E \left[ \sum_{i=1}^T \Delta_i^* \right] \leq E \left[ \sum_{i=1}^T \Delta_i^* I(M_k, M^* \in \mathcal{M}_T) \right] + \tau \sum_{i=1}^T (P(M_k \notin \mathcal{M}_T) + P(M^* \notin \mathcal{M}_T))
$$

$$
\leq E \left[ \sum_{i=1}^T \Delta_i^* I(M_k, M^* \in \mathcal{M}_T) \right] + 2\tau
$$

$$
\leq E \left[ \sum_{i=1}^T \sum_{j=1}^T \left( T_M^{M_k} - T_M^{M^*} \right) \cdot \left( V_M^{M_k} - V_M^{M^*} \right) \bigg| M_k, M^* \in \mathcal{M}_T \right] + 2\tau
$$

where the first step follows by conditioning on event $I(M_k \in \mathcal{M}_T, M^* \in \mathcal{M}_T)$ and its complement, and from the fact that $\Delta_i^* \leq \tau$ as all rewards $R(s, a) \in [0, 1]$. The second step follows from iterated expectations and Lemma C.2 as $P(I(M_k \notin \mathcal{M}_T), P(I(M^* \notin \mathcal{M}_T)) \leq \frac{1}{T}$. The last step follows from the fact that $E[I(M_k, M^* \in \mathcal{M}_T)]$ is bounded by the probabilities of $M_k, M^*$ being in the confidence set. The final expectation is over the distribution of $M_k, M^*$.
We next analyze

\[ \sum_{i=1}^{T} \sum_{j=1}^{\tau} I(\mathbb{P}^*(H_0) \geq \alpha) \left\{ \tilde{R}^{M_k}(s, \pi(s, j)) - \tilde{R}^{M^*}(s, \pi(s, j)) \right\} I(M_k, M^* \in M_k) \]

\[ + \sum_{i=1}^{T} \sum_{j=1}^{\tau} I(\mathbb{P}^*(H_0) > \alpha) \left\{ \sum_{s' \in S} \mathbb{P}_{\tilde{M}_k}(s') - \mathbb{P}^*(s') \right\} V^{M_k}_{\tilde{M}_k, j+1}(s_j+1) I(M_k, M^* \in M_k) \]

\[ + \sum_{i=1}^{T} \sum_{j=1}^{\tau} I(\mathbb{P}^*(H_0) < \alpha) \left\{ \tilde{R}^{M_k}(s, \mu_k(s, j)) - \tilde{R}^{M^*}(s, \mu_k(s, j)) \right\} I(M_k, M^* \in M_k) \]

\[ + 2\tau \leq \sum_{i=1}^{T} \sum_{j=1}^{\tau} \min \{\beta_T(s_{ij}, \pi(s_{ij}, j)), 1\} \]

\[ + \sum_{i=1}^{T} \sum_{j=1}^{\tau} \min \{\beta_T(s_{ij}, \mu_k(s_{ij}, j)), 1\} + 2\tau, \]

where the last step follows by Lemma C.1, next:

\[ \leq \sum_{i=1}^{T} \sum_{j=1}^{\tau} \min \{\beta_T(s_{ij}, \pi(s_{ij}, j)), 1\} \]

\[ + \sum_{i=1}^{T} \sum_{j=1}^{\tau} \min \{\beta_T(s_{ij}, \mu_k(s_{ij}, j)), 1\} + 2\tau \]

\[ \leq M_1 \sqrt{\tau^2 SAT} + M_2 \tau \sqrt{S^2 AT \log(SAT)} + 2\tau \leq M_3 \tau S \sqrt{AT \log(SAT)} + 2\tau, \]

where the last step follows by Appendix B in [14] with constants $M_1, M_2, M_3$.

We next analyze

\[ \mathbb{E} |\Delta_i^j| = \sum_{s \in S} P_0(s) \left( \mathbb{E} \left[ V^{M^*}_{\mu^*}, 1(s) \right] - \mathbb{E} \left[ V^{M^*}_{\pi}, 1(s) \right] \right). \]

We can write the second term as

\[ \mathbb{E} \left[ V^{M^*}_{\mu^*}, 1(s) \right] = \mathbb{E} \left[ \sum_{j=1}^{\tau} I(\hat{\mathbb{P}}(H_0) < \alpha) R^{M^*}(s_j, \mu_k(s_j, j)) + I(\hat{\mathbb{P}}(H_0) \geq \alpha) R^{M^*}(s_j, \pi(s_j, j)) \right] \]

we extend the null probability notation to be explicit on the time index: $\mathbb{P}^*_j(H_0) = \mathbb{P}^*(H_0|s_j, j, D_T), \mathbb{P}^*_j(H_0) = \hat{\mathbb{P}}(H_0|s_j, j, D_T)$. By Lemma 3.2, $\exists \delta > 0$ such that $\mathbb{P}^*_j(H_0) - \mathbb{P}^*_j(H_0) \leq \delta \forall s \in S, j \in \{1, \ldots, \tau\}$ with high probability, therefore

\[ \mathbb{P}^*_j(H_0) < \alpha - \delta \implies \mathbb{P} \left( \hat{\mathbb{P}}_j(H_0) < \alpha \right) \xrightarrow{P} 1, \]

\[ \mathbb{P}^*_j(H_0) \geq \alpha + \delta \implies \mathbb{P} \left( \hat{\mathbb{P}}_j(H_0) \geq \alpha \right) \xrightarrow{P} 1. \]
As $I_1, I_2$ in Algorithm 1 are mutually exclusive, $\hat{P}_j(H_0)$ are independent to $\mu_k(s, j) \forall s \in S, j \in \{1, \ldots, \tau\}$, therefore
\[
\mathbb{E} \left[ V_{\mu,1}^{M^*}(s) \right] = \mathbb{E} \left[ \sum_{j=1}^{\tau} I(\hat{P}_j(H_0) < \alpha - \delta) \left( I\left(\hat{P}(H_0) < \alpha\right) R^M(s_j, \mu_k(s_j, j)) + I\left(\hat{P}(H_0) \geq \alpha\right) R^M(s_j, \pi(s_j, j)) \right) \right]
\]
which follows from Assumption 3.3, as $\mathbb{P} \left(\alpha - \delta \leq \hat{P}_j(H_0) < \alpha + \delta\right) \to 0 \forall s \in S, j \in \{1, \ldots, \tau\}$ as $\delta \to 0$.
Therefore $\mathbb{E} [\Delta^*_t] \xrightarrow{p} 0$.

Putting both terms together we have
\[
\mathbb{E} \left[ \sum_{i=1}^{T} \Delta_t \right] = \mathbb{E} \left[ \sum_{i=1}^{T} \Delta^*_t \right] + \mathbb{E} \left[ \sum_{i=1}^{T} \Delta^*_t \right] = O \left( \tau S \sqrt{AT \log(SAT)} \right)
\]

\[\square\]

C.2 Proofs for other results in main body

Proof of Lemma 3.1. To establish $\hat{Q}_{\hat{\mu},t}(s, a)$ is unbiased, note that for any fixed $(t, s, a)$, $M_k \sim f(\cdot|D_T)$ are iid, now for a given policy function $\tilde{\mu}$:
\[
\mathbb{E} \left[ Q_{\tilde{\mu},t}(s, a)|t, s, a, D_T \right] = \mathbb{E} \left[ \frac{1}{K} \sum_{k=1}^{K} Q_{\tilde{\mu},t}^{(k)}(s, a) \right] t, s, a, D_T = \mathbb{E} \left[ \frac{1}{K} \sum_{k=1}^{K} Q_{\tilde{\mu},t}^{(k)}(s, a) \right] t, s, a, D_T
\]
\[
= \mathbb{E} \left[ Q_{\tilde{\mu},t}^{M^*}(s, a)|t, s, a, D_T \right]
\]
where the last step follows from Lemma B.1 with $g : M \to Q_{\tilde{\mu},t}^M(s, a)$ which is $\sigma(D_T)$ measurable.

To establish the rate, we have that $R^M(s, a) \in [0, 1] \forall (s, a) \in S \times A, t = 1, \ldots, \tau$ thus $Q_{t}^{(k)}(s, a) \leq \tau$. By definition $\hat{Q}_{t}(s, a) - \mathbb{E} \left[ Q_{\tilde{\mu},t}^{M^*}(s, a)|t, s, a, D_T \right] = O_p \left( K^{-\frac{1}{2}} \right)$ if and only if for any $\epsilon > 0$, $\exists M_\epsilon > 0$ such that
\[
\mathbb{P} \left( \hat{Q}_{\hat{\mu},t}(s, a) - \mathbb{E} \left[ Q_{\tilde{\mu},t}^{M^*}(s, a)|t, s, a, D_T \right] > K^{-\frac{1}{2}} M_\epsilon \right) \leq \epsilon \forall K.
\]

Note that for any $M > 0$,
\[
\mathbb{P} \left( \hat{Q}_{\hat{\mu},t}(s, a) - \mathbb{E} \left[ Q_{\tilde{\mu},t}^{M^*}(s, a)|t, s, a, D_T \right] > K^{-\frac{1}{2}} M \right) \leq \exp \left\{ - \frac{2M^2 K^{-1}_K K^2}{K^2} \right\} = \exp \left\{ - \frac{2M^2}{\tau^2} \right\},
\]

which follows from Hoeffding’s inequality as \(\{Q^{(k)}_{\mu,t}(s,a)\}_{k=1}^{K}\) are iid with mean \\
\(\mathbb{E}\left[Q^{M^{*}}_{\mu,t}(s,a)\right] t,s,a,\mathbf{D}_{T}\). The result follows from choosing \(M_{\epsilon} > 0\) large enough such that \(\exp\left\{-\frac{2M^2}{\tau^2}\right\} < \epsilon\).

\(\square\)

**Proof of Lemma 3.2.** To simplify notation, let \(Z^{(k)} \equiv I\left(Q^{(k)}_{\mu^{t}_{k},t}(\mu_{k}(s,t)) - Q^{(k)}_{\mu^{t}_{k},t}(s,\pi(s,t)) \leq 0\right)\), then by definition \(Z^{(k)} - \mathbb{E}\left[Z^{(k)}\right] = O_{p}\left(K^{-\frac{1}{2}}\right)\) if and only if for any \(\epsilon > 0\), \(\exists M_{\epsilon} > 0\) such that
\\
\[\mathbb{P}\left(Z^{(k)} - \mathbb{E}\left[Z^{(k)}\right] > K^{-\frac{1}{2}}M_{\epsilon} \left| t,s,\mathbf{D}_{T}\right.\right) \leq \epsilon \quad \forall K.\]
\\
Note that for any \(M > 0\),
\\
\[\mathbb{P}\left(\|\mathbb{P}(H_{0}|t,s,\mathbf{D}_{T}) - \mathbb{E}\left[Z^{(k)}\right| t,s,\mathbf{D}_{T}\right) > K^{-\frac{1}{2}}M\left| t,s,\mathbf{D}_{T}\right.\right) \]
\\
\[= \mathbb{P}\left(\frac{1}{K} \sum_{k=1}^{K} Z^{(k)} - \mathbb{E}\left[Z^{(k)}\right| t,s,\mathbf{D}_{T}\right) > M K^{-\frac{1}{2}} \left| t,s,\mathbf{D}_{T}\right.\right) \]
\\
\[\leq \exp\left\{-\frac{2M^2K^{-1}K^2}{\tau^2}\right\} = \exp\left\{-\frac{2M^2}{\tau^2}\right\},\]
\\
where the inequality follows from Hoeffding’s inequality as \(\{Z^{(k)}\}_{k=1}^{K}\) are iid with mean \\
\(\mathbb{E}\left[Z^{(k)}\right| t,s,\mathbf{D}_{T}\right]\), since \(\mathcal{I}_{1}, \mathcal{I}_{2}\) in Algorithm 1 are disjoint. We can choose \(M_{\epsilon} > 0\) large enough such that \(\exp\left\{-\frac{2M^2}{\tau^2}\right\} < \epsilon\). Next note that as \(\pi\) is fixed, by Lemma B.1 with \(g : M \mapsto I\left(Q^{M^{*}}_{\mu^{t}_{k},t}(\mu_{k}(s,t)) - Q^{M^{*}}_{\mu^{t}_{k},t}(s,\pi(s,t)) \leq 0\right)\) for any \(M_{k} \sim f(\cdot|\mathbf{D}_{T})\)
\\
\[\mathbb{E}\left[I\left(Q^{(k)}_{\mu^{t}_{k},t}(\mu_{k}(s,t)) - Q^{(k)}_{\mu^{t}_{k},t}(s,\pi(s,t)) \leq 0\right) \left| t,s,\mathbf{D}_{T}\right.\right) \]
\\
\[= \mathbb{E}\left[I\left(Q^{M^{*}}_{\mu^{t}_{k},t}(\mu^{*}(s,t)) - Q^{M^{*}}_{\mu^{t}_{k},t}(s,\pi(s,t)) \leq 0\right) \left| t,s,\mathbf{D}_{T}\right.\right) \]
\\
\[= \mathbb{P}(H_{0}|t,s,\mathbf{D}_{T})\]
\\
which follows from using disjoint sets \(\mathcal{I}_{1}, \mathcal{I}_{2}\) in Algorithm 1. Substituting this in the probability statement gives us
\\
\[\mathbb{P}(H_{0}|t,s,\mathbf{D}_{T}) - \mathbb{P}(H_{0}|t,s,\mathbf{D}_{T}) = O_{p}\left(K^{-\frac{1}{2}}\right),\]
\\
which is our required result. \(\square\)

**Proof of Theorem 4.1.** We start by showing \(\hat{V}_{\hat{\mu}}\) is unbiased:
\\
\[\mathbb{E}\left[\hat{V}_{\hat{\mu}}(s)|\mathbf{D}_{T},\hat{\mu}\right] = \frac{1}{K} \sum_{k=1}^{K} \mathbb{E}\left[V^{(k)}_{\hat{\mu},1}(s)\right|\mathbf{D}_{T}\right].\]
\\
where the first step follows from definition, and the \(M_{k} \sim f(\cdot|\mathbf{D}_{T})\) being iid, now by Lemma B.1 with \(g : M \mapsto V^{M^{*}}_{\mu^{t}_{k},t}\) we have
\\
\[\mathbb{E}\left[\hat{V}_{\hat{\mu}}|\mathbf{D}_{T}\right] = \mathbb{E}\left[V^{M^{*}}_{\mu^{t}_{k},1}(s)|\mathbf{D}_{T}\right].\]
\\
To establish the rate, we have that \(V^{(k)}_{\hat{\mu},1} \leq \tau\) as all rewards are between \([0,1]\) by definition \(\hat{V}_{\hat{\mu}} - \mathbb{E}\left[V^{M^{*}}_{\mu^{t}_{k},1}(s)|\mathbf{D}_{T}\right] = O_{p}\left(K^{-\frac{1}{2}}\right)\) if and only if for any \(\epsilon > 0\), \(\exists M_{\epsilon} > 0\) such that
\\
\[\mathbb{P}\left(\hat{V}_{\hat{\mu}} - \mathbb{E}\left[V^{M^{*}}_{\mu^{t}_{k},1}(s)|\mathbf{D}_{T}\right] > K^{-\frac{1}{2}}M_{\epsilon}\right) \leq \epsilon \quad \forall K.\]
Note that for any \( M > 0 \),
\[
\mathbb{P}\left( \hat{V}_k - \mathbb{E} \left[ V_{M^*}^{M^*} (s) | D_T \right] > K^{-\frac{1}{2}} M \right) = \mathbb{P}\left( \frac{1}{K} \sum_{k=1}^{K} V_{\mu_{\hat{\mu},1}}^{(k)} - \mathbb{E} \left[ V_{M^*}^{M^*} (s) | D_T \right] > K^{-\frac{1}{2}} M \right)
\leq \exp \left\{ - \frac{2M^2K^{-1}K^2}{K^2} \right\} = \exp \left\{ - \frac{2M^2}{\tau^2} \right\},
\]
where the inequality follows from Hoeffding’s inequality as the indicators \( \left\{ V_{\mu_{\hat{\mu},1}}^{(k)} \right\}_{k=1}^{K} \) are iid with mean \( \mathbb{E} \left[ V_{M^*}^{M^*} (s) | D_T \right] \). The result follows from choosing \( M_\epsilon > 0 \) large enough such that \( \exp \left\{ - \frac{2M^2}{\tau^2} \right\} < \epsilon. \)

**Appendix D  Proofs for Supplementary results**

*Proof of Lemma A.1.* First note that conditional on \( D_T \) with \( g : M \mapsto I \left( V_{\mu_1}^M (s) - V_{\mu_2}^M (s) > 0 \right) \), by Lemma B.1,
\[
\mathbb{E} \left[ I \left( V_{\mu_1}^M (s) - V_{\mu_2}^M (s) > 0 \right) \bigg| D_T \right] = \mathbb{E} \left[ I \left( V_{\mu_1}^{M^*} (s) - V_{\mu_2}^{M^*} (s) > 0 \right) \bigg| D_T \right] = \mathbb{P}_\mu \left( H_0 | D_T \right)
\]
By definition \( \mathbb{P}_\mu \left( H_0 | D_T \right) - \mathbb{P}_\mu \left( H_0 | D_T \right) = O_p \left( K^{-\frac{1}{2}} \right) \) if and only if for any \( \epsilon > 0, \exists M_\epsilon > 0 \) such that
\[
\mathbb{P} \left( \mathbb{P}_\mu \left( H_0 | D_T \right) - \mathbb{P}_\mu \left( H_0 | D_T \right) > K^{-\frac{1}{2}} M_\epsilon \bigg| D_T \right) \leq \epsilon \ \forall K.
\]
Now, for any \( M > 0 \),
\[
\mathbb{P} \left( \mathbb{P}_\mu \left( H_0 | D_T \right) - \mathbb{P}_\mu \left( H_0 | D_T \right) > K^{-\frac{1}{2}} M_\epsilon \bigg| D_T \right)
\leq \mathbb{P} \left( \frac{1}{K} \sum_{k=1}^{K} I \left( V_{\mu_1}^{(k)} - V_{\mu_2}^{(k)} > 0 \right) - \mathbb{P}_\mu \left( H_0 | D_T \right) > K^{-\frac{1}{2}} \bigg| D_T \right)
\leq \exp \left\{ - \frac{2M^2K^{-1}K^2}{K^2} \right\} = \exp \left\{ - \frac{2M^2}{\tau^2} \right\},
\]
where the inequality follows from Hoeffding’s inequality as the indicators \( \left\{ I \left( V_{\mu_1}^{(k)} - V_{\mu_2}^{(k)} > 0 \right) \right\}_{k=1}^{K} \) are iid with mean \( \mathbb{P}_\mu \left( H_0 | D_T \right) \). We can choose \( M_\epsilon > 0 \) large enough such that \( \exp \left\{ - \frac{2M^2}{\tau^2} \right\} < \epsilon. \)

*Proof of Lemma C.1.* We first write the estimated regret as a sum of difference in value functions and a Bellman error.

I) We’ll denote the sequence of states for an episode as \( s_1, s_2, \ldots, s_T \), define
\[
W_j = \left( \mathcal{T}_{\mu_k^{M_k} (\cdot, s_j)} - \mathcal{T}_{\mu_k^{M_k^*} (\cdot, s_j)} \right) V_{\mu_k^{M_k}} \left( s_j + 1 \right)
T_j = \mathcal{T}_{\mu_k^{M_k^*} (\cdot, s_j)} \left( V_{\mu_k^{M_k^*}} \left( s_j + 1 \right) - V_{\mu_k^{M_k^*}} \left( s_j \right) \right)
\]
using \( \left( 3 \right) \) we can write
\[
\left( V_{\mu_k^{M_k^*},1} - V_{\mu_k^{M_k^*},1} \right) \left( s_1 \right) = \left( \mathcal{T}_{\mu_k^{M_k} (\cdot, s_1)} V_{\mu_k^{M_k}} \left( s_2 \right) - \mathcal{T}_{\mu_k^{M_k^*} (\cdot, s_1)} V_{\mu_k^{M_k^*}} \left( s_2 \right) \right)
\]
\[
= \left( \mathcal{T}_{\mu_k^{M_k} (\cdot, s_1)} V_{\mu_k^{M_k}} \left( s_2 \right) - \mathcal{T}_{\mu_k^{M_k^*} (\cdot, s_1)} V_{\mu_k^{M_k^*}} \left( s_2 \right) + \mathcal{T}_{\mu_k^{M_k^*} (\cdot, s_1)} V_{\mu_k^{M_k^*}} \left( s_2 \right) - \mathcal{T}_{\mu_k^{M_k^*} (\cdot, s_1)} V_{\mu_k^{M_k^*}} \left( s_2 \right) \right)
= W_1 + T_1,
\]

17
with the same steps we can generalize this to
\[
\left( V^{M_k}_{\mu_k^{*},j} - V^{M^*_k}_{\mu_k^{*},j} \right) (s_j) = W_j + T_j. \tag{4}
\]

Next let
\[
e_j = \left( I (\mathbb{P}^*(H_0) < \alpha) \sum_{s' \in S} P^{M^*}_{\mu_\pi^*(s,j)}(s'|s) + I (\mathbb{P}^*(H_0) \geq \alpha) \sum_{s' \in S} P^{M^*}_{\pi(s,j)}(s'|s) \right)
\times \left( V^{M_k}_{\mu_k^{*},j+1} - V^{M^*_k}_{\mu_k^{*},j+1} \right) (s') - \left( V^{M_k}_{\mu_k^{*},j+1} - V^{M^*_k}_{\mu_k^{*},j+1} \right) (s_j+1),
\]
using the Bellman operator we get
\[
T_j = \left( V^{M_k}_{\mu_k^{*},j+1} - V^{M^*_k}_{\mu_k^{*},j+1} \right) (s_j+1) + e_j,
\]
then we can write \( T_1 = \left( V^{M_k}_{\mu_k^{*},2} - V^{M^*_k}_{\mu_k^{*},2} \right) (s_2) + e_1 \), with the above definitions and repeated use of \( 4 \):
\[
\left( V^{M_k}_{\mu_k^{*},1} - V^{M^*_k}_{\mu_k^{*},1} \right) (s_1) = W_1 + T_1
\]
\[
= W_1 + \left( V^{M_k}_{\mu_k^{*},2} - V^{M^*_k}_{\mu_k^{*},2} \right) (s_2) + e_1
\]
\[
= W_1 + W_2 + \left( V^{M_k}_{\mu_k^{*},3} - V^{M^*_k}_{\mu_k^{*},3} \right) (s_3) + e_1 + e_2
\]
\[ \vdots \]
\[
= \sum_{j=1}^{\tau} W_j + e_j.
\]
II) Next we consider \( \mathbb{E} [e_j | M_k, M^*] \):
\[
\mathbb{E} [e_j | M_k, M^*] = \mathbb{E} \left[ I (\mathbb{P}^*(H_0) < \alpha) \sum_{s' \in S} P^{M^*}_{\mu_\pi^*(s,j)}(s'|s) \left( V^{M_k}_{\mu_k^{*},j+1} - V^{M^*_k}_{\mu_k^{*},j+1} \right) (s') \bigg| M_k, M^* \right]
\]
\[ + \mathbb{E} \left[ I (\mathbb{P}^*(H_0) \geq \alpha) \sum_{s' \in S} P^{M^*}_{\pi(s,j)}(s'|s) \left( V^{M_k}_{\mu_k^{*},j+1} - V^{M^*_k}_{\mu_k^{*},j+1} \right) (s') \bigg| M_k, M^* \right]
\]
\[ - \mathbb{E} \left[ \left( V^{M_k}_{\mu_k^{*},j+1} - V^{M^*_k}_{\mu_k^{*},j+1} \right) (s_j+1) \bigg| M_k, M^* \right]
\]
\[ = \left( I (\mathbb{P}^*(H_0) < \alpha) \sum_{s' \in S} P^{M^*}_{\mu_\pi^*(s,j)}(s'|s) + I (\mathbb{P}^*(H_0) \geq \alpha) \sum_{s' \in S} P^{M^*}_{\pi(s,j)}(s'|s) \right)
\times \left( V^{M_k}_{\mu_k^{*},j+1} - V^{M^*_k}_{\mu_k^{*},j+1} \right) (s')
\]
\[ - \left( I (\mathbb{P}^*(H_0) < \alpha) \sum_{s' \in S} P^{M^*}_{\mu_\pi^*(s,j)}(s'|s) + I (\mathbb{P}^*(H_0) \geq \alpha) \sum_{s' \in S} P^{M^*}_{\pi(s,j)}(s'|s) \right)
\times \left( V^{M_k}_{\mu_k^{*},j+1} - V^{M^*_k}_{\mu_k^{*},j+1} \right) (s')
\]
\[ = 0,
\]
which follows by the expectation conditional on \( M_k, M^* \) and definition of policy \( \mu_k^{*\pi} \).

Putting I) and II) together we get
\[
\mathbb{E} \left[ V^{M_k}_{\mu_k^{*},1} - V^{M^*_k}_{\mu_k^{*},1} \bigg| M^*, M_k \right] = \mathbb{E} \left[ \sum_{j=1}^{\tau} W_j + e_j \bigg| M^*, M_k \right]
\]
\[ = \mathbb{E} \sum_{i=1}^{\tau} \left( T^{M_k}_{\mu_k^{*},(-j)} - T^{M^*_k}_{\mu_k^{*},(-j)} \right) V^{M_k}_{\mu_k^{*},j+1} (s_j) \bigg| M^*, M_k \right]
\]
\[ \square
\]
Proof of Lemma C.2. First consider Azuma-Hoeffding’s Inequality: Let $Z_1, Z_2, \ldots$ be a martingale sequence difference with $|Z_j| \leq c \forall j$. Then $\forall c > 0$ and $n \in \mathbb{N}$ $P \left[ \sum_{i=1}^{n} Z_i > \epsilon \right] \leq \exp \left\{ -\frac{\epsilon^2}{2nc^2} \right\}$.

By definition the difference between the estimated transition and reward functions and their true respective functions are:

\[
\hat{P}_a(s'|s) - P_a^M(s'|s) = \sum_{s' \in S} \left( I(s_{i,j+1} = s') - P_a^M(s'|s) \right) I(s_{ij} = s, a_{ij} = a) / N_T(s,a),
\]

\[
\hat{R}(s,a) - R^M(s,a) = \sum_{s' \in S} \left( r_{ij} - R^M(s,a) \right) I(s_{ij} = s, a_{ij} = a) / N_T(s,a),
\]

now let $\tilde{\beta}_T(s,a) \equiv \sqrt{8ST \log(2TSA)}$, and consider the transition probability function, for a fixed state action pair $(s,a)$, let $\xi = (\xi(s_1), \ldots, \xi(s_2)) \in \{-1, 1\}^S$, we have

\[
P \left( \sum_{s' \in S} \sum_{i,j \in I} \left( I(s_{i,j+1} = s') - P_a^M(s'|s) \right) I(s_{ij} = s, a_{ij} = a) \geq \tilde{\beta}_T(s,a) \right)
\]

\[
\leq \mathbb{P} \left( \max_{\xi \in \{-1,1\}^S} \sum_{s' \in S} \sum_{i,j \in I} \xi(s') \left( I(s_{i,j+1} = s') - P_a^M(s'|s) \right) I(s_{ij} = s, a_{ij} = a) \geq \tilde{\beta}_T(s,a) \right)
\]

\[
\leq 2^S \mathbb{P} \left( \sum_{s' \in S} \sum_{i,j \in I} \xi(s') \left( I(s_{i,j+1} = s') - P_a^M(s'|s) \right) I(s_{ij} = s, a_{ij} = a) \geq \tilde{\beta}_T(s,a) \right)
\]

where the first step follows from multiplying by $N_T(s,a)$, and eliminating the absolute value with $\xi$, we use a union bound for the second step as there are $2^S$ possible $\xi$ for a fixed $(s,a)$ pair. Next we use Azuma-Hoeffding’s inequality to bound the $2^S$ probability terms, note that within the probability function we are summing over $T$ terms:

\[
2^S \mathbb{P} \left( \sum_{s' \in S} \sum_{i,j \in I} \xi(s') \left( I(s_{i,j+1} = s') - P_a^M(s'|s) \right) I(s_{ij} = s, a_{ij} = a) \geq \tilde{\beta}_T(s,a) \right)
\]

\[
\leq 2^S \exp \left\{ -\frac{8ST \log(2TSA)}{2 \times 2^2T} \right\}
\]

\[
\leq 2^S \exp \left\{ \log((2TSA)^{-S}) \right\} = 2^S \frac{1}{(2TSA)^S} < \frac{1}{TSA},
\]

next we sum over all $(s,a)$ pairs and get

\[
P \left( \left\| \hat{P}_a(s'|s) - P_a^M(s'|s) \right\|_1 \geq \tilde{\beta}_T(s,a) \right)
\]

\[
\leq \sum_{s \in S,a \in A} \mathbb{P} \left( \sum_{s' \in S} \sum_{i,j \in I} \left( I(s_{i,j+1} = s') - P_a^M(s'|s) \right) I(s_{ij} = s, a_{ij} = a) \geq \tilde{\beta}_T(s,a) \right)
\]

\[
\leq SA \frac{1}{TSA} = \frac{1}{T},
\]

which follows from using a union bound again. Analogous we can show that

\[
P \left( \left\| \hat{R}(s,a) - R^M(s,a) \right\| \geq \tilde{\beta}_T(s,a) \right) \leq \frac{1}{T},
\]

thus

\[
P \left( M^* \notin \mathcal{M}_T \right), P \left( M_T \notin \mathcal{M}_T \right) < \frac{1}{T},
\]

\[\square\]