THE ENTRIES IN THE LR-TABLEAU

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Abstract. Let $\Gamma$ be the Littlewood-Richardson tableau corresponding to an embedding $M$ of a subgroup in a finite abelian $p$-group. Each individual entry in $\Gamma$ yields information about the homomorphisms from $M$ into a particular subgroup embedding, and hence determines the position of $M$ within the category of subgroup embeddings. Conversely, this category provides a categorification for LR-tableaux in the sense that all subgroup embeddings corresponding to a given LR-tableau share certain homological properties.

Let $\alpha, \beta, \gamma$ be partitions describing the isomorphism types of finite abelian $p$-groups $A$, $B$, $C$. We know from theorems by Green and Klein ([3], [5]) that there is a subgroup embedding $M : (U \subset B)$ where $U \cong A$ and $B/U \cong C$ if and only if there is a Littlewood-Richardson tableau $\Gamma$ of type $(\alpha, \beta, \gamma)$. The LR-coefficient $c_{\alpha, \beta, \gamma}^{\beta}$ counts the number of such LR-tableaux and provides decisive information in surprisingly many areas of modern algebra. In particular, it determines the multiplication of Schur functions, describes the decomposition of tensor products of irreducible polynomial representations of $\text{GL}_n(\mathbb{C})$, detects the possible eigenvalues of sums of Hermitian matrices, and forms the highest coefficient of the classical Hall polynomial. Recent research (see e.g. [1], [4], [2]) uses modern methods in representation theory to study growth and applications of LR-coefficients.

It is the aim of this manuscript to give an interpretation for each individual entry in the LR-tableau $\Gamma$ in terms of properties of the corresponding embedding $M$, when considered as an object in the category of all subgroup embeddings.

Definition. Let $R$ be a (commutative) discrete valuation ring with maximal ideal $(p)$ and residue field $k = R/(p)$. Denote by $S$ the category of all embeddings $(A \subset B)$ of (finite length) $R$-modules; morphisms in $S$ are given by commutative diagrams. Embeddings where $B$ is cyclic

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are called pickets; they are determined uniquely, up to isomorphism, by the lengths $\ell = \text{len } A$ and $m = \text{len } B$:

$$P^m_{\ell} : \left((p^{m-\ell}) \subset R/(p^m)\right).$$

Our main result gives an interpretation of the entries in the LR-tableau in terms of homomorphisms into pickets, and in terms of the picket decomposition of a suitable subquotient of the embedding.

**Theorem 1.** For an embedding $M : (A \subset B)$ with LR-tableau $\Gamma$ and natural numbers $\ell, m$ with $1 \leq \ell \leq m$, the following numbers are equal.

1. The number of boxes $\square$ in the $m$-th row of $\Gamma$.
2. The multiplicity of the picket $P^m_1$ as a direct summand of the subfactor

$$\left(\frac{p^{\ell-1}A}{p^\ell A} \subset B \right).$$

3. The dimension

$$\dim_k \frac{\text{Hom}_S(M, P^m_\ell)}{\text{Im } \text{Hom}_S(M, g^m_{\ell})}.$$

The map $g^m_{\ell}$ is the sink map for $P^m_{\ell}$ in a suitable picket category. It is defined as follows, with each component map an inclusion of pickets:

$$g^m_{\ell} : \begin{cases} 
P^m_{\ell-1} \oplus P^{m-1}_{\ell} & \to P^m_m \quad \text{if } \ell = m \\
P^m_m & \to P^m_{\ell} \quad \text{if } 1 \leq \ell < m \\
P^m_{0-1} & \to P^m_0 \quad \text{if } \ell = 0 
\end{cases}$$

Remark 1. $\text{Im } \text{Hom}(M, g^m_{\ell})$ is the subgroup of $\text{Hom}(M, P^m_{\ell})$ of all maps which factor through $g^m_{\ell}$, or equivalently, the subgroup generated by all maps which factor through any proper inclusion of a picket in $P^m_{\ell}$.

Remark 2. The category $S$ of submodule embeddings provides a categorification for LR-tableaux in the sense that objects in $S$ which correspond to the same LR-tableau share the homological properties given by homomorphisms into pickets (statement 3. in the Theorem). As we will see in the examples in Section 6 such objects are located in the same vicinity in the Auslander-Reiten quiver.

**Definition.** For natural numbers $\ell, n$, let $S_\ell$ and $S(n)$ denote the full subcategories of $S$ of all pairs $(A \subset B)$ which satisfy $p^\ell A = 0$ and $p^n B = 0$, respectively.

**Proposition 1.** Let $1 \leq \ell \leq m \leq n$ be natural numbers. Define $C^m_{\ell} = \tau_{S(n)}^{-1}(P^m_{\ell-1})$ if $\ell < m$, and $C^m_{\ell} = P^m_n$ if $\ell = m$. 
(1) The factor \( \frac{\text{Hom}_S(C^m_\ell, P^m_\ell)}{\text{Im Hom}_S(C^m_\ell, g^m_\ell)} \) is a 1-dimensional \( k \)-vector space.

(2) Suppose that \( M \) is an embedding in \( S(n) \). The bilinear form given by composition,

\[
\frac{\text{Hom}_S(M, P^m_\ell)}{\text{Im Hom}_S(M, g^m_\ell)} \times \frac{\text{Hom}_S(C^m_\ell, M)}{\text{Im Hom}_S(C^m_\ell, g^m_\ell)} \to \frac{\text{Hom}_S(C^m_\ell, P^m_\ell)}{\text{Im Hom}_S(C^m_\ell, g^m_\ell)}
\]

is left non-degenerate.

As a consequence, in each category \( S(n) \), the embeddings \( M \) which have an entry \( \ell \) in the \( m \)-th row of their LR-tableau are characterized by admitting maps \( C^m_\ell \to M \to P^m_\ell \) such that the composition does not factor through \( g^m_\ell \). In this sense, \( M \) is located between \( C^m_\ell \) and \( P^m_\ell \).

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1. Partitions

For \( m \) a natural number, let \( P^m_\ell \) be the cyclic \( R \)-module \( R/(p^m) \). Recall that an arbitrary \( R \)-module is determined uniquely, up to isomorphism, by a partition:

\[
\{ \text{p-modules} \} / \cong \xlongright{\cong^{-1}} \{ \text{partitions} \}
\]

\[
B \quad \longmapsto \quad \text{type}(B) = \beta
\]

if \( \beta'_i = \dim_k (p^{i-1}B/p^iB) \)

\[
M(\beta) = \bigoplus_{i=1}^s P^{\beta_i} \quad \longmapsto \quad \beta = (\beta_1, \ldots, \beta_s)
\]

Here \( \beta' \) denotes the transpose of the partition \( \beta \). Let \( \beta, \gamma \) be partitions. We say \( \gamma \leq \beta \) if \( \gamma_i \leq \beta_i \) holds for each \( i \); in this case \( \beta \) and \( \gamma \) define a skew tableau \( \beta - \gamma \). The skew tableau is a horizontal strip if \( \beta_i - \gamma_i \leq 1 \) holds for each \( i \), and the length \( |\beta - \gamma| \) of the strip is given by the sum \( \sum_i (\beta_i - \gamma_i) \).
Definition. An increasing sequence of partitions $\Gamma = [\gamma^0, \ldots, \gamma^s]$ defines an \textit{LR-tableau of type} $(\alpha, \beta, \gamma)$ if $s = \alpha_1$, $\gamma^0 = \gamma$, $\gamma^s = \beta$ and the following two conditions are satisfied.

1. For each $\ell \geq 1$, the skew tableau $\gamma^\ell - \gamma^{\ell-1}$ is a horizontal strip of length $|\gamma^\ell - \gamma^{\ell-1}| = \alpha'_\ell$.
2. The \textit{lattice permutation property} is satisfied, that is for each $\ell \geq 2$ and each $h$ we have
   \[
   \sum_{i \geq h} (\gamma_i^\ell - \gamma_i^{\ell-1}) \leq \sum_{i \geq h} (\gamma_i^{\ell-1} - \gamma_i^{\ell-2}).
   \]

As usual, we picture an LR-tableau $\Gamma = [\gamma^0, \ldots, \gamma^s]$ by labelling each box in the skew tableau $\gamma^\ell - \gamma^{\ell-1}$ by $\ell$. Note that some authors replace the partitions in the tableau by their transposes; here we follow the convention in \cite{3}.

Given an embedding $M : (A \subset B)$ we obtain an LR-tableau as follows. Put $\alpha = \text{type}(A)$, $\beta = \text{type}(B)$, $\gamma = \text{type}(B/A)$, let $s = \alpha_1$ be the Loewy length of $A$, and define for each $0 \leq \ell \leq s$ the partition $\gamma^\ell = \text{type}(B/p^\ell A)$. According to Green’s Theorem \cite[Theorem 4.1]{3}, the sequence $\Gamma = [\gamma^0, \ldots, \gamma^s]$ forms an LR-tableau of type $(\alpha, \beta, \gamma)$; we say $\Gamma$ is the \textit{LR-tableau for} $M$.

2. \textbf{Semisimple submodules}

The embeddings $(A \subset B)$ of $p$-modules where $A$ is semisimple form the category $S_1$, they are well understood: The indecomposable embeddings are pickets $P^\alpha_m$ with $\ell \leq 1$, and arbitrary embeddings are given by their LR-tableaux \cite[Proposition 3.3]{3}:

**Proposition 2.** The following map defines a one-to-one correspondence between the isomorphism types of embeddings in $S_1$ and horizontal strips:

\[
\begin{array}{rcl}
\{ \text{objects } (A \subset B) \in S_1 \} / \cong & \overset{1-1}{\longleftrightarrow} & \{ \text{horizontal strips} \} \\
(A \subset B) & \mapsto & \text{type}(B) - \text{type}(B/A) \\
\bigoplus P^\beta_{\beta_i - \gamma_i} & \mapsto & \beta - \gamma
\end{array}
\]

**Corollary 1.** Suppose the embedding $(A \subset B) \in S_1$ corresponds to the skew tableau $\beta - \gamma$. The number of boxes in the $m$-th row in $\beta - \gamma$ is equal to the multiplicity of the picket $P^\alpha_m$ as a direct summand of $(A \subset B)$.
The number of boxes in the $m$-th row in $\beta - \gamma$ is equal to
\[ \beta'_m - \gamma'_m = \{ i \mid \beta_i = m \text{ and } \gamma_i = m-1 \}; \]
under the correspondence given above, this is the number of summands $P^m_i$ in the direct sum decomposition for $(A \subset B)$. \qed

As a category, $\mathcal{S}_1$ is an exact Krull-Remak-Schmidt category which has Auslander-Reiten sequences, see [8, Section 3.1]. The Auslander-Reiten quiver consists of one tube that has 2 rays but only 1 coray; in this quiver we represent the indecomposable objects by their LR-tableaux.

The maps $g^m_i$ from the introduction occur as sink maps of the pickets of type $P^m_i$.

3. Categorification

In this section we show Theorem \ref{thm:main}

For $M = (A \subset B)$ an embedding and $\ell$ a natural number let
\[ M'|_\ell^1 = \left( \frac{p^{\ell-1}A}{p^\ell A} \subset \frac{B}{p^\ell A} \right) \]
be the reduced embedding, which is an object in $\mathcal{S}_1$. The following result is clear from the definition.

**Lemma 1.** Suppose $M \in \mathcal{S}$ has LR-tableau $\Gamma = [\gamma^0, \ldots, \gamma^s]$. For $1 \leq \ell \leq s$, the LR-tableau for the reduced embedding $M'|_\ell^1$ is the skew tableau $\Gamma|_{1}^{\ell} = [\gamma^{\ell-1}, \gamma^{\ell}]$. \qed

Combining this result with Corollary \ref{cor:main} we obtain the equality of the numbers in (1) and (2) in Theorem \ref{thm:main}.
Corollary 2. Suppose $M \in \mathcal{S}$ has LR-tableau $\Gamma$. Let $1 \leq \ell \leq m$. The number of boxes $\ell$ in row $m$ in $\Gamma$ equals the multiplicity of $P^m_\ell$ in a direct sum decomposition for $M|_\ell$.

For the proof of Theorem 1 we use the following three observations.

Observation 1. The number of boxes $\ell$ in row $m$ in the LR-tableau $\Gamma = [\gamma^0, \ldots, \gamma^s]$ is

$$(\gamma^\ell)_m - (\gamma^{\ell-1})_m.'$$

Observation 2. Suppose that an $R$-module $B$ has type $\beta$. Then the length of the $m$-th row in the partition $\beta$ is

$$\beta'_m = \text{len} \text{Hom}_R(B, P^m) - \text{len} \text{Hom}_R(B, P^{m-1}).$$

Observation 3. Homomorphisms into pickets can be expressed in terms of homomorphisms between $R$-modules:

$$\text{Hom}_S((A \subset B), P^m_\ell) = \text{Hom}_R(B/p^\ell A, P^m)$$

Proof of Theorem 1. It remains to show the equality of the numbers in (1) and (3). Suppose that the embedding $M : (A \subset B)$ has LR-tableau $\Gamma = [\gamma^0, \ldots, \gamma^s]$.

The number of boxes $\ell$ in the $m$-th row in $\Gamma$ is $(\gamma^\ell)_m - (\gamma^{\ell-1})_m$; this number can be expressed in terms of $R$-homomorphisms as

$$\text{(1)} \quad \text{len} \text{Hom}_R(B/p^\ell A, P^m) - \text{len} \text{Hom}_R(B/p^\ell A, P^{m-1})$$

$$- \text{len} \text{Hom}_R(B/p^{\ell-1} A, P^m) + \text{len} \text{Hom}_R(B/p^{\ell-1} A, P^{m-1})$$

(Observation 2).

We first consider the case where $\ell < m$. Then (1) can be written in terms of homomorphisms into pickets as

$$\text{len} \text{Hom}_S((A \subset B), P^m_\ell) - \text{len} \text{Hom}_S((A \subset B), P^{m-1}_\ell)$$

$$- \text{len} \text{Hom}_S((A \subset B), P^{m-1}_{\ell-1}) + \text{len} \text{Hom}_S((A \subset B), P^{m-1}_{\ell-1})$$

which equals

$$\text{len} \text{Cok} \text{Hom}_S((A \subset B), g^m_\ell)$$

where $g^m_\ell$ is the epimorphism in the short exact sequence

$$0 \longrightarrow P^{m-1}_{\ell-1} \longrightarrow P^{m-1}_\ell \oplus P^{m-1}_{\ell-1} \longrightarrow P^m_\ell \longrightarrow 0.$$
which equals \( \text{len Cok} \text{Hom}_S((A \subset B), g^m_m) \): For this apply the functor \( \text{Hom}_S((A \subset B), -) \) to the monomorphism \( g^m_m : P^m_{m-1} \to P^m_m \).

\( \square \)

Remark 3. Together, the maps of type \( g^m_\ell \) form the sink maps in the Auslander-Reiten quiver for the category \( \mathcal{P}ic \) which has as objects the direct sums of pickets; morphisms are those maps for which each component is either zero or an inclusion between pickets.

\[
\mathcal{P}ic :
\]

In the Remark under Theorem \( \square \) we claim that \( \text{Im} \text{Hom}(M, g^m_\ell) \) is the submodule of \( \text{Hom}(M, P^m_\ell) \) generated by all maps which factor through some proper inclusion of a picket in \( P^m_\ell \). Since any such inclusion factors through \( g^m_\ell \), the claim follows.

4. INTERVALS IN THE AUSLANDER-REITEN QUIVER

We show Proposition \( \square \). Let \( 1 \leq \ell \leq m \leq n \). Define \( C^m_\ell = \tau^{-1}_{S(n)}(P^m_{\ell-1}) \) if \( \ell \leq m \) and \( C^m_\ell = P^n_n \) for \( \ell = m \). We characterize the indecomposable objects \( M \in S(n) \) which have a box \( \square \) in the \( m \)-th row of their LR-tableau by the existence of maps \( f_1 : C^m_\ell \to M, f_2 : M \to P^n_m \) such that \( f_2 f_1 \) does not factor through \( g^m_\ell \). In this sense, \( M \) occurs in the interval from \( C^m_\ell \) to \( P^n_\ell \) in the Auslander-Reiten quiver for \( S(n) \).

First we determine \( C^m_\ell \) in the case where \( \ell < m \) and compute its LR-tableau. We use [6, Theorem 5.2] in which the Auslander-Reiten translation is computed for indecomposable objects in the factor module category \( \mathcal{F}(n) \). The kernel and cokernel functors induce an equivalence between the categories \( \mathcal{F}(n) \) and \( S(n) \) [6, Lemma 1.2 (3)], so we can compute \( \tau^{-1}_{S(n)}(P^m_{\ell-1}) \) as the kernel of the minimal epimorphism representing \( P^m_{\ell-1} \).
For the inclusion $P^{n-1}_{\ell-1} : (P^{\ell} \subseteq P^{m-1})$ the minimal epimorphism is

$$P^n \oplus P^{\ell-1} \xrightarrow{\text{(can, incl)}} P^{m-1},$$

and the map $(\text{incl}, - \text{can}) : P^{n+\ell-m} \to P^n \oplus P^{\ell-1}$ is a kernel. This monomorphism represents the object $C^m_\ell$ in $\mathcal{S}(n)$.

We determine the LR-tableau $\Gamma$ for $C^m_\ell = (A \subset B)$. Let $\alpha = \text{type}(A) = (n + \ell - m)$, $\beta = \text{type}(B) = (n, \ell - 1)$, and $\gamma = \text{type}(B/A) = (m - 1)$.

Since there is only one filling of $\beta - \gamma$ in which each number $1, \ldots, n + \ell - m$ occurs exactly once, the LR-tableau must be as follows:

$$\Gamma :
\begin{array}{c}
\ell \\
m-1 \\
\vdots \\
1
\end{array}
\begin{array}{c}
2 \\
3 \\
\vdots \\
t
\end{array}
\begin{array}{c}
\ell' \\
1
\end{array}
$$

Here, $t = n + \ell - m$ and $\ell' = \ell - 1$. The LR-tableau for the picket $C^m_m = P^n$ is as follows:

$$\Gamma :
\begin{array}{c}
1 \\
\vdots \\
t
\end{array}
\begin{array}{c}
\ell
\end{array}
$$

We can now complete the proof of Proposition 1.

Proof. Let $1 \leq \ell \leq m \leq n$.

1. In each case, there is exactly one box $[\ell]$ in row $m$ in the above LR-tableau for $C^m_\ell$, so $\dim \text{Cok} \text{Hom}(C^m_\ell, g^m_\ell) = 1$, by Theorem 1.

2. We consider first the case where $\ell < m$. Let $\mathcal{A}$ be the sequence

$$\mathcal{A} : 0 \to P^{m-1}_{\ell-1} \to P^m_{\ell-1} \oplus P^{m-1}_{\ell} \xrightarrow{g^m_\ell} P^m_\ell \to 0,$$

and let

$$\mathcal{E} : 0 \to P^{m-1}_{\ell-1} \to B \to C \to 0$$

be the Auslander-Reiten sequence in $\mathcal{S}(n)$ starting at $P^{m-1}_{\ell-1}$, its end term is $C = C^m_\ell$. Since $\mathcal{A}$ is nonsplit, there are maps $h' : B \to P^m_{\ell-1} \oplus P^{m-1}_{\ell}$, $h : C \to P^m_\ell$ which make the upper part of the following
The entries in the LR-tableau are isomorphism invariants for $M$ and can be interpreted in terms of homomorphisms from $M$ into pickets. Also the entries in the LR-tableau for $M^*$ are isomorphism invariants for $M$; they have the following interpretation in terms of homomorphisms from pickets into $M$.

5. Duality

Let $I$ be the injective envelope of the simple $R$-module $P^1$. Then the functor $*=\text{Hom}_R(-,I)$ defines a duality for $R$-modules, which gives rise to a duality for short exact sequences of $R$-modules, and hence yields a duality on $S$.

We have seen that the entries in the LR-tableau for an object $M \in S$ are isomorphism invariants for $M$ and can be interpreted in terms of homomorphisms from $M$ into pickets. Also the entries in the LR-tableau for $M^*$ are isomorphism invariants for $M$; they have the following interpretation in terms of homomorphisms from pickets into $M$. 

\begin{align*}
\mathcal{E}: 0 & \longrightarrow P^{m-1}_{\ell-1} \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0 \\
\mathcal{A}: 0 & \longrightarrow P^{m-1}_{\ell-1} \xrightarrow{f^m} P^{m}_{\ell-1} \oplus P^{m-1}_{\ell-1} \xrightarrow{g^m} P^m_{\ell} \longrightarrow 0 \\
r\mathcal{A}: 0 & \longrightarrow P^{m-1}_{\ell-1} \xrightarrow{s} L \xrightarrow{t} M \longrightarrow 0
\end{align*}

Suppose that $M \in S(n)$. In order to show that the bilinear form given by composition

\[
\frac{\text{Hom}(M, P^m_{\ell})}{\text{Im} \text{Hom}(M, g^m_{\ell})} \times \frac{\text{Hom}(C, M)}{\text{Im} \text{Hom}(C, g^m_{\ell})}
\]

is left non-degenerate, let $r : M \rightarrow P^m_{\ell}$ be a map which does not factor through $g^m_{\ell}$. We will construct $q : C \rightarrow M$ such that $rq$ does not factor through $g^m_{\ell}$. Since $r$ does not factor through $g^m_{\ell}$, the induced sequence at the bottom of the above diagram does not split. Hence the map $s$ factors through $u$: There is a map $q' : B \rightarrow L$ such that $s = q'u$. Let $q : C \rightarrow M$ be the cokernel map, so $qv = tq'$. Then $rqv = rtq' = g^m_{\ell}r'q'$. Since $r'q'u = r's = f^m_{\ell} = h'u$, there exists $z : C \rightarrow P^m_{\ell-1} \oplus P^{m-1}_{\ell}$ such that $zv = r'q' - h'$. So $rqv = g^m_{\ell}r'q' = g^m_{\ell}(zv + h') = (g^m_{\ell}z + h)v$ and since $v$ is onto, $rq = g^m_{\ell}z + h$. Since $E$ is not split exact, $h$ does not factor through $g^m_{\ell}$, and hence $rq$ does not factor over $g^m_{\ell}$.

We deal with the case where $\ell = m$. Let $f_2 : M \rightarrow P^m_{m}$ be a map which does not factor through the inclusion $g^m_m : P^{m-1}_{m} \rightarrow P^m_{m}$ of the maximal submodule, so $f_2$ is an epimorphism. Since $P^m_{n}$ is projective — even in the abelian category of all maps between $R/(p^n)$-modules — the canonical map $P^m_{n} \rightarrow P^m_{m}$ factors through $f_2$. \qed
Theorem 2. For an embedding $M = (A \subset B) \in \mathcal{S}$ and $1 \leq \ell \leq m$, the following numbers are equal.

1. The number of boxes $\blacksquare$ in the $m$-th row of the LR-tableau for the dual module $M^*$.

2. The multiplicity of $P^m_1$ as a direct summand of

$$
\left( \frac{p^{\ell-1}U}{p^{\ell}U} \subset \frac{B^*}{p^{\ell}U} \right)
$$

where $U = \text{ann}_{B^*} A$ is the annihilator of $A$ in $B^*$.

3. The dimension

$$
\dim_k \frac{\text{Hom}_{\mathcal{S}}(P^m_{m-\ell}, M)}{\text{Im} \text{Hom}_{\mathcal{S}}(h^m_{m-\ell}, M)}.
$$

For $1 \leq q \leq m$, the morphism $h^m_q$ is the following map between sums of pickets; each component map is an epimorphism on the total space and on the factor space.

$$
h^m_q : \begin{cases}
  P^m_0 &\to P^m_1, &\text{if } q = 0 \\
  P^m_q &\to P^m_{q-1} \oplus P^m_{q+1}, &\text{if } 1 \leq q < m \\
  P^m_m &\to P^m_{m-1}, &\text{if } q = m
\end{cases}
$$

Proof. The result is an easy consequence of Theorem 1.

For the equality of the numbers in (1) and (2) it suffices to note that if $M$ is the embedding $(A \subset B)$, then the dual $M^*$ is the embedding $(U \subset B^*)$ where $U = \text{ann}_{B^*} A$.

We show the equality of the numbers in (1) and (3) According to Theorem 1 the number in (1) is the dimension

$$
\dim_k \frac{\text{Hom}(M^*, P^m_{m-\ell})}{\text{Im} \text{Hom}(h^m_{m-\ell}, M)}
$$

which is equal to the dimension

$$
\dim_k \frac{\text{Hom}(P^m_{m-\ell}, M)}{\text{Im} \text{Hom}(h^m_{m-\ell}, M)}
$$

since $h^m_{m-\ell}$ is the dual of the map $g^m_{m-\ell}$. □

With the following result we can position $M$ within the category $\mathcal{S}$:

Proposition 3. Suppose that $0 \leq q < m \leq n$. Let $A^m_q = \tau_{\mathcal{S}(n)}P^m_{q-1}$ if $q > 0$ and $A^m_q = P^n_0$ if $q = 0$.

1. The factor

$$
\frac{\text{Hom}(P^m_q, A^m_q)}{\text{Im} \text{Hom}(h^m_q, A^m_q)}
$$

is a 1-dimensional $k$-vector space.
(2) For $M \in S(n)$, the bilinear form given by composition,

\[
\text{Hom}_S(M, A_q^m) \times \frac{\text{Hom}_S(P_q^m, M)}{\text{Im}\text{Hom}_S(h_q^m, M)} \rightarrow \frac{\text{Hom}_S(P_q^m, A_q^m)}{\text{Im}\text{Hom}_S(h_q^m, A_q^m)}
\]

is right non-degenerate.

Proof. This result follows from Proposition 1 by duality. \qed

6. Example: Submodules of $p^5$-bounded modules.

The results in this paper can be visualized on the Auslander-Reiten quivers of categories of embeddings. In this section we consider the category $S(5)$, which among all the categories of type $S(n)$ is the largest of finite representation type \cite{7}. There are 50 indecomposable objects in $S(5)$; we picture here the Auslander-Reiten quiver $\Gamma_{S(5)}$ from \cite{7} (6.5), with the objects represented by their LR-tableaux.

Recall that for $R = k[[T]]$ the power series ring, the homomorphisms between indecomposable modules are given as the linear combinations of paths, modulo mesh relations.
In the second copy of $\Gamma_{S(5)}$, the pickets are encircled, and there is an indecomposable object labelled $M$ which has the following LR-tableau.

\[
M : \\
\begin{array}{ccc}
\ell & 1 & 2 \\
2 & 3 & \\
3 & \\
\end{array}
\]

For each pair $(\ell, m)$ such that

\[
\frac{\text{Hom}_S(M, P^m_\ell)}{\text{Im Hom}_S(M, g^m_\ell)} \neq 0
\]

we indicate a path $M \to P^m_\ell$ representing a map which does not factor through $g^m_\ell$. There are the following five paths:

\[M \to P^1_1, \quad M \to P^2_1, \quad M \to P^3_2, \quad M \to P^4_2, \quad \text{and} \quad M \to P^5_3\]

Corresponding to each such path $M \to P^m_\ell$ there is an entry \(\square\) in the $m$-th row of the LR-tableau for $M$, as predicted by Theorem 1.

Note that nonisomorphic objects in $S$ can have the same LR-tableau. Consider for example the Auslander-Reiten sequence

\[0 \to C^4_2 \to M \to P^3_2 \to 0\]
with middle term $M$. The modules $M$ and $C_2^4 \oplus P_2^3$ have the same LR-tableau, and hence cannot be distinguished by homomorphisms into pickets.

We focus on the case where $\ell = 2$ and $m = 4$. The indecomposables which have an entry $\mathbf{2}$ in the 4-th row in their LR-tableau are in the region labelled $\mathcal{R}$ in the third copy of the Auslander-Reiten quiver $\Gamma_{S(5)}$. Note that the two “eyes” are not part of the region $\mathcal{R}$. Each object $M$ in $\mathcal{R}$ admits a map $t : M \to P_2^4$ which does not factor through $g_2^4$. (In the diagram, the module $P_2^4$ is labelled “$Z$”, while the summands $P_1^4$ and $P_2^3$ of the source of $g_2^4$ are labelled “$Y_1$” and “$Y_2$”.) According to Proposition 1 those modules admit a map $t' : C_2^4 \to M$ such that the composition $tt'$ does not factor through $g_2^4$. The module $C_2^4$ is obtained as follows. Let $\mathcal{E}$ be the short exact sequence given by $g_2^4$:

$$\mathcal{E} : 0 \to P_1^3 \xrightarrow{f_2^3} P_1^4 \oplus P_2^3 \xrightarrow{g_2^4} P_2^4 \to 0$$

and put $C_2^4 = \tau_{S(5)}^{-1}(P_3^4)$ (in the diagram, the modules $C_2^4$ and $P_1^3$ are labelled “$C$” and “$X$”, respectively). As predicted by the Proposition, the indecomposables in $S(5)$ which have a $\mathbf{2}$ in the 4-th row are located between $C_2^4$ and $P_2^4$. 

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**THE ENTRIES IN THE LR-TABLEAU**
We consider duality:
Let us locate those indecomposable objects $M \in S(5)$ for which the LR-tableau of $M^*$ contains a box $\begin{array}{c} 2 \end{array}$ in row 4. Note that duality acts on the above Auslander-Reiten quiver by reflection on the central vertical axis. It follows that the modules $M$ as above are located within the region $\mathcal{R}^*$ encircled by the dotted line, and without the two dotted ellipses. According to Theorem 2 they are characterized in terms of homomorphisms from pickets, as follows. The dual of the above sequence $\mathcal{E}$ is

$$\mathcal{E}^* : 0 \longrightarrow P_2^4 \xrightarrow{h_2^4} P_3^4 \oplus P_1^3 \longrightarrow P_2^3 \longrightarrow 0.$$  

The modules in $\mathcal{R}^*$ admit a map $t : P_2^4 \rightarrow M$ which does not factor over $h_2^4$. Let $A = A_2^4 = \tau_{S(5)}(P_2^3)$ be the $\tau$-translate of the cokernel of $h_2^4$; it is indicated at the right end of the region $\mathcal{R}^*$. According to Proposition 3 the modules $M$ in $\mathcal{R}^*$ are between $P_2^4$ and $A_2^4$ in the sense that given $t$, there is a map $t'' : M \rightarrow A_2^4$ such that $t''t$ does not factor through $h_2^4$.

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