Three-point functions and operator product expansion
in the $SL(2)$ conformal field theory

Yuji Satoh†

Institute of Physics, University of Tsukuba
Tsukuba, Ibaraki 305-8571, Japan

Abstract

In the $SL(2)$ conformal field theory, we write down and analyze the analytic expression of
the three-point functions of generic primary fields with definite $SL(2)$ weights. Using these
results, we discuss the operator product expansion in the $SL(2, R)$ WZW model. We propose
a prescription of the OPE, the classical limit of which is in precise agreement with the tensor
products of the representations of $SL(2, R)$.

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†ysatoh@het.ph.tsukuba.ac.jp
1 Introduction

The conformal field theory with the $\hat{sl}(2)$ symmetry provides us with some of the simplest models beyond the well-studied rational CFT. It is an interesting subject by itself. Moreover, since the $SL(2)$ symmetry is rather general, it appears in various situations, for instance, in studying the strings on $AdS_3$, black holes in string theory and certain problems in condensed matter physics. However, there still remain open questions about the models with the $\hat{sl}(2)$ symmetry.

In this paper, we would first like to discuss the three-point functions in the $SL(2)$ conformal field theory. One of the models with the $\hat{sl}(2)$ symmetry is the $H_3^+$ WZW model [1, 2, 3]. The primary fields which are often used in this model are labeled by the $SL(2)$ spin $j$ as $\Phi_j$. For these primary fields, the two- and three-point functions including normalizations have been obtained [2, 3] (see, also [4, 5]). Since those expressions are analytic in $j$’s (up to delta functions), one expects that the correlation functions in other $SL(2)$ models may be obtained by appropriately continuing the values of the spin. However, in a model where highest(lowest) weight representations appear, it is important to respect the relations between the spin and the $SL(2)$ weights. Thus, one needs the primary fields with definite left and right $SL(2)$ weights, $\Phi_{jmn}^j$. An emphasis has been put on this point in [6]. With the help of an integral formula in [7], in section 2 we write down the analytic expression of the three-point functions of $\Phi_{jmn}^j$ in the $H_3^+$ WZW model. Some properties of these three-point functions are also analyzed.

Based on the results in section 2, we would next like to discuss the operator product expansion in the $SL(2, R)$ WZW model. For this model, it has been a long-standing problem to determine the correct spectrum (see, e.g., [8]-[22] and references therein). The OPE gives us an important clue. Guided by a consideration in the classical limit, in section 3 we propose the OPE of the primary fields in the $SL(2, R)$ WZW model. In the classical limit, we find a complete agreement with the classical tensor products of the representations of $SL(2, R)$. Rather, the full OPE is essentially the same as the classical tensor products. This is natural because one may not have any reason that makes difference, contrary to the $SU(2)$ case. Although we take a different strategy to determine the OPE, the resultant prescription is regarded as a completion of the arguments in [6].

Implications to the spectrum of the $SL(2, R)$ WZW model are briefly discussed in section 4, in particular, in relation to the works in [18]. Some formulas used in the main text and some technical arguments are given in the appendix.
Three-point functions

2.1 $H_3^+$ WZW model

We begin with a brief summary of the correlation functions in the $H_3^+$ ($= SL(2, C)/SU(2)$) WZW model. We mainly follow the notations in [6].

The conformal field theory with the Euclidean $AdS_3$ target space is described by the $H_3^+$ WZW model [1, 2, 3]. This model has an $SL(2, C) \times SU(2)$ affine symmetry, whose currents act on the primary fields as

$$J^a(z)\Phi_j(w,x) \sim -\frac{D^a\Phi_j(w,x)}{z-w},$$

$$D^- = \partial_x, \quad D^3 = x\partial_x - j, \quad D^+ = x^2\partial_x - 2jx,$$

and similarly for $J^a(\bar{z})$. $z, w$ are the world-sheet coordinates. $x$ is a complex parameter. The two- and three-point functions of $\Phi_j$ have been calculated by using symmetries [2, 3], or later by path-integral [4, 5]:

$$\langle \Phi_{j_1}(z_1, x_1)\Phi_{j_2}(z_2, x_2) \rangle = |z_{12}|^{-4h_1}A(j_1)\delta^2(x_{12})\delta(j_1 + j_2 + 1) + B(j_1)|x_{12}|^{4j_1}\delta(j_1 - j_2),$$

$$A(j) = -\frac{\pi^3}{(2j+1)^2}, \quad B(j) = b^2\pi^2[k^{-1}\Delta(b^2)]^{2j+1}\Delta[-b^2(2j+1)],$$

$$\langle \prod_{a=1}^3 \Phi_{j_a}(z_a, x_a) \rangle = D(j_a)\prod_{a<b}|z_{ab}|^{-2h_{ab}}|x_{ab}|^{2j_{ab}},$$

$$D(j_a) = \frac{b^2\pi^2}{2}\frac{[k^{-1}b^{-2k^2}\Delta(b^2)]^{\Sigma j_a+1}\Upsilon[b]\Upsilon[-2j_1b]\Upsilon[-2j_2b]\Upsilon[-2j_3b]}{\Upsilon[-(\Sigma j_a+1)b]\Upsilon[-j_12b]\Upsilon[-j_13b]\Upsilon[-j_23b]}.$$

Here, $x_{ab} = x_a - x_b$, $z_{ab} = z_a - z_b$, $j_{12} = j_1 + j_2 - j_3$, $h_{12} = h_1 + h_2 - h_3$ etc., $h_a = -j_a(j_a+1)b^2$, $b^{-2} = k - 2$, and $\Delta(x) = \Gamma(x)/\Gamma(1-x)$. $k$ is the level of $\hat{sl}(2)$. $\Upsilon(x)$ is a certain function introduced in [23].

In this paper, we are interested in the primary fields with definite $SL(2)$ weights. They are given by the moments with respect to $x$:

$$\Phi_{m\bar{m}}^j(z) = \int d^2x x^j x^{\bar{m}} x^{\bar{m}} \Phi_{-j-1}(z, x).$$

The currents act on $\Phi_{m\bar{m}}^j$ as

$$J^\pm(z)\Phi_{m\bar{m}}^j(w) \sim \mp j + m, \quad J^3(z)\Phi_{m\bar{m}}^j(w) \sim \frac{m}{z-w},$$

1 However, we use slightly different notations. For example, $\Phi_{m\bar{m}}^j, (2\pi)^2\delta^2(\Sigma m_a)$ and $W(j_a; m_a)$ correspond to $\Phi_{m\bar{m}}^{j-1}$, $\pi\delta^2(\Sigma m_a)$ and $F(-j_a; -m_a)$ in [3], respectively.
and similarly for $\bar{J}^a(\bar{z})$. Here and in the following, we assume that $m - \bar{m} \in \mathbb{Z}$, so that the above integral is well-defined. This is the case for the unitary representations on $H^+_3$ (and $SL(2, R)$).

From (2.2)-(2.4), the correlation functions of $\Phi^j_{m\bar{m}}$ are expressed as [24, 18, 23, 14],

$$\langle \Phi^j_{m_1\bar{m}_1} \Phi^j_{m_2\bar{m}_2} \rangle = (2\pi)^2 \delta^2(m_1 + m_2) [A(j_1)\delta(j_1 + j_2 + 1) + B_{\Phi}(j_1, m_1)\delta(j_1 - j_2)],$$

$$B_{\Phi}(j, m) = c_{\bar{m}m}^{-j-1} B(-j - 1), \quad c_{\bar{m}m}^j = \pi \frac{\Delta(2j + 1)\Gamma(-j + m)\Gamma(-j - \bar{m})}{\Gamma(j + 1 + m)\Gamma(j + 1 - \bar{m})},$$

$$\langle \prod_{a=1}^{3} \Phi^{j_a}_{m_a\bar{m}_a} \rangle = (2\pi)^2 \delta^2(\Sigma m_a) W(j_a; m_a) D(-j_a - 1),$$

where

$$W(j_a; m_a) = \int d^2x_1 d^2x_2 x_1^{j_1+m_1} x_2^{j_2+m_2} |1 - x_1|^{-2j_13-2} x_2^{-2j_23-2} |x_1 - x_2|^{-2j_13-2}.$$  

In the above, we have omitted the $z$-dependence. Note that $c_{\bar{m}m}^j = c_{m\bar{m}}^j$ for $m - \bar{m} \in \mathbb{Z}$. 

2.2 Three-point functions of $\Phi^j_{m\bar{m}}$

In a special case with $j_1 + m_1 = j_2 + \bar{m}_2 = 0$, the integral in (2.3) has been carried out in [14]. For the case of $m_a = \bar{m}_a$, see also [20, 23]. In this subsection, we would like to write down the explicit form of $W(j_a; m_a)$ in a generic case.

The calculation proceeds as follows. One can ‘factorize’ the integration over the complex variables, $x_1, x_2$, as in [27]. Then, an integration over one variable gives $\, _2F_1$, and the remaining integration gives $\, _3F_2$, where $\, _pF_q$ are the (generalized) hypergeometric functions. The final result is expressed by certain combinations of $\, _3F_2$. In this course, one needs careful treatment of the integration contours.

Such calculations have been performed in [4]. Thus, we have only to use the simplest formula there given in (A.1), to find that

$$W(j_a; m_a) = \frac{i}{2} \left[ C^{12} \bar{P}^{12} + C^{21} \bar{P}^{21} \right],$$

where

$$\frac{i}{2} \left[ C^{12} \bar{P}^{12} = s(j_1 + m_1)s(j_2 + m_2)C^{31} - s(j_2 + m_2)s(m_1 - j_2 + j_3)C^{13}, \right.$$  

$$C^{12} = \frac{\Gamma(-N)\Gamma(1 + j_3 - m_3)}{\Gamma(-j_3 - m_3)} G \left[ -j_3 - m_3, -j_13, 1 + j_2 + m_2 \right],$$

$$C^{31} = \frac{\Gamma(1 + j_3 + m_3)\Gamma(1 + j_3 - m_3)}{\Gamma(1 + N)} G \left[ 1 + N, 1 + j_1 + m_1, 1 + j_2 - m_2 \right].$$

(2.10)
Furthermore, we can find more evidence for the validity of such a continuation: (i) As the \( SL \) the operators, the continued correlation functions correctly reduce to the known ones for they are analytic in \( j, m \). In addition, the correlation functions discussed so far respect the allowed representations and, hence, the allowed values of the parameters, of the \( SL \) above results \([2, 3, 24, 18, 25, 6]\). In particular, we assume that the correlation functions in the \( \Phi \)

\[
C^{ab} = C^{ba}(j_1, m_1 \leftrightarrow j_2, m_2), \quad P^{ab} = P^{ba}(j_1, m_1 \leftrightarrow j_2, m_2).
\]

We have also used \( \Sigma_a m_a = \Sigma_a \bar{m}_a = 0 \). The expression of \( W(j_a; m_a) \) is analytic in its arguments, and so are the three-point functions of \( \Phi \) except for \( \delta^2(\Sigma m_a) \).

For later use, we further rewrite \( W(j_a; m_a) \). As discussed in appendix B, it turns out that \( P^{12} \) and \( P^{21} \) are expressed in terms of \( C^{12} \) and \( C^{21} \). From the formula \((B.3)\), it follows that

\[
W(j_a; m_a) = D_1 C^{12} \bar{C}^{12} + D_2 C^{21} \bar{C}^{21} + D_3 \left[ C^{12} \bar{C}^{21} + C^{21} \bar{C}^{12} \right],
\]

where \( \bar{C}^{ab} = C^{ab}(m_a \rightarrow \bar{m}_a) \) and

\[
D_1 = \frac{s(j_2 + m_2)s(j_{13})}{s(j_1 - m_1)s(j_2 - m_2)s(j_3 + m_3)} \left[ s(j_1 + m_1)s(j_1 - m_1)s(j_2 + m_2) - s(j_2 - m_2)s(j_2 - j_3 - m_1)s(j_2 + j_3 - m_1) \right],
\]

\[
D_2 = D_1(j_1, m_1 \leftrightarrow j_2, m_2),
\]

\[
D_3 = -\frac{s(j_{13})s(j_{23})s(j_1 + m_1)s(j_2 + m_2)s(j_1 + j_2 + m_3)}{s(j_1 - m_1)s(j_2 - m_2)s(j_3 + m_3)}.
\]

Since \( m_a - \bar{m}_a \in \mathbb{Z} \), \((2.12)\) is symmetric with respect to \( m_a \) and \( \bar{m}_a \), as it should be.

So far, we have discussed the correlation functions in the \( H_3^+ \) WZW model, where the parameters in \( \Phi_{mm} \) have been \( j = -1/2 + i\rho, m = \frac{1}{2}(ip+q), \bar{m} = \frac{1}{2}(ip-q) \) with \( \rho, p \in \mathbb{R}, q \in \mathbb{Z} \).

In the following, we would like to assumed that the correlation functions in other models with an \( \hat{s}(2) \times \hat{s}(2) \) symmetry are obtained by appropriately continuing the parameters in the above results \([2, 3, 24, 18, 25, 6]\). In particular, we assume that the correlation functions in the \( SL(2, R) \) WZW model are obtained by setting \( j, m, \bar{m} \) to be the values of the representations of \( SL(2, R) \).

To understand this, we first note that, from the algebraic point of view, the symmetries of the \( H_3^+ \) and \( SL(2, R) \) WZW models are the same, i.e., \( \hat{s}(2) \times \hat{s}(2) \). The difference is the allowed representations and, hence, the allowed values of the parameters, \( j, m \) and \( \bar{m} \). In addition, the correlation functions discussed so far respect the \( \hat{s}(2) \times \hat{s}(2) \) symmetry, and they are analytic in \( j, m \) and \( \bar{m} \) (up to delta functions). Then, the assumption follows from these facts.

In fact, we will find that, after taking into account the difference of the definitions of the operators, the continued correlation functions correctly reduce to the known ones for the \( SL(2, R) \) WZW model \([24]\) (up to phases) in a special case of \((2.13)\) with \( m_a = \bar{m}_a \). Furthermore, we can find more evidence for the validity of such a continuation: (i) As
discussed in [2] and [6], by using the continued correlators, one can obtain the correct OPE of the models with the \( \hat{sl}(2) \times \hat{sl}(2) \) symmetry other than the \( H_3^+ \) WZW model. (ii) As we will discussed in section 3, the continued correlation functions have the correct pole structure which is required from the representation theory of \( SL(2, R) \). This is quite non-trivial, and implies that they are actually regarded as the correlators of the \( SL(2, R) \) WZW model. It also turns out that the OPE based on these correlators is consistent with the \( SL(2, R) \) symmetry. (iii) If we adopt, e.g., the path integral approach, it is not possible to directly obtain the correlators of the \( SL(2, R) \) WZW model; the integral diverges because of the time like direction. Thus, in analogy to the case of the flat target space-time, it is natural to ‘define’ the correlators of the \( SL(2, R) \) case by some continuation from those in the Euclidean case, i.e., the \( H_3^+ \) case. Such an issue has also been discussed in detail in a recent paper [28]. (iv) Some correlators for other models which are obtained by such a continuation from those in the \( H_3^+ \) case have been used in the literature, e.g., [18, 24], and the results seem to be physically reasonable.

As is discussed shortly, our results pass some other consistency checks in addition to those mentioned above. Therefore, our assumption has fairly good grounds.

In the following, among the \( SL(2, R) \) representations, we concentrate on the normalizable unitary representations, whose parameters are given by

\[
\begin{aligned}
(1) & \quad j < -1/2, \quad m = -j + Z_{\geq 0} \quad \text{for the lowest weight discrete series (} D^+_j \text{);} \\
(2) & \quad j < -1/2, \quad m = j - Z_{\geq 0} \quad \text{for the highest weight discrete series (} D^-_j \text{);} \\
(3) & \quad j = -1/2 + iR_{>0}, \quad m = \alpha + Z \quad (0 \leq \alpha < 1) \quad \text{for the principal continuous series (} C^\alpha_j \text{)}.
\end{aligned}
\]

\( \bar{m} \) takes the values in the same representation as \( m \), because we consider the case where \( \Phi_{m \bar{m}} \) give the diagonal combinations of the left and right \( SL(2, R) \) representations. In this case, \( \Phi_{m \bar{m}} \) correspond to the matrix elements of the above three types of representations, which span the \( L^2 \)-space on an \( SL(2, R) \) manifold.

### 2.3 Case with highest(lowest) weight representations

When \( j \) and \( m \) satisfy a certain relation, the \( \hat{sl}(2) \) representation largely reduces because of the appearance of the null vectors. The correlation functions in such a case may also be quite different from those in a generic case. Given (2.9) and (2.12), we can confirm this.

As an example, we consider the case in which \( \Phi^j_{m \bar{m}} \) belongs to a left and right combination of lowest weight representations:

\[
m_1 = -j_1 + n_1, \quad \bar{m}_1 = -j_1 + \bar{n}_1 \quad (n_1, \bar{n}_1 \in Z_{\geq 0}).
\]

\( D_2 \) and \( D_3 \) in (2.13) then vanish when the parameters are generic except for satisfying (2.14). In addition, the analysis in appendix C shows that \( C^{12} \) and \( C^{21} \) do not give divergences which cancel the zeros in \( D_2 \) and \( D_3 \). Thus, only the first term in (2.12) remains non-vanishing, so
that \( W(j_a; m_a) \) reduces to

\[
W_1(j_a; m_a) = (-)^{m_3 - \bar{m}_3 + \bar{n}_1} \pi^2 \Delta(-N) \Delta(2j_1 + 1) \frac{\Gamma(1 + j_3 - m_3) \Gamma(1 + j_3 - \bar{m}_3)}{\Delta(1 + j_{12}) \Delta(1 + j_{13}) \Gamma(1 + j_3 - m_3 - n_1) \Gamma(1 + j_3 - \bar{m}_3 - \bar{n}_1)} \times \prod_{a=2,3} \frac{\Gamma(1 + j_a + m_a)}{\Gamma(-j_a - \bar{m}_a)} F\left[ -n_1, -j_{12}, 1 + j_{23} \right] -2j_1, 1 + j_3 - m_3 - n_1] F\left[ -\bar{n}_1, -j_{12}, 1 + j_{23} \right] -2j_1, 1 + j_3 - \bar{m}_3 - \bar{n}_1].
\]

(2.15)

Here, we have used the first expression of \( C^{12} \) in (3.4). Note that \( F \)'s above are finite sums, since they have non-positive integers in their upper arguments.

This is in accord with the discussion in [29]: \( W(j_a; m_a) \) is essentially the left and right combination of the \( SL(2) \) Clebsch-Gordan coefficients. Generically, there are two linearly independent solutions to the difference equation for each Clebsch-Gordan coefficients. However, when one of the operator is set to be in a highest(lowest) weight representation, only one solution remains because of boundary conditions. \( C^{12} \) is regarded as such a solution and, hence, only the combination \( D_1 C^{12} \bar{C}^{12} \) remains.

If we further set \( n_1 = \bar{n}_1 = 0 \) in (2.15), the \( F \)'s above become unity. Consequently, we obtain

\[
W_0^0(j_a; m_a) = (-)^{m_3 - \bar{m}_3} \pi^2 \frac{\Delta(-N) \Delta(2j_1 + 1)}{\Delta(1 + j_{12}) \Delta(1 + j_{13})} \prod_{a=2,3} \frac{\Gamma(1 + j_a + m_a)}{\Gamma(-j_a - \bar{m}_a)}. \tag{2.16}
\]

This is in precise agreement with the result in [3], including the phase.

Since \( \Phi_{mn} \) form \( SL(2) \) representations with respect to the zero-modes of the currents, \( J_0^m \) and \( \tilde{J}_0^m \), it should be possible to derive (2.15) conversely from (2.16). One can show this following [26]. In appendix D, we sketch the corresponding argument in our case. We remark that (2.15) does not coincide with the corresponding quantity in [26] even when \( m_a = \bar{m}_a \). This is because the phase of \( W_0^0(j_a; m_a) \) is different, and gives a different analytic expression in a generic case.

3 OPE in the \( SL(2, R) \) WZW model

In this section, we would like to discuss the OPE of the primary fields in the \( SL(2, R) \) WZW model. As mentioned in section 2, we suppose that the correlations functions in the \( SL(2, R) \) WZW model are obtained from those in section 2. It is also understood that the parameters take generic values unless otherwise stated. The cases with special values are obtained by taking appropriate limits. In addition, when singular quantities appear, we regularize them by infinitesimally shifting the parameters, and take limits in the final expressions.

3.1 Preliminary
We start with the following general form of the OPE among $\Phi^j_{m\bar{m}}$:

$$
\Phi^j_{m_1\bar{m}_1}(z_1)\Phi^j_{m_2\bar{m}_2}(z_2) \sim z_1^{-2h_{21}} \sum_{j_3,m_3,\bar{m}_3} Q(j_a;m_a) \Phi^j_{m_3\bar{m}_3}(z_2). \tag{3.1}
$$

$\sum_{j_3,m_3,\bar{m}_3}$ stands for the summation and integration over $j_3, m_3, \bar{m}_3$, whose precise meaning is not determined yet. In our case, $\Phi^j_{m_1\bar{m}_1}$ and $\Phi^j_{m_2\bar{m}_2}$ belong to $D_j^+$ or $C_j^+$.

Here, one might ask what $\Phi^j_{m\bar{m}}$ means after the continuation of the parameters. However, we will see below that the OPE is almost determined by the symmetry and the two- and three-point functions. Thus, the existence of the correct primary fields is enough in our discussion. Namely, we do not need the explicit expressions of $\Phi^j_{m\bar{m}}$, but we need only the fact that $\Phi^j_{m\bar{m}}$ has the correct OPE with the $\hat{sl}(2)$ currents. This is assured by the continuation of the parameters from the $H_3^+$ case, which is in accord with the continuation of the correlators. Moreover, as mentioned in subsection 2.2, the OPE based on such a continuation of $\Phi^j_{m\bar{m}}$ gives the results consistent with the symmetry in the case discussed in [6] and [9]. It also turns out that this is true in our $SL(2,R)$ case. In addition, such a continuation of the operators may be supported by considering the analogy to the flat case, and has been discussed in detail in [23]. Classically, $\Phi^j_{m\bar{m}}$ satisfies the Laplace equation on $SL(2,R)$, of course.

Let us return to (3.3). $Q$ is some coefficient, and it is determined using the two- and three-point functions through

$$
\langle \Phi^j_{m_1\bar{m}_1}(z_1)\Phi^j_{m_2\bar{m}_2}(z_2)\Phi^j_{m_3\bar{m}_3}(z_4) \rangle 
\sim |z_1|^{-2h_{21}} \sum_{j_3,m_3,\bar{m}_3} Q(j_a;m_a) \langle \Phi^j_{m_3\bar{m}_3}(z_2)\Phi^j_{m_4\bar{m}_4}(z_4) \rangle. \tag{3.2}
$$

There are two possible contributions in the right-hand side from $\langle \Phi^j_{m_3\bar{m}_3}\Phi^j_{m_4\bar{m}_4} \rangle$, which are proportional to $\delta(j_3 - j_4)$ and $\delta(j_3 + j_4 + 1)$, respectively. Supposed that $\sum_{j_3,m_3,\bar{m}_3}$ picks up only one of them, as is the case in our later discussion, the solution to the above condition is given by

$$
Q_{\Phi}(j_a;m_1,m_2,m_3) = \delta^2(m_1 + m_2 - m_3)W(j_a;m_1,m_2,-m_3) \frac{D(-j_a - 1)}{B_{\Phi}(j_3,m_3)}. \tag{3.3}
$$

This is easily confirmed when the term proportional to $\delta(j_3 - j_4)$ is picked up. Even when the other term is picked up, the result is the same because of the identity,

$$
Q_{\Phi}(j_1,j_2,j_3;m_a)B_{\Phi}(j_3,m_3) = Q_{\Phi}(j_1,j_2,-j_3 - 1;m_a)A(j_3). \tag{3.4}
$$

This is derived by showing that the four terms in (2.12), $W_a$ ($a = 1 \sim 4$), satisfy

$$
\frac{W_a(j_1,j_2,j_3;m_a)}{W_a(j_1,j_2,-j_3 - 1;m_a)} = \frac{D(-j_1 - 1,-j_2 - 1,j_3)}{D(-j_1 - 1,-j_2 - 1,-j_3 - 1)} \frac{A(j_3)}{B_{\Phi}(-j_3 - 1,m_3)}. \tag{3.5}
$$

To see this, we first make use of (1.7), so that the right-hand side is simplified to

$$
\Delta(-j_13)\Delta(-j_23) \frac{\Gamma(j_3 + 1 + m_3)\Gamma(j_3 + 1 - \bar{m}_3)}{\Gamma(-j_3 + m_3)\Gamma(-j_3 - \bar{m}_3)}. \tag{3.6}
$$
From the second expression of $C_{12}^{12}$ in (B.4) and a similar one for $C_{21}^{21}$, it follows that $C_{12}^{12}$ and $C_{21}^{21}$ are invariant under the exchange of $j_3$ and $-j_3 - 1$ up to factors of the gamma functions. It is then straightforward to evaluate the left-hand side of (3.5), and find that each $W_a$ satisfies (3.5). Thus, even if $\sum_{j_3,m_3,m_3}$ picks up both terms in the right-hand side in (3.2), $Q_\Phi$ is a solution up to a factor one half.\[
\]

Let us move on to a discussion about the analytic structure of $Q_\Phi$, which is needed later. There are two sources of the poles in $j_3$. One is $D(-j_a - 1)$. This has simple poles at \[
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where $S = \{l + nb^{-2} | l, n \in \mathbb{Z}_{\geq 0}\}$. The other source is $W(j_a; m_1, m_2, -m_3)B_{\Phi}^{-1}(j_3, m_3)$. Its analytic structure can be studied using the results in appendix C. Here, we divide the possible cases into the following two: (i) At least one of $\Phi_{m_1 \bar{m}_1}^{11}$ and $\Phi_{m_2 \bar{m}_2}^{22}$ is in $D_j^+$. (The case of $D_j^-$ is similar.) (ii) Both $\Phi_{m_1 \bar{m}_1}^{11}$ and $\Phi_{m_2 \bar{m}_2}^{22}$ are in $C_j^\alpha$.

In case (i), we can choose $j_1, m_1, \bar{m}_1$ so that (2.14) and, hence, $W = W_1$ hold. From the discussion in appendix C, we see that the poles and some of the zeros in $W_1B_{\Phi}^{-1}$ are included in the factor \[
\Delta(-N)\Delta(-j_{12})\Delta(-j_{13})\frac{\Gamma(1 + j_2 + m_2)}{\Gamma(-j_2 - \bar{m}_2)}\frac{\Gamma(-j_3 - m_3)}{\Gamma(1 + j_3 + \bar{m}_3)}.
\]
Thus, $W_1B_{\Phi}^{-1}$ has simple poles at $N, j_{12}, j_{13}, j_3 + m_3 \in \mathbb{Z}_{\geq 0}$, and zeros at $N, j_{12}, j_{13}, j_3 + \bar{m}_3 \in -1 - \mathbb{Z}_{\geq 0}$. Combining these with the poles in $D(-j_a - 1)$, we find the possible poles in $Q_\Phi$: \[
\]
\[
\]
\[
\]
These can collide with each other. In such a case, we regularize the parameters so that they remain simple poles.

In case (ii), it is useful to go back to (2.9), and use the expressions in (B.4) for $C_{31}^{31}$, $C_{12}^{12}$ (the second one), and similar ones for $C_{21}^{21}$ and $C_{13}^{13}$. Then, from the results in appendix C,
the poles in \( W \) are read off from some factors of the gamma functions. Collecting the relevant factors, we see that the poles in \( WB^{-1}_\Phi \) are included in

\[
\Gamma(-N)\Gamma^2(-j_{12})\Gamma(-j_{13})\Gamma(-j_3 - m_3)\Gamma(-j_3 + \bar{m}_3)\Gamma(1 + j_3 + m_3),
\]

(3.10)

and a similar one with \( j_1 \) and \( j_2 \) exchanged. Combining these poles with those in (3.11), we find the possible poles in \( Q_\Phi \) in this case:

\[
\begin{align*}
\dot{j}_3 &= \begin{cases} 
  j_1 + j_2 + 1 + S, & \dot{j}_3 = \begin{cases} 
  -(j_1 + j_2 + 1) + S, \\
  j_1 + j_2 - S^*, \\
  j_1 - j_2 + S, \\
  j_1 - j_2 - 1 - S,
\end{cases} \\
  \dot{j}_3 &= \begin{cases} 
  -(j_1 + j_2 + 1) - 1 - S, \\
  j_2 - j_1 + S, \\
  j_2 - j_1 - 1 - S,
\end{cases}
\end{cases} \\
\dot{j}_3 &= -m_3 + Z_{\geq 0}, \\
\dot{j}_3 &= -m_3 - 1 - Z_{\geq 0}.
\end{align*}
\]

Here, \(*\) indicates that that sequence includes possibly double poles coming from \( \Gamma^2(-j_{12}) \). (It may be possible that several terms sum up to make them simple poles.) The list of the \((m_3, \bar{m}_3)\)-independent poles is the same as the one appeared in the OPE of \( \Phi_j \) up to \(*\).

3.2 Classical limit

The precise meaning of \( \sum_{j_3, m_3, \bar{m}_3} \) is yet to be determined. For this purpose, we consider the classical limit in this subsection. This is along the line of the arguments which determined the OPE in the \( H^+_3 \) WZW model: In [2, 3], the OPE in the \( H^+_3 \) WZW model was proposed by using several arguments, one of which was a consideration of the classical limit. The crossing symmetry of the four-point functions has been shown by using the proposed OPE in [20].

In the classical limit \( k \to \infty \) (or particle limit), the problem reduces to that of zero-modes. The OPE of \( \Phi^{j_3}_{m_3 \bar{m}_3} \) then should represent the decomposition of a product of two ‘wave functions’ \( \Phi^{j_{13}}_{m_1 \bar{m}_1} \times \Phi^{j_{23}}_{m_2 \bar{m}_2} \) into a set of ‘wave functions’ \( \{\Phi^{j_3}_{m_3 \bar{m}_3}\} \). Since \( \Phi^{j_{13}}_{m_1 \bar{m}_1}, \Phi^{j_{23}}_{m_2 \bar{m}_2} \) belong to the normalizable unitary representations of \( SL(2, R) \), they correspond to the elements of the \( L^2 \)-space on \( SL(2, R) \). According to the harmonic analysis, \( \Phi^{j_{13}}_{m_1 \bar{m}_1} \times \Phi^{j_{23}}_{m_2 \bar{m}_2} \) also corresponds to an element of the \( L^2 \)-space and, hence, should be decomposed by the normalizable unitary representations.

Therefore, (at least) in this limit, \( \sum_{j_3, m_3, \bar{m}_3} \) should be the summation over \( \mathcal{D}^\pm_j \) and \( \mathcal{C}^\alpha_j \), namely,

\[
\sum_{j_3, m_3, \bar{m}_3} = \int_{-\infty}^{\infty} dm_3 \int_{-\infty}^{\infty} d\bar{m}_3 \left[ \int_{\mathcal{P}^+} dj_3 + \delta_{\mathcal{P}^+} \int_{\mathcal{C}} dj_3 \right].
\]

(3.12)

Some explanations may be in order here: Together with \( \delta^2(m_1 + m_2 - m_3) \), the integrations over \( m_3 \) and \( \bar{m}_3 \) give the conservation of \( m, \bar{m} \). The integration over \( \mathcal{P}^+ = -1/2 + i\mathbb{R}_{>0} \)
stands for the summation over $C^\alpha_j$. The summation over $\alpha$ is already encoded in that of $m, \bar{m}$. The last contour integral indicates the summation over $D^\pm_j$. $\delta_{D^\pm_j}$ means that $j_3$ is picked up only when $\{j_3, m_3, \bar{m}_3\}$ are the quantum numbers of $D^\pm_j$. In such a case, the corresponding $j_3$ comes from the pole in $Q_\Phi$ picked up by the contour $C$. Note that the range of $j_3$ is $\text{Re} j_3 \leq -1/2$ and $\text{Im} j_3 \geq 0$. This is consistent with the argument which determined $Q_\Phi$, because $\sum_{j_3, m_3, \bar{m}_3}$ picks up only one term in the right-hand side in (3.2).

In addition, in the limit $k \to \infty$, the possible poles in $Q_\Phi$ become

\[
j_3 = \begin{cases} 
-(j_1 + j_2 + 1) + Z_{\geq 0}, \\
j_1 + j_2 - Z_{\geq 0}, \\
j_1 - j_2 - 1 - Z_{\geq 0}, 
\end{cases} \quad j_3 = \begin{cases} 
-j_1 + Z_{\geq 0}, \\
j_1 - j_2 - 1 - Z_{\geq 0}, 
\end{cases} \quad (3.13)
\]

\[
j_3 = -m_3 + Z_{\geq 0}.
\]

in the case where $\Phi^{j_1}_{m_1, \bar{m}_1}$ belongs to $D^+_j$. In the case where both $\Phi^{j_1}_{m_1, \bar{m}_1}$ and $\Phi^{j_2}_{m_2, \bar{m}_2}$ belong to $C^\alpha_j$, the possible poles are obtained simply by replacing $S$ in (3.11) with $Z_{\geq 0}$.

In order to further determine the contour $C$, we need to study which poles are compatible with the quantum numbers of $D^\pm_j$. As an example, let us consider the case where $\Phi^{j_1}_{m_1, \bar{m}_1}, \Phi^{j_2}_{m_2, \bar{m}_2} \in D^+_j \otimes D^+_j$, namely, $m_{1,2} = -j_{1,2} + n_{1,2}$ ($n_{1,2} \in Z_{\geq 0}$) (and similarly for $m_{1,2}$). The conservation of $m, \bar{m}$ then implies that $m_3 = m_1 + m_2 = -j_1 - j_2 + n_1 + n_2$. Thus, for $\Phi^{j_3}_{m_3, \bar{m}_3} \in D^+_j \otimes D^+_j$ with $j_3 = -m_3 + n_3$ ($n_3 \in Z_{\geq 0}$), only $j_3 = -m_3 + Z_{\geq 0}$ and a part of $j_3 = j_1 + j_2 - Z_{\geq 0}$ in (3.13) can satisfy the constraint $\delta_{D^+_j}$. Similarly, for $\Phi^{j_3}_{m_3, \bar{m}_3} \in D^-_j \otimes D^-_j$, the possible sequence is $j_3 = -(j_1 + j_2 + 1) + Z_{\geq 0}$. However, in this case, the residues vanish because of the factor $\Gamma^{-1}(-j_2 - \bar{m}_2)$ in $Q_\Phi$. In this way, we obtain a table of the allowed poles (Table 1).

From this table, we notice that there are two allowed sequences in (1a) and (2a). In (1a), $j_3 = -m_3 + n_3 = j_1 + j_2 - n_1 - n_2 + n_4$. Hence, a part of the two sequences degenerates. We regularize such degeneracy by slightly shifting the parameters. The residues are then non-vanishing only for such degenerating poles because of the factor $\Gamma^{-1}(-j_2 - \bar{m}_2)$ in $Q_\Phi$. To avoid double counting, we take the contour $C$ so that it encloses the poles in the sequence $j_3 = -m_3 + Z_{\geq 0}$. Consequently, $j_3 = j_1 + j_2, j_1 + j_2 - 1, \cdots$ contribute to the OPE.

In (2a), $m_1 = -j_1 + n_1, m_2 = j_2 - n_2$ and $j_3 = -m_3 + n_3 = j_1 - j_2 - n_1 + n_2 + n_3$. Thus, it is possible again that a part of the two sequences degenerates. However, in such a case, the corresponding residues are divergent (assuming the regularization of the degeneracy). Since $Q_\Phi$ is essentially a three-point function, this may indicate that the three-point function is ill-defined. Therefore, we take $C$ so that it encloses the poles in the sequence $j_3 = -m_3 + Z_{\geq 0}$ whose residues are regular. Consequently, $j_3 = j_1 - j_2, j_1 - j_2 - 1, \cdots < -1/2$ contribute to the OPE.

For other cases, we take $C$ so that it simply encloses the poles in Table 1. In sum, the contour $C$ encloses the poles, $j_3 = -m_3 + Z_{\geq 0}, \bar{m}_3 + Z_{\geq 0}, j_2 - j_1 + Z_{\geq 0} < -1/2$, whose residues are not divergent.
| $\Phi_{j_1}^{m_1 \bar{m}_1}$ | $\Phi_{j_2}^{m_2 \bar{m}_2}$ | $\Phi_{j_3}^{m_3 \bar{m}_3}$ | allowed poles ($j_3 < -1/2$) |
|----------------|----------------|----------------|-----------------|
| $\mathcal{D}_j^+$ | $\mathcal{D}_j^+$ | $\mathcal{D}_j^+$ | (1a) $j_3 = -m_3 + Z_{\geq 0}$, $j_3 = j_1 + j_2 - Z_{\geq 0}$ |
| $\mathcal{D}_j^-$ | $\mathcal{D}_j^-$ | $\mathcal{D}_j^-$ | (1b) $-$ |
| $\mathcal{D}_j^+$ | $\mathcal{D}_j^+$ | $\mathcal{D}_j^+$ | (2a) $j_3 = -m_3 + Z_{\geq 0}$, $j_3 = j_1 - j_2 - 1 - Z_{\geq 0}$ |
| $\mathcal{D}_j^-$ | $\mathcal{D}_j^-$ | $\mathcal{D}_j^-$ | (2b) $j_3 = j_2 - j_1 + Z_{\geq 0}$ |
| $\mathcal{D}_j^+$ | $\mathcal{C}_j^a$ | $\mathcal{D}_j^+$ | (3a) $j_3 = -m_3 + Z_{\geq 0}$ |
| $\mathcal{D}_j^-$ | $\mathcal{C}_j^a$ | $\mathcal{D}_j^-$ | (3b) $-$ |
| $\mathcal{C}_j^a$ | $\mathcal{C}_j^a$ | $\mathcal{D}_j^+$ | (4a) $j_3 = -m_3 + Z_{\geq 0}$ |
| $\mathcal{C}_j^a$ | $\mathcal{C}_j^a$ | $\mathcal{D}_j^-$ | (4b) $j_3 = \bar{m}_3 + Z_{\geq 0}$ |

Table 1: The first three columns show the representations of $\Phi_{m_a \bar{m}_a}^{j_a}$ ($a = 1, 2, 3$), respectively. We have used the abbreviations $\mathcal{D}_j^+$ for $\mathcal{D}_j^+ \otimes \mathcal{D}_j^+$, and so on. For the ($m_3, \bar{m}_3$)-independent poles, only a part of the sequences satisfying the conservation of $m, \bar{m}$ is allowed.

Here, we would like to make two remarks. One is about case (2b). In this case, $j_3 = j_2 - j_1 + Z_{\geq 0}$ seems to appear for $n_1 > n_2$, which is out of the range of $\mathcal{D}_j^-$. However, we can show that the residues in such cases automatically vanish: First, for $j_3 = m_3 - 1$, the residue vanishes because of (C.5). Furthermore, using an identity similar to (D.2), we see that the residues for $j_3 - m_3 < -1$ also vanish.

The other remark is about the selection rules from the right sector. In the above, we have concentrated on the left sector. However, since the left and the right sectors are symmetric, so should be the OPE. In particular, $\Phi_{m_3 \bar{m}_3}^{j_3}$ with $j_3 = m_3 - 1$ should not appear. In (1a), (2a) and (3a) where $W = W_1$ and $j_3 + m_3 \in Z_{\geq 0}$, this is accounted by the factor $\Gamma^{-1}(1 + j_3 + \bar{m}_3)$ in $Q_\Phi$. As a consequence, the contributions, for example, in (1a) come from $j_3 = j_1 + j_2, j_1 + j_2 - 1, \ldots, j_1 + j_2 - n$ with $n = \min\{n_1 + n_2, \bar{n}_1 + \bar{n}_2\}$. In addition, the zeros from $\Gamma^{-1}(1 + j_3 + \bar{m}_3)$ can cancel the divergence in the above discussion about (2a). Thus, precisely speaking, we should have said above that $\mathcal{C}$ picks up the poles in the sequence $j_3 = -m_3 + Z_{\geq 0}$ whose residues are regular for generic $\bar{n}_1, \bar{n}_2$.

In (2b) where $W = W_1$ and $j_3 - m_3 \in Z_{\geq 0}$, the mechanism in the above remark about (2b) works also for $(j_3, \bar{m}_3)$. This ensures the correct truncation. For the last two cases in Table 1, we may need to carry out the summation of the terms in $W$ carefully in order to check the absence of the unwanted contributions. However, this is indirectly confirmed by the fact that we can repeat the analysis in this section with $m_a$ and $\bar{m}_a$ exchanged.

Now that the meaning of $\sum_{j_3, m_3, \bar{m}_3}$ has been determined, the OPE is obtained by collecting the contributions from $\mathcal{P}_+$ and $\mathcal{C}$. Since we can exchange the roles of $\mathcal{D}_j^+$ and $\mathcal{D}_j^-$ by redefining $(J_0^+, J_0^3)$ as $(J_0^-, -J_0^3)$, the case with $\mathcal{D}_j^+$ and $\mathcal{D}_j^-$ exchanged is similar. Thus, we find the
following results in the \( k \to \infty \) limit:

\[
[D_{j_1}^\pm \otimes D_{j_1}^\pm] \otimes [D_{j_2}^\pm \otimes D_{j_2}^\pm] \to \sum_{j_3 \leq j_1 + j_2} [D_{j_3}^\pm \otimes D_{j_3}^\pm], \tag{3.14}
\]

\[
[D_{j_1}^+ \otimes D_{j_1}^+] \otimes [D_{j_2}^- \otimes D_{j_2}^-] \to \int_{P_+} dj_3 \left[ C_{j_3}^{\alpha_3} \otimes C_{j_3}^{\alpha_3} \right] + \sum_{j_1-j_2 \leq j_3 \leq -\frac{1}{2}} [D_{j_3}^+ \otimes D_{j_3}^-], \tag{3.15}
\]

\[
[D_{j_1}^\pm \otimes D_{j_1}^\pm] \otimes [C_{j_2}^{\alpha_2} \otimes C_{j_2}^{\alpha_2}] \to \int_{P_+} dj_3 \left[ C_{j_3}^{\alpha_3} \otimes C_{j_3}^{\alpha_3} \right] + \sum_{j_3 < -\frac{1}{2}} [D_{j_3}^+ \otimes D_{j_3}^+] + \sum_{j_3 < -\frac{1}{2}} [D_{j_3}^- \otimes D_{j_3}^-]. \tag{3.16}
\]

\[
[C_{j_1}^{\alpha_1} \otimes C_{j_1}^{\alpha_1}] \otimes [C_{j_2}^{\alpha_2} \otimes C_{j_2}^{\alpha_2}] \to \int_{P_+} dj_3 \left[ C_{j_3}^{\alpha_3} \otimes C_{j_3}^{\alpha_3} \right] + \sum_{j_3 < -\frac{1}{2}} [D_{j_3}^+ \otimes D_{j_3}^+] + \sum_{j_3 < -\frac{1}{2}} [D_{j_3}^- \otimes D_{j_3}^-]. \tag{3.17}
\]

In the above, \( j_3 \) for the discrete series is summed with integer spacing. \( \alpha_3 \) for the continuous series and \( j_3 \) for the discrete series take the values so that they are compatible with the conservation of \( m, \bar{m} \). Note that the contributions from the continuous series are absent in (3.14) because the integrand \( Q_\phi \) vanishes. In (3.17), we have not cared about the multiplicity of the representations in the right-hand side. For the tensor products of the \( SL(2, R) \) representations, the continuous series appears twice in the corresponding case. This is due to the existence of two linearly independent Clebsch-Gordan coefficients. In our case, such multiplicity corresponds to the fact that both \( C^{12} \) and \( C^{21} \) contribute to the OPE. These results are in complete agreement with the tensor products of the \( SL(2, R) \) representations, which are given in (A.8). \footnote{Precisely, we need to confirm that the residues of the contributions to (3.14)-(3.17) are non-vanishing. However, we do not see any special reason for them to vanish.}

### 3.3 Relation to [6]

The OPE in the \( SL(2, R) \) WZW model was also discussed in [6] for finite \( k \), using the three-point functions with \( j_1 + m_1 = j_1 + \bar{m}_1 = 0 \). In this subsection, we would like to discuss the relation between the arguments in the previous subsection and in [6] (after taking the limit \( k \to \infty \)).

First, the primary fields used in [6] were

\[
V_{m\bar{m}}^j = \gamma^{j+m} \bar{\gamma}^{j+\bar{m}} e^{2j\phi}, \tag{3.18}
\]

where \( \gamma, \bar{\gamma}, \phi \) are certain coordinates of \( H^+_3 \) or \( SL(2, R) \). These are nothing but the primary fields in the standard Wakimoto realization of \( \hat{sl}(2) \times \hat{sl}(2) \). Since \( \Phi_j = (|\gamma-x|^2 e^\phi + e^{-\phi})^{2j} \) in terms of \( \gamma, \bar{\gamma}, \phi \), the relation between \( \Phi_{m\bar{m}}^j \) and \( V_{m\bar{m}}^j \) is given by

\[
\Phi_{m\bar{m}}^j |_{\phi \to \infty} = c_{m\bar{m}}^{-j} V_{m\bar{m}}^{-j-1}. \tag{3.19}
\]
Here, $\phi \to \infty$ is the 'free-field' limit (see, e.g., [4, 5]). Note, however, that we do not need such a limit in the discussion using $\Phi^{j}_{m\bar{m}}$. Because of the intertwiner $c^{j}_{m\bar{m}}$, $V^{j}_{m\bar{m}}$ form a representation with spin $j$ (not $-j - 1$) as $\Phi^{j}_{m\bar{m}}$.

The correlation functions of $V^{j}_{m\bar{m}}$ are obtained by substituting the right-hand side of (3.19) into those of $\Phi^{j}_{m\bar{m}}$, namely, (2.6) and (2.7). This is shown by direct calculations [6]. Then, the OPE of $V^{j}_{m\bar{m}}$ can be discussed similarly to the case of $\Phi^{j}_{m\bar{m}}$. In this case, the quantity corresponding to $Q_{\Phi}$ turns out to be

$$Q_{V}(j_{a};m_{a}) = \frac{R(j_{1})R(j_{2})}{R(j_{3})}Q_{\Phi}(j_{a};m_{a}),$$

with $R(j) = B(j)/A(j)$. To derive this relation, we have used (3.5). $Q_{V}$ reduces to the expression in [6] when $j_{1} + m_{1} = j_{1} + \bar{m}_{1} = 0$. Since $R(j)$ has no poles and zeros for generic $j$ and $k$, the discussions of the OPE are essentially the same for $\Phi^{j}_{m\bar{m}}$ and $V^{j}_{m\bar{m}}$.

Second, the main idea in [6] was that the OPE in the $SL(2, R)$ WZW model is obtained from the OPE in the $H^{+}_{3}$ WZW model by continuing the parameters appropriately. This is based on the arguments in [2, 3], which proposed that the OPE in a model with an $\hat{sl}(2)$ symmetry is obtained by such a continuation. With such a prescription, the OPE includes contributions from the continuous series by definition (unless the coefficients vanish), because the OPE in the $H^{+}_{3}$ WZW model is of the form $\Phi_{j_{1}}\Phi_{j_{2}} \sim \int_{P_{+}}dj_{3}\Phi_{j_{3}}$. In [6], it was further argued that the OPE in the $SL(2, R)$ WZW model has other contributions corresponding to case (2b) in Table 1 by continuing $j_{1}$ and $j_{2}$ from the values of the continuous series. However, how to precisely deal with the $(m_{3}, \bar{m}_{3})$-dependent poles, which are absent in the discussion of the $H^{+}_{3}$ WZW model, was an open question.

We have taken an apparently different strategy in this section. Nevertheless, for the cases covered in [6] we notice that our results are obtained by further picking up the appropriate $(m_{3}, \bar{m}_{3})$-dependent poles in [6]. Thus, after all our prescription is regarded as a completion of the arguments in [6]. This means that the arguments in [6] and in this paper support each other. We remark that the issue of the divergence in case (2b) did not appear in [6], because $n_{1} = \bar{n}_{1} = 0$ there.

3.4 Full OPE

Now we would like to consider the full OPE for finite $k$. First, let us summarize our proposal for the OPE. Taking into account the discussion in the previous subsection, it can be stated as follows:

1. The OPE is given by (3.1) with $Q = Q_{\Phi}$ in (3.3) and $\sum_{j_{a}, m_{3}, \bar{m}_{3}}$ in (3.12).

2. The contour $C$ picks up the following poles in the region $j_{3} < -1/2$: (i) the $(m_{3}, \bar{m}_{3})$-dependent poles of the type $j_{3} = \pm m_{3} + Z_{\geq 0}$, (ii) the poles which cross over $P_{+}$ when $j_{1}, j_{2}$ are supposed to be continued from $j_{1}, j_{2} \in P_{+}$.
(3) A pole whose residue is (generically) divergent should not be picked up.

For instance, \( j_3 = j_2 - j_1 + \mathbb{Z}_{\geq 0} \) are on the right-hand side of \( \mathcal{P}_+ \) in the complex \( j_3 \)-plane when \( j_1, j_2 \in \mathcal{P}_+ \). As \( j_1, j_2 \) are continued to \( j_1, j_2 < -1/2 \), some of \( j_3 \)'s above cross over \( \mathcal{P}_+ \), and such poles are picked up. (For more details, see [2, 3, 6].)

Although this prescription was given for \( k \to \infty \), it can be applied to the case of finite \( k \) without modifications. Thus, we take the above (1)-(3) as the definition of the full OPE. The analysis according to this definition is then straightforward. As a consequence, it turns out that all the additional poles appearing for generic finite \( k \) do not contribute because of the conservation of \( m, \overline{m} \). Therefore, we arrive at the results which are essentially the same as those in the \( k \to \infty \) limit. They are given just by replacing \( \hat{D}_j^\pm, \hat{C}_j^\alpha \) in (3.14)-(3.17) with the corresponding affine representations \( \hat{D}_j^\pm, \hat{C}_j^\alpha \).

Those results show that the OPE of the primary fields in the \( SL(2, R) \) WZW model is essentially the same as the classical tensor products. Here, it would be useful to recall the case of the \( SU(2) \) WZW model. In that case, the existence of the non-trivial null states dictates the decoupling of the primary fields with \( SU(2) \) spin \( j > \tilde{k}/2 \). This mechanism makes the difference from the classical tensor products [31]. However, in the case of \( SL(2, R) \), there are no corresponding non-trivial null states. Thus, it is natural that the full OPE (3.14)-(3.17) with \( \hat{D}_j^\pm, \hat{C}_j^\alpha \) is essentially the same as the classical tensor products (A.8). Note that any correct OPE should recover (A.8) in the limit \( k \to \infty \).

4 Discussion

In this paper, we first wrote down the explicit expression of the three-point functions of \( \Phi_{jm, \overline{m}} \), and analyzed its properties. Based on these results, we discussed the OPE in the \( SL(2, R) \) WZW model. Guided by a consideration in the classical (particle) limit, we proposed the full OPE. It was essentially the same as the classical tensor products of the representations of \( SL(2, R) \). This is natural because we may not have any reason that makes difference, contrary to the \( SU(2) \) case. We also noted that our prescription is regarded as a completion of the arguments in [3], though our strategy was apparently different. For a further check of the validity of our OPE, it is desirable to show the crossing symmetry of the four-point functions by using the OPE, as has been done for the \( H_3^+ \) WZW model [30].

A main problem in the \( SL(2, R) \) WZW model has been to determine the correct spectrum. A recent proposal in [18] requires that the spin \( j \) in the discrete series be in the ‘unitarity bound’ \(-(k - 1)/2 < j < -1/2 \). Having a look at the OPE in (3.14)-(3.17) with \( \hat{D}_j^\pm, \hat{C}_j^\alpha \), we then find a puzzle: The OPE of the type (3.14) seems to break the unitarity bound. We remark that such a puzzle exits independently of our prescription, because any correct OPE should include that kind of contributions at least for sufficiently large \( k \). Possibilities of the resolution might be, for example, as follows. (i) As in the case of the partition function [19],
there may be different expressions for the same quantity (up to some formal manipulations),
because there are infinitely many primary fields. Thus, one may be able to rewrite the OPE
of the type (3.14) in a form compatible with the unitarity bound. The spectral-flowed sectors
may appear in such a process. A similar mechanism is discussed about $\Phi_j$ in the $k \to \infty$
limit [4]. (ii) In the string theory context, the $SL(2, R)$ WZW model appears typically for
describing the strings on $AdS_3 \times S^3 \times M$. In such a case, a primary field of the model is a
tensor product of the three parts. For the $SU(2)$ part, there is a bound on the $SU(2)$ spin.
Since the $SU(2)$ and $SL(2, R)$ spins are related by the physical state conditions, the bound
on the $SU(2)$ spin may induce also the bound on the $SL(2, R)$ spin. A related discussion is
found in [32]. Thus, the unitarity bound may be maintained in the case of the string theory
on $AdS_3 \times S^3 \times M$. There is a possibility that we might be missing something important.
In any case, it seems to be important to investigate this problem further.

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I would like to dedicate this paper to the memory of Professor Sung-Kil Yang.

A Useful formulas

In appendix A, we collect the formulas used in the main text.

(1) In [7], the following integral has been carried out:

$$I = \int d^2 z d^2 w z^\alpha (1 - z)^\beta \bar{z}^\alpha (1 - \bar{z})^\beta w'^\gamma (1 - w)' \bar{w}'^\gamma (1 - \bar{w})'|z - w|^{4\sigma}$$

$$= (i/2)^2 \left\{ C^{12}[\alpha_i, \alpha'_i] P^{12}[\bar{\alpha}_i, \bar{\alpha}'_i] + C^{21}[\alpha_i, \alpha'_i] P^{21}[\bar{\alpha}_i, \bar{\alpha}'_i] \right\}. \quad (A.1)$$

Here,

$$C^{ab}[\alpha_i, \alpha'_i] = \frac{\Gamma(1 + \alpha_a + \alpha'_a - k') \Gamma(1 + \alpha_b + \alpha'_b - k') \Gamma(k' - \alpha_c - \alpha'_c)}{\Gamma(k' - \alpha_c - \alpha'_c)} G \left[ \frac{\alpha'_a + 1, \alpha'_b + 1, k' - \alpha_c - \alpha'_c}{\alpha'_a - \alpha_c + 1, \alpha'_b - \alpha'_c + 1} \right],$$

$$G \left[ a, b, c \atop e, f \right] = \frac{\Gamma(a) \Gamma(b) \Gamma(c)}{\Gamma(e) \Gamma(f)} F \left[ a, b, c \atop e, f \right], \quad F \left[ a, b, c \atop e, f \right] = _3F_2(a, b, c; e, f; 1), \quad (A.2)$$

$$\alpha_1 = \alpha, \quad \alpha_2 = \beta, \quad \alpha_3 = \gamma, \quad \alpha + \beta + \gamma + 1 = k' = -2\sigma - 1,$$

$$\alpha'_1 = \alpha', \quad \alpha'_2 = \beta', \quad \alpha'_3 = \gamma', \quad \alpha' + \beta' + \gamma' + 1 = k' = -2\sigma - 1.$$
Among $F$ and $\bar{C}$, $(\gamma, \gamma', \bar{\gamma}, \bar{\gamma'})$ are determined through the above type of equations. $P^{12}$ and $P^{21}$ are expressed by $C^{ab}$ as

$$(i/2)^2 \left[ \frac{P^{12}}{P^{21}} \right] = A_\beta \left[ \frac{C^{23}}{C^{32}} \right] = A^T_{\alpha} \left[ C^{31}_{\alpha} \right], \quad A_\alpha = \left[ \begin{array}{cc} s(\alpha)s(\alpha') & -s(\alpha)s(\alpha' - k') \\ -s(\alpha')s(\alpha - k') & s(\alpha)s(\alpha') \end{array} \right],$$

(A.3)

with $s(x) = \sin(\pi x)$. Note that $C^{ba}$ and, hence, $P^{ba}$ are obtained by exchanging $\alpha_i$ and $\alpha'_i$ in $C^{ab}$ and $P^{ab}$. Namely,

$$C^{ba} = C^{ab}(\alpha_i \leftrightarrow \alpha'_i), \quad P^{ba} = P^{ab}(\alpha_i \leftrightarrow \alpha'_i).$$

(A.4)

(2) Among $F^{[a,b,c]}_{\ [e,f]}$ or $G^{[a,b,c]}_{\ [e,f]}$, many relations are known [33]. For example,

$$G_{\ [a,b,c]}^{\ [e,f]} = \frac{\Gamma(b)\Gamma(c)}{\Gamma(e-a)\Gamma(f-a)} \frac{e-a, f-a, u}{u+b, u+c}$$

$$= \frac{\Gamma(b)\Gamma(c)\Gamma(u)}{\Gamma(f-a)\Gamma(e-b)\Gamma(e-c)} \frac{a, e-b, e-c}{e, a+u},$$

$$G_{\ [a,b,c]}^{\ [e,f]} = \frac{s(e-c)s(f-c)}{s(a)s(b-c)} \frac{b, 1+b-e, 1+b-f}{1+b-c, 1+b-a}$$

$$+ \frac{s(e-c)s(f-c)}{s(a)s(b-c)} \frac{c, 1+c-e, 1+c-f}{1+c-b, 1+c-a},$$

(A.5)

where $u = e + f - a - b - c$. The two-term relations in (A.5) generate identities among ten apparently different expressions. As discussed in appendix C, we need the value of $u$ to study the analytic structure of $F$ and $G$. It changes as $u \rightarrow a \rightarrow f - a$ in (A.5), whereas $u$ is invariant in (A.6). (If we substitute the parameters in (2.8) into $\alpha_i, \alpha'_i$ in (A.2), $u = -j_{12}$ for any $C^{ab}[\alpha_i, \alpha'_i]$.)

(3) From the properties of $\Upsilon(x)$, it follows that

$$D(j_1, j_2, j_3) = \pi \Delta(-j_{13})\Delta(-j_{23})\Delta(2j_3 + 1)R(j_3)D(j_1, j_2, -j_3 - 1),$$

(A.7)

with $R(j) = B(j)/A(j)$. Note that $D(j_0)$ are symmetric with respect to $j_1, j_2, j_3$.

(4) The tensor products of the normalizable unitary representations of $SL(2, R)$ are given by [34, 29, 35]

$${\mathcal D}_{j_1}^{\pm} \otimes {\mathcal D}_{j_2}^{\pm} = \sum_{j \leq j_1 + j_2} \bigoplus {\mathcal D}_{j}^{\pm},$$

$${\mathcal D}_{j_1}^{+} \otimes {\mathcal D}_{j_2}^{-} = \int_{P^+} dj \, C_{j}^{\alpha} \bigoplus \sum_{j_1-j_2 \leq j < -\frac{1}{2}} {\mathcal D}_{j}^{+} \bigoplus \sum_{j_2-j_1 \leq j < -\frac{1}{2}} {\mathcal D}_{j}^{-},$$

$${\mathcal D}_{j_1}^{\pm} \otimes C_{j_2}^{\alpha_2} = \int_{P^+} dj \, C_{j}^{\alpha} \bigoplus \sum_{j < -\frac{1}{2}} {\mathcal D}_{j}^{\pm},$$

$${\mathcal C}_{j_1}^{\alpha_1} \otimes C_{j_2}^{\alpha_2} = \int_{P^+} dj \, C_{j}^{\alpha} \bigoplus \int_{P^+} dj' \, C_{j'}^{\alpha'} \bigoplus \sum_{j < -\frac{1}{2}} \left( {\mathcal D}_{j}^{+} \otimes {\mathcal D}_{j}^{-} \right),$$

(A.8)

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with $\mathcal{P}_+ = -1/2 + i\mathbb{R}_{>0}$. Here, $j$ for the discrete series is summed with integer spacing. $\alpha, \alpha'$ for the continuous series and $j$ for the discrete series should take the values so that they are compatible with the conservation of $J^3_0$.

**B  Rewriting $W(j_a; m_a)$**

We discuss various expressions of $W(j_a; m_a)$ in appendix B. First, we rewrite $I$ in (A.1). Applying the three-term relation (A.6) to $C^{12}$ in (A.2) (with $'a' = k' - \gamma - \gamma'$), we obtain

$$s(\beta)s(\gamma - \beta')C^{12} = s(\gamma)s(\gamma' - k')C^{13} - s(\alpha)s(\beta')C^{21}. $$

A similar expression for $C^{31}$ is also obtained from (A.4). These allow us to express $P^{12}$ in terms of $C^{12}$ and $C^{21}$:

$$(i/2)^2 P^{12} = D_1C^{12} + D_3C^{21}, \quad (B.1)$$

where

$$D_1 = \frac{s(\alpha')s(\beta)}{s(\gamma)s(\gamma')(\gamma + \gamma' - k')} \left\{s(\alpha)s(\alpha')s(\gamma) - s(\gamma)s(\alpha - k')s(\gamma - \beta')\right\},$$

$$D_3 = -\frac{s(\alpha)s(\alpha')s(\beta')s(\gamma + \gamma' + k')}{s(\gamma)s(\gamma')(\gamma + \gamma' - k')}. \quad (B.2)$$

The corresponding expression for $P^{21}$ is given by (A.4). Hence, the integral $I$ becomes

$$I = D_1C^{12}[\alpha_i, \alpha'_i]C^{12}[\bar{\alpha}_i, \bar{\alpha}'_i] + D_2C^{21}[\alpha_i, \alpha'_i]C^{21}[\bar{\alpha}_i, \bar{\alpha}'_i] \quad (B.3)$$

$$+ D_3 \left\{C^{12} [\alpha_i, \alpha'_i]C^{21}[\bar{\alpha}_i, \bar{\alpha}'_i] + C^{21}[\alpha_i, \alpha'_i]C^{12}[\bar{\alpha}_i, \bar{\alpha}'_i] \right\},$$

with $D_2 = D_1(\alpha_i \leftrightarrow \alpha'_i)$. From (B.3), we find that $I$ is symmetric with respect to $(\alpha_i, \alpha'_i)$ and $(\bar{\alpha}_i, \bar{\alpha}'_i)$, as it should be, when $\alpha_i - \bar{\alpha}_i, \alpha'_i - \bar{\alpha}'_i \in \mathbb{Z}$.

Using the above result, $W(j_a; m_a)$ is rewritten as in (2.12). Furthermore, $W(j_a; m_a)$ is expressed in various ways, since $C^{ab}$ can also take various forms because of, e.g., the identities (A.5). Which form is useful depends on each discussion. Thus, we list several ones in addition to those in (2.11):

$$C^{12} = \frac{\Gamma(-N)\Gamma(-j_{12})\Gamma(1 + j_2 + m_2)\Gamma(1 + j_3 - m_3)}{\Gamma(1 + j_{23})\Gamma(-j_1 - m_1)\Gamma(-j_3 - m_3)} \left[\begin{array}{c} -j_1 - m_1, -j_{12}, 1 + j_{23} \\ -2j_1, -j_1 + j_3 + m_2 + 1 \end{array}\right]$$

$$\times G \left[\begin{array}{c} -j_3 - m_3, 1 + j_3 - m_3, -j_1 + m_1 \\ -j_1 + j_2 - m_3 + 1, -j_1 - j_2 - m_3 \end{array}\right].$$

$$C^{31} = \frac{\Gamma(-j_{12})\Gamma(1 + j_1 + m_1)\Gamma(1 + j_2 - m_2)\Gamma(1 + j_3 + m_3)}{\Gamma(1 + N)\Gamma(1 + j_{23})\Gamma(-j_2 - m_2)} \left[\begin{array}{c} 1 + N, 1 + j_{23}, 1 + j_3 - m_3 \\ j_2 + j_3 + m_1 + 2, 2 + 2j_3 \end{array}\right]. \quad (B.4)$$
For $C^{12}$, $u$ are $-j_3 - m_3$ and $-j_1 - m_1$ in the first and second lines, respectively, whereas it is $-j_2 - m_2$ for $C^{31}$. Similar expressions for $C^{21}$ and $C^{13}$ are obtained through (A.4).

C Analytic structure of $F$

Here, we summarize the analytic structure of $F^{[a,b,c]}_{e,f}$. It may be instructive to first consider $\, _2F_1(a, b; c; 1)$ defined by

$$\, _2F_1(a, b; c; 1) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(n+1)}. \quad (C.1)$$

This has obvious poles at $c \in \mathbb{Z}_{\leq 0}$. This series is convergent only if $\text{Re} \,(c-a-b) > 0$, but it can be analytically continued to the region $\text{Re} \,(c-a-b) \leq 0$, e.g., by using its integral representation. One then expects that $\, _2F_1(a, b; c; 1)$ develops additional poles at $c-a-b \in \mathbb{Z}_{\leq 0}$. This is consistent with Gauss’ formula: $\, _2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. Note that the residues can vanish in special cases because of the denominator.

Now, let us turn to the case of $F^{[a,b,c]}_{e,f}$, which is defined by

$$F^{[a,b,c]}_{e,f} = \frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(b)\Gamma(c)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+n)\Gamma(f+n)\Gamma(n+1)}. \quad (C.2)$$

Note that this series generically reduces to a finite sum with $(n+1)$ terms when $a, b$ or $c$ takes a non-positive integer $-n$. A consideration similar to the above shows that the series is convergent only if $\text{Re} \, u > 0$, and generically

$$F^{[a,b,c]}_{e,f} \text{ has simple poles at } e, f, u \in \mathbb{Z}_{\leq 0}. \quad (C.3)$$

This indeed agrees with the poles appearing in the integral expression of $C^{ab}$ in [7]. We remark that the poles coming from $u \in \mathbb{Z}_{\leq 0}$ are absent when the series reduces to a finite sum.

From the above discussion, it follows that: (i) in (2.10), $F$’s in $C^{12}$ and in a similar expression for $C^{21}$ are regular when $j_1 + m_1 \in \mathbb{Z}_{\geq 0}$ and other parameters are generic; (ii) in (2.13), $\frac{\Gamma(1+j_3-m_3)}{\Gamma(1+j_3-m_3-n_3)} \, _2F_1[-n_1-j_2; 1+j_3-m_3-n_3, -2j_1; 1+j_3-m_3-n_3]$ is regular for generic $j_1$; (iii) in (3.4), $F$’s in the second expression of $C^{12}$, in $C^{31}$ and in similar expressions for $C^{21}, C^{13}$ are regular for generic $j_1, j_2 \in \mathcal{P}_+$ and $(j_3, m_3) \in \mathcal{D}_j^\pm$ or $\mathcal{C}_j^\alpha$.

Although we do not have the formula for $\, _3F_2$ corresponding to Gauss’ formula, simple expressions of $\, _3F_2$ in terms of the gamma functions are known in special cases. In particular, from Saalschütz’s theorem, we obtain [33]

$$F^{[a,b,c]}_{e,f} = \frac{\Gamma(e)\Gamma(1+a-f)\Gamma(1+b-f)\Gamma(1+c-f)}{\Gamma(1-f)\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}, \quad (C.4)$$

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when $u = 1$ and one of $a, b, c$ is a negative integer $-n$. This reduces to Gauss’ formula for $n \to \infty$. Applying (C.3) to $F$ in $C^{12}$ gives

$$
\frac{\Gamma(1+j_3-m_3)}{\Gamma(1+j_3-m_3-n_1)} F\left[-n_1, -j_{12}, 1+j_{23}\right] = \frac{\Gamma(1+2j_1-n_1)\Gamma(1+j_{13})\Gamma(1+N)}{\Gamma(1+2j_1)\Gamma(m_2-j_2)\Gamma(1+j_2+m_2)},
$$

(C.5)

for $-j_3 - m_3 = 1$, $j_1 + m_1 = n_1$ and $\Sigma m_a = 0$.

D From $W_1^0$ to $W_1$

In appendix D, we sketch the derivation of (2.13) by acting with $J_0^a$, $J_0^b$ on (2.16). We follow the argument in [20]. First, we note an identity

$$
\left\langle e^{\alpha J_0} \phi_{-j_1-j} e^{-\alpha J_0^+} \phi_{m_2 m_3} \phi_{m_3} \right\rangle = \left\langle \phi_{j_1-j_1} e^{\alpha J_0} \phi_{m_2 m_2} \phi_{m_3} e^{-\alpha J_0^+} \right\rangle .
$$

(D.1)

This gives

$$
\sum_{n_1=0}^{\infty} \frac{(-\alpha)^{n_1}}{n_1!} \frac{\Gamma(2j_1+1)}{\Gamma(2j_1+1-n_1)} \left\langle \phi_{j_1-n_1-j_1} \phi_{j_2 m_2} \phi_{j_3 m_3} \right\rangle
$$

$$
= \sum_{n_2,n_3=0}^{\infty} \frac{\alpha^{n_2+n_3}}{n_2! n_3!} \frac{\Gamma(j_2-m_2+1)}{\Gamma(j_2-m_2+1-n_2)} \frac{\Gamma(j_3-m_3+1)}{\Gamma(j_3-m_3+1-n_3)} \left\langle \phi_{j_1-j_2-n_2} \phi_{j_2 m_2} \phi_{j_3 m_3} \right\rangle .
$$

(D.2)

Equating the terms on both sides with the same power in $\alpha$, we obtain the expression of the three-point functions with $n_1 \neq 0$ in terms of that with $n_1 = 0$. Taking into account the right sector, we find that

$$
\left\langle \phi_{j_1-j_1+n_1-j_2} \phi_{j_2 m_2} \phi_{j_3 m_3} \right\rangle = (2\pi)^2 \delta^2(-j_1 + n_1 + m_2 + m_3) C_1(j_a, m_a) C_2(j_a, m_a)
$$

$$
\times \pi^2 \Delta(-N) \Delta(2j_1+1) \Delta(1+j_{12}) \Delta(1+j_{13}) D(-j_a-1) ,
$$

(D.3)

where

$$
C_1(j_a, m_a) = \frac{\Gamma(-2j_1)}{\Gamma(m_1-j_1)} \frac{\Gamma(j_2+m_2+1)}{\Gamma(j_3-m_3+1-n_1)} \frac{\Gamma(j_3-m_3+1)}{\Gamma(-j_3-m_3-n_1)} \times F\left[-j_1-m_1, -j_2+m_2, j_2+m_2+1\right] \times j_1+j_3+m_2+1, -j_1-j_3+m_2\.view

$$

(D.4)

$$
C_2(j_a, m_a) = \frac{s(j_2+m_2)}{s(j_3+m_3+n_1)} C_1(j_a, m_a).
$$

In obtaining $C_1$, we have used $\frac{\Gamma(\alpha)}{\Gamma(\alpha-n)} = (-)^n \frac{\Gamma(1-\alpha+n)}{\Gamma(1-\alpha)}$ repeatedly for the first line, and the second formula in (A.3) for the second line. Comparing (D.3) with (2.7), we confirm that $W_1$ is correctly recovered.

In [20], the quantity corresponding to $C_1 C_2$ in (D.3) is $|C_2(j_a, m_a)|^2$. This is because the quantity corresponding to $W_1^0$ has a different phase (in addition to $m_a = \bar{m}_a$).
References

[1] K. Gawędzki, Nucl. Phys. B 328 (1989) 733; “Non-compact WZW conformal field theories”, NATO ASI: Cargese 1991: 0247-274, hep-th/9110076.

[2] J. Teschner, Nucl. Phys. B 546 (1999) 390, hep-th/9712256; Nucl. Phys. B 546 (1999) 369, hep-th/9712258.

[3] J. Teschner, Nucl. Phys. B 571 (2000) 555, hep-th/9906215.

[4] N. Ishibashi, K. Okuyama and Y. Satoh, Nucl. Phys. B 588 (2000) 149, hep-th/0005152.

[5] K. Hosomichi, K. Okuyama and Y. Satoh, Nucl. Phys. B 598 (2001) 451, hep-th/0009107.

[6] K. Hosomichi and Y. Satoh, “Operator product expansion in string theory on $AdS_3$”, hep-th/0105283.

[7] T. Fukuda and K. Hosomichi, JHEP 0109 (2001) 003, hep-th/0105217.

[8] J. Balog, L. O’Raifeartaigh, P. Forgacs and A. Wipf, Nucl. Phys. B 325 (1989) 225.

[9] P.M. Petropoulos, Phys. Lett. B 236 (1990) 151.

[10] N. Mohammadi, Int. J. Mod. Phys. A 5 (1990) 3201.

[11] S. Hwang, Nucl. Phys. B 354 (1991) 100; M. Henningson, S. Hwang, P. Roberts and B. Sundborg, Phys. Lett. B 267 (1991) 350.

[12] I. Bars, Phys. Rev. D 53 (1996) 3308, hep-th/9503205.

[13] Y. Satoh, Nucl. Phys. B 513 (1998) 213, hep-th/9705208.

[14] J.M. Evans, M.R. Gaberdiel and M.J. Perry, Nucl. Phys. B 535 (1998) 152, hep-th/9806024.

[15] A. Giveon, D. Kutasov and N. Seiberg, Adv. Theor. Math. Phys. 2 (1998) 733, hep-th/9806194; D. Kutasov and N. Seiberg, JHEP 9904 (1999) 008, hep-th/9903219.

[16] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, JHEP 9812 (1998) 026, hep-th/9812046.

[17] G. Giribet and C. Núñez, JHEP 9911, 031 (1999), hep-th/9909149.

[18] J. Maldacena and H. Ooguri, J. Math. Phys. 42 (2001) 2929, hep-th/0001053; J. Maldacena, H. Ooguri and J. Son, J. Math. Phys. 42 (2001) 2961, hep-th/0005183.

[19] A. Kato and Y. Satoh, Phys. Lett. B486 (2000) 306, hep-th/0001063.

[20] A.L. Larsen and N. Sánchez, Phys. Rev. D 62 (2000) 046003, hep-th/0001180.

[21] I. Pesando, “Some remarks on the free fields realization of the bosonic string on $AdS_3$”, hep-th/0003036.
[22] Y. Hikida, K. Hosomichi and Y. Sugawara, Nucl. Phys. B 589 (2000) 134, hep-th/0005065.
[23] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. B 477 (1996) 577, hep-th/9506136.
[24] A. Giveon and D. Kutasov, JHEP 9910 (1999) 034, hep-th/9909110; JHEP 0001 (2000) 023, hep-th/9911039.
[25] G. Giribet and C. Núñez, JHEP 0106 (2001) 010, hep-th/0105200.
[26] K. Becker and M. Becker, Nucl. Phys. B 418 (1994) 206, hep-th/9310046.
[27] V.S. Dotsenko and V.A. Fateev, Nucl. Phys. B 240 (1984) 312.
[28] J. Maldacena and H. Ooguri, “Strings in AdS3 and the SL(2,R) WZW model. III: Correlation functions”, hep-th/0111180.
[29] W.J. Holman and L.C. Biedenharn, Ann. Phys. 39 (1966) 1; Ann. Phys. 47 (1968) 205.
[30] J. Teschner, Phys. Lett. B 521 (2001) 127, hep-th/0108121.
[31] D. Gepner and E. Witten, Nucl. Phys. B 278 (1986) 493.
[32] P.M. Petropoulos and S. Ribault, JHEP 0107 (2001) 036, hep-th/0105252.
[33] L.J. Slater, “Generalized hypergeometric functions”, Cambridge Univ. Press, Cambridge (1966).
[34] N.Ja. Vilenkin and A.U. Klimyk, “Representation of Lie groups, and special functions”, Vol. 1, Kluwer Academic Publishers, Dordrecht (1991).
[35] K.-H. Wang, J. Math. Phys. 11 (1970) 2077.