Super-resolution in high-contrast media

Habib Ammari and Hai Zhang

Department of Mathematics and Applications, Ecole Normale Supérieure, 45 Rue d’Ulム, Paris 75005, France

A mathematical theory is developed to explain the super-resolution and super-focusing in high-contrast media. The approach is based on the resonance expansion of the Green function associated with the medium. It is shown that the super-resolution is due to sub-wavelength resonant modes excited in the medium which can propagate into the far-field.

1. Introduction

It is well known that the resolution in the homogeneous space for far-field imaging system is limited by half the operating wavelength, which is a direct consequence of Abbe’s diffraction limit. In order to differentiate point sources which are located less than half the wavelength apart, super-resolution techniques have to be used.

While many techniques exist in practice, here we are only interested in the one using resonant media. The resolution enhancement in resonant media has been demonstrated in various recent experiments [1–5]. The basic idea is the following: suppose that we have sources that are placed inside a domain of typical size of the order of the wavelength of the wave the sources can emit, and we want to differentiate them by making measurements in the far-field. While this is impossible in the homogeneous space, it is possible if the medium around these sources is changed so that the point spread function [6, p. 35], which is the imaginary part of the Green function in the new medium, displays a much sharper peak than the homogeneous one and thus can resolve sub-wavelength details [7]. The key issue in such an approach is to design the surrounding medium so that the corresponding Green function has the tailored property.

In this paper, we develop the mathematical theory for realizing this approach by using high-contrast media. We show that in high-contrast media, the super-resolution is due to the propagating sub-wavelength resonant modes excited in the media and is limited by the finest structure in these modes. It is worth emphasizing that this
mechanism is similar to the one using Helmholtz resonators, which was recently investigated in [2,8]. In [9], a simple one-dimensional framework is considered. The resolution enhancement is quantified and the image stability is discussed.

Enhanced refocusing can follow from other mechanisms. The use of an active sink that absorbs the time-reversed wave at the source location allows the resolution limit to be overcome [10]. On the other hand, focusing behind the diffraction limit with far-field time reversal is also possible provided the medium in the near field of the source has a high-effective index. This can be achieved by placing a random distribution of scatterers in the vicinity of the source in order to reduce the effective wavelength; see, for instance, [11].

Super-resolution has many applications in various fields. Besides the well-known ones in fluorescence imaging in molecular biology and in medical imaging [12], it has also been intensively investigated in the field of nanophotonics as a possible technique to focus electromagnetic radiation in a region of the order of a few nanometres beyond the diffraction limit of light and thereby causing an extraordinary enhancement of the electromagnetic fields [13].

The paper is organized as follows. In §2, we recast the imaging problem as an inverse source problem and outline different approaches for solving the inverse source problem. We emphasize that time-reversal is a direct imaging method while \( L^2 \)- and \( L^1 \)-minimization methods are post-imaging processes by using \textit{a priori} information. By investigating the procedures of \( L^1 \) and \( L^2 \) minimization methods, we see that raw images based on time-reversal (or refocusing) are formed first, and then are processed by computational optimization which takes into account the \textit{a priori} information on the objects to be imaged. A proper \textit{a priori} information would yield better images than the raw ones [14]. In §3, we derive expansions of the Green function in a high-contrast medium and provide a mathematical foundation for the super-resolution, which is the counterpart of super-focusing. The paper ends with a short discussion.

2. Inverse source problems

We consider the following inverse source problem in a general medium characterized by refractive index \( n(x) \):

\[
\Delta u + k^2 n(x) u = f, \\
\text{ } u \text{ satisfies the Sommerfeld radiation condition.}
\]

We assume that \( n - 1 \) is compactly supported in a bounded domain \( D \subset \mathbb{R}^d \) for \( d = 2, 3 \), and is assumed to be known. We are interested in imaging \( f \), which can be either a function in \( L^2(D) \) or consists of a finite number of point sources supported in \( D \), from the scattered field \( u \) in the far-field. Denote by \( G(x, y, k) \) the corresponding Green function for the media, that is, the solution to

\[
\Delta G(x, y, k) + k^2 n(x) G(x, y, k) = \delta(x - y), \\
G \text{ satisfies the Sommerfeld radiation condition}
\]

with \( \delta \) being the Dirac mass, we have

\[
u(x) = K_D[f](x) := \int_D G(x, y, k) f(y) \, dy.
\]

The inverse source problem of reconstructing \( f \) from \( u \) for fixed frequency is well known to be ill-posed for general sources; see, for instance, [6,15,16]. While there are many methods of reconstructing \( f \) from \( u \), we concentrate on the following three most common ones in the literature:

(1) time-reversal-based method;
(2) minimum \( L^2 \)-norm solution; and
(3) minimum \( L^1 \)-norm solution.
(a) Time-reversal-based method

We first present some basics about the time-reversal-based method. The imaging functional is given as follows:

\[ \mathcal{I}(x) = \int_{\Gamma} G(x, z, k) u(z) \, ds(z) = K_D^* K_D[f](x), \]  

(2.1)

where \( \Gamma \) is a closed surface in the far-field where the measurements are taken, and \( K_D^* K_D \) is the adjoint of \( K_D \) viewed as a linear operator from the space \( L^2(D) \) to \( L^2(\Gamma) \). The physical meaning of the operator \( K_D^* K_D \) is to time-reversing (or focusing) the observed field. This imaging method is the simplest and perhaps the mostly used one in practice.

The resolution of this imaging method can be derived from the following Helmholtz–Kirchhoff identity:

\[ \int_{\Gamma} \left( \frac{\partial G(x, z, k)}{\partial v} G(y, z, k) - \frac{\partial G(x, z, k)}{\partial v} G(y, z, k) \right) \, ds(z) = -2i \Im G(x, y, k), \quad \forall x, y \in D. \]  

(2.2)

Note that in the far-field, we can use the Sommerfeld radiation condition, as a result, we obtain the following lemma.

**Lemma 2.1.** Let \( G \) be the Green function and let \( \Gamma \) be a smooth closed surface. We have

\[ k \int_{\Gamma} G(x, z, k) G(y, z, k) \, ds(z) = -\Im G(x, y, k) + O \left( \frac{1}{R} \right), \]  

(2.3)

where \( R \) is the distance between the far-field surface \( \Gamma \), where the measurements are taken, and \( D \), where the sources are located.

As a corollary, the following result holds.

**Corollary 2.2.** We have

\[ \mathcal{I}(x) = K_D^* K_D[f](x) \approx -\frac{1}{k} \int_{\mathcal{D}} \Im G(x, y, k) f(y) \, dy. \]

If we take \( f \) to be a point source, we obtain the point spread function of the imaging functional, which shows that the time-reversal based method has resolution limited by \( \Im G(x, y, k) \).

(b) Minimum \( L^2 \)-norm solution

We now consider the second method which is based on \( L^2 \)-minimization. We assume that the source \( f \in L^2(D) \). The method is given as follows:

\[ \min \|g\|_{L^2(D)} \quad \text{subject to} \quad K_D[g] = u, \]  

(2.4)

which can be relaxed in the presence of noise as follows:

\[ \min \|g\|_{L^2(D)} \quad \text{subject to} \quad \|K_D[g] - u\|_{L^2(\Gamma)}^2 < \delta \]  

(2.5)

with \( \delta > 0 \) being a given small parameter.

In order to obtain an explicit formula for this method, we consider the singular value decomposition for the operator

\[ K_D : L^2(D) \to L^2(\Gamma). \]

We have

\[ K_D = \sum_{l \geq 0} \sigma_l P_l, \]

where \( \sigma_l \) is the \( l \)th singular value and \( P_l \) is the associated projection. The ill-posedness of the inverse source problem is due to the fast decay of the singular values to zero; see, for instance, [15,17].
By a direct calculation, one can show that the minimum $L^2$-norm solution to (2.4) is given by

$$I(x) = \sum_{l \geq 0} \frac{\hat{P}_l^* P_l}{\sigma_l^2} K_D^* K_D [f](x),$$

while the regularized one, which is the solution to (2.5) is given by

$$I_\alpha(x) = \sum_{l \geq 0} \frac{\hat{P}_l^* P_l}{\sigma_l^2 + \alpha} K_D^* K_D [f](x),$$

with $\alpha$ as a function of $\delta$ introduced in (2.5) being chosen by Morozov’s discrepancy principle; see, for instance, [18].

(c) Minimum $L^1$-norm solution

The method of minimum $L^1$-norm solution is proposed by Candès & Fernandez-Granda in the recent papers [19,20]. The authors assume that $f$ is equal to superposition of separate point sources. Their method is to solve the minimization problem

$$\min \|g\|_{L^1(D)} \quad \text{subject to} \quad K_D^* K_D [g] = K_D^*[u],$$

or its relaxed version, which reads as

$$\min \|g\|_{L^1(D)} \quad \text{subject to} \quad \|K_D^* K_D [g] - K_D^*[u]\|_{L^2(\Gamma)}^2 < \delta.$$

They show that under a minimum separation condition for the point sources, the inverse source problem is well posed. A main feature of their approach is that the $L^1$-minimization can pull out small spikes even though they may be completely buried in the side lobes of the large ones.

It is worth emphasizing that without any a priori information, the resolution of the raw image, which is obtained by time-reversal method, is determined by the imaginary part of the Green function in the associated media. This can be regarded as a generalization of Abbe’s diffraction limit.

(d) The special case of homogeneous medium

In a homogeneous medium, we have $n \equiv 1$. For simplicity, we consider the case $d = 3$.

$$G(x, y, k) = G_0(x, y, k) = -\frac{e^{ik|x-y|}}{4\pi |x-y|}.$$

In the far-field, where $k|y| = O(1)$ and $k|x| \gg 1$, we have $|x-y| \approx |x| - \hat{x} \cdot y$, where $\hat{x} = x/|x|$. Thus,

$$u(x) = -\int_D \frac{e^{ik|x-y|}}{4\pi |x-y|} f(y) \, dy \approx -\frac{e^{ik|x|}}{4\pi |x|} \hat{f} (\hat{x}),$$

where $\hat{f}$ is the Fourier transform of $f$.

If we make measurements on the surface $\Gamma = S(0, R)$, the sphere of radius $R$ and centre the origin, then we have

$$u(x) = \frac{e^{ikR}}{4\pi R} \hat{f}(\hat{x}).$$

Using the time-reversal method, we have for $R$ large enough

$$I(z) \approx \frac{1}{16\pi^2 R^2} \int_{|x|=R} \int_D e^{ik\hat{x} \cdot (y-z)} f(y) \, dy \, ds(x) = \frac{1}{4\pi} \int_D f(y) \frac{\sin k|z-y|}{k|z-y|} \, dy,$$

where the imaging functional $I$ is defined by (2.1) with $\Gamma = \{|x| = R\}$. 
3. The Green function in high-contrast media

Throughout this section, we put the wavenumber $k$ to be the unit and suppress its presence in what follows. We assume that the wave speed in the free space is one. The free-space wavelength is given by $2\pi$. We consider the following Helmholtz equation with a delta source term:

$$
\Delta_x G(x,x_0) + G(x,x_0) + \tau n(x)\chi_D(x)G(x,x_0) = \delta(x-x_0) \quad \text{in } \mathbb{R}^d,
$$

(3.1)

where $\chi_D$ is the characteristic function of a bounded domain $D \subset \mathbb{R}^d$, which has size of order of the free-space wavelength, $n(x)$ is a positive function of order one in the space of $C^1(D)$ and $\tau \gg 1$ is the contrast. We denote by $G_0(x,x_0)$ the free-space Green’s function.

Write $G = v + G_0$, we can show that

$$
\Delta v + v = -\tau n(x)\chi_D(v + G_0).
$$

(3.2)

Thus,

$$
v(x,x_0) = -\tau \int_D n(y)G_0(x,y)(v(y,x_0) + G_0(y,x_0)) \, dy.
$$

Define

$$
K_D[f](x) = -\int_D n(x)G_0(x,y)f(y) \, dy.
$$

(3.3)

Then, $v = v(x) = v(x,x_0)$ satisfies the following integral equation:

$$(I - \tau K_D)[v] = \tau K_D[G(\cdot,x_0)],
$$

(3.4)

and hence,

$$
v(x) = \left(\frac{1}{\tau} - K_D\right)^{-1} K_D[G(\cdot,x_0)].
$$

We define $H^2(D)$ as the space of functions $u \in L^2(D)$ such that $\partial_i u, \partial_i^2 u \in L^2(D)$. We let $H^2_{\text{loc}}(\mathbb{R}^d)$ to be the set of $H^2(K)$ for any $K \subset \mathbb{R}^d$.

In what follows, we present properties of the integral operator $K_D$.

**Lemma 3.1.** The operator $K_D$ is compact from $L^2(D)$ to $L^2(D)$. In fact, $K_D$ is bounded from $L^2(D)$ to $H^2(D)$. Moreover, $K_D$ is a Hilbert–Schmidt operator.

**Lemma 3.2.** Let $\sigma(K_D)$ be the spectrum of $K_D$ defined by (3.3). We have

(i) $\sigma(K_D) = \{0, \lambda_1, \lambda_2, \ldots, \lambda_m, \ldots\}$, where $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ and $\lambda_m \to 0$;

(ii) $\{0\} = \sigma(K_D) \setminus \sigma_p(K_D)$ with $\sigma_p(K_D)$ being the point spectrum of $K_D$.

Proof. We need only to prove the second assertion. Assume that $K_D[u] = \int_D G_0(x,y)n(y)u(y) \, dy = 0$. We have $0 = (\Delta + 1)K_D[u] = nu$, which shows that $u = 0$. The assertion is then proved.

**Lemma 3.3.** Let $K_D$ be defined by (3.3). Then, $\lambda \in \sigma(K_D)$ if and only if there is a non-trivial solution in $H^2_{\text{loc}}(\mathbb{R}^d)$ to the following problem:

$$
(\Delta + 1)u(x) = \frac{1}{\lambda} n(x)u(x) \quad \text{in } D,
$$

(3.5)

$$
(\Delta + 1)u = 0 \quad \text{in } \mathbb{R}^d \setminus D,
$$

(3.6)

$$
u \text{ satisfies the Sommerfeld radiation condition.}
$$

(3.7)

Proof. Assume that $K_D[u] = \lambda u$. We define $\tilde{u}(x) = \int_D G_0(x,y)n(y)u(y) \, dy$, where $x \in \mathbb{R}^d$. Then $\tilde{u}$ satisfies the required equations.

\[\square\]
We call functions satisfying (3.5) and (3.6) the resonant modes. They have sub-wavelength structures in $D$ for $|\lambda| < 1$ and can propagate into the far-field. It is these sub-wavelength propagating modes that causes super-resolution. We may also call them super-oscillatory modes.

**Remark 3.4.** It is clear that $\lambda$ is a non-zero real eigenvalue for the operator $K_D$ if and only if $1$ is a transmission eigenvalue for the medium characterized by $1 - (1/\lambda)n(x)$, i.e. there exists a non-trivial solution to the following equations:

$$
\begin{align*}
\left(\Delta + 1 - \frac{1}{\lambda}n(x)\right) u(x) &= 0 \quad \text{in } D, \\
(\Delta + 1)u &= 0 \quad \text{in } \mathbb{R}^d \setminus D, \\
u &\in H^2_\text{loc}(\mathbb{R}^d)
\end{align*}
$$

$u$ satisfies the Sommerfeld radiation condition.

We refer to [21] for a discussion on transmission eigenvalue problems.

**Lemma 3.5.** Let $\mathcal{H}_j$ denote the generalized eigenspace of the operator $K_D$ for the eigenvalue $\lambda_j$. The following decomposition holds:

$$L^2(D) = \bigcup_{j=1}^{\infty} \mathcal{H}_j.$$

**Proof.** By the same argument as the one in the proof of lemma 3.2, we can show that $\text{Ker } K_D^{\ast} = \{0\}$. As a result, we have

$$\overline{K_D(L^2(D))} = (\text{Ker } K_D^{\ast})^\perp = L^2(D).$$

The lemma is proved.

**Lemma 3.6.** There exists a basis $\{u_{j,l,k}\}, 1 \leq l \leq m_j, 1 \leq k \leq n_j$ for $\mathcal{H}_j$ such that

$$K_D(u_{j,1,1}, \ldots, u_{j,m_j,n_j}) = (u_{j,1,1}, \ldots, u_{j,m_j,n_j}) J_{j,m_j},$$

where $J_{j,l}$ is the canonical Jordan matrix of size $n_j$ in the form

$$J_{j,l} = \begin{pmatrix}
\lambda_j & 1 & & \\
& \ddots & \ddots & \\
& & \ddots & 1 \\
& & & \lambda_j
\end{pmatrix}.$$

**Proof.** This follows from the Jordan theory applied to the linear operator $K_D|_{\mathcal{H}_j} : \mathcal{H}_j \to \mathcal{H}_j$ on the finite dimensional space $\mathcal{H}_j$.

We denote $\Gamma = \{(j,l,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}; 1 \leq l \leq m_j, 1 \leq k \leq n_j\}$ the set of indices for the basis functions. We introduce a partial order on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Let $\gamma = (j,l) \in \Gamma$, $\gamma' = (j',l',k') \in \Gamma$, we say that $\gamma' \preceq \gamma$ if one of the following conditions are satisfied:

(i) $j > j'$;
(ii) $j = j', l > l'$ and
(iii) $j = j', l = l', k \geq k'$.

By Gram–Schmidt orthonormalization process, the following result is obvious.
Lemma 3.7. There exists orthonormal basis \( \{ e_y : y \in \Gamma \} \) for \( L^2(D) \) such that
\[
e_y = \sum_{y' \leq y} a_{y,y'} u_{y'},
\]
where \( a_{y,y'} \) are constants and \( a_{y,y'} \neq 0 \).

We can regard \( A = \{ a_{y,y'} \}_{y,y' \in \Gamma} \) as a matrix. It is clear that \( A \) is upper-triangular and has non-zero diagonal elements. Its inverse is denoted by \( B = \{ b_{y,y'} \}_{y,y' \in \Gamma} \) which is also upper-triangular and has non-zero diagonal elements. We have
\[
u = \sum_{y' \leq y} b_{y,y'} e_{y'}.
\]

Lemma 3.8. The functions \( \{ e_y(x)e_{y'}(y) \} \) form a normal basis for the Hilbert space \( L^2(D \times D) \). Moreover, the following completeness relation holds:
\[
\delta(x - y) = \sum_y e_y(x)e_{y'}(y).
\]

By standard elliptic theory, we have \( G(x,x_0) \in L^2(D \times D) \) for fixed \( \tau \). Thus, we have
\[
G(x,x_0) = \sum_{y,y'} \alpha_{y,y'} e_y(x)e_{y'}(x_0), \tag{3.8}
\]
for some constants \( \alpha_{y,y'} \) satisfying
\[
\sum_{y,y'} |\alpha_{y,y'}|^2 = \| G(x,x_0) \|_{L^2(D \times D)}^2 < \infty.
\]

To analyse the Green function \( G \), we need to find the constants \( \alpha_{y,y'} \). For doing so, we first note that
\[
G_0(x,x_0) = \frac{1}{n(x_0)} \kappa_D[\delta(\cdot - x_0)].
\]

Thus,
\[
G(x,x_0) = G_0(x,x_0) + \left( \frac{1}{\tau} - \kappa_D \right)^{-1} \kappa_D^2[\delta(\cdot - x_0)]
= G_0(x,x_0) + \frac{1}{n(x_0)} \sum_y \vec{e}_y(x_0) \left( \frac{1}{\tau} - \kappa_D \right)^{-1} \kappa_D^2[\delta_y].
\]

We next compute \( (1/\tau - \kappa_D)^{-1}\kappa_D^2[\delta_y] \). For ease of notation, we define \( u_{j,l,k} = 0 \) for \( k \leq 0 \). We have
\[
\kappa_D[u_{j,l,k}] = \lambda_j u_{j,l,k} + u_{j,l,k-1} \quad \text{for all } j, l, k
\]
and
\[
\kappa_D^2[u_{j,l,k}] = \lambda_j^2 u_{j,l,k} + 2\lambda_j u_{j,l,k-1} + u_{j,l,k-2} \quad \text{for all } j, l, k.
\]

On the other hand, for \( z \notin \sigma(K_D) \), we have
\[
(z - \kappa_D)^{-1}[u_{j,l,k}] = \frac{1}{z - \lambda_j} u_{j,l,k} + \frac{1}{(z - \lambda_j)^2} u_{j,l,k-1} + \cdots + \frac{1}{(z - \lambda_j)^k} u_{j,l,1}.
\]
and therefore, it follows that

\[
(z - \kappa D)^{-1}\kappa_D^{2}[u_{j,l,k}] = \frac{\lambda_j^2}{z - \lambda_j} u_{j,l,k} + \frac{\lambda_j^2}{(z - \lambda_j)^2} u_{j,l,k-1} + \frac{\lambda_j^2}{(z - \lambda_j)^3} u_{j,l,k-2} + \frac{\lambda_j^2}{(z - \lambda_j)^4} u_{j,l,k-3} + \frac{\lambda_j^2}{(z - \lambda_j)^5} u_{j,l,k-4} + \cdots + \frac{\lambda_j^2}{(z - \lambda_j)^k} u_{j,l,1}
\]

then we can obtain an equivalent expansion for the Green function in terms of the basis of bound:

\[
\sum \left\{ d_{j',y} u_{j',y} \right\},
\]

where we have introduced the matrix \( D = \{ d_{j',y} \}_{j',y \in \Gamma} \), which is upper-triangular and has block-structure.

With these calculations, by taking \( z = 1/\tau \), we arrive at the following result.

**Theorem 3.9.** The following expansion holds for the Green function

\[
G(x, x_0) = G_0(x, x_0) + \sum_{y \in \Gamma} \sum_{y'' \in \Gamma} \alpha_{y',y''} e_y(x_0) e_{y''}(x),
\]

(3.9)

where

\[
\alpha_{y',y''} = \frac{1}{n(x_0)} \sum_{y' \leq y, y'' \leq y'} a_{y',y'} d_{y',y''} b_{y'',y''}.
\]

Moreover, for \( \tau^{-1} \) belonging to a compact subset of \( \mathbb{R} \setminus (\mathbb{R} \cap \sigma(K_D)) \), we have the following uniform bound:

\[
\sum_{y',y''} |\alpha_{y',y''}|^2 < \infty.
\]

Alternatively, if we start from the identity,

\[
\delta(x - x_0) = \sum_{y''} e_{y''}(x) e_{y''}(x_0)
\]

\[
= \sum_{y''} \sum_{y' \leq y''} \sum_{y'' \leq y''} \sum_{y''} a_{y',y'} d_{y',y''} b_{y'',y''} u_{y''}(x) u_{y''}(x_0),
\]

then we can obtain an equivalent expansion for the Green function in terms of the basis of resonant modes.

**Theorem 3.10.** The following expansion holds for the Green function:

\[
G(x, x_0) = G_0(x, x_0) + \sum_{y' \in \Gamma} \sum_{y'' \leq y' \leq y } \sum_{y'' \leq y''} \sum_{y''} \beta_{y'',y',y''} u_{y'}(x) u_{y''}(x_0),
\]

(3.10)

where

\[
\beta_{y'',y',y''} = \frac{1}{n(x_0)} \sum_{y' \leq y''} a_{y'',y'} d_{y',y''}.
\]

(3.11)
Here, the infinite summation can be interpreted as follows:

$$\lim_{y_0 \to \infty} \sum_{y' \leq y_0} \sum_{y'' \leq y'} \beta_{y'',y'} u_{y'}(x) \bar{u}_{y''}(x_0) = G(x, x_0) - G_0(x, x_0) \quad \text{in } L^2(D \times D).$$

(3.12)

In order to have some idea of the expansions of the Green function $G(x, y)$, we compare them to the expansion of the Green function in the homogeneous space, i.e. $G_0(x, y)$. For this purpose, we introduce the matrix $H = \{h_{y', y}\}_{y', y \in \Gamma}$, which is defined by

$$K_D[u_y] = \sum_{y'} h_{y', y} u_{y'}.$$ 

In fact, we have

$$h_{i,j,k,l,j,k} = \lambda_j \delta_{i,j} \delta_{l,k} + \delta_{i,j} \delta_{l,k} \delta_{i,j-1,k},$$

where $\delta$ denotes the Kronecker symbol.

**Lemma 3.11.**

(i) In the normal basis $\{e_y\}_{y \in \Gamma}$, the following expansion holds for the Green function $G_0(x, x_0)$:

$$G_0(x, x_0) = \sum_{y \in \Gamma} \sum_{y'' \in \Gamma} \bar{a}_{y, y''} e_y(x_0) e_{y''}(x),$$

where

$$\bar{a}_{y, y''} = \frac{1}{n(x_0)} \sum_{y' \leq y} \sum_{y'' \leq y'} a_{y, y'} h_{y', y''} b_{y'', y''''}.$$

Moreover, we have the following uniform bound:

$$\sum_{y, y'} |\bar{a}_{y, y'}|^2 < C < \infty.$$

(ii) In the basis of resonant modes $\{u_y\}_{y \in \Gamma}$, the following expansion holds for the Green function $G_0(x, x_0)$:

$$G_0(x, x_0) = \sum_{y'' \in \Gamma} \sum_{y \leq y''} \sum_{y' \leq y''} \tilde{\beta}_{y'', y', y''} u_{y'}(x) \bar{u}_{y''}(x_0),$$

where

$$\tilde{\beta}_{y'', y', y''} = \frac{1}{n(x_0)} \sum_{y' \leq y''} \tilde{a}_{y'', y'} a_{y', y''} h_{y', y''}.$$  

(3.15)

Here, the infinite summation can be interpreted as follows:

$$\lim_{y_0 \to \infty} \sum_{y' \leq y_0} \sum_{y'' \leq y'} \tilde{\beta}_{y'', y', y''} u_{y'}(x) \bar{u}_{y''}(x_0) = G_0(x, x_0) \quad \text{in } L^2(D \times D).$$

(3.13)

Based on the resonance expansions of the Green functions in high-contrast media and in the free space, we can now propose an explanation for the super-resolution phenomenon. Observe that the difference between the coefficients $\beta_{y'', y', y''}$ and $\tilde{\beta}_{y'', y', y''}$ in (3.11) and (3.15) are the quantities $d_{y, y'}$ and $h_{y', y'} (a_{y', y'}$ are constants). If, for example, we consider the special case where the spaces $\mathcal{H}_j$ are of dimension one, then we have

$$d_{y, y'} = \delta_{y, y'} \frac{\lambda_j^2}{z - \lambda_j} \quad \text{and} \quad h_{y', y'} = \delta_{y, y'} \lambda_j,$$

and therefore,

$$d_{y, y'} = \frac{1}{z/\lambda_j - 1} h_{y, y'},$$

which shows that the contribution to the Green function $G$ of the sub-wavelength resonant mode $u_y$ is amplified when $z$ is close to $\lambda_j$. 


Therefore, we can see that the imaginary part of $G$ may have sharper peak than the one of $G_0$ due to the excited sub-wavelength resonant modes. When the high contrast is properly chosen (the frequency is fixed), one or several of these sub-wavelength resonance modes can be excited, and they dominate over the other ones in the expansion of the Green function $G$. It is those sub-wavelength modes that essentially determine the behaviour of $G$ and hence the resolution associated in the media. Therefore, we can expect super-resolution to occur in this case.

4. Concluding remarks

In this paper, we provided a mathematical theory to explain the super-resolution and super-focusing mechanisms in high-contrast media. From the expansions (3.10) and (3.12), we proved that the super-resolution is due to propagating sub-wavelength resonant modes. It is worth mentioning that in (3.10) and (3.12), we observed that a phenomenon of mixing of modes occurs. This is essentially due to the non-hermitian nature of the system (the operator $K_D$) we considered. We believe that the mixing of resonant modes is an intrinsic nature of non-hermitian systems, as opposed the eigen-expansion for hermitian systems, because of the rigorous mathematical convergence results. However, this phenomenon is sometimes ignored in physics literature where formal resonance expansions without mixing are proposed without any evidence of convergence.

Our approach in this paper for inverse source problems complements the one recently proposed in [22], which is based on the concept of scattering coefficients and solves the super-resolution problem for inverse scattering problems. Finally, we expect that our present approach could provide a mathematical explanation of the mechanism of super-resolution and super-focusing in other resonant media including negative index materials. The numerical evidence of the principle of super-resolution shown in this paper as well as the generalization to other waves such as elastic waves shall be reported elsewhere.

Data accessibility. There is no supporting data.

Authors’ contributions. The development of this article was a gradual, cooperative process, and each of the authors made significant contributions to every aspect of the paper. The two authors gave final approval for publication.

Competing of interests. The authors have no competing interests.

Funding. This work was supported by the ERC Advanced Grant Project MULTIMOD–267184.

References

1. Arhab S, Soriano G, Ruan Y, Maire G, Talneau A, Sentenac D, Chaumet PC, Belkebir K, Giovannini H. 2013 Nanometric resolution with far-field optical profilometry. Phys. Rev. Lett. 111, 053902. (doi:10.1103/PhysRevLett.111.053902)
2. Lemoult F, Fink M, Lerosey G. 2011 Acoustic resonators for far-field control of sound on a subwavelength scale. Phys. Rev. Lett. 107, 064301. (doi:10.1103/PhysRevLett.107.064301)
3. Lemoult F, Lerosey G, de Rosny J, Fink M. 2011 Time reversal in subwavelength-scaled resonant media: beating the diffraction limit. Int. J. Microw. Sci. Technol. 2011, 425710.
4. Lemoult F, Ourir A, de Rosny J, Tourin A, Fink M, Lerosey G. 2010 Resonant metalenses for breaking the diffraction barrier. Phys. Rev. Lett. 104, 203901. (doi:10.1103/PhysRevLett.104.203901)
5. Lerosey G, de Rosny J, Tourin A, Fink M. 2007 Focusing beyond the diffraction limit with far-field time reversal. Science 315, 1120–1122. (doi:10.1126/science.1134824)
6. Ammari H. 2008 An introduction to mathematics of emerging biomedical imaging. In Math. & Appl., vol. 62. Berlin, Germany: Springer.
7. Ammari H, Bonnetier E, Capdeboscq Y. 2010 Enhanced resolution in structured media. SIAM J. Appl. Math. 70, 1428–1452.
8. Ammari H, Zhang H. 2015 A mathematical theory of super-resolution by using a system of sub-wavelength Helmholtz resonators. Commun. Math. Phys 337, 379–428. (doi:10.1007/s00220-015-2301-4)
9. Ammari H, Garnier J, de Rosny J, Solna K. 2014 Medium induced resolution enhancement for broadband imaging. Inverse Probl. 30, 085006. (doi:10.1088/0266-5611/30/8/085006)
10. de Rosny J, Fink M. 2002 Overcoming the diffraction limit in wave physics using a time-reversal mirror and a novel acoustic sink. *Phys. Rev. Lett.* **89**, 124301. (doi:10.1103/PhysRevLett.89.124301)

11. Blomberg P, Papaioannou G, Zhao HK. 2002 Super-resolution in time-reversal acoustics. *J. Acoust. Soc. Am.* **111**, 230–248. (doi:10.1121/1.1421342)

12. Fernández-Suárez M, Ting AY. 2008 Fluorescent probes for super-resolution imaging in living cells. *Nat. Rev.* **9**, 929–943. (doi:10.1038/nrm2531)

13. Jain PK, Lee KS, El-Sayed IH, El-Sayed MA. 2006 Calculated absorption and scattering properties of gold nanoparticles of different size, shape, and composition: applications in biomedical imaging and biomedicine. *J. Phys. Chem. B* **110**, 7238–7248. (doi:10.1021/jp057170o)

14. Daubechies I, Defrise M, De Mol C. 2004 An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Commun. Pure Appl. Math.* **57**, 1413–1457. (doi:10.1002/cpa.20042)

15. Ammari H, Garnier J, Jing W, Kang H, Lim M, Sølna K, Wang H. 2013 *Mathematical and statistical methods for multistatic imaging*. Lecture Notes in Mathematics, vol. 2098. Cham, Switzerland: Springer.

16. Bao G, Lin J, Triki F. 2010 A multi-frequency inverse source problem. *J. Diff. Equ.* **249**, 3443–3465. (doi:10.1016/j.jde.2010.08.013)

17. Slepian D. 1983 Some comments on Fourier analysis, uncertainty and modeling. *SIAM Rev.* **25**, 379–393. (doi:10.1137/1025078)

18. Grasmair M, Haltmeir M, Scherzer O. 2011 Necessary and sufficient conditions for linear convergence of $l^1$-regularization. *Commun. Pure Appl. Math.* **64**, 161–182. (doi:10.1002/cpa.20350)

19. Candès EJ, Fernandez-Granda C. 2014 Towards a mathematical theory of super-resolution. *Commun. Pure Appl. Math* **67**, 906–956. (doi:10.1002/cpa.21455)

20. Candès EJ, Fernandez-Granda C. 2013 Super-resolution from noisy data. *J. Fourier Anal. Appl.* **19**, 1229–1254. (doi:10.1007/s00041-013-9292-3)

21. Päivärinta L, Sylvester J. 2008 Transmission eigenvalues. *SIAM J. Math. Anal.* **40**, 738–753. (doi:10.1137/070697525)

22. Ammari H, Chow Y, Zou J. 2014 Super-resolution in highly contrasted media from the perspective of scattering coefficients. (http://arxiv.org/abs/1410.1253)