THE VANISHING OF A HIGHER CODIMENSION ANALOGUE OF HOCHSTER’S THETA INVARIANT

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Abstract. We study H. Dao’s invariant $\eta^R_c$ of pairs of modules defined over a complete intersection ring $R$ of codimension $c$ having an isolated singularity. Our main result is that $\eta^R_c$ vanishes for all pairs of modules when $R$ is a graded complete intersection ring of codimension $c > 1$ having an isolated singularity. A consequence of this result is that all pairs of modules over such a ring are $c$-Tor-rigid.

1. Introduction

Let $R$ be an isolated complete intersection singularity, i.e., $R$ is the quotient of a regular local ring $(Q, m)$ by a regular sequence $f_1, \ldots, f_c \in Q$, and $R_p$ is regular for all $p \neq m$. For any pair $(M, N)$ of finitely generated $R$-modules, the Tor modules $\text{Tor}^R_j(M, N)$ have finite length when $j \gg 0$. Moreover, the lengths of the odd and even indexed Tor modules in high degree follow predictable patterns. There are polynomials $P_{\text{ev}}(j) = a_{c-1}j^{c-1} + \cdots + a_1j + a_0$ and $P_{\text{odd}}(j) = b_{c-1}j^{c-1} + \cdots + b_1j + b_0$ of degree at most $c - 1$ such that

$$\text{length } \text{Tor}^R_{2j}(M, N) = P_{\text{ev}}(j) \quad \text{and} \quad \text{length } \text{Tor}^R_{2j+1}(M, N) = P_{\text{odd}}(j)$$

for all $j \gg 0$ (see, for example, [7, Theorem 4.1]).

The polynomials $P_{\text{ev}}$ and $P_{\text{odd}}$ need not be the same, nor is it necessary that the coefficients of $j^{c-1}$ coincide. A natural invariant of the pair $(M, N)$ is thus the difference, $a_{c-1} - b_{c-1}$, of these coefficients. Up to a constant factor, this difference is Dao’s $\eta^R_c$-invariant.

For example, if $c = 1$, then $P_{\text{ev}} = a_0$ and $P_{\text{odd}} = b_0$ are both constants. This reflects the fact that, in our present context, the Eisenbud operator

$$\chi : \text{Tor}^R_j(M, N) \rightarrow \text{Tor}^R_{j-2}(M, N)$$

is an isomorphism for $j \gg 0$. In this case, the invariant $\eta^R_1(M, N)$ is (one half of) the difference $a_0 - b_0$ of these Tor-lengths. This difference is Hochster’s $\theta$-invariant.

In our previous paper [14] we studied Hochster’s $\theta$-invariant in the special case where $R$ is a graded, isolated hypersurface singularity. Now we employ similar

Date: February 18, 2022.
2000 Mathematics Subject Classification. 13D02, 14C35, 19L10.
The fourth author was supported in part by NSF grant DMS-0601666.
techniques to study the invariant $\eta^R_c$ for graded, isolated complete intersection singularities of codimension $c > 1$. That is, we assume

\begin{equation}
R = k[x_0, \ldots, x_{n+c-1}]/(f_1, \ldots, f_c)
\end{equation}

where $k$ is a field, the $f_i$’s are homogeneous polynomials, and $\text{Proj}(R)$ is a smooth $k$-variety. With these assumptions, the irrelevant maximal ideal $m = (x_0, \ldots, x_{n+c-1})$ of $R$ is the only non-regular prime in $\text{Spec} R$. Our main result is that, for such a ring $R$, the invariant $\eta^R_c(M, N)$ vanishes for all $R$-modules $M$ and $N$, provided $c > 1$. See Theorem 4.5.

H. Dao [7, Theorem 6.3] has proven that the vanishing of $\eta^R_c(M, N)$ implies that the pair $(M, N)$ is $c$-Tor-rigid, meaning that if $c$ consecutive Tor modules vanish, then all subsequent Tor modules vanish too. We therefore conclude that all pairs of modules over rings of the form (1.1) having an isolated singularity are $c$-Tor-rigid, provided $c > 1$; our previous result [14, Remark 3.16] shows $c$-Tor-rigidity for $c = 1$ when the dimension of $R$ is even. In general, one only has $(c + 1)$-Tor-rigidity for pairs of modules over a codimension $c$ complete intersection [15].

We conjecture that $\eta^R_c(M, N) = 0$ for all pairs of modules $(M, N)$ over an isolated complete intersection singularity of codimension $c > 1$, and hence that all pairs of modules over such a ring are $c$-Tor-rigid. There are many well-known examples (of isolated hypersurface singularities) where $(M, N)$ is not 1-Tor-rigid, and hence $\theta^R(M, N) \neq 0$.

The invariant $\eta^R_c(M, N)$ is also defined for complete intersection rings that are not isolated singularities, provided the pair $(M, N)$ has the property that $\text{Tor}^R_j(M, N)$ has finite length for all $j \gg 0$. We include an example due to D. Jorgensen and O. Celikbas that shows that $\eta^R_2$ need not vanish for a complete intersection of codimension 2 if the dimension of the singular locus is positive.

2. Dao’s $\eta^R_c$-Invariant

In this section we recall the definition of Dao’s $\eta^R_c$-invariant for complete intersections.

**Proposition 2.1.** Let $R$ be the quotient of a noetherian ring $Q$ by a regular sequence $f_1, \ldots, f_c$. For a pair of finitely generated $R$-modules $M$ and $N$, suppose the $Q$-module $\text{Tor}^Q_j(M, N)$ vanishes for all $j \gg 0$ and that the $R$-modules $\text{Tor}^R_j(M, N)$ are supported on a finite set of maximal ideals $\{m_1, \ldots, m_s\}$ of $R$ for all $j \gg 0$. Then there are polynomials $P_{\text{ev}} = P_{\text{ev}}^R(M, N)$ and $P_{\text{odd}} = P_{\text{odd}}^R(M, N)$ of degree at most $c - 1$ so that

\[
\text{length} \text{Tor}^R_{2^j+1}(M, N) = P_{\text{ev}}(j) \quad \text{and} \quad \text{length} \text{Tor}^R_{2^j+1}(M, N) = P_{\text{odd}}(j)
\]

for all $j \gg 0$.

**Proof.** Apply [7, Theorem 4.1(2)] to each $R_{m_i}$ and add the resulting polynomials to obtain the polynomials here. See Appendix A for an alternative proof of this result. \hfill \Box

The difference of the coefficients of $j^{c-1}$ in $P_{\text{ev}} = P_{\text{ev}}^R(M, N)$ and in $P_{\text{odd}}$ is the basis for an invariant of $(M, N)$. We can obtain these coefficients through the
(c − 1)-st iterated first difference: the first difference of a polynomial \( q(j) \) is the polynomial \( q^{(1)}(j) = q(j) - q(j - 1) \), and recursively one defines \( q^{(i)} = (q^{(i-1)})^{(1)} \).

**Definition 2.2.** In the set up of Proposition 2.1 define
\[
\eta^R_c(M, N) = \frac{(P_{ev} - P_{odd})^{(c-1)}}{2^c \cdot c!}.
\]

This invariant of the pair \((M, N)\) is Dao’s \( \eta^R_c \)-invariant [7, 4.2].

**Remark 2.3.** For a pair of \( R \)-modules, Dao sets
\[
\beta_j(M, N) = \begin{cases} 
\text{length } \text{Tor}_j^R(M, N) & \text{if } \text{length } \text{Tor}_j^R(M, N) < \infty \\
0 & \text{otherwise.}
\end{cases}
\]

It can be shown, as an easy application of Proposition 2.1, that under the assumptions in that result, if \( c > 0 \), then
\[
\eta^R_c(M, N) = \lim_{n \to \infty} \frac{\sum_{j=0}^{n} (-1)^j \beta_j(M, N)}{n^c}.
\]

This limit is the original definition of \( \eta^R_c(M, N) \) due to Dao.

Our main result, Theorem 4.5, suggests the following conjecture:

**Conjecture 2.4.** Suppose \( R = \mathbb{Q}/(f_1, \ldots, f_c) \) with \( \mathbb{Q} \) a regular noetherian ring and \( f_1, \ldots, f_c \) a regular sequence, with \( c > 1 \). If the singular locus of \( R \) consists of a finite number of maximal ideals, then \( \eta^R_c(M, N) = 0 \) for all finitely generated \( R \)-modules \( M \) and \( N \).

**Remark 2.5.** The case \( N = k \) of Conjecture 2.4 follows from a result due to L. Avramov, V. Gasharov, and I. Peeva [2, 8.1]. In this case, the length of the Tor\(_j\) record the Betti numbers of \( M \) over \( R \), and part of their result states that both the even and odd Betti numbers grow at the same polynomial rate and have the same leading coefficient.

**Example 2.6.** The following example is due to D. Jorgensen [13 Example 4.1] and O. Celikbas [5 Example 3.11]. Let \( k \) be a field, and let \( R = k[x, y, z, u]/(xy, zu) \). Then \( R \) is a local complete intersection of codimension two with positive dimensional singular locus. Let \( M = R/(y, u) \), and let \( N \) be the cokernel of the map
\[
R^2 \xrightarrow{\begin{pmatrix} 0 & u \\ -z & x \\ y & 0 \end{pmatrix}} R^3.
\]

Then the pair \((M, N)\) is not 2-Tor-rigid, and hence, by [7 6.3], \( \eta^R_2(M, N) \neq 0 \).

3. The Graded Case

**3.1.** Let \( Q \) be a graded noetherian ring, \( f_1, \ldots, f_c \) be a \( Q \)-regular sequence of homogeneous elements, and \( R = Q/(f_1, \ldots, f_c) \). Then for each pair of finitely generated graded \( R \)-modules \( M \) and \( N \), \( \text{Tor}_j^R(M, N) \) is a graded \( R \)-module for all \( j \). Moreover, with the notation \( d_i = \deg f_i \), the Eisenbud operators [8] \( \chi_1, \ldots, \chi_c \) determine maps of graded \( R \)-modules
\[
\chi_i : \text{Tor}_j^R(M, N) \to \text{Tor}_{j-2}^R(M, N)(-d_i)
\]
for all \( j \), where for a graded \( R \)-module \( T \), we define \( T(m) \) to be the graded \( R \)-module satisfying \( T(m)_k = T_{k+m} \). Since the actions of the \( \chi_i \) commute, we may view \( \bigoplus_{j,l} \text{Tor}_j^R(M,N)_i \) as a bigraded module over the bigraded ring \( S = R[\chi_1, \ldots, \chi_c] \), where the degree of \( \chi_i \) is \((-2,-d_i)\).

The operators \( \chi_i \) first appeared in work of Gulliksen [10] as (co)homology operators (albeit in a different guise), where he proved that \( \text{Tor}^R_j(M,N) \) is artinian over \( S \) if and only if \( \text{Tor}^Q_j(M,N) \) is artinian over \( Q \). Compare our Appendix A.

**Proposition 3.2.** Let \( Q \) and \( R \) be as in paragraph 3.1. For a pair of finitely generated graded \( R \)-modules \( M \) and \( N \), suppose the \( Q \)-module \( \text{Tor}_j^Q(M,N) \) vanishes for all \( j \gg 0 \) and there is a finite set of maximal ideals of \( R \) on which the \( R \)-modules \( \text{Tor}_j^R(M,N) \) are supported for all \( j \gg 0 \). Then the action of the Eisenbud operators induces an exact (Koszul) sequence

\[
0 \rightarrow \text{Tor}_j^R(M,N) \rightarrow \bigoplus_{l=1}^c \text{Tor}_{j-2l}^R(M,N)(-d_l) \rightarrow \\
\bigoplus_{l_1 < l_2} \text{Tor}_{j-4}^R(M,N)(-d_{l_1} - d_{l_2}) \rightarrow \cdots \rightarrow \text{Tor}_{j-2c}^R(M,N)(-d_1 - \cdots - d_c) \rightarrow 0
\]

of graded \( R \)-modules for \( j \gg 0 \).

**Proof.** Within this proof, we use a simplified grading on \( S = R[\chi_1, \ldots, \chi_c] \), effectively ignoring the twists given by the \( d_i \) in 3.1. We let \( R \) lie in degree 0 and let each \( \chi_i \) lie in degree \(-2\). By [7, Lemma 3.2], for a sufficiently large \( J \), the module \( T = \bigoplus_{j \geq J} \text{Tor}_j^R(M,N) \) is graded artinian over the ring \( S \). Consider the Koszul complex \( K = K[\chi_1, \ldots, \chi_c] \otimes_S T \). For \( j \gg 0 \), the complex (3.1) is the \( j \)th graded piece of \( K \). We prove this complex \( K \) is exact in all but finitely many degrees.

As \( T \) is graded artinian over \( S \), the total homology module \( H(K) \) is as well. The descending chain of \( R \)-submodules

\[
H(K) \supseteq \bigoplus_{j \geq 1} H(K)_j \supseteq \bigoplus_{j \geq 2} H(K)_j \supseteq \cdots
\]

intersects to 0. Since \( \chi_i H(K) = 0 \), see [16, IV.A.4], these \( R \)-submodules are in fact \( S \)-submodules. Since \( H(K) \) is artinian over \( S \), the descending chain stabilizes. Thus there exists an \( m > 0 \) such that \( H(K)_j = 0 \) for all \( j \geq m \). \( \square \)

3.3. For the remainder of this section, we assume \( Q = k[x_0, \ldots, x_{n+c-1}] \) is a polynomial ring over a field with each \( x_i \) of degree one, \( f_1, \ldots, f_c \) is a \( Q \)-regular sequence of homogeneous elements, and \( R = Q/(f_1, \ldots, f_c) \). Let \( d_l = \deg(f_l) \). In particular, \( R \) is graded. When \( M \) and \( N \) are finitely generated graded \( R \)-modules, the torsion modules \( \text{Tor}_j^R(M,N) \) are also graded.

**Definition 3.4.** Let \( R \) be as in paragraph 3.3. For finitely generated graded \( R \)-modules \( M \) and \( N \), and an integer \( F \), define

\[
G_F(x,t) = \sum_{l,j \geq 0} \dim_k (\text{Tor}_j^{F+2j}(M,N)_l) t^l x^j \in \mathbb{Q}[[x,t]].
\]

**Remark 3.5.** Note that if for some \( F \gg 0 \), \( \text{Tor}_j^{F+2j}(M,N) \) has finite length for all \( j \geq 0 \), then \( G_F(x,t) \) belongs to \( \mathbb{Q}[[t]][[x]] \).
For a finitely generated graded $R$-module $T$, its Hilbert series is
\[ H_T(t) = \sum_{i \geq 0} \dim_k(T_i)t^i. \]

$H_T(t)$ is a rational function with a pole of order $\dim T$ at $t = 1$. In fact,
\[ H_T(t) = \frac{e_T(t)}{(1-t)^{\dim T}}, \tag{3.2} \]
where $e_T(t)$ is a Laurent polynomial \[ H_{\mathcal{J}} \] (1.1), sometimes called the multiplicity polynomial of $T$. The multiplicity polynomial of $R$ is calculated by using the presentation $R = Q/(f_1, \ldots, f_s)$; explicitly, since $\deg(x_i) = 1$ and $\deg(f_i) = d_i$,
\[ e_R(t) = \prod_{i=1}^c (1 - t^{d_i})/(1-t)^c. \tag{3.3} \]

For graded $R$-modules $M$ and $N$, we write the Hilbert series of $\text{Tor}_j^R(M, N)$ as $H_j(t)$ or just $H_j$. If $M$ and $N$ are such that the $\text{Tor}_j^R(M, N)$ have finite length for $j \gg 0$, then $H_j(t)$, $j \geq 0$, has the property that the number of initial terms of $H_j(t)$ that vanish goes to infinity as $j \to \infty$. It thus makes sense to form the sum
\[ \sum_{j \geq 0} (-1)^j H_j(t), \]
and, more generally, to evaluate $G_F(x, t)$ at $x = 1$ (or any constant; see Remark 3.5). Observe that
\[ \sum_{j \geq 0} (-1)^j H_{j+F}(t) = G_F(1, t) - G_{F+1}(1, t), \tag{3.4} \]
for any $F$.

**Lemma 3.6.** Let $R$ be a graded ring as in paragraph 3.3, and let $M, N$ be finitely generated graded $R$-modules such that $\text{Tor}_j^R(M, N)$ has finite length for $j \gg 0$. For $F \gg 0$ there is a unique polynomial $b_F(x, t) \in \mathbb{Q}[x, t]$ such that
\[ G_F(x, t) = \frac{b_F(x, t)}{\prod_{i=1}^c (1 - t^{d_i}x)}. \tag{3.5} \]

For $E \gg 0$ and even there is a unique polynomial $\eta^R_{c,E}(M, N)(t) \in \mathbb{Q}[t]$ such that
\[ \sum_{j \geq E} (-1)^j H_j(t) = \frac{\eta^R_{c,E}(M, N)(t)c}{e_R(t)(1-t)^c} \quad \text{and} \quad \eta^R_{c,E}(M, N)(1) = 2^c \cdot c! \cdot \eta^R_{c}(M, N), \tag{3.6} \]
where $e_R(t)$ is the multiplicity polynomial of $R$ defined in (3.3).

**Proof.** Let $s_0 = 1$, $s_1 = t^{d_1} + \cdots + t^{d_c}$, and $s_c = t^{d_1 + \cdots + d_c}$; that is, the $s_k$ are elementary symmetric functions on the symbols $t^{d_i}$. The exactness of (3.1) for $j \gg 0$ gives the relation
\[ s_0 H_j - s_1 H_{j-2} + s_2 H_{j-4} + \cdots + (-1)^c s_c H_{j-2c} = 0, \]
from which it follows that, for $F \gg 0$, we have
\[ (1 - s_1 x + s_2 x^2 - \cdots + (-1)^c s_c x^c) G_F(x, t) = b_0 F(t) + b_1 F(t) x + \cdots + b_{c-1} F(t) x^{c-1}, \]
for some polynomials $b_i(t)$. Set $b_F(x, t)$ equal to the right hand side of this equation and use $\sum_{k=0}^n (-1)^k s_k x^k = \prod_{i=1}^c (1 - t^d)$. Then (3.5) follows easily.

To establish (3.6), observe that for $F \gg 0$, we have
\[
\frac{b_F(x, 1)}{(1 - x)^e} = G_F(x, 1) = \sum_{j \geq 0} \dim_k \text{Tor}_{F+2j}^R(M, N)x^j.
\]
Taking $E \gg 0$ to be even, set $\eta^R_{c, E}(M, N)(t)$ to be $b_E(1, t) - b_{E+1}(1, t)$. The first equation in (3.6) follows immediately from (3.3), (3.4), and (3.5).

The leading coefficients of $P_{ev}(M, N)$ and $P_{odd}(M, N)$ are $b_E(1, 1)/(c - 1)!$ and $b_{E+1}(1, 1)/(c - 1)!$, and so the value of $\eta^R_{c, E}(M, N)(t)$ at $t = 1$ is $2^c \cdot c! \cdot \eta^R_{c}(M, N)$. □

Remark 3.7. The polynomials $\eta^R_{c, E}(M, N)(t)$ depend on $E$, but, as Lemma 3.6 shows, they have a common value at $t = 1$.

4. THE VANISHING OF $\eta^R_{c}$

Throughout the remainder of this paper, we use the following notations and assumptions:

- $k$ is a field;
- $R = k[x_0, \ldots, x_{n+c-1}]/(f_1, \ldots, f_c)$, where $\deg x_i = 1$ for all $i$ and the $f_i$'s are homogeneous polynomials in $m = (x_0, \ldots, x_{n+c-1})$ with $d_i = \deg(f_i)$;
- $c > 0$ and $f_1, \ldots, f_c$ forms a regular sequence;
- $X = \text{Proj}(R)$ is a smooth $k$-variety.

Remark 4.1. Recall that the variety $X$ is smooth if and only if $m$ is the radical of the homogeneous ideal generated by the $f_i$'s and the maximal minors of the Jacobian matrix $(\partial f_i / \partial x_i)$. In particular, by the smoothness assumption, $m = (x_0, \ldots, x_{n+c-1})$ is the only non-regular prime of $R$.

For a quasi-projective scheme $Z$ over a field $k$, we let $G(Z)$ and $K(Z)$ denote the Grothendieck groups of coherent sheaves and locally free coherent sheaves, respectively. Recall that if $Z$ is regular (for example, if it is smooth over $k$), then the canonical map $K(Z) \to G(Z)$ is an isomorphism. For further explanation and discussion of the relevant groups and maps, see [14] §2.1.

From the assumptions (4.1), the smooth variety $X = \text{Proj}(R) \subset \mathbb{P}^{n+c-1}$ has dimension $n - 1$ and degree $d = d_1 \cdots d_c$. When $k$ is infinite, there is a linear rational map $\mathbb{P}^{n+c-1} \dashrightarrow \mathbb{P}^{n-1}$ that determines a regular function on an open subset containing $X$, and hence it induces a regular map
\[
\rho: X \to \mathbb{P}^{n-1}
\]
that is finite, flat, and of degree $d$. As $X$ and $\mathbb{P}^{n-1}$ are smooth and $\rho$ is finite, we have the following map and isomorphisms:
\[
\rho_*: K(X) \cong G(X) \to G(\mathbb{P}^{n-1}) \cong K(\mathbb{P}^{n-1}).
\]
We also have the isomorphism
\[
\mathbb{Z}[t]/(1 - t)^n \cong K(\mathbb{P}^{n-1})
\]
Lemma 4.2. \cite{14} Lemma 4.1] Under the assumptions in \cite{4,7} with \( k \) infinite, let \( M \) be a finitely generated graded \( R \)-module and \( \mathcal{M} \) the associated coherent sheaf on \( X \). Then
\[
\rho_*(\mathcal{M}) = (1-t)^n H_M(t)
\]
in \( K(\mathbb{P}^n) \). In particular,
\[
\rho_*(1) = \rho_*(\mathcal{O}_X) = e_R(t) = \prod_{i=1}^c (1 + t + t^2 + \cdots + t^{d_i-1}) \in \mathbb{Q}[t]/(1-t)^n.
\]

Proof. The proof of \cite{14} Lemma 4.1, which is the \( c = 1 \) case, applies verbatim. \( \square \)

Lemma 4.3. \cite{14} Lemma 4.2] Under the assumptions in \cite{4,7} with \( k \) infinite, let \( M \) and \( N \) be finitely generated graded \( R \)-modules. For any sufficiently large even integer \( E \) and for any integer \( m \geq 0 \), the rational function
\[
(1-t)^{n+m-c} \frac{\eta^{R,E}_c(M,N)(t)}{(e_R(t))^2}
\]
does not have a pole at \( t = 1 \). Its image in \( \mathbb{Q}[t]/(1-t)^n \) is
\[
K(\mathbb{P}^n) = (1-t)^n H_M(1) = (1-t)^m \left( \frac{\rho_*(\mathcal{M})}{\rho_*(1)} \cdot \frac{\rho_*(\mathcal{N})}{\rho_*(1)} - \frac{\rho_*(\mathcal{M}) \cdot \rho_*(\mathcal{N})}{\rho_*(1)} \right).
\]
In particular, taking \( m = c - 1 \) yields
\[
(1-t)^{n-1} \eta^{R}_c(M,N) = \frac{1}{2^c \cdot c!} \left( \frac{d \cdot \rho_*(\mathcal{M})}{\rho_*(1)} \cdot \frac{d \cdot \rho_*(\mathcal{N})}{\rho_*(1)} - \frac{d^2 \cdot \rho_*(\mathcal{M}) \cdot \rho_*(\mathcal{N})}{\rho_*(1)} \right),
\]
where \( \deg X = d = d_1 \cdots d_c \).

Proof. As in \cite{14} Lemma 4.2, start with the equation of Hilbert series from \cite{1} Lemma 7], namely
\[
\frac{H_M H_N}{H_R} = \sum_{j \geq 0} (-1)^j H_j.
\]
Splitting the sum at \( E \gg 0 \) and using the first relation in \cite{3,11} gives
\[
\frac{H_M \cdot H_N}{H_R} = \sum_{j=0}^{E-1} (-1)^j H_j + \frac{\eta^{R}_c(M,N)(t)}{e_R(t) \cdot (1-t)^n}.
\]
Upon multiplying by \((1 - t)^m/H_R = (1 - t)^{n+m}/e_R(t)\) and rearranging, this yields
\[
(1 - t)^m \frac{(1 - t)^n H_M}{e_R(t)} - (1 - t)^m \sum_{j=0}^{E-1} (-1)^j \frac{(1 - t)^n H_j}{e_R(t)} = \eta_{c,E}^R(M, N)(t) \cdot \rho_*(1) - \eta_{c,E}^R(M, N)(t)/(e_R(t))^2 - (1 - t)^{n+m-e}.
\]

(4.2)

The first assertion follows from the fact that the expression before the equality in (4.2) does not have a pole at \(t = 1\), using (3.2). Since both sides of this equation are power series in powers of \(1 - t\), we may take their images in \(\mathbb{Q}[t]/(1 - t)^n\). Apply Lemma 4.2. The expression before the equality in (4.2) becomes

\[
(1 - t)^m \frac{\rho_*(\widetilde{[M]}) \rho_*(\widetilde{[N]})}{\rho_*(1)} - (1 - t)^m \sum_{j=0}^{E-1} (-1)^j \frac{\rho_*(\text{Tor}_j^R(M, N))}{\rho_*(1)}.
\]

If \(E\) is large enough so that \(\text{Tor}_j^R(M, N)\) has finite length for \(j \geq E\), then the alternating sum is \(\rho_*(\widetilde{[M] \cdot [N]})/\rho_*(1)\) where \([M] \cdot [N]\) is multiplication in the ring \(K(X)\). This gives the first equation in the Lemma.

For the second equation, set \(m = c - 1\). Define \(g(t) = \eta_{c,E}^R(M, N)(t)/(e_R(t))^2\). As \(e_R(1) = d\) and \(\eta_{c,E}^R(M, N)(t)\) is a polynomial, \(g(t)\) is a rational function without a pole at \(t = 1\). Thus, modulo \((1 - t)^n\),
\[
g(t)(1 - t)^{n-1} = g(1)(1 - t)^{n-1} + \frac{g(t) - g(1)}{1 - t}(1 - t)^n \equiv g(1)(1 - t)^{n-1}.
\]

Multiplication by \(d^2\) establishes the second equation.

□

In the next lemma and in the proof of our main theorem, we use étale cohomology. Assume \(k\) is a separably closed field, fix a prime \(\ell \neq \text{char} k\), and write \(H^*_{\text{ét}}(Z, \mathbb{Q}_\ell(i))\) for the étale cohomology of a scheme \(Z\) with coefficients in \(\mathbb{Q}_\ell(i)\). In addition, write \(H^*_{\text{ét}}(Z, \mathbb{Q}_\ell(\ast))\) for \(\bigoplus_i H^*_{\text{ét}}(Z, \mathbb{Q}_\ell(i))\). This is a commutative ring under cup product. Moreover, the étale Chern character gives a ring homomorphism
\[
ch_{\text{ét}} : K(Z)_Q \rightarrow H^2_{\text{ét}}(Z, \mathbb{Q}_\ell(\ast)).
\]

We refer the reader to [9] for additional background.

**Lemma 4.4** [14] Lemma 4.3] Under the assumptions in (4.1) with \(k\) separably closed, the diagram
\[
\begin{array}{ccc}
K(X)_Q & \xrightarrow{\rho_*(1)} & K(\mathbb{P}^{n-1})_Q \\
\downarrow ch_{\text{ét}} & & \downarrow ch_{\text{ét}} \\
H^2_{\text{ét}}(X, \mathbb{Q}_\ell(\ast)) & \xrightarrow{\rho^*_\text{ét}} & H^2_{\text{ét}}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(\ast))
\end{array}
\]

commutes, where \(\rho^*_\text{ét}\) is the push-forward map for étale cohomology.

**Proof.** The proof of [14] Lemma 4.4, which is the \(c = 1\) case, applies verbatim. □
The following is the main result of this paper.

**Theorem 4.5.** Under the assumptions in (4.1) with k separably closed and infinite, let M and N be finitely generated graded R-modules. For E a sufficiently large even integer, $\eta^R_{c,E}(M, N)(t)$ has a zero at $t = 1$ of order at least $c - 1$. In particular, $\eta^R_c(M, N) = 0$ for $c > 1$.

**Proof.** The claim is vacuous if $c \leq 1$, and so assume $c > 1$. Let $\gamma$ be the element $\omega_{\ell}$ in $H^4_{\ell}(\mathbb{P}^{n-1}, \mathbb{Q}_{\ell}((-c)))$.

We apply $\omega_{\ell}$ to $d^2$ times the first equation in Lemma 4.3 and use the commutative diagram in Lemma 4.4 to obtain

\[
\frac{d^2 \cdot \omega_{\ell}((1-t)^{n+m-c} \cdot \eta^R_{c,E}(M, N)(t))}{c \cdot (e_R(t))^2} = 
\gamma^m \left( \rho^* \cdot \omega_{\ell} \cdot \omega(\mu) \right).
\]

For $\alpha, \beta \in H^2_{\ell}(X, \mathbb{Q}_{\ell}((-c)))$, define

$\Psi_m(\alpha, \beta) = \gamma^m \left( \rho^* \cdot \omega(\mu) \right)$.

We will prove that $\Psi_m$ vanishes for any $m \geq 1$. Using the projection formula $\rho^* \cdot \omega_{\ell}(\mu) = \alpha' \cdot \omega_{\ell}(\mu)$ with $\omega = 1$, and using the fact that $\rho^* \cdot \omega_{\ell}(\mu) = 0$, we see that if $\alpha' = \rho^* \cdot \omega_{\ell}(\mu)$ for some $\alpha' \in H^2_{\ell}(\mathbb{P}^{n-1}, \mathbb{Q}_{\ell}((-c)))$, then

$\Psi_m(\alpha, \beta) = \gamma^m \left( \rho^* \cdot \omega_{\ell}(\mu) \right) = 0$.

Likewise, $\Psi_m(\alpha, \beta) = 0$ if $\beta = \rho^* \cdot \omega_{\ell}(\mu)$. But since $X$ is a complete intersection in projective space, the map

$\rho^{\omega}_{\ell} : H^2_{\ell}(\mathbb{P}^{n-1}, \mathbb{Q}_{\ell}(j)) \to H^2_{\ell}(X, \mathbb{Q}_{\ell}(j))$

is an isomorphism in all degrees except possibly in degree $2j = n - 1$ (see [18 XI.1.6]). So we may assume $n$ is odd and that $\alpha, \beta \in H^{n-1}_{\ell}(X, \mathbb{Q}_{\ell}(\frac{n-1}{2}))$. Noticing that $\gamma$ is in $\bigoplus_{i \geq 1} H^2_{\ell}(X, \mathbb{Q}_{\ell}(i))$, we see that $\gamma^m \rho^* \cdot \omega_{\ell}(\mu)$ and $\gamma^m \rho^* \cdot \omega_{\ell}(\mu)$ belong to

$\bigoplus_{i \geq 0} H^{2n-2+2m+2j}_{\ell}(X, \mathbb{Q}_{\ell}(n-1 + m + j))$.

This group vanishes when $m \geq 1$ because $\dim(X) = n - 1$ and étale cohomology vanishes in degrees more than twice the dimension of a smooth variety over a separably closed field [17 X.4.3]. As $\Psi_m$ is zero for $m \geq 1$, so too is the expression on the left-hand side of (4.4).

We have proven that for $m \geq 1,$

\[
\frac{d^2 \cdot \omega_{\ell}((1-t)^{n+m-c} \cdot \eta^R_{c,E}(M, N)(t))}{c \cdot (e_R(t))^2} = 0
\]

and hence

$\omega_{\ell}((1-t)^{n+m-c} \cdot \eta^R_{c,E}(M, N)(t)) = 0.$
But the Chern character map with \( \mathbb{Q}_\ell \) coefficients induces an isomorphism on projective space,

\[
\chi_{\text{ét}} : K(\mathbb{P}^{n-1}) \otimes \mathbb{Q}_\ell \xrightarrow{\cong} H^{2s}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(*)),
\]

and therefore, \((1 - t)^{n+m-c} \cdot \eta^R_{c,E}(M,N)(t) = 0 \) in \( \mathbb{Q}[t] / (1 - t)^n \). That is, \((1 - t)^n\) divides \((1 - t)^{n+m-c} \cdot \eta^R_{c,E}(M,N)(t) \) in \( \mathbb{Q}[t] \) so that \( \eta^R_{c,E}(M,N)(t) / (1 - t)^{c-m} \) is a polynomial. Taking \( m = 1 < c \) proves the Theorem. \( \square \)

The familiar example below shows that \( \eta^R_c(M,N) \) need not vanish, even under the assumptions (4.1), if \( \dim R \) is odd.

**Example 4.6.** [12] Example 1.5] The assumption that \( c > 1 \) is necessary in the Theorem. Let \( R = \mathbb{C}[x,y,u,v]/(xu+yv) \) and set \( M = R/(x,y), N = R/(u,v) \), and \( L = R/(x,v) \). Then \( \eta^R_c(M,M) = \frac{1}{2}, \eta^R_c(M,N) = \frac{1}{2}, \) and \( \eta^R_c(M,L) = -\frac{1}{2} \).

**Corollary 4.7.** Under the assumptions in (4.1) and for every pair of finitely generated, but not necessarily graded, \( R \)-modules \( M \) and \( N \), if \( \dim R > 0 \), then \( \eta^R_c(M,N) \) vanishes when \( c > 1 \). When \( \dim R = 0 \), then \( \eta^R_c(M,N) \) vanishes for all \( c \).

**Proof.** Upon passing to any faithfully flat field extension \( k' \) of \( k \), the assumptions (4.1) remain valid, and, moreover, for finitely generated \( R \)-modules \( M \) and \( N \), the value of \( \eta^R_c(M,N) \) is unchanged. In more detail, since the lengths involved are dimensions over the field \( k \) for \( \eta^R_c \) and dimensions over the field \( k' \) for \( \eta^R_c \otimes k' \), we have equality

\[
\eta^R_c(M,N) = \eta^R_c \otimes k'(M \otimes_k k', N \otimes_k k').
\]

In particular, by passing to the separable closure of \( k \), we may assume that \( k \) is separably closed.

Since \( \eta^R_c(\cdot, \cdot, \cdot) \) is biadditive [7 Theorem 4.3] and defined for all pairs of finitely generated \( R \)-modules, it follows that \( \eta^R_c \) determines a bilinear pairing on \( G(R) \), and hence on \( G(R)_Q := G(R) \otimes \mathbb{Q} \). It suffices to prove that this latter pairing is zero.

Assume \( \dim R > 0 \). From [14 Section 2.1], we recall the mapping from \( K(X)_Q \) to \( G(R)_Q \) given as follows: if \( T \) is a finitely generated graded \( R \)-module with associated coherent sheaf \( \tilde{T} \) on \( X \), then \( K(X)_Q \rightarrow G(R)_Q \) sends \( [\tilde{T}] \in K(X)_Q \) to \( [T] \in G(R)_Q \). As proven in [14 (2.4)], this mapping is onto, and hence the vector space \( G(R)_Q \) is spanned by classes of graded \( R \)-modules. Therefore, Theorem 4.3 applies to prove the pairing on \( G(R)_Q \) induced by \( \eta^R_c \) is the zero pairing.

Finally, if \( \dim R = 0 \), then \( [R] = \dim_k(R) \cdot [k] \) in \( G(R)_Q \), and hence \( [M] = \text{length}(M) \cdot [k] \) in \( G(R)_Q \). Since \( \eta^R_c(R,R) = 0 \) as \( R \) is projective, it follows that \( \eta^R_c \) vanishes for all pairs. \( \square \)

**Corollary 4.8.** With the assumptions in (4.1), let \( M \) and \( N \) be finitely generated, but not necessarily graded, \( R \)-modules. Then for \( c > 1 \), the pair \( (M,N) \) is \( c \)-tor-rigid; that is, if \( c \) consecutive torsion modules \( \text{Tor}_1^R(M,N), \ldots, \text{Tor}^R_{i+c-1}(M,N) \) all vanish for some \( i \geq 0 \), then \( \text{Tor}_j^R(M,N) = 0 \) for \( j \geq i \).

**Proof.** By Corollary 4.7, \( \eta^R_c(M,N) = 0 \) when \( c > 1 \), and the result immediately follows from [7 Theorem 6.3]. \( \square \)
Appendix A. Adapting Gulliksen’s Work

We show in this appendix how to modify Gulliksen’s work in [10] to give an alternative proof of Proposition 2.1 from the body of this paper. The key result is Proposition A.1 below, which was originally proven by Dao [7, Lemma 3.2]. Using alternative proof of Proposition 2.1 from the body of this paper. The key result is sequence, \( R \), some notation.

Proposition A.1. Assume \( Q \) is a noetherian ring, \( f_1, \ldots, f_c \in Q \) is a regular sequence, \( R = Q/(f_1, \ldots, f_c) \) and \( M \) and \( N \) are finitely generated \( R \)-modules. If for all \( i \gg 0 \), \( \text{Tor}_i^R(M, N) \) has finite length and \( \text{Tor}_i^Q(M, N) = 0 \), then there exists an integer \( j \) such that \( \bigoplus_{i \geq j} \text{Tor}_i^R(M, N) \) is artinian as a module over the polynomial ring \( R[\chi_1, \ldots, \chi_c] \), where the \( \chi_i \)'s act via the Eisenbud operators.

Our proof of this result follows Gulliksen’s proof of [10, Theorem 3.1]; we use two lemmas, both of which are analogues of his results. Our Lemma A.3 sidesteps the issue raised in [7, Example 2.9] while following [10, Lemma 1.2]. First we give some notation.

A.2. Let \( G \) be a \( \mathbb{Z} \)-graded ring concentrated in non-positive degrees (i.e., \( G_i = 0 \) for all \( i > 0 \)). Note that given a graded \( G \)-module \( H \), for any integer \( r \), we have that \( H_{<r} := \bigoplus_{i < r} H_i \) is a \( G \)-graded submodule of \( H \) and \( H_{\geq r} := H/H_{<r} \) is a \( G \)-graded quotient module of \( H \). Following Dao, we say that a graded \( G \)-module \( H \) is almost artinian if there is an integer \( r \) such that \( H_{\geq r} \) is artinian as a \( G \)-module.

Lemma A.3. Let \( G \) and \( H \) be as in paragraph A.2. Assume that \( H_i \) is an artinian \( G_0 \)-module for all \( i \gg 0 \). Let \( X : H \rightarrow H \) be a homogeneous \( G \)-linear map of negative degree. If \( \ker(X) \) is almost artinian as a \( G \)-module, then \( H \) is almost artinian as a \( G[X] \)-module.

Proof. Let \( X \) have degree \( w < 0 \). For each \( r \), there is a map of degree \( w \) on quotient modules given by multiplication by \( X \):

\[
X_{\geq r} : H_{\geq r} \rightarrow H_{\geq r+w}.
\]

Note that \( \ker(X_{\geq r}) = \ker(X)_{\geq r} \), where the latter uses the notation of A.2 and hence \( \ker(X_{\geq r}) \) is artinian as a \( G \)-module for all \( r \gg 0 \), by assumption.

Since \( w < 0 \), there is a canonical surjection \( \pi_{\geq r+w} : H_{\geq r+w} \twoheadrightarrow H_{\geq r} \) of graded \( G \)-modules having degree 0. Define \( Y_{\geq r} = \pi_{\geq r+w} \circ X_{\geq r} \), so that \( Y_{\geq r} \) is the endomorphism of \( H_{\geq r} \) of degree \( w \) given by multiplication by \( X \), and we have the left exact sequence

\[
0 \rightarrow \ker(X_{\geq r}) \rightarrow \ker(Y_{\geq r}) \xrightarrow{X} \ker(\pi_{\geq r+w}).
\]

The module \( \ker(\pi_{\geq r+w}) \) may be regarded as a \( G_0 \)-module via restriction of scalars along the inclusion \( G_0 \hookrightarrow G \). As a \( G_0 \)-module, \( \ker(\pi_{\geq r+w}) \) is a finite direct sum of \( H_i \) for \( i = r + w, \ldots, r \). Hence for \( r \gg 0 \), it is an artinian \( G_0 \)-module by our assumption. So for \( r \gg 0 \), it follows that \( \ker(\pi_{\geq r+w}) \) must also be artinian as a \( G \)-module. Thus for \( r \gg 0 \), the module \( \ker(Y_{\geq r}) \) is an artinian \( G \)-module, as follows from exact sequence A.1.

It follows from [10, Lemma 1.2] that \( H_{\geq r} \) is artinian as a graded \( G[Y_{\geq r}] \)-module, for \( r \gg 0 \); that is, \( H \) is an almost artinian \( G[X] \)-module. □
The Koszul algebra $K$ associated to a regular sequence $(f_1, \ldots, f_c)$ of elements of a commutative ring $Q$ is defined to be the following DG $Q$-algebra: The underlying graded $Q$-algebra is the exterior algebra $\Lambda^*_Q(Q^c)$ on the free $Q$-module $Q^c$, indexed so that $\Lambda^j_Q(Q^c)$ lies in homological degree $j$. Let $T_1, \ldots, T_c$ be the standard basis of $Q^c$. The differential $\partial$ of $K$ is uniquely determined by setting $\partial(T_i) = f_i$ and requiring that it satisfy the Leibniz rule: $\partial(ab) = \partial(a)b + (-1)^{\deg a}a\partial(b)$.

The Koszul algebra comes equipped with a ring map, called the augmentation, to its degree zero homology, namely $H_0(K) = Q/(f_1, \ldots, f_c) =: R$. Since $(f_1, \ldots, f_c)$ is $Q$-regular, the augmentation $K \to R$ is a quasi-isomorphism, so that $K$ is a DG algebra resolution of $R$ over $Q$ that is free as a $Q$-module. Recall that a DG module over $K$ is a graded $K$-module $L$ equipped with a differential $d_L$ of degree minus one so that $d_L(ax) = \partial(a)x + (-1)^{\deg a}ad_L(x)$ holds for all homogeneous elements $a \in K$ and $x \in L$. See [10] for more background material on DG algebras and DG modules.

A.4. Let $K$ be Koszul algebra over $Q$ associated to a regular sequence $f_1, \ldots, f_c$ in $Q$ and let $I = \ker(K \to R)$ be the augmentation ideal. Let $L$ be a DG module over $K$ that is graded free as a module over the graded ring underlying $K$. Let $N$ be a finitely generated $R$-module. Define $Y$ to be $L \otimes_Q R = L/(f_1, \ldots, f_c)L$. For any subset $S = \{i_1, \ldots, i_s\}$ of $\{1, \ldots, c\}$, define $I_S = (T_{i_1}, \ldots, T_{i_s})$ and set $Y^S = Y/I_S$. In particular, $Y^0 = Y$ and $Y^{1, \ldots, c} = Y/IY = L/IL = L \otimes_R R$.

For each $S$, the complex $Y^S$ is a complex of $Q$-modules, and in fact of $R$-modules.

A.5. Gulliksen shows [10] p. 176–8] that for $i \in S$ there is an exact sequence of complexes of $Q$-modules

$$0 \to Y^S \xrightarrow{T_i} Y^{S\setminus\{i\}} \to Y^S \to 0$$

that is degree-wise split exact. It follows that

$$0 \to Y^S \otimes_Q N \to Y^{S\setminus\{i\}} \otimes_Q N \to Y^S \otimes_Q N \to 0$$

is also exact, giving a long exact sequence in homology. The boundary map in this sequence, $H(Y^S \otimes_Q N) \to H(Y^S \otimes_Q N)$, is, up to sign, the action of $X_i$ on $H(Y^S \otimes_Q N)$ as defined by Gulliksen. Thus $X_i$ has degree $-2$, since $T_i$ has degree $+1$. Gulliksen proves that these actions commute: $X_iX_j = X_jX_i$ on $H(Y^S \otimes_Q N)$ when $i, j \in S$. When $S = \{1, \ldots, c\}$, these actions endow $H(L/IL \otimes_R N)$ with the structure of a graded module over the graded ring $R[X_1, \ldots, X_c]$ where each $X_i$ has degree $-2$.

Our next lemma is similar to [10] Lemma 3.2(ii)].

Lemma A.6. With the assumptions in paragraph A.4 and with the $X_i$ from paragraph A.3 if $H_i(L/IL \otimes_R N)$ is artinian as an $R$-module and $H_i(L \otimes_Q N) = 0$ for $i \gg 0$, then $H_i(L/IL \otimes_R N)$ is almost artinian as an $R[X_1, \ldots, X_c]$-module.

Proof. We have that for $i \gg 0$, the $R$-module $H_i(Y^{1, \ldots, c} \otimes_R N) = H_i(L/IL \otimes_R N)$ is artinian and $H_i(Y^0 \otimes_Q N) = H_i(L \otimes_Q N) = 0$.

We first observe that, for every $S \subseteq \{1, \ldots, c\}$, $H_i(Y^S \otimes_Q N)$ is also artinian over $R$, for $i \gg 0$. Indeed, this follows immediately by descending induction on the cardinality of $S$ using the long exact sequence in homology associated to the exact sequence (A.2).
For $S = \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, c\}$, let $G^S = R[X_{i_1}, \ldots, X_{i_s}]$. In particular, $G^\emptyset = R$ and $G^{\{1, \ldots, c\}} = R[X_1, \ldots, X_c]$. We prove $H(Y^S \otimes_Q N)$ is an almost artinian $G^S$-module for each $S$, by induction on the cardinality of $S$. When $S = \emptyset$, then $Y^S = Y^\emptyset = L$ and, by assumption, $H(Y^\emptyset \otimes_Q N) = H(L \otimes_Q N)$ is an almost artinian $R$-module.

Assume that $i \in S$. The exact sequence (A.2) gives an exact homology sequence

$$
H(Y^{S \setminus \{i\}} \otimes_Q N) \longrightarrow H(Y^S \otimes_Q N) \longrightarrow X_i H(Y^S \otimes_Q N).
$$

By the induction hypothesis, $H(Y^{S \setminus \{i\}} \otimes_Q N)$ is almost artinian as a $G^{S \setminus \{i\}}$-module, and since the graded quotient of an almost artinian module is almost artinian, we see that $\ker(X_i)$ is almost artinian. Since $H_s(Y^S \otimes_Q N)$ is artinian over $G_0 = R$, for $q \gg 0$, as was shown above, Lemma A.3 applies to show $H(Y^S \otimes_Q N)$ is an almost artinian $G^S$-module.

**Proof of Proposition A.1.** Regard $M$ as a DG module concentrated in degree 0 via restriction of scalars along the augmentation. Gulliksen shows in [10, Lemma 2.4] how to construct a DG module $L$ over $K$ and a map $L \to M$ of DG modules such that $L$ is free over the graded ring underlying $K$ and the map $L \to M$ is a quasi-isomorphism. Note that $L \to M$ is, in particular, a resolution of $M$ by free $Q$-modules. Moreover, Gulliksen shows [10, Lemma 2.6] that the projection map $L \to L/IL = L \otimes_K R$ is a quasi-isomorphism where $I = \ker(K \to R)$ is the augmentation ideal. In particular, this means that $L/IL$ is an $R$-free resolution of $M$. We therefore obtain the isomorphisms

$$
H_i(L \otimes_Q N) \cong \operatorname{Tor}_i^Q(M, N) \quad \text{and} \quad H_i(L/IL \otimes_R N) \cong \operatorname{Tor}_i^R(M, N)
$$

for any $R$-module $N$.

Lemma A.6 now applies to prove that $\bigoplus_i \operatorname{Tor}_i^R(M, N)$ is almost artinian as a $R[X_1, \ldots, X_c]$-module. Finally, Avramov and Sun [4, §4] prove that the $X_i$’s as constructed by Gulliksen agree with the Eisenbud operators, up to a sign. □

**Acknowledgement.** We thank the referee for helpful comments regarding this paper.

**References**

[1] L. Avramov, R.O. Buchweitz, *Lower bounds for Betti numbers*, Compositio Mathematica 86 (1993) no. 2, 147-158. MR1214454 (94a:13011)

[2] L. Avramov, V. Gasharov, I. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. 86 (1998) 67-114. MR1608565 (99c:13033)

[3] L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118.

[4] L. Avramov, L.C. Sun, *Cohomology operators defined by a deformation*, J. Algebra. 204 (1998), no. 2, 684-710.

[5] O. Celikbas, *Vanishing of Tor over complete intersections*, J. Comm. Alg. 3 (2011) no. 2, 169-206. MR2813471

[6] H. Dao, *Decent intersection and Tor-rigidity for modules over local hypersurfaces*, to appear in Trans. Amer. Math. Soc., preprint http://arxiv.org/pdf/math/0611568v4.pdf

[7] H. Dao, *Asymptotic behavior of Tor over complete intersections and applications*, preprint http://arxiv.org/pdf/0710.5818v1.pdf

[8] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. 260 (1980) no. 1, 35–64. MR0570778 (82d:13013)
14 W. FRANK MOORE, GREG PIEPMeyer, SANDRA SPIROFF, AND MARK E. WALKER

[9] Eberhard Freitag and Reinhardt Kiehl, Étale cohomology and the Weil conjecture, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)] 13, Springer-Verlag, Berlin, 1988, xviii+317. MR 926276 (89f:14017)

[10] T. H. Gulliksen, A change of ring theorem with applications to Poincaré series and intersection multiplicity, Math. Scand. 34 (1974), 167–183. MR0364232 (51 #487)

[11] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics 52, Springer-Verlag, New York, 1977, xvi+496. MR0463157 (57 #3116)

[12] M. Hochster, The dimension of an intersection in an ambient hypersurface, Proceedings of the First Midwest Algebraic Geometry Seminar (Chicago Circle, 1980), Lecture Notes in Mathematics 862, Springer-Verlag, New York, 1981, 93–106. MR0644818 (83g:13017)

[13] D. Jorgensen, Complexity and Tor on a complete intersection, J. Algebra 211 (1999) no. 2, 578-598. MR1666660 (99k:13014)

[14] W. F. Moore, G. Piepmeyer, S. Spiroff, M. E. Walker, Hochster’s theta invariant and the Hodge-Riemann bilinear relations, Advances in Math. 226 (2010) no. 2, 1692-1714. MR2737797 (2011m:13029)

[15] M. P. Murthy, Modules over regular local rings, Illinois J. Math. 7 (1963), 558-565. MR0156883 (28 #126)

[16] J. P. Serre, Algèbre locale. Multiplicités., Cours au Collège de France, 1957–1958, Lecture Notes in Mathematics 11, Springer-Verlag, New York, 1965, xvi+496. MR0201468 (34 #1352)

[17] Théorie des topos et cohomologie étale des schémas, Tome 3, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Lecture Notes in Mathematics 305, Springer-Verlag, Berlin, 1973. MR0354654 (50 #7132)

[18] Groupes de monodromie en géométrie algébrique. II, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Lecture Notes in Mathematics 340, Springer-Verlag, Berlin, 1973. MR0354657 (50 #7135)

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