Non-existence of Mean-Field Models for Particle Orientations in Suspensions

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Abstract
We consider a suspension of spherical inertialess particles in a Stokes flow on the torus $\mathbb{T}^3$. The particles perturb a linear extensional flow due to their rigidity constraint. Due to the singular nature of this perturbation, no mean-field limit for the behavior of the particle orientation can be valid. This contrasts with widely used models in the literature such as the FENE and Doi models and similar models for active suspensions. The proof of this result is based on the study of the mobility problem of a single particle in a non-cubic torus, which we prove to exhibit a nontrivial coupling between the angular velocity and a prescribed strain.

Keyword Mean-field limit · Doi model · oriented particles · suspensions

Mathematics Subject Classification 35Q70 · 70E55 · 76T20 · 74F10

1 Introduction
Recent years have seen considerable progress regarding first-order mean-field limits for singularly interacting particle systems. Hauray showed in Hauray (2009) how to deal with certain mean-field limits with interaction kernels $K_s$ that are $-s$-
homogeneous for \( s < d - 1 \). In the seminal paper (Serfaty 2020), Serfaty proved the mean-field limit for particles interacting through the Coulomb potential (cf. Bresch et al. 2020 for related results with noise and attractive potentials). The results in Serfaty (2020) notably cover super-coulombic repulsive Riesz potentials \( g_s \) with interactions kernels \( K_s = \nabla g_s \) which are \(-s\)-homogeneous as long as \( s < d + 1 \). The proof is based on a modulated energy method.

For many mean-field systems, a suitable energy seems to be lacking, though, and the mean-field limit still remains an open problem. In particular, this is the case for certain problems in the context of suspensions where the particles interact though the fluid with each other in an implicit and singular manner. While certain mean-field problems for suspensions correspond to kernels that are \(-(d - 2)\)-homogeneous and therefore fall into the framework of Hauray (2009), more singular interaction kernels are relevant for problems that involve the particle orientations. In the present paper, we consider a mean-field system that corresponds to an \(-d\)-homogeneous interaction kernel that arises naturally in the context of suspensions. Even though the singularity is weaker than the ones covered in Serfaty (2020), we show that no mean-field limit can exist for this model. We emphasize that the failure is due to the singularity of kernel and not due to initial clustering of the particles.

Interest in mean-field limits for suspensions arises from the observation that particles suspended in a fluid change the rheological properties of the fluid flow. For inertialess rigid passive non-Brownian particles this accounts to an increased viscous stress. In more complex models like active (self-propelled) particles or non-spherical Brownian particles, an additional active or elastic stress arises that renders the fluid viscoelastic. Over the last years, considerable effort has been invested into the rigorous derivation of effective models for suspensions. This has been quite successful regarding the derivation of effective fluid equations in models when only a snapshot in time is studied for a prescribed particle configuration or for certain toy models that do not take into account the effects of the fluid on the particle evolution (see e.g., Haines and Mazzucato 2012; Niethammer and Schubert 2020; Hillairet and Wu 2020; Gérard-Varet and Hillairet 2020; Gérard-Varet and Mecherbet 2022; Duerinckx and Gloria 2020; Gérard-Varet and Höfer 2021; Duerinckx and Gloria 2021; Girodroux-Lavigne 2022; Höfer et al. 2023).

Much less is known regarding the rigorous derivation of fully coupled models between the fluid and dispersed phase, although a number of such models have been proposed a long time ago and some of them have been studied extensively in the mathematical literature. These models typically consist of a transport or Fokker–Planck-type equation for the particle density coupled to a fluid equation incorporating the effective rheological properties.

The rigorous derivation of such models is so far limited to sedimenting spherical particles where the transport Stokes system has been established in Höfer (2018) and Mecherbet (2019) to leading order in the particle volume fraction. This system reads

\[
\begin{cases}
\partial_t \rho + (u + g) \cdot \nabla \rho = 0,

-\Delta u + \nabla p = g \rho, \\
\text{div} u = 0,
\end{cases}
\]

where \( \rho(t, x) \) is the number density of particles, \( g \in \mathbb{R}^3 \) is the constant gravitational acceleration. Here, the gravity is dominating over the change of the rheological prop-

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erties of the fluid, which only appears as a correction to the next order in the particle volume fraction $\phi$. More precisely, as shown in Höfer and Schubert (2021), a more accurate description is given by the system

\[
\begin{align*}
\partial_t \rho + (u + g) \cdot \nabla \rho &= 0, \\
- \text{div}( (2 + 5\phi \rho) Du ) + \nabla p &= g \rho, \quad \text{div} u = 0,
\end{align*}
\] (1.1)

where $Du = 1/2(\nabla u + (\nabla u)^T)$ denotes the symmetric gradient.

For non-spherical particles, the increase in viscous stress depends on the particle orientations (see e.g., Hillairet and Wu 2020 for a rigorous result in the stationary case). Moreover, elastic stresses are observed for non-spherical Brownian particles as well as active stresses for self-propelled particles, both depending on the particle orientation. Therefore, it is necessary to consider models for the evolution of particle densities $f$ that include the particle orientation. In the simplest case of identical axisymmetric particles, the particle orientation can be modeled by a single vector $\xi \in S^2$. The model corresponding to (1.1) then reads

\[
\begin{align*}
\partial_t f + (u + g) \cdot \nabla f + \text{div}_\xi \left( \left( \frac{1}{2} \text{curl} u \wedge \xi + B P_{\xi \perp} Du \xi \right) f \right) &= 0, \\
- \Delta u + \nabla p - \text{div}(\phi M[f] Du) &= g \rho, \quad \text{div} u = 0,
\end{align*}
\] (1.2)

Here, $P_{\xi \perp}$ denotes the orthogonal projection in $\mathbb{R}^3$ to the subspace $\xi \perp$ and $B$ is the Bretherton number that depends only on the particle shape ($B = 0$ for spheres, $B = 1$ in the limit of very elongated particles, see e.g., [Graham 2018, Section 3.8]). Moreover, $M[f]$ is a fourth-order tensor depending on the particle shape and given in terms of moments of $f$. The derivation of (1.2) has been addressed in Duerinckx (2023).

A widely used model for Brownian suspensions of rod-like (Bretherton number $B = 1$) particles at very small particle volume fraction $\phi$ is the so-called Doi model (see e.g., Doi and Edwards 1988; Constantin 2005; Helzel and Otto 2006; Lions and Masmoudi 2007; Constantin et al. 2007; Constantin and Masmoudi 2008; Zhang 2008; Otto and Tzavaras 2008; Constantin and Serre 2009, 2010; Bae and Trivisa 2012, 2013; Helzel and Tzavaras 2017; La 2019) that reads (in the absence of fluid inertia)

\[
\begin{align*}
\partial_t f + \text{div}( u f ) + \text{div}_\xi ( P_{\xi \perp} \nabla_x u \xi f ) &= \frac{1}{\text{De}} \Delta_x f + \frac{\lambda_1}{\text{De}} \text{div}_x ((\text{Id} + \xi \otimes \xi) \nabla_x f), \\
- \Delta u + \nabla p - \text{div} \sigma &= h, \quad \text{div} u = 0, \\
\sigma = \sigma_v + \sigma_e = \phi M[f] Du + \frac{\lambda_2 \phi}{\text{De}} \int_{S^2} (3\xi \otimes \xi - \text{Id}) f \, d\xi.
\end{align*}
\] (1.3)

Here, $\text{De}$ is the Deborah number, $\lambda_1$, $\lambda_2$ are constants that depend on the particle shape, and $h$ is some given source term. Neglecting the effect of the fluid on the particles, the elastic stress $\sigma_e$ in the Doi model has been recently derived in Höfer et al. (2023). There are very similar models for active suspensions (the Doi–Saintillan–Shelley model) and flexible particles, most prominently the FENE model. Well-posedness and behavior
of solutions to such models have been studied for example in Jourdain et al. (2002), Jourdain et al. (2004), Jourdain et al. (2006), Le Bris and Lelièvre (2007), Saintillan and Shelley (2008), Saintillan and Shelley (2008), Le Bris and Lelièvre (2012), Chen and Liu (2013), Masmoudi (2013), Saintillan (2018), Coti Zelati et al. (2022) and Albritton and Ohm (2022).

In this paper, we want to draw attention to the limitations of such fully coupled models like (1.2) and (1.3) regarding the modeling of the particle orientations through the term \( \text{div}_\xi (P_{\xi\|} \nabla_x u_{\xi} f) \) (respectively \( \text{div}_\xi ((1/2 \text{ curl } u \wedge \xi + B P_{\xi\|} D u_{\xi}) f) \) for \( B \neq 1 \)). This term derives from the change of orientation for the particles according to the gradient of the fluid velocity. However, at least partially, this fluid velocity is arising as a perturbation flow due to the presence of the particles themselves that cause the viscous and elastic stresses \( \sigma_v \) and \( \sigma_e \). These perturbations are typically of order \( \phi \).

On the microscopic level, the perturbed fluid velocity is very singular. More precisely, to leading order, it behaves like the sum of stresslets. At the \( i \)-th particle, it is given by

\[
 u^\text{pert}_N (X_i) \approx \sum_{j \neq i} \nabla \Phi(X_i - X_j) : S_j
\]

where the sum runs over all particles \( j \) different from \( i \), \( \Phi \) is the fundamental solution of the Stokes equation, and \( S_j \) is the moment of stress induced at the \( j \)-th particle due to the rigidity constraint (and possibly activeness or flexibility). Consequently, the change of orientation behaves like

\[
 \dot{\xi}_i = P_{\xi_i\|} \xi_i \cdot \nabla u^\text{pert}_N (X_i) \approx P_{\xi_i\|} \xi_i \sum_{j \neq i} \nabla^2 \Phi(X_i - X_j) : S_j
\]

As \( \Phi \) is homogeneous of degree \(-1\), this behavior is too singular to expect the “naive” mean-field limit to be true that would lead to the term \( \text{div}_\xi (P_{\xi\|} \nabla_x u_{\xi} f) \) (respectively \( \text{div}_\xi ((1/2 \text{ curl } u \wedge \xi + B P_{\xi\|} D u_{\xi}) f) \) for \( B \neq 1 \)) in the models (1.2) and (1.3). These models therefore do not seem to describe correctly the behavior of the particle orientations to first order in the particle volume fraction \( \phi \). We emphasize that this is in accordance with the result in Duerinckx (2023): There, roughly speaking, it is shown that (1.2) accurately describes the microscopic fluid velocity field up to errors of order \( o(\phi) \), but the dynamics of the particle density is only captured up to order \( O(\phi) \).

Instead, for an accurate description up to order \( \phi \) for such models, it seems necessary to add to the naive mean-field term another term that captures the singular interactions on small scales. Under suitable assumptions, one could expect this term to be expressed by the microscopic 2-point correlation function. This is reminiscent of the second-order correction in \( \phi \) of the effective viscous stress \( \sigma_v \) (see Gerard-Varet and Hillairet 2020; Gérard-Varet and Mecherbet 2022; Duerinckx and Gloria 2021). For the evolution of the particle orientation, this phenomenon already appears at the first order, since the particle orientations are only sensitive to the gradient of the fluid velocity. We leave the corresponding analysis for the mean-field models addressed here to future research.

In this paper, we will rigorously demonstrate the failure of the mean-field limit to order \( \phi \) for a toy model. More precisely, we consider a model example in which
we show the non-existence of any mean-field model that incorporates the change of orientations to the leading order in the perturbation field of the fluid. This model example consists of a suspension of spherical particles in a background flow which is a linear extensional flow.

In a bounded domain $\Omega \subset \mathbb{R}^3$, the problem would read

\[
\begin{align*}
-\Delta u + \nabla p &= 0, \quad \text{div} \, u = 0 \quad \text{in } \Omega \setminus \bigcup_i B_i, \\
\quad u(x) &= \phi^{-1}Ax \quad \text{on } \partial \Omega, \\
\quad Du &= 0 \quad \text{in } \bigcup_i B_i, \\
\int_{\partial B_i} \sigma[u]n \, dS &= \int_{\partial B_i} (x - X_i) \wedge \sigma[u]n \, dS = 0 \quad \text{for all } 1 \leq i \leq N,
\end{align*}
\]

where $B_i = B_R(X_i)$ denote the spherical particles and $A \in \text{Sym}_0(3)$ is a symmetric tracefree matrix. The rescaling with the volume fraction $\phi = NR^3$ is introduced in order to normalize the perturbation fluid velocity field $u_{\text{per}}$ induced by the particles.

For mathematical convenience, we consider the analogous problem on the torus $\mathbb{T}^3$. We attach (arbitrary) orientations $\xi_i \in S^2$ to the spheres and show that no mean-field limit can exist by proving that for periodically arranged particles on $\mathbb{Z}^3$, the particles do not rotate at all, while for particles arranged periodically on $(2\mathbb{Z}) \times \mathbb{Z}^2$, the particles do rotate with a fixed rate.

1.1 Statement of the Main Results

We will work on the toroidal domains

\[
\mathbb{T} = (\mathbb{R}/\mathbb{Z})^3, \quad \mathbb{T}_L = (\mathbb{R}/2L\mathbb{Z})^3, \quad \mathbb{T}_L = (\mathbb{R}/4L\mathbb{Z}) \times (\mathbb{R}/2L\mathbb{Z})^2.
\]

Furthermore, we set

\[
B = B_1(0), \quad B_R = B_R(0),
\]

and for definiteness,

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

**Theorem 1.1** For $0 < R < 1/2$ and $\Omega = \mathbb{T}_1$ or $\Omega = \mathbb{T}_1$, let $u \in H^1(\Omega)$ be the unique weak solution to the problem

\[
\begin{align*}
-\Delta u + \nabla p &= 0, \quad \text{div} \, u = 0 \quad \text{in } \Omega \setminus B_R, \\
\quad Du &= A \quad \text{in } B_R, \\
\int_{\Omega} u \, dx &= 0, \\
\int_{\partial B_R} \sigma[u]n \, dS &= \int_{\partial B_R} x \wedge \sigma[u]n \, dS = 0.
\end{align*}
\]
(i) If $\Omega = \mathbb{T}_1$, then $\text{curl } u(0) = 0$.
(ii) If $\Omega = \mathbb{T}_1$, then there exists $\tilde{c} > 0$ such that

\[
\left| \text{curl } u(0) - \frac{R^3}{16} \tilde{c} e_3 \right| \leq C R^4
\]  

for a constant $C$ independent of $R$.

Observe that the factor 16 in (1.5) corresponds to the volume of $\mathbb{T}_1$.

As a consequence of Theorem 1.1, we show the negative result stated in Corollary 1.2 that we outlined in the introduction. We will use the following notation. For $N \in \mathbb{N}$, which we think of as the number of particles in a unit cell, we denote by

$$\phi = N R^3$$

the volume fraction of the particles. Both $R$ and $\phi$ may implicitly depend on $N$. We will make this dependence explicit by a subscript $N$ wherever we feel it is necessary for clarity. For $R > 0$ and $N \in \mathbb{N}$, let $X_i \in \mathbb{T}$, $1 \leq i \leq N$ be such that

$$d_{\min} = \min_{i \neq j} |X_i - X_j| > 2R.$$  

(1.6)

This ensures that the balls

$$B_i = B_R(X_i)$$

do not intersect nor touch each other. Moreover, for $1 \leq i \leq N$, let $\xi_i^0 \in \mathbb{S}^2$. The associated initial empirical density $f^0_N \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{S}^2)$ is given by

$$f^0_N = \frac{1}{N} \sum_i \delta_{(X_i, \xi_i^0)}.$$  

Consider the dynamics

$$\frac{d}{dt} \xi_i = \omega_i \wedge \xi_i$$  

(1.7)

where $\omega_i = \frac{1}{2} \text{curl } u(X_i)$ for the solution $u \in \dot{H}^1(\mathbb{T})$ to the problem

\[
\begin{aligned}
-\Delta u + \nabla p &= 0, \quad \text{div } u = 0 \quad \text{in } \mathbb{T} \setminus \bigcup_i B_i, \\
Du &= \phi^{-1} A \quad \text{in } \bigcup_i B_i, \\
\int_{\mathbb{T}} u \, dx &= 0, \\
\int_{\partial B_i} \sigma[u] n \, dS &= \int_{\partial B_i} (x - X_i) \wedge \sigma[u] n \, dS = 0 \quad \text{for all } 1 \leq i \leq N.
\end{aligned}
\]  

(1.8)
Notice that \( u \) is the normalized perturbation field induced by the particles with respect to the background flow \(-Ax\). We then write

\[
f_N(t) := \frac{1}{N} \sum_i \delta(X_i, \xi_i(t)). \tag{1.9}
\]

In the following, we denote by \( W_p, p \in [1, \infty] \) the usual \( p \)-Wasserstein distance (cf. for example Santambrogio 2015) on the space of probability measures \( f \in \mathcal{P}(\mathbb{T} \times \mathbb{S}^2) \).

We recall that \( W_p \leq W_q \) for \( p \leq q \) and that \( W_1 \) metrizes weak convergence of measures.

**Corollary 1.2** For all sequences \( R_N \to 0 \) with \( \phi_N \to 0 \), there exist constants \( c, T > 0 \) and \( (X_1, \cdots, X_N) \in \mathbb{T}^N \) and \( (\xi_0^0, \cdots, \xi_N^0) \in (\mathbb{S}^2)^N, N \in \mathbb{N} \), such that the associated empirical measures \( f_N \in C([0, \infty); \mathcal{P}(\mathbb{T} \times \mathbb{S}^2)) \) defined by (1.9) and (1.7) satisfy the following properties:

(i) With \( d_{\min} \) defined as in (1.6)

\[
d_{\min} \geq cN^{-1/3}.
\]

(ii) There exists \( f^0 \in \mathcal{P}(\mathbb{T} \times \mathbb{S}^2) \cap C^\infty(\mathbb{T} \times \mathbb{S}^2) \) such that \( W_\infty(f_N^0, f^0) \to 0 \).

(iii) There exist at least two distinct accumulation points of \( f_N \). More precisely, there exist subsequences \( f_{N_k}, f_{\bar{N}_k} \) and \( f, \bar{f} \in C^\infty([0, \infty); \mathbb{T} \times \mathbb{S}^2) \) which satisfy

\[
\sup_{t \in [0, T]} W_\infty(f_{N_k}(t), f(t)) + W_\infty(f_{\bar{N}_k}(t), \bar{f}(t)) \to 0, \tag{1.10}
\]

\[
W_1(f(t), \bar{f}(t)) \geq ct \text{ for all } t \leq T. \tag{1.11}
\]

Several remarks are in order.

- Observe that the corollary indeed shows that no general mean-field model can describe the effective behavior of the microscopic system (1.7)–(1.8) since there is a sequence \( f_N \) that on the one hand converges at the initial time to some \( f^0 \) but that on the other hand has at least two distinct accumulation points for \( 0 < t \leq T \). In particular, the “naive” mean-field limit

\[
\begin{aligned}
\partial_t f + \text{div}_\xi \left( \frac{1}{2} \text{curl} u \wedge \xi f \right) &= 0, \\
-\Delta u + \nabla p &= -5 \text{div} \left( A \int_{\mathbb{S}^2} f \, d\xi \right), \quad \text{div} u = 0
\end{aligned} \tag{1.12}
\]

cannot hold true. Here, the factor 5 arises as the relation between the strain and stress of an isolated sphere in an infinite fluid, cf. (1.1). Note that the momentum equation in (1.12) can be obtained from \(- \text{div}(2 + 5\phi \rho)Dv + \nabla p\) from the ansatz \( v(x) = -Ax + \phi u(x) \) upon taking \( \phi \to 0 \).

- Condition (i) ensures that the non-convergence is not caused by particle clusters but appears for well-separated particles.
An adapted version of the statement remains true when one takes into account the time-evolution of the particle positions according to

\[ \frac{d}{dt} X_i = u(X_i). \]

Indeed, as pointed out above, regarding the translations, the “naive” mean-field limit does hold, at least as long as the particles remain well-separated (cf. Höfer and Schubert 2021). In the proof of the corollary, we only consider distributions of particles which are periodic in space. Since periodicity is preserved under the dynamics, such clustering cannot occur. Nevertheless, we restrict ourselves to the case of fixed particle centers for the sake of the simplicity of the presentation.

The regularity of the limiting density \( f \) strengthens the statement. Indeed, the simplest approach would be to consider particles that all have the same orientation, which leads to a delta distribution in orientation for \( f \).

### 2 Proof of Corollary 1.2

Let \( R_N \to 0 \) with \( \phi_N \to 0 \) be given.

Let \( f^0(x, \xi) = h(\xi) \) for some \( h \in \mathcal{P}(S^2) \cap C^\infty(S^2) \) that will be chosen later. For \( k \in \mathbb{N} \), let \( N_k = (k)^3 \) and \( \tilde{N}_k = 4k^3 \) and let \( \{X_i\}_{i=1}^{N_k} = \left(\frac{1}{k}\mathbb{Z}\right)^3 \subset \mathbb{T} \) and \( \{\tilde{X}_i\}_{i=1}^{\tilde{N}_k} = \left(\frac{1}{k}\mathbb{Z}\right) \times \left(\frac{1}{2k}\mathbb{Z}\right)^2 \subset \mathbb{T} \). Define

\[ f^0_{N_k} = \frac{1}{N_k} \sum_i \delta_{(X_i, \xi_i^0)}, \]
\[ f^0_{\tilde{N}_k} = \frac{1}{N_k} \sum_i \delta_{(\tilde{X}_i, \tilde{\xi}_i^0)} \]

where the initial orientations \( \xi_i^0 \) and \( \tilde{\xi}_i^0 \) are chosen in such a way that both \( W^\infty(f^0_{N_k}, f^0) \to 0 \) and \( W^\infty(f^0_{\tilde{N}_k}, f^0) \to 0 \) (for example by taking samples of initial distributions \( \xi_i^0 \) and \( \tilde{\xi}_i^0 \) which are i.i.d. distributed with law \( h \)).

By a suitable choice of \( f^0_N \) for \( N \notin \{k^3 : k \in \mathbb{N}\} \cup \{4k^3 : k \in \mathbb{N}\} \), we can ensure that items (i) and (ii) are satisfied. We will show that item (iii) holds true with \( f(t, x, \xi) := h(\xi) \) and \( \tilde{f}(t, x, \bar{\xi}) := h(e^{-\frac{\bar{e}}{2}M\bar{t}}\bar{\xi}) \), where \( \bar{c} \) is the constant from Theorem 1.1 and

\[ M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

is the unique skew-symmetric matrix satisfying \( Mv = e_3 \wedge v \) for all \( v \in \mathbb{R}^3 \).
In order to show (1.10), it suffices to prove that
\[ \dot{\xi}_i = 0 \quad \text{for all} \quad 1 \leq i \leq N_k, \]
\[ \dot{\bar{\xi}}_i = \frac{\bar{c}}{2} e_3 \wedge \xi_i + O(R_N) \quad \text{for all} \quad 1 \leq i \leq \bar{N}_k. \]

This is an immediate consequence of Theorem 1.1.

It remains to prove (1.11). We use the well-known characterization (cf. [Santambrogio 2015, Equation (3.1)])
\[
W_1(f(t), \bar{f}(t)) = \sup \int_{T \times \mathbb{S}^2} (f(t) - \bar{f}(t)) \varphi \, d\xi \, dx : \varphi: \mathbb{T} \times \mathbb{S}^2 \rightarrow \mathbb{R} \text{ is } 1-Lipschitz.
\]

Choosing \( \varphi(x, \xi) = \xi_1 \) yields
\[
W_1(f(t), \bar{f}(t)) \geq \int_{\mathbb{S}^2} \left( \xi_1 - \left( e^{\frac{\xi_2}{2} M t} \xi_1 \right) \right) h(\xi) \, d\xi =: g(t)
\]

We observe that
\[
g'(0) = \int_{\mathbb{S}^2} \frac{\bar{c}}{2} \xi_2 h(\xi) \, d\xi.
\]
Since \( h \in \mathcal{P}(\mathbb{S}^2) \cap C^\infty(\mathbb{S}^2) \) was arbitrary, we may choose \( h \) in such a way that \( g'(0) > 0 \) (e.g., by taking \( h \) with \( \text{supp } h \subset \{ \xi_2 > 0 \} \)). Then, (1.11) holds.

3 Proof of Theorem 1.1

3.1 Proof of the First Item

For this subsection, denote by \( u_A \) the solution to (1.4) for \( \Omega = \mathbb{T}_1 \), i.e.,
\[
\begin{cases}
-\Delta u_A + \nabla p = 0, & \text{div } u_A = 0 \quad \text{in} \ \mathbb{T}_1 \setminus B_R, \\
Du_A = A & \text{in} \ B_R, \\
\int_{\mathbb{T}_1} u_A = 0, & \int_{\partial B_R} \sigma[u_A] n \, dS = \int_{\partial B_R} x \wedge \sigma[u_A] n \, dS = 0.
\end{cases}
\]

Let \( S \in SO(3) \) be any rotation matrix that leaves the torus \( \mathbb{T}_1 \) invariant. Then, we have
\[ u_{S^{\top}A} = S^{\top} u_A(Sx), \quad u_{-A}(x) = -u_A(x). \]

Therefore, denoting by \( \omega[u_A] := \frac{1}{2} \text{curl } u_A(0) \), the angular velocity of the particle associated with \( u_A \), one can show that
\[
\omega[u_{S^{\top}A}] = S^{\top} \omega[u_A], \quad \omega[u_{-A}] = -\omega[u_A]. \quad (3.1)
\]
Taking $S = S_k$, $k = 1, 2, 3$,

$$
S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

(3.2)

which all have the property $S_k^T A S_k = -A$, we deduce from (3.1) that the three components of $\omega[A]$ are vanishing.

### 3.2 Proof of the Second Item

Let now $u_A$ be the solution to (1.4) with $\Omega = \bar{T}_1$. The fact that the first two components of $\omega[u_A] := \frac{1}{2} \text{curl} \ u_A(0)$ vanish can be shown by the same argument as in Sect. 3.1, considering $S_1$ and $S_2$ in (3.2) that leave $\bar{T}_1$ invariant.

For convenience, we consider the rescaled torus $\bar{T}_{1/R}$ instead of $\bar{T}_1$ and set $L = 1/R$. More precisely, we consider $u$ to be the solution to

$$
\begin{cases}
-\Delta u + \nabla p = 0, & \text{in } \bar{T}_L \setminus B, \\
\text{div } u = 0 & \text{in } B, \\
\int_{\bar{T}_L} u \, dx = 0, \\
\int_{\partial B} \sigma[u] n \, dS = \int_{\partial B} x \wedge \sigma[u] n \, dS = 0.
\end{cases}
$$

(3.3)

By rescaling, it remains to prove the following claim:

$$
|\text{curl } u(0) + \frac{1}{L^3} \tilde{c} e_3| \leq C L^{-4}.
$$

The proof is based on a good explicit approximation of $u$. Here, it is useful to think of $B$ not as a single particle in the torus but as one of infinitely many periodically distributed particles in $\mathbb{R}^3$. Consequently, we will in the following consider functions that a priori are defined on $\mathbb{R}^3$ even if they turn out to be periodic and can thus be considered as functions defined on $\bar{T}_L$. If the volume fraction of the particles is small, the flow field $u$ is well-approximated by the superposition of the single particle solutions $w$ of the problem

$$
\begin{cases}
-\Delta w + \nabla p = 0, & \text{in } \mathbb{R}^3 \setminus B, \\
\text{div } w = 0 & \text{in } B, \\
D w = A & \text{in } B, \\
\int_{\partial B} \sigma[w] n \, dS = \int_{\partial B} x \wedge \sigma[w] n \, dS = 0.
\end{cases}
$$

(3.4)

We emphasize that $w(x) = Ax$ in $B$, and we have (see e.g., [Niethammer and Schubert 2020, Eq. (1.11)])

$$
w(x) = -\frac{20\pi}{3} \nabla \Phi(x) : A + R[A](x)
$$

(3.5)
where $R[A](x)$ is homogeneous of degree $-4$ and $\Phi$ is the fundamental solution of the Stokes equations, i.e.,

$$
\Phi(x) = \frac{1}{8\pi} \left( \frac{\text{Id}}{|x|} + \frac{x \otimes x}{|x|^3} \right),
$$

$$
(\nabla \Phi(x) : A)_i := \partial_k \Phi_{ji}(x) A_{jk} = -\frac{3}{8\pi} \frac{x_i x_j x_k A_{jk}}{|x|^5}.
$$

We set

$$
\Lambda_L = \{(y_1, y_2, y_3), \ y_1 \in 4L\mathbb{Z}, \ y_2 \in 2L\mathbb{Z}, \ y_3 \in 2L\mathbb{Z}\}
$$

and define the superposition of single particle solutions $\tilde{u}$ for $x \in \mathbb{R}^3$ by

$$
\tilde{u}(x) = \sum_{y \in \Lambda_L} \left( w(x - y) + \frac{20\pi}{3} \int_{Q_y} \nabla \Phi(x - z) : A \, dz \right.

- \left. \frac{20\pi}{3} \sum_{y \in \Lambda_L \setminus \{0\}} \left( \nabla \Phi(x - y) - \int_{Q_y} \nabla \Phi(x - z) \, dz \right) \right) dx',
$$

with $Q_y = y + [-2L, 2L] \times [-L, L]^2$. Here, we subtract iterated means (the sum of which formally vanishes because of the symmetries $w(x) = -w(-x)$ and $\nabla \Phi(x) = -\nabla \Phi(-x)$ for all $x \in \mathbb{R}^3$) in order to make the sum absolutely convergent.

The approximation $\tilde{u}$ is convenient because on the one hand, we can profit from the fact that its building blocks satisfy the PDE (3.4) and hence $\tilde{u}$ itself also satisfies a PDE (see Proposition 3.1 below) in order to compare it to $u$ (see Proposition 3.3 below). On the other hand, $w$ and hence $\tilde{u}$ are explicit and, by Lemma 3.2, it is close (inside of $Q_0$) to the following even simpler explicit approximation (shifted by $w$)

$$
\tilde{u}(x) = \frac{20\pi}{3} \int_{Q_0} \nabla \Phi(x - z) : A \, dz

- \frac{20\pi}{3} \sum_{y \in \Lambda_L \setminus \{0\}} \left( \nabla \Phi(x - y) - \int_{Q_y} \nabla \Phi(x - z) \, dz \right) dx'

- \int_{Q_0} \left( \nabla \Phi(x' - y) - \int_{Q_y} \nabla \Phi(x' - z) \, dz \right) dx' : A,
$$

which makes explicit lattice computations accessible.
Proposition 3.1 The functions $\bar{u}$ from (3.8) and $\tilde{u}$ from (3.9) are well-defined and satisfy $\bar{u}, \tilde{u} \in W^{1,\infty}(Q_0)$. Moreover, $\bar{u}$ is periodic and is a weak solution to
\[
\begin{aligned}
-\Delta \bar{u} + \nabla p &= 0, \quad \text{div} \bar{u} = 0 \quad \text{in } \bar{T}_L \setminus B, \\
\int_{\bar{T}_L} \bar{u} \, dx &= 0, \\
\int_{\partial B} \sigma [\bar{u}] n \, dS &= \int_{\partial B} \bar{u} \wedge \sigma [\bar{u}] n \, dS = 0.
\end{aligned}
\] (3.10)

Proof We consider only $\bar{u}$, the argument for $\tilde{u}$ is analogous.

First observe that each term in the series is well-defined. Let $x \in Q_0$. Then, the only term that needs further justification is the term corresponding to $y = 0$. In this case, $w(x) + \frac{20\pi}{3} \int_{Q_0} \nabla \Phi(x - z) : A \, dz$ is well-defined and uniformly bounded with respect to $x \in Q_0$ since $\nabla \Phi(x - z)$ is locally integrable. To see that the derivative is well-defined as well, let us consider the union of the 27 neighboring cells around $Q_0$,$$
\tilde{Q}_0 := \{ z \in \mathbb{R}^3 : z \in Q_y \text{ for some } y \in \Lambda_L \text{ with } Q_y \cap Q_0 \neq \emptyset \} = [-4L, 4L] \times [-2L, 2L]^2.
$$

Then, via integration by parts
\[
\int_{\tilde{Q}_0} \nabla^2 \Phi(x - z) : A \, dz = -\int_{\partial \tilde{Q}_0} \nabla \Phi(x - z)(An) \, dS_z
\]
which is uniformly bounded for $x \in Q_0$.

We now show that the series is absolutely convergent and write
\[
\tilde{u} = \sum_{y \in \Lambda_L} \psi_y
\]
with natural definition of $\psi_y$. Regarding the gradient of each term of the series (3.8), we observe that in view of (3.5), for $|y| > 10L$ (we exclude cells neighboring the origin), it holds that
\[
\| \nabla \psi_y \|_{L^\infty(Q_0)} = \left\| \nabla w(\cdot - y) + \frac{20\pi}{3} \int_{Q_y} \nabla^2 \Phi(\cdot - z) : A \, dz \right\|_{L^\infty(Q_0)} \leq C|A| \frac{L}{|y|^4},
\]
This shows that the sum of the gradients is absolutely convergent. Because each $\psi_y$ has vanishing average over $Q_0$ by definition, we obtain the estimate
\[
\| \psi_y \|_{L^\infty(Q_0)} \leq C L \| \nabla \psi_y \|_{L^\infty(Q_0)} \leq C|A| \frac{L^2}{|y|^4},
\]
and thus also the sum for $\tilde{u}$ itself is absolutely convergent.
The periodicity is immediate from the construction. Moreover, \( \tilde{u} \) has vanishing mean since each term of the sum has vanishing mean in \( Q_0 \). In order to show that \( \bar{u} \) satisfies the other identities in (3.10), we note that both \( w \) and \( \nabla \Phi \) satisfy Stokes equation outside \( B \), and we emphasize that the term \( x \mapsto \int_{Q_y} \nabla \Phi(x - z) : A \, dz \) satisfies \( -\Delta v + \nabla q = \text{div} (1_{\tilde{T}_y} A) \). By summation, no source term is induced in \( Q_0 \).

**Lemma 3.2** For \( \bar{u} \) defined in (3.8) and \( \tilde{u} \) defined in (3.9), the following estimates hold.

\[
\| \nabla (\bar{u} - \tilde{u}) - A \|_{L^\infty(B)} \lesssim L^{-5},
\]

\[
\| \nabla \bar{u} - \nabla \tilde{u}(0) \|_{L^\infty(B)} \lesssim L^{-4},
\]

\[
\| \nabla \tilde{u} \|_{L^\infty(B)} \lesssim L^{-3}.
\]

**Proof** Estimate (3.11) follows immediately from the definitions of \( \bar{u} \) and \( \tilde{u} \) as well as the fact that \( R[A] \) in (3.5) is homogeneous of degree \(-4\). For estimate (3.12), we split as follows for \( x \in B \):

\[
\nabla \tilde{u}(x) - \nabla \tilde{u}(0) = \frac{20\pi}{3} \left( \int_{Q_0} (\nabla^2 \Phi(x - z) - \nabla^2 \Phi(-z)) : A \, dz \right)
+ \frac{20\pi}{3} \sum_{y \in \Lambda_L \setminus \{0\}} \left( (\nabla^2 \Phi(x - y) - \nabla^2 \Phi(-y)) - \int_{Q_y} (\nabla^2 \Phi(x - z) - \nabla^2 \Phi(-z)) \, dz \right) : A.
\]

We deal with the first term by first applying an integration by parts in order to get

\[
\frac{1}{L^3} \left| \int_{\partial Q_0} (\nabla \Phi(x - z) - \nabla \Phi(-z)) \cdot (An) \, dS \right| \leq C \frac{|A|}{L^3} \int_{Q_0} \left( \frac{1}{|x - z|^3} + \frac{1}{|z|^3} \right) |x| \, dS \leq C \frac{|A|}{L^4}.
\]

The remainder in (3.14) can be handled directly by the same estimates as in the proof of Proposition 3.1. Estimate (3.13) is shown analogously.

**Proposition 3.3** For \( u \) satisfying (3.3) and \( \tilde{u} \) defined in (3.8), it holds that

\[
\| \nabla (u - \tilde{u}) \|_{H^1(\tilde{T}_L)} \leq C \| A - D\tilde{u} \|_{L^2(B)}.
\]

**Proof** Let \( v := \tilde{u} - u \). Then, by definition of \( u \) and Proposition 3.1, \( v \) satisfies

\[
\begin{aligned}
-\Delta v + \nabla p &= 0, \quad \text{div} \, v = 0 \quad \text{in} \, \tilde{T}_L \setminus B, \\
\int_{\partial B} \sigma[v]n \, dS &= \int_{\partial B} x \wedge \sigma[v]n \, dS = 0.
\end{aligned}
\]
By standard considerations, \( \| \nabla v \|_{L^2(\overline{T}_L)} \leq \| \nabla w \|_{L^2(\overline{T}_L)} \) for all \( w \in H^1(\overline{T}_L) \) with \( Dv = Dw \) in \( B \), and such a function exists with
\[
\| \nabla w \|_{L^2(\overline{T}_L)} \lesssim \| Dv \|_{L^2(B)} = \| A - D\tilde{u} \|_{L^2(B)}.
\]

We refer to [Niethammer and Schubert 2020, Lemma 4.6] for details. \( \square \)

For the proof of Theorem 1.1, we need the following two technical lemmas which we prove in Sect. 4.

**Lemma 3.4** Let \( w \in H^1(B_R) \) with \( R > 1 \) and satisfying
\[
\begin{align*}
-\Delta w + \nabla p &= 0, \ \text{div} \ u = 0 \quad \text{in} \ B_R \setminus B, \\
\int_{\partial B} x \wedge \sigma[w] n \ dS &= 0.
\end{align*}
\]

Then,
\[
\int_B \text{curl} \ w \ dx = \int_{B_R} \text{curl} \ w \ dx.
\]

**Lemma 3.5** Let \( \Lambda_L \) be the lattice defined in (3.7). There exists a constant \( c_0 > 0 \) such that
\[
\int_{Q_0} \frac{z_1^2 - z_2^2}{|z|^5} \ dz - \sum_{y \in \Lambda_L \setminus \{0\}} \left( \frac{y_1^2 - y_2^2}{|y|^5} - \int_{Q_y} \frac{z_1^2 - z_2^2}{|z|^5} \ dz \right) = \frac{1}{L^3 c_0}.
\]

**Proof of Theorem 1.1(ii)** Direct computation starting with (3.6) yields
\[
\text{curl}(\nabla \Phi(z) : A) = -\frac{3}{4\pi} \frac{Ax \wedge x}{|x|^5}.
\]

Recalling that \( A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), we get
\[
\frac{1}{5} (\text{curl} \ \tilde{u}(0))_3 = \int_{Q_0} \frac{z_1^2 - z_2^2}{|z|^5} \ dz - \sum_{y \in \Lambda_L \setminus \{0\}} \left( \frac{y_1^2 - y_2^2}{|y|^5} - \int_{Q_y} \frac{z_1^2 - z_2^2}{|z|^5} \ dz \right).
\]
Thus, setting $\bar{c} = 5c_0$ with $c_0$ being the constant from Lemma 3.5, we find, using that $\text{curl } u$ is constant in $B$,

$$\left| \text{curl } u(0) - \frac{\bar{c}}{L^3} e_3 \right| \leq \left| \int_B (\text{curl } u - \text{curl } \tilde{u}) \, dx \right| + \left| \int_B (\text{curl } \tilde{u} - \text{curl } \tilde{u}(0)) \, dx \right| .$$

Using Lemma 3.4, as well as Proposition 3.3 and Lemma 3.2, we have

$$\left| \int_B (\text{curl } u - \text{curl } \tilde{u}) \, dx \right| = \left| \int_{B_L} (\text{curl } u - \text{curl } \tilde{u}) \, dx \right| \lesssim L^{-3/2} \| \nabla (u - \tilde{u}) \|_{L^2(B_L)} \lesssim L^{-3/2} \| A - D\tilde{u} \|_{L^2(B)} \lesssim L^{-3/2} \left( \| \nabla (u - \tilde{u}) - A \|_{L^2(B)} + \| \nabla \tilde{u} \|_{L^2(B)} \right) \lesssim L^{-9/2} .$$

Combining this with the estimates for $\int_B |\text{curl } \tilde{u} - \text{curl } \tilde{u}(0)| \, dx$ and $\int_B |\text{curl } \tilde{u} - \text{curl } \tilde{u}(0)| \, dx$ provided by Lemma 3.2, we conclude

$$\left| \text{curl } u(0) - \frac{\bar{c}}{L^3} e_3 \right| \lesssim L^{-4} .$$

This implies the assertion by rescaling to the torus $\tilde{T}$ and the ball $B_R$. $\square$ 

4 Proof of Auxiliary Results

Proof of Lemma 3.4 Let $\omega \in \mathbb{R}^3$ and $\varphi \in H^1_0(B_R)$ the solution to

$$\begin{cases} -\Delta \varphi + \nabla p = 0 \quad , \quad \text{div } \varphi = 0 & \text{in } B_R \setminus B, \\
\varphi = \omega \wedge x & \text{in } B . \end{cases}$$

It is easy to verify that the solution $\varphi$ in $B_R \setminus B$ is given by

$$\varphi(x) = \frac{R^3}{R^3 - 1} \left( \frac{1}{|x|^3} - \frac{1}{R^3} \right) \omega \wedge x .$$

The corresponding normal stress on $B$ and $B_R$ is $\sigma[\varphi]n = -3 \frac{R^3}{R^3 - 1} \omega \wedge n$ and $\sigma[\varphi]n = -3 \frac{1}{R^3 - 1} \omega \wedge n$, where $n$ is the outward unit normal to $B$ and $B_R$, respectively. We compute

Springer
\[ \omega \cdot \int_B \text{curl } w \, dx = \int_{\partial B} \omega \cdot (n \wedge w) \, dS = \int_{\partial B} w \cdot (\omega \wedge n) \, dS \]

\[ = -\frac{R^3 - 1}{3R^3} \int_{\partial B} \mathbf{w} \cdot (\sigma [\varphi] n) \, dS \]

\[ = \frac{R^3 - 1}{3R^3} \int_{B_R \setminus B} \nabla w \cdot \nabla \varphi \, dx - \frac{R^3 - 1}{3R^3} \int_{\partial B_R} w \cdot (\sigma [\varphi] n) \, dS \]

\[ = \frac{R^3 - 1}{3R^3} \int_{\partial B} (\sigma [w] n) \cdot \varphi + \frac{1}{R^3} \int_{\partial B_R} w \cdot (\omega \wedge n) \, dS \]

\[ = \frac{R^3 - 1}{3R^3} \int_{\partial B} (\sigma [w] n) \cdot (\omega \wedge n) + \frac{1}{R^3} \int_{\partial B_R} n \wedge w \, dS \]

\[ = \frac{R^3 - 1}{3R^3} \omega \cdot \int_{\partial B} n \wedge (\sigma [w] n) \, dS + \frac{1}{R^3} \omega \cdot \int_{B_R} \text{curl } w \, dS. \]

Since the first term vanishes and \( \omega \) was arbitrary, this proves the statement. \( \square \)

**Proof of Lemma 3.5** Since the individual terms in the sum decay like \( |y|^{-4} \), the sum exists and converges absolutely.

By homogeneity, it is enough to consider \( L = \frac{1}{2} \). More precisely, we denote the rescaled lattice \( \Lambda = \{(y_1, y_2, y_3) : y_1 \in 2\mathbb{Z}, \ y_2, y_3 \in \mathbb{Z}\} \), the rescaled cells \( Q_y' = y + [-1, 1] \times [-\frac{1}{2}, \frac{1}{2}]^2 \), and for \( y \in \Lambda \)

\[ S(y) = S(y_1, y_2, y_3) = \frac{y_1^2 - y_2^2}{|y|^5}, \quad S'(y) = S(y) - \int_{Q_y'} S(z) \, dz. \]

Then, it is enough to show that

\[ c'_0 := -\int_{Q_0} S(z) \, dz + \sum_{y \in \Lambda \setminus \{0\}} S'(y) < 0. \quad (4.1) \]

The strategy to prove (4.1) is to show that it is enough to sum over all lattice points in a large enough cube and to estimate the remaining sum. We start by proving that if summing over a finite cube, it is possible to ignore the mean integrals in the sum up to a small error. We denote \( |y|_{\infty} = \max\{|y_1|, |y_2|, |y_3|\} \) and rewrite

\[ c'_0 = \lim_{k \to \infty} -\int_{Q'_0} S(z) \, dz + \sum_{y \in \Lambda \setminus \{0\}, |y|_{\infty} \leq 2k} S'(y). \]
The contribution of the mean integral terms at level $k$ is

$$
\int_{Q_0} S(z) \, dz + \sum_{y \in \Lambda \setminus \{0\}, |y|_\infty \leq 2k} \int_{Q_y} S(z) \, dz
= \frac{1}{2} \int_{[-2k-1,2k+1] \times [-2k-\frac{1}{2}, 2k+\frac{3}{2}]} S(z) \, dz
= \frac{1}{2} \int_{[-2k-\frac{1}{2}, 2k+\frac{3}{2}]} S(z) \, dz + \int_{[2k+\frac{1}{2}, 2k+1] \times [-2k-\frac{1}{2}, 2k+\frac{1}{2}]} S(z) \, dz
= \int_{[2k+\frac{1}{2}, 2k+1] \times [-2k-\frac{1}{2}, 2k+\frac{1}{2}]} S(z) \, dz. \tag{4.2}
$$

The last identity holds since $S(y_1, y_2, y_3) = -S(y_2, y_1, y_3)$, and hence, the integral vanishes on every domain that is invariant under the exchange of $y_1, y_2$. Since

$$
\left| \int_{[2k+\frac{1}{2}, 2k+1] \times [-2k-\frac{1}{2}, 2k+\frac{1}{2}]} S(z) \, dz \right| \leq \int_{[2k+\frac{1}{2}, 2k+1] \times [-2k-\frac{1}{2}, 2k+\frac{1}{2}]} \frac{1}{|y|^3} \, dz
\leq \frac{1}{(2k+\frac{1}{2})^3} \int_{[2k+\frac{1}{2}, 2k+1] \times [-2k-\frac{1}{2}, 2k+\frac{1}{2}]} \, dz \leq \frac{4}{(4k+1)}, \tag{4.3}
$$

the combination of (4.3) and (4.2) shows that

$$
\left| -\int_{Q_0} S(z) \, dz + \sum_{y \in \Lambda \setminus \{0\}, |y|_\infty \leq 2k} S'(y) - \sum_{y \in \Lambda \setminus \{0\}, |y|_\infty \leq 2k} S(y) \right| \leq \frac{4}{(4k+1)}. \tag{4.4}
$$

We continue by estimating the parts of the sum in (4.1) that satisfy $|y|_\infty > 2k$. Notice that for $z \in Q'_y$, it holds that $|y-z| \leq \sqrt{\frac{3}{2}} \leq \frac{3}{2}$. Furthermore, the gradient of $S$ satisfies

$$
\nabla S(z) = \frac{1}{|z|^7} \begin{pmatrix}
z_1(-3z_1^2 + 7z_2^2 + 2z_3^2) \\
z_2(-7z_1^2 + 3z_2^2 - 2z_3^2) \\
z_3(-5z_1^2 + 5z_2^2)
\end{pmatrix}
$$

Using the estimate $|S(y) - S(z)| \leq \|\nabla S\|_{L_\infty([y,z])} |y-z|$, where $[y,z] = [\theta y + (1-\theta)z : \theta \in [0,1]]$ is the segment between $y$ and $z$, we infer for all $y \in \Lambda'$, $z \in Q'_y$

$$
|S(y) - S(z)| \leq \frac{3}{2} \frac{7}{|z| - \frac{3}{2}}. \tag{3.2}
$$
We use this to estimate the sum outside a cube:

\[
\left| \sum_{y \in \Lambda, |y|_{\infty} > 2k} S'(y) \right| \leq \frac{21}{2} \int_{|z| > 2k - 1} \frac{1}{|z| - \frac{3}{2}}^4 \, dz \leq \frac{21}{2} \int_{|z| > 2k - 1} \frac{1}{|z| - \frac{3}{2}}^4 \, dz
\]

\[
= 42\pi \int_{2k-1}^{\infty} \frac{r^2}{(r - \frac{3}{2})^4} \, dr \leq 42\pi \frac{(2k - 1)^2}{(2k - \frac{5}{2})^2} \int_{2k-1}^{\infty} \frac{1}{(r - \frac{3}{2})^2} \, dr
\]

\[
= 42\pi \frac{(2k - 1)^2}{(2k - \frac{5}{2})^3} = 84\pi \frac{(4k - 2)^2}{(4k - 5)^3}
\]

(4.5)

Combining (4.4) and (4.5) yields

\[
\left| - \int_{Q_0'} S(z) \, dz + \sum_{y \in \Lambda \setminus \{0\}} S'(y) - \sum_{y \in \Lambda \setminus \{0\}, |y|_{\infty} \leq 2k} S(y) \right| \leq \frac{4}{(4k + 1)} + 84\pi \frac{(4k - 2)^2}{(4k - 5)^3}.
\]

The right-hand side is smaller than 2.1 for \( k = 35 \), and thus the numerical result

\[
\sum_{y \in \Lambda \setminus \{0\}, |y|_{\infty} \leq 70} S(y) \leq -2.25
\]

shows (4.1). The numerical results were obtained with maple. For the source code, we refer to the online appendix.

\[\square\]

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Declarations

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