On the local antimagic labeling of graphs amalgamation

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Abstract. Let $G$ be a simple connected graph, an ordered pair of sets $G(V, E)$, with $V$ is a set of vertices and $E$ is a set of edges. Graph coloring has been one of the most popular branches in topics of graph theory. In 2017 Arumugam et al. developed a new notion of coloring, namely local antimagic coloring of a graph. This concept is a combination of graph labeling and graph coloring. The local antimagic of graphs is one of the colorings of graph theory that is interesting to study. By the definition, the local antimagic coloring is a bijection $f : E(G) \rightarrow \{1, 2, 3, ..., |E(G)|\}$ if for any two adjacent vertices $u$ and $v$, $w(u) = w(v)$, where $w(v) = \sum_{e \in E(v)} f(e)$, and $E(u)$ is the set of edges incident to $u$. Thus any local antimagic labeling induces a proper vertex coloring of $G$ where the vertex $v$ is assigned the color $w(v)$. The local antimagic chromatic number $\chi_{la}(G)$ is the minimum number of colors taken over all colorings induced by local antimagic labelings of $G$. In this paper, we will study the local antimagic coloring of amalgamation of graphs.

1. Introduction
We consider a graph $G$ in this paper is a simple, connected, and finite graph. An ordered pair of sets $G(V, E)$, with $V$ is a set of vertices and $E$ is a set of edges. For detail definition and notation of graph $G$, it can be seen on [3, 4]. In graph theory, the labeling is the popular topic which is quickly growing. Graph labeling is an assignment of labels to the graph elements such as edges or vertices, or both. The labeling concept was firstly introduced by Hartsfield and Ringel [1].

Graph coloring has been one of the most popular branches in topics of graph theory. Graph coloring is assigned colors to the component of the graph. In 2017, local antimagic of a graph firstly introduced by Arumugam et al. [2]. By the definition, local antimagic coloring is a bijection $f : E(G) \rightarrow \{1, 2, 3, ..., |E(G)|\}$ if for any two adjacent vertices $u$ and $v$, $w(u) = w(v)$, where $w(v) = \sum_{e \in E(v)} f(e)$, and $E(u)$ is the set of edges incident
to $u$. Thus any local antimagic labeling induces a proper vertex coloring of $G$ where the vertex $v$ is assigned the color $w(v)$. The local antimagic chromatic number $\chi_{la}(G)$ is the minimum number of colors taken over all colorings induced by local antimagic labelings of $G$. Local antimagic of a graph firstly introduced by Arumugam et al. [2]. They gave a lower bound and an exact value of local antimagic coloring. The other study about local antimagic of graphs can be seen in [3, 4, 5, 6, 7].

In this paper, we study local antimagic coloring of graphs amalgamation. The definition of amalgamation of graphs is taken from [11]. Let $G_i$ be a simple and connected graph, when $i \in \{1, 2, ..., t\}$ and $t \in \mathbb{N}$ and $|V(G_i)| = k_i \geq 2$ for some $k_i \in \mathbb{N}$. For $t \geq 2$ let $G_1, G_2, ..., G_t$ be a finite collection of graphs and each $G_i$, $i \in \{1, 2, ..., t\}$, has a fixed vertex $v$ called a terminal vertex. The amalgamation of graph $G$ is denoted by $Amal(G_i, v)$. The illustration of amalgamation graph can be seen on Figure 1.

![Figure 1. The illustration of graphs based on amalgamation operation](image)

2. Previous Result
In this paper, we use the previous definition, lemma and theorem.

**Definition 2.1.** [2] Let $G = (V, E)$ be a graph of order $n$ and size $m$ having no isolated vertices. A bijection $f : E(G) \rightarrow \{1, 2, 3, ..., |E(G)|\}$ is said to be a local antimagic labeling if for any two adjacent vertices $u$ and $v$, $w(u) \neq w(v)$, where $w(v) = \sum_{e \in E(v)} f(e)$. A graph $G$ is considered to be a local antimagic graph if $G$ has a local antimagic labeling.

**Theorem 2.1.** [2] For any tree $T$ with $l$ leaves, $\chi_{la}(T) \geq l + 1$. 
Theorem 2.2. [2] For the cycle $C_n = (v_1, v_2, ..., v_n, v_1)$, we have $\chi_{la}(C_n) = 3$.

Lemma 2.1. [2] Let $n$ be an odd positive integer. For $1 \leq i \leq n$, the sum of
\[
P^{n\,3}_{3,\,0}(i) = g_1(i), g_2(i), g_3(i).
\]

with
\[
g_1(i) = \begin{cases} \frac{n+1+i}{2}, & i \equiv 0 \pmod{2} \\ \frac{i+1}{2}, & i \equiv 1 \pmod{2} \end{cases}
\]
\[
g_2(i) = \begin{cases} \frac{i}{2}, & i \equiv 0 \pmod{2} \\ \frac{n+i+1}{2}, & i \equiv 1 \pmod{2} \end{cases}
\]
\[
g_3(i) = n + 1 - i
\]

3. Main Result
We will discuss the local antimagic labeling of amalgamation graphs. We then identify the lower bound of amalgamation graph and determine the chromatic number of some graphs as follows $Amal(TB_n, v, m)$, $Amal(S_n, v, m)$ and $Amal(C_n, v, m)$. We present our results as follows.

Remark 3.1. Let $Amal(G, v, m)$ be an amalgamation of graph $G$. The local antimagic chromatic number satisfies $\chi_{la}(Amal(G, v, m)) \geq \chi(Amal(G, v, m))$

Theorem 3.1. Let $Amal(TB_n, v, m)$ be amalgamation of triangular book graph. For any integer number $n \geq 2$ and $m \geq 2$, the local antimagic chromatic number of $Amal(TB_n, v, m)$ is $\chi_{la} Amal(TB_n, v, m) = 3$

Proof. The graph $Amal(TB_n, v, m)$ has the vertex set $V(Amal(TB_n, v, m)) = \{v, u^i, t^j; 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E(Amal(TB_n, v, m)) = \{vu^i, u^it^j, v^j; 1 \leq i \leq n, 1 \leq j \leq m\}$. The cardinality of vertices is $|V(\text{Amal}(TB_n, v, m))| = nm + m + 1$, and the cardinality of edges is $|E(\text{Amal}(TB_n, v, m))| = m(2n + 1)$. The local antimagic chromatic number of $Amal(TB_n, v, m)$ is $\chi_{la}(Amal(TB_n, v, m)) = 3$. First, we will prove that $\chi_{la}(Amal(TB_n, v, m)) \geq 3$. To show the $\chi_{la}(Amal(TB_n, v, m)) \geq 3$, we assign the colors on the vertices such that any two adjacent vertices of a graph $(Amal(TB_n, v, m))$ have different color, i.e. a proper coloring condition. The proper colors needed is 3 colors. It concludes that $\chi_{la}(Amal(TB_n, v, m)) \geq 3$. Secondly, we will show $\chi_{la}(Amal(TB_n, v, m)) \leq 3$. Consider the definition of local antimagic labeling, we will assign the labels into two cases.

Case 1. For $n$ is odd.
Firstly, we assign the labels of edges $u^t_j$. The labeling of edges $u^t_j$ is divided into two parts, for $1 \leq i \leq \frac{n+1}{2}$ and $\frac{n+1}{2} + 1 \leq i \leq n$. The labeling of edges $u^t_j$ for $1 \leq i \leq \frac{n+1}{2}$ and $1 \leq j \leq m$ by the following table.
Table 1. The labeling of edges $w^j t^i_1$ for $1 \leq i \leq \frac{n+1}{2}$ and $1 \leq j \leq m$

| $i$ | 1       | 2       | ... | $j$       | ... | $m-1$ | $m$  |
|-----|---------|---------|-----|----------|-----|-------|------|
| 1   | 1       | 2       | ... | $j$       | ... | $m-1$ | $m$  |
| 2   | $m+1$   | $m+2$   | ... | $m+j$    | ... | $2m-1$| $2m$ |
| 3   | $2m+1$  | $2m+2$  | ... | $2m+j$   | ... | $3m-1$| $3m$ |
| ... | ...     | ...     | ... | ...      | ... | ...   |      |
| $i$ | $(i-1)m+1$ | $(i-1)m+2$ | ... | $(i-1)m+j$ | ... | $m-1$ | $m$  |
| ... | ...     | ...     | ... | ...      | ... | ...   |      |
| $\frac{n-1}{2}$ | $(\frac{n-1}{2})m+1$ | $(\frac{n-1}{2})m+2$ | ... | $(\frac{n-1}{2})m+j$ | ... | $(\frac{n-1}{2})m-1$ | $(\frac{n-1}{2})m$ |
| $\frac{n+1}{2}$ | $(\frac{n+1}{2})m+1$ | $(\frac{n+1}{2})m+2$ | ... | $(\frac{n+1}{2})m+j$ | ... | $(\frac{n+1}{2})m-1$ | $(\frac{n+1}{2})m$ |

The labels of edges $w^j t^i_1$ and $w^j v$ for $\frac{n+1}{2} + 1 \leq i \leq n$, $1 \leq j \leq m$ can be described in Table 2. Suppose $a = \frac{n+1}{2}$.

Table 2. The labeling of edges $w^j t^i_1$ and $w^j v$ for $\frac{n+1}{2} + 1 \leq i \leq n$, $1 \leq j \leq m$

| $i$ | 1       | 2       | ... | $j$       | ... | $m$  |
|-----|---------|---------|-----|----------|-----|------|
| $a+1$ | $(a+1)m$ | $(a+1)m-1$ | ... | $(a+1)m+j$ | ... | $am+1$ |
| $a+2$ | $(a+2)m$ | $(a+2)m-1$ | ... | $(a+1)m+j$ | ... | $(a+1)m+1$ |
| $a+3$ | $(a+3)m$ | $(a+3)m-1$ | ... | $(a+2)m+j$ | ... | $(a+2)m+1$ |
| ... | ...     | ...     | ... | ...      | ... | ...   |
| $a+\frac{n-1}{2}$ | $(a+\frac{n-1}{2})m$ | $(a+\frac{n-1}{2})m-1$ | ... | $(a+\frac{n-1}{2})m+j$ | ... | $(a+\frac{n-1}{2})m+1$ |
| $w^j v$ | $(a+\frac{n+1}{2})m$ | $(a+\frac{n+1}{2})m-1$ | ... | $(a+\frac{n+1}{2})m+j$ | ... | $(a+\frac{n+1}{2})m+1$ |

The labels of edges $vt^i_1$ can be seen on the Table 3.

Table 3. The labeling of edges $w^j t^i_1$ and $w^j v$

| $w^j t^i_1$ | 1       | 2       | ... | $j$       | ... | $(n+1)m$ |
|-----|---------|---------|-----|----------|-----|----------|
| $vt^i_1$ | $2nm+m$ | $2nm+m-1$ | ... | $2nm+m-j+1$ | ... | $(n+1)m+1$ |
| $w(t^i_1)$ | $2nm+m+1$ | $2nm+m+1$ | ... | $2nm+m+1$ | ... | $2nm+m+1$ |

Consider the labels of edges of $Amal(TB_n, v, m)$ above, we have the vertex weights are as follows:
\[ w(u^j) = \frac{n+1}{2}((n+1)m+1), \text{ for } 1 \leq j \leq m \]
\[ w(t_i^j) = 2nm + m + 1, \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m \]
\[ w(v) = \frac{m(n+1)}{2}(3nm + m + 1) \]

In Case 1, based on the above weight, we have \( \chi_{la}(Anal(TB_n, v, m)) \leq 3 \).

**Case 2.** For \( n \) is even and \( m \) is odd.

Firstly, we assign the label of edges \( u^j \). The labeling of edges \( u^j \) is devided into two parts, for \( 1 \leq i \leq \frac{n-2}{2} \) and \( \frac{n-2}{2} + 1 \leq i \leq n-2 \). The labeling of edges \( u^j \) for \( 1 \leq i \leq \frac{n+1}{2} \) and \( 1 \leq j \leq m \) by the following table.

**Table 4.** The labeling of edges \( u^j \) for \( 1 \leq i \leq \frac{n+1}{2} \) and \( 1 \leq j \leq m \)

| \( i \) | \( j \) |
|---|---|
| \( 1 \) | \( 2 \) | \( \ldots \) | \( j \) | \( \ldots \) | \( m-1 \) | \( m \) |
| 1 | \( m+1 \) | \( m+2 \) | \( \ldots \) | \( m+j \) | \( \ldots \) | \( 2m-1 \) | \( 2m \) |
| \( 2 \) | \( 3m+1 \) | \( 3m+2 \) | \( \ldots \) | \( 3m+j \) | \( \ldots \) | \( 3m-1 \) | \( 3m \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( i \) | \( (i-1)m+1 \) | \( (i-1)m+2 \) | \( \ldots \) | \( (i-1)m+j \) | \( \ldots \) | \( im-1 \) | \( im \) |
| \( \frac{n-1}{2} \) | \( \frac{n-1}{2} \) | \( \frac{(n-1)m+1}{2} \) | \( \frac{(n-1)m+2}{2} \) | \( \ldots \) | \( \frac{(n-1)m+j}{2} \) | \( \ldots \) | \( \frac{(n-1)m-1}{2} \) | \( \frac{(n-1)m}{2} \) |
| \( \frac{n+1}{2} \) | \( \frac{n+1}{2} \) | \( \frac{(n+1)m+1}{2} \) | \( \frac{(n+1)m+2}{2} \) | \( \ldots \) | \( \frac{(n+1)m+j}{2} \) | \( \ldots \) | \( \frac{(n+1)m-1}{2} \) | \( \frac{(n+1)m}{2} \) |

The labels of edges \( u^j \) and \( u^v \) for \( \frac{n-2}{2} + 1 \leq i \leq n-2, 1 \leq j \leq m \) can be described in Table 5.

*Suppose \( a = \frac{n-2}{2} \) (See on the Table 5)

**Table 5.** The labeling of edges \( u^j \) and \( u^v \) for \( \frac{n-2}{2} + 1 \leq i \leq n-2, 1 \leq j \leq m \)

| \( i \) | \( j \) |
|---|---|
| \( a+1 \) | \( (a+1)m \) | \( (a+1)m-1 \) | \( \ldots \) | \( am+j \) | \( \ldots \) | \( am+1 \) |
| \( a+2 \) | \( (a+2)m \) | \( (a+2)m-1 \) | \( \ldots \) | \( (a+1)m+j \) | \( \ldots \) | \( (a+1)m+1 \) |
| \( a+3 \) | \( (a+3)m \) | \( (a+3)m-1 \) | \( \ldots \) | \( (a+2)m+j \) | \( \ldots \) | \( (a+2)m+1 \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( a+\frac{n-1}{2} \) | \( (a+\frac{n-1}{2})m \) | \( (a+\frac{n-1}{2})m-1 \) | \( \ldots \) | \( (a+\frac{n-1}{2})m+j \) | \( \ldots \) | \( (a+\frac{n-1}{2})m+1 \) |
| \( a+\frac{n+1}{2} \) | \( (a+\frac{n+1}{2})m \) | \( (a+\frac{n+1}{2})m-1 \) | \( \ldots \) | \( (a+\frac{n+1}{2})m+j \) | \( \ldots \) | \( (a+\frac{n+1}{2})m+1 \) |
To labeling the edges \( u^j t^j_1 \) for \( i = n - 1, \ n \) and \( u^j v \) we use the Lemma 2.1 \[6\]. Consider the Lemma, the labels of edges \( u^j t^j_1 \) for \( i = n - 1, \ n \) and \( u^j v \) by the following function:

\[
f(u^j t^j_{n-1}) = (n - 2)m + (\frac{1+j}{m+1+j}) \text{, for } j \text{ is odd}.
\]

\[
f(u^j t^j_n) = (n - 1)m + (\frac{m+j}{2}) \text{, for } j \text{ is even}.
\]

\[
f(u^j v) = nm \oplus m + 1 - j
\]

The labels of edges \( vt^j_1 \) can be described in Table 6.

| \( u^j t^j_1 \) | 1 | 2 | ... | \( j \) | ... | \( nm \) |
|-----------------|---|---|-----|-----|-----|------|
| \( vt^j_1 \) | \( 2nm + m \) | \( 2nm + m - 1 \) | ... | \( 2nm + m - j + 1 \) | ... | \( (n+1)m + 1 \) |
| \( w(t^j_1) \) | \( 2nm + m + 1 \) | \( 2nm + m + 1 \) | ... | \( 2nm + m + 1 \) | ... | \( 2nm + m + 1 \) |

Consider the labeling edges of \( Amal(TB_n, v, m) \) above, we have the vertex weights are as follows:

\[
w(u^j) = \frac{3(m+1)}{2} + 3m(n-1) + \frac{1}{2}((n-2) + m(n-2)^2), \text{ for } 1 \leq j \leq m
\]

\[
w(t^j_1) = 2nm + m + 1, \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m
\]

\[
w(v) = \frac{m(n+1)}{2}(3nm + m + 1)
\]

In this Case 2, the function of vertex weight we have \( \chi_{la}(Amal(TB_n, v, m)) \leq 3 \).

Based on the function of total vertex weights from Case 1 and Case 2 above, we have \( \chi_{la}(Amal(TB_n, v, m)) \leq 3 \), induces a proper vertex coloring of \( (Amal(TB_n, v, m)) \). If \( \chi_{la}(Amal(TB_n, v, m)) \geq 3 \) and \( \chi_{la}(Amal(TB_n, v, m)) \leq 3 \), then we have the local antimagic chromatic number of \( (Amal(TB_n, v, m)) \) is \( \chi_{la}(Amal(TB_n, v, m)) = 3 \). \( \square \)

**Theorem 3.2.** Let \( Amal(S_n, v, m) \) be amalgamation of star graph. For any integer number \( n \geq 3 \) and \( m \geq 2 \), the local antimagic chromatic number of \( Amal(S_n, v, m) \) is \( (n-1)m + 1 \leq \chi_{la}(Amal(S_n, v, m)) \leq (n-1)m + 2 \).

**Proof.** The graph \( Amal(S_n, v, m) \) have the vertex set \( V(Amal(S_n, v, m)) = \{u, v^j, t^j_1; 1 \leq i \leq n - 1, \ 1 \leq j \leq m\} \) and edge set \( E(Amal(S_n, v, m)) = \{uv^j, w^j t^j_1; 1 \leq i \leq n - 1, \ 1 \leq j \leq m\} \). The cardinality of vertices is \( |V(Amal(S_n, v, m))| = nm + 1 \) and the cardinality of edges is \( |E(Amal(S_n, v, m))| = nm \). The local antimagic chromatic number of \( Amal(S_n, v, m) \) is \( (n-1)m + 1 \leq \chi_{la}(Amal(S_n, v, m)) \leq (n-1)m + 2 \). First, we will prove that \( \chi_{la}(Amal(S_n, v, m)) \geq (n-1)m + 1 \). To show the \( \chi_{la}(Amal(S_n, v, m)) \geq (n-1)m + 1 \) we use the previous result from Arumugam et al \[2\]. The theorem
from Arumugam said that for any tree $T$ with $l$ leaves, the local antimagic chromatic number is $\chi_la(T) \geq l + 1$. Based on the graph $(Amal(S_n, v, m))$ we identify the leaves of this graph is $(n - 1)m$, so the lowerbound of local antimagic chromatic number is $\chi_la(Amal(S_n, v, m)) \geq (n - 1)m + 1$. It concludes that $\chi_la(Amal(S_n, v, m)) \geq (n - 1)m + 1$. Then, we will show $\chi_la(Amal(S_n, v, m)) \leq (n - 1)m + 2$. Consider the definition of local antimagic labeling, we will assign the labels into two cases.

**Case 1.** For $(n - 1)$ is odd.

Firstly, we assign the labels of edges $v^j t^i$. The labeling of edges $v^j t^i$ is divided into two parts, for $1 \leq i \leq \frac{n}{2}$ and $\frac{n}{2} + 1 \leq i \leq n - 1$. The labeling of edges $v^j t^i$ for $1 \leq i \leq \frac{n}{2}$ and $1 \leq j \leq m$ by the following table.

**Table 7.** The labeling of edges $v^j t^i$ for $1 \leq i \leq \frac{n}{2}$ and $1 \leq j \leq m$

| $i$   | 1       | 2       | ... | $j$ | ... | $m-1$ | $m$ |
|-------|---------|---------|-----|-----|-----|-------|-----|
| 1     | 1       | 2       | ... | $j$ | ... | $m-1$ | $m$ |
| 2     | $m+1$   | $m+2$   | ... | $m+j$ | ... | $2m-1$ | $2m$ |
| 3     | $2m+1$  | $2m+2$  | ... | $2m+j$ | ... | $3m-1$ | $3m$ |

The labels of edges $v^j t^i$ and $v^j u$ for $\frac{n}{2} + 1 \leq i \leq n - 1$, $1 \leq j \leq m$ can be described in the Table 8.

*Suppose $a = \frac{n}{2}$ (see on the Table 8)

**Table 8.** The labeling of edges $u^j t^i$ and $u^j v$ for $\frac{n+1}{2} + 1 \leq i \leq n$, $1 \leq j \leq m$

| $i$   | 1       | 2       | ... | $j$ | ... | $m$ |
|-------|---------|---------|-----|-----|-----|-----|
| $a+1$ | $(a+1)m$| $(a+1)m-1$ | ... | $(a+1)m+j$ | ... | $(a+1)m+1$ |
| $a+2$ | $(a+2)m$| $(a+2)m-1$ | ... | $(a+1)m+j$ | ... | $(a+1)m+1$ |
| $a+3$ | $(a+3)m$| $(a+3)m-1$ | ... | $(a+2)m+j$ | ... | $(a+2)m+1$ |
| ...   | ...     | ...     | ... | ... | ... | ... |
| $a+\frac{n}{2}$ | $(a+\frac{n}{2})m$| $(a+\frac{n}{2})m-1$ | ... | $(a+\frac{n}{2})m+j$ | ... | $(a+\frac{n}{2})m+1$ |
| $u^j u$ | $(a+\frac{n}{2})m$| $(a+\frac{n}{2})m-1$ | ... | $(a+\frac{n}{2})m+j$ | ... | $(a+\frac{n}{2})m+1$ |
Consider the labels of edges of $Amal(S_n, v, m)$ above, we have the vertex weights are as follows:

$$w(t^j_i) = 1, 2, 3, 4, \ldots, (n-1)m, \text{ for } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m$$

$$w(v^j) = \frac{n}{2}(nm+1), \text{ for } 1 \leq j \leq m$$

$$w(u) = m(2nm - m + 1)$$

In this Case 1, the function of vertex weight we have $\chi_{la}(Amal(S_n, v, m)) \leq (n-1)m+2$.

**Case 2.** For $n$ is even and $m$ is odd. Firstly, we assign the labels of edges $v^j t^j_i$. The labeling of edges $v^j t^j_i$ is divided into two parts, for $1 \leq i \leq \frac{n-3}{2}$ and $\frac{n-3}{2} + 1 \leq i \leq n - 3$. The labeling of edges $v^j t^j_i$ for $1 \leq i \leq \frac{n-3}{2}$ and $1 \leq j \leq m$ by the following table.

**Table 9.** The labeling of edges $v^j t^j_i$ for $1 \leq i \leq \frac{n-3}{2}$ and $1 \leq j \leq m$

| $i$   | 1      | 2      | $\ldots$ | $j$ | $\ldots$ | $m-1$ | $m$ |
|-------|--------|--------|-----------|-----|-----------|-------|-----|
| 1     | 1      | 2      | $\ldots$ | $j$ | $\ldots$ | $m-1$ | $m$ |
| 2     | $m+1$  | $m+2$  | $\ldots$ | $m+j$ | $\ldots$ | $2m-1$ | $2m$ |
| 3     | $2m+1$ | $2m+2$ | $\ldots$ | $2m+j$ | $\ldots$ | $3m-1$ | $3m$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\vdots$ |
| $i$   | $(i-1)m+1$ | $(i-1)m+2$ | $\ldots$ | $(i-1)m+j$ | $\ldots$ | $im-1$ | $im$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\vdots$ |
| $\frac{n-3}{2}$ | $(\frac{n-i}{2})m+1$ | $(\frac{n-i}{2})m+2$ | $\ldots$ | $(\frac{n-i}{2})m+j$ | $\ldots$ | $(\frac{n-3}{2})m-1$ | $(\frac{n-3}{2})m$ |
| $\frac{n-3}{2}$ | $(\frac{n-i}{2})m+1$ | $(\frac{n-i}{2})m+2$ | $\ldots$ | $(\frac{n-i}{2})m+j$ | $\ldots$ | $(\frac{n-3}{2})m-1$ | $(\frac{n-3}{2})m$ |

The labels of edges $v^j t^j_i$ for $\frac{n-3}{2} + 1 \leq i \leq n - 3$, $1 \leq j \leq m$ can be described in Table 10.

*Suppose $a = \frac{n-3}{2}$ (See on the Table 10)

To labeling the edges $v^j t^j_i$ for $i = n - 2$, $n - 1$ and $v^j u$ we use the Lemma 2.1 [6]. Consider the Lemma, the labeling of edges $v^j t^j_i$ for $i = n - 2$, $n - 1$ and $v^j u$ by the following function:

$$f(v^j t^j_{n-2}) = (n-3)m \oplus \left\{ \begin{array}{ll}
\frac{1+j}{m+1+j}, & \text{for } j \text{ is odd} \\
\frac{m+j}{2}, & \text{for } j \text{ is even}
\end{array} \right.$$  

$$f(v^j t^j_{n-1}) = (n-2)m \oplus \left\{ \begin{array}{ll}
\frac{m+j}{2}, & \text{for } j \text{ is odd} \\
\frac{m}{2}, & \text{for } j \text{ is even}
\end{array} \right.$$  

$$f(v^j u) = (n-1)m \oplus m + 1 - j$$
Consider the labels of edges of $Amal(S_n, v, m)$ above, we have the vertex weights are as follows:

$$w(t^j_i) = 1, 2, 3, 4, \ldots, (n-1)m, \text{ for } 1 \leq i \leq n - 1 \text{ and } 1 \leq j \leq m$$

$$w(v^j) = \frac{3}{2}(m+1) + m(3n-6) + \frac{n-3}{2}((n-3)m+1), \text{ for } 1 \leq j \leq m$$

$$w(u) = \frac{m}{2}(2nm - m + 1)$$

From the vertex weight sets above, we have $\chi_{la}(Amal(S_n, v, m)) \leq (n-1)m + 2$.

Based on the function of total vertex weights from Case 1 and Case 2 above, we have $\chi_{la}(Amal(S_n, v, m)) \leq (n-1)m + 2$, induces a proper vertex coloring of $(Amal(S_n, v, m))$. Since $\chi_{la}(Amal(S_n, v, m)) \geq (n-1)m + 1$ and $\chi_{la}(Amal(S_n, v, m)) \leq (n-1)m + 2$, thus we have the local antimagic chromatic number of $(Amal(S_n, v, m))$ is $(n-1)m + 1 \leq \chi_{la}(Amal(S_n, v, m)) \leq (n-1)m + 2$. $\Box$

**Theorem 3.3.** Let $Amal(C_n, v, m)$ be amalgamation of cycle graph. For any integer number $n \geq 3$ and $m \geq 2$, the local antimagic chromatic number of $Amal(C_n, v, m)$ is $\chi_{la}(Amal(C_n, v, m)) = 3$.

**Proof.** The $Amal(C_n, v, m)$ has the vertex set $V(Amal(C_n, v, m)) = \{x, x^j_i; 1 \leq i \leq n - 1, 1 \leq j \leq m\}$ and edge set $E(Amal(C_n, v, m)) = \{xx^j_i, xx^j_{i+1}, xx^j_{i-1}; 1 \leq i \leq n - 2, 1 \leq j \leq m\}$. The cardinality of vertices is $|V(Amal(C_n, v, m))| = (n-1)m + 1$ and the cardinality of edges is $|E(Amal(C_n, v, m))| = nm$. The local antimagic chromatic number of $Amal(C_n, v, m)$ is $\chi_{la}(Amal(C_n, v, m)) = 3$. First, we will prove that $\chi_{la}(Amal(C_n, v, m)) \geq 3$. To show the $\chi_{la}(Amal(C_n, v, m)) \geq 3$ we assign the colors on the vertices such that any two adjacent vertices of a graph $Amal(C_n, v, m)$ have different color, i.e. a proper coloring condition. The proper colors needed is 3 colors. It concludes that $\chi_{la}(Amal(C_n, v, m)) \geq 3$. Secondly, we will show $\chi_{la}(Amal(C_n, v, m)) \leq 3$. Consider the definition of local antimagic labeling, we will assign the labels into two cases.

**Case 1.** For $n$ is odd.

Firstly, we assign the labels of edges by the following table.
Table 11. The labeling of edges for \( n \) is odd

| \( i \) | \( j \) | \( (n-1)m + m \) | \( (n-1)m + m - 1 \) | \( (n-1)m + j \) | \( (n-1)m + 1 \) |
|---|---|---|---|---|---|
| \( xx^j_1 \) | \( xx^j_2 \) | \( m + 1 \) | \( m + 2 \) | \( m + j \) | \( 2m \) |
| \( xx^j_3 \) | \( xx^j_4 \) | \( (n-2)m + m \) | \( (n-2)m + m - 1 \) | \( (n-2)m + j \) | \( (n-2)m + 1 \) |

From the Table 11, the function of the edges can be show as a follows:

\[
f(x_i^j x_{i+1}^j) = \begin{cases} 
\frac{i+2}{2} - 1) m + j, & \text{for } i \text{ is even} \\
\frac{n+1}{2} m + (\frac{n+1}{2}) m - j + 1, & \text{for } i \text{ is odd}
\end{cases}
\]

Consider the labeling of edges in Case 1, we have the vertex weights are as follows:

\[
w(x_i) = \begin{cases} 
nm + m + 1, & \text{for } i \text{ is even} \\
nm + 1, & \text{for } i \text{ is odd}
\end{cases}
\]

\[
w(x) = \frac{nm^2}{2} + m^2 + m
\]

In Case 1 based on the function of vertex weight we have \( \chi_{la}(Amal(C_n,v,m)) \leq 3 \).

Case 2. For \( n \) is even

For this case, we assign the labels of edges by the following table.

From the Table 12 the function of the edges can be show as a follows:

\[
f(x_i^j x_{i+1}^j) = \begin{cases} 
\frac{i}{2} + j, & \text{for } i \text{ is even} \\
\frac{n-1}{2} m + (\frac{n-1}{2}) m + m - j + 1, & \text{for } i \text{ is odd}
\end{cases}
\]

Consider the labeling of edges in Case 2, we have the vertex weights are as follows:

\[
w(x_i) = \begin{cases} 
nm + m + 1, & \text{for } i \text{ is even} \\
nm + 1, & \text{for } i \text{ is odd}
\end{cases}
\]

\[
w(x) = \frac{nm^2}{2} + m^2 + m
\]
Table 12. The labeling of edges for $n$ is even

| $i$ | $x_1$ | $x_2$ | ... | $x_j$ | ... | $x_m$ |
|-----|-------|-------|-----|-------|-----|-------|
| $j$ | 1     | 2     | ... | $j$   | ... | $m$   |
| $j$ | $(n-1)m+m$ | $(n-1)m+m-1$ | ... | $(n-1)m+j$ | ... | $(n-1)m+1$ |
| $j$ | $m+1$ | $m+2$ | ... | $m+j$ | ... | $2m$ |
| $j$ | $(n-2)m+m$ | $(n-2)m+m-1$ | ... | $(n-2)m+j$ | ... | $(n-2)m+1$ |
| $j$ | $n-1$ | $n-2$ | ... | $n-1$ | ... | $n$ |

In Case 2 based on the function of vertex weight we have $\chi_{la}(Amal(C_n,v,m)) \leq 3$.

Based on the function of total vertex weights from Case 1 and Case 2 above, we have $\chi_{la}(Amal(C_n,v,m)) \leq 3$, induces a proper vertex coloring of $Amal(C_n,v,m))$. Since $\chi_{la}(Amal(C_n,v,m)) \geq 3$ and $\chi_{la}(Amal(C_n,v,m)) \leq 3$, thus we have the local antimagic chromatic number of $Amal(C_n,v,m))$ is $\chi_{la}(Amal(C_n,v,m)) = 3$.

The Illustration of local antimagic labeling of amalgamation graph can be seen on the Figure 2.

Figure 2. The local antimagic labeling of $Amal(C_6,v,7)$
4. Conclusion
Some results in this paper are the local antimagic chromatic number of graphs with amalgamation as follows $\text{Amal}(TB_n, v, m)$, $\text{Amal}(S_n, v, m)$ and $\text{Amal}(C_n, v, m)$.

Open Problem 4.1. find the chromatic number of local antimagic of another operation graphs.

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