Add force and/or change underlying projection method to improve accuracy of Explicit Robin-Neumann and fully decoupled schemes for the coupling of incompressible fluid with thin-walled structure

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Abstract

Enlightened by added-mass effect and viscosity of fluid, in Explicit Robin-Neumann and fully decoupled schemes for the coupling of incompressible fluid with thin-walled structure, the force between fluid and structure corresponding to viscosity is increased. Numerical experiments demonstrate improvement of accuracy under such modification. To further improve accuracy of fully decoupled schemes, the underlying projection method is replaced.
1 Introduction

The coupling of incompressible fluid with thin-walled structure typically arises in subjects like bio-mechanics of blood flow in vessels [10], for which the blood is governed by Navier-Stokes equations, while the structure assumed to be deformable. Generally, like the numerical methods to other fluid-structure interaction problems, there are two types of numerical approaches, namely the monolithic (also named *fully implicit*) and partitioned (also named *decoupled*) [14]. The monolithic is accurate but less efficient, while the partitioned is much more efficient but not that accurate.

Among various partitioned algorithms on this topic [10] [14] [9] [6] [8] [7] [12] [3], Fernandez’s Explicit Robin-Neumann [9] and fully decoupled schemes [6] [8] are exceptional, due to their high efficiency, theoretical and/or numerical stability and applicability to the topic with a vast variety of structure models. The fully decoupled scheme is more efficient than Explicit Robin-Neumann scheme, because the velocity and pressure of fluid are decoupled.

The infamous added-mass effect is known to cause instability to tremendous partitioned algorithms for fluid-structure interaction problems with large fluid/structure density ratio (for many problems the density of fluid is lower than that of structure, so this implies the density of fluid is large enough to be close to that of structure) and long thin geometry [18].

Intuitively, due to viscosity of fluid, when the fluid or structure moves, some amount of fluid is attached to it near the interface, resulting in ”added” mass. By Newton’s Second Law, the mass together with acceleration results in extra force on the structure. By Newton’s Third Law, there is an equal and opposite force on the fluid. Thus, it is reasonable to expect, for a partitioned algorithm, increasing the force due to viscosity of fluid between the fluid and structure can make it more realistic, namely more close to the actual behaviour of the coupling system, which might lead to better accuracy. The amount of extra force resulting from ”added” mass is difficult (if not impossible) to compute; however, it can be approached. Up to certain values, increasing the force gradually should improve accuracy gradually.

The idea is applied to Fernandez’s two schemes mentioned above. Note that, however, this does not mean the two schemes are unstable; in fact, their stabilities have proved by theoretical analysis and/or numerical experiments (availability of theoretical analysis depends
on extrapolation orders) to be free of added-mass effect. In what follows, the two schemes are presented. Afterwards, coefficients of the terms corresponding to force from viscosity are increased, generating Algorithm 1 and Algorithm 2. Fernandez’s fully decoupled scheme is based on Chorin-Temam projection method. To further improve accuracy, the underlying projection method is replaced with Van Kan’s, leading to Algorithm 3.

Numerical results are reported later. The results indicate improvement of accuracy as the force from viscosity increases. Appropriate values for such increment are recommended for practical applications.

Remark 1 Based on the intuition mentioned above, it might not be difficult to understand the reasons why added-mass effect causes instability to tremendous partitioned algorithms for fluid-structure interaction problems with large fluid/structure density ratio and long thin geometry. As the density of fluid increases, the part of fluid attached to the structure due to viscosity influences more seriously on the movement of structure. If the geometry is long and thin, such a part of fluid takes large portion of the whole fluid, which influences seriously on the movement of fluid. If it decouples the fluid and structure without paying sufficient attention to such effect, a partitioned algorithm might be far from the actual movement of the coupling system and thus might fail to finish the simulation, which constitutes stability issues.

2 The simplified model problem

For sake of clarity and simplicity, the simplified test-case used in [5] is adopted. This work is expected to be also applicable to the a bit more general model described in [9] [6] [8]. The fluid dominated by Stokes equations is defined on \( \Omega = [0, L] \times [0, R] \), where \( L = 6, R = 0.5 \) (all the quantities are under CGS system) , with \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \Sigma \cup \Gamma_4 \) (see Figure 1) . The domain is extracted as upper half of the rectangle \( [0, L] \times [-R, R] \) which simulates a tube in two-dimensional space with horizontal centerline \( \Gamma_1 \) and top boundary \( \Sigma \). As a result, \( \Gamma_1 \) is imposed symmetric boundary condition. The structure is assumed to be a generalized string defined on \( \Sigma \) with the two end points ( \( x = 0, L \) ) fixed. When the fluid flows from the left to the right, structure deforms vertically. Equations read as follows.
Find the fluid velocity $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^2$, the fluid pressure $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, the structure vertical displacement $\eta : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$ and the structure vertical velocity $\dot{\eta} : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$ such that

\[
\begin{align*}
\rho_f \partial_t u - \text{div} \sigma(u, p) &= 0 \quad \text{in } \Omega, \\
\text{div} u &= 0 \quad \text{in } \Omega, \\
u \cdot n &= 0, \sigma(u, p)n \cdot t = 0 \quad \text{on } \Gamma_1, \\
\sigma(u, p)n &= -P(t)n \quad \text{on } \Gamma_2, \\
\sigma(u, p)n &= 0 \quad \text{on } \Gamma_4,
\end{align*}
\]

(1)

\[
\begin{align*}
u \cdot n &= \dot{\eta}, u \cdot t &= 0 \quad \text{on } \Sigma, \\
\rho_s \epsilon \partial_t \dot{\eta} - c_1 \partial_{xx} \eta + c_0 \dot{\eta} &= -\sigma(u, p)n \cdot n \quad \text{on } \Sigma, \\
\dot{\eta} &= \partial_t \eta \quad \text{on } \Sigma, \\
\eta &= 0 \quad \text{on } \partial\Sigma,
\end{align*}
\]

(2)

with initial conditions

$u(0) = 0, \eta(0) = 0, \dot{\eta}(0) = 0,$

where normal vector is denoted by $n$, tangent vector is $t$, fluid Cauchy stress tensor $\sigma(u, p) \overset{\text{def}}{=} -pI + 2\mu \varepsilon(u)$, $\varepsilon(u) \overset{\text{def}}{=} \frac{1}{2}(\nabla u + \nabla u^T)$, fluid dynamic viscosity $\mu = 0.035$, fluid density $\rho_f = 1.0$, pressure $P(t) = P_{\text{max}}(1 - \cos(2t\pi/T^*))/2$, $P_{\text{max}} = 2 \times 10^4$ when $0 \leq t \leq T^*$ and $P_{\text{max}} = 0$ when $t > T^*, T^* = 5 \times 10^{-3}$, structure density $\rho_s = 1.1, c_1 \overset{\text{def}}{=} \frac{E\epsilon}{2(1+\nu)}, c_0 \overset{\text{def}}{=} \frac{E\epsilon}{R^2(1-\nu^2)}, \epsilon = 0.1$, Young’s modulus $E = 0.75 \times 10^6$, Poisson’s ratio $\nu = 0.5$. 

3
3 Notations

For all the algorithms mentioned in this work, $\tau$ denotes time step, while $h$ stands for space discretization parameter.

Given arbitrary variable $x$, the notation

$$x^{n,*} \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } r = 0, \\
x^{n-1} & \text{if } r = 1, \\
2x^{n-1} - x^{n-2} & \text{if } r = 2
\end{cases} \quad (3)$$

is used for interface extrapolations of order $r$.

4 Fernandez’s Explicit Robin-Neumann and fully decoupled schemes

The time semi-discrete form of Explicit Robin-Neumann scheme (Fernandez [9]) is cited here.

(Fernandez) Explicit Robin-Neumann scheme (time semi-discrete)

For $n \geq r + 1$, find $u^n : \Omega \to \mathbb{R}^2$, $p^n : \Omega \to \mathbb{R}$, $\eta^n : \Sigma \to \mathbb{R}$ and $\dot{\eta}^n : \Sigma \to \mathbb{R}$ such that

1. Fluid step (interface Robin condition)

$$\begin{cases}
\rho^f \frac{u^n - u^{n-1}}{\tau} - \nabla \cdot \sigma(u^n, p^n) = 0 & \text{in } \Omega, \\
\nabla \cdot u^n = 0 & \text{in } \Omega, \\
u^n \cdot n = 0, \sigma(u^n, p^n) n \cdot t = 0 & \text{on } \Gamma_1, \\
\sigma(u^n, p^n) n = -P(t)n & \text{on } \Gamma_2, \\
\sigma(u^n, p^n) n = 0 & \text{on } \Gamma_4,
\end{cases} \quad (4)$$

$$\sigma(u^n, p^n) n \cdot n + \frac{\rho^s \epsilon}{\tau} u^n \cdot n = \frac{\rho^s \epsilon}{\tau} (\eta^{n-1} + \tau \partial_r \eta^{n,*})$$

$$+ (-p^{n,*} I + 2\mu \epsilon(u^{n,*})) n \cdot n$$

on $\Sigma$, $u^n \cdot t = 0$ on $\Sigma$. 

4
2. Solid step (Neumann condition)

\[
\begin{cases}
\rho^s \varepsilon \frac{\dot{\eta}^n - \dot{\eta}^{n-1}}{\tau} - c_1 \partial_{x\tau} \eta^n + c_0 \eta^n = - (-p^n I + 2 \mu \varepsilon(u^n)) n \cdot n \quad \text{on} \quad \Sigma, \\
\dot{\eta}^n = \partial_{x\tau} \eta^n \quad \text{on} \quad \Sigma, \\
\eta^n = 0 \quad \text{on} \quad \partial \Sigma,
\end{cases}
\]

(5)

The fully decoupled scheme is proposed in Fernandez [6], [8]. There are non-incremental and incremental forms, of which both deliver close numerical results on accuracy. Here only presents the non-incremental form.

(Fernandez) fully decoupled scheme (time semi-discrete)

For \( n \geq r + 1 \),

(1) Fluid viscous sub-step: find \( \tilde{u}^n : \Omega \rightarrow \mathbb{R}^2 \) such that

\[
\begin{cases}
\rho^f \tilde{u}^n - u^{n-1} = - 2 \mu \text{div} \varepsilon(\tilde{u}^n) = 0 \quad \text{in} \quad \Omega, \\
\tilde{u}^n \cdot n = 0, 2 \mu \varepsilon(\tilde{u}^n) n \cdot \tau = 0 \quad \text{on} \quad \Gamma_1, \\
2 \mu \varepsilon(\tilde{u}^n) n \cdot \tau = 0 \quad \text{on} \quad \Gamma_2, \\
2 \mu \varepsilon(\tilde{u}^n) n \cdot \tau = 0 \quad \text{on} \quad \Gamma_4, \\
\tilde{u}^n_1 = 0, 2 \mu \varepsilon(\tilde{u}^n) n \cdot n + \frac{\rho^s \varepsilon}{\tau} \tilde{u}^n \cdot n = \frac{\rho^s \varepsilon}{\tau} \dot{\eta}^{n-1} \text{on} \quad \Sigma,
\end{cases}
\]

(6)

(2) Fluid projection sub-step: find \( \phi^n : \Omega \rightarrow \mathbb{R}^2 \) such that

\[
\begin{cases}
-\frac{\tau}{\rho^f} \Delta \phi^n = - \nabla \cdot \tilde{u}^n \quad \text{in} \quad \Omega, \\
\frac{\partial \phi^n}{\partial n} = 0 \quad \text{on} \quad \Gamma_1, \\
\phi^n = P(t_n) \quad \text{on} \quad \Gamma_2, \\
\phi^n = 0 \quad \text{on} \quad \Gamma_4, \\
\frac{\tau}{\rho^f} \nabla \phi^n \cdot n + \frac{\tau}{\rho^s \varepsilon} \phi^n = \frac{\tau}{\rho^s \varepsilon} \phi^{n,*} + \tilde{u}^{n,*} \cdot n - \dot{\eta}^{n,*} \quad \text{on} \quad \Sigma,
\end{cases}
\]

(7)

Thereafter set \( p^n = \phi^n, u^n = \tilde{u}^n - \frac{\tau}{\rho^f} \nabla \phi^n \) in \( \Omega \).
(3) Solid sub-step: find $\eta^n : \Sigma \to \mathbb{R}^2$ such that

$$
\left\{
\begin{array}{l}
\rho_s \varepsilon \frac{\dot{\eta}^n - \dot{\eta}^{n-1}}{\tau} - c_1 \partial_{xx} \eta^n + c_0 \eta^n = -2\mu \varepsilon (\tilde{u}^n)n \cdot n + p^n \quad \text{on } \Sigma, \\
\dot{\eta}^n = \frac{\eta^n - \eta^{n-1}}{\tau} \quad \text{on } \Sigma, \\
\eta^n = 0 \quad \text{on } \partial \Sigma,
\end{array}
\right.
$$

(8)

**Remark 2** Substituting $u^n = \tilde{u}^n - \frac{\Sigma}{\mu} \nabla \phi^n$ into (6) leads to a more compact style, with $u^n$ eliminated (see [6]).

5 Two schemes with added force

Replacing the coefficient 2 in the term $2\mu \varepsilon (u^{n,*})$ at the right hand side of (4) and the term $2\mu \varepsilon (u^n)$ at the right hand side of (5) with a real number named $\beta$ larger than 2 generates Algorithm 1 as follows. Analogously, substituting the coefficient 2 in the term $-2\mu \varepsilon (\tilde{u}^n)n \cdot n$ at the right hand side of (8) with a real number named $\theta$ larger than 2 leads to Algorithm 2.

**Remark 3** There is no such a term in (5) like $(-p^{n,*}I + 2\mu \varepsilon (u^{n,*}))n \cdot n$ of (4), so there is no such a term in (11) like $(-p^{n,*}I + \beta \mu \varepsilon (u^{n,*}))n \cdot n$ of (9).

Algorithm 1

For real number $\beta > 2$, $n \geq r + 1$, find $u^n : \Omega \to \mathbb{R}^2$, $p^n : \Omega \to \mathbb{R}$, $\eta^n : \Sigma \to \mathbb{R}$ and $\dot{\eta}^n : \Sigma \to \mathbb{R}$ such that
1. Fluid step (interface Robin condition)

\[
\begin{aligned}
\frac{\rho' \mathbf{u}^n - \mathbf{u}^{n-1}}{\tau} - \text{div} \, \sigma(\mathbf{u}^n, p^n) &= 0 \quad \text{in} \; \Omega, \\
\text{div} \, \mathbf{u}^n &= 0 \quad \text{in} \; \Omega, \\
\mathbf{u}^n \cdot \mathbf{n} &= 0, \sigma(\mathbf{u}^n, p^n) \mathbf{n} \cdot \mathbf{t} = 0 \quad \text{on} \; \Gamma_1, \\
\sigma(\mathbf{u}^n, p^n) \mathbf{n} &= -P(t) \mathbf{n} \quad \text{on} \; \Gamma_2, \\
\sigma(\mathbf{u}^n, p^n) \mathbf{n} &= 0 \quad \text{on} \; \Gamma_4,
\end{aligned}
\]

\[
\sigma(\mathbf{u}^n, p^n) \mathbf{n} \cdot \mathbf{n} + \frac{\rho' \epsilon}{\tau} \mathbf{u}^n \cdot \mathbf{n} = \frac{\rho' \epsilon}{\tau} (\ddot{\eta}^{n-1} + \tau \partial_\tau \ddot{\eta}^{n,*}) \\
+ (-p^{n,*} \mathbf{I} + \beta \mu \varepsilon(\mathbf{u}^{n,*}) \mathbf{n} \cdot \mathbf{n} \quad \text{on} \; \Sigma,
\]

\[
\mathbf{u}^n \cdot \mathbf{t} = 0 \quad \text{on} \; \Sigma.
\]

2. Solid step (Neumann condition)

\[
\begin{aligned}
\frac{\rho \epsilon \ddot{\eta}^n - \dot{\eta}^{n-1}}{\tau} - c_1 \partial_{xx} \eta^n + c_0 \eta^n &= -(-p^n \mathbf{I} + \beta \mu \varepsilon(\mathbf{u}^n)) \mathbf{n} \cdot \mathbf{n} \quad \text{on} \; \Sigma, \\
\dot{\eta}^n &= \partial_\tau \eta^n \quad \text{on} \; \Sigma, \\
\eta^n &= 0 \quad \text{on} \; \partial \Sigma.
\end{aligned}
\]

Algorithm 2

For real number \( \theta > 2, n \geq r + 1, \)

(1) Fluid viscous sub-step: find \( \tilde{\mathbf{u}}^n : \Omega \to \mathbb{R}^2 \) such that

\[
\begin{aligned}
\frac{\rho' \tilde{\mathbf{u}}^n - \mathbf{u}^{n-1}}{\tau} - 2 \mu \text{div} \, \varepsilon(\tilde{\mathbf{u}}^n) &= 0 \quad \text{in} \; \Omega, \\
\tilde{\mathbf{u}}^n \cdot \mathbf{n} &= 0, 2 \mu \varepsilon(\tilde{\mathbf{u}}^n) \mathbf{n} \cdot \mathbf{\tau} = 0 \quad \text{on} \; \Gamma_1, \\
2 \mu \varepsilon(\tilde{\mathbf{u}}^n) \mathbf{n} \cdot \mathbf{\tau} &= 0 \quad \text{on} \; \Gamma_2, \\
2 \mu \varepsilon(\tilde{\mathbf{u}}^n) \mathbf{n} \cdot \mathbf{\tau} &= 0 \quad \text{on} \; \Gamma_4, \\
\tilde{u}_1 &= 0, 2 \mu \varepsilon(\tilde{\mathbf{u}}^n) \mathbf{n} \cdot \mathbf{n} + \frac{\rho' \epsilon}{\tau} \tilde{\mathbf{u}}^n \cdot \mathbf{n} = \frac{\rho' \epsilon}{\tau} \ddot{\eta}^{n-1} \quad \text{on} \; \Sigma,
\end{aligned}
\]
(2) Fluid projection sub-step: find $\phi^n : \Omega \to \mathbb{R}^2$ such that

$$\left\{
\begin{array}{l}
-\frac{\tau}{\rho_f} \Delta \phi^n = -\nabla \cdot \tilde{u}^n \quad \text{in} \quad \Omega, \\
\frac{\partial \phi^n}{\partial n} = 0 \quad \text{on} \quad \Gamma_1, \\
\phi^n = P(t_n) \quad \text{on} \quad \Gamma_2, \\
\phi^n = 0 \quad \text{on} \quad \Gamma_4,
\end{array}
\right. \quad (12)$$

$$\frac{\tau}{\rho_f} \nabla \phi^n \cdot n + \frac{\tau}{\rho_s \epsilon} \phi^n = \frac{\tau}{\rho_s \epsilon} \phi^{n,*} + \tilde{u}^{n,*} \cdot n - \dot{\eta}^{n,*} \quad \text{on} \quad \Sigma,$$

Thereafter set $p^n = \phi^n, u^n = \tilde{u}^n - \frac{\tau}{\rho_f} \nabla \phi^n$ in $\Omega$.

(3) Solid sub-step: find $\eta^n : \Sigma \to \mathbb{R}^2$ such that

$$\left\{
\begin{array}{l}
\rho^s \epsilon \frac{\dot{\eta}^n - \dot{\eta}^{n-1}}{\tau} - c_1 \partial_{xx} \eta^n + c_0 \eta^n = -\theta \mu \varepsilon(\tilde{u}^n) n \cdot n + p^n \quad \text{on} \quad \Sigma, \\
\dot{\eta}^n = \frac{\eta^n - \eta^{n-1}}{\tau} \quad \text{on} \quad \Sigma, \\
\eta^n = 0 \quad \text{on} \quad \partial \Sigma,
\end{array}
\right. \quad (13)$$

6 A fully decoupled scheme based on Van Kan’s projection method and with added force

Fernandez’s fully decoupled scheme is based on Chorin-Temam projection method, whose accuracy is of first order in time (see e.g. [16] [13]). It is expected, if the underlying projection method is replaced with Van Kan’s projection method, which is of second order in time (see e.g. [16]), such schemes could be more accurate. This idea produces Algorithms 3.

Algorithm 3

For real number $\xi \geq 2, n \geq r + 1,$
(1) Fluid viscous sub-step: find $\tilde{u}^n : \Omega \to \mathbb{R}^2$ such that

$$
\begin{cases}
\rho^f \frac{\tilde{u}^n - u^{n-1}}{\tau} = -\nabla p^{n-1} + \frac{1}{2} (2\mu \text{div} \varepsilon(\tilde{u}^n) + 2\mu \text{div} \varepsilon(u^{n-1})) & \text{in } \Omega, \\
\tilde{u}^n \cdot n = 0, & \text{on } \Gamma_1, \\
2\mu \varepsilon(\tilde{u}^n) n \cdot \tau = 0 & \text{on } \Gamma_2, \\
2\mu \varepsilon(\tilde{u}^n) n \cdot \tau = 0 & \text{on } \Gamma_4, \\
\tilde{u}^n_1 = 0, & \text{on } \Gamma_3.
\end{cases}
$$

(14)

(2) Fluid projection sub-step: find $\phi^n : \Omega \to \mathbb{R}^2$ such that

$$
\begin{cases}
-\frac{\tau}{\rho^f} \Delta \phi^n = -\nabla \cdot \tilde{u}^n & \text{in } \Omega, \\
\frac{\partial \phi^n}{\partial n} = 0 & \text{on } \Gamma_1, \\
\phi^n = \frac{P(t_n) - P(t_{n-1})}{2} & \text{on } \Gamma_2, \\
\phi^n = 0 & \text{on } \Gamma_4, \\
\tau \rho^f \nabla \phi^n \cdot n + \frac{\tau}{\rho^f} \phi^n = \\
\frac{\tau}{\rho^f} p^{n,*} - \frac{p^{n-1,*}}{2} + \tilde{u}^{n,*} - \tilde{u}^{n-1,*} \cdot n - \frac{\eta^{n,*} - \eta^{n-1,*}}{2} & \text{on } \Sigma.
\end{cases}
$$

(15)

Thereafter set $p^n = p^{n-1} + 2\phi^n$, $u^n = \tilde{u}^n - \frac{\tau}{\rho^f} \nabla \phi^n$ in $\Omega$.

(3) Solid sub-step: find $\eta^n : \Sigma \to \mathbb{R}^2$ such that

$$
\begin{cases}
\rho^s \varepsilon \frac{\dot{\eta}^n - \dot{\eta}^{n-1}}{\tau} - c_1 \partial_{xx} \eta^n + c_0 \eta^n = -\xi \mu \varepsilon(\tilde{u}^n) n \cdot n + p^n & \text{on } \Sigma, \\
\dot{\eta}^n = \frac{\eta^n - \eta^{n-1}}{\tau} & \text{on } \Sigma, \\
\eta^n = 0 & \text{on } \partial \Sigma.
\end{cases}
$$

(16)

Remark 4 Boundary conditions for the fluid projection sub-step of Algorithm 3 are deduced from that of Fernandez’s fully decoupled scheme by noting that $\phi^n = \frac{p^n - p^{n-1}}{2}$ for Algorithm 3 and that $p^n = \phi^n$ for Fernandez’s fully decoupled scheme. For example, on $\Gamma_2$, [7] indicates

$$
p^n = P(t_n)
p^{n-1} = P(t_{n-1})
$$
Taking the difference and divided by 2 yields \( (15)_3 \)

\[
\phi^n = \frac{p^n - p^{n-1}}{2} = \frac{P(t_n) - P(t_{n-1})}{2}
\]

The same procedure applies to the deduction of \((15)_{2,4,5}\)

7 Numerical experiments

Fernandez’s two algorithms and Algorithms 1-3 are all discretized with Galerkin finite element method in space and implemented with FreeFem++ [15] using Lagrange \( P_1 \) element for both the fluid and structure with symmetric pressure stabilization method [2]. In order to observe the order of convergence, the time and space are refined at the same rate,

\[
(\tau, h) = \left( \frac{5 \times 10^{-4}, 0.1}{2^{rate}} \right), \quad rate = 0, 1, 2, 3, 4, 5, ....
\]

The reference solution is generated using monolithic scheme at high time-space grid resolution \( \tau = 10^{-6}, h = 3.125 \times 10^{-3} \). All algorithms run from initial time \( t = 0 \) to final time \( t = 0.015 \). By comparing solutions of the above 5 schemes to reference solution, relative errors in elastic energy norm (see [9]) are computed for structure displacement at final time corresponding to different rates of space and time refinement.

Computation of relative errors and preparation of data for writing are completed with Perl [20] as well as an amount of Perl modules [17] and Bash [11]. Graphs are drew using gnuplot [21]. All codes run on x86_64 Linux 5.6.0 [19] with one Intel® Xeon® E-2186M CPU @ 2.90GHz.

Tables 1, 2 and 3 report relative errors of Fernandez’s two algorithms and Algorithms 1-3 with \( \beta, \theta \) and \( \xi \) ranging from integers 10 to 45 respectively, at refinement rate \( rate = 2, 3, 4, 5 \). The refinement \( rate = 0 \) and 1 are of no interest and not presented, since all of Fernandez’s two algorithms and Algorithms 1-3 perform poorly in accuracy at such low rates. Numerical results of Algorithms 1-3 with \( \beta, \theta \) and \( \xi \) ranging from 2 to 10 are not presented, because they do not yield obvious improvement of accuracy at these intervals.

Both of Fernandez’s two algorithms achieve both highest accuracy and optimal first-order convergence rate in time with first-order extrapolation, so Tables 1, 2 and 3 include
their results at first-order extrapolation only. For purpose of comparison, Algorithm 1 and 2 are also computed with first-order extrapolation. However, Algorithm 3 reaches highest accuracy at zeroth-order extrapolation, so its results at zeroth-order extrapolation are presented.
Table 1: Numerical results of Fernandez Explicit Robin-Neumann scheme (Fern ERN) and Algorithm 1 (Algo 1)

| rate | Fern ERN | Algo 1 | Algo 1 | Algo 1 | Algo 1 | Algo 1 |
|------|----------|--------|--------|--------|--------|--------|
| β = 10 | 0.435176 | 0.423118 | 0.421689 | 0.420281 | 0.418895 | 0.417532 |
| β = 11 | 0.241766 | 0.233158 | 0.232109 | 0.231066 | 0.230030 | 0.229001 |
| β = 12 | 0.128616 | 0.123319 | 0.122668 | 0.122021 | 0.121377 | 0.120735 |
| β = 13 | 0.064847 | 0.061810 | 0.061437 | 0.061066 | 0.060696 | 0.060328 |

| rate | Algo 1 | Algo 1 | Algo 1 | Algo 1 | Algo 1 | Algo 1 |
|------|--------|--------|--------|--------|--------|--------|
| β = 15 | 0.416193 | 0.414879 | 0.413592 | 0.412334 | 0.411104 | 0.409906 |
| β = 16 | 0.227979 | 0.226965 | 0.225959 | 0.224962 | 0.223972 | 0.222992 |
| β = 17 | 0.120097 | 0.119462 | 0.118831 | 0.118202 | 0.117578 | 0.116957 |
| β = 18 | 0.059961 | 0.059597 | 0.059234 | 0.058873 | 0.058514 | 0.058156 |

| rate | Algo 1 | Algo 1 | Algo 1 | Algo 1 | Algo 1 | Algo 1 |
|------|--------|--------|--------|--------|--------|--------|
| β = 21 | 0.408739 | 0.407606 | 0.406510 | 0.405836 | unstable | unstable |
| β = 22 | 0.222021 | 0.221059 | 0.220107 | 0.219165 | 0.218234 | 0.217313 |
| β = 23 | 0.116340 | 0.115727 | 0.115117 | 0.114512 | 0.113911 | 0.113314 |
| β = 24 | 0.057802 | 0.057449 | 0.057097 | 0.056748 | 0.056402 | 0.056057 |
| rate | Algo 1 | Algo 1 | Algo 1 | Algo 1 | Algo 1 | Algo 1 |
|------|--------|--------|--------|--------|--------|--------|
| β = 27 | unstable | unstable | unstable | unstable | unstable | unstable |
| 2 | 0.216403 | 0.215505 | 0.214619 | 0.213744 | 0.212883 | 0.212034 |
| 3 | 0.112722 | 0.112134 | 0.111551 | 0.110973 | 0.110399 | 0.109831 |
| 4 | 0.055715 | 0.055375 | 0.055038 | 0.054703 | 0.054371 | 0.054041 |
| β = 33 | unstable | unstable | unstable | unstable | unstable | unstable |
| β = 34 | unstable | unstable | unstable | unstable | unstable | unstable |
| β = 35 | unstable | unstable | unstable | unstable | unstable | unstable |
| β = 36 | 0.211198 | 0.210376 | 0.209568 | 0.208775 | 0.207996 | 0.207233 |
| β = 37 | 0.109268 | 0.108710 | 0.108157 | 0.107610 | 0.107069 | 0.106536 |
| β = 38 | 0.053714 | 0.053389 | 0.053068 | 0.052749 | 0.052433 | 0.052121 |
| β = 39 | unstable | unstable | unstable | unstable | unstable | unstable |
| β = 40 | unstable | unstable | unstable | unstable | unstable | unstable |
| β = 41 | unstable | unstable | unstable | unstable | unstable | unstable |
| β = 42 | unstable | unstable | unstable | unstable | unstable | unstable |
| β = 43 | unstable | unstable | unstable | unstable | unstable | unstable |
| β = 44 | unstable | unstable | unstable | unstable | unstable | unstable |
| rate | Algo 1 | \( \beta = 45 \) |
|------|--------|------------------|
| 2    | unstable |                  |
| 3    | unstable |                  |
| 4    | 0.102961 |                  |
| 5    | 0.050021 |                  |
Table 2: Numerical results of Fernandez fully decoupled scheme (Fern FD) and Algorithm 2 (Algo 2)

| rate | Fern FD | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 |
|------|---------|--------|--------|--------|--------|--------|
|      | \( \theta = 10 \) | \( \theta = 11 \) | \( \theta = 12 \) | \( \theta = 13 \) | \( \theta = 14 \) |
| 2    | 0.437713 | 0.420421 | 0.418264 | 0.416110 | 0.413961 | 0.411817 |
| 3    | 0.243562 | 0.231346 | 0.229813 | 0.228279 | 0.226744 | 0.225208 |
| 4    | 0.129731 | 0.123637 | 0.122873 | 0.122109 | 0.121345 | 0.120580 |
| 5    | 0.065497 | 0.063052 | 0.062746 | 0.062441 | 0.062136 | 0.061830 |

| rate | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 |
|------|--------|--------|--------|--------|--------|--------|
|      | \( \theta = 15 \) | \( \theta = 16 \) | \( \theta = 17 \) | \( \theta = 18 \) | \( \theta = 19 \) | \( \theta = 20 \) |
| 2    | 0.409654 | unstable | unstable | unstable | unstable | unstable |
| 3    | 0.223672 | 0.222135 | 0.220599 | 0.219063 | 0.217527 | 0.215992 |
| 4    | 0.119816 | 0.119050 | 0.118285 | 0.117519 | 0.116754 | 0.115988 |
| 5    | 0.061525 | 0.061220 | 0.060915 | 0.060610 | 0.060305 | 0.060000 |

| rate | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 |
|------|--------|--------|--------|--------|--------|--------|
|      | \( \theta = 21 \) | \( \theta = 22 \) | \( \theta = 23 \) | \( \theta = 24 \) | \( \theta = 25 \) | \( \theta = 26 \) |
| 2    | unstable | unstable | unstable | unstable | unstable | unstable |
| 3    | 0.214458 | 0.212924 | 0.211393 | 0.209862 | 0.208879 | unstable |
| 4    | 0.115222 | 0.114456 | 0.113690 | 0.112924 | 0.112158 | 0.111392 |
| 5    | 0.059695 | 0.059391 | 0.059086 | 0.058782 | 0.058477 | 0.058173 |
| rate | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 |
|------|--------|--------|--------|--------|--------|--------|
|      | θ = 27 | θ = 28 | θ = 29 | θ = 30 | θ = 31 | θ = 32 |
| 2    | unstable | unstable | unstable | unstable | unstable | unstable |
| 3    | unstable | unstable | unstable | unstable | unstable | unstable |
| 4    | 0.110627 | 0.109861 | 0.109096 | 0.108330 | 0.107566 | 0.106801 |
| 5    | 0.057869 | 0.057565 | 0.057261 | 0.056958 | 0.056654 | 0.056351 |

| rate | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 |
|------|--------|--------|--------|--------|--------|--------|
|      | θ = 33 | θ = 34 | θ = 35 | θ = 36 | θ = 37 | θ = 38 |
| 2    | unstable | unstable | unstable | unstable | unstable | unstable |
| 3    | unstable | unstable | unstable | unstable | unstable | unstable |
| 4    | 0.106036 | 0.105272 | 0.104509 | 0.103746 | 0.102984 | 0.102222 |
| 5    | 0.056048 | 0.055745 | 0.055442 | 0.055139 | 0.054837 | 0.054534 |

| rate | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 | Algo 2 |
|------|--------|--------|--------|--------|--------|--------|
|      | θ = 39 | θ = 40 | θ = 41 | θ = 42 | θ = 43 | θ = 44 |
| 2    | unstable | unstable | unstable | unstable | unstable | unstable |
| 3    | unstable | unstable | unstable | unstable | unstable | unstable |
| 4    | 0.101463 | 0.100700 | 0.099940 | 0.099916 | unstable | unstable |
| 5    | 0.054232 | 0.053931 | 0.053629 | 0.053327 | 0.053027 | 0.052726 |
| rate | Algo 2 |
|------|--------|
| $\theta = 45$ |
| 2    | unstable |
| 3    | unstable |
| 4    | unstable |
| 5    | 0.052425 |
Table 3: Numerical results of Fernandez fully decoupled scheme (Fern FD) and Algorithm 3 (Algo 3)

| rate | Fern FD | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 |
|------|---------|-------|-------|-------|-------|-------|
|      | ξ = 10  | ξ = 11| ξ = 12| ξ = 13| ξ = 14|
| 2    | 0.437713| 0.413443| 0.408376| 0.403425| 0.398606| 0.393939|
| 3    | 0.243562| 0.248517| 0.243651| 0.238858| 0.234148| 0.229535|
| 4    | 0.129731| 0.136471| 0.132807| 0.129173| 0.125574| 0.122016|
| 5    | 0.065497| 0.069889| 0.067621| 0.065367| 0.063132| 0.060917|

| rate | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 |
|------|-------|-------|-------|-------|-------|-------|
|      | ξ = 15 | ξ = 16| ξ = 17| ξ = 18| ξ = 19| ξ = 20|
| 2    | 0.389442| 0.385136| 0.381043| 0.377186| 0.373591| 0.370281|
| 3    | 0.225031| 0.220650| 0.216408| 0.212321| 0.208408| 0.204686|
| 4    | 0.118508| 0.115057| 0.111672| 0.108363| 0.105142| 0.102021|
| 5    | 0.058728| 0.056566| 0.054439| 0.052351| 0.050309| 0.048320|

| rate | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 |
|------|-------|-------|-------|-------|-------|-------|
|      | ξ = 21| ξ = 22| ξ = 23| ξ = 24| ξ = 25| ξ = 26|
| 2    | 0.367286| 0.364632| 0.362348| 0.360462| 0.359005| 0.358005|
| 3    | 0.201178| 0.197904| 0.194886| 0.192147| 0.189712| 0.187604|
| 4    | 0.099015| 0.096139| 0.093410| 0.090847| 0.088472| 0.086299|
| 5    | 0.046394| 0.044540| 0.042769| 0.041095| 0.039531| 0.038095|
| rate | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 |
|------|--------|--------|--------|--------|--------|--------|
|      | $\xi = 27$ | $\xi = 28$ | $\xi = 29$ | $\xi = 30$ | $\xi = 31$ | $\xi = 32$ |
| 2    | 0.357492 | 0.357517 | 0.358058 | 0.359165 | 0.360862 | 0.363169 |
| 3    | 0.185846 | 0.184461 | 0.183471 | 0.182897 | 0.182757 | 0.183065 |
| 4    | 0.084358 | 0.082669 | 0.081254 | 0.080136 | 0.079329 | 0.078857 |
| 5    | 0.036802 | 0.035672 | 0.034723 | 0.033973 | 0.033437 | 0.033131 |

| rate | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 |
|------|--------|--------|--------|--------|--------|--------|
|      | $\xi = 33$ | $\xi = 34$ | $\xi = 35$ | $\xi = 36$ | $\xi = 37$ | $\xi = 38$ |
| 2    | 0.366107 | 0.369690 | 0.373933 | 0.378845 | 0.384418 | 0.392112 |
| 3    | 0.183835 | 0.185076 | 0.186796 | 0.188994 | 0.191672 | 0.194826 |
| 4    | 0.078730 | 0.078957 | 0.079543 | 0.080489 | 0.081789 | 0.083434 |
| 5    | 0.033061 | 0.033234 | 0.033648 | 0.034298 | 0.035172 | 0.036259 |

| rate | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 | Algo 3 |
|------|--------|--------|--------|--------|--------|--------|
|      | $\xi = 39$ | $\xi = 40$ | $\xi = 41$ | $\xi = 42$ | $\xi = 43$ | $\xi = 44$ |
| 2    | unstable | unstable | unstable | unstable | unstable | unstable |
| 3    | 0.198449 | 0.202535 | 0.207070 | 0.212044 | 0.217443 | 0.223253 |
| 4    | 0.085413 | 0.087711 | 0.090310 | 0.093194 | 0.096343 | 0.099741 |
| 5    | 0.037542 | 0.039004 | 0.040630 | 0.042402 | 0.044305 | 0.046326 |
### 8 Conclusions from numerical results

Conclusions can be drawn from Tables 1, 2, and 3 respectively as follows.

#### 8.1 Conclusions for Algorithm 1 from Table 1

Relative errors of Algorithm 1 decrease in a regular manner as $\beta$ or refinement rate increase. All the relative errors are less than that of Fernandez Explicit Robin-Neumann scheme except for unstable ones. All algorithms roughly achieve the same convergence order in time, namely $O(t)$.

Stability is conditional. For a specific value of $\beta$, the algorithm is stable at high refinement rates; for a specific refinement rate, it is stable at small values of $\beta$. At $rate = 2$ and 3, Algorithm 1 is stable up to $\beta = 24$ and 43 respectively. At $rate = 4$ and 5, it is stable for all tested values of $\beta$.

That relative errors keep decreasing as $\beta$ increases up to 45 implies that the amount of force added to the algorithm keeps approaching the actual amount of force resulting from added-mass effect in the coupling system. Based on the intuition mentioned in Section 1, it is guessed continuing increasing $\beta$ up to certain value larger than 45 might decrease the relative errors further. However, since the algorithm performs worse in stability at larger $\beta$ and the stability is already frustrating at $\beta = 45$, it is not worth doing so.
8.2 Conclusions for Algorithm 2 from Table 2

Relative errors of Algorithm 2 decrease in a regular manner as θ or refinement rate increase. All the relative errors are less than that of Fernandez fully decoupled scheme except for unstable ones. All algorithms roughly achieve the same convergence order in time, namely $O(t)$.

Stability is conditional. For a specific value of θ, the algorithm is stable at high refinement rates; for a specific refinement rate, it is stable at small values of θ. At $rate = 2, 3$ and $4$, Algorithm 2 is stable up to $θ = 15, 25$ and $42$ respectively. At $rate = 5$, it is stable for all tested values of θ.

That relative errors keep decreasing as θ increases up to 45 implies that the amount of force added to the algorithm keeps approaching the actual amount of force resulting from added-mass effect in the coupling system. Based on the intuition mentioned in Section 1, it is guessed continuing increasing θ up to certain value larger than 45 might decrease the relative errors further. However, since the algorithm performs worse in stability at larger θ and the stability is already frustrating at $θ = 45$, it is not worth doing so.

8.3 Conclusions for Algorithm 3 from Table 3

Relative errors of Algorithm 3 decrease in a regular manner as ξ or refinement rate increase up to $ξ = 27$ at $rate = 2$, $ξ = 31$ at $rate = 3$, $ξ = 33$ at $rate = 4$ and $ξ = 33$ at $rate = 5$. For larger ξ at that refinement rate, relative errors augment. All the relative errors are less than that of Fernandez fully decoupled scheme except for unstable ones. All algorithms roughly achieve the same convergence order in time, namely $O(t)$.

Stability is conditional. For a specific value of ξ, the algorithm is stable at high refinement rates; for a specific refinement rate, it is stable at small values of ξ. At $rate = 2$, Algorithm 3 is stable up to $ξ = 38$. At $rate = 3, 4$ and $5$, it is stable for all tested values of ξ. Compared with Algorithm 2, Algorithm 3 possesses better stability and accuracy.

That relative errors keep decreasing as ξ increases up to 27 and increasing as ξ increases from 33 complies with the intuition mentioned in Section 1. It is guessed the amount of force added to the algorithm by taking ξ between 27 and 33 is close to the actual amount of
force resulting from added-mass effect in the coupling system.

9 A possible and non-rigorous explanation for the behaviour of stability

Theoretical analysis is not available yet. Here states a possible and non-rigorous explanation. It remains unknown whether such an explanation is correct.

All algorithms mentioned lead to linear equations after space discretizations at each time step. Compared with Fernandez’s two algorithms, Algorithms 1-3 modify the right hand side of those linear equations generated, causing perturbations to the solutions. Since Fernandez’s two algorithms are stable, it is expected solutions still exist and do not change obviously under small perturbations. However, as $\beta, \theta$ or $\xi$ increases, such perturbations become more significant and affect the existence and values of solutions more seriously. As time steps go on, perturbations accumulate and at some time steps cause the solutions to the linear equations generated at that step non-existent. This perhaps explains why Algorithms 1-3 become unstable at large $\beta, \theta$ or $\xi$ for a specific refinement rate.

For a specific value of $\beta, \theta$ or $\xi$, as the refinement rate increases, the number of nodes of mesh enlarges. Let an integer $m$ denote the number of nodes on $\Sigma$, $m = L/h + 1$. The number of nodes on $\Omega$ is approximately $m^2$. Note that the added force is only imposed on the interface. Thus, only $m$ nodes are affected. The ratio of affected and non-affected nodes is approximately $m/(m^2 - m) = 1/(m - 1)$, which decreases as $m$ increases. As refinement rate increases, $m$ increases and therefore the forces added to Algorithms 1-3 disturbs the system less, which yields better stability.

10 Selection of $\beta, \theta$ and $\xi$ for practical applications

Practical applications should take into account both efficiency and accuracy. At refinement rate $2$, time step $\tau = 0.000125$, space discretization parameter $h = 0.025$, it takes no more than 20 seconds to finish computation for any of Fernandez’s two algorithms and Algorithms 1-3 regardless of values of $\beta, \theta$ and $\xi$. It is quite fast. However, all of them are far from
accurate. Therefore, practical applications are not expected to run at such low rate of refinement; it suffices to consider rate = 3, 4 and 5.

Comparing relative errors corresponding to different values at rate = 3, 4 and 5, the values β = 43, θ = 25, 31 ≤ ξ ≤ 33 are recommended for Algorithm 1-3 respectively. Tables 4, 5 and 6 report their values and percents of decrement of relative errors compared with Fernandez’s algorithms respectively. The values of decrement of relative errors equal to relative errors of Fernandez’s algorithms minus that of Algorithm 1-3, while percents equal to values divided by relative errors of Fernandez’s algorithms times 100. Structure displacements are displayed in Figures 2, 3, 4, 5 and 6.

Table 4: Numerical results of Fernandez Explicit Robin-Neumann scheme (Fern ERN) and Algorithm 1 (Algo 1) at selected β

| rate | Fern ERN | Algo 1 | decrement of errors |
|------|----------|--------|---------------------|
|      | β = 43   | values | percents(%)         |
| 2    | 0.435176 | unstable | N/A | N/A |
| 3    | 0.241766 | 0.203850 | 0.037916 | 15.6829 |
| 4    | 0.128616 | 0.103949 | 0.024667 | 19.1788 |
| 5    | 0.064847 | 0.050604 | 0.014243 | 21.9640 |

Table 5: Numerical results of Fernandez fully decoupled scheme (Fern FD) and Algorithm 2 (Algo 2) at selected θ

| rate | Fern FD | Algo 2 | decrement of errors |
|------|---------|--------|---------------------|
|      | θ = 25  | values | percents(%)         |
| 2    | 0.437713 | unstable | N/A | N/A |
| 3    | 0.243562 | 0.208879 | 0.034683 | 14.2399 |
| 4    | 0.129731 | 0.112158 | 0.017573 | 13.5457 |
| 5    | 0.065497 | 0.058477 | 0.00702 | 10.7180 |
Table 6: Numerical results of Fernandez fully decoupled scheme (Fern FD) and Algorithm 3 (Algo 3) at selected $\xi$

| rate | Fern FD | Algo 3 | decrement of errors |
|------|---------|--------|---------------------|
|      |         |        | $\xi = 31$ values | percents(%)       |
| 2    | 0.437713| 0.360862| 0.076851          | 17.5574           |
| 3    | 0.243562| 0.182757| 0.060805          | 24.9649           |
| 4    | 0.129731| 0.079329| 0.050402          | 38.8512           |
| 5    | 0.065497| 0.033437| 0.03206           | 48.9488           |
|      |         |        | $\xi = 32$ values | percents(%)       |
| 2    | 0.437713| 0.363169| 0.074544          | 17.0303           |
| 3    | 0.243562| 0.183065| 0.060497          | 24.8384           |
| 4    | 0.129731| 0.078857| 0.050874          | 39.2150           |
| 5    | 0.065497| 0.033131| 0.032366          | 49.4160           |
|      |         |        | $\xi = 33$ values | percents(%)       |
| 2    | 0.437713| 0.366107| 0.071606          | 16.3591           |
| 3    | 0.243562| 0.183835| 0.059727          | 24.5223           |
| 4    | 0.129731| 0.078730| 0.051001          | 39.3129           |
| 5    | 0.065497| 0.033061| 0.032436          | 49.5229           |
Figure 2: Structure displacement of Fernandez Explicit Robin-Neumann scheme (Fern ERN) and Algorithm 1 (Algo 1) with $\beta = 43$ at final time

11 Discussions and future work

The numerical results validate the ideas that adding force corresponding to viscosity and replacing underlying projection method can improve accuracy; particularly, Table 6 indicates as large improvement as up to 49.5229% for Algorithm 3 with $\xi = 33$ compared with Fernandez fully decoupled scheme at refinement rate $= 5$. It is expected, for other fluid-structure interaction problems, if the fluid is also viscous, adding force might also help with accuracy.

As a direction of future work, it is worth trying investigating how adding force improve accuracy theoretically. Reading works [4] [3] on added-mass effect might benefit such analysis.

This work deals with accuracy. On the other hand, it is possible to improve efficiency by
Figure 3: Structure displacement of Fernandez fully decoupled scheme (Fern FD) and Algorithm 2 (Algo 2) with $\theta = 25$ at final time

parallelism. A choice is to take advantage of extrapolation ($1^{st}$ order might be better than $0^{th}$). To implement such ideas, MPI [1] might work.

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Figure 4: Structure displacement of Fernandez fully decoupled scheme (Fern FD) and Algorithm 3 (Algo 3) with $\xi = 31$ at final time.

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Figure 5: Structure displacement of Fernandez fully decoupled scheme (Fern FD) and Algorithm 3 (Algo 3) with $\xi = 32$ at final time.
Figure 6: Structure displacement of Fernandez fully decoupled scheme (Fern FD) and Algorithm 3 (Algo 3) with $\xi = 33$ at final time
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