We present some results concerning the generalized homologies associated with nilpotent endomorphisms $d$ such that $d^N = 0$ for some integer $N \geq 2$. We then introduce the notion of graded $q$-differential algebra and describe some examples. In particular we construct the $q$-analog of the simplicial differential on forms, the $q$-analog of the Hochschild differential and the $q$-analog of the universal differential envelope of an associative unital algebra.

1 Introduction

Our aim is to discuss properties of nilpotent endomorphisms $d$ such that $d^N = 0$ for some integer $N \geq 2$, of the corresponding generalized homologies and of the associated generalizations of graded differential algebras. The natural setting would be to use a category of modules carrying representations of the group of $N$-th roots of the unity. Here however, for simplicity, we shall work with complex vector spaces and the natural representations of this group in such spaces, (i.e. multiplication by the corresponding complex numbers). Such a representation is characterized by a primitive $N$-root $q$ of the unity. For eventual applications to $q$-deformations, etc., we drop the assumption that $q$ is a root of unity and develop the notion of $q$-differential calculus for $q \in \mathbb{C}$ with $q \neq 0$ and $q \neq 1$ (i.e. $q \in \mathbb{C}\{0,1\}$). The terminology adopted here is influenced by a paper by M. Kapranov on similar topics. Other examples of spaces with $d^N = 0$, etc. can be found in R. Kerner’s contribution to this volume.

2 Generalized homology associated with $d^N = 0$

Let $E$ be a complex vector space equipped with a nilpotent endomorphism $d$ satisfying $d^N = 0$ where $N$ is an integer with $N \geq 2$. One has $\text{Im}(d^{N-k}) \subset \ker(d^k)$ for $k \in \{0,1,\ldots,N\}$ and therefore the vector spaces $H^{(k)} = H^{(k)}(E) = \ker(d^k)/\text{Im}(d^{N-k})$ are well defined. In fact $H^{(0)} = H^{(N)} = \{0\}$ and the $H^{(k)}$ are the generalization of the homology of $E$ for $1 \leq k \leq N-1$.

Let $\ell$ and $m$ be two positive integers such that $\ell + m \leq N$. One has $\ker(d^m) \subset \ker(d^{\ell+m})$ and $\text{Im}(d^{N-m}) \subset \text{Im}(d^{N-(\ell+m)})$ so the inclusion $i^{\ell} :$
\[ \ker(d^m) \rightarrow \ker(d^{\ell+m}) \text{ induces a homomorphism } [i^\ell] : H^{(m)} \rightarrow H^{(\ell+m)}. \]

On the other hand, one has \( d^m(\ker(d^{\ell+m})) \subset \ker(d^\ell) \) and \( d^m(\text{Im}(d^{N-(\ell+m)})) \subset \text{Im}(d^{N-\ell}) \) and therefore \( d^m \) induces a homomorphism \( [d^m] : H^{(\ell+m)} \rightarrow H^{(\ell)}. \)

Notice that \( [i^\ell] = [i^\ell] \) and that \( [d^m] = [d^m] \). One has the following lemma.

**Lemma 1** The hexagon \((H^{\ell,m})\) of homomorphisms

\[
\begin{array}{ccc}
H^{(\ell+m)} & \xrightarrow{[d^m]} & H^{(\ell)} \\
\uparrow & & \uparrow \\
H^{(N-(\ell+m))} & \xrightarrow{[d]} & H^{(N-m)} \\
\downarrow & & \downarrow \\
H^{(\ell)} & \xrightarrow{[i]} & H^{(N-(\ell+m))} \\
\downarrow & & \downarrow \\
H^{(m)} & \xleftarrow{[i]} & H^{(N-\ell)} \\
\end{array}
\]

is exact.

For the proof and for more details, we refer to [1] and [2]. We shall use the following criterion ensuring the vanishing of the \( H^{(k)} \).

**Lemma 2** Let \( q \) be a primitive \( N \)-th root of the unity and assume that there is an endomorphism \( h \) of \( E \) such that \( h \circ d - qd \circ h = I \) where \( I \) denotes the identity mapping of \( E \) onto itself. Then one has \( H^{(k)} = 0 \) for \( k \in \{1, \ldots, N-1\} \).

In fact \( hd - qdh = I \) with \( q \) being a primitive \( N \)-th root of the unity implies, \( 2 \), \( \sum_{k=0}^{N-1} d^{N-1-k}h^{N-1}q^k = [(N-1)!]_q I \), where \( [(N-1)!]_q = [(N-1)]_q \cdots [2]_q \)

with \( [n]_q = 1 + q + \cdots + q^{n-1} \). The result follows since \( [(N-1)!]_q \neq 0 \).

**Example 1: Complex matrix algebras**

Let \( N \) be as above an integer with \( N \geq 2 \) and let \( n_k, \ k \in \{1, \ldots, N\} \) be \( N \) integers greater or equal to 1, \( n_k \geq 1 \), with sum \( S = \sum_{k=1}^{N} n_k \geq N \). The algebra of complex \( S \times S \) matrices will be denoted by \( M_S(\mathbb{C}) \). Associated to the family \( (n_k) \), there is a decomposition into rectangular blocks \( A_j \) of each element \( A \) of \( M_S(\mathbb{C}) : A = (A_j^i) \), \( (i, j = 1, 2, \ldots, N) \), where \( A_j \) is a complex matrix with \( n_i \) lines and \( n_j \) columns. One equips \( M_S(\mathbb{C}) \) with a \( \mathbb{Z}_N \)-graduation, \( M_S(\mathbb{C}) = \bigoplus_{p \in \mathbb{Z}_N} (M_S(\mathbb{C}))^p \), by giving the degree \( j-i \mod(N) \) to the block \( A_j^i \) (i.e. to the matrix which has only \( A_j^i \) as nonzero block). Equipped with this graduation, \( M_S(\mathbb{C}) \) is a \( \mathbb{Z}_N \)-graded algebra: \( (M_S(\mathbb{C}))^a(M_S(\mathbb{C}))^b \subset (M_S(\mathbb{C}))^{a+b}, \forall a, b \in \mathbb{Z}_N \).

Let \( e \in (M_S(\mathbb{C}))^1 \) be such that \( e^N \) is a multiple of the unit \( 1 \in (M_S(\mathbb{C}))^0 \):

\[
e^N = \lambda 1, \ \lambda \in \mathbb{C}
\] (1)
Let $q$ be a primitive $N$-th root of the unity and set $d(A) = eA - q^n Ae$ for $A \in (M_\mathbb{S}(\mathbb{C}))^a$. This defines an endomorphism $d$ of $M_\mathbb{S}(\mathbb{C})$ which is of degree 1 and satisfies

$$d(AB) = d(A)B + q^n Ad(B), \forall A \in (M_\mathbb{S}(\mathbb{C}))^a, \forall B \in M_\mathbb{S}(\mathbb{C})$$

(2)

and $d^N = 0$. Since $d^N = 0$, the $H^{(k)}$, $k \in \{1, \ldots, N - 1\}$, are well defined. In the case $S = N$, i.e. $n_k = 1 \forall k \in \{1, \ldots, N\}$, any $e \in (M_N(\mathbb{C}))^0$ satisfies (1). In this case ($S = N$), if $e^N \neq 0$ then one has $H^{(k)} = 0, \forall k \in \{1, \ldots, N - 1\}$. □

Example 2 : (Co)Presimplicial vector spaces

Although all the constructions described here work as well for presimplicial vector spaces as for copresimplicial vector spaces (dual version), we shall describe them in the copresimplicial setting since most examples we shall meet in the following sections are of this type. Also, it should be clear that we work here with complex vector spaces for simplicity but that we could as well work with an appropriate category of modules carrying representations of the group of $N$-th roots of the unity. For the notion of (co)-presimplicial module, we refer to J.L. Loday’s book [5]. Recall that a copresimplicial vector space $E$ is a family $(E^n)_{n \in \mathbb{N}}$ of (complex) vector spaces together with linear mappings, the cofaces, $f_k : E^n \rightarrow E^{n+1}$, $k \in \{0, \ldots, n+1\}$, such that

$$f_\ell f_k = f_k f_{\ell-1}, \text{ for } k < \ell$$

(3)

As usual, the dependence on $n$ of the $f_k$ has been dropped for notational simplicity. We shall identify $E$ with the $\mathbb{N}$-graded vector space $\oplus_n E^n$. Let $q$ be a complex number with $q \neq 0$ and $q \neq 1$ and let us define two homogeneous endomorphisms of degree 1 of $E$, $d_q$ and $\tilde{d}_q$, by setting for $x \in E^n$

$$d_q(x) = \sum_{k=0}^n q^k f_k(x) - q^n f_{n+1}(x) \text{ and } \tilde{d}_q(x) = \sum_{k=0}^{n+1} q^k f_k(x)$$

(4)

**Lemma 3** Let $N$ be an integer with $N \geq 2$ and assume that $q$ is a primitive $N$-th root of the unity, then one has $(d_q)^N = 0$ and $(\tilde{d}_q)^N$.

Notice that for $N = 2$, i.e. for $q = -1$, these two endomorphisms coincide with the usual differential map of $E$, [3]. Therefore when $q$ is a primitive $N$-th root of the unity, the corresponding $H^{(k)}$ are generically nontrivial. □

In the previous example, in the case where $q$ is a primitive $N$-th root of the unity, $E$ is $\mathbb{N}$-graded and $d$ is homogeneous of degree one with $d^N = 0$. 3
More generally, let $E$ be a $\mathbb{Z}$-graded vector space $E = \oplus_n E^n$ equipped with an endomorphism of degree one $d$ satisfying $d^N = 0$ ($N \in \mathbb{N}\setminus\{0,1\}$). In this case, the $H^{(k)}$ are also $\mathbb{Z}$-graded and the hexagon $(H_{\ell,m}^{(k)})$ of Lemma 1 splits into $N$ long exact sequences $(S_p^{k,m})$, $p \in \{0,1,\ldots,N-1\}$.

$$
\cdots \rightarrow H^{(m),Nr+p}[i^1] \rightarrow H^{(\ell+m),Nr+p}[d^{m}] \rightarrow H^{(\ell),Nr+p+m}[i^{N-(\ell+m)}] \\
\rightarrow H^{(N-m),Nr+p+m}[d^{i}] \rightarrow H^{(N-(\ell+m)),Nr+p+\ell+m}[i^{m}] \rightarrow H^{(N-\ell),Nr+p+\ell+m} \rightarrow H^{(m),N(r+1)+p}[i^1] \rightarrow \cdots
$$

where $H^{(k),n} = \{x \in E^n | d^k(x) = 0\}/q^{N-k}(E_{n+k-N})$. If instead of being $\mathbb{Z}$-graded, $E$ is $\mathbb{Z}_N$-graded as in Example 1, then the $N$ exact sequences $(S_p^{k,m})$ are again $N$ exact hexagons because in $\mathbb{Z}_N$ one has $Nr+p = N(r+1)+p = p$.

### 3 Graded $q$-differential algebras

In the remaining part of the paper, $q$ is a complex number with $q \neq 0$ and $q \neq 1$.

**DEFINITION 1** A graded $q$-differential algebra is a $\mathbb{N}$-graded associative unital $\mathbb{C}$-algebra $\mathfrak{A} = \oplus_n \mathfrak{A}^n$ equipped with an endomorphism $d$ of degree one satisfying $d(\alpha \beta) = d(\alpha) \beta + q^{|\alpha|} \alpha d(\beta)$, $\forall \alpha \in \mathfrak{A}^n$ and $\beta \in \mathfrak{A}$, and such that $d^N = 0$ whenever $q^N = 1$ for $N \in \mathbb{N}$ with $N \geq 2$.

The endomorphism $d$ will be referred to as the $q$-differential of $\mathfrak{A}$ and the above twisted Leibniz rule as the $q$-Leibniz rule. For each $q \in \mathbb{C}\setminus\{0,1\}$ there is a natural notion of homomorphism of graded $q$-differential algebra.

An ordinary graded differential algebra is thus a graded $(-1)$-differential algebra with this terminology. Let us give some other examples.

**Example 3: Complex matrix algebras**

Let us come back to Example 1 of Section 2. The only reason why $M_S(\mathbb{C})$ fails to be a graded $q$-differential algebra, with $q$ being a primitive $N$-th root of the unity, is that it is $\mathbb{Z}_N$-graded instead of being $\mathbb{N}$-graded. There is however an easy way to obtain a graded $q$-differential algebra from it. Let $p : \mathbb{N} \rightarrow \mathbb{Z}_N$ be the canonical projection and define $(p^* M_S(\mathbb{C}))^n$ for $n \in \mathbb{N}$ by $(p^* M_S(\mathbb{C}))^n = (M_S(\mathbb{C}))^{p(n)}$. Then, on the $\mathbb{N}$-graded space $p^* M_S(\mathbb{C}) = \oplus_n (p^* M_S(\mathbb{C}))^n$ there is a unique product of $\mathbb{N}$-graded algebra such that the canonical projection $\pi : p^* M_S(\mathbb{C}) \rightarrow M_S(\mathbb{C})$ is an algebra homomorphism. Furthermore there is a unique endomorphism of degree 1 of $p^* M_S(\mathbb{C})$, again denoted by $d$, such that
π ∘ d = d ∘ π. This endomorphism satisfies the q-Leibniz rule and \( d^N = 0 \). Thus \( p^* M_S(C) \) is a graded q-differential algebra. □

**Example 4: Simplicial forms**

Let \( K \) be a simplicial complex, i.e. a set equipped with a set of non-empty subsets \( \mathcal{S} \) such that \( X \in \mathcal{S} \) and \( Y \subset X \) implies \( Y \in \mathcal{S} \), \( (Y \neq \emptyset) \). For \( n \in \mathbb{N} \), an ordered \( n \)-simplex is a sequence \((x_0, \ldots, x_n)\) of elements of \( K \) such that \( \{x_0, \ldots, x_n\} \in \mathcal{S} \). A simplicial n-form is a complex-valued function \((x_0, \ldots, x_n) \mapsto \omega(x_0, \ldots, x_n)\) on the set of ordered \( n \)-simplices. Let \( \Omega^n_K \) denote the vector space of all simplicial n-forms. The graded vector space \( \Omega_K = \oplus_n \Omega^n_K \) is a \( \mathbb{N} \)-graded unital algebra for the product \((\alpha, \beta) \mapsto \alpha \beta\) defined by \(\alpha \beta(x_0, \ldots, x_{a+b}) = \alpha(x_0, \ldots, x_a) \beta(x_a, \ldots, x_{a+b})\) for \(\alpha \in \Omega^a_K\), \(\beta \in \Omega^b_K\) and any ordered \((a+b)\)-simplex \((x_0, \ldots, x_{a+b})\). One defines for \(q \in \mathbb{C}\{0,1\}\) an endomorphism \(d_q\) of degree 1 of \(\Omega_K\) by setting

\[
d_q(\omega)(x_0, \ldots, x_{n+1}) = \sum_{k=0}^{n} q^k \omega(x_0, \ldots, x_{n+1}) - q^n \omega(x_0, \ldots, x_n)
\]

for \(\omega \in \Omega^n_K\) and any ordered \((n+1)\)-simplex \((x_0, \ldots, x_{n+1})\), where \(\overset{k}{\ldots}\) means omission of \(x_k\). Equipped with \(d_q\), \(\Omega_K\) is a graded q-differential algebra, i.e. \(d_q\) satisfies the q-Leibniz rule and \((d_q)^N = 0\) whenever \(q^N = 1\) for \(N \in \mathbb{N}\) with \(N \geq 2\). The above \(d_q\) is a particular case of the one of Example 2, formula [4], if one takes for the \(f_k\) the dual mappings of the usual simplicial faces. This works as well with \(A\)-valued simplicial forms where \(A\) is a unital associative \(C\)-algebra. □

**Example 5: Hochschild cochains**

Let \(A\) be a unital associative \(C\)-algebra. Recall that a \(A\)-valued Hochschild \(n\)-cochain is a linear mapping of \(\otimes^n A\) into \(A\). The vector space of these \(n\)-cochains is denoted by \(C^n(A, A)\). The \(\mathbb{N}\)-graded vector space \(C(A, A) = \oplus_n C^n(A, A)\) is in a natural way a \(\mathbb{N}\)-graded unital associative algebra with product defined by \((\alpha \beta)(x_1, \ldots, x_{a+b}) = \alpha(x_1, \ldots, x_a) \beta(x_{a+1}, \ldots, x_{a+b})\) for \(\alpha \in C^a(A, A)\), \(\beta \in C^b(A, A)\) and \(x_i \in A\), where on the right hand side the product is the product of \(A\). One defines for \(q \in \mathbb{C}\{0,1\}\) a linear mapping of degree 1 of \(C(A, A)\) in itself \(\delta_q\) by setting for \(\omega \in C^n(A, A)\) and \(x_i \in A\)

\[
\delta_q(\omega)(x_0, \ldots, x_n) = x_0 \omega(x_1, \ldots, x_n) + \sum_{k=1}^{n} q^k \omega(x_0, \ldots, x_{k-1} x_k, \ldots, x_n) - q^n \omega(x_0, \ldots, x_{n-1}) x_n
\]

(5)
One has $\delta_q(\alpha \beta) = \delta_q(\alpha) \beta + q^a \alpha \delta_q(\beta)$ for $\alpha \in C^n(A, A)$ and $\beta \in C(A, A)$. Furthermore, $(\delta_q)^N = 0$ whenever $q^N = 1$ for $N \in \mathbb{N}$ with $N \geq 2$. In other words, equipped with $\delta_q$, $C(A, A)$ is a graded $q$-differential algebra. Again, $\delta_q$ is a particular case of the $d_q$ of Example 2 in Section 2, formula (4), with appropriate cofaces (see in [3]).

More generally, for an arbitrary bimodule $M$ over $A$. Formula (3) still defines a linear mapping homogeneous of degree 1 on the graded vector space $C(A, M)$ of $M$-valued Hochschild cochains which satisfies $\delta_q^N = 0$ whenever $q^N = 1$ ($N \geq 2$) and reduces to the Hochschild coboundary for $q = -1$.

When $q$ is a primitive $N$-th root of the unity, the corresponding $H^{(k), n}(k \in \{1, \ldots, N-1\}, n \in \mathbb{N})$, are given by [4]: $H^{(k), Nm} = H^{2m, k}, N(m+1)-k = H^{2(m+1)-1}$, for $k \in \{1, \ldots, N-1\}$ and $m \in \mathbb{N}$, where the $H^n$ denote the usual Hochschild cohomology spaces (of $C(A, M)$), and the other $H^{(k), n}$ vanish.

Example 6: Dual of a product

Let $A$ be an associative unital $\mathbb{C}$-algebra and let $C(A) = \oplus_n C^n(A)$ be the graded vector space of multilinear forms on $A$; i.e. $C^n(A) = (\otimes^n A)^*$ is the ($\mathbb{C}$-) dual of $\otimes^n A$ and $C^0(A) = \mathbb{C}$. By making the natural identifications $C^n(A) \otimes C^m(A) \subset C^{n+m}(A)$ one sees that $C(A)$ is canonically a $\mathbb{N}$-graded unital $\mathbb{C}$-algebra, (the product being the tensor product over $\mathbb{C}$). By duality, the product $m : A \otimes A \to A$ of $A$ gives a linear mapping $m^*$ of $A^*$ into $(A \otimes A)^*$ i.e. $m^* : C^1(A) \to C^2(A)$.

For $q \in \mathbb{C}\backslash\{0,1\}$, $m^*$ extends into a linear mapping $m_q^* : C(A) \to C(A)$ which satisfies the graded $q$-Leibniz rule with $m_q^*(C^0(A)) = 0$ and

$$m_q^*(\omega)(x_0, \ldots, x_n) = \sum_{k=1}^n q^{k-1}\omega(x_0, \ldots, x_{k-1}x_k, \ldots, x_n)$$

(6)

for $\omega \in C^n(A)$ with $n \geq 1$ and $x_i \in A$. It follows then from the associativity of the product of $A$ that one has $(m_q^*)^N = 0$ whenever $q^N = 1$, $N \in \mathbb{N}\backslash\{0,1\}$. Thus $C(A)$ equipped with $m_q^*$ is a graded $q$-differential algebra.

The mapping $m_q^*$ of (3) is a particular case of the $d_q$ of Example 2 in Section 2, formula (4), with an obvious choice for the cofaces and a shift $-2$ in degree.

Let $h$ be the linear mapping of degree $-1$ of $C^+(A) = \oplus_{n \geq 1} C^n(A)$ into itself defined by $h(\omega)(x_1, \ldots, x_{n-1}) = \omega(1, x_1, \ldots, x_{n-1})$ for $\omega \in C^n(A)$ with $n \geq 2$ and by $h((C^1(A)) = 0$. Then one has on $C^+(A)$: $h \circ m_q^* = qm_q^* \circ h = I$. It follows then from Lemma 2 that if $q$ is a primitive $N$-th root of the unity one has $H^{(k), n} = 0$ for $k \in \{1, \ldots, N-1\}$ and $n \geq 1$. On the other hand, in this case, one obviously has $H^{(k), 0} = \mathbb{C}$ ($k \in \{1, \ldots, N-1\}$).
4 Universal $q$-differential calculus

Our aim in this section is to produce the $q$-analog of the universal differential envelope of a unital associative $C$-algebra $A$ [4]. We start with a new example of graded $q$-differential algebra which is itself of interest.

Example 7: The tensor algebra over $A$ of $A \otimes A$

The tensor product (over $C$) $A \otimes A$ is in a natural way a bimodule over $A$. The tensor algebra over $A$ of the bimodule $A \otimes A$ will be denoted by $T(A) = \bigoplus_n T^n(A)$. This is a unital graded algebra with $T^n(A) = \otimes^{n+1}A$ and product defined by

$$(x_1 \otimes \cdots \otimes x_m)(y_1 \otimes \cdots \otimes y_n) = x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes y_2 \otimes \cdots \otimes y_n$$

for $x_i, y_j \in A$.

In particular $A$ coincides with the subalgebra $T^0(A)$. Since $A$ is unital, $A \otimes A$ is the free bimodule generated by $\tau = 1 \otimes 1$. Hence $T(A)$ is the $\mathbb{N}$-graded algebra generated by $A$ in degree 0 and by a free generator $\tau$ of degree 1. In fact one has $x_0 \otimes \cdots \otimes x_n = x_0 \tau x_1 \cdots \tau x_n, \forall x_i \in A$.

**Lemma 4** There is a unique linear mapping $d_q : T(A) \to T(A)$ homogeneous of degree 1 satisfying the $q$-Leibniz rule such that

$$d_q(x) = 1 \otimes x - x \otimes 1 = x \tau - x \tau, \forall x \in A,$$

and

$$d_q(\tau) = \tau^2, \ (i.e., d_q(1 \otimes 1) = 1 \otimes 1 \otimes 1).$$

Moreover $d_q$ satisfies $d_q^N = 0$ whenever $q^N = 1$ for $N \geq 2, N \in \mathbb{N}$.

The $q$-differential $d_q$ on $T(A)$ defined by the above lemma is given by

$$d_q(x_0 \otimes \cdots \otimes x_n) = \sum_{k=0}^n q^k x_0 \otimes \cdots \otimes x_{k-1} \otimes 1 \otimes x_k \otimes \cdots \otimes x_n - q^n x_0 \otimes \cdots \otimes x_n \otimes 1$$

Thus again this is a particular case of the $d_q$ of Example 2 in Section 2, formula [1], with an obvious choice for the cofaces. By induction on the integer $n$, one obtains the action of the $n$-th power $d_q^n$ of $d_q$ on $\tau = 1 \otimes 1$ and on the elements of $A : d_q^n(\tau) = [n!_q] \tau^{n+1}$ and $d_q^n(x) = [n!_q] \tau^{n-1} d_q(x), \forall x \in A$.

The algebra $T(A)$ equipped with $d_q$ is a graded $q$-differential algebra. □
The universal $q$-differential envelope of $\mathcal{A}$

Let $\Omega_q(\mathcal{A})$ be the smallest $q$-differential subalgebra of $\mathfrak{T}(\mathcal{A})$ equipped with $d_q$ which contains $\mathcal{A}$, i.e. the smallest subalgebra of $\mathfrak{T}(\mathcal{A})$, stable by $d_q$, containing $\mathcal{A} = \mathfrak{T}^0(\mathcal{A})$. This is again a graded $q$-differential algebra which is characterized uniquely up to an isomorphism by the following universal property, [1].

**THEOREM 1** Let $\mathfrak{A} = \oplus_n \mathfrak{A}^n$ be a graded $q$-differential algebra and let $\varphi : \mathcal{A} \to \mathfrak{A}^0$ be a homomorphism of unital algebras. Then there is a unique homomorphism $\varphi : \Omega_q(\mathcal{A}) \to \mathfrak{A}$ of graded $q$-differential algebras inducing $\varphi$.

It is natural to call $\Omega_q(\mathcal{A})$ the universal $q$-differential envelope of $\mathcal{A}$ or the universal $q$-differential calculus over $\mathcal{A}$. Form $q = -1$, $\Omega_{(-1)}(\mathcal{A})$ is just the usual universal envelope $\Omega(\mathcal{A})$ of $\mathcal{A}$ as defined in [3].

Let $q$ be a primitive $N$-th root of the unity and let $E$ be the $\mathbb{Z}$-graded vector space $E = \mathbb{C} e_{-(N-1)} \oplus \cdots \oplus \mathbb{C} e_{-1} \oplus \mathfrak{T}(\mathcal{A})$. One extends $d_q$ to $E$ by setting $d_q e_{-k} = e_{-(k-1)}$ for $N - 1 \geq k \geq 2$. One still has $d_q^N = 0$ on $E$. Let $\omega$ be a linear form on $\mathcal{A}$ satisfying $\omega(1) = 1$ and let us define an endomorphism $h$ of degree $-1$ of $E$ by: $h(x_0 \otimes \cdots \otimes x_n) = \omega(x_0)x_1 \otimes \cdots \otimes x_n$ for $x_i \in \mathcal{A}$ and $n \geq 1$, $h(x_0) = -q^{-1}\omega(x_0)e_{-1}$ for $x_0 \in \mathcal{A}$, $h(e_{-k}) = -q^{-(k+1)}(1 + q + \cdots + q^k)e_{-(k+1)}$ for $N - 2 \geq k \geq 1$ and $h(e_{-(N-1)}) = 0$. One has on $E : h \circ d_q - q d_q \circ h = I$. By Lemma 2, this implies that $H^{(k)}(E) = 0$, $\forall k \in \{1,\ldots,N-1\}$. The subspace $F = \mathbb{C} e_{-(N-1)} \oplus \cdots \oplus \mathbb{C} e_{-1} \oplus \Omega_q(\mathcal{A})$ is stable by $d_q$ and by $h$, [3], therefore $H^{(k)}(F) = 0$, $\forall k \in \{1,\ldots,N-1\}$. This implies $H^{(k),n}(\mathfrak{T}(\mathcal{A}), d_q) = H^{(k),n}(\Omega_q(\mathcal{A})) = 0$ for $n \geq 1$ and $H^{(k),0}(\mathfrak{T}(\mathcal{A}), d_q) = H^{(k),0}(\Omega_q(\mathcal{A})) = \mathbb{C}$, $\forall k \in \{1,\ldots,N-1\}$.

Thus when $q$ is a primitive $N$-th root of the unity, the corresponding generalized cohomologies of $(\mathfrak{T}(\mathcal{A}), d_q)$ and of $\Omega_q(\mathcal{A})$ are trivial. This generalizes a well-known result for $q = -1$.

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