(j, 0) ⊕ (0, j) COVARIANT SPINORS AND CAUSAL PROPAGATORS BASED ON WEINBERG FORMALISM

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ABSTRACT

A pragmatic approach to constructing a covariant phenomenology of the interactions of composite, high–spin hadrons is proposed. Because there are no known wave equations without significant problems, we propose to construct the phenomenology without explicit reference to a wave equation. This is done by constructing the individual pieces of a perturbation theory and then utilizing the perturbation theory as the definition of the phenomenology. The covariant spinors for a particle of spin \( j \) are constructed directly from Lorentz invariance and the basic precepts of quantum mechanics following the logic put forth originally by Wigner and developed by Weinberg. Explicit expressions for the spinors are derived for \( j = 1, 3/2 \) and 2. Field operators are constructed from the spinors and the free–particle propagator is derived from the vacuum expectation value of the time–order product of the field operators. A few simple examples of model interactions are given. This provides all the necessary ingredients to treat at a phenomenological level and in a covariant manner particles of arbitrary spin.
The study of quantum field theories of high–spin particles has a long history. Despite considerable work and progress, there remain fundamental difficulties [1–9] with each of the various theoretical approaches which have so far been proposed. At the same time, there is a need for an internally consistent way of treating high–spin objects in a Lorentz covariant manner. For example, electroproduction of highly excited baryons in the nuclear medium can be studied at CEBAF. This work could be extended at future accelerators such as KAON or PILAC at LAMPF. The known baryons [10] have spin $j$ in the range $1/2 \leq j \leq 15/2$. Higher spins might certainly be found in the future. Studying the final state interaction of an excited baryon with the residual nucleus [11], even at a qualitative level, could yield new insights into the quark structure of these excited hadrons. An alternate application might be to use the strong electromagnetic fields of the relativistic heavy ions at RHIC [12, 13] to produce high–spin mesons (mesons with $j \leq 6$ have been found) through the two–photon mechanism. Of particular interest might be the $f_2(1720)$ (a meson with $j = 2$) whose structure might be predominantly glue as it is seen [14] in ‘gluon–rich’ radiative decays of the $J/\Psi$ and it has a much suppressed [15] electromagnetic coupling.

The history of quantum field theories of high–spin particles is much too extensive to review here. The first work on the subject is that of Dirac [16], published eight years after his classic work [17] on spin one–half particles. In this 1936 paper Dirac makes the following observation, “All the same, it is desirable to have the equations ready for a possible future discovery of an elementary particle with spin greater than a half, or for approximate application to composite particles. Further, the underlying theory is of considerable mathematical interest.” Sixteen years later, in a series of back–to–back letters in Physical Review, Anderson and Fermi et al. [18] reported the existence of an ‘intermediate state’ of spin 3/2, the $\Delta^3(1232)$. Since then, there have been many contributions to the field. A representative sample of the work can be found in Refs. [19–38] and the history can be traced through the references contained therein.

Here, we make a pragmatic proposal for treating composite high–spin particles in an internally consistent and Lorentz covariant manner. We make use of the observation made by Wigner [19] and developed extensively by Weinberg [20] that covariant spinors and field operators follow directly from the basic precepts of quantum mechanics and Poincaré covariance. In this work, we generalize the approach of Ryder [39] to cast the work of Weinberg[20] into a form which allows us to generate explicit expressions for covariant spinors for particles of arbitrary spin. We here produce explicit expressions for spinors with $j = 1, 3/2$ and 2. For $j = 1/2$ the procedure does of course, reproduce the standard Dirac spinors. This
demonstrates the practicality of this approach and provides the needed spinors for our future phenomenological work. The construction of the free-field operators from the covariant spinors follows exactly the same logic as can be used for the Dirac case. The free-particle propagator can be defined in terms of the time-ordered product of the field operators. This is the definition of the propagator which is required for a perturbation theory. We here provide an explicit expression for the propagator so defined. To this, we may add model interactions. These interactions we will take from the simple Lorentz scalars that can be constructed from the field operators and kinematical quantities available for the particular problem being investigated. These interactions will include phenomenological form factors in order to model the compositeness of the interacting hadrons. Combining the covariant spinors, field operators and propagators with the model interactions produces a well-defined perturbation theory. We propose to use this perturbation theory as the basic definition of our phenomenology.

In this construction, we make no reference to any wave equation or to any Lagrangian. This is less than an ideal circumstance. However, there does not exist a wave equation for high-spin particles [1–9] which does not have a fundamental difficulty. Thus, in order to make progress at the phenomenological level, we propose an end run around this difficulty — working without a wave equation. Clearly the lack of a wave equation and a Lagrangian formulation might limit the applicability and generality of our approach.

One might ask, since we make extensive use of the work of Weinberg [20], why do we not utilize the Joos–Weinberg [20,21] equations? In investigating this possibility, we have found [6–9] that the Joos–Weinberg equations, even in the absence of interactions, support unphysical solutions, a situation which we term kinematic acausality. With this difficulty at the free-particle level, attempts to introduce interactions into these equations [40] can be problematic because the interactions could mix in the unphysical solutions.

The essential role of the Poincaré group in constructing Lorentz covariant quantum mechanics [19,20] has long been known. In Section II we briefly review Poincaré invariance. Following Weinberg [20] and generalizing the spin 1/2 work of Ryder [39], we construct the general boost operator for arbitrary spin in a form which allows us to produce explicit algebraic expressions for spinors which describe particles of spin $j$. We present explicit expressions for $j = 1, 3/2, \text{and} 2$. Explicit construction of covariant spinors for any spin is seen to be reduced to a straightforward algebraic exercise. The construction of field operators from the spinors is noted to be the obvious generalization of the Dirac case. In Sec. III, using the results of Sec. II, we derive causal Feynman-Dyson propagators for arbitrary spin. We note in passing that this propagator, which is the propaga-
tor necessary for perturbative calculations, is not equivalent [6] to the Green’s function of the Joos–Weinberg equations. The detailed description of model interactions we leave to future work since the model or models of the interaction are motivated by the problem at hand. As is generally the case with phenomenological work, we choose the model interactions to be the simple Lorentz scalars constructed from the field operators and appropriate kinematical variables. One of the goals of this work is to learn how data could discriminate between the possible model couplings and determine their parameters. In Sec. IV we give conclusions and discuss future applications. The general philosophy of this work was recently published [49] as brief report elsewhere.

1. Construction of the \((j, 0) \oplus (0, j)\) Boost Operator

In this section, we follow the logic of Weinberg [20] and particularly use a generalization of the spin one–half discussion of Ryder [39] to construct the boost operator and the covariant spinors in the \((j, 0) \oplus (0, j)\) representation of the Lorentz group. To set the notation and to make this work self contained, we begin with a brief review of Poincaré covariance. We then briefly summarize the argument [19,20] that allows one to construct the boost operators and hence the covariant spinors directly from invariance principles. We demonstrate how this can be done in practice by producing explicit expressions for the cases \(j = 1, 3/2\) and 2.

1.1 Poincaré Transformations

The Poincaré transformations are defined to be the ten linear and continuous transformations which preserve \(ds^2 = dt^2 - d\vec{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu\). We use the metric and, as far as is possible, the notation of [41]. The ten transformations are three rotations about each of the spacial axes, three boosts along each of the spacial axes, and four spacetime translations. These transformations can be summarized by

\[
x'^\mu = \Lambda^\mu_{\nu} x^\nu + a^\mu.
\]

The inertial frame independence of \(ds^2\) requires that the \(\Lambda\) matrices satisfy the condition

\[
\Lambda^\mu_{\rho} \Lambda^\nu_{\sigma} \eta_{\mu\nu} = \eta_{\rho\sigma}.
\]

In order to exhibit clearly our sign conventions, explicit expressions for the \(\Lambda\) matrices are given in Appendix A. The transformations \(\{\Lambda, a\}\) form a non-abelian group, \([\{\Lambda_1, a_1\}, \{\Lambda_2, a_2\}] = \{(\Lambda_1\Lambda_2 - \Lambda_2\Lambda_1), (\Lambda_1a_2 - \Lambda_2a_1) + (a_1 - a_2)\}\), with the
multiplication law $\{\Lambda, a\} = \{\Lambda a, \Lambda + a\}$, the inverse element $\{\Lambda, a\}^{-1} = \{\Lambda^{-1}, -\Lambda^{-1}a\}$, and identity element $\{I, 0\}$, with $I$ a $4 \times 4$ identity matrix and 0 a zero vector.

For infinitesimal transformations, Eq. (1.1) becomes

$$x'^\mu = (\delta^\mu_\nu + \lambda^\mu_\nu) x^\nu + a^\mu, \quad (1.3)$$

where $\lambda^\mu_\nu$ and $a^\mu$ are infinitesimal constants. The nonvanishing $\lambda^\mu_\nu = \lambda^\mu_\nu \epsilon^{\mu\nu}$ are summarized in Table I. The ten hermitian generators of the Poincaré transformations, $X_\alpha$ corresponding to the parameter $\lambda^\alpha [\lambda^1 = \theta, \ldots; \lambda^4 = \varphi, \ldots; \lambda^7 = a_0, \ldots]$, are defined by

$$X_\alpha \equiv i \frac{\partial x'^\mu}{\partial \lambda^\alpha} \bigg|_{\lambda=0} \frac{\partial}{\partial x^\mu} \quad (\alpha = 1, \ldots, 10). \quad (1.4)$$

The three generators of rotation follow from Eqs. (A1–A3) and are the usual angular momentum operators, $X_{\theta_i} = -L_i, \ i = x, y, z$, The three boosts given by Eqs. (A4-A6) yield the generators of the Lorentz boosts ($X_{\phi_i} = K_i, \ i = x, y, z$)

$$K_x = \left( i \frac{\partial}{\partial x} + x \frac{\partial}{\partial t} \right), \ K_y = \left( i \frac{\partial}{\partial y} + y \frac{\partial}{\partial t} \right), \ K_z = \left( i \frac{\partial}{\partial z} + z \frac{\partial}{\partial t} \right). \quad (1.5)$$

The translations given by $x'^\mu = x^\mu + a^\mu$ (with $a_\mu$ as real constant displacements) are produced by the four generators of translations $P_\mu = i \partial/\partial x^\mu$. It should be explicitly noted that the rotations, boosts and the translations under consideration here are globally constant.

By introducing

$$L_{12} = L_z = -L_2, \ L_{31} = L_y = -L_1, \ L_{23} = L_x = -L_3, \ L_{ij} = \epsilon^{ijk} L_k, \quad (1.6)$$

$$L_{i0} = -L_{0i} = -K_i, \quad (i = 1, 2, 3), \quad (1.7)$$

the algebra associated with these generators can be summarized as follows

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\sigma} L_{\mu\rho}), \quad (1.8)$$

$$[P_\mu, L_{\rho\sigma}] = i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho), \quad (1.9)$$

$$[P_\mu, P_\nu] = 0. \quad (1.10)$$
1.2 Poincaré Transformations and Quantum Mechanical States

The preceding is purely classical physics. What are the implications of Poincaré invariance for quantum mechanics? This question was answered by Wigner [19] and expanded in [20]. We follow these works and introduce a quantum mechanical state \(|\text{state}\rangle\). Let the same system now be observed by another inertial observer characterised by \(\{\Lambda, a\}\). Denote the state as observed by this new observer by \(|\text{state}'\rangle\). In order that \(|\text{state}\rangle\) and \(|\text{state}'\rangle\) be physically acceptable states, they must transform as

\[ |\text{state}'\rangle = U(\{\Lambda, a\}) |\text{state}\rangle, \]  

where \(U(\{\Lambda, a\})\) is an unitary operator constrained to satisfy

\[ U(\{\Lambda, a\}) U(\{\Lambda, a\}) = U(\{\Lambda, a\} + \{\Lambda, a\}), \]  

This is simply the requirement that a Poincaré transformation \(\{\Lambda, a\}\) followed by \(\{\Lambda, a\}\) has the same effect as the Poincaré transformation \(\{\Lambda, a\}\). Strictly speaking (1.12) is true for infinitesimal transformations. The finite Poincaré transformations which are constructed by successive application of infinitesimal transformations will occasionally have a minus sign on the r.h.s of (1.12). The representation is then said to be a representation up to a sign. This situation will arise when considering fermionic representations. In such situations the fermionic fields must be so combined as to yield observables which are even functions of these fields. This point is discussed in more detail in Sec. 2.12 of Ref. [42].

Eqs. (1.11) and (1.12) are sufficient [7, 43] to determine \(U(\{\Lambda, a\})\),

\[ U(\{\Lambda, a\}) = \exp \left[-\frac{i}{2} \lambda^{\mu\nu} J_{\mu\nu} + ia^{\mu} P_{\mu}\right], \]

where the following algebra is associated with the generators inducing the transformation

\[ [J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\sigma} J_{\mu\rho}), \]  

\[ [P_{\mu}, J_{\rho\sigma}] = i(\eta_{\mu\rho} P_{\sigma} - \eta_{\mu\sigma} P_{\rho}), \]  

\[ [P_{\mu}, P_{\nu}] = 0. \]

The algebra given by (1.14)–(1.16) coincides with the algebra associated with the generators of spacetime transformations, (1.8)–(1.10). This does not imply
that $L_{\mu\nu}$ is necessarily identical to $J_{\mu\nu}$. All that is required is that both $L_{\mu\nu}$ and $J_{\mu\nu}$ satisfy the same algebra. Even the $P_{\mu}$ appearing in (1.13) need not coincide with the generators of spacetime translations. Specification of a physical state and determining the effect of a Poincaré transformation $\{\Lambda, a\}$ on that state, therefore, requires an explicit determination of the generators.

1.3 Spin and Angular Momentum

As argued by Weinberg [20], we note that (with the exception of the scalar field) if one wishes to arrive at the particle interpretation within the framework of Poincaré covariant theory of quantum systems, one is forced to incorporate necessarily non–unitary, finite–dimensional representations of the Lorentz group. Since only unitary transformations of physical states allow for a probabilistic interpretation, the representation spaces of finite dimensional representations of the Lorentz group cannot be spanned by “physical states” defined via (1.11). The objects which span the finite dimensional representation spaces are called “matter fields,” just “fields,” or “covariant spinors.” Although the finite–dimensional representations are not unitary, they provide the basic ingredients for the construction of a field theory.

The set of generators $\{\vec{J}, \vec{K}\}$ span a linear vector space \footnote{The vector space of the generators should not be confused with the vector space ($\equiv$ representation space) on which the generators act.} with $\vec{J}$ and $\vec{K}$ as the basis vectors. Since the Lorentz group is non-compact, all its finite dimensional representations are non–unitary. To construct these finite dimensional representations, we explicitly note the algebra associated with the Lorentz group

\begin{align*}
[K_i, J_i] &= 0, \quad i = x, y, z \quad (1.17) \\
[J_x, J_y] &= iJ_z, \quad [K_x, K_y] = -iJ_z, \quad [J_x, K_y] = iK_z, \quad [K_x, J_y] = iK_z, \quad (1.18)
\end{align*}

and cyclic permutations. Next we implement the standard rotation by introducing a new basis:

\begin{align*}
\vec{S}_R &= \frac{1}{2}(\vec{J} + i\vec{K}), \quad \vec{S}_L = \frac{1}{2}(\vec{J} - i\vec{K}). \quad (1.19)
\end{align*}

It follows directly that $\vec{S}_R$ and $\vec{S}_L$ each satisfy the algebra of an $SU(2)$ group

\begin{align*}
[(S_R)_i, (S_L)_j] &= 0, \quad i, j = x, y, z. \quad (1.20) \\
[(S_R)_x, (S_R)_y] &= i(S_R)_z, \quad [(S_L)_x, (S_L)_y] = i(S_L)_z, \quad (1.21)
\end{align*}

and cyclic permutations. The Lorentz group is thus seen to be essentially equiv–
alent to $SU(2)_R \otimes SU(2)_L$. The irreducible representations of $SU(2)_R \otimes SU(2)_L$ are labelled by two numbers $(j_r, j_l)$,

$$(\mathcal{S}_R)^2 \phi_{j_r, \sigma_r} = j_r (j_r + 1) \phi_{j_r, \sigma_r}, \quad (S_R)_{z} \phi_{j_r, \sigma_r} = \sigma_r \phi_{j_r, \sigma_r},$$

with

$$\sigma_r = j_r, j_r - 1, j_r - 2, \ldots, -j_r + 1, -j_r.$$  \hspace{1cm} (1.22)

$$(\mathcal{S}_L)^2 \phi_{j_l, \sigma_l} = j_l (j_l + 1) \phi_{j_l, \sigma_l}, \quad (S_L)_{z} \phi_{j_l, \sigma_l} = \sigma_l \phi_{j_l, \sigma_l},$$

with

$$\sigma_l = j_l, j_l - 1, j_l - 2, \ldots, -j_l + 1, -j_l.$$  \hspace{1cm} (1.23)

At this point we will specialize to the $(j, 0)$ and $(0, j)$ representations. These are the simplest representations and are thus a natural place to start a phenomenology. Results of physical measurements should be independent of the representation chosen [44]. However, arguments of simplicity enter into building model interactions and simplicity is not always representation independent.

Under the parity, $(j, 0) \leftrightarrow (0, j)$. Thus in order to include parity, we are led to consider the $(j, 0) \oplus (0, j)$ representation. This also leads to a theory which avoids any extra degrees of freedom and which naturally incorporates the $2(2j+1)$ spinorial and particle/antiparticle degrees of freedom. We introduce the chiral representation $(j, 0) \oplus (0, j)$ covariant spinors

$$\psi_{CH}(\vec{p}) = \begin{pmatrix} \phi^R(\vec{p}) \\ \phi^L(\vec{p}) \end{pmatrix},$$  \hspace{1cm} (1.24)

where $\phi^R(\vec{p})$ represents functions in the $(j, 0)$ representation space, and $\phi^L(\vec{p})$ represents functions in the $(0, j)$ representation space.

There seems to be some ambiguous statements in the literature [20] and in textbooks [45] concerning the hermiticity of the operators $\mathcal{S}_R$ and $\mathcal{S}_L$. To clarify this we note a basic distinction between the finite dimensional representation of $\{\mathcal{J}, \mathcal{K}\}$ and the infinite dimensional representations of $\{\mathcal{J}, \mathcal{K}\}$. For the $(j, 0)$ representation $\mathcal{K} = -i\mathcal{J}$, since by definition for the $(j, 0)$ representation $\mathcal{S}_R = \mathcal{J}$ and $\mathcal{S}_L = 0$. Similarly for the $(0, j)$ representation $\mathcal{K} = +i\mathcal{J}$. As such both $\mathcal{J} \pm i\mathcal{K}$, $\mathcal{S}_R$ and $\mathcal{S}_L$, are hermitian. On the other hand for the infinite dimensional representations both $\mathcal{J}$ and $\mathcal{K}$ are, (1.4) and (1.5), are hermitian. This makes $\mathcal{J} \pm i\mathcal{K}$, $\mathcal{S}_R$ and $\mathcal{S}_L$, non-hermitian. The hermiticity of $\mathcal{J} \pm i\mathcal{K}$, and hence $\mathcal{S}_R$ and $\mathcal{S}_L$, depends on whether one is concerned with finite dimensional representations or infinite dimensional representations.
There is an additional difference between the finite dimensional and the infinite dimensional representations. This difference arises from the interpretation that spin exists in a separate space, an internal space. The finite dimensional representations thus have spin operators which commute with the generators of translation, $P_\mu$. This is not true for the angular momentum operators $L_i$ as can be seen in (1.9).

1.4 Construction of the $(j, 0) \oplus (0, j)$ Boost Operator

With this background we are now in a position to construct the $(j, 0) \oplus (0, j)$ boost operator. For a particle at rest in the unprimed frame, a Lorentz boost results in a particle with momentum $\vec{p}$. The matter fields or covariant spinors transform as the physical $|state\rangle$’s (see Eq. (1.11)), but with one difference. The $J_{\mu\nu}$ is replaced by its finite dimensional counterpart and the unitary operator $U(\{\Lambda, a\})$ is replaced by the non–unitary $D(\{\Lambda, a\})$ which still satisfy the constraint imposed by Poincaré covariance,

$$D(\{\Lambda, \pi\})D(\{\Lambda, a\}) = D(\{\Lambda\Lambda, \Lambda a + \pi\}).$$

(1.25)

For the $(j, 0)$ and $(0, j)$ representations, we obtain

$$\phi^R (\vec{p}) = \exp[\vec{J} \cdot \vec{\varphi}] \phi^R (\vec{0})$$

(1.26)

$$\phi^L (\vec{p}) = \exp[-\vec{J} \cdot \vec{\varphi}] \phi^L (\vec{0}),$$

(1.27)

where the boost parameter $\vec{\varphi}$ is defined as

$$\cosh(\varphi) = \gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{E}{m}, \quad \sinh(\varphi) = v\gamma = |\vec{p}| = \frac{\vec{p}}{m}, \quad \vec{\varphi} = \frac{\vec{p}}{|\vec{p}|},$$

(1.28)

with $\vec{p}$ the three-momentum of the particle.

As a consequence, the chiral representation $^2 (j, 0) \oplus (0, j)$ relativistic covariant spinors defined by Eq. (1.24) transform as

$$\psi_{CH}(\vec{p}) = \begin{pmatrix} \exp(\vec{J} \cdot \vec{\varphi}) & 0 \\ 0 & \exp(-\vec{J} \cdot \vec{\varphi}) \end{pmatrix} \psi_{CH}(\vec{0}).$$

(1.29)

We also introduce a canonical representation which for spin one–half is equivalent to the canonical representation used in [41]. The transformation matrix $A$

\footnote{We call this representation the “chiral” representation because for $j = 1/2$ the representation coincides with the chiral representation of the Dirac spin one–half formalism.}
which relates these representations is given by

\[ \psi_{CA}(\vec{p}) = A \psi_{CH}(\vec{p}), \quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}. \]  

(1.30)

Each entry \( I \) in the matrix \( A \) represents a \((2j + 1) \times (2j + 1)\) identity matrix.

In the canonical representation, the covariant spinors are

\[ \psi_{CA}(\vec{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi^R (\vec{p}) + \phi^L (\vec{p}) \\ \phi^R (\vec{p}) - \phi^L (\vec{p}) \end{pmatrix}, \]  

(1.31)

with the even and odd under parity components of the spinors separated as the upper and lower components, respectively.

Referring to Eq. (1.29), we identify the chiral representation boost matrix as

\[ \mathbf{M}_{CH}(\vec{p}) = \begin{pmatrix} \exp(\vec{J} \cdot \vec{\varphi}) & 0 \\ 0 & \exp(-\vec{J} \cdot \vec{\varphi}) \end{pmatrix}. \]  

(1.32)

The boost matrix in the canonical representation reads

\[ \mathbf{M}_{CA}(\vec{p}) = \begin{pmatrix} \cosh(\vec{J} \cdot \vec{\varphi}) & \sinh(\vec{J} \cdot \vec{\varphi}) \\ \sinh(\vec{J} \cdot \vec{\varphi}) & \cosh(\vec{J} \cdot \vec{\varphi}) \end{pmatrix}. \]  

(1.33)

If \( \vec{J} \) is set equal to \( \vec{\sigma}/2 \) the boost matrix given by Eq. (1.33) coincides with the boost for Dirac spinors in the standard Bjorken and Drell [41] representation. \( \mathbf{M}_{CA}(\vec{p}) \) contains all the information needed to construct any \((j, 0) \oplus (0, j)\) relativistic covariant spinor. In the next section, we provide the explicit results for \( j = 1, 3/2 \) and 2. The examples are chosen not only to demonstrate the procedure of constructing the arbitrary–spin covariant spinors, for mesons and baryons, but also to provide readily available covariant spinors through spin two. Elsewhere [50], we use the spin-3/2 covariant spinors obtained here to study the scattering of a spin-3/2 baryon from an external Coulomb field.
2. \((j, 0) + (0, j)\) Covariant Spinors

2.1 \((1, 0) \oplus (0, 1)\) Covariant Spinors

The representation space of the \((1, 0) \oplus (0, 1)\) covariant spinors is a six-dimensional internal space. The basis vectors for a particle at rest can be chosen to be, in the canonical representation,

\[
\begin{align*}
    u_{+1}(\vec{0}) &= \begin{pmatrix}
        m \\
        0 \\
        0 \\
        0 \\
        0 \\
        0
    \end{pmatrix}, \\
    u_{0}(\vec{0}) &= \begin{pmatrix}
        0 \\
        m \\
        0 \\
        0 \\
        0 \\
        0
    \end{pmatrix}, \\
    u_{-1}(\vec{0}) &= \begin{pmatrix}
        0 \\
        0 \\
        m \\
        0 \\
        0 \\
        0
    \end{pmatrix}, \\
    v_{+1}(\vec{0}) &= \begin{pmatrix}
        0 \\
        0 \\
        0 \\
        m \\
        0 \\
        0
    \end{pmatrix}, \\
    v_{0}(\vec{0}) &= \begin{pmatrix}
        0 \\
        0 \\
        0 \\
        m \\
        0 \\
        0
    \end{pmatrix}, \\
    v_{-1}(\vec{0}) &= \begin{pmatrix}
        0 \\
        0 \\
        0 \\
        0 \\
        m \\
        0
    \end{pmatrix}.
\end{align*}
\]  

The norm is chosen for convenience in considering the \(m \to 0\) limit. This choice of the basis vectors, and the interpretation attached to them that \(u_\sigma(\vec{0})\) represents a particle at rest with the z-component of its spin to be \(\sigma\) \((\sigma = 0, \pm 1)\) and \(v_\sigma(\vec{0})\) an antiparticle at rest with the z-component of its spin to be \(\sigma\) requires that \(J_z\) be diagonal. This gives [46],

\[
\begin{align*}
    J_x &= \frac{1}{\sqrt{2}} \begin{pmatrix}
        0 & 1 & 0 \\
        1 & 0 & 1 \\
        0 & 1 & 0
    \end{pmatrix}, \\
    J_y &= \frac{1}{\sqrt{2}} \begin{pmatrix}
        0 & -i & 0 \\
        i & 0 & -i \\
        0 & i & 0
    \end{pmatrix}, \\
    J_z &= \begin{pmatrix}
        1 & 0 & 0 \\
        0 & 0 & 0 \\
        0 & 0 & -1
    \end{pmatrix}.
\end{align*}
\]  

The boost matrix \(M_{CA}(\vec{p})\) takes the covariant spinor of a particle at rest, \(\psi_{CA}(\vec{0})\), to \(\psi_{CA}(\vec{p})\), the covariant spinor of the same particle with momentum \(\vec{p}\)

\[
\psi_{CA}(\vec{p}) = M_{CA}(\vec{p}) \psi_{CA}(\vec{0}).
\]
The \( \cosh(\vec{J} \cdot \vec{\varphi}) \) which appears in the covariant spinor boost matrix\(^3\) (1.33) can be expanded to yield

\[
\cosh(\vec{J} \cdot \vec{\varphi}) = \cosh \left( 2\vec{J} \cdot \vec{\varphi} \right) = 1 + 2(\vec{J} \cdot \hat{p})(\vec{J} \cdot \hat{\varphi}) \sinh^2 \left( \frac{\varphi}{2} \right) \tag{2.4}
\]

Note that

\[
\sinh \left( \frac{\varphi}{2} \right) = \left( \frac{E - m}{2m} \right)^{\frac{1}{2}}, \tag{2.5}
\]

and

\[
\vec{J} \cdot \hat{p} = \frac{1}{|\vec{p}|} \vec{J} \cdot \vec{p} = \frac{1}{(E^2 - m^2)^{\frac{1}{2}}} \left( J_x p_x + J_y p_y + J_z p_z \right). \tag{2.6}
\]

Substituting for \( J_i \) from (2.3) gives the matrix \( \vec{J} \cdot \hat{p} \)

\[
\vec{J} \cdot \hat{p} = \frac{1}{(E^2 - m^2)^{\frac{1}{2}}} \begin{pmatrix}
  p_z & \frac{1}{\sqrt{2}}(p_x - ip_y) & 0 \\
  \frac{1}{\sqrt{2}}(p_x + ip_y) & 0 & \frac{1}{\sqrt{2}}(p_x - ip_y) \\
  0 & \frac{1}{\sqrt{2}}(p_x + ip_y) & -p_z
\end{pmatrix} \tag{2.7}
\]

This, when substituted into Eq. (2.4), gives

\[
\cosh(\vec{J} \cdot \vec{\varphi}) = 1 + \frac{1}{m(E + m)} \begin{pmatrix}
  p_z^2 + \frac{1}{2}p_+p_- & \frac{1}{\sqrt{2}}p_zp_+ & \frac{1}{2}p_+^2 \\
  \frac{1}{\sqrt{2}}p_zp_+ & p_+p_- & -\frac{1}{\sqrt{2}}p_zp_- \\
  \frac{1}{2}p_+^2 & -\frac{1}{\sqrt{2}}p_zp_+ & p_z^2 + \frac{1}{2}p_+p_- \end{pmatrix}, \tag{2.8}
\]

where

\[
p_{\pm} \equiv p_x \pm ip_y, \tag{2.9}
\]

Similarly \( \sinh(\vec{J} \cdot \vec{\varphi}) \) which appears in the canonical representation boost matrix

\(^3\) See Appendix B for the general expansions of \( \cosh(\vec{J} \cdot \vec{\varphi}) \) and \( \sinh(\vec{J} \cdot \vec{\varphi}) \)
for the covariant spinors (1.33) can be expanded as

$$\sinh(\vec{J} \cdot \vec{\varphi}) = \sinh \left(2 \vec{J} \cdot \frac{\vec{\varphi}}{2} \right) = 2(\vec{J} \cdot \hat{p}) \cosh \left(\frac{\vec{\varphi}}{2} \right) \sinh \left(\frac{\vec{\varphi}}{2} \right). \quad (2.10)$$

Using Eq. (2.7) and noting that

$$\cosh \left(\frac{\vec{\varphi}}{2} \right) = \left(\frac{E + m}{2m} \right)^{\frac{1}{2}}. \quad (2.11)$$

yields

$$\sinh(\vec{J} \cdot \vec{\varphi}) = \frac{1}{m} \begin{pmatrix} p_z & \frac{1}{\sqrt{2}}p_- & 0 \\ \frac{1}{\sqrt{2}}p_+ & 0 & \frac{1}{\sqrt{2}}p_- \\ 0 & \frac{1}{\sqrt{2}}p_+ & -p_z \end{pmatrix}. \quad (2.12)$$

Substituting $\sinh(\vec{J} \cdot \vec{\varphi})$ and $\cosh(\vec{J} \cdot \vec{\varphi})$ into (1.33) provides an explicit expression for the canonical representation boost operator for the $(1, 0) \oplus (0, 1)$ covariant spinors. Applying the boost operator (1.33) to the rest spinors (2.1) and utilizing the identities (2.8) and (2.12) yields the $(1, 0) \oplus (0, 1)$ covariant spinors

$$u_{+1}(\vec{p}) = \begin{pmatrix} m + \left[ \frac{2p_0^2 + p_+ p_-}{2(E + m)} \right] \\ p_0 + \frac{1}{\sqrt{2}}(E + m) \\ p_+^2 / 2(E + m) \\ p_+ / \sqrt{2} \\ 0 \end{pmatrix}, \quad (2.13)$$
\[ u_o(\vec{p}) = \begin{pmatrix} p_z p_- / \sqrt{2} (E + m) \\ m + [p_+ p_-/(E + m)] \\ -p_z p_+ / \sqrt{2} (E + m) \\ p_- / \sqrt{2} \\ 0 \\ p_+ / \sqrt{2} \end{pmatrix}, \quad (2.14) \]

\[ u_{-1}(\vec{p}) = \begin{pmatrix} p_-^2 / 2 (E + m) \\ -p_z p_- / \sqrt{2} (E + m) \\ m + [(2p_z^2 + p_+ p_-) / 2 (E + m)] \\ 0 \\ p_- / \sqrt{2} \\ -p_z \end{pmatrix}, \quad (2.15) \]
\[ v_{+1}(\vec{p}) = \begin{pmatrix} p_z \\ p_+ / \sqrt{2} \\ 0 \\ m + [(2p_z^2 + p_+ p_-)/2(E + m)] \\ p_z p_+ / \sqrt{2}(E + m) \\ p_+^2 / 2(E + m) \end{pmatrix}, \quad (2.16) \]

\[ v_0(\vec{p}) = \begin{pmatrix} p_- / \sqrt{2} \\ 0 \\ p_+ / \sqrt{2} \\ p_z p_- / \sqrt{2}(E + m) \\ m + [p_+ p_-/(E + m)] \\ -p_z p_+ / \sqrt{2}(E + m) \end{pmatrix}, \quad (2.17) \]
It is important to note that the \( m \to 0 \) limit of these spinors is well-defined and physical. Consider a massless particle travelling along the \( \hat{z} \) axis (for an arbitrary direction the quantization axis for the angular momentum would have to be chosen accordingly). For this case only \( u_{\pm 1}(\vec{p}) \) and \( v_{\pm 1}(\vec{p}) \) survive while \( u_0(\vec{p}) \) and \( v_0(\vec{p}) \) vanish identically. Explicitly the \( m \to 0 \) limits of the covariant spinors are

\[
v_{-1}(\vec{p}) = \begin{pmatrix}
0 \\
p_-/\sqrt{2} \\
-p_z \\
p_-^2/(E + m) \\
-p_z p_-/\sqrt{2}(E + m) \\
m + [(2p_z^2 + p_+ p_-)/(E + m)]
\end{pmatrix},
\]

(2.18)

\[
\begin{align*}
\lim_{m \to 0} u_{+1}(p_z) &= \begin{pmatrix} E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
\lim_{m \to 0} u_0(p_z) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
\lim_{m \to 0} u_{-1}(p_z) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -E \end{pmatrix}, \\
\lim_{m \to 0} u_0(p_z) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
\lim_{m \to 0} u_{-1}(p_z) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -E \end{pmatrix}. 
\end{align*}
\]

(2.19)

\[
\begin{align*}
\lim_{m \to 0} v_{+1}(p_z) &= \begin{pmatrix} E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
\lim_{m \to 0} v_0(p_z) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
\lim_{m \to 0} v_{-1}(p_z) &= \begin{pmatrix} -E \\ 0 \\ 0 \\ 0 \\ 0 \\ E \end{pmatrix}. 
\end{align*}
\]

(2.20)

While \( u_{+1}(p_z) \) and \( v_{+1}(p_z) \) are identical the \( u_{-1}(p_z) \) \( v_{-1}(p_z) \) differ by a relative
minus sign. As one would wish, only the $|m| = j$ spinors remain non-zero in the $m \to 0$ limit.

2.2 $(3/2, 0) \oplus (0, 3/2)$ Covariant Spinors

A covariant description of spin 3/2 baryons, such as the $\Delta (1232)$, can be based on the $(3/2, 0) \oplus (0, 3/2)$ covariant spinors. As for all the $(j, 0) \oplus (0, j)$ spinors, the covariant spinors for spin 3/2 have exactly the correct number of spinorial and particle/antiparticle degrees of freedom. The canonical representation $(3/2) \oplus (0, 3/2)$ covariant spinors are obtained in a similar fashion as were the $(1, 0) \oplus (0, 1)$ covariant spinors in the last section.

We first note that Eqs. (B3) and (B4) for $j = 3/2$ give

$$\cosh(2 \vec{J} \cdot \vec{\varphi}) = \cosh \varphi \left[ I + \frac{1}{2} \left\{ (2 \vec{J} \cdot \vec{\varphi})^2 - I \right\} \right],$$

(2.21)

$$\sinh(2 \vec{J} \cdot \vec{\varphi}) = (2 \vec{J} \cdot \vec{\varphi}) \sinh \varphi \left[ I + \frac{1}{6} \left\{ (2 \vec{J} \cdot \vec{\varphi})^2 - I \right\} \right],$$

(2.22)

Using (2.5), (2.6) and (2.11) gives

$$\cosh(\vec{J} \cdot \vec{\varphi}) = \left( \frac{E + m}{2m} \right)^{1/2} \left[ I + \frac{1}{2} \left\{ \left( \frac{2 \vec{J} \cdot \vec{\varphi}}{E^2 - m^2} \right) - I \right\} \right],$$

(2.23)

$$\sinh(\vec{J} \cdot \vec{\varphi}) = \left( \frac{E + m}{2m} \right)^{1/2} \left[ \frac{2 \vec{J} \cdot \vec{\varphi}}{E + m} + \frac{1}{6} \frac{2 \vec{J} \cdot \vec{\varphi}}{E + m} \left\{ \left( \frac{(2 \vec{J} \cdot \vec{\varphi})^2}{E^2 - m^2} \right) - I \right\} \right].$$

(2.24)

These expansions when substituted in Eq. (1.33) provide the canonical representation boost for the $(3/2, 0) \oplus (0, 3/2)$ covariant spinors.

For the rest spinors we chose the norm such that in the $m \to 0$ limit the rest spinors vanish and the boosted spinors have a non-singular norm. The simplest choice for $u_\sigma(\vec{0})$ and $v_\sigma(\vec{0})$ are eight-element column vectors each with a single entry of $m^{3/2}$ in the appropriate row and zero elsewhere. Interpreting these rest spinors as eigenstates of $J_z$ requires a representation in which $J_z$ is diagonal,

$$J_x = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & \sqrt{3} & 0 & 0 \end{pmatrix},$$

$$J_y = \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix},$$

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Substituting these into (2.23) and (2.24), and putting the result into (1.33) provides an explicit expression for the $(3/2, 0) \oplus (0, 3/2)$ boost operator in the canonical representation, which when applied to the rest spinors, gives
\[
\begin{align*}
\nu_{\frac{1}{3}}(\vec{p}) &= m_{\frac{1}{2}} \left( \frac{E + m_{2}}{2m_{2}} \right)^{\frac{1}{2}} \\
&= \left( \frac{9p_{z}^2 + 3p_{+}p_{-} + 5m^2 + 4Em - E^2}{4(m + E)} \right)^{\frac{1}{2}} \\
&\times \left\{ \right. \\
&\left. \begin{array}{c}
\sqrt{3}p_{+}p_{z}/(m + E) \\
\sqrt{3}p_{z}^2/2(m + E) \\
0 \\
p_{z}(9p_{z}^2 + 7p_{+}p_{-} + 13m^2 + 12Em - E^2)/4(m + E)^2 \\
\sqrt{3}p_{+}(13p_{z}^2 + 7p_{+}p_{-} + 13m^2 + 12Em - E^2)/12(m + E)^2 \\
\sqrt{3}p_{+}^2p_{z}/2(m + E)^2 \\
p_{z}^3/2(m + E)^2 \\
\end{array} \right. \\
\right\} \\
(2.26)
\end{align*}
\]
\[ u_{\frac{1}{2}}(\vec{p}) = m^{\frac{1}{2}} \left( \frac{E + m}{2m} \right)^{\frac{1}{2}} \begin{pmatrix} \sqrt{3}p_z/(m + E) \\ (p_z^2 + 7p_+p_- + 5m^2 + 4Em - E^2)/4(m + E) \\ 0 \\ \sqrt{3}p_z^2/2(m + E) \\ \sqrt{3}p_-(13p_z^2 + 7p_+p_- + 13m^2 + 12Em - E^2)/12(m + E)^2 \\ p_z(p_z^2 + 19p_+p_- + 13m^2 + 12Em - E^2)/12(m + E)^2 \\ p_+(p_z^2 + 10p_+p_- + 13m^2 + 12Em - E^2)/6(m + E)^2 \\ -\sqrt{3}p_z^2p_+/2(m + E)^2 \end{pmatrix} \]
\[ u_{-\frac{1}{2}}(\vec{p}) = m^{\frac{1}{2}} \left( \frac{E + m}{2m} \right)^{\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}p_z^2}{2(m + E)} & 0 & \frac{\sqrt{3}p_z^2}{(m + E)^2} \\ & & \frac{p_-(p_z^2 + 7p_+p_- + 5m^2 + 4Em - E^2)/4(m + E)}{2} \\ & & \frac{-\sqrt{3}p_+p_z/(m + E)}{2} \end{pmatrix} \times \begin{pmatrix} \frac{\sqrt{3}p_z^2}{2(m + E)} & 0 & \frac{\sqrt{3}p_z^2}{(m + E)^2} \\ & & \frac{p_-(p_z^2 + 10p_+p_- + 13m^2 + 12Em - E^2)/6(m + E)^2}{2} \\ & & \frac{-p_z(p_z^2 + 19p_+p_- + 13m^2 + 12Em - E^2)/12(m + E)^2}{2} \end{pmatrix} \left(\begin{array}{c} \sqrt{3}p_+(13p_z^2 + 7p_+p_- + 13m^2 + 12Em - E^2)/12(m + E)^2 \end{array}\right) \]
\[ u_{-\frac{3}{4}}(\vec{p}) = m^\frac{1}{2} \left( \frac{E + m}{2m} \right)^{\frac{1}{4}} \]

\[
\begin{pmatrix}
0 \\
\sqrt{3}p_z^2 / 2(m + E) \\
-\sqrt{3}p_\perp p_z / (m + E) \\
(9p_z^2 + 3p_\perp p_\perp + 5m^2 + 4Em - E^2) / 4(m + E) \\
\sqrt{3}p_\perp (13p_z^2 + 7p_\perp p_\perp + 13m^2 + 12Em - E^2) / 12(m + E)^2 \\
-p_z(9p_z^2 + 7p_\perp p_\perp + 13m^2 + 12Em - E^2) / 4(m + E)^2
\end{pmatrix}
\]

An inspection of the boost given by Eq. (1.33) immediately reveals that the four \( v_\sigma(\vec{p})'s \) can now be readily obtained by interchanging the four bottom elements with the four top elements of the respective \( u_\sigma(\vec{p})'s \), i.e.

\[ v_\sigma(\vec{p}) = \gamma_5 \ u_\sigma(\vec{p}), \]
where $\gamma_5$ is

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

with $I$ the $(2j + 1) \times (2j + 1)$ identity matrix.

It can be readily verified that for a massless particle traveling along the $\hat{z}$ axis, only the $u_{\pm \frac{3}{2}}(\vec{p})$ and $v_{\pm \frac{3}{2}}(\vec{p})$ survive. The $u_{\pm \frac{1}{2}}(\vec{p})$ and $v_{\pm \frac{1}{2}}(\vec{p})$ vanish identically. Explicitly, the non–zero spinors are given by

$$\lim_{m \to 0} u_{+ \frac{3}{2}}(p_z) = \sqrt{2E^2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \lim_{m \to 0} u_{- \frac{3}{2}}(p_z) = \sqrt{2E^2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix},$$

and

$$\lim_{m \to 0} v_{+ \frac{3}{2}}(p_z) = \sqrt{2E^2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \lim_{m \to 0} v_{- \frac{3}{2}}(p_z) = \sqrt{2E^2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}. \quad (2.33)$$

While $u_{+ \frac{3}{2}}(p_z)$ and $v_{+ \frac{3}{2}}(p_z)$ are identical the $u_{- \frac{3}{2}}(p_z)$ $v_{- \frac{3}{2}}(p_z)$ again differ by a minus sign.

For a detailed analysis of certain kinematic acausality [49] in the massless limit of Weinberg’s work [51] and its resolution the reader is referred to one of our recent work [52].

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2.3 \((2, 0) \oplus (0, 2)\) Covariant Spinors

In order to calculate the \((2, 0) \oplus (0, 2)\) covariant spinors we follow the now familiar procedure. The boost is given by (1.33) with

\[
cosh(\vec{J} \cdot \vec{\varphi}) = I + \frac{(\vec{J} \cdot \vec{p})^2}{m(m + E)} + \frac{1}{6} \frac{(\vec{J} \cdot \vec{p})^2((\vec{J} \cdot \vec{p})^2 - \vec{p} \cdot \vec{p})}{m^2(m + E)^2} \tag{2.34}
\]

\[
sinh(\vec{J} \cdot \vec{\varphi}) = \frac{\vec{J} \cdot \vec{p}}{m} + \frac{1}{3} \frac{\vec{J} \cdot \vec{p} ((\vec{J} \cdot \vec{p})^2 - \vec{p} \cdot \vec{p})}{m^2(m + E)} \tag{2.35}
\]

and

\[
J_x = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & \sqrt{3/2} & 0 & 0 \\
0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\
0 & 0 & \sqrt{3/2} & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
J_y = \begin{pmatrix}
0 & -i & 0 & 0 & 0 \\
i & 0 & -i\sqrt{3/2} & 0 & 0 \\
0 & i\sqrt{3/2} & 0 & -i\sqrt{3/2} & 0 \\
0 & 0 & i\sqrt{3/2} & 0 & -i \\
0 & 0 & 0 & i & 0
\end{pmatrix},
\]

\[
J_z = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -2
\end{pmatrix}. \tag{2.36}
\]

Application of this boost to rest spinors \(u_\sigma(\vec{0})\) and \(v_\sigma(\vec{0})\), each in the form of column vectors with a single entry of \(m^2\) in the appropriate row and zero elsewhere, yields the following \((2, 0) \oplus (0, 2)\) covariant spinors.
\[
\begin{align*}
    u_{+2}(\vec{p}) &= \left\{ \begin{array}{l}
    (8p_z^4 + 8(p_- p_+ + 2m^2 + 2Em)p_z^2 + p_-^2 p_z^2 + 4m(m + E)p_- p_+) \\
    + 4m^2(m + E)^2)/4(m + E)^2 \\
    (4p_+ p_z^3 + (3p_- p_+^2 + 6m(m + E)p_+)p_z)/2(m + E)^2 \\
    \sqrt{6}(2p_+^2 p_z^2 + p_- p_+^3 + 2m(m + E)p_+^2)/4(m + E)^2 \\
    p_+^3 p_z/2(m + E)^2 \\
    p_+^4/4(m + E)^2 \\
    (2p_z^3 + (p_- p_+ + 2m^2 + 2Em)p_z)/(m + E) \\
    (4p_+ p_z^2 + p_- p_+^2 + 2m(m + E)p_+)/2(m + E) \\
    \sqrt{6} p_+^2 p_z/2(m + E) \\
    p_+^3/2(m + E) \\
    0
    \end{array} \right.
\end{align*}
\]

(2.37)
\[ u_{+1}(\vec{p}) = \left\{ \begin{array}{l}
(4p_-p_z^3 + 3(p_-^2p_+ + 2m(m + E)p_-)p_z)/2(m + E)^2 \\
(2(2p_-p_+ + m^2 + Em)p_z^2 + 2p_-^2p_+^2 + 5m(m + E)p_-p_+ \\
+ 2m^2(m + E)^2)/2(m + E)^2 \\
\sqrt{6}(p_-p_+^2 + m(m + E)p_+p_z)/2(m + E)^2 \\
(2p_-p_+^3 + 3m(m + E)p_+^2)/2(m + E)^2 \\
p_-^3p_z/2(m + E)^2 \\
(4p_-p_z^2 + p_-^2p_+ + 2m(m + E)p_-)/2(m + E) \\
(2p_-p_+ + m(m + E))p_z/(m + E) \\
\sqrt{6}(p_-p_+^2 + m(m + E)p_+)/2(m + E) \\
0 \\
p_+^3/2(m + E)
\end{array} \right. \]

(2.38)
\[
\begin{align*}
\sqrt{6}(2p_-^2 p_-^2 + p_-^3 p_+ + 2m(m + E)p_-^2) / 4(m + E)^2 \\
\sqrt{6}(p_+^2 p_+ + m(m + E)p_+) p_z / 2(m + E)^2 \\
(3p_-^2 p_+^2 + 6m(m + E)p_- p_+ + 2m^2(m + E)^2) / 2(m + E)^2 \\
-\sqrt{6}(p_- p_+^2 + m(m + E)p_+) p_z / 2(m + E)^2 \\
\sqrt{6}(2p_+^2 p_+^2 + p_-^3 p_+ + 2m(m + E)p_+^2) / 4(m + E)^2 \\
\sqrt{6}p_-^2 p_z / 2(m + E) \\
\sqrt{6}(p_-^2 p_+ + m(m + E)p_-) / 2(m + E) \\
0 \\
\sqrt{6}(p_- p_+^2 + m(m + E)p_+) / 2(m + E) \\
-\sqrt{6}p_+^2 p_z / 2(m + E)
\end{align*}
\]

\[u_0(\vec{p}) = \begin{cases}
\sqrt{6}(2p_-^2 p_-^2 + p_-^3 p_+ + 2m(m + E)p_-^2) / 4(m + E)^2 \\
\sqrt{6}(p_+^2 p_+ + m(m + E)p_+) p_z / 2(m + E)^2 \\
(3p_-^2 p_+^2 + 6m(m + E)p_- p_+ + 2m^2(m + E)^2) / 2(m + E)^2 \\
-\sqrt{6}(p_- p_+^2 + m(m + E)p_+) p_z / 2(m + E)^2 \\
\sqrt{6}(2p_+^2 p_+^2 + p_-^3 p_+ + 2m(m + E)p_+^2) / 4(m + E)^2 \\
\sqrt{6}p_-^2 p_z / 2(m + E) \\
\sqrt{6}(p_-^2 p_+ + m(m + E)p_-) / 2(m + E) \\
0 \\
\sqrt{6}(p_- p_+^2 + m(m + E)p_+) / 2(m + E) \\
-\sqrt{6}p_+^2 p_z / 2(m + E)
\end{cases}\]
\[ u_{-1}(\vec{p}) = \begin{pmatrix} 
\frac{p^3 p_z}{2(m + E)^2} \\
\frac{(2p^3 p_+ + 3m(m + E)p^2)/2(m + E)^2}{2(2p^3 p_+ + m^2 + E m)p_z^2 + 2p^2 p_+^2 + 5m(m + E)p_+ p_+ + 2m^2(m + E)^2)/2(m + E)^2} \\
-\sqrt{6}(p^2 p_+ + m(m + E)p_+ p_+ /2(m + E)^2) \\
-\frac{(4p^3 p_z^3 + 3(p_+ p_+^2 + 2m(m + E)p_+ p_+ p_z)/2(m + E)^2}{2(4p_+ p_z^2 + p_+ p_+^2 + 2m(m + E)p_+ p_+^2)/2(m + E)^2} \\
\sqrt{6}(p^2 p_+ + m(m + E)p_+ )/2(m + E) \\
-\frac{(2p_+ p_+ + m(m + E))p_+ /2(m + E)}{0} \\
\end{pmatrix} \]
\[
  u_{-2}(\bar{p}) = \begin{cases}
    p_-^4/4(m + E)^2 \\
    -p_-^3 p_z/2(m + E)^2 \\
    \sqrt{6}(2p_-^2 p_z^2 + p_-^3 p_+ + 2m(m + E)p_-^2)/4(m + E)^2 \\
    -(4p_- p_z^3 + 3(p_-^2 p_+ + 2m(m + E)p_+)p_z)/2(m + E)^2 \\
    (8p_-^4 + 8(p_- p_+ + 2m(m + E))p_z^2 + p_-^2 p_+^2 + 4m(m + E)p_- p_+ \\
    + 4m^2(m + E)^2)/4(m + E)^2 \\
    0 \\
    p_-^3/2(m + E) \\
    -\sqrt{6}p_-^2 p_z/2(m + E) \\
    (4p_- p_z^2 + p_-^2 p_+ + 2m(m + E)p_-)/2(m + E) \\
    -(2p_z^3 + (p_- p_+ + 2m(m + E))p_z)/(m + E)
  \end{cases}
\]

(2.41)
The antiparticle covariant spinors are
\[ v_\sigma(p) = \gamma_5 u_\sigma(p). \] (2.42)
Just as in the previous cases, for a massless particle travelling along the \( \hat{z} \) direction, only \( u_{\pm 2}(p_z) \) and \( v_{\pm 2}(p_z) \) are found to be non-null. The \( u_{\pm 1,0}(p_z) \) and \( v_{\pm 1,0}(p_z) \) vanish identically. While \( u_{+2}(E) \) and \( v_{+2}(E) \) are identical, the \( u_{-2}(p_z) \) \( v_{-2}(p_z) \) again differ by a sign.

2.4 Orthonormality of \((j, 0) \oplus (0, j)\) Covariant Spinors

We define \( \gamma_6^{CA} \) as the obvious generalization of the Dirac \( \gamma_6 \)
\[ \gamma_6^{CA} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \] (2.43)
and introduce (in the canonical representation)
\[ \bar{\psi}_\sigma(p) = \psi_\sigma^\dagger(p) \gamma_6^{CA}. \] (2.44)
To ensure the correctness of the \( j = 1, 3/2, \) and \( j = 2 \) particle/antiparticle covariant \( u_\sigma(p) \) and \( v_\sigma(p) \) presented here, we have through brute force matrix multiplication verified that
\[ \bar{u}_\sigma(p) u_\sigma'(p) = m^{2j} \delta_{\sigma\sigma'}, \] (2.45)
\[ \bar{v}_\sigma(p) v_\sigma'(p) = -m^{2j} \delta_{\sigma\sigma'}. \] (2.46)
In the canonical representation the origin of the “minus” sign in the \( rhs \) of the orthonormality condition (2.46) can be readily traced back to the structure of \( \gamma_6^{CA} \), and the fact that \( v_\sigma(p) \) are obtained from the \( u_\sigma(p) \) via the matrix \( \gamma_5 \). Symbolically, we have
\[ u \sim \begin{pmatrix} a \\ b \end{pmatrix}, \quad \bar{u} \sim (a^* \quad b^*) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = (a^* \quad -b^*). \] (2.47)
Hence
\[ \bar{u} u \sim a^* a - b^* b. \] (2.48)
While for \( v \)
\[ v \sim \gamma_5 u \Rightarrow v = \begin{pmatrix} b & a \end{pmatrix} \Rightarrow \bar{v} v \sim b^* b - a^* a = -\bar{u} u, \ QED. \] (2.49)
The (relative) minus sign in the \( rhs \) of the orthonormality relations (2.45) and (2.46) is essential for the existence of a conserved charge constructed from the field operators made out of these spinors.
3. Causal Propagators for \((j,0) \oplus (0,j)\) Fields

From the covariant spinors, we can construct field operators. The same arguments \([7]\) which apply for the Dirac case hold here. The field operator for the \((j,0) \oplus (0,j)\) matter fields is

\[
\Psi^{(j,0) \oplus (0,j)}(x) = \sum_{\sigma = -j}^{+j} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \times \left[ u_\sigma(\vec{p}) a(\vec{p}, \sigma) \exp(-ip \cdot x) + v_\sigma(\vec{p}) b^\dagger(\vec{p}, \sigma) \exp(+ip \cdot x) \right],
\]

with

\[
\omega_{\vec{p}} = \sqrt{m^2 + \vec{p}^2}, \quad \Psi^{(j,0) \oplus (0,j)}(x) \equiv \Psi^{(j,0) \oplus (0,j)}\dagger(x) \gamma_\sigma C A.
\]

The object which enters a perturbation calculation as the propagator is the vacuum expectation value of the time–ordered field operators,

\[
\langle x | S_{FD}^j | y \rangle \equiv \langle T[\Psi^{(j,0) \oplus (0,j)}(x) \overline{\Psi^{(j,0) \oplus (0,j)}}(y)] \rangle.
\]

This propagator is not equal to the Green’s function for the Joos–Weinberg equations. This is because these equations support \([6–9]\) spurious and unphysical solutions. A Green’s function constructed from these equations would propagate these extra solutions while (3.3) will propagate only the physical solutions. Using \(\{a_\sigma(\vec{p}), a_\sigma^\dagger(\vec{p}')\} = (2\pi)^3 2\omega_{\vec{p}} \delta_{\sigma\sigma'} \delta(\vec{p} - \vec{p}')\), for fermions and the similar relation for bosons (with the anticommutator replaced by a commutator), we obtain the configuration space Feynman–Dyson propagator for arbitrary spin,

\[
\langle x | S_{FD}^j | y \rangle = \sum_{\sigma = -j}^{+j} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \times \left[ u_\sigma(\vec{p}) \overline{u}_\sigma(\vec{p}) e^{-ip \cdot (x - y)} \theta(x^\circ - y^\circ) + \epsilon v_\sigma(\vec{p}) \overline{v}_\sigma(\vec{p}) e^{+ip \cdot (x - y)} \theta(y^\circ - x^\circ) \right],
\]

with

\[
\epsilon = \begin{cases} 
+1 & \text{for bosons,} \\
-1 & \text{for fermions.}
\end{cases}
\]
The momentum–space Feynman–Dyson propagator is given by

\[
\langle k' | S_{FD}^j | k \rangle = \frac{1}{(2\pi)^3} \int d^4x \int d^4y \frac{e^{ik'\cdot x} e^{-ik\cdot y}}{2\omega_{k'}} \langle x | S_{FD}^j | y \rangle \]

\[
= -i\delta^4(k' - k) \sum_{\sigma = -j}^{+j} \left( \frac{u_\sigma(k)}{k_0 + i\eta - E(k)} - \epsilon \frac{v_\sigma(-k)}{k_0 - i\eta + E(-k)} \right).
\]

The propagator has the structure of a typical particle–hole propagator in non–
relativistic quantum mechanics. Although not a common expression, the stan-
dard Feynman propagator for spin 1/2 \cite{13} can also be written in the form (3.6).

Recently \cite{53} we have inverted the propagator given by Eq. (3.6) and es-
tablished that the resulting wave equation propagates only the kinematically
acceptable solutions. Weinberg \cite{20}, on the other hand, added certain contact
terms to the r.h.s. of (3.3). The resulting Weinberg equation, though manifestly
covariant, propagates kinematically spurious solutions as we have shown in a
recent publication \cite{49}.

The only other element needed in perturbative calculations is an appropriate
model interaction. We will treat model interactions in detail as we make applica-
tions. Phenomenological interactions involving an even number of \((j,0) \oplus (0,j)\)
matter fields and a scalar, pseudoscalar or vector field are straightforward. We
here give some simple examples.

First, the coupling of a scalar with two particles of spin \(j\) can be written as

\[
\mathcal{L}(x) = g_1 \Phi_s(x) \overline{\Psi}^{(j,0)\oplus(0,j)}(x) \Psi^{(j,0)\oplus(0,j)}(x) \\
+ g_2 \partial^\mu \Phi_s(x) \overline{\Psi}^{(j,0)\oplus(0,j)}(x) \partial_\mu \Psi^{(j,0)\oplus(0,j)}(x),
\]

where \(\Phi_s(x)\) is the field operator associated with the scalar particle and \(g_1\) and \(g_2\)
are coupling constants to be determined experimentally. Secondly, the coupling
of a pseudoscalar with two particles of spin \(j\) could be written as

\[
\mathcal{L}(x) = \eta_1 \Phi_p(x) \overline{\Psi}^{(j,0)\oplus(0,j)}(x) \gamma^5 \Psi^{(j,0)\oplus(0,j)}(x) \\
+ \eta_2 \partial^\mu \Phi_p(x) \overline{\Psi}^{(j,0)\oplus(0,j)}(x) \gamma^5 \partial_\mu \Psi^{(j,0)\oplus(0,j)}(x),
\]

where \(\Phi_p(x)\) is the field operator associated with the pseudoscalar particle and \(\eta_1\) and \(\eta_2\) are coupling constants.
Finally, the interactions associated with two–photon production of a spin-2 meson such as the $f_2(1720)$ may be postulated to be of the form

$$\mathcal{L}(x) = \alpha_c A^\mu(x) \overline{\Psi}^{(2,0)\oplus(0,2)}(x) \partial_\mu \Psi^{(2,0)\oplus(0,2)}(x)$$

$$+ \sum_{\{P\}} \alpha_\{P\} A^\mu(x) \overline{\Psi}^{(2,0)\oplus(0,2)}(x) \gamma^-_{\mu\nu\lambda} \partial_\lambda \Psi^{(2,0)\oplus(0,2)}(x),$$

where the summation on $\{P\}$ is a sum on the permutations of the order of the indices of $\gamma^-_{\mu\nu\lambda}$. The $\gamma^-_{\mu\nu\lambda}$ are a set of $10 \times 10$ matrices, related to the spin–2 gamma matrices of Weinberg [20] by

$$\gamma^-_{\mu\nu\lambda} = A \gamma^{Weinberg}_{\mu\nu\lambda} A^{-1}$$

with $A$ given in Eq. (1.30). The explicit form of $\gamma^-_{\mu\nu\lambda}$ can be found in [6] and [7]. The construction of interactions which involve an odd number of $(j, 0) \oplus (0, j)$ matter fields of a given $j$ is a little more involved [47].

4. Conclusions

A totally satisfactory quantum field theory of particles with high spin does not yet exist. This makes it difficult to treat high–spin particles in a covariant manner. We have here taken a different attitude to this subject than seems to have been previously adopted. Not having been able to resolve the difficulties with the existing theories, and indeed having found new problems [6–9], we address the problem from a pragmatic point of view — can one build a covariant phenomenology of high–spin particles which is internally consistent? We here make the proposal of doing this by avoiding any explicit reference to a wave equation and constructing the individual elements needed for a perturbation theory. The first of these, the covariant spinors, can be constructed following the work of Wigner [19] and Weinberg [20]. We have here reviewed their work and provided a practical and detailed technique (a generalization of the spin one–half approach of Ryder [39]) for generating the spinors. Explicit expressions for spinors with $j = 1, 3/2$ and 2 are given. Although the algebra becomes increasingly tedious, the approach can be continued to higher $j$. Field operators can be constructed from the spinors in complete analogy to the spin one–half case. The free–particle propagator is then defined in terms of the vacuum expectation value of the time ordered field operators. We provide explicit expressions for these propagators. The propagator which we define here is not equal (except for the $j = 1/2$ case) to the Green’s function of the Joos–Weinberg equations. Finally, we have provided
some simple examples of model interactions. This provides all of the necessary ingredients to formulate a perturbation theory and thus can form the basis for a phenomenological approach to the interactions of high-spin particles.

Several additional points need to be mentioned. First, although we propose to define our theory as equal to the perturbation theory, calculations need not be done order by order in the expansion. For example, classes of diagrams can be summed through infinite order by making use of integral equations and solving them numerically. The actual implementation of the approach proposed here must be tailored to the particular physical system under investigation. Secondly, if we include phenomenological form factors, all matrix elements will be finite. This does not, however, remove the necessity of renormalizing masses and coupling constants. The calculations need to be executed in terms of physical masses and measurable coupling constants. This approach has been successful for building a phenomenology of the pion–nucleus interaction [48]. We expect that working without a wave equation and an underlying Lagrangian will, at some point, limit the scope of problems which we can undertake. In the mean time, we are examining several applications and have yet to encounter any basic limitations.
APPENDIX A

Three Rotations about each of the \((x, y, z)\)-axes. The transformation matrices relating \(x'\mu\) with \(x^\mu\), \(x'\mu = R^\mu_\nu x^\nu\), are given by

\[ [R^\mu_\nu(\theta_x)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta_x) & -\sin(\theta_x) \\ 0 & 0 & \sin(\theta_x) & \cos(\theta_x) \end{pmatrix}, \]  
(A1)

\[ [R^\mu_\nu(\theta_y)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(\theta_y) & 0 & \cos(\theta_y) \end{pmatrix}, \]  
(A2)

\[ [R^\mu_\nu(\theta_z)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_z) & -\sin(\theta_z) & 0 \\ 0 & \sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]  
(A3)

\([R^\mu_\nu(\theta_i)]\) represents a rotation by \(\theta_i\) about the \(ith\)-axis. The rows and columns are labelled in the order \(0, 1, 2, 3\).

Three Lorentz Boosts along each of the \((x, y, z)\)-axes. The boost matrix for a boost along the positive direction of the unprimed \(x\)-axis, by velocity \(^4v\), is given by

\[ [B^\mu_\nu(\varphi_x)] = \begin{pmatrix} \cosh(\varphi_x) & \sinh(\varphi_x) & 0 & 0 \\ \sinh(\varphi_x) & \cosh(\varphi_x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]  
(A4)

with \(x'^\mu = B^\mu_\nu(\varphi_x)x^\nu\). Similarly

\(^4This\ is\ the\ velocity\ which\ a\ particle\ at\ rest\ in\ the\ unprimed\ frame\ acquires\ when\ seen\ from\ the\ primed\ frame.\)
\[
[B_{\mu \nu}(\varphi_y)] = 
\begin{pmatrix}
\cosh(\varphi_y) & 0 & \sinh(\varphi_y) & 0 \\
0 & 1 & 0 & 0 \\
\sinh(\varphi_y) & 0 & \cosh(\varphi_y) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (A5)
\]

\[
[B_{\mu \nu}(\varphi_z)] = 
\begin{pmatrix}
\cosh(\varphi_z) & 0 & 0 & \sinh(\varphi_z) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh(\varphi_z) & 0 & 0 & \cosh(\varphi_z)
\end{pmatrix}. \quad (A6)
\]

**APPENDIX B**

Here we provide expansions for \(\cosh(2\vec{J} \cdot \vec{\varphi})\) and \(\sinh(2\vec{J} \cdot \vec{\varphi})\). In the identities below we have defined \(\eta = (2\vec{J} \cdot \hat{p})\)

**INTEGER SPIN:**

\[
\cosh(2\vec{J} \cdot \vec{\varphi}) = 1 + \sum_{n=0}^{j-1} \frac{(\eta^2)(\eta^2 - 2^2)(\eta^2 - 4^2) \cdots (\eta^2 - (2n)^2)}{(2n + 2)!} \sinh^{2n+2} \varphi, \quad (B1)
\]

\[
\sinh(2\vec{J} \cdot \vec{\varphi}) = \eta \cosh \varphi \sum_{n=0}^{j-1} \frac{(\eta^2 - 2^2)(\eta^2 - 4^2) \cdots (\eta^2 - (2n)^2)}{(2n + 1)!} \sinh^{2n+1} \varphi. \quad (B2)
\]
HALF INTEGER SPIN:

\[ \cosh(2\vec{J} \cdot \vec{\varphi}) = \cosh \varphi \left[ 1 + \sum_{n=1}^{j-1/2} \frac{(\eta^2 - 1^2)(\eta^2 - 3^2) \ldots (\eta^2 - (2n - 1)^2)}{(2n)!} \sinh^{2n} \varphi \right], \]

\[ \sinh(2\vec{J} \cdot \vec{\varphi}) = \eta \sinh \varphi \left[ 1 + \sum_{n=1}^{j-1/2} \frac{(\eta^2 - 1^2)(\eta^2 - 3^2) \ldots (\eta^2 - (2n - 1)^2)}{(2n + 1)!} \sinh^{2n} \varphi \right]. \]

(B3) 

(B4)
Table I

| Rotation about: | Boost along: |
|-----------------|--------------|
| x–axis          | x–axis       |
| y–axis          | y–axis       |
| z–axis          | z–axis       |
| $\lambda^{23} = -\lambda^{32}$ | $\lambda^{10} = -\lambda^{01}$ |
| $\lambda^{31} = -\lambda^{13}$ | $\lambda^{20} = -\lambda^{02}$ |
| $\lambda^{12} = -\lambda^{21}$ | $\lambda^{30} = -\lambda^{03}$ |

$\theta_x = \theta_y = \theta_z$  
$\phi_x = \phi_y = \phi_z$

Table I. *Nonvanishing* $\lambda^{\mu\nu} = \lambda^{\mu} \epsilon^{\nu\nu}$. Not we only tabulate the nonvanishing $\lambda^{\mu\nu}$, as such, for example $\lambda^{\mu\neq2\nu\neq3} = -\lambda^{\nu\neq3\mu\neq2} = 0$ for a rotation about x-axis. Similar comments apply for other transformations.
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