Saving entanglement via a nonuniform sequence of $\pi$ pulses

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Received 22 April 2010
Accepted for publication 18 May 2010
Published 16 August 2010
Online at stacks.iop.org/PhysScr/82/038103

Abstract
We examine the question of survival of quantum entanglement between the bipartite states and multiparticle states like GHZ states under the action of a dephasing bath by the application of a sequence of $\pi$ pulses. We show the great advantage of the pulse sequence of Uhrig (2007 Phys. Rev. Lett. 98 100504) applied at irregular intervals of time in controlling quantum entanglement. In particular, the death of entanglement could be considerably delayed by pulses. We use quantum optical techniques to obtain exact results.

PACS numbers: 03.67.Pp, 03.65.Yz, 03.65.Ud

(1. Introduction)

It is well known that quantum entanglement deteriorates very fast due to environmental interactions and one would like to find methods that can save or at least slow down the loss of entanglement [1]. It is also now known that quantum entanglement can die much faster than the scale over which dephasing occurs [2]. For example, the coherence of the qubit typically lasts for a time scale of the order of $T_2$ whereas the entanglement can exhibit sudden death and thus it is important to extend the techniques used for single qubits to bipartite and even multipartite systems. In this paper, we examine how the pulse techniques that were developed to examine the issue of dephasing can help in saving the entanglement. The quantum dynamical decoupling [3–5] uses a sequence of control pulses to be used on the system at an interval much less than the time scale of the bath coherence time. In this way, the coupling of the system to the bath can be time reversed and thus canceled. Such a non-Markovian approach has been successfully applied to two-level systems, harmonic oscillators [6]. A different approach was used in [7], where a control pulse was applied to a different transition rather than the relevant two-level transition. This technique shows that the control pulse causes destructive interference between transition amplitudes at different times, which leads to inhibition of the spontaneous emission of an excited atom. Similar techniques could be useful to suppress the decoherence of a qubit coupled to a thermal bath. Other methods for protection against dephasing are known. These include the application of fast modulations to the bath [8] as well as decoherence free subspaces [9]. The dynamical decoupling idea has been implemented in a few recent experiments [10] with excitons in semiconductors, with Rydberg atomic qubits, with solid state qubits and with nuclear spin qubits.

More recent developments primarily due to Uhrig [11–14] go far beyond what has been done earlier on dynamical decoupling. The dynamical decoupling schemes use a series of $\pi$ pulses applied at regular intervals of time. The pulses reverse the evolution given by the Hamiltonian describing the interaction with a dephasing environment. This is because under a $\pi$ pulse the spin operator $S$ reverses sign. Uhrig discovered that $\pi$ pulses applied at irregular intervals of time are much more effective in controlling dephasing. The regular pulse sequence and the Uhrig sequence are given by

$$T_j = \frac{j T}{n+1}, \quad T_j = T \sin^2 \left( \frac{\pi j}{2(n+1)} \right).$$

In this paper, we focus on the utility of the sequence of pulses as discovered by Uhrig in saving quantum entanglement. Unlike other papers that focus on dephasing issues, we concentrate on entanglement. This is important as the dynamical behavior of entanglement could be quite different from that of dephasing. We calculate the concurrence with the sequence of pulses and show the great advantage of the Uhrig pulse sequence.
parameter [15] which characterizes the entanglement between the two qubits. We show the net time evolution of the concurrence parameter under the action of the Uhrig sequence of pulses and compare its evolution with the one given by when the uniform sequence of pulses is applied. We show the great advantage of the Uhrig sequence over the uniform sequence in saving entanglement. A very recent experiment [16] establishes the advantage of Uhrig’s sequence in lengthening the dephasing time of a single qubit. The organization of the paper is as follows: in section 2, we introduce the microscopic model of dephasing and calculate the relevant physical quantities under the influence of the control pulses. In section 3, we show how the coherent state techniques can be used to obtain the dynamical results. In section 4, we calculate the dynamics of entanglement and present numerical results. In section 5, we conclude with possible generalizations of our results on entanglement.

2. Dynamical decay of entanglement under dephasing

Let us consider two qubits in an entangled state [2], which in general could be a mixed state. In terms of the basis states for the two qubits, we choose the initial state as

\[ |1⟩ = |↑⟩_A ⊗ |↑⟩_B, \quad |2⟩ = |↑⟩_A ⊗ |↓⟩_B, \quad |3⟩ = |↓⟩_A ⊗ |↑⟩_B, \quad |4⟩ = |↓⟩_A ⊗ |↓⟩_B. \]  

(2)

\[ \rho = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \]  

(3)

State (3) is positive and normalized if \(a + b + c + d = 1\) and \(bc > |z|^2\). This state has the structure of a Werner state. For \(a = d = 0\), \(b = c = |z| = 1\), it represents a maximally entangled state. The amount of entanglement in the state is given by the concurrence given by

\[ C = \text{Max}(0, \tilde{C}), \quad \tilde{C} = 2|\rho_{23}| - \sqrt{\rho_{11}\rho_{44}} = 2|z|(1 - r), \]  

(4)

Therefore the state is entangled as long as \(|z|\) is greater than \(\sqrt{ad}\). Now, under dephasing the diagonal elements \(a\) and \(d\) do not change. However, the coherence in the qubit decays as \(\exp[-t/T_2]\) and therefore the entanglement survives as long as \(|z|\exp[-2t/T_2] - \sqrt{ad} > 0\) and thus entanglement vanishes if \(t > T_2/2 \ln |z|/\sqrt{ad}\).

We would now examine how the action of pulses can protect the entanglement. We would calculate the time over which entanglement can be made to survive. For this purpose we need to examine the microscopic model of dephasing. We would make the reasonable assumption that each qubit interacts with its own bath. We could then examine the dynamics of the individual qubits and then obtain the evolution of the concurrence.

On a microscopic scale the dephasing can be considered to arise from the interaction of the qubit with a bath of oscillators, i.e. from the Hamiltonian

\[ H = \hbar \sum_i \omega_i a_i^\dagger a_i + \hbar S_z \sum_i g_i (a_i + a_i^\dagger), \]  

(5)

where \(S_z\) is the \(z\) component of the spin operator for the qubit, and the annihilation and creation operators \(a_i\) and \(a_i^\dagger\) represent the oscillators of the Bosonic bath. The bath is taken to have a broad spectrum. In particular for an Ohmic bath we take the spectrum of the bath as

\[ \hbar J \rightarrow \sum_i |g_i|^2 \delta(\omega - \omega_i) = 2\alpha \omega \Theta(\omega_D - \omega), \]  

(6)

where \(\omega_D\) is the cut-off frequency. It essentially determines the correlation time of the bath. Such a bath leads to dephasing, i.e. the spin polarization decays at the rate \(T_2\). The dynamical decoupling schemes use a series of \(\pi\) pulses applied at regular intervals of time, whereas Uhrig applies \(\pi\) pulses at irregular intervals of time. Such nonuniformly spaced pulses are much more effective in controlling dephasing over a time interval determined by the cut-off frequency and number of pulses. The regular sequence of pulses is more effective outside this domain. The pulses reverse the evolution given by the interaction part in Hamiltonian (5) since under a \(\pi\) pulse the spin operator \(S_z\) reverses sign. The regular pulse sequence and the Uhrig sequence are given by equation (1). We need to calculate the dynamical evolution of the off-diagonal element of the density matrix for the qubit. We work in the interaction picture; hence Hamiltonian (5) becomes

\[ H = \hbar S_z \sum_i g_i (a_i e^{-i\omega t} + a_i^\dagger e^{i\omega t}) = \hbar S_z B(t), \]  

(7)

where \(B(t)\) is the bath operator given by

\[ B(t) = \sum_i g_i (a_i e^{-i\omega t} + a_i^\dagger e^{i\omega t}). \]  

(8)

It is easy to see that the off-diagonal element of the single qubit density matrix \(\sigma\) is

\[ \sigma_{1\downarrow}(t) = \text{Tr}_B(\downarrow|U(t)\sigma_B \sigma \sigma B(t) U^\dagger(t)| \uparrow), \]  

(9)

where \(\text{Tr}_B\) is over the initial bath density matrix \(\sigma_B\) and where

\[ U(t) = \text{exp}\left\{-i \int_0^t S_z B(\tau) d\tau\right\}. \]  

(10)

This can be simplified to

\[ \sigma_{1\downarrow}(t) = \sigma_{1\downarrow}(0) \text{Tr}_B \sigma_B \sigma_B V_+(t), \]

\[ = \sigma_{1\downarrow}(0) \langle V_+(t) \rangle, \]  

(11)

where

\[ V_+(t) = \text{exp}\left\{\pm \frac{i}{2} \int_0^t B(t) d\tau\right\}. \]  

(12)

Thus we can write

\[ \sigma_{1\downarrow}(t) = \sigma_{1\downarrow}(0) \zeta(t), \]  

(13)
further simplified to
\[
\int \text{d}ωJ(ω)[n(ω)+\frac{1}{2}]|f(ω)|^2.
\]

Thus
\[
W = \exp(iΦ(t))\Pi_j \exp \left\{ -\int \text{d}ωJ(ω)[n(ω)+\frac{1}{2}]|f(ω)|^2 \right\},
\]

where
\[
f(ω) = -i \left[ 1 + (-1)^{N+1}e^{-iωT} + 2 \sum_{j=1}^{N}(-1)^{j}e^{-iωT_j} \right].
\]

The result (23) is equivalent to equation (8) of Uhrig. We also note that results like (24) appear in the earlier literature [8] dealing especially with non-Markovian master equations.

4. Saving entanglement: numerical results

Since we work under the assumption that each qubit interacts with its own bath, the time-dependent matrix elements of the density matrix in the basis (2) can be obtained by noting that the diagonal elements do not evolve under dephasing. The off-diagonal element \( ρ_{b3}(t) \) is given by
\[
ρ_{b3}(t) = ρ_{b3}(0)|ζ(t)|^2,
\]
where \( ζ(t) \) is defined by equation (13) and its explicit form is given by equation (23). Thus
\[
|ζ(t)|^2 = S(t) = \exp \left\{ -2 \int \text{d}ωJ(ω) \left( n(ω) + \frac{1}{2} \right) |f(ω)|^2 \right\}.
\]

It can then be shown that the time dependence of the concurrence is given by
\[
C(t) = \max\{0, \tilde{C}(t)\},
\]
\[
\tilde{C}(t) = 2|ζ|S(t) - r,
\]
\[
r = \sqrt{q^2} |ζ|^{-1}.
\]
We next discuss the dynamical behavior of the entanglement. The function $f(\omega)$ has been evaluated by Uhrig. For the pulses applied at regular intervals and for $n$ even, we have

$$|f(\omega)|^2 = 4 \tan^2[\omega t/(2n+2)] \cos^2(\omega t/2)/\omega^2 \forall n \text{ even},$$

whereas for Uhrig’s pulse sequence

$$|f(\omega)|^2 \approx 16(n+1)^2 J_{n+1}^2(\omega/2)/\omega^2,$$

where $J_n$ is the Bessel function. The function $S(t)$ is shown in figures 2 and 3 for $n = 10$ and 50. The parameter $\omega_0^{-1}$ is a measure of the bath correlation time. These figures show that the entanglement lives much longer for the Uhrig sequence of pulses applied at nonuniform intervals of time provided that $\omega_0 t \leq 2n$. Thus entanglement can be made to live over times that could be several orders longer than the coherence time of the bath.

5. Conclusions

In conclusion, we have considered how the effects of dephasing on the destruction of entanglement can be considerably slowed by applying the sequence of $\pi$ pulses applied at time intervals given by Uhrig. We demonstrated this explicitly for the case of a mixed entangled state of two qubits. The sequence given by Uhrig is far better in controlling the death of entanglement compared to the sequence applied at regular intervals of time. These conclusions also apply to the multiparticle entangled state like the GHZ state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow \cdots \uparrow\rangle - |\downarrow \cdots \downarrow\rangle)$$

whose entanglement under dephasing would decay as the density matrix at time $t$ would be

$$\rho(t) = \frac{1}{2} |\uparrow \cdots \uparrow\rangle \langle \uparrow \cdots \uparrow| + \frac{1}{2} |\downarrow \cdots \downarrow\rangle \langle \downarrow \cdots \downarrow| - \frac{1}{2} \exp\left(-\frac{t N}{T_2}\right) \left(|\uparrow \cdots \uparrow\rangle \langle \downarrow \cdots \downarrow| + \text{c.c.}\right).$$

Under the application of $\pi$ pulses, the prefactor $\exp[-t N/T_2]$ would be replaced by $(S(t))^{N/2}$. Since $S(t)$ can be made close to unity for times even of the order of the correlation time of the bath, the entanglement of the multiparticle GHZ state would survive over a long time. Note further that Uhrig’s work has been generalized to arbitrary relaxations [12]. Clearly these generalizations should be applicable to the considerations of entanglement. In particular, we hope to examine the protection of the Werner state against different models of the environment.

Finally, we note that our ongoing work also suggests how other methods like the photonic crystal environment can be used to save entanglement.

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