HybridRAM: The first quantum approach for key recovery attacks on Rainbow

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Abstract
A rectangular MinRank attack, proposed by Ward Beullens in 2021[1], reduced the security of Rainbow below the security requirements set out by NIST. If quantum algorithms are applied to perform repeated operations in this attack, the rectangular MinRank attacks may be more threatening and dramatically lower the security level of Rainbow. In this paper, we propose a Hybrid Rank Attack Model called HybridRAM that reduces the computation complexity of rank-based attacks by applying Grover’s quantum search algorithm. We also design a Grover oracle quantum circuit suitable for the rectangular MinRank attack and then propose a Hybrid Rectangular MinRank attack that recovers the keys of Rainbow using the designed quantum circuit. We show that even the parameter set V of Rainbow does not fall short of the 128-bit security level, the minimum security requirement. It means that Rainbow is no longer secure in quantum computing environments.

Keywords: Post Quantum Cryptography, Rainbow signature scheme, Rank attack, Quantum algorithm, Grover’s algorithm, Quantum simulator

1 Introduction
Due to recent advances in the development of quantum computers, such as Google’s 53-qubit quantum processor “Sycamore” [2] and IBM’s 127-qubit quantum processor “Eagle” [3], NIST (National Institute of Standards and Technology) estimates that quantum computers will be capable of breaking RSA-2048 as early as 2026. Therefore, NIST has been fast to conduct a PQC (Post Quantum Cryptography) standardization project that can be securely used in quantum computing environments. The PQC candidates must meet
the security requirements set out by NIST, from the 128-bit security for Level I to the 256-bit security for Level V.

An MQ (Multivariate Quadratic)-based signature scheme, Rainbow, is one of the final standardization candidates for the NIST PQC project. The unforgeability of the Rainbow signature scheme is based on the intractability of the MQ problem (solving multivariate systems of quadratic equations) known to be NP-complete [4]. Even though there were many attempts to forge the Rainbow signature by trying to solve the MQ problem using mathematical techniques such as XL[5] and Gröbner Basis algorithms [6], they have not been successful thus far. However, since Olivier Billet et al. proposed a MinRank attack algorithm to reconstruct private keys from the public keys of the Rainbow in 2006 [7], key recovery attacks such as MinRank have become a potential threat to the Rainbow. In 2020, Bardet et al. reduced the complexity of the MinRank attacks on the Rainbow to 1/3 of existing attacks in the parameter sets III and V of Rainbow [8]. In 2021, Ward Beullens proposed a rectangular MinRank attack [1]. This attack reduces the security level of Rainbow I, III, and V to 127-bit, 177-bit, and 226-bit security levels, respectively. The parameter I of Rainbow does not fall short of the minimum-security requirement, the 128-bit security level, against the rectangular MinRank attack. Hence, the Rainbow team proposed to NIST to replace the parameter set I of Rainbow with the parameter set III and the parameter set III with the parameter set V.

1.1 Our Contributions

In this paper, firstly, we propose a novel quantum rank-based attack model on Rainbow, called a HybridRAM (Hybrid Rank Attack Model), that reduces attack complexity by applying Grover’s quantum search algorithm to existing rank-based attacks. The rank-based attacks on Rainbow consist of three main steps: preprocessing, kernel extraction, and key recovery. The preprocessing step performs mathematical operations to convert the public keys of Rainbow into a single matrix. The mathematical operations are different for each algorithm used by existing rank-based attacks [1, 7–9]. Then, the kernel extraction step repeatedly performs operations finding a kernel of the matrix and computes a linear combination of public keys using the kernel until the rank of the linear combination is smaller than a specific rank (in the case of Rainbow, the rank is less than $v_1$). The private key is recovered during the key recovery step, by constructing a linear combination of the public keys using the kernel. Among all operations in these three steps, the kernel search operation in the kernel extraction step is complicated and time-consuming as it is repeated until the rank condition of the linear combination is satisfied. Our proposed HybridRAM (Hybrid Rank Attack Model) applies Grover’s quantum search algorithm to the kernel extraction step. Since Grover’s quantum search algorithm is capable of parallel processing, the kernel search operation can be performed much faster in our HybridRAM than in the classical rank-based attack model.
After the preprocessing step converts the public keys to suit each attack algorithm, the kernel extraction step is conducted to find a kernel by running Grover’s search algorithm in a quantum computing environment. Our HybridRAM is a general approach that can be applied to any rank-based attack. However, in order to apply our HybridRAM to each existing rank-based attack, it is necessary to design a Grover oracle quantum circuit. This circuit is required to measure the kernel of the matrix converted by the mathematical technique used in each attack algorithm, with a high probability by using appropriate quantum gates. In order to demonstrate the Hybrid Rectangular MinRank Attack, we designed a Grover oracle quantum circuit suitable for the rectangular MinRank attack. Then, we conducted a security analysis on the Hybrid Rectangular MinRank attack, to determine how secure Rainbow is against our proposed attack. Moreover, to verify the feasibility of the Hybrid Rectangular MinRank attack, we implemented the designed quantum circuit with a quantum simulator Qiskit [10], and ProjectQ [11].

Finally, we analyzed the overall complexity of the Hybrid Rectangular MinRank attack and the quantum resources used. According to the results of our security analysis, the complexity of the Hybrid Rectangular MinRank attack is \( \frac{\pi}{4} \sqrt{2^{(o_2-o_1) \log_2 q}} \). Since the parameter set I of the Rainbow has the same number of oil variables in the first and second layers of Rainbow at 32 each, the complexity of the Hybrid Rectangular MinRank attack becomes \( O(1) \). This result means that the kernel can be found with high probability by running the Grover oracle circuit one time only, and the private key of Rainbow using parameter set I can be quickly recovered. Even Parameter V, which provides the highest security level, does not satisfy the 128-bit security level against our Hybrid Rectangular MinRank attack. Therefore, our finding is that all the parameter sets of Rainbow fall short of the 128-bit security level against our Hybrid Rectangular MinRank attack, as shown in Table 1.

### Table 1 A complexity comparison of the MinRank attack [7], the Rectangular MinRank attack [1], and our Hybrid Rectangular MinRank attack

| Parameter set ID | Security complexity |
|------------------|---------------------|
|                  | Rainbow | MinRank attack | Rect. MinRank attack | our Hybrid Rect. MinRank attack |
| I                | \( 2^{128} \) | \( 2^{144} \) | \( 2^{127} \) | \( 1 \) |
| III              | \( 2^{192} \) | \( 2^{544} \) | \( 2^{177} \) | \( 2^{64} \) |
| V                | \( 2^{256} \) | \( 2^{768} \) | \( 2^{226} \) | \( 2^{112} \) |

The summary of the main contributions of this paper are as follows:

- We proposed the HybridRAM (Hybrid rank attack model), the first quantum-based key recovery attack on Rainbow, to perform operations for rank-based attacks rapidly.
• We designed a Grover oracle quantum circuit for applying HybridRAM to Rectangular MinRank attack [1] and then presented the Hybrid Rectangular MinRank attack. We also evaluated the security level of Rainbow against the Hybrid Rectangular MinRank attacks. According to our security analysis, even parameter set V of Rainbow does not satisfy the 128-bit security level against our Hybrid Rectangular MinRank attack.
• We implemented the quantum circuit using various quantum simulators to show the feasibility of our attack. We also analyzed the required quantum resources to evaluate the quantum security of the Rainbow. In order to perform the Hybrid Rectangular MinRank attack and recover private keys of the Rainbow, our quantum circuit require 661, 1832, and 2376 qubits to the parameter set I, III, and V of Rainbow.

1.2 Organization

The remaining paper is organized as follows. Section 2 provides preliminaries of the Rainbow signature scheme, MinRank attack, Grover’s algorithm, and the elementary quantum gates required. In Section 3, we describe a Hybrid rank attack model that applies quantum algorithms to existing rank-based attacks. We also present a Hybrid rectangular MinRank attack in Section 4. Finally, in Section 5, we evaluate the security level of the Rainbow scheme when applying our Hybrid Rectangular MinRank attack and analyze the quantum resources required for it.

2 Preliminaries

We introduce elementary quantum gates of quantum circuits, Rainbow signature scheme, and Grover’s quantum algorithm in this section.

2.1 Elementary Quantum Gates

2.1.1 NOT Gate

The quantum NOT gate is also called the X gate. The operation of the NOT gate in quantum computing, as shown in Figure 1, is the same as the operation of the NOT gate in classical computing.

\[
|x\rangle \xrightarrow{\text{NOT}} |\bar{x}\rangle = |x\rangle \xrightarrow{\text{NOT}} |\bar{x}\rangle
\]

| \hline
| \hline
| \hline

\[\begin{array}{c|c}
0 & 1 \\
1 & 0 \\
\end{array}\]

**Fig. 1** The NOT gate
2.1.2 CNOT Gate

The operation of the CNOT(Controlled-NOT) gate in quantum computing, as shown in Figure 2, is the same as an XOR operation in classical computing. When two qubits $x$ and $y$ are input to this CNOT gate, the CNOT gate results are $x$ for the input $x \oplus y$ for the input $y$.

\[
\begin{array}{c|c}
|x\rangle & |x\rangle \equiv |x\rangle \\
|y\rangle & |y\rangle \equiv |x \oplus y\rangle \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
x & y & x' & y' & x \oplus y & z' \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
\]

Fig. 2 The CNOT gate

2.1.3 $C^N$-NOT Gate

- CCNOT Gate
  The CCNOT(Computed-Controlled-NOT) gate in Figure 3 is also called Toffoli gate ($N = 2$). The Toffoli gate has three inputs, and the output is almost the same as the input, except that the third qubit is flipped only if the first and second qubits are all 1. That is, when three qubits $x$, $y$, and $z$ are input to this Toffoli gate, the result of this Toffoli gate is $x$ and $y$ for the $x$ and $y$, and $z \oplus xy$ for the input $z$.

\[
\begin{array}{c|c|c|c|c|c|c}
x & y & z & x' & y' & z' \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
\end{array}
\]

Fig. 3 The CCNOT gate

- $C^N$-NOT Gate
  The $C^N$-NOT gate in Figure 4 reverses qubit $z$ when $N$ control qubits $c_i$ are all 1. That is, when $N + 1$ qubits $c_0$, $c_1$, ..., $c_{n-1}$, and $z$ are input to this $C^N$-NOT gate, the result of this $C^N$-NOT gate is $c_0$, $c_1$, ..., $c_{n-1}$, and $z' = z \oplus c_0c_1...c_{n-1}$.

2.2 Rainbow Signature Scheme

The Rainbow is one of the Multivariate Quadratic based digital signature schemes, which is NIST Post Quantum Cryptography round 3 candidates.
Rainbow based on the UOV (Unbalanced Oil-Vinegar) problem has the advantage of a relatively short signature and fast signing and verifying algorithm. In this section, we describe the key generation, signature generation, and verification method of the Rainbow signature scheme.

### 2.2.1 Parameters

Rainbow combines five NIST security categories into three, and the parameters used in Rainbow are as follows.

| Parameter sets of Rainbow | NIST security level categories |
|---------------------------|--------------------------------|
| Set ID | $q$ in $\mathbb{F}_q$ | $v_1$ | $o_1$ | $o_2$ | (N-category number) |
| I | $2^4$ | 36 | 32 | 32 | N-I, N-II |
| III | $2^8$ | 68 | 32 | 48 | N-III, N-IV |
| V | $2^8$ | 96 | 36 | 64 | N-V |

The total number of variables used in Rainbow is $n = v_1 + o_1 + o_2$.

### 2.2.2 Key Generation

- **Private Key**

  The private key of the Rainbow consists of two affine maps $S : F^m \to F^m$ and $T : F^n \to F^n$, and central map $F : F^n \to F^n$. The central map $F$ consists of $m(m = n - v_1)$ multivariate equations $f^{(v_1 + 1)}, \ldots, f^{(o_1)}$. When $k \in v_1 + 1, \ldots, n$, $f^{(k)}$ is as follows, and Figure 5 shows the configuration of the central map $F$.

$$
 f^{(k)}(x_1, \ldots, x_n) = \sum_{i,j \in V_l}^{i \leq j} a^{(k)}_{ij} x_i x_j + \sum_{i \in V_l, j \in O_l} \beta^{(k)}_{ij} x_i x_j + \sum_{i \in V_l \cup O_l} \gamma^{(k)}_{i} x_i + \delta^{(k)}
$$

(1)
• Public Key
  The public key of the Rainbow $P$ is the composition of the private keys $S$, $F$, and $T$.

\[ P = S \circ F \circ T : F^n \rightarrow F^m \]  

(2)

2.2.3 Signature Generation and Verification

• Signature Generation
  When the message to be signed is $d$, the hash function is $H : 0,1 \rightarrow F^m$, and the signature is $Z \in F^n$, the signature generation process is as follows.

  1. Compute the hash value $h = H(d) \in F^m$
  2. Compute $x = S^{-1}(h) \in F^m$
  3. Find $y$ that satisfies $F(y) = x$
  4. Compute the signature $z = T^{-1}(y) \in F^n$

• Signature Verification
  Given message $d$ and signature $z$, the signature verification process is as follows.

  1. Compute the hash value $h = H(d) \in F^m$
  2. Compute the $h' = P(z) \in F^m$
  3. If $h = h'$, the signature $z$ is verified

2.3 MinRank Attack

The MinRank problem is to find a linear combination $Q = \sum_{i=1}^{m} \lambda_i Q_i$ with a rank smaller than some rank $r$, given $m (n \times n)$ matrices $Q_1, ..., Q_m$. The MinRank attack finds central map $F$ of Rainbow by solving the MinRank problem. In the case of Rainbow, a linear combination of public keys with a rank of $v_2$, which is the number of vinegar variables of the second layer, corresponds to a linear combination of central maps of the first layer. By finding $o_1$ linear combinations, the number of oil variables in the first layer, the central maps of the first layer can be reconstructed, thereby finding the
secret keys of Rainbow. Algorithm 1 shows the overall process of the MinRank attack on Rainbow.

**Algorithm 1 The MinRank attack**

**Input:** matrices $P^{(v_1+1)},...,P^{(n)}$

**Output:** Linear combination $C = \sum_{i=v_1+1}^{n} c_i \cdot P(i)$ of rank $\leq v_2$

1: \textbf{repeat}
2: \hspace{1em} Choose randomly a vector $\lambda \in \mathbb{F}_m$ and compute $P = \sum_{i=v_1+1}^{n} \lambda \cdot P(i)$
3: \hspace{1em} \textbf{if} $\text{Rank}(P) > 1$ and $\text{Rank}(P) < n$ \textbf{then}
4: \hspace{2em} Choose randomly a vector $\gamma$ from $\text{Ker}(P)$
5: \hspace{2em} $C \leftarrow \sum_{i=v_1+1}^{n} \gamma_i \cdot P(i)$
6: \hspace{1em} \textbf{end if}
7: \hspace{1em} \textbf{until} $\text{Rank}(C) \leq v_2$
8: \textbf{return} $C$

The complexity of the MinRank attack is $o_1 \cdot q^{v_1+1}$. In this attack, it takes $q^{v_1}$ complexity to find a kernel of $P$ (Ker($P$) in Algorithm 1). The complexity of finding a kernel in the Rainbow round 3 parameter takes $2^{144}$ in the parameter set I, $2^{544}$ in the parameter set III, and $2^{768}$ in the parameter set V.

### 2.4 Grover’s algorithm

The Grover’s algorithm uses quantum properties to find the solution in the black box function, in which the size of the input domain is $N (= 2^n)$. While the problem takes the time complexity of $O(N)$ in the classical computer, the Grover’s algorithm reduces the complexity to $O(\sqrt{N})$. Figure 6 shows the entire circuit of the Grover’s algorithm (Algorithm 2).

![Fig. 6 The entire circuit of the Grover’s algorithm](image)

To increase the probability that a solution will be measured, the Grover’s algorithm proceeds by repeatedly using the oracle gate $O_f^\pm$ and Grover diffusion operator $D$. The repetition time ($r_i$) can be selected in two ways. First, if the repetition time is $\sqrt{N}/8$, the probability of measuring a solution exceeds $2/3$ with operating the whole Grover’s algorithm more than 110 times. Secondly, the solution will be measured with high probability when the repetition time is
$\frac{\pi}{4} \sqrt{N}$. In this paper, we repeat the Grover oracle and diffusion gate pair $\frac{\pi}{4} \sqrt{N}$ times to measure the solution at once. The oracle gate should be designed to find a solution to the problem that we want to solve. The oracle gate reverses the qubit $|b\rangle$ only if the input $x$ is the solution. Figure 7 shows how the oracle gate works.

![Fig. 7 The oracle gate](image)

The Grover diffusion operator increases the probability of $x^*$ which is a solution among $2^n$ superpositioned qubit states and is designed as follows.

$$D = H^{\otimes n} Z_0 H^{\otimes n} = H^{\otimes n} (2 |0^n\rangle \langle 0^n| - I) H^{\otimes n} = 2 ((H |0\rangle \langle 0|)^{\otimes n} ((H |0\rangle \langle 0|)^{\otimes n})^\dagger) - H^{\otimes n} H^{\otimes n} = 2 |+^n\rangle \langle +^n| - I = \sum_{x \in 0, 1^n} (2\mu - \alpha_x) |x\rangle$$

(3)

**Algorithm 2** The Grover’s algorithm

**Input:**
- black box function $f$
- $n$-qubit register $x$
- qubit $b$

**Output:** solution $x^*$

1. for $i = 0$ to $n - 1$ do
2. $H(x_i)$
3. end for
4. $H(X(b))$
5. for $i = 1$ to $r_t$ do
6. $O_f^\pm(x, b)$
7. $D(x)$
8. end for
9. solution $x^* \leftarrow Measure(x)$
10. return $x^*$
3 Hybrid Rank Attack Model

In the HybridRAM (Hybrid rank attack model), quantum algorithm is applied to efficiently find kernels, the underlying problem of the rank-based attack. The HybridRAM is the first approach to a key recovery attack using quantum algorithms against Rainbow. HybridRAM threatens the security of Rainbow by effectively recovering the central map of Rainbow to reduce the security level against rank-based attacks. In this section, we introduce how to apply quantum algorithms to rank-based attacks in our HybridRAM. Figure 8 shows our proposed HybridRAM.

![Concept of HybridRAM](image)

The HybridRAM recovers the central map of Rainbow by applying the following three steps. Algorithm 3 shows each step operating in a HybridRAM. CP and QP refer to a classical part operating in a classical computing environment and a quantum part operating in a quantum computing environment, respectively.

![Diagram of Quantum Computing Part](image)

**Algorithm 3 The Hybrid rank attack model**

**Input:** matrices $P^{(v_1+1)}, ..., P^{(n)}$

**Output:** Central Map $F$

1. Convert $P^{(v_1+1)}, ..., P^{(n)}$ to $P$ in CP  
   
2. $\gamma(= Kernel(P)) \leftarrow GroverAlg(P)$ in QP  
   
3. Central Map $F \leftarrow \sum_{i=v_1+1}^{n} \gamma_i \cdot P^{(i)}$ in CP

4. return $F$
• **Preprocessing**

Rainbow has \( m (n \times n) \) matrices \( P^{(v_1+1)}, ..., P^{(n)} \) as public keys. In the preprocessing step, these public keys are converted into another specific \((x \times y)\) matrix \( P \) in a form suitable for the rank-based attack to be performed. In the HighRank attack, the matrices are converted using the greatest common divisor between their determinants. In the MinRank attack, the matrices are converted into a linear combination of them using random vectors. And then, the converted matrix \( P \) is transferred to the quantum kernel extraction step, a quantum part, to efficiently find the kernel.

• **Quantum Kernel Extraction**

To find a kernel of the matrix \( P \), the HybridRAM utilizes Grover’s quantum algorithm. To apply the HybridRAM to rank-based attacks, it is necessary to properly design a Grover oracle quantum circuit to find a kernel of the matrix \( P \) converted in the preprocessing step. In the quantum kernel extraction step, quantum multipliers over \( GF(q) \) are set to multiply \( \log_2 q \)-qubits register \( \gamma_i \) \((0 \leq i < y)\) by elements of matrix \( P \), respectively, to find a \((y \times 1)\) kernel vector \( \gamma \) for \( P \) transferred using quantum algorithms. The example of how to design a Grover’s algorithm for finding the kernel is described in detail in Sect. 4. The kernel of the \( P \) is found through the qubit register \( \gamma \) measured in the quantum algorithm of the quantum kernel extraction step.

• **Key Recovery**

If the kernel \( \gamma \) of the matrix \( P \) is found through the quantum part, the key recovery step recovers the central map of the Rainbow as follows.

\[
CM \leftarrow \sum_{i=v_1+1}^{n} \gamma_i \cdot P^{(i)}
\]  

(4)

Finally, the private key \( F \) is estimated based on the central map.

HybridRAM achieved a quadratic speedup in the complexity of the classical rank-based attacks by applying quantum algorithms to the part of finding a kernel, the main operation of the classical rank-based attacks. HybridRAM allows for finding private keys faster than existing classical rank-based attacks.

### 4 Hybrid Rectangular MinRank Attack

In this section, we propose a Hybrid Rectangular MinRank attack as an example of the application of the HybridRAM. We apply our HybridRAM to a new MinRank attack, the Rectangular MinRank attack [1] by using Grover’s algorithm to find a kernel. The Rectangular MinRank attack reduces the complexity of the MinRank attack by using polar forms of the public keys. It seriously reduces the security level of Rainbow. In the preprocessing step, the \( m (n \times n) \) public keys of Rainbow are converted to the polar form \( P'(e_1, x) \). The polar form of multivariate quadratic polynomial \( p(x) \) is defined as
Hybrid RAM

\[ p'(x, y) := p(x+y) - p(x) - p(y) + p(0) \]  

(5)

For a multivariate quadratic map \( P(x) = p_1(x), \ldots, p_m(x) \), its polar form is defined as

\[ P'(x, y) := p'_1(x, y), \ldots, p'_m(x, y) \]  

(6)

By using the polar form, \( m \times n \) public keys are converted to \( n \times m \) matrices \( (n > m) \). In the preprocessing step, the public keys are converted as shown in Figure 9. Columns at the same position in \( m \) matrices are grouped into one matrix.

![Fig. 9 The conversion to polar form in the preprocessing step](image)

Then, the linear combination of \( n \times m \) converted matrices is computed. After the preprocessing step, if the rank of the linear combination \( P \) is greater than 1 and less than \( n \), the kernel of \( P \) is found using Grover’s algorithm in the quantum kernel extraction step. To find a kernel, we design a Grover oracle circuit as shown in Figure 10. The kernel vector is used for combining the public keys. The combination \( C \) of public keys is a linear combination of the product between the kernel vector and the public key. The preprocessing and quantum kernel extraction steps are repeated until the rank of linear combination \( C \) is less than or equal to \( v_2 \).

Our Grover oracle circuit to find a kernel requires out-of-place quantum-classical multipliers and in-place quantum-quantum adders over \( GF(q) \). The number of qubits required to express a number on \( GF(q) \) is \( n_q = \log_2 q \). The circuits of quantum adder and multiplier over \( GF(q = 2^n) \) are shown in Figures 11 and 12, respectively. For efficient multiplication over large Galois fields, we also use the quantum multiplication circuits proposed in [12]. In our circuit, the quantum dagger circuits of the quantum adder and multiplier are required for the initialization of the quantum circuit. Because the quantum adder and multiplier used in our quantum circuit consist of only CNOT and Toffoli gates, the dagger circuit of the adder and multiplier is the reverse order of the gates of each circuit.

If a kernel vector that meets the conditions is found, the key is recovered using the linear combination \( C \) in the key recovery step. The overall Hybrid
Rectangular MinRank attack algorithm applied to HybridRAM is shown in Algorithm 4 and the design method of grover oracle circuit for the Hybrid Rectangular MinRank attack is shown in Algorithm 5.

For \((n \times m)\) matrix over \(GF(q)\), our Grover oracle circuit requires \(m \cdot n_q\)-qubit registers \(x\), \(n_q\)-qubit register \(t\), \(n \cdot n_q\)-qubit registers \(e\), and qubit \(b\). The required qubit registers are shown in Table 6. By measuring the quantum registers \(x_0, \ldots, x_{m-1}\) after the overall Grover’s algorithm, we get the kernel of the matrix with a high probability.

First, we obtain the result of multiplying the quantum register \(x_0\) by the constant value of row 0, column 0 of the matrix in quantum register \(t\). And then, we add the multiplication result to the quantum register \(e_0\). We repeat the above process for the first row of the matrix and then repeat the above process for the other rows in quantum register \(e_1, \ldots, e_{n-1}\) (line 1~6 in Algorithm 5). And then, take the NOT gates for all quantum registers \(e\) to reverse
Algorithm 4 The Hybrid Rectangular MinRank attack

**Input:** matrices $P^{(v_1+1)}, \ldots, P^{(n)}$

**Output:** Linear combination $C = \sum_{i=v_1+1}^{n} c_i \cdot P^{(i)}$ of rank $\leq v_2$

1: repeat
2: Convert the public keys to polar form $P'(e_i, x)$, $e_i$: basis vector for $\mathbb{F}_q^n$
3: Choose randomly a vector $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\} \in \mathbb{F}_m^n$ and compute $P = \sum_{i=1}^{n} \lambda_i \cdot L_{e_i}$
   $$L_{e_i} = \begin{pmatrix} P'(e_1, e_i) \\ \vdots \\ P'(e_n, e_i) \end{pmatrix}$$
   $\triangleright$ Preprocessing
4: if $\text{Rank}(P) > 1$ and $\text{Rank}(P) < n$ then
5: \quad $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_m\} \leftarrow \text{GroverAlg}(P)$ $\triangleright$ Kernel Extraction
6: \quad $C \leftarrow \sum_{i=v_1+1}^{n} \gamma_i \cdot P^{(i)}$
7: end if
8: until $\text{Rank}(C) \leq v_2$ $\triangleright$ Key Recovery
9: return $C$

Table 3 The qubit registers used in our Grover oracle circuits

| Qubit registers | Number of qubits | Role of qubit registers |
|-----------------|------------------|-------------------------|
| $x$             | $m \cdot n_q$    | At the end of Grover’s algorithm, the kernel is measured at qubit register $x$ |
| $t$             | $n_q$            | The qubit register $t$ is ancillary qubit register. The multiplication of the element of input matrix and qubit register $x$ is stored in the qubit register $t$. |
| $e$             | $n \cdot n_q$    | The sum of multiplications stored in qubit registers $t$, which is the sum of multiplications of each row of input matrix and qubit register $x$, is stored in qubit register $e$. |
| $b$             | 1                | The qubit $b$ is reversed when the qubit register $e$ is all 0. |

all the quantum registers $e$ (line 7~9 in Algorithm 5). We call the above processes FrontOracle (line 1~10 in Algorithm 5). After taking the NOT gates for all quantum registers $e$, the $(n \cdot n_q)$-controlled NOT gate is taken for the qubit $b$ using the quantum registers $e$ as the control qubits to check whether all the $e$ are 0 (line 11 in Algorithm 5). To initialize the quantum registers $e$, dagger circuit of FrontOracle is applied at the end of the Grover oracle circuit (line 12~21 in Algorithm 5).

If we measure the quantum registers $x$ after the Grover oracle and diffusion circuit pairs repeat $\frac{\pi}{4} \sqrt{2^{(o_2-o_1)n_q}}$ times, we can find the kernel vector with high probability.
Algorithm 5 The Grover oracle for the Hybrid Rectangular MinRank attack

Input:
- \((n \times m)\) matrix \(P\)
- \(m \cdot n_q\)-qubit registers \(x\)
- \(n_q\)-qubit register \(t\)
- \(n \cdot n_q\)-qubit registers \(e\)
- qubit \(b\)

Output: \((m \times 1)\) kernel vector \(\gamma\)

1: for \(i = 0\) to \(n - 1\) do
2:     for \(j = 0\) to \(m - 1\) do
3:         \(GF(q)\)-multiply\((x_j, P_{i,j}, t)\)
4:         \(GF(q)\)-add\((t, e_i)\)
5:         \(GF(q)\)-multiply\_dagger\((x_j, P_{i,j}, t)\)
6:     end for
7:     for \(j = 0\) to \(n_q - 1\) do
8:         NOT\((e_{ij})\)
9:     end for
10: end for
11: \((n \cdot n_q)\)-controlled\_NOT\((e_{00} \sim e_{n(n_q-1)}, b)\)
12: for \(i = n - 1\) to 0 do
13:     for \(j = n_q - 1\) to 0 do
14:         NOT\((e_{ij})\)
15:     end for
16:     for \(j = m - 1\) to 0 do
17:         \(GF(q)\)-multiply\((x_j, P_{i,j}, t)\)
18:         \(GF(q)\)-add\((t, e_i)\)
19:         \(GF(q)\)-multiply\_dagger\((x_j, P_{i,j}, t)\)
20: end for
21: \(\gamma \leftarrow Measure(x)\)
22: return \(\gamma\)

After we get the kernel vector \(\gamma\) from the Grover’s algorithm, we compute the linear combination of \(m\) public keys with \(\gamma\) again in the quantum kernel extraction step. If the rank of linear combination \(C\) is less than \(v_2\) (the number of vinegar variables of the second layer), our Hybrid Rectangular MinRank attack outputs the linear combination \(C\) and recovers the central map \(F\) in the key recovery step.

5 Complexity Analysis

In this section, we analyze the security of the Rainbow signature scheme based on our new Hybrid Rectangular MinRank attack. First, we evaluate the security level of the Rainbow by calculating the attack complexity of the Hybrid Rectangular MinRank attack for each Round 3 Rainbow parameter set. Then,
we proceed with the analysis of the quantum resources required for the Hybrid Rectangular MinRank attack.

5.1 Rainbow Parameter Security Level Analysis

Rainbow signature scheme has three security levels with each parameter set shown in Table 2. Its relationship with the NIST security levels is shown in Table 4.

| Security level | Description | Security complexity | Rainbow security levels |
|---------------|-------------|---------------------|------------------------|
| N-I           | hard to break AES128 | $2^{128}$          | I ($2^4, 36, 32, 32$)   |
| N-II          | hard to break SHA256  | $2^{128}$          | I ($2^4, 36, 32, 32$)   |
| N-III         | hard to break AES192  | $2^{192}$          | III ($2^8, 68, 32, 48$) |
| N-IV          | hard to break SHA384  | $2^{192}$          | III ($2^8, 68, 32, 48$) |
| N-V           | hard to break AES256  | $2^{256}$          | V ($2^8, 96, 36, 64$)   |

In the case of Rectangular attack, the public keys of Rainbow are converted to $(n \times m)$ matrices. The number of matrices and the rank of linear combination is also different. Table 5 shows the differences of instances between the existing MinRank attack and the Rectangular MinRank attack.

| Elements of MinRank problem | Known instances | New instances |
|-----------------------------|-----------------|---------------|
| Size of matrices            | $(n \times n)$  | $(n \times m)$ |
| Number of matrices          | $o_2 + 1$       | $n - o_2 + 1$ |
| Rank of linear combination  | $m$             | $o_2$         |

Due to the conversion of the public keys, $v_1$ is equal to $o_2 - o_1$. In this case, since the number of queries in our Grover oracle circuit becomes $\frac{\pi}{4} \sqrt{2^{(o_2-o_1)n_q}}$, if the number of oil variables in the first layer $o_1$ and the number of oil variables in the second layer $o_2$ are the same, the number of queries becomes $O(1)$. Figure 13 shows the comparison of security levels among Rainbow, the MinRank attack, the Rectangular MinRank attack, and our Hybrid Rectangular MinRank attack. The security levels of Rainbow against our attack specified in Figure 13 are calculated based on the complexity equation. All of parameter sets of Rainbow fall short of the 128-bit security level, the minimum security
requirement set out by NIST. It means Rainbow is no longer secure in quantum computing environments.

**Fig. 13** Comparison of security levels among Rainbow, the MinRank attack, the Rectangular MinRank attack, and our Hybrid Rectangular MinRank attack

### 5.2 Quantum Resources

When the parameter of the Rainbow is \((v_1, o_1, o_2)\), the Grover’s algorithm for our Hybrid Rectangular MinRank attack requires total \((n + o_1 + o_2 + 1)n_q + 1\) qubits including \(n\) \(n_q\)-qubit registers \(x\), a \(n_q\)-qubit register \(t\), \((o_1 + o_2)n_q\)-qubit register \(e\), and a qubit \(y\). Table ?? shows the number of qubits required for the Hybrid Rectangular MinRank attack for each Rainbow parameter.

Our Grover oracle quantum circuit requires \(4(o_1 + o_2)n\) quantum multipliers over \(GF(q)\), \(2(o_1 + o_2)n\) quantum adders over \(GF(q)\), \(2(o_1 + o_2)n_q\) NOT gates, and a \(((o_1 + o_2)n_q)\)-controlled NOT gate. There needs to be \(n_q\) NOT gates and \((n_q)^2\) CNOT gates for quantum adder over \(GF(q)\) and quantum multipliers over \(GF(q)\) respectively. Table ?? shows the number of quantum gates required for our Grover oracle circuit for the Hybrid Rectangular MinRank attack.

The Grover diffusion quantum circuit for our attack requires \(2n \cdot n_q\) NOT gates, \(2n \cdot n_q + 2\) Hadamard gates, and a \((2n \cdot n_q - 1)\)-controlled NOT gate. Finally, there needs to be \(\frac{\pi}{2} \sqrt{2^{v_1-n_q}(n + (n + 1)(o_1 + o_2))n_q}\) NOT gates, \(n \cdot n_q + \frac{\pi}{2} \sqrt{2^{v_1-n_q}(n \cdot n_q + 1)}\) Hadamard gates, \(n\pi(o_1 + o_2)\sqrt{2^{v_1-n_q}(n_q)^2}\) CNOT gates, \(\frac{\pi}{2} \sqrt{2^{v_1-n_q}(2n \cdot n_q - 1)}\)-controlled NOT gates, and \(\frac{\pi}{4} \sqrt{2^{v_1-n_q}((o_1 + o_2)n_q)}\)-controlled NOT gates. Table ?? shows the number of quantum gates required for our Grover’s algorithm for the Hybrid Rectangular MinRank attack for each parameter of Rainbow.
Table 6 A comparison of quantum resources for our Hybrid Rectangular MinRank attack

| Parameter Set ID | The number of qubits | The kinds of quantum gates | The number of controlled qubits | The number of gates |
|------------------|----------------------|---------------------------|-------------------------------|-------------------|
| I                | 661                  | H                         | -                             | $3.78 \times 10^{24}$ |
|                  |                      | NOT                       | -                             | $2.47 \times 10^{26}$ |
|                  |                      | $C^N\text{NOT}$           | $1$                           | $1.93 \times 10^{27}$ |
|                  |                      |                           | $2n \cdot n_q - 1$            | $2^{72}$           |
|                  |                      |                           | $(o_1 + o_2)n_q$              | $2^{72}$           |
| III              | 1832                 | H                         | -                             | $1.79 \times 10^{85}$ |
|                  |                      | NOT                       | -                             | $1.46 \times 10^{87}$ |
|                  |                      | $C^N\text{NOT}$           | $1$                           | $2.30 \times 10^{88}$ |
|                  |                      |                           | $2n \cdot n_q - 1$            | $2^{72}$           |
|                  |                      |                           | $(o_1 + o_2)n_q$              | $2^{72}$           |
| V                | 2376                 | H                         | -                             | $1.23 \times 10^{119}$ |
|                  |                      | NOT                       | -                             | $1.25 \times 10^{121}$ |
|                  |                      | $C^N\text{NOT}$           | $1$                           | $1.97 \times 10^{122}$ |
|                  |                      |                           | $2n \cdot n_q - 1$            | $2^{384}$          |
|                  |                      |                           | $(o_1 + o_2)n_q$              | $2^{384}$          |

Table 7 The number of quantum gates required for our Grover oracle circuit for the Hybrid Rectangular MinRank attack

| NOT Gate | CNOT Gate | $((o_1 + o_2)n_q)$-controlled NOT Gate |
|----------|-----------|----------------------------------------|
| $2(o_1 + o_2)n_q$ | $4n(o_1 + o_2)n_q$ | $1$ |

5.3 Quantum Simulation Result

To show the feasibility of our Hybrid Rectangular MinRank attack, we implemented our Grover’s quantum circuit to find a kernel of matrix 7 over $GF(8)$ as a toy example of our model using quantum simulators Qiskit[10] and ProjectQ[11].

$$
\begin{bmatrix}
5 & 2 \\
7 & 1 \\
3 & 4
\end{bmatrix}
$$

(7)

ProjectQ, an open-source software effort for quantum computing, provides a function of drawing quantum circuits. Appendix A shows our Grover oracle quantum circuit of Hybrid Rectangular MinRank attack drawn using ProjectQ. Qiskit, another open-source framework for quantum computing, has the advantage of fast execution speed and supports multi-shot simulation that shows the frequency of measurement when executed several times. We
measured the accuracy of our Grover’s quantum circuit utilizing multi-shot function of Qiskit. Figure 14 shows the output of implementing our Grover’s quantum circuit for the toy example in the Qiskit Aer simulator for 1000 shots.

![Figure 14](image)

**Fig. 14** The histogram obtained by running a quantum circuit in Qiskit Aer simulator

Figure 14 shows a histogram excluding qubit values measured less than 10 times in 1000 shots. Qubit values measured less than 10 times are not kernel values, and the sum of their probabilities is indicated in the bar named ‘NotKernel’. Other bars represent each of the probabilities when the kernels are accurately measured. For example, in our Grover’s quantum circuit, \( x_0 \), \( x_1 \), and \( x_2 \) are measured 120 times out of 1000 times, with kernel values of ’001’, ’101’, and ’110’, respectively. Our Grover’s quantum circuit finds the kernel with 91.4% accuracy.

### 6 Conclusion

In this paper, we have constructed a Hybrid Rank Attack Model, called HybridRAM, that first applied quantum algorithms to key recovery attacks against Rainbow, especially rank-based attacks. Our HybridRAM uses Grover’s algorithm to find kernels faster than classical rank-based attacks. As an application of HybridRAM, we proposed a Hybrid Rectangular MinRank attack by designing a Grover oracle quantum circuit to find a kernel. It is the first approach to applying a quantum algorithm to a key recovery attack on Rainbow. We also showed the reliability of the Hybrid Rectangular MinRank Attack by implementing our quantum circuit using quantum simulators. Our Grover’s quantum search algorithm for the Hybrid Rectangular MinRank
attack finds kernels with 91.4% accuracy. Finally, we evaluated the security level of Rainbow and estimated the quantum resources required for our Hybrid Rectangular MinRank attack. The security levels of all parameter set I, III, and V of Rainbow are less than 128-bit security levels against our Hybrid Rectangular MinRank attack. In order to perform the Hybrid Rectangular MinRank attack for recovering private keys of the Rainbow, our quantum circuit requires 661, 1832, and 2376 qubits to the parameter sets I, III, and V of Rainbow. With nearly 2000-qubit quantum computers, our Grover’s algorithm of Hybrid Rectangular MinRank attack on the third round Rainbow can be performed.
Appendix A  Grover’s algorithm circuit for toy example of Sec. 5.3

Fig. A1 Grover’s algorithm quantum circuit for toy example of Sec. 5.3 when the Grover oracle and diffusion circuits pairs repeat 1
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