CT reconstruction algorithm based on truncated TV

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Abstract. This paper proposes a simultaneous algebraic reconstruction technique (SART) based on truncated total variation (truncated TV), called SART-truncated TV algorithm (SART-TTV). Truncated TV is particularly effective in removing unimportant details while preserving significant edges. Moreover, the truncated TV only penalizes part of the gradient of the image, so it can solve the problem of excessive smoothness caused by the classic TV penalizing larger gradient amplitude. In this paper, the classic Shepp-Logan model is simulated and the results of sparse angle reconstruction are compared and analyzed. The results show that the quality of the reconstructed image obtained by the SART-TTV algorithm is the highest.

1. Introduction

Based on the collected data, a tomographic image of the inside of the scanned object is reconstructed. This technique is called computed tomography (CT). X-ray computed tomography, which is the earliest application of CT, is currently widely used in the medical field. Nowadays, analytical reconstruction algorithms and iterative reconstruction algorithms are two commonly used image reconstruction algorithms. Among them, the filtered back projection (FBP)⁰ is the most commonly used algorithm in analytical reconstruction algorithms. Iterative reconstruction algorithms include: algebraic reconstruction technique (ART)² and simultaneous algebraic reconstruction technique (SART)³. In the case of incomplete projection data or large noise, the iterative method can reconstruct better results than analytical algorithms. However, these basic algorithms have some shortcomings in solving practical problems. Later, various algorithms were proposed one after another, including regularized reconstruction algorithm⁴. The regularized reconstruction algorithm is to add prior knowledge to constrain the reconstruction process, which can effectively reduce the noise. On this basis, Sidky⁵ and others proposed the ART algorithm to add TV regular constraints, using less data to achieve good reconstruction results, but the image edges are blurred. Johnston⁶ and others introduced the SART algorithm into the TV regularization term, and the reconstructed image quality is better, but for images with more image texture details and complex edge structures, it will cause excessive smoothness.

On the basis of the above-mentioned CT reconstruction algorithm, this paper proposes a new regularization reconstruction algorithm. The difference from the above algorithm is that we use truncated TV⁷ as a regular term, combine it with the SART algorithm, and construct a new CT reconstruction model. The SART algorithm and SART-TV algorithm are used to reconstruct the Shepp-Logan model under 60 projections and 90 projections, respectively. The SART-TTV algorithm can effectively suppress noise, reduce artifacts and better restore the edge details of the image. The high-quality images reconstructed by this algorithm can be widely used in medical and other fields.
This article will introduce the SART-TTV algorithm in detail.

2. Method

2.1. SART algorithm

CT image reconstruction is a typical inverse problem, the essence is to solve linear equations

\[ Af = P \]  

(1)

Where \( A = (a_y)_{M \times N} \) is the system projection matrix, and \( a_y \) represents the weight coefficient of the contribution of the \( j \)-th pixel to the corresponding projection value of the \( i \)-th ray.

To solve the problem in Eq. (1), we use SART to enforce consistency with the projection data. The basic idea of the SART reconstruction algorithm is as follows: In the iterative update process, first calculate the error values of all rays passing through a certain pixel at each scan rotation angle, and then perform a weighted average of these error values. The iterative formula of the algorithm can be expressed as follows\[3]\:

\[
    f_j^{(k+1)} = f_j^{(k)} + \frac{\lambda}{\sum_{i=1}^{M} d_{ij}} \left[ \frac{p_i - \sum_{n=1}^{N} a_{in} f_n^{(k)}}{\sum_{n=1}^{N} a_{in}} \right] a_{ij} 
\]

(2)

Where \( \lambda \) is the relaxation parameter and \( 0 < \lambda < 2 \), \( k \) is the number of iterations, \( f_j^{(k)} \) is the pixel value of the \( j \)-th pixel in the \( k \)-th iteration.

2.2. Regular term

2.2.1. TV

The TV of the discretized two-dimensional CT image \( f \) is defined as follows\[9]\:

\[
    TV(f) = \sum_{s,t} \sqrt{(f_{s,t} - f_{s-1,t})^2 + (f_{s,t} - f_{s,t-1})^2} 
\]

(3)

Where \( f \) is a discrete two-dimensional CT image, \( f_{s,t} \) is the gray value of the pixel in the \( s \)-th row and \( t \)-column in the two-dimensional CT image.

It can be seen from the above formula that the value of TV is obtained by finding the \( L_1 \) norm of the image gradient amplitude. The TV regularization method is based on the assumption that the image slices are smooth, but most of the actual images are not smooth. The actual image may contain many fine structures and texture details, and the noise doped in the image is difficult to distinguish, so it is easy to cause the texture details of the reconstructed image to be blurred or too smooth.

2.2.2. Truncated TV

In response to the above TV problems, Dou and others proposed a new regularization method called truncated TV\[7\]. It can effectively reduce unimportant details, preserve prominent edges, and will not cause the image to be too smooth.

Truncated TV is expressed\[7]\:

\[
    U(x) = \int (T(u_x) + T(u_y)) \, dx \, dy 
\]

(4)

\[
    T(u) = \begin{cases} 
    |u|, & |u(x)| < \varepsilon \\
    \varepsilon, & \text{otherwise}
    \end{cases}
\]

(5)
Where $u$ represents the smoothed result graph, $(u_x, u_y)$ is the gradient of $u$. Truncating TV will only penalize gradients whose amplitude is less than the threshold $\varepsilon$, and will not penalize gradients greater than the threshold $\varepsilon$.

2.3. SART-TTV algorithm

The SART-TTV model can be expressed as:

$$\arg \min \| Af - p \|^2 + \lambda \int ((\phi(f_x, l_1) + \phi(f_y, l_2))dxdy$$

(6)

The reconstruction problem is a non-convex problem, and formula (6) is needed to obtain the optimal solution. Therefore, the model is transformed into a separable sub-problem, and each step of the sub-problem is optimized using a two-step alternating iterative algorithm. Since a complete cycle is not completed, for convenience, we use the symbol to represent the output intermediate image of SART. For convenience, the first step of the kth iteration is expressed as follows[10]:

$$f^{k+\frac{1}{2}} = f^k + \frac{\lambda}{\sum d_j} \sum_{j=1}^N \left[ \frac{p_i - \sum_{n=1}^N a_{in} f_n^{(k)}}{\sum_{n=1}^N a_{in}} \right] a_j$$

(7)

The second step is to use $f^{k+\frac{1}{2}}$ as the input image to solve the truncated TV minimum:

$$f^{k+1} = \arg \min \| u - f^{k+\frac{1}{2}} \|^2 + \lambda \int ((\phi(u_x, l_1) + \phi(u_y, l_2))dxdy$$

(8)

It is difficult to solve the minimization function (8) directly because it involves $L_1$ and $L_2$ penalty terms. We adopt the alternate optimization strategy of splitting the Bregman framework. The key idea is to introduce auxiliary variables to expand the original terms and update them alternately. The algorithm solving process is as follows[7]:

We introduce two dual variables corresponding to respectively, and re-express the objective function (8), as shown below:

$$\min \{ ||u - f^{k+\frac{1}{2}}||^2_2 + \lambda \int ((|b_1| + |b_2|)dxdy + \lambda \int (|b_1| + |b_2|)dxdy \}$$

(9)

By using Lagrangian multipliers, (9) is an unconstrained problem:

$$\min \{ ||u - f^{k+\frac{1}{2}}||^2_2 + \lambda \int ((|b_1| + |b_2|)dxdy + \lambda \int (|b_1| + |b_2|)dxdy + \lambda \int (|b_1| + |b_2|)dxdy \}$$

(10)

By introducing Bregman distance, (10) becomes:

$$\min \{ ||u - f^{k+\frac{1}{2}}||^2_2 + \lambda \int ((|b_1| + |b_2|)dxdy + \lambda \int (|b_1| + |b_2|)dxdy + \lambda \int (|b_1| + |b_2|)dxdy \}$$

(11)

The above joint minimization problem can be solved alternately by decomposing it into several sub-problems, as follows[7]:

1. Calculate $u$ subproblem with fixed $l_1, l_2, b_1, b_2, t_1, t_2$

$$u^{k+1} = \min \{ ||u - f^{k+\frac{1}{2}}||^2_2 + \lambda \int ((|b_1| + |b_2|)dxdy \}$$

(12)

We derive the Euler-Lagrange equation as follows:

$$f^{k+\frac{1}{2}} - u + \lambda (\frac{\partial (b_1' + l_1' - t_1')}{\partial x} + \frac{\partial (b_2' + l_2' - t_2')}{\partial y} - \Delta u) = 0$$

(13)

where $\Delta$ is the Laplace operator. Equation (13) can be efficiently solved by using Gauss-Seidel iteration algorithm or FFT operator.
(2) Calculate $b_1$ and $b_2$ with fixed $u, l_1, l_2, t_1, t_2$.

The unique minimizer of this subproblem can be obtained by applying the shrinkage operator:

$$b_i^{i+1} = shrink(u_i^{i+1} - l_i' + t_i' \frac{\lambda}{\lambda_i}) $$  \hspace{1cm} (14)$$

$$b_i^{i+1} = shrink(u_y^{i+1} - l_y' + t_y' \frac{\lambda}{\lambda_l}) $$ \hspace{1cm} (15)

Where $shrink(x, \lambda) = \frac{x}{|x|} \max(x - \lambda, 0)$.

(3) Update $t_1, t_2$ with fixed $u, l_1, l_2, b_1, b_2$

$$t_i^{i+1} = u_i^{i+1} + t_i' - b_i^{i+1} $$ \hspace{1cm} (16)$$

$$t_i^{i+1} = u_y^{i+1} + t_y' - b_i^{i+1} $$ \hspace{1cm} (17)

(4) Update $l_1, l_2$ with fixed $u, t_1, t_2, b_1, b_2$

$$l_i^{i+1}, l_y^{i+1} = \text{argmin}_{l_i, l_y} \left\{ \lambda \int (l_i^{i+1} + l_y^{i+1}) \text{d}x \text{d}y + \lambda \int (l_i - u_i^{i+1} + b_i^{i+1} - t_i^{i+1})^2 + (l_y - u_y^{i+1} + b_y^{i+1} - t_y^{i+1})^2 \text{d}x \text{d}y \right\} $$ \hspace{1cm} (18)

we see that the solution of this subproblem is\([7]\):

$$t_i^{i+1} = \begin{cases} t_i^{i+1} - b_i^{i+1} + u_i^{i+1}, & |t_i^{i+1} - b_i^{i+1} + u_i^{i+1}| < \sqrt{\frac{\lambda E}{\lambda_i}} \\ 0, & \text{otherwise} \end{cases} $$ \hspace{1cm} (19)$$

$$t_y^{i+1} = \begin{cases} t_y^{i+1} - b_y^{i+1} + u_y^{i+1}, & |t_y^{i+1} - b_y^{i+1} + u_y^{i+1}| < \sqrt{\frac{\lambda E}{\lambda_l}} \\ 0, & \text{otherwise} \end{cases} $$ \hspace{1cm} (20)

Table 1. The specific steps of the SART- TTV algorithm.

**Algorithm 1:**

Input: $f, k, \lambda, \lambda_i, b_1 = b_2 = t_1 = t_2 = l_1 = l_2 = 0$

For $i = 1: k$

Solve $f^{i+1}$ by Eq. (7)

For $j = 1: N$

Update $u$ by Eq. (13)

Update $b_1, b_2$ by Eq. (14) and Eq. (15)

Updated $t_1, t_2$ by Eq. (16) and Eq. (17)

Updated $l_1, l_2$ by Eq. (19) and Eq. (20)

Output: the reconstruction results $f^{i+1}$

End

End

3. Evaluation of image results

This article is to objectively evaluate the reconstructed image, which is to use peak signal-to-noise ratio (PSNR)\([11]\) and mean square error (MSE)\([11]\) to analyze the noise reduction ability and accuracy of the reconstructed image. Among them, the larger the value of PSNR, the smaller the value of MSE, and the higher the quality of the reconstructed image. Their calculation formula is as follows\([11]\):

$$MSE = \frac{1}{M \times N} \sum_{j=1}^{N} \sum_{i=1}^{M} (t_{i,j} - r_{i,j}) $$ \hspace{1cm} (21)
\[ PSNR = 10 \times \log \left( \frac{MAX_i^2}{MSE} \right) \] (22)

Where \( t_{ij}, r_{ij} \) represents the pixel value of the original image, \( M \times N \) is the size of the image. \( MAX_i \) is the maximum pixel value of the original image.

4. Experiment and analysis

This paper simulates the Shepp-Logan head model, the operating system is windows10, the processor is AMD Ryzen 5 2500U with Radeon Vega Mobile Gfx 2.00GHz, the memory is 8.00GB, using MATLAB R2018b software.

In this paper, 90 projections and 60 projections are sampled at equal intervals in the range of 180°. Three algorithms of SART, SART-TV and SART-Truncation TV are used for image reconstruction. The size of the Shepp-Logan model is \( 256 \times 256 \), and the number of detection elements is 256. In the SART algorithm, \( K_{SART} = 15, \lambda_{SART} = 0.3 \); in the SART-TV algorithm, \( K_{SART} = 12, \lambda_{SART} = 0.1, \lambda_{TV} = 0.2 \); \( K_{TV} = 20 \); in the SART-TTV algorithm, \( K_{SART} = 5, \lambda_{SART} = 0.1, K_{TTV} = 8, \lambda_{TTV} = 0.6, e = 0.001, \alpha = 0.9 \).

![Figure 1. The reconstruction result and the corresponding partial enlarged image. The reconstruction results of 60 projections and 90 projections are shown in the left panel and right panel, respectively. For each panel, the first line is the reconstructed image. The second line and third line are the partial enlarged images corresponding to the first line of images. For both panels, from left to right, the images reconstructed by SART algorithm, SART-TV algorithm and SART-TTV, respectively.](image)

As shown in Figure 1, the reconstruction result of the SART algorithm has obvious noise. Under 60 projections, the SART-TV algorithm and SART-TTV algorithm can restore the edge structure information of the image, and can effectively reduce the artifacts. Under 90 projections, the SART-TV algorithm and the SART-TTV algorithm have almost no artifacts, but the SART-TTV algorithm can better restore the edge details of the image. The close-up image shows that the image quality reconstructed by the SART-TTV algorithm is the highest, with fewer edge artifacts.

| Evaluation indicators | Algorithms | 60 projections | 90 projections |
|------------------------|------------|---------------|---------------|
| MSE                    | SART       | 0.0039        | 0.0036        |
|                        | SART-TV    | 0.0037        | 0.0034        |
|                        | SART-TTV   | 0.0035        | 0.0033        |
| PSNR                   | SART       | 24.0947       | 24.3969       |
|                        | SART-TV    | 24.2885       | 24.6276       |
|                        | SART-TTV   | 24.5068       | 24.7842       |
Table 2 shows the comparison of the reconstruction performance of the Shepp-Logan head model using various algorithms (SART, SART-TV, SART-TTV) under 60 and 90 projections. It can be clearly seen from Table 2 that the MSE value of the SART-TTV algorithm is the smallest and the PSNR value is the largest. That is to say, among the three algorithms, the reconstructed image effect of the SART-TTV algorithm is the best.

5. Conclusion
This paper studies the problem of image reconstruction under parallel-beam CT scanning. Simulation experiments are carried out with a variety of reconstruction algorithms, image reconstruction is carried out on sparse angle projection data, and the reconstruction effects of various algorithms are compared. The test results show that the SART-TTV algorithm has the best imaging effect, can effectively suppress noise, reduce artifacts, and can better restore the edge details of the image. However, the SART-TTV algorithm contains many parameters, which is time-consuming to debug, and different models have different choices of parameters. On this basis, this article only tests the classic Shepp-Logan head model. In the follow-up research, it is necessary to test various models and adjust the algorithm to achieve better results.

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