THE KINETIC EQUATION FOR THE QUARK WIGNER FUNCTION
IN STRONG GLUON FIELD

A.V. Prozorkevich, S.A. Smolyansky, and S.V. Ilyin

Physics Department, Saratov State University
41026 Saratov, Russian Federation
smol@sgu.ru

Abstract

The Vlasov type quantum kinetic equation for the deconfined quarks in the strong quasi-classical gluon field is derived in the covariant single-time formalism. The equations system for the Wigner function components is obtained as a result of spinor and color decomposition. The field-free and vacuum solutions of the kinetic equation are found, and the conservation laws are derived. The flux-tube configuration of gluon field is discussed in detail.

1 Introduction

The creation and evolution of the quark-gluon plasma (QGP) by ultra-relativistic heavy ion collisions is a very complex process occurring on the time scale of $1\, fm/c$ and at energy density above of $1\, Gev/fm^3$. Parton gas probably comes to equilibrium state in the process of hadronization. Details of this transition are of great importance for interpretation of experimental data. If the QGP is really formed in experiments at the RHIC, its thermalization time must be so small, that it presents a serious problem for theoretical explanation [1]. Several theoretical tools are applied to describe the various physical phenomena accompanying the collision. The kinetic equation is the basic means for investigation of the nonequilibrium evolution [2]-[17]. The flux tube model [18] and the Schwinger mechanism of pair production are often used for the research of an early stage of the QGP formation. Vacuum pair creation is on essentially non-perturbative effect, which description requires the exact solution of the field equations. The resulting source term in the kinetic equation (KE) describes the production rate and momentum spectra of the created particles [19].

The QED example shows that the evolution of a plasma created from vacuum is rather dependent on the structure of the source term [20]-[22]. The semi-phenomenological source term based on the classical Schwinger formula (e.g., [23, 24]) can result in significant inaccuracy for the fast varying electrical field. In particular, such source can not reproduce correctly the relation between the production rates of two components with different masses and statistics [25, 26]. The main reason is that correct source term has a
non-markovian time dependence [19, 20], as against the Schwinger formula. The most interesting effects arising from these feature are the suppression of boson creation with zero kinetic momentum and the suppression of statistical factor influence. The Schwinger-like source contains the statistical factor as multiplier and produce the suppression of fermion creation (Pauli blocking) and enhancement of boson creation. The correct source term [20] contains the statistical factor as integrand on time (non-markovian property) that facilitates the fermion creation and reduces the boson creation by large particle density. The joint action of these factors may cause the effect of "statistic inversion" at the short times scale when the fermions production rate is more greater than bosons one. This effect can be so strong that heavy fermions are created more actively than light bosons.

The other noteworthy feature of a proper source term is that a momentum distribution of the produced fermion pairs is close to the quasi-equilibrium one both in the transverse and in longitudinal directions. The influence of such source term can facilitate more rapid formation of a quasi-equilibrium state of QGP along with the small mean free path compared to the Compton length. The correct source term allows to separate the vacuum polarization effects which can exceed the contribution of real particles in the thermodynamical variables [27].

The kinetic equation with account of vacuum creation effect can be obtained in the Wigner function approach, which is widely used for the description of relativistic quantum systems during the past few decades. It is more convenient to use the single-time Wigner function variety [28, 29] to solve the initial value problem for the collisions at RHIC. A covariant version of this approach has been recently developed in [30] for the QED case within the framework of the proper time method on space-like hyperplanes. We use this approach in the present work for the derivation of the kinetic equation for the covariant single-time Wigner function (WF) in the quark sector of QCD at presence of a strong quasi-classical background gluon field.

The paper is organized as follows. We introduce the equations of motion for quarks and quasi-classical gluon fields in covariant proper time approach in Sect. 2. The single-time Wigner function is introduced here as the basic element of kinetic theory. The physical observables are bilinear compositions of quark field operators, and can be calculated via WF. The kinetic equation for the Wigner function is derived in a matrix form in the spinor and color spaces in Sect. 3. The spinor decomposition of this KE is also fulfilled for the more convenient analysis. This representation of KE is useful for construction of conservation laws, Sect. 5. The vacuum and field-free solutions are obtained in Sect. 4. The vacuum WF play a role of initial value for a solution of the Cauchy problem. The special sample of the kinetic equation is investigated in Sect. 6 which is inspired by the flux-tube [18] model of vacuum quark production under conditions of ultra-relativistic heavy ion collisions. Comparison with QED case is made in Sect. 7. Finally, Sect. 8 summarizes some results of the work. We use the system of units with $\hbar = c = 1$ and the signature of metric tensor $(1, -1, -1, -1)$. 

2
2 Basic Equations

We start with the QCD Lagrange density (for one quark flavor, $g > 0$)

$$L_{QCD} = i \bar{\psi} \gamma^\mu (\partial_\mu - igA_\mu) \psi - m \bar{\psi} \psi - \frac{1}{2} F_{\mu \nu} F^{\mu \nu},$$

(1)

with the quasi-classical gluon field

$$A_\mu = A_\mu^a t^a, \quad F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu], \quad a = 1, 2, \ldots N,$$

(2)

where $t^a = \lambda^a/2$ are generators of SU(N) gauge group in fundamental representation. In particular, $\lambda^a$ are the Gell-Mann matrices for SU(3) and the Pauli ones for the SU(2) group.

The covariant proper time derivative $\partial_\tau$ is defined by means of the space-time parametrization by a family of space-like hyperplanes $\sigma(n, \tau)$

$$\sigma(n, \tau) \perp n^\mu, \quad n^\mu x_\mu = \tau, \quad n^2 = 1.$$

(3)

Hence there is a covariant decomposition

$$x^\mu = \tau n^\mu + x^\mu_\perp, \quad \partial_\mu = n_\mu \partial_\tau + \partial^\perp_\mu,$$

$$\partial_\tau = n^\mu \partial_\mu, \quad \partial^\perp_\mu = \Delta^\mu_\alpha \partial^\alpha, \quad \Delta^\mu_\nu = g^{\mu \nu} - n^\mu n^\nu,$$

(4)

where $n^\mu$ is a unit time-like vector, $\Delta^\mu_\nu$ denotes the transverse projector, and $g^{\mu \nu}$ is the metric tensor. Analogous decomposition is produced for any vector $f^\mu$

$$f^\mu = f^\parallel n^\mu + f^\perp, \quad f^\parallel = f^\mu n_\mu, \quad f^\perp = \Delta^\mu_\alpha f^\alpha,$$

(5)

and for any anti-symmetric tensor $c^{\mu \nu}$

$$c^{\mu \nu} = c^{\parallel \mu} n^\nu + c^{\perp \mu} n^\nu + c^{\perp \nu} n^\mu, \quad c^{\parallel} = n_\nu c^{\nu \mu}, \quad c^{\perp} = \Delta^\mu_\alpha \Delta^\nu_\beta c^{\alpha \beta},$$

(6)

so $c^{\perp}_\mu = c_\mu$. In these terms, the basic equations of motion for quark field operators are

$$\partial_\tau \psi = -\gamma^\parallel \gamma^\perp (\partial^\perp_\mu - igA^\perp_\mu) \psi - im\gamma^\parallel \psi + igA^\parallel \psi,$$

$$\partial_\tau \bar{\psi} = -\bar{\psi}(\partial^\perp_\mu + igA^\perp_\mu) \gamma^\parallel \gamma^\perp + im\bar{\psi} \gamma^\parallel - ig\bar{\psi} A^\parallel.$$

(7)

The mean gluon fields obey the equations

$$\partial_\tau E_\nu = -\partial^\perp_\mu F^\perp_{\mu \nu} + ig \left([A^\perp_\mu, F^\perp_{\mu \nu}] - [A^\parallel, E_\nu]\right) - g J^\perp_\nu,$$

$$\partial^\perp_\mu E_\mu = -ig[A^\perp_\mu, E_\mu] + g J^\parallel,$$

(8)

where $E_\nu = n^\mu F_{\mu \nu}$ and $F_{\mu \nu}^\perp$ represent the color electrical and color magnetic fields, respectively, and $J_\nu$ is a color current

$$J_\nu = J^a_\nu t^a, \quad J^a_\nu = <\bar{\psi} \gamma_\nu t^a \psi >_\sigma,$$

(9)
the symbol \(< \ldots >_\sigma\) designates the average with statistical operator of the system on a hyperplane in the Heisenberg picture.

The basic element of statistical description is the covariant Wigner function (WF) on the space-like hyperplane \(\sigma(\tau)\) [28, 30]:

\[
W_{ik}^{ab}(x, x \pm y/2; \tau) = \int d^4y e^{ip \cdot y} \delta(y \cdot n) \times U^{a'd}(x, x + y/2) \rho_{ik}^{b'}(x, \tau) U^{b'b}(x - y/2, x),
\]

(10)

here the upper and bottom indices are the color and spinor ones, respectively, \(\rho\) is the one-particle density matrix of quarks

\[
\rho_{ik}^{ab}(x, y; \tau) = -\frac{1}{2} \left< \psi_i^a(x + y/2), \bar{\psi}_k^b(x - y/2) \right>_{\sigma},
\]

(11)

and \(U\) is the unitary link operator

\[
U(x_1, x_2) = \exp \left( ig \int_{x_1}^{x_2} dz A_\mu(z) \right),
\]

(12)

provides the gauge invariance of the WF [29]. The integral is taking along straight line between points \(x_1\) and \(x_2\) connected with space-like interval and

\[
U^+(x_1, x_2) = U^{-1}(x_1, x_2) = U(x_2, x_1),
\]

\[
U(x_1, x_2) U(x_2, x_1) = U(x_1, x_1) = \hat{1}.
\]

(13)

The dynamical variables can be expressed in terms of the Wigner function. For example, the color and electromagnetic current densities are

\[
J_\mu^a(x) = \text{Tr} < \gamma_\mu t^a W >,
\]

(14)

\[
j_\mu(x) = \text{Tr} < \gamma_\mu W >,
\]

(15)

where trace \(\text{Tr}\) is carried out on spinor and color indices. The brackets \(< \ldots >\) denote the covariant momentum average on the hyperplane \(p \cdot n = 0\) [30], e.g.

\[
< W > = \int \frac{d^4p}{(2\pi)^3} \delta(p \cdot n) W(x, p; \tau = x \cdot n).
\]

(16)

The integral in (16) provides the locality of corresponding observables, due to that function

\[
\delta_\tau(x) = \int \frac{d^4p}{(2\pi)^3} e^{-ipx} \delta(p \cdot n)
\]

(17)

plays the role of the three-dimensional delta function on a hyperplane \(\sigma(n, \tau)\).

The color WF have a rather complex matrix structure, therefore it is convenient to use the corresponding decompositions in the spinor and color space. The spinor decomposition in a complete basis of the Clifford algebra is

\[
W = aI + b_\mu \gamma^\mu + c_{\mu\nu} \sigma^\mu\nu + d_\mu \gamma^\mu \gamma^5 + ie \gamma^5,
\]

(18)
where the coefficient functions \( a \) (scalar), \( b_\mu \) (vector), \( c_{\mu\nu} \) (antisymmetric tensor), \( d_\mu \) (axial vector), \( e \) (pseudo-scalar) are the hermitian color matrices

\[
\begin{align*}
  a &= \frac{1}{4} \text{tr} W, \quad b_\mu = \frac{1}{4} \text{tr} (\gamma^\mu W), \quad c_{\mu\nu} = \frac{1}{8} \text{tr} (\sigma_{\mu\nu} W), \\
  d_\mu &= -\frac{1}{4} \text{tr} (\gamma^{\mu} \gamma^5 W), \quad e = -\frac{i}{4} \text{tr} (\gamma^5 W),
\end{align*}
\]

the symbol "\( \text{tr} \)" denotes the trace on spinor indices only, and \( \gamma^5 = -i\gamma^0\gamma^2\gamma^2\gamma^3 \).

The algebraic structure of the WF in a color space of \( N \times N \) matrices is represented by the observable color singlet \( W^a \) and the unobservable color multiplet \( W_a \),

\[
W = W^s \cdot \hat{1} + W^a \cdot t^a, \quad a = 1, 2 \ldots (N^2 - 1),
\]

\[
W^s = \left( \frac{1}{N} \right) \text{tr} c W, \quad W^a = 2 \text{tr} c \left( t^a W \right),
\]

where \( \hat{1} \) is unit matrix and \( \text{tr}_c \) is the trace on color indices.

### 3 Kinetic Equation

We calculate the proper time derivative of Eq. (10) for a derivation of the KE for the Wigner function

\[
\partial_\tau W_{ik}^{ab} = \int d^4y e^{ip \cdot y} \delta(y \cdot n) \left\{ U_{aa'}^{\mu}(x, x + y/2) U_{bb'}^{\mu}(x - y/2, x) \partial_\tau \rho_{ik}^{a'b'} + \\
+ \rho_{ik}^{a'b'} \partial_\tau \left[ U_{aa'}^{\mu}(x, x + y/2) U_{bb'}^{\mu}(x - y/2, x) \right] \right\},
\]

and substitute the time derivative of the field operators from the field equations (7). The emerging terms with perpendicular derivative \( \partial_\perp \) are transformed via the integration by parts. The derivative rules for link operator are follow from known formula [5, 29]

\[
\delta U(x_1, x_2) = ig \delta x_1^\mu A_\mu(x_1) U(x_1, x_2) - ig U(x_1, x_2) A_\mu(x_2) \delta x_2^\mu - \\
-ig \int_0^1 ds U(x_1, z(s)) F_{\mu\nu}(z(s)) U(z(s), x_2) \delta x_2^\mu \delta x_2^\nu + s(\delta x_1 - \delta x_2)[\nu],
\]

where \( z(s) = x_2 + s(x_1 - x_2), \) e.g.

\[
\partial_\mu(x) U(x, x + y/2) = -ig \left[ A_{\mu}^{[x]}(x, x + y/2) - A_{\mu}(x) + \\
+ \int_0^{y/2} ds f_{\mu\nu}^{[x]}(x + sy) y^\nu \right] U(x, x + y/2),
\]

where the notation for the "Schwinger string" [29]

\[
A_{\mu}^{[x]}(z) = U(x, z) A_{\mu}(z) U(z, x)
\]
is introduced. Then we express the variable \( y \) as the momentum derivative

\[
y^\mu \exp(i p \cdot y) \delta(y \cdot n) = \delta(y \cdot n) \partial^\mu(p) \exp(i p \cdot y),
\]

and we obtain after some algebra the exact equation of motion for the WF

\[
\partial_t W + \frac{1}{2} \partial^\mu(x) [S^\mu, W] + im[\gamma^\parallel, W] - ip_+^\mu \{S^\mu, W\} +
\]

\[
+ g \partial_\perp(p) \int_0^{1/2} ds \left\{ E^\parallel_\mu(x, \tau - is \partial_\perp(p)) W + W E^\parallel_\mu(x, \tau + is \partial_\perp(p)) +
\right.
\]

\[
+ \frac{1}{2} F^\parallel_\mu \tau, \tau - is \partial_\perp(p) \right) \left( [W, S^\nu] - 2s\{W, S^\nu\} \right) +
\]

\[
+ \frac{1}{2} \left( [W, S^\nu] + 2s\{W, S^\nu\} \right) F^\parallel_\mu \tau, \tau + is \partial_\perp(p) \left) \right. =
\]

\[
i g [A^\parallel(x), W] + \frac{i g}{2} \left[ S^\mu, [A^\perp_\mu, W] \right],
\]

where

\[
S^\mu = \gamma^\parallel \gamma^\perp, \quad \gamma^\parallel = n^\mu \gamma_\mu, \quad \gamma^\perp = \Delta^\mu_\nu \gamma_\nu.
\]

This equation is resulted in a local form by means of the gradient expansion of a gluon field. We obtain in the first order of this procedure the kinetic equation of the Vlasov type, which is correct for a rather slowly changing gluon field

\[
\partial_t W + \frac{1}{2} \partial^\mu(x) [S^\mu, W] +
\]

\[
+ \frac{g}{8} \partial^\mu(p) \left( 4\{W, E^\mu\} + 2 \{F^\mu, [W, S^\nu]\} - [F^\mu, \{W, S^\nu\}] \right) =
\]

\[
= ip_+^\mu \{S^\mu, W\} - im[\gamma^\parallel, W] + ig \left( [A^\parallel(x), W] + [A^\perp_\mu, [S^\mu, W]] \right).
\]

This equation despite of rather compact form is a very complicated matrix one in the direct production of \( 4 \times 4 \) spinor and \( N \times N \) color space. It is convenient to expand the WF in a some basis of this space for the separation of a different physical contributions.

We perform at first the spinor decomposition (18) in the KE. We obtain the equations set calculating the traces of Eq. (28) with the basic matrices of Clifford algebra

\[
\partial_t a + \frac{g}{2} \partial^\alpha(p) \left( \{a, E_\alpha\} + i [F^\alpha_\alpha, c^\beta_\perp] \right) = 4 \rho^\alpha_\perp c^\alpha_\perp + ig [A^\parallel, a],
\]

\[
\partial_t b^\mu + n^\mu \partial^\alpha(p) b^\alpha_\perp - \partial^\mu b_\perp +
\]

\[
+ \frac{g}{2} \partial^\alpha(p) \left( \{b^\mu, E_\alpha\} + g^{\mu\beta} \{b^\parallel, F^\alpha_\beta\} - n^\mu \{b_\perp^\beta, F^\perp_\alpha\} + i \frac{1}{2} [F^\perp_\alpha, d^\parallel] \varepsilon^{\beta\mu} \right) =
\]

\[
= 2 \rho^\alpha_\parallel d^\alpha_\perp \varepsilon^{\alpha\beta\mu} + 4m c_\perp^\mu + ig \left( [A^\parallel, b^\mu] + n^\mu [A^\alpha_\perp, b^\alpha_\perp] - [A^\perp_\alpha, b^\perp] \right),
\]

\[
\partial_t c^\mu + \partial^\alpha(p) \left( n^\nu c^\mu_\perp - n^\mu c^\nu_\alpha \right) + \partial^\mu c^\perp - \partial^\perp c^\mu +
\]
\[ \frac{g}{2} \partial_{\alpha}^\alpha (p) \left( \{ E_{\alpha}, c_{\mu}^{\alpha} \} + \{ F_{a_{\alpha}}^{\perp}, (n^\mu c^\alpha_{\perp} - n^\nu c^\beta_{\perp} + g^\mu_{\alpha} c^\nu - g^\nu_{\alpha} c^\mu) \} + \right. \\
\left. + \frac{i}{4} \left[ F_{a_{\alpha}}, a(n^\mu g^\nu_{\beta} - n^\nu g^\mu_{\beta}) + e\varepsilon^{\beta\mu\nu} \right] \right) = \right. \\
= a(n^\mu p^\nu_{\perp} - n^\nu p^\mu_{\perp}) + e p_{a_{\alpha}}^{\perp} \varepsilon^{\alpha\mu\nu} + m(n^\nu b^\mu - n^\mu b^\nu) + \\
+ i g [A_{\parallel}, c_{\mu}^{\alpha} + ig [A_{\parallel}^{\perp}, (n^\mu c^{\alpha}_{\perp} - n^\nu c^\beta_{\perp} + g^\mu_{\alpha} c^\nu - g^\nu_{\alpha} c^\mu)] , \] 

(31)

\[ \partial_{\tau} d^\mu - \partial_{\tau}^\mu d^\| + n^\mu \partial_{\alpha}^\alpha + \\
\frac{g}{2} \partial_{\alpha}^\alpha (p) \left( \{ d^\mu, E_{\alpha} \} + g^\mu_{\alpha} \{ F_{a_{\alpha}}^{\perp}, d^\| \} - n^\mu \{ F_{a_{\alpha}}^{\perp}, E_{\alpha} \} + \frac{i}{2} F_{a_{\alpha}}^{\perp}, b_{\alpha} \varepsilon^{\beta\mu}\right) = \\
= 2 p_{a_{\alpha}}^{\perp} b^\beta_{\alpha} \varepsilon^{\alpha\beta\mu} + 2 mn^\mu e + i g \left( [A_{\parallel}, d^\mu] + n^\mu [A_{\parallel}^{\perp}, d^\|_{\perp}] - [A_{\parallel}^{\perp}, d^\|_{\perp}] \right) , \] 

(32)

\[ \partial_{\tau} e + \frac{g}{2} \partial_{\alpha}^\alpha (p) \left( \{ e, E_{\alpha} \} - \frac{i}{2} F_{a_{\alpha}}^{\perp}, c_{\beta\rho} \varepsilon^{\beta\rho\mu} \right) = \\
= -2 p_{a_{\alpha}}^{\perp} c_{\beta\rho} \varepsilon^{\alpha\beta\delta} - 2 m d^\| + i g [A_{\parallel}, e] , \] 

(33)

where \( \varepsilon^{\alpha\beta\gamma} \) denote the convolution of the normal \( n^\mu \) with the totally anti-symmetric unit tensor \( \varepsilon^{\alpha\beta\gamma} \), i.e.

\[ \varepsilon^{\alpha\beta\gamma} = n_{\mu} \varepsilon^{\rho\alpha\beta\gamma} \] 

(34)

It is more convenient for the analysis of concrete field configurations to rewrite that system concerning a projections on time-like and space-like directions

\[ \partial_{\tau} a + \frac{g}{2} \partial_{\alpha}^\alpha (p) \left( \{ a, E_{\alpha} \} + i [F_{a_{\alpha}}^{\perp}, c_{\beta}^{\alpha\perp}] \right) = 4 p_{a_{\alpha}}^{\perp} c_{\alpha}^{\perp} + i g [A_{\parallel}, a] , \]

(35)

\[ \partial_{\tau} b_{\|} + \partial_{\alpha}^\alpha (x) b_{\alpha} + \frac{g}{2} \partial_{\alpha}^\alpha (p) \left( \{ b_{\|}, E_{\alpha} \} - \{ b_{\alpha}, F_{a_{\alpha}}^{\perp} \} \right) = i g [A_{\alpha}, b_{\alpha}] , \]

(36)

\[ \partial_{\tau} b_{\perp} + \frac{g}{2} \partial_{\alpha}^\alpha (p) \left( \{ b_{\perp}, E_{\alpha} \} + \{ b_{\|}, F_{a_{\alpha}}^{\perp} \} + \frac{i}{2} F_{a_{\alpha}}^{\perp}, d_{\beta} \varepsilon^{\beta\mu} \right) = \\
= \partial_{\alpha}^\alpha (x) b_{\perp} + 2 p_{a_{\alpha}}^{\perp} d_{\beta} \varepsilon^{\alpha\beta\mu} + 4 m n_{\alpha}^{\mu} + i g \left( [A_{\parallel}, b_{\mu}^{\perp}] - [A_{\perp}^{\mu}, b_{\mu}^{\|}] \right) , \]

(37)

\[ \partial_{\tau} c_{\perp}^{\alpha} + \partial_{\alpha}^\alpha (x) c_{\alpha}^{\mu} + \frac{g}{2} \partial_{\alpha}^\alpha (p) \left( \{ E_{\alpha}, c_{\mu}^{\alpha} \} + \{ F_{a_{\alpha}}^{\perp}, c_{\beta}^{\mu} \} + \frac{i}{4} F_{a_{\alpha}}^{\perp}, c_{\rho}^{\mu} \right) = \\
= p_{a_{\alpha}}^{\perp} a - m b_{\perp} + i g \left( [A_{\parallel}, c_{\perp}^{\mu}] + [A_{\perp}^{\mu}, c_{\mu}^{\alpha}] \right) , \]

(38)

\[ \partial_{\tau} d_{\perp} + \partial_{\alpha}^\alpha (x) d_{\alpha} + \frac{g}{2} \partial_{\alpha}^\alpha (p) \left( \{ d_{\perp}, E_{\alpha} \} - \{ F_{a_{\alpha}}^{\perp}, d_{\beta} \} \right) = \\
= 2 m e + i g \left( [A_{\parallel}, d_{\perp}] + [A_{\perp}^{\mu}, d_{\perp}^{\alpha}] \right) , \]

(40)
\[
\partial_\tau d_\perp^\mu + \frac{g}{2} \partial_\perp^\alpha(p) \left( \{d_\perp^\mu, E_\alpha\} + \{F_\perp^{\mu\beta}, d_\parallel\} + \frac{i}{2} [F_\perp^{\mu\alpha\beta}, b_\parallel^\beta] \varepsilon^{\beta\delta\mu} \right) = \\
= \partial_\perp^\mu(x) d_\parallel + 2p_\mu^\perp b_\alpha^\perp \varepsilon^{\alpha\beta\mu} + ig \left( [A_\parallel, d_\perp^\mu] - [A_\perp^\mu, d_\parallel] \right), 
\]

(41)

\[
\partial_\tau e + \frac{g}{2} \partial_\perp^\alpha(p) \left( \{e, E_\alpha\} - \frac{i}{2} [F_\perp^{\alpha\beta}, c_\perp^\beta] \varepsilon^{\beta\delta} \right) = \\
= -2p_\alpha^\perp c_\beta^\perp \varepsilon^{\alpha\beta\delta} - 2md_\parallel + ig [A_\parallel, e]. 
\]

(42)

The non-abelian character of gluon field is displayed in particular in the presence of the vector potential in right-hand side of these KEs that is necessary for the gauge invariance of the theory.

It is the resulting set of KE for the description of deconfined quarks in a strong quasiclassical gluon field.

4 Field-Free Limit and Vacuum Solution

The equations system (35)-(42) can be solved exactly in the field-free limit \( A_\mu = 0 \). We obtain assuming that all derivatives vanish:

\[
c_\mu = 0, \quad c_\mu^\perp = 0, \quad d_\mu = 0, \quad e = 0,
\]

\[
p_\mu^\perp a(p) - mb_\perp^\mu(p) = 0,
\]

\[
p_\alpha^\perp c_\beta^\perp \varepsilon^{\alpha\beta\mu} = 0.
\]

(43)

The general solution of (43) is

\[
b_\perp^\mu(p) = p_\perp^\mu \frac{a(p)}{m}
\]

(44)

with an arbitrary momentum dependence of \( a(p) \). The equilibrium WF [33] or the vacuum solution follow from the Eq.(44) as a particular cases. We perform the direct calculation of the function (10) to select the vacuum case in the solutions class (44). The fermion operators on the hyperplane obey free anticommutation relations [30, 31]

\[
\{ \psi^a(x_2), \bar{\psi}^b(x_1) \} = \delta^{ab} \gamma_\parallel \delta_\tau(x_1 - x_2).
\]

(45)

Then allows to write down the commutator (11) as

\[
\rho^{ab}(x, y; \tau) = <:\psi^a(x_2) \bar{\psi}^b(x_1):>\tau - \\
-\frac{1}{2} \delta^{ab} \int \frac{d^4k}{(2\pi)^3} e^{ik_\perp \cdot y_\perp} \delta(k^2 - m^2)(m - k \cdot \gamma) [\theta(k_\parallel^\perp) + \theta(-k_\parallel^\perp)],
\]

(46)

where the symbol : : indicates normal ordering. We obtain the vacuum WF by carrying out the averaging on vacuum state and substituting this representation in Eq.(10)

\[
W_{\text{vac}}^{ab} = \frac{1}{2} \delta^{ab} \left( \frac{-m + p_\perp \cdot \gamma_\perp}{\omega(p_\perp)} \right),
\]

(47)
where $\omega(p_\perp) = \sqrt{m^2 - p_\perp^2}$. This function is degenerated in the color space as a consequence of the primary supposition about the Lagrange density structure (1). The WF (47) is the base for a solution of the Cauchy problem for the equations system (35)-(42). It can be proved that the vacuum solution, as well as the free-field ones, gives no contribution to the current densities (14),(15).

The kinetic equation has in the field-free case besides the vacuum solution (47) the another solutions of the type

$$W^a(p) = \eta^a W^s(p), \quad a = 1, 2, \ldots N, \quad W = \{a, b_\perp^\mu, 0 \ldots 0\},$$

(48)

where $\eta^a$ is an arbitrary real numbers.

### 5 Conservation Laws

We calculate the divergence of electromagnetic current (15), using the spinor and color decompositions (18) and (20)

$$\partial^\mu(x) j_\mu(x) = 4N \partial^\mu(x) < b_\perp^s(x) > = 4N[\partial_\tau < b_\parallel^s > + \partial_\perp^\mu < b_\perp^s >].$$

(49)

Performing the momentum averaging procedure (16) and taking the trace in Eq.(36), we obtain the electromagnetic current conservation law

$$\partial^\mu(x) j_\mu(x) = 0.$$

(50)

We multiply the equation (36) on matrix $t^a$ and repeat the same procedure to derive the equation for the color current $J_\mu^a = (1/2) < b_\perp^s >$. The result is

$$\partial^\mu(x) J_\mu^a(x) + g f^{abc} A_\mu^b J_\mu^c = 0.$$

(51)

The energy density $\varepsilon_q$ of quark matter corresponding Eq.(1) is

$$\varepsilon_q \equiv n_\mu n_\nu T^{\mu\nu}_q = i : \bar{\psi} \gamma^\parallel (\partial_\tau - igA_\parallel) \psi :,$$

(52)

where $T^{\mu\nu}_q$ - energy-momentum tensor. Using the field equations (7), we have

$$\varepsilon_q = : i \bar{\psi} \gamma^+ (\partial_\perp - i gA_\perp) \psi + m \bar{\psi} \psi :,$$

(53)

This variable can be written in terms of the WF as

$$\varepsilon_q = \text{Tr} < (m - \gamma^+ p_\perp)W > + \varepsilon_{\text{vac}},$$

(54)

where $\varepsilon_{\text{vac}} = 2N < \omega(p_\perp) >$ is divergent vacuum contribution. We obtain using the spinor and color decompositions

$$\varepsilon_M = 4N [m < a^s > - < p_\mu^\parallel b_\perp^s >] + \varepsilon_{\text{vac}}.$$

(55)

The linear combination of Eqs.(35) and (37) is used to calculate the right side of this equation. The result is

$$\partial_\tau \varepsilon_M + 4 \partial_\perp^\mu(x) < p_\mu^\parallel b_\perp^s > + g J_\mu^a E_\perp^\mu = 0.$$

(56)
6 Space-Homogeneous Color Field

6.1 Flux-tube field configuration

We consider here the “instant” frame of reference where \( n^\mu = (1, 0, 0, 0) \) and the field configuration typical for the flux tube model

\[
A_\mu^a = (0, 0, 0, A^a(t)), \quad F^a_{03} = -\dot{A}^a_3(t) = E^a_3(t) = E^a(t), \quad (57)
\]

where the dot denotes the derivative with respect to time \( \tau = t \) and the Hamilton gauge was selected. We have \( E = (0, 0, E) \), \( H = 0 \) in 3-vector representation

\[
F^{\mu\nu} = (-E, H), \quad c^{\mu\nu} = (-c_1, c_2), \quad c^\mu = (0, -c_1). \quad (58)
\]

We limit oneself to simple sample of \( SU(2) \) group (\( a = 1, 2, 3 \)) below, where

\[
\{t^a, t^b\} = \frac{1}{2} \delta^{ab} \text{I}, \quad [t^a, t^b] = if^{abc} t^c, \quad \text{tr}_c(t^a t^b) = \frac{1}{2} \delta^{ab}, \quad (59)
\]

the totally anti-symmetric structure constants \( f^{abc} \) coincide here with the totally anti-symmetric unit 3-tensor \( \epsilon^{ijk} \), \( f^{123} = +1 \). As a result of color decomposition of the WF, the system \( \{35\} \) is reduced to (\( \partial_p = \partial/\partial p_3 \))

\[
\begin{align*}
\partial_i a^s + \frac{g}{4} E^a \partial_p a^a &= 4pc^i, \\
\partial_i a^a + gE^a \partial_p a^s &= 4pc^i, \\
\partial_i b^s + \frac{g}{4} E^a \partial_p b^a &= 0, \\
\partial_i b^a + gE^a \partial_p b^s &= -gf^{abc} A^b b^c, \\
\partial_i b^a + gE^a \partial_p b^s &= 2p \times d^a + 4mc^i, \\
\partial_i b^a + gE^a \partial_p b^s &= 2p \times d^a - 4mc^i + gf^{abc} A^b b^c, \\
\partial_i c^i + \frac{g}{4} E^a \partial_p c^1 &= -pa^s + mb^s, \\
\partial_i c^i + \frac{g}{4} E^a \partial_p c^1 &= -pa^s + mb^s + gf^{abc}(A^b \times c^2), \\
\partial_i c^2 + \frac{g}{4} E^a \partial_p c^2 &= -pe^a, \\
\partial_i c^2 + \frac{g}{4} E^a \partial_p c^2 &= -pe^a + gf^{abc}(A^b \times c^1), \\
\partial_i d^0 + \frac{g}{4} E^a \partial_p d^0 &= 2me^s, \\
\partial_i d^0 + gE^a \partial_p d^0 &= 2me^s + gf^{abc} A^b c^1, \\
\partial_i d^a + \frac{g}{4} E^a \partial_p d^a &= 2p \times b^a, \\
\partial_i d^a + gE^a \partial_p d^a &= 2p \times b^a + gf^{abc} A^b d^0, \\
\partial_i e^s + \frac{g}{4} E^a \partial_p e^a &= 4pc^2 - 2md^0, \\
\partial_i e^s + \frac{g}{4} E^a \partial_p e^a &= 4pc^2 - 2md^0. \quad (60)
\end{align*}
\]
The important difference of these equations from their QED analogue is the structure of a force terms: the derivatives on time and on momentum act on different parts of WF (singlet and multiplet, respectively). This feature does not allow to reduce the problem to solving of ordinary differential equations even for the simple field configuration (57).

The situation becomes complicated even more in case of SU(3) \((a = 1, 2, \ldots, 8)\) where

\[
\{t^a, t^b\} = \frac{1}{3} \delta^{ab} \hat{1} + d^{abc} t^c, \tag{61}
\]

and \(d^{abc}\) are the totally symmetric structure constants. We shall write out for an illustration two first the equations (60) only:

\[
\begin{align*}
\partial_t a^s + \frac{g}{6} E^a \partial_p a^a &= 4 p c_1, \\
\partial_t a^a + g E^a \partial_p a^s + gd^{abc} E^b \partial_p a^c &= 4 p c_1. \tag{62}
\end{align*}
\]

6.2 Constant Chromo-Magnetic Field

Now we consider the gluon field configuration corresponding to a space-time homogeneous field

\[
A^\mu = (0, A), \quad F^{\mu\nu} = (0, H), \quad A = \text{const}, \quad H = \text{const}. \tag{63}
\]

In this case the system (35)-(42) is reduced to two independent groups of the equations (\(D = gH \times \partial_p\))

\[
\begin{align*}
\{D, b\} &= 2i g[A, b], \\
i[D, c_2] &= -8 p c_2 + 4md^0, \\
\{D \times, c_2\} + i \frac{1}{4}[D, a] &= 2 (pa - mb + ig[A \times, c_2]), \\
\{D, d^0\} + i \frac{1}{2}[D \times, b] &= -4 p \times b - 2ig[A, d^0], \\
\{D, d\} &= -4mc + 2ig[A, d], \\
i[D, c_1] &= -8 p c_1, \\
-i[D \times, d] &= 8 p \times d + 16mc_1 - 4ig[A, b^0], \\
\{D \times, c_1\} + i \frac{1}{4}[D, c] &= 2 (-pe + ig[A \times, c_1]). \tag{64}
\end{align*}
\]

The solution for a corresponding abelian QED case is known [28], but the derivation of a corresponding non-abelian analog is rather time-consuming work. We are limited here with a very simple case that have no analogy in QED. Assuming that \(A^a = A, \forall a\) ("colour democracy"), we have \(H = 0\) and the system (61),(65) is reduced to

\[
[A, b^0] = 0, \quad [A, b] = 0, \quad pa = mb, \tag{66}
\]

all other components are zero. This system allows the solutions with zero colour currents but with non-zero colour charges, for example

\[
b^0 \sim a, \quad a = a_1 \hat{1} + a_2 A, \quad b = ap/m, \tag{67}
\]

where \(a_1, a_2\) are arbitrary constants.
6.3 Chiral limit

The additional simplification is possible in the chiral limit $m \to 0$. In this case, the equations for the vector components of the Wigner function are separated from others. We suppose also the flux-tube symmetry in the electric field direction $\mathbf{n} = \mathbf{E}/E$, then

$$b = b_n + b_\perp \mathbf{p}_\perp, \quad d = d \cdot (\mathbf{n} \times \mathbf{p}_\perp),$$

(68)

because of $b$ is polar and $d$ axial vectors. By the initial conditions of the type (48), it follows from Eq.(60) that $d^0 = 0$ and $b^0 = 0$ (neutral system). Then we obtain

$$\partial_t b^a + \frac{g}{4} E^a \partial_p b^a = 2 \mathbf{p} \times \mathbf{d}^a,$$

$$\partial_t b^a + g E^a \partial_p b^s = 2 \mathbf{p} \times \mathbf{d}^s,$$

$$\partial_t d^a + \frac{g}{4} E^a \partial_p d^a = 2 \mathbf{p} \times b^a,$$

$$\partial_t d^a + g E^a \partial_p d^a = 2 \mathbf{p} \times b^a,$$

$$f^{abc} A^b b^c = 0,$$

$$f^{abc} A^b d^c = 0.$$

(69)

The two last algebraic equations play the role of constraints and have the particular solution ("color democracy")

$$A^a = A, \quad b^a = b, \quad d^a = d, \quad a = 1, 2, 3.$$

(70)

These conditions correspond to the special case of Abelian dominance approximation in relation to the gluon field. If we assume that the conditions (70) are carried out for all others components WF also, the solution get the form (48)

$$W^a(t, p) = \eta W^s(t, p), \quad a = 1, 2, 3,$$

(71)

where $\eta$ is some parameter. We find two admissible values after substituting that in the system (69)

$$\eta = \pm \frac{2}{\sqrt{3}}.$$

(72)

The system (69) is reduced to the three scalar equations at the account of the representation (68)

$$\partial_t b_\parallel + \eta g E \partial_p b_\parallel = 2 p_\perp^2 d,$$

$$\partial_t b_\perp + \eta g E \partial_p b_\perp = -2 p_\parallel d,$$

$$\partial_t d + \eta g E \partial_p d = 2 (p_\parallel b_\perp - b_\parallel).$$

(73)

This equation set can be solved numerically by the characteristics method. The remaining part of Eq.(60) is reduced to

$$\partial_t a + \eta g E \partial_p a = p c_1,$$
\[
\partial_t c_1 + \eta g E \partial_p c_1 = -p_a,
\]
\[
\partial_t c_2 + \eta g E \partial_p c_2 = -p_e,
\]
\[
\partial_t e + \eta g E \partial_p e = 4p c_2.
\]  

(74)

We find two integrals of motion combining these equations in pairs

\[
D_t (a^2 + 4c_1^2) = 0, \quad \rightarrow \quad a^2 + 4c_1^2 = \text{const},
\]
\[
D_t (e^2 + 4c_2^2) = 0, \quad \rightarrow \quad e^2 + 4c_2^2 = \text{const},
\]  

(75)

here \(D_t = \partial_t + \eta g E \partial_p\). It follows for the field-free initial conditions that \(e = 0\) and \(c_1 = 0\). The self-consistent evolution of the mean gluon field obeys the Yang-Mills equation

\[\dot{E} = -2g \int \frac{d^3p}{(2\pi)^3} b.\]  

(76)

The equations (73), (74) and (76) are the closed system for the numerical investigation of initial value problems such as the pair creation in strong field with the account of a back-reaction of the produced particles on the evolution of the mean gluon field.

The simple solutions of the type (71) do not satisfy to the Eq.(62) for the SU(3) case. But the reduction to the ordinary differential equation is possible, nevertheless, at use the more complicated representation of the type (48)

\[W^a(t, p) = \eta^a W(t, p), \quad a = 1, 2 \ldots N.\]  

(77)

We obtain the non-linear equations system for the admissible values of \(\eta^a\) by substituting these relations in Eqs.(62)

\[\eta^a \left( \sum_i \eta^i \right) = 6 \left[ 1 + \sum_i d^{aik} \eta^k \right], \quad a = 1, 2 \ldots 8,\]  

(78)

where \(d^{abc}\) has three independent non-zero values only [34].

7 Comparison with QED

The equations (35)-(42) can be transformed to QED case by formally setting the color matrices \(t^a\) equal to the unit one. Then the commutators of the gauge fields with the spinor components vanish, whereas the anti-commutators give a factor 2:

\[\partial_t a + g E \partial_p \alpha a = 4p \alpha \epsilon,\]
\[\partial_t b^\| + \partial_\alpha (x) b^\| \alpha + g \partial_\alpha (p) \left( b^\| E_\alpha - b^\| F_\alpha \right) = 0,\]
\[\partial_t b^\perp - \partial_\alpha (x) b^\perp + g \partial_\alpha (p) \left( b^\perp E_\alpha + b^\perp F_\alpha \right) = 2p \alpha d_\beta \epsilon^{\alpha\beta\mu} + 4mc^\mu,\]
\[\partial_t c^\mu + \partial_\alpha (x) c^\mu \alpha + g \partial_\alpha (p) \left( E_\alpha c^\mu - F_\alpha c^\mu \right) = p^\mu a - mb^\mu,\]
\[\partial_t c^\perp + \partial_\alpha (x) c^\perp - \partial_\alpha (x) c^\perp + \]

13
\[
\partial_\tau \epsilon_{\perp}^{\alpha}(p) + F_{\alpha}^{\perp \mu} \varepsilon_{\perp}^{\mu} = \epsilon \rho_{\alpha}^{\perp} \varepsilon^{\alpha \mu},
\]
\[
\partial_\tau d_{\perp}^{\parallel} + \partial_{\perp}^{\alpha}(x) d_{\perp}^{\parallel} + g \partial_{\perp}^{\alpha}(p) \left( d_{\parallel}^{\parallel} E_{\alpha} - F_{\alpha \beta} d_{\perp}^{\parallel} \right) = 2m \epsilon,
\]
\[
\partial_\tau d_{\perp}^{\parallel} - \partial_{\perp}^{\mu}(x) d_{\parallel}^{\parallel} + g \partial_{\perp}^{\alpha}(p) \left( d_{\parallel}^{\parallel} E_{\alpha} + F_{\alpha \mu} d_{\perp}^{\parallel} \right) = 2p_{\alpha}^{\perp} b_{\beta}^{\perp} \varepsilon^{\alpha \beta \mu},
\]
\[
\partial_\tau e + g E_{\alpha} \partial_{\perp}^{\alpha}(p) e + 2m d_{\parallel}^{\parallel} = -2p_{\alpha}^{\perp} c_{\beta}^{\perp} \varepsilon^{\alpha \beta \delta}.
\]

These equations are corresponds the formulae (4.9) - (4.16) of the work [30] (part 2). The system (79) can be reduced for the simple field (57) to three scalar ordinary differential equations [28], which allows the simple numerical investigation.

8 Summary

We have derived the system of KE for description of quark-antiquark plasma created from vacuum under action of a strong quasi-classical gluon field. The single-time Wigner function formalism allows in contrast to other approaches of this kind to formulate correctly the Cauchy problem. It is particularly important for non-perturbative description of vacuum particle creation.

We have analyzed some special cases of obtained system of KE (vacuum solution, space-homogeneous time dependent color electric field e.t.c.). It is shown that KE is complex system of the partial differential equations even in the most simple case (chiral limit). That system is rather difficult for the numerical investigation, while the KE in QED allows the reduction to the set of ordinary differential equations [28, 32] for some field configurations. Thus the transition to QCD either sets higher request to level of computer calculations or needs a some additional non-perturbative model assumptions.

Acknowledgments

We thank A. Reichel for helpful discussions. This work was supported partly by the Ministry of Education of the Russian Federation under grant N E02-3.3-210 and Russian Fund of Basic Research (RFBR) under grant 03-02-16877.

References

[1] U. Heinz and P.K. Kolb, hep-ph/0204061
[2] S.K. Wong, Nuovo Chimento, A65, 689 (1970).
[3] H.-Th. Elze and U. Heinz, Phys. Rep. 183, 81 (1989).
[4] U. Heinz, Ann. Phys. 161, 48 (1985); 168, 148 (1986).
[5] H.-Th. Elze, M. Gyulassy, and D. Vasak, Nucl. Phys. B 276, 706 (1986).
[6] M. Gyulassy and A. Iwasaki, Phys. Lett. **B165**, 157 (1985).

[7] H.-Th. Elze, Z. Phys. **C38**, 211 (1988); **C47**, 647 (1990).

[8] S. Mrowczynski, Phys. Rev. **D39**, 1940 (1989).

[9] J.-P. Blaizot and E. Ianku, Nucl. Phys. **B557**, 183 (1999); Phys. Rep. **359**, 355 (2002).

[10] G. C. Nayak, A. Dumitru, L. McLerran and W. Greiner, Nucl. Phys. **A687**, 457 (2001).

[11] A. K. Gangulu, P. K. Kaw, and J. C. Parikh, Phys.Rev. **D48**, R2983 (1993).

[12] D.D. Dietrich, G. C. Nayak, and W. Greiner, Phys. Rev. **D64**, 074006 (2001).

[13] G.C. Nayak and V. Ravishankar, Phys. Rev. **D55**, 6877 (1997); **C58**, 356 (1998).

[14] A. Jain and V. Ravishankar, [hep-ph/0212001](http://arxiv.org/abs/hep-ph/0212001).

[15] D.F. Litim and C. Manuel, Phys. Rept. **364** 451 (2002).

[16] Q. Wang, K. Redlich, H. Stöcker and W. Greiner, Phys.Rev.Lett. **88**, 132303 (2002); J.Phys. **G28**, 2115, (2002).

[17] Yu.A. Markov and M.A. Markova, Transp. Theory Statist. Phys. 28 (1999) 645; J. Phys. **G27**, 1869 (2001); Phys.Rev. **D64**, 105009 (2001).

[18] G. Gatoff, A.K. Kerman, and T. Matsui, Phys. Rev. **36**, 114 (1987).

[19] S.M. Schmidt, D. Blaschke, G. Röpke, S.A. Smolyansky, A.V. Prozorkevich, and V.D. Toneev, Int. J. Mod. Phys. **E** 7, 709 (1998).

[20] S. Schmidt, D. Bläschke, G. Röpke, A. V. Prozorkevich, S. A. Smolyansky, and V. D. Toneev, Phys. Rev. **D59**, 094005 (1999).

[21] J.C. Bloch, V.A. Mizerny, A.V. Prozorkevich, C.D. Roberts, S.M. Schmidt, S.A. Smolyansky, and D.V. Vinnik, Phys. Rev. **D 60** 1160011 (1999).

[22] S.A. Smolyansky, V.A. Mizerny, D.V. Vinnik, A.V. Prozorkevich, and V.D. Toneev, in ”Progress in Nonequilibrium Green’s functions”, M. Bonitz (Ed.), World Scientific Publ., Singapore 2000; Proceedings of the Conference ”Kadanoff-Baym Equations - Progress and Perspectives for Many-Body physics”, p.375, Rostock, 20-24.09.1999.

[23] Y. Kluger, J.M. Eisenberg, and B. Svetitsky, Int. J. Mod. Phys. **E2**, 333 (1993).

[24] Y. Kluger, E. Mottola, and J. Eisenberg, Phys. Rev. **D58**, 5015 (1998).
[25] D.V. Vinnik, R. Alkofer, S.M. Schmidt, S.A. Smolyansky, V.V. Skokov, and A.V. Prozorkevich, Few-Body System, 32, 23 (2002).

[26] V.V. Skokov, S.A. Smolyansky, and V.D. Toneev, hep-ph/0210099.

[27] D.V. Vinnik, A.V. Prozorkevich, S.A. Smolyansky, V.D. Toneev, M.B. Hecht, C.D. Roberts, and S.M. Schmidt, Eur. Phys. J. C22, 341 (2001).

[28] I. Bialynicki-Birula, P. Górnicki, and J. Rafelski, Phys. Rev. D44, 1825 (1991).

[29] S. Ochs and U. Heinz, Ann. Phys. 266, 351 (1998).

[30] A. Höll, V.G. Morozov, and G. Röpke, Theor. Math. Phys. 131, 812 (2002); 132, 1029 (2002); quant-ph/0208083.

[31] S.A. Smolyansky, A.V. Prozorkevich, G. Maino, and S.G. Mashnik, Ann. Phys. 277, 193 (1999).

[32] A.A. Grib, S.G. Mamaev, and V.M. Mostepanenko, Vacuum Quantum Effects in Strong External Fields, (Friedmann Laboratory Publishing, St. Petersburg, 1994).

[33] R. Hakim, Ann. Phys. 139, 230 (1982).

[34] F.J. Yndurain, Quantum Chromodynamics, (Springer-Verlag, N.Y., 1983).