Twilled Lie-Rinehart algebras and differential Batalin-Vilkovisky algebras

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Abstract. Twilled Lie-Rinehart algebras generalize, in the Lie-Rinehart context, complex structures on smooth manifolds. An almost complex manifold determines an “almost twilled pre-LR algebra”, which is a true twilled LR-algebra iff the almost complex structure is integrable. We characterize twilled LR structures in terms of certain associated differential (bi)graded Lie and Gerstenhaber-algebras; in particular the Gerstenhaber algebra arising from an almost complex structure is a d(ifferential) G-algebra iff the almost complex structure is integrable. Such G-algebras, endowed with a generator turning them into a Batalin-Vilkovisky-algebra, occur on the B-side of the mirror conjecture. We generalize a result of Koszul to those dG-algebras which arise from twilled LR-algebras. A special case thereof explains the relationship between holomorphic volume forms and exact generators for the corresponding dG-algebra and thus yields in particular a conceptual proof of the Tian-Todorov lemma.

We give a differential homological algebra interpretation for twilled LR-algebras and by means of it we elucidate the notion of generator in terms of homological duality for differential graded LR-algebras and we indicate how some of our results might be globalized by means of Lie groupoids.

Introduction

A version of the mirror conjecture involves certain differential Batalin-Vilkovisky algebras arising from a Calabi-Yau manifold. A crucial ingredient is what is referred to in the literature as the Tian-Todorov lemma. Our goal is to study such differential Batalin-Vilkovisky algebras and generalizations thereof in the framework of Lie-Rinehart algebras. Now a differential Batalin-Vilkovisky algebra is a Gerstenhaber algebra together with an exact generator, and the underlying Gerstenhaber algebras of interest for us, in turn, arise as (bigraded) algebras of forms on twilled Lie-Rinehart algebras (which we introduce below). A twilled Lie-Rinehart algebra generalizes, in the Lie-Rinehart context, the notion of a complex structure on a

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smooth manifold. One of our results (Theorem 4.4) will say that an “almost twilled pre-Lie-Rinehart algebra” is a true twilled Lie-Rinehart algebra if and only if the corresponding Gerstenhaber algebra is a differential Gerstenhaber algebra. (The wording is somewhat imprecise here and Theorem 4.4 will in fact be phrased in terms of “almost twilled Lie-Rinehart algebras”, to be introduced below.) As a consequence, we deduce that an almost complex structure on a smooth manifold is integrable if and only if the corresponding Gerstenhaber algebra is a differential Gerstenhaber algebra. Now a theorem of Koszul [21] establishes, on an ordinary smooth manifold, a bijective correspondence between generators for the Gerstenhaber algebra of multi vector fields and connections in the top exterior power of the tangent bundle in such a way that exact generators correspond to flat connections. In Theorem 5.4.6 below we will generalize this bijective correspondence to the differential Gerstenhaber algebras arising from twilled Lie-Rinehart algebras; such Gerstenhaber algebras come into play, for example, in the mirror conjecture. What corresponds to a flat connection on the line bundle in Koszul’s theorem is now a holomorphic volume form de termines a generator for the corresponding differential Gerstenhaber algebra turning it into a differential Batalin-Vilkovisky algebra. The resulting differential Batalin-Vilkovisky algebra then generalizes that which underlies what is called the B-model. In particular, as a consequence of our methods, we obtain a new proof of the Tian-Todorov lemma. We will also give a differential homological algebra interpretation of twilled Lie-Rinehart algebras and, furthermore, of a generator for a differential Batalin-Vilkovisky algebra in terms of a suitable notion of homological duality. Finally we indicate how some of our results might be globalized by means of Lie groupoids.

We now give a more detailed outline of the paper. Let $R$ be a commutative ring. A Lie-Rinehart algebra $(A, L)$ consists of a commutative $R$-algebra $A$ and an $R$-Lie algebra $L$ together with an $A$-module structure $A \otimes_R L \to L$ on $L$, written $a \otimes_R \alpha \mapsto a \alpha$, and an action $L \to \text{Der}(A)$ of $L$ on $A$ (which is a morphism of $R$-Lie algebras and) whose adjoint $L \otimes_R A \to A$ is written $\alpha \otimes_R a \mapsto \alpha(a)$; here $a \in A$ and $\alpha \in L$. These mutual actions are required to satisfy certain compatibility properties modeled on $(A, L) = (C^\infty(M), \text{Vect}(M))$ where $C^\infty(M)$ and $\text{Vect}(M)$ refer to the algebra of smooth functions and the Lie algebra of smooth vector fields, respectively, on a smooth manifold $M$. In general, the compatibility conditions read:

\begin{align*}
(0.1) & \quad (a \alpha)b = a\alpha(b), \quad a, b \in A, \quad \alpha \in L, \\
(0.2) & \quad [\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta], \quad a \in A, \quad \alpha, \beta \in L.
\end{align*}

For a Lie-Rinehart algebra $(A, L)$, following [33], we will refer to $L$ as an $(R, A)$-Lie algebra. In differential geometry, $(R, A)$-Lie algebras arise as spaces of sections of Lie algebroids.

Given two Lie-Rinehart algebras $(A, L')$ and $(A, L'')$, together with mutual actions $\cdot : L' \otimes_R L'' \to L''$ and $\cdot : L'' \otimes_R L' \to L'$ which endow $L''$ and $L'$ with an $(A, L')$- and $(A, L'')$-module structure, respectively, we will refer to $(A, L', L'')$ as an almost twilled Lie-Rinehart algebra; we will call it a twilled Lie-Rinehart algebra provided the direct sum $A$-module structure on $L = L' \oplus L''$, the sum $(L' \oplus L'') \otimes_R A \to A$ of the adjoints of the $L'$- and $L''$-actions on $A$, and the bracket $[\cdot, \cdot]$ on $L = L' \oplus L''$
given by

\[(\alpha'', \alpha'), (\beta'', \beta')] = [\alpha'', [\alpha', \beta']] + \alpha'' \cdot \beta' - \beta' \cdot \alpha'' + \alpha' \cdot \beta'' - \beta'' \cdot \alpha'
\]

turn \((A, L)\) into a Lie-Rinehart algebra. We then write \(L = L' \bowtie L''\) and refer to \((A, L)\) as the twilled sum of \((A, L')\) and \((A, L'')\).

For illustration, consider a smooth manifold \(M\) with an almost complex structure, let \(A\) be the algebra of smooth complex functions on \(M\), \(L\) the \((\mathbb{C}, A)\)-Lie algebra of complexified smooth vector fields on \(M\), and consider the ordinary decomposition of the complexified tangent bundle \(\tau^C_M\) as a direct sum \(\tau'_M \oplus \tau''_M\) of the almost holomorphic and almost antiholomorphic tangent bundles \(\tau'_M\) and \(\tau''_M\), respectively; write \(L'\) and \(L''\) for their spaces of smooth sections. Then \((A, L', L'')\), together with the mutual actions coming from \(L\), is a twilled Lie-Rinehart algebra if and only if the almost complex structure is integrable, i.e. a true complex structure; \(\tau'_M\) and \(\tau''_M\) are then the ordinary holomorphic and antiholomorphic tangent bundles, respectively. In Section 1 below we actually show that the precise analogue of an almost complex structure is what we will call an almost twilled pre-Lie-Rinehart algebra structure. A situation similar to that of a complex structure on a smooth manifold and giving rise to a twilled Lie-Rinehart algebra arises from a smooth manifold with two transverse foliations as well as from a Cauchy-Riemann structure; see Section 8 below for some comments about Cauchy-Riemann structures. Lie bialgebras provide another class of examples of twilled Lie-Rinehart algebras; Kosmann-Schwarzbach and F. Magri refer to these objects, or rather to the corresponding twilled sum, as twilled extensions of Lie algebras [20]; Lu and Weinstein call them double Lie algebras [25]; and Majid uses the terminology matched pairs of Lie algebras [29]. Spaces of sections of suitable pairs of Lie algebroids with additional structure lead to yet another class of examples of twilled Lie-Rinehart algebras; these have been studied in the literature under the name matched pairs of Lie algebroids by Mackenzie [26] and Mokri [31].

An almost twilled Lie-Rinehart algebra \((A, L'', L')\) is a true twilled Lie-Rinehart algebra if and only if \((A, L'', L')\) satisfies three compatibility conditions, spelled out in Proposition 1.7 below; this proposition is merely an adaption of earlier results in the literature to our more general situation. We then give another interpretation of the compatibility conditions in terms of annihilation properties of the two operators \(d'\) and \(d''\) which arise as formal extensions of the ordinary Lie-Rinehart differentials with respect to \(L'\) and \(L''\), respectively, on the bigraded algebra \(\text{Alt}^*_A(L'', \text{Alt}^*_A(L', A))\) (but are not necessarily exact); for the twilled Lie-Rinehart algebra arising from the holomorphic and antiholomorphic tangent bundles of a complex manifold, the resulting differential bigraded algebra \((\text{Alt}^*_A(L'', \text{Alt}^*_A(L', A)), d', d'')\) comes down to the ordinary Dolbeault complex. See Theorem 1.15 for details.

In the rest of the paper, we show that other characterizations of twilled Lie-Rinehart algebras explain the differential Batalin-Vilkovisky algebras mentioned before: Let \((A, L'', L')\) be an almost twilled Lie-Rinehart algebra having \(L'\) finitely generated and projective as an \(A\)-module. Write \(\mathcal{A}' = \text{Alt}_A(L'', A)\) and \(\mathcal{L}' = \text{Alt}_A(L', L')\). Now \(\mathcal{A}''\) is a graded commutative \(A\)-algebra and, endowed with the Lie-Rinehart differential \(d''\) (which corresponds to the \((R, A)\)-Lie algebra structure on \(L''\)), \(\mathcal{A}''\) is a differential graded commutative \(R\)-algebra. Moreover, from the
(\(A,L''\))-module structure on \(L'\), \(L'\) inherits an obvious differential graded \(A''\)-module structure. Furthermore, the \((A,L')\)-structure on \(L''\) induces an action of \(L'\) on \(A''\) by graded derivations. The latter, in turn, induces a graded \(R\)-Lie algebra structure on \(L'\) and a pairing

\[ L' \otimes A'' \to A'' \]

which turns \((A'',L')\) into a graded Lie-Rinehart algebra (in an obvious sense); this is in fact the graded crossed product Lie-Rinehart structure. Section 2 below is devoted to differential graded Lie-Rinehart algebras; the differential graded crossed product Lie-Rinehart algebra will be explained in (2.8) and (2.9) below. Now, on \(L' = \text{Alt}_A(L'', L')\) we have the Lie-Rinehart differential \(d''\) which corresponds to the \((R,A)\)-Lie algebra structure on \(L''\) and the \((A,L'')\)-module structure \(L'\). By symmetry, when \(L''\) is finitely generated and projective as an \(A\)-module, we have the same structure, with \(L'\) and \(L''\) interchanged. Now Theorem 3.2 will say that the statements (i), (ii) and (iii) below are equivalent: (i) \((A,L',L'')\) is a true twilled Lie-Rinehart algebra; (ii) \((L',d'') = (\text{Alt}_A(L'', L'), d'')\) is a differential graded \(R\)-Lie algebra; (iii) \((A',L';d'')\) is a differential graded Lie-Rinehart algebra. Thus, under these circumstances, there is a bijective correspondence between twilled Lie-Rinehart algebra and differential graded Lie-Rinehart algebra structures.

We note that, in this situation, the Lie bracket on \(L' = \text{Alt}_A(L'', L')\) does not just come down to the shuffle product of forms on \(L''\) and the Lie bracket on \(L'\); in fact, such a bracket would not even be well defined since the Lie bracket of \(L'\) is not \(A\)-linear, i.e. does not behave as a “tensor”. When \((A,L',L'')\) is the twilled Lie-Rinehart algebra arising from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold \(M\), \((L',d'') = (\text{Alt}_A(L'', L'), d'')\) is what is called the Kodaira-Spencer algebra in the literature; it controls the infinitesimal deformations of the complex structure on \(M\). The cohomology \(\text{H}^*(L'', L')\) then inherits a graded Lie algebra structure and the obstruction to deforming the complex structure is the map \(\text{H}^1(L'', L') \to \text{H}^2(L'', L')\) which sends \(\eta \in \text{H}^1(L'', L')\) to \([\eta,\eta] \in \text{H}^2(L'', L')\).

We now return to a general almost twilled Lie-Rinehart algebra \((A,L',L'')\) having \(L'\) finitely generated and projective as an \(A\)-module and consider the graded crossed product Lie-Rinehart algebra \((A'',L')\). Write \(\Lambda_AL'\) for the exterior \(A\)-algebra on \(L'\); as in the ungraded situation, the graded Lie-Rinehart bracket on \(L' (= \text{Alt}_A(L'', L'))\) extends to a (bigraded) bracket on \(\text{Alt}_A(L'', \Lambda_AL')\) which turns the latter into a bigraded Gerstenhaber algebra; as a bigraded algebra, \(\text{Alt}_A(L'', \Lambda_AL')\) could be thought as of the exterior \(A''\)-algebra on \(L'\), and we will often write

\[ \Lambda_AL' = \text{Alt}_A(L'', \Lambda_AL'). \]

The Lie-Rinehart differential \(d''\) which corresponds to the Lie-Rinehart structure on \(L''\) and the induced graded \((A,L'')\)-module structure on \(\Lambda_AL'\) turn \(\text{Alt}_A(L'', \Lambda_AL')\) into a differential (bi)-graded commutative \(R\)-algebra. By symmetry, when \(L''\) is finitely generated and projective as an \(A\)-module, we have the same structure, with \(L'\) and \(L''\) interchanged. Theorem 4.4 will say that the almost twilled Lie-Rinehart algebra \((A,L'', L')\) is a true twilled Lie-Rinehart algebra if and only if \((\Lambda_AL', d'') = (\text{Alt}_A(L'', \Lambda_AL'), d'')\) is a differential (bi)-graded Gerstenhaber algebra.

When \((A,L',L'')\) arises from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold \(M\), the resulting differential Gerstenhaber algebra
(\text{Alt}_A(L'', \Lambda_A L'), d'') \text{ is that of forms of type } (0, *) \text{ with values in the holomorphic multi vector fields, the operator } d'' \text{ being the Cauchy-Riemann operator (which is more usually written } \overline{\partial} \text{). This differential Gerstenhaber algebra comes into play in the mirror conjecture; it was studied by Barannikov-Kontsevich [1], Manin [30], Witten [38], and others.}

Let now \((A, L'', L')\) be a twilled Lie-Rinehart algebra having \(L'\) finitely generated and projective as an \(A\)-module of constant rank \(n\) (say), and write \(\Lambda^n_A L'\) for the top exterior power of \(L'\) over \(A\). Consider the differential Gerstenhaber algebra \((\text{Alt}_A(L'', \Lambda_A L'), d'')\). Our next aim is to study generators thereof. To this end, we observe that, when \(\text{Alt}_A(L', \Lambda^n_A L')\) is endowed with the obvious graded \((A, L'')\)-module structure induced from the left \((A, L'')\)-module structure on \(L'\) which is part of the structure of twilled Lie-Rinehart algebra, the canonical isomorphism

\[
\text{Alt}_A(L'', \Lambda_A L') \to \text{Alt}_A(L'', \text{Alt}_A(L', \Lambda^n_A L'))
\]

of graded \(A\)-modules is compatible with the differentials which correspond to the Lie-Rinehart structure on \(L''\) and the \((A, L'')\)-module structures on the coefficients on both sides of \((0.4)\); abusing notation, we denote each of these differentials by \(d''\).

Theorem 5.4.6 below says the following: The isomorphism \((0.4)\) furnishes a bijective correspondence between generators of the bigraded Gerstenhaber structure on the left-hand side (of \((0.4)\)) and \((A, L')\)-connections on \(\Lambda^n_A L'\) in such a way that exact generators correspond to \((A, L')\)-module structures (i.e. flat connections). Under this correspondence, generators of the differential bigraded Gerstenhaber structure on the left-hand side correspond to \((A, L')\)-connections on \(\Lambda^n_A L'\) which are compatible with the \((A, L'')\)-module structure on \(\Lambda^n_A L'\). Thus, in particular, exact generators of the differential bigraded Gerstenhaber structure on the left-hand side correspond to \((A, L'')\)-compatible \((A, L')\)-module structures on \(\Lambda^n_A L'\).

When \(L''\) is trivial and \(L'\) the Lie algebra of smooth vector fields on a smooth manifold, the statement of this theorem comes down to the result of Koszul [21] mentioned earlier. Our result not only provides many examples of differential Batalin-Vilkovisky algebras but also explains how every differential Batalin-Vilkovisky algebra having an underlying bigraded \(A\)-algebra of the kind \(\text{Alt}_A(L'', \Lambda_A L')\) arises.

When \((A, L', L'')\) is the twilled Lie-Rinehart algebra which comes from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold \(M\) as explained above, the theorem gives a bijective correspondence between generators of the differential bigraded Gerstenhaber algebra \((\text{Alt}_A(L'', \Lambda_A L'), d'')\) of forms of type \((0, *)\) with values in the holomorphic multi vector fields, the differential \(d''\) being the Cauchy-Riemann operator \(\overline{\partial}\), and holomorphic connections on the highest exterior power of the holomorphic tangent bundle in such a way that exact generators correspond to flat holomorphic connections. In particular, suppose that \(M\) is a Calabi-Yau manifold, that is, admits a holomorphic volume form \(\Omega\) (say). This holomorphic volume form identifies the highest exterior power of the holomorphic tangent bundle with the algebra of smooth complex functions on \(M\) as a module over \(L = L'' \oplus L'\), hence induces a flat holomorphic connection thereupon and hence an exact generator \(\partial_\Omega\) for \((\text{Alt}_A(L'', \Lambda_A L'), d'')\), turning the latter into a differential (bi)graded Batalin-Vilkovisky algebra. This is precisely the differential (bi)graded Batalin-Vilkovisky algebra coming into play on the B-side of the mirror conjecture.
and studied in the cited sources. The fact that the holomorphic volume form induces a generator for the differential Gerstenhaber structure is referred to in the literature as the Tian-Todorov lemma. In our approach, this lemma drops out as a special case of our generalization of Koszul’s theorem to the bigraded setting, and this generalization indeed provides a conceptual proof of the lemma. This lemma implies that, for a Kählerian Calabi-Yau manifold $M$, the deformations of the complex structure are unobstructed, that is to say, there is an open subset of $H^1(M, \tau_M)$ parametrizing the deformations of the complex structure; here $H^1(M, \tau_M)$ is the first cohomology group of $M$ with values in the holomorphic tangent bundle $\tau_M$. Under these circumstances, after a choice of holomorphic volume form $\Omega$ has been made, the canonical isomorphism (0.4), combined with the isomorphism

$$\Omega^\flat \colon \text{Alt}_A(L'', \text{Alt}_A(L', \Lambda^n_L L')) \to \text{Alt}_A(L'', \text{Alt}_A(L', A))$$

induced by $\Omega$ identifies $(\text{Alt}_A(L'', \Lambda_A L'), d'', \partial_\Omega)$ with the Dolbeault complex of $M$ and hence the cohomology $H^\ast(\text{Alt}_A(L'', \Lambda_A L'), d'', \partial_\Omega)$ with the ordinary complex valued cohomology of $M$. This is nowadays well understood; see also 5.4.8 below. The cohomology $H^\ast(\text{Alt}_A(L'', \Lambda_A L'), d'', \partial_\Omega)$ is referred to in the literature as the extended moduli space of complex structures [38]; it underlies what is called the B-model in the theory of mirror symmetry.

In Section 6 we will give differential homological algebra interpretations of some of our earlier results. In particular, we will show that, for a twilled Lie-Rinehart algebra $(A, L', L'')$ having $L'$ finitely generated and projective as an $A$-module, the differential bigraded algebra $(\text{Alt}_A(L'', \Lambda_A L'), d', d'')$ computes the differential graded Lie-Rinehart cohomology $H^\ast(\mathcal{L}', \mathcal{A}'')$, where $(\mathcal{A}'', \mathcal{L}; d'')$ is the differential graded crossed product Lie-Rinehart algebra $(\text{Alt}_A(L'', A), \text{Alt}_A(L'', L'); d'')$ mentioned before. When $L''$ is trivial, so that $H^\ast(\mathcal{L}', \mathcal{A}'')$ is an ordinary (ungraded) Lie-Rinehart algebra $(A, L)$, the differential graded Lie-Rinehart cohomology boils down to the ordinary Lie-Rinehart cohomology $H^\ast(L, A)$. Moreover, for the special case when $A$ and $L$ are the algebra of smooth functions and smooth vector fields on a smooth manifold, the Lie-Rinehart cohomology $H^\ast(L, A)$ amounts to the de Rham cohomology; this fact has been established by Rinehart [33]. In our more general situation, when the twilled Lie-Rinehart algebra $(A, L', L'')$ arises from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold, the complex calculating the differential graded Lie-Rinehart cohomology $H^\ast(\mathcal{L}', \mathcal{A}'')$ of the differential graded crossed product Lie-Rinehart algebra $(\mathcal{A}'', \mathcal{L}; d'') = (\text{Alt}_A(L'', A), \text{Alt}_A(L'', L'); d'')$ is the Dolbeault complex, and the differential graded Lie-Rinehart cohomology amounts to the Dolbeault cohomology. Thus our approach provides, in particular, an interpretation of the Dolbeault complex in the framework of differential homological algebra. In Section 7, generalizing results in our earlier paper [12], we will elucidate the concept of generator of a differential bigraded Batalin-Vilkovisky algebra in the framework of homological duality for differential graded Lie-Rinehart algebras. In particular, we will show that an exact generator amounts to the differential in a standard complex computing differential graded Lie-Rinehart homology (!) with appropriate coefficients; see Proposition 7.13 below. Further, we will see that, when the appropriate additional structure (in terms of Lie-Rinehart differentials and dBV-generators) is taken into account, the above isomorphism (0.4) is essentially just a duality isomorphism in the (co)homology of the differential graded crossed
product Lie-Rinehart algebra \((A'', L')\); see Proposition 7.14 for details. In particular, the Tian-Todorov Lemma comes down to a statement about differential graded (co)homological duality.

Twilled Lie-Rinehart algebras thus generalize Lie bialgebras, and the twilled sum is an analogue, even a generalization, of the Manin double of a Lie bialgebra. The Lie bialgebroids introduced by Mackenzie and Xu \([27]\) generalize Lie bialgebras as well, and there is a corresponding notion of Lie-Rinehart bialgebra, which we explain at the end of Section 4 below. However, twilled Lie-Rinehart algebras and Lie-Rinehart bialgebras are different, in fact non-equivalent notions which both generalize Lie bialgebras. In a sense, Lie-Rinehart bialgebras generalize Poisson and in particular symplectic structures while twilled Lie-Rinehart algebras generalize complex structures. In Theorem 4.8 below we characterize twilled Lie-Rinehart algebras in terms of Lie-Rinehart bialgebras. For the special case where the twilled Lie-Rinehart algebra under consideration arises from a matched pair of Lie algebroids, this characterization may be deduced from what is said in \([26]\). In Section 8 below, we use our characterization of twilled Lie-Rinehart algebras in terms of Lie-Rinehart bialgebras to indicate how some of the results of this paper might be globalized in terms of Lie groupoids.

As in Mac Lane's book \([28]\), (bi)graded objects will always be understood as being externally (bi)graded.

I am indebted to Y. Kosmann-Schwarzbach and K. Mackenzie for discussions, and to J. Stasheff and A. Weinstein for some e-mail correspondence about various topics related with the paper. Most of the results to be given below have been presented at the “Poissonfest” (Warsaw, August 1998), and at that occasion, Y. Kosmann-Schwarzbach introduced me to the recent manuscript \([34]\) which treats topics somewhat related to the present paper. There is little overlap, though.

1. Twilled Lie-Rinehart algebras

The complexified tangent bundle \(\tau^C_M\) of a smooth complex manifold \(M\) decomposes as a direct sum \(\tau'_M \oplus \tau''_M\) of the holomorphic and the antiholomorphic tangent bundles \(\tau'_M\) and \(\tau''_M\), respectively; both \(\tau'_M\) and \(\tau''_M\) yield smooth complex Lie algebroids over \(M\), and the integrability condition amounts to these two Lie algebroid structures being compatible in a very precise sense. Similar situations arise from a smooth manifold with two transverse foliations, from Lie bialgebras, and from a matched pair of Lie algebroids, cf. what has been said in the introduction. We now develop a theory incorporating, generalizing, and unifying these special cases.

As before, let \(R\) be a commutative ring, let \(A\) be a commutative algebra, and let \(L'\) and \(L''\) be two \(A\)-modules. We will study and answer the following two related questions:

**Question 1.1.** Given a Lie-Rinehart algebra \((A, L)\) and a direct sum decomposition \(L = L' \oplus L''\) of \(A\)-modules inducing \((R, A)\)-Lie algebra structures on \(L'\) and \(L''\), what kind of additional structure relates \(L'\) and \(L''\)? The decomposition \(L = L' \oplus L''\) will then be referred to as an integrable decomposition of \(L\).

**Question 1.2.** Given \((R, A)\)-Lie algebra structures on \(L'\) and \(L''\), which kind of additional structure turns the direct sum \(L = L' \oplus L''\) of \(A\)-modules into an \((R, A)\)-Lie algebra in such a way that the action of \(L\) on \(A\) amounts to the sum of the \(L'\) and \(L''\)-actions and that the bracket on \(L\) restricts to the given brackets
on $L'$ and $L''$? The new structure will then be referred to as a twilled $(R, A)$-Lie algebra and the resulting $(R, A)$-Lie algebra will be called the twilled sum of $L'$ and $L''$.

**Example 1.3.** Let $R$ be the ring $\mathbb{C}$ of complex numbers, $A$ the algebra of smooth complex functions on a smooth almost complex manifold $M$, $L$ be the $(\mathbb{C}, A)$-Lie algebra of smooth complexified vector fields, and let $L = L'' \oplus L'$ be the customary eigenspace decomposition (of the spaces of sections of the complexified tangent bundle) arising from the almost complex structure. The $A$-modules $L'$ and $L''$ inherit $(\mathbb{C}, A)$-Lie algebra structures in such a way that $L$ is their twilled sum (in a sense to be made precise) if and only if the almost complex structure is integrable.

We now proceed towards a description of a twilled Lie-Rinehart algebra, the basic concept of the present paper: Let $A$ be a commutative $R$-algebra. Consider two $A$-modules $L'$ and $L''$, together with skew-symmetric $R$-bilinear brackets of the kind (1.4.1)—not necessarily Lie brackets—and $R$-bilinear pairings of the kind (1.4.2'), (1.4.2''), (1.4.3), and (1.4.4) below:

\begin{align*}
(1.4.1') & 
[\cdot, \cdot'] : L' \otimes_R L' \to L' \\
(1.4.1'') & 
[\cdot, \cdot''] : L'' \otimes_R L'' \to L'' \\
(1.4.2') & 
L' \otimes_R A \to A, \quad x \otimes_R a \mapsto x(a), \quad x \in L', \ a \in A \\
(1.4.2'') & 
L'' \otimes_R A \to A, \quad \xi \otimes_R a \mapsto x(a), \quad \xi \in L'', \ a \in A \\
(1.4.3) & 
\cdot : L' \otimes_R L'' \to L'' \\
(1.4.4) & 
\cdot : L'' \otimes_R L' \to L'
\end{align*}

We will refer to a $(A, L', L'')$ as an almost twilled pre-Lie-Rinehart algebra, provided $(A, L', L'')$ satisfies (i), (ii), and (iii) below.

(i) The values of the adjoints $L' \to \text{End}_R(A)$ and $L'' \to \text{End}_R(A)$ of (1.4.2') and (1.4.2'') respectively lie in $\text{Der}_R(A)$;

(ii) (1.4.1'), (1.4.2') and the $A$-module structure on $L'$ and, likewise, (1.4.1''), (1.4.2'') and the $A$-module structure on $L''$ are related by conditions of the kind (0.1) and (0.2);

(iii) (1.4.3) and (1.4.4) behave formally like connections.

Requirement (ii) is made precise by (1.4.5'), (1.4.6'), (1.4.5''), (1.4.6'') below, and (iii) is made precise by (1.4.7) and (1.4.8).

\begin{align*}
(1.4.5') & \quad (ax)(b) = a(x(b)), \quad a, b \in A, \ x \in L', \\
(1.4.6') & \quad [x, ay] = x(a)y + a[x, y], \quad a \in A, \ x, y \in L' \\
(1.4.5'') & \quad (a\xi)(b) = a(\xi(b)), \quad a, b \in A, \ \xi \in L'' \\
(1.4.6'') & \quad [\xi, a\eta] = \xi(a)\eta + a[\xi, \eta], \quad a \in A, \ \xi, \eta \in L'' \\
(1.4.7) & \quad x \cdot (a\xi) = (x(a))\xi + a(x \cdot \xi), \quad (ax) \cdot \xi = a(x \cdot \xi), \quad x \in L', \ \xi \in L'' \\
(1.4.8) & \quad \xi \cdot (ax) = (\xi(a))x + a(\xi \cdot x), \quad (a\xi) \cdot x = a(\xi \cdot x), \quad x \in L', \ \xi \in L''.
\end{align*}

When $(A, L', L'')$ is an almost twilled pre Lie-Rinehart algebra, the pair $(L', L'')$, together with the other structure, will be called an almost twilled pre-$(R, A)$-Lie algebra.
Given an almost twilled pre-Lie-Rinehart algebra \((A, L', L'')\), let \(L = L' \oplus L''\) be the direct sum of \(A\)-modules, and extend the brackets on \(L'\) and \(L''\) to an \(R\)-bilinear alternating bracket

\[
\{\cdot, \cdot\} : L \otimes_R L \rightarrow L
\]

by means of the formula

\[
[(\alpha'', \alpha'), (\beta'', \beta')] = [\alpha'', \beta'']' + [\alpha', \beta']'' + \alpha'' \cdot \beta' - \beta'' \cdot \alpha' - \alpha' \cdot \beta'' - \beta' \cdot \alpha'',
\]

and the two pairings (1.4.2') and (1.4.2'') to a pairing

\[
L \otimes_R A \rightarrow A
\]

in the obvious way, that is, by means of the assignment

\[
(\xi, x) \otimes_R a \mapsto \xi(a) + x(a), \quad x \in L', \, \xi \in L'', \, a \in A.
\]

By construction, the values of the adjoint of (1.5.3) then lie in \(\text{Der}_{R}(A)\), that is this adjoint is then of the form

\[
L = L'' \oplus L' \rightarrow \text{Der}_{R}(A).
\]

An almost twilled pre-Lie-Rinehart algebra \((A, L', L'')\) will be said to be an \textit{almost twilled Lie-Rinehart algebra} provided \((A, L')\), endowed with the structure (1.4.1') and (1.4.2'), and \((A, L'')\), endowed with (1.4.1'') and (1.4.2''), are true Lie-Rinehart algebras and, furthermore, (1.4.3) is a left \((A, L')\)-module structure on \(L''\) and (1.4.4) a left \((A, L'')\)-module structure on \(L'\). The pair \((L', L'')\), together with the two module structures (1.4.3) and (1.4.4), will then be called an \textit{almost twilled \((R, A)\)-Lie algebra}. An almost twilled Lie-Rinehart algebra \((A, L', L'')\) will be said to be a \textit{twilled Lie-Rinehart algebra} provided \((A, L)\), together with the bracket (1.5.1) and the assignment (1.5.4), is a Lie-Rinehart algebra; this Lie-Rinehart algebra will then be called the \textit{twilled sum} of \((A, L')\) and \((A, L'')\); likewise, \((L', L'')\) will then be called a \textit{twilled \((R, A)\)-Lie algebra} and \(L\), written \(L' \bowtie L''\), the \textit{twilled sum} of \(L'\) and \(L''\).

A direct sum decomposition \(L = L' \oplus L''\) of an \((R, A)\)-Lie algebra \(L\) yields in an obvious fashion an almost twilled pre-Lie-Rinehart algebra structure on \((A, L', L'')\): The brackets (1.4.1') and (1.4.1'') result from restriction and projection; the pairings (1.4.2') and (1.4.2'') are as well obtained by restriction; further, the requisite pairings (1.4.3) and (1.4.4) are given by the composites

\[
:L' \otimes_R L'' \xrightarrow{[\cdot, \cdot]|_{L' \otimes_R L''}} L' \oplus L'' \xrightarrow{\text{pr}_{L''}} L''
\]

and

\[
:L'' \otimes_R L' \xrightarrow{[\cdot, \cdot]|_{L'' \otimes_R L'}} L'' \oplus L' \xrightarrow{\text{pr}_{L'}} L'
\]

where, for \(M = L' \otimes_R L''\) and \(M = L'' \otimes_R L'\), \([\cdot, \cdot]|_M\) denotes the restriction of the Lie bracket to \(M\). The formula (1.5.1) is then merely a decomposition of the initially given bracket on \(L\) into components corresponding to the direct sum decomposition of \(L\) into \(L'\) and \(L''\), and (1.5.4) is accordingly a decomposition of the \(L\)-action on \(A\).

The following result is a mere adaption to our situation of earlier results in the literature; it is therefore labelled as a proposition.
Proposition 1.7. An almost twilled Lie-Rinehart algebra \((A, L', L'')\) is a true twilled Lie-Rinehart algebra if and only if the Lie brackets \([·, ·]\) and \([·, ·]'\) on \(L'\) and \(L''\), respectively, and the actions \((1.4.3)\) and \((1.4.4)\) are related by

\[
\begin{align*}
(1.7.1) & \quad \xi(x(a)) - x(\xi(a)) = (\xi \cdot x)(a) - (x \cdot \xi)(a) \\
(1.7.2) & \quad x \cdot [\xi, \eta]' = [x \cdot \xi, \eta]' + [\xi, x \cdot \eta]' - (\xi \cdot x) \cdot \eta + (\eta \cdot x) \cdot \xi \\
(1.7.3) & \quad \xi \cdot [x, y]' = [\xi, x] + [x, \xi] - (\xi \cdot x) \cdot y + (y \cdot \xi) \cdot x,
\end{align*}
\]

where \(a \in A, \ x, y \in L', \ \xi, \eta \in L''\).

An argument for the special case of this proposition where \(L'\) and \(L''\) are ordinary Lie algebras may be found in [20]. In fact, (1.7.2) and (1.7.3) then come down to (1.3.1) and (1.3.2) in [20]. More generally, the case where \(L'\) and \(L''\) arise from two Lie algebroids has been established in Theorem 4.2 of [31].

Proof. The bracket \((1.5.1)\) is plainly skew-symmetric. Hence the proof comes down to relating the Jacobi identity in \(L\) and the Lie-Rinehart compatibility properties with \((1.7.1) - (1.7.3)\).

Thus, suppose that the bracket \([·, ·]\) on \(L = L' \oplus L''\) given by \((1.5.1)\) satisfies the Jacobi identity. Then, with a slight abuse of the notation \([·, ·]\),

\[
x \cdot [\xi, \eta] - [\xi, \eta] \cdot x = [x, [\xi, \eta]]
\]

whence, comparing components in \(L'\) and \(L''\), we conclude

\[
x \cdot [\xi, \eta] = [x \cdot \xi, \eta] + [\xi, x \cdot \eta] - (\xi \cdot x) \cdot \eta + (\eta \cdot x) \cdot \xi
\]

that is, \((1.7.2)\) holds and, furthermore, \(·\) is a left \((A, L'')\)-module structure on \(L'\) (but this is true already by assumption). By symmetry, \((1.7.3)\) holds as well. Conversely, suppose that the two actions are related by \((1.7.2)\) and \((1.7.3)\). We can then read the above calculation backwards and conclude that the bracket \([·, ·]\) on \(L\) satisfies the Jacobi identity.

We leave the rest of the proof to the reader. The arguments given in [31] are actually formal and carry over. □

Theorem 1.7 thus gives a complete answer to Question 1.2, as well as to Question 1.1, as the following shows:

Corollary 1.8. For an integrable decomposition \(L = L' \oplus L''\) of an \((R, A)\)-Lie algebra \(L\), the resulting almost twilled pre-Lie-Rinehart algebra \((A, L', L'')\) is a true twilled Lie-Rinehart algebra.

They reader might ask: Why bother at all? The answer is this: We will show in Section 5 below that the additional structure relating the summands \(L'\) and
of an integrable decomposition explains in particular certain Batalin-Vilkovisky algebras related with the mirror conjecture.

Remark 1.9. Let \( g \) be an ordinary Lie algebra, finitely generated and projective over the ground ring \( R \), with Lie bracket \([\cdot, \cdot]\), let \( \Delta : g \to g \otimes_R g \) be a Lie coalgebra structure on \( g \), write \( \cdot, \cdot \) for the corresponding Lie bracket on the dual \( g^* \), and consider the pair \((g, g^*)\) together with the ordinary actions \( \cdot : g \otimes_R g^* \to g^* \) of \( g \) on \( g^* \) and \( \cdot : g^* \otimes_R g \to g^* \) of \( g^* \) on \( g \) induced by the Lie brackets on \( g \) and \( g^* \), respectively, the \( g \)- and \( g^* \)-actions on the ground ring \( R \) being taken trivial. Then (1.7.2) is equivalent to the customary requirement that \( \Delta \) be a 1-cocycle for \( g \) with values in \( g \otimes_R g \), that is, to

\[
\Delta[x, y] = x \cdot \Delta y - y \cdot \Delta x
\]

or, equivalently, to

\[
d_*[x, y] = [d_* x, y] + [x, d_* y]
\]

where, on the right-hand side, \([\cdot, \cdot]\) refers to the corresponding Gerstenhaber bracket on \( \Lambda_R g \); here \( d_* \) denotes the Chevalley-Eilenberg differential on \( \text{Alt}_R(g^*, R) \cong \Lambda_R g \). Likewise, (1.7.3) is equivalent to the requirement that the dual \( \Delta_* : g^* \to g^* \otimes_R g^* \) of the Lie bracket \([\cdot, \cdot]\) on \( g \) be a 1-cocycle for \( g^* \) with values in \( g^* \otimes_R g^* \), that is, to

\[
\Delta_*[\xi, \eta]_* = \xi \cdot \Delta_* \eta - \eta \cdot \Delta_* \xi
\]

or, equivalently, to

\[
d[\xi, \eta]_* = [d\xi, \eta]_* + [\xi, d\eta]_*
\]

where, on the right-hand side, \([\cdot, \cdot]_*\) refers to the corresponding Gerstenhaber bracket on \( \Lambda_R g^* \), and where \( d \) denotes the Chevalley-Eilenberg differential on \( \text{Alt}_R(g, R) \cong \Lambda_R g^* \). Moreover, (1.9.1) and (1.9.3) are equivalent as well. All these fact are nowadays well known.

Example 1.10. An ordinary Lie bialgebra \((g, g^*)\) is as well a twilled \((R, R)\)-Lie algebra, as the corresponding Manin triple shows.

However, given a Lie-Rinehart algebra \((A, L)\) together with an \((R, A)\)-Lie algebra structure on \( D = \text{Hom}_A(L, A) \), when the action of \( L \) on \( A \) (or that of \( D \) on \( A \), or that of both \( L \) and \( D \) on \( A \)) is non-trivial, (1.9.1) and (1.9.3) will not even make sense, and a twilled Lie-Rinehart algebra structure on \((A, L, D)\) will not satisfy the obvious generalizations of (1.9.2) or (1.9.4). In fact, the obvious generalizations of (1.9.2) or (1.9.4) lead to a different concept, that of what we will call a Lie-Rinehart bialgebra; see the end of Section 4 below and [15]. Lie-Rinehart bialgebras generalize Lie bialgebroids, introduced in [27].

There is yet another way to understand the integrability of a decomposition of a Lie-Rinehart algebra. To explain it, we reproduce briefly the Rinehart complex, having as module variable a graded object: Let \((A, L)\) be an (ungraded) Lie-Rinehart algebra. A graded \( A \)-module \( M \), together with a graded left \( L \)-module structure
\( L \otimes_R M \to M \) is said to be a \textit{graded (left) \((A,L)\)-module}, provided the actions are compatible, that is, for \( \alpha \in L, a \in A, m \in M \), we have

\begin{align*}
(a \alpha)(m) &= a(\alpha(m)), \\
(1.11.1) \\
\alpha(a m) &= a \alpha(m) + \alpha(a)m.
\end{align*}

(1.11.2)

When \( M \) is concentrated in degree zero, we simply talk about a \textit{(left) \((A,L)\)-module}. In particular, with the obvious structure, the algebra \( A \) itself is a \((A,L)\)-module. Given a graded \((A,L)\)-module \( M \), the \((bi)\)-graded \( R \)-multilinear alternating functions from \( L \) into \( M \) with the ordinary \textit{Cartan-Chevalley-Eilenberg} \cite{3} differential given by

\begin{equation}
(df)(\alpha_1, \ldots, \alpha_n) = \begin{cases} 
(-1)^n \sum_{i=1}^n (-1)^{i-1} \alpha_i(f(\alpha_1, \ldots, \overset{\wedge}{\alpha}_i, \ldots, \alpha_n)) \\
+ \sum_{j<k} (-1)^{n+j+k} f([\alpha_j, \alpha_k], \alpha_1, \ldots, \overset{\wedge}{\alpha}_j \ldots \overset{\wedge}{\alpha}_k, \ldots, \alpha_n)
\end{cases}
\end{equation}

constitute a \((graded) chain complex \( \text{Alt}_R(L,M) \) where as \text{usual} \( \overset{\wedge}{\cdot} \) indicates omission of the corresponding term. As observed first by \textit{Palais} \cite{32} (for the ungraded setting), the defining properties (0.1) and (0.2) of a \textit{Lie-Rinehart algebra} entail that the differential \( d \) on \( \text{Alt}_R(L,M) \) passes to an \( R \)-linear differential on the \((bi)\)graded \( A \)-submodule \( \text{Alt}_A(L,M) \) of \( A \)-multilinear functions, written

\begin{equation}
(1.11.4.1) \\
d: \text{Alt}_A(L,M) \to \text{Alt}_A(L,M),
\end{equation}

too, and referred to henceforth as \textit{Lie-Rinehart differential}; this differential will not be \( A \)-linear unless \( L \) acts trivially on \( A \), though. We will call the resulting \((co)chain complex

\begin{equation}
(1.11.4.2) \\
(\text{Alt}_A(L,M),d)
\end{equation}

the \textit{Rinehart complex of \( M \)-valued forms on \( L \)}; often we write this complex more simply in the form \( \text{Alt}_A(L,M) \). For \( M = A \), with its obvious \((A,L)\)-module structure, the differential \( d \) turns \( \text{Alt}_A(L,A) \) into a differential graded commutative \( R \)-algebra, and a general graded \((A,L)\)-module pairing \( M_1 \otimes_A M_2 \to M \) induces a \((bi)\)graded pairing

\begin{equation}
(1.11.5) \\
\text{Alt}_A(L,M_1) \otimes_R \text{Alt}_A(L,M_2) \to \text{Alt}_A(L,M)
\end{equation}

of \( R \)-chain complexes, in fact of differential graded \( \text{Alt}_A(L,A) \)-modules. The sign \((-1)^n \) in (1.11.3) has been introduced according to the customary \text{Eilenberg-Koszul} convention in differential homological algebra, since the Rinehart complex (1.11.4.2) involves graded objects. See also our paper \cite{13}. In the classical approach such a sign does not occur. More generally, given a graded \( A \)-module \( M \), a graded pairing \( L \otimes_R M \to M \), not necessarily a graded left \( L \)-module structure but still satisfying (1.11.1) and (1.11.2), is referred to as an \textit{(A,L)-connection}, cf. \cite{10}, or, somewhat more precisely, as a \textit{graded left \((A,L)\)-connection}; in this language, an \((A,L)\)-module
structure (or a graded one) is a flat \((A, L)-\text{connection}\) (or a graded one). Given a graded \(A\)-module \(M\), together with an \((A, L)\)-connection, we extend the definition of the Lie-Rinehart operator to an operator

\[
d: \text{Alt}_A(L, M) \to \text{Alt}_A(L, M)
\]

by means of the formula (1.11.3), with the \((A, L)\)-connection instead of the \((A, L)\)-action on \(M\). The resulting operator \(d\) is well defined; it is a differential if and only if the \((A, L)\)-connection on \(M\) is flat, i.e. a true \((A, L)\)-module structure.

Let \((A, L', L'')\) be an almost twilled pre-Lie-Rinehart algebra. Consider the bigraded \(A\)-module

\[
\text{Alt}_A^*(L'' \oplus L', A) \cong \text{Alt}_A^*(L'', \text{Alt}_A^*(L', A)).
\]

Henceforth we spell out a particular homogeneous constituent as

\[
\text{Alt}_A^q(L'', \text{Alt}_A^p(L', A)),
\]

keeping in mind that, under the circumstances of (1.3), when the almost complex structure is a true complex structure, the notations \(p\) and \(q\) have become standard for the “holomorphic” and “antiholomorphic” degrees, respectively; for intelligibility, we follow this convention, see below. The pairings (1.4.3) and (1.4.4) induce graded pairings

\[
L' \otimes_R \text{Alt}_A^*(L'', A) \to \text{Alt}_A^*(L'', A)
\]

\[
L'' \otimes_R \text{Alt}_A^*(L', A) \to \text{Alt}_A^*(L', A)
\]

on \(\text{Alt}_A^*(L'', A)\) and \(\text{Alt}_A^*(L', A)\), respectively, when (1.4.3) and (1.4.4) are formally treated like connections. Via (1.11.6), applied formally, that is, by a formal evaluation of the expression given on the right-hand side of (1.11.3), with (1.4.1') and (1.4.1'') instead of the Lie brackets, and (1.4.2') and (1.4.2'') instead of the requisite module structures, these pairings, in turn, induce two operators

\[
d': \text{Alt}_A^q(L'', \text{Alt}_A^p(L', A)) \to \text{Alt}_A^q(L'', \text{Alt}_A^{p+1}(L', A))
\]

\[
d'': \text{Alt}_A^q(L'', \text{Alt}_A^p(L', A)) \to \text{Alt}_A^{q+1}(L'', \text{Alt}_A^p(L', A)).
\]

A little thought reveals that, in view of (1.4.5'), (1.4.5''), (1.4.6'), (1.4.6''), (1.4.7), (1.4.8), these operators, which are at first defined only on the \(R\)-multilinear alternating functions, in fact pass to operators on \(A\)-multilinear alternating functions. Then the requirement that \(d = d' + d''\) be a differential, i.e. that \(dd = 0\), amounts to

\[
d'd' = 0
\]

\[
d''d'' = 0
\]

\[
[d', d''] = 0,
\]

where as usual \([d', d''] = d'd'' + d''d'\); in other words, \(d\) being a differential is equivalent to

\[
(\text{Alt}_A^*(L'', \text{Alt}_A^*(L', A)), d', d'')
\]
being a bicomplex.

An $A$-module $M$ will be said to have property $P$ provided for $x \in M$, $\phi(x) = 0$ for every $\phi : M \to A$ implies that $x$ is zero. For example, a projective $A$-module has property $P$, or a reflexive $A$-module has this property or, more generally, an $A$-module $M$ such that the canonical map from $M$ into its double $A$-dual is injective. On the other hand, for example, for a smooth manifold $X$, the $C^\infty(X)$-module $D$ of formal (= Kähler) differentials does not have property $P$: On the real line, with coordinate $x$, consider the functions $f(x) = \sin x$ and $g(x) = \cos x$. The formal differential $df - gdx$ is non-zero in $D$; however, the $C^\infty(X)$-linear maps from $D$ to $C^\infty(X)$ are the smooth vector fields, whence every such $C^\infty(X)$-linear map annihilates the formal differential $df - gdx$.

**Theorem 1.15.** If $(A, L', L'')$ is a twilled Lie-Rinehart algebra, (1.15.4) is a bicomplex which then necessarily computes the cohomology $H^r(\text{Alt}_A(L,A))$ of the twilled sum $L$ of $L'$ and $L''$. Conversely, $(A, L', L'')$ being an almost twilled pre-Lie-Rinehart algebra, if (1.15.4) is a bicomplex, and if $L'$ and $L''$ have property $P$, $(A, L', L'')$ is a true twilled Lie-Rinehart algebra.

**Proof.** If $(A, L', L'')$ is twilled Lie-Rinehart algebra, (1.15.4) is plainly a bicomplex which then necessarily computes the indicated cohomology. We now prove the converse. Thus suppose that (1.15.4) is a bicomplex. Consider the operator

$$d''d'' : \text{Alt}^j_A(L'', A) \to \text{Alt}^{j+2}_A(L'', A)$$

for $j = 0$ and $j = 1$. Notice that $\text{Alt}^j_A(L'', A)$ equals $\text{Alt}^j_A(L'', \text{Alt}^0_A(L', A))$ and that $\text{Alt}^{j+2}_A(L'', A)$ equals $\text{Alt}^{j+2}_A(L'', \text{Alt}^0_A(L', A))$. For $j = 1$, given $\xi, \eta, \vartheta \in L''$ and $\phi \in \text{Hom}_A(L'', A) = \text{Alt}^1_A(L'', A)$, we find

$$(d''d''\phi)(\xi, \eta, \vartheta) = \phi([[\xi, \eta]''', \vartheta]'' + [[\eta, \vartheta]''', \xi]'' + [[\vartheta, \xi]''', \eta]'').$$

Since $L''$ has property $P$, we conclude that the bracket on $L''$ satisfies the Jacobi identity, that is, $L''$ is an $R$-Lie algebra. Likewise, for $j = 0$, given $\xi, \eta \in L''$ and $a \in A$, we find

$$(d''d''a)(\xi, \eta) = \xi(\eta(a)) - \eta(\xi(a)) - [\xi, \eta](a).$$

Consequently the adjoint $L'' \to \text{Der}_R(A)$ of (1.4.2″) is a morphism of $R$-Lie algebras. In view of (1.4.5″) and (1.4.6″), we conclude that $(A, L'')$ is a Lie-Rinehart algebra. The same kind of reasoning shows that $(A, L')$ is a Lie-Rinehart algebra.

Next, consider the operator

$$d''d' : \text{Alt}^0_A(L'', \text{Alt}^1_A(L', A)) \to \text{Alt}^2_A(L'', \text{Alt}^1_A(L', A)).$$

We note that $\text{Alt}^0_A(L'', \text{Alt}^1_A(L', A)) = \text{Alt}^1_A(L', A) = \text{Hom}_A(L', A)$. Let $x \in L'$, $\xi, \eta \in L''$, and $\phi \in \text{Hom}_A(L', A)$. A straightforward calculation gives

$$(d''d'\phi)(\xi, \eta) = \phi(\eta \cdot (\xi \cdot x) - \xi \cdot (\eta \cdot x) + [\xi, \eta] \cdot x).$$

Since $L'$ is assumed to have property $P$, we conclude that, for every $x \in L'$, $\xi, \eta \in L''$, $[\xi, \eta] \cdot x = \xi \cdot (\eta \cdot x) - \eta \cdot (\xi \cdot x)$,
that is, (1.4.4) is a left \((A, L'')\)-module structure on \(L'\). The same kind of reasoning shows that (1.4.3) is a left \((A, L')\)-module structure on \(L''\).

Pursuing the same kind of reasoning, consider the operator

\[
d' d'' + d'' d' : A = \text{Alt}^0_A(L'', \text{Alt}^0_A(L', A)) \to \text{Alt}^1_A(L'', \text{Alt}^1_A(L', A)).
\]

Let \(a \in A, x \in L', \xi \in L''\). Again a calculation shows that

\[
((d' d'' + d'' d') a)(\xi, x) = \xi(x(a)) - x(\xi(a)) - ((\xi \cdot x)(a) - (x \cdot \xi)(a))
\]

whence the vanishing of \(d' d'' + d'' d'\) in bidegree \((0,0)\) entails the compatibility property (1.7.1). Likewise consider the operator

\[
d' d'' + d'' d' : \text{Hom}_A(L'', A) = \text{Alt}^1_A(L'', \text{Alt}^0_A(L', A)) \to \text{Alt}^2_A(L'', \text{Alt}^1_A(L', A)).
\]

Again a calculation shows that, for \(x \in L', \xi, \eta \in L'', \phi \in \text{Hom}_A(L'', A),\)

\[
((d' d'' + d'' d') \phi)(\xi, x) = \phi(x[x, \xi] + [\xi, x \cdot \eta] - (\xi \cdot x) \cdot \eta + (\eta \cdot x) \cdot \xi)
\]

whence the vanishing of \(d' d'' + d'' d'\) in bidegree \((1,0)\) entails the compatibility property (1.7.2). Likewise, the vanishing of \(d' d'' + d'' d'\) in bidegree \((0,1)\) entails the compatibility property (1.7.3). 

\[\square\]

**Example 1.3** (continued). An almost complex structure determines an almost twilled pre-Lie-Rinehart algebra \((A, L', L'')\), and the almost complex structure is integrable if and only if \(L = L'' \oplus L'\) is an integrable decomposition (whence the terminology). In the integrable case, the operator \(d''\)—the corresponding Cauchy-Riemann operator—defines a holomorphic structure on the manifold \(M\), and the bigraded object

\[
\text{Alt}^*_A(L'' \oplus L', A) \cong \text{Alt}^*_A(L'', \text{Alt}^*_A(L', A))
\]

amounts to the smooth complex valued forms of type \((0,*)\) with values in the exterior powers of the holomorphic cotangent bundle. The resulting bicomplex \(\text{Alt}^*_A(L'', (\text{Alt}^*_A(L', A), d'), d'')\) is the customary Dolbeault complex computing the sheaf hypercohomology of \(M\) with values in the complex of sheaves of germs of holomorphic differential forms on \(M\) and, by virtue of the Poincaré lemma, the total complex of the Dolbeault complex yields a resolution of the constant sheaf of complex numbers whence the cohomology of the Dolbeault complex coincides with the ordinary smooth complex valued (de Rham) cohomology of \(M\), viewed as a real manifold. All this is classical, cf. e. g. [9]. Our description in terms of Lie-Rinehart structures seems to be new, though.

Thus, twilled Lie-Rinehart algebras generalize complex manifolds in the same sense as Lie bialgebroids or more generally Lie-Rinehart bialgebras (see Section 4 below or [15]) generalize Poisson and in particular symplectic structures. Almost twilled pre-Lie-Rinehart algebras have been spelled out above as the exact analogue of almost complex structures. In the rest of the paper, almost twilled pre-Lie-Rinehart algebras will no longer come into play explicitly and only almost twilled and twilled Lie-Rinehart algebras will be considered.
2. Differential graded Lie-Rinehart algebras

There are various concepts of differential graded Lie algebras in the literature. To introduce notation, we reproduce a description tailored to our purposes. To simplify the exposition somewhat, we will assume that the primes 2 and 3 are invertible in the ground ring \( R \). If \( x \) is an element in a graded module then \( |x| \) denotes its degree.

Let \( L \) be a chain complex over \( R \), and let 
\[
\llbracket \cdot, \cdot \rrbracket : L \otimes_R L \to L
\]
be a pairing of chain complexes of degree zero. We will say that \((L, \llbracket \cdot, \cdot \rrbracket)\) is a differential graded Lie algebra provided it is skew-symmetric in the graded sense and satisfies the graded Jacobi identity, that is,
\[
\llbracket x, y \rrbracket = - (-1)^{|y||x|} \llbracket y, x \rrbracket,
\]
for all \( x, y \) in \( L \), \((2.1.1)\)
\[
\llbracket \llbracket x, y \rrbracket, z \rrbracket = \llbracket \llbracket x, y \rrbracket, z \rrbracket + (-1)^{|x||y|} \llbracket y, \llbracket x, z \rrbracket \rrbracket,
\]
for all \( x, y, z \) in \( L \). \((2.1.2)\)

Here are two immediate consequences of the definition:
\[
\llbracket x, x \rrbracket = 0,
\]
for all homogeneous \( x \) in \( L \) of even degree, \((2.1.1.1)\)
\[
\llbracket x, \llbracket x, x \rrbracket \rrbracket = 0,
\]
for all homogeneous \( x \) in \( L \) of odd degree. \((2.1.2.1)\)

The pairing \( \llbracket \cdot, \cdot \rrbracket \) is what is called a (graded) Lie bracket. Given two differential graded Lie algebras \( L \) and \( L' \), a morphism \( \phi : L \to L' \) of differential graded Lie algebras over \( R \) is the obvious thing, i. e. it is a morphism of chain complexes which is compatible with the graded Lie brackets. We note that, when 2 is not invertible in the ground ring, there are two non-equivalent notions of graded Lie algebra depending on whether or not \((2.1.1.1)\) is required to hold and, likewise, when 3 is not invertible in the ground ring, \((2.1.2.1)\) is an additional requirement.

For a differential graded algebra \( U \) over \( R \), the associated differential graded Lie algebra over \( R \), written \( LU \) or, with an abuse of notation, just \( U \), has the same underlying chain complex as \( U \), while its bracket \( \llbracket \cdot, \cdot \rrbracket \) is given by
\[
\llbracket u, v \rrbracket = uv - (-1)^{|u||v|} vu,
\]
for \( u, v \) in \( U \). \((2.2)\)

Whenever we say that a differential graded algebra is viewed as a differential graded Lie algebra, this structure will be the intended one. In particular, for a chain complex \( M \), the object \( \text{End}_R(M) \) is a differential graded algebra, and hence \( L(\text{End}_R(M)) \) is a differential graded Lie algebra. Furthermore, if \( L \) is a differential graded Lie algebra and \( M \) a chain complex, a differential graded \( L \)-module structure on \( M \) is a morphism \( L \to L(\text{End}_R(M)) \) of differential graded Lie algebras.

Let \( U \) be a differential graded algebra over \( R \). Recall that a (homogeneous) derivation of \( U \) is a (homogeneous) morphism \( \delta : U \to U \) of chain complexes so that for \( u, v \in U \),
\[
\delta(uv) = (\delta(u))v + (-1)^{|u||\delta|} u\delta(v).
\]
\((2.3)\)
The graded submodule $\text{Der}(U)$ of derivations of $U$ is a graded submodule of $\text{End}_R(U)$; moreover, it inherits a differential from the latter, and it is well known that the bracket (2.2) induces a bracket

$\langle \cdot, \cdot \rangle : \text{Der}(U) \otimes_R \text{Der}(U) \rightarrow \text{Der}(U)$

for $\text{Der}(U)$ which turns $\text{Der}(U)$ into a differential graded Lie algebra over $R$. Further, if $L$ is a differential graded Lie algebra over $R$ and if $U$ is a differential graded $R$-algebra, as usual, a morphism $L \rightarrow \text{Der}(U)$ of differential graded Lie algebras over $R$ is called an action of $L$ on $U$ (by derivations); on elements, we will always write the adjoint $L \otimes_R U \rightarrow U$ of an $L$-action on $U$ in the form $\alpha \otimes_R x \mapsto \alpha(x)$, $\alpha \in L$, $x \in U$.

Given two differential graded algebras $U$ and $U'$ over $R$, differential graded Lie algebras $L$ and $L'$ over $R$, and actions of $L$ and $L'$ on $U$ and $U'$ respectively, a morphism

$$(\phi, \psi) : (U, L) \rightarrow (U', L')$$

(of actions) is the obvious thing, i.e. it consists of a morphism $\phi : U \rightarrow U'$ of differential graded $R$-algebras and a morphism $\psi : L \rightarrow L'$ of differential graded Lie algebras over $R$, so that the diagram

$$
\begin{array}{ccc}
L \otimes_R U & \longrightarrow & U \\
\psi \otimes_R \phi & \downarrow & \phi \\
L' \otimes_R U' & \longrightarrow & U'
\end{array}
$$

is commutative; here the unlabelled horizontal arrows refer to the corresponding structure maps.

Given a differential graded Lie algebra $L$ and a chain complex $M$ over $R$, as usual, a morphism $L \rightarrow L \text{End}(M)$ of differential graded Lie algebras over $R$ is called an action of $L$ on $M$, and $M$ is said to be a differential graded (left) $L$-module; we will always write the adjoint $L \otimes_R M \rightarrow M$ in the form $\alpha \otimes_R x \mapsto \alpha(x)$, $\alpha \in L$, $x \in M$. The precise definition of the concept of a morphism of differential graded $L$-modules is obvious and left to the reader.

We now generalize the notion of Lie-Rinehart algebra to that of differential graded Lie-Rinehart algebra. For intelligibility, ordinary (ungraded) Lie-Rinehart algebras will be denoted by $(A, L)$ etc. and differential graded Lie-Rinehart algebras by $(\mathcal{A}, \mathcal{L})$, etc.

Let $\mathcal{A}$ be a differential graded commutative $R$-algebra, let $\mathcal{L}$ be a differential graded Lie algebra over $R$, let $\mathcal{A} \otimes_R \mathcal{L} \rightarrow \mathcal{L}$ be a differential graded left $\mathcal{A}$-module structure on $\mathcal{L}$, written $a \otimes_R \alpha \mapsto a \alpha$, and let $\mathcal{L} \rightarrow \text{Der}(\mathcal{A})$ be an action of $\mathcal{L}$ on $\mathcal{A}$ whose adjoint $\mathcal{L} \otimes_R \mathcal{A} \rightarrow \mathcal{A}$ is written $\alpha \otimes_R a \mapsto \alpha(a)$, $\alpha \in \mathcal{L}$, $a \in \mathcal{A}$. We will refer to $\mathcal{L}$ as a differential graded $(R, \mathcal{A})$-Lie algebra, provided

$$(2.5.a) \quad (a \alpha)(b) = a(\alpha(b)), \quad \alpha \in \mathcal{L}, \ a, b \in \mathcal{A},$$
$$(2.5.b) \quad [\alpha, a \beta] = (-1)^{|a||\alpha|} a[\alpha, \beta] + \alpha(a)\beta, \quad \alpha, \beta \in \mathcal{L}, \ a \in \mathcal{A}.$$
Extending terminology introduced in our paper [10] (for the ungraded case), we will refer to a pair \((\mathcal{A}, \mathcal{L})\), where \(\mathcal{A}\) is a differential graded commutative algebra and \(\mathcal{L}\) a differential graded \((R, \mathcal{A})\)-Lie algebra, as a differential graded Lie-Rinehart algebra. An example of a differential graded Lie-Rinehart algebra is the pair \((\mathcal{A}, \text{Der}(\mathcal{A}))\), where \(\mathcal{A}\) is a differential graded commutative algebra and \(\text{Der}(\mathcal{A})\) the differential graded \(\mathcal{A}\)-module of graded derivations of \(\mathcal{A}\), with the obvious structures.

Given two differential graded Lie-Rinehart algebras \((\mathcal{A}, \mathcal{L})\) and \((\mathcal{A}', \mathcal{L}')\), a morphism \((\phi, \psi): (\mathcal{A}, \mathcal{L}) \to (\mathcal{A}', \mathcal{L}')\) of differential graded Lie-Rinehart algebras is the obvious thing, that is, it is a morphism of actions in the above sense so that, in addition, \(\psi: \mathcal{L} \to \mathcal{L}'\) is a morphism of differential graded left \(\mathcal{A}\)-modules where \(\mathcal{A}\) acts on \(\mathcal{L}'\) via \(\phi\).

Let \((\mathcal{A}, \mathcal{L})\) be a differential graded Lie-Rinehart algebra and let \(M\) be a chain complex over \(R\) having a differential graded left \(\mathcal{A}\)-module structure and, furthermore, a differential graded left \(\mathcal{L}\)-module structure. Then \(M\) is said to be a differential graded (left) \((\mathcal{A}, \mathcal{L})\)-module, provided the actions are compatible, that is, for \(\alpha \in \mathcal{L}, x \in \mathcal{A}, m \in M\), we have

\[
(2.6.a) \quad (a \alpha)(m) = a(\alpha(m)),
\]

\[
(2.6.b) \quad \alpha(a \cdot m) = (-1)^{\vert \alpha \vert \cdot \vert a \vert} a \alpha(m) + \alpha(a) \cdot m.
\]

In particular, with the obvious structure, the differential graded algebra \(\mathcal{A}\) itself is a differential graded (left) \((\mathcal{A}, \mathcal{L})\)-module. Furthermore, there is an obvious notion of morphism of modules over differential graded Lie-Rinehart algebras; we leave the details to the reader.

For a differential graded Lie algebra \(L\) over \(R\), given differential graded (left) \(L\)-modules \(M'\) and \(M''\), the customary formula

\[
(2.7) \quad \alpha(x \otimes_R y) = \alpha(x) \otimes_R y + (-1)^{\vert \alpha \vert \cdot \vert x \vert} x \otimes_R \alpha(y), \quad \alpha \in L, x \in M', y \in M'',
\]

endows the differential graded tensor product \(M' \otimes_R M''\) with a differential graded (left) \(L\)-module structure; this is just the ordinary (differential graded) tensor product \(L\)-module structure. If \(M\) is another differential graded \(L\)-module, a pairing \(\mu: M' \otimes_R M'' \to M\) of \(R\)-modules which is a morphism of differential graded \(L\)-modules (with respect to (2.7)) will be said to be a pairing of differential graded \(L\)-modules. For an ungraded Lie-Rinehart algebra \((A, L)\), viewed as a differential graded Lie-Rinehart algebra concentrated in degree zero with zero differential, given differential graded \((A, L)\)-modules \(M'\) and \(M''\), a little thought reveals that the formula (2.7) turns the (graded) tensor product \(M' \otimes_A M''\) into a differential graded \((A, L)\)-module; we refer to \(M' \otimes_A M''\) with this structure as the tensor product of \(M'\) and \(M''\) in the category of differential graded \((A, L)\)-modules. Given differential graded \((A, L)\)-modules \(M, M', \) and \(M'',\) a pairing \(\mu_A: M' \otimes_A M'' \to M\) of \(A\)-modules which is compatible with the differential graded \(L\)-structures will be said to be a pairing of differential graded \((A, L)\)-modules. See our paper [13] for more details.

2.8. THE GRADED CROSSED PRODUCT EXTENSION. For later reference, we reproduce briefly a description of the graded crossed product Lie-Rinehart algebra extension...
tailored to our purposes; see [10] for the ungraded case. Let \((A, L)\) be a Lie-Rinehart algebra, and let \(A\) be a graded commutative \(A\)-algebra which is endowed with a graded \((A, L)\)-module structure in such a way that (i) \(L\) acts on \(A\) by derivations—this is equivalent to requiring the structure map from \(A \otimes_A A\) to \(A\) to be a morphism of graded \((A, L)\)-modules—and that (ii) the canonical map from \(A\) to \(A\) is a morphism of left \((A, L)\)-modules. Let \(\mathcal{L} = A \otimes_A L\), and define a bigraded bracket

\[
[\cdot, \cdot] : \mathcal{L} \otimes_R \mathcal{L} \to \mathcal{L}
\]

of bidegree \((0, -1)\) by means of the formula

\[
[\alpha \otimes_A x, \beta \otimes_A y] = (\alpha \beta) \otimes_A [x, y] + \alpha(x \cdot \beta) \otimes_A y - (-1)^{|\alpha||\beta|}\beta(y \cdot \alpha) \otimes_A x
\]

where \(\alpha, \beta \in A\) and \(x, y \in L\). A calculation shows that, for every \(\beta \in A\) and every \(x, y, z \in L\),

\[
[[x, y], \beta \otimes_A z] - ([x, [y, \beta \otimes_A z]] - [y, [x, \beta \otimes_A z]]) = ([x, y](\beta) - x(y(\beta)) - y(x(\beta))) \otimes_A z,
\]

whence (2.8.1) being a graded Lie bracket is actually equivalent to the structure map \(L \otimes_R A \to A\) being a Lie algebra action. Moreover, let

\[
A \otimes_R \mathcal{L} \to \mathcal{L}
\]

be the obvious graded left \(A\)-module structure arising from extension of scalars, that is from extending \(L\) to a (graded) \(A\)-module, and define a pairing

\[
\mathcal{L} \otimes_R A \to A
\]

by

\[
(\alpha \otimes_A x) \otimes_R \beta \mapsto (\alpha \otimes_A x)(\beta) = \alpha(x(\beta)).
\]

Then \((A, \mathcal{L})\), together with (2.8.1), (2.8.3) and (2.8.4), constitutes a graded Lie-Rinehart algebra. We refer to \((A, \mathcal{L})\) as the crossed product of \(A\) and \((A, L)\) and to the corresponding \((R, A)\)-Lie algebra \(\mathcal{L}\) as the crossed product of \(A\) and \(L\).

**Remark 2.8.6.** We must be a little circumspect here: The three terms on the right-hand side of (2.8.2) are not well defined individually; only their sum is well defined. For example, if we take \(ax\) instead of \(x\), where \(a \in A\), on the left-hand side, \(\alpha \otimes_A (ax)\) equals \((\alpha a) \otimes_A x\) but \((\alpha \beta) \otimes_A [ax, y]\) differs from \((\alpha a \beta) \otimes_A [x, y]\).

(2.9) A special case of the differential graded crossed product arises as follows: Let \((A, L)\) be a Lie-Rinehart algebra, let \(M\) be a left \((A, L)\)-module, and consider the graded \(A\)-algebra \(A = \text{Alt}_A(M, A)\), endowed with the induced left \((A, L)\)-module structure; this is in fact an \(L\)-action on \(A\) by derivations. We then have the crossed product \((R, A)\)-Lie algebra \(\mathcal{L}\) which, as a graded \(A\)-module, has the form \(A \otimes_A L\). When \(L\) is finitely generated and projective, the canonical morphism

\[
A \otimes_A L \to \text{Alt}_A(M, L)
\]
is an isomorphism, and (2.8.3) comes down to the ordinary shuffle pairing

\[(2.9.2) \quad \text{Alt}_A(M, A) \otimes_R \text{Alt}_A(M, L) \to \text{Alt}_A(M, L).\]

### 3. The integrability condition reexamined

Let \((A, L', L'')\) be an almost twilled Lie-Rinehart algebra. Suppose that, as an \(A\)-module, \(L'\) is finitely generated and projective. Let \(A'' = \text{Alt}^*_A(L'', A)\) and, with reference to the left \((A, L')\)-module structure (1.4.3) on \(L''\), consider the graded crossed product Lie-Rinehart algebra

\[(3.1) \quad (A'', L') = (\text{Alt}^*_A(L'', A), \text{Alt}^*_A(L'', A) \otimes_A L') \cong (\text{Alt}^*_A(L'', A), \text{Alt}^*_A(L'', L'))\]

explained in (2.9), \(L''\) playing the role of \(M\) in (2.9). The left \((A, L'')\)-module structure (1.4.4) on \(L''\) induces the corresponding Lie-Rinehart operator \(d''\) (cf. 1.11.4.1) on \(L''\) and, with reference to the graded left \(A''\)-module structure (2.8.3) on \(L''\), \(L''\) is a differential graded \(A''\)-module, \(A''\) being endowed with the ordinary Lie-Rinehart differential (1.11.4.1).

**Theorem 3.2.** Under these circumstances, the following are equivalent.

(i) \((A, L', L'')\) is a twilled Lie-Rinehart algebra.

(ii) \((L', [\cdot, \cdot], d'')\) is a differential graded \(R\)-Lie algebra.

(iii) \((A'', L'; d'')\) is a differential graded Lie-Rinehart algebra.

We note that, when \(L''\) is finitely generated and projective as an \(A\)-module, with the roles of \(L'\) and \(L''\) interchanged, the same statements as those spelled out in Theorem 3.2 are true. Under the circumstances of Theorem 3.2, we will refer to \((A'', L'; d'')\) as a differential graded crossed product Lie-Rinehart algebra. Thus the theorem says that, provided \(L'\) is finitely generated and projective as an \(A\)-module, twilled Lie-Rinehart algebras and differential graded crossed product Lie-Rinehart algebras are equivalent notions.

The proof requires some preparation.

**Lemma 3.3.** Given \(x\) and \(y\) in \(L'

\[(3.3.1') \quad d''[x, y]' = [d''x, y]' + [x, d''y]'\]

if and only if

\[\xi \cdot [x, y]' = [\xi \cdot x, y]' + [x, \xi \cdot y]' - (x \cdot \xi) \cdot y + (y \cdot \xi) \cdot x\]

for every \(\xi \in L''\). Consequently the truth of (3.3.1') for every \(x, y \in L'\) is equivalent to the compatibility condition (1.7.3).

**Proof.** Let \(x, y \in L'\) and write

\[d''x = \sum \alpha_i \otimes_A x_i \in \text{Alt}_A(L'', A) \otimes_A L'\]
\[d''y = \sum \alpha_j \otimes_A x_j \in \text{Alt}_A(L'', A) \otimes_A L'.\]
Then
\[ [d''x, y]' = \sum [\alpha_i \otimes_A x_i, y]' = \sum \alpha_i \otimes_A [x_i, y]' - (y \cdot \alpha_i) \otimes_A x_i \]
\[ [x, d''y]' = \sum [x, \beta_j \otimes_A y_j]' = \sum \beta_j \otimes_A [x, y_j]' + (x \cdot \beta_j) \otimes_A y_j \]

Thus, given \( \xi \in L'' \), we have
\[
[d''x, y]'(\xi) = \sum \alpha_i(\xi) \otimes_A [x_i, y]' - ((y \cdot \alpha_i)(\xi)) \otimes_A x_i
\]
\[
= \sum \alpha_i(\xi) \otimes_A [x_i, y]' - y(\alpha_i(\xi)) \otimes_A x_i + \alpha_i(y \cdot \xi) \otimes_A x_i
\]
\[
= [(d''x)(\xi), y]' + \sum (\alpha_i(y \cdot \xi)) \otimes_A x_i
\]
\[
= [(d''x)(\xi), y]' + (d''x)(y \cdot \xi) = [\xi \cdot x, y]' + (y \cdot \xi) \cdot x
\]
\[
[x, d''y]'(\xi) = \sum \beta_j(\xi) \otimes_A [x, y_j]' + ((x \cdot \beta_j)(\xi)) \otimes_A y_j
\]
\[
= \sum \beta_j(\xi) \otimes_A [x, y_j]' + x(\beta_j(\xi)) \otimes_A y_j - (\beta_j(x \cdot \xi)) \otimes_A y_i
\]
\[
= [x, (d''y)(\xi)]' - \sum (\beta_j(x \cdot \xi)) \otimes_A y_i
\]
\[
= [x, (d''y)(\xi)]' - (d''y)(x \cdot \xi) = [x, \xi \cdot y]' - (x \cdot \xi) \cdot y
\]

Consequently
\[
([d''x, y]' + [x, d''y]')(\xi) = [\xi \cdot x, y]' + (y \cdot \xi) \cdot x + [x, \xi \cdot y]' - (x \cdot \xi) \cdot y
\]
\[
= [\xi \cdot x, y]' + [x, \xi \cdot y]' - (x \cdot \xi) \cdot y + (y \cdot \xi) \cdot x,
\]

On the other hand
\[
(d''[x, y]')(\xi) = \xi \cdot [x, y]'.
\]

Hence
\[
(d''[x, y]' - [d''x, y]' - [x, d''y]')(\xi) = \xi \cdot [x, y]' - ([\xi \cdot x, y]' + [x, \xi \cdot y]' - (x \cdot \xi) \cdot y + (y \cdot \xi) \cdot x)
\]

This completes the proof of the Lemma. \( \square \)

For later reference, from the proof of Lemma 3.3, we record the following formulas
\[
(3.3.2') \quad [d''x, y]'(\xi) = [\xi \cdot x, y]' + (y \cdot \xi) \cdot x
\]
\[
(3.3.3') \quad [x, d''y]'(\xi) = [x, \xi \cdot y]' - (x \cdot \xi) \cdot y,
\]
where \( x, y \in L' \) and \( \xi \in L'' \).
Lemma 3.4. Given \( x \in L' \), the following are equivalent.

(i) For every homogeneous \( \alpha \in A'' = \text{Alt}_A(L'', A) \),

\[
(3.4.1') \quad d''(x \cdot \alpha) = (d''x) \cdot \alpha + x \cdot (d''\alpha).
\]

(ii) For every \( a \in A \) and every \( \xi \in L'' \),

\[
\xi(x(a)) - x(\xi(a)) = (\xi \cdot x - x \cdot \xi)(a)
\]

and, for every \( \xi, \eta \in L'' \), and every \( \beta \in \text{Alt}_A^1(L'', A) = \text{Hom}_A(L'', A) \),

\[
\beta(x \cdot [\xi, \eta] - ([x \cdot \xi, \eta] + [\xi, x \cdot \eta] - (\xi \cdot x) \cdot \eta + (\eta \cdot x) \cdot \xi)) = 0
\]

Consequently, (3.4.1') holds for every \( x \in L' \) and every \( \alpha \in \text{Alt}_A(L'', A) \) if and only if the compatibility conditions (1.7.1) and (1.7.2) are satisfied.

Proof. Let \( \xi \in L'' \), and \( a \in A \). Then

\[
(d''(x(a)))(\xi) = \xi(x(a))
\]

\[
((d''x)(a))(\xi) = ((d''x)(\xi))(a) = (\xi \cdot x)(a)
\]

\[
(x \cdot (d''a))(\xi) = x((d''a)(\xi)) - (d''a)(x \cdot \xi) = x(\xi(a)) - (x \cdot \xi)(a)
\]

whence

\[
(d''(x(a)) - (d''x)(a) - x \cdot (d''a))(\xi)
\]

\[
= \xi(x(a)) - x(\xi(a)) - (\xi \cdot x - x \cdot \xi)(a)
\]

Consequently, given \( a \in A \),

\[
d''(x(a)) = (d''x)(a) + x \cdot (d''a)
\]

if and only if

\[
\xi(x(a)) - x(\xi(a)) = (\xi \cdot x - x \cdot \xi)(a)
\]

for every \( \xi \in L'' \). Thus the statement of the Lemma is true when \( \alpha \) has degree 0.

Let \( \xi, \eta \in L'' \), and \( \beta \in \text{Alt}_A^1(L'', A) = \text{Hom}_A(L'', A) \). Then

\[
(x \cdot \beta)(\xi) = x(\beta(\xi)) - \beta(x \cdot \xi)
\]

\[
(d''(x \cdot \beta))(\xi, \eta) = \xi((x \cdot \beta)(\eta)) - \eta((x \cdot \beta)(\xi)) - (x \cdot \beta)[\xi, \eta]
\]

\[
= \xi(x(\beta(\eta))) - \beta(x \cdot \eta)) - \eta(x(\beta(\xi)) - \beta(x \cdot \xi))
\]

\[
- x(\beta[\xi, \eta]) + \beta(x \cdot [\xi, \eta])
\]

\[
((d''x) \cdot \beta)(\xi, \eta) = ((d''x)(\xi)) \cdot \beta(\eta) - ((d''x)(\eta)) \cdot \beta(\xi)
\]

\[
= (\xi \cdot x \cdot \beta(\eta)) - (\eta \cdot x \cdot \beta(\xi))
\]

\[
= (\xi \cdot x)(\beta(\eta)) - \beta((\xi \cdot x) \cdot \eta) - (\eta \cdot x)(\beta(\xi)) + \beta((\eta \cdot x) \cdot \xi)
\]

\[
(x \cdot (d''\beta))(\xi, \eta) = x((d''\beta)(\xi, \eta)) - (d''\beta)(x \cdot \xi, \eta) - (d''\beta)(\xi, x \cdot \eta)
\]

\[
= x(\xi(\beta(\eta)) - \eta(\beta(\xi)) - \beta[\xi, \eta])
\]

\[
- (x \cdot \xi)(\beta(\eta)) + (x \cdot \eta)(\beta(\xi)) + \beta[x \cdot \xi, \eta]
\]

\[
- \xi(\beta(x \cdot \eta)) + (x \cdot \eta)(\beta(\xi)) + \beta[\xi, x \cdot \eta]
\]
A straightforward comparison of terms gives
\[ (d''(x \cdot \beta) - (d''x) \cdot \beta - x \cdot (d''\beta))(\xi, \eta) \]
\[ = \xi(x(\beta(\eta))) - x(\xi(\beta(\eta))) - (\xi \cdot x - x \cdot \xi)(\beta(\eta)) \]
\[ - (\eta(x(\beta(\xi))) - x(\eta(\beta(\xi))) - (\eta \cdot x - x \cdot \eta)(\beta(\xi))) \]
\[ + \beta(x \cdot [\xi, \eta] - ([x \cdot \xi, \eta] + [\xi, x \cdot \eta] - (\xi \cdot x) \cdot \eta + (\eta \cdot x) \cdot \xi)) \]
This shows that the statement of the Lemma is true when \( \alpha \) has degree 1. Since for two homogeneous elements \( \alpha_1, \alpha_2 \) of \( A'' \)
\[ x(\alpha_1\alpha_2) = (x(\alpha_1))\alpha_2 + \alpha_1(x\alpha_2) \]
the statement of the Lemma is true for homogeneous \( \alpha \) of arbitrary degree. \( \square \)

Proof of Theorem 3.2. Suppose first that \((A, L', L'')\) is a twilled Lie-Rinehart algebra, that is to say, the compatibility conditions (1.7.1) – (1.7.3) are satisfied. Then, by Lemma 3.3, the identity (3.3.1) holds for every \( x, y \in L \) and, by Lemma 3.4, the identity (3.4.1) holds for every \( x \in L \) and every homogeneous \( \alpha \in A'' \). Since as a graded \( A'' = \text{Alt}_A(L'', A) \)-module, in fact as a differential graded \( A'' \)-module, \( L' = \text{Alt}_A(L'', L') \) is generated by \( L' \), this implies that \( L' \) is a differential graded \( R \)-Lie algebra. In fact, a straightforward calculation involving (3.3.1) and (3.4.1) shows that, given \( \alpha, \beta \in A'' \) and \( x, y \in L' \), in view of (2.8.2),
\[ (3.2.1) \quad d''[\alpha \otimes_A x, \beta \otimes_A y]' = [d''(\alpha \otimes_A x), \beta \otimes_A y]' + (-1)^{[\alpha]}[\alpha \otimes_A x, d''(\beta \otimes_A y)]'. \]
Moreover, the truth of the identity (3.4.1) for every \( x \in L \) and every homogeneous \( \alpha \in A'' \) implies that \((A'', L')\) is a differential graded Lie-Rinehart algebra. Conversely, suppose that \( L' \) is a differential graded \( R \)-Lie algebra. Then (3.2.1) is manifestly true for every \( \alpha, \beta \in A'' \) and \( x, y \in L' \). Thus the identity (3.3.1) then holds a fortiori for every \( x, y \in L' \), and from Lemma 3.3 we deduce at once that the compatibility condition (1.7.1) holds. Furthermore, a straightforward comparison of terms involving only the differential graded \( A'' \)-module structure and the identity (3.3.1) gives, for \( x, y \in L' \) and homogeneous \( \alpha \in A'' \),
\[ d''[1 \otimes_A x, \alpha \otimes_A y]' - [d''(1 \otimes_A x), \alpha \otimes_A y]' - [1 \otimes_A x, d''(\alpha \otimes_A y)]' \]
\[ = (d''(x \cdot \alpha) - (d''x) \cdot \alpha) - x \cdot (d''(\beta)) y. \]
Since \( d''[1 \otimes_A x, \alpha \otimes_A y]' - [d''(1 \otimes_A x), \alpha \otimes_A y]' - [1 \otimes_A x, d''(\alpha \otimes_A y)]' \) is actually zero by assumption, \( L' \) being finitely generated and projective as an \( A \)-module, we conclude
\[ d''(x \cdot \alpha) = (d''x) \cdot \alpha + x \cdot (d''\alpha) \]
for every \( x \in L' \) and every homogeneous \( \alpha \in A'' \). Thus the identity (3.4.1) holds for every \( x \in L \) and every homogeneous \( \alpha \in \text{Alt}_A(L'', A) \). From Lemmata 3.3 and 3.4 we deduce at once that the compatibility properties (1.7.1) – (1.7.3) hold. \( \square \)

As for the identity (3.2.1), we must again be a bit circumspect: a comment of formally the same kind as that spelled out in Remark 2.8.6 is to be made here.

We remind the reader that the property P has been introduced before Theorem 1.15.
Corollary 3.5. Under the circumstances of (3.2), if in addition $L''$ has property $P$, each of the three equivalent statements in (3.2) is equivalent to the operator $[d', d'']$ in (1.15.4) being zero.

Proof. This is an immediate consequence of Theorems 1.15 and 3.2. □

Example 3.6. We return to the circumstances of Example 1.3 and consider the twilled Lie-Rinehart algebra $(A, L', L'')$ arising from a complex structure on a smooth manifold $M$. Thus, to adjust the notation, the Cauchy-Riemann operator $\overline{\partial}$ now being identified with the operator $d''$, $L'$ and $L''$ are the spaces of sections of the holomorphic and antiholomorphic tangent bundles of $M$, respectively. The resulting differential graded Lie algebra

$$(\mathcal{L}', [\cdot, \cdot]', d'') = (\text{Alt}_A(L'', L'), [\cdot, \cdot]', d'')$$

spelled out in Theorem 3.2 is what is called the Kodaira-Spencer algebra, cf. e. g. p. 337 of [8]; it controls the infinitesimal deformations of the complex structure of $M$. In particular, $\mathcal{L}'$ is the space of $\overline{\partial}$-forms with values in the holomorphic tangent bundle of $M$, and the differential $d''$ in $\mathcal{L}'$ is the ordinary Cauchy-Riemann operator, more customarily written $\overline{\partial}$. Further, the Lie bracket is given by the formula (2.8.2) above; this bracket is not given just by the shuffle product of $\overline{\partial}$-forms and the Lie-bracket of sections of the holomorphic tangent bundle! Cf. what is said at various places in the literature. In fact, such a bracket would not even be well defined since the Lie bracket of vector fields is not a tensor. It is also worthwhile pointing that, in view of Theorem 3.2, the compatibility properties defining part of the structure of the Kodaira-Spencer algebra are equivalent to the integrability condition of the initially only almost complex structure.

Remark 3.7. By means of a suitable generalization of the graded crossed product Lie-Rinehart algebra extension to the case where, in the notation of (2.9), the left $(A, L)$-module $M$ is no longer assumed to be finitely generated and projective as an $A$-module, in a follow-up paper [14] we will prove that the statement of Theorem 3.2 still holds without the hypothesis that $L'$ be finitely generated and projective as an $A$-module. Thus the bijective correspondence between twilled Lie-Rinehart and differential graded crossed product Lie-Rinehart structures is valid in general.

4. Differential Gerstenhaber algebras

The ground ring $R$ being fixed, recall that a Gerstenhaber algebra is a graded commutative $R$-algebra $A$ together with a graded Lie bracket from $A \otimes_R A$ to $A$ of degree $-1$ (in the sense that, if $A$ is regraded down by one, $[\cdot, \cdot]$ is an ordinary graded Lie bracket) such that, for each homogeneous element $a$ of $A$, $[a, \cdot]$ is a derivation of $A$ of degree $|a| - 1$ where $|a|$ refers to the degree of $a$; see [6] where these objects are called G-algebras, or [17,22,40]; for a Gerstenhaber algebra $A$, the bracket from $A \otimes_R A$ to $A$ will henceforth be referred to as its Gerstenhaber bracket. Below we will interchangeably talk about Gerstenhaber algebras and G-algebras. We recall from Theorem 5 of [6] that (i) the assignment to a Gerstenhaber algebra $A$ of the pair $(A_0, A_1)$ consisting of the homogeneous degree zero and degree one components $A_0$ and $A_1$, respectively, yields a functor from Gerstenhaber algebras to Lie-Rinehart algebras, and that this functor has a left adjoint which assigns the exterior algebra $\Lambda_A L$ in the category of $A$-modules to the Lie-Rinehart algebra $(A, L)$, together
with the obvious bracket operation on $\Lambda_A L$ induced by the Gerstenhaber bracket structure. Here $L$ is viewed to be concentrated in degree one.

**Definition 4.1.** Let $\mathcal{A}$ be a bigraded commutative $R$-algebra. We will say that a bigraded bracket $[\cdot, \cdot]: \mathcal{A} \otimes_R \mathcal{A} \to \mathcal{A}$ of bidegree $(0, -1)$ is a **bigraded Gerstenhaber bracket** provided $[\cdot, \cdot]$ is an ordinary bigraded Lie bracket when the second degree of $\mathcal{A}$ is regraded down by one, the first one being kept, such that, for each homogeneous element $a$ of $\mathcal{A}$ of bidegree $(p, q)$, $[a, \cdot]$ is a derivation of $\mathcal{A}$ of bidegree $(p, q - 1)$; a bigraded $R$-algebra with a bigraded Gerstenhaber bracket will be referred to as a **bigraded Gerstenhaber algebra**.

### 4.2. The bigraded crossed product Gerstenhaber algebra

Let $(A, L)$ be an (ungraded) Lie-Rinehart algebra, and let $\mathcal{A}$ be a graded commutative $A$-algebra together with an $L$-action $L \otimes_R \mathcal{A} \to \mathcal{A}$ by derivations such that the canonical map from $A$ to $\mathcal{A}$ is a morphism of left $(A, L)$-modules. Let $L$ be the corresponding crossed product of $A$ and $L$ given in (2.8); it is a graded $(R, A)$-Lie algebra. Consider the bigraded $A$-algebra $\Lambda_A L = A \otimes A \Lambda_A L$. It is the graded exterior $A$-algebra on $L$ in an obvious sense whence the notation.

The graded Lie bracket $[\cdot, \cdot]$ and the $L$-action on $\mathcal{A}$ induce a bigraded Gerstenhaber bracket

\[
[\cdot, \cdot]: \Lambda_A L \otimes_R \Lambda_A L \to \Lambda_A L
\]

by means of the formulas

\[
[\alpha \beta, \gamma] = \alpha [\beta, \gamma] + (-1)^{|\alpha||\beta|} \beta [\alpha, \gamma], \quad \alpha, \beta, \gamma \in \Lambda_A L,
\]

\[
[x, a] = x(a), \quad x \in L, \ a \in \mathcal{A},
\]

\[
[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)} [\beta, \alpha], \quad \alpha, \beta \in \Lambda_A L,
\]

where $|\cdot|$ refers to the total degree, i.e. the sum of the two bidegree components. We refer to the bracket (4.2.1) as the **bigraded crossed product bracket extension** and to $\Lambda_A L$ as the **bigraded crossed product Gerstenhaber algebra** of $\mathcal{A}$ with the ordinary Gerstenhaber algebra $\Lambda_A L$. In terms of the latter and the graded left $(A, L)$-module structure on $\mathcal{A}$, the bigraded Gerstenhaber bracket (4.2.1) may be described in the following way which, among others, gives an explicit formula: Let $a, b \in \mathcal{A}$ and $u = \alpha_1 \wedge \ldots \wedge \alpha_\ell \in \Lambda_A^\ell L$ and $v = \alpha_{\ell+1} \wedge \ldots \wedge \alpha_n \in \Lambda_A^{n-\ell} L$, where $\alpha_1, \ldots, \alpha_n \in L$; then the ordinary Gerstenhaber bracket $[u, v]$ in $\Lambda_A L$ is given by the expression

\[
[u, v] = (-1)^\ell \sum_{j \leq \ell < k} (-1)^{j+k} [\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \ldots \wedge \widehat{\alpha_j} \ldots \widehat{\alpha_k} \ldots \wedge \alpha_n,
\]

where $\ell = |u|$ is the degree of $u$, cf. [12] (1.1). Writing $au = a \otimes_A u$ and $bv = b \otimes_A v$, 

from (4.2.2), we obtain

\[
[u, b] = [\alpha_1 \wedge \ldots \wedge \alpha_\ell, b] \\
= \sum_{j=1}^{\ell} (-1)^{\ell-j} \alpha_1 \wedge \ldots \hat{\alpha_j} \ldots \wedge \alpha_\ell [\alpha_j, b] \\
= \sum_{j=1}^{\ell} (-1)^{\ell-j+(\ell-1)|b|} \alpha_j(b) \alpha_1 \wedge \ldots \hat{\alpha_j} \ldots \wedge \alpha_\ell
\]

where

\[
[a, v] = [a, \alpha_{\ell+1} \wedge \ldots \wedge \alpha_n] \\
= \sum_{j=1}^{n-\ell} (-1)^{j-1} [a, \alpha_{\ell+j}] \alpha_{\ell+1} \wedge \ldots \hat{\alpha_{\ell+j}} \ldots \wedge \alpha_n
\]

\[
= \sum_{j=1}^{n-\ell} (-1)^{j} \alpha_{\ell+j}(a) \alpha_{\ell+1} \wedge \ldots \hat{\alpha_{\ell+j}} \ldots \wedge \alpha_n
\]

\[
= \sum_{k=\ell+1}^{n} (-1)^{k-\ell} \alpha_k(a) \alpha_{\ell+1} \wedge \ldots \hat{\alpha_k} \ldots \wedge \alpha_n
\]

whence

\[
[au, bv] = (-1)^{(\ell-1)|b|} ab[u, v] + a[u, b]v + (-1)^{(\ell-1+|a|)|b|+(\ell-1)(n-\ell-1)} b[a, v]u
\]

\[
= (-1)^{(\ell-1)|b|} ab[u, v]
\]

\[
+ (-1)^{(\ell-1)|b|} \sum_{j=1}^{\ell} (-1)^{\ell-j} a\alpha_j(b) \alpha_1 \wedge \ldots \hat{\alpha_j} \ldots \wedge \alpha_\ell \wedge v
\]

\[
+ (-1)^{(\ell-1)(|b|+n-\ell-1)} \sum_{k=\ell+1}^{n} (-1)^{k-\ell} \alpha_k(a) b\alpha_{\ell+1} \wedge \ldots \hat{\alpha_k} \ldots \wedge \alpha_n \wedge u.
\]

Here the sign in the expression \((-1)^{(\ell-1+|a|)|b|+(\ell-1)(n-\ell-1)} b[a, v]u\) arises from first interchanging \(b\) with \(au\) and thereafter interchanging \(u\) and \(v\), which necessitate the signs \((-1)^{(\ell-1+|a|)|b|}\) and \((-1)^{(\ell-1)(n-\ell-1)}\), respectively. Consequently

\[
[au, bv] = (-1)^{(\ell-1)|b|} ab[u, v]
\]

\[
+ (-1)^{(\ell-1)|b|} \sum_{j=1}^{\ell} (-1)^{\ell-j} a\alpha_j(b) \alpha_1 \wedge \ldots \hat{\alpha_j} \ldots \wedge \alpha_n
\]

\[
+ (-1)^{(\ell-1)|b|} \sum_{j>\ell} (-1)^{j-\ell} \alpha_j(a) b\alpha_1 \wedge \ldots \hat{\alpha_j} \ldots \wedge \alpha_n.
\]

**Remark 4.2.5.** We note that \([au, bv]\) is the sum of \(\pm ab[u, v]\)—which involves only the product \(ab\) in \(A\) and the Gerstenhaber bracket \([u, v]\) in \(\Lambda_A L\)—and **two other terms**, which involve the action of \(L\) on \(A\) and the product in \(A\). Thus the (crossed product) Gerstenhaber bracket on \(\Lambda_A L\) cannot be written just in terms of the product on \(A\) and the Gerstenhaber bracket on \(\Lambda_A L\).
In particular, under the circumstances of (2.9), with \( \mathcal{A} = \text{Alt}_A(M, A) \), the crossed product Gerstenhaber structure is available for the crossed product \((R, \mathcal{A})\)-Lie algebra \( \mathcal{L} = \mathcal{A} \otimes_A L \) and yields a bigraded Gerstenhaber bracket on the bigraded \( A \)-algebra

\[
\Lambda_A \mathcal{L} = \text{Alt}_A(M, A) \otimes_A \Lambda_A L.
\]

When \( L \) is finitely generated and projective as an \( A \)-module, the bigraded \( A \)-algebra \( \Lambda_A \mathcal{L} \) may in fact be written in the form

\[
\text{Alt}_A(M, \Lambda_A L).
\]

Recall that a differential Gerstenhaber algebra \((\mathcal{A}, [\cdot, \cdot], d)\) consists of a Gerstenhaber algebra \((\mathcal{A}, [\cdot, \cdot])\) together with a differential \( d \) (of degree +1) which endows \( \mathcal{A} \) with a differential graded \( R \)-algebra structure [17], [40]; in [7], these objects are studied under the name braid algebras. We will say that a differential Gerstenhaber algebra \((\mathcal{A}, [\cdot, \cdot], d)\) is strict provided \( d \) behaves as a derivation for the Gerstenhaber bracket \([\cdot, \cdot]\), that is,

\[
d[x, y] = [dx, y] - (-1)^{|x|}[x, dy], \quad x, y \in \mathcal{A}.
\]

**Definition 4.3.** A bigraded Gerstenhaber algebra \((\mathcal{A}, [\cdot, \cdot])\) together with a differential \( d \) of bidegree \((1, 0)\) which endows \( \mathcal{A} \) with a differential graded \( R \)-algebra structure will be said to be a differential bigraded Gerstenhaber algebra (or differential bigraded \( G \)-algebra), written \((\mathcal{A}, [\cdot, \cdot], d)\), provided \( d \) behaves as a derivation for the bigraded Gerstenhaber bracket \([\cdot, \cdot]\), that is,

\[
d[x, y] = [dx, y] - (-1)^{|x|}[x, dy], \quad x, y \in \mathcal{A},
\]

where the total degree \(|x|\) is the sum of the bidegrees.

We now return to the circumstances of (3.2): Thus \((\mathcal{A}, L', L'')\) is an almost twilled Lie-Rinehart algebra, \( L' \) being finitely generated and projective as an \( A \)-module, and

\[
(\mathcal{A}'', L') = (\text{Alt}_A(L'', A), \text{Alt}_A(L'', L'))
\]

is the corresponding graded crossed product Lie-Rinehart algebra, cf. (3.1). Furthermore, \( \mathcal{A}'' = \text{Alt}_A(L'', A) \) is endowed with the ordinary Lie-Rinehart differential \( d'' \) (cf. 1.11.4.1) and \( \mathcal{L}' = \text{Alt}_A(L'', L') \) with the Lie-Rinehart differential \( d'' \) which comes from the given left \((A, L'')\)-module structure (1.4.4) on \( L' \) and, moreover, with the graded Lie bracket \([\cdot, \cdot]'\) given in (2.8.2) which comes from the given left \((A, L')\)-module structure (1.4.3) on \( L''. \) Consider the resulting bigraded crossed product Gerstenhaber algebra (4.2.6), with \( M = L'' \) and \( L = L' \). As a graded \( A \)-module, this Gerstenhaber algebra may be written

\[
\Lambda_{\mathcal{A}''} \mathcal{L}' = \text{Alt}_A(L'', \Lambda_A L'),
\]

and the operator \( d'' \) induced by the given left \((A, L'')\)-module structure on \( L' \) is of bidegree \((1, 0)\) and turns \( \text{Alt}_A(L'', \Lambda_A L') \) into a differential graded \( R \)-algebra. We denote the bigraded Gerstenhaber bracket on the latter by \([\cdot, \cdot]'\).
Theorem 4.4. For an almost twilled Lie-Rinehart algebra \((A, L', L'')\) having \(L'\) finitely generated and projective as an \(A\)-module, \((\text{Alt}_A(L'', \Lambda_A L'), [\cdot, \cdot]', d'')\) is a differential bigraded Gerstenhaber algebra if and only if \((A, L', L'')\) is a twilled Lie-Rinehart algebra.

It is clear that, by symmetry, when \(L''\) is finitely generated (and projective) as an \(A\)-module, with the roles of \(L'\) and \(L''\) interchanged, exactly the same statement as that spelled out in Theorem 4.4 is true.

Proof. This is an immediate consequence of (3.2) above. \(\Box\)

Example 4.5. We return to the circumstances of Example 3.6 and consider the twilled Lie-Rinehart algebra \((A, L', L'')\) arising from a complex structure on a smooth manifold \(M\). The corresponding differential bigraded Gerstenhaber algebra spelled out in (4.4) has as underlying bigraded \(A\)-module, where \(A\) is the algebra of smooth complex functions on \(M\), that of smooth complex valued \(\bar{\partial}\)-forms with values in the exterior powers of the holomorphic tangent bundle; such differential Gerstenhaber algebras were studied in [1] and elsewhere. For reasons explained in (4.2.5) above, the Gerstenhaber bracket does not just involve the shuffle product of \(\bar{\partial}\)-forms and the Schouten-Nijenhuis bracket of sections of the exterior powers of the holomorphic tangent bundle, though, and in the corresponding Gerstenhaber bracket (4.2.1) two others terms come into play, cf. the description (4.2.4). By symmetry, interchanging the roles of the holomorphic and antiholomorphic tangent bundles, we obtain as well a differential (bigraded) Gerstenhaber algebra which consists of smooth complex valued \(\partial\)-forms with values in the exterior powers of the antiholomorphic tangent bundle. It is also worthwhile pointing that, in view of Theorem 4.4, the compatibility properties defining part of the structure of the differential bigraded Gerstenhaber algebra are equivalent to the integrability condition of the initially only almost complex structure.

Remark 4.6. In a follow-up paper [14] we will generalize the bigraded crossed product Gerstenhaber algebra extension to the case where, in the notation of (2.9), the left \((A, L)\)-module \(M\) is no longer assumed to be finitely generated and projective as an \(A\)-module, so that, for arbitrary \(M\), a bigraded crossed product Gerstenhaber algebra structure on a bigraded \(A\)-algebra of the kind (4.2.7) results. By means of this generalization, we will then prove that the statement of Theorem 4.4 still holds without the hypothesis that \(L'\) be finitely generated and projective as an \(A\)-module.

We conclude this section with a description of twilled Lie-Rinehart algebras in terms of Lie-Rinehart bialgebras: Let \(L\) and \(D\) be \((R, A)\)-Lie algebras which, as \(A\)-modules, are finitely generated and projective, in such a way that, as an \(A\)-module, \(D\) is isomorphic to \(L^* = \text{Hom}_A(L, A)\). We say that \(L\) and \(D\) are in duality. We write \(d\) for the differential on \(\text{Alt}_A(L, A) \cong \Lambda_A D\) coming from the Lie-Rinehart structure on \(L\) and \(d_+\) for the differential on \(\text{Alt}_A(D, A) \cong \Lambda_A L\) coming from the Lie-Rinehart structure on \(D\). Likewise we denote the Gerstenhaber bracket on \(\Lambda_A L\) coming from the Lie-Rinehart structure on \(L\) by \([\cdot, \cdot]\) and that on \(\Lambda_A D\) coming from the Lie-Rinehart structure on \(D\) by \([\cdot, \cdot]_+\).

Proposition 4.7. Given \(L\) and \(D\) in duality, the following are equivalent.

\((4.7.1)\) The differential \(d\) on \(\text{Alt}_A(L, A) \cong \Lambda_A D\) and the Gerstenhaber bracket \([\cdot, \cdot]\)
on $\Lambda_A D$ are related by

$$d[x, y] = [dx, y] + [x, dy], \quad x, y \in D.$$  

(4.7.2) The differential $d_*$ on $\text{Alt}_A(D, A) \cong \Lambda_A L$ and the Gerstenhaber bracket $[\cdot, \cdot]$ on $\Lambda_A L$ are related by

$$d_{*}[x, y] = [d_{*}x, y] + [x, d_{*}y], \quad x, y \in L.$$

(4.7.3) The differential $d$ on $\text{Alt}_A(L, A) \cong \Lambda_A D$ behaves as a derivation for the Gerstenhaber bracket $[\cdot, \cdot]$ in all degrees, that is to say

$$d[x, y] = [dx, y] - (-1)^{|x|} [x, dy], \quad x, y \in \Lambda_A D.$$

(4.7.4) The differential $d_*$ on $\text{Alt}_A(D, A) \cong \Lambda_A L$ behaves as a derivation for the Gerstenhaber bracket $[\cdot, \cdot]$ in all degrees, that is to say

$$d_{*}[x, y] = [d_{*}x, y] - (-1)^{|x|} [x, d_{*}y], \quad x, y \in \Lambda_A L.$$

Proof. Since $d$ is a derivation for the algebra structure in $\text{Alt}_A(L, A)$, (4.7.1) and (4.7.3) are manifestly equivalent; likewise, since $d_*$ is a derivation for the algebra structure in $\text{Alt}_A(D, A)$, (4.7.2) and (4.7.4) are equivalent. The equivalence of (4.7.3) and (4.7.4) (the self-duality) is established in [27], [17] (3.3), and [23], for the special case where $L$ and $D$ come from real Lie algebroids. The argument given in these sources is formal and carries over. □

We will say that $(A, L', D)$ constitutes a Lie-Rinehart bialgebra if one (and hence any) of the (equivalent) conditions (4.7.1)–(4.7.4) is satisfied. Thus, for a Lie-Rinehart bialgebra $(A, L, D)$,

$$(\Lambda_A L, [\cdot, \cdot], d_{*}) = (\text{Alt}_A(D, A), [\cdot, \cdot], d_{*})$$

is a strict differential Gerstenhaber algebra, and the same is true of

$$(\Lambda_A D, [\cdot, \cdot], d) = (\text{Alt}_A(L, A), [\cdot, \cdot], d);$$

see [17] (3.5) for details. In fact, a straightforward extension of an observation of Y. Kosmann-Schwarzbach [17] shows that Lie-Rinehart bialgebra structures on $(A, L, D)$ and strict differential Gerstenhaber algebra structures on $(\Lambda_A L, [\cdot, \cdot], d_{*})$ or, what amounts to the same, on $(\Lambda_A D, [\cdot, \cdot], d)$, are equivalent notions. This parallels the well known fact that Lie-Rinehart structures on $(A, L)$ are in bijective correspondence with differential graded $R$-algebra structures on $\text{Alt}_A(L, A)$.

Let $(A, L', L'')$ be an almost twilled Lie-Rinehart algebra, having $L'$ and $L''$ finitely generated and projective as $A$-modules. The $(A, L')$-module structure (1.4.3) on $L''$ induces an $(A, L')$-module on the dual $L''^*$ which, in turn, $L''^*$ being viewed as an abelian Lie algebra and hence abelian $(R, A)$-Lie algebra, gives rise to the semi
direct product $(R,A)$-Lie algebra $L' \ltimes L''*$. Likewise the $(A,L'')$-module structure (1.4.4) on $L'$ determines the corresponding semi direct product $(R,A)$-Lie algebra $L'' \ltimes L''*$. Plainly $L = L' \ltimes L''*$ and $D = L'' \ltimes L''*$ are in duality. Consider the obvious adjointness isomorphisms

\[(4.8.1) \quad \text{Alt}_A(L'', \Lambda_AL') \to \text{Alt}_A(L'' \ltimes L''*, A) = \text{Alt}_A(D, A)\]

and

\[(4.8.2) \quad \Lambda_AL = \Lambda_A(L' \ltimes L''*) \to \text{Alt}_A(L'', \Lambda_AL')\]

of bigraded $A$-algebras; these isomorphisms are independent of the Lie-Rinehart semi direct product constructions and instead of $L' \ltimes L''*$ and $L'' \ltimes L''*$, we could as well have written $L' \oplus L''*$ and $L'' \oplus L''*$, respectively. However, incorporating these semi direct product structures, we see that, under (4.8.1), the Lie-Rinehart differential $d''$ on $\text{Alt}_A(L'', \Lambda(AL'))$ passes to the Lie-Rinehart differential $d_*$ on $\text{Alt}_A(D, A)$ and that under (4.8.2) the (bigraded) Gerstenhaber bracket $[\cdot, \cdot]$ on $\Lambda_AL$ passes to the bigraded Gerstenhaber bracket $[\cdot, \cdot]'$ $\text{Alt}_A(L'', \Lambda(AL'))$. Moreover, by construction, the differentials on both sides of (4.8.1) are derivations with respect to the multiplicative structures.

**Theorem 4.8.** For an almost twilled Lie-Rinehart algebra $(A,L',L'')$ having $L'$ and $L''$ finitely generated and projective as $A$-modules, $(\text{Alt}_A(L'', \Lambda_AL'), [, [, \cdot'], d'')]$ is a differential bigraded Gerstenhaber algebra if and only if $(A,L,D)$ is a Lie-Rinehart bialgebra.

**Proof.** The property (4.7.2) characterizing $(A,L,D)$ to be a Lie-Rinehart bialgebra is plainly equivalent to $(\text{Alt}_A(L'', \Lambda(AL'), [, [, \cdot'], d'']))$ being a differential bigraded Gerstenhaber algebra, cf. (4.3).

The following is now immediate.

**Corollary 4.9.** An almost twilled Lie-Rinehart algebra $(A,L',L'')$ having $L'$ and $L''$ finitely generated and projective as $A$-modules is a true twilled Lie-Rinehart algebra if and only if $(A,L,D) = (A,L' \ltimes L''*, L'' \ltimes L''*)$ is a Lie-Rinehart bialgebra.

This result may be proved directly, i.e. without the intermediate differential bigraded Gerstenhaber algebra in (4.8). The reasoning is formally the same, though. For the special case where $L'$ and $L''$ arise from Lie algebroids, the statement of Corollary 4.9 may be deduced from what is said in [26].

5. Differential Batalin-Vilkovisky algebras

In Section 1 of [12] we obtained an interpretation of the notion of a generator of a Gerstenhaber algebra $\Lambda_AL$ arising from a Lie-Rinehart algebra $(A,L)$ which, for the special case where $A$ is the ring of smooth functions and $L$ the Lie algebra of smooth vector fields on a smooth manifold, comes down to a result of Koszul [21]. In this section, we will first generalize this interpretation to bigraded Gerstenhaber algebras. We will then show that, in the holomorphic context, this extension is crucial for an understanding of the Tian-Todorov Lemma, of the Calabi-Yau condition, and of the Batalin-Vilkovisky algebras arising from the mirror conjecture.
For a bigraded Gerstenhaber algebra \( \mathcal{A} \) over \( R \), with bracket operation written \( [\cdot, \cdot] \), an \( R \)-linear operator \( \Delta \) on \( \mathcal{A} \) of bidegree \((0, -1)\) will be said to generate the Gerstenhaber bracket provided, for every homogeneous \( a, b \in \mathcal{A} \),

\[
[a, b] = (-1)^{|a|} \left( \Delta(ab) - (\Delta a)b - (-1)^{|a|} a(\Delta b) \right);
\]

the operator \( \Delta \) is then called a generator. A generator \( \Delta \) is said to be exact provided \( \Delta \Delta \) is zero, that is, \( \Delta \) is a differential; an exact generator will henceforth be written \( \partial \). A bigraded Gerstenhaber algebra \( \mathcal{A} \) together with a generator \( \Delta \) will be called a weak bigraded Batalin-Vilkovisky algebra (or weak bigraded BV-algebra); when the generator is exact, written \( \partial \), we will refer to \((\mathcal{A}, \partial)\) (more simply) as a bigraded Batalin-Vilkovisky algebra (or bigraded BV-algebra).

It is clear that a generator determines the bigraded Gerstenhaber bracket. An observation due to Koszul [21] (p. 261) carries over to the bigraded case: for any bigraded Batalin-Vilkovisky algebra \((\mathcal{A}, [\cdot, \cdot], \partial)\), the operator \( \partial \) (which is exact by assumption) behaves as a derivation for the bigraded Gerstenhaber bracket \([\cdot, \cdot]\), that is,

\[
\partial[x, y] = [\partial x, y] - (-1)^{|x|}[x, \partial y], \quad x, y \in \mathcal{A}.
\]

An exact generator \( \partial \) does in general not behave as a derivation for the multiplication of \( \mathcal{A} \), though.

### 5.3. The crossed product (weak) Batalin-Vilkovisky algebra

Let \((\mathcal{A}, L)\) be a Lie-Rinehart algebra, and let \( \mathcal{A} \) be a graded commutative \( \mathcal{A} \)-algebra together with an \( L \)-action \( L \otimes_R \mathcal{A} \to \mathcal{A} \) by derivations such that the canonical map from \( A \) to \( \mathcal{A} \) is a morphism of left \((A, L)\)-modules. Consider the crossed product \((R, \mathcal{A})\)-Lie algebra \( \mathcal{L} = \mathcal{A} \otimes_A L \) (given in (2.8)) and the corresponding bigraded crossed product Gerstenhaber algebra \( \Lambda \mathcal{A} \mathcal{L} = \mathcal{A} \otimes_A \Lambda_A L \) introduced in (4.2), with bigraded Gerstenhaber bracket (4.2.1).

**Proposition 5.3.1.** A generator \( D \) for the Gerstenhaber bracket of the ordinary Gerstenhaber algebra \( \Lambda_A L \) admits a unique extension to a generator \( D_\mathcal{A} \) of the bigraded crossed product Gerstenhaber algebra bracket (4.2.1) on \( \Lambda \mathcal{A} \mathcal{L} = \mathcal{A} \otimes_A \Lambda_A L \). This extension may be described by means of the formula

\[
D_\mathcal{A}(a \alpha) = aD(\alpha) + \sum_{i=1}^{n} (-1)^i (\alpha_i(a)) \alpha_1 \wedge \ldots \wedge \hat{\alpha}_i \ldots \wedge \alpha_n
\]

where \( a \in \mathcal{A} \) and \( \alpha = \alpha_1 \wedge \ldots \wedge \alpha_n \in \Lambda_A L \). Further, every generator of the bigraded crossed product Gerstenhaber algebra bracket on \( \Lambda \mathcal{A} \mathcal{L} \) arises in this way.

**Proof.** This is left to the reader. \( \Box \)

Additional insight into the generator \( D_\mathcal{A} \) will be offered in Section 7 below; see in particular Theorem 7.6.

We refer to the generator \( D_\mathcal{A} \) of the bracket on \( \Lambda \mathcal{A} \mathcal{L} \) given by (5.3.2) as the *bigraded crossed product extension* of the generator \( D \) for the bracket on \( \Lambda_A L \). The resulting weak bigraded Batalin-Vilkovisky algebra \((\Lambda \mathcal{A} \mathcal{L}, D_\mathcal{A})\) will be referred to
as the bigraded crossed product of $\mathcal{A}$ and $(\Lambda_L,D)$. A bigraded crossed product of $\mathcal{A}$ and a true Batalin-Vilkovisky algebra $(\Lambda_L,\partial)$ is manifestly a true bigraded Batalin-Vilkovisky algebra $(\Lambda_A\mathcal{L},\partial_A)$.

Suppose that, as an $A$-module, $L$ is finitely generated and projective of finite constant rank, $n$. The canonical pairing

$$\langle \cdot, \cdot \rangle: \Lambda^*_A\mathcal{L} \otimes_A \Lambda^{n-*}_A\mathcal{L} \to \Lambda^n_A\mathcal{L}$$

(5.3.3)

of graded $A$-modules is perfect and its adjoint

$$\Lambda^n_A\mathcal{L} \to \text{Hom}_A(\Lambda^{n-*}_A\mathcal{L},\Lambda^n_A\mathcal{L}) = \text{Alt}^{n-*}_A(L,\Lambda^n_A\mathcal{L})$$

(5.3.4)

is an isomorphism of graded $A$-modules. Given $x \in \Lambda^n_*\mathcal{L}$, write $\phi_x \in \text{Alt}^{n-*}_A(L,\Lambda^n_A\mathcal{L})$ for the image of $x$ under this isomorphism. For an $(A,L)$-connection $\nabla: M \to \text{Hom}_A(L,M)$ on a left $A$-module $M$ we denote its operator of covariant derivative by

$$d\nabla: \text{Alt}_A(L,M) \to \text{Alt}_A(L,M).$$

Proposition 5.3.5. The relationship

$$\phi_{\Delta(x)} = d\nabla \phi_x$$

(5.3.6)

establishes a bijective correspondence between generators $\Delta$ for the (ordinary) Gerstenhaber bracket on $\Lambda_A\mathcal{L}$ and $(A,L)$-connections $\nabla$ on $\Lambda^n_A\mathcal{L}$ in such a way that exact generators $\Delta$ correspond to left $(A,L)$-module structures $\nabla$, i.e., flat $(A,L)$-connections, on $\Lambda^n_A\mathcal{L}$.

Proof. See the Corollary in Section 2 of [12]. □

Combining Propositions 5.3.1 and 5.3.5 we arrive at the following.

Theorem 5.3.7. The relationship (5.3.6), combined with (5.3.2), establishes a bijective correspondence between generators $\Delta_A$ for the bigraded product Gerstenhaber algebra $\Lambda_A\mathcal{L}$ and $(A,L)$-connections $\nabla$ on $\Lambda^n_A\mathcal{L}$ in such a way that exact generators $\Delta_A$ correspond to left $(A,L)$-module structures $\nabla$, i.e., flat $(A,L)$-connections, on $\Lambda^n_A\mathcal{L}$. □

Another proof will be given at the end of Section 7 below; see what is said after Corollary 7.10.

Corollary 5.3.8. When $L$ is finitely generated and projective of constant rank $n$ as an $A$-module, generators for the bigraded Gerstenhaber bracket on the bigraded crossed product $\Lambda_A\mathcal{L}$ always exist.

Proof. In fact, when $L$ is (finitely generated and) projective, so is $\Lambda^n_A\mathcal{L}$, whence $(A,L)$-connections on $\Lambda^n_A\mathcal{L}$ then always exist. □

5.4. Incorporation of differentials

Let $(\mathcal{A},\Delta)$ be a weak bigraded Batalin-Vilkovisky algebra, write $[\cdot,\cdot]$ for the bigraded Gerstenhaber bracket generated by $\Delta$, and let $d$ be a differential of bidegree $(+1,0)$
which endows \((\mathcal{A}, [\cdot, \cdot])\) with a differential bigraded Gerstenhaber algebra structure. Consider the graded commutator
\[
[d, \Delta] = d\Delta + \Delta d
\]
on \mathcal{A}; it is an operator of bidegree \((1, -1)\) and hence of total degree zero. We will say that \((\mathcal{A}, \Delta, d)\) is a weak differential bigraded Batalin-Vilkovisky algebra provided the commutator \([d, \Delta]\) is zero. In particular, a weak differential bigraded Batalin-Vilkovisky algebra \((\mathcal{A}, \partial, d)\) which has \(\partial\) exact will be called a differential bigraded Batalin-Vilkovisky algebra. On the underlying bigraded object \(\mathcal{A}\) of a differential bigraded Batalin-Vilkovisky algebra \((\mathcal{A}, \partial, d)\), the graded commutator \([d, \partial]\) manifestly behaves as a derivation for the bigraded Gerstenhaber bracket since \(d\) and \(\partial\) both behave as derivations for this bracket.

Various notions of differential Batalin-Vilkovisky algebras may be found in the literature, cf. [30] (6.1.1) (dGBV-algebras), [17] (cf. e.g. the differential exact Gerstenhaber algebras on p. 154), [40] (Question 4 in Section 5 where a concept of strong differential BV-algebra occurs). Our notion of differential bigraded Batalin-Vilkovisky algebra does not coincide with any of these.

**Lemma 5.4.2.** Given a weak bigraded Batalin-Vilkovisky algebra \((\mathcal{A}, \Delta)\) and an operator \(\delta\) of bidegree \((1, 0)\) which behaves as a derivation of degree 1 for the bigraded \(R\)-algebra \(\mathcal{A}\), the following are equivalent:

(i) The operator \(\delta\) behaves as a derivation for the bigraded Gerstenhaber bracket \([\cdot, \cdot]\) on \(\mathcal{A}\) generated by \(\Delta\), that is to say, 
\[
\delta[x, y] = [\delta x, y] - (-1)^{|x|}|x, \delta y|, \quad x, y \in \mathcal{A}.
\]

(ii) The graded commutator \([\delta, \Delta]\) behaves as a derivation of degree 0 for the bigraded \(R\)-algebra \(\mathcal{A}\), that is, 
\[
[\delta, \Delta](ab) = ([\delta, \Delta]a)b + a([\delta, \Delta]b), \quad a, b \in \mathcal{A}.
\]

**Proof.** Let \(a, b\) be homogeneous elements of \(\mathcal{A}\). Then
\[
\delta[a, b] = (-1)^{|a|}(\delta \Delta(ab) - \delta((\Delta a)b)) - \delta(a\Delta b)) = (-1)^{|a|}((-\Delta \delta(ab) + [\delta, \Delta](ab) - (\delta \Delta a)b + (-1)^{|a|}(\Delta \delta b))
\]
\[
- \delta(a)(\Delta b) - (-1)^{|a|}a(\delta \Delta b) = (-1)^{|a|}[\Delta((\delta a)b + (-1)^{|a|}a(\delta b)) - [\delta, \Delta](ab) - (\Delta \delta a)b + ([\delta, \Delta]a)b]
\]
\[
+ (\Delta a)(\delta b) - (\delta a)(\Delta b) + (-1)^{|a|}a(\Delta \delta b) - (-1)^{|a|}a([\delta, \Delta]b)
\]
\[
= (-1)^{|a|}([\Delta((\delta a)b) - \Delta(a\delta b) + (-1)^{|a|}(\Delta \delta a)b]
\]
\[
+ (\Delta a)(\delta b) - (\delta a)(\Delta b) + (-1)^{|a|}a(\Delta \delta b)
\]
\[
+ (-1)^{|a|}([\delta, \Delta](ab) - ([\delta, \Delta]a)b - a([\delta, \Delta]b))
\]
\[
= (-1)^{|a|}(\Delta((\delta a)b) - (\Delta \delta a)b) - (\delta a)(\Delta b)
\]
\[
- \Delta(a\delta b) + (\Delta a)(\delta b) + (-1)^{|a|}a(\Delta \delta b)
\]
\[
+ (-1)^{|a|}([\delta, \Delta](ab) - ([\delta, \Delta]a)b - a([\delta, \Delta]b))
\]
\[
= [\delta a, b] - (-1)^{|a|}[a, \delta b] + (-1)^{|a|}([\delta, \Delta](ab) - ([\delta, \Delta]a)b - a([\delta, \Delta]b))
\]
This establishes the claim. □

**Corollary 5.4.3.** For any weak differential bigraded Batalin-Vilkovisky algebra \((A, \Delta, d)\), the differential \(d\) behaves as a derivation for the bigraded Gerstenhaber bracket \([\cdot, \cdot]\) on \(A\) generated by \(\Delta\), that is to say,

\[
d[x, y] = [dx, y] - (-1)^{|x|}[x, dy], \quad x, y \in A.
\]

In other words, \((A, [\cdot, \cdot], d)\) is a differential bigraded Gerstenhaber algebra.

Notice that, under the circumstances of (5.4.3), \(\Delta\) need not behave as a derivation for the bigraded Gerstenhaber bracket unless \(\Delta\) is exact.

**Theorem 5.4.4.** Suppose that \(L'\) is finitely generated and projective as an \(A\)-module, let \(\Delta'\) be a generator for the bigraded Gerstenhaber bracket \([\cdot, \cdot]'\) of the bigraded crossed product Gerstenhaber algebra \((\text{Alt}_A(L'', \Lambda^*_AL'), [\cdot, \cdot]')\), and write \(d''\) for the Lie-Rinehart differential (1.11.4.1) induced by the \((A, L'')\)-action on \(\Lambda^*_AL'\). Then \([d'', \Delta'](= d''\Delta' + \Delta'd'')\) is a derivation (of bidegree \((1, -1)\)) for the bigraded \(R\)-algebra \(\text{Alt}_A(L'', \Lambda^*_AL')\) if and only if \((A, L', L'')\) is a twilled Lie-Rinehart algebra. In particular, when \((\text{Alt}_A(L'', \Lambda^*_AL'), \Delta', d'')\) is a weak differential bigraded Batalin-Vilkovisky algebra (i.e. when \([d'', \Delta']\) is zero), \((A, L', L'')\) is necessarily a twilled Lie-Rinehart algebra.

**Proof.** This is a consequence of Lemma 5.4.2, combined with Theorem 4.4. □

It is clear that, when \(L''\) is finitely generated and projective as an \(A\)-module, the same statements as those given in Theorem 5.4.4 can be made, with the roles of \(L'\) and \(L''\) interchanged. Exploiting the generalization of the bigraded crossed product Gerstenhaber algebra mentioned already in (4.6), in [14], we will prove that the statement of Theorem 5.4.4 holds without the hypothesis that \(L'\) be finitely generated and projective as an \(A\)-module; this then yields a result which is symmetric in \(L'\) and \(L''\).

**Proposition 5.4.5.** Under the circumstances of (5.4.4), the adjoint

\[
\Lambda^*_AL' \to \text{Hom}_A(\Lambda^{n-*}_AL', \Lambda^nAL') = \text{Alt}^{n-*}_A(L', \Lambda^nAL')
\]

of the corresponding pairing (5.3.3) is an isomorphism of graded \((A, L'')\)-modules, whence \((\text{Alt}_A(L'', \Lambda^*_AL'), \Delta', d'')\) is a weak differential bigraded Batalin-Vilkovisky algebra (i.e. \([\Delta', d'']\) is zero but \(\Delta'\) is not necessarily exact) if and only if the \((A, L')\)-connection

\[
\nabla': \Lambda^nAL' \to \text{Hom}_A(L', \Lambda^nAL')
\]

on \(\Lambda^nAL'\) corresponding to \(\Delta'\) (spelled out in (5.3.7)) is \(L''\)-invariant, with reference to the induced \(L''\)-actions on \(\Lambda^nAL'\) and \(\text{Hom}_A(L', \Lambda^nAL')\).

**Proof.** This is straightforward and left to the reader. □

**Theorem 5.4.6.** Under the circumstances of (5.4.4) if, in addition, as an \(A\)-module, \(L'\) (being finitely generated and projective) is of finite constant rank \(n\) (say), the adjoint (5.3.4) induces an isomorphism

\[
(5.4.6.1) \quad (\text{Alt}_A(L'', \Lambda^*_AL'), d'') \to (\text{Alt}_A(L'', \text{Alt}^{n-*}_A(L', \Lambda^nAL')), d'')
\]
of chain complexes which establishes a bijective correspondence between generators $\Delta'$ for the bigraded Gerstenhaber bracket on the left-hand side and operators $\partial_{\nabla'}$ of covariant derivative on the right-hand side, for a uniquely determined $(A,L')$-connection $\nabla'$ on $\Lambda^n_A L'$. Under this correspondence, generators $\Delta'$ turning $(\text{Alt}_A(L'', \Lambda_A L'), \Delta', d'')$ into a weak differential bigraded Batalin-Vilkovisky algebra correspond to $L''$-invariant connections $\nabla'$ (on $\Lambda^n_A L'$) and generators $\Delta'$ turning $(\text{Alt}_A(L'', \Lambda_A L'), \Delta', d'')$ into a true differential bigraded Batalin-Vilkovisky algebra correspond to flat $L''$-invariant connections $\nabla'$.

Proof. This follows readily from Proposition 5.4.5. □

An interpretation of (5.4.6.1) within the framework of (co)homological duality for differential graded Lie-Rinehart algebras will be given in Section 7 below.

A special case is worthwhile spelling out. For this purpose we observe that any closed $L''$-invariant $A$-valued 1-form $\alpha: L' \to A$ on $L'$ determines an $L''$-invariant $(A,L')$-module structure on $A$, i.e. a flat $L''$-invariant $(A,L')$-connection $\nabla_\alpha$ on $A$, whose operator $d'_\alpha$ of covariant derivative is determined by

$$d'_\alpha: A \to \text{Hom}_A(L', A), \quad (d'_\alpha(1))(x) = \alpha(x), \quad x \in L'. \tag{5.4.7.1}$$

This operator plainly extends to the corresponding Lie-Rinehart operator on $\text{Alt}^*_A(L'', \text{Alt}^*_A(L', A))$ (determined by the $L''$-invariant $(A,L')$-module structure on $A$), and we continue to denote this operator by $d'_\alpha$; thus $(\text{Alt}^*_A(L'', \text{Alt}^*_A(L', A)), d'_\alpha, d'')$ is a bicomplex. When $\alpha$ is zero, this is just the ordinary bicomplex of the kind (1.15.4).

**Theorem 5.4.7.** Under the circumstances of (5.4.6), suppose in addition that there is an $A$-valued $n$-form $\Lambda^n_A L' \to A$ on $L'$ yielding an isomorphism of $A$-modules which is invariant under $L''$ (i.e. which is an isomorphism of $(A,L'')$-modules). Then a choice of such an $n$-form $\Omega \in \text{Alt}^*_A(L', A)$ induces an isomorphism

$$\Omega_\alpha: (\text{Alt}^*_A(L'', \Lambda^n_A L'), d'') \to (\text{Alt}^*_A(L'', \text{Alt}^{n-*}_A(L', A)), d'') \tag{5.4.7.2}$$

of chain complexes over the ground ring $R$, in fact, of differential graded $(\text{Alt}^*_A(L'', A), d'')$-modules. Under this isomorphism, the operator $d'$ on the right-hand side $\text{Alt}^*_A(L'', \text{Alt}^{n-*}_A(L', A))$ of (5.4.7.2) corresponds to a uniquely determined exact generator $\partial_{\Omega\alpha}$ for the Gerstenhaber bracket on the differential bigraded Gerstenhaber algebra $(\text{Alt}^*_A(L'', \Lambda^n_A L'), [\cdot, \cdot], d'')$ on the left-hand side of (5.4.7.2). Furthermore, if $\alpha: L' \to A$ is any closed $L''$-invariant $A$-valued 1-form on $L'$, the corresponding Lie-Rinehart operator $d'_\alpha$ on the right-hand side $(\text{Alt}^*_A(L'', \text{Alt}^{n-*}_A(L', A)), d'')$ of (5.4.7.2) induces as well a uniquely determined exact generator $\partial_{\Omega,\alpha}$ (say) for the Gerstenhaber bracket $[\cdot, \cdot]$ on the left-hand side $(\text{Alt}^*_A(L'', \Lambda^n_A L'), [\cdot, \cdot], d'')$ of (5.4.7.2), and every exact generator for this Gerstenhaber bracket arises in this way.

Thus the choice of $\Omega$ enables us to rewrite the differential bigraded algebra $(\text{Alt}^*_A(L'', \text{Alt}^{n-*}_A(L', A)), d', d'')$ as a differential bigraded Batalin-Vilkovisky algebra.

Theorem 5.4.7 is a special case of Theorem 5.4.6, with $\Delta'$ corresponding to a flat $(A,L')$-connection on $\Lambda^n L$ which is invariant under the $L''$-action. A direct proof of Theorem 5.4.7 will be given after (5.4.11).
Let $M$ be a smooth complex $n$-manifold, and write $\tau_M$ and $\overline{\tau}_M$ for the holomorphic and antiholomorphic tangent bundles of $M$. For consistency with notation used in the literature, we momentarily write $\partial$ and $\overline{\partial}$ for the operators which correspond, under our more general circumstances, to our operators $d'$ and $d''$, respectively. Conflict with the notation $\partial$ for an exact generator of a Batalin-Vilkovisky algebra will be avoided since such a generator will be written $\partial_\Omega$ with an appropriate subscript.

**Corollary 5.4.8.** (Tian-Todorov Lemma) If $M$ admits a holomorphic volume form, a choice $\Omega$ of holomorphic volume form induces an isomorphism

$$(5.4.8.1) \quad \Omega_\Omega: (\Gamma(\Lambda^*\tau_M^* \otimes \Lambda^*\tau_M^*), \overline{\partial}) \rightarrow (\Gamma(\Lambda^*\tau_M^* \otimes \Lambda^{n-\ast}\tau_M^*), \overline{\partial})$$

of chain complexes, in fact, of modules over the differential graded algebra $(\Gamma(\Lambda^*\tau_M^*), \overline{\partial})$ of $\overline{\partial}$-forms defined only on the antiholomorphic tangent bundle $\overline{\tau}_M$. Under this isomorphism, the operator $\partial$ on the right-hand side of (5.4.8.1) corresponds, on the left-hand side, to an exact operator $\partial_\Omega$ which turns

$$(\Gamma(\Lambda^*\tau_M^* \otimes \Lambda^*\tau_M^*), \partial_\Omega, \overline{\partial})$$

into a differential bigraded Batalin-Vilkovisky algebra.

**Proof.** This is just a special case of the first statement of (5.4.7). □

The resulting isomorphism

$$(5.4.8.2) \quad \Omega_\Omega: (\Gamma(\Lambda^*\tau_M^* \otimes \Lambda^*\tau_M^*); \overline{\partial}, \partial_\Omega) \rightarrow (\Gamma(\Lambda^*\tau_M^* \otimes \Lambda^{n-\ast}\tau_M^*); \overline{\partial}, \partial)$$

identifies the differential bigraded Batalin-Vilkovisky algebra on the left-hand side with the Dolbeault complex (spelled out on the right-hand side), as pointed out in the introduction.

**Addendum 5.4.9.** Under the circumstances of (5.4.8), if $\alpha$ is any holomorphic 1-form on $M$, the operator $d'_\alpha$ of covariant derivative (given by (5.4.7.1)) on the right-hand side of (5.4.8.1), for the corresponding flat holomorphic connection $\nabla_\alpha$ (say) on $\Lambda^n\tau_M^*$, corresponds as well to a uniquely determined exact generator $\partial_{(\Omega,\alpha)}$ for the Gerstenhaber bracket on the left-hand side of (5.4.8.1), and every exact generator for this Gerstenhaber bracket arises in this way.

**Proof.** This is indeed a special case of the “Furthermore” statement of (5.4.7). □

The corresponding bigraded Gerstenhaber algebra is of course just the corresponding crossed product Gerstenhaber algebra. The existence of a holomorphic volume form is a strong kind of orientability condition; it is implied by the Calabi-Yau condition $c_1 = 0$. The statement of Corollary 5.4.8 includes what is referred to in the literature as the Tian-Todorov lemma [1, 8, 35, 36]. This lemma arises here as a natural consequence of our theory of differential bigraded Batalin-Vilkovisky algebras having as underlying bigraded Gerstenhaber algebra a crossed product Gerstenhaber algebra. Notice that the description (4.2.4) of the Gerstenhaber bracket (4.2.1) generated by $\partial_\Omega$ shows that this bracket does not just involve the shuffle product of $\overline{\partial}$-forms and the Schouten-Nijenhuis bracket of sections of the exterior powers of the
holomorphic tangent bundle, and two other terms (spelled out in greater generality in (4.2.4)) come into play, cf. (4.5) above.

**Remark.** When $M$ is compact, by Serre duality, the statement of Corollary 5.4.8 holds as well with the holomorphic and antiholomorphic tangent bundles interchanged.

Whether or not $M$ has a holomorphic volume form, under the present circumstances, (5.4.6.1) has the form

$$
\tag{5.4.10} (\Gamma(\Lambda^*\tau M \otimes \Lambda^*\tau_M), [\cdot, \cdot], \partial) \rightarrow (\Gamma(\Lambda^*\tau_M \otimes \Lambda^n-\tau_M \otimes \Lambda^n\tau_M), \partial)
$$

and is in fact an isomorphism of chain complexes from the $\partial$-forms with values in the exterior powers of the holomorphic tangent bundle onto the $\partial$-forms with values in the exterior powers of the holomorphic cotangent bundle, tensored with the highest exterior power of the holomorphic tangent bundle.

**Corollary 5.4.11.** Suppose that the highest exterior power of the holomorphic tangent bundle has merely a holomorphic connection $\nabla'$ which is not necessarily flat. Via (5.4.10), the corresponding operator $d\nabla'$ of covariant derivative on the right-hand side of (5.4.10) induces on the left-hand side ($\Gamma(\Lambda^*\tau_M \otimes \Lambda^*\tau_M, \partial)$ (of (5.4.10)), that is, on the differential bigraded Gerstenhaber algebra of $\partial$-forms with values in the exterior powers of the holomorphic tangent bundle, a generator $\Delta'$ so that ($\Gamma(\Lambda^*\tau_M \otimes \Lambda^*\tau_M), \Delta', \partial$) is a weak differential bigraded Batalin-Vilkovisky algebra.

We note that the compatibility condition $[\Delta', \partial] = 0$, which defines part of the structure of the weak differential bigraded Batalin-Vilkovisky algebra occurring in the statement of Corollary 5.4.11, corresponds precisely to the holomorphicity of the connection $\nabla'$.

**Proof.** This follows at once from Theorem 5.4.6. □

**Direct proof of Theorem 5.4.7.** The $n$-form $\Omega$ induces an isomorphism

$$
\tag{5.4.7.3} (\text{Alt}_A^*(L'', \text{Alt}_A^{n-*}(L', \Lambda^n L'))), d'') \rightarrow (\text{Alt}_A^*(L'', \text{Alt}_A^{n-*}(L', A)), d'')
$$

of chain complexes. This relies on the fact that $d''\Omega = 0$ (i.e. the holomorphicity of $\Omega$ under the circumstances of (5.4.8)). The composite of (5.4.7.3) with (5.4.6.1) yields the asserted isomorphism (5.4.7.2) of chain complexes from ($\text{Alt}_A^*(L'', \Lambda^n L'), d'')$ onto ($\text{Alt}_A^*(L'', \text{Alt}_A^{n-*}(L', A)), d'')$. Under this isomorphism, to the operator $d'$ on ($\text{Alt}_A^*(L'', \text{Alt}_A^{n-*}(L', A)), d''$) corresponds to a generator $\partial_\Omega$ for the bigraded Gerstenhaber algebra structure on ($\text{Alt}_A^*(L'', \Lambda^n L'), d''$). Since ($\text{Alt}_A^*(L'', \text{Alt}_A^{n-*}(L', A)), d'', d')$ is a bicomplex, $[d', d''] = 0$. Consequently $[\partial_\Omega, d'] = 0$ on $\text{Alt}_A^*(L'', \Lambda^n L')$, that is, ($\text{Alt}_A^*(L'', \Lambda^n L'), \partial_\Omega, d'')$ is a differential bigraded Batalin-Vilkovisky algebra. □

**Addendum to the proof.** The $n$-form $\Omega$ plainly endows $\Lambda^n L'$ with a left $(A, L')$-module structure. In view of Theorem 5.3.7, $\Omega$ thus induces a generator for the bigraded Gerstenhaber bracket $[\cdot, \cdot]$ on $\text{Alt}_A(L'', \Lambda^n L')$; this generator is just $\partial_\Omega$.

6. Twilled Lie-Rinehart algebras and differential homological algebra

In this section we give an interpretation of twilled Lie-Rinehart algebras in the framework of differential homological algebra. In the next section, we will use
this interpretation to deduce a differential homological algebra interpretation of the generator of a differential bigraded Batalin-Vilkovisky algebra arising from a twilled Lie-Rinehart algebra.

Let \((\mathcal{A}, \mathcal{L})\) be a differential graded Lie-Rinehart algebra. The universal object \((U(\mathcal{A}, \mathcal{L}), \iota_\mathcal{L}, \iota_\mathcal{A})\) for \((\mathcal{A}, \mathcal{L})\) is a differential graded \(R\)-algebra \(U(\mathcal{A}, \mathcal{L})\) together with a morphism \(\iota_\mathcal{A}: \mathcal{A} \to U(\mathcal{A}, \mathcal{L})\) of differential graded \(R\)-algebras and a morphism \(\iota_\mathcal{L}: \mathcal{L} \to U(\mathcal{A}, \mathcal{L})\) of differential graded Lie algebras over \(R\) having the properties

\[
\iota_\mathcal{A}(a)\iota_\mathcal{L}(\alpha) = \iota_\mathcal{L}(a\alpha),
\]

\[
\iota_\mathcal{L}(\alpha)\iota_\mathcal{A}(a) - (-1)^{|\alpha||a|}\iota_\mathcal{A}(a)\iota_\mathcal{L}(\alpha) = \iota_\mathcal{A}(\alpha(a)),
\]

and \((U(\mathcal{A}, \mathcal{L}), \iota_\mathcal{L}, \iota_\mathcal{A})\) is universal among triples \((B, \phi_\mathcal{L}, \phi_\mathcal{A})\) having these properties. More precisely:

**6.1.1.** Given (i) another differential graded \(R\)-algebra \(B\), viewed at the same time as a differential graded Lie algebra over \(R\),

(ii) a morphism \(\phi_\mathcal{L}: \mathcal{L} \to B\) of differential graded Lie algebras over \(R\), and

(iii) a morphism \(\phi_\mathcal{A}: \mathcal{A} \to B\) of differential graded \(R\)-algebras,

so that, for \(\alpha \in \mathcal{L}\), \(a \in \mathcal{A}\),

\[
\phi_\mathcal{A}(a)\phi_\mathcal{L}(\alpha) = \phi_\mathcal{L}(a\alpha),
\]

\[
\phi_\mathcal{L}(\alpha)\phi_\mathcal{A}(a) - (-1)^{|\alpha||a|}\phi_\mathcal{A}(a)\phi_\mathcal{L}(\alpha) = \phi_\mathcal{A}(\alpha(a)),
\]

there is a unique morphism \(\Phi: U(\mathcal{A}, \mathcal{L}) \to B\) of differential graded \(R\)-algebras so that

\[
\Phi \iota_\mathcal{A} = \phi_B, \quad \Phi \iota_\mathcal{L} = \phi_\mathcal{L}.
\]

The universal differential graded algebra \(U(\mathcal{A}, \mathcal{L})\) may be obtained in the customary way as the quotient of the differential graded tensor \(R\)-algebra \(T(\mathcal{A} \oplus \mathcal{L})\) of the direct sum \(\mathcal{A} \oplus \mathcal{L}\), viewed merely as an \(R\)-module, by the differential graded ideal generated in \(T(\mathcal{A} \oplus \mathcal{L})\) by all elements

\[
\alpha \otimes_R \beta - (-1)^{|\alpha||\beta|}\beta \otimes_R \alpha - [\alpha, \beta], \quad \alpha \otimes_R a - (-1)^{|\alpha||a|}a \otimes_R \alpha - \alpha(a),
\]

for \(a \in \mathcal{A}\) and \(\alpha, \beta \in \mathcal{L}\). The morphisms \(\iota_\mathcal{A}\) and \(\iota_\mathcal{L}\) are then the obvious ones. Thus, as a graded \(R\)-algebra, \(U(\mathcal{A}, \mathcal{L})\) is generated by the \(a \in \mathcal{A}\) and the \(\alpha \in \mathcal{L}\) subject to the relations

\[
\alpha \beta - (-1)^{|\alpha||\beta|}\beta \alpha = [\alpha, \beta]
\]

\[
\alpha a - (-1)^{|\alpha||a|}a \alpha = \alpha(a),
\]

for \(a \in \mathcal{A}\) and \(\alpha, \beta \in \mathcal{L}\). Furthermore, since \(U(\mathcal{A}, \mathcal{L})\) is generated by \(\mathcal{A}\) and \(\mathcal{L}\), the differential \(d\) (say) on \(\mathcal{A}\) and \(\mathcal{L}\) extends to a unique differential on \(U(\mathcal{A}, \mathcal{L})\) provided it extends at all. However, the differential is compatible with (6.1.4.1) by assumption, and a little thought reveals that it is compatible with the relations (6.1.4.2), whence the differential extends to a unique differential \(d\) on \(U(\mathcal{A}, \mathcal{L})\). In particular, as a graded \(\mathcal{A}\)-module, \(U(\mathcal{A}, \mathcal{L})\) is generated by monomials of the kind
\(\alpha_1 \ldots \alpha_m\) of arbitrary length \(m\), where \(\alpha_j \in \mathcal{L}\), subject to certain relations involving commutators of various kinds; such a monomial is the class of \(\alpha_1 \otimes_R \cdots \otimes_R \alpha_m\) in \(U(A, \mathcal{L})\). The interpretation of the term “monomial” requires some care, though, since for example when \(\alpha \in \mathcal{L}\) has odd degree, \(\alpha^2\) is zero in \(U(A, \mathcal{L})\). A more explicit description of the universal graded algebra \(U(A, \mathcal{L})\) for the special case where \((A, \mathcal{L})\) is a crossed product Lie-Rinehart algebra will be given below.

If \(A = R\) with trivial \(\mathcal{L}\)-action, so that \(\mathcal{L}\) is just an ordinary differential graded Lie algebra over \(R\), the object \((U(R, \mathcal{L}), \iota_\mathcal{L}, \iota_R)\) is the ordinary universal differential graded algebra for \(\mathcal{L}\) (over \(R\)). If \((A, \mathcal{L})\) is concentrated in degree zero, that is, an ordinary (ungraded) Lie-Rinehart algebra, the universal algebra \(U(A, \mathcal{L})\) comes down to the corresponding ordinary ungraded universal algebra; an explicit description thereof may be found e.g. in [10], [33]. When \(A\) is the algebra of smooth functions and \(L\) the Lie algebra of smooth vector fields on a smooth manifold \(M\), \(U(A, L)\) is the algebra of globally defined differential operators on \(M\).

It is obvious that, for an arbitrary differential graded Lie-Rinehart algebra \((A, \mathcal{L})\), there is a one-one correspondence between differential graded (left) \((A, \mathcal{L})\)-modules and differential graded (left) \(U(A, \mathcal{L})\)-modules; this correspondence is an equivalence of categories. In particular, the obvious differential graded (left) \((A, \mathcal{L})\)-module structure on \(A\) mentioned above turns \(A\) into a differential graded left \(U(A, \mathcal{L})\)-module; the corresponding structure map is given by

\[
U(A, \mathcal{L}) \otimes_R A \longrightarrow A, \quad \alpha \otimes_R a \mapsto \alpha(a),
\]

where \(\alpha \in \mathcal{L}, a \in A\). Next, let \(\varepsilon : U(A, \mathcal{L}) \longrightarrow A\) be the morphism of differential graded left \(U(A, \mathcal{L})\)-modules given by

\[
\varepsilon(a) = a, \quad \varepsilon(aa) = 0, \quad \varepsilon(aa) = \alpha(a).
\]

It is not a morphism of differential graded algebras unless \(\mathcal{L}\) acts trivially on \(A\), and its kernel is the differential graded left ideal in \(U(A, \mathcal{L})\) generated by \(\mathcal{L}\). In particular, the composite \(\varepsilon \iota_A\) is the identity map of \(A\) whence \(\iota_A\) is injective. Henceforth we will identify \(A\) with its image in \(U(A, \mathcal{L})\), and we will not distinguish in notation between the elements of \(A\) and their images in \(U(A, \mathcal{L})\). Furthermore, it is clear that, given two differential graded Lie-Rinehart algebras \((A_1, \mathcal{L}_1)\) and \((A_2, \mathcal{L}_2)\), a morphism

\[
(\phi, \psi) : (A_1, \mathcal{L}_1) \longrightarrow (A_2, \mathcal{L}_2)
\]

of differential graded Lie-Rinehart algebras induces a morphism

\[
U(\phi, \psi) : U(A_1, \mathcal{L}_1) \longrightarrow U(A_2, \mathcal{L}_2)
\]

of differential graded \(R\)-algebras.

Under the circumstances of (2.8), the universal graded algebra \(U = U(A, \mathcal{L})\) (with zero differential) may be obtained as follows: The graded left \((A, L)\)-module structure on \(A\) induces a graded left \(U(A, L)\)-module structure on \(A\), where \(U(A, L)\) refers to the ordinary universal algebra of \((A, L)\) mentioned above. Let

\[
U = A \otimes_A U(A, L);
\]
further, given \( a \in \mathcal{A} \) and \( u \in U(A,L) \), identified in notation with \( a \otimes_A 1 \) and \( 1 \otimes_A u \), respectively, define the product \( au \) in the obvious way and let

\[
(6.2.2) \quad ua = au + u(a)
\]

where \((u,a) \mapsto u(a)\) refers to the \( U(A,L) \)-action on \( \mathcal{A} \). Since \( L \) acts on \( \mathcal{A} \) by derivations, this construction yields a graded \( R \)-algebra structure on \( U = \mathcal{A} \otimes_A U(A,L) \) and, together with the obvious morphisms

\[
(6.2.3) \quad \iota_L : L \to U, \quad \iota_A : \mathcal{A} \to U,
\]

the graded \( R \)-algebra \( U \) is the universal graded algebra \( U(\mathcal{A},L) \) for \( (\mathcal{A},L) \). Thus, since as an \( R \)-algebra, \( U(A,L) \) is generated by the \( a \in A \) and the \( \alpha \in L \) subject to the relations

\[
(6.2.4) \quad \alpha \beta - \beta \alpha = [\alpha, \beta], \quad \alpha a - a \alpha = \alpha(a),
\]

we see that, as a graded \( R \)-algebra, \( U(\mathcal{A},L) \) is generated by the \( a \in \mathcal{A} \) and the \( \alpha \in L \) subject to the relations

\[
(6.2.5) \quad \alpha \beta - \beta \alpha = [\alpha, \beta], \quad \alpha a - a \alpha = \alpha(a),
\]

for \( a \in \mathcal{A} \) and \( \alpha, \beta \in L \). For clarity we point out that the non-trivial fact to be verified here is that the algebra abstractly defined by the generators \( a \in \mathcal{A} \) and \( \alpha \in L \) and the relations \( (6.2.5) \) indeed admits the concrete description given by \( (6.2.1) \) and \( (6.2.2) \).

Under the circumstances of \((2.9)\), given a left \((A,L)\)-module \( M \) which, as an \( A \)-module, is finitely generated and projective, as a graded \( U(A,L) \)-module and as a graded \( \mathcal{A} \)-module, the universal algebra \( U(\mathcal{A},L) \) may be written \( \text{Alt}_A(M,U(A,L)) \). More precisely, given \( \phi \in \text{Alt}_A(M,A) \) and \( w \in U(A,L) \), define \( \phi_w \in \text{Alt}_A(M,U(A,L)) \) by

\[
\phi_w(\xi_1, \ldots, \xi_m) = (\phi(\xi_1, \ldots, \xi_m))w,
\]

where \( \xi_1, \ldots, \xi_m \in M \). It is manifest that the canonical morphism

\[
(6.2.6) \quad \text{Alt}_A(M,A) \otimes_A U(A,L) \to \text{Alt}_A(M,U(A,L))
\]

of graded left \( \text{Alt}_A(M,A) \)-modules and right \( U(A,L) \)-modules given by the assignment to \( \phi \otimes_A w \in \text{Alt}_A(M,A) \otimes_A U(A,L) \) of \( \phi_w \in \text{Alt}_A(M,U(A,L)) \) is an isomorphism. We note that the multiplication in \( \text{Alt}_A(M,U(A,L)) \) is not given a shuffle map, though, and additional terms of the kind spelled out in \((4.2.4)\) come into play; the shuffle map would not even be well defined since \( U(A,L) \) is not an \( A \)-algebra.

Let \((A,L',L'')\) be a twilled Lie-Rinehart algebra having \( L' \) and \( L'' \) finitely generated and projective as \( A \)-modules. By Theorem 3.2, the Lie-Rinehart differential \( d'' \) turns the graded crossed product Lie-Rinehart algebra

\[
(6.3.1) \quad (\mathcal{A}'', \mathcal{L}') = (\text{Alt}_A^*(L'', A), \text{Alt}_A^*(L'', L'))
\]

into a differential graded Lie-Rinehart algebra. Consider its universal differential graded \( R \)-algebra \( U(\mathcal{A}'', \mathcal{L'};d'') \). Its underlying graded \( R \)-algebra structure has been
given in (6.2.1) and (6.2.2) above and, in view of (6.2.6) the underlying graded $A$-module may be written in the form $\text{Alt}_A(L'', U(A, L'))$. We now seek an explicit description of the differential $d''$. Extend the left $(A, L'')$-module structure (1.4.4) on $L'$ (which is part of the structure of a twilled Lie-Rinehart algebra) to a pairing

\[(6.3.2) \quad : L'' \otimes_R U(A, L') \to U(A, L')\]

by means of the recursive formula

\[(6.3.3) \quad \xi \cdot (x_1 \ldots x_m) = (\xi \cdot x_1)x_2 \ldots x_m \]

\[- (x_1 \cdot \xi) (x_2 \ldots x_m) + x_1(\xi \cdot (x_2 \ldots x_m))\]

where $x_1, \ldots, x_m \in L'$ and $\xi \in L''$, where $x_1 \ldots x_m$ and $x_2 \ldots x_m$ refer to the corresponding elements of $U(A, L')$ written out as monomials, and where $(\xi \cdot x_1)x_2 \ldots x_m$ and $x_1(\xi \cdot (x_2 \ldots x_m))$ are the corresponding elements of $U(A, L')$, $\xi \cdot (x_2 \ldots x_m)$ being supposed already defined. We note that, there is initially no need for the expression on the right-hand side of (6.3.3) to be well defined since the element $x_1 \ldots x_m$ of $U(A, L')$ depends on the order of the factors $x_1, \ldots, x_m$. When $(A, L', L'')$ is only a pre-twilled Lie-Rinehart algebra, the right-hand side of (6.3.3) will in general not be well defined.

**Lemma 6.3.** For $x_1, \ldots, x_m \in L'$ and $\xi \in L''$, the differential $d''$ of the universal differential graded algebra $U(A'', L'; d''')$ satisfies the formula

\[(d''(x_1 \ldots x_m))(\xi) = \xi \cdot (x_1 \ldots x_m)\]

and is determined by it.

Before proving the Lemma, we observe that, given $x \in L'$ and $w \in U(A, L')$, under the isomorphism (6.2.6), $(d''x) \otimes w \in \text{Alt}_A(L'', A) \otimes_A U(A, L')$ manifestly goes to $\phi_{x,w} \in \text{Alt}_A(L'', U(A, L'))$ which, for $\xi \in L''$, is defined by $\phi_{x,w}(\xi) = (\xi \cdot x)w$.

**Proof.** Let $x_1, \ldots, x_m \in L'$ and write $x = x_1$ and $w = x_2 \ldots x_m$. With reference to the description (6.2.1) and (6.2.2) of the differential graded algebra $U(A'', L')$, we have

\[d''(x_1 \ldots x_m) = d''(xw) = (d''x)w + x(d''w).\]

Let $\xi \in L''$. In view of the observation made just after the statement of the Lemma,

\[((d''x)w)(\xi) = (\xi \cdot x)w \in U(A, L').\]

We now assert that

\[(x(d''w))(\xi) = -(x \cdot \xi) \cdot w + x(d''w)(\xi).\]

In order to see this, we write $d''w = \sum_j \alpha_j w_j$, for suitable $\alpha_j \in \text{Alt}_A(L'', A)$ and $w_j \in U(A, L')$. Now the product $xd''w(\xi)$ of $x$ and $d''w(\xi)$ in $U(A'', L')$ may be written $xd''w(\xi) = \sum_j x \alpha_j(\xi) w_j$ whence

\[xd''w(\xi) = \sum_j x \alpha_j(\xi) w_j = \sum_j \alpha_j(\xi) x w_j + \sum_j x(\alpha_j(\xi)) w_j,\]

\[(d''w)(x \cdot \xi) = \sum_j \alpha_j(x \cdot \xi) w_j;\]
further, \((x(d''w)) = \sum_j x\alpha_j w_j = \sum_j (\alpha_j x + x \cdot \alpha_j) w_j \) whence
\[
(x(d''w))(\xi) = \sum_j (\alpha_j(\xi)x + (x \cdot \alpha_j)(\xi))w_j \\
= \sum_j \alpha_j(\xi)xw_j + \sum_j x(\alpha_j(\xi))w_j - \sum_j \alpha_j(x \cdot \xi)w_j \\
= x((d''w)(\xi)) - (d''w)(x \cdot \xi).
\]
Hence \((x(d''w))(\xi) = -(x \cdot \xi) \cdot w + x(dw''(\xi))\) as asserted. Consequently
\[
((d''x)w)(\xi) = ((d''x)w)(\xi) + (x(d''w))(\xi) = (\xi \cdot x)w - (x \cdot \xi) \cdot w + x(dw''(\xi)).
\]
By induction on the length of monomials (in the \(x_j\)'s) we may assume that \(dw''(\xi) = \xi \cdot w\) whence the assertion. \(\square\)

Since for an arbitrary Lie-Rinehart algebra \((A,L)\) and an \(A\)-module \(M\), \(L\) and 
\(M\) being projective as \(A\)-modules, the structure of a differential in \(\text{Alt}_A(L,M)\) is 
equivalent to a left \((A,L)\)-module structure on \(M\), the Lemma entails at once the following.

**Theorem 6.4.** For a twilled Lie-Rinehart algebra \((A,L',L'')\) having \(L'\) and \(L''\) 
finitely generated and projective as \(A\)-modules, the pairing \((6.3.2)\) is a left \((A,L'')\)- 
module structure on \(U(A,L')\) which extends the \((A,L'')\)-module structure \((1.4.4)\) on 
\(L'\) (which is part of the structure of a twilled Lie-Rinehart algebra), and the Lie-
Rinehart differential \(d''\) on \(\text{Alt}_A(L'',U(A,L'))\) with respect to this left \((A,L'')\)-module structure on \(U(A,L')\) turns
\[
U(A'',L';d'') = (\text{Alt}_A(L'',U(A,L'))), d'')
\]
into a differential graded \(R\)-algebra in such a way that \((U(A'',L'),d'')\) is the universal 
differential graded algebra for the differential graded crossed product Lie-Rinehart 
algbera \((A'',L';d'')\). \(\square\)

In particular, we see that, when \((A,L',L'')\) is a twilled Lie-Rinehart algebra, 
the expression on the right-hand side of \((6.3.3)\) is well defined. Here and below 
we will write \(U(A'',L';d'')\) for the universal differential graded algebra; when only 
the underlying universal graded algebra is understood (i.e. when differentials are 
ignored), we write \(U(A'',L')\). We note again that, for reasons explained before, the 
multiplication in \(\text{Alt}_A(L'',U(A,L'))\) is not a shuffle product. It is clear that, with 
the roles of \(L'\) and \(L''\) interchanged, the statement of the theorem is true as well.

The left \((A,L'')\)-module structure \((6.3.2)\) on \(U(A,L')\) may be explained in another 
way which we now explain briefly: Let \(L = L'' \cdot L'\) be the twilled sum of \(L'\) and 
\(L''\). From the Poincaré-Birkhoff-Witt theorem for \(U(A,L)\), cf. [33], we deduce at 
once that \(U(A,L)\) may be written as the tensor product over \(A\) of \(U(A,L')\) and 
\(U(A,L'')\). As an \(R\)-algebra, \(U(A,L)\) is what might be called the **twilled product** 
of \(U(A,L'')\) and \(U(A,L')\) but we do not explain this here since we will not need 
it. Let \(\varepsilon'' : U(A,L'') \to A\) be the corresponding morphism of left \(U(A,L'')\)-modules 
troduced (in somewhat greater generality) in \((6.1.6)\) above. Exploiting the fact 
that, as an \(A\)-module, \(U(A,L)\) is generated by monomials \(x_1 x_2 \ldots x_m \xi_1 \xi_2 \ldots \xi_\ell\),
where \( x_1, x_2, \ldots, x_m \in L' \) and \( \xi_1, \xi_2, \ldots, \xi_\ell \in L'' \), we extend \( \varepsilon'' \) to an \( A \)-module surjection

\[(6.3.4) \quad \varepsilon'': U(A, L) \to U(A, L') \]

by means of the assignments

\[
\varepsilon''(x_1x_2 \ldots x_m\xi_1\xi_2 \ldots \xi_\ell) = \begin{cases} 
  x_1x_2 \ldots x_m, & \ell = 0, \\
  0, & \ell \geq 1.
\end{cases}
\]

When \( U(A, L) \) is written as the tensor product \( U(A, L') \otimes_A U(A, L'') \), \( \varepsilon'' \) takes the form \( \text{Id} \otimes_A \varepsilon'' \). The Poincaré-Birkhoff-Witt theorem for Lie-Rinehart algebras implies that \( \varepsilon'' \) is well defined. Moreover, the kernel of \( \varepsilon'' \) is the left ideal of \( U(A, L) \) generated by \( L'' \) whence the assignment

\[
v \cdot w = \varepsilon''(vw), \quad v \in U(A, L), \ w \in U(A, L'),
\]

\( vw \) being the product in \( U(A, L) \), endows \( U(A, L') \) with a left \( (A, L) \)-module structure

\[(6.3.5) \quad \cdot : U(A, L) \otimes_R U(A, L') \to U(A, L') \]

in such a way that, \( U(A, L) \) being endowed with its obvious left \( U(A, L) \)-module structure, \( \varepsilon'' \) is a surjective morphism of \( (A, L) \)-modules. The description of \( \varepsilon'' \) as the tensor product \( \text{Id} \otimes_A \varepsilon'' \) is not compatible with the \( (A, L) \)-module structures, though. For \( w \in U(A, L') \) and \( \xi \in L'' \), the action (6.3.5) plainly satisfies

\[
\xi \cdot w = \varepsilon''[\xi, w],
\]

and the \( (A, L'') \)-module structure (6.3.2) on \( U(A, L') \) resulting from restriction and (6.3.2) may as well be described as the composite

\[(6.3.6) \quad L'' \otimes_R U(A, L') \xrightarrow{\iota_L \otimes \text{Id}} U(A, L'') \otimes_R U(A, L') \xrightarrow{[\cdot, \cdot]} U(A, L) \xrightarrow{\varepsilon''} U(A, L') \]

where \( [\cdot, \cdot] : U(A, L'') \otimes_R U(A, L') \to U(A, L) \) refers to the restriction of the commutator bracket on \( U(A, L) \otimes_R U(A, L') \) to \( U(A, L'') \otimes_R U(A, L') \) (viewed as a subspace of \( U(A, L) \otimes_R U(A, L) \)). It may then readily be seen directly that (6.3.6) yields a left \( (A, L'') \)-module structure on \( U(A, L') \): Since the commutator bracket in \( U(A, L) \) satisfies the Jacobi identity, for \( \xi, \eta \in L'' \) and \( w \in U(A, L') \), we plainly have

\[
[\xi, \eta] \cdot w = \xi \cdot (\eta \cdot w) - \eta \cdot (\xi \cdot w).
\]

Furthermore, given \( a \in A \), \( \xi \in L'' \), and \( w \in U(A, L') \),

\[
(a\xi) \cdot w = \varepsilon''(a\xi w - wa\xi) = \varepsilon''(a\xi w) = a\varepsilon''(\xi w) = a(\xi \cdot w)
\]

\[
\xi \cdot (aw) = \varepsilon''(\xi aw - aw\xi) = \varepsilon''((a\xi + \xi(a))w - aw\xi) = \xi(a)w + a(\xi \cdot w)
\]

whence (6.3.6) yields a left \( (A, L'') \)-module structure on \( U(A, L') \), and this structure plainly extends the left \( (A, L'') \)-module structure (1.4.4) on \( L' \).
We now proceed towards the differential homological algebra interpretation of twilled Lie-Rinehart algebras alluded to earlier. To begin with, let \((A,L)\) be an (ordinary ungraded) Lie-Rinehart algebra. Consider the graded left \(U(A,L)\)-module \(U(A,L) \otimes_A \Lambda_A L\) where \(A\) acts on the right of \(U(A,L)\) by means of the canonical map \(\iota_A: A \rightarrow U(A,L)\). For \(u \in U(A,L)\) and \(\alpha_1, \ldots, \alpha_n \in L\), let

\[
d(u \otimes_A (\alpha_1 \ldots \wedge \alpha_n)) = \sum_{i=1}^{n} (-1)^{(i-1)} u\alpha_i \otimes_A (\alpha_1 \ldots \hat{\alpha}_i \ldots \wedge \alpha_n) + \sum_{j<k} (-1)^{(j+k)} u \otimes_A ([\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \ldots \hat{\alpha}_j \ldots \hat{\alpha}_k \ldots \wedge \alpha_n),
\]

(6.5)

Rinehart [33] has shown that this yields an \(U(A,L)\)-linear differential

\[
d: U(A,L) \otimes_A \Lambda_A L \rightarrow U(A,L) \otimes_A \Lambda_A L,
\]

(6.6)

that is, \(dd\) is zero. The non-trivial fact to be verified here is that the operator \(d\) is well defined. We will refer to

\[
K(A, L) = (U(A,L) \otimes_A \Lambda_A L, d)
\]

(6.7)

as the Rinehart complex for \((A,L)\). Rinehart has also proved that, when \(L\) is projective as an \(A\)-module, \(K(A, L)\) is a projective resolution of \(A\) in the category of left \(U(A,L)\)-modules. This resolution generalizes the Koszul resolution of the ground ring (or ground field) in ordinary Lie algebra cohomology.

Let now \(A\) be a graded commutative \(A\)-algebra which is endowed with a graded left \((A,L)\)-module structure on \(A\) in such a way that the underlying \(L\)-action on \(A\) is by graded derivations and that the canonical map from \(A\) to \(A\) is a morphism of \((A,L)\)-modules. Consider the graded crossed product Lie-Rinehart algebra \((A,L)\) introduced in (2.8) above. Let

\[
K = K(A, L) = A \otimes_A K(A, L).
\]

(6.8)

Plainly, \(K\) is of the form

\[
K = (A \otimes_A U(A,L) \otimes_A \Lambda_A L, d) = (U(A,L) \otimes_A \Lambda_A L, d).
\]

With the obvious induced left \(U(A,L)\)-module structure,

\[
K_n \xrightarrow{d} K_{n-1} \xrightarrow{d} \ldots \xrightarrow{d} K_1 \xrightarrow{d} U(A,L) \xrightarrow{e} A
\]

(6.9)

is then an exact sequence of graded left \(U(A,L)\)-modules. Thus, when \(A\) is projective as an \(A\)-module, (6.8) yields a projective resolution of \(A\) in the category of graded left \(U(A,L)\)-modules. Hence we may then use this resolution to compute, for any graded left \((A,L)\)-module \(M\), the cohomology

\[
H^*(L,M) = \text{Ext}^*_U(A,L)(A,M)
\]

(6.10.1)
of $\mathcal{L}$ with coefficients in $\mathcal{M}$ and, for any graded right $(A,\mathcal{L})$-module $\mathcal{N}$, the homology

$$H_+(\mathcal{L},\mathcal{N}) = \text{Tor}_+^{U(A,\mathcal{L})}(\mathcal{N}, A)$$

of $\mathcal{L}$ with coefficients in $\mathcal{N}$. Since the resolution (6.8) has the form $A \otimes_A K(A, L)$, we see at once that the canonical isomorphisms

$$\text{Hom}_{U(A, L)}(K(A, L), \mathcal{M}) \to \text{Hom}_{U(A, \mathcal{L})}(K(A, \mathcal{L}), \mathcal{M})$$

and

$$\mathcal{N} \otimes_{U(A, L)} K(A, L) \to \mathcal{N} \otimes_{U(A, \mathcal{L})} K(A, \mathcal{L})$$

induces isomorphisms

$$H^+(L, \mathcal{M}) \to H^+(\mathcal{L}, \mathcal{M}) \quad \text{and} \quad H_+(\mathcal{L}) \to H_+(\mathcal{N})$$

of $R$-modules. Thus the homology and cohomology of $\mathcal{L}$ boil down to the homology and cohomology of $L$ with graded coefficients.

Let now $(A, L', L'')$ be a twilled Lie-Rinehart algebra having $L'$ and $L''$ finitely generated and projective as $A$-modules. Consider the differential graded crossed product Lie-Rinehart algebra

$$(A'', \mathcal{L}'') = (\text{Alt}_A^*(L'', A), \text{Alt}_A^*(L'', L'); d'');$$

cf. Theorem 3.2 above. The corresponding Rinehart complex $K(A'', \mathcal{L}''; d'')$, cf. (6.8) above, is plainly of the form

$$(\text{Alt}_A^*(L'', K(A, L')), d') = (\text{Alt}_A^*(L'', U(A, L') \otimes_A \Lambda A L'), d'),$$

that is, may be written

$$(6.13) \quad \ldots \to K_j(A'', \mathcal{L}') \xrightarrow{d'} K_{j-1}(A'', \mathcal{L}') \xrightarrow{d'} \ldots \xrightarrow{d'} K_1(A'', \mathcal{L}') \xrightarrow{d'} K_0(A'', \mathcal{L}')$$

where for $j \geq 0$, $K_j(A'', \mathcal{L}') = \text{Alt}_A(L'', K_j(A, L'))$, and the latter, in turn, is isomorphic to $\text{Alt}_A(L'', U(A, L') \otimes_A \Lambda A L')$; as additional piece of structure, each $K_j(A'', \mathcal{L}')$ now also carries the differential $d''$ (with respect to the Lie-Rinehart structure on $(A, L'')$ and the left $(A, L'')$-module structure (6.3.2) on $U(A, L')$). Plainly, each $\text{Alt}_A(L'', U(A, L') \otimes_A \Lambda A L')$ is isomorphic to $\text{Alt}_A(L'', U(A, L')) \otimes_A \Lambda A L'$ which, in turn, in view of the structure of $U(A'', \mathcal{L}'; d'')$ elucidated in (6.4), is just a rewrite of $U(A'', \mathcal{L}'; d'') \otimes_A \Lambda A L'$. The complex (6.13) (in the category of $R$-chain complexes) is a proper projective resolution (cf. e. g. [28] (XII.11) p. 397 for the notion of a proper projective resolution) of $A'' = (\text{Alt}_A(L'', A), d'')$ in the category of differential graded left $U(A'', \mathcal{L}'; d'')$-modules. Thus, from (6.13) we may compute, for any differential graded left $(A'', \mathcal{L}')$-module $\mathcal{M}$, the cohomology

$$H^*(\mathcal{L}', \mathcal{M}) = \text{Ext}_{U(A'', \mathcal{L}'; d'')}^*(A'', \mathcal{M})$$

of $\mathcal{L}'$ with coefficients in $\mathcal{M}$ and, for any differential graded right $(A'', \mathcal{L}')$-module $\mathcal{N}$, the homology

$$H_+(\mathcal{L}', \mathcal{N}) = \text{Tor}_+^{U(A'', \mathcal{L}'')}\mathcal{N}, A'')$$

of $\mathcal{L}''$ with coefficients in $\mathcal{N}$.  


Theorem 6.15. For a twilled Lie-Rinehart algebra \((A, L', L'')\) having \(L'\) and \(L''\) finitely generated and projective as \(A\)-modules, the graded \(R\)-modules \(H^*(L', A'')\) and \(H^*(L, A) (=\text{Ext}_{U(A, L)}^*(A, A))\) are canonically isomorphic.

It is clear that the statement of the theorem holds as well with the roles of \(L'\) and \(L''\) interchanged.

Proof of the Theorem. Consider the bicomplex

\[
(6.15.1) \quad \text{Hom}^*_U(A'', L'; d'')(K(A'', L'; d''), A'')
\]

computing \(\text{Ext}^*_{U(A'', L'; d'')} (A'', A'')\). The bigraded \(A\)-module which underlies (6.15.1) has the form

\[
\text{Hom}_{A''}(\text{Alt}_A(L'', \Lambda_AL'), A'') \cong \text{Hom}_{A''}(A'' \otimes_A \Lambda_AL', A'')
\]

and this is clearly canonically isomorphic to

\[
(6.15.2) \quad \text{Hom}_A(\Lambda_AL', A'') \cong \text{Alt}_A(L', A'') \cong \text{Alt}_A(L', \text{Alt}_A(L'', A)) \cong \text{Alt}_A(L, A).
\]

The operator \(d'\) on \(\text{Hom}_{A''}(\text{Alt}_A(L'', \Lambda_AL'), A'')\) plainly amounts to the Lie-Rinehart differential \(d'\) on \(\text{Alt}_A(L', A'')\) with reference to the Lie-Rinehart structure on \((A, L')\) and the graded left \((A, L')\)-module structure on \(A''\); this Lie-Rinehart differential, in turn, corresponds to the operator on \(\text{Alt}_A(L, A)\) denoted by the same symbol. By construction, the operator \(d''\) on \(\text{Hom}_{A''}(\text{Alt}_A(L'', \Lambda_AL'), A'')\) is compatible with the operator \(d'\). Moreover, since \(K(A'', L')\) has the form \(\text{Alt}_A(L'', K(A, L'))\), the bigraded \(A\)-module underlying the bicomplex (6.15.1) may as well be written

\[
(6.15.3) \quad \text{Alt}_A(L'', \text{Hom}_{U(A, L')}(K(A, L'), A)) \cong \text{Alt}_A(L'', \text{Alt}_A(L', A))
\]

and, using this description, we see that the operator \(d''\) on (6.15.3), that is, the resulting operator on \(\text{Hom}_{A''}(\text{Alt}_A(L'', \Lambda_AL'), A'')\), amounts to the Lie-Rinehart differential \(d''\) on \(\text{Alt}_A(L'', A')\) with reference to the Lie-Rinehart structure on \((A, L'')\) and the graded left \((A, L'')\)-module structure on \(A' = \text{Alt}_A(L', A);\) thus the operator \(d''\) on (6.15.3) corresponds to the operator on \(\text{Alt}_A(L, A)\) denoted by the same symbol. Consequently the total differential on \(\text{Hom}_{A''}(\text{Alt}_A(L'', \Lambda_AL'), A'')\) amounts to the total differential on \(\text{Alt}_A(L, A)\) arising from the bicomplex (1.15.4). □

The theorem provides an interpretation of the bicomplex (1.15.4): The object in the middle of (6.15.1) is just the bigraded \(A\)-module underlying (1.15.4), with the roles of \(L'\) and \(L''\) interchanged. Thus, the differential \(d'\) (in (1.15.4)) alone computes the graded \(\text{Ext}^*_{U(A'', L')} (A'', A'')\), the differential \(d''\) on every object in sight being ignored. However, the compatibility between \(d'\) and \(d''\) entails that (6.13) is a resolution in the differential graded category, and the bicomplex computing \(\text{Ext}^*_{U(A'', L'; d'')} (A'', A'')\) boils down to (1.15.4). This provides, in particular, a new interpretation of the Dolbeault complex. Rinehart has shown that the ordinary de Rham cohomology groups may be written as \(\text{Ext}\)-groups over the algebra of differential operators. Theorem 6.15 includes the corresponding result for the Dolbeault cohomology groups, which now appear as differential graded \(\text{Ext}\)-groups.
7. Duality and generators of dBV algebras

Let \((A, L)\) be a Lie-Rinehart algebra. In [12] we have shown that an exact generator of a Gerstenhaber algebra of the kind \(\Lambda A L\) yields precisely the differential in the standard complex computing the homology of the Lie-Rinehart algebra \((A, L)\) with values in \(A\), endowed with a right \((A, L)\)-module structure corresponding to the generator. This relies on a notion of homological duality. Our present aim is to generalize this notion and the relationship between exact generators and differentials in the standard complex to the differential graded setting. This will give conceptual explanations of some of the results in earlier sections and will elucidate the at first somewhat mysterious concept of generator of a differential bigraded Gerstenhaber bracket.

Let \((A, L)\) be a Lie-Rinehart algebra. Recall from Section 2 above that a (graded) left \(L\)-module structure \(L \otimes_R M \to M\) on a graded \(A\)-module \(M\), written \((\alpha, x) \mapsto \alpha(x)\), is called a left \((A, L)\)-module structure provided

\[
\begin{align*}
(7.1.1) & \quad \alpha(ax) = \alpha(a)x + a\alpha(x), \\
(7.1.2) & \quad (a\alpha)(x) = a(\alpha(x)),
\end{align*}
\]

where \(a \in A\), \(x \in M\), \(\alpha \in L\). More generally, cf. (1.10), such an assignment \(L \otimes_R M \to M\), not necessarily a left \(L\)-module structure but still satisfying (7.1.1) and (7.1.2), is referred to as an \((A, L)\)-connection, cf. [10], or, somewhat more precisely, left \((A, L)\)-connection; in this language, a (graded) left \((A, L)\)-module structure is a (graded left) flat \((A, L)\)-connection. Likewise, let \(N\) be a graded \(A\)-module, and let there be given an assignment \(N \otimes_R L \to N\), written \((x, \alpha) \mapsto x \circ \alpha\) or, somewhat simpler, \((x, \alpha) \mapsto x\alpha\) (when there is no risk of confusion); it is called a (graded) right \((A, L)\)-module structure provided it is a (graded) right \(L\)-module structure and, moreover, satisfies

\[
\begin{align*}
(7.2.1) & \quad (ax)\alpha = a(x\alpha) - (\alpha(a))x, \\
(7.2.2) & \quad x(a\alpha) = a(x\alpha) - (\alpha(a))x,
\end{align*}
\]

where \(a \in A\), \(x \in N\), \(\alpha \in L\); an assignment \(N \otimes_R L \to N\) of this kind is referred to as a (graded) right \((A, L)\)-connection provided it satisfies only (7.2.1) and (7.2.2) without necessarily being a (graded) right \(L\)-module structure. A (graded) right \((A, L)\)-module structure is also said to be a (graded) flat right \((A, L)\)-connection. Graded left- and right \((A, L)\)-modules correspond to graded left- and right \(U(A, L)\)-modules, and vice versa. More generally, graded left- and right \((A, L)\)-connections may be shown to correspond bijectively to graded left- and right \(U(A, E)\)-module structures, for suitable \((R, A)\)-Lie algebras \(E\) mapping surjectively onto \(L\).

Return to the situation at the beginning of (5.3). Thus \(A\) is a graded commutative \(A\)-algebra together with an \(L\)-action \(L \otimes_R A \to A\) by derivations such that the canonical map from \(A\) to \(A\) is a morphism of left \((A, L)\)- and hence \(U(A, L)\)-modules. A (left) \((A, L)\)-connection

\[
(7.3) \quad L \otimes_R M \to M
\]
on an induced (graded) \(A\)-module of the kind \(M = A \otimes_A M\) where \(M\) is an \(A\)-module will be said to be compatible (with the \(L\)-action on \(A\)) provided

\[
(7.4) \quad \alpha(ax) = \alpha(a)x + a\alpha(x), \quad \alpha \in L, \ a \in A, \ x \in M
\]
where \((\alpha, a) \mapsto \alpha(a)\) refers to the graded left \((A, L)\)-module structure on \(A\) and, accordingly, we define a compatible (left) \((A, L)\)-module structure on an induced (graded) \(A\)-module. Furthermore, a right \((A, L)\)-connection \(M \otimes_R L \rightarrow M\) on an induced (graded) \(A\)-module \(M = A \otimes_A M\), \(M\) being an \(A\)-module, will be said to be compatible (with the \(L\)-action on \(A\)) provided

\[
(ax)\alpha = a(x\alpha) - (\alpha(a))x, \quad \alpha \in L, \quad a \in A, \quad x \in M
\]

where \((\alpha, a) \mapsto \alpha(a)\) still refers to the graded left \((A, L)\)-module structure on \(A\) and we can accordingly talk about a compatible right \((A, L)\)-module structure on an induced (graded) \(A\)-module. It is clear that, given a left \(A\)-module \(M\), any compatible left or right \((A, L)\)-connection or left or right \((A, L)\)-module structure on an induced module of the kind \(A \otimes_A M\), is determined by its restriction to \(M\).

Theorem 1 of [12] now generalizes in the following way.

**Theorem 7.6.** There is a bijective correspondence between right \((A, L)\)-connections on \(A\) and hence compatible right \((A, L)\)-connections on \(A\) and \(R\)-linear operators generating the bigraded Gerstenhaber bracket on \(\Lambda_A L\). Under this correspondence, flat right \((A, L)\)-connections on \(A\) and hence compatible ones on \(A\), that is, compatible right \((A, L)\)-module structures on \(A\), correspond to exact generators. More precisely: Given an \(R\)-linear operator \(D\) generating the bigraded Gerstenhaber bracket on \(\Lambda_A L\), the formula

\[
a \circ \alpha = a(D\alpha) - \alpha(a), \quad a \in A, \quad \alpha \in L.
\]

defines a right \((A, L)\)-connection on \(A\), and this right \((A, L)\)-connection on \(A\) is determined by

\[
a \circ \alpha = a(D\alpha) - \alpha(a), \quad a \in A, \quad \alpha \in L.
\]

Conversely, given a compatible right \((A, L)\)-connection \((a, \alpha) \mapsto a \circ \alpha\) on \(A\) \((a \in A, \alpha \in L)\), the operator \(D\) on \(\Lambda_A L\) defined by means of the formula

\[
D(a\alpha_1 \wedge \ldots \wedge \alpha_n) = \sum_{i=1}^{n} (-1)^{(i-1)} (a \circ \alpha_i) \alpha_1 \wedge \ldots \hat{\alpha}_i \ldots \wedge \alpha_n
\]

\[
+ \sum_{j<k} (-1)^{(j+k)} a[\alpha_j, \alpha_k] \wedge \alpha_1 \ldots \hat{\alpha}_j \ldots \hat{\alpha}_k \ldots \wedge \alpha_n,
\]

where \(a \in A\) and \(\alpha_1, \ldots, \alpha_n \in L\), yields an \(R\)-linear operator \(D\) generating the bigraded Gerstenhaber bracket on \(\Lambda_A L\).

**Proof.** The argument given for the proof of Theorem 1 in [12] carries readily over. In fact, compatible right \((A, L)\)-connections on \(A\) correspond bijectively to right \((A, L)\)-connections on \(A\) and, these, in turn, correspond bijectively to generators for the ordinary Gerstenhaber algebra \(\Lambda_A L\), by virtue of Theorem 1 in [12]. Furthermore, generators for the ordinary Gerstenhaber algebra \(\Lambda_A L\) correspond bijectively to generators for the bigraded Gerstenhaber algebra \(\Lambda_A L\). \(\square\)

We note that, when \(A\) is just \(A\), the statement of the present Theorem 7.6 boils down verbatim to Theorem 1 in [12].

Given an exact generator \(\partial\) for the bigraded Gerstenhaber algebra \(\Lambda_A L\), we will accordingly write \(A_\partial\) for \(A\) together with the graded right \((A, L)\)-module structure given by (7.6.1). Theorem 2 of [12] now extends as follows where \(K(A, L)\) refers to the Rinehart complex for \((A, L)\) (reproduced as (6.7) above).
Theorem 7.7. Given an exact generator $\partial$ for the bigraded Gerstenhaber algebra $\Lambda_A L$, the chain complex underlying the bigraded Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ coincides with the chain complex $(\mathcal{A}_\partial \otimes_{U(A,L)} K(A,L), d)$. In particular, when $L$ is projective as an $A$-module, the bigraded Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ computes

$$H_*(L, \mathcal{A}_\partial) \left(= \text{Tor}_{*}^{U(A,L)}(\mathcal{A}_\partial, A) \right),$$

the homology of $L$ with coefficients in $\mathcal{A}_\partial$.

Proof. The argument given for the proof of Theorem 2 in [12] carries readily over. Details are left to the reader. $\square$

When $\mathcal{A}$ is just $A$, the statement of the present Theorem 7.7 come down verbatim to Theorem 2 in [12].

Remark 7.8. The two above theorems reveal the significance of a generator of the Gerstenhaber bracket of a Gerstenhaber algebra of the kind $\Lambda_A L$ (and in particular of the kind $\Lambda_A$): Indeed, the defining property (5.1) of an exact generator precisely incorporates a description of the Lie-Rinehart differential in the corresponding complex $(\mathcal{A}_\partial \otimes_{U(A,L)} K(A,L), d)$ in terms of the Lie bracket in $L$ and the corresponding right $(A,L)$-module structure on $\mathcal{A}$. We will show below that the extension of this observation to the differential graded setting provides a conceptual explanation of the isomorphism (5.4.6.1). For a generator $\Delta$ which is not necessarily exact, a corresponding remark can still be made: The corresponding right $(A,L)$-connection on $\mathcal{A}$ still induces an operator on $\mathcal{A} \otimes_{U(A,L)} K(A,L)$, and the defining property (5.1) merely yields a description thereof in terms of the Lie-Rinehart structure and the right $(A,L)$-connection on $\mathcal{A}$.

The above considerations entail that, under appropriate circumstances, generators for bigraded Gerstenhaber brackets exist. In order to explain this, we suppose that, as an $A$-module, $L$ is finitely generated and projective of finite constant rank $n$ and, for intelligibility, recall the following, spelled out in [12] as Theorem 3.

Proposition 7.9. There is a bijective correspondence between $(A,L)$-connections on $\Lambda_A^n L$ and right $(A,L)$-connections on $A$. Under this correspondence, left $(A,L)$-module structures on $\Lambda_A^n L$ (i.e. flat connections) correspond to right $(A,L)$-module structures on $A$. More precisely: Given an $(A,L)$-connection $\nabla$ on $\Lambda_A^n L$, the negative of the (generalized) Lie-derivative on $A \cong \text{Hom}_A(\Lambda_A^n L, M)$ with reference to the connection $\nabla$ on $M = \Lambda_A^n L$, that is, the formula

$$\left(\phi_\alpha \right) x = \phi(\alpha x) - \nabla_\alpha(\phi(x)), \quad (7.9.1)$$

where $x \in \Lambda_A^n L$, $\alpha \in L$, $\phi \in \text{Hom}_A(\Lambda_A^n L, \Lambda_A^n L) \cong A$, yields a right $(A,L)$-connection on $A$. Conversely, given a a right $(A,L)$-connection on $A$ (written $(a, \alpha) \mapsto a\alpha$), on $\Lambda_A^n L \cong \text{Hom}_A(C_L, A)$, the assignment

$$L \otimes_R \Lambda_A^n L \to \Lambda_A^n L, \quad (\alpha, \psi) \mapsto \nabla_\alpha \psi$$

where

$$\left(\nabla_\alpha \psi \right) x = \psi(x\alpha) - \psi x\alpha, \quad x \in C_L, \alpha \in L, \psi \in \text{Hom}_A(C_L, A), \quad (7.9.2)$$

yields an $(A,L)$-connection $\nabla$.

Combining Theorem 7.6 with Proposition 7.9, we obtain:
Corollary 7.10. There is a bijective correspondence between \((A,L)-\)connections on \(\Lambda^n_A L\) and linear operators \(D\) generating the Gerstenhaber bracket on \(\Lambda_A L\). Under this correspondence, flat connections correspond to exact generators, that is, to differentials. The relationship is made explicit by means of (7.6.1), (7.6.3), (7.9.1) and (7.9.2). □

The relationship in terms of (7.6.1), (7.6.3), (7.9.1) and (7.9.2) comes down precisely to that spelled out in (5.3.7), which involves (5.3.2) and (5.3.6). This observation provides another proof of Theorem 5.3.7.

Corollary 7.11. As an \(A\)-module, \(L\) being finitely generated and projective of finite constant rank \(n\), the bracket on the bigraded crossed product Gerstenhaber algebra \(\Lambda_A L\) always has a generator, and these generators are classified by \((A,L)-\)connections on \(\Lambda^n_A L\).

Thus, under the circumstances of Theorem 7.7, since \(L\) is just the graded crossed product extension of \(A\) and \(L\), the (graded) right \((A,L)-\)module structure on \(A\) extends to a graded right \((A,L\prime)-\)module structure on \(A\), and the bigraded Batalin-Vilkovisky algebra \((\Lambda_A L, \partial)\) computes as well the homology

\[ H\ast(L, \Lambda_A L, \partial) = \text{Tor}^U_A(A, L, \partial)(A, \Lambda_A L, \partial) \]

of \(L\) with coefficients in \(A\) where \(U(A, L)\) is the universal algebra for the graded Lie-Rinehart algebra \((A, L)\) explained in the previous Section.

With these preparations out of the way, let \((A, L', L'')\) be a twilled Lie-Rinehart algebra having \(L'\) and \(L''\) finitely generated and projective as \(A\)-modules, and consider the corresponding differential graded crossed product Lie-Rinehart algebra

\[(A'', L'; d'') = (\text{Alt}_A(L'', A), \text{Alt}_A(L'', L'; d'')).\]

Suppose that, as an \(A\)-module, \(L'\) (being finitely generated and projective) is of finite constant rank \(n\) (say). The statements of Theorem 7.7, Theorem 7.8, Remark 7.9, Proposition 7.10, and Corollary 7.11, extend to this situation where the role of \(L\) in (7.7) – (7.11) is played by \(L'\), and the statements are in fact compatible with the additional \((A, L'')\)-module structures. Rather than spelling out the details, we confine ourselves to describing the consequences thereof for the differential bigraded Batalin-Vilkovisky algebras studied in earlier sections.

Extending the notion of dualizing module introduced in Section 2 of [11], let

\[ C_L' = \text{Alt}_A(L'', C_L') = \text{Alt}_A(L'', \text{Hom}_A(\Lambda^n_A L', A)). \]

The reasoning in Section 2 of [11], adapted to the present differential bigraded setting, shows that \(C_L'\) is canonically isomorphic to

\[ H^n(L', U(A'', L'; d'')) \]

and hence inherits a differential graded right \(U(A'', L'; d'')\)-module structure from the obvious right \(U(A'', L'; d'')\)-module structure on \(U(A'', L'; d'')\) which remains free when the construction of \(H^n(L', U(A'', L'; d''))\) is carried out. The theory of
homological duality developed in [11] now carries over verbatim and yields natural isomorphisms

\begin{equation}
(7.12.1) \quad H^k(\mathcal{L}', \mathcal{M}) \cong H_{n-k}(\mathcal{L}', C_{\mathcal{L}'} \otimes_{A^n} \mathcal{M})
\end{equation}

for all non-negative integers \(k\) and all left \((A^n, \mathcal{L}')\)-modules \(\mathcal{M}\) and, furthermore, natural isomorphisms

\begin{equation}
(7.12.2) \quad H_k(\mathcal{L}', \mathcal{N}) \cong H^{n-k}(\mathcal{L}, \text{Hom}_{A^n}(C_{\mathcal{L}'}), \mathcal{N}))
\end{equation}

for all non-negative integers \(k\) and all right \((A^n, \mathcal{L}')\)-modules \(\mathcal{N}\). We therefore refer to \(C_{\mathcal{L}'}\) as the dualizing module of \(\mathcal{L}'\).

Under the circumstances of (5.4.6), let \(\partial'\) be an exact generator turning \((\text{Alt}_A(L'', \Lambda_A L'), \partial', d'')\) into a differential bigraded Batalin-Vilkovisky algebra. By Theorem 7.6, this generator \(\partial'\) endows \(A'' = \text{Alt}_A(L'', A)\) with a right \(U(A'', \mathcal{L}')\)-module structure and we denote the resulting right \(U(A'', \mathcal{L}')\)-module by \(A''_{\partial'\mathcal{L}'}\). Since \(\partial'\) turns \((\text{Alt}_A(L'', \Lambda_A L'), \partial', d'')\) into a differential bigraded Batalin-Vilkovisky algebra (not just into a bigraded Batalin-Vilkovisky algebra), the right \(U(A'', \mathcal{L}')\)-module \(A''_{\partial'\mathcal{L}'}\) is a differential graded right \(U(A'', \mathcal{L}'; d'')\)-module. Inspection shows the following:

**Proposition 7.13.** The chain complex \(A''_{\partial'\mathcal{L}'} \otimes_{U(A'', \mathcal{L}'; d'')} K(A'', \mathcal{L}')\) calculating

\[ H_*(\mathcal{L}', A''_{\partial'\mathcal{L}'}) = \text{Tor}_*(U(A'', \mathcal{L}'; d''), (A''_{\partial'\mathcal{L}'}, A'')) \]

boils down to the chain complex which underlies the differential bigraded Batalin-Vilkovisky algebra \((\text{Alt}_A(L'', \Lambda_A L'), \partial', d'')\) (coming into play in (5.4.6)). Thus the exact generator \(\partial'\) amounts to the differential graded Lie-Rinehart differential in the corresponding standard complex \(A''_{\partial'\mathcal{L}'} \otimes_{U(A'', \mathcal{L}'; d'')} K(A'', \mathcal{L}')\), with reference to the differential graded right \(U(A'', \mathcal{L}'; d'')\)-module structure on \(A''_{\partial'\mathcal{L}'}\). □

Likewise, by Corollary 7.10, the generator \(\partial'\) endows \(\Lambda^n_{A^n, L'} = \text{Alt}_A^*(L'', \Lambda^n_A L')\) with a left \((A'', \mathcal{L}')\)-module structure, and we denote the resulting left \((A'', \mathcal{L}')\)-module by \(\Lambda^n_{A^n, \mathcal{L}'_{\partial'}}\). Since \(\partial'\) turns \((\text{Alt}_A(L'', \Lambda_A L'), \partial', d'')\) into a differential bigraded Batalin-Vilkovisky algebra, the left \((A'', \mathcal{L}')\)-module \(\Lambda^n_{A^n, \mathcal{L}'_{\partial'}}\) is a differential graded left \((A'', \mathcal{L}'; d'')\)-module, i.e. a differential graded left \(U(A'', \mathcal{L}'; d'')\)-module. Again inspection shows the following.

**Proposition 7.14.** The chain complex \(\text{Hom}_{U(A'', \mathcal{L}'; d'')} (K(A'', \mathcal{L}'), \Lambda^n_{A^n, \mathcal{L}'_{\partial'}})\) computing

\[ H^*(\mathcal{L}', \Lambda^n_{A^n, \mathcal{L}'_{\partial'}}) = \text{Ext}_*(U(A'', \mathcal{L}'; d''), (A'', \Lambda^n_{A^n, \mathcal{L}'_{\partial'}})) \]

comes down to \((\text{Alt}_A(L'', \text{Alt}_A(L', \Lambda^n_A L')), d', d'')\). Moreover, \(\text{Hom}_{A^n}(C_{\mathcal{L}'}, A''_{\partial'\mathcal{L}'})\) is canonically isomorphic to \(\Lambda^n_{A^n, \mathcal{L}'_{\partial'}}\), and the isomorphism

\begin{equation}
(7.15) \quad (\text{Alt}_A(L'', \Lambda^*_A L'), d', d'') \rightarrow (\text{Alt}_A(L'', \text{Alt}_A^{-*}(L', \Lambda^n_A L')), \partial', d'')
\end{equation}

of chain complexes spelled out as (5.4.6.1) induces the corresponding duality isomorphism

\begin{equation}
(7.16) \quad H_*(\mathcal{L}', A''_{\partial'\mathcal{L}'}) \rightarrow \text{Hom}_{A^n}(C_{\mathcal{L}'}, A''_{\partial'\mathcal{L}'}) \cong H^{n-*}(\mathcal{L}', \Lambda^n_{A^n, \mathcal{L}'_{\partial'}})
\end{equation}
Given as (7.12.2) above, where the roles of \( L, A, N \) in (7.12.2) are played by, respectively, \( L', A', A''_0 \). \( \square \)

Thus, in view of the remarks about the Tian-Todorov Lemma (5.4.8) made in Section 5 above, this Lemma comes down to differential graded homological duality.

We conclude with the following observation: When the twilled Lie-Rinehart algebra \( (A, L', L'') \) has \( L' \) and \( L'' \) abelian, with trivial actions of \( L' \) and \( L'' \) on \( A \) and on \( L'' \) and \( L' \) (respectively), the duality isomorphism (7.16) for this special case is just the isomorphism (7.15), the operators \( d', d'', \partial' \) being ignored. Thus, for a general twilled Lie-Rinehart algebra \( (A, L', L'') \), \( (L', L'') \) still finitely generated and projective as \( A \)-modules and \( L' \) of constant rank), the isomorphism of bigraded \( A \)-modules underlying (7.15) is obtained when the (non-trivial) true twilled Lie-Rinehart structure is ignored. The true twilled Lie-Rinehart structure being considered as a "perturbation" of the trivial twilled Lie-Rinehart structure, for the duality isomorphism (7.16), this perturbation amounts to insertion of the operators \( d', d'', \partial' \) which, in turn, may be viewed as perturbations of the trivial operators.

8. Globalization

Let \( M \) be a smooth manifold, let \( A \) be the ring \( C^\infty M \) of smooth functions on \( M \), and let \( \zeta' \) and \( \zeta'' \) be Lie algebroids over \( M \), that is, \( \zeta' \) and \( \zeta'' \) are smooth real vector bundles together with \((\mathbb{R},A)\)-Lie algebra structures on the spaces of sections \( L' = \Gamma(\zeta') \) and \( L'' = \Gamma(\zeta'') \). Given a twilled Lie-Rinehart algebra structure turning \( (A, L', L'') \) into a twilled Lie-Rinehart algebra, we will say that the pair \( (\zeta', \zeta'') \) is a twilled Lie algebroid. In [26] and in [31] these objects are referred to as matched pairs of Lie algebroids. Likewise, we can consider the ring \( A^C = C^\infty(M,\mathbb{C}) \) of smooth complex functions on \( M \) and two complex vector bundles \( \zeta' \) and \( \zeta'' \); let \( L' = \Gamma(\zeta') \) and \( L'' = \Gamma(\zeta'') \) be their spaces of sections. Given a twilled Lie-Rinehart algebra structure turning \( (A^C, L', L'') \) into a twilled Lie-Rinehart algebra, we will say that the pair \( (\zeta', \zeta'') \) is a complex twilled Lie algebroid. An example of a complex twilled Lie algebroid arises from a complex structure on \( M \). Another example arises from Cauchy-Riemann structures.

Any Lie groupoid \( G \Rightarrow P \) gives rise to a Lie algebroid \( AG \). What is the corresponding object for a twilled Lie algebroid? To provide an answer to this question, we recall that, by Theorem 8.3 of [27], for any Poisson groupoid \( G \Rightarrow P \), the pair \( (AG, A^*G) \) consisting of the Lie algebroid \( AG \) and its dual \( A^*G \) inherits a Lie bialgebroid structure. Let \( (\zeta', \zeta'') \) be a twilled Lie algebroid; in view of Corollary 4.9, \( (\zeta' \ltimes (\zeta'')^*, \zeta'' \ltimes (\zeta')^*) \) then inherits a Lie bialgebroid structure. We define a corresponding Lie groupoid to be a Poisson groupoid \( G \Rightarrow P \) such that the pair \( (AG, A^*G) \) is isomorphic to \( (\zeta' \ltimes (\zeta'')^*, \zeta'' \ltimes (\zeta')^*) \) as a Lie bialgebroid. Such a Poisson groupoid globalizes the notion of twilled Lie algebroid or of matched pair of Lie algebroids. What remains to be done is first to single out explicitly those Poisson groupoids \( G \Rightarrow P \) such that the pair \( (AG, A^*G) \) is of the kind \( (\zeta' \ltimes (\zeta'')^*, \zeta'' \ltimes (\zeta')^*) \), and thereafter to give an intrinsic description of the structure which thus emerges in terms of groupoids alone. We hope to return to this at another occasion. This kind of groupoid might also lead to a concept of groupoid which integrates a general complex Lie algebroid. It will certainly integrate those
complex Lie algebroids $\eta$ which come together with their complex conjugate $\bar{\eta}$ in such a way that $\eta \oplus \bar{\eta}$ carries the Lie algebroid structure which corresponds to a twilled sum, for example those arising from a complex structure on a smooth manifold or from a Cauchy-Riemann structure. See for example [2] (15.4) for a discussion of complex Lie algebroids and how a Cauchy-Riemann structure gives rise to a complex Lie algebroid.

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