ON SMOOTHNESS OF SOLUTIONS TO PROJECTED DIFFERENTIAL EQUATIONS

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Abstract. Projected differential equations are known as fundamental mathematical models in economics, for electric circuits, etc. The present paper studies the (higher order) derivability as well as a generalized type of derivability of solutions of such equations when the set involved for projections is prox-regular with smooth boundary.

1. Introduction. Given a Hilbert space $X$, let us consider the constrained initial value problem:

\[
\begin{aligned}
\dot{x}(t) &= f(t, x(t)) \quad \text{for all } t \in [0, T], \\
x(t) &\in S \quad \text{for all } t \in [0, T], \\
x(0) &= x_0 \in S,
\end{aligned}
\]

(1)

where $T > 0$, $S$ is a closed subset of $X$, and $f : [0, T] \times U \to X$ is a single-valued mapping with $U \subset X$ being an open set containing $S$. Concerning the existence of solutions, Nagumo’s Theorem (see e.g. [4, Theorem 1.2.1]) states that, under some conditions of continuity of $f$ and compactness of $S$, the problem (1) has a solution for any initial value $x_0 \in S$ if and only if the following inclusion holds:

\[
f(t, x) \in T^R(S; x), \quad \text{for all } (t, x) \in [0, T] \times S,
\]

(2)

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where $T^B(S;x)$ stands for the Bouligand tangent cone of $S$ at $x$. When this condition is not fulfilled and $T^B(S;x)$ is convex for all $x \in S$, we still can find generalized solutions of (1) by solving the following relaxed problem:

\[
\begin{aligned}
\dot{x}(t) &= P(f(t,x(t));T^B(x(t);S)), \\
x(0) &= x_0 \in S,
\end{aligned}
\] (3)

where $P(\cdot;K)$ denotes the metric projection to a closed convex set $K$. This type of relaxation was first introduced by C. Henry [23] for $S$ convex, and then studied by B. Cornet [19] with certain nonconvex sets, namely tangentially regular sets. Nowadays, problems of this kind are known as Projected Differential Equations, and they form a particular case of Projected Differential Inclusions (see, e.g., [5]) and Perturbed Sweeping Processes (see, e.g., [18] and the references therein).

Another relaxation of problem (1) is the implicitly constrained differential inclusion:

\[
\begin{aligned}
\dot{x}(t) &\in -N^C(S;x(t)) + f(t,x(t)), \\
x(0) &= x_0 \in S,
\end{aligned}
\] (4)

where $N^C(S;y)$ is the Clarke normal cone of $S$ at $y \in S$ and $N^C(S;y) = \emptyset$ if $y \notin S$. This differential inclusion is known to have diverse applications in electrical circuits [1], crowd motion [26], and hysteresis in elasto-plastic models [24], among others. Further, it is strongly related to the previous relaxed problem (3). Indeed, B. Cornet [19] proved the equivalence between these two problems. Its original proof was done in finite-dimensional spaces and for autonomous systems (that is, when the mapping $f$ depends only on $x$), but the proof is still valid for non-autonomous systems in arbitrary Hilbert spaces. The result is the following:

**Theorem 1.1** (Cornet [19, Theorem 2.3(b)]). Assume that for all $t \in [0,T]$ the closed set $S$ is tangentially regular. Then, the sets of solutions of problems (3) and (4) coincide.

The existence of solutions for problem (4) as well as for its sweeping process extension

\[
\begin{aligned}
\dot{x}(t) &\in -N^C(C(t);x(t)) + f(t,x(t)), \\
x(0) &= x_0 \in C(0),
\end{aligned}
\] (5)

(with closed sets $C(t) \subset X$) is well studied in the literature (see, e.g., [7, 14, 33] and the references therein); the sweeping process itself given by (5) without the perturbation $f(\cdot,\cdot)$ has been introduced for an elasto-plastic mechanical system by J.J. Moreau in [29] who thoroughly developed its study in a series of subsequent fundamental papers. In contrast, little is known about the regularity of those solutions, beyond absolute continuity. In practice, when we deal with some of the aforementioned applications, properties like differentiability of the solutions are needed or at least desired. For example, in the crowd motion model [26], the derivative of the trajectory represents the velocity of people when exiting a congested building. Also, in [3], a certain behavior of the trajectory of a free endpoint Mayer problem for a controlled sweeping process is needed in relation of its interaction with the boundary. Indeed, to derive necessary optimality conditions for this type of control problem, the authors in [3] assume some outward / inward pointing conditions (see conditions $M_1$ and $M_2$ from [3]). These conditions impose a specific structure on the set

\[ I_\partial := \{ t \in [0,T] : x_* \in \text{bd} C(t) \}, \]
where \((x_*, u_*)\) is a solution of the controlled sweeping process, and \(\text{bd } C(t)\) denotes the boundary of the moving set \(C(t)\) (see Propositions 5.2 and 5.3 from [3]). We also refer to [3, 15, 16, 17, 9, 10, 11, 27, 34, 21, 8] for various other first results concerning controlled sweeping processes.

Motivated by the foregoing discussion, in this work we start the study of regularity properties of solutions of problem (5), particularly (thanks to Theorem 1.1) focusing on the stationary case, that is, on problem (3). Here, we study two main properties: differentiability (in some sense) of trajectories of (3); and the structure of points where the trajectory touches the boundary. The study of the first property is motivated by known characterizations of differentiability of the metric projection onto convex sets and, by recent developments, on the differentiability of the metric projection onto nonconvex sets (see [32, 20, 2, 12]). The study of the second property is related to the first one and motivated by the recent works cited above related to the optimal control of sweeping process.

When studying the differentiability of the solutions of problem (3), it is natural to think that this regularity property must be inherited from the regularity of the data of the problem, that is, of the set \(S\) and the mapping \(f\). Based on [20, 32], our study will consider the following two different cases:

1. when \(S\) is a \(C^{p+1}\)-submanifold itself (and therefore, \(S = \text{bd } S\)).
2. when \(S\) is a closed body, meaning that \(\text{int } S \neq \emptyset\), \(S = \text{cl}(\text{int } S)\) and \(\text{bd } S\) is a \(C^{p+1}\)-submanifold of codimension 1.

Related to the mapping \(f : [0, T] \times U \to X\), we will consider the following hypotheses:

\(\mathcal{F}_1\) \(f\) is of class \(C^p\) on \([0, T] \times U\).

\(\mathcal{F}_2\) There exists a non-negative function \(\beta(\cdot) \in L^1([0, T], \mathbb{R})\) such that for all \(t \in [0, T]\) and all \(x \in U\),

\[
\|f(t, x)\| \leq \beta(t)(1 + \|x\|).
\]

Under these hypotheses, at least in the case of a closed body, higher order derivability in the usual sense cannot be directly expected. Indeed, this is illustrated with the situation where, for a solution \(x\), there is a nowhere dense set included in \(x^{-1}(\text{bd } S)\) with positive Lebesgue measure. Thus, we introduce a new notion of derivability of trajectories, called \(\Omega\)-derivability (see Definition 4.5) in order to explore this situation. This notion will help us to derive the main results of the work.

The paper is organized as follows. In Section 2, we set some preliminary notations and definitions needed for the sequels. In Section 3 we study problem (3) when \(S\) is a \(C^{p+1}\)-submanifold, and we deduce that projected differential equations in this case have the same behavior as differential equations on manifolds (see Theorem 3.2). In Section 4 we study the problem (3) when \(S\) is a closed body with \(C^{p+1}\)-boundary. To overcome in this second case the natural difficulty of collisions we introduce the notion of \(\Omega\)-derivability. Our main results are Theorem 4.7, where we deduce the \(\Omega\)-derivability of the solutions, and Corollary 1, where \(\Omega\)-derivability entails a piecewise differentiability in the analytic case. Finally, in Section 5 we derive some applications of our results. In Appendix A, we provide an example of an ill-behaved projected differential equation, where the usual notion of derivability is not well-adapted and the \(\Omega\)-derivability is the appropriate concept.
2. Preliminaries. Throughout this paper, $X$ will be a real Hilbert space endowed with the inner product $\langle \cdot , \cdot \rangle$ and its associated norm $\| \cdot \|$. By Riesz representation theorem, we identify the dual of $X$ with $X$. We denote by $B_X(x, \varepsilon)$, the open ball of radius $\varepsilon$ centered in $x$. We denote by $B_X$ and $S_X$ the closed unit ball and the unit sphere of $X$ centered at zero, respectively. The letter $p$ will always be an integer greater than 1.

Let $S$ be a subset of a Banach space $Y$. We denote by $\text{int} \, S$, $\overline{S}$ and $S^o$ the interior, the closure and the (negative) polar set of $S$, respectively. Recall that the polar set of $S$ is the subset of the dual space $Y^*$ given by
\[ S^o := \{ y^* \in Y^* : \langle y^*, y \rangle \leq 1, \forall y \in S \}, \]
where $\langle \cdot , \cdot \rangle$ stands for the duality product between $Y$ and $Y^*$. We will not distinguish between duality products or inner products, using always the same notation. We will also write (in certain cases) $\overline{S}$ instead of $\overline{S}$ to denote the closure. We say that $S \subset X$ is a closed body if it is connected, closed, and
\[ S = \text{cl}( \text{int} \, S). \] (6)

For two Banach spaces $Y$ and $Z$, we write $\mathcal{L}(Y; Z)$ to denote the space of continuous linear operators from $Y$ to $Z$, endowed with its usual norm. In the case when $Z = Y$, we simply write $\mathcal{L}(Y)$. Also, for $T \in \mathcal{L}(Y; Z)$, we write $T^*$ to denote the adjoint operator of $T$.

For a mapping $f : U \subset Y \to Z$ (where $U$ is an open set) and $u \in U$, we write $Df(u)$ to denote the (Fréchet) derivative of $f$ at $u$. If $Y$ is a Hilbert space and $Z = \mathbb{R}$, we write $\nabla f(u)$ to denote the gradient of $f$ at $u$. Finally, for a curve $\gamma : I \subset \mathbb{R} \to Y$ (where $I$ is an interval of $\mathbb{R}$) and $t \in I$, we will write $\dot{\gamma}(t)$ or $\frac{d}{dt} \gamma(t)$ to denote the derivative of $\gamma$ at $t$. Consequently, we denote by $D^k f(u)$ and by $\frac{d^k}{dt^k} \gamma(t)$ the $k$-th derivative of $f$ at $u$ and of $\gamma$ at $t$, respectively.

Recall that, for a closed bounded interval $I = [a, b]$ with $a < b$ and an integer $p \geq 1$, a mapping $\gamma : I \to Y$ is said to be $p$-continuously differentiable in $I$ if it is of class $C^p$ in $[a, b]$ and for each $k \in \{ 1, \ldots, p \}$ the unilateral derivatives
\[ \frac{d^k \gamma}{dt^k}(a^+) \quad \text{and} \quad \frac{d^k \gamma}{dt^k}(b^-) \]
exist, and the mappings $\frac{d^k \gamma}{dt^k}$ extended from $]a, b]$ to $[a, b]$ in this way are also continuous. In a similar way, we define the $C^p$ property for $\gamma : I \to Y$ when $I = [a, b]$ and $I = ]a, b]$.

For a subset $S$ of $X$ and a point $x \in X$ we denote by $d(x; S)$ or $d_S(x)$ the distance of $x$ from $S$, namely
\[ d(x; S) = \inf_{y \in S} \| x - y \|, \]
and by $\text{Proj}(x; S)$ or $\text{Proj}_{S}(x)$ the set of all nearest points from $S$ to $x$, that is,
\[ \text{Proj}(x; S) := \{ y \in S : \| x - y \| = d(x; S) \}. \]
Whenever $\text{Proj}(x; S)$ is a singleton, we call the only point $\bar{y} \in \text{Proj}(x; S)$ the metric projection of $x$ on $S$, and we denote it by $P(x; S)$ or $P_S(x)$. We will use indistinctly these two notations. Moreover, in the particular case when $S$ is a closed vector subspace of $X$, we will write $\Pi_S(x)$ instead of $P(x; S)$, to emphasize that the metric projection of $x$ coincides with the orthogonal projection onto the subspace.

In what follows, we will write $T^B(x; S)$, $T^C(x; S)$ and $N^C(x; S)$ to denote the Bouligand tangent cone, the Clarke tangent cone and the Clarke normal cone of $x$. 


at $S$, respectively (see [13] for definitions and properties). Recall that for a point $x \in S$, the \textit{proximal normal cone} of $S$ at $x$ is given by

$$N^P(x; S) := \{\xi \in X : \exists t > 0 \text{ such that } x \in \text{Proj}(x + t\xi)\}$$

and that $N^P(S; x) \subset N^C(S; x)$.

Let $\rho : S \rightarrow [0, +\infty]$ be a continuous function. The subset $S$ is said to be $\rho(\cdot)$-prox-regular if for any $x \in S$ and any $\xi \in N^P(S; x)$, one has that

$$x \in \text{Proj}_S(x + t\xi), \quad \text{for any real } t \leq \rho(x).$$

We simply say that $S$ is prox-regular if there exists a continuous function $\rho : S \rightarrow [0, +\infty]$ such that $S$ is $\rho(\cdot)$-prox-regular. In such a case, we define the enlargement

$$U_{\rho(\cdot)}(S) := \{y \in X : \exists x \in S \text{ such that } x \in \text{Proj}_S(y) \text{ and } d_S(y) < \rho(x)\}.$$  

It is known (see, e.g., [18]) that when $S$ is $\rho(\cdot)$-prox-regular, $U_{\rho(\cdot)}(S)$ is an open neighborhood of $S$, and its metric projection $P_S = P(\cdot; S)$ is well-defined and locally Lipschitz-continuous on $U_{\rho(\cdot)}(S)$, and the function $\frac{1}{2}d_S^2$ is of class $C^1$ on $U_{\rho(\cdot)}(S)$ with

$$\frac{1}{2}\nabla d_S^2(x) = x - P_S(x) \quad \text{for all } x \in U_{\rho(\cdot)}(S).$$ \hspace{1cm} (7)

We say that a set $S$ is tangentially regular at $x \in S$ if $T^B(x; S) = T^C(x; S)$. Also, we say that $S$ is normally regular at $x$ if $N^P(x; S) = N^C(x; S)$, so in this case $S$ is also tangentially regular at $x$. Whenever $S$ is a prox-regular set, then it is normally regular (and hence tangentially regular) at each of its points. For further information on prox-regular sets, we refer the reader to [18, 31].

A subset $M$ of $X$ is said to be a $C^p$-submanifold if there exists a closed vector subspace $Z$ of $X$ such that for any point $m \in M$ there exist an open neighborhood $U$ of $m$ and a $C^p$-mapping $\varphi : U \rightarrow X$ with $\varphi(m) = 0$ such that

1. $\varphi$ is a $C^p$-diffeomorphism, that is, $\varphi(U)$ is open, $\varphi : U \rightarrow \varphi(U)$ is invertible and $\varphi^{-1} : \varphi(U) \rightarrow U$ is also of class $C^p$;
2. $\varphi(U \cap M) = \varphi(U) \cap Z$.

In the above definition, the pairs $(U, \varphi)$ are called local charts and $Z$ is called the model space. This local representation of $M$ is not unique, in the sense that they may exist several local charts and model spaces fitting this definition.

For a $C^p$-submanifold and a point $m \in M$ we define the \textit{tangent space} of $M$ at $m$ as

$$T_mM := \{h \in X : \exists \gamma : [-1, 1] \rightarrow M, C^1\text{-curve} \text{ with } \gamma(0) = m \text{ and } \dot{\gamma}(0) = h \}.$$ \hspace{1cm} (8)

which is a closed vector subspace of $X$. Consequently, we define the normal space of $M$ at $m$ as $N_mM = [T_mM]^\perp$. It is not hard to see that a $C^p$-submanifold is tangentially regular (even normally regular when $p \geq 2$) and that

$$T^C(m; M) = T_mM \quad \text{and} \quad N^C(m; M) = N_mM,$$

for every $m \in M$. It is known that if $(U, \varphi)$ is a local chart of $M$ with model space $Z$ and $m \in U \cap M$, then

$$D\varphi(m)T_mM = Z \quad \text{and} \quad D\varphi(m)^*Z^\perp = N_mM.$$ 

Let $M_1$ and $M_2$ be two $C^p$-submanifolds of $X$, with model spaces $Z_1$ and $Z_2$, respectively. A mapping $f : M_1 \rightarrow M_2$ is said to be of class $C^k$ (with $k \in \{1, \ldots, p\}$)
if for any \( m \in M_1 \) and any local charts \((U_1, \varphi_1)\) and \((U_2, \varphi_2)\) of \( M_1 \) and \( M_2 \) with \( m \in U_1 \) and \( f(U_1) \subset U_2 \), we have that

\[
\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U) \cap Z_1 \to Z_2
\]

is of class \( C^k \).

A closed body \( S \) is said to have a \( C^p \)-smooth boundary if \( \text{bd} \ S \) is a \( C^p \)-submanifold. In such a case, it is known that \( \text{bd} \ S \) is a submanifold of \( X \) of codimension 1, that is, any model space \( Z \) used to represent \( \text{bd} \ S \) is a hyperplane. Thus, we can define for \( S \) the mapping \( \hat{n} : \text{bd} \ S \to S_X \) where for any \( x \in \text{bd} \ S \), \( \hat{n}(x) \) is the only element such that

\[
\hat{n}(x) \in N^C(x;S) \cap S_X.
\]

We call the vector \( \hat{n}(x) \) the exterior normal vector of \( S \) at \( x \). It is not hard to prove that the mapping \( \hat{n} \) is of class \( C^{p-1} \) (see, e.g., [20]).

Now, let us recall in the form of a theorem the following properties (a) and (b) from [20] and (c) from [32]. They will serve as key tools in the development in the next sections.

**Theorem 2.1.** Let \( M \) be a \( C^{p+1} \)-submanifold of \( X \) and \( S \) be a closed body in \( X \) with \( C^{p+1} \)-smooth boundary (with \( p \geq 1 \)). Then \( M \) and \( S \) are prox-regular and

(a) \( P(\cdot;M) \) is of class \( C^p \) on \( U_{\rho(\cdot)}(M) \), where \( \rho(\cdot) \) is a prox-regularity function of \( M \);

(b) \( P(\cdot;S) \) is of class \( C^p \) on \( U_{\rho(\cdot)}(S) \setminus S \), where \( \rho(\cdot) \) is a prox-regularity function of \( S \).

(c) Further, for the \( C^{p+1} \)-submanifold \( M \), one also has (see [32]) that

\[
\forall m \in M, \; DP_M(m) = \Pi_{\text{Tr}_m} M.
\]

**Remark 1.** Note that problems (3) and (4) could be defined using other normal cones, like the Proximal normal cone defined above, or the Limiting normal cone (see, e.g., [28] for the definition). However, under the hypothesis of \( C^{p+1} \)-smooth boundary, the set \( S \) becomes prox-regular. This yields that both these normal cones of \( S \) coincide with Clarke normal cone (see, e.g., [18]), and so all these alternative formulations would be equivalent to the ones presented in this work.

Figure 1 shows two examples of prox-regular sets: one with smooth boundary (left set) and the other with non-smooth boundary (right set). The nonsmoothness of the boundary in the right set comes from the corners of it. Note that this set can be recovered as the intersection of two closed bodies with smooth boundary (one closed ball intersected with the complement of another open ball), which yields that nonsmoothness can appear quite easily. However, as we already mention before, in this first study we will focus our attention only to the first example.

Let us also recall what we understand for a solution of problem (3), problem (4) or problem (5). The following definition considers only the perturbed sweeping processes, but we can easily adapt it to projected differential equations.

**Definition 2.2.** A mapping \( x : I \to X \) (where either \( I = [0,t^*] \) or \( I = [0,t^*] \) for some \( t^* \in [0,T] \)) is said to be a **local solution** of (3) if it is a locally absolutely continuous mapping (hence absolutely continuous if \( I = [0,t^*] \)) satisfying

(a) for all \( t \in I \), \( x(t) \in S(t) \) and \( x(0) = x_0 \);

(b) for almost all \( t \in I \), \( \dot{x}(t) \in f(t,x(t)) - N^C(S(t);x(t)) \).

Moreover, \( x \) is said to be a **global solution** if it is a local solution with \( I = [0,T] \).
If \( p \geq 1 \) and \( x : I \to X \) is a solution of (3), (4) or (5) which is of class \( C^p \), we consider its derivative \( \dot{x} : I \to X \) as the only representative of class \( C^{p-1} \) such that
\[
x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(s) \, ds \quad \text{for all } t \in I.
\]

3. The manifold case. In this section we study problem (3) in the case when \( S \) is a \( C^{p+1} \)-submanifold. Before showing the main theorem of this section, we will need the following lemma:

**Lemma 3.1.** Let \( M \) be a \( C^{p+1} \)-submanifold of a Hilbert space \( Y \). Then, the mapping \( A : M \to L(Y) \) given by \( A(m) := \Pi_{Z} \circ D\phi(m)^* \) is of class \( C^p \) (in the sense of manifolds). Moreover, the mapping \( \tilde{A} : U_{\rho(\cdot)}(M) \to L(Y) \) given by \( \tilde{A}(u) := \Pi_{S} \circ \Pi_{Z} \circ D\phi(m)^* \), where \( \rho(\cdot) \) is the prox-regularity function associated with \( M \).

**Proof.** Let \( (U, \varphi) \) be a local chart of \( M \) and let \( Z \) be the model space of \( M \). To abbreviate notation, let us denote \( \phi := \varphi^{-1} \). For every \( m \in M \cap U \), let us define the continuous linear operator \( L_m : Z \to Y \) given by \( L_m := D\phi(m)|_Z \). It is easy to see that \( L_m^* = \Pi_{Z} \circ D\phi(m)^* \) and also, since \( D\phi(m)^* (N_m M) = Z^\perp \), that \( \text{Ker}(L_m^*) = N_m M \).

Noting that \( Y = T_m M \oplus N_m M \) and that \( L_m(Z) = T_m M \), we deduce that \( L_m^* L_m \) is an automorphism of \( Z \), and therefore we can define the continuous linear operator \( A_m := L_m \circ (L_m^* L_m)^{-1} \circ L_m^* \in L(Y) \).

It is not hard to see that for every \( m \in M \cap U \), \( A_m = A_m^2 = A_m^* \), which yields that \( A_m \) is the orthogonal projection from \( Y \) onto \( A_m(Y) \). Furthermore, we have that the mapping \( m \mapsto A_m \) is a \( C^p \)-mapping from \( U \cap M \) to \( L(Y) \), due to the fact that \( \varphi \) is a \( C^{p+1} \)-diffeomorphism.

Finally, since for every \( m \in M \cap U \) we have that \( D\phi(m)^* (N_m M) = Z^\perp \) and \( D\phi(m)|_Z \) is an isomorphism between \( Z \) and \( T_m M \), we deduce that \( A_m(Y) = T_m M \). Since \( \Pi_{T_m M} = \Pi_{T_m M} \) by Theorem 2.1(c), we conclude that \( A_m = A(m) \), finishing the first part of the proof.

The second part of the proof follows as a simple application of the chain rule and Theorem 2.1(a).

**Theorem 3.2.** Consider the initial value problem (3) and assume that for some integer \( p \geq 1 \):
(i) $S$ is a $C^{p+1}$-submanifold;
(ii) $f$ is of class $C^p$ in $[0, T] \times U$, where $U$ is an open set satisfying
\[ S \subset U \subset U_{\rho(\cdot)}(S), \]
where $\rho : S \to [0, +\infty]$ stands for the prox-regularity function of $S$.

Then, there exists a local solution $x : I \to X$ of $(3)$, and it is unique in $I$. Furthermore, the solution $x(\cdot)$ is of class $C^{p+1}$ in $I$.

If in addition the mapping $f$ also satisfies the linear growth condition $(F_2)$, that is,
\[ \|f(t, u)\| \leq \beta(t)(1 + \|u\|) \text{ for all } (t, u) \in [0, T] \times U \]
with $\beta(\cdot) \in L^1([0, T], \mathbb{R})$, then the solution $x(\cdot)$ is global, that is, $I = [0, T]$.

**Proof.** Since $X$ is a metric space, there exists an open set $V \subset X$ such that $S \subset V$ and $\nabla \subset U$. Since Hilbert spaces admit $C\infty$-partitions of unity (see [35]), there exists a $C\infty$-mapping $\varphi$ from $X$ to $[0, 1]$ such that
\[ \varphi^{-1}([0, +\infty]) \subset V \quad \text{and} \quad \varphi|_S \equiv 1. \]

Let us define then the function
\[ g : [0, T] \times X \to X \]
\[ (t, x) \mapsto g(t, x) := \begin{cases} \varphi(x) \cdot DP_S(P_S(x))f(t, x) & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases} \]

Applying Lemma 3.1 and recalling that $P_S$ is of class $C^p$ in $U_{\rho(\cdot)}(S)$ by Theorem 2.1(a), we have that $g$ is of class $C^p$ on $[0, T] \times X$. In particular, it is also locally Lipschitz. Consider the auxiliary differential equation
\[ \begin{align*}
\dot{x}(t) &= g(t, x(t)), \quad \forall t \in [0, T], \\
x(0) &= x_0.
\end{align*} \tag{9} \]

This problem has a local solution $x : I \to X$ which is unique in $I$, where $I$ is an interval closed at its left endpoint 0, and $x$ is $(p+1)$-continuously differentiable.

To conclude, it is enough to prove that $x(t) \in S$ for all $t \in I$. Let $\Psi : I \to \mathbb{R}$ be the function given by $\Psi(t) = \frac{1}{2}d_S^2(x(t))^2$. Since $d_S^2$ is of class $C^1$ on $U_{\rho(\cdot)}(S)$ with $\frac{1}{2}d_S^2(u) = u - P_S(u)$ for all $u$ therein (see (7)), for every $t \in I$ with $x(t) \in U$, we can write
\[ \frac{d}{dt}\Psi(t) = \frac{1}{2}(\nabla(d_S^2)(x(t)), \dot{x}(t)) = (x(t) - P_S(x(t)), g(t, x(t))) = \varphi(x(t)) \cdot (x(t) - P_S(x(t)), DP_S(P_S(x(t)))f(t, x(t))) = 0, \]
where the last equality to 0 follows from the inclusion $DP_S(P_S(y))(v) \in T_yS$ for all $y \in S$, $v \in X$ and from the inclusion $u - P_S(u) \in N_{P_S(u)}S$ for all $u \in U$. Also, if $x(t) \notin U$, then there exists an interval $J = [t - \varepsilon, t + \varepsilon \cap I$ such that $x(s) \notin \nabla$ for all $s \in J$. Thus, $\dot{x}(s) = 0$ for all $s \in J$, which yields that $x(s)$ is constant in $J$. In particular, $\frac{d}{dt}\Psi(t) = 0$.

In any case, we get that $\frac{d}{dt}\Psi \equiv 0$, which yields that $\Psi$ is constant in $I$. So, since $\Psi(0) = 0$, we conclude that $\Psi \equiv 0$. The conclusion follows.

Now, assume in addition that $f$ satisfies the growth condition $(F_2)$. Then, the mapping $g$ also satisfies on the whole set $[0, T] \times X$ the same growth condition.
Indeed, recalling that for every \( x \in S \) and \( DP_S(x) = \Pi_{T_xS} \) (see Theorem 2.1(c)), we can write

\[
\|g(t, x)\| = \begin{cases} 
\|\varphi(x)DP_S(P_S(x))f(t, x)\| \leq \|f(t, x)\| & \text{if } (t, x) \in [0, T] \times U, \\
0 & \text{otherwise.}
\end{cases}
\]

Then, since \( \beta(t) \) is always nonnegative, we deduce that \( \|g(t, x)\| \leq \beta(t)(1 + \|x\|) \) for all \((t, x) \in [0, T] \times X\). By a classical application of Grönwall’s lemma (see, e.g., [25, Chapter 2.4]), the linear growth condition of \( g \) yields that the local solution \( x \) is Lipschitz-continuous on \( I \), and so, if we consider \( x : I \to X \) as the maximal solution of problem (9), then it has to be global (namely, \( I = [0, T] \)). Otherwise, again by classic arguments, we could extend the solution \( x \) which would contradict its maximality. This finishes the proof. \( \square \)

**Remark 2.** Note that Theorem 3.2 remains true if we replace problem (3) by

\[
\begin{cases}
\dot{x}(t) = P\left(f(t, x(t)); T^B(x(t); S)\right), & \text{a.e. } t \in [0, T], \\
x(t_0) = x_0 \in S,
\end{cases}
\]

where \( t_0 \in [0, T] \). Indeed, we may consider any initial condition \( x(t_0) = x_0 \) maintaining the same proof. In such a case (when \( t_0 \neq 0 \)), we understand a local solution \( x : I \to X \) as in Definition 2.2, but with

\[
I = [t_0 - \delta, t_0 + \delta \cap [0, T],
\]

for some \( \delta > 0 \).

This result should be compared with [14, Theorem 3.1], where local existence results for projected differential equations are obtained. A small modification also allows us to derive local results without considering global prox-regularity. The proof of our theorem is rather simple in comparison to the one of the theorem cited above, thanks to the advantage of smoothness of the manifold.

This result should also be compared with classic ordinary differential equations on manifolds (see, e.g., [25, Ch. 9]): We could approach the proof of Theorem 3.2 by showing that \((t, m) \mapsto DP_M(m) \circ f(t, m)\) is a time-dependent flow in the manifold \( M \). This is possible based on the recent results in [32, 20] and Lemma 3.1. However, the present proof has the interest of using only the tools of variational analysis and classic differential equations.

4. **The nonconvex body case.** Now, we study the case when \( S \) is a nonconvex body. The main difference between this case and the previous one (when \( S \) is a manifold itself) is that, for a solution \( x : I \to X \) of problem (3), the instants when the trajectory meets the boundary of \( S \) furnish a discontinuity of the derivative \( \dot{x}(\cdot) \). In what follows, we will refer to these instants in \( I \) as the **collisions** of the trajectory. To better understand this difficulty, we can divide the interval of the solution as follows:

**Definition 4.1.** Let \( x : I \to X \) be a solution of (3) where \( S \) is a closed body. Then, we define the sets

- \( A(x) := x^{-1}(\text{int } S) = \{ t \in I : x(t) \in \text{int } S \} \);
- \( B_{\text{int}}(x) := \text{int}(x^{-1}(\text{bd } S)) \): the set of instants \( t \in I \) for which there exists \( \delta > 0 \) such that \( x([t - \delta, t + \delta] \cap I) \subset \text{bd } S \);
- \( B_{\text{iso}}(x) := \) the set of instants \( t \in I \) for which there exists \( \delta > 0 \) such that \([t - \delta, t + \delta] \subset I \) and \([t - \delta, t + \delta] \cap x^{-1}(\text{bd } S) = \{t\} \).
• \( C_{\text{in}}(x) \): the set of instants \( t \in I \) for which there exists \( \delta > 0 \) such that \( x([t, t + \delta] \cap I) \subset \text{bd} \, S \) and \( x([t - \delta, t]) \subset \text{int} \, S \);
• \( C_{\text{out}}(x) \): the set of instants \( t \in I \) for which there exists \( \delta > 0 \) such that \( x([t - \delta, t] \cap I) \subset \text{bd} \, S \) and \( x([t, t + \delta]) \subset \text{int} \, S \);
• \( C_{\text{irr}}(x) := x^{-1}(\text{bd} \, S) \setminus (B_{\text{int}}(x) \cup B_{\text{iso}}(x) \cup C_{\text{in}}(x) \cup C_{\text{out}}(x)) \).

To simplify notation, we will write also \( B(x) = B_{\text{int}}(x) \cup B_{\text{iso}}(x) \) and \( C_{\text{reg}}(x) = C_{\text{in}}(x) \cup C_{\text{out}}(x) \).

Figure 2 shows a schema of how the different sets described in Definition 4.1 look like, for a trajectory \( x(\cdot) \) in \( \mathbb{R}^2 \). The set \( A(x) \) is easy to understand: it stands for the instants \( t \in I \) when the trajectory \( x(\cdot) \) lies in the interior of \( S \). For each \( t \in A(x) \) where \( \dot{x}(t) \) exists, we see from the inclusion \( x(t) \in \text{int} \, S \) that \( T^{B}(S; x(t)) = X \), and hence

\[
\dot{x}(t) = f(t, x(t)).
\]  

(11)

The set \( B_{\text{int}}(x) \) are the instants \( t \in I \) when the trajectory lies locally in the boundary of \( S \), that is, when there exists a neighborhood \( V \) around \( t \) such that \( x|_{V} \) is a curve in \( \text{bd} \, S \). The set \( B_{\text{iso}}(x) \) are the instants when the trajectory touches the boundary coming from \( \text{int} \, S \), and then goes immediately back to \( \text{int} \, S \) again. The set \( B_{\text{iso}}(x) \) is by definition isolated. The set \( C_{\text{in}}(x) \) stands for the instants of the incoming collisions: the trajectory is coming from the interior \( \text{int} \, S \), meets \( \text{bd} \, S \) and then remains a while in \( \text{bd} \, S \). Similarly, the set \( C_{\text{out}}(x) \) stands for the instants of the outgoing collisions: the trajectory is coming from the \( \text{bd} \, S \) and then it enters in \( \text{int} \, S \).

The sets \( C_{\text{in}}(x) \) and \( C_{\text{out}}(x) \) are easy to picture, and so we will call their union the set of instants of regular collisions, denoted by \( C_{\text{reg}}(x) \). Finally, the set \( C_{\text{irr}}(x) \) stands for the instants of irregular collisions, in the sense that they represent the moments when the trajectory makes infinitely many exchanges between \( \text{int} \, S \) and \( \text{bd} \, S \), before and/or after each one of them. Namely, an instant \( \bar{t} \in I \) belongs to \( C_{\text{irr}}(x) \) if and only if there exist a monotone (i.e. either increasing or decreasing) sequence \((t_n)\) in \( I \setminus \{\bar{t}\}\) converging to \( \bar{t} \) such that for all \( n \in \mathbb{N} \)

\[
\begin{cases} 
  x(t_n) \in \text{int} \, S & \text{if } n \text{ is even,} \\
  x(t_n) \in \text{bd} \, S & \text{if } n \text{ is odd.}
\end{cases}
\]

(12)
We will study differentiability properties of $x$ for each one of these sets. In the rest of this section we will adopt the following notation concerning the boundary of $S$: $\rho : \text{bd } S \rightarrow [0, +\infty]$ will be the prox-regularity function of $\text{bd } S$ and we will write $\tilde{\rho} : \text{bd } S \rightarrow \mathbb{S}_X$ to denote the mapping assigning to each $x \in \text{bd } S$ the unit exterior normal vector of $S$ at $x$ (see Section 2). Also, we will simply write $P$ instead of $P_{\text{bd } S}$.

Throughout the rest of this section we assume that $S$ is a closed body whose boundary is a $C^{p+1}$-submanifold and we also assume that $f : [0, T] \times U \rightarrow X$ is of class $C^p$ on $[0, T] \times U$, where $U$ is an open set containing $S$.

**Lemma 4.2.** Let $x : I \rightarrow X$ be a locally absolutely continuous mapping with $d(\cdot ; \text{bd } S)$ Fréchet differentiable at each point in $x(I)$ and let $G : I \rightarrow \mathbb{R}_+$ be defined by $G(t) := d(x(t); \text{bd } S)$. Let $\tau < \theta$ in $I$ be such that $x(\tau) \in \text{int } S$ and $x(\theta) \in \text{bd } S$ (resp. $x(\tau) \in \text{bd } S$ and $x(\theta) \in \text{int } S$). Then the set of $\sigma \in [\tau, \theta]$ such that $\dot{x}(\sigma)$ exists and $\tilde{G}(\sigma) < 0$ (resp. $G(\sigma) > 0$) has positive Lebesgue measure.

**Proof.** Assume that $x(\tau) \in \text{int } S$ and $x(\theta) \in \text{bd } S$, so $G(\tau) > 0$ and $G(\theta) = 0$. Since $d(\cdot ; \text{bd } S)$ is Lipschitz, the function $G$ is locally absolutely continuous on $I$. Further, at each $t \in I$ where $x$ is derivable, $G$ is derivable too. Writing

$$0 > G(\theta) - G(\tau) = \int_{\tau}^{\theta} \tilde{G}(t) \, dt,$$

it results that the set of $t \in [\tau, \theta]$ where $\dot{x}(t)$ exists and $\dot{G}(t) < 0$ has positive Lebesgue measure. The other case is similar. 

Recall that we are studying the differentiability of the solutions $x : I \rightarrow X$ in the sense of Definition 2.2. In particular, $I$ is always a subinterval of $[0, T]$ closed at its left endpoint 0.

**Lemma 4.3.** Let $x : I \rightarrow X$ be a solution of problem (3) and $\bar{t} \in I$. If $\bar{t} \in A(x) \cup B(x)$, then $x(\cdot)$ is of class $C^{p+1}$ near $\bar{t}$.

**Proof.** On the one hand, if $\bar{t} \in A(x)$, then there exists $\delta > 0$ such that $J = [\bar{t} - \delta, \bar{t} + \delta] \cap I \subset A(x)$. Then, $x|_J$ is a local solution of the classic problem

$$\begin{cases}
\dot{y}(t) = f(t, y(t)), & t \in I, \\
y(\bar{t}) = x(\bar{t}).
\end{cases}$$

This yields that $x(\cdot)$ is of class $C^{p+1}$ near $\bar{t}$. On the other hand, if $\bar{t} \in B_{\text{int}}(x)$, there exists $\delta > 0$ such that $J = [\bar{t} - \delta, \bar{t} + \delta] \cap I \subset B_{\text{int}}(x)$. Then, $x|_J$ is a local solution of

$$\begin{cases}
\dot{y}(t) = P \left( f(t, y(t)); T^B(\text{bd } S; y(t)) \right), & \text{a.e. } t \in I \\
y(\bar{t}) = x(\bar{t}) \in \text{bd } S.
\end{cases}$$

According to Theorem 3.2 and Remark 2, $x|_J$ is of class $C^{p+1}$, which yields that $x(\cdot)$ is of class $C^{p+1}$ near $\bar{t}$.

Finally, assume that $\bar{t} \in B_{\text{iso}}(x)$. Note that in such a case, $\bar{t}$ cannot be an extreme point of $I$. Thus, by continuity of $x(\cdot)$ and the definition of $B_{\text{iso}}(x)$, we may choose an interval $J = [\bar{t} - \delta, \bar{t} + \delta] \subset I$ such that $x(J) \subset S \cap U_{\rho(\cdot)}(\text{bd } S)$ and $J \cap x^{-1}(\text{bd } S) = \{ \bar{t} \}$. Then, for all $t \in J$ we can write

$$f(t, x(t)) = DP(P(x(t)))f(t, x(t)) + \frac{\langle f(t, x(t)), \tilde{\rho}(P(x(t))) \rangle}{\rho(t)} \rho(P(x(t))),$$

where $\rho(t)$ is the prox-regularity function of $\text{bd } S$. 

This completes the proof of Lemma 4.3.
where as said above \( \vec{n} : \partial S \to \mathbb{S}_X \) is the mapping assigning at each point in \( \partial S \) the unit exterior normal vector of \( \partial S \) at this point. Since \( x(t) \in S \) for all \( t \in J \), it results that \( x(t) \in \text{int} S \) for all \( t \in J \setminus \{ \tilde{t} \} \). We also note for each \( t \in J \setminus \{ \tilde{t} \} \) that

\[
g(t) = \langle \dot{x}(t), -\nabla d_{\partial S}(x(t)) \rangle = -\frac{d}{dt} d(x(t); \partial S) = -\dot{G}(t),
\]

where \( G(\cdot) := d(x(\cdot); \partial S) \). By Lemma 4.2 we can find two sequences \( (t^-_n) \subset [\tilde{t} - \delta, \tilde{t}] \) and \( (t^+_n) \subset [\tilde{t}, \tilde{t} + \delta] \) converging to \( \tilde{t} \) such that

\[
g(t^-_n) > 0 \quad \text{and} \quad g(t^+_n) < 0, \quad \text{for all} \ n \in \mathbb{N}.
\]

Since \( g \) is continuous, we deduce that \( g(\tilde{t}) = 0 \), and so, \( x|_J \) is a local solution of

\[
\begin{align*}
\dot{y}(t) &= f(t, y(t)), \quad t \in I, \\
y(\tilde{t}) &= x(\tilde{t}).
\end{align*}
\]

Then, as for the case \( \tilde{t} \in C_{\text{reg}}(x) \), we deduce that \( x(\cdot) \) is of class \( C^{p+1} \) near \( \tilde{t} \). This finishes the proof.

Clearly, whenever \( \tilde{t} \in C_{\text{reg}}(x) \), there is no hope to have continuous derivability of \( x \) at \( \tilde{t} \), since the derivative \( \dot{x} \) is changing discontinuously due to the effect of the boundary. Nevertheless, we still can have piecewise differentiability, if the set \( C_{\text{reg}}(x) \) is small. The following lemma ensures precisely this feature.

**Lemma 4.4.** Let \( x : I \to X \) be a solution of problem (3). The set \( C_{\text{reg}}(x) \) is at most countable.

**Proof.** We will show that \( C_{\text{in}}(x) \) must be at most countable. The proof for \( C_{\text{out}}(x) \) is similar. Fix \( t_1, t_2 \in C_{\text{in}}(x) \) and let \( \delta_1, \delta_2 > 0 \) be such that \( |t_1, t_2, \delta_1| \subset B_{\text{int}}(x) \) (with \( i = 1, 2 \)). Without losing generality, assume that \( t_1 < t_2 \). It is not hard to see that \( t_2 \notin |t_1, t_1 + \delta_1| \), which yields that

\[
|t_1, t_1 + \delta_1| \cap |t_2, t_2 + \delta_2| = \emptyset.
\]

Now, assume that \( C_{\text{in}}(x) = \{ t_\alpha \}_{\alpha \in \Lambda} \) with \( t_\alpha \neq t_{\alpha'} \) for \( \alpha \neq \alpha' \), and let \( \{ \delta_\alpha \}_{\alpha \in \Lambda} \) be a family of positive values such that \( J_\alpha := |t_\alpha, t_\alpha + \delta_\alpha| \subset B_{\text{int}}(x) \). From the reasoning above, we have that \( \{ J_\alpha : \alpha \in \Lambda \} \) is a pairwise disjoint family of subsets of the separable metric space \( I \), thus the set \( \Lambda \) is at most countable.

Now, let us study the set \( C_{\text{irr}}(x) \), which does not seem to be a set of regularity of the solution \( x(\cdot) \). It would be tempting to try to prove that \( C_{\text{irr}}(x) \) is “small”, like the set \( C_{\text{reg}}(x) \). Unfortunately, in Appendix A we present an example for which \( C_{\text{irr}}(x) \) is in fact large (with positive Lebesgue measure). Furthermore, in this example, the trajectory \( x(\cdot) \) fails to be continuously differentiable at each point of \( C_{\text{irr}}(x) \).

Nevertheless, the main problem when looking for smoothness of \( x(\cdot) \) is not the set \( C_{\text{irr}}(x) \) itself, but the density of the set of instants of regular collisions. Thus, to continue our study of smoothness in \( C_{\text{irr}}(x) \), we would like to avoid the set \( C_{\text{reg}}(x) \). To do so, we need to introduce a weaker notion of differentiability.

**Definition 4.5** ((\( \Omega, k \))-continuous derivability). Let \( Y \) be a Banach space, \( I \) be an interval of \( \mathbb{R} \), \( \Omega \) be a subset of \( I \) and \( k \) be a nonnegative integer. Let us consider a curve \( y : \Omega \to Y \). We say that the curve \( y \) is
1. **Ω-continuous** at \( t \) if \( y \) is continuous at \( t \) with respect to the induced topology on \( \Omega \). Similarly, for an open set \( U \) (relative to \( I \)) we say that \( y \) is \( \Omega \)-continuous in \( U \) if it is so for each \( t \in U \cap \Omega \).

2. **Ω-derivable** at \( t \in \Omega \) if there exists a unique element \( d \in Y \) such that

\[
\lim_{\Omega; t' \to t} \frac{y(t') - y(t) - (t' - t)d}{t' - t} = 0. \tag{14}
\]

In such a case, we call \( d \) the \( \Omega \)-derivative of \( y \) at \( t \), and we denote it by \( D_\Omega y(t) \); we also notice in this case that \( y \) is \( \Omega \)-continuous at \( t \).

3. **(Ω, k)-derivable** at \( t \in \Omega \), with \( k \geq 2 \), if there exist a sequence of curves \( y_j : \Omega \to Y \) with \( j = 0, 1, \ldots, k - 1 \) and a neighborhood \( U \) of \( t \) (relative to \( I \)) such that

- \( y_0 = y \);
- For every \( j \), \( y_j \) is \( \Omega \)-continuous in \( U \);
- For every \( j \in \{1, \ldots, k - 1\} \) and every \( t' \in U \cap \Omega \), \( y_j(t') = D_\Omega y_{j-1}(t') \).
- The curve \( y_{k-1} \) is \( \Omega \)-derivable at \( t \).

In such a case, the \( \Omega \)-derivative of \( y_{k-1} \) at \( t \) is uniquely determined by the curve \( y \), and we call it the \((\Omega, k)\)-derivative of \( y \) at \( t \), denoted by \( D^k_{\Omega} y(t) \). By convention, we will say that a curve \( y \) is \((\Omega, 0)\)-derivable (resp. \((\Omega, 1)\)-derivable) at \( t \) if it is \( \Omega \)-continuous (resp. \( \Omega \)-derivable) at \( t \).

4. **(Ω, k)-continuously derivable** at \( t \) if there exist a curve \( \tilde{y} : \Omega \to Y \) and a neighborhood \( U \) of \( t \) (relative to \( I \)) such that

- For every \( t' \in U \), \( y \) is \((\Omega, k)\)-derivable at \( t' \), and \( \tilde{y}(t') = D^k_{\Omega} y(t') \); and
- the curve \( \tilde{y} \) is \( \Omega \)-continuous at \( t \).

We simply say that \( y \) is \((\Omega, k)\)-derivable (resp. \( \Omega \)-continuous, \((\Omega, k)\)-continuously derivable) if it is so for each \( t \in \Omega \). If we consider a curve \( y : I \to Y \), we will say that \( y \) is \((\Omega, k)\)-derivable (resp. \( \Omega \)-continuous, \((\Omega, k)\)-continuously derivable) at \( t \in \Omega \) whenever \( y|_t \) is so.

The definition above has no interest when the set \( \Omega \) is “too small”. In fact, if \( \Omega \) has an isolated point, no curve can be \( \Omega \)-derivable at that point. Nevertheless, if \( \Omega \) is large enough, we may obtain interesting properties. The following proposition shows that the \( \Omega \)-derivability has reasonable stability properties when \( \Omega \) is a dense set of \( I \).

**Proposition 1.** Let \( Y \) be a Banach space and let \( \Omega \) be a subset of an interval \( I \). If \( \Omega \) has no isolated points, then the following properties hold:

1. If \( t \in \Omega \) and \( y : I \to Y \) is an \( \Omega \)-continuous curve such that

\[
d = \lim_{\Omega; t' \to t} \frac{y(t') - y(t)}{t' - t}
\]

exists, then \( y \) is \( \Omega \)-derivable at \( t \) with \( D_\Omega y(t) = sd \).

2. If \( y_1 \) and \( y_2 \) are two \((\Omega, k)\)-continuously derivable curves in \( Y \), and \( \alpha : \Omega \to \mathbb{R} \) is also \((\Omega, k)\)-continuously derivable, then the mapping \( t \mapsto \alpha(t)y_1(t) + y_2(t) \) is \((\Omega, k)\)-continuously derivable.

3. If \( y \) is an \((\Omega, k)\)-continuously derivable curve in \( Y \) and \( G : Y \to Z \) is a mapping of class \( C^k \), then \( G \circ y \) is an \((\Omega, k)\)-continuously differentiable curve in \( Z \).

4. If \( y : I \to Y \) is derivable at each \( t \in \Omega \), then it is \( \Omega \)-derivable and \( D_\Omega y(t) = \dot{y}(t) \) for each \( t \in \Omega \).
5. If \( y : I \to Y \) is a curve of class \( C^k \), then it is \((\Omega, k)\)-continuously derivable and \( D^j_{\Omega y}(t) = \frac{d^j}{dt^j} y(t) \) for each \( j \in \{1, \ldots, k\} \) and each \( t \in \Omega \).

**Proof.** We will prove the first statement. All other properties follow directly from it just repeating the classic proofs for the usual definition of derivatives.

It is clear that \( d = \lim_{\Omega \ni t', t \to t} \frac{y(t') - y(t)}{t' - t} \) satisfies equation (14). Thus, we only need to prove the uniqueness. Assume that there exists another vector \( d' \in Y \) satisfying equation (14). Since \( \Omega \) is not isolated, then there exists a sequence \((t_n) \subset \Omega \setminus \{t\}\) converging to \( t \), and so we can write

\[
0 = \lim_{n} \frac{y(t_n) - y(t) - (t_n - t)d'}{t_n - t} = \left( \lim_{n} \frac{y(t_n) - y(t)}{t_n - t} \right) - d' = d - d'.
\]

Then, \( d' = d \), finishing the proof. \( \square \)

With the notion of \( \Omega \)-derivability, we now continue the study of the smoothness of \( x \) in \( C_{irr}(x) \). To do so, we will need the following lemma describing the behavior of the normal component of the velocity of \( x \) in such moments.

**Lemma 4.6.** Let \( U \) be an open set of \( X \) containing \( \text{bd} S \) on which \( P(\cdot) \) is well defined and \( p \)-continuously differentiable. Let \( x : I \to X \) be a solution of (3) with \( x(I) \subset U \). Let \( \Omega = I \setminus C_{reg}(x) \) and let the function \( g : I \to \mathbb{R} \) be defined by

\[
t \mapsto g(t) := \langle f(t, x(t)), n(P(x(t))) \rangle.
\]

Then, for each \( \bar{t} \in C_{irr}(x) \), we have that \( g(\bar{t}) = 0 \). Furthermore, if \( x(\cdot) \) is \((\Omega, k)\)-continuously derivable with \( 1 \leq k \leq p \), then \( g \) is \((\Omega, k)\)-continuously derivable with

\[
D^j_{\Omega} g(\bar{t}) = 0, \quad \forall j \in \{1, \ldots, k\}.
\]

**Proof.** Fix \( \bar{t} \in C_{irr}(x) \). Without loss of generality, there exists an increasing sequence \((\bar{t}_n)\) converging to \( \bar{t} \) such that \( x(\bar{t}_n) \in \text{int} S \) when \( n \) is even and \( x(\bar{t}_n) \in \text{bd} S \) if \( n \) is odd, for which we may assume that \((\bar{t}_n) \subset \Omega \).

Now, as seen in (13) for each \( t \) where \( x(t) \) exists and \( x(t) \in \text{int} S \) one has

\[
g(t) = \langle \dot{x}(t), \frac{P(x(t)) - x(t)}{d(x(t)); \text{bd} S} \rangle = -\frac{d}{dt} d(x(t)); \text{bd} S),
\]

hence by Lemma 4.2 we can construct a sequence \((s_n) \subset \Omega \) such that \( \bar{t}_n \leq s_n \leq \bar{t}_{n+1} \) and

- \( g(s_n) > 0 \) if \( n \) is even, and
- \( g(s_n) < 0 \) if \( n \) is odd.

Since \( g \) is continuous, we deduce that \( g(\bar{t}) = 0 \).

Let us assume now that \( x(\cdot) \) is \((\Omega, k)\)-continuously derivable. Then, since \( \Omega \) has no isolated points, we can apply chain rule of Proposition 1 to deduce that \( g \) is \((\Omega, k)\)-continuously derivable.

We claim that the following statement holds: for each \( j \in \{0, \ldots, k\} \) there exists a mapping \( h_j : I \times U \to X \) of class \( C^{k-j} \) such that

\[
h_j(t, x(t)) = D^j_{\Omega} g(t), \quad \forall t \in A(x),
\]

and such that there exists a sequence \((t_n) \subset \Omega \) with \( t_n \not\to \bar{t} \) and satisfying

- \( h_j(t_n, x(t_n)) > 0 \) if \( n \) is even, and
- \( h_j(t_n, x(t_n)) < 0 \) if \( n \) is odd.
Let us prove it inductively. Suppose first that such a condition holds for \( j \in \{0, \ldots, k-1\} \) and fix \( n \in \mathbb{N} \). If \( n \) is even and noting that the mapping \( t \mapsto h_j(t, x(t)) \) is at least locally absolutely continuous, we have that

\[
0 > h_j(t_{n+1}, x(t_{n+1})) - h_j(t_n, x(t_n)) = \int_{t_n}^{t_{n+1}} \frac{d}{dt} h_j(t, x(t)) \, dt
\]

Thus, we can construct a sequence \( s_n \) in \( \Omega \) such that \( t_n < s_n < t_{n+1} \) and such that
- \( \frac{d}{dt} h_j(s_n, x(s_n)) < 0 \) if \( n \) is even, and
- \( \frac{d}{dt} h_j(s_n, x(s_n)) > 0 \) if \( n \) is odd.

Let us consider also the set \( S := \{ s_n : n \in \mathbb{N} \} \cap B_{\text{int}}(x) \). Since \( (s_n) \) is an increasing sequence, for each \( s \in S \) we can choose \( \delta > 0 \) such that the neighborhood \( V_s := [s - \delta, s + \delta] \cap I \) is included in \( B_{\text{int}}(x) \) and also \( V_s \cap S = \{ s \} \) and \( V_s \cap V_{s'} = \emptyset \) if \( s \neq s' \) in \( S \). Further, for each \( s \in S \) we can construct a \( C^\infty \)-function \( \varphi_s : I \to [0, 1] \) such that \( \varphi_s(s) = 1 \) and \( \varphi_s(t) = 0 \) for each \( t \in I \setminus V_s \).

With this construction, and denoting \( V = \bigcup_{s \in S} V_s \) and \( \varphi := \sum_{s \in S} \varphi_s \), we define the function \( h_{j+1} : I \times U \to \mathbb{R} \) given by

\[
h_{j+1}(t, u) = \begin{cases} 
1 & \text{if } t \in V \\
Dh_j(t, u) \left(f(t, u) - \varphi(t)(f(t, u), \tilde{n}(P(u)))\tilde{n}(P(u))\right) & \text{if } t \in I \setminus V.
\end{cases}
\]

Note that \( h_{j+1} \) is of class \( C^{k-(j+1)} \) (in particular, it is continuous) and that for each \( t \in A(x) \) where \( \dot{x}(t) \) exists we have \( t \in I \setminus V \) along with \( x(t) \in \text{int} \, S \). Hence \( \dot{x}(t) = f(t, x(t)) \) by (11), which yields

\[
h_{j+1}(t, x(t)) = D_1 h_j(t, x(t))(1) + D_2 h_j(t, x(t))(f(t, x(t))
\]

\[
= D_1 h_j(t, x(t))(1) + D_2 h_j(t, x(t))(\dot{x}(t))
\]

\[
= \frac{d}{dt} h_j(t, x(t)) = D_{j+1}^2 g(t).
\]

Furthermore, for each \( n \in \mathbb{N} \), we have that

\[
\frac{d}{dt} h_j(s_n, x(s_n)) = \begin{cases} 
Dh_j(s_n, x(s_n)) \left(DP(x(s_n))f(s_n, x(s_n))\right) & \text{if } s_n \in S \\
\lim_{A(x) \ni t \to s_n} \frac{d}{dt} h_j(t, x(t)) & \text{otherwise.}
\end{cases}
\]

In both cases, we get that \( \frac{d}{dt} h_j(s_n, x(s_n)) = h_{j+1}(s_n, x(s_n)) \). Then, we deduce that \( h_{j+1}(s_n, x(s_n)) < 0 \) if \( n \) is even and \( h_{j+1}(s_n, x(s_n)) > 0 \) if \( n \) is odd. By replacing \( t_n = s_{n+1} \) for each \( n \in \mathbb{N} \), we conclude that the statement holds true for \( j + 1 \).
For $j = 0$, just consider $h_0 : I \times U \to \mathbb{R}$ given by

$$h_0(t,u) = \left\langle f(t,u), \hat{\mathbf{n}}(P(u)) \right\rangle,$$

and take for $(t_n)$ the sequence $(s_n)$ constructed in the first part of the proof. The claim is then proved.

Now, to conclude, fix $j \in \{0, \ldots, k-1\}$ and consider $h_j$ and a sequence $(t_n)$ as in the claim. Since $h_j$ is at least continuous, $h_j(\bar{t}, x(\bar{t})) = \lim_n h_j(t_n, x(t_n)) = 0$. Then, consider the sequence $(\bar{t}_n)$ defined at the beginning of the proof. Recalling that $\bar{t}_{2n} \in A(x)$ for every $n \in \mathbb{N}$, we can write

$$D_{\bar{t}_n}^tg(\bar{t}) = \lim_{n} D_{\bar{t}_n}^tg(\bar{t}_{2n}) = \lim_{n} h_j(\bar{t}_{2n}, x(\bar{t}_{2n})) = h_j(\bar{t}, x(\bar{t})) = 0,$$

which proves the second part of the lemma. Finally, the case where the sequence $(\bar{t}_n)$ decreases to $\bar{t}$ is similar, and so the proof is finished. \hfill \square

Now, we present the main theorem of this section, where we prove the $(\Omega, p+1)$-continuous derivability of the trajectory $x$, with $\Omega = I \setminus C_{\text{reg}}(x)$.

**Theorem 4.7.** Let us consider problem (3) and suppose that the set $S$ is a non-convex body with $C^{p+1}$-smooth boundary. If the mapping $f$ satisfies $(F_1)$, that is, $f$ is of class $C^p$ on $I \times U$, then problem (3) admits at least a local solution and any such solution $x : I \to X$ is $(\Omega, p+1)$-continuously derivable with $\Omega = I \setminus C_{\text{reg}}(x)$.

**Proof.** By Theorem 1.1 and [14, Theorem 3.1], problem (3) has at least a local solution. Now let $x : I \to X$ be a solution of (3) and fix $\bar{t} \in \Omega$. If $\bar{t} \notin A(x)$ or $B(x)$, we can apply Lemma 4.3 to deduce that $x(\cdot)$ is of class $C^{p+1}$ near $\bar{t}$. Without loss of generality, we may suppose that $x(I) \subset U_{\rho(\cdot)}(bd S)$. If not, we can replace problem (3) by

$$\begin{cases}
\dot{y}(t) = P(f(t,y(t)); T^B(y(t); S)), & \text{a.e. } t \in J := [\bar{t} - \delta, \bar{t} + \delta] \cap I, \\
y(t_0) = x(t_0) \in S,
\end{cases}$$

with $t_0 = \min J$, and $\delta > 0$ small enough such that $x(J) \subset U_{\rho(\cdot)}(bd S)$.

Then, let us suppose that $\bar{t} \in C_{\text{irr}}(x)$. Since $x$ is locally absolutely continuous, we can write

$$\frac{x(t) - x(\bar{t})}{t - \bar{t}} = \frac{1}{t - \bar{t}} \int_{\bar{t}}^{t} \dot{x}(s) \, ds = \frac{1}{t - \bar{t}} \int_{\bar{t}}^{t} D_P(P(x(s))) f(s, x(s)) \, ds + \frac{1}{t - \bar{t}} \int_{A(x) \cap [\bar{t}, t]} g(s) \hat{n}(P(x(s))) \, ds,$$

where $g : I \to \mathbb{R}$ is the function given in (15). Now, we can write

$$\frac{1}{t - \bar{t}} \int_{A(x) \cap [\bar{t}, t]} g(s) \hat{n}(P(x(s))) \, ds \rightarrow \frac{1}{t - \bar{t}} \int_{\bar{t}}^{t} |g(s)| \, ds \quad \text{as } t \to \bar{t},$$

Thus, since $\frac{1}{t - \bar{t}} \int_{\bar{t}}^{t} D_P(P(x(s))) f(s, x(s)) \, ds \rightarrow D_P(x(\bar{t})) f(\bar{t}, x(\bar{t}))$ as $t \to \bar{t}$, we deduce that

$$\lim_{t \to \bar{t}} \frac{x(t) - x(\bar{t})}{t - \bar{t}} = D_P(x(\bar{t})) f(\bar{t}, x(\bar{t})).$$
which yields that \( x(\cdot) \) is derivable at \( \bar{t} \), with derivative \( DP(x(\bar{t}))f(\bar{t}, x(\bar{t})) \). Since \( \bar{t} \) is arbitrary in \( \text{C}_{\text{irr}}(x) \), we deduce that \( x(\cdot) \) is derivable at each element in \( \Omega \) and so, it is \( \Omega \)-derivable. Furthermore, if we consider a sequence \((t_n) \subset \Omega \) converging to \( \bar{t} \) we have that
\[
\dot{x}(t_n) = \begin{cases} 
DP(x(t_n))f(t_n, x(t_n)) & \text{if } x(t_n) \in \text{bd } S, \\
DP(P(x(t_n)))f(t_n, x(t_n)) + g(t_n)\hat{n}(P(x(t_n))) & \text{if } x(t_n) \in \text{int } S,
\end{cases}
\]
and so, recalling that \( g(t_n) \to 0 \) when \( n \to \infty \), we get that \( \dot{x}(t_n) \to \dot{x}(\bar{t}) \), proving that \( \dot{x} \) is \( \Omega \)-continuous at \( \bar{t} \). Again, since \( \bar{t} \in \text{C}_{\text{irr}}(x) \) is arbitrary, we conclude that \( x \) is \((\Omega, 1)\)-continuously derivable.

Now, let us suppose that for \( k \in \{1, \ldots, p\} \), \( x(\cdot) \) is \((\Omega, k)\)-continuously derivable and fix again \( \bar{t} \in \text{C}_{\text{irr}}(x) \). By Lemma 4.6, we know that \( g \) is also \((\Omega, k)\)-continuously derivable and \( D_{\Omega}^k g(\bar{t}) = 0 \) for all \( j \in \{1, \ldots, k\} \). Let us define the function \( \gamma : \Omega \to X \) by
\[
\gamma(t) := \begin{cases} 
0 & \text{if } x(t) \in \text{bd } S, \\
D_{\Omega}^{k-1}(g(t)\hat{n}(P(x(t)))) & \text{if } x(t) \in \text{int } S.
\end{cases}
\]
It is not hard to see that \((t-\bar{t})^{-1}\gamma(t) \to 0 \) when \( t \to \bar{t} \). Indeed, it is enough to note that
\[
\|((t-\bar{t})^{-1}\gamma(t))\| \leq \|(t-\bar{t})^{-1}D_{\Omega}^{k-1}g(t)\| \xrightarrow{\Omega \ni t \to \bar{t}} \|D_{\Omega}^{k}g(\bar{t})\| = 0.
\]
Thus, for \( t \in \Omega \setminus \{\bar{t}\} \) we can write
\[
\frac{D_{\Omega}^{k}x(t) - D_{\Omega}^{k}x(\bar{t})}{t-\bar{t}} = \frac{D_{\Omega}^{k-1}\dot{x}(t) - D_{\Omega}^{k-1}\dot{x}(\bar{t})}{t-\bar{t}} = \frac{D_{\Omega}^{k-1}\left(DP(x(t))f(t, x(t))\right) - D_{\Omega}^{k-1}\left(DP(x(\bar{t}))f(\bar{t}, x(\bar{t}))\right) + \gamma(t)}{t-\bar{t}} \xrightarrow{\Omega \ni t \to \bar{t}} D_{\Omega}^{k}\left(DP(x(\bar{t}))f(\bar{t}, x(\bar{t}))\right).
\]
This yields that \( x \) is \((\Omega, k+1)\)-derivable with
\[
D_{\Omega}^{k+1}x(t) = \begin{cases} 
D_{\Omega}^{k}\left(DP(x(t))f(t, x(t))\right) & \text{if } x(t) \in \text{bd } S, \\
D_{\Omega}^{k}\left(DP(P(x(t)))f(t, x(t)) + g(t)\hat{n}(P(x(t)))\right) & \text{if } x(t) \in \text{int } S.
\end{cases}
\]
Noting that the mapping \( t \mapsto D_{\Omega}^{k}\left(g(t)\hat{n}(P(x(t)))\right) \) is \( \Omega \)-continuous with
\[
D_{\Omega}^{k}\left(g(t)\hat{n}(P(x(t)))\right) = 0
\]
for each \( t \in \text{C}_{\text{irr}}(x) \), it is not hard to realize that \( D_{\Omega}^{k+1}x(\cdot) \) is \( \Omega \)-continuous and so, \( x(\cdot) \) is \((\Omega, k+1)\)-continuously derivable.

By induction, we conclude that \( x(\cdot) \) is \((\Omega, p+1)\)-continuously derivable, finishing the proof.

An interesting corollary of the above theorem, is the case of analytic equations in finite dimensions, that is when the boundary of \( S \) is analytic and when the perturbation function \( f \) is also analytic. In such a case, we can assure that there are merely finitely many collisions, entailing that the solution \( x \) must be piecewise analytic.
Corollary 1. [Analytic case] Let $X = \mathbb{R}^n$ and suppose that the set $S$ is a nonconvex body with analytic boundary. If the vector field $J$ is also analytic, then for any solution $x : [0, t^*] \to X$ (with $0 < t^* \leq T$) of problem (3), $C_{\text{reg}}(x)$ is finite and $C_{\text{irr}}(x) = \emptyset$.

Thus, $x$ is piecewise analytic, that is, there exist a finite sequence $0 = t_0 < t_1 < \ldots < t_n = t^*$ such that

$$
\forall i \in \{0, \ldots, n-1\}, \quad x|_{t_i} \text{ is analytic, where } t_i = [t_i, t_{i+1}].
$$

Proof. Since $I = [0, t^*]$ is compact, if $C_{\text{reg}}(x)$ is infinite, any cluster point of $C_{\text{reg}}(x)$ would be an element of $C_{\text{irr}}(x)$. So, we only need to prove that $C_{\text{irr}}(x) = \emptyset$. By contradiction, let us suppose that there exists $\bar{t} \in C_{\text{irr}}(x)$.

Let $U = U_\rho(\partial S)$, where $\rho(\cdot)$ is the prox-regularity function of $\partial S$. Let $J_0 := [\bar{t} - \delta_0, \bar{t} + \delta_0] \cap I$ small enough such that $x(J_0) \subseteq U$. By Lemma 4.6, the function $g(t) = \langle f(t,x(t)), \bar{n}(P(x(t))) \rangle$ is $(\Omega, k)$-continuously derivable for every $k \in \mathbb{N}$ with $\Omega = I \setminus C_{\text{reg}}(x)$, and

$$
D_{\Omega}^k g(\bar{t}) = 0, \quad \forall k \in \mathbb{N}.
$$

Now, consider the auxiliary problem

$$
geq \begin{cases}
\dot{y}(t) = f(t, y(t)), & \forall t \in \mathbb{R} \\
y(\bar{t}) = x(\bar{t}),
\end{cases}
$$

and let $y : J_1 \to X$ be a local solution, which is analytic by the Cauchy-Kowalevski theorem (see, e.g. [22, Ch. 1.D]). Without lose of generality, we may assume that $J_1$ is small enough such that $y(J_1) \subseteq U$. Let us define the function $h : J_1 \to \mathbb{R}$ by $h(t) = \langle f(t, y(t)), \bar{n}(P(y(t))) \rangle$ for all $t \in J_1$. Note that $h$ is analytic since $\bar{n}$ is analytic by hypothesis, and $P = P_{\partial S}$ is also analytic (see [30, Lemma 1]).

We claim that for each $k \in \mathbb{N}$, there exists a $C^\infty$-function $F_k : J_1 \times U \to \mathbb{R}$ such that

$$
F_k(t, y(t)) = \frac{d^k}{dt^k} h(t), \quad \forall t \in J_1 \quad \text{and} \quad F_k(t, x(t)) = D_{\Omega}^k g(t), \quad \forall t \in A(x) \cap J_1.
$$

The case $k = 0$ is direct: Consider $F_0(t, u) = \langle f(t, u), \bar{n}(P(u)) \rangle$. Now, fix $k \in \mathbb{N}$ and assume that such a function $F_k$ exists. As in Lemma 4.6, we note that

$$
\frac{d^{k+1}}{dt^{k+1}} h(t) = \frac{d}{dt} F_k(t, y(t)) = D F_k(t, y(t)) \left( \frac{1}{\dot{y}(t)} \right) = D F_k(t, y(t)) \left( \frac{1}{f(t, y(t))} \right)
$$

and we define the $C^\infty$-function $F_{k+1}$ as $F_{k+1}(t, u) = D F_k(t, u) \left( \frac{1}{f(t, u)} \right)$. Then, by chain rule

$$
D_{\Omega}^{k+1} y(t) = D F_k(t, x(t)) \left( \frac{1}{\dot{x}(t)} \right) = F_{k+1}(t, x(t)), \quad \forall t \in A(x) \cap J,
$$

and so, our claim is proved for $k + 1$. This finishes the proof of the claim. Now, since $\bar{t} \in C_{\text{irr}}(x)$, there exists a sequence $(t_n) \subseteq A(x) \cap J$ converging to $\bar{t}$. Thus, for each $k \in \mathbb{N}$ we can write

$$
\frac{d^k}{dt^k} h(\bar{t}) = F_k(\bar{t}, y(\bar{t})) = F_k(\bar{t}, x(\bar{t})) = \lim_{n} F_k(t_n, x(t_n)) = \lim_{n} D_{\Omega}^k g(t_n) = D_{\Omega}^k g(\bar{t}) = 0.
$$

Thus, $\frac{d^k}{dt^k} h(\bar{t}) = 0$ for each $k \in \mathbb{N}$. Since $h$ is analytic, there exists an open interval $J_2 = [\bar{t} - \delta_2, \bar{t} + \delta_2] \subseteq J_1$ on which $h$ is identically zero. This yields that $y|_{J_2}$ is a
solution of the projected differential equation
\[
\begin{align*}
\dot{y}(t) &= DP(P(y(t)))f(t, y(t)), \quad \forall t \in \mathbb{R} \\
y(\bar{t}) &= x(\bar{t}),
\end{align*}
\]
which yields that \(y\big|_{J_2}\) is a curve in \(S\). Since
\[
f(t, y(t)) = \dot{y}(t) = DP(P(y(t)))f(t, y(t)) = P\left(f(t, y(t)); T^C(y(t)); \text{bd } S\right), \quad \forall t \in J_2,
\]
then \(y\big|_{J_2 \cap I}\) is a local solution of problem (10) with \(t_0 = \bar{t}\) (see Remark 2), which yields that \(y\big|_{J_2 \cap I} = x\big|_{J_2 \cap I}\). Since \(J_2 \cap I\) is relatively open in \(I\), we conclude that \(\bar{t} \in B_{\text{int}}(x)\), which is a contradiction since \(B(x) \cap C_{\text{irr}}(x) = \emptyset\). We deduce that \(C_{\text{irr}}(x)\) is empty, which finishes the proof. \(\square\)

5. Applications to sweeping processes. In this section, we illustrate how Theorem 4.7 and Corollary 1 can be applied to a particular case of sweeping processes, namely those where the moving set \(S : [0, T] \to X\) can be described as a translation and homothetic transformation of the starting set \(S(0) = S_0\).

Real applications can be found in electrical circuits. Indeed, several nonsmooth electrical circuits can be written as perturbed sweeping processes (see, e.g., [1, 7]). For example, let us consider a circuit with an ideal diode, an inductor and a current source (see Figure 3), where \(x\) is the current through the inductance and a current source \(i(t)\).

![Figure 3. A circuit with an ideal diode, an inductor and a current source.](image)

The dynamic is given by
\[
\begin{align*}
\dot{x}(t) &= u(t) \\
y(t) &= x(t) - i(t) \\
\mathbb{R}_+ \ni y(t) \perp u(t) &\in \mathbb{R}_+ \\
x(0) &= 0.
\end{align*}
\]
(16)

The third relation in (16) is a complementarity relation which can be written equivalently as \(u(t) \in -\mathcal{N}(\mathbb{R}_+: y(t))\). Therefore, the system (16) is equivalent to the
following sweeping process:
\[
\begin{aligned}
\dot{x}(t) & \in -N(C(t); x(t)), \\
x(0) & = 0,
\end{aligned}
\]
where \(C(t) := i(t) + \mathbb{R}_+\).

**Proposition 2.** Let \(\alpha : [0, T] \to [0, +\infty[\) and \(\xi : [0, T] \to X\) be two mappings of class \(C^{p+1}\), let \(S_0 \subset X\), and let \(S : [0, T] \to X\) be the set-valued mapping given by
\[
S(t) := \alpha(t)S_0 + \xi(t).
\]
Assume \(S_0\) is a closed body with \(C^{p+1}\)-smooth boundary and \(f : [0, T] \times X \to X\) satisfies the hypothesis \((F_1)\). Then, problem \((5)\) has a maximal solution \(x : I \to X\) which is \((\Omega, p + 1)\)-continuously derivable, \(\Omega = I \setminus N\) and \(N := C_{\text{reg}}(x)\) is at most a countable subset of \(I\).

Furthermore, if \(X\) is finite dimensional, \(\alpha, \xi\) and \(f\) are analytic and \(S_0\) has analytic boundary, then the solution \(x\) is piecewise analytic.

Finally, if \(f\) also satisfies the hypothesis \((F_2)\), that is,
\[
\|f(t, u)\| \leq \beta(t)(1 + \|u\|)\ 
\text{for all } (t, u) \in [0, T] \times U
\]
with \(\beta(\cdot) \in L^1([0, T], \mathbb{R})\), then the solution \(x\) is global.

**Proof.** Without loss of generality, we may assume that \(\alpha(0) = 1\) and \(\xi(0) = 0\). Fix \(t \in [0, T]\) and \(u \in S(t)\). By elementary calculations (see, e.g., [13]) of Clarke subdifferentials, and recalling that the Clarke normal cone is the Clarke subdifferential of the associated indicator function, we know that
\[
N^C(S(t); u) = N^C(\alpha(t)S_0 + \xi(t); u)
\]
\[= \frac{1}{\alpha(t)} \cdot N^C \left( S_0; \frac{u - \xi(t)}{\alpha(t)} \right) = N^C \left( S_0; \frac{u - \xi(t)}{\alpha(t)} \right). \]

For the change of variables given by \(y(t) = \alpha(t)^{-1} \cdot (x(t) - \xi(t))\), we can write
\[
\dot{y}(t) = \frac{\alpha(t)(\dot{x}(t) - \dot{\xi}(t)) - \alpha(t)(x(t) - \xi(t))}{\alpha(t)^2} = -\frac{\dot{\alpha}(t)y(t) + \dot{\xi}(t)}{\alpha(t)} - \frac{\dot{\alpha}(t)\xi(t)}{\alpha(t)} + \frac{1}{\alpha(t)}\dot{x}(t).
\]

Defining \(g : [0, T] \times X \to X\) by \(g(t, z) = \alpha(t)^{-1} \left( -\dot{\alpha}(t)z - \dot{\xi}(t) + f(t, \alpha(t)z + \xi(t)) \right)\) for all \((t, z) \in [0, T] \times X\), which is evidently of class \(C^p\), by the above computation we can rewrite problem \((5)\) as
\[
\begin{aligned}
\dot{y}(t) & \in g(t, y(t)) - N^C(S_0, y(t)) \quad \text{for almost all } t \in [0, T], \\
y(0) & = \alpha(0)^{-1}x_0 - \xi(0) = x_0 \in S_0.
\end{aligned}
\]
(17)

Applying for example [14, Theorem 3.1] and Theorem 1.1, we deduce that problem \((17)\) has a maximal solution \(y : I \to X\), which, by Theorem 4.7, is \((\Omega, p + 1)\)-continuously derivable with \(\Omega := I \setminus N\) and where \(N\) is at most a countable subset of \(I\).

It is not hard to realize that \(x : I \to X\) given by \(x(t) = \alpha(t)y(t) + \xi(t)\) is a maximal solution of problem \((5)\), and thanks to Proposition 1, we deduce that it is also \((\Omega, p + 1)\)-continuously derivable.

The second conclusion follows similarly, applying Corollary 1, instead of Theorem 4.7.
Proposition 2 can be directly applied by setting $f \equiv R$ collisions nor outgoing ones), we can see that and outgoing collisions alternate (it is not possible to have two consecutive incoming

Now, let us assume that $f$ satisfies (F2). Then, the above mapping $g$ also satisfies this condition. Indeed, noting that $\alpha, \alpha^{-1}, \xi, \dot{\alpha}$ and $\dot{\xi}$ are all bounded and integrable we can write for all $z \in X$

$$
\|g(t, z)\| \leq \alpha(t)^{-1}(\|z\| + \|f(t, \alpha(t)z + \xi(t))\| + \|\dot{\xi}(t)\|)
$$

$$
\leq \alpha(t)^{-1}(\|z\| + \beta(t) + \beta(t)\alpha(t)\|z\| + \beta(t)\|\xi(t)\| + \|\dot{\xi}(t)\|)
$$

$$
\leq \alpha(t)^{-1}\max\left\{\beta(t)\alpha(t) + 1, \beta(t)(1 + \|\xi(t)\|) + \|\dot{\xi}(t)\|\right\}(1 + \|z\|).
$$

$$
=:\gamma(t) \in L^1[0, T]
$$

Then, since

$$
\dot{g}(t) = \begin{cases} 
DP_{S_0}(y(t))g(t, y(t)) & \text{if } y(t) \in \text{bd } S_0 \\
g(t, y(t)) & \text{if } y(t) \in \text{int } S_0,
\end{cases}
$$

Grönwall’s lemma allows us to ensure that the solution $y$ of Problem (17) has to be global (namely, $I = [0, T]$). The same applies then for the solution $x$ of the original problem. This finishes the proof.

Observe that for the example of electrical circuits given by the system (16), Proposition 2 can be directly applied by setting $f \equiv 0$, $\alpha \equiv 1$, $\xi \equiv i(\cdot)$ and $S_0 = \mathbb{R}_+$. Furthermore, in this particular example, it is not hard to see that the set $N$, given by the regular collisions of $x(\cdot)$, is related to the sign changes of the function $\frac{di}{dt}$. Indeed, rewriting (17) for the change of variables $y(t) = x(t) - i(t)$, we get

$$
\begin{cases}
\dot{y}(t) \in -\frac{di}{dt}(t) - N^C(\mathbb{R}_+, y(t)) & \text{for almost all } t \in [0, T], \\
y(0) = 0.
\end{cases}
$$

(18)

This dynamical system is easier to understand in terms of the collisions: An incoming collision $\bar{t} \in C_{in}(y)$ requires $\frac{di}{dt}(t)$ to be strictly positive in the interval $[\bar{t} - \delta, \bar{t}]$ for some $\delta > 0$. Similarly, an outgoing collision $\bar{t} \in C_{out}(y)$ requires $\frac{di}{dt}(t)$ to be strictly negative in the interval $[\bar{t}, \bar{t} + \delta]$ for some $\delta > 0$. Intuitively, since incoming and outgoing collisions alternate (it is not possible to have two consecutive incoming collisions nor outgoing ones), we can see that

$$
|N| \leq 2 \cdot \left(\# \text{ Sign changes of } \frac{d}{dt}i\right).
$$

Formally, defining the sign function $\text{sgn} : \mathbb{R} \to \mathbb{R}$ by

$$
\text{sgn}(t) = \begin{cases} 
1 & \text{if } t \geq 0, \\
-1 & \text{if } t < 0,
\end{cases}
$$

we can write the following proposition:

**Proposition 3.** Let us consider the dynamical system given by (16) and suppose that $i : [0, T] \to \mathbb{R}$ is of class $C^p$. Then, there exists a unique global solution $x : [0, T] \to \mathbb{R}$ of (16) and we have that

$$
\text{sgn} \circ \frac{d}{dt}i \text{ has bounded variation in } [0, T] \implies |C_{\text{reg}}(x)| < \infty.
$$

In such a case, the solution $x$ is piecewise $(p + 1)$-continuously differentiable.
Proof. Noting from the proof of Proposition 2 that $C_{\text{reg}}(x) = C_{\text{reg}}(y)$, where $y(t) := x(t) - i(t)$, it is enough to prove the proposition for the solution $y$ of the system (18).

First, noting that $i : [0, T] \to \mathbb{R}$ is at least continuously differentiable (and therefore Lipschitz-continuous), the solution $y$ has to be global. Also, if $C_{\text{reg}}(y)$ is finite, the piecewise $(p + 1)$-continuous differentiability can be obtained following the same reasoning as in the proof of Corollary 1.

Now, assume that the function $\text{sgn} \circ \frac{d}{dt} i$ has bounded variation in the interval $[0, T]$, and let us suppose that $|C_{\text{reg}}(y)| = \infty$. We denote by $V \left( \text{sgn} \circ \frac{d}{dt} i; [0, T] \right)$ the variation of $\text{sgn} \circ \frac{d}{dt} i$ in $[0, T]$.

Without losing generality, we may assume that there exists an increasing sequence $(t_n) \subset [0, T]$ such that

- $y(t_n) > 0$ if $n$ is even (that is, $y(t_n) \in \text{int} \mathbb{R}_+$); and
- $y(t_n) = 0$ if $n$ is odd (that is, $y(t_n) \in \text{bd} \mathbb{R}_+$).

Fix $n \in \mathbb{N}$ even. Since the solution $y : [0, T] \to \mathbb{R}_+$ is continuous, there exists $\varepsilon > 0$ such that $y(t) > 0$ for every $t \in [t_n - \varepsilon, t_n + \varepsilon]$. Define then

$$t_n^- = \sup \{ t < t_n : y(t) = 0 \} \quad \text{and} \quad t_n^+ = \inf \{ t > t_n : y(t) = 0 \}.$$

It is not hard to see that $t_{n-1}^- \leq t_n^- < t_n < t_n^+ \leq t_{n+1}$, and so we can define the nonempty open set

$$A_n = ]t_n^-, t_n^+[.$$

By construction, $A_n \subset A(y)$ and so $y$ is continuously differentiable in $A_n$ with $\dot{y}(s) = -\frac{d}{dt} i(s)$ for all $s \in A_n$. Noting that $y(t) = d(y(t), \text{bd} \mathbb{R}_+)$, we can apply directly Lemma 4.2 to $t_n^-$ and $t_n$ and to $t_n$ and $t_{n+1}$ to conclude that there exist $s^- \in ]t_n^-, t_n[$ and $s^+ \in ]t_n, t_{n+1}[$ such that

$$- \frac{d}{dt} i(s^-) = \dot{y}(s^-) > 0 \quad \text{and} \quad - \frac{d}{dt} i(s^+) = \dot{y}(s^+) < 0.$$

This yields that

$$\sup_{s \in A_n} \text{sgn} \left( \frac{d}{dt} i(s) \right) - \inf_{s \in A_n} \text{sgn} \left( \frac{d}{dt} i(s) \right) = 2.$$

Since for every $n \in \mathbb{N}$ even the set $A_n$ is measurable, we have that

$$V \left( \text{sgn} \circ \frac{d}{dt} i; [0, T] \right) \geq \sum_{k=1}^m \left( \sup_{t \in A_{2k}} \text{sgn} \left( \frac{d}{dt} i(t) \right) - \inf_{t \in A_{2k}} \text{sgn} \left( \frac{d}{dt} i(t) \right) \right) \geq 2m.$$

We deduce that $V \left( \text{sgn} \circ \frac{d}{dt} i; [0, T] \right) = +\infty$, which is a contradiction. This finishes the proof. \hfill \Box

The above proposition gives a very natural (and easily verifiable) sufficient condition to get piecewise smoothness of the solution of system (18). However, this condition of bounded variation of the sign function relies in the particularities of this example: the function $i(\cdot)$ depends only on the time and in addition problem (18) is one-dimensional. Thus, it would be very interesting to give similar conditions for more complicated electrical circuits. This is one of the perspectives for future research.
6. Final comments. In this work, we investigate the regularity of the solutions, when we deal with a smooth projected differential equation, in the sense that the perturbation \( f \) and the set \( S \) are smooth. The notion of \( \Omega \)-derivability allowed us to deduce Theorem 4.7 and Corollary 1, which are the main results of this work. Furthermore, we presented applications of our results to an important class of perturbed sweeping processes, studied in the literature. We hope that these fundamental results can bring to the community some new perspectives on the different situations that require smoothness of the solutions or regularity of the collisions of the trajectories.

The natural continuation of this work is to address the general perturbed sweeping process. To replicate the results that we already have for the stationary case, we need to introduce a suitable notion of “smooth movement” for the set-valued map \( S : [0, T] \rightrightarrows X \).

Another important problem is to study the case when the set \( S_0 \) can be described as an intersection of finite closed bodies with smooth boundary, which is the situation of most of the applications. A particular case of such sets are the semi-algebraic sets, which have shown to be very pertinent objects when dealing with applications outside the manifold setting. We will treat these problems in a future work. Applications to optimal control will also be explored.

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Appendix A. An example of a large irregular collision set. This appendix is devoted to provide an example of a projected system whose set of irregular collisions of its solution is large. The aim of this example is to show the necessity of introducing the notion of \( \Omega \)-derivability given in Definition 4.5.

Let us denote by \( \lambda \) the Lebesgue measure and let us consider the \( \frac{1}{4} \)-Cantor-like set (also known as Smith-Volterra-Cantor set, see, e.g., [6, Chapter 4]), denoted by \( \mathcal{C}(1/4) \), which is constructed as follows:

1. Define \( J_{1,1} := \left[ \frac{3}{8}, \frac{5}{8} \right] \) and \( \ell_1 = \frac{1}{4} \). Let \( I_{1,1} = \left[ \frac{3}{8}, \frac{5}{8} \right] \) and \( I_{1,2} = \left[ \frac{7}{8}, 1 \right] \).

2. For each integer \( n > 1 \) and each \( k \in \{1, \ldots, 2^n-1\} \), suppose that the closed interval \( I_{n-1,k} \) is defined. Let \( \ell_{n-1} = \lambda(I_{n-1,k}) \), \( a_{n-1,k} = \min I_{n-1,k} \) and \( b_{n-1,k} = \max I_{n-1,k} \), so we have \( I_{n-1,k} = [a_{n-1,k}, b_{n-1,k}] \) with \( b_{n-1,k} = a_{n-1,k} + \ell_{n-1} \). Define
   - \( J_{n,k} = \left[ a_{n-1,k} + \frac{\ell_{n-1}}{2}, a_{n-1,k} + \frac{\ell_{n-1}}{2} + \frac{1}{2^{2n}} \right] \);
   - \( I_{n,2k-1} = \left[ a_{n-1,k}, a_{n-1,k} + \frac{\ell_{n-1}}{2} - \frac{1}{2^{2n}} \right] \); and
   - \( I_{n,2k} = \left[ b_{n-1,k} - \frac{\ell_{n-1}}{2} + \frac{1}{2^{2n}}, b_{n-1,k} \right] \).

Observe that this construction is well defined, since for each \( n \in \mathbb{N} \) and each \( k \in \{1, \ldots, 2^n\} \), we have that \( \lambda(I_{n,k}) > \frac{1}{4^n} \). Indeed, for the case \( n = 1 \) we have that \( \lambda(I_{1,1}) = \lambda(I_{1,2}) = \frac{3}{4} > \frac{1}{2} \). Now, assume that this estimate is true for \( n-1 \) and each \( k \in \{1, \ldots, 2^{n-1}\} \). Then, for each \( k \in \{1, \ldots, 2^{n-1}\} \) we have that

\[
\lambda(I_{n,2k-1}) = \lambda(I_{n,2k}) = \frac{1}{2} (\lambda(I_{n-1,k}) - \lambda(J_{n,k})) > \frac{1}{2} \left( \frac{1}{4^{n-1}} - \frac{1}{4^n} \right) > \frac{1}{4^n}.
\]
Thus, the inequality \( \lambda(I_{n,k}) > \frac{1}{4^n} \) holds for every \( n \in \mathbb{N} \) and any \( k \in \{1, \ldots, 2^n\} \). With this construction, the \( \frac{1}{4^n} \)-Cantor-like set is defined as

\[
\mathcal{C}(1/4) := [0,1] \setminus \left( \bigcup_{n \geq 1} \bigcup_{k=1}^{2^n - 1} J_{n,k} \right) = \bigcap_{n \geq 1} \left( \bigcup_{k=1}^{2^n} J_{n,k} \right).
\]

(19)

Since all sets \( J_{n,k} \) are disjoint, we have that

\[
\lambda(\mathcal{C}(1/4)) = 1 - \sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} \lambda(J_{n,k}) = 1 - \sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} \frac{1}{4^n} = 1 - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.
\]

It is known (and not hard to see) that the set \( \mathcal{C}(1/4) \) is closed, it has empty interior and it is uncountable. This is a classic example of a (nowhere dense) set whose boundary has positive Lebesgue measure. In what follows we will put

\[
\begin{align*}
\alpha_{n,k} &= a_{n,k} + \frac{1}{2^n} \lambda(J_{n,k}) - \frac{1}{2^{n+1}}, \\
\beta_{n,k} &= a_{n,k} + \frac{1}{2^n},
\end{align*}
\]

which are the end-points of the interval \( J_{n,k} \), that is

\[
\forall n \geq 1, \forall k \in \{1, \ldots, 2^{n-1}\}, J_{n,k} = [\alpha_{n,k}, \beta_{n,k}].
\]

Also, for each \( n \in \mathbb{N} \) and each \( k \in \{1, \ldots, 2^{n-1}\} \) we will divide the interval \( J_{n,k} \) into the following two intervals:

\[
J_{n,k}^- := [a_{m,k} + \frac{1}{2^{m+1}}, \beta_{m,k}] \quad \text{and} \quad J_{n,k}^+ := [a_{m,k} + \frac{1}{2^{m+1}}, \beta_{m,k}].
\]

We will construct a continuously differentiable function \( f : [0,1] \to \mathbb{R} \) for which the set of its critical points contains the \( \mathcal{C}(1/4) \) set. This function will be used in Example A.1, below.

1. Fix first \( f_0 \equiv 0 \).
2. For each \( n \geq 1 \), define the function \( f_n : [0,1] \to \mathbb{R} \) as follows:
   - For each \( m \in \{1, \ldots, n\} \) and each \( k \in \{1, \ldots, 2^m-1\} \),
     \[
     f_n(t) := \begin{cases} 
     \frac{1}{2^{m+1}} \sin^3(4^m \cdot 2\pi(t - a_{m,k})) & \text{if } t \in J_{n,k}^- \\
     \frac{2^{m+1}}{3^{m+1}} \sin^3(4^m \cdot 2\pi(t - a_{m,k})) & \text{if } t \in J_{n,k}^+
     \end{cases}
     \]
   - \( f_n(t) = 0 \) if \( t \in [0,1] \setminus \left( \bigcup_{m=1}^{n} \bigcup_{k=1}^{2^{m-1}} J_{n,k} \right) \).
   - Note that for each \( n \in \mathbb{N} \), each \( m \in \{1, \ldots, n\} \) and each \( k \in \{1, \ldots, 2^{m-1}\} \), the function \( f_n \) satisfies \( f_n(a_{m,k}) = f_n(a_{m,k} + \frac{1}{2^{m+1}}) = f_n(\beta_{m,k}) = 0 \), from which we deduce the continuity of \( f_n \). Furthermore,
     \[
     f_n'(t) := \begin{cases} 
     \frac{6}{2^{m+1}} \sin^2(4^m \cdot 2\pi(t - a_{m,k})) \cos(4^m \cdot 2\pi(t - a_{m,k})) & \text{if } t \in J_{n,k}^- \\
     \frac{12}{3^{m+1}} \sin^2(4^m \cdot 2\pi(t - a_{m,k})) \cos(4^m \cdot 2\pi(t - a_{m,k})) & \text{if } t \in J_{n,k}^+
     \end{cases}
     \]

and \( f_n'(t) = 0 \) if \( t \in [0,1] \setminus \left( \bigcup_{m=1}^{n} \bigcup_{k=1}^{2^{m-1}} J_{n,k} \right) \). Finally, it is not hard to see that

\[
\begin{align*}
f_n'(a_{m,k}) &= f_n'(a_{m,k} + \frac{1}{2^{m+1}}) = f_n'(\beta_{m,k}) = 0, \quad \forall m \leq n, \forall k \in \{1, \ldots, 2^{m-1}\},
\end{align*}
\]

which yields that \( f_n \) is continuously differentiable. To illustrate how the construction is, Figure 4 presents the graphs of \( f_2 \) and \( f_2' \).

Note that \( f_1 \) and \( f_2 \) (and so \( f_1' \) and \( f_2' \)) coincide in \( J_{1,1} \). More generally, the construction leads that for each \( n \in \mathbb{N} \) and each \( k \in \{1, \ldots, 2^{n-1}\} \) we have

\[
f_{m+n}(t) = f_n(t) \quad \text{and} \quad f'_{m+n}(t) = f'_n(t), \quad \forall t \in J_{n,k}, \forall m \in \mathbb{N}.
\]

(20)
Now, consider the space $C^1([0,1],\mathbb{R})$ endowed with the norm $\| \cdot \| : C^1([0,1],\mathbb{R}) \to \mathbb{R}_+$ given by
$$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$ It is known that $(C^1([0,1],\mathbb{R}),\|\cdot\|)$ is a Banach space. We claim that $(f_n)$ is a Cauchy sequence in this space. Fix $n > 1$. By relation (20), it is not hard to see that
$$\|f_{n+1} - f_n\| = \sup_{k \in \{1, \ldots, 2^n\}} \sup_{t,s \in J_{n+1,k}} \|f_{n+1}(t)\| + \|f'_{n+1}(s)\|$$
$$\leq \frac{2}{4^{2(n+1)}} + \frac{12\pi}{4^{4(n+1)}} \leq \frac{24\pi}{4^{n+1}},$$
and so the series $\sum_{n=1}^{\infty} \|f_n - f_{n+1}\|$ is convergent, proving that $(f_n)$ is a Cauchy sequence as claimed. Let $f$ be the limit function of $(f_n)$ in $C^1([0,1],\mathbb{R})$. Using again equation (20) and the fact that $(f_n)$ converges pointwise to $f$, we can deduce that
$$f(t) = \begin{cases} f_n(t) & \text{if } t \in J_{n,k} \\ 0 & \text{if } t \in \mathcal{C}(1/4). \end{cases}$$
and
$$f'(t) = \begin{cases} f'_n(t) & \text{if } t \in J_{n,k} \\ 0 & \text{if } t \in \mathcal{C}(1/4). \end{cases}$$

Denoting by $K = \mathcal{C}(1/4) \cup \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} \{\alpha_{n,k} + \frac{1}{4^{n+1}}, \alpha_{n,k} + \frac{2}{4^{n+1}}, \alpha_{n,k} + \frac{3}{4^{n+1}}\}$, it is not hard to see that
$$f^{-1}(0) = \mathcal{C}(1/4) \cup \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} \{\alpha_{n,k} + \frac{2}{4^{n+1}}\} \subset K = (f')^{-1}(0),$$
finishing the construction. In the next example we use this function $f$ to construct an example of a projected differential inclusion whose solution $x(\cdot)$ has $\mathcal{C}(1/4)$ as its set of irregular collisions $C_{\text{irr}}(x)$. Furthermore, we will see that $x(\cdot)$ fails to be continuously differentiable at each point of $C_{\text{irr}}(x)$.

**Example A.1.** There exists a convex body $S \subset \mathbb{R}^2$ with $C^\infty$-boundary and a Lipschitz continuous and continuously differentiable mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ such
that the differential inclusion
\[
\begin{cases}
\dot{x}(t) \in F(x(t)) - NC(S; x(t)) & \text{a.e. } t \in [0, 1], \\
x_0 \in S,
\end{cases}
\] (23)
has a unique absolutely continuous solution \( x : [0, 1] \to \mathbb{R}^2 \) which fails to be twice-
continuously differentiable near each point of the \( \mathcal{C}(1/4) \) set.

**Proof.** Let \( f : [0, 1] \to \mathbb{R} \) be the function constructed above, and let \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) given
by \( \tilde{f}(t) = f(\pi_{\mathbb{R}/\mathbb{Z}}(t)) \), where \( \pi_{\mathbb{R}/\mathbb{Z}} \) is the quotient map from \( \mathbb{R} \) to \([0, 1]\). Namely, \( \tilde{f} \)
is the periodic extension of \( f \). Let the mapping \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by
\[
F(x_1, x_2) := (1, \tilde{f}(x_1)).
\]
Since the derivative of \( f \) is continuous and uniformly bounded, it is not hard to see
that \( F \) is Lipschitz continuous and continuously differentiable.

Now, consider \( S \) as the half-space \( S = \{(x_1, x_2) : x_2 \geq 0\} \). Note that the normal
cone \( N(S; x) \) at each point of \( x \in \text{bd} S \) is the vector
\[
\hat{n} = (0, -1).
\]
Consider also the curve \( x : [0, 1] \to \mathbb{R}^2 \) defined as follows:

1. For each \( n > 0 \) and each \( k \in \{1, \ldots, 2^n - 1\} \), define \( x(t) \) in \( J_{n,k} \) as
\[
x(t) := \begin{cases}
(t, \int_{0}^{t} f(s) ds) & \text{if } t \in [\alpha_{n,k}, z_{n,k}]
\\
(t, 0) & \text{if } t \in [z_{n,k}, \beta_{n,k}],
\end{cases}
\]
where \( z_{n,k} \) is the first zero of the function \( t \mapsto \int_{0}^{t} f(s) ds \) after the middle point
\[
m_{n,k} := \alpha_{n,k} + 2^{-1}(\beta_{n,k} - \alpha_{n,k}) \text{ of } [\alpha_{n,k}, \beta_{n,k}].
\]
We claim that \( z_{n,k} \) exists and \( z_{n,k} \in [m_{n,k}, \beta_{n,k}] \). Indeed, by construction of the function \( f \), it is easy to see
that in every interval \( J_{n,k} \) we have that
- \( f > 0 \) in \( [\alpha_{n,k}, m_{n,k}] \) and \( f < 0 \) in \( [m_{n,k}, \beta_{n,k}] \);
- \( \int_{\alpha_{n,k}}^{m_{n,k}} f(s) ds > 0 \);
- \( \int_{\alpha_{n,k}}^{\beta_{n,k}} f(s) ds = -\frac{1}{2} \int_{m_{n,k}}^{\beta_{n,k}} f(s) ds \), and so \( \int_{\alpha_{n,k}}^{\beta_{n,k}} f(s) ds < 0 \).

We deduce that there exists a unique point \( z_{n,k} \in [m_{n,k}, \beta_{n,k}] \) such that
\[
f_{\alpha_{n,k}} z_{n,k} f(s) ds = 0,
\]
as claimed. Thus, \( x(\cdot) \) is well defined and continuous on \( [\alpha_{n,k}, \beta_{n,k}] \).

2. For each \( n \in \mathcal{C}(1/4) \), we set \( x(t) = (t, 0) \).

To illustrate the trajectory \( x(\cdot) \), Figure 5 shows the trajectory (green curve) when
we use for \( f \) the function \( f_2 \) defined in the iterative construction. The red stars are
the collisions of \( x \) with \( \text{bd} S \). The dashed green curves represent the trajectory that
\( x \) would follow if we remove the constraint set \( S \).

We claim that \( x(\cdot) \) is a solution for the differential inclusion (23). By construction,
It is not hard to see that \( x \) is absolutely continuous (even Lipschitz continuous)
since, if we define
\[
J := \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^n-1} [\alpha_{n,k}, z_{n,k}],
\]
we can write
\[
x(t) = \left( t, \sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} \int_{0}^{t} I_{[0,t]}(s) \cdot f(s) ds \right) = \int_{0}^{t} (1, I_{J}(s) f(s)) ds.
\]
Furthermore, we have for almost every \( t \in [0, 1] \)
\[
\dot{x}(t) = (1, 1, f(t)) = (1, 1, f(x_1(t)) = (1, 1, f(x(t)) = (1, 1, f(x(t)) = (1, f(x(t)) = (1, f(x(t)) = (1, f(x(t))) = (1, f(x(t))) = (1, f(x(t))) = (1, f(x(t))).
\]
We also notice for each \( t \in [0, 1] \) that \( x_2(t) \geq 0 \) by construction, so \( x(t) \in S \). Fix any \( t \in [0, 1] \) where \( \dot{x}(t) \) exists. We have the following cases:

1. There exists \( n \geq 1 \) and \( k \in \{1, \ldots, 2^{n-1}\} \) such that \( t \in \alpha_{n,k}, z_{n,k} \). Then, \( x(t) \in \text{int} S \) and so
\[
\dot{x}(t) = (1, f(x_1(t)) = (1, f(x(t)) = (1, f(x(t)) = (1, f(x(t)) = (1, f(x(t)) = (1, f(x(t)) = (1, f(x(t)) = (1, f(x(t)) = (1, f(x(t)).
\]
2. There exists \( n \geq 1 \) and \( k \in \{1, \ldots, 2^{n-1}\} \) such that \( t \in [z_{n,k}, \beta_{n,k}] \). Then, \( x(t) \in \text{bd} S \) and \( f(t) \leq 0 \) which yields
\[
\dot{x}(t) = (1, 0) = F(x(t)) - |f(x_1(t))|(0, -1) = F(x(t)) - N(S; x(t)).
\]
3. The inclusion \( t \in C(1/4) \) holds. Then, \( x(t) \in \text{bd} S \) and \( f(t) = 0 \), so
\[
\dot{x}(t) = (1, 0) = (1, f(x_1(t))) = F(x(t)) = F(x(t)) = F(x(t)) = F(x(t)) = F(x(t)).
\]
Thus, \( x(\cdot) \) is the solution of problem (23), as we claimed.

Now, suppose that there exists \( t \in C(1/4) \) such that \( x \) is twice-continuously differentiable near \( t \). Then, there exists a neighborhood \( U = ]t - \varepsilon, t + \varepsilon[ \cap [0, 1] \) on which \( x \) is twice-differentiable and \( \dot{x}(\cdot) \) is continuous on \( U \). However, by construction, there exists \( n > 1 \) and \( k \in \{1, \ldots, 2^n\} \) large enough such that \( t \in I_{n-1,k} \) (see the definition of \( I_{n-1,k} \) above equation (19)) and \( \lambda(I_{n-1,k}) < \varepsilon \). This yields that \( z_{n,k} \in J_{n,k} \subset I_{n-1,k} \subset U \), but the function \( 1_{J_{n,k}} f(\cdot) \) is not continuous at \( z_{n,k} \). Thus, \( \dot{x}(\cdot) \) is not continuous at \( z_{n,k} \), and so the second derivative \( \ddot{x} \) does not exist at \( z_{n,k} \), which is a contradiction. The proof is now finished.

\[\Box\]
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