POLARIZED PUSHOUTS OVER FINITE FIELDS

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To Steve Kleiman on his sixtieth birthday

Abstract. Let \( p : Y \to X \) be a surjection between schemes projective over the algebraic closure of a finite field. Let \( L \) be a line bundle on \( X \) such that \( p^*(L) \) is globally generated. A natural necessary and sufficient condition is given under which some positive tensor power of \( L \) is globally generated. An application is a sufficient condition for semi-ampleness of nef line bundles on \( \overline{M}_{g,n} \) in positive characteristic, which implies the semi-ampleness of \( \lambda, \psi_i \) and myriad other nef line bundles.

§0 Introduction and statement of results.

Here I consider the following question: Let \( p : Y \to X \) be a proper surjection. Suppose \( p^*(L) \) is globally generated. Under what conditions does it follow that \( L \) is semi-ample (ie. some positive tensor power is globally generated)? This always holds when \( X \) is normal, so the interesting case is when \( p \) is the normalisation.

0.0 Definition. Let \( L \) be a nef line bundle on a complete algebraic space \( X \). Two closed points \( x_1, x_2 \in X \) are \( L \)-equivalent if there is a connected closed curve \( x_1, x_2 \in C \subset X \) with \( L \cdot C = 0 \). We say that \( L \)-equivalence is bounded if there is an integer \( m > 0 \) so that any two \( L \)-equivalent points lie on a connected curve \( C \) as above, with \( C \) having at most \( m \) irreducible components.

The main technical result of this paper is the following:

0.1 Theorem. Let \( p : \tilde{X} \to X \) be a surjection between algebraic spaces proper over the algebraic closure of a finite field. Let \( L \) be a line bundle on \( X \). \( L \) is semi-ample iff \( p^*(L) \) is semi-ample and \( L \)-equivalence is bounded.

(0.1) fails for all other fields, take for example the normalisation of a rational curve with one ordinary node, and \( L \) a degree zero line bundle. The
pullback is always trivial, but $L$ is semi-ample iff torsion. One cannot drop the boundedness assumption in (0.1), see §5.

Here are some applications of (0.1).

[Keel99] gives a general basepoint freeness result in positive characteristic (that fails in characteristic zero) namely a nef line bundle $L$ is semi-ample iff its restriction to the exceptional locus, $E(L)$, semi-ample, where $E(L)$ is the union of $L$-exceptional subvarieties, and a subvariety $Z \subset X$ is called $L$-exceptional if $c_1(L)^{\dim(Z)} \cup Z = 0$ (or equivalently, $L|_Z$ is not big). The main difficulty in applying this in practice is that $E(L)$ can have bad singularities, e.g. it is frequently reducible, so one would like to pass to a desingularisation. An immediate corollary of (0.1) is:

**0.3 Corollary.** Let $L$ be a nef line bundle on a scheme $X$ projective over the algebraic closure of a finite field. Then $L$ is semi-ample iff the restriction of $L$ to the normalisation of the exceptional locus is semi-ample and $L$-equivalence is bounded.

(0.3) has applications to $\overline{M}_{g,n}$.

**0.3.1 Corollary.** Characteristic $p > 2$. Let $L$ be a nef line bundle on $\overline{M}_{g,n}$. $L$ is semi-ample iff $L$-equivalence is bounded, and $r^*(L)$ is semi-ample, for $r : \overline{M}_{0,2g+n} \to \overline{M}_{g,n}$ the natural finite map whose image is the locus of curves with only rational components.

The boundedness condition is a kind of combinatorial question. I expect it always holds, see (0.6), and can show it does in many cases. A precise statement requires some notation:

For a subset $S \subset N := \{1, 2, 3, \ldots, n\}$ $\pi_S : \overline{M}_{g,N} \to \overline{M}_{g,S}$ is the tautological map dropping the points labeled by $S$ (and stabilizing). Let $C$ be a stable pointed curve. Varying the moduli of the normalisation of one of the irreducible components (together with its distinguished points), keeping the other components fixed, induces a natural finite map: $h : \overline{M}_{r,m} \to \overline{M}_{g,n}$ (where $r$ is the geometric genus of the varying component). The product of these maps over all components of $C$ has image the closed stratum of topological type $C$. For details see [GKM00].

**0.4 Corollary.** Characteristic $p > 2$. Let $L$ be a nef line bundle on $\overline{M}_{g,n}$. Assume for each map $h : \overline{M}_{r,m} \to \overline{M}_{g,n}$ as above with $r \leq 1$ that either $h^*(L)$ is numerically trivial, or $L = \pi_S^*(L_S)$ for a (necessarily nef) line bundle $L_S$ on $\overline{M}_{r,S}$ with $E(L_S) \subset \partial \overline{M}_{r,S}$. Then $L$ is semi-ample.

(0.3.1) and (0.4) are proved in §3, see (3.6). (0.4) covers myriad cases. Here are just a few:
0.5 Corollary. In characteristic $p > 0$ the line bundles $\lambda$, $\psi_i$, and $D_{g,n}$ of [GKM00,4.8], are all semi-ample.

Recall $\lambda$ is the determinant of the Hodge bundle (which has fibre at a curve the vector space of global 1-forms). Semi-ampleness of $\lambda$ is known in all characteristics. In characteristic $p > 0$ it is due to Szpiro (by a completely different argument), [Szpiro90]. The associated map is to the Satake compactification of $A_g$.

$\psi_i$ is the bundle with fibre the cotangent bundle of the curve at the $i^{th}$ point. It is nef and big in all characteristics, but fails to be semi-ample in characteristic zero, see [Keel99]. I claimed the semi-ampleness for $p > 0$ in [Keel99], but the proof contains a gap – I use implicitly the fact that for a product of $M_{g_i,n_i}$, the boundary is connected, which fails for $M_{0,4}$ (in fact this is the only case where it fails).

$D_{g,n}$ is introduced in [GKM00]. The semi-ampleness of $D_{g,n}$ gives an interesting birational contraction: Let $F_{g,n} \subset M_{g,n}$ be the locus of flag curves, i.e. the image map $f : M_{0,g+n}/S_g \to M_{g,n}$ induced by gluing on $g$ copies of a one pointed irreducible rational curve with a single node (any two are isomorphic) at $g$ points (which the symmetric group $S_g \subset S_{g+n}$ permutes). $f$ is the normalization of $F_{g,n}$. The flag locus is important from several points of view, see e.g. [HarrisMorrison98], [Logan00]. It is particularly important in the Mori theory of $M_{g,n}$: By [GKM00,0.7], $D_{g,n}$ is nef and the corresponding face of the (Kleiman) Mori cone is

$$D_{g,n}^+ \cap NE_1(M_{g,n}) = NE_1(M_{0,g+n}/S_g).$$

The face is natural both geometrically and purely combinatorially, and outside of this face the Mori cone is completely known. The semi-ampleness of $D_{g,n}$ means this face can be blowdown.

0.5.1 Corollary. In positive characteristic there is a birational map

$$q : M_{g,n} \to Q_{g,n}$$

such that $q(C)$ is a point iff $C$ is numerically equivalent to a curve in $F_{g,n}$. The relative Mori cone $NE_1(M_{g,n}/Q_{g,n})$ is naturally identified with $NE_1(M_{0,g+n}/S_g)$.

By [GKM00] the analog of the assumption in (0.4) holds for any nef line bundle $L$ on $M_{r,m}$ for $r \geq 2$. A slight generalization holds for $r = 1$, namely any $L$ is a tensor product of $L_S$, and I believe (0.4) could be generalized to allow for this. I do not know of a nef line bundle on $M_{0,M}$ which does not satisfy this generalized condition.

I propose the following:
0.6 Conjecture. Every nef line bundle on $\overline{M}_{g,n}$ is semi-ample in positive characteristic.

Note that together with [GKM00,0.2] this gives the conjecture:

0.7 Conjecture. The Mori cone of $\overline{M}_{g,n}$ is generated by finitely many rational curves and in characteristic $p > 0$ every face has an associated contraction.

Thus in characteristic $p > 0$ the Mori cone behaves (conjecturally) like that of a Fano variety, although $\overline{M}_{g,n}$ is usually of general type.

0.8.1 Question. If $L$ is a non-trivial nef line bundle on $\overline{M}_{0,n}$ can it be written as a tensor product of line bundles of the form $\pi^*_S(L_S)$ for $L_S$ as in (0.4)?

It is natural to wonder quite generally:

0.8.2 Question. Over finite fields, if $L$-equivalence is bounded does it follow that $L$ is semi-ample?

A special case of (0.8.2):

0.9 Question. Let $S$ be smooth surface over a finite field. Let $D$ be a divisor on $S$. If $D \cdot C > 0$ for every irreducible curve $C \subset S$, does it follow that $D$ is ample (or equivalently, that $D^2 > 0$)?

(0.1) has a (to me) surprising consequence. By a map related to $L$ I mean a proper map $g : X \to Z$ between algebraic spaces whose exceptional subvarieties (i.e. the subvarieties $W$ such that $\dim(W) > \dim(g(W))$ are precisely the $L$-exceptional subvarieties. The Stein factorisations of any two related maps are the same, and (if it exists) is called the map associated to $L$. For elementary functorial properties of the associated map see [Keel99,1.0]. If $L$ has an associated map we say $L$ is endowed with a map (EWM). A semi-ample line bundle is EWM – the associated map is given by sections of a sufficiently high tensor power, but the converse fails. However:

0.10 Theorem. A nef line bundle on a scheme projective over the algebraic closure of a finite field is semi-ample iff it has a related map.

(0.1) is a formal consequence of the existence of certain pushout diagrams, (0.17) which though technical is I think of independent interest. I will turn to this next.

Content overview. The remainder of §0 is devoted to pushouts. (0.17) is proven in §1. (0.1) is proven in §2. The results for $\overline{M}_{g,n}$ are proven in §3. (0.10) is proven in §4. An interesting example of Kollár is given in §5.
Thanks: I thank James McKernan with whom I discussed many aspects of this paper, and who stimulated my interest by raising the question of whether or not nef and semi-ample are the same over a finite fields; János Kollár, who gave me a counter-example, (5.2), to an overly optimistic version of (0.1); Joe Harris, for inviting me (or more precisely, accepting my self invitation) to Harvard for a year, and Tom Graber, with whom I jointly observed (1.9). I especially thank Dan Abramovich for his very careful reading of the paper. He found lots of mistakes, and many places where the exposition could be simplified and or otherwise improved.

Notations and conventions: Throughout the note all spaces considered are algebraic spaces of finite type over a field \( k \), and all maps are \( k \)-linear. I will often replace line bundles by positive tensor powers without remark.

By a contraction we mean a proper map \( f : X \to Y \), with \( f_* (\mathcal{O}_X) = \mathcal{O}_Y \).

I use the following definitions and conventions of [Keel99]. For a line bundle \( L \) on \( X \) and a map \( h : Y \to X \) we write \( L|_Y \) (or sometimes \( L_Y \)) for \( h^*(L) \). If \( L \) on \( X \) is EWM then the associated map is denoted \( g_X : X \to Z_X \).

Pushouts.

0.12 Pushout Notation.

We assume we are given \((X, L_X)\), with \( L_X \) nef, all spaces proper, maps \( f_i : X \to X_i, i = 1, 2 \), and line bundles \( L_{X_i} \) on \( X_i \) with \( L_{X_i}|_X = L_X \).

Our goal will be to find a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & X_1 \\
\downarrow{f_2} & & \downarrow{g_1} \\
X_2 & \xrightarrow{g_2} & P
\end{array}
\]

and a line bundle \( L_P \) on \( P \) such that \( L_P|_{X_i} = L_X \otimes^m \) for \( i = 1, 2 \) and some \( m > 0 \). Furthermore we require that \( L_P \) is ample, or equivalently, by [Keel99,1.0], that \( g_i \) is a related map for \( L_{X_i}, i = 1, 2 \). As a shorthand for the existence of this diagram we will say that a polarized pushout for \( L_{X_1}, L_{X_2}, f_1, f_2 \) (or just \( f_1, f_2 \)) exists.

Remarks. If a polarized pushout exists, then \( L_X, L_{X_i} \) are semi-ample. If \( L_X \) is ample, and \( f_i \) are surjective then \( f_i \) is finite, and \( L_{X_i} \) is ample.

By an equivalence relation on a space \( X \) we mean an equivalence relation on the \( k \)-points of \( X \). We call the relation algebraic if it is the set of \( k \)-points of a closed subset \( R \subset X \times X \).

We say an equivalence relation on \( X \) dominates a map \( f : X \to Y \) if (\( k \)-points of) fibres of \( f \) are contained in \( R \)-equivalence classes. Of course if \( R \) is algebraic this holds iff \( R \) contains the reduction of \( X \times_Y X \).
For a map \( h : W \to X \), and an equivalence relation \( R \) on \( X \) by the restriction \( R|_W \) we mean the relation where \( w_1 \) is equivalent to \( w_2 \) iff their images are \( R \)-equivalent. If \( R \) is algebraic this is given by the reduction of \((h \times h)^{-1}(R)\). For a general discussion of equivalence relations see [Keel-Mori97].

The following is immediate:

\[ 0.13 \text{ Definition-Lemma.} \text{ If } f : X \to Y \text{ is a map and } R \text{ an equivalence relation on } X \text{, then } R \text{ dominates } f \text{ iff } R \text{ is the restriction of an equivalence relation on } Y. \text{ If } f \text{ is surjective this equivalence relation is unique, we denote it by } R_Y. \text{ In this case } R_Y = (f \times f)(R), \text{ and } R = R_Y|_X. \]

Note that for a map \( h : W \to X \), the restriction of \( L_X \)-equivalence to \( W \) is refined by, but in general coarser than, \( L_W \)-equivalence. It’s convenient to have a variant that is preserved by restriction:

\[ 0.14 \text{ Definitions.} \text{ We say that a closed subset } R \subset X \times X \text{ is related to a nef line bundle } L \text{ iff } R \text{ contains any pair of } L \text{-equivalent } k \text{ points, and } p_2^*(L) \text{ is numerically trivial on the fibres of the first projection } p_1 : R \to X. \]

(Note in (0.14) we do not require that \( R \) is an equivalence relation. This (slight) additional flexibility will be useful in the proof of (3.6).)

\[ 0.15 \text{ Lemma.} \text{ The following hold for a closed algebraic equivalence relation } R \subset X \times X, \text{ a nef line bundle } L \text{ on } X \text{ and a map } h : W \to X, \text{ with } X \text{ and } W \text{ proper (over } k). \]

(1) If \( L \)-equivalence is algebraic, then \( L \) is numerically trivial on \( L \)-equivalence classes.

(2) If \( L_X \) is semi-ample and \( R \) is refined by \( L \)-equivalence, then \( R \) dominates \( g_X \), and is the restriction along the associated map of an equivalence relation \( R_{Z_X} \). If furthermore \( R \) is related to \( L \) then either projection \( R_{Z_X} \to Z_X \) is finite.

(3) If \( R \) is related to \( L \) then \( R|_W \) is related to \( L|_W \).

Proof. (1) follows from [BCEKPRSW,2.4]. (3) is obvious. For (2) note that \( L \)-equivalence is defined by (the reduction of) \( X \times_{Z_X} X \), so obviously \( R \) dominates \( Z_X \) and so is a restriction by (0.13). Now suppose \( R \) is related to \( L \), and let \( G \) be a connected component of an \( R \)-equivalence class. Then \( L|_G \) is numerically trivial, and so \( g_X(G) \) is a point. Thus the images of \( R \)-equivalence classes are finite, and the result follows, since these are the \( R_{Z_X} \) equivalence classes (and so the fibres of \( R_{Z_X} \) under either projection). \( \square \)

\[ 0.16 \text{ Lemma.} \text{ Let } R \to X \text{ be an algebraic equivalence relation, with both projections } R \to X \text{ finite, and } X \text{ reduced. Let } Z \subset R \text{ be a closed subset such that both projections } Z \to X \text{ are generically étale. After replacing } X \text{ by a dense } R \text{-invariant open set (and } R \text{ and } Z \text{ by their restrictions), the} \]
equivalence relation generated by closed points of $Z$ is algebraic, given by a closed subset of $R$, étale over $X$.

Proof. Note, since the projections are finite, that $R$-invariant open sets form a base for the Zariski topology on $X$, and furthermore finiteness of $R$ is preserved by restriction to such sets. Replacing $Z$ by its union with the diagonal, and its image under the involution $i : R \to R$ (that switches the factors), we may assume $Z$ contains the diagonal, and is symmetric. We inductively define an increasing sequence of closed subsets $Z_i \subset R$, that are symmetric, contain the diagonal, and are generically étale over $X$. Let $Z_1 = Z$ and let $Z_{i+1}$ be the reduction of

$$Z_i \times_{(p_2,p_1)} Z_i \subset Z_i \times Z_i.$$

Note $Z_{i+1}$ consists of 4-tuples $(a,b,b,c) \in X^4$ with $(a,b),(b,c) \in Z_i$. Let $Z_{i+1} \subset X \times X$ be the reduction of the image under the (finite) map sending $(a,b,b,c)$ to $(a,c)$. It’s clear that $Z_i \subset Z_{i+1}$, and $Z_{i+1}$ is symmetric, contains the diagonal, and is generically étale over $X$. Over generic points of $X$ the degree of $Z_i \to X$ is bounded, thus, after shrinking $X$, the process terminates, in an étale equivalence relation, which is by construction the relation generated by $Z$. □

The main technical observation of this note is the following:

0.17 Theorem: Polarized pushouts over finite fields. If the base field $k$ is the algebraic closure of a finite field, then given $L_X, f_i$ as in (0.12), with $L_X$ semi-ample, a polarized pushout exists iff there is an algebraic equivalence relation $R$ related to $L$ and dominating $f_1, f_2$. Furthermore, in this case there exists a pushout with $X \to P$ dominated by $R$.

To understand the proof of (0.17) the reader may wish to look first at the proof of the special case (1.9) where the role of the assumptions, especially in light of example (5.2), becomes clear. (0.17) is reduced to (1.9) by a series of formal manipulations of the sort used in [Keel99, §2], and [Kollár97].

§1 Proof of (0.17).

1.1 Lemma. To prove (0.17) in general, it is enough to prove it in the case when $L_X$ and $L_{X_i}$ are all ample.

Proof. Suppose (0.17) holds in the ample case. Consider the general case. Note $g_X \circ f_i$ is related to $L_X$ and so factors through $g_X$. Thus we have a pair of finite maps $f'_i : Z_X \to Z_{X_i}$ with $f'_i \circ g_X = g_X \circ f_i$. By (0.15) $R$ is the restriction of some $R_{Z_X}$ on $Z_X$, easily seen to dominate $f'_i$. $L_X, L_{X_i}$ are pullbacks of ample line bundles $L_{Z_X}, L_{Z_{X_i}}$. The assumptions of (0.17) are
thus satisfied for the induced diagram
\[
\begin{array}{ccc}
Z_X & \xrightarrow{f'_1} & Z_{X_1} \\
\downarrow{f'_2} & & \downarrow{g'_1} \\
Z_{X_2} & & 
\end{array}
\]

So by the ample case we have a polarized pushout
\[
\begin{array}{ccc}
Z_X & \xrightarrow{f'_1} & Z_{X_1} \\
\downarrow{f'_2} & & \downarrow{g'_1} \\
Z_{X_2} & \xrightarrow{g'_2} & P 
\end{array}
\]

with the line bundles pulled back from ample \(L_P\) on \(P\), and \(R_{Z_X}\) dominating \(P\). As \(R\) is restricted from \(Z_X\) it dominates \(P\) as well, so \(g'_1 \circ g_X\), give the required polarized pushout. \(\square\)

1.3 Lemma. Let \(f : Y \rightarrow X\) be a finite map, with \(Y\) quasi-projective (and \(X\) an algebraic space). Then the natural map \(H^1(f_*(\mathcal{O}_Y^*)) \rightarrow \text{Pic}(Y)\) is an isomorphism (where \(H^1\) means étale cohomology).

Proof. By the Leray spectral sequence it’s enough to show for each \(x \in X\) the restriction to the stalk
\[
\text{Pic}(Y) \rightarrow R^1f_*(\mathcal{O}_Y^*)_x
\]
is trivial. For this we may replace \(X\) by an étale neighborhood of \(x\) and so may assume \(X\), and hence \(Y\), are affine. But now given any line bundle \(A\) we can find a section \(\sigma\) non-vanishing at any point of (the finite set) \(f^{-1}(x)\). Let \(Z\) be the zero locus of \(\sigma\).

\[
x \in U := f(Z)^c \subset X
\]

and \(A|_{f^{-1}(U)}\) is trivial. \(\square\)

1.4 Lemma. If \(f_1\) is a closed embedding, \(f_2\) is finite and the \(L_{X_i}\) are ample then a polarized pushout \(P\) exists, with \(g_2\) a closed embedding and \(g_1\) finite. The pushout diagram is a pullback (i.e. \(X = g_1^{-1}(X_2)\)), and outside of \(X\), \(g_1\) is an isomorphism onto its image. Any equivalence relation dominating \(f_2\) will dominate \(P\).

Proof. The final claim follows from the claim that precedes it. Replacing \(X_2\) by the scheme-theoretic image of \(f_2\), and running the argument twice (the
second time with each \( f_i \) a closed embedding) we can assume \( f_2 \) is surjective. By [Artin70,6.1] a universal pushout exists with the required properties in the category of algebraic spaces. Thus it’s enough to show that some positive tensor power of \( L_{X_1} \) descends to \( P \). Consider the complex of étale sheaves of Abelian groups

\[
1 \rightarrow O_P^* \rightarrow g_2^*(O_{X_2}^*) \times g_1^*(O_{X_1}^*) \rightarrow (g_1 \circ f_1)_*(O_X^*) \rightarrow 1.
\]

By (1.3) and the long exact cohomology sequence it is enough to show this is exact. At the left and middle this follows from the universal property of \( P \) (and its restriction to étale open sets) applied to maps to \( \mathbb{A}^1 \), see e.g. [Kollár97,8.1.3]. For exactness on the right, it’s enough to show

\[
g_1^*(O_{X_1}^*) \rightarrow (g_1 \circ f_1)_*(O_X)
\]
is surjective. For this we can assume \( P \), and hence all the spaces, are affine. A unit on \( X \) lifts to a function on \( X_1 \), invertible in a neighborhood of \( X \). Since \( X = g_1^{-1}(X_2) \), this neighborhood is the inverse image of a neighborhood of \( X_2 \) in \( P \). □

1.5 Lemma. Assume positive characteristic. If \( f_1 \) is a finite universal homeomorphism, and the line bundles \( L_X, L_{X_1} \) are all ample, then a polarized pushout exists, with \( g_2 \) a finite universal homeomorphism. Any equivalence relation dominating \( f_2 \) will dominate \( P \).

Proof. By [Kollár97,8.4] the pushout exists, and line bundles descend by [Keel99,1.4]. The condition that an equivalence relation dominate \( P \) is set-theoretical, so since \( P \) is homeomorphic to \( x_2 \), the final claim is clear. □

1.6 Lemma. Assume positive characteristic. Suppose there is a finite universal homeomorphism \( h : X' \rightarrow X \) such that a polarized pushout for \( h \circ f_i, L_{X_1} \big|_{X'} \) exists. Then a polarized pushout for \( f_i, L_{X_1} \) exists.

Proof. Immediate from [Keel99,2.1] and [Keel99,1.4]. □

1.7 Lemma. Assume \( k \) is the algebraic closure of a finite field. Let \( f : Y \rightarrow X \) be a proper surjection between projective schemes. The kernel of \( f^* : \text{Pic}(X) \rightarrow \text{Pic}(Y) \) is torsion.

Proof. Any line bundle in the kernel will be numerically trivial and thus torsion, see e.g. [Keel99,2.16]. □

1.8 Lemma. Assume \( k \) is the algebraic closure of a finite field. Suppose there is a proper surjection \( h : X' \rightarrow X \) such that a polarized pushout for \( h \circ f_i, L_{X'} \big|_{X'} \) exists. Then a polarized pushout for \( f_i, L_{X_1} \) exists. Furthermore if there is an equivalence relation \( R \) on \( X \) dominating \( f_i \), and there is a
pushout for \( h \circ f_i \) dominated by \( R|_X' \), then there is a pushout for \( f_i, L_{X_i} \), dominated by \( R \).

**Proof.** Assume we have a polarized pushout \( g_1,g_2 \) for \( h \circ f_i \). To obtain the desired polarized pushout we need only show that \( g_1 \circ f_1 = g_2 \circ f_2 \), and \( L_P|_X = L_X \). By (1.7) and [Keel99,2.1] these are achieved after replacing line bundles by powers, and \( P \) by a finite universal homeomorphism \( P \to P' \).

Now the final remarks with respect to \( R \) follow (as we have only modified the spaces by universal homeomorphism and the conditions on the equivalence relations are set-theoretical). □

**1.9 Lemma.** (0.17) holds over any field assuming \( X, X_i \) are normal, \( L_{X_i} \) are ample, and \( f_i \) are finite surjections.

**Proof.** We are free to replace \( X \) by a finite cover \( h : X' \to X \): Since \( X \) is normal the kernel of the pullback on Picard groups is torsion, and since \( X \) is reduced

\[
\text{Hom}(X, T) \xrightarrow{h^*} \text{Hom}(X', T)
\]

is injective for any \( T \), so we can argue as in the proof of (1.8).

Note since \( R \) is related to an ample line bundle, either projection \( R \to X \) is finite.

By (1.5), factoring the field extensions into separable and purely inseparable parts, we may assume the field extensions \( K(X_i) \subset K(X) \) are separable.

Let \( K \) be the intersection of function fields \( K(X_1) \cap K(X_2) \subset K(X) \). Suppose for the moment that \( K \subset K(X) \) is finite. Let \( X' \) the integral closure of \( X \) in the Galois closure of \( K(X) \) over \( K \). We may replace \( X \) by \( X' \), thus we may assume \( K(X_i) \subset K(X) \) are Galois extensions, and so \( f_i \) is the quotient by the Galois group, \( G_i \). Let \( H \) be the subgroup of \( \text{Aut}(X) \) generated by \( G_1 \) and \( G_2 \). \( H \) is a subgroup of \( \text{Aut}(K(X), K) \) and so finite.

We can take the geometric quotient \( P = X/H \). Some power of \( L_X \) will descend to \( P \). \( R \) will contain the equivalence relation generated by \( G_1, G_2 \) which is exactly (the reduction of) \( X \times_X X \).

Thus it is enough to show that \( K \subset K(X) \) is a finite extension. Since \( R \) dominates \( f_i \) any \( R \)-invariant open set is the inverse image of its image under \( f_i \), so to check the finiteness of \( K \subset K(X) \) we can replace \( X \) by any non-empty \( R \)-invariant set, and such sets form a base for the topology, e.g. as either projection \( R \to X \) is finite. Note also that the finiteness of \( R \to X \) is preserved by restriction to \( R \)-invariant open sets. Let \( Z \subset R \) be the reduction of the union of \( X \times_{X_1} X \) and \( X \times_{X_2} X \). Either projection \( Z \to X \) is generically étale, since the extensions \( K(X_i) \subset K(X) \) are separable, so by (0.16) we may assume there is an algebraic equivalence relation \( R' \subset R \) dominating \( f_1, f_2 \), with \( R' \to X \) finite and étale. A geometric quotient by \( R' \) exists (in fact it’s just the algebraic space defined by \( R' \)), see e.g.
[KeelMori97,4.8,5.1], a finite map $q : X \to X/R'$. Since $R'$ dominates $f_i$, $q$ factors through $f_i$ and so $K(X/R') \subseteq K$. Since $K(X/R') \subseteq K(X)$ is a finite extension, so is $K \subseteq K(X)$. \hfill $\Box$

Proof of (0.17). We prove the result by induction on the dimension of $X$. By (1.1) we may assume all of $L_X$, $L_X^i$ are ample. By (1.8) we can assume $X$ is normal. By (1.4) applied to scheme-theoretic images, we may assume the $f_i$ are surjective and $X^i$ reduced. By (1.9) we reduce to the case when $f_1$ is the normalization: Indeed by (1.9) we have a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\tilde{f}_1} & \tilde{X}_1 & \xrightarrow{p_1} & X_1 \\
\downarrow \tilde{f}_2 & & \downarrow \tilde{g}_1 & & \\
\tilde{X}_2 & \xrightarrow{\tilde{g}_2} & P & & \\
\downarrow p_2 & & & & \\
X_2 & & & & \\
\end{array}
\]

with $p_i$ the normalization, where the square is a polarized pushout relative to $R$. By (0.15), $R = R_{\tilde{X}_1} \mid_X$ for a finite equivalence relation on $\tilde{X}_1$ dominating $p_1, \tilde{g}_1$. So by (0.15), it’s enough to construct a polarized pushout $h : P \to P'$ of $p_1, \tilde{g}_1$, dominated by $R_{\tilde{X}_1}$, and then a polarized pushout of $p_2, h \circ \tilde{g}_2$, dominated by $R_{\tilde{X}_2}$.

We write $p = \tilde{f}_1$. Recall that the conductor is the sheaf of functions on $X$ which multiply $\mathcal{O}_X$ into $\mathcal{O}_{X_1}$. This is at once an ideal of $\mathcal{O}_X$ and of $\mathcal{O}_{X_1}$. Let $C \subset X$, $D \subset X_1$ be the associated subspaces. The definitions imply that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{i} & X \\
p \downarrow & & p \downarrow \\
D & \xrightarrow{i} & X_1 \\
\end{array}
\]  

(1.10)

(where the maps $i$ are the closed embeddings defined by the conductor ideals) is a fibre diagram, and a universal pushout, see e.g. [Reid94,2.1.]. By induction on dimension we have a polarized pushout diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f_2 \circ i} & f_2(C) \\
p \downarrow & & \downarrow \\
D & \rightarrow & P' \\
\end{array}
\]
for $L_{X_2} , L_{X_1}|_D$, dominated by $R|_C$. Further by (1.4) we have a polarized pushout

\[
\begin{array}{ccc}
f_2(C) & \xrightarrow{i} & X_2 \\
p & & p \\
P' & \xrightarrow{i} & P
\end{array}
\]

with $i$ closed embeddings, $p$ finite surjections, which is also a pullback diagram, with $X_2 \to P$ an isomorphism outside of $f_2(C)$. In particular we have a commutative diagram of finite maps

\[
\begin{array}{ccc}
C & \xrightarrow{f_2 \circ i} & X_2 \\
p & & p \\
D & \xrightarrow{} & P
\end{array}
\]

and compatible ample $L_P$. Since the conductor diagram is a universal pushout, the composition

$X \to X_2 \to P$

factors through $p: X \to X_1$. $L_{X_1}$ descends to $L_P$ (after taking powers) by (1.7). Finally we check this pushout is dominated by $R$. Suppose $x_1, x_2 \in X$ are distinct points that have the same image in $P$. We show they are $R$-equivalent. We may assume $f_2(x_1), f_2(x_2)$ are distinct (or the $x_i$ are $R$-equivalent by assumption). Since $p: X_2 \to P$ is an isomorphism outside of $f_2(C)$, and (*) is a pullback, we have $f_2(x_i) \in f_2(C)$. So replacing $x_i$ by $X \times_{X_2} X$, and hence $R$, equivalent points, we can assume $x_i \in C$ and so $R|_C$ equivalent. \(\square\)

\section*{2 Proof of (0.1).} The following is clear from the definitions:

\begin{theorem}
\textbf{Let $W \to X$ be a map between proper algebraic spaces. Let $L$ be a nef line bundle on $X$. $L_W$ equivalence is a refinement of the restriction of $L_X$-equivalence.}
\end{theorem}

\begin{lemma}
\textbf{Let $p: \tilde{X} \to X$ be a surjection between algebraic spaces proper over the algebraic closure of a finite field. Let $L$ be a line bundle on $X$. Then $L$ is semi-ample iff the following two conditions hold:}

(1) $p^*(L)$ is semi-ample, and

(2) there is a closed algebraic equivalence relation $R \subset X \times X$ related to $L$ (i.e. $R$ is refined by $L$-equivalence, and $L$ is is numerically trivial on $R$-equivalence classes).}
\end{lemma}
Proof. If $L$ is semi-ample then clearly the conditions must hold – for $R$ one takes $X \times_{\tilde{X}} X$. We prove the other implication by induction on the dimension of $X$. Since we may replace $\tilde{X}$ by anything that surjects properly onto it, we can assume $\tilde{X}$ is normal. By [Keel99.1.4] (passing to the reduction) we may assume $X$ is reduced. (0.1) holds for $X$ normal, over any field, and without the boundedness assumption, see e.g. the proof of [Keel99.2.10]. (Alternatively, in positive characteristic, one can apply [Keel99.2.10] to the case where the base space is normal and $D$ and $C$ are empty.) So the restriction of $L$ to the normalisation of $X$ is semi-ample, and thus we may assume from the start that $X$ is reduced and $\tilde{X}$ is the normalisation.

We let $C, D$ be the conductors as in (1.10) (but here $p: \tilde{X} \to X$ replaces $p: X \to X_1$). By (2.1) $R|_D$ is refined by $L_D$-equivalence, and $L_D$ is clearly numerically trivial on $R|_D$ equivalence classes. Thus by assumption and induction, $L_{\tilde{X}}, L_C, L_D$ are all semi-ample. Any two points in the same fibre of the composition

$$g_{\tilde{X}} \circ i: C \to \tilde{X} \to Z_{\tilde{X}}$$

map to $L_X$-equivalent points of $X$, and thus are $R|_C$ equivalent, i.e. $R|_C$ dominates $Z_{\tilde{X}}$. $R|_C$ obviously dominates $C \to D$ (since $R|_C$ is restricted from $D$). Thus by (0.17) there is a polarized pushout

$$
\begin{array}{ccc}
C & \longrightarrow & Z_{\tilde{X}} \\
\downarrow & & \downarrow q \\
D & \longrightarrow & P
\end{array}
$$

with $L_{Z_{\tilde{X}}}$ (and thus $L_{\tilde{X}}$) pulled back from an ample line bundle $L_P$. Since the conductor diagram is a universal pushout, $q \circ g_{\tilde{X}}$ factors through $X$. By (1.7), as a $\mathbb{Q}$-line bundle $L_X$ is the pullback of $L_P$, thus semi-ample. □

2.3 Lemma. Let $p: W \to X$ be a surjection between proper algebraic spaces. If $p^*(L)$-equivalence is algebraic and $L$-equivalence is bounded, then $L$-equivalence is algebraic.

Proof. Let $m$ be the bound for $X$. Let $S' \subset W \times W$ define $L_W$ equivalence, and let $S \subset X \times X$ be its image. Clearly the $k$-points of $S$ are contained in the union of $L$-equivalent pairs. Note that if $x_1, x_2 \in C$ and $C \cdot L = 0$ for irreducible $C$, then $(x_1, x_2) \in S$, since we can lift $C$ to an $L_W$-trivial irreducible curve. So by assumption, $x, y$ are $L$-equivalent if we can find $x = x_1, x_2, \ldots, x_m = y$ with $(x_i, x_{i+1}) \in S$. It’s clear this defines a closed subset of $X \times X$. □

Proof of (0.1). Immediate from (2.2 - 2.3). □
2.4 Lemma. Let $L$ be a nef line bundle on a complete algebraic space $X$. If there is a closed subset $W \subset X \times X$ related to $L$, then $L$-equivalence is bounded.

Proof. We can take for the bound $m$ the maximum number of irreducible components in any fibre of the first projection $W \to X$. \qed

§3 Semi-ampleness on $\overline{M}_{g,n}$.

Assumptions Global to §3 Throughout this section $L$ is a nef line bundle on $\overline{M}_{g,n}$. $k$ is a fixed finite field. All spaces are assumed to be of finite type over $k$ and all morphisms are $k$-linear.

3.0 Topological Stratification and Product Decompositions: $\overline{M}_{g,n}$ has a tautological stratification by topological type. Here we recall the main points, see [Keel99,pg. 274] for more precise details. A codim $i$-stratum is a connected component of the locus corresponding to stable pointed curves with precisely $i$-singular points. I will abuse language and refer to the closure of a stratum as a closed stratum. There is one stratum for each topological type of stable pointed curve, or equivalently, for each isomorphism class of decorated dual curve – where by decorated I mean the vertices are labeled by genera, and labeled points are indicated by labeled edges, incident to a unique vertex (of course corresponding to the component of the curve containing the labeled point). For a stable pointed curve $[E] \in \overline{M}_{g,n}$, we write $X_E$ for the unique closed stratum containing $[E]$ in its interior. Of course this depends only on the topological type of $E$. We write $X^o_E$ for the (open) stratum.

The normalisation of each closed stratum is a natural projective moduli space. More convenient for our purposes is a certain branched cover which we call the product decomposition of the stratum: Let $E$ be a stable pointed curve, and $X = X_E$. Let $\Gamma$ be the dual graph of $E$. Let $P_X$ be the product of $\overline{M}_{g_i,N_i}$, one for each vertex, with $g_i$ the genus of the vertex, and $N_i$ the (ordered) set of incident edges. In the notation of [Keel99,pg. 274],

$$P_X = \overline{M}_E^{X \cup Y}$$

$$N_i = X_i \cup Y_i.$$ 

Note formation of $P_X$ requires ordering the vertices of $\Gamma$, and then for each vertex the collection of unlabeled incident edges. In this sense it is not unique, though any two choices are isomorphic. $H = \text{Aut}(\Gamma)$ (meaning automorphisms of the decorated graph) acts naturally on $P_X$ and the quotient is $\tilde{X}$, the normalisation of $X$. See e.g. [HainLooijenga97,§4], or [Graber-Pandharipande, §A]. The quotient of the interior of $P_X$ is $X^o$ (where the interior of $P_X$ means the product of the interiors of its factors).

We note that $\overline{M}_{g,n}$, all of its $\mathbb{Q}$-line bundles, the stratification by topological type, the closed strata, their normalisations, and product decompositions
of all strata, are all defined over $k$ (indeed over $\mathbb{Z}$), see [DeligneMumford69], [Moriwaki01].

In various notations we replace $X_E$ by $E$, e.g. we will sometimes write $P_E$ for the product decomposition $P_{X_E}$.

**3.0.1:** Suppose that $F$ is a degeneration of the stable pointed curve $E$, so $X_F \subset X_E$. Observe that $P_F \to X_F \subset X_E$ factors (non canonically) through $P_E \to X_E$: Let $E_i$ be the irreducible components of $E$. $F$ can be written as a union of connected (not necessarily irreducible) curves $F_i$, with $F_i$ a degeneration of $E_i$. Let $\tilde{E}_i$ be the normalisation of $E_i$. There is a corresponding connected partial normalisation $\tilde{F}_i$ of $F_i$, $\tilde{F}_i$ a degeneration of $\tilde{E}_i$. Each $E_i$ corresponds to a factor, $M_{g_i,N_i}$, of $P_E$. $\tilde{F}_i$ gives a closed stratum $X_{\tilde{F}_i}$ of $M_{g_i,N_i}$. The composition

$$\times_i P_{\tilde{F}_i} \to \times_i \overline{M}_{g_i,N_i} = P_E \to X_E$$

has image $X_F$. $\times_i P_{\tilde{F}_i} \to X_F$ is the product decomposition $P_F$.

The goal here is to give a necessary and sufficient condition for $L$ to be semi-ample. We will use (0.1), and (2.4), and so look for an algebraic equivalence relation $R$ related to $L$ (see (0.14) for a reminder of what it means for a closed subset to be related to a nef line bundle). As the argument is a bit technical, let me begin by sketching the general philosophy:

If one can guess the set theoretic fibres of the associated map, then one can write down (a candidate) $R$ directly (note we only need $R$ related to $L$-equivalence, so its equivalence classes need not be precisely the fibres of the associated map). For example, for $\lambda$, the associated map is to the Satake compactification of the moduli of Abelian varieties, so we say $[C], [D]$ are $R$-equivalent iff the union of the non-rational components of their respective normalisations are isomorphic. For $\psi$: For a pointed curve $C$ let $C_i$ be the irreducible component containing the $i^{th}$ point, and let $\tilde{C}_i, S$ be its normalisation, together with the (unordered) set $S$ of all its distinguished points, i.e. all the labeled points lying on $C_i$, together with the inverse images of all the singular points of $C$. Then we say $[C], [D]$ are $R$-equivalent iff $(\tilde{C}_i, S)$ and $(\tilde{D}_i, S)$ are isomorphic. (In fact (a slight variant) of this equivalence relation was introduced purely topologically (in characteristic 0) by Kontsevich. Looijenga observed that it was algebraic, and in fact defined $\psi$ equivalence, see [Looijenga95].) In these examples one can argue $R$ is algebraic by using the associated moduli spaces. E.g. for $L = \psi$, let $Q$ be the disjoint union of $\overline{M}_{g,n}/S_n$, one copy for each $g, n$, and $X^o$ the disjoint union of the (open) strata. We have a natural map $X^o \to Q$, sending $[C]$ to $[\tilde{C}_i, S]$ (notation as above). Since $Q$ is the moduli space of stable unordered pointed curves, two points of $\overline{M}_{g,n}$ are (by definition) $R$-equivalent iff they have the same images in $Q$, thus $R$ is the set of closed points of $X^o \times_Q X^o \subset \overline{M}_{g,n} \times \overline{M}_{g,n}$. (Of
course it remains to show that $R$ is closed and related to $\psi_i$, this requires some work, see the proof of (3.6).

For general $L$ we do something similar. We will know by induction on dimension that $L$ is semi-ample on the normalisation of any proper closed stratum. Then we can form a space analogous to $Q$ above out of the images of the associated maps. To do so we make use of a simple construction:

By a polarised variety we mean a pair $(Z, L_Z)$ of a projective $Z$ and an ample $Q$-line bundle $L_Z$. Note since $k$ is finite, a $Q$-line bundle is the same as its Chern class in the Neron-Severi group with rational coefficients.

3.1 Lemma. Let $(Z, L_Z)$ be a polarized variety (projective over $k$). The discrete group $G = \text{Aut}(Z, L_Z)(k)$ is finite. Let $Q_Z = Z/G$. $L_Z$ descends (as a $Q$-line bundle) to an ample $L_{Q_Z}$. Let $f : (Z, L_Z) \to (W, L_W)$ be an isomorphism of polarised varieties. Then there is an induced isomorphism of polarised varieties $Q_Z \to Q_W$, which is independent of the choice of $f$.

Proof. $\text{Aut}(Z, L_Z)$ is of finite type (for any polarized variety), so its set of rational points over the finite field $k$ is finite. The rest is easy. □

3.2 Definition. Let $X$ be a closed stratum such that $L_{PX}$ is semi-ample. We define

$$(Q_X, L_{Q_X}) := (Q_{Z_{PX}}, L_{Q_{ZPX}})$$

where the right hand side uses the notation of (3.1).

3.3 Lemma. Notation as in (3.2). The composition

$$q_{PX} : PX \to Z_{PX} \to Q_X$$

factors through $PX \to \tilde{X}$. In particular there is an induced map $q_X : X^o \to Q_X$. $q$ and $(Q_X, L_{Q_X})$ depend only on $X$ and $L$.

Proof. Let $X = X_E$, for a topological type of stable pointed curve $E$. Let $H$ be, as in (3.0), the automorphisms of the dual graph of $E$. Since $L_{PX}$ is pulled back from $X$ (and so from $\tilde{X}$) it has a natural $H$ linearization. Thus $H$ acts on $(Z_{PX}, L_{Z_{PX}})$. The quotient is $Z_{\tilde{X}}$. Indeed:

$$Z_{PX}/H = \text{Proj}(R(P_X, L_{PX})^H)$$

$$= \text{Proj}(R(\tilde{X}, L_{\tilde{X}}))$$

$$= Z_{\tilde{X}}$$

(where $R$ indicates the graded ring of sections). Thus $Z_{PX} \to Q_X$ factors through $Z_{\tilde{X}}$, and thus $PX \to Q_X$ factors through $\tilde{X}$, inducing $q : \tilde{X} \to Q_X$. The choices here were the orderings (of factors, and marked points) in the
construction of $P_X$. Different choices yield isomorphic polarized varieties and so, by (3.1), the same $Q_X$. □

Now in the proof of (3.6) we will define $[E]$ and $[F]$ to be $R$-equivalent (assuming $L_{PE}$ and $L_{PF}$ are semi-ample) iff $Z_{PE}$ and $Z_{PF}$ are isomorphic polarized varieties, and $q_E([E])$ and $q_F([F])$, ($q_E = q_X$ of (3.3)) are identified under the canonical identification $Q_E = Q_F$ of (3.1-2). To show $R$ is closed and related to $L$ we will need some further conditions, which we formalize as follows:

3.4 Compatibility Condition. Let $E$ be the topological type of a stable pointed curve. We say (3.4) holds for $E$ if there exists a contraction $p : P_E \to W$ with the following properties:

1. $L_{PE}$ is pulled back from $W$.
2. $p$ contracts any irreducible $L_{PE}$-exceptional curve which meets the interior of $P_E$.
3. Assume $L_{PE}$ is semi-ample. Let $[F] \in X_E$, and let $[\tilde{F}], [\tilde{E}] \in P_E$ be points with image $[F], [E] \in X_E$. If $p([\tilde{F}]) = p([\tilde{E}])$ then the polarized varieties $Z_{PE}$ and $Z_{PF}$ are isomorphic and $q_F([F]) = q_E([E])$ in $Q_F = Q_E$, under the identification (3.1) and the maps, $q_E, q_F$, of (3.3).

Remark: Note in (3.4.3) that since $[F] \in P_E$, there is, by (3.0.1), a factorisation $P_E \to P_F$, and thus $L_{PE}$ is semi-ample as well, so $q_F : X^0_F \to Q_F$ is defined.

3.5 Lemma. Let $[E] \in \overline{M}_{g,n}$ be a stable pointed curve. Suppose for each $h : \overline{M}_{r,N} \to \overline{M}_{g,n}$ coming from a factor of the product decomposition $P_E \to X_E$, either $h^*(L)$ is numerically trivial, or there is a subset $S \subset N$ such that

$$h^*(L) = \pi^*_S(L_S)$$

for a nef line bundle $L_S$ on $\overline{M}_{r,S}$ with $E(L_S) \subset \partial \overline{M}_{r,S}$ (here we require that $S$ is such that $\overline{M}_{r,S}$ is defined, i.e. if $r = 1$, $S$ is non-empty, and if $r = 0$, $S$ has at least 3 elements).

Then (3.4) holds for $E$.

Proof. If $L_{PE}$ is numerically trivial we take $W$ to be a point and there is nothing to check (the spaces $Q_F$ and $Q_E$ in (3.4.3) will both be points). So we assume otherwise.

Let $W$ be the product of the $\overline{M}_{r,S}$, over those factors of the product decomposition where $p^*(L)$ is not numerically trivial. Let $p : P_E \to W$ be the product of (projection to the factor composed with) the maps $\pi_S$. By [GKM00,1.1], $L_{PE}$ is a tensor product of (the pullbacks of) the $h^*(L)$ (as $h$ varies over the factors of $P_E$). Thus (3.4.1) holds.
Let $C$ be an irreducible $L_{P_E}$ exceptional curve that meets the interior of $P_E$. Note

$$\pi_S^{-1}(\partial \overline{M}_{r,S}) \subset \partial \overline{M}_{r,N}.$$ 

Thus $\pi(C)$ meets the interior of $\overline{M}_{r,S}$. By the projection formula $c_1(L_S) \cdot \pi_S(C) = 0$ for all $S$. So by the assumption $E(S_L) \subset \partial \overline{M}_{r,S}$, $\pi_S(C)$ is a point for all $S$, whence (3.4.2).

Now suppose $L_{P_E}$, or equivalently, $L_W$, is semi-ample, and $\pi_S(\{\tilde{F}\}) = \pi_S(\{\tilde{E}\})$, for $F$ a degeneration of $E$. Choose a factorisation $P_F \to P_E$ as in (3.0.1). Assume for the moment that the composition $P_F \to W$ is a contraction:

Then since $L_{P_E}$ and $L_{P_E}$ are pulled back from $L_W$ (the tensor product of the $L_S$) along contractions, the images of the associated maps, $Z_{P_F}$, $Z_{P_E}$, and $Z_W$, are isomorphic polarised varieties. Thus $Q_F$ and $Q_E$ are canonically identified by (3.1-2). By definition (3.3) the images $q_F(\{E\})$ and $q_F(\{F\})$ in $Q_F = Q_E$ are $q_{P_E}(\{\tilde{E}\})$ and $q_{P_E}(\{\tilde{F}\})$. $q_{P_E}$ and $q_{P_E}$ factor through $p : P_E \to W$, and $p(\{\tilde{E}\}) = p(\{\tilde{F}\})$ by assumption, whence (3.4.3).

So it’s enough to show $P_F \to W$ is a contraction. We first reduce to the case when $E$ is irreducible. We follow the notation of (3.0.1), so $P_E = \times_i \overline{M}_{g_i,N_i}$. For factors with $L|_{\overline{M}_{g_i,N_i}}$ non-trivial, let $S_i \subset N_i$ be the corresponding subset (called $S$ in the statement of (3.5)). Let $W_i = \overline{M}_{g_i,S_i}$ and let $p_i : \tilde{P}_{F_i} \to W_i$ be the composition of

$$P_{\tilde{F}_i} \to X_{\tilde{F}_i} \subset \overline{M}_{g_i,N_i},$$

with

$$\pi_{S_i} : \overline{M}_{g_i,N_i} \to \overline{M}_{g_i,S_i}.$$ 

$p_F = \times_i p_{\tilde{F}_i}$ and $P_F \to W$ is the product of the $p_i$. So it’s enough to show each $p_i$ is a contraction. But note if from the start we replace $\overline{M}_{g,n}$ by $\overline{M}_{g_i,N_i}$ and $E,F$ by $\tilde{E}_i,\tilde{F}_i$, then our construction replaces $P_F \to W$ by $p_i$. So we may assume from the start that $E$ is irreducible.

Now $P_E$ has a single factor $\overline{M}_{r,N}$, $p = \pi_S$ and $W = \overline{M}_{r,S}$.

$\pi_S$ sends $[C,N]$ to the stabilisation of $[C,S]$. Note since $E$ is irreducible, $[\tilde{E},S]$ is stable, and by assumption, is the stabilisation of $[\tilde{F},S]$. So by the definition of stabilisation there is a unique irreducible component $\tilde{F}_S$ of $\tilde{F}$ such that: each connected component of $\tilde{F} \setminus \tilde{F}_S$ meets $S$ in at most one point, $\tilde{F}_S$ is smooth, $\tilde{F} \setminus \tilde{F}_S$ is a (possibly disconnected) tree of smooth rational curves, and $\tilde{F}_S$ is the underlying curve of $\tilde{E}$. It follows that the closed stratum $X_E \subset \overline{M}_{r,N}$ is normal, equal to its product decomposition, $F_F$, and is a product of $\overline{M}_{r,S,T}$ with a product of various $\overline{M}_{0,N_i}$, where $T$ are the labels in $N \setminus S$ that lie on $\tilde{F}_S$, and under this identification $X_{\tilde{F}} \to \overline{M}_{r,S}$ is a
contraction, the composition of projection onto the $\overline{M}_{r,S\cup T}$ factor with the natural contraction $\overline{M}_{r,S\cup T} \to \overline{M}_{r,S}$. □

The next proposition is a restatement of (0.3.1) and (0.4):

3.6 Proposition. Let $L$ be a nef line bundle on $\overline{M}_{g,n}$. Assume the characteristic $p > 2$. $L$ is semi-ample provided one of the following holds:

1. $L$-equivalence is bounded and $r^*(L)$ is semi-ample for

$$r : \overline{M}_{0,2g+n} \to \overline{M}_{g,n}$$

the natural map given by the rational locus, or

2. For each $h : \overline{M}_{r,N} \to \overline{M}_{g,n}$ a component of a product decomposition of a closed stratum, with $r \leq 1$, either $h^*(L)$ is trivial, or $h^*(L) = \pi_S^*(L_S)$ for some subset $S \subset N$ and some nef line bundle $L_S$ with $E(L_S) \subset \partial \overline{M}_{r,S}$.

Remark: In (3.6.2) we require, as in (3.5) that $S$ is such that $\overline{M}_{r,S}$ is defined. We allow the possibility that $S = N$, in which case the assumption is that $E(h^*(L)) \subset \partial \overline{M}_{r,N}$.

Proof. We induct on the dimension of $\overline{M}_{g,n}$. It’s clear that the assumptions apply to $h^*(L)$ for any component of a product decomposition of a proper stratum. $L_{P_X}$ is a tensor product of nef line bundles pulled back from the factors, see [GKM00,1.1]. Thus $L_{P_X}$ is by induction semi-ample for any proper closed stratum. In particular the restriction of $L$ to the normalisation of $\partial \overline{M}_{g,n}$ is semi-ample.

We can assume that $L$ is big and $E(L) \subset \partial \overline{M}_{g,n}$: Suppose not. If $g \geq 1$ then by [GKM00,0.9], $L$ is pulled back from some lower dimensional $\overline{M}_{g,S}$, and we can apply induction. If $g = 0$, case (1) is vacuous. In case (2), by the assumption applied to the entire space, $L$ is again pulled back.

Thus the restriction of $L$ to the normalisation of $E(L)$ is semi-ample, so by (0.3) it’s enough to show that $L$-equivalence is bounded. This completes the proof for (1), so we can assume that we have (2). Let $E$ be a pointed stable curve. Note the conditions of (3.5) apply – by assumption (3.6.2) if $r \leq 1$ and otherwise by [GKM00,0.9]. Thus by (3.5), (3.4) holds for all $E$.

(As foretold) we define two pointed curves $F, E$, corresponding to closed boundary points of $\overline{M}_{g,n}$ to be $R$-equivalent if the polarised varieties $Z_{P_F}$ and $Z_{P_E}$ are isomorphic, and $q_F([F]) = q_E([E])$ in $Q_E = Q_F$ under the identification of (3.1), and the map of (3.3) (note this makes sense since $L_{P_X}$ is semi-ample for any proper closed stratum). We define a closed interior point to be $R$-equivalent to itself, but not to any other point. $R$ is clearly an equivalence relation on the set of closed points.
Choose a product decomposition for each closed stratum and let $\mathcal{P}$ be their disjoint union. Let $\mathcal{X}$ be the disjoint union of the normalisations of all closed strata, and $\mathcal{X}^0 \subset \mathcal{X}$ the disjoint union of all (open) strata. Let $\mathcal{Q}$ be the disjoint union of $\overline{M}_{g,n}$ together with the disjoint union of $Q_X$, one for each isomorphism class of polarised varieties $Z_{P_X}$, for $X$ a proper closed stratum. The maps $q_{P_X}$ and $q_X$ of (3.3) give maps

$$q_P : \mathcal{P} \to \mathcal{Q}$$

$$q_{\tilde{X}} : \tilde{\mathcal{X}} \to \mathcal{Q}$$

$$q : \mathcal{X}^0 \to \mathcal{Q}$$

with $q_P$ the composition of $q_{\tilde{X}}$ with $\mathcal{X} \to \mathcal{Q}$. $L|_P$ is pulled back from $Q$.

Let $\mathcal{R}$ be the image of $\mathcal{P} \times \mathcal{Q} : \mathcal{P} \to \overline{M}_{g,n} \times \overline{M}_{g,n}$. Since $q_P$ factors through $q_{\tilde{X}}$, $\mathcal{R}$ is also the image of $\tilde{\mathcal{X}} \times \mathcal{Q} : \tilde{\mathcal{X}} \to \overline{M}_{g,n} \times \overline{M}_{g,n}$.

$R$ is by construction the set of closed points of

$$\mathcal{X}^0 \times \mathcal{Q} : \mathcal{X}^0 \to \overline{M}_{g,n} \times \overline{M}_{g,n}$$

and so contained in the closed points of $\mathcal{R}$. $\mathcal{R}$ is projective so $\mathcal{R} \subset \overline{M}_{g,n} \times \overline{M}_{g,n}$ is closed (we will not need to know whether or not it’s an equivalence relation). Since $L|_P$ is pulled back from $Q$ and $\mathcal{P} \to \overline{M}_{g,n}$ is finite, it follows that $p^*_2(L)$ is numerically trivial on fibres of the first projection $\mathcal{R} \to \overline{M}_{g,n}$.

Any fibre of the first projection $\mathcal{R} \to \overline{M}_{g,n}$ is the union of images of finitely many fibres of the first projection $\mathcal{P} \times \mathcal{Q} \to \mathcal{P}$, and thus of finitely many fibres of $\mathcal{P} \to Q$. $L_P$ is numerically trivial on fibres of $\mathcal{P} \to Q$, since it is pulled back from $Q$.

Thus by (2.4) it’s enough to show $L$ equivalence is a refinement of $R$ (for then $\mathcal{R}$ is related to $L$) and for this, since $R$ is an equivalence relation, it’s enough to show that for $Y \subset \overline{M}_{g,n}$ an irreducible curve, with $L \cdot Y = 0$, any two points of $Y$ are $R$-equivalent. $Y \subset E(L) \subset \partial \overline{M}_{g,n}$. $Y$ meets the interior of a unique closed stratum. Clearly it’s enough to consider a point $[E] \in Y$ in the interior of this stratum and an arbitrary point $[F] \in Y$. $F$ is a degeneration of $E$ as in (3.0.1). Let $Y' \subset P_E$ be an irreducible curve lifting $Y$, and $[\tilde{F}], [\tilde{E}] \in Y'$ points mapping to $[F], [E]$. Let $p : P_E \to W$ be the map of (3.4). $L_{P_E} \cdot Y' = 0$, and $Y'$ meets the interior of $P_E$, so by
(3.4.2), \( p(Y') \) is a point, and in particular \( p([F]) = p([E]) \). So by (3.4.3), \( F \) and \( E \) are \( R \)-equivalent. □

3.7 Remark. If one has (3.6.2) for all \( r \), then (3.6) holds also for \( p = 2 \) – in the proof the assumption \( p > 2 \) is used only to obtain (3.6.2) (via [GKM00,0.9]) for \( r > 1 \).

Proof of (0.5). We use (3.7). So we need to check that the conditions of (3.6.2) hold for each of the line bundles in question, and all \( r \geq 0 \). For \( \psi_i \) apply [Keel99,4.9] (where \( \psi_i \) on \( \overline{M}_{g,n+1} \) is denoted \( L_{g,n} \)). For \( D_{g,n} \) apply [GKM00,4.7-8]. For \( \lambda \): \( \lambda \) is trivial on \( \overline{M}_{0,n} \). \( \lambda \) on \( \overline{M}_{g,n} \) for \( g \geq 2 \) (resp. for \( g = 1 \)) is the pullback of \( \lambda \) from \( \overline{M}_g \) (resp. \( \overline{M}_{1,1} \)). Finally \( \mathbb{E}(\lambda_{\overline{M}_g}) \subset \partial \overline{M}_g \), which follows e.g. from Mumford’s formula 12 \( \lambda = \kappa + \delta \) and the well known ampleness of \( \kappa \). □

§4 Semi-ample and EWM. 4.0 Assumptions: Throughout this section \( k \) is the algebraic closure of a finite field. \( L \) is a nef line bundle. All spaces are proper algebraic spaces over \( k \). Throughout we work with \( \mathbb{Q} \)-line bundles. Note by the base field assumption, that a \( \mathbb{Q} \)-line bundle is the same as its Chern class in the Neron-Severi group.

4.1 Theorem. Let \( L \) be a nef line bundle on an algebraic space \( X \) proper over \( k \), the algebraic closure of a finite field. Then \( L \) is semi-ample iff \( L \) is EWM.

4.1.1 Remark. Note (4.1) is equivalent to the statement that the class \( c_1(L) \) in the Neron-Severi group is pulled back from \( Z \) for \( g : X \to Z \) the associated map. This formulation looks at first reasonable for any field, but unfortunately fails: Take for example a smooth curve \( C \subset X \) of negative self intersection on a projective surface so that the contraction, \( g \), of \( C \) is not projective. Take \( D \) an effective combination of \( C \) and ample \( H \) so that \( D \cdot C = 0 \). Then \( g \) is the map associated to \( D \), but the first Chern class (in the Neron-Severi group) cannot descend, as it would represent an ample divisor on the image. I suspect there are also counter-examples with \( Z \) projective.

We use the following in proving (4.1):

4.2 Lemma. Let \( g : X \to Z \) be a proper surjection with geometrically connected fibres, with \( X \) projective. Assume \( L \) is nef, EWM and \( g \)-numerically trivial. Assume there is a proper \( Y \to X \), with \( Y \to Z \) surjective and a line bundle \( L_Y \) so that \( L_Y|_Y = L_Z|_Y \). Then \( L = L_Z|_X \) (as \( \mathbb{Q} \)-line bundles).

Proof. It’s enough by (1.7) to show \( L \) is pulled back from \( Z \). As we work in the Neron-Severi group, to check \( L = L_Z|_X \) it’s enough to check the intersection number of either side with a given subcurve \( C \subset X \). We can replace \( Z \) by the image of \( C \) and everything else by the obvious pullbacks.
We can replace $X$ by a general hyperplane section containing $C$, so long as $g$ has fibre dimension at least 2 (for then the restriction of $g$ to the hyperplane section still has connected fibres).

So we can assume $X$ has dimension at most two and $Z$ is a curve. It’s enough, by (1.7), to show $L$ is semi-ample, for then, since $g$ has connected fibres, the associated map will the composition of $g$ with a finite universal homeomorphism. Since $L$ is EWM, $L$-equivalence is bounded, so by (0.1) to check semi-ampleness we can replace $X$ by a connected component of a desingularisation. Any nef line bundle on a curve over a finite field is semi-ample so the next result applies: □

For the next lemma we allow arbitrary base field.

4.4 Lemma. Let $p : S \to C$ be a map from an irreducible non-singular projective surface to a normal curve. Let $L$ be a nef, $p$-numerically trivial line bundle. The class $c_1(L)$ in the Neron-Severi group is pulled back from $C$.

Proof. We can assume $p$ is surjective, otherwise $L$ is numerically trivial and the result is obvious. Then apply the next lemma with $\beta$ the class of a general fibre and $\gamma = c_1(L)$. □

4.5 Lemma. Let $S$ be a smooth projective surface and $\beta \in \text{NS}^1(S)$ such that $\beta^2 = 0$ and $\beta \cdot H > 0$, for some ample class $H$. If $\gamma \cdot \beta = 0$ then $\gamma^2 \leq 0$, with equality iff $\gamma$ is equivalent to a multiple of $\beta$.

Proof. This is immediate from the hodge index theorem. Indeed suppose $\gamma \cdot \beta = 0$. Then

$$\langle \gamma - \frac{\gamma \cdot H}{\beta \cdot H} \beta, H \rangle = 0$$

$$\langle \gamma - \frac{\gamma \cdot H}{\beta \cdot H} \beta, \beta \rangle^2 = \gamma^2. \quad \square$$

Proof of 4.1. We induct on the dimension of $X$. Since $L$ is EWM, $L$-equivalence is algebraic. Thus by (0.1) we may replace $X$ by anything of the same dimension that surjects properly onto it, and so may assume $X$ is normal, irreducible, and projective. Let $g : X \to Z$ be the map associated to $L$.

Suppose first the relative dimension is at least 2. Let $Y \subset X$ be an ample hypersurface. $Y \to Z$ is a surjection with geometrically connected fibres, the Stein factorisation $Y \to Z_Y$ is the map associated to $L_Y$ and $Z_Y \to Z$ is a finite universal homeomorphism. Thus by induction and [Keel99,1.4], $L_Y$ is pulled back from $Z$, necessarily from an ample line bundle. Now apply (4.2).
Suppose next that \( g \) is birational. Then \( L \) is big. \( L|_{\mathbb{E}(L)} \) is semi-ample by induction, so \( L \) is semi-ample by [Keel99,0.2].

So we can assume \( g \) has relative dimension one.

Let \( Y' \subset X \) be a general hyperplane section, and let \( Y \to Y' \) be a finite cover so the (finite) field extension \( K(Z) \subset K(Y) \) is normal, with group Galois group \( G \). Let \( g_Y : Y \to Z_Y \) be the Stein factorisation of \( Y \to Z \). Note it is birational, and precisely the map associated to \( L_Y \). By induction, \( L_Y \) is pulled back from ample \( L_Z \). \( G \) acts on \( (Z_Y,L_Z) \). Since \( g_Y \) is birational, \( G \) is the Galois group of the normal field extension \( K(Z_Y)/K(Z) \). Thus, since \( Z_Y \to Z \) is finite, the quotient \( Z_Y/G \) is a finite purely inseparable cover of \( Z \). By (4.2) it’s enough to show the \( \mathbb{Q} \)-line bundle \( L_Z \) is pulled back from \( Z \), and so, by [Keel99,1.4], it’s enough to show \( c_1(L_Y) \) is preserved by \( G \).

Thus we need only check

\[
\gamma^*(L_{Z_Y}) \cdot C = L_{Z_Y} \cdot C
\]

for a given \( \gamma \in G \) and irreducible \( C \subset Z_Y \). Let \( f \) be the map \( Z_Y \to Z \) and \( D = f(C) \). Note for (4.6) it’s enough (since \( f \) is \( G \) invariant), to show \( L_{Z_Y}|_{f^{-1}(D)} \) is pulled back from \( D \) and for this, by (1.7), it’s enough to show \( L|_{g^{-1}(D)} \) is pulled back from \( D \). Since \( g \) (and so \( g|_{g^{-1}(D)} \)) has geometrically connected fibres, it’s enough, by [Keel99,1.4], to show \( L|_{g^{-1}(D)} \) is semi-ample (for then the associated map will be the composition of (the restriction of) \( g \) with a finite universal homeomorphism). Thus we can replace \( X \) by \( g^{-1}(D) \), and so reduce to the case when \( Z \) (the image of the associated map) is a curve. Now return to the beginning of the proof and retrace the argument. We reduce to the case \( X \) an irreducible normal surface and \( Z \) a normal curve. We can replace \( X \) by a desingularisation, and apply (4.4). \( \square \)

§5 counter example.

We begin with a general construction. Let \( f_i : X \to X_i, L_{X_i} \) be as in (0.12) with \( L_{X_i} \) ample. Let \( Y \) be obtained by gluing \( Y_1 := X \times X_1 \) to \( Y_2 := X \times X_2 \) along the graphs of \( f_1, f_2 \). The line bundles \( L_Y := \pi_Y^*(L_{X_i}) \) are obviously semi-ample and glue to give a nef line bundle \( L_Y \) on \( Y \), with \( L_Y|_X = L_X \).

5.1 Lemma. Notation as above. Assume \( X \) is geometrically connected. If \( L_Y \) is EWM (resp semi-ample), then a finite pushout (resp. polarised pushout) exists for \( f_1, f_2 \) (resp \( (f_i, L_{X_i}) \)).

Proof. Let \( g : Y \to Z \) be the map associated to \( L_Y \). Note \( \pi_{X_i} \) is the map associated to \( L_Y \); thus \( g|_{Y_i} \) factors through \( \pi_{X_i} \). In particular the finite map \( g|_X : X \to Z \) factors through \( f_1 \) and through \( f_2 \), and is thus a pushout for \( f_1, f_2 \). If \( L_Y \) is pulled back from \( L_Z \), then the pushout is polarised. \( \square \)
5.2 Lemma (Kollár). Let $k$ be the algebraic closure of $\mathbb{F}_p$. The intersection of the subfields $k(T^p + T^{p-1})$ and $k(T^{p-1})$ of the polynomial field $k(T)$ is $k$. In particular the corresponding pair of endomorphisms of $\mathbb{P}^1$ has no finite pushout.

Proof. Suppose on the contrary the fields have nontrivial intersection. Then there exist monic polynomials $f_i, g_i$, with

$$(f_1(T), f_2(T)) = (g_1(T), g_2(T)) = 1.$$ 

such that

$$(5.2.1)\quad f_1(T^p + T^{p-1})/f_2(T^p + T^{p-1}) = g_1(T^{p-1})/g_2(T^{p-1})$$

with $f_1, g_1$ non-constant. There exist $a_1, a_2$ with

$$a_1(T)f_1(T) + a_2(T)f_2(T) = 1.$$ 

Replacing $T$ by $T^p + T^{p-1}$ we conclude the $f_i(T^p + T^{p-1})$ are relatively prime. Similarly the $g_i(T^{p-1})$ are relatively prime. Now (5.2.1) implies $f_i(T^p + T^{p-1}) = g_i(T^{p-1})$. Thus we can assume $f_2 = g_2 = 1$. Now choose $f$ and $g$ of minimal positive degree so that $f(T^p + T^{p-1}) - g(T^{p-1})$ is of degree at most one, say $aT + b$. We can absorb $b$ into $f$, and assume $b = 0$. Differentiating (and using characteristic $p$) we see that

$$(p - 1)T^{p-2}(f'(T^p + T^{p-1}) - g'(T^p)) = a$$

is constant. We conclude that $a = 0$, and so (by minimality of degree) $f' = g' = c$ is constant. Thus for some polynomials $F$ and $G$, $f(T) = F(T)^p + cT$, $g(T) = G(T)^p + cT$, and we have

$$F(T^p + T^{p-1})^p + cT^p = G(T^{p-1})^p$$

But then

$$F(T^p + T^{p-1}) - G(T^{p-1})$$

is linear, contradicting the minimality. □

Now take $X = X_1 = X_2 = \mathbb{P}^1$ and $L_{X_i} = \mathcal{O}(1)$, and $f_1, f_2$ the endomorphisms of (5.2). The construction of (5.1) gives a non semi-ample line bundle on a surface defined over a finite field whose restriction to the normalisation is semi-ample.
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