ON CERTAIN LABELLED DIRECTED GRAPHS
OF SYMBOLIC DYNAMICS

WOLFGANG KRIEGER

Abstract. We describe classes of semisynchronizing, non-synchronizing subshifts, that are closed under topological conjugacy. The subshifts that belong to these classes satisfy a strong hypothesis of context-freeness and are canonically presented by means of Shannon graphs that have the structure of a pd.

1. Introduction

Let $\Sigma$ be a finite alphabet. The dynamical systems that are obtained by restricting the left shift on $\Sigma^\mathbb{Z}$ to one of its closed invariant sets are called subshifts. For introductory expositions if their theory see [LM] or [K].

A finite word in the symbols of $\Sigma$ is called admissible for the subshift $X \subset \Sigma^\mathbb{Z}$ if it appears somewhere in a point of $X$. We denote he set of admissible words of a subshift $X \subset \Sigma^\mathbb{Z}$ by $\mathcal{L}(X)$. A subshift $X \subset \Sigma^\mathbb{Z}$ is uniquely determined by $\mathcal{L}(X)$, and one can therefore define a subshift $X \subset \Sigma^\mathbb{Z}$ by specifying $\mathcal{L}(X)$. One way to do this is by means of directed graphs or labelled directed graphs.

For directed graphs with vertex set $V$ and edge set $E$ we use the notation $G(V,E)$. We consider finite or countably infinite directed graphs $G(V,E)$ such that every vertex has a finite positive number of incoming edges, and a finite positive number of outgoing edges. We denote the set of finite directed paths in $G(V,E)$ by $\Pi_{G(V,E)}$. If $G(V,E)$ is finite, then $\Pi_{G(V,E)}$ is the set of admissible words of a topological Markov shift, that is called the edge shift of $\Pi_{G(V,E)}$.

For the labelled directed graph, that is obtained from a directed graph $G(V,E)$ by providing a label alphabet $\Sigma$ and a label map $\lambda: E \rightarrow \Sigma$, we use the notation $G_{\Sigma,\lambda}(V,E)$. The label map $\lambda$ extends to a label map on $\Pi_{G(V,E)}$, that we also denote by $\lambda$, by

$$\lambda((b_i)_{1 \leq i \leq I}) = (\lambda(b_i))_{1 \leq i \leq I}, \quad (b_i)_{1 \leq i \leq I} \in \Pi_{G(V,E)}, \quad I \in \mathbb{N}.$$ 

A labelled directed graph defines a subshift $X(G_{\Sigma,\lambda}(V,E))$ by

$$\mathcal{L}(X(G_{\Sigma,\lambda}(V,E))) = \{\lambda(a) : a \in \Pi_{G_{\Sigma,\lambda}(V,E)}\}.$$ 

One says that $G_{\Sigma,\lambda}(V,E)$ presents $X(G_{\Sigma,\lambda}(V,E))$. A finite labelled directed graph $G_{\Sigma,\lambda}(V,E)$ has a labelled edge shift that has as its set of admissible words the set $\{(a,\lambda(a)) : a \in \Pi_{G(V,E)}\}$ of finite labelled directed paths in $G_{\Sigma,\lambda}(V,E)$. The labelled edge shift of the finite labelled directed graph $G_{\Sigma,\lambda}(V,E)$ projects onto $X(G_{\Sigma,\lambda}(V,E))$.

A labelled directed graph $G_{\Sigma,\lambda}(V,E)$ is called a Shannon graph, if for $V \in V$ and for $\sigma \in \Sigma$ there is at most one $e \in E$, that leaves $V$, and that carries the label $\sigma$. Semisynchronizing shifts $X \subset \Sigma^\mathbb{Z}$ are canonically presented by their semisynchronizing Shannon graphs $G_{s-syn}(X) = G_{\Sigma,\lambda}(V_{s-syn}(X),E_{s-syn}(X))$, and a-synchronizing shifts $X \subset \Sigma^\mathbb{Z}$ are canonically presented by their a-synchronizing Shannon graphs $G_{a-syn}(X) = G_{\Sigma,\lambda}(V_{a-syn}(X),E_{a-syn}(X))$ [Kr4, Kr5]. This extends the case of topologically transitive sofic systems [Kr2, Kr3]. In this paper we identify classes of semisynchronizing, non-synchronizing subshifts, that...
are closed under topological conjugacy by imposing structural hypotheses on their semisynchronizing Shannon graphs. The key hypothesis is a strong hypothesis of context-freeness. Semisynchronizing, non-synchronizing subshifts, that satisfy this key hypothesis, we call \((C - F)\)-semisynchronizing.

Before introducing the notion of a \((C - F)\)-semisynchronizing shift in section 5, we introduce in a preliminary section 2 additional terminology and notation and recall the necessary tools. Also, in section 3, we prove auxiliary result about countably infinite directed graphs \(G(V, E)\). In these we distinguish a finite subset \(B\), that we refer to as a boundary. For the resulting structure we use the notation \(G(V, E, B)\). Given the directed graph w.r.b. \(G(V, E, B)\) we denote for \(V \in V \setminus B\) by \(D_B(V)\) the shortest length of a directed path in \(G(V, E, B)\) from \(V\) to \(B\), and for \(V \in B\) we set \(D_B = 0\).

We say that a directed path \((e_i)_{i \geq 1}\) in \(G(V, E, B)\) approaches \(B\) from infinity if

\[
\lim_{i \to \infty} D_B(s(e_i)) = \infty.
\]

We say, that a finite path in \(G(V, E, B)\) approaches \(B\) directly, if one has for every edge \(e\), that is traversed by the path, that \(D_B(t(e)) = D_B(s(e)) - 1\). We say, that an infinite path in \(G(V, E, B)\) approaches \(B\) directly from infinity, if one has for every edge \(e\), that is traversed by the path, that \(D_B(t(e)) = D_B(s(e)) - 1\), and we say that an infinite path in \(G(V, E, B)\) approaches \(B\) almost directly from infinity, if one has for all but finitely many edges \(e\), that are traversed by the path, that \(D_B(t(e)) = D_B(s(e)) - 1\). We say, that a vertex \(V \in V\) is accessible (almost directly accessible, directly accessible) from infinity, if there exists a directed path \((e_i)_{i \geq 1}\) in \(G(V, E, B)\), that approaches \(B\) from infinity (almost directly from infinity, directly from infinity), that has \(V\) as its target vertex. Given a directed graph w.b. \(G(V, E, B)\) we denote by \(V^i(V^i, V^i)\) the set of vertices, that are accessible (almost directly accessible, directly accessible) from infinity. We also set

\[
S_{V, B}(K) = \{V \in V : D_B(V) \leq K\},
\]

\[
S_{V, B}^i(K) = \{V \in V : D_B(V) = K\}, \quad K \in \mathbb{Z}_+.
\]

Given the directed graph w.b. \(G(V, E, B)\) and \(r \in \mathbb{N}\) we use the notation

\[
\gamma(V \setminus S_{V, B}(r - 1), S_{V, B}^i(r))
\]

for the set of directed graphs w.b. \(G(C, B_C)\), with a connected component \(C\) of \(V \setminus S_{V, B}(r - 1)\) as vertex set, and with the boundary \(B_C = C \cap S_{V, B}^i(r)\). The edge set of \(G(C, B_C)\), that we suppress in the notation, is the edge set \(E_C\) of the full subgraph of \(G(V, E)\), that has \(C\) as vertex set.

We introduce the notion of a reference set \(R\) of a directed graph \(G(V, E)\), by which we mean a boundary \(R\) of \(G(V, E)\), such that the directed paths in \(V \setminus B\), that approach \(R\) directly from infinity, behave in a coordinated way (in a sense to be made precise). The existence of a reference set will be a standing hypothesis on the infinite directed graphs that we consider. It is essential that the objects and notions that are introduced by using a reference set do in fact not depend on the choice of the reference set. We therefore suppress the reference set in the notation. For directed graphs w.r.s. we formulate two structural hypotheses (HDG1) and (HDG2) that concern the behavior of the directed paths in the graph with respect to a reference set.

In section 4 we consider Shannon graphs w.r.s. \(G_{\Sigma, \lambda}(V, E)\). (We mean a Shannon graph \(G_{\Sigma, \lambda}(V, E)\) such that the underlying directed graph \(G(V, E)\) has a reference set. We maintain this logic in our terminology throughout the paper). To a Shannon graphs w.r.s. \(G_{\Sigma, \lambda}(V, E)\) there is invariantly associated a subshift \(Y(G_{\Sigma, \lambda}(V, E))\),
that has as its set of admissible words the set of words that appear at an arbitrary large distance from the reference set in the label sequences of infinite directed paths in $G_{\Sigma,\lambda}(V, \mathcal{E})$, that approach the reference set directly from infinity.

For a Shannon graph w.r.t. $G_{\Sigma,\lambda}(V, \mathcal{E})$ we formulate three hypotheses. We formulate a hypothesis on $G_{\Sigma,\lambda}(V, \mathcal{E})$, that we call the visibility hypothesis. To satisfy this hypothesis means for $G_{\Sigma,\lambda}(V, \mathcal{E})$, that, in case, that a sufficiently long word in $L(Y(G_{\Sigma,\lambda}(V, \mathcal{E})))$, is the label sequence of a path in $G_{\Sigma,\lambda}(V, \mathcal{E})$, that has sufficient distance from a reference set, that this path approaches the reference set directly, except, possibly, in initial or final segments of negligible length. (For the choice of terminology compare [BBD1])

We formulate a hypothesis on $G_{\Sigma,\lambda}(V, \mathcal{E})$, that we call the coherence hypothesis, that asks for a reference set $\mathcal{R}$ of $G_{\Sigma,\lambda}(V, \mathcal{E})$ and for sufficiently large $r$, and for every

\[(1) \quad G(C^i, B^i) \in \gamma(V \setminus S_{V,R}(r - 1), S_{V,R}(r)),\]

that every automorphism of $G(C^i, B^i)$ leaves all elements of

$$\gamma(V \setminus S_{C_i, B_i}^i(k - 1), S_{C_i, B_i}^i(k)), \quad k \in \mathbb{N},$$

fixed, and also every vertex in $B_{C_i}$, that can be reached directly from infinity. We will use the coherence hypothesis in conjunction with (DGH2), to determine up to isomorphism the location of vertices within infinite graphs $G(C, B_C)$ w.b as in (1) by data that are intrinsic to the graphs.

We formulate a hypothesis on $G_{\Sigma,\lambda}(V, \mathcal{E})$, that we call the discordance hypothesis. To satisfy this hypothesis means for $G_{\Sigma,\lambda}(V, \mathcal{E})$ that $X(G_{\Sigma,\lambda}(V, \mathcal{E}))$ imitates the Dyck shifts in an essential way.

Given a Shannon graph w.r.s. $G_{\Sigma,\lambda}(V, \mathcal{E})$ we denote for a reference set $\mathcal{R}$ of $G_{\Sigma,\lambda}(V, \mathcal{E})$ and for $r \in \mathbb{N}$ by $\hat{\gamma}(V \setminus S_{V,R}(r - 1), S_{V,R}(r))$ the set of boundary isomorphism types of the elements of $\gamma(V \setminus S_{V,R}(r - 1), S_{V,R}(r))$. In section 5 we introduce the $(C - F)$-semisynchronizing shifts as the semisynchronizing non-synchronizing shifts $X \subset \Sigma^Z$ such that $G_{s-syn}(X)$ has a reference set, and such that for a reference set $\mathcal{R}$ of $G_{s-syn}(X)$ the set

$$\hat{\gamma}(X) = \bigcap_{q \in \mathbb{N}} \bigcup_{r \geq q} \hat{\gamma}(V \setminus S_{V,syn}(X), R(r - 1), S_{V,syn}(X), R(r))$$

is non-empty and finite. The notion of a $(C - F)$-a-synchronizing shift is analogue. For $(C - F)$-semisynchronizing shifts and $(C - F)$-a-synchronizing shifts, one can choose a reference set $\mathcal{R}$ such that

$$\hat{\gamma}(X) = \bigcup_{r \in \mathbb{N}} \hat{\gamma}(V \setminus S_{V,syn}(X), R(r - 1), S_{V,syn}(X), R(r)),$$

and we will assume that such a choice is made. We prove that the class of $(C - F)$-semisynchronizing shifts is closed under topological conjugacy. The same holds for the class of a-synchronizing shifts. The semisynchronizing Shannon graphs of $(C - F)$-semisynchronizing shifts and the a-synchronizing Shannon graphs of $(C - F)$-a-synchronizing shifts have a structure that is similar to the structure of a context-free graph as described by Muller and Schupp [MS] (Motivated by the problems of symbolic dynamics we consider directed graphs. The place of the generating vertex of a context-free graph is taken by any reference set).

Given a $(C - F)$-semisynchronizing shift $X$, retaining only the infinite elements of $\hat{\gamma}(X)$, and and in these only the vertices, that can be reached from infinity, one arrives at a set $\hat{\gamma}^{-1}(X)$ of boundary isomorphism types, that is the vertex set of a strongly connected directed graph with a naturally defined edge set $\delta^{-1}(X)$. The edge shift of $G(\hat{\gamma}^{-1}(X), \delta^{-1}(X))$ is invariantly associated to $X$.
For \((C - F)\)-semisynchronizing shift \(X\), that satisfy the coherence hypothesis, there is also a finite labelled directed graph, that we denote by \(G_{\Sigma, \lambda}(\mathcal{B}^\mathbb{Z}, \mathcal{F}^\mathbb{Z})(X)\), that is invariantly associated to \(X\), and that presents \(Y\left(G_{\text{a-syn}}(X)\right)\). For these \((C - F)\)-synchronizing shifts \(X\), the subshift \(Y\left(G_{\text{a-syn}}(X)\right)\) is therefore sofic. We say that a \((C - F)\)-semisynchronizing shift \(X\) satisfies the projection hypothesis if the projection of the labelled edge shift of \(G_{\Sigma, \lambda}(\mathcal{B}^\mathbb{Z}, \mathcal{F}^\mathbb{Z})(X)\) onto \(Y\left(G_{\text{a-syn}}(X)\right)\) is a topological conjugacy.

For \((C - F)\)-semisynchronizing shifts \(X\), that satisfy the coherence hypothesis and (DGH2) we formulate a hypothesis on \(G_{\Sigma, \lambda}(V, E)\), that we call the resolution hypothesis. This hypothesis expresses a quality of backward resolution of the infinite paths in \(G_{\Sigma, \lambda}(V, E)\) that approach their target vertex from infinity, and we formulate a hypothesis \((h)\) that expresses a quality of forward resolution of the paths in \(G_{\text{a-syn}}(X)\), that approach their target vertex from infinity.

For a \((C - F)\)-semisynchronizing shift \(X\) we define a resolving word for \(G_{\text{a-syn}}(X)\) as a word in \(\mathcal{L}(X)\), that, when labelling a directed path \(b\) in \(G_{\text{a-syn}}(X)\) determines for the specific vertex \(V\), that is traversed by \(b\) at coordinate zero, and with a reference set \(\mathcal{R}\) of \(G_{\text{a-syn}}(X)\), the boundary isomorphism type of the connected component of

\[V_{\text{a-syn}}(X) \setminus \mathcal{S}_{V_{\text{a-syn}}(X), \mathcal{R}}(\Delta_{\mathcal{R}}(V) - 1), \mathcal{S}_{V_{\text{a-syn}}(X), \mathcal{R}}(\Delta_{\mathcal{R}}(V)),\]

that contains \(V\), and also, up to boundary isomorphism, the location of \(V\) within this component. For \((C - F)\)-semisynchronizing shifts \(X\), that satisfy (HDG1) and (HDG2), and that satisfy the coherence, discordance, visibility and projection hypotheses, and hypothesis \((h)\), we describe, relying essentially on resolving words and hypothesis \((h)\), the construction of \(G_{\text{a-syn}}(X)\) as an automaton. (This is a real time deterministic pda with all states initial and final. The set of states with empty stack acts as a reference set and the directed paths in \(G(\hat{\gamma}(X), \hat{\delta}(X))\) appear in the stack). However, hypothesis \((h)\), in contrast to the other hypothesis, that we consider, is not an invariant of topological conjugacy. We do not investigate here, to what extent the various assumptions, that enter into the construction of the automaton, are independent.

In section 6 we consider the class of \((C - F)\)-semisynchronizing shifts \(X\), that are also \((C - F)\)-a-synchronizing, and such that \(G_{\text{a-syn}}(X)\) satisfies the visibility and projection hypotheses. This class is closed under topological conjugacy. We give a complete description of the automata that arise as the a-synchronizing Shannon graphs of the subshifts in this class. This includes a necessary and sufficient condition for the candidate automata to be forward separated.

Every continuous shift commuting map \(\varphi\) that sends a subshift \(X \subset \Sigma^\mathbb{Z}\) into a subshift \(\bar{X} \subset \bar{\Sigma}^\mathbb{Z}\) is given for some \(L \in \mathbb{Z}_+\) by a block map \(\Phi\),

\[\varphi((x_i)_{i \in \mathbb{Z}}) = (\Phi((x_j)_{i-L \leq j \leq i+L}))_{i \in \mathbb{Z}}.\]

If for a labelled directed graph \(G_{\Sigma, \lambda}(V, E)\) there is given a finite alphabet \(\bar{\Sigma}\) and a conjugacy \(\varphi\) of \(X(G_{\Sigma, \lambda}(V, E))\) onto a subshift \(\bar{X} \subset \bar{\Sigma}^\mathbb{Z}\), that is given by a one-block map \(\Phi : \Sigma \to \bar{\Sigma}\), then we say, that \(G_{\Sigma, \lambda}(V, E)\) presents \(\bar{X}\) via the one-block conjugacy \(\varphi\). In section 7, following a pattern, that was used in the definition of finite-type-Dyck shifts by Béna, Blockelet and Dima [BBD1], we consider the class of a-synchronizing \((C - F)\)-semi-synchronizing shifts \(X\), that satisfy the discordance and resolution hypotheses, and that are such that every word in \(\mathcal{L}(X)\) with sufficient anticipation and memory is a resolving word for \(X\). This class is closed under topological conjugacy. Subshifts in this class satisfy the visibility and projection hypotheses. Subshifts \(X\) in this class allow a recoding as a \((C - F)\)-semisynchronizing shift \(\bar{X}\), that satisfies hypothesis \((h)\), and that yields a canonical
presentation of $X$ by $G_{a-syn}(\tilde{X})$ via a one-block conjugacy. As $X$ is topologically conjugate to its recoding $\tilde{X}$, information about the topological conjugacy class of $X$ can be obtained from $\tilde{X}$. This class contains a set of subshifts $X$ that recode as finite-type-Dyck shifts $\tilde{X}$. Concerning invariants of topological conjugacy for finite-type-Dyck shifts see [BBD2].

In section 8 we list examples.

2. Preliminaries

We introduce additional terminology and notation and recall the necessary tools concerning directed graphs, labelled directed graphs and subshifts. We also make some comments.

We denote the source vertex of an edge (or of a path) in a directed graph $G(V, E)$ by $s$, and its target vertex by $t$. The empty path we denote by $\epsilon$.

As we are concerned in this paper with directed graphs $G(V, E, B)$, we want to point out at this point that the connectedness of a set $C \subset V \setminus B$ can be described in terms of directed paths as follows: $C$ is connected if and only if for $V, V' \in V, V \neq V'$ there is an $I \in \mathbb{N}$, and $V_i \in V, 1 \leq i \leq I, V_1 = V, V_I = V'$, and directed paths $b_i, 1 \leq i \leq I$, in $V \setminus B$ such that

$$s(b_i) = V_i, \quad 1 \leq i \leq I,$$

and

$$t(b_i) = t(b_{i+1}) \in B, \quad 1 \leq i < I.$$

A directed graph $G(V, E)$ is called bipartite, if its vertex set is a disjoint union $V \cup \tilde{V}$ and its edge set is a disjoint union $F \cup \tilde{F}$, such that

$$s(f) \in V, t(f) \in \tilde{V}, \quad f \in F,$$

$$s(\tilde{f}) \in \tilde{V}, t(\tilde{f}) \in V, \quad \tilde{f} \in \tilde{F}.$$

For a bipartite directed graph $G(V \cup \tilde{V}, F \cup \tilde{F})$ we denote by $\Pi(2)(G(V \cup \tilde{V}, F \cup \tilde{F}))$ (or $\Pi(2)(G(V \cup \tilde{V}, F \cup \tilde{F}))$) the set of paths in $G(V \cup \tilde{V}, F \cup \tilde{F})$ of length two, that start in $V(\tilde{V})$.

We denote the adjacency matrix of a directed graph $G(V, E)$, in which every vertex has a finite positive number if incoming and of outgoing edges, by

$$A_{G(V, E)} = (A_{G(V, E)}(V, V'))_{V, V' \in V},$$

$$A_{G(V, E)}(V, V') = \text{card}\{e \in E : s(e) = V, t(e) = V'\}.$$ Directed graphs $G(V, E)$ and $G(\tilde{V}, \tilde{E})$ are bipartitely related if there exists a bipartite directed graph $G(V \cup \tilde{V}, F \cup \tilde{F})$ together with bijections

$$\Psi : E \to \Pi(2)(G(V \cup \tilde{V}, F \cup \tilde{F}), \quad \tilde{\Psi} : \tilde{E} \to \tilde{\Pi}(2)(G(V \cup \tilde{V}, F \cup \tilde{F})),$$

such that

$$s(e) = s(\Psi(e)), \quad t(e) = t(\Psi(e)), \quad e \in E,$$

$$s(\tilde{e}) = s(\tilde{\Psi}(\tilde{e})), \quad t(\tilde{e}) = t(\tilde{\Psi}(\tilde{e})), \quad \tilde{e} \in \tilde{E}.$$

For the matrices $B = (B(V, \tilde{V}))_{V, \tilde{V} \in V}$,

$$B(V, \tilde{V}) = \text{card}\{f \in F : s(f) = V, t(f) = \tilde{V}\} \quad V \in V, \tilde{V} \in \tilde{V},$$

and $\tilde{B} = (B(\tilde{V}, V))_{\tilde{V}, \tilde{V} \in V}$,

$$B(\tilde{V}, V) = \text{card}\{f \in F : s(\tilde{f}) = \tilde{V}, t(\tilde{f}) = V\} \quad \tilde{V} \in \tilde{V}, V \in V,$$

one has

$$A_{G(V, E)} = B\tilde{B}, \quad A_{G(\tilde{V}, \tilde{E})} = \tilde{B}B.$$
In other words, $A_{G(V,E)}$ and $A_{G(V,E)}'$ are strong shift equivalent in one step. Conversely, if the matrices $A_{G(V,E)}$ and $A_{G(V,E)}'$ are strong shift equivalent in one step, then the graphs $G(V,E)$ and $G(V,E')$ can be put into a bipartite relation. Infinite directed graphs, such that every vertex has a positive finite number of incoming and outgoing edges, can be said to be strong shift equivalent, if their adjacency matrices are strong shift equivalent. Results on invariance under strong shift equivalence of infinite directed graphs, such that every vertex has a positive finite number of incoming and outgoing edges, can be interpreted as belonging to the theory of locally compact countable state topological Markov shifts.

A labelled directed graph is called bipartite, if its vertex set is a disjoint union $V \cup \tilde{V}$, its edge set is a disjoint union $F \cup \tilde{F}$, and its label alphabet is a disjoint union $\Sigma \cup \tilde{\Sigma}$ such that

$$s(f) \in V, t(f) \in \tilde{V}, \lambda(f) \in \Delta, \quad f \in F,$$

$$s(\tilde{f}) \in \tilde{V}, t(\tilde{f}) \in V, \lambda(\tilde{f}) \in \tilde{\Delta}, \quad \tilde{f} \in \tilde{F}.$$

Labelled directed graphs $G_{\Sigma,\Lambda}(V,E)$ and $G_{\tilde{\Sigma},\tilde{\Lambda}}(\tilde{V},\tilde{E})$ are bipartitely related if there exists a labelled bipartite directed graph $G_{\Delta \cup \tilde{\Delta},\tilde{\Lambda}}(V \cup \tilde{V}, F \cup \tilde{F})$ together with bijections

$$\Psi : E \to \Pi(2)(G(V \cup \tilde{V}, F \cup \tilde{F}), \quad \tilde{\Psi} : \tilde{E} \to \tilde{\Pi}(2)(G(\tilde{V} \cup V, F \cup \tilde{F})),$$

$$\Lambda : \Sigma \to \lambda(\Pi(2)(G(V \cup \tilde{V}, F \cup \tilde{F}))), \quad \tilde{\Lambda} : \tilde{\Sigma} \to \tilde{\lambda}(\tilde{\Pi}(2)(G(V \cup \tilde{V}, F \cup \tilde{F}))),$$

such that

$$s(e) = s(\Psi(e)), t(e) = t(\Psi(e)), \tilde{\lambda}(\Psi(e)) = \Lambda(\lambda(e)), \quad e \in E,$$

$$s(\tilde{e}) = s(\tilde{\Psi}(\tilde{e})), t(\tilde{e}) = t(\tilde{\Psi}(\tilde{e})), \tilde{\lambda}(\tilde{\Psi}(\tilde{e})) = \tilde{\lambda}(\tilde{\lambda}(\tilde{e})), \quad \tilde{e} \in \tilde{E}.$$

Given a Shannon graph $G_{\Sigma,\Lambda}(V,E)$ we denote for $V \in \mathcal{V}$ by $\Gamma_1^+(V)$ the set of $\sigma \in \Sigma$, such that there is an edge $e \in E$, that leaves $V$ and that carries the label $\sigma$. We call $\Gamma_1^+(V)$ the acceptance set of the vertex $V$.

For $V \in \mathcal{V}$ and $\sigma \in \Gamma_1^+(V)$ we denote the edge $e \in E$, that leaves $V$ and that carries the label $\sigma$ by $e_\sigma(V)$, and we use the notation $\Theta_\sigma(V)$ for $t(e_\sigma(V))$. We refer to the mappings $\Theta$ as the transition functions of the Shannon graph. Given its vertex set $\mathcal{V}$ the sets $\Gamma_1^+(V), V \in \mathcal{V}$ together with the mappings $\Theta_\sigma(V), \sigma \in \Gamma_1^+(V), V \in \mathcal{V}$ determine the structure of a Shannon graph $G_{\Sigma,\Lambda}(\mathcal{V},E)$. For $V \in \mathcal{V} \setminus B$ we set

$$\Gamma_1^{+,1-}(V) = \{ \sigma \in \Gamma_1^+(V) : \Delta_B(\Theta_\sigma(V)) = \Delta_B(B) - 1 \}.$$ 

For a Shannon graph $G_{\Sigma,\Lambda}(\mathcal{V},E)$ we also denote by $\Gamma^+(V)$ the set of label sequences that are carried by the finite paths that leave $V$, and for $a \in \Gamma^+(V)$ we denote by $\Theta_a(V)$ the target vertex of the path that leaves $V$ and that carries the label sequence $a$. We say that vertices $V, V' \in \mathcal{V}, V \neq V'$ can be separated if $\Gamma^+(V) \neq \Gamma^+(V')$. A Shannon graph $G_{\Sigma,\Lambda}(\mathcal{V},E)$ is called forward separated if all pairs $V, V' \in \mathcal{V}, V \neq V'$, can be separated.

When we say that Shannon graphs $G_{\Sigma,\Lambda}(\mathcal{V},E)$ and $G_{\tilde{\Sigma},\tilde{\Lambda}}(\tilde{V},\tilde{E})$ are bipartitely related, then we mean that the bipartite relation is implemented by a Shannon graph $G_{\Delta \cup \tilde{\Delta},\tilde{\Lambda}}(V \cup \tilde{V}, F \cup \tilde{F})$ together with bijections

$$\Psi : E \to \Pi(2)(G(V \cup \tilde{V}, F \cup \tilde{F}), \quad \tilde{\Psi} : \tilde{E} \to \tilde{\Pi}(2)(G(\tilde{V} \cup V, F \cup \tilde{F})),$$

$$\Lambda : \Sigma \to \lambda(\Pi(2)(G(V \cup \tilde{V}, F \cup \tilde{F}))), \quad \tilde{\Lambda} : \tilde{\Sigma} \to \tilde{\lambda}(\tilde{\Pi}(2)(G(V \cup \tilde{V}, F \cup \tilde{F}))).$$

In this case the maps $\Psi$ and $\tilde{\Psi}$ are determined by the maps $\Lambda$ and $\tilde{\Lambda}$ : Setting

$$f(e) = f_{\Lambda(e)}(s(e)), \quad \tilde{f}(e) = \tilde{f}_{\tilde{\Lambda}(e)}(t(f(e))), \quad e \in E,$$
A word \( b \) that is admissible for a topologically transitive subshift \( X \subset \Sigma^\mathbb{Z} \) is called a synchronizing word of \( X \), if for all \( c \in \Gamma^-(b), d \in \Gamma^+(b) \) one has that \( cdb \in \omega^-(b) \). Equivalently, a synchronizing word of \( X \) can be defined as a word \( b \in L(X) \) such that \( \Gamma^-(b) = \omega^-(b) \), or, such that \( \Gamma^+(b) = \omega^+(b) \). We denote the set of synchronizing words of a topologically transitive subshift \( X \subset \Sigma^\mathbb{Z} \) by \( L_{\text{syn}}(X) \). A topologically transitive subshift \( X \subset \Sigma^\mathbb{Z} \) that has a synchronizing word is called synchronizing.

A word \( b \) that is admissible for a topologically transitive subshift \( X \subset \Sigma^\mathbb{Z} \) is called an \( s \)-synchronizing word of \( X \), if for all \( c \in L(X) \) there exists a \( d \in \Gamma^+(c) \) such that \( cd \in \omega^+(b) \). A synchronizing word is semisynchronizing. We denote the set of semisynchronizing words of a topologically transitive subshift \( X \subset \Sigma^\mathbb{Z} \) by \( L_{s\text{-syn}}(X) \). A topologically transitive subshift \( X \subset \Sigma^\mathbb{Z} \) that has an semisynchronizing word is called semisynchronizing (see [Kr3]).

A word \( b \) that is admissible for a topologically transitive subshift \( X \subset \Sigma^\mathbb{Z} \) is called an \( a \)-synchronizing word for \( X \) if \( b \) satisfies the following two conditions a-s(1) and a-s(2):

| Condition a-s(1) | For \( c \in L(X) \) there exists a \( d \in \Gamma^-(c) \) such that \( dc \in \omega^+(b) \).
| Condition a-s(2) | For \( c \in \omega^+_\infty(b) \) there exists a \( d \in \Gamma^+(c) \) such that \( cd \in \Gamma^-(b) \) and such that \( cdb \in \omega^+(b) \) and \( \omega^+_\infty(bcd) = \omega^+_\infty(b) \).
A topologically transitive subshift is called semisynchronizing if it has a semisynchronizing word. Semisynchronization is an invariant of topological conjugacy. We denote the set of semisynchronizing words of a semisynchronizing subshift $X$ by $\mathcal{L}_{a\text{-syn}}(X)$. For $a \in \mathcal{L}_{a\text{-syn}}(X)$ and $\sigma \in \Gamma_+^1(a)$ one has $a\sigma \in \mathcal{L}_{a\text{-syn}}(X)$. For a semisynchronizing subshift $X$ we set

$$\mathcal{V}_{a\text{-syn}}(X) = \{\Gamma_+^1(a) = a \in \mathcal{L}_{a\text{-syn}}(X)\},$$

For $V \in \mathcal{V}_{a\text{-syn}}(X)$ we denote by $\Gamma_+^1(V)$ the set of $\sigma \in \Sigma$ such that $V$ contains a right-infinite word that begins with $\sigma$, and we set

$$\Phi_{a\text{-syn}}(\sigma)(X) = \{x^+ \in \Gamma_+^1(\sigma) : \sigma x^+ \in V\}, \quad \sigma \in \Gamma_+^1(V)$$

The semisynchronizing Shannon graph

$$G_{a\text{-syn}}(X) = G_{\Sigma,\lambda^{vn}}(\mathcal{V}_{a\text{-syn}}(X), \mathcal{E}_{a\text{-syn}}(X))$$

of a semisynchronizing subshift $X$ is defined as the Shannon graph with vertex set $\mathcal{V}_{a\text{-syn}}(X)$, that has the transition function $\Phi(X)$. The $a$-synchronizing Shannon graph

$$G_{a\text{-syn}}(X) = G_{\Sigma,\lambda^{vn}}(\mathcal{V}_{a\text{-syn}}(X), \mathcal{E}_{a\text{-syn}}(X))$$

is defined similarly.

A subshift is said to be bipartite if its alphabet is the disjoint union of alphabets $\Delta$ and $\tilde{\Delta}$ such that its admissible words of length two are contained in $\Delta\tilde{\Delta} \cup \tilde{\Delta}\Delta$. For a bipartite subshift $\tilde{Y} \subset (\Delta \cup \tilde{\Delta})^\mathbb{Z}$ set

$$C_{\tilde{Y}}(\Delta) = \{\tilde{y} \in \tilde{Y} : \tilde{y}_0 \in \Delta\}, \quad C_{\tilde{Y}}(\tilde{\Delta}) = \{\tilde{y} \in \tilde{Y} : \tilde{y}_0 \in \tilde{\Delta}\}.$$ 

The sets $C_{\tilde{Y}}(\Delta)$ and $C_{\tilde{Y}}(\tilde{\Delta})$ are invariant under $S_{\tilde{Y}}^2$. Given alphabets $\Sigma$ and $\tilde{\Sigma}$ and bijections

$$\Psi : \Sigma \to \mathcal{L}_2(\tilde{Y}) \cap \Delta\tilde{\Delta}, \quad \tilde{\Psi} : \tilde{\Sigma} \to \mathcal{L}_2(\tilde{Y}) \cap \tilde{\Delta}\Delta,$$

denote the mapping that carries a point $\tilde{y} \in C_{\tilde{Y}}(\Delta)$ into the point

$$(\Psi^{-1}(\tilde{y}_{2i-1}\tilde{y}_{2i}))_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z},$$

by $\varphi$, denote the mapping that carries a point $\tilde{y} \in C_{\tilde{Y}}(\tilde{\Delta})$ into the point

$$(\tilde{\Psi}^{-1}(\tilde{\tilde{y}}_{2i-1}\tilde{y}_{2i}))_{i \in \mathbb{Z}} \in \tilde{\Sigma}^\mathbb{Z}$$

by $\tilde{\varphi}$, and set

$$Y = \varphi(C_{\tilde{Y}}(\Delta)), \quad \tilde{Y} = \tilde{\varphi}(C_{\tilde{Y}}(\tilde{\Delta})).$$

The mapping $\varphi^{-1}$ followed by the restriction of $S_{\tilde{Y}}$ to $C_{\tilde{Y}}(\Delta)$ and by the mapping $\tilde{\varphi}$ is a topological conjugacy of the subshift $Y$ onto the subshift $\tilde{Y}$. In the case of a semisynchronizing bipartite subshift $\tilde{Y}$ the Shannon graph $G_{a\text{-syn}}(\tilde{Y})$ sets up a bipartite relation between the Shannon graphs $G_{a\text{-syn}}(Y)$ and $G_{a\text{-syn}}(\tilde{Y})$. Similarly in the a-synchronizing case. By a theorem of Nasu [N] for topologically conjugate subshifts $Y \subset \Sigma^\mathbb{Z}, \tilde{Y} \subset \tilde{\Sigma}^\mathbb{Z}$ there exist subshifts $Y_k \subset \Sigma_k^\mathbb{Z}, 1 \leq k < K \in \mathbb{N}, Y_1 = Y, Y_K = \tilde{Y}$, such that $Y_k$ and $Y_{k+1}$ are bipartitely related, $1 \leq k < K$.

3. Directed graphs

Let $G(\mathcal{V}, \mathcal{E})$ be a countably infinite directed graph such that every vertex has a finite positive number of incoming edges and finite positive number of outgoing edges. Let $A \subset \mathcal{V}$ be finite. For $H \in \mathbb{N}$, and $r \geq H$, and for $V \in S_{\mathcal{V},A}(H + r)$ we denote by $\Pi_{\mathcal{V},A,H}(V)$ the set of directed paths of length $H$ in $G(\mathcal{V}, \mathcal{E})$ that start at $V$ and that approach $A$ directly. We say that $A$ is an $H$-reference set of $G(\mathcal{V}, \mathcal{E})$ if the following holds: There exists an $R \in \mathbb{N}$ such that for $r \geq R + H$ and for

$$G(\mathcal{C}, \mathcal{B}_C) \in \gamma(\mathcal{V} \setminus S_{\mathcal{V},A}(r - 1), S_{\mathcal{V},A}(r)),$$
one has that
\[ t(\Pi_{V,A,H}^g(V)) = t(\Pi_{V,A,H}^g(V')), \quad V, V' \in \mathcal{B}_C. \]

**Lemma 3.1.** Let \( H \in \mathbb{N}, \) and let \( A \subset \mathcal{V} \) be an \( H \)-reference set of \( G(\mathcal{V}, \mathcal{E}). \) Let \( A' \subset \mathcal{V} \) be finite. Set
\[ K = \min\{ K \in \mathbb{Z}_+ : \mathcal{S}_{V,A}(K) \supset A' \}. \]
Then there exists a \( Q > K \) such that
\[ \gamma(\mathcal{V} \setminus \mathcal{S}_{V,A}(q-1), \mathcal{S}_{V,A}^g(q)) \subset \bigcup_{q' > q-K} \gamma(\mathcal{V} \setminus \mathcal{S}_{V,A}(q'-1), \mathcal{S}_{V,A}^g(q')). \]

**Proof.** Let \( R \in \mathbb{N} \) be such that one has for \( r \geq R, \) and for
\[ G(\mathcal{C}_c, \mathcal{B}_c) \in \gamma(\mathcal{V} \setminus \mathcal{S}_{V,A}(r-1), \mathcal{S}_{V,A}^g(r)), \]
that
\[ t(\Pi_{V,A,H}^g(U)) = t(\Pi_{V,A,H}^g(V')), \quad U, W \in \mathcal{S}_{V,A}^g(\mathcal{B}_c, \mathcal{B}_c)(H). \]
Set
\[ D = \max \{ D_{A'}(V) : V \in \mathcal{S}_{V,A}(K + R) \}. \]
Also set
\[ Q = K + R + (D + 1)H + H + D. \]
One has
\[ (1) \quad D_{A'}(V) \leq D + (D + 1)H, \quad V \in \mathcal{S}_{V,A}^g(K + R + (D + 1)H). \]
Let
\[ V \in \mathcal{S}_{V,A}^g(K + R + (D + 1)H), \]
and let \((e_i)_{D_{A'}(V) \geq 1} \) be a directed path in \( G(\mathcal{V}, \mathcal{E}), \) such that
\[ s(e_{D_{A'}}) = V, \quad t(e_1) \in A', \]
that reaches \( A' \) directly from \( V. \) There is an \( I \in [1, D + 1] \) such that the path
\[ (e_i)_{D_{A'}(V) -(i-1)H \geq D_{A'}(V) - IH} \]
approaches \( A \) directly, for otherwise one would have that
\[ D_{A'}(V) \geq (D + 1)(H + 1), \]
which contradicts (1). In view of the assumptions on \( A \) and \( R \) we have the following consequence: For \( q \geq Q \) and for any given connected component w.b.
\[ G(\mathcal{C}_c, \mathcal{B}_c) \in \gamma(\mathcal{V} \setminus \mathcal{S}_{V,A}(q-1), \mathcal{S}_{V,A}^g(q)) \]
one has that
\[ t(\Pi_{V,A,q-K,R}^g(U)) = t(\Pi_{V,A,q-K,R}^g(W)), \quad U, W \in \mathcal{B}. \]
For
\[ q' = q - Q + (D + 2)H + D + \min_{V \in \mathcal{B}} \{ \min_{V' \in \mathcal{B}} \{ t(\Pi_{V,A,q-K,R}^g(V')) \} \}, \]
this means that
\[ (2) \quad D_{A'} = q', \quad V \in \mathcal{B}. \]
One has that
\[ (3) \quad q' \geq R + (D + 2)H + D = Q - K. \]
Moreover, for the connected component w.b.
\[ G(\mathcal{C}_c, \mathcal{B}_c) \in \gamma(\mathcal{V} \setminus \mathcal{S}_{V,A}(Q-1), \mathcal{S}_{V,A}^g(Q)), \]
such that \( \tilde{C} \subset \mathcal{C}, \) one has as a consequence that
\[ (4) \quad D_{A'}(V) = q' + q'', \quad V \in \mathcal{S}_{\tilde{C}, \mathcal{B}}^g(q' + q''), -D < q'' < \infty, \]
Lemma 3.2. Let the directed graphs \( \mathcal{G} \).

It follows from (3), (4), (5) and (6), that

\[
(6) \quad D_A(V) \leq D + (D + 1)H, \quad V \in S^{\circ}_{V, A}(K + R + (D + 1)H).
\]

It follows from (3), (4), (5) and (6), that

\[
C_B \in \gamma(V \setminus S_{V, A}(q' - 1), S^{\circ}_{V, A}(q')). \quad \square
\]

**Lemma 3.2.** Let the directed graphs \( G(V, E) \) and \( G(\overline{V}, \overline{E}) \) be bipartitely related by the bipartite graph \( G(V \cup \overline{V}, F \cup \overline{F}) \), and let \( R \subset V \) be a reference set for \( G(V, E) \). Then \( S^\circ_{V, \overline{V}, R}(1) \) is a reference set for \( G(\overline{V}, \overline{E}) \).

**Proof.** Let \( R \subset V \) be an \( H \)-reference set for \( G(V, E) \). Set

\[
\tilde{H} = H + 1.
\]

We prove that \( \tilde{R} = S^\circ_{V, \overline{V}, R}(1) \) is an \( \tilde{H} \)-reference set for \( G(\overline{V}, \overline{E}) \).

Let \( R \in \mathbb{N} \) be such that for \( r \geq R + H \) and for

\[
G(C, B_C) \in \gamma(V \setminus S_{V, A}(r - 1), S^\circ_{V, A}(r)),
\]

one has that

\[
t(P^\circ_{V, A, H}(\tilde{V})) = t(P^\circ_{V, A, H}(\tilde{V}')), \quad V, V' \in B_C.
\]

Set

\[
\tilde{r} > R + \tilde{H},
\]

let

\[
G(C, B_{\tilde{C}}) \in \gamma(V \setminus S_{V, \overline{R}}(\tilde{r} - 1), S^\circ_{V, A}(\tilde{r})),
\]

and let

\[
\tilde{V}, \tilde{V}' \in B_{\tilde{C}}.
\]

Let

\[
G(C, B_C) \in \gamma(V \setminus S_{V, A}(\tilde{r} - 2), S^\circ_{V, A}(\tilde{r} - 1)),
\]

be such that \( \tilde{C} \subset C \). One has that

\[
t(P^\circ_{V \cup \overline{V}, R, 1}(\tilde{V})), \quad t(P^\circ_{V \cup \overline{V}, R, 1}(\tilde{V}')) \subset B_{\tilde{C}},
\]

and it follows for

\[
V \in t(P^\circ_{V \cup \overline{V}, R, 1}(\tilde{V})), \quad V' \in t(P^\circ_{V \cup \overline{V}, R, 1}(\tilde{V}')),
\]

that

\[
t(P^\circ_{V \cup \overline{V}, \overline{R}, 2H}(\tilde{V})) = t(P^\circ_{V \cup \overline{V}, R, H}(\tilde{V}')) = t(P^\circ_{V \cup \overline{V}, R, 2H}(\tilde{V}')),
\]

and therefore

\[
t(P^\circ_{V \cup \overline{V}, R, 2H + 1}(\tilde{V})) = \bigcup_{V \in t(P^\circ_{V \cup \overline{V}, R, 2H}(\tilde{V}))} t(P^\circ_{V \cup \overline{V}, R, 2H}(\tilde{V}')) = \bigcup_{V' \in t(P^\circ_{V \cup \overline{V}, R, 2H + 1}(\tilde{V}'))} t(P^\circ_{V \cup \overline{V}, \overline{R}, 2H + 1}(\tilde{V}')).
\]
and then
\[ t(\Pi_{\tilde{V},\tilde{R},\tilde{H}}^{\tilde{V}}) = t(\Pi_{\tilde{V},\tilde{R},\tilde{H}}^\tilde{V}) = \bigcup_{V \in t(\Pi_{\tilde{V},\tilde{R},\tilde{H}}^{\tilde{V}})} t(\Pi_{\tilde{V},\tilde{R},\tilde{H}}^1(V)) = \bigcup_{V' \in t(\Pi_{\tilde{V},\tilde{R},\tilde{H}}^{\tilde{V}})} t(\Pi_{\tilde{V},\tilde{R},\tilde{H}}^1(V')) = t(\Pi_{\tilde{V},\tilde{R},H}^{\tilde{V}}(V')) = t(\Pi_{\tilde{V},\tilde{R},H}^{\tilde{V}}(\tilde{V}')) \]

We note that as a consequence of this lemma the smallest \( H \) such that a reference set is an \( H \)-reference set, is the same for all reference sets. It follows from the lemma that the existence of a reference set is an invariant of strong shift equivalence. We adopt the existence of a reference set as a standing hypothesis. For the various hypotheses that we will introduce for directed graphs with reference set, and that will always involve a reference set, it will follow from the lemma that the validity of the hypothesis does in fact not depend on the reference set.

We have two conditions \((DG1)\) and \((DG1)\) on a directed graph w.r.b. \( G_B(C) \):
\[(DG1)\quad \text{card}(C \setminus C^i) < \infty.\]
\[(DG2)\quad C^i \subset C^{i+1}.\]

We say that a directed graph w.r.s. \( G(V, E) \) satisfies hypothesis \((HDG1)\), \((HDG2)\) if for a reference set \( R \) of \( G(V, E) \), and therefore for all reference sets \( R \) of \( G(V, E) \), there is an \( R \in \mathbb{N} \), such that \((DG1)((DG2))\) holds for all connected components w.r.b. \( G(B, C) \in \gamma(V \setminus S_{\tilde{V},R}(R - 1), S_{\tilde{V},R}(R))\).

the hypotheses \((HDG1)\) and \((HDG2)\) are invariant under topological conjugacy.

4. Shannon graphs

We associate to a Shannon graph w.r.s. \( G_{\Sigma,\lambda}(V, E) \), a subshift \( Y(G_{\Sigma,\lambda}(V, E)) \), that has as its set of admissible words the set of words \( w \in \mathcal{X}(G_{\Sigma,\lambda}(V, E)) \) such that there is a reference set \( R \) and an \( R \in \mathbb{N} \) such that \( w \) appears in the label sequence of of a directed path in \( G_{S_{\tilde{V},R}(R)}(V \setminus S_{\tilde{V},R}(R - 1)) \), that approaches \( R \) directly from infinity. A bipartite Shannon graph that sets up a bipartite equivalence between Shannon graph w.r.s. \( G_{\Sigma,\lambda}(V, E) \) and \( G_{\Sigma,\lambda}(V, E) \), also yields a bipartite coding between \( Y(G_{\Sigma,\lambda}(V, E)) \) and \( Y(G_{\Sigma,\lambda}(V, E)) \).

We say that a Shannon graph w.r.s. \( G_{\Sigma,\lambda}(V, E) \) satisfies the coherence hypothesis if the following holds: For a reference set \( R \), there is an \( R \in \mathbb{N} \), such that one has for all infinite
\[ G_{\Sigma,\lambda}(V, E \cap S_{\tilde{V},R}(r)) \subset \gamma(V \setminus S_{\tilde{V},R}(r - 1), S_{\tilde{V},R}(r)), \quad r \geq R,\]
that every boundary automorphism \( \alpha \) of \( G_{\Sigma,\lambda}(V, E, V \cap S_{\tilde{V},R}(r)) \) leaves every element of \( \gamma(V \setminus V \cap S_{\tilde{V},R}(r), S_{\tilde{V},V \cap S_{\tilde{V},R}(r)}(1)) \) invariant and every vertex in \( V \cap S_{\tilde{V},R}(r) \) fixed.

We say that a Shannon graph w.r.s. \( G_{\Sigma,\lambda}(V, E) \) satisfies the visibility hypothesis if the following holds: For \( K \in \mathbb{N} \) there is an \( M > 2K \) such that for a reference set \( R \) of \( G_{\Sigma,\lambda}(V, E) \) there is an \( R \in \mathbb{N} \), such that one has for a directed path \((e_i)_{M \geq 2K} \) in \( G_{S_{\tilde{V},R}(R)}(V \setminus S_{\tilde{V},R}(R - 1)) \) that is labelled by a word in \( C(Y(G_{\Sigma,\lambda}(V, E))) \), that the directed path \((e_i)_{M \geq 2K} \) approaches \( R \) directly. For Shannon graphs w.r.s. \( G_{\Sigma,\lambda}(V, E) \) and \( G_{\Sigma,\lambda}(V, E) \), that are bipartitely related, it can be shown by bipartite coding that \( G_{\Sigma,\lambda}(V, E) \) satisfies the visibility hypothesis if and only if \( G_{\Sigma,\lambda}(V, E) \) satisfies the visibility hypothesis.
We say that a Shannon graph w.r.s. \( G_{\Sigma, \lambda}(V, \mathcal{E}) \) satisfies the discordance hypothesis if the following holds: for \( I, M \in \mathbb{N} \), there is a \( K \in \mathbb{N} \), such that for a reference set \( R \) of \( G_{\Sigma, \lambda}(V, \mathcal{E}) \) there is an \( \tilde{R} \in \mathbb{R} \), such that one has for a directed path \( (e_i)_{i+K \geq 1} \) in \( G_{\tilde{V}, \tilde{R}}(V \setminus \mathcal{S}_{\tilde{V}, \tilde{R}}(R-1)) \), such that \( s(e_{i+K}) \) is accessible from infinity, and such that the directed path \( (e_i)_{i \geq 1} \) approaches \( R \) directly, that for the word \( v \), that labels the directed path \( (e_i)_{i+K \geq 1} \) and the word \( w \) that labels the directed path \( (e_i)_{i \geq 1} \), there is a word \( u \in \mathcal{L}(G_{\Sigma, \lambda}(V, \mathcal{E})) \), such that \( \ell(u) > M \), and

\[
  uv \in \mathcal{L}(G_{\Sigma, \lambda}(V, \mathcal{E})), \quad uuv \notin \mathcal{L}(G_{\Sigma, \lambda}(V, \mathcal{E})).
\]

For Shannon graphs w.r.s. \( G_{\Sigma, \lambda}(V, \mathcal{E}) \) and \( \tilde{G}_{\Sigma, \lambda}(\tilde{V}, \tilde{\mathcal{E}}) \), that are bipartitely related, it can be shown by bipartite coding that \( G_{\Sigma, \lambda}(V, \mathcal{E}) \) satisfies the discordance hypothesis if and only if \( \tilde{G}_{\Sigma, \lambda}(\tilde{V}, \tilde{\mathcal{E}}) \) satisfies the discordance hypothesis.

We say that a Shannon graph w.r.s. \( G_{\Sigma, \lambda}(V, \mathcal{E}) \) satisfies the resolution hypothesis, if for a reference set \( \mathcal{R} \) of \( G_{\Sigma, \lambda}(V, \mathcal{E}) \) there is an \( \mathcal{R} \in \mathbb{R} \), such that any two infinite paths in \( G_{\Sigma, \lambda}(V, \mathcal{E}) \) that approach a common target vertex from infinity, and that carry the same label sequence, are identical. For Shannon graphs w.r.s. \( G_{\Sigma, \lambda}(V, \mathcal{E}) \) and \( \tilde{G}_{\Sigma, \lambda}(\tilde{V}, \tilde{\mathcal{E}}) \), that are bipartitely related, it can be shown by bipartite coding that \( G_{\Sigma, \lambda}(V, \mathcal{E}) \) satisfies the resolution hypothesis if and only if \( \tilde{G}_{\Sigma, \lambda}(\tilde{V}, \tilde{\mathcal{E}}) \) satisfies the resolution hypothesis.

5. \((C - F)\)-semisynchronizing shifts

We say, that a semi-synchronizing, non-synchronizing subshift \( X \subset \Sigma^2 \) is a \((C - F)\)-semi-synchronizing shift, if its semisynchronizing Shannon graph has a reference set, and if \( \hat{\gamma}(X) \) is finite. We view the elements of \( \hat{\gamma}(X) \) as canonical models that represent boundary isomorphism types, and we use them for the notation \( G_{\Sigma, \lambda}(\hat{C}, \mathcal{B}_{\hat{C}}) \). Note that the \( G_{\Sigma, \lambda}(\hat{C}, \mathcal{B}_{\hat{C}}) \) can be taken as finite structures.

For a word \( W \in \mathcal{L}(X) \) that begins with \( \sigma \in \Sigma \) we denote by \( \Phi_\sigma(W) \) the word that is obtained from \( W \) by removing the initial \( \sigma \). Denote the empty word by \( \epsilon \), and for \( \mathcal{L} \subset (\Sigma \cup \{\epsilon\}) \cup \emptyset \), and \( \sigma \in \Sigma \), denote by \( \mathcal{L}_\sigma \) the set of words in \( \mathcal{L} \) beginning with \( \sigma \), and set

\[
  \Phi_\sigma(\mathcal{L}) = \mathcal{L}_\sigma.
\]

With the convention that \( \Theta_\sigma(V) = V, V \in \mathcal{B}_C \), we denote for \( G_{\Sigma, \lambda}(C, \mathcal{B}_C) \) and \( U \in \mathcal{C} \) and \( V \in \mathcal{B}_C \) by \( \mathcal{L}[U, V] \) the set of words \( z \in \mathcal{L}(X) \cup \{\epsilon\} \) such that \( \Theta_\sigma(U) = V \).

The structure \( \mathcal{L}[U, V] \) for \( U \in \mathcal{B}_C \) we refer to as the data of the vertex \( U \).

**Lemma 5.1.** Let \( X \subset \Sigma^2 \) be a \((C - F)\)-semisynchronizing shift, and let \( G_{\Sigma, \lambda}(\hat{C}, \mathcal{B}_{\hat{C}}) \in \hat{\gamma}(X) \). Then

(a) Vertices in \( \hat{C} \), that have the same data, are identical.

(b) For \( \hat{U} \in \hat{C} \), \( \sigma \in \Sigma(\hat{U}) \) and \( \hat{W} \in \hat{C} \), one has that \( \Phi_\sigma(\hat{U}) = \hat{W} \), if and only if

\[
  \mathcal{L}[\hat{W}, \hat{V}] = \Phi_\sigma(\mathcal{L}[\hat{U}, \hat{V}]), \quad \hat{V} \in \mathcal{B}_{\hat{C}}.
\]

**Proof.** Let \( \mathcal{R} \) be a reference set of \( G_{s\text{-syn}}(X) \), let \( r \in \mathbb{N} \), and let

\[
  G_{\Sigma, \lambda}(\mathcal{C}, \mathcal{B}_C) \in \gamma(V, \mathcal{B}_C) \setminus \mathcal{S}_{V, \mathcal{B}_C}(X, \mathcal{R}(r-1)), \mathcal{S}_{\mathcal{B}_C}(X, \mathcal{R}(r)).
\]

The Shannon graph \( G_{s\text{-syn}}(X) \) is forward separated, which means that vertices \( V \) and \( V' \) in \( \mathcal{C} \) are identical, if \( \Gamma^+(V) = \Gamma^+(V') \), which is the case if and only if the data of \( V \) and \( V' \) are the same. This proves (a). (b) follows from (a). □
Lemma 5.2. For a \((C - F)\)-semisynchronizing shift \(X \subset \Sigma^Z\) the boundary isomorphism groups of the Shannon graphs w.r.t. \(G_{\hat{\gamma}}(C, \mathcal{D}) \in \hat{\gamma}(X)\) \(G_{\Sigma, \lambda}(\tilde{C}, \mathcal{B}_C) \in \hat{\gamma}(X)\) are finite.

Proof. Let \(\mathcal{R}\) be a reference set of \(G_{s-syn}(X)\), let \(r \in \mathbb{N}\), and let

\[
G_{\Sigma, \lambda}(\mathcal{C}, B_C) \in \gamma(V_{s-syn}(X) \setminus S_{V_{s-syn}(X), \mathcal{R}(r - 1)}, S_{V_{s-syn}(X), \mathcal{R}(r)}).
\]

Let \(\zeta\) be a boundary isomorphism of \(G_{\Sigma, \lambda}(\mathcal{C}, B_C)\) that leaves every vertex in \(B_C\) fixed. Then for all \(U \in \mathcal{C}\),

\[
\mathcal{L}[\zeta(U), V] = \mathcal{L}[\zeta(U), \zeta(V)] = \mathcal{L}[U, V], \quad V \in B_C,
\]

and it follows by Lemma (5.1), that \(\zeta = \text{id}\). This implies that every boundary isomorphism of \(G_{\Sigma, \lambda}(\mathcal{C}, B_C)\) is determined by its restriction to the boundary, and as a consequence every boundary isomorphism of \(G_{\Sigma, \lambda}(\tilde{C}, B_C)\) is also determined by its restriction to the boundary.

\(\square\)

Theorem 5.3. The class of \((C - F)\)-semisynchronizing shifts is closed under topological conjugacy.

Proof. Consider the situation, that there is given a \((C - F)\)-semisynchronizing shift \(X \subset \Sigma^Z\) that is bipartitely related to a subshift \(\tilde{X} \subset \tilde{\Sigma}^Z\), and let the bipartite relation between \(X\) and \(\tilde{X}\) be implemented by a bipartite subshift \(\tilde{X} \subset (\Delta \cup \Delta)^\tilde{Z}\) with the bijections

\[
\Psi : \mathcal{E} \to \Pi^{(2)}(G(V \cup \tilde{V}, F \cup \tilde{F})), \quad \tilde{\Psi} : \tilde{\mathcal{E}} \to \tilde{\Pi}^{(2)}(G(V \cup \tilde{V}, F \cup \tilde{F})),
\]

\[
\lambda : \Sigma \to \tilde{\lambda}(\Pi^{(2)}(G(V \cup \tilde{V}, F \cup \tilde{F}))), \quad \tilde{\lambda} : \tilde{\Sigma} \to \tilde{\lambda}(\tilde{\Pi}^{(2)}(G(V \cup \tilde{V}, F \cup \tilde{F}))).
\]

We prove that \(\tilde{X}\) is also \((C - F)\)-semisynchronizing. Set

\[
(1) \quad J = \max_{G_{\Sigma, \lambda}(\mathcal{C}, B_C) \in \hat{\gamma}(X)} \max_{V \in B_C, \sigma \in \Sigma(V)} \Delta_{\mathcal{B}_C}(\Psi_{\sigma}(V)).
\]

Let \(\mathcal{R}\) be a reference set for \(G_{s-syn}(X)\). Then \(\tilde{X} = \Psi(\mathcal{R})\) is a reference set for \(\tilde{X}\), and a reference set \(\tilde{\mathcal{R}}\) for \(\tilde{X}\) is given by

\[
S_{V_{s-syn}(\tilde{X}), \tilde{\mathcal{R}}}(1) = \tilde{\Psi}(\tilde{\mathcal{R}}).
\]

Let \(G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}) \in \hat{\gamma}(X)\), and choose for some \(r \in \mathbb{N}\), and for some

\[
G_{\Sigma, \lambda}(\mathcal{C}, B_C) \in \gamma(V_{s-syn}(X) \setminus S_{V_{s-syn}(X), \mathcal{R}(r)}, S_{V_{s-syn}(X), \mathcal{R}(r - 1)}),
\]

such that

\[
G_{\Sigma, \lambda}(\mathcal{C}, B_C) \sim G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}),
\]

a boundary isomorphism \(\beta\) of \(G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}})\) onto \(G_{\Sigma, \lambda}(\mathcal{C}, B_C)\). Set

\[
\tilde{C}^\sim(k) = \{\tilde{V} \in S_{V_{s-syn}(\tilde{X}), \tilde{\mathcal{R}}}(2r + 2k + 1) : \Delta_{\mathcal{B}_{\tilde{C}}}(\tilde{V}) = 2k + 1\}, \quad k \in \mathbb{Z}_+, \quad \tilde{C}^\sim = \bigcup_{k \in \mathbb{Z}_+} \tilde{C}^\sim(k), \quad B_{\tilde{C}} = \Psi(B_{\tilde{C}}).
\]

One has that

\[
G_{\Delta \cup \Delta}(\tilde{C}, B_{\tilde{C}}) \in \gamma(V_{s-syn}(\tilde{X}) \setminus S_{V_{s-syn}(\tilde{X}), \tilde{\mathcal{R}}}(2r - 1), S_{V_{s-syn}(\tilde{X}), \tilde{\mathcal{R}}}(2r)).
\]

We set

\[
\tilde{C}^\sim = \bigcup_{k \geq J} \tilde{C}^\sim(k), \quad \tilde{B}^\sim = \tilde{C}^\sim(J).
\]

The definition of \(J\) in (1) is such that

\[
(2) \quad \{\Phi_{\sigma}(\sigma)(V) : V \in \mathcal{C}, \sigma \in \Sigma(V)\} \supset \tilde{C}^\sim.
\]
For the Shannon graph w.b. \( G_{\Delta \cup \Delta} (\mathcal{C}, \mathcal{B}^-) \) it follows from Lemma (5.1) that \( G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \) has a collection of data \(((\mathcal{L}[V, U])_{V \in \mathcal{C}})_{V \in \mathcal{C}}\) from which one can through \( \Psi \) obtain the collection of data \(((\mathcal{L}[V, U])_{\bar{V}, \bar{U})_{\bar{V} \in \mathcal{C}}\} of \ G_{\Delta \cup \Delta} (\mathcal{C}, \mathcal{B}^-) \). Moreover, also by Lemma (5.1), the vertices \( \hat{V} \in \hat{C}^- (k), k \geq J \), can be identified by means of their data. It follows that \( G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \) together with the boundary isomorphism \( \beta \) determines the boundary isomorphism class of \( G_{\Delta \cup \Delta} (\mathcal{C}, \mathcal{B}^-) \), and for \( \hat{C} \) and \( \mathcal{B}^- \) given by

\[
\Psi (\hat{C}) = \hat{C}^-, \quad \Psi (\mathcal{B}^-) = \mathcal{B}^-,
\]

it follows then, that the boundary isomorphism class of \( G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \) is also determined by \( G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \) together with the boundary isomorphism \( \beta \). We denote the subset of

\[
\bigcup_{r \in \mathbb{N}} \gamma (V_{s \cdot \text{syn}} (\hat{X}) \setminus S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J - 1), S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J)) \subset
gamma (V_{s \cdot \text{syn}} (\hat{X}) \setminus S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J - 1), S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J)).
\]

that contains the boundary isomorphism types of the connected components of the \( G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \), that are constructed in this way from \( G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \), by \( \epsilon (G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-)) \). The set \( \epsilon (G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-)) \) is finite, since by Lemma (5.2) there are finitely many choices for \( \beta \).

We complete the proof by showing that

\[
\bigcup_{r \in \mathbb{N}} \gamma (V_{s \cdot \text{syn}} (\hat{X}) \setminus S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J - 1), S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J)) \subset
gamma (V_{s \cdot \text{syn}} (\hat{X}) \setminus S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J - 1), S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J)).
\]

Let \( r \in \mathbb{N} \), and let

\[
G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) = G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \in
\gamma (V_{s \cdot \text{syn}} (\hat{X}) \setminus S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J - 1), S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r + J)).
\]

Then let

\[
G_{\Delta \cup \Delta} (\mathcal{C}, \mathcal{B}^-) \in \gamma (V_{s \cdot \text{syn}} (\hat{X}) \setminus S_{V_{s \cdot \text{syn}}, \mathcal{R}} (2r - 1), S_{V_{s \cdot \text{syn}}, \mathcal{R}} (2r))
\]

such that

\[
\hat{C} \supset \Psi (\mathcal{B}^-),
\]

and let

\[
G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \in \gamma (V_{s \cdot \text{syn}} (\hat{X}) \setminus S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r - 1), S_{V_{s \cdot \text{syn}}, \mathcal{R}} (r)),
\]

such that

\[
\Psi (\mathcal{B}^+) = \mathcal{B}^+.
\]

One has for

\[
G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \in \gamma (X)
\]

such that

\[
G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \in \gamma (G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-))
\]

that

\[
G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-) \in \gamma (G_{\Sigma, \lambda} (\mathcal{C}, \mathcal{B}^-)).
\]

\[\Box\]
Lemma 5.4. Let $\alpha$ be a $(C-F)$-semisynchronizing shift that satisfies (HDG1) and the coherence hypothesis. Let $G_{\Sigma,\lambda}(\hat{C}, B_{\hat{C}}) \in \hat{\gamma}^4(X)$, and let $\alpha$ be a boundary automorphism of $G_{\Sigma,\lambda}(\hat{C}, B_{\hat{C}})$. Then $\alpha(\hat{V}) = \hat{V}$, $\hat{V} \in B_{\hat{C}}$.

Proof. For $\hat{V} \in B_{\hat{C}}$ there exists by (HDG1) a $k \in \mathbb{N}$, a $\hat{W} \in B_{\hat{C}}^k$, and a word $c \in L(X)$, such that $\Theta_c(\hat{W}) = \hat{V}$. By the coherence hypothesis $\alpha$ leaves $\hat{W}$ fixed, and it follows, that $\alpha(\hat{V}) = \Theta_c(\alpha(W)) = \Theta_c(W) = \hat{V}$. □
We introduce a directed graph $G(\hat{\delta}^+(X), \hat{\delta}^+(X))$, with the edge set $\hat{\delta}^+(X)$ given by
\[
\{d \in \hat{\delta}^+(X) : s(d) = G_{\Sigma, \hat{\lambda}}(\hat{C}_d, B_{\hat{C}_d}), t(d) = G_{\Sigma, \hat{\lambda}}(\hat{C}_d, B_{\hat{C}_d})\} = \\
\{G_{\Sigma, \lambda}(\hat{C}_d, B_{\hat{C}_d}) \in \gamma(\hat{C}_d \setminus B_{\hat{C}_d}, S_{\hat{C}_d}^\beta, S_{\hat{C}_d}^\beta(1)) : G_{\Sigma, \lambda}(\hat{C}_d, B_{\hat{C}_d}) \sim G_{\Sigma, \lambda}(\hat{C}_d, B_{\hat{C}_d})\},
\]

By Lemma (5.4) this description of $G(\hat{\delta}^+(X), \hat{\delta}^+(X))$ is meaningful.

Given a $G_{\Sigma, \lambda}(\hat{C}, B_{\hat{C}}) \in \hat{\delta}^+(X)$, and $d \in \hat{\delta}^+(X)$, such that $t(d) = G_{\Sigma, \lambda}(\hat{C}, B_{\hat{C}})$, we set
\[
\Gamma^+_d(\hat{V}, \hat{d}) = \Gamma^+_d(\beta[G_{\Sigma, \lambda}(\hat{C}, B_{\hat{C}}), \hat{d}(\hat{V}))).
\]

Given a $(C - F)$-semisynchronizing shift $X \subset \Sigma^\omega$, that satisfies (HDG1) and the coherence hypothesis, we say that $X$ satisfies hypothesis (h) if it holds for $G_{\Sigma, \lambda}(\hat{C}, B_{\hat{C}}) \in \hat{\delta}^+(X), V \in B_{\hat{C}}$, and for $d, d' \in \hat{\delta}^+(X), d \neq d'$, such that
\[
t(d) = t(d') = G_{\Sigma, \lambda}(\hat{C}, B_{\hat{C}}),
\]

that
\[
\Gamma^+_d(\hat{V}, \hat{d}) \cap \Gamma^+_d(\hat{V}, \hat{d}') = \emptyset.
\]

We consider a $(C - F)$-semisynchronizing subshift $X \subset \Sigma^\omega$, that satisfies (HDG1) and (HDG2) and the coherence, the visibility and the projection hypotheses, and also hypothesis (h). Using the data that are provided by $\hat{\delta}^+(X)$ and by hypothesis (h), we construct a pda $G_{\Sigma, \lambda}(V, E, B)(X)$, that we call the automaton of $X$. Under the assumption, that $X$ also satisfies the discordance hypothesis the automaton of $X$ will be shown to be isomorphic to the semisynchronizing Shannon graph of $X$.

For $G_{\Sigma, \lambda}(\hat{C}, B_{\hat{C}}) \in \hat{\delta}^+(X)$ and $\hat{V} \in \hat{C}_1$, and for $I \in [1, \Delta_{B_{\hat{C}_1}}(\hat{V})]$ we denote by $G_{\Sigma, \lambda}(\hat{C}(V, I), B_{\hat{C}(V, I)})$ the element of $\gamma(\hat{C}_1 \setminus S_{\hat{C}_1} \setminus (I-1), S_{\hat{C}_1}^\beta \setminus (I))$ that contains $\hat{V}$. We also denote for $G_{\Sigma, \lambda}(\hat{C}, B_{\hat{C}}) \in \hat{\delta}^+(X)$ the set of $\hat{V} \in \hat{C}_1$ that are not accessible from infinity in $G_{\Sigma, \lambda}(\hat{C}(V, 1), B_{\hat{C}(V, 1)})$ by $\hat{C}_1^\beta$. By HDG2 $\hat{C}_1^\beta$ is a finite set. We assign to $\hat{V} \in \hat{C}_1$ an $I(\hat{V}) \in [0, \Delta_{B_{\hat{C}_1}}(\hat{V})]$. If $\hat{V} \in B_{\hat{C}_1} \cup \hat{C}_1^\beta$, we set $I(\hat{V}) = 0$, and otherwise we set $I(\hat{V})$ equal to the maximal $I \in [1, \Delta_{B_{\hat{C}_1}}(\hat{V})]$, such that $\hat{V}$ is accessible from infinity in $G_{\Sigma, \lambda}(\hat{C}(V, I), B_{\hat{C}(V, I)})$. Let $G_{\Sigma, \lambda}(\hat{C}(V, I), B_{\hat{C}(V, I)}) \in \hat{\delta}^+(X)$, such that $G_{\Sigma, \lambda}(\hat{C}(V, I), B_{\hat{C}(V, I)}) \sim G_{\Sigma, \lambda}(\hat{C}(V, I), B_{\hat{C}(V, I)}), 1 \leq I \leq I(\hat{V})$. We then assign to $\hat{V} \in \hat{C}_1 \setminus (B_{\hat{C}_1} \cup \hat{C}_1^\beta)$ the path $b(\hat{V})$ of length $I(\hat{V})$ in $G(\hat{\delta}^+(X), \hat{\delta}^+(X))$, that is given by
\[
b_1(\hat{V}) = G_{\Sigma, \lambda}(\hat{C}(V, 1), B_{\hat{C}(V, 1)}),
\]
and if $I(\hat{V}) > 1$, also by
\[
b_I(\hat{V}) = \\
\beta[G_{\Sigma, \lambda}(\hat{C}(V, I-1), B_{\hat{C}(V, I-1)}), G_{\Sigma, \lambda}(\hat{C}(V, I), B_{\hat{C}(V, I)})(G_{\Sigma, \lambda}(\hat{C}(V, I), B_{\hat{C}(V, I)})), 1 \leq I \leq I(\hat{V}).
\]
We also set
\[
b(\hat{V}) = 1, \quad \hat{V} \in B_{\hat{C}_1} \cup \hat{C}_1^\beta.
\]

To obtain the automaton of $X$ we first construct an auxiliary Shannon graph $G_{\Sigma, \lambda}(V, E, B)(X)$. We set
\[
\hat{B} = \bigcup_{G_{\Sigma, \lambda}(\hat{C}, B_{\hat{C}}) \in \hat{\delta}^+(X)} B_{\hat{C}}.
\]
With the notation $\epsilon$ or the empty path and with $G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}) \in \tilde{V} \in B_{\tilde{C}}$, we write the vertices in $B$ as triples $(\epsilon, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V})$. We write $V \setminus B$ as a disjoint union, 
\[ \tilde{V} \setminus B = V_0 \cup \tilde{V}_{+, 0} \cup \tilde{V}_{+, +}. \]

We take for $\tilde{V}_0$ the set of triples $(\tilde{a}, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V})$ with 
\[ (\tilde{a}, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}})) \in \Pi(\tilde{\gamma}^{-\epsilon}(X), \tilde{\gamma}^{\epsilon}(X)) \]
such that 
\[ t(\tilde{a}) = G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \]
and with 
\[ \tilde{V} \in B_{\tilde{C}}. \]

For $\tilde{V}_{+, 0}$ we take the set of triples $(\epsilon, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V})$ with 
\[ G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}) \in \tilde{\gamma}^{\epsilon}(X), \]
and with 
\[ \tilde{V} \in B_{\tilde{C}} \setminus \tilde{B}_{\tilde{C}}, \]
and for $\tilde{V}_{+, +}$ we take the set of triples $(\tilde{a}, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V})$ with 
\[ (\tilde{a}, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}})) \in \Pi(\tilde{\gamma}^{-\epsilon}(X), \tilde{\gamma}^{\epsilon}(X)) \]
such that 
\[ t(\tilde{a}) = G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \]
and with 
\[ \tilde{V} \in B_{\tilde{C}} \setminus \tilde{B}_{\tilde{C}}. \]

We denote the acceptance sets of $G_{\Sigma, \lambda}(\tilde{V}, \tilde{B}, \tilde{B})(X)$ by $\Gamma^+_1$. We set 
\[ \Gamma^+_1((d_k)_{1 \leq k \leq K}, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V}) = \Gamma^+_1(\tilde{V}), \]
\[ (\epsilon, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V}) \in \tilde{V}_{+, +}, \]
and 
\[ \Gamma^+_1((d_k)_{1 \leq k \leq K}, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V}) = \Gamma^+_1(\tilde{V}) \cup \Gamma^+_{1, -}(d_K, \tilde{V}), \]
\[ (\epsilon, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V}) \in \tilde{V}_{+, 0}, \]
and 
\[ \Gamma^+_1((d_k)_{1 \leq k \leq K}, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V}) = \Gamma^+_1(\tilde{V}) \cup \bigcup_{d \in \tilde{\delta}^{-\epsilon}(X), t(d) = G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}})} \Gamma^+_{1, -}(d, \tilde{V}), \]
\[ (\epsilon, G_{\Sigma, \lambda}(\tilde{C}, B_{\tilde{C}}), \tilde{V}) \in \tilde{B}. \]

We complete the description of $G_{\Sigma, \lambda}(\tilde{V}, \tilde{B}, \tilde{B})$ by specifying its transition function, that we denote by $\tilde{\Theta}$. For $G_{\Sigma, \lambda}(\tilde{C}_{\tilde{b}}, B_{\tilde{C}_{\tilde{b}}}) \in \tilde{V}_0$ we let $G_{\Sigma, \lambda}(\tilde{C}_{\tilde{b}}, B_{\tilde{C}_{\tilde{b}}}) \in \tilde{\gamma}^{\pm}(X)$ be given by, 
\[ G_{\Sigma, \lambda}(\tilde{C}_{\tilde{b}}, B_{\tilde{C}_{\tilde{b}}}) \simeq G_{\Sigma, \lambda}(\tilde{C}_{\tilde{b}}, \tilde{B}_{\tilde{C}_{\tilde{b}}}, \tilde{V}_0), \]
and we set 
\[ \tilde{V}_1 = \beta(G_{\Sigma, \lambda}(\tilde{C}_{\tilde{b}}, \tilde{B}_{\tilde{C}_{\tilde{b}}}, \tilde{V}_0), G_{\Sigma, \lambda}(\tilde{C}_{\tilde{b}}, \tilde{B}_{\tilde{C}_{\tilde{b}}}, \tilde{V}_1)), \]
and we set 
(I) \[ \tilde{\Theta}(\epsilon, G_{\Sigma, \lambda}(\tilde{C}_{\tilde{b}}, B_{\tilde{C}_{\tilde{b}}}, \tilde{V}_0)) = (b(\tilde{V}_0), G_{\Sigma, \lambda}(\tilde{C}_{\tilde{b}}, B_{\tilde{C}_{\tilde{b}}}, \tilde{V}_1)), \]
\[ (\epsilon, G_{\Sigma, \lambda}(\tilde{C}_{\tilde{b}}, B_{\tilde{C}_{\tilde{b}}}, \tilde{V}_0)) \in \tilde{V}_0 \cup \tilde{V}_{+, 0}, \sigma \in \Gamma^+_{1, \pm}(\tilde{V}_0), \]
(II) \( \Theta_{\sigma}(\hat{d}, G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0) = (\hat{d}h(\hat{V}_0), G_{\Sigma,\hat{\lambda}}(\hat{C}_1^i, B_{\hat{c}_i}), \hat{V}_1) \),
\( (\epsilon, G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0) \in \tilde{V}_{++}, \sigma \in \Gamma^+_1(\hat{V}_0) \).

(III) \( \Theta_{\sigma}(\epsilon, G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0) = (\epsilon, s(\hat{d}), \Theta_{\sigma}(\beta^{-1}[\hat{d}, G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i})](\hat{V})) \),
\( (\epsilon, G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0) \in \tilde{B} \cup \tilde{V}_0, \hat{d} \in \bar{\delta}^i(X), t(\hat{d}) = G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \sigma \in \Gamma^+_1(\hat{d}, \hat{V}) \).

(IV) \( \Theta_{\sigma}((d_k)_{1 \leq k \leq K}, G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0) = 
((d_k)_{1 \leq k \leq K}, \Theta_{\sigma}(\beta^{-1}[d_K, G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i})](\hat{V})) \),
\( ((d_k)_{1 \leq k \leq K}, G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0) \in \tilde{V}_0, \sigma \in \Gamma^+_1(\hat{d}_K, \hat{V}). \)

The coherence hypothesis assures, that the last component of the target vertex in the transitions I - IV is correctly identified as a vertex of the canonical model, that is the second component if the target vertex.

It follows from the defining property of a reference set and from (HDG1) that \( G_{\Sigma,\lambda}(\tilde{V}, \tilde{E}, \tilde{\delta})(X) \) has an irreducible component, that contains \( \tilde{B} \). We call this irreducible component the automaton of \( X \) and we denote it by \( G_{\Sigma,\hat{\lambda}}(\tilde{V}, \tilde{E}, \tilde{\delta})(X) \) (\( \tilde{V} = \tilde{B} \)). We denote the transition function of \( G_{\Sigma,\hat{\lambda}}(\tilde{V}, \tilde{E}, \tilde{\delta})(X) \) by \( \Theta \).

We introduce at this point a labelled directed graph \( G_{\Sigma,\hat{\lambda}}(\tilde{V}_*, \tilde{E}_*)(X) \). The vertex set \( \tilde{V}_* \) contains the pairs \( (G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}) \), with \( G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}) \in \tilde{\gamma}^i(X) \) and \( \hat{V} \in \tilde{B}_i \cup \tilde{C}_i^- \). We denote the acceptance sets of \( G_{\Sigma,\hat{\lambda}}(\tilde{V}_*, \tilde{E}_*)(X) \) by \( \tilde{T}_* \), and we set
\( \tilde{T}_1^+(((G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i, B_{\hat{c}_i}), \hat{V})) = \Gamma_1^+(\epsilon, G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i, B_{\hat{c}_i}), \hat{V})), \)
\( (G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}) \in \tilde{V}_* \).

For \( \sigma \in \Sigma \) and \( (G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0), (G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_1) \in \tilde{V}_* \) there is an edge \( e_* \in \tilde{E}_*(X) \), such that
\( s(e_*) = (G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0), t(e_*) = (G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_1), \)
that carries the label \( \sigma \) precisely, if either \( \epsilon \in \Gamma_1^+(\hat{V}_0) \), or \( \hat{V}_0 \in \tilde{B}_i^+ \), and there is a \( \hat{d} \in \tilde{\gamma}^i(X) \), such that \( t(\hat{d}) = (G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0), (G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_1) \) is obtained from \( \Theta_{\sigma}(\epsilon, G_{\Sigma,\hat{\lambda}}(\hat{C}_0^i, B_{\hat{c}_i}), \hat{V}_0) \) by deleting the first component. We say that \( X \) (more precisely \( G_{\delta-syn}(X) \)) satisfies the resolution hypothesis, if the reversed graph of \( G_{\Sigma,\hat{\lambda}}(\tilde{V}_*, \tilde{E}_*)(X) \) is Shannon. (To ask that \( G_{\Sigma,\hat{\lambda}}(\tilde{V}_*, \tilde{E}_*)(X) \) itself is Shannon is equivalent to asking that hypothesis \( h \) is satisfied.)

Given a \( (C - F) \)-semisynchronizing shift \( X \in \Sigma^\mathbb{N} \), we say that a word \( c \in X_{[-M,M]} \in \mathcal{L}(X) \) is a resolving word for \( G_{\delta-syn}(X) \), if there is an
\( (\epsilon, G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i(c), B_{\hat{c}_i(c)}), \hat{V}(c)) \in \tilde{B}(X), \)
such that the following holds for \( G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i, B_{\hat{c}_i}) \in \tilde{\gamma}^i(X) \): If \( (\hat{c}_m)_{-M \leq m \leq M} \) is a path in \( G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i, B_{\hat{c}_i}) \), that carries the label sequence \( c \), then
\( G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i(\hat{V}, \Delta_{\hat{c}_i(c)}(\hat{V})), B_{\hat{c}_i(c)}(\hat{V}, \Delta_{\hat{c}_i(c)}(\hat{V}))), G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i(c), B_{\hat{c}_i(c)})) = G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i(c), B_{\hat{c}_i(c)})), \)
and
\( \hat{V}(c) = \hat{\beta}[G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i(\hat{V}, \Delta_{\hat{c}_i(c)}(\hat{V})), B_{\hat{c}_i(c)}(\hat{V}, \Delta_{\hat{c}_i(c)}(\hat{V}))), G_{\Sigma,\hat{\lambda}}(\hat{C}_i^i(c), B_{\hat{c}_i(c)}))](s(\hat{e}_0), \)
and in this case we say, that the vertex \((e, G_{\Sigma,\lambda}(\mathcal{C}^i(c), \mathcal{B}_{\mathcal{C}^i}(c)), \hat{V}(c))\) is assigned to the word \(c\).

**Theorem 5.5.** The semisynchronizing Shannon graph of a \((C,F)\)-semisynchronizing shift \(X \subseteq \Sigma^\mathbb{Z}\), that satisfies (HDG1) and (HDG2), and the coherence, visibility, discordance and projection hypotheses, and hypothesis \((h)_1\), is isomorphic to the automaton of \(X\).

**Proof.** By the visibility hypothesis and by the projection hypothesis there is for some \(M \in \mathbb{N}\) a resolving word \(c \in X_{[-M,M]} \in L(X)\) for \(X\). Denote by \(V(c)\) the vertex in \(\mathcal{B}(X)\), that is assigned to \(c\). For a transitive \(x^- \in X_{-\infty,-M}\) set

\[ I(x^-) = \{ i \in (-\infty,M] : x^-_{i-M,i+M} = c \}, \]

enumerate

\[ I(x^-) = \{ i-k(x^-) : k \in \mathbb{N} \}, \]

\[ i-k(x^-) < i_{k-1}(x^-), \quad k \in \mathbb{N}, \]

and let

\[ V_{k,i}(x^-) \in \mathcal{V}(X), \quad i_k \geq i \geq 0, k \in \mathbb{N}, \]

be given by

\[ V_{k,i} = \Theta_{x^-_{i-k-1}}(V_{k,i-1}), \quad i_k \geq i > 0, \quad k \in \mathbb{N}. \]

For

\[ (\hat{a}, G_{\Sigma,\lambda}(\mathcal{C}^i, \mathcal{B}_{\mathcal{C}^i}), \hat{V}) \in \mathcal{V}(X), \quad \sigma \in \Gamma^+_1(\hat{a}, G_{\Sigma,\lambda}(\mathcal{C}^i, \mathcal{B}_{\mathcal{C}^i}), \hat{V}), \]

set

\[ \eta(\hat{a}, G_{\Sigma,\lambda}(\mathcal{C}^i, \mathcal{B}_{\mathcal{C}^i}), \hat{V}, \sigma) = \begin{cases} I(\Theta_{x^-}(\hat{V})), & \text{if } \sigma \in \Gamma^+_1(\hat{V}), \\ -1, & \text{if } \sigma \not\in \Gamma^+_1(\hat{V}). \end{cases} \]

We consider the case, that

\[ \lim_{k \to \infty} \sum_{i_k \geq i \geq 0} \eta(V_{k,i}(x^-), x^-_{i+1}) = -\infty. \]

In this case there is a unique sequence

\[ V_i(x^-) = (\hat{a}, G_{\Sigma,\lambda}(\mathcal{C}^i, \mathcal{B}_{\mathcal{C}^i})(k_i), \hat{V}(k_i)) \in \mathcal{V}(X), i \in \mathbb{Z}_-, \]

that satisfies

\[ (e, G_{\Sigma,\lambda}(\mathcal{C}^i, \mathcal{B}_{\mathcal{C}^i})(i), \hat{V}(i)) = V(c), \quad i \in \mathbb{Z}_-, \]

\[ V_{k,i} = \Theta_{x^-_{i-k-1}}(V_{k,i-1}), \quad i_k \geq i > 0, \quad k \in \mathbb{N}. \]

One has that

\[ S(x^-) = \lim_{k \to \infty} \sum_{-i \geq 0} \eta(V_{k,i}(x^-), x^-_{i+1}) < \infty, \]

and with

\[ I(x^-) = \max \{ I \in \mathbb{Z}_- : S(x^-) = \sum_{-i \geq 0} \eta(V_{k,i}(x^-), x^-_{i+1}) \}, \]

one finds, that

\[ x^-_{-\infty,-I} \in \omega^-(x^-_{-[I,0]}), \]

which means, that \(x^-_{-[I,0]}\) is a semisynchronizing word of \(X\). The discordance hypothesis implies, that

\[ \Gamma^+(x^-_{-[I(x^-),0]}) = \Gamma^+(V_0(x^-)), \]

and and together with the strong connectedness of \(G_{s,\text{syn}}(X)\) it also implies, that exactly the semisynchronizing words of \(X\) appear in this way as a suffix of a transitive \(x^- \in X_{-\infty,0}\). This concludes the proof. \(\square\)
6. A CLASS OF $(C - F)$-SEMSYNCHRONIZING SHIFTS I

In this section we consider the class of $(C - F)$-semisynchronizing and $(C - F)$-a-synchronizing shifts, that satisfy the visibility and discordance as well as the projection hypothesis. For a $(C - F)$-semi-synchronizing shift $X$, that is also $(C - F)$-a-synchronizing, the directed graph $G(\hat{\gamma}, \delta)(X)$ is a one-vertex graph. Accordingly, we start the construction of the automata, that arise as the Shannon graphs $G_{a-syn}(X)$ for subshifts $X$ in this class, with a finite alphabet $\Sigma$, a finite set $B$, and a one vertex directed graph $G(\{B\}, D)$. For $V \in B$ we assume given disjoint subsets $\Sigma^-(V)$ and $\Sigma^+(V)$ of $\Sigma$, and non-empty subsets $\Sigma_d^-(V), d \in D$, of $\Sigma^-(V)$, such that

(a) \[ \Sigma^-(V) = \bigcup_{d \in D} \Sigma_d^-(V). \]

We assume given for $V \in B$ and $\sigma \in \Sigma^+(V)$ a pair

\[ (\pi(V, \sigma), \chi(V, \sigma)) \in \{\epsilon\} \cup \Pi_{G(\{B\}, D)} \times B, \]

and we assume given for $V \in B, d \in D$ and $\sigma \in \Sigma_d^{-}(V)$ a $\chi_d(V, \sigma)) \in B$.

We construct an automaton

\[ G_{\Sigma, \lambda}(V, E) = G_{\Sigma, \lambda}((\{\epsilon\} \cup \Pi_{G(\{B\}, D)} \times B, E) \]

with boundary $\{\epsilon\} \times B$ by setting

\[ \Gamma_1^+((\epsilon, V)) = \Sigma^+(V), \quad V \in B, \]

\[ \Gamma_1^+((ad, V)) = \Sigma_d^-(V) \cup \Sigma^+(V), \quad a \in \{\epsilon\} \cup \Pi_{G(\{B\}, D)}, d \in D, \quad V \in B, \]

and by adopting the following two transition rules:

(I) When in state $(a, V), a \in \{\epsilon\} \cup \Pi_{G(\{B\}, D)}, V \in B$, the automaton $G_{\Sigma, \lambda}(V, E)$ accepts an input $\sigma \in \Sigma^+(V)$ and transits to state $(a\pi(V, \sigma), \chi(V, \sigma))$.

(II) When in state $(ad, V), a \in \{\epsilon\} \cup \Pi_{G(\{B\}, D)}, d \in D, V \in B$, the automaton $G_{\Sigma, \lambda}(V, E)$ accepts an input $\sigma \in \Sigma_d^-(V)$ and transits to state $(a, \chi_d(V, \sigma))$.

The strong connectedness of $G_{\Sigma, \lambda}(V, E)$ can be tested. It can also be tested if $\{\epsilon\} \times B$ is a reference set of $G_{\Sigma, \lambda}(V, E)$. The validity of the visibility hypothesis and also of the projection hypothesis can be tested.

We require the following condition (b) to hold

(b) \[ \bigcap_{d \in D} \Sigma_d^-(V) = \emptyset, \quad V \in B. \]

Also we require the following condition (c) to hold:

(c) For $V \in B$ and for $d, d' \in D, d \neq d'$, one has either

\[ \Sigma_d^-(V) \neq \Sigma_{d'}^-(V), \]

or there is a $\sigma \in \Sigma_d^-(V) = \Sigma_{d'}^-(V)$, such that

\[ \chi_d(V, \sigma) \neq \chi_{d'}(V, \sigma). \]

The discordance hypothesis is equivalent to (b). We have imposed the condition (c) to make it possible to separate the states $(d, V)$ and $(d', V), d, d' \in D, d \neq d', V \in B$.

We give a necessary and sufficient condition for $G_{\Sigma, \lambda}(V, E)$ to be forward separated. By the visibility hypothesis and the projection hypothesis there is an $M \in \mathbb{N}$ together with an $M_0 \in [1, M]$, such that the following condition (A) holds:
The acceptance set of a state \((\Delta, U)\) is defined as the set of all words \(w \in \mathcal{L}_M(Y(G_{\Sigma,\lambda}(V, E)))\) such that, if a word \(w \in \mathcal{L}_M(Y(G_{\Sigma,\lambda}(V, E)))\) labels a path \(a = (e_m)_{1 \leq m \leq M} \in \Pi G_{\Sigma,\lambda}(V, E)\), then

\[ t(a) = V(w), \]

and the path \((e_m)_{1 \leq m \leq M}\) approaches \(\{\epsilon\} \times \mathcal{B}\) directly.

The smallest \(M \in \mathbb{N}\) such that (A) holds, we denote by \(M(G_{\Sigma,\lambda}(V, E))\), and the smallest \(M_0 \in [1, M]\), such that (A) holds for \(M = M(G_{\Sigma,\lambda}(V, E))\), we denote by \(M_0(G_{\Sigma,\lambda}(V, E))\). We set

\[ J_{G_{\Sigma,\lambda}(V, E)} = \max \{\ell((\pi(V, \sigma)) : V \in \mathcal{B}, \sigma \in \Sigma\}, \]

\[ M_{G_{\Sigma,\lambda}(V, E)} = M_0(G_{\Sigma,\lambda}(V, E)) + J_{G_{\Sigma,\lambda}(V, E)}. \]

**Lemma 6.1.** States \((a, U), (b, V) \in V\), such that \(\ell(a) > \ell(b) > M_{G_{\Sigma,\lambda}(V, E)}\), can be separated.

**Proof.** There are two cases. In the case that \(\ell(b) \geq M(G)\), let

\[ z \in \mathcal{L}_{\ell(b)}(Y(G_{\Sigma,\lambda}(V, E))) \]

be accepted by \((b, V)\) such that \(\ell(\Theta_z((b, V))) = 0\). In the case that \(z\) is also accepted by \((a, U)\) one has a \(\tilde{a} \in \Pi\) and a \(W \in \mathcal{B}\), such that

\[ \Theta_z((b, V)) = (\tilde{a}, W), \quad \Theta_z((b, V)) = (\epsilon, W), \]

and the states \((\tilde{a}, W)\) and \((\epsilon, W)\) can be separated.

In the case that \(\ell(b) \geq M(G)\), let

\[ z \in \mathcal{L}_{\ell(b)}(Y(G_{\Sigma,\lambda}(V, E))) \]

be accepted by \((a, V)\) such that \(\ell(\Theta_z((a, V))) = 0\). Let \(\tilde{z}\) be the prefix of length \(M_{G_{\Sigma,\lambda}(V, E)}\) of \(z\) and let \(\tilde{a}\) be the prefix of length \(\ell(a) - M_{G_{\Sigma,\lambda}(V, E)}\) of \(a\). In the case that \(z\) is also accepted by \((b, V)\) one has a \(W \in \mathcal{B}\) and a \(\tilde{b} \in \Pi\) of length less than \(M_0(G_{\Sigma,\lambda}(V, E)) + 1\), such that

\[ \Theta_z((a, U)) = (\tilde{a}, W), \quad \Theta_z((b, V)) = (\tilde{b}, W), \]

and it follows that \((b, V)\) cannot accept \(z\). \(\square\)

We introduce a deterministic transition system \(\mathcal{P}\) with input alphabet \(\Sigma\). The state space of \(\mathcal{P}\) contains the pairs

\[ ((a, U), (b, V)) \in V \times V, \]

such that

\[ (a, U) \neq (b, V), \]

and

\[ \Sigma((a, U)) = \Sigma((b, V)), \quad \ell(a) = \ell(b). \]

The acceptance set of a state \(((a, U), (b, V)) \in \mathcal{P}\) contains the symbols \(\sigma\) in the common acceptance set of \((a, U)\) and \((b, V)\), such that

\[ (\Theta_\sigma((a, U)), \Theta_\sigma((b, V))) \in \mathcal{P}. \]

When in state \(((a, U), (b, V))\) the transition system \(\mathcal{P}\) accepts a input \(\sigma \in \Sigma(((a, U), (b, V)))\) and transits to state \(((\Theta_\sigma((a, U)), \Theta_\sigma((b, V))))\). We set

\[ \mathcal{P}^{(0)} = \{((a, U), (b, V)) \in \mathcal{P} : \ell((a, U), (b, V)) = 0\}, \quad \mathcal{P}^{(+)} = \mathcal{P} \setminus \mathcal{P}^{(0)}. \]

Denote for

\[ (((\epsilon, U), (\epsilon, V)), ((\epsilon, U'), (\epsilon, V'))) \in \mathcal{P}^{(0)} \times \mathcal{P}^{(0)} \]
by $E^{(0)}(((\epsilon, U), (\epsilon, V)), ((\epsilon, U'), (\epsilon, V'))) \cup F^{(0)}$ the set of words $w \in L(G_{\Sigma}(V, E))$ such that

$$\Theta_w(((\epsilon, U), (\epsilon, V))) = ((\epsilon, U'), (\epsilon, V')),$$

and denote by $F^{(0)}$ the set of

$$(((\epsilon, U), (\epsilon, V)), ((\epsilon, U'), (\epsilon, V'))) \in \mathcal{P}^{(0)} \times \mathcal{P}^{(0)}$$

such that

$$E^{(0)}(((\epsilon, U), (\epsilon, V)), ((\epsilon, U'), (\epsilon, V')))) \neq \emptyset.$$

Set

$$\Lambda^{(0)} = \max_{w \in F^{(0)}} \max_{(((\epsilon, U), (\epsilon, V)), ((\epsilon, U'), (\epsilon, V'))) \in E^{(0)} \cup F^{(0)}} \ell(w).$$

Denote for

$$(((\epsilon, U'), (\epsilon, V')), ((\epsilon, U'), (\epsilon, V')))) \in \mathcal{P}^{(0)} \times \mathcal{P}^{(+)}$$

by $E^{(+)}(((\epsilon, U), (\epsilon, V)), ((\epsilon, U'), (\epsilon, V'))))$ the set of words

$$w = (w_i)_{1 \leq i \leq I} \in L(G_{\Sigma}(V, E)), I \in \mathbb{N},$$

such that

$$\Theta_w(((\epsilon, U), (\epsilon, V))) = ((\epsilon, U'), (\epsilon, V')),$$

and such that

$$\ell(\Theta(w_i)_{1 \leq i \leq J, ((\epsilon, U'), (\epsilon, V'))}) \geq \ell(((\epsilon, U'), (\epsilon, V'))), \quad 1 \leq J \leq I.$$

Denote by $F^{(+)}$ the set of

$$(((\epsilon, U), (\epsilon, V)), ((\epsilon, U'), (\epsilon, V')))) \in \mathcal{P}^{(0)} \times \mathcal{P}^{(+)}$$

such that

$$E^{(+)}(((\epsilon, U), (\epsilon, V)), ((\epsilon, U'), (\epsilon, V')))) \neq \emptyset.$$

and set

$$\Lambda^{(+)} = \max_{w \in F^{(0)}} \max_{(((\epsilon, U), (\epsilon, V)), ((\epsilon, U'), (\epsilon, V'))) \in E^{(+)} \cup F^{(0)}} \ell(w).$$

We denote by $\Xi_{\mathcal{P}}$ the set of $((a, U), (b, V)) \in \mathcal{P}$, such that there is a $\sigma$ in the common acceptance set of $(a, U)$ and of $(b, V)$ such that

$$\Sigma(\Theta_{\sigma}((a, U))) \neq \Sigma(\Theta_{\sigma}((b, V))).$$

and we denote by $\Xi'_{\mathcal{P}}$ the set of $((a, U), (b, V)) \in \mathcal{P}$, such that there is a $\sigma$ in the common acceptance set of $(a, U)$ and of $(b, V)$ such that

$$\ell(\Theta_{\sigma}((a, U))) \neq \ell(\Theta_{\sigma}((b, V))).$$

Denote by $A_{\mathcal{P}}$ the set of $((\sigma, U), (\sigma, V)) \in \mathcal{P} \setminus (\Xi_{\mathcal{P}} \cup \Xi'_{\mathcal{P}})$ such that there is a word $w \in L(G_{\Sigma}(V, E))$ such that

$$\Theta_w(((\sigma, U), (\sigma, V))) \in \Xi_{\mathcal{P}} \cup \Xi'_{\mathcal{P}}.$$

We construct a (possibly non-deterministic) transition system with input alphabet $\Sigma$ and state space $Q$, that contains the pairs

$$((a, U), (b, V)) \in V \times V,$$

such that

$$U \neq V,$$

and such that

$$\Sigma((a, U)) = \Sigma((b, V)).$$

To prepare the construction denote by $\mathcal{M}$ the set of

$$((a, W), (b, W)) \in V \times V$$
such that
\[ a \neq b, \]
and
\[ \Sigma((a, W)) = \Sigma((b, W)). \]

We associate to an element \(((a, W), (b, W)) \in \mathcal{M}\) sets
\[ \mathcal{Y}_I((a, W_0), (b, W_0)) \subset \mathcal{L}_I(Y(X)), \quad 0 \leq I < \min\{\ell(a), \ell(b)\}. \]
The set \(\mathcal{Y}_0((a, W_0), (b, W_0))\) contains the empty word if and only if
\[ \Sigma((-)((a_1, W_0)) = \Sigma((-)((b_1, W_0)), \]
and a word
\[ y = (y_j)_{1 \leq j \leq I} \subset \mathcal{L}_I(Y(X)), \quad 1 \leq I < \min\{\ell(a), \ell(b)\}, \]
is in \(\mathcal{Y}_I((a, W_0), (b, W_0))\) if and only if there exist
\[ W_j((y_j)_{1 \leq j \leq I}) \in \mathcal{B}, \]
such that
\[ y_1 \in \Sigma((-)((a_1, W_0)) = \Sigma((-)((b_1, W_0)), \]
\[ W_1(y_1) = \chi_{y_1}(a_1, W_0) = \chi_{y_1}(b_1, W_0), \]
\[ y_j \in \Sigma((-)((a_j, W_{j-1}((y_{i_11})_{1 \leq i_1 < j}))) = \Sigma((-)((b_j, W_{j-1}((y_{i_11})_{1 \leq i_1 < j}))), \]
\[ W_j((y_j)_{1 \leq i < j}) = \chi_{y_j}(a_j, W_{j-1}((y_{i_11})_{1 \leq i_1 < j}))) = \chi_{y_j}(b_j, W_{j-1}((y_{i_11})_{1 \leq i_1 < j})), \]
\[ 1 < j \leq I, \]
\[ \Sigma((-)((a_{I+1}, W_I(y))) = \Sigma((-)((b_{I+1}, W_I(y))), \]
and such that one has for
\[ \tilde{a}(I) = (a_i)_{\ell(a) \geq i > I+1}, \quad \tilde{b}(I) = (b_i)_{\ell(b) \geq i > I+1}, \]
that
\[ \tilde{a}(I) \neq \tilde{b}(I). \]

For
\[ y \in \mathcal{Y}_I((a, W_0), (b, W_0)), \quad 1 \leq I < \min\{\ell(a), \ell(a)\}, \]
we set
\[ \mathcal{Q}_I((a, W_0), (b, W_0), y) = \]
\[ \mathcal{Q} \cap \{\Theta_\sigma(\tilde{a}(I), W_I(y), \Theta_\sigma(\tilde{b}(I), W_I(y)) : \sigma \in \Sigma((-)(\tilde{a}(I), W_I(y)) = \Sigma((-)(\tilde{b}(I), W_I(y)))}, \]
and we set
\[ \mathcal{Q}((a, W_0), (b, W_0))) = \]
\[ \bigcup_{0 \leq I < \min\{\ell(a), \ell(b)\}} \bigcup_{y \in \mathcal{Y}_I((a, W_0), (b, W_0))} \mathcal{Q}_I((a, W_0), (b, W_0), y). \]

We denote by \(\mathcal{M}'\) the set of \(((a, W_0), (b, W_0)) \in \mathcal{M}\) such that there are \(I \in [0, \min\{\ell(a), \ell(b)\}])\), and \(y \in \mathcal{Y}_I((a, W_0), (b, W_0))\), and a
\[ \sigma \in \Sigma((-)(\tilde{a}(I), W_I(y)) = \Sigma((-)(\tilde{b}(I), W_I(y)) \]
such that
\[ \Sigma(\Theta_\sigma(\tilde{a}(I), W_I(y)) \neq \Sigma(\Theta_\sigma(\tilde{b}(I), W_I(y))). \]

Also set
\[ \mathcal{M}_Q = \{((a, W)) \neq (b, W)) \in \mathcal{M} \setminus \mathcal{M}' : \mathcal{Q}((a, W), (b, W)) \neq \emptyset). \]

For \(((a, U), (b, V)) \in \mathcal{Q}\) denote by \(\Sigma_\sigma(((a, U), (b, V))))\) the set of \(\sigma\) in the common acceptance set of \((a, U)\) and \((b, V)\), such that
\[ (\Theta_\sigma(a, U), \Theta_\sigma(b, V)) \in \mathcal{Q}, \]
and by $\Sigma_\mathcal{Q}(([a, U]), (b, V)))$ the set of $\sigma$ in the common acceptance set of $(a, U)$ and $(b, V)$, such that

$$(\Theta_\sigma(a, U), \Theta_\sigma((b, V))) \in \mathcal{M}_\mathcal{Q}.$$  

We define the acceptance set

$$\Sigma(([a, U]), (b, V)) = \Sigma_\sigma(([a, U]), (b, V))) \cup \Sigma_\mathcal{Q}(([a, U]), (b, V))).$$

When accepting the symbol $\sigma\in \Sigma_\mathcal{Q}(([a, U]), (b, V)))$ as input the transition system $\mathcal{Q}$ transits to state $(\Theta_\sigma(a, U), \Theta_\sigma((b, V)))$, and when accepting the symbol $\sigma\in \Sigma_\mathcal{Q}(([a, U]), (b, V)))$ as input the transition system $\mathcal{Q}$ transits to any of the states in $\mathcal{Q}((\Theta_\sigma(a, U), \Theta_\sigma((b, V))).$

We denote by $\mathcal{Q}[M_G]$ the transition system, that has as state space the set

$$\{((a, U)), (b, V)) \in \mathcal{Q} : \ell(a), \ell(a) < M_G\}
$$

an inherits its transition rules from $\mathcal{Q}$ We denote by $\Xi$ the set of $((a, U)), (b, V)) \in \mathcal{Q}$ such that there is a $\sigma$ in the common acceptance set of $(a, U)$ and $(b, V)$, such that $\Sigma(([\Theta_\sigma(a, U))), (b, V)))$, and by $\Xi'$ the set of $((a, U)), (b, V)) \in \mathcal{Q}$ such that there is a $\sigma$ in the common acceptance set of $(a, U)$ and $(b, V)$, such that

$$\ell(\Theta_\sigma((a, U)), (b, V))) > M_G.$$

**Theorem 6.2.** The Shannon graph $G(V, \cdot)$ is forward separated if and only if following conditions (a) and (b) are satisfied:

(a) For all $U, V \in \mathcal{B}, U \neq V$, there exists in $\mathcal{Q}[M_G]$ a directed path from $((\epsilon, U), (\epsilon, V))$ to $\Xi_\mathcal{Q} \cup \Xi_\mathcal{Q}'$.

(b) For all $U, V \in \mathcal{B}, U \neq V$, there is a word $w \in \mathcal{L}(G_{\Sigma, \lambda}(V, E))$ such that $\Theta_w((\epsilon, U), (\epsilon, V)) \in \Xi_\mathcal{P} \cup \Xi_\mathcal{P}'$.

**Proof.** If (a) or (b) does not hold for a pair $U, V$ of distinct vertices in $\mathcal{B}$, then $(\epsilon, U)$ and $(\epsilon, V)$ cannot be separated. This proves necessity.

Assume (a) and (b). Let $U, V \in \mathcal{B}, U \neq V$. If there exists in $\mathcal{Q}[M_G]$ a directed path from $((\epsilon, U), (\epsilon, V))$ to $\Xi_\mathcal{Q}$, then $(\epsilon, U)$ and $(\epsilon, V)$ can be separated by the construction of $\Xi_\mathcal{Q}$. If there exists in $\mathcal{Q}[M_G]$ a directed path $q$ from $((\epsilon, U), (\epsilon, V))$ to $((a, U'), (b, V')) \in \Xi_\mathcal{Q}$, and if $\ell(a) \neq \ell(b)$, then $(\epsilon, U)$ and $(\epsilon, V)$ can be separated by Lemma (6.1). In the case, that $\ell(a) = \ell(b)$, let $w \in \mathcal{L}(G_{\Sigma, \lambda}(V, E)$ be such that $\Theta_w((\epsilon, U), (\epsilon, V)) \in \Xi_\mathcal{P} \cup \Xi_\mathcal{P}'$, and concatenate the directed path $r$ with the directed path $q$ from $\Theta_w((\epsilon, U), (\epsilon, V)) \in \Xi_\mathcal{P} \cup \Xi_\mathcal{P}'$ that is labelled by $w$. If the target of $r$ is in $\Xi_\mathcal{P}$ then $(\epsilon, U)$ and $(\epsilon, V)$ can be separated by the construction of $\Xi_\mathcal{P}$, and if the target of $r$ is in $\Xi_\mathcal{P}'$, then $(\epsilon, U)$ and $(\epsilon, V)$ can be separated by Lemma (6.1).

We remark that the criterion of Theorem (6.2) is effective.

**7. A CLASS OF $(C - F)$-SEMISYNCHRONIZING SHIFTS II**

We consider in this section the class of a-synchronizing $(C - F)$-semisynchronizing shifts $X$, that satisfy the discordance and resolution hypotheses, and that are such that there is an $M \in \mathbb{Z}_+$, such that every word in $X_{[-M, M]}$ is resolving for $G_{\text{a-syn}}(X)$. This class is closed under topological conjugacy. Subshifts in this class satisfy the visibility and projection hypotheses.

We can assume that the a-synchronizing Shannon graph of subshift $X$ in this class is given as a forward separated Shannon graph as constructed in section 6 from a finite set $\mathcal{B}$, and a one vertex directed graph $G(\mathcal{B}, \mathcal{D})$, and sets $\Sigma^-(V), \Sigma^+(V) \subset \Sigma, \Sigma^-(V), \Sigma^+(V) \neq \emptyset$ and $\Sigma^-(V)$, $\Sigma^+(V) \subset \Sigma^-(V), d \in \mathcal{D}$, and also from a set of pairs

$$(\pi(V, \sigma), \chi(V, \sigma)) \in \{\epsilon\} \cup \Pi_G(\mathcal{B}, \mathcal{D}) \times \mathcal{B},$$
and a set of vertices $\chi_d(V,\sigma) \in B$, that satisfy conditions (a), (b) and (c), and which is denoted by $G_{\Sigma,\lambda}((\{e\} \cup \Pi G([B], D)) \times B, e, \{e\} \times B)$.

For $a \in X_{[-M,M]}$, $M \in \mathbb{Z}_+$, to be a resolving word for $G_{a-syn}(X)$ means in this case, that there is a $V(a) \in B$ and a path $\pi(a) \in \Pi G([B], D)$, such that the following holds: For a path $(e_m)_{-M \leq m \leq M}$ in $G_{\Sigma,\lambda}(V, e, \{e\} \times B)$, that carries the label sequence $a$, there exists a path $b$ in $G([B], D)$, such that $s(e_0) = (b, V(a))$.

Under the assumption, that for some $M \in \mathbb{Z}_+$ every word in $X_{[-M,M]}$ is a resolving word for $(G_{a-syn}(X))$, one can recode $X$ with an appropriately chosen subset of $\Sigma \times \{e, i, r\}$ as label alphabet $\tilde{\Sigma}$, as a subshift $\tilde{X} \subset \tilde{\Sigma}^\mathbb{Z}$ by using the block map $\Phi$, that is given by

$$\Phi(w) = \begin{cases} (a_0, c), & \text{if } a_0 \in \Sigma^+(V(a)), \pi(V(a), a_0) \neq \epsilon, \\ (a_0, i), & \text{if } a_0 \in \Sigma^+(V(a)), \pi(V(a), a_0) = \epsilon, \\ (a_0, r), & \text{if } a_0 \in \Sigma^+(V(a)), \end{cases}$$

$a \in X_{[-M,M]}$.

The subshift $X$ is canonically presented by $G_{s-syn}(\tilde{X})$ via the one block conjugacy, that drops the second component of the symbols in $\tilde{\Sigma}$. As the subshift $\tilde{X}$ is topologically conjugate to the subshift $X$, the subshift $\tilde{X}$ yields the invariants of the subshift $X$. In the case, that the length of the paths $\pi(V, \sigma), V \in B, \sigma \in \Sigma^+(V)$ is at most one, $\tilde{X}$ is the finite-type-Dyck shift, that is given by the deterministic Dyck automaton with $B$ as vertex set, with the set

$$\{(V, \sigma, \chi(V, \sigma)) : V \in B, \sigma \in \Sigma^+(V)\} \cup \bigcup_{C \in \gamma} \{(V, \sigma, \chi_\delta(V, \sigma)) : V \in B_C, d \in \delta, t(\delta) = C, \sigma \in \Sigma_d(V)\}$$

as edge set, and with the set

$$\{(V, \sigma, \chi(V, \sigma)), (V', \sigma', \chi(V', \sigma)) : V \in B, \sigma \in B_{t(\pi(V, \sigma))}, \pi(V, \sigma) \neq \epsilon, V' \in \sigma, \sigma' \in \sigma_{\pi(V, \sigma)}(V')\}$$

as set of matching pairs. For information about invariants of topological conjugacy of finite-type-Dyck shifts see [BBD2].

8. Examples

For a finite directed graph $G(V, E)$ we recall the construction of its graph inverse semigroup $S(G(V, E))$ (see [L, Section 10.7]). Let $E^- = \{e^- : e \in E\}$ be a copy of $E$. Reverse the directions of the edges in $E$ to obtain the edge set $E^+ = \{e^+ : e \in E\}$ of the reversed graph of $G(V, E^-)$.

With idempotents $1_V, V \in V$, the set $E^- \cup \{1_V : V \in V\} \cup E^+$ is a generating set of $S(G(V, E))$. Besides $1_V^2 = 1_V, V \in V$, the relations are

$$1_U 1_W = 0, \quad U, W \in V, U \neq W,$$

$$f^- g^+ = \begin{cases} 1_{s(f)}, & \text{if } f = g, \\ 0, & \text{if } f \neq g, \quad f, g \in E, \end{cases}$$

$$1_{s(f)} f^- = f^- 1_{t(f)}, \quad 1_{t(f)} f^+ = f^+ 1_{s(f)}, \quad f \in E.$$

8.1. Prototypical examples of $(C - F)$-semisynchronizing, a-synchronizing shifts are the Dyck shifts, that were introduced in [KR1]. By starting from a directed graph $(\{B\}, D), \text{card}(D) > 1$, with a singleton set $B = \{V\}$, the Dyck shift with alphabet $\Sigma = \{d^-, d^+ : d \in D\}$ is obtained within the setting of section 6 (or section 7) by setting

$$\Sigma^+(V) = \{d^- : d \in D\},$$

$$\pi(V, d^-) = d, \quad \Sigma^d(V) = \{d^+\}, \quad d \in D.$$
8.2. For a more general construction of $(C-F)$-semisynchronizing, a-synchronizing shifts, let there be given a finite aperiodic directed graph $G(B,\mathcal{H})$ and consider the directed graph $\{\{B\},D\}$, $\text{card}(D) > 1$, and set
\[
\Sigma = \{h^-,h^+: h \in \mathcal{H}\} \times D,
\]
\[
\Sigma^+(V) = \{(h^-,d): h \in \mathcal{H}, d \in D, s(h) = V\}, \quad V \in B,
\]
\[
\Sigma_{\alpha}^{-}(V) = \{(h^+,d): h \in \mathcal{H}, d \in D, s(h) = V\}, \quad V \in B,
\]
\[
\pi(V,(h^-,d)) = d, \chi(V,(h^-,d)) = t(h), \quad V \in B,(h^-,d) \in \Sigma^+(V),
\]
\[
\chi_{d}(V,(h^+,d)) = s(h), \quad V \in B,(h^+,d) \in \Sigma_{\alpha}^{-}(V).
\]
The state space of the automaton is $(\{\epsilon\} \cup \Pi_{\{\{B\}\},D}) \times \{V\}$.

8.3. We describe the subshifts, that were introduced by Béal and Heller in [BH, Section 6], within the framework of of section 6 (or section 7). For this purpose let $\mathcal{K}$ be a finite nonempty set, and let $I_k \in \mathbb{N}, k \in \mathcal{K}$, We set
\[
\mathcal{B} = \{V\}, \quad D = \{d_{K,i}: K \in \mathcal{K}, 1 \leq i \leq I_K\},
\]
\[
\Sigma = \{\alpha_K^+,\alpha_K^-: K \in \mathcal{K}\} \cup \{\alpha_{K,i}^-: K \in \mathcal{K}, 1 \leq i \leq I_K\},
\]
\[
\Sigma^+(V) = \{\alpha_K^-: K \in \mathcal{K}\},
\]
\[
\Sigma_{\alpha}^{-}(V) = \{\alpha_K^+: K \in \mathcal{K}\},
\]
\[
\pi(\alpha_K^+,V) = d_{K}, \quad K \in \mathcal{K}.
\]
The state space of the automaton is the set of pairs $(a,V)$, where $a$ is a concatenation of the words $(d_{K,i})_{1 \leq i \leq I_K, 1 \leq I \leq I_K}$. The case $K > 1, I_K = 1, K \in \mathcal{K}$, yields the Dyck shifts. Under the assumption
\[
\sum_{K \in \mathcal{K}} I_K > 1,
\]
the shifts satisfy the discordance hypothesis.

8.4. We give an example of a $(C-F)$-semisynchronizing, a-synchronizing shift, such that its automaton has states, that are not directly accessible from infinity. We set
\[
\mathcal{B} = \{V,V'\}, \quad D = \{d_0,d_1\},
\]
\[
\Sigma = \{\alpha_0^-,\alpha_0^+,\alpha_1^+,\alpha_1^-,\alpha\},
\]
\[
\Sigma^+(V') = \{\alpha_0^-\}, \quad \Sigma^+(V') = \emptyset,
\]
\[
\Sigma_{\alpha}^{-}(V') = \{\alpha_0^+,\alpha_1^+,\alpha\}, \quad \Sigma_{\alpha}^{-}(V') = \emptyset,
\]
\[
\pi(V,\alpha_0^-) = d_0, \chi(V,\alpha_0^-) = V, \quad \pi(V,\alpha_1^-) = d_1, \chi(V,\alpha_1^-) = V,
\]
\[
\pi(V,\alpha) = \epsilon, \quad \chi(V,\alpha) = V',
\]
\[
\chi_{d_0}(V,\alpha_0^+) = \chi_{d_1}(V,\alpha_1^+) = \chi_{d_0}(V,\alpha_0^+) = \chi_{d_1}(V,\alpha_1^+) = V.
\]
The state space of the semisynchronizing automaton is
\[
(\{\epsilon\} \cup \Pi_{G(\mathcal{B}),D}) \times \{V,V'\}.
\]
The states $(a,V'), a \in (\{\epsilon\} \cup \Pi_{G(\mathcal{B}),D})$, are not directly accessible from infinity. To obtain the a-synchronizing automaton remove the state $(\epsilon,V')$ from the semisynchronizing automaton.
8.5. Prototypical examples of \((C - F)\)-semisynchronizing are the Markov-Dyck shifts (see [M]). Before we describe them we need to extend the framework of Section 6. We start with a finite alphabet \(\Sigma\), a finite directed graph \(G(V, \mathcal{E})\), and finite non-empty sets \(B_V, V \in \mathcal{V}\). We assume given for \(U \in B_V, V \in \mathcal{V}\), disjoint sets \(\Sigma^+(U) \subseteq \Sigma\) and \(\Sigma^-(U) \subseteq \Sigma\) and for \(U \in B_V, V \in \mathcal{V}\), and \(e \in \mathcal{E}\), such that \(t(e) = V\), non-empty sets \(\Sigma^-\{U\} \subseteq \Sigma\), such that
\[
\Sigma^-\{U\} = \bigcup_{e \in \mathcal{E}, t(e) = V} \Sigma^-\{U\}.
\]
We assume given for \(U \in B_V, V \in \mathcal{V}\), and \(\sigma \in \Sigma^+(U)\) a path \((\pi(U, \sigma)) \in \{e\} \cup \Pi_{G(V, \mathcal{E})}\) and a vertex \(\chi(U, \sigma)\) \(\in t(\pi(U, \sigma))\) and we assume given for \(V \in B\), and \(e \in \mathcal{E}\) such that \(t(e) = U\) and \(\sigma \in \Sigma^-\{U\}\) a vertex \(\chi_e(U, \sigma)\) \(\in B_{\mathcal{E}(e)}\).

We construct an automaton with state space
\[
\{(b, U) \in (\{e\} \cup \Pi_{G(V, \mathcal{E})}) \times B : U \in B_{\mathcal{E}(b)}\}
\]
by setting
\[
\Gamma^+_U (ae, U) = \Sigma^+(U), \quad U \in B_V, V \in \mathcal{V},
\]
\[
\Gamma^-_U (ae, U) = \Sigma^-(U) \cup \Sigma^+(U), \quad a \in \{e\} \cup \Pi_{G(V, \mathcal{E})}, e \in \mathcal{E}, U \in B_{\mathcal{E}(e)},
\]
and by adopting the following two transition rules:

(I) When in state \((a, t(a))\), \(a \in \{e\} \cup \Pi_{G(V, \mathcal{E})}\), \(U \in B_{\mathcal{E}(a)}\), the automaton accepts an input \(\sigma \in \Sigma^+(U)\) and transits to state \((a\pi(U, \sigma), \chi(U, \sigma))\).

(II) When in state \((ae, U)\), \(a \in \{e\} \cup \Pi_{G(V, \mathcal{E})}, e \in \mathcal{E}, U \in B_{\mathcal{E}(e)}\), the automaton accepts an input \(\sigma \in \Sigma^-\{U\}\) and transits to state \((a, \chi_e(U, \sigma))\).

To obtain the Markov-Dyck shift of the graph \(G(V, \mathcal{E})\) within this extended framework set
\[
B_V = \{V\}, \quad V \in \mathcal{V},
\]
\[
\Sigma = \{e^- : e \in \mathcal{E}\} \cup \{e^+ : e \in \mathcal{E}\},
\]
\[
\Sigma^+(V) = \{e^- : e \in \mathcal{E}, s(e) = V\},
\]
\[
\Sigma^-(V) = \{e^+\}, \quad e \in \mathcal{E}, t(e) = V,
\]
\[
\chi(V, e^+) = s(e), \quad V \in \mathcal{V}, V \in B_{\mathcal{E}(e)},
\]
\[
\pi(V, e^-) = e, \quad V \in \mathcal{V}, V \in B_{\mathcal{E}(e)},
\]
\[
\chi(V, e^-) = t(e).
\]
The state space of the automaton of the Markov-Dyck shift is \(\{(a, t(a)) : a \in \Pi_{G(V, \mathcal{E})}\}\). The Dyck shifts are Markov-Dyck shifts of one-vertex directed graphs.

8.6. Within the extended framework of Section 8.5 a special case arises by combining the construction of the Markov-Dyck shifts with the construction of Section 8.2. Let there be given a finite irreducible directed graph \(G(V, \mathcal{E})\) and a finite aperiodic directed graph \(G(A, \mathcal{H})\). Set
\[
B_V = B \times \{V\}, \quad V \in \mathcal{V},
\]
\[
\Sigma = \{(h^-, e^-) : (h, e) \in \mathcal{H} \times \mathcal{E}\} \cup \{(h^+, e^+) : (h, e) \in \mathcal{H} \times \mathcal{E}\},
\]
\[
\Sigma^+(U, V) = \{(h^-, e^-) : s(h) = U, s(e) = V\}, \quad U \in A, V \in \mathcal{V},
\]
\[
\Sigma^-\{U, V\} = \{(h^+, e^+) : h \in \mathcal{H}\}, \quad U \in A, V \in \mathcal{V}, e \in \mathcal{E}, t(e) = V,
\]
and
\[
\pi((U, V), (h^-, e^-)) = (h, e), \quad \chi((U, V), (h^-, e^-)) = (t(h), t(e)),
\]
\[
(h^-, e^-) \in \Sigma^+(U, V), U \in A, V \in \mathcal{V},
\]
and set
\[
\chi_e((U, V), (h^+, e^+)) = (s(h), s(e)), \quad U \in A, V \in \mathcal{V}.
\]
The state space of the automaton is
\[ \{ ((b, a), (t(b), t(a))) : (b, a) \in \Pi_{G(V, E)} \times \Pi_{G(V, E)} \}. \]

8.7. Also the R-graph shifts of [Kr6] provide examples of \((C-F)\)-semisynchronizing shifts

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Wolfgang Krieger
Institute for Applied Mathematics,
University of Heidelberg,
Im Neuenheimer Feld 205,
69120 Heidelberg,
Germany
krieger@math.uni-heidelberg.de