Two mass conjectures on axially symmetric black hole–disk systems.

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We analyze stationary self-gravitating disks around spinning black holes that satisfy the recently found general-relativistic Keplerian rotation law. There is a numerical evidence that the angular velocity, circumferential radius and angular momenta yield a bound onto the asymptotic mass of the system. This bound is proven analytically in the special case of massless disks of dust in the Kerr spacetime.

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I. INTRODUCTION

Angular velocities $\Omega$ of test bodies in a nonrelativistic Keplerian circular motion encode information about central masses. The central mass is inferred to be $\Omega^2 r_C^3 / G$, where $G$ is the gravitational coupling constant and $r_C$ denotes the circumferential radius of the orbit of the body. Recent numerical analysis suggests that one can estimate the total mass $m$ also for a system with selfgravitating fluids in Keplerian rotation around a spherical centre, $\Omega^2 r_C^3 / G \leq m$ [1, 2]. In this paper we shall investigate general-relativistic selfgravitating disks or toroids around spinning central black holes. We shall propose two modified inequalities for selfgravitating rotating systems in the general-relativistic Keplerian motion [3, 4]: they imply $\Omega^3 r_C^3 / G \leq \sqrt{G M_{\text{ADM}}} + 2a / \sqrt{r_C}$, where $a$ and $M_{\text{ADM}}$ are the spin parameter of the black hole and the asymptotic mass of the whole configuration, respectively. There is a rich numerical evidence, some reported in what follows, that supports our conjecture.

The question of estimating masses or determining rotation curves in toroidal systems has been investigated since early 1970’s — primarily a rigid rotation [19, 20] and its modifications [21, 22]. New general-relativistic differential rotation laws $j = j(\Omega)$, where $j$ denotes the specific angular momentum, have recently been found for stationary systems consisting of self-gravitating toroids around spinless [24, 25] and spinning black holes [3, 4]. They describe, in particular, the motion of a massless disks of dust, with $\Omega = \sqrt{G M_{\text{ADM}} r_C^3 / 2}$ in the case of the Schwarzschild geometry. In the nonrelativistic limit one has the monomial rotation law $\Omega = w / r_C^2$ (0 $\leq \lambda \leq 2$); that obviously includes the Keplerian rotation law.

The order of the main part of this paper is as follows. Section II describes formalism. Section III contains two main conjectures concerning masses of axially symmetric stationary systems with black holes. In Section IV we show that test-like disks of dust satisfy the two mass conjectures in Schwarzschild and Kerr geometries. The proof concerning the Kerr spacetime is relegated to the Appendix. Section V presents a sample of numerical examples that confirm our two conjectures.

II. EQUATIONS

We assume a stationary metric of the form

$$ds^2 = -\alpha^2 dt^2 + r^2 \sin^2 \theta \psi^4 (d\varphi + \beta dt)^2 + \psi^4 r^2 \left(dr^2 + r^2 d\theta^2 \right).$$

(1)

Here $t$ is the time coordinate, and $r$, $\theta$, $\varphi$ are spherical coordinates. In the general-relativistic part of this paper the gravitational constant $G = 1$ and the speed of light $c = 1$. We assume axial symmetry and employ the stress-momentum tensor

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + pg^{\alpha\beta},$$

where $\rho$ is the baryonic rest-mass density, $h$ is the specific enthalpy, and $p$ is the pressure. Metric functions $\alpha$, $\psi$, $q$ and $\beta$ in (1) depend on $r$ and $\theta$ only.

The following method can be applied to any barotropic equation of state, but we will deal with polytropes $p(\rho) = K \rho^n$. Then one has the specific enthalpy

$$h(\rho) = 1 + \frac{np}{(\gamma - 1)\rho},$$
The 4-velocity \((u^a) = (u^t, 0, 0, u^r)\) is normalized, \(g_{\alpha\beta}u^\alpha u^\beta = -1\). The coordinate angular velocity reads
\[
\Omega = \frac{u^\phi}{u^t}.
\]
We define the angular momentum per unit inertial mass \(\rho \Omega \) [26]
\[
j \equiv u_\phi u^t.
\]
It is known since early 1970’s that general-relativistic Euler equations are solvable under the condition that the squared linear velocity is given by
\[
\nu^2 = \theta^2 (\Omega + \beta)^2 \psi^4 \, \frac{\alpha^2}{\beta}. \tag{3}
\]
In this construction the angular momentum is given rigidly on the event horizon \(S_{BH}\), in terms of data taken from the Kerr solution and independently of the content of mass in a torus, \(J_{BH} = m \tilde{a} [13]\). The mass of the black hole is then defined in terms of the irreducible mass and the angular momentum,
\[
M_{BH} = M_{irr} \sqrt{1 + \frac{J_{BH}^2}{4M_{irr}^4}}. \tag{9}
\]
We define the black hole spin parameter as \(a = J_{BH}/M_{BH}\). If the disk is sufficiently massive (self-gravitating), we have in general \(a \neq \tilde{a} [3, 4]\). If the self-gravity of the torus can be neglected, then \(M_{BH} = m\), \(a = \tilde{a}\), and the metric of the spacetime coincides (by construction) with the Kerr solution.

Asymptotic (total) mass \(M_{ADM}\) and angular momentum \(J_{ADM}\) can be defined as appropriate Arnowitt-Deser-Misner charges, and they can be computed by means of corresponding volume integrals [13].

A circumferential radius corresponding to the circle \(r = \text{const on the symmetry plane } \theta = \pi/2\) is given by
\[
r_C = r_C^2. \tag{10}
\]

The numerical part of this paper is based on the scheme described in [3, 4]. In the rest of the paper we always assume that \(\Omega > 0\). Corotating disks have \(a > 0\), while counterrotating disks have negative spins: \(a < 0\).

### III. TWO MASS CONJECTURES

General-relativistic Keplerian systems with tori are characterized by their asymptotic masses \(M_{ADM}\) and angular momenta \(J_{ADM}\), and the quasilocal characteristics of central black holes—the mass \(M_{BH}\) and the angular momentum \(J_{BH}\). It is clear that in the Newtonian limit we should get \(\Omega_C^{3/2} \leq \sqrt{m} [12]\). From this, and from the dimensional analysis, one may guess that \(\Omega_C^{3/2} \leq \sqrt{M_{ADM} + 2 \frac{|x|}{m v_x \sqrt{\alpha}}\sqrt{\psi}}\), where \(X, Y = BH, ADM\). The asymptotic mass is larger than the mass of the central black hole, but the angular momentum is not monotonic — the asymptotic angular momentum might be smaller than \(J_{BH}\). That means that there are two independent possibilities.

**Conjecture 1.**
\[ \Omega_{C}^{3/2} \leq \sqrt{M_{\text{ADM}}} + \frac{2|J_{\text{BH}}|}{M_{\text{ADM}}\sqrt{r_{C}}} \]  

Conjecture 2.

\[ \Omega_{C}^{3/2} \leq \sqrt{M_{\text{ADM}}} + \frac{2|J_{\text{BH}}|}{M_{\text{ADM}}\sqrt{r_{C}}} \]  

Notice that \( \frac{|J_{\text{BH}}|}{M_{\text{ADM}}} \leq |a| \), where \( a \) is the black hole spin parameter. Thus Conjecture 2 implies \( \Omega_{C}^{3/2} \leq \sqrt{M_{\text{ADM}}} + \frac{2|a|}{\sqrt{r_{C}}} \).

Numerical calculations reported in Sec. VI suggest the validity of both Conjectures. They coincide for test disks of dust in the Kerr geometry, since then \( M_{\text{BH}} = M_{\text{ADM}} \) and \( J_{\text{ADM}} = J_{\text{BH}} \). The inequality reduces to the equality in the Schwarzschild spacetime and it is satisfied in Kerr spacetimes (see the Appendix for an algebraic proof).

The inspection of both inequalities (11) and (12) shows that far from the center the corresponding expressions on the right-hand sides are expected to be constant; if a toroid is light and large, then \( \Omega_{C}^{3/2} \) is expected to be close to \( \sqrt{M_{\text{ADM}}} \) (see Sec. V). The inspection of these inequalities in the interior might give some information about angular momentum.

IV. ANGULAR MOMENTUM AND MASS ESTIMATES IN KERR AND SCHWARZSCHILD SPACETIMES

A. Kerr geometry in conformal coordinates

Define

\[ r_{K} = r \left( 1 + \frac{m}{r} + \frac{m^{2} - a^{2}}{4r^{2}} \right), \]  

\[ \Delta_{K} = r_{K}^{2} - 2r_{K} + a^{2}, \]  

and

\[ \Sigma_{K} = r_{K}^{2} + a^{2}\cos^{2}\theta. \]

The Kerr metric can be written in form (1) as follows

\[ \psi_{K} = \frac{1}{\sqrt{r}} \left( \frac{r_{K}^{2} + a^{2} + 2ma^{2}r_{K}\sin^{2}\theta}{\Sigma_{K}} \right)^{1/4}. \]  

The only component \( \beta_{K} \) of the shift vector is given by

\[ \beta_{K} = -\frac{2ma}{r}\frac{r_{K}}{r_{K}^{2} + a^{2}}\Sigma_{K} + 2ma^{2}r_{K}\sin^{2}\theta. \]

Finally, the functions \( \alpha_{K} \) and \( q_{K} \) are defined as

\[ \alpha_{K} = \left( \frac{\Sigma_{K}\Delta_{K}}{(r_{K}^{2} + a^{2})\Sigma_{K} + 2ma^{2}r_{K}\sin^{2}\theta} \right)^{1/2}, \]

\[ e^{\delta_{K}} = \frac{\Sigma_{K}}{\sqrt{(r_{K}^{2} + a^{2})\Sigma_{K} + 2ma^{2}r_{K}\sin^{2}\theta}}. \]  

The surface \( r = r_{s} \equiv \sqrt{\frac{m^{2} - a^{2}}{2}} \) is an apparent horizon, that coincides with the event horizon.

Test particles can rotate along circular orbits \( r = \text{const} \) in the Kerr geometry. That implies the existence of a test-like disk made of dust, that moves circularly and lies in the plane \( \theta = \pi/2 \). Its angular velocity reads

\[ \Omega(r) = \frac{8\sqrt{r}}{((2r + 1)^{2} - a^{2})^{3/2} + 8ar^{3/2}}. \]  

The dragging angular velocity of the Kerr space-time is given by \( \Omega_{d} = -\beta_{K} \), that is

\[ \Omega_{d} = \frac{2ma}{r_{K}(r_{K}^{2} + a^{2}) + 2ma^{2}}. \]  

The circumferential radius of this circular orbit is equal to

\[ r_{C} = r_{K}\psi^{2} = \sqrt{r_{K}^{2} + a^{2} + 2ma^{2}r_{K}}. \]

It is clear that

\[ \Omega_{d} = \frac{2ma}{r_{K}r_{C}^{2}}. \]  

It appears that the product \( \Omega_{C}^{3/2} \) exceeds the value \( \sqrt{m} \) for all strictly negative spin parameters \( a \) and for \( a > 0.9525 \). There exists, however, a modified inequality that takes into account the dragging effects and that is always true:

\[ \Omega_{C}^{3/2} \leq \sqrt{m} + |\Omega_{d}|^{3/2} C = \sqrt{m} + \frac{2m|a|}{r_{K}r_{C}^{1/2}}. \]  

The proof of (21) is relegated to the Appendix.

The quantity \( m|a| \) is the absolute value \( |J_{\text{BH}}| \) of the angular momentum, while \( r_{K} \geq m \) outside the region encircled by the trapped surface \( r = r_{s} \). Thus for a disk in Keplerian motion around a Kerr black hole we have

\[ \Omega_{r_{C}^{3/2}} \leq \sqrt{m} + 2\frac{J_{\text{BH}}}{m\sqrt{r_{C}}} = \sqrt{m} + 2\frac{|a|}{\sqrt{r_{C}}}. \]  

This agrees with both inequalities (11) and (12), since they coincide in this case.
B. Schwarzschild geometry

Schwarzschild space-time in conformal coordinates is given by metric functions of the preceding subsection assuming the spin parameter $a = 0$. Test bodies in a Schwarzschild space-time can move on circular orbits with the angular velocity $\Omega = \frac{\sqrt{m}}{r^2}$. Thus there exist massless disks, consisting of particles of dust, with the rotation law $\Omega = \frac{\sqrt{m}}{r^2}$, which implies the strict equality $\Omega r_{C}^{3/2} = \sqrt{m}$; the inequalities (11) and (12) coincide, and they are saturated.

V. NUMERICAL RESULTS

In this section we deal with numerical solutions describing self-gravitating fluids around black holes. The numerical method was described in detail in [3, 4]. It is convenient to rewrite inequalities (11) and (12) in the following form.

i) Conjecture 1.

$$\max\left(\left(\Omega - \Omega_{d,1}\right)r_{C}^{3/2}\right) \leq \sqrt{M_{\text{ADM}}};$$  \hspace{1cm} (23)

here $\Omega_{d,1} = \frac{2J_{M_{\text{ADM}}}}{M_{\text{ADM}}r_{C}^{2}}$.

ii) Conjecture 2.

$$\max\left(\left(\Omega - \Omega_{d,2}\right)r_{C}^{3/2}\right) \leq \sqrt{M_{\text{ADM}}};$$  \hspace{1cm} (24)

here $\Omega_{d,2} = \frac{2J_{M_{\text{ADM}}}}{M_{\text{ADM}}r_{C}^{2}}$.

We performed a large number of numerical calculations, with equations of state $p = K\rho^{1/3}$ and $p = K\rho^{5/3}$, for a range of values of the spin parameter $a$, the radius of the inner boundary $r_{1}$ and for several values of the maximal mass density $\rho_{\text{max}}$. In this paper we report only results concerning the polytropes with the polytropic index $\gamma = 4/3$, but the other case gives similar results. In numerical solutions of Figs. 1 and 2 we fix the coordinate radius of the disk’s outer boundary $r_{2} = 20$, but the inner radius $r_{1}$ is changed, within limits shown in captions of Figures. Figures 3–5 are dedicated to the analysis of $\Omega r_{C}^{3/2}$ in specific solutions. In all cases we calculated values of the left hand sides of (23) and (24) in the symmetry plane $\theta = \pi/2$.

Figure 1 summarizes results of more than four hundred (counter-rotating) numerical solutions. The diagram shows maximal values of the left hand sides of (23) and (24), in units of the square root of the asymptotic mass. There is a sharp spike in the diagram corresponding to Conjecture 1 (see (23)); around $r_{1} = 11.7$ the angular momentum of the disk cancels the black hole spin. The asymptotic angular momentum $J_{\text{ADM}}$ decreases, vanishes at the top of the spike and becomes more and more negative. Conjecture 2 is also satisfied, but the expression $\max((\Omega - \Omega_{d,2})r_{C}^{3/2})/\sqrt{M_{\text{ADM}}}$ changes only moderately. Figure 2 presents results of more than seven hundred (co-rotating) numerical solutions. It is clear that both proposed conjectures, 1 and 2, are valid in our numerical calculations, for co- and counter-rotating systems.

![Figure 1](image1)

**FIG. 1.** Maximal values of left hand-sides of Conjectures 1 (conj. 1) and 2 (conj. 2) for $\gamma = 4/3$, $\tilde{a} = -0.99$ and $\rho_{\text{max}} = 3.0 \times 10^{-4}$. Here $r_{2} = 20$ and $r_{1}$ varies from 6.01 to 18.97. Asymptotic masses are in the range $(1.005, 1.46)$. Each point on the diagram corresponds to a solution.

![Figure 2](image2)

**FIG. 2.** Maximal values of left hand-sides of Conjectures 1 and 2 for $\gamma = 4/3$, $\bar{a} = 0.99$ and $\rho_{\text{max}} = 3.0 \times 10^{-4}$. Here $r_{2} = 20$ and $r_{1}$ varies from 0.805 to 19.05. Asymptotic masses are in the range $(1.005, 1.46)$. Each point on the diagram corresponds to a solution.

Figure 3 shows the behaviour of the product $\Omega r_{C}^{3/2}$ on...
the symmetry plane of a compact disk—its outer circumferential radius is less than 21.6 and the asymptotic mass $M_{\text{ADM}} \in (1.45, 1.47)$ while the mass of the central black hole is $M_{\text{BH}} = 1.02$. We display two curves, for $a = 0.49$ and $a = -0.485$. The dependence on the spin is more pronounced in disk’s interior and becomes negligible in disk’s peripherals.

Figure 4 shows the behaviour of the product $\Omega r_C^3/2$ on the symmetry plane of the disk. The disk is quite extended and somewhat heavier —its outer circumferential radius is larger than 100 and the asymptotic mass $M_{\text{ADM}} = 2$ while the mass of the black hole is $M_{\text{BH}} = 1.02$. We display two curves, for $a = 0.49$ and $a = -0.49$. The dependence on the spin is seen in disk’s interior and becomes negligible in disk’s peripherals.

Figure 5 shows the same as Fig. 4, but the spin is higher. Again the disk is extended but it is light—its outer circumferential radius is larger than 100 and the asymptotic mass $M_{\text{ADM}} = 1.1$ while the mass of the black hole is $M_{\text{BH}} = 1.00$. We display two curves, for $a = 0.99$ and $a = -0.99$. A strong dependence on the spin is seen in disk’s interior, but it is noticeable even in disk’s peripherals. Notice that $\text{sup}(\Omega r_C^3/2) > 1.1$ for the two counter-rotating branches, but only the massless co-rotating branch has $\Omega r_C^3/2$ exceeding 1.

It is instructive to compare the two self-gravitating solutions in Fig. 3 with the two massless disks of dust in the Kerr geometry. It is clear that the self-gravity merely pushes down the value of $\Omega r_C^3/2/M_{\text{ADM}} - 1$ obtained for dust solutions in Kerr space-times. The branch corresponding to the counter-rotation remains above the co-rotating one, but they almost coincide at the outer boundary. A similar picture is seen in Fig. 5. We see the same effect in Fig. 4, but with a more pronounced shift downwards. The co-rotating and counter-rotating branches cross around $r_C \approx 30$ and then their positions do reverse. Thus the more massive is the rotating toroid, the stronger the self-gravity impacts the quantity $\Omega r_C^3/2$. Figure 5 demonstrates yet another influence of the self-gravity; the inner boundary of the co-rotating toroid shifts upwards from $r_C = 2.108$ (in the Kerr spacetime) to $r_C = 2.32$ and of the counter-rotating toroid moves downwards from $r_C = 9.04$ (in the Kerr spacetime) to $r_C = 4.68$.

**VI. SUMMARY**

The two conjectures on masses of rotating systems are based partly on analytic arguments and partly on extensive numerical data. Their proof would pose a serious analytic challenge.

We envisage two further applications. The first concerns astrophysics. Rotating axially symmetric systems are quite common. They are known to exist in some active galactic nuclei. Particularly interesting are those containing supermasers. Our inequalities might be useful in extracting information about masses and angular momenta, provided that more information on modelling of AGN’s with toroids becomes available.

The two statements of [11] and [12] can be useful in estimating the amount of angular momentum within a fixed volume. There is already a formidable work done in
Ω = \frac{u_\phi}{a^2} = \frac{\sqrt{m}}{r_K^{3/2} + a\sqrt{m}} \quad (26)

The circular radius \( r_C \) can be expressed as follows, in the symmetry plane:

\[ r_C = \sqrt{r_K^2 + a^2 + \frac{2ma^2}{r_K}}. \quad (27) \]

The angular velocity due to dragging plays a significant role in the calculation. It is given by

\[ \Omega_d = \frac{2ma}{r_C^2 r_K}. \quad (28) \]

The irreducible mass of the Kerr black hole reads

\[ M_{\text{irr}} = \frac{m}{2} \sqrt{2 \left( 1 + \sqrt{1 - \frac{a^2}{m^2}} \right)} \quad (29) \]

The quantity \( r_{\text{isco}} \) denotes the coordinate radius of the innermost stable circular orbit (ISCO), that depends on \( a \) and \( m \). In the case of co-rotation \( r_{\text{isco}} \geq m \) (with the equality when \( a = 1 \)) while in the case of counter-rotation \( 6m \leq r_{\text{isco}} \leq 9m \) (with the upper bound saturated for \( a = -1 \)).

### B. Proofs

We shall prove analytically the validity of the following inequality, provided that the areal radius is not smaller than \( r_{\text{isco}} \): \( r_K \geq r_{\text{isco}} \):

\[ \Omega r_C^{3/2} \leq \sqrt{m} + |\Omega_d| r_C^{3/2}. \quad (30) \]

For the simplicity, but without the loss of generality, we shall put \( m = 1 \).

We divide both sides of (30) by \( r_C^{3/2} \), that leads to

\[ \Omega - |\Omega_d| \leq \sqrt{\frac{1}{r_C}}. \quad (31) \]

Using (13, 26, 29), one arrives at

\[ \frac{1}{r_K^{3/2} + a} - \frac{2|a|}{r_K (r_K^2 + a^2) + 2a^2} \leq \frac{a^2}{r_K^2} \left( \frac{2a}{r_K^2 + r_K^2} \right)^{-3/4}. \quad (32) \]

a. The case \( a = 0 \) corresponds to the Schwarzschild geometry and it has been already discussed.
b. In the corotating case, \( a > 0 \), inequality (32) takes the form

\[
\frac{1}{r_K^{3/2} + a} - \frac{2a}{r_K (r_K^2 + a^2) + 2a^2} \leq \left( a^2 + \frac{2a^2}{r_K} + r_K^2 \right)^{-3/4}.
\]  

(33)

It is easy to show that the left hand side of this inequality is non-negative, since \((r_K^2 - 2a\sqrt{r_K + a^2}) \geq (r_K - a)^2\), which is obviously non-negative. That in turn implies the non-negativity of (34). Therefore the sign of the inequality (35) does not reverse when we calculate the quartic power of both sides,

\[
0 \leq \frac{r_K^3}{(r_K^2 + a^2) + 2a^2} (a^2 + \frac{2a^2}{r_K} + r_K^2)^4 \times 
\left( r_K (r_K^2 + a^2) + 2a^2 \right)^{-3/4} 
- r_K \left( 2a r_K^{3/2} + a^2 \right)^4.
\]  

(35)

The denominator of (35) is positive for \( r_K \geq r_{\text{ISCO}} \geq 1 \). One can find out, using the REDUCE function of Mathematica [28], that the numerator of (35) is also non-negative, if \( r_K \) is not smaller than 1.

c. The counter-rotating case \( a < 0 \). Changing \( a \rightarrow -|a| \) in the inequality (32), one arrives at

\[
\frac{1}{r_K^{3/2} + |a|} - \frac{2|a|}{r_K (r_K^2 + |a|^2) + 2|a|^2} \leq \left( a^2 + \frac{2a^2}{r_K} + r_K^2 \right)^{-3/4}.
\]  

(36)

It is easy to show that the left hand side of this inequality is nonpositive, if and only if

\[
r_K^3 - 2a r_K^{3/2} + a^2 (4 + r_K) \geq 0.
\]  

(37)

However for \( r_K \geq 0 \):

\[
r_K^3 - 2a r_K^{3/2} + a^2 (4 + r_K) \geq 0 
\geq r_K^3 - 2a |r_K^{3/2} + 4a^2 = (r_K^{3/2} - |a|)^2 + 3a^2,
\]

which is manifestly non-negative. Thus the left-hand side of (36) is non-negative for \( r \geq r_K \). Again we can calculate the quartic power of both sides,

\[
0 \leq \frac{1}{(r_K (r_K^2 + |a|^2) + 2|a|^2)^4 \left( r_K^{3/2} - |a| \right)^4 - 
\left( r_K^3 - 2a r_K^{3/2} + a^2 (4 + r_K) \right)^4}.
\]  

(38)

The denominator of (38) is positive for \( r_K \geq 1 \). One can find out, using the REDUCE function of Mathematica [28], that the numerator is also non-negative, if \( r_K \) is not smaller than 6. In conclusion, in the interval of interest for counter-rotation \( (r_K > r_{\text{ISCO}} \geq 6) \) inequality (36) holds true.

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