Matter effects for ‘just so’ oscillations

Mohan Narayan and S. Uma Sankar

Department of Physics, I.I.T. Powai, Mumbai 400076, India

(November 8, 2018)

Abstract

We study the effect of the matter term on the evolution of the solar neutrinos when the neutrino parameters are those of the ‘just-so’ case. The extreme non-adiabatic effects at the edge of the sun reduce the expression for the survival probability in the just-so case to that of the vacuum case. This conclusion is independent of the width of the extreme non-adiabatic region, which is a function of the density profile of the sun beyond \( r > 0.9 R_s \). However, in its propagation, neutrino encounters regions of moderate (non-extreme) non-adiabaticity. Neutrino traversal through these regions give corrections to the survival probability which are profile dependent.

I. INTRODUCTION

The solar neutrino data of Super-Kamiokande has lead to a reexamination of all the solutions of the solar neutrino problem [1]. In addition to the overall suppression rate, Super-Kamiokande has measured the day-night asymmetry and the spectrum of the scattered electrons. These two measurements are independent of the overall normalization of the \(^8\text{B}\) neutrino flux. This normalization has the largest uncertainty among all the predictions of the solar models [2,3]. Hence data, which are independent of it, are extremely important in studying the properties of \(^8\text{B}\) neutrinos from the sun. The present data, especially the recoil spectrum, favor ‘just-so’ oscillation solution to the solar neutrino problem [4,5].
In analyzing the solar neutrino data in terms of ‘just-so’ oscillations, the expression used for electron neutrino survival probability $P_{ee}$ is simply the vacuum survival probability. Given that the matter term in the sun is several orders of magnitude greater than the mass-squared difference, the question arises whether the use of vacuum survival probability is justified. A very simple justification can be given the following way: The mass-squared difference $\delta$ required for just-so oscillations is about $10^{-10}$ eV$^2$. The matter term

$$A \text{ (in eV}^2\text{)} = 0.76 \times 10^{-7} \rho \text{ (in gm/cc)} E \text{ (in MeV)},$$

(1)
is much larger than $\delta$ as long as the density $\rho$ is greater than 0.01 gm/cc. First we assume that the density is spherically symmetric and it falls abruptly to 0 once it decreases to a value of 0.01 gm/cc. As long as the neutrino is in the sun, $A$ completely overwhelms $\delta$ and the electron neutrino in the sun is essentially the higher mass eigenstate. Hence during its travel through the sun, the neutrino does not oscillate and it simply acquires a phase. Thus throughout the travel through the sun, an electron neutrino remains an electron neutrino. When the neutrino comes out of the sun, it passes through the abrupt change in density from 0.01 to 0. In such a case, the flavour composition is unchanged. Since the neutrino in the sun remained an electron neutrino, we have an electron neutrino coming out of the sun. Thus the starting point of neutrino evolution is transferred from the core of the sun to the edge of the sun [6].

But now we can raise the question: Suppose the solar matter density does not change abruptly to zero but goes to zero smoothly. Then at some point in its propagation the neutrino will pass through a region where the density is equal to the resonant density. In such a situation, how is the neutrino oscillation probability modified? We address the question below.

**II. JUST-SO SURVIVAL PROBABILITY**

For simplicity here we consider only two flavor oscillations. The flavor states $\nu_e$ and $\nu_\mu$ are linear combinations of the two mass eigenstates $\nu_1$ and $\nu_2$. 
\[ \nu_e = \cos \theta \, \nu_1 + \sin \theta \, \nu_2 \]
\[ \nu_\mu = -\sin \theta \, \nu_1 + \cos \theta \, \nu_2. \quad (2) \]

Without loss of generality we can take \( \nu_2 \) to be more massive than \( \nu_1 \) and hence \( \delta = m_2^2 - m_1^2 > 0 \). If \( \theta < \pi/4 \), then \( \nu_e \) is predominantly the lighter state. If \( \theta > \pi/4 \), then \( \nu_e \) is predominantly the heavier state. Thus the two physically distinguishable possibilities are covered by taking the range of \( \theta \) to be \((0, \pi/2)\). Since we are considering ‘just-so’ oscillations, we will assume that \( \delta = 10^{-10} \text{eV}^2 \). We will also assume that the value of \( \theta \) is moderately large, \textit{i.e.} \( \theta \) is not finetuned to be very small or be very close to \( \pi/4 \).

Electron neutrinos are produced in the core of the sun. Since the solar matter near the core is very dense, one must take the matter term \( A \) into account in determining the mass eigenstates in the core. We can define instantaneous matter dependent mass eigenstates

\[ \nu_e = \cos \theta_m \, \nu_{1m} + \sin \theta_m \, \nu_{2m} \]
\[ \nu_\mu = -\sin \theta_m \, \nu_{1m} + \cos \theta_m \, \nu_{2m}, \quad (3) \]

where \( \theta_m \) is the matter dependent mixing angle and is given by

\[ \cos 2\theta_m = \frac{\delta \cos 2\theta - A}{\delta_m}. \quad (4) \]

\( \delta_m \) is the matter dependent mass-square difference and is given by

\[ \delta_m = \sqrt{(\delta \cos 2\theta - A)^2 + (\delta \sin 2\theta)^2}. \quad (5) \]

As the neutrino travels through the sun, it encounters matter of constantly decreasing density. In the basis of matter dependent mass eigenstates, the evolution equation for the neutrino is

\[ \begin{pmatrix} i \frac{d}{dt} \nu_{1m} \\ i \frac{d}{dt} \nu_{2m} \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\delta_m(t) & -4i E \hat{\theta}_m(t) \\ 4i E \hat{\theta}_m(t) & \delta_m(t) \end{pmatrix} \begin{pmatrix} \nu_{1m} \\ \nu_{2m} \end{pmatrix}. \quad (6) \]

The propagation of neutrino is adiabatic as long as the off-diagonal terms are smaller than the difference of the diagonal terms in the above equation, \textit{i.e.}
\[ \delta_m \gg \frac{2EA\delta \sin 2\theta}{\delta^2_m} \left| \frac{\dot{A}}{A} \right|. \] (7)

In the body of the sun the density profile is of the form

\[ \rho(r) = \rho_0 \exp(-10.54r/R_s) \] (8)

where \( \rho_0 \) is the core density (about 140 gm/cc) and \( R_s \) is the radius of the sun (about \( 7 \times 10^5 \) km) [2]. Therefore

\[ \left| \frac{\dot{A}}{A} \right| = \left| \frac{\dot{\rho}}{\rho} \right| = \frac{10.54}{R_s} = 3 \times 10^{-15} \text{ eV}. \] (9)

As long as \( \rho \geq 0.01 \text{ gm/cc} \) (or \( r \leq 0.9R_s \)) the matter term \( A \) is much larger than \( \delta \), hence \( \delta_m \approx A \). According to our assumption on \( \theta \), \( \sin 2\theta \approx 1 \). Substituting all these in Eq. (7), the adiabatic condition becomes

\[ A^2 \gg 6 \times 10^{-19}(E/\text{MeV}). \] (10)

The most compelling reason for considering just-so oscillations as a solution to the solar neutrino problem is Super-Kamiokande data on electron spectrum due to \(^8\text{B}\) neutrinos. The observed range of these neutrinos is 6-14 MeV and they are peaked around 10 MeV. Henceforth, we will take \( E = 10 \text{ MeV} \) for illustrative purpose. Conservatively we require that \( A^2 \) should be at least 10 times the value on RHS. With this requirement, we find that the adiabatic condition is satisfied if

\[ \rho \geq 0.01 \text{ gm/cc}. \] (11)

The adiabatic condition is likely to break down if \( \rho \leq 0.01 \text{ gm/cc} \), or for radial distances greater than \( r > 0.9R_s \).

When the neutrino is well out of the sun the matter term is much smaller than \( \delta \) and adiabaticity is restored. In such a case we have, essentially, vacuum propagation. The radial distance at which this happens is a function of the density profile of the sun beyond 0.9\( R_s \). For example, if the density falls linearly beyond 0.9\( R_s \), vacuum propagation starts around
\( r = R_s \). If the density fall continues to be the exponential form shown in Eq. (8), then vacuum propagation starts only around \( 2R_s \). Between the time of breakdown of the adiabaticity and the time of its restoration, the off-diagonal terms in the equation of propagation, Eq. (9), are comparable to the diagonal terms. In fact, for some intermediate region, the off-diagonal terms completely dominate, making the propagation extremely non-adiabatic. These extreme non-adiabatic effects cause the just-so oscillation probability to reduce to vacuum oscillation probability.

An electron neutrino is produced in the core of the sun at time \( t_0 \) and propagates adiabatically up to \( t_1 \). Between times \( t_1 \) and \( t_2 \) its propagation is non-adiabatic. Beyond \( t_2 \) the neutrino propagates in vacuum and is detected at \( t_3 \). Its state vector at \( t_3 \) is

\[
|\Psi_e(t_3)\rangle = \sum_{j,i} |\nu_j\rangle \exp(-i\varepsilon_j(t_3 - t_2)) M_{ij} \exp\left(-i \int_{t_0}^{t_1} \varepsilon_i^S(t) dt\right) U_{e_i}^C, \tag{12}
\]

where \( \varepsilon_i^S(t) \) are the matter dependent energy eigenvalues in the sun and \( \varepsilon_i \) are the energy eigenvalues in vacuum. \( M_{ij} \) is the amplitude for non-adiabatic transition from mass eigenstate \( \nu_{im} \) at \( t = t_1 \) to mass eigenstate \( \nu_j \) at \( t = t_2 \). \( U_{e_i}^C \) is the \( e - i \) element of the matter dependent mixing matrix at the core of the sun. From the above equation, we obtain the electron neutrino survival probability to be

\[
P_{ee} = |\langle \nu_e |\Psi_e(t_3)\rangle|^2
= \sum_{j,j'} \sum_{j'j} U_{ej} M_{ij} U_{e_i}^C U_{e_j'} M_{j'j} U_{e_i}^C \exp\left(i \left(\int_{t_0}^{t_1} (\varepsilon_i^S(t) - \varepsilon_j^S(t)) dt\right)\right) \exp\left\{i(\varepsilon_{j'} - \varepsilon_j)(t_3 - t_2)\right\}. \tag{13}
\]

Because of the variation in the production region, the phase picked up in the time from \( t_0 \) to \( t_1 \) can be averaged out. So \( \exp\left(i \int_{t_0}^{t_1} (\varepsilon_i^S(t) - \varepsilon_j^S(t)) dt\right) \) can be replaced by \( \delta_{ii'} \). Thus, \( P_{ee} \) is simplified to

\[
P_{ee} = \sum_{i,j,j'} U_{ej} M_{ij} U_{e_i}^C U_{e_j'} M_{j'j} U_{e_i}^C \exp\left\{i(\varepsilon_{j'} - \varepsilon_j)(t_3 - t_2)\right\}. \tag{14}
\]

The mixing angle at the core of the sun \( \theta_C = \pi/2 \), because the electron neutrino is the heavier mass eigenstate due to the dominance of the matter term. In vacuum, of course, it is \( \theta \). Substituting these in the expression for \( P_{ee} \), we get
\[ P_{ee} = x_{12} \cos^2 \theta + (1 - x_{12}) \sin^2 \theta + 2 \sin \theta \cos \theta R (M_{12}' M_{22}) \cos \left( 2.54 \frac{\delta E}{E} \right), \] (15)

where \( x_{12} = |M_{12}|^2 \) is the probability for the \( \nu_{1m}(t_1) \) evolve into \( \nu_2 \) at \( t_2 \). In the above equation we have also made the extreme non-relativistic approximation for neutrinos and replaced the difference in energies by the mass-squared difference and the time time of travel \((t_3 - t_2)\) by the distance of travel \(x\).

The survival probability given in Eq. (15) should give us vacuum survival probability if the density falls abruptly to 0, that is \( t_2 - t_1 \) is extremely small. In such a case, it was shown that \( M_{12} = (U_S^\dagger U_{12}) = \sin(\theta_S - \theta) = \cos \theta \) because \( \theta_S \), the mixing angle in the sun, is \( \pi/2 \) [6]. For \( M_{12} = \cos \theta \), \( P_{ee} \) immediately reduces to vacuum survival probability. Now the question is: what is \( M_{12} \) if the density falls smoothly to zero and the distance between the point of breakdown of adiabaticity and the point of start of vacuum propagation is a significant fraction of solar radius. In such a case, the calculation of \( M_{12} \) is complicated and depends on the density profile of the sun beyond \( 0.9R_s \). Let us calculate \( M_{12} \) in a simplified situation where we will assume that the off-diagonal terms are much larger than the diagonal terms between the point of breakdown of adiabaticity \((t_1)\) and the point of restoration of adiabaticity \((t_2)\). That is, in the region where adiabatic approximation is not valid, the evolution is extremely non-adiabatic. Then the evolution equation is

\[
\frac{d}{dt} \begin{bmatrix} \nu_{1m} \\ \nu_{2m} \end{bmatrix} = \frac{1}{4E} \begin{bmatrix} 0 & -4iE \dot{\theta}_m(t) \\ 4iE \dot{\theta}_m(t) & 0 \end{bmatrix} \begin{bmatrix} \nu_{1m} \\ \nu_{2m} \end{bmatrix}. \] (16)

Integrating this equation from \( t_1 \) to \( t_2 \), we get

\[ M_{12} = \sin (\theta(t_1) - \theta(t_2)) . \] (17)

As mentioned above, until the time \( t_1 \), \( A \gg \delta \), hence \( \theta(t_1) = \pi/2 \). At time \( t_2 \), vacuum propagation starts, hence \( \theta(t_2) = \theta \). So we get \( M_{12} = \cos \theta \), which reduces \( P_{ee} \) to vacuum survival probability. Note that we have not made any assumption about how the density varies to zero. The width of non-adiabatic region \((\text{given by } (t_1 - t_2))\) is irrelevant. In
case $\theta \leq \pi/4$ there may be a point within this non-adiabatic region where the resonance condition is satisfied. However, the discussion above is completely independent of whether a resonance exists or not. The reason for this is that the breakdown of adiabaticity condition is independent of the existence of resonance. This is in contrast to the usual MSW effect where the adiabaticity condition breaks down only in the neighbourhood of a resonance.

The above simplification is not very good because, just after the breakdown of adiabaticity and also just before the restoration of the adiabaticity, the diagonal and off-diagonal terms in the evolution matrix are comparable. We need to consider these regions of ‘moderate non-adiabaticity’ separately. To study how good the above assumption of extreme non-adiabaticity throughout the region of non-adiabaticity is, we define a quantity Non-Adiabaticity Paramater (NAP). It is the ratio of the non-diagonal term to the diagonal term in the evolution matrix (Eq. (3))

$$NAP = \frac{4E\dot{\theta}_m}{\delta_m} = \frac{2E\delta \sin 2\theta \dot{A}}{\delta^3_m}. \quad (18)$$

Note that NAP is proportional to $d\rho/dt$ and is much smaller than 1 for regions of adiabaticity and is comparable to or greater than 1 in the region where adiabatic approximation breaks down. The region where NAP is much greater than 1 is the region of extreme non-adiabaticity. Suppose density profile varies smoothly upto the edge of the sun and then abruptly falls to zero. If the adiabatic approximation holds till the edge of the sun, the graph of NAP vs $r$ looks like a Dirac delta function at $r = R_s$. In the discussion below, we consider two different density profiles.

- Exponential density fall (given in Eq. (3)) for all $r$, even beyond $r = 0.9R_s$.

- Exponential density fall upto $r = 0.9R_s$ and then a linear density fall beyond $r = 0.9R_s$.

The equation of the linear fall is determined by requiring that the value of the density and its first derivative should match at $r = 0.9R_s$. 


We see that for linear fall, NAP is zero until \( r \simeq R_s \) and then rises sharply at \( r = R_s \), which resembles the NAP graph for an abrupt density change. Hence for linear fall at the edge of the sun, the just-so oscillation probability reduces to the vacuum oscillation probability. The region of moderate non-adiabaticity is very narrow (\( r = 0.957R_s \) to \( r = 0.991R_s \)).

However for exponential fall, NAP rises slowly, becomes greater than 0.3 (breakdown of adiabatic approximation) at \( r = 0.96R_s \), crosses 10 around \( r = 1.12R_s \) and falls below 10 again for \( r = 1.71R_s \). The adiabatic approximation (NAP \( \leq 0.3 \)) is restored only for \( r = 2.04R_s \). We see that regions of moderate non-adiabaticity are reasonably large both during increasing NAP (0.16\( R_s \)) and during decreasing NAP 0.33\( R_s \). The transition amplitude between the two edges of extreme non-adiabaticity can be easily calculated, as shown in Eqs. (16) and (17). However, the effects of propagation through the regions of moderate non-adiabaticity (0.3 < NAP < 10) must be taken into account. Since the regions of moderate non-adiabaticity are wide, these can be calculated only by numerical integration of the evolution equation in these regions. These moderate non-adiabatic effects can lead to the energy dependence of the just-so survival probability to be different from that of the vacuum survival probability. This investigation is in progress.

As mentioned above, for exponential density fall for all \( r \), the survival probability for just-so oscillations is likely to be different from that of vacuum oscillations. Recently a calculation of such a modified survival probability was presented \([8]\). We find the calculation in \([8]\) untenable for the following reasons.

1. The calculation uses an expression for non-adiabatic jump probability which was calculated under the assumption that the physical region in the sun of extreme non-adiabaticity is very small so that it can be taken to be a point (namely the resonance point) and the non-adiabatic jump is calculated only at this point. However, as can be seen from Figure 1, the physical region of non-adiabaticity is of the order of solar radius, hence the assumption of non-adiabatic jump occurring at a point is unreasonable.
2. In ref [8] it was pointed out that the survival probability is asymmetric under the exchange $\theta \leftrightarrow \pi/2 - \theta$. The expression for survival probability uses a standard formula for non-adiabatic jump probability. However, this formula is derived assuming the existence of a resonance [9]. Hence it cannot be used for scenarios where $\theta > \pi/4$.

III. CONCLUSIONS

- If the solar density abruptly falls to zero at the edge of the sun, just-so oscillation probability reduces to vacuum oscillation probability.

- If the density fall is smooth, the just-so oscillation probability reduces to vacuum oscillation probability if one assumes that the evolution is extremely non-adiabatic throughout the region where adiabatic condition is not valid. This conclusion is independent of the exact density profile near the edge of the sun or the exact width of the region of non-adiabaticity.

- The regions of moderate non-adiabaticity can give corrections to the simple vacuum oscillation probability. However, for linear density fall these corrections are negligible.

- If the recoil spectrum analysis of Super Kamiokande points towards a departure from the simple vacuum oscillation probability for just-so parameters, these departures can be used to obtain information on the density profile at the edge of the sun.

Acknowledgements: We thank Prof. Paul Langacker for a helpful discussion.
FIG. 1. Plot of NAP vs $x (= r/R_s)$, for linear density fall (solid line) and exponential density fall (dotted line).
REFERENCES

[1] Super-Kamiokande Collaboration: Y. Fukuda et al, Phys. Rev. Lett. 81, 1158 (1998); E 81, 4279 (1998); 82, 2430 (1999).

[2] J. N. Bahcall, Neutrino Astrophysics, Cambridge University Press, Cambridge (1987).

[3] J. N. Bahcall and M. H. Pinnsonneault, Rev. Mod. Phys. 67, 781 (1995).

[4] J. N. Bahcall, P. I. Krastev and A. Yu. Smirnov, Phys. Rev. D58, 096016 (1998).

[5] M. B. Smy, hep-ex/9903034, Talk given at DPF 99.

[6] We thank Prof. Paul Langacker for pointing this to us.

[7] T. K. Kuo and J. Pantaleone, Rev. Mod. Phys. 61, 937 (1989).

[8] A. Friedland, hep-ph/0002063.

[9] S. T. Petcov, Phys. Lett. 200 B, 373 (1988); P. I. Krastev and S. T. Petcov, Phys. Lett. 207 B, 64 (1988).