Chiral symmetry breaking from Ginsparg-Wilson fermions

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We calculate the large-volume and small-mass dependences of the quark condensate in quenched QCD using Neuberger’s operator. We find good agreement with the predictions of quenched chiral perturbation theory, enabling a determination of the chiral lagrangian parameter Σ, up to a multiplicative renormalization.

1. Introduction

Spontaneous chiral symmetry breaking (SχSB) is fundamental to our understanding of low energy hadronic phenomena and it is thus important to demonstrate quantitatively that it is a consequence of QCD. A natural candidate for such investigations is the numerical simulation of QCD on a spacetime lattice. SχSB, however, presents the lattice approach with a twofold challenge.

The first is that spontaneous symmetry breaking does not occur in a finite volume. In QCD, a possible signal of SχSB is the presence of a non-vanishing quark condensate defined as:

$$\Sigma = \lim_{m \to 0} \lim_{V \to \infty} \langle \bar{q}q \rangle_{m,V},$$

where $\langle \bar{q}q \rangle_{m,V}$ is the condensate for finite volume $V$ and mass $m$. The double limit in Eq. (1) is rather challenging numerically! To get around this problem, we resort to a finite-size scaling analysis. This involves studying the scaling of the condensate with $V$ and $m$ as the limit of restoration of $\chi S$ is approached ($m \to 0$, $V$ finite).

Such a study requires good control over the chiral properties of the theory, which is the second challenge. Indeed, at finite lattice spacing, “reasonable” discretizations of fermions either break continuum $\chi S$ explicitly or lead to extraneous fermion species [4]. To minimize this problem, we resort to recently rediscovered [3] Ginsparg-Wilson (GW) fermions [5] which break continuum $\chi S$ in a very mild and controlled fashion and actually have a slightly generalized $\chi S$ even at finite lattice spacing [6].

2. Light quarks on a torus

In a large periodic box of volume $V = L^4$ such that $F_\pi L \gg 1$, for small quark masses and assuming the standard pattern of SχSB with $N_f \geq 2$, the QCD partition function is dominated by the nearly massless pions; the system can be described with the first few terms of a chiral lagrangian [7]. If, in addition, $m \to 0$ so that $M_\pi L \approx \sqrt{2m} \ll 1$, the global mode of the chiral lagrangian $U \in SU(N_f)$ dominates the partition function, leading to a regime of restoration of $\chi S$ [8].

In the quenched approximation to which we restrict here, topological zero modes of the Dirac operator induce $1/m$ singularities in $\langle \bar{q}q \rangle_{m,V}$ as $m \to 0$. To subtract these contributions, we work in sectors of fixed topological charge. Generalizing the line of argument given above, the partition function $Z_\nu$, in a sector of topological charge $\nu$, was recently evaluated [9] for the quenched case[9]. The quark condensate in sector $\nu$, proportional to the derivative of $\ln (Z_\nu)$ w.r.t. $m$, is then

$$-\frac{\Sigma_\nu}{\Sigma} = z [I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_{\nu-1}(z)] \frac{\nu}{z} ,$$

where $I_n$ and $K_n$ are modified Bessel functions of the first and second kind, respectively.

We assume here for simplicity that the $N_f$ flavors all have mass $m$.

5. The original unquenched treatment is given in [10].
where \( z \equiv m\Sigma V \) and \( L_\nu(z), K_\nu(z) \) are the modified Bessel functions. As advertised, there is a divergence \( \sim 1/m \) in sectors with topology. These terms, however, are independent of \( \Sigma \).

Eq. (2) summarizes the scaling of the quark condensate with the volume and quark mass in the global mode regime, as a function of only one non-perturbative parameter: \( \Sigma \). Thus, by fitting the dependence of the finite-volume condensate in quark mass and volume to Monte Carlo data, we can extract \( \Sigma \) in a perfectly controlled manner.

3. Ginsparg-Wilson fermions

To perform the finite-volume scaling analysis outlined above, we need to be able to reach the chiral restoration regime without excessive fine-tuning and we need an index theorem to control the contribution of topological zero modes. Both these requirements are satisfied by GW fermions \[1\]. In particular, the leading cubic UV divergence of the condensate is known analytically for GW fermions and can thus be subtracted exactly. The resulting subtracted condensate, \( \Sigma^{\text{sub}} \), however, is still divergent:

\[
\Sigma^{\text{sub}}(a) = C_2 \frac{m^2}{a^2} + \cdots + \Sigma_\nu, \tag{3}
\]

where \( a \) is the lattice spacing. The coefficients of the divergences are not known a priori and have to be determined, preferably non-perturbatively. For the values of \( m \) and \( a \) considered below, however, only the quadratic divergence is important numerically, weaker divergences being suppressed by higher powers of \( m \). A final multiplicative renormalization is still required to eliminate a residual logarithmic UV divergence in \( \Sigma_\nu \).

In the present work, we use Neuberger’s implementation of GW fermions encoded in the Dirac operator \[13\]:

\[
aD_N = (1 + s) \left[ 1 - A/\sqrt{A^\dagger A} \right], \tag{4}
\]

with \( A = 1 + s - aD_W \) where \( D_W \) is the standard Wilson-Dirac operator. The parameter \( s \) must satisfy \(|s| < 1\).

4. Numerical results

We work in the quenched approximation on hypercubic lattices with periodic boundary conditions for gauge and fermion fields. We choose \( \beta = 5.85 \), which corresponds to \( a^{-1} \approx 1.5 \text{ GeV} \) \[14\], and use standard methods to obtain decorrelated gauge-field configurations.

To evaluate \( 1/\sqrt{A^\dagger A} \) in Eq. (4), we use a Chebyshev approximation, \( P_n(x) \), where \( P_n(x) \) is a polynomial of degree \( n \), which gives an exponentially converging approximation to \( 1/\sqrt{x} \) for \( x \in [\epsilon, 1] \) \[14\]. The cost of a multiplication by \( D_N \) is linear in \( n \) and becomes rapidly high. To reduce \( n \) substantially, we perform the improvements described in \[1\]. We take \( s = 0.6 \), a value at which Neuberger’s operator is nearly optimally local for \( \beta = 5.85 \) \[13\].

To determine whether a gauge configuration belongs to the \( \nu = 0 \) or \( \pm 1 \) sectors, we compute the few lowest eigenvalues of \( D_N^\dagger D_N \) by minimizing the relevant Ritz functional \[13\]. As pointed out in \[15\], it is advantageous for this computation, as well as for the inversion of \( (D_N^\dagger + m)(D_N + m) \), to stay in a given chiral subspace. Having determined the topological charge of a configuration, we then obtain the condensate of Eq. (3) in three volumes (8\(^4\), 10\(^4\) and 12\(^4\)) by computing

\[
\Sigma^{\text{sub}}_\nu = \frac{1}{V} \left\{ \text{Tr} \left( \frac{1}{D_N + m} + \text{h.c.} - \frac{a}{1 + s} \right) \right\}_\nu, \tag{5}
\]

where the trace is taken in the chiral sector opposite to that with the zero modes \[13\] and the gauge average is performed in a sector of fixed topology \( \nu \). With this definition, terms \( \sim 1/m \) in Eq. (3) are absent\[14\]. Three gaussian sources and a multimass solver \[17\] were used to compute the trace in Eq. (5) for seven values of \( m \).

We show in Fig. 1 our results for \( a^3 \Sigma^{\text{sub}}_{\nu = \pm 1}/am \) as a function of bare quark mass. We have 15, 10 and 7 gauge configurations on our 8\(^4\), 10\(^4\) and 12\(^4\) lattices, respectively\[15\]. The solid lines are a fit of the data to Eqs. (3) and (4) for all volumes and

\[\text{Table} 1\] for some selected volumes and \( m \).
masses. This fit has only two parameters, namely $\Sigma$ and the coefficient of the quadratic divergence. We find $a^3\Sigma = 0.0032(4)$ and $C_2 = -0.914(8)$.

![Figure 1. Mass dependence of the condensate for the 8$^4$ (circles), 10$^4$ (squares) and 12$^4$ (triangles) lattices. The curves result from a fit to Eqs. (3) and (2).](image)

Clearly, the formulae derived in quenched $\chi$PT give a very good description of the numerical data. The value of $\Sigma$ that we extract is, in physical units, $\Sigma(\mu \sim 1.5 \text{ GeV}) = (221^{+8}_{-9} \text{ MeV})^3$, up to a multiplicative renormalization constant, which has not been computed yet for Neuberger's operator. The quoted error on $\Sigma$ is purely statistical and the statistics are rather small. Quenching and discretization errors, for instance, as well as possible contributions from higher orders in $\chi$PT are not accounted for. Nevertheless, the value obtained and the agreement with $q\chi$PT support the standard scenario of $S\chi$SB.

We further consider the mean value of the lowest non-zero eigenvalue of $\sqrt{D_N^\dagger D_N}$ in different topological sectors. In Random Matrix Theory the distributions of these eigenvalues are given solely in terms of $\Sigma$ [18]. Our determination of $\Sigma$ therefore yields predictions for the mean values. These can then be compared to the average values obtained in simulation. With our 8$^4$ results, we find agreement within roughly one standard deviation for $|\nu| = 1$ (29 configurations) and two for $\nu = 0$ (41 configurations) [19].

Note: Related work with Neuberger’s operator can be found in [15,19,20].

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