A Higher Order GUP with Minimal Length Uncertainty and Maximal Momentum

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Abstract
We present a higher order generalized (gravitational) uncertainty principle (GUP) in the form \([X,P] = i\hbar/(1 - \beta P^2)\). This form of GUP is consistent with various proposals of quantum gravity such as string theory, loop quantum gravity, doubly special relativity, and predicts both a minimal length uncertainty and a maximal observable momentum. We show that the presence of the maximal momentum results in an upper bound on the energy spectrum of the momentum eigenstates and the harmonic oscillator.

Keywords: quantum gravity, generalized uncertainty principle, minimal length uncertainty, maximal momentum.

1. Introduction

In recent years, there is a great interest to study the effects of the Generalized Uncertainty Principle (GUP) and the Modified Dispersion Relation (MDR) on various quantum mechanical systems (see [1] and the references therein). Indeed, the ideas of GUP and MDR arise naturally from various candidates of quantum gravity such as string theory [2–5], loop quantum gravity [6], noncommutative spacetime [7–9], and black holes gedanken experiments [10, 11]. These theories indicate that the Heisenberg uncertainty principle should be modified to incorporate additional constraints in the presence of the gravitational field.

The existence of a minimal length scale of the order of the Planck length \(\ell_{\text{Pl}} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35}\) m is one of the main outcomes of various GUP proposals where \(G\) is Newton’s gravitational constant. In fact, beyond the Planck energy scale, the effects of gravity are so important which would result in discreteness of the very spacetime. Notably, the quantum field theory in curved background can be renormalizable by introducing a minimal observable length as an effective cutoff in the ultraviolet domain. Also, in the string-theoretic argument, we can say that the string cannot probe distances smaller than its own length.

The introduction of this idea has drawn much attention in the literature to study the effects of GUP on small scale and large scale systems [12–31]. It is also possible to incorporate the idea of a maximal observable momentum into this scenario. In fact, in doubly special relativity (DSR) theories, we consider the Planck energy (Planck Momentum) as an additional invariant other than the velocity of light [32–34]. Recently, the construction of a perturbative GUP which is consistent with DSR theories is also discussed in Refs. [35–40]. It is also shown that a minimum uncertainty in momentum can arise from curvature, as part of a study that indicated that curvature and noncommutativity can be seen as dual to each other [41].

In this Letter, we investigate the effects of a new generalized uncertainty principle to all orders in the Planck length on some quantum mechanical systems. This form of GUP implies the existence of a minimal length uncertainty and a maximal momentum in agreement with various theories of quantum gravity. Here, we study the problems of the eigenstates of the position operator, maximal localized states, and the harmonic oscillator in this framework and obtain their energy spectrum, which as we shall see, are bounded from above.

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2. The Generalized Uncertainty Principle

First let us consider a generalized uncertainty principle proposed by Kempf, Mangano and Mann (KMM) and results in a minimum observable length

$$\Delta X \Delta P \geq \frac{\hbar}{2} (1 + \beta (\Delta P)^2 + \zeta) ,$$  \hspace{1cm} (1)

where \(\beta\) is the GUP parameter and \(\zeta\) is a positive constant that depends on the expectation values of the momentum operator, i.e., \(\zeta = \beta \langle P \rangle^2\). We also have \(\beta = \beta_0 / (M_{Pl} c)\) where \(M_{Pl}\) is the Planck mass and \(\beta_0\) is of the order of the unity. It is straightforward to check that the inequality relation (1) implies the existence of a minimum observable length as \((\Delta X)_{KMM}^{\min} = \hbar \sqrt{\beta}\). In one-dimension, the above uncertainty relation can be obtained from the following deformed commutation relation:

\[
[X, P] = i \hbar (1 + \beta P^2).
\]  \hspace{1cm} (2)

As KMM have indicated in their seminal paper, we can write \(X\) and \(P\) in momentum space representation as

\[
P \phi(p) = p \phi(p),
\]  \hspace{1cm} (3)
\[
X \phi(p) = i \hbar (1 + \beta p^2) \partial_p \phi(p).
\]  \hspace{1cm} (4)

Now the symmetricity condition of the position operator implies the following modified completeness relation and scalar product:

\[
\langle \psi|\phi \rangle = \int_{-\infty}^{+\infty} dp \frac{\psi^*(p)}{1 + \beta p^2} \phi(p),
\]  \hspace{1cm} (5)

where \(\int_{-\infty}^{+\infty} dp \frac{1}{1 + \beta p^2} \langle p|p \rangle = 1\) and \(\langle p|p' \rangle = (1 + \beta p^2) \delta(p - p')\). With this definition, the commutation relation (2) is exactly satisfied.

Based on the field theory on nonanticommutative superspace, Nouicer has suggested the following higher order GUP which agrees with (2) to the leading order and also predicts a minimal length uncertainty

\[
[X, P] = i \hbar \exp (\beta P^2) .
\]  \hspace{1cm} (6)

This algebra can be satisfied from the following representation of the position and momentum operators:

\[
P \phi(p) = p \phi(p),
\]  \hspace{1cm} (7)
\[
X \phi(p) = i \hbar \exp (\beta p^2) \partial_p \phi(p).
\]  \hspace{1cm} (8)

Now the symmetricity condition of the position operator implies the following modified completeness relation and scalar product

\[
\langle \psi|\phi \rangle = \int_{-\infty}^{+\infty} dp \exp (-\beta p^2) \psi^*(p) \phi(p),
\]  \hspace{1cm} (9)
\[
\langle p|p' \rangle = \exp (\beta p^2) \delta(p - p').
\]  \hspace{1cm} (10)

Also, the absolutely smallest uncertainty in position is given by \((\Delta X)^{\text{Nouicer}}_{\text{min}} = \hbar \sqrt{\beta}\).

To incorporate the idea of the maximal momentum, Ali, Das and Vagenas have proposed the following modified commutation relation

\[
[X_i, P_j] = i \hbar \left[ \delta_{ij} - \alpha \left( P \delta_{ij} + \frac{P_i P_j}{p} \right) + \alpha^2 \left( P^2 \delta_{ij} + 3 P_i P_j \right) \right],
\]  \hspace{1cm} (11)
where $\alpha = \alpha_0/M_{Pl}c = \alpha_0\ell_{Pl}/\hbar$ is the GUP parameter, $P^2 = \sum_{j=1}^{3} P_j P_j$, $M_{Pl}$ is the Planck mass, and $M_{Pl}c^2 \sim 10^{19}\text{GeV}$ is the Planck energy. This form of GUP implies both a minimal length uncertainty and a maximal momentum uncertainty, namely

$$\Delta X \geq (\Delta X)_{\text{min}} \approx \alpha_0 \ell_{Pl} = \bar{\hbar} \alpha_0,$$

(12)

$$\Delta P \leq (\Delta P)_{\text{max}} \approx \frac{M_{Pl}c}{\alpha_0} = 1/\alpha.$$

(13)

The commutation relation (11) is approximately satisfied by the following representation

$$X_i = x_i,$$

(14)

$$P_i = p_i \left(1 - \alpha p + 2\alpha^2 p^2\right),$$

(15)

where $x_i$ and $p_i$ obey the usual commutation relations $[x_i, p_j] = i\bar{\hbar}\delta_{ij}$ and $p$ is the magnitude of $\vec{p}$. Now Eq. (12) implies $\alpha \approx \sqrt{\beta}$. However, this proposal has the following difficulties:

- It is perturbative, i.e., it is only valid for small values of the GUP parameter.
- Although the minimal length uncertainty can be interpreted as the minimal length, the maximal momentum uncertainty differs from the idea of the maximal momentum which is required in DSR theories. Indeed Eq. (13) puts an upper bound on the uncertainty of the momentum measurement, not on the value of the observed momentum.
- It does not imply noncommutative geometry, because $[X_i, X_j] = 0$ [see Eq. (14)].

To overcome these problems, consider the following higher order generalized uncertainty principle (GUP*) which implies both the minimal length uncertainty and the maximal observable momentum

$$[X, P] = \frac{i\hbar}{1 - \beta P^2}.$$  

(16)

This commutation relation agrees with KMM’s and Noucier’s proposals to the leading order and contains a singularity at $P^2 = 1/\beta$. This fact shows that the momentum of the particle cannot exceed $1/\sqrt{\beta} \approx 1/\alpha$ which agrees formally with Eq. (13). As stated before, Eqs. (13) and (16) imply two basically different quantities. However, the presence of an upper bound on the momentum properly agrees with DSR theories. As we shall see, the physical observables such as energy and momentum are not only nonsingular, but also are bounded from above.

Note that, this choice is the simplest choice (using rational functions) that implements momentum cutoff at the commutation relation level, and which reduces to KMM proposal. One way of getting rational approximations from a truncated power series is by using Padé resummation. Indeed, the Padé approximant is the best approximation of a function by a rational function and for a series expansion $f(P) = \sum_{i=0}^{k} f_i P^i + \cdots$ up to the order $k$ is presented by

$$[m/n] = \frac{a_0 + a_1 P + \cdots + a_m P^m}{1 + b_1 P + \cdots + b_n P^n}, \quad m + n = k,$$

(17)

where $a_i$ and $b_i$ are found such that the series expansion of $[m/n]$ up to $O(k)$ equals the original series, namely

$$\sum_{i=0}^{k} f_i P^i = [m/n] + O(m + n + 1).$$

(18)

So the $m + n + 1$ unknown coefficients are given uniquely by the $k + 1$ coefficients $f_i$. Now if one treats the KMM relation (2) as a low momentum $[2/0]$ approximation of the ultimate GUP proposal $[X, P] = i\hbar f(P)$,
then its \([0/2]\) Padé approximant gives Eq. (16) which also contains an additional property, i.e., the momentum cutoff. Of course Padé resummations are approximations and not a rigorous justification but they are popular in many fields in estimating “nonperturbative” effects.

On the other hand, and from a physical viewpoint, GUPs are common phenomenological aspects of all promising candidates of quantum gravity. Adopting a mathematically well-motivated and nonperturbative GUP has the potential to shed light on even more phenomenological aspects of the mentioned candidates. Especially, the relatively different algebraic structure of the GUP* (16) has new implications on the Hilbert space representation of quantum mechanics that overcomes some conceptual problems raised in the original KMM formalism such as the divergence of the energy spectrum of the eigenfunctions of the position operator. Unlike the KMM case that the energy of the short wavelength modes are divergent, it is straightforward to show that in our case there is no divergence in the energy spectrum for short wavelengths [see Eq. (39)].

Also, the different Hilbert space structure may have some new implications on measurement theory in this framework.

To satisfy the above commutation relation, we can write the position and momentum operators in the momentum space representation as

\[
P \phi(p) = p \phi(p),
\]

\[
X \phi(p) = \frac{i\hbar}{1 - \beta p^2} \partial_p \phi(p).
\]

Using the symmetricity condition of the position operator the modified completeness relation and scalar product can be written as

\[
\langle \psi|\phi \rangle = \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} dp \ (1 - \beta p^2) \ \psi^*(p) \phi(p),
\]

\[
\langle p|p' \rangle = \delta(p - p') \ \frac{1}{1 - \beta p^2}.
\]

The uncertainty relation that arises from GUP* is given by

\[
(\Delta X)(\Delta P) \geq \frac{\hbar/2}{1 - \beta (P^2)^2},
\]

\[
\geq \frac{\hbar}{2} \left( 1 + \beta \langle P^2 \rangle + \beta^2 \langle P^4 \rangle + \beta^3 \langle P^6 \rangle + \cdots \right),
\]

\[
\geq \frac{\hbar}{2} \left( 1 + \beta \langle P^2 \rangle + \beta^2 \langle P^2 \rangle^2 + \beta^3 \langle P^2 \rangle^3 + \cdots \right),
\]

\[
\geq \frac{\hbar}{2} \left( 1 + \beta \left( [\Delta P]^2 + \langle P \rangle^2 \right) + \beta^2 \left( [\Delta P]^2 + \langle P \rangle^2 \right)^2 + \beta^3 \left( [\Delta P]^2 + \langle P \rangle^2 \right)^3 + \cdots \right),
\]

\[
\geq \frac{\hbar/2}{1 - \beta \left( [\Delta P]^2 + \langle P \rangle^2 \right)}. \quad (23)
\]

where we have used the property \(\langle P^{2n} \rangle \geq (P^2)^n\). In order to find the minimal length uncertainty of this deformed algebra, we consider the physical states for which we have \(\langle P \rangle = 0\) and solve the following saturate GUP* for \(\Delta P\)

\[
(\Delta X)(\Delta P) = \frac{\hbar/2}{1 - \beta (\Delta P)^2},
\]

which has a minimum at \(\Delta P = 1/\sqrt{3\beta}\). So the absolutely smallest uncertainty in position is given by

\[
(\Delta X)_{\text{min}}^* = \frac{3\sqrt{3}}{4} \hbar \sqrt{\beta}. \quad (25)
\]

In Table \(\text{I}\) we have compared minimal length uncertainties from various GUP scenarios. These results show that \((\Delta X)_{\text{min}}^{\text{KMM}} < (\Delta X)_{\text{min}}^{\text{Noucier}} < (\Delta X)_{\text{min}}^*\).
3. Functional analysis of the position operator

The eigenvalue problem for the position operator in the GUP* framework and in the momentum space is given by

\[ \frac{i\hbar}{1 - \beta p^2} \partial_p \psi_\lambda(p) = \lambda \psi_\lambda(p). \]  

(26)

This equation can be solved to obtain the position eigenvectors

\[ \psi_\lambda(p) = c \exp \left( -\frac{i\lambda p}{\hbar} \left( 1 - \frac{\beta}{3} p^2 \right) \right). \]  

(27)

The eigenfunctions are normalizable

\[ 1 = |c|^2 \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \left( 1 - \beta p^2 \right) \exp \left( -i\lambda p \frac{h}{\beta} \right) dp = \frac{4|c|^2}{3\sqrt{\beta}}. \]  

(28)

Therefore

\[ \psi_\lambda(p) = \frac{\sqrt{3\sqrt{\beta}}}{2} \exp \left( -\frac{i\lambda p}{\hbar} \left( 1 - \frac{\beta}{3} p^2 \right) \right). \]  

(29)

Now we calculate the scalar product of the position eigenstates

\[ \langle \psi_\lambda | \psi_{\lambda'} \rangle = \frac{3\sqrt{\beta}}{4} \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} (1 - \beta p^2) \exp \left( i(\lambda - \lambda') p \left( \frac{1 - \beta}{3} p^2 \right) \right) dp, \]

\[ = \frac{3\hbar}{2(\lambda - \lambda')} \sin \left( \frac{2(\lambda - \lambda')}{3\hbar} \right). \]  

(30)

Thus, similar to the KMM scenario, the position eigenstates are generally no longer orthogonal. In Fig. 1, we have depicted \( \langle \psi_\lambda | \psi_{\lambda'} \rangle \) for the KMM proposal and GUP*.

Although this quantity in both models has a same functional form, it is more oscillatory in the KMM framework.

3.1. Maximal localization states

The maximal localization states \( \psi_{\lambda}^{ML} \) are defined with the properties

\[ \langle \psi_{\lambda}^{ML} | \lambda | \psi_{\lambda}^{ML} \rangle = \xi, \]  

(31)

and

\[ \Delta \lambda_{\psi_{\lambda}^{ML}} = (\Delta \lambda)_{\min}. \]  

(32)

These states also satisfy

\[ \left( X - \langle X \rangle + \frac{\langle [X, P] \rangle}{2(\Delta P)^2} (P - \langle P \rangle) \right) |\psi\rangle = 0. \]  

(33)
To proceed further we need to express $\langle [X, P] \rangle$ in terms of $\Delta P$ and $\langle P \rangle$. However, since $\langle [X, P] \rangle$ also depends on $\langle P^4 \rangle$, $\langle P^6 \rangle$, etc., and these quantities cannot be calculated before specifying $|\psi\rangle$, to first order in the GUP parameter we can use the approximate relation $\langle [X, P] \rangle \simeq i\hbar(1 + \beta(\Delta P)^2 + \beta\langle P \rangle^2)$. So, in momentum space, the above equation takes the form

$$
\left( \frac{i\hbar}{1 - \beta p^2} \frac{\partial}{\partial p} - \langle X \rangle + i\hbar \frac{1 + \beta(\Delta P)^2 + \beta\langle P \rangle^2}{2(\Delta P)^2}(p - \langle P \rangle) \right)\psi(p) \simeq 0,
$$

which has the solution

$$
\psi(p) \simeq N \exp \left[ \left( \frac{i}{\hbar}(X) + \frac{1 + \beta(\Delta P)^2 + \beta\langle P \rangle^2}{2(\Delta P)^2} \langle P \rangle \right) \times \left( p - \frac{\beta}{3} p^3 \right) - \frac{1 + \beta(\Delta P)^2 + \beta\langle P \rangle^2}{4(\Delta P)^2} \left( p^2 - \frac{\beta}{2} p^4 \right) \right].
$$

To find the absolutely maximal localization states we need to choose the critical momentum uncertainty $\Delta P = 1/\sqrt{3\beta}$ that gives the minimal length uncertainty and take $\langle P \rangle = 0$, i.e.,

$$
\psi_{\xi}^{\text{ML}}(p) \simeq N \exp \left[ -\frac{i}{\hbar}\xi \left( p - \frac{\beta}{3} p^3 \right) - \beta \left( p^2 - \frac{\beta}{2} p^4 \right) \right],
$$

where the normalization factor is given by

$$
1 = NN^* \int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}} dp \left( 1 - \beta p^2 \right) \exp \left( 2\beta p^2 - \beta^2 p^4 \right),
$$

$$
1 = 1.0123\sqrt{\frac{N^2}{\sqrt{\beta}}}. \tag{37}
$$

Note that $\psi_{\xi}^{\text{ML}}(p)$ exactly satisfies Eq. (31). However, because of the approximation that assumed to find $\psi_{\xi}^{\text{ML}}(p)$ (36), it approximately obeys relation (32), i.e.,

$$
\Delta X_{|\psi_{\xi}^{\text{ML}}\rangle} = 1.0998(\Delta X)_{\text{min}}^*, \tag{38}
$$

which shows an error less than 10%. Also, because of the fuzziness of space, these maximal localization states are not mutually orthogonal. It is worth to mention that, in this framework, the expectation value of the kinetic energy operator $P^2/2m$ is finite for both $|\psi\rangle$ and $|\psi_{\xi}^{\text{ML}}\rangle$. Indeed we have

$$
\langle \psi_{\lambda} | \frac{P^2}{2m} | \psi_{\lambda} \rangle = \frac{1}{10m\beta^2}, \tag{39}
$$
and

\[ \langle \psi_{\xi}^M | \frac{P^2}{2m} | \psi_{\xi}^M \rangle = 0.7345 \frac{10}{10m\beta} \]  

(40)

These quantities for the KMM proposal are \( \infty \) and \( 1/2m\beta \), respectively.

To find the quasiposition wave function \( \psi(\xi) \), we define

\[ \psi(\xi) \equiv \langle \psi_{\xi}^M | \psi \rangle, \]  

(41)

where in the limit \( \beta \to 0 \) it goes to the ordinary position wave function \( \psi(\xi) = \langle \xi | \psi \rangle \). Now the transformation of the wave function in the momentum representation into its counterpart quasiposition wave function is

\[ \psi(\xi) = N \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} dp (1 - \beta p^2) \exp \left[ \frac{i}{\hbar} \xi \left( p - \frac{\beta}{3} p^3 \right) - \beta \left( p^2 - \beta \frac{3}{2} p^4 \right) \right] \psi(p). \]  

(42)

This relation shows that similar to the ordinary quantum mechanics and the KMM proposal, the quasiposition wave function of a momentum eigenstate \( \tilde{\psi}_p(p) = \delta(p - \tilde{p}) \) with energy \( E = \tilde{p}^2/2m \) is still a plane wave but with a modified dispersion relation

\[ \lambda(E) = \frac{2\pi \hbar}{\sqrt{2mE}} \left( 1 - \frac{2}{3} m\beta E \right) = \frac{\lambda_{\text{ord}}(E)}{1 - \frac{4}{3} m\beta E}, \]  

(43)

where \( \lambda_{\text{ord}}(E) = 2\pi \hbar/\sqrt{2mE} \) is the wavelength in the absence of GUP. In Fig. 2 we have depicted \( \lambda \) versus \( mE \) in various scenarios. Since Eq. (43) is bounded from below, there exists a nonzero minimal wavelength. So the wavelength components smaller than

\[ \lambda_0 = 3\pi \hbar \sqrt{\beta} = \frac{3}{4} \pi \lambda_{0,\text{KMM}}, \]  

(44)

are absent in the Fourier decomposition of the quasiposition wave function of the physical states. Therefore, the maximal energy of a momentum eigenstate is

\[ E_{\text{max}} = \frac{3}{2m\beta}. \]  

(45)

Since the transformation (42) as the generalized Fourier transformation is invertible, the transformation of a quasiposition wave function into a momentum space wave function is given by

\[ \psi(p) = \frac{N^{-1}}{2\pi \hbar} \int_{-\infty}^{+\infty} d\xi \exp \left[ \beta \left( p^2 - \frac{\beta}{3} p^4 \right) \right] \exp \left[ -\frac{i}{\hbar} \xi \left( p - \frac{\beta}{3} p^3 \right) \right] \psi(\xi). \]  

(46)

Now the scalar product of states in terms of the quasiposition wave functions reads

\[ \langle \phi | \psi \rangle = \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} dp (1 - \beta p^2) \phi^*(p) \psi(p), \]

\[ = \left( \frac{N^{-1}}{2\pi \hbar} \right)^2 \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} d\xi d\xi' \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dp dp' \frac{d\xi}{d\xi'} \exp \left[ 2\beta \left( p^2 - \frac{\beta}{3} p^4 \right) \right] \exp \left[ -\frac{i}{\hbar} \left( \xi - \xi' \right) \left( p - \frac{\beta}{3} p^3 \right) \right] \phi^*(\xi) \psi(\xi'). \]  

(47)
4. Harmonic oscillator

In this section, we apply the developed formalism to the case of a linear harmonic oscillator. Using the expression for the Hamiltonian

\[ H = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 X^2, \quad (48) \]

and the representation for \( X \) and \( P \), we obtain the following form for the stationary state Schrödinger equation:

\[ \frac{d^2 \psi(p)}{dp^2} + \frac{2 \beta p}{1 - \beta p^2} \frac{d \psi(p)}{dp} + (1 - \beta p^2)^2 (\epsilon - \eta^2 p^2) \psi(p) = 0, \quad (49) \]

where \(-1/\sqrt{\beta} \leq p \leq 1/\sqrt{\beta}\) and

\[ \epsilon = \frac{2E}{\hbar \omega^2}, \quad \eta = \frac{1}{\hbar \omega}. \quad (50) \]

4.1. The quantum mechanical solution

Using the dimensionless variable \( u = \sqrt{\beta} p \), Eq. (49) can be written as

\[ \frac{d^2 \psi(u)}{du^2} + \frac{2u}{1 - u^2} \frac{d \psi(u)}{du} + (1 - u^2)^2 (\epsilon' - \eta^2 u^2) \psi(u) = 0, \quad (51) \]

where \(-1 \leq u \leq 1\) and

\[ \epsilon' = \frac{\epsilon}{\beta}, \quad \eta' = \frac{\eta}{\beta}. \quad (52) \]

Now by changing the variable to \( x = u - (1/3) u^3 \) we have

\[ -\frac{d^2 \psi(x)}{dx^2} + \eta'^2 V(x) \psi(x) = \epsilon' \psi(x), \quad (53) \]

where \(-2/3 \leq x \leq 2/3\) and

\[ V(x) = \left[ \frac{1 - i \sqrt{3} + (-2)^{1/3} \left(3x + \sqrt{9x^2 - 4}\right)^{2/3}}{2^{2/3} \left(3x + \sqrt{9x^2 - 4}\right)^{1/3}} \right]^2, \quad (54) \]
is the effective potential which is real in this domain (see Fig. 3). The boundary condition now reads

$$\psi(x)\bigg|_{\pm \frac{2}{3}L} = 0.$$  \hspace{1cm} (55)

To solve Eq. (53), we can expand the wave function in terms of the particle in a box eigenfunctions. Since the potential term $V(x)$ is an even function of $x$, to avoid large matrices, we use

$$\phi^e_m(x) = \sqrt{\frac{1}{L}} \cos \left( (m - \frac{1}{2}) \frac{\pi x}{L} \right),$$  \hspace{1cm} (56)

and

$$\phi^o_m(x) = \sqrt{\frac{1}{L}} \sin \left( \frac{m\pi x}{L} \right),$$  \hspace{1cm} (57)

basis functions ($m = 1, 2, \ldots$) for even and odd parity solutions, respectively, and write the wave function as $\psi(x) = \sum_m A_m \phi_m(x)$ which vanishes at $\pm L$. Now the boundary condition (55) reads $L = 2/3$.

The approximate solutions are the eigenvalues and the eigenfunctions of the $(N \times N)$ Hamiltonian matrix $H_N$ in the form

$$H_{mm} = \left( m - \frac{1}{2} \right)^2 \frac{\pi^2}{L^2} \delta_{mn} + D_{mn}^{\text{even}},$$

and

$$H_{mm} = m^2 \frac{\pi^2}{L^2} \delta_{mn} + D_{mn}^{\text{odd}},$$  \hspace{1cm} (58)

for even and odd states, respectively. Here, $\delta_{mn}$ is the kronecker’s delta and

$$D_{mn}^{\text{even}} = \frac{\eta^2}{L} \int_{-L}^{L} V(x) \cos \left( (m - \frac{1}{2}) \frac{\pi x}{L} \right) \cos \left( (n - \frac{1}{2}) \frac{\pi x}{L} \right) \, dx,$$  \hspace{1cm} (59)

$$D_{mn}^{\text{odd}} = \frac{\eta^2}{L} \int_{-L}^{L} V(x) \sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \, dx,$$  \hspace{1cm} (60)

where $m$ and $n$ run from 1 to $N$. In the usual diagonalization scheme with the particle in box basis functions, we need to adjust the domain $L$ with respect to the number of basis functions in such way that the total error to be minimized. However, for our case, since the boundary condition (55) has fixed the domain, i.e., $L = 2/3$, the accuracy of the solutions grows as the number of the basis increases. In Table 2 we have reported the first ten energy eigenvalues of the harmonic oscillator in the GUP* framework. Indeed $N = 30$ basis functions suffices to obtain nearly accurate results for the low lying energy eigenstates.
\[ E_n = \frac{\hbar \omega}{2} x^2 (1 - \beta p^2)^2 \]

(61)

To find the approximate energy eigenvalues of the above Hamiltonian, we use the Wilson-Sommerfeld quantization rule in the form

\[ \oint x \, dp = \left( n + \frac{1}{2} \right) \hbar, \quad n = 0, 1, 2, \ldots, \]

(62)

where we have used \( \oint d(xp) = 0 = \oint x \, dp + \oint p \, dx \). This integral can be written as

\[ \oint x \, dp = \frac{2}{m\omega} \int_{-z}^{z} (1 - \beta p^2) \sqrt{z^2 - p^2} \, dp, \]

(63)

where \( z = \sqrt{2mE} \). So the semiclassical energy spectrum is given by

\[ E_n^{(SC)} = 1 - \sqrt{1 - 2m\beta \hbar \omega (n + \frac{1}{2})}, \]

(64)

\[ E_n^{(SC)} = \frac{1 - \frac{1}{8} \gamma \hbar \omega + \hbar \omega \left( n + \frac{1}{2} \right) \left( 1 + \frac{\gamma}{2} \right) + \frac{1}{2} \gamma^2 \hbar \omega n^2 + \frac{1}{2} \gamma^2 \hbar \omega \left( n + \frac{1}{2} \right)^3}{m\beta} + \mathcal{O}(\gamma^3), \]

(65)

where \( \gamma = \beta \hbar \omega = \eta'^{-1} \). As it is shown in the appendix, the first three terms are similar to the energy spectrum of the harmonic oscillator in the KMM framework. In Fig. 4 we have depicted the energy spectrum in both GUP\textsuperscript{KMM} and GUP\textsuperscript{*} frameworks. Note that, in GUP\textsuperscript{*} scenario, the energy is also bounded from above. Indeed, the maximum possible energy for the harmonic oscillator is

\[ E_{max}^{(SC)} = \frac{1}{m\beta}, \]

(66)

and the number of states \( (N = n + 1) \) is finite, namely

\[ n_{max} = \left\lfloor \frac{1}{2\gamma} - \frac{1}{2} \right\rfloor, \]

(67)

where \( \lfloor x \rfloor \) denotes the largest integer not greater than \( x \). So, to have at least one state, we should have

\[ \gamma \leq 1. \]

(68)

Table 2: The energy eigenvalues of the harmonic oscillator in the GUP\textsuperscript{*} framework. Here \( \varepsilon_n = \varepsilon_n'/\eta' = \varepsilon_n/\eta = 2E_n/\hbar \omega, N = 30, \) and \( \eta' = 100. \)

| \( n \) | \( \varepsilon_n' \) | \( \varepsilon_n^{(SC)} \) | \( \varepsilon_n \) | \( |\varepsilon_n - \varepsilon_n^{(SC)}|/\varepsilon_n \) |
|---|---|---|---|---|
| 0 | 1.00251 | 1.00509 | 2.6 \times 10^{-4} |
| 1 | 3.02284 | 3.02559 | 9.1 \times 10^{-4} |
| 2 | 5.06411 | 5.06704 | 5.8 \times 10^{-4} |
| 3 | 7.12698 | 7.13011 | 4.4 \times 10^{-4} |
| 4 | 9.21216 | 9.21550 | 3.6 \times 10^{-4} |
| 5 | 11.3204 | 11.3240 | 3.2 \times 10^{-4} |
| 6 | 13.4524 | 13.4563 | 2.9 \times 10^{-4} |
| 7 | 15.6091 | 15.6133 | 2.7 \times 10^{-4} |
| 8 | 17.7913 | 17.7958 | 2.5 \times 10^{-4} |
| 9 | 20.0000 | 20.0049 | 2.4 \times 10^{-4} |
As Fig. 4 shows, we have 50 states for $\gamma = 0.01$. The first ten semiclassical energy eigenvalues are presented in Table 2. As the table shows the semiclassical results agree well with the quantum mechanical energy spectrum. In fact, the relative error is less than $3 \times 10^{-3}$ even for the ground state. It is worth to mention that a model with analogous properties has been studied in the context of the nonrelativistic Snyder model in curved space [44]. Moreover, in the KMM framework, the bound states of the relativistic particle in a box problem is also finite [51].

4.3. The classical solution

In the classical domain, the equations of motion are

$$\dot{X} = \{X, H\} = \frac{P}{m(1 - \beta P^2)},$$

$$\dot{P} = \{P, H\} = -\frac{m\omega^2 X}{1 - \beta P^2}. \tag{70}$$

The solutions to these equations are

$$\omega t = \left(1 - \frac{\epsilon}{2}\right) \arccos \left(\frac{P(t)}{P_{\text{max}}}\right) - \frac{\beta}{2} P(t) \sqrt{P_{\text{max}}^2 - P^2(t)}, \tag{71}$$

$$X(t) = -\frac{1 - \beta P^2(t)}{m\omega^2} \frac{dP(t)}{dt}, \tag{72}$$

where

$$\epsilon = 2m\beta E, \quad P_{\text{max}} = \sqrt{2mE}. \tag{73}$$

To first-order in $\beta$ we have

$$P(t) = P_{\text{max}} \left\{ \cos \left[ \left(1 + \frac{\epsilon}{2}\right) \omega t \right] - \frac{\epsilon}{2} \sin^2 \omega t \cos \omega t \right\}, \tag{74}$$

$$X(t) = X_{\text{max}} \left\{ \left(1 + \frac{\epsilon}{2}\right) \cos \left[ \left(1 + \frac{\epsilon}{2}\right) \omega t \right] - \frac{\epsilon}{2} \sin^3 \omega t \right\}, \tag{75}$$

where $X_{\text{max}} = \sqrt{2E/m\omega^2}$. As we have expected these results agree with the KMM proposal to $O(\beta)$ [45].

\footnote{We used the relation $e_n^{(SC)} = 2n' \left(1 - \sqrt{1 - 2n'^{-1}(n + 1/2)}\right)$.}
It is straightforward to show that the infinitesimal phase space volume between equal energy contours $E$ and $E + dE$, and equal time contours $t$ and $t + dt$ can be written as

$$dE \, dt = (1 - \beta P^2) dX \, dP.$$  \hfill (76)

Now, since by definition the left hand side of this equation is time independent, the right hand is also time independent.

5. Conclusions

In this Letter, we have presented a higher order generalized uncertainty principle that implies both a minimal length uncertainty and a maximal momentum proportional to $\hbar \sqrt{\beta}$ and $1/\sqrt{\beta}$, respectively. We found the exact eigenfunctions of the position operator and the quantum mechanical and semiclassical energy spectrum of the harmonic oscillator and showed that the energy spectrum is also bounded from above. Here we implemented a momentum cutoff not through terms like $P^2$ on the right hand side of the commutation relations. Instead, we implemented the momentum cutoff through a function of $P$ with a singularity. So the momentum space is cut into several sectors that decouple from each other. The sectors are separated from each other at the singularities of the function of $P$ that is used. Technically, we have inequivalent irreducible representations of the commutation relations, one each in each sector \[10\]. This type of issue with the various sectors can be avoided, as it is indicated in Ref. \[11\]. Applied to our case, the trick would be to write the right hand side of the commutation relation not as a fraction but instead to expand it out as a geometric series. It has a finite radius of convergence and that rules out all representations beyond the singularity. The generalization of this GUP to $D$ dimensions which is noncommutative, its invariant density of states, and its effects on the blackbody radiation spectrum and the cosmological constant problem are discussed in \[12\].

Appendix A. Harmonic oscillator spectrum in the KMM framework

In the context of the KMM proposal, the total energy in terms of ordinary variables is given by

$$E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 (1 + \beta p^2)^2 x^2.$$  \hfill (A.1)

Now the Wilson-Sommerfeld integral can be written as

$$\oint x \, dp = \frac{2}{m \omega} \int_{-z}^{z} \frac{\sqrt{z^2 - p^2}}{1 + \beta p^2} \, dp = \left( n + \frac{1}{2} \right) \hbar,$$  \hfill (A.2)

where $z = \sqrt{2mE}$. So the semiclassical energy spectrum is given by

$$E_n^{(SC)} = -\frac{1}{8} \gamma \hbar \omega + \hbar \omega \left( n + \frac{1}{2} \right) \left( 1 + \frac{\gamma}{2} \right) + \frac{1}{2} \gamma \hbar \omega n^2,$$  \hfill (A.3)

where $\gamma = \beta m \omega$. This result agrees (up to a constant) with the exact solution to first order of the GUP parameter \[8\]

$$E_n^{exact} = \hbar \omega \left( n + \frac{1}{2} \right) \left( \sqrt{1 + \gamma^2/4} + \gamma/2 \right) + \frac{1}{2} \gamma \hbar \omega n^2$$

$$= E_n^{(SC)} + \frac{1}{8} \gamma \hbar \omega + O(\gamma^2),$$  \hfill (A.4)

and gives the correct $n^2$ dependence behavior. An alternative derivation of Eq. (A.3) is also presented in Ref. \[30\].
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