Abstract
Since Bustince et al. introduced the concepts of overlap and grouping functions, these two types of aggregation functions have attracted a lot of interest in both theory and applications. In this paper, the depiction of $(O, G)$-granular variable precision fuzzy rough sets ($(O, G)$-GVPFRSs for short) is first given based on overlap and grouping functions. Meanwhile, to work out the approximation operators efficiently, we give another expression of upper and lower approximation operators by means of fuzzy implications and co-implications. Furthermore, starting from the perspective of construction methods, $(O, G)$-GVPFRSs are represented under diverse fuzzy relations. Finally, some conclusions on the granular variable precision fuzzy rough sets (GVPFRSs for short) are extended to $(O, G)$-GVPFRSs under some additional conditions.

Keywords  Grouping function · Overlap function · Granular variable precision fuzzy rough set · Fuzzy rough set

1 Introduction

1.1 Brief review of fuzzy rough sets

Rough set, as a way to portray uncertainty problems, was originally proposed by Pawlak (1982, 1991), and it has been extensively developed in the fields of knowledge discovery...
(Polkowski and Skowron 1998) and data mining. Rough set theory uses indistinguishable relations to divide the knowledge of research domain, thus forming a system of knowledge representation that approximates an arbitrary subset of the universe by defining upper and lower approximation operators (Chen et al. 2006). As a generalization of classical theory, Zadeh introduced fuzzy set theory (Zadeh 1965) in 1965, where objects can be owned by different sets with different membership functions. Since rough sets are defined based on equivalence relations, they are mainly used to process qualitative (discrete) data (Jensen and Shen 2004), and there are greater restrictions on the processing of real-valued data sets in the database. In particular, fuzzy sets can solve this problem by dealing with fuzzy concepts. Therefore, complementing the features of rough sets and fuzzy sets with each other constitutes a new research hotspot.

In 1990, Dubois and Prade (1990) described fuzzy rough set, which can be viewed as the combination of two uncertainty models. As another innovation of rough set, Ziarko presented the variable precision rough set (Ziarko 1993), which mainly solved the classification problem of uncertain and inaccurate information with an effective error-tolerance competence. More details about variable precision rough sets can refer to Mieszkowicz-Rolka and Rolka (2004a, b) and Zhang et al. (2008). In addition, since the upper and lower approximation operators of fuzzy rough sets are defined according to membership functions, while rough sets are described based on the union of some sets, there exists a significant difference in their granular structures. To overcome this limitation, Chen et al. (2011) explored the concept and related properties of granular fuzzy sets based on fuzzy similarity relations. Furthermore, from the perspective of granular computing, the granular fuzzy set is used to characterize the granular structure of upper and lower approximations. However, the above model cannot tolerate even minor errors and is not suited to handle uncertain information well. Lots of extended fuzzy rough sets are applied to solve these problems, but some studies still have problems in dealing with mislabeled samples (see, e.g., Hu et al. 2010, 2011; Zhao et al. 2009), and others have only considered the relative error cases (Salido and Murakami 2003; Mieszkowicz-Rolka and Rolka 2004a).

To fill these loopholes, the model of variable precision \((\theta, \sigma)\)-fuzzy rough sets over fuzzy granules were presented by Yao et al. (2014). However, the above model is based on fuzzy \(\ast\)-similarity relation, satisfying reflexivity, symmetry and \(\ast\)-transitivity, which is too strict to facilitate generalized conclusions. Thus, Wang and Hu (2015) studied the GVPFRSs and then the equivalent expressions of the approximation operators are given with fuzzy implications and co-implications over arbitrary fuzzy relations. Subsequently, they gave the properties of GVPFRSs on different fuzzy relations. In addition, compared with the unit closed interval \([0, 1]\), the complete lattice has a wider structure, so Qiao and Hu expanded the content of Yao et al. (2014) and Wang and Hu (2015), and further discussed the concept of granular variable precision \(L\)-fuzzy rough sets based on residuated lattices.

In fact, both Wang and Hu (2015) and Qiao and Hu (2018) are based on \(t\)-norm (\(t\)-conorm), which satisfying associative, commutative, increasing in each argument and has an identity element 1 (resp. 0). However, there are various applications (Fodor and Keresztfalvi 1995; Bustince et al. 2010, 2011) in which the associativity property of the \(t\)-norm (resp. \(t\)-conorm) is not necessary, such as classification problem, face recognition and image processing.

### 1.2 Brief analysis of overlap and grouping functions

Bustince et al. described the axiomatic definitions of overlap and grouping functions (Bustince et al. 2009, 2011), which stem from some practical problems in image processing and classi-
Fig. 1 The relationship between t-norms and overlap functions (Qiao 2019)

Classification. In fact, in some situations, the associativity of t-norm and t-conorm usually does not work. Therefore, as two types of nonassociative fuzzy logic connectives, overlap and grouping functions have made rapid development in theoretical research and practical applications.

In theory, there exists many studies involving overlap and grouping functions, such as crucial properties (Bedregal et al. 2013; Dimuro and Bedregal 2014; Wang and Liu 2019), corresponding implications (Dimuro et al. 2014; Dimuro and Bedregal 2015; Ti and Zhou 2018; Zhang et al. 2022), additive generator pairs (Dimuro et al. 2016), interval overlap functions and grouping functions (Bedregal et al. 2017; Liang and Zhang 2022; Qiao and Hu 2017), distributive equations (Liu and Zhao 2020; Zhang and Qin 2020; Zhang et al. 2019) and concept extensions (De Miguel et al. 2019; Liang and Zhang 2022; Zhou and Yan 2021). At the same time, the concept of overlap (grouping) function is extended to more general structure, such as quasi-overlap (grouping) functions on bounded partially ordered sets (Qiao 2022) and overlap (grouping) functions on complete lattices (Qiao 2021) appear successively. On the other hand, Zhang et al. (2022) removed the commutative law of the original overlap (grouping) function, constructed pseudo overlap (grouping) functions and gave their corresponding construction methods and applications. From an application point of view, overlap and grouping functions can find interesting applications in classification (Lucca et al. 2017; Paternain et al. 2016), image processing (Bustince et al. 2007, 2010; Jurio et al. 2013), fuzzy community detection problem (Bustince et al. 2007) and decision making (Bustince et al. 2011; Wen and Zhang 2021; Elkano et al. 2018). In particular, Wen and Zhang (2021) proposed fuzzy $\beta$-covering fuzzy rough set model based on the overlap function and their multi-granulation case, which can effectively deal with multi-criteria (group) decision-making problem.

1.3 The motivation of this paper

Note that $O : [0, 1]^2 \rightarrow [0, 1]$ is an associative overlap function (resp. grouping function) if and only if $O$ is a continuous and positive $t$-norm (resp. $t$-conorm), which can be shown in Fig. 1 (Qiao 2019; Adamatti 2014). On the other side, as a kind of not necessarily associative binary aggregation function, overlap and grouping function have been widely studied and applied in recent years, and the combining of overlap and grouping function with fuzzy rough set theory plays a pivotal role in practical problems (Jiang and Hu 2022). The main way of combination is to use overlap and grouping function instead of classical logical connective $\wedge$ and $\vee$ on the unit closed interval.

Therefore, based on the aforementioned consideration, this paper proposed the concepts of $(O, G)$-GVFRSs based on overlap and grouping functions, which can be viewed as a generalization of GVFRSs based on $t$-norm and $t$-conorm (Wang and Hu 2015). It should be pointed out that the present paper further enriches the application of overlap and grouping functions. In addition, it makes the research on fuzzy rough sets more complete.
The rest of this paper is arranged as follows. Section 2 enumerates some fundamental concepts which are necessary to understand this paper. In Sect. 3, the concepts of \((O, G)\)-GVPFRSs with general fuzzy relations are proposed, and then, an equivalent characterization is constructed to efficiently calculate the approximation operators. Furthermore, we study the model of \((O, G)\)-GVPFRSs under the conditions of crisp relations and crisp sets and draw corresponding conclusions. Section 4 represents the \((O, G)\)-GVPFRSs on diverse fuzzy relations. In particular, some special conclusions are given under additional conditions. In Sect. 5, conclusions on our research are given.

2 Preliminaries

In this section, we recapitulate some fundamental notions which shall be used in the sequel.

**Definition 1** (Bustince et al. 2010) An overlap function is a binary function \(O : [0, 1]^2 \rightarrow [0, 1]\) which satisfies the following conditions for all \(x, y \in [0, 1]\):

(O1) \(O(x, y) = O(y, x)\);
(O2) \(O(x, y) = 0\) if and only if \(xy = 0\);
(O3) \(O(x, y) = 1\) if and only if \(xy = 1\);
(O4) \(O\) is non-decreasing;
(O5) \(O\) is continuous.

Furthermore, an overlap function \(O\) fulfills the exchange principle (Dimuro and Bedregal 2015) if

(O6) \(\forall x, y, u \in [0, 1]: O(x, O(y, u)) = O(y, O(x, u))\).

**Definition 2** (Bustince et al. 2011) A grouping function is a binary function \(G : [0, 1]^2 \rightarrow [0, 1]\) which satisfies the following conditions for all \(x, y \in [0, 1]\):

(G1) \(G(x, y) = G(y, x)\);
(G2) \(G(x, y) = 0\) if and only if \(x = y = 0\);
(G3) \(G(x, y) = 1\) if and only if \(x = 1\) or \(y = 1\);
(G4) \(G\) is non-decreasing;
(G5) \(G\) is continuous.

Furthermore, a grouping function \(G\) fulfills the exchange principle (Dimuro and Bedregal 2015) if

(G6) \(\forall x, y, u \in [0, 1]: G(x, G(y, u)) = G(y, G(x, u))\).

**Remark 1** (Dimuro and Bedregal 2015) Notice that a commutative function \(H : [0, 1]^2 \rightarrow [0, 1]\) is associative if and only if \(H\) satisfies the exchange principle. It is obvious that an overlap function \(O\) (resp. a grouping function \(G\)) is associative if and only if it satisfies (O6) (resp. (G6)).

**Remark 2** (Dimuro and Bedregal 2015; Dimuro et al. 2016) Suppose that overlap function \(O\) satisfies (O6), then 1 is the identity element of \(O\), similarly, when a grouping function \(G\) satisfies (G6), then 0 is the identity element of \(G\).

Next, some common overlap and grouping functions are listed in Bedregal et al. (2013) and Bedregal et al. (2016).
Example 1: (1) Any continuous $t$-norm with no non-trivial zero divisors is an overlap function.

(2) The function $O_p : [0, 1]^2 \rightarrow [0, 1]$ given by

$$O_p(x, y) = x^p y^p$$

is an overlap function for any $p > 0$ and $p \neq 1$. Since it neither satisfies the associative law nor takes 1 as identity element, it is not a $t$-norm.

(3) The function $O_{DB} : [0, 1]^2 \rightarrow [0, 1]$ given by

$$O_{DB} = \begin{cases} 
\frac{2xy}{x+y}, & \text{if } x + y \neq 0, \\
0, & \text{if } x + y = 0 
\end{cases}$$

is an overlap function.

(4) Any continuous $t$-conorm with no non-trivial one divisors is a grouping function.

(5) The function $G_p : [0, 1]^2 \rightarrow [0, 1]$ given by

$$G_p(x, y) = 1 - (1 - x)^p (1 - y)^p$$

is a grouping function for $p > 1$. Since it neither satisfies the associative law nor takes 0 as identity element, it is not a $t$-conorm.

In the following, we give the definitions of fuzzy implication and fuzzy co-implication on the basis of overlap and grouping function.

A fuzzy implication $I_O : [0, 1]^2 \rightarrow [0, 1]$ given by

$$I_O(x, y) = \max\{z \in [0, 1] : O(x, z) \leq y\}$$

for all $x, y \in [0, 1]$. In (Dimuro and Bedregal 2015), Dimuro et al. had proven $O$ and $I_O$ form an adjoint pair, if they satisfy the residuation property:

$$\forall x, y, u \in [0, 1] : O(x, u) \leq y \iff I_O(x, y) \geq u.$$ 

Furthermore, $I_O$ satisfies the exchange principle (Dimuro and Bedregal 2015) if and only if

$$\forall x, y, z \in [0, 1], \; I_O(x, I_O(y, z)) = I_O(y, I_O(x, z)).$$

Fuzzy implication $I_O$ was introduced in Dimuro and Onresidual (2015) and fuzzy co-implication $I^G$ was discussed in De Baets (1997). Since $O$ and $G$ are dual w.r.t. $N$, we can easily deduce the properties of fuzzy co-implication $I^G$.

A fuzzy co-implication $I^G : [0, 1]^2 \rightarrow [0, 1]$ given by

$$I^G(x, y) = \min\{z \in [0, 1] : y \leq G(x, z)\}$$

for all $x, y \in [0, 1]$. Similarly, the following hold:

$$\forall x, y, u \in [0, 1] : y \leq G(x, u) \iff I^G(x, y) \leq u.$$ 

Moreover, $I^G$ satisfies the exchange principle if and only if

$$\forall x, y, z \in [0, 1], \; I^G(x, I^G(y, z)) = I^G(y, I^G(x, z)).$$

If $O$ and $G$ are dual w.r.t. $N$, then for all $x, y \in [0, 1],$

$$I_O(x, y) = N(I^G(N(x), N(y))),$$

$$I^G(x, y) = N(I_O(N(x), N(y))).$$
According to the definition of $I_O$ that for all $x, y \in [0, 1]$,

$$I_O(x, y) = \max\{z \in [0, 1] : O(x, z) \leq y\}$$

$$= \max\{z \in [0, 1] : N(G(N(x), N(z))) \leq y\}$$

$$= \max\{z \in [0, 1] : G(N(x), N(z)) \geq N(y)\}$$

$$= \max\{N(z) \in [0, 1] : G(N(x), z) \geq N(y)\}$$

$$= N(\min\{z \in [0, 1] : G(N(x), z) \geq N(y)\})$$

$$= N(I^G(N(x), N(y))).$$

Similarly, the following equation can be obtained.

$$I^G(x, y) = \min\{z \in [0, 1] : y \leq G(x, z)\}$$

$$= \min\{z \in [0, 1] : y \leq N(O(N(x), N(z)))\}$$

$$= \min\{z \in [0, 1] : N(y) \geq O(N(x), N(z))\}$$

$$= \min\{N(z) \in [0, 1] : N(y) \geq O(N(x), z)\}$$

$$= N(\max\{z \in [0, 1] : N(y) \geq O(N(x), z)\})$$

$$= N(I_O(N(x), N(y))).$$

**Remark 3** (Dimuro and Bedregal 2015) $I_O$ satisfies the exchange property if and only if $O$ satisfies (O6), similarly, $I^G$ satisfies the exchange property if and only if $G$ satisfies (G6).

**Lemma 1** (Qiao 2021) Let $x, y, z \in [0, 1]$ and $\{x_i\}_{i \in \Lambda} \subseteq [0, 1]$. Then

(1) $O(x, I_O(x, y)) \leq y$ and $y \leq G(I^G(x, y), x)$;

(2) $I_O(y, (\bigwedge_{i \in \Lambda} x_i)) = \bigwedge_{i \in \Lambda} I_O(y, x_i)$ and $I^G(y, (\bigvee_{i \in \Lambda} x_i)) = \bigvee_{i \in \Lambda} I^G(y, x_i)$;

(3) $I_O(y, (\bigvee_{i \in \Lambda} x_i)) = \bigvee_{i \in \Lambda} I_O(y, x_i)$ and $I^G(y, (\bigwedge_{i \in \Lambda} x_i)) = \bigwedge_{i \in \Lambda} I^G(y, x_i)$;

(4) $I_O(\bigvee_{i \in \Lambda} x_i, y) = \bigvee_{i \in \Lambda} I_O(x_i, y)$ and $I^G(\bigwedge_{i \in \Lambda} x_i, y) = \bigwedge_{i \in \Lambda} I^G(x_i, y)$;

(5) $I_O(x, I_O(y, z)) = I_O(O(x, y), z)$ if and only if $O$ satisfies (O6) and $I^G(x, I^G(y, z)) = I^G(G(x, y), z)$ if and only if $G$ satisfies (G6).

**Lemma 2** (Dimuro and Bedregal 2015) Let overlap function $O$ have identity element 1, and grouping function $G$ have identity element 0. For any $x, y, z \in [0, 1]$, the following statements hold.

(1) $I_O(1, x) = x$ and $I^G(0, x) = x$;

(2) $x \leq y$ if and only if $I_O(x, y) = 1$ if and only if $I^G(y, x) = 0$;

(3) $x \leq I_O(y, x)$ and $x \geq I^G(y, x)$.

**Lemma 3** Let overlap function $O : [0, 1]^2 \rightarrow [0, 1]$ (resp. grouping function $G : [0, 1]^2 \rightarrow [0, 1]$) satisfy (O6) (resp. (G6)). For any $x, y, z \in [0, 1]$, the following statements hold.

(1) $O(x, I_O(y, z)) \leq I_O(y, O(x, z))$ and $I^G(y, G(x, z)) \leq G(x, I^G(y, z))$;

(2) $I_O(y, z) \leq I_O(I_O(x, y), I_O(x, z))$ and $I^G(G(x, y), I^G(x, z)) \leq I^G(x, z)$.

**Proof** It is obvious that $O$ becomes a $t$-norm when it satisfies (O6), we can immediately obtain that $O(x, I_O(y, z)) \leq I_O(y, O(x, z))$ and $I_O(y, z) \leq I_O(I_O(x, y), I_O(x, z))$. The equations about $G$ can be derived similarly.

In the following, some basics about fuzzy sets are given.
Let finite set $X$ be universe, and the family of all fuzzy sets on $X$ is denoted $\mathcal{F}(X)$. The fuzzy set $A$ defined as $A(x) = \alpha$ for any $A \in \mathcal{F}(X)$ and $x \in X$, is a constant and further called $\alpha_X$. In addition, a fuzzy point $A$ is tagged with $y_A$, if for all $x \in X$,

$$A(x) = \begin{cases} \alpha, & x = y; \\ 0, & x \neq y. \end{cases}$$

Further, $|A|$ denotes the cardinality of the set $A$ for all crisp sets $A$.

**Definition 3** A function $N : [0, 1] \rightarrow [0, 1]$ is a fuzzy negation, if it satisfies the following conditions:

1. If $x < y$, then $N(x) > N(y)$, for all $x, y \in [0, 1]$.
2. $N(0) = 1$ and $N(1) = 0$.

More specifically, $N$ is called an involutive negation, if $N(N(x)) = x$ holds for all $x \in [0, 1]$ and the standard negation, $N(x) = 1 - x$ for all $x \in [0, 1]$, is a special case of involutive negation $N$.

The operations on fuzzy sets are defined as follows: for all $A, B \in \mathcal{F}(X)$ and $x \in X$,

1. $A^N(x) = N(A(x))$;
2. $O(A, B)(x) = O(A(x), B(x))$;
3. $G(A, B)(x) = G(A(x), B(x))$;
4. $I_O(A, B)(x) = I_O(A(x), B(x))$;
5. $I^G(A, B)(x) = I^G(A(x), B(x))$.

If for all $x, y \in [0, 1]$, $N(x \oplus y) = N(x) \odot N(y)$, then the two binary operations $\oplus$ and $\odot$ are dual with respect to (w.r.t., for short) $N$. Especially, $(A^f)(x) = 1 - A(x)$ and $A \subseteq B$ defined as $A(x) \leq B(x)$ for all $x \in X$. In addition, a fuzzy relation on $X$ is a fuzzy set $R \in \mathcal{F}(X \times X)$ and $R^{-1}$ is defined as $R^{-1}(x, y) = R(y, x)$ for all $x, y \in X$.

**Definition 4** Let $R$ be a fuzzy relation on $X$ and for all $x, y, z \in X$, $R$ satisfies

1. seriality: $\forall y \in X, R(x, y) = 1$;
2. reflexivity: $R(x, x) = 1$;
3. symmetry: $R(x, y) = R(y, x)$;
4. $O$-transitivity: $O(R(x, y), R(y, z)) \leq R(x, z)$.

For sake of simplicity, $\wedge$-transitive is called transitive. $R$ is a fuzzy $O$-preorder relation when it satisfies reflexivity and $O$-transitivity and a fuzzy $O$-similarity relation when it satisfies reflexivity, symmetry and $O$-transitivity.

Next, the model of GVPFRSs which proposed by Wang and Hu (2015) will be given below.

**Definition 5** (Wang and Hu 2015) Let $R$ be a fuzzy relation on $X$, $\beta \in [0, 1]$ and $\mathcal{F}_\beta(X) = \{X_i \subseteq X : |X_i| \geq \beta |X| \}$. Then for all $A \in \mathcal{F}(X)$, two fuzzy operators $R^\beta$ and $\bar{R}^\beta$ are defined as follows. 

$$R^\beta(A) = \bigcup \{[x_y]_R^\beta : x \in X, \gamma \in [0, 1], \{y \in X : [x_y]_R^\beta(y) \leq A(y)\} \in \mathcal{F}_\beta(X)\},$$

$$\bar{R}^\beta(A) = \bigcap \{[x_y]_R^\beta : x \in X, \gamma \in [0, 1], \{y \in X : A(y) \leq [x_y]_R^\beta(y)\} \in \mathcal{F}_\beta(X)\},$$

Then $R^\beta$ (resp. $\bar{R}^\beta$) is the generalized granular variable precision lower (resp. upper) approximation operator and the pair $(R^\beta(A), \bar{R}^\beta(A))$ is GVPFRS of fuzzy set $A$. 

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3 \((O, G)\)-granular variable precision fuzzy rough sets based on overlap and grouping functions

In the following, we give the model of \((O, G)\)-GVPFRSs and then utilize fuzzy implication and co-implication to compute the approximation operators more efficiently. In addition, we continue to study the related properties of degenerated \((O, G)\)-GVPFRSs under the condition of crisp relations and crisp sets, respectively.

**Definition 6** Let \(R\) be a fuzzy relation on \(X\). Fuzzy granules \([x_\lambda]_R^O\) and \([x_\lambda]_R^G\) are denoted as following.

\[
[x_\lambda]_R^O(y) = O(R(x, y), \lambda) \text{ and } [x_\lambda]_R^G(y) = G(R^N(x, y), \lambda),
\]

where \(x, y \in X, \lambda \in [0, 1]\) and \(N\) is an involutive negation.

Dimuro et al. (2014) had defined the class of fuzzy implications called \((G, N)\)-implications, where \(G\) and \(N\) are grouping functions and fuzzy negations respectively. Detailed definition of \((G, N)\)-implications is introduced as follows:

Let \(G : [0, 1]^2 \rightarrow [0, 1]\) be a grouping function and \(N : [0, 1] \rightarrow [0, 1]\) be a fuzzy negation, the function \(I_{G,N}\) denoted by

\[
I_{G,N}(a, b) = G(N(a), b),
\]

for all \(a, b \in [0, 1]\).

Then, according to Definition 6, we have that

\[
[x_\lambda]_R^G(y) = G(R^N(x, y), \lambda) = I_{G,N}(R(x, y), \lambda).
\]

3.1 \((O, G)\)-granular variable precision fuzzy rough sets based on overlap and grouping functions

**Definition 7** Let \(R\) be a fuzzy relation on \(X\), \(\beta \in [0, 1]\) and \(\mathcal{F}_\beta(X) = \{X_i \subseteq X : |X_i| \geq \beta |X|\}\). Then, for all \(A \in \mathcal{F}(X)\), two fuzzy operators from \(\mathcal{F}(X)\) to \(\mathcal{F}(X)\) are defined as follows:

\[
R_\beta^O(A) = \bigcup \{[x_\lambda]_R^O : x \in X, \lambda \in [0, 1], \{y \in X : [x_\lambda]_R^O(y) \leq A(y)\} \in \mathcal{F}_\beta(X)\},
\]

\[
\overline{R}_\beta^G(A) = \bigcap \{[x_\lambda]_R^G : x \in X, \lambda \in [0, 1], \{y \in X : A(y) \leq [x_\lambda]_R^G(y)\} \in \mathcal{F}_\beta(X)\},
\]

then \(R_\beta^O\) and \(\overline{R}_\beta^G\) are called the \(O\)-granular variable precision lower approximation operator and \(G\)-granular variable precision upper approximation operator, respectively. The pair \((R_\beta^O(A), \overline{R}_\beta^G(A))\) is called the \((O, G)\)-granular variable precision fuzzy rough set of fuzzy set \(A\).

**Remark 4** If \(t\)-norm (resp. \(t\)-conorm) in Wang and Hu (2015) is continuous and positive, then generalized granular variable precision lower approximation operator (resp. generalized granular variable precision upper approximation operator) defined by Wang and Hu is equal to the \(O\)-granular variable precision lower approximation operator (resp. \(G\)-granular variable precision upper approximation operator) in Definition 7. The \((O, G)\)-granular variable precision fuzzy rough sets are defined on arbitrary fuzzy relations unlike the variable precision \((\theta, \sigma)\)-fuzzy rough sets in Yao et al. (2014) only defined on fuzzy \(\ast\)-similarity relations. Furthermore, the overlap function \(O\) and the grouping function \(G\) in \((O, G)\)-GVPFRSs do not need to be dual w.r.t. the standard negation \(N\).
In the following two propositions, we give the equivalent expressions of the $O$-granular variable precision lower approximation operator and $G$-granular variable precision upper approximation operator, respectively.

**Proposition 1** Let $R$ be a fuzzy relation on $X$. For all $A \in \mathcal{P}(X)$, $x \in X$ and $X_i \in \mathcal{P}\beta(X)$, define that

$$g^i_A(x) = \bigcap_{y \in X_i} I_O(R(x, y), A(y))$$

$$g_A(x) = \bigvee_{X_i \in \mathcal{P}\beta(X)} g^i_A(x).$$

Then, it always holds

$$R^\beta_O(A) = \bigcup \{[x_{g_A(x)}]^O_R : x \in X\} \text{ and } \{y : [x_{g_A(x)}]^O_R(y) \leq A(y)\} \in \mathcal{P}\beta(X)$$

where $x \in X$ and $A \in \mathcal{P}(X)$.

**Proof** On the one hand, let $x \in X$, $\lambda \in [0, 1]$, and $\{y \in X : [x_{\lambda}]^O_R(y) \leq A(y)\}$ be written as $Y$, while $Y \in \mathcal{P}\beta(X)$. Then for all $y \in Y$, consider the following equivalences,

$$[x_{\lambda}]^O_R(y) \leq A(y) \iff O(R(x, y), \lambda) \leq A(y) \iff \lambda \leq I_O(R(x, y), A(y)).$$

Hence, we obtain $\lambda \leq g_A(x)$. Then, for all $A \in \mathcal{P}(X)$, it always holds $R^\beta_O(A) \subseteq \bigcup \{[x_{g_A(x)}]^O_R : x \in X\}$ by Definition 7.

On the other hand, let $x \in X$. Then there exists $X_i \in \mathcal{P}\beta(X)$ such that $g_A(x) = g^i_A(x)$. For all $y \in X_i$, we get that

$$[x_{g_A(x)}]^O_R(y) = O(R(x, y), g^i_A(x))$$

$$= O(R(x, y), \bigcap_{z \in X_i} I_O(R(x, z), A(z)))$$

$$\leq O(R(x, y), I_O(R(x, y), A(y)))$$

$$\leq A(y).$$

Thus, $X_i \subseteq \{y \in X : [x_{g_A(x)}]^O_R(y) \leq A(y)\}$ and $R^\beta_O(A) \subseteq \bigcup \{[x_{g_A(x)}]^O_R : x \in X\}$ hold.

In summary, for all $x \in X$ and $A \in \mathcal{P}(X)$, $R^\beta_O(A) = \bigcup \{[x_{g_A(x)}]^O_R : x \in X\}$ and $\{y : [x_{g_A(x)}]^O_R(y) \leq A(y)\} \in \mathcal{P}\beta(X)$ hold. $\square$

**Proposition 2** Let $R$ be a fuzzy relation on $X$. For all $A \in \mathcal{P}(X)$, $x \in X$ and $X_i \in \mathcal{P}\beta(X)$, define

$$h^i_A(x) = \bigvee_{y \in X_i} I^G(R^N(x, y), A(y))$$

$$h_A(x) = \bigwedge_{X_i \in \mathcal{P}\beta(X)} h^i_A(x).$$

Then, it always holds

$$R^\beta_O(A) = \bigcap \{[x_{h_A(x)}]^G_R : x \in X\} \text{ and } \{y : A(y) \leq [x_{h_A(x)}]^G_R(y)\} \in \mathcal{P}\beta(X),$$

where $x \in X$ and $A \in \mathcal{P}(X)$. 

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Proof On the one hand, let \( x \in X, \lambda \in [0, 1] \) and \( \{y \in X : A(y) \leq [x_\lambda]_R^G(y)\} \) be written as \( Y \), while \( Y \in \mathcal{F}_\beta(X) \). Then for all \( y \in Y \), consider the following equivalences,

\[
A(y) \leq [x_\lambda]_R^G(y) \iff A(y) \leq G(R^N(x, y), \lambda) \iff I^G(R^N(x, y), A(y)) \leq \lambda.
\]

Hence, we obtain \( h_A(x) \leq \lambda \). Thus, for all \( A \in \mathcal{F}(X) \), it always holds \( \overline{R}_G^\beta(A) \supseteq \bigcap \{[x_{h_A(x)}]_R^G : x \in X\} \) by Definition 7.

On the other hand, let \( x \in X \). Then there exists \( X_i \in \mathcal{F}_{\beta}(X) \) such that \( h_A(x) = h_A^{(i)}(x) \). For all \( y \in X_i \), we get that

\[
[x_{h_A(x)}]_R^G(y) = G(R^N(x, y), h_A^{(i)}(x))
\]

\[
= G(R^N(x, y), \bigvee_{z \in X_i} I^G(R^N(x, z), A(z)))
\]

\[
\geq G(R^N(x, y), I^G(R^N(x, y), A(y)))
\]

\[
\geq A(y).
\]

Thus, \( X_i \subseteq \{y \in X : A(y) \leq [x_{h_A(x)}]_R^G(y)\} \) and \( \overline{R}_G^\beta(A) \subseteq \bigcap \{[x_{h_A(x)}]_R^G : x \in X\} \) hold.

In summary, for all \( x \in X \) and \( A \in \mathcal{F}(X) \), \( \overline{R}_G^\beta(A) = \bigcap \{[x_{h_A(x)}]_R^G : x \in X\} \) and \( \{y : A(y) \leq [x_{h_A(x)}]_R^G(y)\} \in \mathcal{F}_{\beta}(X) \) hold. \( \square \)

Remark 5 Propositions 1 and 2 provide the equivalent expressions of \( \overline{R}_O^\beta \) and \( \overline{R}_G^\beta \) with \( g_A \) and \( h_A \) on arbitrary fuzzy relation, which can efficiently calculate the approximation operators and no longer to consider fuzzy granule \( [x_\lambda]_R^O \) or \( [x_\lambda]_R^G \) for all \( x \in X \). Note that the proofs of Propositions 1 and 2 are similar. Therefore, in the following we only give the proof of the lower approximation operator \( \overline{R}_O^\beta \), and the proof of the upper approximation operator \( \overline{R}_G^\beta \) can be derived in a similar way.

Proposition 3 Let \( R \) be a fuzzy relation on \( X \). If overlap function \( O \) and grouping function \( G \) are dual w.r.t. \( N \), then the following statements hold for all \( A \in \mathcal{F}(X) \),

\[
(g_A)^N = h_A^N \quad \text{and} \quad (h_A)^N = g_A^N.
\]

And then, we have

\[
(\overline{R}_O^\beta(A))^N = \overline{R}_G^\beta(A^N) \quad \text{and} \quad (\overline{R}_G^\beta(A))^N = \overline{R}_O^\beta(A^N).
\]

Proof If \( O \) and \( G \) are dual w.r.t. \( N \), then for any \( A \in \mathcal{F}(X) \) and \( x \in X \),

\[
(g_A)^N(x) = \bigwedge_{X_i \in \mathcal{F}_{\beta}(X)} N(g_A^{(i)}(x))
\]

\[
= \bigwedge_{X_i \in \mathcal{F}_{\beta}(X)} \bigvee_{y \in X_i} N(I_O(R(x, y), A(y)))
\]

\[
= \bigwedge_{X_i \in \mathcal{F}_{\beta}(X)} \bigvee_{y \in X_i} I^G(R^N(x, y), A^N(y))
\]

\[
= \bigwedge_{X_i \in \mathcal{F}_{\beta}(X)} h_A^{(i)}(x)
\]

\[
= h_A^N(x).
\]
Hence, it always holds \((g_A)^N = h_{AN}\). In a similar way, we obtain \((h_A)^N = g_{AN}\).

For any \(A \in \mathcal{F}(X)\) and \(y \in X\), the following equations hold according to Propositions 1 and 2.

\[
\left( R^\beta_O(A) \right)^N(y) = \bigwedge_{x \in X} N \left( \left[ x_{g_A(x)} \right]_R \right)
\]

\[
= \bigwedge_{x \in X} N \left( O \left( R(x, y), g_A(x) \right) \right)
\]

\[
= \bigwedge_{x \in X} G \left( R^N(x, y), (g_A(x))^N \right)
\]

\[
= \bigwedge_{x \in X} G \left( R^N(x, y), h_{AN}(x) \right)
\]

\[
= \bigwedge_{x \in X} \left[ x_{h_{AN}(x)} \right]_R
\]

\[
= R^\beta_G(A^N)(y).
\]

Therefore, we know that \(\left( R^\beta_O(A) \right)^N = R^\beta_G(A^N)\). Similarly, \(\left( R^\beta_G(A) \right)^N = R^\beta_O(A^N)\) holds. \(\square\)

The comparable property, as a fundamental property between lower rough approximation operator and upper rough approximation operator is discussed in Ciucci (2009), Csajb and Fromvagueness (2014), Yao (1996). Next, we study the case of \((O, G)\)-granular variable precision fuzzy rough sets meet this fundamental property.

**Remark 6** Based on the variable precision \(\beta\), the comparable property of \(O\)-granular variable precision lower approximation operator and \(G\)-granular variable precision upper approximation operator are discussed in three cases.

- Variable precision \(\beta = 1\)

In particular, when the value of \(\beta\) is 1, we have \(\mathcal{F}_\beta(X) = \{X\}\). Then for all \(A \in \mathcal{F}(X)\) and \(x \in X\),

\[
g_A(x) = \bigwedge_{y \in X} I_O(R(x, y), A(y)).
\]

It follows from Proposition 1 that for all \(z \in X\),

\[
R^\beta_O(A)(z) = \bigvee_{x \in X} O \left( R(x, z), g_A(x) \right)
\]

\[
= \bigvee_{x \in X} O \left( R(x, z), \bigwedge_{y \in X} I_O(R(x, y), A(y)) \right)
\]

\[
\leq \bigvee_{x \in X} O \left( R(x, z), I_O(R(x, z), A(z)) \right)
\]

\[
\leq A(z).
\]

Hence, if \(\beta = 1\), it always holds that \(R^\beta_O(A) \subseteq A\) and \(R^\beta_G(A) \supseteq A\) can be proven in a similar way.

In the sequel, we further assume that \(R\) is a fuzzy \(O\)-similarity relation and \(O\) (resp. \(G\))
satisfies (O6) (resp.(G6)), then by Theorem 4.1.3 in Chen et al. (2011), we can obtain that

\[ R^β_O(A)(x) = \bigwedge_{y \in X} I_O(R(x, y), A(y)) \quad \text{and} \quad \overline{R}^β_G(A) = \bigvee_{y \in X} I^G(R^N(x, y), A(y)), \]

for all \( A \in \mathcal{F}(X) \) and \( x \in X \).

Due to \( R \) is reflexive, it follows from Lemma 2 (1) that for all \( A \in \mathcal{F}(X) \) and \( x \in X \),

\[ R^β_O(A)(x) \leq I_O(R(x, x), A(x)) = I_O(1, A(x)) = A(x), \]

\[ \overline{R}^β_G(A)(x) \geq I^G(R^N(x, x), A(x)) = I^G(0, A(x)) = A(x). \]

Therefore, if \( \frac{|X|-1}{|X|} < β \leq 1 \), then \( R^β_O(A) \subseteq A \subseteq \overline{R}^β_G(A) \) holds for all \( A \in \mathcal{F}(X) \).

- Arbitrary variable precision \( β \) and fuzzy \( O \)-similarity relation \( R \)
  Even if overlap function \( O \) and grouping function \( G \) are dual w.r.t the standard negation \( N \) for all \( a \in [0, 1] \), \( R^β_O(A) \) and \( \overline{R}^β_G(A) \) do not have comparable properties. A specific example is given below.
  Consider \( X = \{x_1, x_2, x_3\} \) and fuzzy relation \( R \) on \( X \) as

\[
R = \begin{bmatrix}
1 & 0.6 & 1 \\
0.6 & 1 & 0.6 \\
1 & 0.6 & 1 \\
\end{bmatrix}.
\]

Here, the overlap function \( O \) and its fuzzy implication \( I_O \) are defined as follows,

\[ O(x, y) = xy \quad \text{and} \quad I_O(x, y) = \begin{cases} 
\frac{x}{y} \wedge 1, & x \neq 0 \\
1, & x = 0
\end{cases} \quad \text{for any } x, y \in [0, 1]. \]

It is easy to see that fuzzy relation \( R \) is a fuzzy \( O \)-similarity relation for overlap function \( O \). Let \( A = \frac{0.8}{x_1} + \frac{0.1}{x_2} + \frac{0.6}{x_3} \) and \( β = 0.5 \). By Theorem 2 in Yao et al. (2014), it holds that

\[ R^β_O(A) = g(A) = \frac{0.6}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3}. \]

According to Theorem 3(1) in Yao et al. (2014) or Proposition 3, we can obtain

\[ \overline{R}^β_G(A) = (R^β_O(A^N))^N = \frac{0.4}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3}, \]

where \( N \) is the standard negation \( N(x) = 1 - x \) for all \( x \in [0, 1] \) and grouping function \( G \) is defined as \( G(x, y) = 1 - (1 - x)(1 - y) \) for all \( x, y \in [0, 1] \).

- Arbitrary variable precision \( β \) and fuzzy relation \( R \)
  Consider \( X = \{x_1, x_2, x_3\} \) and fuzzy relation \( R \) on \( X \) as

\[
R = \begin{bmatrix}
0 & 0.2 & 0.8 \\
1 & 0 & 1 \\
0 & 0.1 & 0 \\
\end{bmatrix},
\]

where the fuzzy relation \( R \) is neither reflexive nor symmetric. Since

\[ \bigvee_{y \in X} \{ O(R(x_1, y), R(y, x_1)) \} = 0.2 > R(x_1, x_1), \]

\( R \) is not \( O \)-transitive. Next, the expression of \((O, G)\)-GVPFRSs is given according to Definition 7.
Here, the overlap function $O$ and its the fuzzy implication $I_O$ use the expression in the above case. Let $A = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3}$ and $\beta = 0.5$, by Proposition 1, it holds that

$$g_A = \frac{0.75}{x_1} + \frac{0.6}{x_2} + \frac{1}{x_3}. $$

So $O$-granular variable precision lower approximation operator is

$$R^L_O(A) = \frac{0.6}{x_1} + \frac{0.15}{x_2} + \frac{0.6}{x_3}. $$

Next, we calculate the $G$-granular variable precision upper approximation operator with $N(x) = 1 - x$ for all $x \in [0, 1]$, $G(x, y) = \max\{x, y\}$ and $IG(x, y) = \left\{ \begin{array}{ll} y, & x < y \\ 0, & x \geq y \end{array} \right.$

for all $x, y \in [0, 1]$. It follows from Proposition 2 that

$$R^U_G(A) = \frac{0.2}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3}. $$

Hence, $R^L_O$ and $R^U_G$ are not comparable, where $R^L_O(A)$ and $R^U_G(A)$ are not dual w.r.t. the standard negation $N$.

### 3.2 The degenerated $(O, G)$-granular variable precision fuzzy rough sets

We define $[x]_R = \{y : R(x, y) = 1\}$ when $R$ is a crisp relation on $X$. In particular, if fuzzy relations and fuzzy sets degenerate into crisp relations and crisp sets, we call the existing models as degenerated $(O, G)$-GVPFRSs.

**Lemma 4** Let $R$ be a crisp relation on $X$. Then for all $A \in \mathcal{F}(X)$,

$$(R^L_O(A))^N = R^U_G(A^N) \text{ and } (R^U_G(A))^N = R^L_O(A^N).$$

**Proof** Since $R$ is a crisp relation, for all $x \in X$ and $\lambda \in [0, 1]$,

$$[x \lambda]_R^O = [x \lambda]_R^G \text{ and } [x \lambda]_R^G = [x \lambda]_R^G. $$

Due to the duality of minimum and maximum w.r.t. $N$, it follows from Proposition 3 that for all $A \in \mathcal{F}(X)$,

$$(R^L_O(A))^N = R^U_G(A^N) \text{ and } (R^U_G(A))^N = R^L_O(A^N).$$

\[\Box\]

**Proposition 4** Let $R$ be a crisp relation on $X$. Then for all crisp sets $A \subseteq X$,

$$R^L_O(A) = \bigcup\{[x]_R \in X, [x]_R \cap A \leq (1-\beta) \mid X \},$$

$$R^U_G(A) = \bigcap\{[x]_R^C \in X, [x]_R \cap A \leq (1-\beta) \mid X \}. $$

**Proof** For any $\lambda \in (0, 1]$ and crisp sets $A \subseteq X$, we will prove the following statement.

$$\{y : [x]_R^O(y) \leq A(y)\} = \{y : [x]_R(y) \leq A(y)\}. $$

Let $O(R(x, y), \lambda) \leq A(y)$ if $A(y) = 1$, it is obvious that $R(x, y) \leq A(y)$. If $A(y) = 0$, we can get that $R(x, y) = 0$, if not, $O(R(x, y), \lambda) = \lambda \leq 0$, which contradicts with $\lambda \in (0, 1]$. Thus, $\{y : [x]_R^O(y) \leq A(y)\} \subseteq \{y : [x]_R(y) \leq A(y)\}$. 

\(\square\)
On the other side, \( \{ y : [x]^O_R(y) \leq A(y) \} \supseteq \{ y : [x]_R(y) \leq A(y) \} \) can hold apparently. Hence, it always holds that \( \{ y : [x]^O_R(y) \leq A(y) \} = \{ y : [x]_R(y) \leq A(y) \} \) for all crisp sets \( A \subseteq X \). Furthermore, by Definition 2, we have

\[
R^\beta_O(A) = \bigcup \{ [x]_R : x \in X, \lambda \in [0, 1], \{ y : [x\lambda]_R(y) \leq A(y) \} \in \mathcal{F}_\beta(X) \}
\]

\[
= \bigcup \{ [x]_R : x \in X, \{ y : [x]_R(y) \leq A(y) \} \in \mathcal{F}_\beta(X) \}.
\]

Because of the following equivalences,

\[
\{ y : [x]_R(y) \leq A(y) \} \in \mathcal{F}_\beta(X) \iff A \cup (A^c \cap [x]_R) \in \mathcal{F}_\beta(X)
\]

\[
\iff | A \cup (A^c \cap [x]_R) | \geq \beta | X |
\]

\[
\iff | A^c \cap [x]_R | \leq (1 - \beta) | X |,
\]

then \( R^\beta_O(A) = \bigcup \{ [x]_R : x \in X, | A^c \cap [x]_R | \leq (1 - \beta) | X | \} \) holds for all crisp sets \( A \subseteq X \). Since \( R \) and \( A \) are crisp relation and crisp set, respectively, \( R^N = R^c \) and \( A^N = A^c \). Therefore, the other equation can be obtained by Lemma 1.

\[\square\]

**Proposition 5** Let \( R \) and \( A \) be crisp relation and crisp subset on \( X \), the following statements hold.

1. If \( R \) is reflexive, then

\[
R^\beta_O(A) \supseteq \{ x : | [x]_R \cap A^c | \leq (1 - \beta) | X | \},
\]

\[
\overline{R}^\beta_G(A) \subseteq \{ x : | [x]_R \cap A | > (1 - \beta) | X | \}.
\]

2. If \( R \) is transitive, then

\[
R^\beta_O(A) \subseteq \{ x : | [x]_R \cap A^c | \leq (1 - \beta) | X | \},
\]

\[
\overline{R}^\beta_G(A) \supseteq \{ x : | [x]_R \cap A | > (1 - \beta) | X | \}.
\]

3. If \( R \) is a preorder, then

\[
R^\beta_O(A) = \{ x : | [x]_R \cap A^c | \leq (1 - \beta) | X | \},
\]

\[
\overline{R}^\beta_G(A) = \{ x : | [x]_R \cap A | > (1 - \beta) | X | \}.
\]

**Proof** (1) If \( R \) is reflexive, by Proposition 4, we get that

\[
\{ x : | [x]_R \cap A^c | \leq (1 - \beta) | X | \} \subseteq \bigcup \{ x : | [x]_R \cap A^c | \leq (1 - \beta) | X | \} = R^\beta_O(A).
\]

Further according to Lemma 4, then

\[
\overline{R}^\beta_G(A) = (R^\beta_O(A^c))^c \subseteq \{ x : | [x]_R \cap A | \leq (1 - \beta) | X | \}^c
\]

\[
= \{ x : | [x]_R \cap A | > (1 - \beta) | X | \}.
\]

(2) Let \( R \) be transitive and \( w \in R^\beta_O(A) \). There exits an \( x \in X \) such that \( w \in [x]_R \) and \( | [x]_R \cap A^c | \leq (1 - \beta) | X | \). Since \( R \) is transitive, \( R(w, y) \leq R(x, y) \) holds for all \( y \in X \). Hence, we obtain that \( [w]_R \cap A^c \subseteq [x]_R \cap A^c \). By Proposition 4, we conclude that

\[
w \in \{ x : | [x]_R \cap A^c | \leq (1 - \beta) | X | \}.
\]
So $R^\beta_O(A) \subseteq \{x : |[x]_R \cap A^c| \leq (1 - \beta) |X|\}$ holds. According to Lemma 4, it holds that

$$\overline{R}^\beta_G(A) = (R^\beta_O(A^c))^c = \left(\{x : |[x]_R \cap A| \leq (1 - \beta) |X|\}\right)^c = \{x : |[x]_R \cap A| > (1 - \beta) |X|\}.$$

(3) It can be proven by items (1) and (2).

\[\square\]

4 Characterizations of $(O, G)$-granular variable precision fuzzy rough sets

By Remark 5, we realize that two fuzzy sets $g_A$ and $h_A$ are vital to calculate the upper approximation operator $\overline{R}^\beta_G$ and the lower approximation operator $R^\beta_O$, respectively. Thus, we start this section with discussing the properties of $\overline{R}^\beta_G$ and $R^\beta_O$, and then, some related conclusions are drawn.

4.1 Some conclusions based on arbitrary fuzzy relations

**Lemma 5** Let $R$ be a fuzzy relation on $X$. Then the following statements hold.

1. $g^{(i)}_{(\cap_{k \in I} A_k)} = \bigcap_{k \in I} g^{(i)}_{A_k}$ and $h^{(i)}_{(\cup_{k \in I} A_k)} = \bigcup_{k \in I} h^{(i)}_{A_k}$ for all $X_i \in \mathcal{F}_\beta(X)$ and $\{A_k\}_{k \in I} \subseteq \mathcal{F}(X)$.

2. $A \subseteq B$ implies $g_A \subseteq g_B$ and $h_A \subseteq h_B$ for all $A, B \in \mathcal{F}(X)$.

**Proof** (1) It follows from Lemma 1(2) that for all $x \in X$ and $X_i \in \mathcal{F}_\beta(X)$,

$$g^{(i)}_{(\cap_{k \in I} A_k)}(x) = \bigwedge_{y \in X_i} I_O^\beta(R(x, y), (\bigwedge_{k \in I} A_k(x)))$$

$$= \bigwedge_{y \in X_i} \bigwedge_{k \in I} I_O^\beta(R(x, y), A_k(x))$$

$$= \bigwedge_{k \in I} \bigwedge_{y \in X_i} I_O^\beta(R(x, y), A_k(x))$$

$$= \bigwedge_{k \in I} g^{(i)}_{A_k}(x)$$

$$= \left(\bigcap_{k \in I} g^{(i)}_{A_k}\right)(x).$$

Hence, we get $g^{(i)}_{(\cap_{k \in I} A_k)} = \bigcap_{k \in I} g^{(i)}_{A_k}$. In a similar way, $h^{(i)}_{(\cup_{k \in I} A_k)} = \bigcup_{k \in I} h^{(i)}_{A_k}$ holds for all $X_i \in \mathcal{F}_\beta(X)$ and $\{A_k\}_{k \in I} \subseteq \mathcal{F}(X)$.

(2) According to item (1), it can be directly proven.

\[\square\]

**Lemma 6** Let $R$ be a fuzzy relation on $X$, overlap function $O$ and grouping function $G$ have 1 and 0 as identity element, respectively. Then the following statements hold.
(1) \( g_X = X \) and \( h_\emptyset = \emptyset \).
(2) \( \alpha_X \subseteq g_{\alpha_X} \) and \( h_{\alpha_X} \subseteq \alpha_X \) for all \( \alpha \in [0, 1] \).
(3) If \( A = I_O(y, \alpha_X) \) and \( \gamma = 1 \), then
\[
g_A(x) = \begin{cases} 
1, & 0 \leq \beta \leq \frac{|X| - 1}{|X|}, \\
I_O(R(x, y), \alpha), & \frac{|X| - 1}{|X|} < \beta \leq 1.
\end{cases}
\]
(4) If \( A = y_{\alpha} \), then
\[
h_{A}(x) = \begin{cases} 
0, & 0 \leq \beta \leq \frac{|X| - 1}{|X|}, \\
I^G(R^N(x, y), \alpha), & \frac{|X| - 1}{|X|} < \beta \leq 1.
\end{cases}
\]

**Proof**

(1) It is easy to get that \( I_O(\alpha, 1) = 1 \) and \( I^G(\alpha, 0) = 0 \) for all \( \alpha \in [0, 1] \) by Lemma 2(1). Thus for all \( x \in X \),
\[
g_X(x) = \bigvee_{X_i \in \mathcal{F}_\beta(X)} \bigwedge_{y \in X_i} (I_O(R(x, y), X(y)))
= \bigvee_{X_i \in \mathcal{F}_\beta(X)} \bigwedge_{y \in X_i} (I_O(R(x, y), 1)) = 1,
\]
\[
h_\emptyset(x) = \bigwedge_{X_i \in \mathcal{F}_\beta(X)} \bigvee_{y \in X_i} (I^G(R^N(x, y), \emptyset(y)))
= \bigwedge_{X_i \in \mathcal{F}_\beta(X)} \bigvee_{y \in X_i} (I^G(R^N(x, y), 0)) = 0.
\]

Hence, we obtain that \( g_X = X \) and \( h_\emptyset = \emptyset \).

(2) In light of Lemma 2(3), it holds that
\[
g_{\alpha_X}(x) = \bigvee_{X_i \in \mathcal{F}_\beta(X)} \bigwedge_{y \in X_i} (I_O(R(x, y), \alpha_X(x)))
= \bigvee_{X_i \in \mathcal{F}_\beta(X)} \bigwedge_{y \in X_i} (I_O(R(x, y), \alpha))
\geq \alpha_X(x).
\]

Hence, we get that \( \alpha_X \subseteq g_{\alpha_X} \). In a similar way, \( h_{\alpha_X} \subseteq \alpha_X \) holds.

(3) If \( \frac{|X| - 1}{|X|} < \beta \leq 1 \), then we obtain that \( \mathcal{F}_\beta(X) = \{X\} \). Let \( A = I_O(y, \alpha_X) \) and \( \gamma = 1 \), for any \( x \in X \),
\[
g_A(x) = \bigwedge_{z \in X} I_O(R(x, z), A(z))
= I_O(R(x, y), I_O(y, \alpha_X(y)))
= I_O(R(x, y), I_O(1, \alpha))
= I_O(R(x, y), \alpha).
\]

Otherwise, if \( 0 \leq \beta \leq \frac{|X| - 1}{|X|} \), we have \( X - \{y\} \in \mathcal{F}_\beta(X). \) For all \( x \in X \), it holds that
\[
g_A(x) \geq \bigwedge_{z \in X - \{y\}} I_O(R(x, z), A(z))
\]
(4) The proof is similar to item (3).

\[\square\]

Lemma 7 Let \( R \) be a fuzzy relation on \( X \), overlap function \( O \) and grouping function \( G \) satisfy (O6) and (G6), respectively. Then the following statements hold.

(1) \( g(I_O(\alpha, A)) = I_O(\alpha, g A) \) and \( h(I_G(\alpha, A)) = I_G^G(\alpha, h A) \) for all \( \alpha \in [0, 1] \) and \( A \in \mathcal{F}(X) \).

(2) \( O(\alpha, g A) \subseteq g(O(\alpha, A)) \) and \( G(\alpha, h A) \supseteq h(G(\alpha, A)) \) for all \( \alpha \in [0, 1] \) and \( A \in \mathcal{F}(X) \).

Proof (1) By Lemma 1(2) and (5), the following equation holds for all \( x \in X, \alpha \in [0, 1] \) and \( X_i \in \mathcal{F}_\beta(X) \),

\[
g_{IO(\alpha, A)}^{(i)}(x) = \bigwedge_{y \in X_i} I_O(R(x, y), I_O(\alpha, A(y)))
\]

\[
= \bigwedge_{y \in X_i} I_O(O(\alpha, R(x, y)), A(y))
\]

\[
= \bigwedge_{y \in X_i} I_O(\alpha, I_O(R(x, y), A(y)))
\]

\[
= I_O\left(\alpha, \bigwedge_{y \in X_i} (I_O(R(x, y), A(y)))\right)
\]

\[
= I_O(\alpha, g_A^{(i)}(x))
\]

\[
= I_O(\alpha, g_A^{(i)}(x)).
\]

Since \( \mathcal{F}_\beta(X) \) is finite, it follows from Lemma 1(3) that for all \( x \in X \) and \( A \in \mathcal{F}(X) \),

\[
g(IO(\alpha, A))(x) = \bigvee_{X_i \in \mathcal{F}_\beta(X)} g_{IO(\alpha, A)}^{(i)}(x)
\]

\[
= \bigvee_{X_i \in \mathcal{F}_\beta(X)} I_O(\alpha, g_A^{(i)}(x))
\]

\[
= I_O(\alpha, \bigvee_{X_i \in \mathcal{F}_\beta(X)} g_A^{(i)}(x))
\]

\[
= I_O(\alpha, g_A(x))
\]

\[
= I_O(\alpha, g_A(x)).
\]

Hence, we get that \( g(IO(\alpha, A)) = I_O(\alpha, g_A) \). In a similar way, \( h(IO(\alpha, A)) = I_G^G(\alpha, h_A) \) holds.

(2) It follows from Lemma 3(1) that for all \( x \in X \),

\[
g(O(\alpha, A))(x) = \bigvee_{X_i \in \mathcal{F}_\beta(X)} \bigwedge_{y \in X_i} I_O(R(x, y), O(\alpha, A(y)))
\]
\[ \geq \bigvee_{X_i \in \mathcal{F}(X)} \bigwedge_{y \in X_i} O(\alpha, I_O(R(x, y), A(y))) \]
\[ = O(\alpha, \bigvee_{X_i \in \mathcal{F}(X)} \bigwedge_{y \in X_i} I_O(R(x, y), A(y))) \]
\[ = O(\alpha_X, g_A(x)). \]

Then we conclude that \( O(\alpha_X, g_A) \subseteq g(\alpha_X, A). \) In a similar way, \( G(\alpha_X, h_A) \supseteq h(\alpha_X, A) \) holds.

\[ \square \]

**Proposition 6** Let \( R \) be a fuzzy relation on \( X \). Then the following statements hold.

1. \( A \subseteq B \) implies \( R^\beta_O(A) \subseteq R^\beta_O(B) \) and \( \overline{R}^\beta_G(A) \subseteq \overline{R}^\beta_G(B) \) for all \( A, B \in \mathcal{F}(X) \).
2. If \( \beta > 0.5 \), then for all \( A, B \in \mathcal{F}(X) \),
\[ R^\beta_O(A) \cup R^\beta_O(B) \subseteq R^{(2\beta-1)}_O(A \cup B) \] and \( \overline{R}^{(2\beta-1)}_G(A \cap B) \subseteq \overline{R}^\beta_G(A) \cap \overline{R}^\beta_G(B) \).

**Proof** (1) According to Propositions 1, 2 and Lemma 5(2), it can be directly proven.
(2) According to Proposition 1, it always holds that for all \( A, B \in \mathcal{F}(X) \),
\[ R^\beta_O(A) \cup R^\beta_O(B) = \bigcup \{ [x_{g_A(x)}]^O_R \cup [x_{g_B(x)}]^O_R : x \in X \} \]
there exist \( X_i, X_j \in \mathcal{F}(X) \) such that \( g_A(x) = g_A^i(x) \) and \( g_B(x) = g_B^j(x) \). Hence, we have \( |X_i \cap X_j| \geq (2\beta - 1) \) \(|X|\) \( , \) \( i.e., \) \( X_i \cap X_j \in \mathcal{F}(2\beta-1)(X) \). Then for all \( y \in X_i \cap X_j \),
\[ ([x_{g_A(x)}]^O_R \cup [x_{g_B(x)}]^O_R)(y) = (O(R(x, y), g_A^i(x))) \cup (O(R(x, y), g_B^j(x))) \]
\[ \leq A(y) \cup B(y). \]
Thus, we obtain that \( R^\beta_O(A) \cup R^\beta_O(B) \subseteq R^{(2\beta-1)}_O(A \cup B) \). In a similar way, \( \overline{R}^{(2\beta-1)}_G(A \cap B) \subseteq \overline{R}^\beta_G(A) \cap \overline{R}^\beta_G(B) \) holds.

\[ \square \]

**Proposition 7** Let \( R \) be a fuzzy relation on \( X \), overlap function \( O \) and grouping function \( G \) have 1 and 0 as identity element, respectively. Then the following statements hold.

1. For all \( \alpha \in [0, 1] \) and \( z \in X \), \( O(\alpha, \sqrt{\bigwedge_{x \in X} R(x, z)}) \leq R^\beta_O(\alpha_X)(z) \) and \( \overline{R}^\beta_G(\alpha_X)(z) \leq G(\alpha, \bigwedge_{x \in X} R^\beta_N(x, z)) \).
2. If crisp set \( Y \subseteq X \) and \( \beta = \frac{|Y|}{|X|} \), then the following statements hold for any \( y \in X \),
\[ R^\beta_O(Y)(z) \geq \bigvee_{x \in X} R(x, z) \] and \( \overline{R}^\beta_G(Y^C)(z) \leq \bigwedge_{x \in X} R^\beta_N(x, z) \).
3. If \( A = I_O(y \gamma, \alpha_X) \) and \( \gamma = 1 \), then
\[ R^\beta_O(A)(z) = \begin{cases} \bigvee_{x \in X} R(x, z), & 0 \leq \beta \leq \frac{|X| - 1}{|X|}, \\ \bigwedge_{x \in X} O(R(x, z), I_O(R(x, y), \alpha)), & \frac{|X| - 1}{|X|} < \beta \leq 1. \end{cases} \]
(4) If $A = y_\alpha$, then
\[
\overline{R}_G^\beta(A)(z) = \begin{cases} \\
\bigwedge_{x \in X} R^N(x, z), \\
\bigwedge_{x \in X} (G(R^N(x, z), I^G(R^N(x, y), \alpha))), \\
0 \leq \beta \leq \frac{|X| - 1}{|X|}, \\
|X| - 1 < \beta \leq 1.
\end{cases}
\]

**Proof** (1) According to Lemma 6(2), it can be directly proven.
(2) If $\beta = \frac{|Y|}{|X|}$, then we have $Y \in F_\beta(X)$ and for all $x \in X$,
\[
g_Y(x) \geq \bigwedge_{y \in Y} I_O(R(x, y), Y(y)) = 1.
\]
Hence, for all $z \in X$,
\[
\overline{R}_O^\beta(Y)(z) = \bigvee_{x \in X} O(R(x, z), g_Y(x))
= \bigvee_{x \in X} O(R(x, z), 1)
= \bigvee_{x \in X} R(x, z).
\]
(3) According to Lemma 6(3), it can be directly proven.
(4) According to Lemma 6(4), it can be directly proven.

\[\square\]

**Remark 7** Let $X = \{x_1, x_2, x_3\}$ and the fuzzy relation $R$ on $X$ as
\[
R = \begin{bmatrix}
0 & 0.4 & 0.4 \\
0.2 & 0 & 0.2 \\
0.2 & 0.2 & 0
\end{bmatrix}
\]
Here, we use overlap function $O_{DB}$ and fuzzy implication $I_O$ defined as follows, respectively,
\[
O_{DB}(x, y) = \begin{cases}
\frac{2xy}{x + y}, & \text{if } x + y \neq 0, \\
0, & \text{if } x + y = 0,
\end{cases}
I_O(x, y) = \begin{cases}
\frac{xy}{2x - y}, & \text{if } y < \frac{2x}{x + 1}, \\
1, & \text{if } y \geq \frac{2x}{x + 1}.
\end{cases}
\]
for all $x, y \in [0, 1]$. Let $A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0}{x_3}, \alpha = 1$ and $\beta = 0.5$. Then it follows from Proposition 1 that
\[
\overline{R}_O^\beta(O(\alpha_X, A)) = \frac{1}{x_1} + \frac{4}{x_2} + \frac{4}{x_3},
\]
and
\[
O(\alpha_X, \overline{R}_O^\beta(A)) = \frac{1}{x_1} + \frac{4}{x_2} + \frac{4}{x_3}.
\]
By comparison, we get $O(\alpha_X, \overline{R}_O^\beta(A)) \supseteq \overline{R}_O^\beta(O(\alpha_X, A))$. In this example, the overlap function $O_{DB}$ does not satisfy the associative law. Furthermore, according to the above
conditions, we get that \( R_\beta^O(A) = \frac{1}{x_1} + \frac{2}{x_2} + \frac{1}{x_3} \). In particular, we take \( \alpha = 1 \). It follows from Proposition 1 that

\[
I_O(\alpha_X, R_\beta^O(A)) = \frac{3}{x_1} + \frac{1}{x_2} + \frac{1}{x_3},
\]

and

\[
R_\beta^O(I_O(\alpha_X, A)) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}.
\]

Hence, \( R_\beta^O(I_O(\alpha_X, A)) \supseteq I_O(\alpha_X, R_\beta^O(A)) \).

In particular, some special conclusions of the above example are given when overlap functions or grouping functions satisfy the associative law.

**Proposition 8** Let \( R \) be a fuzzy relation on \( X \), overlap function \( O \) and grouping function \( G \) satisfy (O6) and (G6), respectively. Then the following statements hold.

1. \( R_\beta^O(I_O(\alpha_X, A)) \subseteq I_O(\alpha_X, R_\beta^O(A)) \) and \( I^G(\alpha_X, R_\beta^G(A)) \subseteq R_\beta^G(I^G(\alpha_X, A)) \) for all \( \alpha \in [0, 1] \) and \( A \in \mathcal{F}(X) \). Especially,

   \[
   R_\beta^O(I_O(\alpha_X, \emptyset)) = I_O(\alpha_X, \emptyset) \text{ implies } R_\beta^O(\emptyset) = \emptyset;
   \]

   \[
   R_\beta^O(I^G(\alpha_X, X)) = I^G(\alpha_X, X) \text{ implies } R_\beta^G(X) = X.
   \]

2. \( O(\alpha_X, R_\beta^O(A)) \subseteq R_\beta^O(O(\alpha_X, A)) \) and \( R_\beta^G(G(\alpha_X, A)) \subseteq G(\alpha_X, R_\beta^G(A)) \) for all \( \alpha \in [0, 1] \) and \( A \in \mathcal{F}(X) \).

**Proof** (1) According to Lemma 1(3), 3(1) and 7(1), it holds that for any \( z \in X \),

\[
R_\beta^O(I_O(\alpha_X, A))(z) = \bigvee_{x \in X} O(R(x, z), g(I_O(\alpha_X, A))(x))
= \bigvee_{x \in X} O(R(x, z), I_O(\alpha, g_A(x)))
\leq \bigvee_{x \in X} I_O(\alpha, O(R(x, z), g_A(x)))
= I_O\left(\alpha, \bigvee_{x \in X} O(R(x, z), g_A(x))\right)
= I_O(\alpha, R_\beta^O(A)(z)).
\]

Hence, we get that \( R_\beta^O(I_O(\alpha_X, A)) \subseteq I_O(\alpha_X, R_\beta^O(A)) \). In a similar way, \( I^G(\alpha_X, R_\beta^G(A)) \subseteq R_\beta^G(I^G(\alpha_X, A)) \) holds.

(2) It follows from Lemma 7(2) and the associativity of overlap function that

\[
R_\beta^O(O(\alpha_X, A))(z) = \bigvee_{x \in X} O(R(x, z), g(O(\alpha_X, g_A(x))))
\geq \bigvee_{x \in X} O(R(x, z), O(\alpha, g_A(x))
= \bigvee_{x \in X} O(\alpha, O(R(x, z), g_A(x)))
\]
\[
= O \left( \alpha, \bigvee_{x \in X} O(R(x, z), g_A(x)) \right) \\
= O(\alpha_X, R^O_G(A))(z).
\]

Hence, we get that \( R^O_G(O(\alpha_X, A)) \supseteq O(\alpha_X, R^O_G(A)) \). In a similar way, \( R^G_G(G(\alpha_X, A)) \subseteq G(\alpha_X, R^G_G(A)) \) holds for all \( \alpha \in [0, 1] \) and \( A \in \mathcal{F}(X) \).

\[\square\]

### 4.2 Some new conclusions based on special fuzzy relations

**Proposition 9** Let \( R \) be a fuzzy relation on \( X \), overlap function \( O \) and grouping function \( G \) have 1 and 0 as identity element, respectively. Then the following statements are equivalent:

1. \( R^{-1} \) is serial.
2. \( X = R^O_G(X) \).
3. \( \emptyset = R^G_G(\emptyset) \).
4. \( \alpha_X \subseteq R^O_G(\alpha_X) \) for all \( \alpha \in [0, 1] \).
5. \( \alpha_X \subseteq R^G_G(\alpha_X) \) for all \( \alpha \in [0, 1] \).
6. A crisp set \( Y \subseteq X \) and \( \beta = \frac{|Y|}{|X|} \) imply \( R^O_G(Y) = X \).
7. A crisp set \( Y \subseteq X \) and \( \beta = \frac{|Y|}{|X|} \) imply \( R^G_G(Y^c) = \emptyset \).
8. \( \beta \leq \frac{|X| - 1}{|X|} \) and \( \gamma = 1 \) imply \( R^O_G(I_O(y, \alpha_X)) = X \) for all \( y \in X \) and \( \alpha \in [0, 1] \).
9. \( \beta \leq \frac{|X| - 1}{|X|} \) implies \( R^G_G(y) = \emptyset \) for all \( y \in X \) and \( \alpha \in [0, 1] \).

**Proof** By Propositions 7(2), (3) and (4), it holds that

\[(1) \iff (6) \iff (7) \iff (8) \iff (9).\]

Furthermore, we can obtain that \((1) \Rightarrow (4) \Rightarrow (2)\) by Proposition 7(3). Next, the proof of \((2) \Rightarrow (1)\) will be verified.

Suppose \( R^{-1} \) is not serial, then there exists a \( z_0 \in X \) such that \( \bigvee_{x \in X} R(x, z_0) < 1 \). It follows from Lemma 6(1) that

\[
R^O_G(X)(z_0) = \bigvee_{x \in X} O(R(x, z_0), g_X(x)) \\
= \bigvee_{x \in X} O(R(x, z_0), 1) \\
= O \left( \bigvee_{x \in X} R(x, z_0), 1 \right) \\
< O(1, 1) = 1,
\]

which leads to a contradiction. Therefore \( R^{-1} \) is serial. Further, we have \((1) \Rightarrow (5) \Rightarrow (3) \Rightarrow (1)\) in a similar way.

By Proposition 9 that we can obtain the equivalent expressions of the seriality property for the relation \( R \) by replacing \( R \) (resp. \( R^{-1} \)) with \( R^{-1} \) (resp. \( R \)).

**Proposition 10** Let \( R \) be reflexive. Then the following statements hold.
(1) \( g_A \subseteq R_\beta^\alpha(A) \) and \( \overline{R}_G^\beta(A) \subseteq h_A \) for all \( A \in \mathcal{F}(X) \).
(2) \( R_\beta^\alpha(X) = X \) and \( \overline{R}_G^\beta(\emptyset) = \emptyset \).
(3) \( \alpha_X \subseteq R_\beta^\alpha(\alpha_X) \) and \( \overline{R}_G^\beta(\alpha_X) \subseteq \alpha_X \) for all \( \alpha \in [0, 1] \).
(4) A crisp set \( Y \subseteq X \) and \( \beta = \frac{|Y|}{|X|} \) imply \( R_\beta^\alpha(Y) = X \) and \( \overline{R}_G^\beta(Y^c) = \emptyset \).
(5) \( \beta \leq \frac{|X|}{|Y|} - 1 \) and \( \gamma = 1 \) imply \( R_\beta^\alpha(I_O(y, \alpha_X)) = X \) and \( \overline{R}_G^\beta(y_\alpha) = \emptyset \) for all \( y \in X \) and \( \alpha \in [0, 1] \).

**Proof** It follows immediately from Propositions 1, 2 and 9. \( \square \)

**Proposition 11** Let \( R \) be symmetric. Then the following statements hold for all \( A \in \mathcal{F}(X) \),

\[
R_\beta^\alpha(A) = (R^{-1})_\beta^\alpha(A) \quad \text{and} \quad \overline{R}_G^\beta(A) = (R^{-1})_G^\beta(A).
\]

**Proof** The proof is trivial. \( \square \)

**Remark 8** Consider \( X = \{x_1, x_2, x_3\} \) and crisp relation \( R \) on \( X \) as

\[
R = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}.
\]

It is easy to conclude that crisp relation \( R \) is a fuzzy \( O \)-similarity relation for arbitrary overlap function \( O \).

Here, we use the overlap function \( O_{DB} \) and its residual implication \( I_O \) defined as Remark 7.

Let \( A = \frac{0.2}{x_1} + \frac{0}{x_2} + \frac{0.5}{x_3} \), \( \alpha = \bigwedge_{x \in X} R(x, x) \) and \( \beta = 0.5 \). Then it follows from Proposition 1 that

\[
g_A = \frac{1}{3} + \frac{1}{2} + \frac{1}{3}.
\]

Hence, the \( O \)-granular variable precision lower approximation operator is

\[
R_\beta^\alpha(A) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}.
\]

Furthermore, we obtain the following conclusions,

\[
O(\alpha_X, R_\beta^\alpha(A)) = \frac{2}{3} x_1 + \frac{1}{x_2} + \frac{2}{3} x_3,
\]

and

\[
R_\beta^\alpha(R_\beta^\alpha(A)) = \frac{1}{3} x_1 + \frac{1}{x_2} + \frac{1}{3} x_3.
\]

The above example shows that \( g_A \subseteq R_\beta^\alpha(A) \) and \( O(\alpha_X, R_\beta^\alpha(A)) \supseteq R_\beta^\alpha(R_\beta^\alpha(A)) \). However, the opposite conclusion can be obtained when \( O \) satisfies (O6) as follows.

**Proposition 12** Let \( R \) be \( O \)-transitive, overlap function \( O \) satisfy (O6) and \( \alpha = \bigwedge_{x \in X} R(x, x) \).

Then the following statements hold for arbitrary \( A \in \mathcal{F}(X) \),

(1) \( R_\beta^\alpha(A) \subseteq g_A \) and \( O(\alpha_X, R_\beta^\alpha(A)) \subseteq R_\beta^\alpha(R_\beta^\alpha(A)) \).
(2) \( h_A \subseteq \overline{R}_G^\beta(A) \) and \( \overline{R}_G^\beta(\overline{R}_G^\beta(A)) \subseteq \overline{G((\alpha_X)^N, \overline{R}_G^\beta(A))} \), if \( O \) and \( G \) are dual w.r.t. \( N \).
\textbf{Proof} (1) For any $z \in X$, it follows from Proposition 1 that there are $x \in X$ and $X_i \in \mathcal{F}_\beta(X)$ such that

$$R^\beta_O(A)(z) = O(R(x, z), g_A^{(i)}(x)).$$

Furthermore, in terms of Lemma 1(1) and (O6), for all $y \in X_i$, it holds that

$$O(R(z, y), R^\beta_O(A)(z)) = O(R(z, y), O(R(x, z), g_A^{(i)}(x)))$$

$$= O(O(R(z, y), R(x, z)), g_A^{(i)}(x))$$

$$\leq O(R(x, y), g_A^{(i)}(x))$$

$$\leq O(R(x, y), I_O(R(x, y), A(y)))$$

$$\leq A(y).$$

Hence, we obtain that $R^\beta_O(A)(z) \leq g_A^{(i)}(z) \leq g_A(z)$, that is, $R^\beta_O(A) \subseteq g_A$.

Let $B = R^\beta_O(A)$. By Proposition 1, it follows that for all $y \in X$ and $X_i \in \mathcal{F}_\beta(X)$,

$$B(y) = R^\beta_O(A)(y)$$

$$= \bigvee_{x \in X} O(R(x, y), g_A(x))$$

$$\geq O(R(y, y), g_A(y))$$

$$= O\left(R(y, y), \bigvee_{x \in X} g_A^{(i)}(y)\right)$$

$$\geq O(R(y, y), g_A^{(i)}(y)).$$

According to Lemma 3(1) and (2), the following statement holds for all $x \in X$ and $X_i \in \mathcal{F}_\beta(X)$,

$$g_B^{(i)}(x) = \bigwedge_{y \in X_i} I_O(R(x, y), B(y))$$

$$\geq \bigwedge_{y \in X_i} I_O(R(x, y), O(R(y, y), g_A^{(i)}(y)))$$

$$\geq \bigwedge_{y \in X_i} O(R(y, y), I_O(R(x, y), g_A^{(i)}(y)))$$

$$= \bigwedge_{y \in X_i} O\left(R(y, y), \bigwedge_{z \in X_i} I_O(R(x, y), I_O(R(y, z), A(z))\right)$$

$$= \bigwedge_{y \in X_i} O\left(R(y, y), \bigwedge_{z \in X_i} I_O(O(R(x, y), R(y, z)), A(z))\right)$$

$$\geq \bigwedge_{y \in X_i} O\left(R(y, y), \bigwedge_{z \in X_i} I_O(O(R(x, z), A(z))\right)$$

$$= \bigwedge_{y \in X_i} O(R(y, y), g_A^{(i)}(x))$$
According to item (1) and Proposition 3, it can be directly proven.

(2) According to item (1) and Proposition 3, it can be directly proven.

Considering fuzzy O-preorders, we obtain the following conclusions.

**Proposition 13** Let R be a fuzzy O-preorder, overlap function O and grouping function G satisfy (O6) and (G6), respectively. Then the following statements hold for any \( A \in \mathcal{F}(X) \).

1. \( R^\alpha_O(A) = g_A \) and \( R^\alpha_O(A) \subseteq \bigcup_i \bigcap_{y \in X_i} R(x, y) \).
2. \( R^\alpha_G(A) = h_A \) and \( R^\alpha_G(A) \subseteq R^\alpha_O(A) \), if O and G are dual w.r.t. N.

**Proof** It follows immediately from Propositions 3, 10 and 12.

**Proposition 14** Let overlap function O and grouping function G be dual w.r.t. N, which satisfy (O6) and (G6), respectively, and R be a fuzzy O-preorder. Then the following statements hold.

1. \( R^\beta_O(I_O(\alpha X, A)) = I_O(\alpha X, R^\beta_O(A)) \) and \( R^\beta_G(\lambda G(\alpha X, A)) = \lambda G(\alpha X, R^\beta_G(A)) \) for all \( \alpha \in [0, 1] \) and \( A \in \mathcal{F}(X) \).
2. \( R^\beta_O(\emptyset) = \emptyset \) if and only if \( R^\beta_O(I_O(\alpha X, \emptyset)) = I_O(\alpha X, \emptyset) \) for all \( \alpha \in [0, 1] \).
3. \( R^\beta_G(X) = X \) if and only if \( R^\beta_G(I_G(\alpha X, X)) = I_G(\alpha X, X) \) for all \( \alpha \in [0, 1] \).
4. If \( \beta > 0.5 \), then for all \( A, B \in \mathcal{F}(X) \),
   
   \[
   R^\beta_O(A) \cap R^\beta_O(B) \subseteq R^\beta_O(A \cap B), \quad \overline{R^\beta_G(A \cap B)} \subseteq \overline{R^\beta_G(A)} \cap \overline{R^\beta_G(B)}; \\
   R^\beta_O(A) \cup R^\beta_O(B) \subseteq R^\beta_O(A \cup B), \quad \overline{R^\beta_G(A \cup B)} \subseteq \overline{R^\beta_G(A)} \cup \overline{R^\beta_G(B)}. 
   \]

**Proof** (1) Let \( x \in X \) and \( \lambda = I_O(\alpha, R^\beta_O(A)(x)) \). Then we have \( O(\alpha, \lambda) \leq R^\beta_O(A)(x) \). It follows from Propositions 1 and 13(1) that there is an \( X_\lambda \in \mathcal{F}(X) \) such that

\[
O(\alpha, \lambda) \leq R^\beta_O(A)(x) = g_A(x) = g_A^{(i)}(x) = \bigwedge_{y \in X_\lambda} I_O(R(x, y), A(y)).
\]

So we have the following equivalences for all \( y \in X_\lambda \),

\[
O(\alpha, \lambda) \leq I_O(R(x, y), A(y)) \iff O(\alpha, \lambda) \leq A(y) \iff O(\alpha, O(\lambda, R(x, y))) \leq A(y) \iff [x_\lambda]_R^{(i)}(y) \leq I_O(\alpha, A(y)).
\]

According to Definition 7, we obtain that \([x_\lambda]_R^{(i)} \subseteq R^\beta_O(I_O(\alpha X, A)) \). Since R is a fuzzy O-preorder, we get that \( \lambda = [x_\lambda]_R^{(i)} \subseteq R^\beta_O(I_O(\alpha X, A)) \).
If \( O \) and \( G \) are dual w.r.t. \( N \), and \( R \) \( _N \), it can be directly proven.

(2) According to item (1) and Proposition 8(1), it can be directly proven.

(3) According to item (1) and Proposition 8(1), it can be directly proven.

(4) Let \( x \in X \), by Proposition 13(1), we get that \( R_\beta (A)(x) = g_A(x) \) and \( R_\beta (B)(x) = g_B(x) \), then

\[
X_i = \{ y : [x g_A(x)]_R(y) \subseteq A(y) \} \quad \text{and} \quad X_j = \{ y : [x g_B(x)]_R(y) \subseteq B(y) \}.
\]

Hence, \( X_i, X_j \in \mathcal{F}_\beta (X) \) by Proposition 1, we have that \( X_i \cap X_j \in \mathcal{F}(2\beta - 1)(X) \). The following statement holds for all \( y \in X_i \cap X_j \),

\[
[x (g_A(x) \cap g_B(x))]_R(y) = O(R(x, y), g_A(x)) \cap O(R(x, y), g_B(x)) \subseteq A(y) \land B(y).
\]

Thus, we get that \([x (g_A(x) \cap g_B(x))]_R(y) \subseteq R_\beta (A \cap B) (A \cap B) \). Since \( R \) is reflexive and \( O \) has 1 as identity, one concludes that

\[
R_\beta (A)(x) \land R_\beta (B)(x) = g_A(x) \land g_B(x)
\]

\[
= [x (g_A(x) \land g_B(x))]_R(x)
\]

\[
\leq R_\beta (A \cap B)(x).
\]

Therefore, we obtain that \( R_\beta (A) \cap R_\beta (B) \subseteq R_\beta (A \cap B) \). Similarly, we have that \( R_\beta (B) (A) \cup B \subseteq R_\beta (A) \cup B \). The others can be proven by Proposition 6(2).

The characteristics of \( R_\beta (R_\beta (A)) \) and \( R_\beta (R_\beta (A)) \) are further explored when considering some special fuzzy relations.

**Proposition 15** Let \( \alpha = \bigwedge_{x \in X} R(x, x) \) and overlap function \( O \) satisfy (O6). Then for any \( A \in \mathcal{F}(X) \), the following statements hold.

(1) If \( R(x, y) \leq I_O(A, A)(y) \) for all \( x, y \in X \), then

\[
R_\beta (O(\alpha X, A)) \subseteq R_\beta (O(\alpha X, A)).
\]

(2) If \( O \) and \( G \) are dual w.r.t. \( N \), and \( R^N(x, y) \geq I^G(A(x), A(y)) \) for all \( x, y \in X \), then

\[
R_\beta (R_\beta (A)) \subseteq R_\beta (R_\beta (A)).
\]

**Proof** (1) Let \( X_i \in \mathcal{F}_\beta (X) \), \( B = O(\alpha \beta) \) and \( C = R_\beta (A) \). Then it follows from Lemmas 1(2) and 3(2) that for all \( x \in X \),

\[
I_O(g_B^{(i)}(x), g_C^{(i)}(x)) = I_O(g_B^{(i)}(x), \bigwedge_{y \in X_i} I_O(R(x, y), C(y)))
\]

\[
= \bigwedge_{y \in X_i} I_O(g_B^{(i)}(x), I_O(R(x, y), C(y)))
\]

\[
\geq \bigwedge_{y \in X_i} I_O(I_O(R(x, y), O(\alpha, A(y))), I_O(R(x, y), C(y)))
\]

\[
\geq \bigwedge_{y \in X_i} I_O(O(\alpha, A(y)), C(y))
\]
If $S_R(y, y)$ hold.

Lemma 8 Let $R$, $S$ be fuzzy $O$-preorders, $S \subseteq R$, when overlap function $O$ and grouping function $G$ satisfy $(O6)$ and $(G6)$, respectively. Then the following statements hold for all $A \in \mathcal{P}(X)$,

$R_O^\beta(A) \subseteq S_O^\beta(A)$ and $S_G^\beta(A) \subseteq R_G^\beta(A)$, if $O$ and $G$ are dual w.r.t. $N$.

Proof According to Lemma 1(4) and Proposition 13, it can be directly proven.

At the end of this section, we give the sufficient and necessary conditions when $(O, G)$-GVPFRSs on two different fuzzy relations are equal to each other.

Proposition 16 Let $R$ be fuzzy $O$-transitive, overlap function $O$ and grouping function $G$ satisfy $(O6)$ and $(G6)$, respectively. If $O$ and $G$ are dual w.r.t. $N$, then the following statements hold.

1. If $S_O^\beta(A)(x) = R_O^\beta(A)(x)$, then \( \{ y : [x S_O^\beta(A)(x)]_R^O(y) \leq A(y) \} \in \mathcal{P}_N(X) \).

2. If $S_G^\beta(A)(x) = R_G^\beta(A)(x)$, then \( \{ y : A(y) \leq [x S_G^\beta(A)(x)]_R^G(y) \} \in \mathcal{P}_N(X) \).

Proof (1) Since $R$ is $O$-transitive, it follows from Proposition 12(1) that $R_O^\beta(A)(x) \leq g_A(x)$ for all $x \in X$. By $S_O^\beta(A)(x) = R_O^\beta(A)(x)$, we obtain that

\[
\{ y : [x S_O^\beta(A)(x)]_R^O(y) \leq A(y) \} = \{ y : [x R_O^\beta(A)(x)]_R^O(y) \leq A(y) \} \subseteq \{ y : [x g_A(x)]_R^O(y) \leq A(y) \}.
\]

Hence, we have that \( \{ y : [x S_G^\beta(A)(x)]_R^G(y) \leq A(y) \} \in \mathcal{P}_N(X) \) by Proposition 1.

(2) The proof is similar to item (1).
If \( \{ y : [x_{\Sigma_{O}(A)(x)}]^G_R(y) \leq A(y) \} \in \mathcal{F}_\beta(X) \) for all \( x \in X \), then \( S^\beta_O(A) = R^\beta_O(A) \).

(2) \( If \{ y : A(y) \leq [x_{\Sigma_{G}(A)(x)}]^G_R(y) \} \in \mathcal{F}_\beta(X) \) for all \( x \in X \), then \( S^\beta_G(A) = \overline{R^\beta_G}(A) \).

**Proof** (1) Let \( X_i = \{ y : [x_{\Sigma_{O}(A)(x)}]^G_R(y) \leq A(y) \} \). For any \( x \in X \), it holds that
\[
S^\beta_O(A)(x) \subseteq \bigwedge_{y \in X_i} I_O(R(x, y), A(y)) = g^O_A(x) \leq g_A(x) = R^\beta_O(A)(x).
\]
Hence, we obtain that \( S^\beta_O(A) = R^\beta_O(A) \) by Lemma 8.

(2) The proof is similar to item (1).

Combining Propositions 16 and 17, we obtain the sufficient and necessary condition for the identity of \((O, G)\)-GVPFRSs on two different fuzzy relations as follows.

**Proposition 18** Let \( S, R \) be fuzzy \( O \)-preorders, \( S \subseteq R \), overlap function \( O \) and grouping function \( G \) satisfy (O6) and (G6). If \( O \) and \( G \) are dual w.r.t. \( N \), then the following statements hold for all \( A \in X \).
\[
S^\beta_O(A) = R^\beta_O(A) \iff \{ y : [x_{\Sigma_{O}(A)(x)}]^O_R(y) \leq A(y) \} \in \mathcal{F}_\beta(X),
\]
\[
S^\beta_G(A) = \overline{R^\beta_G}(A) \iff \{ y : A(y) \leq [x_{\Sigma_{G}(A)(x)}]^G_R(y) \} \in \mathcal{F}_\beta(X).
\]

Moreover, when the fuzzy sets degenerate into crisp sets, we have the following conclusions.

**Proposition 19** Let \( S, R \) be fuzzy \( O \)-preorders, \( S \subseteq R \), overlap function \( O \) and grouping function \( G \) satisfy (O6) and (G6). If \( O \) and \( G \) are dual w.r.t. \( N \), then the following statements hold for all crisp set \( A \subseteq X \).
\[
S^\beta_O(A) = R^\beta_O(A) \iff | \{ y : y \notin A, [x_{\Sigma_{O}(A)(x)}]^O_R(y) = 0 \} | \geq \beta | X | - | A |, 
\]
\[
S^\beta_G(A) = \overline{R^\beta_G}(A) \iff | \{ y : y \in A, [x_{\Sigma_{G}(A)(x)}]^G_R(y) = 1 \} | \geq | A | + (\beta - 1) | X | .
\]

**Proof** Since \( A \) is a crisp set, the following statement holds for all \( x \in X \),
\[
\{ y : [x_{\Sigma_{O}(A)(x)}]^O_R(y) \leq A(y) \} = A \bigcup \{ y : y \notin A, [x_{\Sigma_{O}(A)(x)}]^O_R(y) = 0 \}.
\]
Hence, it follows from Proposition 18 that
\[
S^\beta_O(A) = R^\beta_O(A) \iff | \{ y : y \notin A, [x_{\Sigma_{O}(A)(x)}]^O_R(y) = 0 \} | \geq \beta | X | - | A | .
\]
The proof of another formula is similar.

\( \square \)

### 5 Conclusions

In this paper, a new type of fuzzy rough set model on arbitrary fuzzy relations has been defined using overlap and grouping functions, which called \((O, G)\)-GVPFRSs. Meanwhile, we give two equivalent expressions of the upper and lower approximation operators with the help of fuzzy implications and co-implications, which facilitate more efficient calculations. In particular, some special conclusions are further discussed, when fuzzy relations and sets
degenerate into crisp relations and sets. In addition, we characterize the \((O, G)\)-GVPFRSs based on different fuzzy relations. Finally, the richer conclusions about \((O, G)\)-GVPFRSs are given. In summary, this paper further explores GVPFRSs from a theoretical perspective based on overlap and grouping functions. In the future, \((O, G)\)-GVPFRSs can be further considered based on the idea of three-way decisions (Hu 2014).

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