HOMOLOGY OF THE KATSURA-EXEL-PARDO GROUPOID

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Abstract. We compute the homology of the groupoid associated to the Katsura algebras, and show that they capture the $K$-theory of the $C^*$-algebras, and hence satisfying the (HK) conjecture posted by Matui. Moreover, we show that several classifiable simple $C^*$-algebras are groupoid $C^*$-algebras of this class.

Introduction

In [8] Katsura defined a nice class of $C^*$-algebras that exhausts all the Kirchberg algebras in the UCT class. The construction of these $C^*$-algebras has two layers: the first is the graph skeleton, that gives to the $C^*$-algebra most of the desired structural properties, and the second layer that consists of partial unitaries associated to every vertex, that provide the necessary richness in $K$-theory. These two layers are given by two equal size square matrices $A$ and $B$.

Later in [5] Exel and Pardo, while studying the $C^*$-algebras associated to self-similar graphs, realized that the $C^*$-algebras constructed by Katsura were prominent examples. The advantage of Exel and Pardo approach is that they described these algebras as groupoids $C^*$-algebras of combinatorial origin, and managed to give beautiful characterizations of the most fundamental properties of groupoids. In particular, for the Katsura algebras they construct an amenable groupoid $G_{A,B}$ such that $C^*(G_{A,B})$ is the desired $C^*$-algebra and give conditions, in most of the cases equivalent conditions, in terms of the matrices $A$ and $B$ for Hausdorffness, effectiveness and minimality of the groupoid. Because Katsura found the $C^*$-algebra, but Exel and Pardo gave the description as a grupoid $C^*$-algebras, is the reason why we will call $G_{A,B}$ the Katsura-Exel-Pardo groupoid.

As mentioned above, Katsura computed the $K$-theory of $C^*(G_{A,B})$ in terms of the matrices $A$, $B$, that is

$$K_0(C^*(G_{A,B})) \cong \ker(I-A) \oplus \ker(I-B) \quad \text{and} \quad K_1(C^*(G_{A,B})) \cong \ker(I-B) \oplus \ker(I-A),$$

and showed that given any two countably generated abelian groups $G_0$ and $G_1$ there exist matrices $A$ and $B$ such that $K_0(C^*(G_{A,B})) \cong G_0$ and $K_1(C^*(G_{A,B})) \cong G_1$.

In [11, 12] Matui started an exhaustive study of étale groupoids with totally disconnected unit space, and showed how their homology reflects dynamical properties of their topological full groups. He later conjectured in [13] that the homology groups of a minimal effective, étale groupoid totally captures the $K$-theory of their associated reduced groupoid $C^*$-algebra, and called it the (HK) conjecture. He verified that the (HK) conjecture is true for important classes of groupoids, like the transformation groupoids of Cantor minimal systems, Cuntz-Krieger groupoids and products of Cuntz-Krieger groupoids.

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In the present paper, we verify the conjecture for the class of Katsura-Exel-Pardo groupoids, that is, we compute all the homology groups of the groupoid $G_{A,B}$, and show that they sum up the $K$-theory of the $C^*$-algebra. Furthermore, we see that homology groups provide a refinement of the $K$-theory allowing us to define invariants for the Kakutani equivalence class of the groupoid $G_{A,B}$ that could not be found just looking at the $K$-theory of the associated $C^*$-algebras.

It was proved by Matsumoto and Matui [10, Corollary 3.8] that given two irreducible matrices $A$ and $A'$, the Cuntz-Krieger groupoids $G_{A,0}$ and $G_{A',0}$ are equivalent if and only if $\text{coker}(I - A) \cong \text{coker}(I - A')$ and $\det(I - A) = \det(I - A')$, it then looks natural to go for a classification result for the Katsura-Exel-Pardo groupoids. The main result of this paper (Theorem 2.5) states that the kernel and the cokernel of the matrices $I - A$ and $I - B$ are an invariant of the Kakutani equivalence class of the groupoid $G_{A,B}$. In the Cuntz-Krieger case, the part of the invariant involving the determinant is contained in the first cohomology group of the groupoid, that is isomorphic to the Boyle-Handelman group [10, Proposition 3.4], while the cohomology of the Katsura-Exel-Pardo groupoid is much bigger and containing parts of the Boyle-Handelman group. So further study of this group is needed. It is then the aim of this paper to set the first step in a future classification of the groupoids $G_{A,B}$ analyzing the combinatorial structure that they posses.

Recently it has been a big interest in finding which classifiable $C^*$-algebras can be realized as étale groupoid $C^*$-algebras (see [7, 15]). In order to do that one wants to construct groupoids for whose associated $C^*$-algebras exhaust the possible Elliott invariants. Here is where étale groupoids satisfying the (HK) conjecture gain importance, since in general $K$-theory is an important part of this invariant. Therefore, it is here where Katsura-Exel-Pardo groupoids plays an important role too, because of the richness and variety of their homology groups.

The paper is organized as follows. In section 1 we give the preliminaries on étale groupoids and their homology. Here is where we state Lemma 1.4 that is the analog of the Pimsner-Voiculescu 6-terms exact sequence of $K$-theory, but for the homology of étale groupoids with a $\mathbb{Z}$-cocyle. This Lemma will be the crucial technical tool for the computation of the homology of the Katsura-Exel-Pardo groupoid. In section 2 we introduce the Katsura-Exel-Pardo groupoid, that is, a groupoid associated to a self-similar graph introduced by Exel and Pardo in [5] that realizes the $C^*$-algebra defined by Katsura [8]. After a quick overview of the basics properties of this groupoid found in [5, Section 18], we move to the computation of the homology. This is done in two steps: the first computes the homology of the kernel groupoid of the natural $\mathbb{Z}$-cocyle of $G_{A,B}$, denoted by $H_{A,B}$. The second step is to use the long exact sequence found in Lemma 1.4 to compute the homology of $G_{A,B}$. This long exact sequence contains the homology groups of $H_{A,B}$ with maps induced by the dual action of the $\mathbb{Z}$-cocyle. Thanks to the nice description of these maps given in Proposition 2.4 and the nature of the homology groups of $H_{A,B}$ the homology groups of $G_{A,B}$ fits in short exact sequences and hence be computed. Finally in section 3 we use Theorem 2.5 to construct a variety of étale groupoids whose associated $C^*$-algebra fall in a classifiable class and with a prescribed $K$-theoretical invariant.
1. Basics on groupoid homology.

In this section we will recall the basic definitions and results on groupoid homology that one can find in [11], and we will state the conjecture of study in this paper.

A groupoid is a small category of isomorphisms, that is, a set $G$ (the morphisms, or arrows in the category) equipped with a partially defined multiplication $g_1g_2 \mapsto g_1 \cdot g_2$ for a distinguished subset $G^{(2)} \subseteq G \times G$, and everywhere defined involution $g \mapsto g^{-1}$ satisfying the following axioms:

(1) If $g_1g_2$ and $(g_1g_2)g_3$ are defined, then $g_2g_3$ and $g_1(g_2g_3)$ are defined and $(g_1g_2)g_3 = g_1(g_2g_3)$.

(2) The products $gg^{-1}$ and $g^{-1}g$ are always defined. If $g_1g_2$ is defined, then $g_1 = g_1g_2g_2^{-1}$ and $g_2 = g_1^{-1}g_1g_2$.

A topological groupoid is a groupoid together with a topology on it such that the operations of multiplication and taking inverse are continuous.

The elements of the form $gg^{-1}$ are called units. We denote the set of units of a groupoid $G$ by $G^{(0)}$, and refer to this as the unit space. We always think of the unit space as a topological space equipped with the relative topology from $G$. The source and range maps are

$$s(g) := g^{-1}g \quad \text{and} \quad r(g) := gg^{-1}$$

for $g \in G$.

An étale groupoid is a topological groupoid where the range map (and necessarily the source map) is a local homeomorphism (as a map from $G$ to $G$). The unit space $G^{(0)}$ of an etale groupoid is always an open subset of $G$.

Definition 1.1. Let $G$ be an étale groupoid. A bisection is an open subset $U \subseteq G$ such that $s$ and $r$ are both injective when restricted to $U$.

Two units $x,y \in G^{(0)}$ belong to the same $G$-orbit if there exists $g \in G$ such that $s(g) = x$ and $r(g) = y$. We denote by $\text{orb}_G(x)$ the $G$-orbit of $x$. When every $G$-orbit is dense in $G^{(0)}$, $G$ is called minimal. An open set $A$ is called $G$-full if for every $x \in G^{(0)}$, then $\text{orb}_G(x) \cap A \neq \emptyset$.

For an open subset $A \subseteq G^{(0)}$ we denote by $G_A$ the subgroupoid $\{ g \in G \mid s(g), r(g) \in A \}$, called the restriction of $G$ to $A$. When $G$ is étale, the restriction $G_A$ is an open étale subgroupoid.

The isotropy group of a unit $x \in G^{(0)}$ is the group $G^x := \{ g \in G \mid s(g) = r(g) = x \}$, and the isotropy bundle is

$$G' := \{ g \in G \mid s(g) = r(g) \} = \bigcup_{x \in G^{(0)}} G^x.$$

A groupoid $G$ is said to be principal if all isotropy groups are trivial, or equivalently, $G' = G^{(0)}$. We say that $G$ is effective if the interior of $G'$ equals $G^{(0)}$.

Definition 1.2. We say that a groupoid is elementary if it is compact and principal. A groupoid $G$ is an AF groupoid if there exists an ascending chain of open elementary subgroupoids $K_1, K_2, \ldots$ such that $G = \bigcup_{i=1}^\infty K_i$. 

Let \( \pi : X \to Y \) be a local homeomorphism between to locally compact Hausdorff space, then given any \( f \in C_c(X, \mathbb{Z}) \) we define
\[
\pi_*(f)(y) := \sum_{\pi(x) = y} f(x).
\]
It is not hard to show that \( \pi_*(f) \in C_c(Y, \mathbb{Z}) \).

Given an étale groupoid \( G \) and \( n \in \mathbb{N} \) we write \( G^{(n)} \) for the space of composable strings of \( n \) elements in \( G \) with the product topology. For \( i = 0, \ldots, n \), we let \( d_i : G^{(n)} \to G^{(n-1)} \) be a map defined by
\[
d_i(g_1, g_2, \ldots, g_n) = \begin{cases} (g_2, g_3, \ldots, g_n) & \text{if } i = 0, \\ (g_1, \ldots, g_{i+1}, \ldots, g_n) & \text{if } 1 \leq i \leq n-1, \\ (g_1, g_2, \ldots, g_{n-1}) & \text{if } i = n.
\end{cases}
\]

Then we define the homomorphism \( \delta_n : C_*(G^{(n)}, \mathbb{Z}) \to C_*(G^{(n-1)}, \mathbb{Z}) \) given by
\[
\delta_1 = s_* - r_* \quad \text{and} \quad \delta_n = \sum_{i=0}^{n} (-1)^i d_*.\]

Then we define the homology \( H_*(G) \) as the homology groups of the chain complex \( C_*(G, \mathbb{Z}) \) by
\[
0 \leftarrow C_*(G^{(0)}, \mathbb{Z}) \leftarrow \delta_1 C_*(G^{(1)}, \mathbb{Z}) \leftarrow \delta_2 C_*(G^{(2)}, \mathbb{Z}) \leftarrow \cdots.
\]

The following conjecture, posted in [13], states that the homology of the groupoid refines the \( K \)-theory of the reduced groupoid \( C^* \)-algebra.

\textbf{(HK) conjecture:} Let \( G \) be a minimal, effective, étale groupoid with \( G^{(0)} \) homeomorphic to the Cantor space, then
\[
K_i(C^*_r(G)) \cong \bigoplus_{n=0}^{\infty} H_{2n+1}(G), \quad \text{for } i = 0, 1.
\]

The conjecture was confirmed for the \( AF \)-groupoids, transformation groupoids of Cantor minimal systems, groupoids of shifts of finite type and products of groupoids of shifts of finite type (see [11, 13]).

Now we are going to collect some results from [11] that will allows us to compute the homology of a groupoid.

Let \( \Gamma \) be a countable discrete group and \( G \) an étale groupoid. When \( \rho : G \to \Gamma \) is a groupoid homomorphism, the \textit{skew product} \( G \times^\rho \Gamma \) is \( G \times \Gamma \) with the following groupoid structure: \((g, \gamma)\) and \((g', \gamma')\) are composable if and only if \( g \) and \( g' \) are composable and \( \gamma \rho(g) = \gamma' \), and
\[
(g, \gamma) \cdot (g', \gamma \rho(g)) = (gg', \gamma) \quad \text{and} \quad (g, \gamma)^{-1} = (g^{-1}, \gamma \rho(g)).
\]
We can define the action \( \hat{\rho} : \Gamma \curvearrowright G \times^\rho \Gamma \) by \( \hat{\rho}^\gamma(g', \gamma') = (g', \gamma' \rho) \).

\textbf{Lemma 1.3} (cf. [11, Lemma 4.13]). Let \( G \) be an étale groupoid, let \( \rho : G \to \mathbb{Z} \) be a groupoid homomorphism, and let \( Y = G^{(0)} \times \{0\} \). Then \( Y \) is a \((G \times^\rho \mathbb{Z})\)-full open subspace of \((G \times^\rho \mathbb{Z})^{(0)}\) and \((G \times^\rho \mathbb{Z})_Y \cong \ker \rho \). In particular \( G \times^\rho \mathbb{Z} \) is Morita equivalent to \( \ker \rho \).
In order to compute the homology of groupoids $\mathcal{G}$ that have a groupoid homomorphism $\rho: \mathcal{G} \to \mathbb{Z}$, Matui uses the spectral sequence

$$E^2_{pq} = H_p(\mathbb{Z}, H_q(\mathcal{G} \times_\rho \mathbb{Z})) \Rightarrow H_{p+q}(\mathcal{G}).$$

However, we are going to use a long exact sequence that relates the homology groups of $\mathcal{G}$ and $\mathcal{G} \times_\rho \mathbb{Z}$. This sequence might be known to experts but since I could not find any reference we state it for completeness. I would like to thank Jamie Gabe for suggesting me the use of this long exact sequence.

**Lemma 1.4.** Let $\mathcal{G}$ be an étale groupoid with $\mathcal{G}^{(0)}$ a locally compact, Hausdorff and totally disconnected space, and let $\rho: \mathcal{G} \to \mathbb{Z}$ be a group homomorphism. Then there exists a long exact sequence

$$0 \longrightarrow H_0(\mathcal{G}) \longrightarrow H_0(\mathcal{G} \times_\rho \mathbb{Z}) \overset{1-\hat{\rho}^1}{\longrightarrow} H_0(\mathcal{G} \times_\rho \mathbb{Z}) \longrightarrow H_1(\mathcal{G}) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H_n(\mathcal{G}) \longrightarrow H_n(\mathcal{G} \times_\rho \mathbb{Z}) \overset{1-\hat{\rho}^1}{\longrightarrow} H_n(\mathcal{G} \times_\rho \mathbb{Z}) \longrightarrow H_{n+1}(\mathcal{G}) \longrightarrow \cdots$$

where $\hat{\rho}$ is the induced map by the action $\hat{\rho}: \mathbb{Z} \curvearrowright \mathcal{G} \times_\rho \mathbb{Z}$.

**Proof.** Let $\hat{\rho}^1: \mathcal{G} \times_\rho \mathbb{Z} \to \mathcal{G} \times_\rho \mathbb{Z}$ given by $g \times \{i\} \mapsto g \times \{i+1\}$ for $g \in \mathcal{G}$ and $i \in \mathbb{Z}$, then we define the short exact sequence

$$(1) \quad 0 \longrightarrow C_*(\mathcal{G} \times_\rho \mathbb{Z}, \mathbb{Z}) \overset{1-\hat{\rho}^1}{\longrightarrow} C_*(\mathcal{G} \times_\rho \mathbb{Z}, \mathbb{Z}) \overset{\hat{\pi}}{\longrightarrow} C_*(\mathcal{G}, \mathbb{Z}) \longrightarrow 0,$$

where $\hat{\pi}$ is induced by the map $\pi: \mathcal{G} \times_\rho \mathbb{Z} \to \mathcal{G}$ given by $g \times \{i\} \mapsto g$. It is clear that $1 - \hat{\rho}^1$ is an injective map, $\hat{\pi}$ is a surjective map and that $\text{im}\ (1 - \hat{\rho}^1) \subseteq \ker \hat{\pi}$. So it is enough to check that $\text{im}\ (1 - \hat{\rho}^1) \supseteq \ker \hat{\pi}$. Let $f \in C_c((\mathcal{G} \times_\rho \mathbb{Z})^{(n)}, \mathbb{Z})$ such that $\hat{\pi}(f) = 0$. Observe that we can write $f = \sum_{i=-m}^{m} f_i$ where $f_i \in C_c(\mathcal{G}^{(n)} \times \{i\}, \mathbb{Z})$ such that $\sum_{i=-m}^{m} \hat{\pi}(f_i) = 0$. Let $A$ be the compact support of $f$, and let $B := \pi(A)$ a compact subset of $\mathcal{G}^{(n)}$. Let $B_1, \ldots, B_k$ be clopen partition of $B$ such that $(f_i)_{|B_i \times \{i\}}$ is constant for every $-m \leq i \leq m$, so let $\lambda_{ij}$ be the integer number such that $(f_i)_{|B_i \times \{i\}} = \lambda_{ij}$. But then for every $1 \leq j \leq k$ we have that $\sum_{i=-m}^{m} \lambda_{ij} = 0$. Thus, we can write $f = \sum_{j=1}^{k} \sum_{i=-m}^{m} \lambda_{ij} 1_{B_j \times \{i\}}$. Finally from the observation that for any clopen $C$ of $\mathcal{G}^{(n)}$ the map $1_{C \times \{i\}} - 1_{C \times \{0\}} \in \text{im}\ (1 - \hat{\rho}^1)$ it is straightforward to check that $f$ belongs to $\text{im}\ (1 - \hat{\rho}^1)$, as desired.

Then, the long exact sequence of homology of the exact sequence (1), give us the desired sequence

$$0 \longrightarrow H_0(\mathcal{G}) \longrightarrow H_0(\mathcal{G} \times_\rho \mathbb{Z}) \overset{1-\hat{\rho}^1}{\longrightarrow} H_0(\mathcal{G} \times_\rho \mathbb{Z}) \longrightarrow H_1(\mathcal{G}) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H_n(\mathcal{G}) \longrightarrow H_n(\mathcal{G} \times_\rho \mathbb{Z}) \overset{1-\hat{\rho}^1}{\longrightarrow} H_n(\mathcal{G} \times_\rho \mathbb{Z}) \longrightarrow H_{n+1}(\mathcal{G}) \longrightarrow \cdots$$

$\square$

The following Lemma is straightforward to prove.
Lemma 1.5. Let $G$ be a locally compact, étale groupoid with $G^{(0)}$ a totally disconnected Hausdorff space. Let us suppose that there exists a sequence $G_1, G_2, \ldots$ of open sub-groupoids of $G$, such that $G_i \subseteq G_{i+1}$ with $\bigcup_{i=1}^{\infty} G_i = G$. Then $H_*(G) \cong \lim_{\to} H_*(G_i)$ where the maps $H_*(G_i) \to H_*(G_{i+1})$ are induced by the natural inclusions $G_i \to G_{i+1}$.

2. The Katsura groupoid.

Let $N \in \mathbb{N} \cup \{\infty\}$, and let $A$ and $B$ be two $N \times N$ row-finite matrices with integer entries, and such that $A_{i,j} \geq 0$ for all $i$ and $j$. We define

$$\Omega_A := \{(i,j) \in \{1,\ldots, N\}^2 : A_{i,j} \neq 0\}.$$ 

Throughout the paper we will assume that $A$ has no identically zero rows. Let $E_A$ be the graph with $E_A^0 = \{1,\ldots, N\}$, and such that the set of edges from vertex $i$ to vertex $j$ is a set of $A_{i,j}$ elements, say

$$E_A^1 := \{e_{i,j,n} : 0 \leq n < A_{i,j}\},$$

with source map given by $s(e_{i,j,n}) = i$ and range map by $r(e_{i,j,n}) = j$. A path of length $n$, is a concatenation of edges $\alpha_1 \cdots \alpha_n$ with $r(\alpha_i) = s(\alpha_{i+1})$, and we denote by $E_A^n$ the set of all paths of length $n$. Given a path $\alpha$ we denote by $|\alpha|$ its length. It follows from the assumption that $E_A$ has no sinks. We denote by $E_A^I$ the set of all finite length paths (including the length zero paths, that are the vertices), and let $E_A^\infty$ be the infinity path space with the product topology. So given a finite path $\alpha \in E_A^n$ we define the compact and open set $Z(\alpha) := \{\alpha x : x \in E_A^\infty \text{ with } s(x) = r(\alpha)\}$. This family of sets is a basis of clopen and compact subsets, and hence $E_A^\infty$ is a totally disconnected and locally compact space. Observe that $E_A^\infty$ is compact if and only if $N \in \mathbb{N}$.

Given $\alpha \in E_A^n$ we define $A_{\alpha} := A_{s(\alpha),r(\alpha)}$ and $B_{\alpha} := B_{s(\alpha),r(\alpha)}$. Then given $y = y_1 y_2 \cdots \in E_A^\infty$ with $y_i \in E_A^n$ and $n \in \mathbb{N}$ we define $y|_n = y_1y_2\cdots y_n \in E_A^n$, $A_{y|_n} := A_{y_1} \cdots A_{y_n}$ and $B_{y|_n} := B_{y_1} \cdots B_{y_n}$.

We define an action $\kappa$ of $\mathbb{Z}$ on $E_A$ which is trivial in $E_A^0$, and which acts on edges as follows: given $m \in \mathbb{Z}$, and $e_{i,j,n} \in E_A^n$, let $(k,l)$ be the unique pair of integers such that

$$mB_{i,j} + n = kA_{i,j} + l \quad \text{and} \quad 0 \leq l < A_{i,j}.$$ 

We then define

$$\kappa_m(e_{i,j,n}) = e_{i,j,l}.$$ 

Moreover we define the cocycle $\varphi : \mathbb{Z} \times E_A^1 \to \mathbb{Z}$ as $\varphi(m,e_{i,j,n}) = k$.

Then we can extend the action of $\kappa$ and the cocycle $\varphi$ to the paths $E_A^2$ as follows: given $\alpha, \beta \in E_A^n$ with $r(\alpha) = s(\beta)$ we set

$$\kappa_m(\alpha \beta) = \kappa_m(\alpha)\kappa_{\varphi(m,\alpha)}(\beta),$$

and

$$\varphi(m, \alpha \beta) = \varphi(\varphi(m, \alpha), \beta).$$

In a similar way the action $\kappa$ and the cocycle $\varphi$ can be extended to paths of arbitrary finite length, and in particular $\kappa$ to an action of $E_A^\infty$.

Now we denote by $S_{A,B}$ the set of triples $(\alpha, m, \beta)$ where $\beta, \alpha \in E_A^n$ with $r(\alpha) = r(\beta)$ and $m \in \mathbb{Z}$. In [5] it was given $S_{A,B}$ a structure of inverse $\ast$-semigroup, and it constructed the groupoid of certain partial action of $S_{A,B}$ on $E_A^\infty$. Here we will avoid to explain all the construction and defined only the resulting groupoid.
We define the equivalent relation in the set of quadruples of the form $(\alpha, m, \beta; x)$ where $(\alpha, m, \beta) \in S_{AB}$ and $x \in \mathbb{Z}(\beta)$ generated by the relation:

$$(\alpha, m, \beta; x) \sim (\alpha \kappa_m(\gamma), \varphi(m, \gamma), \beta \gamma; x),$$

where $x = \beta \gamma y$ for $\gamma \in E_{A}^\infty$ with $s(\gamma) = r(\beta)$ and $y \in E_{A}^\infty$ with $s(y) = r(\gamma)$. We denote by $[\alpha, n, \beta; x]$ the equivalence class under the above equivalent relation.

Then we define the **Katsura-Exel-Pardo groupoid**

$$G_{AB} := \{ [\alpha, m, \beta; x] : (\alpha, m, \beta) \in S_{AB} \text{ and } x \in \mathbb{Z}(\beta) \},$$

with product defined

$$[\eta, m', \gamma; z] [\alpha, m, \beta; x] = [\alpha \kappa_m(\eta'), \varphi(m', \gamma') + m, \beta; x]$$

if $x = \beta y$, $z = \alpha \kappa_m(y)$ and $\alpha = \gamma \gamma'$

and

$$[\eta, m', \gamma; z] [\alpha, m, \beta; x] = [\alpha \varphi(m, \alpha') + m', \beta \kappa_m(\alpha'); x]$$

if $x = \beta y$, $z = \alpha \kappa_m(y)$ and $\gamma = \alpha \alpha'$,

and inverse

$$[\alpha, m, \beta; x]^{-1} = [\beta, -m, \alpha; \alpha \kappa_m(y)],$$

if $x = \beta y$.

Therefore if we identify $G_{AB}^{(0)}$ with $E_{A}^\infty$ via the map $[v, 0, v; x] \mapsto x$ for $v \in E_{A}^0$ and $x \in \mathbb{Z}(v)$, then the range and the source map can be defined as

$$s([\alpha, m, \beta; x]) = x \quad \text{and} \quad r([\alpha, m, \beta; x]) = \alpha \kappa_m(y),$$

if $x = \beta y$.

With the topology given by the set of open and compact subsets

$$Z(\alpha, m, \beta; U) := \{ [\alpha, m, \beta; x] : x \in U \},$$

where $U$ is an open and compact subset of $\mathbb{Z}(\beta)$, the groupoid $G_{AB}$ is étale with unit space $G_{AB}^{(0)}$ a locally compact, totally disconnected space. Observe that the sets of the form $Z(\alpha, m, \beta; Z(\beta))$ forms a basis for the topology.

In [5] and later in [6] it was shown that $C^*(G_{AB})$ is isomorphic to the $C^*$-algebra $O_{AB}$ constructed in [8].

Now we summarize the properties of the groupoid $G_{AB}$ (see [5] Section 18)):

1. $G_{AB}$ is an étale, locally compact, amenable groupoid,
2. $G_{AB}^{(0)}$ is a locally compact, totally disconnected Hausdorff space, and it is compact if and only if the matrices $A$ and $B$ are finite,
3. $G_{AB}$ is effective if
   a. every circuit in $E_{A}$ has an exit,
   b. for every $1 \leq i \leq N$ and $l \in \mathbb{Z}$, there exists $x \in Z(i)$ such that $\lim_{n \to \infty} l_{ \frac{B_{E_{A}}}{A_{E_{A}}}}^{B_{E_{A}}-l} = 0$,
4. if the matrix $A$ is irreducible and it is not a permutation matrix, then $G_{AB}$ is minimal and purely infinite [12] Definition 4.9.

In [5] Theorem 18.6] there were given additional conditions for $G_{AB}$ being a Hausdorff groupoid. Katsura showed that

$$K_0(C^*(G_{AB})) \cong \ker(1-A) \oplus \ker(1-B) \quad \text{and} \quad K_1(C^*(G_{AB})) \cong \ker(1-B) \oplus \ker(1-A),$$

and that given two countably generated abelian groups $G_0$ and $G_1$ there exists an irreducible matrix $A$ and a matrix $B$ satisfying condition

$$(O) \quad A_{i,j} \geq 2 \quad \text{and} \quad A_{i,j} > |B_{i,j}| \text{ for every } i,$$
such that \( G_0 \cong \text{coker}(1-A) \oplus \ker(1-B) \) and \( G_1 \cong \text{coker}(1-B) \oplus \ker(1-A) \) \cite{9}, Proposition 4.5, and hence \( G_{A,B} \) is an effective, minimal and purely infinite groupoid.

We define the homomorphism \( \rho : G_{A,B} \to \mathbb{Z} \) given by \([\alpha, n, \beta; x] \mapsto |\alpha| - |\beta|\), and we define the subgroupoid \( H_{A,B} := \ker \rho \). By Lemma \[1.3\] we have that \( G_{A,B} \times_\rho \mathbb{Z} \) is Morita equivalent to \( H_{A,B} \). Now given \( n \in \mathbb{N} \) we define the open subgroupoid

\[
H_{A,B,n} := \{ [\alpha, m, \beta; x] \in H_{A,B} : |\alpha| = |\beta| = n \},
\]

and then the map \( \eta_n : H_{A,B,n} \to \mathbb{Z} \) given by \([\alpha, m, \beta; x] \mapsto m \) is a well-defined groupoid homomorphism. Since \( A \) has no zero rows, given \( \beta, \alpha \in E^*_A \) with \( r(\alpha) = r(\beta), m \in \mathbb{Z} \) we have that

\[
[\alpha, m, \beta; x] = [\alpha \kappa_m(\gamma), \varphi(m, \gamma), \beta \gamma; x],
\]

if \( x = \beta \gamma y \) for some \( y \in E^*_A \) and \( \gamma \in E^*_A \) with \( s(\gamma) = r(\beta) \) and \( s(y) = r(\gamma) \), then it follows that \( H_{A,B,n} \subseteq H_{A,B,n+1} \) for every \( n \in \mathbb{N} \), moreover \( H_{A,B} = \bigcup_{n=0}^{\infty} H_{A,B,n} \).

We define the groupoid \( R_{A,B,n} := \ker \eta_n \), that is an open subgroupoid of \( R_{A,B} \) and Morita equivalent to \( H_{A,B,n} \times_{\eta_n} \mathbb{Z} \) (Lemma \[1.3\]).

**Lemma 2.1.** Let \( N \in \mathbb{N} \cup \{\infty\} \), and let \( A \) and \( B \) be two \( N \times N \) matrices with integer entries, and such that \( A_{i,j} \geq 0 \) for all \( i \) and \( j \). Let \( H_{A,B} = \ker \rho \), then \( H_i(H_{A,B}) = 0 \) for \( i \geq 2 \).

**Proof.** Given \( n \in \mathbb{N} \), \( R_{A,B,n} \) is an AF groupoid, and hence \( H_i(R_{A,B,n}) = 0 \) for \( i \geq 1 \). Then using Lemma \[1.4\] we have the exact sequence

\[
0 \longrightarrow H_i(H_{A,B,n}) \longrightarrow H_0(H_{A,B,n} \times_{\eta_n} \mathbb{Z}) \longrightarrow H_0(H_{A,B,n} \times_{\eta_n} \mathbb{Z}) \longrightarrow H_0(H_{A,B,n}) \longrightarrow 0,
\]

and \( H_i(H_{A,B,n}) = 0 \) for \( i \geq 2 \). Therefore by Lemma \[1.5\] it follows that \( H_i(H_{A,B}) = 0 \) for \( i \geq 2 \).

Now we are going to give an explicit isomorphism of the lower degree homology groups of \( H_{A,B} \).

Given groups \( G_1, G_2, G_3, \ldots \) and maps \( \varphi_{i,i+1} : G_i \to G_{i+1} \), we denote by \( \text{lim}_i(G_i, \varphi_{i,i+1}) \) its inductive limit, and the maps \( \varphi_{i,\infty} : G_i \to \text{lim}_i(G_i, \varphi_{i,i+1}) \) the canonical ones.

Given \( i \in \mathbb{N} \cup \{0\} \) we define \( G_i := \bigoplus_{v \in E^0_A} \mathbb{Z}_v \) with generators \( 1_v \), for \( v \in E^0_A \), maps \( \varphi^A_{i,i+1} : G_i \to G_{i+1} \) given by \( 1_v \mapsto \sum_{w \in E^0_A} |\varphi^A_{i,i+1}(1_w)1_v = \sum_{w \in E^0_A} A_{v,w}1_w \). Then we define by \( \mathbb{Z}^N_A \) the inductive limit \( \text{lim}_i(G_i, \varphi^A_{i,i+1}) \) and by \( \varphi_{i,\infty}^A : \mathbb{Z}^N_A \rightarrow \mathbb{Z}^N_A \) the canonical map.

**Proposition 2.2.** Let \( N \in \mathbb{N} \cup \{\infty\} \), and let \( A \) and \( B \) be two \( N \times N \) row-finite matrices with integer entries, and such that \( A_{i,j} \geq 0 \) for all \( i \) and \( j \). Then there exists a group isomorphism \( \Phi_A : H_0(H_{A,B}) \to \mathbb{Z}^N_A \) given by the map

\[
[1_{Z(\alpha,0,0;Z(\alpha))}] \mapsto \varphi_{i,\infty}^A(1_v),
\]

where \( \alpha \in E^*_A \) and \( r(\alpha) = v \).

**Proof.** Let \( \Phi_A : H_0(H_{A,B}) \to \mathbb{Z}^N_A \) be the above defined map. First recall that the boundary map \( \delta_1 : C_c(H_{A,B}) \to C_c(H_{A,B}^{(0)}) \) sends \( 1_{Z(\alpha,0,0;Z(\alpha))} \mapsto 1_{Z(\beta,0,0;Z(\beta))} - 1_{Z(\alpha,0,0;Z(\alpha))} \) for \( \alpha, \beta \in E^*_A \) with \( |\alpha| = |\beta| \) and \( r(\alpha) = r(\beta) \), and \( m \in \mathbb{Z} \). Therefore \( 1_{Z(\alpha,0,0;Z(\alpha))} = \)
Proposition 2.3. \( \delta \)-ary map \( \Phi \)orphism. □

Given \( i \in \mathbb{N} \cup \{0\} \), let \( G_i := \bigoplus_{v \in E^n_A} \mathbb{Z} \), with generators \( v \) for \( v \in E^0_A \), and let \( \varphi^B_{i,i+1} : G_i \to G_{i+1} \) be the maps given by \( v \mapsto \sum_{w \in E^0_A} B_{v,w} w \). Then we define by \( \mathbb{Z}^N_B \) the inductive limit \( \lim_{\rightarrow \gamma}(G_i, \varphi^B_{i,i+1}) \) and by \( \varphi^B_{r,i} : \mathbb{Z}^N \to \mathbb{Z}^N_B \) the canonical map.

Proposition 2.3. Let \( N \in \mathbb{N} \cup \{\infty\} \), and let \( A \) and \( B \) be two \( N \times N \) row-finite matrices with integer entries, and such that \( A_{i,j} \geq 0 \) for all \( i \) and \( j \). Then there exists a group isomorphism \( \Phi_B : H_1(\mathcal{H}_{A,B}) \to \mathbb{Z}^N_B \) given by the map

\[
[1_{\mathbb{Z}(\alpha,1;\alpha)}] \mapsto \varphi^B_{n,\infty}(1_v)
\]

where \( \alpha \in E^n_A \) and \( r(\alpha) = v \).

Proof. Let \( \Phi_B : H_1(\mathcal{H}_{A,B}) \to \mathbb{Z}^N_B \) be the above defined map. First recall that the boundary map \( \delta_2 : C_c(\mathcal{H}^{(2)}_{A,B}, \mathbb{Z}) \to C_c(\mathcal{H}_{A,B}, \mathbb{Z}) \) sends

\[
[1_{\mathbb{Z}(\alpha, m;\alpha)}] \mapsto [1_{\mathbb{Z}(\alpha, m;\alpha)}] - [1_{\mathbb{Z}(\alpha, 0;\beta)}] + [1_{\mathbb{Z}(\alpha, m + n;\gamma)}],
\]

for \( \alpha, \beta, \gamma \in E^n_A \) with \( |\alpha| = |\beta| = |\gamma| \) and \( r(\alpha) = r(\beta) = r(\gamma) \), and \( m, n \in \mathbb{Z} \). In particular we have that

\[
[1_{\mathbb{Z}(\alpha, m;\beta)}] = [1_{\mathbb{Z}(\alpha, m;\alpha)}] - [1_{\mathbb{Z}(\alpha, 0;\beta)}] + [1_{\mathbb{Z}(\alpha, m + n;\gamma)}],
\]

in \( C_c(\mathcal{H}_{A,B}, \mathbb{Z})/\text{im}(\delta_2) \), from where we can deduce that

\[
[1_{\mathbb{Z}(\alpha, 0;\alpha)}] = 0,
\]

\[
[1_{\mathbb{Z}(\alpha, 0;\beta)}] = -[1_{\mathbb{Z}(\beta, 0;\alpha)}],
\]

\[
[1_{\mathbb{Z}(\alpha, m;\alpha)}] = [1_{\mathbb{Z}(\beta, m;\beta)}],
\]

\[
[1_{\mathbb{Z}(\alpha, m;\alpha)}] = m \cdot [1_{\mathbb{Z}(\alpha, 1;\alpha)}].
\]

in \( C_c(\mathcal{H}_{A,B}, \mathbb{Z})/\text{im}(\delta_2) \). Let \( \alpha, \beta \in E^n_A \) with \( r(\alpha) = r(\beta) \), then we have that

\[
Z(\alpha, 1, \beta; Z(\beta)) = \bigcup_{e \in r(\alpha) E^n_A} Z(\alpha \kappa_1(e), \varphi(1, e), \beta e; Z(\beta e)).
\]
Now let $f \in C_*(\mathcal{H}_{A,B}, \mathbb{Z})$ with $\delta_2(f) = 0$, then by the above we can assume that

$$f = \sum_{i=1}^{k} \lambda_i \cdot 1_{Z(\alpha_i, m_i, \beta_i; Z(\beta_i))},$$

with $\alpha_i, \beta_i \in E^n_A$ for some $n \in \mathbb{N}$, and $m_i \in \mathbb{Z}$. For every $v \in E^0_A$ we choose a $\alpha_v \in E^n_A$ with $r(\alpha_v)$, then by the above relations we can assume that

$$f = \sum_{v \in E^0_A} \lambda_v 1_{Z(\alpha_v, 1, \alpha_v; Z(\alpha_v))} + \sum_{v \in E^1_A} \sum_{\gamma \in E^n_A \setminus \{\alpha_v\}} \xi_{\gamma} 1_{Z(\gamma, 0, \alpha_v; Z(\alpha_v))}.$$

But then

$$\delta_2(f) = \delta_2 \left( \sum_{v \in E^0_A} \sum_{\gamma \in E^n_A \setminus \{\alpha_v\}} \xi_{\gamma} 1_{Z(\gamma, 0, \alpha_v; Z(\alpha_v))} \right)$$

$$= \sum_{v \in E^0_A} \sum_{\gamma \in E^n_A \setminus \{\alpha_v\}} \xi_{\gamma} \left( 1_{Z(\alpha_v, 0, \alpha_v; Z(\alpha_v))} - 1_{Z(\gamma, 0; Z(\gamma))} \right) = 0,$$

but this implies that $\xi_{\gamma} = 0$ for every $\gamma \in E^n_A$. Thus we can assume that

$$f = \sum_{v \in E^0_A} \lambda_v 1_{Z(\alpha_v, 1, \alpha_v; Z(\alpha_v))}.$$

Then if for every $w \in E^0_A$ we choose $\beta_w \in E^{n+1}_A$ with $r(\beta_w) = w$, and the observation $\sum_{e \in v E^1_A} \varphi(1, e) = B_{v, w}$, we have that

$$\varphi^B_{n+1, \infty} \left( \sum_{v \in E^0_A} \lambda_v 1_v \right) = \Phi_B([f]) = \Phi_B \left( \left[ \sum_{v \in E^0_A} \lambda_v 1_{Z(\alpha_v, 1, \alpha_v; Z(\alpha_v))} \right] \right)$$

$$= \Phi_B \left( \left[ \sum_{v \in E^0_A} \lambda_v \sum_{w \in E^0_A} \sum_{e \in v E^1_A} 1_{Z(\alpha_v, \kappa_1(e), \varphi(1, e), \alpha_v; Z(\alpha_v))} \right] \right)$$

$$= \Phi_B \left( \sum_{v \in E^0_A} \lambda_v \sum_{w \in E^0_A} \sum_{e \in v E^1_A} \left[ 1_{Z(\beta_w, \varphi(1, e), \beta_w; Z(\beta_w))} \right] \right)$$

$$= \Phi_B \left( \sum_{v \in E^0_A} \lambda_v \sum_{w \in E^0_A} \sum_{e \in v E^1_A} \varphi(1, e) \left[ 1_{Z(\beta_w, 1, \beta_w; Z(\beta_w))} \right] \right)$$

$$= \Phi_B \left( \sum_{v \in E^0_A} \lambda_v \sum_{w \in E^0_A} B_{v, w} \left[ 1_{Z(\beta_w, 1, \beta_w; Z(\beta_w))} \right] \right)$$

$$= \varphi^B_{n+1, \infty} \left( \sum_{v \in E^n_A} \lambda_v \sum_{w \in E^0_A} B_{v, w} 1_w \right).$$

Now let $\alpha \in E^n_A$ and let $m \in \mathbb{Z}$, then $Z(\alpha, m, \alpha; Z(\alpha)) = Z(\alpha, 0, \alpha; Z(\alpha))$ if and only if for every $x \in Z(r(\alpha))$ there exists $k \in \mathbb{N}$ such that $B_{x, k} = 0$ if and only if $\varphi^B_{n, \infty}(1_{r(\alpha)}) = 0.$
Therefore, \( \Phi_B \) is a well-defined map. Clearly \( \Phi_B \) is an exhaustive map, and injectivity follows since by the above argument the inverse map \( \Phi_B^{-1} \) is also well-defined. \( \square \)

Now using Lemma 1.3, we have the following long exact sequence
\[
\begin{align*}
0 \rightarrow H_0(G_{AB}) & \rightarrow H_0(G_{AB} \times \rho Z) \rightarrow H_0(G_{AB} \times \rho Z) \\
& \rightarrow H_1(G_{AB}) \rightarrow H_1(G_{AB} \times \rho Z) \rightarrow H_1(G_{AB} \times \rho Z) \\
& \rightarrow H_2(G_{AB}) \rightarrow 0,
\end{align*}
\]
and \( H_i(G_{AB}) = 0 \) for \( i \geq 3 \).

It is then enough to describe the action \( \tilde{\rho} : Z \rightarrow H_i(G_{AB} \times \rho Z) \) for \( i = 0, 1 \). Observe that \( Y := G_{AB}^{(0)} \times \{0\} \) is a full open subset, and that \( (G_{AB} \times \rho Z)_Y \cong H_{AB} \). Then for every \( x \in H_i(G_{AB} \times \rho Z) \) there exists \( f \in C_c(H_{AB}^{(i)}; Z) \) such that \( [f] = x \) in \( H_i(G_{AB} \times \rho Z) \), so the assignment \( x \rightarrow [f] \) gives the group isomorphism \( \Psi : H_i(G_{AB} \times \rho Z) \rightarrow H_i(H_{AB}) \). Then the action \( \tilde{\rho} : Z \rightarrow H_i(H_{AB}) \) is defined as the unique action that makes the diagram
\[
\begin{array}{ccc}
H_i(G_{AB} \times \rho Z) & \xrightarrow{\rho^1} & H_i(G_{AB} \times \rho Z) \\
\downarrow \Psi & & \downarrow \Psi \\
H_i(H_{AB}) & \xrightarrow{\tilde{\rho}^1} & H_i(H_{AB})
\end{array}
\]
commutative.

**Proposition 2.4.** Let \( N \in \mathbb{N} \cup \{\infty\} \), and let \( A \) and \( B \) be two \( N \times N \) row-finite matrices with integer entries, and such that \( A_{i,j} \geq 0 \) for all \( i \) and \( j \). Then \( \Phi_A \circ \rho_1 \circ \Phi_A^{-1} : Z_A^N \rightarrow Z_A^N \) is given by \( \varphi^A_{i,\infty}(x) \mapsto \varphi^A_{i+1,\infty}(x) \) for every \( x \in Z_A^N \). Moreover, \( \Phi_B \circ \rho_1 \circ \Phi_B^{-1} : Z_B^N \rightarrow Z_B^N \) is given by \( \varphi^B_{i,\infty}(x) \mapsto \varphi^B_{i+1,\infty}(x) \) for every \( x \in Z_B^N \).

**Proof.** First recall that the homeomorphism \( \tilde{\rho}^1 : G_{AB} \times \rho Z \rightarrow G_{AB} \times \rho Z \) is given by \( g \times \{k\} \mapsto g \times \{k+1\} \) for \( g \in G_{AB} \) and \( k \in Z \). Now let \( Z(\alpha, \beta; Z(\beta)) \times \{0\} \) be a clopen bisection of \( (G_{AB} \times \rho Z)_Y \cong H_{AB} \), then \( \tilde{\rho}^1(\{Z(\alpha, \beta; Z(\beta)) \times \{0\}\}) = Z(\alpha, \beta; Z(\beta)) \times \{1\} \subseteq G_{AB} \times \rho Z \), so the induced map \( \tilde{\rho}^1 : C_c(H_{AB}; Z) \rightarrow C_c(G_{AB} \times \rho Z, Z) \) is given by \( 1_{Z(\alpha, \beta; Z(\beta)) \times \{0\}} \mapsto 1_{Z(\alpha, \beta; Z(\beta)) \times \{1\}} \).

Thus, we need to find the equivalent function of \( 1_{Z(\alpha, \beta; Z(\beta)) \times \{1\}} \) in \( C_c(\mathcal{H}_{AB}; Z) \). First observe that
\[
\tilde{\rho}^1 : C_c(H_{AB}^{(0)}; Z) \rightarrow C_c((G_{AB} \times \rho Z)^{(0)}, Z) \quad \text{is given by} \quad 1_{Z(\alpha, 0; Z(\alpha)) \times \{0\}} \mapsto 1_{Z(\alpha, 0; \beta; Z(\beta)) \times \{1\}},
\]
and that given any \( \beta \in E_{AB}^{(1)} \) with \( s(\alpha) = r(\beta) \) we have that
\[
\delta_1(1_{Z(\beta, 0, 0; Z(\beta)) \times \{0\}}) = 1_{Z(0, 0; Z(\alpha)) \times \{1\}} - 1_{Z(\beta, 0, 0; Z(\alpha)) \times \{0\}},
\]
so \( 1_{Z(\alpha, 0; Z(\alpha)) \times \{1\}} = (1_{Z(\beta, 0, 0; Z(\alpha)) \times \{0\}}) \) in \( H_0(G_{AB} \times \rho Z) \). Then,
\[
\tilde{\rho}^1(\Phi_A^{-1}(\varphi^A_{i,\infty}(1_{v}))) = \tilde{\rho} \left( \left[1_{Z(\alpha, 0; Z(\alpha)) \times \{0\}}\right] \right) = \left[1_{Z(\beta, 0, 0; Z(\alpha)) \times \{0\}}\right]
\]
\[
= \left[1_{Z(\beta, 0, 0; Z(\alpha)) \times \{0\}}\right] = \Phi_A^{-1}(\varphi^A_{i+1,\infty}(1_{v})),
\]
as desired.
Now, on the other hand given any $\alpha \in E^n_A$ and any $\beta \in E^1_A$ with $r(\hat{\beta}) = s(\alpha)$, we can define the functions in $C_\ast((G_{A,B} \times \rho) \langle 2 \rangle, \mathbb{Z})$

$$
\begin{align*}
f_1 &= 1_{(Z(a,1,\alpha;Z(\alpha)) \times \{1\}) \times (Z(\alpha,0,\beta;Z(\beta)) \times \{1\})}, \\
f_2 &= 1_{(Z(\beta,0,0,\alpha;Z(\alpha)) \times \{0\}) \times (Z(\alpha,1,\beta;Z(\beta)) \times \{1\})}, \\
f_3 &= 1_{(Z(\beta,0,0,\alpha;Z(\alpha)) \times \{0\}) \times (Z(\alpha,0,\beta;Z(\beta)) \times \{1\})}, \\
f_4 &= 1_{(Z(\beta,0,0,\alpha;Z(\alpha)) \times \{0\}) \times (Z(\beta,0,\beta;Z(\beta)) \times \{0\})},
\end{align*}
$$

that satisfy

$$
\delta_2(f_1 - f_2 + f_3 + f_4) = 1_{Z(\alpha,1,\alpha;Z(\alpha)) \times \{1\}} - 1_{Z(\beta,1,\beta;Z(\beta)) \times \{0\}}.
$$

Then,

$$
\hat{\rho}^1(\varphi^{-1}_{B,\infty}(1_v)) = \hat{\rho}^1([1_{Z(\alpha,1,\alpha;Z(\alpha)) \times \{0\}}]) = [1_{Z(\alpha,1,\alpha;Z(\alpha)) \times \{1\}}] = \varphi^{-1}_{B,\infty}(1_v),
$$

as desired.

\[ \square \]

**Theorem 2.5.** Let $N \in \mathbb{N} \cup \{\infty\}$, and let $A$ and $B$ be two $N \times N$ row-finite matrices with integer entries, and such that $A_{i,j} \geq 0$ for all $i$ and $j$. Then

$$
\begin{align*}
H_0(G_{A,B}) &\cong \text{coker}(1 - A) & H_1(G_{A,B}) &\cong \ker(1 - A) \oplus \text{coker}(1 - B) \\
H_2(G_{A,B}) &\cong \ker(1 - B), & H_i(G_{A,B}) &= 0 \text{ for } i \geq 3.
\end{align*}
$$

Therefore, $G_{A,B}$ satisfies the (HK) conjecture.

**Proof.** By Lemma 1.4 we have the long exact sequence

$$
\begin{array}{cccccccccc}
0 & \longrightarrow & H_0(G_{A,B}) & \longrightarrow & H_0(G_{A,B} \times \rho \mathbb{Z}) & \overset{1 - \hat{\rho}^1}{\longrightarrow} & H_0(G_{A,B} \times \rho \mathbb{Z}) & \longrightarrow & H_1(G_{A,B}) & \longrightarrow & \cdots,
\end{array}
$$

$$
\begin{array}{cccccccccc}
\cdots & \longrightarrow & H_n(G_{A,B}) & \longrightarrow & H_n(G_{A,B} \times \rho \mathbb{Z}) & \overset{1 - \hat{\rho}^1}{\longrightarrow} & H_n(G_{A,B} \times \rho \mathbb{Z}) & \longrightarrow & H_{n+1}(G_{A,B}) & \longrightarrow & \cdots
\end{array}
$$

where $\hat{\rho}^1$ is the induced map by the action $\hat{\rho} : \mathbb{Z} \curvearrowright G \times \rho \mathbb{Z}$. Since by Lemma 1.3 the groupoids $H_{A,B}$ and $G_{A,B} \times \rho \mathbb{Z}$ are Morita equivalent, then Lemma 2.1 says that $H_i(G_{A,B} \times \rho \mathbb{Z}) = 0$ for $i \geq 2$. Then we have the following long exact sequence

$$
\begin{array}{cccccccccc}
0 & \longrightarrow & H_0(G_{A,B}) & \longrightarrow & H_0(G_{A,B} \times \rho \mathbb{Z}) & \overset{1 - \hat{\rho}^1}{\longrightarrow} & H_0(G_{A,B} \times \rho \mathbb{Z}) & \longrightarrow & H_1(G_{A,B}) & \longrightarrow & 0
\end{array},
$$

$$
\begin{array}{cccccccccc}
\longrightarrow & H_1(G_{A,B}) & \longrightarrow & H_1(G_{A,B} \times \rho \mathbb{Z}) & \overset{1 - \hat{\rho}^1}{\longrightarrow} & H_1(G_{A,B} \times \rho \mathbb{Z}) & \longrightarrow & H_2(G_{A,B}) & \longrightarrow & 0
\end{array}
$$

and $H_i(G_{A,B}) = 0$ for $i \geq 3$. But by Proposition 2.4 we have that that

$$
\begin{align*}
\ker(1 - \hat{\rho}^1) : H_0(G_{A,B} \times \rho \mathbb{Z}) &\rightarrow H_0(G_{A,B} \times \rho \mathbb{Z}) \cong \ker(1 - A), \\
\text{coker}(1 - \hat{\rho}^1) : H_0(G_{A,B} \times \rho \mathbb{Z}) &\rightarrow H_0(G_{A,B} \times \rho \mathbb{Z}) \cong \ker(1 - A), \\
\ker(1 - \hat{\rho}^1) : H_1(G_{A,B} \times \rho \mathbb{Z}) &\rightarrow H_1(G_{A,B} \times \rho \mathbb{Z}) \cong \ker(1 - B), \\
\text{coker}(1 - \hat{\rho}^1) : H_0(G_{A,B} \times \rho \mathbb{Z}) &\rightarrow H_0(G_{A,B} \times \rho \mathbb{Z}) \cong \ker(1 - B).
\end{align*}
$$
Since \( \ker(I - A) \) and \( \ker(I - B) \) are free abelian groups, then the exact sequence splits in the short exact sequences

\[
0 \rightarrow \text{coker}(I - A) \rightarrow H_0(G_{A,B}) \rightarrow 0,
\]

\[
0 \rightarrow \text{coker}(I - B) \rightarrow H_1(G_{A,B}) \rightarrow \ker(I - A) \rightarrow 0,
\]

\[
0 \rightarrow H_2(G_{A,B}) \rightarrow \ker(I - B) \rightarrow 0,
\]
as desired. \( \square \)

**Remark 2.6.** We would like to point out that the exact sequences at the end of the proof of Theorem 2.5 are the same that one gets when using the spectral sequence described in [11] as this was our initial strategy. But now our proof uses a more primitive but intuitive method in homological algebra, without an extra cost in the computations.

We say that two étale groupoids \( G \) and \( H \) with \( \sigma \)-compact, Hausdorff, totally disconnected unit spaces are *Kakutani equivalent*, or *Morita equivalent*, if there exist clopen full subsets \( A \subseteq G^{(0)} \) and \( B \subseteq H^{(0)} \) such that \( G_A \cong H_B \). It was proved in [11, Theorem 3.6(2)] that two Kakutani equivalent groupoids have the same homology groups.

**Corollary 2.7.** Let \( N, N' \in \mathbb{N} \), and let \( A, B \in M_N(\mathbb{Z}) \) and \( A', B' \in M_{N'}(\mathbb{Z}) \), such that \( A_{i,j}, A'_{i,j} \geq 0 \) for all \( i \) and \( j \). Suppose that \( G_{A,B} \) and \( G_{A',B'} \) are Kakutani equivalent, then \( \text{coker}(I - A) \cong \text{coker}(I - A') \) and \( \text{coker}(I - B) \cong \text{coker}(I - B') \).

**Example 2.8.** Let \( A = (2) \) and \( B = (1) \), and let

\[
A' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

then we have that \( G_{A,B} \) and \( G_{A',B'} \) are minimal, Hausdorff, effective and purely infinite étale groupoids with compact unit space, and

\[
K_0(C^*(G_{A,B})) = K_0(C^*(G_{A',B'})) \cong \mathbb{Z} \quad \text{and} \quad K_1(C^*(G_{A,B})) = K_1(C^*(G_{A',B'})) \cong \mathbb{Z},
\]

so \( \mathcal{O}_{A,B} \) and \( \mathcal{O}_{A',B'} \) are stable isomorphic. But then by Theorem 2.5 we have that

\[
H_0(G_{A,B}) = 0, \quad H_1(G_{A,B}) \cong \mathbb{Z} \quad \text{and} \quad H_2(G_{A,B}) \cong \mathbb{Z},
\]

while

\[
H_0(G_{A',B'}) \cong \mathbb{Z}, \quad H_1(G_{A',B'}) \cong \mathbb{Z} \quad \text{and} \quad H_2(G_{A',B'}) = 0,
\]

and therefore \( G_{A,B} \) and \( G_{A',B'} \) cannot be equivalent. In particular, does not exists any diagonal preserving isomorphism between the stabilizations of \( \mathcal{O}_{A,B} \) and \( \mathcal{O}_{A',B'} \).

In a private correspondence, Enrique Pardo showed me how to prove using [4] that the isotropy groups of \( G_{A,B} \) are isomorphic either to 0 or \( \mathbb{Z} \). Therefore, homology is the invariant that distinguishes the equivalence classes of these groupoids.
3. Final remarks

In this final section we will use the previous computations on the homology of the groupoid $G_{A,B}$ to give examples of groupoids with prescribed homology and satisfying the (HK) conjecture, whose associated groupoid $C^*$-algebra falls in a classifiable class.

**Lemma 3.1.** Let $N \in \mathbb{N} \cup \{\infty\}$, and let $A$ and $B$ be two $N \times N$ row-finite matrices with integer entries, and such that $A_{i,j} \geq 0$ for all $i$ and $j$, and $|B_{i,j}| < A_{i,j}$ for every $(i, j) \in \Omega_A$ and $E_A$ is acyclic. Then the groupoid $G_{A,B}$ is principal.

**Proof.** Let $g = [\alpha, m, \beta; \beta x] \in G_{A,B}$ with $r(g) = s(g) = \beta x$. Since $E_A$ is acyclic we can assume that $\alpha = \beta$. Then $r(g) = s(g) = \beta x$ if and only if $\kappa_m(x) = x$. Now by [5, Lemma 18.4] given $x \in E_A^\infty$ and $m \in \mathbb{Z}$, $\kappa_m(x) = x$ if and only if $m_{A,m}^{B,I} \in \mathbb{Z}$ for every $l \in \mathbb{N}$. But then by hypothesis it is clear that for every $x \in E_A^\infty$ and $m \in \mathbb{Z}$ there exists $t \in \mathbb{N}$ such that $m_{A,m}^{B,I} \notin \mathbb{Z}$. \hfill $\Box$

Given a simple, dimension group $G_0$, and any dimension group $G_1$ one can find Bratteli diagrams $(V, E)$ and $(W, F)$ such that the associated AF-algebras have $K_0$ groups $G_0$ and $G_1$ respectively. Let $A$ be the adjacency matrix of $(V, E)$ and let $B$ be the adjacency matrix of $(W, F)$ (see for example [15]). Telescoping and out-splitting $(V, E)$ we can assume that $|B_{i,j}| < A_{i,j}$ for every $(i, j) \in \Omega_A$.

**Proposition 3.2.** Let $G_0$ be a simple, dimension group, and let $G_1$ be any dimension group. Then there exist $N \in \mathbb{N} \cup \{\infty\}$ and $N \times N$ row-finite matrices $A$ and $B$ with natural entries, such that $G_{A,B}$ is an amenable, Hausdorff, principal, minimal étale groupoid with

$$H_0(G_{A,B}) \cong G_0 \quad \text{and} \quad H_0(G_{A,B}) \cong G_1.$$ 

In particular, $C^*(G_{A,B})$ is a simple AT-algebra.

**Proof.** Let us consider $A$ and $B$ as explained before the Lemma, then $G_{A,B}$ is an amenable, Hausdorff and minimal groupoid groupoid [5, Section 18], and by Lemma 3.1 it is also principal. The homology is computed in Theorem 2.5 so we only need to see that $C^*(G_{A,B})$ is an AT-algebra. Let $(V, E)$ be the Bratteli diagram with incidence matrix $A$, and let $V = \bigcup_{i \geq 0} V_i$ be the level decomposition of the diagram. Then given $n \in \mathbb{N}$ we define

$$G_{A,B,n} = \{[\alpha, n, \beta; x] \in G_{A,B} : r(\alpha) = r(\beta) \in V_n\},$$

with the subspace topology, it is an open subgroupoid of $G_{A,B}$ and we have that $G_{A,B} = \bigcup_{n=0}^\infty G_{A,B,n}$. Given $v \in E_A^\infty$ let $u_v$ be the partial unitary $1_{Z(v)} \in C^*(G_{A,B})$, then we have that $C^*(G_{A,B,n}) \cong \bigoplus_{v \in V_n} M_{n_v}(C(\text{spec}(u_v))) \otimes C(Z(v))$ where $n_v = |\{\alpha \in E_A : r(\alpha) = v\}|$, that is an AT-algebra. Then by [15] Proposition 1.9 we have that $C^*(G_{A,B,n})$ is a subalgebra of $C^*(G_{A,B})$, and hence $C^*(G_{A,B}) = \bigcup_{n \geq 0} C^*(G_{A,B,n})$, whence $C^*(G_{A,B})$ is an AT-algebra. \hfill $\Box$

For the rest of the section we will assume that $A$ and $B$ are the incidence matrices of two Bratteli diagrams $(V, E)$ and $(W, F)$ respectively satisfying that $|B_{i,j}| < A_{i,j}$ for every $(i, j) \in \Omega_A$. 
In general, the unit space of the groupoid $G_{A,B}$ is not compact, and hence $C^*(G_{A,B})$ is not a unital $C^*$-algebra. We can define the groupoid $\tilde{G}_{A,B} := (G_{A,B}, Z_{(\nu_0)}$ where $\nu_0$ is the initial vertex of the Bratteli diagram $(V, E)$. Then the groupoid $\tilde{G}_{A,B}$ is amenable, Hausdorff, principal, minimal and étale, and has a compact unit space homeomorphic to $Z(\nu_0)$. Moreover, since $Z(\nu_0) \subseteq G_{A,B}$-full we have that $H_i(G_{A,B}) \cong H_i(\tilde{G}_{A,B})$, and $C^*(\tilde{G}_{A,B})$ is a unital AT-algebra Morita equivalent to $C^*(G_{A,B})$.

Given a groupoid $G$ with compact unit space $G^{(0)}$, we denote by $M(G)$ the set of probability measures $\mu$ of $G^{(0)}$ such that given any bisection $U \subseteq G$ we have that $\mu(s(U)) = \mu(r(U))$.

**Lemma 3.3.** Let $A$ and $B$ be the incidence matrices of two Bratteli diagrams $(V, E)$ and $(W, F)$ respectively, satisfying that $|B_{i,j}| < A_{i,j}$ for every $(i, j) \in \Omega_A$. Then $M(\tilde{G}_{A,B}) = M(\tilde{G}_{A,0})$.

**Proof.** Clearly $M(\tilde{G}_{A,B}) \supseteq M(\tilde{G}_{A,0})$. On the other hand given $\eta \in M(\tilde{G}_{A,0})$, $\alpha \in E^*_A$ with $s(\alpha) = \nu_0$, and $m \in Z$, we have that $\eta(Z(\alpha)) = \eta(Z(\kappa_m(\alpha)))$ because the bisection $U = Z(\kappa_m(\alpha), 0, \alpha; Z(\alpha)) \subseteq \tilde{G}_{A,0}$ is such $s(U) = Z(\alpha)$ and $r(Z(\kappa_m(\alpha)))$. \qed

Every $\mu \in M(G)$ induces a trace $\tau \circ E$ on $C^*_r(G)$ (viewing $\mu$ as a state of $C(G^{(0)})$), where $E : C^*_r(G) \to C(G^{(0)})$ is the canonical conditional expectation. Moreover, if $G$ is a principal groupoid every trace $\tau$ of $C^*_r(G)$ satisfies $\tau \circ E = \tau$ (see [7, Lemma 4.3] for example). Observe that given two different $\mu_1, \mu_2 \in M(G)$ induce two different traces $\mu_1 \circ E$ and $\mu_2 \circ E$ of $C^*_r(G)$. Therefore, we have bijection between $M(G)$ and $T(C^*_r(G))$, the traces of $C^*_r(G)$.

Then given row-finite matrices $A$ and $B$, there is a bijection between $T(C^*_r(\tilde{G}_{A,0}))$ and $M(\tilde{G}_{A,B})$. But $C^*(\tilde{G}_{A,0})$ is a simple unital $AF$-algebra, and hence by [11] for each metrizable Choquet simplex $\Delta$ there exists $A$ such that $T(C^*(\tilde{G}_{A,0}))$ is homeomorphic to $\Delta$.

Finally, we present a last example of a minimal, purely infinite étale groupoid with a prescribed homology. The example covers partially the result of Li and Renault [7, Lemma 5.5].

**Proposition 3.4.** Let $G_0$ be a simple, dimension group, and let $G_1$ be any dimension group. Then there exist an amenable, Hausdorff, effective, purely infinite, minimal étale groupoid $G$ with unit space homeomorphic to the Cantor space and isotropy groups isomorphic either to 0 or to $Z$, that satisfies the (HK) conjecture, and with $K_0(C^*_r(G)) \cong H_0(G) \cong G_0$ and $K_1(C^*_r(G)) \cong H_1(G) \cong G_1$.

**Proof.** Let $A$ and $B$ be from Proposition 3.2 and let $\tilde{G}_{A,B}$, that is a principal étale groupoid, with $\tilde{G}_{A,B}^{(0)}$ homeomorphic to the Cantor space, and with $H_0(\tilde{G}_{A,B}) \cong G_0$ and $H_1(\tilde{G}_{A,B}) \cong G_1$. Now let $G_{\infty}$ be any graph groupoid such that $C^*_r(G_{\infty}) \cong O_{\infty}$, that is an amenable, Hausdorff, minimal, effective and purely infinite étale groupoid, with $G_{\infty}^{(0)}$ homeomorphic to the Cantor space and isotropy groups isomorphic to either 0 or $Z$. It is computed in [3] that $H_0(G_{\infty}) \cong Z$ and $H_1(G_{\infty}) = 0$ for $i \geq 1$. Then the groupoid $G := G_{A,B} \times G_{\infty}$ is an amenable, Hausdorff, minimal, effective and purely infinite, with $G^{(0)}$ homeomorphic to the Cantor space and isotropy groups isomorphic to either 0 or $Z$, and by [13, Theorem 2.4 & Theorem 2.8] it follows the rest of the statement. \qed
The groupoids constructed in the above Proposition have much simple isotropy groups than the general groupoids $G_{A,B}$ [1]

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