On a compactification of a Hurwitz space in the wild case

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Abstract

Let $g$ and $g'$ be two integers and $p$ a prime number. Denote by $H_{g,g',p}$ the moduli space of morphisms of degree $p$ between smooth curves of genus $g$ and $g'$ and with constant ramification. The purpose of this article is to construct and describe a modular compactification of this space.

1 Introduction

The aim of this work is to construct a compactification of one Hurwitz space in the wild case. By Hurwitz spaces, we mean spaces classifying finite morphisms between curves with some condition (for example morphisms of a fixed degree, morphisms to $\mathbb{P}^1$, ...). In the tame case, we already know a good modular compactification of those spaces (see for example the work of Harris and Mumford [HM82], Mochizuki [Moc95] or Wewers [Wew98]). On the other hand, we are far from having a good compactification in the wild case. Let us illustrate the problems that can arise on an example. Let $g, g', d \in \mathbb{N}$ with $g \geq 2$ and consider the moduli space $H_{g,g',d}$ of geometrically separable morphisms of degree $d$ between proper smooth curves of genus $g$ and $g'$. Then it is relatively easy to prove that $H_{g,g',d}$ is representable by an algebraic stack that is formally smooth over $\mathbb{Z}$. Using stable curves, it is possible to embed this space in a proper space over $\mathbb{Z}$. Indeed, consider the moduli space $\bar{H}_{g,g',d}$ of “stable” morphisms of degree $d$ between semi-stable curves of genus $g$ and $g'$ (the “stable” condition is just a minimality condition that ensures the space $\bar{H}_{g,g',d}$ to be separated, cf. [Liu]). In general the space $H_{g,g',d}$ contains morphisms between singular curves and inseparable morphisms (through a “stable reduction”-like theorem, we can see that this phenomenon is necessary in order to have a proper space). Then it is possible to prove that this space is representable by a proper algebraic stack over $\mathbb{Z}$. The problems are that in general, $H_{g,g',d}$ fails to be dense in $\bar{H}_{g,g',d}$ and that $\bar{H}_{g,g',d}$ may have nasty singularities. Moreover, it does not seem easy to give a thorough description of the geometric points of the closure of $\bar{H}_{g,g',d}$ (in particular, this closure have probably no “simple” interpretation).

Our aim is to construct a compactification with mild singularities of a variation of $H_{g,g',p}$ for a prime $p$. The restriction on $p$ is not just a convenience matter. There are many instances throughout the paper where this hypothesis is fundamental. Although the construction can be done in the general case, it would fail to produce a compactification in most of the cases. To avoid some technical complication, we will restrict ourselves to moduli spaces over $\mathbb{Z}_p$ (so
that we can focus on only one prime). This restriction can be lifted but only at the cost of even more technicalities in the definition, which did not seem worth the effort. More precisely, let \( g, g', p \in \mathbb{N} \) with \( p \) prime and \( g \geq 2 \) and denote by \( \mathcal{H}^c_{g,g',p} \), the moduli space over \( \mathbb{Z}_p \) of morphisms of degree \( p \) between smooth curves of genus \( g \) and \( g' \) with constant ramification. That is, the degree \( p \) geometrically separable morphisms \( f : X \to Y \) between proper smooth \( S \)-curves, where \( S \) is a \( \mathbb{Z}_p \)-scheme, \( g(X) = g \) and \( g(Y) = g' \) and such that \( \text{coker} \, df \) is locally of the form \( \mathcal{O}_S[x]/(x^n) \) for the finite flat topology. In particular, we have a canonical morphism \( \mathcal{H}^c_{g,g',p} \to \mathcal{H}^c_{g',g,p} \) which is easily proved to be bijective on the geometric points, but is not an isomorphism. Hence we can consider \( \mathcal{H}^c_{g,g',p} \) as a stratification of \( \mathcal{H}^c_{g',g,p} \).

It is relatively easy to see that \( \mathcal{H}^c_{g,g',p} \) splits in two components: one that lives over \( \mathbb{Z}_p \) and one that lives over \( \mathbb{F}_p \). More precisely, denote by \( \mathcal{H}^{c,\infty}_{g,g',p} \) the substack of \( \mathcal{H}^c_{g,g',p} \) parameterising morphisms \( f : X \to Y \) over \( S \) such that there exists a point \( p \in X \) at which \( f \) is of degree \( p \) and \( \text{coker} \, df \) is, locally at \( p \) for the finite flat topology on \( X \), of the form \( \mathcal{O}_S[x]/(x^n) \) with \( n > p - 1 \). In particular, if \( S \) is a field, a morphism \( f \) is in \( \mathcal{H}^{c,\infty}_{g,g',p} \) if and only if \( S \) is of characteristic \( p \) and there exists a point at which \( f \) is of degree \( p \). As we require the ramification locus to be constant, such a morphism cannot be lifted to characteristic 0. In particular, this defines an open and closed substack of \( \mathcal{H}^c_{g,g',p} \). We will denote by \( \mathcal{H}^{c,\leq \infty}_{g,g',p} \) the complement of \( \mathcal{H}^{c,\infty}_{g,g',p} \) in \( \mathcal{H}^c_{g,g',p} \).

In this article, we construct a modular compactification of \( \mathcal{H}^c_{g,g',p} \) with only mild (and completely explicit) singularities. This is achieved using a quite technical definition of the admissible covering which generalises the existing ones. The main ingredient is the concept of separating data which is a generalisation of the different in the global case and over a general basis. Another important feature of the admissible covering is the concept of the Hurwitz graph which was introduced originally by Henrio (cf. [Hen00]) but is vastly modified here to fit our purpose. The Hurwitz graph is a combinatorial data which is attached on each geometric fiber of the morphism and which we need to ensure the existence of a smooth deformation. What makes the definition quite tricky is that we need quite a large amount of compatibility conditions between the separating data and the Hurwitz graph. Roughly speaking, all those conditions are necessary to ensure that the tangent space of the deformation functor is as little as possible, i.e. to ensure that the singularities of the moduli space are as mild as possible.

Let’s now describe the structure of the article. In the second section, we introduce and study some morphisms between differential modules. The original problem is that, for a morphism \( f : X \to Y \) over a scheme \( S \), the homomorphism \( df : \Omega_{Y/S} \to f_*\Omega_{X/S} \) may be zero at some points. The \( p' \)-earnest morphisms defined in this section are a mean to bypass this problem and to keep track of the differential information.

In the third section, we turn to the definition and the basic properties of the admissible covers. The definition is highly technical and requires many steps and many intermediary objects. This definition was thought out to enjoy good degeneracy and deformation properties (in fact, it was really constructed by
looking precisely at the degeneracy of morphisms and keeping the properties that can be deformed in a nice way). At the end of the section, we compare our notion with the already existing notions of admissible covers.

The fourth section is entirely devoted to the reduction of separable morphisms between smooth curves. More precisely, we prove that admissible coverings are stable by degeneracy. The first subsection of this section consists of the study of morphisms between formal annuli; the second one uses the results of the first one to prove the main result of this section.

The fifth section is concerned with the deformation theory of admissible covers and the deformation functor \( D_{\text{adm}} \). For this, we construct another functor \( D_{\text{abs}} \) and a morphism \( \text{abs} : D_{\text{adm}} \to D_{\text{abs}} \). The results are then the following: the functor \( D_{\text{abs}} \) has a versal deformation, which is explicit and allows us to give a thorough description of the singularities. Furthermore the morphism \( \text{abs} \) admits a relative versal deformation, which will simultaneously be proved to be smooth. The relative versality of the morphism \( \text{abs} \) is obtained by a good deformation theory and theorems à la Schlessinger, adapted to the relative situation.

In the sixth part, we define the moduli space of admissible covers, prove its representability by an algebraic stack (using Artin’s theorem), the density of \( \mathcal{M}_{g,g',p} \) and its properness.

Finally, in the appendix, we gather some results about various deformation theories.

**Notation and convention:** The letter \( p \) will denote a fixed prime number and \( \nu_p \) will be the \( p \)-adic valuation on \( \mathbb{Z}_p \) normalized by \( \nu_p(p) = 1 \).

All the schemes and rings in this article will be \( \mathbb{Z}_p \)-schemes or \( \mathbb{Z}_p \)-algebras.

Let \( \ell \) be an integer and \( X \) be a scheme in which \( p\ell = 0 \). We will denote by \( F_\ell \) the absolute Frobenius defined by \( x \mapsto x^{p^\ell} \). This morphism is a homeomorphism. In particular, the functor \( F_\ell^* \) is exact in the category of sheaves of abelian groups.

When we speak of a semi-stable curve, we don’t always mean it is proper.

Let \( A \) be a complete local ring and \( B \) the completion of a singular double point over \( A \). A set of coordinates of \( B \) is a couple \( (x, y) \in B^2 \) such that \( b := xy \in A \) and the morphism \( A[[X,Y]]/(XY - b) \to B \) defined by \( X \mapsto x, Y \mapsto y \) is an isomorphism.

All the considered graphs will be directed. For any edge \( e \) we will denote by \( o(e) \) the origin and \( t(e) \) the target.

Let \( i, j \in \mathbb{Z} \), \( i \neq 0 \) and write \( r := \frac{j}{i} \). We will say that an element \( \alpha \in \mathbb{Z}_p \) is a \( r \)-th root of \( p \) if \( \alpha^i - p^j = 0 \). Two \( r \)-th roots of \( p \) differ from unit (in particular, for divisibility conditions, we can pick anyone). We will denote by \( p^r \) one of these elements and denote by \( \mathbb{Z}_p[p^r] \) the \( \mathbb{Z}_p \)-sub-algebra of \( \mathbb{Z} \) generated by \( p^r \).

By abuse of notation, we will denote by \( p^{\infty} = 0 \) and \( \mathbb{Z}_p[p^{\infty}] = \mathbb{F}_p \).

## 2 Sheaf of \( p^r \)-earnest morphism

For this whole section, we will fix an \( r \in \mathbb{Q}_+ \cup \{\infty\} \) verifying \( r \leq 1 \) or \( r = \infty \). All considered schemes will be \( \mathbb{Z}_p[p^r] \)-schemes. Let us fix a scheme \( S \), a finite
morphism $f : X \to Y$ of degree $p$ between smooth and not necessarily proper curves. Our goal is to study a certain class of homomorphisms which belongs to $\text{Hom}_{\mathcal{O}_X}(f^*\Omega^1_{Y/S},\Omega^1_{X/S})$ (philosophically, if $r \leq \infty$ those are the homomorphisms of the form $\frac{df}{p^r}$ up to a lifting in characteristic 0).

The first step of the definition comes from an explicit description of the formal power series case. This definition could probably be done in a more abstract (and perhaps canonical) way. The general definition follows from algebraisation and flat base change.

**Definition 2.1 (Formal power series)**

Let $r \in (\mathbb{Q} \cap [0,1]) \cup \{\infty\}$ and $A$ be a $\mathbb{Z}_p[p^r]$-algebra, $B = A[[x]]$, $C = A[[u]]$, $f : B \to C$ an $A$-morphism and $\hat{\Omega}_{B/A}$ (resp. $\hat{\Omega}_{C/A}$) the completion (for the $(x)$-adic (resp. $(u)$-adic) topology) of the module of differentials. Write $f(x) = \sum_i a_i x^i$. Let $\delta : \hat{\Omega}_{B/A} \to \hat{\Omega}_{C/A}$ be a $B$-homomorphism and write $\delta(dx) = \left(\sum_j b_j x^j \right)du$. If $r < \infty$, we will say that $\delta$ is $p^r$-earnest if for all $i \in \mathbb{N}$ the following property is true

$$
\begin{cases}
    \text{if } i \neq 0 \mod p & \text{then } a_i = \frac{p^r b_i}{i} \\
    \text{if } i = 0 \mod p & \text{then } b_i = \left(\frac{1}{p^r}\right) a_i.
\end{cases}
$$

If $r = \infty$ (in particular $A$ is a $\mathbb{F}_p$-algebra) and $g \in A$, we will say that $\delta$ is $p^\infty$-earnest for $g$ if for all $i \neq 0 \mod p$ we have $a_i = \frac{p^r b_i}{i}$ and $b_i = 0$ if $i = 0 \mod p$.

We will denote by $\Xi^r_A$ the set of $p^r$-earnest homomorphisms.

The following lemma explains the behaviour of $\Xi^r_A$ with respect to base change and its link to the elements of the form $\frac{df}{p^r}$. In particular, it proves that this notion is well defined (i.e. does not depend on the choices).

**Lemma 2.2** With the notation of the definition, the following properties are true.

1. Let $A \to A'$ be a faithfully flat morphism, then we have an exact sequence

$$
0 \to \Xi^r_A \to \Xi^r_{A'} \Rightarrow \Xi^r_{A' \otimes_A A'}.
$$

2. Suppose that $A$ is local noetherian and complete with residue field of characteristic $p$ and that $r < \infty$. Let $W(A)$ be a complete local noetherian $\mathbb{Z}_p[p^r]$-algebra endowed with a surjective morphism $W(A) \to A$ such that $p^r$ is regular in $W(A)$ (such an algebra exists thanks to [Bou83, Théorème IX.5.3]). Then an element $\delta \in \text{Hom}_B(\hat{\Omega}_{B/A},\hat{\Omega}_{C/A})$ is $p^r$-earnest if and only if there exists a lifting $\tilde{f} : W(A)[[x]] \to W(A)[[u]]$ of $f$ such that $p^r|d\tilde{f}$ and $\delta$ is the reduction of $\frac{df}{p^r}$.

3. Suppose that $A$ is local noetherian and complete, of characteristic $p$ and that $r = \infty$. Let $g \in A$ and let $W(A)$ be a complete local noetherian ring of characteristic $p$ endowed with a surjective morphism $W(A) \to A$
and a lifting $\tilde{g}$ of $g$ which is regular in $W(A)$. Then an element $\delta \in \text{Hom}_B(\hat{\Omega}_{B/A}, \hat{\Omega}_{C/A})$ is $p^\infty$-earnest if and only if there exists a lifting $\tilde{f} : W(A)[[x]] \to W(A)[[u]]$ of $f$ such that $\tilde{g}|d\tilde{f}$ and $\delta$ is the reduction of $d\tilde{f}$.  

4. If $A$ is local noetherian, then the property to be $p^r$-exact is independent of the uniformising parameters $x$ and $u$. 

Proof: 1. Write $A'' = A' \otimes_A A'$. By the faithful flatness of $A \to A'$ we have an exact sequence 

$$0 \to \text{Hom}_B(\hat{\Omega}_{B/A}, \hat{\Omega}_{C/A}) \to \text{Hom}_{B \otimes A A'}(\hat{\Omega}_{B \otimes A A'}/A', \hat{\Omega}_{C \otimes A A'}/A')$$

$$\Rightarrow \text{Hom}_{B \otimes A A''}(\hat{\Omega}_{B \otimes A A''}/A'', \hat{\Omega}_{C \otimes A A''}/A'').$$

(even though the rings and modules do not commute with base change). In particular, we get the exactness on the left in (2.1). Then we only have to prove that the divisibility conditions descend but this is a classical result of faithful flatness. 

2. The “if” part is trivial (this is the definition of $p^r$-earnestness). Let us prove the other direction. For all $i \not\equiv 0 \mod p$, let $\tilde{b}_i$ be a lifting of $b_i$ in $W(A)$ and for all $i \equiv 0 \mod p$ let $\tilde{a}_i$ be a lifting of $a_i$ in $W(A)$. Then define 

$$\tilde{f} = \sum_{i \not\equiv 0 \mod p} p^i b_i u^i + \sum_{i \equiv 0 \mod p} \tilde{a}_i u^i,$$

this is a lifting of $f$. Then we see that, by definition, 

$$d\tilde{f} = p^r \left( \sum_{i \not\equiv 0 \mod p} b_i u^{i-1} + \sum_{i \equiv 0 \mod p} \frac{i}{p^r} \tilde{a}_i \right) du$$

hence the desired property is satisfied. 

3. The proof is similar to the second case. 

4. Thanks to the first property, we can perform a base change and suppose that $A$ is complete. Then the result is obtained by the second and third assertions. 

Thanks to this lemma, we can extend the notion of $p^r$-earnestness to points which are not necessarily rational in the following way. 

**Definition 2.3** Let $A$ be a complete local ring, $B \to C$ a morphism of local complete $A$-algebras which are formally smooth and of relative dimension 1. Let $\hat{\Omega}_{B/A}$ and $\hat{\Omega}_{C/A}$ be the completion of the differential modules, $r \in \mathbb{Q}_+ \cup \{\infty\}$ and $\delta \in \text{Hom}_B(\hat{\Omega}_{B/A}, \hat{\Omega}_{C/A})$. Then $\delta$ is said to be $p^r$-earnest if there exists a finite faithfully flat $A[p^r]$-algebra $A'$ so that the closed points of $C \otimes_A A'$ are rational and the image of $\delta$ in $\text{Hom}_{B \otimes A A'}(\hat{\Omega}_{B \otimes A A'}/A', \hat{\Omega}_{C \otimes A A'}/A')$ is $p^r$-earnest. 

The lemma 2.2 ensures that this notion is well defined. 

We are now able to give a definition in the general situation.
Definition 2.4 Let $S$ be a scheme and $f : X \to Y$ a finite morphism of degree $p$ between smooth curves. Let $p \in X$ and $\delta \in \text{Hom}_{\mathcal{O}_Y}(\Omega_{Y/S}, f_*\Omega_{X/S})$. We will say that $\delta$ is $p^r$-earnest if it is $p^r$-earnest at the completion at each point. We will denote by $\Xi^r_{X/S}$ the sub-sheaf of $\text{Hom}_{\mathcal{O}_Y}(\Omega_{Y/S}, f_*\Omega_{X/S})$ composed by the homomorphisms which are $p^r$-exact at each point.

Remark 2.5 If $r = 0$, we see that the set of $p^r$-earnest morphism is reduced to $df$ and is therefore of no significant interest.

If $r > 0$ we see that $p^r|df$, hence the morphism $f$ should be purely inseparable where $p = 0$. Contrary to the case $r = 0$, the $p^r$-earnest morphism need no longer be unique.

Let us state some properties of $p^r$-earnest morphisms which are formal consequences of the lemma 2.2.

Proposition 2.6 Let $S$ be a scheme and $f : X \to Y$ be a morphism of smooth $S$-curves. Then the notion of $p^r$-earnestness is local for the finite flat topology on $Y$ and the flat topology on $S$.

Next thing we would like to give a description of the $p^r$-earnest homomorphisms in the global case. For that, it is convenient to work with the Frobenius (when available).

Let $p$ be a prime and $S$ be a scheme where $p^\ell = 0$. Then we can define a Frobenius morphism $F^\ell : S \to S$ by $\alpha \mapsto \alpha^{p^\ell}$. As in the case of a field, we can define a relative Frobenius $F^\ell_S : X \to X^\ell$ on each $S$-scheme. We have the following lemma which gives us the structure of this morphism.

Lemma 2.7 Let $X \to S$ be a smooth curve. Then the morphism $F^\ell_S : X \to X^\ell$ is locally (over $X^\ell$) of the form $B \to B[u]/(u^{p^\ell} - v)$ where $dv$ is a basis of $\Omega_{B/S}$.

Proof: Locally on $X^\ell$, we can find $v \in \mathcal{O}_{X^\ell}$ which is a basis of $\Omega_{X^\ell/S}$ and which is a $p^\ell$-th power in $\mathcal{O}_X$. The lemma follows easily from this remark. \(\square\)

In particular, consider the case of a local artinian base of residue characteristic $p$ and a finite morphism $f : X \to Y$ of degree $p$ between smooth curves which is purely inseparable at the special fiber. Then writing things explicitly, we see that the induced morphism $X \to Y^\ell$ can be identified with $F^{\ell+1}$. That is, for all such $f$ we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{F^{\ell+1}} & & \downarrow{F^\ell} \\
Y^\ell & & \\
\end{array}
\]

We can now give the following “global” criterion for a homomorphism to be $p^r$-earnest.
Lemma 2.8 Let $S$ be a local scheme where $p^r = 0$ and $f : X → Y$ a finite morphism of degree $p$ between $S$-smooth curves which is purely inseparable at the special fiber. Suppose that $Y^r = \text{Spec}B$ and that we can write $X = \text{Spec}B[u]/(u^{p^{r+1}} - v)$ and $Y = \text{Spec}B[x]/(x^{p^r} - v)$. In particular, $dx$ is a basis of $\Omega_{Y/S}$. Let $δ ∈ \text{Hom}_{\mathcal{O}_Y}(\Omega_{Y/S}, f_\ast\Omega_{X/S})$ and write $f(x) = \sum_{i=0}^{p^{r+1}-1} a_i u^i$ and $δ(dx) = (\sum_{i=0}^{p^{r+1}-1} b_i u^i)du$. If $r < ∞$, then $δ$ is $p^r$-earnest if and only if for all $i$ the following is true

$$\begin{cases} 
\text{if } i \neq 0 \mod p & \text{then } a_i = \frac{p^r b_i}{i} \\
\text{if } i = 0 \mod p & \text{then } b_i = \frac{1}{p^r}a_i.
\end{cases}$$

If $r = ∞$ then $δ$ is $p^∞$-earnest if for all $i \neq 0 \mod p$ we have $a_i = \frac{p^r b_i}{i}$.

A decomposition of $X$ and $Y$ as required in the lemma always exists locally on $Y^r$, cf. lemma 2.7 and the discussion above.

Proof : The lemma is trivially true where $x$ is a uniformising parameter. Therefore it is true over all the points of $Y^r$ where $v$ is a parameter. Let $p$ be a point of $Y^r$. By definition, the notion of $p^r$-earnestness can be checked after and tale base change on $Y^r$ (hence we can suppose that $Y^r$ is an open subset of the affine line $\text{Spec}A[x]$ where $x$ is the parameter above) and stable under finite base change of $S$, hence we can suppose that $p$ is rational. Thus there is a parameter of $Y^r$ at $p$ which is of the form $x - α$ with $α ∈ A$. The result can be obtained in a straightforward way.

Corollary 2.9 Let $A' → A$ be a surjective morphism of local artinian rings with residue characteristic $p$. Let $f : X → Y$ be a finite morphism between smooth $A$-curves which is purely inseparable at the special fiber and let $δ ∈ \text{Hom}_{\mathcal{O}_Y}(\Omega_{Y/S}, f_\ast\Omega_{X/S})$ be a $p^r$-earnest homomorphism. Then there exists locally on $Y$ a lifting $f' : X' → Y'$ of $f$ and a lifting $δ'$ of $δ$ such that $δ'$ is $p^r$-earnest.

Proof : Use the explicit description given by the lemma above and then lift explicitly as in the proof of the lemma 2.2.

In the following, we will need a lemma concerning exact forms which will allow us to classify the possible liftings of $p^r$-earnest homomorphisms. Since the techniques involved in the proof are close to the one used in this chapter, we state it here.

Lemma 2.10 Let $k$ be a field of characteristic $p$, $X$ a smooth curve over $k$ and $ω ∈ Ω_{X/k}$. Then $ω$ is exact if and only if for all $p ∈ X$, $ω$ is exact in $Ω_{X/k,p}^r$.

Proof : Denote by $Ω_{X/S}^c$ the sheaf of locally exact differential forms and $F : X → X$ the absolute Frobenius. We are going to prove that $F_\ast Ω_{X/S}^c$ is coherent (then using the general properties of coherent modules, it is easy to get the result).

Denote by $d : \mathcal{O}_X → Ω_{X/S}$ the universal derivation. By definition, $Ω_{X/S}^c$ is the image sheaf of $d$. As $F_\ast$ is an exact functor on the category of abelian sheaves
(F is an homeomorphism) we see that $F^*\Omega^c_{X/S}$ is the image of the map $F_*d$.
But $F_*d$ append to be $O_X$-linear. Hence $F^*\Omega^c_{X/k}$ is coherent being the image
of a homomorphism between coherent modules. □

3 Basic properties of ∂-morphisms

The present section defines the notion of ∂-morphisms. Roughly speaking, those morphisms are morphisms between semi-stable curves endowed with an additional differential structure. This allows one to deform these morphisms generically in a unique way, although the considered morphism may be inseparable when restricted to some irreducible component. More precisely, the ∂-morphisms will be morphisms with additional data satisfying some conditions “formal locally”. The following sections will be devoted to degeneracy and deformations of ∂-morphisms.

Before we can give the definition of ∂-morphism, we need to introduce some other concepts.

All the considered graphs will be directed. For a directed graph $\Gamma$, we will denote by $\mathbf{Ar}(\Gamma)$ the set of directed edges (positive and negative), by $\mathbf{Ar}^+(\Gamma)$ the set of positive edges and by $\mathbf{Som}(\Gamma)$ the set of vertices. For any directed edge $e$ we will denote by $\bar{e}$ the edge with the opposite direction.

Definition 3.1 (Hurwitz graph) Let $\Gamma$ be a directed graph and $k$ a field. A $k$-Hurwitz data on $\Gamma$ is a couple $(m, r)$ with

i) $m : \mathbf{Ar}(\Gamma) \to \mathbb{Z}$ (the conductor);

ii) $r : \mathbf{Som}(\Gamma) \to \mathbb{Q}^+ \cup \{\infty\}$ (the earnestness degree)

satisfying the following properties

1. $\forall e \in \mathbf{Ar}(\Gamma)$ we have $m(\bar{e}) = -m(e)$;

2. $\forall e \in \mathbf{Ar}^+(\Gamma)$ we have $r(o(e)) \leq r(t(e))$ with equality if $m(e) = 0$;

3. we have either $\text{Im} \ r \subset [0, 1]$ or $\text{Im} \ r \subset \{0, \infty\}$;

4. for all $e \in \mathbf{Ar}^+(\Gamma)$ one has $m(e) \geq 0$;

5. if $k$ is of characteristic $p$ then $m(e)$ is either 0 or prime to $p$ for all $e \in \mathbf{Ar}(\Gamma)$;

6. if $k$ is of characteristic 0 then $m = 0$ and $r = 0$.

This definition is inspired by Henrio (cf. [Hen00]) a modification of which was used in [Mau03]. In fact, our definition defers hugely from the original (we lost 5 functions out of 6 and introduced a new one) but we give it the same name because the purpose of those data remains the same (namely, to ensure the existence of a lifting).

The definition above will be useful to study the deformation problem but will not be enough to ensure a lifting. For a lifting to exist, we need an additional condition on the graph which is given through the following definition.
Definition 3.2 (Reduced Hurwitz graph)
Let \((\Gamma, m, r)\) be a Hurwitz graph. Denote by \(\Gamma_{\text{red}}\) the graph obtained from \(\Gamma\) after contracting the edges \(e\) which verifies \(m(e) = 0\) and identifying the vertices which are linked by such edges. The functions \(m\) and \(r\) then descend to functions on \(\text{Ar}(\Gamma_{\text{red}})\) and \(\text{Som}(\Gamma_{\text{red}})\) (through the property 2 of the definition 3.1) which we will denote by \(m_{\text{red}}\) and \(r_{\text{red}}\), and \(\Gamma_{\text{red}}\) inherits an orientation from \(\Gamma\). It is easily seen that \((\Gamma_{\text{red}}, m_{\text{red}}, r_{\text{red}})\) is a Hurwitz graph. It will be called the reduced Hurwitz graph attached to \((\Gamma, m, r)\). By definition the reduced Hurwitz graph verifies \(\forall e \, m_{\text{red}}(e) \neq 0\).

We call the reduced Hurwitz graph good if the orientation on \(\Gamma_{\text{red}}\) defines an order on \(\text{Som}(\Gamma_{\text{red}})\) (saying that \(t(e) > o(e)\) for any \(e \in \text{Ar}^+(\Gamma_{\text{red}})\)).

Remark 3.3 We can quite easily prove that a reduced Hurwitz graph is good if and only if there exists a function \(\ell : \text{Som}(\Gamma_{\text{red}}) \to \mathbb{N}\) that verifies \(\ell(t(e)) > \ell(o(e))\) for any \(e \in \text{Ar}^+(\Gamma_{\text{red}})\) and \(\ell(s) = 0\) for \(s\) minimal. For the lifting property, the existence of such a function will be fundamental.

If \(k\) is of characteristic 0, then the reduced graph is trivial.

The following definition is mainly a computational help, all the rest may be independent of it, but it will simplify some of the later proofs.

Definition 3.4 (Distended morphism) Let \(S\) be a scheme and \(X \to S, Y \to S\) be \(S\)-semistable curves. Denote by \(\omega_{X/S}\) and \(\omega_{Y/S}\) the relative dualising sheaves. In particular we have canonical inclusions \(\Omega_{X/S} \to \omega_{X/S}\) and \(\Omega_{Y/S} \to \omega_{Y/S}\). A distended morphism is a couple \((f, \partial f)\) where \(f : X \to Y\) is an \(S\)-morphism and \(\partial f : \omega_{Y/S} \to f_* \omega_{X/S}\) is a homomorphism of \(\mathcal{O}_Y\)-module such that the following diagram is commutative

\[
\begin{array}{ccc}
\Omega_{Y/S} & \xrightarrow{\partial f} & f_* \Omega_{X/S} \\
\downarrow & & \downarrow \\
\omega_{Y/S} & \xrightarrow{df} & f_* \omega_{X/S}.
\end{array}
\]

Very often, the morphism \(\partial f\) will also be denoted by \(df\) (although \(df\) may not determine \(\partial f\) in a unique way).

A morphism between smooth curves is automatically endowed with a unique distended structure but this fail to be true when one considers semistable curves. For example, consider the morphism \(f : k[x, y]/(xy) \to k[u, v]/(uv)\) defined by \(x = u^n\) and \(y = v^m\) with \(n \neq m\) in \(k\). The morphism \(df\) cannot be extended to a morphism between dualising sheaves.

Moreover, an existing distended structure may not be unique.

In the definition of \(\partial\)-morphisms, we will need some invertible sheaves on curves which are trivial on the smooth locus. The best way to formalise this for our purpose is given by the following definition.
Definition 3.5 (ftsl-invertible sheaf)
Let $S$ be a scheme, $X \to S$ a semistable curve and $\mathcal{L}$ an invertible sheaf on $X$. Denote by $U$ the smooth locus of $X \to S$. We will say that $\mathcal{L}$ is formally trivial on the smooth locus (ftsl for short) if for all point $s$ of $S$, the invertible sheaf induced by $\mathcal{L}$ on the formal completion of $U \times_S \text{Spec} \mathcal{O}_{S,s}$ along $s$ is trivial. That is, if we denote by $m_s$ the maximal ideal of $\mathcal{O}_{S,s}$, there exists an isomorphism

$$\phi_U : \lim \left( \mathcal{L}|_{U \times_S \mathcal{O}_{S,s}/m_s} \right) \to \lim \left( \mathcal{O}_U \otimes_S \mathcal{O}_{S,s}/m_s \right)$$

Let us choose, for any singular point $p \in X_s$, a trivialisation $\phi_p : \mathcal{L}_p \otimes_{\mathcal{O}_{X,p}} \widehat{\mathcal{O}}_{X,p} \to \widehat{\mathcal{O}}_{X,p}$. The data $(\phi_U, \phi_p)$ will be called a formal trivialisation of $\mathcal{L}$ at $s$.

Suppose that $S$ is local and that $U$ is affine. It’s easy to see that $\mathcal{L}$ is ftsl if and only if the restriction of $\mathcal{L}_U$ to the fiber over $s$ is trivial. Indeed in this case the deformations of $\mathcal{L}_s$ are classified by the sheaf $H^1(U_s, \mathcal{O}_{U_s})$ which is zero because $U_s$ is affine.

Definition 3.6 (Unfolded separating data)
Let $S$ be a scheme and $f : X \to Y$ a distended morphism between $S$-semistable curves. An unfolded separating data relative to $f$ is a triplet $(\mathcal{L}, \delta, g)$ where $\mathcal{L}$ is a ftsl invertible sheaf on $X$, $g : \mathcal{L} \to \mathcal{O}_X$ is a homomorphism of $\mathcal{O}_X$-module and $\delta : \omega_{Y/S} \to f^* (\omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{L})$ is a homomorphism of $\mathcal{O}_Y$-module, satisfying the following conditions

a) the diagram

$$\begin{array}{ccc}
\omega_{Y/S} & \xrightarrow{df} & f^* \omega_{X/S} \\
\delta & \downarrow & \quad \\
& f^*(\omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{L}) & \quad f^*(\text{Id} \otimes g)
\end{array}$$

is commutative;

b) $f^* \delta$ induces an injective morphism in every fiber (which is equivalent to say that the cokernel of $\delta$ is finite and flat over $S$);

c) $f^* \delta$ induces an isomorphism in a neighbourhood of all singular point;

d) $\text{coker } f^* \delta$ is locally (for the finite flat topology) isomorphic to $\mathcal{O}_S[x]/(x^n)$ at a point $p$. The integer $n$ is called the horizontal ramification degree at $p$. If $f$ is of degree prime to $p$ at $p$ then we see that $n = \deg_p f - 1$;

e) the homomorphism $g$ is zero over no point (although it can be zero in restriction to some irreducible component, hence not injective).

The unfolded data will be called finite (resp. infinite) if for any point $p$ such that $f$ is of degree $p$ at $p$, the horizontal ramification degree at $p$ is $\leq p$ (resp. $\geq p$ and $\neq -1 \mod p$).
The set of unfolded separating data is naturally endowed with an action of \( O^*_S \) defined by \( \alpha.(\mathcal{L}, g, \delta) := (\mathcal{L}, \alpha g, \alpha^{-1} \delta) \). The separating data will later be defined as an equivalence class of unfolded separating data under this action.

**Remark 3.7** The separating data are here to remember the information about the ramification which is normally encoded in \( df \) but can be trivial in the inseparable case. More precisely, the “horizontal” ramification is encoded in \( \text{coker} \delta \) and the “vertical” ramification is encoded in \( \text{coker} g \).

Let \( S \) be a scheme, \( X \to S \) a semistable curve and suppose that for any geometric point \( \bar{s} \to S \) we are given a structure of a Hurwitz graph on the dual graph \( \Gamma_S \) of \( X_S \) (in particular, we are given an orientation on \( \Gamma_S \)). Then the oriented edges of \( \Gamma_S \) are naturally in bijection with the branches at singular points of \( X_S \). We can thus talk of the origin and target branches. When referring to a set of coordinates \((u_o, u_t)\) at a singular point, \( u_o \) will be a parameter of the origin branch and \( u_t \) of the target branch. The singular points will be identified with the positive edge.

**Definition 3.8 (Unfolded \( \delta \)-morphism)**

Let \( S \) be the spectrum of a local ring, \( X \to S \) and \( Y \to S \) semi-stable curves such that the singular points of the special fiber of \( X \) are rational, \( f : X \to Y \) a distended morphism, \((m, r)\) a Hurwitz structure on the dual graph \( \Gamma \) of the special fiber of \( X \) and \((\mathcal{L}, g, \delta)\) an unfolded separating data relative to \( f \). Denote with \( \hat{\alpha} \) the data obtained by completion along the special fiber and \( U \) the smooth locus of \( X \to S \). We will say that \((f, m, r, \mathcal{L}, g, \delta)\) is an unfolded \( \delta \)-morphism at the closed point of \( S \) if there exists a formal trivialisation \((\hat{\phi}_U, \hat{\phi}_p)\) of \( \mathcal{L} \) and for any singular point \( p \) of \( X \) there exists a set of coordinates \((u_o, u_t)\) of \( X \) at \( p \) such that

i) for any connected open formal subscheme \( V \) of \( \hat{U} \) the homomorphism \( g \circ \hat{\phi}_U^{-1}|_V \) is given by an element of \( A \) and if \( r(V) < \infty \) then this scalar is in \( p^{r(V)} A^* \);

ii) for all singular point \( p \) corresponding to a positive edge \( e \), the morphism \( g \circ \hat{\phi}_p^{-1} \) is given by an element of the form \( \delta u_0^{m(e)} \) with \( \delta \in A \);

iii) for all singular point \( p \), the morphism \( f \) is given in a formal neighborhood of \( p \) by \( x_o \mapsto u_o^n \alpha, x_t \mapsto u_t^n \alpha^{-1} \) where \((x_o, x_t)\) is a set of coordinates of \( Y \) at \( f(p) \);

iv) for any singular point \( p \) corresponding to a positive edge \( e \), the automorphism \( \phi_p \circ \phi_U^{-1} \) of \( \text{Frac}\hat{\mathcal{O}}_{X,p} \) is defined on the origin branch (resp. target branch) by the multiplication by \( u_o^{m(e)} \) (resp. \( u_t^{m(e)} \));

v) for any irreducible open subset \( V \) of \( \hat{U} \) the homomorphism \( \delta \circ f_* (Id \otimes \hat{\phi}_U) : \omega_f(V) \to f_* \omega_V \) is \( p^{r(V)} \)-horizontal.

vi) Let \( p \) be a double point and \( r_o \) and \( r_t \) the degree of earnestness of \( \delta \) on the origin and terminal branches. Then \((u_0^{m(p)} \delta)|_{u_0 \neq 0} \) is \( p^{r_o} \)-earnest and \((u_t^{-m(p)} \delta)|_{u_t \neq 0} \) is \( p^{r_t} \)-earnest.
The unfolded $\mathfrak{d}$-morphism $(f, m, r, \mathcal{L}, g, \delta)$ will be called finite if

1. $\text{Im } r \subset \mathbb{Q}_+$;

2. $(\mathcal{L}, g, \delta)$ is a finite unfolded separating data;

It will be called infinite if

1. $S$ is an $\mathbb{F}_p$-scheme;

2. $\text{Im } r \subset \{0, \infty\}$;

3. $(\mathcal{L}, g, \delta)$ is an infinite unfolded separating data;

Let $S$ be a scheme and $f : X \to Y$ a distended morphism between semistable $S$-curves. For any geometric point $\bar{s} \to S$ let us a Hurwitz data $(m_{\bar{s}}, r_{\bar{s}}, \mathcal{L}, g, \delta)$ on the dual graph of $X_{\bar{s}}$ and an unfolded separating data $(\mathcal{L}, g, \delta)$ relative to $f$. We will say that $(f, m_{\bar{s}}, r_{\bar{s}}, \mathcal{L}, g, \delta)$ is an unfolded $\mathfrak{d}$-morphism if it’s a $\mathfrak{d}$-morphism at each point after eventually a finite flat base change. It’s easily seen that the definition is independent of this base change.

The morphism $f$ will be called the underlying morphism of $(f, m_{\bar{s}}, r_{\bar{s}})$.

Moreover, $(f, m_{\bar{s}}, r_{\bar{s}}, \mathcal{L}, g, \delta)$ will be called a structure of unfolded $\mathfrak{d}$-morphism relative to $f$.

**Definition 3.9 ($\mathfrak{d}$-morphism)**

A $\mathfrak{d}$-morphism is an equivalence class of unfolded $\mathfrak{d}$-morphism between proper semistable curves under the action of $\mathbb{G}_m$ on unfolded separating data.

The next step is to prove the compatibility of this definition with those already existing in the literature. Let’s do it first for separable morphisms between smooth curves satisfying an extra condition.

**Proposition 3.10** Let $S$ be a scheme, $f : X \to Y$ a morphism between proper smooth curves such that for any point $s \in S$, the morphism $f : X_s \to Y_s$ is separable. Suppose that $\text{coker } df$ satisfies condition (A) of definition 3.6. Then $f$ admits a unique structure of $\mathfrak{d}$-morphism. Moreover, if $S$ is of equal characteristic $p$ then this $\mathfrak{d}$-morphism is infinite and if $S$ has no point of characteristic $p$ then it is finite.

**Proof:** The morphism $f$ is trivially distended (and admits a unique structure of a distended morphism). Moreover, it is easily seen that $(f, 0, 0, \mathcal{O}_X, 1, df)$ is a $\mathfrak{d}$-morphism because, $f$ being separable in each fiber, the homomorphism $df$ is injective in each fiber.

Let us suppose that $(f, m, r, \mathcal{L}, g, \delta)$ is another $\mathfrak{d}$-morphism. By definition, we have $g \circ \delta = df$ and as $\delta$ and $df$ are injective, so is $g$. Moreover, using formal trivialisation, we see that $g$ is an isomorphism in each fiber. In particular, we can assume $\mathcal{L} = \mathcal{O}_X$. Then we can see $g$ as a global automorphism of $\mathcal{O}_X$, that is an element of $H^0(X, \mathcal{O}_X^*)$. But since $X$ is proper and smooth over $S$, we get that $g$ is an element of $H^0(S, \mathcal{O}_S^*)$. 

With almost the same proof, we can see that the moderated admissible covering (cf. for example [Wew98]) admits a unique structure of a $\mathfrak{d}$-morphism. Namely we have the following proposition.
Proposition 3.11 Let $S$ be a scheme, $f : X \to Y$ a moderated admissible covering between proper semistable curves. Then there exists a unique structure of $d$-morphism on $f$.

Proof: The proof is similar to the case of smooth curves when taking into account the description of the admissible covering in neighborhoods of singular points. Here we don’t need the extra condition on $\text{coker } df$ because of the tameness of the morphism. □

Then we can turn to the definition of admissible coverings which are $d$-morphisms with a minimality condition.

Definition 3.12 (Admissible covering of degree $p$)
Let $S$ be a scheme, $X \to S$ and $Y \to S$ semistable curves. Let $f$ be a $d$-morphism between $X$ and $Y$ such that the underlying morphism is finite of degree $p$. Then $f$ will be called admissible if the following conditions are true

i) for any geometric point $\bar{s} \to S$ the reduced Hurwitz graph of $(\Gamma_{\bar{s}}, m_{\bar{s}}, r_{\bar{s}})$ is good;

ii) for any geometric point $\bar{s} \to S$ and any irreducible component $V \in X_{\bar{s}}$ of genus 0, $V$ meets the rest of $X_{\bar{s}}$ in at least $3 - \#\text{supp}(\text{coker } \delta|_V)$ points.

4 Degeneracy of unfolded $d$-morphism
The goal of this section is to prove the following “stable reduction” theorem.

Theorem 4.1 Let $R$ be a discrete valuation ring of fraction field $K$ and $f : X \to Y$ an admissible covering of degree $p$ between smooth proper $K$-curves with $g(X) \geq 2$ and separable underlying morphism. Then there exists a unique model of $f$ proper over $R$ which admits a unique structure of a $d$-morphism extending $f$. Moreover, if $f$ is finite (resp. infinite) then this morphism is also finite (resp. infinite).

To prove this theorem, we first need to conduct a thorough study of morphism between formal annuli. Then we will proceed with the proof.

4.1 Morphisms between formal annuli
Let $R$ be a complete discrete valuation ring, $B := R[[x, y]]/(xy - b)$, $C := R[[u, v]]/(uv - c)$ and $f : \text{Spec} C \to \text{Spec} B$ a finite morphism of degree $n$ étale at the generic fiber. We restrict ourselves to the case where the residue field of $R$ is of characteristic $p > 1$ and $n$ is a multiple of $p$ (otherwise the results are well known, see for example [Wew98]). We will denote by $\varpi$ a uniformising parameter of $R$, $\Omega_B$ (resp. $\Omega_C$) the completion of the differential modules of $\text{Spec} B \to \text{Spec} R$ (resp. $\text{Spec} C \to \text{Spec} R$) and $\omega_B$ (resp. $\omega_C$) the completion of the canonical morphism. In particular, we have a natural injection $\Omega_C \subset \omega_C$ and $\frac{du}{u} = -\frac{dv}{v}$ is a basis of $\omega_C$. 
Finally, for the special fiber we have \( f(x) \in (u^n) \) or \( f(x) \in (v^n) \). We can assume that \( f(x) \in (u^n) \) and thus that \( f(y) \in (v^n) \).

The goal of this subsection is to prove the following proposition which does not require that \( n = p \).

**Proposition 4.2** Suppose that \( p | n \). Up to the multiplication of \( x \) by an element of \( \mathbb{R}^* \) there exists \( \alpha \in C^* \) such that \( f(x) = u^n \alpha \) and \( f(y) = v^n \alpha^{-1} \). In particular, \( df \) can be extended to a homomorphism \( \omega_B \to \omega_C \) in a unique way.

Moreover, there exists a \( \omega \in R \) and \( m \in \mathbb{N} \) such that the homomorphism \( df : \omega_B \to \omega_C \) is the composition of a morphism \( \delta : \omega_B \to \omega_C \) which induces an isomorphism \( \omega_B \otimes_B C \to \omega_C \), and the multiplication by \( \omega \) of \( u^m \) or \( v^m \).

Finally, we have the following alternative

(A) - if \( R \) is of equal characteristic \( p \) then \( m \) is prime to \( p \);

(B) - if \( R \) is of unequal characteristic then \( \nu_p(m) < \nu_p(n) \). Moreover \( m = 0 \) if and only if \( (\omega) = (n) \) in \( R \).

This proposition will allow us to prove the properties concerning double points (that is properties (a) of definition 3.8 and properties (b), (c), (d) and (e) of definition 3.8). It also gives the definition and properties of the function \( m \) of a Hurwitz graph if \( \nu_p(n) \leq 1 \).

The rest of this subsection is devoted to the proof of the above proposition and is broken down in several steps.

First of all, using the norm homomorphism \( \text{Frac}(C)^* \to \text{Frac}(D)^* \), we can prove that \( b = c^n \) up to an invertible element of \( R \). Thus, after eventually replacing \( x \) by a multiple, we can assume that \( b = c^n \). In particular \( b = 0 \) mod \( c \).

\( \triangleright \) We can write \( x = uP_u + cP_v \) with \( P_u \in R[[u]] \) and \( P_v \in R[[v]] \).

As \( uv = c \) in \( C \), it is enough to prove that \( x = uP_u \) mod \( c \). We are going to prove this by induction on \( q \). For \( q = 1 \) the result is already known.

Suppose it is true for \( q \) and that \( c \equiv 0 \) mod \( \varpi^{q+1} \).

Write \( x = u(\sum_{i \geq 0} a_i u^i) + \varpi^{q}(\sum_{j \geq 0} b_j v^j), y = v(\sum_{j \geq 0} c_j v^j) + \varpi^{q}(\sum_{i \geq 0} d_i u^i) \) (such a decomposition of \( y \) exists by symmetry).

As \( 0 = b = c = uv = \varpi^{2q} \) = mod \( \varpi^{q+1} \) we get

\[
0 = xy = \varpi^{q+1} u \sum_{i \geq 0} d_i u^i \sum_{i \geq 0} a_i u^i + \varpi^{q} v \sum_{j \geq 0} b_j v^j \sum_{j \geq 0} c_j v^j \quad \text{mod } \varpi^{q+1}.
\]

In particular, \( \varpi^{q} v \sum_{j \geq 0} b_j v^j \sum_{j \geq 0} c_j v^j = 0 \) in \( R/(\varpi^{q+1})[[v]] \) but in this ring, \( v \sum_{j \geq 0} c_j v^j \) is regular (because \( y \not\equiv 0 \) mod \( \varpi \)). Hence we get \( \varpi^{q} \sum_{j \geq 0} b_j v^j = 0 \) mod \( \varpi^{q+1} \).

\( \triangleright \) The element \( dx \) is a multiple of \( du \) in \( \Omega_C \).

Write \( x = uP_u + cP_v \) as above. Then we have \( dx = \frac{\partial(uP_u)}{\partial u} du + c \frac{\partial P_v}{\partial v} dv \). But on the other hand, we have \( cdv = uv dv = -v^2 du \) in \( \Omega_C \). Hence we get \( dx = (\frac{\partial(uP_u)}{\partial u} - v^2 \frac{\partial P_v}{\partial v}) du \).

Before continuing, we need a lemma which is probably well known but will be used in several occasions.
Lemma 4.3 Let $P = \sum a_i u^i \in R[u]$ be a unitary polynomial. Consider it to be a function on $\text{Spec} C$ (via $R[u] \subset R[[uv]]/(uv - c)$) and assume that it has no zeros on $\text{Spec} C \otimes_R K$. Then there exists $\alpha \in C^*$ such that $P = u^d \alpha$.

Proof: Since the condition $u^d | P$ is stable under base change, it is enough to prove the result where $P$ splits. Let us then write $P = \prod (u - \alpha_i)$ with $\alpha_i \in R$. The annulus $C$ can be described (rigidly) as $\{ u \mid |c| < |u| < 1 \}$. Since $P$ has no roots in $\text{Spec} C \otimes_R K$, any root of $P$ either verifies $|\alpha_i| \leq |c|$ in which case $(u - \alpha_i) = (u - c \frac{\alpha_i}{c}) = u(1 - v \frac{\alpha_i}{c})$ the element $(1 - v \frac{\alpha_i}{c})$ being invertible in $C$, or $|\alpha_i| = 1$ in which case $u - \alpha_i$ is invertible in $C$. This proves the lemma. \(\square\)

The Weierstrass preparation theorem of Henriqo (cf. [Hen01], Lemme 1.6) allows one to generalize in the following direction.

Corollary 4.4 Let $f \in C$ be non zero at the special fiber. Suppose that $f$ has no zeros on the generic fiber. Then $f$ is of the form $u^d \beta$ or $v^d \beta$ with $\beta \in C^*$.

Proof: Direct consequence of the above lemma and [Hen01], Lemme 1.6. \(\square\)

\(\triangleright\) There exists $\alpha \in C^*$ such that $y = v^m \alpha$.

Let us write $dx = h du$, which is possible due to the previous step. Using the fact that $xy = b$ and $uw = c$, we get

$$bdy = hc \left( \frac{y}{v} \right)^2 dv.$$ 

The quantity $\left( \frac{y}{v} \right)$ has a meaning in $C$ due to the first step of the proof.

By hypothesis, the morphism $f$ is étale on the generic fiber, hence $dy$ is a basis of $\Omega_C \otimes_R K$. In particular, $\left( \frac{y}{v} \right)$ has no zero on the generic fiber. By corollary [Hen01], we get that $y$ is of the form $v^m \alpha$ for an $\alpha \in C^*$. Here we use that $y$ is of the form $v^m \alpha$ in the special fiber.

\(\triangleright\) The homomorphism $df$ extends in a unique way to $\omega_B \rightarrow \omega_C$ and there exists $d \in R$ and $m \in \mathbb{N}$, $\beta \in C^*$ such that $\frac{dx}{x} = d u^m \beta \frac{du}{v}$ or $\frac{dx}{x} = d v^m \beta \frac{dv}{v}$.

Since $C$ is integral and $\omega_C \otimes_R K = \Omega_C \otimes_R K$ if the morphism $df$ extends to $\omega_B \rightarrow \omega_C$, this happens in a unique way. The fact that $\frac{dy}{y}$ is a regular element of $\omega_C$ is a formal consequence of the previous step which tells us that $y = v^m \alpha$.

Let us now prove the last assertion. We can write $\frac{dy}{v} = h \frac{dv}{v}$. But by hypothesis, $\frac{dy}{v}$ is a basis of $\Omega_C \otimes_R K$ because $f$ is étale at the generic fiber, hence $h$ has no zero in $\text{Spec} C \otimes_R K$. The corollary [Hen01] yields the result.

\(\triangleright\) The alternative (A) is true.

Assume that $\frac{dy}{y} = d v^m \beta \frac{dv}{v}$ (cf. previous step).

We first consider the case of an equal characteristic ring and write $y = v^m \alpha$. By hypothesis $p | n$, hence we have $\frac{dy}{y} = (d\alpha) \alpha^{-1} = d v^m \frac{dv}{v}$. Therefore we see that $m - 1$ is the vanishing order of $\frac{dy}{y}$. As $R$ is of equal characteristic $p$, $d\alpha$ has not term of order $n$ with $n = -1 \mod p$. Hence $m - 1 \neq -1 \mod p$, which is what we claimed.

Now we continue with the more complicated unequal characteristic case.
For this, we work directly in $R[[v, \frac{w}{v}]]$ and write $y = v^n\alpha$ and $\alpha = \sum_i a_i \left(\frac{w}{v}\right)^i + \sum_j b_j v^j$. As $\frac{dy}{y} = n\frac{dv}{v} + \frac{d\alpha}{\alpha} = dv^m\beta\frac{w}{v}$ we get

$$dv^m\beta\alpha = n + \alpha^{-1}\left(-\sum_i i a_i \left(\frac{w}{v}\right)^i + \sum_j j b_j v^j\right)$$

(4.1)

As $\beta\alpha$ is invertible, we see that if $m = 0$ then $(\varnothing) = (n)$ in $R$.

Suppose that $\nu_p(m) \geq \nu_p(n)$. Looking at the degree 0 term in (4.1) we get that $\varnothing n$ in $R$. But by definition, we know that $(\varnothing) = (mb_m)$ in $R$ hence $m|\varnothing$ and by hypothesis, we find that $(n) = (\varnothing)$. But the order of $\varnothing\left(n + \alpha^{-1}\left(-\sum_i i a_i \left(\frac{w}{v}\right)^i + \sum_j j b_j v^j\right)\right)$ mod $\varnothing$ (which is supposed to be $m$) is then equal to 0. Thus we have $m = 0$.

We have also proved that $m = 0$ if $(\varnothing) = (p^{\nu_p(n)})$. □

### 4.2 Proof of Theorem 4.1

First we are going to choose the different data and prove their uniqueness. Then we will prove that they satisfy the condition of the definition.

▷ Choice and uniqueness of a model of $f$

Let us first recall a definition (cf. [Liu]): a semistable model of $f$ is a finite morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ between semistable proper $R$-curves the generic fiber of which is isomorphic to $f$. Such a model is called a stable model if it is minimal (for the domination relation) between all the semistable models.

As $g(X) \geq 2$, $f$ admits (up to a finite extension of $K$) a stable model $f_0 : X_0 \to Y_0$, cf. [Liu], Corollary 4.6. Up to another finite extension of $K$, we can assume the ramification locus to be composed of rational points $e_i \in X(K)$. Denote by $X_1$ the minimal semistable model obtained from $X_0$ and unfolding the specialisation of the $e_i$ such that the $e_i$ specialize in the smooth locus. For this model to be semistable, we may need another extension of $R$. Then it is easily seen that there exists a model of $Y_1$ dominating $Y_0$ and a finite morphism $f_1 : X_1 \to Y_1$ which extends $f_1$.

Moreover, as this model is minimal, we see that it verifies the condition $\text{ii}$ of the definition 3.12. Indeed, if $V \subset X_0$ is an irreducible component of the special fiber which does not satisfy this condition, then $f(V)$ is an irreducible component of genus 0 which intersects the rest of $Y_0$ in at most 2 points. Hence $V$ and $f(V)$ can be simultaneously contracted and thus define a model of $f$, contradicting the minimality of $f_1$.

Finally, if there exists an admissible covering $f$ which is a model of $f$, then the underlying morphism must be the morphism we just constructed because the condition $\text{ii}$ of the definition 3.6 ensures that the $e_i$ specialize in the smooth locus and condition $\text{ii}$ of the definition 3.6 says that the specialisation of the $e_i$ in the special fiber are distinct.

By definition, for any singular point $p \in X_1$ the restriction of the morphism $f_1$ to the generic fiber of $\text{Spec} \mathcal{O}_{X_1, p}$ is tame; in particular the proposition 4.2 proves that $f_1$ is distended and that there exists a unique morphism $\omega_{Y_1/S} \to f_1^* \omega_{X_1/S}$ extending $df$. 16
\(\triangleright\) Choice and uniqueness of the separating data

Denote by \(D\) the divisor of ramification on \(X\) and by \(\bar{D}\) its closure in \(X_1\). As the \(e_i\) specialize in the smooth locus, \(\bar{D}\) is a Cartier divisor. Denote by \(L = \omega_{X_1/S} \otimes f_!^* \omega_{Y_1/S}(\bar{D})\). Using the homomorphism \(df : f^* \omega_{Y_1/S} \to \omega_{X_1/S}\), we get a natural morphism \(O_{X_1} \to L^{-1}\) and thus \(g : L \to O_{X_1}\) which is an isomorphism at the generic fiber.

In particular, we see that the Weil divisor associated to \(g : L \to O_{X_1}\) is precisely the vertical ramification of the morphism \(f_1\) (and only the vertical part).

By definition, we see that \(df\) factorises through \(\omega_{X_1/S} \otimes L \xrightarrow{\bar{g}} \omega_{X_1/S}\).

Suppose now that there exists an admissible covering \(\bar{f}\) which is a model of \(f\). We have already seen that the underlying morphism is then \(f_1\). Let \((L', g', \delta')\) be a separating data relative to \(\bar{f}\). Then the fact that \(df_1 = g' \circ \delta\), and \(\delta'\) is injective in the special fiber (cf. definition \(3.6\) property \(i\)) and that \(g'\) is an isomorphism in the generic fiber (which comes from the property \(i\) of the definition \(3.8\)) implies that the Weil divisor associated to \(g' : L' \to O_{X_1}\) is the vertical ramification of the morphism \(f_1\). Hence we see that \(g\) and \(g'\) are isomorphic. As \(X_1\) is a semi-stable proper curve on \(R\), there exists an \(\alpha \in R^* = H^0(X_1, O_{X_1})\) such that \(\alpha(L, g, \delta) = (L', g', \delta')\). This proves the uniqueness.

\(\triangleright\) Choice and uniqueness of the Hurwitz graph

For any geometric point \(\bar{s} \to S\) and for any singular point \(p \in X_{1, \bar{s}}\), the proposition \(\mathfrak{a}\) gives a function \(m\). With the notation of this proposition, if \(\frac{dv}{v} = v^{-m} \delta\left(\frac{dv}{v}\right)\) and \(e\) is the edge at \(p\) corresponding to the branch \(v\) then \(m(e) = m\) and \(m(\bar{e}) = -m\). The properties \(\mathfrak{b}\) and \(\mathfrak{v}\) of definition \(3.8\) ensure the uniqueness of \(m\).

Moreover, we can define an orientation on the graph so that it is compatible with \(m\). This orientation is unique only at the vertices where \(m \neq 0\).

Let us now define the function \(r\). If \(R\) is of equal characteristic, and \(V\) is an irreducible component of the smooth locus of \(X\), we define \(r(V)\) to be 0 if \(g|_V\) is invertible and \(r(V) = \infty\) otherwise. In this case it is easily seen that \(r\) is then uniquely determined.

Let us assume that \(R\) is of unequal characteristic. Let \(\eta\) be the generic point of an irreducible component \(V\) of the special fiber of \(X_1\). Denote by \(\varpi\) a uniformising parameter of \(R\). As the special fiber of \(X_1\) is reduced, \(\varpi\) is a uniformising parameter of the discrete valuation ring \(O_{X_1, \eta}\). Thus, there exists an element \(\delta_V \in R\) (the different of the extension) which is a generating parameter of the ideal \(L_\eta \subset O_{X_1, \eta}\). Denote by \(\nu_{\varpi}\) a valuation on \(R\). We define \(r(V) := \frac{\nu_{\varpi}(\delta_V)}{\nu_{\varpi}(p)} \in \mathbb{Q}_+\). The property \(\mathfrak{b}\) of definition \(3.8\) ensures the uniqueness of \(r\) because \(\delta\) has to be injective and \(g\delta = df\).

Now we only need to check the properties required for \((f_1, m, r, L, g, \delta)\) to be an unfolded \(\varpi\)-morphism.

\(\triangleright\) The data \((m, r)\) define a Hurwitz graph

We only have to check properties \(\mathfrak{a} \mathfrak{b}\) and \(\mathfrak{v}\) of definition \(3.1\). Concerning property \(\mathfrak{a}\), this comes from the fact that we are looking at morphisms of degree
of at most \( p \) and that \( r(V) \) is defined to be the different.

Property 2 comes from the explicit description of the data in a neighbourhood of a singular point given by proposition 1.2.

The last property comes from the alternative (A) of proposition 4.2 using the fact that the degree of the morphism is \( \leq p \).

The data \((L, g, \delta)\) is an unfolded separating data.

Let us prove first that \( L \) is ftsl. For that, we can restrict ourselves to an open smooth irreducible formal subscheme \( W \) of the formal completion of \( X_1 \). Let \( \eta \) be its generic point. Via the morphism \( g \), we can look at \( L_\eta \) as an ideal of \( O_{W,\eta} \). As we have seen before, denoting \( \varpi \) a uniformising parameter of \( R \), \( \varpi \) is a uniformising parameter or \( O_{W,\eta} \). Thus there exists \( \ell \in \mathbb{N} \) such that \( L_\eta = \varpi^\ell O_{W,\eta} \). It is then easy to prove that \( L_W = \varpi^\ell O_W \) (via \( g \)) because both Weil divisors are equal at each point of codimension 1 (they are both trivial at the generic fiber) and \( U \) is normal.

The fact that \( g \circ \delta = df \) comes from the definition of these objects. Moreover, again from the definition, we see that \( \delta \) is injective.

The homomorphism \( f^*_U \delta \) induces an isomorphism in a neighborhood of each singular point due to the explicit description of proposition 1.2. Moreover, as we have taken a model that splits the specialization of the ramification locus, it follows the fact that \( \text{coker} \, \delta \) is of the form prescribed by the property 3 of definition 3.6. The bound appearing in the definition comes from the bound in the hypothesis of our morphism.

Let us prove now that \( g \) is not zero at the special fiber. For that, assume the contrary. Then we get in particular that \( df = 0 \) at the special fiber. As the morphism is of degree \( p \), it follows that \( f \) is an homeomorphism. Thus \( g(X) = g(Y) \) but the Hurwitz formula tells us that \( 2g(X) - 2 - p(2g(Y) - 2) \geq 0 \), which means that \((1 - p)(2g(X) - 2) \geq 0 \). This is absurd because \( p \geq 2 \) and \( g(X) \geq 2 \).

The data \((f_1, m, r, L, g, \delta)\) is an unfolded \( \mathfrak{d} \)-morphism.

First let us take for \( \phi_U \) the trivialisation of \( L \) on the smooth locus obtained above such that \( g \circ \phi_U^{-1} \) is given by an element of \( R \). Up to a finite extension of \( R \) it is easy to find trivialisation of \( L_p \) for each singular point \( p \) such that the conditions 1 and 2 of definition 3.8 are satisfied. Simply look at the definition of \((L, g, \delta)\) in a neighborhood of \( p \) using the local description given by the proposition 1.2. This procedure may require a change of the set of coordinate, thus leading to the choice of \( u_o \) and \( u_t \).

Property 3 of definition 3.8 follows directly from proposition 1.2.

Finally, let \( V \subset \hat{X}_1 \) be an open connected formal subscheme and \( \mathfrak{d} \) be a generator of \( L|_V \subset O_V \) which is in \( R \). Suppose that \( R \) is of unequal characteristic, then up to the multiplication by an invertible element of \( R \) we can assume that \( \mathfrak{d} = p^r(V) \). We see that, with respect to the trivialisation \( \phi_U \), we have \( \delta = \frac{df}{\mathfrak{d}} \) and this implies \( \delta \) being \( p^r(V) \)-earnest.

If \( R \) is of equal characteristic, the corresponding property is proved in the same way.

The data \((f_1, m, r, L, g, \delta)\) define an admissible covering.
We already saw that \((f_1, L, \delta)\) satisfies the property ii) of definition 3.12. It is thus enough to look at the other condition.

Thanks to remark 3.3, it is enough to construct an “increasing” (relatively to the orientation) function on the graph of the special fiber.

Choose a valuation \(\nu_\omega\) on \(R\). Then for any irreducible component of the special fiber of \(X_1\) denote by \(\delta_Y \in R\) the element corresponding to \((g \circ \phi^{-1})|_{Y}\). Then define \(\ell(V) := \nu_\omega(\delta_Y)\). Proposition 4.2 shows that the function \(\ell\) has the desired property. This proposition describes precisely how the different changes going from one branch to the other.

5 Deformation Theory of \(\mathcal{d}\)-morphisms

In this section, we want to prove the prorepresentability of the functor of deformations of admissible covers. More precisely, let \(k\) be a field and \(f_0\) be an admissible covering. If \(f_0\) is an infinite covering, denote by \(A\) the category of artinian local \(k\)-algebras with residue field \(k\), if \(f_0\) is finite, denote by \(A\) the category of artinian local rings whose residue field is \(k\). Then we can define a functor \(D_{adm}: A \rightarrow \text{Set}\) by

\[
D_{adm}(A) = \{\text{admissible covers } f \text{ over } A \text{ with special fiber } f_0\}/\text{isom}.
\]

The aim of this section is to prove the following theorem.

**Theorem 5.1** Suppose that \(p \neq 2\), then the functor \(D_{adm}\) has a universal deformation.

The proof of this will be decomposed into several steps which will give more information about this functor. In particular, we will get a description of its singularities.

The first step is to prove that there is no infinitesimal automorphism. This is done in the following lemma.

**Lemma 5.2** Suppose \(p \neq 2\). Let \(A' \rightarrow A\) be a small extension of artinian local rings with residue field \(k\) and kernel \(a\) and \(f': X' \rightarrow Y'\) a deformation of \(f_0\). Then \(f'\) has no non trivial automorphism as lifting of \(f' \otimes_{A'} A\).

**Proof:** Let \(\sigma\) be an automorphism of \(f'\). Then \(\sigma\) can be decomposed in \((\sigma_X, \sigma_Y, \sigma_L)\) where \(\sigma_X\) is an automorphism of \(X\), \(\sigma_Y\) is an automorphism of \(Y\) and \(\sigma_L\) is an automorphism of \(L\), satisfying compatibility conditions (in particular, \(\sigma\) should fix \(\delta\)). As \(X_0\) is proper and semi-stable, \(\sigma_L\) can be thought of as an element of \(1 + a\). Using usual deformation theory of local complete intersection morphisms, one sees that \(\sigma_X\) and \(\sigma_Y\) are given by vector fields \(\chi_X\) and \(\chi_Y\) defined on the special fiber. Let \(V\) be an irreducible component of \(X\). If \(V\) is of genus \(\geq 1\), we see that \(\chi_X\) must be zero. This implies that \(\chi_Y = 0\). If \(f\) is inseparable this comes from the fact that \(f(V)\) is also of genus \(\geq 1\). If \(f\) is separable this comes from the compatibility condition as expressed in theorem A.1.
Thus we can assume $V$ to be of genus 0. Assume first that $f|_V$ is of degree $< p$. As $\sigma$ must preserve $\delta$, we see that $\chi_X$ must have a zero at each point of the support of $\text{coker} \, \delta_0$. Thus by the condition on the number of points of this support, we get that $\chi_X = 0$. As before, we deduce from this that $\chi_Y = 0$ because $f$ is separable.

Suppose now that $V$ is of genus 0 and that $f|_V$ is of degree $p$. Then two cases can occur: either $f|_V$ is separable or inseparable. If $f|_V$ is inseparable, it is easy to see (for example by the theorem [A.1]) that $\chi_X = 0$. Moreover, if $V$ meets the rest of $X_0$ in at least 3 points, then it is a classical result that $\chi_Y = 0$. Thus we can suppose that $V$ intersects the rest of $X_0$ in less than 3 points.

In particular, as $f|_0$ is admissible, the support of $\text{coker} \, \delta_0$ is not empty. Let us choose an isomorphism $V \rightarrow \mathbb{P}^1_k$ such that the point 0 is a point where $V$ meets the rest of $X_0$ and the point at infinity is not a point of the support of $\text{coker} \, \delta_0$. Write $V \setminus \{\infty\} = \text{Spec} k[x]$ such that $x = 0$ is a point where $V$ meets the rest of $X_0$ and write $\chi_Y = (a_0 + a_1 x + a_2 x^2) \frac{\partial}{\partial x}$. As $X_0$ has a singular point at $x = 0$, $\chi_Y$ must be zero at 0. Hence $a_0 = 0$. Then writing that $\sigma_Y$ fixes $\delta$, one get $a_1 = 2a_2 = 0$ (by writing things explicitly). That is, $\chi_Y = 0$.

If $f|_V$ is separable of degree $p$, then the result follows by analysing the different cases. For example, suppose that $V$ meets the rest of $X_0$ in only one point. Then by the Hurwitz formula and the fact that $\text{coker} \, \delta$ has its support in at least 2 points one gets the existence of at least one point in the support of $\text{coker} \, \delta$ at which $f$ is of degree $< p$. In particular, $\chi_X$ must have a zero at this point. Suppose that $f$ is of degree $p$ at another point (otherwise the result follows trivially). Using the compatibility of $\sigma$ and $\delta$ one gets that $\chi_X = -\chi_Y$. But the compatibility with $f$ says that $\chi_X = \chi_Y$ thus (as $p \neq 2$) one has $\chi_Y = \chi_X = 0$.

A problem that arises when studying this functor is the need to speak about $p^{r(e)}$-earnest morphisms and it is not easy to describe these morphisms if $p^{r(e)}$ is not in the considered ring. To bypass this problem, we can proceed as follows. Let $k$ be a field and $f|_0$ be a finite $\delta$-morphism (we don’t need $f|_0$ to be an admissible covering). Let $r_0 \in \mathbb{Q}_+$ such that all the $r(e)$ ($e \in \text{Som}(\Gamma)$) are integer multiples of $r_0$. Denote by $\mathfrak{A}_{r_0}$ the category of $\mathbb{Z}_p[p^{r_0}]$-algebras which are local artinian and with residue field $k$.

If $f|_0$ is an infinite $\delta$-morphism, we define $r_0 = \infty$ and $\mathfrak{A}_{r_0}$ to be the category of artinian $k$-algebras with residue field $k$.

Then consider the functor $D_0 : \mathfrak{A}_{r_0} \rightarrow \text{Set}$ defined by

$$D_0(A) = \{\delta\text{-morphisms } f \text{ over } A \text{ with special fiber } f|_0\}/\text{isom}.$$ 

**Theorem 5.3** The functor $D_0$ admits a formal versal deformation. Moreover, if $p \neq 2$ and $f|_0$ is an admissible covering, then $D_0$ admits a universal deformation.

Theorem 5.3 then follows by descent theory and the fact that $D_0$ has a universal deformation in the admissible covering case, which follows from the lemma above.
To study $D_\varnothing$, we are going to define a functor $D_{\varnothing-\text{abs}} : \mathfrak{M}_0 \to \text{Set}$ endowed with a natural morphism $\text{abs} : D_\varnothing \to D_{\varnothing-\text{abs}}$. We then prove that the functor $D_{\varnothing-\text{abs}}$ has a versal deformation by exhibiting it, which allows us to precise its geometry. The last subsection is devoted to the morphism $\text{abs}$. Namely we prove that it admits a relative formal versal deformation using Schlessinger’s criterion adapted to the relative situation and deformation theory. A step of the proof is the formal smoothness of $D_{\varnothing-\text{abs}}$.

5.1 Versality of the functor $D_{\varnothing-\text{abs}}$

We define at this point the functor $D_{\varnothing-\text{abs}}$. Roughly speaking, it encodes the information about the ftsl sheaf $L$, the homomorphism $g$ and the singularities of $X$. If the source $X$ of the $\varnothing$-morphism $f$ is smooth, then we know that $L$ is trivial and $g$ is an isomorphism. In particular, we can assume that $L = \mathcal{O}_X$ and $g = \text{Id}$. Thus from now on we will assume $X$ to be not smooth. In particular, the scheme $U := X \setminus \text{Sing}(X)$ is affine.

Before going into the definition, we need a result which will enable us to eliminate $L$. This lemma will be a corollary of a formal patching result of which we need only a particular case.

**Theorem 5.4** Let $A$ be a noetherian complete local ring, $X \to \text{Spec} A$ a proper semi-stable curve. Denote by $\mathfrak{M}$ the category of invertible sheaves on $X$, $U$ the smooth locus of $X \to \text{Spec} A$, $\mathfrak{M}^\circ$ the category composed by the triples $(L_U, (L_p)_{p \in \text{Sing}(X)}, (\phi_p)_{p \in \text{Sing}(X)})$ where $L_U$ is an invertible $\mathcal{O}_U$-module, $L_p$ is an invertible $\hat{\mathcal{O}}_{X,p}$-module and $\phi_p : (L_U \otimes \mathcal{O}_U \text{Frac}\hat{\mathcal{O}}_{X,p}) \to (L_p \otimes \hat{\mathcal{O}}_{X,p} \text{Frac}\hat{\mathcal{O}}_{X,p})$ is an isomorphism of $\text{Frac}\hat{\mathcal{O}}_{X,p}$-module (the isomorphism of such triples being defined in an obvious way). Then the canonical morphism

$$L \mapsto \left( L|_U, (\hat{L}_p), (L|_U \otimes \mathcal{O}_U \text{Frac}\hat{\mathcal{O}}_{X,p}) \xrightarrow{L|_U} \hat{L}_p \otimes \hat{\mathcal{O}}_{X,p} \text{Frac}\hat{\mathcal{O}}_{X,p} \right)$$

is an equivalence of category.

**Proof:** The proof is essentially similar to the one in [HS99] Theorem 1 but we need to remove the hypothesis that $R$ is a ring of formal power series.

More precisely, denote by $i_U : U \to X$ and $i_p : p \to X$, then the result relies on the exact sequence

$$0 \to L \to i_{U*}L|_U \oplus \bigoplus_p i_{p*}\hat{L}_p \to \bigoplus_p i_{p*}\left( \hat{L}_p \otimes \hat{\mathcal{O}}_{X,p} \text{Frac}\hat{\mathcal{O}}_{X,p} \right) \to 0$$

For a proof of the exactness of this sequence over a complete local ring, see for example [Mau03], Lemmes 4.5 and 4.6. □

Let $f_1$ and $f_2$ be two deformations of $f_0$ over a base $A$ with isomorphic sources $X$. Denote by $L_1$ and $L_2$ the underlying ftsl invertible sheaves and chose a trivialisation as in definition $\S\S$ At each singular point $p$ there exists sets of coordinates $(u_1, v_1)$ and $(u_2, v_2)$ such that the transition functions are of the form prescribed by property $(\S\S\S)$ of definition $\S\S$ with the same $m$ since
it depends only on the Hurwitz graph. In particular, if we suppose that \( k \) is algebraically closed, there exists an automorphism of \( \hat{O}_{X,p} \) which sends \( u_1^m \) to \( u_2^m \) and \( v_1^m \) to \( v_2^m \) because \( m \) is prime to the characteristic of \( k \) or equal to 0, in which case we have nothing to do. Then the formal patching theorem above shows that \( L_1 \) and \( L_2 \) are isomorphic. Let’s denote it by \( L \).

Let’s chose, for each singular point \( p \in X \), a set of coordinate \( (u_o,p,u_t,p) \) and denote by \( \varpi_p = u_o,pu_t,p \in A \) the thickness. As \( L \) is ftsl, we have \( L|_U \cong \mathcal{O}_U \) and the theorem above yields an identification

\[
\text{Hom}_{\mathcal{O}_X}(L,\mathcal{O}_X) \cong \left\{ (g_U,g_p)|g_U \in \mathcal{O}_U, g_p \in \hat{O}_{X,p} \text{ satisfying } \forall b \in \{o,t\} \right. \\
\left. \forall p \in \text{Sing}(X) \ g_U = g_p u_{b,p}^{m(b(p))} \text{ in } \hat{O}_{X,p}[u_{b,p}^{-1}] \right\}.
\]

Denote by \( \Gamma \) the dual graph of the special fiber of \( X \) and let us take an element \( (g_U,g_p) \) of the right hand side set. Then \( g_U \) can naturally be decomposed into \( (g_v)_{v \in \text{Som}(\Gamma)} \).

Then we see that an element \( (g_U,g_p) \) defines an element of \( \text{Hom}_{\mathcal{O}_X}(L,\mathcal{O}_X) \) by the above identification if and only if for all \( p \in \text{Ar}^+(\Gamma) \) we have \( g_o(p)\varpi^{m(p)} = g_t(p) \) and \( g_p = g_o(p)u_{o(p)}^{m(p)} \).

Thus we can define a bijection

\[
\text{Hom}_{\mathcal{O}_X}(L,\mathcal{O}_X) \cong \left\{ (g_v)_{v \in \text{Som}(\Gamma)} | g_v \in \mathcal{O}_{\text{U}^c}, \forall p \in \text{Ar}^+(\Gamma) \ g_o(p)\varpi^{m(p)} = g_t(p) \right\}
\]

i.e. we can “forget” the information about \( g_p \) without actually forgetting anything.

Let us now define a functor \( D_{\mathcal{L}-\text{hom}} : \mathfrak{A}_m \to \text{Set} \). For any \( A \in \mathfrak{A}_m \), the set \( D_{\mathcal{L}-\text{hom}}(A) \) is composed by the isomorphism classes of triples \((U,(X_p),(g_v))\) where \( U \) is a deformation of \( U := X \setminus X_{\text{sing}} \), \( X_p \) is a deformation of \( \text{Spec}\hat{O}_{X,p} \) and \((g_v)\) is an element of the set \( \text{Hom}_{\mathcal{O}_X}(L,\mathcal{O}_X) \) as described above which is a deformation of the morphism \( f \) defined by \( f \).

Let’s define a subfunctor \( D_{\mathcal{L},g} \) of \( D_{\mathcal{L}-\text{hom}} \). If \( X \) is smooth, then \( D_{\mathcal{L},g} \) is trivial. If \( X \) is not smooth then for all singular point \( p \in X \), fix a “universal” thickness of the versal deformation of \( \text{Spec}\hat{O}_{X,p} \) i.e. an isomorphism between the versal deformation and \( W(k)[[\varpi_p]] \) and set \( D_{\mathcal{L},g}(A) = (U_0 \times \text{Spec }A, X_p, (g_v)) \) with \( g_v \in A \) for all \( A \) satisfying the conditions

\[
\begin{cases}
\text{if } r(v) < \infty \text{ then } g_v = p^{r(v)} \\
\forall p \in \text{Ar}^+(\Gamma) \ g_o(p)\varpi^{m(p)} = g_t(p).
\end{cases}
\]

Then, by definition of an unfolded \( \mathfrak{d} \)-morphism the canonical morphism \( D_0 \to D_{\mathcal{L}-\text{hom}} \) factorises through \( D_{\mathcal{L},g} \to D_{\mathcal{L}-\text{hom}} \) thus giving rise to a morphism \( D_0 \to D_{\mathcal{L},g} \).

**Proposition 5.5** The functor \( D_{\mathcal{L},g} \) admits a versal deformation.

**Proof**: This is clear if \( X \) is smooth. If \( X \) is not smooth (in which case \( U \) is affine) then all the deformations of \( U \) are trivial. Hence it’s easily seen that a versal deformation of \( D_{\mathcal{L},g} \) is given by the completion at “\( g_v \)” of

\[
W(k)[p^{r_o}][[(\varpi_p)]][g_v])/I,
\]

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where $I$ is given by the conditions above.

Moreover, we see that the singularities are explicit and depend only on the geometry of the reduced Hurwitz graph.

Finally let us choose, for any singular point $p$, a thickness $\varpi_p$ of the universal deformation of $\text{Spec} \hat{O}_{X,p}$ and denote by $n_p$ the degree of $f_0$ at $p$. Then define a subfunctor $D_{abs}$ of $D_{\mathcal{L},g}$ by the condition that $\forall \ p, p'$ such that $f_0(p) = f_0(p')$ one has $\varpi_p^{n_p} = \varpi_p'^{n_p'}$.

Then we immediately see that $D_d \to D_{\mathcal{L},g}$ factorises through $D_{abs}$ giving rise to a morphism $\text{abs} : D_d \to D_{abs}$, and that $D_{abs}$ has a versal deformation.

5.2 Versality of the morphism $\text{abs}$

The aim of this subsection is to prove that the natural map $\text{abs}$ defined in the preceding subsection admits a relative formal versal deformation. The principal tool here is Schlessinger’s deformation theory (cf. [Sch68]). Although the original article does not deal with the relative case but only the absolute, the definitions and results can be generalised by the same methods.

It’s therefore sufficient to prove the following theorem.

**Theorem 5.6** Let $k$ be an algebraically closed field, $f$ a $\mathfrak{d}$-morphism. Then there exists finite dimensional $k$-vector spaces $T^0$ and $T^1$ such that for all $A \in \mathfrak{A}_{r_0}$, any small extension $A' \to A$ in $\mathfrak{A}_{r_0}$ of kernel $\mathfrak{a}$, any deformation $f_0$ of $\underline{f}$ over $A$ and any $g' \in D_{\mathcal{L},g}(A')$ whose image in $D_{\mathcal{L},g}(A)$ is $\text{abs}(\underline{f})$, the following properties are true

i) there exists a lifting $f'$ of $\underline{f}$ over $A'$ such that $\text{abs}(\underline{f'}) = g'$;

ii) the set of isomorphism classes of liftings of $\underline{f}$ to $A'$ with image $g'$ is endowed with a canonical action of $\mathfrak{a} \otimes_k T^1$ making it a principal homogeneous space;

iii) the group of isomorphisms of a lifting of $\underline{f}$ to $A'$ is canonically isomorphic to $\mathfrak{a} \otimes T^0$.

In particular, by the first property, $\text{abs}$ is formally smooth.

We are going to prove this theorem without the assumption that the underlying curves are proper. Thus we will need to remove, at first, the finiteness of the vector spaces. The condition on the image of the deformation by $\text{abs}$ still makes sense since it is equivalent to conditions on $\mathcal{L}$, $g$ and conditions on the deformation of the double points.

Let us first take care of the automorphism group of a lifting (third part of the theorem above). For any invertible sheaf $\mathcal{F}_0$ on $X_0$ and any section $s_0 \in H^0(X_0, \mathcal{F}_0)$ denote by

$$T^0_{X_0, \mathcal{F}_0, s_0} := \ker \left( \text{Ext}_{\mathcal{O}_X}^0(\mathcal{P}^1_{X_0/k}(\mathcal{F}_0), \mathcal{F}_0) \xrightarrow{d^1_{s_0}} H^0(X_0, \mathcal{F}_0) \right)$$

This sheaf classifies the automorphism of a deformation of $(X_0, \mathcal{F}_0, s_0)$ (see proposition [A.3]). In particular, we have a canonical morphism, the forgetful
functor \((X_0, F_0, s_0) \mapsto X_0,\)

\[ T_{X_0,F_0,s_0}^0 \to \text{Ext}^0_{O_{X_0}}(\Omega_{X_0/k}, O_{X_0}). \]

Define the space \(T_{X_0,L_0,g_0,\delta_0}^0\) by the following Cartesian diagram:

\[ \begin{array}{ccc}
T_{X_0,L_0,g_0,\delta_0}^0 & \to & T_{X_0,L_0^{-1},g_0}^0 \\
\downarrow & & \downarrow \\
T_{X_0,f^*\omega_{Y_0}^{-1}(k) \otimes \omega_{X_0/k} \otimes L_0,\delta_0}^0 & \to & \text{Ext}^0_{O_{X_0}}(\Omega_{X_0/k}, O_{X_0}).
\end{array} \]

We see that it classifies the automorphism of a lifting of \((X_0, L_0, g_0, \delta_0)\). Moreover, we have a canonical morphism, the forgetful morphism \((X_0, L_0, g_0, \delta_0) \mapsto \text{Ext}^0_{O_{X_0}}(\Omega_{X_0/k}, O_{X_0}).\)

Then define \(T_{L_0}^0\) by the following Cartesian diagram:

\[ \begin{array}{ccc}
T_{L_0}^0 & \to & T_{X_0,L_0,g_0,\delta_0}^0 \times \text{Ext}^0_{O_{Y_0}}(\Omega_{Y_0/k}, O_{Y_0}) \\
\downarrow & & \downarrow \\
\text{Ext}^0(df_0, f_0) & \to & \text{Ext}^0_{O_{X_0}}(\Omega_{X_0/k}, O_{X_0}) \times_k \text{Ext}^0_{O_{Y_0}}(\Omega_{Y_0/k}, O_{Y_0}).
\end{array} \]

Theorem A.1 explains the bottom row. It is then easily seen by construction that \(T_{L_0}^0\) satisfies the desired property. Moreover, through the explicit description above we see that if \(X_0\) and \(Y_0\) are proper, then \(T_{L_0}^0\) is finite dimensional.

We will denote by \(\mathcal{P}_{L_0}\) the sheaf on \(Y_0\) defined for all \(U\) by \(T_{L_0}^0|_{f^{-1}(U)}\).

We turn now to the two other parts of the theorem. To prove those, we’re going to prove that in each of the following cases, there exists a deformation which is unique up to isomorphism:

1. the affine smooth \(U \subset X_0\) such that \(f_0|_U\) is separable;

2. the affine smooth \(U \subset X_0\) such that \(f_0|_U\) is purely inseparable (as \(f_0\) is of degree \(p\), only those two cases can occur in the smooth case);

3. the singular locus.

Then as usual the global result follows defining \(T_{L_0}^1 = H^1(Y_0, \mathcal{P}_{L_0}).\)

The first point above is classical and can easily be deduced from the usual deformation theory of morphisms as stated in theorem A.1. Hence we’ll focus on the other two cases.
5.2.1 Deformation of the inseparable smooth locus

First, by corollary 2.9 there exists a lifting (at least locally on $Y$). Here we assume $X_0$ and $Y_0$ to be smooth, affine, irreducible and that $\Omega_{Y_0/k}$ admits a basis which is exact. Let $f'_1$ and $f'_2$ be two liftings of $f$ with the same image under $\text{abs}$. We are going to prove that they are isomorphic. Considering the deformation theory of smooth affine pointed curves (which is trivial) we see that we can set $Y' := Y'_1 = Y'_2$, $X' := X'_1 = X'_2$ and that $\text{coker} \delta_1 = \text{coker} \delta_2$.

Consider $g'_1$ and $g'_2$ as elements of $A'$; we can do so because $X_0$ is smooth and irreducible. By hypothesis, $\text{abs}(f'_1) = \text{abs}(f'_2)$ hence $g' := g'_1 = g'_2$. Moreover, as $f_0$ is inseparable the element $g'$ is in the maximal ideal of $A'$. Hence $df'_1 = g' \delta_1 = g' \delta_2 = df'_2$. Then proposition A.5 tells us that $f'_1 = f'_2$ up to an isomorphism of $Y'$.

What is left to prove is that $\delta_1 = \delta_2$. This will be up to an isomorphism of $X'$. Denote by $\varepsilon$ a generator of the ideal $a$ and $r$ the exactness degree of $\delta_0$.

First, it is convenient to recall how an infinitesimal automorphism of $X'$ acts on $\delta_1$. For this, suppose that $\Omega_{X'/A'}$ admits a basis of the form $du$ and denote by $\partial_{\alpha} = \partial_{u^\alpha}$ the dual basis of $du$. Then any infinitesimal automorphism $\chi$ is given by a vector field of the form $\chi = \varepsilon \alpha \partial_{\alpha}$. Suppose moreover that $\Omega_{Y'/A'}$ admits a basis of the form $dx$ and write $\delta_1(dx) = \beta du$. Then it is easy to see that

$$(\chi \delta_1)(dx) = \chi(\beta du) = \delta_1(dx) + \varepsilon d(\alpha \beta).$$

As $\text{coker} \delta_1 = \text{coker} \delta_2$ we see that $\delta_1(dx) = \gamma \delta_2(dx)$ because they have the same zeros with $\gamma = 1 + \varepsilon \zeta$ (because $\delta_1$ and $\delta_2$ are liftings of $\delta$). Therefore we get $\delta_1(dx) = \delta_2(dx) + \varepsilon \zeta \delta_0(dx)$. As $\delta_1$ and $\delta_2$ are earnest $\beta \delta_0(dx)$ is at least locally exact; to see this, prove it after completion and algebraise with the lemma 2.10. Thus we can write $\beta \delta_0(dx)$ as $dh$. Moreover, after eventually shrinking $X'$, we can choose $h$ such that for all $p \in X'$ one has $\text{ord}_p h \geq \text{ord}_p \delta_0(dx)$. Denote by $\beta_0$ the reduction of $\beta$ modulo the maximal ideal of $A'$ so that $\delta_0(dx) = \beta_0 du$ and write $\alpha := \frac{h}{\delta_0}$. Then $\alpha$ is regular and the automorphism defined by $\varepsilon \alpha \partial_{\alpha}$ sends $\delta_1(dx)$ to $\delta_2(dx)$. As $dx$ is a basis, it sends $\delta_1$ to $\delta_2$.

5.2.2 Deformation of the singular locus

Through descent theory, we are reduced to the case of morphisms $A[[x, y]]/(xy - a) \to A[[u, v]]/(uv - b)$ and by hypothesis this morphism is of the form $x \mapsto u \alpha$, $y \mapsto v^d \alpha^{-1}$ with $\alpha \in A[[u, v]]/(uv - b)$ invertible. Denote by $\beta \in A[[u, v]]/(uv - b)$ the invertible such that $\beta \frac{du}{u} = \delta \left( \frac{dx}{x} \right)$. We’ll suppose, to fix the ideas, that $g$ is of the form $\partial u^m$. Since we are looking at elements with fixed image under $\text{abs}$, the data $\alpha'$ and the lifting $\beta'$ of $b$ (the thickness of the lifting) are fixed. Then we are looking for the liftings of $\alpha$ and $\beta$ satisfying $d = g \delta$. Denote by $\gamma := \alpha \beta$, and by $r_\alpha$ and $r_\beta$ the degree of earnestness of $\delta$ on the origin (say $(x)$) and the terminal (say $(y)$) branches.

Let us write $\alpha = \sum_{i \geq 0} \zeta_i u^i + \sum_{j > 0} \eta_j v^j$. Then it is easy to see through an explicit computation that $d = g \delta$ if and only if

$$\sum_{i \geq 0} \zeta_i (p + i) u^i + \sum_{j > 0} \eta_j (p - j) v^j = \partial u^m \gamma.$$
Using the fact that \((u^m \delta)_{x \neq 0}\) is \(p^r\)-earnest and \((v^{-m} \delta)_{y \neq 0}\) is \(p^r\)-earnest, it is easy to see that there exists a lifting of the morphism and of \(\delta\) which are compatible.

Although having to consider several cases, uniqueness of the lifting can then be proved similar to the uniqueness in the smooth case.

6 Properties of the moduli space of admissible covering

In this section, we finally introduce the moduli space of admissible covers, prove that it is representable by an algebraic stack and that it is proper.

Let \(g, g', p \in \mathbb{N}\) with \(g \geq 2\) and \(p\) prime be distinct from \(2\). Denote by \(\mathcal{H}^c_{g, g', p}\) the groupoid over \(\mathbb{Z}_p\) classifying the admissible covers of degree \(p\) between curves of genus \(g\) and \(g'\). By the usual descent theory, it is easy to see that \(\mathcal{H}^c_{g, g', p}\) is actually a \(\mathbb{Z}_p\)-stack.

Our main theorem is the following

**Theorem 6.1** The stack \(\mathcal{H}^c_{g, g', p}\) is an algebraic Deligne-Mumford stack, contains \(\mathcal{H}^c_{g, g', p}\) as an open dense substack, and is proper over \(\mathbb{Z}_p\).

**Proof**: The fact that the diagonal morphism

\[
\Delta : \mathcal{H}^c_{g, g', p} \to \mathcal{H}^c_{g, g', p} \times_{\mathbb{Z}_p} \mathcal{H}^c_{g, g', p}
\]

is representable, separated and quasi-compact follows from the usual properties of \(\text{Isom}\) schemes of curves.

Moreover we can prove by usual techniques that \(\mathcal{H}^c_{g, g', p}\) is locally of finite presentation. Then theorem 5.1 together with Artin’s algebrasation theorem (see for example [LMB00], Corollaire 10.11) shows that \(\mathcal{H}^c_{g, g', p}\) is an algebraic stack. Moreover, by the fact that an admissible covering has no infinitesimal automorphism (cf. lemma 5.2 and the general deformation theory) one gets that the diagonal morphism \(\Delta\) above is non ramified and \(\mathcal{H}^c_{g, g', p}\) is Deligne-Mumford.

It is easy to see that \(\mathcal{H}^c_{g, g', p}\) is open in \(\mathcal{H}^c_{g, g', p}'\). We now prove its density. Let \(k\) be a field and \(f_0\) an admissible covering over \(k\). We have to prove that \(f_0\) can be deformed to a morphism between smooth curves. In fact, by definition of an unfolded separating data which tells us that the different \(g\) is zero over no point, it is enough to prove that \(f_0\) can be deformed into a morphism between smooth curves. This morphism will automatically be separable. The problem for \(k\) of characteristic 0 is easy to solve, we assume \(k\) to be of characteristic \(p\).

By the formal smoothness of the functor \(\text{abs}\) (cf. theorem 5.6), it is enough to see that there exists a lifting of \(\text{abs}(f_0)\) over a complete discrete valuation ring \(R\) such that the thickness at each point is non zero. If \(f_0\) is finite, then choose a ring \(R\) of unequal characteristic with residue field \(k\), if \(R\) is infinite then choose \(R = k[[t]]\). As there is, up to an extension of \(R\) no problem to deform the tame part of \(\text{abs}(f_0)\) (that is, the singular points where \(f\) is of degree < \(p\)) we can focus on the wild part. As stated after proposition 5.3, the
problem depends only on the reduced Hurwitz graph, which we know is good by definition of an admissible cover. Suppose first that $f_0$ is infinite (i.e. $R$ is of equal characteristic). In particular, due to remark 3.3 there exists a function $\ell : \text{Som}(\Gamma_{\text{red}}) \to \mathbb{N}$ which verifies $\ell(t(e)) > \ell(o(e))$. Moreover, we can suppose that $\ell$ is zero at the minimal points. Those are precisely the points where $g$ is invertible. Let $\pi \in R$ be an element with positive valuation. Then define, for all $v \in \text{Som}(\Gamma_{\text{red}})$, $g_v := \pi^{\ell(v)}$. Denote by $m_0$ the gcd of all the $m(e)$ for $e$ an edge of the reduced Hurwitz graph. Then, up to an extension of $R$, we can suppose that there exists $\pi' \in R$ such that $\pi^{m_0} = \pi$. Then for any edge $e$ of the reduced Hurwitz graph, chose $\pi^{m(e)}$ for the thickness. It is then easily seen that it defines a lifting of $\text{abs}(f_0)$.

In the finite case, repeat the procedure with $\ell = r$.

What is left to prove is that $\mathcal{H}_{g,g',p}^c$ is proper over $\mathbb{Z}_p$. For this, we want to use the valuative criterion of properness. As in [DM69] (remarque after the theorem 4.19, see also [LM100], Remarque 7.12.4) it is enough to prove that for any discrete valuation ring $R$ with quotient field $K$ and any morphism $\text{Spec}K \to \mathcal{H}_{g,g',p}^c$, this morphism can be prolonged up to a finite extension of $K$ in a unique way to a morphism $\text{Spec}R \to \mathcal{H}_{g,g',p}^c$. This version of the valuative criterion for properness is valid here because we have just proved that $\mathcal{H}_{g,g',p}^c$ is dense in $\mathcal{H}_{g,g',p}^c$.

With this criterion in hand, the properness of $\mathcal{H}_{g,g',p}^c$ follows from theorem 4.1.

As in the case of $\mathcal{H}_{g,g',p}^c$, we can decompose the algebraic stack $\mathcal{H}_{g,g',p}^{c,\infty}$ in two open and closed components : one containing the finite morphism $\mathcal{H}_{g,g',p}^{c,<\infty}$ and one containing the infinite part $\mathcal{H}_{g,g',p}^{c,\infty}$ (this last part lives only in characteristic $p$).

A Some results about classical deformations

Let us first state a generalisation of a result of Ziv Ran (cf. [Ran89]) concerning the deformation theory of morphisms between pointed semistable curves.

**Theorem A.1** Let $k$ be a field and $f_0 : X_0 \to Y_0$ a morphism between semistable (not necessarily proper) curves. Let $A$ and $A'$ be a local artinian rings with residue field $k$, $f : X \to Y$ a deformation of $f_0$ and $A' \to A$ a small extension with kernel $a$. Then there exist $k$-vector fields $\text{Ext}^i(df_0, f_0)$ fitting in a long exact sequence

$$\cdots \to \text{Ext}^{i-1}(\Omega_{Y_0/k}, f_*O_{X_0}) \to \text{Ext}^i(df_0, f_0) \to$$

$$\text{Ext}^i(\Omega_{Y_0/k}, O_{Y_0}) \oplus \text{Ext}^i(\Omega_{X_0/k}, O_{X_0}) \to \text{Ext}^{i+1}(\Omega_{Y_0/k}, f_*O_{X_0}) \to \cdots \quad (A.1)$$

such that

1. there exists a canonical element $\omega \in a \otimes_k \text{Ext}^2(df_0, f_0)$, the vanishing of which is necessary and sufficient for a lifting of $f$ to $A'$ to exist;
2. if there exists a lifting of $f$ to $A'$, then the set of isomorphism classes of liftings is naturally a principal homogeneous space under the action of $a \otimes_k \mathrm{Ext}^1(d_{f_0}, f_0)$;

3. if a lifting exists, then the automorphism group of liftings of an element is canonically isomorphic to $a \otimes_k \mathrm{Ext}^0(d_{f_0}, f_0)$.

Proof: cf. [Ran89].

In the following, we will also need results about deformation of curves endowed with an invertible sheaf and a global section of it. Before stating the result, we need to introduce some notation.

For any scheme $X/S$ denote by $\mathcal{P}^n_{X/S}$ the sheaf of principal part of order $n$ of $\mathcal{O}_X$ and for any invertible sheaf on $X$, write $\mathcal{P}^n_{X/S}(\mathcal{L})$ for the sheaf of principal part of order $n$ of $\mathcal{L}$ (in particular, we have $\mathcal{P}^n_{X/S}(\mathcal{L}) = \mathcal{P}^n_{X/S} \otimes_X \mathcal{L}$, the module structure on $\mathcal{P}^n_{X/S}$ being given by $d^n : \mathcal{O}_X \to \mathcal{P}^n_{X/S}$ (cf. [EGA] IV, chapitre 16). In particular, one has $\mathrm{Ext}^0_{\mathcal{O}_X}(\mathcal{P}^1_{X/S}(\mathcal{L}), \mathcal{L}) = \text{Diff}(\mathcal{L}, \mathcal{L})$.

**Theorem A.2** Let $k$ be a field, $X_0/k$ be a local complete intersection morphism and $\mathcal{L}_0$ be an invertible sheaf on $X$. Let $A$ and $A'$ be local artinian ring with residue field $k$, $A' \to A$ a small extension with kernel $a$ and $(X, \mathcal{L})$ be a deformation of $(X_0, \mathcal{L}_0)$ over $A$. Then the following is true

1. there exists a canonical element $\omega \in a \otimes_k \mathrm{Ext}^2(\mathcal{P}^1_{X_0/k}(\mathcal{L}), \mathcal{L})$ which vanishes if and only if there exists a lifting of $(X, \mathcal{L})$ to $A'$;

2. if there exists a lifting of $(X, \mathcal{L})$ to $A'$, then the set of isomorphism classes of liftings is naturally a principal homogeneous space under the action of $a \otimes_k \mathrm{Ext}^1(\mathcal{P}^1_{X_0/k}(\mathcal{L}), \mathcal{L})$;

3. if there exists a lifting $(X', \mathcal{L}')$ of $(X, \mathcal{L})$ to $A'$ then the automorphism group of liftings of $(X', \mathcal{L}')$ is isomorphic to $a \otimes_k \mathrm{Ext}^0(\mathcal{P}^1_{X/S}(\mathcal{L}), \mathcal{L})$.

Proof: This is a generalization to the case of local complete intersection morphism of the results of [KS88] (see also [Gro95], page 13, corollaire 2).

We can generalize results about the deformation of schemes endowed with an invertible sheaf and a global section of this sheaf.

**Proposition A.3** Let $k$ be a field, $X_0/k$ a local complete intersection morphism, $\mathcal{L}_0$ an invertible sheaf on $X_0$ and $s_0 \in H^0(X_0, \mathcal{L}_0)$. Then for any small extension $A' \to A$ with kernel $a$ between local artinian rings with residue field $k$, any deformation $(X, \mathcal{L}, s)$ of $(X_0, \mathcal{L}_0, s_0)$ over $A$ and any lifting $(X', \mathcal{L}', s')$ of $(X, \mathcal{L}, s)$ to $A'$, the automorphism group of lifting of $(X', \mathcal{L}', s')$ is canonically isomorphic to

$$a \otimes_k \ker \left( \mathrm{Ext}^0_{\mathcal{O}_X}(\mathcal{P}^1_{X_0/k}(\mathcal{L}), \mathcal{L}) \xrightarrow{d^1_0} H^0(X_0, \mathcal{L}_0) \right)$$

where $d^1_{s_0}(D) = Ds$.  

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Proof: Follow the idea of Welters in [Wel83], chapter 1.

Next we need some results about the deformation of purely inseparable morphism.

Lemma A.4 Let $A$ be a complete local ring and $f : \text{Spec} B \to \text{Spec} C$ a finite morphism between smooth $A$-curves which is purely inseparable of degree $p$ and suppose that $\Omega_{C/A}$ admits a basis which is exact. Then $f$ is of the form $C \to C[Y]/(P)$ the reduction of $P$ modulo the maximal ideal being of the form $Y^p - x$.

In particular, $\Omega_{X/A} = \bigoplus_{i=0}^{p-1} y^i Cy$.

Proof: The general case is deduced from the case of a field using flatness of $B$ over $A$. Hence we can assume $A$ to be a field so that $f$ can be identified to the Frobenius. Then choose a basis $dx$ of $\Omega_{Y/A}$ which is exact. As $f$ is the Frobenius, $x$ is a $p$-th power in $B$, denote by $y \in B$ an element such that $y^p = x$. Since $dx$ is a basis of $\Omega_Y/k$, Spec $C[y]/(y^p - x)$ is smooth over $k$, hence normal. Thus the result.

Proposition A.5 Let $A' \to A$ be a small extension of local artinian rings with same residue field $k$, $f : X \to Y$ a morphism of smooth curves satisfying the hypothesis of lemma A.4 (in particular, there exists a basis $dx$ of $\Omega_{Y/A}$). Let $X'$ and $Y'$ be liftings of $X$ and $Y$ respectively and $f_1'$ and $f_2'$ be two liftings of $f$ between $X'$ and $Y'$ such that there exists $dx' \in \Omega_{Y'/A'}$ satisfying $df_1'(dx') = df_2'(dx')$. Then $f_1'$ and $f_2'$ are equal up to an isomorphism of $Y'$.

Proof: Let $\varepsilon$ a generator of the kernel $A' \to A$. Denote by $m'$ the maximal ideal of $A'$, $k = A'/m'$ and $f_0 : X_0 \to Y_0$ the special fiber of $f$. As $f_1'$ and $f_2'$ are liftings of $f$, we have $f_1'(x') - f_2'(x') = \varepsilon \mu$ with $\mu \in \mathcal{O}_{X_0}$. By hypothesis, we have $\varepsilon d\mu = d(\varepsilon \mu) = 0$. Writing $\mathcal{O}_{X_0} = \bigoplus_{i=0}^{p-1} y^i \mathcal{O}_{Y_0}$ (such a decomposition exists by lemma A.4) we see that we have naturally $\mu \in \mathcal{O}_{Y_0}$. Then we see that the automorphism of $Y'$ defined by

$$\varepsilon \mu \frac{\partial}{\partial x'} \in (\varepsilon) \otimes_k \mathcal{Ext}^0_{\mathcal{O}_{Y_0}}(\Omega_{Y_0/k}, \mathcal{O}_{Y_0})$$

sends $f_1'(x')$ to $f_2'(x')$, and thus sends $f_1'$ to $f_2'$.

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