Constraint preserving boundary conditions for the linearized BSSN formulation

Alexander M. Alekseenko
Department of Mathematics, California State University Northridge, Northridge, California 91330
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We derive two sets of explicit algebraic constraint preserving boundary conditions for the linearized BSSN system. The approach can be generalized to inhomogeneous differential and evolution conditions, the examples of which are given. The proposed conditions are justified by an energy estimate on the original BSSN variables.

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I. INTRODUCTION

The widely used treatment of Einstein’s equations in numerical relativity is to cast them to the form of a nonlinear hyperbolic system with constraints (e.g., \[1, 2, 3, 4, 5\]) and solve by employing sophisticated discretization techniques. In the course of solution, the constraint part is either monitored, or explicitly imposed. It was observed, that the solution of the evolution part with no constraints produces a violation which grows rapidly breaking computations in a short time \[6, 7\]. An attempt to control constraint violation, by projecting the solution, or by incorporating constraint quantities in the evolution equations, results in a longer life time of calculations as a rule of thumb (e.g., \[8, 9, 10\]). It was found in \[8\], that exponentially growing constraint violating solutions converge to unstable solutions of the dynamic equations, which suggests that the constraint violation is closely related to loss of stability in the system.

An exact solution to evolution equations in the entire space has a property that it satisfies constraint equations automatically as long as it satisfies them initially. However, in numerical simulations, because of the roundoff and truncation errors, one cannot hope for automatic constraint compliance. Instead, care must be taken to ensure that the inserted perturbations are small, and remain small during the evolution.

The behavior of the solution can be improved significantly \[8, 11\] by introducing special sets of boundary data, the so-called constraint-preserving boundary conditions, or conditions that imply trivial evolution of constraints. Several sets of such data were proposed for various first order formulations of Einstein’s equations (e.g., \[8, 11, 12, 13\]). These conditions are typically written as a system of partial differential equations restricted to the boundary, and in cases when the equations are time dependent and decouple from the bulk system, the equations may be integrated in time to produce regular Dirichlet data that is compatible with constraints \[12\].

In this work, two sets of well-posed homogeneous algebraic constraint-preserving boundary conditions for the linearized Baumgarte-Shapiro-Shibata-Nakamura formulation \[4, 5\] are constructed. As is common, our derivation starts from considering the evolution equations for constraint quantities and looks for sets of data for the variables of the main system that guarantee zero Dirichlet data for the constraint quantities. The procedure is similar to the procedure found in \[12\] but a) does not employ reduction to first order, and b) does not involve integration of equations in time along the boundary. Instead, following \[12, 16\], we rewrite the equations in a special form to find well-posed constraint-preserving boundary conditions by direct inspection. The approach can be generalized to produce boundary conditions of the evolving type (see, \[12\]) and the differential type \[8, 11\]. To further justify the proposed conditions, we derive an energy estimate for the nonlinear BSSN system with boundaries extending the results of \[14, 17\], and demonstrate that the nonlinear estimate has the same boundary terms as in the linearized case.

In Section 2 we recall the derivation of the BSSN formulation and use the opportunity to discuss choices of lapse and shift most commonly found in numerical relativity. Section 3 describes the linearization of the BSSN equations. In Section 4, the constraint preserving boundary conditions are derived, and the generalization to evolving and differential boundary conditions is discussed. In Sections 5, and 6 the initial-boundary problem is defined using the derived conditions. Also, in Section 6, a set of boundary conditions for the dynamic part of the BSSN system, which is a system first order in time, second order in space, is formulated. Section 7 describes an energy estimate for the BSSN system not involving first order in space or second order in time reductions.

II. THE TRACE-FREE DECOMPOSITION OF THE ADM SYSTEM

To point out some facts about the nature and properties of the BSSN formulation (see \[4, 5\] also, some special cases are in \[8, 15\]), let us briefly recall the derivation in the case of vacuum fields where the right hand side of
Einstein’s equation is zero.

The derivation starts from the Arnowitt–Deser–Misner 3 + 1 decomposition [14, 20],

\[
\partial_b h_{ij} = -2ak_{ij} + 2h_{ij}(\partial_j b^l), \tag{1}
\]
\[
\partial_b k_i = a[R_{ij} + (k_i^j)k_j - 2k_{ij}k_j^l] + k_{ij}\partial_j b^l + k_{ji}\partial_i b^l - D_i D_j a,
\tag{2}
\]
\[
R_i^j + (k_i^j)^2 - k_{ij}k^{ij} = 0, \tag{3}
\]
\[
D_j k_i - D_i k_j^l = 0. \tag{4}
\]

Here \(a\) denotes the lapse, the \(b_i\) are the components of the shift vector \(b\), \(h_{ij}\) are the components of the spatial metric \(h\). The components of the 4-dimensional metric \(g\) are given by

\[
g_{00} = -a^2 + b_i b_j h^{ij}, \quad g_{0i} = b_i, \quad g_{ij} = h_{ij}.
\]

\(h^{ij}\) denotes the matrix inverse to \(h_{ij}\), indices are raised and traces taken with respect to the spatial metric; \(\partial_h := (\partial_t - b^s \partial_s)\) is the convective derivative, \(D_i\) is the covariant derivative operator associated to the spatial metric; the extrinsic curvature \(k_{ij}\) is defined by equation (4); we assume that global Cartesian coordinates \(t = x_0, x_1, x_2, x_3\) are specified; \(R_{ij}\) are the components of the spatial Ricci tensor

\[
R_{ij} = \frac{1}{2} h^{pq} (\partial_p \partial_q h_{ij} + \partial_i \partial_p h_{qj} - \partial_j \partial_q h_{ip} - \partial_i \partial_j h_{pq}) + h^{pq} h^{rs} (\Gamma_{ipr} \Gamma_{qjr} - \Gamma_{ipq} \Gamma_{ijr}),
\]

where \(\Gamma_{ijk}\) are the spatial Christoffel symbols defined by \(\Gamma_{ijk} = (\partial_i h_{kj} + \partial_j h_{ik} - \partial_k h_{ij})/2\).

The operator \(R_{ij}\) in (1) contains second order spatial derivatives of unknown fields and is very difficult to analyze. As a result, it is difficult to judge about properties of equation (1), and properties of \(k_{ij}\) in general. However, for the trace of the extrinsic curvature \(k = k_i^i\), the situation is different. Taking the trace of (1) and using (3, 4) we find

\[
\partial_b h^l_i = a k^l_i k_i^l - D^l D_l a. \tag{5}
\]

The remarkable simplicity of the latter equation suggests to separate the evolution of the trace of the extrinsic curvature from the system. Specifically, we introduce the trace of the extrinsic curvature \(k = k_i^i\) and the trace-free part of the extrinsic curvature \(A_{ij} = k_{ij} - (1/3)h_{ij}k\) as new variables. Then (3) yields

\[
\partial_b k = \frac{1}{3}ak^2 + a A^{lm} A_{lm} - D^l D_l a. \tag{6}
\]

Unless the lapse function \(a\) is chosen with care, equation (6) is expected to be unstable. For example, for a spatially independent lapse and zero shift vector, equation (6) yields an estimate \(\partial_t k \geq \frac{1}{4} ak^2\) which implies that \(k \geq \frac{(1/3) \int_0^1 a(\tau) d\tau}{1/k(0)}\), or that the solution \(k\) is unbounded in a finite time, which is a well-known example of a coordinate singularity. The problem can be solved, for example, by imposing maximal slicing in the BSSN formulation [15]

\[
D^l D_l a = ak^{lm} k_{lm}.
\]

With this condition equation (6) reduces to \(\partial_b k = 0\).

Alternatively, it is often proposed to use harmonic slicing [2] which corresponds to setting

\[
\partial_b a = -a^2 k. \tag{7}
\]

The equation on \(A\) is obtained from (2, 4), and (1) as

\[
\partial_b A_{ij} = a R_{ij} + \frac{1}{3}ak A_{ij} - 2a A_{il} A^l_j + \frac{2}{9} ak^2 h_{ij} - \frac{1}{3} a A^{lm} A_{lm} h_{ij} + A_{il} \partial_j b^l + A_{ij} \partial_i b^l - D_i D_j a + \frac{1}{3} h_{ij} D^l D_l a. \tag{8}
\]

To proceed with the derivation we need a splitting for the spatial metric \(h\) compatible to the splitting of \(k_{ij}\) into \(k\) and \(A_{ij}\). In the BSSN formulation, the desired splitting is achieved by introducing the conformal factor \(\varphi = (1/12) \ln(\det(h_{ij}))\) and the conformal metric \(\tilde{h}_{ij} = e^{4\varphi} h_{ij}\), \(\tilde{h}^{ij} = e^{4\varphi} h^{ij}\). Using Leibniz formula for differentiating the determinant of a matrix

\[
\partial \det(h_{ij}) = \det(h_{ij}) \partial h_{lm} \tag{9}
\]

one finds that the derivative of the conformal metric is trace-free:

\[
\partial \tilde{h}_{ij} = e^{-4\varphi}[\partial h_{ij} - \frac{1}{3} h_{ij} \partial h_{lm}]. \tag{10}
\]

By applying operator \(\partial_b\) on the definition of \(\varphi\) and using (9, 10) we get the second equation of our system

\[
\partial_b \varphi = -\frac{1}{6} ak + \frac{1}{6} \partial_b b. \tag{11}
\]

Now using (10) and (11) we obtain the third equation

\[
\partial_t \tilde{h}_{ij} = -2 a \tilde{A}_{ij} + 2 \tilde{h}_{ij}(\partial_j \tilde{b}^l - \frac{2}{3} \tilde{h}_{ij} \partial_k \tilde{b}^k), \tag{12}
\]

where \(\tilde{A}_{ij} = e^{-4\varphi} A_{ij}\), \(\tilde{b}_i = e^{-4\varphi} b_i\) are the conformal analogs of the variables \(A\) and \(b\). Beginning from the last equation, indices are lowered and raised with the

3 Harmonic slicing is a particular case of Bona-Masso family of \(k\)-driving slicing conditions \((\partial_t - b^l D_l) a = -a^2 f(a) k, f(a) > 0\) [21, 22].
conformal metric $\tilde{h}_{ij}$ and its inverse $\tilde{h}^{ij} = e^{2\varphi}h^{ij}$ (in this case $b^* = b^*$, and it is easy to redefine $\partial_0 = \partial_t - b^*\partial_8$).

The remaining two equations can be obtained from (13) which can be rewritten in terms of $\tilde{A}$ as

$$\partial_0 \tilde{A}_{ij} = a e^{-4\varphi} R_{ij} + a(k \tilde{A}_{ij} - 2 \tilde{A}_{il} \tilde{A}^l_j + \frac{2}{9} k^2 \tilde{h}_{ij}$$

$$= \frac{1}{3} \tilde{A}^m_{im} \tilde{A}^l_{mj} + \tilde{A}_{ij} \partial_t \tilde{b}^l + \frac{2}{3} \tilde{A}_{ij} \partial_t \tilde{b}^l$$

$$- e^{-4\varphi} D_i D_j a + e^{-4\varphi} \frac{1}{3} \tilde{h}_{ij} D^l D_l a.$$  

(13)

The Ricci tensor in terms of the conformal metric reads

$$R_{ij} = \frac{1}{2} \tilde{h}^{pq}(\partial_p \partial_j \tilde{h}_{iq} + \partial_i \partial_p \tilde{h}_{qj} - \partial_p \partial_i \tilde{h}_{qj} - \partial_i \partial_j \tilde{h}_{pq})$$

$$- 2 \tilde{D}_i \tilde{D}_j \varphi - 2 \tilde{h}_{ij} \tilde{h}^{pq} \tilde{D}_p \tilde{D}_q \varphi$$

$$+ \tilde{h}^{pq} \tilde{h}^{rs} (\Gamma^i_{pq} \tilde{\Gamma}^j_{rj} - \Gamma^p_{qj} \tilde{\Gamma}^i_{jq})$$

$$+ 4 \partial_i \varphi \partial_j \varphi - 4 \tilde{h}_{ij} \tilde{h}^{pq} \partial_p \varphi \partial_q \varphi.$$  

(14)

Here $\tilde{\Gamma}_{ijk} = (\partial_i \tilde{h}_{kj} + \partial_j \tilde{h}_{ik} - \partial_k \tilde{h}_{ij})/2$; $\tilde{D}_i \varphi = \partial_i \varphi - \tilde{h}^{pq} \Gamma_{ijp} \tilde{\Gamma}_{qij}$ is the covariant derivative associated with the conformal metric. The first line in (14) can be rewritten

$$R_{ij} = - \frac{1}{2} \tilde{h}^{pq} \partial_p \partial_i \tilde{h}_{qj} + \partial_j \tilde{h}^{pq} \tilde{\Gamma}_{ipq} + \Gamma^q_{pq} (\partial_j \tilde{h}^{pq}$$

$$- 2 \tilde{D}_i \tilde{D}_j \varphi - 2 \tilde{h}_{ij} \tilde{h}^{pq} \tilde{D}_p \tilde{D}_q \varphi + \ldots.$$  

(15)

This suggests to introduce a new variable

$$\tilde{\Gamma}_j = \tilde{h}^{pq} \tilde{\Gamma}_{pqj} = \tilde{h}^{pq} \partial_p \tilde{h}_{qj}.$$  

(16)

Substituting (16) in (13) one gets the fourth evolution equation

$$\partial_0 \tilde{A}_{ij} = a e^{-4\varphi} \tilde{\Gamma}_{pq}(\partial_j \tilde{h}^{pq}$$

$$+ \frac{1}{3} \tilde{h}^{pq} \tilde{h}^{rs} (\tilde{\Gamma}^i_{pq} \tilde{\Gamma}^j_{rj} - \tilde{\Gamma}^p_{qj} \tilde{\Gamma}^i_{jq})$$

$$+ 4a e^{-4\varphi} \partial_i \partial_j \varphi - 4 e^{-4\varphi} \tilde{h}^{pq} \partial_p \varphi \partial_q \varphi$$

$$+ a(k \tilde{A}_{ij} - 2 \tilde{A}_{il} \tilde{A}^l_j + \frac{2}{9} k^2 \tilde{h}_{ij} - \tilde{A}^m_{im} \tilde{A}^l_{mj})$$

$$+ \tilde{A}_{il} \partial_j \tilde{b}^l + \tilde{A}_{jl} \partial_i \tilde{b}^l - \frac{2}{3} \tilde{A}_{ij} \partial_t \tilde{b}^l.$$  

(17)

where

$$W_{ij} = a e^{-4\varphi} \tilde{\Gamma}_{pq}(\partial_j \tilde{h}^{pq}$$

$$+ \frac{1}{2} \tilde{h}^{pq} \tilde{h}^{rs} (\tilde{\Gamma}^i_{pq} \tilde{\Gamma}^j_{rj} - \tilde{\Gamma}^p_{qj} \tilde{\Gamma}^i_{jq})$$

$$+ 4a e^{-4\varphi} \partial_i \partial_j \varphi$$

$$+ a(k \tilde{A}_{ij} - 2 \tilde{A}_{il} \tilde{A}^l_j + \frac{2}{9} k^2 \tilde{h}_{ij} - \tilde{A}^m_{im} \tilde{A}^l_{mj})$$

$$+ \tilde{A}_{il} \partial_j \tilde{b}^l + \tilde{A}_{jl} \partial_i \tilde{b}^l - \frac{2}{3} \tilde{A}_{ij} \partial_t \tilde{b}^l.$$  

The evolution equation on $\tilde{\Gamma}_i$ is obtained by differentiating its definition and using the momentum constraint. Namely, we apply operator $\partial_0$ on (10) to get

$$\partial_0 \tilde{\Gamma}_i = - 2a \partial_t \tilde{A}_i - \tilde{h}^{pq}(\partial_p \partial_q \tilde{A}_{iq} + 2 \tilde{A}^{pq}(\partial_p \tilde{h}_{qj}$$

$$+ \Gamma_i \partial_t \tilde{b}^l + \frac{1}{3} \partial_i \partial_t \tilde{b}^l + \tilde{h}_{ij} \varphi \partial_i \tilde{b}^l.$$  

(18)

Then we notice that $h^{pq} \tilde{D}_p A_{iq} = \partial^q \tilde{A}_{pi} - \tilde{\Gamma}_s \tilde{A}^s_{pi} + 6(\partial_s \varphi) \tilde{A}^s_{pi}$, and thus (11) reduces to

$$\partial^q \tilde{A}_{pi} - \frac{2}{3} \partial_t k - \tilde{\Gamma}_s \tilde{A}^s_{pi} + 6(\partial_s \varphi) \tilde{A}^s_{pi} = 0.$$  

Solving this equation for $\partial^q \tilde{A}_i$ and substituting the result in (15) we derive the fifth equation of the BSSN system

$$\partial_0 \tilde{\Gamma}_i = - \frac{4}{3} a \partial_t k + S_i,$$  

(19)

where

$$S_i = - 2a \tilde{\Gamma}_s \tilde{A}^s_i + 12a(\partial_s \varphi) \tilde{A}^s_i - 2 \tilde{h}^{pq}(\partial_p \partial_q \tilde{A}_{ij}$$

$$+ 2 \tilde{A}^{pq}(\partial_p \tilde{h}_{qj}) + \Gamma_i \partial_t \tilde{b}^l + \frac{1}{3} \partial_t \partial_t \tilde{h} - \tilde{h}_{ij} \varphi \partial_t \tilde{b}^l.$$  

Equations (11), (13), (12), (17), and (19) constitute the core of the BSSN formulation. These equations are usually supplemented by one or more equations describing the choice of the gauge functions. Thus, in most cases the lapse and the shift are not known but dynamically depend on the metric and other quantities. In this work, we will assume the harmonic lapse condition (17). Further we consider either a prescribed shift $b_i$ or a shift that follows from the gamma-freezing condition $\partial_t \Gamma_i = 0$ (see, for example, (18)).

III. LINEARIZATION AROUND MINKOWSKI SPACE

Minkowski spacetime in Cartesian coordinates is represented by the trivial solution to ADM system: $h_{ij} = \delta_{ij}$, $k_{ij} = 0$, $a = 1$, $b_i = 0$. Consider perturbations of ADM variables around the Minkowski spacetime: $h_{ij} = \delta_{ij} + \gamma_{ij}$, $k_{ij} = \kappa_{ij}$, $a = a_1 + \alpha$, $b_i = \beta_i$, with the $\gamma_{ij}$, $\kappa_{ij}$, $\alpha$, and $\beta_i$ supposed to be small. Substituting these expressions into the definitions of the BSSN variables and neglecting terms of second and higher order in perturbations we get

$$\det(h_{ij}) = 1 + \gamma_{ij}, \quad \varphi = \frac{1}{12} \gamma_{ij},$$

$$e^{-4\varphi} = 1 - \frac{1}{4} \gamma_{ij}, \quad e^{4\varphi} = 1 + \frac{1}{4} \gamma_{ij},$$

$$\tilde{h}_{ij} = \delta_{ij} + \gamma_{ij} - \frac{1}{3} \delta_{ij} \gamma_{ij} =: \delta_{ij} + \tilde{\gamma}_{ij},$$

$$k = \kappa = \kappa_{ij}, \quad A_{ij} = \tilde{A}_{ij} = \kappa_{ij} - \frac{1}{3} \delta_{ij} \kappa,$$

$$\Gamma_i =: \tilde{\Gamma}_i = \partial^q \tilde{\gamma}_{iq}.$$  

(20)

Substituting these quantities in equations (11), (12), (17), (19), and (18) and ignoring the terms which are at least second order in $\varphi$, $\gamma_{ij}$, $k$, $A_{ij}$, and $\Gamma_i$ we derive the
linearized BSSN system
\[ \partial_t \phi = -\frac{1}{6} \kappa + \frac{1}{6} \partial^i \beta_i, \]  
(21)
\[ \partial_t \alpha = -\kappa, \]  
(22)
\[ \partial_t \kappa = -\partial^i \partial_i \alpha, \]  
(23)
\[ \partial_t \tilde{\gamma}_{ij} = -2A_{ij} + 2\partial_{(i} \beta_{j)} - \frac{2}{3} \delta_{ij} \partial^k \beta_k, \]  
(24)
\[ \partial_t A_{ij} = -\frac{1}{2} \partial_i \partial_j \phi - \partial_{(i} \partial_j \alpha + \frac{1}{3} \delta_{ij} \partial^k \partial_k \alpha, \]  
(25)
\[ \partial_t \Gamma_i = -\frac{4}{3} \partial_i \kappa + \frac{1}{3} \partial_i \partial^\rho \beta_\rho + \partial^\rho \partial_\rho \beta_i. \]  
(26)

Notice that the linearized harmonic lapse condition is included in this system in the form of equation (22). In fact, this condition will be used in the hyperbolic reduction which does not seem possible in general.

Linearization of Hamiltonian and the momentum constraint equations yields correspondingly
\[ \partial^i \partial^j \tilde{\gamma}_{ij} - 8 \partial^i \partial_i \phi = 0, \quad \partial^i \Gamma_i - 8 \partial^i \partial_i \phi = 0, \]  
(27)
\[ \partial^i A_{il} - \frac{2}{3} \partial_i \kappa = 0. \]  
(28)

Hamiltonian constraint appears in two versions since it can be written both in terms of \( \dot{\gamma} \) and \( \Gamma \). Also, introducing the new variable \( \Gamma \) entails an artificial constraint
\[ \Gamma_i = \partial^j \tilde{\gamma}_{il}. \]  
(29)

The linearized problem then consists of determining \( \phi, \alpha, \kappa, \dot{\gamma}, A, \Gamma \) from equations (24)–(28) provided initial data and admissible boundary data. The constraint equations (24)–(28) may or may not be imposed during the evolution. The initial data \( \phi(0), \alpha(0), \kappa(0), \dot{\gamma}(0), A(0), \Gamma(0) \) is determined from \( \gamma(0) \) and \( \kappa(0) \) using (20). It can be checked that if \( \gamma(0) \) and \( \kappa(0) \) satisfy the linearized Hamiltonian and momentum constraints in the ADM system, then \( \phi(0), \kappa(0), \dot{\gamma}(0), A(0), \Gamma(0) \) satisfy the constraint equations (24)–(28).

IV. CONSTRAINT PRESERVING BOUNDARY CONDITIONS

The BSSN system is a constrained evolution system in the sense that it has the dynamic part (21)–(26) and the constraint part (27)–(29). It was assumed for a long time, that for the right boundary data, a solution to (21)–(26) will satisfy the constraints automatically once it satisfies them initially, however, examples of such data were constructed only recently \[14\] for a first order reduction. Still it remains a question, which we are trying to address in this paper, whether a set of constraint preserving boundary conditions can be proposed for the original BSSN variables.

Let us notice that, for a solution of (21)–(26), constraints (27) and (29) are consequences of (28), so we can focus just on the last one. Indeed, in view of (21), (25), and (26), the following equations hold for the time derivative of (24)
\[ \partial_t (\partial^j \partial^i \tilde{\gamma}_{ij} - 8 \partial^i \partial_i \phi) = \partial^i (\partial^j A_{il} - \frac{2}{3} \partial_i \kappa), \]  
\[ \partial_t (\partial^i \Gamma_i - 8 \partial^i \partial_i \phi) = 0. \]

These equations state that both parts of (24) are satisfied as long as they are satisfied initially, and (28) is true. Similarly, if (25) is satisfied, then the time derivative of (29) is zero in view of (24), (26). Thus (29) remains zero provided it is zero initially.

We will now construct boundary conditions for system (21)–(26) that preserve (28). We introduce a new variable
\[ M_i = \partial^j A_{il} - \frac{2}{3} \partial_i \kappa. \]  
(30)

Equation (28) is satisfied iff \( M_i = 0 \), in other words, the condition in question must guarantee \( M_i = 0 \).

By differentiating (30) twice in time and substituting time derivatives of equations (25) and (26) for \( \partial_t^2 A, \partial_t^2 \kappa \), we derive (terms in \( \phi, \alpha, \kappa, \beta, \Gamma \) cancel in view of (21)–(26))
\[ \partial_t^2 M_i = \partial^i \partial_j M_i. \]  
(31)

Initial values \( M(0) \) can be determined from \( A(0), \kappa(0) \) using (20) and must be zero for physical initial data. The initial values for \( \partial_t M_i \) can be calculated by differentiating (30) in time and substituting (25) and (26) for \( \partial_t \kappa, \partial_t A_{ij} \),
\[ \partial_t M_i = -\frac{1}{2} \partial^j \partial_i \partial^m \tilde{\gamma}_{jm} + \frac{1}{2} \partial_i \partial_j \Gamma_i + \frac{1}{2} \partial^j \partial_i \Gamma_i - 4 \partial_i \partial^j \partial_j \phi. \]

It can be verified by substitution, that if \( \tilde{\gamma}(0), \Gamma(0), \phi(0) \) satisfy (27), (29), then \( \partial_i M(0) = 0 \).

It remains to select the boundary conditions on \( M \) that imply trivial evolution of (31). However, we notice that the boundary data on \( M \) is expected not to be given freely but determined by the boundary conditions on \( A \) and \( \kappa \), similar to the way \( M(0), \partial_t M(0) \) is determined by the main variables \( A(0), \kappa(0), \tilde{\gamma}(0), \Gamma(0), \phi(0) \). But we do not know how to specify the boundary conditions on \( A \) and \( \kappa \) either! Here is the key: we will select that data now by observing its relationship with the boundary data on \( \Gamma \). For the boundary conditions, again, we expect both definition (30) and the evolution equations (21)–(26) to contribute into the relationship.

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4 The linearized Hamiltonian and momentum constraints in the ADM system are \( \partial^i \partial^j \tilde{\gamma}_{ij} - \partial^j \partial^i \tilde{\gamma}_{ij} = 0 \) and \( \partial^i \kappa_{il} - \partial_i \kappa_l = 0 \).
We introduce scalar products \((v_i, u_i) = \int_\Omega v_i u_i^* d\Omega d\sigma\) and \((\rho_{ij}, \sigma_{ij}) = \int_\Omega \rho_{ij} \sigma_{ij}^* d\Omega d\sigma\) for the spaces of vector fields and matrix fields on \(\Omega\) correspondingly. The \(L_2\) norms naturally associated with these scalar products, \(\|u\|^2 = (u_i, u_i)\) and \(\|ho\|^2 = (\rho_{ij}, \rho_{ij})\), are denoted by \(\| \cdot \|^2\). We introduce the energy of system \(37\),

\[
\epsilon = \|\partial_t M\|^2 + \|\partial_t M\|^2
\]

If we prove that the energy \(\epsilon\) remains zero at all times, then, in view of the trivial initial data, this will imply that \(\|M\| = 0\) (e.g., \(23\)).

Differentiating \(\epsilon\) in time, using the Green’s First Identity component-wise to transfer the spatial derivative in the second term, and also \(37\), we obtain

\[
\partial_t \epsilon = \int_{\partial \Omega} \left( \frac{\partial}{\partial n} M_i \right) (\partial_t M^i) d\sigma.
\]

(32)

The energy \(\epsilon\) is not increasing if

\[
\left( \frac{\partial}{\partial n} M_i \right) (\partial_t M^i) \leq 0 \quad \text{on} \quad \partial \Omega.
\]

(33)

The desired boundary conditions on \(A\) and \(\kappa\) will follow immediately once we rewrite \(33\) in terms of the main variables.

We assume that the boundary \(\partial \Omega\) is a combination of arbitrarily oriented planes and consider any of its faces. Let vector \(n_i\) be the unit vector perpendicular to the face, \(m_i\), \(l_i\) complement \(n_i\) to an orthonormal triple (for example, \(m_i\) is any unit vector parallel to the boundary and \(l_i\) is the cross product of \(n_i\) and \(m_i\), \(l_i = \varepsilon_i^j n_j m_k = (n \times m)_i\)). At the flat boundary, the divergence of a vector field can be expressed in terms of the directional derivatives along vectors \(n, m,\) and \(l\), as

\[
\partial^j v_i = n^i \frac{\partial}{\partial n} v_i + m^i \frac{\partial}{\partial m} v_i + l^i \frac{\partial}{\partial l} v_i.
\]

(34)

Similarly, the gradient of a scalar field \(\psi\) reads

\[
\partial_i \psi = n_i \frac{\partial}{\partial n} \psi + m_i \frac{\partial}{\partial m} \psi + l_i \frac{\partial}{\partial l} \psi.
\]

(35)

Next we note that, at any point of the boundary a symmetric trace free matrix is spanned by

\[n_i(m_j), n_i(l_j), l_i(m_j), l_i l_j - m_i m_j, 2n_i n_j - l_i l_j - m_i m_j.\]

Introducing scalar functions

\[
A_1 = 2A^{ij} n_i(m_j), \quad A_2 = 2A^{ij} n_i(l_j),
\]

\[
A_3 = 2A^{ij} l_i(m_j), \quad A_4 = \frac{1}{2} A^{ij}(l_i l_j - m_i m_j),
\]

\[
A_5 = \frac{1}{6} A^{ij} (2n_i n_j - l_i l_j - m_i m_j)
\]

(36)

we restate \(A\) as

\[
A_{ij} = A_1 (n_i(m_j)) + A_2 (n_i(l_j)) + A_3 (l_i(m_j)) + A_4 (l_i l_j - m_i m_j) + A_5 (2n_i n_j - l_i l_j - m_i m_j).
\]

(37)

Substituting \(37\) into \(33\) and using \(34\) and \(35\) to replace partial derivatives with the directional derivatives, we get

\[
M_i = \frac{1}{2} \frac{\partial}{\partial m} A_1 + \frac{1}{2} \frac{\partial}{\partial l} A_2 + 2 \frac{\partial}{\partial n} A_5 - 2 \frac{\partial}{\partial m} \kappa |n_i|
\]

\[
+ \left[ \frac{1}{2} \frac{\partial}{\partial n} A_1 + \frac{1}{2} \frac{\partial}{\partial l} A_3 - \frac{\partial}{\partial m} A_4 - \frac{\partial}{\partial m} A_5 - 2 \frac{\partial}{\partial m} \kappa |l_i| \right]
\]

\[
+ \left[ \frac{1}{2} \frac{\partial}{\partial n} A_2 + \frac{1}{2} \frac{\partial}{\partial m} A_3 + \frac{\partial}{\partial l} A_4 - \frac{\partial}{\partial l} A_5 - 2 \frac{\partial}{\partial l} \kappa |l_i| \right].
\]

(38)

The last equation implies that

\[
\left( \frac{\partial}{\partial n} M_i \right) (\partial_t M^i)
\]

\[
= \frac{\partial}{\partial n} \left[ \frac{1}{2} \frac{\partial}{\partial m} A_1 + \frac{1}{2} \frac{\partial}{\partial l} A_2 + 2 \frac{\partial}{\partial n} A_5 - 2 \frac{\partial}{\partial m} \kappa \right] \]

\[
\times \partial_t \left[ \frac{1}{2} \frac{\partial}{\partial m} A_1 + \frac{1}{2} \frac{\partial}{\partial l} A_2 + 2 \frac{\partial}{\partial n} A_5 - 2 \frac{\partial}{\partial m} \kappa \right]
\]

\[
+ \frac{\partial}{\partial n} \left[ \frac{1}{2} \frac{\partial}{\partial m} A_1 + \frac{1}{2} \frac{\partial}{\partial l} A_3 - \frac{\partial}{\partial m} A_4 - \frac{\partial}{\partial m} A_5 - 2 \frac{\partial}{\partial m} \kappa \right]
\]

\[
\times \partial_t \left[ \frac{1}{2} \frac{\partial}{\partial m} A_1 + \frac{1}{2} \frac{\partial}{\partial l} A_3 - \frac{\partial}{\partial m} A_4 - \frac{\partial}{\partial m} A_5 - 2 \frac{\partial}{\partial m} \kappa \right]
\]

\[
+ \frac{\partial}{\partial n} \left[ \frac{1}{2} \frac{\partial}{\partial m} A_2 + \frac{1}{2} \frac{\partial}{\partial m} A_3 + \frac{\partial}{\partial l} A_4 - \frac{\partial}{\partial l} A_5 - 2 \frac{\partial}{\partial l} \kappa \right]
\]

\[
\times \partial_t \left[ \frac{1}{2} \frac{\partial}{\partial m} A_2 + \frac{1}{2} \frac{\partial}{\partial m} A_3 + \frac{\partial}{\partial l} A_4 - \frac{\partial}{\partial l} A_5 - 2 \frac{\partial}{\partial l} \kappa \right].
\]

(39)

Either of the two sets of boundary conditions imply \(((\partial/\partial n) A^i) (\partial_t M_i) = 0\) on \(\partial \Omega\):

\[
A_1 = 0, \quad A_2 = 0, \quad \frac{\partial}{\partial n} A_3 = 0, \quad \frac{\partial}{\partial n} A_4 = 0,
\]

\[
\frac{\partial}{\partial n} A_5 = 0, \quad \frac{\partial}{\partial n} \kappa = 0.
\]

(40)

\[
\frac{\partial}{\partial n} A_1 = 0, \quad \frac{\partial}{\partial n} A_2 = 0, \quad A_3 = 0, \quad A_4 = 0,
\]

\[
A_5 = 0, \quad \kappa = 0.
\]

(41)

In particular, \(40\) eliminates the second multiplier in the first term of \(37\) and the first multipliers in the second and third terms (by commuting partial derivatives and using \(10\)). Condition \(41\) is verified in a similar way.

More examples of constraint-preserving boundary conditions can be proposed by inspection of \(37\). For example, the condition \(M_i |_{\partial \Omega} = 0\) according to \(35\), is equivalent to the set of differential boundary conditions that
can be implemented numerically [8, 11]:

\[
\begin{align*}
\frac{1}{2} \frac{\partial}{\partial m} A_1 + \frac{1}{2} \frac{\partial}{\partial l} A_2 + 2 \frac{\partial}{\partial m} A_3 - \frac{2}{3} \frac{\partial}{\partial n} \kappa &= 0, \\
\frac{1}{2} \frac{\partial}{\partial n} A_1 + \frac{1}{2} \frac{\partial}{\partial l} A_3 - \frac{\partial}{\partial m} A_4 - \frac{2}{3} \frac{\partial}{\partial n} \kappa &= 0, \\
\frac{1}{2} \frac{\partial}{\partial n} A_2 + \frac{1}{2} \frac{\partial}{\partial m} A_3 + \frac{\partial}{\partial l} A_4 - \frac{2}{3} \frac{\partial}{\partial m} \kappa &= 0.
\end{align*}
\]

(42)

Namely, one could prescribe Dirichlet data on \(A_3, A_4\), and \(\kappa\). Then, (42) gives mixed conditions on \(A_1, A_2, A_5\). The problem with this condition is that it is not obvious if it leads to a well-posed evolution of (46) (the next example, however, contains an idea on how the well-posedness can be established).

Furthermore, one could have considered a combination of Neumann and Dirichlet conditions

\[
\frac{\partial}{\partial n} M_i n^i = 0, \quad M_i l^i = 0, \quad M_i m^i = 0.
\]

(43)

Applying \(\partial/\partial m\) to the second, \(\partial/\partial l\) to the third equation of (42) and subtracting the results from the normal derivative of the first one, we derive (using (10), (43) to eliminate \((\partial^2/\partial n^2)\) derivatives) an evolution equation defined on the boundary

\[
2 \frac{\partial^2}{\partial t^2} A_5 - \left(\frac{\partial^2}{\partial l^2} + \frac{\partial^2}{\partial m^2}\right) A_5 - 2 \frac{\partial^2}{\partial t^2} \kappa + 4 \left(\frac{\partial^2}{\partial l^2} + \frac{\partial^2}{\partial m^2}\right) \kappa = \left(\frac{\partial^2}{\partial l^2} - \frac{\partial^2}{\partial m^2}\right) A_4 + \frac{\partial}{\partial l} A_3.
\]

(44)

Equation (44) can be used to find Dirichlet data on \(A_5\), \(\kappa\), provided Dirichlet data for \(A_3, A_4, \kappa\) is given (see (12)). Once \(A_5, \kappa\) are known, the values of \((\partial/\partial n) A_1, (\partial/\partial n) A_2\) can be determined from the last two equations of (42). The corresponding inhomogeneous algebraic conditions on \(A_{ij}\) then would read

\[
2 \left(\frac{\partial/\partial n}{\partial n}\right) A_{ij} n_i (m_j) = \left(\frac{\partial/\partial n}{\partial n}\right) A_1,
\]

\[
2 \left(\frac{\partial/\partial n}{\partial n}\right) A_{ij} n_i (l_j) = \left(\frac{\partial/\partial n}{\partial n}\right) A_2,
\]

\[
2 A_{ij} l_i (m_j) = A_3,
\]

\[
1(2) A_{ij} (l_i l_j - m_i m_j) = A_4,
\]

\[
1(6) A_{ij} (2n_i n_j - l_i l_j - m_i m_j) = A_5.
\]

(45)

The boundary conditions (45) are constraint-preserving and represent an analog of conditions introduced in (12, 11).

V. EVOLUTION OF \(A\) AND \(\kappa\). SECOND ORDER IN TIME REDUCTION

We will argue that the boundary conditions (41) and (43) lead to a well-posed problem for the linearized BSSN system. By differentiating equation (25) in time and substituting (21) for \(\partial_t \gamma_{ij}\) in the result, we obtain (terms in \(\varphi, \alpha, \kappa, \beta, \Gamma\) cancel in view of (21–23, 20)

\[
\partial_t^2 A_{ij} = \partial_t \partial_t A_{ij}.
\]

(46)

We assume that the initial values \(A(0)\) and \(\partial_t A(0)\) are determined from (20) and (26) correspondingly, and that either of the conditions (40) or (41) is given at the domain boundary.

We introduce scalar products \((\mu, \nu) = \int_O \mu d\Omega d\nu d\Omega\) and \((u_{i,j,k}, v_{i,j,k}) = \int_O u_{i,j,k} v_{i,j,k} d\Omega d\Omega\) for the spaces of scalar fields and triple indexed fields on \(\Omega\). The \(L_2\) norms naturally associated with these scalar products are \(\|\mu\|^2 = (\mu, \mu)\) and \(\|u\|^2 = (u_{i,j,k}, u_{i,j,k})\). The energy of system (46) is defined as

\[
\epsilon_1 = \frac{1}{2} \left(\|\partial_t A\|^2 + \|\partial_t^2 A\|^2\right).
\]

Similar to Section 4, by differentiating \(\epsilon_1\) in time, integrating terms with spatial derivatives by parts, and using (40), we obtain

\[
\partial_t \epsilon_1 = \int_{\partial \Omega} \left(\frac{\partial}{\partial n} A_{ij} (\partial_t A^{ij})\right) d\sigma,
\]

(47)

Since the right side of (47) is zero for either (40) or (41), we conclude that \(\epsilon_1\), and therefore, \(A\) remains bounded.

Similarly, by differentiating (20) and substituting (22) for \(\partial_t \alpha\) one derives the equation for \(\kappa\)

\[
\partial_t^2 \kappa = \partial_t \partial_t \kappa.
\]

(48)

The boundary conditions (40) or (41) imply trivial Neumann and Dirichlet data on \(\kappa\) correspondingly.

Finally, assuming the shift perturbation \(\beta\) is known, the matrix \(A\) is computed from (40), and \(\kappa\) is determined from (45), variables \(\varphi, \alpha, \gamma, \Gamma\) can be determined from (21, 22, 24, 20) by integration in time.

VI. INITIAL BOUNDARY VALUE PROBLEM FOR THE LINEARIZED BSSN. PRESCRIBED AND GAMMA-FREEZING SHIFT

Let system (21–26) be provided with relevant initial data, and \(\beta\) be given. If the boundary conditions for \(A\) and \(\kappa\) are taken in either form (40) or (41) (conditions (42) and (46) can be formally imposed and treated similarly but contain derivatives of the unknown fields), then boundary conditions on variables \(\gamma, \varphi, \alpha, \Gamma\) can be obtained by integration of (21, 24, 22, 20) as (the
projection operator $N_{ij}^{pq}$ is defined below)

$$
N_{ij}^{pq} \tilde{\gamma}_{pq} + (\delta^p \delta_j - N_{ij}^{pq}) \frac{\partial}{\partial n} \tilde{\gamma}_{pq} = N_{ij}^{pq} \tilde{\gamma}_{pq}(0)
$$

$$
+ (\delta^p \delta_j - N_{ij}^{pq}) \frac{\partial}{\partial n} \tilde{\gamma}_{pq}(0) + \int_0^t (N_{ij}^{pq})
$$

$$
+ (\delta^p \delta_j - N_{ij}^{pq}) \frac{\partial}{\partial n} (2\partial_{(p} \beta_{q)}) - \frac{2}{3} \delta_{pq} \partial^\alpha \beta_\alpha,
$$

(49)

$$
\mu \varphi + (1 - \mu) \frac{\partial}{\partial n} \varphi = \mu \varphi(0) + (1 - \mu) \frac{\partial}{\partial n} \varphi(0)
$$

$$
+ \int_0^t \frac{1}{6} (1 - \mu) \frac{\partial}{\partial n} \beta_i.
$$

(50)

$$
\mu \alpha + (1 - \mu) \frac{\partial}{\partial n} \alpha = \mu \alpha(0) + (1 - \mu) \frac{\partial}{\partial n} \alpha(0),
$$

(51)

$$
n^i \Gamma_i = n^i \Gamma_i(0) + \int_0^t \left[ -\frac{4}{3} \frac{\partial}{\partial n} \kappa + \frac{1}{3} \frac{\partial}{\partial n} \beta_p + \partial^p \partial_p \beta_\alpha \right],
$$

$$
\tau^i \Gamma_i = \tau^i \Gamma_i(0) + \int_0^t \left[ -\frac{4}{3} \frac{\partial}{\partial t} \kappa + \frac{1}{3} \frac{\partial}{\partial t} \beta_p + \partial^p \partial_p \tau^i \beta_i \right].
$$

(52)

The projection operator $N_{ij}^{pq}$ corresponding to (40) is

$$
N_{ij}^{pq} = 2n(p m^q) n_i m_j + 2n(p m^q) n_i l_j
$$

and the one corresponding to (11) is

$$
N_{ik}^{pq} = 2l^i m^q l_i m_j + \frac{1}{2}(p m^q - m^q m^q)(l_i l_j - m_i m_j)
$$

$$
+ \frac{1}{6}(2 n^p n^q - p m^q) (2 n_i n_j - l_i l_j - m_i m_j),
$$

$\mu = 0$ corresponds to (40) and $\mu = 1$ to (11). In (52), $\tau_i$ stands for vectors $l_i$, $m_i$. Unless the Neumann data is given on $\kappa$, the equation $\partial^i \Gamma_i$ couples $\Gamma$ and $\kappa$ through an integral equation. If the Neumann data is specified for $\kappa$, for example in (40), the last equation can be replaced with

$$
\tau^i \frac{\partial}{\partial n} \Gamma_i = \tau^i \frac{\partial}{\partial n} \Gamma_i(0) + \int_0^t \left[ -\frac{4}{3} \frac{\partial}{\partial t} \frac{\partial}{\partial n} \kappa
$$

$$
+ \frac{1}{3} \frac{\partial}{\partial t} \frac{\partial}{\partial n} \beta_p + \partial^p \partial_p \tau^i \beta_i \right].
$$

(53)

**Theorem 1.** Let $A$ and $\kappa$ be smooth solutions of (40) and (41) corresponding to boundary data (40) (or (41)) and the initial data $A(0), \kappa(0), \partial_t A(0), \partial_t \kappa(0)$ (the last two are determined from equations (23) and (26)) and the initial data $\tilde{\gamma}(0), \Gamma(0), \alpha(0), \varphi(0)$ as

$$
\partial_t \tilde{\gamma}_{ij}(0) = -\frac{1}{2} \delta^i \partial_t \tilde{\gamma}_{ij}(0) + \partial_i \Gamma_j(0) - 2 \partial_t \delta_{ij} \varphi(0)
$$

$$
- 2 \delta_{ij} \partial_t \partial_t \varphi(0) - \partial_t \partial_t \kappa(0) + \frac{1}{3} \delta_{ij} \partial_t \partial_t \alpha(0),
$$

$$
\partial_t \kappa(0) = -\delta^i \partial_t \alpha(0).
$$

(54)

Then a solution to (21) - (23) satisfying boundary data (47) (or (48)), (49) - (52) is given by

$$
\varphi = \varphi(0) + \int_0^t \left[ \frac{1}{6} \kappa + \frac{1}{3} \partial_t \beta_\alpha \right],
$$

$$
\alpha = \alpha(0) - \int_0^t \kappa,
$$

$$
\tilde{\gamma}_{ij} = \tilde{\gamma}_{ij}(0) + \int_0^t \left[ -2 A_{ij} + 2 \partial_i \beta_j - \frac{2}{3} \delta_{ij} \partial_t \beta_\alpha \right],
$$

$$
\Gamma_i = \Gamma_i(0) + \int_0^t \left[ \frac{4}{3} \partial_t \kappa + \frac{1}{3} \partial_t \partial_t \beta_p + \partial_t \partial_t \partial_t \beta_i \right].
$$

(55)

Moreover, if $A$ and $\kappa$ satisfy constraint equation (46), then $\tilde{\gamma}, \Gamma,$ and $\varphi$ defined by (47) - (52), satisfy constraints (29) and (28) as long as they satisfy them at the initial time.

**Proof.** Equations (21), (22), (24), and (26) are verified by substitution. Replacing $\tilde{\gamma}, \Gamma,$ and $\varphi$ in (28) and (29) by their expressions from (55) and using (53) and (54) we obtain

$$
\partial_t A_{ij} = \partial_t A_{ij}(0) + \int_0^t \partial^t \partial_t A_{ij},
$$

$$
\partial_t \kappa = \partial_t \kappa(0) + \int_0^t \partial^t \partial_t \kappa
$$

which is a consequence of (48) and (49). Substituting (55) into (49) - (52) we verify the boundary conditions.

Now consider constraints (27). Replacing $\gamma, \varphi,$ and $\Gamma$ with their expressions from (55) we obtain

$$
\partial^t \partial_t \tilde{\gamma}_{ij} - 8 \partial_t \partial_t \varphi = \partial^t \partial_t \tilde{\gamma}_{ij}(0) - 8 \partial_t \partial_t \varphi(0)
$$

$$
- \int_0^t 2 \partial^t (\partial_t A_{ij} - \frac{2}{3} \partial_t k),
$$

$$
\partial_t \delta_{ij} \partial_t \varphi = \partial_t \delta_{ij}(0) - 8 \partial_t \partial_t \varphi(0),
$$

(56)

from which it follows that (27) is met as long as it is satisfied initially and (28) is true. Constraint (28) follows similarly.

The situation is similar when $\beta$ is to be determined from the gamma-freezing condition $\partial_t \Gamma_i = 0$ which yields an elliptic equation for $\beta$ that can be solved at each time step

$$
\frac{1}{3} \partial_t \partial_t \beta_p + \partial_t \partial_t \partial_t \beta_i - \frac{4}{3} \partial_t \partial_t \kappa = 0.
$$

(57)

The boundary conditions on $\beta$ can be taken, for example,

$$
\beta_i m_i = 0, \quad \beta_i l_i = 0, \quad \frac{\partial}{\partial n} \beta_i m_i = 0,
$$

(58)

or,

$$
\frac{\partial}{\partial n} \beta_i m_i = 0, \quad \beta_i l_i = 0, \quad \beta_i n_i = 0.
$$

(59)

It is beneficial for the computation purposes to replace the elliptic equation (57) with the hyperbolic equation (24)

$$
\partial_0^2 \beta = \frac{1}{3} \partial_t \partial_t \beta_p + \partial_t \partial_t \partial_t \beta_i - \frac{4}{3} \partial_t \partial_t \kappa.
$$

(60)
which corresponds to a dynamic gamma-freezing condition \( \partial_t \Gamma_i = \partial_t^2 \beta_i \). In this case, in addition to (52) and (53), one can consider either of two sets of the radiative boundary conditions

\[
\beta_i m^i = 0, \quad \beta_i n^i = 0, \quad (\partial_t \beta_i + \frac{\partial}{\partial n} \beta_i) m^i = 0,
\]

\[
(\partial_t \beta_i + \frac{\partial}{\partial n} \beta_i) n^i = 0, \quad (\partial_t \beta_i + \frac{\partial}{\partial n} \beta_i) l^i = 0, \quad \beta_i l^i = 0.
\]

After the boundary conditions for \( \beta, A, \) and \( \kappa \) are chosen, one can set \( \Gamma_i - \Gamma(0) = 0 \) on the boundary, then compatible conditions for the rest of the variables follow from integration of the system as in the examples above.

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VII. APPENDIX: ENERGY ESTIMATES FOR THE BSSN SYSTEM WITH BOUNDARIES

We use differentiation in time to propose boundary conditions (40) (or (41)), but one will not be able to prove well-posedness of the new system by repeating the argument of Sections 4–6. In this appendix we try to fix this flaw by establishing a well posed energy estimate without reduction to second order in time. We use approach proposed by Gundlach and Martin-Garcia \cite{14,17} and modify the proof for the case of a bounded domain.

Following \cite{14}, we use the densitized lapse

\[
a = e^{\delta \varphi} Q,
\]

which results in significant simplifications in equations. The result, however, is expected to extend to harmonic slicing and \( k \)-driving slicing as well.

We illustrate the idea in the linear case, first. Under the densitized lapse assumption, equation \((63)\) is replaced with the linearized densitized lapse condition, the latter in the special case \( Q = 1 \), reduces to \( \alpha = 6 \varphi \). Assuming for further simplicity zero shift perturbation \( \beta_i = 0 \), the equations \((52), (53) - (56)\), are restated as

\[
\begin{align*}
\partial_t \varphi & = -\frac{1}{6} \kappa, \\
\partial_t \kappa & = -6 \partial_t \varphi, \\
\partial_t \gamma_{ij} & = -2 A_{ij}, \\
\partial_t A_{ij} & = -\frac{1}{2} \partial_t \gamma_{ij} + \partial_i \Gamma_j - 8 \partial_j \varphi, \\
\partial_t \Gamma_i & = -\frac{4}{3} \partial_t \kappa.
\end{align*}
\]

Taking scalar product of (53) with \( \kappa \), integrating the result by parts, and using (52) to replace \( \kappa \) with \( \partial_t \varphi \), we obtain

\[
\frac{1}{2} \partial_t ||[\kappa||^2 + 36 ||\partial_t \varphi||^2| = -6 \int_{\partial \Omega} \left( \frac{\partial}{\partial n} \varphi \right) \kappa.
\]

Also, we conclude from (52) that \( 36 ||\partial_t \varphi||^2 = ||\kappa\|^2 \).

From (52) and (66) we observe that \( \partial_t (\Gamma_i - 8 \partial_j \varphi) = 0 \), which implies that

\[
\partial_t ||\Gamma_i - 8 \partial_j \varphi||^2 = 0, \quad (\partial_t (\Gamma_i - 8 \partial_j \varphi), \partial_t \gamma_{ij}) = 0.
\]

Next we rewrite the right side of (54) in divergence form,

\[
\partial_t A_{ij} = \partial_l \left[ -\frac{1}{2} \partial_t \gamma_{ij} + \delta_{l(i}(\Gamma_j) - 8 \partial_j \varphi) \right],
\]

then, take scalar product with \( A_{ij} \), and integrate by parts:

\[
\frac{1}{2} \partial_t ||A||^2 + \left( -\frac{1}{2} \partial_t \gamma_{ij} + \delta_{l(l} (\Gamma_j) - 8 \partial_j \varphi), \partial_t A_{ij} \right)
\]

\[
= \int_{\partial \Omega} \left[ -\frac{1}{2} \frac{\partial}{\partial n} \gamma_{ij} + n_{j} (\Gamma_j) - 8 \partial_j \varphi | A_{ij} \right].
\]

Replacing \( A_{ij} \) with (64) in the second term and using \( \partial_t \delta_{l(i}(\Gamma_j) - 8 \partial_j \varphi) = 0 \), we replace the last identity with

\[
\frac{1}{2} \partial_t ||A||^2 + \frac{1}{2} \partial_t \gamma_{ij} - \delta_{l(l} (\Gamma_j) - 8 \partial_j \varphi) ||\|^2
\]

\[
= \int_{\partial \Omega} \left[ -\frac{1}{2} \frac{\partial}{\partial n} \gamma_{ij} + n_{j} (\Gamma_j) - 8 \partial_j \varphi \right] A_{ij}.
\]

From (67) one observes that the energy

\[
\epsilon = ||\kappa\|^2 + 36 ||\partial_t \varphi||^2 + ||\Gamma_i - 8 \partial_j \varphi||^2 + ||A||^2
\]

\[
+ \frac{1}{2} \partial_t \gamma_{ji} - \delta_{l(i} (\Gamma_j) - 8 \partial_j \varphi) ||\|^2,
\]

has its growth determined by the boundary terms:

\[
\partial_t \epsilon = -6 \int_{\partial \Omega} \left( \frac{\partial}{\partial n} \varphi \right) \kappa - \int_{\partial \Omega} \left( \frac{\partial}{\partial n} \gamma_{ij} \right) A_{ij}
\]

\[
+ 2 \int_{\partial \Omega} n_{l} (\Gamma_j) - 8 \partial_j \varphi A_{ij}.
\]

\footnote{The difficulty appears to be the extra derivatives of the inverse metric resulting from differentiation of (17) that contaminate principal part and break the similarity with the linear case.}
Expression (71) can be used to propose examples of new energy (or constraint) preserving boundary conditions for the linearized BSSN formulation. However, here we will use (71) to discuss the meaning of conditions (40) and (41) proposed in Section 6. Under assumption of (40) and (41) can be treated similar) and the associated conditions (10), (30), (32), expression (71) reduces to

\[ \partial_t \epsilon = \int_{\partial \Omega} \left[ -\frac{\partial}{\partial n} \phi(0) \kappa - \frac{1}{2} \frac{\partial}{\partial n} \tilde{\gamma}^3(0) A3 - 2 \frac{\partial}{\partial n} \tilde{\gamma}^4(0) A4 \right] - \frac{6}{\partial n} \tilde{\gamma}^5(0) A5 + 4 \left( n^1 \Gamma_i(0) - 8 \frac{\partial}{\partial n} \phi(0) \right) A5, \] 

(72)

where \( \tilde{\gamma}^1 \sim 5 \) are the coefficients of decomposition

\[ \tilde{\gamma}^2 = \tilde{\gamma}^1 \left( n_i (m_j) + \tilde{\gamma}^2 \left( n_i (l_j) + \tilde{\gamma}^3 \left( l_i (m_j) \right) + \tilde{\gamma}^4 \left( l_i (l_j - m_j - m_j) \right) + \tilde{\gamma}^5 \left( 2 n_i n_j - l_i l_j - m_j \right) \right) \right). \]

The energy (70) is conserved if the initial data is chosen to satisfy \( (\partial / \partial n) \phi(0) = 0, \tilde{\gamma}^1 (0) = 0, \) and \( (\partial / \partial n) \tilde{\gamma}^5 (0) = 0 \) at the boundary (notice that \( \Gamma_i (0) \) can not be given freely but is expected to be subject to constraint (29)).

Similarly, conditions (41) are constraint and energy preserving for (29 - 35) if \( \phi = 0, (\partial / \partial n) \tilde{\gamma}^1 (0) = 0, \) and \( \tilde{\gamma}^5 (0) = 0 \) on \( \partial \Omega. \)

We are now ready to establish the nonlinear analog of (71). Substituting (41) into the BSSN equations, distributing and expanding covariant derivatives, we rewrite the system,

\[ e^{6 \varphi} \partial_0 k = -6 a \partial^p \partial_q \varphi + F, \] 

(73)

\[ \partial_0 \varphi = -\frac{1}{6} a - \frac{1}{6} \tilde{b}^l l, \] 

(74)

\[ \partial_0 \tilde{h}_{ij} = -2 a \tilde{A}_{ij} + 2 \tilde{h}_{ij} \tilde{b}^l l - \frac{2}{3} \tilde{h}_{ij} \partial_0 \tilde{b}^l, \] 

(75)

\[ e^{6 \varphi} \partial_0 \tilde{A}_{ij} = -a \partial^p \left[ \frac{1}{2} \partial_p \tilde{h}_{ij} - \tilde{h}_{pi} (\tilde{\Gamma}_j - 8 \partial_j \varphi) \right] + G_{ij}, \] 

(76)

\[ \partial_0 \tilde{\Gamma}_i = -\frac{4}{3} a \partial_q k + S_i. \] 

(77)

where

\[ F = 6 a \tilde{p} q \tilde{t} \tilde{p} i q \partial_0 \varphi - 48 a \partial^p \partial_q \varphi (\partial_0 \varphi) - 14 e^{6 \varphi} (\partial_0 \varphi) (\partial_0 Q) - 2 e^{6 \varphi} (\tilde{D}^p \tilde{D}_p Q) + \frac{1}{3} e^{6 \varphi} a k^2 + e^{6 \varphi} a \tilde{A}^2 q \tilde{p} q, \]

\[ G_{ij} = -a (\partial^p \tilde{h}_{pi} (\tilde{\Gamma}_j - 8 \partial_j \varphi) + 8 a \tilde{A}^p \partial_0 \varphi - 12 a (\partial_0 \varphi) (\partial_0 \varphi) + 4 a \tilde{h}_{ij} (\partial^p \varphi) (\partial_0 \varphi) - 8 e^{6 \varphi} (\partial_0 \varphi) (\partial_0 Q) - e^{6 \varphi} (\tilde{D}_i \tilde{D}_j Q) + \frac{8}{3} e^{6 \varphi} h_{ij} (\partial^p \varphi) (\partial_0 Q) + e^{6 \varphi} \tilde{h}_{ij} (\tilde{D}^p \tilde{D}_p Q) + e^{6 \varphi} W_{ij}. \]

Taking scalar product of (70) with \( k \) and integrating by parts in the right side, we get \( (\tilde{n}_i \) is the outward normal vector to the boundary in the sense of the conformal metric \( \tilde{h} ),

\[ \int_{\Omega} \left( \partial_0 k \right) e^{6 \varphi} = 6 \int_{\Omega} \left( \partial_0 \tilde{p} q \right) (\partial_0 (ak)) \tilde{h}^{pq} - 6 \int_{\partial \Omega} \left( \tilde{n}^p \partial_0 \varphi \right) ak + \int_{\Omega} \left( 6 (\partial_0 \varphi) ak \tilde{h}_p q + F k. \right) \]

Substituting (71) for ak in the second term and re-grouping we obtain our first energy identity:

\[ \frac{1}{2} \partial_0 \| k \|^2 + 36 \| \partial_0 \varphi \|^2 = -6 \int_{\partial \Omega} \left( \tilde{n}^p \partial_0 \varphi \right) ak + \int_{\Omega} H, \]

(78)

where \( \| k \|^2 = \int_{\Omega} k^2 e^{6 \varphi}, \) and

\[ H = 6 (\partial_0 \varphi) [\partial_0 \tilde{b}^s + 6 (\partial_0 \tilde{b}^s) (\partial_0 \varphi)] \tilde{h}^{pq} + 36 (\partial_0 \varphi) (\partial_0 \varphi) (a \tilde{A}^p q - \tilde{D}^p \tilde{D}_p q) + \frac{1}{3} \tilde{h}^{pq} \partial_0 \tilde{b}^s + k^2 e^{6 \varphi} \frac{1}{6} (\tilde{b}^s) (\partial_0 \varphi) (a \tilde{A}^p q - \tilde{D}^p \tilde{D}_p q + F k. \]

Also, from (74) it follows that

\[ \frac{1}{2} \partial_0 \| \Gamma_i - 8 \partial_0 \varphi \|^2 = \int_{\Omega} \left[ \frac{1}{6} a + \frac{1}{6} \tilde{b}^l l \right] \varphi. \] 

(79)

Next, we notice from (78) and (79) that

\[ \partial_0 (\Gamma_i - 8 \partial_0 \varphi) = 8 a (\partial_0 \varphi) k + \frac{4}{3} e^{6 \varphi} (\partial_0 Q) k - \frac{4}{3} \partial_0 \tilde{b}^s, \]

\[ -8 (\partial_0 \tilde{b}^s) (\partial_0 \varphi) + S_i. \]

Therefore,

\[ \frac{1}{2} \partial_0 \| \Gamma_i - 8 \partial_0 \varphi \|^2 = \int_{\Omega} J, \] 

(80)

\[ J = [8 a (\partial_0 \varphi) k + \frac{4}{3} e^{6 \varphi} (\partial_0 Q) k - \frac{4}{3} \partial_0 \tilde{b}^s, \]

\[ -8 (\partial_0 \tilde{b}^s) (\partial_0 \varphi) + S_p |(\tilde{t} - 8 \partial_0 \tilde{b}^s) \tilde{h}^{pq} + \frac{1}{2} (\tilde{p} - 8 \partial_0 \varphi) (\tilde{t} - 8 \partial_0 \varphi) |(2 a \tilde{A}^p q - \tilde{D}^p \tilde{D}_p q) + \frac{2}{3} \tilde{h}^{pq} \partial_0 \tilde{b}^s]). \]

Finally, taking scalar product of (70) with \( A, \) integrating by parts in the right side, using (40) to replace \( A \) with \( \partial_0 \tilde{h}, \) and re-grouping, we derive our last energy identity:

\[ \frac{1}{2} \partial_0 \| \tilde{A}^l l + \| \frac{1}{2} \partial_0 \tilde{h}_{ij} - \tilde{h}_{ij} (\tilde{\Gamma}_j - 8 \partial_j \varphi) \|^2 \] 

\[ = - \int_{\partial \Omega} \left[ \frac{1}{2} (\tilde{n}^p \partial_0 \tilde{h}_{ij} - \tilde{n}_{ij} (\tilde{\Gamma}_j - 8 \partial_j \varphi) a \tilde{A}^l l + \int_{\Omega} K, \right) \] 

(81)
where \( \|A\|^2 = \int_O A_{ij} A^{ij} e^{4\varphi} \), and

\[
K = \left[ \frac{1}{2} \partial_\nu \tilde{h}_{ij} - \tilde{h}_{ij} \partial_\nu (\tilde{\Gamma}_{ij} - 8 \partial_j \varphi) \right] \times \{ \partial_i (\tilde{h}_{n(m} \partial_n \tilde{b}^a - \frac{1}{3} \tilde{h}_{mn} \partial_a \tilde{b}^n) + (\partial_i \tilde{b}^a)(\partial_a \tilde{h}_{mn}) \}
\]

\[-2\tilde{A}_q(m) \tilde{h}_{ij} \partial_i \varphi k + \frac{4}{3} e^{6\varphi} (\partial_i Q) k - \frac{4}{3} \partial_j \tilde{h}^a \partial_a \tilde{b}^n - 8(\partial_n \tilde{b}^a)(\partial_a \varphi + S_n))(\tilde{h}_{pq} \tilde{h}_{im} \tilde{h}^jn
+a \tilde{A}_{mn} \tilde{A}_q(\tilde{h}_{pq} \tilde{h}^{im} \tilde{h}^jn) \} + G_{ij} A^{ij} \text{.}
\]

Notice that boundary terms in (78) and (81) differ from that of linear case in spatial metric only. Notice also, that right sides of (78)−(81) are combinations of \( \varphi, k, \tilde{h}, \tilde{A}, \tilde{\Gamma} \), \( \partial \varphi, \partial \tilde{h} \), but not derivatives of \( k, \tilde{A}, \tilde{\Gamma} \), which implies that (78)−(81) is a closed estimate and may be proposed for proving local well-posedness of initial-boundary value problem for the BSSN system. In this derivation, we assumed that covariant components of the conformal shift vector, \( \tilde{b}^a \), are prescribed, otherwise, terms \( \partial_i \partial_j \tilde{b}^a \) has to be expanded to ensure that they do not contribute to the principal part of the equations.

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