CONFIGURATION SPACES OF $\mathbb{R}^n$ BY WAY OF EXIT-PATH $\infty$-CATEGORIES

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ABSTRACT. We approach manifold topology by examining configurations of finite subsets of manifolds within the homotopy-theoretic context of $\infty$-categories by way of stratified spaces. Through these higher categorical means, we identify the homotopy types of such configuration spaces in the case of Euclidean space in terms of the category $\Theta_n$.

CONTENTS

Introduction 1
0.1. Stratified spaces and their exit-path $\infty$-categories 3
0.2. The category $\Theta_n$ and finite subsets of $\mathbb{R}^n$ 4
0.3. Approach 8
Motivation and conjectures 8
Use of $\infty$-categories 10
Linear overview 11
Acknowledgements 12
1. The category $\Theta_n$ 12
1.1. The subcategory of active morphisms in $\Theta_n$ 14
2. The exit-path $\infty$-category of the unital Ran Space 16
2.1. Proving $\text{Exit}(\text{Ran}^u(M))$ is an $\infty$-category 17
2.2. The main result, Theorem 2.2.1 20
3. Part 1 of the proof of Theorem 2.2.1: Refining $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ 21
3.1. The exit-path $\infty$-category of the refined unital Ran space of $\mathbb{R}^n$ 21
3.2. Proving the exit-path $\infty$-category of the refined unital Ran space of $\mathbb{R}^n$ is an $\infty$-category 25
3.3. Identifying the exit-path $\infty$-category of the refined unital Ran space of $\mathbb{R}^n$ as $\Theta_n^{\text{ext}}$ 29
4. Part 2 of the proof of Theorem 2.2.1: Localizing to $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ 31
4.1. Identifying the space of objects of the localization of Lemma 4.0.2 31
4.2. Identifying the space of morphisms of the localization of Lemma 4.0.2 33
4.3. Proving Lemma 4.0.8 37
5. A corollary to Theorem 2.2.1: Identifying $\text{Exit}(\text{Ran}(\mathbb{R}^n))$ combinatorially 59
5.1. The exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$ 59
5.2. Proving Corollary 5.1.2 61
References 66

INTRODUCTION

Broadly, we ask:

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Question 0.0.1. How well does the homotopy type of the configuration space of finite subsets of a nonempty smooth manifold distinguish the homeomorphism type of that manifold?

Given a manifold \( M \), the configuration space of finite subsets of \( k \) points of \( M \) is defined to be the sub topological space of \( M^k \)

\[
\text{Conf}_k(M) := \{(x_1, \ldots, x_k) \in M^k \mid x_i \neq x_j \text{ if } i \neq j\}
\]
equipped with the subspace topology. It was previously conjectured that the homotopy type of \( \text{Conf}_k(M) \) for a closed compact smooth manifold \( M \) depends only on the homotopy type of \( M \) (see [15]\(^1\)). However, in [15] Longoni and Salvatore show, by example of certain Lens spaces, that this is not the case: the homotopy type of configuration spaces can distinguish some manifolds that are homotopy equivalent and yet not homeomorphic.

A first take on Question 0.0.1 is to examine these configuration spaces in the case of \( \mathbb{R}^n \), since manifolds are locally Euclidean. In and of itself, this case is of much interest for its inherent relationship to a plethora of topics in topology and geometry - see [16] for a lovely, comprehensive exposition; notable for the scope of this paper is the relationship between configuration spaces of unordered points in \( \mathbb{R}^n \) and spaces of framed embeddings from disjoint copies of \( \mathbb{R}^n \) into \( \mathbb{R}^n \) (i.e., embeddings which respect the canonical trivializations of the tangent bundles). Explicitly, for a smooth nonempty manifold \( M \), there is an evident action of \( \Sigma_k \) on \( \text{Conf}_k(M) \) given by permuting the order of the indexing. The resulting quotient space

\[
\text{Conf}_k(M)_{\Sigma_k} := \text{Conf}_k(M)/\Sigma_k = \{S \subset M \mid |S| = k\}
\]
is called the unordered configuration space of finite subsets of \( k \) points of \( M \). There is an evident homotopy equivalence between the space of framed embeddings of \( k \) disjoint copies of \( \mathbb{R}^n \) into \( \mathbb{R}^n \) and \( \text{Conf}_k(\mathbb{R}^n)_{\Sigma_k} \) given by evaluation at the origin.

This homotopy equivalence reveals that the homotopy types of the whole collection of these configuration spaces of \( \mathbb{R}^n \) as we vary \( k \) naturally carries the algebraic structure of an operad; this is known as the 'little disks' or \( \mathcal{E}_n \)-operad. Topologically speaking, considering the whole family of these unordered configuration spaces as we vary \( k \) is quite natural; intuitively, they organize together as a topological space. For example, let \( v \neq w \in \mathbb{R}^n \) be distinct points. Define the half-open path

\[
(0, 1] \rightarrow \text{Conf}_2(\mathbb{R}^n)_{\Sigma_2}
\]
in the unordered configuration space of two points in \( \mathbb{R}^n \) by \( t \mapsto \{tv, tw\} \). The limit of the reverse of this path as \( t \) approaches 0 is a point in the unordered configurations of one point in \( \mathbb{R}^n \). This indicates that there is a natural ambient topological space that contains both \( \text{Conf}_2(\mathbb{R}^n)_{\Sigma_2} \) and \( \text{Conf}_1(\mathbb{R}^n)_{\Sigma_1} \) as subspaces. More generally, for a fixed nonempty, smooth connected manifold \( M \), there is an ambient topological space that contains, for all \( k \geq 1 \), \( \text{Conf}_k(M)_{\Sigma_k} \) as subspaces.

This ambient space, called the Ran space (named after Ziv Ran), was introduced by Borsuk and Ulam in [11]. Following that of Beilinson and Drinfeld in §3.4.1 of [8], we define the Ran space as follows.

Definition 0.0.2. Let \( M \) be a connected manifold. The Ran space of \( M \) is the topological space whose underlying set is

\[
\text{Ran}(M) := \{S \subset M \mid S \text{ is finite and nonempty}\}
\]
and whose topology is the finest for which the map

\[
M^k \rightarrow \text{Ran}(M)
\]
given by \( (m_1, \ldots, m_k) \mapsto \{m_1, \ldots, m_k\} \), is continuous for each \( k \geq 1 \).\(^2\)

\(^1\)It is stated in [17] that this was a ‘long standing conjecture’, though we have not found it explicitly stated until [15], which came after [12].

\(^2\)There are other natural topologies that one can consider for the Ran space of a fixed manifold \( M \). We highlight Definition 5.5.1.2 defined in [19], which is equivalent to the Hausdorff topology, if \( M \) is given a metric inducing its topology (Remark 5.5.1.5 in [19]). This topology and the one defined in Definition 0.0.2 are not equivalent, but are comparable; the topology given in Definition 0.0.2 contains that of Definition 5.5.1.2 in [19]. There is another topology on the Ran space of \( M \) given in Remark 5.5.1.13 in [19]. It is not hard to prove that...
Henceforth, we will say manifold to mean a smooth, nonempty manifold. Though the Ran space of a manifold is weakly contractable (\cite{12} and \cite{8}), one can recover the homotopy types of the unordered configuration spaces $\text{Conf}_k(M)$ using the \textit{exit-path $\infty$-category} construction of \cite{19} and \cite{4}; this construction applied to the Ran space of $\mathbb{R}^n$ is at the heart of this work. Indeed, enabled by the developments of \cite{4} on exit-path $\infty$-categories, we define an $\infty$-category, denoted $\text{Exit}(\text{Ran}^u(M))$, which codifies the homotopy types of these unordered configuration spaces for a connected manifold $M$. Our main result identifies this $\infty$-category combinatorially in terms of Joyal’s category $\Theta_n$ in the case of $\mathbb{R}^n$; it is the following.

**Theorem 0.0.3.** For $n \geq 1$, there is a localization\footnote{This is not a reflective, nor a coreflective localization; see Definition 4.0.1} of $\infty$-categories

$$
\Theta_n^{\text{act}} \to \text{Exit}(\text{Ran}^u(\mathbb{R}^n))
$$

over $\text{Fin}^{op}$ from the subcategory $\Theta_n^{\text{act}}$ of $\Theta_n$, to the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$.

The statement of this theorem requires some explanation, which will be supplied in the following two sections. Notably, we will define the Ran space as a \textit{stratified space}, \textit{exit-path $\infty$-categories}, and the category $\Theta_n$.

### 0.1. Stratified spaces and their exit-path $\infty$-categories

Notice that, for a connected manifold $M$, there is a natural filtration of $\text{Ran}(M)$ by bounding cardinality; so the filtration complements are the unordered configuration spaces. This filtration is, in fact, an example of a \textit{stratified space}, which, after Lurie in \cite{19}, is the following.

**Definition 0.1.1** (A.5.1 in \cite{19}).

- (Topological structure on a poset) Let $P$ be a partially ordered set. We equip $P$ with the topology that defines $U \subset P$ to be open if and only if it is \textit{closed upwards}: that is, if $a \in U$, then every $b \geq a$ is also in $U$.
- A \textit{$P$-stratified space} $X \xrightarrow{\sigma} P$ is a paracompact, Hausdorff topological space $X$ together with a poset $P$ and a continuous map $\sigma$ such that for each $p \in P$, the fiber over $p$ is nonempty and connected. The fiber over $p \in P$ is called the \textit{$p$-stratum}, which we denote by $X_p$.

We will denote a $P$-stratified space $X \xrightarrow{S} P$ by its underlying topological space $X$, if we expect $S$ and $P$ to be understood.

**Example 0.1.2.** We stratify the topological space $\mathbb{R}^2$: Define the poset $P$ to be

\[
\begin{array}{c}
1^+ \\
2^0 \\
2^+ \\
1^0 \\
2^- \\
1^-
\end{array}
\]

where the arrows indicate the ordering. The following figure depicts a stratified space $\mathbb{R}^2 \to P$, with each stratum labeled. The positioning and coloring of the figure and $P$ are coordinated as to indicate the stratifying map, as well.

\footnote{Similarly, the topology on the Ran space of $M$ given in Definition 3.7.1 in \cite{6} is equivalent to that of Definition 0.0.2.}
Example 0.1.3. The standard stratification of the topological $k$-simplex 
\[ \Delta^k := \{ (t_0, \ldots, t_k) \in [0,1]^{k+1} \mid \sum_{i=0}^k t_i = 1 \} \to [k] := \{0 < \cdots < k\} \]
is given by 
\[ (t_0, \ldots, t_k) \mapsto \max \{ i \mid t_i \neq 0 \}. \]
This stratification for $\Delta^0$, $\Delta^1$, and $\Delta^2$ is depicted below, from left to right, respectively:

\[
\begin{array}{ccc}
0 & \rightarrow & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & 1 & \rightarrow & 2 \\
\end{array}
\]

Definition 0.1.4 (2.1.10 in [6]). For stratified spaces $X \to P$ and $Y \to Q$, a continuous stratified map from $(X \to P)$ to $(Y \to Q)$ is a continuous map $X \xrightarrow{f} Y$ such that

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
P & \rightarrow & Q
\end{array}
\]
is a commutative diagram of topological spaces.

Example 0.1.5. The Ran space of a connected manifold $M$ emits a stratification over the natural numbers by cardinality. Namely, the stratified space

\[ \text{Ran}(M) \to \mathbb{N} \]
assigns to each point $S \subset M$ in $\text{Ran}(M)$ the cardinality of the underlying set $S$.
For $p \in \mathbb{N}$, the $p$-stratum of $\text{Ran}(M)$ is precisely $\text{Conf}_p(M)_{\Sigma p}$.

We recover the homotopy type of the unordered configuration spaces of a connected manifold $M$, which arise as the strata of the Ran space of $M$, as the underlying $\infty$-groupoid of the exit-path $\infty$-category of the Ran space of $M$. More generally, we can construct the exit-path $\infty$-category of any stratified space. To give an idea of this construction, recall the fundamental $\infty$-groupoid of a topological space $X$: Informally, an object is a point in $X$ and a morphism is a path in $X$. The exit-path $\infty$-category is an analogue for stratified spaces: An object is a point in the underlying topological space and a morphism is a path in the space subject to a certain ‘exiting’ constraint which depends on the stratification of the space. Following [19], we define the following.

Definition 0.1.6. For a stratified space $X \xrightarrow{f} P$, the exit-path $\infty$-category of $X$, denoted $\text{Exit}(X)$, is the simplicial set whose value on $[k]$ is the set

\[ \{ \Delta^k \xrightarrow{\sigma} X \mid \sigma \text{ is a stratified map} \}. \]

\footnote{The simplicial set $\text{Exit}(X)$ is equivalent to $\text{Sing}^{\infty}(X)$ (called the $\infty$-category of exit-paths in $X$) defined in Definition A.6.2 of [19]. The simplicial set $\text{Sing}^{\infty}(X)$ is defined to be the subsimplicial set of $\text{Sing}(X)$ in which a $[k]$-point $\Delta^k \xrightarrow{\sigma} X$ satisfies the condition that there exists a sequence $p_0 \leq \cdots \leq p_k$ in $P$ such that for each $(t_0, \ldots, t_k \neq 0, \ldots, 0) \in \Delta^k, f(\sigma(t_0, \ldots, t_k, 0, \ldots, 0)) = a_i$. It is a simple observation that the existence of such a sequence in $P$ for each $[k]$-point $\sigma$ is equivalent to the data of $\sigma$ as a stratified map, implementing the equivalence between these simplicial sets.}
MacPherson first introduced the notion of an ‘exit-path category’ and Treumann extended this notion to the ‘exit-path 2-category’ \[28\]. A similar notion to the exit-path $\infty$-category, called a ‘configuration category’ is given in \[1\] and further developed in \[10\].

Informally, an object of the exit-path $\infty$-category is a point in $X$ and a morphism is a path in $X$ which is allowed to ‘exit’ from a deeper stratum to a less-deep stratum, but not vice-versa. In other words, these ‘exit-paths’ swim along with the ordering of the poset $P$, but not against it. In the case of the Ran space of a connected manifold $M$, an object of the exit-path $\infty$-category of the Ran space of $M$ is a finite subset of $M$ and a morphism witnesses anticollision of points in $M$.

**0.2. The category $\Theta_n$ and finite subsets of $\mathbb{R}^n$.** In \[7\], Ayala and Hepworth show that the homotopy types of the configuration spaces of ordered points in $\mathbb{R}^n$ are naturally encoded by Joyal’s category $\Theta_n$: a category in which an object is a finite, $n$-ordered set (explained below). They conjecture that the exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$ is a localization of a subcategory of $\Theta_n$. Their methods do not extend toward a proof of this conjecture, so a new idea is required, which is supplied in this paper.

First, let us give a description of the objects of $\Theta_n$ using finite rooted planar level trees, following §2 of \[7\]. With this description in hand, we will be able to give some intuition for why the exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$ has a combinatorial description in terms of $\Theta_n$, at least on the level of objects.

**Definition 0.2.1.**
- A *level tree* is a finite, rooted tree, the root of which is a choice of vertex thereof. The choice of root uniquely determines a direction to each edge such that there is a unique directed path from each vertex to the root.
- For each vertex, we may equip the set of edges directed toward the vertex with a linear order. A tree is called a *planar level tree* if such an order is specified with respect to each vertex.
- A vertex is at *level* $i$ if the directed path from the vertex to the root counts $i$ edges.
- A tree has *height* $n$ if the maximum level of all the vertices is $n$.
- A vertex is a *leaf* if it has no edges directed towards it.
- A planar level tree of height $n$ is *healthy* if all of its leaves are at level $n$.

We will say *tree* to mean finite rooted planar level tree. If a tree is not healthy, we call it *unhealthy*.

The following are depictions of trees, where the root is at the bottom of each diagram; on the left, $T_1$ is a healthy tree of height one, in the middle, $T_2$ and $T_2'$ are a healthy trees of height two, and on the far right, $T_2^{uh}$ is a unhealthy tree of height two.

![Figure 1](image-url)

In particular, the set of leaves of a healthy tree of height $n$ naturally emits an $n$-order. Formally, an $n$-order on a finite set $S$ is a sequence of surjective maps

$S \xrightarrow{\sigma_{n-1}} S_{n-1} \rightarrow \cdots \xrightarrow{\sigma_1} S_1$

among finite sets together with a linear order on $S_1$ and a linear order on each fiber of each map in the sequence; that is, for each $1 \leq i \leq n - 1$, a linear order on $\sigma_i^{-1}(s)$ for each $s \in S_i$. A 1-order on a set $S$, then, is just a linear order on $S$. 

5
A healthy tree $T$ of height $n$ naturally encodes an $n$-order on its set of leaves as follows. The $i$th set of the sequence is the set of vertices at level $i$. The assignment from the set of vertices at level $i$ to the set of vertices at level $i - 1$ is canonical, given by assigning to each vertex at level $i$ the vertex at level $i - 1$ that is adjacent to it. Such a sequence of maps, then, is surjective precisely because $T$ is healthy. The planar condition (the second bullet point in Definition 0.2.1) fixes the set of incoming edges of each vertex with a linear order; this condition induces a linear order on each fiber in the sequence and on the set of vertices at level 1. As a convention, we will fix these linear orders to be read from left to right on trees. For now, we will informally regard the objects of the category $\Theta_n$ as trees of height $n$. Unfortunately, with this description of the objects, the morphisms of $\Theta_n$ are not straightforward. For now, we will leave them as a mystery; in §1, we give a full definition of $\Theta_n$ as the $n$-fold wreath product of the simplex category $\Delta$ with itself, following Berger in [9].

In [7], Ayala and Hepworth show that $\text{Conf}_k(\mathbb{R}^n)$ is homotopy equivalent to the classifying space of a certain subcategory of $\Theta_n$ consisting of all those healthy trees with $k$ leaves. Their method of proof does not extend to the unordered case wherein the homotopy type of the unordered configuration spaces of $\mathbb{R}^n$ are encoded by the exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$, as they conjecture. The main result of this paper, Theorem 0.0.3, provides a generalization of this conjecture, one consequence of which is Corollary 0.2.2, which proves their conjecture.

**Corollary 0.2.2.** For $n \geq 1$, there is a localization

$$\Theta_n^{\text{exit}} \to \text{Exit}(\text{Ran}(\mathbb{R}^n))$$

over $(\text{Fin}^{\text{surj}})^{\text{op}}$ from the subcategory $\Theta_n^{\text{exit}}$ of $\Theta_n$, to the exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$.

The category $\Theta_n^{\text{exit}}$ is given in Definition 5.1.1. Informally, its objects are all those trees that are healthy (e.g., $T_2$ and $T_2'$ from [1]).

Towards our more general result, Theorem 0.0.3, we introduce the exit-path $\infty$-category of the unital Ran space of a connected manifold $M$, $\text{Exit}(\text{Ran}^u(M))$: an $\infty$-category that contains the exit-path $\infty$-category of the Ran space of $M$ as an $\infty$-subcategory. Heuristically, the objects of $\text{Exit}(\text{Ran}^u(M))$ are the same as in $\text{Exit}(\text{Ran}(M))$ together with the empty subset of $M$; a morphism in $\text{Exit}(\text{Ran}^u(M))$ witnesses anti-collision of points, just like in $\text{Exit}(\text{Ran}(M))$, but also ‘disappearances’ of points, unlike in $\text{Exit}(\text{Ran}(M))$. Explicitly, we define $\text{Exit}(\text{Ran}^u(M))$ to be the simplicial space in which an object is a finite, possibly empty subset $S$ of $M$, and a morphism from $S \subset M$ to $T \subset M$ is a map between sets $T \to S$ together with an injection $(S \coprod (T \times \Delta^1)) \hookrightarrow M \times \Delta^1$ over $\Delta^1$.

**Example 0.2.3.** Consider the map of sets $T = \{t, t', t''\} \to S = \{s, s', s''\}$ given by $t, t' \mapsto s$ and $t'' \mapsto s'$. The coproduct $S \coprod (T \times \Delta^1)$ over $\Delta^1$ is depicted by the following.

![Figure 2](image-url)
We formally define the exit-path ∞-category of the unital Ran space of $M$ in §2. Our main result, Theorem 0.0.3 identifies this ∞-category combinatorially in the case of $\mathbb{R}^n$ as a localization of a certain subcategory of $\Theta_n$. The superscript ‘u’ stands for ‘unital’, which, through Theorem 0.0.3, makes reference to the role of degeneracy morphisms in $\Theta_n$.

0.2.1. Intuition for the main results. With the following informal discussion, we hope reveal some of the natural intuition of the localizations of Theorem 0.0.3 and Corollary 0.2.2. Heuristically, an ∞-category $\mathcal{C}$ localizes to an ∞-category $\mathcal{D}$ if there is some collection of morphisms in $\mathcal{C}$ such that the ∞-category obtained from $\mathcal{C}$ by formally inverting each morphism in this collection is equivalent to $\mathcal{D}$; for a formal definition, see Definition 4.0.1. Roughly speaking, this means that our results Corollary 0.2.2 and Theorem 0.0.3 state that upon inverting some collection of morphisms in $\Theta_n^{\text{exit}}$, we obtain an ∞-category which is equivalent to the exit-path ∞-category of the (unital) Ran space of $\mathbb{R}^n$. On the level of objects, a relationship is no surprise since finite subsets of $\mathbb{R}^n$ naturally inherit an $n$-order from $\mathbb{R}^n$. Namely, a finite subset $S \subset \mathbb{R}^n$ inherits the $n$-order given by the sequence of projection maps $S \twoheadrightarrow \text{pr}_{<n}(S) \twoheadrightarrow \cdots \twoheadrightarrow \text{pr}_{<2}(S)$ each of which projects off the last coordinate. The linear order on the base set and on each fiber is canonically inherited from $\mathbb{R}^n$ with a fixed orientation.

First, we consider the assignment on the level of objects given by the equivalence induced upon localizing in Corollary 0.2.2 for the cases $n = 1, 2$. We will use the healthy trees $T_1$ and $T_2$ and $T_2'$ from [1] regarded as objects of $\Theta_1^{\text{exit}}$ and $\Theta_2^{\text{exit}}$, respectively, as examples for each case, respectively. Note these examples also apply to Theorem 0.0.3 since the localization of Corollary 0.2.2 is a restriction of that of Theorem 0.0.3.

For the case $n = 1$, fix an object $\text{Exit}(\text{Ran}(\mathbb{R}))$ of cardinality two. As a subset of $\mathbb{R}$, it inherits a linear order (a.k.a., a 1-order) from $\mathbb{R}$ equipped with a fixed orientation. This object is identified with its underlying linearly ordered set, which is codified by the tree $T_1$ from [1]. More generally, any subset of $\mathbb{R}$ with cardinality $k$ is identified with the tree of height 1 that has $k$ leaves. This identification indicates the canonical assignment of Corollary 0.2.2 on the level of objects for the case $n = 1$. We would like to note, in fact, that the localization is trivial in the case $n = 1$.

For the case $n = 2$, the subset of $\mathbb{R}^2$ depicted on the left below is identified with the tree $T_2$ from [1], and the subset on the right is identified with $T_2'$. (Note that in these pictures we let the horizontal axis be the first coordinate of $\mathbb{R}^2$.)

![Figure 3](image)

In other words, these finite subsets of $\mathbb{R}^2$, which are objects in $\text{Exit}(\text{Ran}(\mathbb{R}^2))$, are assigned to the objects $T_2$ and $T_2'$, respectively, in the equivalence induced upon localizing in Corollary 0.2.2 for the case $n = 2$. Note that the objects of $\text{Exit}(\text{Ran}(\mathbb{R}^2))$ depicted in [3] are isomorphic precisely because, as points in $\text{Ran}(\mathbb{R}^2)$, they are in the same stratum. Thus, it is the case that $T_2$ and $T_2'$ become isomorphic upon localizing $\Theta_n^{\text{exit}}$ in Corollary 0.2.2. In general, the only isomorphisms in $\Theta_n$ are automorphisms, whereas in the exit-path ∞-category of the Ran space of $\mathbb{R}^n$, this is not the case: any path in $\text{Ran}(\mathbb{R}^n)$ that stays in the same stratum is an invertable morphism. Informally, then, the localization of Corollary 0.2.2 formally inverts all those morphisms of $\Theta_n^{\text{exit}}$ that induce bijections between their sets of leaves.
Next, consider the object $T_2^{\text{sth}}$ of $\Theta_n^{\text{act}}$ from \[\Box\]. Note that this example only applies for the more general of our two results, Theorem 0.0.3 because $T_2^{\text{sth}}$ is unhealthy and is therefore not an object of $\Theta_n^{\text{exit}}$. The object of $\text{Exit}(\text{Ran}^n(\mathbb{R}^2))$ depicted on the left in \[\Box\] is identified with $T_2^{\text{sth}}$ through the localization of Theorem 0.0.3. The same subset, however, is identified with $T_2'$ from \[\Box\] as previously discussed; this exemplifies that the trees $T_2$ and $T_2^{\text{sth}}$ also become isomorphic (and therefore $T_2'$ as well) upon localizing $\Theta_n^{\text{act}}$ in Theorem 0.0.3. This example indicates a heuristic for the morphisms in the localizing subcategory of Theorem 0.0.3 similar to that of Corollary 0.2.2; namely, all those morphisms in $\Theta_n^{\text{act}}$ which induce bijections between the sets of leaves are localized.

0.3. Approach. As defined in Definition 0.1.6, $\text{Exit}(X)$ for a $P$-stratified space $X \to P$ is not necessarily a quasi-category. Indeed, the definition of a $P$-stratified space previously given in Definition 0.1.1 is not strong enough to guarantee that $\text{Exit}(X)$ satisfies the inner-horn filling condition. Rather, a stronger notion of a stratified space, namely, a conically stratified space (Definition A.5.5 in [19]), is needed to guarantee that the exit-path $\infty$-category is indeed an $\infty$-category (Theorem A.6.4 in [19]).

Ayala-Francis-Tanaka in [6] also develop a theory of stratified spaces, coined conically smooth stratified spaces, which is a notion of a stratified space even stronger than that of [19]. The defining feature of a conically smooth stratified space is that it is locally $\mathbb{R}^n$ producted with the open cone of some stratified space of a lower dimension; such structure is used for the construction of ‘links’ between strata.

Further, in [4] Ayala-Francis-Rozenblyum define the exit-path $\infty$-category of a stack on stratified spaces; an extension of that of Definition 0.1.6 in that, it agrees with Definition 0.1.6 for a conically smooth stratified space $X$ (Lemma 3.3.9 of [4]).

The codomain of our main result Theorem 0.0.3, the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$, is conceived from [4]. In particular, we define $\text{Exit}(\text{Ran}^n(M))$ as a simplicial space (Definition 2.0.3) and show that it can be derived through formal constructions among the $\infty$-categories developed in [4].

The key idea behind the localization of Theorem 0.0.3 from $\Theta_n^{\text{act}}$ to $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ is that the exit-path $\infty$-category construction of [4] carries refinements of stratified spaces (Definition 2.1.4) to localizations of their exit-path $\infty$-categories (Theorem 3.3.12 of [4]). This idea does not directly apply to the case at hand however, since $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ is not the exit-path $\infty$-category of a stratified space. Rather, this idea motivates our method of proof; namely, we introduce an $\infty$-category, $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$, defined in Definition 3.1.2 whose $[p]$-spaces are inspired by refining the stratified space $\text{Ran}^n(\mathbb{R}^n)$ and we prove an equivalence between $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ and $\Theta_n^{\text{act}}$. Then, we prove that $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ localizes to $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$. To do this, we use Theorem 4.0.6 of [22] which gives a method for identifying localizations in favorable cases.

Much of the proof of Corollary 0.2.2 is extrapolated from the proof of Theorem 0.0.3 since both the domain $\Theta_n^{\text{act}}$ and codomain $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ of the localization in Corollary 0.2.2 are sub $\infty$-categories (respectively) of the domain $\Theta_n^{\text{act}}$ and codomain $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ of the localization in Theorem 0.0.3.

Motivation and conjectures

Rezk used the category $\Theta_n$ to define $(\infty, n)$-categories as certain functors from $\Theta_n^{op}$ to Spaces; a direct generalization of his theory of complete Segal spaces [25]. In light of this, this work stands

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5 The Ran space of a connected manifold $M$ is not a conically smooth stratified space precisely because it is not locally compact. However, its subspaces of bounded cardinality are examples of conically smooth stratified spaces (Proposition 3.7.5 of [6]). We expect that the Ran space of a connected manifold is an example of a conically stratified space, though we are unaware of a proof in the literature.

6 Through the lens of the aforementioned objects and morphisms of the exit-path $\infty$-category of the unital Ran space of a connected manifold $M$, let us entertain for a moment what the ‘unital Ran space’ of $M$, $\text{Ran}^u(M)$, as a stratified space would be: one would need to put a topology on the set $\text{Ran}(M)$ II $\{\emptyset\}$ such that the aforementioned morphisms of $\text{Exit}(\text{Ran}^u(M))$ are continuous paths in $\text{Ran}^u(M)$ - oh no! It is possible however, to define the unital Ran space of $M$ as a stack on stratified spaces so that its resulting value under the exit-path $\infty$-category construction of [4] is $\text{Exit}(\text{Ran}^u(M))$, though we do not do so in this paper.
as a new perspective on a bridge that is conjectured to exist between configuration spaces of finite subsets in $\mathbb{R}^n$ and the combinatorics of $(\infty, n)$-categories. Broadly, we ask:

**Question 0.3.1.** How are the combinatorics of $(\infty, n)$-categories described by configuration spaces of finite subsets of $\mathbb{R}^n$?

As a first take, Corollary 0.2.2 relates the exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$ to presheaves on $\Theta_n$ through the identification of $\text{Exit}(\text{Ran}(\mathbb{R}^n))$ in terms of the subcategory $\Theta_n^{\text{exit}}$. Namely, there is a fully faithful functor

\[ \text{Cat}(\infty, n) \longrightarrow \text{Fun}(\text{Exit}(\text{Ran}(\mathbb{R}^n))^{\text{op}}, \text{Spaces}) \]

(1)

Recall that the collection of unordered configuration spaces of finite subsets of $\mathbb{R}^n$ exhibit the algebraic structure of the $E_n$-operad. In fact, Lurie identifies a relationship $\text{Alg}_{E_n}(\text{Spaces}) \rightarrow \text{Shv}_{\text{Spaces}}(\text{Ran}(\mathbb{R}^n))$ from $E_n$-algebras valued in $\text{Spaces}$ to constructable cosheaves on the Ran space of $\mathbb{R}^n$ (Theorem 5.5.4.10 [19]). (Roughly, constructible means locally constant on each stratum.)

Corollary 3.3.11 of [4] yields an equivalence

\[ \text{Shv}_{\text{Spaces}}(\text{Ran}(\mathbb{R}^n)) \cong \text{Fun}(\text{Exit}(\text{Ran}(\mathbb{R}^n)), \text{Spaces}) \]

between space-valued constructible cosheaves on $\text{Ran}(\mathbb{R}^n)$ and functors from $\text{Exit}(\text{Ran}(\mathbb{R}^n))$ to $\text{Spaces}$. Thus, there is a relationship $\text{Alg}_{E_n}(\text{Spaces}) \rightarrow \text{Fun}(\text{Exit}(\text{Ran}(\mathbb{R}^n)), \text{Spaces})$ which articulates the relationship between $E_n$-algebras and $\text{Exit}(\text{Ran}(\mathbb{R}^n))$.

More generally, let $\mathcal{V}$ be a symmetric monoidal $\infty$-category. Consider the $\infty$-category $\text{Alg}_{E_n}(\mathcal{V})$ of $E_n$-algebras in $\mathcal{V}$. Consider the $\infty$-category $\text{Cat}(\infty, n)(\mathcal{V})$ of $\mathcal{V}$-enriched ($\infty, n$)-categories. Consider the object $1 \in \text{Cat}(\infty, n)(\mathcal{V})$ whose underlying ($\infty, n-1$)-category $1_{\leq n} \simeq *$ is terminal, and whose $n\text{End}_1(*) = 1 \in \mathcal{V}$.

Say a morphism $A \xrightarrow{F} \mathcal{B}$ between ($\infty, n$)-categories is $n$-connected if, for each $0 \leq k \leq n$, each diagram among ($\infty, n$)-categories

\[
\begin{array}{ccc}
\partial c_k & \longrightarrow & A \\
\downarrow & & \downarrow F \\
\downarrow c_k & \longrightarrow & B
\end{array}
\]

admits a filler. Here, $c_k$ is the ($\infty, n$)-category corepresenting the space of $k$-morphisms, and $\partial c_k \rightarrow c_k$ is the inclusion of its maximal ($\infty, k-1$)-subcategory.
Conjecture 0.3.2. Let $\mathcal{V}$ be a symmetric monoidal $\infty$-category. There is a fully faithful left adjoint $\mathfrak{B}^n : \text{Alg}_{\mathcal{E}_n}(\mathcal{V}) \to \text{Cat}_{(\infty,n)}(\mathcal{V})^1$; the image consists of those morphisms $1 \to \mathcal{C}$ between $\mathcal{V}$-enriched $(\infty,n)$-categories whose underlying morphism $\ast \to \mathcal{C}_{\leq n}$ between $(\infty,n-1)$-categories is $(n-1)$-connected.

If we specialize to the case that $(\mathcal{V}, \otimes) = (\text{Spaces}, \times)$, the results Theorem 0.0.3 and Corollary 0.2.2 address this conjecture in that they yield fully faithful functors fitting into the following diagram.

$$
\begin{array}{ccc}
\text{Alg}_{\mathcal{E}_n}(\text{Spaces}) & \xrightarrow{\mathfrak{B}^n} & \text{Cat}_{(\infty,n)}(\text{Spaces})^1 \\
\text{forget} & & \text{forget (fully faithful, by def)} \\
\text{Fun}\left(\text{Exit}\left(\text{Ran}^u(\mathbb{R}^n)^\text{op}, \text{Spaces}\right)\right) & \xrightarrow{\text{Thm 0.0.3}} & \text{Fun}\left(\Theta_n^{\text{op}}, \text{Spaces}\right) \\
\text{forget} & & \text{forget} \\
\text{Fun}\left(\text{Exit}\left(\text{Ran}^u(\mathbb{R}^n)^\text{op}, \text{Spaces}\right)\right) & \xrightarrow{\text{Cor 0.2.2}} & \text{Fun}\left(\Theta_n^{\text{ex}}, \text{Spaces}\right) \\
\text{forget} & & \text{forget} \\
\text{coShv}_{\text{Spaces}}(\text{Ran}(\mathbb{R}^n)) & & \text{coShv}_{\text{Spaces}}(\text{Ran}(\mathbb{R}^n)) \\
\end{array}
$$

We are presently working on a generalization of Theorem 0.0.3 that identifies an $\infty$-category $\text{Exit}\left(\text{Ran}^u(\mathbb{R}^n)^\text{op}\right)$, defined similar to $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$, as a localization of $\Theta_n$.

**USE OF $\infty$-CATEGORIES**

In this paper, we use $(\infty,1)$-categories, or simply, `$\infty$-categories', to package the homotopy type of configuration spaces. There are many models for $\infty$-categories; notable for the scope of this paper is the work of Lurie in [19] on the theory of quasi-categories and the work of Rezk in [26] on the theory of complete Segal spaces.

**Quasicategories.** We use Joyal’s quasi-category model of $\infty$-categories from [14], wherein a quasi-category is defined to be a simplicial set, that is, a functor from the opposite of the simplex category $\Delta^\text{op}$ (Definition 1.0.1) to the category of sets $\text{Set}$, that satisfies a certain condition called the inner-horn filling condition. For the definition of this condition, together with other basic notions regarding quasi-categories see Rezk’s friendly exposition [27].

**Complete Segal spaces.** We use Rezk’s complete Segal spaces to model $\infty$-categories [26]. The $\infty$-category of spaces $\text{Spaces}$ is the localization of the category of topological spaces that admit a CW structure and continuous maps thereof, localized on (weak) homotopy equivalences. We call its objects spaces, i.e., CW complexes. A complete Segal space is a simplicial space, that is, a functor from the opposite of the simplex category $\Delta^\text{op}$ to the $\infty$-category of spaces, $\text{Spaces}$, that satisfies the completeness and Segal conditions (Definition 2.1.1). We refer the reader to [26] for any additional information regarding complete Segal spaces.
**The nerve functor.** There is a construction which takes an ordinary category $\mathcal{C}$ and produces an $\infty$-category $\mathcal{NC}$ called the (ordinary) nerve of $\mathcal{C}$. This construction is explicated by a fully faithful functor from the category of categories to the category of simplicial sets, through which each category is carried to a quasi-category. In light of the fully faithfulness of this functor, we refer to an ordinary category as an $\infty$-category without any reference to its nerve, whenever appropriate within the context. For a definition of the nerve, see Definition 3.1 in [27].

**Model independence.** In this paper, we work model independently, which, by the work of Joyal and Tierney in [15], is a valid approach, since quasi-categories are shown to be equivalent to complete Segal spaces. Model independence is exercised in this paper, for example, in that the hom-$\infty$-groupoid with fixed source and target of a quasi-category is equivalent to a space (i.e., CW complex) by way of the equivalence of quasi-categories and complete Segal spaces. Throughout this work, we are liberal with our use of model independence and typically do not give forewarning of its implementation.

**Linear overview**

§ 1. We begin by defining the category $\Theta_n$ as the $n$-fold wreath product of the simplex category $\Delta$ with itself. This definition yields a description of the objects which is naturally equivalent to that of finite rooted planar level trees. We conclude this section defining and recording some observations about the domain of our main result Theorem 0.0.3, the subcategory $\Theta_n^{\text{act}}$ of active morphisms in $\Theta_n$.

§ 2. This section is devoted to the codomain of our main result Theorem 0.0.3, $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$. We define it as a simplicial space in Definition 2.0.3, and then go on to prove that it is an $\infty$-category. At the end of this section, we restate our main result, Theorem 2.2.1, that there is a localization $\Theta_n^{\text{act}} \to \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$.

§ 3. This section is the first part (of two) of the proof of the main result Theorem 2.2.1. Our method of proof is motivated by Theorem 3.3.12 of [4] which states that the exit-path $\infty$-category construction carries refinements of stratified spaces to localizations. Towards the idea of a refinement, we define an $\infty$-category $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ (Definition 3.1.2). The main result of this section is Lemma 3.3.2 which states that there is an equivalence $\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{\sim} \Theta_n^{\text{act}}$.

Under the equivalence, $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ acts as a sort of middleman between the combinatorial domain $\Theta_n^{\text{act}}$ and the topological codomain $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ of our main result Theorem 2.2.1.

§ 4. This section is the second part (of two) of the proof of the main result Theorem 0.0.3. In it we show that there is a localization $\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \to \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ by a natural forgetful functor; this is articulated as Lemma 4.0.2. Our argument is technical, built around Theorem 4.0.6 from [22] which identifies localizations of $\infty$-categories in favorable cases. In light of this theorem, we prove two lemmas, Lemma 4.0.7 and Lemma 4.0.8 together which compile to prove that $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ localizes to $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$.

Through the equivalence between $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ and $\Theta_n^{\text{act}}$ of Lemma 3.3.2, then, our main result, Theorem 0.0.3 which states that $\Theta_n^{\text{act}}$ localizes to $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$, is established.

§ 5. This section is devoted to the main corollary of our result, Corollary 5.1.2 which states that there is a localization $\Theta_n^{\text{exit}} \to \text{Exit}(\text{Ran}(\mathbb{R}^n))$ to the exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$. Our method of proof mainly extrapolates that of Theorem 2.2.1 in that it, too, is built from Theorem 4.0.6 of [22].
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1. The category $\Theta_n$

In this section, we define the domain of the main result, Theorem 0.0.3. Namely, we use Berger’s definition from [9] to define Joyal’s category $\Theta_n$ as the $n$-fold wreath product of the simplex category $\Delta$ with itself.

Definition 1.0.1. The simplex category $\Delta$ is the category in which an object is a nonempty, finite, linearly ordered set and in which a morphism is a non-decreasing map of sets. Composition is composition of maps between sets.

For each object $S$ of $\Delta$, there is a unique non-negative integer $p$ such that $S$ is canonically isomorphic to the linearly ordered set $[p] := \{0 < \cdots < p\}$. We call $[p]$ the $p$-simplex and will henceforth refer to the objects of $\Delta$ as $p$-simplicies.

The category of posets $\text{Poset}$ has an evident fully faithfully embedding into the category of categories $\text{Cat}$. In light of this fully faithful functor, we refer to $[p]$ as either a linearly ordered set or as the category whose objects are $\{0, 1, \ldots, p\}$ and in which there is a unique morphism from $i$ to $j$ precisely when $i \leq j$, and no morphism otherwise.

Definition 1.0.2. The category of pointed, finite sets $\text{Fin}_*$ is the category in which an object is a finite, pointed set and a morphism is a pointed map; composition is evident.

Notation 1.0.3. Given a finite set $S$, let $S^*$ denote the finite, pointed set $S \amalg \{\ast\}$.

Definition 1.0.4. The wreath product $\text{Fin}_* \bowtie \mathcal{D}$ for an arbitrary category $\mathcal{D}$ is the category defined as follows: An object is a symbol $S(d_s)$ where $S$ is a finite set and $(d_s)_{s \in S}$ is a tuple of objects in $\mathcal{D}$ indexed by $S$. A morphism $S(d_s) \to T(e_t)$ consists of a pair of data:

i) A morphism $S \delta \to T_\ast$ in $\text{Fin}_*$

ii) For each pair $(s \in S, t \in T)$ such that $\delta(s) = t$, a morphism $d_s \delta_{st} \to e_t$ in $\mathcal{D}$.

Composition is given by composition in $\text{Fin}_*$ and $\mathcal{D}$.

Observation 1.0.5. There is a forgetful functor $\text{Fin}_* \bowtie \mathcal{D} \to \text{Fin}_*$ given by $S(d_s) \mapsto S$; its value on morphisms is evident.

Definition 1.0.6. Given a category $\mathcal{C} \to \text{Fin}_*$ over the category of based finite sets and a category $\mathcal{D}$, the wreath product $\mathcal{C} \bowtie \mathcal{D}$ is the pullback of categories

\[
\begin{array}{ccc}
\mathcal{C} \bowtie \mathcal{D} & \longrightarrow & \text{Fin}_* \bowtie \mathcal{D} \\
\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & \text{Fin}_*
\end{array}
\]

where the vertical arrow on the right is the forgetful functor from Observation 1.0.5.

We take advantage of the previous observation and define Joyal’s category $\Theta_n$ over $\text{Fin}_*^{op}$ inductively as the $n$-fold wreath product of the simplex category $\Delta$ with itself.

Definition 1.0.7. The assembly functor

$\text{Fin}_* \bowtie \text{Fin}_* \overset{\nu}{\longrightarrow} \text{Fin}_*$

is given by the wedge sum. Explicitly, the value of $\nu$ on an object $S((T_s)_s)$ is the wedge sum $\bigvee_{s \in S} (T_s)_s$. Its value on a morphism $S((T_s)_s) \to S'((T'_{s'})_{s'})$ given by

i) A morphism $S \delta \to S'$
ii) For each pair \((s, s')\) such that \(\delta(s) = s'\), a morphism \((T_s)_* \xrightarrow{\delta_{ss'}} (T_{s'})_*\), is
\[
\bigvee_{s \in S} (T_s)_* \to \bigvee_{s' \in S'} (T_{s'})_*
\]
defined by \(t \in T_s \mapsto \delta_{ss'}(t)\) for every pair \((s, s')\) such that \(\delta(s) = s'\).

**Definition 1.0.8.** The simplicial circle is the functor
\[
\Delta \xrightarrow{\gamma} \text{Fin}_*^{\text{op}}
\]
the value of which on an object \([p]\) is the quotient morphism set \(\Delta([p],[1])\)/\{\{0\}, \{1\}\}, where \{i\} denotes the constant map at \(i\); \{0\} \sim \{1\} is the evident basepoint of the image. The value of \(\gamma\) on a morphism \([p] \xrightarrow{f} [q]\) is precomposition.

**Observation 1.0.9.** The map induced by \(\gamma\) between each hom-set is injective. This observation comes down to the fact that on morphisms \(\gamma\) is given by precomposition and composition is unique in categories.

**Observation 1.0.10.** There is an evident isomorphism \(\gamma([p]) \xrightarrow{\approx} \{1, \ldots, p\}_*\) in \(\text{Fin}_*\). Let \(\nu_j\) denote the morphism that assigns each \(0 \leq i \leq j - 1\) to 0 and each \(j \leq i \leq p\) to 1 (i.e., a unique composite of degeneracy maps). The value of \(\nu_j\) under the isomorphism is \(j\). Its assignment on morphisms, then, is evident.

**Terminology 1.0.11.** In light of the previous observation, we will freely refer to a non-basepoint value in the pointed set \(\gamma([p])\) by \(j\) for some \(1 \leq j \leq p\).

**Definition 1.0.12.** For each integer \(n \geq 1\), the categories \(\Theta_n\) are defined inductively by setting
\[
\Theta_1 := \Delta \quad \text{and} \quad \Theta_n := \Theta_1 \wr \Theta_{n-1}
\]
where the assembly functors \(\Theta_n \xrightarrow{\gamma_n} \text{Fin}_*^{\text{op}}\) are also defined inductively by setting
\[
\gamma_1 := \gamma \quad \text{and} \quad \gamma_n := \nu \circ (\gamma_1 \wr \gamma_{n-1}).
\]
Recall that a planar level tree of height \(n\) is a finite rooted tree with an \(n\)-order on its set of leaves (Definition 0.2.1). These trees, in fact, naturally describe the objects of \(\Theta_n\), as we will explain in the following observation.

**Observation 1.0.13 (4.5 in [7]).** Finite rooted planar level trees of height \(n\) naturally describe the objects of \(\Theta_n\) as follows: When \(n = 1\), the object \([p]\) corresponds to the tree of height 1 that has \(p\) leaves. For \(n > 1\), the object \([p](T_1, \ldots, T_p)\) of \(\Theta_n\) corresponds to the tree described as follows: The tree which has \(p\) vertices at level 1; the \(i\)th vertex at level 1 (according to the linear order) is the root of the tree which corresponds to \(T_i\), i.e., it is the subtree consisting of those vertices for which the unique directed path from each one to the root intersects the \(i\)th vertex at level 1. In terms of trees, then, the assembly functor \(\gamma_n\) assigns a tree to its set of leaves.

In this paper, we will use this description of the objects of \(\Theta_n\) whenever convenient. In fact, we often prefer it since it makes the objects of \(\Theta_n\) so accessible.

**Example 1.0.14.** The object, \([3](1, [3], [0])\) in \(\Theta_2\), corresponds to the following planar level tree of height 2.
Remark 1.0.15. Alternatively (but equivalently), Rezk defined the category $\Theta_n$ as the full subcategory of the category of strict $n$-categories, $\mathbf{Cat}_n$, in which an object is a pasting diagram \((15)\). For example, the object \([3][1, 3, 0]\) in $\Theta_2$ corresponds to the pasting diagram

\[
\begin{array}{ccc}
0 & \downarrow & 1 \\
\downarrow & \Psi & \downarrow \\
1 & \Psi & 2 \\
\Psi & \downarrow & 3
\end{array}
\]

1.1. The subcategory of active morphisms in $\Theta_n$. Our main result identifies the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$ with the subcategory $\Theta_n^{\text{act}}$ of active morphisms of $\Theta_n$. More generally, we define the active subcategory of a category over pointed, finite sets.

Definition 1.1.1. $\text{Fin}$ is the category in which an object is a finite set and a morphism is a map of sets; composition is evident.

Definition 1.1.2. Given a category $\mathcal{C} \xrightarrow{F} \text{Fin}_*$ over pointed, finite sets, a morphism $\sigma$ in $\mathcal{C}$ is active if $(F(\sigma))^{-1}(* \setminus \{\ast\}) = \{\ast\}$. The subcategory $\mathcal{C}^{\text{act}}$ of $\mathcal{C}$ is defined to be the pullback $\mathcal{C}^{\text{act}} \xrightarrow{\frgt} \text{Fin} \xleftarrow{\text{Fin}} \text{Fin}_*$.

For a category $\mathcal{D}$, there is a monomorphism $\text{Fin} \wr \mathcal{D} \to \text{Fin}_* \wr \mathcal{D}$ and thus, the category $\text{Fin}_* \wr \mathcal{D}$ has an explicit description of objects and morphisms similar to that of $\text{Fin} \wr \mathcal{D}$ as given in Definition 1.0.4. We will use this description of $\text{Fin} \wr \mathcal{D}$ whenever convenient. Additionally, we would like to point out that given a category $\mathcal{C} \xrightarrow{F} \text{Fin}_*$ over pointed, finite sets, $F$ restricts to a functor $\mathcal{C}^{\text{act}} \to \text{Fin}_*$ which, by definition, factors through $\text{Fin}$. Evidently then, the category $\mathcal{C}^{\text{act}} \wr \text{Fin}$ is canonically isomorphic to $\mathcal{C}^{\text{act}} \wr \text{Fin}_*$.

The following alternative definition of $\Theta_n^{\text{act}}$ coincides with that of Definition 1.1.2.

Definition 1.1.3. For each integer $n \geq 1$, the categories $\Theta_n^{\text{act}}$ are the subcategories of $\Theta_n$ defined inductively by setting $\Theta_1^{\text{act}} := \Delta^{\text{act}}$ and defining $\Theta_n^{\text{act}} := \Theta_1^{\text{act}} \wr \Theta_{n-1}^{\text{act}}$, i.e., the pullback

\[
\begin{array}{ccc}
\Theta_n^{\text{act}} & \xrightarrow{\gamma_1} & \text{Fin}^\text{op} \wr \Theta_{n-1}^{\text{act}} \\
\downarrow & & \downarrow \\
\Theta_1^{\text{act}} & \xrightarrow{\frgt} & \text{Fin}^\text{op}
\end{array}
\]

(3)

The benefit of this particular formulation of $\Theta_n^{\text{act}}$ is that it makes the following facts straightforward.

Observation 1.1.4. Because the wreath product is associative, equivalently $\Theta_n^{\text{act}} := \Theta_{n-1}^{\text{act}} \wr \Theta_1^{\text{act}}$.

Observation 1.1.5. For each $n$, there is a natural forgetful functor

$\Theta_n^{\text{act}} \xrightarrow{\text{tr}} \Theta_{n-1}^{\text{act}}$.

The value on an object $T_{n-1}([m_k]) \in \Theta_{n-1}^{\text{act}} \wr \Theta_1^{\text{act}} =: \Theta_n^{\text{act}}$ is $T_{n-1}$.

Let $T_{n-1}([m_k]) \xrightarrow{\sigma'} W_{n-1}([p_l])$ be a morphism in $\Theta_n^{\text{act}}$ defined by

i) a morphism $T_{n-1} \xrightarrow{\sigma'} W_{n-1}$ in $\Theta_{n-1}^{\text{act}}$ and,

ii) a morphism $[m_k] \xrightarrow{\gamma_{n-1}} [p_l]$ in $\Theta_1^{\text{act}}$ for each pair $(k, l)$ such that $\gamma_{n-1}(l) = k$.

The value of $\sigma$ under $\text{tr}$ is $\sigma'$. 

14
**Example 1.1.6.** Let $T$ be the object of $\Theta_3$ depicted as the far left tree in the figure below. We depict two iterations of the truncation functor $\text{tr}$ on $T$:

![Diagram of trees]

**Notation 1.1.7.** For each $1 \leq i \leq n-1$, denote the $(n-i)$-fold composite of the truncation functor $\text{tr}$ by $\text{tr}_i : \Theta_i^{\text{act}} \to \Theta_i^{\text{act}}$.

**Observation 1.1.8.** For each $1 \leq i \leq n-1$, there is a natural transformation from $\Theta_i^{\text{act}} \stackrel{\gamma_i}{\longrightarrow} \text{Fin}^{\text{op}}$ to the composite $\gamma_i \circ \text{tr}_i$,

$$
\Theta_i^{\text{act}} \xrightarrow{\text{tr}_i} \Theta_i^{\text{act}} \xrightarrow{\gamma_i} \text{Fin}^{\text{op}}.
$$

For each tree $T$, the natural transformation $\epsilon$ is given by the natural map $\gamma_n(T) \xrightarrow{\epsilon} \gamma_i(\text{tr}_i(T))$ from the leaves of $T$ to the leaves of the truncation of $T$ to height $i$, the assignment of which is the evident one given by the structure of the tree $T$. It is straightforward to verify that $\epsilon$ does indeed define a natural transformation.

**Definition 1.1.9.** Given a category $\mathcal{C}$, we define $\text{Fun}(\{1 < \cdots < n\}, \mathcal{C})$ to be the category in which an object is a functor $\{1 < \cdots < n\} \to \mathcal{C}$ which selects out a sequence of composable morphisms in $\mathcal{C}$:

$$
c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_n
$$

and a morphism from $c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_n$ to $d_1 \rightarrow d_2 \rightarrow \cdots \rightarrow d_n$ is a commutative diagram in $\mathcal{C}$:

$$
c_1 \rightarrow d_1 \\
c_2 \downarrow \downarrow \\
\vdots \\
c_n \rightarrow d_n.
$$

Composition is evident.

**Observation 1.1.10.** We use Observation 1.1.8 to define the natural functor

$$
\Theta_n^{\text{act}} \stackrel{\gamma_n}{\longrightarrow} \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})
$$

the value of which on an object $T$ is the functor which selects out the sequence of composable maps of sets

$$
\gamma_n(T) \xrightarrow{\epsilon} \gamma_{n-1}(\text{tr}(T)) \xrightarrow{\epsilon_{\text{tr}(T)}} \gamma_{n-2}(\text{tr}_{n-2}(T)) \rightarrow \cdots \rightarrow \gamma_1(\text{tr}_1(T))
$$
and on a morphism $T \xrightarrow{f} S$ is the diagram of finite sets

\[
\begin{array}{ccc}
\gamma_n(S) & \xrightarrow{\gamma_n(f)} & \gamma_n(T) \\
\epsilon_S \downarrow & & \downarrow \epsilon_T \\
\gamma_{n-1}(\text{tr}_{n-1}(S)) & \xrightarrow{\gamma_{n-1}(\text{tr}_{n-1}(f))} & \gamma_{n-1}(\text{tr}_{n-1}(T)) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\gamma_1(\text{tr}_1(S)) & \xrightarrow{\gamma_1(\text{tr}_1(f))} & \gamma_1(\text{tr}_1(T))
\end{array}
\]

which is guaranteed to commute in $\text{Fin}$ because the downward arrows in the diagram are given by the natural transformation $\epsilon$ from Observation 1.1.8.

2. The exit-path $\infty$-category of the unital Ran space

In this section, we articulate the sense in which we encode the homotopy type of unordered configurations of points in $\mathbb{R}^n$ with an $\infty$-category. Namely, we define the codomain of our main result, Theorem 0.0.3, as a simplicial space and then show that it is an $\infty$-category.

**Definition 2.0.1** (6.6.12 in [4]). The reversed cylinder of a map between finite sets $T \to S$ is

\[
\text{cylr}(T \to S) := S \coprod_{T \times \{0\}} T \times \Delta^1.
\]

More generally, the reversed cylinder of a composable sequence of maps between finite sets $S_p \to S_{p-1} \to \cdots \to S_0$ is

\[
\text{cylr}(S_p \to S_{p-1} \to \cdots \to S_0) := S_0 \coprod_{S_1 \times \{0\}} S_1 \times \Delta^1 \coprod_{S_2 \times \Delta^1} \cdots \coprod_{S_p \times \Delta^{p-1}} S_p \times \Delta^p.
\]

For an example of the cylinder construction for an explicit map of sets, see Example 0.2.3.

**Observation 2.0.2.** Recall that by definition the nerve of $\text{Fin}^{op}$ is a presheaf on $\Delta$. Thus, we may define the $\infty$-category $\Delta$ slice over $\text{Fin}^{op}$ as the pullback of $\infty$-categories

\[
\Delta/\text{Fin}^{op} \longrightarrow \text{PShv}(\Delta)/\text{Fin}^{op}
\]

\[
\Delta \xrightarrow{\text{Yoneda}} \text{PShv}(\Delta)
\]

where $\text{PShv}(\Delta)$ is the category of presheaves on the simplex category.

**Definition 2.0.3.** For a smooth, connected manifold $M$, the exit-path $\infty$-category of the unital Ran space of $M$, $\text{Exit}(\text{Ran}^{u}(M))$, is the simplicial space over $\text{Fin}^{op}$ representing the presheaf on $\Delta/\text{Fin}^{op}$ whose value on an object $[p] \xrightarrow{\leq \sigma} \text{Fin}^{op}$ which selects out a sequence of maps among finite sets $\sigma : S_p \to \cdots \to S_0$, is the space of embeddings $\text{cylr}(\sigma) \hookrightarrow M \times \Delta^p$ over $\Delta^p$ equipped with the compact-open topology; the structure maps are evident.

**Observation 2.0.4.** Explicitly, an object of $\text{Exit}(\text{Ran}^{u}(M))$ in the fiber over the finite (possibly empty) set $S$ is an embedding,

\[
S \hookrightarrow M.
\]
A morphism from $S \xhookrightarrow{e} M$ to $T \xhookrightarrow{d} M$ over the map of finite sets $T \xrightarrow{a} S$ is an embedding,

$$\text{cylr}(T \xrightarrow{a} S) \xhookrightarrow{E} M \times \Delta^1$$

over $\Delta^1$ such that $E|_S = e$ and $E|_{T \times 1} = d$.

**Observation 2.0.5.** There is a natural forgetful functor

$$\text{Exit}(\text{Ran}^u(M)) \xrightarrow{\phi} \text{Fin}^{op}$$

the value of which on an object $S \hookrightarrow M$ is $S$ and on a morphism $\text{cylr}(J \xrightarrow{\sigma} S) \hookrightarrow M \times \Delta^1$ is $\sigma$.

Recall Example 0.2.3 wherein we gave an example of the cylinder of a fixed map of finite sets. Any embedding of this cylinder (depicted by (2)) into $M \times \Delta^1$ over their projections to $\Delta^1$ is an example of a morphism in $\text{Exit}(\text{Ran}^u(M))$. Heuristically, we can think of such a morphism simply as a collection of paths in $M$ parameterized by $\Delta^1$ each of which is pairwise disjoint in $M$ for all $0 < t \leq 1$ in $\Delta^1$ (i.e., ‘anticollision of points’ is allowed) together with some points which ‘disappear’ after $t = 0 \in \Delta^1$. Such a morphism, however, is not a path in the Ran space of $M$ (Definition 0.0.2) precisely because the map of sets $T \to S$ in Example 0.2.3 is not surjective. We are led to observe the following characterisation of the exit-path $\infty$-category of the Ran space of $M$ relative to the exit-path $\infty$-category of the unital Ran space of $M$.

**Observation 2.0.6.** There is a natural forgetful functor

$$\text{Exit}(\text{Ran}(M)) \xrightarrow{\phi} \text{Fin}^{op}$$

whose value on an object $S \subset M$ is the underlying set $S$ and whose value on a morphism $\Delta^1 \xrightarrow{f} \text{Ran}(M)$ is the map of finite sets $f(\{1\}) \to f(\{0\})$ given by the reverse of the path in $M$. Note that, in particular, the image is a surjection. With the functor $\phi$ in mind then, observe that a point $\Delta^k \xrightarrow{f} \text{Ran}(M)$ in the set of $[k]$-values of $\text{Exit}(\text{Ran}(M))$ is precisely an embedding

$$\text{cylr}(S_k \hookrightarrow \cdots \hookrightarrow S_0) \hookrightarrow M \times \Delta^k$$

over $\Delta^k$ of the reverse cylinder of the sequence of surjective maps $S_k \to \cdots \to S_0$ defined to be the image of $f$ under $\phi$. Thus, the exit-path $\infty$-category of the Ran space of $M$ is a sub simplicial space of the exit-path $\infty$-category of the unital Ran space of $M$, described as the following pullback of simplicial spaces:

$$\text{Exit}(\text{Ran}(M)) \xleftarrow{\phi} \text{Exit}(\text{Ran}^u(M))$$

$$(\text{Fin}^{\text{surj}})^{op} \xleftarrow{} \text{Fin}^{op}$$

where $\text{Fin}^{\text{surj}}_{\neq \emptyset}$ is the subcategory of finite sets consisting of nonempty sets and all those morphisms that are surjections.

**2.1. Proving $\text{Exit}(\text{Ran}^u(M))$ is an $\infty$-category.** Of central importance in this paper is that the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$ is, in fact, an $\infty$-category. This subsection is devoted to proving the technical result that for a connected manifold $M$, $\text{Exit}(\text{Ran}^u(M))$ is a complete Segal space. First, we give a definition of a complete Segal space following (3.3.1) of [4].

**Definition 2.1.1 (after [26]).** A simplicial space $\Delta^{op} \xrightarrow{E} \text{Spaces}$ is a complete Segal space if it satisfies the following two conditions:
i) (Segal Condition) For each \( p > 1 \), the diagram of spaces

\[
\begin{array}{ccc}
F[p] & \xrightarrow{\gamma} & F\{p-1 < p\} \\
\downarrow & & \downarrow \\
F\{0 < \cdots < p-1\} & \longrightarrow & F\{p-1\}
\end{array}
\]

is a pullback.

ii) (Completeness Condition) The diagram of spaces

\[
\begin{array}{ccc}
F(\ast) & \xleftarrow{\gamma} & F[3] \\
\downarrow & & \downarrow \\
F\{0 < 2\} & \longrightarrow & F\{1 < 3\}
\end{array}
\]

is a limit.

**Proposition 2.1.2.** The simplicial space \( \text{Exit}(\text{Ran}^u(M)) \) satisfies the Segal and completeness conditions.

**Corollary 2.1.3.** The simplicial space \( \text{Exit}(\text{Ran}(M)) \) satisfies the Segal and completeness conditions.

**Proof.** In Observation 2.0.6 we observed the pullback diagram among simplicial spaces

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}(M)) & \xleftarrow{\gamma} & \text{Exit}(\text{Ran}^u(M)) \\
\downarrow & & \downarrow \\
(\text{Fin}^\text{surj})^\text{op} & \longrightarrow & \text{Fin}^\text{op}
\end{array}
\]

The result follows because the full \( \infty \)-subcategory of simplicial spaces consisting of the complete Segal spaces is closed under the formation of pullbacks. \( \square \)

We allow ourselves, in this subsection, to freely use notation and results from [4]. The idea for this proof is to witness the simplicial space \( \text{Exit}(\text{Ran}^u(M)) \) as one derived through formal constructions among complete Segal spaces from a complete Segal space \( \text{Bun} \), defined in §6 of [4].

Namely, the simplicial space \( \text{Bun} : \Delta^\text{op} \to \text{Spaces} \) is that for which the value on \([p] \) is the moduli space of constructible bundles over \( \Delta^p \) with its standard stratification (see Example 0.1.3). The simplicial structure maps are implemented by base change of constructible bundles. Section §6 of [4] is devoted to the proof that this simplicial space satisfies the Segal and completeness conditions, which is to say \( \text{Bun} \) is an \( \infty \)-category.

So the space of objects in \( \text{Bun} \) is the moduli space of constructible bundles over \( \Delta^0 = \ast \), which is simply the moduli space of stratified spaces. So an object of \( \text{Bun} \) is simply a stratified space. In particular, a finite set is an example of an object in \( \text{Bun} \), and a smooth manifold is an example of an object in \( \text{Bun} \) as well. Lemma 6.3.11 of [4] uses the reverse cylinder (Definition 2.0.1) construction to construct a fully faithful functor

\[
\text{Fin}^\text{op} \longrightarrow \text{Bun} ,
\]

whose image consists of finite sets. In particular, there is a composite monomorphism between \( \infty \)-categories:

\[
(5) \quad \text{Fin}^\text{op} \xrightarrow{(-)_+} \text{Fin}^\text{op}_\ast \longrightarrow \text{Bun} .
\]
Note that a \([p]\)-point \(X \to \Delta^p\) of \(\text{Bun}\) factors through this monomorphism (5) if and only if \(X \to \Delta^p\) is a finite proper constructible bundle.

Lemma 3.31 of [5] constructs, for each dimension \(k\), the \(k\)-skeleton functor\(\text{sk}_k : \text{Bun} \to \text{Bun}.\)

Explicitly, the value on a stratified space \(X\) is the proper constructible stratified subspace \(\text{sk}_k(X) \subset X\) that is the union of the strata whose dimension is at most \(k\). The value of \(\text{sk}_k\) on a \([p]\)-point \(X \to \Delta^p\) of \(\text{Bun}\) is the constructible bundle
\[
\text{sk}_k^{\text{fib}}(X) \longrightarrow \Delta^p
\]
which is the fiberwise \(k\)-skeleton of \(X\) – it is the union of those strata of \(X\) whose projection to \(\Delta^p\) have fiber-dimension at most \(k\).

Note, then, that \(\text{sk}_0\) factors through \(\text{Fin}_{\ast}^{\text{op}}:\)
\[
\text{sk}_0 : \text{Bun} \longrightarrow \text{Fin}_{\ast}^{\text{op}} \hookrightarrow \text{Bun}.\]

Consider the \(\infty\)-category \(\text{Strat}^{\text{ref}}\) underlying the topological category in which an object is a stratified space and the space of morphisms is that of refinements.

**Definition 2.1.4** (3.6.1 in [6]). A map of stratified spaces \((X \to P) \\{\xrightarrow{f}\} (Y \to Q)\) is a refinement if \(f\) is a homeomorphism between the underlying topological spaces, and if for each \(p \in P\) the restriction of \(f\) to each stratum \(X_p\) is an embedding into \(Y\).

Section §6.6 of [4] constructs the open cylinder functor between \(\infty\)-categories
\[
\text{Cylo} : \text{Strat}^{\text{ref}} \longrightarrow \text{Bun}
\]
which is an equivalence on spaces of objects. Theorem 6.6.15 of [4] verifies that this functor is a monomorphism. So each refinement between stratified spaces defines a morphism in \(\text{Bun}\). As a matter of notation, a morphism in \(\text{Bun}\) that is in the image of this functor is called a refinement; the \(\infty\)-category of refinement arrows in \(\text{Bun}\) is the full \(\infty\)-subcategory
\[
\text{Ar}^{\text{ref}}(\text{Bun}) \subset \text{Ar}(\text{Bun})
\]
consisting of the refinements arrows. Evaluation at source-target defines a functor\(\text{(ev}_s,\text{ev}_t) : \text{Ar}^{\text{ref}}(\text{Bun}) \longrightarrow \text{Bun} \times \text{Bun}.\)

Denote the pullback \(\infty\)-category:
\[
\begin{array}{ccc}
\text{Ref}^0(M) & \longrightarrow & \text{Ar}^{\text{ref}}(\text{Bun}) \\
\downarrow & & \downarrow \text{(ev}_s,\text{ev}_t) \\
\text{Fin}^{\text{op}} & \longrightarrow & \text{Bun} \times \text{Bun} \\
\downarrow & & \downarrow \text{sk}_{n-1} \times \text{Id} \\
\text{Fin}^{\text{op}} & \longrightarrow & \text{Fin}^{\text{op}} \times *\text{((M))}_p \times \text{Bun} \times \text{Bun}
\end{array}
\]

Unpacking this definition (and using the open cylinder construction of [4] referenced above in (5)) \(\text{Ref}^0(M)\) is a simplicial space whose value on \([p] \in \Delta\) is the moduli space of

- constructible bundles \(X \longrightarrow \Delta^p\) for which the \((n-1)\)-skeleton of each fiber of which is a finite set,
- together with a refinement \(X \xrightarrow{\text{refinement}} M \times \Delta^p\) over \(\Delta^p\).
We will denote such a \([p]\)-point of \(\text{Ref}^0(M)\) simply as \((X \xrightarrow{\text{ref}} M \times \Delta^p)\). Example 2.1.7 of [6] shows that the product of stratified spaces is naturally a stratified space. We consider \(M \times \Delta^p\) as a product stratified space, where \(M\) is trivially stratified over the poset with a singleton, and \(\Delta^p\) is given the standard stratification (see Example 0.1.3).

**Remark 2.1.5.** Informally, an object in \(\text{Ref}^0(M)\) is a refinement of \(M\) in which the \((n - 1)\)-skeleton of the domain is a finite set, and a morphism in \(\text{Ref}^0(M)\) is a path of such refinements of \(M\) witnessing anti-collision of strata and disappearences of strata.

**Observation 2.1.6.** There is a natural forgetful functor

\[
\text{Ref}^0(M) \rightarrow \text{Fin}^{op}
\]

to the opposite of the category of finite sets, the value of which on an object \(X \xrightarrow{\text{ref}} M\) is the underlying set of the \((n - 1)\)-skeleton of \(X\), and on a morphism \(X \xrightarrow{\text{ref}} M \times \Delta^1\) is the canonical assignment between sets, from the \((n - 1)\)-skeleton of the fiber of \(X \rightarrow \Delta^1\) over \(\{1\}\) in \(\Delta^1\), implemented by taking connected components of the \((n - 1)\)-skeleton of \(X\). In other words, taking connected components of the fiberwise \((n - 1)\)-skeleton of \(X\) induces a canonical assignment of sets

\[
\text{sk}_{n-1}^{\text{fib}}(X_{11}) \rightarrow \text{sk}_{n-1}^{\text{fib}}(X_{00})
\]

from the \((n - 1)\)-skeleton of the target of \(X\) to the \((n - 1)\)-skeleton of the source of \(X\).

**Lemma 2.1.7.** There is a canonical equivalence between simplicial spaces over \(\text{Fin}^{op}\):

\[
\text{Ref}^0(M) \simeq \text{Exit}(\text{Ran}^u(M))
\]

**Proof.** A rightward morphism is implemented by, for each \([p] \in \Delta\), the assignment,

\[
(X \xrightarrow{\text{ref}} M \times \Delta^p) \mapsto (\text{sk}_{n-1}^{\text{fib}}(X) \hookrightarrow X \rightarrow M \times \Delta^p)
\]

whose value is the embedding over \(\Delta^p\) from the fiberwise \((n - 1)\)-skeleton, which maps to \(\Delta^p\) as a finite proper constructible bundle. A leftward morphism is implemented by, for each \([p] \in \Delta\), the assignment,

\[
(\text{Cyl}(\sigma) \hookrightarrow M \times \Delta^p) \mapsto \left(\left((\text{Cyl}(\sigma) \subset M \times \Delta^p) \xrightarrow{\text{ref}} M \times \Delta^p\right)\right)
\]

whose value is the coarsest refinement of \(M \times \Delta^p\) for which the embedding from \(\text{Cyl}(\sigma)\) is a proper and constructible. (Such a refinement exists because the image of this embedding is, by definition, a properly embedded stratified subspace.)

It is straightforward to verify that these two assignments are mutually inverse to one another, and further, that they are both over \(\text{Fin}^{op}\). \(\square\)

**Proof of Proposition 2.1.2.** Being an \(\infty\)-category, the simplicial space \(\text{Ref}^0(M)\) satisfies the Segal and completeness conditions. Through the equivalence of Lemma 2.1.7, then so does the simplicial space \(\text{Exit}(\text{Ran}^u(M))\). \(\square\)

2.2. **The main result, Theorem 2.2.1.** We restate our main result Theorem 0.0.3 as follows.

**Theorem 2.2.1.** For \(n \geq 1\), there is a localization of \(\infty\)-categories

\[
\Theta_n^{\text{act}} \rightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^n))
\]

over \(\text{Fin}^{op}\) from the subcategory \(\Theta_n^{\text{act}}\) of active morphisms of \(\Theta_n\), to the exit-path \(\infty\)-category of the unital Ran space of \(\mathbb{R}^n\).

Our method of proof for Theorem 2.2.1 is motivated by the result of [4] that the exit-path \(\infty\)-category construction carries refinements of stratified spaces to localizations. Towards the idea of a refinement, in §3.1 we define the simplicial space, \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\), which inherently has a topological flavor closer to that of \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\) in that its \([p]\)-spaces are built with embeddings into Euclidean
space. In particular, we design its \([p]\)-spaces to be reminiscent of a refinement of the stratified space \(\text{Ran}(\mathbb{R}^n)\) by coordinate coincidence, in addition to cardinality. This refining of \(\text{Ran}(\mathbb{R}^n)\) by keeping track of coordinate coincidence is inspired by the inherent relationship between the objects of \(\Theta_n\) as finite rooted planar level trees and finite subsets of \(\mathbb{R}^n\) as \(n\)-ordered sets. And indeed, we show that this simplicial space, \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\), is equivalent to \(\Theta_n^{\text{act}}\) in \(\S 3.3\). In this way, \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\) acts as a ‘refined’ version of \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\), and as such, is the middleman between the combinatorial domain \(\Theta_n^{\text{act}}\) and the topological codomain \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\) of our main result Theorem 2.2.1. Then, in \(\S 4\) we show that \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\) localizes to \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\).

3. Part 1 of the proof of Theorem 2.2.1: Refining \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\)

The goal of this section is to prove an equivalence between the subcategory of active morphisms of \(\Theta_n\) and the \(\infty\)-category \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\). We define this \(\infty\)-category next in \(\S 3.1\). Then, in \(\S 3.2\) we prove that it is an \(\infty\)-category, and lastly, in \(\S 3.3\) we identify this \(\infty\)-category as equivalent to \(\Theta_n^{\text{act}}\).

3.1. The exit-path \(\infty\)-category of the refined unital Ran space of \(\mathbb{R}^n\).

In this subsection we define the simplicial space \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\), whose \([p]\)-spaces are inspired by the idea of keeping track of the coordinate coincidence of finite subsets of \(\mathbb{R}^n\).

**Observation 3.1.1.** Recall that by definition the nerve of \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\) is a presheaf on \(\Delta\). Thus, we may define the \(\infty\)-category \(\Delta\) slice over \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\) as the following pullback

\[
\begin{array}{ccc}
\Delta/\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}) & \longrightarrow & \text{PShv}(\Delta)/\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}) \\
\downarrow & & \downarrow \\
\Delta & \longrightarrow & \text{PShv}(\Delta)
\end{array}
\]

where \(\text{PShv}(\Delta)\) is the category of presheaves on the simplex category.

**Definition 3.1.2.** The exit-path \(\infty\)-category of the refined unital Ran space of \(\mathbb{R}^n\) \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\) is the simplicial space over \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\) representing the presheaf on \(\Delta/\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\) whose value on an object

\[
[p] \rightarrow \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})
\]

which selects a diagram of finite sets

\[
\begin{array}{ccc}
s_n : S_n^p & \longrightarrow & \cdots & \longrightarrow & S_n^0 \\
\downarrow & & & & \downarrow \\
\vdots & & & & \vdots \\
\downarrow & & & & \downarrow \\
s_1^p & \longrightarrow & \cdots & \longrightarrow & S_1^0
\end{array}
\]

is the space of compatible embeddings.
where each embedding is over $\Delta^p$ and the downward arrows on the lefthand side are induced by the downward arrows of (8). This embedding space is given the compact-open topology; the structure maps are evident. Observation 3.1.5 defines a canonical functor $\text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \rightarrow \text{Fin}^{\mathbb{R}^n}$.

**Observation 3.1.3.** Explicitly, an object of $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ over the sequence of finite sets,

$$
S_n \xrightarrow{\tau_{n-1}} S_{n-1} \xrightarrow{\tau_{n-2}} \cdots \rightarrow S_1
$$

is a sequence of embeddings,

$$
S_n \xrightarrow{e_n} \mathbb{R}^n
$$

When the context is clear, we denote (11) by $S \xrightarrow{\varepsilon} \mathbb{R}^n$ or just $\varepsilon$. A morphism from $S \xrightarrow{\varepsilon} \mathbb{R}^n$ to $T \xrightarrow{d} \mathbb{R}^n$ over the diagram of finite sets,
is a sequence of embeddings,

\[
\begin{array}{ccc}
\text{cylr}(\sigma_n) & \xleftarrow{E_n} & \mathbb{R}^n \times \Delta^1 \\
\downarrow & & \downarrow \\
\text{cylr}(\sigma_{n-1}) & \xleftarrow{E_{n-1}} & \mathbb{R}^{n-1} \times \Delta^1 \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\text{cylr}(\sigma_1) & \xleftarrow{E_1} & \mathbb{R} \times \Delta^1 \\
\end{array}
\]

(13)

over \(\Delta^1\) such that \(E_i|_{S_i} = e_i\) and \(E_i|_{T_i \times \{1\}} = d_i\), for each \(1 \leq i \leq n\). When the context is clear, we denote (13) by \(\text{cylr}(\sigma) \xleftarrow{E} \mathbb{R}^n \times \Delta^1\) or \(E\).

Heuristically, a morphism in the exit-path \(\infty\)-category of the refined unital Ran space of \(\mathbb{R}^n\) consists of a morphism in \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\) for each \(1 \leq i \leq n\), the collection of which are compatible under, but not limited to projection.

**Notation 3.1.4.** We denote a point in the \([p]\)-space of \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\) over (12) by \(\text{cylr}(\sigma) \xleftarrow{E} \mathbb{R}^n \times \Delta^p\).

**Observation 3.1.5.** For each \(1 \leq i \leq n\), there is a natural forgetful functor to finite sets

\[\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{\phi_i} \text{Fin}^{op}\]

that forgets all but the set data at the \(\mathbb{R}^i\) level. Its value on an object \(S \xleftarrow{E} \mathbb{R}^n\) is \(S_i\) and on a morphism

\[\text{cylr}(T \xleftarrow{E} S) \xleftarrow{E} \mathbb{R}^n \times \Delta^1\]

from \(S \xleftarrow{E} \mathbb{R}^n\) to \(T \xleftarrow{E} \mathbb{R}^n\) is the map of finite sets

\[T_i \xrightarrow{\sigma_i} S_i.\]

The collection of functors \(\{\phi_i\}_{i=1}^n\) naturally compile to name the canonical functor

\[\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{\phi} \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{op})\]

which just remembers the underlying set data, i.e., its value on an object \(S \xleftarrow{E} \mathbb{R}^n\) is the functor which selects out the composable sequence of maps of finite sets \(S\) and its value on a morphism \(\text{cylr}(T \xleftarrow{E} S) \xleftarrow{E} \mathbb{R}^n \times \Delta^1\) is the commutative diagram of finite sets \(T \xleftarrow{E} S\). .

**Observation 3.1.6.** There is a natural forgetful functor

\[\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \rightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\]

over \(\text{Fin}^{op}\) induced by the functor from \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{op})\) to \(\text{Fin}^{op}\) that evaluates on \(\{n\}\).

The value of a \([p]\)-value \(\text{cylr}(\sigma) \xleftarrow{E} \mathbb{R}^n \times \Delta^p\) over \(T\) under this forgetful functor is the embedding of \(E\) at the \(\mathbb{R}^n\) level

\[\text{cylr}(\sigma_n) \xleftarrow{E_n} \mathbb{R}^n \times \Delta^p\]

over \(\sigma_n : S^p_n \rightarrow \cdots \rightarrow S^0_n\).
Observation 3.1.7. There is a natural forgetful functor
\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \overset{\rho}{\longrightarrow} \text{Exit}(\text{Ran}^u(\mathbb{R}))
\]
that forgets all but the first coordinate data. The image of a \([p]\)-value
\[
\text{cyl}(\sigma) \overset{E}{\hookrightarrow} \mathbb{R}^n \times \Delta^p
\]
over \([S]\) under \(\rho\) is
\[
\text{cyl}(\sigma_1) \overset{E_1}{\hookrightarrow} \mathbb{R} \times \Delta^p
\]
defined over \(\sigma_1 : S_1^p \to \cdots \to S_0^p\).

Observation 3.1.8. There is a natural functor
\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \overset{\pi}{\longrightarrow} \text{Fin}^{op} \backslash \text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1}))
\]
the value of which on an object \(S \overset{\xi}{\hookrightarrow} \mathbb{R}^n\) is
\[
S_1((S)_s \overset{\xi_s}{\hookrightarrow} \mathbb{R}^{n-1})
\]
where for each \(s \in S_1\), \((S)_s \overset{\xi_s}{\hookrightarrow} \mathbb{R}^{n-1}\) denotes the object of \(\text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1}))\) determined by the compatible sequence of embeddings of pullbacks over \(s\):
\[
\begin{align*}
(S_n)_s & \hookrightarrow S_n \overset{e_n}{\longrightarrow} \mathbb{R}^n \\
\downarrow & \downarrow \tau_{n-1} \\
(S_{n-1})_s & \hookrightarrow S_{n-1} \overset{e_{n-1}}{\longrightarrow} \mathbb{R}^{n-1} \\
\downarrow & \downarrow \tau_{n-2} \\
& \vdots \\
(S_2)_s & \hookrightarrow S_2 \overset{e_2}{\longrightarrow} \mathbb{R}^2 \\
\downarrow & \downarrow \tau_1 \\
\{s\} & \hookrightarrow S_1 \overset{e_1}{\longrightarrow} \mathbb{R}.
\end{align*}
\]
Note that for each \(1 \leq i \leq n\), each embedding \((S_i)_s \hookrightarrow S_i \hookrightarrow \mathbb{R}^i\) agrees on its first coordinate and thus, canonically factors through \(\mathbb{R}^{i-1}\), which in particular means that \(14\) yields an object of \(\text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1}))\).

The value of \(\pi\) on a morphism \(\text{cyl}(\sigma) \overset{E}{\hookrightarrow} \mathbb{R}^n \times \Delta^1\) is:

i. the morphism \(T_1 \overset{\sigma_1}{\longrightarrow} S_1\) in \(\text{Fin}\)
ii. for each pair $r \in T_1, s \in S_1$ such that $\sigma_1(r) = s$, the morphism

$$
cylr(\sigma_n)|_{\sigma_1 \mapsto s} \longrightarrow cyl(\sigma_n) \longrightarrow \Bbb{R}^n
$$

$$
\downarrow \quad \quad \downarrow \quad \quad \downarrow
$$

$$
cylr(\sigma_{n-1})|_{\sigma_1 \mapsto s} \longrightarrow cylr(\sigma_{n-1}) \longrightarrow \Bbb{R}^{n-1}
$$

$$
\downarrow \quad \quad \downarrow \quad \quad \downarrow
$$

$$
\vdots \quad \quad \vdots \quad \quad \vdots
$$

$$
cylr(\sigma_2)|_{\sigma_1 \mapsto s} \longrightarrow cylr(\sigma_2) \longrightarrow \Bbb{R}^2
$$

$$
\downarrow \quad \quad \downarrow \quad \quad \downarrow
$$

$$
cylr(\sigma_1)|_{s} \simeq \Delta^1 \longrightarrow cyl(\sigma_1) \longrightarrow \Bbb{R}
$$

in $\text{Exit}(\text{Ran}^\infty(\Bbb{R}^{n-1}))$, where for each $1 \leq i \leq n$, each embedding

$$
cylr(\sigma_i)|_{\sigma_1 \mapsto s} \hookrightarrow \Bbb{R}^i \times \Delta^1
$$

canonically factors through $\Bbb{R}^{i-1} \times \Delta^1$ and thus, (15) yields a morphism in $\text{Exit}(\text{Ran}^\infty(\Bbb{R}^{n-1}))$.

3.2. Proving the exit-path $\infty$-category of the refined unital Ran space of $\Bbb{R}^n$ is an $\infty$-category. This subsection is devoted to proving that the simplicial space $\text{Exit}(\text{Ran}^\infty(\Bbb{R}^n))$ is a complete Segal space. We build directly off of Section 2.1 wherein we showed that for a connected, smooth manifold $M$, $\text{Exit}(\text{Ran}^\infty(M))$ is a complete Segal space. Our approach was to witness the simplicial space $\text{Exit}(\text{Ran}^\infty(M))$ as one derived through formal constructions among complete Segal spaces from the complete Segal space $\text{Bun}$, defined in §6 of [4]. Our approach in this subsection is similar in that we show that $\text{Exit}(\text{Ran}^\infty(\Bbb{R}^n))$, too, can be derived through formal constructions among complete Segal spaces from $\text{Bun}$.

Proposition 3.2.1. The simplicial space $\text{Exit}(\text{Ran}^\infty(\Bbb{R}^n))$ satisfies the Segal and completeness conditions.

We build off of §2.1 and freely use notation and results from [4]. Recall the $\infty$-category $\text{Ref}^0(\Bbb{R})$ defined as the pullback in (7). Heuristically, an object is a refinement of $\Bbb{R}$ for which the 0-skeleton of the domain is a finite set, and a morphism is a path of such refinements of $\Bbb{R}$ witnessing anti-collision of strata and disappearances of strata. Observe that $\text{Ref}^0(\Bbb{R}) \simeq \text{Ar}^{\text{ref}}(\text{Bun})_R$ is equivalent to the $\infty$-category of refinement arrows in $\text{Bun}$ with the target fixed as the trivially stratified space $\Bbb{R}$. This is because every refinement of $\Bbb{R}$ has as its 0-skeleton, a finite (possibly empty) set. The equivalence is given by the functor from $\text{Ar}^{\text{ref}}(\text{Bun})_R$ to $\text{Ref}^0(\Bbb{R})$ which forgets the target. Let $\text{Ref}(\Bbb{R}^n)$ denote the $\infty$-category of refinement arrows in $\text{Bun}$ with the target fixed as $\Bbb{R}^n$ equipped with the trivial stratification. Explicitly, $\text{Ref}(\Bbb{R}^n)$ is the simplicial space whose value on $[p] \in \Delta$ is the moduli space of

- constructible bundles
  $$
  Y \longrightarrow \Delta^p
  $$

- together with a refinement
  $$
  Y \underset{\text{refinement}}{\longrightarrow} \Bbb{R}^n \times \Delta^p
  $$

over $\Delta^p$.

We will denote such a $[p]$-point of $\text{Ref}(\Bbb{R}^n)$ simply as $(Y \underset{\text{ref}}{\longrightarrow} \Bbb{R}^n \times \Delta^p)$. Note that $\Bbb{R}^n \times \Delta^p$ is stratified as a product stratified space, where $\Bbb{R}^n$ is trivially stratified over the poset consisting of a singleton, and $\Delta^p$ is given the standard stratification, defined in Example 0.1.3.
For $n \geq 2$, define the functor

$$F_n : \text{Ref}^0(\mathbb{R}) \to \text{Ref}(\mathbb{R}^n)$$

from refinements of $\mathbb{R}$ to refinements of $\mathbb{R}^n$ as follows: The value of a $[p]$-point $(X \xrightarrow{\text{ref}} \mathbb{R} \times \Delta^p)$ under $F_n$ is the refinement of $\mathbb{R}^n \times \Delta^p$ defined as the pullback of stratified spaces

$$F_n(X) \xrightarrow{\alpha} \mathbb{R}^n \times \Delta^p \xrightarrow{\text{ref}} \mathbb{R} \times \Delta^p.$$  

(16)

It is straightforward to check that the value $F_n(X)$ is a refinement of $\mathbb{R}^n \times \Delta^p$ by virtue of it being a pullback. Explicitly, $(F_n(X) \to \mathbb{R}^n \times \Delta^p)$ is

$$((\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^0_{\text{fib}}(X)) \subset \mathbb{R}^n \times \Delta^p) \xrightarrow{\text{ref}} \mathbb{R}^n \times \Delta^p$$

which denotes the coarsest refinement of $\mathbb{R}^n \times \Delta^p$ for which the embedding from $((\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^0_{\text{fib}}(X)))$ is proper and constructible. As a technicality (that we will use later) define $F_1$ to be the identity on $\text{Ref}^0(\mathbb{R})$. The following figure is a sketch of the values of an object and a morphism under

$$F_2 : \text{Ref}^0(\mathbb{R}) \to \text{Ref}(\mathbb{R}^2).$$

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure4}
\caption{The values of a $[0]$-point and a $[1]$-point in $\text{Ref}^0(\mathbb{R})$ under $F_2$.}
\end{figure}

Denote the pullback $\infty$-category

$$\tilde{\text{Ref}}(\mathbb{R}^n) \longrightarrow \text{Ar}_{\text{ref}}(\text{Ref}(\mathbb{R}^n))$$

(17)

$$\xrightarrow{\text{target}} \text{Ref}^0(\mathbb{R}) \xrightarrow{F_n} \text{Ref}(\mathbb{R}^n)$$

where $\text{Ar}_{\text{ref}}(\text{Ref}(\mathbb{R}^n))$ is the $\infty$-category of refinement arrows of $\text{Ref}(\mathbb{R}^n)$; that is, it is the full $\infty$-subcategory of the $\infty$-category of arrows of $\text{Ref}(\mathbb{R}^n)$ consisting of the refinement arrows.
We employ the open cylinder construction of \[4\] (previously referenced in §2.1) to determine that \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) is the following simplicial space: Its value on \([p] \in \Delta\) is the moduli space of

- pairs of constructible bundles
  \(\left((X \to \Delta^p), (Y \to \Delta^p)\right)\)

- together with a pair of refinements of stratified spaces
  \(\left((X \xrightarrow{\text{refinement}} \mathbb{R} \times \Delta^p), (Y \xrightarrow{\text{refinement}} F_n(X))\right)\)

each of which is over \(\Delta^p\).

To keep in mind that \(Y\) is, in particular, a refinement of \(\mathbb{R}^n \times \Delta^p\), we denote such a \([p]\)-point of \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) by

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{ref}} & F_n(X) \\
\downarrow & \downarrow & \downarrow \\
\mathbb{R} \times \Delta^p & \xrightarrow{\text{ref}} & F_n(X)
\end{array}
\]

The following figure is a sketch of an object \((p = 0)\) and a morphism \((p = 1)\) in \(\widetilde{\text{Ref}}(\mathbb{R}^2)\).

![Figure 5. A [0]-point and a [1]-point in \(\widetilde{\text{Ref}}(\mathbb{R}^2)\).](image)
For each such \([p]\)-point of \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) as in \([18]\) above, there is a canonical stratified map of stratified spaces from \(Y\) to \(X\) defined to be the composite of the stratified maps \(Y \xrightarrow{\text{ref}} F_n(X) \xrightarrow{\alpha} X\), where \(\alpha\) is the stratified map in \([16]\), which is given by virtue of \(F_n(X)\) being defined as a pullback. The map of underlying topological spaces is projection onto the first Euclidean coordinate product with the identity on \(\Delta^p\). We denote this stratified projection map \(Y \xrightarrow{\text{pr}_2} X\).

**Definition 3.2.2.** For \(n \geq 2\), the \(\infty\)-category \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) is defined inductively on \(n\):

\(\widetilde{\text{Ref}}(\mathbb{R}^2)\) is the full \(\infty\)-subcategory of \(\widetilde{\text{Ref}}(\mathbb{R}^2)\) consisting of those objects

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{ref}} & F_2(X) \\
\downarrow & & \downarrow \text{ref} \\
\mathbb{R}^2 & &
\end{array}
\]

for which the 1-skeleton of the open cylinder of \((Y \xrightarrow{\text{ref}} F_2(X))\) is a refinement morphism in \(\text{Bun}\).

\(\widetilde{\text{Ref}}(\mathbb{R}^n)\) is the full \(\infty\)-subcategory of \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) consisting of those objects

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{ref}} & F_n(X) \\
\downarrow & & \downarrow \text{ref} \\
\mathbb{R}^n & &
\end{array}
\]

such that

i. the \((n - 1)\)-skeleton of the open cylinder of \((Y \xrightarrow{\text{ref}} F_n(X))\) is a refinement morphism in \(\text{Bun}\).

ii. the fiber of the stratified projection map \(Y \xrightarrow{\text{pr}_2} X\) over each point in the 0-skeleton of \(X\) is an object in \(\widetilde{\text{Ref}}(\mathbb{R}^{n-1})\).

Explicitly, \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) is the simplicial space whose value on \([p] \in \Delta\) is the moduli space of

- pairs of constructible bundles \((X \rightarrow \Delta^p), (Y \rightarrow \Delta^p)\)
- together with a pair of refinements among stratified spaces \((X \xrightarrow{\text{refinement}} \mathbb{R} \times \Delta^p), (Y \xrightarrow{\text{refinement}} F_n(X))\)

each of which is over \(\Delta^p\)

satisfying the conditions

i. the fiberwise \((n - 1)\)-skeleton of the open cylinder of \((Y \xrightarrow{\text{ref}} F_n(X))\) is a refinement morphism in \(\text{Bun}\).

ii. the fiber of the stratified projection map \((Y \xrightarrow{\text{pr}_2} X)\) over each point in the 0-skeleton of \(X\) is an object in \(\widetilde{\text{Ref}}(\mathbb{R}^{n-1})\).

We will denote such a \([p]\)-point in \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) as

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{ref}} & F_n(X) \\
\downarrow & & \downarrow \text{ref} \\
\mathbb{R}^n \times \Delta^p & &
\end{array}
\]

The following figure is a sketch of an object \((p = 0)\) and a morphism \((p = 1)\) in \(\widetilde{\text{Ref}}(\mathbb{R}^2)\).
Lemma 3.2.3. For each integer \( n \geq 1 \) and \( 1 \leq k \leq n - 1 \), there is a canonical functor
\[
\text{pr}_k : \mathcal{Rf}(\mathbb{R}^n) \to \mathcal{Rf}(\mathbb{R}^k)
\]
induced by projection from \( \mathbb{R}^n \) onto the first \( k \)-coordinates. The functor is given by, for each \( [p] \in \Delta \), assigning to the \([p]\)-point
\[
Y \xrightarrow{\text{ref}} F_n(X)
\]
the refinement
\[
\left( \text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}_{n-k}^\text{fib}(Y)) \right) \subset \cdots \subset \left( \text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}_{n-1}^\text{fib}(Y)) \right) \subset \mathbb{R}^k \times \Delta^p
\]
the value of which is the coarsest refinement of \( \mathbb{R}^k \times \Delta^p \) for which the embeddings from \( \text{pr}_{<k+1}(\text{sk}_i^\text{fib}(Y)) \) into \( \mathbb{R}^k \times \Delta^p \) for each \( n - k \leq i \leq n - 1 \) are proper and constructible.
We denote such a value by

\[ Y_k \xrightarrow{\text{ref}} F_k(X) \xrightarrow{\text{ref}} \mathbb{R}^k \times \Delta^p \]

Before we proceed with the proof, we provide a sketch of the values of an object in \( \widetilde{\text{Ref}}(\mathbb{R}^3) \) under \( \text{pr}_2 \) and \( \text{pr}_1 \) in the following figure.

\[ \rho = \circ : \]

Figure 7. The values of a \([0]\)-point of \( \widetilde{\text{Ref}}(\mathbb{R}^3) \) under \( \text{pr}_2 \) and \( \text{pr}_1 \).

Proof of Lemma 3.2.3. We proceed by induction on \( k \).

\( k = 1 \) : First, recall the definition of \( \widetilde{\text{Ref}}(\mathbb{R}^n) \) as the pullback in (17). Observe that the target \( \text{Ref}^0(\mathbb{R}) \) of the leftmost vertical functor in (17) is, by definition, \( \text{Ref}(\mathbb{R}) \). We claim that \( \text{pr}_1 \) is precisely this functor upon restricting the domain to \( \widetilde{\text{Ref}}(\mathbb{R}^n) \). To verify this claim, first note that the functor in (17), by virtue of \( \text{Ref}(\mathbb{R}^n) \) being a pullback, is given by the assignment, for each \( [p] \in \Delta \),

\[ X \xrightarrow{\text{ref}} F_n(X) \xrightarrow{\text{ref}} \mathbb{R}^n \times \Delta^p \]

Therefore, we must show that the value of such a \([p]\)-point under \( \text{pr}_1 \),

\[ Y_1 \xrightarrow{\text{ref}} F_1(X) \xrightarrow{\text{ref}} \mathbb{R} \times \Delta^p \]

is \((X \xrightarrow{\text{ref}} \mathbb{R} \times \Delta^p)\). Recall that (as a technicality) \( F_1 \) was previously defined to be the identity on \( \text{Ref}^0(\mathbb{R}) \). Thus, we need to verify that \( Y_1 \) and \( X \) are equivalent refinements of \( \mathbb{R} \times \Delta^p \). By virtue of being a \([p]\)-point of \( \widetilde{\text{Ref}}(\mathbb{R}^n) \), the fiberwise \((n - 1)\)-skeleton of \( Y \) refines the fiberwise \((n - 1)\)-skeleton of \( F_n(X) \). This means that, in particular, the underlying topological spaces of the \((n - 1)\)-skeletons of \( Y \) and \( F_n(X) \) are homeomorphic. Thus, their projections onto the first Euclidean coordinate product with the identity on \( \Delta^p \)

\[ \text{pr}_{<2} \times \text{Id}_{\Delta^p}(\text{sk}_{n-1}^{\text{fib}}(Y)) \cong \text{pr}_{<2}(\text{sk}_{n-1}^{\text{fib}}(F_n(X))) \]

are homeomorphic subspaces of \( \mathbb{R} \times \Delta^p \) over \( \Delta^p \). This simply means that the fibers of each over the same point in \( \Delta^p \) have the same cardinality.

Previously, we observed that explicitly \( F_n(X) \) is the coarsest refinement

\[ ((\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}_{n-1}^{\text{fib}}(X))) \subset \mathbb{R}^n \times \Delta^p \]
of \(\mathbb{R}^n \times \Delta^p\) for which the embedding from \((\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^{fib}_0(X))\) is proper and constructible. This means that the fiberwise \((n - 1)\)-skeleton of \(F_n(X)\) is \((\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^{fib}_0(X))\). Thus, the projection of the \((n - 1)\)-skeleton onto the first Euclidean coordinate product with the identity on \(\Delta^p\) is simply the fiberwise 0-skeleton of \(X\), i.e.,

\[
\text{pr}_{<2} \times \text{Id}_{\Delta^p}(\text{sk}^{fib}_{n-1}(F_n(X))) = \text{sk}^{fib}_0(X).
\]

Therefore, through the previous equivalence \([19]\), the fiberwise 0-skeleton of \(X\) is homeomorphic to \(\text{pr}_{<2} \times \text{Id}_{\Delta^p}(\text{sk}^{fib}_{n-1}(Y))\). Upon making the (somewhat trivial) observation that \((X \xrightarrow{\text{ref}} \mathbb{R} \times \Delta^p)\) has the explicit description as the coarsest refinement \((\text{sk}^{fib}_0(X)) \subset \mathbb{R} \times \Delta^p\) of \(\mathbb{R} \times \Delta^p\) for which the embedding from the fiberwise 0-skeleton of \(X\) is proper and constructible, we conclude that \(Y_1 := (\text{pr}_{<2}(\text{sk}^{fib}_{n-1}(Y)) \subset \mathbb{R} \times \Delta^p)\) is equivalent to \(X\) as a refinement of \(\mathbb{R} \times \Delta^p\).

**[General case]:** We need to check that the value

\[
Y_k \xrightarrow{\text{ref}} F_k(X) \xrightarrow{\text{ref}} \mathbb{R}^k \times \Delta^p
\]

is in fact a \([p]\)-value in \(\widehat{\text{Ref}}(\mathbb{R}^k)\). Thus, we must verify three things:

(a) \(Y_k\) refines \(F_k(X)\),

(b) the \((k - 1)\)-skeleton of the open cylinder of \((Y_k \xrightarrow{\text{ref}} F_k(X))\) is a refinement, and

(c) the fiber of the stratified projection map \(Y_k \xrightarrow{\text{pr}_{<2}} X\) over each point in the 0-skeleton of \(X\) is an object in \(\widehat{\text{Ref}}(\mathbb{R}^{k-1})\).

(a) Recall that \(F_n(X)\) is the coarsest refinement \(((\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^{fib}_0(X)) \subset \mathbb{R}^n \times \Delta^p)\) of \(\mathbb{R}^n \times \Delta^p\) for which the embedding from \((\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^{fib}_0(X))\) is proper and constructible. Projection of \(F_n(X)\) onto its first \(k\) Euclidean coordinates (product with the identity on \(\Delta^p\)) yields an explicit description

\[
F_k(X) = (\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}^{fib}_{n-1}(F_n(X)))) \subset \mathbb{R}^k \times \Delta^p
\]

which denotes the coarsest refinement of \(\mathbb{R}^k \times \Delta^p\) for which the embedding from \(\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}^{fib}_{n-1}(F_n(X)))\) is proper and constructible. By definition of a \([p]\)-point of \(\widehat{\text{Ref}}(\mathbb{R}^n)\),

\[
Y \xrightarrow{\text{ref}} F_n(X) \xrightarrow{\text{ref}} \mathbb{R}^n \times \Delta^p
\]

satisfies that the fiberwise \((n - 1)\)-skeleton of \(Y\) is a refinement of the fiberwise \((n - 1)\)-skeleton of \(F_n(X)\). In particular then, the underlying topological spaces of the \((n - 1)\)-skeletons are homeomorphic. Thus, so are their projections onto the first \(k\) Euclidean coordinates (product with the identity on \(\Delta^p\)), i.e.,

\[
\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}_{n-1}^{fib}(Y)) \equiv \text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}_{n-1}^{fib}(F_n(X))).
\]

By definition, \(Y_k\) refines the coarsest refinement \(\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}_{n-1}^{fib}(Y))\) of \(\mathbb{R}^k \times \Delta^p\) for which the embedding from \(\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}_{n-1}^{fib}(Y))\) is proper and constructible. Thus, through the equivalences \([21]\) and \([22]\) above, we conclude that \(Y_k\) refines \(F_k(X)\).

(b) By definition, the fiberwise \((k - 1)\)-skeleton of \(Y_k\) is a refinement of \(\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}_{n-1}^{fib}(Y))\). Through the equivalences \([21]\) and \([22]\) above, then, the fiberwise \((k - 1)\)-skeleton of \(Y_k\) refines the fiberwise \((k - 1)\)-skeleton of \(F_k(X)\). That is to say, the refinement \((Y_k \xrightarrow{\text{ref}} F_k(X))\) of stratified
spaces restricts to a refinement \((\text{sk}_{k-1}(Y_k) \xrightarrow{\text{ref}} \text{sk}_{k-1}(F_k(X)))\) of stratified spaces between the \((k-1)\)-skeletons. Thus, the \((k-1)\)-skeleton of the open cylinder of \((Y_k \xrightarrow{\text{ref}} F_k(X))\) is a refinement morphism in \(\mathcal{B}\text{un}\).

(c) The fiber of \(Y_k \xrightarrow{pr_{<2}} X\) over a point in the 0-skeleton of \(X\) can, by definition of \(Y_k\), be described in terms of projection of the fiber of \(Y \xrightarrow{pr_{<2}} X\) over the same point as follows: Let \(x\) be a point in the 0-skeleton of \(X\) and let \(Y_x\) denote the fiber over \(x\) in \(Y\). The fiber over \(x\) in \(Y_k\) is

\[
(Y_k)_x := (pr_{<k} \times \text{Id}_{\Delta^p})(\text{sk}^\text{fib}_{n-2}(Y_x)) \subset \cdots \subset pr_{<k} \times \text{Id}_{\Delta^p}(\text{sk}^\text{fib}_{n-2}(Y_{n-2})) \xrightarrow{\text{ref}} \mathbb{R}^{k-1}
\]

which denotes the coarsest refinement of \(\mathbb{R}^{k-1}\) for which the embedding from \(pr_{<k} \times \text{Id}_{\Delta^p}(\text{sk}_i(Y_x))\) into \(\mathbb{R}^{k-1}\), for each \(n-k \leq i \leq n-2\), is proper and constructible. By definition, \((Y_k)_x\) is the value of \(Y_x\) under the projection functor \(\overline{\text{Ref}}(\mathbb{R}^n) \xrightarrow{pr_{<k-1}} \overline{\text{Ref}}(\mathbb{R}^{k-1})\), which exists by the inductive step, and therefore identifies \((Y_k)_x\) as an object in \(\overline{\text{Ref}}(\mathbb{R}^{k-1})\), as desired.

\[\square\]

**Observation 3.2.4.** There is a natural functor

\[
\overline{\text{Ref}}(\mathbb{R}^n) \longrightarrow \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})
\]

the value of which on an object

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{ref}} & F_n(X) \\
& \downarrow{\text{ref}} & \\
& \mathbb{R}^n
\end{array}
\]

is the functor from \(\{1 < \cdots < n\}\) to \(\text{Fin}^{\text{op}}\) that selects out the sequence

\[
\text{sk}^\text{fib}_0(Y) \xrightarrow{pr_{<n}} \text{sk}^\text{fib}_0(Y_{n-1}) \rightarrow \cdots \xrightarrow{pr_{<2}} \text{sk}^\text{fib}_0(X)
\]

of finite sets.

The value on a morphism

\[
\begin{array}{ccc}
Y' & \xrightarrow{\text{ref}} & F_n(X) \\
& \downarrow{\text{ref}} & \\
& \mathbb{R}^n \times \Delta^1
\end{array}
\]

is given by selecting out the diagram

\[
\begin{array}{cccc}
\text{sk}^\text{fib}_0(Y_{[1]}) & \longrightarrow & \text{sk}^\text{fib}_0(Y_{[0]}) \\
\downarrow{pr_{<n}} & & \downarrow{pr_{<n}} \\
\text{sk}^\text{fib}_0((Y_{n-1})_{[1]}) & \longrightarrow & \text{sk}^\text{fib}_0((Y_{n-1})_{[0]}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\text{sk}^\text{fib}_0((X)_{[1]}) & \longrightarrow & \text{sk}^\text{fib}_0((X)_{[0]}) \\
\downarrow{pr_{<2}} & & \downarrow{pr_{<2}}
\end{array}
\]

among finite sets, where, for each \(1 \leq k \leq n\), the horizontal arrow is from the (0)-skeleton of the fiber of \(Y_k \rightarrow \Delta^1\) over \(\{1\} \in \Delta^1\) to the (0)-skeleton of the fiber of \(Y_k \rightarrow \Delta^p\) over \(\{0\} \in \Delta^1\), and is a canonical map of sets implemented by taking connected components of the fiberwise (0)-skeleton of \(X\).
Lemma 3.2.5. There is a canonical equivalence between simplicial spaces over 
\( \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}) \):
\[
\widetilde{\text{Ref}}(\mathbb{R}^n) \simeq \text{Exit}(\text{Ran}^u(\mathbb{R}^n)).
\]

Proof. A rightward morphism is implemented by, for each \([p] \in \Delta\), the assignment,
\[
\begin{array}{ccc}
\text{sk}_0(Y) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\mathbb{R}^n \times \Delta^p & \longrightarrow & \mathbb{R}^n \times \Delta^p
\end{array}
\]
whose value is the sequence of embeddings over \(\Delta^p\), each of which is from the fiberwise \((0)\)-skeleton of the value \(Y_k\) of \(Y\) under the functor \(\text{pr}_k\) \([3.2.3]\), which maps to \(\Delta^p\) as a finite proper constructible bundle.

A leftward morphism is given by assigning to each \([p]-point
\[
\begin{array}{ccc}
\text{cylr}(\sigma_n) & \longrightarrow & \mathbb{R}^n \times \Delta^p \\
\downarrow & & \downarrow \\
\mathbb{R}^n \times \Delta^p & \longrightarrow & \mathbb{R}^n \times \Delta^p
\end{array}
\]
the refinement
\[
(\text{cylr}(\sigma_n) \subset (\text{pr}_n \times \text{Id}_{\Delta^p})^{-1}(\text{cylr}(\sigma_{n-1})) \subset \cdots \subset (\text{pr}_n \times \text{Id}_{\Delta^p})^{-1}(\text{cylr}(\sigma_1)) \subset \mathbb{R}^n \times \Delta^p) \longrightarrow \mathbb{R}^n \times \Delta^p
\]
whose value is the coarsest refinement of \(\mathbb{R}^n \times \Delta^p\) for which the embeddings from \(\text{cylr}(\sigma_n)\) and each prevalue \((\text{pr}_i \times \text{Id}_{\Delta^p})^{-1}(\text{cylr}(\sigma_{i-1}))\) for \(2 \leq i \leq n\) are proper and constructible. (Such a refinement exists because the value of each embedding is, by definition, a properly embedded stratified subspace.)

It is straightforward to verify that these two assignments are mutually inverse to one another, and further, that they are over \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\).

Proof of Proposition 3.2.1. The simplicial space \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) is a pullback of complete Segal spaces and is therefore also a complete Segal space, since the full \(\infty\)-subcategory of simplicial spaces consisting of the complete Segal spaces is closed under formation of pullbacks. By virtue of \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) being a full subsimplicial space of \(\widetilde{\text{Ref}}(\mathbb{R}^n)\), it too satisfies the Segal and completeness conditions. Through the equivalence of Lemma 3.2.5, then so does the simplicial space \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\).

3.3. Identifying the exit-path \(\infty\)-category of the refined unital Ran space of \(\mathbb{R}^n\) as \(\Theta_n^{\text{act}}\). This subsection is devoted to proving an equivalence between the space of active morphisms of \(\Theta_n\) and the \(\infty\)-category \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\). First, in Construction 3.3.1, we construct the functor and then, in Lemma 3.3.2, we show that it is an equivalence.
Construction 3.3.1. For $n \geq 1$, there is a functor
\[ G_n : \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \rightarrow \Theta_n^{\text{act}} \]
over $\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})$, from the exit-path $\infty$-category of the refined unital Ran space of $\mathbb{R}^n$ to the subcategory of active morphisms of $\Theta_n$.

Proof. We proceed by induction on $n$.

$n = 1$ : We seek to define a functor $G_1$ over $\text{Fin}^{\text{op}}$:

\[ \text{Exit}(\text{Ran}^n(\mathbb{R})) \xrightarrow{\phi_1} \Delta^{\text{act}} \xrightarrow{\gamma_1} \text{Fin}^{\text{op}}. \]

A functor from an $\infty$-category to the nerve of a (small) category is completely determined by its assignment on objects, morphisms and the requirement that composition is respected. This is due to the fact that the nerve of a small category is completely determined by its values on $[i]$ for $0 \leq i \leq 2$. See the proof of Lemma 3.5 in [13] for more details. Thus, since $\Delta^{\text{act}}$ is an ordinary category, we will simply define $G_1$ on objects and morphisms and check that composition is respected.

Let $S \hookrightarrow \mathbb{R}$ be an object in $\text{Exit}(\text{Ran}^n(\mathbb{R}))$. The value of $G_1$ on $e$ is the linearly ordered set of connected components of the complement of $e(S)$ in $\mathbb{R}$

\[ G_1 : e \mapsto \pi_o(\mathbb{R} - e(S)) \]

the linear order of which is inherited from the linear order on $\mathbb{R}$.

Let $\text{cylr}(T \xrightarrow{\sigma} S) \xleftarrow{\epsilon} \mathbb{R} \times \Delta^1$ be a morphism from $S \xleftarrow{\epsilon} \mathbb{R}$ to $T \xrightarrow{d} \mathbb{R}$ in $\text{Exit}(\text{Ran}^n(\mathbb{R}))$. Let $C_E$ denote the compliment of the image of the embedding of $E$,

\[ C_E := (\mathbb{R} \times \Delta^1) - E(\text{cylr}(\sigma)). \]

Before we name the value of $G_1$ on $E$, we make three observations:

1. Consider the inclusion $i_1 : (\mathbb{R} - d(T)) \hookrightarrow C_E$ given by $x \mapsto (x, \{1\})$. Taking connected components induces an inclusion of sets

\[ \pi_o(i_1) : \pi_o(\mathbb{R} - d(T)) \hookrightarrow \pi_o(C_E). \]

It is easy to see that $\pi_o(i_1)$ is, in particular, a bijection. We denote its inverse $\pi_o(i_1)^{-1}$.

2. Taking connected components of the inclusion $i_0 : (\mathbb{R} - e(S)) \hookrightarrow C_E$ given by $x \mapsto (x, \{0\})$ induces a map between sets

\[ \pi_o(i_0) : \pi_o(\mathbb{R} - e(S)) \hookrightarrow \pi_o(C_E). \]

Note that $\pi_o(i_0)$ is not necessarily injective nor surjective because $\sigma$ is not necessarily injective nor surjective.

3. $\pi_o(i_1)$ determines a linear order on $\pi_o(C_E)$ and thus, $\pi_o(C_E)$ is an object in $\Delta$.

Then, the value of $G_1$ on $\text{cylr}(T \xrightarrow{\sigma} S) \xleftarrow{\epsilon} \mathbb{R} \times \Delta^1$ is the composite

\[
\begin{align*}
\pi_o(\mathbb{R} - e(S)) &\xrightarrow{\pi_o(i_0)} \pi_o(C_E) \\
G_1(E) &\xrightarrow{\pi_o(i_1)^{-1}} \pi_o(\mathbb{R} - d(T))
\end{align*}
\]

in $\Delta^{\text{act}}$. It must be checked that $G_1(E)$ is linear and active. We do this by verifying that each morphism in the composite is linear and active. $\pi_o(i_1)^{-1}$ is a linear map because it defines the linear order of $\pi_o(C_E)$. Bijectivity of $\pi_o(i_1)^{-1}$ implies that it is active. Similarly, it is easy to see that $\pi_o(i_0)$ is order-preserving and sends unbounded components to unbounded components thereby being active.
Next, we will show that $G_1$ respects composition by showing that the diagram of $\infty$-categories commutes on the level of objects and morphisms:

$$\begin{array}{ccc}
\text{Exit}(\text{Ran}^u(\mathbb{R})) & \xrightarrow{G_1} & \Delta^{\text{act}} \\
\downarrow \phi_1 & & \downarrow \gamma_1 \\
\text{Fin}^{\text{op}} & &
\end{array}$$

Indeed, if (24) commutes, then faithfulness of $\gamma_1$ together with functorality of $\phi_1$ guarantee that $G_1$ respects composition. More precisely, let

$$\begin{array}{ccc}
a & \xrightarrow{f} & b \\
\downarrow h & & \downarrow g \\
c & &
\end{array}$$

denote a commutative triangle in $\text{Exit}(\text{Ran}^u(\mathbb{R}))$. We will show that if (24) commutes, then $G_1$ carries the composite $h$ in (25) to $G_1(g) \circ G_1(f)$. First, note that functorality of $\phi_1$ implies that $\phi_1(h) = \phi_1(g) \circ \phi_1(f)$. Commutativity of (24) guarantees that the morphisms $\gamma_1(G_1(h)), \gamma_1(G_1(f))$ and $\gamma_1(G_1(g))$ are equivalent (upto composition with canonical isomorphisms) to $\phi_1(h), \phi_1(f)$ and $\phi_1(g)$ respectively, in $\text{Fin}$. Thus,

$$\gamma_1(G_1(h)) = \gamma_1(G_1(g)) \circ \gamma_1(G_1(f)) = \gamma_1(G_1(g) \circ G_1(f)).$$

Then, faithfulness of $\gamma_1$ guarantees that $G_1(h) = G_1(g) \circ G_1(f)$, as desired.

Now, we verify commutativity of (24) on objects and morphisms. Let $S \xrightarrow{\epsilon} \mathbb{R}$ be an object in $\text{Exit}(\text{Ran}^u(\mathbb{R}))$. There is a canonical bijection of sets

$$\gamma_1(G_1(e)) \xrightarrow{\cong} \phi_1(e) := S$$

in $\text{Fin}$ given by

$$\left(\pi_0(\mathbb{R} - e(S)) \xrightarrow{\alpha} [1]\right) \mapsto \inf\{x \in \bigsqcup_{U \in \alpha^{-1}(1)} U\}$$

verifying commutativity of (24) on objects.

Let $\text{cylr}(T \xrightarrow{\sigma} S) \xrightarrow{E} \mathbb{R} \times \Delta^1$ be a morphism from $S \xrightarrow{\epsilon} \mathbb{R}$ to $T \xrightarrow{d} \mathbb{R}$ in $\text{Exit}(\text{Ran}^u(\mathbb{R}))$. We consider the canonical bijections of the source and target of $\gamma_1(G_1(E))$, and the corresponding composite, $\alpha$, in $\text{Fin}$ from $T$ to $S$:

$$\begin{array}{ccc}
\gamma_1(G_1(d)) & \xrightarrow{\gamma_1(G_1(E))} & \gamma_1(G_1(e)) \\
\uparrow \cong & & \uparrow \cong \\
T & \xrightarrow{\alpha} & S
\end{array}$$

By definition, the value of $\alpha$ on $r \in T$ is

$$\alpha(r) := \inf\{x \in \bigsqcup_{U \in S^r} U\},$$

where $S^r := \{U \in \pi_0(\mathbb{R} - e(S))| \inf\{y \in G_1(E)(U)\} \geq r\}$. The composite $\alpha$ agrees with $\phi_1(E) := \sigma$, as desired. Indeed, if $U \in S^r$, then

$$\inf\{x \in U\} = \sigma(r) \text{ or } \inf\{x \in U\} = \sigma(r'),$$

35
for some \( r' > r \). But \( \sigma(r') \geq \sigma(r) \) whenever \( r' > r \), which implies
\[
\inf \{ x \in \bigcup_{U \in \mathcal{S}^r} U \} = \sigma(r).
\]

In summary, we have just shown that (24) commutes on objects and morphisms, which, as previously argued, implies that \( G_1 \) respects composition. Therefore, \( G_1 \) is a functor, and moreover is defined naturally over \( \text{Fin}^{\text{op}} \).

[General case]: In the inductive step, we assume the existence of a functor over \( \text{Fun}(\{1 < \cdots < n-1\}, \text{Fin}^{\text{op}}) \)

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1})) & \xrightarrow{G_{n-1}} & \Theta_{n-1}^{\text{act}} \\
\Phi_{n-1} & \downarrow & \tau_{n-1} \\
\text{Fun}(\{1 < \cdots < n-1\}, \text{Fin}^{\text{op}}) & \xrightarrow{} & \end{array}
\]

In particular, this implies that \( G_{n-1} \) is over \( \text{Fin}^{\text{op}} \) for each \( 1 \leq i \leq n-1 \); i.e., the following diagram commutes

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1})) & \xrightarrow{G_{n-1}} & \Theta_{n-1}^{\text{act}} \\
\Phi_i & \downarrow & \theta_i \\
\text{Fin}^{\text{op}} & \xrightarrow{\gamma_i} & \end{array}
\]

for each \( 1 \leq i \leq n-1 \), where, recall that \( \tau_i \) denotes the \((n-1-i)\)-fold self-composite of the truncation map \( \tau \) defined in Observation 1.1.5.

We define the functor \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \rightarrow \Theta_n^{\text{act}} \) by defining \( \Psi \) and \( \Gamma \) such that

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) & \xrightarrow{\Psi} & \text{Fin}^{\text{op}} \downarrow \Theta_n^{\text{act}} \\
\xrightarrow{G_n} & & \downarrow \text{frgt} \\
\Theta_n^{\text{act}} & \xrightarrow{} & \text{Fin}^{\text{op}}.
\end{array}
\]

\( \Gamma \) is defined to be the composite of the forgetful functor \( \rho \) followed by \( G_1 \), \( G_1 \circ \rho \), where \( \rho \) is the functor defined in Observation 3.1.7 which forgets all but the first coordinate data.

\( \Psi \) is defined to be the composite of the functor

\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \rightarrow \text{Fin}^{\text{op}} \downarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1}))
\]

which was defined in Observation 3.1.8 followed by the functor

\[
\text{Fin}^{\text{op}} \downarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1})) \rightarrow \text{Fin}^{\text{op}} \downarrow \Theta_{n-1}^{\text{act}}
\]
determined by the identity on \( \text{Fin}^{\text{op}} \) and the functor \( G_{n-1} \) given by the inductive step. Thus, \( G_n \) is a well-defined functor.

Unwinding the above definition of \( G_n \), an inductive description of \( G_n \) is apparent. We explicate this inductive description on objects and morphisms: Let \( S \xrightarrow{\xi} \mathbb{R}^n \) be an object in \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \). Its value under \( G_n \) is inductively defined as

\[
G_n(S \xrightarrow{\xi} \mathbb{R}^n) := G_1(S_1 \xrightarrow{e_1} \mathbb{R})(G_{n-1}((S) \xrightarrow{\xi(S)} \mathbb{R}^{n-1}))
\]
where $(S)_s \xrightarrow{\xi(S)_s} \mathbb{R}^{n-1}$ denotes the object of $\text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1}))$ determined by (14).

Let $\text{cyl}(T) \xrightarrow{c} S \xleftarrow{\epsilon} \mathbb{R}^n \times \Delta^1$ be a morphism from $S \xleftarrow{\epsilon} \mathbb{R}^n$ to $T \xrightarrow{c} \mathbb{R}^n$ in $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$. Its value under $G_n$, is inductively defined by:

i) the morphism $G_1(S_1 \xrightarrow{c_1} \mathbb{R}) \xrightarrow{G_1(cyl(\sigma_1)) \xrightarrow{\epsilon(\sigma_1)}} G_1(T_1 \xrightarrow{d_1} \mathbb{R})$ in $\Delta^\text{act}$

ii) for each pair $(t \in T_1, s \in S_1)$ such that $\sigma_1(t) = s$, the morphism given by the image of $1$ under $G_{n-1}$ in $\Theta^\text{act}_{n-1}$.

Next, we will show that for each $1 \leq i \leq n$,

$$\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{G_n} \Theta^\text{act}_n \xrightarrow{\phi_i} \Theta^\text{act}_i \xrightarrow{\gamma_i} \text{Fin}^{\text{op}}. \quad (29)$$

For the cases $1 \leq i \leq n-1$, this diagram follows by the inductive step wherein we assume commutativity of (28). For the remaining case, $i = n$, we use the inductive definitions of $G_n$ and $\gamma_n$ in terms of $G_1$ and $G_{n-1}$, and $\gamma_1$ and $\gamma_{n-1}$, respectively. Then, indeed, in employing the commutativity of (24) and (28) for $i = n-1$, we see that for the case $i = n$, (29) must commute. Through Observation 3.1.5 wherein the functor $\Phi_n$ was defined in terms of $\phi_i$ for $1 \leq i \leq n$, commutativity of this diagram for each $1 \leq i \leq n$ compiles to prove that $G_n$ is over $\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})$,

$$\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{G_n} \Theta^\text{act}_n \xrightarrow{\Phi_n} \text{Fin}^{\text{op}}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}).$$

We have just defined a functor from the exit-path $\infty$-category of the refined unital Ran space of $\mathbb{R}^n$ to the category $\Theta^\text{act}_n$. In the next lemma, we show that this functor is an equivalence by showing that $G_n$ is essentially surjective and fully faithful.

**Lemma 3.3.2.** For each $n \geq 1$, the functor from Construction 3.3.1 over $\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})$

$$G_n : \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{\simeq} \Theta^\text{act}_n$$

is an equivalence of $\infty$-categories from the exit-path $\infty$-category of the refined unital Ran space of $\mathbb{R}^n$ to the subcategory of active morphisms of $\Theta_n$.

**Proof.** We proceed by induction on $n$.

$n = 1$ : We will show that $G_1$ is essentially surjective and fully faithful; the former follows easily:

Let $[p] \in \Delta^\text{act}$. Define the set $T_p := \{1, 2, \ldots, p\}$ together with the object $T_p \xrightarrow{d} \mathbb{R}$ in $\text{Exit}(\text{Ran}^u(\mathbb{R}))$, given by $i \mapsto i$. Then, $[p]$ is isomorphic to $G_1(T_p) := \pi_o(\mathbb{R} - d(T_p))$ in $\Delta$, with the isomorphism given by $i \mapsto [i + \frac{1}{2}]$.

Fix a pair of objects $S \xrightarrow{c} \mathbb{R}$ and $T \xrightarrow{d} \mathbb{R}$ in $\text{Exit}(\text{Ran}^u(\mathbb{R}))$. Showing fully faithfulness of $G_1$ amounts to showing that the map induced by $G_1$ between corresponding hom-spaces

$$(30) \quad \text{Hom}_{\text{Exit}(\text{Ran}^u(\mathbb{R}))}(e, d) \xrightarrow{G_1} \text{Hom}_{\Delta^\text{act}}(\pi_o(\mathbb{R} - e(S)), \pi_o(\mathbb{R} - d(T)))$$

is a surjection on connected components with contractible fibers.
Fix a morphism \( \pi_0(\mathbb{R} - e(S)) \xrightarrow{\gamma_1} \pi_0(\mathbb{R} - d(T)) \) in \( \Delta^{\text{act}} \). Any morphism \( (31) \)

\[
\text{cylr}(T) \xrightarrow{\gamma_1(\varphi)} S \xrightarrow{E} \mathbb{R} \times \Delta^1
\]

in \( \text{Hom}_{\text{Exit}(\text{Ran}^n(\mathbb{R}))}(e, d) \) is in the fiber of \( G_1 \) over \( \varphi \). Indeed, we observed in Observation 1.0.9 that \( \gamma_1 \) is injective on hom-sets. Thus, commutativity of (24) guarantees that \( E \) is in the fiber of \( G_1 \) over \( \varphi \). Hence, (30) is a surjection on connected components.

The fiber of (30) over \( \varphi \) is the topological space of embeddings \( \text{cylr}(\gamma_1 \circ \varphi) \hookrightarrow \mathbb{R} \times \Delta^1 \) such that \( E|_S = e \) and \( E|_{T \times \{1\}} = d \), which we denote by

\[
G_1^{-1}(\varphi) \cong \text{Emb}^{e,d}_{\Delta^1}(\text{cylr}(\gamma_1(\varphi)), \mathbb{R} \times \Delta^1)
\]

under the compact-open topology. We will show that this space is contractible. Fix an embedding \( \tilde{E} \) in the fiber of \( G_1 \) over \( \varphi \). Let \( S^k \xrightarrow{\psi} G_1^{-1}(\varphi) \) be continuous and based at \( \tilde{E} \). We construct a null-homotopy of \( \psi \). For each \( z \in S^n \), denote the image of \( z \) under \( \psi \) by \( \psi_z \). The straight-line homotopy, \( H_z \), from \( \psi_z \) to \( \tilde{E} \) defined by

\[
H_z(x, t) = (1 - t)\psi_z(x) + t\tilde{E}(x)
\]

names a path from \( \psi_z \) to \( \tilde{E} \) in \( G_1^{-1}(\varphi) \). For each \( z \in S^n \), we let each path \( H_z \) run simultaneously to name a null-homotopy of \( \psi \) to the constant path at \( \{\tilde{E}\} \). Explicitly, the null-homotopy \( S^k \times [0, 1] \to G_1^{-1}(\varphi) \) is given by \( (z, t) \mapsto H_z(-, t) \).

[General case]: We will show that \( G_n \) is essentially surjective and fully faithful. Let \( [k](T_s) \) be an object in \( \Theta^{\text{act}} \). Because \( G_1 \) is essentially surjective, we may choose an object of \( \text{Exit}(\text{Ran}^n(\mathbb{R})) \) that is in the fiber of \( G_1 \) over \( [k] \):

\[
(32) \quad \{1, \ldots, k\} \xrightarrow{\psi} \mathbb{R}.
\]

Likewise, by essential surjectivity of \( G_{n-1} \), for each \( s \in \{1, \ldots, k\} \), we may choose an object of \( \text{Exit}(\text{Ran}^n(\mathbb{R}^{n-1})) \) that is in the fiber of \( G_{n-1} \) over \( T_s \):

\[
(33) \quad \begin{array}{ccc}
(S_{n-1})_s & \xrightarrow{(e_{n-1})_s} & \mathbb{R}^{n-1} \\
\downarrow (\tau_{n-2})_s & & \downarrow \text{pr}_{e,n-1} \\
(S_{n-2})_s & \xrightarrow{(e_{n-2})_s} & \mathbb{R}^{n-2} \\
\downarrow (\tau_{n-3})_s & & \downarrow \text{pr}_{e,n-2} \\
\vdots & & \vdots \\
(S_1)_s & \xrightarrow{(e_1)_s} & \mathbb{R}.
\end{array}
\]
The choices (32) and (33) for each $s$, uniquely determine an object of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ that is in the fiber of $G_n$ over $[k](T_s)$:

$$\prod_{1 \leq s \leq k} (S_{n-1})_s \xrightarrow{\prod e(s) \times (e_{n-2})} \mathbb{R} \times \mathbb{R}^{n-1}$$

$$\prod_{1 \leq s \leq k} (S_{n-2})_s \xrightarrow{\prod e(s) \times (e_{n-2})} \mathbb{R} \times \mathbb{R}^{n-2}$$

$$\vdots$$

$$\prod_{1 \leq s \leq k} (S_1)_s \xrightarrow{\prod e(s) \times (e_1)} \mathbb{R} \times \mathbb{R}$$

$$(34)$$

where each map defined in terms of a coproduct is indexed over $1 \leq s \leq k$, and $\{e(s)\}$ and $\{s\}$ denote the constant maps at $e(s)$ and $s$, respectively.

Fix a pair of objects $T \xrightarrow{d} \mathbb{R}^n$ and $S \xrightarrow{e} \mathbb{R}^n$ in $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$. We will show fully faithfulness of $G_n$ by showing that the map induced by $G_n$ between hom-spaces

$$(35) \quad \text{Hom}_{\text{Exit}(\text{Ran}^u(\mathbb{R}^n))}(T, S) \xrightarrow{G_n} \text{Hom}_{\Theta_n^{\text{act}}}(G_n(T), G_n(S))$$

is a surjection on connected components with contractible fibers.

Fix a morphism $G_n(T) \xrightarrow{\varphi} G_n(S)$ in $\Theta_n^{\text{act}}$. Using the inductive description of $G_n$, $\varphi$ is given by:

i) a morphism $G_1(S_1) \xrightarrow{\varphi_1} G_1(T_1)$ in $\Delta^{\text{act}}$

ii) for each pair $(r \in T_1, s \in S_1)$ such that $\gamma_1(\varphi_1(r)) = s$, a morphism $G_{n-1}(S_1) \xrightarrow{\varphi(r)} G_{n-1}(T_1)$ in $\Theta_{n-1}^{\text{act}}$.

Using the basecase and inductive step, we define a morphism that is in the fiber of $G_n$ over $\varphi$: By fullness of $G_1$, we may choose a morphism in the fiber of $G_1$ over $\varphi_1$,

$$(36) \quad \text{cylr}(\gamma_1(\varphi_1)) : \mathbb{R} \times \Delta^1$$

which is defined over the map of finite sets $T_1 \xrightarrow{\gamma_1 \circ \varphi_1} S_1$.

By fullness of $G_{n-1}$ as assumed in the inductive step, for each pair $(r \in T_1, s \in S_1)$ such that $\gamma_1(\varphi_1(r)) = s$, we may choose a morphism in the fiber of $G_{n-1}$ over $\varphi_r$, 






















Using (36) and (37), we define a morphism in the fiber of $G_n$ over $\varphi$:

$$
cyl(\bigoplus_{r \in T_1} \gamma_{n-1} \circ \varphi_r) \xrightarrow{(E_n)_r} \mathbb{R}^{n-1} \times \Delta^1
$$

$$
cyl(\bigoplus_{r \in T_1} \gamma_{n-2} \circ \text{tr} \circ \varphi_r) \xrightarrow{(E_{n-1})_r} \mathbb{R}^{n-2} \times \Delta^1
$$

$$
\vdots
$$

$$
cyl(\gamma_1 \circ \varphi_1) \xrightarrow{E_1} \mathbb{R} \times \Delta^1
$$

where $\{r\}$ and $\{E_1(r)\}$ denote the constant map at $r$ and $E_1(r)$, respectively, and (39) is defined over the diagram of finite sets,
\[
T_n = \prod_{r \in T_1} (T_n)_r \xrightarrow{\prod \gamma_{n-1} \circ \varphi_r} \prod_{s \in S_1} (S_n)_s = S_n
\]

\[
T_{n-1} = \prod_{r \in T_1} (T_{n-1})_r \xrightarrow{\prod \gamma_{n-2} \circ \text{tr}_{n-2} \circ \varphi_r} \prod_{s \in S_1} (S_{n-1})_s = S_{n-1}
\]

\[
T_2 = \prod_{r \in T_1} (T_2)_r \xrightarrow{\prod \gamma_1 \circ \varphi_r} \prod_{s \in S_1} (S_2)_s = S_2
\]

\[
T_1 = \prod_{r \in T_1} r \xrightarrow{\gamma_1 \circ \varphi_1} \prod_{s \in S_1} s = S_1.
\]

Lastly, we will show that each fiber of (35) is contractible. The fiber of \(G_n\) in (35) over \(\varphi\) is, under the compact-open topology, the topological space of compatible embeddings

\[
\begin{align*}
\text{cyl}(\gamma_n \circ \varphi) & \xrightarrow{E_n} \mathbb{R}^n \times \Delta^1 \\
\text{cyl}(\gamma'_n \circ \text{tr}_{n-1} \circ \varphi) & \xrightarrow{E_{n-1}} \mathbb{R}^{n-1} \times \Delta^1 \\
\text{cyl}(\gamma_1 \circ \text{tr}_1 \circ \varphi) & \xrightarrow{E_1} \mathbb{R} \times \Delta^1
\end{align*}
\]

over \(\Delta^1\) such that \(E|_{S_n} = e_n\) and \(E|_{T_n \times \{1\}} = d_n\). Note that (41) guarantees that each morphism in \(G^{-1}_n(\varphi)\) is defined over the diagram of finite sets

\[
\begin{align*}
T_n & \xrightarrow{\gamma'_n \circ \varphi} S_n \\
T_{n-1} & \xrightarrow{\gamma'_n \circ \text{tr}_{n-1} \circ \varphi} S_{n-1} \\
T_1 & \xrightarrow{\gamma_1 \circ \text{tr}_1 \circ \varphi} S_1.
\end{align*}
\]

Fix an embedding \(E\) in the fiber of \(G_n\) over \(\varphi\). Let \(S^h \xrightarrow{\psi} G^{-1}_n(\varphi)\) be continuous and based at \(E\). We construct a null-homotopy of \(\psi\): For each \(z \in S^n\), denote the image of \(z\) under \(\psi\) by \(\psi_z\). The straight-line homotopy, \(H_z\), from \(\psi_z\) to \(E\) defined by

\[
H_z(x, t) = (1 - t)\psi_z(x) + tE(x)
\]
names a path from \( \psi_z \) to \( E \) in \( G_n^{-1} (\varphi) \). For each \( z \in S^k \), we let each path \( H_z \) run simultaneously to name a null-homotopy of \( \psi \) to the constant path at \( \{ E \} \). Explicitly, the null-homotopy \( S^k \times [0, 1] \to G_n^{-1} (\varphi) \) is given by \( (z, t) \mapsto H_z (-, t) \).

\[ \square \]

We have just identified the domain \( \Theta_n^{\text{fin}} \) of our main result, Theorem 2.2.1 as the \( \infty \)-category \( \text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \); an \( \infty \)-category which, in particular, has an inherent relationship to the codomain \( \text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \) of Theorem 2.2.1 by forgetting, as observed in Observation 3.1.6: Explicitly, there is a natural forgetful functor

\[
\text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \to \text{Exit}(\text{Ran}^u (\mathbb{R}^n))
\]

over \( \text{Fin}^{\text{op}} \) induced by the functor from \( \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}) \) to \( \text{Fin}^{\text{op}} \) that evaluates on \( \{n\} \). The value on an object \( S \xrightarrow{\xi} \mathbb{R}^n \), where \( S := S_n \to \cdots \to S_1 \) is a sequence of finite sets, is \( S_n \xrightarrow{\xi_n} \mathbb{R}^n \).

The value on a morphism \( \text{cylr}(T \xrightarrow{\sigma} S) \xrightarrow{E} \mathbb{R}^n \times \Delta^1 \) is \( \text{cylr}(T_n \xrightarrow{\sigma_n} S_n) \xrightarrow{E_n} \mathbb{R}^n \). The next section is devoted to proving that this natural forgetful functor is a localization to \( \text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \).

4. Part 2 of the proof of Theorem 2.2.1: Localizing to \( \text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \)

The main goal of this section is to prove Lemma 4.0.2, which states that \( \text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \) localizes to \( \text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \) by the forgetful functor. We define the notion of a localization of \( \infty \)-categories as follows.

**Definition 4.0.1.** Let \( \mathcal{C} \) be an \( \infty \)-category and let \( W \) be a \( \infty \)-subcategory of \( \mathcal{C} \) which contains the maximal \( \infty \)-subcategory \( \mathcal{C}^{-} \) of \( \mathcal{C} \). The localization of \( \mathcal{C} \) on \( W \) is an \( \infty \)-category \( \mathcal{C}[W^{-1}] \) and a functor \( \mathcal{C} \xrightarrow{\xi} \mathcal{C}[W^{-1}] \) satisfying the following universal property: For any \( \infty \)-category \( \mathcal{D} \), any functor \( F \) from \( \mathcal{C} \) to \( \mathcal{D} \) uniquely factors through \( L \) if and only if \( F \) maps each morphism in \( W \) to an isomorphism in \( \mathcal{D} \); otherwise, there is no filler

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow & \mu \text{ or } \emptyset & \\
\mathcal{C}[W^{-1}].
\end{array}
\]

Heuristically, a localization means that upon formally inverting some collection of morphisms in the domain \( \mathcal{C} \) there results an \( \infty \)-category which is equivalent to the codomain \( \mathcal{D} \). The focus of this section is the following lemma, which states that the natural forgetful functor from Observation 3.1.6 localizes \( \text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \) to \( \text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \).

**Lemma 4.0.2.** The forgetful functor is a localization of \( \infty \)-categories

\[
\text{Exit}(\text{Ran}^u (\mathbb{R}^n)) \to \text{Exit}(\text{Ran}^u (\mathbb{R}^n))
\]

over \( \text{Fin}^{\text{op}} \) from the exit-path \( \infty \)-category of the refined unital Ran space of \( \mathbb{R}^n \) to the exit-path \( \infty \)-category of the unital Ran space of \( \mathbb{R}^n \).

Heuristically, this means that upon formally inverting a certain subclass of morphisms in \( \text{Exit}(\mathbb{R}^n) \), we obtain an \( \infty \)-category which, through the forgetful functor, is equivalent to \( \text{Exit}(\mathbb{R}^n) \). We define the localizing \( \infty \)-subcategory of \( \text{Exit}(\mathbb{R}^n) \) next.

**Definition 4.0.3.** \( W_n \) is the \( \infty \)-subcategory of \( \text{Exit}(\mathbb{R}^n) \) defined to be the pullback

\[
\begin{array}{ccc}
W_n & \xrightarrow{\phi_n} & \text{Exit}(\mathbb{R}^n) \\
\downarrow & & \downarrow \\
\text{Fin}^{n\text{-bij}}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}) & \xrightarrow{\phi_n} & \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})
\end{array}
\]
where \( \text{Fun}^{n, \text{bij}} \{ 1 < \cdots < n \}, \text{Fin}^\text{op} \) is the subcategory of \( \text{Fun} \{ 1 < \cdots < n \}, \text{Fin}^\text{op} \) in which the objects are the same and a morphism must satisfy that its value under evaluation at \( n \) is a bijection, and \( \Phi_n \) is the forgetful functor from Observation 3.1.5 that simply remembers the underlying data of sets at each level \( 1 \leq i \leq n \).

Heuristically, \( W_n \) has the same objects as \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) and all those morphisms whose values under \( \phi_n \) from Observation 3.1.5 are bijections. Intuitively then, \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) localizing on \( W_n \) to \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) is no surprise. Indeed, in formally declaring all those morphisms in \( W_n \) to be isomorphisms, we forget the restriction by coordinate coincidence which defines morphisms in \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) and only remember cardinality, which is the defining restriction of morphisms in \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \).

When convenient, we consider \( W_n \) as a subcategory of \( \Theta_n^\text{ext} \) in lieu of the equivalence

\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \simeq \Theta_n^\text{ext}
\]

from in Lemma 3.3.2.

Our procedure for showing Lemma 4.0.2 is technical, built around Theorem 4.0.6 from [23]. First, we define the following notion used in the statement of Theorem 4.0.6.

**Definition 4.0.4.** Given an \( \infty \)-category \( \mathcal{C} \) and an \( \infty \)-subcategory \( W \hookrightarrow \mathcal{C} \), \( \text{Fun}^W([p], \mathcal{C}) \) is defined to be the pullback of \( \infty \)-categories

\[
\begin{array}{ccc}
\text{Fun}^W([p], \mathcal{C}) & \longrightarrow & \text{Fun}([p], \mathcal{C}) \\
\downarrow & & \downarrow \\
\text{Fun}([p]^\sim, W) & \longrightarrow & \text{Fun}([p]^\sim, \mathcal{C})
\end{array}
\]

where \([p]^\sim\) denotes the underlying maximal \( \infty \)-subgroupoid of \([p]\).

**Observation 4.0.5.** In the case \( p = 0 \), \( \text{Fun}^W([0], \mathcal{C}) \) is equivalent to \( W \). Indeed, an object is a functor \([0] \to \mathcal{C}\) selects out an object of \( W \), which is precisely an object of \( \mathcal{C} \); a morphism is a natural transformation between any two such functors, which is precisely determined by a morphism in \( W \).

A similar examination of the \( p = 1 \) case identifies that \( \text{Fun}^W([1], \mathcal{C}) \) is the \( \infty \)-category whose objects are morphisms of \( \mathcal{C} \) and whose morphisms are all those natural transformations given by morphisms in \( W \), i.e., a morphism from \( c \to d \) in \( \mathcal{C} \) to \( c' \to d' \) in \( \mathcal{C} \) is a commutative square in \( \mathcal{C} \)

\[
\begin{array}{ccc}
c & \longrightarrow & c' \\
\downarrow & & \downarrow \\
d & \longrightarrow & d'
\end{array}
\]

such that both horizontal arrows are morphisms in \( W \).

The following theorem identifies localization of \( \infty \)-categories in favorable cases. It will be our route for identifying the localization of Lemma 4.0.2.

**Theorem 4.0.6** (3.8 in [23]). For an \( \infty \)-category \( \mathcal{C} \) and an \( \infty \)-subcategory containing the maximal \( \infty \)-subgroupoid of \( \mathcal{C} \), \( \mathcal{C}^\sim \subset W \subset \mathcal{C} \), if the classifying space of \( \text{Fun}^W([\bullet], \mathcal{C}) \) is a complete Segal space, then it is equivalent as a simplicial space to the localization of \( \mathcal{C} \) on \( W \),

\[
\mathbf{\text{Fun}}^W([\bullet], \mathcal{C}) \simeq \mathcal{C}^{[W^{-1}]}.
\]

Through Theorem 4.0.6 Lemma 4.0.2 follows from the following two lemmas.

**Lemma 4.0.7.** For \( p = 0, 1 \), there is an equivalence of spaces

\[
\mathbf{\text{Fun}}^{W_n}([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \simeq \text{Hom}_{\text{Cat}_\infty}([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]

between the classifying space of the \( \infty \)-category \( \text{Fun}^{W_n}([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \) and the hom-space in \( \infty \)-categories from \([p]\) to the exit-path \( \infty \)-category of the unital Ran space of \( \mathbb{R}^n \).

43
Lemma 4.0.8. The classifying space of $\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$ is a complete Segal space.

Let us explain how it is that Lemma 4.0.2 follows from these two lemmas; we will give a formal proof at the end of this section. First, note that Lemma 4.0.8 verifies the hypothesis of Theorem 4.0.6. In particular, $\mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$ is determined by its values on $[0]$ and $[1]$ because it satisfies the Segal condition (Definition 2.1.1). Through Theorem 4.0.6, Lemma 4.0.7 identifies the space of objects of the localization of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ on $W_n$ as the space of objects of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ and the space of morphisms of the localization as the space of morphisms of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$. The Segal condition, then, implies the desired result, that $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))[W_n^{-1}]$ is equivalent to $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$.

We organize our proofs of Lemma 4.0.7 and Lemma 4.0.8 as follows. Lemma 4.0.7 naturally decomposes into its two cases, $p = 0$ and $p = 1$; we make each case into a lemma, each of which is stated and proven in the subsequent subsections §4.1 and §4.2. And lastly, §4.3 is devoted to the proof of Lemma 4.0.8.

4.1. Identifying the space of objects of the localization of Lemma 4.0.2. This subsection is devoted to proving Lemma 4.1.1, wherein, through Theorem 4.0.6, the space of objects of the localization of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ to $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ by the forgetful functor is identified.

In Observation 4.0.5, we observed that in general, $\text{Fun}^W_n([0], C) \simeq W$. In light of this, we rephrase the $p = 0$ case of Lemma 4.0.7 as follows.

**Lemma 4.1.1** (Lemma 4.0.7, $p = 0$). There is an equivalence of spaces $\mathcal{B}W_n \simeq \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ from the classifying space of the $\infty$-subcategory $W_n$ of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ to the maximal $\infty$-subcategory of the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$.

We first compile two key facts: Lemma 4.1.3 and Corollary 4.1.7. In Lemma 4.1.3, we show that there is an adjunction between $W_n$ and the subcategory of $W_n$ consisting of healthy trees. Explicitly, this subcategory is defined as follows.

**Definition 4.1.2.** $W^n_{\text{ht}}$ is the subcategory of $W_n$ defined to be the pullback

$$
\begin{array}{ccc}
W^n_{\text{ht}} & \longrightarrow & W_n \\
\downarrow & & \downarrow \\
\text{Fun}^n_{\text{bij}}(\{1 < \cdots < n\}, (\text{Fin}^\text{surj})^{\text{op}}) & \longrightarrow & \text{Fun}^n_{\text{bij}}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})
\end{array}
$$

of categories.

Informally, the category $W^n_{\text{ht}}$ is the full subcategory of $W_n$ consisting of all those objects of $\Theta_n$ that are healthy trees.

**Lemma 4.1.3.** The inclusion functor $W^n_{\text{ht}} \hookrightarrow W_n$ is a right adjoint.

To prove this lemma, we need the following construction.

**Construction 4.1.4** (The Pruning Functor, $P_n$). For each $n \geq 1$, we define a canonical functor $P_n : W_n \to W^n_{\text{ht}}$.

For $n = 1$, $P_1 := \text{Id}_{W_1}$ since $W_1 = W^1_{\text{ht}}$.

For $n \geq 2$, we define $P_n$ inductively. First, for each object $T = [p](T_i) \in W_n$ for $n \geq 2$, define the sub-linearly ordered set $N_T := \left\{0 = i_0 < i_1 < \cdots < i_k \mid \begin{array}{l}i_j \in \{1, \ldots, p\} \forall 1 \leq j \leq k \\
T_i = \emptyset \iff \exists 1 \leq j \leq k \text{ s.t. } i_j = i\end{array} \right\} \subset [p]$. 

44
Further, respecting composition because restriction respects composition. On objects; the value of a morphism under $P$ is again determined by restriction of that morphism to $N_T$ together with $P_n-1$. Composition is preserved by $P_n$ because restriction and $P_n-1$ both respect composition.

Proof of Lemma 4.1.3. We use Lemma 2.17 from [3] which states that for a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of $\infty$-categories, $F$ is a right adjoint if and only if for each object $d \in \mathcal{D}$, the $\infty$-undercategory $\mathcal{C}^{d/}$ has an initial object, and verify that for each $T \in W_n$, the undercategory $W_{\text{hlt}}^{T/}$ has an initial object. To define such an initial object, we use the canonical morphism defined as follows:

For each object $T \in W_n$, we define a morphism $T \xrightarrow{\alpha_T} P_n(T)$ in $W_n$ such that any morphism $T \xrightarrow{f} S$ in $W$ to a healthy tree $S$ uniquely factors through $\alpha_T$. For $n = 1$, $T = P_1(T)$ for each $T \in W_1$ and thus we define $\alpha_T := \text{Id}_T$.

To define $\alpha_T$ for $T \in W_n$ for $n \geq 2$, we proceed by induction. Fix an object $T \in W_2$. Define $\alpha_T : T = [p]([q_i]) \rightarrow N_T([q_i])$ by

i) $[p] \rightarrow N_T$ is given by the assignment $i \mapsto \begin{cases} i_j, & \text{if } \exists 0 \leq j \leq k-1 \text{ s.t. } i_j \leq i < i_{j+1} \\ i_k, & \text{if } i \geq i_k. \end{cases}$

ii) For each pair $(i, i_j)$ such that $i = i_j$, $[q_i] \xrightarrow{\text{Id}} [q_{i_j}]$.

For the general case, fix an object $T \in W_n$. Define $\alpha_T : T = [p](T_i) \rightarrow N_T(P_{n-1}(T_i))$ by

i) $[p] \rightarrow N_T$ is the same as i) for $n = 2$.

ii) For each pair $(i, i_j)$ such that $i = i_j$, $T_i \xrightarrow{\alpha_{T_i}} P_{n-1}(T_{i_j})$, where $\alpha_{T_i}$ is guaranteed by the inductive step.

Next, we observe that by design each morphism $T \xrightarrow{f} S$ in $W$ to a healthy tree $S$ factors through $\alpha_T$ via $P_n(f)$:

\[
\begin{array}{ccc}
T & \xrightarrow{\alpha_T} & S \\
\downarrow^f & & \downarrow^S \\
\downarrow_{P_n(f)} & & \downarrow_{P_n(T)} \\
\end{array}
\]

Further, $P_n(f)$ uniquely fills (43).

We have just verified that for each fixed object $T \in W$, the initial object of $W_{\text{hlt}}^{T/}$ is $(P_n(T), T \xrightarrow{\alpha_T} P_n(T))$.

\[\square\]

We have just proven the first of the two key facts that we will use to prove Lemma 4.1.1. The second fact, Corollary 4.1.7, is a corollary to the main result in [7]. Namely, Theorem A in [7] identifies the homotopy type of the configuration space of a fixed number of ordered points in $\mathbb{R}^n$; we extend this identification in Corollary 4.1.7 to the unordered case. The next subsection 4.1.1 is devoted to proving this corollary. We access the proof of Corollary 4.1.7 through the developments of [7] and thus, 4.1.1 also serves as the explicit bridge between their work and ours. The tools developed in it articulate the sense in which the main result of this paper, Theorem 2.2.1, is a generalization of that of [7].
4.1.1. Identifying the space of configurations of $r$ unordered points in $\mathbb{R}^n$. The ultimate goal of this subsection is to prove Corollary 4.1.7, wherein we identify that the configuration space of $r$ unordered points in $\mathbb{R}^n$ is homotopy equivalent to the classifying space of the following $\infty$-category:

**Definition 4.1.5.** For $r \geq 0$, $\text{Exit}(\text{Conf}_r(\mathbb{R}^n))_{\Sigma_r}$ is the $\infty$-subcategory of the exit-path $\infty$-category of the refined unital Ran space of $\mathbb{R}^n$ defined to be the pullback

$$
\begin{align*}
\text{Exit}(\text{Conf}_r(\mathbb{R}^n))_{\Sigma_r} & \hookrightarrow \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \\
\downarrow & \\
\text{Fun}^{r,n}(\{1 < \cdots < n\}, (\text{Fin}^\text{surj})^\text{op}) & \hookleftarrow \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^\text{op}).
\end{align*}
$$

where $\text{Fun}^{r,n}(\{1 < \cdots < n\}, (\text{Fin}^\text{surj})^\text{op})$ is the subcategory of $\text{Fun}(\{1 < \cdots < n\}, (\text{Fin}^\text{surj})^\text{op})$ in which the value of an object upon evaluation at $n$ has cardinality $r$.

**Observation 4.1.6.** It follows from the previous definition that the colimit of $\infty$-categories

$$
\coprod_{r \geq 0} \text{Exit}(\text{Conf}_r(\mathbb{R}^n))_{\Sigma_r}
$$

is equivalent to the category $W_{\mathbb{R}^n}^{\text{hlt}}$.

As previously mentioned, the following is a corollary to Theorem A in [7] which identifies the homotopy type of the configuration space of $r$ ordered points in $\mathbb{R}^n$ as the classifying space of certain subcategory of $\Theta_n$.

**Corollary 4.1.7.** There is a homotopy equivalence

$$
\text{B} \text{Exit}(\text{Conf}_r(\mathbb{R}^n))_{\Sigma_r} \simeq \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}
$$

from the classifying space of the $\infty$-category $\text{Exit}(\text{Conf}_r(\mathbb{R}^n))_{\Sigma_r}$ to the configuration space of $r$ unordered points in $\mathbb{R}^n$.

In order to prove Corollary 4.1.7 we must first translate the language of [7] into that of this paper; namely, that of exit-path $\infty$-categories. The following $\infty$-categories do just that, as we will explain in Observation 4.1.9.

**Definition 4.1.8.** Fix a natural number $r$ and a set $S$ with cardinality $r$.

- The category $\Theta_n^{\text{hlt, act}}$ is the full subcategory of $\Theta_n^{\text{act}}$ consisting of all those objects that are healthy trees.
- The category $\Theta_n^{\text{hlt}}$ is the subcategory of $\Theta_n^{\text{hlt, act}}$ defined to be the pullback

$$
\begin{align*}
\Theta_n^{\text{hlt, act}} & \twoheadrightarrow \Theta_n^{\text{hlt}} \\
\downarrow & \\
(\text{Fin}_r)^\text{op} & \twoheadrightarrow \text{Fin}^\text{op}
\end{align*}
$$

of categories, where $\text{Fin}_r$ is the full subcategory of $\text{Fin}$ consisting of all those finite sets with cardinality $r$.
- The category $\Theta_n^{\text{hlt}}(S)$ is defined to be the pullback

$$
\begin{align*}
\Theta_n^{\text{hlt}}(S) & \twoheadrightarrow \Theta_n^{\text{hlt, act}} \\
\downarrow & \\
\langle S \rangle & \twoheadrightarrow (\text{Fin}_k)^\text{op}
\end{align*}
$$

of categories.
• The $\infty$-category $\text{Exit}(\text{Conf}_S(\mathbb{R}^n))$ is defined to be the pullback

$$
\begin{align*}
\text{Exit}(\text{Conf}_S(\mathbb{R}^n)) & \longrightarrow \text{Exit}(\text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}) \\
\downarrow & \\
\Theta_n^{\text{hit}}(S) & \longrightarrow \Theta_n^{\text{hit},r}
\end{align*}
$$

of $\infty$-categories.

The objects of $\Theta_n^{\text{hit},r}$ are healthy trees whose set of leaves has cardinality $r$, whereas the objects of $\Theta_n^{\text{hit}}(S)$ are healthy trees whose set of leaves is labeled by the set $S$ of cardinality $r$.

Heuristically, an object of $\text{Exit}(\text{Conf}_S(\mathbb{R}^n))$ is an embedding of the set $S$ into $\mathbb{R}^n$ and a morphism is a path in $\text{Conf}_S(\mathbb{R}^n)$, the image of which, after projecting off the last $i$ coordinates, for each $0 \leq i \leq n-1$, is an ‘exit-path’ in $\text{Conf}_*(\mathbb{R}^{n-i})$; i.e., it allows anticollision of points, but does not allow collision of points.

**Observation 4.1.9.** The category $\Theta_n^{\text{hit}}(S)$ is equivalent to the category in [7] referred to as the poset of $n$-orderings of $S$, denoted $n\text{Ord}(S)$, in which an object is a healthy tree of height $n$ whose set of leaves is labeled by the set $S$, and a morphism is an active morphism in $\Theta_n$ which satisfies the branching condition (see Definition 8 in [7]).

The following result is essentially a restatement of Theorem A of [7] in the language of exit-path $\infty$-categories.

**Corollary 4.1.10 (Theorem A of [7]).** There is a homotopy equivalence

$$
\mathbb{B} \text{Exit}(\text{Conf}_S(\mathbb{R}^n)) \simeq \text{Conf}_S(\mathbb{R}^n)
$$

between the classifying space of the $\infty$-category $\text{Exit}(\text{Conf}_S(\mathbb{R}^n))$ and the configuration space of points in $\mathbb{R}^n$ marked by the set $S$ of cardinality $r$.

**Proof.** First, observe that the following diagram is a pullback of $\infty$-categories:

$$
\begin{align*}
\text{Exit}(\text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}) & \longrightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \\
\downarrow & \\
\Theta_n^{\text{hit},r} & \longrightarrow \Theta_n^{\text{act}}
\end{align*}
$$

where $G_n$ was defined in Construction 3.3.1.

Combining (44) with the definition of $\text{Exit}(\text{Conf}_S(\mathbb{R}^n))$, we obtain the following diagram of $\infty$-categories:

$$
\begin{align*}
\text{Exit}(\text{Conf}_S(\mathbb{R}^n)) & \longrightarrow \text{Exit}(\text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}) \longrightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \\
\downarrow & \\
\Theta_n^{\text{hit}}(S) & \longrightarrow \Theta_n^{\text{hit},r} \longrightarrow \Theta_n^{\text{act}}
\end{align*}
$$

Lemma 3.3.2 showed that the functor $G_n$ is an equivalence, which implies the other two downward vertical arrows in (45) are both equivalences of $\infty$-categories as well. In particular, we note the equivalence $\text{Exit}(\text{Conf}_S(\mathbb{R}^n)) \simeq \Theta_n^{\text{hit}}(S)$. Then, using the homotopy equivalence between the classifying space of $\Theta_n^{\text{hit}}(S)$ and $\text{Conf}_S(\mathbb{R}^n)$ established in Theorem A of [7], we have the desired result, namely,

$$
\mathbb{B} \text{Exit}(\text{Conf}_S(\mathbb{R}^n)) \simeq \mathbb{B}\Theta_n^{\text{hit}}(S) \simeq \text{Conf}_S(\mathbb{R}^n).
$$

□

We are now equipped to prove Corollary 4.1.7, which is the unordered version of the previous result Corollary 4.1.10.
Proof of Corollary 4.1.7. Fix a natural number $r$ and a set $S$ with cardinality $r$. Observe that the equivalence in Corollary 4.1.10 is $\Sigma_S$-equivariant, and thus,

$$\left( B \text{ Exit}(\text{Conf}_S(\mathbb{R}^n)) \right)_{\Sigma_S} \simeq \text{Conf}_S(\mathbb{R}^n)_{\Sigma_S}.$$ 

In chapter 4 of [21], it is shown that the classifying space of a colimit is equivalent to the colimit of the classifying space. Thus, since the quotient is a colimit, the quotient of the classifying space of

$$\text{Exit}(\text{Conf}_S(\mathbb{R}^n)) \Sigma_S \simeq \text{Conf}_S(\mathbb{R}^n) \Sigma_S,$$

is equivalent to the classifying space of

$$\text{Exit}(\text{Conf}_S(\mathbb{R}^n)) \Sigma_S \simeq \text{Conf}_S(\mathbb{R}^n) \Sigma_S.$$ 

□

Now we will compile Lemma 4.1.3 and Corollary 4.1.7 to prove Lemma 4.1.1 that there is an equivalence

$$B \text{ Exit}(\text{Conf}_S(\mathbb{R}^n)) \Sigma_r \simeq \text{Conf}_S(\mathbb{R}^n) \Sigma_r.$$ 

Proof of Lemma 4.1.1. Corollary 2.1.28 in [22] states that an adjunction between $\infty$-categories yields an equivalence between their classifying spaces. We apply this result to the adjunction from Lemma 4.1.3 to obtain an equivalence of the classifying spaces,

$$B W_n \simeq B W^\text{bit}.$$ 

In Observation 4.1.6, we observed that

$$B W^\text{bit} \simeq \prod_{r \geq 0} B \text{ Exit}(\text{Conf}_r(\mathbb{R}^n)) \Sigma_r.$$ 

By Corollary 4.1.7

$$\prod_{r \geq 0} B \text{ Exit}(\text{Conf}_r(\mathbb{R}^n)) \Sigma_r \simeq \prod_{r \geq 0} \text{Conf}_r(\mathbb{R}^n) \Sigma_r.$$ 

Lastly, $\prod_{r \geq 0} \text{Conf}_r(\mathbb{R}^n) \Sigma_r$ is, by definition, equivalent to $\text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n)) \sim$ the maximal $\infty$-subgroupoid of $\text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n)).$ □

In summary, we have just proven the $p = 0$ case of Lemma 4.0.7 which was restated as Lemma 4.1.1. By Theorem 4.0.6 this result identifies the space of objects of the localization of $\text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n))$ on $W_n$, as the space of objects of the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$.

4.2. Identifying the space of morphisms of the localization of Lemma 4.0.2. This subsection is devoted to proving the $p = 1$ case of Lemma 4.0.7. We restate it as follows.

Lemma 4.2.1 (Lemma 4.0.7; $p = 1$). There is an equivalence of spaces

$$\mathcal{B} \text{ Fun}^W_n \left( [1], \text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n)) \right) \simeq \text{mor} \left( \text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n)) \right)$$

induced by the forgetful functor $\text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n)) \to \text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n))$.

The core idea of the proof is the following: Both spaces of the equivalence in Lemma 4.2.1 naturally assemble as the fibrations depicted in (46) below. We will use the induced long exact sequence of homotopy groups to show a weak homotopy equivalence of total spaces by showing a homotopy equivalence between the base spaces and between the fibers. By Whitehead Theorem then, we obtain a homotopy equivalence of the total spaces since they are CW complexes.

\[ \begin{array}{ccc}
    \mathcal{B} \text{ Fun}^W_n \left( [1], \text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n)) \right) & \xrightarrow{\text{frg}} & \text{mor} \left( \text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n)) \right) \\
    \mathcal{B} \text{ Fun}^W_n \left( [0], \text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n)) \right) & \xrightarrow{\text{ev}_0} & \text{Exit}(\text{Ran}^\text{u}(\mathbb{R}^n)) \sim \\
\end{array} \]
Since we already showed that the base spaces are equivalent in Lemma 4.1.1, our work lies in showing an equivalence on the level of fibers. We identify the fibers throughout the course of three lemmas: Lemma 4.2.7 identifies the fibers of $ev_0$, and Lemma 4.2.8 and Lemma 4.2.11 identify the fibers of $\mathcal{B}_{ev_0}$. Each lemma is technical, relying on the concept of a Cartesian fibration. In the next subsection, we follow [3] to recall this technical machinery and further tailor it to the situation at hand, towards the goal of proving Lemma 4.2.7, Lemma 4.2.8, and Lemma 4.2.11.

4.2.1. Cartesian fibrations.

**Definition 4.2.2** (2.1 in [3]). Let $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ be a functor between $\infty$-categories. A morphism $c = (\langle e, \phi \rangle, \langle e', \phi' \rangle) \xrightarrow{\phi \circ \pi} \mathcal{E}$ is $\pi$-Cartesian if the diagram of $\infty$-overcategories

$$
\begin{array}{ccc}
\mathcal{E}_{e} & \xrightarrow{\phi \circ \pi} & \mathcal{E}_{e'} \\
\downarrow & & \downarrow \\
\mathcal{B}_{\pi(e)} & \xrightarrow{\pi(e) \circ \phi} & \mathcal{B}_{\pi(e')}
\end{array}
$$

is a pullback.

$\pi$ is a Cartesian fibration if for every solid square

$$
\begin{array}{ccc}
\ast & \xrightarrow{\pi} & \mathcal{E} \\
\downarrow & & \downarrow \\
c_1 & \xrightarrow{\pi} & \mathcal{B}
\end{array}
$$

there is a $\pi$-Cartesian filler.

**Observation 4.2.3.** The map

$$
\text{Fun}^W_{\mathcal{B}}\left([1], \text{Exit}(\text{Ran}^n(\mathbb{P}^n))\right) \xrightarrow{ev_0} \text{Fun}^W_{\mathcal{B}}\left([0], \text{Exit}(\text{Ran}^n(\mathbb{P}^n))\right)
$$

is a Cartesian fibration. The proof is straightforward, using Example 2.5 in [3], wherein it is shown that for an $\infty$-category $\mathcal{C}$, the functor given by evaluation at 0, $\text{Fun}([1], \mathcal{C}) \xrightarrow{ev_0} \text{Fun}([0], \mathcal{C})$ is a Cartesian fibration.

**Observation 4.2.4.** Let $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ be a Cartesian fibration. For each object $b \in \mathcal{B}$ there is a canonical inclusion $\pi^{-1}(b) \hookrightarrow \mathcal{E}^b$ from the fiber of $\pi$ over $b$ to the undercategory of $\mathcal{E}$ under $b$. Its value on an object $e \in \pi^{-1}(b)$ such that $b \cong \pi(e)$ is the equivalence $b \xrightarrow{\pi(e)} \pi(e)$. Its value on a morphism $e \xrightarrow{f} e'$ is $\pi(f)$.

**Definition 4.2.5** (Cartesian Monodromy Functor). Let $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ be a Cartesian fibration. For each morphism $b \xrightarrow{f} b'$ in $\mathcal{B}$, the induced Cartesian monodromy functor $f^* : \pi^{-1}(b') \to \pi^{-1}(b)$ from the fiber over $b'$ to the fiber over $b$ is defined to be the threefold composite

$$
\begin{array}{ccc}
\pi^{-1}(b') & \xrightarrow{f^*} & \pi^{-1}(b) \\
\downarrow & & \downarrow \\
\mathcal{E}^{b'} & \xrightarrow{- \circ f} & \mathcal{E}^b
\end{array}
$$

where $\mu$ is right adjoint to the inclusion functor $\pi^{-1}(b) \hookrightarrow \mathcal{E}^b$, the existence of which is guaranteed by Lemma 2.20 in [3].

**Observation 4.2.6.** Given a diagram of $\infty$-categories

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{G} & \mathcal{E}' \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow{F} & \mathcal{B}'
\end{array}
$$

(47)
in which \( \pi \) and \( \pi' \) are Cartesian fibrations, for each morphism \( b \to b' \) in \( \mathcal{B} \), \( G \) carries the induced monodromy functor \( \alpha^* \) to the monodromy functor induced by \( F(\alpha) \),

\[
\begin{array}{ccc}
\pi^{-1}(b') & \xrightarrow{\alpha^*} & \pi^{-1}(b) \\
\downarrow & & \downarrow \\
\pi'^{-1}(F(b')) & \xrightarrow{F(\alpha)^*} & \pi'^{-1}(F(b)).
\end{array}
\]

(48)

Further, if (47) is a pullback of \( \infty \)-categories, then the downward vertical arrows of (48) are equivalences.

4.2.2. Three lemmas to prove Lemma 4.2.1. This subsection is devoted to the proof of Lemma 4.2.1 wherein we show an equivalence

\[
\mathcal{B}\text{Fun}^{W_n}\left([1], \text{Exit}(\text{ Ran}^u(\mathbb{R}^n))\right) \simeq \text{mor}\left(\text{Exit}(\text{ Ran}^u(\mathbb{R}^n))\right)
\]

of spaces. The proof relies on three lemmas, each of which is presented and proven in this subsection, together which identify the fibers of (46).

For the remainder of this paper, we implement the following notational changes:

- \( \text{Exit}(\text{ Ran}^u(\mathbb{R}^n))\): We denote an object \( S \hookrightarrow \mathbb{R}^n \) by \( S \), by which we mean the image of \( S \) in \( \mathbb{R}^n \) under the embedding \( e \).
- \( \text{Exit}(\text{ Ran}^u(\mathbb{R}^n))\): We denote an object \( S \hookrightarrow \mathbb{R}^n \) by \( S \) by \( S = S_1 \to \cdots \to S_n \), by which we mean the images of \( S_i \) under \( e_i \) for each \( 1 \leq i \leq n \) together with the coordinate projection data given by the sequence of maps of finite sets \( S_n \to \cdots \to S_1 \).

We denote a morphism \( \text{cyl}(S') : S \to S' \) by \( S \to S' \) by simply an arrow \( S \to S' \).

The first lemma identifies the fiber of \( \text{ev}_0 \) in (46) as follows.

**Lemma 4.2.7.** The fiber of the map of spaces

\[
\text{mor}\left(\text{Exit}(\text{ Ran}^u(\mathbb{R}^n))\right) \xrightarrow{\text{ev}_0} \text{Exit}(\text{ Ran}^u(\mathbb{R}^n))
\]

from the space of morphisms of \( \text{Exit}(\text{ Ran}^u(\mathbb{R}^n)) \) to the maximal \( \infty \)-subgroupoid of \( \text{Exit}(\text{ Ran}^u(\mathbb{R}^n)) \) over an object \( S \subset \mathbb{R}^n \) of \( \text{Exit}(\text{ Ran}^u(\mathbb{R}^n)) \) is

\[
\prod_{s \in S} \text{Exit}(\text{ Ran}^u(T_s \mathbb{R}^n))
\]

the product space indexed by \( S \) of the maximal \( \infty \)-subgroupoid of the exit-path \( \infty \)-category of the unital Ran space of the tangent space of \( \mathbb{R}^n \) at \( s \in S \).

**Proof.** First, recall that the maximal \( \infty \)-subgroupoid \( \text{Exit}(\text{ Ran}^u(\mathbb{R}^n)) \) of \( \text{Exit}(\text{ Ran}^u(\mathbb{R}^n)) \) is equivalent to the disjoint union of configuration spaces \( \bigoplus_{r \geq 0} \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r} \).

Next, we define three maps.

1. Fix a continuous map \( \epsilon : \bigoplus_{r \geq 0} \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r} \to \mathbb{R}_{>0} \)

such that for each pair of distinct points \( s \neq s' \) in \( S \), the open \( n \)-dimensional cubes \( \text{Box}_{\epsilon(S)}(-) \cong (-\epsilon(S), \epsilon(S))^n \) of volume \( (2\epsilon(S))^n \), centered at each point do not intersect,

\[
\text{Box}_{\epsilon(S)}(s) \cap \text{Box}_{\epsilon(S)}(s') = \emptyset.
\]

Note that such a continuous map exists because \( \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r} \) is Hausdorff.

2. For each \( S \in \bigoplus_{r \geq 0} \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r} \) and each \( s \in S \), define a homeomorphism

\[
D_{s \in S} : \mathbb{R}^n \to \text{Box}_{\epsilon(S)}(s)
\]
by the composite
\[(49) \quad \mathbb{R}^n \xrightarrow{\sim} (-\epsilon(S), \epsilon(S))^n \xrightarrow{\sim} \text{Box}_{\epsilon(S)}(s)\]
the first homeomorphism of which is the product $\eta^{\times n}$ where
\[\mathbb{R} \xrightarrow{\eta} (-\epsilon(S), \epsilon(S))\]
is the homeomorphism given by $2\epsilon(S)\arctan(-)$; the second homeomorphism of \[(49)\] is translation by $s$, that is, $(-) + s$.

3. For $S \in \coprod_{r \geq 0} \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$, let
\[\text{Emb}^S_{\Delta^1}(cylr(f), R^n \times \Delta^1)\]
denote the subspace (with the subspace topology) of $\text{Emb}_{\Delta^1}(cylr(f), R^n \times \Delta^1)$ with the compact-open topology, consisting of those embeddings $E$ for which the image of $E|_S$ is the given subset $S \subset \mathbb{R}^n$.

Then, for each $S \in \coprod_{r \geq 0} \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$, fix a continuous map
\[\delta_S : \text{Emb}^S_{\Delta^1}(cylr(f), R^n \times \Delta^1) \to (0, 1]\]
such that for each point $cylr(T \xrightarrow{f} S) \xrightarrow{E} \mathbb{R}^n \times \Delta^1$ in the domain,
\[E(f^{-1}(s) \times \delta) \subset \text{Box}_{\epsilon(S)}(s)\]
for each $0 < \delta \leq \delta_S(E)$ and for each $s \in S$. Note that such a continuous map exists because $\text{Emb}^S_{\Delta^1}(cylr(f), R^n \times \Delta^1)$ is Hausdorff.

Next, fix an object $S$ in $\coprod_{r \geq 0} \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$. We will show that the map
\[(50) \quad \prod_{s \in S} \left( \prod_{k \geq 0} \text{Conf}_k(T_s \mathbb{R}^n)_{\Sigma_k} \right) \to e^{-1}(S)\]
given by
\[(51) \quad (R_s \subset T_s \mathbb{R}^n)_{s \in S} \mapsto \left( \text{cylr} \left( \prod_{s \in S} D_s(R_s) \xrightarrow{\text{index}} S \right) \xrightarrow{E^{\text{straight}}} \mathbb{R}^n \times \Delta^1 \right)\]
is a homotopy equivalence, where index is the map given by assigning $r \in D_s(R_s)$ to its index, $s \in S$, and $E^{\text{straight}}$ is the embedding given by straight-line paths; namely, for each pair $(s \in S, r \in D_s(R_s))$, the embedding $E^{\text{straight}}$ restricted to the segment $\text{cylr}(r \mapsto s) \simeq \Delta^1$ into $\mathbb{R}^n$ is given by
\[t \in \Delta^1 \mapsto r(1 - t) + st.\]

Consider a map in the other direction
\[e^{-1}(S) \to \prod_{s \in S} \left( \prod_{k \geq 0} \text{Conf}_k(T_s \mathbb{R}^n)_{\Sigma_k} \right)\]
defined by
\[(52) \quad (\text{cylr}(T \xrightarrow{f} S) \xrightarrow{E} \mathbb{R}^n \times \Delta^1) \mapsto (E_{|_{\text{ev}_1}}(f^{-1}(s)) \subset \mathbb{R}^n = T_s \mathbb{R}^n)_{s \in S}.\]

The homotopy from the identity of the left-hand side of \[(50)\] to the composite of \[(51)\] followed by \[(52)\] is given by applying the collection of homeomorphisms $\{D_s\}_{s \in S}$.

The homotopy from the identity on the fiber over $S$ to the other composite is given by concatenating the following four homotopies together in the specified order. For each $\text{cylr}(R \xrightarrow{f} S) \xrightarrow{E} \mathbb{R}^n \times \Delta^1$ in $e^{-1}(S)$,
1. Simultaneously run the paths of $E$ backwards until $t = \delta_S(E)$
2. For each $s \in S$, simultaneously run straight-line paths from each $r \in E(f^{-1}(s) \times \delta_S(E)) \subset \mathbb{R}^n$ to $D_{s \in S}(r)$
3. For each $s \in S$, simultaneously run the paths given by the composite $D_s \circ E_{\psi(t^{-1}(x) \to x)}$ from $t = \delta_S(E)$ to $t = 1$
4. For each $r \in R$, simultaneously straighten each path by the Alexandar trick.

The second lemma identifies the classifying space of the fiber of the map from $\text{Fun}^{W_n}([1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$ to $\text{Fun}^{W_n}([0], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$ given by evaluation at 0 as follows.

**Lemma 4.2.8.** The classifying space of the fiber of $\text{Fun}^{W_n}([1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$ to $\text{Fun}^{W_n}([0], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$

over an object $S := S_n \to S_{n-1} \to \cdots \to S_1$ is

$$\prod_{s \in S_n} \text{Exit}(\text{Ran}^n(T_s \mathbb{R}^n))$$

the product space indexed by $S_n$ of the maximal $\infty$-subgroupoid of the exit-path $\infty$-category of the unital Ran space of the tangent space of $\mathbb{R}^n$ at $s \in S_n$.

**Proof.** Fix an object $S$ in the base space $\text{Fun}([0], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$ of $ev_0$. We will show that there exists a refinement of the stratified space

$$\prod_{s \in S_n} \left( \prod_{k \geq 0} \text{Conf}_k(T_s \mathbb{R}^n)_{\Sigma_r} \right)$$

such that there exists an adjunction between the exit-path $\infty$-category of that refinement and the fiber of $ev_0$ over $S$. First, we define the desired refinement. Similar to $\epsilon$ and $D_{s \in S}$ as defined in the proof of Lemma 4.2.7, we define two maps:

1. A continuous map $\epsilon : \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \to \mathbb{R}_{>0}$ such that for each object $S := S_n \to \cdots \to S_1$ in $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ and for each pair of distinct points $s \neq s'$ in $S_n$,

$$\text{Box}_{\epsilon(S)}(s) \cap \text{Box}_{\epsilon(S)}(s') = \emptyset.$$

2. For each object $S := S_n \to \cdots \to S_1$ in $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ and for each $s \in S_n$, a homeomorphism $D_{s \in S_n} : \mathbb{R}^n \to \text{Box}_{\epsilon(S)}(s)$ from $n$-Euclidean space to the box of size $(2\epsilon(S))^n$ centered at $s$ defined exactly as in the proof of Lemma 4.2.7.

Consider the sub-stratified space of $\prod_{k \geq 0} \text{Conf}_k(\mathbb{R}^n)_{\Sigma_k}$,

$$\prod_{k \geq 0} \text{Conf}_k(S)(\mathbb{R}^n)_{\Sigma_k} := \left\{ R \subset \prod_{s \in S_n} \text{Box}_{\epsilon(S)}(s) \mid R \text{ is finite} \right\}.$$

For each $s \in S_n$, define a continuous map between topological spaces $\Phi_s : \prod_{k \geq 0} \text{Conf}_k(S)(\mathbb{R}^n)_{\Sigma_k} \to \prod_{r \geq 0} \text{Conf}_r(T_s \mathbb{R}^n)_{\Sigma_r}$ by the assignment

$$R \mapsto D_{s}^{-1}(R \cap \text{Box}_{\epsilon(S)}(s)).$$

52
Take the product of the $\Phi_s$ over all $s \in S_n$ to define a homeomorphism of topological spaces
\[
\Phi : \prod_{k \geq 0} \text{Conf}^k_\Sigma(R^n)_{\Sigma_k} \to \prod_{s \in S_n} \left( \prod_{r \geq 0} \text{Conf}_r(T_sR^n)_{\Sigma_r} \right)
\]
the inverse of which is given by the assignment
\[
(R_s \subset T_sR^n)_{s \in S_n} \mapsto \prod_{s \in S_n} D_s^{-1}(R_s).
\]
Observe that each stratum of $\prod_{k \geq 0} \text{Conf}^k_\Sigma(R^n)_{\Sigma_k}$ is carried by $\Phi$ into a stratum of the stratified space $\prod_{s \in S_n} \left( \prod_{r \geq 0} \text{Conf}_r(T_sR^n)_{\Sigma_r} \right)$, whose stratification is given as the product of the stratified spaces $\prod_{r \geq 0} \text{Conf}_r(T_sR^n)_{\Sigma_r}$. As such, the homeomorphism $\Phi$ is a refinement of $\prod_{k \geq 0} \text{Conf}^k_\Sigma(R^n)_{\Sigma_k}$ to $\prod_{s \in S_n} \left( \prod_{r \geq 0} \text{Conf}_r(T_sR^n)_{\Sigma_r} \right)$.

Consider the functor
\[
\text{Exit} \left( \prod_{k \geq 0} \text{Conf}^k_\Sigma(R^n)_{\Sigma_k} \right) \xrightarrow{\iota} \text{ev}^{-1}_0(\Sigma)
\]
whose value on object $R \subset \prod_{s \in S_n} \text{Box}_e(\Sigma)(s)$ is
\[
\text{cylr} \left( \prod_{s \in S_n} (R \cap \text{Box}_e(\Sigma)(s)) \to S_n \right) \xrightarrow{E^\text{straight}} \mathbb{R}^n \times \Delta^1
\]
where $E^\text{straight}$ is the embedding given by straight-line paths. Further, using Lemma 2.17 from [3], showing that $\iota$ is a right adjoint follows directly from Lemma 2.1.3. By Corollary 2.1.28 of [22] and Corollary 1.2.7 of [6], therefore,
\[
\text{Bev}^{-1}_0(\Sigma) \simeq \text{B Exit} \left( \prod_{k \geq 0} \text{Conf}^k_\Sigma(R^n)_{\Sigma_k} \right) \simeq \prod_{s \in S_n} \left( \prod_{r \geq 0} \text{Conf}_r(T_sR^n)_{\Sigma_r} \right).
\]

Let us briefly review: We seek to show an equivalence on the level of fibers in [40]. We have just shown that the classifying space of the fiber of
\[
\text{ev}_0 : \text{Fun}^W_n([1], \text{Exit} (\text{Ran}^u(R^n))) \to \text{Fun}^W_n([0], \text{Exit} (\text{Ran}^u(R^n)))
\]
on $\Sigma$ is equivalent to the fiber of
\[
\text{ev}_0 : \text{mor} \left( \text{Exit}(\text{Ran}^u(R^n)) \right) \to \text{Exit}(\text{Ran}^u(R^n))^\sim
\]
on $\Sigma$, as identified in Lemma 1.2.7. Thus, we have yet to show that this fiber over $\Sigma$ is equivalent to the fiber of
\[
\text{Bev}_0 : \text{B Fun}^W_n([1], \text{Exit} (\text{Ran}^u(R^n))) \to \text{B Fun}^W_n([0], \text{Exit} (\text{Ran}^u(R^n)))
\]
on $\Sigma$, since, in general, the classifying space of a fiber is not inherently equivalent to the fiber of the map induced between classifying spaces. That is to say, for a functor of $\infty$-categories $\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D}$, the classifying space $\mathcal{B}(\mathcal{F}^{-1}d)$ of the fiber of $\mathcal{F}$ over $d \in \mathcal{D}$ is not necessarily equivalent to the homotopy fiber $(\mathcal{B} \mathcal{F})^{-1}(d)$ of the map induced between the classifying spaces $\mathcal{B} \mathcal{C} \xrightarrow{\mathcal{B} \mathcal{F}} \mathcal{B} \mathcal{D}$ over $d \in \mathcal{B} \mathcal{D}$.
Quillen’s Theorem B identifies the homotopy fibers of the map induced between classifying spaces in favorable cases.

**Theorem 4.2.9 (Quillen’s Theorem B).** Given a functor between $\infty$-categories $\mathcal{C} \xrightarrow{F} \mathcal{D}$, if each morphism $d \xrightarrow{F} d'$ in $\mathcal{D}$ induces a weak equivalence $\mathcal{B}(\mathcal{C}/d') \xrightarrow{\simeq} \mathcal{B}(\mathcal{C}/d)$ between the classifying spaces of the induced $\infty$-undercategories, then $\mathcal{B}(\mathcal{C}/d)$ is the homotopy fiber of $\mathcal{B}F$ over $d$ and thus, $\mathcal{B}(\mathcal{C}/d) \xrightarrow{\simeq} \mathcal{B}\mathcal{C} \xrightarrow{\mathcal{B}F} \mathcal{B}\mathcal{D}$ is a fiber sequence.

**Remark 4.2.10.** Quillen originally proved Theorem B in [24] for categories. Theorem 5.16 in [3] generalizes the result for $\infty$-categories, which is the statement of Quillen’s Theorem B given above.

We apply Quillen’s Theorem B for the situation at hand and identify the homotopy fibers of $\mathcal{B}\text{ev}_0$ as follows.

**Lemma 4.2.11.** The fiber of the map of spaces

$$\mathcal{B}\text{Fun}^W_n\left(\left[1, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right]\right) \xrightarrow{\mathcal{B}\text{ev}_0} \mathcal{B}\text{Fun}^W_n\left(\left[0, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right]\right)$$

over an object $S = S_n \rightarrow \cdots \rightarrow S_1$ is equivalent to the classifying space of the fiber of $\text{ev}_0$ over $S$.

$$(\mathcal{B}\text{ev}_0)^{-1}(S) \simeq \mathcal{B}(\text{ev}_0^{-1}(S))$$

**Proof.** Fix a morphism $S \xrightarrow{\alpha} S'$ in $\text{Fun}^W_n\left(\left[0, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right]\right)$. Recall that the induced monodromy functor $\alpha^*$ is defined as the composite

$$\text{ev}_0^{-1}(S') \xrightarrow{\alpha^*} \text{ev}_0^{-1}(S)$$

where $\mu$ is right adjoint to inclusion (which exists by Lemma 2.20 of [3]). The diagram induced upon taking classifying spaces

$$\mathcal{B}\text{ev}_0^{-1}(S') \xrightarrow{\mathcal{B}\alpha^*} \mathcal{B}\text{ev}_0^{-1}(S)$$

will follow by Quillen’s Theorem B; let us explain further. If $\mathcal{B}\alpha^*$ is an equivalence, then, according to [33], $\mathcal{B}(\circ \circ \alpha)$ is an equivalence. By Quillen’s Theorem B, $\mathcal{B}(\circ \circ \alpha)$ being an equivalence yields an equivalence

$$(\mathcal{B}\text{ev}_0)^{-1}(S) \simeq \mathcal{B}\text{Fun}^W_n\left(\left[1, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right]\right)\xrightarrow{\sim} \mathcal{B}\text{Fun}^W_n\left(\left[0, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right]\right)$$

The upward vertical arrow in (33) is an equivalence between $\mathcal{B}\text{Fun}^W_n\left(\left[1, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right]\right)$ and $\mathcal{B}\text{ev}_0^{-1}(S)$. Thus, $(\mathcal{B}\text{ev}_0)^{-1}(S)$ is equivalent to $\mathcal{B}\text{ev}_0^{-1}(S)$.

First, consider the diagram

$$\text{Fun}^W_n\left(\left[1, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right]\right) \xrightarrow{\text{frtg}} \text{mor}\left(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$$

$$\text{Fun}^W_n\left(\left[0, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right]\right) \xrightarrow{\sim} \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$$

(54)
in which each vertical arrow is a Cartesian fibration. By Observation 4.2.6, the induced monodromy functor \( \alpha^* \) is carried by the forgetful functor to the induced monodromy functor of the image of \( \alpha \) under the forgetful functor, \( \text{frgt}(\alpha)^* \):

\[
\begin{array}{ccc}
\text{ev}_0^{-1}(S') & \xrightarrow{\alpha^*} & \text{ev}_0^{-1}(S) \\
\downarrow \text{frgt} & & \downarrow \text{frgt} \\
\text{ev}_0^{-1}(S'_n) & \xrightarrow{\text{frgt}(\alpha)^*} & \text{ev}_0^{-1}(S_n).
\end{array}
\]

Observe that \( \text{frgt}(\alpha)^* \) is an equivalence precisely because the image of the morphism \( \alpha \) under the forgetful functor, \( \alpha_n : S_n \to S'_n \), is a bijection. We apply the universal property of localization to the canonical localization \( \text{ev}_0^{-1}(S) \to \text{Bev}_0^{-1}(S) \) to obtain

\[
\begin{array}{ccc}
\text{ev}_0^{-1}(S) & \xrightarrow{\text{frgt}} & \text{ev}_0^{-1}(S_n) \\
\downarrow & & \downarrow \\
\text{Bev}_0^{-1}(S) & \xrightarrow{\exists!} & \text{Bev}_0^{-1}(S_n),
\end{array}
\]

and observe that such a filler must be an equivalence. We paste (55) and (56) for \( S \) and \( S' \) together to see that \( \mathcal{B}\alpha^* \) is an equivalence:

\[
\begin{array}{ccc}
\text{ev}_0^{-1}(S') & \xrightarrow{\alpha^*} & \text{ev}_0^{-1}(S) \\
\downarrow \text{frgt} & & \downarrow \text{frgt} \\
\text{ev}_0^{-1}(S'_n) & \xrightarrow{\mathcal{B}\alpha^*} & \text{ev}_0^{-1}(S_n).
\end{array}
\]

According to (53), the map \( \mathcal{B}\alpha^* \) being an equivalence implies that the map \( \mathcal{B}(- \circ \alpha) \) is an equivalence. Thus, by Quillen’s Theorem B, the fiber of \( \text{Bev}_0 \) over \( S \) is identified as \( \mathcal{B}\text{Fun}^W([1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \), which, according to (53), is equivalent to \( \text{Bev}_0^{-1}(S) \).

\( \square \)

With Lemma 4.2.7, Lemma 4.2.8, and Lemma 4.2.11, we have just identified that the map induced between the fibers of 46 is an equivalence. Lemma 4.2.1, which states that the map induced by the forgetful functor

\[
\mathcal{B}\text{Fun}^W([1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \to \text{mor}(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]

is an equivalence, follows almost immediately. The proof is as follows.

**Proof of Lemma 4.2.1.** We first recall (46):

\[
\begin{array}{ccc}
\mathcal{B}\text{Fun}^W([1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) & \xrightarrow{\text{frgt}} & \text{mor}(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \\
\downarrow \text{Bevo} & & \downarrow \text{ev}_0 \\
\mathcal{B}\text{Fun}^W([0], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) & \xrightarrow{\sim} & \text{Exit}(\text{Ran}^u(\mathbb{R}^n)).
\end{array}
\]

The equivalence in Lemma 4.1.1

\[
\mathcal{B}W_n \simeq \text{Exit}(\text{Ran}^u(\mathbb{R}^n))
\]
is induced by the forgetful functor from $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ to $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$; this is precisely the map in (57), since the map between total spaces is induced by the forgetful functor, and the domain $\mathcal{B} \text{Fun}^W_n([0], \text{Exit}(\text{Ran}^n(\mathbb{R}^n)))$ is inherently equivalent to $\mathcal{B} W_n$.

In Lemma 4.2.11 through Lemma 4.2.11 we identified the fiber of the vertical map on the left of (57), $\text{Bevo}$, over $S \in \mathcal{B} \text{Fun}^W_n([0], \text{Exit}(\text{Ran}^n(\mathbb{R}^n)))$ as the product $\prod_{s \in S_n} \text{Exit}(\text{Ran}^n(T_s \mathbb{R}^n))$, where $T_s \mathbb{R}^n$ is the tangent space of $\mathbb{R}^n$ at $s \in S_n \subset \mathbb{R}^n$. In Lemma 4.2.7 we identified the fiber of the map on the right of (57), $e_{vo}$, over the image $S_n \in \text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ of $S$ under the forgetful map as the product $\prod_{s \in S_n} \text{Exit}(\text{Ran}^n(T_s \mathbb{R}^n))$.

Thus, the induced long exact sequence in homotopy of each fibration induces a weak homotopy equivalence between the total spaces, which is, in fact, a homotopy equivalence since the total spaces are CW complexes. □

This result, together with Lemma 4.1.1 wherein we identified the space of objects $\mathcal{B} \text{Fun}^W_n([0], \text{Exit}(\text{Ran}^n(\mathbb{R}^n)))$ as the space of objects $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$, prove Lemma 4.0.7. This lemma is the first of two lemmas that we will use to prove the main lemma of this section, Lemma 4.0.8, which states that the forgetful functor from the exit-path $\infty$-category of the refined unital Ran space of $\mathbb{R}^n$ is a localization on $W_n$ to the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$. The key to our proof of Lemma 4.0.2 is Theorem 4.0.6, which identifies a localization in favorable situations. The second lemma that we will use to prove Lemma 4.0.2 states that the hypothesis of Theorem 4.0.6 is satisfied; it is the focus of the next subsection.

4.3. **Proving Lemma 4.0.8** The main goal of this subsection is to prove Lemma 4.0.8 which states that the simplicial space $\mathcal{B} \text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^n(\mathbb{R}^n)))$ is a complete Segal space (Definition 2.1.1). The proof is technical, also using Cartesian fibrations and Quillen’s Theorem B, and builds off of arguments developed in § 4.2. Before we can prove Lemma 4.0.8 we need the following lemma.

**Lemma 4.3.1.** Given a pullback of $\infty$-categories

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{G} & \mathcal{E}' \\
\downarrow \iota & & \downarrow \pi' \\
\mathcal{B} & \xrightarrow{F} & \mathcal{B}'
\end{array}
\]

in which $\pi'$ is a Cartesian fibration, if $\pi'$ satisfies Quillen’s Theorem B, then so does $\pi$.

**Proof.** Let $b \xrightarrow{f} b'$ be a morphism in $\mathcal{B}$. Lemma 2.8 of [3] states that Cartesian fibrations are closed under base change. Thus, $\pi'$ being a Cartesian fibration and (58) being a pullback implies that $\pi$ is a Cartesian fibration. By Observation 4.2.6, then, $G$ carries the induced monodromy functor $f^*$ to the induced monodromy functor $F(f)^*$ and yields equivalences between the fibers:

\[
\begin{array}{ccc}
\pi^{-1}(b') & \xrightarrow{f^*} & \pi^{-1}(b) \\
\downarrow \simeq & & \downarrow \simeq \\
\pi'^{-1}(F(b')) & \xrightarrow{F(f)^*} & \pi'^{-1}(F(b))
\end{array}
\]
Combining this diagram with the definition of the monodromy functor yields

\[
\begin{array}{ccc}
\mathcal{E}^{b'} & \xrightarrow{-\circ f} & \mathcal{E}^b \\
\downarrow & & \downarrow \\
\pi^{-1}(b') & \xrightarrow{f^*} & \pi^{-1}(b) \\
\cong & & \cong \\
\pi'^{-1}(F(b')) & \xrightarrow{F(f)^*} & \pi^{-1}(F(b)) \\
\downarrow & & \downarrow \\
\mathcal{E}^F(b') & \xrightarrow{-\circ F(f)} & \mathcal{E}^F(b) \\
\end{array}
\]

(59)

The diagram induced upon taking classifying spaces yields the desired result. Indeed, \(\pi'\) satisfying Quillen’s Theorem B implies \(\mathcal{B}(-\circ F(f))\) is a equivalence and thus, each horizontal arrow resulting between classifying spaces is an equivalence, which in particular means \(\mathcal{B}(-\circ f)\) is an equivalence:

\[
\begin{array}{ccc}
\mathcal{B}\mathcal{E}^{b'} & \xrightarrow{\mathcal{B}(-\circ f)} & \mathcal{B}\mathcal{E}^b \\
\cong & & \cong \\
\mathcal{B}\pi^{-1}(b') & \xrightarrow{\mathcal{B}f^*} & \mathcal{B}\pi^{-1}(b) \\
\cong & & \cong \\
\mathcal{B}\pi'^{-1}(F(b')) & \xrightarrow{\mathcal{B}F(f)^*} & \mathcal{B}\pi^{-1}(F(b)) \\
\cong & & \cong \\
\mathcal{B}\mathcal{E}^F(b') & \xrightarrow{\mathcal{B}(-\circ F(f))} & \mathcal{B}\mathcal{E}^F(b) \\
\end{array}
\]

\[\Box\]

**Observation 4.3.2.** The following diagram of \(\infty\)-categories is a pullback:

\[
\begin{array}{ccc}
\text{Fun}_{\infty}^W([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) & \xrightarrow{\tau} & \text{Fun}_{\infty}^W([1 - p < p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \\
\downarrow & & \downarrow \\
\text{Fun}_{\infty}^W([0 < \cdots < p - 1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) & \xrightarrow{\sigma} & \text{Fun}_{\infty}^W([p - 1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \\
\end{array}
\]

(60)

Indeed, for an \(\infty\)-category \(\mathcal{C}\), \(\text{Fun}(\bullet, \mathcal{C})\) satisfies the Segal condition, i.e., for each \(p \geq 2\), the diagram obtained by replacing \(\text{Fun}_{\infty}^W([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) with \(\text{Fun}([p], \mathcal{C})\) in (60) is pullback. Using this, it is straightforward to show (60) is pullback.

We now prove Lemma 4.0.8 which states that the simplicial space \(\mathcal{B}\text{Fun}_{\infty}^W([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) is a complete Segal space.

**Proof of Lemma 4.0.8.** First, we will show that \(\mathcal{B}\text{Fun}_{\infty}^W([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) satisfies the Segal condition. Consider the diagram of spaces obtained by taking the classifying spaces of (60)

\[
\begin{array}{ccc}
\mathcal{B}\text{Fun}_{\infty}^W([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) & \xrightarrow{\mathcal{B}\tau} & \mathcal{B}\text{Fun}_{\infty}^W([p - 1 < p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \\
\downarrow & & \downarrow \\
\mathcal{B}\text{Fun}_{\infty}^W([0 < \cdots < p - 1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) & \xrightarrow{\mathcal{B}\sigma} & \mathcal{B}\text{Fun}_{\infty}^W([p - 1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \\
\end{array}
\]

(61)
To show that this diagram is a pullback, we will show that the map induced between fibers of \((61)\) is an equivalence. By Observation 4.2.3 the functor \(s\) in \((60)\) is a Cartesian fibration. Further, in the proof of Lemma 4.2.11 we showed that \(s\) satisfies Quillen’s Theorem B. Thus, \((60)\) satisfies the hypothesis’ of Lemma 4.3.1 and we identify the fibers of \(B\sigma\) and \(Bs\) over the objects

\[
\mathcal{S}_0 \to \cdots \to \mathcal{S}_{p-1} \quad \text{in} \quad \mathcal{B}\text{Fun}^W_n(\{0 < \cdots < p - 1\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]

and

\[
\mathcal{S}_{p-1} \quad \text{in} \quad \mathcal{B}\text{Fun}^W_n(\{p - 1\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]

respectively, as the classifying spaces of the fibers of \(\sigma\) and \(s\) over \(\mathcal{S}_0 \to \cdots \to \mathcal{S}_{p-1}\) and \(\mathcal{S}_{p-1}\), respectively:

\[
(\mathcal{B}\sigma)^{-1}(\mathcal{S}_0 \to \cdots \to \mathcal{S}_{p-1}) \simeq \mathcal{B}\sigma^{-1}(\mathcal{S}_0 \to \cdots \to \mathcal{S}_{p-1})
\]

and

\[
(\mathcal{B}s)^{-1}(\mathcal{S}_{p-1}) \simeq \mathcal{B}s^{-1}(\mathcal{S}_{p-1}).
\]

Therefore, because \((60)\) being a pullback implies an equivalence between fibers induced by \(\tau\)

\[
\tau_\ast : \sigma^{-1}(\mathcal{S}_0 \to \cdots \to \mathcal{S}_{p-1}) \overset{\simeq}{\rightarrow} \mathcal{S}_{p-1}
\]

there results an equivalence between fibers of \((61)\) given by \(\mathcal{B}\tau_\ast\)

\[
(\mathcal{B}\sigma)^{-1}(\mathcal{S}_0 \to \cdots \to \mathcal{S}_{p-1}) \simeq (\mathcal{S}_0 \to \cdots \to \mathcal{S}_{p-1}) \overset{\simeq}{\rightarrow} (\mathcal{B}s)^{-1}(\mathcal{S}_{p-1})
\]

which verifies that \((61)\) is a pullback.

Then, Lemma 4.0.7 extends to an equivalence of spaces

\[
\mathcal{B}\text{Fun}^W_n([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \simeq \text{Hom}_{\text{Cat}_\infty}([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]

for each \(p \geq 0\), since \(\mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) satisfying the Segal condition means that its values on \([0]\) and \([1]\) determine all of its higher \([p]\) values. This, in particular, implies that \(\mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) is a complete Segal space, since \(\text{Hom}_{\text{Cat}_\infty}([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) is a complete Segal space precisely because \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) is a complete Segal space.

We have just verified the hypothesis of Theorem 4.0.6 which means that the localization of \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) on \(W_n\) is equivalent to the simplicial space \(\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\). Using Lemma 4.0.7 we now prove Lemma 4.0.2 which states that the forgetful functor localizes \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) on \(W_n\) to \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\).

**Proof of Lemma 4.0.2** In the proof of the previous Lemma 4.0.8 we showed

\[
(62) \quad \mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \simeq \text{Hom}_{\text{Cat}_\infty}([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]

which, in particular means that the hypothesis of Theorem 4.0.6 is satisfied. Thus, by Theorem 4.0.6

\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))|_{W_n^{-1}} \simeq \mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]

Then, by the equivalence \((62)\), we have an equivalence of simplicial spaces

\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))|_{W_n^{-1}} \simeq \text{Hom}_{\text{Cat}_\infty}([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]

which establishes that \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) localizes on \(W_n\) to \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\).

Note that this localization is given by the forgetful functor from \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) to \(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) because our identification of \(\mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) with \(\text{Hom}_{\text{Cat}_\infty}([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))\) was induced by the forgetful functor; this was proven in Lemma 4.2.1.
Lastly, we will show this localization is over \( \text{Fin}^{\text{op}} \). In Observation 3.1.6, we observed that the forgetful functor from \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) to \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) is naturally over \( \text{Fin}^{\text{op}} \) by just remembering the data of underlying sets at the \( \mathbb{R}^n \) level. Then, by the universal property of localization, we have:

\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{\text{frg}} \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{\phi_n} \text{Fin}^{\text{op}} \xrightarrow{\exists !} \text{Exit}(\text{Ran}^u(\mathbb{R}^n))[W_n^{-1}]
\]

The unique existence of such a filler is guaranteed because each morphism in \( W_n \) gets carried to isomorphisms in \( \text{Fin}^{\text{op}} \) under \( \phi_n \). Thus, we see that the forgetful functor from \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) to \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) yields a localization over \( \text{Fin}^{\text{op}} \).

\[\square\]

We have just proved that the exit-path \( \infty \)-category of the refined unital Ran space of \( \mathbb{R}^n \) localizes to the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \). Through the equivalence of Lemma 3.3.2 of \( \text{[3.3]} \) between the exit-path \( \infty \)-category of the refined unital Ran space of \( \mathbb{R}^n \) and the subcategory of active morphisms of \( \Theta_n \), then, we have proven the main result of this paper, Theorem 2.2.1, which states that the subcategory of active morphisms of \( \Theta_n \) localizes to the exit-path \( \infty \)-category of the unital Ran space of \( \mathbb{R}^n \).

An immediate corollary to Theorem 2.2.1 is that for \( \mathbb{R}^2 \), in particular, the exit-path \( \infty \)-category of the unital Ran space construction allows us to understand all the unordered configurations of finite subsets of \( \mathbb{R}^2 \) in terms of that of \( \mathbb{R} \) and \( \mathbb{R} \).

**Corollary 4.3.3.** There is a localization of \( \infty \)-categories

\[
\text{Exit}(\text{Ran}^u(\mathbb{R})) \downarrow \text{Exit}(\text{Ran}^u(\mathbb{R})) \rightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^2))
\]

from the two fold wreath product of the exit-path \( \infty \)-category of the unital Ran space of \( \mathbb{R} \) with itself and the exit-path \( \infty \)-category of the unital Ran space of the product \( \mathbb{R}^2 \).

In light of this corollary, we conjecture the following more general statement.

**Conjecture 4.3.4.** For smooth connected manifolds \( M \) and \( N \), there is a localization of \( \infty \)-categories

\[
\text{Exit}(\text{Ran}^u(M)) \downarrow \text{Exit}(\text{Ran}^u(N)) \rightarrow \text{Exit}(\text{Ran}^u(M \times N))
\]

from the wreath product of the exit-path \( \infty \)-category of the unital Ran space of \( M \) with that of \( N \) and the exit-path \( \infty \)-category of the unital Ran space of the product \( M \times N \).

There is another corollary to Theorem 2.2.1 that considers a restriction of the localization \( \Theta_n^{\text{act}} \rightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) to a certain subcategory of the domain. The next section is devoted to proving this corollary.

5. A COROLLARY TO THEOREM 2.2.1 IDENTIFYING \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \) COMBINATORIALLY

In this section, we identify the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \) as a localization of a certain subcategory of \( \Theta_n^{\text{act}} \). This result specializes the localization of Theorem 2.2.1 to a localization of the \( \infty \)-subcategory \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \) of \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) by restricting to this subcategory. To begin, we review the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \).

5.1. The exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \). Following Definition 0.1.6 the exit-path \( \infty \)-category of the Ran space of a smooth connected manifold \( M \) is the simplicial set whose value on a simplex \( [k] \in \Delta \) is

\[
\{ \Delta^k \rightarrow \text{Ran}(M) \mid f \text{ is a stratified map} \}
\]

the set of continuous, stratified maps from the topological \( k \)-simplex stratified over \( [k] \) to the Ran space of \( M \) stratified over \( N \). So, heuristically, an object of \( \text{Exit}(\text{Ran}(M)) \) is a finite subset \( S \subseteq M \) and a morphism is a path in \( \text{Ran}(M) \) that witnesses anticollision of points.
In Observation 2.0.6, we observed that for a connected manifold $M$ \( \text{Exit}(\text{Ran}(M)) \) is equivalently the sub simplicial space of \( \text{Exit}(\text{Ran}^u(M)) \) defined as the pullback of simplicial spaces

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}(M)) & \longrightarrow & \text{Exit}(\text{Ran}^u(M)) \\
\downarrow & \swarrow & \downarrow \\
(\text{Fin}_{\neq \emptyset}^{\text{surj}})^{\text{op}} & \longrightarrow & \text{Fin}^{\text{op}} \\
\end{array}
\]

where \( \text{Fin}_{\neq \emptyset}^{\text{surj}} \) is the subcategory of finite sets consisting of nonempty sets and all those morphisms that are surjections. In light of this observation, we deduced that \( \text{Exit}(\text{Ran}(M)) \) is an \( \infty \)-category by virtue of it being a pullback of \( \infty \)-categories (Corollary 2.1.3).

The main result of this section identifies the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \) as a localization of the following subcategory of \( \Theta_n^{\text{act}} \).

**Definition 5.1.1.** The category \( \Theta_n^{\text{exit}} \) is the subcategory of \( \Theta_n^{\text{act}} \) defined as the pullback

\[
\begin{array}{ccc}
\Theta_n^{\text{exit}} & \longrightarrow & \Theta_n^{\text{act}} \\
\downarrow & \swarrow & \downarrow \\
\text{Fun}(\{1 < \cdots < n\}, (\text{Fin}_{\neq \emptyset}^{\text{surj}})^{\text{op}}) & \longrightarrow & \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}) \\
\end{array}
\]

where we recall the functor \( \tau \) from Observation 1.1.10 (defined by the truncation functor \( \text{tr}_i \) and \( \gamma_n \)).

Heuristically, \( \Theta_n^{\text{exit}} \) consists of healthy trees as its objects and all those morphisms that induce surjections between the sets of leaves. We state the main result of this section, which articulates the sense in which the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \) is identified combinatorially in terms of \( \Theta_n \).

**Corollary 5.1.2.** There is a localization

\[
\Theta_n^{\text{exit}} \rightarrow \text{Exit}(\text{Ran}(\mathbb{R}^n))
\]

over \( (\text{Fin}_{\neq \emptyset}^{\text{surj}})^{\text{op}} \) from the subcategory \( \Theta_n^{\text{exit}} \) of \( \Theta_n^{\text{act}} \) to the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \).

The localizing subcategory of \( \Theta_n^{\text{exit}} \) is nearly equivalent to the subcategory \( W^{\text{hlt}}_n \) of \( \Theta_n^{\text{act}} \) from Definition 4.1.2, which consists of healthy trees as its objects and all those morphisms in \( \Theta_n^{\text{act}} \) that induce bijections on the sets of leaves under \( \gamma_n \) to \( \text{Fin} \). We implement a slight abuse of notation as we will denote the localizing subcategory of Corollary 5.1.2 as \( W^{\text{hlt}}_n \) as well, which is distinct from \( W^{\text{hlt}}_n \subset \Theta_n^{\text{hlt}} \) only in that it does not have the empty tree as an object. We will always clarify which category \( W^{\text{hlt}}_n \) we mean by indicating contextually which category it is a subcategory of, namely, \( W^{\text{hlt}}_n \subset \Theta_n^{\text{exit}} \) or \( W^{\text{hlt}}_n \subset \Theta_n^{\text{act,hlt}} \). Formally, we define the localizing subcategory of Corollary 5.1.2 as follows.

**Definition 5.1.3.** The subcategory \( W^{\text{hlt}}_n \) of \( \Theta_n^{\text{exit}} \) is defined to be the pullback

\[
\begin{array}{ccc}
W^{\text{hlt}}_n & \longrightarrow & W^{\text{hlt}}_n \\
\downarrow & \swarrow & \downarrow \\
\Theta_n^{\text{exit}} & \longrightarrow & \Theta_n^{\text{act,hlt}} \\
\end{array}
\]

of categories.

Heuristically then, \( W^{\text{hlt}}_n \subset \Theta_n^{\text{exit}} \) has all those nonempty, healthy trees of \( \Theta_n \) as its objects and only those morphisms that induce bijections between the sets of leaves.

---

7Note that we are considering \( W^{\text{hlt}}_n \) as a subset of \( \Theta_n^{\text{act,hlt}} \) in light of the equivalence \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \simeq \Theta_n^{\text{act,hlt}} \) of Lemma 3.3.2.
5.2. Proving Corollary 5.1.2. The proof of Corollary 5.1.2 is similar to that of Lemma 4.0.2 in that we identify the localization of $\Theta_n^{\text{exit}}$ on $W_n^{\text{ht}}$ using Theorem 4.0.6, which states that if the simplicial space $\mathcal{B}\text{Fun}^{W_n^{\text{ht}}}(\Theta_n^{\text{exit}})$ is a complete Segal space, then it is equivalent to the localization of $\Theta_n^{\text{exit}}$ on $W_n^{\text{ht}}$. We do this by extrapolating the argument of Lemma 4.0.2 using the fact that the domain and codomain of Corollary 5.1.2 are sub $\infty$-categories of the domain and codomain of the localization of Lemma 2.2.1.

First, we compile the following lemma, which follows (not immediately) from Lemma 4.1.3 wherein we showed that there is an adjunction between $W_n^{\text{ht}}$ and $W_n$.

Lemma 5.2.1. For each $p \geq 0$, the inclusion functor between $\infty$-categories

$$\text{Fun}^{W_n^{\text{ht}}}(p, \Theta_n^{\text{act,ht}}) \hookrightarrow \text{Fun}^{W_n}(p, \Theta_n^{\text{act}})$$

induces an equivalence between their classifying spaces

$$\mathcal{B}\text{Fun}^{W_n^{\text{ht}}}(p, \Theta_n^{\text{act,ht}}) \simeq \mathcal{B}\text{Fun}^{W_n}(p, \Theta_n^{\text{act}}).$$

Proof. First, observe that we can describe the subcategory $W_n^{\text{ht}}$ of $W_n$ as the following pullback of categories over $\Theta_n^{\text{act,ht}}$:

$$\begin{array}{ccc}
W_n^{\text{ht}} & \longrightarrow & W_n \\
\downarrow & & \downarrow \\
\Theta_n^{\text{act,ht}} & \longrightarrow & \Theta_n^{\text{act}}
\end{array}$$

where we recall that $W_n$ consists of all the same objects as $\Theta_n^{\text{act}}$ and all those morphisms that induce bijections on the sets of leaves, and $W_n^{\text{ht}}$ is the full subcategory consisting of only those trees that are healthy.

In Lemma 4.1.3, we showed that the inclusion functor $W_n^{\text{ht}} \hookrightarrow W_n$ is a right adjoint. The reader may observe that no where in the proof did we use that the morphisms of $W_n$ and $W_n^{\text{ht}}$ induce bijection between their sets of leaves. Thus, Lemma 4.1.3 immediately extends to an adjunction between $\Theta_n^{\text{act,ht}}$ and $\Theta_n^{\text{act}}$ whose right adjoint is given by inclusion. Further, observe that the unit transformation of this adjunction is given by morphisms in $W_n$. Indeed, for each tree $T \in \Theta_n^{\text{act}}$, the morphism assigned to $T$ by the unit is $T \xrightarrow{T} P_n(T)$, which, in particular, induces a bijection on the leaves, and is thus in $W_n$. In identifying that the unit of the right adjoint $\Theta_n^{\text{act,ht}} \hookrightarrow \Theta_n^{\text{act}}$ is given by morphisms in $W_n$, we may extend this adjunction to an adjunction between $\text{Fun}^{W_n^{\text{ht}}}(p, \Theta_n^{\text{act,ht}})$ and $\text{Fun}^{W_n}(p, \Theta_n^{\text{act}})$ whose right adjoint is inclusion.

Recall that Corollary 2.1.28 in [22] states that the classifying space of an adjunction is an equivalence of spaces. Thus, upon taking the classifying space of the right adjoint

$$\text{Fun}^{W_n^{\text{ht}}}(p, \Theta_n^{\text{act,ht}}) \hookrightarrow \text{Fun}^{W_n}(p, \Theta_n^{\text{act}}).$$

There results the desired equivalence of spaces. \[
\]

We will compile one more lemma before commencing with the proof of Corollary 5.1.2. First, note that the inclusion of the subcategory $\Theta_n^{\text{exit}} \hookrightarrow \Theta_n^{\text{act,ht}}$ together with the induced inclusion of the respective subcategories $W_n^{\text{ht}} \hookrightarrow W_n^{\text{ht}}$ guarantees that for each $p \geq 0$, the induced map $\text{Fun}^{W_n^{\text{ht}}}(p, \Theta_n^{\text{exit}}) \hookrightarrow \text{Fun}^{W_n}(p, \Theta_n^{\text{act,ht}})$ is also an inclusion of a $\infty$-subcategory. Towards the proof of Corollary 5.1.2, we need that the map induced between the classifying spaces of this inclusion is, in particular, a monomorphism. This is articulated by the following lemma.

Lemma 5.2.2. For each $p \geq 0$, the inclusion functor

$$\text{Fun}^{W_n^{\text{ht}}}(p, \Theta_n^{\text{exit}}) \hookrightarrow \text{Fun}^{W_n}(p, \Theta_n^{\text{act,ht}})$$

induces a monomorphism between classifying spaces

$$\mathcal{B}\text{Fun}^{W_n^{\text{ht}}}(p, \Theta_n^{\text{exit}}) \hookrightarrow \mathcal{B}\text{Fun}^{W_n}(p, \Theta_n^{\text{act,ht}}).$$
Towards the proof of this lemma, we record a technical result, Lemma 2.1.2, which involves the following notion.

**Definition 5.2.3.** A functor $C \to D$ between $\infty$-categories is an *inclusion of a cofactor* if there is an $\infty$-category $E$ and an equivalence between $\infty$-categories under $C$:

$$C \coprod E \cong D.$$ 

**Lemma 5.2.4.** A functor $C \xrightarrow{F} D$ is an inclusion of a cofactor if and only if $F$ is a monomorphism and for each solid commutative square

$$
\begin{array}{ccc}
[0] & \rightarrow & C \\
\nu \downarrow & \searrow & \downarrow F \\
[1] & \rightarrow & D
\end{array}
$$

for either $\nu := \langle 0 \rangle$ or $\nu := \langle 1 \rangle$, there exists a filler.

**Proof.** First, notice that if $F$ is a monomorphism and $\nu$ is satisfied (with the two possible lifts), then $C \xrightarrow{F} D$ is fully faithful. Consider the full $\infty$-subcategory $E \subset D$ consisting of those objects that are not isomorphic to objects in the image of $C \to D$. Consider the canonical functor

$$C \coprod E \to D,$$

which is canonically under $C$. By design, this functor is essentially surjective, and fully faithful. This established the implication that $F$ being a monomorphism and satisfying $\nu$ implies $C \xrightarrow{F} D$ is an inclusion of a cofactor.

We now show the converse. Suppose there is an $\infty$-category $E$ together with an equivalence $C \coprod E \cong D$ under $C$. Consider a solid diagram

$$
\begin{array}{ccc}
[0] & \rightarrow & C \\
\nu \downarrow & \searrow & \downarrow \coprod E \\
[1] & \rightarrow & C \coprod E
\end{array}
$$

The functor $[0] \to [1]$ has the feature that every object in $[1]$ admits a morphism to or from an object in the image of $\nu$. It follows that there is a unique filler, as desired. □

**Proof of Lemma 5.2.4.** First, we will verify that the functor

$$\text{Fun}^{\text{Whlt}}_W ([p] , \Theta^\text{exit} \cap \text{hlt}_n) \hookrightarrow \text{Fun}^{\text{Whlt}}_W ([p] , \Theta^\text{exit} \cap \text{hit}_n)$$

is an inclusion of a cofactor; that is, we will show that this functor is a monomorphism and satsifies $\nu$.

First, note that because $\text{Fun}^{\text{Whlt}}_W ([p] , \Theta^\text{exit} \cap \text{hlt}_n) \hookrightarrow \text{Fun}^{\text{Whlt}}_W ([p] , \Theta^\text{exit} \cap \text{hit}_n)$ is an inclusion of $\infty$-categories, it is in particular a monomorphism.

Similar to Observation 4.3.2, it is straightforward to verify that for each $p \geq 0$ the following diagram of $\infty$-categories

$$
\begin{array}{ccc}
\text{Fun}^{\text{Whlt}}_W ([p] , \Theta^\text{exit} \cap \text{hlt}_n) & \rightarrow & \text{Fun}^{\text{Whlt}}_W ([1 - p < p] , \Theta^\text{exit} \cap \text{hlt}_n) \\
\downarrow & \nearrow & \downarrow \\
\text{Fun}^{\text{Whlt}}_W ([0 < \cdots < p - 1] , \Theta^\text{exit} \cap \text{hlt}_n) & \rightarrow & \text{Fun}^{\text{Whlt}}_W ([p - 1] , \Theta^\text{exit} \cap \text{hlt}_n).
\end{array}
$$
is pullback, as is the diagram obtained by just replacing $\Theta_n^{\text{exit}}$ with $\Theta_n^{\text{act, hlt.}}$. Thus, to show that (63) for our situation, namely

$$
\begin{array}{c}
0 \\
\downarrow \cong \\
1
\end{array} \quad \begin{array}{c}
\text{Fun}^W_n([p], \Theta_n^{\text{exit}}) \\
\text{Fun}^W_n([p], \Theta_n^{\text{act, hlt.}})
\end{array}
$$

(65)

is satisfied for all $p \geq 0$, it suffices to show for the cases for $p = 0, 1$. Indeed, it is straightforward to verify that the cases $p = 0, 1$ imply each $p \geq 0$ case upon applying the universal property of pullback from (64) and the diagram obtained by replacing $\Theta_n^{\text{exit}}$ with $\Theta_n^{\text{act, hlt.}}$ in (60).

Both cases, $p = 0, 1$, come down to the following observation: Each morphism in $W_n^{\text{hlt}}$ between objects in $\Theta_n^{\text{act, hlt.}}$ is a morphism in $\Theta_n^{\text{exit}}$. The root reason for this is that surjections enjoy the ‘2 out of 3’ property, that is, for any commutative triangle of morphisms among sets in which two of the morphisms are surjections, the third map is necessarily a surjection as well. For our situation, any morphism $T \xrightarrow{f} T'$ in $W_n^{\text{hlt}}$ yields a bijection between the sets of leaves, $\gamma_n(f) : \gamma_n(T') \cong \gamma_n(T)$. For any $1 \leq i \leq n - 1$, in applying the natural transformation $\epsilon$ from Observation 1.1.8 whose value on $T$ is the natural map between sets of leaves $\gamma_n(T) \xrightarrow{\epsilon_T} \gamma_n(\text{tr}_i(T))$ induced by the structure of $T$, we obtain the following diagram among sets:

$$
\begin{array}{c}
\gamma_n(T') \\
\downarrow \epsilon_T \\
\gamma_n(\text{tr}_i(T'))
\end{array} \quad \begin{array}{c}
\cong \\
\uparrow \gamma_i(\text{tr}_i(T)) \\
\gamma_n(\text{tr}_i(T))
\end{array}
$$

Observe that because $T$ and $T'$ are healthy trees, both $\epsilon_T$ and $\epsilon_T'$ are surjections. Then, by the ‘2 out of 3’ property, $\gamma_i(\text{tr}_i(T')) \xrightarrow{\gamma_i(\text{tr}_i(f))} \gamma_i(\text{tr}_i(T))$ is a surjection. Such a surjection at each $i$ guarantees that the image of $f$ under the functor $\Theta_n^{\text{act}} \rightarrow \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})$ lands in $\text{Fun}(\{1 < \cdots < n\}, (\text{Fin}^{\text{surj}})^{\text{op}})$ and is thus a morphism in $\Theta_n^{\text{exit}}$. Using this observation, we now verify (65) for the cases $p = 0, 1$:

For $p = 0$, the desired lift in

$$
\begin{array}{c}
0 \\
\downarrow \cong \\
1
\end{array} \quad \begin{array}{c}
(T) \xrightarrow{f} W_n^{\text{hlt}} \\
\text{Fun}^W_n([1], \Theta_n^{\text{exit}})
\end{array}
$$

is given by selecting out the morphism $T \xrightarrow{f} T'$, which is in $\Theta_n^{\text{exit}}$ because each morphism in $W_n^{\text{hlt}}$ between objects in $\Theta_n^{\text{act, hlt.}}$ is a morphism in $\Theta_n^{\text{exit}}$, as previously discussed. A similar argument yields a lift for the square whose downward arrow on the left is (1).

For $p = 1$, the desired lift in

$$
\begin{array}{c}
0 \\
\downarrow \cong \\
1
\end{array} \quad \begin{array}{c}
\text{Fun}^W_n([1], \Theta_n^{\text{exit}}) \\
\text{Fun}^W_n([1], \Theta_n^{\text{act, hlt.}})
\end{array}
$$

is again given by $\alpha$, which is straightforward to check upon applying the fact discussed above, that each morphism in $W_n^{\text{hlt}}$ between objects in $\Theta_n^{\text{act, hlt.}}$ is a morphism in $\Theta_n^{\text{exit}}$. A similar argument applies for the square whose downward arrow on the left is (1).
Thus, \( \text{Fun}^W_n([p], \Theta_n^{\text{exit}}) \hookrightarrow \text{Fun}^W_n([p], \Theta_n^{\text{act,ht}}) \) is an inclusion of a cofactor, meaning the target is equivalent to a coproduct, one term of which is the source. Thus, because the classifying space respects colimits, the induced map between classifying spaces \( \mathcal{B} \text{Fun}^W_n([p], \Theta_n^{\text{exit}}) \hookrightarrow \mathcal{B} \text{Fun}^W_n([p], \Theta_n^{\text{act,ht}}) \) is, in particular, still a monomorphism.

Finally, we are equipped to prove Corollary 5.1.2 by showing that the category \( \Theta_n^{\text{exit}} \) localizes on \( W_n \) to the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \).

**Proof.** Corollary 5.1.2 Similar to the proof of Lemma 4.0.2, we will use Theorem 1.0.6 that if the classifying space of \( \text{Fun}^W_n([\bullet], \Theta_n^{\text{exit}}) \) is a complete Segal space, then it is equivalent to the localization \( \Theta_n^{\text{exit}} \) about \( W_n \). First, we will show that there is an equivalence of simplicial spaces from the classifying space of \( \text{Fun}^W_n([\bullet], \Theta_n^{\text{exit}}) \) to \( \text{Hom}_{\mathcal{C}^{\infty}}(\bullet, \text{Exit}(\text{Ran}(\mathbb{R}^n))) \). To do this, we use the following diagram of simplicial spaces

\[
\begin{array}{ccc}
\mathcal{B} \text{Fun}^W_n([\bullet], \Theta_n^{\text{exit}}) & \hookrightarrow & \mathcal{B} \text{Fun}^W_n([\bullet], \Theta_n^{\text{act,ht}}) \\
\downarrow & \downarrow & \downarrow \\
\text{Hom}_{\mathcal{C}^{\infty}}(\bullet, \text{Exit}(\text{Ran}(\mathbb{R}^n))) & \hookrightarrow & \text{Hom}_{\mathcal{C}^{\infty}}(\bullet, \text{Exit}(\text{Ran}(\mathbb{R}^n)))
\end{array}
\]

(66)

This diagram needs some explanation. The top horizontal arrow on the left is a monomorphism by Lemma 5.2.2. We showed the top horizontal arrow on the left to be an equivalence in Lemma 5.2.1. Because \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \) is a \( \infty \)-subcategory of \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \), the bottom horizontal arrow is a monomorphism. We showed the downward equivalence in the proof of Lemma 4.0.8.

Lastly, to define the induced downward functor on the left of (66), first recall Observation 2.0.6 where we witnessed that the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \) is naturally a \( \infty \)-subcategory of the exit-path \( \infty \)-category of the unital Ran space of \( \mathbb{R}^n \), given as a pullback over surjective finite sets. Then, the induced downward arrow in (66) from \( \mathcal{B} \text{Fun}^W_n([\bullet], \Theta_n^{\text{exit}}) \) to \( \text{Hom}_{\mathcal{C}^{\infty}}(\bullet, \text{Exit}(\text{Ran}(\mathbb{R}^n))) \) is induced by the unique (up to a contractible space of choices) functor given by the universal property of pullback in the following diagram of \( \infty \)-categories:

\[
\begin{array}{ccc}
\Theta_n^{\text{exit}} & \longrightarrow & \text{Exit}(\text{Ran}(\mathbb{R}^n)) \\
\downarrow & \downarrow & \downarrow \\
\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}) & \longrightarrow & \text{Fin}^{\text{op}} \\
\downarrow & \downarrow & \downarrow \\
\text{Fun}(\{1 < \cdots < n\}, (\text{Fin}^{\text{surj}}_{\neq \emptyset})^{\text{op}}) & \longrightarrow & (\text{Fin}^{\text{surj}}_{\neq \emptyset})^{\text{op}}
\end{array}
\]

(67)

where we note that the top, back horizontal functor is the localization from Theorem 2.2.1 and the square on the right wall is the pullback that we just observed in (66). Also note that we apply the universal property of the classifying space to ensure that the unique functor in (67) from \( \Theta_n^{\text{exit}} \) to \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \) induces a functor from \( \mathcal{B} \text{Fun}^W_n([\bullet], \Theta_n^{\text{exit}}) \) to \( \text{Hom}_{\mathcal{C}^{\infty}}(\bullet, \text{Exit}(\text{Ran}(\mathbb{R}^n))) \) in (66).

We wish to show that this induced functor

\( \mathcal{B} \text{Fun}^W_n([\bullet], \Theta_n^{\text{exit}}) \xrightarrow{\kappa} \text{Hom}_{\mathcal{C}^{\infty}}(\bullet, \text{Exit}(\text{Ran}(\mathbb{R}^n))) \)

in (66) is an equivalence. First, observe that monomorphisms enjoy the ‘2 out of 3’ property (by Observation 5.4 in [5]) and thus, \( \kappa \) is a monomorphism.
All that remains to be shown then, in showing that $\kappa$ is an equivalence of simplicial spaces, is to show that $\kappa$ induces a surjection on path components between each space given by the value on $[p]$, 

$$\mathcal{B} \text{Fun}^{\text{wh}}_n ([p], \Theta_n^{\text{exit}}) \xrightarrow{\sim} \text{Hom}_{\text{Cat}^\infty} \left( [p], \text{Exit} (\text{Ran}^n) \right).$$

Recall in the proof of Lemma 4.0.8 where we show that $\mathcal{B} \text{Fun}^{\text{wh}}_n ([\bullet], \text{Exit} (\text{Ran}^n))$ satisfies the Segal condition. Observe that the same argument applies to $\mathcal{B} \text{Fun}^{\text{wh}}_n ([\bullet], \Theta_n^{\text{exit}})$ to show that it, too, satisfies the Segal condition. Thus, to show $\kappa$ is a surjection on path components, it suffices to show it for the cases $p = 0, 1$.

For the case $p = 0$, we wish to show that the map of spaces 

$$\mathcal{B}W_n^{\text{wh}} \xrightarrow{\kappa} \coprod_{r \geq 1} \text{Conf}_r (\mathbb{R}^n) \Sigma_r$$

is a surjection on path components. Indeed, this follows by Lemma 4.1.1 wherein we showed a homotopy equivalence between the classifying space of the subcategory $W_n^{\text{wh}}$ of $\Theta_n^{\text{act,wh}}$ and the coproduct $\coprod_{r \geq 0} \text{Conf}_r (\mathbb{R}^n) \Sigma_r$; the only difference here is that $r = 0$ is allowed.

For the case $p = 1$, we wish to show that the map of spaces 

$$\mathcal{B} \text{Fun}^{\text{wh}}_n ([1], \Theta_n^{\text{exit}}) \xrightarrow{\kappa} \text{mor} \left( \text{Exit} (\text{Ran}^n) \right)$$

is a surjection on path components. Let $\text{cylr} (S^1 \overset{f}{\to} T) \xrightarrow{E} \mathbb{R}^n \times \Delta^1$ be a point in the target. Recall that from (67), $\kappa$ is determined by $\Theta_n^{\text{exit}} \to \Theta_n^{\text{act}} \simeq \text{Exit} (\text{Ran}^n (\mathbb{R}^n))$ over $\text{Fun} \left( \{ 1 < \cdots < n \}, (\text{Fin}^\text{surj})^{\text{op}} \right)$. Thus, we will identify a point in the fiber over $E$ under $\kappa$ by identifying a morphism in $\text{Exit} (\text{Ran}^n (\mathbb{R}^n))$ over $(\text{Fin}^\text{surj})^{\text{op}}$. Such a morphism is precisely obtained by naming the projection data of $E$, namely:

$$\text{cylr} (S^1 \overset{f}{\to} T) \xrightarrow{E} \mathbb{R}^n \times \Delta^1$$

$$\xrightarrow{pr_1} \mathbb{R} \times \Delta^1$$

the value of which under the functor $\text{Exit} (\text{Ran}^n (\mathbb{R}^n)) \to \text{Fun} \left( \{ 1 < \cdots < n \}, (\text{Fin}^\text{surj})^{\text{op}} \right)$ precisely because $\text{Exit} (\text{Ran}^n (\mathbb{R}^n))$ is naturally over $\text{Fun} \left( \{ 1 < \cdots < n \}, (\text{Fin}^\text{surj})^{\text{op}} \right)$ as observed from its pullback description in (4). Thus, this morphism in $\text{Exit} (\text{Ran}^n (\mathbb{R}^n))$ defines a morphism in $\Theta_n^{\text{exit}}$ whose value under $\kappa$ is $E$. Thus, $\kappa$ for the case $p = 1$ is a surjection on path components which, as previously argued, implies that $\kappa$ is an equivalence of simplicial spaces. The target of $\kappa$, $\text{Hom}_{\text{Cat}^\infty} \left( [\bullet], \text{Exit} (\text{Ran}^n) \right)$ is, in particular, a complete Segal space, and hence, $\mathcal{B} \text{Fun}^{\text{wh}}_n ([\bullet], \Theta_n^{\text{exit}})$ is a complete Segal space. As such, the hypothesis of Theorem 4.0.6 is satisfied and we establish that $\Theta_n^{\text{exit}}$ localizes on $W_n^{\text{wh}}$ to the exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$.

Lastly, to see that this localization is over $(\text{Fin}^\text{surj})^{\text{op}}$, we recall that the equivalence 

$$\mathcal{B} \text{Fun}^{\text{wh}}_n ([\bullet], \Theta_n^{\text{exit}}) \simeq \text{Hom}_{\text{Cat}^\infty} \left( [\bullet], \text{Exit} (\text{Ran}^n) \right)$$

65
was induced by the functor $\Theta^\text{exit}_n : \text{Exit}(\text{ Ran}(\mathbb{R}^n)) \to \mathbb{N}$, which is in particular over $\text{Fin}^{\text{surj}}$, which implies that the localization, too, is over $\text{Fin}^{\text{surj}}$. □

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