1. INTRODUCTION AND MOTIVATION

The problem addressed in this paper is how to visualize flows generated by the collisionless Boltzmann equation (CBE), i.e., the gravitational analog of the electrostatic Vlasov equation from plasma physics.

It is generally accepted that many physical problems arising in galactic dynamics and cosmology can be modeled in terms of the CBE, perhaps allowing also for low-amplitude discreteness effects, modeled as noise or phase mixing through the formulation of a Fokker-Planck equation, or for a coupling to a dissipative fluid described, e.g., by the Navier-Stokes equation. Astronomers recognize that an evolution described completely by the CBE is special because of the constraints associated with Liouville's theorem and that, at some level, the flow must be Hamiltonian, which precludes the possibility of any pointwise approach toward a time-independent equilibrium: in the absence of dissipation, one can speak meaningfully only of a coarse-grained approach toward equilibrium. However, there does not seem to be a clear sense of exactly how one should approach toward a time-independent equilibrium: in the context of the full noncanonical Hamiltonian dynamics and in terms of a simpler canonical Hamiltonian structure associated with the tangent dynamics, i.e., identifying explicitly a set of canonically conjugate variables in terms of which, in many cases, would be expected to exhibit linear Landau damping. If, instead, the initial condition is far from any stable extremal, the flow will be more complicated, but, in general, one might anticipate that the evolution can be visualized as involving nonlinear oscillations about some lower energy $f_0$. In this picture, the coarse-grained approach toward equilibrium usually termed violent relaxation is interpreted as nonlinear Landau damping. Evolution of a generic initial condition involves a coherent initial excitation $\delta f(0) = f(0) - f_0$, not necessarily small, being converted into incoherent motion associated with nonlinear oscillations about some $f_0$ which, in general, will exhibit destructive interference. This picture allows for distinctions between regular and chaotic "orbits" in $\Gamma$: stable extremals $f_0$ all have vanishing Lyapunov exponents, even though "orbits" oscillating about $f_0$ may well correspond to chaotic trajectories with one or more positive Lyapunov exponents.

Subject headings: galaxies: evolution — galaxies: kinematics and dynamics — galaxies: structure
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process: even though a perturbation cannot “die away” in
any pointwise sense, one may expect a coarse-grained
approach toward equilibrium in which observables like the
density perturbation $\delta \rho$ eventually decay to zero.

Section 4 generalizes the preceding to the case of nonlinear
stability, allowing for perturbations $\delta \theta$ away from some
equilibrium $f_0$, which are not necessarily small. The intuition
derived from that problem is then used to motivate one
possible way in which to visualize the flow associated with a
generic initial $f(t = 0)$. The obvious point is that a generic
initial $f(0)$ can be viewed as a (possibly strongly nonlinear)
perturbation of some equilibrium $f_0$, the form of which,
however, need not be known explicitly. To the extent that
this interpretation is accepted, those aspects of the flow
typically denoted violent relaxation should be viewed as
nonlinear Landau damping/phase mixing (see Kandrup
1998). Section 5 concludes by describing the mathematical
issues which must be resolved to make the preceding dis-
cussion rigorous and complete.

A simple mechanical model, which can help in visualizing
the basic ideas described in this paper, is the following:
consider a point particle moving in some complicated,
many-dimensional potential $V(r)$ which is characterized
genERICALLY by multiple extremal points but which, being
bounded from below, will have a (in general, nondegenerate)
lowest global minimum. If one chooses initial data
corresponding to a configuration space point $r$ close to but
slightly above some local minimum $r_0$ and a velocity $v$
whose magnitude is very small, the subsequent evolution
will involve linear oscillations about $r_0$, whether or not that
point corresponds to a global minimum. The trajectory of
the point particle thus corresponds to a regular orbit in
what appears locally as a harmonic potential. If the initial
deVIation from the extremal point becomes somewhat larger because $|r - r_0|$ and/or $|v|$ is bigger, one would still
anticipate oscillations around $r_0$, but these will now become
nonlinear and the particle trajectory may well correspond
to a chaotic orbit. Suppose, however, that $r_0$ is not the
global minimum. In this case, one would expect that, for
initial data sufficiently far from $r_0$, the particle will have left
the “basin of attraction” associated with the local
minimum and will instead (generically) exhibit strongly
nonlinear oscillations about the global minimum (it could,
of course, oscillate around a different non-global
minimum). In the absence of dissipation, there is no point-
wise sense in which the particle evolves toward the global
minimum. However, the nonlinear oscillations in different
directions will, in general, interfere destructively, so that any
initial coherence between motions in different directions
will eventually be lost (at least for times short compared with
the Poincaré recurrence time). It is this loss of coher-
ence which, for the CBE, gives rise to (linear or nonlinear)
Landau damping.

2. THE NONCANONICAL HAMILTONIAN FORMULATION

If one considers the Liouville equation appropriate for a
collection of noninteracting particles evolving in a fixed
potential $\Phi(x)$, the natural phase space is the six-
dimensional phase space associated with the canonical pair
$(x, \nu)$. If however, one considers the full CBE, allowing for a
self-consistent potential $\Phi[f(x, \nu)]$ determined by the free-
streaming particles, this is no longer so. In this case, the
fundamental dynamical variable is the distribution function
In general, it is not easy to identify conjugate coordinates and
momenta in this phase space so as to rewrite the CBE in
the form of Hamilton’s equations. However, one can still
capture the Hamiltonian character at a formal algebraic
level through the identification of an appropriate cosym-
pletic structure (see Arnold 1989).

In this context, manifesting the Hamiltonian character of
the flow entails identifying a Lie bracket $[\cdot,\cdot]$, defined on
pairs of phase-space functionals $\mathcal{A}[f]$ and $\mathcal{B}[f]$, and
a Hamiltonian functional $\mathcal{H}[f]$, so chosen that the CBE

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} - \nabla \Phi \cdot \frac{\partial f}{\partial \nu} = 0,$$

with $\Phi(x, t)$ the self-consistent potential satisfying

$$\nabla^2 \Phi[f] = 4\pi G \rho \equiv \int d^3v \nu,$$

can be written in the form

$$\frac{\partial f}{\partial t} + \mathcal{H}[f] = 0.$$

The Hamiltonian $\mathcal{H}$ may be taken as

$$\mathcal{H}[f] = \frac{1}{2} \int d\Gamma v^2 f(x, \nu) - \frac{G}{2} \int d\Gamma \int d\Gamma' \frac{f(x, \nu)f(x', \nu)}{|x - x'|},$$

with $d\Gamma \equiv d^3x d^3v$, which corresponds to the obvious mean
field energy, as identified, e.g., by Lynden-Bell & Sanitt
(1969). The bracket is then chosen to satisfy (Morrison

1 One example of a noncanonical Hamiltonian system, well known to
astronomers, is rigid body rotations described by the standard Euler equa-
tions (see Landau & Lifshitz 1960). Specifically, as discussed and gen-
eralized, e.g., in Kandrup (1990) and Kandrup & Morrison (1993, the Euler
equations constitute a Hamiltonian system, formulated in the three-
dimensional phase space coordinatized by the three components of angular
momentum $J_i$ (i = 1, 2, 3), with the Hamiltonian

$$H[J_i] = \sum_{i=1}^3 J_i^2 / 2I_i$$

(along the analog of eq. [4]) defined in terms of the principal
moments of inertia, and the Lie bracket (the analog of eq. [5]) given as
the natural bracket associated with the three-dimensional rotation group,

$$[a, b] = \sum_{i, j, k} \epsilon_{ijk} J_i \left( \frac{\partial a}{\partial J_j} \frac{\partial b}{\partial J_k} \right)$$

for functions $a(J)$ and $b(J)$. As for the CBE, there is also a Casimir (the
analog of eq. [9]), namely, $C[J_i] = \sum_{i=1}^3 J_i^2$, which restricts motion to
the two-dimensional constant $C$ surface in the three-dimensional phase space.

Astronomers are also acquainted with infinite-dimensional Hamiltonian
systems, at least those realizable in canonical coordinates, one simple
example being the scalar wave equation $\partial^2_t \Psi - \nabla^2 \Psi = 0$, which derives
from the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int d^3x [\Pi^2(x) + |\nabla \Psi(x)|^2],$$

where $\Psi$ and $\Pi$ are canonically conjugate.
where \[\{g, h\}\] denotes the ordinary Poisson bracket acting on functions \(g(x, v)\) and \(h(x, v)\), and \(\delta \delta f/\delta f\) denotes a functional derivative. It is straightforward to show that the operation defined by equation (5) is a skew-symmetric, bilinear form, satisfying the Jacobi identity

\[
[g, [h, k]] + [h, [k, g]] + [k, [g, h]] = 0 ,
\]
so that it defines a bona fide Lie bracket. However, for this bracket one verifies immediately that equation (3) reduces to the CBE in the form

\[
\frac{df}{dt} - \{E, f\} = 0 ,
\]
where \(E\) represents the energy of a unit mass test particle, i.e.,

\[
E = \frac{1}{2}v^2 + \Phi(x, t)
\]

A flow governed by the CBE is strongly constrained by Liouville’s theorem, which implies the existence of an infinite number of conserved quantities, the so-called Casimirs \(\mathcal{Q}(f)\). Specifically, the flow has the property that, for any function \(\chi(f)\), the value of the phase-space integral,

\[
\mathcal{Q}(f) = \int d\Gamma \chi(f) ,
\]
is invariant under time translation, i.e., \(d\mathcal{Q}/dt = 0\). The simplest case corresponds to the choice \(\chi = f\), which leads to conservation of number (or mass):

\[
\frac{d}{dt} \int d\Gamma f \equiv 0 .
\]

By analogy with finite-dimensional systems, where Noether’s theorem relates conserved quantities to continuous symmetries, these Casimirs reflect internal symmetries in the infinite-dimensional phase space \(\Gamma\) (Morrison & Eliezer 1986).

The Casimirs play an important role in analyzing the stability of equilibrium solution \(f_0\), where one must restrict attention to perturbations \(\delta f\) that satisfy \(\delta \mathcal{Q} \equiv 0\) for all possible choices of \(\chi\). As first noted by Bartholomew (1971), this demand implies that any allowed perturbation \(\delta f\) is related to \(f_0\) by a canonical transformation induced by some generating function \(g\), i.e.,

\[
f \equiv f_0 + \delta f = \exp \left( (g, \cdot) \right) f_0 .
\]

In addition to the Casimirs, there is also at least one other conserved quantity, namely, the mean field energy \(\mathcal{H}(f)\). Specifically, it follows from the CBE that \(d\mathcal{H}/dt \equiv 0\). If one considers initial data \(f(0)\) characterized by a high degree of symmetry, other conserved quantities may also exist. For example, if the initial data correspond to a potential \(\Phi\) which is spherically symmetric, it follows that the numerical value of the angular momentum,

\[
\mathbf{J} \equiv \int d^3x d^3v f(x) \times v ,
\]
is necessarily conserved. However, these conserved quantities, if they exist, are on a different footing from the Casimirs since they reflect symmetries in the particle phase space, rather than internal symmetries associated with the infinite-dimensional phase space of distribution functions.

Because of the infinite number of constraints associated with the Casimirs, the evolution of \(f\) is reduced to a lower (but presumably still infinite-) dimensional phase space hypersurface, say \(\gamma\). One might naively believe that, in the same sense as, e.g., for the Kortweg–de Vries equation (see Arnold 1989), the flow associated with the CBE is integrable. In point of fact, however, this is almost certainly not so (see Morrison 1987), the important point being that the Casimirs associated with the CBE are all “ultralocal” quantities which do not involve derivatives of \(f\).

At the present time, there is no universally accepted notion of what precisely one should mean by chaos in an infinite-dimensional Hamiltonian system. However, one obvious tack entails comparing initially nearby flows and asking whether, for some given \(f(t = 0)\), there exist perturbations \(\delta f(t = 0)\) which grow exponentially. This leads naturally to the notion of a functional Lyapunov exponent which, at least formally, can be defined by analogy with the definition of an ordinary Lyapunov exponent in a finite-dimensional system (see Lichtenberg & Lieberman 1992). Specifically, given the introduction of an appropriate norm denoted by \(\| \|\), one can write

\[
\chi = \lim_{t \to \infty} \lim_{\delta f(0) \to 0} \frac{1}{t} \| \delta f(t) \| .
\]

For finite-dimensional systems one knows that, independent of the choice of norm, the analog of equation (13) will, for a generic phase space perturbation \(\delta f\), converge toward the largest Lyapunov exponent. Much less is known about the infinite-dimensional case. For specificity, it thus seems reasonable to choose the norm denoted by \(\| \|\) as corresponding to a (possibly weighted) \(L^2\) norm defined in the phase space of distribution functions, i.e.,

\[
\| \delta f \| \equiv \int d\Gamma M(x, v) |\delta f(x, v)|^2 ,
\]

where \(M\) denotes a specified function of \(x\) and \(v\). This is, e.g., the type of norm that has been used in proving theorems about linear stability.

3. LINEAR STABILITY AND GRAVITATIONAL LANDAU DAMPING

The key fact underlying the interpretation of flows described by the CBE and, especially, the problem of stability, is that every time-independent equilibrium \(f_0\) is an energy extremal with respect to “symplectic” perturbations \(\delta f\) of the form given by equation (11) which preserve the numerical values of every Casimir. This implies that if one restricts attention to the reduced phase space \(\gamma\) obtained by freezing the value of each Casimir at its equilibrium value \(\mathcal{H}(f_0)\), every equilibrium \(f_0\) corresponds to an isolated fixed point: to lowest order, the quantity \(\delta \mathcal{H} \equiv 0\) for any symplectic \(\delta f\). As explained below, if \(f_0\) is a local energy minimum, so that, to next leading order, \(\delta \mathcal{H} \geq 0\), \(f_0\) must be linearly stable. Alternatively, if \(f_0\) corresponds to a saddle point, so that \(\mathcal{H}\) increases for some perturbations but decreases for others, linear stability is no longer guaranteed.
although one cannot necessarily infer that $f_0$ must be linearly unstable.

The proof that, to lowest order, $\delta H$ vanishes for any perturbation of the form given by equation (11) and the computation of $\delta H$ to higher order are straightforward if one expands equation (11) perturbatively to infer that

$$\delta f = \langle g, f_0 \rangle + \frac{1}{2} \langle g, \{ g, f_0 \} \rangle + \ldots \equiv \delta^{(1)} f + \delta^{(2)} f + \ldots .$$

(15)

It is easy to see that, for any $\delta^{(1)} f$, the first variation $\delta^{(1)} H$ becomes

$$\delta^{(1)} H = \int d\Gamma \left( \frac{1}{2} v^2 - G \int d\gamma \frac{f_0}{|x - x'|} \right) \delta^{(1)} f$$

$$= \int d\Gamma E_0 \delta^{(1)} f ,$$

(16)

where $E_0$ is the particle energy associated with $f_0$. However, by combining equations (15) and (16) and then integrating by parts, one finds that

$$\delta^{(1)} H = \int \delta E_0 \langle g, f_0 \rangle = - \int d\Gamma g \{ E_0, f_0 \} \equiv 0 ,$$

(17)

where (see eq. [7]) the final equality follows from the fact that $f_0$ is time-independent. Extending this calculation to one higher order shows that the second variation

$$\delta^{(2)} H = - \frac{1}{2} \int d\Gamma \langle gj, f_0 \rangle \langle g, E_0 \rangle - G \frac{1}{2} \int d\Gamma$$

$$\times \int d\Gamma \{ g, f_0 \} \langle g, f_0 \rangle \langle f_0 \rangle \frac{f_0}{|x - x'|} .$$

(18)

To help visualize what is going on and to understand why linear stability follows if $\delta^{(2)} H$ is positive for all symplectic perturbations of the form given by equation (11), suppose that, in ordinary three-dimensional space, the $x$-$y$ plane corresponds to a hypersurface in the reduced $\gamma$-space of distribution functions. One can then “warp” this plane into a curved two-dimensional surface by assigning to each $x$-$y$ pair a coordinate $z$ which corresponds to the numerical value assumed by the energy $H$. On this warped surface, the equilibrium points correspond to those pairs $(x_0, y_0)$ which are external in $z$, so that any infinitesimally displaced point $(x_0 + \delta x, y_0 + \delta y)$ assumes a new value $z + \delta z$.

If the equilibrium point is a local energy minimum, any infinitesimal displacement on the surface necessarily increases the value of $z$, so that, in the neighborhood of $(x_0, y_0)$, the surface has the geometry of an upward-opening paraboloid. Any perturbation comes with positive energy and corresponds to bounded motion on the paraboloid. Thus the equilibrium is linearly stable. In principle, the same conclusion also obtains if the extremal point is a local maximum, although one can show that, for realistic equilibria, $\delta^{(2)} H$ is never strictly negative. If, however, the equilibrium corresponds to a saddle point, so that $z$ increases in some directions but decreases in others, the situation becomes more complicated. In this case, the linearized dynamics implies that it is possible to combine a very large negative energy perturbation in one direction with a very large positive energy perturbation in another to generate a total perturbation with vanishing energy. In itself, this does not guarantee a linear instability, but the simple geometric argument for stability that holds for a local minimum is no longer applicable.2

That saddle points need not imply linear instability may seem surprising at first glance. However, the following two-dimensional example makes clear exactly what can go wrong:

$$H = \frac{1}{2}(v_1^2 + \omega_1^2 x_1^2) - \frac{1}{2}(v_2^2 + \omega_2^2 x_2^2) .$$

(19)

Here $x_1 = v_1 = x_2 = v_2 = 0$ is a time-independent extremal point in the phase space which corresponds to a saddle, but, nevertheless, the equilibrium is clearly stable. This model may seem somewhat contrived, but, as discussed in § 5 of Kandrup & Morrison (1993), such stable saddle points are not uncommon in various infinite-dimensional Hamiltonian systems.

The preceding argument for stability or lack thereof may seem somewhat unusual because it is formulated abstractly in phase space, without the introduction of conjugate coordinates and momenta. One might therefore hope that, by identifying an appropriate set of conjugate variables, a more intuitive proof could be derived. In certain cases, this is, in fact, possible. One knows that, when formulated in the full $1$-space, the dynamics cannot be decomposed completely into canonical variables because of the existence of the Casimirs, which correspond to null vectors of the cosymplectic structure. If, however, one passes to the reduced $\gamma$ space, where the values of all the Casimirs are frozen, one might expect that, at least locally, conjugate variables do exist. Indeed, for finite-dimensional systems it follows from Darboux’s theorem (see Arnold 1989) that, if the cosymplectic structure has vanishing determinant, i.e., if there are no null eigenvectors, it is always possible to find a set of canonically conjugate variables, at least locally (see § 5 of Kandrup & Morrison 1993 for a detailed discussion of this point).

One setting in which such a canonical formulation is possible is for the special case of linear perturbations of an equilibrium $f_0$ which is a function only of the one-particle energy $E$, i.e., $f_0 = f_0(E)$, and for which the partial derivative $F_E \equiv \partial f_0 / \partial E$ is strictly negative. Physically the latter restriction implies that the system does not exhibit a population inversion; mathematically it ensures that division by $F_E$ is well defined. The basic idea, due originally to Antonov (1960), is to split the linearized perturbation $f$ into two pieces, $f_\uparrow$ and $f_\downarrow$, respectively even and odd under a velocity inversion $v \rightarrow -v$, and to view the single linearized perturbation equation for $f$ as a coupled system for $f_\uparrow$.

When linearized about some equilibrium $f_0$, the CBE reduces to

$$\partial_t f - \{ E, f \} - \Phi[f, f_0] = 0 ,$$

(20)

where $E$ is the particle energy associated with $f_0$ and $\Phi[f, f_0]$ denotes the gravitational potential “sourced” (cf. eq. [2]) by the perturbation $f$. If one observes that $E$ is an even function of $v$, that the Poisson bracket is odd under velocity

2 Strictly speaking, the application of this finite-dimensional argument to an infinite-dimensional Hamiltonian system requires that the reduced phase space $\gamma$ be endowed with a metric, so that one knows what is meant by distance between points. In practice, this can be done by introducing an appropriate $E$ norm, which provides the natural extension of the Euclidean notion of distance to an infinite-dimensional space. In this context, a proof of stability entails showing that $||df(t)||$ remains bounded for all times.
inversion, and that $\Phi[\delta f_-]$ vanishes identically, it is clear that equation (20) is equivalent to the coupled system,

$$\dot{\delta f}_+ - \{E, \delta f_+\} = 0$$

and

$$\dot{\delta f}_- - \{E, \delta f_-\} - \Phi[\delta f_+], f_0 = 0. \quad (21)$$

However, if one differentiates the second of these relations with respect to $t$ and uses the first to eliminate $\dot{\delta f}_+$, it follows that

$$\dot{\delta f}_- = \{E, \{E, \delta f_-\}\} + \Phi[\{E, \delta f_-\}], f_0 \equiv F_\mathcal{A} \delta f_- , \quad (22)$$

where $\mathcal{A}$ denotes a linear operator. One can then show that, given the identification of $\delta f_-$ and $\dot{\delta f}_+$ as conjugate variables, the equation

$$( - F_\mathcal{A} )^{-1} \dot{\delta f}_- = - \mathcal{A} \delta f_- \quad (23)$$

can be derived from the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int \frac{d\Gamma}{( - F_\mathcal{A} )} (\dot{\delta f}_-)^2 + \frac{1}{2} \int d\Gamma \delta f_\mathcal{A} \delta f_- \quad (24)$$

where $\mathcal{H}$ and the energy $\delta^{(2)} \mathcal{H}$ associated with a small symplectic perturbation is discussed in Kandrup (1990). In particular, one can show that $\mathcal{H} > 0$ for all $\delta f_-$ if and only if $\delta^{(2)} \mathcal{H} > 0$ for all symplectic perturbations.

The fact that $\mathcal{A}$ is a symmetric (i.e., Hermitian) operator facilitates a proof that the equilibrium $f_\mathcal{A}(E)$ is linearly stable if and only if $\mathcal{H}$ and $\delta^{(2)} \mathcal{H}$ is positive. Specifically, a simple energy argument (see Laval, Mercier, & Pellat 1965) implies that the magnitude of $\delta f_-$, and hence $\delta f_\mathcal{A}$, is bounded in time if $\mathcal{A}$ is a positive operator, so that $\delta^{(2)} \mathcal{H} > 0$, whereas the possibility of perturbations with $\delta t \delta f_- \mathcal{A} \delta f_- < 0$ implies the existence of solutions that grow exponentially. This is easy to understand in the language of normal modes. Since $\mathcal{A}$ is symmetric, it is clear that all solutions $\delta f_- \propto \exp (\sigma t)$ have $\sigma^2$ real, so that the evolution is either purely oscillatory or purely exponential. If $\mathcal{A}$ is positive, $\sigma^2$ must be negative, so that the modes are purely oscillatory. If, however, $\mathcal{A}$ is not a positive operator, there exist modes with $\sigma^2 > 0$, which implies an exponential instability.\footnote{In point of fact, one anticipates that, for this simple case, $\mathcal{A}$ is guaranteed to be positive. Perez & Aly (1996) have proved that any $f_\mathcal{A}$ depending only on $E$ corresponds to a spherically symmetric configuration, but assuming that the mass density $\rho$ associated with $f_\mathcal{A}$ is spherical one can prove that $\mathcal{A}$ is indeed positive (see Kandrup 1990).}

This sort of normal mode expansion facilitates a simple geometric picture of an infinite-dimensional configuration space of perturbations $\delta f_-$ which is (locally) embeddable in the reduced $\gamma$-space. The equilibrium $f_\mathcal{A}$, which is necessarily an extremal point of the full Hamiltonian $\mathcal{H}$, satisfies $\delta f_- \equiv \dot{\delta f}_- = 0$. An arbitrary initial perturbation entails a kinetic energy $\mathcal{H} = \int dt ( - F_\mathcal{A} )^{-1} (\dot{\delta f}_-)^2$ which is necessarily positive and a potential energy $\delta \mathcal{H} = \int dt \delta f_- \mathcal{A} \delta f_-$, with a sign which depends on the properties of $\mathcal{A}$. If $\mathcal{A}$ is a positive operator, the evolution in configuration space involves a particle with “mass” $(- F_\mathcal{A})^{-1}$ moving in an infinite-dimensional harmonic potential which corresponds to an upward-opening paraboloid. Linear stability is therefore assured. If, however, $\mathcal{A}$ is not always positive, $\delta f_- \equiv 0$ corresponds to a saddle point, rather than a local minimum, and the flow is linearly unstable.

In visualizing all of this, there is the strong temptation to think of the normal modes as being discrete, i.e., corresponding to honest square integrable eigenfunctions rather than singular eigendistributions. This, however, is not necessarily justified.

Assuming completeness, one can always view any linear perturbation of an equilibrium $f_\mathcal{A}(E)$ with $F_\mathcal{A} < 0$ as a superposition of normal modes, writing $\delta f$ as a formal sum

$$\delta f(x, v, t) = \sum_\sigma A_\sigma g_\sigma(x, v) \exp (i\sigma t), \quad (25)$$

where $g_\sigma$ labels the eigenvector, $\sigma$ is the corresponding frequency, which is necessarily real, and $A_\sigma$ is an expansion coefficient.\footnote{Strictly speaking, this sum must be interpreted (see Riesz & Nagy 1955) as a Stieltjes integral.} Modulo largely important technical details, the modes then divide into two types, namely (1) a countable set of discrete frequencies belonging to the point spectrum, for which the corresponding eigenvectors are well-behaved (e.g., square-integrable) eigenfunctions; and (2) a continuous set of frequencies belonging to the continuous spectrum, for which the eigenvectors are singular eigendistributions.

The distinction between these two types of modes is extremely important (see Habib et al. 1986). Because true eigenfunctions are nonsingular, they can in principle be triggered individually, i.e., one can choose a reasonable initial $\delta f$ which populates only a single discrete mode. By contrast, because eigendistributions are singular, one cannot sample a single continuous mode. Rather, any smooth $\delta f$ sampling the continuous spectrum must really be constructed as a wavepacket comprised of a continuous set of modes. The important point then is that, when evolved into the future, such a wavepacket implies a damping of coarse-grained observables like the density $\rho$. In other words, if the modes are continuous there is a precise sense in which the perturbation “dies away,” and the system exhibits a coarse-grained approach toward the original equilibrium $f_\mathcal{A}$.

The physics here is analogous to what arises in ordinary quantum mechanics. If, in that setting, one considers a physical observable like angular momentum with a discrete spectrum, one can construct well-behaved eigenstates which, when evolved into the future, maintain their coherence for all time: the only effect of the evolution is a coherently oscillating phase. If, however, one considers an observable like position or linear momentum, where the spectrum is continuous, this is no longer so. In this case, a normalizable initial state must be constructed from a continuous set of singular eigendistributions, so that the best one can do is build a localized (e.g., minimum uncertainty) wavepacket. However, when evolved into the future such a wavepacket will necessarily spread because different eigendistributions have different phase velocities.

It is this loss of coherence associated with the spreading of a wavepacket that corresponds to (linear) Landau damping. In the context of plasma physics, Landau damping was derived originally (Landau 1946) in a very
4. NONLINEAR STABILITY AND GLOBAL EVOLUTION

Suppose, once again, that attention is focused on some linearly stable equilibrium \( f_0(E) \) with \( F < 0 \), but that one is now interested in the effects of larger perturbations \( \delta f \), i.e., the problem of nonlinear stability. To the extent that the normal modes of the linear problem remain complete, one can still envision evolution in terms of these modes, the important point, however, being that because of nonlinearities the modes will now interact. This is, e.g., the basis for the standard quasi-linear analyses implemented in plasma physics, which allow for the effects of the quadratic term \( \Phi \cdot (\partial \delta f / \partial v) \) which is ignored when considering linear perturbations.

Mode-mode couplings are important in that they facilitate the transfer of energy between different modes, which makes the physics more complicated. However, one might still anticipate that, if the modes are continuous, Landau damping can and will occur. Because of the interactions between modes, the simple model of a dispersing quantum mechanical wavepacket is no longer directly applicable, but the basic phenomenon of loss of coherence is robust. Indeed, there are many examples in nonlinear dynamics of flows satisfying nonlinear evolution equations where phase mixing occurs. It thus seems reasonable to suppose that, when considering the nonlinear evolution of some perturbation \( \delta f \), one will encounter nonlinear Landau damping. For the case of an electrostatic plasma, nonlinear Landau damping is a well known, and reasonably well understood, phenomenon (see Davidson 1972, and references therein). Indeed, there are simple geometries where the nonlinear evolution can be computed explicitly in the context of a systematic perturbation expansion, thus facilitating analytic formulae for exactly how this phenomenon works (see Montgomery 1963).

Mode-mode couplings can also lead to another important possibility, namely, the onset of chaos. Because \( f_0 \) is a
local energy minimum, one knows that any infinitesimal perturbation $\delta f$ will simply oscillate, each eigenvector corresponding to motion in a “direction” in configuration space that is orthogonal to the motion of all the other eigenvectors. This implies that, for the fixed point $f_0$, the Lyapunov exponents, which were defined in equation (13) as probing the average linear instability of the orbit generated from some initial $f(0)$, must all vanish identically. One might anticipate further that, when evolved into the future, other phase-space points sufficiently close to will also might anticipate further that, when evolved into the future, other phase-space points sufficiently close to $f_0$ will also correspond to regular orbits with vanishing Lyapunov exponents. Thus, e.g., for finite-dimensional systems one knows that there is a regular phase-space region of finite measure surrounding every stable periodic orbit. However, for sufficiently large $\delta f$, where mode-mode couplings become significant and the motion cannot be well approximated by orthogonal harmonic oscillations, one might anticipate that many, if not all, perturbations will evolve chaotically. If true, this would suggest that a “typical” perturbation with $\delta \mathcal{H} = \mathcal{H}[f_0 + \delta f] - \mathcal{H}[f_0]$ will evolve ergodically on (some subset of) the constant energy hypersurface in the $\gamma$-space with energy $\mathcal{H}[f_0 + \delta f]$.

This idea of the onset and development of chaos is an infinite-dimensional generalization of what is typically found when considering the motion of a point mass in a multidimensional nonlinear potential which has only one extremal point, a global minimum. Low-energy orbits sufficiently close to the pit of the potential move in what is essentially a harmonic potential, so that their motion is regular. If, however, the energy is raised one finds generically that, unless the motions in different directions remain completely decoupled, there is an onset of global stochasticity which leads, for sufficiently high energies, to well-developed chaotic regions.

This configuration space distribution is not appropriate when considering generic equilibria, where the energy $\mathcal{H}$ associated with a small perturbation cannot be written easily as a functional of conjugate variables, and there is no guarantee that $\mathcal{H}$ can be written as a simple sum of kinetic and potential contributions, $\mathcal{H}$ and $\mathcal{W}$. Modulo technical details, one might expect that canonical phase-space coordinates do exist, at least in principle, but the energy $\mathcal{H}$ associated with the tangent dynamics could in general be an arbitrary quadratic functional $\mathcal{H}[q, p]$ of the conjugate variables $q$ and $p$. Moreover, even for the simple model of an equilibrium $f_0(E)$ with $F_k < 0$, it may not be possible to extend the canonical description to allow for arbitrarily large perturbations $\delta f$. One really needs to return to a full phase-space description.

As discussed in § 3, if for some equilibrium $f_0$, the second variation $\delta^{(2)} \mathcal{H}$ is positive for all $\delta f$, a linearized perturbation corresponds in phase space to stable motion on an upward-opening infinite-dimensional paraboloid. As long as this surface remains convex, one would anticipate that stability will persist, and, as such, one would expect intuitively that the equilibrium could remain nonlinearly stable even for small but finite $\delta f$. The normal modes of the linearized problem become coupled, but the geometric argument for stability should remain valid. In particular, one can presumably visualize the evolution of $\delta f$ as involving nonlinear phase-space oscillations about the equilibrium point $f_0$.

If, however, $f_0$ corresponds to a stable saddle, one might suppose that even the smallest nonlinearities could trigger an instability (see Moser 1968; Morrison 1987). Thus, e.g., for the simple toy model of two stable oscillators described by equation (19), it is possible to trigger an instability by introducing even very tiny mode-mode couplings which allow energy to be transferred between modes. Indeed, as noted by Cherry (1925), if the two frequencies are in an appropriate resonance, e.g., $\omega_2 = 2\omega_1$, the introduction of a simple cubic coupling implies that initial data arbitrarily close to $x_1 = v_1 = x_2 = v_2 = 0$ can lead to solutions in which $x_1, x_2, v_1$, and $v_2$ all diverge in a finite time. If true, this expectation about saddle points would suggest that, even though they can be linearly stable, they cannot represent reasonable candidate equilibria in terms of which to model real astronomical objects.

If a linearly stable $f_0$ corresponds to a unique extremal point in the $\gamma$-space, the surface which near $f_0$ is a paraboloid will remain upward opening even if $\delta f$ is very large, so that stability should persist for arbitrarily large perturbations. In other words, one would expect that the equilibrium $f_0$ is globally stable: in this case, any phase-space deformation $\delta f$ increases the energy, and the evolution of an initial $\delta f(0)$ will involve nonlinear phase-space oscillations around the unique stable fixed point.

If, however, there exist multiple extremal points in the $\gamma$-space, each corresponding to a local energy minimum, the situation is much more complicated. In this case, one would expect that, for sufficiently large $\delta f$, the distribution function can actually be transferred from the “basin of attraction” of one equilibrium $f_0$ to the “basin” of another $f_1$. In other words, the evolution of $\delta f(0)$ could yield oscillations around $f_1$, rather than $f_0$. By suitably fine-tuning the perturbation, one can in principle displace the system from any one basin to any other. However, by analogy with the behavior observed in finite-dimensional systems, one might expect generically that, if the perturbation is sufficiently large, its motion can be interpreted as involving nonlinear phase-space oscillations about the global energy minimum. To the extent that this is true, one would anticipate that a sufficiently large perturbation will tend generically to push $f$ into the “basin of attraction” of the equilibrium $f_0$ that corresponds to a global energy minimum.

If one considers an initial perturbation $\delta f(0)$ that is sufficiently large, the subsequent evolution will in general be almost completely unrelated to the initial equilibrium $f_0$ and, as such, the way in which one visualizes the evolution of $\delta f(0)$ is really no different from the way in which one can, and arguably should, envision the evolution of a generic $f(0)$. In other words, the physical picture described above can be used equally well to visualize generic flows associated with the initial value problem, the only difference being that, in general, one may know nothing at all about what time-independent equilibria $f_0$ actually exist.

Specification of an initial $f(0)$ fixes the values of all the Casimirs for all times, thus determining $\gamma$, the reduced infinite-dimensional phase space which constitutes the natural arena of physics. This $f(0)$ also fixes the numerical value of the conserved energy $\mathcal{H}$ and, as such, determines the constant energy hypersurface in the $\gamma$-space to which the flow is necessarily restricted. By analogy with finite-dimensional Hamiltonian systems (see Kandrup & Mahon 1994), one might expect that, when evolved into the future,
f(0) will exhibit a coarse-grained approach toward an invariant measure on this hypersurface, i.e., a suitably defined microcanonical distribution. If the flow associated with f(0) is chaotic, one might anticipate an approach toward this invariant measure that is exponential in time. If, alternatively, the flow is regular, one might instead expect a power-law approach. However, in either case one might anticipate an approach toward a “phase-mixed” invariant measure. In this context, the crucial question is then: to what extent can this invariant measure be interpreted as corresponding to a distribution function f executing phase-space oscillations about one or more equilibrium solutions f₀?

It is easy to see that, in the γ-space, there must exist one or more extremal points with δ^(1)H = 0, these corresponding to equilibrium solutions f₀ for which all the Casimirs share the same values as the Casimirs associated with f(0). Indeed, one knows that, for sufficiently smooth initial data, the CBE admits global existence (see Pfaffelmoser 1992; Schaeffer 1991), so that δf cannot diverge and, presumably, the Hamiltonian is bounded from below. However, this implies that there must exist at least one f₀, namely, the global energy minimum (although in principle the global minimum could be degenerate). The question therefore becomes: in the basin of which f₀ (or f₀'s) does the flow reside?

In principle, the evolved distribution function f could execute phase-space oscillations about any f₀ with lower energy, which one presumably depending on the initial f(0). However, one might conjecture that, if the initial f(0) is sufficiently far from any equilibrium f₀, it will execute oscillations around the global minimum f₀. The initial f(0) cannot exhibit a pointwise approach toward this, or any other, f₀. However, one might expect that, in general, the initial deviation ∂f(0) = f(0) − f₀ will exhibit nonlinear Landau damping so that, in terms of observables like the density ρ, ∂f does indeed “die away,” and one can speak of a coarse-grained approach toward the equilibrium f₀.

5. CONCLUSIONS AND UNANSWERED QUESTIONS

The aim of this paper has been to suggest a potentially fruitful way in which to visualize flows described by the CBE and, in particular, the expected coarse-grained approach toward an equilibrium. No claim is made regarding mathematical rigor, and it is not clear that all the details have been synthesized in an optimal fashion. However, the viewpoint developed here does have the advantage that it incorporates what is known rigorously about the CBE and that it provides a framework in terms of which to pose precise, well-defined questions. In this context, there are at least three basic questions which, if answered satisfactorily, would yield important insights into the physical properties of a flow generated by the CBE:

1. Will generic initial conditions exhibit effective Landau damping, thus allowing one to speak of an efficient coarse-grained evolution toward some equilibrium f₀? In the context of linear Landau damping, the answer to this question depends on the spectral properties of the linearized evolution equation. If the modes are all continuous, every initial perturbation will eventually phase mix away, so that physical observables like the density will damp to zero. If, however, some of the modes are discrete, it is possible to construct initial perturbations that do not damp away. At the present time, it is not clear whether, for realistic galactic models, the spectrum is purely continuous, although the investigation of various toy models is currently underway (D. Lynden-Bell 1997; private communication).

To the extent that N-body simulations are reliable and that, for sufficiently large N, they capture the same physics as the CBE, the fact that most initial conditions yield an efficient approach toward some statistical equilibrium can be interpreted as evidence that nonlinear Landau damping is in general very effective. However, there do exist toy models like one-dimensional gravity where one ends up with undamped oscillations. For example, the evolution of counterstreaming initial conditions in one-dimensional systems (either gravitational or electrostatic) can lead to a final state which corresponds seemingly to a distribution function f exhibiting finite-amplitude undamped oscillations about a (near-) equilibrium f₀ (see Mineau, Feix, & Rouet 1990). This toy model actually corroborates the physical intuition described in this paper in the sense that, as one would expect, the phase space contains a large “hole,” i.e., a region where f₀ → 0. Whether or not analogous results obtain for two- and three-dimensional systems is as yet unclear, although the problem is currently under investigation (Habib et al. 1998).

2. Are functional Lyapunov exponents the most appropriate way in which to identify chaos in infinite-dimensional systems and, assuming that they are, will a generic flow associated with the CBE be chaotic? Given this definition, will standard results from finite-dimensional chaos remain at least approximately valid? Although not proved for generic finite-dimensional systems, there is the physical expectation that, when evolved into the future, a chaotic initial condition will evolve toward an invariant distribution on a timescale that is related somehow to the spectrum of Lyapunov exponents. This implies however that, at asymptotically late times, one can visualize the flow as densely filling a chaotic phase-space region of finite measure. Assuming, however, that this is true, the Ergodic theorem provides important information about the statistical properties of the flow, implying the equivalence of time and phase-space averages (see Lichtenberg & Lieberman 1992).

One other point about chaos in the CBE should be stressed: the definition proposed in this paper is, at least superficially, completely decoupled from the (also interesting) question (see Udry & Pfenniger 1988; Martinet & Udry 1990; Merritt & Fridman 1996) of whether individual orbits in a self-consistent potential generated from the CBE are, or are not, chaotic. This latter question refers to the behavior of nearby trajectories in the six-dimensional particle phase space. The “natural” definition of chaos for the CBE should presumably reflect properties of the flow in the infinite-dimensional phase space of distribution functions.

3. For a specified initial f(0), toward which equilibrium f₀ will the system evolve? Given f(0), one can compute the numerical value of all possible Casimirs, thus identifying explicitly the γ-space to which the evolution is restricted. The obvious problem, then, is to identify all time-independent equilibria f₀ in γ and to determine which initial conditions correspond to which equilibria. Although unquestionably difficult, this is a problem that is both well defined mathematically and well motivated physically. Finding all equilibria is equivalent mathematically to
finding all extremal points in $\gamma$. However, to the extent that one chooses to visualize the flow as involving oscillations in the $\gamma$-space, there is no question physically but that the extremal points define "basins of attraction" associated with the oscillations.

The basic points described in this paper are easily summarized:

1. The CBE is a Hamiltonian system, albeit an unusual one. The fundamental dynamical variable is the distribution function $f$, not the particle $x$'s and $v$'s and it is not always possible (at least easily) to identify canonically conjugate variables.

2. Because the CBE is Hamiltonian, there can be no pointwise approach toward equilibrium. The best for which one can hope is a coarse-grained approach toward equilibrium.

3. Even though the phase space $\gamma$ associated with the dynamics is infinite-dimensional, one might expect that much of one's intuition from finite-dimensional systems remains valid. In particular, one might anticipate an asymptotic approach toward an invariant measure, and one might hope to make meaningful distinctions between regular and chaotic flows.

4. The phenomenon normally designated as linear Landau damping can be interpreted as a phase mixing of a continuous set of normal modes. Whether a small initial perturbation will always eventually Landau damp/phase mix away depends on whether the normal modes for the linearized perturbation equation are discrete or continuous.

5. To the extent that one's ordinary intuition about finite-dimensional phase spaces remains approximately valid, the evolution of generic initial data should be interpreted as involving nonlinear (phase space) oscillations about one or more energy extremals, which correspond to time-independent equilibria $f_0$. The phenomenon of violent relaxation should thus be interpreted as nonlinear phase-mixing/Landau damping which, if efficient, will facilitate a coarse-grained approach toward equilibrium.

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