Clustered Colouring of Graph Classes with Bounded Treedepth or Pathwidth

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Abstract. The clustered chromatic number of a class of graphs is the minimum integer \( k \) such that for some integer \( c \) every graph in the class is \( k \)-colourable with monochromatic components of size at most \( c \). We determine the clustered chromatic number of any minor-closed class with bounded treedepth, and prove a best possible upper bound on the clustered chromatic number of any minor-closed class with bounded pathwidth. As a consequence, we determine the fractional clustered chromatic number of every minor-closed class.

1 Introduction

This paper studies improper vertex colourings of graphs with bounded monochromatic degree or bounded monochromatic component size. This topic has been extensively studied recently [1–6, 8, 10, 12–21, 23–25]; see [26] for a survey.

A \( k \)-colouring of a graph \( G \) is a function that assigns one of \( k \) colours to each vertex of \( G \). In a coloured graph, a monochromatic component is a connected component of the subgraph induced by all the vertices of one colour.

A colouring has defect \( d \) if each monochromatic component has maximum degree at most \( d \). The defective chromatic number of a graph class \( \mathcal{G} \), denoted by \( \chi_{\Delta}(\mathcal{G}) \), is the minimum integer \( k \) such that, for some integer \( d \), every graph in \( \mathcal{G} \) is \( k \)-colourable with defect \( d \).

A colouring has clustering \( c \) if each monochromatic component has at most \( c \) vertices. The clustered chromatic number of a graph class \( \mathcal{G} \), denoted by \( \chi_{\star}(\mathcal{G}) \), is the minimum integer \( k \) such that, for some integer \( c \), every graph in \( \mathcal{G} \) has a \( k \)-colouring with clustering \( c \). We shall consider such colourings, where the goal is to minimise the number of colours, without optimising the clustering value.

Every colouring of a graph with clustering \( c \) has defect \( c - 1 \). Thus \( \chi_{\Delta}(\mathcal{G}) \leq \chi_{\star}(\mathcal{G}) \) for every class \( \mathcal{G} \).

The following is a well-known and important example in defective and clustered graph colouring. Let \( T \) be a rooted tree. The depth of \( T \) is the maximum number of vertices
on a root–to–leaf path in $T$. The closure of $T$ is obtained from $T$ by adding an edge between every ancestor and descendant in $T$. For $h, k \geq 1$, let $C(h, k)$ be the closure of the complete $k$-ary tree of depth $h$, as illustrated in Figure 1.

Figure 1: The standard example $C(4, 2)$.

It is well known and easily proved (see [26]) that there is no $(h - 1)$-colouring of $C(h, k)$ with defect $k - 1$, which implies there is no $(h - 1)$-colouring of $C(h, k)$ with clustering $k$. This says that if a graph class $\mathcal{G}$ includes $C(h, k)$ for all $k$, then the defective chromatic number and the clustered chromatic number are at least $h$. Put another way, define the tree-closure-number of a graph class $\mathcal{G}$ to be

$$\text{tcn}(\mathcal{G}) := \min \{ h : \exists k \ C(h, k) \notin \mathcal{G} \} = \max \{ h : \forall k \ C(h, k) \in \mathcal{G} \} + 1;$$

then

$$\chi_\star(\mathcal{G}) \geq \chi_\Delta(\mathcal{G}) \geq \text{tcn}(\mathcal{G}) - 1.$$

Our main result, Theorem 1 below, establishes a converse result for minor-closed classes with bounded treedepth. First we explain these terms. A graph $H$ is a minor of a graph $G$ if a graph isomorphic to $H$ can be obtained from some subgraph of $G$ by contracting edges. A class of graphs $\mathcal{M}$ is minor-closed if for every graph $G \in \mathcal{M}$ every minor of $G$ is in $\mathcal{M}$, and $\mathcal{M}$ is proper minor-closed if, in addition, some graph is not in $\mathcal{M}$. The connected treedepth of a graph $H$, denoted by $\overline{\text{td}}(H)$, is the minimum depth of a rooted tree $T$ such that $H$ is a subgraph of the closure of $T$. This definition is a variant of the more commonly used definition of the treedepth of $H$, denoted by $\text{td}(H)$, which equals the maximum connected treedepth of the connected components of $H$. (See [22] for background on treedepth.) If $H$ is connected, then $\text{td}(H) = \overline{\text{td}}(H)$.

In fact, $\text{td}(H) = \overline{\text{td}}(H)$ unless $H$ has two connected components $H_1$ and $H_2$ with $\text{td}(H_1) = \text{td}(H_2) = \text{td}(H)$, in which case $\overline{\text{td}}(H) = \text{td}(H) + 1$. It is convenient to work with connected treedepth to avoid this distinction. A class of graphs has bounded treedepth if there exists a constant $c$ such that every graph in the class has treedepth at most $c$. 

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Theorem 1. For every minor-closed class $\mathcal{G}$ with bounded treedepth,

$$
\chi_\Delta(\mathcal{G}) = \chi_*(\mathcal{G}) = \text{tcn}(\mathcal{G}) - 1.
$$

Our second result concerns pathwidth. A *path-decomposition* of a graph $G$ consists of a sequence $(B_1, \ldots, B_n)$, where each $B_i$ is a subset of $V(G)$ called a *bag*, such that for every vertex $v \in V(G)$, the set $\{i \in [1,n] : v \in B_i\}$ is an interval, and for every edge $vw \in E(G)$ there is a bag $B_i$ containing both $v$ and $w$. Here $[a,b] := \{a, a+1, \ldots, b\}$. The *width* of a path decomposition $(B_1, \ldots, B_n)$ is $\max\{|B_i| : i \in [1,n]\} - 1$. The *pathwidth* of a graph $G$ is the minimum width of a path-decomposition of $G$. Note that paths (and more generally caterpillars) have pathwidth 1. A class of graphs has *bounded pathwidth* if there exists a constant $c$ such that every graph in the class has pathwidth at most $c$.

Theorem 2. For every minor-closed class $\mathcal{G}$ with bounded pathwidth,

$$
\chi_\Delta(\mathcal{G}) \leq \chi_*(\mathcal{G}) \leq 2 \text{tcn}(\mathcal{G}) - 2.
$$

Theorems 1 and 2 are respectively proved in Sections 2 and 3. These results are best possible and partially resolve a number of conjectures from the literature, as we now explain.

Ossona de Mendez et al. [24] studied the defective chromatic number of minor-closed classes. For a graph $H$, let $\mathcal{M}_H$ be the class of $H$-minor-free graphs (that is, not containing $H$ as a minor). Ossona de Mendez et al. [24] proved the lower bound, $\chi_\Delta(\mathcal{M}_H) \geq \overline{td}(H) - 1$ and conjectured that equality holds.

Conjecture 3 ([24]). For every graph $H$,

$$
\chi_\Delta(\mathcal{M}_H) = \overline{td}(H) - 1.
$$

Conjecture 3 is known to hold in some special cases. Edwards et al. [10] proved it if $H = K_t$; that is, $\chi_\Delta(\mathcal{M}_{K_t}) = t - 1$, which can be thought of as a defective version of Hadwiger’s Conjecture; see [25] for an improved bound on the defect in this case. Ossona de Mendez et al. [24] proved Conjecture 3 if $\overline{td}(H) \leq 3$ or if $H$ is a complete bipartite graph. In particular, $\chi_\Delta(\mathcal{M}_{K_{s,t}}) = \min\{s,t\}$.

Norin et al. [23] studied the clustered chromatic number of minor-closed classes. They showed that for each $k \geq 2$, there is a graph $H$ with treedepth $k$ and connected treedepth $k$ such that $\chi_*(\mathcal{M}_H) \geq 2k - 2$. Their proof in fact constructs a set $\mathcal{X}$ of graphs in $\mathcal{M}_H$ with bounded pathwidth (at most $2k - 3$ to be precise) such that $\chi_*(\mathcal{X}) \geq 2k - 2$. Thus the upper bound on $\chi_*(\mathcal{G})$ in Theorem 2 is best possible.

Norin et al. [23] conjectured the following converse upper bound (analogous to Conjecture 3):

Conjecture 4 ([23]). For every graph $H$,

$$
\chi_*(\mathcal{M}_H) \leq 2 \overline{td}(H) - 2.
$$
While Conjectures 3 and 4 remain open, Norin et al. [23] showed in the following theorem that $\chi_\Delta(M_H)$ and $\chi_\star(M_H)$ are controlled by the treedepth of $H$:

**Theorem 5 ([23]).** For every graph $H$, $\chi_\star(M_H)$ is tied to the (connected) treedepth of $H$. In particular,

$$\text{td}(H) - 1 \leq \chi_\star(M_H) \leq 2^{\text{td}(H)+1} - 4.$$  

Theorem 1 gives a much more precise bound than Theorem 5 under the extra assumption of bounded treedepth.

Our third main result concerns fractional colourings. For real $t \geq 1$, a graph $G$ is **fractionally $t$-colourable with clustering $c$** if there exist $Y_1, Y_2, \ldots, Y_s \subseteq V(G)$ and $\alpha_1, \ldots, \alpha_s \in [0, 1]$ such that

1. Every component of $G[Y_i]$ has at most $c$ vertices,
2. $\sum_{i=1}^s \alpha_i \leq t$,
3. $\sum_{v \in Y_i} \alpha_i \geq 1$ for every $v \in V(G)$.

The **fractional clustered chromatic number** $\chi_\Delta^f(G)$ of a graph class $\mathcal{G}$ is the infimum of $t > 0$ such that there exists $c = c(t, \mathcal{G})$ such that every $G \in \mathcal{G}$ is fractionally $t$-colourable with clustering $c$.

**Fractionally $t$-colourable with defect $d$** and **fractional defective chromatic number** $\chi_\Delta^f(\mathcal{G})$ are defined in exactly the same way, except the condition on the component size of $G[Y_i]$ is replaced by “the maximum degree of $G[Y_i]$ is at most $d$”.

The following theorem determines the fractional clustered chromatic number and fractional defective chromatic number of any proper minor-closed class.

**Theorem 6.** For every proper minor-closed class $\mathcal{G}$,

$$\chi_\Delta^f(\mathcal{G}) = \chi_\star^f(\mathcal{G}) = \text{tcn}(\mathcal{G}) - 1.$$  

This result is proved in Section 4.

We now give an interesting example of Theorem 6.

**Corollary 7.** For every surface $\Sigma$, if $\mathcal{G}_\Sigma$ is the class of graphs embeddable in $\Sigma$, then

$$\chi_\Delta^f(\mathcal{G}_\Sigma) = \chi_\star^f(\mathcal{G}_\Sigma) = 3.$$  

**Proof.** Note that $C(3, k)$ is planar for all $k$. Thus $\text{tcn}(\mathcal{G}_\Sigma) \geq 4$. Say $\Sigma$ has Euler genus $g$. It follows from Euler’s formula that $K_{3, 2g+3} \not\in \mathcal{G}_\Sigma$. Since $K_{3, 2g+3} \subseteq C(4, 2g + 3)$, we have $C(4, 2g + 3) \not\in \mathcal{G}_\Sigma$. Thus $\text{tcn}(\mathcal{G}_\Sigma) = 4$. The result follows from Theorem 6. \qed

In contrast to Corollary 7, Dvořák and Norin [8] proved that $\chi_\star(\mathcal{G}_\Sigma) = 4$. Note that Archdeacon [2] proved that $\chi_\Delta(\mathcal{G}_\Sigma) = 3$; see [5] for an improved bound on the defect.

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1 If $c = 1$, then this corresponds to a (proper) fractional $t$-colouring, and if the $\alpha_i$ are integral, then this yields a $t$-colouring with clustering $c$.  

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2 Treedepth

Say $G$ is a subgraph of the closure of some rooted tree $T$. For each vertex $v \in V(T)$, let $T_v$ be the maximal subtree of $T$ rooted at $v$ (consisting of $v$ and all its descendants), and let $G[T_v]$ be the subgraph of $G$ induced by $V(T_v)$.

The weak closure of a rooted tree $T$ is the graph $G$ with vertex set $V(T)$, where two vertices $v, w \in V(T)$ are adjacent in $G$ whenever $v$ is a leaf of $T$ and $w$ is an ancestor of $v$ in $T$. As illustrated in Figure 2, let $W(h, k)$ be the weak closure of the complete $k$-ary tree of height $h$.

Figure 2: The weak closure $W(4, 2)$.

Note that $W(h, k)$ is a proper subgraph of $C(h, k)$ for $h \geq 3$. On the other hand, Norin et al. [23] showed that $W(h, k)$ contains $C(h, k - 1)$ as a minor for all $h, k \geq 2$. Therefore Theorem 1 is an immediate consequence of the following lemma.

**Lemma 8.** For all $d, k, h \in \mathbb{N}$ there exists $c = c(d, k, h) \in \mathbb{N}$ such that for every graph $G$ with treedepth at most $d$, either $G$ contains a $W(h, k)$-minor or $G$ is $(h - 1)$-colourable with clustering $c$.

**Proof.** Throughout this proof, $d, k$ and $h$ are fixed, and we make no attempt to optimise $c$.

We may assume that $G$ is connected. So $G$ is a subgraph of the closure of some rooted tree of depth at most $d$. Choose a tree $T$ of depth at most $d$ rooted at some vertex $r$, such that $G$ is a subgraph of the closure of $T$, and subject to this, $\sum_{v \in V(T)} \text{dist}_T(v, r)$ is minimal. Suppose that $G[T_v]$ is disconnected for some vertex $v$ in $T$. Choose such a vertex $v$ at maximum distance from $r$. Since $G$ is connected, $v \neq r$. By the choice of $v$, for each child $w$ of $v$, the subgraph $G[T_w]$ is connected. Thus, for some child $w$ of $v$, there is no edge in $G$ joining $v$ and $G[T_w]$. Let $u$ be the parent of $v$. Let $T'$ be obtained from $T$ by deleting the edge $vw$ and adding the edge $uw$, so that $w$ is a child of $u$ in $T'$. Note that $G$ is a subgraph of the closure of $T'$ (since $v$ has no neighbour in $G[T_w]$). Moreover,
dist\(_T(v, r)\) = dist\(_T(x, r)\) - 1 for every vertex \(x \in V(T_w)\), and dist\(_T(y, r)\) = dist\(_T(y, r)\) for every vertex \(y \in V(T) \setminus V(T_w)\). Hence \(\sum_{v \in V(T')} \text{dist}_T(v, r) < \sum_{v \in V(T)} \text{dist}_T(v, r)\), which contradicts our choice of \(T\). Therefore \(G[T_v]\) is connected for every vertex \(v\) of \(T\).

Consider each vertex \(v \in V(T)\). Define the level \(\ell(v) := \text{dist}_T(r, v) \in [0, d - 1]\). Let \(T_v^+\) be the subtree of \(T\) consisting of \(T_v\) plus the \(vr\)-path in \(T\), and let \(G[T_v^+]\) be the subgraph of \(G\) induced by \(V(T_v^+)\). For a subtree \(X\) of \(T\) rooted at vertex \(v\), define the level \(\ell(X) := \ell(v)\).

A ranked graph (for fixed \(d\)) is a triple \((H, L, \preceq)\) where:

- \(H\) is a graph,
- \(L : V(H) \to [0, d - 1]\) is a function,
- \(\preceq\) is a partial order on \(V(H)\) such that \(L(v) < L(w)\) whenever \(v < w\).

Here and throughout this proof, \(v < w\) means that \(v \leq w\) and \(v \neq w\). Up to isomorphism, the number of ranked graphs on \(n\) vertices is at most \(2^{(\binom{n}{2})} \cdot 3^{(\binom{n}{2})}\). For a vertex \(v\) of \(T\), a ranked graph \((H, L, \preceq)\) is said to be contained in \(G[T_v^+]\) if there is an isomorphism \(\phi\) from \(H\) to some subgraph of \(G[T_v^+]\) such that:

(A) for each vertex \(v \in V(H)\) we have \(L(v) = \ell(\phi(v))\), and

(B) for all distinct vertices \(v, w \in V(H)\) we have that \(v < w\) if and only if \(\phi(v)\) is an ancestor of \(\phi(w)\) in \(T\).

Say \((H, L, \preceq)\) is a ranked graph and \(i \in [0, d - 1]\). Below we define the \(i\)-splice of \((H, L, \preceq)\) to be a particular ranked graph \((H', L', \preceq')\), which (intuitively speaking) is obtained from \((H, L, \preceq)\) by copying \(k\) times the subgraph of \(H\) induced by the vertices \(v\) with \(L(v) > i\). Formally, let

\[
V(H') := \{\langle v, 0 \rangle : v \in V(H), L(v) \in [0, i]\} \cup \{\langle v, j \rangle : v \in V(H), L(v) \in [i + 1, d], j \in [1, k]\}.
\]

\[
E(H') := \{\langle v, 0 \rangle \langle w, 0 \rangle : vw \in E(H), L(v) \in [0, i], L(w) \in [0, i]\} \cup \{\langle v, 0 \rangle \langle w, j \rangle : vw \in E(H), L(v) \in [0, i], L(w) \in [i + 1, d], j \in [1, k]\} \cup \{\langle v, j \rangle \langle w, j \rangle : vw \in E(H), L(v) \in [i + 1, d], L(w) \in [i + 1, d], j \in [1, k]\}.
\]

Define \(L'(v, j) := L(v)\) for every vertex \((v, j) \in V(H')\). Now define the following partial order \(\preceq'\) on \(V(H'):\)

- \((v, j) \preceq' (v, j)\) for all \((v, j) \in V(H')\);
- if \(v < w\) and \(L(v), L(w) \in [0, i]\), then \((v, i) \preceq' (w, 0)\);
- if \(v < w\) and \(L(v) \in [0, i]\) and \(L(w) \in [i + 1, d]\), then \((v, 0) \preceq' (w, j)\) for all \(j \in [1, k]\); and
- if \(v < w\) and \(L(v), L(w) \in [i + 1, d]\), then \((v, j) \preceq' (w, j)\) for all \(j \in [1, k]\).

Note that if \((v, a) \preceq' (w, b)\), then \(a \leq b\) and \(v < w\) (implying \(L(v) < L(w)\)). It follows that \(\preceq'\) is a partial order on \(V(H')\) such that \(L'((v, a)) < L'((w, b))\) whenever \((v, a) \preceq' (w, b)\). Thus \((H', L', \preceq')\) is a ranked graph.
For $\ell \in [0, d - 1]$, let
\[ N_\ell := (d + 1)(h - 1)(k + 1)^{d-1-\ell}. \]
For each vertex $v$ of $T$, define the profile of $v$ to be the set of all ranked graphs $(H, L, \preceq)$ contained in $G[T^+_v]$ such that $|V(H)| \leq N_\ell(v)$. Note that if $v$ is a descendant of $u$, then the profile of $v$ is a subset of the profile of $u$. For $\ell \in [0, d - 1]$, if $N = N_\ell$ then let
\[ M_\ell := 2^{(\frac{d}{2})}d^\frac{\ell}{2}(\frac{3}{2}). \]
Then there are at most $M_\ell$ possible profiles of a vertex at level $\ell$.

We now partition $V(T)$ into subtrees. Each subtree is called a group. (At the end of the proof, vertices in a single group will be assigned the same colour.) We assign vertices to groups in non-increasing order of their distance from the root. Initialise this process by placing each leaf $v$ of $T$ into a singleton group. We now show how to determine the group of a non-leaf vertex. Let $v$ be a vertex not assigned to a group at maximum distance from $r$. So each child of $v$ is assigned to a group. Let $Y_v$ be the set of children $y$ of $v$, such that the number of children of $v$ that have the same profile as $y$ is in the range $[1, k - 1]$. If $Y_v = \emptyset$ start a new singleton group $\{v\}$. If $Y_v \neq \emptyset$ then merge all the groups rooted at vertices in $Y_v$ into one group including $v$. This defines our partition of $V(T)$ into groups. Each group $X$ is rooted at the vertex in $X$ closest to $r$ in $T$. A group $Y$ is above a distinct group $X$ if the root of $Y$ is on the path in $T$ from the root of $X$ to $r$.

The next claim is the key to the remainder of the proof.

**Claim 1.** Let $wv \in E(T)$ where $w$ is the parent of $v$, and $u$ is in a different group to $v$. Then for every ranked graph $(H, L, \preceq)$ in the profile of $v$, the $\ell(u)$-splice of $(H, L, \preceq)$ is in the profile of $u$.

**Proof.** Since $(H, L, \preceq)$ is in the profile of $v$, there is an isomorphism $\phi$ from $H$ to some subgraph of $G[T^+_v]$ such that for each vertex $x \in V(H)$ we have $L(x) = \ell(\phi(x))$, and for all distinct vertices $x, y \in V(H)$ we have that $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi(y)$ in $T$.

Since $u$ and $v$ are in different groups, there are $k$ children $y_1, \ldots, y_k$ of $u$ (one of which is $v$) such that the profiles of $y_1, \ldots, y_k$ are equal. Thus $(H, L, \preceq)$ is in the profile of each of $y_1, \ldots, y_k$. That is, for each $j \in [1, k]$, there is an isomorphism $\phi_j$ from $H$ to some subgraph of $G[T^+_{y_j}]$ such that for each vertex $x \in V(H)$ we have $L(x) = \ell(\phi_j(x))$, and for all distinct vertices $x, y \in V(H)$ we have that $x \prec y$ if and only if $\phi_j(x)$ is an ancestor of $\phi_j(y)$ in $T$.

Let $(H', L', \preceq')$ be the $\ell(u)$-splice of $(H, L, \preceq)$. We now define a function $\phi'$ from $V(H')$ to $V(G[T^+_u])$. For each vertex $(x, 0)$ of $H'$ (thus with $x \in V(H)$ and $L(x) \in [0, \ell(u)]$), define $\phi'((x, 0)) := \phi(x)$. For every other vertex $(x, j)$ of $H'$ (thus with $x \in V(H)$ and $L(x) \in [\ell(u) + 1, d - 1]$ and $j \in [1, k]$), define $\phi'((x, j)) := \phi_j(x)$.

We now show that $\phi'$ is an isomorphism from $H'$ to a subgraph of $G[T^+_u]$. Consider an edge $(x, a)(y, b)$ of $H'$. Thus $xy \in E(H)$. It suffices to show that $\phi'((x, a))\phi'((y, b)) \in G[T^+_u]$. If $x \prec y$, then $x$ is an ancestor of $y$ in the profile of $u$. Thus $\phi'(x) \preceq \phi'(y)$, and $\phi'(x) \neq \phi'(y)$. Thus there are at most $M_\ell$ different profiles for vertices at level $\ell$. Therefore, for each profile $\phi'(x)$, there are at most $\binom{k}{\ell}$ different profiles $\phi'(y)$ such that $\phi'(x) \preceq \phi'(y)$ and $\phi'(x) \neq \phi'(y)$.
Thus $0((x, a)) = 0(x)$ and $0((y, b)) = 0(y)$. Since $0$ is an isomorphism to a subgraph of $0[0^+]\), we have $0(x)0(y) \in E(0[0^+])$, which is a subgraph of $0[0^+]\). Hence $0((x, a))0((y, b)) \in E(0[0^+])$, as desired. Now suppose that $a = 0$ and $b \in [1, k]$. Thus $0((x, a)) = 0(x)$ and $0((y, b)) = 0(y)$. Moreover, both $0(0(x))$ and $0(0(y))$ equal $0(x) \in [0, 0(u)]$. There is only vertex $z$ in $0^+$ with $0(z)$ equal to a specific number in $[0, 0(u)]$. Thus $0((x, a)) = 0(x) = 0(y) = z$. Since $0(y)$ is an isomorphism to a subgraph of $0[0^+]\), we have $0(y)0(y) \in E(0[0^+])$, which is a subgraph of $0[0^+]\). Hence $0((x, a))0((y, b)) \in E(0[0^+])$, as desired. Finally, suppose that $a = b \in [1, k]$. Thus $0((x, a)) = 0_a(x)$ and $0((y, b)) = 0_a(y)$. Since $0_a$ is an isomorphism to a subgraph of $0[0^+]\), we have $0_a(x)0_a(y) \in E(0[0^+])$, which is a subgraph of $0[0^+]\). Hence $0((x, a))0((y, b)) \in E(0[0^+])$, as desired. This shows that $0$ is an isomorphism from $0'$ to a subgraph of $0[0^+]\).

We now verify property (A) for $(0', L', \preceq')$. For each vertex $(x, 0)$ of $0'$ (thus with $x \in V(0)$ and $0(x) \in [0, 0(u)])$ we have $0'(0((x, 0))) = 0(x) = 0(0((x, 0)))$, as desired. For every other vertex $(x, j)$ of $0'$ (thus with $x \in V(0)$ and $0(x) \in [0(u) + 1, d^d] - 1$ and $j \in [1, k]$) we have $0'(0((x, j))) = 0(x) = 0(0((x, j)))$, as desired. Hence property (A) is satisfied for $(0', L', \preceq')$.

We now verify property (B) for $(0', L', \preceq')$. Consider distinct vertices $(x, a), (y, b) \in V(0')$. First suppose that $a = 0$ and $b = 0$. Then $(x, a) \not\preceq' (y, b)$ if and only if $x < y$ if and only if $0(x)$ is an ancestor of $0(y)$ in $0$ if and only if $0((x, a))$ is an ancestor of $0((y, b))$ in $0$, as desired. Now suppose that $a = 0$ and $b \in [1, k]$. Then $(x, a) \not\preceq' (y, b)$ if and only if $x < y$ if and only if $0(x)$ is an ancestor of $0(y)$ in $0$ if and only if $0((x, a))$ is an ancestor of $0((y, b))$ in $0$, as desired. Now suppose that $a = b \in [1, k]$. Then $(x, a) \not\preceq' (y, b)$ if and only if $x < y$ if and only if $0(x)$ is an ancestor of $0(y)$ in $0$ if and only if $0((x, a))$ is an ancestor of $0((y, b))$ in $0$, as desired. Finally, suppose that $a, b \in [1, k]$ and $a \neq b$. Then $(x, a)$ and $(y, b)$ are incomparable under $\preceq'$, and $0((x, a))$ and $0((y, b))$ in $0$ are unrelated in $0$, as desired. Hence property (B) is satisfied for $(0', L', \preceq')$.

So $0'$ is an isomorphism from $0'$ to a subgraph of $0[0^+]\]$ satisfying properties (A) and (B). Thus $(0', L', \preceq')$ is contained in $0[0^+]\]$, as desired. Since $(0, L, \preceq)$ is in the profile of $\nu$, we have $|V(0)| \leq (d + 1)(h - 1)(k + 1)^h - \ell(v)$. Since $|V(0')| \leq (k + 1)|V(0)|$ and $\ell(u) = \ell(v) - 1$, we have $|V(0')| \leq (d + 1)(h - 1)(k + 1)^h - \ell(v) = (d + 1)(h - 1)(k + 1)^h - \ell(u)$. Thus $(0', L', \preceq')$ is in the profile of $u$. 

The proof now divides into two cases. If some group $X_0$ is adjacent in $0$ to at least $h - 1$ other groups above $X_0$, then we show that $0$ contains $W(h, k)$ as a minor. Otherwise, every group $X$ is adjacent in $0$ to at most $h - 2$ other groups above $X$, in which case we show that $0$ is $(h - 1)$-colourable with bounded clustering.
Finding the Minor

Suppose that some group \( X_0 \) is adjacent in \( G \) to at least \( h - 1 \) other groups \( X_1, \ldots, X_{h-1} \) above \( X_0 \). We now show that \( G \) contains \( W(h, k) \) as a minor; refer to Figure 3. For \( i \in [1, h - 1] \), since \( X_i \) is above \( X_0 \), the root \( v_i \) of \( X_i \) is on the \( v_0r \)-path in \( T \). Without loss of generality, \( v_0, v_1, \ldots, v_{h-1} \) appear in this order on the \( v_0r \)-path in \( T \). For \( i \in [1, h - 1] \), let \( w_i \) be a vertex in \( X_i \) adjacent to some vertex \( z_i \) in \( X_0 \); since \( G \) is a subgraph of the closure of \( T \), \( w_i \) is on the \( v_0r \)-path in \( T \). For \( i \in [0, h - 2] \), let \( u_i \) be the parent of \( v_i \) in \( T \) (which exists since \( v_{h-2} \neq r \)). So \( u_i \) is not in \( X_i \) (but may be in \( X_{i+1} \)). Note that \( v_0, u_0, w_1, v_1, u_1, \ldots, w_{h-2}, v_{h-2}, u_{h-2}, w_{h-1}, v_{h-1} \) appear in this order on the \( v_0r \)-path in \( T \), where \( v_0, v_1, \ldots, v_{h-1} \) are distinct (since they are in distinct groups).

![Figure 3: Construction of a \( W(4, k) \) minor (where \( u_i \) might be in \( X_{i+1} \)).](image)

Let \( P_j \) be the \( z_jr \)-path in \( T \) for \( j \in [1, h - 1] \). Let \( H_0 \) be the graph with \( V(H_0) := V(P_1 \cup \cdots \cup P_{h-1}) \) and \( E(H_0) := \{ z_jw_j : j \in [1, h - 1] \} \). Define the function \( L_0 : V(H_0) \to [0, d - 1] \) by \( L_0(x) := \ell(x) \) for each \( x \in V(H_0) \). Define the partial order \( \preceq_0 \) on \( V(H_0) \), where \( x \preceq_0 y \) if and only if \( x \) is ancestor of \( y \) in \( T \). Thus \( (H_0, L_0, \preceq_0) \) is a ranked graph. By construction, \( (H_0, L_0, \preceq_0) \) is contained in \( G[T_{v_0}^+] \). Since \( H_0 \) has less than \( (d + 1)(h - 1) \) vertices, \( H_0 \) is in the profile of \( v_0 \). For \( i = 0, 1, \ldots, h - 2 \), let \( (H_{i+1}, L_{i+1}, \preceq_{i+1}) \) be the \( \ell(u_i) \)-splice of \( (H_i, L_i, \preceq_i) \).

By induction on \( i \), using Claim 1 at each step and since \( G[T_{u_i}^+] \subseteq G[T_{v_i}^+] \), we conclude that for each \( i \in [0, h - 1] \), the ranked graph \( (H_i, L_i, \preceq_i) \) is in the profile of \( v_i \). In particular, \( (H_{h-1}, L_{h-1}, \preceq_{h-1}) \) is in the profile of \( v_{h-1} \), and \( H_{h-1} \) is isomorphic to a subgraph
of $G$. Note that each vertex of $H_{h-1}$ is of the form $((\ldots (x, d_1), d_2), \ldots, d_{h-1})$ for some $x \in V(H_0)$ and $d_1, \ldots, d_{h-1} \in [0, k]$. For brevity, call such a vertex $x\langle d_1, \ldots, d_{h-1}\rangle$. Note that if $x = w_j$ for some $j \in [1, h-1]$, then $d_1 = \cdots = d_j = 0$ (since $w_j$ is above $u_i$ whenever $i < j$, and $(H_{i+1}, L_{i+1}, \preceq_{i+1})$ is the $\ell(u_i)$-splice of $(H_i, L_i, \preceq_i)$).

For $x \in V(H_0)$, let $\Lambda_x$ be the set of vertices $x\langle d_1, \ldots, d_{h-1}\rangle$ in $H_{h-1}$. By construction, no two vertices in $\Lambda_x$ are comparable under $\preceq_{h-1}$. Therefore, by property (B), $V(T_a) \cap V(T_b) = \emptyset$ for all distinct $a, b \in \Lambda_x$. In particular, $V(T_a) \cap V(T_h) = \emptyset$ for all distinct $a, b \in \Lambda_{v_0}$. As proved above, $G[T_a]$ is connected for each $a \in V(T)$. Let $G'$ be the graph obtained from $G$ by contracting $G[T_a]$ into a single vertex $\alpha\langle d_1, \ldots, d_{h-1}\rangle$, for each $a = v_0\langle d_1, \ldots, d_{h-1}\rangle \in \Lambda_{v_0}$. So $G'$ is a minor of $G$.

Let $U$ be the tree with vertex set \[
\{(d_1, \ldots, d_{h-1}) : \exists j \in [0, h-1] d_1 = \cdots = d_j = 0 \text{ and } d_{j+1}, \ldots, d_{h-1} \in [1, k]\},
\]
where the parent of $(0, \ldots, 0, d_{j+1}, d_{j+2}, \ldots, d_{h-1})$ is $(0, \ldots, 0, d_{j+2}, \ldots, d_{h-1})$. Then $U$ is isomorphic to the complete $k$-tree of height $h$ rooted at $(0, \ldots, 0)$. We now show that the weak closure of $U$ is a subgraph of $G'$, where each vertex $(0, \ldots, 0, d_{j+1}, \ldots, d_{h-1})$ of $U$ with $j \in [1, h-1]$ is mapped to vertex $w_j\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\rangle$ of $G'$, and each other vertex $(d_1, \ldots, d_{h-1})$ of $U$ is mapped to $\alpha\langle d_1, \ldots, d_{h-1}\rangle$ of $G'$. For all $d_1, \ldots, d_{h-1} \in [1, k]$ and $j \in [1, h-1]$ the vertex $z_j\langle d_1, \ldots, d_{h-1}\rangle$ of $G$ is contracted into the vertex $\alpha\langle d_1, \ldots, d_{h-1}\rangle$ of $G'$. By construction, $z_j\langle d_1, \ldots, d_{h-1}\rangle$ is adjacent to $w_j\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\rangle$ in $G$. So $\alpha\langle d_1, \ldots, d_{h-1}\rangle$ is adjacent to $w_j\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\rangle$ in $G'$. This implies that the weak closure of $U$ (that is, $W(h, k)$) is isomorphic to a subgraph of $G'$, and is therefore a minor of $G$.

Finding the Colouring

Now assume that every group $X$ is adjacent in $G$ to at most $h - 2$ other groups above $X$. Then $(h - 1)$-colour the groups in order of distance from the root, such that every group $X$ is assigned a colour different from the colours assigned to the neighbouring groups above $X$. Assign each vertex within a group the same colour as that assigned to the whole group. This defines an $(h - 1)$-colouring of $G$.

Consider the function $s : [0, d - 1] \to \mathbb{N}$ recursively defined by
\[
s(\ell) := \begin{cases} 
1 & \text{if } \ell = d - 1 \\
(k - 1) \cdot M_{\ell+1} \cdot s(\ell + 1) & \text{if } \ell \in [0, d - 2].
\end{cases}
\]

Then every group at level $\ell$ has at most $s(\ell)$ vertices. By construction, our $(h - 1)$-colouring of $G$ has clustering $s(0)$, which is bounded by a function of $d, k$ and $h$, as desired. \hfill \Box

3 Pathwidth

The following lemma of independent interest is the key to proving Theorem 2. Note that Eppstein [11] independently discovered the same result (with a slightly weaker bound
Every graph with pathwidth at most $w$ has a vertex 2-colouring such that each monochromatic path has at most $(w + 3)^w$ vertices. 

Proof. We proceed by induction on $w \geq 1$. Every graph with pathwidth 1 is a caterpillar, and is thus properly 2-colourable. Now assume $w \geq 2$ and the result holds for graphs with pathwidth at most $w - 1$. Let $G$ be a graph with pathwidth at most $w$. Let $(B_1, \ldots, B_n)$ be a path-decomposition of $G$ with width at most $w$. Let $t_1, t_2, \ldots, t_m$ be a maximal sequence such that $t_1 = 1$ and for each $i \geq 2$, $t_i$ is the minimum integer such that $B_{t_i} \cap B_{t_i-1} = \emptyset$. For odd $i$, colour every vertex in $B_i$ ‘red’. For even $i$, colour every vertex in $B_i$ ‘blue’. Since $B_i \cap B_{t_i-1} = \emptyset$ for $i \geq 2$, no vertex is coloured twice. Let $G'$ be the subgraph of $G$ induced by the uncoloured vertices. By the choice of $B_i$, for $i \geq 2$ each bag $B_j$ with $j \in [t_i-1+1, t_i-1]$ intersects $B_{t_i-1}$. Thus $(B_1 \cap V(G'), \ldots, B_n \cap V(G'))$ is a path-decomposition of $G'$ of width at most $w - 1$. By induction, $G'$ has a vertex 2-colouring such that each monochromatic path has at most $(w + 3)^{w-1}$ vertices. Since $B_{t_i} \cup B_{t_i+2}$ separates $B_{t_{i+1}} \cup \cdots \cup B_{t_{i+2}-1}$ from the rest of $G$, each monochromatic component of $G$ is contained in $B_{t_{i+1}} \cup \cdots \cup B_{t_{i+2}-1}$ for some $i \in [0, n - 2]$. Consider a monochromatic path $P$ in $G[B_{t_{i+1}} \cup \cdots \cup B_{t_{i+2}-1}]$. Then $P$ has at most $w + 1$ vertices in $B_{t_{i+1}}$. Note that $P - B_{t_{i+1}}$ is contained in $G'$. Thus $P$ consists of up to $w + 2$ monochromatic subpaths in $G'$ plus $w + 1$ vertices in $B_{t_{i+1}}$. Hence $P$ has at most $(w + 2)(w + 3)^{w-1} + (w + 1) < (w + 3)^w$ vertices. 

Nešetřil and Ossona de Mendez [22] showed that if a graph $G$ contains no path on $k$ vertices, then $td(G) < k$ (since $G$ is a subgraph of the closure of a DFS spanning tree with height at most $k$). Thus Lemma 9 implies:

Corollary 10. Every graph with pathwidth at most $w$ has a vertex 2-colouring such that each monochromatic component has treedepth at most $(w + 3)^w$.

Proof of Theorem 2. Let $\mathcal{G}$ be a minor-closed class of graphs, each with pathwidth at most $w$. Let $h$ be the minimum integer such that $C(h, k) \notin \mathcal{G}$ for some $k \in \mathbb{N}$. Consider $G \in \mathcal{G}$. Thus $W(h, k + 1)$ is not a minor of $G$ (since $C(h, k)$ is a minor of $W(h, k + 1)$, as noted above). By Corollary 10, $G$ has a vertex 2-colouring such that each monochromatic component $H$ of $G$ has treedepth at most $(w + 3)^w$. Thus $W(h, k + 1)$ is not a minor of $H$. By Lemma 8, $H$ is $(h - 1)$-colourable with clustering $c((w + 3)^w, k + 1, h)$. Taking a product colouring, $G$ is $(2h - 2)$-colourable with clustering $c((w + 3)^w, k + 1, h)$. Hence $\chi_{\Delta}(\mathcal{G}) \leq \chi_4(\mathcal{G}) \leq 2h - 2$. 

Note that Lemma 9 cannot be extended to the setting of bounded tree-width graphs: Esperet and Joret (see [17, Theorem 4.1]) proved that for all positive integers $w$ and $d$ there exists a graph $G$ with tree-width at most $w$ such that for every $w$-colouring of $G$ there exists a monochromatic component of $G$ with diameter greater than $d$ (and thus with a monochromatic path on more than $d$ vertices, and thus with treedepth at least $\log_2 d$).
4 Fractional Colouring

This section proves Theorem 6. The starting point is the following key result of Dvořák and Sereni [9].

**Theorem 11 ([9]).** For every proper minor-closed class $\mathcal{G}$ and every $\delta > 0$ there exists $d \in \mathbb{N}$ satisfying the following. For every $G \in \mathcal{G}$ there exist $s \in \mathbb{N}$ and $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that:

- $td(G[X_i]) \leq d$, and
- every $v \in V(G)$ belongs to at least $(1 - \delta)s$ of these sets.

We now prove a lower bound on the fractional defective chromatic number of the closure of complete trees of given height.

**Lemma 12.** Let $\mathcal{C}_h := \{C(h, k)\}_{k \in \mathbb{N}}$. Then $\chi^f_d(\mathcal{C}_h) \geq h$.

**Proof.** We show by induction on $h$ that if $C(h, k)$ is fractionally $t$-colourable with defect $d$, then $t \geq h - (h - 1)d/k$. This clearly implies the lemma. The base case $h = 1$ is trivial.

For the induction step, suppose that $G := C(h, k)$ is fractionally $t$-colourable with defect $d$. Thus there exist $Y_1, Y_2, \ldots, Y_s \subseteq V(G)$ and $\alpha_1, \ldots, \alpha_s \in [0, 1]$ such that:

- every component of $G[Y_i]$ has maximum degree at most $d$,
- $\sum_{i=1}^s \alpha_i \leq t$, and
- $\sum_{v \in Y_i} \alpha_i \geq 1$ for every $v \in V(G)$.

Let $r$ be the vertex of $G$ corresponding to the root of the complete $k$-ary tree and let $H_1, \ldots, H_k$ be the components of $G - r$. Then each $H_i$ is isomorphic to $C(h - 1, k)$. Let $J_0 := \{ j : r \in Y_j \}$, and let $J_i := \{ j : Y_j \cap V(H_i) \neq \emptyset \}$ for $i \in [1, k]$. Denote $\sum_{j \in J_i} \alpha_j$ by $\alpha(J_i)$ for brevity. Thus $\alpha(J_0) \geq 1$. For $i \in [1, k]$, the subgraph $H_i$ is isomorphic to $C(h, k)$ and thus $\alpha(J_i) \geq h - 1 - (h - 2)d/k$ by the induction hypothesis. Thus

\[(k - d)\alpha(J_0) + \sum_{i=1}^k \alpha(J_i) \geq (k - d) + k(h - 1) - (h - 2)d = kh - (h - 1)d.\]

If $j \in J_0$ then $Y_j$ intersects at most $d$ of $H_1, \ldots, H_k$ (since $G[Y_j]$ has maximum degree at most $d$). Thus every $\alpha_j$ appears with coefficient at most $k$ in the left side of the above inequality, implying

\[(k - d)\alpha(J_0) + \sum_{i=1}^k \alpha(J_i) \leq k \sum_{i=1}^s \alpha_i \leq kt.\]

Combining the above inequalities yields the claimed bound on $t$. \hfill $\Box$

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1 Dvořák and Sereni [9] expressed their result in the terms of "treedepth fragility". The sentence "proper minor-closed classes are fractionally treedepth-fragile" after Theorem 31 in [9] is equivalent to Theorem 11. Informally speaking, Theorem 11 shows that the fractional "treedepth" chromatic number of every minor-closed class equals 1.
Proof of Theorem 6. By Lemma 12,
\[ \chi^{f}_{\Delta}(G) \geq \chi_{\Delta}(G) \geq tcn(G) - 1. \]
It remains to show that \( \chi^{f}_{\Delta}(G) \leq tcn(G) - 1 \). Equivalently, we need to show that for all \( h, k \in \mathbb{N} \) and \( \varepsilon > 0 \), if \( C(h, k) \notin G \) then there exists \( c \) such that every graph in \( G \) is fractionally \( (h - 1 + \varepsilon) \)-colourable with clustering \( c \). This is trivial for \( h = 1 \), and so we assume \( h \geq 2 \).

Let \( d \in \mathbb{N} \) satisfy the conclusion of Theorem 11 for the class \( G \) and \( \delta = 1 - \frac{1}{1+\varepsilon/(h-1)}. \) Choose \( c = c(d, k + 1, h) \) to satisfy the conclusion of Lemma 8. We show that \( c \) is as desired.

Consider \( G \in G \). By the choice of \( d \) there exists \( s \in \mathbb{N} \) and \( X_1, X_2, \ldots, X_s \subseteq V(G) \) such that:
- \( td(G[X_i]) \leq d \), and
- every \( v \in V(G) \) belongs to at least \( (1 - \delta)s \) of these sets.

Since \( C(h, k) \notin G \), we have \( W(h, k + 1) \notin G \), and by the choice of \( c \), for each \( i \in [1, s] \) there exists a partition \( (Y^1_i, Y^2_i, \ldots, Y^{h-1}_i) \) of \( X_i \) such that every component of \( G[Y^j_i] \) has at most \( c \) vertices. Every vertex of \( G \) belongs to at least \( (1 - \delta)s \) sets \( Y^j_i \) where \( i \in [1, s] \) and \( j \in [1, h - 1] \). Considering these sets with equal coefficients \( \alpha^j_i := \frac{1}{(1 - \delta)s} \), we conclude that \( G \) is fractionally \( \frac{h-1}{1+\varepsilon} \)-colourable with clustering \( c \), as desired (since \( \frac{h-1}{1+\varepsilon} = h - 1 + \varepsilon \)). \( \square \)

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