Symmetry analysis for the 2 + 1 generalized quantum Zakharov-Kuznetsov equation

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Abstract

We solve the group classification problem for the 2+1 generalized quantum Zakharov-Kuznetsov equation. Particularly we consider the generalized equation \( u_t + f(u) u_z + u_{zzz} + u_{xxz} = 0 \), and the time-dependent Zakharov-Kuznetsov equation \( u_t + \delta(t) u u_z + \lambda(t) u_{zzz} + \varepsilon(t) u_{xxz} = 0 \). Function \( f(u) \) and \( \delta(t) \), \( \lambda(t) \), \( \varepsilon(t) \) are determine in order the equations to admit additional Lie symmetries. Finally, we apply the Lie invariants to find similarity solutions for the generalized quantum Zakharov-Kuznetsov equation.

Keywords: Quantum Zakharov-Kuznetsov equation, Lie symmetries; Similarity transformations; exact solutions.

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1 Introduction

Lie symmetry analysis is a powerful tool for the study of nonlinear differential equations [1–4]. The pioneer approach established by Sophus Lie is based on the determination of one-parameter point transformations which leave invariant a given differential equation. The existence of transformations which leave invariant a differential equation indicates the existence of invariant functions which can be used to write the corresponding differential into a simpler form or into the form of another, well-known, differential equation. The theory of symmetries provide a systematic way which has been applied the last decades in a plethora of differential equations in all areas of applied mathematics, we refer the reader to [5–16] and references therein. For other methods on the derivation of analytic solutions for differential equations we refer the reader in [17–21] and references therein.

In [22], Ovsiannikov classified all forms of the nonlinear heat equation $u_t = (f(u) u_x)_x$ according to the admitted Lie algebra. Since then, the classification problem has been widely studied in the literature [23–31].

In this work we are interested on the Lie symmetry analysis for the 2 + 1 quantum Zakharov-Kuznetsov (qZK)

$$u_t + uu_z + u_{zzz} + u_{xxx} = 0.$$ (1)

The qZK equation describes weakly nonlinear ion–acoustic waves in the presence of an uniform dense magnetic field. The quantum plasma has various applications in many physical systems. Hence the qZK is an equation of special interest. The Lie symmetry analysis for the Zakharov-Kuznetsov equation, without the quantum terms, has been studied before in [32]. The Lie symmetries for the fractional differential Zakharov-Kuznetsov (ZK) were found in [33], while for a modified ZK equation the symmetry analysis was performed in [34]. As far as the 3 + 1 qZK equation is concerned, the Lie point symmetries were found for the first time in [35]. Finally the conservation laws for the qZK were constructed for the first time in [36].

In this work we extend our analysis and inspired by [37] we consider the generalized 2 + 1 qZK equation

$$u_t + f(u) u_z + u_{zzz} + u_{xxx} = 0,$$ (2)

where $f(u)$ is an arbitrary function. Function $f(u)$ is determined by the group properties of the differential equation (2) as established by Ovsiannikov.

In addition we consider the 2 + 1 qZK equation with time-varying coefficients defined as [38]

$$u_t + \delta (t) uu_z + \lambda (t) u_{zzz} + \varepsilon (t) u_{xxx} = 0.$$ (3)

Again the time-varying coefficients are constrained according to the admitted Lie symmetries. The plan of the paper is as follows.

In Section 3 we present the basic properties and definitions for the theory of symmetries for differential equations. The Lie point symmetries for the 2 + 1 qZK equation are determined in Section 3. We find that the 2 + 1 qZK equation admits five Lie point symmetries. The commutators and the adjoint representation of the admitted Lie symmetries are calculated and are used to write the one-dimensional optimal system. The symmetry vectors are used to define similarity transformations and to write closed-form solutions. Specifically, the similarity transformations are used to reduce the number of independent variables in the given differential equation. By applying two similarity transformations we end with an ordinary differential equation. We show that periodic solutions which belong to the family to travelling-wave solutions exist. In Section 4 we present the
complete classification scheme for the generalized 2 + 1 qZK equation (2). The results are given in a proposition and a table. As far as the time-dependent 2 + 1 qZK equation (3) is concerned the Lie point symmetries are studied in Section 5. Finally, in Section 6 we summarize our results and we draw our conclusions.

2 Preliminaries

In this Section we present the basic properties and definitions for the theory of Lie symmetries of differential equations. Consider the function \( \Phi \) which describes the map of an one-parameter point transformation such as

\[
\begin{align*}
t' &= t' + \varepsilon \xi (t, x^i, u) \\
x'^i &= x^i + \varepsilon \xi^i (t, x^i, u) \\
u' &= u + \varepsilon \eta (t, x^i, u)
\end{align*}
\]

and generator

\[
X = \frac{\partial t'}{\partial \varepsilon} \partial_t + \frac{\partial x'^i}{\partial \varepsilon} \partial_x + \frac{\partial u'}{\partial \varepsilon} \partial_u,
\]

where \( \varepsilon \) is the parameter of smallness; \( x^i = (x, z) \), where \( u (t, x^i) \) is the dependent function and \( (t, x, z) \) are the independent variables.

Let \( u (t, x^i) \) be a solution for the differential equation \( \mathcal{H} (u, u_t, u_x, ...) = 0 \). Therefore under the one-parameter map \( \Phi \), function \( u' (x'^i) = \Phi (u (x^i)) \) is a solution for the differential equation \( \mathcal{H} = 0 \), if and only if the differential equation is also invariant under the action of the map, \( \Phi \), that is, the following condition holds

\[
\Phi (\mathcal{H} (u, u_t, u_x, ...)) = 0.
\]

For every map \( \Phi \) in which the latter condition holds it means that the generator \( X \) is a Lie point symmetry for the differential equation while

\[
X^{[n]} (\mathcal{H}) = 0
\]

holds, where \( X^{[n]} \) describes the \( n^{th} \) prolongation/extension of the symmetry vector in the jet-space of variables, \( \{ t, x^i, u, u_t, u_x, u_{ij}, ... \} \).

The importance of the existence of a Lie symmetry for a given differential equation is that from the associated Lagrange’s system,

\[
\frac{dt}{\xi^t} = \frac{dx^i}{\xi^i} = \frac{du}{\eta},
\]

invariants, \( U^{[0]} (t, x^i, u) \) are able to be determined which can be used to reduce the number of the independent variables of the differential equation and lead to the construction of similarity solutions. As far as partial differential equations are concerned, the application of the Lie invariants reduces the number of the independent variables. On the other hand, in the case of ordinary differential equations the Lie invariants are applied to reduce the order for the differential equation.

The admitted symmetry vectors of a given set of differential equations constitute a closed-group known as a Lie group. The main application of the Lie symmetries is the determination of solutions known as similarity solutions and follow from the application of the Lie invariants in the differential equations. However, in order
to classify all the possible similarity transformations and solutions the one-dimensional optimal system should be calculated [3].

Assume the $n$-dimensional Lie algebra $G_n$ with elements \( \{X_1, X_2, \ldots, X_n\} \) and structure constants $C_{ijk}^j$. We define the two symmetry vectors

\[
Z = \sum_{i=1}^{n} a_i X_i, \quad W = \sum_{i=1}^{n} b_i X_i, \quad a_i, b_i \text{ are constants.} \tag{11}
\]

and we define the operator

\[
Ad(\exp(\epsilon X_i)) X_j = X_j - \epsilon [X_i, X_j] + \frac{1}{2}\epsilon^2 [X_i, [X_i, X_j]] + \ldots \tag{12}
\]

known as the adjoint representation, in which $[X_i, X_j]$ is the Lie Bracket.

We say that the vectors $Z$ and $W$ are equivalent if and only if [4]

\[
W = \sum_{j=i}^{n} Ad(\exp(\epsilon_i X_i)) Z \tag{13}
\]

or

\[
W = cZ, \quad c = \text{const} \quad \text{that is } \quad b_i = ca_i. \tag{14}
\]

The one-dimensional subalgebras of $G_n$ which are not related through the adjoint representation form the one-dimensional optimal system. The determination of the one-dimensional system is essential in order to perform a complete classification of all the possible similarity transformations and solutions.

### 3 Symmetry analysis for the qZK

For the qZK equation (1) the application of the Lie theory provides that qZK admits as Lie symmetries the elements of the five dimensional Lie algebra

\[
X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_z, \\
X_4 = t\partial_z + \partial_u, \quad X_5 = 3t\partial_t + x\partial_x + z\partial_z.
\]

The commutators and the adjoint representation for the admitted Lie symmetries are presented in Tables 1 and 2 respectively. We observe that the Lie symmetries form the $A_4 \oplus A_1$ Lie algebra in the Morozov-Mubarakzyanov classification [39,42].

The one-dimensional optimal system consists of the following vector fields

\[
\{X_1\}, \quad \{X_2\}, \quad \{X_3\}, \quad \{X_4\}, \quad \{X_5\}, \\
\{X_1 + \alpha X_2\}, \quad \{X_1 + \alpha X_3\}, \quad \{X_1 + \alpha X_4\}, \\
\{X_2 + \alpha X_3\}, \quad \{X_2 + \alpha X_4\}, \quad \{X_3 + \alpha X_4\}, \\
\{X_1 + \alpha X_2 + \beta X_4\}, \quad \{X_2 + \alpha X_3 + \beta X_4\}, \\
\{X_1 + \alpha X_2 + \beta X_3\}.
\]

We proceed with our analysis by applying the Lie symmetry vectors in order to reduce the partial differential equation (1) into an ordinary differential equation. Indeed, in order to perform such reduction we should apply Lie point symmetries to perform the reduction process. Some closed-form similarity solutions are presented.
Table 1: Commutator table for the admitted Lie point symmetries of the qKZ equation

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|--------------|-------|-------|-------|-------|-------|
| $X_1$        | 0     | 0     | 0     | $X_3$ | $3X_1$ |
| $X_2$        | 0     | 0     | 0     | $X_4$ | $X_2$  |
| $X_3$        | 0     | 0     | 0     | $X_5$ | $X_3$  |
| $X_4$        | $-X_3$| 0     | 0     | 0     | $-2X_4$|
| $X_5$        | $-3X_1$| $-X_2$| $-X_3$| 2$X_4$| 0     |

Table 2: Adjoint representation for the admitted Lie point symmetries of the qKZ equation

| $Ad (e^{(\varepsilon X_i)})$ $X_j$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|-------------------------------------|-------|-------|-------|-------|-------|
| $X_1$                               | $X_1$ | $X_2$ | $X_3$ | $X_4 - \varepsilon X_3$ | $X_5 - 3\varepsilon X_1$ |
| $X_2$                               | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5 - \varepsilon X_2$ |
| $X_3$                               | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5 - \varepsilon X_3$ |
| $X_4$                               | $X_1 + \varepsilon X_3$ | $X_2$ | $X_3$ | $X_4$ | $X_5 + 2\varepsilon X_4$ |
| $X_5$                               | $e^{3\varepsilon X_1}$ | $e^{\varepsilon X_2}$ | $e^{\varepsilon X_3}$ | $e^{-2\varepsilon X_4}$ | $X_5$ |

3.1 Similarity transformations

We proceed by presenting the similarity transformations which follow by the two-dimensional Lie algebras $\{X_4, X_5\}, \{X_1 + \beta X_2, X_1 + \gamma X_3\}$. The solutions that we present are those for which $u$ is function of all the variables $\{t, x, z\}$.

3.1.1 Solution $\{X_4, X_5\}$

By using the Lie symmetry vectors $\{X_4, X_5\}$ we end to the following ordinary differential equation $U_\zeta \zeta - U = 0$, where $\zeta = xt^{-\frac{1}{2}}$ and $u(t, x, z) = \zeta + U(\zeta) t^{-\frac{1}{2}}$. Therefore the similarity solution is derived to be

$$u(t, x, z) = \frac{z + x}{t}. \quad (15)$$

3.1.2 Solution $\{X_1 + \beta X_2, X_1 + \gamma X_3\}$

Reduction with the symmetry vectors $\{X_1 + \beta X_2, X_1 + \gamma X_3\}$ gives the travelling-wave solution $u = U(y)$, $y = \beta z - \gamma t + \gamma x$, where $U(y)$ satisfies the following differential equation

$$(\beta^2 + \gamma^2) U_{yy} - \gamma \beta^2 U + \frac{\beta^2}{2} U^2 - U_1 = 0 \quad (16)$$

or, equivalently,

$$\frac{(\beta^2 + \gamma^2)}{2} U_y^2 - \frac{\gamma \beta^2}{2} U^2 + \frac{\beta^2}{6} U^3 - U_1 y - U_0 = 0. \quad (17)$$

The latter equation can be integrated by quadratures.
| $f(u)$ | Lie Algebra | $\dim G_n$ | Elements of $G_n$ |
|-------|-------------|------------|------------------|
| Arbitrary | $3A_1$ | 3 | $X_1, X_2, X_3$ |
| $u$ | $A_{4,2} \oplus A_1$ | 5 | $X_1, X_2, X_3, X_4, X_5$ |
| $u_0$ | $A_{3,1} \oplus 2A_1$ | 5 & $\infty$ | $X_1, X_2, X_3, X_4^A, X_5^A, X_b$ |
| $u^\mu + u_0$ | $A_{4,2}$ | 4 | $X_1, X_2, X_3, X_4^P$ |
| $u + \kappa u^2 + u_0$ | $A_{4,2}$ | 4 | $X_1, X_2, X_3, X_4^C$ |
| $e^{\mu u} + u_0$ | $A_{4,2}$ | 4 | $X_1, X_2, X_3, X_4^D$ |
| $\ln u + u_0$ | $A_{4,2}$ | 4 | $X_1, X_2, X_3, X_4$ |

Equation (16) can be written as

$$U_y = V, \quad V_y = \gamma \beta^2 U - \frac{\beta^2}{2} U^2 + U_1. \tag{18}$$

System (18) admits two stationary points, they are

$$U_A^\pm = \gamma \pm \sqrt{\gamma^2 - \frac{2U_1}{\beta^2}}.$$

These points are real when $\gamma^2 \geq \frac{2U_1}{\beta^2}$. Easily we find that $U_+^\pm$ is always a source while $U_-^A$ is always a centre point and describes periodic solutions.

4 Group classification for the generalized qZK

For the generalized 2+1 qZK equation we find that for arbitrary function $f(u)$ the admitted Lie symmetries are the $\{X_1, X_2, X_3\}$. However, for other functional forms of $f(u)$ equation (2) admits additional Lie symmetry vectors. Hence for the Lie symmetry classification of the generalized 2+1 qZK equation the following proposition follows.

**Proposition 1** The generalized 2 + 1 qZK equation (2) for an arbitrary function $f(u)$ admits three Lie point symmetries which form an Abelian Lie algebra, $3A_1$. Furthermore, for $f(u) = u_0$ the generalized 2 + 1 qZK equation admits an infinite number of Lie point symmetries with finite algebra the $A_{4,2} \oplus A_1$. For $f(u) = u$ it admits five Lie point symmetries which form the $A_{4,2} \oplus A_1$ Lie algebra. Finally Moreover, for the following functional forms of $f(u)$, that is, $f_B(u) = u^\mu + u_0$, $f_C(u) = u + \kappa u^2 + u_0$, $f_D(u) = e^{\mu u} + u_0$ and $f_E(u) = \ln u + u_0$ the generalized 2 + 1 qZK equation is invariant under four-dimensional Lie algebras as they are presented in Table 3.
Table 4: Commutator table for the admitted Lie point symmetries of the generalized qKZ equation with \( f(u) = f_0 \)

| \([X_i, X_j]\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4^A\) | \(X_5^A\) |
|-----------------|--------|--------|--------|----------|----------|
| \(X_1\)        | 0      | 0      | 0      | 0        | \(3X_1 + 2X_3\) |
| \(X_2\)        | 0      | 0      | 0      | 0        | \(X_2\)     |
| \(X_3\)        | 0      | 0      | 0      | 0        | \(X_3\)     |
| \(X_4^A\)      | 0      | 0      | 0      | 0        | 0          |
| \(X_5^A\)      | \(-3X_1 - 2X_3\) | \(-X_2\) | \(-X_3\) | 0        | 0          |

Table 5: Adjoint representation for the admitted Lie point symmetries of the generalized qKZ equation with \( f(u) = f_0 \)

| \(Ad(e^{\varepsilon X_i})\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4^A\) | \(X_5^A\) |
|-------------------------------|--------|--------|--------|----------|----------|
| \(X_1\)                      | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4^A\) | \(X_5^A - 3\varepsilon X_1 - 2\varepsilon X_3\) |
| \(X_2\)                      | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4^A\) | \(X_5^A - \varepsilon X_2\) |
| \(X_3\)                      | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4^A\) | \(X_5^A - \varepsilon X_3\) |
| \(X_4^A\)                    | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4^A\) | \(X_5^A\) |
| \(X_5^A\)                    | \(e^{3\varepsilon} (X_1 + X_3) - \varepsilon X_3\) | \(e^{\varepsilon} X_2\) | \(e^{\varepsilon} X_3\) | \(X_4^A\) | \(X_5^A\) |

4.1 Case A: \( f(u) = u_0 \)

For a constant function \( f(u) = u_0 \), where without loss of generality we assume \( u_0 = 1 \), equation (2) admits as Lie symmetries the vector fields

\[ X_1, X_2, X_3, X_4^A = u \partial_u, X_5^A = 3t \partial_t + x \partial_x + (2t + z) \partial_z \] and \( X_6^A = b(t, x, z) \partial_u \)

in which \( b \) is a solution of the original equation. The symmetry vectors \( X_4^A, X_5^A \) indicate the linearity for the partial differential equation. The commutators and the adjoint representation for the admitted Lie symmetries are presented in Tables 4 and 5 respectively. The Lie symmetries form the \( A_{3,1} \oplus 2A_1 \) Lie algebra.

We observe that the Lie point symmetries for this case form a different Lie algebra from that of equation (2). Hence, the resulting one-dimensional optimal system is determined to consist of the symmetry vectors

\[ \{X_1\}, \{X_2\}, \{X_3\}, \{X_4\}, \{X_5\}, \]
\[ \{X_1 + \alpha X_2\}, \{X_1 + \alpha X_3\}, \{X_1 + \alpha X_4\}, \]
\[ \{X_2 + \alpha X_3\}, \{X_2 + \alpha X_4\}, \{X_3 + \alpha X_4\}, \]
\[ \{X_1 + \alpha X_2 + \beta X_4\}, \{X_2 + \alpha X_3 + \beta X_4\}, \]
\[ \{X_1 + \alpha X_2 + \beta X_3\}, \{X_1 + \alpha X_2 + \beta X_3 + \gamma X_4\}. \]
Table 6: Commutator table for the admitted Lie point symmetries of the generalized qKZ equation with \( f(u) = u^\mu + u_0 \)

| \([X_i, X_j]\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X^B_4\) |
|-----------------|-------|-------|-------|-------|
| \(X_1\)        | 0     | 0     | 0     | 3\(X_1 + 2u_0X_3\) |
| \(X_2\)        | 0     | 0     | 0     | \(X_2\) |
| \(X_3\)        | 0     | 0     | 0     | \(X_3\) |
| \(X_4^B\)      | \(-3X_1 - 2u_0X_3\) | \(-X_2\) | \(-X_3\) | 0 |

Table 7: Adjoint representation for the admitted Lie point symmetries of the generalized qKZ equation with \( f(u) = u^\mu + u_0 \)

| \(Ad(e^{\epsilon X_i})X_j\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X^B_4\) |
|-----------------------------|-------|-------|-------|-------|
| \(X_1\)                     | \(X_1\) | \(X_2\) | \(X_3\) | \(X^B_4 - 3\varepsilon X_1 - 2u_0\varepsilon X_3\) |
| \(X_2\)                     | \(X_1\) | \(X_2\) | \(X_3\) | \(X^B_4 - \varepsilon X_2\) |
| \(X_3\)                     | \(X_1\) | \(X_2\) | \(X_3\) | \(X^B_4 - \varepsilon X_3\) |
| \(X^B_4\)                  | \(e^{3\varepsilon}(X_1 + u_0X_3) - u_0\varepsilon X_3\) | \(e^{\varepsilon}X_2\) | \(e^{\varepsilon}X_3\) | \(X^B_4\) |

4.2 Case B: \( f(u) = u^\mu + u_0 \)

For \( f(u) = u^\mu + u_0 \) the admitted Lie point symmetries are

\[
X_1, X_2, X_3, X^B_4 = (3t\partial_t + x\partial_x + (z + 2u_0t)\partial_z - \frac{u}{\mu}\partial_u).
\]

The commutators and the adjoint representation for the admitted four-dimensional Lie algebra are presented in Tables 4 and 5 respectively. By using the results of these Tables we can calculate easily the one-dimensional optimal system composed of the one-dimensional Lie algebras

\[
\{X_1\}, \{X_2\}, \{X_3\}, \{X^B_4\},
\]

\[
\{X_1 + \alpha X_2\}, \{X_1 + \alpha X_3\}, \{X_1 + \beta X^B_4\},
\]

\[
\{X_2 + \alpha X_3\}, \{X_1 + \alpha X_2 + \beta X_3\},
\]

while the Lie symmetries form the \( A_{4,2} \) Lie algebra.

4.3 Case C: \( f(u) = u + \kappa u^2 + u_0 \)

For \( f(u) = u + \kappa u^2 \) the admitted Lie point symmetries are

\[
X_1, X_2, X_3, X^C_4 = 6\kappa t\partial_t + 2\kappa x\partial_x + (2\kappa z - t + 4u_0\kappa t)\partial_z - (1 + 2\kappa u)\partial_u.
\]

The commutators and the adjoint representation for these four-dimensional Lie algebra are presented in Tables 8 and 9. From these two tables we observe that the admitted Lie algebra is the same as that of case B, i.e.
Table 8: Commutator table for the admitted Lie point symmetries of the generalized qKZ equation with \( f(u) = u + \kappa u^2 + u_0 \)

| \([X_i, X_j]\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4^C\) |
|-----------------|-------------|-------------|-------------|-------------|
| \(X_1\)        | 0           | 0           | 0           | 6\(\kappa X_1 + (4\kappa u_0 - 1) X_3\) |
| \(X_2\)        | 0           | 0           | 0           | 2\(\kappa X_2\) |
| \(X_3\)        | 0           | 0           | 0           | 2\(\kappa X_3\) |
| \(X_4^B\)      | -6\(\kappa X_1 - (4\kappa u_0 - 1) X_3\) | -2\(\kappa X_2\) | -2\(\kappa X_3\) | 0 |

Table 9: Adjoint representation for the admitted Lie point symmetries of the generalized qKZ equation with \( f(u) = u + \kappa u^2 + u_0 \)

| \(\text{Ad}(e^{\varepsilon X_i})\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4^C\) |
|-------------------------------|-------------|-------------|-------------|-------------|
| \(X_1\)                      | \(X_1\)    | \(X_2\)    | \(X_3\)    | \(X_4^B - 6\kappa \varepsilon X_1 - (4\kappa u_0 - 1) \varepsilon X_3\) |
| \(X_2\)                      | \(X_1\)    | \(X_2\)    | \(X_3\)    | \(X_4^B - 2\kappa \varepsilon X_2\) |
| \(X_3\)                      | \(X_1\)    | \(X_2\)    | \(X_3\)    | \(X_4^B - 2\kappa \varepsilon X_3\) |
| \(X_4^B\)                    | \(e^{6\kappa \varepsilon} (X_1 + (u_0 - \frac{1}{4\kappa}) X_3) - (u_0 - \frac{1}{4\kappa}) e^{2\varepsilon} X_3\) | \(e^{\varepsilon} X_2\) | \(e^{\varepsilon} X_3\) | \(X_4^B\) |

the \(A_{4,2}\), however, in a different representation. Thus, the one-dimensional optimal system is comprised of the same one-dimensional Lie algebras as that of case B.

4.4 Case D: \( f(u) = e^{\mu u} + u_0 \)

For \( f(u) = e^{\mu u} + u_0 \) the admitted Lie point symmetries are

\[
X_1, X_2, X_3, X_4^D = \left(3t \partial_t + x \partial_x + (z + 2u_0 t) \partial_z - \frac{2}{\mu} \partial_u\right).
\]

The commutators and the adjoint representation are exactly the same as those of case B presented in Tables 6 and 7. Therefore, the one-dimensional system is composed of the same one-dimensional Lie algebras.

4.5 Case E: \( f(u) = \ln u + u_0 \)

For \( f(u) = \ln u + u_0 \) the admitted Lie point symmetries are

\[
X_1, X_2, X_3, X_4 = t \partial_t + u \partial_u.
\]

We observe that this is the fourth dimensional sub-algebra of the original equation. Therefore, the commutators and the adjoint representation are given in Tables 1 and 2 respectively. Moreover, the one-dimensional optimal system is that for the qZK equation 1 except that here the vector field is \(X_5\).
5 Lie symmetries for the time-varying coefficient qZK

In this Section we extend our analysis by studying the Lie symmetries for the time-varying $2 + 1$ qZK equation (3). Without loss of generality we can select $\delta (t) = 1$. That it can be seen easily by change the time variable $t \to T (\tau)$ and define new coefficient functions. Thus in the following we assume $\delta (t) = 1$.

We apply the Lie symmetry condition and we summarize the results in the following proposition.

**Proposition 2** The time-varying $2 + 1$ qZK equation (3), for which without loss of generality we have assumed $\delta (t) = 1$, for arbitrary functions $\lambda (t)$ and $\varepsilon (t)$. Equation (3) admits a three-dimensional Lie algebra comprising the symmetry vectors $X_2 = \partial_x$, $X_3 = \partial_z$ and $X_4 = t \partial_z + \partial_u$. However, when $B (t) = t^p$, $C (t) = t^q$, an additional symmetry vector exists, namely, $X_1^p = t \partial_t - \frac{(p-3q-2)}{6} x \partial_x + \frac{p+1}{3} z \partial_z + \frac{p-2}{3} u \partial_u$, while, when $B (t) = e^{pt}$ and $C (t) = e^{qt}$, the additional symmetry vector is $X_1^q = \partial_t - \frac{p-3q}{6} x \partial_x + \frac{q}{3} z \partial_z + \frac{q}{6} u \partial_u$.

The proof of this proposition is omitted. As far as the nonzero commutators of the Lie symmetries are concerned for the time-dependent qZK equation we find

$$[X_2, X_1^p] = -\frac{(p-3q-2)}{6} X_2, \quad [X_3, X_1^p] = \frac{p+1}{3} X_3, \quad [X_4, X_1^p] = X_4,$$

and

$$[X_2, X_1^q] = -\frac{p-3q}{6} X_2, \quad [X_3, X_1^q] = \frac{p}{3} X_3, \quad [X_4, X_1^q] = -X_2 + \frac{p}{3} X_4.$$

6 Conclusions

In this piece of work, we studied the algebraic properties of the $2 + 1$ qZK equation. In particular we solved the classification problem for the partial differential equation (2) by determining all the functional forms of $f (u)$ for which the equation admits Lie symmetries. For an arbitrary function $f (u)$ the differential equation admits three Lie point symmetries, while for linear function $f (u)$ admits five nontrivial Lie point symmetries. Moreover, for the following cases, $f_B (u) = u^2 + u_0$, $f_C (u) = u + \kappa u^2 + u_0$, $f_D (u) = e^{au} + u_0$ and $f_E (u) = \ln u + u_0$, the differential equation admits four Lie point symmetries which form the Lie Algebra $A_{4,2}$. The results are summarized in Proposition 1.

In addition, we consider the time-varying equation (2) with nonconstant coefficients, and we classified the time-dependent coefficients according to the admitted Lie point symmetries. Indeed, in the general case the equation admits three Lie point symmetries, However, for the two special cases described by Proposition 2 additional symmetries follow.

For the linear function $f (u) = u$ we applied the Lie invariants in order to define similarity transformations and to reduce the differential equation to an ordinary differential equation. We were able to find a scaling solution and to prove the existence of travelling-wave solutions. We do not proceed with the investigation of travelling-wave solutions for the general case of arbitrary function $f (u)$ for equation (2).

For an arbitrary function $f (u)$ the application of the Lie point symmetries $\{X_1 + \beta X_2, X_1 + \gamma X_3\}$ reduces equation (2) to the partial differential equation

$$\left(\frac{\beta^2 + \gamma^2}{\beta^2}\right) U_{yy} - (\gamma - f (U)) U_y = 0,$$

(19)
where \( u = U(y) \) and \( y = \beta z - \gamma t + \gamma x \). The third-order differential equation can be integrated easily as

\[
(\beta^2 + \gamma^2) U_{yy} - \beta^2 (\gamma U - F(U)) U - U_1 = 0 \quad f(U) = \frac{dF(U)}{dU} 
\]  

or, equivalently,

\[
U_y = V \quad V_y = \gamma \beta^2 U - \beta^2 F(U) + U_1. 
\]  

Therefore, in order for the equation to admit periodic solution it should follow that the latter system admits at least a stationary point \( U_P \) in which \( \gamma \beta^2 U_P - \beta^2 F(U_P) + U_1 \) and \( f(U_P) > \gamma \).

This work contributes to the subject of the group properties of differential equations and specifically of plasma physics differential equations. In a future work we plan to investigate the derivation of conservation laws for the generalized \( 2 + 1 \) \( qZK \) equation.

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