The flare Package for High Dimensional Linear Regression and Precision Matrix Estimation in $\mathbb{R}^*$

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Abstract

This paper describes an R package named flare, which implements a family of new high dimensional regression methods (LAD Lasso, SQRT Lasso, $\ell_q$ Lasso, and Dantzig selector) and their extensions to sparse precision matrix estimation (TIGER and CLIME). These methods exploit different nonsmooth loss functions to gain modeling flexibility, estimation robustness, and tuning insensitiveness. The developed solver is based on the alternating direction method of multipliers (ADMM). The package flare is coded in double precision C, and called from R by a user-friendly interface. The memory usage is optimized by using the sparse matrix output. The experiments show that flare is efficient and can scale up to large problems.

Keywords: sparse linear regression, sparse precision matrix estimation, alternating direction method of multipliers, robustness, tuning insensitiveness

1. Introduction

As a popular sparse linear regression method for high dimensional data analysis, Lasso has been extensively studied by machine learning scientists (Tibshirani, 1996). It adopts the $\ell_1$-regularized least square formulation to select and estimate nonzero parameters simultaneously. Software packages such as glmnet and huge have been developed to efficiently

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solve large problems (Friedman et al., 2010; Zhao et al., 2012, 2014). Lasso further yields a
wide range of research interests, and motivates many variants by exploiting nonsmooth loss
functions to gain modeling flexibility, estimation robustness, and tuning insensitivity (See
more details in the package vignette, Zhao and Liu (2014); Liu et al. (2014a)). These nonsmooth
loss functions pose a great challenge to computation. To the best of our knowledge,
no efficient solver has been developed so far for these Lasso variants.

In this report, we describe a newly developed R package named flare (Family of Lasso
Regression). The flare package implements a family of linear regression methods in cluding:
(1) LAD Lasso, which is robust to heavy tail random noise and outliers (Wang, 2013); (2)
SQRT Lasso, which is tuning insensitive (the optimal regularization parameter selection
does not depend on any unknown parameter, Belloni et al. (2011)); (3) \ell_q Lasso, which
shares the advantage of LAD Lasso and SQRT Lasso; (4) Dantzig selector, which can
tolerate missing values in the design matrix and response vector (Candes and Tao, 2007).
By adopting the column by column regression scheme, we further extend these regression
methods to sparse precision matrix estimation, including: (5) TIGER, which is tuning
insensitive (Liu and Wang, 2012); (6) CLIME, which can tolerate missing values in the
data matrix (Cai et al., 2011). The developed solver is based on the alternating direction
method of multipliers (ADMM), which is further accelerated by a multistage screening
approach (Boyd et al., 2011; Liu et al., 2014b). The global convergence result of ADMM
has been established in He and Yuan (2015, 2012). The numerical simulations show that
the flare package is efficient and can scale up to large problems.

2. Algorithm

We are interested in solving convex programs in the following generic form

\[ \hat{\beta} = \arg\min_{\beta, \alpha} L_\lambda(\alpha) + \|\beta\|_1 \quad \text{subject to } r - A\beta = \alpha. \]  

where \( \lambda > 0 \) is the regularization parameter. The possible choices of \( L_\lambda(\alpha) \), \( A \), and \( r \) for
different regression methods are listed in Table 1. Note that LAD Lasso and SQRT Lasso
are special examples of \( \ell_q \) Lasso for \( q = 1 \) and \( q = 2 \) respectively.

All methods in Table 1 can be efficiently solved by the iterative scheme as follows

\[ \alpha^{t+1} = \arg\min_{\alpha} \frac{1}{2} \| u^t + r - A\beta^t - \alpha \|_2^2 + \frac{1}{\rho} L_\lambda(\alpha), \]  

\[ \beta^{t+1} = \arg\min_{\beta} \frac{1}{2} \| u^t - \alpha^{t+1} + r - A\beta \|_2^2 + \frac{1}{\rho} \| \beta \|_1, \]  

\[ u^{t+1} = u^t + (r - \alpha^{t+1} - A\beta^{t+1}), \]  

where \( u \) is the rescaled Lagrange multiplier (Boyd et al., 2011), and \( \rho > 0 \) is the penalty
parameter. For LAD Lasso, SQRT Lasso, or Dantzig selector, (2) has a closed form solution
via the winsorization, soft thresholding, and group soft thresholding operators respectively.
For \( L_q \) Lasso with \( 1 < q < 2 \), (2) can be solved by the bisection-based root finding algorithm.
(3) is a Lasso problem, which can be (approximately) solved by linearization or coordinate
descent. Besides the pathwise optimization scheme and the active set trick, we also adopt
the multistage screening approach to speedup the computation. In particular, we first
select \( k \) nested subsets of coordinates \( A_1 \subseteq A_2 \subseteq \ldots \subseteq A_k = \mathbb{R}^d \) by the marginal correlation between the covariates and responses. Then the algorithm iterates over these nested subsets of coordinates to obtain the solution. The multistage screening approach can greatly boost the empirical performance, especially for Dantzig selector.

| Method        | Loss function \( L_\lambda(\alpha) = \frac{1}{\sqrt{n\lambda}} \|\alpha\|_q \) | \( A \)  | \( r \) | Existing solver |
|---------------|-------------------------------------------------|--------|--------|-----------------|
| \( L_q \) Lasso | \( L_\lambda(\alpha) = \begin{cases} \infty & \text{if } \|\alpha\|_\infty > \lambda \\ 0 & \text{otherwise} \end{cases} \) | \( \frac{1}{n}X^TX \) | \( \frac{1}{n}X^Ty \) | L.P. or S.O.C.P. |
| Dantzig selector | \( L_\lambda(\alpha) = \begin{cases} \infty & \text{if } \|\alpha\|_\infty > \lambda \\ 0 & \text{otherwise} \end{cases} \) | \( \frac{1}{n}X^TX \) | \( \frac{1}{n}X^Ty \) | L.P. |

Table 1: All regression methods provided in the \texttt{flare} package. \( X \in \mathbb{R}^{n \times d} \) denotes the design matrix, and \( y \in \mathbb{R}^n \) denotes the response vector. “L.P.” denotes the general linear programming solver, and “S.O.C.P” denotes the second-order cone programming solver.

3. Examples

We illustrate the user interface by analyzing the eye disease data set in \texttt{flare}.

```r
> # Load the data set
> library(flare); data(eyedata)
> # SQRT Lasso
> out1 = slim(x,y,method="lq",nlambda=40,lambda.min.value=sqrt(log(200)/120))
> # Dantzig Selector
> out2 = slim(x,y,method="dantzig",nlambda=40,lambda.min.ratio=0.35)
```

The program automatically generates a sequence of 40 regularization parameters and estimates the corresponding solution paths of SQRT Lasso and the Dantzig selector. For the Dantzig selector, the optimal regularization parameter is usually selected based on some model selection procedures, such as cross validation. Note that Belloni et al. (2011) has shown that the theoretically consistent regularization parameter of SQRT Lasso is \( C \sqrt{\log d/n} \), where \( C \) is some constant. Thus we manually choose its minimum regularization parameter to be \( \sqrt{\log(d)/n} = \sqrt{\log(200)/120} \). The minimum regularization parameter yields 19 nonzero coefficients out of 200.

4. Numerical Simulation

All experiments below are carried out on a PC with Intel Core i5 3.3GHz processor, and the convergence threshold of \texttt{flare} is chosen to be \( 10^{-5} \). Timings (in seconds) are averaged over 100 replications using 20 regularization parameters, and the range of regularization parameters is chosen so that each method produces approximately the same number of nonzero estimates.

We first evaluate the timing performance of \texttt{flare} for sparse linear regression. We set \( n = 100 \) and vary \( d \) from 375 to 3000 as is shown in Table 2. We independently generate
each row of the design matrix from a $d$-dimensional normal distribution $N(0, \Sigma)$, where $\Sigma_{jk} = 0.5^{|j-k|}$. Then we generate the response vector using $y_i = 3X_{i1} + 2X_{i2} + 1.5X_{i4} + \epsilon_i$, where $\epsilon_i$ is independently generated from $N(0,1)$. From Table 2, we see that all methods achieve good timing performance. Dantzig selector and $\ell_q$ Lasso are slower than the others due to more difficult computational formulations.

We then evaluate the timing performance of `flare` for sparse precision matrix estimation. We set $n = 100$ and vary $d$ from 100 to 400 as is shown in Table 2. We independently generate the data from a $d$-dimensional normal distribution $N(0, \Sigma)$, where $\Sigma_{jk} = 0.5^{|j-k|}$. The corresponding precision matrix $\Omega = \Sigma^{-1}$ has $\Omega_{jj} = 1.3333$, $\Omega_{jk} = -0.6667$ for all $j,k = 1, \ldots, d$ and $|j-k| = 1$, and all other entries are 0. From Table 2, we see that both TIGER and CLIME achieve good timing performance, and CLIME is slower than TIGER due to a more difficult computational formulation.

| Sparse Linear Regression |
|--------------------------|
| Method                   | $d = 375$    | $d = 750$    | $d = 1500$   | $d = 3000$   |
| LAD Lasso                | 1.1713(0.2915) | 1.1046(0.3640) | 1.8103(0.2919) | 3.1378(0.7753) |
| SQRT Lasso               | 0.4888(0.0264) | 0.7330(0.1234) | 0.9485(0.2167) | 1.2761(0.1510) |
| $\ell_{1.5}$ Lasso       | 12.995(0.5535) | 14.071(0.5966) | 14.382(0.7390) | 16.936(0.5696) |
| Dantzig selector         | 0.3245(0.1871) | 1.5360(1.8566) | 4.4669(5.9929) | 17.034(23.202) |

| Sparse Precision Matrix Estimation |
|------------------------------------|
| Method                   | $d = 100$ | $d = 200$ | $d = 300$ | $d = 400$ |
| TIGER                    | 1.0637(0.0361) | 4.6251(0.0807) | 7.1860(0.0795) | 11.085(0.1715) |
| CLIME                    | 2.5761(0.3807) | 20.137(3.2258) | 42.882(18.188) | 112.50(11.561) |

Table 2: Average timing performance (in seconds) with standard errors in the parentheses on sparse linear regression and sparse precision matrix estimation.

### 5. Discussion and Conclusions

Though the `glmnet` package cannot handle nonsmooth loss functions, it is much faster than `flare` for solving Lasso\(^1\) and the `glmnet` package can also be applied to solve $\ell_1$ regularized generalized linear model estimation problems, which `flare` cannot. Overall speaking, the `flare` package serves as an efficient complement to the `glmnet` package for high dimensional data analysis. We will continue to maintain and support this package.

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1. See more detail in the package vignette.
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