MONADIC VS ADJOINT DECOMPOSITION

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Abstract. It is known that the so-called monadic decomposition, applied to the adjunction connecting the category of bialgebras to the category of vector spaces via the tensor and the primitive functors, returns the usual adjunction between bialgebras and (restricted) Lie algebras. Moreover, in this framework, the notions of heavily separable functor and combinatorial rank play a central role. In order to set these results into a wider context, we are led to substitute the monadic decomposition by what we call the adjoint decomposition. This construction has the advantage of reducing the computational complexity when compared to the first one. We connect the two decompositions by means of an embedding and we investigate its properties by using a relative version of Grothendieck fibration. As an application, in this wider setting, by using the notion of heavily separable functor, we introduce a notion of combinatorial rank that, among other things, is expected to give some hints on the length of the monadic decomposition.

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Introduction

Let $A$ be a category with all coequalizers and let $L \dashv R : A \rightarrow B$ be an adjunction with unit $\eta : \text{Id} \rightarrow RL$ and counit $\epsilon : LR \rightarrow \text{Id}$. Consider the Eilenberg-Moore category $B_1$ of algebras over the monad $(RL,R\epsilon L,\eta)$. Then the comparison functor $R_1 : A \rightarrow B_1$ has a left adjoint $L_1$, with unit $\eta_1 : \text{Id} \rightarrow R_1 L_1$ and counit $\epsilon_1 : L_1 R_1 \rightarrow \text{Id}$, and we can compute the Eilenberg-Moore category $B_2$ of algebras over the monad $(R_1 L_1,R_1 \epsilon_1 L_1,\eta_1)$. Going on this way we obtain a tower

\[
\begin{array}{ccccccccc}
A & \xrightarrow{id_A} & A & \xrightarrow{id_A} & A & \xrightarrow{id_A} & \cdots & \cdots & \xrightarrow{id_A} & A \\
L & \xrightarrow{R} & B & \xrightarrow{U_{n,1}} & B_1 & \xrightarrow{R_1} & B_2 & \xrightarrow{R_2} & B_{N-1} & \xrightarrow{R_N} B_N
\end{array}
\]

where $U_{n,n+1}$ denotes the forgetful functor and $U_{n,n+1} \circ R_{n+1} = R_n$. If this process stops exactly after $N$ steps, meaning that $N$ is the smallest positive integer such that $U_{N,N+1}$ is a category isomorphism, then $R$ is said to have a monadic decomposition of monadic length $N$. For relevant

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outcomes of this notion we refer to [4, 6, 10]. We just mention here how our interest in this construction stems from the case when \((L, R)\) is the adjunction \((\tilde{T}, P)\), where \(P\) is the functor that associates to any bialgebra, over a base field \(k\), its space of primitive elements and its left adjoint \(\tilde{T}\) associates to a vector space \(V\) its tensor algebra \(TV\) endowed with the usual bialgebra structure in which the elements of \(V\) are primitive. One of the outcomes of the papers quoted above is the existence of an equivalence \(\Lambda\) between the category \(\text{Vec}_2\) and the category \(\text{Lie}\) of either Lie algebras, if \(\text{char}(k) = 0\), or restricted Lie algebras, if \(\text{char}(k) > 0\). Moreover one has \(\Lambda \circ P_2 = \mathcal{P} \) and \(H \circ \Lambda = U_{0,1}U_{1,2}\). Here \((\mathcal{U}, \mathcal{P})\) is the usual adjunction between \(\text{Bialg}\) and \(\text{Lie}\) given by the (restricted) universal enveloping algebra functor and the primitive functor.

\[
\begin{array}{ccc}
\text{Bialg} & \overset{\tilde{T}}{\longrightarrow} & \text{Vec} \\
\downarrow & & \downarrow \U_0,1 \\
\text{Vec}_1 & \overset{\mathcal{P}}{\longrightarrow} & \text{Lie} \\
\downarrow & & \downarrow \Lambda \\
\text{Vec}_2 & \overset{\gamma}{\longrightarrow} & \mathcal{P} \\
\end{array}
\]

The starting functor \(P\) comes out to have monadic decomposition of monadic length at most 2 and this reflects the fact that the functor \(\mathcal{U}\) is fully faithful, or equivalently the unit \(\text{Id} \rightarrow \mathcal{P}\mathcal{U}\) of the adjunction \((\mathcal{U}, \mathcal{P})\) is invertible, which is part of the so-called Milnor-Moore theorem. Thus if the input \((L, R)\) of the monadic decomposition procedure is the adjunction \((\tilde{T}, P)\), then the corresponding output, when the iteration stops, is the adjunction \((\mathcal{U}, \mathcal{P})\) up to equivalence.

We point out that unit \(\tilde{\eta}\) of the adjunction \((\tilde{T}, P)\) splits via a suitable natural retraction \(\gamma : P\tilde{T} \rightarrow \text{Id}\) (whose existence amounts to the \textit{heavy separability} of the functor \(\tilde{T}\)) that allows to define a functor \(\Gamma_1 : \text{Vec} \rightarrow \text{Vec}_1\), \(V \mapsto (V, \gamma V)\). The composite functor \(S_1 := \tilde{T}_1\Gamma_1 : \text{Vec} \rightarrow \text{Bialg}\) associates to a vector space \(V\) the tensor bialgebra \(\tilde{T}V\) factored out by the ideal generated by its homogeneous primitive elements of degree at least two. If \(\tilde{\eta}_1\) denotes the unit of \((\tilde{T}_1, P_1)\), it comes out that \(\tilde{\eta}_1S_1V\) is invertible for every \(V\) (this is equivalent to ask that \(V\) has \textit{combinatorial rank} at most one) and this plays a central role in proving that the iteration stops after two steps.

It is natural to wonder what happens to monadic decomposition if we substitute the category of vector spaces over \(k\) and the category of bialgebras over \(k\) by an arbitrary braided monoidal category \(\mathcal{M}\) and the category \(\text{Bialg}(\mathcal{M})\) of bialgebras in \(\mathcal{M}\) respectively, once we made the proper assumptions on \(\mathcal{M}\) to have an analogue of the adjunction \((\tilde{T}, P)\). Partial results have been obtained in [10] giving rise to the notion of Milnor-Moore Category. It is worth to notice that, to the best of our knowledge, even in the more restrictive case when \(\mathcal{M}\) is a symmetric monoidal category it is an open problem whether the monadic length is still at most 2.

In order look at the problem from a more general perspective, unconstrained by the particular features of the examples considered above, we think one has to investigate the stationarity of monadic decomposition at the level of an arbitrary adjunction \((L, R)\). The notions of heavily separable functor and of combinatorial rank, mentioned above, are expected to play a relevant role in the picture. Moreover, since the procedure may, in principle, stop at some level higher than 2, the functor \(\Gamma_1\), arising from the heavy separability of \(\tilde{T}\), should be extended to some functor \(\Gamma_n : \mathcal{B} \rightarrow \mathcal{B}_n\). A first attempt to define such a functor shows how it is inconvenient to prove that the candidate object \(\Gamma_nB\) belongs to \(\mathcal{B}_n\), for every \(B\) in \(\mathcal{B}\). This is due to the fact that to test if an object belongs to this category several equalities have to be checked. The first aim of this paper is to reduce drastically the number of equalities to verify by replacing the category \(\mathcal{B}_n\) by a new category \(\mathcal{B}_{[n]}\). More precisely, we construct a kind of monadic decomposition that we call an \textit{adjoint decomposition} as follows,

\[
\begin{array}{cccccccc}
A & \overset{\text{Id}}{\longrightarrow} & A & \overset{\text{Id}}{\longrightarrow} & A & \overset{\text{Id}}{\longrightarrow} & \cdots & \overset{\text{Id}}{\longrightarrow} & A \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
L & \overset{R}{\rightarrow} & L_{[1]} & \overset{R_{[1]}}{\rightarrow} & L_{[2]} & \overset{R_{[2]}}{\rightarrow} & \cdots & \overset{R_{[n]}}{\rightarrow} & L_{[n]} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
B & \overset{U_{[0,1]}}{\longrightarrow} & B_{[1]} & \overset{U_{[1,2]}}{\longrightarrow} & B_{[2]} & \overset{U_{[2,3]}}{\longrightarrow} & \cdots & \overset{U_{[n-1,n]}}{\longrightarrow} & B_{[n]} \\
\end{array}
\]
where \((L[n], R[n], ε[n], η[n])\) is a suitable adjunction. Denote by \(U[α, β] : B[β] → B[α]\) the composite functor \(U[α, α+1] ∘ U[α+1, α+2] ∘ \cdots ∘ U[β−2, β−1] ∘ U[β−1, β]\) for all \(α ≤ β\). An object in \(B[n]\) is a pair \((V[n], b_{n[−1]}n)\) where \(V[n−1]\) is an object in \(B[n]\) and \(b_{n[−1]}n : RL[n−1]V[n−1] → U[0,n−1]V[n]\) is a morphism in \(B\). Thus it can be regarded as a datum \((V[n]:=(V[0], b[0], b[1], \ldots, b[n−1]),\) where \(b[t] : RL[t]V[t] → V[0]\) and \(V[t] := U[t,n]V[n],\) for each \(t \in \{0, \ldots, n−1\}\). A morphism \(f[n] : V[n] → V'[n]\) in \(B[n]\) is a morphism \(f[n−1] : V[n−1] → V'[n−1]\) in \(B[n−1]\) such that \(U[0,n−1]f[n−1] ∘ b[n−1] = b'[n−1] ∘ RL[n−1]f[n−1]\).

For every \(n ≥ 1\), we can construct a fully faithful functor \(Λ_n : B_n → B[n]\) which satisfies the equalities \(Λ_n ∘ R_n = R[n]\) and \(U[n−1, n] ∘ Λ_n = Λ_{n−1} ∘ U[n−1, n]\) i.e. that makes commutative the solid faces of the following diagram.

\[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{cc}
L[n−1]
\end{array}
\begin{array}{c}
B[n−1]
\end{array}
\begin{array}{c}
\begin{array}{c}
L_n
A
\end{array}
\end{array}
\begin{array}{c}
B_n
\end{array}
\begin{array}{c}
\begin{array}{c}
R_n
\end{array}
\end{array}
\begin{array}{c}
U[n−1, n]
\end{array}
\begin{array}{c}
\begin{array}{c}
R[n−1]
\end{array}
\end{array}
\end{array}
\]

Furthermore we have an isomorphism \(λ_n : L[n]Λ_n → L_n\). By means of a relative version of Grothendieck fibration, we are able to give sufficient conditions for an object in \(B[n]\) to be the image through \(Λ_n\) of an object in \(B_n\). As an instance of how this strategy works we construct, under appropriate conditions, involving the heavy separability of \(L\), a family of functors \(Γ[n] : B → B[n]\), \(n ∈ N\), that factor through \(Λ_n\) returning the desired functor of \(Γ_n : B → B_n\). These constructions apply to the adjunction \(T → P : \text{Biaig}(M) → M\). In the particular case when \(M\) is the category \(H^N\text{Fd}\) of Yetter-Drinfeld modules over a Hopf algebra \(H\), we obtain an explicit description of the functors \(S[n] := \overline{T_n}Γ[n] \cong \overline{T_n}Γ_n =: S_n\), which extend the functor \(S_1\) mentioned above. The combinatorial rank of an object \(V\) in \(H^N\text{Fd}\), regarded as a braided vector space through the braiding of \(H^N\text{Fd}\), is exactly the smallest \(n\) such that the canonical projection \(S[n]V → S[n+1]V\) is invertible and in this case \(S[n]V\) is isomorphic to the Nichols algebra of \(V\). Since the previous projection makes sense also if we start from a general adjunction \(L → R : A → B\) and an object \(B\) in \(B\), we are led to a notion of combinatorial rank in this wide setting that, among other things, is expected to give some hints on the length of the monadic decomposition. Finally we propose possible lines of future investigation.

**Description of main results and applications.** The paper is organized as follows.

In Section 1 we recall the notion of monadic decomposition and the definition of inserter category together with its properties needed in the paper.

In Section 2 we revise the notion of Adjoint triangle introduced by Dubuc. In Proposition 2.5, we give a procedure to associate a new adjoint triangle to a given one. By means of this result, we construct iteratively the adjoint decomposition.

In Section 3 we compare monadic and adjoint decompositions. More explicitly, we construct a fully faithful functor \(Λ_n : B_n → B[n]\), which is injective on objects, connecting the two decompositions. This is obtained in Remark 3.6 by applying iteratively Proposition 3.5.

In Section 4, we investigate a relative version of Grothendieck fibrations. As a byproduct, we deduce other properties of the functor \(Λ_n\). In particular, in Theorem 4.19, we prove it is an \(M(U[n])\)-fibration, where \(M(U[n])\) stands for the class of morphisms in \(B[n]\) whose image in \(B\) via the forgetful functor \(U[n] : B[n] → B\) is monomorphisms. As a consequence, in Theorem 4.20, we give conditions guaranteeing that an object \(B[n] ∈ B[n]\) is image of an object in \(B_n\) though \(Λ_n\). These conditions enable to reduce the number of equalities to check in order to establish that an object lives inside \(B_n\). In Corollary 4.21 we are able to prove that if \(L(N)\) is fully faithful for some \(N\), then \(R\) has a monadic decomposition of monadic length at most \(N\).

In Section 5 we connect these results to the notion of heavily separable functor introduced in [11]. Explicitly, given a suitable diagram involving two adjunctions \((L, R)\) and \((L', R')\), in Theorem
5.2, we prove that under certain assumptions, if $L'$ is heavily separable so is $L$ and we can construct a family of functors $\Gamma_n : B \rightarrow B[n]$, $n \in \mathbb{N}$. Any object of the form $\Gamma_n B \in B[n]$, with $B \in B$, fulfills the conditions mentioned above and hence it belongs to the image of $\Lambda_n$. As a consequence $\Gamma_n$ factors through $\Lambda_n$, see Proposition 5.3.

In Section 6, we study our prototype example for Theorem 5.2 which also explains the relevant role played by the functors $\Gamma_n$. Given a preadditive braided monoidal category $\mathcal{M}$ having equalizers, denumerable coproducts and coequalizers of reflexive pairs of morphisms and such that all of them are preserved by the tensor products, we construct a diagram, as in Theorem 5.2,

\[
\begin{array}{cccc}
\text{Bialg}(\mathcal{M}) & \xrightarrow{\Omega^+} & \text{Alg}^+(\mathcal{M}) \\
\xrightarrow{\bar{T}} & \xrightarrow{P} & \xrightarrow{T^+} & \xrightarrow{\Omega^+} \\
\mathcal{M} & \xrightarrow{\text{id}} & \mathcal{M}
\end{array}
\]

where $\text{Bialg}(\mathcal{M})$ is the category of bialgebras in $\mathcal{M}$, $\text{Alg}^+(\mathcal{M})$ is the category of augmented algebras in $\mathcal{M}$, $\bar{T}$ is the tensor bialgebra functor, $P$ is the primitive functor, $T^+$ is essentially the tensor algebra functor and $\Omega^+$ associates to an augmented algebra $(A, \varepsilon)$ the kernel in $\mathcal{M}$ of its augmentation $\varepsilon$. The functor $\bar{\Omega}^+$ is just the forgetful functor. By the foregoing we get a family of functors $\Gamma_n : \mathcal{M} \rightarrow \mathcal{M}[n]$, $n \in \mathbb{N}$, that factor through $\Lambda_n$, as desired.

In Section 7 we describe explicitly these functors $\Gamma_n$ in the case when $\mathcal{M}$ is the category $H^H\text{YD}$ of Yetter-Drinfeld modules over a finite-dimensional Hopf algebra $H$, the particular case of the category Vec of vector spaces being obtained by taking $H = k$. Concurrently we are lead to define a possible analogue of the notion of combinatorial rank $\kappa(V,c)$ of a braided vector space $(V,c)$ as defined in [2, Section 5] by mimicking V. K. Kharchenko’s definition in [20, Definition 5.4]. We refer to [19] for an overview on the notion of combinatorial rank and its importance. Recall that a braided vector space $(V,c)$ is a vector space $V$ endowed with a braiding $c : V \otimes V \rightarrow V \otimes V$. The tensor algebra $TV$ can be endowed with a braided bialgebra structure (this means to have a braided vector space endowed with an algebra and a coalgebra structure suitably compatible with the braiding), arising from the braiding of $V$, that we denote by $T(V,c)$. If we divide out $T(V,c)$ by the ideal generated by its homogeneous primitive elements of degree at least two we obtain a new braided bialgebra, say $S[\Delta](V,c)$. We can repeat the same procedure on this braided bialgebra obtaining a new quotient braided bialgebra $S[\Sigma](V,c)$ and go on this way. At the limit this procedure yields the so-called Nichols algebra $B(V,c)$ and the number of steps occurred is exactly $\kappa(V,c)$.

Now it is well-known that under some finiteness conditions, a braided vector space $(V,c)$ can be realized as an object in the category $H^H\text{YD}$ for some Hopf algebra $H$ and $c$ becomes the braiding $c_{V,V}$ of $H^H\text{YD}$ applied to $V$, see [32, 3.2.9]. On the other hand $H^H\text{YD}$ is a braided monoidal category and any bialgebra in it becomes a braided bialgebra in the above sense if we forget the Yetter-Drinfeld module structure and we just keep the underlying braiding, algebra and coalgebra structures. In particular $\tilde{T}V \in \text{Bialg}(H^H\text{YD})$ becomes the braided tensor algebra $T(V,c)$ mentioned above. Define the functors $S[n] := \tilde{T}[n]_{\Gamma_n} : H^H\text{YD} \rightarrow \text{Bialg}(H^H\text{YD})$. In Example 7.1, we show that $S[n]V \in \text{Bialg}(H^H\text{YD})$ becomes the braided bialgebra $S[n](V,c)$ mentioned above, for each $n \in \mathbb{N}$. As a consequence the combinatorial rank of $V$, regarded as braided vector space through the braiding $c = c_{V,V}$ of $H^H\text{YD}$ as above, is the smallest $n$ such that the canonical projection $S[n]V \rightarrow S[n+1]V$ is invertible, if such an $n$ exists, and in this case we have $S[n]V = B(V,c)$.

Since, in the setting of Theorem 5.2, we can always define $S[n] := L[n]_{\Gamma_n} : B \rightarrow A$, for every $B \in B$ we are lead to define (see Definition 7.3) the combinatorial rank of an object $B \in B$, with respect to the adjunction $(L, R)$, to be the smallest $n$ such that the canonical projection $S[n]B \rightarrow S[n+1]B$ is invertible (see Lemma 5.4), if such an $n$ exists. Thus a concept of combinatorial rank can be introduced and investigated in this very general setting in which there is neither a bialgebra nor a braided vector spaces but just an adjunction $(L, R)$ as in Theorem 5.2. In the case when $\mathcal{M}$ is the category Vec of vector spaces and the adjunction is $(\bar{T}, P)$, every object in Vec has combinatorial rank at most one (Example 7.7), but this is not true for an arbitrary $\mathcal{M}$, e.g. the
Corollary 7.6. for the combinatorial rank of objects in a proper category the notion of combinatorial rank is settled here, can give new hints on the existence of some bound.

A possible idea for a future investigation is to establish whether the general framework, in which the notion of combinatorial rank is settled here, can give new hints on the existence of some bound for the combinatorial rank of objects in a proper category $\mathcal{B}$ with respect to an adjunction $(L, R)$ (or more specifically in a category $\mathcal{M}$ with respect to the adjunction $(\tilde{T}, P)$) as it happens in $\text{Vec}$. The fact that all objects in $\text{Vec}$ have combinatorial rank at most one constitutes one of the main ingredients in [4] to prove that the monadic decomposition of $P : \text{Bialg}(\text{Vec}) \to \text{Vec}$ has length at most two. A natural question, that we also leave to future investigations, is to determine whether a similar result still holds in the setting of $\mathcal{B}$ as above for the functor $R$. Such a result would be related to an analogue of the so-called Milnor-Moore theorem, see Remark 7.8. More generally one can ask whether the length of the monadic decomposition of the functor $R$ is upper-bounded in case the combinatorial rank of objects in $\mathcal{B}$ with respect to $(L, R)$ is upper-bounded.

1. Monadic Decomposition and Inserter Category

Throughout this paper $\kappa$ will denote a field. All vector spaces and (co)algebras will be defined over $\kappa$. The unadorned tensor product $\otimes$ will denote the tensor product over $\kappa$ if not stated otherwise. We denote either by $\mathfrak{M}$ or $\text{Vec}$ the category of vector spaces.

Definition 1.1. Recall that a monad on a category $\mathcal{A}$ is a triple $Q := (Q, m, u)$, where $Q : \mathcal{A} \to \mathcal{A}$ is a functor, $m : QQ \to Q$ and $u : A \to Q$ are functorial morphisms satisfying the associativity and the unitarity conditions $m \circ mQ = m \circ Qm$ and $m \circ Qu = \text{Id}_Q = m \circ uQ$. An algebra over a monad $Q$ on $\mathcal{A}$ (or simply a $Q$-algebra) is a pair $(X, \mu)$ where $X \in \mathcal{A}$ and $\mu : QX \to X$ is a morphism in $\mathcal{A}$ such that $\mu \circ Q\mu = \mu \circ mX$ and $\mu \circ uX = \text{Id}_X$. A morphism between two $Q$-algebras $(X, \mu)$ and $(X', \mu')$ is a morphism $f : X \to X'$ in $\mathcal{A}$ such that $\mu' \circ Qf = f \circ \mu$. We will denote by $\mathcal{Q}\mathcal{A}$ the category of $Q$-algebras and their morphisms. This is the so-called Eilenberg-Moore category of the monad $Q$ (which is sometimes also denoted by $\mathcal{A}^Q$ in the literature). When the multiplication and unit of the monad are clear from the context, we will just write $Q$ instead of $Q$.

A monad $Q$ on $\mathcal{A}$ gives rise to an adjunction $(F, U) := (\mu, \eta)$ where $U : Q\mathcal{A} \to \mathcal{A}$ is the forgetful functor and $F : \mathcal{A} \to Q\mathcal{A}$ is the free functor. Explicitly:

$$U(X, \mu) := X, \quad UF := f \quad \text{and} \quad FX := (QX, mX), \quad Ff := Qf.$$ 

Note that $UF = Q$. The unit of the adjunction $(F, U)$ is given by the unit $u : A \to UF = Q$ of the monad $Q$. The counit $\lambda : FU \to Q\mathcal{A}$ of this adjunction is uniquely determined by the equality $\lambda(X, \mu) = \mu$ for every $(X, \mu) \in Q\mathcal{A}$. It is well-known that the forgetful functor $U : Q\mathcal{A} \to \mathcal{A}$ is faithful and reflects isomorphisms (see e.g. [14, Proposition 4.1.4]).

Let $L \dashv R : \mathcal{A} \to \mathcal{B}$ be an adjunction with unit $\eta : \text{Id}_\mathcal{B} \to RL$ and counit $\epsilon : LR \to \text{Id}_\mathcal{A}$. Then $(RL, RL, \eta)$ is a monad on $\mathcal{B}$ and we can consider the so-called comparison functor $K : A \to RLB$ which is defined by $KX := (RX, R\epsilon)$ and $KF := RF$. Note that $RLU \circ K = R$.

Definition 1.2. An adjunction $L \dashv R : \mathcal{A} \to \mathcal{B}$ is called monadic (tripleable in Beck's terminology [12, Definition 3, page 8]) whenever the comparison functor $K : \mathcal{A} \to RL\mathcal{B}$ is an equivalence of categories. A functor $R$ is called monadic if it has a left adjoint $L$ such that the adjunction $(L, R)$ is monadic, see [12, Definition 3', page 8].

Definition 1.3. (See [4, Definition 2.7], [8, Definition 2.1] and [23, Definitions 2.10 and 2.14]) Fix a $N \in \mathbb{N}$. We say that a functor $R$ has a monadic decomposition of monadic length $N$ whenever there exists a sequence $(R_n)_{n \leq N}$ of functors $R_n$ such that

1. $R_0 = R$;
2. for $0 \leq n \leq N$, the functor $R_n$ has a left adjoint functor $L_n$;
3) for $0 \leq n \leq N - 1$, the functor $R_{n+1}$ is the comparison functor induced by the adjunction $(L_n, R_n)$ with respect to its associated monad;

4) $L_N$ is fully faithful while $L_n$ is not fully faithful for $0 \leq n \leq N - 1$.

Compare with the construction performed in [24, 1.5.5, page 49].

For $R : A \to B$, as above we have a diagram

\[
\begin{array}{cccccccc}
A & \xrightarrow{\Id_A} & A & \xrightarrow{\Id_A} & A & \cdots & A & \xrightarrow{\Id_A} & A \\
L_0 \downarrow R_0 & & L_1 \downarrow R_1 & & L_2 \downarrow R_2 & & \cdots & & L_N \downarrow R_N \\
B_0 = B & & B_1 & & B_2 & & \cdots & & B_{N-1} = B \\
\end{array}
\]

where $B_0 = B$ and, for $1 \leq n \leq N$,

- $B_n$ is the category of $Q_{n-1}$-algebras $Q_{n-1}B_{n-1}$, where $Q_{n-1} := R_{n-1}L_{n-1}$;
- $U_{n-1,n} : B_n \to B_{n-1}$ is the forgetful functor $Q_{n-1}U$.

We will denote by $\eta_n : \Id B_n \to R_nL_n$ and $\epsilon_n : L_nR_n \to \Id A$ the unit and counit of the adjunction $(L_n, R_n)$ respectively for $0 \leq n \leq N$. Note that one can introduce the forgetful functor $U_{m,n} : B_n \to B_m$ for all $m \leq n$ with $0 \leq m, n \leq N$.

We point out that $L_N$ is full and faithful is equivalent to the fact that the forgetful functor $U_{N,N+1}$ is a category isomorphism, see e.g. [4, Remark 2.4].

We refer to [4, Remarks 2.8 and 2.10] for further comments on monadic decompositions.

**Definition 1.4.** Let $F, G : A \to B$ be functors. The inserter category $(F | G)$ has objects the pairs $(A, \alpha_A)$ where $A \in A$ and $\alpha_A : FA \to GA$ is a morphism in $B$. A morphism $f : (A, \alpha_A) \to (A', \alpha_{A'})$ is a morphism $f : A \to A'$ in $A$ such that the following diagram commutes

\[
\begin{array}{ccc}
FA & \xrightarrow{Ff} & FA' \\
\alpha_A \downarrow & & \downarrow \alpha_{A'} \\
GA & \xrightarrow{Gf} & GA'
\end{array}
\]

If we denote by

\[
P = P_{(F|G)} : (F | G) \to A, \quad (A, \alpha_A) \mapsto A, \quad f \mapsto f
\]

the forgetful functor, then there is a natural transformation

\[
\psi := \psi_{(F|G)} : FP \to GP
\]

which is defined by $\psi(A, \alpha) = \alpha$ for every $(A, \alpha) \in (F \downarrow G)$.

Given functors $F, G, F', G' : A \to B$ and natural transformations $\phi : F' \to F$ and $\gamma : G \to G'$ we can define the functor

\[
\langle \phi|\gamma \rangle : (F | G) \to (F' | G') , \quad \left( A, FA \xrightarrow{\phi A} GA \right) \mapsto \left( A, F'A \xrightarrow{\phi A} FA \xrightarrow{\gamma A} GA' \right), \quad f \mapsto f.
\]

**Remark 1.5.** We point out that $(F | G)$ is exactly the inserter category $\text{Insert}(F, G)$ in the 2-category $\text{Cat}$, see e.g. [15, page 157].

**Lemma 1.6.** 1) Let $F, G : A \to B$ be functors and let $Q : Q \to A$ be a functor endowed with a natural transformation $q : FQ \to GQ$. Then there is a unique functor $Q[q] : Q \to (F | G)$ such that $P \circ Q[q] = Q$ and $\psi Q[q] = q$. Explicitly $Q[q]X := (QX, qX) \in (F | G)$ for every $X \in Q$. Clearly any functor $N : Q \to (F | G)$ is of the form $Q[q]$ for $Q = PN$ and $q = \psi N$.

2) Let $Q[q], K[k] : Q \to (F | G)$ be functors and let $\pi : Q \to K$ be such that $G\pi \circ q = k \circ F\pi$. Then there is a unique natural transformation $\tilde{\pi} : Q[q] \to K[k]$ such that $P\tilde{\pi} = \pi$. Clearly any natural transformation $\nu : Q[q] \to K[k]$ is of the form $\tilde{\pi}$ for $\pi = P\nu$.

3) Let $\phi : F' \to F$ and $\gamma : G \to G'$. Then $\langle \phi|\gamma \rangle \circ Q[q] = Q[\gamma Q \circ q \circ \phi Q]$.

**Proof.** 1) For every $X \in Q$ define $Q[q]X := (QX, qX : FQX \to GQX) \in (F | G)$. Given $f : X \to Y$ in $Q$, by naturality of $q$ we have $qY \circ FQf = GQf \circ qX$ so that $f$ induces a morphism $Q[q]f : Q[q]X \to Q[q]Y$ such that $PQ[q]f = Qf$. Thus the functor $Q[q] : Q \to (F | G)$ is defined. Moreover $\psi Q[q]X = \psi (QX, qX) = qX$ so that $\psi Q[q] = q$. Let us check that $Q[q]$ is unique. Given a functor $N : Q \to (F | G)$ such that $PN = Q$ and $\psi N = q$ we have that
\[ \text{PNX} = QX \] so that \( N X = (QX, \alpha) \) for some \( \alpha \). Moreover \( qX = \psi N X = \psi (QX, \alpha) = \alpha \) and hence \( N X = (QX, qX) = Q[q] X \). Moreover, given \( f : X \to Y \) in \( Q \), we have \( P NF = Q f = PQ[q] f \). Since \( P \) is faithful, we deduce \( NF = Q[q] f \) and hence \( W = Q[q] \).

2.1 Definition of the Eilenberg-Moore category. For this purpose we will use the notion of adjoint triangle. Composition that will be called the adjoint decomposition. Our first aim is to obtain an analogue

\[ \langle \psi \rangle \]

so that \( \langle \psi \rangle = \langle \eta \rangle = \eta \).

\[ \text{Proposition 1.7. Consider the forgetful functor } P = P(F|G) : (F|G) \to A. \text{ Let } f : (A, a) \to (C, c) \text{ and } g : (B, b) \to (C, c) \text{ morphisms in } (F|G) \text{ and let } h : A \to B \text{ be a morphism in } A \text{ such that } P f = P g \circ h. \text{ If } G P g \text{ is a monomorphism, then there is a (unique) morphism } h' : (A, a) \to (B, b) \text{ such that } P h' = h \text{ and } f = g \circ h'. \]

Proof. Consider the following diagram

Since \( f \) is a morphism in \( (F|G) \), then the external diagram commutes, and since \( g \) is a morphism in \( (F|G) \), so does the right square. Using that \( G P g \) is a monomorphism, we deduce that the left square commutes as well. Hence \( h \) induces a morphism \( h' : (A, a) \to (B, b) \) such that \( P h' = h \). Now \( P f = P g \circ h = P (g \circ h') \) and \( P \) is faithful imply \( f = g \circ h' \). Since \( P \) is faithful, \( h' \) is unique.

Example 1.8. Recall from [29, Definition 2.2.2], that given an endofunctor \( F : A \to A \), the category of \( F \)-algebras (not to be confused with an Eilenberg-Moore algebra) is \( F \)-Alg = \( (F|\text{Id}_A) \).

Let \( F, G : A \to B \) be functors and let \( \epsilon : F \to G \) be a natural transformation. If \( A \) has coequalizers we can define the functor

\[ \mathcal{U}(\epsilon) : (F|G) \to B \]

by the following coequalizer of natural transformations

\[ \text{2. Adjoint Triangles and Adjoint Decomposition} \]

In this section we construct iteratively the category \( B[n] \) and an analogue of the monadic decomposition that will be called the adjoint decomposition. Our first aim is to obtain an analogue of the Eilenberg-Moore category. For this purpose we will use the notion of adjoint triangle.

Definition 2.1. [16, Definition 1] By an \textit{adjoint triangle}, we mean a diagram of functors

where \((L, R, \eta, \epsilon)\) and \((L', R', \eta', \epsilon')\) are adjunctions and \( GR = R' \).

The letter \( \zeta \) inserted in (5) is the unique natural transformation \( \zeta : L'G \to L \) such that \( c \circ \zeta R = \epsilon' \) namely \( \zeta := \epsilon' L \circ L'G \eta \). It will be useful to state our results. It is easy to check that

\[ R' \zeta \circ \eta' G = G \eta. \]
Note that diagram (5) has been drawn as a square to make it more readable, although the two copies of $A$ on the top can be glued together to give rise, in fact, to a triangle.

**Remark 2.2.** As a particular case of horizontal composition of adjoint squares (see [17, I.6.8]), we can define the (horizontal) composition $\mathcal{T}'' := \mathcal{T}' \circ \mathcal{T}$ of two adjoint triangles $\mathcal{T}'$ and $\mathcal{T}$ by

$$
\mathcal{T} := \begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
\theta & \xrightarrow{\zeta} & \zeta \\
B & \xrightarrow{G} & B',
\end{array} \quad \mathcal{T}' := \begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
L' & \xrightarrow{R'} & L' \\
B & \xrightarrow{\Theta} & B',
\end{array} \quad \mathcal{T}'' := \begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
L'' & \xrightarrow{R''} & L'' \\
B & \xrightarrow{G''} & B',
\end{array}
$$

where $\theta \ast \zeta := \left(L'G\Theta \xrightarrow{\zeta} L\Theta \xrightarrow{\theta} L''\right)$. In fact $(G\Theta)R'' = GR = R'$ and

$$
e'' \circ (\theta \ast \zeta) R'' = \epsilon'' \circ \theta R'' \circ \zeta \Theta R'' = \epsilon \circ \zeta R = \epsilon'.
$$

To any adjoint triangle $\mathcal{T}$ as in (5) we would like to associate a new adjoint triangle $\mathcal{T}^2$ as follows

$$
\mathcal{T} := \begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
\zeta & \xrightarrow{\zeta} & \zeta \\
B & \xrightarrow{G} & B',
\end{array} \quad \mathcal{T}' := \begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
L & \xrightarrow{R} & L \\
B & \xrightarrow{H} & B',
\end{array} \quad \mathcal{T}^2 := \begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
R'' & \xrightarrow{R''} & R'' \\
B & \xrightarrow{G''} & B',
\end{array}
$$

First we have to introduce the category

$$I(\mathcal{T}) := \langle R' L \rangle.
$$

For any category $\mathcal{A}$ we can consider the functor

$$D : \mathcal{A} \to \langle \text{Id}_A | \text{Id}_A \rangle, \quad A \mapsto (A, \text{Id}_A), \quad h \mapsto h.
$$

If $F, G : \mathcal{A} \to \mathcal{B}$, $H : \mathcal{B} \to \mathcal{B}'$ and $K : \mathcal{A}' \to \mathcal{A}$ are functors, we can define

$$S^H : \langle F | G \rangle \to \langle HF | HG \rangle, \quad (A, \alpha_A : FA \to GA) \mapsto (A, H\alpha_A : HFA \to HGA), \quad f \mapsto f,$n

$$D^K : \langle FK | GK \rangle \to \langle F | G \rangle, \quad (A', \alpha_{A'} : FKA' \to GKA') \mapsto (KA', \alpha_{A'} : FKKA' \to GKKA'), \quad f \mapsto Kf.
$$

Given $\epsilon : F \to G$ we define the functor

$$S(\epsilon) := \left(\mathcal{A} \xrightarrow{D} \langle \text{Id}_A | \text{Id}_A \rangle \xrightarrow{S^G} \langle G | G \rangle \xrightarrow{\langle \text{Id}_A \rangle} \langle F | G \rangle\right).
$$

Explicitly, by the notation of Lemma 1.6, we have

$$S(\epsilon) = \text{Id}_A [\epsilon] : \mathcal{A} \to \langle F | G \rangle, \quad A \mapsto (A, \epsilon A), \quad f \mapsto f.
$$

Let us show how $S(\epsilon)$ relates to the functor $U(\epsilon)$ of (3) in the particular case when $G = \text{Id}_A$.

**Lemma 2.3.** Let $F : \mathcal{A} \to \mathcal{A}$ be a functor and let $\epsilon : F \to \text{Id}_A$ be a natural transformation. Assume $\mathcal{A}$ has coequalizers. Then $\pi(\epsilon)S(\epsilon)$ is invertible and $\left(U(\epsilon), S(\epsilon), \eta(\epsilon), (\pi(\epsilon)S(\epsilon))^{-1}\right)$ is an adjunction where $\eta(\epsilon) : \text{Id}_{(F|\text{Id}_A)} \to S(\epsilon)U(\epsilon)$ is uniquely determined by $P\eta(\epsilon) = \pi(\epsilon)$.

**Proof.** Set $\pi := \pi(\epsilon)$. Note that $\psi S(\epsilon) = \epsilon$ and $P \circ S(\epsilon) = \text{Id}_A$. Thus, if we evaluate the left-hand side coequalizer below on $S(\epsilon)$, we obtain the right-hand side one.

$$FP \xrightarrow{\psi} P \xrightarrow{\pi} U(\epsilon) \quad F \xrightarrow{\epsilon} \text{Id}_A \xrightarrow{\pi S(\epsilon)} U(\epsilon)S(\epsilon)
$$

This means $\pi S(\epsilon)$ is invertible. Let us check that there is $\eta(\epsilon) : \text{Id}_{(F|\text{Id}_A)} \to S(\epsilon)U(\epsilon)$ such that $P\eta(\epsilon) = \pi$. We have $\pi \circ \psi = \pi \circ \epsilon P = dU(\epsilon) \circ F\pi = \psi S(\epsilon)U(\epsilon) \circ F\pi = dU(\epsilon) \circ F\pi$.

Since $\text{Id}_{(F|\text{Id}_A)} = P[v]$ and $S(\epsilon)U(\epsilon) = U(\epsilon) [\psi S(\epsilon)U(\epsilon)] = U(\epsilon) [dU(\epsilon)]$, by Lemma 1.6 there is a unique natural transformation $\eta(\epsilon) : \text{Id}_{(F|\text{Id}_A)} \to S(\epsilon)U(\epsilon)$ such that $P\eta(\epsilon) = \pi$. We have $P\eta(\epsilon)S(\epsilon) = \pi S(\epsilon) = PS(\epsilon)\pi S(\epsilon)$.
Moreover

\[ U(e) \eta(e) \circ \pi = \pi S(e) U(e) \circ P \eta(e) = \pi S(e) U(e) \circ \pi \]

and hence \( U(e) \eta(e) \circ \pi = \pi S(e) U(e) \). Therefore \( \left( U(e), S(e), \eta(e), (\pi(e) S(e))^{-1} \right) \) is an adjunction. \( \square \)

**Lemma 2.4.** Let \( F, F' : A \to A \) be functors. Let \( \epsilon : F \to \text{Id}_A \) and \( \phi : F' \to F \) be natural transformations. Set \( \epsilon' := \epsilon \circ \phi : F' \to \text{Id}_A \). If \( A \) has coequalizers, we have the adjoint triangle

\[
\begin{array}{ccc}
A & \xrightarrow{\pi} & A \\
\downarrow U(e) & & \downarrow U(e) \\
(F|\text{Id}) & \xrightarrow{(\phi|\text{Id})} & (F'|\text{Id})
\end{array}
\]

Moreover the natural transformation \( \phi^* : U(\epsilon') \circ (\phi|\text{Id}) \to U(e) \) satisfies

\[ \phi^* \circ \pi'(\epsilon') (\phi|\text{Id}) = \pi(e) \]

and \( \phi^* S(e) \circ \pi'S(e') = \pi S(e) \). In particular \( \phi^* S(e) \) is invertible.

**Proof.** By Lemma 2.3, we have that \( U(e) \dashv S(e) \) and \( U(e') \dashv S(e') \).

Set \( P := P(F|\text{Id}_A) \) and \( P' := P(F'|\text{Id}_A) \). Set also \( \pi = \pi(e) \) and \( \pi' = \pi'(e') \). By Lemma 1.6, we have

\[ (\phi|\text{Id}) \circ S(e) = (\phi|\text{Id}) \circ \text{Id}_A \circ \epsilon = \text{Id}_A \circ \phi = \text{Id}_A \circ \epsilon' = S(e') \]

so that \( \phi|\text{Id} \circ S(e) = S(e') \) and hence the diagram in the statement is an adjoint triangle.

By definition \( \phi^* = (\pi'(e') S(e') U(e))^{-1} \circ U(e') (\phi|\text{Id}) \eta(e) \). Then

\[
\phi^* \circ \pi'(\epsilon') (\phi|\text{Id}) = (\pi'S(e') U(e))^{-1} \circ U(e') (\phi|\text{Id}) \eta(e) \circ \pi' \circ (\phi|\text{Id})
\]

\[
= (\pi'S(e') U(e))^{-1} \circ \pi' \circ (\phi|\text{Id}) S(e) U(e) \circ P' \circ (\phi|\text{Id}) \eta(e)
\]

\[
= (\pi'S(e') U(e))^{-1} \circ \pi'S(e') U(e) \circ P \eta(e) = \pi.
\]

Moreover

\[
\phi^* S(e) \circ \pi'S(e') = \phi^* S(e) \circ \pi' \circ (\phi|\text{Id}) S(e) = (\phi^* \circ \pi' \circ (\phi|\text{Id})) S(e) = \pi S(e).
\]

Since \( \pi' S(e') \) and \( \pi S(e) \) are invertible so is \( \phi^* S(e) \). \( \square \)

Assume \( A \) has coequalizers and consider the following diagram where we apply Lemma 2.4 to \( \epsilon : LR \to \text{Id}_A \) and \( \phi := \zeta R : L'R' \to LR \) to get the adjoint triangle with \( (\zeta R)^* \) in the middle.

\[
\begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
\downarrow U(e) & & \downarrow U(e) \\
(LR|\text{Id}) & \xrightarrow{(\zeta R|\text{Id})} & (L'R'|\text{Id})
\end{array}
\]

The functors \( L(T) \) and \( R(T) \) appearing in the diagram above are defined as follows

\[
L(T) = L[\zeta P \circ L'\psi] : I(T) := (LR|\text{Id}_A) \xrightarrow{S_L} (L'R'L|\text{Id}_A) \xrightarrow{\text{Id}_A \circ L'R'} (L'R|\text{Id}_A),
\]

\[
(B, b) \mapsto (LB, a(B, b), := \zeta B \circ L'b), \quad h \mapsto Lh
\]

\[
R(T) = R[R'\psi] : (LR|\text{Id}_A) \xrightarrow{S_R} (R' LR|\text{Id}_A) \xrightarrow{\text{Id}_A \circ L'R} (R|\text{Id}_A),
\]

\[
(A, a_A : LRA \to A) \mapsto (RA, R'a_A : R'LRA \to R'A), \quad f \mapsto Rf.
\]

Set

\[
R(T) = R[R'] = R(T) \circ S(e) : A \to I(T), \quad A \mapsto (RA, R'e_A), \quad f \mapsto Rf,
\]
Consider also the forgetful functor
\[ P(T) = P_{1(T)} : I(T) \to B, \quad (B, b) \mapsto B, \quad f \mapsto f \]
and the functor
\[ G(T) := G \circ P(T), \quad I(T) \to B' : (B, b) \mapsto GB, \quad h \mapsto Gh. \]
Note that, if we set \( P' := P_{\langle L' R' \mid \Id_A \rangle} \), we get

\begin{align}
(8) & \quad P(T) R(T) = R, \\
(9) & \quad P' L(T) R(T) = LR
\end{align}

We are now ready to construct the adjoint triangle \( T^2 \) announced in (7).

**Proposition 2.5.** Assume \( A \) has coequalizers. Given an adjoint triangle \( T \) as in (5), then

\[ T^2 := \begin{array}{c}
\text{A} \\
L(T) \\
I(T) \\
P(T)
\end{array}
\begin{array}{cccc}
\nearrow & \searrow & & \searrow \\
\pi' & \eta & \Id & \circ\epsilon \\
R(T) & \Leftarrow & \circ\epsilon & \Leftarrow \\
\text{B} & \Downarrow & \downarrow
\end{array}
\begin{array}{c}
\text{A} \\
\text{B}
\end{array}
\]

is an adjoint triangle where the adjunction \((L(T), R(T), \eta(T), \epsilon(T))\) is uniquely determined by the following equalities

\begin{align}
(10) & \quad P(T) \eta(T) = R\pi' \circ \epsilon(T), \\
(11) & \quad \epsilon(T) = \pi' \circ \epsilon(T) \circ R(T) = \epsilon.
\end{align}

**Proof.** Set \( \psi' := \psi_{\langle L' R' \mid \Id_A \rangle} \) and \( \pi' := \pi(\epsilon') \) and let us construct \( \epsilon(T) \).

One easily checks that \( L(T) R(T) = \langle L \rangle R \circ L' R' \rangle \). Moreover, \( S(\epsilon') = \Id_A [\epsilon'] \).

Since \( \epsilon \circ (\zeta R \circ L' R') = \epsilon' \circ L' R' \epsilon \), by Lemma 1.6, the natural transformation \( \epsilon \) induces \( \bar{\epsilon} : L(T) R(T) \to S(\epsilon') \) such that \( P' \bar{\epsilon} = \epsilon \). Define

\[ \epsilon(T) := (\pi(\epsilon') S(\epsilon'))^{-1} \circ \bar{\epsilon} \circ \epsilon' : L(T) R(T) \to \Id_A. \]

Now we define \( \eta(T) : \Id_{I(T)} \to R(T) L(T) \). One easily checks that \( R(T) L(T) = \langle RL(T) \rangle [R' \epsilon L(T)] \) and \( \Id_{I(T)} = P(T) [\psi(\epsilon(T))] \).

We compute

\begin{align}
\epsilon' L(T) & \circ L' \left( G \alpha \circ \psi_{1(T)} \right) \\
\quad & = \epsilon' L(T) \circ L' G \alpha \circ L' \psi_{1(T)} \\
\quad & = \epsilon' L(T) \circ L' G R \pi' (\epsilon') \circ L' \psi_{1(T)} \\
\quad & = \epsilon' L(T) \circ L' R' \pi' (\epsilon') \circ L' \psi_{1(T)} \\
\quad & = \pi(\epsilon') L(T) \circ L' G \pi(\epsilon') \circ L' \psi_{1(T)} \\
\quad & = \pi(\epsilon') L(T) \circ L' \psi_{1(T)} \\
\quad & = \pi(\epsilon') L(T) \circ \psi L(T) \\
\quad & = \epsilon' L(T) \circ \psi L(T)
\end{align}

\begin{align}
\eta(T) & \circ \epsilon' L(T) \\
\quad & = \eta(T) \circ \epsilon' L(T) \\
\quad & = \psi' L(T) \\
\quad & = \epsilon' L(T) \circ \psi' L(T)
\end{align}

\begin{align}
\alpha & := \left( P(T) \eta P(T) \Rightleftharpoons RLP(T) \Rightarrow R (\epsilon') L(T) \Rightarrow RL(T) \right)
\end{align}
\[ \epsilon'(L(T)) \circ L' \epsilon'(\epsilon L(T) \circ L \alpha) = \epsilon' L(T) \circ L' (R' \epsilon L(T) \circ R' L \alpha) \]

where (*) follows by the equality \( \zeta P(T) = L' \psi_{1}(T) = \psi' L(T) \) that can be easily checked.

We have so proved that \( \epsilon'(L(T)) \circ L' \left( G \alpha \circ \psi_{1}(T) \right) = \epsilon' L(T) \circ L' \left( R' \epsilon L(T) \circ R' L \alpha \right) \). By the adjunction this is equivalent to \( G \alpha \circ \psi_{1}(T) = R' \epsilon L(T) \circ R' L \alpha \). By Lemma 1.6, the map \( \alpha \) induces \( \eta(T) : \text{Id}_{L(T)} \to R(T) L(T) \) such that \( P(T) \eta(T) = \alpha \). We compute

\[
\begin{align*}
\epsilon (T) L(T) \circ L(T) \eta(T) \circ \pi (\epsilon') L(T) \\
= (\pi (\epsilon') S(\epsilon') L(T))^{-1} \circ \Upsilon(\epsilon') \epsilon L(T) \circ \Upsilon(\epsilon') L(T) \eta(T) \circ \pi (\epsilon') L(T) \\
= P' \epsilon L(T) \circ LP(T) \eta(T) = L \alpha \\
= \epsilon L(T) \circ LR \pi (\epsilon') L(T) \circ L \eta P(T) \\
= \pi (\epsilon') L(T) \circ \epsilon L P(T) \circ L \eta P(T) = \pi (\epsilon') L(T)
\end{align*}
\]

so that \( \epsilon(T) L(T) \circ L(T) \eta(T) = \text{Id}_{L(T)} \).

We compute

\[
\begin{align*}
P(T) (R(T) \epsilon(T) \circ \eta(T) R(T)) \\
= P(T) R(T) \epsilon(T) \circ P(T) \eta(T) R(T) \\
= R \epsilon(T) \circ \alpha R(T) \\
= (R \pi (\epsilon') S(\epsilon'))^{-1} \circ R \Upsilon (\epsilon') \epsilon \circ R \pi (\epsilon') L(T) R(T) \circ \eta P(T) R(T) \\
= (R \pi (\epsilon') S(\epsilon'))^{-1} \circ R \pi (\epsilon') S(\epsilon') \circ R P' \epsilon \circ \eta R \\
= R \epsilon \circ \eta R = \text{Id}_R.
\end{align*}
\]

We have so proved that \( (L(T), R(T), \eta(T), \epsilon(T)) \) is an adjunction. We compute

\[
\begin{align*}
\epsilon L(T) \circ LP(T) \eta(T) \\
= \epsilon L(T) \circ L \alpha \\
= \epsilon L(T) \circ LR \pi (\epsilon') L(T) \circ L \eta P(T) \\
= \pi (\epsilon') L(T) \circ \epsilon L P(T) \circ L \eta P(T) = \pi (\epsilon') L(T)
\end{align*}
\]

Thus, since \( P(T) R(T) = R \), the diagram in the statement is an adjoint triangle and the natural transformation inside it is the correct one. Note that (11) follows by definition of adjoint triangle and, since \( \pi (\epsilon') L(T) R(T) \) is an epimorphism, it uniquely determines \( \epsilon(T) \).

Starting from an adjunction \( (L, R, \eta, \epsilon) \), with \( R : A \to B \) and where \( A \) has all coequalizers, we are now able to construct a kind of monadic decomposition that will be called an adjoint decomposition as follows, where we set \( L[0] := L, R[0] := R \) and \( B[0] := B \).

\[
\begin{align*}
\begin{array}{c}
\text{T[0,1]} \\
\text{T[1,2]} \\
\text{T[2,3]} \\
\text{T[n-1,n]}
\end{array}
& \begin{array}{c}
A \\
B
\end{array}
& \begin{array}{c}
A \\
B
\end{array}
& \begin{array}{c}
A \\
B
\end{array}
& \begin{array}{c}
A \\
B
\end{array}
& \begin{array}{c}
A \\
B
\end{array}
\end{align*}
\]

In the diagram above we label by \( T_{[0,1]} \) the first adjoint triangle from left, by \( T_{[1,2]} \) the second one and in general by \( T_{[n-1,n]} \) the \( n \)-th one. Denote by \( T_{[n]} \) the composition of the first \( n \) adjoint triangles. They are constructed iteratively as follows. The adjoint triangle \( T_{[0]} \) is defined as in the following diagram while, for \( n > 0 \), we set \( T_{[n-1,n]} := (T_{[n-1]})^2 \) (see Proposition 2.5) and \( T_{[n]} := T_{[n-1,n]} \ast T_{[n-1]} \).

\[
\begin{align*}
T_{[0]} := \begin{array}{c}
A \\
B
\end{array} \\
\text{R}_{[0]} := \text{Id}_{L} \\
\text{U}_{[0,1]}
\end{align*}
\]

\[
\begin{align*}
T_{[n]} := \begin{array}{c}
A \\
B
\end{array} \\
\text{R}_{[n]} := \text{Id}_{L} \\
\text{U}_{[n,\infty]}
\end{align*}
\]
Explicitly $B_{[1]} = I(T_{[0]}) = (RL)G = (RL)Id_g$, $U_{[1]} = U_{[0,1]} = P(T_{[0]})$ the forgetful functor.

\[
\begin{align*}
\mathcal{L}(T_{[0]}) : & \quad B_{[1]} \rightarrow (LR)(Id_A), \quad (B,b) \mapsto (LB,\zeta B \circ Lb) = (LB,Lb), \quad h \mapsto Lh, \\
R_{[1]} & = R(T_{[0]}) = R(T_{[0]}) \circ S(\epsilon) : A \rightarrow I(T_{[0]}), \quad A \mapsto (RA,ReA), \quad \quad f \mapsto Rf, \\
L_{[1]} & = L(T_{[0]}) = \mathcal{U}(\epsilon) \circ \mathcal{L}(T_{[0]}), \quad (B,b) \mapsto \mathcal{U}(\epsilon)(LB,Lb), \quad h \mapsto \mathcal{U}(\epsilon)(Lh), \\
\pi_{[1]} & = \pi_{[0,1]} = \zeta(T_{[0]}) = \pi(\epsilon) \mathcal{L}(T_{[0]}) \circ \pi_{[0]} P(T_{[0]}) \pi(\epsilon) \mathcal{L}(T_{[0]}).
\end{align*}
\]

The unit $\eta_{[1]} = \eta(T_{[0]})$ and the counit $\epsilon_{[1]} = \epsilon(T_{[0]})$ of the adjunction $(L_{[1]},R_{[1]})$ are uniquely defined by

\[
U_{[1]}\eta_{[1]} = R\pi_{[1]} \circ \eta U_{[1]} \quad \text{and} \quad \epsilon_{[1]} \circ \pi_{[1]} R_{[1]} = \epsilon = \epsilon_{[0]}.
\]

Note that for every $B_{[1]} := (B,b) \in B_{[1]}$ we have the following coequalizer

\[
\begin{array}{c}
\xymatrix{ LRLB \ar[r]^{\epsilon_{LB}} & LB \ar[r]^{\pi_{[1]}B_{[1]}} & L_{[1]}B_{[1]} }
\end{array}
\]

Next $B_{[2]} = I(T_{[1]}) = (RL)U_{[1]}, U_{[1,2]} := P(T_{[1]})$ and $U_{[2]} = U_{[0,1]} \circ U_{[1,2]}$. Moreover

\[
\begin{align*}
\mathcal{L}(T_{[1]}) : & \quad B_{[2]} \rightarrow (LR)(Id_A), \quad (B_{[1]},b_{[1]}) \mapsto (L_{[1]}B_{[1]},\pi_{[1]}B_{[1]} \circ Lb_{[1]}), \quad h \mapsto L_{[1]}h, \\
R_{[2]} & = R(T_{[1]}) = R(T_{[1]}) \circ S(\epsilon_{[1]}) : A \rightarrow B_{[2]}, \quad A \mapsto (R_{[1]}A,Re_{[1]}A), \quad f \mapsto R_{[1]}f, \\
L_{[2]} & = L(T_{[1]}) = \mathcal{U}(\epsilon) \circ \mathcal{L}(T_{[1]}), B_{[2]} \rightarrow A, \quad \quad (B_{[1]},h_{[1]}) \mapsto \mathcal{U}(\epsilon)(L_{[1]}B_{[1]},\pi_{[1]}B_{[1]} \circ Lb_{[1]}), \quad h \mapsto \mathcal{U}(\epsilon)(L_{[1]}h), \\
\pi_{[1,2]} & = \pi(\epsilon) \mathcal{L}(T_{[1]}), \\
\pi_{[2]} & = \pi_{[2,1]} \ast \pi_{[1,0]} = \pi_{[1,2]} \circ \pi_{[1]}U_{[1,2]}
\end{align*}
\]

The unit $\eta_{[2]} = \eta(T_{[1]})$ and the counit $\epsilon_{[2]} = \epsilon(T_{[1]})$ of the adjunction $(L_{[2]},R_{[2]})$ are uniquely determined by

\[
U_{[1,2]}\eta_{[2]} = R_{[1]}\pi_{[2]} \circ \eta_{[1]}U_{[1,2]} \quad \text{and} \quad \epsilon_{[2]} \circ \pi_{[2]} R_{[2]} = \epsilon_{[1]}.
\]

Note that for every $B_{[2]} := (B_{[1]},b_{[1]}) \in B_{[2]}$ we have the following coequalizer

\[
\begin{array}{c}
\xymatrix{ LRL_{[1]}B_{[1]} \ar[r]^{\epsilon_{L_{[1]}B_{[1]}}} & L_{[1]}B_{[1]} \ar[r]^{\pi_{[1,2]}B_{[2]}} & L_{[2]}B_{[2]} }
\end{array}
\]

Finally $B_{[n+1]} = I(T_{[n]}) = (RL)(U_{[n]}), U_{[n,n+1]} = P(T_{[n]})$ and $U_{[n+1]} = U_{[n]} \circ U_{[n,n+1]}$. Moreover

\[
\begin{align*}
\mathcal{L}(T_{[n]}) : & \quad B_{[n+1]} \rightarrow (LR)(Id_A), \quad (B_{[n]},b_{[n]}) \mapsto (L_{[n]}B_{[n]},\pi_{[n]}B_{[n]} \circ Lb_{[n]}), \quad h \mapsto L_{[n]}h, \\
R_{[n+1]} & = R(T_{[n]}) = R(T_{[n]}) \circ S(\epsilon_{[n]}) : A \rightarrow B_{[n+1]}, \quad A \mapsto (R_{[n]}A,Re_{[n]}A), \quad f \mapsto R_{[n]}f, \\
L_{[n+1]} & = L(T_{[n]}) = \mathcal{U}(\epsilon) \circ \mathcal{L}(T_{[n]}), \quad \quad B_{[n+1]} \rightarrow A, \quad (B_{[n]},h_{[n]}) \mapsto \mathcal{U}(\epsilon)(L_{[n]}B_{[n]},\pi_{[n]}B_{[n]} \circ Lb_{[n]}), \quad h \mapsto \mathcal{U}(\epsilon)(L_{[n]}h), \\
\pi_{[n,n+1]} & = \pi(\epsilon) \mathcal{L}(T_{[n]}), \\
\pi_{[n+1]} & = \pi_{[n,n+1]} \ast \pi_{[n]} = \pi_{[n,n+1]} \circ \pi_{[n]}U_{[n,n+1]}
\end{align*}
\]

The unit $\eta_{[n+1]} = \eta(T_{[n]})$ and the counit $\epsilon_{[n+1]} = \epsilon(T_{[n]})$ of the adjunction $(L_{[n+1]},R_{[n+1]})$ are uniquely determined by

\[
U_{[n,n+1]}\eta_{[n+1]} = R_{[n]}\pi_{[n,n+1]} \circ \eta_{[n]}U_{[n,n+1]} \quad \text{and} \quad \epsilon_{[n+1]} \circ \pi_{[n,n+1]} R_{[n+1]} = \epsilon_{[n]}.
\]

Note that for every $B_{[n+1]} := (B_{[n]},b_{[n]}) \in B_{[n+1]}$ we have the following coequalizer

\[
\begin{array}{c}
\xymatrix{ LRL_{[n]}B_{[n]} \ar[r]^{\epsilon_{L_{[n]}B_{[n]}}} & L_{[n]}B_{[n]} \ar[r]^{\pi_{[n,n+1]}B_{[n+1]}} & L_{[n+1]}B_{[n+1]} }
\end{array}
\]
so that
\[ \pi_{[2]}B_{[2]} \circ Lb[n] = \pi_{[2]}B_{[2]} \circ \epsilon L[n]B[n] \]

By composing the functors on the bottom of (12) and the corresponding natural transformations
e one defines, for \( 0 \leq t \leq n, \)
\[
\begin{align*}
U_{[t,n]} &= U_{[t,t+1]} \circ U_{[t+1,t+2]} \circ \cdots \circ U_{[n-2,n-1]} \circ U_{[n-1,n]}, \\
\pi_{[t,n]} &= \pi_{[n-1,n]} \circ \pi_{[n-2,n-1]} \circ \cdots \circ \pi_{[t+1,t+2]} \circ \pi_{[t,t+1]} \\
&= \pi_{[n-1,n]} \circ \pi_{[n-2,n-1]} \circ \cdots \circ \pi_{[t+1,t+1]}U_{[t+1,n]}.
\end{align*}
\]

Let us give a more explicit description of objects and morphisms in the category \( B[n] \) for \( n \in \mathbb{N}. \)
First \( B[0] = B. \) An object in \( B[1] \) is a pair \( V[1] = (V, b[0] : RL[0]V \to V) \) where \( V \in B, b[0] \in B. \) An object in \( B[2] \) is a pair \( V[2] = (V[1], b[1] : RL[1]V[1] \to V), \) where \( V[1] = (V, b[0]) \in B[1], b[1] \in B[1]. \)
Thus we can regard \( V[2] \) as the tern \( (V, b[0] : RL[0]V \to V, b[1] : RL[1]V[1] \to V). \) Going on this way,
an object in \( B[n] \) has the form \( V[n] = (V, b[0], b[1], \ldots, b[n-1]) \) where \( b[t] : RL[t]V[t] \to V \) and \( V[t] = U[t,n]V[n] \) for each \( t \in \{0, \ldots, n-1\}. \)

The lower case \( n = 0 \) can also be included in the notation \( V[n] = (V, b[0], b[1], \ldots, b[n-1]) \) by thinking that the \( b[i] \)'s disappear. A datum such as \( (b[0], b[1], \ldots, b[n-1]) \) is called a \( R \)-structured sink in the literature.

A morphism \( f[1] : V[1] \to V'[1] \) in \( B[1] \) means a morphism \( f = U[1,f[1]] : V \to V' \) such that
\[
\begin{CD}
RL[0]V @> RL[0]f >> RL[0]V' \\
V @. @VV b[0] \downarrow \ \\
@. V'
\end{CD}
\]

For \( n > 1, \) a morphism \( f[n] : V[n] \to V'[n] \) in \( B[n] \) is a morphism \( f[n-1] = U[n-1,n,f[n]] : V[n-1] \to V'[n-1] \) such that
\[
\begin{CD}
RL[n-1]V[n-1] @> RL[n-1]f[n-1] >> RL[n-1]V'[n-1] \\
V[n-1] @. @VV b[n-1] \downarrow \ \\
@. V'[n-1]
\end{CD}
\]

3. Comparing monadic and adjoint decompositions

Next aim is to connect the monadic and adjoint decompositions by constructing functors \( (\Lambda_n)_{n \in \mathbb{N}} \)
making commutative the solid faces of diagram (1) for every \( n \geq 1. \)

To this aim we first prove some technical results needed to obtain Proposition 3.5 which is the main tool to iteratively construct \( (\Lambda_n)_{n \in \mathbb{N}} \) in Remark 3.6.

**Proposition 3.1.** Assume \( A \) has coequalizers and consider the two adjoint triangles \( T, T' \) and their composition \( T'' \) of Remark 2.2. Then there is an adjoint triangle
\[
\begin{CD}
A @> \text{Id} >> A \\
\text{L}(T'') @. @ARL(T'')A \\
\text{L}(T) @AA \text{R}(T)A \\
\text{I}(T'') @> \text{I}(\theta) >> \text{I}(T)
\end{CD}
\]

The functor \( I(\theta) \) is defined by
\[ I(\theta) : I(T'') \to I(T), \quad (B'', \mu'') \mapsto (\Theta B'', \mu'' \circ R\theta B''), \quad f \mapsto \Theta P(T'') f \]
and it satisfies
\[ P(T) \circ I(\theta) = \Theta \circ P(T'') \quad \text{and} \quad G(T) \circ I(\theta) = G'' \circ P(T''). \]
The natural transformation \( \theta : L(T)I(\theta) \to L(T') \) appearing inside the adjoint triangle is defined by \( U[T, \theta] = \theta P(T'') \).
If \( \Theta \) is faithful so is \( I(\theta) \).
If \( \theta \) is invertible (resp. the identity) so is \( \Theta \).

**Proof.** The functor \( I(\theta) \) can be more properly defined as follows:

\[
I(\theta) := D^\Theta \circ (R'\theta | \mathrm{Id}_{G''}) : I(T'') \rightarrow I(T) = \langle R'L'' | G'' \rangle,
\]

\[
\left( B'', R'L'' B'' \rightarrow G'' B'' \right) \mapsto \left( \Theta B'', R'L\Theta B'' R'\theta B'' \rightarrow R'L'' B'' \rightarrow G'' B'' = G\Theta B'' \right)
\]

We compute:

\[
P(T) \circ I(\theta) = P(T) \circ D^\Theta \circ (R'\theta | \mathrm{Id}_{G''}) = \Theta \circ P_{(R'\theta | G'')} \circ (R'\theta | \mathrm{Id}_{G''}) = \Theta \circ P(T''),
\]

\[
G(T) \circ I(\theta) = G \circ P(T) \circ I(\theta) = G \circ \Theta \circ P(T'') = G'' \circ P(T'').
\]

Let us construct \( \Theta : \mathcal{L}(T) I(\theta) \rightarrow \mathcal{L}(T') \). It is easy to check that

\[
\mathcal{L}(T) I(\theta) = (L\Theta P(T'')) \left[ \zeta \Theta P(T'') \circ L'\psi_{I(T'')} \circ L' R'\theta P(T'') \right],
\]

\[
\mathcal{L}(T') = (L'' P(T'')) \left[ \theta P(T'') \circ \zeta \Theta P(T'') \circ L' \psi_{I(T'')} \right].
\]

Since

\[
\theta P(T'') \circ \left( \zeta \Theta P(T'') \circ L'\psi_{I(T'')} \circ L' R'\theta P(T'') \right) = \left( \theta P(T'') \circ \zeta \Theta P(T'') \circ L' \psi_{I(T'')} \right) \circ L' R'\theta P(T''),
\]

by Lemma 1.6, there is a unique \( \Theta : \mathcal{L}(T) I(\theta) \rightarrow \mathcal{L}(T') \) such that \( U_{[1]} \Theta = \theta P(T'') \).

Consider \( U(\epsilon') \theta : U(\epsilon') \mathcal{L}(T) I(\theta) \rightarrow U(\epsilon') \mathcal{L}(T') \) i.e. \( U(\epsilon') \theta : L(T) I(\theta) \rightarrow L(T') \). This gives rise to the adjoint triangle (19). In fact we have

\[
P(T) \circ I(\theta) \circ R(T'') = \Theta \circ P(T'') \circ R(T'') = \Theta \circ R'' = R = P(T) \circ R(T),
\]

\[
\psi_{I(T)} I(\theta) R(T'') = \left( \psi_{I(T'')} \circ R'\theta P(T'') \right) R(T'') = R' R'' = R'(\epsilon'' \circ \theta R'') = R'(\epsilon'' \circ \theta R'') = R'(\epsilon = \psi_{I(T)} R(T).
\]

By Lemma 1.6, we get \( I(\theta) \circ R(T'') = R(T). \) Moreover

\[
\epsilon(T'') \circ U(\epsilon') \theta R(T'') \circ \pi' L(T) I(\theta) R(T'') = \epsilon(T'') \circ \pi' L(T'') R(T'') \circ U_{[1]} \theta R(T'')
\]

\[
\left( \epsilon \right. \epsilon'' \circ \theta P(T'') R(T'') \epsilon'' \circ \theta R'' = \epsilon(\pi' L(T) R(T)
\]

so that \( \epsilon(T'') \circ U(\epsilon') \theta R(T'') = \epsilon(T) \) which means that \( U(\epsilon') \theta \) is the correct natural transformation to put inside the adjoint triangle.

If \( \Theta \) is faithful, from \( P(T) \circ I(\theta) = \Theta \circ P(T'') \) and the fact that \( P(T'') \) is faithful we get that \( I(\theta) \) is faithful too.

If \( \theta \) is invertible, from \( U_{[1]} \theta = \theta P(T'') \) and the fact that \( U_{[1]} \) reflects isomorphisms, we deduce that \( \Theta \) is invertible as well.

If \( \theta \) is the identity, then \( L = L'' \) so that, by the foregoing, we get

\[
\mathcal{L}(T) I(\theta) \left( B'', \beta'' \right) = \left( L\Theta B'', \zeta \Theta B'' \circ L'\beta'' \circ L' R'\theta B'' \right) \left( L'' B'', \theta B'' \circ \zeta \Theta B'' \circ L' \beta'' \right) = \mathcal{L}(T') \left( B'', \beta'' \right).
\]

Hence the domain and codomain of \( \theta \left( B'', \beta'' \right) \) are the same. Thus, since \( U_{[1]} \theta \left( B'', \beta'' \right) = \theta B'' = \mathrm{Id}_{L'' B''} = U_{[1]} \mathrm{Id}_{\mathcal{L}(T') \left( B'', \beta'' \right)} \) and \( U_{[1]} \) is faithful, we obtain \( \theta \left( B'', \beta'' \right) = \mathrm{Id}_{\mathcal{L}(T') \left( B'', \beta'' \right)} \). \( \square \)

**Proposition 3.2.** Assume \( \mathcal{A} \) has coequalizers. Given an adjoint triangle \( T \) as in (5), then

\[
\begin{array}{ccc}
A & \xrightarrow{\mathrm{Id}} & A \\
\mathrm{L}_{[1]} & \downarrow & \mathrm{L}(T) \\
B_{[1]} & \xrightarrow{S^G} & I(T)
\end{array}
\]

is an adjoint triangle too. If \( \zeta R \) is epimorphism on each component then \( \sigma_{[1]} \) is invertible.
Proposition 3.3. Assume $\mathcal{A}$ has coequalizers. Consider the two adjoint triangles $\mathcal{T}$, $\mathcal{T}'$ and their composition $\mathcal{T}''$ of Remark 2.2. Then we can define a new adjoint triangle

$$
\begin{array}{ccc}
\mathcal{T}'' & \cong & \mathcal{T}'' \\
\Theta_{[1]} : B''_{[1]} & \rightarrow & \mathcal{I}(\mathcal{T}) \\
\end{array}
$$

where

$$
\Theta_{[1]} (V'', \mu'') = I (\mathcal{T}) , \quad (V'', \mu'') \mapsto (\Theta V'', G'' \mu'' \circ R' \theta V'') , \quad f \mapsto \Theta U'', f,
$$

and such that

$$
P (\mathcal{T}) \Theta_{[1]} = \Theta' U''_{[1]} \quad \text{and} \quad G' (\mathcal{T}) \Theta_{[1]} = G'' U''_{[1]} .
$$

1) If $\Theta$ is faithful so is $\Theta_{[1]}$.

2) If $\theta$ is invertible and any component of $\zeta R$ is an epimorphism, then $\theta_{[1]}$ is invertible.

Proof. By composing the two adjoint triangles obtained in Proposition 3.1 and Proposition 3.2, the latter applied to $\mathcal{T}''$, i.e.

$$
\begin{array}{ccc}
\mathcal{T}'' & \cong & \mathcal{T}'' \\
B''_{[1]} = \langle R'' | \id \rangle & \cong & \mathcal{I}(\mathcal{T}) \\
\end{array}
$$

we obtain the triangle (20) with

$$
\Theta_{[1]} = \sigma''_{[1]} \circ U' (\varepsilon') \theta = \sigma''_{[1]} \circ U (\varepsilon') \theta S'' G'' , \quad \text{and} \quad \Theta_{[1]} = I (\mathcal{T}) \circ S'' G'' .
$$

Explicitly for every $(V'', \mu'') \in B''_{[1]}$ we have

$$
\Theta_{[1]} (V'', \mu'') = I (\mathcal{T}) S'' G'' (V'', \mu'') = I (\mathcal{T}) (V'', G'' \mu'') = (\Theta V'', G'' \mu'') = (\Theta V'', G'' \mu'') \circ R' \theta V''
$$
and for every morphism \( f'' \in B''_{1} \), we have \( \Theta_{[1]} f = I(\theta) S^{G''} f = \Theta U''_{0,1} f \).

If \( \Theta \) is faithful, by Proposition 3.1, so is \( I(\theta) \). Since \( S^{G''} \) acts as the identity on morphisms it is faithful too and we get that \( \Theta_{[1]} \) is faithful as a composition of faithful functors.

Assume \( \theta \) is invertible and that any component of \( \zeta R \) is an epimorphism. By Proposition 3.1, \( \theta \) is invertible. Now
\[
\zeta'' R'' = (\theta \ast \zeta) R'' = \theta R'' \circ \zeta \Theta R'' = \theta R'' \circ \zeta R
\]
which is an epimorphism on each component. Thus, by Proposition 3.2, we get that \( \sigma'' \) is invertible. Hence \( \Theta_{[1]} \) is invertible as composition of invertible.

\[\Box\]

**Proposition 3.4.** Assume \( A \) has coequalizers. Consider an adjunction \( L \dashv R : A \to B \). Then the inclusion functor \( \Lambda_{1} : B_{1} \to B_{[1]} \) gives rise to an adjoint triangle

\[
\begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
\downarrow L_{[1]} & & \downarrow L_{[1]} \\
B_{1} & \xrightarrow{\Lambda_{1}} & B_{[1]} = (RL | \text{Id})
\end{array}
\]

such that \( U_{[1]} \circ \Lambda_{1} = U_{0,1} \) and \( L_{[1]} \circ \Lambda_{1} = L_{1} \).

**Proof.** Clearly \( B_{1} \) is a full subcategory of \( B_{[1]} \) through \( \Lambda_{1} \) and one has \( \Lambda_{1} \circ R_{1} = R_{[1]} \). By construction we have that \( L_{1} = L_{[1]} \circ \Lambda_{1} \) and \( \pi_{[1]} A_{1} = \pi_{1} : L U_{1} \to L_{1} \) is the canonical projection. Note that \( L_{[1]} R_{1} = L_{[1]} \Lambda_{1} R_{1} = L_{1} R_{1} \) and
\[
epsilon_{[1]} \circ \pi_{1} R_{1} = \epsilon_{[1]} \circ \pi_{1} \Lambda_{1} R_{1} = \epsilon_{[1]} \circ \pi_{[1]} R_{[1]} \equiv \epsilon = \epsilon_{1} \circ \pi_{1} R_{1}
\]
so that \( \epsilon_{[1]} = \epsilon_{1} \). Note that the last equality, in the above displayed formula, is just the definition of the counit \( \epsilon_{1} \) of \( (L_{1}, R_{1}) \). As a consequence \( \lambda_{1} = \epsilon_{[1]} L_{1} \circ L_{[1]} \Lambda_{1} \eta_{1} = \epsilon_{1} L_{1} \circ L_{1} \eta_{1} = \text{Id}_{L_{1}} \).

\[\Box\]

**Proposition 3.5.** Assume \( A \) has coequalizers. Consider the two adjoint triangles \( T, T' \) and their composition \( T'' \) of Remark 2.2. Then we can define a new adjoint triangle

\[
\begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
\downarrow L''_{[1]} & & \downarrow L''_{[1]} \\
B''_{1} & \xrightarrow{\theta_{1}} & B''_{[1]} = (R''L'' | \text{Id})
\end{array}
\]

where
\[
\begin{array}{ccc}
\Theta_{1} : B''_{1} & \to & I(T) \\
V''_{1}, \mu'' & \to & (\Theta V'', \mu'' \circ R'' \theta V''), \\
\theta & \to & \Theta U''_{0,1} f,
\end{array}
\]
is such that
\[
P(T) \Theta_{1} = \Theta U''_{0,1} \quad \text{and} \quad G(T) \Theta_{1} = G'' U''_{0,1}.
\]

1) If \( \Theta \) is faithful so is \( \Theta_{1} \).

2) Assume that \( \theta \) is invertible and that any component of \( \zeta R \) is an epimorphism.

Then \( \theta_{1} \) is invertible. Moreover if \( \Theta \) is full so is \( \Theta_{1} \) and if \( \Theta \) is injective on objects so is \( \Theta_{1} \).

**Proof.** Compose the adjoint triangles of Proposition 3.4 (applied to the adjunction \( (L'', R'') \)) and Proposition 3.3

\[
\begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
\downarrow L''_{[1]} & & \downarrow L''_{[1]} \\
B''_{1} & \xrightarrow{\Lambda_{1}} & B''_{[1]} = (R''L'' | \text{Id})
\end{array}
\]

and the adjoint triangle in the present statement. Thus \( \Theta_{1} = \Theta_{[1]} \circ \Lambda_{1} \) and \( \theta_{1} = \lambda_{1} \ast \theta_{[1]} = \lambda_{1} \circ \theta_{[1]} \Lambda_{1} = \theta_{[1]} \Lambda_{1} \).
If $\Theta$ is faithful, by Proposition 3.3, so is $\Theta_1$. Since the inclusion functor $A_1$ is faithful it is then clear that $\Theta_1$ is faithful too as a composition of faithful functors.

Assume that $\theta$ is invertible and that any component of $\zeta R$ is an epimorphism. Still by Proposition 3.3, we deduce that $\theta_{1}^{R}A_{1}$ is invertible too.

It remains to prove that if $\Theta$ is full so is $\Theta_1$. Let $\xi : \Theta_1(W'',\mu_{W''}) \to \Theta_1(W'',\mu_{W''})$ be a morphism in $\text{I}(\mathcal{T}) = \langle R'\mathcal{L}(G) \rangle$. This means to have a morphism $h := P(\mathcal{T}) \xi : \Theta V'' \to \Theta W''$ such that

\[Gh \circ (G''\mu_{V''} \circ R'\theta V'') = (G''\mu_{W''} \circ R'\theta W'') \circ R'\xiW.
\]

Since $G'' = G\Theta$, $R = \Theta R''$ and $R' = GR$ we can rewrite this equality as

\[G(h \circ \Theta\mu_{V''} \circ \Theta R''\theta V'') = G(\Theta\mu_{W''} \circ \Theta R''\theta W'' \circ \Theta R''\xiW).
\]

Since $\Theta$ is full, there is a morphism $g : V'' \to W''$ such that $h = \Theta g$ so that, using $G'' = G\Theta$, we can further rewrite

\[G''(g \circ \mu_{V''}) = G''(\mu_{W''} \circ R''L''g)\]

By naturality of $\theta$ we have $\mu_{W''} \circ R''\theta W'' \circ R''L'\Theta g = \mu_{W''} \circ R''L''g \circ R''\theta V''$ so that, since $\theta$ is invertible, we obtain

\[G''(g \circ \mu_{V''}) = G''(\mu_{W''} \circ R''L''g)\]

and hence

\[L''(g \circ \mu_{V''}) \circ \zeta''R''L''V'' \overset{\text{nat.}}{=} \zeta''W'' \circ L''(g \circ \mu_{V''}) = \zeta''W'' \circ L''(\mu_{W''} \circ R''L''g) \overset{\text{nat.}}{=} \zeta''L''(\mu_{W''} \circ R''L''g) \circ \zeta''R''L''V''.
\]

Now $\zeta'' = \theta \ast \zeta = \theta \circ \Theta \xi$ so that $\zeta''R'' = \theta R'' \circ \zeta R'' = \theta R'' \circ \zeta R$ which is an epimorphism on each component. Thus we arrive at

\[L''(g \circ \mu_{V''}) = L''(\mu_{W''} \circ R''L''g).
\]

Using this equality we compute

\[g \circ \mu_{V''} = \mu_{W''} \circ \eta''W'' \circ g \circ \mu_{V''} = \mu_{W''} \circ R''L''(g \circ \mu_{V''}) \circ \eta''R''L''V'' = \mu_{W''} \circ R''L''(\mu_{W''} \circ R''L''g) \circ \eta''R''L''V'' = \mu_{W''} \circ \eta''W'' \circ \mu_{W''} \circ R''L''g = \mu_{W''} \circ R''L''g
\]

This means there is a morphism $g_1 : (V'',\mu_{V''}) \to (W'',\mu_{W''})$ such that $U''_{0,1}g_1 = g$. We have

\[P(\mathcal{T})\Theta_1g_1 = \Theta U''_{0,1}g_1 = \Theta g = h = P(\mathcal{T})\xi.
\]

Since $P(\mathcal{T})$ is faithful, we deduce that $\Theta_1g_1 = \xi$. Thus $\Theta_1$ is full.

Assume that $\Theta$ is injective on objects and $\Theta_1(V'',\mu_{V''}) = \Theta_1(W'',\mu_{W''})$. Then we can apply the above argument for $\xi := \text{Id}$. In this case $h := P(\mathcal{T})\xi = \text{Id} : \Theta V'' \to \Theta W''$. The fact that $\Theta$ is injective on objects tells that $V'' = W''$ so that we write $h = \Theta g$ for $g = \text{Id}$ (and the above assumption that $\Theta$ is full can be dropped out). As above we arrive at $g \circ \mu_{V''} = \mu_{W''} \circ R''L''g$ i.e. $\mu_{V''} = \mu_{W''}$. We have so proved that $(V'',\mu_{V''}) = (W'',\mu_{W''})$ and hence $\Theta_1$ is injective on objects.

\[\text{Remark 3.6.} \text{ Consider an adjunction } (L, R, \eta, \epsilon) \text{ and assume } A \text{ has coequalizers.}
\]

Apply Proposition 3.5 to $\mathcal{T} = \mathcal{T}' = \mathcal{T}_{[0]}$ to obtain the adjoint triangle $\Lambda_{[1]}$.

\[\Lambda_{[1]} := L_{1}[\mathcal{T}] \to R_{1}[\mathcal{T}] \to \Lambda_{[1]} \to L_{1}[\mathcal{T}] \to R_{1}[\mathcal{T}]
\]

Since the natural transformations inside the adjoint triangles $\mathcal{T}$ and $\mathcal{T}'$ are the identities, by the same results, we get that $\lambda_1 : L_{[1]}A_1 \to L_1$ is invertible. Moreover $U_{0,1}^{[1]}A_1 = U_{0,1}^{[1]}A_{1}$ and $\Lambda_1$ is fully faithful and injective on objects. Recall that any component of $\pi_{[1]}^{[1]}R_{[1]}$ is an epimorphism.
Hence all the conditions in Proposition 3.5 are verified for $T' = \Lambda_{[1]}$ and $T = T_{[1]}$ and we obtain the adjoint triangle $\Lambda_{[2]}$

where $\lambda_2 : L_{[2]} \Lambda_2 \to L_2$ is invertible. Moreover $U_{[1,2]} \Lambda_2 = \Lambda_1 U_{1,2}$ and $U_{[2]} \Lambda_2 = U_{[0,1]} \Lambda_1 U_{1,2}$ i.e. $U_{[0,2]} \Lambda_2 = U_{[0,1]} \Lambda_1 U_{1,2} = U_{[0,2]}$. Furthermore $\Lambda_2$ is fully faithful and injective on objects. Recall that any component of $\pi_{[2]} R_{[2]}$ is an epimorphism.

Going on this way we construct iteratively $(\Lambda_n)_{n\in\mathbb{N}}$ such that $\Lambda_0 := \text{Id}$ and $U_{[n-1,n]} \Lambda_n = \Lambda_{n-1} U_{n-1,n}$, for every $n \geq 1$. Moreover $\lambda_n : L_{[n]} \Lambda_n \to L_n$ is invertible, $U_{[0,n]} \Lambda_n = U_{0,n}$ for every $n \in \mathbb{N}$ and $\Lambda_n$ is fully faithful and injective on objects.

**Remark** Note that, by construction $\Lambda_n$ is defined as follows

$\Lambda_n : B_n \to B_{[n]}$, $(V_{n-1}, \mu_{n-1}) \mapsto (\Lambda_n \mu_{n-1} \circ R\lambda_{n-1} V_{n-1})$, $f \mapsto \Lambda_{n-1} U_{n-1,n} f$

i.e.

$\Lambda_n : B_n \to B_{[n]}$, $(V_{n-1}, \mu_{n-1}) \mapsto (\Lambda_n \mu_{n-1} \circ R\lambda_{n-1} V_{n-1})$, $f \mapsto \Lambda_{n-1} U_{n-1,n} f$.

Since $\Lambda_n : B_n \to B_{[n]}$ is fully faithful, we get that $B_n$ is equivalent to the essential image of $\Lambda_n$. Later on we will look for handy criteria for an object in $B_{[n]}$ to belong to the image of $\Lambda_n$.

4. Relative Grothendieck fibrations

In order to deduce properties of the functors $\Lambda_n$, a relative version of the notion of Grothendieck fibration is needed. We collect here its definition and properties.

**Definition 4.1.** Let $F : A \to B$ be a functor and let $M$ be a class of morphisms in $B$.

We say that a morphism $f \in A$ is cartesian (with respect to $F$) over a morphism $f' \in B$ whenever $F f = f'$ and given $g \in A$ and $h \in B$ such that $F f \circ h = F g$, then there exists a unique morphism $k \in A$ such that $F k = h$ and $f \circ k = g$ [14, Definition 8.1.2].

![Diagram](image)

We say that $F$ is an $M$-fibration if every morphism $f' : B \to FA$ in $M$ there is $f : A' \to A$ which is cartesian over $f'$. When $M$ is the class of all morphisms in $B$ we recover the notion of fibration, see [14, Definition 8.1.3].

**Remark 4.2.** A morphism $f : X \to Y$ is cartesian over $F f$ if and only if following diagram is a pullback for every object $Z$, where the vertical maps are obtained by evaluating $F$.

![Pullback Diagram](image)

In fact the map $\text{Hom}(Z, X) \to \text{Hom}(Z, Y) \times_{\text{Hom}(FZ, FY)} \text{Hom}(FZ, FX) : k \mapsto (f \circ k, Fk)$ into the pullback becomes bijective. This fact is well-known, see e.g. [34, Definition 4.32.1].

**Lemma 4.3** (Cf. [35, Proposition 3.4(ii)]). Being cartesian is transitive.

**Proof.** Since the vertical composition of pullbacks is a pullback ([13, Proposition 2.5.9]), it follows from Remark 4.2. □

Recall that an isofibration (called transportable functor in [18, Corollaire 4.4]) is a functor $F : A \to B$ such that for any object $A \in A$ and any isomorphism $f' : B \to FA$, there exists an isomorphism $f : A' \to A$ such that $F f = f'$. A discrete isofibration is an isofibration such that $f$ is unique (see [22, page 13]).

It is known that every fibration is an isofibration. Let us prove a relative version of this result.

**Proposition 4.4.** Let $\text{Iso}$ be the class of all isomorphisms in $B$. Then $F : A \to B$ is an iso-fibration if and only if it is an isofibration.
Hence we get is there exists an isomorphism $f : A' \to A$ which is cartesian over $f'$. In particular $Ff = f'$ and $FA' = B$. From $Ff \circ (f')^{-1} = FId_A$ we deduce that there exists a unique morphism $k : A \to A'$ such that $Fk = (f')^{-1}$ and $f \circ k = Id_A$. Similarly, from $Ff \circ Id_B = Ff$, we get a unique morphism $\lambda : A' \to A'$ such that $F\lambda = Id_B$ and $f \circ \lambda = f$. Since $F(k \circ f) = Fk \circ Ff = (f')^{-1} \circ f' = Id$ and $f \circ (k \circ f) = f$, we get $\lambda = k \circ f$. On the other hand since $FId_A = Id_B$ and $f \circ Id_A = f$ we also have $\lambda = Id_A$. Hence $k \circ f = Id$. We have so proved that $f$ is an isomorphism. Thus $F$ is an isofibration.

$(\Leftarrow)$ Let $f' : B \to FA$ be in $\text{Iso}$. Then it is an isomorphism. Since $F$ is an isofibration there is there exists an isomorphism $f : A' \to A$ such that $Ff = f'$. Let $g \in A$ and $h \in B$ such that $Ff \circ h = Fg$. Then we can take $k = f^{-1} \circ g$ to get $Fk = F(f^{-1}) \circ F(g) = F(f^{-1}) \circ Ff \circ h = h$ and $f \circ k = g$. On the other hand any morphism $k$ such that $Fk = h$ and $f \circ k = g$, from the latter equality is $f^{-1} \circ g$. We have so proved that $f$ is cartesian over $f'$. Hence $F$ is an $M$-fibration. □

**Corollary 4.5.** If $M \supseteq \text{Iso}$ and $F : A \to B$ is an $M$-fibration then $F$ is an isofibration.

**Proof.** Clearly from $M \supseteq \text{Iso}$ we deduce that $F$ is $M$-fibration implies $F$ is an $\text{Iso}$-fibration. By Proposition 4.4, $F$ is an isofibration. □

**Remark 4.6.** If $F$ is an isofibration which is faithful and injective on objects then $F$ is a discrete isofibration. In fact, if there is another $t : A'' \to A$ such that $Ft = f'$, then $FA'' = B = FA'$ so that $A = A'$. Moreover $Ft = f' = Ff$ so that $t = f$. Hence $f$ is unique.

**Definition 4.7.** Given a functor $F : A \to B$, consider the following categories.

- $\text{Im}(F)$, the image of $F$, i.e. the subcategory of objects in $B$ of the form $FA$ for some $A \in A$ and of morphisms in $B$ of the form $Ff$ for some $f \in A$.
- $\text{Eim}(F)$, the essential image of $F$, i.e. the full subcategory of objects in $B$ isomorphic to $FA$ for some $A \in A$.
- $\text{Im}'(F)$, i.e. the full subcategory of objects in $B$ of the form $FA$ for some $A \in A$.

Clearly $\text{Im}(F) \subseteq \text{Eim}(F)$ and $\text{Im}(F) \subseteq \text{Im}'(F)$ hold always.

**Lemma 4.8.** Let $F : A \to B$ be a functor.

1) If $F$ is an iso isofibration. Then $\text{Eim}(F) \subseteq \text{Im}'(F)$.

2) If $F$ is full. Then $\text{Im}'(F) = \text{Im}(F)$.

**Proof.** 1). Given an object $B$ in $\text{Eim}(F)$ then $B \in B$ is endowed with an isomorphism $f' : B \to FA$ for some $A \in A$. Since $F$ is an isofibration we get an isomorphism $f : A' \to A$ such that $Ff = f'$. In particular $FA' = B$ whence $B \in \text{Im}(F)$. Since $\text{Eim}(F)$ and $\text{Im}'(F)$ at both full subcategories of $B$ we get $\text{Eim}(F) \subseteq \text{Im}'(F)$.

2). The two categories have the same objects. Let $g : FA' \to FA$ be a morphism in $\text{Im}'(F)$. Since $F$ is full, there is $f : A' \to A$ such that $g = Ff$. Then $g$ is a morphism in $\text{Im}(F)$. Since the two categories share the same objects we get $\text{Im}'(F) \subseteq \text{Im}(F)$. Hence the equality holds. □

**Lemma 4.9.** Let $F : A \to B$ be a fully faithful functor. Then $F$ is an $M$-fibration if and only if for every morphism $f' : B \to FA$ in $M$ there is $A' \in A$ such that $FA' = B$.

**Proof.** Assume that for every morphism $f' : B \to FA$ in $M$ there is $A' \in A$ such that $FA' = B$.

Since $F$ is fully faithful there is $f : A' \to A$ such that $Ff = f'$. In order to prove that $f : A' \to A$ is cartesian over $f'$, let $g : C \to A$ in $A$ and $h : FC \to FA'$ in $B$ such that $Ff \circ h = Fg$. Since $F$ is fully faithful there exists a unique morphism $k \in A$ such that $Fk = h$ and from $Ff \circ h = Fg$ and faithfulness of $F$ we conclude that $f \circ k = g$ as desired. □

**Remark 4.10.** Proposition 1.7 states that a morphism $g \in \{F|G\}$ is cartesian over $Pg$ whenever $GPG$ is a monomorphism. In other words any morphism $g \in M(GP)$ is cartesian over $Pg$ where

$$M(F) = \{f \in A \mid Ff \text{ is a monomorphism}\}.$$

**Proposition 4.11.** For every $n \in \mathbb{N}$, every morphism $g \in M(U[n])$ is cartesian over $U[n]g$.
Proof. We proceed by induction on \( n \in \mathbb{N} \). The first step is trivially true since \( \mathcal{U}[0] = \text{Id}_\mathcal{B} \).

Let \( n \geq 1 \) and assume the statement true for \( n - 1 \). Let \( g \) be a morphism in \( \mathcal{M} (\mathcal{U}[n]) \). Since \( \mathcal{U}[n] = \mathcal{U}[n-1]\mathcal{U}[n-1,n] \) we get that \( \mathcal{U}[n-1,n]g \in \mathcal{M} (\mathcal{U}[n-1]) \). By inductive hypothesis we have that \( \mathcal{U}[n-1,n]g \) is cartesian over \( \mathcal{U}[n-1]\mathcal{U}[n-1,n]g = \mathcal{U}[n]g \). By Lemma 4.3, it remains to prove that \( g \) is cartesian over \( \mathcal{U}[n-1,n]g \) in order to conclude. To this aim observe that \( \mathcal{U}[n-1,n] = \mathcal{P} (\mathcal{T}[n-1]) \).

Thus, by Remark 4.10 applied to \( P = \mathcal{P} (\mathcal{T}[n-1]) \), \( F = RL[n-1] \), \( G = \mathcal{U}[n-1] \), we get that \( g \) is cartesian over \( \mathcal{U}[n-1,n]g \) as \( g \in \mathcal{M} (\mathcal{U}[n-1]\mathcal{U}[n-1,n]) \).

\textbf{Proposition 4.12.} Let \( (L, R) \), with \( R : \mathcal{A} \to \mathcal{B} \), be an adjunction with unit \( \eta \) and counit \( \epsilon \).

(1) A morphism \( f \) is cartesian with respect to \( L \) over \( Lf \) if and only if the following diagram is a pullback.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\eta_X \downarrow & & \downarrow\eta_Y \\
RLX & \xrightarrow{RLf} & RLY
\end{array}
\]

(2) A morphism \( f \) is cartesian with respect to \( R \) over \( Rf \) if and only if \( \epsilon Z \perp f \) (that is \( \epsilon Z \) and \( f \) are orthogonal) for every \( Z \in \mathcal{A} \) i.e. any commutative diagram as follows admits a unique diagonal filler \( k \) making both triangles commute.

\[
\begin{array}{ccc}
LRZ & \xrightarrow{\epsilon Z} & Z \\
\downarrow k & & \downarrow v \\
X & \xrightarrow{f} & Y
\end{array}
\]

Proof. (1). Assume that \( f : X \to Y \) is cartesian over \( Lf \) and let us prove that (21) is a pullback. Let \( u : Z \to Y \) and \( v : Z \to RLX \) be such that \( \eta_Y \circ u = RLf \circ v \). We have

\[
Lf \circ LX \circ Lf = \epsilon LY \circ LRLf \circ LV = \epsilon LY \circ L (RLf \circ v) = \epsilon LY \circ L (\eta Y \circ u) = \epsilon LY \circ L \eta Y \circ Lu = Lu
\]

so that the following diagram commutes.

\[
\begin{array}{ccc}
& & LZ \\
& \xRightarrow{\epsilon Z} & \\
LRZ & \xrightarrow{\epsilon L} & Z \\
\downarrow k & & \downarrow v \\
& & v \\
LX & \xrightarrow{Lf} & LRY
\end{array}
\]

Since \( f \) is cartesian over \( Lf \), there is a unique \( k : Z \to X \) such that \( Lk = \epsilon LX \circ LV \) and \( f \circ k = u \). The condition \( Lk = \epsilon LX \circ LV \), via the adjunction isomorphism, is equivalent to \( RRLk \circ \eta Z = \mathcal{R} (\eta Z \circ LV) \circ \eta Z \) i.e. \( \eta X \circ k = ReLX \circ RLv \circ \eta Z \) i.e. \( \eta X \circ k = ReLX \circ \eta RLX \circ v \) i.e. \( \eta X \circ k = v \). Thus there is a unique \( k : Z \to X \) such that \( \eta X \circ k = v \) and \( f \circ k = u \). In other words (21) is a pullback.

Conversely assume that (21) is a pullback and let us prove that \( f : X \to Y \) is cartesian over \( Lf \). Let \( u : Z \to Y \) and \( h : LZ \to LX \) be such that \( Lf \circ h = Lu \). Set \( v := Rh \circ \eta Z \). Then

\[
RLf \circ v = RLf \circ Rh \circ \eta Z = RL \circ \eta Z = v
\]

By the universal property of the pullback there is a unique morphism \( k : Z \to X \) such that \( \eta X \circ k = v \) and \( f \circ k = u \). The condition \( \eta X \circ k = v \), via the adjunction isomorphism, is equivalent to \( \epsilon LX \circ L (\eta X \circ k) = \epsilon LX \circ LV \) i.e. \( \epsilon LX \circ L \eta X \circ Lk = \epsilon LX \circ L (Rh \circ \eta Z) \) i.e. \( Lk = \epsilon LX \circ LRh \circ L\eta Z = h \circ \epsilon LZ \circ L\eta Z = h \). Thus there is a unique \( k : Z \to X \) such that \( Lk = h \) and \( f \circ k = u \). In other words \( f \) is cartesian over \( Lf \).

(2). Assume that \( f : X \to Y \) is cartesian over \( Rf \) and let us prove that \( \epsilon Z \perp f \). Consider a commutative square as in (22). Set \( h := Ru \circ \eta RZ \). Then \( Rf \circ h = Rf \circ Ru \circ \eta RZ = R \circ ReLZ \circ \eta RZ \). Since \( f \) is cartesian over \( Rf \), there is a unique \( k : Z \to X \) such that \( Rk = h \) and \( f \circ k = v \). By the adjunction, the condition \( Rk = h \) is equivalent to \( \epsilon X \circ LRk = \epsilon X \circ Lh \) i.e. \( k \circ \epsilon Z = \epsilon X \circ L (Ru \circ \eta RZ) \) i.e. \( k \circ \epsilon Z = u \circ \epsilon LRZ \circ L \eta RZ = u \). Thus there is a unique \( k : Z \to X \) such that \( k \circ \epsilon Z = u \) and \( f \circ k = v \). Hence \( \epsilon Z \) and \( f \) are orthogonal.
Conversely suppose $\epsilon Z \bot f$ and let us prove $f$ is cartesian over $Rf$. Let $v : Z \to Y$ and $h : RZ \to RX$ be such that $Rf \circ h = Rv$. By the adjunction the later equality is equivalent to $\epsilon Y \circ L(Rf \circ h) = \epsilon Y \circ LRv$ i.e. $f \circ \epsilon X \circ Lh = v \circ \epsilon Z$. Thus, if we set $u := \epsilon X \circ Lh$, by the orthogonality there is a unique morphism $k : Z \to X$ such that $k \circ \epsilon Z = u$ and $f \circ k = v$.

By the adjunction the condition $k \circ \epsilon Z = u$ is equivalent to $R(k \circ \epsilon Z) \circ \eta RZ = Ru \circ \eta RZ$ i.e. $Rk \circ ReZ \circ \eta RZ = R(\epsilon X \circ Lh) \circ \eta RZ$ i.e. $Rk = ReX \circ RLh \circ \eta RZ = ReX \circ \eta RX \circ h = h$. Thus there is a unique $k : Z \to X$ such that $k \circ \epsilon Z = u$ and $Rk = h$ i.e. $f$ is cartesian over $Rf$.

A somewhat faster proof could be obtained by applying Remark 4.2.

If $F = L$ or $F = R$ we can rewrite the corresponding pullback by means of the adjunction obtaining respectively the diagrams

$$\begin{array}{ccc}
\text{ Hom}(Z,X) & \xrightarrow{\text{Hom}(Z,f)} & \text{ Hom}(Z,Y) \\
\text{ Hom}(Z,RLX) & \xrightarrow{\text{Hom}(Z,RLf)} & \text{ Hom}(Z,RLY)
\end{array}$$

The fact that the left-hand side diagram is a pullback means that (21) is a pullback, while the fact that the right-hand side diagram is a pullback means that $\epsilon Z \bot f$.

\textbf{Lemma 4.13.} The functor $\Lambda_1 : B_1 \to B_{[1]}$ of Remark 3.6 is an $\mathcal{M} \langle U_{[1]} \rangle$-fibration.

\textbf{Proof.} Let $f_{[1]} : B_{[1]} \to \Lambda_1 C_1$ be a morphism in $\mathcal{M} \langle U_{[1]} \rangle$ i.e. such that $U_{[1]}f_{[1]}$ is a monomorphism. Since $\Lambda_1$ is fully faithful, in order to conclude, by Lemma 4.9, it suffices to prove that there is $B_1 \in B_1$ such that $\Lambda_1 B_1 = B_{[1]}$.

Write $B_{[1]} = (B,b : RLB \to B)$, $C_1 = (C,c : RLC \to C)$ and note that $C_{[1]} := \Lambda_1 C_1 = (C,c : RLC \to C)$ this time regarded as an object in $B_{[1]}$. Set $f := U_{[1]}f_{[1]} : B \to C$ and consider the following diagrams.

$$\begin{array}{ccc}
RLRLB & \xrightarrow{RLb} & RLB \\
RLRLf & \xrightarrow{RLc} & RLC
\end{array}$$

$$\begin{array}{ccc}
RLB & \xrightarrow{b} & B \\
RLf & \xrightarrow{c} & C
\end{array}$$

The left-hand side one serially commutes since $f$ induces the morphism $f_{[1]}$ and by naturality of $c$. Since $C_1 \in B_1$, we also have that $c \circ RLC = c \circ RLC$. Since $f$ is a monomorphism, we deduce that $b \circ RLb = b \circ RLC$. A similar argument as above, but applied on the right-hand side diagram, shows that $b \circ \eta B = \text{Id}_B$. This means that $B_1 := (B,b : RLB \to B) \in B_1$. By definition of $\Lambda_1$, we have that $\Lambda_1 B_1 = B_{[1]}$.

\textbf{Lemma 4.14.} 1) Let $F : A \to B$ be a functor and let $f \in \mathcal{M}(F)$ be cartesian over $Ff$. Then $f$ is a monomorphism.

2) Let $F : A \to B$ be an $\mathcal{M}(G)$-fibration. Then any morphism $f \in \mathcal{M}(F) \cap \mathcal{M}(GF)$ factors as $f = g \circ k$, where $g$ is a monomorphism and $Fk = \text{Id}$.

\textbf{Proof.} 1) Let $f : A \to A'$ in $\mathcal{M}(F)$ be cartesian over $Ff$. By definition of $\mathcal{M}(F)$ we have that $Ff$ is a monomorphism. Let $a,b : A' \to A$ be such that $f \circ a = f \circ b$. Then $Ff \circ Fa = Ff \circ Fb$. Since $Ff$ is a monomorphism, we get $Fa = Fb$. Call $h$ this morphism and $g := f \circ a$. Then $Ff \circ h = Fg$. Since $f$ is cartesian over $Ff$, there exists a unique morphism $k \in A$ such that $Fk = h$ and $f \circ k = g$. Hence $a = k = b$.

2) Let $f : A \to A'$ be in $\mathcal{M}(F) \cap \mathcal{M}(GF)$. Then $Ff$ and $GFf$ are both monomorphisms.

In particular $f' := Ff : FA \to FA'$ is in $\mathcal{M}(G)$ so that, by definition of $\mathcal{M}(G)$-fibration, there is $g : E \to A'$ which is cartesian over $f'$. Since $Fg \circ \text{Id}_{FA} = Ff$, there is a unique $k : A \to E$ such that $Fk = \text{Id}_{FA}$ and $g \circ k = f$.

Moreover $g \in \mathcal{M}(F)$ is cartesian over $Fg$ so that, by 1), we get that $g$ is a monomorphism.
We are now going to prove Propositions 4.15, Proposition 4.16 and Theorem 4.17. These results will be used to obtain Theorem 4.18 that is our main tool to get Theorem 4.19 where the embedding $\Lambda_n$ is shown to be an $M\left(U\left[n\right]\right)$-fibration.

**Proposition 4.15.** In the setting of Proposition 3.2, assume that any morphism $g \in M\left(G\right)$ is cartesian over $Gg$. Then for every morphism $f' : (B, \beta) \to S^G\alpha[1]$ in $M\left(GP\left(T\right)\right)$ there is a morphism $f : B[1] \to C[1]$ in $M\left(GU[1]\right)$ which is cartesian with respect to $S^G : B[1] \to \langle GRL | G \rangle = I\left(T\right)$ over $f'$. In particular $S^G$ is an $M\left(G \circ P\left(T\right)\right)$-fibration.

**Proof.** Let $f' : (B, \beta) \to S^G\alpha[1]$ be a morphism in $M\left(GP\left(T\right)\right)$, in particular it is a morphism in $I\left(T\right)$. Write $C[1] = (C, c : RL \to C)$ so that $S^G\alpha[1] = (C, Gc)$. The fact that $f'$ is a morphism $I\left(T\right)$ means that the following left-hand side diagram commutes

\[
\begin{array}{ccc}
GRLB & \xrightarrow{\text{GRPL}(T)f'} & GRLC \\
\beta \downarrow & & \downarrow \text{Gr} \\
GB & \xrightarrow{\text{GP}(T)f'} & GC
\end{array}
\]

(23)

Since $f' \in M\left(GP\left(T\right)\right)$, we get $P\left(T\right)f' \in M\left(G\right)$. By hypothesis $P\left(T\right)f'$ is cartesian with respect to $G$ over $GP\left(T\right)f'$. As a consequence the diagram above implies there is a unique morphism $b : RL \to B$ such that $Gb = \beta$ and the right-hand side diagram in (23) commutes.

Set $B[1] = (B, b)$ and $B[1] = (B, \beta)$. Then the last diagram means that there is a unique morphism $f := P\left(T\right)f'[1] : B[1] \to C[1]$ such that $U[1]f = P\left(T\right)f'$. Hence $GU[1]f = GP\left(T\right)f'$ is a monomorphism so that $f \in M\left(GU[1]\right)$.

Let us check that $f$ is cartesian with respect to $S^G$ over $f'$.

Note that $S^G\alpha[1] = (B, GB) = (B, \beta)$ so that $S^Gf$ has the same domain and codomain of $f' : (B, \beta) \to S^G\alpha[1]$. Thus we get the equality $S^Gf = f'$ by the following computation

\[
P\left(T\right)S^Gf = U[1]f = P\left(T\right)f'
\]

and the fact that $P\left(T\right)$ is faithful. Consider $g[1]$ and $h$ as in the following left-hand side diagram.

\[
\begin{array}{ccc}
S^G\alpha[1] & \xrightarrow{s^Gf} & S^G\alpha[1] \\
\uparrow s^Gg[1] & & \uparrow U[1]g[1] \\
U[1]B[1] & \xrightarrow{P\left(T\right)h} & U[1]C[1]
\end{array}
\]

By applying $P\left(T\right)$ we get the right-hand side diagram above. We know that $U[1]f = P\left(T\right)f' \in M\left(G\right)$. Then, by hypothesis $U[1]f$ is then cartesian with respect to $G$ over $GU[1]f$. Thus, by Lemma 4.14, we get that $U[1]f$ is a monomorphism. Thus $f \in M\left(U[1]\right)$. By Proposition 4.11, we get that $f$ is cartesian over $U[1]f$. As a consequence, the right-hand side diagram above implies there is a unique morphism $d[1] : D[1] \to B[1]$ such that $U[1]d[1] = P\left(T\right)h$ and $f \circ d[1] = g[1]$. Note that

\[
P\left(T\right)S^Gd[1] = U[1]d[1] = P\left(T\right)h
\]

and hence $S^Gd[1] = h$. It remains to prove the uniqueness of $d[1]$. If there is another $k[1] : D[1] \to B[1]$ such that $S^Gk[1] = h$ and $f \circ k[1] = g[1]$. Then

\[
U[1]k[1] = P\left(T\right)S^Gk[1] = P\left(T\right)h = U[1]d[1]
\]

so that $k[1] = d[1]$ as $U[1]$ is faithful. \hfill $\square$

When $G = \text{Id}$, $L' = L$, $R' = R$, the functor $\Theta[1]$ will be denoted by $\Theta'[1] : B[1] \to B[1]$.

**Proposition 4.16.** Assume that

\begin{itemize}
  \item any morphism $g \in M\left(G\right)$ is cartesian over $Gg$;
  \item $\Theta : B' \to B$ is an $M\left(G\right)$-fibration,
  \item $\theta$ is invertible.
\end{itemize}
If $\Theta : \mathcal{B}'' \to \mathcal{B}$ is an $\mathcal{M}(G)$-fibration, then for every morphism $f : B_{[1]} \to \Theta'_{[1]} C_{[1]}''$ in $\mathcal{M}(GU_{[1]})$ there is $f''_{[1]} : B'_{[1]} \to C'_{[1]}''$ in $\mathcal{M}
abla{U''_{[1]}}$ which is cartesian with respect to $\Theta'_{[1]}$ over $f$. In particular $\Theta'_{[1]} : B'_{[1]} \to B_{[1]}$ is an $\mathcal{M}(GU_{[1]})$-fibration.

Proof. Let $f : B_{[1]} \to \Theta'_{[1]} C_{[1]}''$ be a morphism in $\mathcal{M}(GU_{[1]})$. Write $B_{[1]} = (B, b : \mathcal{R}LB \to B)$ and $C_{[1]}'' = (C'', c'' : \mathcal{R}L''C'' \to C'')$. We have  
\[ \Theta'_{[1]} C_{[1]}'' = \Theta_{[1]} (C'', c'') = (\Theta C'', G\Theta c'' \circ GR\theta C'') = (\Theta C'', \Theta c'' \circ R\theta C'') . \]
The fact that $f \in B_{[1]}$ means that the first diagram in (24) commutes.

\[
\begin{array}{cccc}
RLB & \xrightarrow{RLU_{[1]} f} & RL\Theta C'' & \\
B & \xrightarrow{U_{[1]} f} & \Theta C'' & \\
\end{array}
\]

Since $f \in \mathcal{M}(GU_{[1]})$, we have that $U_{[1]} f \in \mathcal{M}(G)$. Since $\Theta : \mathcal{B}'' \to \mathcal{B}$ is an $\mathcal{M}(G)$-fibration, there is a morphism $f'' : B'' \to C''$ which is cartesian (with respect to $\Theta$) over $U_{[1]} f$. In particular $\Theta B'' = B$ and $\Theta f'' = U_{[1]} f$. Note that $R = \Theta R''$ so that the first diagram in (24) rewrites as the second one therein. Since $f''$ is cartesian (with respect to $\Theta$) over $U_{[1]} f = \Theta f''$, this diagram implies there is a unique morphism $\tau'' : R'' LB \to B''$ such that the third diagram in (24) commutes.

Set $b'' := \tau'' \circ (R'' \theta B'')^{-1}$ and $B'' := (B'', b'')$. Then
\[ f'' \circ b'' \circ R'' \theta B'' = f'' \circ \tau'' = c'' \circ R'' \theta C'' \circ R'' \Theta f'' = c'' \circ R'' L'' f'' \circ R'' \theta B'' \]
and hence $f'' \circ b'' = c'' \circ R'' L'' f''$. As a consequence $f''$ induces a morphism $f''_{[1]} : (B'', b'') \to (C'', c'')$ such that $U_{[1]} f''_{[1]} = f''$. We compute
\[ U_{[1]} \Theta'_{[1]} f''_{[1]} = \Theta U''_{[1]} f''_{[1]} = \Theta f''_{[1]} = U_{[1]} f \]
Since
\[ \Theta'_{[1]} f''_{[1]} = (\Theta B'', \Theta b'' \circ R\theta B'') = (\Theta B'', \Theta (b'' \circ R'' \theta B'')) = (\Theta B'', \Theta r'') = (B, b) = B_{[1]} \]
we have that $\Theta'_{[1]} f''_{[1]}$ and $f$ have the same domain (and codomain). Since $U_{[1]} f$ is faithful, we get $\Theta'_{[1]} f''_{[1]} = f$.

Since $U_{[1]} \Theta'_{[1]} f''_{[1]} = U_{[1]} f \in \mathcal{M}(G)$, by hypothesis $U_{[1]} \Theta'_{[1]} f''_{[1]}$ is cartesian over $GU_{[1]} \Theta'_{[1]} f''_{[1]}$. By Lemma 4.14, we deduce that $U_{[1]} \Theta'_{[1]} f''_{[1]}$ is a monomorphism. Since $U_{[1]} \Theta'_{[1]} = U_{[1]} \Theta''_{[1]}$, we get that $U_{[1]} f''_{[1]} \in \mathcal{M}(\Theta)$. The latter morphism is $f''$ which is cartesian (with respect to $\Theta$) over $\Theta f''$. Again, by Lemma 4.14, we deduce that $f'' = U_{[1]} f''_{[1]}$ is a monomorphism i.e. $f''_{[1]} : B''_{[1]} \to C''_{[1]}$ in $\mathcal{M}
abla{U''_{[1]}}$ as desired.

Let us check that $f''_{[1]} : B''_{[1]} \to C''_{[1]}$ is cartesian with respect to $\Theta_{[1]}$ over $f$.

Let $g''_{[1]} : D''_{[1]} \to C''_{[1]}$ in $B''_{[1]}$ and $h : \Theta_{[1]} D''_{[1]} \to B_{[1]}$ be such that $\Theta_{[1]} g''_{[1]} \circ h = \Theta_{[1]} g''_{[1]}$. By applying on both sides $U_{[1]}$, we get $U_{[1]} \Theta_{[1]} g''_{[1]} \circ U_{[1]} h = U_{[1]} \Theta_{[1]} g''_{[1]}$, i.e. $\Theta'_{[1]} g''_{[1]} \circ U_{[1]} h = \Theta'_{[1]} g''_{[1]}$.

Since $f'' : B'' \to C''$ is cartesian over $U_{[1]} f = \Theta f''$, we get that there is a unique morphism $k'' : D'' \to B''$ in $B''$ such that $U_{[1]} h = \Theta k''$ and $f'' \circ k'' = U_{[1]} g''_{[1]}$. Let us check that $k''$ induces a morphism $k''_{[1]} : D''_{[1]} \to B''_{[1]}$ such that $\Theta_{[1]} k''_{[1]} = h$. Write $D''_{[1]} = (D'', d'') : \mathcal{R}L''L''D'' \to \mathcal{R}L''D''$.

Knowing that $f'' \circ b'' = c'' \circ R'' L'' f''$ and that $g''_{[1]} : D''_{[1]} \to C''_{[1]}$ belongs to $B''_{[1]}$, we obtain
\[ f'' \circ b'' \circ R'' L'' k'' = c'' \circ R'' L'' f'' \circ R'' L'' k'' = c'' \circ R'' L'' U_{[1]} g''_{[1]} = U_{[1]} g''_{[1]} \circ d'' = f'' \circ k'' \circ d'' . \]

Since we proved that $f''$ is a monomorphism, we get that $b'' \circ R'' L'' k'' = k'' \circ d''$ i.e. that $k$ induces a morphism $k''_{[1]} : D''_{[1]} \to B''_{[1]}$ such that $U_{[1]} k''_{[1]} = k''$. We have
\[ U_{[1]} \Theta_{[1]} k''_{[1]} = U_{[1]} g''_{[1]} = \Theta k'' = U_{[1]} h . \]
Since $\Theta'_1|B''_1 = B_1$ we have that $\Theta'_1|k''_1$ and $h$ have the same domain and codomain. Since $U_1$ is faithful, we obtain that $\Theta_1|k''_n = h$. Moreover

$$U''_1 \left( f''_1 \circ k''_1 \right) = U''_1 f''_1 \circ U''_1 k''_1 = f'' \circ k'' = U''_1 g''_1.$$  

Since $U''_1$ is faithful, we get $f''_1 \circ k''_1 = g''_1$. Moreover $k''_1$ is unique since $U''_1$ is faithful and $f''$ is a monomorphism. We have so proved that $f''_1 : B''_1 \to C''_1$ is cartesian over $f$. □

**Theorem 4.17.** In the setting of Proposition 3.5, assume that

- any morphism $g \in M(G)$ is cartesian over $Gg$;
- $\Theta : B'' \to B$ is an $M(G)$-fibration,
- $\theta$ is invertible.

Then for every morphism $f' : (B, \beta) \to \Theta_1|C''_1$ in $M(G \circ P(T))$ there is $f''_1 : B''_1 \to C''_1$ in $M\left(U''_1\right)$ which is cartesian with respect to $\Theta_1$ over $f'$. In particular $\Theta_1 : B''_1 \to I(T)$ is an $M\left(G \circ P(T)\right)$-fibration.

Proof. First note that $\Theta_1 = S^G\Theta'_1$ as they coincide on morphisms and for every $C''_1 = (C'', c' : R''L''C'' \to C'') \in B''_1$, we have

$$\Theta_1|C''_1 = \Theta_1(C'', c'') = (\Theta C'', G'' c'' \circ R'' \theta C'') = (\Theta C'', G\Theta c'' \circ GR\theta C'').$$

Let $f' : (B, \beta) \to \Theta_1|C''_1$ in $M(G \circ P(T))$. Since $\Theta_1|C''_1 = S^G\Theta'_1|C''_1$, by Proposition 4.15, there is a morphism $f : B_1 \to \Theta'_1|C'_1$, in $M\left(GU_1\right)$ which is cartesian with respect to $S^G$ over $f'$. By Proposition 4.16, there is $f''_1 : B''_1 \to C''_1$ in $M\left(U''_1\right)$ which is cartesian with respect to $\Theta'_1$ over $f$. By Lemma 4.3, the morphism $f''_1$ is cartesian with respect to $\Theta_1 = S^G\Theta'_1$ over $f'$. □

**Theorem 4.18.** In the setting of Proposition 3.5, assume that

- any morphism $g \in M(G)$ is cartesian over $Gg$;
- $\Theta : B'' \to B$ is an $M(G)$-fibration,
- $\theta$ is invertible.

Then $\Theta_1 : B'' \to I(T)$ is an $M\left(G \circ P(T)\right)$-fibration.

Proof. Let $f' : (B, \beta : R'L'B \to GB) \to \Theta_1|C''_1$ be a morphism in $M\left(G \circ P(T)\right)$ i.e. a morphism in $I(T)$ such that $GP(T)f' : GB \to GP(T)\Theta_1|C''_1 = G\Theta C''$ is a monomorphism, where $C''_1 = (C'', c' : R''L''C'' \to C'')$. Since $\Theta_1 = \Theta_1|\Lambda_1$, by Theorem 4.17, there is a morphism $f''_1 : (B'', b'') \to (C'', c'')$ in $M\left(U''_1\right)$ which is cartesian with respect to $\Theta_1$ over $f'$. In particular $\Theta_1|B''_1 = (B, \beta)$ and $\Theta_1|f''_1 = f'$. Since $f''_1 \in M\left(U''_1\right)$, by Lemma 4.13, there is $f_1 : B_1 \to C'_1$ which is cartesian with respect to $\Lambda_1 : B_1 \to B_1$ over $f''_1$. We compute

$$\Theta_1 f_1 = \Theta_1|\Lambda_1 f_1 = \Theta_1|f''_1 = f'.$$

Since $f_1$ is cartesian with respect to $\Lambda_1$ over $f''_1$ and $f''_1$ is cartesian with respect to $\Theta_1$ over $f'$, by Lemma 4.3, $f_1$ is cartesian with respect to $\Theta_1 = \Theta_1|\Lambda_1$ over $f'$. □

**Theorem 4.19.** Let $n \in \mathbb{N}$. The functor $\Lambda_n : B_n \to B_{[n]}$ of Remark 3.6 is an $M\left(U_{[n]}\right)$-fibration. Moreover it is a discrete isofibration and $Eim\left(\Lambda_n\right) = \text{Im}\left(\Lambda_n\right) = \text{Im}'\left(\Lambda_n\right)$ (see Definition 4.7).

Proof. We proceed by induction on $n \in \mathbb{N}$. The first step is trivially true since $\Lambda_0 = \text{Id}_G = U_{[0]}$.

Let $n \geq 1$ and assume the statement true for $n - 1$. Apply Theorem 4.18 to $\Theta := \Lambda_{n-1}, G := U_{[n-1]} : T = T_{[n-1]}$ by noting that $\Lambda_n = (\Lambda_{n-1})_1$, that $\Theta$ fulfills the required conditions by inductive hypothesis, $G$ also fulfills them by Proposition 4.11 and $\theta = \lambda_{n-1}$ is invertible by Remark 3.6. Thus $\Lambda_n$ is an $M\left(U_{[n]}\right)$-fibration.
Any isomorphism in $\mathcal{B}_{[n]}$ belongs trivially to $M(U_{[n]})$. Moreover, by Remark 3.6, we know that $\Lambda_n$ is (fully) faithful and injective on objects. Thus we can apply Corollary 4.5 and Remark 4.6 to obtain that $\Lambda_n$ is a discrete isofibration. From Lemma 4.8 and the fact that $\Lambda_n$ is full, we get that $\text{Eim}(\Lambda_n) \subseteq \text{Im}(\Lambda_n)$. We know that $\text{Im}(\Lambda_n) \subseteq \text{Eim}(\Lambda_n)$ holds always.

The following result gives conditions for an object in $\mathcal{B}_{[n]}$ to be image via $\Lambda_n$ of an object in $\mathcal{B}_n$.

**Theorem 4.20.** Fix $n \in \mathbb{N}$ consider the functors $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$ and $U_{[n]} : \mathcal{B}_{[n]} \to \mathcal{B}$.

1. For every morphism $B_{[n]} \to \Lambda_n C_n$ in $M(U_{[n]})$ we have $B_{[n]} \in \text{Im}(\Lambda_n)$.
2. Let $B_{[n]} \in \mathcal{B}_{[n]}$ be such that $\eta_{[n]} B_{[n]}$ is in $M(U_{[n]})$. Then $B_{[n]} \in \text{Im}(\Lambda_n)$.
3. Let $(B_{[n]}, b_{[n]}) \in \text{Im}(\Lambda_n + 1)$. Then $\eta_{[n]} B_{[n]}$ is in $M(U_{[n]})$.

**Proof.** 1) By Theorem 4.19, the functor $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$ is an $M(U_{[n]})$-fibration. Thus, for every morphism $B_{[n]} \to \Lambda_n C_n$ in $M(U_{[n]})$ there is $B_n \in \mathcal{B}_n$ such that $\Lambda_n B_n = B_{[n]}$.
2) Since $\eta_{[n]} B_{[n]}$ is a morphism $B_{[n]} \to \Lambda_n L_n B_{[n]} = \Lambda_n R_n L_n B_{[n]}$, we conclude by 1).
3) Since $(B_{[n]}, b_{[n]}) \in \text{Im}(\Lambda_n + 1)$, there is $B_{n+1} = (B_n, \mu_n : R_n L_n B_n \to B_n) \in \mathcal{B}_{n+1}$ such that $(B_{[n]}, b_{[n]}) = \Lambda_n + 1 B_{n+1}$. Then $\mu_n \circ \eta_{[n]} B_n = \text{Id}_{B_n}$ and $B_{[n]} = U_{[n+1]}(B_n, b_{[n]}) = U_{[n+1]} \Lambda_{n+1} B_{n+1} = \Lambda_n U_{n+1} B_{n+1} = \Lambda_n B_{[n]}$.

By definition, we have $\lambda_n = \epsilon_{[n]} L_n \circ \Lambda_n \eta_{[n]}$ so that

$$R_{[n]} \lambda_n \circ \eta_{[n]} \Lambda_n = R_{[n]} \epsilon_{[n]} L_n \circ R_{[n]} L_n \Lambda_n \eta_{[n]} = R_{[n]} \epsilon_{[n]} L_n \Lambda_n R_n L_n \circ \Lambda_n \eta_{[n]} = R_{[n]} \epsilon_{[n]} L_n \Lambda_n \eta_{[n]} = \lambda_n \mu_n \circ \Lambda_n \eta_{[n]} B_n = \text{Id}_{\Lambda_n B_n}.$$ 

As a consequence we obtain

$$\lambda_n \mu_n \circ \lambda_n B_n \circ \eta_{[n]} \Lambda_n B_n = \lambda_n \mu_n \circ \Lambda_n \eta_{[n]} B_n = \text{Id}_{\Lambda_n B_n}.$$ 

In particular $\eta_{[n]} B_{[n]} = \eta_{[n]} \Lambda_n B_n$ is in $M(U_{[n]})$.

**Corollary 4.21.** Fix $n \in \mathbb{N}$. If the functor $L_{[n]}$ is fully faithful so is $L_n$ and $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$ is a category isomorphism. In particular $R_n$ has a monadic decomposition of monadic length at most $n$.

**Proof.** We have the isomorphism $\lambda_n : L_{[n]} \circ \Lambda_n \to L_n$. Thus if $L_{[n]}$ is fully faithful we get that $L_n$ is fully faithful being isomorphic to a composition of fully faithful functors. By the dual version of [13, Proposition 3.4.1], we have that $\eta_{[n]}$ is invertible and hence $\eta_{[n]} B_{[n]}$ is in $M(U_{[n]})$ for every $B_{[n]} \in \mathcal{B}_{[n]}$. By Theorem 4.20, $B_{[n]} \in \text{Im}(\Lambda_n)$. Thus $\Lambda_n$ is surjective on objects. We already know that $\Lambda_n$ is injective on objects, see Remark 3.6, thus it is bijective on objects. Since we know it is also fully faithful, we deduce that it is an isomorphism.

**Corollary 4.22.** Consider an adjunction $(L, R)$ such that $L_{[1]}$ and $L_1$ exist. If $R_{[1]}$ is an equivalence of categories then $R$ is monadic. Moreover $\Lambda_1 : \mathcal{B}_1 \to \mathcal{B}_{[1]}$ is a category isomorphism.

**Proof.** If $R_{[1]}$ is an equivalence of categories then $L_{[1]}$ is an equivalence of categories and hence, by Corollary 4.21, $L_1$ is fully faithful and $\Lambda_1 : \mathcal{B}_1 \to \mathcal{B}_{[1]}$ is a category isomorphism. Since $R_{[1]} = \Lambda_1 \circ R_1$ we get that $R_1$ is an equivalence of categories. Equivalently $R$ is monadic.

**Example 4.23.** Let us show that the converse of Corollary 4.22 is not true, in general. Let $\mathcal{B} = \text{Vec}_k$. As a starting adjunction consider $(T, \Omega)$ where $T : \mathcal{B} \to \text{Alg}_k$ is the tensor algebra functor and $\Omega : \text{Alg}_k \to \mathcal{B}$ is the forgetful functor. It is well-known that $\Omega$ is strictly monadic i.e. the comparison functor $\Omega_1 : \text{Alg}_k \to \mathcal{B}_1$ is a category isomorphism, see [10, Theorem A.6]. Given $B \in \mathcal{B}$, consider the zero map $b : \Omega TB \to B$. Then $(B, b) \in \langle \Omega T, \text{Id} \rangle$ but $(B, b) \notin \text{Im}(\Lambda_1)$ since $b \circ \eta B \neq \text{Id}_B$, where $\eta$ is the unit of the adjunction $(T, \Omega)$. Thus $\Lambda_1$ is not surjective whence not even a category isomorphism. By Corollary 4.22, we conclude that $\Omega_{[1]}$ is not an equivalence.

$$\begin{array}{cccc}
\text{Alg}_k & \xrightarrow{\text{Id}} & \text{Alg}_k \\
\xrightarrow{T} & \xrightarrow{\lambda_1} & \xrightarrow{T_{[1]} \rho_{[1]}} & \xrightarrow{\rho_{[1]}} \\
\text{B}_1 & \xrightarrow{\Lambda_1} & \text{B}_{[1]} = \langle \Omega T, \text{Id} \rangle
\end{array}$$
We have so proved that $R$ is monadic although $R_{[1]}$ is not an equivalence for $R = \Omega$.

5. Connection to heavy separability

As an application of Theorem 4.20, in this section we show how to construct some functors $\Gamma_n : B \to B_{[n]}$ that factor through $\Lambda_n : B_n \to B_{[n]}$. The existence of $\Gamma_n$ is related to the notion of heavily separable functor recalled in the following definition.

**Definition 5.1.** [11, Definition 1.1] For every functor $F : B \to A$ we set

$$F_{X,Y} : \text{Hom}_B(X,Y) \to \text{Hom}_A(FX,FY) : f \mapsto Ff$$

Recall that $F$ is called **separable** if the canonical map $F_{X,Y}$ cosplits naturally i.e. there is a natural transformation $P_{X,Y} := P^F_{X,Y} : \text{Hom}_A(FX,FY) \to \text{Hom}_B(X,Y)$ such that $P_{X,Y} \circ F_{X,Y} = \text{Id}$ for every $X,Y \in B$. We say that $F$ is **heavily separable** (h-separable for short) if it is separable and one also has the equality $P_{X,Y}(f \circ g) = P_{Y,Z}(f) \circ P_{X,Y}(g)$, for every $X,Y \in B$ and morphisms $g : FX \to FY$ and $f : FY \to FZ$.

We point out that, given an adjunction $(L,R)$, then the left adjoint $L$ is h-separable if and only if the associated monad $(RL, R\epsilon L, \eta)$ has an augmentation, see [11, Corollary 2.7].

**Theorem 5.2.** Consider a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{F} & A' \\
\downarrow{L} & & \downarrow{L'} \\
B & \xrightarrow{R} & B'
\end{array}
$$

where $(L,R,\eta,\epsilon)$ and $(L',R',\eta',\epsilon')$ are adjunctions such that $F \circ L = L'$. Define $\xi : R \to R'F$ by

$$R \eta^R \Rightarrow R'F'L'R = R'F\pi_L R' \Rightarrow R'F\epsilon' \Rightarrow R'F.$$

Then $\xi L : RL \to R'L'$ is a morphism of monads such that

$$\epsilon'F \circ L'\xi = F\epsilon.$$

Assume that:

1) $A$ has all coequalizers and that $F$ preserves them;
2) $R'$ preserves coequalizers of pairs $(fe,f)$ where $f$ is composition of regular epimorphisms and $e$ is an idempotent morphism;
3) $R'$ preserves regular epimorphisms;
4) the monad $R'L'$ has an augmentation $\gamma' : R'L' \to \text{Id}_B$.

Then the monad $RL$ is augmented via $\gamma := \gamma' \circ \xi L : RL \to \text{Id}_B$. For every $n \in \mathbb{N}$, there are a functor $\Gamma_{[n]} : B \to B_{[n]}$ and a natural transformation $\gamma_{[n]} : RL_{[n]} \Gamma_{[n]} \to \text{Id}$, such that

$$\Gamma_{[0]} := \text{Id}_B, \gamma_{[0]} := \gamma \text{ and, for } n \geq 0,$$

$$\Gamma_{[n+1]}B = \left(\Gamma_{[n]}B, \gamma_{[n]}B\right) \in B_{[n+1]}, \quad \gamma_{[n]} \circ U_{[n]}\eta_{[n]} \Gamma_{[n]} = \text{Id}, \quad \gamma_{[n+1]} \circ R\pi_{[n,n+1]} \Gamma_{[n+1]} = \gamma_{[n]}.$$

Moreover $U_{[n,n+1]} \circ \Gamma_{[n+1]} = \Gamma_{[n]}$.

**Proof.** First we have

$$\epsilon'F \circ L'\xi = \epsilon'F \circ L'R'F\epsilon \circ L'\eta'R = F\epsilon \circ \epsilon'L'R \circ L'\eta'R = F\epsilon$$

so that (26) holds true. It is easy to check that $\xi L : RL \to R'FL = R'L'$ is a morphism of monads.

Since $\xi L : RL \to R'L'$ is a morphism of monads and $R'L'$ is augmented via $\gamma' : R'L' \to \text{Id}_B$, we get that $\gamma := \gamma' \circ \xi L : RL \to \text{Id}_B$ is an augmentation for $RL$.

We set $S_{[n]} := L_{[n]} \Gamma_{[n]}$ and we define iteratively $\Gamma_{[n]}, \gamma'_{[n]} : R'FS_{[n]} \to \text{Id}$ and

$$\gamma_{[n]} := \gamma'_{[n]} \circ \xi S_{[n]} : RS_{[n]} \to \text{Id}$$

such that

$$\gamma_{[n]} \circ U_{[n]}\eta_{[n]} \Gamma_{[n]} = \text{Id} \quad \text{and} \quad \gamma'_{[n]} \circ R'F\pi_{[n]} \Gamma_{[n]} = \gamma'.$$
as follows.

For \( n = 0 \), we set \( \Gamma_{[0]} := \text{Id}_B \), \( \gamma_0' := \gamma' \), \( \gamma_{[0]} := \gamma \) as required.

Let \( n \geq 0 \). Suppose that \( \Gamma_{[n]} \), \( \gamma_{[n]}' \) such that (27) hold are given and let us construct \( \Gamma_{[n+1]}, \gamma_{[n+1]}' \) such that \( \gamma_{[n+1]}' \circ U_{[n+1]} \eta_{[n+1]} \Gamma_{[n+1]} = \text{Id} \) and \( \gamma_{[n+1]}' \circ RF \pi_{[n+1]} \Gamma_{[n+1]} = \gamma' \).

Since \( B_{[n+1]} = (RL_{[n]}U_{[n]}) \) we can apply Lemma 1.6, taking \( Q = \Gamma_{[n]} \) and \( q = \gamma_{[n]} \), to construct a unique functor \( \Gamma_{[n+1]} = \Gamma_{[n]} : B \to B_{[n+1]} \) such that \( U_{[n+1]} \circ \Gamma_{[n+1]} = \Gamma_{[n]} \) and \( \psi_{\Gamma_{[n+1]}} = \gamma_{[n]} \).

Explicitly \( \Gamma_{[n+1]} B = (\Gamma_{[n]} B, \gamma_{[n]} B) \) as desired.

For \( B \in B \) consider the coequalizer (17) taking \( B_{[n+1]} := \Gamma_{[n+1]} B = (\Gamma_{[n]} B, \gamma_{[n]} B) \) :

\[
(28) \quad LRS_{[n]} B \xrightarrow{\pi_{[n]} \Gamma_{[n]} B \circ L \gamma_{[n]} B} S_{[n]} B \xrightarrow{\epsilon S_{[n]} B} S_{[n+1]} B
\]

Set \( e_{[n]} := U_{[n]} \eta_{[n]} \Gamma_{[n]} \circ \gamma_{[n]} \). Then \( e_{[n]} \) is an idempotent natural transformation. Moreover, since \( B_{[n]} \) is an augmented triangle, we have \( \epsilon = e_{[n]} \circ \pi_{[n]} R_{[n]} \) so that

\[
\epsilon S_{[n]} B \circ Le_{[n]} = e_{[n]} S_{[n]} B \circ \pi_{[n]} R_{[n]} S_{[n]} B \circ LU_{[n]} \eta_{[n]} \Gamma_{[n]} \circ L \gamma_{[n]} = e_{[n]} S_{[n]} B \circ L \eta_{[n]} \Gamma_{[n]} \circ \pi_{[n]} \Gamma_{[n]} \circ L \gamma_{[n]} = \pi_{[n]} \Gamma_{[n]} \circ L \gamma_{[n]}
\]

and hence \( (\pi_{[n]} \Gamma_{[n]} B \circ L \gamma_{[n]} B, \epsilon S_{[n]} B) = (\epsilon S_{[n]} B \circ Le_{[n]} B, \epsilon S_{[n]} B) \).

Moreover

\[
\epsilon S_{[n]} B = \epsilon_{[n]} S_{[n]} B \circ \pi_{[n]} R_{[n]} S_{[n]} B
\]

is a composition of regular epimorphisms as \( \epsilon_{[n]} S_{[n]} B \) is the coequalizer of the parallel pair of morphisms \( (L_{[n]} R_{[n]} e_{[n]} S_{[n]} B, \epsilon_{[n]} L_{[n]} R_{[n]} \pi_{[n]} S_{[n]} B) \) (a split coequalizer, as \( S_{[n]} = L_{[n]} \pi_{[n]} \Gamma_{[n]} \) and \( \pi_{[n]} R_{[n]} S_{[n]} B = \pi_{[0, n]} R_{[n]} S_{[n]} B = \pi_{[n-1, n]} R_{[n]} S_{[n]} B \circ \pi_{[n-2, n-1]} R_{[n-1]} S_{[n]} B \circ \cdots \circ \pi_{[0, 1]} R_{[1]} S_{[n]} B \).

Since, by hypothesis, \( F \) preserves coequalizers, we get that \( (FS_{[n+1]} B, F \pi_{[n+1]} \Gamma_{[n+1]} B) \) is the coequalizer of \( \left( F \left( \pi_{[n]} \Gamma_{[n]} B \circ L \gamma_{[n]} B \right), F \epsilon S_{[n]} B \right) = (F \epsilon S_{[n]} B \circ F L e_{[n]} B, F \epsilon S_{[n]} B) \) where \( FLe_{[n]} B \) is still idempotent and \( FFS_{[n]} B \) is still composition of regular epimorphism.

By the hypothesis, the latter coequalizer is preserved by \( R' \). Thus we get the coequalizer

\[
R' RFS_{[n]} B \xrightarrow{R' RFS_{[n]} B} R' FS_{[n]} B \xrightarrow{R' \pi_{[n]} \Gamma_{[n]} B \circ L \gamma_{[n]} B} R' FS_{[n]} B
\]

Let us check that \( \gamma_{[n]}' B : R' FS_{[n]} B \to B \) coequalizes the parallel pair above i.e. that

\[
(29) \quad \gamma_{[n]}' \circ R' F \Gamma_{[n]} \Gamma_{[n]} \circ R' F L \gamma_{[n]} = \gamma_{[n]}' \circ R' F e S_{[n]}
\]

To this aim we first compute

\[
\gamma_{[n]}' \circ R' F \Gamma_{[n]} \Gamma_{[n]} \circ R' F L \gamma_{[n]}' = R' F R' F \Gamma_{[n]}
\]

\[
(27) \quad \gamma' \circ R' F L \gamma' = \gamma' \gamma'
\]

\[
\gamma_{[n]}' \circ R' F e' L = \gamma_{[n]}' \circ R' F e' L
\]

\[
(27) \quad \gamma_{[n]}' \circ R' F e' \Gamma_{[n]} \Gamma_{[n]} \circ R' F e' L = \gamma_{[n]}' \circ R' F e' \Gamma_{[n]} \Gamma_{[n]} \circ R' F e' L
\]

\[
\text{nat.'} \quad \gamma_{[n]}' \circ R' F e' FS_{[n]} \circ R' F L' R' F \pi_{[n]} \Gamma_{[n]} = \gamma_{[n]}' \circ R' F e' FS_{[n]} \circ R' F L' R' F \pi_{[n]} \Gamma_{[n]}
\]

Since \( R', F \) and \( L \) preserve regular epimorphisms and \( \pi_{[n]} \Gamma_{[n]} \) is a regular epimorphism, we get that \( R' F L' R' F \pi_{[n]} \Gamma_{[n]} \) is a regular epimorphism and hence

\[
\gamma_{[n]}' \circ R' F \Gamma_{[n]} \Gamma_{[n]} \circ R' F L \gamma_{[n]}' = \gamma_{[n]}' \circ R' F e S_{[n]}
\]
Coming back to the equality (29), we compute
\[
\gamma'_{[n]} \circ R'F \pi_{[n]} \Gamma_{[n]} \circ R'FL \gamma_{[n]} = \gamma'_{[n]} \circ R'F \pi_{[n]} \Gamma_{[n]} \circ R'FL \xi S_{[n]}
\]
\[
= \gamma'_{[n]} \circ R'F \xi S_{[n]} \circ R'FL \xi S_{[n]}
\]
\[
= \gamma'_{[n]} \circ R'F \xi S_{[n]} \circ R'F \xi S_{[n]} = \gamma'_{[n]} \circ R' \xi S_{[n]}
\]
Thus \(\gamma'_{[n]}\) coequalizes and hence, the universal property of the above coequalizer yields a unique natural transformation \(\gamma'_{[n+1]} : R'F \xi S_{[n+1]} \to \text{Id}\) such that
\[
\gamma'_{[n+1]} \circ R'F \pi_{[n,n+1]} \Gamma_{[n+1]} = \gamma'_{[n]}
\]
We compute
\[
\gamma'_{[n+1]} \circ R'F \pi_{[n,n+1]} \Gamma_{[n+1]} = \gamma'_{[n+1]} \circ R'F \left(\pi_{[n,n+1]} \circ \pi_{[n]} U_{[n,n+1]} \right) \Gamma_{[n+1]}
\]
\[
= \gamma'_{[n+1]} \circ R'F \pi_{[n,n+1]} \Gamma_{[n+1]} \circ R'F \pi_{[n]} U_{[n,n+1]} \Gamma_{[n]}
\]
\[
= \gamma'_{[n]} \circ R'F \pi_{[n]} \Gamma_{[n]} \quad \text{(27)}
\]
We also have
\[
\gamma_{[n+1]} \circ U_{[n+1]} \eta_{[n+1]} \Gamma_{[n+1]} = \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ U_{[n+1]} \eta_{[n+1]} \Gamma_{[n+1]}
\]
\[
= \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ U_{[n]} \left( R_{[n]} \pi_{[n,n+1]} \circ \eta_{[n]} U_{[n,n+1]} \right) \Gamma_{[n+1]}
\]
\[
= \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ U_{[n]} R_{[n]} \pi_{[n,n+1]} \Gamma_{[n+1]} \circ U_{[n]} \eta_{[n]} U_{[n,n+1]} \Gamma_{[n]}
\]
\[
= \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ R \pi_{[n,n+1]} \Gamma_{[n+1]} \circ U_{[n]} \eta_{[n]} \Gamma_{[n]}
\]
\[
= \gamma'_{[n+1]} \circ R'F \pi_{[n,n+1]} \Gamma_{[n+1]} \circ \xi S_{[n]} \circ U_{[n]} \eta_{[n]} \Gamma_{[n]}
\]
\[
= \gamma'_{[n]} \circ \xi S_{[n]} \circ U_{[n]} \eta_{[n]} \Gamma_{[n]} \quad \text{(27)}
\]
Finally
\[
\gamma_{[n+1]} \circ R \pi_{[n,n+1]} \Gamma_{[n+1]} = \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ R \pi_{[n,n+1]} \Gamma_{[n+1]}
\]
\[
= \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ R'F \pi_{[n,n+1]} \Gamma_{[n+1]} \circ \xi S_{[n]} = \gamma'_{[n]} \circ \xi S_{[n]} = \gamma_{[n]}
\]
\[
\square
\]
**Proposition 5.3.** The functor \(\Gamma_{[n]} : B \to B_{[n]}\) induces a functor \(\Gamma_{n} : B \to B_{n}\) such that \(\Lambda_{n} \circ \Gamma_{n} = \Gamma_{[n]}\) and \(U_{n,n+1} \circ \Gamma_{n+1} = \Gamma_{n}\). Moreover there is \(\gamma_{n} : R_{n} L_{n} \Gamma_{n} \to \Gamma_{n}\) such that \(\Gamma_{n+1} B = (\Gamma_{n} B, \gamma_{n} B)\), for all \(B \in B\), and \(U_{n} \gamma_{n} \circ R \lambda_{n} \Gamma_{n} = \gamma_{[n]}\). Note that \(L_{[n]} \Gamma_{n} = L_{[n]} A_{n} \Gamma_{n}\).\(\lambda_{n+1} \circ \lambda_{n} \circ L_{n} \Gamma_{n}\) is invertible.

**Proof.** The condition \(\gamma_{[n]} \circ U_{[n]} \eta_{[n]} \Gamma_{[n]} = \text{Id}\), given in Theorem 5.2, implies that \(\eta_{[n]} \Gamma_{[n]} B \in M (U_{[n]} B)\) for every \(B \in B\). By Theorem 4.20, we have that \(\Gamma_{[n]} B \in \text{Im}(\Lambda_{n}) = \text{Im}'(\Lambda_{n})\). Thus \(\text{Im}'(\Gamma_{[n]}) \subseteq \text{Im}'(\Lambda_{n})\). Since \(\Lambda_{n}\) is fully faithful and injective on objects, by [10, Lemma 1.12], there is a functor \(\Gamma_{n} : B \to B_{n}\) such that \(\Lambda_{n} \circ \Gamma_{n} = \Gamma_{[n]}\). We compute
\[
\Lambda_{n} \circ U_{n,n+1} \circ \Gamma_{n+1} = U_{[n,n+1]} \circ \Lambda_{n+1} \circ \Gamma_{n+1} = U_{[n,n+1]} \circ \Gamma_{[n+1]} = \Gamma_{[n]} = \Lambda_{n} \circ \Gamma_{n}.
\]
Since \(\Lambda_{n}\) is faithful and injective on objects, we get \(U_{n,n+1} \circ \Gamma_{n+1} = \Gamma_{n}\). Moreover since \(\Gamma_{n+1} B \in B_{n+1} \) and \(U_{n,n+1} \Gamma_{n+1} B = \Gamma_{n} B\), there is \(\gamma_{B} : R_{n} L_{n} \Gamma_{n} B \to \Gamma_{n} B\) such that \(\Gamma_{n+1} B = (\Gamma_{n} B, \gamma_{n} B)\). From \(\Lambda_{n} \circ \Gamma_{n+1} = \Gamma_{[n]}\), we get \(U \gamma_{n} \circ R \lambda_{n} \Gamma_{n} = \gamma_{[n]}\). The last part follows by Remark 3.6. \(\square

**Lemma 5.4.** In the setting of Theorem 5.2, define \(S_{[n]} := L_{[n]} \Gamma_{n} : B \to A\). Given \(B \in B\), a morphism \(f : S_{[n]} B \to A\) coequalizes the pair \(\pi_{[n]} B \circ L_{[n]} \eta_{[n]} B, c S_{[n]} B\) if and only if \(R f\) coequalizes the pair \((e_{[n]} B, \text{Id} RS_{[n]} B)\), where \(e_{[n]} := U_{[n]} \eta_{[n]} \Gamma_{[n]} \circ \gamma_{[n]}\).

As a consequence, \(\pi_{[n+1]} B : S_{[n]} B \to S_{[n+1]} B\) is invertible if and only if either \(\gamma_{[n]} B\) or \(\eta_{[n]} B\) is invertible. If \(\pi_{[n+1]} B\) is invertible so is \(\pi_{[m,n+1]} B \) for all \(m \geq n\).
Proof. In the proof of Theorem 5.2, we have seen that \( \pi_{[n]} \Gamma_{[n]} \circ L_{\gamma_{[n]}} = \epsilon L_{[n]} \Gamma_{[n]} \circ \alpha_{[n]} = cS_{[n]} \circ \alpha_{[n]} \) where \( e_{[n]} := U_{[n]} \eta_{[n]} \Gamma_{[n]} \circ \gamma_{[n]} \). As a consequence, \( f : S_{[n]}B \to A \) coequalizes the pair \( (\pi_{[n]} \Gamma_{[n]}B \circ L_{\gamma_{[n]}B}, \epsilon S_{[n]}B) \) if and only if \( f \circ cS_{[n]}B \circ \alpha_{[n]}B = f \circ \epsilon S_{[n]}B \) if and only if \( RF \circ R\alpha_{[n]}B \circ R\alpha_{[n]}S_{[n]}B = RF \circ R\alpha_{[n]}B \circ \eta_{[n]}RS_{[n]}B \) if and only if \( RF \circ R\alpha_{[n]}B \circ \eta_{[n]}RS_{[n]}B \circ e_{[n]}B = RF \circ e_{[n]}B = RF \) if and only if \( RF \circ cS_{[n]}B \circ \alpha_{[n]}B = RF \circ \epsilon S_{[n]}B \circ \alpha_{[n]}B = \epsilon S_{[n]}B \) if and only if \( f = \text{Id} \) coequalizes the pair \( (\pi_{[n]} \Gamma_{[n]}B \circ L_{\gamma_{[n]}B}, \epsilon S_{[n]}B) \). By the foregoing this is equivalent to \( RF \circ e_{[n]}B \circ \alpha_{[n]}B = \text{Id} \) i.e. \( U_{[n]} \eta_{[n]} \Gamma_{[n]}B \circ \gamma_{[n]}B = \text{Id} \).

Since \( \gamma_{[n]}B \circ U_{[n]} \eta_{[n]} \Gamma_{[n]}B = \text{Id} \), we conclude that \( \pi_{[n]} \Gamma_{[n+1]}B \) is invertible if and only if either \( \gamma_{[n]}B \) or \( U_{[n]} \eta_{[n]} \Gamma_{[n]}B \) is invertible. Since \( U_{[n]} \) reflects isomorphism, we have that \( U_{[n]} \eta_{[n]} \Gamma_{[n]}B \) is invertible if and only if \( \gamma_{[n]}B \) is invertible.

If \( \pi_{[n]} \Gamma_{[n+1]}B \) is invertible, then \( \gamma_{[n]}B \) is invertible. Since \( \gamma_{[n]}B \circ R\pi_{[n]} \Gamma_{[n+1]} \Gamma_{[n+1]}B = \gamma_{[n]}B \), we obtain that \( \gamma_{[n+1]}B \) is invertible. As a consequence \( \pi_{[n+1]} \Gamma_{[n+2]}B \) is invertible. Going on this way, we obtain that \( \pi_{[m,m]} \Gamma_{[m+1]}B \) is invertible for all \( m \geq n \).

6. Example on monoidal categories

Given a category \( A \) and an object \( X \in A \) we denote by \( A/X \) the corresponding slice category consisting of pairs \( (A,tA : A \to X) \) and where a morphism \( f : (A,tA) \to (B,tB) \) is a morphism \( f : A \to B \) such that \( tB \circ f = tA \).

Let \( B \) be a category with pullbacks. It is known that any adjunction \((L,R)\) with unit \( \eta \) and counit \( \epsilon \) and object \( 1 \in B \) induces an adjunction \((L/1,R/1)\) as in the following left-hand side diagram where \( U_A \) and \( U_B \) are the obvious forgetful functors and \( U_A \circ L/1 = L \circ U_B \).

\[
\begin{array}{ccc}
A/L1 & \xrightarrow{U_A} & A \\
\downarrow{L/1} & & \downarrow{R/1} \\
B/1 & \xrightarrow{U_B} & B
\end{array}
\quad
\begin{array}{ccc}
KA & \xrightarrow{tKA} & 1 \\
\downarrow{kA} & & \downarrow{\eta_1} \\
RA & \xrightarrow{RLA} & RL1
\end{array}
\]

Explicitly \( (L/1)(B,tB : B \to 1) := (LB,LtB) \) and \( (L/1)f = Lf \). The functor \( R/1 \) associates to an object \( (A,tA : A \to L1) \) the pair \( (KA,tKA) \) given by the pullback in the right-hand side diagram above.

Given a morphism \( f : (A,tA : A \to L1) \to (A',tA' : A' \to L1) \) then \( (R/1)f : (KA,tKA) \to (K'A',tK'A') \) is defined by the universal property of the pullback as the unique morphism such that

\[
kA' \circ U_B (R/1)f = RU_A f \circ kA.
\]

The unit \( \eta/1 \) and counit \( \epsilon/1 \) of the adjunction are uniquely defined by the following equalities

\[
kLB \circ U_B (\eta/1)(B,tB) = \eta B \\
U_A (\epsilon/1)(A,tA) = \epsilon A \circ LkA.
\]

Remark 6.1. As mentioned, the construction above is well-known. It can be recovered as follows.

For every morphism \( f : X \to Y \) in a category \( C \) with pullbacks consider the functor \( C/X \xrightarrow{L} C/Y \) defined on objects by \((C,g) : (C,f \circ g) \) and as the identity on morphisms. It is well-known that this functor has a right adjoint \( f^* \) given by pullbacks along \( f \) in the underlying category (see e.g. [33, 16.8.5]). Now note that the functor \( L/1 : B/1 \to A/L1 \) can be written as the composition

\[
B/1 \xrightarrow{(\eta/1)^*} \xrightarrow{RL/1} A/LRL1 \xrightarrow{(\epsilon/1)^*} A/L1.
\]

By [33, 16.8.7] the functor \( R/L1 : A/L1 \to B/RL1 : (A,a) \to (RA,Ra), \alpha \mapsto R\alpha \) is a right adjoint for the composition \( (\epsilon/1)^* \circ L/RL1 \). As a consequence we get that a right adjoint of \( L/1 \) is given by the composition \( (\eta/1)^* \circ R/RL1 \) which is exactly the functor \( R/1 \) defined above.

Lemma 6.2. Let \( C \) be a category and \( 1 \in C \). The forgetful functor \( U_C : C/1 \to C \) creates colimits.

Proof. Cf. [13, Proposition 2.16.3] or dual version of [25, Exercise 1, page 108].
Remark 6.3. Since $\mathcal{B}$ is a category with pullbacks, if $1 \in \mathcal{B}$ is a terminal object then $\mathcal{B}$ would be finitely complete by [13, Proposition 2.8.2]. As a consequence $\mathcal{B}/1$ is finitely complete whenever $\mathcal{B}$ is a category with pullbacks (cf. [13, Proposition 2.16.3]).

In the rest of this section $\mathcal{M}$ is a non-empty preadditive braided monoidal category such that

- $\mathcal{M}$ has equalizers, denumerable coproducts and coequalizers of reflexive pairs of morphisms;
- the tensor products are additive and preserve equalizers, denumerable coproducts and coequalizers of reflexive pairs of morphisms.

We include here a well-known result we need.

Lemma 6.4. A non-empty preadditive category $\mathcal{C}$ with equalizers has a zero object.

Proof. For every $A, B$ in $\mathcal{C}$, the set $\text{Hom}_\mathcal{C}(A, B)$ contains a zero morphism i.e. $\mathcal{C}$ is a pointed category. For any morphism $f : A \to B$ we can compute the equalizer of $f$ and the zero morphism $A \to B$, i.e. the kernel of $f$. By [7C(d), page 127], the category $\mathcal{C}$ has a zero object. □

Remark 6.5. Let us show that under the hypotheses above, the category $\mathcal{M}$ is a pre-abelian, see [30, pag 24]. First we see it is additive. By Lemma 6.4, the category $\mathcal{M}$, being non-empty and preadditive, admits a zero object, say $0$. Given two objects $X_1, X_2$ in $\mathcal{M}$ we can set $X_n := 0$ for all $n \in \mathbb{N}$ with $n > 2$. Then the denumerable coproduct $\bigsqcup_{n \in \mathbb{N}} X_n$, which exists by assumption, is just the coproduct of $X_1, X_2$. By [14, Proposition 1.2.4 and Definition 1.2.5], the category $\mathcal{M}$ has binary biproducts. Since $\mathcal{M}$ has a zero object, then $\mathcal{M}$ is additive, see e.g. [14, Definition 1.2.6].

By hypotheses $\mathcal{M}$ has all equalizers. Moreover, since $\mathcal{M}$ has binary coproducts and coequalizers of reflexive pairs, then $\mathcal{M}$ has all coequalizers: to check this one has to apply the procedure mentioned in [21, page 20] to replace a pair of morphisms by a reflexive pair with the same coequalizer. Since $\mathcal{M}$ has a zero object, we get that $\mathcal{M}$ has all kernels and cokernels. Thus $\mathcal{M}$ is a preabelian category. We point out that, by [13, Proposition 2.8.2] and its dual form, the category $\mathcal{M}$ is finitely complete and finitely cocomplete (this makes sense since the dual of a preabelian category is preabelian, as observed in [30, page 24]).

We point out that, since, by hypothesis, denumerable coproducts and coequalizers of reflexive pairs are preserved by tensor products, all coequalizers are preserved too by [5, Lemma 2.3].

By the assumptions above, we can apply [9, Theorem 4.6] to give an explicit description of an adjunction $\bar{T} \dashv P : \text{Bialg}(\mathcal{M}) \to \mathcal{M}$. Note that $1$ is a terminal object in $\text{Bialg}(\mathcal{M})$ so that $0 := P1$ is a terminal object in $\mathcal{M}$, as right adjoint functors preserve the terminal object. It is indeed a zero object in $\mathcal{M}$ by Lemma 6.4.

As a particular case of the constructions above, consider the following left-hand side diagram

\[
\begin{array}{ccc}
\text{Alg}(\mathcal{M})/T0 & \xrightarrow{U_{\text{Alg}}(\mathcal{M})} & \text{Alg}(\mathcal{M}) \\
T/0 & \xrightarrow{\Omega/0} & T/\Omega \\
\mathcal{M}/0 & \xrightarrow{U_\mathcal{M}} & \mathcal{M}
\end{array}
\]

Since left adjoint functors preserve the initial object, we get that $T0$ is initial. By uniqueness of initial object, we get $T0 \cong 1$ as $1$ is the initial object in $\text{Alg}(\mathcal{M})$. Thus $\text{Alg}(\mathcal{M})/T0$ is $\text{Alg}(\mathcal{M})/1$ i.e. the category of augmented algebras that will be denoted by $\text{Alg}^+(\mathcal{M})$. Note also that the functor $U_\mathcal{M} : \mathcal{M}/0 \to \mathcal{M}$ is a category isomorphism because $0$ is terminal in $\mathcal{M}$. In light of these observations we can rewrite the starting diagram as the right-hand side one above where $T^+ := (T/0) \circ (U_\mathcal{M})^{-1}$ and $\Omega^+ := U_\mathcal{M} \circ (\Omega/0)$. Explicitly

\[
T^+ M = (T/0)(U_\mathcal{M})^{-1} M = (T/0)(M,tM) = (TM,TtM) = (TM,\varepsilon_{TM}),
\]

\[
\Omega^+(A,\varepsilon) = U_\mathcal{M}(\Omega/0)(A,\varepsilon) = U_\mathcal{M}(KA,tKA) = KA
\]
where $\Omega/0$ associates $(A, \varepsilon) \in \text{Alg}^+(\mathcal{M})$ the pair $(KA, tKA)$ defined by the pullback in $\mathcal{M}$

$$
\begin{array}{c}
KA \\
\downarrow kA \\
\Omega A \\
\downarrow \Omega e \\
\Omega \Theta \cong \Theta 0
\end{array}
\quad \vspace{1em}
\begin{array}{c}
kA \\
\downarrow kA \\
\Omega A \\
\downarrow \Omega e \\
\Omega \Theta \cong \Theta 0
\end{array}
$$

where $iX : 0 \rightarrow X$ is the unique morphism from the initial object 0 and $tX : X \rightarrow 0$ the unique one into the terminal object. This means that $(KA, kA) = \text{Ker}(\Omega e)$. Hence

$$T^+ M = (TM, e_{TM}), \quad (\Omega^+ (A, e), kA) = \text{Ker}(\Omega e).$$

Given a morphism $f : (A, e) \rightarrow (A', e')$ then $\Omega^+ f : \Omega^+ (A, e) \rightarrow \Omega^+ (A', e')$ is defined by

$$kA' \circ \Omega^+ f = \Omega Uf \circ kA.$$

The unit $\eta^+ := (U_M)(\eta/0)(U_M)^{-1} : \text{Id} \rightarrow \Omega^+ T^+$ and the counit $\epsilon^+ := (\epsilon/0) : T^+ \Omega^+ \rightarrow \text{Id}$ are uniquely determined by the following equalities

$$kTB \circ \eta^+ B = \eta B \quad U \epsilon^+ (A, e) = \epsilon A \circ TkA.$$

Next aim is to show that the left-hand side diagram below fits into the setting of Theorem 5.2.

$$
\begin{array}{c}
\text{Bialg}(\mathcal{M}) \\
\downarrow U := \text{UAlg}(\mathcal{M}) \\
\text{Alg}(\mathcal{M})
\end{array}
\quad \vspace{1em}
\begin{array}{c}
\Omega^+ \\
\downarrow \Omega \\
\text{Alg}(\mathcal{M})
\end{array}
\quad \vspace{1em}
\begin{array}{c}
\text{Alg}(\mathcal{M}) \\
\downarrow \text{Id} \\
\mathcal{M}
\end{array}
\quad \vspace{1em}
\begin{array}{c}
\text{Alg}(\mathcal{M}) \\
\downarrow \text{Id} \\
\mathcal{M}
\end{array}
$$

By construction of $\overline{T}$ we have $\overline{U} \circ \overline{T} = T$ and $\overline{U}^+$ is the obvious forgetful functor such that $U \circ \overline{U}^+ = U$ and $\overline{U}^+ \circ \overline{T} = T^+$.

Moreover the assumptions guarantees that the category $\text{Alg}(\mathcal{M})$ has coequalizers, see e.g. [5, Proposition 2.5], see also [31, Theorem 2.3]. Since Bialg$(\mathcal{M}) = \text{Coalg}(\text{Alg}(\mathcal{M}))$ we can apply [28, Proposition 2.5] to $\mathcal{C} := \text{Alg}(\mathcal{M})^{\text{op}}$ to obtain that $\overline{U}^{\text{op}} : \text{Bialg}(\mathcal{M})^{\text{op}} \rightarrow \text{Alg}(\mathcal{M})^{\text{op}}$ creates limits, equivalently $\overline{U} : \text{Bialg}(\mathcal{M}) \rightarrow \text{Alg}(\mathcal{M})$ creates colimits. Since $\text{Alg}(\mathcal{M})$ has coequalizers, we deduce that Bialg$(\mathcal{M})$ has coequalizers and $\overline{U}$ preserves coequalizers. On the other hand the functor $\Omega : \text{Alg}(\mathcal{M}) \rightarrow \mathcal{M}$ needs not to preserve coequalizers. Nevertheless $\Omega$ preserves the coequalizers of reflexive pairs of morphisms, see e.g. [10, Corollary A.10]. It is noteworthy that, since $\Omega$ has a left adjoint $T$, then $\Omega$ is strictly monadic (the comparison functor is a category isomorphism), see [10, Theorem A.6].

Let $V \in \mathcal{M}$. By construction $\Omega TV = \bigoplus_{n \in \mathbb{N}} V^\otimes n$, see [9, Remark 1.2]. Leto$_n V : V^\otimes n \rightarrow \Omega TV$ denote the canonical inclusion. The unit of the adjunction $(T, \Omega)$ is $\eta : \text{Id}_\mathcal{M} \rightarrow \Omega T$ defined by $\eta V := \alpha_1 V$ while the counit $\epsilon : T \Omega \rightarrow \text{Id}$ is uniquely defined by the equality

$$\Omega \epsilon (A, m, u) \circ \alpha_n A = m^{n-1} \text{ for every } n \in \mathbb{N}$$

where $m^{n-1} : A^{\otimes n} \rightarrow A$ is the iterated multiplication of an algebra $(A, m, u)$ defined by $m^1 = u$, $m^0 = \text{Id}_A$ and, for $n \geq 2$, by $m^{n-1} = m \circ (m^{n-2} \otimes A)$.

Denote by $\overline{\eta}, \overline{\epsilon}$ the unit and counit of the adjunction $(\overline{T}, P)$.

**Lemma 6.6.** ([31, Theorem 2.3]) The functor $\Omega : \text{Alg}(\mathcal{M}) \rightarrow \mathcal{M}$ preserves regular epimorphisms.

Not that the previous result does not mean that $\Omega$ preserves coequalizers.

**Lemma 6.7.** 1) Let $e : A \rightarrow A$ and $f : A \rightarrow A'$ be morphisms in $\mathcal{M}$ such that $A \otimes f, f \otimes A$ are epimorphisms and $f \circ e \circ e = f \circ e$. Then $(-)^{\otimes 2} : \mathcal{M} \rightarrow \mathcal{M}$ preserves the coequalizer of $(fe, f)$.

2) Let $a, b : A \rightarrow B$ be a reflexive pair of morphisms in $\mathcal{M}$ and let $g : B \rightarrow A'$ be a morphism in $\mathcal{M}$ such that $A \otimes g, g \otimes A$ are epimorphisms. Then the functor $(-)^{\otimes 2} : \mathcal{M} \rightarrow \mathcal{M}$ preserves the coequalizer of $(ga, gb)$.

3) $\Omega : \text{Alg}(\mathcal{M}) \rightarrow \mathcal{M}$ creates the coequalizers of those parallel pairs $(f, g)$ such that the functor $(-)^{\otimes 2} : \mathcal{M} \rightarrow \mathcal{M}$ preserves the coequalizer of $(\Omega f, \Omega g)$.
Proof. Recall that \( \mathcal{M} \) has all coequalizers and the tensor products preserve coequalizers.  

1) Consider the following left-hand side coequalizer 

\[
\begin{array}{c}
A \xrightarrow{f} A' \xrightarrow{p} V \\
\end{array}
\] 

Let us check that the right-hand side one is a coequalizer too.  

Let \( \zeta : A' \otimes A' \to Z \) be such that \( \zeta \circ (fe \otimes fe) = \zeta \circ (f \otimes f) \). Then 

\[
\begin{align*}
\zeta \circ (fe \otimes A') & = \zeta \circ (f \otimes f) \circ (e \otimes A) = \zeta \circ (fe \otimes fe) \circ (e \otimes A) \\
& = \zeta \circ (fe \otimes fe) = \zeta \circ (f \otimes f) = \zeta \circ (f \otimes A') \circ (A \otimes 1) \\
\end{align*}
\]

Since \( A \otimes f \) is an epimorphism, we deduce that \( \zeta \circ (fe \otimes A') = \zeta \circ (f \otimes A') \).

Since the tensor products preserve coequalizers, the following are both coequalizers. 

\[
\begin{array}{c}
A \otimes A' \xrightarrow{f \otimes f} A' \otimes A' \xrightarrow{p \otimes p} V \otimes V \\
\end{array}
\]

By using the left one there is a morphism \( \zeta_1 : V \otimes A' \to Z \) such that \( \zeta_1 \circ (p' \otimes A') = \zeta \). We have 

\[
\begin{align*}
\zeta_1 \circ (V \otimes f) \circ (pf \otimes A) & = \zeta_1 \circ (p \otimes A') \circ (f \otimes f) \circ (A \otimes e) = \zeta \circ (f \otimes f) \circ (A \otimes e) \\
& = \zeta \circ (fe \otimes fe) \circ (A \otimes e) = \zeta \circ (f \otimes fe) = \zeta \circ (f \otimes f) \\
& = \zeta_1 \circ (p \otimes A') \circ (f \otimes f) \circ (V \otimes f) \circ (pf \otimes A) \\
\end{align*}
\]

Now, \( f \otimes A \) is an epimorphism by assumption. Moreover \( p \otimes A \) is an epimorphism because the tensor products preserve coequalizers. Thus, from the chain of equalities above, we deduce that \( \zeta_1 \circ (V \otimes f) = \zeta_1 \circ (V \otimes f) \). Since the right-hand side diagram in (37) is a coequalizer, there is a morphism \( \zeta_2 : V \otimes A \to Z \) such that \( \zeta_2 \circ (V \otimes p) = \zeta_1 \). Therefore 

\[
\zeta_2 \circ (p \otimes p) = \zeta_2 \circ (V \otimes p) \circ (p \otimes A') = \zeta \circ (p \otimes A') = \zeta.
\]

Note also that \( p \otimes p = (V \otimes p) \circ (p \otimes A') \) is an epimorphism. Thus the right-hand side diagram in (36) is a coequalizer.

2) Let \( s : A' \to A \) be such that \( a \circ s = 1d = b \circ s \). Set \( f := g \circ b \) and \( e := s \circ a \). Then \( f \circ e = g \circ b \circ s \circ a = g \circ a \) so that \( (ge, gb) = (fe, f) \) and we can apply 1). We just point out that \( f \otimes A \circ (g \otimes A) \) is an epimorphism as a composition of the epimorphism \( g \otimes A \) by the split-epimorphism \( b \otimes A \). Similarly \( A \otimes f \) is an epimorphism.

3) It is straightforward. 

\[\square\]

Lemma 6.8. The forgetful functor \( U : \text{Alg}^+(\mathcal{M}) \to \text{Alg}(\mathcal{M}) \) creates colimits. Moreover the category \( \text{Alg}^+(\mathcal{M}) \) has coequalizers and \( U \) preserves all coequalizers. 

Proof. By Lemma 6.2, the forgetful functor \( U : \text{Alg}^+(\mathcal{M}) \to \text{Alg}(\mathcal{M}) \) creates colimits and since \( \text{Alg}(\mathcal{M}) \) has coequalizers, we deduce that \( \text{Alg}^+(\mathcal{M}) \) has coequalizers and \( U \) preserves them.  

\[\square\]

Lemma 6.9. The forgetful functor \( \mathcal{U}^+ : \text{Bialg}(\mathcal{M}) \to \text{Alg}^+(\mathcal{M}) \) preserves coequalizers. 

Proof. Since \( \mathcal{U} : \text{Bialg}(\mathcal{M}) \to \text{Alg}(\mathcal{M}) \) creates colimits, we have that \( \mathcal{U} \) preserves all colimits that exist in \( \text{Alg}(\mathcal{M}) \), see [26, 11.5, page 106]. Since \( \text{Alg}(\mathcal{M}) \) has all coequalizers, we get that \( \mathcal{U} \) preserves all coequalizers. Since \( \mathcal{U} = \mathcal{U}^+ \) and, by Lemma 6.8, \( U \) creates, whence reflects coequalizers [25, Exercice 1, page 150], we get that \( \mathcal{U}^+ \) preserves coequalizers as desired. 

\[\square\]

Corollary 6.10. \( \mathcal{U}^+ : \text{Alg}^+(\mathcal{M}) \to \mathcal{M} \) preserves coequalizers for pairs \( (fe, f) \) where \( f : A \to A' \) is composition of regular epimorphisms in \( \text{Alg}^+(\mathcal{M}) \) and \( e : A \to A \) is a morphism in \( \text{Alg}^+(\mathcal{M}) \) such that \( f \circ e \circ e = f \circ e \).

Proof. Consider in \( \text{Alg}^+(\mathcal{M}) \) the following left-hand side coequalizer. 

\[
\begin{array}{c}
A \xrightarrow{f \circ e} A' \xrightarrow{p} C \\
\end{array}
\]

By Lemma 6.8, \( U \) preserves coequalizers so that also the right-hand side one is a coequalizer.
Since $\Omega U$ preserves the regular epimorphisms (as $U$ preserves coequalizers and $\Omega$ preserves regular epimorphisms by Lemma 6.6), we get that $\Omega UF$ is composition of regular epimorphisms. Since the tensor products preserves coequalizers, $\Omega U A \otimes \Omega U f$ and $\Omega U f \otimes \Omega U A$ are epimorphisms.

Since $\Omega U f \otimes \Omega U e \otimes \Omega U e = \Omega U f \otimes \Omega U e$ and $\Omega U A \otimes \Omega U f, \Omega U f \otimes \Omega U A$ are epimorphisms, we can apply Lemma 6.7-1 to $\pi f = \Omega U f$ and $\pi e = \Omega U e$ to get that the coequalizer of $(\Omega U f \otimes \Omega U e, \Omega U f)$ is preserved by $(-)^{\otimes 2}$ and hence, by Lemma 6.7-3, $\Omega$ creates the coequalizer of $(U f \otimes U e, U f)$. As a consequence the above right-hand side displayed coequalizer is preserved by $\Omega$. Hence $\Omega U$ preserves the starting coequalizer.

**Proposition 6.11.** Let $\nu : F \to G$ and $\tau : G \to F$ be natural transformations such that $\tau \circ \nu = \text{Id}$. Then $F$ preserves those colimits which are preserved by $G$. Moreover $F$ preserves regular epimorphisms which are preserved by $G$.

**Proof.** Let $F, G : A \to B$ be as above and let $H : S \to A$ be a functor with $S$ small. Consider the leftmost diagram below, where $(p_S : HS \to C)_{S \in S}$ is a family of morphisms with the same target and the property that two morphisms $g_1, g_2 : GC \to X$ are equal as long as $g_1 \circ Gp_S = g_2 \circ Gp_S$ for all $S \in S$. Let us prove that this diagram is a kind of pushout. Let $g, (f_S)_{S \in S}$ be morphisms such that $g \circ Gp_S = f_S \circ \tau HS$, for all $S \in S$, and set $f := g \circ \nu C$. We get $f \circ Gp_S = g \circ \nu C \circ Fp_S = g \circ Gp_S \circ HS = f_S \circ \tau HS \circ \nu HS = f_S$ and $f \circ \tau C \circ Gp_S = f \circ Fp_S \circ \tau HS = f_S \circ \tau HS = g \circ Gp_S$. The hypothesis on $Gp_S$ forces $f \circ \tau C = g$. Thus $f$ is unique as $\tau Z$ is a split-epimorphism.

Let now $(C, (p_S : HS \to C)_{S \in S})$ be a colimit for $H$ and assume that it is preserved by $G$ i.e. that the datum $(GC, (Gp_S)_{S \in S})$ is a colimit for $G H : S \to B$. In this setting, the property proved above can be rephrased by saying that the central diagram above is a pullback. Indeed, by construction, the pullback is the set $P$ of pairs $(g,(f_S)_{S \in S})$ such that $(X,(f_S)_{S \in S})$ is a cocone and $g \circ Gp_S = f_S \circ \tau HS$, for all $S \in S$; moreover the map $\text{Hom}(FC,X) \to P : f \mapsto (f \circ \tau C, f \circ Fp_S)$ is bijective.

The fact that $(GC,(Gp_S)_{S \in S})$ is a colimit for $GH$ means that the diagonal map $\Delta \text{Hom}(Gp_S,X)$ is bijective for all $X$. By [13, Proposition 2.5.3], the isomorphisms are stable under pullbacks so that the diagonal map $\Delta \text{Hom}(Fp_S,X)$ is also bijective for all $X$. Hence $(FC,(Fp_S)_{S \in S})$ is a colimit for $F H$.

Assume that $G$ preserves regular epimorphisms and let us check that also $F$ does. To this aim we will prove that given a morphism $p : A \to C$ such that $Gp$ is a regular epimorphism, then also $Fp$ is a regular epimorphism. If we apply the first part of the proof to the discrete category $S$ with $\text{Ob}(S) = \{S\}$ taking the functor $H$ defined by $H(S) = A$ and $p_S = p$, the requirement on $p_S$ means that $Gp$ is an epimorphism and hence the rightmost diagram above is a pushout. By [13, Proposition 4.3.8], since $Gp$ is a regular epimorphism, so is $Fp$. 

Consider the natural transformation $\xi : P \to \Omega \Omega \tilde{D}$ defined, as in Theorem 5.2, by

$$ P \xrightarrow{\eta P} \Omega T P = \Omega \Omega \tilde{D} P \xrightarrow{\Omega \eta P} \Omega \Omega \tilde{D}. $$

As in the proof of the above theorem, we have (26) that in local notations becomes

$$ \epsilon \tilde{D} \circ T \xi = \Omega \tilde{D}. $$

so that $\xi$ is exactly the natural transformation of [9, Theorem 4.6], whose components are the canonical inclusions of the subobject of primitives of a bialgebra $B$ in $M$ into $\Omega \tilde{D} B$ and hence they are regular monomorphisms.

Since $UT^+ = T$, we can define

$$ \zeta := \left( \Omega^+ \xrightarrow{\eta T^+} \Omega T \Omega^+ = \Omega U T^+ \Omega^+ \xrightarrow{\Omega \eta T^+} \Omega \Omega^+ \right). $$
Given \((A, \varepsilon) \in \text{Alg}^+(\mathcal{M})\), we have \(\zeta(A, \varepsilon) : \text{Ker}(\Omega \varepsilon) \to \Omega A\).

**Remark 6.12.** We compute
\[
\zeta(A, \varepsilon) = (\Omega U e^+ \circ \eta \Omega^+)(A, \varepsilon) = \Omega U e^+(A, \varepsilon) \circ \eta \Omega^+(A, \varepsilon) \\
= \Omega(eA \circ TkA) \circ \eta \Omega^+(A, \varepsilon) = \Omega eA \circ \Omega TkA \circ \eta \Omega^+(A, \varepsilon) = \Omega eA \circ \eta eA \circ kA = kA.
\]
where \(kA\) is the morphism in diagram (31). Thus
\[
(39) \\
\zeta(A, \varepsilon) = kA.
\]

**Lemma 6.13.** There is a natural transformation \(\tau : \Omega U \to \Omega^+\) such that \(\tau \circ \zeta = \text{Id}\). As a consequence \(\Omega^+ : \text{Alg}^+(\mathcal{M}) \to \mathcal{M}\) preserves coequalizers for pairs \((f, e)\) where \(f : A \to A'\) is composition of regular epimorphisms in \(\text{Alg}^+(\mathcal{M})\) and \(e : A \to A\) is a morphism in \(\text{Alg}^+(\mathcal{M})\) such that \(f \circ e \circ e = f \circ e\). Moreover \(\Omega^+\) preserves regular epimorphisms.

**Proof.** Let \((A, \varepsilon) \in \text{Alg}^+(\mathcal{M})\). As observed, the pullback (31) means that \(\Omega^+(A, \varepsilon) = ka = \text{Ker}(\Omega \varepsilon)\). The canonical inclusion is \(kA\) which by (39) equals \(\zeta(A, \varepsilon)\). Thus we have the following kernel in \(\mathcal{M}\).
\[
0 \to \Omega^+(A, \varepsilon) \xrightarrow{\zeta(A, \varepsilon)} \Omega A \xrightarrow{\Omega \varepsilon} \Omega 1 = 1
\]
Since \(\varepsilon\) is an algebra morphism, we have \(\Omega \varepsilon \circ u_{\Omega A} = \text{Id}\). Hence \(\Omega \varepsilon \circ (\text{Id}_{\Omega A} - u_{\Omega A} \circ \Omega \varepsilon) = 0\) so that, by the universal property of the kernel we get a unique morphism \(\tau(A, \varepsilon) : \Omega A \to \Omega^+(A, \varepsilon)\) such that \(\zeta(A, \varepsilon) \circ \tau(A, \varepsilon) = \text{Id}_{\Omega A} - u_{\Omega A} \circ \Omega \varepsilon\). Moreover \(\tau(A, \varepsilon) \circ \zeta(A, \varepsilon) = \text{Id}_{\Omega^+(A, \varepsilon)}\).

It remains to check that \(\tau(A, \varepsilon)\) is natural in \((A, \varepsilon)\). To this aim, first let \(f : (A, \varepsilon_A) \to (B, \varepsilon_B)\) be a morphism in \(\text{Alg}^+(\mathcal{M})\) and compute
\[
\zeta(B, \varepsilon_B) \circ \tau(B, \varepsilon_B) \circ \Omega U f = (\text{Id}_{\Omega B} - u_{\Omega B} \circ \Omega \varepsilon_B) \circ \Omega U f = \Omega U f - u_{\Omega B} \circ \Omega \varepsilon_B \circ \Omega U f \\
= \Omega U f - u_{\Omega B} \circ \Omega (\varepsilon_B \circ U f) = \Omega U f - u_{\Omega B} \circ \Omega \varepsilon_A \\
= \Omega U f - \Omega U f \circ u_{\Omega A} \circ \Omega \varepsilon_A = \Omega U f \circ (\text{Id}_{\Omega A} - u_{\Omega A} \circ \Omega \varepsilon_A) \\
= \Omega U f - \varepsilon(A, \varepsilon) \circ \tau(A, \varepsilon) = \Omega U f - kA \circ \tau(A, \varepsilon) \\
= (32) \\
kA' \circ \Omega^+ f \circ \tau(A, \varepsilon) = kA \circ \Omega^+ f \circ \tau(A, \varepsilon).
\]
Since \(\zeta(B, \varepsilon_B)\) is a monomorphism we deduce \(\tau(B, \varepsilon_B) \circ \Omega U f = \Omega^+ f \circ \tau(A, \varepsilon)\) which means that \(\tau\) is natural. Thus \(\zeta : \Omega^+ \to \Omega U\) cosplits via \(\tau : \Omega U \to \Omega^+\) i.e. \(\tau \circ \zeta = \text{Id}\).

Now, by Lemma 6.6, the functor \(\Omega : \text{Alg}(\mathcal{M}) \to \mathcal{M}\) preserves regular epimorphisms.

By Lemma 6.8, the forgetful functor \(U : \text{Alg}^+(\mathcal{M}) \to \text{Alg}(\mathcal{M})\) creates colimits and preserves all coequalizers. As a consequence \(\Omega U : \text{Alg}^+(\mathcal{M}) \to \mathcal{M}\) preserves regular epimorphisms. Hence by Proposition 6.11 also \(\Omega^+\) preserves regular epimorphisms. By Corollary 6.10, the functor \(\Omega U : \text{Alg}^+(\mathcal{M}) \to \mathcal{M}\) preserves coequalizers for pairs \((f, e)\) where \(f : A \to A'\) is composition of regular epimorphisms in \(\text{Alg}^+(\mathcal{M})\) and \(e : A \to A\) is a morphism in \(\text{Alg}^+(\mathcal{M})\) such that \(f \circ e \circ e = f \circ e\). By Proposition 6.11, the functor \(\Omega^+\) preserves the same type of coequalizers.

Next aim is to show that the functor \(T^+ : \mathcal{M} \to \text{Alg}^+(\mathcal{M})\) is h-separable. First note that there is a unique morphism \(\omega V : \Omega TV \to V\) such that
\[
(40) \\
\omega V \circ \alpha_n V = \delta_{n, 1} \text{Id}_V.
\]
Given \(f : V \to W\) a morphism in \(\mathcal{M}\), we get for every \(n \in \mathbb{N}\),
\[
\omega W \circ \Omega TV f \circ \alpha_n V = \omega W \circ \alpha_n W \circ f \circ \omega V = \delta_{n, 1} f \circ \omega V = f \circ \omega V \circ \alpha_n V
\]
so that \(\omega W \circ \Omega TV f = f \circ \omega V\) which means that \(\omega := (\omega V)_{V \in \mathcal{M}}\) is a natural transformation \(\omega : \Omega T \to \text{Id}_\mathcal{M}\).

**Lemma 6.14.** The natural transformation \(\omega\) fulfills \(\omega \circ \eta = \text{Id}\) and
\[
(41) \\
\omega \omega \circ \Omega TV \zeta T^+ = \omega \circ \Omega T \circ \Omega TV \zeta T^+.
\]
Proof. In [11, Lemma 5.2] we prove that \( \omega \circ \Omega T \circ \zeta T = \omega \circ \Omega T \circ \Omega T \circ \zeta T \) where \( \zeta' : E \to \Omega \Omega \) is a natural transformation whose domain is the functor \( E : \text{Bialg}(M) \to M \) assigning to each bialgebra \( A \) the kernel \( (E A, \zeta' A : E A \to \Omega \Omega A) \) in \( M \) of its counit \( \Omega \varepsilon_{UA} \), where here \( \varepsilon_{UA} \) is regarded as an algebra map. Then, for every \( M \in M \), we have \[
(E \tilde{T} M, \zeta' \tilde{T} M) = \text{Ker}(\Omega \varepsilon_{\Omega T M}) = \text{Ker}(\Omega \varepsilon_{TM}) = (\Omega^+ (TM, \varepsilon_{TM}), kTM) \quad (39) = (\Omega^+ T^+ M, \zeta T^+ M).
\]
Moreover, given \( f : M \to N \), since, by (39) we have \( kTM = \zeta T^+ M \), we obtain \[
\zeta T^+ N \circ \Omega^+ T^+ f = kTN \circ \Omega^+ T^+ f = \Omega UT^+ f \circ kTM = \Omega T^+ f \circ \zeta T^+ M = \Omega \tilde{T}N \circ \zeta' \tilde{T} M = \zeta' \tilde{T}N \circ E \tilde{T} f
\]
so that \( \Omega^+ T^+ f = E \tilde{T} f \). As a consequence \( E \tilde{T} = \Omega^+ T^+ \) and \( \zeta' \tilde{T} = \zeta T^+ \). If we substitute \( \zeta' \tilde{T} \) by \( \zeta T^+ \) in the starting equality we obtain the desired one. \( \square \)

The following result shows that \( T^+ \) is h-separable.

**Lemma 6.15.** \( \zeta T^+ : \Omega^+ T^+ \to \Omega T \) is a monad morphism between the monads associated to \( (T^+, \Omega^+) \) and \( (T, \Omega) \). Moreover \( \omega^+ := \omega \circ \zeta T^+ : \Omega^+ T^+ \to \text{Id} \) is an augmentation for the monad associated to the adjunction \( (T^+, \Omega^+) \). Equivalently \( T^+ \) is h-separable.

**Proof.** The monads to consider are \((\Omega^+ T^+, \Omega^+ e^+ T^+, \eta^+)\) and \((\Omega T, \Omega e T, \eta)\). We compute \[\zeta T^+ o \Omega^+ e^+ T^+ n_{\eta^+} = \zeta T^+ o \Omega U e^+ T^+ o \zeta T^+ o \Omega^+ T^+ = kU T^+ o \zeta T^+ = \zeta T^+ o \Omega e^+ T^+ = \Omega^+ T^+ o \zeta T^+ \quad (33) \equiv (39) \quad \Omega e^+ T^+ o \zeta T^+ = \Omega^+ T^+ o \zeta T^+ = \zeta T^+ o \eta^+ = kT o \eta^+ \equiv \eta.
\]
We have so proved that \( \zeta T^+ : \Omega^+ T^+ \to \Omega T \) is a morphism of monads. We compute \[
\omega^+ o \omega^+ = \omega \circ \zeta T^+ o \Omega^+ T^+ = \omega \circ \Omega T o \zeta T^+ = \Omega e^+ T^+ o \zeta T^+ o \Omega^+ T^+ = \omega \circ \Omega e T o \zeta T^+ o \zeta T^+ o \Omega^+ T^+ = \omega \circ \Omega e T o \zeta T^+ = \omega \circ \zeta T^+ o \Omega^+ e^+ T^+ = \omega \circ \zeta T^+ o \eta^+ = \omega \circ \eta = \text{Id}.
\]
Moreover \( \omega^+ o \eta^+ = \omega \circ \zeta T^+ o \eta^+ = \omega \circ \eta = \text{Id} \). Thus \( \omega^+ \) is an augmentation for the monad \((\Omega^+ T^+, \Omega^+ e^+ T^+, \eta^+)\). By [11, Corollary 2.7], this means that \( T^+ \) is h-separable. \( \square \)

As a consequence of the results above, Theorem 5.2 applies to the leftmost diagram in (34).

**7. Conclusions**

In this section we collect some fallouts of Theorem 5.2. We describe explicitly the functor \( \Gamma_{[n]} \) in case of Yetter-Drinfeld modules and in particular of vector spaces. We infer an analogue of the notion of combinatorial rank and we propose possible lines of future investigation on the subject.

**Example 7.1.** Let \( H \) be a finite-dimensional Hopf algebra over a field \( k \). We want to apply the results of the previous sections in the case when \( M \) is the category \( \text{Alg}^+(\mathcal{YD}) \) of (left-left) Yetter-Drinfeld modules over \( H \). This category is braided as the antipode of \( H \) is invertible. Moreover \( \text{Alg}^+(\mathcal{YD}) \) satisfies all the requirements of Section 6. The related diagram rewrites as follows and fulfills the assumptions of Theorem 5.2.

\[
\text{Bialg}(\mathcal{YD}) \xrightarrow{\text{Alg}^+} \mathcal{YD}^+ \xrightarrow{\text{Alg}^+} \mathcal{YD}^+.
\]

Let \( V \in \mathcal{YD}^+ \) and \((A, \varepsilon) \in \text{Alg}^+(\mathcal{YD}) \). The object \( TV \) is the usual tensor algebra \( TV \) that becomes the tensor algebra in \( \mathcal{YD}^+ \), as \( V \) belongs to \( \mathcal{YD}^+ \), and that is endowed with a braided bialgebra structure by means of its universal property and the braiding of \( V \). By definition \( T^+ V = (TV, \varepsilon_{TV}) \), \((\Omega^+ (A, \varepsilon), kA) = \text{Ker}(\Omega \varepsilon) \).

Thus \( \Omega^+ (A, \varepsilon) \) is nothing but the augmentation ideal \( A^+ \) regarded as an object in \( \mathcal{YD}^+ \) being the kernel of \( \varepsilon \) which is a morphism in this category. The monad \( \Omega^+ T^+ \) is augmented via the morphism
define the ideal generated by the primitive elements in \( S_V \), the braiding of \( \omega \) denoted by \( S \).

By Theorem 5.2, also the monad \( P \) is augmented via \( \gamma := \omega \circ \xi : PT \to \text{Id} \). Explicitly \( \gamma V \) is the restriction of \( \omega V \) to the Yetter-Drinfeld submodule of primitive elements of \( TV \). For every \( n \in \mathbb{N} \), there are a functor \( \Gamma[n] : \frac{H}{H} \text{YD} \to \frac{H}{H} \text{YD}[n] \) and a natural transformation \( \gamma[n] : P \Gamma[n] \to \text{Id} \), such that \( \Gamma[0] := \text{Id}, \gamma[0] := \gamma \) and, for \( n \geq 0, \Gamma[n+1] V = (\Gamma[n] V, \gamma[n] V) \in \frac{H}{H} \text{YD}[n+1] \) and \( \gamma[n] \circ U[n] \eta[n] \Gamma[n] = \text{Id} \). Let us describe explicitly the functor \( S[n] := \tilde{T}[n] \Gamma[n] : \frac{H}{H} \text{YD} \to \text{Bialg} \left( \frac{H}{H} \text{YD} \right) \).

For \( n = 0 \) we have \( S[0] := \tilde{T}[0] \Gamma[0] = \tilde{T} \). Moreover \( S[n+1] V \) is given by the coequalizer, see (28):

\[
\tilde{T} P S[n] V \overrightarrow{\pi[n] \Gamma[n] V \circ \tilde{\eta}[n] V} S[n] V \overrightarrow{\pi[n,n+1] \Gamma[n+1] V} S[n+1] V
\]

By Lemma 5.4, a bialgebra map \( f : S[n] V \to B \) in \( \frac{H}{H} \text{YD} \) coequalizes the above pair if and only if \( P f \circ e[n] V = P f \), where \( e[n] := U[n] \eta[n] \Gamma[n] \circ \gamma[n] \). Since \( P f : P S[n] V \to PB \) is just the restriction of \( f \) to the primitive elements, we get that \( S[n+1] V \) is obtained by factoring out \( S[n] V \) by its two-sided ideal generated by \( \text{Im} (\text{Id} - e[n] V) \). Since \( e[n] V \) is idempotent, we have that \( \text{Im} (\text{Id} - e[n] V) = \text{Ker} (e[n] V) \). By definition of \( e[n] V \) and since \( U[n] \eta[n] \Gamma[n] V \) is split-injective, its retraction being \( \gamma[n] V \), we get \( \text{Im} (\text{Id} - e[n] V) = \text{Ker} \left( \gamma[n] V \right) \). Hence \( S[n+1] V = \frac{S[n] V}{\text{Ker} (\gamma[n] V)} \).

In order to give explicitly \( \text{Ker} (\gamma[n] V) \) and to get a complete description of the functors \( \Gamma[n] \), let us take a closer look at \( \gamma[n] \). By construction (see Theorem 5.2), we have that \( \gamma[n] := \omega[n] \circ \xi S[n] : P S[n] \to \text{Id} \) where \( \omega[n] := \Omega^{+} \check{S} \to \text{Id} \) (denoted by \( \gamma[n] \) in the quotient theorem as it stems from \( \gamma' = \omega' \) is defined iteratively by the following equality \( \omega[n+1] \circ \Omega^{+} \check{S} \pi[n,n+1] \Gamma[n+1] = \omega[n] \).

Since \( \Omega^{+} \check{S} \pi[n,n+1] \Gamma[n+1] \) is surjective, we get that \( \omega[n] V : \Omega^{+} \check{S} \pi[n,n+1] \Gamma[n+1] V = (S[n] V)^{+} \to V \) is just the projection onto \( V \) passed to the quotient. Since \( \gamma[n] := \omega[n] \circ \xi S[n] \) we get that \( \gamma[n] V : P S[n] V \to V \) is still the projection onto \( V \). As a consequence \( \text{Ker} (\gamma[n] V) \) is spanned by the homogeneous elements of \( P S[n] V \) of degree at least two.

Note that, if we forget the structure of YD module and we just keep the underlying braided bialgebra structure, the braided bialgebra \( S[n] V \) is exactly what in [2, Definition 3.10] was denoted by \( S[n] (B) \) for \( B := \tilde{T} V \). As a consequence, the direct limit of the direct system

\[
\tilde{T} V \to S[1] V \to S[2] V \to \cdots
\]

is the Nichols algebra \( B (V, c) \) ([2, Corollary 3.17 and Remark 5.4]), where \( c : V \otimes V \to V \otimes V \) is the braiding of \( V \) in \( \frac{H}{H} \text{YD} \).

**Remark 7.2.** In the previous example \( S[n+1] V \) is obtained by factoring out \( S[n] V \) by the two-sided ideal generated by the primitive elements in \( S[n] V \) of degree at least two. Following [2, Definition 4.1 and Section 5], we get that the combinatorial rank of \( V \), regarded as braided vector space through the braiding \( c \) of \( \frac{H}{H} \text{YD} \) as above, is the smallest \( n \) such that \( \pi[n,n+1] \Gamma[n+1] V : S[n] V \to S[n+1] V \) is invertible, if such an \( n \) exists. In this case obviously \( S[n] V = B (V, c) \).

Since, in the setting of Theorem 5.2, we can always define \( S[n] := L[n] \Gamma[n] : B \to A \), for every \( B \in B \) we are lead to the following definition.

**Definition 7.3.** In the setting of Theorem 5.2, consider the functor \( S[n] := L[n] \Gamma[n] : B \to A \). We define the **combinatorial rank** of an object \( B \in B \) (with respect to the adjunction \( (L, R) \)) to be the smallest \( n \) such that \( \pi[n,n+1] \Gamma[n+1] B : S[n] B \to S[n+1] B \) is invertible, if such an \( n \) exists.
Remark 7.4. Thus a concept of combinatorial rank can be introduced and investigated in this very general setting in which there are neither bialgebras nor braided vector spaces but just an adjunction $(L, R)$ as in Theorem 5.2. Note that, by Lemma 5.4, the morphism $\pi_{[n,n+1]} \Gamma_{[n+1]} B$ is invertible if and only if either $\gamma_{[n]} B$ or $\eta_{[n]} \Gamma_{[n]} B$ is invertible.

As we will see below, a case of interest is the one in which all objects in $\mathcal{B}$ have combinatorial rank at most one, equivalently $\eta_{[1]} \Gamma_{[1]} : \Gamma_{[1]} \to R_{[1]} S_{[1]}$ is invertible. Since, to this aim, only the functor $\Gamma_{[1]}$ is needed, we can even more relax our assumptions by taking just an adjunction $(L, R)$ with an augmentation $\gamma : RL \to \text{Id}$ for the associated monad, avoiding the setting of Theorem 5.2 and define directly $\Gamma_{[1]}$ by $\Gamma_{[1]} B := (B, \gamma B)$.

Theorem 7.5. In the setting of Theorem 5.2, if the adjunction $(L_N, R_N)$ is idempotent for some $N \in \mathbb{N}$, then every object in $\mathcal{B}$ has combinatorial rank at most $N$ with respect to the adjunction $(L, R)$. In particular the length of the monadic decomposition of $R : \mathcal{A} \to \mathcal{B}$ is an upper bound for the combinatorial rank of objects in $\mathcal{B}$ with respect to the adjunction $(L, R)$.

Proof. The fact that the adjunction $(L_N, R_N)$ is idempotent is equivalent to require that $\eta_N U_{N,N+1}$ is an isomorphism. By Proposition 5.3, we have that $\Gamma_{[N]} = \Lambda_N \Gamma_N$ and $U_{N,N+1} \circ \Gamma_{N+1} = \Gamma_N$. Thus $\eta_N \Gamma_N = \eta_N U_{N,N+1} \Gamma_{N+1}$ is an isomorphism. As in the proof of Theorem 4.20, we get $R_{[N]} \lambda_N \gamma_{[N]} \lambda_N = \Lambda_N \eta_N$. In particular we get $R_{[N]} \lambda_N \gamma_{[N]} \lambda_N = \Lambda_N \eta_N \Gamma_N$ i.e. $R_{[N]} \lambda_N \gamma_{[N]} \lambda_N = \Lambda_N$. Since $\eta_N \Gamma_N$ and $\lambda_N$ are invertible, we get that $\eta_N \Gamma_{[N]}$ is invertible. By the foregoing, every object in $\mathcal{B}$ has combinatorial rank at most $N$.

If $R$ has a monadic decomposition of monadic length $N$. Then $L_N$ is fully faithful i.e. $\eta_N$ is invertible. Thus, in particular, $\eta_N U_{N,N+1}$ is an isomorphism and hence $(L_N, R_N)$ is idempotent. As a consequence every object in $\mathcal{B}$ has combinatorial rank at most $N$. \hfill $\square$

Corollary 7.6. Let $\mathcal{M}$ be a symmetric MM-category in the sense of [10, Definition 7.4]. Then every object in $\mathcal{M}$ has combinatorial rank at most one with respect to the adjunction $(\bar{T}, P)$.

Proof. By hypothesis all the requirements of Section 6 are satisfied so that the adjunction $(\bar{T}, P)$ is in the setting of Theorem 5.2. By [10, Theorem 7.2] the adjunction $(\bar{T}, P_1)$ is idempotent. We conclude by Theorem 7.5. \hfill $\square$

As a consequence all the symmetric MM-categories given in [10, Section 9] have objects with combinatorial rank at most one.

Example 7.7. Consider the particular case when $\mathcal{M}$ is the category $\text{Vec}$ of vector spaces over a field $k$. Since $\text{Vec}$ is just $\mathcal{H}_{\text{alg}} \downarrow \mathbb{D}$ in case $\mathcal{H}$ is the trivial Hopf algebra $k$, this is a particular case of Example 7.1. The diagram above can be more easily written as follows

\[
\begin{array}{ccc}
\text{Bialg} & \xrightarrow{\bar{\gamma}^+} & \text{Alg}^+ \\
\bar{T} \downarrow P & & \downarrow \bar{T}^+ \\
\text{Vec} & \xrightarrow{\text{Id}} & \text{Vec}
\end{array}
\]

As above we can define $S_{[n]} := \bar{T}_{[n]} \Gamma_{[n]} : \text{Vec} \to \text{Bialg}$. Thus $S_{[0]} := \bar{T}$ and $S_{[n+1]} V = \frac{S_{[n]} V}{(\ker (\gamma_{[n]} V))}$ is obtained by factoring out $S_{[n]} V$ by the two-sided ideal generated by the homogeneous primitive elements in $S_{[n]} V$ of degree at least two. Note that the procedure we used to compute $S_{[1]} V = \frac{\bar{T} V}{(\ker (\gamma_{[1]} V))}$ is essentially the same used to compute $L_{1} V_{1}$ in the proof of [4, Theorem 3.4].

By [2, Definition 6.8 and Theorem 6.13], if $\text{char}(k) = 0$, and [3, Example 3.13], if $\text{char}(k) = p$, we get that $V$, regarded as a braided vector space via the braiding $c : V \otimes V \to V \otimes V : x \otimes y \mapsto y \otimes x$ of $\text{Vec}$, has combinatorial rank at most one. Thus $\text{Vec}$ is an example of braided monoidal category where every object has combinatorial rank at most one with respect to the adjunction $(\bar{T}, P)$.

By the foregoing $S_{[1]} V$ coincides with the Nichols algebra $\mathcal{B}(V, c)$ and all the maps $\pi_{[1,2]} \Gamma_{[2]} V : S_{[1]} V \to S_{[2]} V$, $\gamma_{[1]} V : PS_{[1]} V \to V$ and $U_{[1]} \eta_{[1]} \Gamma_{[1]} V : V \to RS_{[1]} V$ are invertible. By Lemma 5.4, we have that $\pi_{[n,n+1]} \Gamma_{[n+1]} B$ is invertible for all $n \geq 1$ and hence $\gamma_{[n]} V$ is invertible for all $n \geq 1$. 
In Example 7.7 we observed that \( \gamma_{1[V]} : \psiS[1][V] \to V \) (equivalently \( U[1][\eta_{1}]\Gamma[1][V] : V \to \psiS[1][V] \)) is an isomorphism for \( \mathcal{M} = \text{Vec} \). This fact may fail to be true if we change \( \mathcal{M} \). For instance, let us come back to the category \( \H{H}{\YD} \). By the foregoing we have \( S[1][V] = \frac{TV}{\ker(\gamma_{V})} \) where \( \gamma : PT\overline{V} \to V \) is the projection on degree one and hence \( \ker(\gamma_{V}) \) are the elements of \( PTV \) of degree at least two. In order to see that the projection \( \gamma_{1[V]} : \psiS[1][V] \to V \) and the injection \( U[1][\eta_{1}]\Gamma[1][V] : V \to \psiS[1][V] \) need not to be invertible we refer to [2, Section 7] where examples of braided vector spaces of combinatorial rank greater than two, arising as object in \( \H{H}{\YD} \) and braided via the braiding of \( \H{H}{\YD} \), are given.

It would be of interest to determine which conditions on \( H \) guarantee that \( \psiS[1][V] : V \to \psiS[1][V] \) is always invertible for every \( V \in \H{H}{\YD} \), equivalently any object in \( \H{H}{\YD} \) has combinatorial rank ant most one.

**Remark 7.8.** In [4, Theorem 3.4] we showed that the functor \( P \) in case \( \mathcal{M} = \text{Vec} \) admits a monadic decomposition of length at most two, represented in the following diagram.

\[
\begin{array}{ccc}
\text{Bialg} & \xrightarrow{\Id} & \text{Bialg} \\
\xrightarrow{\overline{T}} & \xrightarrow{P} & \xrightarrow{\overline{T}_1} \\
\text{Vec} & \xrightarrow{V_{0,1}} & \text{Vec} \\
\end{array}
\]

This result was obtained by proving first that the adjunction \( (\overline{T}_1, P_1) \) is idempotent or equivalently that \( \overline{n}_1U_{1,2} \) is an isomorphism. Note that, by [4, Proposition 2.3], we can take \( \overline{T}_2 := \overline{T}_1U_{1,2} \), \( U_{1,2}\overline{n}_2 = \overline{n}_1U_{1,2} \) and \( \overline{\epsilon}_2 = \epsilon_2 \). We have seen in [10, Theorems 7.2 and 8.1] and [6, Theorem 3.3] that the category \( \text{Vec}_2 \) is equivalent to the category \( \text{Lie} \) of (restricted) Lie algebras over \( k \) and that the adjunction \( (\overline{T}_2, P_2) \) plays the role of the usual adjunction, between the categories Bialg and Lie, given by the (restricted) universal enveloping algebra functor and the primitive functor. The fact that the monadic decomposition has length at most two means that the unit \( \overline{n}_2 : \Id \to P_2\overline{T}_2 \) is invertible. In view of the identifications we mentioned, this is the counterpart of half of the Milnor–Moore theorem [27, Theorems 5.18(1) and 6.11(1)]. Now, given \( V_2 = (V, \mu, \mu_1) \in \text{Vec}_2 \), with \( \mu : VTV \to V, V_1 := (V, \mu) \) and \( \mu_1 : P_1\overline{T}_1V_1 \to V_1 \), one has \( \mu_1 \circ \overline{n}_1 = \Id \) and hence \( \mu_1 = (\overline{n}_1V_1)^{-1} \) (note that \( \overline{n}_1V_1 = \overline{n}_1U_{1,2}V_2 \) is invertible). Moreover \( \overline{T}_2V_2 = \overline{T}_1U_{1,2}V_2 = \overline{T}_1V_1 \). Following the proof of [4, Theorem 3.4], we can compute explicitly \( \overline{T}_1V_1 \) as \( \frac{TV}{(z-\mu(z))_{z \in EV}} \), where \( EV \) denotes the subspace of \( \overline{TV} \) spanned by element of homogeneous degree greater than one, and hence we obtain that \( \overline{T}_1V_1 = U(V, c, \mu) \) in the sense of [1, Definition 3.5], where \( c : V \otimes V \to V \otimes V : x \otimes y \mapsto y \otimes x \) is the braiding of \( \text{Vec} \).

Note that, in the same quoted definition, it is set \( S(V, c) := U(V, c, 0) = \frac{TV}{(z-\mu(z))_{z \in EV}} \). Clearly \( S(V, c) \) coincides with \( S[1][V] \) of Example 7.7. In [1, Corollary 5.5] it is proved that \( PS(V, c, \mu) \cong V \) using the fact that \( PS(V, c) \cong V \). In view of the above identifications, the latter isomorphism means that \( U[1][\eta_{1}][\Gamma][1]V : V \to P\overline{L}[1][\Gamma][1]V = \psiS[1][V] \) is invertible and we already observed that this is another way to say that \( V \) has combinatorial rank at most one (the primitive elements in \( P\overline{S}[1][V] \) are concentrated in degree one). On the other hand, the first isomorphism implies that \( U_{1}\overline{n}_1V_1 : V \to P\overline{T}_1V_1 \) is invertible for any \( V_2 \in \text{Vec}_2 \). Equivalently \( U_1\overline{n}_1U_{1,2} \) is invertible which is the same as requiring that \( \overline{n}_1U_{1,2} \) is invertible i.e. the condition, recalled above, saying that the adjunction \( (\overline{T}_1, P_1) \) is idempotent. Summing up, using that any object in \( \text{Vec} \) has combinatorial rank at most one, we can prove that \( (\overline{T}_1, P_1) \) is idempotent and hence that \( P \) has monadic decomposition of length at most two.

As mentioned, we can consider an adjunction \( (L, R) \) whose associated monad is augmented. If every object in \( \mathcal{B} \) has combinatorial rank at most one, it is natural to wonder if, also in this wider setting, it is true that \( (L_1, R_1) \) is idempotent and hence \( R \) has monadic decomposition of length at most two. In this way the adjunction \( (L_2, R_2) \) would be involved in an analogue of the Milnor–Moore theorem in the above sense. More generally one can ask whether \( (L_N, R_N) \) is idempotent
in case the combinatorial rank of objects in $\mathcal{B}$ for an adjunction $(L, R)$ as in Theorem 5.2 is at most $N \in \mathbb{N}$. This would provide an inverse of Theorem 7.5.

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