OPERATORS ASSOCIATED WITH A STOCHASTIC DIFFERENTIAL EQUATION
DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

FABRICE BAUDOIN, LAURE COUTIN

Laboratoire de Probabilités et Statistiques
Université Paul Sabatier
31062 TOULOUSE Cedex 9 France
fbaudoin@cict.fr, coutin@cict.fr

Abstract. In this paper, by using a Taylor development type formula, we show how it is possible to
associate differential operators with stochastic differential equations driven by a fractional Brownian
motion. As an application, we deduce that invariant measures for such SDEs must satisfy an infinite
dimensional system of partial differential equations.

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1. INTRODUCTION AND MAIN RESULT

A $d$-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is a Gaussian process

$$B_t = (B^1_t, ..., B^d_t), \ t \geq 0,$$
where $B_1, \ldots, B^d$ are $d$ independent centered Gaussian processes with covariance function

$$R(t, s) = \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right).$$

It can be shown that such a process admits a continuous version whose paths have $p$ finite variation for $1/p < H$. Let us observe that for $H = \frac{1}{2}$, $B$ is a standard Brownian motion.

In this paper, we are interested in the study in small times of stochastic differential equations on $\mathbb{R}^n$

\begin{equation}
X_t^{x_0} = x_0 + \sum_{i=1}^d \int_0^t V_i(X_s^{x_0}) dB^i_s
\end{equation}

where the $V_i$’s are $C^\infty$-bounded vector fields on $\mathbb{R}^n$ and $B$ is a $d$ dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{3}$.

Let us recall that a smooth vector field $V$ on $\mathbb{R}^n$ is simply a smooth map

$$V : \mathbb{R}^n \to \mathbb{R}^n \quad \text{such that} \quad x \mapsto (v_1(x), \ldots, v_n(x)).$$

It defines a differential operator acting on the smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$(V f)(x) = \sum_{i=1}^n v_i(x) \frac{\partial f}{\partial x_i}.$$  

With this notation, we observe that $V$ is a derivation, that is a map on $C^\infty(\mathbb{R}^n, \mathbb{R})$, linear over $\mathbb{R}$, satisfying for $f, g \in C^\infty(\mathbb{R}^n, \mathbb{R})$,

$$V(fg) = (V f)g + f(V g).$$

If $H > \frac{1}{2}$, the integrals

$$\int_0^t V_i(X_s^{x_0}) dB^i_s$$

are understood in the sense of Young’s integration; see [15], [21] and [22]. But if $H > \frac{1}{3}$ the integrals that appear in [13] are understood in the rough paths sense of Lyons (see [9]). For the convenience of the reader, we included in an appendix at the end of this paper some results of rough paths theory that are used in our proofs.

By using [22] and Theorem 6.3.1. pp. 179 of [13], it is possible to show the existence and the uniqueness of a process $(X_t^{x_0})_{t \geq 0}$ solving (1.1). Observe that from the change of variable formula the process $(X_t^{x_0})_{t \geq 0}$ is such that for every smooth function $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(X_t^{x_0}) = f(x_0) + \sum_{i=1}^d \int_0^t (V_i f)(X_s^{x_0}) dB^i_s.$$
We denote by $C^\infty_b(\mathbb{R}^n, \mathbb{R})$ the set of compactly supported smooth functions $\mathbb{R}^n \to \mathbb{R}$. If $f \in C^\infty_b(\mathbb{R}^n, \mathbb{R})$, let us denote

$$P_t f(x_0) = E(f(X^{x_0}_t)), \ t \geq 0,$$

where $X^{x_0}_t$ is the solution of (1.1) at time $t$.

Our main result is the following:

**Theorem 1.** Assume $H > \frac{1}{2}$. There exists a family $(\Gamma^H_k)_{k \geq 0}$ of differential operators such that:

1. If $f \in C^\infty_b(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$, then for every $N \geq 0$, when $t \to 0$

   $$P_t f(x) = \sum_{k=0}^{N} t^{2kH} (\Gamma^H_k f)(x) + o(t^{(2N+1)H});$$

2. $$\Gamma^H_1 = \frac{1}{2} \sum_{i=1}^{d} V_i^2;$$

3. $$\Gamma^H_2 = \frac{H}{4} \beta(2d, 2H) \sum_{i,j=1}^{d} V_i^2 V_j^2 + \frac{2H - 1}{8(4H - 1)} \sum_{i,j=1}^{d} V_i V_j^2 V_i$$

   $$+ \left( \frac{H}{4(4H - 1)} - \frac{H}{4} \beta(2d, 2H) \right) \sum_{i,j=1}^{d} (V_i V_j)^2,$$

   where $\beta(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} dx$;

4. More generally, $\Gamma^H_k$ is a homogeneous polynomial in the $V_i$'s of degree $2k$;

5. If $H = \frac{1}{2}$ or $V_i V_j = V_j V_i$ for every $1 \leq i, j \leq d$, then

   $$\Gamma^H_k = \frac{1}{k!2^k} \left( \sum_{i=1}^{d} V_i^2 \right)^k.$$  

**Remark 2.** The proof of this Theorem relies on the explicit bound of modulus of continuity of the Itô map (see the appendix). In the book of [13], this is done only for sample paths with $p$ finite variation, with $2 \leq p < 3$. This is the only reason why this Theorem is stated for $H > \frac{1}{3}$, but it certainly also holds true for $H \in \left( \frac{1}{4}, \frac{1}{3} \right]$.

**Remark 3.** In the case of Brownian motion, some more precise results are available in [1], [2] and [4].
2. Commutative case

In this section, we investigate the simplest case which is the commutative case. More precisely, we assume throughout the section that the Lie brackets $[V_i, V_j] = V_i V_j - V_j V_i = 0$, $1 \leq i, j \leq d$.

For $i = 1, ..., d$, let us denote by $(e^{t V_i})_{t \in \mathbb{R}}$ the (deterministic) flow associated with the ordinary differential equation

$$\frac{dx}{dt} = V_i(x_t).$$

**Proposition 4.** The flow $\Phi_t$ associated with equation (1.1) is given by the formula

$$\Phi_t = e^{V_1 B_1 t} \circ \cdots \circ e^{V_d B_d t}.$$

**Proof.** Observe first that since the vector fields $V_i$’s are commuting, the flows $(e^{t V_i})_{t \in \mathbb{R}}$ are also commuting.

We set now for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^d$,

$$F(x, y) = (e^{y_1 V_1} \circ \cdots \circ e^{y_d V_d})(x).$$

By applying the change of variable formula, we easily see that the process $(e^{B_d t V_d x_0})_{t \geq 0}$ is solution of the equation

$$d \left( e^{B_d t V_d x_0} \right) = V_d \left( e^{B_d t V_d x_0} \right) dB^d_t.$$

A new application of Itô’s formula shows now that, since $V_d$ and $V_{d-1}$ are commuting,

$$d \left( e^{B_d^{-1} V_{d-1} (e^{B_d^+ V_d x_0})} \right) = V_{d-1} \left( e^{B_d^{-1} V_{d-1} (e^{B_d^+ V_d x_0})} \right) dB^{d-1}_t + V_d \left( e^{B_d^{-1} V_{d-1} (e^{B_d^+ V_d x_0})} \right) dB^d_t.$$

We deduce hence, by an iterative application of the change of variable formula that the process $(F(x_0, B_t))_{t \geq 0}$ satisfies

$$dF(x_0, B_t) = \sum_{i=1}^d V_i(F(x_0, B_t)) dB^i_t.$$

Thus, by pathwise uniqueness for the equation (1.1), we conclude that

$$X^{x_0}_t = F(x_0, B_t), \ t \geq 0.$$

□

**Remark 5.** Observe that the expression

$$e^{V_1 B_1 t} \circ \cdots \circ e^{V_d B_d t}$$

satisfies the equation (1.1).
is actually defined for every $H \in (0, 1)$. Therefore, in the commutative case, it makes sense to define solutions of stochastic differential equations driven by fractional Brownian motions without restriction on the values of the Hurst parameter $H$, and without using rough paths theory. For instance, solutions of one-dimensional equations like

$$dX_t = \sigma(X_t)dB_t$$

are well defined for any value of the Hurst parameter, see for instance [14].

**Corollary 6.** For any smooth $f : \mathbb{R}^n \to \mathbb{R}$,

$$E(f(X_t^{x_0})) = \left(\exp\left(\frac{1}{2} t^{2H} \sum_{i=1}^{d} V_i^2\right) f\right)(x_0).$$

That is, the function

$$\varphi(t, x) = E(f(X_t^x)),$$

satisfies the partial differential equation

$$\frac{\partial \varphi}{\partial t} = H t^{2H-1} \sum_{i=1}^{d} (V_i f),$$

associated with the initial condition

$$\varphi(0, x) = f(x).$$

**Proof.** Observe first that from Itô’s formula for the fractional Brownian motion, see [7]

$$E\left(e^{V_iB_i t} f(x_0)\right) = f(x_0) + H \int_0^t s^{2H-1} E\left(e^{V_iB_i s} V_i^2 f(x_0)\right) ds.$$

Therefore,

$$E\left(e^{V_iB_i t} f(x_0)\right) = \left(\exp\left(\frac{1}{2} t^{2H} V_i^2\right) f\right)(x_0).$$

It has been seen that (Proposition 4)

$$\Phi_t = e^{V_iB_i t} \circ \cdots \circ e^{V_iB_i t}.$$ 

Thus

$$E(f(X_t^{x_0})) = \left(\exp\left(\frac{1}{2} t^{2H} \sum_{i=1}^{d} V_i^2\right) f\right)(x_0).$$

**Remark 7.** So, in the commutative case, there is a Feynman-Kac type formula for solutions of equations driven by fractional Brownian motions. It shall be shown later that this type of formula only holds in the commutative case.

□
Example 8. Let us consider a one-dimensional stochastic differential equation of the type

\[(2.1) \quad X_t^{x_0} = x_0 + \int_0^t \sigma(X_s^{x_0}) dB_s \]

where \( \sigma : \mathbb{R} \to \mathbb{R} \) is a \( C^\infty \) bounded function and \( B \) is a fractional Brownian motion with Hurst parameter \( H \in (0, 1) \). Then, the function

\[ \varphi(t, x) = \mathbb{E}(f(X_t^x)), \]

satisfies the partial differential equation

\[ \frac{\partial \varphi}{\partial t} = H t^{2H-1} \sigma^2(x) \frac{\partial^2 \varphi}{\partial x^2}, \]

associated with the initial condition

\[ \varphi(0, x) = f(x). \]

3. Asymptotic development in small times of \( P_t \)

We now study the generic case of non-commuting vector fields. Throughout this section we assume \( H > \frac{1}{3} \) and introduce the following notations:

1. \( \Delta^k[0, t] = \{(t_1, ..., t_k) \in [0, t]^k, t_1 < ... < t_k\}; \)

2. If \( I = (i_1, ..., i_k) \in \{1, ..., d\}^k \) is a word with length \( k \),

\[ \int_{\Delta^k[0, t]} dB^I = \int_{0 < t_1 < ... < t_k \leq t} dB_{i_1}^{t_{i_1}}...dB_{i_k}^{t_{i_k}}. \]

Theorem 9. For \( f \in C^\infty_b(\mathbb{R}^n, \mathbb{R}) \), \( x \in \mathbb{R}^n \), and \( N \geq 0 \), when \( t \to 0 \),

\[ f(X_t) = f(x) + \sum_{k=1}^N t^{2kH} \sum_{l=(i_1, ..., i_{2k})} (V_{i_1}...V_{i_{2k}} f)(x) \int_{\Delta^{2k}[0, 1]} dB^I + o(t^{(2N+1)H}), \]

and

\[ P_t f(x) = f(x) + \sum_{k=1}^N t^{2kH} \sum_{l=(i_1, ..., i_{2k})} (V_{i_1}...V_{i_{2k}} f)(x) \mathbb{E} \left( \int_{\Delta^{2k}[0, 1]} dB^I \right) + o(t^{(2N+1)H}). \]

Proof. Let us denote by \( B^m \) the sequel of linear interpolations of \( B \) along the dyadic subdivision of mesh \( m \), that is if \( t^m_i = i2^{-m} \) for \( i = 0, ..., 2^m \) then for \( t \in [t^m_i, t^m_{i+1}] \),

\[ B^m_{ti} = B^m_{ti} + 2^m(t - t^m_i) \left( B^m_{t_{i+1}} - B^m_{t_i} \right). \]

Consider now the equation

\[(3.1) \quad X^{m,x}_t = x + \sum_{i=1}^d \int_0^t V_i(X^{m,x}_s) dB^{i,m}_s. \]
The process $X^{m,x}$, defined in (3.1), has Lipschitz continuous sample paths. Let $p > \frac{1}{H}$. According to Theorem 5 of (9), $(X^{m,x}_t, \ t \in [0,1])$ converges to $(X_t, \ t \in [0,1])$ in the distance of $p$ variation (see Appendix 6 (6.5) for the definition of this distance).

Let $f$ be in $C_{\infty}^k(\mathbb{R}^n, \mathbb{R})$. Using $2N + 1$ times the change of variable formula

$$f(X^{m,x}_t) = f(x) + \sum_{i=1}^d \int_0^t (V_i f)(X^{m,x}_s) dB^{i,m}_s,$$

we obtain

$$f(X^{m,x}_t) = f(x) + \sum_{i=1}^d \int_0^t (V_i f)(x) \int_{\Delta^k[0,t]} dB^{i,m}_s$$

$$+ \sum_{I=(i_1, \ldots, i_k)} \int_0^t (V_{i_1} \cdots V_{i_k} f)(X^{m,x}_s) dB_{u_1}^{i_1} \cdots dB_{u_{2N+2}}^{i_{2N+2}}.$$

By taking the expectation we obtain therefore:

$$\mathbb{E}(f(X^{m,x}_t)) = f(x) + \sum_{i=1}^d \int_0^t (V_i f)(x) \mathbb{E} \left( \int_{\Delta^k[0,t]} dB^{i,m}_s \right)$$

$$+ \sum_{I=(i_1, \ldots, i_k)} \mathbb{E} \left( \int_0^t (V_{i_1} \cdots V_{i_k} f)(X^{m,x}_s) dB_{u_1}^{i_1} \cdots dB_{u_{2N+2}}^{i_{2N+2}} \right).$$

Since $f$ is continuous and bounded the left member of (3.3) converges to $\mathbb{P}_t f(x)$ when $m$ goes to infinity:

$$\lim_{m \to \infty} \mathbb{E}(f(X^{m,x}_t)) = \mathbb{P}_t f(x).$$

Let now $B^m = (1, B^{m,1}, \ldots, B^{m,2N+1})$ be the smooth functional over $B^m$ in the sense of Definition 3.1.1 page 30 of (13) (see also the Example 21 of the Appendix 6). For $k \leq 2N + 1$, $I = (i_1, \ldots, i_k)$ we have therefore

$$\int_{\Delta^k[0,t]} dB^{m,I} = B^{m,I}.$$

According to Theorem 4 of (9), $B^{m,2}$ converges, in the distance of $p$ variation $p > \frac{1}{H}$, almost surely and in $L^2$ to the geometric functional denoted by $B^2$. For $k \leq 2$, we thus have

$$\int_{\Delta^k[0,t]} dB^I = B^{2,I}.$$

According to Theorem 3.1.3 of (13) or Theorem 26 of the Appendix 6, the geometric functional $B^2$ has a unique extension in $C_{0,p}(T^{(2N+1)}(\mathbb{R}^n))$ denoted by $B^{2N+1}$, and for $k \leq 2N + 1$, $I = (i_1, \ldots, i_k)$ almost surely

$$\int_{\Delta^k[0,t]} dB^I = B^{2N+1,I} = \lim_{m \to \infty} \int_{\Delta^k[0,t]} dB^{m,I}.$$
Since \( \int_{\Delta^{[0,t]}} dB^{m,I} \) belongs to the \( k \)th Wiener chaos of \( B \), the limit also holds in \( L^1 \) according to 3. That is, for \( k \leq 2N + 1 \), \( I = (i_1, \ldots, i_k) \)

\[
\mathbb{E} \left( \int_{\Delta^{[0,t]}} dB^I \right) = \mathbb{E} \left( \mathbb{B}^{2N+1,I} \right) = \mathbb{E} \left( \lim_{m \to \infty} \int_{\Delta^{[0,t]}} dB^{m,I} \right).
\]

Let us now observe that by symmetry for \( k \) an odd integer

\[
\mathbb{E} \left( \int_{\Delta^{[0,t]}} dB^{m,I} \right) = 0.
\]

Using scaling property of fractional Brownian motion, we have for \( k \) an even integer

\[
\mathbb{E} \left( \int_{\Delta^{[0,t]}} dB^I \right) = t^{H_k} \mathbb{E} \left( \int_{\Delta^{[0,1]}} dB^I \right).
\]

Define for \( k = 1, \ldots, N \)

\[
\Gamma^H_k = \sum_{I=(i_1, \ldots, i_{2k})} \mathbb{E} \left( \int_{\Delta^{[2k,0,[1]}} dB^I \right) V_{i_1} \ldots V_{i_{2k}},
\]

\[
= \lim_{m \to \infty} \sum_{I=(i_1, \ldots, i_{2k})} \mathbb{E} \left( \int_{\Delta^{[2k,0,1]}} dB^{m,I} \right) V_{i_1} \ldots V_{i_{2k}}.
\]

Then, the first sum in the right member of (3.6) converges to \( \sum_{k=1}^{N} t^{2kH} (\Gamma^H_k f)(x) \), that is:

\[
\lim_{m \to \infty} \sum_{k=1}^{2N+1} \sum_{I=(i_1, \ldots, i_k)} (V_{i_1} \ldots V_{i_k} f)(x) \mathbb{E} \left( \int_{\Delta^{[0,t]}} dB^{I,m} \right) = \sum_{k=1}^{N} t^{2kH} (\Gamma^H_k f)(x).
\]

According to Theorem 17 of 8, for any \( q \geq 1 \) and \( p > \frac{1}{2} \) there exists a random variable \( C_p \) belonging to \( L^p \) such that for any \( m, k = 1, 2, I = (i_1, i_k) \) and \( (s, t) \in \Delta^{[0,1]} \),

\[
\|\mathbb{E}^{m,I}_{s,t}\| \leq C_p |t - s|^{k/p}.
\]

In what follows \( \theta \) and \( \kappa \) may vary from lines to lines. According to Theorem 37 there exist \( \theta \geq 1, \kappa \geq 1 \) (depending on \( x, V_i, i = 1, \ldots, d \) and \( p \)), such that the geometric functional \( Z^{m,2} \) over \( (B^m, X^{m,x}) \) is controlled by \( \omega(t, s) = \kappa(1 + C_p^\theta)|t - s| \) for any \( m \in \mathbb{N} \). Then, according to Theorem 35 applied to

\[
\alpha(b, x)(v, w) = (v, f(x)v) \forall (b, x) \in \mathbb{R}^{d+n}, \forall (v, w) \in \mathbb{R}^{d+n}
\]

there exist \( \theta \geq 1, \kappa \geq 1 \) (depending on \( x, f, V_i, i = 1, \ldots, d \) and \( p \)), such that the geometric functional \( G^{m,2} \) over \( (B^m, f(X^{m,x})) \) is controlled by \( \omega(t, s) = \kappa(1 + C_p^\theta)|t - s| \), for any \( m \in \mathbb{N} \). Therefore, according to Theorem 28 there exist \( \theta \geq 1, \kappa \geq 1 \) (depending on \( x, f, V_i, i = 1, \ldots, d, N \) and \( p \)), such that for any \( I = (i_1, \ldots, i_{2N+2}), m \in \mathbb{N} \), and \( t \)

\[
\left| \int_{0 \leq u_1 < \ldots < u_{2N+2} < t} f(X^{m,x}_{u_1}) dB_{u_1}^{m,i_1} \ldots dB_{u_{2N+2}}^{m,i_{2N+2}} \right| \leq \kappa(1 + C_p^\theta)t^{(2N+2)/p}.
\]
By taking the expectation of each members of (3.7) and using the fact that \( C_\theta^p \) belongs to \( L^1 \), we deduce that there exists a constant \( \kappa \) depending only on \( x, f, V_i, i = 1, \ldots, N \) and \( p \), such that for \( I = (i_1, \ldots, i_{2N+2}) \), \( m \in \mathbb{N} \), and \( t \)

\[
\mathbb{E} \left( \int_{0 < u_1 < \ldots < u_{2N+2}} f(X_{u_1}^m, x) dB_{u_1}^m \ldots dB_{u_{2N+2}}^m \right) \leq \kappa |t|^{(2N+2)/p}.
\]

Finally, by taking the limit of each term of (3.3) and by using (3.4), (3.6) and (3.8) we get Theorem 9. \( \Box \)

4. Expectation of iterated integrals of the fractional Brownian motion

In this section we analyse the coefficients appearing in the differential operators

\[
\Gamma^H_k = \sum_{I = (i_1, \ldots, i_{2k})} \mathbb{E} \left( \int_{\Delta^{2k}[0,1]} dB^I \right) V_{i_1} \ldots V_{i_{2k}}, \quad H > \frac{1}{4}.
\]

We shall need the following lemma

**Lemma 10.** Let \( G = (G_1, \ldots, G_{2k}) \) be a centered Gaussian vector. We have

\[
\mathbb{E} (G_1 \ldots G_{2k}) = \frac{1}{k!2^k} \sum_{\sigma \in \mathfrak{S}_{2k}} \prod_{l=1}^k \mathbb{E} (G_{\sigma(2l)} G_{\sigma(2l-1)}) ,
\]

where \( \mathfrak{S}_{2k} \) is the group of the permutations of the set \( \{1, \ldots, 2k\} \).

**Theorem 11.** Assume \( H > \frac{1}{2} \). Let \( I = (i_1, \ldots, i_{2k}) \) be a word, then

\[
\mathbb{E} \left( \int_{\Delta^{2k}[0,1]} dB^I \right) = \frac{H^k}{k!2^k} (2H-1)^k \sum_{\sigma \in \mathfrak{S}_{2k}} \prod_{l=1}^k \delta_{\sigma(2l-1), i_{\sigma(2l)}} \left| t_{\sigma(2l)} - t_{\sigma(2l-1)} \right|^{2H-2} dt_{1} \ldots dt_{2k}
\]

where \( \delta_{i,j} \) is the Kronecker’s symbol.

**Proof.** Let us first hint how this formula works with a heuristic argument. We have

\[
\mathbb{E} \left( \int_{\Delta^{2k}[0,1]} dB^I \right) = \int_{\Delta^{2k}[0,1]} \mathbb{E} (dB^I) .
\]

But, from the Lemma 10

\[
\mathbb{E} (dB^I) = \frac{1}{k!2^k} \sum_{\sigma \in \mathfrak{S}_{2k}} \prod_{l=1}^k \mathbb{E} \left( dB_{\sigma(2l)}^I dB_{\sigma(2l-1)}^I \right) .
\]

By using the covariance function of the fractional Brownian motion, we get therefore

\[
\mathbb{E} \left( dB_{\sigma(2l)}^I dB_{\sigma(2l-1)}^I \right) = \delta_{\sigma(2l-1), i_{\sigma(2l)}} H(2H-1) \left| t_{\sigma(2l)} - t_{\sigma(2l-1)} \right|^{2H-2} dt_{\sigma(2l)} dt_{\sigma(2l-1)} .
\]
which leads to the expected result.

We now turn to the rigorous proof. Let us again denote by $B^m$ the sequel of linear interpolation of $B$ along the dyadic subdivision of mesh $m$ and recall that

$$E \left( \int_{\Delta^k[0,t]} dB^l \right) = \lim_{m \to \infty} E \left( \int_{\Delta^k[0,t]} dB^{m,l} \right).$$

¿From the Lemma 10, we have

$$E \left( \int_{\Delta^k[0,t]} dB^{m,l} \right) = \frac{1}{k!2^k} \sum_{\sigma \in \mathcal{E}_{2k}} \int_{\Delta^k[0,t]} \prod_{k} \left( \frac{dB^{m,i\sigma(2l)} dB^{m,i\sigma(2l-1)}}{dt_{\sigma(2l)} dt_{\sigma(2l-1)}} \right) dt_1...dt_{2k}.$$ 

If $t_{\sigma(2l)} \in [t^m_{i+1}, t^m_i]$ and $t_{\sigma(2l-1)} \in [t^m_j, t^m_{j+1}]$, using the expression of $B^m$, we get

$$E \left( \frac{dB^{m,i\sigma(2l)} dB^{m,i\sigma(2l-1)}}{dt_{\sigma(2l)} dt_{\sigma(2l-1)}} \right) = \delta_{i\sigma(2l-1),i\sigma(2l)} 2^{2m-2Hm}, \text{ for } i = j;$$

$$= \delta_{i\sigma(2l-1),i\sigma(2l)} 2^{m} E(\Delta^m_i B \Delta^m_{i-1} B), \text{ for } j = i - 1;$$

$$= \delta_{i\sigma(2l-1),i\sigma(2l)} H(2H-1) 2^{2m} \int \int_{[t^m_{i+1}, t^m_{i+1}] \times [t^m_{j}, t^m_{j+1}]} |x-y|^{2H-2} dxdy \text{ for } j < i - 1.$$ 

Here, $\Delta^m_i B$ is the increment of $B$ between $t^m_i$ and $t^m_{i+1}$ that is, $\Delta^m_i B = B(t^m_{i+1}) - B(t^m_i)$. By using Cauchy-Schwartz inequality, we have

$$|E(\Delta^m_i B \Delta^m_{i-1} B^m)| \leq 2^{-2mH}.$$ 

and the result follows from the Lebesgue convergence dominated theorem.

□

**Lemma 12.** Let $H > \frac{1}{2}$.

$$\int_{\Delta^4[0,1]} (t_2 - t_1)^{2H-2} dt_1 dt_2 = \frac{1}{2H(2H-1)}$$

$$\int_{\Delta^4[0,1]} (t_4 - t_3)^{2H-2}(t_2 - t_1)^{2H-2} dt_1 dt_2 dt_3 dt_4 = \frac{\beta(2H, 2H)}{4H(2H-1)^2}$$

$$\int_{\Delta^4[0,1]} (t_4 - t_1)^{2H-2}(t_3 - t_2)^{2H-2} dt_1 dt_2 dt_3 dt_4 = \frac{1}{8H^2(2H-1)(4H-1)}$$

$$\int_{\Delta^4[0,1]} (t_4 - t_2)^{2H-2}(t_3 - t_1)^{2H-2} dt_1 dt_2 dt_3 dt_4 = \frac{1}{4H(4H-1)(2H-1)^2} - \frac{\beta(2H, 2H)}{4H(2H-1)^2}$$

**Proof.** The result follows by direct but tedious computations.

First of all, we recall that the beta function

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

satisfies:

$$\beta(a + 1, b) = \frac{a}{b} \beta(a, b + 1)$$
\[ \beta(a, b) = \beta(a + 1, b) + \beta(a, b + 1). \]

(1) For the first integral:
\[ \int_{\Delta^*[0,1]} (t_2 - t_1)^{2H-2} dt_1 dt_2 = \int_0^1 \int_0^{t_2} (t_2 - t_1)^{2H-2} dt_1 dt_2 = \frac{1}{2H-1} \int_0^1 t_2^{2H-1} dt_2 = \frac{1}{2H(2H-1)}. \]

(2) For the second integral:
\[ \int_{\Delta^*[0,1]} (t_4 - t_3)^{2H-2} (t_2 - t_1)^{2H-2} dt_1 dt_2 dt_3 dt_4 = \frac{1}{2H(2H-1)} \int_0^1 \int_0^{t_3} \int_0^{t_4} (t_4 - t_3)^{2H-2} (t_2 - t_1)^{2H-2} dt_1 dt_2 dt_3 dt_4 \]
\[ = \frac{1}{8H^2(2H-1)} \int_0^1 \int_0^{t_3} (1 - s)^{2H-2} s^{2H} ds \quad (t_3 = s) \]
\[ = \frac{\beta(2H, 2H)}{4H(2H-1)^2}. \]

(3) For the third integral
\[ \int_{\Delta^*[0,3]} (t_4 - t_1)^{2H-2} (t_3 - t_2)^{2H-2} dt_1 dt_2 dt_3 dt_4 = \frac{1}{2H-1} \int_0^1 \int_0^{t_4} \int_0^{t_3} \int_0^{t_1} (t_4 - t_1)^{2H-1} (t_3 - t_2)^{2H-1} (t_1 - t_2)^{2H-1} dt_1 dt_2 dt_3 dt_4 \]
On one hand,
\[ \frac{1}{2H-1} \int_0^1 \int_0^{t_4} \int_0^{t_3} t_4^{2H-1} (t_3 - t_2)^{2H-2} dt_2 dt_3 dt_4 \]
\[ = \frac{1}{(2H-1)^2} \int_0^1 \int_0^{t_4} t_4^{2H-1} t_3^{2H-1} dt_3 dt_4 \]
\[ = \frac{1}{8H^2(2H-1)} \]
On the other hand,
\[ \frac{1}{2H-1} \int_0^1 \int_0^{t_4} \int_0^{t_3} (t_4 - t_2)^{2H-1} (t_3 - t_2)^{2H-2} dt_2 dt_3 dt_4 \]
\[ = \frac{1}{2H(2H-1)} \int_0^1 \int_0^{t_3} (t_3 - t_2)^{2H-2} (1 - t_2)^{2H} dt_2 dt_3 dt_4 \]
\[ = \frac{1}{2H(2H-1)^2} \int_0^1 (1 - t_2)^{4H-1} dt_2 - \frac{1}{2H(2H-1)(4H-1)} \int_0^1 (1 - t_2)^{4H-1} dt_2. \]
By putting things together, we obtain the expected result.

(4) For the last integral, the computation follows the same lines so that we do not enter into details.
As an immediate corollary, we deduce

**Corollary 13.** Assume $H > \frac{1}{2}$.

\[
\Gamma^H_1 = \frac{1}{2} \sum_{i=1}^{d} V_i^2
\]

\[
\Gamma^H_2 = \frac{H}{4} \beta(2H,2H) \sum_{i,j=1}^{d} V_i^2 V_j^2 + \frac{2H-1}{8(4H-1)} \sum_{i,j=1}^{d} V_i V_j^2 V_i + \left( \frac{H}{4(4H-1)} - \frac{H}{4} \beta(2H,2H) \right) \sum_{i,j=1}^{d} (V_i V_j)^2.
\]

It is interesting to observe that the previous corollary makes sense and actually also holds true for $H > \frac{1}{4}$.

**Theorem 14.** Assume $H > \frac{1}{4}$, then the conclusions of Corollary 13 are still true.

We devote now the end of the section to the proof of this theorem. The trick is to perform transformations on

\[
\mathbb{E} \left( \int_{\Delta^2[0,t]} dB^{m,I} \right)
\]

before passing to the limit.

Showing that for $H > \frac{1}{4}$,

\[
\Gamma^H_1 = \frac{1}{2} \sum_{i=1}^{d} V_i^2
\]

is easy. Indeed, let $(i_1, i_2) \in \{1, ..., d\}^2$ such that $i_1 \neq i_2$. It is easily seen that $B^{m,i_1}$ and $B^{m,i_2}$ are independent, thus

\[
\mathbb{E} \left( \int_{\Delta^2[0,1]} dB^{m,I} \right) = 0.
\]

Now, if $i \in \{1, ..., d\}$ and $I = (i, i)$ then

\[
\mathbb{E} \left( \int_{\Delta^2[0,1]} dB^{m,I} \right) = \frac{1}{2} \mathbb{E} \left( (B^{m,i}(1))^2 \right) = \frac{1}{2}.
\]

Since $\Gamma^H_1 = \sum_{I = (i,j)} \lim_{m \to \infty} \mathbb{E} \left( \int_{\Delta^2[0,1]} dB^{m,I} \right) V_j V_i$, we get the expected result.

Let now $I = (i, j, k, l) \in \{1, ..., d\}^4$. Recall that $B^m$ is absolutely continuous with respect to the Lebesgue measure, and the fourth iterated integral is

\[
\int_{\Delta^4[0,1]} dB^{I,m} = \int_{0 < u_1 < u_2 < u_3 < u_4 < 1} \frac{dB^{m,i}}{du}(u_1) \frac{dB^{m,j}}{du}(u_2) \frac{dB^{m,k}}{du}(u_3) \frac{dB^{m,l}}{du}(u_4) du_1 du_2 du_3 du_4.
\]
Applying Lemma 10 yields

\[ E \left( \int_{\Delta^4[0,1]} dB^{l,m} \right) = \delta_{i,j} \delta_{l,k} A^{m,1} + \delta_{i,k} \delta_{l,j} A^{m,2} + \delta_{i,l} \delta_{j,k} A^{m,3}, \]

where

\[
A^{m,1} = \int_0^1 du_4 \int_0^{u_4} du_3 \mathbb{E} \left( \frac{dB^m}{du}(u_3) \frac{dB^m}{du}(u_4) \right) \int_0^{u_3} du_2 \int_0^{u_2} du_1 \mathbb{E} \left( \frac{dB^m}{du}(u_2) \frac{dB^m}{du}(u_1) \right);
\]

\[
A^{m,2} = \int_0^1 du_4 \int_0^{u_4} du_2 \mathbb{E} \left( \frac{dB^m}{du}(u_2) \frac{dB^m}{du}(u_4) \right) \int_0^{u_4} du_3 \int_0^{u_3} du_1 \mathbb{E} \left( \frac{dB^m}{du}(u_3) \frac{dB^m}{du}(u_1) \right);
\]

\[
A^{m,3} = \int_0^1 du_4 \int_0^{u_4} du_1 \mathbb{E} \left( \frac{dB^m}{du}(u_4) \frac{dB^m}{du}(u_1) \right) \int_0^{u_4} du_3 \int_0^{u_3} du_2 \mathbb{E} \left( \frac{dB^m}{du}(u_3) \frac{dB^m}{du}(u_2) \right).
\]

Therefore, we have to prove that

\[
\lim_{m \to \infty} A^{m,1} = \frac{H}{4} \beta(2H, 2H);
\]

\[
\lim_{m \to \infty} A^{m,2} = \frac{H}{4(4H - 1)} - \frac{H}{4} \beta(2H, 2H);
\]

and

\[
\lim_{m \to \infty} A^{m,3} = \frac{2H - 1}{8(4H - 1)}.
\]

The proof relies on several Lemma.

**Lemma 15.** For any \( \alpha < H \), the random variable

\[
\sup_m \| B^m - B \|_\alpha = \sup_{m \in \mathbb{N}} \sup_{(s,t) \in [0,1]^2, s < t} \frac{|B(t) - B^m(t)| - |B(s) - B^m(s)|}{|t - s|^{\alpha}}
\]

has a finite exponential moment.

**Proof.** According to Kolmogorov lemma, the random variable \( \sup_{(s,t) \in [0,1]^2, s < t} \frac{|B(t) - B^m(t)| - |B(s) - B^m(s)|}{|t - s|^{\alpha}} \) is finite. Since \( B \) is Gaussian process, according to Theorem 1.2.3 of [12], it has a finite exponential moment. Then the Lemma is a consequence of results on linear interpolation of Hölder functions. \( \square \)

**Lemma 16.** For \( H > \frac{1}{4} \),

\[
\lim_{m \to \infty} A^{m,1} = \frac{H}{4} \beta(2H, 2H).
\]

**Proof.** Integrating with respect to \( u_1 \) and \( u_2 \) in the expression of \( A^{m,1} \) leads to

\[
A^{m,1} = \frac{1}{2} \int_0^1 du_4 \int_0^{u_4} du_3 \frac{d^2}{du_3 du_4} \mathbb{E} \left( B^m(u_3) B^m(u_4) \right) \mathbb{E} \left( B^m(u_3)^2 \right).
\]
Using Fubini’s theorem, we integrate first with respect to $u_4$:

$$A_{m,1} = \frac{1}{2} \int_0^1 du_3 \mathbb{E}[[B^m(1) - B^m(u_3)] \frac{d}{du} B^m(u_3)] \mathbb{E}(B^m(u_3)^2);$$

$$= \frac{1}{2} \left[ \int_0^1 du_3 \frac{d}{du} B^m(u_3) B^m(1) \mathbb{E}(B^m(u_3)^2) - \frac{1}{4} \mathbb{E}(B^m(1)^2)^2 \right].$$

Now, using the expression of $B^m$ we obtain that for $u \in [t^m_i, t^m_{i+1}[$,

$$\frac{d}{du} \mathbb{E}(B^m(u) B^m(1)) = 2^m \mathbb{E}(\Delta^m_i BB(1))$$

$$= \frac{2^m}{2} |t^m_{i+1}|^{2H} - |1 - t^m_{i+1}|^{2H} - (|t^m_i|^{2H} - |1 - t^m_i|^{2H})]$$

$$= H 2^m \int_{t^m_i}^{t^m_{i+1}} [r^{2H-1} + |1 - r|^{2H-1}] dr.$$

Then, using Fubini’s theorem, we get

$$\int_0^1 du \mathbb{E}(B^m(1) \frac{d}{du} B^m(u)) \mathbb{E}(B^m(u)^2) = H \int_0^1 [r^{2H-1} + |1 - r|^{2H-1}] a^m(r) dr$$

where

$$a^m(r) = \sum_{i=0}^{2^m-1} 2^m \int_{t^m_i}^{t^m_{i+1}} \mathbb{E}(B^m(u)^2) du 1_{t^m_i, t^m_{i+1}[}(r).$$

For all $r \in [0, 1]$, $a^m(r)$ converges to $\mathbb{E}(B(r)^2) = r^{2H}$. Following Lemma 14, it is bounded uniformly on $m$ and $r$. Then, using dominated convergence Lebesgue theorem we obtain

$$\lim_{m \to \infty} \int_0^1 du \mathbb{E}(B^m(1) \frac{d}{du} B^m(u)) \mathbb{E}(B^m(u)^2) = H \int_0^1 [r^{2H-1} + |1 - r|^{2H-1}] r^{2H} dr$$

$$= \frac{1}{4} + \frac{H}{2} \beta(2H, 2H)$$

because $\beta(2H, 2H + 1) = \frac{1}{2} \beta(2H, 2H)$. We conclude that

$$\lim_{m \to \infty} A_{m,1} = \frac{H}{4} \beta(2H, 2H).$$

\[\square\]

**Lemma 17.** For $H > \frac{1}{4}$,

$$\lim_{m \to \infty} A_{m,3} = \frac{2H - 1}{8(4H - 1)}.$$

**Proof.** Recall that

$$A_{m,3} = \int_0^1 du_4 \int_0^{u_4} du_1 \mathbb{E}\left( \frac{dB^m}{du}(u_4) \frac{dB^m}{du}(u_1) \right) \int_0^{u_4} du_3 \int_{u_4}^{u_3} du_2 \mathbb{E}\left( \frac{dB^m}{du}(u_3) \frac{dB^m}{du}(u_2) \right).$$
Integrating with respect to $u_2$ and $u_3$ we obtain

$$A^{m,3} = \frac{1}{2} \int_0^1 du_4 \int_0^{u_4} du_1 \mathbb{E}\left(\frac{dB^m}{du}(u_4)\frac{dB^m}{du}(u_1)\right) \mathbb{E}\left([B^m(u_4) - B^m(u_1)]^2\right).$$

We introduce the indices $i, j$ such that $u_4 \in [t_i^m, t_{i+1}^m]$, and $u_1 \in [t_j^m, t_{j+1}^m]$. Recall that

$$\mathbb{E}\left(\frac{dB^m}{du}(u_1)\frac{dB^m}{du}(u_4)\right) = 2^{m-2H}, \text{ for } i = j;$$

$$= 2^m \mathbb{E}(\Delta_i^m B \Delta_{i-1}^m B) \text{ for } j = i - 1;$$

$$= H(2H - 1)2^m \int_{t_i^m}^{t_{i+1}^m} \int_{t_j^m}^{t_{j+1}^m} |x - y|^{2H-2} dx \, dy \text{ for } j < i - 1.$$ 

Then, we split $A^{m,3}$ into three parts

$$A^{m,3} = \frac{1}{2} \left(A^{3,1,m} + A^{3,2,m} + A^{3,3,m}\right)$$

where

$$A^{3,1,m} = \sum_{i=0}^{m-1} 2^{4m-2Hm} \int_{t_i^m}^{t_{i+1}^m} \int_{t_i^m}^{t_{i+1}^m} du \, dv (u - v)^2 \mathbb{E}(\Delta_i^m B^2);$$

$$A^{3,2,m} = \sum_{i=0}^{m-1} 2^m \mathbb{E}(\Delta_i^m B \Delta_{i-1}^m B) \int_{t_i^m}^{t_{i+1}^m} \int_{t_{i-1}^m}^{t_i^m} dv \mathbb{E}([B^m(u) - B^m(v)]^2);$$

and

$$A^{3,3,m} = H(2H - 1) \sum_{i=2}^{m-1} \sum_{j=0}^{i-2} 2^{2m-i} \int_{t_i^m}^{t_{i+1}^m} dx \int_{t_j^m}^{t_{j+1}^m} dy (x - y)^{2H-2} \int_{t_i^m}^{t_{i+1}^m} \int_{t_j^m}^{t_{j+1}^m} du \, dv \mathbb{E}([B^m(u) - B^m(v)]^2).$$

We have to prove that for $k = 1, 2$

$$\lim_{m \to \infty} A^{3,k,m} = 0$$

and

$$\lim_{m \to \infty} A^{3,3,m} = \frac{2H - 1}{4(4H - 1)}.$$ 

First we prove (4.4). Integrating with respect to $u$ and $v$ yields

$$A^{3,1,m} = \frac{1}{12} \sum_{i=0}^{m-1} 2^{-4Hm}.$$ 

Since $H > \frac{1}{4}$, we deduce that $\lim_{m \to \infty} A^{3,1,m} = 0$.

According to Lemma 13, we have for $\alpha < H$

$$|A^{3,1,m}| \leq \frac{2}{(2\alpha + 1)(2\alpha + 2)} 2^{(1-2H-2\alpha)m} \mathbb{E}(\sup_{m} \sup_{(s,t) \in [0,1]^2, s < t} \frac{|B(s) - B(t)|^2}{|t - s|^{2\alpha}}).$$
Since \( H > \frac{1}{4} \), we deduce that \( \lim_{m \to \infty} A^{3,2,m} = 0 \).

Now, we prove 155. Indeed using Fubini’s theorem, we have

\[
A^{3,3,m} = H(2H - 1) \int_0^1 dx \int_0^x (x - y)^{2H - 2} a^{3,3,m}(x, y) dy.
\]

where for \( 0 \leq j < i - 1 \), and \( (x, y) \in [t^m_i, t^m_{i+1}] \times [t^m_j, t^m_{j+1}] \)

\[
a^{3,3,m}(x, y) = 2^{2m} \int_{t^m_i}^{t^m_{i+1}} du \int_{t^m_j}^{t^m_{j+1}} dv \mathbb{E}((B^m(v) - B^m(u))^2)
\]

and \( a^{3,3,m}(x, y) = 0 \) elsewhere. For all \( x, y \) \( a^{2,3,m}(x, y) \) converges almost surely to \( \mathbb{E}([B(x) - B(y)]^2) \).

Moreover, since \( |t^m_{i+1} - t^m_j| \leq 2|x - y| \), \( a^{3,3,m} \) is bounded by

\[
|a^{3,3,m}(x, y)| \leq 2\mathbb{E}(\sup_m \sup_{s < t} \frac{|B(s) - B(t)|}{|t - s|^{\alpha}})^2 |2(x - y)|^{2\alpha}.
\]

Using Lemma 15 and dominated Lebesgue convergence theorem, we take the limit when \( m \) goes to infinity for \( H > \frac{1}{4} \):

\[
\lim_{m \to \infty} A^{3,3,m} = H(2H - 1) \int_0^1 dx \int_0^x (x - y)^{2H - 2} \mathbb{E}([B(x) - B(y)]^2)dy
\]

\[
= \frac{2H - 1}{4(4H - 1)}.
\]

Finally,

**Lemma 18.** For \( H > \frac{1}{4} \),

\[
\lim_{m \to \infty} A^{m,2} = \frac{H}{4(4H - 1)} - \frac{H}{4} \beta(2H, 2H)
\]

**Proof.** We know that if the vector fields commute then

\[
\Gamma^2 = \frac{1}{8} \left( \sum_{i=1}^d V_i^2 \right)^2.
\]

Therefore

\[
\lim_{m \to \infty} A^{m,1} + A^{m,2} + A^{m,3} = \frac{1}{8}.
\]

\( \square \)
5. Application to the study of invariant measures

Consider the stochastic differential equation on \( \mathbb{R}^n \)

\[
dX_t = V_i(X_t^n)dB_i^t
\]

where the \( V_i \)'s are \( C^\infty \)-bounded vector fields on \( \mathbb{R}^n \) and \( B \) is a \( d \) dimensional fractional Brownian motion with Hurst parameter \( H > \frac{1}{3} \).

**Proposition 19.** Assume that \( \mu \) is a probability measure on \( \mathbb{R}^n \) that is invariant by (5.1). In the sense of distributions, we have for every \( k \geq 1 \),

\[
(\Gamma^H_k)^* \mu = 0,
\]

where \( (\Gamma^H_k)^* \) is the formal adjoint of \( \Gamma^H_k \).

**Proof.** Let \( f \in C^\infty_b(\mathbb{R}^n, \mathbb{R}) \). We have for every \( N \geq 0 \), when \( t \to 0 \)

\[
E \left( \int_{\mathbb{R}^n} f(X_t^n)\mu(dx) \right) = \sum_{k=0}^N t^{2kH} \int_{\mathbb{R}^n} (\Gamma^H_k f)(x)\mu(dx) + o(t^{(2N+1)H}).
\]

But since \( \mu \) is invariant,

\[
E \left( \int_{\mathbb{R}^n} f(X_t^n)\mu(dx) \right) = \int_{\mathbb{R}^n} f(x)\mu(dx).
\]

The result follows therefore. \( \square \)

6. Appendix : On rough path theory

We recall some definitions and results on rough path theory, see [13]. Indeed, we precise how all the constants appearing in the continuity Theorem of the Itô map depend on the vectors fields and the control of the paths.

6.1. Basic definitions and properties. We work on \( V = \mathbb{R}^d \) endowed with the Euclidean norm. The tensor product is \( V^\otimes k = V \otimes \ldots \otimes V \) (of \( k \) copies of \( V \)) endowed with a norm \( |.|_k \) compatible with the tensor product that is

\[
|\xi \otimes \eta|_{k+l} \leq |\xi|_k|\eta|_l, \ \forall \xi \in V^\otimes k, \ \forall \eta \in V^\otimes l.
\]

For each \( n \in \mathbb{N} \), the truncated tensor algebra \( T^{(n)}(V) \) is

\[
T^{(n)}(V) = \sum_{k=0}^n V^\otimes k, \ V^\otimes 0 = \mathbb{R}.
\]
Its multiplication is
\[
(\xi \otimes \eta)^k = \sum_{j=0}^{k} \xi^j \otimes \eta^{k-j}, \quad k = 0, \ldots, n, \quad \forall \xi, \eta \in T^{(n)}(V).
\]

The norm \(|\cdot|\) on \(T^{(n)}(V)\) is defined by
\[
|\xi| = \sum_{i=0}^{n} |\xi^i|, \quad \text{if} \ \xi = (\xi^0, \ldots, \xi^n).
\]

The pair \((T^{(n)}(V), |\cdot|)\) is a tensor algebra with identity element \((1, 0, \ldots, 0)\) and for \(\xi, \eta \in T^{(n)}(V)\), \(|\xi \otimes \eta| \leq |\xi||\eta|\).

The tensor algebra \(T^{(\infty)}(V)\) is
\[
T^{(\infty)}(V) = \sum_{k=0}^{\infty} V \otimes^k, \quad V \otimes^0 = \mathbb{R}.
\]

We use \(\Delta^2_{[0,T]}\) to denote the simplex \(\{(s, t), \ 0 \leq s \leq t \leq T\}\). Recall that a control \(\omega\) is a continuous, supper additive function on \(\Delta^2_{[0,T]}\) with values in \([0, +\infty[\) such that \(\omega(t, t) = 0\). Therefore
\[
\omega(s, t) + \omega(t, u) \leq \omega(s, u) \ \forall (s, t), (t, u) \in \Delta^2_{[0,T]}.
\]

**Definition 20.** A continuous map \(X\) from the simplex \(\Delta^2_{[0,T]}\) into a truncated tensor algebra \(T^{(n)}(V)\), and written as \(X_{s,t} = (X_{s,t}^0, \ldots, X_{s,t}^n)\) with \(X_{s,t}^k \in V \otimes^k\) for any \((s, t) \in \Delta^2_{[0,T]}\), \(k = 1, \ldots, n\) is called a multiplicative functional of degree \(n\) \((n \in \mathbb{N}^*)\) if
\[
X_{s,t}^0 = 1
\]
\[
(6.1) \quad X_{s,t} \otimes X_{t,u} = X_{s,u}, \quad \forall (s, t), (t, u) \in \Delta^2_{[0,T]},
\]

where the tensor product \(\otimes\) is taken in \(T^{(n)}(V)\).

Equality (6.1) is called Chen identity, although it appears long before Chen’s fundamental works in which a connection is made from iterated path integrals along smooth paths to a class of differential forms on a space of loops on manifold.

**Example 21.** Let \(x : [0, T] \rightarrow V\) be a continuous path. If \(x\) is Lipschitz path, then we may build a sequence of iterated integral \(X_{s,t}^k = \int_{s<t_1<\ldots<t_k<t} dx_{t_1} \otimes \ldots \otimes dx_{t_k}\). In this case identity (6.1) is equivalent to the additive property of iterated path integrals over different domains.
Definition 22. Let $p \geq 1$ be a constant. We say that a map $X : [0,T] \to T^{(n)}(V)$ possesses finite $p$ variation of

$$|X^i_{s,t}| \leq \omega^{i/p}(s,t), \quad \forall i = 1,\ldots,n, \quad \forall (s,t) \in \Delta^2_{[0,T]}$$

for some control $\omega$.

Definition 23. A multiplicative functional with finite $p$ variation on $T^{([p])}(V)$ is called a rough path (of roughness $p$). We say that a rough path (of roughness $p$) $X$ in $T^{([p])}(V)$ is controlled by $\omega$ if

$$|X^i_{s,t}| \leq \omega(s,t)^{i/p}, \quad \forall i = 1,\ldots,[p], \quad \forall (s,t) \in \Delta^2_{[0,T]}.$$

The set of all rough path with roughness $p$ in $T^{([p])}(V)$ will be denoted by $\Omega_p(V)$.

Definition 24. A smooth rough path $X$ is an element of $\Omega_p(V)$ such that there exists a Lipschitz $x : [0,T] \to V$ Lipschitz such that

$$X^k_{s,t} = \int_{s<t_1<\ldots<t_k<t} dx_{t_1} \otimes \ldots \otimes dx_{t_k}, \quad k = 1,\ldots,[p], \quad \forall (s,t) \in \Delta^2_{[0,T]}.$$

6.2. Extension of rough path. The following theorem shows that the higher (than $[p]$) order terms $X^k (k > [p])$ are determined uniquely by $X^i (i \leq [p])$ among all possible extensions to a multiplicative functional which possess finite $p-$ variation.

Theorem 25. Let $p \geq 1$, and let $X : \Delta^2_{[0,T]} \to T^{(n)}(V)$ be a multiplicative functional with finite $p$ variation so that

$$|X^i_{s,t}| \leq \omega(s,t)^{i/p}, \quad \forall i = 1,\ldots,n \quad \text{and} \quad \forall (s,t) \in \Delta^2_{[0,T]}.$$

for some control $\omega$. If $n \geq [p]$, then we may uniquely extend $X$ to be a multiplicative functional in $T^{(\infty)}(V)$ with finite $p$ variation. Moreover, if $\omega$ is a control such that

$$|X^i_{s,t}| \leq \frac{\omega(s,t)^{i/p}}{\beta([p])}, \quad \forall i = 1,\ldots,[p], \quad \text{and} \quad \forall (s,t) \in \Delta^2_{[0,T]},$$

(6.2)

where $\beta$ is a constant such that

$$\beta \geq 2p^2[1 + \sum_{r=3}^{\infty} \left( \frac{2}{r-2} \right)^{[r]/2}],$$

then (6.2) remains true for all $i > [p]$.

The extension of a rough path $X$ to a higher-order multiplicative functional is continuous in $p-$ variation distance.
Theorem 26. Theorem 3.1.3 p 39 of [13]

Let \( X \) and \( Y \) be two rough path of roughness \( p \) and let \( \beta \) be a constant such that \( \beta \geq 2p^2[1 + \sum_{r=3}^{\infty} \left( \frac{2}{r-2} \right)^{r+1}] \).

If \( \omega \) is a control such that

\[
|X_s^i|, |Y_s^i| \leq \frac{\omega(s,t)^{i/p}}{\beta(\frac{i}{p})!}, \quad \forall i = 1, ..., [p], \quad \text{and all } (s,t) \in \Delta^2_{[0,T]},
\]

\[
|X_s^i - Y_s^i| \leq \frac{\epsilon \omega(s,t)^{i/p}}{\beta(\frac{i}{p})!}, \quad \forall i = 1, ..., [p], \quad \text{and all } (s,t) \in \Delta^2_{[0,T]},
\]

then (6.3) and (6.4) hold for all \( i \).

6.2.1. Almost rough path. In this section we give a method of constructing rough paths.

Definition 27. Let \( p \geq 1 \) be a constant. A function \( X : \Delta^2 \to T^{(p)}(V) \) is called an almost rough path (of roughness \( p \)) if it is of finite \( p \)-variation, \( X^0_{s,t} = 1 \) and for some control \( \omega \) and some constant \( \theta > 1 \),

\[
|(X_s^i \otimes X_t^u)^i - X_s^i| \leq \omega(s,u)^\theta
\]

for all \( (s,t), (t,u) \in \Delta^2_{[0,T]} \) and \( i = 1, ..., [p] \).

The following theorem justifies the name of almost rough path.

Theorem 28. If \( X : \Delta^2 \to T^{(p)}(V) \) is an almost rough path of roughness \( p \), controlled by \( \omega \) and \( \theta \) then there exists an unique rough path \( \hat{X} \) (with roughness \( p \)) in \( T^{(p)}(V) \) such that

\[
|\hat{X}^i_{s,t} - X^i_{s,t}| \leq K_i \omega(s,t)^\theta, \quad \forall 1 \leq i \leq [p], \quad \forall (s,t) \in \Delta^2_{[0,T]},
\]

for \( K_i \) defined by induction

\[
K_0 = \max \omega \lor 1,
\]

\[
K_1 = 1 + \sum_{r=3}^{\infty} \left( \frac{2}{r-2} \right)^{r+1},
\]

\[
K_i = K_1 \left[ 1 + \sum_{l=1}^{i} \left( 2K_l^{1/p}K_{k+1-l} + K_lK_{k+1-l}K_0^\theta \right) \right].
\]

The following theorem shows that in fact the map \( X \to \hat{X} \) is continuous.

Theorem 29. Let \( X \) and \( Y \) be two almost rough paths of roughness \( p \) in \( T^{(p)}(V) \), both of which controlled by a control \( \omega \), that is

\[
|X^i_{s,t}|, |Y^i_{s,t}| \leq \omega(s,t)^{i/p}, \quad \forall i = 1, ..., [p], \quad \forall (s,t) \in \Delta^2_{[0,T]}.
\]
and for some $\theta > 1 \left| (X_{s,t} \otimes X_{t,u})^{i/p} - X_{s,u}^{i/p} \right| \leq \omega(s,u)^{\theta}$ for all $(s,t), (t,u) \in \Delta^2_{[0,T]}, i = 1, \ldots, [p]$, with the same inequality also holding for $Y$. Suppose that

$$|X_{s,t}^i - Y_{s,t}^i| \leq \varepsilon \omega(s,t)^{i/p}, \forall i = 1, \ldots, [p], \forall (s,t) \in \Delta^2_{[0,T]},$$

then

$$|\hat{X}_{s,t}^i - \hat{Y}_{s,t}^i| \leq B_i(\varepsilon)\omega(s,t)^{i/p}, \forall i = 1, \ldots, [p], \forall (s,t) \in \Delta^2_{[0,T]},$$

where the $B_i$ are defined inductively by

$$B_1(\varepsilon) = \varepsilon + 3 \left\{ \sum_{r=3}^{\infty} \left\lbrack \varepsilon \left( \frac{2}{r-2} \right)^{1/p} \right\rbrack \wedge \left\lbrack \left( \frac{2}{r-2} \right)^{\theta} K_0^{\theta} \right\rbrack + \varepsilon \wedge K_0^{\theta} \right\}$$

and

$$B_{k+1}(\varepsilon) = \varepsilon + 3 \sum_{r=2}^{\infty} A_k(r, \varepsilon)$$

and

$$A_k(r, \varepsilon) = \min \left\{ K_0^{\theta} \left( \frac{2}{r-2} \right)^{\theta} \left( 1 + 2 \sum_{i=1}^{k} K_{k+1-i} K_0^{i/p} + K_i K_{k+1-i} K_0^{\theta} \right), \right.$$

$$\left. \left( \frac{2}{r-2} \right)^{(k+1)/p} \sum_{i=1}^{k} \left| B_i(\varepsilon) (1 + K_{k+1-i} + K \varepsilon + \varepsilon) \right| \right\}$$

$$K = \max_{i=1, \ldots, [p]} K_i.$$  

6.2.2. Spaces of rough path. Let $C(\Delta^2_{[0,T]}, T^{(n)}(V))$ denote the set of all continuous functions from the simplex $\Delta^2_{[0,T]}$ into the truncated tensor algebra $(T^{(n)}(V), |.|)$. If $X \in C(\Delta^2_{[0,T]}, T^{(n)}(V))$, then we may write

$$X_{s,t} = (X_{s,t}^0, \ldots, X_{s,t}^n), \forall (s,t) \in \Delta^2_{[0,T]},$$

where $X_{s,t}^i \in V^{\otimes i}$ is the $i$th component of $X$ (also called the $i$th level path of $X$). The subset of the functions of $C(\Delta^2_{[0,T]}, T^{(n)}(V))$ such that $X_{s,t}^0 = 1$ is denoted $C^0(\Delta^2_{[0,T]}, T^{(n)}(V))$.

**Definition 30.** A function $X \in C^0(\Delta^2_{[0,T]}, T^{(n)}(V))$ is said to have finite total $p$-variation if

$$\sup_D \sum_i |X_{i-1,i}^i|^{p/i} < \infty, \ i = 1, \ldots, n,$$

where $\sup_D$ runs over all finite subdivisions of $[0,T]$. 
Proposition 31. Let $p \geq 1$ be a constant, and let $X \in C_0(\Delta^2_{[0,T]}, T^{(n)}(V))$ satisfy Chen’s identity (6.1) (i.e. $X$ is a multiplicative functional in $T^{(n)}(V)$ of order $n$). If $X$ has a finite $p$-variation, then

$$\omega(s,t) = \sum_{i=1}^{n} \sup_{D_{s,t}} \sum_{t} |X_{t_{i-1},t_i}|^{p/i}, \forall (s,t) \in \Delta^2_{[0,T]}$$

is a control function, and

$$|X_{s,t}^i| \leq \omega(s,t)^{i/p}, \forall i = 1, \ldots, n, \forall (s,t) \in \Delta^2_{[0,T]}.$$ 

Let $C_{0,p}(\Delta^2_{[0,T]}, T^{(n)}(V))$ denote the subspace of all $X \in C_0(\Delta^2_{[0,T]}, T^{(n)}(V))$ with finite $p$ variation. It is clear that $C_{0,p}(\Delta^2_{[0,T]}, T^{(n)}(V))$ is a metric space. The $p$ variation metric $d_p$ on $C_{0,p}(\Delta^2_{[0,T]}, T^{(n)}(V))$ is defined by

$$d_p(X,Y) = \max_{i=1,\ldots,|p|} \sup_{D} \left( \sum_{t} |X_{t_{i-1},t_i}^i - Y_{t_{i-1},t_i}^i|^{p/i} \right)^{i/p}. \tag{6.5}$$

The space $(\Omega_p(V),d_p)$ is a complete metric space.

Definition 32. Geometric rough paths with roughness $p$ are the rough paths in the closure of smooth rough path under the $p-$ variation distance.

The space of all geometric rough paths with roughness $p$ is denoted by $G\Omega_p(V)$.

6.3. Integration theory degree 2. Let $p \in [2,3]$. Let $W$ be $\mathbb{R}^n$, $n \geq 1$. Let $\alpha : V \to L(V,W)$ be a function which sends elements of $V$ linearly to $W$-valued one-forms on $V$. Suppose that $\alpha$ possesses all $k$th continuous derivatives $d^k\alpha$ up to the degree 3 and denote $\alpha_i = d^i\alpha$, $i = 1, 2, 3$.

Let $X \in \Omega_p(V)$ and let $X_{s,t} = (1, X^1_{s,t}, X^2_{s,t})$. The almost rough path which defines the path integral $\int \alpha(X) dX$ is $Y \in C_0(\Delta^2_{[0,T]}, T^{(2)}(W))$ where $Y_{s,t} = (1, Y^1_{s,t}, Y^2_{s,t})$ and

$$Y^1_{s,t} = \alpha^1(X^1_{s,t})X^1_{s,t} + \alpha^2(X^1_{s,t})X^2_{s,t},$$

$$Y^2_{s,t} = \alpha^1(X^1_{s,t}) \otimes \alpha^1(X^1_{s,t})X^2_{s,t}.$$ 

Theorem 33. Theorem 5.2.1 and remark 5.3.1 of [13]

Let $\alpha : V \to L(V,W)$. Suppose that $\alpha$ possesses all $k$th continuous derivatives $d^k\alpha$ up to the degree 3 and

$$|d^i\alpha(\xi)|_{L(V \times \ldots \times V,W)} \leq M(1 + |\xi|), \forall \xi \in V, \quad i = 1, \ldots, 3.$$
Assume that $X \in \Omega_p(V)$ is controlled by $\omega$, namely

$$|X^i_{s,t}| \leq \omega(s,t)^{i/p}, \ i = 1, 2, \ \forall(s,t) \in \Delta^2_{[0,T]}.$$ 

Then $Y$ is an almost rough path with roughness $p$ in $T^{(2)}(W)$ with control $\omega$ and $\theta = 3/p$ i.e. there exists a universal constant $C$ such that

$$|(Y_{s,t} \otimes Y_{t,u})^i - Y^i_{s,u}| \leq CM\omega(s,u)^{3/p}.$$ 

**Definition 34.** Let $X \in \Omega_p(V)$. Then the integral of the one-form $\alpha$ against the rough path $X$, denoted by $\int \alpha(X) dX$, is the unique rough path with roughness $p$ in $T^{(2)}(W)$ associated to the almost rough path $Y \in C_0(\Delta^2_{[0,T]}, T^{(2)}(W))$ where $Y_{s,t} = (1, Y^1_{s,t}, Y^2_{s,t})$ and

$$Y^1_{s,t} = \alpha^1(X_{0,s}^1).X^1_{s,t} + \alpha^2(X_{0,s}^1).X^2_{s,t},$$
$$Y^2_{s,t} = \alpha^1(X_{0,s}^1) \otimes \alpha^1(X_{0,s}^1).X^2_{s,t}.$$ 

**Theorem 35.** Let $\alpha : V \to \mathbf{L}(V,W)$. Suppose that $\alpha$ possesses all $k$th continuous derivatives $d^k \alpha$ up to the degree 3 and

$$|d^i \alpha(\xi)|_{\mathbf{L}(V \times \cdots \times V;W)} \leq M(1 + |\xi|), \ i = 1, \ldots, 3, \ \forall \xi \in V.$$ 

Assume that $X, \hat{X} \in \Omega_p(V)$ is controlled by $\omega$, namely

$$|X^i_{s,t}|, |\hat{X}^i_{s,t}| \leq \omega(s,t)^{i/p}, \ i = 1, 2, \ \forall(s,t) \in \Delta^2_{[0,T]},$$ 

and

$$|X^i_{s,t} - \hat{X}^i_{s,t}| \leq \varepsilon \omega(s,t)^{i/p}, \ i = 1, 2, \ \forall(s,t) \in \Delta^2_{[0,T]}.$$ 

Then

$$|\int_s^t \alpha(X_{0,u})dX_u^i - \int_s^t \alpha(\hat{X}_{0,u})d\hat{X}_u^i| \leq K\varepsilon M\omega(s,u)^{j/p},$$

for all $(s,t) \in \Delta^2_{[0,T]}$ and $i = 1, 2$, where $K$ is a constant which is polynomial in $M, \max \omega$.

6.4. Itô maps: rough path with $2 \leq p < 3$. 

6.4.1. Framework. Let $V = \mathbb{R}^d$ and $W = \mathbb{R}^n$. Let $f : V \to \mathcal{L}(V, W)$ be a function, which can be viewed as a map sending vector of $V$ linearly to a vector field on $W$. Consider the following differential equation (initial value problem)

$$dY_t = f(Y_t)dX_t, \quad Y_0 = y_0.$$ \hspace{1cm} (6.6)

Since the integral $\int f(Y)dX$ for rough path $X, Y$ make no sense generally we are not able to iterate differential equation eqn (6.6) to obtain the unique solution directly. To overcome this difficulty, the idea is to combine $X$ and $Y$ together as a new path. We view equation (6.6) as

$$dX_t = dX_t, \quad \quad dY_t = f(Y_t)dX_t, \quad Y_0 = y_0.$$ \hspace{1cm} (6.7)

The initial condition of $X$ is irrelevant, and therefore we simply take $X_0 = 0$. Define $\hat{f} : V \oplus W \to \mathcal{L}(V \oplus W; V \oplus W)$ ($V \oplus W$ is the direct sum of $V$ and $W$) by

$$\hat{f}(x, y)(v, w) = (v, f(y + y_0) . v), \quad \forall (x, y) \in V \oplus W, \quad \forall (v, w) \in V \oplus W.$$ \hspace{1cm} (6.8)

Then eqn (6.7) can be written in the following more appreciating form

$$dZ_t = \hat{f}(Z_t)dZ_t.$$ \hspace{1cm} (6.9)

Given a rough path $X$ in $V$, we said that a geometric rough path $Z$ in $V \oplus W$ is a solution to (6.7) if

$$\Pi_V(Z) = X, \quad \quad Z = \int \hat{f}(Z)dZ,$$

where $\Pi_V$ is the projector on $T^{[p]}(V)$.

6.4.2. Existence and uniqueness results. One can summarize the results proved in pages 149 to 162 of [13], when the control is $\omega(t, s) = C|t - s|$ and the vector field, $f$, and its derivatives are bounded in the following way.

**Theorem 36.** Let $f \in C^3(W; \mathcal{L}(V, W))$ be a vector field and let $M$ be a constant such that

$$|d^i f(\xi)| \leq M, \quad \forall \xi \in W, \quad i = 0, 1, 2, 3;$$

$$|d^i f(\xi) - d^i f(\eta)| \leq M|\xi - \eta|, \quad \forall \xi, \eta \in W, \quad i = 0, 1, 2, 3.$$
Let \( X \) be a rough path in \( T^{(2)}(V) \) with roughness \( 2 \leq p < 3 \) controlled by \( \omega \), where for a constant \( C_p \), 
\[
\omega(t, s) = C_p|t - s|, \quad (s, t) \in \Delta^2_{[0,1]}. 
\]

Then, there exist some constants \( \kappa \), and \( \tilde{\theta} \) depending only on \( M, p \) such that if
\[
T_1 = \frac{\kappa}{1 + C_p^\theta} < 1
\]
(6.9)

there exists a unique \( Z \in \Omega_p(V \oplus W) \) such that \( \Pi_V(Z) = X \) and \( Z \) satisfies the following integral equation :
\[
Z_{s,t}^i = \int_s^t \hat{f}(Z)dZ^i, \quad i = 1, 2, \quad \forall (s, t) \in \Delta^2_{[0,T_1]}.
\]

Moreover, the following estimation holds
\[
|Z_{s,t}^i| \leq (\frac{1}{2}|t - s|)^{i/p}, \quad i = 1, 2, \quad \forall (s, t) \in \Delta^2_{[0,T_1]}.
\]
(6.10)

Finally, we may extend the solution to the whole interval \([0,1]\).

**Theorem 37.** Let \( f \in C^3(W; \mathbb{L}(V, W)) \) be a vector field and let \( M \) be a constant such that
\[
|d^i f(\xi)| \leq M, \quad \forall \xi \in W, \quad i = 0, 1, 2, 3;
\]
\[
|d^i f(\xi) - d^i f(\eta)| \leq M|\xi - \eta|, \quad \forall \xi, \eta \in W, \quad i = 0, 1, 2, 3.
\]

Let \( X \) be a rough path in \( T^{(2)}(V) \) with roughness \( 2 \leq p < 3 \) controlled by \( \omega \), where for a constant \( C_p \), 
\[
\omega(t, s) = C_p|t - s|, \quad (s, t) \in \Delta^2_{[0,1]}. 
\]

Then, there exists a unique \( Z \in \Omega_p(V \oplus W) \) such that \( \Pi_V(Z) = X \) and \( Z \) satisfies the following integral equation :
\[
Z_{s,t}^i = \int_s^t \hat{f}(Z)dZ^i, \quad i = 1, 2, \quad \forall (s, t) \in \Delta^2_{[0,1]}.
\]

Moreover, there exist some constants \( \kappa \) and \( \tilde{\theta} \) depending only on \( M, p \) such that the following estimation holds
\[
|Z_{s,t}^i| \leq \kappa(1 + C_p^\theta)(|t - s|)^{i/p}, \quad i = 1, 2, \quad \forall (s, t) \in \Delta^2_{[0,1]}.
\]
(6.11)

**Proof.** For example, we may solve the integral equation beyond \( T_1 \) by replacing the initial condition \( y_0 \) by \( Y_{T_1} = \Pi_W(Z)_{0,T_1} \). Then, the solution is defined up to the time 
\[
S_2 = 2T_1
\]

Moreover, we have
\[
|Y_{S_2}| \leq 2\left(\frac{T_1}{2}\right)^{1/p} + |y_0|.
\]
By an iteration procedure, the solution is defined up to the time $1$. Let $N$ be an integer such that such that $N\kappa \frac{1}{1+C_p} > 1$. Then $N$ is bounded by a polynomial in $C_p$. Then using the Chen rules, identity (6.1) and several times estimation (6.10) we obtain

$$|Z_{s,t}^i| \leq \kappa (1 + C_p^\delta) \mathcal{P}(N)(|t-s|)^{1/p}, \ i = 1, 2, \ \forall (s,t) \in \Delta_{[0,1]}^2,$$

where $\mathcal{P}(N)$ is polynomial in $N$ and then in $C_p$.  

\[\square\]

\textbf{REFERENCES}

[1] F. Baudoin: An Introduction to the Geometry of Stochastic Flows, Imperial College Press, (2004).
[2] G. Ben Arous: Flots et sèries de Taylor stochastiques, Probab. Theory Relat. Fields, 81, 29-77, (1989).
[3] Borell, Christer: On polynomial chaos and integrability. Probab. Math. Statist. 3 (1984), no. 2, 191–203.
[4] F. Castell: Asymptotic expansion of stochastic flows, Probab. Theory Relat. Fields, 96, 225-239, (1993).
[5] K.T. Chen: Integration of paths, Geometric invariants and a Generalized Baker-Hausdorff formula, Annals of Mathematics, 65, n1, (1957).
[6] K.T. Chen: Formal differential equations, Ann. Math. 73, 110-133, (1961).
[7] P. Cheridito and D. Nualart: Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H$ in $(0,1/2)$, preprint.
[8] L. Coutin, P Friz and N. Victoir, Good Rough Path Sequences ans Applications to Anticipating and Fractional Stochastic Calculus; preprint 2005.
[9] L. Coutin, Z. Qian: Stochastic rough path analysis and fractional Brownian motion, Probab. Theory Relat. Fields 122, 108-140, (2002).
[10] L. Decreusefond and A. S. Ùstünel, Stochastic analysis of the fractional Brownian motion, Potential Analysis. 10, 177-214, (1998).
[11] H. Doss: Lien entre équations différentielles stochastiques et ordinaires, Ann. Inst. H. Poincaré, Prob. Stat. 13, 99-125, (1977).
[12] X. M. Fernique, Régularité des trajectoires des fonctions aléatoires gaussiennes, ecole d’été de probabilités de Saint-Flour 1974, L. N. in Math 480, 1-96.
[13] T. Lyons, Differential Equations Driven by Rough Signals, Revista Mathemática Iberio Americana, Vol 14, No 2, 215 - 310, (1998).
[14] I. Nourdin : One-dimensional differential equations driven by a fractional Brownian motion with any Hurst index $H \in (0, 1)$, preprint (2003).
[15] D. Nualart and A. Răşcanu : Differential equations driven by fractional Brownian motion, Collect. Math. 53, 1 , 55-81,(2002).
[16] D. Nualart, B. Saussereau: Malliavin calculus for stochastic differential equation driven by a fractional Brownian motion, preprint.
[17] Rogers L.C.G., Williams D.: Diffusions, Markov processes and Martingales, Vol. 1, second edition, Cambridge university press, (2000).
[18] R.S. Strichartz: The Campbell-Baker-Hausdorff-Dynkin formula and solutions of differential equations, Jour. Func. Anal., 72, 320-345, (1987).
[19] H. Stüssmann: On the gap between deterministic and stochastic ordinary differential equations. Ann. Probab. 6, 19-41, (1978).
[20] Y. Yamato: Stochastic differential equations and nilpotent Lie algebras. Z. Wahrscheinlichkeitstheorie. Verw. Geb., 47, 213-229, (1979).
[21] Young L.C. (1936): An inequality of the Hölder type connected with Stieltjes integration. Acta Math., 67, 251-282.
[22] Zähle M. (1998): Integration with respect to fractal functions and stochastic calculus I. Prob. Theory Rel. Fields, 111, 333-374.