A General Theorem of Gauß Using Pure Measures

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Abstract

This paper shows that finitely additive measures occur naturally in very general Divergence Theorems. The main results are two such theorems. The first proves the existence of pure normal measures for sets of finite perimeter, which yield a Gauß formula for essentially bounded vector fields having divergence measure. The second extends a result of Silhavy [19] on normal traces. In particular, it is shown that a Gauß Theorem for unbounded vector fields having divergence measure necessitates the use of pure measures acting on the gradient of the scalar field. All of these measures are shown to have their core on the boundary of the domain of integration.

1 Introduction

The Divergence Theorem, or Theorem of Gauß, is a very important theorem in real analysis. It connects the integral over a volume with the integral over the bounding surface of said volume. Its classic form for smooth $\Omega \subset \mathbb{R}^n$ and smooth vector fields $F: \Omega \rightarrow \mathbb{R}^n$ is

$$\int_{\Omega} \text{div} F \, d\mathcal{L}^n = \int_{\partial \Omega} F \cdot \nu \, d\mathcal{H}^{n-1},$$

where $\mathcal{L}^n$ is the Lebesgue measure, $\mathcal{H}^{n-1}$ the $n-1$-dimensional Hausdorff measure and $\nu$ the unit outer normal to $\Omega$. In the theory of Continuum Mechanics, but also in general analytical problems it is desirable to have the theorem at hand for very general vector fields and very general domains of
integration. The main problem is to find a useful substitute for the area integral. Up to now, the area integral was substituted for a continuous linear functional on a function space on the boundary (cf. Silhavy [19]) or for a normal trace given by a measure on the boundary, in the case of the vector fields considered being essentially bounded (cf. Chen [7], [2], [3], [4])). In Schuricht [18], a general limit formula is proved.

In this paper, it will be shown that pure measures are a natural substitute for area measure, when vector fields having divergence measure are considered. There are two main results. The first one is a theorem for essentially bounded vector fields having divergence measure. It is proved that for sets of finite perimeter there is a normal measure which gives rise to a very general Gauß formula. The second theorem treats unbounded vector fields and open sets. A result due to Silhavy [19] is extended and it is proved that a Gauß-Green formula for these vector fields necessarily contains a pure measure acting on the values of the gradient of the scalar field near the boundary.

The first section is a primer on pure measures. Many properties and integration with respect to pure measures are laid out.

The second section contains the results for essentially bounded vector fields.

The third section contains the theorem for unbounded vector fields and open sets.

Concerning notation, in the following $n \in \mathbb{N}$ denotes a positive natural number and $\mathbb{R}^n$ the vector space of real $n$-tuples. For a set $\Omega \subset \mathbb{R}^n$ the set $\Omega_\delta$ denotes the open $\delta$-neighbourhood of $\Omega$. Open balls with radius $\delta > 0$ and centre $x \in \mathbb{R}^n$ are written $B_\delta(x) = \{x\}_\delta$. The Borel subsets of $\Omega$, i.e. the $\sigma$-measure generated by all relatively open sets in $\Omega$, is denoted by $\mathcal{B}(\Omega)$. $\mathcal{L}^n$ is the Lebesgue measure and $\mathcal{H}^d$ the $d$-dimensional Hausdorff measure. For set function $\mu$ on $\Omega$, $\mu|_A$ denotes the restriction of $\mu$ to $A$. The Banach space of equivalence classes of $p$-integrable functions is denoted by $L^p(\Omega, \mathcal{L}^n)$ and $p'$ denotes the Hölder-conjugate of $p$. (Weak) Derivates of functions $f$ are written $Df$. The divergence of a vector field $F$, be it classical or distributional, is denoted by $\text{div} F$. 
2 A Short Primer On Pure Measures

In this article, set functions \( \mu : \mathcal{A} \subset 2^\Omega \rightarrow \mathbb{R} \) will be called measure, if for all \( m \in \mathbb{N} \) and every pairwise disjoint \( \{A_k\}_{k=1}^m \subset \mathcal{A} \) with \( \bigcup_{k=1}^m A_k \in \mathcal{A} \)

\[
\mu \left( \bigcup_{k=1}^m A_k \right) = \sum_{k=1}^m \mu(A_k).
\]

If this holds with \( m = \infty \), the measure is called \( \sigma \)-measure. A measure is called bounded if

\[
\sup_{A \in \mathcal{A}} |\mu(A)| < \infty.
\]

An algebra is a class of sets which is stable under union, intersection and differences and contains at least \( \emptyset \). The spaces of measures considered in this thesis are defined in accordance with [16].

**Definition 2.1.** Let \( \Omega \subset \mathbb{R}^n \) and \( \mathcal{A} \subset 2^\Omega \) be an algebra. The set of all bounded measures \( \mu : \mathcal{A} \rightarrow \mathbb{R} \) is denoted by \( \text{ba}(\Omega, \mathcal{A}) \).

The set of all bounded \( \sigma \)-measures \( \sigma : \mathcal{A} \rightarrow \mathbb{R} \) is denoted by \( \text{ca}(\Omega, \mathcal{A}) \).

The following proposition is an application of Riesz’s Decomposition Theorem (cf. [16, p. 241]).

**Proposition 2.2.** Let \( \Omega \subset \mathbb{R}^n \) and \( \mathcal{A} \subset 2^\Omega \) be an algebra. Then every \( \mu \in \text{ba}(\Omega, \mathcal{A}) \) can uniquely be decomposed into \( \mu_c \in \text{ca}(\Omega, \mathcal{A}) \) and \( \mu_p \in \text{ba}(\Omega, \mathcal{A}) \) such that

\[
\mu = \mu_c + \mu_p
\]

and for every \( \sigma \in \text{ca}(\Omega, \mathcal{A}) \)

\[
0 \leq \sigma \leq |\mu_p| \implies \sigma = 0.
\]

**Definition 2.3.** Let \( \Omega \subset \mathbb{R}^n \) and \( \mathcal{A} \subset 2^\Omega \) be an algebra. Then every such measure \( \mu_p \) is called pure. Notice that \( \mu_p \) is not \( \sigma \)-additive, by definition.

One important example of measures that are pure are density measures. The following new example presents a particular density measure, namely a density at zero. In the literature, examples of pure measure are only known for \( \Omega = \mathbb{N} \) (cf. [16, p. 247]), they are defined on very small algebras (cf. [16, p. 246]) or they are constructed in such a way that the measure cannot be computed explicitly, even on simple sets (cf. [20, p. 57f]). The example given here is constructed on \( \Omega = \mathbb{R}^n \) and lives on the Borel subsets of \( \Omega \).
Example 2.4. Let $\Omega := B_1(0) \subset \mathbb{R}^n$ be open. Then there exists $\mu \in \text{ba}(\Omega, B(\Omega))$, $\mu \geq 0$ such that for every $B \in B(\Omega)$

$$
\mu(B) = \lim_{\delta \downarrow 0} \frac{\mathcal{L}^n(B \cap B_\delta(0))}{\mathcal{L}^n(B_\delta(0))}
$$

if this limit exists. This measure is non-unique. Its existence is shown Proposition 5.7 in [17, p. 25] (take $\lambda := \mathcal{L}^n$ and $C = \{0\}$).

It is shown in Example 2.7 that $\mu$ is indeed pure. Figure 1 shows the family $\{A_k\}_{k \in \mathbb{N}} \subset B(\Omega)$

$$
A_k := \left[ \frac{1}{k+2}, \frac{1}{k+1} \right) \times [-1,1]^{n-1}.
$$

For this family

$$
\sum_{k \in \mathbb{N}} \mu(A_k \cap \Omega) = 0 \neq \mu \left( \left( 0, \frac{1}{2} \right) \times [-1,1]^{n-1} \cap \Omega \right) = \mu \left( \bigcup_{k=1}^{\infty} A_k \cap \Omega \right).
$$

Hence, $\mu$ is not a $\sigma$-measure.

Figure 1: A family of sets on which $\mu$ is not $\sigma$-additive

Measures that do not charge sets of Lebesgue measure zero are of special interest, because these measures lend themselves naturally to the integration of functions that are only defined outside of a set of measure zero. When treating non $\sigma$-additive measures, one carefully has to distinguish the following two notions (cf. [16, p. 159]).

Definition 2.5. Let $\Omega \subset \mathbb{R}^n, \mathcal{A} \subset 2^\Omega$ be an algebra and $\lambda \in \text{ba}(\Omega, \mathcal{A})$. Then $\mu \in \text{ba}(\Omega, \mathcal{A})$ is called
1. **absolutely continuous** with respect to $\lambda$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{A}$

$$|\lambda|(A) < \delta \implies |\mu(A)| < \varepsilon.$$ 

In this case, write $\mu \ll \lambda$.

2. **weakly absolutely continuous** with respect to $\lambda$, if for every $A \in \mathcal{A}$

$$|\lambda|(A) = 0 \implies \mu(A) = 0.$$ 

In this case, write $\mu \ll^w \lambda$.

The set of all weakly absolutely continuous measures in $\text{ba}(\Omega, \mathcal{A})$ is denoted by

$$\text{ba}(\Omega, \mathcal{A}, \lambda).$$

As in Proposition 2.2, weakly absolutely continuous measures can be decomposed into pure and $\sigma$-additive parts.

**Proposition 2.6.** Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\lambda \in \text{ba}(\Omega, \mathcal{A})$.

Then for every $\mu \in \text{ba}(\Omega, \mathcal{A}, \lambda)$ there exist unique $\mu_c \in \text{ca}(\Omega, \mathcal{A}) \cap \text{ba}(\Omega, \mathcal{A}, \lambda)$, $\mu_p \in \text{ba}(\Omega, \mathcal{A}, \lambda)$ such that

$$\mu = \mu_c + \mu_p$$

such that for all $\sigma \in \text{ca}(\Omega, \mathcal{A})$

$$0 \leq \sigma \leq |\mu_p| \implies \sigma = 0.$$ 

**Proof.** See Proposition 3.7 in [17, p. 8].

**Example 2.7.** Since the measure $\mu$ from Example 2.4 is positive and $\mu_c \perp \mu_p$, using the additivity of the total variation on orthogonal element (cf. [16, p. 25], [17, p. 3]) yields

$$0 \leq |\mu_c| \leq |\mu_c| + |\mu_p| = |\mu| = \mu.$$ 

Hence, for every $\delta > 0$

$$|\mu_c|(B_\delta(0)^c) = 0.$$ 

Thus

$$|\mu_c|(\Omega \setminus \{0\}) = \lim_{\delta \downarrow 0} |\mu_c|(B_\delta(0)^c) = 0.$$ 

But $|\mu_c|(\{0\}) \leq \mu(\{0\}) = 0$. Hence

$$|\mu_c|((0) = 0$$

and $\mu = \mu_p$ is pure.
The structure of $\mu_p$ is described by the following proposition taken from [16, p. 244] (cf. [20, p. 56]).

**Proposition 2.8.** Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a $\sigma$-algebra and $\sigma \in \text{ca}(\Omega, \Sigma)$, $\sigma \geq 0$. Then $\mu \in \text{ba}(\Omega, \Sigma, \sigma)$ is pure if and only if there exists a decreasing sequence $\{A_k\}_{k \in \mathbb{N}} \subset \Sigma$ such that

$$\sigma(A_k) \xrightarrow{k \to \infty} 0$$

and for all $k \in \mathbb{N}$

$$|\mu_p|(A_k^c) = 0.$$

**Definition 2.9.** Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a $\sigma$-algebra, $\sigma \in \text{ca}(\Omega, \Sigma)$, $\sigma \geq 0$ and $\mu_p \in \text{ba}(\Omega, \Sigma, \sigma)$ be pure. Then every $A \in \Sigma$ such that

$$|\mu_p|(A^c) = 0$$

is called aura of $\mu_p$.

Any decreasing sequence $\{A_k\}_{k \in \mathbb{N}} \subset \Sigma$ of auras for $\mu_p$ such that

$$\sigma(A_k) \xrightarrow{k \to \infty} 0$$

is called aura sequence.

Intuitively speaking, weakly absolutely continuous measures are pure if and only if they concentrate in the vicinity of a set of measure zero. Reviewing Example 2.4, the support (cf. [1, p. 30]) of the measure can be seen to lie outside of $\Omega \setminus \{0\}$. Yet the construction of the measure would still work on this set. Hence, it is possible for a pure measure to have support outside of its domain of definition. This necessitates the following definition of core.

**Definition 2.10.** Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra containing every relatively open set in $\Omega$. Furthermore let $\mu \in \text{ba}(\Omega, \mathcal{A})$. Then the set

$$\text{core} \mu := \{x \in \mathbb{R}^n \mid |\mu|(V \cap \Omega) > 0, \forall V \subset \mathbb{R}^n, V \text{ open}, x \in V\}$$

is called core of $\mu$.

Let $d \in [0, n]$ be the Hausdorff dimension of core $\mu$. Then $d$ is called core dimension of $\mu$ and $\mu$ is called $d$-dimensional.

**Proposition 2.11.** Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ and $\mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$.

If $\text{core} \mu \cap \Omega$ is a $\mathcal{L}^n$-null set then $\mu$ is pure.
Now, integration with respect to measure which are not necessarily \(\sigma\)-additive is outlined. Measurability of functions is not defined through the regularity of preimages but by approximability by simple functions in measure. In this definition, the measure is needed on possibly non-measurable sets. Hence, an outer measure has to be used. This outer measure is defined as in the case of \(\sigma\)-measures (cf. [16, p. 86], [14, p. 42]).

**Definition 2.12.** Let \(\Omega \subset \mathbb{R}^n\) and \(A \subset 2^{\Omega}\) be an algebra. For \(\mu \in \text{ba}(\Omega, A)\), \(\mu \geq 0\) the outer measure of \(\mu\) is defined for \(B \in 2^\Omega\) by

\[
\mu^*(B) := \inf_{A \in A, B \subset A} \mu(A).
\]

Now, convergence in measure can be defined. The definition is taken from [16, p. 92] (cf. [14, p. 91]).

**Definition 2.13.** Let \(\Omega \subset \mathbb{R}^n\) and \(A \subset 2^{\Omega}\) be an algebra and \(\mu : A \to \mathbb{R}\) be a measure. A sequence \(\{f_k\}_{k \in \mathbb{N}}\) of functions \(f_k : \Omega \to \mathbb{R}\) is said to converge in measure to a function \(f : \Omega \to \mathbb{R}\) if for every \(\varepsilon > 0\)

\[
\lim_{k \to \infty} \left| \mu^* \{ x \in \Omega \mid |f_k(x) - f(x)| > \varepsilon \} \right| = 0.
\]

In this case, write

\[
f_k \xrightarrow{\mu} f.
\]

Note that the limit in measure is not unique, yet. Therefore, the following notion of equality almost everywhere is needed. The definition is taken from [16, p. 88].

**Definition 2.14.** Let \(\Omega \subset \mathbb{R}^n\), \(A \subset 2^{\Omega}\) and \(\mu : A \to \mathbb{R}\) be a measure.

Then \(f : \Omega \to \mathbb{R}\) is called null function, if for every \(\varepsilon > 0\)

\[
|\mu|^* (\{ x \in \Omega \mid |f(x)| > \varepsilon \}) = 0.
\]

Two functions \(f_1 : \Omega \to \mathbb{R}\), \(f_2 : \Omega \to \mathbb{R}\) are called equal almost everywhere (a.e.) with respect to \(\mu\), if \(f_1 - f_2\) is a null function.

In this case, write

\[
f_1 = f_2 \mu\text{-a.e.}
\]

**Remark 2.15.** If \(f : \Omega \to \mathbb{R}\) is a null function, then it need not be true that

\[
|\mu|^* (\{ x \in \Omega \mid f(x) \neq 0 \}) = 0. \tag{1}
\]
Take e.g. the density measure $\mu$ introduced in Example 2.4 and \( f(x) := |x| \). Then \( f \) is a null function but

\[ |\mu|^\ast(\{x \in \mathbb{R}^n | f(x) \neq 0\}) = \mu(B_1(0) \setminus \{0\}) = 1 > 0. \]

This entails that the notion of equality almost everywhere that was defined above does not imply the existence of a null set such that \( f_1 = f_2 \) outside of that set. Take e.g. the density measure introduced in Example 2.4, \( f_1(x) := |x| \) and \( f_2(x) := 2f_1(x) \).

On the other hand, if \( \mu \) is a $\sigma$-measure and \( A \) a $\sigma$-algebra, then Equation (11) is equivalent to \( f \) being a null function (cf. [16, p. 89]).

The limit in measure turns out to be unique in the sense of almost equality. This is stated in the following proposition taken from [16, p. 92].

**Proposition 2.16.** Let \( \Omega \subset \mathbb{R}^n \), \( A \subset 2^\Omega \) be an algebra and \( \mu : A \to \mathbb{R} \) be a measure. Furthermore let \( \{f_k\}_{k \in \mathbb{N}} \) be a sequence of functions \( f_k : \Omega \to \mathbb{R} \) and \( f, \tilde{f} : \Omega \to \mathbb{R} \) be functions such that

\[ f_k \mu \rightharpoonup f. \]

Then

\[ f_k \mu \rightharpoonup \tilde{f} \iff f = \tilde{f} \mu\text{-a.e.} \]

Now, the notion of measurability is introduced. The definition is similar to the definition of $T_1$-measurability in [16, p. 101].

**Definition 2.17.** Let \( \Omega \subset \mathbb{R}^n \) and \( A \subset 2^\Omega \) be an algebra and \( \mu : A \to \mathbb{R} \) be a measure. A function \( f : \Omega \to \mathbb{R} \) is called measurable if there exists a sequence \( \{h_k\}_{k \in \mathbb{N}} \) of simple functions \( h_k : \Omega \to \mathbb{R} \) such that

\[ h_k \mu \rightharpoonup f. \]

The integral for measurable functions can now be defined via $L^1$-Chauchy sequences. This is of course well-defined (cf. [16, p. 102]).

**Definition 2.18.** Let \( \Omega \subset \mathbb{R}^n \), \( A \subset 2^\Omega \) be an algebra and \( \mu : A \to \mathbb{R} \) be a measure. A function \( f : \Omega \to \mathbb{R} \) is said to be integrable if there exists a sequence \( \{h_k\}_{k \in \mathbb{N}} \) of integrable simple functions \( h_k : \Omega \to \mathbb{R} \) such that

\[ 1. \ h_k \mu \rightharpoonup f, \]

\[ 2. \ \lim_{k,l \to \infty} \int_{\Omega} |h_k - h_l| \, d|\mu| = 0. \]
In this case, denote
\[ \int_{\Omega} f \, d\mu := \lim_{k \to \infty} \int_{\Omega} h_k \, d\mu. \]
The sequence \( \{h_k\}_{k \in \mathbb{N}} \) is called **determining sequence** for the integral of \( f \).

**Remark 2.19.** In particular, integrable functions are measurable. This notion of integral is also called Daniell-Integral in the literature (cf. [16]).

The \( L^p \)-spaces are defined in the usual way (cf. [16, p. 121]).

**Definition 2.20.** Let \( \Omega \subset \mathbb{R}^n, A \subset 2^\Omega \) be an algebra, \( \mu : A \to \mathbb{R} \) be a measure and \( p \in [1, \infty) \). Then the set of all measurable functions \( f : \Omega \to \mathbb{R} \) such that \(|f|^p \) is \( |\mu| \)-integrable is denoted by
\[ L^p(\Omega, A, \mu). \]
If \( A = \mathcal{B}(\Omega) \), write
\[ L^p(\Omega, \mu). \]
For \( f_1, f_2 \in L^p(\Omega, A, \mu) \)
\[ f_1 = f_2 \ \mu\text{-a.e.} \]
defines an equivalence relation. The set of all equivalence classes of this relation is denoted by
\[ L^p(\Omega, A, \mu). \]
If \( A = \mathcal{B}(\Omega) \), write
\[ L^p(\Omega, \mu). \]

**Definition 2.21.** Let \( \Omega \subset \mathbb{R}^n, A \subset 2^\Omega \) be an algebra and \( \mu : A \to \mathbb{R} \) a measure. Then for every \( p \in [1, \infty) \) and \( f \in L^p(\Omega, A, \mu) \) write
\[ \|f\|_p := \left( \int_{\Omega} |f|^p \, d|\mu| \right)^{\frac{1}{p}}. \]
Furthermore, for measurable \( f : \Omega \to \mathbb{R} \) define
\[ \text{ess sup } f := \inf \{ K \in \mathbb{R} \mid |\mu|\left( \{ x \in \Omega \mid f(x) > K \} \right) = 0 \} \]
and
\[ \|f\|_{\infty} := \text{ess sup } |f|. \]
The set of all measurable functions \( f : \Omega \to \mathbb{R} \) such that
\[ \|f\|_{\infty} < \infty \]
is denoted by
\[ L^\infty (\Omega, \mathcal{A}, \mu) . \]

As in the case \( p \in [1, \infty) \),
\[ \mathcal{L}^\infty (\Omega, \mathcal{A}, \mu) \]
denotes the set of all equivalence classes in \( L^\infty (\Omega, \mathcal{A}, \mu) \) with respect to equality almost everywhere.

In the case \( \mathcal{A} = \mathcal{B}(\Omega) \), only write
\[ L^\infty (\Omega, \mu) \text{ and } \mathcal{L}^\infty (\Omega, \mu) \text{ respectively.} \]

The integral defined in this way shares many properties of the Lebesgue-integral. The Hölder and Minkowski inequality hold true. Furthermore, dominated convergence is available when using convergence in measure instead of pointwise convergence (cf. [16, p. 105ff]).

Before proceeding to the characterisation of the dual of \( L^\infty \), a new integral symbol is introduced, which gives formulas traces and integrals over pure measures a more pleasing shape.

**Definition 2.22.** Let \( \Omega \subset \mathbb{R}^n \) be bounded and \( C \subset \overline{\Omega} \) be closed. Then for every \( \mu \in \text{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \) such that
\[ \text{core } \mu \subset C, \]
every \( f \in \mathcal{L}^1 (\Omega, \mu) \) and \( \delta > 0 \) write
\[ \int_C f \, d\mu := \int_{C_\delta \cap \Omega} f \, d\mu . \]

**Remark 2.23.** This notion of integral is well-defined since the definition of core \( \mu \) yields
\[ |\mu| \left( (C_\delta)^c \right) = 0 \]
for any \( \delta > 0 \).

The following proposition is a specialised version of the proposition from [16, p. 139] (cf. [20, p. 53]).

**Proposition 2.24.** Let \( \Omega \subset \mathbb{R}^n \), \( \Sigma \subset 2^\Omega \) be a \( \sigma \)-algebra and \( \sigma : \Sigma \to \mathbb{R} \) be a \( \sigma \)-measure.

Then for every \( u^* \in (\mathcal{L}^\infty (\Omega, \Sigma, \sigma))^* \) there exists a unique \( \mu \in \text{ba}(\Omega, \Sigma, \sigma) \) such that
\[ \langle u^*, f \rangle = \int_\Omega f \, d\mu \]
for every \( f \in \mathcal{L}^\infty(\Omega, \Sigma) \) and
\[
\|u^*\| = \|\mu\| = |\mu|(\Omega).
\]
On the other hand, every \( \mu \in \text{ba}(\Omega, \Sigma) \) defines \( u^* \in \mathcal{L}^\infty(\Omega, \Sigma, \sigma)^* \).

Hence, \( \mathcal{L}^\infty(\Omega, \Sigma, \sigma)^* \) and \( \text{ba}(\Omega, \Sigma, \sigma) \) can be identified.

Using the decomposition Proposition 2.6 from above, one obtains a more refined characterisation of the dual of \( \mathcal{L}^\infty(\Omega, \Sigma, \sigma) \). In particular, every element of the dual space is the sum of a \( \sigma \)-measure with \( \mathcal{L}^1 \)-density and a pure measure. In contrast to the literature, this makes the intuitive idea of the dual of \( \mathcal{L}^\infty \) being \( \mathcal{L}^1 \) plus something which is not weakly absolutely continuous with respect to Lebesgue measure precise.

**Proposition 2.25.** Let \( \Omega \subset \mathbb{R}^n \) and \( \Sigma \subset 2^\Omega \) be a \( \sigma \)-algebra and \( \sigma : \Sigma \to \mathbb{R} \) be a \( \sigma \)-measure. Then for every \( u^* \in \mathcal{L}^\infty(\Omega, \Sigma, \sigma)^* \) there exists a unique pure \( \mu_p \in \text{ba}(\Omega, \Sigma, \sigma) \) and a unique \( h \in \mathcal{L}^1(\Omega, \Sigma, \sigma) \) such that
\[
\langle u^*, f \rangle = \int_{\Omega} fh \, d\mathcal{L}^n + \int_{\Omega} f \, d\mu_p
\]
for every \( f \in \mathcal{L}^\infty(\Omega, \Sigma, \sigma) \).

**Proof.** See Theorem 4.14 in [17, p. 21]. \( \Box \)

**Remark 2.26.** Note that the \( \mathcal{L} \)-space over a measure \( \mu \geq 0 \) is in general not complete. Nevertheless, the completion is known to be the set of all absolutely continuous measures whose \( p \)-norm is finite, i.e. all bounded measures \( \lambda \) with \( \lambda \ll \mu \) and
\[
\lim_{P \in \mathcal{P}} \sum_{A \in P \mu(A) \neq 0} \left| \frac{\lambda(A)}{\mu(A)} \right|^p \mu(A) < \infty.
\]
Here, the limit is taken over the directed set \( \mathcal{P} \) of all partitions \( P \) of \( \Omega \). See [16, p. 185ff] for reference. Using the convention \( \frac{0}{0} = 0 \), this limit is the same as the refinement integral
\[
\int_{\Omega} \left| \frac{\lambda}{\mu} \right|^p \mu
\]
as defined by Kolmogoroff in [15].

The following proposition states that every pure measure induces a Radon measure on its core.
Proposition 2.27. Let $\Omega \in B(\mathbb{R}^n)$ be bounded and $\mu \in ba(\Omega, B(\Omega), \mathcal{L}^n)$. Then there exists a Radon measure $\sigma$ supported on $\text{core } \mu \subset \overline{\Omega}$ such that for every $\phi \in C(\Omega)$

$$\int_{\Omega} \phi \, d\mu = \int_{\text{core } \mu} \phi \, d\sigma.$$

Proof. See Proposition 5.24 in [17, p. 36].

Remark 2.28. In the setting of the proposition above, $\sigma$ is said to be a representation of $\mu$ on $\text{core } \mu$.

The next proposition gives a partial inverse to the statement of the proposition above. In particular, any Radon measure can be extended to a measure on all of its domain.

Proposition 2.29. Let $\Omega \in B(\mathbb{R}^n)$ be bounded and $C \subset \overline{\Omega}$ be closed such that for every $x \in C$ and every $\delta > 0$

$$\mathcal{L}^n(B_{\delta}(x) \cap \Omega) > 0.$$

Furthermore, let $\sigma$ be a Radon measure on $C$. Then there exists $\mu \in ba(\Omega, B(\Omega), \mathcal{L}^n)$ such that for every $\phi \in C(\Omega)$

$$\int_{\Omega} \phi \, d\mu = \int_{C} \phi \, d\sigma.$$

In particular,

\[ \text{core } \mu \subset C \]

and

\[ |\mu|(\Omega) = |\sigma|(C). \]

Remark 2.30. The conditions of the statement are satisfied if, for example, $C \subset \partial^* \Omega \cup \Omega_{int}$.

Proof. See Proposition 5.26 in [17, p. 37].

The measure from the preceding proposition is pure if the Radon measure is singular with respect to Lebesgue measure.

Corollary 2.31. Let $\Omega \in B(\mathbb{R}^n)$ be bounded and $C \subset \overline{\Omega}$ be closed such that for every $x \in C$ and $\delta > 0$

$$\mathcal{L}^n(B_{\delta}(x) \cap \Omega) > 0.$$
and
\[ \mathcal{L}^n(C \cap \Omega) = 0. \]

Furthermore, let \( \sigma \) be a Radon measure on \( C \).
Then there exists \( \mu \in \text{ba}(\Omega, B(\Omega), \mathcal{L}^n) \) such that for all \( \phi \in C_0(\Omega) \)
\[ \int_{\Omega} \phi \, d\mu = \int_{C} \phi \, d\sigma. \]

Furthermore,
\[ |\mu|(\Omega) = |\sigma|(C) \]
and \( \mu \) is pure.

Proof. The preceding proposition and Proposition 2.11 yield the statement. \( \square \)

The following example presents another way to construct a density at zero.

Example 2.32. Let \( \Omega \in B(\mathbb{R}^n) \) be bounded and \( x \in \overline{\Omega} \) such that for every \( \delta > 0 \)
\[ \mathcal{L}^n(B_\delta(x) \cap \Omega) > 0. \]
Then there exists a pure \( \mu \in \text{ba}(\Omega, B(\Omega), \mathcal{L}^n) \) such that for every \( \phi \in C(\overline{\Omega}) \)
\[ \int_{\Omega} \phi \, d\mu = \phi(x). \]

The next example shows an extension for \( \mathcal{H}^{n-1} \).

Example 2.33. Let \( \Omega \in B(\mathbb{R}^n) \) be open, bounded and have smooth boundary. Then \( \mathcal{L}^n(\partial \Omega) = 0 \) and \( C = \partial \Omega \) satisfies the assumptions of Proposition 2.29. Hence, there exists \( \mu \in \text{ba}(\Omega, B(\Omega), \mathcal{L}^n) \) such that for all \( \phi \in C(\overline{\Omega}) \)
\[ \int_{\partial \Omega} \phi \, d\mathcal{H}^{n-1} = \int_{\Omega} \phi \, d\mu. \]

The following example shows, that the surface part of a Gauss formula can be expressed as an integral with respect to a pure measure. In Section 3 this is extended to vector fields having divergence measure.

Example 2.34. Let \( \Omega \in B(\mathbb{R}^n) \) be a bounded set with smooth boundary. Then \( C = \partial \Omega \subset \overline{\Omega} \) is a closed set and for every \( k \in \mathbb{N} \) such that \( 1 \leq k \leq n \)
\[ \nu^k \cdot \mathcal{H}^{n-1}|\partial \Omega \]
is a Radon measure on $C$. By Proposition 2.29 there exists $\mu_k \in \text{ba} (\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $\phi \in C(\overline{\Omega})$

$$\int_{\partial\Omega} \phi \cdot \nu^k \, d\mathcal{H}^{n-1} = \int_{\Omega} \phi \, d\mu_k = \int_{\partial\Omega} \phi \, d\mu_k$$

and

$$\text{core } \mu_k \subset \partial \Omega.$$

Hence, there exists $\mu \in (\text{ba} (\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$ such that for all $\phi \in C^1 (\overline{\Omega}, \mathbb{R}^n)$

$$\int_{\partial\Omega} \phi \, d\mu = \int_{\Omega} \phi \, d\mu = \int_{\partial\Omega} \phi \cdot \nu \, d\mathcal{H}^{n-1} = \int_{\Omega} \text{div } \phi \, d\mathcal{L}^n,$$

where the Gauß formula for sets with finite perimeter from Evans [11, p. 209] was used. Furthermore,

$$\text{core } \mu \subset \partial \Omega$$

and $\mu$ is pure by Proposition 2.11.

3 Bounded Vector Fields having Divergence Measure

The following lemma enables the use of the characterisation of the dual of $L^\infty$ in the following theorem. The key point of this statement is that the dual of a product space is essentially the product of the dual spaces.

Nevertheless, a self-contained proof is given.

Proposition 3.1. Let $\Omega \subset \mathbb{R}^n$. The dual space of

$$L^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n)$$

equipped with the norm

$$\| F \| := \sup_{k \in \mathbb{N}} \| F_k \|_\infty \quad \text{for} \quad F \in L^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n)$$

is the space

$$(\text{ba} (\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$$

equipped with the norm

$$\| \nu \| = \sum_{k=1}^n |\nu_k| (\Omega) \quad \text{for } \nu \in (\text{ba} (\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n.$$
Proof. Let $\nu \in (\text{ba} (\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$. Then $u^* : \mathcal{L}^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n) \to \mathbb{R}$ defined by

$$
\langle u^*, F \rangle = \sum_{k=1}^n \int_\Omega F_k \, d\nu_k
$$

for $F \in \mathcal{L}^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n)$ is obviously a linear functional on $\mathcal{L}^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n)$. Furthermore

$$
|\langle u^*, F \rangle| \leq \sum_{k=1}^n \|F_k\|_\infty |\nu_k| (\Omega) \leq \|F\| \|\nu\|
$$

for $F \in \mathcal{L}^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n)$, where the norms are defined as in the statement of the proposition.

Now let $u^* \in \mathcal{L}^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n)^*$. Then for every $k \in \mathbb{N}, 1 \leq k \leq n$

$$
u_k^* : \mathcal{L}^\infty (\Omega, \mathcal{L}^n) \to \mathbb{R} : f \mapsto \langle u^*, fe_k \rangle
$$

is a continuous linear functional on $\mathcal{L}^\infty (\Omega, \mathcal{L}^n)$. By Proposition 2.24 there exist $\nu_k \in \text{ba} (\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that

$$
\langle u_k^*, f \rangle = \int_\Omega f \, d\nu_k
$$

for every $f \in \mathcal{L}^\infty (\Omega, \mathcal{L}^n)$. Hence for every $F \in \mathcal{L}^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n)$

$$
\langle u^*, F \rangle = \sum_{k=1}^n \langle u_k^*, F_k \rangle = \sum_{k=1}^n \int_\Omega F_k \, d\nu_k = \int_\Omega F \, d\nu,
$$

where $\nu = (\nu_1, ..., \nu_n)$.

Now let $\nu \in (\text{ba} (\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$. Then

$$
\sup_{F \in \mathcal{L}^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n), \|F\| \leq 1} \left| \int_\Omega F \, d\nu \right| = \sup_{F \in \mathcal{L}^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n), \|F\| \leq 1} \int_\Omega F \, d\nu
$$

$$
= \sup_{F \in \mathcal{L}^\infty (\Omega, \mathbb{R}^n, \mathcal{L}^n), \|F\| \leq 1} \sum_{k=1}^n \int_\Omega F_k \, d\nu_k
$$

$$
= \sum_{k=1}^n \sup_{F_k \in \mathcal{L}^\infty (\Omega, \mathcal{L}^n), \|F_k\| \leq 1} \int_\Omega F_k \, d\nu_k
$$

$$
= \sum_{k=1}^n |\nu_k| (\Omega) = \|\nu\| .
$$

This finishes the proof. □
The proof of the upcoming Gauss Theorem relies on the following notion of approximation of the domain $\Omega$. It turns out that this is not only a technical necessity but gives the obtained Gauss formulas a more flexible shape.

**Definition 3.2.** Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(U)$ with $\text{dist}_{\partial U}(\Omega) > 0$.

A sequence $\{\chi_k\}_{k \in \mathbb{N}} \subset W^{1,\infty}(U, [0, 1])$ of Lipschitz continuous real functions with compact support in $U$ is called **good approximation for** $\chi_\Omega$ with limit function $\chi$, if

1. $\lim_{k \to \infty} \chi_k(x) =: \chi(x)$ exists $\mathcal{H}^{n-1}$-a.e. on $U$

2. $\chi = 1$ $\mathcal{H}^{n-1}$-a.e. on $\text{int} \Omega$

3. $\chi = 0$ $\mathcal{H}^{n-1}$-a.e. on $(\Omega)^c$

4. $\sup_{k \in \mathbb{N}} \|D\chi_k\|_1 < \infty$.

A necessary condition for $\Omega$ to allow a good approximation is given in the next proposition.

**Proposition 3.3.** Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(U)$ be bounded such that $\text{dist}_{\Omega}(\partial U) > 0$. If there is a good approximation for $\chi_\Omega$ with $\|\chi - \chi_\Omega\|_1 = 0$. Then $\Omega$ is a set of finite perimeter.

**Proof.** Let $\{\chi_k\}_{k \in \mathbb{N}}$ be a good approximation for $\chi_\Omega$. Since $\Omega$ is bounded and every $\mathcal{H}^{n-1}$-null set is a $\mathcal{L}^n$-null set,

$\chi_k \overset{L^1}{\to} \chi_\Omega$.

Since the total variation is lower semi continuous,

$|D\chi_{\Omega}|(U) \leq \liminf_{k \to \infty} \|D\chi_k\|_1 < \infty$.

This proves the statement.

**Remark 3.4.** In Example 3.10 it is shown that every set of finite perimeter allows a good approximation.

Now, the Gauss Theorem can be proved using good approximations and the characterisation of the dual of $\mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$.
Theorem 3.5. Let $U \subset \mathbb{R}^n$ be open, $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter such that $\text{dist}_\Omega(\partial U) > 0$. Furthermore, let $\{\chi_k\}_{k \in \mathbb{N}}$ be a good approximation with limit $\chi$. Then there exists $\nu \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ such that for every $k \in \mathbb{N}, 1 \leq k \leq n$

$$\text{core} \nu_k \subset \partial \Omega.$$ 

and the Gauß formula

$$\text{div } F(\text{int } \Omega) + \int_{\partial \Omega} \chi \text{ d} \text{div } F = \int_{\partial \Omega} F \text{ d}\nu$$

(2)

holds for every $F \in \mathcal{D}\mathcal{M}^\infty(U, \mathbb{R}^n)$. The measure $\nu$ is minimal in the norm, i.e. if $\nu' \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ satisfies (2) for every $F \in \mathcal{D}\mathcal{M}^\infty(U, \mathbb{R}^n)$, then

$$\|\nu\| \leq \|\nu'\|.$$

In addition, for every set of finite perimeter $B \in \mathcal{B}(U)$

$$\nu(B) = -\lim_{k \to \infty} \int_{B \cap \text{supp } D\chi_k} D\chi_k \text{ d}\mathcal{L}^n = -\int_{\partial^*B \cap \Omega} \chi \cdot \nu^B \text{ d}\mathcal{H}^{n-1}.$$ 

The preceding new Gauß Theorem sets itself apart from the literature by introducing normal measures. In the literature, Gauß formulas for sets of finite perimeter and essentially bounded vector field can be found in the form of functionals on a function space on the boundary (cf. [19, p. 448]) or as functions on the boundary which are obtained by mollification (cf. [7, p. 262f]). The approach chosen here enables a clean separation of geometry and vector field. In plus, it yields the existence of a normal measure which is defined on all Borel subsets.

Definition 3.6. $\nu \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ satisfying (2) for all $F \in \mathcal{D}\mathcal{M}^\infty(U, \mathbb{R}^n)$ with some limit $\chi$ of a good approximation of $\chi_\Omega$ that is minimal in the sense of Theorem 3.5 is called normal measure of $\Omega$ related to $\chi$.

Now, the proof of Theorem 3.5 is given.

Proof. Let $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter with $\text{dist}_\Omega(\partial U) > 0$ and let $\{\chi_k\}_{k \in \mathbb{N}} \subset \mathcal{W}^{1,\infty}(U, [0,1])$ be an associated good approximation with limit function $\chi$.

Now, let $F \in \mathcal{D}\mathcal{M}^\infty(U, \mathbb{R}^n)$. Then by the Dominated Convergence Theorem (cf. [11, p. 20])

$$\int_U \chi_k \text{ d} \text{div } F \overset{k \to \infty}{\longrightarrow} \text{div } F(\text{int } \Omega) + \int_{\partial \Omega} \chi \text{ d} \text{div } F.$$ 

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Note that $\text{div } F \ll^w \mathcal{H}^{n-1}$ (cf. \cite{6} p. 1014). On the other hand, 

$$F \cdot \chi_k \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$$

by \cite{2} p. 101. Furthermore, $F \cdot \chi_k$ is compactly supported in $U$. Thus by \cite{5} p. 252 for every $k \in \mathbb{N}$

$$\int_U \chi_k \, d\text{div } F = -\int_U D\chi_k \cdot F \, d\mathcal{L}^n.$$ 

Hence for every $k \in \mathbb{N}$

$$\left| \int_U \chi_k \, d\text{div } F \right| \leq \|D\chi_k\|_{L^1} \|F\|_{\infty, \text{supp } \chi_k}.$$ 

This implies

$$\left| \text{div } F(\text{int } \Omega) + \int_{\partial \Omega} \chi \, d\text{div } F \right| \leq \limsup_{k \to \infty} \|D\chi_k\|_{L^1} \|F\|_{\infty, \text{supp } \chi_k} \leq \sup_{k \in \mathbb{N}} \|D\chi_k\|_1 \|F\|_{\infty}.$$ 

Hence

$$u^*_0 : \mathcal{DM}^\infty(U, \mathbb{R}^n) \to \mathbb{R} : F \mapsto \text{div } F(\text{int } \Omega) + \int_{\partial \Omega} \chi \, d\text{div } F$$

is a continuous linear functional on a subspace of $\mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$. By the Hahn-Banach Theorem \cite{10} p. 63 there exists a continuous linear extension $u^*$ of $u^*_0$ to all of $\mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$ such that $\|u^*\| = \|u^*_0\|$. In particular, this extension is minimal in the norm. By Proposition \ref{prop3.1} there exists a $\nu \in (\text{ba } (U, \mathcal{B}(U), \mathcal{L}^n))^n$ such that for all $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$

$$\text{div } F(\text{int } \Omega) + \int_{\partial \Omega} \chi \, d\text{div } F = \int_U F \, d\nu.$$ 

and $\|\nu\| = \|u^*_0\|$. Furthermore

$$\sum_{k=1}^n |\nu_k| (\Omega) = \|u^*_0\| = \|u^*\|.$$ 

Note that by the Coarea Formula (cf. \cite{11} p. 112), for a.e. $0 < \delta < \text{dist}_\Omega(\partial U)$ the neighbourhood $\Omega_\delta$ is a set of finite perimeter. By \cite{2} p. 101

$$F \cdot \chi_{\Omega_\delta} \in \mathcal{DM}^\infty(U, \mathbb{R}^n).$$

But $F \cdot \chi_{\Omega_\delta}$ and $F$ agree on a neighbourhood of $\Omega$, whence

$$\text{div } (F \cdot \chi_{\Omega_\delta})(\text{int } \Omega) + \int_{\partial \Omega} \chi \, d\text{div } (F \cdot \chi_{\Omega_\delta}) = \text{div } F(\text{int } \Omega) + \int_{\partial \Omega} \chi \, d\text{div } F.$$
whence
\[ \int_{U \setminus \Omega_\delta} F \, d\nu = 0 \]
for every \( F \in \mathcal{DM}^\infty(U, \mathbb{R}^n) \) and almost every \( 0 < \delta < \text{dist}_\Omega(\partial U) \). Thus for almost every such \( \delta > 0 \) and \( F \in \mathcal{DM}^\infty(U, \mathbb{R}^n) \)
\[ \langle u^*, F \rangle = \int_{\Omega} F \, d\nu_{\Omega_\delta}. \]
This implies
\[ \|u_0^*\| \leq \|\nu_{\Omega_\delta}\| \leq \|\nu\| = \|u_0^*\| \]
and thus
\[ \sum_{k=1}^{n} |\nu_k|(U \setminus \Omega_\delta)(\Omega) = \sum_{k=1}^{n} |\nu_k|(U) - |\nu_k|_{\Omega_\delta}(\Omega) = \|u_0^*\| - \|u_0^*\| = 0 \]
whence
\[ |\nu_k|(U \setminus \Omega_\delta) = 0 \]
for almost every \( 0 < \delta < \text{dist}_\Omega(\partial U) \). Since \( |\nu_k| \) is monotone, the statement follows for all \( 0 < \delta < \text{dist}_\Omega(\partial U) \).

Note that by the Coarea formula (cf. [11, p. 112]) \( \Omega_{-\delta} \) is a set of finite perimeter for almost every \( \delta > 0 \). By [2, p. 101]
\[ F \cdot \chi_{\Omega_{-\delta}} \in \mathcal{DM}^\infty(U, \mathbb{R}^n). \]
Since \( \Omega \) is bounded, \( F \cdot \chi_{\Omega_{-\delta}} \) is compactly supported in \( \Omega \) and thus by [5, p. 252]
\[ \text{div} F \cdot \chi_{\Omega_{-\delta}}(\text{int}\, \Omega) + \int_{\partial\Omega} \chi \, d\text{div} (F \cdot \chi_{\Omega_{-\delta}}) = 0. \]
This implies
\[ \int_{\Omega} F \, d\nu_{\Omega_{-\delta}} = 0 \]
for every such \( \delta > 0 \) and \( F \in \mathcal{DM}^\infty(U, \mathbb{R}^n) \). Hence
\[ \|u_0^*\| \leq \|\nu_{U \setminus \Omega_{-\delta}}\| \leq \|\nu\| = \|u_0^*\| \]
and analogously to the reasoning for \( \Omega_\delta \) one deduces
\[ |\nu_k|(\Omega_{-\delta}) = 0 \]
for every \( k \in \mathbb{N}, 1 \leq k \leq n \).
Now for every $\delta > 0$, $\delta < \text{dist}_\Omega(\partial U)$ and every $k \in \mathbb{N}, 1 \leq k \leq n$

$$|\nu_k| \left( (U \setminus \Omega_\delta) \cup \Omega_{-\delta} \right) = 0.$$

This implies

$$\text{core} \nu_k \subset \partial \Omega$$

for every $k \in \mathbb{N}, 1 \leq k \leq n$. This establishes Equation (2).

Now, let $B \in \mathcal{B}(U)$ be a set of finite perimeter. Then for $k \in \mathbb{N}, 1 \leq k \leq n$

$$e_k \cdot \chi_B \in \mathcal{DM}^\infty(U, \mathbb{R}^n).$$

Note that

$$\text{div} (e_k \cdot \chi_B) = \partial_k \chi_B = -(\nu^B)_k \mathcal{H}^{n-1}|_{\partial^* B}.$$

The established Gauß formula yields

$$\int_U \chi_B \, d\nu_k = \int_U e_k \cdot \chi_B \, d\nu$$

$$= \text{div} (e_k \cdot \chi_B)(\text{int } \Omega) + \int_{\partial \Omega} \chi \, d\text{div} (e_k \cdot \chi_B)$$

$$= \int_U \chi \, d\text{div} (e_k \cdot \chi_B)$$

$$= \lim_{l \to \infty} \int_U \chi_l \, d\text{div} (e_k \cdot \chi_B)$$

$$= \lim_{l \to \infty} - \int_B e_k \cdot D\chi_l \, d\mathcal{L}^n$$

Since $k \in \mathbb{N}, 1 \leq k \leq n$ was arbitrary

$$\nu(B) = - \lim_{k \to \infty} \int_{B \cap \text{supp } D\chi_k} D\chi_k \, d\mathcal{L}^n.$$

On the other hand, for every set of finite perimeter $B \in \mathcal{B}(U)$ and for $k \in \mathbb{N}, 1 \leq k \leq n$

$$\nu_k(B) = \lim_{l \to \infty} \int_U \chi_l \, d\text{div} (e_k \cdot \chi_B)$$

$$= \lim_{l \to \infty} \int_U \chi_l \, d\partial_k(\chi_B)$$

$$= - \lim_{l \to \infty} \int_{\partial^* B} \chi_l \cdot (\nu^B)_k \, d\mathcal{H}^{n-1}$$

$$= - \int_{\partial^* B} \chi \cdot (\nu^B)_k \, d\mathcal{H}^{n-1}.$$
Hence
\[ \nu(B) = -\int_{\partial^* B \cap \Omega} \chi \cdot \nu^B \, d\mathcal{H}^{n-1}. \]

Given a good approximation of \( \chi_\Omega \), normal measures are uniquely defined on sets of finite perimeter.

**Proposition 3.7.** Let \( U \subset \mathbb{R}^n \) be open and \( \Omega \in \mathcal{B}(U) \) be a bounded set of finite perimeter such that \( \text{dist}_\Omega(\partial U) > 0 \). Let \( \{ \chi_k \}_{k \in \mathbb{N}} \) be a good approximation with limit \( \chi \). Let \( \nu \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n \) be an associated normal measure. Then for every set of finite perimeter \( B \in \mathcal{B}(U) \) there exists a Lebesgue null set \( N \subset \mathbb{R} \)

\[ \nu(B) = \int_{\partial^* B \cap \Omega} -\chi \nu^B \, d\mathcal{H}^{n-1} + \lim_{\delta \downarrow 0} \int_{B_{\text{int}} \cap \partial^* \Omega_{-\delta}} \nu^{\Omega_{-\delta}} \, d\mathcal{H}^{n-1}. \]

**Proof.** Note that \( B \) and \( B_{\text{int}} \) only differ by a \( \mathcal{L}^n \)-null set (cf. [11, p. 43]). Hence \( B_{\text{int}} \) is also a set of finite perimeter. W.l.o.g. \( B = B_{\text{int}} \). The Coarea Formula (cf. [11, p. 112]) implies that for a.e. \( \delta > 0 \) the set

\[ \Omega_\delta \setminus \Omega_{-\delta} \]

has finite perimeter. Then \((\Omega_\delta \setminus \Omega_{-\delta})_{\text{int}}\) is also a set of finite perimeter. By [13, p. 5],

\[ B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{\text{int}} \]

is also a set of finite perimeter. Note that the Coarea Formula also implies that for a.e. \( \delta > 0 \)

\[ \mathcal{H}^{n-1}(\partial_+ B \cap \partial(\Omega_\delta \setminus \Omega_{-\delta})) = 0. \]

Using this and [9, p. 199]

\[ \partial_+(B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{\text{int}}) \]

diffs from

\[ (\partial_+ B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{\text{int}}) \cup (B \cap \partial_+(\Omega_\delta \setminus \Omega_{-\delta})) \]

only by a \( \mathcal{H}^{n-1} \)-null set. Since \( \Omega_\delta \setminus \Omega_{-\delta} \) has density 1 at points of its measure theoretic interior and the measure theoretic normal is characterised by the halfspace it generates (cf. [11, p. 203]), one sees that

\[ \nu^{B \cap (\Omega_\delta \setminus \Omega_{-\delta})} = \nu^B \quad \text{on} \quad \partial^* B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{\text{int}} \cap \partial^*(B \cap (\Omega \setminus \Omega_{-\delta})). \]
Theorem 3.5 states core $\nu \subset \partial \Omega$, thus
\[

\nu(B) = \nu(B \cap (\Omega_\delta \setminus \Omega_{-\delta}))_{\text{int}} = -\int_{\partial^{*}(B \cap (\Omega_\delta \setminus \Omega_{-\delta}))_{\text{int}}} \chi_{\nu}^{B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{\text{int}}} \, dH^{n-1}.
\]
for a.e. $\delta > 0$. The integral on the right hand side is for a.e. $\delta > 0$ equal to
\[

\int_{\partial^{*}B \cap \Omega \cap (\Omega_\delta \setminus \Omega_{-\delta})_{\text{int}}} -\chi \nu^{B} \, dH^{n-1} + \int_{\partial^{*}B \cap \partial^{*}(\Omega_{-\delta})} \chi_{\nu}^{\Omega_{-\delta}} \, dH^{n-1}.
\]
Noting that
\[

\int_{\partial^{*}B \cap \Omega \cap A} -\chi \nu^{B} \, dH^{n-1}
\]
defines a $\sigma$-measure in $A$ and using continuity from above yields
\[

\lim_{\delta \downarrow 0} \int_{\partial^{*}B \cap \Omega \cap (\Omega_\delta \setminus \Omega_{-\delta})_{\text{int}}} -\chi \nu^{B} \, dH^{n-1} = \int_{\partial^{*}B \cap \partial \Omega} -\chi \nu^{B} \, dH^{n-1}
\]
On the other hand $\partial^{*}(\Omega_{-\delta}) \subset \overline{\Omega}$. Hence
\[

\partial^{*}(\Omega_{-\delta}) \cap \overline{\Omega} \cap B = B \cap \partial^{*}(\Omega_{-\delta})
\]
Furthermore $\chi = 1$ on $\text{int} \Omega$. This finishes the proof.

The following picture illustrates the representation of a normal measure from the preceding proposition.

![Figure 2: Domain of influence for a normal measure and a set of finite perimeter $B$](image)

The relation of $H^{n-1}|\partial^{*}\Omega$ and $|\nu|$ is treated in the next proposition.
Proposition 3.8. Let \( U \subset \mathbb{R}^n \) be open, \( \Omega \in \mathcal{B}(U) \) be a bounded set of finite perimeter such that \( \text{dist}_\Omega(\partial U) > 0 \). Furthermore, let \( \{\chi_k\}_{k \in \mathbb{N}} \) be a good approximation with limit \( \chi \) and let \( \nu \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n \) be the associated normal measure.

If \( \|\chi - \chi_\Omega\|_1 = 0 \), then for every open set \( B \subset U \)

\[
|\nu| (B) \geq (\mathcal{H}^{n-1}[\partial^* \Omega])(B).
\]

Remark 3.9. Note that \( \mathcal{L}^n(\partial \Omega) = 0 \) implies \( \|\chi - \chi_\Omega\|_1 = 0 \).

Proof. Let \( \phi \in C_0^1(U, \mathbb{R}^n) \). Then using the Gauß Theorem from Evans [11, p. 209]

\[
\int_U \phi \, d\nu = \int_{\text{int} \Omega} \text{div} \phi \, d\mathcal{L}^n + \int_{\partial \Omega} \chi \text{div} \phi \, d\mathcal{L}^n
\]

\[
= \int_{\partial \Omega} \text{div} \phi \, d\mathcal{L}^n
\]

\[
= \int_{\partial^* \Omega} \phi \cdot \nu^\Omega \, d\mathcal{H}^{n-1}.
\]

Hence for every open set \( B \subset U \)

\[
|\nu| (B) \geq \sup_{\phi \in C_0^1(B, \mathbb{R}^n), \|\phi\|_\infty \leq 1} \int_U \phi \, d\nu
\]

\[
\geq \sup_{\phi \in C_0^1(B, \mathbb{R}^n), \|\phi\|_C \leq 1} \int_{\partial^* \Omega} \phi \cdot \nu^\Omega \, d\mathcal{H}^{n-1}
\]

\[
= |D\chi_\Omega| (B)
\]

\[
= (\mathcal{H}^{n-1}[\partial^* \Omega])(B).
\]

For the last equality, see e.g. [11, p. 205].

Since \( B \in \mathcal{B}(U) \) was arbitrary, this finishes the proof. \( \square \)

The following example shows that for every set of finite perimeter there exists a canonical normal measure. Hence, Theorem 3.5 is always applicable.

Example 3.10. Canonical normal measure

Let \( U \subset \mathbb{R}^n \) be open and \( \Omega \in \mathcal{B}(U) \) be a bounded set of finite perimeter such that \( \text{dist}_\Omega(\partial U) > 0 \). Furthermore, let \( \rho \in C_0^\infty(\mathbb{R}^n) \) be the standard mollification kernel (cf. [11, p. 122]). Then

\[
\chi_k(x) := \int_{\mathbb{R}^n} \frac{1}{k^n} \rho(k(y - x)) \chi_\Omega(x) \, dy.
\]
is a good approximation for $\chi_\Omega$. The limit function $\chi$ satisfies

$$\chi = \chi_{\Omega_{\text{int}}} + \frac{1}{2} \chi_{\partial^* \Omega} \mathcal{H}^{n-1} \text{-a.e.}.$$ 

See [1, p. 175] for reference. Hence, there exists a normal measure $\nu \in (\text{ba}(U, B(U), \mathcal{L}^n))^n$ such that for every $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$ the following Gauß formula holds

$$\text{div } F(\Omega_{\text{int}}) + \frac{1}{2} \text{div } F(\partial^* \Omega) = \int_{\partial \Omega} F \, d\nu.$$ 

Furthermore,

$$\text{core } \nu \subset \partial \Omega.$$ 

The divergence on the regular boundary of $\Omega$, weighted with $\frac{1}{2}$, cannot be found in the literature. This is due the fact, that the majority of the texts prohibit the vector fields under consideration from exhibiting such concentrations. The remaining sources treat settings similar to the one of Theorem 3.18 below. The weight $\frac{1}{2}$ appears plausible, when interpreting the divergence as source strength of the field $F$. At points of the regular boundary, $\Omega$ geometrically resembles a half-space. Then half of the source strength can be seen to flow into the domain and the other half flows outwards.

The next example shows that for many closed sets of finite perimeter a more familiar form of the Gauß Theorem can be derived.

**Example 3.11.** Outer normal measure

Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(\Omega)$ be a bounded, closed set of finite perimeter such that $\delta_0 := \text{dist}_\Omega(\partial U) > 0$. Furthermore, let there be a sequence $\{\delta_k\}_{k \in \mathbb{N}} \subset (0, \infty)$ such that $\lim_{k \to \infty} \delta_k = 0$ and

$$\sup_{k \in \mathbb{N}} \int_{(0, \delta_k)} \mathcal{H}^{n-1}(\partial \Omega_{\delta}) \, d\delta < \infty.$$ 

This is the case if, e.g.

$$\lim_{\delta \downarrow 0} \mathcal{H}^{n-1}(\partial \Omega_{\delta}) = \mathcal{H}^{n-1}(\partial^* \Omega).$$ 

For $x \in U$ and $k \in \mathbb{N}$ set

$$\chi_k(x) := \chi_{\Omega_{\delta_k}}(x) \left(1 - \frac{1}{\delta_k} \chi_{\Omega_{\delta_k} \setminus \Omega} \text{dist}_\Omega(x)\right)$$

$$= \max \left\{0, \min \left\{1, 1 - \frac{1}{\delta_k} \text{dist}_\Omega\right\}\right\}.$$ 

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Then \( \chi_k \in W^{1,\infty}(U,[0,1]) \) is Lipschitz continuous (cf. [8, p. 47]). These functions are called (outer) **Portmanteau functions**. Note that by the Coarea formula for functions of bounded variation (cf. [11, p. 185])

\[
\|D\chi_k\|_1 = \int_{(0,1)} \mathcal{H}^{n-1}(\chi_k^{-1}(\delta)) \, d\delta = \int_{(0,\delta_k)} \mathcal{H}^{n-1}(\partial \Omega_\delta) \, d\delta.
\]

Hence, the sequence \( \{\chi_k\}_{k \in \mathbb{N}} \) is a good approximation for \( \chi_\Omega \) and the limit function is

\[
\chi = \chi_\Omega.
\]

Thus, there exists a normal measure \( \nu \in (ba(U,\mathcal{B}(U),\mathcal{L}^n))^n \) such that for every \( F \in DM^{\infty}(U,\mathbb{R}^n) \) the following **Gauß formula** holds

\[
\text{div } F(\Omega) = \int_{\partial \Omega} F \, d\nu.
\]

Open set of finite perimeter can be treated similarly, as the following example shows.

**Example 3.12.** Inner normal measure

Let \( U \subset \mathbb{R}^n \) be open and \( \Omega \subset U \) be a bounded, open set of finite perimeter such that \( \text{dist}_\Omega(\partial U) > 0 \). Furthermore, assume there exists \( \{\delta_k\}_{k \in \mathbb{N}} \subset (0,\infty) \) such that \( \lim_{k \to \infty} \delta_k = 0 \) and

\[
\sup_{k \in \mathbb{N}} \int_{(0,\delta_k)} \mathcal{H}^{n-1}(\partial \Omega_\delta) \, d\delta < \infty.
\]

For \( k \in \mathbb{N} \) and \( x \in U \) set

\[
\chi_k(x) := \chi_{\Omega_{-\delta_k}}(x) + \frac{1}{\delta_k} \text{dist}_{\Omega^c}(x)\chi_{\Omega \setminus \Omega_{-\delta_k}}
\]

\[
= \min\left\{1, \max\left\{0, \frac{1}{\delta_k} \text{dist}_{\Omega^c}\right\}\right\}.
\]

Then \( \chi_k \in W^{1,\infty}(U,[0,1]) \) is Lipschitz continuous (cf. [8, p. 47]). Then as in Example 3.11 the sequence \( \{\chi_k\}_{k \in \mathbb{N}} \) is a good approximation for \( \chi_\Omega \) and the limit function is

\[
\chi = \chi_\Omega.
\]

These functions are called (inner) **Portmanteau functions**. Hence there exists a normal measure \( \nu \in (ba(U,\mathcal{B}(U),\mathcal{L}^n))^n \) such that for every vector field \( F \in DM^{\infty}(U,\mathbb{R}^n) \) the following **Gauß formula** holds

\[
\text{div } F(\Omega) = \int_{\partial \Omega} F \, d\nu.
\]
The subsequent corollary illustrates the dependence of the integral with respect to normal measure on the good approximation of $\chi\Omega$.

**Corollary 3.13.** Let $U \subset \mathbb{R}^n$ and $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter such that $\text{dist}_{\mathcal{O}}(\partial U) > 0$. Then for any two good approximations $\{\chi^k_1\}_{k \in \mathbb{N}}, \{\chi^k_2\}_{k \in \mathbb{N}} \subset W^{1,\infty}(U, [0, 1])$ for $\chi\Omega$, associated normal measures $\nu_1, \nu_2 \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ and any $F \in \mathcal{D}\mathcal{M}^\infty(U, \mathbb{R}^n)$

$$\int_{\partial \Omega} F \, d\nu_1 - \nu_2 = \int_{\partial \Omega} \chi_1 - \chi_2 \, d\text{div} F$$

where $\chi_1$ and $\chi_2$ are the limit functions for $\{\chi^k_1\}_{k \in \mathbb{N}}$ and $\{\chi^k_2\}_{k \in \mathbb{N}}$ respectively.

**Remark 3.14.** In particular, if $|\text{div} F| (\partial \Omega) = 0$,

$$\int_{\partial \Omega} F \, d\nu$$

is independent of the choice of the good approximation.

Since $\nu$ is a bounded measure, all essentially bounded vector fields $F$ are integrable with respect to this measure. This leads to the question, whether $F \in L^1(U, \nu)$ for unbounded vector fields. The next example answers this negatively. The function is similar to the one in [2, p. 100].

**Example 3.15.** Let $U := (0, 1)^2 \subset \mathbb{R}^2$ and $\Omega := \{(x, y) \in \mathbb{R}^2 | x \leq y\} \cap B_{1/2}(1, 1)$. Furthermore let $\nu \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ be a normal measure for $\Omega$ and $F \in \mathcal{D}\mathcal{M}^1(U, \mathbb{R}^n)$ defined by

$$F(x, y) := |x - y|^{-\frac{1}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for $x \neq y$. Then $\text{div} F = 0$. In order to see that, let $\Delta := \{(x, x) \in U | x \in \mathbb{R}\}$, $1 > \delta > 0$ and note that for $\phi \in C^1_0(U)$

$$\int_U F \cdot D\phi \, d\mathcal{L}^n = \int_{U \cap \Delta_\delta} F \cdot D\phi \, d\mathcal{L}^n + \int_{U \setminus \Delta_\delta} F \cdot D\phi \, d\mathcal{L}^n$$

$$= \int_{U \cap \Delta_\delta} F \cdot D\phi \, d\mathcal{L}^n - \int_{U \setminus \Delta_\delta} \phi \text{div} F \, d\mathcal{L}^n - \int_{\partial(U \setminus \Delta_\delta)} \phi F \cdot \nu_{\Delta_\delta} \, d\mathcal{H}^1$$

Since $\text{div} F = 0$ outside of $\Delta_\delta$ and $F \cdot \nu_{\Delta_\delta} = \pm \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot F = 0$ on $U$

$$\int_U F \cdot D\phi \, d\mathcal{L}^n = \int_{U \cap \Delta_\delta} F \cdot D\phi \, d\mathcal{L}^n \xrightarrow{\delta \downarrow 0} 0.$$
Note that for every $c > 0$ with $\frac{1}{c^2} > \delta > 0$

$$\Omega_\delta \cap \{(x, y) \in U \mid |F(x, y)| \geq c\} \supset \Delta_\delta \cap \Omega_\delta$$

Hence for every $F' \in \mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$

$$|\nu| (\Omega_\delta \cap \{(x, y) \in U \mid |F(x, y) - F'(x, y)| \geq \varepsilon\}) \geq |\nu| (\Omega_\delta \cap \{(x, y) \in U \mid |F(x, y)| \geq \|F\|_\infty + \varepsilon\}) \geq |\nu| (\Delta_\delta \cap \Omega_\delta) \geq \mathcal{H}^1\left(\Delta \cap B_1\left(\frac{1}{2}(1, 1)\right)\right) = \frac{1}{2} > 0$$

for every $0 < \delta < \frac{1}{(\varepsilon + \|F\|_\infty)^2}$.

Hence, there is no sequence $\{F_k\}_{k \in \mathbb{N}} \subset \mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$ converging in measure to $F$. In particular, $F$ cannot be approximated in measure by simple functions.

**Remark 3.16.** The preceding example indeed works for $U = (0, 1)^2$ and every $F \in \mathcal{D}M^1(U, \mathbb{R}^n)$ such that for some $\phi : \mathbb{R} \to \mathbb{R}$ satisfying

1. $\phi$ is continuously differentiable on $\mathbb{R} \setminus \{0\}$
2. $\lim_{x \to 0} \phi(x) = \infty$
3. $g : U \to \mathbb{R} : (x, y) \mapsto \phi(x - y)$ is integrable on $U$

it holds

$$F = g \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

The essential point is that $F$ is tangential to the curve where it is unbounded. Hence, there are many vector fields which cannot even be approximated in measure.

The following example gives a vector field that only blows up at one point and still is not integrable with respect to normal measure. The function is the same as in [3, p. 403].

**Example 3.17.** Let $n = 2$, $U := B_1(0) \subset \mathbb{R}^2$ and

$$\Omega := B_{\frac{1}{2}}(0) \cap \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}.$$ 

Furthermore, let

$$F : U \to \mathbb{R}^2 : x \mapsto \frac{1}{2\pi |x|^2} \cdot \frac{x}{|x|^2}.$$
Then \( F \in \mathcal{DM}^1(U, \mathbb{R}^n) \) and \( \text{div} F = \delta_0 \). Let \( \{ \chi_k \}_{k \in \mathbb{N}} \) be the canonical good approximation from Example 3.10. Let \( \nu \in (\mathcal{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n \) be the normal measure associated with this good approximation. Assume that \( F \in L^1(U, \nu) \). Then

\[
\int_U |F| \, d\nu < \infty.
\]

But

\[
\int_U |F| \, d\nu \geq \frac{1}{2\pi} \int_{\partial \Omega} \frac{1}{|x|} \, d\mathcal{H}^1 \geq \frac{1}{2\pi} \int_{(0, \frac{1}{2})} \frac{1}{t} \, dt = \infty,
\]

a contradiction. Hence \( F \notin L^1(U, \nu) \).

Up to now, the Gauss Theorem was given for sets that have a positive distance to the boundary. In order to complement this result, the following theorem states the theorem for the whole set, in the case of \( U = \Omega \).

**Theorem 3.18.** Let \( \Omega \in \mathcal{B}(\mathbb{R}^n) \) be a bounded open set of finite perimeter. If there exists \( \delta_0 > 0 \) and \( c > 0 \) such that for almost every \( \delta \in (0, \delta_0) \)

\[
\mathcal{H}^{n-1}(\partial \Omega - \delta) \leq c,
\]

then there exists \( \nu \in (\mathcal{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n \) such that for every \( k \in \mathbb{N}, 1 \leq k \leq n \)

\[
\text{core} \, \nu_k \subset \partial \Omega
\]

and for all \( F \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n) \) the following Gauss formula holds

\[
\int_{\partial \Omega} F \, d\nu = \text{div} F(\Omega).
\]

and for every open set \( B \subset \mathbb{R}^n \)

\[
|\nu|(B \cap \Omega) \geq (\mathcal{H}^{n-1}|_{\partial^* \Omega})(B).
\]

Furthermore, \( \nu \) is minimal in the sense, that if \( \nu' \in (\mathcal{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n \) satisfies the equations above, then

\[
\|\nu\| \leq \|\nu'\|.
\]

For every \( B \in \mathcal{B}(\Omega) \) having finite perimeter in \( \mathbb{R}^n \)

\[
\nu(B) = -\int_{\partial^* B \cap \Omega} \nu^B \, d\mathcal{H}^{n-1}.
\]
Remark 3.19. Note that if $\Omega \in \mathcal{B}(\Omega)$ is only supposed to be open and $$\mathcal{H}^{n-1}(\partial \Omega_{-\delta}) \leq c$$ is required, then $\Omega$ is necessarily a set of finite perimeter, due to the total variation being lower semi-continuous.

On the other hand, this condition loosely resembles the definition of Lipschitz deformable boundaries defined in [2, p. 94], but is much more general.

Proof. Let $\{\chi_k\}_{k \in \mathbb{N}} \subset W^{1,\infty}(\mathbb{R}^n, [0, 1])$ be such that
\[
\chi_k := \chi_{\Omega - \frac{1}{k}} + \chi_{(\Omega - \frac{1}{k}) \setminus \Omega - \frac{1}{k} + 1} (k \text{ dist}_{\partial \Omega} - 1)
\]
\[
= \min \{1, \max \{0, k \text{ dist}_{\partial \Omega} - 1\}\}.
\]
See [8, p. 47] for reference. Then
\[
|D\chi_k| = k\chi_{(\Omega - \frac{1}{k}) \setminus \Omega - \frac{1}{k} + 1} .
\]
Then the Coarea Formula [11, p. 112] implies
\[
\|D\chi_k\|_1 = \int_{\Omega - \frac{1}{k} \setminus \Omega - \frac{1}{k} + 1} k \, d\mathcal{L}^n = \int_{\frac{1}{k} \setminus \Omega - \frac{1}{k}} \mathcal{H}^{n-1}(\partial \Omega_{-\delta}) \, d\delta \leq c .
\]
As in the proof of Theorem 3.5
\[
\lim_{k \to \infty} \int_{\Omega} F \cdot D\chi_k \, d\mathcal{L}^n = -\lim_{k \to \infty} \int_{\Omega} \chi_k \, d\text{div} F = -\int_{\Omega} 1 \, d\text{div} F = \text{div} F(\Omega) .
\]
On the other hand, for every $k \in \mathbb{N}$
\[
\left| \int_{\Omega} F \cdot D\chi_k \, d\mathcal{L}^n \right| \leq \|F\|_\infty \|D\chi_k\|_1 \leq \|F\|_\infty \sup_{k \in \mathbb{N}} \|D\chi_k\|_1 \leq c \|F\|_\infty .
\]
Hence
\[
u_0^* : \mathcal{DM}^\infty(\Omega, \mathbb{R}^n) \to \mathbb{R} : F \mapsto \text{div} F(\Omega)
\]
is a continuous linear functional on a subspace of $\mathcal{L}^\infty(\Omega, \mathbb{R}^n, \mathcal{L}^n)$. The Hahn-Banach Theorem (cf. [10, p.63]) implies the existence of a measure $\nu \in (\text{ba} (\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^\ast$ such that for all $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$
\[
\text{div} F(\Omega) = \int_{\Omega} F \, d\nu .
\]
Furthermore, $\|\nu\| = \|u_0^*\|$, implying minimality in the norm. Now by [2, p. 101], for almost every $\delta > 0$ and every $F \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$

$$F \cdot \chi_{\Omega_{-\delta}} \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$$

and $F \cdot \chi_{\Omega_{-\delta}}$ has compact support in $\Omega$. By [3, p. 252]

$$\text{div} (F \cdot \chi_{\Omega_{-\delta}})(\Omega) = 0.$$

Thus, for every $F \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$

$$\int_\Omega F \, d\nu = \text{div} F(\Omega) = \text{div} (F \cdot \chi_{\Omega_{-\delta}})(\Omega) + \text{div} (F \cdot \chi_{\Omega_{-\delta}})(\Omega) = \int_\Omega F \, d\nu[\Omega \setminus \Omega_{-\delta}].$$

Thus, $\nu[\Omega \setminus \Omega_{-\delta}]$ also satisfies Equation (3). The minimality of $\|\nu\|$ then implies

$$\|\nu[\Omega_{-\delta}]\| = 0.$$

Since $\delta > 0$ can be arbitrarily small

$$\text{core} \, \nu \subset \partial \Omega.$$

Note that for $B \in \mathcal{B}(\Omega)$ having finite perimeter in $\mathbb{R}^n$

$$e_k \chi_B \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n).$$

In order to see this, compute

$$\text{div} (e_k \cdot \chi_B) = \partial_k \chi_B = -\nu_k^B \mathcal{H}^{n-1}|\partial^* B.$$

In particular

$$\nu(B) = -\int_{\partial^* B \setminus \Omega} \nu^B \, d\mathcal{H}^{n-1}.$$
\[ |\nu| (B \cap \Omega) \geq \sup_{\phi \in C^1_0(B, \mathbb{R}^n), \|\phi\|_\infty \leq 1} \int_{\Omega} \phi \, d\nu \]

\[ \geq \sup_{\phi \in C^1_0(B, \mathbb{R}^n), \|\phi\|_C \leq 1} \int_{\Omega} \phi \, d\nu \]

\[ = \sup_{\phi \in C^1_0(B, \mathbb{R}^n), \|\phi\|_C \leq 1} \int_{\partial^* \Omega} \phi \cdot \nu \, d\mathcal{H}^{n-1} \]

\[ = |D\chi_\Omega| (B) = (\mathcal{H}^{n-1}[\partial^* \Omega])(B). \]

**Remark 3.20.** Theorem 3.18 still holds true for open \( \Omega \) such that there exists \( \{\delta_k\}_{k \in \mathbb{N}} \subset (0, \infty) \) such that \( \lim_{k \to \infty} \delta_k = 0 \) and

\[ \sup_{k \in \mathbb{N}} \int_{(\Omega - \delta_k)} \mathcal{H}^{n-1}(\Omega - \delta) \, d\delta < \infty. \]

The arguments are the same as in Example 3.11 and Example 3.12.

The following proposition is a new Gauß-Green formula for essentially bounded functions of bounded variation and essentially bounded vector fields having divergence measure. In contrast to the literature, where only continuous scalar fields were treated (cf. [19, p. 448], [6, p. 1014]), this is a new quality.

**Proposition 3.21.** Let \( U \subset \mathbb{R}^n \) be open and \( \Omega \in B(U) \) be a bounded set of finite perimeter such that \( \text{dist}_{\Omega}(\partial U) > 0 \). Furthermore, let \( \{\chi_k\}_{k \in \mathbb{N}} \subset W^{1,\infty}(U, [0, 1]) \) be a good approximation for \( \chi_\Omega \) and \( \nu \in (\text{ba}(U, B(U), \mathcal{L}^n))^n \) be an associated normal measure.

Then for every \( F \in \mathcal{DM}^\infty(U, \mathbb{R}^n) \) the set function

\[ F^\nu : B(U) \to \mathbb{R} : B \mapsto \int_B F \, d\nu \]
is an element of $\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n)$ with
\[
\text{core } F' \subset \partial \Omega.
\]
and for every compactly supported $f \in \mathcal{BV}(U) \cap \mathcal{L}^\infty(U, \mathcal{L}^n)$ the following Gauß formula holds
\[
\text{div} (f \cdot F)(\text{int } \Omega) + \int_{\partial \Omega} \chi \text{div} (f \cdot F) - \int_{\partial \Omega} f \cdot F \, d\nu.
\]
Call $F'$ \textbf{normal trace of $F$ on $\partial \Omega$}.

\textbf{Proof.} Note that
\[
f \cdot F \in \mathcal{D}\mathcal{M}^\infty(U, \mathbb{R}^n) \cdot
\]
See [6, p. 1014] for reference. Hence
\[
\text{div} (f \cdot F) \text{div} \nu = \int_{\partial \Omega} f \cdot F \, d\nu.
\]
Note that for every $B \in \mathcal{B}(U)$
\[
\left| \int_B F \, d\nu \right| \leq \|F\|_\infty |\nu|(B),
\]
whence
\[
F' \in \text{ba}(U, \mathcal{B}(U), \mathcal{L}^n) \cdot
\]
Since for every $B \in \mathcal{B}(U)$
\[
F'(B) = \int_B F \, d\nu = \int_{B \cap (\Omega \setminus \Omega_-)} F \, d\nu
\]
the core of $F'$ is a subset of $\partial \Omega$.

Let $\varepsilon > 0$. Since $f \in \mathcal{L}^\infty(U, \mathcal{L}^n)$, there exist $m \in \mathbb{N}$, \{\$y_k$\}$_{k=0}^m \subset \mathbb{R}$ and \{\$B_k$\}$_{k=0}^m$ pairwise disjoint, such that
\[
\|y_k - f \cdot \chi_{B_k}\|_\infty \leq \varepsilon \text{ and } \bigcup_{k=0}^m B_k = U
\]
Set $h := \sum_{k=0}^m y_k \chi_{B_k}$. Then
\[
\left| \int_{\partial \Omega} f F \, d\nu - \int_{\partial \Omega} f \, dF' \right| \leq \left| \int_{\partial \Omega} (f - h) F \, d\nu \right| + \left| \int_{\partial \Omega} h F \, d\nu - \int_{\partial \Omega} h \, dF' \right| \\
+ \left| \int_{\partial \Omega} f - h \, dF' \right| \\
\leq \varepsilon \|F\|_\infty |\nu|(U) + 0 + \varepsilon |F'| (U).
Since \( \varepsilon > 0 \) was arbitrary

\[
\int_{\partial \Omega} f F \, d\nu = \int_{\partial \Omega} f F' \, d\nu.
\]

\( \square \)

4 Unbounded Vector Fields and Open Sets

In the previous section, general Gauß formulas for essentially bounded vector fields having divergence measure were presented. Example 3.15 and 3.17 showed that it is in general not possible to integrate unbounded vector fields with respect to the normal measures obtained. In Proposition 3.21, the measure \( F' \) was presented as a notion of normal trace.

In the following, this is carried over to the case of unbounded vector fields. Therefore, a result due to Silhavy (cf. [19]) is improved upon. In particular, Silhavy proved that for \( F \in DM^1(U, \mathbb{R}^n) \) there exists a continuous linear functional on \( \text{Lip}(\partial \Omega) \), the space of Lipschitz continuous functions on \( \partial \Omega \), balancing the volume part of the Gauß formula. The following exposition proves that this functional can be represented by the sum of a Radon measure \( F' \) and a measure \( \mu_F \in (\mathcal{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n \) with core on the boundary. The arguments from Silhavy are retraced, in order to give a self-contained proof of the main theorem.

Throughout this section, for \( F \in (\mathcal{L}^\infty(U, \mathcal{L}^n))^n \) and \( V \subset U \) open, set

\[
\|F\|_{\infty, V} := \text{esssup}_V |F|.
\]

It is essential for the subsequent proofs to be able to compare the Lipschitz constant of a function by the norm of its gradient. The following lemma enables this comparison on balls.

**Lemma 4.1.** Let \( U \subset \mathbb{R}^n \) be open and \( f \in W^{1, \infty}(U, \mathbb{R}) \). Then for every \( x_0 \in U \) and \( 0 < \delta < \frac{1}{2} \text{dist}_{x_0}(\partial U) \) with \( B_\delta(x_0) \subset U \)

\[
\sup_{x, y \in B_\delta(x_0)} \frac{|f(x) - f(y)|}{|x - y|} \leq \|Df\|_{\infty, B_{2\delta}(x_0)}.
\]

**Proof.** Let \( \varepsilon < \delta \). For \( x \in B_\delta(x_0) \) set

\[
f_\varepsilon(x) := \int_{\mathbb{R}^n} \rho_\varepsilon(y - x) f(y) \, dy = \rho_\varepsilon * f(x),
\]
where \( \rho_\varepsilon \) is a scaled standard mollification kernel. Then as in Evans [11, p. 123]

\[ Df_\varepsilon = \rho_\varepsilon * Df. \]

Note that \( f_\varepsilon \to f \) point wise (cf. [11, p. 123]). Hence, for every \( x, y \in B_\delta (x_0) \) with \( x \neq y \)

\[ |f(x) - f(y)| = \lim_{\varepsilon \downarrow 0} |f_\varepsilon(x) - f_\varepsilon(y)| \]

\[ \leq \lim inf_{\varepsilon \downarrow 0} \|Df_\varepsilon\|_\infty |x - y|. \]

Now for every \( x \in B_\delta (x_0) \)

\[ |Df_\varepsilon(x)| \leq \int_{\mathbb{R}^n} |\rho_\varepsilon(y - x)||Df(y)||d\mathcal{L}^n \leq \|Df\|_{\infty, B_{\delta + \varepsilon}(x_0)}. \]

Thus

\[ \sup_{x, y \in B_\delta (x_0) \atop x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim inf_{\varepsilon \downarrow 0} \|Df\|_{\infty, B_{\delta + \varepsilon}(x_0)} \leq \|Df\|_{\infty, B_{2\delta}(x_0)}. \]

Once the estimate on balls is obtained, it is possible to prove the statement for path-connected sets.

**Lemma 4.2.** Let \( U \subset \mathbb{R}^n \) be open and \( C \subset U \) be compact and path-connected. Furthermore let \( 0 < \delta < \text{dist}_C(\partial U) \).

Then there exists \( c > 0 \) depending only on \( C \) and \( \delta \) such that for every \( f \in W^{1,\infty} (U, \mathbb{R}) \)

\[ \sup_{x, y \in C \atop x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leq c \|Df\|_{\infty, C_\delta}. \]

**Proof.** First, let \( \delta < \frac{1}{6} \text{dist}_C(\partial U) \). Note that

\[ C \subset \bigcup_{x \in C} B_\delta (x). \]

Since \( C \) is compact, there exists \( m \in \mathbb{N} \) and \( \{x_k\}_{k=0}^m \subset C \) such that

\[ C \subset \bigcup_{k=0}^m B_\delta (x_k). \]
Now, let \( x, y \in C \) with \( x \neq y \) be such that \( |x - y| < \delta \). Then
\[
y \in \mathcal{B}_{\delta}(x) \subset \mathcal{B}_{2\delta}(x) \subset C_{2\delta} \subset U.
\]

By Lemma 4.1
\[
\frac{|f(x) - f(y)|}{|x - y|} \leq \|Df\|_{\infty, \mathcal{B}_{2\delta}(x)} \leq \|Df\|_{\infty, C_{2\delta}}.
\]

Now, assume \( |x - y| \geq \delta \). Then there exists a continuous \( \gamma : [0, 1] \to C \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \). Let \( 0 \leq k \leq m \) such that
\[
x \in \mathcal{B}_{\delta}(x_k)
\]
and set
\[
t_0 := \sup\{t \in [0, 1] \mid \gamma(t) \in \mathcal{B}_{\delta}(x_k)\}.
\]
If \( \gamma(t_0) \in \mathcal{B}_{\delta}(x_k) \), then \( t_0 = 1 \) and \( y \in \mathcal{B}_{\delta}(x_k) \). Hence
\[
\frac{|f(x) - f(x)|}{|x - y|} \leq \|Df\|_{\infty, \mathcal{B}_{2\delta}(x)} \leq \|Df\|_{\infty, C_{2\delta}}.
\]
Otherwise, \( \gamma(t_0) \notin \mathcal{B}_{\delta}(x_k) \). But then there exists \( 0 \leq l \leq m \) and \( l \neq k \) such that
\[
\gamma(t_0) \in \mathcal{B}_{\delta}(x_l)
\]
and
\[
\gamma(t) \notin \mathcal{B}_{\delta}(x_k) \text{ for all } t \geq t_0.
\]
Set
\[
\gamma^0(t) = \begin{cases} 
  x + \frac{1}{t_0}(\gamma(t_0) - x) & \text{for } t \leq t_0 \\
  \gamma(t) & \text{otherwise.}
\end{cases}
\]

In essence, \( \gamma^0 \) is a shortcut in \( \overline{\mathcal{B}_{\delta}(x_k)} \) to the last point where \( \gamma \) is in \( \mathcal{B}_{\delta}(x_k) \).

Repeating the steps above, induction yields a continuous path \( \overline{\gamma} : [0, 1] \to \overline{C}_{\delta}, 0 \leq m' \leq m \) and \( \{t_l\}^{m'}_{l=1} \subset [0, 1] \) such that
\[
|\overline{\gamma}(t_l) - \overline{\gamma}(t_{l+1})| \leq 2\delta \text{ for } l = 0, ..., m' - 1
\]
and
\[
|x - \overline{\gamma}(t_0)| \leq 2\delta \text{ and } |y - \overline{\gamma}(t_{m'})| \leq 2\delta.
\]
Using Lemma 4.1 again for balls of radius $3\delta$

$$|f(x) - f(y)| \leq |f(x) - f(\gamma(t_0))| + \sum_{l=0}^{m'-1} |f(\gamma(t_l)) - f(\gamma(t_{l+1}))| + |f(\gamma_{m'}) - f(y)|$$

$$\leq \|DF\|_{\infty,B_{6\delta}(x)} |x - \gamma(t_0)| + \sum_{l=0}^{m'-1} \|DF\|_{\infty,B_{6\delta}(\gamma(t_l))} |\gamma(t_l) - \gamma(t_{l+1})|$$

$$+ \|DF\|_{\infty,B_{6\delta}(y)} |y - \gamma(t_{m'})|$$

$$\leq \|DF\|_{\infty,C_{6\delta}} 2\delta(m' + 2)$$

$$\leq 2(m + 2) \|DF\|_{\infty,C_{6\delta}} |x - y|.$$ 

Since $x, y \in C$ were arbitrary

$$\sup_{x, y \in C, x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leq 2(m + 2) \|DF\|_{\infty,C_{6\delta}}.$$

Note that $m$ only depends on $C$ and $\delta$. Finally, for $0 < \delta < \text{dist}_C(\partial U)$ set

$$\overline{\delta} := \frac{1}{6}\delta.$$

Then the inequality above yields

$$\sup_{x, y \in C, x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leq 2(m + 2) \|DF\|_{\infty,C_{\delta}}.$$

This finishes the proof.

$\square$

**Remark 4.3.** The requirement that $C$ is path-connected cannot be dropped. In order to see this, let $U := \mathbb{R}^2$ and

$$C := [-1, 1] \times \{-2, 2\}.$$

Let $f \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R})$ be a Lipschitz continuous function such that

$$f := \begin{cases} 1 & \text{on } [-2, 2] \times [1, 3] \\ 0 & \text{on } [-2, 2] \times [-3, -1] \end{cases}.$$

Then for $0 < \delta < 1$

$$\|DF\|_{\infty,C_{\delta}} = 0$$

but

$$\sup_{x, y \in C, x \neq y} \frac{|f(x) - f(y)|}{|x - y|} > 0.$$
Since the trace operator of Silhavy [19] is defined on the space of Lipschitz continuous functions, this space needs to be introduced now.

**Definition 4.4.** Let $\Omega \subset \mathbb{R}^n$. Let

$$\text{Lip}(\Omega)$$

 denote the set of all Lipschitz continuous functions on $\Omega$. For $f \in \text{Lip}(\Omega)$ set

$$\|f\|_{\text{Lip}} := \|f\|_C + \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$  

The following result is a slight variation of Lemma 3.2 in Silhavy [19, p. 451]. It states that the Gauß formula yields zero, if the scalar field is zero on the boundary.

**Proposition 4.5.** Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $F \in \mathcal{D}M^1(\Omega, \mathbb{R}^n)$ and $f \in \text{Lip}(\overline{\Omega})$ be such that $f|_{\partial \Omega} = 0$. Then

$$\int_{\Omega} F \cdot Df \, d\mathcal{L}^n + \int_{\Omega} f \, d\text{div} F = 0.$$  

**Proof.** First, suppose that $\text{supp} \, f \subset \subset \Omega$. By [5, p. 252] and Silhavy [19, p. 448] (cf. [5, p. 250])

$$\int_{\Omega} 1 \, d\text{div} (F \cdot f) = \int_{\Omega} f \, d\text{div} F + \int_{\Omega} F \cdot Df \, d\mathcal{L}^n = 0.$$  

For the general case, let

$$\chi_k := \chi_{\Omega - \mathbb{R}^n} + (k \text{dist}_{\partial \Omega} - 1) \chi_{\Omega - \mathbb{R}^n \setminus \Omega - \mathbb{R}^n} = \min \{1, \max \{0, k \text{dist}_{\partial \Omega} - 1\}\} \in \text{Lip}(\overline{\Omega}).$$

Then $f \cdot \chi_k \in \text{Lip}(\overline{\Omega})$ (cf. [8, p. 48]). In order to estimate the norm independently of $k \in \mathbb{N}$, let $x, y \in \Omega$. If $x, y \in \Omega - \mathbb{R}^n$ then

$$|f(x)\chi_k(x) - f(y)\chi_k(y)| = |f(x) - f(y)| \leq \|f\|_{\text{Lip}} |x - y|.$$  

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Otherwise, w.l.o.g. $x \in \Omega \setminus \Omega_{-\frac{2}{k}}$ and

$$|f(x) \cdot \chi_k(x) - f(y)\chi_k(y)| \leq |f(x)| |\chi_k(x) - \chi_k(y)| + |\chi_k(y)||f(x) - f(y)|$$

$$\leq \|f\|_{C(\Omega \setminus \Omega_{-\frac{2}{k}})} |\chi_k(x) - \chi_k(y)| + |f(x) - f(y)|$$

$$\leq \left( \sup_{0 \leq \text{dist}_{\partial \Omega}(x) \leq \frac{2}{k}} |f(x)| k + \|f\|_{\text{Lip}} \right) |x - y|.$$ 

Since $f$ vanishes on $\partial \Omega$

$$\sup_{0 \leq \text{dist}_{\partial \Omega}(x) \leq \frac{2}{k}} |f(x) - 0| \leq \|f\|_{\text{Lip}} \frac{2}{k},$$

whence

$$\|f \cdot \chi_k\|_{\text{Lip}} \leq 3 \|f\|_{\text{Lip}}.$$ 

Furthermore, for every $k \in \mathbb{N}$

$$\text{supp } f \cdot \chi_k \subset \subset \Omega.$$ 

Hence, for every $k \in \mathbb{N}$

$$\int_{\Omega} F \cdot D(f \cdot \chi_k) \, d\mathcal{L}^n + \int_{\Omega} f \cdot \chi_k \, d\operatorname{div} F = 0.$$ 

First note that

$$\int_{\Omega} f \cdot \chi_k \, d\operatorname{div} f \xrightarrow{k \to \infty} \int_{\Omega} f \, d\operatorname{div} F$$

by the Dominated Convergence Theorem (cf. [11, p. 20]).

On the other hand, since $\|D(f \cdot \chi_k)\|_{\infty} \leq \|f \cdot \chi_k\|_{\text{Lip}}$ is bounded independently of $k \in \mathbb{N}$ the Dominated Convergence Theorem also yields

$$\int_{\Omega} F \cdot D(f \cdot \chi_k) \, d\mathcal{L}^n \xrightarrow{k \to \infty} \int_{\Omega} F \cdot D(f) \, d\mathcal{L}^n.$$ 

Hence

$$\int_{\Omega} F \cdot D f \, d\mathcal{L}^n + \int_{\Omega} f \, d\operatorname{div} F \xrightarrow{k \to \infty} \int_{\Omega} F \cdot D(f \cdot \chi_k) \, d\mathcal{L}^n + \int_{\Omega} f \cdot \chi_k \, d\operatorname{div} F = 0.$$ 

$$\square$$

The following proposition is a specialised version of Theorem 2.3 in Silhavy [19, p. 448]. It states that the volume part of a Gauß formula only depends on the boundary values of the Lipschitz continuous scalar function.
Proposition 4.6. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $F \in \mathcal{DM}^1(\Omega, \mathbb{R}^n)$. Then there exists a continuous linear functional

$$\mathcal{NT}_F(\Omega) : \text{Lip}(\partial \Omega) \to \mathbb{R}$$

such that for every $f \in \text{Lip}(\Omega)$

$$\mathcal{NT}_F(\Omega)(f|_{\partial \Omega}) = \int_{\Omega} f \, d\text{div} F + \int_{\Omega} F \cdot Df \, d\mathcal{L}^n.$$

Furthermore

$$\|\mathcal{NT}_F(\Omega)\| \leq \|F\|_{\mathcal{DM}^1}.$$

Proof. The proof follows the same lines as the one in [19, p. 452]. Let $f \in \text{Lip}(\partial \Omega)$ and $f_1, f_2 \in \text{Lip}(\mathbb{R}^n)$ be extensions of $f$ to all of $\mathbb{R}^n$ (cf. [12, p. 201]). Note that $(f_1 - f_2)|_{\partial \Omega} = 0$. Then by Proposition 4.5

$$\int_{\Omega} f_1 \, d\text{div} F + \int_{\Omega} F \cdot Df_1 \, d\mathcal{L}^n = \int_{\Omega} f_2 \, d\text{div} F + \int_{\Omega} F \cdot Df_2 \, d\mathcal{L}^n.$$

For $f \in \text{Lip}(\partial \Omega)$ and any extension $\overline{f} \in \text{Lip}(\mathbb{R}^n)$ of $f$ define

$$\mathcal{NT}_F(\Omega)(f) := \int_{\Omega} \overline{f} \, d\text{div} F + \int_{\Omega} F \cdot D\overline{f} \, d\mathcal{L}^n.$$

Then $\mathcal{NT}_F(\Omega) : \text{Lip}(\partial \Omega) \to \mathbb{R}$ is well-defined and a linear functional. For $f \in \text{Lip}(\partial \Omega)$ there exists an extension $\overline{f} \in \text{Lip}(\mathbb{R}^n)$ such that

$$\|\overline{f}\|_{\text{Lip}} = \|f\|_{\text{Lip}}.$$

See Silhavy [19, p. 452] and Federer [12, p. 201] for reference. With this extension

$$|\mathcal{NT}_F(\Omega)(f)| \leq \|\text{div} F\|_{\Omega} \|\overline{f}\|_C + \|F\|_1 \|D\overline{f}\|_\infty$$

$$\leq \|F\|_{\mathcal{DM}^1} \|\overline{f}\|_{\text{Lip}}$$

$$= \|F\|_{\mathcal{DM}^1} \|f\|_{\text{Lip}}.$$

Up to now, the arguments from Silhavy [19] were retraced. Now, the representation of $\mathcal{NT}_F(\Omega)$ by the sum of a Radon measure and a measure $\mu_F \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ is proved. This result is new because it gives the abstract functionals found in the literature a concrete representation as integral functionals.
Theorem 4.7. Gauß Theorem

Let $U \subset \mathbb{R}^n$ be open, $\Omega \subset U$ be open with $\overline{\Omega} \subset U$ compact and $\partial \Omega$ path-connected. Furthermore, let $F \in \mathcal{D'}(U, \mathbb{R}^n)$.

Then there exists a Radon measure $F^\nu$ on $\partial \Omega$ and $\mu_F \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ with
\[
\text{core } \mu_F \subset \partial \Omega
\]
such that for all $f \in W^{1, \infty}(U, \mathbb{R})$ the following Gauß-Green formula holds
\[
\int_{\partial \Omega} f \, d F^\nu + \int_{\partial \Omega} D f \, d \mu_F = \int_{\Omega} f \, d \text{div } F + \int_{\Omega} F \cdot D f \, d \mathcal{L}^n.
\]

Remark 4.8. We think that the arguments in the proof can easily be adapted to prove the theorem for arbitrary open $\Omega \subset \subset U$.

Note that the existence of the measures in the above theorem is trivial, neglecting the core and the support, $\mu_F = F \mathcal{L}^n$ and $F^\nu = \text{div } F$ would be viable choices. The difficulty lies in the localisation of $\text{core } \mu_F \subset \partial \Omega$ and the support of $F^\nu$.

Proof. By Proposition 4.6 there exists a continuous linear functional $\mathcal{N} T_F(\Omega)$ on $\text{Lip}(\partial \Omega)$ such that for every $f \in W^{1, \infty}(U, \mathbb{R})$
\[
\mathcal{N} T_F(\Omega)(f|_{\partial \Omega}) = \int_{\Omega} f \, d \text{div } F + \int_{\Omega} F \cdot D f \, d \mathcal{L}^n.
\]

Let $0 < \delta < \text{dist}_\Omega(\partial U)$. Note that by [11, p. 131f] every $f \in W^{1, \infty}(U, [0,1])$ is locally Lipschitz continuous. Since $\overline{\Omega}$ is compact and path-connected, $f \in \text{Lip}(\overline{\Omega})$ (cf. Lemma 4.2). Then by Lemma 4.2 for every $f \in W^{1, \infty}(U, \mathbb{R})$
\[
\|f|_{\partial \Omega}\|_{\text{Lip}} \leq \|f|_{\partial \Omega}\|_C + c \|D f\|_{\infty,(\partial \Omega)_\delta},
\]
with $c > 0$ depending only on $\delta$ and $\partial \Omega$. Note that
\[
\iota : W^{1, \infty}(U, \mathbb{R}) \rightarrow C_0(\partial \Omega) \times \mathcal{L}^\infty((\partial \Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n) \text{ with } \iota(f) = (f|_{\partial \Omega}, D f|_{(\partial \Omega)_\delta})
\]
is continuous and linear. Set
\[
X_0 := \iota(W^{1, \infty}(U, \mathbb{R}))
\]
and define
\[
u_0^* : X_0 \to \mathbb{R} \text{ with } \langle \nu_0^*, (f|_{\partial \Omega}, D f|_{(\partial \Omega)_\delta}) \rangle = \mathcal{N} T_F(\Omega)(f|_{\partial \Omega}).
\]
Then $\nu_0^*$ is a continuous linear functional on the linear space $X_0 \subset C_0(\partial \Omega) \times \mathcal{L}^\infty((\partial \Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n)$ by equation (4).
By the Hahn-Banach Theorem (cf. [10, p. 63]) there exists a continuous linear extension $u^*$ of $u_0^*$ to all of $C_0(\Omega) \times \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n)$ with $||u^*|| = ||u_0^*||$. Note that the dual of a product space can be identified with the product of the dual spaces. Hence, as in Proposition 3.1 there exist a Radon measure $F^\nu$ on $\partial\Omega$ and a measure $\mu \in (\text{ba}((\partial\Omega)_\delta, \mathcal{B}((\partial\Omega)_\delta), \mathcal{L}^n))^n$ such that for all $f \in W^{1,\infty}(U, \mathbb{R})$

$$\mathcal{N}\mathcal{T}_F(\Omega)(f|_{\partial\Omega}) = \langle u^*, (f|_{\partial\Omega}, Df|_{(\partial\Omega)_\delta}) \rangle = \int_{\partial\Omega} f \, dF^\nu + \int_{(\partial\Omega)_\delta} Df \, d\mu.$$  

This proves that there is $\mu \in (\text{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ with core $\mu \subset (\partial\Omega)_\delta$ such that the above equation is satisfied. It remains to show that there exists $\mu_F$ with core $\mu_F \subset \partial\Omega$ satisfying the same equation. Now, let

$$X_1 := \{ \tilde{F} \in \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n) \mid \exists F \in W^{1,\infty}(U, \mathbb{R}) \exists F_1, F_2 \in \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n) : F = 0 \text{ on } (\partial\Omega)_{\tilde{\delta}} \text{ for some } 0 < \tilde{\delta} < \delta \}
\tilde{F} = Df + F \}$$

Then $u_1^* : X_1 \to \mathbb{R}$ with

$$u_1^*(\tilde{F}) := \int_{(\partial\Omega)_\delta} Df \, d\mu$$

defines a linear functional on $X_1$ with

$$||u_1^*|| \leq ||\mu||.$$  

First, it is shown that the definition is independent of the decomposition of $\tilde{F}$. Therefore, let $\tilde{F} \in X_1$ and $f_1, f_2 \in W^{1,\infty}(U, \mathbb{R}), F_1, F_2 \in \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n)$ be such that for some $0 < \tilde{\delta} < \delta$

$$F_1 = F_2 = 0 \text{ on } (\partial\Omega)_{\tilde{\delta}}$$

and

$$\tilde{F} = Df_1 + F_1 = Df_2 + F_2.$$  

Then

$$Df_1 = Df_2 \text{ on } (\partial\Omega)_{\tilde{\delta}}.$$  

Since $\partial\Omega$ is path-connected, $(\partial\Omega)_{\tilde{\delta}}$ is path-connected. Hence $f_1 - f_2$ is constant on $(\partial\Omega)_{\tilde{\delta}}$. Note that for $\tilde{c} \in \mathbb{R}$

$$\mathcal{N}\mathcal{T}_F(\Omega)(\tilde{c}) = \int_{\partial\Omega} \tilde{c} \, dF^\nu + \int_{(\partial\Omega)_\delta} 0 \, d\mu = \int_{\partial\Omega} \tilde{c} \, dF^\nu.$$  

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Hence Equation (5) yields
\[ \int_{(\partial\Omega)_\delta} D(f_1 - f_2) \, d\mu = \mathcal{N}T_F(\Omega)((f_1 - f_2)|_{\partial\Omega}) - \int_{\partial\Omega} f_1 - f_2 \, dF^\nu = 0. \]
This shows that \( u_1^* \) is well-defined.

Since \( X_1 \subset L^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n) \) is a linear subspace, the Hahn-Banach Theorem (cf. [10, p. 63]) yields an extension of \( u_1^* \) to all of \( L^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n) \) and by Proposition 3.1 a measure \( \mu_F \in (\text{ba}((\partial\Omega)_\delta, \mathcal{B}((\partial\Omega)_\delta), \mathcal{L}^n))^n \) with
\[ \langle u_1^*, \tilde{F} \rangle = \int_{(\partial\Omega)_\delta} \tilde{F} \, d\mu_F. \]
By definition,
\[ \int_{(\partial\Omega)_\delta} F \, d\mu_F = \langle u_1^*, 0 + F \rangle = 0 \]
for \( F \in L^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n) \) with \( F = 0 \) on \( (\partial\Omega)_\delta \) for some \( 0 < \delta < \delta \). Hence, \( \text{core} \mu_F \subset \partial\Omega \).

Since for every \( f \in W^{1,\infty}(U, \mathbb{R}) \)
\[ \int_{(\partial\Omega)_\delta} Df \, d\mu = \langle u_1^*, Df + 0 \rangle = \int_{(\partial\Omega)_\delta} Df \, d\mu_F \]
by definition, the statement of the theorem follows.

**Remark 4.9.** Note that the measure \( \mu_F \) is a direct result of the analysis. In regular settings, this measure is expected to be zero (see also the following examples). For \( F \in DM^\infty(U, \mathbb{R}^n) \) and open \( \Omega \in \mathcal{B}(U) \) having finite perimeter such that the inner normal measure exists (see Example 3.12), Proposition 3.21 and Proposition 2.27 yield the existence of a Radon measure \( F^\nu \) on \( \partial\Omega \) such that for all compactly supported continuous functions \( f \in BV(U) \)
\[ \text{div}(f \cdot F)(\Omega) = \int_{\partial\Omega} f \, dF^\nu. \]
In particular, \( \mu_F = 0 \).

For the general case note that for \( k \in \mathbb{N} \)
\[ \chi_k := \chi_{(\partial\Omega)_\delta} \frac{1}{k} (1 - k \text{dist}\partial\Omega) \in W^{1,\infty}(U, \mathbb{R}). \]
Since \( \chi_k = 1 \) on \( \partial\Omega \), Proposition 4.5 yields
\[ \int_{\Omega} f \chi_k \, d\text{div} F + \int_{\Omega} F \cdot D(f \cdot \chi_k) \, d\mathcal{L}^n = \int_{\Omega} f \, d\text{div} F + \int_{\Omega} F \cdot Df \, d\mathcal{L}^n. \]
Using Domated Convergence (cf. [11, p. 20]) yields
\[
\int_{\Omega} f \chi_k \, dF \xrightarrow{k \to \infty} 0
\]
\[
\int_{\Omega} F \cdot D(f \chi_k) \, d\mathcal{L}^n = \int_{\Omega} \chi_k F Df + f F D\chi_k \, d\mathcal{L}^n \xrightarrow{k \to \infty} 0 + \lim_{k \to \infty} \int_{\Omega} f F D\chi_k \, d\mathcal{L}^n,
\]
where the last limit exists because the other addends tend to zero and their sum is constant. Hence
\[
\lim_{k \to \infty} \int_{\Omega} f F D\chi_k \, d\mathcal{L}^n = \int_{\partial\Omega} f \, dF' + \int_{\partial\Omega} Df \, d\mu_F.
\]
Note that the left-hand side is essentially the same as in Schuricht [18, p. 534] (cf. [19, p. 449]).

The following examples proves, that $\mu_F$ can be non-zero. The function in the following example is the same as in [19, p. 449f].

**Example 4.10.** Let $n = 2$ and $U = B_2(0) \subset \mathbb{R}^2$. Furthermore, let $\Omega = (0, 1)^2$ and $F \in DM^1(U, \mathbb{R}^n)$ be defined via
\[
F(x, y) := \frac{1}{x^2 + y^2} \begin{pmatrix} y \\ -x \end{pmatrix}.
\]
Note that $\text{div} F$ is the zero measure. In order to see this, let $\phi \in C^1_0(U)$. Then
\[
\int_U F \cdot D\phi \, d\mathcal{L}^n = \lim_{\delta \downarrow 0} \int_{U \setminus B_\delta(0)} F \cdot D\phi \, d\mathcal{L}^n
= \lim_{\delta \downarrow 0} \int_{\partial B_\delta(0)} \phi F \cdot \nu \, d\mathcal{H}^{n-1} - \int_{U \setminus B_\delta(0)} \phi \, d\text{div} F \, d\mathcal{L}^n.
\]
But $F \cdot \nu = 0$ on $\partial B_\delta(0)$ and $\text{div} F = 0$ on $U \setminus B_\delta(0)$.

Now, set
\[
f_k := \chi_{(\frac{1}{k}, \infty)} \times \mathbb{R} + \chi_{(0, \frac{1}{k})} \times \mathbb{R} k \, \text{dist}_{\{0\}} \times \mathbb{R} \in W^{1, \infty}(U, [0, 1]) .
\]
Then
\[
Df_k = \chi_{(0, \frac{1}{k})} \times \mathbb{R} k e_1.
\]
By Theorem 4.7, there exists a Radon measure $F^\nu$ on $\partial \Omega$ and $\mu_F \in (\text{ba } (U, \mathcal{B}(U), \mathcal{L}^n))^n$ with core $\mu_F \subset \partial \Omega$ such that for $k \in \mathbb{N}$

$$\int_\Omega F \cdot Df_k \, d\mathcal{L}^n + \int_\Omega f_k \, d\text{div} F = \int_{\partial \Omega} Df_k \, d\mu_F + \int_{\partial \Omega} f_k \, dF^\nu.$$ 

But $f_k = 0$ on $\partial \Omega$ and $\text{div} F = 0$. Hence

$$\int_\Omega F \cdot Df_k \, d\mathcal{L}^n = \int_{\partial \Omega} Df_k \, d\mu_F.$$ 

Furthermore

$$\int_\Omega F \cdot Df_k \, d\mathcal{L}^n = \int_{(0,1)} \frac{y}{x^2 + y^2} \, dy \, dx$$

$$= \int_{(0,1)} \left[ \frac{1}{2} \ln(x^2 + y^2) \right]_0^1 \, dx$$

$$= \int_{(0,1)} \frac{1}{2} \ln \left( \frac{1}{x^2 + 1} \right) \, dx$$

$$\geq \frac{1}{2} \ln(k^2 + 1) \xrightarrow{k \to \infty} \infty.$$

The example above shows that $\mu_F$ can actually be non-zero, if the concentrations of the vector field $F$ are sufficiently large near $\partial \Omega$. Thus, $\mu_F$ is indeed necessary for the characterisation of the Gauss-Green formula.

**Example 4.11.** Revisiting Example 3.17 let $n = 2$ and $U := B_2(0) \subset \mathbb{R}^n$. Furthermore, let $\Omega := (0, 1) \times (-1, 1)$ and $F \in DM^1(U, \mathbb{R}^n)$ be defined by

$$F(x, y) := \frac{1}{2\pi x^2 + y^2} \left( \frac{x}{y} \right).$$

Recall that div $F = \delta_0$. For $k \in \mathbb{N}$ let $f_k \in W^{1,\infty}(U, [0, 1])$ be defined by

$$f_k := \chi_{(\frac{1}{k}, \infty)} \times \mathbb{R} + k \text{ dist}(0) \times \chi_{(0, \frac{1}{k})} \times \mathbb{R}.$$ 

Then

$$\int_\Omega F \cdot Df_k \, d\mathcal{L}^n = \int_{(0,1)} \frac{x}{2\pi(x^2 + y^2)} \, dy \, dx$$

$$= \frac{1}{2\pi} \int_{(0,1)} \left[ \arctan \frac{y}{x} \right]_1^1 \, dx$$

$$= \frac{1}{2\pi} \int_{(0,1)} 2 \arctan \frac{1}{x} \, dx$$

$$\xrightarrow{k \to \infty} \frac{1}{2}.$$
In contrast to the previous example, this example shows a vector field with a strongly concentrated divergence in zero, yet $\mu_F$ seems to be zero. Indeed, in [19, p. 449] Silhavy shows that the normal trace can be represented by a Radon measure, if

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{\Omega \setminus \Omega_{\delta}} |F \cdot D\text{dist}_{\partial \Omega}| \, d\mathcal{L}^n < \infty.$$ 

This holds true in the last example and thus $\mu_F = 0$.

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