Towards an efficient method to compute two-loop scalar amplitudes

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Abstract. We report on an ongoing work initiated by Prof. Shimizu, proposing a method to numerically compute two-loop scalar integrals as sums of two-dimensional integrals of generalised one-loop $N$-point functions analytically computed and integrated over some simple weight functions. The analytic computation of the generalised one-loop $N$-point functions in a systematic way motivates a novel approach sketched in this talk.

1. Introduction
After several years of run, the LHC delivered high quality data which push theorists to improve the accuracy of their predictions up to NNLO. A key ingredient in an automated evaluation of two-loop multileg processes is a fast and numerically stable evaluation of scalar Feynman integrals. The derivation of a fully analytic result remains beyond reach so far in the general mass case. On the opposite side, in particular for the calculation of two-loop three- and four-point functions in the general complex mass case relying on multidimensional numerical integration by means of sector decomposition \cite{1–5} a reliable result has a high computing cost. Approaches based on Mellin-Barnes techniques \cite{6–10} allow to perform part of the integrals analytically, yet, as far as we know, the number of integrals left over for numerical quadratures depends on the topologies considered and can remain rather costly. It would therefore be useful to perform part of the Feynman parameter integrations analytically in a systematic way to reduce the number of numerical quadratures.

2. Structure of a two-loop amplitude
Let us consider an arbitrary two-loop Feynman diagram with topology $\mathcal{T}$ involving $N$ external legs with external momenta $\{p_i, i = 1, \cdots, N\}$ and $I$ internal lines with internal masses $\{m_k^2, k = 1, \cdots, I\}$. To simplify we stick here to a scalar function. After the integration over the two loop momenta is performed, the integral representation of the diagram is given by:

\begin{equation}
^{(2)}I_N^n (\{p_i\}; \mathcal{T}) = \int_{(\mathbb{R}^+)^I} \left( \prod_{k=1}^I d\tau_k \right) \delta \left( 1 - \sum_{l=1}^I \tau_l \right) \left[ \text{det}(A) \right]^{I-\frac{d}{2}} \left[ \mathcal{F}(\{\tau_k\}) - i \lambda \right]^{n-I}
\end{equation}

Y. Shimizu passed away during the completion of this work.
with

\[ \mathcal{F}(\{\tau_k\}) = \left\{ \sum_{i,j=1}^{2} \text{Cof}[A]_{ij} (r_i \cdot r_j) \right\} - \det(A) \mathcal{C} \tag{2} \]

The matrix \( A \), the momenta \( r_i \) and the scalar function \( \mathcal{C} \) are defined by the form of the integrand before the integration over the loop momenta, namely:

\[ [k_1 \quad k_2] \cdot A \cdot \left[ \begin{array}{c} k_1 \\ k_2 \end{array} \right] + 2 \left[ r_1 \quad r_2 \right] \cdot \left[ \begin{array}{c} k_1 \\ k_2 \end{array} \right] + \mathcal{C} \]

and \( \text{Cof}[A] \) is the matrix of cofactors of \( A \). \( \mathcal{F} \) is homogeneous of degree 3 in the \( \tau_k \)'s and it depends on the external momenta \( \{p_j\} \), the internal masses \( \{m_k^2\} \) and the topology \( T \) of the diagram. The parametric representation (1) is the actual starting point of this article.

Let us partition the set of Feynman parameter labels \( \{1, \cdots, I\} \) into three subsets \( S_j \) and define three auxiliary parameters \( \rho_j \), \( j = 1, 2, 3 \) accordingly as follows: i) \( S_1 \) contains the labels of the internal lines involving only \( k_1 \) not \( k_2 \), to \( S_1 \) is associated \( \rho_1 \equiv \sum_{i \in S_1} \tau_i \); ii) \( S_2 \) contains the labels of internal lines involving only \( k_2 \) not \( k_1 \), to \( S_2 \) is associated \( \rho_2 \equiv \sum_{i \in S_2} \tau_i \); iii) \( S_3 \) contains the labels of internal lines common to the two overlapping loops. Each of these lines involves the same combination\(^5\) \( k_1 + k_2 \), so that the matrix element \( A_{12} \) weighting the scalar product \( (k_1 \cdot k_2) \) is equal to the combination \( \rho_3 \equiv \sum_{i \in S_3} \tau_i \). The \( \rho_j \)'s thus fulfil the constrain

\[ \rho_1 + \rho_2 + \rho_3 = \sum_{j=1}^{I} \tau_j = 1 \tag{3} \]

The elements of the matrix \( A \) read:

\[ A_{12} = \rho_3, \quad A_{11} = \rho_1 + \rho_3, \quad \text{and} \quad A_{22} = \rho_2 + \rho_3 \tag{4} \]

Hence:

\[ \det(A) = \rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1 \tag{5} \]

The determinant \( \det(A) \) is clearly non negative. Let \( |S_j| \) be the number of elements of \( S_j \), with \( |S_1| + |S_2| + |S_3| = I \). Let us introduce \( |S_j| \) parameters \( u_{kj} \) with \( k_j \in S_j \) so as to reparametrise the \( \tau_{kj} \) summing up into \( \rho_j \) as follows:

\[ \tau_{kj} = \rho_j u_{kj}, \quad \text{with the constraint} \quad \sum_{k_j \in S_j} u_{kj} = 1 \tag{6} \]

Accordingly the reparametrised integration measure takes the following factorised form:

\[ \left[ \prod_{j=1}^{I} d\tau_j \right] \delta \left( 1 - \sum_{i=1}^{I} \tau_i \right) = \prod_{k=1}^{3} \left\{ dp_k \rho_k |S_k|^{-1} \prod_{j_k \in S_k} \left[ du_{j_k} \delta \left( 1 - \sum_{i \in S_k} u_i \right) \right] \right\} \delta \left( 1 - \sum_{i=1}^{3} \rho_i \right) \tag{7} \]

\(^{5}\) It could alternatively involve \( k_1 - k_2 \) in every internal line common to the two overlapping loops, depending on the convention adopted for the orientations of the loop momenta.
With this reparametrisation, the elements of the $A$ matrix depend only on the parameters $\rho_j$ and on none of the $u_i$'s, so do $\text{CoF}[A]$ and $\det(A)$. In $\mathcal{F}$, the dependence in the $u_i$'s enters through the factors $(r_i \cdot r_j)$, quadratically, and through the term $C$, linearly. The term $\mathcal{F}$ may thus be seen as a polynomial of second degree in the $u_i$'s and can thus be interpreted as building up the integrand of a “generalised” one-loop function represented as a Feynman integral over the $u_i$'s. The integral representation of the two-loop diagram $(2)I^\rho_N(\{p_j\}; \mathcal{T})$ can thus be recast in the following form:

$$(2)I^\rho_N(\{p_j\}; \mathcal{T}) = \int_{(\mathbb{R}^+)^3} \prod_{k=1}^3 d\rho_k \rho_k^{S_k-1} \delta \left( 1 - \sum_{l=1}^3 \rho_l \right) \left[ \rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1 \right]^{1-\frac{3n}{2}} \left(1\right)\tilde{I}^\rho_N(1)$$

where we have introduced

$$(1)\tilde{I}^\rho_N = \int_{(\mathbb{R}^+)^3} \prod_{k=1}^3 \prod_{j \in S_k} du j \delta \left( 1 - \sum_{l \in S_k} u_l \right) \left[ \mathcal{F}(\{u_k\}, \{\rho_l\}) - i \lambda \right]^{n-I} \left(9\right)$$

with $\mathcal{F}(\{u_k\}, \{\rho_l\}) = \mathcal{F}(\{\tau_i(\{u_k\}, \{\rho_l\})\})$ and we have set $N' = I - 2$ and $n' = 2(n - 2)$. The reparametrisation of $(2)I^\rho_N(\{p_j\}; \mathcal{T})$ according to eqs. (8), (9) has already been used in the literature [11–14] in order to perform the integration over all Feynman parameters fully numerically. We alternatively wish to advocate here the separate identification of $(1)\tilde{I}^\rho_N'$ in eq. (9) with $n - I = N' + n' / 2$ as a $N'$-point function of “generalised one-loop type” in $n'$ dimensions, and the possibility to compute $(1)\tilde{I}^\rho_N'$ analytically.

The above qualitative “generalised one-loop type” refers to two kinds of generalisations.

1. After integrating over three of the $u_i$'s in order to eliminate the $\delta(1 - \sum_{l \in S_k} u_l)$-constraints, the effective kinematics of the “generalised” one-loop $N'$-point function in $n'$ dimensions is encoded in a $(I - 3) \times (I - 3)$ matrix $G = G(\{p_j\}, \{\rho_l\})$, a column $(I - 3)$-vector $V = V(\{p_j\}, \{\rho_l\})$ and a scalar function $C = C(\{p_j\}, \{\rho_l\})$, all of which functions of the external momenta $\{p_j\}$ and of the integration variables $\{\rho_k\}$ seen as external parameters. Let us note that this effective kinematics of the “generalised” one-loop function depends on the $\rho_j$ seen as “external” parameters beside the external momenta $p_k$'s, and that it may span a larger parameter space than the one involved in standard one-loop $N'$-point functions involved in collider processes at one loop.

2. Unlike for the standard one-loop function, the integration domain of the parameters $u_k$'s is not the usual $(I - 3)$-simplex defined by $\Sigma_{(I-3)} = \{u_k \geq 0, k = 1, \cdots, I - 3\mid \sum_{k=1}^{I-3} u_k = 1\}$ but instead the polysimplicial set $\Sigma \times \Sigma_{(I-3)} \times \Sigma_{(I-3)}$. The quantity $\mathcal{F}$ formally reads:

$$\mathcal{F} = U^T \cdot G \cdot U - 2 V^T \cdot U - C \left(10\right)$$

where $U$ is the column $(I - 3)$-vector gathering the yet unintegrated $(I - 3)$ variables $u_k$ parametrising the polysimplicial integration domain $\Sigma_{(I-3)} \times \Sigma_{(I-3)} \times \Sigma_{(I-3)}$.

3. “Generalised” one-loop functions

Although long-tested standard techniques developed for the genuine one-loop case [15,16] might be customised to treat the new ones at hand, the two issues mentioned in the previous section motivate the development of a novel approach which tackles both these issues in a systematic way.

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6 The polysimplicial set depends on the topology $\mathcal{T}$ of the two-loop diagram considered. It is understood that, in case some of the $|S_j|$ equals 1, the corresponding trivial set factor $\Sigma_{(I-3)}$ shall be omitted.
and straightforward way while computing the “generalised 1-loop type functions”. This novel method is described in details in ref. [17] for the real mass case and has been extended for complex masses in ref. [18] and for infrared divergent cases in ref. [19].

The key point of this novel method is an extensive use of a “Stokes-type” identity:

\[
\frac{1}{D^{\alpha+1}\{\{u_k\}\}} = \frac{1}{2\alpha \Delta_d} \left[ \frac{d-2\alpha}{D^\alpha\{\{u_k\}\}} - \nabla_u^T \left( \frac{U - G^{-1} V}{D^\alpha\{\{u_k\}\}} \right) \right]
\]  

(11)

where \(D\{\{u_k\}\}\) is a second order polynomial in the \(d\) variables \(u_k\) which has the structure defined in eq. (10) and the quantity \(\Delta_d\) is given by:

\[
\Delta_d = V^T G^{-1} V + C
\]

The relation (11) is not very useful unless the power \(\alpha\) is such that \(d-2\alpha = 0\), in this case only the boundary term remains which enables to perform one integration trivially. In general, the power with which the polynomial \(D\{\{u_k\}\}\) appears in a one-loop \(N\)-point function is not equal to \(d/2\). So we need a formula to shift the power of \(D\) given by:

\[
\int_0^\infty \frac{d\xi}{(D + \xi^\nu)^\mu} = \frac{1}{\nu} B \left( \frac{1}{\nu}, \mu - \frac{1}{\nu} \right) \frac{1}{D^{\mu-1/\nu}}
\]  

(12)

where \(B(x, y)\) is the Euler beta function, provided that \(1/\nu, \mu\) and \(\mu - 1/\nu\) are non-negative integers.

As a proof of concept, the three- and four-point one-loop function have been recalculated yet with systematic analytic continuation to arbitrary kinematics i.e. not only those restricted to one-loop processes. As an example, let us sketch the computation of a one-loop four-point function, more details are given in refs. [17–19]. After the integration over the loop momentum, the one-loop four-point function can be written as:

\[
^{(1)}I_4^4 = \sum_{i \in S_4} \sum_{j \in S_4 \setminus \{i\}} \sum_{k \in S_4 \setminus \{i,j\}} \bar{b}_i \bar{b}_j^{\{i\}} \bar{b}_k^{\{i,j\}} \frac{1}{\det(G) \det(G^{[i]} \det(G^{[i,j]})} \times L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk})
\]  

(14)

\[
^{(1)}I_4^4 = \sum_{i \in S_4} \sum_{j \in S_4 \setminus \{i\}} \sum_{k \in S_4 \setminus \{i,j\}} \bar{b}_i \bar{b}_j^{\{i\}} \bar{b}_k^{\{i,j\}} \frac{1}{\det(G) \det(G^{[i]} \det(G^{[i,j]})} \times L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk})
\]  

(15)

\[7\) Depending on the parametrisation, it exists several Gram matrices but they all have the same determinant.
with
\[
L_f^4(\Delta_3, \Delta_2^{ij}, \Delta_1^{ij}, \tilde{D}_{ijk})
= \kappa \int_{0}^{+\infty} \frac{d\xi}{(\xi^2 - \Delta_3 - i\lambda)} \int_{0}^{+\infty} \frac{d\rho}{(\rho^2 + \xi^2 - \Delta_2^{ij} - i\lambda)} \int_{0}^{+\infty} \frac{d\sigma}{(\sigma^2 + \rho^2 + \xi^2 - \Delta_1^{ij} - i\lambda)(\sigma^2 + \rho^2 + \xi^2 - \tilde{D}_{ijk} - i\lambda)^{1/2}}
\]
(16)

where:
\[
\Delta_3 = V^T \cdot G^{-1} \cdot V + C \quad \text{and} \quad \bar{b}_i = (G^{-1} \cdot V)_i
\]
(18)

The quantities \(\Delta_2^{ij}\) (resp. \(\Delta_1^{ij}\)) and the coefficients \(\bar{b}_j^{(i)}\) (resp. \(\bar{b}_k^{(ij)}\)) obey to formulae of the same type as eq. (18) but which now involve the reduced Gram matrices \(G^{(i)}\) and the reduced vectors \(V^{(i)}\) (resp. the reduced Gram matrices \(G^{(ij)}\) and the reduced vectors \(V^{(ij)}\)). These reduced matrices (resp. vectors) are constructed from the original Gram matrix (resp. vector \(V\)) by removing the line(s) and the column(s) (resp. the line(s)) whose label(s) belong(s) to the set defined in the exponent. The integration variables \(\xi, \rho\) and \(\sigma\) are the three extra variables introduced to adjust the power of the denominator. One can argue that we have progressed next to nothing but the integrals defined in eq. (17) are simpler concerning the analyticity than the original integral over the simplex. The \(\Delta\)'s and the \(\tilde{D}\)'s are complex numbers in the case of complex masses and the result of the integrations will depend on the sign of the imaginary parts of these latter quantities. There are eight cases to distinguish but they share a common structure. Let us give two examples.

1) case \(\text{Im}(\Delta_3) < 0, \text{Im}(\Delta_2) > 0, \text{Im}(\Delta_1^{ij}) > 0, \text{Im}(\tilde{D}_{ijk}) < 0\)

\[
L_f^4(\Delta_3, \Delta_2^{ij}, \Delta_1^{ij}, \tilde{D}_{ijk})
= - \left\{ \int_{0}^{1} \frac{du}{u^2 P_{ijk} Q_i - R_{ij}} \right. \left[ - \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln \left( T (P_{ijk} + R_{ij}) (R_{ij} + Q_i) \right) \right] + \left. \int_{0}^{1} \frac{du}{u^2 P_{ijk} Q_i - R_{ij}} \right. \left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln \left( (R_{ij} + Q_i) \right) \right) \right. \left. - \ln (u^2 Q_i + T) \right. \left. - \ln \left( T \left( P_{ijk} + R_{ij} \right) \right) \right) + \eta \left( (R_{ij} + Q_i) \right) \right) \right. \left. - \eta \left( T \left( P_{ijk} + R_{ij} \right) \right) \right) - \int_{1}^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij}} \eta \left( T \left( P_{ijk} + R_{ij} \right) \right) \right\}
\]
(19)

where the quantities \(T, Q_i, R_{ij}\) and \(P_{ijk}\) are linear combinations of \(\Delta_3, \Delta_2^{ij}, \Delta_1^{ij}\) and \(\tilde{D}_{ijk}\).
2) case $\text{Im}(\Delta_3) > 0$, $\text{Im}(\Delta_2^{(i)}) > 0$, $\text{Im}(\Delta_1^{(ij)}) > 0$, $\text{Im}(\tilde{D}_{ijk}) < 0$

\[
L^i_1(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(ij)}, \tilde{D}_{ijk}) = - \int_0^1 du \int_0^1 P_{ijk} Q_i - R_{ij} T \left[ \ln\left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln\left( \frac{(R_{ij} + Q_i)}{Q_i} (Q_i + T) \right) \right. \\
- \ln\left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln\left( T \frac{(P_{ijk} + R_{ij}) (R_{ij} + Q_i)}{P_{ijk}} \right) + \ln\left( u^2 Q_i + T \right) - \ln\left( T \frac{P_{ijk} (P_{ijk} + R_{ij})}{P_{ijk}} \right) + \eta \left( \frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T) \right) - \eta \left( T \frac{(P_{ijk} + R_{ij}) (R_{ij} + Q_i)}{P_{ijk}} \right) \left. \right] \tag{20}
\]

The structure of eq. (19) and eq. (20) are similar. Note that in the former equation, the contour of integration of the first term is no more along the real axis but has to be deformed in order to prevent of being crossed by the cut of the logarithm $\ln(u^2 (P_{ijk} + R_{ij} + Q_i) + T)$. Using Cauchy’s theorem, the contour deformation can be written as a part along the imaginary positive axis plus another part along the real axis between 1 and $+\infty$.

4. Conclusion

We sketched a novel method to compute the scalar massive two-loop $N$-point functions using analytically computed one-loop building blocks. A long way shall still be scouted out to extend it to full-fledged two-loop tensor integrals appearing in general gauge theories.

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