ALGEBRAIC THEORIES OF BRACKETS AND RELATED (CO)HOMOLOGIES

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Abstract A general theory of the Frölicher–Nijenhuis and Schouten–Nijenhuis brackets in the category of modules over a commutative algebra is described. Some related structures and (co)homology invariants are discussed, as well as applications to geometry.

Keywords Frölicher–Nijenhuis bracket · Schouten–Nijenhuis bracket · Poisson structures · Integrability · Nonlinear differential equations · Hamiltonian formalism · Algebraic approach

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1 Introduction

Bracket structures play an important role in classical differential geometry (see, for example, Refs. [3,4,19,20,21,22,23]), Poisson geometry (e.g., [17]), and the theory of integrable systems (Refs. [5,6,9]). Being initially of a geometrical nature, these brackets found exact counterparts in abstract algebra, in the framework of Vinogradov’s theory of algebraic differential operators, [25] (see also book [15]). It became clear that many geometrical constructions (such as the ones we meet in Hamiltonian mechanics or in partial differential equations; cf. with Refs. [27] and [7], respectively) may be more or less exactly expressed using the language of commutative algebra.

In this paper, I collected together my old results on the algebraic theory of the Frölicher–Nijenhuis and Schouten–Nijenhuis brackets and related homological and cohomological theories (for shortness, I call these brackets Nijenhuis and Schouten ones). These results were initially published in papers [7,8,9,13]. The results exposed below are easily generalized to the case of super-commutative algebras (see [10]) and, being slightly modified, can be incorporated in Lychagin’s “colored calculus” (see Ref. [16]).

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To simplify exposition, I shall always assume that the algebra $A$ is such that the module $A^1(A)$ (see Sec. 3.2) of 1-forms is projective and of finite type.

2 A general scheme

This scheme was first presented in Ref. [26]. Let $k$ be a field, $\text{char} k \neq 2$. Let also $P = \sum_{k \in \mathbb{Z}} P_k$, $Q = \sum_{k \in \mathbb{Z}} Q_k$ be graded vector spaces and $Q$ be endowed with a differential $d : Q_k \to Q_{k+1}$, $d^2 = 0$.

Assume that there exists a graded monomorphism

$$\varphi : P \to \text{Hom}_k^\varphi(Q, Q), \quad \varphi(P_\alpha) \subset \text{Hom}_k^{\alpha+\beta}(Q, Q), \quad \beta = \text{gr} \varphi,$$

and define the “Lie derivative”

$$L^\varphi_p = [d, \varphi(p)], \quad p \in P.$$

Here and everywhere below $[\cdot, \cdot]$ denotes the graded commutator. If we are lucky then for two elements $p \in P_\alpha, p' \in P_{\alpha'}$ we can define their $\varphi$-bracket by

$$L^\varphi_{[p, p']} = [L^\varphi_p, L^\varphi_{p'}], \quad [p, p'] \in P_{\alpha + \alpha' + \beta + 1}.$$

In some interesting cases we are lucky indeed.

3 Algebraic calculus

Let us introduce the basic notions of the calculus over commutative algebras that will be needed below (see Refs. [12, 15, 25] for details).

3.1 Differential operators

Consider a unitary commutative associative $k$-algebra $A$ and $A$-modules $P$ and $Q$.

**Definition 1** A $k$-linear map $\Delta : P \to Q$ is a differential operator (DO) of order $\leq k$ if

$$[a_0, [a_1, \ldots, [a_k, \Delta], \ldots]] = 0$$

for all $a_0, \ldots, a_k \in A$.

The set of all DOs $P \to Q$ forms an $A$-bimodule denoted by $\text{Diff}_A(P, Q)$. An operator $X : A \to P$ is called a $P$-valued derivation if

$$X(ab) = aX(b) + bX(a), \quad a, b \in A.$$

The module of these derivations is denoted by $D_1(P)$. Define by induction the modules

$$D_i(P) = \{ X \in D_1(D_{i-1}(P)) \mid X(a, b) + X(b, a) = 0 \}, \quad i \geq 2,$$

and set formally $D_0(P) = P$. Elements of $D_1(P)$ are called multiderivations.

**Remark 1** Let $M$ be a smooth manifold and $A = C^\infty(M)$. Let also $\pi$ and $\xi$ be vector bundles over $M$ and $P = \Gamma(\pi)$, $Q = \Gamma(\xi)$ be the modules of their smooth sections. Then Definition 1 gives the classical notion of a linear differential operator.
3.2 Differential forms

**Proposition 1** The correspondence $P \Rightarrow D_i(P)$ is a representable functor from the category of $A$-modules to itself. The corresponding representative object is denoted by $\Lambda^i = \Lambda^i(A)$ and called the module of differential $i$-forms of the algebra $A$. In particular, there exists a natural derivation $d: A \rightarrow \Lambda^1$ such that any derivation $X \in D_1(P)$ uniquely decomposes as $X = \psi_X \circ d$, where $\psi_X \in \text{Hom}_A(\Lambda^1, P)$. The module $\Lambda^1$ is generated by the elements of the form $da$, $a \in A$, while $\Lambda^i$ are exterior powers of $\Lambda^1$. This leads to the complex

\[0 \rightarrow A \xrightarrow{d} \Lambda^1 \rightarrow \cdots \rightarrow \Lambda^i \xrightarrow{d} \Lambda^{i+1} \rightarrow \cdots\]

called the de Rham complex of $A$.

3.3 Exterior products

Due to the above formulated proposition, the module $\Lambda^* = \sum_i \Lambda^i$ is a Grassmannian algebra with the exterior, or wedge product

$\wedge: \Lambda^i \otimes_A \Lambda^j \rightarrow \Lambda^{i+j}$.

A similar operation

$\wedge: D_i(A) \otimes_A D_j(P) \rightarrow D_{i+j}(P)$

is introduced by induction in $D_*(P) = \sum_i D_i(P)$. Namely, for $i + j = 0$ we set

$a \wedge p = ap$, \quad $a \in D_0(A) = A, \quad p \in D_0(P) = P$,

and

$(X \wedge Y)(a) = X \wedge Y(a) + (-1)^j X(a) \wedge Y, \quad i + j > 0, \quad i > 0$,

$X \in D_i(A), Y \in D_j(P)$. In this way, $D_*(A)$ becomes a Grassmannian algebra, $D_*(P)$ being a module over $D_*(A)$.

3.4 Inner product

The inner product

$i: D_i(P) \otimes_A \Lambda^j \rightarrow \begin{cases} P \otimes_A \Lambda^{i-j}, & j \geq i, \\ D_{i-j}(P), & j \leq i, \end{cases}$

is defined by induction. If $i = 0$ we set

$i(p \otimes \omega) = p \otimes \omega$, \quad $p \in P = D_0(P), \quad \omega \in \Lambda^j$,

and for $j = 0$ we set

$i(X \otimes a) = aX$, \quad $a \in A = \Lambda^0, \quad X \in D_i(P)$.

If $i, j > 0$ we set

$i(X \otimes da \wedge \omega) = i(X(\omega) \otimes \omega)$.

We shall use the notation

\[i_X \omega = \begin{cases} i(X \otimes \omega), & i \geq j, \\ 0, & i < j \end{cases}, \quad i_\omega X = \begin{cases} i(X \otimes \omega), & i \leq j, \\ 0, & i > j. \end{cases}\]
Remark 2 When $P = \Lambda^k$ and $j \geq i$, the inner product may be completed to the following operation

$$D_i(\Lambda^k) \otimes_A \Lambda^j \rightarrow \Lambda^k \otimes_A \Lambda^{j-i} \rightarrow \Lambda^{k+j-i}$$

which will be also called the inner product.

Remark 3 Let $X \in D_i(P)$ and $\omega \in \Lambda^j$. Then the maps

$$i_X : \Lambda^* \rightarrow P \otimes_A \Lambda^*$$

and

$$i_\omega : D_\ast(P) \rightarrow D_\ast(P)$$

are super-differential operators of order $i$ and $j$, respectfully.

4 The Schouten bracket

We define here the Schouten bracket, describe its properties and related (co)homologies. Some applications are also discussed.

4.1 Definition and existence

Let $X \in D_i(A)$. Consider the Lie derivative

$$L_X = d \circ i_X - (-1)^i i_X \circ d = [d, i_X] : \Lambda^i \rightarrow \Lambda^{i-i}.$$

Theorem 1 For any two elements $X \in D_i(A)$ and $X' \in D_{i'}(A)$ there exists a uniquely defined element $[[X, X']]^* \in D_{i+i'-1}(A)$ such that

$$[L_X, L_{X'}] = L_{[[X, X']]^*}.$$

This element is called the Schouten bracket of $X$ and $X'$.

Proof We establish existence of $[[X, X']]^*$ by induction. For $i' = 0$ we set

$$[[X, a]]^* = X(a), \quad a \in A = D_0(A),$$

and similarly for $i = 0$

$$[[a, X']]^* = (-1)^i X'(a).$$

If $i, i' > 0$ we set

$$[[X, X']]^*(a) = [[X, X'(a)]]^* + (-1)^{i'} [[X(a), X']]^*.$$

It is easily checked that thus defined bracket enjoys the needed property.
4.2 Properties

**Proposition 2** Let $X, X', X'' \in D_*(A)$ be multiderivations of degree $i, i'$ and $i''$ respectively. Then:

1. $[X, X']^s + (-1)^{(i-1)(i'-1)} [X', X]^s = 0.$
2. $[X, [X', X'']^s] = [[X, X']^s, X'']^s + (-1)^{(i-1)(i'-1)} [X', [X, X'']^s],$
3. $[X, X' \wedge X''] = [X, X']^s \wedge X'' + (-1)^{(i-1)(i'-1)} X' \wedge [X, X'']^s,$
4. $[X, X''] = [X, X'],$ if $i = i' = 1,$
5. $i_{[X, X']^s} = [L X, i_{X'}].$

5 Poisson structures

To any bivector $\mathcal{P} \in D_2(A),$ one can put into correspondence a skew-symmetric bracket $\{a, b\}_\mathcal{P} = \mathcal{P}(a, b), a, b \in A.$

**Proposition 3** The following statements are equivalent:

1. $\{a, b\}_\mathcal{P}$ satisfies the Jacobi identity;
2. $[\mathcal{P}, \mathcal{P}]^s = 0;$
3. $\partial_{\mathcal{P}} \circ \partial_{\mathcal{P}} = 0,$ where $\partial_{\mathcal{P}} = [[\mathcal{P}, \cdot]]^s.$

**Definition 2** If one of the previous conditions fulfills then:

1. $\mathcal{P}$ is called a Poisson structure and a pair $(A, \mathcal{P})$ is a Poisson algebra.
2. $\{\cdot, \cdot\}_\mathcal{P}$ is the Poisson bracket associated with $\mathcal{P}.$
3. $\mathcal{P}(a) \in D_1(A)$ are Hamiltonian derivations.
4. Derivations $X$ satisfying $X \{a, b\}_\mathcal{P} = \{X a, b\}_\mathcal{P} + \{a, X b\}_\mathcal{P}$ are canonical derivations.

5.1 Example: algebraic $T^*$ (see Ref. [27])

Let $\text{Diff}_k(A) = \bigcup_{k \geq 0} \text{Diff}_k(A)$ denote the algebra of all DOs $A \rightarrow A.$ For any $\Delta \in \text{Diff}_k(A)$ the coset $[\Delta]_k = \Delta \mod \text{Diff}_{k-1}(A)$ is called its symbol.

If $s_1 = [\Delta_1]_{k_1}, s_2 = [\Delta_2]_{k_2}$ are two symbols we define their product by

$$s_1 \cdot s_2 = [\Delta_1 \circ \Delta_2]_{k_1 + k_2}$$

and their bracket by

$$\{s_1, s_2\} = [\Delta_1 \circ \Delta_2 - \Delta_2 \circ \Delta_1]_{k_1 + k_2 - 1}.$$

In such a way we obtain the algebra of symbols

$$S_*(A) = \sum_k \frac{\text{Diff}_k(A)}{\text{Diff}_{k-1}(A)}.$$

**Proposition 4** The above introduced algebra of symbols $S_*(A)$ is a graded commutative algebra with a graded Poisson bracket $\{\cdot, \cdot\}.$ In the case $A = C^\infty(M)$ it coincides with the algebra of smooth functions on $T^*M$ polynomial along the fibers, while the bracket is the one defined by the canonical symplectic form $\Omega = dp \wedge dq.$
Remark 4 The parallel between geometrical constructions and the corresponding algebraic modules is even deeper, though perhaps not so straightforward. As an example, let us describe how the canonical form $\rho = pdq$ is defined within the model under consideration (other illustrations can be found, e.g., in Refs. [11][14]).

Note first that exactly in the same way as it was done above one can define symbols of arbitrary operators $A \in \text{Diff}_r(P, Q)$. Moreover, under the assumption of Sec. 1, one has the isomorphism
\[ S_\omega(P, Q) = S_\omega(A) \otimes A \text{Hom}_A(P, Q). \] (1)

Now, to define a 1-form, one needs to evaluate all derivations on this form. Let $X : S_\omega(A) \rightarrow R$ be such a derivation, $R$ being an $S_\omega(A)$-module. Since $A = S_0(A) \subset S_\omega(A)$, one can consider the restriction $X = X|_A : A \rightarrow R$. Due to Eq. (1), one has
\[ [X] \in S_\omega(A, R) = S_\omega(A) \otimes_A R, \]
and we set
\[ i_X \rho = \mu_S(X), \]
where $\mu_S : S_\omega(A) \otimes_A R \rightarrow R$ is the multiplication. Consequently, we can define the form $\Omega = d\rho$, but for general algebras it may be degenerate, contrary to the geometric case.

Remark 5 It may be also appropriate to discuss another parallel here. Namely, in geometry, 1-forms are identified with sections of the cotangent bundle. In algebra, the notion of section is generated by arbitrary operators $s \in \Delta$. Therefore, any element $s \in S_\omega(A)$ is of the form $s = \sum [X_{a_1}] \cdots [X_{a_k}]$, where $X_{a}$ are derivations. Then we set
\[ \varphi_\omega(s) = \sum i_{X_{a_1}}(\omega) \cdots i_{X_{a_k}}(\omega). \]
Conversely, let $\varphi : S_\omega(A) \rightarrow A$ be a homomorphism. To define the corresponding 1-form $\omega_\varphi$, we need to evaluate an arbitrary derivation $X : A \rightarrow P$ at it, where $P$ is an $A$-module. But
\[ [X] \in S_\omega(A, P) = S_\omega(A) \otimes_A P, \]
and we set
\[ i_X(\omega_\varphi) = \mu_A(\varphi \otimes \text{id}_P([X])), \]
where $\mu_A : A \otimes_k P \rightarrow P$ is the multiplication.

5.2 Poisson cohomologies

Let $(A, \mathcal{P})$ be a Poisson algebra. The sequence
\[ 0 \rightarrow A \xrightarrow{\partial_\rho} D_1(A) \xrightarrow{\partial_\rho} \cdots \xrightarrow{\partial_\rho} D_i(P) \xrightarrow{\partial_\rho} D_{i+1}(P) \xrightarrow{\partial_\rho} \cdots, \]
where $\partial_\rho = [\mathcal{P}, \cdot]$; is the Poisson complex of $A$ and its cohomologies $H^i(A, \mathcal{P})$ are the Poisson cohomologies.

**Proposition 5** 1. $H^0(A; \mathcal{P})$ consists of Casimirs of $\mathcal{P}$ and coincides with the Poisson center of $A$. 

2. $H^1(A; \mathcal{P}) = \text{Can}(\mathcal{P})/\text{Ham}(\mathcal{P})$, where $\text{Can}(\mathcal{P})$ is the space of canonical derivations and $\text{Ham}(\mathcal{P})$ consists of the Hamiltonian ones.

3. $H^2(A; \mathcal{P})$ coincides with the set of classes of nontrivial infinitesimal deformations of the Poisson structure $\mathcal{P}$.

4. $H^3(A; \mathcal{P})$ contains obstructions to prolongation of infinitesimal deformations up to formal ones.

5.3 Poisson homologies (see Ref. [2])

Take a Poisson algebra $(A, \mathcal{P})$ and consider the operator

$$d_{\mathcal{P}} = L_{\mathcal{P}} = [d, i_{\mathcal{P}}] : \Lambda^j \rightarrow \Lambda^{j-1}.$$ 

By definition of the Poisson structure, one has

$$2d_{\mathcal{P}} \circ d_{\mathcal{P}} = [d_{\mathcal{P}}, d_{\mathcal{P}}] = [L_{\mathcal{P}}, L_{\mathcal{P}}] = L_{[\mathcal{P}, \mathcal{P}]} = 0$$

and one gets the complex

$$\cdots \rightarrow \Lambda^j \xrightarrow{d_{\mathcal{P}}} \Lambda^{j-1} \rightarrow \cdots \rightarrow \Lambda^1 \xrightarrow{d_{\mathcal{P}}} A \rightarrow 0,$$

whose homologies $H_j(A, \mathcal{P})$ are called the Poisson homologies of $(A, \mathcal{P})$. The action of $d_{\mathcal{P}}$ is fully defined by the following two properties:

$$d_{\mathcal{P}}(\omega \wedge \omega') = (d_{\mathcal{P}}\omega) \wedge \omega' + (-1)^j \omega \wedge d_{\mathcal{P}}\omega'$$

and

$$d_{\mathcal{P}}(ab) = \{a, b\}_{\mathcal{P}}, \quad a, b \in A.$$ 

5.4 Hamiltonian filtrations

Let $\mathcal{H}^1 \subset D_0(A)$ be the ideal generated by Hamiltonian derivations. Let

$$\mathcal{H}^p = \underbrace{\mathcal{H}^1 \wedge \cdots \wedge \mathcal{H}^1}_{p \text{ times}}$$

be its powers. Then

$$D_*(A) = \mathcal{H}^0 \supset \mathcal{H}^1 \supset \cdots \supset \mathcal{H}^p \supset \mathcal{H}^{p+1} \supset \cdots$$

is a filtration that generates a spectral sequence for Poisson cohomologies. In a dual way, the filtration

$$0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_p \subset \mathcal{H}_{p+1} \subset \cdots$$

where

$$\mathcal{H}_p = \{ \omega \in \Lambda^* | i_{X_1} \cdots (i_{X_p}(\omega)) \cdots = 0 \ \forall X_i \in \text{Ham}(\mathcal{P}) \}$$

gives rise to a spectral sequence for Poisson homologies.
6 Extended Poisson bracket

The Poisson bracket defined by a Poissonian bi-vector $\mathcal{P}$ can be extended to a super-bracket on the Grassmannian algebra $\Lambda^*(A)$.

Consider the differential $\partial_{\mathcal{P}} : D_i(A) \to D_{i+1}(A)$ and a form $\omega \in \Lambda^j$. Then we have the map

$$L_{\omega}^{\mathcal{P}} = [\partial_{\mathcal{P}}, i_\omega] : D_i(A) \to D_{i-j+1}(A).$$

**Proposition 6** For any two forms $\omega \in \Lambda^j$, $\omega' \in \Lambda^j$ the equality

$$i_{(\omega, \omega')}^{\mathcal{P}} = [L_{\omega}^{\mathcal{P}}, i_{\omega'}]$$

uniquely determines a form $\{\omega, \omega'\}^{\mathcal{P}} \in \Lambda^{j+i-1}$, which is called their Poisson bracket.

**Proposition 7** The Poisson bracket of forms enjoys the following properties:

1. $\{ab\}^{\mathcal{P}} = -\{b,a\}^{\mathcal{P}}$.
2. $\{da, db\}^{\mathcal{P}} = d\{a, b\}^{\mathcal{P}}$.
3. $\{\omega, \omega' \wedge \omega''\}^{\mathcal{P}} = \{\omega, \omega'\}^{\mathcal{P}} \wedge \omega'' + (-1)^{(j-1)f} \omega' \wedge \{\omega, \omega''\}^{\mathcal{P}}$.
4. $\{\omega, \{\omega', \omega''\}^{\mathcal{P}}\}^{\mathcal{P}} = \{\{\omega, \omega'\}^{\mathcal{P}}, \omega''\}^{\mathcal{P}} + (-1)^{(j-1)(f-1)} \{\omega', \{\omega, \omega''\}^{\mathcal{P}}\}^{\mathcal{P}}$.
5. $\{\omega, \omega'\}^{\mathcal{P}} = -(-1)^{(j-1)(f-1)} \{\omega', \omega\}^{\mathcal{P}}$.
6. $L_{\{\omega, \omega'\}^{\mathcal{P}}} = [L_{\omega}, L_{\omega'}]^{\mathcal{P}}$.

Note that the first three properties may be taken for the constructive definition of the extended bracket.

7 Commuting structures

Two Poisson structures $\mathcal{P}$ and $\mathcal{P}'$ commute, or are compatible if $\|\mathcal{P}, \mathcal{P}'\|^s = 0$. This is equivalent to

$$\partial_{\mathcal{P}} \circ \partial_{\mathcal{P}'} = 0$$

or to the fact that $\mu \mathcal{P} + \mu' \mathcal{P}'$ (the Poisson pencil) is a Poisson structure for all $\mu$, $\mu' \in \mathbb{k}$.

The Magri scheme (see Ref [13]) that establishes existence of infinite series of commuting conservation laws for bi-Hamiltonian systems has an exact algebraic counterpart:

**Theorem 2** Let $A$ be an algebra with two commuting Poisson structures $\mathcal{P}$ and $\mathcal{P}'$ and assume that $H^1(A; \mathcal{P}') = 0$. Assume also that two elements $a_1, a_2 \in A$ are given, such that $\partial_{\mathcal{P}}(a_1) = \partial_{\mathcal{P}'}(a_2)$. Then:

1. There exist elements $a_3, \ldots, a_{l} \in A$ satisfying

$$\partial_{\mathcal{P}}(a_i) = \partial_{\mathcal{P}'}(a_{i+1}).$$

2. All elements $a_1, \ldots, a_{l}$ are in involution with respect to both Poisson structures, i.e.,

$$\{a_{\alpha}, a_{\beta}\}^{\mathcal{P}} = \{a_{\alpha}, a_{\beta}\}^{\mathcal{P}'} = 0$$

for all $\alpha, \beta \geq 1$. 

8 The Nijenhuis bracket

Consider a form-valued derivation $\Omega \in D_1(A^k)$ and the Lie derivative

$$L_{\Omega} = [d, i_{\Omega}]: A^j \to A^{k+j},$$

where $i_{\Omega}$ is defined by the composition

$$D_1(A^k) \otimes_A A^j \xrightarrow{i} A^k \otimes_A A^j \to A^{k+j}.$$

**Proposition 8** The above Lie derivative possesses the following properties:

1. $L_{\Omega}(\omega \wedge \omega') = L_{\Omega}(\omega) \wedge \omega' + (-1)^{kj} \omega \wedge L_{\Omega}(\omega'),$
2. $[L_{\Omega}, d] = 0,$
3. $L_{\omega \wedge \Omega} = \omega \wedge L_{\Omega} + (-1)^{kj} d\omega \wedge i_{\Omega}.$

Here $\omega \in A^j$, $\omega' \in A^{j'}, \Omega \in D_1(A^k).$

Other basic properties of the Nijenhuis bracket are presented in the following

**Proposition 9** Let $\Omega \in D_1(A^j)$, $\Omega' \in D_1(A^{j'})$, $\Omega'' \in D_1(A^{j''})$, and $\omega \in A^j$. Then:

1. $[[\Omega, \Omega'', \omega]]^n = (-1)^{jj'}[\Omega', \omega]' = 0,$
2. $[[\Omega, [\Omega', \omega]]^n] = [[[\Omega, [\Omega', \omega]]^n, [\Omega'', \omega]]^n] = 0,$
3. $[[\Omega, \omega \wedge [\Omega', \omega]']^n = L_{\Omega}(\omega) \wedge [\Omega', \omega]' - (-1)^{jj'} \omega \wedge \omega' \wedge i_{\Omega} = (-1)^{jj'} \omega \wedge L_{\Omega}(\omega') + (-1)^{jj'} [\Omega', \omega'] \wedge i_{\Omega}.$
4. $[L_{\Omega}, i_{\Omega}'] = (-1)^{jj'} [L_{\Omega}, \omega] + i_{\Omega} \omega.$
5. $i_{\Omega}[[\Omega', \omega]']^n = i_{\Omega}[[\Omega', \omega]'^n] = [i_{\Omega} \omega', \omega]'^n = 0.$

On decomposable elements the Nijenhuis bracket acts as follows. Let $\omega \in A^j$, $\omega' \in A^{j'}, X, X' \in D_1(A).$ Then

$$[[\omega \wedge X, \omega' \wedge X']^n = \omega \wedge \omega' \wedge [X, X'] + L_{\omega \wedge X}(\omega') \wedge X' = (-1)^{jj'} L_{\omega \wedge X}(\omega') \wedge X,$$

$$+ \omega \wedge \omega' \wedge X - [L_{\omega \wedge X}(\omega') \wedge X' - L_{\omega \wedge X}(\omega') \wedge \omega' \wedge X$$

$$+ (-1)^{ij} d\omega \wedge i_{X}(\omega') \wedge X' + (-1)^{ij} i_{X}(\omega') \wedge d\omega' \wedge X.$$

A derivation $\mathcal{N} \in D_1(A^1)$ is called integrable if

$$[[\mathcal{N}', \mathcal{N}]]^n = 0.$$

With any integrable derivation one can associate a complex

$$0 \to D_1(B) \xrightarrow{\partial_x} \cdots \to D_1(A^j(B)) \xrightarrow{\partial_y} D_1(A^{j+1}(B)) \to \cdots$$

where $\partial_x = [[\mathcal{N}', \cdot]]^n.$

Such structures, in particular, arise in an algebraic model of flat connections.
9 Flat connections

Let $A$ and $B$ be $k$-algebras and $\gamma: A \to B$ be a homomorphism. Then $B$ is an $A$-algebra and one can consider the module $D_1(A,B)$ of $B$-valued derivations $A \to B$. For any $X \in D_1(B)$ denote by $X|_A \in D_1(A,B)$ its restriction to $A$.

A connection is a $B$-homomorphism $\nabla: D_1(A,B) \to D_1(B)$ such that $\nabla(X)|_A = X$. A vector-valued form $U \in D_1(B)$ defined by

$$i_X(U \nabla) = X - \nabla(X|_A), \quad X \in D_1(B),$$

is called the connection form. For any two derivations $X, X' \in D_1(A,B)$ we set

$$R_\nabla(X, X') = [\nabla(X), \nabla(X')] - \nabla([X, X']) - \nabla(X') \circ X - \nabla(X) \circ X';$$

$R_\nabla$ is the curvature of $\nabla$. A connection is flat if $R_\nabla = 0$.

9.1 Nijenhuis cohomologies associated to a connection

**Theorem 3** Let $\nabla$ be a connection. Then

$$i_X(i_X([U_\nabla, U_\nabla]|^n)) = 2R_\nabla(X|_A, X'|_A)$$

for any $X, X' \in D_1(B)$.

Hence, to any flat connection, i.e., to a connection whose curvature vanishes, we associate a Nijenhuis complex with $\mathcal{N} = U_\nabla$. In applications, its vertical subcomplex is useful:

$$0 \to D_1^0(B) \overset{\partial}{\to} D_1^1(B) \overset{\partial}{\to} D_1^2(B) \overset{\partial}{\to} \cdots,$$

where $D_1^p(P) = \{ X \in D_1(P) \mid X|_A = 0 \}$. Denote its cohomologies by $H^i(B, \nabla)$.

9.2 Nijenhuis cohomologies: $H^0, H^1, and \ H^2$

**Theorem 4** Let $\nabla$ be a flat connection. Then:

1. The cohomology groups $H^i(B, \nabla)$ inherit the inner product operation,

   $$i: H^i(B, \nabla) \times H^j(B, \nabla) \to H^{i+j-1}(B, \nabla).$$

   In particular, the group $H^1(B, \nabla)$ is an associative algebra represented in endomorphisms of $H^0(B, \nabla)$:

   $$i: H^1(B, \nabla) \times H^1(B, \nabla) \to H^1(B, \nabla),$$

   $$i: H^1(B, \nabla) \times H^0(B, \nabla) \to H^0(B, \nabla).$$

2. The cohomology groups $H^i(B, \nabla)$ inherit the Nijenhuis bracket,

   $$[[\cdot, \cdot]]^n: H^i(B, \nabla) \times H^j(B, \nabla) \to H^{i+j}(B, \nabla).$$

   In particular, $H^0(B, \nabla)$ is a Lie algebra:

   $$[[\cdot, \cdot]]^n: H^0(B, \nabla) \times H^0(B, \nabla) \to H^0(B, \nabla).$$
9.3 Application to differential equations

Let \( \mathcal{E} \subset J^\infty(\pi) \) be an infinitely prolonged differential equation in the jet bundle of a bundle \( \pi: E \to M \). The bundle \( \pi_{\infty}: \mathcal{E} \to M \) is always endowed with a natural flat connection \( \mathcal{E} \) (the Cartan connection, see Refs [115]) and taking

\[
\gamma = \pi_{\infty}^*: A = C^\infty(M) \to B = C^\infty(\mathcal{E})
\]

we obtain the picture considered above.

Let us use the notation \( H^j(\mathcal{E}, \mathcal{E}) \) for the cohomology groups arising in this case.

9.4 The main result

**Theorem 5** For any formally integrable equation \( \mathcal{E} \) that surjectively projects to \( J^0(\pi) \) one has:

1. \( H^0(\mathcal{E}, \mathcal{E}) \) coincides with the Lie algebra sym\( \mathcal{E} \) of higher symmetries of \( \mathcal{E} \).
2. Elements of \( H^1(\mathcal{E}, \mathcal{E}) \) act on sym\( \mathcal{E} \) and thus are identified with recursion operators for symmetries.
3. On the other hand, elements of \( H^1(\mathcal{E}, \mathcal{E}) \) can be understood as classes of nontrivial infinitesimal deformations of the equation structure.
4. \( H^2(\mathcal{E}, \mathcal{E}) \) contains obstructions to prolongation of infinitesimal deformations up to formal ones.

9.5 Commutative hierarchies

Let \( (B, V) \) be an algebra with flat connection. For \( X = X_0 \in H^0(B, V) \) and \( R \in H^1(B, V) \), use the notation \( R(X) = i_X(R), X_n = R^n(X), n = 0, 1, 2\ldots \)

**Theorem 6** Assume that \( H^2(B, V) = 0 \). Then for any \( X, Y \in H^0(B, V) \) and \( R \in H^1(B, V) \) for all \( m, n \in \mathbb{Z}_+ \) one has

\[
[X_m, Y_n] = [X, Y]_{m+n} + \sum_{j=0}^{n-1} ([X, R]^n(Y_j))_{m+n-j-1} - \sum_{j=0}^{m-1} ([Y, R]^n(X_j))_{m+n-j-1}.
\]

**Corollary 1** If \( \|[X, R]^n = [Y, R]^n = 0 \) and \( [X, Y] = 0 \) then \( [X_m, Y_n] = 0 \) for all \( m, n \in \mathbb{Z}_+ \).

**Remark 6** If \( \mathcal{E} \) is a scalar evolutionary equation of order \( > 1 \) then \( H^2(\mathcal{E}, \mathcal{E}) = 0 \).

9.6 Bi-complex

Let \( \mathcal{N} \in D_1(A^1) \) be an integrable element, i.e., \( \|[\mathcal{N}, \mathcal{N}] = 0 \). Then the operator

\[
d_{\mathcal{N}} = L_{\mathcal{N}}: A^{j} \to A^{j+1}
\]

is a differential: \( d_{\mathcal{N}} \circ d_{\mathcal{N}} = 0 \). Moreover, one has

\[
[d, d_{\mathcal{N}}] = 0
\]

and consequently the pair \( (d_{\mathcal{N}}, \tilde{d}_{\mathcal{N}}) \), where \( \tilde{d}_{\mathcal{N}} = d - d_{\mathcal{N}} \), constitutes a bi-complex that converges to the de Rham cohomologies of \( B \).

In the case of differential equations (\( A = C^\infty(M) \), \( B = C^\infty(\mathcal{E}) \), and \( \mathcal{N} \) is the connection form of the Cartan connection in the bundle \( \pi_{\infty}: \mathcal{E}^{\infty} \to M \)), this bi-complex coincides with the variational bi-complex, or Vinogradov’s \( \mathcal{C} \)-spectral sequence, see Ref. [24][28].
10 More brackets…

To conclude, note that several more brackets can be constructed in a similar way.

1. First, mention the Nijenhuis–Richardson bracket

\[
\llbracket \cdot, \cdot \rrbracket^f: D_1(A^j) \otimes_A D_1(A^j) \to D_1(A^{j+j-1})
\]

that can be defined by

\[
[i_\Omega, i_\Omega'] = i_{[\Omega, \Omega']^f}
\]

and is of the form

\[
\llbracket \Omega, \Omega' \rrbracket^f = i_\Omega(\Omega') - (-1)^{(i-1)(j-1)}i_{\Omega'}(\Omega)
\]

and is one of the classical and well known brackets.

Two more brackets arise also if we fix a Poisson structure \( \mathcal{P} \in D_2(A) \) or a Nijenhuis

structure \( \mathcal{N} \in D_1(A) \):

2. Consider the inner product

\[
i: D_i(A^1) \otimes_A D_k(A) \to D_{i+k-1}(A).
\]

Then for \( \Omega \in D_i(A^1) \) the following “Lie derivative” arises:

\[
L^\mathcal{P}_{\Omega} = [\partial_{\mathcal{P}}, i_\Omega]: D_k(A) \to D_{k+i}(A)
\]

and one can introduce a bracket

\[
\llbracket \cdot, \cdot \rrbracket^\mathcal{P}: D_i(A^1) \times D_k(A^1) \to D_{i+k}(A^1)
\]

by

\[
L^\mathcal{P}_{[\Omega, \Omega]} = [L^\mathcal{P}_\Omega, L^\mathcal{P}_{\Omega'}].
\]

3. In a similar way, one can consider the inner product

\[
i: D_1(A^i) \otimes_A D_1(A^k) \to D_1(A^{i+k-1})
\]

and the “Lie derivative”

\[
L^\mathcal{N}_\Omega = [\partial_{\mathcal{N}}, i_\Omega]: D_1(A^i) \to D_1(A^{k+i}),
\]

\( \Omega \in D_1(A^i) \). Then a new bracket

\[
\llbracket \cdot, \cdot \rrbracket^\mathcal{N}: D_i(A^i) \times D_1(A^i) \to D_1(A^{i+i})
\]

is defined by

\[
L^\mathcal{N}_{[\Omega, \Omega']} = [L^\mathcal{N}_\Omega, L^\mathcal{N}_{\Omega'}].
\]
11 . . . and when brackets fail to arise

One can also define the inner products

\[ i: D^i(A^j) \otimes_A A^k \rightarrow A^{k+j-i} \]

and

\[ i: D^i(A^j) \otimes_A D_k(A) \rightarrow D_{k-j+i}(A) \]

together with the corresponding Lie actions

\[ L_{\Omega} = [d, i_\Omega]: A^k \rightarrow A^{k+j-i+1} \]

and

\[ L_{\Omega}^\rho = [\partial_\rho, i_\Omega]: D_k(A) \rightarrow D_{k-j+i+1}(A), \]

where \( \Omega \in D^i(A^j) \). Of course, it is tempting to find the elements \([\Omega, \Omega']\] and \([\Omega, \Omega']^\rho\) such that

\[ L_{[\Omega, \Omega']} = [L_\Omega, L_{\Omega'}], \quad L_{[\Omega, \Omega']}^\rho = [L_\Omega^\rho, L_{\Omega'}^\rho], \]

but in general such elements do not exist (see discussion of these matters in Ref. [26]).

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