SYMmetric and Exterior Squares of Hook Representations

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Abstract. We determine the multiplicities of irreducible summands in the symmetric and the exterior squares of hook representations of symmetric groups over an algebraically closed field of characteristic zero.

1. Introduction

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero and $n$ a positive integer. Denote by $M^\lambda$ the irreducible right $\mathbb{F}$-representation of the symmetric group $S_n$ corresponding to the Young diagram $\lambda \vdash n$. Let $V = M^{(n-1,1)}$ be the standard representation of degree $n-1$.

In [3, Thm. 1.2] J. B. Remmel determined the multiplicities of irreducible summands of the tensor square of the so called hook representations $\Lambda^k V \cong M^{(n-k,1^k)}$, for all $n,k \in \mathbb{N}^+$. The factors appearing in the decomposition are either hook representations i.e. $\lambda_2 \leq 1$ or double hook representation i.e. $\lambda_3 \leq 2$ (but $\lambda_2 > 1$).

In this paper, we refine this decomposition by determining the multiplicities of irreducible factors in the representations $\text{Sym}^2(\Lambda^k V)$ and $\Lambda^2(\Lambda^k V)$.

Consider $\text{Sym}^2(\Lambda^k V)$ and $\Lambda^2(\Lambda^k V)$ as complementary subspaces of $(\Lambda^k V)^\otimes 2$. We show the following:

**Theorem 1.1.** Let $\lambda \vdash n$ be a Young diagram and $M^\lambda$ be an irreducible summand of $(\Lambda^k V)^\otimes 2$. Then

- if $\lambda = (q,p,2^{d_2},1^{d_1})$ where $q \geq p \geq 2$ and
  - $d_1 \equiv 0 \mod 4$ then every $M^\lambda$ factor is contained in $\text{Sym}^2(\Lambda^k V)$,
  - $d_1 \equiv 2 \mod 4$ then every $M^\lambda$ factor is contained in $\Lambda^2(\Lambda^k V)$,
  - $2 \nmid d_1$ then the multiplicity of $M^\lambda$ is 1 in $\text{Sym}^2(\Lambda^k V)$ and in $\Lambda^2(\Lambda^k V)$,
- if $\lambda = (n-m,1^m)$ where $0 \leq m \leq n-1$ and
  - $m \equiv 0$ or $1 \mod 4$ then every $M^\lambda$ factor is contained in $\text{Sym}^2(\Lambda^k V)$,
  - $m \equiv 2$ or $3 \mod 4$ then every $M^\lambda$ factor is contained in $\Lambda^2(\Lambda^k V)$.

In particular, the multiplicities depend only on the modulo 4 value of $d_1$, the tail of the Young diagram, given the multiplicities in the tensor square $(\Lambda^k V)^\otimes 2$.

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Our approach may be summarized as follows. If $d_1$ is even, we consider a subspace $A$ in the covering representation:

$$A \subseteq (\Lambda^k(V \oplus 1))^\otimes 2 \xrightarrow{f} (\Lambda^kV)^\otimes 2$$

where $1 = M^{(n)}$ denotes the trivial representation, and $f$ is the natural projection induced from $V \oplus 1 \to V$. Denote by $c_\lambda$ the Young symmetrizer corresponding to the standard Young tableau of $\lambda$. Let $T$ be the flip of the two tensor components. The subspace is chosen so that $f|_A$ is surjective and

$$(1.1) \quad ac_\lambda = \varepsilon_\lambda T(a)c_\lambda \quad (\forall a \in A)$$

for some $\varepsilon_\lambda \in \{−1, +1\}$ depending on the mod 4 value of $d_1$.

If $ac_\lambda = −T(a)c_\lambda$ then $M^\lambda$ cannot be a summand of $\text{Sym}^2(\Lambda^kV)$, and similarly for $\Lambda^2(\Lambda^kV)$ in the positive case. The statements in Theorem 1.1 are exactly of this form if $d_1$ is even, so it is sufficient to prove such skew-symmetry relations as in Eq. 1.1. These relations are proved by an induction-type argument, starting from the diagrams $\lambda = (2, 2, 2)$ and $\lambda = (2, 2, 1, 1)$, using combinatorial arguments on colored Young diagrams, see Lemma 3.3.

If $d_1$ is odd then we use the branching rule of $S_n$ representations, and an induction-restriction argument to derive the result from the even case and from Remmel’s theorem, recalled in Theorem 4.4.

Since the multiplicities are known for $(\Lambda^kV)^\otimes 2$, we get:

**Corollary 1.2.** Let $\lambda \vdash n$ be a Young diagram. Then the multiplicities of $M^\lambda$ in $\text{Sym}^2(\Lambda^kV)$ (resp. $\Lambda^2(\Lambda^kV)$) are the following:

- if $\lambda = (q, p, 2d_2, 1^{d_1})$ is a double hook for some $q \geq p \geq 2$, then  
  - $2$, if $|2k+1-n| \leq q−p$ and $d_1 \equiv 0$ (resp. $2$) mod 4,
  - $1$, if $|2k+1−n| \leq q−p$ and $2 \nmid d_1$,
- $1$, if $|2k+1−n| = q−p + 1$ and $d_1 \equiv 0$ (resp. $2$) mod 4,

- if $\lambda = (n, m, 1^m)$ is a hook where $0 \leq m \leq 2 \min(k, n−k−1)$ and $m \equiv 0$ or $1$ mod 4 (resp. $m \equiv 2$ or $3$ mod 4),

- $0$ otherwise.

Let us note that the decomposition of tensor products is not known for all irreducible $S_n$-representations, or at least not in a combinatorially tractable way. A recent result in [1] describes the case where one of the terms is a hook representation. About the symmetric and exterior squares even less is known. In [4] it is characterized whether an irreducible representation is a quotient of its own exterior square, assuming its Young diagram is of height two, of width two, or a hook diagram.

The article is organized as follows: In Section 2 we introduce the notation on bases in the product representations, moreover we derive some observations on how the tensor-component flipping $T$, and the dualization $P$ act on the subrepresentations of $(\Lambda^k(V \oplus 1))^\otimes 2$. We also introduce the notion of proper swaps in Lemma 2.4 to simplify calculations in the subsequent sections. In Section 3 (resp. 4) we prove the case of Theorem 1.1 when $d_1$ is even (resp. odd), see Prop. 3.1 (resp. Prop. 4.1). The case of hook representations is a by-product of the argument in Sec. 3 (see Cor. 3.13).
2. Preliminaries

Let \( n \in \mathbb{N}^+ = \{1, 2, \ldots\} \) be fixed and consider the \( n \)-dimensional permutation representation of the symmetric group \( \mathfrak{S}_n \):

\[
U \overset{\text{def}}{=} V \oplus 1
\]

where \( V = M^{(n-1,1)} \) is the standard and \( 1 = M^{(n)} \) is the trivial representation. The standard \( \mathbb{F} \)-basis elements of \( \Lambda^k U \otimes \Lambda^l U \) for any \( k, l \in \mathbb{N}^+ \) are denoted as

\[
(2.1) \quad u_I \otimes u_J = (u_{i_1} \land \cdots \land u_{i_k}) \otimes (u_{j_1} \land \cdots \land u_{j_l})
\]

where \( I = \{i_1, \ldots, i_k\} \) and \( J = \{j_1, \ldots, j_l\} \) for some \( 1 \leq i_1 < \cdots < i_k \leq n \) and \( 1 \leq j_1 < \cdots < j_l \leq n \). The action of \( s \in \mathfrak{S}_n \) is defined as

\[
(u_I \otimes u_J) s = (u_{i_1 s} \land \cdots \land u_{i_k s}) \otimes (u_{j_1 s} \land \cdots \land u_{j_l s}).
\]

Consequently, the basis of \( \Lambda^k U \otimes \Lambda^l U \) defined above may be indexed with 4-colorings as follows. Define the set of all 4-colorings as

\[
X \overset{\text{def}}{=} \{ x : [n] \to \{0, 1, 2, 3\} \}
\]

where \([n] = \{1, 2, \ldots, n\}\) and \( n \) is assumed to be fixed, hence omitted from the notation. Then let

\[
X_{k,l} \overset{\text{def}}{=} \{ x \in X \mid |x^{-1}(\{1, 3\})| = k, \mid x^{-1}(\{2, 3\})| = l \}.
\]

We claim that there is a bijection between the given basis of \( \Lambda^k U \otimes \Lambda^l U \) and \( X_{k,l} \), based on the four subsets \( I \cap J, I \setminus J, J \setminus I \) and \([n] \setminus (I \cup J)\). Indeed, for any \( x \in X_{k,l} \) define

\[
w_x \overset{\text{def}}{=} u_{x^{-1}(\{1,3\})} \otimes u_{x^{-1}(\{2,3\})} \in \Lambda^k U \otimes \Lambda^l U.
\]

Clearly, \( \{ w_x \mid x \in X_{k,l} \} \) is the standard basis of \( \Lambda^k U \otimes \Lambda^l U \) as defined in Eq. 2.1.

Define the right action of \( \mathfrak{S}_n \) on \( X \) as

\[
x s = (m \mapsto x(ms^{-1})) \quad (m \in [n])
\]

for all \( x \in X \). Note that even though there is a bijection on the \( \mathfrak{S}_n \)-sets \( \{ w_x \mid x \in X_{k,l} \} \) and \( X_{k,l} \subseteq X \), \( w_x s \) does not necessarily equal the basis element \( w_{xs} \). Instead,

\[
(2.2) \quad w_{xs} = \varepsilon_{x,s} w_{xs}
\]

for an appropriate choice of \( \varepsilon_{x,s} \in \{1, -1\} \) for any \( x \in X_{k,l} \) and \( s \in \mathfrak{S}_n \).

More explicitly, we may express these signs using inversion numbers as

\[
(2.3) \quad \varepsilon_{x,s} = (-1)^{N_1(x,s) + N_2(x,s)}
\]

where

\[
(2.4) \quad N_c(x,s) = \left\{ \{(p, q) \in [n]^2 \mid p < q, \ \text{ps} > qs, \ x(ps), x(qs) \in \{c, 3\}\} \right\}
\]

for \( c \in \{1, 2\} \). Indeed, if \( w_x = u_I \otimes u_J \) as above, then

\[
u_{I s} = u_{i_1 s} \land \cdots \land u_{i_k s} = (-1)^{N_1(x,s)} u_{I s}
\]

where \( I_s = \{i s \mid i \in I\} \), since \( N_1(x,s) \) is the inversion number of the permutation required to sort the sequence \((i_1 s, \ldots, i_k s)\) increasingly. Similarly, \( u_{J s} = (-1)^{N_2(x,s)} u_{J s} \), hence Eq. 2.3 holds.
2.1. **Color-switch (12).** Denote by \( t = (1, 2) \) the transposition of 1 and 2 on the set \( \{0, 1, 2, 3\} \). Then \( x \mapsto t \circ x \) gives a bijection \( X \to X \) that commutes with the \( \mathfrak{S}_n \)-action i.e. \( (t \circ x) s = t \circ (x s) \) for any \( s \in \mathfrak{S}_n \).

We will need the following elementary properties of \( \varepsilon_{x,s} \) defined in Eq. 2.2:

**Lemma 2.1.** Let \( x \in X \) and \( s \in \mathfrak{S}_n \). Then

1. \( \varepsilon_{tox,s} = \varepsilon_{x,s} \),
2. If \( s \) is a transposition \((i, j)\) such that \( x(i) = x(j) \in \{1, 2\} \) then \( \varepsilon_{x,s} = -1 \),
3. If \( s \) is a transposition \((i, j)\) such that \( x(i) = x(j) \in \{0, 3\} \) then \( \varepsilon_{x,s} = 1 \).

**Proof.** Let \( N_1(x, s) \) and \( N_2(x, s) \) as in Eq. 2.4. Then \( N_2(t \circ x, s) = N_1(x, s) \) and \( N_1(t \circ x, s) = N_2(x, s) \) by definition, hence \( \varepsilon_{tox} = \varepsilon_{x,s} \) holds.

For the second statement, it is enough to prove the case of \( x(i) = x(j) = 1 \) and \( i < j \) by symmetry. Then we may note that \( N_2(x, s) = 0 \) and

\[
N_1(x, s) = 1 + 2\{m \in [n] | i < m < j, x(m) \in \{1, 3\}\}
\]

hence \( \varepsilon_{x,s} = -1 \). The proof of the last statement follows similarly. \( \square \)

**Corollary 2.2.** The linear extension of \( t \),

\[
T : \Lambda^k U \otimes \Lambda^l U \to \Lambda^U \otimes \Lambda^k U \quad w_x \mapsto w_{tox}
\]

is an \( \mathfrak{F}\mathfrak{S}_n \)-module isomorphism.

**Proof.** Indeed, as \( t \) commutes with the group action, Lemma 2.1/1 implies \( T(w_x)s = w_{tox}s = \varepsilon_{tox,sw_{tox}s} = \varepsilon_{x,s}w_{tox}s = \varepsilon_{x,s}T(w_s) = T(w_s) \) so the claim follows. \( \square \)

2.2. **Color-switch (03)(12).** In the previous subsection we introduced the isomorphism \( T \), that can be interpreted combinatorially as switching the colors 1 and 2 for the elements of \( X_{k,l} \), which parametrize the standard basis of \( \Lambda^k U \otimes \Lambda^l U \). Now we define a similar isomorphism, switching color 1 with 2 and color 0 with 3, at the cost of an extra sign.

Define \( p : \{0, 1, 2, 3\} \to \{0, 1, 2, 3\} \) as \( p(c) = 3 - c \). Clearly, \( p \circ (x s) = (p \circ x) s \) for any \( x \in X_{k,l} \). By definition we may write \( w_x = u_I \otimes u_J \) for some \( I, J \subseteq [n] \). Then

\[
w_{pox} = u_{I^c} \otimes u_{J^c}
\]

where \( I^c = [n] \setminus I \).

Consider the linear bijection

\[
P : \Lambda^k U \otimes \Lambda^l U \to \Lambda^{n-k} U \otimes \Lambda^{n-l} U \quad w_x \mapsto h(x) w_{pox}
\]

where we set \( h(x) = (-1)^{\sum_{i=1}^k (i \circ -\alpha) + \sum_{i=1}^l (j \circ -\alpha)} \) for any \( x \in X_{k,l} \).

**Lemma 2.3.** \( P \) is an \( \mathfrak{F}\mathfrak{S}_n \)-isomorphism.

**Proof.** First we show that

\[
\hat{P}_k : \Lambda^k U \to \Lambda^{n-k} U \otimes \Lambda^n U \quad u_I \mapsto u_{I^c} \otimes u_I \wedge u_{I^c}
\]

is an \( \mathfrak{F}\mathfrak{S}_n \)-isomorphism.

Indeed, define \( \text{sign}_I(s) \in \{1, -1\} \) by the equation \( u_I s = \text{sign}_I(s) u_{I^c} \). Then

\[
\hat{P}_k(u_I s) = \hat{P}_k(\text{sign}_I(s) u_{I^c}) = \text{sign}_I(s) u_{I^c} \otimes u_I \wedge u_{I^c}
\]
On the other hand,
\[
\hat{P}_k(u_I)s = (u_{I_1} \otimes u_{I_2} \wedge u_{I_3})s \\
= (\text{sign}_{I_1}(s)u_{I_1}s) \otimes (\text{sign}_{I_2}(s)u_{I_2}s) \wedge (\text{sign}_{I_3}(s)u_{I_3}s) \\
= \text{sign}(s)u_{I_1}s \otimes u_{I_2}s \wedge u_{I_3}s
\]

Therefore, \( \hat{P}_k \) is indeed \( \mathfrak{S}_n \)-equivariant.

Note that we may express \( \hat{P}_k(u_I) \) equivalently as
\[
\hat{P}_k(u_I) = (-1)^{\sum_{\alpha=1}^k (i_\alpha - \alpha)} u_{I_1} \otimes (u_1 \wedge u_2 \wedge \cdots \wedge u_n)
\]

because we may sort the components of \( u_I \wedge u_{I'} \) using \( \sum_{\alpha=1}^k (i_\alpha - \alpha) \) transpositions.

Now consider the tensor product of \( \hat{P}_k \) and \( \hat{P}_l \):
\[
\hat{P}_k \otimes \hat{P}_l : \Lambda^n U \otimes \Lambda^l U \rightarrow \Lambda^{n-k} U \otimes \Lambda^l U \otimes \Lambda^{n-l} U \otimes \Lambda^n U \\
\quad u_I \otimes u_J \mapsto (-1)^{\sum_{\alpha=1}^k (i_\alpha - \alpha) + \sum_{\beta=1}^l (j_\beta - \beta)} u_{I_1} \otimes \kappa \otimes u_{I'} \otimes \kappa
\]

where \( \kappa = u_1 \wedge u_2 \wedge \cdots \wedge u_n \). However, \( \Lambda^n U \) is the sign representation of \( \mathfrak{S}_n \) hence its square is the identity. Therefore, \( \hat{P}_k \otimes \hat{P}_l \) is the same as \( P \) if we identify \( \mathcal{F} \) with \( (\Lambda^n U)^{\otimes 2} \) using \( 1 \mapsto \kappa \otimes \kappa \), in particular, \( P \) is an isomorphism. \( \square \)

2.3. Proper swaps. We will need another statement about \( \varepsilon_{x,s} \) in Sec. 3. First let us illustrate it on an example. Let \( n = 9 \) and
\[
(1, 2, 1, 2, 1, 3, 1, 2) \in X_{5,5} \quad s = (1, 2)(4, 6)(5, 8) \in \mathfrak{S}_9
\]
i.e. \( s \) is switching cells of color 1 with cells of color 2 in an order-preserving way, such that no cell of color 1 or 2 is missed between them. Note that outside of switches there may be cells of color 1 or 2. In other words, we may split \( x \) into blocks like \((1, 2, 1, 2, 1, 3, 1, 2)\) where there are blocks where we switch all the 1’s and 2’s, and there are blocks not moved by the permutation.

One can check that \( \varepsilon_{x,s} = 1 \), since each transposition contributes to the inversion numbers \( N_1(x, s) \) and \( N_2(x, s) \) with the same amount. In the next lemma we generalize this example.

For \( i, j \) integers denote
\[
[[i, j]] \overset{\text{def}}{=} \begin{cases} [i, j] \cap \mathbb{Z} & \text{if } i < j \\ [j, i] \cap \mathbb{Z} & \text{otherwise.} \end{cases}
\]

We show the following:

**Lemma 2.4 (Proper Swap Lemma).** Let \( x \in X \) and \( s \in \mathfrak{S}_n \) such that
\begin{itemize}
  \item \( s = \prod_{\ell=1}^{m} (i_\ell, j_\ell) \) is a product of \( m \) disjoint transpositions for some \( 1 \leq i_1 < \cdots < i_m \leq n \) and \( 1 \leq j_1 < \cdots < j_m \leq n \),
  \item for all \( \ell \in [m] \), \( x(i_\ell) = 1 \) and \( x(j_\ell) = 2 \), and
  \item for all \( \ell \in [m] \),
\end{itemize}
\[
|\{ \nu \in [[i_\ell, j_\ell]] \mid \nu s = \nu, x(\nu) = 1 \}| \equiv |\{ \nu \in [[i_\ell, j_\ell]] \mid \nu s = \nu, x(\nu) = 2 \}| \pmod{2}
\]
Then \( \varepsilon_{x,s} = 1 \).

We call \( s \) a proper swap with respect to \( x \) if the assumptions of Lemma 2.4 hold.
Proof. Let \( p, q \in [n]^2 \) such that \( x(ps), x(qs) \in \{1, 3\} \). If they are both fixed points of \( s \) then they clearly don’t contribute to \( N_1(x, s) \) by \( ps = p q = qs \). Similarly, if they are non-fixed points, then \( p = i_{\ell_1} \) and \( q = i_{\ell_2} \) for some \( 1 \leq \ell_1 < \ell_2 \leq m \), hence \( ps = j_{\ell_1} < j_{\ell_2} = qs \), and again they don’t contribute.

Now let \( p \) be a non-fixed point and \( q \) a fixed point. Then \( p = i_{\ell} \) for some \( \ell \), and the pair contributes to \( N_1(x, s) \) if and only if \( i_{\ell} = ps \leq qs = q < p = j_{\ell} \) i.e. if \( q \in [i_{\ell}, j_{\ell}] \). Similarly, if \( p \) is a fixed point and \( q \) is a non-fixed point, then they contribute to \( N_1(x, s) \) if and only if \( j_{\ell} = qs < ps = p < q = i_{\ell} \) for some \( \ell \). In short,

\[
N_1(x, s) = \sum_{\ell} \left( \left| \{ \nu \in [i_{\ell}, j_{\ell}] \mid \nu s = \nu, x(\nu) = 1 \} \right| + \left| \{ \nu \in [i_{\ell}, j_{\ell}] \mid x(\nu) = 3 \} \right| \right)
\]

The same holds for \( N_2(x, s) \) if we replace \( x(\nu) = 1 \) by \( x(\nu) = 2 \). The claim follows by the third assumption.

2.4. Standard Young symmetrizers. Let \( \lambda \vdash n \) be a Young diagram with rows of length \( (\lambda_1, \ldots, \lambda_h) \) for some height \( h \in \mathbb{N}^+ \) and consider the subgroup of row-preserving permutations

\[
R_\lambda = \mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_h} \subseteq \mathfrak{S}_n.
\]

Similarly, denote by \( C_\lambda = R_\lambda^T \) the subgroup of column-preserving permutations, where \( \lambda^T \) denotes the transpose of the diagram \( \lambda \).

We define the Young symmetrizer corresponding to (the standard Young tableau of) \( \lambda \) as

\[
c_\lambda = \sum_{a \in R_\lambda} a \sum_{b \in C_\lambda} \text{sign}(b) b \in \mathcal{F} \mathfrak{S}_n.
\]

Given a fixed Young diagram, e.g. \( \lambda = (5, 3, 2) \), we may visualize an element such as \( x = (0, 1, 3, 0, 3, 2, 0, 1, 0, 2) \in X_{4,4} \) as a coloring of the Young diagram using the set of colors \( \{0, 1, 2, 3\} \):

\[
\begin{array}{cccc}
0 & 1 & 3 & 0 & 3 \\
2 & 0 & 1 & & \\
0 & 2 & & & \\
\end{array}
\]

This terminology implied by the visualization makes it easier to formulate statements such as “there are two 3’s in the first row” as a shorthand for \( \exists i, j \leq \lambda_1 : x(i) = x(j) = 3 \).

3. Double Hooks with Even Length Tail

In this section we prove the case of Theorem 1.1 where \( \lambda \) is a double hook \((q, p, 2d_2, 1^{d_1})\) and the length of its “tail” \( d_1 \) is even:

**Proposition 3.1.** Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (q, p, 2d_2, 1^{d_1}) \) for some \( q \geq p \geq 2 \). If \( d_1 \equiv 2 \pmod{4} \) then the multiplicity of \( M^\lambda \) in \( \text{Sym}^2(\Lambda^k V) \) is zero. Similarly, if \( d_1 \equiv 0 \pmod{4} \), then the multiplicity of \( M^\lambda \) in \( \Lambda^2(\Lambda^k V) \) is zero.

Equivalently, we prove that \( \text{Sym}^2(\Lambda^k V)c_\lambda = 0 \) if \( d_1 \equiv 2 \pmod{4} \), where \( c_\lambda \) is the (standard) Young-symmetrizer corresponding to \( \lambda \), hence \( M^\lambda \) is not a summand of \( \text{Sym}^2(\Lambda^k V) \), and similarly for the exterior square.

The steps of the proof are the following: First, in Lemma 3.3 we show a skew-symmetry relation for \( w_c c_\lambda \) in the case of \( n = 6 \), using the observations of Lemma 3.2. Then we prove Lemma 3.6 so we may induce these skew-symmetries for larger diagrams. This induction is carried out in Prop. 3.10, under the assumption that
the first row contains no 1’s or 2’s. (Note that the cases where \( n \leq 5 \) are covered in Prop. 3.10, case III.) Finally, in Lemma 3.11 we show that it was enough to prove under the assumption on the first row, as the images of the basis elements of this form under the projection \((\Lambda^k U) \otimes 2 \rightarrow (\Lambda^k V) \otimes 2\) are generating \((\Lambda^k V) \otimes 2\).

3.1. Base case. Let us make some simple observations on the annihilation of \(c_\lambda\) on the basis vectors. Recall from Subsec. 2.4 that for a given \(\lambda\) we may visualize \(x\) as a 4-colored Young diagram. Figures here are for illustration purposes only.

**Lemma 3.2.** Let \(x \in X\) and \(\lambda \vdash n\) a Young diagram.

1. If there are two 1’s or two 2’s in the same row (i.e. there exist distinct \(\lambda_f < i, j \leq \lambda_{f+1}\) for some \(f\) such that \(x(i) = x(j) \in \{1, 2\}\)), then \(w_x c_\lambda = 0\).

2. If there are two 0’s or 3’s in the same columns (i.e. there exist distinct \(\lambda_f < i \leq \lambda_{f+1}, \lambda_g < j \leq \lambda_{g+1}\) for some \(f, g\) such that \(i - \lambda_f = j - \lambda_g\) and \(x(i) = x(j) \in \{0, 3\}\)), then \(w_x \sum_{b \in C_\lambda} \text{sign}(b)b = 0\).

3. If for every \(a \in R_\lambda\) there are three (resp. five) of 0’s and 3’s in the first (resp. first two) columns of \(xa\) in total then \(w_x c_\lambda = 0\).

**Proof.** Let \(a_0 \in R_\lambda\) be the transposition \((i, j)\). The first statement follows from \(c_\lambda = a_0 c_\lambda\) (by the definition of \(c_\lambda\)) and \(w_x a_0 = -w_x a_0 = -w_x\) by Lemma 2.1/2.

For the second statement denote \(s = \sum_{b \in C_\lambda} \text{sign}(b)b\) and let \(b_0 \in C_\lambda\) be the transposition \((i, j)\). Then we have \(s = -b_0 s\) and \(w_x b_0 = w_x b_0 = w_x\) by Lemma 2.1/3.

The last statement follows from the previous one directly. \(\square\)

**Lemma 3.3.** Let \(\lambda \vdash 6\) be a Young diagram with \(\lambda_1 = 2\). Let \(x \in X_{k,l}\) be a coloring such that in the first row of \(\lambda\) there are no elements of color 1 or 2, and in every other row with length at least two, there is at least one element of color 0 or 3.

1. If \(\lambda\) is \((2, 2, 2)\), then \(w_x c_\lambda = w_{102} c_\lambda\).

2. If \(\lambda\) is \((2, 2, 1, 1)\), \(k = l\) and at least one of the elements on the tail is of color 0 or 3, then \(w_x c_\lambda = -w_{102} c_\lambda\).

**Remark 3.4.** Equations 3.1 and 3.2 can be checked one by one for the finitely many basis vectors, thus we could safely ignore the proof of Lemma 3.3. Nonetheless, we have decided to present a proof in detail, so that we can provide some explicit calculations using the notation introduced in Section 2, which might prove useful later in following the general argument.
Proof: We start the proof of the lemma with four general observations on the action $w_x c_\lambda$ to reduce the number of cases where equations 3.1 and 3.2 are needed to be checked.

**Preliminary observations:**

First, $w_x c_\lambda = 0$ implies $w_{10x} c_\lambda = 0$, since $T : \Lambda^k U \otimes \Lambda^k U \to \Lambda^k U \otimes \Lambda^k U$ is a $\mathbb{F}S_n$-module isomorphism (Cor. 2.2).

Second, if $w_x c_\lambda = \rho w_{10x} c_\lambda$ for some $\rho \in \{1, -1\}$ and $a \in R_\lambda$ is an arbitrary row permutation, then $w_{xa} c_\lambda = \rho w_{10xa} c_\lambda$. Indeed

$$w_{xa} c_\lambda = \varepsilon_{x,a} w_x a c_\lambda = \varepsilon_{x,a} w_x c_\lambda = \varepsilon_{x,a} \rho w_{10x} c_\lambda = \varepsilon_{x,a} \rho w_{10xa} c_\lambda = \rho w_{10xa} c_\lambda,$$

where equation $ac_\lambda = c_\lambda$ was used in the second and fourth step, and Lemma 2.1/1 in the last step. Our second observation means that if the statement of the lemma holds for some $w_x$ corresponding to a coloring $x$, then it will also hold for any other $w_{x'}$, where $x'$ is derived from $x$ by rearranging the colors of $x$ in the rows of $\lambda$.

Third, using the fact that $w_{10x} c_\lambda = w_x c_\lambda$, it can be assumed that the coloring $x$ contains at least as many elements of color 1 as of color 2. Moreover, in case of nonzero equality, the first element of color 1 is smaller than the first element of color 2 (that is $\min(x^{-1}\{1\}) < \min(x^{-1}\{2\})$).

Fourth, if Lemma 3.3 holds for some basis vector $w_x$ then it also holds for $w_{p0tx}$ i.e. when we swap the colors 0 and 3. Recall that $p(c) = 3 - c$ for any $c \in \{0, 1, 2, 3\}$.

$$w_x c_\lambda = \rho w_{10x} c_\lambda \implies w_{p0tx} c_\lambda = \rho w_{p0tx} c_\lambda \implies w_{p0tx} c_\lambda = \rho w_{p0tx} c_\lambda$$

where $\rho \in \{1, -1\}$. Indeed, by Lemma 2.3 $w_x \mapsto h(x) w_{p0x}$ is an $\mathbb{F}S_n$-isomorphism. By definition $h(x) = h(t \circ x)$ so the first implication holds. For the second implication we used the fact that actions $p$ and $t$ are commuting on the colorings.

**Proof of Equation 3.1:**

Let's determine those $w_x$ vectors which shall be investigated in order the prove Eq. 3.1. By the hypotheses, it is clear that coloring $x$ contains at least four elements of color 0 or 3. Combining Lemma 3.2/3 with the first observation, we get that it will be enough to investigate those colorings $x$, which satisfy $|x^{-1}\{0\}| = 2$ and $|x^{-1}\{3\}| = 2$. By the second observation, we may assume that the elements of color 0 or 3 are on the following positions of $\lambda$:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

We show in the next paragraph that it is sufficient to check Equation 3.1 for the following colorings:

\[
\begin{array}{cccc}
x_1 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, & x_2 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, & x_3 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 3 & 1 \\ 4 & 5 \end{bmatrix}, & x_4 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 3 & 1 \\ 4 & 5 \end{bmatrix}
\end{array}
\]

Two cases are distinguished based on whether the colors in the first row are equal or not. If they equal, then by the fourth observation we may assume that $x(1) = x(2) = 0$ and $x(4) = x(6) = 3$. If they are not equal, then with the help of the second and fourth observations we might assume that $x(1) = 0$, $x(2) = 3$, $x(4) = 0$ and $x(6) = 3$. The two remaining entries are of color 1 and 2. They might have identical colors or different colors. Using the third observation two cases shall be
follows for 

\[ w_{x_1} = u_{3456} \otimes u_{46}, \quad w_{x_2} = u_{346} \otimes u_{456}, \quad w_{x_3} = u_{2356} \otimes u_{26}, \quad w_{x_4} = u_{236} \otimes u_{256}. \]

We will show in details how to handle the actions \( w_{x_1}, c_\lambda \) and \( w_{x_4}, c_\lambda \), as the investigation of these actions contain all the necessary type of computational steps needed for the remaining two cases. We start with expanding the Young symmetrizer \( c_\lambda = a_\lambda b_\lambda \). Here \( a_\lambda = (1 + (12))(1 + (34))(1 + (56)) \) by definition, so 

\[ w_{x_1} c_\lambda = w_{x_1} a_\lambda b_\lambda = 2w_{x_1}(1 + (34))(1 + (56))b_\lambda. \]

Now we check the action of \((1 + (34))(1 + (56))\) on the coloring \( x_1 \):

\[
\begin{bmatrix}
0 & 0 \\
1 & 3 \\
3 & 1 \\
1 & 3
\end{bmatrix}
\]

\( x_1(34) = \begin{bmatrix} 0 & 0 \end{bmatrix} \), \( x_1(56) = \begin{bmatrix} 0 & 0 \end{bmatrix} \), \( x_1(34)(56) = \begin{bmatrix} 0 & 0 \end{bmatrix} \).

By Lemma 3.2/2, \( b_\lambda \) will annihilate the terms coming from \( w_{x_1} \) and \( w_{x_1}(34)(56) \), as the result of these actions on \( x_1 \) contain two elements of color 3 in the same column. Using this fact, and that \((35)(46)b_\lambda = b_\lambda \), we get:

\[
\begin{align*}
2w_{x_1}(1 + (34))(1 + (56))b_\lambda &= 2w_{x_1}(34 + (56))b_\lambda = \\
2u_{3456} \otimes u_{46}(34)b_\lambda + 2u_{3456} \otimes u_{46}(56)(35)(46)b_\lambda &= \\
2u_{4356} \otimes u_{36}b_\lambda + 2u_{56}b_\lambda = -2u_{3456} \otimes u_{36}b_\lambda + 2u_{4356} \otimes u_{36}b_\lambda = 0.
\end{align*}
\]

By the first observation Equation 3.1 follows for \( w_{x_1} \).

Now let us consider \( w_{x_4}, c_\lambda \). The annihilation property of \( b_\lambda \) stated in Lemma 3.2/2, allows us to consider only two summands of \( a_\lambda \): \((56)\) and \((12)(34)\). By this fact and that \((135)(264)b_\lambda = (153)(246)b_\lambda = b_\lambda \) we get:

\[
\begin{align*}
w_{x_4} c_\lambda &= w_{x_4}(1 + (12))(1 + (34))(1 + (56))b_\lambda = w_{x_4}(56 + (12)(34))b_\lambda = \\
u_{236} \otimes u_{256}(56) + (12)(34)b_\lambda &= u_{235} \otimes u_{265}b_\lambda + u_{146} \otimes u_{156}b_\lambda = \\
u_{235} \otimes u_{265}(135)(264)b_\lambda + u_{146} \otimes u_{156}(153)(246)b_\lambda &= \\
u_{651} \otimes u_{541}b_\lambda + u_{562} \otimes u_{532}b_\lambda &= u_{156} \otimes u_{146}b_\lambda - u_{256} \otimes u_{235}b_\lambda.
\end{align*}
\]

On the other hand

\[
t \circ x_4 = \begin{bmatrix} 0 & 3 \\
2 & 0 \\
1 & 3
\end{bmatrix}
\]

and \( w_{10x_4} = u_{256} \otimes u_{236} \),

so

\[
w_{10x_4} c_\lambda = w_{10x_4}(1 + (12))(1 + (34))(1 + (56))b_\lambda = w_{10x_4}(56 + (12)(34))b_\lambda = \\
u_{265} \otimes u_{235}b_\lambda + u_{156} \otimes u_{146}b_\lambda = -u_{256} \otimes u_{235}b_\lambda + u_{156} \otimes u_{146}b_\lambda,
\]

which means that Equation 3.1 holds for \( w_{x_1} \). For the remaining two basis vectors we provide the raw computations.

\[
w_{x_2} c_\lambda = 2u_{346} \otimes u_{456}(34 + (56))(35)(46)b_\lambda = 2(-u_{456} \otimes u_{345} - u_{356} \otimes u_{346})b_\lambda,
\]

\[
w_{10x_2} c_\lambda = 2u_{456} \otimes u_{346}(34 + (56))b_\lambda = 2(-u_{356} \otimes u_{346} - u_{456} \otimes u_{345})b_\lambda,
\]

\[
w_{x_3} c_\lambda = u_{2356} \otimes u_{26}(12)(34)b_\lambda + u_{2356} \otimes u_{26}(56)(135)(264)b_\lambda = 0.
\]
**Proof of Equation 3.2:**

For the second part of the lemma, take $\lambda = (2, 2, 1, 1)$. The hypotheses combined with Lemma 3.2/3 and the first observation give, that it is enough to investigate those colorings $x$, which satisfy $|x^{-1}({0})| = |x^{-1}({3})| = 2$ and $|x^{-1}({1})| = |x^{-1}({2})| = 1$. Using the second observation, there are two possible configurations for the location of elements of color 0 and 3:

```
+ + +
+ + +
+ + +
+ + +
```

However, it will be sufficient to prove Eq. 3.2 for the first configuration, as we can act with $(56)$ on 3.2, providing a proof for the second type of configurations. Here we used the fact that $(56)$ commutes with $c_{\lambda}$. Now we can list the critical colorings that are needed to be checked, just as in the first part of the lemma:

$x_1 = \begin{pmatrix} 0 & 0 \\ 1 & 3 \\ 3 & 2 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 3 \\ 1 & 0 \\ 3 & 2 \end{pmatrix}$

The corresponding basis vectors are the following:

$$w_{x_1} = u_{345} \otimes u_{456}, \quad w_{x_2} = u_{235} \otimes u_{256}.$$  

The computations are a bit simpler than the ones we have already seen in the first part of the lemma.

$$w_{x_1}c_{\lambda} = -2u_{345} \otimes u_{456}((36))b_{\lambda} = -2u_{456} \otimes u_{345}b_{\lambda}.$$  

$$w_{10x_1}c_{\lambda} = 2u_{456} \otimes u_{345}b_{\lambda}.$$  

$$w_{x_2}c_{\lambda} = -u_{123} \otimes u_{126}(36)b_{\lambda} = -u_{126} \otimes u_{123}b_{\lambda}.$$  

$$w_{10x_2}c_{\lambda} = u_{126} \otimes u_{123}b_{\lambda}.$$  

The proof of Lemma 3.3 is complete. $\square$

3.2. **Technical lemmas for the induction step.** In this subsection we prove Lemma 3.6 that is used in the inductive step of the proof of Prop. 3.1. First, let us consider the following simplified version of the lemma.

Recall the definition of proper swap from Subsec. 2.3.

**Lemma 3.5 (Simplified Induction Lemma).** Let $\lambda \vdash n$ a Young diagram, $x \in X$, $a_0 \in R_\lambda$ and $b_0 \in C_\lambda$ such that

1. $a_0b_0$ is a proper swap with respect to $x$,
2. $t \circ x = xa_0b_0$, and
3. $b_0$ centralizes $R_\lambda$.

Then $w_{x}c_{\lambda} = \text{sign}(b_{0})w_{10x}c_{\lambda}$.  

In more colorful language, the lemma says that if we may mimic the action of $t$ on $x$ by a proper swap $a_0b_0 \in R_\lambda C_\lambda$ where $b_0$ only moves the tail of $\lambda$, then an
skew-symmetry relation holds. For example, if $\lambda$ and $x$ are visualized as

\[
\begin{array}{ccc}
3 & 1 & 0 \\
0 & 1 & 2 \\
2 & 3 & 1
\end{array}
\]

then $a_0 = (2,4)(6,7)$, $b_0 = (8,10)$ satisfies the assumptions of the lemma and hence $w_x c_\lambda = -w_t o x c_\lambda$. Note that this may happen only if $x \in X_{k,k}$ for some $k \in \mathbb{N}^+$. 

**Proof.** Denote $\text{sign}(b_0) = \delta$. As $b_0$ centralizes $R_\lambda$ we obtain

\[
w_x c_\lambda = \delta w_x \sum_{a \in R_\lambda} a_0 a \sum_{b \in C_\lambda} \text{sign}(b) b_0 b = \delta w_x a_0 b_0 c_\lambda.
\]

We may apply Lemma 2.4 to $a_0 b_0$, so $\varepsilon_{x,a_0 b_0} = 1$. Hence, we may continue as

\[
= \delta w_x a_0 b_0 c_\lambda = \delta w_t o x c_\lambda
\]

as we claimed. \qed

For the generalization of Lemma 3.5, let us define restrictions of Young symmetrizers. Let $H \subseteq [n]$, $\lambda \vdash n$ a Young diagram and $x \in X$. Recall that $T_\lambda$ denotes its standard Young tableau i.e. $T_\lambda$ is $\lambda$ filled with $1,2,\ldots,n$ row-continuously from left to right, from top to bottom. We will say that $H$ is compatible with $\lambda$ if the subset of $T_\lambda$ determined by $H$ is left-aligned and has non-increasing row lengths, precisely

\[
\forall i \in [n], \forall h \in H, \lambda_u < i \leq h \leq \lambda_{u+1} \Rightarrow i \in H \\
\forall i \in [n], \forall g, h \in H, \lambda_u < g \leq \lambda_{u+1} \leq \lambda_v < h \leq \lambda_{v+1} \Rightarrow \lambda_u + (h - \lambda_v) \in H
\]

For example, the cells marked with ‘$H$’ in

\[
\begin{array}{cccc}
H & H & H & 1 \\
H & H & & \\
\end{array}
\]

form a subset compatible with $\lambda$.

For any $H \subseteq [n]$ denote by $R_\lambda(H)$ (resp. $C_\lambda(H)$) the pointwise stabilizer of $[n]\setminus H$ in $R_\lambda$ (resp. $C_\lambda$). Denote the complement of $H$ by $H^c = [n]\setminus H$ and let

\[
R_\lambda(H^c) = \text{Stab}_{R_\lambda}(H) \quad C_\lambda(H^c) = \text{Stab}_{C_\lambda}(H)
\]

the pointwise stabilizer of $H$ in $R_\lambda$ and $C_\lambda$ respectively.

Define the $H$-restricted Young symmetrizer as

\[
e_{\lambda,H} \overset{\text{def}}{=} \sum_{a \in R_\lambda(H)} a \sum_{b \in C_\lambda(H)} \text{sign}(b) \in \text{Sym}(H) \subseteq \mathfrak{S}_n
\]

We will also need an $H$-restricted notion of the color-swap $x \mapsto t \circ x$ defined in Subsec. 2.1. Denote by $t_H : X \to X$ the map

\[
t_H(x)(i) = \begin{cases} 
1 & \text{if } i \in H \text{ and } x(i) = 2 \\
2 & \text{if } i \in H \text{ and } x(i) = 1 \\
x(i) & \text{otherwise}
\end{cases}
\]
Note that while \((t \circ x)s = t \circ (xs)\) for any \(s \in \mathfrak{S}_n\), the same does not hold for \(t_H\).
Let us illustrate the action of \(t_H\) using the example for \(H\) given above and some \(x \in X_{3,5}:

\[
\begin{array}{cccc}
0 & 1 & 2 & 123 \\
0 & 0 & 0 & 0 \\
2 & 2 & 0 & \\
\end{array}
\quad \xrightarrow{t_H} \quad
\begin{array}{cccc}
0 & 2 & 1 & 23 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & \\
\end{array}
\]

The following lemma helps us to deduce relations of the form \(w_xc_\lambda = \pm w_{10x}c_\lambda\) given similar relations with \(c_\lambda,H\).

**Lemma 3.6 (Induction Lemma).** Let \(\lambda \vdash n\) a Young diagram, \(x \in X\) and \(\rho, \delta \in \{1, -1\}\). Assume that \(H \subseteq \{\lambda\}\) is compatible with \(\lambda\). Moreover, assume that for all \(r \in R_\lambda\) if \(w_x c_\lambda,H\) is non-zero, then there exists \(a_r \in R_\lambda(H^c)\) and \(b_r \in C_\lambda(H^c)\) such that

1. \(w_x c_\lambda,H = \rho w_{t_H(xr)}c_\lambda,H\),
2. \(a_r b_r\) is a proper swap with respect to \(t_H(xr)\),
3. \(t_H(xr)a_r b_r = t \circ xr\),
4. \(b_r\) centralizes \(R_\lambda(H^c)\) and \(\text{sign}(b_r) = \delta\).

Then \(w_x c_\lambda = (\rho \delta) w_{10x}c_\lambda\).

In short, if we may supplement the restricted color-swap \(t_H\) with an element \(a_r b_r \in R_\lambda(H^c) C_\lambda(H^c)\) such that they together mimic the action of \(t\) on \(xr\), where \(b_r\) only moves the tail of \(\lambda\) outside of \(H\) (and this holds for all \(r \in R_\lambda\)), then we may lift the skew-symmetry relation (given in (1)) from \(H\) to \([n]\).

**Proof.** Let \((r_i)_{i \in I}\) resp. \((s_j)_{j \in J}\) be a set of representatives for the left cosets resp. right cosets of the subgroups

\[R_\lambda(H) \times R_\lambda(H^c) \subseteq R_\lambda\] resp. \[C_\lambda(H) \times C_\lambda(H^c) \subseteq C_\lambda.\]

In particular, we may write

\[
\sum_{r \in R_\lambda} r = \sum_{i \in I} \sum_{a \in R_\lambda(H)} \sum_{a' \in R_\lambda(H^c)} r_i a a' \sum_{c \in C_\lambda} \text{sign}(c)c = \sum_{j \in J} \sum_{b \in C_\lambda(H)} \sum_{b' \in C_\lambda(H^c)} \text{sign}(bb' s_j) b b' s_j.
\]

Let’s start to compute \(w_x c_\lambda\). Since \(R_\lambda(H^c)\) centralizes \(C_\lambda(H)\) we have

\[
(3.3) \quad w_x c_\lambda = w_x \sum_{a,b} r_i a a' \sum_{b',j} \text{sign}(bb' s_j) b b' s_j =
\]

\[
= \sum_{i \in I} \varepsilon_{x,r_i} w_{xr_i} \sum_{a,b} \text{sign}(b)ab \sum_{a',b',j} \text{sign}(b' s_j) a' b' s_j
\]

Denote the terms

\[
c_{\lambda,H} = \sum_{a,b} \text{sign}(b)ab \quad \text{and} \quad s_0 = \sum_{a',b',j} \text{sign}(b' s_j) a' b' s_j,
\]

Clearly, if we denote by \(I'\) the set of \(i \in I\) such that \(w_{xr_i} c_{\lambda,H} \neq 0\), then Eq. 3.3 holds with summation index \(i \in I'\) as well.

For each \(i \in I'\) we may choose \(a_i \in R_\lambda(H^c)\) and \(b_i \in C_\lambda(H^c)\) corresponding to \(r = r_i\) as in our assumptions. Note that for these we have

\[
(3.4) \quad s_0 = \delta a_i b_i s_0
\]
for any \( i \in I' \), using assumption (4). Note also that
\[
(3.5) \quad c_{\lambda, H} a_i b_i = a_i b_i c_{\lambda, H}
\]
as \( \text{Sym}(H) \) centralizes \( \text{Sym}(H^c) \). Recall that condition (2) in the statement implies \( w_{t_H(x)} a_i b_i = w_{t_H(x)} a_i b_i \) by Lemma 2.4. Therefore we obtain
\[
w_x c_\lambda \overset{\text{Eq. 3.3}}{=} \sum_{i \in I'} \varepsilon_{x, r_i} w_{x, r_i} c_{\lambda, H} s_0 \quad \text{Cond. (1)}
\]
\[
= \sum_{i \in I'} \varepsilon_{x, r_i} \rho w_{t_H(x)} c_{\lambda, H} s_0 = \quad \overset{\text{Eq. 3.4}}{=}
\]
\[
(\rho \delta) \sum_{i \in I'} \varepsilon_{t_{ox}, r_i} w_{t_H(x)} a_i b_i c_{\lambda, H} s_0 \quad \overset{\text{Eq. 3.5}}{=} \quad (\rho \delta) \sum_{i \in I'} \varepsilon_{t_{ox}, r_i} w_{t_{ox}} c_{\lambda, H} s_0
\]

Lemma \( \overset{\text{Cond. (2)}}{=}, (\rho \delta) \sum_{i \in I'} \varepsilon_{t_{ox}, r_i} w_{t_H(x)} a_i b_i c_{\lambda, H} s_0 \quad \overset{\text{Cond. (3)}}{=} \quad (\rho \delta) \sum_{i \in I'} \varepsilon_{t_{ox}, r_i} w_{t_{ox}} c_{\lambda, H} s_0
\]

Finally, note that if \( w_{x, r_i} c_{\lambda, H} = 0 \) then \( w_{t_{ox}, r_i} c_{\lambda, H} = 0 \) holds as well by Cor. 2.2 and 2.1/1. Hence we may apply Eq. 3.3 for \( t \circ x \), that gives
\[
w_{t_{ox}} c_\lambda = \sum_{i \in I'} \varepsilon_{t_{ox}, r_i} w_{t_{ox}} c_{\lambda, H} s_0.
\]

where \( I' \) is still defined as above. The claim follows. \( \square \)

As assumption (1) of Lemma 3.6 is non-trivial to prove, we claim the following:

**Lemma 3.7.** Let \( \lambda \vdash n \), \( x \in X_{k, l} \) a coloring of \([n]\) and \( H \subseteq [n] \) of size \( m \) that is compatible with \( \lambda \). Denote by \( \lambda' \) the partition corresponding to \( H \), and by \( E \) the unique monotonically increasing \([m] \to H \) function. Define the coloring \( x' = (x| H \circ E) : [m] \to \{0, 1, 2, 3\} \).

Assume that for each \( h_1, h_2 \in H \) such that \( \{x(h_1), x(h_2)\} = \{1, 2\} \) we have
\[
(3.6) \quad |\{\nu \in H^c \mid h_1 < \nu < h_2, \ x(\nu) = 1\}| = |\{\nu \in H^c \mid h_1 < \nu < h_2, \ x(\nu) = 2\}|.
\]

Then
\[
(3.7) \quad w_{x'} c_{\lambda'} = \rho w_{t_{ox}'} c_{\lambda'} \implies w_x c_{\lambda, H} = \rho w_{t_H(x)} c_{\lambda, H}.
\]

**Proof.** Denote by \( f_H \) the composition of the group homomorphisms
\[
\mathfrak{S}_m \xrightarrow{\sim} \text{Sym}(H) \longrightarrow \mathfrak{S}_n
\]
where the first is induced by \( E \) and the second by the inclusion \( H \hookrightarrow [n] \). The map \( f_H \) induces an \( \mathfrak{S}_m \)-module structure on \( \Lambda^k U \otimes \Lambda^l U \), denote this module structure as \( v * r = v f_H(r) \) for any \( r \in \mathfrak{S}_m \). This definition assures that
\[
(3.8) \quad v * c_{\lambda'} = v c_{\lambda, H}.
\]
for any \( v \in \Lambda^k U \otimes \Lambda^l U \).

Denote by \( U' \) the representation \( U \) but for \( \mathfrak{S}_m \) instead of \( \mathfrak{S}_n \), and assume that \( w_{x', r} \in \Lambda^k U \otimes \Lambda^l U' \). Then we may consider the map of \( \mathfrak{S}_m \)-modules
\[
F : \Lambda^k U' \otimes \Lambda^l U' \longrightarrow \Lambda^k U \otimes \Lambda^l U \quad w_{x', r} \mapsto w_x * r
\]
that can be checked to be a well-defined \( \mathfrak{S}_m \)-homomorphism, using that \( w_{x', r} \) generates \( \Lambda^k U' \otimes \Lambda^l U' \).
First assume that $k' \neq l'$. In this case $w_{lH(x)} \notin \Lambda^{k}U \otimes \Lambda^{l}U$ so both sides of the claim are zero.

Assume that $k' = l'$. By the definition of $F$ it is clear that $F(w_{tox'})$ is either $w_{tH(x)}$ or $-w_{tH(x)}$. We prove that it is always the former. (Without Condition 3.6 the negative case could also happen, see Example 3.8.) This is sufficient, as then

$$w_x c_{\lambda,H} = w_x * c_{\lambda'} = F(w_x c_{\lambda'}) \overset{3.7}{=} F(\rho w_{tox'}c_{\lambda'}) =$$

$$= \rho F(w_{tox'}) * c_{\lambda'} = \rho w_{tH(x)} * c_{\lambda'} = \rho w_{tH(x)} c_{\lambda,H}$$

so the statement will follow in this case.

To show that $F(w_{tox'}) = w_{tH(x)}$, note that there exists a proper swap $s$ with respect to $x'$ that satisfies $t \circ x' = x's$ (simply swap the first 1 with the first 2, the second 1 with the second 2, etc.). By Lemma 2.4, $w_{tox'} = w_{x's} = w_{x'}s$. Applying $F$ gives $F(w_{tox'}) = F(w_{x's}) = w_x * s = w_x f_H(s)$. Notice that $f_H(s)$ is a proper swap with respect to $x$. Indeed, $f_H(s)$ fulfills the first two conditions of proper swaps by the inheritance of $s$, and fulfills the third condition of proper swaps by Condition 3.6 of the present lemma. Applying Lemma 2.4 on $f_H(s)$ gives $w_x f_H(s) = w_x f_H(s) = w_{tH(x)}$, and this is exactly our claim.

\[\square\]

**Example 3.8.** Without Condition 3.6, it may happen that $F(w_{tox'}) = -w_{tH(x)}$. Define $\lambda$ and $x$ as

$$x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$$

and let $H = \{1, 3, 5\}$. Then, $w_x = u_1 \land u_4 \otimes u_5 \land u_6$, $w_{tH(x)} = u_4 \land u_5 \otimes u_1 \land u_6$ and

$$F(w_{tox'}) = F(w_x (13)) = w_x * (13) = w_x (15) = u_1 \land u_4 \otimes u_5 \land u_6 (15)$$

$$= u_5 \land u_4 \otimes u_1 \land u_6 = -u_4 \land u_5 \otimes u_1 \land u_6.$$ 

hence Condition 3.6 is indeed required.

We will also need an $H$-restricted version of Lemma 3.2/3 in the next subsection.

**Lemma 3.9.** Let $\lambda \vdash n$, $x \in X$ and $H \subseteq [n]$ a subset of the first two columns of $\lambda$ that is compatible with $\lambda$. If there are at least five elements of $H$ that are of color 0 or 3, then $w_x c_{\lambda,H} = 0$.

**Proof.** Denote $b_0 = \sum_{b \in C_{\lambda}(H)} \text{sign}(b)b$. It is enough to show that $w_xb_0 = 0$. Indeed, then

$$w_x c_{\lambda,H} = w_x \sum_{a \in R_{\lambda}(H)} ab_0 = \sum_{a \in R_{\lambda}} \varepsilon_{x,a} w_x a b_0 = 0$$

as we may apply $w_x b_0 = 0$ for $xa$ instead of $x$ as the assumptions of the lemma are invariant under $x \rightarrow xa$ by $a \in R_{\lambda}(H)$.

To prove $w_x b_0 = 0$, take a transposition $s = (i,j) \in C_{\lambda}(H)$ such that $x(i) = \varepsilon_{x,s} w_x b_0 = -w_x b_0$

hence $w_x b_0 = 0$, as we claimed. \[\square\]
3.3. Inductive step. Now that all the necessary technical machinery is available, we can take a direct step toward the proof of Prop. 3.1. In this subsection we apply Lemma 3.6 (Induction Lemma) on the double hook Young diagrams with no 1’s or 2’s in the first row.

Proposition 3.10. Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (q, p, 2d_2, 1^{d_1}) \) for some \( q \geq p \geq 2, d_1, d_2 \geq 0 \). Let \( x \in X_{k,k} \) for some \( k \) such that \( x(\{1, 2, \ldots, q\}) \subseteq \{0, 3\} \). If \( d_1 \) is even, then

\[
(3.9) \quad w_{x_\lambda}c_\lambda = (-1)^{\frac{d_1}{2}} w_{t_{0x}}c_\lambda
\]

holds.

Proof. We may assume that \( w_{x_\lambda}c_\lambda \neq 0 \). Indeed, otherwise \( w_{t_{0x}}c_\lambda = 0 \) also holds by Cor. 2.2. We may also assume that there is at most one cell of color 1 and at most one cell of color 2 in each row in \( \lambda \), for the given coloring \( x \), by Lemma 3.2/1.

Let us call \( m \in [n] \) unpaired if it is the only element in its row of color 1 or 2, and the row is of length at least two. Our tactic will be to choose an \( H \subseteq [n] \) that covers as many unpaired elements as possible, and satisfies the assumptions of Lemma 3.6 (Induction Lemma).

There are at most two unpaired elements in total, by Lemma 3.2/3 and the assumption on the first row, so we may distinguish three cases based on the number of unpaired elements.

**Case I:** Assume that there are two unpaired elements. For example,

\[
\begin{array}{ccc}
0 & 0 & 0 \\
1 & 3 & 0 \\
1 & 2 & \\
2 & 3 & \\
\end{array}
\]

Let their rows be the \( m \)-th and the \( m' \)-th row of \( \lambda \). Then we may define \( H = \{1, 2, m_1, m_2, m'_1, m'_2\} \) where \( m_1 \) and \( m_2 \) denotes the first and second elements of the \( m \)-th row, and similarly for \( m' \).

Let us check the assumptions of Lemma 3.6. Let \( r \in R_\lambda \) and assume that \( w_{x_{\lambda_0}}c_{\lambda, H} \neq 0 \). If \( H \) contains at least five elements of color 0 or 3 in \( xr \), then we would have \( w_{x_\lambda_0}c_{\lambda, H} = 0 \) by Lemma 3.9. Consequently, all unpaired elements of \( xr \) are contained in \( H \). Observe that for each \( h_1, h_2 \in H \subseteq [n] \) we have

\[
|\{ \nu \in H^c \mid h_1 < \nu < h_2, \ (xr)(\nu) = 1 \}| = |\{ \nu \in H^c \mid h_1 < \nu < h_2, \ (xr)(\nu) = 2 \}|
\]

Indeed, all the unpaired elements are contained in \( H \) so every other row between the elements of \( H \) either doesn’t contain any 1’s or 2’s, or it does contain one of each. This equation and Lemma 3.1/1 assures that Lemma 3.7 can be applied to the coloring \( xr \) and the given choice of \( H \). The lemma gives \( w_{x_{\lambda_0}}c_{\lambda, H} = w_{t_{H(xr)}}c_{\lambda, H} \), hence condition (1) of Lemma 3.6 is verified with \( \rho = 1 \).

Let us check the remaining assumptions of Lemma 3.6. If there are different number of 1’s as 2’s in \( H \) then \( w_{x_\lambda}c_\lambda \) would be zero by Lemma 3.3/1, hence \( w_{x_\lambda}c_{\lambda, H} = 0 \) too, by Lemma 3.7. As we assumed this is not the case, we may define \( a_r \in R_\lambda(H^c) \) as the product of disjoint transpositions swapping the elements of color 1 and 2 that are in the same row, and similarly \( b_r \in C_\lambda(H^c) \) as the product of disjoint transpositions swapping the 1’s and 2’s on the tail of \( \lambda \). The permutation \( b_r \) has to be chosen in a monotonic way so \( a_r b_r \) is a proper swap. Then it is straightforward to
check the assumptions (2), (3) and (4) of Lemma 3.6 with $\delta = \text{sign}(b_r) = (-1)^{\frac{d^2}{2}}$ by Lemma 3.2/3. The proof of this case follows from Lemma 3.6 (Induction Lemma).

**Case II:** Assume that there is exactly one unpaired element and it is of color 1 (the case of color 2 is analogous). For example,
\[
\begin{matrix}
0 & 0 & 0 & 3 & 3 \\
1 & 3 & 3 &   \\
1 &   &   &   \\
2 &   &   &   \\
2 & 3 &   &   
\end{matrix}
\]
Let the row of the unpaired element be the $m$-th row of $\lambda$. Then by Lemma 3.2/3 and that $d_1$ is even, there is exactly one element $l$ on the tail (i.e. $l > n - d_1$) that is of color 0 or 3. Moreover, as the number of 1’s and 2’s in $x$ agree (i.e. $x \in X_{k,k}$ for some $k$), there is one more element of color 2 on the tail, than of color 1.

Define $j$ as the least element such that it is on the tail (i.e. $n - d_1 < j \leq n$), is of color 2 and there are the same number of elements of color 1 as of color 2 strictly between $n - d_1$ and $j$. Then we may define $H = \{1, 2, m_1, m_2, j, l\}$ where $m_1$ and $m_2$ denotes the first and second elements of the $m$-th row.

Let us repeat the argument of the previous case. Let $r \in R_\lambda$ and assume that $w_{xr}c_{\lambda,H} \neq 0$. Then $H$ contains at most four elements of color 0 or 3 by the same argument using Lemma 3.9. In particular, the unpaired element in row $m$ is contained in $H$. The definition of $j$ assures that for each $h_1, h_2 \in H \subseteq [n]$ such that $\{x(h_1), x(h_2)\} = \{1, 2\}$ we have
\[
|\{\nu \in H^c \mid h_1 < \nu < h_2, \ (xr)(\nu) = 1\}| = |\{\nu \in H^c \mid h_1 < \nu < h_2, \ (xr)(\nu) = 2\}|.
\]
Just as in the previous case, this equation and Lemma 3.1/2 assures that Lemma 3.7 can be applied to coloring $xr$ and restriction $H$. It gives $w_{xr}c_{\lambda,H} = -w_{1H}c_{H^c}$ hence condition (1) of Lemma 3.6 (Induction Lemma) is verified with $\rho = -1$.

Define $a_r \in R_\lambda(H^c)$ (resp. $b_r \in C_\lambda(H^c)$) exactly the same way as in the previous case, in particular $b_r$ is defined in a monotonic way. Then it is straightforward to check the assumptions (2), (3) and (4) of Lemma 3.6 with $\delta = \text{sign}(b_r) = (-1)^{\frac{d^2}{2} - 2}$, hence $\rho \delta = (-1)^{\frac{d^2}{2}}$. The proof of this case follows analogously.

**Case III:** Assume that there are no unpaired elements, e.g.
\[
\begin{matrix}
0 & 0 & 0 & 3 & 3 \\
1 & 2 &   &   &   \\
1 &   &   &   &   \\
2 &   &   &   &   
\end{matrix}
\]
Then we may simply define $a_0 \in R_\lambda$ (resp. $b_0 \in C_\lambda$) as the product of disjoint transpositions swapping the elements of color 1 and 2 that are in the same row (resp. on the tail of $\lambda$), where $b_0$ (in fact both) are chosen in an order-preserving way. Then the claim follows by Lemma 3.5. \qed
3.4. **Proof of Proposition 3.1.** We show that it was enough to prove for the case of no 1's or 2's in the first row, that is done in Prop. 3.10.

For a given \( \lambda \vdash n \) Young diagram and \( x \in X \) coloring, denote by \( \ell(x) \) the number of 1's and 2's in total appearing in the first row i.e. \( \ell(x) = |x^{-1}(\{1,2\}) \cap [\lambda_1]| \), and let

\[
W_{\ell(x)} = \text{Span}(w_y \mid y \in X_{k,k}, \ell(y) < \ell(x)) \leq (\Lambda^k U)^{\otimes 2}.
\]

The plan is to prove Prop. 3.1 for \( x \) by induction on \( \ell(x) \).

Consider the natural projection

\[
U \to U / \left( \sum_{i=1}^{n} u_i \right) \cong V.
\]

It induces a projection \( \Lambda^k U \to \Lambda^k V \) and also a projection \( (\Lambda^k U)^{\otimes 2} \to (\Lambda^k V)^{\otimes 2} \).

Denote the kernel of the latter by \( K \). For the inductive step, we prove the following:

**Lemma 3.11.** Let \( \lambda \vdash n \) and \( x \in X_{k,k} \). If \( \ell(x) \geq 1 \) then

\[
w_x c_{\lambda} \in W_{\ell(x)} c_{\lambda} + K.
\]

Note that the statement is interesting only for \( \ell(x) \leq 2 \) by Lemma 3.2/1.

**Proof.** Let \( m \in [\lambda_1] \) such that \( x(m) = 1 \) (the case of \( x(m) = 2 \) is analogous). Write

\[
w_x = \varepsilon \cdot (u_m \wedge u_I) \otimes u_J
\]

where \( \varepsilon \in \{1, -1\} \) depending on the position of \( m \) and \( m \not\in I \cup J \subseteq [n] \) by \( x(m) = 1 \).

Consider the element:

\[
(3.10) \quad K \ni \left( \sum_{i=1}^{n} \varepsilon \cdot u_i \wedge u_I \right) \otimes u_J = w_x + \sum_{i \neq m} \varepsilon \cdot (u_i \wedge u_I) \otimes u_J.
\]

We show the following:

**Claim 3.12.** For given \( i \neq m \) the element \((u_i \wedge u_I) \otimes u_J c_{\lambda}\) either equals \( \varepsilon w_x c_{\lambda}\) or is contained in \( W_{\ell(x)} c_{\lambda}\).

The lemma follows by the claim, because then Eq. 3.10 gives

\[
z w_x c_{\lambda} \in W_{\ell(x)} c_{\lambda} + K
\]

for some positive integer \( z \).

**Proof of the claim.** If \( x(i) \in \{1,3\} \) then \( u_i \wedge u_I = 0 \) so \( (u_i \wedge u_I) \otimes u_J c_{\lambda} = 0 \in W_{\ell(x)} c_{\lambda} \). If \( x(i) = 2 \), then \( (u_i \wedge u_I) \otimes u_J = \pm w_y \) where \( y \in X_{k,k} \) is defined as

\[
y(j) = \begin{cases} 
0 & \text{if } j = m \\
3 & \text{if } j = i \\
x(j) & \text{otherwise}
\end{cases}
\]

Notice that \( \ell(y) = \ell(x) - 1 \), in particular \((u_i \wedge u_I) \otimes u_J \in W_{\ell(x)} \).

If \( i > \lambda_1 \) then \((u_i \wedge u_I) \otimes u_J \in W_{\ell(x)} \) by definition. If \( i \leq \lambda_1 \) and \( x(i) = 0 \), then \((u_i \wedge u_I) \otimes u_J s = (u_m \wedge u_I) \otimes u_J \) where \( s = (i, m) \in R_\lambda \). Consequently,

\[
((u_i \wedge u_I) \otimes u_J) c_{\lambda} = ((u_i \wedge u_I) \otimes u_J) s c_{\lambda} = ((u_m \wedge u_I) \otimes u_J) c_{\lambda} = \varepsilon w_x c_{\lambda}.
\]

The claim follows.

The claim proves Lemma 3.11. \( \square \)
Corollary 3.13. Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (n - m, 1^m) \) and \( x \in X_{k,k} \). Then
\[
w_x c_\lambda \equiv (-1)^{\frac{\ell(x)}{2}} w_{t_{ox}} c_\lambda \mod K.
\]

Proof. By Lemma 3.11, it is enough to prove for the case when every cell in the first row of \( \lambda \) is of color 0 or 3. Indeed, if we show that then we may prove the statement by induction on \( \ell(x) \). The case of \( \ell(x) = 0 \) is our assumption. If we know the statement for all \( y \) with \( \ell(y) < \ell(x) \) then we may express \( w_x c_\lambda \) as a sum of \( w_y c_\lambda \) modulo \( K \).

Assume that every cell in the first row of \( \lambda \) is of color 0 or 3. The number of cells of color 1 and 2 on the tail of \( \lambda \) are the same, by \( x \in X_{k,k} \). Therefore, we may apply Lemma 3.5 with \( a_0 = \text{id} \) and \( b_0 \) the product of disjoint transpositions swapping the cells of color 1 with the cells of color 2 in a monotonic way. To determine \( \text{sign}(b_0) \), note that there are at most two cells in the first column of \( \lambda \) that are of color 0 or 3, by Lemma 3.2/2, but at least one, by the assumption on the first row. Therefore, \( \text{sign}(b_0) = (-1)^{\frac{\ell(x)}{2}} \) and the claim follows. \( \square \)

Note that Cor. 3.13 is directly connected to Prop. 3.1 through the following statement:

Lemma 3.14. Assume that \( w_x c_\lambda = (-1)^a w_{t_{ox}} c_\lambda \mod K \) for some \( a \in \mathbb{Z} \). If \( a \) is odd then the image of \( w_x c_\lambda \) in \( \text{Sym}^2(\Lambda^k V) \) is zero. Similarly, if \( a \) is even then the image of \( w_x c_\lambda \) in \( \Lambda^2(\Lambda^k V) \) is zero.

Proof. It follows from the fact that the image of \( w_x \) and \( w_{t_{ox}} \) (resp. \( -w_{t_{ox}} \)) are the same in \( \text{Sym}^2(\Lambda^k V) \) (resp. \( \Lambda^2(\Lambda^k V) \)). \( \square \)

Proof of Prop. 3.1. By Lemma 3.14 it is enough to show that
\[
w_x c_\lambda \equiv (-1)^{\frac{\ell(x)}{2}} w_{t_{ox}} c_\lambda \mod K.
\]
If \( \ell(x) = 0 \) then the statement is proved in Proposition 3.10. If \( \ell(x) > 0 \) then the claim is proved by induction using Lemma 3.11, analogously to the proof of Corollary 3.13. \( \square \)

4. Double Hooks with Odd Tail

In this section we prove the second case of Theorem 1.1 by showing the following:

Proposition 4.1. Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (q,p,2d_2,1^{d_1}) \) for some \( q \geq p \geq 2 \). If \( d_1 \) is odd then the multiplicity of \( M^\lambda \) in \( \text{Sym}^2(\Lambda^k V) \) equals the multiplicity of \( M^\lambda \) in \( \Lambda^2(\Lambda^k V) \).

The proof is based on Frobenius reciprocity, the branching rule, the fact that we already proved the case of even length tails, and that the exact multiplicities of \( (\Lambda^k V)^{\otimes 2} \) are known by Remmel’s theorem.

4.1. Branching Argument. Let \( \mu \vdash n-1 \) be a Young diagram such that the number of rows of length one is even, i.e. it has even length tail. Denote \( \text{Ind} = \text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n-1}} \), similarly for \( \text{Res} \), and let \( \langle M,N \rangle \) be the usual inner product of \( \mathfrak{S}_n \)-representations, i.e. \( \langle M,N \rangle = \dim \text{Hom}_{\mathfrak{S}_n}(M,N) \).

By Frobenius reciprocity we have
\[
\langle \text{Ind} M^\mu, F(\Lambda^k V) \rangle = \langle M^\mu, \text{Res} F(\Lambda^k V) \rangle \quad F \in \{\text{Sym}^2, \Lambda^2\}.
\]
By the branching rule of $S_n$-representations, we may decompose the left hand side of the equation as follows. For two Young diagrams $\mu \vdash (n-1)$ and $\lambda \vdash n$ let us write $\mu \nrightarrow \lambda$ if and only if $\lambda$ may be obtained from $\mu$ by adding a single box to it. With this notation the branching rule states that

\begin{equation}
\text{Ind } M^\mu = \sum_{\lambda: \mu \nrightarrow \lambda} M^\lambda
\end{equation}

Denote by $\mu[i]$ the Young diagram obtained from $\mu$ by adding a box to the $i$-th column, if it exists. Let

\begin{equation}
\lambda = (q, p, 2d_2, 1^{d_1}) \quad \text{and} \quad \mu = (q, p, 2d_2, 1^{d_1-1})
\end{equation}

for some odd $d_1$ and $q \geq p \geq 2$. Then we have $\mu[1] = \lambda$ and $\mu[2] = (q, p, 2d_2+1, 1^{d_1-2})$. By Eq. 4.2 we get:

\begin{equation}
\text{Ind } M^\mu = M^\lambda \oplus (M^\mu[2] \text{ if } d_1 > 1) \oplus (M^\mu[3] \text{ if } p > 2) \\
\quad \oplus (M^\mu[p+1] \text{ if } q > p) \oplus M^\mu[q+1]
\end{equation}

The conditional terms are defined to be zero if the condition fails.

**Lemma 4.2.** Denote by $V_{n-1}$ the standard $(n-2)$-dimensional irreducible representation of $S_{n-1}$. Then

\[ \text{Res } F(\Lambda^k V) \cong F(\Lambda^k V_{n-1}) \oplus F(\Lambda^{k-1} V_{n-1}) \oplus (\Lambda^k V_{n-1} \otimes \Lambda^{k-1} V_{n-1}) \]

for $F \in \{\text{Sym}^2, \Lambda^2\}$.

It is at least plausible that Eq. 4.1 together with Eq. 4.4 and Lemma 4.2 completely determines the multiplicities for double hooks $\lambda$ with odd length tail. We will prove this in the next subsection.

**Proof of Lemma 4.2.** As Res commutes with $F$ and $\Lambda^k$ we have

\[ \text{Res } F(\Lambda^k V) = F(\Lambda^k \text{Res } V) \]

Denote by $1_{n-1}$ the trivial representation of $S_{n-1}$. By definition Res $V \cong V_{n-1} \oplus 1_{n-1}$. Moreover $\Lambda^k (N \oplus 1) \cong \Lambda^k N \oplus \Lambda^{k-1} N$ for any $N$, hence:

\[ \cong F(\Lambda^k (V_{n-1} \oplus 1_{n-1})) \cong F(\Lambda^k V_{n-1} \oplus \Lambda^{k-1} V_{n-1}) \]

Finally, one can observe that $F(N_1 \oplus N_2) = F(N_1) \oplus F(N_2) \oplus (N_1 \otimes N_2)$ for any $N_1, N_2$ and $F \in \{\text{Sym}^2, \Lambda^2\}$, hence the claim of the lemma follows. \qed

**Remark 4.3.** The argument given above is not dependent on the parity of $d_1$ i.e. with induction-restriction we may get similar equations for $d_1$ even. In the end, one could combine this argument with a simultaneous induction on 4 variables $(n, q, p, \text{ and } d_1, \text{ descending on } q \text{ and } p)$ and derive some parts of Prop. 3.1 too.

This approach would have two serious drawbacks: on one hand it wouldn’t solve the case of $\lambda = (q, p, 2d_2)$, where we would need a proof similar to the one given in Sec. 3. Moreover, it wouldn’t explain why the mod 4 value of $d_1$ appears in the answer, while we think that Lemma 3.3 and 3.5 are more insightful in this regard.
4.2. Application of Remmel’s theorem. First let us recall Remmel’s theorem:

**Theorem 4.4** (Remmel [3], Rosas [2]). Let $n, k, l \in \mathbb{N}^*$ and $\lambda \vdash n$ a Young diagram. Then the multiplicities of $M^\lambda$ in $\Lambda^k V \otimes \Lambda^l V$ are the following:

- if $\lambda = (q, p, 2d_2, 1^{d_1})$, $q \geq p \geq 2$ is a double hook then
  - 2, if $|k-l| \leq d_1$ and $|k+l+1-n| \leq q-p$,
  - 1, if $|k-l| \leq d_1$ and $|k+l+1-n| = q-p+1$,
  - 1, if $|k-l| = d_1 + 1$ and $|k+l+1-n| \leq q-p$,
- if $\lambda = (n-m, 1^m)$ is a hook where $|k'-l'| \leq m^{k,l} \leq k'+l'$, using the notation $u' = \min(u, n-u-1)$ and
  $$m^{k,l} = \begin{cases} 
m & \text{if } (k = k' \text{ and } l = l') \text{ or } (k \neq k' \text{ and } l \neq l') 
\n-m-1 & \text{otherwise}, \end{cases}$$
- 0 otherwise.

**Remark 4.5.** The notation of the statement is an alternative version of the one used in [2, Thm. 3] by M. H. Rosas, where she characterized the case of multiplicity 2 as

$$|k-l| \leq d_1 \quad \text{and} \quad 2p-1 \leq k+l-2d_2-d_1 \leq 2q-1.$$ 

The latter is equivalent to $|k+l+1-n| \leq q-p$ by $q+p+2d_2+d_1 = n$.

Note also that Remmel’s formulation in [3, Thm. 2.1(b)] contains a mathematical typo on the case of $\lambda = (r, 1^{n-r})$, as he writes $c_\lambda = \chi(s+t-n-1 \leq r \leq s+n-t)$ instead of $c_\lambda = \chi(s+t-n \leq r \leq s+n-t)$, where the characteristic function $\chi$ is defined below.

Let us apply the theorem for some special cases. For any statement $P$ define $\chi(P) = 1$ if $P$ is true, and 0 otherwise, in particular $\chi(a \leq b) = 1$ if and only if $a \leq b$. Moreover, denote

$$\psi(a, b) = \begin{cases} 
2 & \text{if } |a| < b \\
1 & \text{if } |a| = b \\
0 & \text{otherwise} \end{cases}$$

Recall the definition of $\lambda$ and $\mu[i]$ from Eq. 4.3. Using the notation of the previous paragraph, by Theorem 4.4, we have

$$\langle M^\lambda, (\Lambda^k V)^{\otimes 2} \rangle = \psi(2k+1-n, q-p+1)$$

Moreover,

$$\langle M^\mu, \Lambda^k V_{n-1} \otimes \Lambda^{k-1} V_{n-1} \rangle = \begin{cases} 
\psi(2k+1-n, q-p+1) & \text{if } d_1 \geq 2 \\
\chi(2k+1-n \leq q-p) & \text{if } d_1 = 1 \end{cases}$$

where $V_{n-1}$ is the standard $(n-2)$-dimensional representation of $\mathfrak{S}_{n-1}$.

**Corollary 4.6.** Let $\mu = (q, p, 2d_2, 1^{d_1-1})$ for some $q \geq p \geq 2$, $d_1$ odd. If $q > p$ then

$$\langle M^{[q+1]} \oplus M^{[p+1]}, F(\Lambda^k V) \rangle = \langle M^\mu, F(\Lambda^{k-1} V_{n-1}) \oplus F(\Lambda^k V_{n-1}) \rangle$$

Moreover, if $q = p$ then

$$\langle M^{[q+1]}, F(\Lambda^k V) \rangle = \langle M^\mu, F(\Lambda^{k-1} V_{n-1}) \oplus F(\Lambda^k V_{n-1}) \rangle.$$
Proof. First assume that either $F = \text{Sym}^2$ and $(d_1 - 1) \equiv 2 \mod 4$ or $F = \Lambda^2$ and $(d_1 - 1) \equiv 0 \mod 4$. As $\mu$ has even tail we may apply by Prop. 3.1, and so both sides of Eq. 4.7 and 4.8 are zero.

Now assume that $F$ and $d_1$ are not as above. Then by Prop. 3.1:

$$\langle M^\mu, F(\Lambda^k V_{n-1}) \rangle = \langle M^\mu, (\Lambda^k V_{n-1})^{\otimes 2} \rangle.$$ 

Therefore, by Eq. 4.5 we get

$$\langle M^{\mu[p+1]}, F(\Lambda^k V) \rangle = \psi(2k + 1 - n, (q + 1) - p + 1)$$

$$\langle M^\mu, F(\Lambda^{k-1} V_{n-1}) \rangle = \psi(2k - 1 + 1 - (n - 1), q - p + 1)$$

It is easy to see that if $a, b$ are integers such that $b \geq 2$ then

$$\psi(a, b) - \psi(a - 1, b - 1) = \psi(a + b - 1)$$

Hence, we get

$$\langle M^{\mu[q+1]}, F(\Lambda^k V) \rangle - \langle M^\mu, F(\Lambda^{k-1} V_{n-1}) \rangle = \psi(2k - n + q - p + 2, 1)$$

Similarly, we have

$$\langle M^{\mu[p+1]}, F(\Lambda^k V) \rangle - \langle M^\mu, F(\Lambda^k V_{n-1}) \rangle =$$

$$\psi(2k + 1 - n, q - p) - \psi(2k + 2 - n, q - p + 1)$$

$$= - \psi(2k - n + q - p + 2, 1)$$

so the first statement follows.

For Eq. 4.8 an analogous computations yields

$$\langle M^{\mu[q+1]}, F(\Lambda^k V) \rangle - \langle M^{\mu[p+1]}, F(\Lambda^k V_{n-1}) \rangle =$$

$$\psi(2k + 1 - n, 2) - \psi(2k - n, 1) - \psi(2k + 2 - n, 1)$$

so the claim follows from $\psi(x, 2) - \psi(x - 2, 1) - \psi(x + 1, 1) = 0$. \hfill \Box

Now we may prove the main proposition of the section:

Proof of Prop. 4.1. Let us derive recursive equations on the multiplicities. First assume that $d_1 > 1$, $F \in \{\text{Sym}^2, \Lambda^2\}$, and consider Eq. 4.1:

$$\langle \text{Ind} M^\mu, F(\Lambda^k V) \rangle = \langle M^\mu, \text{Res} F(\Lambda^k V) \rangle.$$ 

Expand the left hand side by Eq. 4.4 and the right hand side by Lemma 4.2, and subtract the appropriate equation in Cor. 4.6 (depending on whether $q = p$):

$$\langle M^{\lambda} \oplus M^{\mu[2]}, F(\Lambda^k V) \rangle = \langle M^\mu, \Lambda^k V_{n-1} \otimes \Lambda^{k-1} V_{n-1} \rangle =$$

$$\text{4.6} \psi(2k + 1 - n, q - p + 1)$$

using that $\mu[1] = \lambda$ and $\langle M^{\mu[3]}, F(\Lambda^k V) \rangle = 0$ by Theorem 4.4 (assuming $p > 2$ so $M^{\mu[3]}$ is defined). Similarly, if $d_1 = 1$ then we get

$$\langle M^{\lambda}, F(\Lambda^k V) \rangle = \chi(|2k + 1 - n| \leq q - p)$$

Note that the right hand sides of Eq. 4.9 and 4.10 are independent of whether $F = \text{Sym}^2$ or $F = \Lambda^2$. As these equations uniquely determine each multiplicity by induction on $d_1$, the claim follows. \hfill \Box
4.3. **Proof of the main theorem.**

*Proof of Theorem 1.1.* Let $\lambda \vdash n$ be a Young diagram of the form $\lambda = (q, p, 2^{d_2}, 1^{d_1})$. If $d_1$ is even, then the statement follows from Prop. 3.1.

If $d_1$ is odd, then by Theorem 4.4 we know that $\langle M^K, (\Lambda^k V) \otimes^2 \rangle \leq 2$. On the other hand, by Prop. 4.1 the multiplicity of the symmetric and the exterior part are the same, so the multiplicity is either zero or one in both.

If $\lambda = (n - m, 1^m)$ then by Corollary 3.13 and Lemma 3.14 we get that the multiplicity of $M^K$ in $\text{Sym}^2(\Lambda^k V)$ is zero if $\lfloor \frac{m}{2} \rfloor$ is odd, and similarly the multiplicity in $\Lambda^2(\Lambda^k V)$ is zero if the $\lfloor \frac{m}{2} \rfloor$ is even. The claim follows. □

Corollary 1.2 is directly implied by Theorem 1.1 and Remmel’s Theorem 4.4.

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