On congruences of Galois representations of number fields

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Abstract

We give a criterion for two \( \ell \)-adic Galois representations of an algebraic number field to be isomorphic when restricted to a decomposition group, in terms of the global representations mod \( \ell \). This is applied to prove a generalization of a conjecture of Rasmussen-Tamagawa [14] under a semistability condition, extending some results [12] of one of the authors. It is also applied to prove a congruence result on the Fourier coefficients of modular forms.

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1 Introduction

Let \( K \) be an algebraic number field (= finite extension of \( \mathbb{Q} \)) and let \( G_K = \text{Gal}(\bar{K}/K) \) denote its absolute Galois group, where \( \bar{K} \) is a fixed algebraic closure of \( K \). Choosing an extension of \( v \) to \( \bar{K} \), we denote by \( G_v \) (resp. \( I_v \)) the decomposition (resp. inertia) group of \( v \) in \( G_K \). Let \( E \) be another algebraic number field, \( \lambda \) a finite place of \( E \), \( E_\lambda \) the completion of \( E \) at \( \lambda \). We denote by \( \mathcal{O}_E \) and \( \mathcal{O}_{E_\lambda} \) the integer rings of \( E \) and \( E_\lambda \), respectively. Let \( f_\lambda \) denote the absolute residue degree of \( \lambda \). We identify any finite place \( v \) of an algebraic number field with the corresponding prime ideal, and denote its residue field by \( k_v \) and put \( q_v := \# k_v \). Throughout the paper, we fix \( K, E, \) and a finite place \( v \) of \( K \), and let the finite place \( \lambda \) of \( E \) of residue characteristic \( \ell \) vary. We denote by \( \ell \) the residue characteristic of \( \lambda \), and assume \( v \nmid \ell \), while \( u \) will denote another finite place of \( K \) lying above \( \ell \). All representations of Galois groups denoted \( V \) are either \( \mathbb{Q}_\ell \)- or
$E_{\lambda}$-linear of finite dimension, and assumed to be continuous with respect to the natural topologies. Their “reductions” will be denoted by $\bar{V}$.

In the following, $n$ and $e$ are fixed integers $\geq 1$ and $e$ is assumed to be divisible by the absolute ramification index $e(K_u/Q_{\ell})$ of $K_u/Q_{\ell}$. For $K, u, v, E, \lambda, n, e$ as above and a real number $b$, let $\text{Rep}_{E,\lambda,n}^{(G)}(K; u, b, e, v)$ denote the set of $n$-dimensional $E_{\lambda}$-linear representations $V$ of $G_K$ which have the following properties:

- $V$ is semistable at $v$ (in the sense that the action of the inertia is unipotent (including the case where it is trivial)),
- $V$ is $E$-integral at $v$ in the sense of Definition 2.2,
- $V$ becomes semistable (in the sense of Fontaine [7]) over a finite extension $K'_{u'}$ of $K_u$ whose absolute ramification index $e(K'_{u'}/Q_{\ell})$ divides $e$,
- $V$ has Hodge-Tate weights $\subset [0, b]$ at $u$, and
- $V$ is of type (G) in the sense of Definition 2.4,

Our first main result is:

**Theorem 1.1.** For any $K, E, n, b, v$ as above, there exists a constant $C = C([E:Q], n, b, e, q_v)$ such that the following holds: For any prime number $\ell > C$, any places $u$ of $K$ and $\lambda$ of $E$ both lying above $\ell$, and any representations $V \in \text{Rep}_{E,\lambda,n}^{(G)}(K; u, b, e, v)$ and $V' \in \text{Rep}_{E,\lambda,n}^{(G)}(K; u, (\ell - 2)/e^2, e, v)$, if one has $V \equiv_{ss} V'$ (mod $\lambda$) both as $G_u$-representations and $G_v$-representations, then one has $V \simeq_{ss} V'$ as $G_v$-representations. (In particular, if $V \equiv_{ss} V'$ (mod $\lambda$) as $G_K$-representations, then $V \simeq_{ss} V'$ as $G_v$-representations.)

The constant $C$ can be taken explicitly to be

$$C := \max\{e^2b + 1, \left(2 \frac{n}{\lfloor n/2 \rfloor}\right)^{[E:Q]/f_{\lambda}}\},$$

where $[x]$ denotes the largest integer not exceeding $x$.

Here, the meaning of the notations $\equiv_{ss}$ and $\simeq_{ss}$ is as follows: we say $V \equiv_{ss} V'$ (mod $\lambda$) as $G_v$-representations if $T$ and $T'$ are $G_v$-stable $O_{E_{\lambda}}$-lattices in $V$ and $V'$, respectively, and the semisimplifications $(T/\lambda T)^{ss}$ and $(T'/\lambda T')^{ss}$ are isomorphic as $k_{\lambda}$-linear representations of $G_v$ (this definition does not depend on the choice of the lattices). We say also $V \simeq_{ss} V'$ as $G_v$-representations if their semisimplifications are isomorphic as $E_{\lambda}$-linear representations of $G_v$.

To state a variant of this theorem, let $\text{Rep}_{E,\lambda,n}^{(G)}(K; u, b, e, v)'$ be the set of $n$-dimensional $E_{\lambda}$-linear representations $V$ of $G_K$ which have the following properties:
- $V$ is $E$-integral at $v$.
- $V$ becomes semistable over a finite extension $K_v'$ of $K_u$ whose absolute ramification index $e(K_v'/\mathbb{Q}_\ell)$ divides $e$,
- $V$ has Hodge-Tate weights $\subset [0,b]$ at $u$, and
- $V$ is of type (G).

Thus $\text{Rep}_{E,\lambda,n}^{(G)}(K;u,b,e,v)'$ contains $\text{Rep}_{E,\lambda,n}^{(G)}(K;u,b,e,v)$, and the difference is that the elements $V$ of the former are not assumed to be semistable at $v$.

Let $W_v(V)$ denote the multi-set of Weil weights of $V$ (Def. 2.1) considered as a $\mathbb{Q}_\ell$-linear representation of $G_v$.

**Theorem 1.2.** For $K,E,n,b,v$ as above, the following holds with the same constant $C = C([E : \mathbb{Q}],n,b,e,q_v)$ as in Theorem 1.1: For any prime number $\ell > C$, any places $u$ of $K$ and $\lambda$ of $E$ both lying above $\ell$, and any representations $V \in \text{Rep}_{E,\lambda,n}^{(G)}(K;u,b,e,v)'$ and $V' \in \text{Rep}_{E,\lambda,n}^{(G)}(K;u,(\ell - 2)/e^2,v)'$, if one has $V \equiv \text{ss} V'$ (mod $\lambda$) both as $G_u$-representations and $G_v$-representations, then one has $W_v(V) = W_v(V')$. [In particular, if $V \equiv \text{ss} V'$ (mod $\lambda$) as $G_K$-representations, then $W_v(V) = W_v(V')$.]

**Remark.** If we consider representations of type (W) at all places $v|q$ for a fixed prime number $q$ and of Hodge-Tate type at all places $u|\ell$, we can prove versions of Theorems 1.1 and 1.2 without assuming “type (G)” but with a larger constant

$$C' := \max\{e^2b + 1, \left(2\left(\frac{n}{\lceil n/2 \rceil}\right)q^{nb[K:Q]/[K_v:Q_v]}\right)^{[E:Q]/f_\lambda}\}.$$ 

The proofs are basically the same as in the case of type (G) but use Proposition 2.8 instead of the equality (G) in Definition 2.4.

The constant $C = C([E : \mathbb{Q}],n,b,e,q_v)$ above depends on the coefficient field $E$. By working mod $\ell$ rather than mod $\lambda$, however, we can suppress this dependence on $E$ as follows:

**Theorem 1.3.** For any $K,E,n,b,v$ as above, there exists a constant $\tilde{C} = \tilde{C}(n,b,e,q_v)$ such that the following holds: For any prime number $\ell > C$, any places $u$ of $K$ and $\lambda$ of $E$ both lying above $\ell$, and any representations $V \in \text{Rep}_{E,\lambda,n}^{(G)}(K;u,b,e,v)$ and $V' \in \text{Rep}_{E,\lambda,n}^{(G)}(K;u,(\ell - 2)/e^2,e,v)$, if one has $V \equiv \text{ss} V'$ (mod $\lambda$) as $G_u$-representations and $\det(T - \text{Frob}_v|V) \equiv \det(T - \text{Frob}_v|V')$ (mod $\mathcal{O}_E$), then one has $V \simeq \text{ss} V'$ as $G_v$-representations. [In
particular, if \( V \equiv_{ss} V' \pmod{\ell} \) as \( G_K \)-representations, then \( V \simeq_{ss} V' \) as \( G_{v'} \)-representations.

The constant \( \tilde{C} \) can be taken explicitly to be

\[
\tilde{C} := \max\{e^2b + 1, 2\left(\frac{n}{[n/2]}\right)q_v^{nb}\}.
\]

After recalling some notions and results on Galois representations in Section 2, we give proofs of the above theorems in Section 3 and several corollaries of Theorem 1.2 in Section 4. In Section 5, we apply Theorem 1.3 with \( E \) a Hecke field to prove a congruence result on the Fourier coefficients of modular forms of various levels, where the “independence of \( E \)” in the theorem plays a significant role.

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## 2 Weights

2.1. Weil weights. Let \( V \) be a \( \mathbb{Q}_\ell \)-linear representation of \( G_v \). Choose a lift \( \sigma_v \in G_v \) of the \( q_v \)-th power Frobenius \( \text{Frob}_v \in G_{k_v} \) and let \( P(T) = \text{det}(T - \sigma_v|V) \) be the characteristic polynomial of \( \sigma_v \) acting on \( V \). Recall that an algebraic integer \( \alpha \) is said to be a \( q \)-Weil integer of weight \( w \) if \( |\iota(\alpha)| = q^{w/2} \) for any field embedding \( \mathbb{Q} \hookrightarrow \mathbb{C} \), where \( |\cdot| \) denotes the absolute value of \( \mathbb{C} \).

**Definition 2.1.** We say that \( V \) is of type \( (W) \) at \( v \) if all the roots of \( P(T) \) are \( q_v \)-Weil integers. If this is the case, we call the weights of the roots of \( P(T) \) the **Weil weights** of \( V \) at \( v \), and denote by \( W_v(V) \) the multi-set consisting of them.

This definition does not depend on the choice of the Frobenius lift \( \sigma_v \). Also, the multi-set \( W_v(V) \) is unchanged by a finite extension of the base field \( K_v \).
Now suppose $V$ is an $E_{\lambda}$-linear representation of $G_v$. The action of the inertia subgroup $I_v$ on $V$ is quasi-unipotent ([22], Appendix); thus there exists a finite extension $K'_v/K_v$ such that the inertia subgroup $I_v$ for $K'_v$ acts unipotently on $V$ (or equivalently, trivially on the semisimplification $V^{ss}$ as an $E_{\lambda}[G_v]$-module). Hence we can consider the characteristic polynomial $P'(T) = \det(T - \text{Frob}_v|V^{ss})$ of the Frobenius $\text{Frob}_v$ acting on $V^{ss}$. (Note that the characteristic polynomial taken with $V^{ss}$ viewed as a $Q_{\ell}$-vector space is the product of the “conjugates” of this $P'(T)$.)

**Definition 2.2.** An $E_{\lambda}$-linear representation $V$ of $G_v$ is said to be $E$-integral at $v$ if, for any finite extension $K'_v/K_v$ for which the inertia action on $V$ is unipotent, the characteristic polynomial $P'(T)$ defined as above has coefficients in $O_E$.

Note that an $E$-integral representation of type $(W)$ at $v$ has Weil weights $\geq 0$ at $v$.

For example, if $X$ is a proper smooth variety over $K_v$, then the $Q_{\ell}$-linear dual $V = H^r_{\text{et}}(X_K, Q_{\ell})^*$ of the $r$-th $\ell$-adic étale cohomology group of $X_K := X \otimes_{K_v} K_v$ is conjectured to be $Q$-integral (cf. [18], Cor. 0.6 (1)). This conjecture is known to be true under the assumption of the existence of the Künneth projector ([16], Cor. 0.6 (1)).

We note here that, by the next lemma, there are totally ramified extensions among the finite extensions $K'_v/K_v$ as above (so that, when we want to compare the characteristic polynomials $P'(T)$ for different $V$’s, we can use a $K'_v$ with residue degree $f = 1$):

**Lemma 2.3.** If $L/K_v$ is a finite Galois extension, then there exists a totally ramified subextension $L'/K_v$ of $L/K_v$ such that $L = L'L_0$, where $L_0$ is the maximal unramified subextension of $L/K_v$.

**Proof.** If $L/K_v$ is abelian, this is a consequence of local class field theory. Suppose $L/K_v$ is non-abelian. We proceed by induction on the extension degree $[L : K_v]$. Let $\sigma$ be a lift in $G := \text{Gal}(L/K_v)$ of the Frobenius in $\text{Gal}(L_0/K_v)$, and set $H := \langle \sigma \rangle$. Then we have $H \subseteq G$, and the extension $L^H/K_v$ is a non-trivial totally ramified subextension of $L/K_v$. Repeating this process with $L/K_v$ replaced by $L/L^H$, we are reduced to the case of abelian $L/K_v$. \hfill $\square$

2.2. Hodge-Tate weights. Recall that $u$ is a finite place of $K$ lying above $\ell$. A $Q_{\ell}$-linear representation $V$ of $G_u$ is said (cf. [7]) to be of Hodge-Tate type of
Hodge-Tate weights $h_1, \ldots, h_n$, where $n = \dim_{\mathbb{Q}_\ell}(V)$ and $h_i$ are integers, if one has $V \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell \cong \mathbb{C}_\ell(h_1) \oplus \cdots \oplus \mathbb{C}_\ell(h_n)$ as a $\mathbb{C}_\ell$-semilinear $G_u$-representation, where $\mathbb{C}_\ell(h)$ denotes the $h$-th Tate twist of the completion $\mathbb{C}_\ell$ of a fixed algebraic closure $\bar{\mathbb{Q}}_\ell$. If this is the case, let $\text{HT}_u(V)$ denote the multiset of Hodge-Tate weights of $V$. Note that $\text{HT}_u(V)$ is unchanged by a finite extension of the base field $K_u$.

2.3. Tame inertia weights. Let $I_{u}^{\text{tame}}$ the tame inertia group of $K$ at $u$ (= the quotient of the inertia group $I_u$ at $u$ by its maximal pro-$\ell$ subgroup). A character $\varphi : I_{u}^{\text{tame}} \to \mathbb{F}_{\ell}^\times$ can be written in the form $\varphi = \psi_1^{t_1} \cdots \psi_h^{t_h}$, where $\psi_i$ are the fundamental characters of level $h$ ([19], §1.7) and $0 \leq t_i \leq \ell - 1$. Then we set $\text{TI}_u(\varphi) := \{t_1/e, \ldots, t_h/e\}$ (as a multi-set), where $e = e(K_u/\mathbb{Q}_\ell)$ is the ramification index of $K/\mathbb{Q}$ at $u$. Note that, by §1.4 of [19], $\text{TI}_u(\varphi)$ is unchanged by a “moderately” ramified extension of $K_u$; precisely speaking, if $K'_u/K_u$ is a finite extension of ramification index $e(K'_u/K_u) < (\ell - 1)/\max\{t_j | 1 \leq j \leq h\}$, then we have $\text{TI}_{u'}(\varphi|_{I_u^{\text{tame}}}) = \text{TI}_u(\varphi)$.

Let $V$ be a $\mathbb{Q}_\ell$-linear representation of $G_u$, and $T$ a $G_u$-stable $\mathbb{Z}_\ell$-lattice of $V$. Set $\bar{T} := T/\ell T$. Then its semisimplification $\bar{T}^{\text{ss}}$ (as an $\mathbb{F}_\ell[G_u]$-module) is tamely ramified (note that its isomorphism class does not depend on the choice of $T$), and the action of the tame inertia group $I_{u}^{\text{tame}}$ is described by a sum of characters $\varphi_i : I_{v}^{\text{tame}} \to \mathbb{F}_{\ell}^\times$. Then we define $\text{TI}_u(V)$ (as a multi-set) to be the union of the $\text{TI}_u(\varphi_i)$ for all $i$.

2.4. Weights of geometric Galois representations. Let $V$ be a $\mathbb{Q}_\ell$-linear representation of $G_K$. For any multi-set $X$, we write $\Sigma(X) := \sum_{x \in X} x$, whenever the sum on the right-hand side has a meaning.

**Definition 2.4.** We say that $V$ is of type $(G)$ if it is of type $(W)$ at $v$, of Hodge-Tate type at $u$, and one has

$$(G) \quad \Sigma(W_v(V)) = 2\Sigma(\text{HT}_u(V)).$$

If this is the case, we denote this value by $w(V)$ and call it the total weight of $V$. 

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Note that $\Sigma(W_v(V))$ and $\Sigma(HT_u(V))$ are respectively the Weil and Hodge-Tate weights of $\text{det}\mathbb{Q}_\ell(V)$.

Typical examples of $V$ of type (G) include the Tate twists $\mathbb{Q}_\ell(r)$ for $r \in \mathbb{Z}$ and their twists by characters of finite order; their total weights are $2r$.

A priori, the notion of type (G) depends on the places $v \nmid \ell$ and $u \mid \ell$ (so it should be called, say, type $(G_u,v)$), but in practice (i.e., in case $V$ comes from algebraic geometry), it should be independent of the places. The proof of the following proposition, which is modeled on the proof of Lemma 2.1 of [17], has been communicated to us by Yoichi Mieda, to whom we are grateful:

**Proposition 2.5.** Let $X$ be a proper smooth variety over $K$. Let $V = H^r_{\text{et}}(X_K, \mathbb{Q}_\ell)^* \otimes K$ be the $\mathbb{Q}_\ell$-linear dual of the $r$-th $\ell$-adic étale cohomology group of $X_K := X \otimes_K K$, and put $n = \dim_{\mathbb{Q}_\ell}(V)$. Then we have:

(i) $\det(V)$ is isomorphic to the twist of $\mathbb{Q}_\ell(nr/2)$ by a character $\varepsilon$ of order at most 2. If $r$ is odd, then $\varepsilon = 1$.

(ii) $V$ is of type (G) with respect to any finite places $u \mid \ell$ and $v \nmid \ell$ of $K$.

Note that, in (i), the Betti number $n$ is even if $r$ is odd by, say, the Hodge symmetry.

**Proof.** (ii) follows from (i) immediately. To show (i), consider the character $\varepsilon : G_K \to \mathbb{Q}_\ell^\times$ defined by $\det(V)(-nr/2)$, where $(-nr/2)$ denotes the $(-nr/2)$-th Tate twist. If $v$ is a finite place of $K$ where $X$ has good reduction, then by [5] $V$ is $\mathbb{Q}$-integral and has all Weil weights equal to $r$. Hence $\varepsilon(\text{Frob}_v)$ is a Weil integer in $\mathbb{Q}$ of weight 0, i.e., a unit of $\mathbb{Z}$. Since $\text{Frob}_v$’s for such $v$’s are dense in $G_K$, we see that $\varepsilon$ takes values in $\mathbb{Z}^\times$. The second statement of (i) follows from Corollary 3.3.5 of [23].

In some cases, we can expect the total weight $w(V)$ to be equal also to $2\Sigma(TI_u(V))$:

**Proposition 2.6.** Let $V$ be a $\mathbb{Q}_\ell$-linear semistable representation of $G_u$ with $HT_u(V) \subset [0,b]$. If $e(K_u/\mathbb{Q}_\ell)b < \ell - 1$, then we have:

(i) ([3], Thms. 1.0.3 and 1.0.5) $TI_u(V) \subset [0,b]$.

(ii) ([4], Thm. 1) $\Sigma(HT_u(V)) = \Sigma(TI_u(V))$.

The equality (G) holds in general if $K = \mathbb{Q}$:

**Lemma 2.7.** Let $q$ be a prime number $\neq \ell$. If $V$ is a $\mathbb{Q}_\ell$-linear representation of $G_q$ which is of type (W) at $q$ and of Hodge-Tate type at $\ell$, then $V$ is of type (G).
Proof. By taking the determinant, we are reduced to the case \( \dim_{\mathbb{Q}_\ell}(V) = 1 \). Then \( V \) is geometric (in the sense of Fontaine-Mazur [9] (note that a one-dimensional \( \mathbb{Q}_\ell \)-representation is de Rham if and only if it is Hodge-Tate) and hence is a twist by a finite character of \( \mathbb{Q}_\ell(r) \) for some integer \( r \). Thus (G) holds for \( V \).

If \( K \neq \mathbb{Q} \), the equality (G) may not hold even for a geometric representation. For example, let \( K \) be an imaginary quadratic field, \( E \) an elliptic curve over \( K \) such that \( \text{End}_K(E) \otimes \mathbb{Z} \mathbb{Q} \simeq K \), and \( \ell \) a prime number which splits in \( K \) as \( \ell = \lambda \lambda' \). Let \( V \) be a one dimensional \( G_K \)-subrepresentation of the \( \ell \)-adic Tate module \( T_\ell(E) \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell \) of \( E \). Then \( V \) is of type (W) of Weil weight 1 at any \( v \nmid \ell \), while it is of Hodge-Tate type of Hodge-Tate weight 0 or 1 at \( \lambda \).

If we do not assume the equality (G), we can in fact prove an equality which is fairly close to (G) under a mild condition:

**Proposition 2.8.** Let \( V \) be a \( \mathbb{Q}_\ell \)-linear representation of \( G_K \) and \( q \) a prime number \( \neq \ell \). Assume \( V \) is of type (W) at all places \( v \mid q \) and of Hodge-Tate type at all places \( u \mid \ell \). Then we have

\[
\sum_{v \mid q} [K_v : \mathbb{Q}_q] \Sigma(W_v(V)) = 2 \sum_{u \mid \ell} [K_u : \mathbb{Q}_\ell] \Sigma(HT_u(V)).
\]

**Proof.** The induced representation \( \text{Ind}^{G_\mathbb{Q}}_{G_K}(V) \) is a representation of \( G_\mathbb{Q} \) which is of type (W) at \( q \) and of Hodge-Tate type at \( \ell \), and hence we have

\[
\Sigma(W_q(\text{Ind}^{G_\mathbb{Q}}_{G_K}(V))) = 2\Sigma(HT_\ell(\text{Ind}^{G_\mathbb{Q}}_{G_K}(V)))
\]

by Lemma 2.7. We then observe that

\[
W_q(\text{Ind}^{G_\mathbb{Q}}_{G_K}(V)) = \prod_{v \mid q} [K_v : \mathbb{Q}_q] W_v(V),
\]

\[
HT_\ell(\text{Ind}^{G_\mathbb{Q}}_{G_K}(V)) = \prod_{u \mid \ell} [K_u : \mathbb{Q}_\ell] HT_\ell(V),
\]

where the multiple \( mX \) of a multi-set \( X \) by a positive integer \( m \) is defined in the obvious manner. Indeed, we have

\[
(\text{Ind}^{G_\mathbb{Q}}_{G_K}(V))|_{G_q} = \bigoplus_{v \mid q} \text{Ind}^{G_\mathbb{Q}}_{G_v}(V|_{G_v})
\]
by Mackey’s formula ([21], Section 7.3, Proposition 22), and
\[
W_q(\text{Ind}_{G_v}^G(V|_{G_v})) = [K_v : Q_q]W_v(V|_{G_v})
\]
by definition of the induced representation and by the invariance of the Weil weights by finite extensions of the base field. Similar equalities hold for \(u|\ell\) and \(\text{Ind}_{G_u}^G(V|_{G_u})\).

\[\boxdot\]

3 Proof of the theorems

We begin with a version of the gap principle:

**Lemma 3.1.** Let \(E, n, v\) be as before, and let \(w \in \mathbb{R}_{\geq 0}\) be given. Then there exists a constant \(C_1 = C_1([E : Q], n, q_v^w) > 0\) such that, for any prime \(\ell > C_1\) and for any \(n\)-dimensional \(E_\lambda\)-linear representations \(V, V'\) of \(G_v\) which are of type (W), \(E\)-integral at \(v\) and such that \(\Sigma(W_v(V)), \Sigma(W_v(V'))\) are in \([0, [E_\lambda : Q_\ell] \cdot w]\), the following (i) and (ii) hold:

(i) If \(V \equiv_{ss} V' (mod \lambda)\) as \(G_v\)-representations, then \(W_v(V) = W_v(V')\).

(ii) Assume further that \(V^{ss}\) and \((V')^{ss}\) are unramified. If \(V \equiv_{ss} V' (mod \lambda)\) as \(G_v\)-representations, then \(V \cong_{ss} V'\) as \(G_v\)-representations.

The constant \(C_1\) can be taken explicitly to be

\[
C_1 := 2\left(\frac{n}{[n/2]}\right)^{q_v^w/2} [E : Q]/f_\lambda.
\]

We have also the following mod \(\ell\) version of (ii) above, in which the constant is independent of \([E : Q]\):

**Lemma 3.2.** Let \(E, n, v\) be as before, and let \(w \in \mathbb{R}_{\geq 0}\) be given. Then there exists a constant \(\tilde{C}_1 = \tilde{C}_1(n, q_v^w) > 0\) such that, for any prime \(\ell > C_1\) and for any \(n\)-dimensional \(E_\lambda\)-linear representations \(V, V'\) of \(G_v\) such that \(V^{ss}, (V')^{ss}\) are unramified and which are of type (W), \(E\)-integral at \(v\) and such that \(\Sigma(W_v(V)), \Sigma(W_v(V'))\) are in \([0, [E_\lambda : Q_\ell] \cdot w]\), the following holds: If \(\det(T - \text{Frob}_v|V) \equiv \det(T - \text{Frob}_v|V') (mod \ell \mathcal{O}_E)\), then one has \(V \cong_{ss} V'\) as \(G_v\)-representations.

The constant \(\tilde{C}_1\) can be taken explicitly to be

\[
\tilde{C}_1 := 2\left(\frac{n}{[n/2]}\right)^{q_v^w/2}.
\]
Proof. As the proofs are similar, we only give a proof of Lemma 3.1. Choose a totally ramified extension $K'/K$ over which $V$ and $V'$ become semistable (cf. Lem. 2.3). Let $P(T) = \det(T - \operatorname{Frob}_v | V^{ss})$ and $P'(T) = \det(T - \operatorname{Frob}_{v'} | (V')^{ss})$ be the characteristic polynomials (taken as $E_\lambda$-linear representations) of the Frobenius $\operatorname{Frob}_v$ at $v'$ acting on the semisimplifications $V^{ss}$ and $(V')^{ss}$, respectively. By assumption, they have coefficients in $O_E$.

By assumption on the weights, for any embedding $E \hookrightarrow \mathbb{C}$, the terms of $T^{n-i}$ have coefficients of absolute value $\leq \binom{n}{i} q_v^{w/2}$ Note that $\Sigma(W_v(V))$ is the sum of the Weil weights of $V$ as a $\mathbb{Q}_\ell$-linear representation, and hence the sum of the Weil weights of the roots of $P(T)$ is in $[0, w]$. Set $C_1 := (2 \max_{0 \leq i \leq n} \binom{n}{i} q_v^{w/2})^{[E:Q]/f_\lambda} = (2 \binom{n}{\lfloor n/2 \rfloor} q_v^{w/2})^{[E:Q]/f_\lambda}$. If then $\ell > C_1$, we have

$$V \equiv_{ss} V' \pmod{\lambda} \quad \text{as } G_v\text{-representations}$$

$$\iff P(T) \equiv P'(T) \pmod{\lambda}$$

$$\iff P(T) = P'(T).$$

Here, the last equivalence follows from the next lemma. This implies that $W_v(V) = W_v(V')$. If $V^{ss}$ and $(V')^{ss}$ are unramified, then they are determined by the actions of $\operatorname{Frob}_v$, and hence the equality $P(T) = P'(T)$ is equivalent to $V \simeq_{ss} V'$.

Lemma 3.3. Let $a$ be a non-zero integer of $E$ and $C_0$ a real number $> 0$. If $a \equiv 0 \pmod{\lambda}$ (resp. $a \equiv 0 \pmod{\ell O_E}$) and $|\ell(a)| \leq C_0$ for any embedding $\iota : E \hookrightarrow \mathbb{C}$, then we have $\ell \leq C_0^{[E:Q]/f_\lambda}$ (resp. $\ell \leq C_0$).

Proof. If $\lambda | a$ (resp. $\ell | a$ in $O_E$), then by taking the norm $N : E^\times \to \mathbb{Q}^\times$, we have $\ell^f_\lambda \leq |N(a)|$ (resp. $\ell^{[E:Q]} \leq |N(a)|$). If $|\ell(a)| \leq C_0$, then by taking the norm (or product over all $\iota$), we have $|N(a)| \leq C_0^{[E:Q]}$. The required inequality follows from these two inequalities. \qed

We need one more lemma:

Lemma 3.4. Let $G$ be a profinite group and $T, T'$ be free $O_{E_\lambda}$-modules on which $G$ acts continuously and $O_{E_\lambda}$-linearly. Let $(T/\ell^e T)^{ss}$ and $(T/\ell T)^{ss}$ be the semisimplifications of $T/\ell T$ and $T/\ell T$ as $k_\lambda[G]$-modules, respectively. Let $e$ be the ramification index of $E_\lambda/Q_\ell$. Then we have:

(i) $(T/\ell T)^{ss}$ is isomorphic to the direct-sum of $e$ copies of $(T/\ell T)^{ss}$.

(ii) If $(T/\ell T)^{ss} \simeq (T'/\ell T')^{ss}$, then $(T/\ell T)^{ss} \simeq (T'/\ell T')^{ss}$. 

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Proof. Part (ii) follows from Part (i) immediately. To prove (i), consider the filtration

\[ T/\ell T = T/\lambda^{i}T \supset \lambda^{i+1}T/\lambda^{i}T \supset \cdots \supset \lambda^{e}T/\lambda^{e}T = 0. \]

Then “multiplication by \( \lambda \)” (where \( \lambda \) is identified with a uniformizer at \( \lambda \)) induces isomorphisms \( \lambda^{i}T/\lambda^{i+1}T \rightarrow \lambda^{i+1}T/\lambda^{i+2}T \) of the graded quotients as \( k_{\lambda}[G] \)-modules. It then follows that \( (T/\ell T)^{\mathrm{ss}} \simeq ((T/\lambda T)^{\mathrm{ss}})^{\otimes e} \). \( \square \)

Now we can prove the theorems. We only prove Theorem 1.1 and 1.2, the proof of Theorem 1.3 being similar. Let \( C = \max \{ e^{2}b+1, (2\left( \frac{n}{2} \right) q_{v}^{\ell} \lambda^{b})^{[E:Q]/f_{s}} \} \), as in Theorem 1.1. Choose a finite totally ramified extension \( K'/u \), with absolute ramification index \( e^{2} \), over which \( V \) and \( V' \) become semistable (cf. Lem. 2.3). If \( \ell > C \), then \( e^{2}b < \ell - 1 \). Take \( K' \) a finite extension of \( K \) and \( \nu' \) a place of \( K' \) such that the completion of \( K' \) at \( \nu' \) is \( K'_{\nu'} \).

By assumption, we have \( \mathrm{HT}_{\nu}(V) \subset [0, b] \). Then by (i) of the Proposition 2.6, we have \( \mathrm{TL}_{\nu}(V) \subset [0, b] \). The same holds for \( V' \), since we have \( \mathrm{TL}_{\nu'}(V) = \mathrm{TL}_{\nu'}(V') \) by the assumption \( V \equiv_{\mathrm{ss}} V' \pmod{\lambda} \) as \( G_{\nu} \)-representations (Note that, by Lemma 3.4, we have also \( V \equiv_{\mathrm{ss}} V' \pmod{\ell} \) as \( F_{\ell}[G_{u}]-\mathrm{modules} \), where \( V \) and \( V' \) are now regarded as \( Q_{\ell} \)-linear representations, so that the definition of \( \mathrm{TL}_{u} \) and Proposition 2.6 are applicable). Now we recall that \( V \) and \( V' \) are of type \( (G) \). By (ii) of Proposition 2.6, we have \( \Sigma(\mathrm{TL}_{\nu}(V)) = \Sigma(\mathrm{HT}_{\nu}(V)) = \Sigma(\mathrm{HT}_{\nu}(V')) = (1/2)\Sigma(W_{v}(V)) \), and these are also equal to \( \Sigma(\mathrm{TL}_{\nu'}(V')) = \Sigma(W_{v}(V')) \). Since \( \mathrm{HT}_{\nu}(V) \subset [0, b] \), these are bounded by \( [E_{\lambda} : Q_{\ell}].nb \). In particular, total weights \( \Sigma(W_{v}(V)) \) and \( \Sigma(W_{v}(V')) \) are \( \leq [E_{\lambda} : Q_{\ell}].2nb \). By (i) (resp. (ii)) of Lemma 3.1, the assumption that \( V \equiv_{\mathrm{ss}} V' \pmod{\lambda} \) as \( G_{\nu} \)-representations implies that \( W_{v}(V) = W_{v}(V') \) (resp. \( V \simeq_{\mathrm{ss}} V' \) as \( G_{v} \)-representations) if \( \ell > (2\left( \frac{n}{2} \right) q_{v}^{\ell} \lambda^{b})^{[E:Q]/f_{s}} \). \( \square \)

4 Corollaries

Here we give several corollaries of Theorem 1.2, which are motivated by a conjecture of Rasmussen and Tamagawa ([14]; see also [2], [12], [13] and [15]). The notations \( (K, E, n, b, e, v, u, \ell, \lambda, C = C([E:Q], n, b, e, q_{v}), ...) \) are the same as in the theorem. In this section, \( V = V^{+}_{X} \) will be the \( E_{\lambda} \)-linear dual \( H^{*}_{\alpha}(X_{K}, E_{\lambda})^{*} \) of the \( r \)-th \( \lambda \)-adic étale cohomology group, where \( X \) is a smooth proper variety (variety := separated scheme of finite type over a field) over \( K \).
and $X_K$ denotes its base extension to $\overline{K}$. We set $\overline{V} = \overline{V}_X := T/\lambda T$, choosing a $G_K$-stable $O_{E_\lambda}$-lattice in $V$, and let $\overline{V}^{ss} = \overline{V}_X^{r,ss}$ be its semisimplification as a $k_\lambda[G_K]$-module ($\overline{V}_X^{r,ss}$ does not depend on the choice of $T$). To state the first corollary, we make the following hypothesis on $\overline{V}^{ss}$:

**Hypothesis (H).** Each simple factor $\overline{W}$ of $\overline{V}^{ss}$ lifts to an $E_\lambda$-linear representation $\overline{W}$ of $G_K$ of the form $H_{et}^s(Y_K, E_\lambda)^*$ which is semistable at all $u \mid \ell$ and $HT_u(W) \subset [0, \ell - 2]$, where $Y$ is a proper smooth variety over $K$ and $s$ is some non-negative integer.

**Corollary 4.1.** For any prime $\ell > C$, any odd integer $r$ with $1 \leq r \leq b$, any places $u$ of $K$ and $\lambda$ of $E$ both lying above $\ell$, and any smooth proper variety $X$ which has the $r$-th Betti number $\leq n$, has potentially good reduction at $v$, and has semistable reduction at some place $u \mid \ell$, if (H) is true for $\overline{V}_X^{r,ss}$, then none of the simple factors of $\overline{V}_X^{r,ss}$ are of odd dimension.

**Proof.** Note first that, if $s$ is odd, then $H_{et}^s(Y_K, E_\lambda)$ has even dimension by (GAGA and) Hodge theory. Now, let $W_1, \ldots, W_k$ be the simple factors of $\overline{V}^{ss}$. By (H), each $W_i$ lifts to a geometric $W_i$ with $HT_u(W_i) \subset [0, \ell - 2]$. If one of the $W_i$ has odd dimension, then it must have even weight, while $V$ has odd weight $r$, since $X$ has potentially good reduction at $v$. Thus the corollary follows from Theorem 1.2 by putting $V' := W_1 \oplus \cdots \oplus W_k$. □

As a special case where the Hypothesis (H) holds, we have:

**Corollary 4.2.** For any prime number $\ell > C$, any odd integer $r$ with $1 \leq r \leq b$, any places $u$ of $K$ and $\lambda$ of $E$ both lying above $\ell$, and any smooth proper variety $X$ over $K$ which has $r$-th Betti number $\leq n$, has potentially good reduction at $v$, and has semistable reduction at $u$, the Galois representation on $\overline{V}_X^{r,ss}$ is not the sum of integral powers mod $\ell$ cyclotomic characters.

In fact, we can generalize this a bit as follows. Let $\chi$ and $\bar{\chi}$ denote respectively the $\ell$-adic and mod $\ell$ cyclotomic characters of $G_K$.

**Corollary 4.3.** Assume $E$ contains the $\ell^2$-th roots of unity. Then for any prime number $\ell > C$ such that $\ell \equiv 1 \pmod{\ell^2}$, any odd integer $r$ with $1 \leq r \leq b$, any places $u$ of $K$ and $\lambda$ of $E$ both lying above $\ell$, and any smooth proper variety $X$ over $K$ which has $r$-th Betti number $\leq n$, has potentially good reduction at $v$, and acquires semistable reduction over a finite extension $K'_u/K_u$ with absolute ramification index $\epsilon(K'_u/Q_\ell)$ dividing $e$, the Galois representation $\overline{V}_X^{r,ss}$ is not the sum of characters of $G_K$ of the form $\bar{\varepsilon}_i \chi^{b_i}$. 

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where $\bar{\varepsilon}_i : G_K \to k_\lambda^\times$ are characters unramified at $u$ and of finite order dividing the order of the group of roots of unity in $E$, and $b_i$ are integers.

Proof. Suppose $X$ has semistable reduction over $K'_{\ell'}$ with $e(K'_{\ell'}/\mathbb{Q}_\ell) | e$. We may assume $e(K'_{\ell'}/\mathbb{Q}_\ell) = e$. Suppose $\bar{V}^{ss}$ is the sum of the characters $\bar{\varepsilon}_i \chi^{b_i}$ as above. Then the action of the tame inertia group $I_{u'}^{tame}$ at $u'$ on the $i$-th factor is via $\bar{\chi}^{b_i}$, which equals $\theta^{eb_i}$, where $\theta$ is the fundamental character of $I_{u'}^{tame}$ of level 1 ([19], Sect. 1.8, Prop. 8). By (i) of Proposition 2.6, we have $eb_i \equiv c_i \pmod{\ell - 1}$ with $0 \leq c_i \leq eb$. Since $e^2 | \ell - 1$, we have $b_i = b_{0i} + \frac{e^2 - 1}{e^2} j$ with $0 \leq b_{0i} \leq b$ and $0 \leq j < e^2$. Set $\bar{\kappa} := \bar{\chi}^{(\ell-1)/\ell^2}$ and let $\kappa : G_K \to E_\lambda^\times$ be its Teichmüller lift. Since the $e^2$-th power of $\kappa$ is trivial, it takes values in $E^\times$. Similarly, the Teichmüller lift $\bar{\varepsilon}_i$ of $\bar{\varepsilon}_i$ has also values in $E^\times$. Now each character $\bar{\varepsilon}_i \chi^{b_i} = \bar{\varepsilon}_i \bar{\kappa}^{b_{0i}}$ lifts to the character $\varepsilon_i \kappa^j \chi^{b_{0i}} : G_K \to E_\lambda^\times$, or to the 1-dimensional $E_\lambda$-linear $E$-integral geometric representation $E_\lambda(\varepsilon_i \kappa^j) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(b_{0i})$, where $E_\lambda(\varepsilon_i \kappa^j)$ is the twist of the trivial representation $E_\lambda$ by the finite character $\varepsilon_i \kappa^j$ and $\mathbb{Q}_\ell(b_{0i})$ denotes the $b_{0i}$-th Tate twist. Let $V'$ be the direct-sum of these representations. By Theorem 1.2, we have $W_v(V) = W_{\bar{v}}(V')$, but $W_v(V) = \{r,...,r\}$ (since $X$ has potentially good reduction at $v$) while $W_{\bar{v}}(V') = \{2b_{01},...,2b_{0m}\}$, which is a contradiction if $r$ is odd.

Specializing further, we have:

Corollary 4.4. Let $K = \mathbb{Q}$. Assume $E$ contains the $e^2$-th roots of unity. Then for any prime number $\ell > C$ such that $\ell \equiv 1 \pmod{e^2}$, for any odd integer $r$ with $1 \leq r \leq b$, and for any smooth proper variety $X$ over $\mathbb{Q}$ which has $r$-th Betti number $\leq n$, has good reduction outside $\ell$ and acquires semistable reduction over a finite extension $K'_{\ell'}/\mathbb{Q}_\ell$ with absolute ramification index $e(K'_{\ell'}/\mathbb{Q}_\ell)$ dividing $e$, the Galois representation on $\bar{V}$ is not Borel.

Here, we say that the representation $\bar{V}$ is Borel if the action of $G_{\mathbb{Q}}$ is given by upper-triangular matrices with respect to a suitable $k_\lambda$-basis of $\bar{V}$.

Proof. Indeed, if it is Borel, its semisimplification is a sum of characters, which are unramified outside $\ell$ by assumption. Since the base field is $\mathbb{Q}$, they are powers of the mod $\ell$ cyclotomic character. Now the the result follows from the previous corollary. 

\[\square\]
5 Congruences of modular forms

We use the same notations as in the Introduction, except that we always suppose $K = \mathbb{Q}$ and write $q$ for $q_v$ in this section. We put $\varphi(N) = \#(\mathbb{Z}/N\mathbb{Z})^\times$ for any positive integer $N$ and denote by $\mathbb{Z}$ the integer ring of $\mathbb{Q}$. The goal of this section is to give a proof of the following congruence result on the Fourier coefficients of modular forms. For any integers $k, N \geq 1$ and a character $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, let $S_k(N, \epsilon)$ denote the $\mathbb{C}$-vector space of cusp forms of weight $k$, level $N$ and Nebentypus character $\epsilon$. For a normalized Hecke eigenform $f(z) = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z} \in S_k(N, \epsilon)$, integers $i, j$ and a prime number $\ell$, consider the following condition on the Fourier coefficients $a_p(f)$ of $f$:

\[(C_{i,j;\ell}) \quad a_p(f) \equiv p^i + p^j \pmod{\ell \mathbb{Z}} \quad \text{for all but finitely many primes } p \nmid \ell N.\]

For fixed $k$ and $N$, it is well known (cf. e.g. Thm. 10 of [20] and the Introduction of [11]) that there are only finitely many exceptional primes, and a fortiori finitely many primes $\ell$ for which $(C_{i,j;\ell})$ hold for some $i, j$ and $f \in S_k(N, \epsilon)$. Until recently, however, the situation had not been very clear when we let $k$ and $N$ vary; as for recent works, see [10] for the case of modular Abelian varieties and [1] for the case of modular forms on $\Gamma_0(N)$. In this vein, we show the following by using Theorem 1.3:

**Theorem 5.1.** Fix a prime number $q$. For any integer $k \geq 1$, any prime $\ell > 4q^2(k-1)$, any integer $N$ such that $q \nmid N$, $\ell \nmid \varphi(N)$ and $\ell^2 \nmid N$, any character $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, and any normalized Hecke eigenform $f \in S_k(N, \epsilon)$, we have the following:

(i) The condition $(C_{i,j;\ell})$ can hold only if $i \equiv j \equiv (k - 1)/2 \pmod{\ell - 1}$.

(ii) The condition $(C_{i,j;\ell})$ holds for no $i$ and $j$ if either $k = 1$, $k$ is even, or $\ell \nmid N$.

We begin by proving a lemma. For any $f$ as in the theorem, we denote by $E = \mathbb{Q}_f$ the field obtained by adjoining all Fourier coefficients of $f$ to $\mathbb{Q}$, which is a finite extension of $\mathbb{Q}$. We regard $\epsilon$ as a character with values in $\mathcal{O}_E^\times$. Denote by $\bar{\epsilon}$ (resp. $\bar{\epsilon}_\lambda$) the composite $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}_E^\times \overset{\mod \ell}{\to} (\mathcal{O}_E/\ell \mathcal{O}_E)^\times$ (resp. $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}_E^\times \overset{\mod \lambda}{\to} (\mathcal{O}_E/\lambda \mathcal{O}_E)^\times$). Let

\[\rho_{f,\lambda} : G_\mathbb{Q} \to \text{GL}_E(V_{f,\lambda})\]
be the 2-dimensional $E_\lambda$-linear representation of $G_Q$ associated with $f$. Thus if $p \nmid \ell N$, then $V_{f,\lambda}$ is unramified at $p$ and one has

\[
\det(T - \text{Frob}_p|V_{f,\lambda}) = T^2 - a_p(f)T + \epsilon(p)p^{k-1}.
\]

In particular, it is $E$-integral at $p$ in the sense of Definition 2.2. One has $W_p(V_{f,\lambda}) = \{(k-1)/2, (k-1)/2\}$. It is crystalline (resp. semistable) at $\ell$ if $\ell \nmid N$ (resp. $\ell^2 \nmid N$).

**Lemma 5.2.** Suppose $\ell > 2$. Let $k \geq 1$ and $N \geq 1$ be integers with $\ell \nmid \varphi(N)$. Let $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a character. Suppose that a normalized Hecke eigenform $f \in S_k(N,\epsilon)$ satisfies the condition $(C_{i,j;\ell})$ for some $i, j$. Then $\bar{\epsilon}$ has values in fact in the canonical image of $F^\times \ell$ in $(\mathcal{O}_E/\ell \mathcal{O}_E)^\times$. Moreover, the following holds:

(i) We have $\bar{\epsilon}(x \pmod{N}) = x^{i+j-(k-1)} \pmod{\ell}$ for any $x$ prime to $N$.

(ii) If $\ell \nmid N$, then we have $i + j \equiv k - 1 \pmod{\ell - 1}$ and $\bar{\epsilon} = 1$.

**Proof.** By assumption, we have $\text{Tr}(\text{Frob}_p|V_{f,\lambda}) \equiv p^i + p^j \pmod{\ell \mathcal{O}_E}$ for all but finitely many $p \mid \ell N$. In particular, we have

\[
(1) \quad \rho_{f,\lambda} \equiv_{ss} \chi^i \oplus \chi^j \pmod{\lambda}
\]

as $k_\lambda$-linear representations of $G_Q$ (This holds because $\ell > \dim \rho_{f,\lambda}$; see e.g. Lemma 2.10 of [13]), and then we have also $\epsilon(p)p^{k-1} \equiv p^{i+j} \pmod{\lambda}$. Hence we see that

\[
(2) \quad \bar{\epsilon}_\lambda(x \pmod{N}) = x^{i+j-(k-1)} \pmod{\lambda}
\]

for any $\lambda | \ell$ and any integer $x$ prime to $N$.

(i) Since the kernel of the projection $(\mathcal{O}_E/\ell \mathcal{O}_E)^\times \to \prod_{\lambda | \ell} (\mathcal{O}_E/\lambda \mathcal{O}_E)^\times$ has $\ell$-power order, if $\ell \nmid \varphi(N)$, then the homomorphism $\prod_{\lambda | \ell} \bar{\epsilon}_\lambda : (\mathbb{Z}/N\mathbb{Z})^\times \to \prod_{\lambda | \ell} (\mathcal{O}_E/\lambda \mathcal{O}_E)^\times$ lifts uniquely to a homomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathcal{O}_E/\ell \mathcal{O}_E)^\times$, which is $\bar{\epsilon}$. According to (2), it is given by

\[
(3) \quad \bar{\epsilon}(x \pmod{N}) = x^{i+j-(k-1)} \pmod{\ell \mathcal{O}_E}
\]

for any integer $x$ prime to $N$.

(ii) Suppose $\ell \nmid N$. Then (3) must hold for $x = \ell$, which is possible only if $i + j \equiv k - 1 \pmod{\ell - 1}$. In particular, we obtain $\bar{\epsilon} = 1$. \qed
Proof of Theorem 5.1. (i) Suppose $\ell \nmid \varphi(N)$ and $\ell^2 \nmid N$. Then $\rho_{f,\lambda}$ is semistable at $\ell$. By assumption, we have $\text{Tr}(\text{Frob}_q|V_{f,\lambda}) \equiv q^i + q^j$ (mod $\ell \mathcal{O}_E$). Combining this with Lemma 5.2 (i), we obtain $\det(T - \text{Frob}_q|V_{f,\lambda}) \equiv \det(T - \text{Frob}_q|\chi^i \oplus \chi^j)$ (mod $\ell \mathcal{O}_E$). We also have the congruence (1). Therefore, if $\ell > 4q^2(k - 1)$, it follows from Theorem 1.3 (applied with $V' = \chi^{i'} \oplus \chi^{j'}$, where $i', j'$ are integers in $[0, \ell - 2]$ such that $i' \equiv i, j' \equiv j$ (mod $\ell - 1$)) that $\rho_{f,\lambda} \simeq_{ss} \chi^i \oplus \chi^j$ as $E_\lambda$-linear representations of the decomposition group $G_q$ of $q$. Looking at the Weil weights, we obtain $i \equiv j \equiv (k - 1)/2$ (mod $\ell - 1$).

(ii) If $k$ is even, then the impossibility of $(C_{i,j;\ell})$ follows from Part (i).

If $k = 1$ and the congruence condition (C_{i,j;\ell}) holds, then Part (i) together with (1) implies that $\tilde{\rho}_{f,\lambda} := \rho_{f,\lambda} \pmod{\lambda}$ is unipotent and, in particular, $\text{Im}(\tilde{\rho}_{f,\lambda})$ is an $\ell$-group. On the other hand, if $k = 1$, then by [6], $\text{Im}(\rho_{f,\lambda})$ is finite and its image in $\text{PGL}_2(\mathcal{O}_{E_\lambda})$ is either dihedral, $A_4$, $S_4$ or $A_5$. Since the kernel of the reduction map $\text{GL}_2(\mathcal{O}_{E_\lambda}) \to \text{GL}_2(k_\lambda)$ is pro-$\ell$, the representation $\tilde{\rho}_{f,\lambda}$ cannot be unipotent if $\ell \geq 3$.

Finally, assume $\ell \nmid N$. Then $\rho_{f,\lambda}$ is crystalline at $\ell$, and thus the Fontaine-Laffaille theory [8] implies that the tame inertia weights and the Hodge-Tate weights of $\rho_{f,\lambda}$ coincide with each other. Hence it follows from (1) that $\{i, j\} \equiv \{0, k - 1\}$ (mod $\ell - 1$). Since $\ell > k$, we obtain $\{(k - 1)/2, (k - 1)/2\} = \{0, k - 1\}$, which is impossible unless $k = 1$. ∎

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