Thermal Bosonic Green Functions Near Zero Energy

T.S. Evans
Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ U.K.

Abstract

The properties of the various types of bosonic Green functions at finite temperature in the zero energy limit are considered in the light of recent work.

Nous discutons, à la lumière de récents travaux, des propriétés de certains types de fonctions de Green à température finie, dans le cas limite où l’énergie est nulle.

1 Introduction

Zero energy Green functions are used in several important physical problems. The most obvious is in calculations of the free energy, the finite temperature generalisation of the effective potential. This can be thought of as the generating functional of the 1PI diagrams with zero four-momenta on external legs [1]. The zero-energy bosonic self-energies are also crucial when trying to improve standard one-loop calculations of the free energy. More generally, the effective potential can be thought of as the lowest order term in a derivative expansion of the effective action [1]. Effective actions are useful tools in their own right and to understand whether or not such a derivative expansion can be performed at non-zero temperature means we must know something about the analyticity of the Green functions near zero four-momenta.

*E-mail: UMAPT85@UK.AC.IC.CC.VAXA
The purpose of this paper is to look at the properties of various types of bosonic Green functions near zero energy in the light of recent work \[2, 3, 4\], in which differences between retarded and time-ordered Green functions etc. were highlighted. This will be done with particular reference to a calculation of Bedaque and Das \[3\] and one in \[4\].

In ITF (Imaginary Time Formalism) \[4, 5\] the energy is discrete, \(E = 2\pi \nu / \beta\) where \(\nu\) is an integer for bosons. Thus one can calculate the free energy directly as the zero energy value of bosonic Green functions is required in all methods. The alternative approach is to calculate the Green functions at general discrete energy and then make an analytic continuation to complex energies (requiring boundary conditions at large energies to be imposed on the Green functions \[6\]). One can then in principle investigate the analyticity of the Green functions and so the effective action near zero energy. In zero temperature Euclidean field theory one would obtain the same result with this method as when one calculates the Green functions at zero energies, i.e. the Green functions are analytic near zero energies at zero. One point worth noting is that after the continuation to real energies has been made in ITF the Green functions obtained always include the retarded and advanced functions which are quite different at non-zero temperature from the time-ordered functions normally encountered in modern field theory \[2, 3, 4\].

An alternative approach to thermal field theories are the RTF (Real-Time Formalisms) \[1, 6, 8, 9\]. These involve fields at real physical times and energies, unlike ITF. To achieve this one has to double the degrees of freedom so that the vertices have a thermal label attached \((a = 1, 2)\) and the propagators are two-by-two matrices. The diagrams with all external legs or vertices type one (that is the thermal label is one) are by definition in RTF time-ordered thermal expectation values so that the connection with standard Minkowskii zero temperature field theory is much closer with RTF than with ITF.

However, even in the early days of RTF, it was noticed that diagrams can contain singular terms. These come about because the non-zero temperature corrections to the propagators are always proportional to an on-shell delta function (particles in the heat bath are real not virtual particles). For instance the propagator for a scalar field can be written as

\[
\begin{align*}
\imath \Delta \Phi_{ab}(k_0, \vec{k}) &= \imath \tilde{\Delta} \Phi_{ab}(k_0, \vec{k}) + 2\pi \delta(k\Phi_2 - m\Phi_2) n(|k_0|) D\Phi_{ab}(k_0, \vec{k}), \\
\imath \tilde{\Delta} \Phi_{ab}(k_0, \vec{k}) &= \begin{pmatrix} \imath \tilde{\Delta} & 0 \\ 0 & -\imath \tilde{\Delta} \Phi^* \end{pmatrix} \Phi_{ab}, \\
D\Phi_{ab}(k_0, \vec{k}) &= 2\pi \delta(k\Phi_2 - m\Phi_2) n(|k_0|) \begin{pmatrix} 1 & \exp(-\beta|k_0|/2) \\ \exp(-\beta|k_0|/2) & 1 \end{pmatrix} \Phi_{ab}, \\
n(z) &= [\exp(\beta z) - 1] \Phi_{ab}, \quad (1. 2)
\end{align*}
\]

where

\[
\tilde{\Delta} = (k\Phi_2 - m\Phi_2 + \imath \epsilon) \Phi_{ab} - 1, \quad n(z) = [\exp(\beta z) - 1] \Phi_{ab} - 1.
\]
It is possible for two or more lines in a diagram to carry the same four-momenta and thus for singular terms of the form \[\delta(k\Phi^2 - m\Phi^2)\Phi^N \geq 2\] to appear. This can happen when a section of a diagram corresponds to a self-energy insertion. This leads to contribution of the form

\[
\sum_{c,d=1,2} n(|k_0|)\Phi^2[\delta(k\Phi^2 - m\Phi^2)]\Phi^2 D\Phi^1 c R\Phi^d D\Phi^d
\]

where \(D\) is the matrix coefficient of the \(n\delta\) term in the propagator and \(\Sigma\) is the matrix self-energy insertion. With internal vertices one has to sum over all values of the thermal labels attached to the vertices, and this is the sum over \(c, d\). Any one term in the sum in (1.3) contains singular contributions to the integrand of the form noted above. However when all contributions from all the different vertex labelling are included, the singular part is cancelled leaving a regular contribution to the Feynman diagram. Using the matrix structure of the propagators and self energy insertions, which follow from the fundamental properties of thermal Green functions (KMS condition for two-point functions), one can show that such singularities always cancel [6, 8, 10].

The other situation where these singularities occur is when the external leg of a diagram carries zero four-momenta. Then we find we have a contribution of the form

\[
\sum_{c,d=1,2} n(|k_0|)\Phi^2[\delta(k\Phi^2 - m\Phi^2)]\Phi^2 D\Phi^1 c R\Phi^d D\Phi^d
\]

where \(R\) represents the rest of the diagram. Note that if the whole diagram is a tadpole then \(R\) is a self-energy diagram and one can use the same proof as with general self-energy insertions to show that in this case the singularity cancels. For more general cases there is no cancellation. It is not surprising that the tadpole is a special case, it is the only case where the external leg must carry zero four-momenta, i.e. it is not telling us anything about analyticity near zero energy. From this discussion, it is clear that the question of the singularities in RTF is closely related to the behaviour of Green functions near zero four-momenta.

Originally, such singularities in RTF zero four-momenta diagrams were dealt with by the introduction of an ‘ad-hoc’ rule [11], namely that one should keep only one external vertex fixed to be type one and sum over all possible labelling of the remaining vertices, internal and external. Further this then gave the same result as would be obtained when using ITF on these diagrams. The first question that we shall answer is this rule really ‘ad-hoc’ and what exactly is it doing?

With regards to the apparent singularities encountered in RTF zero four-momenta diagrams, there is suggestion that with a suitable regularisation there is no problem with these diagrams. In the work of Bedaque and Das [5] and Landsman and van Weert [6], the suggestion is that one must keep the \(\epsilon\), from the \(\nu\epsilon\) terms in the propagators, finite.
The $\epsilon \to 0$ limit is only taken at the very end of a calculation. Thus the delta functions of RTF should be replaced as
\[
\delta(k\Phi^2 - m\Phi^2) \to \frac{1}{\pi} \frac{\epsilon}{(k\Phi^2 - m\Phi^2)\Phi^2 + \epsilon\Phi^2}.
\]

In [5, 6] one-loop two-point 1PI diagrams in RTF with both vertices fixed to be type one (so that it is a time-ordered function) are considered. In the case of [5] the real part only is investigated in a cubic self-interacting scalar theory. The real part of the diagram is claimed to be finite, identical to the ITF result and analytic in both energy and three momenta near zero energy. In [6], the one loop gauge boson self-energy in the Feynman gauge of a pure $SU(N)$ gauge theory is calculated, both real and imaginary parts. Again the result quoted for zero four-momenta is finite, pure real and identical with the result of ITF.

The fact that the real parts are claimed to be identical with the ITF result is not surprising in that it is an identity that even at finite temperature, that retarded, advanced and time-ordered functions have identical real parts for all four-momenta (see below or [6]). Further the $[\delta(k\Phi^2 - m\Phi^2)]\Phi^2 N \geq 2$ singularity only appears in the imaginary part so the only remarkable aspect of the calculation in [6] is the analyticity of the result. The interesting aspect of the [6] result is that it is identical to the ITF result, and in particular has a finite real part and a zero not infinite imaginary part. The results of these papers suggest that if we use this $\epsilon$ regularisation, then one can calculate the time-ordered function and it is equal to the retarded functions (the latter is obtained using is obtained directly in ITF or the extra or ad-hoc rule in RTF).

The results obtained using this $\epsilon$ regularisation again raise the question, what is happening when the extra rule for such zero energy calculations in RTF is used? However, three further questions come to mind. Are the real parts of the time-ordered functions always equal to retarded and advanced functions at zero energy? This is equivalent to asking are the usual results for the real parts of diagrams in RTF and ITF at zero energy the same. Next, are the time-ordered or retarded functions analytic near zero four-momenta? Lastly are the time-ordered functions, the usual RTF result, finite at zero four-momenta and therefore calculable with a suitable regularisation scheme? We shall try to answer all but the last question in the rest of this paper.

## 2 The Extra Rule In RTF

We shall first look at exactly what is going on with this mysterious ‘ad-hoc’ rule for zero energy diagrams in RTF. This ‘ad-hoc’ rule was first derived by Matsumoto et al. [12] using the Thermo Field Dynamics approach to RTF for the case where the Free energy is being calculated. These authors laid the emphasis on the fact that the free energy was not
a Green function and so one should not necessarily expect the same rules to be used when calculating them. Later it was shown within the path integral approach to RTF that the generating functional, or equivalently the partition function, for time-independent external sources could not be factorised in the normal manner \[3, 14\]. However, it was shown there that the problems due to these additional contributions to the generating functional could be avoided by using a simple trick. Alternatively, one can completely avoid the need to factorise out unwanted contributions to the partition function of RTF in the path integral approach by using a different path in the complex time plane \[9\]. In either case, the derivations in \[3, 13\] prove that the complete partition function in RTF can be calculated using the standard Feynman rules provided an additional rule was used. This is precisely the rule that had been introduced ad-hoc elsewhere \[11\]. An important point in this derivation was that the same rule was essential for calculating the partition function even when the external sources were static in time but varying in space \[14\]. In terms of Feynman diagrams the contributions to the partition function comes from diagrams other than the ones with all external vertices are type one even when the external legs carry non-zero three-momentum. These additional contributions are in general non-zero in such situations though they are often zero for very simple diagrams. None of the diagrams are singular for general non-zero external three-momenta as this ensures that every leg carries a unique four-momenta.

It is to be emphasised that the extra Feynman rule for zero four-momenta is not ad-hoc. In \[12, 13, 14\] it was derived in very precise situations. In particular these papers constitute the only derivations within RTF of how to calculate the free energy. What is being demonstrated by this extra rule is that there is a different relationship between Green functions and the effective potential at non-zero temperature from that which exists at zero temperature.

The recent work on the relationships between the various types of real-time Green functions \[2, 3, 4\] throw some further light on this extra rule. The time-ordered expectation value of fields at real-times is simply the RTF function with all external legs or vertices fixed to be type one. The \(N\) retarded \(N\)-point Green functions, \(R_a\), are multiple commutators. For pure bosonic fields they are defined to be

\[
R_a(t_1, t_2, ..., t_N) = \sum_{\text{perm.} \{a\}|a_N = a} \prod_{j=1}^{N} \Phi a - 1\theta(t_{a_{j+1}} - t_a)[[[[\phi_a, \phi_{a_{N-1}}], \phi_{a_{N-2}}], \ldots], \phi_{a_1}] \quad (2.6)
\]

where \(\phi_a = \phi_a(t_a, \vec{x}_a)\). The sum is over \(\{a_j\}, j = 1\) to \(N\), running through all permutations of the numbers 1 to \(N\) subject to the constraint \(a_N = a\). The advanced functions \(A_a\) are defined in a similar manner except we replace all the \(\theta(t)\) by \(\theta(-t)\) and add an overall factor of \((-1)^{\Phi a - 1}\), c.f. the two-point case in \[3, 7\]. For fields of mixed statistics see \[3\]. There is a direct link between ITF and the retarded and advanced functions. If the
result of an ITF calculation is $\Phi\{z = 2\pi\nu/\beta\}$ then after a suitable analytic continuation to real energies, $\{E\}$, we find that

$$
\Phi\Phi_N(\{E + i\epsilon\}|\epsilon_a > 0, \epsilon_{other} < 0) = R_a(\{E\}), \\
\Phi\Phi_N(\{E + i\epsilon\}|\epsilon_a < 0, \epsilon_{other} > 0) = A_a(\{E\}),
$$

(2.7)

where $\{\epsilon\}$ are infinitesimal and real.

The relevant point here is that one can also relate all the RTF functions to each of the retarded and advanced functions [3]. For instance for truncated RTF functions, $\Gamma_{tr}\Phi_{\nu_1 \ldots \nu_N}$ we have

$$
R_{tr,a}(\{E\}) = \sum_{\mu_j=1,2|\mu_a=1} \prod_{j=1}^N \Phi_N(1,0) e^{\Phi\bar{\alpha}E_j} \Phi_{\mu_j \nu_j} \Gamma_{\mu_1 \nu_1 \ldots \nu_N}(\{E\})
$$

(2.8)

$$
A_{tr,a}(\{E\}) = \sum_{\mu_j=1,2|\mu_a=1} \prod_{j=1}^N \Phi_N(1,0) \sigma_j e^{-\Phi\bar{\alpha}E_j} \Phi_{\mu_j \nu_j} \Gamma_{tr,\nu_1 \ldots \nu_N}(\{E\})
$$

(2.9)

The parameter $\bar{\alpha} = 1 - \alpha$ describes how far below the real axis the return section of the RTF curve runs.

As it is more usual to look at 1PI functions we define

$$
R\Phi(N)_{tr,a}(\{E\}) = [R\Phi(2)_{a}(E_a) \prod_{b \neq a} R\Phi(2)_{b}(E_b)]\Phi - 1R\Phi(N)_a(\{E\})
$$

$$
A\Phi(N)_{tr,a}(\{E\}) = [R\Phi(2)_{a}(E_a) \prod_{b \neq a} R\Phi(2)_{b}(E_b)]\Phi - 1A\Phi(N)_a(\{E\})
$$

(2.10)

where all the energies, $\{E\}$, are flowing into the diagrams, $R\Phi(N)_a$ and $R\Phi(N)_{tr,a}$ are the $N$-point connected and 1PI retarded functions respectively, with the $a$-th leg having the largest time. The $R\Phi(2)_a$ are the retarded propagators associated with the fields of the $a$-th leg. Likewise for the advanced functions. Such a definition ensures that these are the functions that obtains when one calculates a 1PI diagram in ITF with the usual continuation to real energies.

Now if we specialise to the case of bosonic functions and take the zero energy limit these expressions simplify considerably to give

$$
R_{tr,a}(\{E = 0\}) = A_{tr,a}(\{E = 0\}) = \sum_{\mu_j=1,2|\mu_a=1} \Gamma_{tr,\mu_1 \ldots \mu_N}(\{E = 0\}).
$$

(2.11)

Thus it is clear that when using the extra zero-energy rule in RTF one is calculating retarded or advanced Green functions (they are all the same at zero energy as this proof shows) not time-ordered functions that are normally considered in RTF, the $\Gamma\Phi_{11 \ldots 1}$.  


In terms of the types of Green functions being calculated this means that at non-zero temperature the free energy is a generating function for the retarded and advanced functions at zero four-momenta. This is to be contrasted with the situation at zero temperature where the effective potential is generating time-ordered functions. As the non-zero temperature proof holds at all temperatures, the zero temperature limit tells us that the time-ordered, retarded and advanced functions at zero four-momenta and zero energy are all identical. In moving to non-zero temperature, it is well known that big differences appear between the time ordered functions and the retarded and advanced functions [2, 3]. The only derivations of the free energy at non-zero temperature [12, 13, 13] tell us that correct generalisation of the zero-temperature expansion of the effective potential in terms of real-time Green functions is in terms of retarded and advanced functions and not time-ordered functions. The ‘ad-hoc’ or ‘extra’ rule is therefore really neither of these things, it is the zero-energy limit of the prescription for obtaining retarded or advanced functions directly in RTF as (2. 11) shows.

3 Differences between Green Functions

The methods of [3] can be used to look at the relation between time-ordered, retarded and advanced thermal Green functions near zero energy. In particular are they equal or are the real parts equal in general?

The bosonic two-point connected time ordered function is $i\Pi_{\mu\nu}\Phi11(t)$, where

$$i\Pi_{\mu\nu}\Phi11 = \langle\langle T\phi_\mu(t)\phi_\nu\Phi^\dagger(0)\rangle\rangle$$

(3. 12)

and the $\mu, \nu$ indices represent any indices on the fields. No thermal labels are needed as the type one field are identical to the real-time fields. The time-ordered propagator is related to the retarded and advanced functions by

$$\Pi_{\mu\nu}\Phi11(E) = R_{\mu\nu}\Phi(2)(E) + n(E)(R_{\mu\nu}\Phi(2)(E) + A_{\mu\nu}\Phi(2)(E)).$$

(3. 13)

From the definition of the retarded and advanced functions we have that

$$iR_{\mu\nu}\Phi(2)(E)+iA_{\mu\nu}\Phi(2)(E) = \rho_{\mu\nu}(E) = \int dt e\Phi t E t \left[ \langle\langle \phi_\mu(t)\phi_\nu\Phi^\dagger(0)\rangle\rangle - \langle\langle \phi_\nu\Phi^\dagger(0)\phi_\mu(t)\rangle\rangle \right].$$

(3. 14)

Using the cyclicity of the trace in the thermal expectation values $\langle\langle . . . \rangle\rangle$ gives the KMS relation between the thermal Wightman functions on the right hand side. Inserting into (3. 13) this gives

$$\lim_{E \to 0} (i\Pi_{\mu\nu}\Phi11(E, \vec{x}) - iR_{\mu\nu}\Phi(2)(E, \vec{x})) = \int dt \langle\langle \phi_\mu(t, \vec{x})\phi_\nu\Phi^\dagger(0, \vec{0})\rangle\rangle.$$
Thus only if $\rho_{\mu\nu}(E = 0)/(\beta E)$ is zero can the time-ordered and retarded and advanced functions be equal. From (2.11) we have that $R_{\mu\nu}\Phi(2)(E = 0, k) = A_{\mu\nu}\Phi(2)(E = 0, k)$ and so $\rho_{\mu\nu}(E = 0) = 0$. For instance this can be shown using spectral representation [6]. However $\rho$ may still be linear in $E$ near zero energy.

Inverting the two-point spectral representation of RTF gives

$$\Pi_{\mu\nu}^1 \Phi(2)(E) = (1 + n(E))R_{\mu\nu} - n(E)A_{\mu\nu}(E)$$

and similar manipulations as before tell us that

$$\lim_{E \to 0}[(\Pi_{\mu\nu}^1 - 1)\Phi_{11}(E) - R_{\mu\nu}^{-1} - n(E)A_{\mu\nu}^{-1}] = -i \int dt \langle\langle \phi(0) | \Phi(0) \rangle \phi(t) | \Phi(0) \rangle}.$$  

Looking at $\gamma(0, \vec{x}) = \int dt \langle\langle \phi(0, \vec{x}) | \Phi(0) \rangle \phi(t) | \Phi(0) \rangle}$ one can insert a complete set of energy eigenstates, $\{|n, i\}$, and find

$$\gamma(0, \vec{p}) = Z\Phi^{-1} \sum_n e^{E_n} \{ \sum_{i,j} |\langle n, i | \phi(0, \vec{0}) | n, j \rangle \} |\Phi \geq 0, \gamma(0, \vec{p}) \in \Re,$$

where we have simplified to the case of a scalar field. The sums over $n$ run through all the distinct energy values and $g(E_n)$ is the degeneracy of the $E_n$ energy level. The sums over $i$ and $j$ take the bra and ket states through all possible states with the same energy, $E_n$ but which differ in three-momentum by $\vec{p}$. At zero temperature we can thus conclude that this thermal Wightman function is zero as there are no vacuum to one particle processes. However at non-zero temperature one has a non-zero $\gamma(0)$ if there are $n$ to $n+1$ particle processes. In this case the heat bath is providing a background of real physical particles which can participate in reactions precisely in this way and thus $\gamma(0)$ is non-zero. It is exactly this sort involvement of arbitrary numbers of particles coming from the heat bath in any given process that distinguishes zero and non-zero temperature field theory. This is the physics that underlies the existence of Landau damping.

Mathematically, for non-zero three-momenta, there is always a cut across along the real axis running across the zero energy point in the complex energy plane of non-zero temperature self-energies because of this Landau damping. One sees it even at the one-loop level in simple scalar theories [13]. While the discontinuity across the cut is zero at zero energy $\rho(E) \to \beta E\gamma(0)$, the Landau damping processes still ensure $\gamma$ is non-zero. The infra-red divergence of the Bose-Einstein distribution cancels the $E$ factor leaving the non-zero derivative of the spectral function at zero energy, $\gamma(0, \vec{p})$. This ensures that there is a difference between the time-ordered and retarded two-point functions at zero energy and non-zero temperature. Note that in the above discussion we have kept
the spatial or three-momenta dependence arbitrary. Also note that $\gamma(0)$ is real so that $(\Pi\Phi - 1)\Phi_{11}$ and $R\Phi - 1$ differ only in the imaginary part.

A similar analysis can be performed on bosonic three-point functions. This is essential as the work of \cite{2, 3, 4} shows that the relation between retarded and time-ordered functions is much more complicated for general functions than the special case of two-point functions would suggest. The pure bosonic time-ordered connected function, $G_{\Phi 111}$, is related to the pure bosonic three-point retarded and advanced functions, $R\Phi(3)_{a}, A\Phi(3)_{a}$, through the relation

$$G_{\Phi 111}(E_{1}, E_{2}) = \sum_{cycle} n(E_{2})n(E_{3})(R\Phi(3)_{1} + e\Phi - \beta E_{1}A\Phi(3)_{1}), \quad (3.19)$$

where it is implicit that the third energy variable is given by $E_{3} = -E_{1} - E_{2}$. The sum is taken over cycles of the 1, 2, 3 indices. For general energies this suggests that the bosonic time-ordered connected function is $O(T\Phi 2)$ bigger than the bosonic retarded and advanced functions at high temperatures. Again we use the spectral representations for the retarded and advanced functions. There is a unique function of two independent complex energy variables $\Phi(z_{1}, z_{2})$ given by

$$\Phi(z_{1}, z_{2}) = (2\pi)^{3} \int d\Phi 3\vec{k} \sum_{cycle} \rho_{1}(k_{1}, k_{2}, k_{3}) \frac{1}{z_{2} - k_{2} z_{3} - k_{3}}, \quad \rho_{1}(t_{1}, t_{2}, t_{3}) = \langle\langle \phi_{3}(t_{3})\phi_{1}(t_{1})\phi_{2}(t_{2})\rangle\rangle - \langle\langle \phi_{2}(t_{2})\phi_{1}(t_{1})\phi_{3}(t_{3})\rangle\rangle \quad (3.20)$$

where $z_{3} = -z_{1} - z_{2}$ has been defined to make the formula tidy. The $a$-th retarded (advanced) function is obtained by letting the $a$-th energy variable approach the real axis from above (below) and the remaining two approach their real axes from the opposite side as \cite{2, 3} shows. One includes the third redundant variable $z_{3}$ in this scheme. For instance if the first field has the largest time we have the retarded function

$$R\Phi(3)_{1}(E_{1}, E_{2}) = \Phi(z_{1} = E_{1} + 2\epsilon, z_{2} = E_{2} - \epsilon) \quad (3.21)$$

where $\epsilon$ is a positive infinitesimal quantity. It is easy to show that using the cyclicity of the trace one can relate the thermal Wightman functions in the $\rho$’s of \cite{3, 20} in a generalisation of the KMS condition on two-point functions \cite{2, 3}. Taking the zero energy limit as before, we look at the real part of the connected time ordered functions $\frac{1}{2}(G_{\Phi 111} + G_{\Phi 222})$ in the notation used here. At tree level in a theory with a cubic interaction of strength $g$ between three relativistic scalar fields with dispersion relations $\omega_{a}(\vec{p}_{a})$ this would be

$$\frac{1}{2}(G_{\Phi 111}(0, \{\vec{p}\}) + G_{\Phi 222}(0, \{\vec{p}\})) = \frac{g}{\omega_{1}\Phi 2 \omega_{2}\Phi 2 \omega_{3}\Phi 2} \quad (3.22)$$
For the full connected three-point functions at zero energy but arbitrary three-momenta, we find

\[ \frac{1}{2}[G\Phi111(\{0\}, \{\vec{p}\}) + G\Phi222(\{0\}, \{\vec{p}\})] = \frac{1}{2}[R\Phi(3)_a(\{0\}, \{\vec{p}\}) + A\Phi(3)_a(\{0\}, \{\vec{p}\})] + \rho_a(\{0\}, \{\vec{p}\}), \]

\[ R\Phi(3)_1(\{0\}, \{\vec{p}\}) = R\Phi(3)_2(\{0\}, \{\vec{p}\}) = R\Phi(3)_3(\{0\}, \{\vec{p}\}) = A\Phi(3)_1(\{0\}, \{\vec{p}\}) = A\Phi(3)_2(\{0\}, \{\vec{p}\}) = A\Phi(3)_3(\{0\}, \{\vec{p}\}). \]

\[ \rho_1(\{0\}, \{\vec{p}\}) = \rho_2(\{0\}, \{\vec{p}\}) = \rho_3(\{0\}, \{\vec{p}\}). \] (3.23)

It is not obvious if \( \rho_a(\{0\}, \{\vec{p}\}) \) should be zero when quantum and thermal corrections are included. Inserting a complete set of energy and three-momentum eigenstates we see that the three-point thermal Wightman functions at zero energy are given by

\[ \gamma_{312}(\{0\}, \vec{p}_1, \vec{p}_2) = \int d\Phi 4x_1 d\Phi 4x_2 d\Phi 4x_3 \langle\phi_3(t_3, \vec{x}_3)\phi_1(t_1, \vec{x}_1)\phi_2(t_2, \vec{x}_2)\rangle \prod_{j=1}^3 e^{\Phi - \Phij \cdot \vec{x}_j} \]

\[ = Z\Phi^{-1} \sum_n e^{\Phi - \beta E_n} Tr \{V_3 V_1 V_2\} \] (3.24)

\[ (V_a)_{ij} = \langle i | \Phi_a(0, \vec{0}) | j \rangle. \] (3.25)

Again the sum over \( n \) runs through all the possible energy values and the trace runs through all the degenerate energy states of energy \( E_n \) and where the three-momentum of the bra state differs by \( p_a \) from that of the ket state in the definition of the \( V_a \) matrix in (3.25). Just as in the two-point case, it is the Landau damping phenomena present at all non-zero temperatures that ensures that the thermal Wightman functions are not zero.

However, unlike the two-point case it is a spectral function \( \rho_a \) and not the thermal Wightman functions that appear as the difference between the retarded and time-ordered Green functions at zero energy. All the spectral functions are equal and can be written as

\[ \rho_a(\{E\} = 0, \vec{p_1}, \vec{p_2}) = Z\Phi^{-1} \sum_n e^{\Phi - \beta E_n} Tr \{V_3 [V_1, V_2]\}. \] (3.26)

The trace can be rewritten as any one of the \( V \) matrices multiplied by the commutator of other two. In general the \( V \) matrices will be non-zero, unequal and non-commuting. If however two of the fields are equal and the three-momenta are zero then it is an identity that two of the matrices are equal.

Therefore at zero energy the real part of the connected three-point functions are not generally equal to the real part of the retarded and advanced three-point functions at
non-zero temperatures. One special case is where two of the fields are identical and the three-momenta are zero when it is an identity that the spectral functions are zero. This is especially relevant as this is precisely what appears in free energy calculations as the external legs of relevant diagrams all correspond to the scalar field associated with the order parameter.

While the only derivations of the free energy at non-zero temperature link it to retarded and advanced functions \([8, 12, 13, 14]\), it may, on the evidence of two-and three-point functions, be possible to express it in terms of the real parts of the time-ordered functions. It is to be stressed that such a relationship between time ordered functions and the free energy has yet to be proven.

4 Analyticity

The result of \([5]\) was that the one-loop bubble diagram in a pure self-interacting cubic scalar theory was analytic about zero four-momentum. In considering the analyticity of Green functions near zero energy, it is simplest to look at the \(\Phi\) function that contains all the retarded and advanced functions. At least for two- and three-point functions, these can then be related to the time-ordered functions \([2, 3]\). What we have shown is that the zero energy limit is very special as there the retarded and advanced functions are all equal, the discontinuities across the cuts on the real energy axes disappear. However, \(\Phi\) is \emph{not} analytic at \(\{E\} = 0\). The discontinuities across the cuts are not in general zero at infinitesimal real energies. Thus there is no neighbourhood of the zero energy point in which the function is analytic and so the function is not analytic there. Indeed even if the discontinuity is exactly zero when at the zero energy point, the cuts are still running across the zero energy point and there may well be discontinuities in derivatives at zero energy. All that has been shown is that the zero energy limit of \(\Phi\), with the other variables such as three-momenta held fixed, has a unique value however it is approached, despite the fact that the function is not analytic. Again it is the Landau damping processes that are causing this strange behaviour at non-zero temperatures. Thus the analyticity in energy reported in \([5]\) for the simple bubble diagram must be incorrect as there are Landau damping processes giving cuts across zero energy in this simple diagram \([15]\).

The calculation by Weldon \([15]\) clearly shows this cut for the case of the one-loop scalar bubble diagram, \(B\), when using ITF. The bubble diagram is defined to be the one of the generic form

\[-\Phi(B(E, p)) = \frac{1}{2}(-ig)\Phi(2\pi)\Phi-4 \int d\Phi4k \, \Delta(k; m_1)\Delta(k + p; m_2)\]

with appropriate integration and propagators for whatever formalism is being used. Even when the masses in the bubble diagram are equal and the three-momentum is taken to
zero as well as the energy (which is when the $[\delta(k\Phi - m\Phi)]\Phi$ singularities appear in the RTF calculation) one may or may not be sitting on an infinitesimally short cut depending on how this limit is taken i.e. the function does not appear to be analytic. Two simple examples of the real part of a bosonic scalar self-energy already exist and they both show this lack of analyticity. In [14] the contribution to the retarded function coming from the scalar bubble diagram was calculated in the zero four-momentum limit and was found to be

$$
\lim_{E, p \to 0} B_r(E, p) = (T = 0) + (4\pi \Phi^2)\Phi^{-1} \int_{x = \Phi^2}^\infty d\xi n(\beta m\xi) \frac{(x\Phi^2 - 1)\Phi^{1/2}}{x\Phi^2 + v\Phi^2(1 - v\Phi^2)} \Phi^{-1/2} x\Phi^2 + v\Phi^2 \gamma \Phi^2\Phi^{-1} + \ldots \tag{4.28}
$$

The lack of analyticity of bosonic two-point functions can also be seen in the leading term of the high-temperature expansion of the one-loop self-energy of a gauge boson in any gauge theory. This is given in many places and is found to be of the form (e.g. see [16])

$$
\Pi_t(E, p) \propto g\Phi^2 T\Phi^2[v\Phi^{-2} + \frac{v\Phi^2 - 1}{2v\Phi^2}] \log \left(\frac{v + 1}{v - 1}\right) + \ldots \tag{4.30}
$$

$$
\Pi_r(E, p) \propto g\Phi^2 T\Phi^2[1 - v\Phi^{-2} + \frac{1}{2v\Phi^2}] \log \left(\frac{v + 1}{v - 1}\right) + \ldots \tag{4.31}
$$

The factor of proportionality depends on the details of the gauge theory. These results are valid when all parameters are smaller than the temperatures so that $E, p$ need not be infinitesimal, they could be comparable with the zero temperature masses of any particles in the theory.

In both cases these one-loop self-energies are clearly not analytic at zero four-momentum. No problems with any sort of singularities are encountered in the calculations and the real parts are identical to the time-ordered functions, the 11 diagrams of RTF. Thus the real parts of 11 RTF diagrams are not analytic at zero four-momentum, in contradiction with the results of [14].

5 Conclusions

In this paper three questions were studied. The ‘ad-hoc’ rule for zero-energy diagrams in RTF was shown to be nothing more than the rule needed at zero energy to calculate retarded and advanced functions. It was also noted that the partition function is related to an expansion of retarded and advanced functions and not time-ordered Green functions with which one is familiar in zero-temperature field theory.

The difference between Green functions were then studied without any approximation being made. For the two-point functions it was shown that the time-ordered and retarded
functions differ in their imaginary part by a non-zero quantity. This contradicts the results for the one-loop QCD self-energy given in [6]. In the case of the real part of connected Green functions a difference was found between time-ordered and retarded functions. Generally it is not zero but in the case where two of the fields are identical, such as is relevant to free energy calculations the difference appears to be zero. There is therefore a possibility that the real parts of time-ordered and retarded bosonic Green functions may be equal in special cases. In particular one might be able to use the time-ordered Green functions to calculate free-energies directly but this has yet to be shown.

Finally, it was noted that one-loop self-energy diagrams sometimes show explicitly a lack of analyticity at zero four-momentum. This is understandable on physical grounds and one will always expect to find it at some order in the calculation. This calls into question derivative expansions of effective actions at finite temperature.

Overall it would seem that the ǫ regularisation stressed in [6, 5] will not help. The ‘problems’ of analyticity and equality of the different types of thermal bosonic Green functions at zero energy should be there and reflect genuine new physics not present in the more familiar zero temperature field theory.

Since this work was completed, the ǫ regularisation scheme has been studied by Weldon [17] and shown to be more intricate than suggested in [3, 4].

Acknowledgements

I would like to thank R. Kobes, G. Kunstatter and the Institute for Theoretical Physics at the University of Winnipeg for their hospitality. Illuminating discussions with M.A. van Eijck, R.L. Kobes, J.C. Taylor and H.A. Weldon are gratefully acknowledged. I would like to thank the Central Research Fund of the University of London for financial support that made my visit to Winnipeg possible.

References

[1] R.J. Rivers, Path Integral Methods in Quantum Field Theory (Cambridge University Press, Cambridge, 1987).

[2] T.S. Evans, Phys. Lett. B249 (1990) 286; “N-Point Functions in Imaginary- and Real-Time at Finite Temperature”, pp.441, in ‘Thermal Field Theories’, ed. H. Ezawa, T. Arimitsu and Y. Hashimoto (North Holland, Amsterdam, 1991).

[3] T.S. Evans, Nucl. Phys. B374 (1992) 340.
[4] R. Kobes, Phys. Rev. D42 (1990) 562; *ibid* D43 (1991) 1269; Phys. Rev. Lett. 67 (1991) 1347.

[5] P.F. Bedaque and A. Das, Phys. Rev. D45 (1992) 2906.

[6] N.P. Landsman and Ch.G. van Weert, Phys. Rep. 145 (1987) 141.

[7] A.A. Abrikosov, L.P. Gor’kov and I.Ye. Dzyaloshinskii, Quantum Field Theoretical Methods in Statistical Physics (Pergamon Press, Oxford, 1965); A. Fetter and J. Walecka, Quantum Theory of Many-Particle Systems (McGraw-Hill, New York, 1971); J.I. Kapusta, Finite Temperature Field Theory (Cambridge University Press, Cambridge, 1989).

[8] A.J. Niemi, G.W. Semenoff, Ann. Phys. 152 (1984) 305; Nucl. Phys. B220 (1984) 181.

[9] T.S. Evans, “A New Time Contour for Equilibrium Real-Time Thermal Theories”, Imperial College preprint, Imperial/TP/91-92/36, September 1992.

[10] H. Matsumoto, I. Ojima and H. Umezawa, Ann. Phys. 152 (1984) 348.

[11] Y. Fujimoto and R. Grigjanis, Z. Phys.C 28 (1985) 395; Y. Fujimoto and R. Grigjanis, Prog. Th. Phys. 74 (1985) 1105; Y. Fujimoto, *in* ‘Quantum Field Theory’, ed. F. mancini, Elsevier Sci. Pub, (1986); I. Ojima, *ibid*.

[12] H. Matsumoto, Y. Nakano and H. Umezawa, Phys. Rev. D31 (1985) 1495.

[13] T.S. Evans, Z.Phys.C 36 (1987) 153;

[14] T.S. Evans, Z.Phys.C 41 (1988) 333.

[15] H.A. Weldon, Phys. Rev. D28 (1983) 2007.

[16] H.A. Weldon, Phys. Rev. D26 (1982) 1394.

[17] H.A. Weldon, “Mishaps with Feynman parametrization at finite temperature”, West Virginia Univ. preprint WVU-922 (August, 1992).