The example of a self-similar continuum which is not an attractor of any zipper.

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Let $S$ be a system $\{S_1, \ldots, S_m\}$ of injective contraction maps of a complete metric space $(X, d)$ to itself and let $K$ be it’s invariant set, i.e. such a nonempty compact set $K$ that satisfies $K = \bigcup_{i=1}^{m} S_i(K)$. The set $K$ is also called the attractor of the system $S$. A natural construction allowing to obtain the systems $S$ with a connected (and therefore arcwise connected) invariant set is called a self-similar zipper and it goes back to the works of Thurston [4] and Astala [2] and was analyzed in detail by Aseev, Kravtchenko and Tetenov in [5]. Namely,

**Definition 0.1** A system $S = \{S_1, \ldots, S_m\}$ of injective contraction maps of complete metric space $X$ to itself is called a zipper with vertices $(z_0, \ldots, z_m)$ and signature $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m$ if for any $j = 1, \ldots, m$ the following equalities hold: 1. $S_j(z_0) = z_{j-1+\varepsilon_j}$; 2. $S_j(z_m) = z_{j-\varepsilon_j}$.

If the maps $S_i$ are similarities (or affine maps) the zipper is called self-similar (correspondingly self-affine).

We shall call the points $z_0$ and $z_m$ the initial and the final point of the zipper respectively.

The simplest example of a self-similar zipper may be obtained if we take a partition $P$, $0 = x_0 < x_1 < \ldots < x_m = 1$ of the segment $I = [0, 1]$ into $m$ pieces and put $T_i = x_{i-1+\varepsilon_i}(1-t) + x_{i-\varepsilon_i}t$. This zipper $\{T_1, \ldots, T_m\}$ will be denoted by $S_{P, \vec{\varepsilon}}$.

**Theorem 0.2** (see [5]). For any zipper $S = \{S_1, \ldots, S_m\}$ with vertices $\{z_0, \ldots, z_m\}$ and signature $\vec{\varepsilon}$ in a complete metric space $(X, d)$ and for any partition $0 = x_0 < x_1 < \ldots < x_m = 1$ of the segment $I = [0, 1]$ into $m$ pieces there exists unique map $\gamma : I \rightarrow K(S)$ such that for each $i = 1, \ldots, m$, $\gamma(x_i) = z_i$ and $S_i \cdot \gamma = \gamma \cdot T_i$ (where $T_i \in S_{P, \vec{\varepsilon}}$). Moreover, the map $\gamma$ is Hölder continuous.
The mapping $\gamma$ in the Theorem is called a linear parametrization of the zipper $S$. Thus, the attractor $K$ of any zipper $S$ is an arcwise connected set, whereas the linear parametrization $\gamma$ may be viewed as a self-similar Peano curve, filling the continuum $K$.

**Some Peano curves.**

a) The attractor $K$ of a self-similar zipper $S$ with vertices $(0, 0), (1/4, \sqrt{3}/4), (3/4, \sqrt{3}/4), (1, 0)$ and signature $(1, 0, 1)$ is the Sierpinsky gasket.

![Figure 1: 1,2,4, and 8 iterations in the construction of the Peano curve for Sierpinsky gasket.](image)

b) A self-similar zipper with vertices $(0, 0), (0, 1/2), (1/2, 1/2), (1, 0)$ and signature $(1, 0, 0, 1)$ produces a self-similar Peano curve for the square $[0, 1] \times [0, 1]$

![Figure 2: Iterations for square-filling Peano curve.](image)

c) A self-similar zipper with vertices $(0, 0), (0, 1/3), (1/3, 1/3), (1/3, 2/3), (1/3, 1), (2/3, 1), (2/3, 2/3), (2/3, 1/3), (2/3, 0), (1, 0)$ and signature $(0, 1, 0, 0, 1, 0, 0, 1, 0)$ gives a Peano curve for Sierpinsky carpet.

d) The attractor of a zipper with vertices $(0, 0), (1, 0), (1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (2/3, 1), (2/3, 2/3), (2/3, 1/3), (2/3, 0), (1, 0)$ and signature $(0, 1, 0, 0, 1, 0, 0, 1, 0)$ gives a Peano curve for Sierpinsky carpet.
(2,1), (2,0), (3,0) and signature (0,0,1,1,1,0,0) is a dendrite.

Figure 3: A zipper whose attractor is a dendrite.

The main example.

The following example shows that there do exist self-similar continua which cannot be represented as an attractor of a self-similar zipper.

Let \( S \) be a system of contraction similarities \( g_k \) in \( \mathbb{R}^2 \) where \( S_2(\vec{x}) = \vec{x}/2 + (2,0) \), and \( S_k(\vec{x}) = \vec{x}/4 + \vec{a}_k \) where \( \vec{a}_k \) run through the set \{ (0,0), (3,0), (1,2h), (3/2,3h) \}, \( h = \sqrt{3}/2 \) for \( k = 1, 3, 4, 5 \). Let \( K \) be the attractor of the system \( S \) and \( T \) – the Hutchinson operator of the system \( S \) defined by \( T(A) = \bigcup_{j=1}^{5} S_j(A) \).

We shall use the following notation: By \( \Delta \) we denote the triangle with vertices \( A = (0,0) \), \( B = (2,2\sqrt{3}) \) and \( C = (4,0) \). The point \( (2,0) \) is denoted by \( D \). For a multiindex \( i = i_1...i_k \) we denote \( S_i = S_{i_1}...S_{i_k} \), \( \Delta_i = S_i(\Delta) \), \( K_i = S_i(K) \), \( A_i = S_i(A) \), etc.

1. The set \( K \) is a dendrite. The way the system \( S \) is defined (see [3, Thm.1.6.2]) guarantees the arcwise connectedness of \( K \). Since for each \( n \) the set \( T^n(\Delta) \) is simply-connected, the set \( K \) contains no cycles and therefore \( K \) is a dendryte. Each point of \( K \) has the order 2 or 3. If a point \( x \) has the order 3, it is an image \( S_i(D) \) of the point \( D \) for some multiindex \( i \). Any path in \( K \) connecting a point \( \xi \in J \) with a point \( \eta \in \Delta_i, i = 4, 5, 24, 25, 224, 225, .., \).
passes through the point $D$.

2. Each non-degenerate line segment $J$ contained in $K$, is parallel to $x$ axis and is contained in some maximal segment in $K$ which has the length $4^{1-n}$.

Consider a non-degenerate linear segment $J \subset K$. There is such multiindex $i$, that $J$ meets the boundary of $S_i(\Delta)$ in two different points which lie on different sides of $S_i(\Delta)$ and do not lie in the same subcopy of $K_1$. Then $J' = g_i^{-1}(J \cap K_i)$ is a segment in $K$ with the endpoints lying on different sides of $D$ which is not contained in neither of subcopies $K_1, \ldots, K_5$ of $K$. Then $J' = [0, 4]$. Since a part of $J$ is a base of some triangle $S_i(\Delta)$, the length of the maximal segment in $K$ containing $J$ is $4^{1-n}$ where $n \leq |i|$.

3. Any injective affine mapping $f$ of $K$ to itself is one of the similarities $S_1 = S_{i_1} \cdot \ldots \cdot S_{i_k}$. Since $f$ maps $[0, 4]$ to some $J \subset S_1([0, 4])$ for some $i$, it is of the form $f(x, y) = (ax + b_1y + c_1, b_2y + c_2)$, with positive $b_2$. Choosing appropriate composition $S_{i_1}^{-1} \cdot f \cdot S_{i_2}(K)$ we obtain a map of $K$ to itself sending $[0, 4]$ to some subset of $[0, 4]$.

Therefore we may suppose that $f(x, y) = (ax + b_1y + c_1, b_2y)$, and that
the image $f(\Delta)$ is contained in $\Delta$ and is not contained in any $\Delta_i, i = 1, \ldots, 5$.

If $f(B) \in \Delta_i, i = 4, 5, 24, 25$, then, since every path from $J$ to $f(B)$ passes through $D, f(D) = D$ and therefore $c_1 = 2 - a$.

If $f(B) \in \Delta_i, i = 4, 5$, then $1/2 \leq b_2 \leq 1$. In this case $y$–coordinates of the points $f(B_1), f(B_3)$ are greater than $\sqrt{3}/4$, so they are contained in $\Delta_1$ and $\Delta_3$, therefore the map $f$ either keeps the points $D_1, D_3$ invariant, or transposes them. In each case $|a| = 1$ and $f(\{A, C\}) = \{A, C\}$. If in this case $f(B) \neq B$, then $f(A)$ cannot be contained in $T(\Delta)$. The same argument shows that if $f(B) = B$, then $f(A) \neq C$. Therefore $f = \text{Id}$.

Suppose $f(B) \in \Delta_i, i = 24, 25$ and $a > 1/2$. Then the points $f(B_1), f(B_3)$ are contained in $\Delta_1$ and $\Delta_3$, therefore the map $f$ either keeps the points $D_1, D_3$ invariant, or transposes them, so $|a| = 1$ and $f(\{A, C\}) = \{A, C\}$. Considering the intersections of the line segments $[A, f(B)]$ and $[f(B), C]$ with the boundary of $T(\Delta)$ and $T^2(\Delta)$ we see that either $f(A)$ or $f(C)$ is not contained in $T^2(\Delta)$, which is impossible.

Therefore, either $a \leq 1/2$ or $f = \text{Id}$. The first means that $f(\Delta) \subset \Delta_2$, which contradicts the original assumption, so $f = \text{Id}$.

4. The set $K$ cannot be an attractor of a zipper. Let $\Sigma = \{\varphi_1, \ldots, \varphi_m\}$ be a zipper whose invariant set is $K$. Let $x_0, x_1$ be the initial and final points of the zipper $\Sigma$. Let $\gamma$ be a path in $K$ connecting $x_0$ and $x_1$. Since for every $i = 1, \ldots, m$ the map $\varphi_i$ is equal to some $S_i$, the sets $\varphi_i(K)$ are the subcopies of $K$, therefore for each $i$ at least one the images $\varphi_i(x_0), \varphi_i(x_1)$ is contained in the intersection of $\varphi_i(K)$ with adjacent copies of $K$. Consider the path $\tilde{\gamma} = T_{\Sigma}(\gamma) = \bigcup_{i=1}^{m} \varphi_i(\gamma)$. It starts from the point $x_0$, ends at $x_1$ and passes through all copies $K'_j$ of $K$. Each of the points $C_1 = A_2, C_2 = A_3, B_2 = C_4$ and $B_4 = A_5$ splits $K$ to two components, therefore is contained in $\tilde{\gamma}$ and is a common point for the copies $\varphi_i(\gamma), \varphi_{i+1}(\gamma)$ for some $i$. Therefore one of the points $x_0, x_1$ must be $A$, one of the points $x_0, x_1$ must be $B$, and one of the points $x_0, x_1$ must be $C$, which is impossible.

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