Improvement of a Theorem of Lorentz (1963) and its Generalization to the Multivariate Case

Zhong Guan
Department of Mathematical Sciences
Indiana University South Bend
South Bend, IN 46634, USA
zguan@iusb.edu
Tao Wang
School of Mathematical Sciences
Harbin Normal University
Harbin, 150025, China

July 29, 2018

Abstract
In this short note we have proved an enhanced version of a theorem of Lorentz [1] and its generalization to the multivariate case which gives a non-uniform estimate of degree of approximation by a polynomial with positive coefficients. The performance of the approximation at the vertices of $[0,1]^d$ is more precisely characterized by the improved result and its multivariate generalization. The latter provides mathematical foundation on which multivariate density approximation by a polynomial with positive coefficients can be established.

Key Words and Phrases: Degree of Approximation, Non-uniform estimate, Polynomial with positive coefficients.

2010 MSC: 41A10

1 Introduction

The polynomial of degree $n$ with positive coefficients studied by Lorentz [1] can be uniquely represented as $P_n(x) = \sum_{k=0}^{n} b_k p_{nk}(x)$, $b_k \geq 0$, where $p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0, \ldots, n$, which are linearly independent. An example is the Bernstein polynomial $B_n f(x) = \sum_{k=0}^{n} f(k/n) p_{nk}(x)$, for a nonnegative function $f$ defined on $[0,1]$. Recently, Bernstein polynomial and its generalization, polynomial with positive coefficients, find their important applications in nonparametric statistics for estimating an unknown probability density function $f$ which is assumed to have support on a closed interval $[0,1]$. The traditional
density estimate is the kernel density \([4, 5]\) which targets the convolution of the unknown density function and a scaled kernel function, instead of the unknown density \(f\), and suffers from serious boundary effects in the sense that the estimation errors are usually large at the endpoints of the closed supporting interval. In order that the method of Guan \([3]\), which is much better than the kernel method, can be applied to estimate the multivariate density functions, we shall extend results of Lorentz \([1]\) to the multivariate case in this note. Even better, we actually are able to prove an improved generalization of Lorentz’s result \([1]\) which clarifies mathematically why the method of \([3]\) and other methods \([6, 7]\) using Bernstein polynomials can very much reduce the boundary effects.

2 Multivariate Polynomials with Positive Coefficients

Throughout the paper, we use bold face letters to denote vectors. For example, \(\mathbf{x} = (x_1, \ldots, x_d)^T\) is a \(d\)-dimensional vector. We denote the taxicab norm by \(|\mathbf{x}| = \sum_{i=1}^d |x_i|\). Inequality \(\mathbf{x} \leq \mathbf{y}\) is understood componentwise, i.e., \(x_j \leq y_j\) for all \(j = 1, \ldots, d\). The strict inequality \(\mathbf{x} < \mathbf{y}\) means \(\mathbf{x} \leq \mathbf{y}\) but \(\mathbf{x} \neq \mathbf{y}\).

Let \(C([0, 1]^d)\) denote the class of functions \(f\) on the unit hypercube \([0, 1]^d\) that have continuous partial derivatives \(f^{(t)}(t) = \partial^{(t)} f(t)/\partial t_1^{i_1} \cdots \partial t_d^{i_d}\), where \(0 \leq \langle t \rangle \equiv l_1 + \cdots + l_d \leq k\) and \(C([0, 1]^d) = C(0)[0, 1]^d\). Define the modulus of continuity for \(f^{(r)}\) by \(\omega_r(h) = \omega(f^{(r)}, h)\), where \(\omega(f, h) = \max_{|s-t|<h} |f(s) - f(t)|\), \(h > 0\). For each \(r \geq 0\), let \(\omega^{(r)}(h) = \max_{0 \leq r < h} \omega_r(h)\).

One version of the multivariate Bernstein polynomial approximation \([8, 9]\) for \(f \in C([0, 1]^d)\) is defined as \(B_n^f(x) = \sum_{t \in \mathcal{T}} f(x(t)) \cdot \prod_{j=1}^d p_{n, i_j}(x_j) = \sum_{t \in \mathcal{T}} f(x(t)) \cdot \sum_{i=0}^n f_{m, i}\cdot p_{n, i}(x)\), where \(x = (x_1, \ldots, x_d) \in [0, 1]^d\), \(n = (n_1, \ldots, n_d)\), \(i = (i_1, \ldots, i_d)\), \(\frac{n}{m} = (\frac{n_1}{m_1}, \ldots, \frac{n_d}{m_d})\), \(p_{n, i}(x) = \prod_{j=1}^d p_{n, i_j}(x_j)\), and \(\sum_{i=0}^n a(i) = 1\). The multivariate polynomial with positive coefficients can then be defined as \(P_n(x) = \sum_{i=0}^n a(i) \cdot p_{n, i}(x)\) with \(a(\mathbf{i}) = a(i_1, \ldots, i_d) \geq 0\).

3 Degree of Approximation

To generalize the results of Lorentz \([1]\) in a convenient way, we define \(\Delta_r = \Delta^{(d)}(m, \mathcal{M})\), where \(\mathcal{M}_0 = \mathcal{M}_1 = M_0\) and \(\mathcal{M}_r = (M_0, M_4, 2 \leq \langle i \rangle \leq r)\) for \(r \geq 2\), as the class of functions \(f(t)\) in \(C([0, 1]^d)\) such that \(m \leq f(t) \leq M_0\), \(f^{(i)}(t) \leq M_i\), \(t \in [0, 1]^d\), for some \(m > 0\), \(M_i\), and all \(i\) such that \(2 \leq \langle i \rangle \leq r\).

Using the notations of \([1]\), we define \(\Delta_n = \Delta_n(t) = \max\{1/n, \delta_n(t)\}\), \(\delta_n = \delta_n(t) = \sqrt{t(1-t)/n}\), and \(T_{ns}(x) = \sum_{k=0}^n (k-nx)^s p_{nk}(x), s = 0, 1, \ldots\). It is clear that \(\Delta_n(t) = n^{-1}\) if \(n \leq 4\). For \(n > 4\), if \(|t-0.5| \leq 0.5 \sqrt{1 - 4/n}\) then \(\Delta_n(t) = \delta_n(t)\), or \(= n^{-1}\). If \(d = 1\) and \(f \in C([0, 1])\) for \(r \geq 2\), Theorem 1 of \([1]\) gives the estimate \(|f(t) - P_n(t)| \leq C_n \Delta_n(t) \omega_r(\Delta_n(t))\), for \(t \in [0, 1]\),
and $n \geq 1$. So when $|t - 0.5| > 0.5 \sqrt{1 - 4/n}$ this estimate becomes uniform

$f(t) - P_n(t) \leq C, n \rightarrow \omega_r(1/n)$.

In order to get a non-uniform estimate, we need to prove an improved version of Lemma 1 of [1]. It is convenient to denote $\tilde{T}_{ns}(x) = n^{-s}T_{ns}(x)$ and $\tilde{T}^{*}_{ns}(x) = n^{-s}\sum_{k=0}^{n} |k - nx|^s p_{nk}(x)$, $s = 0, 1, \ldots$.

**Lemma 1.** For $s \geq 0$ and some constant $A_s$

\[ \tilde{T}^{*}_{ns}(x) \leq A_s \delta_n^{2/s}(x) \Delta_n^{4v(s-2)}(x), \]

where $\vee = \max(a, b)$, and $\wedge = \min(a, b)$. Particularly $A_0 = A_1 = A_2 = 1$, $A_3 = 2$ and $A_4 = 4$. The equality holds when $s = 0, 2$.

**Remark 3.1.** Lemma 1 of [3] gives $\tilde{T}^{*}_{ns}(x) \leq A_s \Delta^s_n(x)$, $s \geq 1$, which does not imply zero estimates at $x = 0, 1$.

**Proof:** The special results for $s = 0, 1, 2$ are obvious. By the formulas on P. 14 of [10] we have $\tilde{T}^{*}_{ns}(x) = \tilde{T}_{ns}(x) = n^{-2} \delta_n^2(x) [3n(n-2) \delta_n^2(x) + 1] \leq 4 \delta_n^2(x) \Delta_n^2(x)$. By the Schwartz inequality $\tilde{T}^{*}_{n3}(x) \leq [\tilde{T}^{*}_{n2}(x) \tilde{T}^{*}_{n4}(x)]^{1/2} = \delta_n(x) [\tilde{T}^{*}_{n4}(x)]^{1/2} \leq 2 \delta_n^2(x) \Delta_n(x)$. For $s \geq 4$, Romanovsky [see Eq. 5 of [11]] has proved that both $\tilde{T}^{*}_{n2r}(x)$ and $\tilde{T}^{*}_{n2r+1}(x)$ can be expressed as $nx(1-x) \sum_{i=0}^{r-1} [nx(1-x)]^{t} Q_{r}(x)$, where $Q_{r}(x)$ are polynomials in $x$ with coefficients depending on $r$ and $t$ only. Similar to [3], this implies that $\tilde{T}^{*}_{n2r}(x) = \tilde{T}^{*}_{n2r+1}(x) \leq A_{2r} \delta_n^2(x) \Delta_n^{2r-2}(x)$. By Schwartz inequality again $\tilde{T}^{*}_{n2r+1}(x) \leq [\tilde{T}^{*}_{n2}(x) \tilde{T}^{*}_{n4}(x)]^{1/2} \leq A_{2r+1} \delta_n^2(x) \Delta_n^{2r-1}(x)$. The proof of the Lemma is complete.

Now, Theorem 1 of [3] can be enhanced and generalized as follows.

**Theorem 2.** (i) If $f \in C^{(r)}[0, 1]^d$, $r = 0, 1$, then

\[ |f(x) - B^r_n(x)| \leq (d+1) \omega^{(r)} \left( \frac{\delta_n(x)}{\max_{1 \leq j \leq d} \delta_n(x_j)} \right)^r, \quad 0 \leq x \leq 1. \]  

(ii) Let $r \geq 0$, $m > 0$, $M_i \geq 0$, be given. Then there exists a constant $C_{r,d} = C_{r,d}(m, \mathcal{M}_r)$ such that for each function $f(x) \in \Lambda_r^{(d)}(m, \mathcal{M}_r)$ one can find a sequence $P_n(x)$, $n \geq 1$, of polynomials with positive coefficients of degree $n$, satisfying

\[ |f(x) - P_n(x)| \leq C_{r,d} \omega^{(r)} \Lambda_n^{r-2}(x) \left( \sum_{j=1}^{d} \delta_n(x_j) \right)^2, \quad 0 \leq x \leq 1. \]  

where $D_n(x) = \max_{1 \leq j \leq d} \Delta_n(x_j)$.

**Remark 3.2.** Estimates in [2] are generalizations of (6) and (7) of [3]. If $d = 1$ and $r \geq 2$, then [3] is an improved version of Theorem 1 of [3]:

\[ |f(t) - P_n(t)| \leq C_r \delta_n^{2}(t) \Lambda_n^{r-2}(t) \omega_r(\Lambda_n(t)), \quad 0 \leq t \leq 1, \quad n = 1, \ldots. \]  

This indicates that the approximation $P_n$ for $f$ performs especially good at the boundaries because the errors are zero at $t = 0, 1$. However, results of [3] do not imply this when $r \geq 2$.  

3
Proof: Similar to (1), we want to prove that, for \( r > 0 \), there exist polynomials of the form

\[
Q_{n,r}^f(x) = \sum_{k=0}^{n} \left\{ f\left( \frac{k}{n} \right) + \sum_{i=2}^{r} \frac{1}{i!} \sum_{(i)_{i=1}} \binom{i}{i} f^{(i)}\left( \frac{k}{n} \right) \prod_{j=1}^{d} \frac{1}{n_{j}} \tau_{r_{j}}(x_{j}, n_{j}) \right\} p_{n,k}(x), \quad (5)
\]

where \( \binom{i}{i} = \binom{i}{i_{1}, \ldots, i_{d}} \) is the multinomial coefficient, and \( \tau_{r}(x, n) \)’s are polynomials, independent of \( f \), in \( x \) of degree \( i \), in \( n \) of degree \( \lceil i/2 \rceil \), such that for each function \( f \in C^{(r)}[0, 1]^d \),

\[
|f(x) - Q_{n,r}^f(x)| \leq C_{r,d}^{(r)} D_{n}(x)|D_{n}^{0}(r_{2})(x)\left[ \sum_{j=1}^{d} \delta_{n_{j}}(x_{j}) \right]^{2n_{r}} \quad (6)
\]

with \( C_{r,d}^{(r)} \) depending only on \( r \) and \( d \).

If \( f \in C^{(r)}[0, 1]^d \), \( r > 1 \), by the Taylor expansion of \( f(k/n) \) at \( x \), we have

\[
f(x) = f\left( \frac{k}{n} \right) - \sum_{i=1}^{r} \frac{1}{i!} \sum_{(i)_{i=1}} \binom{i}{i} \prod_{j=1}^{d} \frac{k_{j}}{n_{j}} - x_{j} f^{(i)}(x) \]

\[
+ \frac{1}{r!} \sum_{(i)_{i=r}} \prod_{j=1}^{d} \frac{k_{j}}{n_{j}} - x_{j} f^{(i)}(x) - f^{(i)}(\xi_{k}^{(r)}) \},
\]

where \( \xi_{k}^{(r)} \) is on the line segment connecting \( x \) and \( k/n \). This equation is also true when \( r = 0 \) by defining \( \xi_{k}^{(0)} = k/n \) and the empty sum to be zero. Multiplying both sides by \( p_{n,k}(x) \) and taking summation over \( 0 \leq k \leq n \), we obtain

\[
f(x) = B_{r}^{f}(x) - \sum_{i=2}^{r} \frac{1}{i!} \sum_{(i)_{i=1}} \binom{i}{i} \prod_{j=1}^{d} T_{n_{j}}(x_{j}) f^{(i)}(x) + R_{n}^{(r)}(x), \quad (7)
\]

where \( r > 0 \), empty sum is zero, and

\[
R_{n}^{(r)} = \frac{1}{r!} \sum_{(i)_{i=r}} \binom{r}{i} \sum_{k=0}^{n} \prod_{j=1}^{d} \frac{1}{n_{j}} (k_{j} - n_{j} x_{j})^{i_{j}} p_{n,k_{j}}(x_{j}) [f^{(i)}(x) - f^{(i)}(\xi_{k}^{(r)})].
\]

For each \( \delta > 0 \), define \( \lambda = \lambda(x, y; \delta) = |x - y|/\delta \), where \( |x| \) is the integer part of \( x \). Then \( \lambda \delta \leq |x - y| < (\lambda + 1)\delta \), and for \( g \in C[0, 1]^d \), \( |g(x) - g(y)| \leq (\lambda + 1) |\omega(g, \delta)| \).

If \( f \in C^{(r)}[0, 1]^d \), \( r = 0, 1 \), then similar to the proofs of Theorems 1.6.1 and 1.6.2 of [10], pp. 20–21 and by (7) we have \( |f(x) - B_{r}^{f}(x)| = |R_{n}^{(r)}(x)| \). Because
Choosing $\delta$ and $Q$ such that all $R(x) \leq n$, by Lemma 1 with $s = 0, 1, 2$, we have

$$|f(x) - B_0(x)| \leq \sum_{(i) = r} \omega_i(\delta) \left[ \prod_{j=1}^d T_{n, i_j}^* (x_j) + \frac{1}{\delta} \sum_{l=1}^d \prod_{1 \leq j \leq d, j \neq l} \prod_{n \leq i \leq \omega \leq 1} \delta_{n}(x_j) \right]$$

$$\leq \sum_{(i) = r} \omega_i(\delta) \left[ \prod_{j=1}^d \delta_{n}(x_j) + \frac{1}{\delta} \sum_{l=1}^d \prod_{1 \leq j \leq d, j \neq l} \prod_{n \leq i \leq \omega \leq 1} \delta_{n}(x_j) \right]. \quad (8)$$

The estimates in (2) follow from (3) with $\delta = \max_{1 \leq j \leq d} \delta_{n}(x_j)$. This also proves (3) with $r = 0, 1$ and $Q_{n, r} = B_f^r$.

If $r \geq 2$, then we have

$$|R_n^{(r)}| \leq \frac{1}{r!} \left\{ \sum_{(i) = r} \left( \sum_{(j) = r} \right) \omega_i(\delta) \left[ \prod_{j=1}^d T_{n, i_j}^* (x_j) + \frac{1}{\delta} \sum_{l=1}^d \prod_{1 \leq j \leq d, j \neq l} \prod_{n \leq i \leq \omega \leq 1} \delta_{n}(x_j) \right] \right\}$$

$$\leq \frac{1}{r!} \left\{ \sum_{(i) = r} \left( \sum_{(j) = r} \right) \omega_i(\delta) \left[ \prod_{j=1}^d A_{i_j} \delta_{n_j}^{2^l}(x_j) \Delta_{n_j}^{0^l}(x_j) \right]$$

$$+ \frac{1}{\delta} \sum_{l=1}^d A_{i_l+1} \delta_{n_l}^{2^l}(x_l) \Delta_{n_l}^{0^l}(x_l) \prod_{1 \leq j \leq d, j \neq l} \prod_{n \leq i \leq \omega \leq 1} \delta_{n_j}(x_j) \right\}. \quad \text{Choosing} \quad \delta = D_n(x), \quad \text{we have}$$

$$|R_n^{(r)}| \leq \omega(r)(\delta) \frac{1}{r!} \left\{ \sum_{(i) = r} \left( \sum_{(j) = r} \right) \prod_{j=1}^d A_{i_j} \delta_{n_j}^{2^l}(x_j) \Delta_{n_j}^{0^l}(x_j) \right\}$$

$$+ \sum_{l=1}^d \sum_{(i) = r} \left( \sum_{(j) = r} \right) A_{i_l+1} \delta_{n_l}^{2^l}(x_l) \Delta_{n_l}^{0^l}(x_l) \prod_{1 \leq j \leq d, j \neq l} \prod_{n \leq i \leq \omega \leq 1} \delta_{n_j}(x_j) \right\}$$

$$\leq C(r, d) \omega(r)(\delta) \max_{1 \leq j \leq d} \Delta_{n_j}^{-2}(x_j) \left[ \sum_{j=1}^d \delta_{n_j}(x_j) \right]^2.$$
By Lemma 1 and the inductive assumption, (6) is satisfied by (9) as following.

\[ |f(x) - Q^f_{nr}(x)| \leq \sum_{i=2}^r \frac{1}{i!} \sum_i \binom{d}{i} \prod_{j=1}^d T_{n_i j_j}^* \big|f^{(i)}(x) - Q^{f^{(i)}}_{n_i r_i-1}(x)\big| + |R^f_n(x)| \]

\[ \leq \sum_{i=2}^r \frac{C^m_{r,d}}{i!} D_n(x)^{i-2} \left( \sum_{j=1}^d \delta_{n_j}(x_j) \right)^2 \cdot \omega^{(r)} \left[D_n(x)\right] D_n^{0\nu(r-i-2)}(x) \left[ \sum_{j=1}^d \delta_{n_j}(x_j) \right]^{2 \wedge (r-i)} + |R^f_n(x)| \]

\[ \leq C^m_{r,d}\omega^{(r)} \left[D_n(x)\right] D_n^{0\nu(r-2)}(x) \left[ \sum_{j=1}^d \delta_{n_j}(x_j) \right]^2 \cdot \omega^{(r)} \left[D_n(x)\right] D_n^{0\nu(r-2)}(x) \left[ \sum_{j=1}^d \delta_{n_j}(x_j) \right]^{2 \wedge (r-i)} + |R^f_n(x)| \]

Since \( f(x) \geq m > 0 \), by an obvious generalization of remark (a) on p. 241 of [1] with \( h = 1/n \) we know that \( P_{n+r}(x) = Q^f_{nr}(x) \) is a \( d \)-variate polynomial of degree \( n+r = (n_1+r, \ldots, n_d+r) \) with positive coefficients for all \( n \geq n_r(m, \mathcal{M}_r) \) so that

\[ |f(x) - P_{n+r}(x)| \leq C_{r,d}\omega^{(r)} \left[D_n(x)\right] D_n^{r-2}(x) \left[ \sum_{j=1}^d \delta_{n_j}(x_j) \right]^2 \cdot \omega^{(r)} \left[D_n(x)\right] D_n^{r-2}(x) \left[ \sum_{j=1}^d \delta_{n_j}(x_j) \right]^{2 \wedge (r-i)} + |R^f_n(x)| \]

Then (3) follows for all \( n \) and a larger \( C_{r,d} \) from \( \Delta_{n_r} = O(\Delta_{n_r+r}) \) for all \( r \geq 0 \).

### 4 Conclusion

We have generalized the univariate polynomials with positive coefficients to the multivariate ones and proved an enhanced generalization of Theorem 1 of G. G. Lorentz [1]. The estimation of the degree of approximation of \( f \in C^{\nu \nu}[0,1]^d \) by the polynomials with positive coefficients contains a factor \( \frac{\sum_{j=1}^d \delta_{n_j}(x_j) \right|^2 = \left[ \sum_{j=1}^d \sqrt{x_j(1-x_j)/n_j} \right]^{2\nu \nu} \) when \( r \geq 1 \) which is non-uniform even for \( x \) close to the vertices of the unit hypercube \([0,1]^d\).

### References

**References**

[1] George Gunther Lorentz. The degree of approximation by polynomials with positive coefficients. *Math. Ann.* 151:239–251, 1963.

[2] S. N. Bernstein. Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités. *Comm. Soc. Math. Kharkov*, 13:1–2, 1912.

[3] Zhong Guan. Efficient and robust density estimation using bernstein type polynomials. *Journal of Nonparametric Statistics*, 28(2):250–271, 2016.
[4] Murray Rosenblatt. Remarks on some nonparametric estimates of a density function. *Ann. Math. Statist.*, 27:832–837, 1956. ISSN 0003-4851.

[5] Emanuel Parzen. On estimation of a probability density function and mode. *Ann. Math. Statist.*, 33(3):1065–1076, 1962.

[6] Richard A. Vitale. Bernstein polynomial approach to density function estimation. In *Statistical Inference and Related Topics (Proc. Summer Res. Inst. Statist. Inference for Stochastic Processes, Indiana Univ., Bloomington, Ind., 1974, Vol. 2; dedicated to Z. W. Birnbaum)*, pages 87–99. Academic Press, New York, 1975.

[7] Zhong Guan, Baolin Wu, and Hongyu Zhao. Nonparametric estimator of false discovery rate based on Bernstein polynomials. *Statist. Sinica*, 18(3):905–923, 2008. ISSN 1017-0405.

[8] TH Hildebrandt and IJ Schoenberg. On linear functional operations and the moment problem for a finite interval in one or several dimensions. *Ann. of Math.*, 34(2):317–328, 1933.

[9] P. L. Butzer. On two-dimensional Bernstein polynomials. *Canadian J. Math.*, 5:107–113, 1953. ISSN 0008-414X.

[10] George Gunther Lorentz. Bernstein polynomials. Chelsea Publishing Co., New York, second edition, 1986. ISBN 0-8284-0323-6.

[11] V. Romanovsky. Note on the moments of a binomial $(p + q)^n$ about its mean. *Biometrika*, 15(3–4):410–412, 1923.