A Version of the Goldman-Millson Theorem for Filtered $L_\infty$-Algebras

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Abstract

In this paper we consider $L_\infty$-algebras equipped with complete descending filtrations. We prove that, under some mild conditions, an $L_\infty$ quasi-isomorphism $U : L \to \tilde{L}$ induces a weak equivalence between the Deligne-Getzler-Hinich (DGH) $\infty$-groupoids corresponding to $L$ and $\tilde{L}$, respectively. This paper may be considered as a modest addition to foundational paper [10] by Ezra Getzler.

1 Introduction

The Getzler-Hinich construction [10], [13] assigns to every nilpotent $L_\infty$-algebra $L$ a Kan complex $\mathcal{MC}_\bullet(L)$. This Kan complex is a natural generalization of the Deligne groupoid [9], [12] from dg Lie algebras to nilpotent $L_\infty$-algebras. Various versions of the Deligne-Getzler-Hinich (DGH) groupoid are used in deformation theory [2], [12], [14], [18], [20], rational homotopy theory [1], [15], [16], and derived algebraic geometry [18]. In this paper we consider a version of the Kan complex $\mathcal{MC}_\bullet(L)$ for $L_\infty$-algebras equipped with a complete descending filtration. We prove that, under some mild conditions, an $L_\infty$ quasi-isomorphism $U : L \to \tilde{L}$ of such filtered $L_\infty$-algebras induces a weak equivalence of simplicial sets

$$\mathcal{MC}_\bullet(U) : \mathcal{MC}_\bullet(L) \to \mathcal{MC}_\bullet(\tilde{L}).$$

Let $L$ be a cochain complex of $k$-vector spaces. We recall that an $L_\infty$-structure $Q$ of the cocommutative coalgebra

$$\mathcal{S}(s^{-1} L) := s^{-1} L \oplus S^2(s^{-1} L) \oplus S^3(s^{-1} L) \oplus \ldots$$

satisfying the equation

$$Q^2 = 0,$$

and the condition $Q(s^{-1} v) = -s^{-1} \partial v$, where $\partial$ is the differential on $L$ and $(s^{-1} L)^* = L^{*+1}$.

An $\infty$-morphism $U$ from an $L_\infty$-algebra $(L, Q)$ to an $L_\infty$-algebra $(\tilde{L}, \tilde{Q})$ is a homomorphism of the cocommutative coalgebras

$$U : \mathcal{S}(s^{-1} L) \to \mathcal{S}(s^{-1} \tilde{L})$$

In this paper, we assume that char($k$) = 0.
which intertwines the coderivations $Q$ and $\tilde{Q}$:

$$U \circ Q = \tilde{Q} \circ U.$$ 

Let us also recall that every coderivation $Q$ of $S(s^{-1} L)$ is uniquely determined by its composition

$$Q' := p_{s^{-1} L} \circ Q : S(s^{-1} L) \to s^{-1} L$$

with the canonical projection $p_{s^{-1} L} : S(s^{-1} L) \to s^{-1} L$. Similarly, every homomorphism (1.1) is uniquely determined by its composition

$$U' := p_{s^{-1} L} \circ U : S(s^{-1} L) \to s^{-1} \tilde{L}.$$

The restriction of $U'$ to $s^{-1} L$ gives us a chain map

$$U'\bigg|_{s^{-1} L} : s^{-1} L \to s^{-1} \tilde{L} \quad (1.2)$$

which we call the *linear term* of the $\infty$-morphism $U$. We recall that an $\infty$-morphism $U$ is an $\infty$ *quasi-isomorphism* if its linear term is a quasi-isomorphism of cochain complexes $s^{-1} L \to s^{-1} \tilde{L}$.

In this paper, we deal with *filtered* $L_\infty$-algebras $L$, i.e. $L_\infty$-algebras $L$ equipped with a complete descending filtration

$$L = F_1 L \supset F_2 L \supset F_3 L \supset \cdots, \quad (1.3)$$

$$L = \lim_{\leftarrow} L/F_k L \quad (1.4)$$

such that

$$Q'(F_{i_1} s^{-1} L \otimes F_{i_2} s^{-1} L \otimes \cdots \otimes F_{i_m} s^{-1} L) \subset F_{i_1 + i_2 + \cdots + i_m} s^{-1} L \quad \forall \; m \geq 1. \quad (1.5)$$

Furthermore, we assume that all $\infty$-morphisms $U$ in question are also compatible with the filtrations in the sense that

$$U'(F_{i_1} s^{-1} L \otimes F_{i_2} s^{-1} L \otimes \cdots \otimes F_{i_m} s^{-1} L) \subset F_{i_1 + i_2 + \cdots + i_m} s^{-1} \tilde{L} \quad \forall \; m \geq 1. \quad (1.6)$$

Let us recall that a Maurer-Cartan (MC) element of $L$ is a degree 1 element $\alpha \in L$ satisfying the equation

$$\sum_{m=1}^{\infty} \frac{1}{m!} Q'((s^{-1} \alpha)^m) = 0, \quad (1.7)$$

where the infinite series in the left hand side makes sense due to conditions $L = F_1 L$, (1.4), and (1.5).

We also recall that every $\infty$-morphism (1.1) of $L_\infty$-algebras gives us a map of sets

$$U_* : MC(L) \to MC(\tilde{L})$$

given by the formula

$$U_*(\alpha) := \sum_{m=1}^{\infty} \frac{1}{m!} s U'((s^{-1} \alpha)^m), \quad (1.8)$$

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where $\text{MC}(L)$ (resp. $\text{MC}(\tilde{L})$) denotes the set of MC elements of $L$ (resp. $\tilde{L}$).

It is natural to view MC elements of $L$ as zero-cells of the simplicial set $\text{MC}_\bullet(L)$ with

$$\text{MC}_n(L) := \text{MC}(L \otimes \Omega_n)$$

where $\Omega_n$ the de Rham-Sullivan algebra of polynomial differential forms on the geometric simplex $\Delta^n$ with coefficients in $k$ and

$$L \otimes \Omega_n := \lim_{\leftarrow k} (L/F_k L \otimes \Omega_n).$$

Due to [6, Proposition 4.1], the simplicial set $\text{MC}_\bullet(L)$ is a Kan complex (a.k.a. an $\infty$-groupoid).

It is easy to see that the assignment $L \mapsto \text{MC}_\bullet(L)$ upgrades to a functor from the category of (filtered) $L_\infty$-algebras (with morphisms being $\infty$-morphisms) to the category of simplicial sets.

The goal of this paper is to prove the following version\(^2\) of the Goldman-Millson theorem:

**Theorem 1.1** Let $L$ and $\tilde{L}$ be filtered $L_\infty$-algebras and $U$ be an $\infty$-morphism from $L$ to $\tilde{L}$ compatible with the filtrations in the sense of (1.6). If the linear term $\varphi : L \to \tilde{L}$ of $U$ gives us a quasi-isomorphism

$$\varphi|_{\mathcal{F}_m L} : \mathcal{F}_m L \to \mathcal{F}_m \tilde{L}$$

for every $m \geq 1$ then

$$\text{MC}_\bullet(U) : \text{MC}_\bullet(L) \to \text{MC}_\bullet(\tilde{L})$$

is a weak equivalence of simplicial sets.

We would like to remark that Theorem 1.1 is a generalization of Proposition 4.9 from paper [10].\(^3\)

Since it is more convenient to deal with shifted $L_\infty$-algebras (a.k.a. $S\text{Lie}_\infty$-algebras) than with usual $L_\infty$-algebras, the bulk of the presentation is given in the setting of shifted $L_\infty$-algebras which are briefly reviewed in Section 2. In this section, we also formulate the main result of this paper (see Theorem 2.2), prove an important particular case of this theorem (see Proposition 2.4) and outline the structure of the proof of the main result. In this respect, Section 2 may be viewed as more detailed introduction to our paper.

The proof of the main result occupies Sections 3 and 4. In Section 3 we take care of the induced map from $\pi_0(\text{MC}_\bullet(L))$ to $\pi_0(\text{MC}_\bullet(\tilde{L}))$ and, in Section 4, we take care of the corresponding map for higher homotopy groups of the simplicial sets $\text{MC}_\bullet(L)$ and $\text{MC}_\bullet(\tilde{L})$.

The three appendices at the end of the paper are devoted to proofs of various technical statements used in the bulk of the paper. In Appendix A we recall the operators $h_n^i : \Omega_n^i \to \Omega_n^{i-1}$ used in [8], [10] and prove a version of Lemma 4.6 from [10] for filtered $L_\infty$-algebras. In Appendix B we introduce the notion of rectified 1-cell in $\text{MC}_\bullet(L)$ and prove that if two 0-cells are connected in $\text{MC}_\bullet(L)$ then they can be connected by a rectified 1-cell. Finally,\(^2\)This theorem is already used in papers [4] and [19].

\(^3\)We recently found a very nice proof of [10, Proposition 4.9] in [2]. See Proposition 3.4 in [2].
in Appendix C, we prove that, under certain conditions, an infinite sequence of 1-cells in $\mathcal{M}_\bullet(L)$ can be “composed”. We believe that this appendix has an independent value.

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**Notation and conventions**

A big part of our conventions is borrowed from our paper [6]. Thus, we work in the setting of unbounded cochain complexes of $k$-vector spaces where $k$ is any field of characteristic zero. We will frequently use the ubiquitous abbreviation “dg” (differential graded) to refer to algebraic objects in the category of such cochain complexes. For a cochain complex $V$ we denote by $sV$ (resp. by $s^{-1}V$) the suspension (resp. the desuspension) of $V$. In other words,

$$(sV)^* = V^{*-1}, \quad (s^{-1}V)^* = V^{*+1}. $$

The notation $\mathcal{Z}^d(V)$ is reserved for the subspace of degree $d$ cocycles in $V$.

The notation $S_n$ is reserved for the symmetric group on $n$ letters and $\text{Sh}_{p_1,\ldots,p_k}$ denotes the subset of $(p_1,\ldots,p_k)$-shuffles in $S_n$, i.e. $\text{Sh}_{p_1,\ldots,p_k}$ consists of elements $\sigma \in S_n$, $n = p_1 + p_2 + \cdots + p_k$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p_1),$$

$$\sigma(p_1 + 1) < \sigma(p_1 + 2) < \cdots < \sigma(p_1 + p_2),$$

$$\cdots$$

$$\sigma(n - p_k + 1) < \sigma(n - p_k + 2) < \cdots < \sigma(n).$$

For a graded vector space (or a cochain complex) $V$ the notation $S(V)$ (resp. $S(V)$) is reserved for the underlying vector space of the symmetric algebra (resp. the truncated symmetric algebra) of $V$:

$$S(V) = k \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \ldots,$$

$$\underline{S}(V) = V \oplus S^2(V) \oplus S^3(V) \oplus \ldots,$$

where

$$S^n(V) = (V^\otimes k^n)^{S_n}$$

is the subspace of coinvariants with respect to the obvious action of $S_n$.

The graded vector space $S(V)$ is usually considered as the cocommutative coalgebra without counit and with the comultiplication $\Delta$ given by the formula

$$\Delta(v_1, v_2, \ldots, v_n) := \sum_{p=1}^{n} \sum_{\sigma \in \text{Sh}_{p,n-p}} (-1)^c(\sigma; v_1, \ldots, v_m) (v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \otimes (v_{\sigma(p+1)}, \ldots, v_{\sigma(n)}),$$

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where \((-1)^{\varepsilon(\sigma;v_1,\ldots,v_m)}\) is the Koszul sign factor
\[
(-1)^{\varepsilon(\sigma;v_1,\ldots,v_m)} := \prod_{i<j} (-1)^{|v_i||v_j|}
\]
and the product in (1.10) is taken over all inversions \((i < j)\) of \(\sigma \in S_m\).

We often use the plain arrow \(\rightarrow\) for \(\infty\)-morphisms of \(L_\infty\)-algebras (or shifted \(L_\infty\)-algebras). Of course, it should be kept in mind that, in general, such morphisms are maps of the corresponding coalgebras but not the underlying cochain complexes.

The abbreviation “MC” is reserved for the term “Maurer-Cartan”.

We denote by \(\Omega_n\) Sullivan’s polynomial de Rham complex on the geometric \(n\)-simplex \(\Delta^n\) with coefficients in \(k\) and recall that the collection \(\{\Omega_n\}_{n \geq 0}\) form a simplicial dg commutative \(k\)-algebra. (See, for example, Section 3 in [10].) The notation \(d\) is reserved for the de Rham differential on \(\Omega_n\).

### 2 Preliminaries

For technical reasons it is more convenient to deal with shifted \(L_\infty\)-algebras (as in [4] and [6]) than with usual \(L_\infty\)-algebras. This is why we now “shift gears” and pass to the setting of shifted \(L_\infty\)-algebras for the rest of the paper. In this section, we briefly review shifted \(L_\infty\)-algebras, introduce the operation \(\text{curv}\) (see (2.12)) and formulate the main result of this paper (Theorem 2.2). We also prove an important particular case of the main theorem (Proposition 2.4) and outline the structure of the proof of Theorem 2.2.

Let us recall [6] that a shifted \(L_\infty\)-algebra (a.k.a. \(\mathcal{S}\text{Lie}_\infty\)-algebra) is a cochain complex \((L, \partial)\) for which the cocommutative coalgebra \(\mathcal{S}(L)\) (2.1) carries a degree 1 coderivation \(Q\) satisfying
\[
Q^2 = 0
\]
and the condition \(Q(v) = \partial(v)\) \(\forall\ v \in L\).

Since every coderivation \(Q\) of the coalgebra \(\mathcal{S}(L)\) is uniquely determined by its composition
\[
Q' := p_L \circ Q : \mathcal{S}(L) \rightarrow L
\]
with the projection \(p_L : \mathcal{S}(L) \rightarrow L\), an \(\mathcal{S}\text{Lie}_\infty\)-algebra structure on a cochain complex \(L\) is uniquely determined by the sequence of degree 1 multi-brackets \((m \geq 2)\)
\[
\{\ldots, , \ldots, \}_m : S^m(L) \rightarrow L
\]
\[
\{v_1, v_2, \ldots, v_m\}_m = Q'(v_1v_2\ldots v_m), \quad v_j \in L.
\]

Furthermore, equation (2.2) is equivalent to the following sequence of relations
\[
\partial\{v_1, v_2, \ldots, v_m\}_m + \sum_{i=1}^m (-1)^{|v_1|+\cdots+|v_{i-1}|} \{v_1, \ldots, v_{i-1}, \partial v_i, v_{i+1}, \ldots, v_m\}_m \\
+ \sum_{k=2}^{m-1} \sum_{\sigma \in \text{Sh}_{k,m-k}} (-1)^{\varepsilon(\sigma;v_1,\ldots,v_m)} \{\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}_k, v_{\sigma(k+1)}, \ldots, v_{\sigma(m)}\}_{m-k+1} = 0,
\]
where \((-1)^{\varepsilon(\sigma; v_1, \ldots, v_m)}\) is the Koszul sign factor defined in (1.10).

An \(\infty\)-morphism \(U\) from a \(\mathfrak{G}\text{Lie}_\infty\)-algebra \((L, Q)\) to a \(\mathfrak{G}\text{Lie}_\infty\)-algebra \((\tilde{L}, \tilde{Q})\) is a homomorphism
\[
U : (S(L), Q) \to (S(\tilde{L}), \tilde{Q}) \tag{2.6}
\]
of the corresponding dg cocommutative coalgebras.

We recall that any such coalgebra homomorphism \(U\) is uniquely determined by its composition \(U'\) with the projection \(p_{\tilde{L}} : S(\tilde{L}) \to \tilde{L}:\)
\[
U' := p_{\tilde{L}} \circ U : S(L) \to \tilde{L}. \tag{2.6}
\]

For every \(\infty\)-morphism (2.6) the map \(\varphi := U'\big|_L : L \to \tilde{L}\)\) is a morphism of cochain complexes and we call \(\varphi\) the linear term of \(U\). An \(\infty\)-morphism \(U\) for which \(\varphi\) induced an isomorphism \(H^* (L) \to H^* (\tilde{L})\) is called an \(\infty\) quasi-isomorphism of \(\mathfrak{G}\text{Lie}_\infty\)-algebras.

Following [6], we call a \(\mathfrak{G}\text{Lie}_\infty\)-algebra \(L\) filtered if the underlying cochain complex \((L, \partial)\) is equipped with a complete descending filtration,
\[
L = F_1 L \supset F_2 L \supset F_3 L \cdots \tag{2.8}
\]
which is compatible with the brackets, i.e.
\[
\left\{ F_{i_1} L, F_{i_2} L, \ldots, F_{i_m} L \right\}_m \subseteq F_{i_1 + i_2 + \ldots + i_m} L \quad \forall \ m > 1. \tag{2.9}
\]

In this paper, we consider \(\infty\)-morphisms (2.6) of filtered \(\mathfrak{G}\text{Lie}_\infty\)-algebras which are compatible with the filtrations, i.e.
\[
U'(F_{i_1} L \otimes F_{i_2} L \otimes \cdots \otimes F_{i_m} L) \subset F_{i_1 + i_2 + \ldots + i_m} \tilde{L}, \tag{2.10}
\]

Since \(L = F_1 L\), condition (2.10) guarantees that, for every degree zero vector \(\alpha \in L\), the infinite sum
\[
\sum_{m \geq 1} \frac{1}{m!} U'(\alpha^m)
\]
is a well defined element of \(\tilde{L}\). We denote by \(U_*\) the map of sets \(L^0 \mapsto \tilde{L}^0\) given by this formula
\[
U_*(\alpha) := \sum_{m \geq 1} \frac{1}{m!} U'(\alpha^m). \tag{2.11}
\]

We denote by \(\text{curv}\) the map of sets \(L^0 \mapsto L^1\) given by the formula
\[
\text{curv}(\alpha) := \partial \alpha + \sum_{m \geq 1} \frac{1}{m!} \{\alpha, \ldots, \alpha\}_m. \tag{2.12}
\]

For example, elements \(\alpha \in L^0\) satisfying \(\text{curv}(\alpha) = 0\) are precisely \(\text{MC}\) elements of the \(\mathfrak{G}\text{Lie}_\infty\)-algebra \(L\). Various useful properties of the operation \(\text{curv}\) are listed in the following proposition, whose proof is given in [6].
Proposition 2.1 (Prop. 2.2 [6]) Let \( L \) and \( \tilde{L} \) be filtered \( \mathfrak{glie}_\infty \)-algebras and \( U \) be an \( \infty \)-morphism from \( L \) to \( \tilde{L} \) satisfying (2.10). Then for every \( \alpha, \beta \in L^0, v \in L \) we have

\[
\partial(\text{curv}(\alpha)) + \sum_{m=1}^{\infty} \frac{1}{m!} \{\alpha, \ldots, \alpha, \text{curv}(\alpha)\}_{m+1} = 0, \tag{2.13}
\]

\[
\text{curv}(U_*(\alpha)) = \sum_{m \geq 0} \frac{1}{m!} U'(\alpha^m \text{curv}(\alpha)), \tag{2.14}
\]

\[
\partial^\alpha \circ \partial^\alpha(v) = -\{\text{curv}(\alpha), v\}_2, \tag{2.15}
\]

\[
\text{curv}(\alpha + \beta) = \text{curv}(\alpha) + \partial^\alpha(\beta) + \sum_{m=2}^{\infty} \frac{1}{m!} \{\beta, \ldots, \beta\}_m^\alpha, \tag{2.16}
\]

where \( \partial^\alpha := \partial + \sum_{m \geq 1} \frac{1}{m!} \{\alpha, \ldots, \alpha, \cdot\}_{m+1} \) and \( \{\cdot, \ldots, \cdot\}_n^\alpha := \sum_{m \geq 1} \frac{1}{m!} \{\alpha, \ldots, \alpha, \cdot, \ldots, \cdot\}_{m+n} \).

2.1 Twisting of \( \mathfrak{glie}_\infty \)-algebra structures by MC elements

Let us recall [3, Section 2.4], [7], [10, Section 4] that, given a MC element \( \alpha \) of a filtered \( \mathfrak{glie}_\infty \)-algebra \( L \) we can form a new filtered \( \mathfrak{glie}_\infty \)-algebra \( L^\alpha \). As a graded vector space with a filtration, \( L^\alpha = L \); the differential \( \partial^\alpha \) and the multi-brackets \( \{\cdot, \ldots, \cdot\}_m^\alpha \) on \( L^\alpha \) are defined by the formulas

\[
\partial^\alpha(v) := \partial(v) + \sum_{k=1}^{\infty} \frac{1}{k!} \{\alpha, \ldots, \alpha, v\}_{k+1}, \tag{2.17}
\]

\[
\{v_1, v_2, \ldots, v_m\}_m^\alpha := \sum_{k=0}^{\infty} \frac{1}{k!} \{\alpha, \ldots, \alpha, v_1, v_2, \ldots, v_m\}_{k+m}. \tag{2.18}
\]

Equation (2.14) implies that, for every \( \infty \)-morphism \( U \) (2.6) compatible with the filtrations and for every MC element \( \alpha \in L \),

\[
U_*(\alpha)
\]

is a MC element of \( \tilde{L} \). Finally, for every \( \infty \)-morphism \( U \) (2.6) compatible with the filtrations and for every MC element \( \alpha \in L \), we can construct a new \( \infty \)-morphism

\[
U^\alpha : L^\alpha \rightarrow \tilde{L}^{U_*(\alpha)}
\]

with

\[
p_L \circ U^\alpha(v_1 v_2 \ldots v_m) := \sum_{k=0}^{\infty} \frac{1}{k!} U'(\alpha^k v_1 v_2 \ldots v_m). \tag{2.19}
\]
2.2 The Deligne-Getzler-Hinich (DGH) $\infty$-groupoid and Theorem

For every filtered $\mathfrak{S}\text{Lie}_\infty$-algebra $L$ we introduce the following collection of filtered $\mathfrak{S}\text{Lie}_\infty$-algebras

$$L \hat{\otimes} \Omega_n, \quad n \geq 0$$

where $L$ is considered with the topology coming from the filtration and the dg commutative algebra $\Omega_n$ is considered with the discrete topology.

The simplicial set $\mathcal{MC}_\bullet(L)$ with

$$MC_n(L) := MC(L \hat{\otimes} \Omega_n)$$

is the main hero of this paper. Due to [6, Proposition 4.1], $\mathcal{MC}_\bullet(L)$ is a Kan complex (a.k.a. an $\infty$-groupoid) for every filtered $\mathfrak{S}\text{Lie}_\infty$-algebra $L$. We call this simplicial set the Deligne-Getzler-Hinich (DGH) $\infty$-groupoid of $L$.

For example, 0-cells of $\mathcal{MC}_\bullet(L)$ are precisely MC elements of $L$ and 1-cells in $\mathcal{MC}_\bullet(L)$ are elements

$$\beta = \beta_0(t_0) + dt_0 \beta_1(t_0), \quad \beta_0(t_0) \in L^0 \hat{\otimes} k[t_0], \quad \beta_1(t_0) \in L^{-1} \hat{\otimes} k[t_0],$$

satisfying the equations

$$\text{curv}(\beta_0(t_0)) = 0 \quad (2.20)$$

and

$$\frac{d}{dt_0} \beta_0(t_0) = \partial^{\beta_0(t_0)} \beta_1(t_0), \quad (2.21)$$

where $\partial^{\beta_0(t_0)}$ denotes the differential on $L \hat{\otimes} k[t_0]$ twisted by the MC element $\beta_0(t_0)$ (as in Sec. 2.1). The zeroth face $\partial_0(\beta)$ (resp. the first face $\partial_1(\beta)$) of $\beta$ is the MC element $\beta_0(0)$ (resp. $\beta_0(1)$) of $L$.

Let us recall that any $\infty$-morphism $U$ (2.6) of filtered $\mathfrak{S}\text{Lie}_\infty$-algebras satisfying (2.10) gives us the collection of $\infty$-morphisms of $\mathfrak{S}\text{Lie}_\infty$-algebras

$$U^{(n)} : L \hat{\otimes} \Omega_n \to \tilde{L} \hat{\otimes} \Omega_n$$

$$U^{(n)}(v_1 \otimes \omega_1, v_2 \otimes \omega_2, \ldots, v_m \otimes \omega_m) = \pm U(v_1, v_2, \ldots, v_m) \otimes \omega_1 \omega_2 \cdots \omega_m, \quad (2.22)$$

where $v_i \in L$, $\omega_i \in \Omega_n$, and $\pm$ is the usual Koszul sign. This collection is obviously compatible with all the faces and all the degeneracies. Hence, $U$ induces a morphism of simplicial sets

$$\mathcal{MC}_\bullet(U) : \mathcal{MC}_\bullet(L) \to \mathcal{MC}_\bullet(\tilde{L}) \quad (2.23)$$

given by the formula

$$\mathcal{MC}_n(U)(\alpha) := U^{(n)}(\alpha). \quad (2.24)$$

The main result of this paper is the following version of the Goldman-Millson theorem [12]:

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Theorem 2.2 Let $L$ and $\tilde{L}$ be filtered $\mathcal{S}\mathfrak{Lie}_\infty$-algebras and $U : L \to \tilde{L}$ be an $\infty$-morphism compatible with the filtrations on $L$ and $\tilde{L}$. If the linear term $\varphi$ of $U$ induces a quasi-isomorphism of cochain complexes

$$\varphi \vert_{F_n L} : F_n L \to F_n \tilde{L}$$

(2.25)

for every $n$, then $\mathcal{MC}_\bullet(U)$ is a homotopy equivalence of simplicial sets.

Remark 2.3 Due to the obvious equivalence between the category of filtered $\mathcal{S}\mathfrak{Lie}_\infty$-algebras and the category of filtered $L_\infty$-algebras, Theorem 1.1 stated in the Introduction is equivalent to Theorem 2.2.

2.3 What if we deal with abelian $\mathcal{S}\mathfrak{Lie}_\infty$-algebras?

If $\{\ldots, \}$ is $0$ for all $m \geq 2$, the $\mathcal{S}\mathfrak{Lie}_\infty$-algebra $L$ is nothing but an unbounded cochain complex. In this case we may consider the simplicial set

$$\mathcal{MC}_n(L) := MC(L \otimes \Omega_n)$$

with usual tensor product $\otimes$ which is equivalent to considering $L$ with the silly filtrations:

$$L = F_1 L \supset F_2 L = F_3 L = \cdots = 0.$$

Since all the brackets are zero, the set $MC(L \otimes \Omega_n)$ is precisely the subspace of degree zero cocycle in $L \otimes \Omega_n$:

$$MC(L \otimes \Omega_n) = Z^0(L \otimes \Omega_n, \partial + d).$$

(2.26)

In particular, the simplicial set $\mathcal{MC}_\bullet(L)$ is a simplicial vector space.

Let us consider the very special case of Theorem 2.2 in which both $L$ and $\tilde{L}$ are abelian (with the above silly filtrations) and $U$ is a strict morphism $\varphi : L \to \tilde{L}$, i.e.

$$U'(v_1 v_2 \ldots v_m) = 0 \text{ if } m \geq 2.$$

In this case we have

Proposition 2.4 If $\varphi : L \to \tilde{L}$ is a quasi-isomorphism of cochain complex and $L$, $\tilde{L}$ are viewed as the abelian $\mathcal{S}\mathfrak{Lie}_\infty$-algebras then the induced map

$$\mathcal{MC}_\bullet(\varphi) : \mathcal{MC}_\bullet(L) \to \mathcal{MC}_\bullet(\tilde{L})$$

(2.27)

is a weak equivalence of simplicial vector spaces.

Remark 2.5 In principle, Proposition 2.4 is consequence of [10, Proposition 5.1] and [10, Corollary 5.11]. The proof given below is essentially a more detailed presentation of an argument from the proof of [10, Corollary 5.11].
Proof of Proposition 2.4. First, we recall that for every simplicial vector space $V_ullet$
\[ \pi_0(V_ullet) = \pi_0(M(V_ullet)) \quad \pi_i(V_ullet) = \pi_i(M(V_ullet)), \quad \forall \ i \geq 1, \quad (2.28) \]
where $M(V_ullet)$ is the Moore chain complex of the simplicial vector space $V_ullet$:
\[ M(V_ullet) := \bigoplus_n s^nV_n \quad (2.29) \]
with the differential
\[ \partial := \sum_{i=0}^n (-1)^i \partial_i : s^nV_n \to s^{n-1}V_{n-1}. \]
Furthermore, $\pi_i(V_ullet, v)$ is naturally isomorphic to $\pi_i(V_ullet, 0)$ for every base point $v \in V_0$.

Therefore, a map $f : V_ullet \to W_ullet$ of simplicial vector spaces is a weak equivalence if and only if the induced map
\[ M(f) : M(V_ullet) \to M(W_ullet) \]
is a quasi-isomorphism of chain complexes.

Using the elementary forms introduced on page 282 of [10, Section 3], one can identify the simplicial cochain complex
\[ C_n := C_{\text{simpl}}(\Delta^n, k) \quad (2.30) \]
with a subcomplex of $\Omega_n$.

Eq. (3-6) from [10, Section 3] defines projections
\[ P_n : \Omega_n \to C_n \quad (2.31) \]
which assemble into a map of simplicial cochain complexes.

Due to [10, Theorem 3.7] there exists a simplicial endomorphism
\[ s_n : \Omega_n^\bullet \to \Omega_n^{\bullet-1} \quad (2.32) \]
which satisfies
\[ \text{id} - P_n = ds_n + s_nd \quad (2.33) \]
and
\[ P_n \circ s_n = 0, \quad \forall \ n \geq 0. \quad (2.34) \]
We call $s_\bullet$ the Dupont operator.$^5$

We now extend the operators $P_n$, $s_n$, in the obvious way, to
\[ L \otimes \Omega_n \]
for any cochain complex $(L, \partial)$. Then, equation (2.33) becomes
\[ \text{id} - P_n = (\partial + d)s_n + s_n(\partial + d). \quad (2.35) \]

---

$^4$See, for example, Corollary 2.5 in [11, Sec. III.2]

$^5$A very similar operator was introduced in Dupont’s proof [8, Eq. (2.25), page 25] of the de Rham theorem.
Equations (2.34) and (2.35) imply that the projection $P_\bullet$ gives us the following short exact sequence of simplicial vector spaces

$$0 \rightarrow (\partial + d)_{\bullet}((L \otimes \Omega_\bullet)^0) \rightarrow Z^0(L \otimes \Omega_\bullet) \xrightarrow{P_\bullet} Z^0(L \otimes C_\bullet) \rightarrow 0$$

(2.36)

Using equations (2.34) and (2.35) again, we get

$$( (\partial + d)_{\bullet} )^2 = (\partial + d)_{\bullet}(\partial + d)_{\bullet} = (\partial + d)((\partial + d)_{\bullet} (\partial + d)_{\bullet} + (\partial + d)_{\bullet})_{\bullet}$$

$$= (\partial + d)(\text{id} - P_\bullet)_{\bullet} = (\partial + d)_{\bullet}$$

which means that the operator $(\partial + d)_{\bullet}$ is a retract of simplicial vector spaces

$$(L \otimes \Omega_\bullet)^0 \xrightarrow{(\partial + d)_{\bullet}} (\partial + d)_{\bullet}((L \otimes \Omega_\bullet)^0).$$

(2.37)

On the other hand, Lemma 3.2 from [10] implies that $(L \otimes \Omega_\bullet)^0$ has an acyclic Moore complex. Therefore, the Moore complex

$$\mathcal{M} \left( (\partial + d)_{\bullet}((L \otimes \Omega_\bullet)^0) \right)$$

of the simplicial vector space $(\partial + d)_{\bullet}((L \otimes \Omega_\bullet)^0)$ is also acyclic.

Thus we conclude that the projection $P_\bullet$ gives us a quasi-isomorphism of chain complexes:

$$\mathcal{M}(P_\bullet) : \mathcal{M}(Z^0(L \otimes \Omega_\bullet)) \rightarrow \mathcal{M}(Z^0(L \otimes C_\bullet)).$$

(2.38)

Let us now observe that the simplicial vector space

$$Z^0(L \otimes C_\bullet)$$

is precisely the result of applying the Dold-Kan functor to the chain complex which is obtained from the truncation

$$\cdots \rightarrow L^{-2} \xrightarrow{\partial} L^{-1} \xrightarrow{\partial} Z^0(L)$$

(2.39)

of $(L, \partial)$ by reversing the grading.

Since $\varphi : L \rightarrow \tilde{L}$ is a quasi-isomorphism of cochain complexes, it also induces a quasi-isomorphism of the corresponding truncations of $L$ and $\tilde{L}$.

Thus we get a commutative diagram of chain complexes

$$\begin{array}{ccc}
\mathcal{M}(Z^0(L \otimes \Omega_\bullet)) & \xrightarrow{\mathcal{M}(P_\bullet)} & \mathcal{M}(Z^0(L \otimes C_\bullet)) \\
\psi \downarrow & & \downarrow \varphi^* \\
\mathcal{M}(Z^0(\tilde{L} \otimes \Omega_\bullet)) & \xrightarrow{\mathcal{M}(P_\bullet)} & \mathcal{M}(Z^0(\tilde{L} \otimes C_\bullet))
\end{array}$$

(2.40)

where $\psi := \mathcal{M} \circ \mathcal{Mc}_\bullet(\varphi)$.

We already proved that the horizontal arrows in (2.40) are quasi-isomorphisms. Furthermore, the right vertical arrow is a quasi-isomorphism by the Dold-Kan correspondence.

Thus $\mathcal{M} \circ \mathcal{Mc}_\bullet(\varphi)$ is also a quasi-isomorphism of chain complexes and Proposition 2.4 follows.
2.4 The structure of the proof of Theorem 2.2

By the definition of weak equivalence of simplicial sets, we have to show that

- the map $MC\bullet(U)$ induces a bijection
  \[ \pi_0(MC\bullet(L)) \rightarrow \pi_0(MC\bullet(\tilde{L})) \] (2.41)

- and for every 0-cell $\alpha$ in $MC\bullet(L)$ the map $MC\bullet(U)$ induces group isomorphisms
  \[ \pi_i(MC\bullet(L), \alpha) \rightarrow \pi_i(MC\bullet(\tilde{L}), U_\ast(\alpha)), \quad \forall \ i \geq 1. \] (2.42)

In Section 3 we give a detailed proof of the fact that map (2.41) is a bijection of sets. In Section 4 we apply Proposition 2.4 to the map of simplicial sets

\[ MC\bullet(L/F_nL) \rightarrow MC\bullet(\tilde{L}/\tilde{F}_n\tilde{L}) \]

and then prove, by induction on $n$, that $MC\bullet(U)$ induces an isomorphism of groups

\[ \pi_i(MC\bullet(L/F_nL), 0) \rightarrow \pi_i(MC\bullet(\tilde{L}/\tilde{F}_n\tilde{L}), 0) \]

for every $i \geq 1$ and $n \geq 2$.

Then technical Lemma 4.1 implies that the corresponding map

\[ \pi_i(MC\bullet(L/F_nL), \alpha) \rightarrow \pi_i(MC\bullet(\tilde{L}/\tilde{F}_n\tilde{L}), U_\ast(\alpha)) \]

is an isomorphism of groups for every $\alpha \in MC(L)$, $i \geq 1$, and $n \geq 2$.

Combining this statement with the results of Section 3 applied to the corresponding $\infty$-morphism of quotients

\[ L/F_nL \rightarrow \tilde{L}/\tilde{F}_n\tilde{L} \]

we conclude that $MC\bullet(U)$ induces a weak equivalence of simplicial sets

\[ MC\bullet(L/F_nL) \rightarrow MC\bullet(\tilde{L}/\tilde{F}_n\tilde{L}) \]

for all $n \geq 2$.

Finally, we deduce Theorem 2.2 from standard facts about maps of towers of simplicial sets from [11, Section VI].

Let us remark that the explicit constructions given in the proof of injectivity and in the proof of surjectivity of (2.41) have an independent value. These constructions may be used in establishing various properties of homotopy algebraic structures or $\infty$-morphisms of homotopy algebras [4], [5], [17] for which only the existence statements are proved.

3 $MC\bullet(U)$ induces a bijection on the level of $\pi_0$

We start the proof of the fact that (2.41) is a bijection with two obvious observations. Since the linear term $\varphi$ of $U$ induces quasi-isomorphism (2.25) for every $n$, $\varphi$ induces quasi-isomorphisms

\[ L/F_nL \rightarrow \tilde{L}/\tilde{F}_n\tilde{L} \] (3.1)

\[ F_nL/F_{n+1}L \rightarrow F_n\tilde{L}/F_{n+1}\tilde{L} \] (3.2)

for every $n$. 
3.1 Map (2.41) is injective

To prove that map (2.41) induced by $U_*$ is injective, we denote by $\alpha$ and $\alpha'$ MC elements of $L$ for which $U_* (\alpha)$ and $U_* (\alpha')$ are connected by a 1-cell in $\mathcal{MC}_*(\tilde{L})$. From this, we will construct a sequence of MC elements in $L$

$$\{\alpha(n)\}_{n \geq 1}, \quad (3.3)$$

a sequence of rectified 1-cells

$$\{\rho(n) = \rho_0(n)(t_0) + dt_0 \rho_1(n)\}_{n \geq 1} \quad (3.4)$$

in $\mathcal{MC}_*(L)$ and a sequence of rectified 1-cells

$$\{\tilde{\beta}(n) = \tilde{\beta}_0(n)(t_0) + dt_0 \tilde{\beta}_1(n)\}_{n \geq 1} \quad (3.5)$$

in $\mathcal{MC}_*(\tilde{L})$ satisfying the following properties:

$$\alpha^{(1)} = \alpha, \quad (3.6)$$

$$\alpha' - \alpha^{(n)} \in F_n L, \quad (3.7)$$

$$\rho_1^{(n)} \in F_n L, \quad (3.8)$$

$$\rho_0^{(n)}(0) = \alpha^{(n)}, \quad \rho_0^{(n)}(1) = \alpha^{(n+1)}, \quad (3.9)$$

$$\tilde{\beta}_1^{(n)} \in F_n \tilde{L}, \quad (3.10)$$

and

$$\tilde{\beta}_0^{(n)}(0) = U_*(\alpha^{(n)}), \quad \tilde{\beta}_0^{(n)}(1) = U_*(\alpha'). \quad (3.11)$$

Then, by concatenating together the infinite sequence of 1-cells $\rho(n)$, we will obtain a 1-cell in $\mathcal{MC}_*(L)$ connecting $\alpha$ with $\alpha'$.

3.1.1 The inductive construction of sequences (3.3), (3.4), and (3.5)

Since $U_*(\alpha)$ and $U_*(\alpha')$ are connected by a 1-cell in $\mathcal{MC}_*(\tilde{L})$, Lemma B.2 from Appendix B implies that there exists a rectified 1-cell

$$\tilde{\beta}^{(1)} = \tilde{\beta}_0^{(1)}(t_0) + dt_0 \tilde{\beta}_1^{(1)} \quad (3.12)$$

such that

$$\tilde{\beta}_0^{(1)}(0) = U_*(\alpha), \quad (3.13)$$

and

$$\tilde{\beta}_0^{(1)}(1) = U_*(\alpha'). \quad (3.14)$$

The desired condition $\tilde{\beta}_1^{(1)} \in F_1 \tilde{L}$ is satisfied automatically since $\tilde{L} = F_1 \tilde{L}$.

Thus we take $\alpha^{(1)} := \alpha$ as the base of our induction and assume that we already constructed MC elements

$$\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)},$$
In the inductive step, we construct 1-cells \(\rho(1), \rho(2), \ldots, \rho(n-1)\) in \(MC_* (L)\) and 1-cells \(\tilde{\beta}(1), \tilde{\beta}(2), \ldots, \tilde{\beta}(n)\) in \(MC_* (\tilde{L})\) satisfying all the desired properties.

Our goal is to construct a rectified 1-cell
\[
\rho(n) = \rho_0^n (t_0) + dt_0 \rho_1^n
\]
(3.15)
in \(MC_* (L)\) and a rectified 1-cell
\[
\tilde{\beta}(n+1) = \tilde{\beta}_0^{(n+1)} (t_0) + dt_0 \tilde{\beta}_1^{(n+1)}
\]
(3.16)
such that \(\rho_1^n \in F_n L, \tilde{\beta}_1^{(n+1)} \in F_{n+1} \tilde{L}\),

\[
\rho_0^n (0) = \alpha^{(n)},
\]
(3.17)
\[
\alpha' - \rho_0^n (1) \in F_{n+1} L,
\]
(3.18)
and
\[
\tilde{\beta}_0^{(n+1)} (0) = U_* (\rho_0^n (1))
\]
(3.19)

Then setting \(\alpha^{(n+1)} := \rho_0^n (1)\) would complete the inductive step.

Diagram (3.21) below shows all the links between 0-cells and 1-cells under consideration. The elements which should be constructed in the inductive step are shown in blue.

\[
\begin{align*}
\alpha = \alpha(1) &\xrightarrow{\rho(1)} \alpha(2) \xrightarrow{\rho(2)} \alpha(3) \quad \cdots \quad \alpha(n-1) &\xrightarrow{\rho(n-1)} \alpha(n) \xrightarrow{\rho(n)} \alpha(n+1) \\
U_* (\alpha(1)) &\xrightarrow{\tilde{\beta}(1)} U_* (\alpha(2)) \xrightarrow{\tilde{\beta}(2)} \cdots \xrightarrow{\tilde{\beta}(n-1)} U_* (\alpha(n)) \xrightarrow{\tilde{\beta}(n)} U_* (\alpha(n+1)) \\
\end{align*}
\]
(3.21)

Since both \(\alpha^{(n)}\) and \(\alpha'\) satisfy the MC equation and \(\alpha' - \alpha^{(n)} \in F_n L\), equation (2.16) implies that the difference
\[
\alpha' - \alpha^{(n)}
\]
(3.22)
represents a cocycle in the quotient complex
\[
F_n L / F_{n+1} L.
\]

Let us prove that
Claim 3.1 The difference in (3.22) represents a coboundary in $F_nL/F_{n+1}L$.

Proof. Let us consider the 1-cell $\tilde{\beta}^{(n)}$ in $MC_{\ast}(\tilde{L})$ which connects $U_*\alpha^{(n)}$ to $U_*\alpha'$.

The MC equation for $\tilde{\beta}^{(n)}$ is equivalent to the pair of equations

$$\text{curv}(\tilde{\beta}_0^{(n)}(t)) = 0 \quad (3.23)$$

and

$$\frac{d}{dt}\tilde{\beta}_0^{(n)}(t) = \partial \tilde{\beta}_0^{(n)}(t)\tilde{\beta}_1^{(n)}, \quad (3.24)$$

where $\partial \tilde{\beta}_0^{(n)}(t)$ denotes the differential twisted by $\tilde{\beta}_0^{(n)}(t)$ (see Section 2.1).

Integrating both sides of (3.24), we deduce that

$$U_*\alpha' - U_*\alpha^{(n)} = \int_0^1 dt \partial \tilde{\beta}_0^{(n)}(t)\tilde{\beta}_1^{(n)}$$

$$= \partial \tilde{\beta}_1^{(n)} + \sum_{m \geq 1} \frac{1}{m!} \int_0^1 dt \{\tilde{\beta}_0^{(n)}(t), \ldots, \tilde{\beta}_0^{(n)}(t), \tilde{\beta}_1^{(n)}\}_{m+1}. \quad (3.25)$$

Thus the difference $U_*\alpha' - U_*\alpha^{(n)}$ represents a coboundary in the quotient $F_n\tilde{L}/F_{n+1}\tilde{L}$.

On the other hand,

$$U_*\alpha' - U_*\alpha^{(n)} - \varphi(\alpha' - \alpha^{(n)}) \in F_{n+1}\tilde{L}. \quad (3.26)$$

Therefore, since $\varphi$ induces quasi-isomorphism (3.2), $\alpha' - \alpha^{(n)}$ indeed represents a coboundary in $F_nL/F_{n+1}L$.

Claim 3.1 implies that there exists

$$\rho_1^{(n)} \in F_nL \quad (3.27)$$

such that

$$\alpha' - \alpha^{(n)} - \partial \rho_1^{(n)} \in F_{n+1}L. \quad (3.28)$$

Let us denote by $\rho^{(n)}$ the 1-cell

$$\rho^{(n)} = \rho_0^{(n)}(t) + dt\rho_1^{(n)}, \quad (3.29)$$

where $\rho_0^{(n)}(t)$ is the unique solution to this integral equation:

$$\rho_0^{(n)}(t) = \alpha^{(n)} + \int_0^t dt_1 \partial \rho_0^{(n)}(t_1)(\rho_1^{(n)}). \quad (3.30)$$

It is easy to see that the new MC element

$$\alpha^{(n+1)} := \rho_0^{(n)}(1) \quad (3.31)$$

satisfies the condition

$$\alpha^{(n+1)} - \alpha^{(n)} - \partial \rho_1 \in F_{n+1}L$$

and hence

$$\alpha' - \alpha^{(n+1)} \in F_{n+1}L. \quad (3.32)$$

We claim that
Claim 3.2 The element $\rho_1^{(n)}$ in (3.27) can be chosen in such a way that

$$\tilde{\beta}_1^{(n)} - \varphi(\rho_1^{(n)}) - \partial(\tilde{\gamma}) \in F_{n+1}\tilde{L}$$  \hspace{1cm} (3.33)

for some $\tilde{\gamma} \in F_n\tilde{L}$. Here $\tilde{\beta}_1^{(n)}$ is the degree $-1$ component of the 1-cell $\tilde{\beta}^{(n)}$ which connects $U_*(\alpha^{(n)})$ to $U_*(\alpha')$.

Proof. Indeed, due to (3.25) and inclusion (3.26), we have

$$\varphi(\alpha') - \varphi(\alpha^{(n)}) - \partial(\tilde{\beta}_1^{(n)}) \in F_{n+1}\tilde{L}.$$  \hspace{1cm} (3.34)

Combining this inclusion with (3.28), we conclude that

$$\partial(\tilde{\beta}_1^{(n)} - \rho_1^{(n)}) \in F_{n+1}\tilde{L}.$$  \hspace{1cm} (3.35)

Inclusion (3.34) implies that we can safely replace $\rho_1^{(n)}$ in (3.27) without violating inclusion (3.28).

Claim (3.2) is proved. $\square$

Thus we constructed the desired “blue” elements $\rho^{(n)}$ and $\alpha^{(n+1)}$ in diagram (3.21).

Let us now consider the 1-cell

$$\tilde{\rho}^{(n)} = \rho_0^{(n)}(t) + dt\rho_1^{(n)}(t) := U_*(\rho_0^{(n)}(t) + dt\rho_1^{(n)})$$  \hspace{1cm} (3.36)

in $MC_{\bullet}(\tilde{L})$ connecting $U_*(\alpha^{(n)})$ to $U_*(\alpha^{(n+1)})$.

Combining (3.36) with the 1-cell $\tilde{\beta}^{(n)}$ in $MC_{\bullet}(\tilde{L})$ we get the 1-dimensional horn in $MC_{\bullet}(\tilde{L})$ shown on figure 1.

We claim that
Proposition 3.3 There exists a MC element 
\[ \tilde{\eta} \in \hat{L} \otimes \Omega \]
which fills in the horn shown on figure [4] and such that the component \[ \tilde{\eta}'(t_0) \] of
\[ \tilde{\eta}'_0(t_0) + dt_0 \tilde{\eta}'_1(t_0) := \tilde{\eta}\big|_{t_1=0} \] (3.37)
satisfies
\[ \tilde{\eta}'_1(t_0) \in F_{n+1} \hat{L} \otimes k[t_0] \] (3.38)

Proof. In this proof we will use the material presented in Appendix A. For example, we will need the operators \( h^1_1 \) and \( h^1_2 \) on \( \hat{L} \otimes \Omega \) and \( \hat{L} \otimes \Omega \), respectively, and we will need Lemma A.1.

According to Lemma A.1, the 1-cell \( \tilde{\beta}^{(n)} \) is uniquely determined by \[ \tilde{\beta}^{(n)}_0(0) = U^*(\alpha^{(n)}) \] and its “stub”
\[ (\partial + d)h^1_1(\tilde{\beta}^{(n)}) = (\partial + d)(t_0 \tilde{\beta}^{(n)}_1) . \] (3.39)

Similarly, 1-cell (3.36) is uniquely determined by \( \tilde{\rho}^{(n)}_0(0) = U^*(\alpha^{(n)}) \) and its “stub”
\[ (\partial + d)(\tilde{\xi}(t_0)) , \] (3.40)

where
\[ \tilde{\xi}(t_0) := h^1_1(\tilde{\rho}^{(n)}) \in \hat{L} \otimes k[t_0] . \] (3.41)

Since \( U \) is compatible with the filtrations and \( \rho^{(n)}_1 \in F L \)
\[ \tilde{\rho}^{(n)}_1(t_0) - \varphi(\rho^{(n)}_1) \in F_{n+1} \hat{L} \otimes k[t_0] \]
and hence
\[ \tilde{\xi}(t_0) - t_0 \varphi(\rho^{(n)}_1) \in F_{n+1} \hat{L} \otimes k[t_0] . \] (3.42)

Our goal is to construct a MC element
\[ \tilde{\eta} \in \hat{L} \otimes \Omega \] (3.43)
such that
\[ \tilde{\eta}\big|_{t_2=0} = \tilde{\rho}^{(n)}_0(t_0) + dt_0 \tilde{\beta}^{(n)}_1 , \] (3.44)
\[ \tilde{\eta}\big|_{t_0=0} = \tilde{\rho}^{(n)}_0(t_2) + dt_2 \tilde{\rho}^{(n)}_1(t_2) , \] (3.45)
and such that condition (3.38) is satisfied.

For this purpose, we introduce the following element in \( \text{Stub}^1_2(\hat{L}) \)
\[ \tilde{\nu} := (\partial + d)(\tilde{\xi}(t_2) + t_0 \tilde{\beta}^{(n)}_1 + (t_2 dt_0 - t_0 dt_2) \tilde{\gamma}) , \] (3.46)

\textsuperscript{6}See Remark B.1
\textsuperscript{7}The notation \( \text{Stub}^i_n \) is introduced in (A.3) in Appendix A.
where $\tilde{\gamma}$ is a degree $-2$ element in $\mathcal{F}_n \tilde{L}$ introduced in Claim 3.2 and $\tilde{\xi}$ is defined in (3.41).

It is clear that
$$\tilde{\nu} \bigg|_{t_1 = 1} = (\partial + d) \left( \tilde{\xi}(t_2) \right) \bigg|_{t_2 = 0}.$$ 

Furthermore,
$$(\partial + d) \left( \tilde{\xi}(t_0) \right) \bigg|_{t_0 = 0} = 0$$

since $(\partial + d) \left( \tilde{\xi}(t_0) \right)$ is the “stub” of the 1-cell $\tilde{\rho}^{(n)}$.

Thus $\tilde{\nu}$ is indeed an element of $\text{Stub}_2(\tilde{L})$.

Next we set
$$\tilde{\eta}^{(0)} := U_* (\alpha^{(n)}) + \tilde{\nu} \quad (3.47)$$

and define $\tilde{\eta}$ (3.43) as the limiting element of the sequence
$$\{ \tilde{\eta}^{(k)} \}_{k \geq 0}$$

defined by the recursive procedure in (A.7)
$$\tilde{\eta}^{(k+1)} := \tilde{\eta}^{(0)} - \sum_{m=2} \frac{1}{m!} \frac{1}{2} \{ \tilde{\eta}^{(k)}, \ldots, \tilde{\eta}^{(k)} \}_m. \quad (3.48)$$

Conditions (3.44) and (3.45) are satisfied due to Lemma A.1.

So it remains to prove that
$$\tilde{\eta}'_1(t_0) \in \mathcal{F}_{n+1} \tilde{L} \otimes k[t_0], \quad (3.49)$$

where $\tilde{\eta}'_1(t_0)$ is the degree $-1$ component of the 1-cell
$$\tilde{\eta}'_0(t_0) + dt_0 \tilde{\eta}'_1(t_0) := \tilde{\eta} \bigg|_{t_1 = 0}. \quad (3.49)$$

To prove (3.49), we observe that $\tilde{\nu} \in \mathcal{F}_n \tilde{L}$ and hence
$$\tilde{\eta} - U_* (\alpha^{(n)}) - \tilde{\nu} \in \mathcal{F}_{n+1} \tilde{L} \otimes \Omega_2 \quad (3.50)$$

On the other hand,
$$\tilde{\nu} \bigg|_{t_1 = 0} = (\partial + d) \left( \tilde{\xi}(1 - t_0) + t_0 \tilde{\beta}^{(n)}_1 + dt_0 \tilde{\gamma} \right).$$

Therefore, using (3.42), we get
$$\tilde{\nu} \bigg|_{t_1 = 0} = \left( t_0 (\partial \tilde{\beta}^{(n)}_1 - \partial \varphi^{(n)}_1) + dt_0 (\tilde{\beta}^{(n)}_1 - \varphi^{(n)}_1 - \partial \tilde{\gamma}) \right) \in \mathcal{F}_{n+1} \tilde{L} \otimes \Omega_1.$$

Hence
$$\tilde{\eta}'_1(t_0) - (\tilde{\beta}^{(n)}_1 - \varphi^{(n)}_1 - \partial \tilde{\gamma}) \in \mathcal{F}_{n+1} \tilde{L} \otimes k[t_0].$$

Combining the latter inclusion with (3.33) we immediately deduce the desired inclusion $\tilde{\eta}'_1(t_0) \in \mathcal{F}_{n+1} \tilde{L} \otimes k[t_0].$
Proposition 3.3 is proved.

The existence of the desired 1-cell \( \tilde{\beta}^{(n+1)} \) in diagram (3.21) follows immediately from Proposition 3.3 and Lemma B.2 from Appendix B.

Thus desired sequences (3.3), (3.4), and (3.5) are constructed.

Since the sequence of 1-cells (3.4) satisfies (3.8) and (3.9), Lemma C.1 from Appendix C implies that there exists a 1-cell \( \gamma \) in \( \mathcal{MC}_0(L) \) which connects \( \alpha = \alpha^{(1)} \) to the limiting MC element \( \alpha_{lim} \) of sequence (3.3).

On the other hand, condition (3.7) implies that \( \alpha_{lim} = \alpha' \).

Thus the 0-cells \( \alpha, \alpha' \in \mathcal{MC}_0(L) \) are indeed connected by a 1-cell. The injectivity of map (2.41) is proved.

3.2 Map (2.41) is surjective

To establish that map (2.41) is surjective, we will prove inductively the following proposition:

**Proposition 3.4** For every MC element \( \tilde{\alpha} \) in \( \tilde{L} \), there exists a sequence of degree zero elements

\[
\{\alpha^{(n)}\}_{n \geq 0} \subset L
\]  

(3.51)

and a sequence of rectified 1-cells in \( \mathcal{MC}_\cdot(\tilde{L}) \)

\[
\tilde{\beta}^{(n)} = \tilde{\beta}_0^{(n)}(t_0) + dt_0 \tilde{\beta}_1^{(n)}
\]  

(3.52)

such that for every \( n \geq 0 \)

\[
\text{curv}(\alpha^{(n)}) \in \mathcal{F}_{n+2}L,
\]  

(3.53)

\[
\alpha^{(n+1)} - \alpha^{(n)} \in \mathcal{F}_{n+1}L,
\]  

(3.54)

\[
\tilde{\beta}_0^{(n)}(0) = \tilde{\alpha},
\]  

(3.55)

\[
\tilde{\beta}^{(n+1)} - \tilde{\beta}^{(n)} \in \mathcal{F}_{n+1}\tilde{L} \otimes \Omega_1
\]  

(3.56)

and

\[
\tilde{\beta}_0^{(n)}(1) - U_*(\alpha^{(n)}) \in \mathcal{F}_{n+1}\tilde{L}.
\]  

(3.57)

Proof. For \( n = 0 \) we set and

\[
\alpha^{(0)} = 0 \quad \text{and} \quad \tilde{\beta}^{(0)} := \tilde{\alpha}.
\]

In this case, the only non-trivial condition is (3.57) and it holds because \( \tilde{L} = \mathcal{F}_1\tilde{L} \).

Let us now assume that we constructed elements

\[
\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(n)}
\]

and

\[
\tilde{\beta}^{(0)}, \tilde{\beta}^{(1)}, \ldots, \tilde{\beta}^{(n)}
\]
with the desired properties.

Our goal is to construct a degree zero element \( \alpha^{(n+1)} \in L \) and a MC element
\[
\tilde{\beta}^{(n+1)} = \tilde{\beta}_0^{(n+1)}(t_0) + dt_0 \tilde{\beta}_1^{(n+1)} \in \tilde{L} \otimes \Omega_1
\]
such that (3.54) and (3.56) hold,

\[
\text{curv}(\alpha^{(n+1)}) \in \mathcal{F}_{n+3}L, \quad (3.58)
\]

\[
\tilde{\beta}_0^{(n+1)}(0) = \tilde{\alpha}, \quad (3.59)
\]

and

\[
\tilde{\beta}_0^{(n+1)}(1) - U_*(\alpha^{(n+1)}) \in \mathcal{F}_{n+2}\tilde{L}. \quad (3.60)
\]

Equation (2.14) and inclusion (3.53) imply that
\[
\text{curv}(U_*(\alpha^{(n)})) \in \mathcal{F}_{n+2}\tilde{L}.
\]

Hence, using (2.16), (3.57) and the fact that \( \tilde{\beta}^{(n)} \) is a MC element of \( \tilde{L} \otimes \Omega_1 \) we conclude that

\[
\tilde{\beta}^{(n)}(1) - U_*(\alpha^{(n)})
\]
represents a cocycle in the quotient complex

\[
\mathcal{F}_{n+1}\tilde{L} / \mathcal{F}_{n+2}\tilde{L}.
\]

Since the map \( \varphi \) induces quasi-isomorphism (3.2), there exists a degree 0 element \( \gamma \in \mathcal{F}_{n+1}L \) and a degree \(-1\) element \( \tilde{\xi} \in \mathcal{F}_{n+1}\tilde{L} \) such that

\[
\partial(\gamma) \in \mathcal{F}_{n+2}L \quad (3.62)
\]

and

\[
\tilde{\beta}_0^{(n)}(1) + \partial(\tilde{\xi}) - (U_*(\alpha^{(n)}) + \varphi(\gamma)) \in \mathcal{F}_{n+2}\tilde{L}. \quad (3.63)
\]

So we introduce the new degree 0 element
\[
\alpha := \alpha^{(n)} + \gamma \in L
\]

and the new (rectified) 1-cell
\[
\tilde{\beta}^{(n+1)} = \tilde{\beta}_0^{(n+1)}(t_0) + dt_0 \tilde{\beta}_1^{(n+1)}, \quad (3.65)
\]
in \( \mathcal{MC}_*(\tilde{L}) \), where
\[
\tilde{\beta}_1^{(n+1)} = \tilde{\beta}_1^{(n)} + \tilde{\xi}, \quad (3.66)
\]

\( \tilde{\beta}_0^{(n+1)}(t_0) \) is the unique solution of the integral equation

\[
\tilde{\beta}_0^{(n+1)}(t_0) = \tilde{\alpha} + \int_0^{t_0} \partial \tilde{\beta}_0^{(n+1)}(u) (\tilde{\beta}_1^{(n)} + \tilde{\xi}) du, \quad (3.67)
\]

and \( \partial \tilde{\beta}_0^{(n+1)}(u) \) denotes the differential twisted by the MC element \( \tilde{\beta}_0^{(n+1)}(u) \).
Inclusions (3.53), $\gamma \in \mathcal{F}_{n+1}L$, and (3.62) imply that
\[ \text{curv}(\alpha) \in \mathcal{F}_{n+2}L. \] (3.68)
Similarly, inclusions $\tilde{\xi} \in \mathcal{F}_{n+1}\tilde{L}$ and $\gamma \in \mathcal{F}_{n+1}L$ imply that
\[ \tilde{\beta}_0^{(n)}(1) + \partial(\tilde{\xi}) - \tilde{\beta}_0^{(n+1)}(1) \in \mathcal{F}_{n+2}\tilde{L}. \] (3.69)
and
\[ U_*(\alpha^{(n)}) + \varphi(\gamma) - U_*(\alpha) \in \mathcal{F}_{n+2}\tilde{L}. \] (3.70)
Thus, combining (3.63), (3.69), and (3.70), we conclude that
\[ \tilde{\beta}_0^{(n+1)}(1) - U_*(\alpha) \in \mathcal{F}_{n+2}\tilde{L}. \] (3.71)
We will now use $\alpha$ to construct a degree 0 element $\alpha_{(n+1)}^0$ satisfying (3.54), (3.58), and (3.60).

Due to (2.13) and (3.68), the degree 1 element $\text{curv}(\alpha)$ represents a cocycle in the quotient complex
\[ \mathcal{F}_{n+2}L/\mathcal{F}_{n+3}L. \]
Let us prove that

**Claim 3.5** The element $\text{curv}(\alpha)$ represents a coboundary in the quotient complex $\mathcal{F}_{n+2}L/\mathcal{F}_{n+3}L$.

**Proof of Claim 3.5** Due to (2.14)
\[ \text{curv}(U_*(\alpha)) = \sum_{m \geq 0} \frac{1}{m!} U^m(\alpha^m \text{curv}(\alpha)) \in \mathcal{F}_{n+2}\tilde{L}. \]
Hence, inclusion (3.68) implies that
\[ \varphi(\text{curv}(\alpha)) - \text{curv}(U_*(\alpha)) \in \mathcal{F}_{n+3}\tilde{L}. \] (3.72)
On the other hand, equation (2.16) implies that
\[ \text{curv}(U_*(\alpha)) = \text{curv}(\tilde{\beta}_0^{(n+1)}(1) + (U_*(\alpha) - \tilde{\beta}_0^{(n+1)}(1))) = \text{curv}(\tilde{\beta}_0^{(n+1)}(1)) + \text{curv}(U_*(\alpha) - \tilde{\beta}_0^{(n+1)}(1)), \]
where
\[ \text{curv}^\beta \alpha := \partial^\beta(\alpha) + \sum_{m=2}^{\infty} \frac{1}{m!} \{\alpha, \ldots, \alpha\}_m^\beta, \]
and $\partial^\beta$ and $\{\cdot, \ldots, \cdot\}_m^\beta$ denote the differential and the multi-brackets of the $\mathcal{G}\text{Lie}_\infty$-structure twisted by $\beta$.
Hence, using (3.71) and the fact that $\tilde{\beta}_0^{(n+1)}(1)$ is a MC element of $\tilde{L}$ we conclude that
\[ \text{curv}(U_*(\alpha)) - \partial(U_*(\alpha) - \tilde{\beta}_0^{(n+1)}(1)) \in \mathcal{F}_{n+3}\tilde{L}. \]
Combining this observation with inclusion (3.72), we conclude that the element $\varphi(\text{curv}(\alpha))$ represents a coboundary in the quotient complex

$$\mathcal{F}_{n+2}\tilde{L}/\mathcal{F}_{n+3}\tilde{L}.$$  

Thus the desired statement follows from the fact that $\varphi$ induces quasi-isomorphism (3.2). 

Due to Claim 3.5 there exists a degree 0 element $\sigma \in \mathcal{F}_{n+2}\tilde{L}$ such that

$$\text{curv}(\alpha) + \partial(\sigma) \in \mathcal{F}_{n+3}\tilde{L}.$$  

Thus, setting

$$\alpha^{(n+1)} := \alpha + \sigma$$

we get a degree 0 element of $L$ satisfying desired properties (3.54), (3.58), and (3.60).

Proposition 3.4 is proved. □

Let us denote by $\alpha_{\text{lim}} \in L$ and $\tilde{\beta} \in \tilde{L} \otimes \Omega_1$ the limiting elements of sequences (3.51) and (3.52), respectively.

Due equation (3.55) and inclusions (3.53), (3.57), $\alpha_{\text{lim}}$ is a MC element of $L$ and $\tilde{\beta}$ is a 1-cell of $\mathcal{MC}_*(\tilde{L})$ which connects the MC elements $\tilde{\alpha}$ and $U_*(\alpha_{\text{lim}})$ of $\tilde{L}$. Thus we proved that map (2.41) is surjective.

4 Taking care of higher homotopy groups

Let us start this section by recalling a lemma from [6] which allows us to reduce questions about homotopy groups $\pi_i(\mathcal{MC}_*(L), \alpha)$ of $\mathcal{MC}_*(L)$ with an arbitrary base point $\alpha \in \text{MC}(L)$ to the corresponding questions about homotopy groups of $\mathcal{MC}_*(L^\alpha)$ with the zero base point:

Lemma 4.1 (Lem. 4.3 [6]) Let $\alpha$ be a MC element in $L$ and $L^\alpha$ be the filtered $\mathcal{S}\text{Lie}_\infty$ algebra which is obtained from $L$ via twisting by $\alpha$. Then the following assignment

$$\beta \in \text{MC}(L^\alpha \otimes \Omega_n) \mapsto \alpha + \beta \in \text{MC}(L \otimes \Omega_n) \quad (4.1)$$

is an isomorphism of simplicial sets

$$\text{Shift}_\alpha : \mathcal{MC}_*(L^\alpha) \to \mathcal{MC}_*(L) \quad (4.2)$$

which sends the zero MC element of $L^\alpha$ to the MC element $\alpha$ in $L$. For every $\infty$-morphism $U$ of filtered $\mathcal{S}\text{Lie}_\infty$-algebras $L \to \tilde{L}$ the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{MC}_*(L^\alpha) & \xrightarrow{\text{Shift}_\alpha} & \mathcal{MC}_*(L) \\
\mathcal{MC}_*(U^\alpha) \downarrow & & \downarrow \mathcal{MC}_*(U) \\
\mathcal{MC}_*(\tilde{L}^{U_*(\alpha)}) & \xrightarrow{\text{Shift}_{U_*(\alpha)}} & \mathcal{MC}_*(\tilde{L})
\end{array} \quad (4.3)$$

where $U^\alpha$ denotes the $\infty$-morphism $L^\alpha \to \tilde{L}^{U_*(\alpha)}$ which is obtained from $U$ via twisting by the MC element $\alpha$.  

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Let us now prove that for every \( n \geq 2 \) the map \( \mathcal{MC}_\bullet(U) \) induces a weak equivalence of simplicial sets:

\[
\mathcal{MC}_\bullet(L/F_nL) \longrightarrow \mathcal{MC}_\bullet(L/F_n\bar{L}) \tag{4.4}
\]

Since \( S\text{Lie}_\infty \)-algebras \( L/F_2L \) and \( L/F_2\bar{L} \) are abelian, Proposition \[2.4\] implies that \( (4.4) \) is indeed a weak equivalence for \( n = 2 \).

So we take \( n = 2 \) as the base of our induction and assume that

\[
\mathcal{MC}_\bullet(L/F_mL) \longrightarrow \mathcal{MC}_\bullet(L/F_m\bar{L})
\]

is a weak equivalence of simplicial sets for every \( m \leq n \).

Our goal is to prove that

\[
\mathcal{MC}_\bullet(L/F_{n+1}L) \longrightarrow \mathcal{MC}_\bullet(L/F_{n+1}\bar{L}) \tag{4.5}
\]

is also a weak equivalence of simplicial sets.

Due to the results of Section 3 and Lemma 4.1, it suffices to prove that

\[
\pi_i\left(\mathcal{MC}_\bullet(L/F_{n+1}L), 0\right) \longrightarrow \pi_i\left(\mathcal{MC}_\bullet(L/F_{n+1}\bar{L}), 0\right) \tag{4.6}
\]

is an isomorphism of groups for every \( i \geq 1 \).

For this purpose, we consider the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & F_nL/F_{n+1}L & \longrightarrow & L/F_{n+1}L & \longrightarrow & L/F_nL & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_n\bar{L}/F_{n+1}\bar{L} & \longrightarrow & \bar{L}/F_{n+1}\bar{L} & \longrightarrow & \bar{L}/F_n\bar{L} & \longrightarrow & 0,
\end{array}
\tag{4.7}
\]

where the rows form short exact sequences of strict morphisms of \( S\text{Lie}_\infty \)-algebras, the leftmost vertical arrow is the strict morphism induced by the linear term \( \varphi \) of \( U \), and the other two vertical arrows are \( \infty \)-morphisms induced by \( U \).

Since the morphisms \( L/F_{n+1}L \to L/F_nL \) and \( \bar{L}/F_{n+1}\bar{L} \to \bar{L}/F_n\bar{L} \) are surjective, Proposition 4.7 from [10] implies that the corresponding maps

\[
\mathcal{MC}_\bullet(L/F_{n+1}L) \to \mathcal{MC}_\bullet(L/F_nL) \quad \text{and} \quad \mathcal{MC}_\bullet(L/F_{n+1}\bar{L}) \to \mathcal{MC}_\bullet(L/F_n\bar{L}) \tag{4.8}
\]

are fibrations of simplicial sets with the fibers over the base point \( 0 \in L/F_nL \) (resp. \( 0 \in \bar{L}/F_n\bar{L} \)) being \( \mathcal{MC}_\bullet(F_nL/F_{n+1}L) \) (resp. \( \mathcal{MC}_\bullet(F_n\bar{L}/F_{n+1}\bar{L}) \)).

Therefore, the rows of diagram (4.7) give us the long “exact” sequence\(^8\) of homotopy groups (see, for example, [11, Lemma 7.3]):

\[
\ldots \longrightarrow \pi_1(F_nL/F_{n+1}L) \longrightarrow \pi_1(L/F_{n+1}L) \longrightarrow \pi_1(L/F_nL) \\
\pi_0(F_nL/F_{n+1}L) \longrightarrow \pi_0(L/F_{n+1}L) \longrightarrow \pi_0(L/F_nL),
\tag{4.9}
\]

\(^8\)The words exact is put in quotation marks because the \( \pi_0 \) portion of sequence (4.9) is not an exact sequence of groups. Moreover, in general, the map \( \pi_0(L/F_{n+1}L) \to \pi_0(L/F_nL) \) is not a surjective map of sets.
\[
\ldots \to \pi_1(F_n L/F_{n+1} L) \to \pi_1(L/F_{n+1} L) \to \pi_1(L/F_n L) \\
\overset{\text{where by abuse of notation}}{\longrightarrow} \pi_0(F_n L/F_{n+1} L) \to \pi_0(L/F_{n+1} L) \to \pi_0(L/F_n L),
\]

(4.10)

Furthermore, the $\infty$-morphism $U$ induces a map from sequence (4.9) to sequence (4.10). For our purposes, we only need certain truncations of sequences (4.9), (4.10). So we denote by

\[
b(L, F_n L, F_{n+1} L)
\]

(4.11)

the image of the map

\[
\pi_1(L/F_n L) \to \pi_0(F_n L/F_{n+1} L).
\]

In other words, $b(L, F_n L, F_{n+1} L)$ consists of cohomology classes $c \in H^0(F_n L/F_{n+1} L)$ satisfying the following property:

**Property 4.2** For $c$, there exists a MC element

\[
\beta = \beta_0(t_0) + dt_0 \beta_1(t_0) \in (L/F_{n+1} L) \otimes \Omega_1
\]

such that $\beta_0(0) = 0$ in $L/F_{n+1} L$ and $\beta_0(1)$ is a cocycle in $F_n L/F_{n+1} L$ representing the class $c$.

Let us now consider the following commutative diagram

\[
\ldots \to \pi_1(F_n L/F_{n+1} L) \to \pi_1(L/F_{n+1} L) \to \pi_1(L/F_n L) \to b(L, F_n L, F_{n+1} L), \\
\ldots \to \pi_1(F_n L/F_{n+1} L) \to \pi_1(L/F_{n+1} L) \to \pi_1(L/F_n L) \to b(L, F_n L, F_{n+1} L),
\]

(4.12)

where vertical arrows are induced by $\mathfrak{MC}(U)$, rows

\[
\ldots \to \pi_2(L/F_n L) \to \pi_1(F_n L/F_{n+1} L) \to \pi_1(L/F_{n+1} L) \to \pi_1(L/F_n L) \\
\ldots \to \pi_2(L/F_n L) \to \pi_1(F_n L/F_{n+1} L) \to \pi_1(L/F_{n+1} L) \to \pi_1(L/F_n L)
\]

(4.13)

are sequences of groups which are exact in terms

\[
\pi_1(L/F_{n+1} L), \ \pi_1(F_n L/F_{n+1} L), \ \pi_2(L/F_n L), \ldots
\]

and

\[
\pi_1(L/F_{n+1} L), \ \pi_1(F_n L/F_{n+1} L), \ \pi_2(L/F_n L), \ldots,
\]

respectively.

We claim that

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Proposition 4.3 If $\mathcal{M}_\bullet(U)$ induces an isomorphism of groups
\[ \pi_i(\mathcal{M}_\bullet(L/F_nL), 0) \to \pi_i(\mathcal{M}_\bullet(\bar{L}/F_n\bar{L}), 0) \]
for every $i \geq 1$ then the corresponding homomorphism of groups
\[ \pi_i(\mathcal{M}_\bullet(L/F_{n+1}L), 0) \to \pi_i(\mathcal{M}_\bullet(\bar{L}/F_{n+1}\bar{L}), 0) \]
is also an isomorphism for all $i \geq 1$.

Proof. The most subtle part of the proof is the surjectivity of the homomorphism
\[ \pi_1(\mathcal{M}_\bullet(F_nL/F_{n+1}L)) \to \pi_1(\bar{L}/F_{n+1}\bar{L}) \]
(4.15)
So we will start with proving this fact.

Let $\tilde{g}$ be an element of $\pi_1(\bar{L}/F_{n+1}\bar{L})$ and $\tilde{h}$ be the corresponding element in $\pi_1(\bar{L}/F_n\bar{L})$.

Since the homomorphism
\[ \pi_1(\mathcal{M}_\bullet(L/F_nL)) \to \pi_1(\mathcal{M}_\bullet(\bar{L}/F_{n+1}\bar{L})) \]
is an isomorphism there exists (a unique) element $h \in \pi_1(L/F_nL)$ which is sent to $\tilde{h}$ via (4.16).

Let us denote by $c$ (resp. $\tilde{c}$) the image of $h \in \pi_1(L/F_nL)$ in $\mathcal{M}_\bullet(\mathcal{F}_nL/F_{n+1}L)$ (resp. $\mathcal{M}_\bullet(\mathcal{F}_{n+1}L/F_{n+1}\bar{L})$).

We know that $c$ is a cohomology class in $H^0(F_nL/F_{n+1}L)$ for which there exists a $\mathcal{M}_\bullet$ element
\[ \beta = \beta_0(t) + dt\beta_1(t) \in (L/F_{n+1}L) \otimes \Omega_1 \] (4.17)
such that
\begin{itemize}
  \item $\beta_0(0) = 0$ in $L/F_{n+1}L$,
  \item $\beta_0(1)$ is a cocycle in $\mathcal{F}_nL/F_{n+1}L$ representing the class $c$,
  \item the image of 1-cell (4.17) in $\mathcal{M}_\bullet(L/F_nL)$ is a loop representing $h \in \pi_1(L/F_nL)$.
\end{itemize}

Since $\tilde{h}$ comes from an element of $\tilde{g} \in \pi_1(\bar{L}/F_{n+1}\bar{L})$, the class $\tilde{c} \in H^0(\mathcal{F}_{n+1}L/F_{n+1}\bar{L})$ is zero. Hence, since diagram (4.12) commutes and $\varphi$ induces an isomorphism
\[ H^\bullet(\mathcal{F}_nL/F_{n+1}L) \to H^\bullet(\mathcal{F}_{n+1}L/F_{n+1}\bar{L}) \]
the class $c$ is also zero.

Therefore, there exists an element $\xi \in \mathcal{F}_nL/F_{n+1}L$ such that
\[ \beta_0(1) + \partial \xi = 0 \quad \text{in} \quad \mathcal{F}_nL/F_{n+1}L. \] (4.18)

Using the inclusion $\xi \in \mathcal{F}_nL/F_{n+1}L$ and (4.18), it is easy to show that the degree 0 element
\[ \beta' := \beta_0(t) + t\partial \xi + dt(\beta_1(t) + \xi) \in (L/F_{n+1}L) \otimes \Omega_1 \] (4.19)
satisfies these properties:
i) $\beta'$ is a MC element in $(L/F_{n+1}L) \otimes \Omega_1$,

ii) $\beta'|_{t=0} = \beta'|_{t=1} = 0$, and

iii) the image of 1-cell $\beta'$ in $\mathcal{MC}_*(L/F_nL)$ is a loop representing $h \in \pi_1(L/F_nL)$.

On the other hand, property ii) means that the 1-cell $\beta'$ is also a loop in $\mathcal{MC}_*(L/F_{n+1}L)$. Thus we proved that there exists an element

$$g' \in \pi_1(L/F_{n+1}L) \quad \text{(4.20)}$$

whose image in $\pi_1(L/F_nL)$ is $h$.

Let us denote by $\tilde{g}'$ the image of $g'$ in $\pi_1(\tilde{L}/F_{n+1}\tilde{L})$. Since diagram (4.12) commutes, we the element

$$\tilde{g}'\tilde{g}^{-1} \quad \text{(4.21)}$$

belongs to the kernel of the map

$$\pi_1(\tilde{L}/F_{n+1}\tilde{L}) \to \pi_1(L/F_{n+1}L).$$

Therefore, since sequence (4.14) is exact in $\pi_1(\tilde{L}/F_{n+1}\tilde{L})$ and

$$\pi_1(F_nL/F_{n+1}L) \to \pi_1(F_n\tilde{L}/F_{n+1}\tilde{L})$$

is an isomorphism of groups, there exists $f \in \pi_1(F_nL/F_{n+1}L)$ which is sent to $\tilde{g}(\tilde{g}')^{-1}$ via the composition

$$\pi_1(F_nL/F_{n+1}L) \to \pi_1(L/F_{n+1}L) \to \pi_1(\tilde{L}/F_{n+1}\tilde{L}).$$

Thus, if $f'$ is the image of $f$ in $\pi_1(L/F_{n+1}L)$ then the image of $f'g'$ in $\pi_1(\tilde{L}/F_{n+1}\tilde{L})$ coincides with $\tilde{g}$. We proved that homomorphism (4.15) is surjective.

The proof of injectivity of (4.15) is much easier so we leave it to the reader.

It remains to prove that

$$\pi_i(\mathcal{MC}_*(L/F_{n+1}L), 0) \to \pi_i(\mathcal{MC}_*(\tilde{L}/F_{n+1}\tilde{L}), 0)$$

is an isomorphism for all $i \geq 2$. This is done by induction on $i$ using the exactness of sequences (4.13) and (4.14) in terms

$$\pi_1(L/F_{n+1}L), \pi_1(F_nL/F_{n+1}L), \pi_2(L/F_nL), \ldots$$

and

$$\pi_1(\tilde{L}/F_{n+1}\tilde{L}), \pi_1(F_n\tilde{L}/F_{n+1}\tilde{L}), \pi_2(\tilde{L}/F_n\tilde{L}), \ldots,$$

respectively. Here we also use the fact that $\mathcal{MC}_*(U)$ induces an isomorphism of groups

$$\pi_i(\mathcal{MC}_*(F_nL/F_{n+1}L), 0) \to \pi_i(\mathcal{MC}_*(F_n\tilde{L}/F_{n+1}\tilde{L}), 0)$$

for every $i \geq 1$. This fact follows from Proposition 2.4 since both $F_nL/F_{n+1}L$ and $F_n\tilde{L}/F_{n+1}\tilde{L}$ are abelian $\mathcal{GLie}_\infty$-algebras.

Proposition 4.3 is proved. \[\square\]
4.1 The end of the proof of Theorem 2.2

The results presented in Section 3, Lemma 4.1, and Proposition 4.3 imply that $U$ induces a morphism between towers of Kan complexes:

$$
\begin{array}{ccc}
\cdots & \cdots & \\
MC^\bullet(L/F_{n+1}L) & \sim & MC^\bullet(\tilde{L}/F_{n+1}\tilde{L}) \\
| & & | \\
MC^\bullet(L/F_nL) & \sim & MC^\bullet(\tilde{L}/F_n\tilde{L}) \\
| & & | \\
MC^\bullet(L/F_{n-1}L) & \sim & MC^\bullet(\tilde{L}/F_{n-1}\tilde{L}) \\
| & & | \\
\cdots & \cdots & \\
\end{array}
$$

in which every vertical arrow is a fibration and every horizontal arrow is a weak equivalence. Hence, we have a weak equivalence between fibrant objects in the model category of towers of simplicial sets (Definition 1.1, Proposition 1.3, and Remark 1.5 in [11, Section VI]). The inverse limit is right adjoint to the constant tower functor, which preserves cofibrations and weak equivalences. Hence, the inverse limit preserves weak equivalences between fibrant objects, and thus we conclude that the map

$$
MC^\bullet(U) : \lim_{\leftarrow}MC^\bullet(L/F_nL) \to \lim_{\leftarrow}MC^\bullet(\tilde{L}/F_n\tilde{L})
$$

is a homotopy equivalence of simplicial sets.

A The homotopy operators $h^i_n$ and Lemma 4.6 from [10]

Let, as above, $\Delta^n$ denote the geometric $n$-simplex and $\Omega_n$ denote Sullivan’s polynomial de Rham complex of $\Delta^n$ with coefficients in $\mathbb{k}$.

For $0 \leq i \leq n$, let $\phi_i : [0,1] \times \Delta^n \to \Delta^n$ be the map

$$
\phi_i(u, \vec{t}) = u\vec{t} + (1-u)\vec{e}_i,
$$

where $\vec{e}_i$ denotes the $i$th vertex of $\Delta^n$. Let

$$
\pi_* : \Omega^\bullet([0,1] \times \Delta^n) \to \Omega^\bullet_{n-1}
$$

denote the integration over the fibers of the projection $\pi : [0,1] \times \Delta^n \to \Delta^n$. As in [10 Sec. 4], we define the homotopy operators

$$
h^i_n : \Omega^\bullet_n \to \Omega^\bullet_{n-1} \\
h^i_n \omega = \pi_\ast \phi^\ast_i \omega. \quad (A.1)
$$
Explicitly, if
\[ \omega = f(t_0, \ldots, t_n)dt_{k_1}dt_{k_2}\cdots dt_{k_m} \]
is a \( m \)-form on \( \Delta^n \), then
\[ h^i_n \omega = \sum_{j=1}^{m} (-1)^{j-1}(t_{kj} - \delta_{ik_j})dt_{k_1}dt_{k_2}\cdots \hat{dt}_{k_j} \cdots dt_{k_m} \int_0^1 u^{m-1}(f \circ \phi) \ du, \quad (A.2) \]
where \( \delta_{ik_j} \) is the Kronecker delta.

One can show that \( h^i_n \) is a chain homotopy between \( \varepsilon^i_n \) and the identity \( \text{id}_{\Omega_n} \):
\[ dh^i_n + h^i_n d = \text{id}_{\Omega_n} - \varepsilon^i_n, \quad (A.3) \]
where \( \varepsilon^i_n : \Omega_n \to \mathbb{K} \) is the evaluation at \( \overline{\varepsilon}_i \in \Delta^n \).

Due to [10, Lemma 3.5], the operator \( h^i_n \) satisfies the property
\[ h^i_n \circ h^i_n = 0. \quad (A.4) \]

For a filtered \( \mathfrak{G}\text{Lie}_\infty \)-algebra \( L \), a positive integer \( n \), and an integer \( 0 \leq i \leq n \), we consider the following subspace of \( (L \hat{\otimes} \Omega_n)^0 \)
\[ \text{Stub}^i_n(L) := \{ (\partial + d)\xi \mid \xi \in (L \hat{\otimes} \Omega_n)^{-1} \text{, such that } \partial(\xi) \big|_{t_i=1} = 0 \} . \quad (A.5) \]

For example, from (A.2) we see that for every MC element \( \alpha \in L \hat{\otimes} \Omega_n \), the element
\[ (\partial + d) \circ h^i_n(\alpha) \in \text{Stub}^i_n(L). \quad (A.6) \]

Given a pair \((\mu, \nu) \in \text{MC}(L) \times \text{Stub}^i_n(L)\), we define the following sequence \(\{\alpha^{(k)}\}_{k \geq 0}\) of degree 0 elements in \( L \hat{\otimes} \Omega_n \)
\[ \alpha^{(0)} := \mu + \nu, \]
\[ \alpha^{(k+1)} := \alpha^{(0)} - \sum_{m=2}^{\infty} \frac{1}{m!} h^i_n \{\alpha^{(k)}, \ldots, \alpha^{(k)}\}_m. \quad (A.7) \]

A simple inductive argument shows that \( \alpha^{(k+1)} - \alpha^{(k)} \in \mathcal{F}_{k+2}L \hat{\otimes} \Omega_n \). Hence, the sequence converges to \( \alpha = \lim \alpha^{(k)} \), which satisfies
\[ \alpha = \mu + \nu - \sum_{m \geq 2} \frac{1}{m!} h^i_n \{\alpha, \ldots, \alpha\}_m. \quad (A.8) \]

Combining (A.2) with (A.8) and the definition of \( \text{Stub}^i_n(L) \), we see that
\[ \varepsilon^i_n(\alpha) = \mu. \quad (A.9) \]
Furthermore, combining (A.3) with (A.4) and (A.8), we deduce that
\[ (\partial + d) \circ h^i_n(\alpha) = \nu. \quad (A.10) \]
Let us now show that $\alpha$ is a MC element of $L \hat{\otimes} \Omega_n$. Using (A.3), (A.9) and equation (2.13) from Proposition 2.1 we see that

$$\text{curv}(\alpha) = \text{curv}(\mu) + \sum_{m \geq 2} \frac{1}{m!} h_n^i (\partial + d) \{ \alpha, \ldots, \alpha \}_m$$

$$= h_n^i (\partial + d) \text{curv}(\alpha)$$

$$= - \sum_{m \geq 1} \frac{1}{m!} h_n^i \{ \alpha, \ldots, \alpha, \text{curv}(\alpha) \}_{m+1} \quad (A.11)$$

Therefore

$$\text{curv}(\alpha) \in \mathcal{F}_q L \hat{\otimes} \Omega_n$$

for all $q \geq 1$ and hence $\alpha$ is indeed a MC element of $L \hat{\otimes} \Omega_n$.

Let us now prove that every MC element $\alpha$ of $L \hat{\otimes} \Omega_n$ is determined uniquely by the pair $(\mu, \nu) \in \text{MC}(L) \times \text{Stub}^i_n(L)$, where

$$\mu = \varepsilon_n^i (\alpha), \quad \text{and} \quad \nu = (\partial + d) \circ h_n^i (\alpha).$$

Indeed, if $\beta$ is another MC element of $L \hat{\otimes} \Omega_n$ such that

$$\varepsilon_n^i \alpha = \varepsilon_n^i \beta, \quad \text{and} \quad (\partial + d) \circ h_n^i \alpha = (\partial + d) \circ h_n^i \beta$$

then, using (A.3), we see that

$$\alpha - \beta = - \sum_{m \geq 2} \frac{1}{m!} h_n^i \{ \alpha, \ldots, \alpha \}_m - \{ \beta, \ldots, \beta \}_m$$

which means that

$$\alpha - \beta \in \mathcal{F}_q L \hat{\otimes} \Omega_n$$

for all $q \geq 1$ and hence $\alpha = \beta$.

Combining this observation with the above consideration of the limiting element of sequence (A.7), we get the following version of [10, Lemma 4.6]:

**Lemma A.1** Let $n \geq 1$ and $0 \leq i \leq n$. For every filtered $\mathfrak{S}\text{Lie}_\infty$-algebra $L$ the assignment

$$\alpha \mapsto (\varepsilon_n^i \alpha, (\partial + d) \circ h_n^i (\alpha)) \quad (A.12)$$

is a bijection of sets

$$\text{MC}(L \hat{\otimes} \Omega_n) \cong \text{MC}(L) \times \text{Stub}^i_n(L).$$

The inverse of (A.12) assigns to a pair $(\mu, \nu) \in \text{MC}(L) \times \text{Stub}^i_n(L)$ the limiting element of sequence (A.7).

*In the definition of the corresponding subspace $\text{Stub}^i_n(L)$ (A.5) in paper [10], the condition $\partial(\xi)|_{t_i=1} = 0$ is probably omitted by mistake. We are sure that this condition is unavoidable.*
B  Every 1-cell can be “rectified”

Let $L$ be a filtered $\mathcal{S}\text{Lie}_\infty$-algebra and $\mathcal{MC}_\bullet(L)$ be the corresponding DGH $\infty$-groupoid. In some cases, it is convenient to deal with 1-cells in $\mathcal{MC}_\bullet(L)$ of the form

$$\beta = \beta_0(t_0) + dt_0 \beta_1,$$  \hspace{1cm} (B.1)

where $\beta_1$ is a vector in $L^{-1} \subseteq L^{-1} \otimes \mathbb{k}[t_0]$, i.e. the component $\beta_1$ does not “depend” on $t_0$. We call such 1-cells rectified$^{[10]}$.

Remark B.1 It is not hard to see that under the bijection established in Lemma A.1 rectified 1-cells (B.1) in $\mathcal{MC}_\bullet(L)$ correspond to pairs $(\mu, \nu) \in \text{MC}(L) \times \text{Stub}_1^\text{L}$ where

$$\mu = \beta_0(0) \quad \text{and} \quad \nu = (\partial + d)(t_0 \beta_1).$$

In this appendix we prove the following lemma:

Lemma B.2 Let $L$ be a filtered $\mathcal{S}\text{Lie}_\infty$-algebra $\alpha = \alpha_0(t) + dt \alpha_1(t)$ be a 1-cell of $\mathcal{MC}_\bullet(L)$ such that $\alpha_1(t) \in \mathcal{F}_k L^{-1} \otimes \mathbb{k}[t]$ for some $k \geq 1$. Then there exists 1-cell of $\mathcal{MC}_\bullet(L)$

$$\beta = \beta_0(t) + dt \beta_1$$  \hspace{1cm} (B.2)

with $\beta_1 \in \mathcal{F}_k L^{-1}$ such that

$$\beta_0(0) = \alpha_0(0) \quad \text{and} \quad \beta_0(1) = \alpha_0(1).$$  \hspace{1cm} (B.3)

Proof. We will prove by induction that there exists a sequence of MC elements $(m \geq k)$

$$\gamma^m \in \text{MC}(L \otimes \Omega_2)$$  \hspace{1cm} (B.4)

such that

$$\gamma^m|_{t_0=0} = \alpha_0(t_2) + dt_2 \alpha_1(t_2),$$  \hspace{1cm} (B.5)

the 1-cell

$$\beta^m = \beta_0^m(t_0) + dt_0 \beta_1^m := \gamma^m|_{t_2=0}$$  \hspace{1cm} (B.6)

is rectified and$^{[1]}$

$$\beta_1^m \in \mathcal{F}_k L, \quad \forall \ m \geq k.$$  \hspace{1cm} (B.7)

Finally the 1-cell

$$\sigma_0^m(t_2) + dt_2 \sigma_1^m(t_2) := \gamma^m|_{t_1=0}$$  \hspace{1cm} (B.8)

satisfies the condition

$$\sigma_1^m(t_2) \in \mathcal{F}_m L \otimes \mathbb{k}[t_2]$$  \hspace{1cm} (B.9)

and, for every $m \geq k$, we have

$$\gamma^{m+1} - \gamma^m \in \mathcal{F}_m L \otimes \Omega_2.$$  \hspace{1cm} (B.10)
The links between the 2-cell $\gamma^m$ and 1-cells $\alpha_0(t_2) + dt_2 \alpha_1(t_2), \beta_0^m(t_0) + dt_0 \beta_1^m, \sigma_0^m(t_2) + dt_2 \sigma_1^m(t_2)$ are presented graphically on figure 2.

Let us denote by $\rho(0)$ the following degree 0 element in $L \hat{\otimes} \Omega_2$:

$$\rho(0) := \alpha_0(0) + (\partial + d) \circ h_2^1(\gamma^m).$$

According to Lemma A.1, $\gamma^m$ is the limiting element of the convergent sequence $\{\rho^{(p)}\}_{p \geq 0}$ defined inductively by

$$\rho^{(p+1)} := \rho^{(p)} - \sum_{l=2}^{\infty} \frac{1}{l!} h_2^1 \{\rho^{(p)}, \ldots, \rho^{(p)}\}_l.$$

Let $\{\tilde{\rho}^{(p)}\}_{p \geq 0}$ be the following sequence of degree 0 elements in $L \hat{\otimes} \Omega_2$:

$$\tilde{\rho}(0) := \alpha_0(0) + (\partial + d) \left( \Delta + h_2^1(\gamma^m) \right),$$

$$\tilde{\rho}^{(p+1)} := \tilde{\rho}^{(p)} - \sum_{l=2}^{\infty} \frac{1}{l!} h_2^1 \{\tilde{\rho}^{(p)}, \ldots, \tilde{\rho}^{(p)}\}_l,$$

where $\Delta$ is a degree $-1$ element in $\mathcal{F}_m L^{-1} \otimes \mathbb{k}[t_0, t_2]$ of the form

$$\Delta := \sum_{s=0}^{N} \Delta_s t_0 t_2^s.$$

10Note that rectified 1-cells are precisely 1-cells in the Kan complex $\gamma_\bullet(L)$ introduced in [10, Section 5].

11For $m = k$ we will actually have $\beta_1^k \equiv 0$. 

Figure 2: Links between different cells
Let us denote by $\gamma^{m+1}$ the limiting element of the sequence $\{\tilde{\rho}^{(p)}\}_{p \geq 0}$ and show that, for an appropriate choice of coefficients $\Delta_s$ in (B.16) the element $\gamma^{m+1}$ satisfies all the desired properties.

Since $(\partial + d)\Delta$ disappears on the 0-th face,
\[
\gamma^{m+1}\big|_{t_0=0} = \gamma^m \big|_{t_0=0} = \alpha_0(t_2) + dt_2 \alpha_1(t_2).
\]
We also have
\[
(\partial + d)\Delta \big|_{t_2=0} = (\partial + d)(\Delta_0 t_0).
\]
Hence, due to Remark [B.1], the restriction of $\gamma^{m+1}$ to the second face is a rectified 1-cell
\[
\beta^{m+1}_0(t_0) + dt_0 \beta^{m+1}_1(t_0).
\]
Inclusion $\beta^{m+1}_1 \in F_kL$ holds because $\Delta_0 \in F_mL$ with $m \geq k$.

Since $\Delta \in F_mL \otimes k[t_0, t_2]$, condition (B.10) is obviously satisfied. Moreover,
\[
\gamma^{m+1} - \gamma^m - (\partial + d) \Delta \in F_{m+1}L \otimes \Omega_2.
\]
Hence for
\[
\sigma^{m+1} = \sigma^{m+1}_0(t_2) + dt_2 \sigma^{m+1}_1(t_2) := \gamma^{m+1}\big|_{t_1=0}
\]
we have\(^{12}\)
\[
\sigma^{m+1}_1(t_2) - \sigma^m_1(t_2) - \frac{\partial}{\partial t_2} \sum_{s=0}^N \Delta_s (1 - t_2) t^s_2 \in F_{m+1}L \otimes k[t_2].
\]
Since $\sigma^m_1(t_2) \in F_mL \otimes k[t_2]$, we have
\[
\sigma^m_1(t_2) = \sum_{s=0}^N \sigma^{m,s}_1 t^s_2 + \ldots,
\]
where each $\sigma^{m,s}_1 \in F_mL$ and $\ldots$ denotes terms in $F_{m+1}L \otimes k[t_2]$.

Therefore, using inclusion (B.18) and obvious identity
\[
\frac{\partial}{\partial t_2} \sum_{s=0}^N \Delta_s (1 - t_2) t^s_2 = \sum_{s=0}^N (s + 1)(\Delta_{s+1} - \Delta_s) t^s_2
\]
with $\Delta_{N+1} = 0$, we conclude that
\[
\sigma^{m+1}_1(t_2) \in F_{m+1}L \otimes k[t_2]
\]
provided the coefficients $\{\Delta_s\}_{0 \leq s \leq N}$ satisfy the linear equations:
\[
\Delta_s - \Delta_{s+1} = \frac{\sigma^{m,s}}{s+1}, \quad 0 \leq s \leq N, \quad \Delta_{N+1} = 0.
\]
\(^{12}\)Observe that when $t_1 = 0$, $t_0 = 1 - t_2$. 

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Since linear system (B.21) has the obvious solution
\[ \Delta_N = \frac{\sigma_{1,N}^m}{N+1}, \quad \text{and} \quad \Delta_s = \frac{\sigma_{1,s}^m}{s+1} + \Delta_{s+1}, \quad \forall \ 0 \leq s \leq N-1 \] (B.22)
we conclude that the desired MC element $\gamma_{m+1}^m$ can be constructed.

Let us now show that the existence of sequence (B.4) satisfying all the above conditions implies the lemma.

Indeed, due to inclusions (B.9) and (B.10) the sequence \( \{\gamma_m\}_{m \geq k} \) converges to a MC element
\[ \gamma \in L \otimes \Omega \]
for which the 1-cell
\[ \sigma_0(t_2) + dt_2 \sigma_1(t_2) := \gamma \big|_{t_1=0} \]
has the property $\sigma_1(t_2) \equiv 0$.

Therefore, we have
\[ \alpha_0(1) = \gamma \big|_{t_2=1} = \gamma \big|_{t_0=1} = \beta_0(1). \]

Thus, since $\beta_0(0) = \alpha_0(0)$ by construction, Lemma B.2 follows. \( \square \)

Remark B.3 Lemma B.2 can be deduced from [10, Corollary 5.11] using twisting by the MC element $\alpha_0$ and some basic facts proved in [11, Section VI]. However, we decided to give a proof which bypasses the introduction of an additional Kan complex $\gamma_\bullet(L)$ from [10, Section 5].

C Concatenating infinitely many 1-cells

Let $L$ be a filtered $\mathcal{S}\text{Lie}_\infty$-algebra and
\[ \{ \rho^{(n)} = \rho_{0}^{(n)}(t_0) + dt_0 \rho_{1}^{(n)}(t_0) \}_{n \geq 1} \] (C.1)
be a sequence of 1-cells in $MC_\bullet(L)$ satisfying the following conditions:
\[ \rho_{1}^{(n)}(t_0) \in F_n L \otimes \mathbb{k}[t_0], \] (C.2)
and
\[ \rho_{0}^{(n)}(1) = \rho_{0}^{(n+1)}(0), \quad \forall \ n \geq 1. \] (C.3)

The following lemma implies that there exists a 1-cell in $MC_\bullet(L)$ which may be viewed as the result of concatenating the sequence of 1-cells (C.1).

Lemma C.1 Under the above conditions on 1-cells (C.1), one can construct a sequence of 1-cells
\[ \{ \gamma^{(n)} = \gamma_{0}^{(n)}(t_0) + dt_0 \gamma_{1}^{(n)}(t_0) \}_{n \geq 1} \] (C.4)
in $MC_\bullet(L)$ such that
\[ \gamma_{0}^{(n)}(0) = \rho_{0}^{(1)}(0), \quad \gamma_{0}^{(n)}(1) = \rho_{0}^{(n)}(1), \] (C.5)
and
\[ \gamma^{(n+1)} - \gamma^{(n)} \in F_{n+1} L \otimes \Omega. \] (C.6)
Proof. Setting $\gamma^{(1)} := \rho^{(1)}$

we get the base of the induction on $n$. So let us assume that we already constructed the desired 1-cells

$\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)}$.

Since

$\gamma_0^{(n)}(1) = \rho_0^{n+1}(0),$

the 1-cells $\gamma^{(n)}$ and $\rho^{(n+1)}$ give us the following horn:

$$
\begin{align*}
1 & \mapsto \rho_0^{(n+1)}(0) \\
2 & \mapsto \rho_1^{(1)}(0) \\
0 & \mapsto \rho_0^{(n+1)}(1)
\end{align*}
$$

(C.7)

Let us prove that there exists a MC element

$$\eta \in L \hat{\otimes} \Omega_2$$

such that

$$\eta \big|_{t_2=0} = \rho_0^{(n+1)}(t_0) + dt_0 \rho_1^{(n+1)}(t_0),$$

(C.9)

$$\eta \big|_{t_0=0} = \gamma_0^{(n)}(t_1) + dt_1 \gamma_1^{(n)}(t_1),$$

(C.10)

and the 1-cell

$$\gamma_0^{(n+1)}(t_0) + dt_0 \gamma_1^{(n+1)}(t_0) := \eta \big|_{t_1=0}$$

satisfies the property

$$\gamma^{(n+1)} - \gamma^{(n)} \in F_{n+1} L \hat{\otimes} \Omega_1.$$  

(C.12)

To construct $\eta$, we denote by $\gamma$ and $\rho$ the degree $-1$ elements

$$\gamma(t_0) := h_1^0(\gamma(n)) \in L \hat{\otimes} k[t_0]$$

(C.13)

and

$$\rho(t_0) := h_1^1(\rho^{(n+1)}) \in L \hat{\otimes} k[t_0]$$

(C.14)

respectively.

In other words, $(\partial + d)\gamma(t_0)$ (resp. $(\partial + d)\rho(t_0)$) is the “stub” of the 1-cell $\gamma^{(n)}$ (resp. $\rho^{(n+1)}$) in the sense of Lemma [A.1].

Next, we consider the following degree 0 element of $L \hat{\otimes} \Omega_2$

$$\nu := (\partial + d) \left( \gamma(1 - t_2) + \rho(t_0) \right).$$

(C.15)

The element $\nu$ belongs to $\text{Stub}^1_2(L)$ since, at vertex $1$, $1 - t_2 = 1$ and $t_0 = 0$.  

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Therefore, according to Lemma A.1, the sequence of degree zero elements
\[ \{\eta^{(k)}\}_{k\geq 0} \]
defined by
\[ \eta^{(0)} := \rho_0^{(n+1)}(0) + \nu \]
and
\[ \eta^{(k+1)} := \eta^{(0)} - \sum_{m=2}^{\infty} \frac{1}{m!} \{\eta^{(k)}, \ldots, \eta^{(k)}\}_m \]
converges to a MC element \( \eta \in L \hat{\otimes} \Omega_2 \).

Let us prove that the limiting element \( \eta \) satisfies all the desired properties.

Indeed, (C.9) and (C.10) follow from Lemma A.1 and the obvious equations
\[ \nu \big|_{t_2=0} = (\partial + d)\left(\rho(t_0)\right), \quad \text{and} \quad \nu \big|_{t_0=0} = (\partial + d)\left(\gamma(1 - t^2)\right) = (\partial + d)\left(\gamma(t_1)\right). \]

Inclusion (C.2) implies that
\[ \nu - (\partial + d)\left(\gamma(1 - t_2)\right) \in \mathcal{F}_{n+1}L \hat{\otimes} \Omega_2. \]

On the other hand a direction computation shows that
\[ \gamma_0(n)(1 - t_2) - dt_2\gamma_1(n)(1 - t_2) = \gamma(1 - t_2), \]
where \( \gamma_0(n)(1 - t_2) - dt_2\gamma_1(n)(1 - t_2) \) is viewed as an element of \( L \hat{\otimes} \Omega_2 \) in the obvious way.

Hence, applying Lemma A.1 to MC elements of \( (L/\mathcal{F}_{n+1}L) \otimes \Omega_2 \), and using (C.18) we conclude that
\[ \eta - \gamma_0(n)(1 - t_2) + dt_2\gamma_1(n)(1 - t_2) \in \mathcal{F}_{n+1}L \hat{\otimes} \Omega_2. \]

Thus 1-cell (C.11) indeed satisfies (C.12).

Lemma C.1 is proved. \( \square \)

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