On Arbitrary Dimension Hermitian Hull linear codes from Hermitian self-orthogonal codes and New Hermitian Self-Orthogonal GRS Codes

Yang Li\textsuperscript{a,}\* \\
\textsuperscript{a}School of Mathematics, HeFei University of Technology, Hefei 230009, China

Abstract

Due to the important role of hulls of linear codes in coding theory, the problem about constructing arbitrary dimension hull linear codes has become a hot issue. In this paper, we generalize conclusions in \cite{42} and \cite{43} and prove that the Hermitian self-orthogonal codes of length $n$ can construct linear codes of length $n + 2i$ and $n + 2i + 1$ with arbitrary-dimensional Hermitian hulls, where $i \geq 0$ is an integer. Then four new classes of Hermitian self-orthogonal GRS or extend GRS codes are constructed via two known multiplicative coset decompositions of $F_{q^2}$. The codes we constructed can be used to obtain new arbitrary dimension Galois hull linear codes by Theorems 11 and 12 in \cite{22} and finally we get many new EAQECCs whose code lengths can take $n + 2i$ and $n + 2i + 1$.

Keywords: extend Generalized Reed-Solomon codes, Hermitian self-orthogonal, coset decomposition, Galois hull, EAQECCs

2010 MSC: 94B05, 81p70

1. Introduction

In 1996, Calderbank et al. \cite{1} and Steane \cite{2} independently proposed a systematic method for constructing quantum error correction codes (for short...
QECCs), commonly referred to as CSS construction. About ten years later, Ketkar et al. [3] and Aly et al. [4] made an effective generalization of this, and their work allowed us to construct quantum stabilizer codes from any given Hermitian self-orthogonal codes. In the same year, Burn et al. [5] introduced the concept of maximally entangled state $c$ and proposed the new entanglement-assisted quantum error correction code (for short EAQECCs), which allows people to abandon the traditional self-orthogonal condition and construct quantum codes through any linear codes. After this, people worked to study how to determine $c$, and found the relationship between $c$ and $\text{dim}(\text{Hull}(C))$, where the hull denoted by $\text{Hull}(C) = C \cap C^\perp$ is firstly given by Assume [6] and can be discussed on different inner products, such as the Euclidean inner product [7], Hermitian inner product [7] and Galois inner product [8].

Besides being used to construct EAQECCs, the Hull of linear codes also plays an important role in determining the algorithm complexity [9, 10], calculating the automorphism group of linear codes [9, 11], and determining the equivalence of linear codes [12, 13, 14]. And in fact, if we describe self-orthogonal codes in Hull’s language, then a linear code $C$ with $k$ dimension is a self-orthogonal code if and only if $\text{dim}(\text{Hull}(C)) = k$.

Based on these effects, researches on the Hull of linear codes have become a hot issue in recent years. Generally speaking, when people construct EAQECCs, they usually firstly construct linear codes (actually MDS codes) with arbitrary-dimensional hulls including the Euclidean, Hermitian and Galois cases, such as [15, 16, 17, 18, 19, 20, 21, 22, 47]. In particular, in [21] and [22], Fang et al. and Li et al. constructed MDS codes with arbitrary-dimensional $e$-Galois hull by Euclidean self-orthogonal (extend) GRS codes and Hermitian self-orthogonal (extend) GRS codes for some $e$, respectively. In [49], Sok proved that 1-dimensional Euclidean hull codes of length $n$ and $n+1$ can be constructed by Euclidean self-orthogonal codes of length $n$, and further explained that they can also take arbitrary-dimensional Euclidean hulls in fact. And in [42] and [43], Chen and Sok proved that for a linear Hermitian self-orthogonal $[n, k]_{q^2}$ code $C$, there exists $h$-dimension Hermitian hull linear codes where $0 \leq h \leq k$, ...
independently.

Therefore, considering the important applications of hull and quantum codes, it is of great significance to study Hermitian self-orthogonal codes, especially Hermitian self-orthogonal (extend) GRS codes for the extra application in Galois hulls, which have been studied in [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 46]. For other researches on the construction of Hermitian self-orthogonal codes from cyclic codes, constacyclic codes, negacyclic codes, graph theory and so on, we refer to [37, 38, 39, 40, 41].

In this paper, we firstly prove that Hermitian self-orthogonal codes of length $n$ can obtain codes of length $n + 2i$ and $n + 2i + 1$ with arbitrary-dimensional Hermitian hulls, where $i \geq 0$ is an integer. Then four new classes of Hermitian self-orthogonal (extend) GRS codes are constructed via two known multiplicative coset decompositions of $F_{q^2}$. For reference, we list some known Hermitian self-orthogonal (extend) GRS codes and the new ones we construct in Table 1.

The rest of this paper is organized as follows. In Section 2, we recall the basic knowledge of (extend) generalized Reed-Solomon codes. In Section 3, we prove that Hermitian self-orthogonal codes of length $n$ can obtain codes of length $n + 2i$ and $n + 2i + 1$ with arbitrary-dimensional Hermitian hulls for an integer $i \geq 0$ and present a new construction of EAQECCs. In Section 4, we construct four new Hermitian self-orthogonal (extend) GRS codes and list some examples calculated by Magma. In Section 5, We use the conclusions in Section 2 and Section 3 to construct many new EAQECCs. And finally, Section 6 concludes this paper.

2. Preliminary

Let $q = p^m$ be a prime power and $F_{q^2}$ be the finite field with $q^2$ elements. Then all nonzero elements in $F_{q^2}$ can form a multiplicative group of order $q^2 - 1$, denoted by $F_{q^2}^*$. And if we assume $\omega$ as a primitive element of $F_{q^2}$, then $F_{q^2}^* = \langle \omega \rangle$. A linear code of length $n$ over $F_{q^2}$ is called an $[n, k, d]_{q^2}$ code with dimension $k$ and minimum distance $d$, which can be seen as an $F_{q^2}$-subspace of $F_{q^2}^n$. For
| Code   | Length | Dimension | Type | Reference |
|--------|--------|-----------|------|-----------|
|        |        |           |      |           |

Table 1: Some known Hermitian self-orthogonal GRS and extend GRS codes
any $\alpha \in F_{q^2}^*$, we define the conjugation of $\alpha$ by $\overline{\alpha} = \alpha^q$.

Let any two vectors $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in F_{q^2}^n$, then we can define the Hermitian inner products of $x, y$ as

$$(x, y)_H = \sum_{i=1}^{n} x_i \overline{y}_i. \quad (1)$$

Then the Hermitian dual code of $C$ can be defined as

$$C^\perp_H = \{ x | x \in F_{q^2}^n, (x, y)_H = 0, \text{ for all } y \in C \}. \quad (2)$$

If $C \subseteq C^\perp_H$, then $C$ is called a Hermitian self-orthogonal code.

Now, we recall the definitions of GRS codes and extend GRS codes which are well-known MDS codes. For a positive integer $k$, let

$$F_{q^2}[x]_k = \{ f(x) \in F_{q^2}[x] | \text{deg}(f(x)) \leq k - 1 \} \quad (3)$$

be a set of polynomials with degree less than $k - 1$ over $F_{q^2}$. Let $a = (a_1, a_2, \ldots, a_n) \in F_{q^2}^n$ with $a_1, a_2, \ldots, a_n$ distinct elements, $v = (v_1, v_2, \ldots, v_n) \in F_{q^2}^*$. Then the GRS code associated to $a$ and $v$ can be denoted as

$$GRS_k(a, v) = \{(v_1 f(a_1), v_2 f(a_2), \ldots, v_n f(a_n)) | f(x) \in F_{q^2}[x]_k \}. \quad (4)$$

It is well known that both $GRS_k(a, v)$ and its dual are MDS codes. And if we add an extra coordinate to the GRS code above, we can get the extend GRS code associated to $a$ and $v$, which is defined as

$$GRS_k(a, v, \infty) = \{v_1 f(a_1), v_2 f(a_2), \ldots, v_n f(a_n), f_{k-1} | f(x) \in F_{q^2}[x]_k \}, \quad (5)$$

where $f_{k-1}$ is the coefficient of $x^{k-1}$ in $f(x)$. It is also well known that $GRS_k(a, v, \infty)$ is an $[n + 1, k, n - k + 2]_{q^2}$ MDS code and its dual is also an MDS code.

As in other studies, for $1 \leq i \leq n$, we also denote

$$u_i = \prod_{1 \leq j \leq n, i \neq j} (a_i - a_j)^{-1}. \quad (6)$$

Based on the above concepts and notations, we can use the following two lemmas to construct Hermitian self-orthogonal GRS and extend GRS codes.
Lemma 1. ([14], Corollary 1) A codeword $c = (v_1 f(a_1), \ldots, v_n f(a_n))$ of $GRS_k(a, v)$ is contained in $GRS_k(a, v)^{\perp_H}$ if and only if there exists a monic polynomial $h(x) \in F_{q^2}[x]$ with $\deg(h(x)f^q(x)) \leq n - k - 1$ such that
\[
\lambda u_i h(a_i) = v_i^{q+1}, \quad 1 \leq i \leq n,
\] where $\lambda \in F_{q^2}^\times$.

Lemma 2. ([22], Corollary 12) A codeword $c = (v_1 f(a_1), \ldots, v_n f(a_n), f_{k-1})$ of $GRS_k(a, v, \infty)$ is contained in $GRS_k(a, v, \infty)^{\perp_H}$ if and only if there exists a monic polynomial $h(x) \in F_{q^2}[x]$ with $\deg(h(x)f^q(x)) \leq n - k$, such that
\[
-u_i h(a_i) = v_i^{q+1}, \quad 1 \leq i \leq n.
\]

3. Arbitrary-dimensional Hermitian hull linear codes from Hermitian self-orthogonal codes

In this section, we generalize conclusions in [42] and [43], which has been briefly described in Section 1, and prove that arbitrary dimension Hermitian hull linear codes of length $n + 2i$ and $n + 2i + 1$ can be obtained from Hermitian self-orthogonal codes of length $n$, where $i \geq 0$ is an integer.

Before we start, we give a useful notation and lemma that can helps us calculate the dimension of hull of a linear code. If $M = (m_{ij})$, then we denote the conjugate transposition of $M$ as $M^\dagger = (m_{ji}^q)$.

Lemma 3. ([48], Proposition 3.3) Let $C$ be an $[n, k, d]_{q^2}$ linear code. Assume that $H$ is a parity check matrix and $G$ is a generator matrix of $C$. Then
\[
\text{rank}(HH^\dagger) = n - k - \dim(Hull_H(C)) = n - k - \dim(Hull_H(C^{\perp_H}))
\]
and
\[
\text{rank}(GG^\dagger) = k - \dim(Hull_H(C)) = k - \dim(Hull_H(C^{\perp_H})).
\]

Lemma 4. If $C$ is an $[n, k, d]_{q^2}$ Hermitian self-orthogonal code, then there exists a 1-dim Hermitian hull $[n+1, k, \geq d]_{q^2}$ code $C_1$. 

6
Proof. Assume that $C$ is an $[n, k, d]_{q^2}$ Hermitian self-orthogonal code with the generator matrix $G$. Then in the sense of equivalence, we can set $G = [I_k|A]$. Hence $C^\perp_H$ has $G^\dagger = \begin{bmatrix} I_k \\ A^\dagger \end{bmatrix}$ as the generator matrix. So we have

$$GG^\dagger = I_k + AA^\dagger = 0 \implies AA^\dagger = -I_k.$$ 

Consider $\alpha, \beta, \lambda \in F_{q^2}^*$ and $\beta^{q+1}, \lambda^{q+1} \neq 1$. Then

$$(1 - \beta^{q+1})^2 = 1 - \beta^{q^2-1+q+1} = 1 - \beta^{q+1} \implies 1 - \beta^{q+1} \in F_{q^2}^*.$$ 

So there exists $\alpha \in F_{q^2}^*$, such that $1 - \beta^{q+1} = \alpha^{q+1}$, i.e. $\alpha^{q+1} + \beta^{q+1} = 1$.

Now, let $G_1 = [G'|A|a]$ be the generator matrix of $C_1$, where $G' = \text{diag}(\lambda, \lambda, \ldots, \lambda, \beta)$, $a = (0, 0, \ldots, 0, \alpha)^T$ and both of $\lambda$ and $0$ appear $k-1$ times. Then

$$G_1G_1^\dagger = [G'|A|a] \begin{bmatrix} G^\dagger \\ A^\dagger \\ a^\dagger \end{bmatrix} = G'G'^\dagger + AA^\dagger + aa^\dagger$$

$$= \text{diag}(\lambda^{q+1}, \lambda^{q+1}, \ldots, \lambda^{q+1}, \beta^{q+1}) - I_k + \text{diag}(0, 0, \ldots, 0, \alpha^{q+1})$$

$$= \text{diag}(\lambda^{q+1} - 1, \lambda^{q+1} - 1, \ldots, \lambda^{q+1} - 1, 0),$$

which means $\text{rank}(G_1G_1^\dagger) = k-1$ and $\text{dim}(\text{Hull}_H(C_1)) = k - \text{rank}(G_1G_1^\dagger) = 1$ from Lemma 4. This completes the proof.

Lemma 5. If $C$ is an $[n, k, d]_{q^2}$ Hermitian self-orthogonal code, then there exists a 1-dim Hermitian hull $[n+2i, k, \geq d]_{q^2}$ code $C_{2i}$ and 1-dim Hermitian hull $[n+2i+1, k, \geq d]_{q^2}$ code $C_{2i+1}$, where $i \geq 1$ is an integer.

Proof. Keep notations as Lemma 4. We need to discuss the parity of $q$ for both cases $n+2i$ and $n+2i+1$.

- The case $n+2i$:

  If $q$ is even, then $q+1$ is odd. When we choose $\alpha_1, \alpha_2, \ldots, \alpha_{2i}, \beta, \lambda \in F_{q^2}^*$ with $\lambda^{q+1} \neq 1$ and $\alpha_1^{q+1} + \alpha_2^{q+1} + \ldots, \alpha_{2i}^{q+1} + \beta^{q+1} = 1$. In fact, we can take $\alpha_1 = -\alpha_2, \alpha_3 = -\alpha_4, \ldots, \alpha_{2i-1} = -\alpha_{2i}$ with $\alpha_1, \alpha_3, \ldots, \alpha_{2i-1} \in F_{q^2}^*$ and $\beta = 1$. 

- The case $n+2i+1$:

  If $q$ is odd, then $q+1$ is even. When we choose $\alpha_1, \alpha_2, \ldots, \alpha_{2i+1}$, $\beta, \lambda \in F_{q^2}^*$ with $\lambda^{q+1} \neq 1$ and $\alpha_1^{q+1} + \alpha_2^{q+1} + \ldots, \alpha_{2i+1}^{q+1} + \beta^{q+1} = 1$. In fact, we can take $\alpha_1 = -\alpha_2, \alpha_3 = -\alpha_4, \ldots, \alpha_{2i} = -\alpha_{2i+1}$ with $\alpha_1, \alpha_3, \ldots, \alpha_{2i} \in F_{q^2}^*$ and $\beta = 1$. 

7
then \(\alpha q^{+1} + \alpha q^{+1} + \cdots + \alpha q^{+1} + \beta q^{+1} = 1 - 1 + \cdots + 1 + 1 + 1 = 1\) satisfying the condition.

If \(q\) is odd, then \(\frac{q^{-1}}{2}\) and \(\frac{q^2-1}{2}\) are integers. Since \(\omega\) is a primitive element of \(F_{q^2}\), we have \(\omega \frac{q^2-1}{2} = -1\). Similarly, when we choose \(\alpha_1, \alpha_2, \ldots, \alpha_{2i}, \beta, \lambda \in F_{q^2}^*\) with \(\lambda^{q+1} \neq 1\) and \(\alpha_1 q^{+1} + \alpha_2 q^{+1} + \cdots \alpha_{2i} q^{+1} + \beta q^{+1} = 1\). Hence, for example, we can take \(\alpha_1 = \alpha_2 = \cdots = \alpha_i = \omega \frac{q^2-1}{2} \neq 0\) and \(\alpha_{i+1} = \alpha_{i+2} = \cdots = \alpha_{2i} = \beta = 1\), then \(\alpha_1 q^{+1} + \alpha_2 q^{+1} + \cdots + \alpha_i q^{+1} + \alpha_{i+1} q^{+1} + \alpha_{i+2} q^{+1} + \cdots + \alpha_{2i} q^{+1} + \beta q^{+1} = 1 - 1 - \cdots - 1 + 1 + \cdots + 1 + 1 = 1\) satisfying the condition as well.

Now we suppose that \(G_{2i} = [G' | A| a_1 | a_2]| \cdots | a_{2i}]\) is the generator matrix of \(C_{2i}\), where \(G' = diag(\lambda, \lambda, \ldots, \lambda, \beta)\), \(a_1 = (0, 0, \ldots, 0, \alpha_1)^T\), \(a_2 = (0, 0, \ldots, 0, \alpha_2)^T\), \ldots, \(a_{2i} = (0, 0, \ldots, 0, \alpha_{2i})^T\) and both of \(\lambda\) and 0 appear \(k-1\) times. Then taking a calculation similar to Lemma 4 we know that \(C_{2i}\) is a 1-dim Hermitian hull linear code.

- The case \(n + 2i + 1\):

If \(q\) is even, when we choose \(\alpha_1, \alpha_2, \ldots, \alpha_{2i+1}, \beta, \lambda \in F_{q^2}^*\) with \(\lambda^{q+1} \neq 1\) and \(\alpha_1 q^{+1} + \alpha_2 q^{+1} + \cdots \alpha_{2i} q^{+1} + \beta q^{+1} = 1\). In fact, we only need to take \(\alpha_1 = -\alpha_2, \alpha_3 = -\alpha_4, \ldots, \alpha_{2i-1} = -\alpha_{2i}\) with \(\alpha_1, \alpha_3, \ldots, \alpha_{2i-1} \in F_{q^2}^*\) and \(\alpha_{2i+1} q^{+1} + \beta q^{+1} = 1\), where \(\alpha q^{+1} \neq 1\). Then by Lemma 4 we can see that the condition is satisfied.

If \(q\) is odd, when we choose \(\alpha_1, \alpha_2, \ldots, \alpha_{2i}, \beta, \lambda \in F_{q^2}^*\) with \(\lambda^{q+1} \neq 1\) and \(\alpha_1 q^{+1} + \alpha_2 q^{+1} + \cdots \alpha_{2i} q^{+1} + \beta q^{+1} = 1\). Hence, for example, we can take \(\alpha_1 = \alpha_2 = \cdots = \alpha_i = \omega \frac{q^2-1}{2} \neq 0\) and \(\alpha_{i+1} = \alpha_{i+2} = 1\) and \(\alpha_{2i+1} q^{+1} + \beta q^{+1} = 1\). Then by Lemma 4 again, we know that the condition is satisfied as well.

Now we suppose that \(G_{2i+1} = [G' | A| a_1 | a_2]| \cdots | a_{2i+1}]\) is the generator matrix of \(C_{2i+1}\), where \(G' = diag(\lambda, \lambda, \ldots, \lambda, \beta)\), \(a_1 = (0, 0, \ldots, 0, \alpha_1)^T\), \(a_2 = (0, 0, \ldots, 0, \alpha_2)^T\), \ldots, \(a_{2i+1} = (0, 0, \ldots, 0, \alpha_{2i+1})^T\) and both of \(\lambda\) and 0 appear \(k-1\) times. Then taking a calculation similar to Lemma 4 we know that \(C_{2i+1}\) is a 1-dim Hermitian hull linear code.

**Corollary 6.** If \(C\) is an \([n, k, d]_{q^2}\) Hermitian self-orthogonal code, then there exist a \(l\)-dim Hermitian hull \([n + 2i, k, \geq d]_{q^2}\) code \(C_{2i}\) and a \(l\)-dim Hermitian hull \([n + 2i + 1, k, \geq d]_{q^2}\) code \(C_{2i+1}\) for \(0 \leq l \leq k\), where \(i \geq 0\) is an integer.
Proof. Considering the different occurrences of $\lambda$ and $\alpha_1, \alpha_2, \ldots, \alpha_{2i+1}$ in the proofs of Lemma 5 and combining Lemma 4 and conclusions in [42] and [43], the corollary can be obtained immediately by taking the same method. □

Remark 1. (1) From Corollary 6 we can know that the results in [42] and [43] are indeed a special case, so our work gives a more general result.

(2) By Corollary 6, we can construct arbitrary dimension $e$-Galois hull linear codes like Theorems 11 and 12 in [22]. And according to a well known result in [44], we can construct EAQECCs with parameters $[[n, k - l, d; n - k - l]]_q$ and $[[n, n - k - l, d^\perp; k - l]]_q$ from any $[n, k, d]_q^2$ code $C$, where $l$ is the hull of dimension of the linear code $C$ and $d^\perp$ is the dual distance of $C$. Then by Corollary 6 we can obtain many new EAQECCs for all the code lengths $n + 2i$ and $n + 2i + 1$ from the Hermitian self-orthogonal code with length $n$. We write this useful conclusion in the form of a theorem.

Theorem 7. Let $q$ be a prime power, if there is a $[n, k, d]_q^2$ Hermitian self-orthogonal code, then we can obtain EAQECCs with parameters $[[n + 2i, k - l, d; n + 2i - k - l]]_q$, $[[n + 2i, n + 2i - k - l, d^\perp; k - l]]_q$, $[[n + 2i + 1, k - l, d; n + 2i + 1 - k - l]]_q$ and $[[n + 2i + 1, n + 2i + 1 - k - l, d^\perp; k - l]]_q$, where $0 \leq l \leq k$ and $i \geq 0$ is an integer.

4. Constructions of new Hermitian self-orthogonal codes

In this section, we present four classes of new constructions of Hermitian self-orthogonal (extend) GRS codes. They are based on two known multiplicative coset decompositions of $F_{q^2}$.

4.1. $n = m(q - 1), 1 \leq m \leq q$

Now we consider the first multiplicative coset decomposition of $F_{q^2}$ in [40]. Assume $n = m(q - 1), 1 \leq m \leq q$. Set

$$I_t = \{x \cdot \omega | x \in F_q^*\}, \ 1 \leq t \leq q \quad (8)$$
In other words, we have

\[ I_t = \{ \omega^{t-1}, \omega^{q+1+t-1}, \ldots, \omega^{(q+1)i+t-1}, \ldots, \omega^{(q+1)(q+2)+t-1} \}. \]

Denote \( I = \bigcup_{t=1}^{n} I_t \) and label the elements of the set \( I \) as \( a_1, a_2, \ldots, a_n \), where \( a_i = \omega^{(q+1)i+t-1} \). Then from \( [46] \), we have the following lemma.

**Lemma 8.** (\([46]\)) Keep the notation as above, then

1. If \( q \) is even,
   \[
   u_i = (q - 1)^{-1} a_i^2 - q, \omega^{(t-1)(m-2) + \frac{m(m+1)}{2} - (q+1)i_0} \tag{9}
   \]
   for some integer \( i_0 \).

2. If \( q \) is odd,
   \[
   u_i = (q - 1)^{-1} a_i^2 - q, \omega^{(t-1)(m-2) + \frac{(m-q-1)(m-1)}{2} - (q+1)i_0} \tag{10}
   \]
   for some integer \( i_0 \).

**Theorem 9.** Let \( q \) be a prime power and \( 1 \leq m \leq q \).

1. If \( q \) is even and \( (q+1) \mid \frac{m(m+1)}{2} \), then there exists an \([m(q-1) + 1, k, m(q-1) - k + 2]_{q^2}\) Hermitian self-orthogonal extend GRS code, where \( 1 \leq k \leq \lfloor \frac{mq}{q+1} \rfloor \).

2. If \( q \) is odd and \( (q+1) \mid \frac{(m-q-1)(m-1)}{2} \), then there exists an \([m(q-1) + 1, k, m(q-1) - k + 2]_{q^2}\) Hermitian self-orthogonal extend GRS code, where \( 1 \leq k \leq \lfloor \frac{mq}{q+1} \rfloor \).

**Proof.** Keep notations as above.

1. When \( q \) is even, by Eq.\( [9] \)
   \[
   u_i = (q - 1)^{-1} a_i^2 - q, \omega^{(t-1)(m-2) + \frac{m(m+1)}{2} - (q+1)i_0}
   \]
   for some integer \( i_0 \). Let \( h(x) = x^{q-2} \), so \( h(a_i) = a_i^{q-2} a_i^{2-m} \). Hence
   \[
   -u_i h(a_i) = - (q - 1)^{-1} \omega^{(t-1)(m-2) + \frac{m(m+1)}{2} - (q+1)i_0} \\
   \omega^{(q+1)(2-m) + (t-1)(2-m)} \\
   = - (q - 1)^{-1} \omega^{-\lfloor i(m-2) + i_0 \rfloor(q+1) + \frac{m(m+1)}{2}}.
   \]
It is obviously that
\[
(q - 1)^{-1} \omega^{-i[(m-2)+i_0](q+1)} = [(q - 1)\omega^{i[(m-2)+i_0](q+1)}]^{-1} \in F_q^*
\]
for \(\omega^{i[(m-2)+i_0](q+1)} \in F_q^*\). And
\[
(q + 1)\frac{m(m + 1)}{2} \implies [\omega^{\frac{m(m+1)}{2}}]q = \omega^{\frac{m(m+1)}{2}},
\]
i.e. \(\omega^{\frac{m(m+1)}{2}} \in F_q^*\). So it deduces that \(-u_i h(a_i) \in F_q^*\) and there exists \(v_i \in F_q^*\)
such that \(v_i^{q+1} = -u_i h(a_i)\). Set \(v = (v_1, v_2, \ldots, v_n)\).

For \(1 \leq k \leq \lfloor \frac{m}{q} \rfloor\), for any codeword \(c = (v_1 f(a_1), v_2 f(a_2), \ldots, v_n f(a_n), f_{k-1})\)
\(\in GRS_k(a, v, \infty)\) with \(\deg(f(x)) \leq k - 1\). Let \(g(x) = -h(x)f^q(x)\), then it is
easy to see that
\[
\deg(g(x)) \leq q - m + q(k - 1) \leq n - k.
\]
According to Lemma 2, \(c \in GRS_k(a, v, \infty)^{\perp \perp}\). Thus \(GRS_k(a, v, \infty) \subseteq GRS_k\)
\((a, v, \infty)^{\perp \perp}\). This completes the proof.

(2) When \(q\) is odd, by Eq.11
\[
u_i = (q - 1)^{-1} a_i^{2-q} \omega^{t-1} \frac{m(q-1)(m-1)(m+1)}{2} - (q+1)i_0
\]
for some integer \(i_0\). Let \(h(x) = x^{q-m}\), so \(h(a_i) = a_i^{q-2} a_i^{2-m}.\) Hence
\[
-u_i h(a_i) = - (q - 1)^{-1} \omega^{(t-1)(m-2)+\frac{(m-q-1)(m-1)}{2} - (q+1)i_0}
\cdot \omega^{(q+1)i_1(t-2)(m-2) - (t-1)(2-m)}
\]
\[
= - (q - 1)^{-1} \omega^{-i[(m-2)+i_0](q+1)+\frac{(m-q-1)(m-1)}{2}}.
\]
It is obviously that
\[
(q - 1)^{-1} \omega^{-i[(m-2)+i_0](q+1)} = [(q - 1)\omega^{i[(m-2)+i_0](q+1)}]^{-1} \in F_q^*
\]
for \(\omega^{i[(m-2)+i_0](q+1)} \in F_q^*\). And
\[
(q + 1)\frac{(m - q - 1)(m-1)}{2} \implies [\omega^{\frac{(m-q-1)(m-1)}{2}}]q = \omega^{\frac{(m-q-1)(m-1)}{2}},
\]
i.e. \(\omega^{\frac{(m-q-1)(m-1)}{2}} \in F_q^*\). So it deduces that \(-u_i h(a_i) \in F_q^*\) and there exists
\(v_i \in F_q^*\) such that \(v_i^{q+1} = -u_i h(a_i)\). Set \(v = (v_1, v_2, \ldots, v_n)\).

Taking the same calculation as in (1), and it follows from Lemma 2. \(\square\)

11
### Table 2: Some new Hermitian self-orthogonal extend GRS codes from Theorem 9 (1)

| $q$  | $m$  | $n$  | $k_3$ | $k_4$   | $k$       | best class          |
|------|------|------|------|--------|----------|---------------------|
| 3    | 4    | 13   | [1, 2]| no value | [1, 3]   | Theorem 9 (1)       |
| 8    | 8    | 57   | [1, 4]| no value | [1, 37]  | Theorem 9 (1)       |
| 16   | 16   | 241  | [1, 82]| no value | [1, 15]  | Theorem 9 (1)       |
| 32   | 32   | 993  | [1, 16]| no value | [1, 31]  | Theorem 9 (1)       |
| 64   | 32   | 2548 | [1, 16]| [1, 44]  | [1, 24]  | Class 4             |
| 64   | 39   | 652  | [1, 32]| [1, 51]  | [1, 38]  | Class 4             |
| 64   | 64   | 4033 | [1, 32]| no value | [1, 63]  | Theorem 9 (1)       |
| 16384| 1130 | 18512791 | [1, 8192]| no value | [1, 1129]| Class 1             |
| 16384| 6554 | 107374183 | [1, 8192]| no value | [1, 6553]| Class 1             |
| 16384| 7684 | 125866973 | [1, 8192]| no value | [1, 7683]| Class 1             |
| 16384| 8700 | 142532101 | [1, 8192]| no value | [1, 8699]| Theorem 9 (1)       |
| 16384| 9830 | 161044891 | [1, 8192]| no value | [1, 9829]| Theorem 9 (1)       |
| 16384| 15254| 249906283 | [1, 8192]| no value | [1, 15253]| Theorem 9 (1)       |
| 16384| 16384| 268419073 | [1, 8192]| no value | [1, 16383]| Theorem 9 (1)       |

**Example 10.** (1) Compared with Class 39 in Table 1, the Hermitian self-orthogonal extend GRS codes obtained by Theorem 9 are all new. And, in Table 1, we can find that when taking $2s + 1 = q + 1$ in both Class 3 and Class 4 and $2s = q + 1$ in both Class 5 and 6, we can get the code length with the same form $1 + m(q - 1)$, but by comparing carefully, we can find that these six classes of codes have a big difference in the range of dimension $k$ under the same length. We list some comparison results among them in Tables 2 and 3 for even $q$ and odd $q$, respectively, and thus show that Theorem 9 can generate many new codes. And it’s worth noting that a simple search with Magma will reveal many more new Hermitian self-orthogonal extend GRS codes.

(2) Likewise, by Remark 1 (2), we can construct new EAQECCs with flexible parameters and arbitrary dimension Galois hull linear codes via these codes we constructed. Generally, when $q \geq 3$, $m \geq 2$, the length of codes from Theorem 9 is larger than $q + 1$.

\[ n = tn', 1 \leq t \leq \frac{q^2 - 1}{n_1}, n' | (q^2 - 1) \text{ and } n_1 = \frac{q^2 - 1}{\gcd(n', q + 1)} \]

Now we consider the second multiplicative coset decomposition of $F_{q^2}$ in $[47]$. Suppose $n' | (q^2 - 1)$ and $n'$ can be written as $n' = \frac{n'}{\gcd(n', q + 1)} \cdot \gcd(n', q + 1)$. For simplicity, we denote $n_1 = \frac{n'}{\gcd(n', q + 1)}$ and $n_2 = \frac{n'}{n_1} = \gcd(n', q + 1)$. According
Table 3: Some new Hermitian self-orthogonal extend GRS codes from Theorem 9 (2)

| $q$ | $m$ | $n$ | $k_{l_0}$ | $k_{l_1}$ | $k_0$ | $k_1$ | best class |
|-----|-----|-----|-----------|-----------|-----|-----|------------|
| 11  | 9   | 91  | [1, 6]    | no value  | [1, 6] | Theorem 9 (2) |
| 17  | 9   | 145 | [1, 9]    | no value  | [1, 8] | Class 5 |
| 29  | 6   | 169 | [1, 15]   | [1, 17]   | [1, 5] | Class 6 |
| 29  | 10  | 281 | [1, 15]   | [1, 19]   | [1, 9] | Class 6 |
| 29  | 21  | 589 | [1, 15]   | no value  | [1, 20] | Theorem 9 (2) |
| 29  | 25  | 701 | [1, 15]   | no value  | [1, 24] | Class 6 |
| 289 | 261 | 75169 | [1, 145] | no value | [1, 8] | Class 5 |
| 729 | 585 | 425881 | [1, 365] | no value | [1, 584] | Theorem 9 (2) |
| 3125 | 522 | 1630729 | [1, 1563] | no value | [1, 1563] | Class 6 |
| 3125 | 2085 | 6513541 | [1, 1563] | no value | [1, 2084] | Theorem 9 (2) |
| 3125 | 2605 | 8138021 | [1, 1563] | no value | [1, 2604] | Theorem 9 (2) |
| 289 | 261 | 75169 | [1, 145] | no value | [1, 8] | Class 5 |
| 117649 | 4525 | 532457201 | [1, 58825] | no value | [1, 4524] | Class 5 |
| 117649 | 63350 | 7453000801 | [1, 58825] | [1, 90499] | [1, 63349] | Class 6 |
| 117649 | 73125 | 8603010001 | [1, 58825] | no value | [1, 73124] | Theorem 9 (2) |
| 1771561 | 1090193 | 1931342311081 | [1, 885781] | no value | [1, 1090192] | Theorem 9 (2) |

Now, let $n = tn'$, where $1 \leq t \leq q - 1$. Set $A_b = \beta_bG$ and $A = \bigcup_{b=1}^{t} A_b$.

Suppose

$$A = \{a_1, a_2, \ldots, a_n\}. \quad (11)$$

Then based on Lemma 3.7 of [47], we only need to modify it slightly to get the following lemma, which is helpful for our construction.

**Lemma 11.** ([47], Lemma 3.7) Keep the notations as above. Given $1 \leq i \leq n$, suppose $a_i \in A_b$ for some $1 \leq b \leq t$, then

$$u_i = n'^{-1}a_i\beta_b^{-n'} \prod_{1 \leq s \leq t, s \neq b}(\beta_b^n - \beta_s^n)^{-1}. \quad (12)$$

Moreover, $a_i^{n'-1}u_i \in F_q^w$.

**Proof.** Similar to Lemma 3.7 of [47], let $\theta = \omega_{q^2-1}^{n'}$ be the generator of $G$ and set $a_i = \beta_{b}^{l} \theta^l$ for some $1 \leq l \leq n'$. Then by Lemma 3.7 of [47], we get Eq.12.
This completes the proof. Moreover, we have
\[
a_i u_i^{-1} = a_i^{-1} \cdot a_i^{n'} \cdot n'^{-1} a_i \beta_b^{-n'} \prod_{1 \leq s \leq t, s \neq b} (\beta_b^{n'} - \beta_s^{n'})^{-1}
\]
= \((\beta_b^{n'})^{n'} \cdot n'^{-1} \beta_b^{-n'} \prod_{1 \leq s \leq t, s \neq b} (\beta_b^{n'} - \beta_s^{n'})^{-1}
\]
= \(n'^{-1} \prod_{1 \leq s \leq t, s \neq b} (\beta_b^{n'} - \beta_s^{n'})^{-1}.
\]

By the proof of Lemma 3.7 in \[47\] again, we can know that \(\beta_s^{n'} \in F_q^n\). Hence \(a_i u_i^{-1} \in F_q^n\) and this completes the proof.

\[Q.E.D.\]

**Theorem 12.** Let \(q > 2\) be a prime power and \(n' \mid (q^2 - 1)\). Let \(n = tn'\), where \(1 \leq t \leq \frac{q-1}{n-1}\) and \(n_1 = \frac{n}{\gcd(n', q+1)}\). Then there exists a \([tn', k, tn' - k + 1]_{q^2}\) Hermitian self-orthogonal GRS code, where \(1 \leq k \leq \lfloor \frac{n-n'+2}{q+1} \rfloor\).

**Proof.** Keep notations as above, then by Eq.12
\[
u_i = n'^{-1} a_i \beta_b^{-n'} \prod_{1 \leq s \leq t, s \neq b} (\beta_b^{n'} - \beta_s^{n'})^{-1}.
\]

Set \(\lambda = 1 \in F_q^n\), and let \(h(x) = x^{n'-1}\). Then according to Lemma 11 we have \(\lambda u_i h(a_i) = a_i^{n'-1} u_i \in F_q^n\). So there exists \(v_i \in F_q^n\) such that \(\lambda u_i h(a_i) = v_i^{q+1}\). Set \(v = (v_1, v_2, \ldots, v_n)\).

For \(1 \leq k \leq \lfloor \frac{n-n'+q}{q+1} \rfloor\), for any codeword \(c = (v_1 f(a_1), v_2 f(a_2), \ldots, v_n f(a_n)) \in GRS_k(a, v)\) with \(\deg(f(x)) \leq k - 1\). Let \(g(x) = \lambda h(x) f^\varphi(x)\), then it is easy to see that
\[
\deg(g(x)) \leq q(k-1) \leq n - k - 1.
\]
According to Lemma 11 \(c \in GRS_k(a, v)^{\perp-H}\). Thus \(GRS_k(a, v) \subseteq GRS_k(a, v)^{\perp-H}\). This completes the proof.

**Example 13.** (1) In \[47\], for the same code length \(tn'\), the authors prove that for each \(1 \leq k \leq \lfloor \frac{n+n}{q+1} \rfloor\), there exists a \(q^2\)-ary \([n, k]\) MDS code \(C\) with \(\dim(\Hull_H(C)) = l\), where \(0 \leq l \leq k - 1\). So Theorem 12 means that there also
Table 4: Some Hermitian self-orthogonal GRS codes from Theorem 12 for $q = 9$

| $n'$  | $t$  | $[n, k, d]_{q^2}$ | $k$  | $n'$  | $t$  | $[n, k, d]_{q^2}$ | $k$  |
|-------|------|-------------------|-----|-------|------|-------------------|-----|
| 2     | 7    | $[14, k, 15-k]_{q^2}$ | $1 \leq k \leq 2$ | 2     | 8    | $[16, k, 17-k]_{q^2}$ | $1 \leq k \leq 2$ |
| 5     | 4    | $[20, k, 21-k]_{q^2}$ | $1 \leq k \leq 2$ | 5     | 5    | $[25, k, 26-k]_{q^2}$ | $1 \leq k \leq 2$ |
| 5     | 6    | $[30, k, 31-k]_{q^2}$ | $1 \leq k \leq 3$ | 5     | 7    | $[35, k, 36-k]_{q^2}$ | $1 \leq k \leq 3$ |
| 5     | 8    | $[40, k, 41-k]_{q^2}$ | $1 \leq k \leq 4$ | 10    | 5    | $[50, k, 51-k]_{q^2}$ | $1 \leq k \leq 4$ |
| 10    | 6    | $[60, k, 61-k]_{q^2}$ | $1 \leq k \leq 5$ | 10    | 7    | $[70, k, 71-k]_{q^2}$ | $1 \leq k \leq 5$ |
| 10    | 8    | $[80, k, 81-k]_{q^2}$ | $1 \leq k \leq 7$ |

exists a $q^2$-ary MDS code for $l = k$. Based on this, Theorem 12 produces a class of new Hermitian self-orthogonal GRS codes.

(2) We will not repeat the content of EAQECCs and arbitrary dimension Galois hull linear codes, but in Table 4 we list some Hermitian self-orthogonal GRS codes for $q = 9$ from Theorem 12 which are calculated by Magma.

Remark 2. (1) In Table 4 we omit all codes whose dimension can only take $k = 1$ and for the same length, we only keep the codes whose dimension $k$ can take the largest case. For example, considering the code with parameters $[40, k, 41-k]$, if we take $n' = 5$, $t = 8$, then $1 \leq k \leq 4$. But if we take $n' = 10$, $t = 4$, then $1 \leq k \leq 3$. So we only list the case $n' = 5$, $t = 8$.

(2) Looking closely at Table 4 we can see that the constructions in Theorem 12 can contain all the results in Theorem 3 of [46], and can also generate additional codes such as $[14, k, 15-k]$, $[16, k, 17-k]$, $[25, k, 26-k]$ and $[35, k, 36-k]$. Of course, the reason for this phenomenon is also simple, we only need to take $n' = q + 1$, then $n_1 = \frac{q+1}{gcd(q+1, q+1)} = 1$, and it deduces that $1 \leq t \leq q - 1$. So in this case, we can get $[t(q+1), k, t(q+1) + 1 - k]_{q^2}$ Hermitian self-orthogonal GRS codes, where $1 \leq t \leq q - 1$ and $1 \leq k \leq t - 1$ and these are all the results given by Theorem 3 of [46].

(3) In particular, based on the discussion of (1), for some same lengths, the dimensions achieved in our construction can be more flexible and $[40, k, 41-k]$, $1 \leq k \leq 4$ in Table 4 is a specific example because of $1 \leq k \leq 3$ in [46].

In Theorem 12 by adding the zero element and extending it, we can obtain
a family of new Hermitian self-orthogonal extend GRS codes with length $n + 2$ from the following lemma, which is proposed during the proof of Theorem 3.9 in [47].

**Lemma 14.** ([47], Theorem 3.9) Let $a_1, a_2, \ldots, a_n$ be defined as Eq. 11 and put $a_{n+1} = 0$. Then for any $1 \leq i \leq n$,

$$u_i = \prod_{1 \leq j \leq n+1, j \neq i} (a_i - a_j)^{-1} = a_i^{-1} \prod_{1 \leq j \leq n, j \neq i} (a_i - a_j)^{-1} \in F_q^*, \quad (13)$$

and for $i = n + 1$,

$$u_{n+1} = \prod_{1 \leq j \leq n} (a_{n+1} - a_j)^{-1} = (-1)^{n+1+n'} \prod_{1 \leq j \leq n} \beta_j^{-n'} \in F_q^*. \quad (14)$$

Hence, $u_i \in F_q^*$ for $1 \leq i \leq n + 1$.

**Theorem 15.** Keep notations as in Theorem 12. Then there exists a $[tn' + 2, k, tn' - k + 3]_{q^2}$ Hermitian self-orthogonal extend GRS code, where $1 \leq k \leq \lfloor \frac{n+q+1}{q+1} \rfloor$.

**Proof.** Let $h(x) = 1$. Then by Lemma 14, $-u_i h(a_i) \in F_q^*$ for $1 \leq i \leq n + 1$. So there exists $v_i \in F_q^*$ such that $-u_i h(a_i) = v_i^{q+1}$. Set $v = (v_1, v_2, \ldots, v_{n+1})$.

For $1 \leq k \leq \lfloor \frac{n+q+1}{q+1} \rfloor$, taking a similar calculation to Theorem 9, we can know that $GRS_k(a, v, \infty) \subseteq GRS_k(a, v, \infty)^{\perp_{H}}$. This completes the proof. \(\square\)

**Example 16.** (1) Same as Example 13 in [44], for the same code length $tn' + 2$, the authors prove that for each $1 \leq k \leq \lfloor \frac{n+2}{q+1} \rfloor$, there exists a $q^2$-ary $[n, k]$ MDS code $C$ with $\dim(Hull_H(C)) = l$, where $0 \leq l \leq k - 1$. So Theorem 12 means that there also exists a $q^2$-ary MDS code for $l = k$. So Theorem 12 produces a class of new Hermitian self-orthogonal extend GRS codes.

(2) Similarly, we list some Hermitian self-orthogonal GRS codes for $q = 9$ from Theorem 12 in Table 5, which are calculated by Magma and omit those codes that can only take $k = 1$ or the same parameters.
Table 5: Some new Hermitian self-orthogonal extend GRS codes from Theorem 15 for $q = 9$

| $n'$ | $t$ | $[n, k, d]_{q^2}$ | $k$ | $n'$ | $t$ | $[n, k, d]_{q^2}$ | $k$ |
|------|-----|------------------|-----|------|-----|------------------|-----|
| 2    | 6   | $[14, k, 15 - k]_{81}$ | $1 \leq k \leq 2$ | 2    | 7   | $[16, k, 17 - k]_{81}$ | $1 \leq k \leq 2$ |
| 2    | 8   | $[18, k, 19 - k]_{81}$ | $1 \leq k \leq 2$ | 5    | 3   | $[17, k, 18 - k]_{81}$ | $1 \leq k \leq 2$ |
| 5    | 4   | $[22, k, 23 - k]_{81}$ | $1 \leq k \leq 2$ | 5    | 5   | $[27, k, 28 - k]_{81}$ | $1 \leq k \leq 3$ |
| 5    | 6   | $[32, k, 33 - k]_{81}$ | $1 \leq k \leq 3$ | 5    | 7   | $[37, k, 38 - k]_{81}$ | $1 \leq k \leq 4$ |
| 5    | 8   | $[42, k, 43 - k]_{81}$ | $1 \leq k \leq 4$ | 10   | 5   | $[52, k, 53 - k]_{81}$ | $1 \leq k \leq 5$ |
| 10   | 6   | $[62, k, 63 - k]_{81}$ | $1 \leq k \leq 6$ | 10   | 7   | $[72, k, 73 - k]_{81}$ | $1 \leq k \leq 7$ |
| 10   | 8   | $[82, k, 83 - k]_{81}$ | $1 \leq k \leq 8$ |      |      |                   |     |

5. The new EAQMDS codes

As we all know, as the same with classical linear codes, there exists many bounds in EAQECCs and one of the bounds we discuss most often is entanglement-assisted quantum Singleton bound.

**Lemma 17.** (Quantum Singleton Bound [45]) Let $Q$ be an $[[n, k, d]; c]_q$ entanglement-assisted quantum error-correcting code. If $2d \leq n + 2$, then

$$k \leq n + c - 2(d - 1).$$

**Remark 3.** (1) In particularly, an EAQECC for which equality holds in this bound is called an MDS EAQECC (for short EAQMDS).

(2) It is well-known that if a classical linear code $C$ is an MDS code and $d \leq \frac{n + 2}{2}$, then the EAQECC constructed by it is EAQMDS.

So combining the four new Hermitian self-orthogonal (extend) GRS codes and Theorem 15, we can get more new EAQMDS codes with flexible parameters.

**Theorem 18.** Let $q$ be a prime power and $1 \leq m \leq q$. Set $n = m(q - 1) + 1$

(1) If $q$ is even and $(q + 1)\lfloor\frac{m(m + 1)}{2}\rfloor$, then we can obtain EAQECCs with parameters $[[n + 2i, k - l, n + 2i - k + 1; n + 2i - k - l]]_q$, $[[n + 2i + 1, k - l, n - k + 2i + 2; n + 2i + 1 - k - l]]_q$ and EAQMDS codes with parameters $[[n + 2i, n + 2i - k - l, k + 1; k - l]]_q$ and $[[n + 2i + 1, n + 2i + 1 - k - l, k + 1; k - l]]_q$, where $0 \leq l \leq k$ and $i \geq 0$ is an integer.

(2) If $q$ is odd and $(q + 1)\lfloor\frac{(m - 1)(m - 1)}{2}\rfloor$, then we can obtain EAQECCs with
parameters \([n + 2i, k - l, n + 2i - k + 1; n + 2i - k - l]_q\), \([n + 2i + 1, k - l, n - k + 2i + 2; n + 2i + 1 - k - l]_q\) and EAQMDS codes with parameters \([n + 2i, n + 2i - k - l, k + 1; k - l]_q\) and \([n + 2i + 1, n + 2i + 1 - k - l, k + 1; k - l]_q\), where \(0 \leq l \leq k\) and \(i \geq 0\) is an integer.

**Theorem 19.** Let \(q > 2\) be a prime power and \(n' | (q^2 - 1)\). Let \(N_1 = tn'\), \(N_2 = tn' + 2\), where \(1 \leq t \leq \frac{q - 1}{n_1}\) and \(n_1 = \frac{n'}{\gcd(n' + q + 1)}\). Then we can obtain EAQECCs with parameters \([N_1 + 2i, k - l, N_1 + 2i - k + 1; N_1 + 2i - k - l]_q\), \([N_1 + 2i + 1, k - l, N_1 - k + 2i + 2; N_1 + 2i + 1 - k - l]_q\), \([N_2 + 2i, k - l, N_2 + 2i - k + 1; N_2 + 2i - k - l]_q\), \([N_2 + 2i + 1, k - l, N_2 - k + 2i + 2; N_2 + 2i + 1 - k - l]_q\) and EAQMDS codes with parameters \([N_1 + 2i, N_1 + 2i - k - l, k + 1; k - l]_q\), \([N_1 + 2i + 1, N_1 + 2i + 1 - k - l, k + 1; k - l]_q\), \([N_2 + 2i, N_2 + 2i - k - l, k + 1; k - l]_q\), \([N_2 + 2i + 1, N_2 + 2i + 1 - k - l, k + 1; k - l]_q\), where \(0 \leq l \leq k\) and \(i \geq 0\) is an integer.

**Example 20.** Considering that we have discussed the parameters of the four classes of Hermitian self-orthogonal (extend) GRS codes we constructed in Examples 10 13 16, and the construction of EAQECCs is clear according to Theorems 18 and 19, we will not give a detailed example here.

6. Conclusions

In this paper, we propose a unified method to construct linear codes of length \(n + 2i\) and \(n + 2i + 1\) with arbitrary-dimensional Hermitian hulls, where \(i \geq 0\) is an integer. Furthermore, we construct four new classes of Hermitian self-orthogonal GRS and extend GRS codes by two known multiplicative coset decompositions of \(F_{q^2}\). Finally applying them to EAQECCs, we get new EAQECCs where the code lengths can take \(n + 2i\) and \(n + 2i + 1\). In the future, we will try to use other new coset decompositions to construct more Hermitian self-orthogonal GRS and extend GRS codes with new parameters so that we can obtain more new EAQECCs with flexible parameters and arbitrary dimension Galois hull linear codes.
Acknowledgments

This research was supported by the National Natural Science Foundation of China (No.U21A20428 and 12171134).

References

[1] A. Calderbank, P. Shor, Good quantum error-correcting codes exist, Phys. Rev. A, Gen. Phys. 54(2) (1996),1098–1105.

[2] A. M. Steane, Error correcting codes in quantum theory, Phys. Rev. Lett. 77(5) (1996),793–797.

[3] A. Ketkar, A. Klappenecker, S. Kumar, Nonbinary stabilizer codes over finite fields, IEEE Trans. Inf. Theory. 52 (2006),4892–4914.

[4] S.A. Aly, A. Klappenecker, S. Kumar, P.K. Sarvepalli, On quantum and classical BCH codes, IEEE Trans. Inf. Theory. 53 (2007),1183–1188.

[5] T. Brun, I. Devetak, M. Hsieh, Correcting quantum errors with entanglement, Science. 314(5798) (2006),436–439.

[6] E.F. Assmus, J.D. Key, Designs and Their Codes, Cambridge Univ, Press. (1993),103.

[7] K. Guenda, S. Jitman, T. A. Gulliver, Constructions of good entanglement-assisted quantum error correcting codes, Des. Codes Cryptogr. 86 (2018),121–136.

[8] X. Liu, H. Liu, L. Yu, New EAQEC codes constructed from Galois LCD codes, Quantum Inf. Process. 19(20) (2020).

[9] J.S. Leon, An algorithm for computing the automorphism group of a Hadamard matrix, J. Comb. Theory, Ser. A. 27(3) (1979),289–306.
[10] J.S. Leon, Permutation group algorithms based on partition, Theory and algorithms. J. Symb. Comput. 12 (1991), 533–583.

[11] J.S. Leon, Computing automorphism groups of error-correcting codes, IEEE Trans. Inf. Theory. 28(3) (1982), 496–511.

[12] E. Petrank, R.M. Roth, Is code equivalence easy to decide, IEEE Trans. Inf. Theory. 43(5) (1997), 1602–1604.

[13] N. Sendrier, Finding the permutation between equivalent binary code, Proceedings of IEEE ISIT 1997, Ulm, Germany. (1997), 367.

[14] N. Sendrier, Finding the permutation between equivalent codes: The support splitting algorithm, IEEE Trans. Inf. Theory. 46(4) (2000), 1193–1203.

[15] G. Luo, X. Cao, X. Chen, MDS codes with hulls of arbitrary dimensions and their quantum error correction, IEEE Trans. Inf. Theory. 65(5) (2018), 2944–2952.

[16] L. Li, S. Zhu, L. Liu, and X. Kai, Entanglement-assisted quantum MDS codes from generalized Reed-Solomon codes, Quantum Inf. Process. 18(5) (2019), 153.

[17] Y. Li, S. Zhu. Several Classes of New MDS Codes with Arbitrary-Dimensional Hulls and Their Applications, arXiv:2206.06615 [cs.IT]. (2022).

[18] N. Gao, J. Li, S. Huang, Hermitian Hulls of Constacyclic Codes and Their Applications to Quantum Codes, International Journal of Theoretical Physics. 61(3) (2022), 1-14.

[19] X. Liu, H. Liu, L. Yu, New EAQEC codes constructed from Galois LCD codes, Quantum Inf. Process. 19(20) (2020).

[20] M. Cao, MDS Codes With Galois Hulls of Arbitrary Dimensions and the Related Entanglement-Assisted Quantum Error Correction, IEEE Trans. Inf. Theory. 67(12) (2021), 7964-7984.
[21] X. Fang, R. Jin, J. Luo, W. M, New Galois Hulls of GRS Codes and Application to EAQECCs, Cryptogr. Commun. 14 (2022),145–159.

[22] Y. Li, On Constructions of New MDS Codes with Arbitrary-Dimensional Galois Hulls, arXiv:2207.01424 [cs.IT]. (2022).

[23] Z. Li, L. Xing, X. Wang, Quantum generalized Reed-Solomon codes: Unified framework for quantum maximum-distance-separable codes, Physical Review A. 77(1) (2008).

[24] W. Fang, F. Fu, Some New Constructions of Quantum MDS Codes, IEEE Trans. Inf. Theory. 65(12) (2019),7840-7847.

[25] W. Fang, F. Fu, Two new classes of quantum MDS codes, Finite Fields and Their Applications. 53 (2018),85-98.

[26] X. Fang, J. Luo, New quantum MDS codes over finite fields, Quantum Inf Process. 19(16) (2020).

[27] L. Jin, C. Xing, A Construction of New Quantum MDS Codes, IEEE Trans. Inf. Theory. 60(5) (2014),2921-2925.

[28] L. Jin, H. Kan, J. Wen, Quantum MDS codes with relatively large minimum distance from Hermitian self-orthogonal codes, Des. Codes Cryptogr. 84 (2017),463–471.

[29] X. Shi, Q. Yue, Y. Chang, Some quantum MDS codes with large minimum distance from generalized Reed-Solomon codes, Cryptogr. Commun. 10 (2018),1165–1182.

[30] X. Shi, Q. Yue, X. Zhu, Construction of some new quantum MDS codes, Finite Fields and Their Applications. 46 (2017),347-362.

[31] F. Tian, S. Zhu, Some new quantum MDS codes from generalized Reed-Solomon codes, Discrete Mathematics. 342(12) (2019),111593.
[32] C. Galindo, F. Hernando, On the generalization of the construction of quantum codes from Hermitian self-orthogonal codes. Des. Codes Cryptogr. 90 (2022),1103–1112

[33] T. Zhang, G. Ge, Quantum MDS codes with large minimum distance, Des. Codes Cryptogr. 83 (2017),503–517.

[34] X. He, L. Xu, H. Chen, New q-ary quantum MDS codes with distances bigger than $\frac{q}{2}$, Quantum Inf Process. 15 (2016),2745–2758.

[35] H. Liu, X. Liu, Constructions of quantum MDS codes, Quantum Inf Process. 20(14) (2021).

[36] X. Shi, Q. Yue, Y. Wu, New quantum MDS codes with large minimum distance and short length from generalized Reed–Solomon codes, Discrete Mathematics. 342(7) (2019),1989-2001.

[37] M. Grassl, T. Beth, M. Rötteler, On optimal quantum codes, Int. J. Quantum Inf. 2(1) (2004),55–64.

[38] X. Kai, S. Zhu, P. Li, Constacyclic codes and some new quantum MDS codes, IEEE Trans. Inf. Theory. 60(4) (2014),2080–2086.

[39] L. Wang, S. Zhu, New quantum MDS codes derived from constacyclic codes, Quantum Inf. Process. 14(3) (2015),881–889.

[40] X. Kai, S. Zhu, New quantum MDS codes from negacyclic codes, IEEE Trans. Inf. Theory. 59(2) (2013),1193–1197.

[41] K. Feng, Quantum codes $[[6, 2, 3]]p$ and $[[7, 3, 3]]p(p \geq 3)$ exist, IEEE Trans. Inf. Theory. 48(8) (2002),2384–2391.

[42] H. Chen, New MDS Entanglement-Assisted Quantum Codes from MDS Hermitian Self-Orthogonal Codes, arXiv:2206.13995 [cs.IT]. (2022).

[43] L. Sok, Explicit constructions of optimal linear codes with Hermitian hulls and their application to quantum codes, arXiv:2105.00513 [cs.IT]. (2021).
[44] C. Galindo, F. Hernando, R. Matsumoto, D. Ruano, Entanglement-assisted quantum error-correcting codes over arbitrary finite fields, Quantum Inf. Process. 18(4) (2019),1-18.

[45] A. Allahmadi, A. AlKenani, R. Hijazi, N. Muthana, F. Özbudak, P. Solé, New constructions of entanglement-assisted quantum codes, Cryptography and Communications. 14(1) (2022),15-37.

[46] G. Guo, R. Li, Y. Liu, Application of Hermitian self-orthogonal GRS codes to some quantum MDS codes, Finite Fields and Their Applications. 76 (2021).

[47] W. Fang, F. Fu, L. Li, S. Zhu, Euclidean and hermitian hulls of mds codes and their applications to eaqeccs, IEEE Trans. Inf. Theory. 66(6) (2020),3527–3527.

[48] K. Guenda, S. Jitman, T. A. Gulliver, Constructions of good entanglement-assisted quantum error correcting codes, Des. Codes Cryptogr. 86 (2018),121–136.

[49] L. Sok, A new construction of linear codes with one-dimensional hull, Designs, Codes and Cryptography. (2022),1-17.