Nonequilibrium dynamics of charged particles in a quantized electromagnetic field: causal, stable and self-consistent dynamics from $1/c$ expansion

C H Fleming$^{1}$, P R Johnson$^{2}$ and B L Hu$^{3}$

$^{1}$ Joint Quantum Institute and Department of Physics, University of Maryland, College Park, MD 20742, USA
$^{2}$ Department of Physics, American University, Washington, DC 20016, USA
$^{3}$ Joint Quantum Institute and Maryland Center for Fundamental Physics, University of Maryland, College Park, MD 20742, USA

E-mail: hfleming@physics.umd.edu, pjohnson@american.edu and blhu@physics.umd.edu

Received 24 December 2011, in final form 19 March 2012
Published 31 May 2012
Online at stacks.iop.org/JPhysA/45/255002

Abstract

We derive a set of stochastic equations of motion for a system of ordinary quantum-mechanical, spinless charged particles in a second-quantized electromagnetic field based on a consistent application of a dimensionful $1/c$ expansion, which is analogous to the post-Newtonian expansion in gravity. All relativistic corrections up to order $1/c^3$ are found, including electrostatic interactions (Coulomb), magnetostatic backreaction (Biot–Savart), dissipative backreaction (Abraham–Lorentz) and quantum-field fluctuations at zero and finite temperatures. With self-consistent backreaction of the EM field included, we show that this approach yields causal and runaway-free equations of motion, provides new insights into charged-particle backreaction and naturally leads to equations consistent with the (classical) Darwin Hamiltonian, and has quantum operator ordering consistent with the Breit Hamiltonian. To order $1/c^3$ the approach leads to a nonstandard mass renormalization which is associated with magnetostatic self-interactions, and no cutoff is required to prevent runaways. Our results also show that the pathologies of the standard Abraham–Lorentz equations can be seen as a consequence of applying an inconsistent (i.e. incomplete, mixed-order) expansion in $1/c$, if, from the start, the analysis is viewed as generating an effective theory with all relativistic corrections considered perturbatively. Finally, we show that the $1/c$ expansion within a Hamiltonian framework yields well-behaved noise and dissipation, in addition to the multiple-particle interactions.

PACS numbers: 05.40.−a, 05.70.Ln, 41.60.−m

(Some figures may appear in colour only in the online journal)
1. Introduction

1.1. Background

The backreaction of a charged particle interacting with the electromagnetic field involves famous problems including acausality, in the form of pre-acceleration, and runaway solutions to the Abraham–Lorentz equation [1, 2]. A number of approaches to resolving these problems, based upon either various types of particle structure, modifications to the particle–field interaction such as a high-energy cutoff in the field modes or perturbative approximation schemes have been developed. Examples include treating the electromagnetic field interaction perturbatively and truncating at a specific order [3, 4], replacing point particles with extended objects [5–7], replacing time-local differential equations of motion with nonlocal integro-differential equations of motion [4, 8] or simply assuming a sufficiently small cutoff in the field. In the latter case, to avoid runaways the cutoff (or inverse radius) must be small enough such that given a positive renormalized mass the implied bare mass of the particle is positive [4, 9].

In this work, we reanalyze the problem of charged-particle backreaction as an expansion in $1/c$, where $c$ is the speed of light. Although the $1/c$ expansion is commonly used for other problems, particularly for gravity [10–13], somewhat surprisingly it appears that it has not been applied to the problem of charged-particle backreaction, beyond the energy-conserving terms, such as those first analyzed by Darwin [1, 2, 14]. To order $1/c^3$ we find that our method is consistent with the Eliezer and Breit equations, without needing any additional assumptions or order reduction. This provides evidence that the lowest orders of the $1/c$ expansion correctly capture the physics of charged-particle backreaction, without the need for additional approximations or considerations. Perhaps most importantly, we find that the dipole approximation, which is commonly made in backreaction calculations, is inconsistent with the results of the $1/c$ expansion and, we claim, is therefore inconsistent when used in an analysis of backreaction. The incorrect use of the dipole approximation leads to the need for further approximations to the equations of motion via order reduction. The advantage of the $1/c$ expansion is that it provides a unified approximation method that correctly describes multiparticle interactions and quantum fluctuations.

Our $1/c$ expansion perturbs off of the nonrelativistic electrostatic theory and into the weakly relativistic regime. Therefore, it is important to note that, so far, fully covariant treatments have not clarified any of the underlying issues present in the nonrelativistic analysis. In particular, the fully covariant Abraham–Lorentz–Dirac (ALD) equation [15] has either runaway or acausal solutions. Stable and causal weak-field approximations to the ALD equation have been provided by Eliezer [3] and later by Landau and Lifshitz [1], and although these solutions are perturbatively consistent with the ALD equation and Maxwell’s equations, neither can be considered the final word because at higher orders both approaches disagree with Maxwell’s equations and lead to physically incorrect predictions (e.g. see [16] for the Landau–Lifshitz equation).

The calculation we present here should be viewed as a promising first step in a rigorous analysis of radiation reaction. Ultimately, we expect that the classical theory of a structureless charged particle should sum to the ALD equation in any perturbative expansion. We also expect that full QED is implicitly well behaved because it has a ground state and therefore it cannot permit runaway solutions. Last, we know that the ‘true’ solution, as given by QED, must have a classical limit consistent with Maxwell’s equations, and thus it belongs to the family of equations characterized by Dirac and Bhabha [15, 17]. What we demonstrate in this work is that electrodynamics has well-behaved backreaction to order $1/c^3$, without imposing finite
size, spin or a quantized matter field. The question is then, what ingredients are necessary to retain this stability as the order of perturbation is increased and how best can we approximate the true solution.

1.2. Summary of results

Before we present the full details, let us first summarize our method of analysis and its expected domain of validity. We begin with the standard Hamiltonian for a system of spinless, charged particles coupled via their current to the electromagnetic field. In the Coulomb gauge, in addition to the electrostatic interactions, the Hamiltonian includes both \((e/m)p \cdot A(x)\) and \((e^2/2m)A(x)^2\) interaction terms. Our analysis shows that in the (lowest order in \(1/c\)) dipole approximation the \(p \cdot A\) interaction generates order \(1/c^2\) magnetostatic interactions and mass renormalization, and order \(1/c^3\) backreaction and noise correlation for the particle dynamics; in contrast, we show that the dipole approximation to the \(A^2\) interaction generates terms starting at order \(1/c^4\). Therefore, to order \(1/c^3\) the correct effective Hamiltonian drops the \(A^2\) interaction from the beginning and keeps only the \(p \cdot A\) interaction.

If, on the other hand, we want to include the lowest order in \(1/c\) backreaction generated by the \(A^2\) interaction term we must work at order \(1/c^4\) (or higher) and include all terms at the same order in \(1/c\), and these will necessarily include post-dipole terms generated by the \(p \cdot A\) interaction. Traditional calculations, which retain and treat both \(p \cdot A\) and \(A^2\) in the dipole approximation, are ‘mixed order’ with respect to the \(1/c\) expansion; that is, they inconsistently keep some but not all \(O(1/c^4)\) contributions. This is one of the most important observations of this paper. The \(1/c\) expansion and the dipole approximation fundamentally disagree as to which terms should be included at a given order, and, we claim, the \(1/c\) expansion is fully consistent. The dipole approximation takes the limit \(1/c \rightarrow 0\) in the particle–field interaction at an intermediate step in the calculation, whereas the \(1/c\) expansion is a well-controlled perturbative expansion of the stochastic dynamics that must yield the same result regardless of how or where the limit is taken. Moreover, the \(1/c\) expansion provides a method to include higher-order corrections including the appropriate (at each order) multipole terms and relativistic corrections. In concordance with the above facts, we show that the consistently obtained \(O(1/c^3)\) results are well-behaved (runaway-free and casual) without need for any additional order reduction. We show that the quantum noise and dissipation is also consistent and well-behaved. We show that at order \(1/c^4\) there are post-dipole terms from the \(p \cdot A\) interaction which have previously been (we claim inconsistently) ignored. We do not obtain the full \(O(1/c^4)\) solutions—that is an important goal for future work—but our conjecture is that the solutions should remain non-pathological order by order as long as all post-dipole terms and all appropriate physics are included at a given order in \(1/c\). In this work, the only ingredients we have included beyond ALD theory is that we have quantized the electromagnetic field and particle motion, and stability is achieved at the order of \(1/c^3\) regardless of these extra ingredients. Finally, we note that in other contexts mixed-order equations are often pathological (e.g. \[18, 19\]) and therefore our approach provides a significant new perspective as to why some traditional analyses have pathologies, despite the fact that our final result seems so simple.

In this paper, we work in SI units and standard notation, which has the disadvantage that factors of \(c\) are sometimes hidden or implicit within other variables. To assist in power counting in the subsequent analysis, we summarize here where all factors of \(c\) arise. We consider the Coulomb potential to be purely nonrelativistic, and therefore define the vacuum permittivity to be orderless: \(\varepsilon_0 = O(1/c^0)\). Since \(\varepsilon_0 \mu_0 = 1/c^2\), the vacuum permeability governing the magnetic interactions between the particles is therefore relativistic: \(\mu_0 = O(1/c^2)\). It follows
that all remaining contributions from the transverse degrees of freedom of the electromagnetic field are relativistic in their influence upon the system, and more specifically that all remaining contributions from the transverse degrees of freedom of the electromagnetic field are relativistic in their influence upon the system, and more specifically that all remaining contributions from the transverse degrees of freedom of the electromagnetic field.

The intrinsic backreaction timescale is

\[ \tau_m = e^2 / 6\pi \varepsilon_0 mc^3 \sim \alpha \hbar / mc^2, \]

where \( \alpha = e^2 / 4\pi \varepsilon_0 hc \) is the fine-structure constant, and expansion to order \( 1/c^3 \) is essentially equivalent to \( \tau_m \) being small compared to all other timescales in the problem. For backreaction involving bound states, the effective small dimensionless parameter is \( \alpha \Delta E / mc^2 \sim 1/c^3 \), where \( \Delta E \) denotes the relevant energy-level transitions of the system. For backreaction with external driving forces, the small dimensionless expansion parameter is \( a\hbar \omega / mc^2 \sim 1/c^3 \), where \( \omega \) is the driving frequency. Finally, the fact that post-dipole terms arise as higher-order corrections can be seen in a simplified way from the expansion of the electromagnetic field modes as

\[ e^{ik \cdot x} = 1 + i\omega_k / c k \cdot x - \frac{1}{2} (k \cdot x)^2 + \cdots, \]

\[ = 1 + i\omega_k / c k \cdot x - \frac{1}{2} \frac{\omega_k^2}{c^2} (k \cdot x)^2 + O(1/c^3), \]

where \( k \) are the wave vectors, \( \omega_k = ck \) are the frequencies and \( e^{ik \cdot x} = 1 \) is essentially the dipole approximation. We provide a fully rigorous treatment of this physics in the subsequent analysis. Remarkably, we find that all physical processes in the system, ranging from radiation reaction to particle–particle interactions to noise, are naturally and consistently organized in terms of powers of \( 1/c \). This illustrates why, as we have already noted, it is essential to include all terms at a given order in \( 1/c \).

Our analysis of the single-particle dynamics also highlights a subtle but important connection between dissipation and noise. There are two standard nonrelativistic calculations for radiation reaction arising from working in either the Coulomb or electric-dipole gauge. We show that the former (see, e.g., [20] for an example) has a problematic description of noise: it is not manifestly thermal in the sense that the stochastic process present in the Langevin equation is not sampled from an independent thermal distribution constructed from the field’s Hamiltonian acting as a reservoir. The latter (see, e.g., [21] for an example) has ordinary noise but involves integration kernels which are approximately given by the second derivatives of a delta function, making them more pathological than those in the Coulomb gauge. In other words, either the noise or dissipation has undesirable features in the usual derivations. We show that the \( 1/c \) expansion in the Coulomb gauge, with a correct delineation of system and environment, naturally avoids these problems, yielding well-behaved noise and dissipation in addition to the multiparticle interactions within a unified framework.

In section 2, we first analyze nonrelativistic quantum particles in the electromagnetic field starting from the usual Hamiltonian but instead of taking the dipole approximation we expand in orders of powers in \( 1/c \). Continuing with the usual calculation we reproduce the standard results, but eventually see that they are both ‘mixed order’ in powers of \( 1/c \), and have
undesirable features in terms of either noise or dissipation. In section 4, we instead analyze the physics applying a consistent $1/c$ expansion at order $1/c^3$ from the beginning, and obtain the multiparticle stochastic equations of motion with stable backreaction and self-consistent noise which are a main result of this work. Section 5 summarizes our results, discusses the regime of validity of our analysis and discusses future directions. In the appendix, we review the derivation for backreaction in quantum Brownian motion (QBM) which parallels our analysis in many respects. Other characteristics of the QBM model which are used here, such as the renormalization and integration kernels, are more thoroughly discussed in [22].

Our analysis results in causal (i.e. no pre-acceleration) and stable (i.e. runaway-free) backreaction, and automatically leads to equations with a form consistent with the (classical) Darwin Hamiltonian and operator ordering consistent with the Breit Hamiltonian. To order $1/c^3$ the approach leads to a nonstandard mass renormalization that is associated with magnetostatic self-interactions, and (to this order) no cutoff is required to prevent runaways.$^4$

2. Standard ‘nonrelativistic’ radiation reaction

We first review the derivation of the standard Hamiltonian for the motion of a system of $N$ particles, while also defining the notation we will use in the rest of the paper. We start with the Lagrangian $\mathcal{L}_{\text{sys}}$ for a system of charged particles $j=1,\ldots,N$, where $m_j$ is the mass and $e_j$ the charge of the $j$th particle, and the Lagrangian $\mathcal{L}_{\text{env}}$ for an electro ($\mathbf{E}$)–magnetic ($\mathbf{B}$) field acting as its environment:

$$\mathcal{L} = \mathcal{L}_{\text{sys}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{env}},$$

$$\mathcal{L}_{\text{sys}} \equiv \sum_j \frac{1}{2} m_j \dot{x}_j^2,$$

$$\mathcal{L}_{\text{env}} = \int d^3x' \frac{\varepsilon_0}{2} \left\{ \mathbf{E}(\mathbf{x}')^2 - c^2 \mathbf{B}(\mathbf{x}')^2 \right\}.$$

The interaction between the system of charged particles and the EM field is given by

$$\mathcal{L}_{\text{int}} = \int d^3x' \left( \mathbf{J}(\mathbf{x}') \cdot \mathbf{A}(\mathbf{x}') - \rho(\mathbf{x}') \phi(\mathbf{x}') \right),$$

$$= \sum_j e_j \left[ \dot{x}_j \cdot \mathbf{A}(\mathbf{x}_j) - \phi(\mathbf{x}_j) \right],$$

where $\rho$ and $\mathbf{J}$ are the charge and current density coupled to the scalar and vector potentials $\phi$, $\mathbf{A}$ respectively which are related to the electromagnetic fields by

$$\mathbf{E}(\mathbf{x}) = -\nabla \phi(\mathbf{x}) - \frac{\partial}{\partial t} \mathbf{A}(\mathbf{x}),$$

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}).$$

Expressing the vector potential in terms of the spatial Fourier modes with wavevector $\mathbf{k}$ and polarization $\epsilon^\mu$ gives

$$\mathbf{A}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \sum_{\epsilon^\mu} \mathbf{A}_{\mathbf{k},\epsilon^\mu}(\mathbf{x}).$$

$^4$ By electrostatic and magnetostatic we refer to the lowest order (in $1/c$) electric and magnetic fields. These fields are instantaneous and accompany even static charge and current distributions, though our system is not static. The higher-order relativistic contributions contain retarded and radiative effects.
\[ A_{k,\epsilon}(x) = \frac{\epsilon_k}{\sqrt{2\varepsilon_0 \omega_k}} \{ e^{i\kappa \cdot x} a_{k,\epsilon} + e^{-i\kappa \cdot x} a_{k,\epsilon}^\dagger \}, \] 

(10)

where \( \omega_k = c k \). To satisfy the commutation relations, the conjugate momentum of the field is then given by

\[ \pi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \sum_{\epsilon_i} \pi_{k,\epsilon_i}(x), \]

(11)

\[ \pi_{k,\epsilon_i}(x) = \frac{-i\epsilon_k}{\sqrt{2\varepsilon_0 \omega_k}} \{ e^{i\kappa \cdot x} a_{k,\epsilon_i} - e^{-i\kappa \cdot x} a_{k,\epsilon_i}^\dagger \}. \]

(12)

To systematically treat the position dependence of the coupling, we write

\[ A_{k,\epsilon}(x) = \cos(k \cdot x) A_{k,\epsilon} - \sin(k \cdot x) \frac{\pi_{k,\epsilon}}{\varepsilon_0 \omega_k}, \]

(13)

\[ \pi_{k,\epsilon}(x) = \cos(k \cdot x) \pi_{k,\epsilon} + \sin(k \cdot x) \varepsilon_0 \omega_k A_{k,\epsilon}, \]

(14)

where

\[ A_{k,\epsilon} \equiv A_{k,\epsilon}(0), \]

(15)

\[ \pi_{k,\epsilon} \equiv \pi_{k,\epsilon}(0). \]

(16)

are \( x \)-independent field operators that more closely correspond to the ‘positions’ and ‘momenta’, \( q_k \) and \( \pi_k \), of the reservoir oscillators for QBM, defined in the appendix.

### 2.1. Coulomb-gauge Hamiltonian

Choosing the Coulomb gauge \( \nabla \cdot A = 0 \), the scalar potential \( \phi \) becomes the nondynamical Coulomb potential and the vector potential \( A \) is purely transverse, i.e. \( k \cdot \epsilon_k = 0 \). The system plus environment Hamiltonian may then be expressed as

\[ H_A = \sum_j [\frac{p_j - e_j A(x_j)}{2m_j}]^2 + \sum_{i<j} U_{ij}^E + H_{\text{field}}, \]

(17)

\[ U_{ij}^E = \frac{1}{4\pi \varepsilon_0 |r_{ij}|} \]

(18)

\[ H_{\text{field}} = \int d^3x' \frac{1}{2} \varepsilon_0^{-1} \pi(x')^2 + \varepsilon_0 c^2 [\nabla \times A(x')]^2, \]

(19)

where \( r_{ij} = x_i - x_j \) is the separation vector operator. Expanding in field modes, the Hamiltonian is

\[ H_{\text{field}} = \int d^3k \sum_{\epsilon_i} \frac{1}{2} \{ \varepsilon_0^{-1} \pi_{k,\epsilon_i}^2 + \varepsilon_0 \omega_k^2 A_{k,\epsilon_i}^2 \}. \]

(20)

The Heisenberg equations of motion for the system as driven by the field are then

\[ \dot{x}_j = \frac{p_j}{m_j} - \frac{e_j}{m_j} A(x_j), \]

(21)

\[ \dot{p}_j = -V_j \sum_{i \neq j} U_{ij}^E + \frac{1}{2m_j} V [p_j - e_j A(x_j)]^2, \]

(22)
and the Heisenberg equations of motion for the environment as driven by the system are given by

$$\dot{A}_{k,\ell} = \frac{1}{\varepsilon_0} \pi_{k,\ell} + \frac{1}{\varepsilon_0 \omega_k} \sum_j \frac{e_j}{2} \{\sin(k \cdot x_j), \dot{x}_j\},$$

$$\dot{\pi}_{k,\ell} = -\varepsilon_0 \omega_k^2 A_{k,\ell} + \sum_j \frac{e_j}{2} \{\cos(k \cdot x_j), \dot{x}_j\},$$

(23)

(24)

where $[A, B] = AB + BA$ is the anticommutator. To avoid confusion, we note that the particle and field variables, $x, A$, etc., are quantum operators, not c-numbers. From section 3.1, the driven solution can be expressed in the manifestly Hermitian form

$$A(x_t, t) = \xi^A_t(t) - \sum_j e_j \left\{\left(\mu^A_j \ast \dot{x}_j\right)(t) + \left(\mu^A_j \ast \dot{x}_j\right)\right\},$$

(25)

with the convolutions defined as

$$\left(\mu^A_j \ast \dot{x}_j\right)(t) = \int_0^t d\tau' \mu^A_j[x_j(t), x_j(t')]; t - t' [x_j(t')]$$

where the dissipation kernel $\mu^A_j$ and its corresponding damping kernel $\gamma^A_j$ are resolved to order $1/c^2$ in section 3.1. The dissipation-kernel integral arises from the system driving the field, whereas the operator $\xi^A_t(t)$ corresponds to the homogeneous evolution of the field.

Multiparticle quantum Langevin equations of motion can be obtained by substituting equation (25) for $A$ back into the system equations of motion (21)–(22). We will not write down the multiparticle equations in this approach, however, since they are more cumbersome than the results we obtain in section 4. We analyze the single-particle case in section 3.5. This gives the Abraham–Lorentz Langevin result in [20].

Although the Langevin equation derived from equation (25) is valid as an equation of motion for the Heisenberg operators, proper global boundary conditions must be applied so that the homogeneous-evolution operator $\xi^A_t(t)$ can be given the interpretation of independently sampled thermal noise. (For the simpler case of QBM these details are described in appendices A.3 and A.4.) To obtain statistically independent thermal noise, the Langevin equation must be derived assuming an initially factorized system and environment state $\rho_{\text{sys}+\text{env}} = \rho_{\text{sys}} \otimes \rho_{\text{env}}$ with the environment $\rho_{\text{env}}$ initially in its equilibrium state. (Of course the full, factorized state $\rho_{\text{sys}+\text{env}}$ is not an equilibrium state at the initial time.) Because the field equations in equation (25) are expressed in terms of the particle velocity $\dot{x}_j(t)$ rather than the canonical momentum $p_j(t)$, this implies that the initial state of the system plus environment cannot immediately be placed into the standard form with respect to the Langevin equation.

To see this, note that for a Langevin equation sourced with $\dot{x}_j(t)$ we must supply initial data $x_j(0), \dot{x}_j(0)$, etc, for the particle that is independent of (uncorrelated with) the initial state of the environment. Most naively, such an initial state would have the factorized form

$$\rho_{\text{sys}+\text{env}} = \rho_{\text{sys}}(x, \dot{x}) \otimes \rho_{\text{env}}(A, \pi).$$

However, because the velocity $\dot{x}_j(t)$ does not commute with the canonical field momentum $\pi$, such a state with $\rho_{\text{env}}(A, \pi)$ cannot represent a (field) equilibrium state with respect to Hamiltonian (20). It is also likely that such a ‘factorized’ initial state is not even a proper density matrix as the product of two non-commuting positive-definite matrices is not necessarily positive definite. A nontrivial initial state that did take the above form would necessarily involve, therefore, an initial nonequilibrium state of the environment that implicitly depends on $\rho_{\text{sys}}(x, \dot{x})$. The operator-valued quantum noise $\xi^A_t(t)$ would then be sampled from this initial nonequilibrium state of the environment and consequently it would not represent
standard thermal noise. This would also necessarily imply that the initial state of the system would always be dependent upon the realization of the noise. To obtain standard (initially uncorrelated) noise from an initially equilibrium environment, the initial state of the system must be of the form \( \rho_{\text{sys}}(x, p) \), represented in terms of canonical coordinates \( x \) and \( p \).

### 2.2. Electric-dipole gauge Hamiltonian

To obtain a Langevin equation with standard, statistically independent noise, we can make the canonical transformation

\[
e \frac{\partial}{\partial t} x \cdot A(x) = e \frac{d}{dt} [x \cdot A(x)] - e \nabla \cdot A(x),
\]

and neglect the total derivative in the action (see, e.g. [21]). The Hamiltonian becomes

\[
H_{\pi} = \sum_j \frac{p_j^2}{2m_j} + \sum_{i<j} U_{ij}^{\text{bare}} + \int d^3 x \left\{ \frac{1}{2\epsilon_0} [\pi(x') + \rho(x')x]^2 + \frac{\epsilon_0 c^2}{2} \left[ \nabla \times A(x') \right]^2 \right\},
\]

which is analogous to the QBM Hamiltonian (A.6). The interaction is

\[
H_{\text{int}} = \sum_j e_j x_j \cdot \pi(x_j),
\]

after the \( \rho^2 x^2 \) term is included in the renormalization of the potential \( \phi \).

The equations of motion for the system driven by the field are

\[
\dot{x}_j = \frac{p_j}{m_j}, \quad (27)
\]

\[
\dot{p}_j = -\nabla_j \sum_{i \neq j} U_{ij}^{\text{bare}} - e_j [x_j \cdot \pi(x_j)], \quad (28)
\]

where we take the bare potential to include the divergent contribution from \( \rho^2 x^2 \) in the Hamiltonian. The analogous analysis for QBM is described in the appendix.

The equations of motion for the field driven by the particle are

\[
\dot{A}_{k, \epsilon} = + \frac{1}{\epsilon_0} \pi_{k, \epsilon} + \sum_j e_j \cos(k \cdot x_j) x_j, \quad (29)
\]

\[
\dot{\pi}_{k, \epsilon} = -\epsilon_0 \omega_k^2 A_{k, \epsilon} - \epsilon_0 \omega_0 \sum_j e_j \sin(k \cdot x_j) x_j. \quad (30)
\]

As calculated in section 3.1, the driven solution expressed in the manifestly Hermitian form is

\[
\pi(x_0, t) = \xi_\pi(t) - \sum_j e_j \left\{ \left[ \mu_{ij}^\pi \ast x_j \right](t) + \left[ \mu_{ij}^\pi \ast x_j \right]^\dagger(t) \right\},
\]

where the dissipation kernel \( \mu_{ij}^\pi \) and its corresponding damping kernel \( \gamma_{ij}^\pi \) can be resolved to order \( 1/c^3 \) as is done for the Coulomb gauge in section 3.1.

In parallel to the analysis of the previous section, multiparticle quantum Langevin equations of motion can be obtained by substituting equation (31) for \( \pi \) into the system equations of motion (27)–(28). Again, we delay writing them down because in this approach the multiple-particle equations of motion are more cumbersome than the results found in section 4. The single-particle case, however, is analyzed in section 3.5.

Unlike the analysis from the previous subsection, the field variables are now expressed in terms of the canonical variable \( x_j \), in contrast to \( \dot{x}_j \), and consequently the stochastic variable \( \xi_\pi(t) \) can be interpreted as statistically independent noise. Issues due to non-commutativity

5 In fact, \( \xi_\pi(t) \) can be (perturbatively) identified with a stochastic electric field, though \( \pi \) cannot be identified with the electric field as it does not contain the electrostatic fields and necessarily contains the magnetostatic fields.
of system (particle) and environment (field) coordinates discussed in section 2.1 do not occur in this gauge since an initial state of the form
\[ \rho_{\text{sys+env}} = \rho_{\text{sys}}(\mathbf{x}, \mathbf{p}) \otimes \rho_{\text{env}}(\mathbf{A}, \pi) \] (32)
is consistent with the required initial data for particle position in the Langevin equation, with \( \rho_{\text{env}}(\mathbf{A}, \pi) \) at the same time an equilibrium state of the field Hamiltonian. As will be reviewed in the next section, however, the electric-dipole gauge damping kernel for \( \mu_0^f \) is more pathological than the Coulomb-gauge damping kernel for \( \mu_0^j \). The latter is approximately a delta function (Ohmic), whereas the former is approximately the second derivative of a delta function (supra-Ohmic).

To summarize, stochastic equations of motion obtained for the Coulomb gauge with \( \mathbf{x} \cdot \mathbf{A} \) coupling in the particle–field interaction have noise that is not statistically independent of the particle’s requisite initial data, and this will lead to severe complications in both the interpretation and evaluation of the resulting Langevin equations. In contrast, stochastic equations of motion for the electric-dipole gauge with \( \mathbf{x} \cdot \pi \) coupling have statistically independent noise, but more pathological damping, as described in the next section.

3. Electromagnetic damping kernels

3.1. Explicit calculation of driven quantum field

Here we will exactly derive the integration kernels stated in section 2. Most simply, let us consider the field degrees of freedom as driven by one particle, given an \( \mathbf{A} \cdot \mathbf{p} \) interaction:
\[ \dot{\mathbf{a}}_{k, \epsilon} = -i\hbar \omega_k \mathbf{a}_{k, \epsilon} + i \sum_j \frac{\epsilon_j}{2\hbar} \left\{ \mathbf{e}^{-i\mathbf{k} \cdot \mathbf{x}_j}, (\mathbf{e}_k \cdot \mathbf{p}_j) \right\}. \] (33)

Hamiltonian (17) will yield analogous equations of motion, but with the field sourced by the system velocity instead of momentum. The driven solutions of each field mode are therefore given by
\[ \mathbf{a}_{k, \epsilon}(t) = e^{-i\omega_k t} \mathbf{a}_{k, \epsilon}(0) + i \sum_j \frac{\epsilon_j}{2\hbar} \int_0^t dt' e^{-i\omega_k (t-t')} \left\{ e^{-i\mathbf{k} \cdot \mathbf{x}_j(t')}, \mathbf{e}_k \cdot \mathbf{p}_j(t') \right\}. \] (34)

In calculating \( \mathbf{A}(\mathbf{x}_i) \) (10) we require evaluation of the field modes at the system location
\[ e^{+i\mathbf{k} \cdot \mathbf{x}_i(t)} \mathbf{a}_{k, \epsilon}(t) = \frac{1}{2} \left\{ e^{+i\mathbf{k} \cdot \mathbf{x}_i(t)}, \mathbf{a}_{k, \epsilon}(0) \right\}, \] (35)
which we have placed into symmetric form using the commutativity of the system and field operators. We now must consider the driven mode
\[ e^{+i\mathbf{k} \cdot \mathbf{x}_i(t)} \mathbf{a}_{k, \epsilon}(t) = \frac{1}{2} \left\{ e^{-i\omega_k (t-k \cdot \mathbf{x}_i(t))}, \mathbf{a}_{k, \epsilon}(0) \right\} \]
\[ + i \sum_j \frac{\epsilon_j}{2\hbar} \int_0^t dt' \left\{ e^{-i\omega_k (t'-k \cdot \mathbf{x}_j(t'))}, e^{+i\omega_k (t'-k \cdot \mathbf{x}_j(t'))}, \mathbf{e}_k \cdot \mathbf{p}_j(t') \right\}, \] (36)
which we have also placed into a manifestly Hermitian form. Note that any function \( f[\mathbf{k} \cdot \mathbf{x}(t')] \) commutes with \( \mathbf{e}_k \cdot \mathbf{p}(t') \) given the Coulomb-gauge constraint \( \mathbf{k} \cdot \mathbf{e}_k \). Therefore we may move \( \mathbf{e}_k \cdot \mathbf{p}(t') \) to the most suitable side of \( e^{+i\mathbf{k} \cdot \mathbf{x}(t')} \). For the velocity source, one can first decompose the velocity into momentum \( \mathbf{e}_k \cdot \mathbf{p}(t') \) and field \( \mathbf{e}_k \cdot \mathbf{A} \mathbf{x}(t') \) and then note that by the previous argument both terms commute with any function \( f[\mathbf{k} \cdot \mathbf{x}(t')] \). For the position source, commutativity follows trivially.
In any case, one can produce relations such as (73) by substitution of the above expression into equation (10):

$$\mathbf{A}(\mathbf{x}_i, t) = \xi_i^a(t) - \sum_j \frac{\epsilon_j}{m_j} \left\{ (\mathbf{\mu}_{ij} \ast \mathbf{p}_j)(t) + (\mathbf{\mu}_{ij} \ast \mathbf{p}_j)^\dagger(t) \right\},$$  

where the dissipation convolution is given by

$$\left(\mathbf{\mu}_{ij} \ast \mathbf{p}_j\right)(t) = \int_0^t dt' \mathbf{\mu}_{ij}^a(t, t') \mathbf{p}_j(t'),$$

and with the noise and dissipation kernel microscopically determined to be

$$\xi_i^a(t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \sum_{\epsilon_{ik}} \frac{\epsilon_{ik}}{2\pi \epsilon_{ik}} \frac{1}{2} \left\{ e^{-i(\omega t - k \cdot \mathbf{x}_i)} \mathbf{a}_{k,\epsilon}(0) \right\} + \left\{ e^{i(\omega t - k \cdot \mathbf{x}_i)} \mathbf{a}_{k,\epsilon}^\dagger(0) \right\},$$

$$\mathbf{\mu}_{ij}^a(t, t') = \frac{1}{(2\pi)^{3/2}} \int d^3k \sum_{\epsilon_{ik}} \frac{\epsilon_{ik}}{2\pi \epsilon_{ik}} \frac{1}{2} \left\{ e^{i(\omega t - k \cdot \mathbf{x}_i)} e^{-i(\omega t' - k \cdot \mathbf{x}_j(t'))} + e^{-i(\omega t - k \cdot \mathbf{x}_i)} e^{i(\omega t' - k \cdot \mathbf{x}_j(t'))} \right\},$$

$$\mathbf{\mu}_{ij}^a(t, t') = -\frac{1}{(2\pi)^{3/2}} \int d^3k \sum_{\epsilon_{ik}} \frac{\epsilon_{ik} \epsilon_{jk}}{2\pi \epsilon_{ik}} \sin [\omega_k (t - t') - \mathbf{k} \cdot (\mathbf{x}_i(t) - \mathbf{x}_j(t'))] + \mathcal{O}(h^2 k^3).$$

In the classical calculation the position operators in equation (40) commute, however in the quantum-relativistic regime they give rise to nontrivial interference terms as can be seen in equation (41). The $\mathcal{O}(k)$ phases in (41) contribute to the $1/c^3$ dissipative forces (given microscopic structure) and $1/c^2$ nondissipative forces (given multiple particles). These phases are discarded in the dipole approximation. The next-order $\mathcal{O}(h^2 k^3)$ quantum phase corrections are determined by two-time commutators of the system trajectories and are negligible to the order we work at. By $\mathcal{O}(h)$ we only mean to keep track of the fact that these corrections contain commutator dependence, and vanish for the classical equations of motion. Planck’s constant is not an expansion parameter which we consider in this work.

Recognizing the familiar form of (41), one might notice the classically appealing expression

$$\mathbf{\mu}_a[\mathbf{x}(t), \mathbf{y}(t'); t, t'] \approx \left\{ \frac{1}{2\hbar} \left[ \mathbf{A}^{[\mathbf{x}(t), t]} \mathbf{A}^{[\mathbf{y}(t'), t']} \right] \right\}_{kmv},$$

which is exact when ignoring the above-mentioned quantum-relativistic phase corrections. The commutators of vector operators are defined

$$\left\| \mathbf{X}, \mathbf{Y} \right\| = \mathbf{X}^T \mathbf{Y} - \mathbf{Y}^T \mathbf{X}.$$  

This object is a matrix whose entries are ordinary commutators.

Finally, if we take the anti-derivative of the dissipation kernel in equation (40), when ignoring the time dependence in $\mathbf{x}$, then we obtain the damping kernel:

$$\nu_i^a(t, t') = \frac{1}{(2\pi)^{3/2}} \int d^3k \sum_{\epsilon_{ik}} \frac{\epsilon_{ik} \epsilon_{jk}}{2\pi \epsilon_{ik}} \frac{1}{2} \left\{ e^{-i(\omega t - k \cdot \mathbf{x}_i)} e^{i(\omega t' - k \cdot \mathbf{x}_j(t'))} + e^{i(\omega t - k \cdot \mathbf{x}_i)} e^{-i(\omega t' - k \cdot \mathbf{x}_j(t'))} \right\},$$

$$\nu_i^a(t, t') = \frac{1}{(2\pi)^{3/2}} \int d^3k \sum_{\epsilon_{ik}} \frac{\epsilon_{ik} \epsilon_{jk}}{2\pi \epsilon_{ik}} \cos [\omega_k (t - t') - \mathbf{k} \cdot (\mathbf{x}_i(t) - \mathbf{x}_j(t'))] + \mathcal{O}(\hbar^2 k^3).$$
which has the important property of being positive definite in the sense of
\[ \sum_{ij} \int_0^t dt_1 \int_0^t dt_1 \mathbf{p}_i(t) \mathbf{p}_j(t) > 0. \] (44)
A positive-definite damping kernel is one of the necessary ingredients for stability.

3.2. 1/c expansion of the dissipation integrals

Let us denote the time dependence in the field operators which arises from \( x(t) \) as intrinsic and the explicit time dependence in \( A(r, t) \) at some fixed location \( r \) as extrinsic. We now express the integration kernels in a manner that distinguishes the two causes of time dependence:
\[ \mu_{ij}(t, t') = \mu[x_i(t), x_j(t') ; t, t']. \] (45)
With this distinction, the dissipation kernel is stationary with regard to extrinsic time dependence, i.e.
\[ \mu[x(t), y(t'); t, t'] = \mu[x(t), y(t'); t-t']. \]
It is not stationary, however, with regard to the intrinsic time dependence of \( x(t) \) and \( y(t') \).
The quantum dissipation kernel is only approximately spatially stationary in the sense that
\[ \mu[x(t), y(t'); t, t'] \approx \mu[x(t) - y(t'); t-t'], \]
with \( \mathcal{O}(\hbar^2/c^3) \) phase discrepancies as discussed following equation (41).
The positive-definite damping kernel can now be related by
\[ \mu[x(t), y(t'); t, t'] = -\frac{\partial}{\partial t'} \mathcal{Y}[x(t), y(t'); t, t'], \] (46)
\[ \tilde{\mu}[x(t), y(t'); \omega] = \text{io} \mathcal{F}[\mathcal{Y}[x(t), y(t'); t, t']], \] (47)
where the partial derivative and Fourier transform neglect any intrinsic time dependence in \( x(t) \) and \( y(t') \), similar to the analysis in [23].

To reduce our Langevin equations to damping-kernel integrals, we require relation to the total derivative, as given by
\[ \mu_{ij}(t, t') = -\frac{d}{dt'} \mathcal{Y}_{ij}(t, t') + \delta \mu_{ij}(t, t'), \] (48)
where the last term \( \delta \mu \) is a relativistic correction such that
\[ \mathcal{O}(\delta \mu) = \mathcal{O}(\mu/c), \] (49)
and which exists even classically and would be neglected if the dipole approximation is taken earlier in the calculation. For the Coulomb gauge this kernel is given by
\[ \delta \mu_{ij}(t, t') = \frac{1}{(2 \pi)^{3/2}} \int d^3 k \sum_{\epsilon_i} \frac{\epsilon_i \epsilon_j^*}{2 \epsilon_0 \omega_k^2} \frac{t}{4} e^{i(\epsilon_0 t' - k \cdot x(t'))} \{ e^{-i(\epsilon_0 t - k \cdot x(t))}, k \cdot \dot{x}_j(t') \}
- e^{-i(\epsilon_0 t - k \cdot x(t))} \{ e^{i(\epsilon_0 t' - k \cdot x(t'))}, k \cdot \dot{x}_j(t') \}. \] (50)
In a similar manner we can integrate this kernel by parts to obtain even higher-order corrections:
\[ \delta \mu_{ij}(t, t') = -\frac{d}{dt} \mathcal{Y}_{ij}(t, t') + \delta \delta \mu_{ij}(t, t'), \] (51)
and so on for \( \delta \delta \mu \). The partitioning is unique given that we require \( \delta \mathcal{Y} \) to have the same order and frequency sensitivity as \( \delta \mu \), and we require \( \delta \delta \mu \) to be higher order in \( 1/c \). The expansion is never ending, but we will only need a finite number of terms for any finite order of analysis. At
some point in the series it will become sufficient to take the trajectories to be quasi-stationary within the integral, with the nonstationary correction being of an order in $1/c$ that we neglect.

Thus, by iteration, we may obtain the total derivative of the dissipation kernel in terms of the relativistic expansion

$$\mu_{ij}(t, t') = \frac{d}{dt'} \eta_{ij}(t, t'),$$

(52)

$$\eta_{ij}(t, t') \equiv \gamma_{ij}(t, t') + \delta \gamma_{ij}(t, t') + \delta \delta \gamma_{ij}(t, t') + \cdots,$$

(53)

where each term of the expansion comes in increasing orders of $1/c$. Given this expansion, we can now reduce arbitrary dissipation integrals to

$$(\mu_{ij} * \dot{z})(t) = (\eta_{ij} * \dot{z})(t) - \eta_{ij}(t, t')z(t) + \eta_{ij}(t, 0)z(0),$$

(54)

where the final term is a transient slip. Although the slip evolution arises during the same period of time as the conventional Abraham–Lorentz acausality, its role is well understood from QBM studies where it has been shown to be unproblematic (see the appendix and [22] where the initial evolution for initial factorized system plus environment states in terms of so-called slip and jolt evolution is reviewed). It may therefore be discarded for our purposes; doing so is equivalent to choosing an initial state which is more properly correlated to the environment. The difference between QBM and electrodynamics is that with the relativistic field, not only are there instantaneous self-transients from $\gamma_{ii}(t, 0)$, there are also retarded cross-transients from $\gamma_{ij}(t, 0)$. Essentially, for factorized states of the particles and field, the particles are completely unaware of each other’s existence until photons travel between them and suddenly (and violently) establish correlations at their first mediation.

Seeing that we may safely neglect the final term of equation (54), this leaves two terms to evaluate. The second single-time term denotes instantaneous forces (a result of the Coulomb gauge) and these terms can all be integrated exactly using the standard commutation relations. The largest challenge is to resolve the first integral terms, which we will consider in the following section.

### 3.3. Lowest order calculations

Fourier transforming with respect to the extrinsic time variables, the classical or quasistatic $A$-coupling damping kernel (without cutoff) is

$$\tilde{\gamma}_A[r, \omega] = \frac{1}{6\pi \varepsilon_0 c^3} \left[ \tilde{S}_1 \left( \frac{\omega}{c} \right) \hat{r} \hat{r}^T \right],$$

(55)

in terms of the functions

$$\tilde{S}_1(z) = \frac{3(z^2 - 1)}{2} \sin(z) + z \cos(z) z^3,$$

(56)

$$\tilde{S}_0(z) = -\frac{3}{2} \left( z^2 - 3 \right) \sin(z) + 3z \cos(z) z^3.$$

(57)

A cutoff regulator $\chi$ can be easily inserted by multiplying the right-hand side of equation (55) with a function

$$\chi(\omega/\Lambda) : [0, \infty) \rightarrow [1, 0),$$

(58)

that vanishes sufficiently fast.

In the coincidence limit we recover the usual Ohmic damping:

$$\lim_{r \to 0} \tilde{\gamma}_A[r, \omega] = \frac{1}{6\pi \varepsilon_0 c^3},$$

(59)
Figure 1. Comparison of sinc (bold), $\tilde{S}_1$ and $\tilde{S}_0$ (dashed). Sinc and $\tilde{S}_1$ are extremely qualitatively similar, both being unity at zero whereas $S_0$ vanishes at zero.

Figure 2. The same functions as in figure 1, but in the time domain: the rectilinear distribution (bold), $S_0$ and $S_1$ (dashed).

$$\lim_{r \to 0} \gamma_A[\mathbf{r}; t] = \frac{1}{6\Omega \varepsilon_0 c^3} \delta(t), \quad (60)$$

and for particles that are far separated all cross correlations vanish:

$$\lim_{r \to \infty} \tilde{\gamma}_A[\mathbf{r}; \omega] = 0, \quad (61)$$

$$\lim_{r \to \infty} \gamma_A[\mathbf{r}; t] = 0. \quad (62)$$

In figure 1 we compare these special functions to sinc($\zeta$), which is the result obtained from the simpler but analogous case of coupling to a scalar field. In figure 2 we compare these functions in the time domain. The frequency domain is useful for noting the Markovian limit at $r\omega = 0$. 

13
and the decorrelation which occurs for $r_\omega \to \infty$. The time domain is useful for illustrating that the damping kernel is (perturbatively) causal and constrained to the light cone. As is well known, this latter property is not true of the quantum noise kernel. From equation (55) and the preceding analysis, it is now clear that under integration the damping kernel is an order $1/c^3$ delta function. In addition to higher-order effects of kernel (55) in the damping integrals, which are not relevant for point particles, following section 3.2 there are higher-order integrals we have neglected. These integrals are implicitly discarded when taking the dipole approximation earlier in the calculation.

The remaining terms we consider are the instantaneous forces generated by $\gamma_{ij}(t, t)$ and higher-order terms. Unlike QBM, $\gamma_{ij}[r_i(t); 0]$ contains both self-interactions, which renormalize the mass, and field-mediated interactions. Using equation (55), this term becomes

$$\gamma_{ij}[r; 0] = \frac{1}{16\pi \epsilon_0 c^2} \frac{1 + \hat{r}^T}{|r|} = \frac{\mu_0}{16\pi} \frac{1 + \hat{r}^T}{|r|},$$

and we see the emergence of the Darwin Hamiltonian (e.g. see Jackson [2, section 12.7]). The classical magnetostatic energy of the Darwin Hamiltonian is known to be given by

$$U^\theta_{ij} \text{classical} = -\frac{\mu_0}{8\pi} \frac{e_i}{m_j} \frac{1 + \hat{r}^T e_j}{|r|} p_j.$$  

The quantum extension of the Darwin Hamiltonian requires an operator-ordering prescription that our analysis will provide upon applying these results to the Langevin equation. In principle there could also be $1/c^3$ forces arising from $\delta \gamma$, as described in section 3.2. However, we have calculated that this order of forces vanishes upon integration, in agreement with the next order of known forces being $(v/c)^4$. These results exemplify how the equations of motion that result from an A-coupling interaction give consistent formulas for damping and backreaction, despite the noise being problematic as previously discussed.

3.4. Dissipation and damping with electric-dipole coupling

In this subsection we describe the sense in which the electric-dipole interaction gives a less desirable description of damping and backreaction, despite providing a consistent description of thermal noise, as previously described. For $\pi$-coupling to the field, the dissipation kernel is approximately given by

$$\mu_{\pi}[x(t), y(t'); t, t'] \approx \frac{1}{2i\hbar} \left[ \mathbf{\pi}(x(t), t), \mathbf{\pi}(y(t'), t') \right]_{\text{env}},$$

with the same $O(\hbar^2/c^3)$ phase discrepancies discussed in section 3.1. The $\pi$-coupling dissipation kernel is related to the A-coupling dissipation kernel via

$$\gamma_{ij}[r; t', t] = -\frac{\partial^2}{\partial \omega^2} P_i[r; t', t].$$

This result is equivalent to dissipation from a supra-Ohmic bath in QBM. As an integration kernel $\gamma_{ij}$ is relatively pathological and must be integrated by parts twice to obtain the well-behaved kernel $\gamma_{ij}$. Such an integration is straightforward for the single-particle theory in the dipole approximation, but for the multiparticle and higher-order relativistic theory this generates many additional terms and limits which must be carefully analyzed in constructing the Langevin equation. Although, with care, it should be possible to proceed in this way, our approach in the next section is more straightforward.
3.5. Standard dipole calculation

Before presenting our effective equations of motion in section 4, we review the derivation of the standard results for a single particle. Taking the dipole approximation we drop all position dependence in the integration kernels. Substituting equation (31) (integrated by parts once) into equations (27) and (28) we obtain the Langevin equation

\[ m\ddot{x}(t) = -\nabla U(x) - 2e^2 (\gamma^A \ast \dot{x})(t) + e\xi^A(t), \]

(65)
discarding the transient terms. This equation of motion is runaway free for bare mass \( m > 0 \) as reviewed in appendix A.5.

Integrating by parts the \( \pi \)-damping integral in the Langevin equation (65) two times, to obtain \( A \)-coupling damping, discarding additional transient terms, and grouping like terms, we obtain the standard Abraham–Lorentz–Langevin equation

\[ m_{\text{ren}}\ddot{x}(t) = -\nabla U(x) + 2e^2 (\gamma^A \ast \dot{x})(t) - e\dot{\xi}^A(t), \]

(66)

\[ m_{\text{ren}} = m + 2e^2 \gamma^A(0), \]

(67)

where \( m_{\text{ren}} \) is the renormalized particle mass. The same result is obtained by integrating by parts the dissipation integral in the Langevin equations (21) and (22). Note that

\[(\gamma^A \ast \dot{x})(t) \approx \frac{1}{12\pi \varepsilon_0 c^3 m_{\text{ren}}},\]

in the high cutoff (‘point-particle’) limit. Positive bare mass \( m \), which is required for runaway-free motion, requires a finite cutoff \( \Lambda \) in the field modes such that \( 2e^2 \gamma^A(0) < m_{\text{ren}} \), noting that \( \gamma^A(0) \propto \Lambda \). Therefore positive bare mass requires an ultraviolet cutoff \( \Lambda \), or equivalently a form factor that cuts off the particle field coupling, on the order of \( \varepsilon_0 c^3 m_{\text{ren}}/e^2 \) or smaller. This is directly seen from our stability analysis in appendix A.5, or simply by noting that with a negative bare mass the system + environment Hamiltonian no longer has a lower bound in its energy spectrum.

As mentioned in section 3.2, by taking the dipole approximation prior to integrating by parts, a host of higher-order terms are implicitly discarded in the standard calculation. This would not be problematic, except that in keeping the \( A^2 \) terms in the Hamiltonian (which are \( O(1/c^4) \)) one is preferring one set of higher-order terms over another, and ultimately obtaining a mixed-order result. Also note that the electric-dipole calculation requires integration by parts thrice; this calculation would be extremely difficult to do order by order.

4. \( 1/c^3 \) theory

In the previous section, we showed how to carry out a \( 1/c \) expansion in evaluating electromagnetic kernels, and how the result differs from the traditional dipole approximation. We now apply these results to the consistent application of a dimensionful \( 1/c \) expansion around the Hamiltonian

\[ H_{\text{NR}} = \sum_i \frac{p_i^2}{2m_i} + \sum_{i<j} U_{ij}^E, \quad U_{ij}^E \equiv \frac{e_ie_j}{4\pi \varepsilon_0 |\mathbf{r}_{ij}|}. \]

We view \( H_{\text{NR}} \) (modulo any external fields) as the lowest order, ‘nonrelativistic’ Hamiltonian in this approach. All additional terms that arise in the Hamiltonian are then viewed as perturbations, whose order is based on the powers of \( 1/c \).

We begin with the \( A \)-coupling, Coulomb-gauge Hamiltonian in (17). From the previous analysis we observe that the self-field generates, at lowest order \( O(1/c^2) \), magnetostatic...
and renormalization terms (see section 3.3). Because our approximation is based on a systematic expansion in powers of \(1/c\), and here we work to \(O(1/c^3)\), we drop the \(A^2\) system–field interaction terms in the Hamiltonian\(^6\). These neglected terms are all of order \((v/c)^4\) \(a (\Delta E/mc^2) (v/c)^2\), and so, making them \(O(1/c^3)\). For now we assume the usual nonrelativistic kinetic energy for the particles, but for consistency we include the relativistic corrections at the appropriate order in section 4.3. Therefore, the consistent \(O(1/c^4)\) effective Hamiltonian in the Coulomb gauge is

\[
H_{\text{eff}} = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + \sum_{i<j} \mathbf{U}_{ij}^E - \sum_i \frac{e_i^2}{2m_i} \left[ \mathbf{p}_i \cdot \mathbf{A}(x_i) + \mathbf{A}(x_i) \cdot \mathbf{p}_i \right] + H_{\text{field}} + O(1/c^4).
\]

The Heisenberg equations of motion for the system driven by the field for the effective Hamiltonian \(H_{\text{eff}}\) are

\[
\dot{x}_i = \frac{\mathbf{p}_i}{m_i} - \frac{e_i}{m_i} \mathbf{A}(x_i),
\]

and the Heisenberg equations of motion for the environment driven by the system are

\[
\dot{A}_{k,i} = \frac{1}{\varepsilon_{0}} \mathbf{\pi}_{k,i} + \frac{1}{\varepsilon_{0} \varepsilon_{k}} \sum_{j} \frac{\varepsilon_{j}}{2m_{j}} \left[ \sin(k \cdot x_j), \mathbf{p}_j \right],
\]

\[
\dot{\mathbf{\pi}}_{k,i} = -\varepsilon_{0} \varepsilon_{k}^2 A_{k,i} + \sum_{j} \frac{\varepsilon_{j}}{2m_{j}} \left[ \cos(k \cdot x_j), \mathbf{p}_j \right].
\]

As derived in section 3.1, the driven solution expressed in manifestly Hermitian form is

\[
\mathbf{A}(x_i, t) = \xi_i(t) - \sum_i \frac{\varepsilon_i}{m_i} \left\{ (\mathbf{\mu}_i^\dagger \cdot \mathbf{p}_i)(t) + (\mathbf{\mu}_i \cdot \mathbf{p}_i^\dagger)(t) \right\},
\]

where the \(\xi_i(t)\) are uncorrelated processes for \(i \neq j\). Following the approach in section 3.3, integrating the dissipation kernel by parts gives

\[
\mathbf{A}[x_i(t), t] = \xi_i(t) = \frac{e_i}{m_i} \frac{1}{6 \pi \varepsilon_0 c^3} \mathbf{p}_i(t) + \sum_j \left\{ \mathbf{y}[r_{ij}(t); 0]\frac{\varepsilon_j}{m_j} \mathbf{p}_j(t) + \left( \mathbf{y}[r_{ij}(t); 0]\frac{\varepsilon_j}{m_j} \mathbf{p}_j(t) \right)^\dagger \right\}.
\]

Here we have neglected the transient slip terms (which, as noted before and as is reviewed in the appendix, can be verified to not induce runaways or acausal behavior) and corrections of higher order in \(v/c\).

Substituting the driven field solution into the system equations of motion gives the renormalized multiple-particle Langevin equation

\[
\mathbf{x}_i = \frac{\mathbf{p}_i}{m_i} - \frac{e_i}{m_i} \mathbf{\xi} + \frac{e_i^2}{m_i} \frac{1}{6 \pi \varepsilon_0 c^3} \mathbf{p}_i + \nabla_x \sum_{i \neq j} \mathbf{U}_{ij}^\theta, \quad \text{damping}
\]

\[
\dot{\mathbf{p}}_i = -\nabla_x \left[ \mathbf{U}_{ij}^E + \mathbf{U}_{ij}^\theta \right], \quad \text{electric, magnetic}
\]

\(^6\) For external fields this approximation is not made, and any \(\mathbf{A}_{\text{ext}}\) should be included by appropriately translating the system momenta.
These are one of the main results of this paper. In the name of renormalization we have removed by hand all magnetostatic self-interactions $U^B_{ii}$ that arise in equation (75); more rigorous details will be given in section 4.2. The quantum magnetostatic potential $U^B_{ij}$ is given by

$$U^B_{ij} = -\frac{1}{2} \frac{\epsilon_i \epsilon_j}{m_i m_j} \left[ p_i^\dagger p_j + p_j^\dagger p_i - p_i p_j \right] + \frac{1}{2} \frac{\epsilon_i \epsilon_j}{m_i m_j} \text{Tr}_i \left[ p_i^\dagger p_j^\dagger \gamma[r_{ij};0] + \gamma[r_{ij};0] p_i p_j \right],$$

(77)
in terms of the kernel

$$\gamma[r;0] = \frac{1}{16\pi \epsilon_0 c^2} \frac{1 + \hat{r} \hat{r}^T}{r} = \frac{\mu_0}{16\pi} \frac{1 + \hat{r} \hat{r}^T}{|r|}.$$ 

The last two terms of the magnetostatic potential (77) have been expressed with a spatial trace

$$\text{Tr}_i[M] \equiv M_{xx} + M_{yy} + M_{zz},$$

(78)
to keep the proper Hilbert-space operator ordering. The complexity of this expression is due to the non-commutativity from both Hilbert space and spatial (3-vector and $3 \times 3$ matrix) operations. This quantum magnetostatic potential is consistent with the (classical) Darwin magnetostatic potential

$$U^B_{ij} \big|_{\text{classical}} = -2 \frac{\epsilon_i}{m_i} p_i^\dagger p_j + \frac{\epsilon_j}{m_j} p_j,$$

(79)

$$= -\frac{\mu_0}{8\pi} \frac{\epsilon_i}{m_i} p_i^\dagger p_j^\dagger \frac{1 + \hat{r} \hat{r}^T}{|r|} e_j p_j.$$ 

(80)

Classically, the Darwin magnetostatic potential generates the Biot–Savart force to lowest order in $v/c$ (see Jackson [2, section 12.7]). Moreover, the operator ordering in equation (77) is consistent with the Breit equation for spin-1/2 particles in QED [19] (often named ‘orbit–orbit interaction’ in that context), when one ignores all of the Pauli spin matrices. Due to operator ordering, the quantum Biot–Savart law is a rather complicated expression.

Equations (75) and (76) are multiparticle stochastic equations of motion consistent through order $O(1/c^3)$ in the field influences with, as we will show, stable (runaway-free) backreaction even as the field cutoff goes to infinity. ($O(1/c^3)$ kinematics are given in section 4.3.) Note that our results show that the mass renormalization associated with the dissipative backreaction, at $O(1/c^3)$, is due to magnetostatic self-interactions. Our effective-Hamiltonian treatment also correctly produces the second-order magnetostatic corrections without extraneous fourth-order terms (as Breit accidentally included in his first calculation [18, 19]). This result is not present in the standard treatments. Note that in our formalism, to order $1/c^3$ there are no corrections to the damping and noise due to multiple-particle interactions. This is not expected to hold at higher orders.

If instead we had used the standard velocity driving of the field then we would have not obtained standard thermal noise, as discussed in section 2.1. Sourcing the field appropriately with the system momenta, we have obtained thermal noise. Driving the field with system momenta, if we had used the standard Hamiltonian and dipole approximation then we would also have obtained the traditional, and pathological, result. However, from the perspective of momentum driving, this result would be mixed order. This analysis therefore suggests that the pathologies of the standard Abraham–Lorentz equations can be viewed, from the perspective of a $1/c$ effective-theory expansion, as a consequence of performing a mixed-order calculation.

Let us summarize this important argument. In terms of a $1/c$ expansion starting from the Coulomb-gauge Hamiltonian, there are $O(1/c^3)$ post-dipole-expansion terms in the $\mathbf{p} \cdot \mathbf{A}(\mathbf{x})$
interaction terms that are of the same order as the lowest-order dipole terms present in the $A(x)^2$ interaction. This reveals that uniform application of the dipole approximation by itself to the Coulomb-gauge Hamiltonian is inconsistent with the $1/c$ expansion, in the sense that it keeps some terms of $O(1/c^3)$, but discards others. For a consistent $O(1/c^3)$ expansion, effectively the dipole approximation to the $p \cdot A(x)$ interaction is kept and the $A(x)^2$ interaction is dropped entirely, as we have done. Thus we may say that the pathologies of the standard Abraham–Lorentz equation (and the structure dependence of Ford-O’Connell’s integro-differential Langevin equation) are comparable to terms neglected in the dipole approximation.

Combining our equations (75) and (76) for a single particle results in

$$m\ddot{x} = \left\{ 1 + \frac{e^2}{m} \frac{1}{6\pi\varepsilon_0 c^3} \frac{d}{dt} \right\} F_{\text{ext}}(x) - e\dot{\xi}(t).$$

(81)

This reproduces the same damping as the Eliezer and order-reduced Ford-O’Connell equation, showing that the $1/c$ expansion also reproduces that universal result for lowest order single-particle backreaction. In the previous derivations, this result is obtained by order reduction, starting from structure-dependent equations. In contrast, the $1/c$ expansion achieves this result automatically at order $1/c^3$, without any higher-order terms to be order reduced, further illustrating the power of this approach. Thus, in terms of a $1/c$ expansion, the damping in the order-reduced equation is strictly $1/c^3$ (and thus consistent), whereas the pathological backreaction in the Abraham–Lorentz equation is of mixed order. There is an interesting parallel here to Breit’s original calculation [18], where pathological equations resulted when Breit accidentally included a fourth-order operator in a calculation which was only accurate to second order.

4.1. Noise and backreaction stability

As previously discussed in section 2.1 the ‘noise’ for the standard $v \cdot A$ coupling Langevin equation cannot be independently sampled from a thermal distribution. The standard calculation only gives well-behaved noise within the electric-dipole gauge. In contrast, our analysis, to order $O(1/c^3)$ starting from the Coulomb gauge, gives well-behaved noise and dissipation. Comparing the derivations in sections 2.1 and 4, we see that our noise and the standard electric-dipole gauge noise only differ by contributions from the $A(x)^2$ interaction of the order $1/c^4$ and higher, or more specifically $(v/c)^2, \alpha(\Delta E/mc^2)(v/c)^2$, etc.

It is interesting to consider further why the noise in the standard (mixed-order) Coulomb-gauge calculation is problematic. Examining equation (25), we see that the backreaction (which depends on velocity $v$ in this expression) contains some perturbative amount of the ‘$p$-noise’, which is true thermal noise, obtained in our calculation. Similarly, inspection also shows that the noise in the standard (mixed-order) Coulomb-gauge calculation contains some perturbative amount of ‘$p$-backreaction’, which is the resistive damping that accompanies thermal noise, obtained in our calculation. In other words, the (non-thermal) noise in the standard (mixed-order) Coulomb-gauge calculation contains some backreaction, and the backreaction contains some noise. Therefore, even in the (classical) limit of vanishing noise for (25) we would have to enforce an artificial constraint upon the $p$-backreaction implying that the backreaction in the Coulomb gauge includes a non-zero average value of the $p$-noise.

The physics may be clearer from another perspective. Recalling that $mv = p - eA$ such that the velocity $v$ depends on both the particle and field canonical coordinates, we see that velocity driving entails that the field is driven by both the system (particle) and itself. In other words, velocity driving of the field implies a perturbative feedback loop in the environment dynamics. Essentially, the field can self-generate stronger and stronger field excitations, albeit at an order beyond which the theory is accurate, and this can result in runaways. In contrast,
in our new equations of motion, equation (73), the field is driven by the canonical momentum $\mathbf{p}$ rather than the velocity $\mathbf{v}$, and such pathological processes do not occur.

Our next task is to demonstrate that the equations of motion are nonperturbatively stable. When combined, relations (75) and (76) reproduce the same damping as the Eliezer equation, which is already known to be stable. We can gain additional insight, however, by comparing our analysis to the analysis leading to the same result in [4]. In this paper, we begin with an effective Hamiltonian, and work consistently to $O(1/c^3)$. With regard to the self-force this is essentially equivalent to taking both the dipole approximation and neglecting the $A^2$ term in the Hamiltonian. At $O(1/c^3)$, we find that there is no constraint on the cutoff and the backreaction is fully insensitive to particle structure. In contrast, in [8, 4] the dipole approximation is made to the full Hamiltonian including the $A^2$ term, and instead a particle-structure form factor is assumed, which gives an effective cutoff in the particle–field interaction. The equations of motion are then order reduced, effectively to $O(1/c^3)$, and the resulting structure-independent equations of motion are found to be runaway free. The ultimate agreement between these approaches is consistent with the fact that the contribution to backreaction from the $A^2$ term in the Hamiltonian is effectively negligible in the regime where the two methods of approximation are valid.

Although it is already known by correspondence that (81) is runaway free and causal, we now reanalyze these properties within the present framework to see how this physics arises from within a purely effective theory framework. We will show that the dynamics of the open system are dissipative and stable in a manner analogous to the QBM analysis given in appendix A.5. Consider a single particle under the influence of some potential $U$ and denote the system Hamiltonian

$$H_{\text{sys}} = \frac{\mathbf{p}^2}{2m} + U(\mathbf{x}).$$

The following analysis can be easily extended to multiple-particle systems. An energy constraint can be obtained from either the Heisenberg equations of motion for $H_{\text{sys}}(t)$ or by integrating the (classical) Langevin equation (88) along with the second (89). Discarding the irrelevant transient terms (which can be shown to be bounded and runaway-free) we obtain the relation

$$H_{\text{sys}}(t) = H_{\text{sys}}(0) + H_{\gamma}(t) + H_{\xi}(t),$$

where

$$H_{\gamma}(t) = -\frac{e^2}{m^2} \frac{1}{6\pi \varepsilon_0 c^3} \int_0^t dt' \dot{\mathbf{p}}(t')^2$$

is the energy lost to damping and

$$H_{\xi}(t) = \frac{e}{m} \int_0^t dt' \frac{1}{2} \left\{ \mathbf{\xi}(t') \cdot \dot{\mathbf{p}}(t') + \dot{\mathbf{p}}(t') \cdot \mathbf{\xi}(t') \right\}$$

is the work done by the noise $\mathbf{\xi}(t')$. The contribution from damping is manifestly a negative quantity. The noise is random and may do positive or negative work, but the damping only removes energy from the system (and delivers it to the environment and interaction). At least at this order, the damping is strictly local and so energy is lost to dissipation in a strictly uniform manner.

It is important to note that the system here is given by the canonical variables $(\mathbf{x}, \mathbf{p})$, and $H_{\text{sys}}$ does not correspond to the mechanical energy of the particle, for the same reason that $\mathbf{p}^2/2m$ is not the mechanical kinetic energy. From equation (21) the velocity and momentum differ by the vector potential, which includes both backreaction and noise. If
the system momentum $\mathbf{p}$ relaxes under dissipative motion, then so does $\dot{\mathbf{p}}$ and by extension the backreaction. Given noise, the system velocity fluctuates around the average

$$\langle m \mathbf{v} \rangle = \left( \mathbf{p} + \frac{e^2}{m} \frac{1}{6\pi \varepsilon_0 c^3} \mathbf{p} \right).$$

(84)

implying that the system velocity is damped. If no external forces are applied, the canonical and mechanical momenta approach each other (on average) in the late-time limit.

4.2. Mass renormalization

In our Langevin equations (75)–(76), the $\mathbf{p}^2/2m$ mass renormalization takes the form of magnetostatic self-interaction, which is ordinarily discarded in classical electrodynamics. Here we examine the counter terms involved in the renormalization and contrast them to the standard mass renormalization previously discussed in section 3.5.

Consider most simply the single-particle theory in the dipole approximation, with some potential $U$. The resultant open-system equations of motion are then given by

$$\dot{\mathbf{x}}(t) = \frac{\mathbf{p}(t)}{m_{\text{ren}}} + \frac{e^2}{m_{\text{bare}}^2} \frac{1}{6\pi \varepsilon_0 c^3} \mathbf{p}(t) - \frac{e}{m_{\text{bare}}} \xi(t),$$

(85)

$$\dot{\mathbf{p}}(t) = -\nabla U(\mathbf{x}),$$

(86)

in terms of the renormalized mass

$$\frac{1}{m_{\text{ren}}} = \frac{1}{m_{\text{bare}}} + 2 \frac{e^2}{m_{\text{bare}}^3} \gamma(0).$$

(87)

Consistent with this order of perturbative analysis, we may express our Langevin equations as

$$\dot{\mathbf{x}}(t) = \frac{\mathbf{p}(t)}{m_{\text{ren}}} + \frac{e^2}{m_{\text{ren}}^2} \frac{1}{6\pi \varepsilon_0 c^3} \mathbf{p}(t) - \frac{e}{m_{\text{ren}}} \xi(t),$$

(88)

$$\dot{\mathbf{p}}(t) = -\nabla U(\mathbf{x}).$$

(89)

In this perspective, the renormalization of the $\mathbf{p}^2/2m$ mass and the $e/m (\mathbf{p} \cdot \mathbf{A})$ mass enter at different orders.

It is well known that the instability of the Abraham–Lorentz equation arises from its implied negative bare mass for the system, which in turn comes about if the high-frequency cutoff $\Lambda$ (or reciprocal radius $c/r$) exceeds the characteristic frequency $\tau_m^{-1}$ [4, 9]. Yet to lowest order in $\tau_m$ (equivalently $1/c^3$), the dynamics of charged-particle motion is insensitive to the high-energy details of the theory and is thus not problematic in that regime [4]. It is therefore not surprising that our perturbative approach yields cutoff-insensitive behavior, however, the manner in which cutoff sensitivity is avoided is interesting. Whereas the standard radiation-reaction calculations involve a mass renormalization of $m\mathbf{v}^2/2$ given by equation (87), which runs the bare mass to negative infinity in the high cutoff limit, our calculation runs the bare mass to positive zero in the high cutoff limit, and no pathological behavior is induced for any finite cutoff.
4.3. Relativistic kinematics

For consistency in our quasirelativistic expansion of $1/c$, we must include the relativistic corrections to the single-particle kinetic energy, as is standard in the Darwin Hamiltonian. Expanding the relativistic kinetic energy gives

$$H_{\text{rel}} = c\sqrt{(mc)^2 + (p - eA)^2},$$

$$\approx mc^2 + \frac{(p - eA)^2}{2m} - \frac{(p - eA)^4}{8m^3c^2} + \cdots,$$

keeping terms of order $O(1/c^2)$. In the absence of external fields, this generates a second-order correction to equation (75) giving

$$\dot{x}(t) = \left[1 - \frac{1}{2} \left(\frac{p(t)}{mc}\right)^2\right] \frac{p(t)}{m} + \frac{e^2}{m^2} \frac{1}{6\pi\varepsilon_0 c^3} \dot{p}(t) - \frac{e}{m} \xi(t),$$

and otherwise the Langevin equations are identical. As in the Darwin Hamiltonian, this relativistic correction must be considered perturbatively. At the present order of perturbation theory, however, we can resum the term into the free velocity

$$\left[1 - \frac{1}{2} \left(\frac{p}{mc}\right)^2\right] \frac{p}{m} = \frac{p}{\sqrt{1 + (p/m)^2}} + O\left(\frac{1}{c^3}\right),$$

which is more convenient and better behaved in the equations of motion. This can also be done in the effective Hamiltonian to ensure that the energy spectrum has a lower bound.

Note that even nonperturbatively these kinematic corrections included in the stability analysis given in section 4.1 will still yield dissipative backreaction, as all of these terms commute with the $A \cdot p$ interaction at the relevant order. The only modification will be in the definition of the canonical system Hamiltonian (now relativistic), and the noise average of the velocity will be given by

$$\langle m \dot{v} \rangle_\xi = \left\langle \frac{p}{\sqrt{1 + (p/m)^2}} + \frac{e^2}{m \cdot \varepsilon_0 c^3} \dot{p} \right\rangle_\xi.$$

4.4. Electromagnetic damping is relativistic

The standard Abraham–Lorentz equation is commonly referred to as ‘nonrelativistic’. Within the framework of a $1/c$ expansion, this equation is more accurately described as quasirelativistic. From this perspective, the damping force, which is $O(1/c^3)$, is intrinsically a relativistic correction to the particle dynamics. Let us re-express the $(v/c)^2$ magnetostatic and $1/c^3$ damping forces in equation (75) as both arising from dynamical generators

$$\dot{x}_i = \frac{p_i - e_i \xi}{m_i} + \nabla_s \left\{ \Gamma_i + \sum_{j \neq i} U^\#_{ij} \right\},$$

$$\Gamma_i = \frac{1}{12\pi e_i c^3} \frac{d}{dt} \left( \frac{e_i}{m_i} p_T - e_i p_i \right).$$

By comparison with equations (79) and (80), if the magnetostatic potential $U^\#$ is considered $(v/c)^2$, then the damping generator $\Gamma$ is of relative order $(r/c)(d/dt)(v/c)^2$. Thus the damping force can be interpreted as a more dynamical and more relativistic correction. Given that both
of these generators arise from the same integration kernel without the full dipole approximation (see section 3.1), a higher-order ‘post-dipole’ expansion in $1/c$ can expect to include many more such terms of higher order. The relativistic nature of the expansion terms will be even clearer at higher orders.

5. Summary and discussions

We have derived new stochastic equations of motion (75)–(76) and (92) for multiple charged particles in the electromagnetic field. These equations of motion incorporate the known relativistic corrections to the electrodynamics of spinless charged particles to order $\mathcal{O}(1/c^3)$, including the electrostatic, magnetostatic, electromagnetic damping forces and field fluctuations. Moreover the equations of motion are manifestly causal and runaway-free. Our analysis shows that a $1/c$ expansion to the Coulomb-gauge Hamiltonian describes consistent and well-behaved nonequilibrium electrodynamics for spinless charged particles. Our consistent-order equations of motion have a close correspondence with the order-reduced Eliezer equation, which is known to be well behaved. Whereas traditionally the Abraham–Lorentz equation has been considered an ‘exact nonrelativistic’ equation, from the perspective of our analysis, pathologies in the standard Abraham–Lorentz equation are associated with inconsistent, mixed order in $1/c$, approximations. Our view is that radiation reaction is best thought of as intrinsically relativistic, though the Abraham–Lorentz equation may only fully capture its effect to lowest order in $1/c$, and the order reduction used in deriving the Eliezer equation actually serves to rid the dynamics of inappropriate mixed-order contributions.

At $\mathcal{O}(1/c^3)$, mass renormalization is identified with the magnetostatic self-interaction and, at this order, the bare mass is positive for all cutoffs and the backreaction is fully insensitive to particle structure. An interesting question for future research is whether the cutoff insensitivity is preserved at higher orders in the $1/c$ expansion, with and without the inclusion of spin and other features from full QED. For a single particle, we see that only at $\mathcal{O}(1/c^4)$ does the (dipole approximation to the) $A(x)^2$ interaction play a role, but at the same order post-dipole terms in the $p \cdot A(x)$ interaction must also be included for consistency. Because the traditional (mixed-order) analysis show that inclusion of the $A(x)^2$ interaction plus dipole approximation results in pathological equations of motion, it will be very interesting to see in detail whether including the $\mathcal{O}(1/c^4)$ post-dipole terms in $p \cdot A(x)$ resolves these pathologies automatically, giving runaway-free behavior for any cutoff (i.e. full particle-structure insensitivity), or if a finite (or bounded) cutoff will also be required to avoid pathologies at higher order. Although it will be complicated, our method provides a systematic method to investigate this question.

Also of interest is the comparison of our methods and results to those that can be obtained from super-operator [24, 25] and influence-functional [26, 27] formalisms, which provide a more natural framework for fully relativistic physics. Future work should also incorporate spin degrees of freedom. Relativistic effects like particle creation, however, will probably be more naturally described by a relativistic matter field. The advantage of the formalism presented here, in contrast, is that it naturally describes particle trajectories within a familiar nonrelativistic framework but allows for the systematic and consistent inclusion of higher-order, relativistic corrections.

In conclusion, our results show that a $1/c$ effective-theory expansion provides a powerful method for making a consistent, order-by-order approximation to backreaction and interactions between multiple charged particles, within a nonequilibrium yet Hamiltonian framework that incorporates a well-defined description of noise induced by the electromagnetic field. To order $1/c^3$, the expansion automatically generates both the correct, causal and cutoff-insensitive runaway-free radiation reaction and particle–particle interactions, and consistent noise. These
results make it promising that the $1/c$ expansion will also correctly capture the physics at higher orders. Finally, applications of the results in this paper also allow for self-consistent studies of the nonequilibrium dynamics of ensembles of charged particles.

Acknowledgments

We would like to thank Albert Roura for discussions of his related work on the radiation-reaction problem. CHF and BLH are supported partially by the National Science Foundation under grant PHY-0801368 to the University of Maryland. PRJ acknowledges support from the US Army Research Office under Grant 60661PH and the Research Corporation for Science Advancement.

Appendix. Quantum Brownian motion

We begin our discussion with the QBM Lagrangian which we have adapted in form and notation to better mirror the problem of backreaction in the electromagnetic field. This Lagrangian describes a quantum system bilinearly coupled to a bosonic bath of harmonic oscillators and is traditionally used to model ordinary motional damping in quantum mechanics:

$$\mathcal{L}_{\text{QBM}} = \mathcal{L}_{\text{sys}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{env}},$$ (A.1)

$$\mathcal{L}_{\text{sys}} = \frac{1}{2} m \dot{x}^2 - U(x),$$ (A.2)

$$\mathcal{L}_{\text{int}} = e x \cdot \mathbf{Q},$$ (A.3)

$$\mathcal{L}_{\text{env}} = \int_0^\infty dk \left\{ \frac{\dot{q}_k^2}{2} - \omega_k^2 q_k^2 \right\},$$ (A.4)

where $x$ denotes the system position, $q_k$ denote the field-mode ‘positions’ and $\mathbf{Q}$ is the collective field operator

$$\mathbf{Q} = \int_0^\infty dk g_k \mathbf{q}_k.$$ (A.5)

The system + environment Hamiltonian is then given by

$$\mathcal{H}_{\text{QBM}} = \mathcal{H}_{\text{sys}} + \int_0^\infty dk \left\{ \frac{\pi_k^2}{2} + e g_k x \right\}^2 + \omega_k^2 q_k^2 \right\},$$ (A.6)

$$\mathcal{H}_{\text{sys}} = \frac{p^2}{2m} + U(x),$$ (A.7)

where, as determined by the gauge of our Lagrangian, $p$ is the system momentum conjugate to $x$ and $\pi$ is the field ‘momentum’ conjugate to $\mathbf{q}$.

Note that for $m \geq 0$ and $U(x)$ sufficiently well behaved, Hamiltonian (A.6) is bounded from below in its energy spectrum. Therefore, under these conditions runaway solutions will not occur when the environment is initially described by a thermal state.

Additionally note that the ‘bare’ system potential in equation (A.6) is given by

$$U_{\text{bare}}(x) = U(x) + \left( e^2 \int_0^\infty dk \frac{g_k^2}{2} \right) x^2,$$ (A.8)

and that the system + environment Hamiltonian can also be expressed as

$$\mathcal{H}_{\text{QBM}} = \mathcal{H}_{\text{sys}} - e x \cdot \pi + \int_0^\infty dk \left\{ \frac{\pi_k^2}{2} + \omega_k^2 q_k^2 \right\},$$ (A.9)
in terms of the collective field operator
\[
\pi = \int_0^\infty dk g_k \pi_k.
\]

The resulting Heisenberg equations of motion then dictate that the system is driven by the field
\[
\dot{x} = \frac{p}{m}, \quad (A.9)
\]
\[
\dot{p} = -\nabla U_{\text{bare}}(x) + e\pi, \quad (A.10)
\]
whereas the field modes are driven by the system
\[
\dot{q}_k = \pi_k - eg_k x, \quad (A.11)
\]
\[
\dot{\pi}_k = -\omega_k^2 q_k. \quad (A.12)
\]

Solving for the field-mode evolution as driven by the system, we obtain the homogeneous + driven solution
\[
\pi_k(t) = \xi_k(t) + \frac{eg_k \omega_k^2}{2} \int_0^t dt' \xi_k(t'),
\]
\[
\xi_k(t) \equiv \dot{G}_k(t) \pi_k(0) + \dot{G}_k(t) q_k(0), \quad (A.14)
\]
\[
G_k(t) \equiv \frac{\sin(\omega_k t)}{\omega_k}, \quad (A.15)
\]
where the * product denotes the Laplace convolution
\[
(A*B)(t) \equiv \int_0^t dt' A(t-t') B(t'). \quad (A.16)
\]
The time-evolving field operator is then given by
\[
\pi(t) = \xi(t) - \frac{2e(\mu * x)(t)}{\text{noise}}, \quad (A.17)
\]
\[
\xi(t) \equiv \int_0^\infty dk g_k \xi_k(t), \quad (A.18)
\]
\[
\mu(t) \equiv -\int_0^\infty dk g_k^2 \omega_k^2 \int_0^t dt' \xi_k(t'), \quad (A.19)
\]
where \(\mu(t, t') = \mu(t-t')\) is the stationary dissipation kernel and \(\xi(t)\) is a Gaussian stochastic process for the initial conditions we assume: a factorized state of the system and environment, with the environment in a thermal state.

Next we introduce the related damping kernel
\[
\mu(t, t') = -\frac{\partial}{\partial t'} \gamma(t, t'), \quad (A.20)
\]
\[
\gamma(t, t') \equiv \int_0^\infty dk g_k^2 \frac{\omega_k^2}{2} \cos[\omega_k(t-t')], \quad (A.21)
\]
which is necessarily positive definite and independent of the (factorized) initial state of the environment. The backreaction can then be expressed

\[ (\mu \ast x)(t) = (\gamma \ast \dot{x})(t) + \gamma(t, 0) x(0) - \gamma(t, t) x(t), \]

(A.22)

in terms of the positive-definite damping and where we have labeled the terms corresponding to the renormalization and initial short-time slip dynamics associated with the factorization of the initial state (see appendix A.3).

Substituting our field solutions into the system equations of motion, we obtain the quantum Langevin equation

\[ m \ddot{x}(t) + 2e^2 (\gamma \ast \dot{x})(t) + \nabla U(x) = e \xi(t) - e^2 \gamma(t) x(0), \]

which reduces to

\[ m \ddot{x}(t) + 2e^2 (\gamma \ast \dot{x})(t) + \nabla U(x) = e \xi(t), \]

(A.23)
after the transient slip is taken into account.

A.1. Ohmic coupling and local damping

Considering the damping kernel, which is given by

\[ \gamma(t) = \int_0^{\infty} dk \frac{g^2}{2} k \cos(\omega_k t). \]

(A.24)

If assume \( \omega_k = c k \) and \( g_k \approx g \) up to some high-frequency cutoff \( \Lambda \), then we may evaluate the integral as

\[ \gamma(t) = \frac{g^2}{2c} \int_0^\Lambda d\omega \cos(\omega t) = \frac{g^2}{2c} \sin(\Lambda t) \frac{t}{\Lambda}, \]

(A.25)

The damping kernel may then be expressed

\[ \gamma(t) = 2\gamma_0 \delta_\Lambda(t), \]

(A.26)

\[ \gamma_0 \equiv \frac{\pi g^2}{4c}, \]

(A.27)

\[ \delta_\Lambda(t) \equiv \frac{\sin(\Lambda t)}{\pi t}, \]

(A.28)
in terms of the Dirac delta \( \delta_\Lambda(t) \). In the high-frequency limit, the damping contribution to the Langevin equation becomes

\[ \lim_{\Lambda \to \infty} (\gamma \ast \dot{x})(t) = \gamma_0 \dot{x}(t), \]

(A.29)
or local damping.

A.2. Renormalization

For Ohmic coupling or local damping the quantum Langevin equation described by equation (A.23) is phenomenological, in the sense that its various parameters correspond to the physically measurable parameters at low energy. Assuming the Langevin equation to be phenomenological, note that the bare system potential in Hamiltonian perspective (A.8) and (A.10) as compared to the phenomenological system potential \( U(x) \) is

\[ U_{\text{bare}}(x) = U(x) + 2e^2 \gamma(0) x^2, \]

where \( \gamma(0) = \frac{g^2}{2} \Lambda \) for local damping with a hard cutoff regulator. The renormalization is a quadratic term, regardless of whether or not the original model contained such a term. The QBM model typically proceeds from an \( x \cdot Q \) interaction, where this renormalization does not naturally result from the Lagrangian theory.
A.3. Factorized initial conditions

If the operator noise $\xi(t)$ in our Langevin equation is to be sampled from a thermal distribution which is (initially) statistically independent from the system, then the initial state of the system and environment must be a product state of the form $\rho_{\text{sys}+\text{env}} = \rho_{\text{sys}} \otimes \rho_{\text{env}}$, or a product of marginal phase-space distributions in the classical regime, and with the environment initially in equilibrium. This is an important simplification in our (and most other, e.g., [28–30]) analyses of the nonequilibrium dynamics of open systems.

The consequence of assuming an initially uncorrelated system and environment must be carefully examined when studying radiation reaction, however, because acausal behaviors arise during the same very short time scale where the unphysical nature of a factorized state is relevant. It is therefore an important aspect of our analysis that we are also able to apply recent results [22] showing that for classical or high-temperature electromagnetic noise ($\hbar \omega_{\text{sys}} \ll k_B T$ in equation (A.32)) the initial evolution of factorized states (or distributions) quickly leads to physical, dressed particle states without reintroducing the pathologies or instabilities in the dynamics that our analysis is intended to avoid. In the semiclassical or quantum regime, use of properly correlated initial states can mitigate the unphysical aspects of assuming initially factorized states entirely, without otherwise spoiling the results in this paper [31].

A.3.1. The slip. The transient slip in our Langevin equation is an initial-time pathology associated with vanishing correlations in the factorized initial conditions despite non-vanishing interaction strength between the system and field. In addition to the slip, there is a diffusive initial-time pathology, called jolts, which arise from correlation with the (quantum) zero-point fluctuations of the environment. The slip in particular was thoroughly analyzed in [22], where it was pointed out to generate the linear dynamical map

$$\rho \rightarrow e^{\frac{\mathrm{i}}{2}\gamma_0 x^2} \rho e^{-\frac{\mathrm{i}}{2}\gamma_0 x^2},$$

(A.30)

which maps states in a unitary fashion and preserves all kinematic moment invariants [32], including the uncertainty function. Therefore one can identify the post-slip state as a 'renormalized' initial state which is more properly correlated with the environment and the pre-slip state as a 'bare' initial state. If one only considers the classical regime, then jolting is not severe due to the lack of zero-point fluctuations in the environment. Moreover, for a classical zero-temperature environment there is no noise causing any diffusion. Thus for this case one only needs to consider the renormalized initial states, effectively discarding the slip term entirely.

A.4. The fluctuation–dissipation relation

The Gaussian process $\xi(t)$ has the noise kernel

$$\nu(t, t') = \frac{1}{2} \langle [\xi(t), \quad \xi(t')] \rangle_{\xi} = \nu(t - t'),$$

(A.31)

which is stationary and positive definite for any stationary initial state of the environment. For an equilibrium initial state of the environment the noise kernel is related to damping kernel by the (quantum) fluctuation–dissipation relation (FDR)

$$\tilde{\nu}(\omega) = \tilde{\gamma}(\omega) \hbar \omega \coth \left( \frac{\hbar \omega}{2 \kappa_B T} \right),$$

(A.32)

where \( \tilde{\gamma}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \ e^{-\mathrm{i}\omega t} \gamma(t) \). Essentially, the damping kernel and temperature completely characterize Gaussian, thermal noise.
Also note that as the coupling and environment are dynamically linear, the damping kernel, being determined by the commutator, is independent of the state of the environment and it is the same whether in the classical or quantum regimes. In the classical regime we have the limit
\[ \lim_{\hbar \to 0} \tilde{\nu}(\omega) = \tilde{\gamma}(\omega) \sqrt{\frac{k_B T}{2}}, \] (A.33)
and the classical fluctuations vanish in the zero-temperature limit. In this limit (the classical vacuum) we can neglect the stochastic process \(\xi(t)\).

In the quantum regime, the anti-commutator expectation value (A.31) is not sufficient to describe the statistics of the operator-valued stochastic process \(\xi(t)\). One additionally requires the commutator expectation value
\[ \mu(t, t') = \frac{1}{2i\hbar} \langle [\xi(t), \xi(t')] \rangle = \mu(t-t'), \] (A.34)
which is given by the dissipation kernel, a state-independent quantity. Here the dissipation kernel is not generating backreaction, but consistent time evolution for the non-commuting stochastic process. The full quantum correlation is therefore given by
\[ \alpha(t, t') \equiv \langle \pi(t)\pi(t') \rangle_{\text{env}} = \langle \xi(t)\xi(t') \rangle_{\xi}, \] (A.35)
\[ = v(t-t') + i\hbar\mu(t-t'), \] (A.36)
where \(\pi(t)\) denotes the interaction-picture or Dirac-picture field operator and not the Heisenberg-picture field operator which we have already denoted \(\pi(t)\).

A.5. Stability analysis

We will now show that the dynamics of the system are dissipative and stable under the very same conditions for which the system + environment Hamiltonian (A.6) has a lower bound in its energy spectrum. Let us denote the canonical system Hamiltonian
\[ H_{\text{sys}} \equiv \frac{\mathbf{p}^2}{2m} + U(x). \] (A.37)
One may then calculate an energy constraint from either the Heisenberg equations of motion for \(H_{\text{sys}}(t)\) or by integrating the (classical) Langevin equation (A.23) along with velocity. Accounting for the slip in our initial state, which only produces a finite change in energy, we obtain the relation
\[ H_{\text{sys}}(t) = H_{\text{sys}}(0) - H_{\gamma}(t) + H_{\xi}(t), \] (A.38)
in terms of the energy lost to damping \(H_{\gamma}(t)\) and energy generated by noise \(H_{\xi}(t)\)
\[ H_{\gamma}(t) = e^2 \int_0^t dt' \int_0^{t'} \gamma(t', t'') \frac{1}{2} \left\{ \dot{x}(t') \cdot \dot{x}(t'') + \dot{x}(t'') \cdot \dot{x}(t') \right\}, \] (A.39)
\[ H_{\xi}(t) = e^2 \int_0^t dt' \frac{1}{2} \left\{ \xi(t') \cdot \dot{x}(t') + \dot{x}(t') \cdot \xi(t') \right\}. \] (A.40)
The contribution from damping is a negative quantity as the damping kernel is a positive-definite kernel in a quadratic form. The noise is random and may drive the system erratically, but the damping may only remove energy from the system (and deliver it to the environment and interaction). Therefore it is imperative that \(H_{\text{sys}}\) have a lower bound in its energy spectrum. For our model, this necessarily implies that the system + environment Hamiltonian (A.6) also has a lower bound in its energy spectrum. If this is the case then true runaway motion cannot
occur. In the classical-vacuum limit, energy is continually siphoned from $H_{\text{sys}}(t)$ until all motion ceases.

Locally damped energy is additionally simplified to

$$\dot{H}_{\gamma}(t) = 2e^2 \gamma x(t)^2,$$

(A.41)

which monotonically dissipates energy in time. Nonlocal damping can produce an instantaneous increase in system energy, though the cumulative effect is always dissipative.

References

[1] Landau L D and Lifshitz E M 1962 The Classical Theory of Fields 2nd edn (London: Pergamon)
[2] Jackson J D 1998 Classical Electrodynamics 3rd edn (New York: Wiley)
[3] Eliezer C J 1948 Proc. R. Soc. A 194 543
[4] Ford G and O’Connell R 1991 Phys. Lett. A 157 217
[5] Yaghjian A D 1992 Relativistic Dynamics of a Charged Sphere: Updating the Lorentz–Abraham Model (Berlin: Springer)
[6] Rohrlich F 1997 Am. J. Phys. 65 1051
[7] Medina R 2006 J. Phys. A: Math. Gen. 39 3801
[8] Ford G W, Lewis J T and O’Connell R F 1988 Phys. Rev. A 37 4419
[9] Moniz E J and Sharp D H 1977 Phys. Rev. D 15 2850
[10] Lorentz H A and Droste J 1917 Versl. K. Akad. Wet. Amsterdam 26 392
[11] Lorentz H A and Droste J 1917 Versl. K. Akad. Wet. Amsterdam 26 649
[12] Einstein A, Infeld L and Hoffmann B 1938 Ann. Math. 39 65
[13] Blanchet L 2006 Living Rev. Rel. 9 4
[14] Darwin C G 1920 Phil. Mag. 39 537
[15] Dirac P A M 1938 Proc. R. Soc. A 167 148
[16] Rivera R and Villarroel D 2002 Phys. Rev. E 66 046618
[17] Bhabha H 1939 Proc. Math. Sci. 10 324
[18] Breit G 1929 Phys. Rev. 34 553
[19] Breit G 1930 Phys. Rev. 36 383
[20] Dalibard J, Dupont-Roc J and Cohen-Tannoudji C 1982 J. Phys. France 43 1617
[21] Barone P M V B and Caldeira A O 1991 Phys. Rev. A 43 57
[22] Fleming C H, Roura A and Hu B L 2011 Ann. Phys. 326 1207
[23] Hsiang J T, Wu T H and Lee D S 2011 Found. Phys. 41 77
[24] Díosi L 1990 Found. Phys. 20 63
[25] Breuer H P and Petruccione F 2000 Relativistic Quantum Measurement and Decoherence (Lecture Notes in Physics vol 559) ed H P Breuer and F Petruccione (Berlin: Springer) pp 31–65 (arXiv:quant-ph/0210013)
[26] Anastopoulos C and Zoupas A 1998 Phys. Rev. D 58 105006
[27] Haba Z and Kleinert H 2001 Eur. Phys. J. B 21 553
[28] Feynman R P and Vernon F L 1963 Ann. Phys. 24 118
[29] Caldeira A O and Leggett A J 1983 Physica A 121 587
[30] Hu B L, Paz J P and Zhang Y 1992 Phys. Rev. D 45 2584
[31] Fleming C H, Roura A and Hu B L 2011 Phys. Rev. E 84 021106
[32] Dragt A J, Neri F and Rangarajan G 1992 Phys. Rev. A 45 2572