LINEAR MEASURE AND $K$-QUASICONFORMAL HARMONIC MAPPINGS

SHAOLIN CHEN, GANG LIU, AND SAMINATHAN PONNUSAMY

Abstract. In this paper, we investigate the relationships between linear measure and harmonic mappings.

1. Preliminaries and main results

For $a \in \mathbb{C}$ and $r > 0$, we let $D(a, r) = \{ z : |z - a| < r \}$ so that $D_r := D(0, r)$ and thus, $\mathbb{D} := D_1$ denotes the open unit disk in the complex plane $\mathbb{C}$. Let $\partial \mathbb{D}$ be the boundary of $\mathbb{D}$. For a real $2 \times 2$ matrix $A$, we use the matrix norm $\| A \| = \sup\{ |Az| : |z| = 1 \}$ and the matrix function $\lambda(A) = \inf\{ |Az| : |z| = 1 \}$. For $z = x + iy \in \mathbb{C}$, the formal derivative of the complex-valued functions $f = u + iv$ is given by

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

so that $\| Df \| = |f_z| + |f_{\bar{z}}|$ and $\lambda(Df) = ||f_z| - |f_{\bar{z}}||$, where $f_z = (1/2)(f_x - if_y)$ and $f_{\bar{z}} = (1/2)(f_x + if_y)$ are partial derivatives.

Let $\Omega$ be a domain in $\mathbb{C}$, with non-empty boundary. A sense-preserving homeomorphism $f$ from a domain $\Omega$ onto $\Omega'$, contained in the Sobolev class $W^{1,2}_{loc}(\Omega)$, is said to be a $K$-quasiconformal mapping if, for $z \in \Omega$,

$$|Df(z)||^2 \leq K \det Df(z), \text{ i.e., } \| Df(z) \| \leq K\lambda(Df(z)),$$

where $K \geq 1$ and $\det Df$ denotes the determinant of $Df$ (cf. [9, 12, 18]). We note that $\det Df = |f_z|^2 - |f_{\bar{z}}|^2$, the Jacobian of $f$, is usually denoted by $J_f$.

A complex-valued function $f$ defined in a simply connected subdomain $G$ of $\mathbb{C}$ is called a harmonic mapping in $G$ if and only if both the real and the imaginary parts of $f$ are real harmonic in $G$. It is well known that every harmonic mapping $f$ in $G$ admits a decomposition $f = h + \overline{g}$, where $h$ and $g$ are analytic in $G$. Throughout we use this representation. Without loss of generality, we assume $0 \in G$. If we choose the additive constant such that $g(0) = 0$, then the decomposition is unique. Because $J_f = |h'|^2 - |g'|^2$, it follows that $f$ is locally univalent and sense-preserving in $G$ if and only if $|g'(z)| < |h'(z)|$ in $G$; or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property that $|\omega(z)| < 1$ in $G$ (see [6] and also [13]).

2010 Mathematics Subject Classification. Primary: 30C65, 30C75; Secondary: 30C20, 30C45, 30H10.

Key words and phrases. $K$-quasiconformal harmonic mappings, area, length distortion.
Let $\gamma : \varphi(t), \ t \in [\alpha, \beta]$, be a curve in $\mathbb{C}$. Its length $\ell(\gamma)$ is defined by

$$(1.1) \quad \ell(\gamma) = \sup \sum_{k=1}^{n} |\varphi(t_k) - \varphi(t_{k-1})|,$$

where the supremum is taken over all partitions $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ and all $n \in \{1, 2, \ldots \}$. We call $\gamma$ rectifiable if $\ell(\gamma) < +\infty$. It is clear from (1.1) that $\text{diam} \gamma \leq \ell(\gamma)$. In the case of a closed curve $\gamma : \psi(\zeta), \ \zeta \in \mathbb{T}$, with piecewise continuously differentiable $\psi$, we can write

$$\ell(\gamma) = \int_{\mathbb{T}} |\psi'(\zeta)| \, |d\zeta|,$$

where $\mathbb{T} = \partial \mathbb{D}$. If $\gamma_1, \gamma_2, \ldots$ are disjoint curves, then we define

$$\ell(\bigcup_{k=1}^{\infty} \gamma_k) = \sum_{k=1}^{\infty} \ell(\gamma_k).$$

In particular, $\ell(\emptyset) = 0$ (see [16, p. 3]).

For $p \in (0, \infty]$, the generalized Hardy space $H^p_g(\mathbb{D})$ consists of all those functions $f : \mathbb{D} \to \mathbb{C}$ such that $f$ is measurable, $M_p(r, f)$ exists for all $r \in (0, 1)$ and $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup_{0 < r < 1} M_p(r, f) & \text{if } p \in (0, \infty) \\ \sup_{z \in \mathbb{D}} |f(z)| & \text{if } p = \infty \end{cases}, \quad \text{and} \quad M^p_p(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

**Proposition 1.** Let $f$ be a $K$-quasiconformal harmonic mapping of $\mathbb{D}$ onto an inner domain of Jordan curve $\gamma$. Then $\ell(\gamma) < +\infty$ if and only if $\|D_f\| \in H^1_g(\mathbb{D})$.

**Proof.** Assume that $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\mathbb{D}$. We first prove the necessity. Let $\ell(\gamma) < +\infty$. Then, by Lemma B (in Section 2), we see that

$$F_n(z) = \sum_{k=1}^{n} |f(z e^{2\pi i k/n}) - f(z e^{2\pi (k-1)/n})|$$

is subharmonic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. It follows from the maximum principle and (1.1) that $F_n(z) \leq \ell(\gamma)$ for $z \in \mathbb{D}$, and thus, for $r \in [0, 1)$, we have

$$\frac{r}{K} \int_{0}^{2\pi} \|D_f(re^{it})\| \, dt \leq r \int_{0}^{2\pi} \left| h'(re^{it}) - e^{-2it} \overline{g'(re^{it})} \right| \, dt = \int_{0}^{2\pi} |df(re^{it})| = \lim_{n \to \infty} F_n(r) \leq \ell(\gamma),$$

which implies that $\|D_f\| \in H^1_g(\mathbb{D})$.

Next, we prove the sufficiency part. For $r \in [0, 1)$, let $\gamma_r = \{f(re^{i\theta}) : \theta \in [0, 2\pi]\}$. Since $|zh'(z) - \overline{g'}(z)|$ is subharmonic in $\mathbb{D}$, we see that

$$\ell(\gamma_r) = r \int_{0}^{2\pi} \left| h'(re^{it}) - e^{-2it} \overline{g'(re^{it})} \right| \, dt$$

In particular, $\ell(\emptyset) = 0$ (see [16, p. 3]).
is an increasing function of \( r \) on \([0, 1)\). By calculations, we get
\[
\ell(\gamma_r) \leq \int_0^{2\pi} \| D_f(re^{it}) \| \, dt \leq \sup_{0 < r < 1} \int_0^{2\pi} \| D_f(re^{it}) \| \, dt < +\infty,
\]
which, together with the monotonicity, yields that \( \lim_{r \to 1^-} \ell(\gamma_r) \) does exist and thus,
\[
\ell(\gamma) \leq \lim_{r \to 1^-} \ell(\gamma_r) < +\infty,
\]
as desired. \( \square \)

In [11], Lavrentiev proved that if \( f \) maps \( \mathbb{D} \) conformally onto the inner domain of Jordan curve \( \gamma \) of finite length, then, for any \( E \subset \gamma \), \( \ell(E) > 0 \) implies that \( \ell(f(E)) > 0 \). For univalent harmonic mappings, we obtain the following result.

**Theorem 1.** Let \( f \) be a sense-preserving and univalent harmonic mapping from \( \mathbb{D} \) onto a domain \( \Omega \) and \( \partial \Omega \) is a rectifiable Jordan curve. Furthermore, let \( E \subset \mathbb{T} \) is measurable with \( \ell(E) > 0 \). Then, we have
\[
\ell(f(E)) \geq \frac{\ell(\partial\Omega) \ell(E)}{2\pi - \ell(E)} \left[ \frac{(|f_z(0)| - |f_\bar{z}(0)|)(2\pi - \ell(E))}{\ell(\partial\Omega)} \right]^{\frac{2\pi}{\ell(E)}}.
\]
This estimate is sharp as \( \ell(E) \to 2\pi^- \) and the extreme univalent harmonic mapping is
\[
f(z) = \frac{M}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} e^{i\varphi(t)} \, dt
\]
satisfying \( |f_z(0)| - |f_\bar{z}(0)| = M \), where \( M \) is a positive constant and \( \varphi(t) \) is a continuously increasing function in \([0, 2\pi]\) with \( \varphi(2\pi) - \varphi(0) = 2\pi \).

**Corollary 1.** Under the hypotheses of Theorem 1, we have \( \ell(f(E)) > 0 \).

Let \( \gamma \) be a rectifiable Jordan curve. The shorter arc between \( z \) and \( w \) in \( \gamma \) will be denoted by \( \gamma[z, w] \). We say that \( \gamma \) is a \( M \)-Lavrentiev curve if there is a constant \( M > 1 \) such that \( \ell(\gamma[z, w]) \leq M|z - w| \) for each \( z, w \in \gamma \). The inner domain of a \( M \)-Lavrentiev curve is called a \( M \)-Lavrentiev domain (cf. [7, 9, 16, 19]).

The following result is considered to be a Schwarz-type lemma of subharmonic functions.

**Theorem A.** ([1, Theorem 2]) Let \( \phi \) be subharmonic in \( \mathbb{D} \). If, for all \( r \in [0, 1) \),
\[
A(r) = \sup_{\theta \in [0, 2\pi]} \int_0^r \phi(\rho e^{i\theta}) \, d\rho \leq 1,
\]
then \( A(r) \leq r \).

Analogy to Theorem A, applying some part of proof technique of [19, Lemma 1], we obtain a Schwarz type estimate on the length function of \( K \)-quasiconformal harmonic mappings.
\textbf{Theorem 2.} Suppose that \( f \) is a \( K \)-quasiconformal harmonic mapping of \( \mathbb{D} \) onto a \( M \)-Lavrentiev domain \( \Omega \). Then, for \( r \in (0, 2] \), \( \rho \in (0, r] \) and any fixed \( \zeta_0 \in \partial \mathbb{D} \),
\[
\int_0^r \ell(f(\Gamma_\rho)) \, d\rho \leq \sqrt{\frac{K \pi A(\Omega)}{3}} \frac{r^{3/2}}{\ell_{\Pi}^*(1 - 1/2)} \leq \sqrt{\frac{K \pi A(\Omega)}{3}} r^{3/2},
\]
where \( \alpha = 4/[K(1 + M)^2] \), \( A(\Omega) \) is the area of \( \Omega \) and \( \Gamma_\rho \) is the arc of the circle \( \partial \mathbb{D}(\zeta_0, \rho) \) which lies in \( \mathbb{D} \).

We say that a simply connected domain \( G \subset \mathbb{C} \) is \( M \)-linearly connected if, for any two points \( z_1, z_2 \in G \), there is a curve \( \gamma \subset G \) and a constant \( M \geq 1 \) such that
\[
\text{diam}\gamma \leq M|z_1 - z_2|.
\]

Zinsmeister [20] obtained an analytic characterization of \( M \)-Lavrentiev domains (see also [16, Chapter 7]). The following result is an analogous result to the analytic characterization of \( M \)-Lavrentiev domains.

\textbf{Theorem 3.} Let \( f \) be a \( K \)-quasiconformal harmonic mapping from \( \mathbb{D} \) onto the inner domain \( G \) of a rectifiable Jordan curve. If \( G \) is a \( M_1 \)-Lavrentiev domain and for each \( \zeta \in \partial \mathbb{D} \),
\[
\| D_f(\rho \zeta) \| \leq M'_1 \| D_f(r \zeta) \| \left( \frac{1 - \rho}{1 - r} \right)^{\delta - 1} \quad (0 \leq r \leq \rho < 1),
\]
then \( G \) is \( M_2 \)-linearly connected and, for all \( z \in \mathbb{D} \), there is a constant \( M' \) such that
\[
\frac{1}{\ell(I(z))} \int_{I(z)} \| D_f(\zeta) \| \, |d\zeta| \leq M' \| D_f(z) \|,
\]
where \( I(z) = \{ \zeta \in \mathbb{T} : |\arg \zeta - \arg z| \leq \pi(1 - |z|) \} \) and \( \delta \in (0, 1) \), \( M'_1, M_1, M_2 \) are constants.

For any fixed \( \theta \in [0, 2\pi] \), the radial length of the curve \( C_\theta(r) = \{ w = f(\rho e^{i\theta}) : 0 \leq \rho \leq r \} \) with counting multiplicity is defined by
\[
\ell^*_\theta(\theta, r) = \int_0^r |df(\rho e^{i\theta})| = \int_0^r |f_z(\rho e^{i\theta}) + e^{-2i\theta} f_\theta(\rho e^{i\theta})| \, d\rho,
\]
where \( r \in [0, 1) \) and \( f \) is a harmonic mapping defined in \( \mathbb{D} \). In particular, let
\[
\ell^*_\theta(\theta, 1) = \sup_{0 < r < 1} \ell^*_\theta(\theta, r).
\]

\textbf{Proposition 2.} Suppose that \( f \) is a bounded harmonic mapping in \( \mathbb{D} \) and \( r_0 \in (0, 1) \). For \( \zeta \in \mathbb{D} \), let \( F(\zeta) = f(r_0 \zeta) \). Then, for \( \rho \in [0, 1) \) and \( \theta \in [0, 2\pi] \),
\[
\ell^*_\theta(\theta, r) = \int_0^r r_0 |f_z(\rho r_0 e^{i\theta}) + e^{-2i\theta} f_\theta(\rho r_0 e^{i\theta})| \, d\rho \leq Mr,
\]
where \( r \in (0, 1) \) and \( M = \frac{2\alpha}{\pi} \sup_{z \in \mathbb{D}} |f(z)| \log((1 + r_0)/(1 - r_0)) \).
Proof. For \( \zeta \in \mathbb{D} \), let \( F(\zeta) = f(r_0 \zeta) \). By [5, Theorem 3], we get
\[
\ell_F^*(\theta, r) = \int_0^r r_0 |f_z(r_0 e^{i\theta}) + e^{-2i\theta} f_z(r_0 e^{i\theta})| \, d\rho \\
\leq \int_0^r r_0 \|D_f(r_0 e^{i\theta})\| \, d\rho \\
\leq \frac{4r_0 \sup_{z \in \mathbb{D}} |f(z)|}{\pi} \int_0^r \frac{d\rho}{1 - r_0^2 \rho^2} \\
= \frac{2r_0 \sup_{z \in \mathbb{D}} |f(z)|}{\pi} \log \left( \frac{1 + r_0 r}{1 - r_0 r} \right) \\
\leq \frac{2r_0 \sup_{z \in \mathbb{D}} |f(z)|}{\pi} \log \left( \frac{1 + r_0}{1 - r_0} \right) = M.
\]

By the subharmonicity of \( \|D_f(r_0 e^{i\theta})\| \) and Theorem A, we see that, for \( \theta \in [0, 2\pi] \),
\[
\ell_F^*(\theta, r) \leq Mr, \quad \text{where } r \in (0, 1).
\]

In [3], the authors obtained the coefficient estimates of a class of \( K \)-quasiconformal harmonic mapping with a finite perimeter length. In the following, we will investigate the coefficient estimates on a class of \( K \)-quasiconformal harmonic mapping with the finite radial length.

**Theorem 4.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n \) be a \( K \)-quasiconformal harmonic mapping on \( \mathbb{D} \). If, for all \( \theta \in [0, 2\pi] \), \( \ell_F^*(\theta, 1) \leq M \) for some positive constant \( M \), then
\[
|a_n| + |b_n| \leq KM \quad \text{for } n \geq 1,
\]
In particular, if \( K = 1 \), then the estimate (1.5) is sharp and the extreme function is \( f(z) = Mz \).

Let \( d_\Omega(z) \) be the Euclidean distance from \( z \) to the boundary \( \partial \Omega \) of the domain \( \Omega \). In the following, we investigate the behaviour on the ratio of the radial length and the perimeter length on \( K \)-quasiconformal harmonic mappings.

**Theorem 5.** Let \( f \) be a \( K \)-quasiconformal harmonic mapping of \( \mathbb{D} \) onto a bounded domain \( D \). Then, for \( r \in (0, 1) \) and all \( \theta \in [0, 2\pi] \),
\[
\frac{\ell_F^*(\theta, r)}{\ell(r)} \leq \frac{32r(1 + r)K^3 \sup_{z \in \mathbb{D}} |f(z)|}{\int_0^{2\pi} d_\mathbb{D}(f(re^{it})) \, dt}
\]
and
\[
\lim_{r \to 0^+} \left\{ \sup_{\theta \in [0, 2\pi]} \frac{\ell_F^*(\theta, r)}{\ell(r)} \right\} = 0.
\]

The proofs of Theorems 1, 2, 3, 4 and 5 will be presented in Section 2.
2. The proofs of the main results

The following lemmas are well-known.

**Lemma B.** (see [8], [9, Proposition 1.8] and [15]) If \( f = h + g \) is a \( K \)-quasiconformal harmonic mapping of \( \mathbb{D} \) onto a Jordan domain with a rectifiable boundary, then \( h \) and \( g \) have absolutely continuous extension to \( \mathbb{T} \).

**Lemma C.** (cf. [17]) Let \((\Omega, A, \mu)\) be a measure space such that \( \mu(\Omega) = 1 \). If \( g \) is a real-valued function that is \( \mu \)-integrable, and if \( \varphi \) is a convex function on the real line, then
\[
\varphi \left( \int_{\Omega} g \, d\mu \right) \leq \int_{\Omega} \varphi \circ g \, d\mu.
\]

**Lemma D.** (cf. [2]) Among all rectifiable Jordan curves of a given length, the circle has the maximum interior area.

**Proof of Theorem 1.** Let \( f = h + g \) be a sense-preserving and univalent harmonic mapping in \( \mathbb{D} \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). Then, \( h'(z) \neq 0 \) for \( z \in \mathbb{D} \) and the dilatation \( \omega \) defined by \( \omega = g'/h' \) is analytic and \( |\omega(z)| < 1 \) in \( \mathbb{D} \). Since \( \log |h'| \) is harmonic in \( \mathbb{D} \) and \( \log(1 - |\omega|) \) is subharmonic in \( \mathbb{D} \), we see that, for \( r \in (0, 1) \),
\[
\log |h'(0)| = \frac{1}{2\pi} \int_E \log |h'(rz)| |dz| + \frac{1}{2\pi} \int_{T \setminus E} \log |h'(rz)| |dz|
\]
and
\[
\log(1 - |\omega(0)|) \leq \frac{1}{2\pi} \int_E \log(1 - |\omega(rz)|) |dz| + \frac{1}{2\pi} \int_{T \setminus E} \log(1 - |\omega(rz)|) |dz|,
\]
which, together with Lemma C (Jensen’s inequality), imply that
\[
2\pi \log \left[ |h'(0)|(1 - |\omega(0)|) \right] \\
\leq \int_E \log \left[ |h'(rz)|(1 - |\omega(rz)|) \right] |dz| + \int_{T \setminus E} \log \left[ |h'(rz)|(1 - |\omega(rz)|) \right] |dz|
\]
\[
\leq \ell(E) \log \left[ \frac{1}{\ell(E)} \int_E (|h'(rz)| - |g'(rz)|) |dz| \right]
\]
\[
+ (2\pi - \ell(E)) \log \left[ \frac{1}{2\pi - \ell(E)} \int_{T \setminus E} (|h'(rz)| - |g'(rz)|) |dz| \right]
\]
\[
\leq \ell(E) \log \left[ \frac{1}{r\ell(E)} \int_E r|h'(rz) - g'(rz)\overline{z}^2| |dz| \right]
\]
\[
+ (2\pi - \ell(E)) \log \left[ \frac{1}{r[2\pi - \ell(E)]} \int_{T \setminus E} r|h'(rz) - g'(rz)\overline{z}^2| |dz| \right].
\]
By letting $r \to 1^-$ and Lemma B, we have

$$2\pi \log \left[ |h'(0)|(1 - |\omega(0)|) \right] \leq \frac{\ell(E) \log \ell(f(E))}{2\pi - \ell(E)} + \frac{(2\pi - \ell(E)) \log \ell(f(T \setminus E))}{(2\pi - \ell(E))} \leq \frac{\ell(E) \log \ell(f(E)) - \ell(E) \log \ell(E) - (2\pi - \ell(E)) \log(2\pi - \ell(E)) + (2\pi - \ell(E)) \log L,}$$

which implies that

$$\ell(f(E)) \geq \frac{L\ell(E)}{2\pi - \ell(E)} \left[ \frac{|h'(0)| - |g'(0)|}{(2\pi - \ell(E))} \right]^{\frac{2\pi}{2\pi - \ell(E)}},$$

where $\ell(\partial \Omega) = L$. Now we prove the sharpness part. By calculation, we have

$$\lim_{\ell(E) \to 2\pi^-} \frac{L\ell(E)}{2\pi - \ell(E)} \left[ \frac{|h'(0)| - |g'(0)|}{(2\pi - \ell(E))} \right]^{\frac{2\pi}{2\pi - \ell(E)}} = 2\pi(|h'(0)| - |g'(0)|).$$

Let

$$f(z) = \frac{M}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} e^{i\varphi(t)} \, dt$$

satisfying $|f_z(0)| - |f_\varphi(0)| = M$, where $M$ is a positive constant and $\varphi(t)$ is a continuously increasing function in $[0, 2\pi]$ with $\varphi(2\pi) - \varphi(0) = 2\pi$. If $\ell(E) \to 2\pi^-$, then the sense-preserving and univalent harmonic mapping $f$ of (2.2) shows that the estimate of (2.1) is sharp. \qed

**Proof of Theorem 2.** Let $f = h + \overline{g}$ satisfy the assumption, where $h$ and $g$ are analytic in $\mathbb{D}$. By Lemma B, we know that $h$ and $g$ can be absolutely continuous extension to $\mathbb{T}$. For $r \in (0, 2]$ and $\rho \in (0, r]$, let $\Delta_\rho = \{z : |z - \zeta_0| \leq \rho$ and $|z| \leq 1\}$. Let $\Gamma_\rho$ denote the arc of the circle $\partial \mathbb{D}(\zeta_0, \rho)$ which lies in $\mathbb{D}$. Then we have

$$\ell^2(f(\Gamma_\rho)) = \left( \int_{\Gamma_\rho} |f_z(\zeta_0 + \rho e^{it}) - e^{-2it} f_\varphi(\zeta_0 + \rho e^{it})| \rho \, dt \right)^2$$

$$\leq \left( \int_{\Gamma_\rho} \rho \, dt \right) \left( \int_{\Gamma_\rho} |f_z(\zeta_0 + \rho e^{it}) - e^{-2it} f_\varphi(\zeta_0 + \rho e^{it})|^2 \rho \, dt \right)$$

$$= \ell(\Gamma_\rho) \rho \int_{\Gamma_\rho} |f_z(\zeta_0 + \rho e^{it}) - e^{-2it} f_\varphi(\zeta_0 + \rho e^{it})|^2 \rho \, dt$$

$$\leq 2\rho^2 \arccos \frac{\rho}{2} \int_{\Gamma_\rho} \|D_f(\zeta_0 + \rho e^{it})\|^2 \rho \, dt$$

$$\leq K\pi \rho^2 \int_{\Gamma_\rho} J_f(\zeta_0 + \rho e^{it}) \rho \, dt,$$

which implies that

$$(2.3) \quad P(r) = \int_0^r \frac{\ell^2(f(\Gamma_\rho))}{\rho^2} \, d\rho \leq K\pi \int_0^r \int_{\Gamma_\rho} J_f(\zeta_0 + \rho e^{it}) \rho \, dt \, d\rho$$

$$\leq K\pi A_f(r),$$
where \( z = \zeta_0 + \rho e^{it} \) and \( A_f(r) \) denotes the area of \( f(\Delta_r) \). Since the boundary of \( \Omega \) is a \( M \)-Lavrentiev curve, we see that
\[
\ell(f(\partial \Delta_r)) \leq \ell(f(\Gamma_r)) + M\ell(f(\Gamma_r)) = (1 + M)\ell(f(\Gamma_r))
\]
and thus, by Lemma D, we have
\[
A_f(r) \leq \frac{(1 + M)^2}{4\pi}\ell^2(f(\Gamma_r)). \tag{2.4}
\]
By (2.3) and (2.4), we obtain
\[
P(r) \leq \frac{K(1 + M)^2}{4}\ell^2(f(\Gamma_r)).
\]
By calculations, for \( r \in (0, 2] \), we have
\[
\frac{4}{K(1 + M)^2}P(r) \leq \ell^2(f(\Gamma_r)) = r^2P'(r),
\]
which gives that
\[
\frac{\alpha}{r^2} \leq \frac{P'(r)}{P(r)}, \tag{2.5}
\]
where \( \alpha = 4/[K(1 + M)^2] \). By (2.5), we get
\[
\int_r^2 \frac{\alpha}{\rho^2} d\rho \leq \int_r^2 \frac{P'(\rho)}{P(\rho)} d\rho,
\]
which by integration, together with (2.3), yield that
\[
P(r) \leq \frac{P(2)}{e^{\alpha(\frac{1}{r} - \frac{1}{2})}} \leq \frac{K\pi A(\Omega)}{e^{\alpha(\frac{1}{r} - \frac{1}{2})}}, \tag{2.6}
\]
where \( A(\Omega) \) is the area of \( \Omega \).

By Cauchy-Schwarz’s inequality and (2.6), we have
\[
\left( \int_0^r \ell(f(\Gamma_\rho)) d\rho \right)^2 \leq P(r) \int_0^r \rho^2 d\rho \leq \frac{K\pi A(\Omega)r^3}{3e^{\alpha(\frac{1}{r} - \frac{1}{2})}},
\]
which gives that
\[
\int_0^r \ell(f(\Gamma_\rho)) d\rho \leq \sqrt{\frac{K\pi A(\Omega)}{3}} \frac{r^{\frac{3}{2}}}{e^{\alpha(\frac{1}{r} - \frac{1}{2})}} \leq \sqrt{\frac{K\pi A(\Omega)}{3}} r^{\frac{3}{2}}.
\]
The proof of this theorem is complete. \( \square \)

A Jordan curve \( J \) is said to be a \( M \)-quasicircle if, for any \( z_1, z_2 \in J \), there is a constant \( M \geq 1 \) such that
\[
diam J(z_1, z_2) \leq M|z_1 - z_2|. \tag{2.7}
\]
The inner domain \( G \) of a quasicircle \( J \) is called a \( M \)-quasidisk. We say that the curve \( \gamma \subset \mathbb{C} \) is Ahlfors-regular if there is a positive constant \( M \) such that
\[
\ell(\gamma \cap \mathbb{D}(w, r)) \leq Mr,
\]
where \( r \in (0, \infty) \) and \( w \in \mathbb{C} \) (cf. [16]).
Lemma E. ([16, Proposition 7.7]) A domain is a $M_3$-Laurentiev domain if and only if it is an $M_4$-Ahlfors-regular quasidisk, where $M_3$ and $M_4$ are positive constants.

Proof of Theorem 3. Assume that $f = h + \overline{g}$ is a $K$-quasiconformal harmonic mapping from $\mathbb{D}$ onto the inner domain $G$ of a rectifiable Jordan curve, where $h$ and $g$ are analytic in $\mathbb{D}$. Let $w_1, w_2 \in G$ and let $[w_1, w_2]$ denote the line segment with endpoints $w_1$ and $w_2$. If $[w_1, w_2] \subset G$, then (1.2) holds. Without loss of generality, we assume that $[w_1, w_2] \notin G$. Let $f(\xi_k)$ be the boundary point on $[w_1, w_2]$ nearest to $w_k$, where $k = 1, 2$ and $\xi_k \in \mathbb{T}$. By (2.7), we see that one of the arcs $\mathbb{T}_{[\xi_1, \xi_2]}$ of $\mathbb{T}$ from $\xi_1$ to $\xi_2$ satisfies

$$\text{diam} f(\mathbb{T}_{[\xi_1, \xi_2]}) \leq M_1 |f(\xi_1) - f(\xi_2)|.$$ 

If $r$ is close enough to 1, then $f(r\mathbb{T}_{[\xi_1, \xi_2]})$ is a curve in $G$ of

$$\text{diam} f(r\mathbb{T}_{[\xi_1, \xi_2]}) \leq 2M_1 |f(\xi_1) - f(\xi_2)| < 2M_1 |w_1 - w_2|$$

which can be connected within $G$ by curves $\gamma_k$ to $w_k$ satisfying

$$\ell(\gamma_k) < |w_1 - w_2|.$$

Then $\gamma = \gamma_1 \cup f(r\mathbb{T}_{[\xi_1, \xi_2]}) \cup \gamma_2$ is curve in $G$ from $w_1$ to $w_2$ satisfying

$$\text{diam}\gamma \leq (2M_1 + 2)|w_1 - w_2|,$$

which implies that $G$ is a $(2M_1 + 2)$-linearly connected domain.

Now we prove (1.4). Let $I(z) = \{\zeta \in \mathbb{T} : |\arg \zeta - \arg z| \leq \pi(1 - |z|)\}$. By (1.3) and [4, Theorems 1 and 2], we see that there is a constant $M^*$ such that

$$\text{diam} f(I(z)) \leq M^*d_G(z).$$

Applying the inequality (2.3) in [4], we get

$$d_G(z) \leq \frac{2K}{1 + K}\|D_f(z)\|(1 - |z|^2).$$

It follows from (2.8) and (2.9) that

$$\text{diam} f(I(z)) \leq \frac{2KM^*}{1 + K}\|D_f(z)\|(1 - |z|^2) = r_z,$$

which implies that $f(I(z))$ lies in $\mathbb{D}_{r_z}$. By (2.10) and Lemma E, we conclude that there is a constant $M'$ such that

$$\frac{1}{K} \int_{I(z)} \|D_f(\zeta)\| \, |d\zeta| \leq \int_{I(z)} \left| h'(\zeta) - \zeta^{-2}g'(\zeta) \right| |d\zeta| = \ell(f(I(z)))$$

$$\leq M'\|D_f(z)\||(1 - |z|^2)$$

$$\leq M'\ell(I(z))\|D_f(z)\|.$$

The proof of the theorem is complete. 

Theorem F. ([10, Proposition 3.1] and [10, Theorem 3.2]) Let $f$ be a $K$-quasiconformal harmonic mapping from $\mathbb{D}$ onto itself. Then for all $z \in \mathbb{D}$, we have

$$\frac{1 + K}{2K} \left( \frac{1 - |f(z)|^2}{1 - |z|^2} \right) \leq |f_z(z)| \leq \frac{K + 1}{2} \left( \frac{1 - |f(z)|^2}{1 - |z|^2} \right).$$
Proof of Theorem 4. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1} b_n \overline{z}^n \) be a \( K \)-quasiconformal harmonic mapping on \( \mathbb{D} \). Then, by Cauchy’s integral formula, for \( \rho \in (0, 1) \) and \( n \geq 1 \), we get
\[
na_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f_z(z)}{z^n} \, dz \quad \text{and} \quad nb_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{\overline{f_z(z)}}{z^n} \, dz,
\]
which imply that
\[
n(|a_n| + |b_n|) = \frac{1}{2\pi} \left| \int_{|z|=\rho} \frac{f_z(z)}{z^n} \, dz \right| + \frac{1}{2\pi} \left| \int_{|z|=\rho} \frac{\overline{f_z(z)}}{z^n} \, dz \right| \leq \frac{1}{2\pi \rho^{n-1}} \int_0^{2\pi} \|D_f(\rho e^{i\theta})\| \, d\theta. \tag{2.11}
\]
By calculations, for \( \theta \in [0, 2\pi] \), we obtain
\[
\ell_f^*(\theta, r) = \int_0^r \left| f_z(\rho e^{i\theta}) + e^{-2i\theta} f_{\overline{z}}(\rho e^{i\theta}) \right| \, d\rho \\
\geq \int_0^r \lambda(D_f)(\rho e^{i\theta}) \, d\rho \\
\geq \frac{1}{K} \int_0^r \|D_f(\rho e^{i\theta})\| \, d\rho,
\]
which gives
\[
\int_0^r \|D_f(\rho e^{i\theta})\| \, d\rho \leq K \ell_f^*(\theta, r) \leq KM. \tag{2.12}
\]
By (2.12), the subharmonicity of \( D_f(\rho e^{i\theta}) \) and Theorem A, we have
\[
\int_0^r \|D_f(\rho e^{i\theta})\| \, d\rho \leq KM r. \tag{2.13}
\]
By (2.11) and (2.13), we get
\[
2\pi n(|a_n| + |b_n|) \int_0^r \rho^{n-1} \, d\rho = \int_0^r \left( \int_0^{2\pi} \|D_f(\rho e^{i\theta})\| \, d\theta \right) \, d\rho \\
\leq \int_0^{2\pi} \left( \int_0^r \|D_f(\rho e^{i\theta})\| \, d\rho \right) \, d\theta \\
\leq 2\pi KM r,
\]
which yields that
\[|a_n| + |b_n| \leq \frac{KM}{r^{n-1}} \text{ for } n \geq 1.\]
Since this is true for each \( r < 1 \), the desired bound follows by letting \( r \to 1^- \). \( \square \)
Proof of Theorem 5. Let \( f = h + \overline{g} \) be a \( K \)-quasiconformal harmonic mapping of \( \mathbb{D} \) onto a bounded domain \( D \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). By [14, Proposition 13] and Theorem F, for \( r \in (0, 1) \) and all \( \theta \in [0, 2\pi] \), we obtain

\[
\ell_f^*(\theta, r) = \int_0^r |df(re^{i\theta})| = \int_0^r \left| h'(re^{i\theta}) + e^{-2i\theta}g'(re^{i\theta}) \right| d\rho 
\leq \int_0^r \| D_f(re^{i\theta}) \| d\rho 
\leq 16K \int_0^r \frac{d_D(f(re^{i\theta}))}{1 - \rho^2} d\rho 
\leq 16K \sup_{z \in \mathbb{D}} |f(z)| \int_0^r \frac{1}{1 - \rho^2} d\rho 
= 8K \sup_{z \in \mathbb{D}} |f(z)| \log \frac{1 + r}{1 - r} 
\]

and

\[
\ell(r) = \int_0^{2\pi} r \left| h'(re^{i\theta}) - e^{-2i\theta}g'(re^{i\theta}) \right| dt 
\geq r \int_0^{2\pi} \lambda(D_f)(re^{i\theta}) dt 
\geq \frac{r}{K} \int_0^{2\pi} \| D_f(re^{i\theta}) \| dt 
\geq \frac{r(1 + K)}{2K^2(1 - r^2)} \int_0^{2\pi} d_D(f(re^{i\theta})) dt 
\]

where the last inequaility is a consequence of [4, Inequality (2.3)]. Equations (2.14) and (2.15) imply that

\[
\frac{\ell_f^*(\theta, r)}{\ell(r)} \leq 16K^3 \sup_{z \in \mathbb{D}} |f(z)| \int_0^{2\pi} d_D(f(re^{i\theta})) dt \leq \frac{32r(1 + r)K^3 \sup_{z \in \mathbb{D}} |f(z)|}{\int_0^{2\pi} d_D(f(re^{i\theta})) dt} 
\]

and, for all \( \theta \in [0, 2\pi] \),

\[
\lim_{r \to 0^+} \frac{\ell_f^*(\theta, r)}{\ell(r)} = 0. 
\]

The proof of this theorem is complete. \( \square \)

Acknowledgements: This research was partly supported by the National Natural Science Foundation of China (No. 11571216 and No. 11401184), the Hunan Province Natural Science Foundation of China (No. 2015JJ3025), the Excellent Doctoral Dissertation of Special Foundation of Hunan Province (higher education 2050205), the Construct Program of the Key Discipline in Hunan Province. The third is on leave from IIT Madras. The second author thanks Indian Statistical Institute, Chennai Centre for the hospitality and the support during the period of my visit to India.
References

1. E. F. Beckenbach, A relative of the lemma of Schwarz, *Bull. Amer. Math. Soc.*, 44 (1938), 698–707.
2. T. Carleman, Zur Theorie der Minimalflächen, *Math. Z.*, 9 (1921), 154–160.
3. S. Chen, S. Ponnusamy and A. Rasila, Lengths, areas and Lipschitz-type spaces of planar harmonic mappings, *Nonlinear Anal.*, 115 (2015), 62–70.
4. S. Chen and S. Ponnusamy, John disks and K-quasiconformal harmonic mappings, *J. Geom. Anal.*, 2016, DOI: 10.1007/s12220-016-9727-6, published online.
5. F. Colonna, The Bloch constant of bounded harmonic mappings, *Indiana Univ. Math. J.*, 38 (1989), 829–840.
6. P. Duren, *Harmonic mappings in the plane*, Cambridge Univ. Press, 2004.
7. D. S. Jerison and C. E. Kenig, Hardy spaces, $A_\infty$ and singular integrals in chord-arc domains, *Math. Scand.*, 50 (1982), 221–247.
8. D. Kalaj, M. Marković and M. Mateljević, Carathéodory and Simirnov type theorems for harmonic mappings of the unit disk onto surfaces, *Ann. Acad. Sci. Fenn. Math.*, 38 (2013), 565–580.
9. D. Kalaj, Muckenhoupt weights and Lindelöf theorem for harmonic mappings, *Adv. Math.*, 280 (2015), 301–321.
10. M. Knežević and M. Mateljević, On the quasi-isometries of harmonic quasiconformal mappings, *J. Math. Anal. Appl.*, 334 (2007), 404–413.
11. M. Lavrentiev, Boundary problems in the theory of univalent functions, *Amer. Math. Soc. Transl. Ser.*, 32 (1963), 1–35.
12. O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, Springer Verlag, 1973.
13. H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.*, 42 (1936), 689–692.
14. M. Mateljević, Distortion of quasiregular mappings and equivalent norms on Lipschitz-type spaces, *Abstr. Appl. Anal.*, Volume 2014, Article ID 895074, 20 pages.
15. M. Pavlović, Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disc, *Ann. Acad. Sci. Fenn. Math.*, 27 (2002), 365–372.
16. Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, 1992.
17. W. Rudin, Real and Complex Analysis. McGraw-Hill. ISBN 0-07-054234-1,1987.
18. M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture Notes in Math. Vol. 1319, Springer-Verlag, 1988.
19. S. E. Warschawski, On differentiability at the boundary in conformal mapping, *Proc. Amer. Math. Soc.*, 12 (1961), 614–620.
20. M. Zinsmeister, *Domaines de Laurentiev*, Publications Math. Orsay, 162pp, no.3, 1985.

College of Mathematics and Statistics, Hengyang Normal University, Hengyang, Hunan 421008, People’s Republic of China.

E-mail address: mathechen@126.com

G. Liu, College of Mathematics and Statistics, Hengyang Normal University, Hengyang, Hunan 421008, People’s Republic of China.

E-mail address: liugangmath@sina.cn

S. Ponnusamy, Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and Security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India.

E-mail address: samy@isichennai.res.in, samy@iitm.ac.in