Research article

On coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives

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Abstract: In this paper, we investigate the existence, uniqueness and stability of coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives. To prove the existence and uniqueness results for afore mentioned system, we use the techniques of Kransnoselskii’s type fixed point theorem. Furthermore, different kinds of Ulam stabilities are discussed along with examples, to demonstrate the validity of main results.

Keywords: Riemann-Liouville fractional derivatives; fractional integro-differential equations; coupled system; impulses; Ulam stability

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1. Introduction

Fractional order differential equations are the generalizations of the classical integer order differential equations. The idea about the fractional order derivative was introduced at the end of the sixteenth century (1695) when Leibniz used the notation $\frac{d^n}{dx^n}$ for $n^{th}$ order derivative. By writing a letter to him, L’Hospital asked the question: what would be the result if $n = \frac{1}{2}$? Leibniz answered in such words, “An apparent Paradox, from which one day useful consequences will be drawn”, and this question became the foundation of fractional calculus. Fractional calculus has become a speedily developing area and its applications can be found in diverse fields ranging from physical sciences, porous media, electrochemistry, economics, electromagnetics, medicine and engineering to biological sciences. Progressively, fractional differential equations play a very important role in fields such as thermodynamics, statistical physics, nonlinear oscillation of earthquakes, viscoelasticity, defence, optics, control, signal processing, electrical circuits, astronomy etc. There are some outstanding articles which provide the main theoretical tools for the qualitative analysis of this research field, and
at the same time, shows the interconnection as well as the distinction between integral models, classical and fractional differential equations, see [14, 16, 18, 19, 22, 25, 26, 28, 30].

Impulsive fractional differential equations are used to describe both physical, social sciences and many dynamical systems such as evolution processes pharmacotherapy. There are two types of impulsive fractional differential equations the first one is instantaneous impulsive fractional differential equations while the other one is non-instantaneous impulsive fractional differential equations. In last few decades, the theory of impulsive fractional differential equations are well utilized in medicine, mechanical engineering, ecology, biology and astronomy etc. There are some remarkable monographs [3, 6, 8, 15, 20, 23, 33, 34], considering fractional differential equations with impulses.

The most preferable research area in the field of fractional differential equations (FDE's), which received great attention from the researchers is the theory regarding the existence of solutions. Many researchers developed some interesting results about the existence of solutions of different boundary value problems (BVPs) using different fixed point theorems. For details we refer the reader to [2, 7, 9–11, 13, 27]. Most of the time, it is quite intricate to find the exact solutions of nonlinear differential equations, in such a situation different approximation techniques are introduced. The difference between exact and approximate solutions is nowadays dealt with using Hyers-Ulam (HU) type stabilities, which were first introduced in 1940 by Ulam [29] and then answered by Hyers in the following year in the context of Banach spaces. Many researchers investigated HU type stabilities for different problems with different approaches [12, 17, 31–37, 39, 40].

Zada and Dayyan [38], investigated the existence, uniqueness and Ulam’s type stability for the implicit fractional differential equation with instantaneous impulses and Riemann-Liouville fractional integral boundary conditions having the following form

\[
\begin{align*}
&\{^cD_{0+}^\alpha u(\sigma) - \phi_1(\sigma, u(\sigma)), ^cD^\alpha u(\sigma)\} = 0, \quad \sigma \neq \sigma_j \in I, \quad 0 < \alpha \leq 1, \\
&\{\Delta u(\sigma_j) - E_j(u(\sigma_j))\} = 0, \quad j = 1, 2, \ldots, q - 1, \\
&\{\eta_1 u(\sigma)|_{\sigma=0} + \xi_1 I^\gamma u(\sigma)|_{\sigma=0} = \nu_1, \quad \eta_2 u(\sigma)|_{\sigma=T} + \xi_2 I^\gamma u(\sigma)|_{\sigma=T} = \nu_2,
\end{align*}
\]

where \( I = [0, T], \) and \(^cD_{0+}^\alpha\) is a generalization of classical Caputo derivative of order \( \alpha \) with lower bound at 0, \( \phi_1 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a continuous function. Furthermore, \( u(\sigma^+) \) and \( u(\sigma^-) \) represent the right-sided and left-sided limits respectively at \( \sigma = \sigma_j \) for \( j = 1, 2, \ldots, q - 1 \).

Ali et al. [4], studied a coupled system for the existence and uniqueness of solution using Riemann-Liouville derivative

\[
\begin{align*}
&D^\alpha u(\sigma) = \phi_1(\sigma, v(\sigma), D^\alpha v(\sigma)), \quad D^\beta v(\sigma) = \phi_2(\sigma, u(\sigma), D^\beta u(\sigma)), \quad \sigma \in J, \\
&D^{\alpha-1} u(0^+) = \beta_1 D^{\alpha-1} u(\Gamma^-), \quad D^{\alpha-1} v(0^+) = \gamma_1 D^{\alpha-1} v(\Gamma^-), \\
&D^{\beta-1} v(0^+) = \beta_2 D^{\beta-1} v(\Gamma^-), \quad D^{\beta-1} v(0^+) = \gamma_2 D^{\beta-1} v(\Gamma^-),
\end{align*}
\]

where \( \sigma \in J = [0, T], T > 0, \alpha, \beta \in (1, 2], \) and \( \beta_1, \beta_2, \gamma_1, \gamma_2 \neq 1. \) \( D^\alpha, D^\beta \) are the Riemann-Liouville fractional derivatives and \( \phi_1, \phi_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are continuous functions.

Wang et al. [32], presented stability of the following coupled system of implicit fractional integro-
differential equations having anti-periodic boundary conditions:

\[
\begin{aligned}
&\begin{cases}
\mathcal{D}^\alpha u(\sigma) - \phi_1(\sigma, v(\sigma), \mathcal{D}^\beta u(\sigma)) = \frac{1}{\Gamma(\gamma_1)} \int_0^\sigma (\sigma - s)^{\gamma_1 - 1} f(s, v(s), \mathcal{D}^\beta u(s)) ds = 0, \quad \forall \sigma \in \mathcal{I}, \\
\mathcal{D}^\beta v(\sigma) - \phi_2(\sigma, u(\sigma), \mathcal{D}^\beta v(\sigma)) = \frac{1}{\Gamma(\gamma_2)} \int_0^\sigma (\sigma - s)^{\gamma_2 - 1} g(s, u(s), \mathcal{D}^\beta v(s)) ds = 0, \quad \forall \sigma \in \mathcal{I},
\end{cases}
\end{aligned}
\]

where \(1 < \alpha, \beta \leq 2, 0 \leq r_1, r_2 \leq 2, \gamma_1, \gamma_2 > 0, \text{ and } \mathcal{I} = [0, T], T > 0. \phi_1, \phi_2, f, g : \mathcal{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are continuous functions.

Motivated by the above work, we focus our attention on the following coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives of the form:

\[
\begin{aligned}
&\begin{cases}
\mathcal{D}^\alpha u(\sigma) - \phi_1(\sigma, \mathcal{I}^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma)) = 0, \quad \sigma \in \omega, \quad \sigma \neq \sigma_j, \quad j = 1, 2, \ldots, p, \\
\Delta u(\sigma_j) - E_j(u(\sigma_j)) = 0, \quad \Delta u'(\sigma_j) - E_j'(u(\sigma_j)) = 0, \quad j = 1, 2, \ldots, p, \\
v_1 \mathcal{D}^{\alpha - 2} u(\sigma)|_{\sigma = 0} = u_1, \quad \mu_1 u(\sigma)|_{\sigma = T} + v_2 \mathcal{I}^{\alpha - 1} u(\sigma)|_{\sigma = T} = u_2, \\
\mathcal{D}^\beta v(\sigma) - \phi_2(\sigma, \mathcal{I}^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma)) = 0, \quad \sigma \in \omega, \quad \sigma \neq \sigma_k, \quad k = 1, 2, \ldots, q, \\
\Delta v(\sigma_k) - E_k(v(\sigma_k)) = 0, \quad \Delta v'(\sigma_k) - E_k'(v(\sigma_k)) = 0, \quad k = 1, 2, \ldots, q, \\
v_3 \mathcal{D}^{\beta - 2} v(\sigma)|_{\sigma = 0} = v_1, \quad \mu_2 v(\sigma)|_{\sigma = T} + v_4 \mathcal{I}^{\beta - 1} v(\sigma)|_{\sigma = T} = v_2,
\end{cases}
\end{aligned}
\]

where \(1 < \alpha, \beta \leq 2, \phi_1, \phi_2 : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) being continuous functions and

\[
\begin{aligned}
\Delta u(\sigma_j) &= u(\sigma_j^+) - u(\sigma_j^-), \quad \Delta u'(\sigma_j) = u'(\sigma_j^+) - u'(\sigma_j^-), \\
\Delta v(\sigma_k) &= v(\sigma_k^+) - v(\sigma_k^-), \quad \Delta v'(\sigma_k) = v'(\sigma_k^+) - v'(\sigma_k^-),
\end{aligned}
\]

where \(u(\sigma_j^+), v(\sigma_k^+) \) and \(u(\sigma_j^-), v(\sigma_k^-) \) are the right limits and left limits respectively, \(E_j, E_j^*, E_k, E_k^* : \mathbb{R} \to \mathbb{R} \) are continuous functions, and \(\mathcal{D}^\alpha, \mathcal{I}^\alpha \) are the \(\alpha\)-order Riemann-Liouville fractional derivative and integral operators respectively.

The remaining article is arranged as follows: In Section 2, we present some basic definitions, theorems, and lemmas that will be used in our main results. In Section 3, we use suitable cases for the existence and uniqueness of solution for the proposed system (1.1) using Kransnoselskii’s type fixed point theorem. In Section 4, we discuss different kinds of stabilities in the sense of Ulam under certain conditions. In Section 5, an example is given to support the main results.

### 2. Auxiliary results

In this section, we present some basics notations, definitions, and results that are used in the whole article.

Let \(T > 0, \omega = [0, T] \). The Banach space of all continuous functions from \(\omega\) into \(\mathbb{R}\) is denoted by \(C(\omega, \mathbb{R})\) with the norm

\[
||u|| = \sup \{|u(\sigma)| : \sigma \in \omega\}
\]
and the product of these spaces is also a Banach space with the norm

\[ \| (u, v) \| = |u| + |v|. \]

The piecewise continuous functions with \( 1 < \alpha, \beta \leq 2 \) are denoted as follows:

\[ \vartheta_1 = \mathcal{PC}_{2-\alpha}(\omega, \mathbb{R}^+) = \{ u : \omega \to \mathbb{R}^+, u(\sigma^+), u(\sigma^-) \text{ and } \Delta u'(\sigma^+), u'(\sigma^-) \text{ exist for } j = 1, 2, \ldots, p \}, \]

\[ \vartheta_2 = \mathcal{PC}_{2-\beta}(\omega, \mathbb{R}^+) = \{ v : \omega \to \mathbb{R}^+, v(\sigma^+), v(\sigma^-) \text{ and } \Delta v'(\sigma^+), v'(\sigma^-) \text{ exist for } k = 1, 2, \ldots, q \}, \]

with the norms

\[ \| u \|_{\vartheta_1} = \sup \{|\sigma^{2-\alpha}u(\sigma)| : \sigma \in \omega\}, \]

\[ \| v \|_{\vartheta_2} = \sup \{|\sigma^{2-\beta}v(\sigma)| : \sigma \in \omega\}, \]

respectively. Their product \( \vartheta = \vartheta_1 \times \vartheta_2 \) is also a Banach space with the norm \( \|(u, v)\|_{\vartheta} = \|u\|_{\vartheta_1} + \|v\|_{\vartheta_2} \).

**Definition 2.1.** [1] The Riemann-Liouville fractional integral of order \( \alpha > 0 \) for a function \( u : \mathbb{R}^+ \to \mathbb{R} \) is defined as

\[ \mathcal{I}^\alpha u(\sigma) = \frac{1}{\Gamma(\alpha)} \int_0^\sigma (\sigma - \tau)^{\alpha-1} u(\tau)d\tau, \]

where \( \Gamma(\cdot) \) is the Euler gamma function defined by \( \Gamma(\alpha) = \int_0^\infty e^{-\tau}\tau^{\alpha-1}d\tau, \ \alpha > 0. \)

**Definition 2.2.** For a function \( u : \mathbb{R}^+ \to \mathbb{R} \), the Riemann-Liouville derivative of fractional order \( \alpha > 0 \), \( p = [\alpha] + 1 \), is defined as

\[ \mathcal{D}_\alpha^p u(\sigma) = \frac{1}{\Gamma(p-\alpha)} \left( \frac{d}{d\sigma} \right)^p \int_0^\sigma (\sigma - \tau)^{\alpha-p-1} u(\tau)d\tau, \]

provided that integral on the right side exists. \([\alpha] \) denotes the integer part of the real number \( \alpha \). For more properties, the reader may refer to [1].

**Lemma 2.1.** [1] Let \( u \) be any function, and let \( \alpha > 0 \), then the Riemann-Liouville fractional derivative for the Homogeneous differential equation

\[ \mathcal{D}_\alpha^p u(\sigma) = 0, \ \alpha > 0, \]

has a solution

\[ u(\sigma) = c_1 \sigma^{\alpha-1} + c_2 \sigma^{\alpha-2} + \cdots + c_{p-1} \sigma^{\alpha-p} + c_p \sigma^{-p}, \]

and for non-homogeneous differential equation

\[ \mathcal{D}_\alpha^p u(\sigma) = \phi(\sigma), \ \alpha > 0, \]

has a solution

\[ I^\alpha \mathcal{D}_\alpha^p u(\sigma) = I^\alpha \phi(\sigma) + c_1 \sigma^{\alpha-1} + c_2 \sigma^{\alpha-2} + \cdots + c_{p-1} \sigma^{\alpha-p} + c_p \sigma^{-p}, \]

where \( p = [\alpha] + 1 \) and \( c_i, i = 1, 2, \ldots, p, \) are real constants.

**Theorem 2.1.** (Altman [5]) Let \( \Lambda \neq 0 \) be a convex and closed subset of Banach space \( \vartheta \). Consider two operators \( \mathcal{S}_1, \mathcal{S}_2 \) such that

1. \( \mathcal{S}_1(u, v) + \mathcal{S}_2(u, v) \in \Lambda; \)
2. \( \mathcal{S}_1 \) is a contractive operator;
3. \( \mathcal{S}_2 \) is a compact and continuous operator.

Then there exists \( (u, v) \in \Lambda \) such that \( \mathcal{S}_1(u, v) + \mathcal{S}_2(u, v) = (u, v) \in \vartheta. \)
2.1. Ulam’s Stabilities and Remarks

The following definitions and remarks are taken from [21, 24].

Definition 2.3. The given system \((1.1)\) is \(HU\) stable if there exists \(N_{\alpha,\beta} = \max\{N_\alpha, N_\beta\} > 0\) such that, for \(\kappa = \max\{\kappa_\alpha, \kappa_\beta\} > 0\) and for every solution \((\xi, \zeta) \in \vartheta\) of the inequality

\[
\begin{align*}
|D^\alpha \xi(\sigma) - \phi_1(\sigma, I^\alpha \xi(\sigma), I^\beta \xi(\sigma))| & \leq \kappa_\alpha, \ \sigma \in \omega, \\
|\Delta \xi(\sigma_j) - E_j(\xi(\sigma_j))| & \leq \kappa_\alpha, \ j = 1, 2, \ldots, p, \\
|\Delta \xi(\sigma_j) - E_j(\xi(\sigma_j))| & \leq \kappa_\beta, \ j = 1, 2, \ldots, p,
\end{align*}
\]

(2.1)

there exists a solution \((u, v) \in \vartheta\) with

\[||(u, v) - (\xi, \zeta)||_\vartheta \leq N_{\alpha,\beta}\kappa, \ \sigma \in \omega.\]

Definition 2.4. The given system \((1.1)\) is generalized \(HU\) stable if there exists \(N' \in C(\mathbb{R}^+, \mathbb{R}^+)\) with \(N'(0) = 0\) such that, for any approximate solution \((\xi, \zeta) \in \vartheta\) of inequality (2.1), there exists a solution \((u, v) \in \vartheta\) of \((1.1)\) satisfying

\[||(u, v) - (\xi, \zeta)||_\vartheta \leq N'(\kappa), \ \sigma \in \omega.\]

Definition 2.5. The given system \((1.1)\) is \(HUR\) stable with respect to \(\psi_{\alpha,\beta} = \max\{\psi_\alpha, \psi_\beta\}\) with \(\psi_{\alpha,\beta} \in C(\omega, \mathbb{R})\) if there exists a constant \(N_{\psi_{\alpha,\beta}} = \max\{N_{\psi_{\alpha}}, N_{\psi_{\beta}}\} > 0\) such that, for any \(\kappa = \max\{\kappa_\alpha, \kappa_\beta\} > 0\) and for any approximate solution \((\xi, \zeta) \in \vartheta\) of the inequality

\[
\begin{align*}
|D^\alpha \xi(\sigma) - \phi_2(\sigma, I^\alpha \xi(\sigma), I^\beta \xi(\sigma))| & \leq \psi_\alpha(\sigma)\kappa_\alpha, \ \sigma \in \omega, \\
|\Delta \xi(\sigma_j) - E_j(\xi(\sigma_j))| & \leq \psi_\alpha(\sigma)\kappa_\alpha, \ j = 1, 2, \ldots, p, \\
|\Delta \xi(\sigma_j) - E_j(\xi(\sigma_j))| & \leq \psi_\beta(\sigma)\kappa_\beta, \ j = 1, 2, \ldots, p,
\end{align*}
\]

(2.2)

there exists a solution \((u, v) \in \vartheta\) with

\[||(u, v) - (\xi, \zeta)||_\vartheta \leq N_{\psi_{\alpha,\beta}}\psi_{\alpha,\beta}(\sigma)\kappa, \ \sigma \in \omega.\]

Definition 2.6. The given system \((1.1)\) is generalized \(HUR\) stable with respect to \(\psi_{\alpha,\beta} = \max\{\psi_\alpha, \psi_\beta\}\) with \(\psi_{\alpha,\beta} \in C(\omega, \mathbb{R})\) if there exists a constant \(N_{\psi_{\alpha,\beta}} = \max\{N_{\psi_\alpha}, N_{\psi_\beta}\} > 0\) such that, for any approximate solution \((\xi, \zeta) \in \vartheta\) of inequality (2.2), there exists a solution \((u, v) \in \vartheta\) of \((1.1)\) satisfying

\[||(u, v) - (\xi, \zeta)||_\vartheta \leq N_{\psi_{\alpha,\beta}}\psi_{\alpha,\beta}(\sigma), \ \sigma \in \omega.\]

Remark 2.1. Let \((\xi, \zeta) \in \vartheta\) be a solution of inequalities (2.1) if there exist functions \(R_{\phi_1}, R_{\phi_2} \in C(\omega, \mathbb{R})\) depending on \(\xi, \zeta\) respectively such that
\begin{align}
|\mathcal{R}_\phi_1(\sigma)| &\leq \kappa_\alpha, \quad |\mathcal{Q}_{\phi_2}(\sigma)| \leq \kappa_\beta, \quad \sigma \in \omega; \\
(2) \quad \begin{cases}
\mathcal{D}^\alpha \xi(\sigma) = \phi_1(\sigma, I^\alpha \xi(\sigma), I^\beta \xi(\sigma)) + \mathcal{R}_{\phi_1}(\sigma), \\
\Delta \xi(\sigma_j) = E_j(\xi(\sigma_j)) + \mathcal{R}_{\phi_1}, \quad j = 1, 2, \ldots, p, \\
\Delta \xi'(\sigma_j) = E_j'(\xi(\sigma_j)) + \mathcal{R}_{\phi_1}, \quad j = 1, 2, \ldots, p, \\
\mathcal{D}^\beta \xi(\sigma) = \phi_2(t, I^\alpha \xi(\sigma), I^\beta \xi(\sigma)) + \Omega_{\phi_2}(\sigma), \\
\Delta \zeta(\sigma_k) = E_k(\zeta(\sigma_k)) + \Omega_{\phi_2}, \quad k = 1, 2, \ldots, q, \\
\Delta \zeta'(\sigma_k) = E_k'(\zeta(\sigma_k)) + \Omega_{\phi_2}, \quad k = 1, 2, \ldots, q.
\end{cases}
\end{align}

3. Existence and uniqueness

In this section, we discuss the existence and uniqueness of solution of the proposed system (1.1).

**Theorem 3.1.** Let \( \alpha, \beta \in (1, 2] \) and \( \phi_1 \) be any linear and continuous function. The fractional impulsive differential equation

\[
\begin{cases}
\mathcal{D}^\alpha u(\sigma) = \phi_1(\sigma, I^\alpha u(\sigma), I^\beta v(\sigma)), \quad \sigma \in \omega, \quad \sigma \neq \sigma_j, \quad j = 1, 2, \ldots, p, \\
\Delta u(\sigma_j) = E_j(u(\sigma_j)), \quad \Delta u'(\sigma_j) = E_j'(u(\sigma_j)), \quad j = 1, 2, \ldots, p, \\
\nu_1 \mathcal{D}^{\alpha-2} u(\sigma)|_{\sigma=0} = u_1, \quad \mu_1 u(\sigma)|_{\tau=T} + \nu_2 I^{\alpha-1} u(\sigma)|_{\tau=T} = u_2,
\end{cases}
\]

has a solution
\[
\begin{align*}
\text{Proof.} & \quad \text{Consider} \quad \sigma \\
& \quad \text{For} \quad \sigma \\
& \quad \text{Proof.} \quad \text{Consider} \quad \sigma \quad \text{Lemma 2.1 gives} \\
& \quad \text{For} \quad \sigma \\
\end{align*}
\]
Again, for \( \sigma \in (\sigma_1, \sigma_2] \), Lemma 2.1 gives

\[
\begin{align*}
\begin{cases}
\ u(\sigma) &= \frac{1}{\Gamma(\alpha)} \int_{\sigma_1}^{\sigma} (\sigma - \pi)^{\alpha-1} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi + b_1 \sigma_1^{\alpha-1} + b_2 \sigma_1^{\alpha-2}, \\
\ u'(\sigma) &= \frac{1}{\Gamma(\alpha - 1)} \int_{\sigma_1}^{\sigma} (\sigma - \pi)^{\alpha-2} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi + b_1 (\alpha - 1) \sigma_1^{\alpha-2} + b_2 (\alpha - 2) \sigma_1^{\alpha-3}.
\end{cases}
\end{align*}
\] (3.5)

Hence it follows that

\[
\begin{align*}
\ u(\sigma_1^-) &= \frac{1}{\Gamma(\alpha)} \int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-1} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi + a_1 \sigma_1^{\alpha-1} + a_2 \sigma_1^{\alpha-2}, \\
\ u(\sigma_1^+) &= b_1 \sigma_1^{\alpha-1} + b_2 \sigma_1^{\alpha-2}, \\
\ u'(\sigma_1^-) &= \frac{1}{\Gamma(\alpha - 1)} \int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-2} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi + a_1 (\alpha - 1) \sigma_1^{\alpha-2} + a_2 (\alpha - 2) \sigma_1^{\alpha-3}, \\
\ u'(\sigma_1^+) &= b_1 (\alpha - 1) \sigma_1^{\alpha-2} + b_2 (\alpha - 2) \sigma_1^{\alpha-3}.
\end{align*}
\]

Using

\[
\begin{align*}
\Delta u(\sigma_1) &= u(\sigma_1^+) - u(\sigma_1^-) = E_1(u(\sigma_1)), \\
\Delta u'(\sigma_1) &= u'(\sigma_1^+) - u'(\sigma_1^-) = E_1'(u(\sigma_1)),
\end{align*}
\]

we obtain

\[
\begin{align*}
\begin{cases}
\ b_1 &= a_1 - (\alpha - 2) \sigma_1^{1-\alpha} E_1(u(\sigma_1)) + \sigma_1^{2-\alpha} E_1'(u(\sigma_1)) + \frac{\sigma_1^{2-\alpha}}{\Gamma(\alpha - 1)} \int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-2} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
&\quad - \frac{(\alpha - 2) \sigma_1^{1-\alpha}}{\Gamma(\alpha)} \int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-1} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi, \\
\ b_2 &= a_2 + (\alpha - 1) \sigma_1^{2-\alpha} E_1(u(\sigma_1)) - \sigma_1^{3-\alpha} E_1'(u(\sigma_1)) - \frac{\sigma_1^{3-\alpha}}{\Gamma(\alpha - 1)} \int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-2} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
&\quad + \frac{(\alpha - 1) \sigma_1^{2-\alpha}}{\Gamma(\alpha)} \int_{0}^{\sigma_1} (\sigma_1 - \pi)^{\alpha-1} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi.
\end{cases}
\end{align*}
\]

Substituting the values of \( b_1, b_2 \) in (3.5), we get
Finally, after applying conditions

Similarly, for $\sigma \in (\sigma_j, \sigma_{j+1}]$, 

Finally, after applying conditions $v_1 D^{\alpha-2} u(\sigma)|_{\sigma = 0} = u_1$, and $\mu_1 u(\sigma)|_{\sigma = T} + v_2 I^{\alpha-1} u(\sigma)|_{\sigma = T} = u_2$ to (3.6) and finding the values of $a_1$ and $a_2$, we obtain Eq (2.2).
Corollary 1. In view of Theorem 3.1, our coupled system (1.1) has the following solution:

\[
\begin{align*}
\sigma^{-1} u_2 &= \sigma^{-1} u_1 + \sigma^{-2} u_1 + \Gamma(\alpha) \int_0^{\tau} (\sigma - \tau)^{\alpha-1} \phi_1(\tau, I^\alpha u(\tau), I^\beta v(\tau)) d\tau \\
- \frac{\sigma^{-1} \tau^{1-\alpha}}{\Gamma(\alpha)} &\int_0^{\tau} (\tau - \tau)^{\alpha-1} \phi_1(\tau, I^\alpha u(\tau), I^\beta v(\tau)) d\tau - \frac{\mu_1 \sigma^{-1} \tau^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\tau} (\tau - \tau)^{\alpha-2} u(\tau) d\tau \\
- \frac{\sigma^{-1}}{\Gamma(\alpha)} \sum_{j=1}^{z} \left[ (\tau - \tau)^{\alpha-1} \phi_j(\tau, I^\alpha u(\tau), I^\beta v(\tau)) \right] \\
+ \frac{(\tau - \sigma_j) \sigma^{-2} \tau^{1-\alpha}}{\Gamma(\alpha)} &\int_0^{\tau} (\tau - \tau)^{\alpha-2} \phi_1(\tau, I^\alpha u(\tau), I^\beta v(\tau)) d\tau \\
+ \frac{(\tau - \sigma_j) \sigma^{-2} \tau^{1-\alpha}}{\Gamma(\alpha)} &\int_0^{\tau} (\tau - \tau)^{\alpha-2} \phi_1(\tau, I^\alpha u(\tau), I^\beta v(\tau)) d\tau \\
&\quad + \sum_{j=1}^{z} \left[ (\tau - \tau)^{\alpha-1} \phi_j(\tau, I^\alpha u(\tau), I^\beta v(\tau)) \right] \\
\sigma &\in (\sigma_j, \sigma_{j+1}); \quad z = 1, 2, \ldots, p.
\end{align*}
\]
Now, for transformation of the given system (1.1) into a fixed point problem, let the operators $\mathcal{S}_1, \mathcal{S}_2 : \theta \to \theta$ be define as follows:

$$
\mathcal{S}_1(u, v)(\sigma) = (\mathcal{S}_1^u(u(\sigma)), \mathcal{S}_1^v(v(\sigma))),
\mathcal{S}_2(u, v)(\sigma) = (\mathcal{S}_2^u(u(\sigma)), \mathcal{S}_2^v(v(\sigma))).
$$
\begin{align*}
\mathcal{J}_1^*(u) &= \begin{cases}
\frac{\sigma^{a-1}u_2}{\mu_1 T^{a-1}} - \frac{\sigma^{a-2}u_1}{v_1 \Gamma(a-1)} - \frac{v_2 \sigma^{a-1}T^{1-a}}{\Gamma(a-1)} \int_0^T (T-\pi)^{a-2}u(\pi)d\pi \\
- \frac{\sigma^{a-1}}{T} \sum_{j=1}^z \left[ ((\alpha - 1) - (\alpha - 2)\Gamma(\alpha^{-1})) \sigma_j^{2-a} E_j(u(\sigma_j)) + (T - \sigma_j) \sigma_j^{2-a} E_j^*(u(\sigma_j)) \right] \\
+ \sum_{j=1}^z \left[ ((\alpha - 1) - (\alpha - 2)\sigma_j^{-1}) \sigma_j^{a-2} \sigma_j^{2-a} E_j(u(\sigma_j)) + (\sigma - \sigma_j) \sigma_j^{a-2} \sigma_j^{2-a} E_j^*(u(\sigma_j)) \right], \\
\sigma &\in (\sigma_j, \sigma_{j+1}); \quad z = 1, 2, \ldots, p,
\end{cases}

\mathcal{J}_1^*(v) &= \begin{cases}
\frac{\sigma^{\beta-1}v_2}{\mu_2 T^{\beta-1}} - \frac{\sigma^{\beta-2}v_1}{v_3 \Gamma(\beta-1)} - \frac{v_4 \sigma^{\beta-1}T^{1-\beta}}{\Gamma(\beta-1)} \int_0^T (T-\pi)^{\beta-2}v(\pi)d\pi \\
- \frac{\sigma^{\beta-1}}{T} \sum_{k=1}^z \left[ ((\beta - 1) - (\beta - 2)\Gamma(\beta^{-1})) \sigma_k^{2-\beta} E_k(v(\sigma_k)) + (T - \sigma_k) \sigma_k^{2-\beta} E_k^*(v(\sigma_k)) \right] \\
+ \sum_{k=1}^z \left[ ((\beta - 1) - (\beta - 2)\sigma_k^{-1}) \sigma_k^{\beta-2} \sigma_k^{2-\beta} E_k(v(\sigma_k)) + (\sigma - \sigma_k) \sigma_k^{\beta-2} \sigma_k^{2-\beta} E_k^*(v(\sigma_k)) \right], \\
\sigma &\in (\sigma_k, \sigma_{k+1}); \quad z = 1, 2, \ldots, q,
\end{cases}
\end{align*}

and
For additional analysis, the following hypothesis needs to hold:

\[(H_1) \quad \text{For } \sigma \in \omega \text{ there exist bounded functions } o, \tau, \nu \in \theta \text{ such that} \]

\[|\phi_1(\sigma, x_1(\sigma), x_2(\sigma))| \leq o(\sigma) + \tau(\sigma)|x_1(\sigma)| + \nu(\sigma)|x_2(\sigma)|\]

with \(o_1 = \sup_{\sigma \in \omega} o(\sigma), \tau_1 = \sup_{\sigma \in \omega} \tau(\sigma), \) and \(\nu_1 = \sup_{\sigma \in \omega} \nu(\sigma) < 1.\)
Similarly, for \( \sigma \in \omega \) there exist bounded functions \( o^*, \tau^*, \upsilon^* \in \hat{\theta} \) such that

\[
|\phi_2(\sigma, x_1(\sigma), x_2(\sigma))| \leq o^*(\sigma) + \tau^*(\sigma)|x_1(\sigma)| + \upsilon^*(\sigma)|x_2(\sigma)|
\]

with

\[
o_2 = \sup_{\sigma \in \omega} o^*(\sigma), \quad \tau_2 = \sup_{\sigma \in \omega} \tau^*(\sigma), \quad \upsilon_2 = \sup_{\sigma \in \omega} \upsilon^*(\sigma) < 1.
\]

\((H_2)\) \( E_j, E_j^* : \mathbb{R} \to \mathbb{R} \) are continuous and there exist constants \( G_{E}, G_{E}^*, G_{E}^2, G_{E}^3, G_{E}^4, G_{E}^5, G_{E}^6, G_{E}^7, G_{E}^8 > 0 \) such that, for any \((u, v) \in \hat{\theta}, \)

\[
|E_z(u)| \leq G_E|u| + G_{E}^*, \quad |E_z(v)| \leq G_E|v| + G_{E}^*,
\]

where \( z = 1, 2, \ldots, p. \)

\((H_3)\) For all \( x_1, x_2, x_1^*, x_2^* \in \mathbb{R} \) and for each \( \sigma \in \omega \), there exist constants \( L_{\phi_1} > 0, 0 < L_{\phi_1}^* < 1 \) such that

\[
|\phi_1(\sigma, x_1, x_2) - \phi_1(\sigma, x_1^*, x_2^*)| \leq L_{\phi_1}|x_1 - x_1^*| + L_{\phi_1}^*|x_2 - x_2^*|.
\]

Similarly, for all \( x_1, x_2, x_1^*, x_2^* \in \mathbb{R} \) and for each \( \sigma \in \omega \), there exist constants \( L_{\phi_2} > 0, 0 < L_{\phi_2}^* < 1 \) such that

\[
|\phi_2(\sigma, x_1, x_2) - \phi_2(\sigma, x_1^*, x_2^*)| \leq L_{\phi_2}|x_1 - x_1^*| + L_{\phi_2}^*|x_2 - x_2^*|.
\]

\((H_4)\) \( E_z, E_z^* : \mathbb{R} \to \mathbb{R} \) are continuous and there exist constants \( L_E, L_E^*, L_E^1, L_E^2 \) such that, for any \((u, v), (u^*, v^*) \in \hat{\theta}, \)

\[
|E_z(u(\sigma)) - E_z(u^*(\sigma))| \leq L_E|u - u^*|, \quad |E_z^*(v(\sigma)) - E_z^*(v^*(\sigma))| \leq L_E^*|v - v^*|,
\]

\[
|E_z(u(\sigma)) - E_z^*(u^*(\sigma))| \leq L_E^1|u - u^*|, \quad |E_z^*(v(\sigma)) - E_z^*(v^*(\sigma))| \leq L_E^2|v - v^*|.
\]

Here we use Kransnoselskii’s fixed point theorem to show that the operator \( \mathcal{J}_1 + \mathcal{J}_2 \) has at least one fixed point. Therefore, we choose a closed ball

\[
\hat{\theta}_r = \{(u, v) \in \hat{\theta}, \|(u, v)\| \leq r, \|u\| \leq \frac{r}{2} \text{ and } \|v\| \leq \frac{r}{2}\} \subset \hat{\theta},
\]

where

\[
r \geq \frac{G_1 + G_1^* + o_1 G_3 + o_2 G_3^*}{1 - (G_2 + G_2^* + G_3 G_4 + G_3^* G_4^*)}.
\]

**Theorem 3.2.** If hypotheses \((H_1)-(H_4)\) are hold, then the given system (1.1) has at least one solution.

**Proof.** 1) For any \((u, v) \in \hat{\theta}_r, \) we have

\[
\|\mathcal{J}_1(u, v) + \mathcal{J}_2(u, v)\|_{\hat{\theta}_r} \leq \|\mathcal{J}_1^*(u)\|_{\hat{\theta}_1} + \|\mathcal{J}_2^*(v)\|_{\hat{\theta}_1} + \|\mathcal{J}_2^*(u, v)\|_{\hat{\theta}_1} + \|\mathcal{J}_2^*(u, v)\|_{\hat{\theta}_2}.
\]  

From (3.9), we get

\[
|\sigma^{2-a} \mathcal{J}_1^*(u(\sigma))| \leq \left| \frac{\sigma u_2}{\mu_1 T^{a-1}} \right| + \left| \frac{\sigma u_1}{T v_1 \Gamma(a-1)} \right| + \left| \frac{u_1}{v_1 \Gamma(a-1)} \right| + \frac{v_2}{\mu_1 \Gamma(a-1)} \int_0^T \left| (T - \tau)^{a-2} \right| |u(\tau)| d\tau
\]
\[
+ \sum_{j=1}^{z} \left[ (\alpha - 1) - (\alpha - 2)\sigma \sigma^{-1}_{j} \right] - \frac{\sigma}{T} \left[ (\alpha - 1) - (\alpha - 2)T \sigma^{-1}_{j} \right] \left| \sigma^{2-\alpha}_{j} \right| \left| \mathcal{E}_{j}(u(\sigma_{j})) \right|
\]
\[
+ \sum_{j=1}^{z} \left( \sigma - \sigma_{j} \right) - \frac{\sigma}{T} \left( T - \sigma_{j} \right) \left| \sigma^{2-\alpha}_{j} \right| \left| \mathcal{E}_{j}(u(\sigma_{j})) \right|
\]
\[
z = 1, 2, \ldots, p.
\]

This implies that
\[
\| \mathcal{J}_{1}^{*}(u) \|_{\theta_{1}} \leq \left| \frac{\sigma u_{2}}{\mu_{1} T^{\alpha-1}} \right| + \left| \frac{\sigma u_{1}}{T \nu_{1} \Gamma(\alpha - 1)} \right| + \left| \frac{u_{1}}{\nu_{1} \Gamma(\alpha - 1)} \right| + \frac{v_{2} |\sigma|}{\mu_{1} \Gamma(\alpha)} \|u\| + \frac{v_{1} |\sigma|}{\Gamma(\alpha - 1)} \|u\| + \|G_{1} + \mathcal{G}_{2}||u||
\]
\[
\leq \mathcal{G}_{1} + \mathcal{G}_{2} ||u|| \]

Similarly, we can obtain
\[
\| \mathcal{J}_{1}^{*}(v) \|_{\theta_{2}} \leq \mathcal{G}_{1}^{*} + \mathcal{G}_{2}^{*} ||v||,
\]

where
\[
\mathcal{G}_{1} = z \mathcal{G}_{E}^{*}(\alpha - 1) |\sigma^{2-\alpha}_{z}| \left| 1 - \frac{\sigma}{T} \right| + z \mathcal{G}_{E}^{*} |\sigma^{3-\alpha}_{z}| \left| \frac{\sigma}{T} - 1 \right| + \frac{\sigma u_{2}}{\mu_{1} T^{\alpha-1}} + \frac{\sigma u_{1}}{T \nu_{1} \Gamma(\alpha - 1)} + \frac{u_{1}}{\nu_{1} \Gamma(\alpha - 1)} \left| \sigma \right|,
\]
\[
\mathcal{G}_{2} = z \mathcal{G}_{E}^{*}(\alpha - 1) |\sigma^{2-\alpha}_{z}| \left| 1 - \frac{\sigma}{T} \right| + z \mathcal{G}_{E}^{*} |\sigma^{3-\alpha}_{z}| \left| \frac{\sigma}{T} - 1 \right| + \frac{\sigma u_{2}}{\mu_{1} T^{\alpha-1}} + \frac{\sigma u_{1}}{T \nu_{1} \Gamma(\alpha - 1)} + \frac{u_{1}}{\nu_{1} \Gamma(\alpha - 1)} \left| \sigma \right|, \quad \text{for } z = 1, 2, \ldots, p,
\]
\[
\mathcal{G}_{1}^{*} = z \mathcal{G}_{E}^{*}(\beta - 1) |\sigma^{2-\beta}_{z}| \left| 1 - \frac{\sigma}{T} \right| + z \mathcal{G}_{E}^{*} |\sigma^{3-\beta}_{z}| \left| \frac{\sigma}{T} - 1 \right| + \frac{\sigma v_{2}}{\mu_{2} \Gamma(\beta - 1)} + \frac{\sigma v_{1}}{T \nu_{2} \Gamma(\beta - 1)} + \frac{v_{1}}{\nu_{2} \Gamma(\beta - 1)} \left| \sigma \right|,
\]
\[
\mathcal{G}_{2}^{*} = z \mathcal{G}_{E}^{*}(\beta - 1) |\sigma^{2-\beta}_{z}| \left| 1 - \frac{\sigma}{T} \right| + z \mathcal{G}_{E}^{*} |\sigma^{3-\beta}_{z}| \left| \frac{\sigma}{T} - 1 \right| + \frac{\sigma v_{2}}{\mu_{2} \Gamma(\beta - 1)} + \frac{\sigma v_{1}}{T \nu_{2} \Gamma(\beta - 1)} + \frac{v_{1}}{\nu_{2} \Gamma(\beta - 1)} \left| \sigma \right|, \quad \text{for } z = 1, 2, \ldots, q.
\]

Also, we have
\[
|^{2-\alpha}_{2} \mathcal{J}_{2}(u, v) \| \leq \frac{|^{2-\alpha}_{2}}{\Gamma(\alpha)} \int_{\pi}^{\alpha} \int_{\pi}^{\alpha} |(\alpha - \pi)^{a-1}| \left| y(\pi) \right| d\pi + \frac{|^{2-\alpha}_{2}}{\Gamma(\alpha)} \int_{\pi}^{\alpha} |(\alpha - \pi)^{a-1}| \left| y(\pi) \right| d\pi
\]
\[
+ \frac{\sigma}{T} \sum_{j=1}^{z} \left| \frac{T - \sigma}{T \Gamma(\alpha - 1)} \right| \int_{\pi}^{\alpha} \left| (\alpha - \pi)^{a-1} \right| \left| y(\pi) \right| d\pi
\]
\[
+ \frac{\sigma}{T} \sum_{j=1}^{z} \left| \frac{T - \sigma}{T \Gamma(\alpha - 1)} \right| \int_{\pi}^{\alpha} \left| (\alpha - \pi)^{a-1} \right| \left| y(\pi) \right| d\pi
\]
\[
+ \frac{\sigma}{T} \sum_{j=1}^{z} \left| \frac{T - \sigma}{T \Gamma(\alpha - 1)} \right| \int_{\pi}^{\alpha} \left| (\alpha - \pi)^{a-1} \right| \left| y(\pi) \right| d\pi
\]
Similarly, we have

\[
|y(\sigma)| = |\phi_1(\sigma, I^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma))| \\
\leq o(\sigma) + \tau(\sigma)|I^\alpha u(\sigma)| + v(\sigma)|\mathcal{I}^\beta v(\sigma)| \\
\leq o(\sigma) + \tau(\sigma)(\frac{1}{\Gamma(\alpha)}\int_0^\sigma |(\sigma - \pi)^{\alpha-1}| |u(\pi)|d\pi + v(\sigma)(\frac{1}{\Gamma(\beta)}\int_0^\sigma |(\sigma - \pi)^{\beta-1}| |v(\pi)|d\pi).
\]

Now, taking sup_{\sigma \in \omega} on both sides, we get

\[
||y|| \leq o_1 + \tau_1\frac{|\sigma^\alpha||u||}{\Gamma(\alpha + 1)} + v_1\frac{|\sigma^\beta||v||}{\Gamma(\beta + 1)}.
\]

(3.16)

Now taking sup_{\sigma \in \omega} of (3.15) and using (3.16) in (3.15), we get

\[
||\mathfrak{I}_2(u, v)||_{\theta_1} \leq \left( o_1 + \tau_1\frac{|\sigma^\alpha||u||}{\Gamma(\alpha + 1)} + v_1\frac{|\sigma^\beta||v||}{\Gamma(\beta + 1)} \right) \left( \frac{|\sigma^{2-\alpha}|||\sigma - \sigma_1||}{\Gamma(\alpha)} + \frac{|\sigma^{1-\alpha}|||\sigma - \sigma_1||}{\Gamma(\alpha + 1)} \right) \\
+ \frac{z|\sigma|\left|\frac{\sigma^2-\alpha}{\Gamma(\alpha)}\right|\left|\frac{\sigma^{1-\alpha}}{\Gamma(\beta)}\right|\left|\frac{\sigma^2-\alpha}{\Gamma(\alpha + 1)}\right|\left|\frac{\sigma^{1-\alpha}}{\Gamma(\beta + 1)}\right|}{\Gamma(\alpha)} \\
+ \frac{z|\sigma - \sigma_1|\left|\frac{\sigma^2-\alpha}{\Gamma(\alpha)}\right|\left|\frac{\sigma^{1-\alpha}}{\Gamma(\beta)}\right|\left|\frac{\sigma^2-\alpha}{\Gamma(\alpha + 1)}\right|\left|\frac{\sigma^{1-\alpha}}{\Gamma(\beta + 1)}\right|}{\Gamma(\alpha)} \\
\leq o_1G_3 + \tau_1\frac{|\sigma^\alpha||u||G_3}{\Gamma(\alpha + 1)} + v_1\frac{|\sigma^\beta||v||G_3}{\Gamma(\beta + 1)} \\
\leq o_1G_3 + G_3G_4||u, v||.
\]

(3.17)

Similarly,

\[
||\mathfrak{I}_2^*(u, v)||_{\theta_2} \leq o_2G_3 + G_3G_5||u, v||,
\]

(3.18)

where

\[
G_3 = \frac{|\sigma^{2-\alpha}|||\sigma - \sigma_1||}{\Gamma(\alpha + 1)} + \frac{|\sigma^{1-\alpha}|||\sigma - \sigma_1||}{\Gamma(\alpha + 1)} \\
+ \frac{z|\sigma|\left|\frac{\sigma^2-\alpha}{\Gamma(\alpha)}\right|\left|\frac{\sigma^{1-\alpha}}{\Gamma(\beta)}\right|\left|\frac{\sigma^2-\alpha}{\Gamma(\alpha + 1)}\right|\left|\frac{\sigma^{1-\alpha}}{\Gamma(\beta + 1)}\right|}{\Gamma(\alpha)} \\
+ \frac{z|\sigma - \sigma_1|\left|\frac{\sigma^2-\alpha}{\Gamma(\alpha)}\right|\left|\frac{\sigma^{1-\alpha}}{\Gamma(\beta)}\right|\left|\frac{\sigma^2-\alpha}{\Gamma(\alpha + 1)}\right|\left|\frac{\sigma^{1-\alpha}}{\Gamma(\beta + 1)}\right|}{\Gamma(\alpha)} \\
, z = 1, 2, \ldots, p,
\]

\[
G_5 = \frac{|\sigma^{2-\beta}|||\sigma - \sigma_1||}{\Gamma(\beta + 1)} + \frac{|\sigma^{1-\beta}|||\sigma - \sigma_1||}{\Gamma(\beta + 1)}.
\]
Putting (3.13), (3.14), (3.17) and (3.18) in (3.11), we get

\[
\begin{align*}
\frac{z|\sigma|-|\sigma\sigma_z|}{T} \left[ \frac{1}{\Gamma(\beta)} \left[ (T - \sigma_z) |(\sigma_z - \sigma_{z-1})^{\beta-1}| \right] \right] \\
+ \frac{z|\sigma|-|\sigma\sigma_z|}{T} \left[ \frac{1}{\Gamma(\beta)} \left[ (\beta - 1 - (\beta - 2) T \sigma_z^{\beta-1}) |(\sigma_z - \sigma_{z-1})^{\beta}| \right] \right] \\
+ \frac{z|\sigma|-|\sigma\sigma_z|}{T} \left[ \frac{1}{\Gamma(\beta)} \left[ (\beta - 1 - (\beta - 2) T \sigma_z^{\beta-1}) |(\sigma_z - \sigma_{z-1})^{\beta-1}| \right] \right] \leq \frac{G}{\lambda_{\sigma-}}
\end{align*}
\]

\[z = 1, 2, \ldots, q,\]

\[G_4 = \max \left\{ \frac{|\sigma|}{\Gamma(\alpha + 1)}, u_1 \left\{ \frac{|\sigma|}{\Gamma(\alpha + 1)} \right\} \right\} \text{ and} \]

\[G_4^* = \max \left\{ \frac{|\sigma|}{\Gamma(\alpha + 1)}, u_2 \left\{ \frac{|\sigma|}{\Gamma(\alpha + 1)} \right\} \right\} .\]

Putting (3.13), (3.14), (3.17) and (3.18) in (3.11), we get

\[
\begin{align*}
||S_1(u, v) + S_2(u, v)||_{\theta} &\leq \mathcal{G}_1 + \mathcal{G}_2 |u| + \mathcal{G}_3^* |v| + \mathcal{G}_2 |E_1(u, v)| + \mathcal{G}_3^* |E_2(u, v)| + \mathcal{G}_2^* |E_3(u, v)| + \mathcal{G}_3^* |E_4(u, v)| \\
&\leq \mathcal{G}_1 + \mathcal{G}_2^* + \mathcal{G}_3^* + \mathcal{G}_2^* \mathcal{G}_4 \|u, v\| \\
&\leq r.
\end{align*}
\]

Hence, \( ||S_1(u, v) + S_2(u, v)||_{\theta} \in \theta. \)

2) Next, for any \( \sigma \in \omega, (u, v), (\xi, \zeta) \in \theta \)

\[
\begin{align*}
||S_1(u, v) - S_1(u, \xi)||_{\theta} &\leq ||S_1(u) - S_1(u, \xi)||_{\theta_1} + ||S_1^*(v) - S_1^*(v, \xi)||_{\theta_2} \\
&\leq \frac{|v_2| |\sigma|^{|T|^{\alpha-\beta}|}}{\mu_1 |\Gamma(\alpha + 1)|} \int_0^T \left| (T - \pi)^{\alpha - 2} |u(\pi) - \xi(\pi)| \right| d\pi \\
&+ \sum_{j=1}^\infty \left| \left( (\alpha - 1) - (\alpha - 2) T \sigma_j^{\beta-1} \right) - \frac{\sigma}{T} \left( (\alpha - 1) - (\alpha - 2) T \sigma_j^{\beta-1} \right) \right| \\
&\times |\sigma_j^{-1}| |E_j(u(\sigma)) - E_j(\xi(\sigma))| \\
&+ \sum_{j=1}^\infty \left| (\sigma - \sigma_j) - \frac{\sigma}{T} (T - \sigma_j) \right| |\sigma_j^{-1}| |E_j(u(\sigma)) - E_j(\xi(\sigma))| \\
&\leq \frac{|v_4| |T|^{\alpha-\beta}|}{\mu_2 |\Gamma(\beta + 1)|} \int_0^T \left| (T - \pi)^{\alpha - 2} |v(\pi) - \zeta(\pi)| \right| d\pi \\
&+ \sum_{k=1}^\infty \left| (\beta - 1) - (\beta - 2) T \sigma_k^{\beta-1} \right| - \frac{\sigma}{T} \left( (\beta - 1) - (\beta - 2) T \sigma_k^{\beta-1} \right) \\
&\times |\sigma_k^{-1}| |E_k(v(\sigma)) - E_k(\zeta(\sigma))| \\
&\leq \left( z(\alpha - 1) |\sigma_z^{-1}| \right) \left| 1 - \frac{\sigma}{T} \right| L_E + |z| |\sigma_z^{-1}| \left| 1 - \frac{\sigma}{T} \right| L_E + \frac{|v_2| |\sigma|}{\mu_1 |\Gamma(\alpha)|} \|u - \xi\| \\
&\left( z(\beta - 1) |\sigma_z^{-2}| \right) \left| 1 - \frac{\sigma}{T} \right| L_E + |z| |\sigma_z^{-2}| \left| 1 - \frac{\sigma}{T} \right| L_E + \frac{|v_4| |\sigma|}{\mu_2 |\Gamma(\beta)|} \|v - \zeta\|.
\end{align*}
\]
\[ \leq L(g_1 + g_2)\|(u - \xi, v - \zeta)\|. \]

Here \( L = \max\{L_\varepsilon, L_{\varepsilon'}, L_{\varepsilon''}, L_{\varepsilon'''}\}, \)

\[ g_1 = z(\alpha - 1)|\sigma_{z}^{\alpha - 1}| \left( 1 - \frac{\sigma}{T} \right) + z|\sigma_{z}^{\alpha - 1}| \left| \frac{\sigma}{T} - 1 \right| + \frac{|v_2| |\sigma|}{|\mu_1| \Gamma(\alpha)}, \quad z = 1, 2, \ldots, p, \]

and

\[ g_2 = z(\beta - 1)|\sigma_{z}^{\beta - 1}| \left( 1 - \frac{\sigma}{T} \right) + z|\sigma_{z}^{\beta - 1}| \left| \frac{\sigma}{T} - 1 \right| + \frac{|v_4| |\sigma|}{|\mu_2| \Gamma(\beta)}, \quad z = 1, 2, \ldots, q. \]

Therefore, \( \mathcal{S}_1 \) is a contractive operator.

3) Now, for the continuity and compactness of \( \mathcal{S}_2 \), we make a sequence \( T_s = (u_s, v_s) \) in \( \theta_r \) such that \( (u_s, v_s) \to (u, v) \) for \( s \to \infty \) in \( \theta_r \). Thus, we have

\[ \|\mathcal{S}_2(u_s, v_s) - \mathcal{S}_2(u, v)\|_\theta \leq \left( \frac{L_{\phi_1} |\sigma^\alpha| \|u_s - u\|}{\Gamma(\alpha + 1)} + \frac{L_{\phi_2} |\sigma^\beta| \|v_s - v\|}{\Gamma(\beta + 1)} \right) \left( \frac{|\sigma^\alpha| |(\sigma - \sigma_z)^{\alpha - 1}|}{\Gamma(\alpha + 1)} + \frac{|\sigma^\beta| |(\sigma - \sigma_z)^{\beta - 1}|}{\Gamma(\beta + 1)} \right) \]

This implies \( \|\mathcal{S}_2(u_s, v_s) - \mathcal{S}_2(u, v)\|_\theta \to 0 \) as \( s \to \infty \), therefore \( \mathcal{S}_2 \) is continuous.

Next, we show that \( \mathcal{S}_2 \) is uniformly bounded on \( \theta_r \). From (3.17) and (3.18), we have

\[ \|\mathcal{S}_2(u, v)\|_\theta \leq \|\mathcal{S}_2^*(u, v)\|_{\theta_1} + \|\mathcal{S}_2^{**}(u, v)\|_{\theta_2} \]

\[ \leq o_1G_3 + o_2G_4 + (G_3G_4 + G_3G_4)||u, v|| \]

\[ \leq r. \]

Thus, \( \mathcal{S}_2 \) is uniformly bounded on \( \theta_r \).
For equicontinuity, suppose \( \eta_1, \eta_2 \in \omega \) with \( \eta_1 < \eta_2 \), and for any \( (u, v) \in \vartheta_r \subset \vartheta \) where \( \vartheta_r \) is clearly bounded, we have

\[
\| \mathcal{J}_2(u, v)(\eta_1) - \mathcal{J}_2(u, v)(\eta_2) \|_{\vartheta_r} = \max |\sigma^{\alpha-\beta}(\mathcal{J}_2(u, v)(\eta_1) - \mathcal{J}_2(u, v)(\eta_2))| \\
\leq \left( \alpha_1 + \tau_1 \frac{|\sigma^{\alpha-\beta}|}{\Gamma(\alpha + 1)} + v_1 \frac{|\sigma^{\alpha-\beta}|}{\Gamma(\beta + 1)} \right) \left( |(\eta_1 - \sigma_\alpha)^\alpha - (\eta_2 - \sigma_\alpha)^\alpha| \right) \\
+ \frac{|\sigma^{\alpha-\beta}| \eta_1^{\alpha-1} - \eta_2^{\alpha-1}|}{\Gamma(\alpha + 1)} \left| |(T - \sigma_\alpha)^\alpha| \right| + \left| \left( \eta_1^{\alpha-2} - \eta_2^{\alpha-2} \right) \right| \\
\times \left( z(\alpha - 1) |\sigma_\alpha^{\alpha-\beta}| \left| |(\sigma_\alpha - \sigma_{\alpha-1})^{\alpha-1} | \right| + z(\alpha - 1) |\sigma_\alpha^{\alpha-\beta}| \left| |(\sigma_\alpha - \sigma_{\alpha-1})^{\alpha} | \right| \right).
\]

This implies that

\[
\| \mathcal{J}_2(u, v)(\eta_1) - \mathcal{J}_2(u, v)(\eta_2) \|_{\vartheta_r} \rightarrow 0 \quad \text{as} \quad \eta_1 \rightarrow \eta_2.
\]

In the same way, we have

\[
\| \mathcal{J}_2(u, v)(\eta_1) - \mathcal{J}_2(u, v)(\eta_2) \|_{\vartheta_\sigma} \rightarrow 0 \quad \text{as} \quad \eta_1 \rightarrow \eta_2.
\]

Hence

\[
\| \mathcal{J}_2(u, v)(\eta_1) - \mathcal{J}_2(u, v)(\eta_2) \|_{\vartheta} \rightarrow 0 \quad \text{as} \quad \eta_1 \rightarrow \eta_2.
\]

Thus, \( \mathcal{J}_2 \) is equicontinuous. So \( \mathcal{J}_2 \) is relatively compact on \( \vartheta_r \). Hence, by the Arzelà–Ascoli Theorem, \( \mathcal{J}_2 \) is compact on \( \vartheta_r \). Thus all the condition of Theorem 2.1 are satisfied. So the given system (1.1) has at least one solution. \( \square \)

**Theorem 3.3.** Let hypotheses \((H_3), (H_4)\) be satisfied with

\[
\Delta_1 + \Delta_3 + \frac{(\Delta_2 L^{\alpha}_{\phi_1} + \Delta_4 L^\beta_{\phi_2})|\sigma^{\alpha}|}{\Gamma(\alpha + 1)} + \frac{(\Delta_2 L^{\alpha}_{\phi_1} + \Delta_4 L^\beta_{\phi_2})|\sigma^{\beta}|}{\Gamma(\beta + 1)} < 1,
\]

(3.19)

then the given system (1.1) has unique solution.

**Proof.** First we define an operator \( \varphi = (\varphi_1, \varphi_2) : \vartheta \rightarrow \vartheta \), i.e., \( \varphi(u, v)(\sigma) = (\varphi_1(u, v), \varphi_2(u, v))(\sigma) \), where

\[
\varphi_1(u, v)(\sigma) = \frac{\sigma^{\alpha-1}u_2}{\mu_1 \Gamma(\alpha - 1)} - \frac{\sigma^{\alpha-1}u_1}{v_1 \Gamma(\alpha - 1)} + \frac{\sigma^{\alpha-2}u_2}{v_1 \Gamma(\alpha - 1)} + \frac{1}{\Gamma(\alpha)} \int_{\sigma_\alpha}^{\sigma} (\sigma - \tau)^{\alpha-1} \phi_1(\pi, T^\alpha u(\pi), T^\beta v(\pi))d\pi \\
- \frac{\sigma^{\alpha-1}(T - \tau)^{\alpha-1}}{\Gamma(\alpha)} \int_{\sigma_\alpha}^{\tau} (\tau - \sigma)^{\alpha-1} \phi_1(\pi, T^\alpha u(\pi), T^\beta v(\pi))d\pi + u_2 \frac{\sigma^{\alpha-1}(T - \tau)^{\alpha-1}}{\mu_1 \Gamma(\alpha - 1)} \int_{\sigma_\alpha}^{\tau} (\tau - \sigma)^{\alpha-2} u(\pi)d\pi \\
- \frac{\sigma^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{\infty} \left( (\alpha - 1) - (\alpha - 2)T^\alpha \sigma_j^{-1} \right) \sigma_j^{\alpha-2} E_j(u(\sigma_j)) + (T - \sigma_j) \sigma_j^{\alpha-2} E_j(u(\sigma_j)) \\
+ \frac{(T - \sigma_j) \sigma_j^{\alpha-2}}{\Gamma(\alpha - 1)} \int_{\sigma_{j-1}}^{\sigma_j} (\sigma - \pi)^{\alpha-2} \phi_1(\pi, T^\alpha u(\pi), T^\beta v(\pi))d\pi.
\]
\begin{align*}
&+ \frac{(\alpha - 1) - (\alpha - 2)T^2\sigma_{j-1}^{-1}}{\Gamma(\alpha)} \sum_{j=1}^{\infty} \left( \sigma_j - \pi \right)^{\alpha-1} \phi_1(\pi, I^\sigma u(\pi), I^\beta v(\pi))d\pi
&+ \frac{(\sigma - \sigma_j)\sigma^{-2}\sigma_j^{-1}}{\Gamma(\alpha - 1)} \sum_{j=1}^{\infty} \left( \sigma_j - \pi \right)^{\alpha-2} \phi_1(\pi, I^\sigma u(\pi), I^\beta v(\pi))d\pi
&+ \frac{(\alpha - 1) - (\alpha - 2)\sigma_{j-1}^{-1}}{\Gamma(\alpha)} \sum_{j=1}^{\infty} \left( \sigma_j - \pi \right)^{\alpha-1} \phi_1(\pi, I^\sigma u(\pi), I^\beta v(\pi))d\pi
&& \text{for } z = 1, 2, \ldots, p, \tag{3.2}

\text{and}

\varphi_2(u, v)(\sigma) = \frac{\sigma^\beta - 1}{\mu_2 T^\beta - 1} \frac{\sigma^\beta - 1}{\nu_3 \Gamma(\beta - 1)} + \frac{\sigma^\beta - 1}{\nu_3 \Gamma(\beta - 1)} + \frac{1}{\Gamma(\beta)} \int_{\pi}^{\infty} (\sigma - \pi)^{\beta-1} \phi_2(\pi, I^\sigma u(\pi), I^\beta v(\pi))d\pi
&- \frac{\sigma^\beta - 1}{\Gamma(\beta)} \int_{\pi}^{\infty} (\sigma - \pi)^{\beta-2} \phi_2(\pi, I^\sigma u(\pi), I^\beta v(\pi))d\pi
&- \frac{\sigma^\beta - 1}{\Gamma(\beta)} \int_{\pi}^{\infty} (\sigma - \pi)^{\beta-2} \phi_2(\pi, I^\sigma u(\pi), I^\beta v(\pi))d\pi
&+ \frac{(\beta - 1) - (\beta - 2)T^2\sigma_k^{-1}}{\Gamma(\beta - 1)} \sum_{k=1}^{\infty} \left( \sigma_k - \pi \right)^{\beta-2} \phi_2(\pi, I^\sigma u(\pi), I^\beta v(\pi))d\pi
&+ \frac{(\beta - 1) - (\beta - 2)T^2\sigma_k^{-1}}{\Gamma(\beta - 1)} \sum_{k=1}^{\infty} \left( \sigma_k - \pi \right)^{\beta-2} \phi_2(\pi, I^\sigma u(\pi), I^\beta v(\pi))d\pi
&& \text{for } z = 1, 2, \ldots, q. \tag{3.3}
\end{align*}

In view of Theorem 3.2, we have

\begin{align*}
|\sigma^{\alpha-2}(\varphi_1(u, v) - \varphi_1(\xi, \zeta))| & \leq \left( \frac{L^*_{\varphi_1}}{\Gamma(\beta + 1)} \right) \left( \frac{\sigma^{\alpha-2} \left| (\sigma - \sigma_j)^\alpha \right|}{\Gamma(\alpha + 1)} + \frac{\sigma \left| (T^2 - \sigma_k)^\alpha \right|}{\Gamma(\alpha + 1)} \right)
&+ \frac{z |\sigma| \sigma_{z-1}^{-\alpha}}{\Gamma(\alpha + 1)} \left( \frac{\left| (T^2 - \sigma_k)^\alpha \right|}{\Gamma(\alpha + 1)} + \frac{\left| (\alpha - 1) - (\alpha - 2)T^2\sigma_k^{-1}\right|}{\Gamma(\alpha + 1)} \frac{\left| (\sigma_j - \sigma_{z-1})^{\alpha-1} \right|}{\Gamma(\alpha + 1)} \right)
\end{align*}
\[
\begin{align*}
&z[(\sigma - \sigma_\zeta)|\sigma_z^{2-a}] + z[(\alpha - 1) - (\alpha - 2)\sigma_\zeta^{-1}]\frac{|(\sigma_z - \sigma_{-1})^{a-1}|}{\Gamma(\alpha)} + \left(\frac{\mathcal{L}_\phi|\sigma^a|}{\Gamma(\alpha + 1)} + \left(\frac{(\sigma - \sigma_\zeta)^a}{\Gamma(\alpha + 1)} + \frac{|\sigma|\mathcal{T}^{1-a} - (\sigma - \sigma_\zeta)^a}{\Gamma(\alpha + 1)}\right)
\right)\nu - \zeta |.
\end{align*}
\]

Taking \(\sup_{\sigma \in \omega}\), we get

\[
\|\varphi_1(u, v) - \varphi_1(\xi, \zeta)\|_{\phi_1} \leq \left(\Delta_1 + \frac{\Delta_2\mathcal{L}_\phi|\sigma^a|}{\Gamma(\alpha + 1)} + \frac{\Delta_2\mathcal{L}_\phi|\sigma^a|}{\Gamma(\beta + 1)}\right)\|u, v) - (\xi, \zeta)\|
\]

for \(z = 1, 2, \ldots, p\),

where

\[
\begin{align*}
\Delta_1 &= (\sigma - \sigma_\zeta)|\sigma_z^{2-a}] + \frac{\frac{\mathcal{L}_\phi|\sigma^a|}{\Gamma(\alpha + 1)} + \frac{|\sigma|\mathcal{T}^{1-a} - (\sigma - \sigma_\zeta)^a}{\Gamma(\alpha + 1)}\right)\nu - \zeta |,
\end{align*}
\]

\[
\begin{align*}
\Delta_2 &= \frac{|\sigma^2 - a|}{\Gamma(\alpha + 1)} + \frac{|\sigma|\mathcal{T}^{1-a} - (\sigma - \sigma_\zeta)^a}{\Gamma(\alpha + 1)}
\end{align*}
\]

for \(z = 1, 2, \ldots, p\).

Similarly,

\[
\|\varphi_2(u, v) - \varphi_2(\xi, \zeta)\|_{\phi_2} \leq \left(\Delta_3 + \frac{\Delta_4\mathcal{L}_\phi|\sigma^a|}{\Gamma(\alpha + 1)} + \frac{\Delta_4\mathcal{L}_\phi|\sigma^a|}{\Gamma(\beta + 1)}\right)\|u, v) - (\xi, \zeta)\|
\]

for \(z = 1, 2, \ldots, q\),

where

\[
\begin{align*}
\Delta_3 &= (\sigma - \sigma_\zeta)|\sigma_z^{2-a}] + \frac{\frac{\mathcal{L}_\phi|\sigma^a|}{\Gamma(\alpha + 1)} + \frac{|\sigma|\mathcal{T}^{1-a} - (\sigma - \sigma_\zeta)^a}{\Gamma(\alpha + 1)}\right)\nu - \zeta |,
\end{align*}
\]

\[
\begin{align*}
\Delta_4 &= \frac{|\sigma^2 - a|}{\Gamma(\alpha + 1)} + \frac{|\sigma|\mathcal{T}^{1-a} - (\sigma - \sigma_\zeta)^a}{\Gamma(\alpha + 1)}
\end{align*}
\]

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By Remark 2.1 we have

\[ HUR \]

This implies that the operator \( \varphi \) is a contraction. Therefore, (1.1) has a unique solution. \( \square \)

4. Ulam’s stability analysis

In this section, we study different kinds of stabilities, like \( HU \), generalized \( HU \), \( HUR \), and generalized \( HUR \) stability of the proposed system.

**Theorem 4.1.** If assumptions \((H_3), (H_4)\) and inequality (3.19) are satisfied and

\[
\mathcal{F} = 1 - \frac{\left( \frac{\Delta_2 L_{\phi_1}}{\Gamma(\beta + 1)} \right) \left( \frac{\Delta_4 L_{\phi_2}}{\Gamma(\beta + 1)} \right)\| \varphi \|}{1 - \left( \frac{\Delta_1}{\Gamma(\alpha + 1)} \right)\| \varphi \|} > 0,
\]

then the unique solution of the coupled system (1.1) is \( HU \) stable and consequently generalized \( HU \) stable.

**Proof.** Let \((\xi, \zeta) \in \vartheta\) is a solution of inequality (2.1), and let \((u, v) \in \vartheta\) be the unique solution of the coupled system given by

\[
\begin{align*}
\mathcal{D}^\alpha u(\sigma) - \phi_1(\sigma, \mathcal{I}^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma)) &= 0, \quad \sigma \in \omega, \quad \sigma \neq \sigma_j, \quad j = 1, 2, \ldots, p, \\
\Delta u(\sigma_j) - E_j(\varphi(\sigma_j)) &= 0, \quad j = 1, 2, \ldots, p, \\
\nu_1 \mathcal{D}^{\alpha-2} u|_{\sigma=\tau} &= u_1, \\
\mathcal{D}^{\beta} v(\sigma) - \phi_2(\sigma, \mathcal{I}^\alpha u(\sigma), \mathcal{I}^\beta v(\sigma)) &= 0, \quad \sigma \in \omega, \quad \sigma \neq \sigma_k, \quad k = 1, 2, \ldots, q, \\
\Delta v(\sigma_k) - E_k(\varphi(\sigma_k)) &= 0, \quad k = 1, 2, \ldots, q, \\
\nu_3 \mathcal{D}^{\beta-2} v(\sigma)|_{\sigma=\tau} &= v_1.
\end{align*}
\]

By Remark 2.1 we have

\[
\begin{align*}
\mathcal{D}^\alpha \xi(\sigma) &= \phi_1(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \xi(\sigma)) + \mathcal{R}_1(\sigma), \\
\Delta \xi(\sigma_j) &= E_j(\xi(\sigma_j)) + \mathcal{R}_1, \quad j = 1, 2, \ldots, p, \\
\Delta \xi(\sigma_j) &= E_j^*(\xi(\sigma_j)) + \mathcal{R}_1, \quad j = 1, 2, \ldots, p, \\
\mathcal{D}^\beta \xi(\sigma) &= \phi_2(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \xi(\sigma)) + \mathcal{R}_2(\sigma), \\
\Delta \zeta(\sigma_k) &= E_k(\zeta(\sigma_k)) + \mathcal{R}_2, \quad k = 1, 2, \ldots, q, \\
\Delta \zeta(\sigma_k) &= E_k^*(\zeta(\sigma_k)) + \mathcal{R}_2, \quad k = 1, 2, \ldots, q.
\end{align*}
\]
By Corollary 1, the solution of problem (4.2) is

\[\xi(\sigma) = \frac{\alpha^{-1}u_2 - \alpha^{-1}v_1}{\mu_1 \Gamma(\alpha - 1)} + \frac{\alpha^{-2}u_1}{v_1 \Gamma(\alpha - 1)} - \frac{\nu \alpha^{-1} u_1}{\mu_1 \Gamma(\alpha - 1)} \int_0^\pi (\mathbb{T} - \pi)^{-\alpha-2} \xi(\sigma) d\pi + \frac{1}{\Gamma(\alpha)} \int_0^\pi (\mathbb{T} - \pi)^{-\alpha-1}(\phi_1(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{R}_0(\pi)) d\pi - \frac{\alpha^{-2}}{\Gamma(\alpha)} \int_0^\pi (\mathbb{T} - \pi)^{-\alpha-1}(\phi_1(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{R}_0(\pi)) d\pi \]

\[+ \sum_{j=1}^z \left[ \frac{\alpha^{-1}}{\Gamma(\alpha - 1)} \int_{\pi_j}^{\sigma_j} (\mathbb{T} - \pi)^{-\alpha-2} (\phi_1(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{R}_0(\pi)) d\pi \right] \]

\[+ \frac{\alpha^{-2}}{\Gamma(\alpha)} \int_{\pi_j}^{\sigma_j} (\mathbb{T} - \pi)^{-\alpha-2} (\phi_1(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{R}_0(\pi)) d\pi \]

\[+ \frac{(\mathbb{T} - \sigma_j)^{-1}}{\Gamma(\alpha - 1)} \int_{\pi_j}^{\sigma_j} (\mathbb{T} - \pi)^{-\alpha-2} (\phi_1(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{R}_0(\pi)) d\pi \]

\[+ \frac{(\mathbb{T} - \sigma_j)^{-2}}{\Gamma(\alpha - 1)} \int_{\pi_j}^{\sigma_j} (\mathbb{T} - \pi)^{-\alpha-1} (\phi_1(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{R}_0(\pi)) d\pi \]

for \(z = 1, 2, \ldots, p,\) \hspace{1cm} (4.3)

and

\[\xi(\sigma) = \frac{\beta^{-1}v_2 - \beta^{-1}v_1}{\mu_2 \Gamma(\beta - 1)} + \frac{\beta^{-2}v_1}{\nu \Gamma(\beta - 1)} - \frac{\nu \beta^{-1} v_1}{\mu_2 \Gamma(\beta - 1)} \int_0^\pi (\mathbb{T} - \pi)^{-\beta-2} \xi(\sigma) d\pi + \frac{1}{\Gamma(\beta)} \int_0^\pi (\mathbb{T} - \pi)^{-\beta-1}(\phi_2(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{Q}_0(\pi)) d\pi - \frac{\beta^{-1}}{\Gamma(\beta)} \int_{\pi_k}^{\sigma_k} (\mathbb{T} - \pi)^{-\beta-1}(\phi_2(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{Q}_0(\pi)) d\pi \]

\[+ \sum_{k=1}^z \left[ \frac{(\beta - 1)^{-1}}{\Gamma(\beta - 1)} \int_{\pi_k}^{\sigma_k} (\mathbb{T} - \pi)^{-\beta-2} (\phi_2(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{Q}_0(\pi)) d\pi \right] \]

\[+ \frac{(\beta - 1)^{-2}}{\Gamma(\beta - 1)} \int_{\pi_k}^{\sigma_k} (\mathbb{T} - \pi)^{-\beta-1} (\phi_2(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{Q}_0(\pi)) d\pi \]

\[+ \sum_{k=1}^z \left[ \frac{(\beta - 1)^{2}}{\Gamma(\beta)} \int_{\pi_k}^{\sigma_k} (\mathbb{T} - \pi)^{-\beta-2} (\phi_2(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{Q}_0(\pi)) d\pi \right] \]

\[+ \sum_{k=1}^z \left[ \frac{(\beta - 1)^{-1}}{\Gamma(\beta - 1)} \int_{\pi_k}^{\sigma_k} (\mathbb{T} - \pi)^{-\beta-2} (\phi_2(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{Q}_0(\pi)) d\pi \right] \]

\[+ \sum_{k=1}^z \left[ \frac{(\beta - 1)^{-2}}{\Gamma(\beta - 1)} \int_{\pi_k}^{\sigma_k} (\mathbb{T} - \pi)^{-\beta-2} (\phi_2(\pi, \mathbb{T} \xi(\pi), \mathbb{T} \xi(\pi)) + \mathbb{Q}_0(\pi)) d\pi \right] \]
We consider

\[
\begin{align*}
&\frac{(\sigma - \sigma_j)\sigma^{\beta-2}\sigma_j^{2-\beta}}{\Gamma(\beta - 1)} \int_{\sigma_{j-1}}^{\sigma_j} (\sigma - \pi)^{\beta-2} (\phi_2(\pi, I^a\xi(\pi), I^b\zeta(\pi)) + \mathcal{L}_{\phi_2}(\pi))d\pi \\
&+ \frac{(\beta - 1) - (\beta - 2)\sigma\sigma_j^{-1}}{\Gamma(\beta)} \int_{\sigma_{j-1}}^{\sigma_j} (\sigma - \pi)^{\beta-1} (\phi_2(\pi, I^a\xi(\pi), I^b\zeta(\pi)) + \mathcal{L}_{\phi_2}(\pi))d\pi,
\end{align*}
\]

\[z = 1, 2, \ldots, q.\]  

(4.4)

We consider

\[
|\sigma^{2-a}(u(\sigma) - \xi(\sigma))| \leq \frac{|\sigma^{2-a}|}{\Gamma(\alpha)} \int_{\sigma_j}^{\sigma} (\sigma - \tau)^{\alpha-1} \left|\phi_1(\tau, I^a\xi(\tau), I^b\zeta(\tau)) - \phi_1(\pi, I^a\xi(\pi), I^b\zeta(\pi))\right|d\tau \\
+ \frac{|\sigma|^{[1-a]}}{\Gamma(\alpha)} \int_{\sigma_j}^{\sigma} (\sigma - \tau)^{\alpha-1} \left|\phi_1(\tau, I^a\xi(\tau), I^b\zeta(\tau)) - \phi_1(\pi, I^a\xi(\pi), I^b\zeta(\pi))\right|d\tau \\
+ \frac{|\nu_2|\sigma|^{[1-a]}}{|\mu_1|\Gamma(\alpha - 1)} \int_{0}^{\sigma_j} (\tau - \tau)^{\alpha-1} |u(\tau) - \xi(\tau)|d\tau \\
+ \sum_{j=1}^{z} \left|\left((\sigma - \sigma_j) - \sigma_j\left(\frac{\alpha - 1}{\sigma_j} - (\alpha - 2)\sigma_j^{-1}\right)\right)\right| |\sigma_j^{2-a} - \sigma_j^{2-a}| |\phi_1(\tau, I^a\xi(\tau), I^b\zeta(\tau)) - \phi_1(\pi, I^a\xi(\pi), I^b\zeta(\pi))|d\tau \\
+ \frac{|\sigma|}{\Gamma(\alpha)} \sum_{j=1}^{z} \left|\left(\frac{\tau - \sigma_j}{\sigma_j}\right)\right| |\sigma_j^{2-a} - \sigma_j^{2-a}| \\
\times \int_{\sigma_{j-1}}^{\sigma_j} (\sigma - \tau)^{\alpha-1} \left|\phi_1(\tau, I^a\xi(\tau), I^b\zeta(\tau)) - \phi_1(\pi, I^a\xi(\pi), I^b\zeta(\pi))\right|d\tau \\
+ \frac{|\sigma|}{\Gamma(\alpha)} \sum_{j=1}^{z} \left|\left(\frac{\tau - \sigma_j}{\sigma_j}\right)\right| |\sigma_j^{2-a} - \sigma_j^{2-a}| \\
\times \int_{\sigma_{j-1}}^{\sigma_j} (\sigma - \tau)^{\alpha-1} \left|\phi_1(\tau, I^a\xi(\tau), I^b\zeta(\tau)) - \phi_1(\pi, I^a\xi(\pi), I^b\zeta(\pi))\right|d\tau \\
+ \sum_{j=1}^{z} \left|\left(\frac{\tau - \sigma_j}{\sigma_j}\right)\right| |\sigma_j^{2-a} - \sigma_j^{2-a}| \\
\times \int_{\sigma_{j-1}}^{\sigma_j} (\sigma - \tau)^{\alpha-1} \left|\phi_1(\tau, I^a\xi(\tau), I^b\zeta(\tau)) - \phi_1(\pi, I^a\xi(\pi), I^b\zeta(\pi))\right|d\tau \\
+ \frac{|\sigma|}{\Gamma(\alpha)} \sum_{j=1}^{z} \left|\left(\frac{\tau - \sigma_j}{\sigma_j}\right)\right| |\sigma_j^{2-a} - \sigma_j^{2-a}| \\
\times \int_{\sigma_{j-1}}^{\sigma_j} (\sigma - \tau)^{\alpha-1} \left|\phi_1(\tau, I^a\xi(\tau), I^b\zeta(\tau)) - \phi_1(\pi, I^a\xi(\pi), I^b\zeta(\pi))\right|d\tau \\
+ \sum_{j=1}^{z} \left|\left(\frac{\tau - \sigma_j}{\sigma_j}\right)\right| |\sigma_j^{2-a} - \sigma_j^{2-a}| \\
\times \int_{\sigma_{j-1}}^{\sigma_j} (\sigma - \tau)^{\alpha-1} \left|\phi_1(\tau, I^a\xi(\tau), I^b\zeta(\tau)) - \phi_1(\pi, I^a\xi(\pi), I^b\zeta(\pi))\right|d\tau.
\]
\[
\frac{|\sigma^{2-a}|}{\Gamma(\alpha)} \int_{\mathcal{C}} \left| (\alpha - \pi)^{a-1} \right| |\mathbf{R}_{\phi_i}(\pi)| \, d\pi + \frac{|\sigma|}{\Gamma(1-a)} \int_{\mathcal{C}} \left| (\mathcal{T} - \pi)^{a-1} \right| |\mathbf{R}_{\phi_i}(\pi)| \, d\pi \\
+ \sum_{j=1}^{z} \left| (\alpha - 1) - (\alpha - 2)\sigma \sigma_j^{-1} \right| - \frac{\sigma}{\mathcal{T}} \left| (\alpha - 1) - (\alpha - 2)\mathcal{T} \sigma_j^{-1} \right| |\sigma^{2-a}_j| |\mathbf{R}_{\phi_i}| \\
+ \sum_{j=1}^{z} \left| (\sigma - \sigma_j) - \frac{\sigma}{\mathcal{T}} (\mathcal{T} - \sigma_j) \right| |\sigma^{2-a}_j| |\mathbf{R}_{\phi_i}| \\
+ \frac{\sigma}{\mathcal{T}} \sum_{j=1}^{z} \left| \frac{\left( \mathcal{T} - \sigma_j \right) |\sigma^{2-a}_j|}{\Gamma(\alpha - 1)} \right| \int_{\sigma_{j-1}}^{\sigma_j} \left| (\sigma_j - \pi)^{a-2} \right| |\mathbf{R}_{\phi_i}(\pi)| \, d\pi \\
+ \frac{|(\alpha - 1) - (\alpha - 2)\mathcal{T} \sigma_j^{-1}| |\sigma^{2-a}_j|}{\Gamma(\alpha)} \int_{\sigma_{j-1}}^{\sigma_j} \left| (\sigma_j - \pi)^{a-1} \right| |\mathbf{R}_{\phi_i}(\pi)| \, d\pi \\
+ \sum_{j=1}^{z} \left| \frac{(\mathcal{T} - \sigma_j) |\sigma^{2-a}_j|}{\Gamma(\alpha - 1)} \right| \int_{\sigma_{j-1}}^{\sigma_j} \left| (\sigma_j - \pi)^{a-2} \right| |\mathbf{R}_{\phi_i}(\pi)| \, d\pi \\
+ \frac{|(\alpha - 1) - (\alpha - 2)\sigma \sigma_j^{-1}| |\sigma^{2-a}_j|}{\Gamma(\alpha)} \int_{\sigma_{j-1}}^{\sigma_j} \left| (\sigma_j - \pi)^{a-1} \right| |\mathbf{R}_{\phi_i}(\pi)| \, d\pi.
\]

As in Theorem 3.3, we get

\[
\|u - \xi\|_{\theta_1} \leq \left( \Delta_1 + \frac{\Delta_2 \mathcal{L}_{\phi_i} |\sigma^a|}{\Gamma(\alpha + 1)} \right) \|u - \xi\|_{\theta_1} + \left( \Delta_2 \mathcal{L}_{\phi_i}^* |\sigma^\beta| \right) \|v - \zeta\|_{\theta_1} \\
+ \left( \Delta_2 + z(\alpha - 1) |\sigma^{2-a}_z| \right) \left| 1 - \frac{\sigma}{\mathcal{T}} + z |\sigma^{3-a}_z| \left| \frac{\sigma}{\mathcal{T}} - 1 \right| + \frac{|v_2| |\sigma|}{|\mu_1| \Gamma(\alpha)} \right) \kappa_\alpha,
\]

\[z = 1, 2, \ldots , p, \quad (4.5)\]

and

\[
\|v - \zeta\|_{\theta_2} \leq \left( \Delta_3 + \frac{\Delta_3 \mathcal{L}_{\phi_i} |\sigma^\beta|}{\Gamma(\beta + 1)} \right) \|u - \xi\|_{\theta_2} + \left( \Delta_3 \mathcal{L}_{\phi_i}^* |\sigma^\beta| \right) \|v - \zeta\|_{\theta_2} \\
+ \left( \Delta_4 + z(\beta - 1) |\sigma^{2-\beta}_z| \right) \left| 1 - \frac{\sigma}{\mathcal{T}} + z |\sigma^{3-\beta}_z| \left| \frac{\sigma}{\mathcal{T}} - 1 \right| + \frac{|v_4| |\sigma|}{|\mu_2| \Gamma(\beta)} \right) \kappa_\beta,
\]

\[z = 1, 2, \ldots , q. \quad (4.6)\]

From (4.5) and (4.6), we have

\[
\|u - \xi\|_{\theta_1} - \frac{\left( \Delta_2 \mathcal{L}_{\phi_i}^* |\sigma^\beta| \right)}{1 - \left( \Delta_1 + \frac{\Delta_2 \mathcal{L}_{\phi_i} |\sigma^a|}{\Gamma(\alpha + 1)} \right)} \|v - \zeta\|_{\theta_1} \\
\leq \left( \Delta_2 + z(\alpha - 1) |\sigma^{2-a}_z| \right) \left| 1 - \frac{\sigma}{\mathcal{T}} + z |\sigma^{3-a}_z| \left| \frac{\sigma}{\mathcal{T}} - 1 \right| + \frac{|v_2| |\sigma|}{|\mu_1| \Gamma(\alpha)} \right) \kappa_\alpha \\
1 - \left( \Delta_1 + \frac{\Delta_2 \mathcal{L}_{\phi_i} |\sigma^a|}{\Gamma(\alpha + 1)} \right)
\]
where

\[
\left\| \mathbf{v} - \zeta \right\|_{\theta_2} = \frac{\left( \Delta_4 L_{\delta_1} [r_1] \right)}{1 - \left( \Delta_3 + \frac{\Delta_4 L_{\delta_1} [r_1]}{1 + (\beta + 1)} \right)} \left\| \mathbf{u} - \xi \right\|_{\theta_2} \leq \frac{\left( \Delta_3 + \frac{\Delta_4 L_{\delta_1} [r_1]}{1 + (\beta + 1)} \right)}{1 - \left( \Delta_3 + \frac{\Delta_4 L_{\delta_1} [r_1]}{1 + (\beta + 1)} \right)} \left\| \mathbf{v} - \zeta \right\|_{\theta_2}
\]

respectively. Let

\[
P_1 = \frac{\left( \Delta_3 + \frac{\Delta_4 L_{\delta_1} [r_1]}{1 + (\beta + 1)} \right)}{1 - \left( \Delta_1 + \frac{\Delta_3 + \frac{\Delta_4 L_{\delta_1} [r_1]}{1 + (\beta + 1)} \right)}
\]

\[
P_2 = \frac{\left( \Delta_2 + \frac{\Delta_3 + \frac{\Delta_4 L_{\delta_1} [r_1]}{1 + (\beta + 1)} \right)}{1 - \left( \Delta_1 + \frac{\Delta_3 + \frac{\Delta_4 L_{\delta_1} [r_1]}{1 + (\beta + 1)} \right)}
\]

Then the last two inequalities can be written in a matrix form as follows:

\[
\begin{bmatrix}
1 & -P_1 & \left\| \mathbf{v} - \zeta \right\|_{\theta_2} \\
-\mathcal{F} & 1 & 1 & \left\| \mathbf{u} - \xi \right\|_{\theta_1}
\end{bmatrix}
\leq
\begin{bmatrix}
P_2 \mathcal{K}_a \\
P_3 \mathcal{K}_b
\end{bmatrix}
\]

\[
\begin{bmatrix}
\left\| \mathbf{u} - \xi \right\|_{\theta_1} \\
\left\| \mathbf{v} - \zeta \right\|_{\theta_2}
\end{bmatrix}
\leq
\begin{bmatrix}
P_1 \mathcal{F} \\
P_3 \mathcal{F}
\end{bmatrix}
\begin{bmatrix}
P_2 \mathcal{K}_a \\
P_3 \mathcal{K}_b
\end{bmatrix}. \quad (4.7)
\]

where

\[
\mathcal{F} = 1 - \left( \frac{\Delta_4 L_{\delta_1} [r_1]}{1 + (\beta + 1)} \right) \left\| \mathbf{v} - \zeta \right\|_{\theta_2} > 0.
\]

From system (4.7) we have

\[
\left\| \mathbf{u} - \xi \right\|_{\theta_1} \leq \frac{P_2 \mathcal{K}_a}{\mathcal{F}} + \frac{P_3 \mathcal{K}_b}{\mathcal{F}},
\]

\[
\left\| \mathbf{v} - \zeta \right\|_{\theta_2} \leq \frac{P_2 \mathcal{F} \mathcal{K}_a}{\mathcal{F}} + \frac{P_3 \mathcal{F} \mathcal{K}_b}{\mathcal{F}},
\]

which implies that

\[
\left\| \mathbf{u} - \xi \right\|_{\theta_1} + \left\| \mathbf{v} - \zeta \right\|_{\theta_2} \leq \frac{P_2 \mathcal{K}_a}{\mathcal{F}} + \frac{P_3 \mathcal{K}_b}{\mathcal{F}} + \frac{P_2 \mathcal{F} \mathcal{K}_a}{\mathcal{F}} + \frac{P_3 \mathcal{F} \mathcal{K}_b}{\mathcal{F}}.
\]
If $\kappa = \max\{\kappa_{\alpha}, \kappa_{\beta}\}$ and $N_{\alpha,\beta} = \frac{p_{2}}{\bar{F}} + \frac{p_{3}p_{4}}{\bar{F}} + \frac{p_{5}}{\bar{F}} + \frac{p_{6}}{\bar{F}}$, then

$$
\|(u, v) - (\xi, \zeta)\|_\theta \leq N_{\alpha,\beta} \kappa.
$$

Thus system (1.1) is $HU$ stable. Also, if

$$
\|(u, v) - (\xi, \zeta)\|_\theta \leq N_{\alpha,\beta} \kappa',
$$

with $N'(0) = 0$, then the given system (1.1) is generalized $HU$ stable.

For the next result, we assume the following:

(H$_5$) Let there exists two nondecreasing functions $w_{\alpha}, w_{\beta} \in C(\omega, \mathbb{R}^+)$ such that

$$
I^n w_{\alpha}(\sigma) \leq L_{\alpha} w_{\alpha}(\sigma) \quad \text{and} \quad I^\beta w_{\beta}(\sigma) \leq L_{\beta} w_{\beta}(\sigma), \quad \text{where} \quad L_{\alpha}, L_{\beta} > 0.
$$

Theorem 4.2. If assumptions (H$_3$)–(H$_5$) and inequality (3.19) are satisfied and

$$
F = 1 - \frac{\left(\Delta_{2}L_{\alpha} |\sigma|^\alpha \Gamma(\alpha + 1) \right) \left(\Delta_{4}L_{\beta} |\sigma|^\beta \Gamma(\beta + 1) \right)}{\left[1 - \left(\Delta_{1} + \frac{\Delta_{2}L_{\alpha} |\sigma|^\alpha \Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \right) \right] \left[1 - \left(\Delta_{3} + \frac{\Delta_{4}L_{\beta} |\sigma|^\beta \Gamma(\beta + 1)}{\Gamma(\beta + 1)} \right) \right]} > 0,
$$

then the unique solution of the given system (1.1) is $HUR$ stable and accordingly generalized $HUR$ stable.

Proof. With the help of Definitions 2.5 and 2.6, we can achieve our result doing the same steps as in Theorem 4.1.

5. Example

Here we present a specific example, as follows.
Example 5.1. Let

\[
\begin{align*}
\mathcal{D}^\delta u(\sigma) - \frac{2 + I^\delta u(\sigma) + I^\delta v(\sigma)}{80e^{\sigma+90}(1 + I^\delta u(\sigma) + I^\delta v(\sigma))} &= 0, \quad \sigma \neq \frac{3}{2}, \\
\Delta u \left( \frac{3}{2} \right) = \mathcal{E}_1 \left( \frac{u \left( \frac{3}{2} \right)}{2} \right) &= \frac{|u(\frac{3}{2})|}{70 + |u(\frac{3}{2})|}, \\
\Delta u' \left( \frac{3}{2} \right) = \mathcal{E}_1' \left( \frac{u \left( \frac{3}{2} \right)}{2} \right) &= \frac{|u(\frac{3}{2})|}{70 + |u(\frac{3}{2})|}, \\
\mathcal{D}^{-1} u(\sigma)|_{r=0} &= u_1, \\
\frac{-50u(\sigma)|_{r=c} + \frac{1}{85} I^\delta u(\sigma)|_{r=c} = u_2, \\
\mathcal{D}^\delta v(\sigma) - \frac{\sigma \cos(u(\sigma)) - v(\sigma) \sin(\sigma)}{95} - \frac{u(\sigma)}{95 + u(\sigma)} &= 0, \quad \sigma \neq \frac{3}{2}, \\
\Delta v \left( \frac{3}{2} \right) = \mathcal{E}_1 \left( \frac{v \left( \frac{3}{2} \right)}{2} \right) &= \frac{|v(\frac{3}{2})|}{70 + |v(\frac{3}{2})|}, \\
\Delta v' \left( \frac{3}{2} \right) = \mathcal{E}_1' \left( \frac{v \left( \frac{3}{2} \right)}{2} \right) &= \frac{|v(\frac{3}{2})|}{70 + |v(\frac{3}{2})|}, \\
\mathcal{D}^{-1} v(\sigma)|_{r=0} &= v_1, \\
\frac{-50v(\sigma)|_{r=c} + \frac{1}{85} I^\delta v(\sigma)|_{r=c} = v_2.
\end{align*}
\]

From system (5.1), we see that \( \alpha = \frac{6}{5}, \beta = \frac{5}{4}, \mu_1 = \mu_2 = -50, v_1 = v_3 = 1, v_2 = v_4 = \frac{1}{85}, T = e, \sigma_1 = \frac{3}{2}, \) and \( u_1, u_2, v_1, v_2 \in \mathbb{R}. \)

Set

\[
\begin{align*}
\phi_1(\sigma, u, v) &= \frac{2 + I^\delta u(\sigma) + I^\delta v(\sigma)}{80e^{\sigma+90}(1 + I^\delta u(\sigma) + I^\delta v(\sigma))}, \\
\phi_2(\sigma, u, v) &= \frac{\sigma \cos(u(\sigma)) - v(\sigma) \sin(\sigma)}{95} - \frac{u(\sigma)}{95 + u(\sigma)}.
\end{align*}
\]

Now, for all \( u, u', v, v' \in \mathbb{R}, \) and \( \sigma \in [0, e], \) we obtain

\[
|\phi_1(\sigma, u, v) - \phi_1(\sigma, u', v')| = \frac{1}{80e^{90}}|u - u'| + \frac{1}{80e^{90}}|v - v'|
\]

and

\[
|\phi_2(\sigma, u, v) - \phi_1(\sigma, u', v')| = \frac{1}{95}|u - u'| + \frac{1}{95}|v - v'|.
\]

These satisfy condition \((H_3)\) with \( \mathcal{L}_{\phi_1} = \mathcal{L}_{\phi_1}' = \frac{1}{80e^{90}}, \mathcal{L}_{\phi_2} = \mathcal{L}_{\phi_2}' = \frac{1}{95}. \)

Set

\[
\begin{align*}
\mathcal{E}_1 \left( \frac{u \left( \frac{3}{2} \right)}{2} \right) &= \frac{|u(\frac{3}{2})|}{70 + |u(\frac{3}{2})|}, \\
\mathcal{E}_1' \left( \frac{u \left( \frac{3}{2} \right)}{2} \right) &= \frac{|u(\frac{3}{2})|}{70 + |u(\frac{3}{2})|}, \\
\mathcal{E}_1 \left( \frac{v \left( \frac{3}{2} \right)}{2} \right) &= \frac{|v(\frac{3}{2})|}{70 + |v(\frac{3}{2})|}, \\
\mathcal{E}_1' \left( \frac{v \left( \frac{3}{2} \right)}{2} \right) &= \frac{|v(\frac{3}{2})|}{70 + |v(\frac{3}{2})|}.
\end{align*}
\]
Then we have

\[
\left| \mathcal{E}_1 \left( u \left( \frac{3}{2} \right) \right) - \mathcal{E}_1 \left( u^* \left( \frac{3}{2} \right) \right) \right| = \frac{1}{70} |u - u^*|, \quad \left| \mathcal{E}_1 \left( u \left( \frac{3}{2} \right) \right) - \mathcal{E}_1^* \left( u^* \left( \frac{3}{2} \right) \right) \right| = \frac{1}{70} |u - u^*|.
\]

\[
\left| \mathcal{E}_1 \left( v \left( \frac{3}{2} \right) \right) - \mathcal{E}_1 \left( v^* \left( \frac{3}{2} \right) \right) \right| = \frac{1}{70} |v - v^*| \quad \text{and} \quad \left| \mathcal{E}_1 \left( v \left( \frac{3}{2} \right) \right) - \mathcal{E}_1^* \left( v^* \left( \frac{3}{2} \right) \right) \right| = \frac{1}{70} |v - v^*|.
\]

These satisfy condition $(H_4)$ with $L = L^* = L^* = \frac{1}{70}$.

From Theorem 3.3, we use the inequality and get

\[
\Delta_1 + \Delta_3 + \frac{(\Delta_2 L_{\phi_1} + \Delta_4 L_{\phi_2})|\sigma|}{\Gamma(\alpha + 1)} + \frac{(\Delta_2 L^*_{\phi_1} + \Delta_4 L^*_{\phi_2})|\sigma^*|}{\Gamma(\beta + 1)} \approx 0.976847 < 1,
\]

hence (5.1) has a unique solution, so (5.1) has a solution $(u, v) \in \varnothing$. The solution of (5.1) is given by...
\[
\begin{align*}
\frac{\sigma^2 u_2}{50e^{\frac{\sigma}{4}}} - \frac{\sigma^2 u_1}{\Gamma(\frac{1}{2})} + \frac{\sigma^{-\frac{1}{2}} u_1}{\Gamma(\frac{1}{2})} + \frac{1}{\Gamma(\frac{5}{2})} \int_0^\infty (\sigma - \pi)^{\frac{3}{2}} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
- \frac{\sigma^2 e^{-\frac{\sigma}{4}}}{\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{\frac{1}{2}} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi + \frac{1}{8\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{-\frac{1}{2}} u(\pi) d\pi \\
- \frac{\sigma^2}{e} \left[ \left( \left( \frac{1}{4} \right) + e \left( \frac{1}{2} \right) \frac{3}{2} \right) \frac{3}{2} \right] \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
+ \frac{(e^{\frac{3}{2}} \left( \frac{1}{2} \right)^{\frac{3}{2}})}{\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{\frac{3}{2}} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
+ \frac{\sigma}{e} \left[ \left( \left( \frac{3}{4} \right) + e \left( \frac{3}{2} \right) \frac{1}{2} \right) \frac{3}{2} \right] \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
+ \frac{(e^{\frac{3}{2}} \left( \frac{1}{2} \right)^{\frac{3}{2}})}{\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{\frac{3}{2}} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
+ \frac{1}{8\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{-\frac{1}{2}} u(\pi) d\pi \\
\frac{\sigma^2 u_2}{50e^{\frac{\sigma}{4}}} - \frac{\sigma^2 u_1}{\Gamma(\frac{1}{2})} + \frac{\sigma^{-\frac{1}{2}} u_1}{\Gamma(\frac{1}{2})} + \frac{1}{\Gamma(\frac{5}{2})} \int_0^\infty (\sigma - \pi)^{\frac{3}{2}} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
- \frac{\sigma^2 e^{-\frac{\sigma}{4}}}{\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{\frac{1}{2}} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi + \frac{1}{8\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{-\frac{1}{2}} u(\pi) d\pi \\
- \frac{\sigma^2}{e} \left[ \left( \left( \frac{1}{4} \right) + e \left( \frac{1}{2} \right) \frac{3}{2} \right) \frac{3}{2} \right] \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
+ \frac{(e^{\frac{3}{2}} \left( \frac{1}{2} \right)^{\frac{3}{2}})}{\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{\frac{3}{2}} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
+ \frac{\sigma}{e} \left[ \left( \left( \frac{3}{4} \right) + e \left( \frac{3}{2} \right) \frac{1}{2} \right) \frac{3}{2} \right] \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
+ \frac{(e^{\frac{3}{2}} \left( \frac{1}{2} \right)^{\frac{3}{2}})}{\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{\frac{3}{2}} \phi_1(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
+ \frac{1}{8\Gamma(\frac{5}{2})} \int_0^\infty (e - \pi)^{-\frac{1}{2}} u(\pi) d\pi \\
\sigma \in \left[ \frac{3}{2}, e \right]
\end{align*}
\]
\[
\sigma^2 = \begin{cases} 
\sigma^2_{v_2} - \frac{\sigma^2_{v_1}}{e^{\Gamma(\frac{1}{2})}} + \frac{\sigma^{-\frac{3}{2}}}{\Gamma(\frac{1}{2})} & + \frac{1}{\Gamma(\frac{1}{2})} \int_0^{\infty} (\sigma - \pi)^{\frac{1}{2}} \phi_2(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi \\
-50e^{-\frac{e^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}} \int_0^{\infty} (e - \pi)^{\frac{1}{2}} \phi_2(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi & + \frac{1}{85} \int_0^{\infty} (e - \pi)^{-\frac{3}{2}} v(\pi) d\pi \\
\int_0^{\infty} (\sigma - \pi)^{-\frac{1}{2}} \phi_2(\pi, I^\alpha u(\pi), I^\beta v(\pi)) d\pi & - \frac{1}{50} \int_0^{\infty} (e - \pi)^{-\frac{3}{2}} v(\pi) d\pi \\
\end{cases}
\]

\[
\sigma \in \left[0, \frac{3}{2}\right], \\
\sigma \in \left[\frac{3}{2}, e\right].
\]
(i) If we take
\[ \phi_1(\sigma, \Gamma^\alpha u(\sigma), \Gamma^\beta v(\sigma)) = \frac{1}{80e^{\sigma + 90}}, \quad \phi_2(\sigma, \Gamma^\alpha u(\sigma), \Gamma^\beta v(\sigma)) = \frac{\sigma \cos(\sigma) - \sin(\sigma)}{95} - \frac{1}{95}, \]
\[ E_1(u(\sigma)) = E_1(v(\sigma)) = \frac{1}{70}, \quad \text{and} \quad u(\sigma) = v(\sigma) = \sigma \]
then with the constant values \( u_1 = v_1 = \frac{1}{15}, \quad u_2 = v_2 = 2 \), the graph of the solution is shown in Figure 1.

(ii) If we take
\[ \phi_1(\sigma, \Gamma^\alpha u(\sigma), \Gamma^\beta v(\sigma)) = \frac{\sigma + 1}{80e^{\sigma + 90}}, \quad \phi_2(\sigma, \Gamma^\alpha u(\sigma), \Gamma^\beta v(\sigma)) = \frac{\sigma^2 + 1}{95} - \frac{\sigma}{95}, \]
\[ E_1(u(\sigma)) = E_1(v(\sigma)) = \frac{1}{70}, \quad \text{and} \quad u(\sigma) = v(\sigma) = \sigma \]
then with the constant values \( u_1 = v_1 = -\frac{1}{15}, \quad u_2 = v_2 = -2 \), the graph of the solution is shown in Figure 2.

From Theorem 4.1, we use the inequality and get
\[ \mathcal{F} = 1 - \frac{\Delta_1 L_1 [r^m]}{1 + (\alpha + 1)} \left[ 1 - \left( \Delta_3 + \frac{\Delta_4 L_4 [r^n]}{1 + (\beta + 1)} \right) \right] \approx 1 > 0, \]
thus, the given system (5.1) is \(\mathcal{HU}\) stable and also generalized \(\mathcal{HU}\) stable. Likewise, we can justify the condition of Theorems 3.2 and 4.2.

6. Conclusion

In this article, we used the Kransnoselskii’s fixed point theorem and acquired the necessary cases for the existence and uniqueness of solution for the given fractional integro-differential Eqs (1.1). Furthermore, under specific assumptions and conditions, we proved different kinds of Ulam’s stability of system (1.1). The concept of Ulam’s stability is very important because it gives a relationship between approximate and exact solutions, so our results may be very helpful in approximation theory and numerical analysis. The mentioned stability is rarely investigated for impulsive fractional integro-differential equations. Finally, we illustrated the main results by giving a suitable example.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. B. Ahmad, J. J. Nieto, Riemann-Liouville fractional differential equations with fractional boundary conditions, *Fixed Point Theor.*, 13 (2012), 329–336.
2. B. Ahmad, S. Sivasundaram, On four point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order, *Appl. Math. Comput.*, 217 (2010), 480–487.
3. N. Ahmad, Z. Ali, K. Shah, A. Zada, G. Rahman, Analysis of implicit type nonlinear dynamical problem of impulsive fractional differential equations, *Complexity*, 2018 (2018), 1–15.
4. Z. Ali, A. Zada, K. Shah, Ulam stability to a toppled systems of nonlinear implicit fractional order boundary value problem, *Bound. Value Probl.*, 2018 (2018), 1–16.
5. M. Altman, A fixed point theorem for completely continuous operators in Banach spaces, *Bull. Acad. Polon. Sci. Cl. III*, 3 (1955), 409–413.
6. M. Benchohra, D. Seba, Impulsive fractional differential equations in Banach spaces, *Electron. J. Qual. Theo.*, 8 (2009), 1–14.
7. M. El-Shahed, J. J. Nieto, Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order, *Comput. Math. Appl.*, 59 (2010), 3438–3443.
8. M. Feckan, Y. Zhou, J. Wang, On the concept and existence of solution for impulsive fractional differential equations, *Commun. Nonlinear Sci.*, 17 (2012), 3050–3060.
9. X. Hao, L. Zhang, L. Liu, Positive solutions of higher order fractional integral boundary value problem with a parameter, *Nonlinear Anal-Model.*, **24** (2019), 210–223.

10. X. Hao, L. Zhang, Positive solutions of a fractional thermostat model with a parameter, *Symmetry*, **11** (2019), 1–9.

11. X. Hao, H. Sun, L. Liu, Existence results for fractional integral boundary value problem involving fractional derivatives on an infinite interval, *Math. Meth. Appl. Sci.*, **41** (2018), 6984–6996.

12. S. M. Jung, Hyers-Ulam stability of linear differential equations of first order, *Appl. Math. Lett.*, **19** (2006), 854–858.

13. R. A. Khan, K. Shah, Existence and uniqueness of solutions to fractional order multi-point boundary value problems, *Commun. Appl. Anal.*, **19** (2015), 515–526.

14. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and application of fractional differential equation*, North-Holl and Mathematics Studies, 2006, 1–523.

15. N. Kosmatov, Initial value problems of fractional order with fractional impulsive conditions, *Results Math.*, **63** (2013), 1289–1310.

16. V. Lakshmikantham, S. Leela, J. V. Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009, 1–170.

17. M. Obloza, Hyers stability of the linear differential equation, *Rocznik Nauk-Dydakt, Prace Mat.*, **13** (1993), 259–270.

18. K. B. Oldham, Fractional differential equations in electrochemistry, *Adv. Eng. softw.*, **41** (2010), 9–12.

19. I. Podlubny, *Fractional differential equations*, Academic Press, 1998.

20. U. Riaz, A. Zada, Z. Ali, Y. Cui, J. Xu, Analysis of nonlinear coupled systems of impulsive fractional differential equations with Hadamard derivatives, *Math. Probl. Eng.*, **2019** (2019), 1–20.

21. U. Riaz, A. Zada, Z. Ali, Y. Cui, J. Xu, Analysis of coupled systems of implicit impulsive fractional differential equations involving Hadamard derivatives, *Adv. Differ. Equ.*, **2019** (2019), 1–27.

22. F. A. Rihan, Numerical Modeling of Fractional Order Biological Systems, *Abstr. Appl. Anal.*, **2013** (2013), 1–11.

23. R. Rizwan, A. Zada, X. Wang, Stability analysis of non linear implicit fractional Langevin equation with non-instantaneous impulses, *Adv. Differ. Equ.*, **2019** (2019), 1–31.

24. I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathina J. Math.*, **26** (2010), 103–107.

25. J. Sabatier, O. P. Agrawal, J. A. T. Machado, *Advances in fractional calculus*, Springer, 2007.

26. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach Science, 1993.

27. K. Shah, R. A. Khan, Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti-periodic boundary conditions, *Differ. Equ. Appl.*, **7** (2015), 245–262.
28. V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of particles, Fields and Media*, Springer, 2010.

29. S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience, 1960.

30. B. M. Vintagre, I. Podlybni, A. Hernandez, V. Feliu, Some approximations of fractional order operators used in control theory and applications, *Fract. Calc. Appl. Anal.*, 4 (2000), 47–66.

31. J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron. J. Qual. Theo.*, 2011 (2011), 1–10.

32. J. Wang, A. Zada, H. Waheed, Stability analysis of a coupled system of nonlinear implicit fractional anti-periodic boundary value problem, *Math. Method. Appl. Sci.*, 42 (2019), 6706–6732.

33. A. Zada, S. Ali, Stability Analysis of multi-point boundary value problem for sequential fractional differential equations with non-instantaneous impulses, *Int. J. Nonlin. Sci. Num.*, 19 (2018), 763–774.

34. A. Zada, S. Ali, Stability of integral Caputo-type boundary value problem with non-instantaneous impulses, *Int. J. Appl. Comput. Math.*, 5 (2019), 55.

35. A. Zada, W. Ali, S. Farina, Hyers-Ulam stability of nonlinear differential equations with fractional integrable impulses, *Math. Method. App. Sci.*, 40 (2017), 5502–5514.

36. A. Zada, S. Ali, Y. Li, Ulam-type stability for a class of implicit fractional differential equations with non-instantaneous integral impulses and boundary condition, *Adv. Differ. Equ.*, 2017 (2017), 1–26.

37. A. Zada, W. Ali, C. Park, Ulam’s type stability of higher order nonlinear delay differential equations via integral inequality of Grönwall-Bellman-Bihari’s type, *Appl. Math. Comput.*, 350 (2019), 60–65.

38. A. Zada, B. Dayyan, Stability analysis for a class of implicit fractional differential equations with instantaneous impulses and Riemann-Liouville boundary conditions, *Annals of the University of Craiova, Mathematics and Computer Science Series*, 47 (2020), 88–110.

39. A. Zada, F. Khan, U. Riaz, T. Li, Hyers-Ulam stability of linear summation equations, *Punjab University Journal of Mathematics*, 49 (2017), 19–24.

40. A. Zada, U. Riaz, F. Khan, Hyers-Ulam stability of impulsive integral equations, *B. Unione Mat. Ital.*, 12 (2019), 453–467.