Adversarial Risk Bounds for Binary Classification via Function Transformation

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Abstract

We derive new bounds for a notion of adversarial risk, characterizing the robustness of binary classifiers. Specifically, we study the cases of linear classifiers and neural network classifiers, and introduce transformations with the property that the risk of the transformed functions upper-bounds the adversarial risk of the original functions. This reduces the problem of deriving adversarial risk bounds to the problem of deriving risk bounds using standard learning-theoretic techniques. We then derive bounds on the Rademacher complexities of the transformed function classes, obtaining error rates on the same order as the generalization error of the original function classes. Finally, we provide two algorithms for optimizing the adversarial risk bounds in the linear case, and discuss connections to regularization and distributional robustness.

1 Introduction

Deep learning systems are becoming ubiquitous in everyday life. From virtual assistants on phones to image search and translation, neural networks have vastly improved the performance of many computerized systems in a short amount of time (Goodfellow et al., 2016). However, neural networks have a variety of shortcomings: A peculiarity that has gained much attention over the past few years has been the apparent lack of robustness of neural network classifiers to adversarial perturbations. Szegedy et al. (2013) noticed that small perturbations to images could cause neural network classifiers to predict the wrong class. Further, these perturbations could be carefully chosen so as to be imperceptible to humans.

Such observations have instigated a deluge of research in finding adversarial attacks (Athalye et al., 2018; Goodfellow et al., 2014; Papernot et al., 2016; Szegedy et al., 2013), defenses against adversaries for neural networks (Madry et al., 2018; Raghunathan et al., 2018; Sinha et al., 2018; Wong and Kolter, 2018), evidence that adversarial examples are inevitable (Shafahi et al., 2018), and theory suggesting that constructing robust classifiers is computationally infeasible (Bubeck et al., 2018). Attacks are usually constructed assuming a white-box framework, in which the adversary has access to the network, and adversarial examples are generated using a perturbation roughly in the direction of the gradient of the loss function with respect to a training data point. This idea generally produces adversarial examples that can break ad-hoc defenses in image classification.

Currently, strategies for creating robust classification algorithms are much more limited. One approach (Madry et al., 2018; Suggala et al., 2018) is to formalize the problem of robustifying the network as a novel optimization problem, where the objective function is the expected loss of

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a supremum over possible perturbations. However, Madry et al. (2018) note that the objective function is often not concave in the perturbation. Other authors (Raghunathan et al., 2018; Wong and Kolter, 2018) have leveraged convex relaxations to provide optimization-based certificates on the adversarial loss of the training data. However, the generalization performance of the training error to unseen examples is still not understood.

The optimization community has long been interested in constructing robust solutions for various problems, such as portfolio management (Ben-Tal et al., 2009), and deriving theoretical guarantees. Robust optimization has been studied in the context of regression and classification (Trafalis and Gilbert, 2007; Xu et al., 2009a,b). More recently, a notion of robustness that attempts to minimize the risk with respect to the worst-case distribution close to the empirical distribution has been the subject of extensive work (Ben-Tal et al., 2013; Namkoong and Duchi, 2016, 2017). Researchers have also considered a formulation known as distributionally robust optimization, using the Wasserstein distance as a metric between distributions (Esfahani and Kuhn, 2015; Blanchet and Kang, 2017; Gao et al., 2017; Sinha et al., 2018). With the exception of Sinha et al. (2018), generalization bounds of a learning-theoretic nature are nonexistent, with most papers focusing on studying properties of a regularized reformulation of the problem. Sinha et al. (2018) provide bounds for Wasserstein distributionally robust generalization error based on covering numbers for sufficiently small perturbations. This is sufficient for ensuring a small amount of adversarial robustness and is quite general; but for classification using neural networks, known covering number bounds (Bartlett et al., 2017) are substantially weaker than Rademacher complexity bounds (Golowich et al., 2018).

Although neural networks are rightly the subject of attention due to their ubiquity and utility, the theory that has been developed to explain the phenomena arising from adversarial examples is still far from complete. For example, Goodfellow et al. (2014) argue that non-robustness may be due to the linear nature of neural networks. However, attempts at understanding linear classifiers (Fawzi et al., 2018) argue against linearity, i.e., the function classes should be more expressive than linear classification.

In this paper, we provide upper bounds for a notion of adversarial risk in the case of linear classifiers and neural networks. These bounds may be viewed as a sample-based guarantee on the risk of a trained classifier, even in the presence of adversarial perturbations on the inputs. The key step is to transform a classifier $f$ into an “adversarially-perturbed” classifier $\Phi f$ by modifying the loss function. The risk of the function $\Phi f$ can then be analyzed in place of the adversarial risk of $f$; in particular, we can more easily provide bounds on the Rademacher complexities necessary for bounding the robust risk. Finally, our transformations suggest algorithms for minimizing the adversarially robust empirical risk. Thus, from the theory developed in this paper, we can show that adversarial perturbations have somewhat limited effects from the point of view of generalization error.

This paper is organized as follows: We introduce the precise mathematical framework in Section 2. In Section 3, we discuss our main results. In Section 4, we provide results on optimizing the adversarial risk bounds. In Section 5, we prove our key theoretical contributions. Finally, we conclude with a discussion of future avenues of research in Section 6.

**Notation:** For a matrix $A$, we write $\|A\|_{\infty}$ to denote the $\ell_{\infty}$-operator norm. We write $\|A\|_F$ to denote the Frobenius norm. For a vector $v$, we write $\|v\|_p$ to denote the $\ell_p$-norm.

## 2 Setup

We consider a standard statistical learning setup. Let $\mathcal{X} \subseteq \mathbb{R}^m$ be a space of covariates, and define the space of labels to be $\mathcal{Y} = \{+1, -1\}$. Let $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$. Suppose we have $n$ observations...
\( z_1 = (x_1, y_1), \ldots, z_n = (x_n, y_n) \), drawn i.i.d. according to some unknown distribution \( P \). We write \( S = \{z_1, \ldots, z_n\} \).

A classifier corresponds to a function \( f : \mathcal{X} \rightarrow \mathcal{D} \), where \( \mathcal{Y} \subseteq \mathcal{D} \). Thus, the function \( f \) may express uncertainty in its decision; e.g., prediction in \( \mathcal{D} = [-1, +1] \) allows the classifier to select an expected outcome.

### 2.1 Risk and Losses

Given a loss function \( \ell : \mathcal{D} \times \mathcal{Z} \rightarrow \mathbb{R}^+ \), our goal is to minimize the **adversarially robust risk**, defined by

\[
R_{\text{rob}}(\ell, f) = \mathbb{E}_{z \sim P} \left[ \sup_{w \in B(\varepsilon)} \ell(f, z + w) \right],
\]

where \( w \) is an adversarially chosen perturbation in the \( \ell_p \)-ball \( B(\varepsilon) \subseteq \mathbb{R}^m \) of radius \( \varepsilon \). For simplicity, we write \( z + w = (x + w, y) \), so the input is perturbed by a vector in the \( \ell_p \)-ball of radius \( \varepsilon \), but still classified according to \( f(x) \). Usually in the literature, \( p \) is taken to be 1, 2, or \( \infty \); the case \( p = \infty \) has received particular interest. Also note that if \( \varepsilon = 0 \), the adversarial risk reduces to the usual statistical risk, for which upper bounds based on the empirical risk are known as generalization error bounds. For some discussion of the relationship between the adversarial risk to the distributionally robust risk, see Appendix E.

We now define a few specific loss functions. The indicator loss

\[
\ell_{01}(f, z) = 1 \{ \text{sgn} f(x) = y \}
\]

is of primary interest in classification; in both the linear classifier and neural network classification settings, we will primarily be interested in bounding the adversarial risk with respect to the indicator loss. As is standard in linear classification, we also define the hinge loss

\[
\ell_h(f, z) = \max \{0, 1 - y f(x)\},
\]

which is a convex surrogate for the indicator loss, and will appear in some of our bounds. We also introduce the indicator of whether the hinge loss is positive, defined by

\[
\ell_{h,01}(f, z) = 1 \{ \ell_h(f, z) > 0 \}.
\]

For analyzing neural networks, we will also employ the cross-entropy loss, defined by

\[
\ell_{xe}(f, z) = - \left( \frac{1 + y}{2} \right) \log_2 \left( \frac{1 + \delta(f(x))}{2} \right) - \left( \frac{1 - y}{2} \right) \log_2 \left( \frac{1 - \delta(f(x))}{2} \right),
\]

where \( \delta \) is the softmax function:

\[
\delta(w) = \frac{\exp(w) - 1}{\exp(w) + 1}.
\]

Note that in all of the cases above, we can also write the loss \( \ell(f, z) = \bar{\ell}(f(x), y) \), for an appropriately defined loss \( \bar{\ell} : \mathcal{D} \rightarrow \mathbb{R}^+ \). Furthermore, \( \ell_h \) and \( \ell_{xe} \) are 1-Lipschitz.

### 2.2 Function Classes and Rademacher Complexity

We are particularly interested in two function classes: linear classifiers and neural networks. We denote the first class by \( \mathcal{F}_{\text{lin}} \), and we write an element \( f \) of \( \mathcal{F}_{\text{lin}} \), parametrized by \( \theta \in \mathbb{R}^m \) and \( b \in \mathbb{R} \), as

\[
f(x) = \theta^\top x + b.
\]
We similarly denote the class of neural networks as $\mathcal{F}_{nn}$, and we write a neural network $f$, parametrized by $\{A_k\}$ and $\{s_k\}$, as

$$f(x) = A_{d+1}s_d(A_ds_{d-1}(\ldots s_1(A_1x))),$$

where each $A_k$ is a matrix and each $s_k$ is a monotonically increasing 1-Lipschitz activation function applied elementwise to vectors, such that $s_k(0) = 0$. For example, we might have $s_k(u) = \max\{0, u\}$, which is the ReLU function. The matrix $A_k$ is of dimension $J_k \times J_{k-1}$, where $J_0 = m$ and $J_{d+1} = 1$.

We use $(a_{jk}^r)$ to denote the $j$th row of $A_k$, with $r$th entry $a_{jk}^r$. Also, when discussing indices, we write $j_{2:d+1}$ as shorthand for $j_2, \ldots, j_{d+1}$.

A standard measure of the complexity of a class of functions is the Rademacher complexity. The empirical Rademacher complexity of a function class $\mathcal{F}$ and a sample $S$ is

$$\hat{\mathcal{R}}_n(\mathcal{F}) = \frac{1}{n} \mathbb{E}_\sigma \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^n \sigma_i f(x_i) \right] ,$$

where the $\sigma_i$’s are i.i.d. Rademacher random variables; i.e., the $\sigma_i$’s are random variables taking the values $+1$ and $-1$, each with probability $1/2$. Note that $\mathbb{E}_\sigma$ denotes the expectation with respect to the $\sigma_i$’s. Finally, we note that the standard Rademacher complexity is obtained by taking an expectation over the data: $\mathcal{R}_n(\mathcal{F}) = \mathbb{E}\hat{\mathcal{R}}_n(\mathcal{F})$.

3 Main Results

We introduce our main results in this section. The trick is to push the supremum through the loss and incorporate it into the function $f$, yielding a transformed function $\Phi f$. We require this transformation to satisfy

$$\sup_{w \in B(\varepsilon)} \ell(f, z + w) \leq \ell(\Phi f, z),$$

so an upper bound on the transformed risk leads to an upper bound on the adversarial risk. We call the proposed functions $\Phi$ the supremum transformation and tree transformation in the cases of linear classifiers and neural networks, respectively.

In both cases, we have to make a minor assumption about the loss. The assumption is that $\ell(f, z)$ is monotonically decreasing in $yf(x)$: Specifically, $\ell(f(x), +1)$ is decreasing in $f(x)$ and $\ell(f(x), -1)$ is increasing in $f(x)$. This is not a stringent assumption, and is satisfied by all of the loss functions mentioned earlier.

One technicality is that the transformed function $\Phi f$ needs to be a function of both $x$ and $y$; i.e., we have $\Phi f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{D}$. Thus, the loss of a transformed function is $\ell(\Phi f, z) = \ell(\Phi f(x, y), y)$. We now define the essential transformations studied in our paper.

**Definition 1.** The supremum (sup) transform $\Psi$ is defined by

$$\Psi f(x, y) := -y \sup_{w \in B(\varepsilon)} (-y)f(x + w).$$

Additionally, we define $\Psi \mathcal{F}$ to be the transformed function class

$$\Psi \mathcal{F} := \{\Psi f : f \in \mathcal{F}\}.$$
Proposition 1. Let $\ell(f, z)$ be a loss function that is monotonically decreasing in $y f(x)$. Then
\[
\sup_{w \in B(\varepsilon)} \ell(f, z + w) = \ell(\Psi f, z).
\]

Remark 1. The consequence of the supremum transformation can be seen by taking the expectation:
\[
\mathbb{E}_P \sup_{w \in B(\varepsilon)} \ell(f, z) = \mathbb{E}_P \ell(\Psi f, z).
\]
Thus, we can bound the adversarial risk of a function $f$ with a bound on the usual risk of $\Psi f$ via Rademacher complexities. For linear classifiers, we shall see momentarily that the supremum transformation can be calculated exactly.

3.1 The Supremum Transformation and Linear Classification

We start with an explicit formula for the supremum transform.

Proposition 2. Let $f(x) = \theta^\top x + b$. Then the supremum transformation takes the explicit form
\[
\Psi f(x, y) = \theta^\top x + b - y \varepsilon \|\theta\|_q,
\]
where $q$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

The proof is contained in Section 5.

Next, the key ingredient to a generalization bound is an upper bound on the Rademacher complexity of $\Psi \mathcal{F}$.

Lemma 1. Let $\mathcal{F}_{\text{lin}}$ be a compact linear function class such that $\|\theta\|_2 \leq M_2$ and $\|\theta\|_q \leq M_q$ for all $f \in \mathcal{F}_{\text{lin}}$, where $f(x) = \theta^\top x + b$. Suppose $\|x_i\|_2 \leq R$ for all $i$. Then we have
\[
\hat{\mathcal{R}}_n(\Psi \mathcal{F}_{\text{lin}}) \leq \frac{M_2 R}{\sqrt{n}} + \varepsilon M_q \sqrt{n}.
\]

This leads to the following upper bound on adversarial risk, proved in Appendix C:

Corollary 1. Let $\mathcal{F}_{\text{lin}}$ be a collection of linear classifiers such that, for any classifier $f(x) = \theta^\top x + b$ in $\mathcal{F}_{\text{lin}}$, we have $\|\theta\|_2 \leq M_2$ and $\|\theta\|_q \leq M_q$. Let $R$ be a constant such that $\|x_i\|_2 \leq R$ for all $i$. Then for any $f \in \mathcal{F}_{\text{lin}}$, we have
\[
R_{\text{rob}}(\ell_{01}, f) = \mathbb{E}_P \ell_{01}(\Psi f, z) \leq \frac{1}{n} \sum_{i=1}^{n} \ell_{h}(\Psi f, z_i) + 2 \frac{M_2 R}{\sqrt{n}} + \varepsilon M_q \sqrt{n} + 3 \sqrt{\frac{\log \frac{2}{\delta}}{2n}} \tag{2}
\]
and
\[
R_{\text{rob}}(\ell_{01}, f) \leq \frac{1}{n} \sum_{i=1}^{n} \ell_{h}(f, z_i) + \varepsilon \|\theta\|_q \frac{1}{n} \sum_{i=1}^{n} \ell_{h,01}(\Psi f, z_i) + 2 \frac{M_2 R}{\sqrt{n}} + \varepsilon M_q + 3 \sqrt{\frac{\log \frac{2}{\delta}}{2n}}, \tag{3}
\]
with probability at least $1 - \delta$. 

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As seen in the proof of Corollary 1, the loss involved in defining the adversarial risk could be replaced by another loss, which would then need to be upper-bounded by a Lipschitz loss function (in this case, the hinge loss). The empirical version of the latter loss would then appear on the right-hand side of the bounds.

**Remark 2.** An immediate question is how our adversarial risk bounds compare with the case when perturbations are absent. Plugging $\varepsilon = 0$ into the equations above yields the usual generalization bounds of the form

$$E_P \ell_{01}(f, z) \leq \frac{1}{n} \sum_{i=1}^{n} \ell(f, z_i) + \frac{C_1}{\sqrt{n}},$$

so the effect of an adversarial perturbation is essentially to introduce an additional $O(n^{-1/2})$ term as well as an additional contribution to the empirical risk that depends linearly on $\varepsilon$. The additional empirical risk term vanishes if $f$ classifies adversarially perturbed points $z + w$ correctly, since $\ell_{0,1}(\Psi f, z) = 0$ in that case.

**Remark 3.** Clearly, we could further upper-bound the regularization term in equation (3) by $\varepsilon \|\theta\|_q$. This is essentially the bound obtained for the empirical risk for Wasserstein distributionally robust linear classification (Gao et al., 2017). However, this bound is loose when a good robust linear classifier exists, i.e., when $\sum_{i=1}^{n} \ell_{01}(\Psi f, z_i)$ is small relative to $n$. Thus, when good robust classifiers exist, distributional robustness is relatively conservative for solving the adversarially robust problem (cf. Appendix E).

### 3.2 The Tree Transformation and Neural Networks

In this section, we consider adversarial risk bounds for neural networks. We begin by introducing the tree transformation, which unravels the neural network into a tree in some sense.

**Definition 2.** Let $f$ be a neural network given by

$$f(x) = A^{(d+1)} s_d \left( A^{(d)} s_{d-1} \left( \ldots A^{(1)} x \right) \right).$$

Define the terms $w_f^{(j_2,d+1)}$ and $\text{sgn}(f, j_2,d+1)$ by

$$w_f^{(j_2,d+1)} := -y \text{sgn}(f, j_2,d+1) \varepsilon \left\| a_{j_d}^{(1)} \right\|_q$$

and

$$\text{sgn}(f, j_2,d+1) := \text{sgn} \left( \prod_{k=2}^{d+1} a_{j_{k+1},j_k}^{(k)} \right).$$

Then the tree transform $Tf$ is defined by

$$Tf(x, y) := \sum_{j_{d+1}=1}^{J_{d+1}} a_{1,j_{d+1}+1}^{(d+1)} s_d \left( \sum_{j_d=1}^{J_d} a_{j_{d+1},j_d}^{(d)} s_{d-1} \ldots \sum_{j_2=1}^{J_2} a_{j_3,j_2}^{(2)} s_1 \left( \left( a_{j_2}^{(1)} \right)^\top x + w_f^{(j_2,d+1)} \right) \right).$$

Intuitively, the tree transform (5) can be thought of as a new neural network classifier where the adversary can select a different worst-case perturbation $w_f$ for each path through the neural network from the input to the output indexed by $(j_2, \ldots, j_{d+1})$. This leads to $\prod_{k=2}^{d+1} J_k$ distinct
paths through the network for given inputs $x$ and $w_i^{(j_2,d+1)}$, and if these paths were laid out, they would form a tree (see Section 3.3).

Next, we show that the risk of the tree transform upper-bounds the adversarial risk of the original neural network.

**Proposition 3.** Let $\ell(f, z)$ be monotonically decreasing in $y f(x)$. Then we have the inequality

$$\sup_{w \in B(\varepsilon)} \ell(f, z + w) = \ell(\Psi f, z) \leq \ell(T f, z).$$

As an immediate corollary, we obtain

$$E \sup_{w \in B(\varepsilon)} \ell(f, z + w) \leq E \ell(T f, z),$$

so it suffices to bound this latter expectation. We have the following bound on the Rademacher complexity of $T \mathcal{F}_{nn}$:

**Lemma 2.** Let $\mathcal{F}$ be a class of neural networks of depth $d$ satisfying $\|A_j\|_\infty \leq \alpha_j$ and $\|A_j\|_F \leq \alpha_{1,F}$, for each $j = 1, \ldots, d + 1$, and let $a = \prod_{j=1}^{d+1} \alpha_j$. Additionally, suppose $\max_{j=1,\ldots,J_1} \|a_j^{(1)}\|_q \leq \alpha_{1,q}$ and $\|x_i\|_2 \leq R$ for all $i$. Then we have the bound

$$\hat{R}_n(T, F) \leq a \left( \frac{\alpha_{1,F}}{\alpha_1} R + \frac{\alpha_{1,q}}{\alpha_1} \varepsilon \right) \cdot \frac{\sqrt{2d \log 2} + 1}{\sqrt{n}}.$$

Finally, we have our adversarial risk bounds for neural networks. The proof is contained in Appendix C.

**Corollary 2.** Let $\mathcal{F}_{nn}$ be a class of neural networks of depth $d$. Let $g_i(a) = \ell_{xe}(\delta(a), y_i)$. Under the same assumptions as Lemma 2, for any $f \in \mathcal{F}_{nn}$, we have the upper bounds

$$R_{rob}(\ell_0, f) = E \ell_{01}(T f, z) \leq \frac{1}{n} \sum_{i=1}^{n} \ell_{xe}(T f, z_i) + 3 \sqrt{\frac{\log 2}{2n}} + 2 \alpha \left( \frac{\alpha_{1,F}}{\alpha_1} R + \frac{\alpha_{1,q}}{\alpha_1} \varepsilon \right) \frac{\sqrt{2d \log 2} + 1}{\sqrt{n}}$$

and

$$R_{rob}(\ell_0, f) \leq \frac{1}{n} \sum_{i=1}^{n} \ell_{xe}(f, z_i) + 3 \sqrt{\frac{\log 2}{2n}} + \varepsilon \max_{j=1,\ldots,J_1} \|a_j^{(1)}\|_q \prod_{j=2}^{d+1} \|A_j\|_\infty \frac{1}{n} \sum_{i=1}^{n} |g_i(T f(x_i, y_i))|$$

$$+ 2 \alpha \left( \frac{\alpha_{1,F}}{\alpha_1} R + \frac{\alpha_{1,q}}{\alpha_1} \varepsilon \right) \frac{\sqrt{2d \log 2} + 1}{\sqrt{n}},$$

with probability at least $1 - \delta$.

**Remark 4.** As in the linear case, we can essentially recover pre-existing non-adversarial risk bounds by setting $\varepsilon = 0$ (Bartlett et al., 2017; Golowich et al., 2018). Again, the effect of adversarial perturbations on the adversarial risk is the addition of $O \left( n^{-1/2} \right)$ on top of the empirical risk bounds for the unperturbed loss. Finally, the bound (6) includes an extra perturbation term that is linear in $\varepsilon$, with coefficient reflecting the Lipschitz coefficient of the neural network, as well as a term $\frac{1}{n} \sum_{i=1}^{n} |g_i'(T f(x_i, y_i))|$, which decreases as $T f$ improves as a classifier because $|g_i'(T f(x_i, y_i))|$ is small when $\ell_{xe}(T f, z_i)$ is small. A similar term appears in the bound (3).
### 3.3 A Visualization of the Tree Transform

In this section, we provide a few pictures to illustrate the tree transform. Consider the following two-layer network with two hidden units per layer:

\[ f(x) = A^{(3)} s_2(A^{(2)} s_1(A^{(1)} x)) . \]

We begin by visualizing \( \sup_{w \in B(\varepsilon)} f(x + w) \) in Figure 1.

**Figure 1.** A visualization of \( f(x + w) \). The input \( x + w \) is fed up through the network.

Next, we examine what happens when the supremum is taken inside the first layer. The resulting transformed function (cf. Lemma 3 in Section 5) becomes

\[
g(x, y) = \sum_{j_3=1}^{2} a_{1,j_3}^{(3)} s_2 \left( \text{sgn} \left( -y a_{1,j_3}^{(3)} \right) \sup_{w(j_3) \in B(\varepsilon)} \text{sgn} \left( -y a_{1,j_3}^{(3)} \right) A^{(2)} s_1 \left( A^{(1)} \left( x + w(j_3) \right) \right) \right) . \tag{7} \]

The corresponding network is shown in Figure 2.

**Figure 2.** A visualization of the function \( g(x, y) \) of equation (7). Note that two different perturbations, \( w^{(1)} \) and \( w^{(2)} \), are fed upward through different paths in the network.
Finally, we examine the entire tree transform. This is
\[
T_f(x, y) = \sum_{j_2=1}^{J_2} a^{(2)}_{j_2} s_1 \left( \sgn \left( -y a^{(3)}_{j_3} a^{(2)}_{j_3,j_2} \right) \sup_{w_{j_2,j_3}} \left( a^{(1)}_{j_2} \right)^T (x + w_{j_2,j_3}) \right)
\]
(8)

the result, shown in Figure 3, yields a tree-structured network.

In particular, we note that now the visualization of the network reveals a tree. This is the reason that \( T \) is called the tree transform.

4 Optimization of Risk Bounds

In practice, our sample-based upper bounds on adversarial risk suggest the strategy of optimizing the bounds in the corollaries, rather than simply the empirical risk, to achieve robustness of the trained networks against adversarial perturbations. Accordingly, we provide two algorithms for optimizing the upper bounds appearing in Corollary 1. One idea is to optimize the first bound (2) directly. Recalling the form of \( \Psi \), this leads to the following optimization problem:
\[
\min_{\theta, b} \sum_{i=1}^{n} \max \{0, 1 - y_i (\theta^T x_i + b) + \varepsilon \|\theta\|_q\}.
\]
(9)

Note that the optimization problem of equation (9) is convex in \( \theta \) and \( b \); therefore, this is a computationally tractable problem. We summarize this approach in Algorithm 1.

\begin{algorithm}
\caption{Convex risk}
\begin{algorithmic}
\State \textbf{Input}: Data \( z_1, \ldots, z_n \); function class \( F_{\text{lin}} \).
\State Solve equation (9) to obtain \((\hat{\theta}, \hat{b})\).
\State Return the resulting classifier \( \sgn(\hat{f}) \), where \( \hat{f}(x) = \hat{\theta} x + \hat{b} \).
\end{algorithmic}
\end{algorithm}

The second approach involves optimizing the second adversarial risk bound (3). Although this bound is generally looser than the bound (2), we comment on optimization due to the fact that
regularization has been suggested as a way to encourage generalization. However, note that the regularization coefficient in the bound (3) depends on \( f \). Thus, we propose to perform a grid search over the value of the regularization parameter.

Specifically, define
\[
\gamma_{\text{lin}}(f) := n \sum_{i=1}^{n} \ell_{h,01}(\Psi f, z_i).
\] (10)

We then have the optimization problem
\[
\min_{\theta,b} \sum_{i=1}^{n} \max\{0, 1 - y_i (\theta^T x_i + b)\} + \varepsilon \|\theta\|_q \gamma_{\text{lin}}(f).
\] (11)

Note, however, that \( \gamma_{\text{lin}}(f) \) is nonconvex, and the form as a function of \( \theta \) and \( b \) is complicated. We propose to take \( \gamma_i = i/n \) for \( i = 0, \ldots, n \) and solve
\[
\min_{\theta,b} \sum_{j=1}^{n} \max\{0, 1 - y_j (\theta^T x_j + b)\} + \varepsilon \|\theta\|_q \gamma_i.
\] (12)

At the end, we simply pick the solution minimizing the objective function in equation (11) over all \( i \). Note that this involves evaluating equation (10), but this is easy to do in the linear case. This method is summarized in Algorithm 2.

Algorithm 2: Regularized risk

\begin{algorithm}
\textbf{Input:} Data \( z_1, \ldots, z_n \), function class \( F_{\text{lin}} \).
\begin{algorithmic}[1]
\For {\( i = 0, \ldots, n \)}
\State Set \( \gamma_i = i/n \).
\State Calculate the \( f_i \) minimizing equation (12).
\State Save the robust empirical risk, the objective of equation (11), of \( f_i \) as \( R_i \).
\EndFor
\State Return the \( f_i \) with the minimum \( R_i \).
\end{algorithmic}
\end{algorithm}

5 Proofs

We now present the proofs of our core theoretical results regarding the transform functions \( \Psi \) and \( T \).

Proof of Proposition 1. We break our analysis into two cases. If \( y = +1 \), then \( \tilde{\ell}(f(x), +1) \) is decreasing in \( f(x) \). Thus, we have
\[
\sup_{w \in B(\varepsilon)} \tilde{\ell}(f(x+w), +1) = \tilde{\ell} \left( \inf_{w \in B(\varepsilon)} f(x+w), +1 \right) = \tilde{\ell} \left( -1 \sup_{w \in B(\varepsilon)} (-1) f(x+w), +1 \right)
= \ell (\Psi f, (x, +1)).
\]

If instead \( y = -1 \), then \( \tilde{\ell}(f(x), -1) \) is increasing in \( f(x) \), so
\[
\sup_{w \in B(\varepsilon)} \tilde{\ell}(f(x+w), -1) = \tilde{\ell} \left( \sup_{w \in B(\varepsilon)} f(x+w), -1 \right) = \tilde{\ell} \left( 1 \sup_{w \in B(\varepsilon)} (1) f(x+w), -1 \right)
= \ell (\Psi f, (x, -1)).
\]

This completes the proof. \( \square \)
Proof of Proposition 2. Using the definition of the sup transform, we have
\[
\Psi f(x, y) = -y \sup_{w \in B(\varepsilon)} (-y)(\theta^T x + b + \theta^T w)
\]
\[
= \theta^T x + b - y \sup_{w \in B(\varepsilon)} (-y)\theta^T w
\]
\[
= \theta^T x + b - y\|\theta\|_q,
\]
which completes the proof.

Before we begin the proof of Proposition 3, we state, prove, and remark upon a helpful lemma.

Lemma 3. Let \( g : \mathcal{X} \rightarrow \mathbb{R}^J \) be a function and define \( s : \mathbb{R} \rightarrow \mathbb{R} \) to be a monotonically increasing function applied elementwise to vectors. Then we have the inequality
\[
\sup_{w \in B(\varepsilon)} \sum_{j=1}^J b_j s(a_j^T g(x + w)) \leq \sum_{j=1}^J b_j s \left( sgn(b_j) \sup_{w(j) \in B(\varepsilon)} sgn(b_j) a_j^T g \left( x + w^{(j)} \right) \right).
\]

Proof. Denote the left hand-side of the desired inequality by \( L \). First, we can push the supremum inside the sum to obtain
\[
L \leq \sum_{j=1}^J \sup_{w(j) \in B(\varepsilon)} b_j s \left( a_j^T g \left( x + w^{(j)} \right) \right).
\]
Next, note that
\[
\sup_{w(j) \in B(\varepsilon)} b_j s \left( a_j^T g \left( x + w^{(j)} \right) \right) = \sup_{w(j) \in B(\varepsilon)} b_j s \left( sgn(b_j) sgn(b_j) a_j^T g \left( x + w^{(j)} \right) \right). \tag{13}
\]
Since \( s \) is monotonically increasing, we see that the map \( x \mapsto b_j s(\text{sgn}(b_j)x) \) is monotonically increasing, as well. Thus, the supremum in equation (13) is obtained when \( sgn(b_j) a_j^T g(x + w^{(j)}) \) is maximized. Hence, we obtain
\[
L \leq \sum_{j=1}^J b_j s \left( sgn(b_j) \sup_{w(j) \in B(\varepsilon)} sgn(b_j) a_j^T g \left( x + w^{(j)} \right) \right),
\]
which completes the proof.

Remark 5. Note that if \( f(x) = b^T s(A^T h(x)) \), where \( g(x) = s'(A^T h(x)) \), this lemma yields
\[
L \leq \sum_{j=1}^J b_j s \left( sgn(b_j) \sup_{w(j) \in B(\varepsilon)} \sum_{k=1}^K sgn(b_j) a_{j,k} s' \left( (a_k')^T h \left( x + w^{(j)} \right) \right) \right).
\]
If we apply Lemma 3 again, we obtain
\[
L \leq \sum_{j=1}^J b_j s \left( sgn(b_j) \sum_{k=1}^K sgn(b_j) a_{j,k} s' \left( sgn(b_j a_{j,k}) \sup_{w(j,k) \in B(\varepsilon)} sgn(b_j a_{j,k}) (a_k')^T h \left( x + w^{(j,k)} \right) \right) \right)
\]
\[
= \sum_{j=1}^J b_j s \left( \sum_{k=1}^K a_{j,k} s' \left( sgn(b_j a_{j,k}) \sup_{w(j,k) \in B(\varepsilon)} sgn(b_j a_{j,k}) (a_k')^T h \left( x + w^{(j,k)} \right) \right) \right).
\]
In particular, we note that the sign terms accumulate within the supremum, but when we take the supremum inside another layer, the sign terms \( \text{sgn}(b_j) \) remaining in the previous layers cancel out and are incorporated into the \( \text{sgn}(b_j a_{j,k}) \) of the next layer.
Proof of Proposition 3. First note that the assumption that $\ell$ is monotonically decreasing in $yf(x)$ is equivalent to $\ell$ being monotonically increasing in $-yf(x)$. As in the proof of Proposition 1, if $y = +1$, we want to show that $\Psi f(x, y) \geq Tf(x, y)$; if $y = -1$, we want to show that $\Psi f(x, y) \leq Tf(x, y)$. Thus, it is our goal to establish the inequality

$$-y\Psi f(x, y) \leq -yTf(x, y).$$

(14)

We define $L := -y\Psi f(x, y)$ and show how to take the supremum inside each layer of the neural network to yield $-yTf(x, y)$. To this end, we simply apply Lemma 3 and Remark 5 iteratively until the remaining function is linear. Thus, we see that

$$L \leq -y \sum_{j_{d+1}=1}^{J_{d+1}} a_{1,j_{d+1}}^{(d+1)} s_d \left( \sum_{j_d=1}^{J_d} a_{j_{d+1},j_d}^{(d)} s_{d-1} \left( \sum_{j_{d-1}=1}^{J_{d-1}} a_{j_{d+1},j_{d-1}}^{(d-1)} s_{d-2} \left( \ldots s_1 \left( \text{sgn} \left( -ya_1^{(d+1)} a_1^{(d)} \ldots a_j^{(2)} \right) \right) \right) \right) \right),$$

and simplifying gives

$$L \leq -y \sum_{j_{d+1}=1}^{J_{d+1}} a_{1,j_{d+1}}^{(d+1)} s_d \left( \sum_{j_d=1}^{J_d} a_{j_{d+1},j_d}^{(d)} s_{d-1} \left( \sum_{j_{d-1}=1}^{J_{d-1}} a_{j_{d+1},j_{d-1}}^{(d-1)} s_{d-2} \left( \ldots s_1 \left( a_1^{(d)} \right) \right) \right) \right) + \text{sgn} \left( -ya_1^{(d+1)} a_1^{(d)} \ldots a_j^{(2)} \right) \times \sup_{w^{(j_{d+1})} \in B(\varepsilon)} \left( -ya_1^{(d+1)} a_1^{(d)} \ldots a_j^{(2)} \right) \left( a_j^{(1)} \right)^T \left( x + w^{(j_{d+1})} \right),$$

The final supremum clearly evaluates to $\varepsilon\|a_j^{(1)}\|_q$. Recalling the definition (4) of $w^{(j_{d+1})}$, we then have

$$-y\Psi f(x, y) \leq -y \sum_{j_{d+1}=1}^{J_{d+1}} a_{1,j_{d+1}}^{(d+1)} s_d \left( \sum_{j_d=1}^{J_d} a_{j_{d+1},j_d}^{(d)} s_{d-1} \left( \ldots s_1 \left( \left( a_j^{(1)} \right)^T x + w^{(j_{d+1})} \right) \right) \right) = -yTf(x, y),$$

which proves the proposition.

\[\square\]

6 Discussion

We have presented a method of transforming classifiers to obtain upper bounds on the adversarial risk. We have shown that bounding the generalization error of the transformed classifiers may be performed using similar machinery for obtaining traditional generalization bounds in the case of linear classifiers and neural network classifiers. In particular, since the Rademacher complexity of neural networks only has a small additional term due to adversarial perturbations, generalization even in the presence of adversarial perturbations should not be impossibly difficult for binary classification.

We mention several future directions for research. First, one might be interested in extending the supremum transformation to other types of classifiers. The most interesting avenues would include calculating explicit representations as in the case of linear classifiers, suitable alternative
transformations as in the case of neural networks, and bounds on the resulting Rademacher complexities.

A second direction is to understand the tree transformation better and develop algorithms for optimizing the resulting adversarial risk bounds. One view that we have taken in this paper is to bound the difference between the empirical risk of $Tf$ and $f$ as a regularization term, but one could also optimize the empirical risk of $Tf$ directly. An immediate idea would be to train a good $Tf$ and then use the resulting $f$, since the empirical risk of $Tf$ provides an upper bound on the adversarial risk of $f$. For computational reasons, this may not be practical for the tree transform, in which case one might need to explore alternative transformations.

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H. Xu, C. Caramanis, and S. Mannor. Robustness and regularization of support vector machines. *Journal of Machine Learning Research*, 10(Jul):1485–1510, 2009b.
A Rademacher Complexity Proofs

In this section, we prove Lemmas 1 and 2, which are the bounds on the empirical Rademacher complexities of $\Psi F_{\text{lin}}$ and $T F_{\text{nn}}$. The proofs are largely based on pre-existing proofs for bounding the empirical Rademacher complexities of $F_{\text{lin}}$ and $F_{\text{nn}}$, and this simplicity is part of what makes $\Psi$ and $T$ attractive.

Proof of Lemma 1. Using Proposition 2, we have

$$n\hat{R}_n(\Psi F_{\text{lin}}) = \mathbb{E}_\sigma \left[ \sup_{f \in F} \sum_{i=1}^n \sigma_i \Psi f(x, y) \right]$$

$$= \mathbb{E}_\sigma \left[ \sup_{f \in F} \sum_{i=1}^n \sigma_i (\theta^T x_i + b - y_i \varepsilon \|\theta\|_q) \right]$$

$$\leq \mathbb{E}_\sigma \left[ \sup_{f \in F} \sum_{i=1}^n \sigma_i (\theta^T x_i + b) \right]$$

$$+ \mathbb{E}_\sigma \left[ \sup_{f \in F} \sum_{i=1}^n \sigma_i (-y_i \varepsilon \|\theta\|_q) \right]$$

$$= n\hat{R}_n(F_{\text{lin}}) + \varepsilon \mathbb{E}_\sigma \left[ \sup_{f \in F} \|\theta\|_q \sum_{i=1}^n \sigma_i \right].$$

By Lemma 10, the empirical Rademacher complexity of a linear function class is given by

$$\hat{R}_n(F_{\text{lin}}) \leq \frac{M_2 R}{\sqrt{n}}.$$ 

Thus, it remains to analyze the second term in the upper bound.

If the sum of the $\sigma_i$'s is negative, the $\theta$ maximizing the supremum is the zero vector. Alternatively, if the sum is positive, we clearly have the upper bound $M_q \sum_{i=1}^n \sigma_i$. Thus, we have

$$\varepsilon \mathbb{E}_\sigma \left[ \sup_{f \in F} \|\theta\|_q \sum_{i=1}^n \sigma_i \right] \leq \varepsilon \mathbb{E}_\sigma \left[ M_q \sum_{i=1}^n \sigma_i 1 \left\{ \sum_{i=1}^n \sigma_i > 0 \right\} \right]$$

$$\leq \varepsilon M_q \mathbb{E} \left[ \sum_{i=1}^n \sigma_i \right] \leq \varepsilon M_q \left( \mathbb{E} \left[ \left( \sum_{i=1}^n \sigma_i \right)^2 \right] \right)^{\frac{1}{2}},$$

where (a) follows because $\sigma_i$ and $-\sigma_i$ have the same distribution, and the last inequality follows by Jensen’s inequality. The last term is equal to $\frac{\varepsilon M_q}{2} \sqrt{n}$, using the fact that the $\sigma_i$'s are independent, zero-mean, and unit-variance random variables. Putting everything together yields

$$\hat{R}_n(\Psi F_{\text{lin}}) \leq \frac{M_2 R}{\sqrt{n}} + \frac{\varepsilon M_q}{2\sqrt{n}},$$

which completes the proof.

Proof of Lemma 2. Our broad goal is to peel off the layers of the neural network one at a time. Most of the work is done by Lemma 7. The proof is essentially the same as the Rademacher complexity bounds on neural networks of Golowich et al. (2018) until we reach the underlying linear classifier. We then bound the action of the adversary in an analogous manner to the linear case.
We write

\[ n\hat{\mathcal{R}}(T\mathcal{F}_{nn}) = \frac{1}{\lambda} \log \exp \left( \lambda \mathbb{E} \left[ \sup_{f \in \mathcal{F}_{nn}} \sum_{i=1}^{n} \sigma_i T f(x_i, y_i) \right] \right) \]

\[ \leq \frac{1}{\lambda} \log \mathbb{E} \left[ \sup_{f \in \mathcal{F}_{nn}} \exp \left( \lambda \sum_{i=1}^{n} \sigma_i T f(x_i, y_i) \right) \right]. \]

Recalling the form of \( T f \) from equation (5), we can apply Lemma 7 successively \( d \) times with \( G(x) = \exp((\lambda \prod_{j \in J} \alpha_j)x) \) for various \( J \) in order to remove the layers of the neural network. Specifically, we use \( J = \emptyset, J = \{d + 1\}, J = \{d + 1, d\} \), up to \( J = \{d + 1, \ldots, 3\} \), as we peel away the layers and retain the bounds \( \alpha_j \) on the matrix norms from the layers that we have removed. This implies

\[ n\hat{\mathcal{R}}(\mathcal{F}_{nn}) \leq \frac{1}{\lambda} \log 2^d \mathbb{E} \left[ \sup_{f \in \mathcal{F}_{nn}} \max_{j_2, \ldots, j_{d+1}} \exp \left( \frac{\alpha \lambda}{\alpha_1} \left( a^{(1)}_{j_2} \right) \sum_{i=1}^{n} \sigma_i x_i + \frac{\alpha \lambda}{\alpha_1} \sum_{i=1}^{n} \sigma_i y_i^{(j_2, d+1)} \right) \right] \]

\[ = \frac{1}{\lambda} \log 2^d \mathbb{E} \left[ \sup_{f \in \mathcal{F}_{nn}} \max_{j_2, \ldots, j_{d+1}} \exp \left( \frac{\alpha \lambda}{\alpha_1} \left( a^{(1)}_{j_2} \right) \sum_{i=1}^{n} \sigma_i x_i - \frac{\alpha \lambda}{\alpha_1} \varepsilon \left\| a^{(1)}_{j_2} \right\|_q \sum_{i=1}^{n} \sigma_i y_i \right) \right]. \]

Note that the maxima over \( j_2, \ldots, j_{d+1} \) are accumulated from each application of Lemma 7. These maxima correspond to taking a worst-case path through the tree. To bound the first term, we apply the Cauchy-Schwarz inequality. To bound the second term, we use the inequality

\[ -\text{sgn}(f, j_{d+1}) \sum_{i=1}^{n} \sigma_i y_i \leq \left\| \sum_{i=1}^{n} \sigma_i y_i \right\|. \]

Thus, we have

\[ n\hat{\mathcal{R}}(\mathcal{F}_{nn}) \leq \frac{1}{\lambda} \log 2^d \mathbb{E} \left[ \sup_{f \in \mathcal{F}_{nn}} \max_{j_2, \ldots, j_{d+1}} \exp \left( \frac{\alpha \lambda}{\alpha_1} \left\| a^{(1)}_{j_2} \right\|_q \left\| \sum_{i=1}^{n} \sigma_i x_i \right\|_2 + \frac{\alpha \lambda}{\alpha_1} \varepsilon \left\| a^{(1)}_{j_2} \right\|_q \sum_{i=1}^{n} \sigma_i y_i \right) \right] \]

\[ \leq \frac{1}{\lambda} \log 2^d \left[ \exp \left( \frac{\alpha \lambda \gamma}{\alpha_1} \left\| \sum_{i=1}^{n} \sigma_i x_i \right\|_2 + \frac{\alpha \lambda \gamma}{\alpha_1} \varepsilon \left\| \sum_{i=1}^{n} \sigma_i y_i \right\| \right) \right]. \]

In order to bound the final expectation, we define

\[ Z(\sigma) := \alpha \left( \frac{\alpha_1 \gamma}{\alpha_1} \left\| \sum_{i=1}^{n} \sigma_i x_i \right\|_2 + \frac{\alpha_1 \gamma}{\alpha_1} \varepsilon \left\| \sum_{i=1}^{n} \sigma_i y_i \right\| \right), \]

where we view \( Z \) as a function of the \( \sigma_i \)’s. Now we have

\[ n\hat{\mathcal{R}}(\mathcal{F}_{nn}) \leq \frac{1}{\lambda} d \log 2 + \log \mathbb{E} [\exp(\lambda(Z - \mathbb{E}[Z])) \exp(\lambda\mathbb{E}[Z])] \]

\[ = \frac{1}{\lambda} d \log 2 + \frac{1}{\lambda} \log \mathbb{E} [\exp(\lambda(Z - \mathbb{E}[Z]))] + \mathbb{E}[Z]. \] (15)

Thus, it remains to compute the last two terms on the right-hand side. We start with the expectation
of \( Z \). By Jensen’s inequality,

\[
\mathbb{E}[Z] \leq \frac{\alpha}{\alpha_1} F \left( \mathbb{E} \left[ \left\| \sum_{i=1}^{n} \sigma_i x_i \right\|^2 \right] \right)^{\frac{1}{2}} + \frac{\alpha_1 q}{\alpha_1} \varepsilon \left( \mathbb{E} \left[ \left\| \sum_{i=1}^{n} \sigma_i y_i \right\|^2 \right] \right)^{\frac{1}{2}}
\]

\[
= \frac{\alpha}{\alpha_1} F \left( \sum_{i,j=1}^{n} \sigma_i \sigma_j x_i^T x_j \right)^{\frac{1}{2}} + \frac{\alpha_1 q}{\alpha_1} \varepsilon n^{\frac{1}{2}}
\]

\[
\leq \alpha \left( \frac{\alpha_1 F}{\alpha_1} R + \frac{\alpha_1 q}{\alpha_1} \varepsilon \right) \sqrt{n}
\]

\[= C \sqrt{n}. \]

Next, we need to handle the middle term in inequality (15). The idea is to use standard bounds employed in concentration inequalities. Let \( \sigma_i' = \sigma_i \) for all \( i = 1, \ldots, n \), except for one, where \( \sigma_j' = -\sigma_j \). Treating \( Z \) as a function of the \( \sigma_i \)'s, we obtain

\[
Z(\sigma) - Z(\sigma') = \alpha \frac{\alpha_1 F}{\alpha_1} \left( \left\| \sum_{i=1}^{n} \sigma_i x_i \right\|^2 - \left\| \sum_{i=1}^{n} \sigma_i' x_i \right\|^2 \right) + \frac{\alpha_1 q}{\alpha_1} \varepsilon \left( \left\| \sum_{i=1}^{n} \sigma_i y_i \right\| - \left\| \sum_{i=1}^{n} \sigma_i' y_i \right\| \right)
\]

\[\leq 2 \alpha \left( \frac{\alpha_1 F}{\alpha_1} R + \frac{\alpha_1 q}{\alpha_1} \varepsilon \right)
\]

\[\leq 2C. \]

Thus, the variance factor in the bounded differences inequality (Lemma 12) is

\[
v = \frac{1}{4} \sum_{i=1}^{n} (2C)^2 = C^2 n.
\]

This yields

\[
\frac{1}{\lambda} \log \mathbb{E} \exp(\lambda (Z - \mathbb{E}[Z])) \leq \frac{1}{\lambda} \frac{\lambda^2 C^2 n}{2} = \frac{\lambda C^2 n}{2}.
\]

Finally, putting everything together, we have

\[
n \mathcal{R}(\mathcal{T}_{\mathcal{F}}) \leq \frac{1}{\lambda} d \log 2 + \frac{\lambda C^2 n}{2} + C \sqrt{n} = C \left( \sqrt{2d \log 2} + 1 \right) \sqrt{n},
\]

where in the last equality, we set the free parameter to be \( \lambda = (2d \log(2)/(C^2 n))^{1/2} \) to minimize the bound. This completes the proof. \( \square \)

## B Additional Lemmas

In this appendix, we provide additional lemmas used in the proofs of our main results.
and the lower bound

\[ \frac{1}{n} \sum_{i=1}^{n} (\ell_h(\Psi f, z_i) - \ell_h(f, z_i)) \geq \varepsilon \| \theta \|_q \frac{1}{n} \sum_{i=1}^{n} \ell_{h,01}(f, z_i). \]

**Proof of Lemma 4.** Using Proposition 2 and the fact that \( \ell_h(f, z) = f(x)\ell_{h,01}(f, z) \), we write the difference in losses as

\[ \ell_h(\Psi f, z) - \ell_h(f, z) = (1 - y(\theta^T x + b) + \varepsilon \| \theta \|_q) \ell_{h,01}(\Psi f, z) - (1 - y(\theta^T x + b)) \ell_{h,01}(f, z). \] (16)

We start by proving the upper bound. Suppose \( \ell_{h,01}(f, z) = 1 \). Then \( \ell_{h,01}(\Psi f, z) \geq \ell_{h,01}(f, z) \), so \( \ell_{h,01}(\Psi f, z) = 1 \), as well, which means that

\[ \ell_h(\Psi f, z) - \ell_h(f, z) = \varepsilon \| \theta \|_q \ell_{h,01}(\Psi f, z). \] (17)

If instead \( \ell_{h,01}(f, z) = 0 \), we have

\[ (1 - y(\theta^T x + b) + \varepsilon \| \theta \|_q) \leq 0, \]

so by equation (16), we have

\[ \ell_h(\Psi f, z) - \ell_h(f, z) \leq (1 - y(\theta^T x + b) + \varepsilon \| \theta \|_q) \ell_{h,01}(\Psi f, z) - (1 - y(\theta^T x + b)) \ell_{h,01}(\Psi f, z) \]

\[ = \varepsilon \| \theta \|_q \ell_{h,01}(\Psi f, z). \] (18)

Averaging over all \( i \) completes the upper bound.

The lower bound is very similar. In detail, consider the case \( \ell_{h,01}(f, z) = 1 \). Once again, we have \( \ell_{h,01}(\Psi f, z) = 1 \), so

\[ \ell_h(\Psi f, z) - \ell_h(f, z) = \varepsilon \| \theta \|_q \ell_{h,01}(f, z). \]

Next, suppose \( \ell_{h,01}(f, z) = 0 \). Clearly, we then have

\[ \ell_h(\Psi f, z) - \ell_h(f, z) \geq 0 = \varepsilon \| \theta \|_q \ell_{h,01}(f, z). \]

Averaging over all \( i \) completes the lower bound and the proof.

**B.2 Neural Network Lemmas**

In this section, we collect lemmas for neural networks. We start with a bound on the difference between the empirical risks of \( Tf \) and \( f \).

**Lemma 5.** Let \( f(x) = A^{(d+1)}s_d(\ldots s_1(A^{(1)}x)) \) be a neural network with 1-Lipschitz activation functions \( s_j \), applied elementwise. Let \( g_i(a) = \ell_{xe}(a, y_i) \). Then

\[ \frac{1}{n} \sum_{i=1}^{n} (\ell_{xe}(Tf, z_i) - \ell_{xe}(f, z_i)) \leq \varepsilon \max_{j_2=1,\ldots,j_2} \left\| a^{(1)}_{j_2} \right\|_q \prod_{j=2}^{d+1} \left\| A^{(j)} \right\|_\infty \frac{1}{n} \sum_{i=1}^{n} |g_i(Tf(x_i, y_i))|. \]
Proof of Lemma 5. We only need to prove the bound for a single summand, since we sum and then divide by \( n \). By Lemma 9, we have the inequality
\[
g_i(b) - g_i(a) \leq |g'_i(b)||b - a|.
\]
It follows that
\[
L_i := \ell_{xe}(Tf, z_i) - \ell_{xe}(f, z_i) = g_i(Tf(x_i, y_i)) - g_i(f(x_i))
\]
\[
\leq |g'_i(Tf(x_i, y_i))| \left( \sum_{j:d+1} a_{j_{d+1}}^{d+1} s_d \left( \sum_{j:d=1} a_{1,j_d}^{d} s_{d-1} \left( \cdots s_1 \left( (a_{j_2}^{1})^t x_i + w_{j_{d+1}}^{j_{d+1}} \right) \right) \right) \right)
\]
\[
- \sum_{j:d+1=1} a_{1,j_d}^{d+1} s_d \left( \sum_{j:d=1} a_{1,j_d}^{d} s_{d-1} \left( \cdots s_1 \left( (a_{j_2}^{1})^t x_i \right) \right) \right).
\]
Now we need to peel off the layers of our neural networks. Applying Lemma 6 a total of \( d \) times, we have
\[
L_i \leq |g'_i(Tf(x_i, y_i))| \left( \prod_{j=2}^{d+1} \left\| A_j \right\|_\infty \right) \max_{j_2,\ldots,j_{d+1}} \left( a_{j_2}^{1} \right)^t x_i + w_{j_{d+1}}^{j_{d+1}} - \left( a_{j_2}^{1} \right)^t x_i
\]
\[
= \left( \prod_{j=2}^{d+1} \left\| A_j \right\|_\infty \right) \max_{j_2,\ldots,j_{d+1}} w_{j_2}^{j_{d+1}} \left| g'_i(Tf(x_i)) \right|
\]
\[
= \varepsilon \max_{j_2=1,\ldots,j_{d+1}} \left( a_{j_2}^{1} \right)^t \left\| \prod_{j=2}^{d+1} A_j \right\|_\infty \left| g'_i(Tf(x_i)) \right|
\]
where the last equality follows by the definition of the \( w_{j_2}^{j_{d+1}} \). Summing over \( i \) and averaging proves the lemma.

Next, we have two lemmas for peeling back layers of a neural network.

Lemma 6. Let \( s : \mathbb{R} \rightarrow \mathbb{R} \) be a 1-Lipschitz function applied elementwise to vectors. Let \( a_{j_2}^j \) denote the \( j \)th row of \( A \), and let \( b_{j_2}^j \) denote the \( j \)th row of \( B \). Let \( f_{j,j'} \) and \( f_{j,j'}' \) be functions from \( \mathbb{R}^m \) to \( \mathbb{R}^K \), for \( j = 1, \ldots, J \) and \( j' = 1, \ldots, J' \). Then we have
\[
\max_{j=1,\ldots,J} \sum_{j'=1}^{J'} b_{j,j'}^j s \left( a_{j}^j f_{j,j'}(x) \right) - \sum_{j'=1}^{J'} b_{j,j'}^j s \left( a_{j}^j f_{j,j'}'(x) \right)
\]
\[
\leq \left\| B \right\|_\infty \max_{j=1,\ldots,J} \max_{j'=1,\ldots,J'} \left| a_{j}^j f_{j,j'}(x) - a_{j}^j f_{j,j'}'(x) \right|.
\]

Proof. Let \( L \) denote the left-hand side of the inequality. Applying Hölder’s inequality and using the fact that \( s \) is 1-Lipschitz, we obtain
\[
L \leq \max_{j=1,\ldots,J} \max_{j'=1,\ldots,J'} \left( \sum_{j'=1}^{J'} |b_{j,j'}^j| s \left( a_{j}^j f_{j,j'}(x) \right) - s \left( a_{j}^j f_{j,j'}'(x) \right) \right)
\]
\[
\leq \left\| B \right\|_\infty \max_{j=1,\ldots,J} \max_{j'=1,\ldots,J'} \left| a_{j}^j f_{j,j'}(x) - a_{j}^j f_{j,j'}'(x) \right|.
\]
This establishes the lemma.
The next lemma deals with the Rademacher complexity. This is essentially the same as the lemmas of Golowich et al. (2018).

**Lemma 7.** Let \( \{b_j\} \) be vectors such that \( \|b_j\|_1 \leq \beta \), and let \( \{a_j\} \) denote the rows of \( A \). Let \( s \) be a 1-Lipschitz activation function applied elementwise to vectors, such that \( s(0) = 0 \). Let \( G \) be a convex, increasing, positive function. Finally, let the \( f_{j,j'} : \mathbb{R}^m \to \mathbb{R}^K \) be functions. Then we have

\[
\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} G \left( \sum_{i=1}^n \sigma_i \sum_{j'=1}^{j'} b_{j,j'} s \left( a_{j,j'}^T f_{j,j'}(x_i) \right) \right) \right] 
\leq 2 \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} G \left( \sum_{i=1}^n \sigma_i a_{j,j'}^T f_{j,j'}(x_i) \right) \right].
\]

**Proof.** Let \( L \) denote the left-hand side in the statement of the lemma. Using Hölder’s inequality and the assumption that \( G \) is increasing, we have

\[
L := \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} G \left( \sum_{j'=1}^{J'} b_{j,j'} \left( \sum_{i=1}^n \sigma_i s \left( a_{j,j'}^T f_{j,j'}(x_i) \right) \right) \right) \right] 
\leq \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} G \left( \|b_j\|_1 \max_{j'=1, \ldots, J'} \left( \sum_{i=1}^n \sigma_i s \left( a_{j,j'}^T f_{j,j'}(x_i) \right) \right) \right) \right] 
\leq \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} \max_{j'=1, \ldots, J'} G \left( \beta \sum_{i=1}^n \sigma_i s \left( a_{j,j'}^T f_{j,j'}(x_i) \right) \right) \right].
\]

Now we perform a symmetrization step. Since \( G \) is positive and monotone, we have \( G(|x|) \leq G(x) + G(-x) \). Combining this with the fact that \( \sigma_i \) and \(-\sigma_i\) have the same distribution, we obtain

\[
L \leq \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} \max_{j'=1, \ldots, J'} G \left( -\beta \sum_{i=1}^n \sigma_i s \left( a_{j,j'}^T f_{j,j'}(x_i) \right) \right) + G \left( \beta \sum_{i=1}^n \sigma_i s \left( a_{j,j'}^T f_{j,j'}(x_i) \right) \right) \right] 
\leq \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} \max_{j'=1, \ldots, J'} G \left( -\beta \sum_{i=1}^n \sigma_i s \left( a_{j,j'}^T f_{j,j'}(x_i) \right) \right) \right] 
+ \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} \max_{j'=1, \ldots, J'} G \left( \beta \sum_{i=1}^n \sigma_i s \left( a_{j,j'}^T f_{j,j'}(x_i) \right) \right) \right] 
= 2 \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} \max_{j'=1, \ldots, J'} G \left( \beta \sum_{i=1}^n \sigma_i s \left( a_{j,j'}^T f_{j,j'}(x_i) \right) \right) \right].
\]

Finally, we apply Lemma 12 to obtain

\[
L \leq 2 \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \max_{j=1, \ldots, J} \max_{j'=1, \ldots, J'} G \left( \beta \sum_{i=1}^n \sigma_i a_{j,j'}^T f_{j,j'}(x_i) \right) \right].
\]

This completes the proof. \( \square \)

## C Proofs of Corollaries 1 and 2

**Proof of Corollary 1.** Define the bounded hinge loss \( \ell_h^x(f, z) = \min\{1, \ell_h(f, z)\} \). This bounded hinge loss also bounds the indicator loss, so we have

\[
\mathbb{E}_P \ell_0 (\Psi f, z) \leq \mathbb{E}_P \ell_h^x (\Psi f, z).
\]
Applying Lemma 8 in Appendix D and the Rademacher complexity bound of Lemma 1 immediately gives

\[
\mathbb{E}_P \ell_{01}(\Psi f, z) \leq \frac{1}{n} \sum_{i=1}^{n} \ell_t(\Psi f, z_i) + 2 \frac{M_2 R}{\sqrt{n}} + \frac{\varepsilon M_q}{\sqrt{n}} + 3 \sqrt{\frac{\log \frac{2}{\delta}}{2n}}.
\]

This is the first of the two bounds that we wished to prove. To prove the second bound, we simply apply Lemma 4.

**Proof of Corollary 2.** We define the bounded cross-entropy loss to be \( \ell_{xe}^t(f, z) = \min\{1, \ell_{xe}(f, z)\} \). Then the bounded cross-entropy loss also bounds the indicator loss, so we have

\[
\mathbb{E}_P \ell_{01}(f, z) \leq \mathbb{E}_P \ell_{xe}^t(f, z).
\]

Applying Lemma 8 and the Rademacher complexity bound of Lemma 2 gives

\[
\mathbb{E}_P \ell_{01}(T f, z) \leq \frac{1}{n} \sum_{i=1}^{n} \ell_{xe}(T f, z_i) + 3 \sqrt{\frac{\log \frac{2}{\delta}}{2n}} + 2\alpha \left( \frac{\alpha_{1,F}}{\alpha_1} R + \frac{\alpha_{1,q}}{\alpha_1} \varepsilon \right) \frac{\sqrt{2d \log \frac{2}{\delta} + 1}}{\sqrt{n}}.
\]

This is the first desired generalization bound. To obtain the second bound, we apply Lemma 5.

**D Auxiliary Lemmas**

In this section, we collect auxiliary results. We start with a standard generalization bound (Mohri et al., 2012).

**Lemma 8.** Let \( \mathcal{F} \) be a class of functions. Let \( \ell \) be a loss function that takes values in \([0, 1]\) and is 1-Lipschitz in \( f(x) \). With probability at least \( 1 - \delta \), we have

\[
\mathbb{E}_P [\ell(f, z)] \leq \frac{1}{n} \sum_{i=1}^{n} \ell(f, Z_i) + 2 \mathcal{R}_n(\mathcal{F}) + 3 \sqrt{\frac{\log \frac{2}{\delta}}{2n}}.
\]

Next, we derive a result concerning the Lipschitz continuity of the cross-entropy loss composed with a softmax activation function.

**Lemma 9.** Define the function \( g_y(a) = \ell_{xe}(a, y) \). The derivative is given by

\[
g_y'(a) = \begin{cases} 
\frac{-1}{\exp(a+1)}, & y = +1 \\
\frac{1}{\exp(a+1)}, & y = -1.
\end{cases}
\]

In particular, the function \( g_y'(a) \) is monotonic and bounded in magnitude by 1, and

\[
g_y(a) - g_y(b) \leq |g_y'(b)| \cdot |b - a|,
\]

for all \( a, b \in \mathbb{R} \).
Proof. Substituting the expression for \( \delta(a) \) into the loss \( \bar{\ell}_{xe} \), we have

\[
g_y(a) = \begin{cases} 
- \log \left( \frac{\exp(a)}{\exp(a) + 1} \right), & y = +1 \\
- \log \left( \frac{1}{\exp(a) + 1} \right), & y = -1.
\end{cases}
\]

Thus, \( g_y \) is monotonically decreasing when \( y = +1 \), and monotonically increasing when \( y = -1 \), yielding equation (19). Differentiating yields the desired expression for \( g'_y \), and it is easy to see that the function is always monotonic and bounded by 1, as claimed.

We also derive a bound on the empirical Rademacher complexity of a linear classifier.

**Lemma 10.** Suppose \( \|x_i\|_2 \leq R \) for all \( i \). Let \( \mathcal{F}_{\text{lin}} \) be a class of linear functions of the form \( f(x) = \theta^T x + b \). If \( \|\theta\|_2 \leq M_2 \) for all \( f \) in \( \mathcal{F}_{\text{lin}} \), then the empirical Rademacher complexity satisfies

\[
\hat{R}_n(\mathcal{F}_{\text{lin}}) \leq \frac{M_2 R}{\sqrt{n}}.
\]

**Proof.** Using the Cauchy-Schwarz inequality and Jensen’s inequality, we obtain

\[
\hat{R}(\mathcal{F}) = \frac{1}{n} \mathbb{E}_\sigma \left[ \sup_{\theta \in \mathcal{F}} \theta^T \left( \sum_{i=1}^n \sigma_i x_i \right) \right] \leq \frac{1}{n} \mathbb{E}_\sigma \left[ \sup_{\theta \in \mathcal{F}} \|\theta\|_2 \left\| \sum_{i=1}^n \sigma_i x_i \right\|_2 \right] \\
\leq \frac{M_2}{n} \mathbb{E}_\sigma \left\| \sum_{i=1}^n \sigma_i x_i \right\|_2 \leq \frac{M_2}{n} \left( \mathbb{E}_\sigma \left[ \left\| \sum_{i=1}^n \sigma_i x_i \right\|_2^2 \right] \right)^{1/2}.
\]

Further note that

\[
\mathbb{E}_\sigma \left\| \sum_{i=1}^n \sigma_i x_i \right\|_2^2 = \mathbb{E}_\sigma \left[ \sum_{i,j=1}^n \sigma_i \sigma_j x_i^T x_j \right] = \mathbb{E}_\sigma \left[ \sum_{i=1}^n \|x_i\|_2^2 \right] \leq n R^2.
\]

Putting everything together gives

\[
\hat{R}(\mathcal{F}) \leq \frac{M_2}{n} \sqrt{n R^2} = \frac{M_2 R}{\sqrt{n}},
\]

as desired.

We also provide a bound on the cumulant generating function of a centered random variable and the resulting bounded differences inequality, which is given as Theorem 6.2 of Boucheron et al. (2013).

**Lemma 11.** Let \( f : \mathcal{X}^n \to \mathbb{R} \) be a function satisfying the bounded differences assumption

\[
f(x_1, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n) \leq c_i
\]

for all \( x_i \) and \( x_i' \) in \( \mathcal{X} \). Define the variance factor

\[
v := \frac{1}{4} \sum_{i=1}^n c_i^2.
\]
Let \( Z = f(x_1, \ldots, x_n) \), where the \( x_i \)'s are independent random variables. Then
\[
\log \mathbb{E} \exp(\lambda(Z - \mathbb{E}[Z])) \leq \frac{\lambda^2 \nu}{2}
\]
and
\[
P \{ Z - \mathbb{E}Z > t \} \leq e^{-\frac{t^2}{2\nu}}.
\]

Finally, we provide Talagrand’s contraction lemma. The term “contraction” refers to a 1-Lipschitz function, although one can easily extend the result to any \( L \)-Lipschitz function. The version stated here appears as equation (4.20) in Ledoux and Talagrand (1991). A similar statement appears as Proposition 4 of Ledoux and Talagrand (1989).

**Lemma 12.** Let \( G \) be a convex, increasing function. Let \( \phi_i : \mathbb{R} \to \mathbb{R} \) be 1-Lipschitz functions such that \( \phi_i(0) = 0 \). Let \( T \) be a compact subset of \( \mathbb{R}^n \). Then
\[
\mathbb{E} G \left( \sup_{t \in T} \sum_{i=1}^{n} \sigma_i \phi_i(t_i) \right) \leq \mathbb{E} G \left( \sup_{t \in T} \sum_{i=1}^{n} \sigma_i t_i \right).
\]

**E Adversarial Versus Distributional Robustness**

In this appendix, we discuss the relationship between adversarial and distributional robustness. Specifically, we see that Wasserstein distributional robustness of the kind usually considered is a stronger notion than adversarial robustness.

**E.1 Definitions**

Let \( P \) and \( Q \) be probability measures over \( \mathbb{R}^d \), and let \( \Gamma(P,Q) \) denote the set of all couplings of \( P \) and \( Q \). In more detail, if \( P \) and \( Q \) are probability measures defined over the \( \sigma \)-field \( \mathcal{G} \), a probability measure \( \mu : \mathcal{G} \times \mathcal{G} \to [0,1] \) is an element of \( \Gamma(P,Q) \) if for any event \( A \) in \( \mathcal{G} \), we have \( \mu(A, R^d) = P(A) \) and \( \mu(R^d, A) = Q(A) \). Given a metric \( d(\cdot, \cdot) \) on \( \mathbb{R}^d \) and \( 1 \leq s \leq \infty \), the Wasserstein distance is defined as
\[
W_s(P,Q) = \left\{ \begin{array}{ll}
\inf_{\mu \in \Gamma(P,Q)} \mathbb{E}_{(z,z')} \left[ d(z, z')^s \right]^{rac{1}{s}}, & s < \infty \\
\text{ess sup} \ d(z, z'), & s = \infty,
\end{array} \right.
\]
where \( \text{ess sup} f \) denotes the essential supremum of \( f \). We denote the set of distributions within an \( s \)-Wasserstein distance of \( P \) by
\[
\mathcal{P}(P,\varepsilon, s) = \{ Q : W_s(P,Q) \leq \varepsilon \}.
\]
The goal in distributionally robust learning is to control a worst-case risk of the form
\[
\sup_{Q \in \mathcal{P}(P,\varepsilon, s)} \mathbb{E}_{z \sim Q} [\ell(f,z)], \tag{20}
\]
where we take \( P \) to be the true distribution in our discussion.
E.2 Two Simple Relations

We now rigorously derive the fact that the distributionally robust risk upper-bounds the adversarial risk studied in this paper. Thus, adversarial robustness is a less stringent condition than Wasserstein distributional robustness, which is also reflected in the regularization terms appearing in our bounds. We start by showing an equivalence between adversarial robustness and distributional robustness in the case $s = \infty$.

**Lemma 13.** Let $P$ be a distribution. Suppose $\ell$ is continuous or takes finitely many values. Then

$$
\mathbb{E}_P \left[ \sup_{w \in B(\varepsilon)} \ell(f, z + w) \right] = \sup_{Q \in \mathcal{P}(P, \varepsilon, \infty)} \mathbb{E}_{z' \sim Q} \ell(f, z').
$$

**Proof.** Let $\ell$ and $f$ be given. We start by proving that

$$
\mathbb{E}_P \left[ \sup_{w \in B(\varepsilon)} \ell(f, z + w) \right] \leq \sup_{Q \in \mathcal{P}(P, \varepsilon)} \mathbb{E}_Q \ell(f, z').
$$

Let $w^*$ be a random variable maximizing the supremum on the left-hand side. Since $\ell(f, \cdot)$ is either continuous or takes finitely many values and $B(\varepsilon)$ is compact, such a random variable exists.

Define $Q$ such that when $z' \sim Q$, we have $z' = z + w^*$. Since

$$
\mathbb{E}_P \left[ \sup_{w \in B(\varepsilon)} \ell(f, z + w) \right] = \mathbb{E}_Q \ell(f, z'),
$$

we only need to prove that $Q$ is in $\mathcal{P}(P, \varepsilon)$. Since we have

$$
\text{ess sup} \ d(z, z') = \text{ess sup} \ ||x - x'||_p = \text{ess sup} \ ||w^*||_p \leq \varepsilon,
$$

this completes the first direction.

We now prove the reverse inequality:

$$
\mathbb{E}_P \left[ \sup_{w \in B(\varepsilon)} \ell(f, z + w) \right] \geq \sup_{Q \in \mathcal{P}(P, \varepsilon)} \mathbb{E}_Q \ell(f, z').
$$

Let $Q$ be an element of $\mathcal{P}(P, \varepsilon)$. Then we can find a sequence of couplings $(\mu_k)_{k=1}^\infty$ such that when $(z_k, z'_k) \sim \mu_k$, we have

$$
\max ||x'_k - x_k||_p = \max d(z_k, z'_k) \leq \varepsilon + \frac{1}{k}.
$$

Define $w_k = x'_k - x_k$. Since all the $w_k$’s are elements of the compact ball $B(\varepsilon + 1)$, there is a subsequence $w_{k_j}$ that converges almost surely to some $w_{\infty}$. Moreover, we see that $||w_{\infty}||_p \leq \varepsilon$, so $w_{\infty}$ is always in $B(\varepsilon)$. Denote the limiting measure by $\mu_{\infty}$. We then have

$$
\mathbb{E}_Q \ell(f, z') = \mathbb{E}_{\mu_{\infty}} \ell(f, z + w_{\infty}) \leq \mathbb{E}_P \left[ \sup_{w \in B(\varepsilon)} \ell(f, z + w) \right].
$$

In particular, taking a supremum over $Q \in \mathcal{P}(P, \varepsilon)$ on the left-hand side proves the desired inequality. 

The second lemma simply states that the robust risk under the $W_\infty$ distance is bounded by the robust risk under the $W_s$ distance for $s < \infty$. 

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**Lemma 14.** Let $P$ be a distribution. Then for any $s$ in $[1, \infty]$, we have

$$\sup_{Q \in \mathcal{P}(P, \varepsilon, \infty)} \mathbb{E}_Q \ell(f, z') \leq \sup_{Q \in \mathcal{P}(P, \varepsilon, s)} \mathbb{E}_Q \ell(f, z').$$

*Proof.* It suffices to show that $W_\infty(P, Q) \leq \varepsilon$ implies $W_s(P, Q) \leq \varepsilon$. If $s = \infty$, the inequality is trivial, so assume $s < \infty$. If $P$ and $Q$ are distributions such that $W_\infty(P, Q) \leq \varepsilon$, we can find a sequence of couplings $\{\mu_k\}_{k=1}^\infty$ such that, for $(z_k, z_k') \sim \mu_k$, we have

$$\text{ess sup } d(z_k, z_k') \leq \varepsilon + \frac{1}{k}.$$ 

As a result, we have

$$W_s(P, Q) \leq \mathbb{E}_{\mu_k} [d(z_k, z_k')^s]^{\frac{1}{s}} \leq \text{ess sup } [d(z_k, z_k')^s]^{\frac{1}{s}} \leq \varepsilon + \frac{1}{k}.$$ 

Taking the limit as $k \to \infty$ shows that $W_s(P, Q) \leq \varepsilon$, proving the lemma. \qed