ON MODELLING OF SINGULARITIES AND THEIR PRODUCTS IN
COLOMBEAU ALGEBRA $\mathcal{G}(\mathbb{R})$

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Modelling of singularities given by discontinuous functions or distributions by means of generalized functions has proved useful in many problems posed by physical phenomena. We introduce in a systematic way generalized functions of Colombeau that model singularities given by distributions with singular point support. Moreover, we evaluate various products of such generalized models whenever the results admit associated distributions. The results obtained follow the idea of a well-known result of Jan Mikusiński on balancing of singular distributional products.

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1 INTRODUCTION

The Colombeau algebra of generalized functions $\mathcal{G}$ [1] have become a powerful tool for treating differential equations with singular coefficients and data as well as singular products of Schwartz distributions. The flexibility of Colombeau theory allows to model such singularities by means of appropriately chosen generalized functions, treat them in this framework and obtain results on distributional level, using the association process in $\mathcal{G}$.

In particular, Colombeau functions have proved useful in studying Euler-Lagrange equations for classical particle in $\delta$-type potential as well as the geodesic equation for impulsive gravitational waves; see [6, §1.5, §5.3]. Generalized models in $\mathcal{G}$ of Heaviside step-function $\theta$ were successfully applied to solving problems arising in Mathematical Physics [2]. Other examples involving $\theta$- and $\delta$-type singularities that describe realistic physical phenomena are jump conditions in hyperbolic systems leading to travelling $\delta$-waves solutions [3], controlled hybrid systems [7], geodesics for impulsive gravitational waves [12]. A detailed presentation of results on this topic and list of citations can be found in [11] and [6].

Recall further the well-known result published by Jan Mikusiński in [10]:

$$x^{-1} \cdot x^{-1} - \pi^2 \delta(x) \cdot \delta(x) = x^{-2}, \quad x \in \mathbb{R}. \quad (1)$$

Though, neither of the products on the left-hand side here exists, their difference still has a correct meaning in the distribution space $\mathcal{D}'(\mathbb{R})$. Formulas including balanced singular products of distributions can be found in the mathematical and physical literature. For

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balanced products of this kind, we used the name ‘products of Mikusiński type’ in a previous paper \cite{4}, where we derived a generalization in Colombeau algebra of equation (1) so that the distributions $x^{-p}$ and $\delta^{(q)}$ for arbitrary natural $p, q$ were involved.

Motivated by the aforementioned works on generalized models in the algebra $\mathcal{G}$, we have introduced in a {	extit{unified way}} generalized functions of Colombeau that model singularities of certain type and have additional properties \cite{5}. The singularities we considered were given by distributions with singular support (the complement to the maximal open set where the distribution is a $C^\infty$-function) in a point $x$ on the real line $\mathbb{R}$. For $x = 0$, such are Dirac $\delta$-function and its derivatives, Heaviside step function, the non-differentiable functions $x^p$, and the distributions $x^p_\pm$, $a \in \mathbb{R}\setminus\mathbb{Z}$.

In this paper, we study generalized models in $\mathcal{G}$ of the distributions $x^{-p}_\pm$, $p \in \mathbb{N}$ and evaluate various products of such models whenever the result admits associated distribution. When computed for the canonical embedding of the distributions in $\mathcal{G}$, none of the computed here singular products admits associated distribution.

2 NOTATION AND DEFINITIONS

2.1. We recall first the basic definitions of Colombeau algebra $\mathcal{G}(\mathbb{R})$ \cite{1}.

\textit{Notation 1.} Let $\mathbb{N}$ denote the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\delta_{ij} = \{1$ if $i = j$, $0$ if $i \neq j\}$, for $i, j \in \mathbb{N}_0$. Then we put for arbitrary $q \in \mathbb{N}_0$:

$$A_q(\mathbb{R}) = \{\varphi(x) \in \mathcal{D}(\mathbb{R}) : \int_{\mathbb{R}} x^j \varphi(x) \, dx = \delta_{0j}, \ j = 0, 1, \ldots, q\},$$

where $\mathcal{D}(\mathbb{R})$ is the space of infinitely differentiable functions with compact support. For $\varphi \in A_0(\mathbb{R})$ and $\varepsilon > 0$, we will use the following notation throughout the paper: $\varphi_\varepsilon = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ and $s \equiv s(\varphi) := \sup \{|x| : \varphi(x) \neq 0\}$. Then clearly $s(\varphi_\varepsilon) = \varepsilon s(\varphi)$, and denoting $\sigma \equiv \sigma(\varphi, \varepsilon) := s(\varphi_\varepsilon) > 0$, we have $\sigma := \varepsilon s = O(\varepsilon)$, as $\varepsilon \to 0$, for each $\varphi \in A_0(\mathbb{R})$.

Finally, the shorthand notation $\partial_x = d/dx$ will be used in the one-dimensional case too.

\textbf{Definition 1.} Let $\mathcal{E}[\mathbb{R}]$ be the algebra of functions $F(\varphi, x) : A_0(\mathbb{R}) \times \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable for fixed ‘parameter’ $\varphi$. Then the generalized functions of Colombeau are elements of the quotient algebra $\mathcal{G} \equiv \mathcal{G}(\mathbb{R}) = \mathcal{E}_M[\mathbb{R}] / \mathcal{I}[\mathbb{R}]$. Here $\mathcal{E}_M[\mathbb{R}]$ is the subalgebra of ‘moderate’ functions such that for each compact subset $K$ of $\mathbb{R}$ and $p \in \mathbb{N}_0$ there is a $q \in \mathbb{N}$ such that, for each $\varphi \in A_q(\mathbb{R})$, $\sup_{x \in K} |\partial^p F(\varphi_\varepsilon, x)| = O(\varepsilon^{-q})$, as $\varepsilon \to 0_+$, where $\partial^p$ denotes the derivative of order $p$. The ideal $\mathcal{I}[\mathbb{R}]$ of $\mathcal{E}_M[\mathbb{R}]$ consists of all functions such that for each compact $K \subset \mathbb{R}$ and any $p \in \mathbb{N}_0$ there is a $q \in \mathbb{N}$ such that, for every $r \geq q$ and $\varphi \in A_r(\mathbb{R})$, $\sup_{x \in K} |\partial^p F(\varphi_\varepsilon, x)| = O(\varepsilon^{r-q})$, as $\varepsilon \to 0_+$. 

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The differential algebra $\mathcal{G}(\mathbb{R})$ contains the distributions on $\mathbb{R}$, canonically embedded as a $\mathbb{C}$-vector subspace by the map
\[ i : \mathcal{D}'(\mathbb{R}) \to \mathcal{G} : u \mapsto \tilde{u} = \{ \tilde{u}(\varphi, x) := (u * \tilde{\varphi})(x) | \varphi \in A_q(\mathbb{R}) \}, \text{ where } \tilde{\varphi}(x) = \varphi(-x). \]

The equality of generalized functions in $\mathcal{G}$ is very strict and a weaker form of equality in the sense of *association* is introduced, which plays a fundamental role in Colombeau theory.

**Definition 2** (a) Two generalized functions $F, G \in \mathcal{G}(\mathbb{R})$ are said to be ‘associated’, denoted $F \approx G$, if for some representatives $F(\varphi_\varepsilon, x), G(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}_0$, such that for any $\varphi(x) \in A_q(\mathbb{R})$, $\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} [F(\varphi_\varepsilon, x) - G(\varphi_\varepsilon, x)] \psi(x) \, dx = 0$.

(b) A generalized function $F \in \mathcal{G}(\mathbb{R})$ is said to be ‘associated’ with a distribution $u \in \mathcal{D}'(\mathbb{R})$, denoted $F \approx u$, if for some representative $F(\varphi_\varepsilon, x)$, and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}_0$, such that for any $\varphi(x) \in A_q(\mathbb{R})$, $\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x) \psi(x) \, dx = \langle u, \psi \rangle$.

These definitions are independent of the representatives chosen, and the association is a faithful generalization of the equality of distributions. The following relations hold in $\mathcal{G}$:

\[ F \approx u \quad \& \quad F_1 \approx u_1 \implies F + F_1 \approx u + u_1, \quad \partial F \approx \partial u. \quad (2) \]

**Remark.** The equation $F \approx u$ is asymmetric in the sense that the terms cannot be moved over the $\approx$-sign: on the r.h.s. of it there stands a distribution. Of course, its equivalent relation $F \approx \tilde{u}$ in $\mathcal{G}$ is symmetric (and can be written as $F - \tilde{u} \approx 0$ as well). We prefer however the first, simpler and suggesting, notation for the associated distribution.

**2.2.** We next recall the definition of some distributions to be used in the sequel.

**Notation 2.** If $a \in \mathbb{C}$ and $\Re a > -1$, denote as usual the locally-integrable functions:

\[ x_+^a = \{ x^a \text{ if } x > 0, \quad 0 \text{ if } x < 0 \}, \quad x_-^a = \{ (-x)^a \text{ if } x < 0, \quad 0 \text{ if } x > 0 \}. \]

\[ \ln x_+ = \{ \ln x \text{ if } x > 0, \quad 0 \text{ if } x < 0 \}, \quad \ln x_- = \{ \ln(-x) \text{ if } x < 0, \quad 0 \text{ if } x > 0 \}. \]

\[ \ln|\!|x|\!| = \ln x_+ + \ln x_-, \quad \ln|\!|x|\!| \text{sgn } x = \ln x_+ - \ln x_- . \]

The distributions $x_+^a$ are defined for any $a \in \Omega := \{ a \in \mathbb{R} : a \neq -1, -2, \ldots \}$, by setting

\[ x_+^a = \partial^r x_+^{a+r}(x), \quad x_-^a = (-1)^r \partial^r x_-^{a+r}(x), \]

where $r \in \mathbb{N}_0$ is such that $a + r > -1$ and the derivatives are in distributional sense.

This definition can be extended also for negative integer values of $a$ by a procedure essentially due to M. Riesz (see [R § 3.2]). For each $\psi(x) \in \mathcal{D}(\mathbb{R})$, $a \mapsto \langle x_+^a, \psi \rangle$ is an
analytic function of \( a \) on the set \( \Omega \). The excluded points are simple poles of this function. For any \( p \in \mathbb{N}_0 \), the residue at \( a = -p - 1 \) is \( \lim_{a \to -p - 1} (a + p + 1) \langle x_+^a, \psi \rangle = \psi^{(p)}(0)/p! \). Subtracting the singular part, one gets for any \( p \in \mathbb{N}_0 \):

\[
\lim_{a \to -p - 1} \langle x_+^a, \psi \rangle - \frac{1}{p!} \frac{\psi^{(p)}(0)}{a + p + 1} = -\frac{1}{p!} \int_0^\infty \ln x \psi^{(p+1)}(x) dx + \frac{\psi^{(p)}(0)}{p!} \sum_{k=1}^p \frac{1}{k}.
\]

The right-hand side of this equation, which is the principal part of the Laurent expansion, was proposed by Hörmander in [S] to define the distribution \( x_+^{-p-1} \), acting here on the test-function \( \psi(x) \). In view of the notation in 2.2, this is equivalent to the following definition of \( x_+^{-p-1} \) for arbitrary \( p \in \mathbb{N}_0 \) (\( x \in \mathbb{R} \)):

\[
x_+^{-p-1} = \frac{(-1)^p}{p!} \partial_x^{p+1} \ln x_+ + \frac{1}{p!} \frac{\kappa_p \delta(x)}{p!}.
\]

We have introduced here the shorthand notation \( \kappa_p := \sum_{k=1}^p 1/k \) (\( p \in \mathbb{N}_0 \)); note that \( \kappa_0 = 0 \). Similar consideration leads to the defining equation

\[
x_-^{-p-1} = -\frac{1}{p!} \partial_x^{p+1} \ln x_- + \frac{\kappa_p}{p!} \delta(x).
\]

One checks that the distributions \( x_\pm^{-p} \) satisfy:

\[
\partial_x x_+^{-p} = -p x_+^{-p-1} + \frac{(-1)^p}{p!} \delta(x), \quad \partial_x x_-^{-p} = p x_-^{-p-1} - \frac{1}{p!} \delta(x).
\]

Moreover, it follows immediately that

\[
x_+^{-p} \big|_{x \to -x} = x_-^{-p} \quad \text{and also} \quad x_+^{-p} + (-1)^p x_-^{-p} = x_-^{-p}, \quad (p \in \mathbb{N}),
\]

where \( x_-^{-p} \) is defined, as usual, as a distributional derivative of order \( p \) of \( \ln |x| \).

Alternatively, we define the distribution

\[
x_-^{-p} \text{sgn} x := x_+^{-p} - (-1)^p x_-^{-p}, \quad (p \in \mathbb{N}_0).
\]

Note that \( x_-^{-p} \text{sgn} x \neq x_-^{-p} \) for arbitrary \( p \in \mathbb{N}_0 \); it also differs from the ‘odd’ and ‘even’ compositions \( x|^{-p} \text{sgn} x := x_+^{-p} + x_-^{-p} = x_-^{-p} \) for odd natural \( p \) and \( x|^{-p} := x_+^{-p} - x_-^{-p} = x_-^{-p} \) for even \( p \).

Recall finally the definition of the distributions \((x \pm i0)^{-p-1}\) for \( p \in \mathbb{N}_0 \):

\[
(x \pm i0)^{-p-1} := \lim_{y \to 0^+} (x \pm iy)^{-p-1} = x_-^{-p-1} \mp \frac{(-1)^p}{p!} \frac{i \pi}{p} \delta(x), \quad x \in \mathbb{R}.
\]
3 MODELLING OF SINGULARITIES IN COLOMBEAU ALGEBRA \( \mathcal{S}(\mathbb{R}) \)

Consider first generalized functions that model the \( \delta \)-type singularity in the sense of association, i.e. being associated with the \( \delta \)-function. Since there is an abundant variety of such functions (together with the canonical imbedding \( \tilde{\delta} \) in \( \mathcal{S} \) of the distribution \( \delta \)), we can put on the generalized functions in question an additional requirement. So define, following [11, §10], a generalized function \( D \in \mathcal{S} \) with the properties:

\[ D \approx \delta, \quad D^2 \approx \delta. \]  

To this aim, we let \( \varphi \in A_0(\mathbb{R}), \ s \equiv s(\varphi), \) and \( \sigma = s(\varphi_\varepsilon) = \varepsilon s \) be as in Notation 1, and \( D \in \mathcal{S} \) be the class \([\varphi \mapsto D(s(\varphi), x)]\). We specify further that \( D(s, x) = f(x) + \lambda s g(x) \), where \( f, g \in \mathcal{D}(\mathbb{R}) \) are real-valued, symmetric, with disjoint support, and satisfying:

\[ \int_{\mathbb{R}} f(x) dx = 1, \quad \int_{\mathbb{R}} g(x) dx = 0, \quad \text{and} \quad \lambda^2 = \frac{s - \int f^2(x) dx}{\int g^2(x) dx}. \]

It is not difficult to check that, for each \( \varphi \in A_0(\mathbb{R}) \), the representative \( D(s, x) \) of the generalized function \( D \) satisfies the conditions:

\[ D(., x) \in \mathcal{D}(\mathbb{R}), \quad D(., -x) = D(., x), \quad \frac{1}{s} \int_{\mathbb{R}} D^2(s, x) \ dx = \int_{\mathbb{R}} D(s, x) dx = 1, \]  

for each real positive value of the parameter \( s \). Moreover, the generalized function \( D \) so defined satisfies the association relations (8). To show this, denote by

\[ D_\sigma(x) := \frac{1}{\sigma} D\left(\sigma, \frac{x}{\sigma} \right), \quad \text{where} \quad \sigma = s(\varphi_\varepsilon). \]

Now, for an arbitrary test-function \( \psi \in \mathcal{D}(\mathbb{R}) \), evaluate the functional values

\[ I_1(\sigma) = \langle D_\sigma(x), \psi(x) \rangle, \quad I_2(\sigma) = \langle D_\sigma^2(x), \psi(x) \rangle, \]

as \( \varepsilon \to 0_+ \), or equivalently, as \( \sigma \to 0_+ \). But in view of (11), it is immediate to see that

\[ \lim_{\sigma \to 0_+} I_1(\sigma) = \lim_{\sigma \to 0_+} I_2(\sigma) = \langle \delta, \psi \rangle; \]

which according to Definition 2(b) gives (8).

The first equation in (8) is in consistency with the observation that \( D_\sigma(x) \) is a strict \( \delta \)-net as defined in distribution theory [11, §7]. But notice that \( D \) is not the canonical embedding \( \tilde{\delta} \) of the \( \delta \)-function since \( \tilde{\delta}^2 \) does not admit associated distribution.

The flexible approach to modelling singularities allowed by generalized functions in \( \mathcal{S} \), so that the models satisfy auxiliary conditions, can be systematically applied to defining generalized models of particular singularities. We will consider models of singularities given by
distributions with singular point support. For their definition, we intend to take advantage of the properties of \( \delta \text{-modelling function} \ D \). Observe that it holds

\[
(\delta \star D(s, .))(x) = \langle \delta_y, D(s, x - y) \rangle = D(s, x).
\]

\[
(\delta' \star D(s, .))(x) = \langle \delta'_y, D(s, x - y) \rangle = -\langle \delta_y, \partial_y D(s, x - y) \rangle = D'(s, x).
\]

This can be continued by induction for any derivative to define a generalized function \( D^{(p)}(s, .) \) that models \( \delta^{(p)}(s, .) \) and has representative \( D^{(p)}(s, x) = (\delta^{(p)} \star D(s, .))(x) \).

Clearly, this definition is in consistency with the differentiation: \( \partial_x D^{(p)}(x) = D^{(p+1)}(x) \), \( p \in \mathbb{N}_0 \). Moreover,

\[
D^{(p)}(-x) = (-1)^p D^{(p)}(x). \tag{11}
\]

In [5] we have employed such procedure for a unified modelling of singularities given by distributions with singular point support, i.e. (besides \( \delta^{(p)} \)) the distributions \( x^a_\pm \), \( a \in \Omega \). Namely, choosing an arbitrary generalized function \( D \) with representative \( D(s, x) \) that satisfies (9) for each \( \varphi \in A_0(\mathbb{R}) \), we have introduced generalized functions \( X^a_\pm(x) \), modelling the above singularities, with representatives

\[
X^a_\pm(s, x) := (y^a_\pm \star D(s, y))(x), \quad a \in \Omega. \tag{12}
\]

This is consistent with the differentiation: \( \partial_x X^a_\pm(x) = aX^{a-1}_\pm(x) \); in particular, \( H' = D \), where \( H \in \mathcal{G} \) is model of the step-function \( \theta \), with representative \( H(s, x) = \theta \star D(s, .)(x) \).

Extending now definition (12) to the distributions \( x^{-p-1}_\pm \), \( p \in \mathbb{N}_0 \), we can write

\[
X^{-p-1}_\pm(s, x) := (y^{-p-1}_\pm \star D(s, y))(x). \tag{13}
\]

Similarly, we put \( \ln x_\pm := \ln y^1_\pm \star D(s, y)(x) \).

We note that generalized functions so introduced are indeed models of the corresponding singularities. It is not difficult to show that for each \( a \in \Omega \)

\[
X^a_\pm(x) \approx x^a_\pm(x); \quad \text{in particular, } H \approx \theta, \text{ and } H^p \approx \theta \text{ for each } p \in \mathbb{N}.
\]

Also, it was proved in [5] that — as it can be expected — the functions \( H \) and \( D \) that model correspondingly the \( \theta \)- and \( \delta \)-type singularities satisfy the relation \( H \cdot D \approx \frac{1}{2} \delta \).

Moreover, these generalized models were proved to satisfy

\[
H \cdot D' \approx -\delta + \frac{1}{2} \delta'. \tag{14}
\]

Concerning the singularities given by the distributions \( x^{-p}_\pm \), \( p \in \mathbb{N} \), it can be easily checked that \( \ln x_\pm \approx \ln x_\pm x \) for the latter locally-integrable function. Then the modelling property
for the generalized functions \( X_{\pm}^{-p}(x) \) follows in view of relation (2) for consistency between differentiation and association in \( S \).

Finally, we need to compute the representatives of the generalized models when they depend on \( \varphi_\varepsilon \), or rather on the value \( s(\varphi_\varepsilon) = \varepsilon s(\varphi) = \sigma \). In view of equations (3), (4), (10), (12), and (13), we obtain for the corresponding representatives \((p \in \mathbb{N}_0)\) :

\[
X_{+\sigma}^p(x) = \frac{1}{\sigma} \int_0^\infty y^p D\left(\sigma, \frac{x-y}{\sigma}\right) dy, \quad X_{-\sigma}^p(x) = \frac{1}{\sigma} \int_{-\infty}^0 (-y)^p D\left(\sigma, \frac{x-y}{\sigma}\right) dy.
\]

(15)

\[
X_{+\sigma}^{-p-1}(x) = \frac{(-1)^p}{\sigma^{p+2}p!} \int_0^\infty \ln y D^{(p+1)}\left(\sigma, \frac{x-y}{\sigma}\right) dy + \frac{(-1)^p}{\sigma^{p+1}p!} D^{(p)}\left(\sigma, \frac{x}{\sigma}\right).
\]

(16)

4 PRODUCTS OF SOME SINGULARITIES MODELLED IN \( S(\mathbb{R}) \)

The models of singularities we consider all have products in Colombeau algebra as generalized functions, but we are seeking results that can be evaluated back in terms of distributions, i.e. such that admit an associated distribution. We will establish first certain balanced products of generalized models in the algebra \( S(\mathbb{R}) \) that exist on distributional level, proving the following.

**Theorem 1.** The generalized models of the distributions \( x_{\pm}^{-2}, 0, \bar{0}, \delta'(x) \) satisfy:

\[
X_{-\sigma}^{-2}. H - \ln x_+ . D' \approx - \delta.
\]

(17)

\[
X_{+\sigma}^{-2}. H + \ln x_- . D' \approx - \delta.
\]

(18)

**Proof:** (i) For an arbitrary test-function \( \psi(x) \in \mathcal{D}(\mathbb{R}) \), denote \( I(\sigma) := \langle X_{-\sigma}^{-2}. H_{\sigma}, \psi(x) \rangle \).

Suppose (without loss of generality) that \( \text{supp} D(\sigma, x) \subseteq [-l, l] \) for some \( l \in \mathbb{R}_+ \); then \( -l \leq x/\sigma \leq l \) implies \( -l\sigma \leq x \leq l\sigma \). Now from equations (10) for \( p = 1 \) and (15) for \( p = 0 \), we get on transforming the variables \( y = \sigma u + x, z = \sigma v + x, \) and \( x = -\sigma w \):

\[
I(\sigma) = -\frac{1}{\sigma^4} \int_{-\sigma l}^{\sigma l} dx \psi(x) \int_0^{\sigma l+x} dy D\left(\sigma, \frac{x-y}{\sigma}\right) \int_{-\sigma l+x}^0 \ln(-z) D''\left(\sigma, \frac{x-z}{\sigma}\right) dz
\]

\[
+ \frac{1}{\sigma^3} \int_{-\sigma l}^{\sigma l} dx \psi(x) D'\left(\sigma, \frac{x}{\sigma}\right) \int_{-\sigma l+x}^{\sigma l+x} D\left(\sigma, \frac{x-y}{\sigma}\right) dy
\]

\[
= -\frac{1}{\sigma} \int_{-l}^{l} dw \psi(-\sigma w) \int_{-l}^{l} du D(\sigma, u) \int_{-\sigma l}^{w} \ln(\sigma w - \sigma v) D''(\sigma, v) dv
\]

\[
+ \frac{1}{\sigma} \int_{-l}^{l} dw \psi(-\sigma w) D'(\sigma, w) \int_{-l}^{l} D(\sigma, u) du =: I_1 + I_2.
\]

(19)
Applying Taylor theorem to the test-function \( \psi \) and changing the order of integration (which is permissible here), we get further

\[
I_1 = -\frac{\psi(0)}{\sigma} \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} dv D''(\sigma, v) \int_{v}^{u} \ln(\sigma w - \sigma v) \, dw \\
+ \psi'(0) \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} dv D''(\sigma, v) \int_{v}^{u} \ln(\sigma w - \sigma v) \, w \, dw + o(1)
\]

Here the Landau symbol \( o(1) \) stands for an arbitrary function of asymptotic order less than any constant, and the asymptotic evaluation is obtained taking into account that the third term in the Taylor expansion is multiplied by definite integrals majorizable by constants.

Now the substitution \( w \rightarrow t = (w - v)/(u - v) \), together with \( w - v = (u - v)t \), yields

\[
I_1 = -\frac{\psi(0)}{\sigma} \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} dv D''(\sigma, v) (u - v) \left[ \ln(\sigma u - \sigma v) + \int_{0}^{1} \ln t \, dt \right] \\
+ \psi'(0) \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} dv D''(\sigma, v) (u - v)^2 \left[ \frac{1}{2} \ln(\sigma u - \sigma v) + \int_{0}^{1} t \ln t \, dt \right] \\
- \psi'(0) \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} dv D''(\sigma, v) (u - v)^2 \left[ \ln(\sigma u - \sigma v) + \int_{0}^{1} \ln t \, dt \right] \\
+ \psi'(0) \int_{-l}^{l} du \, u D(\sigma, u) \int_{-l}^{u} dv D''(\sigma, v) (u - v) \left[ \ln(\sigma u - \sigma v) + \int_{0}^{1} \ln t \, dt \right] + o(1).
\]

Calculating the integrals \( \int_{0}^{1} \ln t \, dt = -1 \), \( \int_{0}^{1} \ln t \, dt = -1/4 \), replacing \( v = u - (u - v) \), and integrating by parts in the variable \( v \) (the integrated part being 0) we get

\[
I_1 = -\frac{\psi(0)}{\sigma} \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} \ln(\sigma u - \sigma v) \, D'(\sigma, v) \, dv - 2\psi(0) \\
+ \psi'(0) \int_{-l}^{l} du \, u D(\sigma, u) \int_{-l}^{u} \ln(\sigma u - \sigma v) \, D'(\sigma, v) \, dv \\
- \psi'(0) \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} \ln(\sigma u - \sigma v) \, D'(\sigma, v) \, dv + o(1). \tag{20}
\]

To obtain the latter result, we have used equation (19) and also that

\[
\int_{-l}^{l} du \, D(\sigma, u) \int_{-l}^{u} D(\sigma, v) \, dv = \frac{1}{2}. \tag{21}
\]

Applying again Taylor theorem to the function \( \psi \), changing the order of integration, and integrating by parts in the variable \( w \), we obtain for the second term in (19):

\[
I_2 = \frac{\psi(0)}{\sigma} \int_{-l}^{l} dw \, D'(\sigma, w) \int_{-l}^{l} D(\sigma, u) \, du - \psi'(0) \int_{-l}^{l} dw \, w \, D'(\sigma, w) \int_{-l}^{l} D(\sigma, u) \, du \\
= \frac{\psi(0)}{\sigma} \int_{-l}^{l} du \, D(\sigma, u) \int_{-l}^{u} D'(\sigma, w) \, dw - \psi'(0) \int_{-l}^{l} du \, D(\sigma, u) \int_{-l}^{u} \, w \, D'(\sigma, w) \, dw + o(1) \\
= \psi(0) - \frac{1}{2} \psi'(0) + o(1),
\]

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where equation (21) is used again. Therefore

\[ I(\sigma) = -\frac{\psi(0)}{\sigma} \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} \ln(\sigma u - \sigma v) D'(\sigma, v) \, dv \]

\[ + \psi'(0) \int_{-l}^{l} du u D(\sigma, u) \int_{-l}^{u} \ln(\sigma u - \sigma v) D'(\sigma, v) \, dv \]

\[ - \psi'(0) \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} \ln(\sigma u - \sigma v) D(\sigma, v) \, dv - \psi(0) - 1/2 \psi'(0) + o(1). \tag{22} \]

(ii) On the other hand, denoting

\[ J(\sigma) := \langle \text{Ln } x_+ \sigma, D', \psi(x) \rangle, \]

we obtain on transforming the variables \( y = \sigma u + x \) and \( x = -\sigma v \), applying Taylor theorem to the test-function \( \psi \), and changing the order of integration:

\[ J(\sigma) = \frac{1}{\sigma^3} \int_{-\sigma}^{\sigma} dx \psi(x) D' \left( \sigma, \frac{x}{\sigma} \right) \int_{-\sigma}^{\sigma} \ln y D \left( \sigma, \frac{x-y}{\sigma} \right) dy \]

\[ = -\frac{1}{\sigma} \int_{-l}^{l} dv \psi(-\sigma v) D'(\sigma, v) \int_{v}^{l} \ln(\sigma v - \sigma u) D(\sigma, u) \, du \]

\[ = -\frac{\psi(0)}{\sigma} \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} \ln(\sigma v - \sigma u) D'(\sigma, v) \, dv \]

\[ + \psi'(0) \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} \ln(\sigma v - \sigma u) D(\sigma, v) \, dv + o(1). \]

Replacing now \( v = u + (v - u) \) in the last term and integrating by parts the third term so obtained, we get

\[ J(\sigma) = -\frac{\psi(0)}{\sigma} \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} \ln(\sigma v - \sigma u) D'(\sigma, v) \, dv \]

\[ + \psi'(0) \int_{-l}^{l} du u D(\sigma, u) \int_{-l}^{u} \ln(\sigma v - \sigma u) D'(\sigma, v) \, dv \]

\[ - \psi'(0) \int_{-l}^{l} du D(\sigma, u) \int_{-l}^{u} \ln(\sigma v - \sigma u) D(\sigma, v) \, dv - \frac{1}{2} \psi'(0) + o(1). \tag{23} \]

Combining now equations (22) and (23), we obtain by linearity

\[ \lim_{\sigma \to 0_+} \int_{\mathbb{R}} \psi(x) \left[ X_{-\sigma}^{-2}(x) \cdot H_\sigma(x) - \text{Ln } x_+ \sigma(x) \cdot D'_\sigma(x) \right] \, dx = -\psi(0) = -\langle \delta, \psi \rangle. \]

According to Definition 2(b), this proves the first equation in (17). The second equation follows on replacing \( x \to -x \) in the first one and taking into account equations (5) and (11). This completes the proof.

The above balanced products of the functions \( X_{\pm}^{-2} \) supported in the corresponding real half-lines can be employed further to get results on singular products of the generalized modelling functions \( X^{-2} \text{sgn } x \) and \( X^{-2} \) (obtained from equations (6), (5), and (13)).
Corollary 1. The following balanced product holds for the generalized models of the distribution $x^{-2} \text{sgn} x$, $\theta$, and $\delta'$:

$$X^{-2} \text{sgn} x \cdot H + \ln |x| \cdot \text{sgn} x \cdot D' \approx x^{-2} + 2\delta. \quad (24)$$

Proof: Consider the following chain of identities and associations in $\mathcal{G}(\mathbb{R})$, taking into account equation (18) and the relation $H + \hat{H} \approx 1$:

$$X_{-}^{-2} \cdot H = X_{-}^{-2} \cdot (1 - \hat{H}) = X_{-}^{-2} - X_{+}^{-2} \cdot \hat{H} \approx X_{-}^{-2} + \ln x_+ \cdot D' + \delta.$$ 

Thus

$$X_{-}^{-2} \cdot H - \ln x_+ \cdot D' \approx X_{+}^{-2} + \delta,$$

which, in view of the association $X_{-}^{-2} \approx x_{-}^{-2}$ and the linearity by (2) of the association in $\mathcal{G}$ leads to the balanced product

$$X_{-}^{-2} \cdot H - \ln x_+ \cdot D' \approx x_{+}^{-2} + \delta. \quad (25)$$

Further, equation (6) for $p = 2$, as well as equations (17) and (25), will all yield

$$X^{-2} \text{sgn} x \cdot H = (X_{-}^{-2} - X_{+}^{-2}) \cdot H \approx \ln x_- \cdot D' + x_{+}^{-2} + \delta - \ln x_+ \cdot D' + \delta.$$ 

In view of relation (2) for linearity of the association, this proves equation (24).

Other consequences from the above results are given by this.

Corollary 2. The generalized models in $\mathcal{G}$ of the distributions $(x \pm i0)^{-2}$, $\theta$, and $\delta'$ satisfy

$$(X \pm i0)^{-2} \cdot H - \ln |x| \cdot D' \approx x_{\pm}^{-2} \pm i\pi \delta(x) \pm \frac{i\pi}{2} \delta'.$$ \quad (26)

Proof: The second equation in (5), as well as equations (17) and (25), now give

$$X^{-2} \cdot H = (X_{+}^{-2} + X_{-}^{-2}) \cdot H \approx \ln x_- \cdot D' + x_{+}^{-2} + \delta + \ln x_+ \cdot D' - \delta.$$ 

In view of (2), this yields

$$X^{-2} \cdot H - \ln |x| \cdot D' \approx x_{+}^{-2}. \quad (27)$$

Employing further equations (7), (27) and (14), we get

$$(X \pm i0)^{-1} \cdot H = X^{-2} \cdot H \pm i\pi D'(x) \cdot H \approx \ln |x| \cdot D' + x_{+}^{-2} \pm i\pi \delta \pm \frac{i\pi}{2} \delta'.$$
which in view of linearity of association in $G$ proves equation (26).

Finally, we will evaluate some products of singularities given by the non-differentiable functions $x_{\pm}$ modelled by the generalized functions $X_{\pm}$ with derivatives of $D$. They only exist as balanced products, as demonstrated by this.

**Theorem 2.** The following balanced products hold for the modelling generalized function $X_{\pm}, H$ and $D$:

\[
X_{+} \cdot D^{(4)} + H \cdot D^{(3)} \approx \frac{5}{2} \delta'' - \frac{3}{2} \delta'''
\]

\[
X_{-} \cdot D^{(4)} + \tilde{H} \cdot D^{(3)} \approx \frac{5}{2} \delta'' + \frac{3}{2} \delta'''
\]

**Proof:** For an arbitrary $\psi(x) \in D(\mathbb{R})$, we denote $I(\sigma) := \langle X_{+}(x) \cdot D^{(4)}(x), \psi(x) \rangle$. From equations (10) and (15), we get on transforming the variables $y = \sigma v + x$, $x = -\sigma u$, changing the order of integration, and applying Taylor theorem

\[
I(\sigma) = \frac{1}{\sigma^3} \int_{-\ell}^{\ell} du \psi(-\sigma u)D^{(4)}(\sigma, u) \int_{u}^{\ell} (v-u)D(\sigma, v) d\sigma
\]

\[
= \frac{\psi(0)}{\sigma^3} \int_{-\ell}^{\ell} dv D(\sigma, v) \int_{-\ell}^{v} (v-u)D^{(4)}(\sigma, u) du
\]

\[- \frac{\psi'(0)}{\sigma^2} \int_{-\ell}^{\ell} dv D(\sigma, v) \int_{-\ell}^{v} u (v-u)D^{(4)}(\sigma, u) du
\]

\[+ \frac{\psi''(0)}{2 \sigma} \int_{-\ell}^{\ell} dv D(\sigma, v) \int_{-\ell}^{v} u^2 (v-u)D^{(4)}(\sigma, u) du
\]

\[- \frac{\psi'''(0)}{6} \int_{-\ell}^{\ell} dv D(\sigma, v) \int_{-\ell}^{v} u^3 (v-u)D^{(4)}(\sigma, u) du + O(\sigma)
\]

\[= \psi(0) I_0 + \psi'(0) I_1 + \psi''(0) I_2 + \psi'''(0) I_3 + O(\sigma).
\]

Denote further $J(\sigma) := \langle H(x) \cdot D^{(3)}(x), \psi(x) \rangle$. Proceeding as above, we get

\[
J(\sigma) = - \frac{\psi(0)}{\sigma^3} \int_{-\ell}^{\ell} dv D(\sigma, v) \int_{-\ell}^{v} D^{(3)}(\sigma, u) du
\]

\[+ \frac{\psi'(0)}{\sigma^2} \int_{-\ell}^{\ell} dv D(\sigma, v) \int_{-\ell}^{v} u D^{(3)}(\sigma, u) du
\]

\[- \frac{\psi''(0)}{2 \sigma} \int_{-\ell}^{\ell} dv D(\sigma, v) \int_{-\ell}^{v} u^2 D^{(3)}(\sigma, u) du
\]

\[+ \frac{\psi'''(0)}{6} \int_{-\ell}^{\ell} dv D(\sigma, v) \int_{-\ell}^{v} u^3 D^{(3)}(\sigma, u) du + O(\sigma)
\]

\[= \psi(0) J_0 + \psi'(0) J_1 + \psi''(0) J_2 + \psi'''(0) J_3 + O(\sigma).
\]
We next compute the terms \( I_k, \ k = (0, 1, 2, 3) \). During the calculations we shall use equations (9), (21), as well as that
\[
\frac{1}{\sigma} \int_{-l}^{l} v D(\sigma, v) D'(\sigma, v) \, dv = - \frac{1}{2\sigma} \int_{-l}^{l} D^2(\sigma, v) \, dv = - \frac{1}{2}.
\]
Also, due to the equality \( D'(., -x) = -D'(., x) \), the following equations hold
\[
\int_{-l}^{l} D(\sigma, v) D'(\sigma, v) \, dv = \int_{-l}^{l} v D^2(\sigma, v) \, dv = \int_{-l}^{l} v^2 D(\sigma, v) D'(\sigma, v) \, dv = 0.
\]
Integrating now by parts in the variable \( u \), the integrated part being 0 each time, we obtain:
\[
I_0 = \frac{1}{\sigma^3} \int_{-l}^{l} du \int_{-l}^{v} D^{(3)}(\sigma, u) \, du = - J_0,
\]
\[
I_1 = - \frac{1}{\sigma^2} \int_{-l}^{l} du \int_{-l}^{v} u D^{(3)}(\sigma, u) \, du + \frac{1}{\sigma^2} \int_{-l}^{l} du \int_{-l}^{v} (v - u) D^{(3)}(\sigma, u) \, du = - J_1 + \frac{1}{\sigma^2} \int_{-l}^{l} D(\sigma, v) D'(\sigma, v) \, dv = - J_1.
\]
\[
I_2 = \frac{1}{2\sigma} \int_{-l}^{l} du \int_{-l}^{v} u^2 D^{(3)}(\sigma, u) \, du - \frac{1}{\sigma} \int_{-l}^{l} du \int_{-l}^{v} u(v - u) D^{(3)}(\sigma, u) \, du = - J_2 + I_2',
\]
where
\[
I_2' = \frac{1}{\sigma} \int_{-l}^{l} du \int_{-l}^{v} (v - u)^2 D^{(3)}(\sigma, u) \, du - \frac{1}{\sigma} \int_{-l}^{l} du \int_{-l}^{v} v D(\sigma, v) D'(\sigma, v) \, dv = \frac{5}{2}.
\]
\[
I_3 = \frac{1}{6} \int_{-l}^{l} du \int_{-l}^{v} u^3 D^{(3)}(\sigma, u) \, du + \frac{1}{2} \int_{-l}^{l} du \int_{-l}^{v} u^2 (v - u) D^{(3)}(\sigma, u) \, du = - J_3 + I_3'.
\]
Here
\[
I_3' = \frac{1}{2} \int_{-l}^{l} du \int_{-l}^{v} (v - u)^3 D^{(3)}(\sigma, u) \, du + \int_{-l}^{l} du \int_{-l}^{v} u (v - u) D^{(3)}(\sigma, u) \, du
- \frac{1}{2} \int_{-l}^{l} du \int_{-l}^{v} v^2 D(\sigma, v) D^{(3)}(\sigma, u) \, du = 3 \int_{-l}^{l} du \int_{-l}^{v} D(\sigma, v) D(\sigma, u) \, du
- \int_{-l}^{l} du \int_{-l}^{v} (v - u)^2 D^{(3)}(\sigma, u) \, du + \int_{-l}^{l} du \int_{-l}^{v} v^2 D(\sigma, v) D(\sigma, u) \, du = \frac{3}{2}.
\]
Summing up, we get
\[
I(\sigma) = -\psi(0) J_0 - \psi'(0) J_1 - \psi''(0) J_2 - \psi'''(0) J_3 + \frac{5}{2} \psi''(0) + \frac{3}{2} \psi'''(0) + O(\sigma). \quad (31)
\]
Now from equations (30) and (31), we obtain by linearity
\[
\lim_{\sigma \to 0^+} \int_{\mathbb{R}} \psi(x) \left[ X_+ \sigma(x) \cdot D^{(4)}_{\sigma}(x) + H \sigma(x) \cdot D^{(3)}_{\sigma}(x) \right] dx = \langle \frac{5}{2} \delta'' - \frac{3}{2} \delta''' , \psi \rangle.
\]
According to Definition 2(b), this proves equation (28), whereas equation (29) follows on replacing \( x \to -x \) in the former. The proof is complete.

**Remark.** When computed for the canonical embedding of the distributions in \( \mathcal{G} \), none of the above singular products can be balanced so as to admit associated distribution.

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