Aspects of $\sigma$ Models

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Abstract

Some aspects and applications of $\sigma$-models in particle and condensed matter physics are briefly reviewed.

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1 Introduction

σ-models appear in many places in particle and condensed matter physics. The $O(4)$ invariant linear and non-linear $\sigma$-models were originally introduced by Gell-Man and Levy as a simple framework within which the $SU(2)_L \times SU(2)_R$ chiral symmetry and the partial conservation of the axial vector current were realized \cite{1}. The linear model is also a good testing ground for some of the general ideas concerning renormalizability, symmetries and their spontaneous breakdown. The non-linear models on the other hand, incorporate the non-linear realization of symmetries and are used as phenomenological models whose tree graphs generate the soft pion amplitudes \cite{2}. In general the dynamical fields are maps from the space-time to some target manifold. When the target space itself is a Lie group, the corresponding $\sigma$ model is also called a principal chiral model. In this case the coordinates of the group manifold are identified with the Goldstone bosons of the spontaneous breakdown of a $G \times G$ symmetry to the subgroup $G$.

In string theory applications the target manifolds have a more complicated structure. Their geometry is fixed by some consistency requirements of the string theory itself \cite{3}.

The $\sigma$ models are also encountered in the studies of the magnetic properties of matter in condensed matter physics. In this case the coordinates of the target space are somehow related to the magnetization order field \cite{4}. Some of the behaviour of the magnetic systems near their phase transition points are understood by applying the renormalization group techniques to its non-linear $\sigma$-model description \cite{5}.

Finally one should also mention the unique perspective offered by these models in the study of the role of geometry and topology in the theory of the quantized fields.

The aim of this contribution is to briefly review some of these topics. No derivations of the results will be presented. The plan of the paper is as follows: in section 2 we mention some of the places where the principal chiral models and their various descendents appear. In section 3 we recount some condensed matter physics applications. Finally in section 4 we summarize some of the results contained in ref \cite{6} and \cite{7}.
2 The principal chiral model and its generalizations

The principal chiral models are effective field theories which encapsulate and systematize the results of the current algebras on soft meson physics. At the moment there is no rigourous derivation of these models from an underlying QCD Lagrangian. However the following argument makes their formulation plausible \[2,8\].

In the limit of vanishing (light) quark masses the QCD Lagrangian has a global symmetry group which acts independently on the left and the right handed massless fermions. For \(N\) flavours of massless quarks this group is \(U(N)_L \times U(N)_R\). However the hadronic spectrum shows only an approximate \(SU(N)\) flavour symmetry and there is no parity doubling of the meson multiplets. Therefore this symmetry must be broken. The breakdown takes place by two different mechanisms. Firstly a \(U(1)\) subgroup is broken by chiral anomalies. Then it is assumed that the remaining \(SU(N)_L \times SU(N)_R \times U(1)_V\) subgroup breaks down spontaneously to \(SU(N)_V \times U(1)_V\). The Goldstone bosons produced by this symmetry breaking are identified with the light scalar pseudo mesons. The symmetry in this phase is assumed to be realized non-linearly, namely the pseudo scalar Goldstone bosons are described by the coordinates of the manifold of the \(SU(N)\) group. Together with some physical assumptions such as the pion pole dominance one lands almost uniquely at an action of the form

\[
S = \frac{1}{4f^2} \int d^4x \ tr(\partial_\mu U \partial_\mu U^{-1}) + ... \tag{1}
\]

where \(f\) is the pion decay constant and \(U \in SU(N)\). The ... in equ. \[3\] indicate all \(G\)-invariant terms involving more than two derivatives of \(U\). For the moment we shall ignore these extra terms.

To develop a perturbation theory one writes

\[
U(x) = e^{i f \pi(x)} \pi(x) = \pi(x)^a Q_a
\]

where \(Q_a\) are the hermitian generators of \(SU(N)\). By substituting from equ. \[2\] into equ.
and expanding in powers of $\pi$ we obtain

$$S = \int d^4x \left[ -\frac{1}{4} tr(\partial_\mu \pi \partial^\mu \pi) + \frac{f^2}{48} tr((\partial_\mu \pi, \pi)^2) + O(f^4) \right]$$

The theory obtained in this way is not power counting renormalizable in 4-dimensions. This should not come as a surprise, if one remembers that it is only the tree diagrams of the action which should be considered for the description of the low energy meson physics. The action of equ.(1) displays only the leading term in an expansion in powers of derivative. In principle it is possible to add infinite number of terms involving higher derivatives of $U$, compatible with the $SU(N) \times SU(N)$ symmetry. The resulting Lagrangian will contain an infinite number of phenomenological constants and may be renormalizable in some generalized sense. The action of equ. (1) also does not generate all the amplitudes of the fundamental QCD Lagrangian. For example to reproduce the processes which are mediated through the anomalous triangular diagrams of QCD we need to include a term which involves four derivatives of $U$. This is the the Wess-Zumino term which is also needed if one requires that the discrete symmetries of the effective theory described by equ. (1) to coincide exactly with those of the fundamental QCD Lagrangian.

The Wess-Zumino term, $\Gamma$ can not be written as a manifestly $SU(N) \times SU(N)$ invariant contribution to equ.(1) in 4-dimensions. To put it in a manifestly invariant form one needs to embed the Euclidean compactified 4-dimensional space time manifold as the boundary of a 5-dimensional space in the group manifold. One then writes $\Gamma = k \int \omega$, where $\omega$ is a closed 5-form on the group manifold. To extend the group valued function $U(x)$ from the 4-dimensional boundary to the 5-dimensional space creates some ambiguity in physics which can be removed by assuming that the coupling constant $k$ is an integer. Witten has argued that this integer should equal the $n$ of the colour $SU(n)$ group. The addition of this term has interesting consequences both in 4-as well as in 2-dimensional space-times. For example in the 4-dimensional case, Witten has shown that the solitons of the $\sigma$-model will behave as fermions (bosons) if $k = n$ is odd (even).

In 2-space-time dimensions the theory defined by equ.(1) is renormalizable. There is a logarithmic divergence at the 1-loop order, which can be defined away by a simple
coupling constant renormalization

\[ f = f_0 + \frac{Nf_0^3}{4\pi} \ln \left( \frac{\mu_0}{\mu} \right) \]

where \( \mu \) is the scale of definition of \( f \). The running of the coupling is therefore governed by the renormalization group equation

\[ \mu \frac{df}{d\mu} = \beta(f) = -\frac{Nf^3}{4\pi} \]

The \( \beta(f) \) is negative and \( f = 0 \) is an UV stable fixed point. The asymptotic freedom of the model makes it an interesting analogue of more realistic models like QCD in \( D = 4 \).

The inclusion of the WZ-term to the 2-dimensional action changes the picture dramatically \cite{12}. Indeed starting from

\[ S = \frac{1}{4f^2} \int d^2x \, tr(\partial_\mu U \partial_\mu U) + k\Gamma_{WZW} \]  

Witten has shown that

\[ \beta(f) = -\frac{Nf^3}{4\pi} \left[ 1 - \left( \frac{f^2k}{2\pi} \right)^2 \right] \]

In equ.( 3) \( \Gamma_{WZW} \) is the Wess-Zumino-Witten term which is defined in a manner analogous to the 4-dimensional case, namely as an integral over a 3-dimensional ball whose boundary is identified with the compactified 2-dimensional space-time \cite{10}

\[ \Gamma_{WZW} = \frac{1}{24\pi} \int_{B^3} d^2x \, \varepsilon^{\mu\nu\lambda} tr(g^{-1}\partial_\mu gg^{-1}\partial_\nu gg^{-1}\partial_\lambda g) \]  

For the same reason as in 4-dimensions, the constant \( k \) must assume integer values only.

Now returning to equ.( 4) we see that in addition to the origin the \( \beta \) function vanishes also at a non-zero value of the coupling constant, namely at \( f_c^2 = 2\pi/|k| \) Unlike the origin, the new fixed point \( f_c \) is an IR stable point. The theory defined at \( f_c \) is conformally invariant and at this value of \( f \) the physics can be described in terms of the representation theory of two copies of commuting Kac-Moody algebras. These algebras are generated by the separately conserved left and right moving Noether currents derived from the action integral of equ. \( (3) \) \cite{12}.
An obvious generalization of the model defined by equ.(1) is obtained by replacing the target space $G$ by one of the factor spaces $G/H$ of $G$. For example for $G/H = SU(2)/U(1) = CP^1$ the Lagrangian may be defined by

$$L = \frac{1}{2f^2} \partial_\mu \mu \partial^\mu \mu$$

where $\mu^2 = 1$ is a vector describing a $S^2 = CP^1$. The Theory is renormalizable at $D = 2$, where it is also asymptotically free. For $D > 2$ it has been shown that $\beta(f) = (D - 2)f - \frac{1}{2\pi} f^2$

Thus for $D > 2$ the asymptotic freedom is lost and a non-trivial ultraviolet stable fixed point is generated at $f_c = 2\pi(D - 2)$. If the Euclidean Lagrangian of equ.(6) is interpreted as the Hamiltonian of a classical Heisenberg ferromagnet, then $f_c$ will correspond to a critical temperature for the transition to a disordered phase.

Of course the most ambitious generalization of the action of equ.(1) is constructed by replacing the target space by an arbitrary manifold $M$ with coordinates $\phi^a$, $a = 1, 2, \ldots \text{dim} M$, and by writing the most general action

$$S = \int d^Dx \left( \frac{1}{2} g_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b + A_{ab}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b \varepsilon^{\mu \nu} + \ldots \right)$$

where $g_{ab}(\phi)$, $A_{ab}(\phi)$, $\ldots$ are tensor fields on $M$. The $D = 2$ version of equ.(8) has been used to obtain effective field theoretical description of the low energy dynamics of the massless modes of the string theory. In this case the field equations determining the dynamics of $g_{ab}$, $A_{ab}$, $\ldots$ are obtained by the requirement of the conformal invariance of the theory defined by equ. (8) [3].

3 $SU(2)$- Quantum Magnets and the $O(3)$ invariant $\sigma$-model

Now we turn to a very brief discussion of the magnetic systems. The analysis of the spin Hamiltonians has been a source of many interesting and important results in physics and mathematics. Physical concepts like spin waves, magnons and mathematical ideas like
Yang-Baxter equations and the quantum groups in fact have their roots in the studies of such systems.

Consider a lattice of points \( \mathbf{r} \in \mathbb{Z}^n \) and attach the \( SU(2) \) generators \( S^\alpha_{\mathbf{r}}, \alpha = 1, 2, 3 \) to the sites \( \mathbf{r} \) of the lattice. The Heisenberg Hamiltonian is defined by

\[
H = \frac{1}{2} \sum_{\mathbf{r}, \mathbf{r}'} J_{\mathbf{r}, \mathbf{r}'} S^1_{\mathbf{r}} S^1_{\mathbf{r}'}
\] (9)

where \( J_{\mathbf{r}, \mathbf{r}'} \) are assumed to be translationally invariant exchange energies. The generators of course satisfy the usual \( SU(2) \) Lie algebra at each site, viz,

\[
[S^\alpha_{\mathbf{r}}, S^\beta_{\mathbf{r}'}] = i\varepsilon^{\alpha\beta\gamma} S^\gamma_{\mathbf{r}} \delta_{\mathbf{r}, \mathbf{r}'}
\] (10)

Exact solutions of the Hamiltonian of equ.( 9) are known only in \( D = 1 \) \([13]\). Therefore to understand the physics in higher dimensions the best we can do at present is to resort to approximation schemes. The spin wave analysis is one such method which can be applied in any number of dimensions to understand the longwavelength physics of the fluctuations around a semi-classical ground state \([14]\).

Let us consider an isotropic \( D = 1 \) magnet with a nearest neighbour interaction,

\[
H = \frac{J}{2} \sum_{\mathbf{r}} S^1_{\mathbf{r}} S^1_{\mathbf{r}+1}
\] (11)

Assume that at each site the \( S^1_{\mathbf{r}} \) are given by a set of Pauli matrices. Then for \( J \) negative, the ground state will be given by a configuration in which all the spins are parallel,

\[
\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow
\] (12)

The state is an exact (ferromagnetic)-eigenstate of the Hamiltonian of equ.( 11). For \( J \) positive on the other hand, the construction of an exact ground state is too complicated. However a semi-classical (anti-ferromagnetic) ground state, called the Neel state, can be envisaged in which the pairwise neighbouring spins point in opposite directions.

\[
\uparrow \downarrow \downarrow \uparrow \downarrow \downarrow \uparrow \downarrow
\] (13)
Such configurations can be constructed on any bi-partite lattices. One can also generalize them to the spin chains with arbitrary values of the spin $s$.

To study the effects of the thermal and quantum fluctuations on the magnetic order, one defines an order parameter $n$, the so called staggered magnetization

$$n = \sum_x (-)^{r_x} S_x \hat{n}$$

(14)

Clearly in the semi-classical ground state we have

$$\langle 0 | n | 0 \rangle \neq 0$$

(15)

This could indicate the spontaneous breakdown of the rotational symmetry of the system. The corresponding Goldstone bosons are the spin waves or the magnons[4].

So far we have only summarized the standard lore. However in 1983 there was an interesting observation by Haldane [15] that the long wave-length fluctuations of $n$ in the $s \to \infty$ limit of the Heisenberg chain equ.(11), can be described by an $O(3)$ invariant $\sigma$-model with a topological term. The corresponding Lgrangian is given by

$$L = \frac{1}{2f^2} (\partial_\mu \underline{u})^2 + \theta \varepsilon^{\mu\nu\rho} \partial_\mu \underline{u} \times \partial_\nu \underline{u}$$

(16)

together with $\underline{u}^2 = 1$. Here $f^2 = 2/s$ and $\theta = s/4$. Noting that

$$\int d^2 x \varepsilon^{\mu\nu\rho} \partial_\mu \underline{u} \times \partial_\nu \underline{u} = 8\pi .integer$$

(17)

we see that the $\theta$ term in equ.(16) can be relevant only if $s$ is a half integer number. It is known that for $\theta = 0$ the model described by equ. (16) is always disordered [4]. The spectrum consists of a triplet of massive states. It was conjectured by Haldane that the $s = \text{half integer}$ chain is gapless. Furthermore Affleck and Haldane argued [16] that the universality class of the critical chain is a $SU(2)$ Wess-Zumino-Witten model of the type given by equ. (3) with $k = 1$.

One might expect that starting from a 2-dimensional lattice rather than a chain, one should obtain a (2+1)- dimensional field theory analogous to the one defined by equ. (16).
In this case it is better to introduce a complex 2-vector
\[ z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \]
which is related to \[ n^\alpha = z^\dagger \sigma^\alpha z \] where \( \sigma^\alpha, \alpha = 1, 2, 3 \) are the Pauli spin matrices. It is also assumed that \( z^\dagger z = 1 \). Thus a phase redefinition of \( z \) will leave \( n \) invariant. Therefore \( z \) can be regarded as the homogeneous coordinate of a point on \( CP^1 \) which has the same topology as a 2-dimensional sphere \( S^2 \), described by the unit vector \( n \). It can be shown by direct calculation that
\[
\frac{1}{4} (\partial_\mu z) e^2 = |(\partial_\mu - i A_\mu) z|^2 \quad (18)
\]
where
\[
A_\mu = -\frac{i}{2} (z^\dagger \partial_\mu z - \partial_\mu z^\dagger z) \quad (19)
\]
In 2+1 dimensions there is a topologically conserved current, viz,
\[
J^\mu = \frac{1}{8\pi} \varepsilon^{\mu\nu\lambda} n^\nu \partial_\lambda n \times \partial_\nu n
= \frac{1}{4\pi} \varepsilon^{\mu\nu\lambda} F_{\nu\lambda} \quad (20)
\]
where
\[
F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu
= -i(\partial_\nu z^\dagger \partial_\lambda z - \partial_\lambda z^\dagger \partial_\nu z)
\]
Therefore a natural (2+1)-dimensional generalization of equ.(16) would be
\[
L = \frac{1}{2 f^2} (\partial_\mu n)^2 + \theta J_\mu A^\mu
= \frac{2}{f^2} |(\partial_\mu - i A_\mu) z|^2 + \frac{\theta}{4\pi} \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} \quad (22)
\]
This model has soliton like solutions [17] and it has been shown by Wilczek and Zee [18] that they behave like bosons for \( \theta = 0 \) and like fermions for \( \theta = \pi \). The cause of this phenomenon is basically the non-local interaction which is obtained by solving the algebraic field equations of the \( A \)-field and substituting the result back in equ. (22).
For arbitrary values of $\theta$ the statistics of the solitons should interpolate between the ones of Bose-Einstein and Fermi-Dirac, which is also called anyon statistics [19]. The mechanism of statistics change could induce interesting physical effects such as superconductivity. Unfortunately, however, unlike equ. (16), the term in equ. (20) does not seem to be present in the large $s$-field theoretical description of the planer magnets [20].

It is interesting to note that the conserved charge $Q$ of the topological current $J^\mu$ of equ. (20) is given by,

$$Q = \frac{1}{8\pi} \int d^2x \, \varepsilon_{\mu\nu} n.\partial_\mu n \times \partial_\nu n$$

By virtue of equ.(17) this is an integer.

4 Some generalizations

Affleck [21] and independently Read and Sachdev [22] have generalized the group $SU(2)$ to $SU(N)$ by simply regarding the $S_r^\alpha$ in equ.(11) as the $SU(N)$ generators. In this case of course the range of the index $\alpha$ will be from 1 to $N^2 - 1$. It is also assumed that the representation content of the generators is independent from the site index $r$.

This generalization opens up interesting new possibilities. For example by regarding $N$ as a new independent parameter one can now apply the well known method of $1/N$ expansion to study the system’s behaviour for large values of $N$. In some sense this is similar to the large $s$ approximation of the ordinary $SU(2)$ model. Indeed interesting results have been obtained in this way about the critical behaviour of the system by Affleck and Marston [23] and, by Read and Sachdev [22].

It is possible to give a generalization of the concepts of ferromagnetic as well as of the anti-ferromagnetic orders in this broader context. The dynamics of the long-wavelength fluctuations will still be governed by a field theory for which the dynamical variables are maps from the space-time manifold to a factor space $G/H$ of $G$. The subgroup $H$ is the invariance group of the generalized magnetic order. Here we shall summarize some of the results of this study as carried out in [3] and [4]. To this end the coordinates of $G/H$ will
be denoted by $\phi^\mu$ where $\mu = 1, 2, ... \dim \frac{G}{H}$. It will always be assumed that $H$ contains the maximal Abelian subgroup of $G$. The coset corresponding to the point $\phi \in G/H$ can be represented by a group element $L(\phi)$. This representation is of course not unique. The action of $g \in G$ on $G/H$ which maps $\phi \rightarrow \phi' = \phi'(\phi, g)$ is defined by $gL(\phi) = L(\phi')h$ where $h = h(\phi, g) \in H$. The detailed form of the functions $\phi'(\phi, g)$ and $h(\phi, g)$ depend on the representative element $L(\phi)$. The $G$-invariant geometry of $G/H$ can be constructed from the pull-back of the Mauer-Cartan forms $L(\phi)^{-1}dL(\phi)$ . These forms take their values in the Lie algebra of $G$. Thus, they may be expanded on a basis $Q^\alpha$ of this algebra

$$L(\phi)^{-1}dL(\phi) = e^\alpha Q_\alpha = -A^j H_j + e^\alpha Q_\bar{\alpha}$$

where the generators $Q_\alpha$ satisfy the commutation relations

$$[Q_\alpha, Q_\beta] = c_{\alpha\beta}^\gamma Q_\gamma$$

The generators of the Cartan subalgebra are denoted by $H_j$ and the $Q_\bar{\alpha}$ are the remaining generators.

Now to each site $r$ of the lattice attach the weight vectors $\Lambda_r$. These are eigenvectors of the generators $H_j$, viz,

$$H_j|\Lambda_z > = |\Lambda_z > H_j$$

The semi classical ground states of the system are defined in terms of a distributions of these vectors on the lattice. We assume that in the ground states the lattice can be decomposed into translationally invariant sublattices such that in each one of them the distribution of the $\Lambda$’s is site independent. If there is only one translationally invariant sublattice we call the corresponding ground state ferromagnetic. The generalized anti-ferromagnetic ordering corresponds to the presence of more than one translationally invariant sublattice.

With this notational background we are now in a position to state some of our results.

1) For the generalized ferromagnetic ordering the dynamics of the long-wavelength fluctuations are governed by a set of non-relativistic field equations which reduce to the
Landau-Lifschitz equations in the special case of $G = SU(2)$ and $H = U(1)$. These equations have the following form:

$$\frac{\hbar}{i} F_{\mu \nu} \partial_t \phi^\nu = \frac{J}{2} k_{\mu \nu} \Delta \phi^\nu$$

where we have assumed an isotropic coupling $J < 0$ and where the "Laplacian" $\Delta$ is defined by

$$\Delta \phi^\nu = \partial_i^2 \phi^\nu + k^{\nu \rho}(\partial_\lambda k_{\rho \mu} + \partial_\mu k_{\rho \lambda} - \partial_\rho k_{\lambda \mu}) \partial_i \phi^\lambda \partial_i \phi^\mu$$

Here $k_{\mu \nu}$ is a $G$ invariant tensor on $G/H$ and $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A_\mu = A_\mu^j \Lambda_j$

Thus it is seen that the equations are intrinsically non-relativistic, i.e. they involve a first order time - and second order space derivatives.

It is not difficult to verify that for $G/H = SU(2)/U(1)$ these equations reduce to the phenomenological Landau-Lifshitz equations,

$$\hbar \partial_t n = -\frac{J}{2} s n \times \nabla^2 n$$

where $s$ is the spin and $n$ is a unit vector describing the target space $S^2$.

2) For the generalized anti-ferromagnetic order we also in general obtain non-relativistic low energy effective field equations. However in this case, unlike the ferromagnetic ordering, the low energy dispersion relations of the small oscillations have a linear form, viz, $\omega \sim |k|$ as $|k| \to 0$. These models become relativistic only if $G/H$ is a symmetric homogenous space, such as the Grassmanians $\frac{SU(N+M)}{SU(N) \times SU(M) \times U(1)}$.

With two translationally invariant sublattice in $D = 1$, the dynamics of the long wavelength fluctuations of the staggered magnetization is governed by

$$S = \int d^2x \left[ \frac{1}{2} g_{ab} \partial_t \phi^a \partial_t \phi^b + k_{ab} \partial_x \phi^a \partial_x \phi^b \right]$$

where $g_{ab}(\phi)$ is another $G$ invariant tensor field on $G/H$. If the space is symmetric, then there will be -up to a normalization- a unique second rank symmetric $G$-invariant tensor field on $G/H$. This means that the tensors $g_{ab}$ and $k_{ab}$ will be proportional and
therefore the action will be relativistic. In general however there are more than one such
fields and the model -as stated above- is not Lorentz invariant.

The tensors $g_{ab}$ and $k_{ab}$ are the coupling constants of our system and one may ask
about their running under the renormalization group transformations. Indeed the 1-loop
RG equations can be set up in quite general terms. As an example we have examined these
equations for the flag manifolds of the form $\frac{SU(N)}{U(1) \times \cdots \times U(1)}$ and shown that in $D = 2 + \varepsilon$
the equations admit a non-trivial fixed point at which the theory becomes relativistic,
i.e. the tensors $g_{ab}$ and $k_{ab}$ become proportional. This point is a generalization of the
Polyakov fixed point given by equ.(17). At this point there is an enlargement of the global
symmetries. One can find new discrete coordinate transformations on $G/H$ which leave
the effective fixed point theory invariant.

3) The $\theta$ term is also generated for the generalized 1-dimensional antiferromagnetic
systems. This term is rather like the $\theta$ term in QCD and for a system with two trans-
lationally invariant sublattices it is given by $\frac{1}{2\pi}\int F$, where $F$ is as defined above the
pull-back of the canonical $U(1)$ connections. The weight vector defining the representa-
tion content of semi-classical antiferromagnetic ground state entering into the definition
of $F$ is $\Lambda = \Lambda_1 = -\Lambda_2$, where $\Lambda_1$ and $\Lambda_2$ are the two weight vectors associated with the
sublattices 1 and 2 respectively.

4) The models admit finite action solution of Euclidean field equations with non-
zero values of $\int F$. These solutions can be regarded as the generalizations of the soliton
solutions of the $CP^N$ models.

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