Can baryogenesis occur on the lattice?

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We examine the question of how baryogenesis can occur in lattice models of the Standard Model where there is a global $U(1)$ symmetry which is accompanied by an exactly conserved fermion number. We demonstrate that fermion creation and annihilation can occur in these models despite this exact fermion number conservation, by explicitly computing the spectral flow of the Hamiltonian in the two dimensional $U(1)$ axial model with Wilson fermions. For comparison we also study the closely related Schwinger model where a similar mechanism gives rise to anomalous particle creation and annihilation.

1. INTRODUCTION

It is well known that fermion number is not conserved in the Standard Model (StM) due to the anomaly in the baryon and lepton currents which arises from the rich vacuum structure of non-abelian gauge theories. Topologically distinct vacua which are characterized by their integer Chern class $n$ are separated by a potential barrier whose minimal height is given by the so-called sphaleron energy, $E_{\text{sph}}$. Transitions from one vacuum to another are accompanied by a change in baryon and lepton numbers, $\Delta Q_{\text{B,L}} = N_G \Delta n$, with $N_G$ the number of generations. It was shown in ref. \[1\] that the tunneling transition rate at zero temperature is enormously suppressed by the small exponential factor $e^{-4\pi/a_W} \approx e^{-150}$, $a_W$ being the weak fine structure constant. As was first pointed out in ref. \[2\] though, the transition rate gets substantially enhanced at large temperatures, since the system can travel classically from one vacuum to another by thermal excitation instead of having to penetrate the barrier by tunneling. This happens when the thermal energy approaches the sphaleron energy and the transition rate is then proportional to the Boltzmann factor $\exp(-E_{\text{sph}}(T)/T)$.

The existing semiclassical calculations of the transition rate are based on certain assumptions and it is important to check the results by a calculation from the first principles and the only possibility at present to address such a non-perturbative problem is to use lattice Monte Carlo calculations. Some first attempts in this direction have been performed in the SU(2) gauge-Higgs sector of the StM \[3\] and in the two dimensional abelian Higgs model \[4\]. However these simulations must be extended to the full StM including fermions. A potential problem however is that a satisfactory formulation of the StM is still lacking. On the lattice each fermion is accompanied by spurious species doublers of opposite chirality which render chiral gauge theories vector-like. In the past three years most models, which have been proposed so far to overcome this phenomenon, have been shown to fail. The situation looks even worse with regard to the issue of fermion non-conservation. It has been argued in ref. \[5\] that currently existing models are unable to reproduce the fermion number non-conservation of the StM, since the action which in general is of the form

$$S = -\sum_{x,y} \bar{\psi}_x D(A)_{xy} \psi_y$$

(1)

and also the measure in the path integral are invariant under a global $U(1)$ symmetry $\psi_x \rightarrow \exp(i\alpha)\psi_x$, $\bar{\psi}_x \rightarrow \bar{\psi}_x \exp(-i\alpha)$. One therefore expects this exact global $U(1)$ symmetry to be accompanied by an exactly conserved fermion number, and hence no baryogenesis on the lattice.
when fermions are included in the above simulations.

We shall outline in this contribution how anomalous fermion creation and annihilation can occur on the lattice despite the exact fermion number conservation. For the sake of simplicity we have illustrated our reasoning for the case of a two dimensional axial model with Wilson fermions, however we see no reason that the results would not generalize to four dimensions, or to models with Higgs fields. Most of this work has been previously presented in ref. [6] using staggered fermions.

2. AXIAL AND VECTOR QED

The massless two-dimensional axial model is given by the following (real time) continuum action

\[ S = -\int d^2x \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + S_F, \]  

(2)

\[ S_F = -\int d^2x \overline{\psi} i\gamma^\mu (\partial_\mu + iA_\mu) \psi, \]  

(3)

where \( \gamma^1 = \gamma_1 = \sigma_1, \gamma^0 = -\gamma_0 = -i\sigma_2, \) \( \gamma_5 = \sigma_3. \) We take \( A_\mu \) to be an external gauge field; only the fermion fields are quantized. The model is invariant under local axial gauge transformations: \( \psi(x) \rightarrow \exp(i\omega(x)\gamma_5)\psi(x), \overline{\psi}(x) \rightarrow \overline{\psi}(x)\exp(i\omega(x)\gamma_5), \) \( A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \omega(x), \) with gauge current \( j^5_\psi = i\overline{\psi} \gamma^\mu \gamma_5 \psi. \) The action is furthermore invariant under the global U(1) symmetry \( \psi(x) \rightarrow \exp(\alpha)\psi(x), \overline{\psi}(x) \rightarrow \overline{\psi}(x)\exp(-\alpha), \) with a corresponding globally conserved vector current \( j^\mu = i\overline{\psi} \gamma^\mu \psi. \) According to standard lore the (local) gauge current has to be anomaly free whereas the vector current becomes anomalous, \( \partial_\mu j^\mu = -2q \) where \( q = \frac{\pi}{4} \epsilon_{\mu\nu\rho} F^{\mu\nu} \) denotes the topological charge density and \( C^\mu(x) = (1/2\pi) \epsilon^{\mu\nu\rho} A_\nu(x) \) is the Chern-Simons current. The axial and vector charges are defined as \( Q_5 = \int dx^1 j^5_\psi(x) \) and \( Q = \int dx^1 j^0(x) \), and the Chern-Simons number as \( C = \int dx^1 C^0(x) \). From the current divergence equation one sees that

\[ Q(t) - Q(0) = -2(C(t) - C(0)) \]  

(4)

which relates a time dependent change in the Chern-Simons number to a time dependent change in the fermion number. Thus a change in the topological charge of the gauge field \( \Delta C = -1 \) gives rise to the creation of a left and right-handed fermion. The aim of this contribution is to investigate whether, in lattice models which have exact global U(1) symmetry (i.e. the vector current is divergence free in contrast to the above continuum example), a change in \( C \) due to a sphaleron transition can still give rise to a change in the fermion number in accordance with eq. (4).

A nice feature of the two-dimensional axial model is that it is equivalent to the massless Schwinger model (QED₂) which is well studied. By performing a charge conjugation transformation on only the right (or left)-handed fermion fields in eq. (3), i.e. \( \psi_R = (\overline{\psi}_R C)^T, \overline{\psi}_R = -(\overline{C}^\dagger \psi'_R)^T, \psi_L = \overline{\psi}_L, \overline{\psi}_L = \psi'_L \) one finds that eq. (3) turns into

\[ S'_f = -\int d^2x \overline{\psi} \gamma^\mu (\partial_\mu - iA_\mu) \psi', \]  

(5)

which is the action of vector QED with massless fermions in two dimensions. We put a prime in the sequel to label quantities of the vector model. Under the same charge conjugation it can be shown that the axial current turns into the vector current, \( j^5_\psi \rightarrow -i\overline{\psi} \gamma^\mu \psi' = -j'^\mu, \) which is now the divergence free gauge current, \( \partial_\mu j'^\mu = 0. \) The global vector current becomes the global axial current \( j^\mu \rightarrow -i\overline{\psi} \gamma^\mu \gamma_5 \psi' = -j'^\mu \) and is anomalous, \( \partial_\mu j'^\mu = 2q(\chi). \) Correspondingly \( Q \rightarrow Q'_5, Q_5 \rightarrow Q'_5 \) for the charges. From the divergence relation for \( j^5_\psi \) we can derive the following formula

\[ Q'_5(t) - Q'_5(0) = 2(C(t) - C(0)) \]  

(6)

A change in Chern-Simons number \( \Delta C = -1 \) gives rise to the creation of a left-handed fermion and a right-handed antifermion. It is therefore instructive to study the spectral flow of the vector model in conjunction to the axial model.

In our earlier publication we used staggered fermions to transcribe the two models to the lattice [6]. Here we shall use instead Wilson fermions which may be somewhat more transparent. The euclidean lattice path integral for external gauge field configuration is then given by

\[ Z = \int \prod_x D\overline{\psi}_x D\psi_x \exp[-(S_F + S_W)], \]  

(7)
with
\[ S'_F = \frac{1}{2} \sum_{\mu x} (\bar{\psi}_x \gamma_\mu U_{\mu x} \psi_{x+\mu} - \bar{\psi}_{x+\mu} \gamma_\mu U^*_{\mu x} \psi'_x) \] (8)
for the vector QED$_2$ and

\[ S_F = \frac{1}{2} \sum_{\mu x} (\bar{\psi}_x \gamma_\mu (U_{\mu x} P_L + U^*_{\mu x} P_R) \psi_{x+\mu} - \bar{\psi}_{x+\mu} \gamma_\mu (U^*_{\mu x} P_L + U_{\mu x} P_R) \psi'_x) \] (9)

for the axial model. We use lattice units in which the lattice distance \( a = 1 \). \( U_{\mu x} = \exp(-iA_{\mu x}) \) is the lattice gauge field and \( P_{R,L} = \frac{1}{2}(1 \pm \gamma_5) \) are the chiral projectors. We have added in (8) a Wilson term of the form

\[ S_W = r \sum_{\mu x} \left( \bar{\psi}_x U_{\mu x} \psi'_x - \frac{1}{2} \left( \bar{\psi}_x U_{\mu x} \psi'_x + \bar{\psi}'_x U^*_{\mu x} \psi_x \right) \right) \] (10)

(and for the axial model with \( \psi'_x \to \psi_x \)) which for external gauge fields leaves the physical fermion at momentum \((0,0)\) massless and removes the three species doublers, that are located in the two dimensional Brillouin zone at momenta \((0,\pi)\), \((\pi,0)\) and \((\pi,\pi)\), from the spectrum by giving them a mass of the order of the cutoff.

The vector model is invariant under local vector gauge transformations \( \bar{\psi}_x \to \bar{\psi}_x \Omega_{x}^*, \psi_x \to \Omega_x \psi_x, U_{\mu x} \to \Omega_x U_{\mu x} \Omega_{x+\mu}^* \). The Wilson term however breaks the global chiral invariance but it has been shown ref. [7] that the violation of this global symmetry turns, in the scaling region, into the usual chiral anomaly. We investigate in the following how this manifests itself in the spectral flow of the fermionic hamiltonian when the gauge field is slowly changed from one vacuum configuration to a topologically distinct one. An earlier analysis of the spectral flow in vector QED$_2$ with Wilson fermions has been presented in ref. [8].

The path integral of the axial model is invariant under the unwanted global U(1) symmetry. There however the (local) chiral gauge invariance is broken by the Wilson term. It has been shown in ref. [8] that gauge invariance can be restored in the perturbative regime by adding a mass counterterm for the gauge field to the action, however a complete chirally gauge invariant lattice regulator is still lacking and is a fundamental problem of lattice gauge theory. As mentioned earlier most of the very promising proposals for a lattice formulation of chiral gauge theories have been shown to fail. Much work remains to be done in this direction, and our present study of the subtleties of anomalous fermion production seems to be a necessary step, which is also of interest in its own right.

3. INVESTIGATION OF THE SPECTRAL FLOW

In this section we study the spectrum of the fermionic hamiltonian in an external gauge field, which starts out as a vacuum gauge field, goes through a sphaleron like configuration and ends up as a different vacuum gauge field configuration that is related to the first one by a topologically non-trivial (‘large’) gauge transformation. We shall only consider gauge fields which vary very slowly in time so that we can make use of the adiabatic theorem and determine the spectrum of the hamiltonian by continuity. From the euclidean path integral one can derive the transfer operator \( T, Z = \text{Tr} T^N, \) where \( N \) is the number
of time slices in the lattice. The hamiltonian is defined in terms of \( T \) by \( H = \exp(-\tilde{H}) \). We shall consider in the following the simple gauge field \( A_{1z} = A(t) \), \( A_{0x} = 0 \) (i.e. \( U_0 = 1 \)). Then the time dependence of the hamiltonian is expressed as \( A \) dependence. The Chern-Simons number is given by \( C = -LA/2\pi \). With this choice of the gauge field configuration we can diagonalize the transfer operator explicitly and then determine from its eigenvalues the spectrum of the associated hamiltonian. We shall furthermore use anti-periodic boundary conditions for the fermions in the spatial momenta which implies that the spatial momenta are periodic boundary conditions for the fermions in space, and that fermions are excitations relative to the vacuum, which implies that the change in the axial charge is given by

\[ \Delta Q_5' \equiv (Q_{5,\Psi}' - Q_{5,0}')_{\text{final}} - (Q_{5,\Psi}' - Q_{5,0}')_{\text{initial}}. \]  

The axial charge of the ground state \( Q_{5,0}' \) changes by two units when changing \( C \) from 0 to \(-1\) since it looses an \( L \) state and gains an \( R \) state, i.e. \( (Q_{5,0}')_{\text{initial}} = 0 \), but \( (Q_{5,0}')_{\text{final}} = -2 \). The result is that \( \Delta Q_5' = -2 \) which is in concordance with eq. (13) since \( \Delta C = -1 \) in our case. Similarly it can be shown also that \( \Delta Q' = 0 \). Vector QED is gauge invariant and the vacua at \( A = 0 \) and \( A = 2\pi/L \) are related by a large gauge transformation. In Hilbert state, these gauge transformations induce unitary transformations on the hamiltonian which implies that its spectra coincide at \( A = 0 \) and \( A = 2\pi/L \). We also mention that there is a peculiarity about the lowest state.

3.1. Spectral flow in vector QED

In the vector model we derived for the energy spectrum \( \epsilon(p, A) \) of the hamiltonian the following expression

\[ \epsilon(p, A) = \epsilon_0(p - A) \]
\[ \cosh \epsilon_0(p) = \frac{1 + 4 \sin^2 \frac{p}{2}}{1 + 2 \sin^2 \frac{p}{2}}, \]

where \( \epsilon_0(p) \) are energies for free Wilson fermions. For momenta \( p \) and gauge fields \( A \) which are small compared to the cutoff \( \pi \) (11) reduces to the continuum expression \( \epsilon(p, A) = \pm(p - A) \) for linearly spaced modes \( p \). The physical vacuum (Dirac sea) is the state of lowest energy in which all levels with negative energy are occupied. The spectrum of the hamiltonian has been plotted in fig. 1 as a function of \( LA/2\pi = -C \) using a lattice of size \( L = 16 \). For clarity we have displayed only the states in and close to the surface of the Dirac sea. The \( L \) states move upwards whereas the \( R \) state move downwards. Midway between \( A = 0 \) and \( A = 2\pi/L \) the Dirac sea loses an \( L \) state and gains an \( R \) state (coming from the Dirac sky).

Consider now the flow of the physical state \( |\Psi, A\rangle \) which at \( A = 0 \) starts out as the vacuum state \( |0, A\rangle \). The occupied levels of that state are represented by the solid and dashed lines. Now assign the quantum numbers \( Q' \) and \( Q_5' \) to this state. The axial charge \( Q_5' \) is given by the number of \( R \) states minus the number of \( L \) states and the fermion number \( Q' \) is given by the sum of \( L \) and \( R \) states minus half the total number of states. From this it is clear that the charges \( Q_{5,\Psi}' \) and \( Q_{5}' \) of the physical state are both zero in the initial state at \( A = 0 \) and the final state at \( A = 2\pi/L \). The crucial observation however is that fermions are excitations relative to the vacuum, which implies that the change in the axial charge is given by

\[ \Delta Q_5' \equiv (Q_{5,\Psi}' - Q_{5,0}')_{\text{final}} - (Q_{5,\Psi}' - Q_{5,0}')_{\text{initial}}. \]  

The axial charge of the ground state \( Q_{5,0}' \) changes by two units when changing \( C \) from 0 to \(-1\) since it looses an \( L \) state and gains an \( R \) state, i.e. \( (Q_{5,0}')_{\text{initial}} = 0 \), but \( (Q_{5,0}')_{\text{final}} = -2 \). The result is that \( \Delta Q_5' = -2 \) which is in concordance with eq. (13) since \( \Delta C = -1 \) in our case. Similarly it can be shown also that \( \Delta Q' = 0 \). Vector QED

![Figure 2. Spectral flow for axial QED](image-url)
in the spectrum which corresponds to the largest momentum. It starts out as an \( L \) state (moving downwards) and ends up as an \( R \) state (moving upwards). This change of chirality appears to be possible because the Wilson term mixes \( L \) and \( R \) states. The chirality flip occurs however only for the largest momentum and hence has to be regarded as a lattice artefact.

3.2. Spectral flow in axial QED\(_2\)

The energy spectrum in the axial model is given by the formula

\[
\epsilon(p, A) = \ln \frac{1 + \sin A \cos p + 2 \sin^2 \frac{p}{2}}{1 - \sin A \cos p + 2 \sin^2 \frac{p}{2}} + \\
\cosh^{-1} \frac{1 + 4 \sin^2 \frac{p}{2} - \frac{1}{2} \sin^2 A}{2 \sqrt{(1 + 2 \sin^2 \frac{p}{2} - \sin^2 A(1 - 2 \sin^2 \frac{p}{2}))}}
\]

(14)

where we have dropped the gauge fields from the Wilson term in eq. (10). Both the Wilson terms (with and without \( U \) fields) break the (local) axial gauge invariance. Eq. (14) reduces for momenta \( p \) and gauge fields \( A \), which are small compared to the cutoff, to the continuum expression

\[
\epsilon(p, A) = \pm p + A
\]

for linearly spaced modes \( p \). The energies \( \epsilon(p, A) \) have been displayed in fig. 2, again plotted as a function of \( L A/2\pi \). Each state is doubly degenerate since it involves an \( R \) and an \( L \) state.

Consider now the flow of the state \(|\Psi, A\rangle\) which starts out at \( A = 0 \) as the vacuum state \(|0, A\rangle\). Like in the vector model, the state \(|\Psi, A\rangle\) does not change quantum numbers. i.e. \( (Q_\Psi)_{\text{initial}} = 0 \), \( (Q_\Psi)_{\text{final}} = 0 \). The vacuum however changes fermion number since it looses two states half way between \( A = 0 \) and \( A = 2\pi/L \) where an \( L \) state and an \( R \) state cross the surface of the Dirac sea. For the vacuum state we find \( (Q_0)_{\text{initial}} = 0 \), \( (Q_0)_{\text{final}} = -2 \) and hence

\[
\Delta Q \equiv (Q_\Psi - Q_0)_{\text{final}} - (Q_\Psi - Q_0)_{\text{initial}}.
\]

\[
= +2.
\]

(15)

which agrees with the continuum result (4). In contrast to the vector case the spectra at \( A = 0 \) and \( A = 2\pi/L \) are not identical which is due to lack of gauge invariance. For modes near the maximal negative or positive momenta we see clearly regularization effects, while the energy of modes near the surface of the Dirac sea return roughly to their original energies. Fig. 2 shows that the slope of the lines decreases when increasing \( |p| \) and finally even becomes negative. We should stress that such an effect has not been observed with staggered fermions showing that the regularization effects are smaller for staggered fermions.

Summary: We have shown based on the investigation of the spectral flow in the external field approximation, that anomalous fermion creation and annihilation can occur, even when fermion number is conserved. The crucial observation is that fermions are excitations relative to the vacuum. The global U(1) symmetry prohibits a state from changing its fermion number, however nothing prevents the ground state from doing so.

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