The classical Hochschild–Kostant–Rosenberg (HKR) theorem computes the Hochschild homology and cohomology of smooth commutative algebras. In this paper, we generalise this result to other kinds of algebraic structures. Our main insight is that producing HKR isomorphisms for other types of algebras is directly related to computing quasi-free resolutions in the category of left modules over an operad; we establish that an HKR-type result follows as soon as this resolution is diagonally pure.

As examples we obtain a permutative and a pre-Lie HKR theorem for smooth commutative and smooth brace algebras, respectively. We also prove an HKR theorem for operads obtained from a filtered distributive law, which recovers, in particular, all the aspects of the classical HKR theorem. Finally, we show that this property is Koszul dual to the operadic PBW property defined by V. Dotsenko and the second author.

**MSC 2020:** 18M70; 18N40, 13D03, 13N05.

1 Introduction

Hochschild homology is a classical homology theory for associative algebras[29] dating back to 1945. Originally conceived by Hochschild to obtain a cohomological proof of Wedderburn’s theorem [32], this cohomology theory plays nowadays important roles in representation theory [2], deformation theory [17, 20, 24], derived geometry [44], factorisation homology [23], and formality results [28], among others.

While Hochschild homology of an associative $k$-algebra $A$ is in general difficult to compute, in the case where $k$ is a field of characteristic zero and $A$ is commutative and smooth (for example if it is the coordinate ring of a smooth algebraic variety) the celebrated Hochschild–Kostant–Rosenberg (HKR) theorem [30] identifies the Hochschild homology of $A$ with its module $\Omega_A^*$ of algebraic differential forms, which is nothing but a free commutative algebra over the module $\Omega_A^1$ of Kähler differentials of $A$. In fact, this result is also used the other way around: it provides us with a way to generalise geometrical results, usually stated in terms of differential forms and fields on manifolds, to non-commutative or non-smooth algebras by replacing these geometrical objects.
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with Hochschild homology and cohomology. This philosophy falls under the general theory of non-commutative geometry; see [7, 9, 25, 43].

The HKR theorem depends on two hypothesis on the underlying algebra that are of very different flavours: while smoothness is a property that concerns certain geometric regularity of the algebra itself, and is thus intrinsic to the category of commutative algebras, the constraint that the associative algebra be commutative for one to obtain a description of its cohomology involves, perhaps in a more mysterious way, the interplay between the category of commutative algebras and the one of associative algebras.

Recently, V. Dotsenko and the second author have shown in [16] that one can produce, using the language of operads, what they consider the ‘bare-bones’ framework for Poincaré–Birkhoff–Witt (PBW) type theorems about universal enveloping algebras of types of algebras. There, they have shown that one can understand PBW-type results by way of studying a homological property between morphisms of operads: the universal enveloping algebra functor associated to a map of algebraic operads satisfies a PBW-type property if and only if it makes its codomain a free right module.

We pursue this philosophy here, by considering the question of the existence of an HKR-like theorem for operadic algebras. Since the ingredients we will need are slightly more involved than those in [16], let us first recall these.

Given an operad \( P \) and an algebra \( A \) over \( P \), we can consider its cotangent homology [41], which we will write \( H^P_*(A,M) \), and which corresponds to the Hochschild homology when \( P \) is the operad governing associative algebras, to Chevalley–Eilenberg homology when \( P \) is the operad governing Lie algebras, and to Harrison homology when \( P \) is the operad governing commutative algebras over a field of characteristic zero. Any map of algebraic operads \( f : P \longrightarrow Q \) induces a restriction functor \( f^* : Q-\text{Alg} \longrightarrow P-\text{Alg} \) and, in turn, a map

\[
H_*^Q(A) \longrightarrow H_*^P(A),
\]

and an HKR theorem can be seen as a way to promote this map to an isomorphism, by applying an appropriate functor to the codomain; the resulting object in the codomain deserves to be thought as “differential forms” on \( A \). With this in mind, our first step towards obtaining an HKR-type formalism is the following result. It says that promoting the map \( H_*(f) \) to a possible candidate for an HKR isomorphism can be done as soon as one produces a quasi-free resolution of \( Q \) in left \( P \)-modules. This resolution will have the form \( (P \circ Y, d) \) for some graded symmetric sequence \( Y \) of generators, and these will play the central role of the “functor of differential forms for \( f \)”. We will see it is convenient to phrase our result in terms of the complexes \( \text{Def}_Q^\infty(A) \) and \( \text{Def}_P^\infty(A) \) computing the homology groups above.

**Theorem.** Let \( \mathcal{F} = (P \circ Y, d) \) be a quasi-free resolution of \( Q \) in left \( P \)-modules. Then there exists a functorial complex \( \Omega_{\mathcal{F},A}^* \) of ‘differential forms’ on \( A \) associated to \( f \), depending on \( \mathcal{F} \) and \( \text{Def}_P^\infty(A) \), and a morphism of complexes

\[
\text{HKR}_{\mathcal{F},A} : \text{Def}_P^\infty(A) \longrightarrow \Omega_{\mathcal{F},A}^*. 
\]
There is a well developed theory of Kähler differentials $\Omega^1_A$ for operadic algebras [39], that is, algebraic 1-forms, which we use to construct the module of differential forms $\Omega^*_F,A$. The construction $\Omega^1_A \mapsto \Omega^*_F,A$ is not intrinsic to $P$-algebras but depends on the homotopical properties of the morphism $P \mapsto Q$ and, up to quasi-isomorphism, on a choice of resolution $\mathcal{F}$ of $Q$, as we explain in Section 3.4. With this construction at hand, we consider the notion of smoothness in Section 3.2 as a generalisation of one of the equivalent notions of smoothness in the commutative case, which we recall from the excellent monograph [36]. This allows us to make the following definition, central to our paper:

**Definition.** The map $f : P \mapsto Q$ has the Hochschild–Kostant–Rosenberg property if for every smooth $Q$-algebra $A$, the map $\text{HKR}_{\mathcal{F},A}$ is a quasi-isomorphism.

The main result of this paper is that the PBW property is, in the sense made precise below, Koszul dual to the HKR property, as we record in Corollary 3.23. We cannot avoid to note this follows the ‘mantra’ pursued by B. Ward in [45], that it is desirable to consider Koszul duality not as an aspect of categories separately, but rather as a construction which intertwines functors between them.

**Theorem.** Let $f : P \mapsto Q$ be a map of Koszul operads. Then $f$ has the HKR property if the morphism of Koszul dual operads $Q' \mapsto P'$ enjoys the PBW property.

This gives us a short and conceptual proof of the classical HKR theorem: the maps $\text{Ass} \mapsto \text{Com}$ and $\text{Lie} \mapsto \text{Ass}$ are Koszul dual, so that the classical PBW theorem implies, in this way, the HKR theorem. With generous hindsight, this comes as no surprise: apart from using standard techniques of localisation to reduce the proof of the HKR theorem to smooth local commutative algebras, a straightforward way to prove that the HKR theorem holds is by use of the Koszul complex of the symmetric algebra $S(V)$ and its Koszul dual coalgebra $S^c(V[-1])$.

Having settled the above, we then observe there are several examples of maps of operads satisfying the HKR property and, in Section 4.2, we explore some of them. Of particular interest is the map $\text{Perm} \mapsto \text{Com}$ which factors the projection of the associative operad onto the commutative operad by passing through permutative algebras [5]. The Koszul dual map to the projection $\text{Perm} \mapsto \text{Com}$ is the inclusion $\text{Lie} \mapsto \text{PreLie}$, which is known to enjoy the PBW property by [16]. In Corollary 4.3 we conclude that the following HKR-type theorem holds, providing us with the computation of the cotangent homology of a smooth commutative algebra $A$ seen as a permutative algebra.

**Theorem.** The permutative cotangent homology of a smooth commutative algebra $A$ is given by a module $RT_{\mathcal{F}1}(\Omega^*_A)$ which is spanned by rooted trees whose vertices are labeled by elements of the classical space of forms $\Omega^*_A$ and no vertex has exactly one child.

Finally, we offer a technique to compute the tangent cohomology of a $\mathcal{P}$ algebra coming from a smooth $Q$-algebra under the projection $f : \mathcal{P} \mapsto Q$ in case $\mathcal{P}$ is obtained from a filtered distributive law [13] between $Q$ and $R$ as originally defined by V. Dotsenko in [10]. The shining example of this phenomenon is the way in which the operad $\text{Ass}$ is obtained from $\text{Com}$ and $\text{Lie}$; in this way, the reader may think of the following filtered HKR theorem as another ‘ultimate’ generalization to algebraic operads of the classical HKR theorem for the map $\text{Ass} \mapsto \text{Com}$. Indeed, in this case, the functor $\mathcal{R}'$ below is precisely $V \mapsto S^c(V[-1])[1]$. 

**Theorem.** The filtered permutative cohomology of a smooth commutative algebra $A$ is given by a module $FR\mathcal{F}1(\Omega^*_A)$ which is spanned by rooted trees whose vertices are labeled by elements of the classical space of forms $\Omega^*_A$ and no vertex has exactly one child.
Theorem. Suppose $\mathcal{P}$ is obtained from Koszul operads $\mathcal{Q}$ and $\mathcal{R}$ by a filtered distributive law, so that $\mathcal{P}$ is isomorphic to $\mathcal{Q} \circ \mathcal{R}$ as a right $\mathcal{R}$-module. Then for every smooth $\mathcal{Q}$-algebra $A$ the cotangent homology of $f^*A$ is given by the endofunctor

$$A \longrightarrow \mathcal{R}^i(\Omega^1_A)$$

of "$\mathcal{R}^i$-enriched differential forms" on $A$.

Dually to the result of homology, we were able to obtain a result for tangent cohomology. In this case, a choice of quasi-free resolution $\mathcal{F}$ gives us a functor of "poly-vector fields" $A \longrightarrow \text{Poly}^*(A)$, and we obtain the following:

Theorem. If $f$ satisfies the HKR property then for every smooth $\mathcal{Q}$-algebra $A$ there is a quasi-isomorphism of complexes:

$$\text{HKR}^i : \text{Def}^i(f^*A) \longrightarrow \text{Poly}^*(A).$$

In case of cohomology, our result on filtered distributive laws says that the tangent cohomology of $f^*A$ is given by the endofunctor $\mathcal{R}^i(\text{Der}(A)[1])[−1]$. In the classical case, we recover the Lie structure on tangent homology, since $\text{Der}(A)$ is a Lie algebra. Indeed, since $\mathcal{R}^i = \text{Com}$, the usual distributive law allows us to give $\mathcal{R}^i(\text{Der}(A)[1])[−1]$ a Lie algebra structure isomorphic to the one on $\mathcal{H}_{A_{\text{Ass}}}^*(A)$. It is unclear, however, how one could attempt to obtain the Lie algebra structure on tangent cohomology in a more general situation.

Structure. The paper is organised as follows. In Section 2 we recall the usual HKR theorem for smooth commutative algebras, in a way that suits our operadic approach that follows, and hoping that it will be useful for the reader to incorporate the new formalism that we then develop in Section 3. Here, we recall the notions of (co)tangent (co)homology and introduce the relevant notions of smoothness and the “full” module of differential forms. With this at hand, we introduce the HKR property and prove our main theorem. In Section 4.2, where we focus on applications, we show how to recover the classical HKR theorem from our main result and apply it to obtain new examples: we obtain a “permutative” HKR theorem for smooth commutative algebras and a “pre-Lie” HKR theorem for smooth braces algebras. In the process of drawing some connections of our work to that of J. Griffin [26], we obtain an HKR theorem for operads obtained from filtered distributive laws, and briefly outline how it recovers the HKR isomorphism at the level of Lie algebras. Finally, with the purpose of making this paper better self-contained, we collect some useful results in an Appendix about algebraic operads, their algebras and their Kähler differentials, hoping it will be of use for a reader with some background in algebraic operads.

Notation and conventions. For references on operads and their modules we point the reader to [18, 39], and to [4, 36, 46] for homological algebra. We allow operads to be homologically graded, but will make it clear when we require operads to be dg. We assume that algebras over operads are non-dg, and we fix a closed symmetric monoidal category $C$ like Vect over which our algebras are defined; we always work over a field of zero characteristic. Most arguments we make actually hold for dg algebras, taking into the account the given bigrading of the resulting
objects. For simplicity, we work with non-dg algebras. We write \# for the forgetful functor from algebras to \( \mathcal{C} \). If \( V \) is a chain complex and \( p \in \mathbb{Z} \), we write \( V[p] \) for the chain complex for which \( V[n]_p = V_{n-p} \) for each \( n \in \mathbb{Z} \), and whose differential changes sign according to the parity of \( p \). Accordingly, if \( \Omega \) is an operad, we write \( \Omega\{p\} \) for the operad uniquely defined by the condition that a \( \Omega\{p\} \)-algebra structure on \( V \) is the same as a \( \Omega \)-algebra structure on \( V[p] \).

Throughout, for two quadratic operads given by quadratic data \((V, R)\) and \((V', R')\), we will only consider maps of operads induced by a map of quadratic data \( V \rightarrow V' \) such that the induced map \( T(V) \rightarrow T(V') \) sends \( (R) \) to \( (R') \). We remind the reader that the data \((V, R)\) may contain non-binary generators and that, in this case, the weight and arity gradings in \( T(V) \) may not coincide; see [39, §7.1.3]. Moreover, we confine ourselves to the category of weight graded operads and their weight graded algebras and modules. We distinguish the weight degree from the homological degree by using parentheses. Hence, while \( X_3 \) denotes a component of homological degree 3, we write \( X(3) \) for a component of weight degree 3.

**Acknowledgements.** We kindly thank B. Keller for explaining to us the very short proof of Lemma 3.6 which we reproduced here. We also thank V. Dotsenko, J. Bellier-Millès, N. Combe and J. Nuiten for useful conversations, comments and suggestions.

## 2 The case of commutative algebras

This section serves to recall the objects and results related to the classical Hochschild–Kostant–Rosenberg theorem for commutative algebras. Such objects will be presented in the way that we find best suited for the operadic generalisation and the main results appearing in Section 3. For a classical approach we recommend both Chapter 3 and Appendix E of [36].

### 2.1 The classical HKR morphism

Throughout, fix a non-unital commutative algebra \( A \), and let us recall how to construct a natural map that relates the homology of \( A \) as a commutative algebra, its Harrison homology, and the homology of \( A \) as an associative algebra, its Hochschild homology, through a particular functor. This is the well known *Hochschild–Kostant–Rosenberg map*

\[
\text{HKR}_A : C^\bullet (A, A) \rightarrow \Omega^A_1
\]

where the left hand side is the cyclic Hochschild complex of \( A \) considered as an associative algebra and \( \Omega^A_1 \) is the space of *differential forms* on \( A \). We now recall the details necessary to construct this map.

For any commutative algebra \( A \), the module of *Kähler differentials* \( \Omega^1_A \) of \( A \) is the symmetric \( A \)-bimodule representing the functor of derivations

\[
M \rightarrow \text{Der}(A, M).
\]
Recall that we have a natural isomorphism of symmetric \( A \)-bimodules
\[
\frac{I}{I^2} \longrightarrow \Omega^1_A,
\]
where \( I = \ker(\mu : A \otimes A \longrightarrow A) \)
such that \( 1 \otimes x - x \otimes 1 + I^2 \longrightarrow dx \).

**Definition 2.1** Let be \( J \) the kernel of the multiplication of a cofibrant replacement \( QA \) of \( A \). The \textit{cotangent complex} of \( A \) with coefficients in a symmetric \( A \)-bimodule \( M \) is by definition
\[
\text{Def}^*(-(A,M)) = \frac{J}{J^2} \otimes_{QA} M.
\]

The cotangent homology of \( A \) with coefficients in \( M \) is, by definition, the homology of this complex, and we write it \( \mathcal{H}_*(A,M) \).

In other words, this is the non-abelian derived functor of \( M \longrightarrow \Omega^1_A \otimes_A M \). Dually, we have a \textit{tangent complex} of \( A \) with values in \( M \)
\[
\text{Def}^*(A,M) = \text{hom}_A\left( \frac{J}{J^2}, M \right)
\]
and the tangent cohomology of \( A \) with values in \( M \) is, by definition, the homology of this complex, and we write it \( \mathcal{H}^*(A,M) \).

**Definition 2.2** We say \( A \) is a \textit{smooth commutative algebra} if for every \( A \)-module \( M \),
\[
\mathcal{H}^1(A,M) = 0.
\]

For our convenience and that of the reader, we record now some equivalent definitions of smoothness, which in particular show that the cotangent homology \( \mathcal{H}_*(A,A) \) of \( A \) is very simple in case it is smooth: it is concentrated in degree zero where it equals the module of Kähler differentials \( \Omega^1_A \).

We remind the reader from the Appendix that one can also consider relative versions of the homology and cohomology theories above for a morphism of algebras. In particular, since \( A \) is a commutative algebra, we can consider the (co)homology of \( A \) relative to \( A \otimes A \) through the multipication map.

**Proposition 2.3** Let \( A \) be a finitely generated commutative algebra over a field of characteristic zero, and let \( B = A \otimes A \). Then the following conditions are equivalent:

1. \( \mathcal{H}^1(A,M) = 0 \) for any symmetric \( A \)-bimodule \( M \),
2. \( \mathcal{H}^1(A,A) = 0 \) and \( \Omega^1_A \) is a projective \( A \)-module,
3. \( \mathcal{H}^2(A|B,N) = 0 \) for any \( A \)-module \( N \),
4. \( \mathcal{H}^2(A|B,A) = 0 \) and \( \Omega^1_A \) is a projective \( A \)-module.

**Proof.** See [36, Appendix E].

As we noted, if \( A \) is a smooth commutative algebra, the fact \( \Omega^1_A \) is projective, implies that \( \mathcal{H}_*(A,A) \) is concentrated in degree zero and
\[
\mathcal{H}_0(A,A) = \Omega^1_A.
\]
This receives a map $A \to \Omega^1_A$, the universal derivation, and we can then form the non-unital symmetric algebra $S_A(\Omega^1_A[-1])$ of $\Omega^1_A$ under $A$. We call this the space of differential forms on $A$ and write it $\Omega^*_A$. Finally, let us recall that the cyclic Hochschild complex $C_*(A,A)$ of $A$ is given for each $n \in \mathbb{N}$ by $C_n(A,A) = A \otimes \mathbb{A}^\otimes n$ and we write a generic element in here by $a[a_1|\cdots|a_n]$. There is a map of complexes

$$HKR_A : C_*(A,A) \to \Omega^*_A$$

such that $a[a_1|\cdots|a_n] \mapsto ada_1 \cdots da_n$. It will be useful to note that this map is the identity of $A$ in degree 0, and in fact split as a map of complexes as a sum of this map and the remaining part

$$C_*(A,A)^+ \to \overline{S}_A(\Omega^1_A).$$

where the right hand side uses the non-unital symmetric algebra functor under $A$. Over a field of characteristic zero, the HKR map is a split injection. The Hochschild–Kostant–Rosenberg theorem \cite{30} asserts the following stronger conclusion in case $A$ is smooth:

\begin{theorem}
For every smooth commutative algebra $A$ of finite type over a field $k$ the morphism

$$HKR_A : C_*(A,A) \to \Omega^*_A$$

is a quasi-isomorphism. \hfill \Box
\end{theorem}

\section{The HKR isomorphism}

There are many proofs of Theorem 2.4 in the literature. Because it serves to illustrate the general formalism that we will develop later, let us give a non-standard proof of this theorem: it will follow once we show that (in the dg setting), the HKR map is a quasi-isomorphism for cofibrant algebras that resolve smooth algebras.

To see why this is enough, observe that HKR$_A$ is natural, in the sense that given a map of algebras $f : B \to A$ we have that $\mathcal{H}_*(f) \circ HKR_B = HKR_A \circ \Omega^*_f$ or, what is the same, there is a commutative diagram

$$
\begin{array}{ccc}
C_*(B,B) & \longrightarrow & C_*(A,A) \\
\downarrow & & \downarrow \\
\Omega^*_B & \longrightarrow & \Omega^*_A.
\end{array}
$$

Since the functor $\mathcal{H}_*$, by its very definition, preserves quasi-isomorphisms, and since the HKR map is a quasi-isomorphism for cofibrant algebras that resolve smooth algebras,

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$$
\begin{array}{ccc}
C_*(B,B) & \longrightarrow & C_*(A,A) \\
\downarrow & & \downarrow \\
\Omega^*_B & \longrightarrow & \Omega^*_A.
\end{array}
$$

Since the functor $\mathcal{H}_*$, by its very definition, preserves quasi-isomorphisms, and since the HKR map is a quasi-isomorphism for cofibrant algebras, we deduce the following interesting lemma. Its content is central to develop our operadic formalism later.

\begin{lemma}
The map HKR$_A$ is a quasi-isomorphism if and only if the functor of differential forms $A \to \Omega^*_A$ preserves quasi-isomorphisms $Q \to A$ for $Q$ a cofibrant resolution of an arbitrary smooth algebra $A$. \hfill \Box
\end{lemma}

To proceed with the proof, let us first recall that we can express $C_*(A,A)$ as a twisted tensor product $A \otimes BA$ where $BA = (T^*(\wedge A), \delta)$ is the bar construction of $A$, arising from the fact the associative
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An operad is Koszul self-dual. Similarly, there is a commutative-Lie bar-cobar adjunction arising from Koszul duality between the category of conilpotent Lie coalgebras and the category of commutative algebras

$$\mathcal{C} : \text{Lie-Cog} \rightleftarrows \text{Com-Alg} : \mathcal{L}.$$ 

The only properties of this adjunction that we need are the following:

- the counit of the adjunction $\mathcal{C} \mathcal{L} \to \text{id}$ is a quasi-isomorphism and,
- the commutative cobar construction of a Lie coalgebra is given by the quasi-free commutative algebra $\mathcal{C}(g) = (S(g[1]), \delta)$, where $\delta$ is a differential extending the cobracket of $g$.

These two adjunctions interact in the following way: the restriction functor from conilpotent associative coalgebras to conilpotent Lie coalgebras $\text{Ass-Cog} \to \text{Lie-Cog}$ has a left adjoint $U_c$, the universal enveloping coalgebra, which satisfies $B \pi^* = U_c \mathcal{L}$, where $\pi^*$ is the forgetful functor from commutative algebras to associative algebras, see Lemma 3.25.

**Lemma 2.6** The HKR map is a quasi-isomorphism for any free commutative dga algebra.

**Proof.** A derivation on a free commutative algebra $A = S(V)$ is uniquely determined by specifying the values on generators, from where it follows that $\Omega^1_A = A \otimes V$. In this way, we obtain an identification

$$\Omega^*_A = A \otimes S(V[-1]).$$

At the same time, $A = C(V[-1])$ is the commutative cobar construction of the abelian Lie coalgebra $g = V[-1]$. It follows that as chain complexes

$$C_*(A, A) = A \otimes \mathcal{C}(V[-1])$$

$$= A \otimes \mathcal{U}^c \mathcal{L}(V[-1])$$

$$\simeq A \otimes \mathcal{U}^c(V[-1])$$

$$= A \otimes S(V[-1]).$$

On the third line we used that $\mathcal{U}^c$ preserves quasi-isomorphisms, which is a consequence of the PBW theorem [40]. The resulting anti-symmetrization quasi-isomorphism $\varphi_A : \Omega^*_A \longrightarrow C_*(A, A)$ gives us an inverse to HKR$_A$ when taking homology, which proves our claim.

A cobar algebra is any commutative algebra $\mathcal{C}(g)$ obtained via a cobar construction of a (shifted) Lie coalgebra $g$. Recall that cobar algebras are triangulated and hence cofibrant and that every algebra admits a cofibrant replacement given by a cobar algebra, as explained in Corollary 11.3.5 and Proposition B.6.6 of [39].

**Proposition 2.7** The map HKR$_A$ is a quasi-isomorphism for any commutative cobar algebra.

**Proof.** The differential of the cobar construction $A = \mathcal{C}(g)$ splits as $d_g + \delta$. As in the previous lemma, we have that

$$\Omega^1_A = (A \otimes g[-1], d_A \otimes 1 + 1 \otimes d_g + \overline{\delta}),$$
where we notice that the differential has an external component induced by $\delta$, such that if we write $\delta x = x_{(1)} \otimes x_{(2)}$ in Sweedler notation, then $\overline{\delta}(a \otimes x) = ax_{(1)} \otimes x_{(2)}$. On the other hand, the same computation as in the previous lemma shows that there is a morphism of $A$-modules,

$$\varphi_A : \Omega^* A \longrightarrow C_*(A,A).$$

The result now follows from taking the spectral sequence associated to the PBW filtration: the associated morphism to $\varphi_A$ is equal, in homology, to the desired inverse of the HKR map corresponding to the commutative algebra $S(g^e)$ where $g^e$ is the Lie algebra $g$ with zero bracket.

**Theorem 2.8** The HKR map is a quasi-isomorphism for any smooth commutative algebra.

**Proof.** Let $A$ be a smooth commutative algebra. Since $A$ is smooth it follows in particular that $\mathcal{H}_1(A,M) = 0$ for every $A$-module $M$. Setting $A = M$ we see that $\mathcal{H}_*(A,A) = \Omega^1_A$, from where it follows that for any cofibrant replacement $p : Q \longrightarrow A$

$$\Omega^1_p : A \otimes_Q \Omega^1_Q \longrightarrow \Omega^1_A$$

is a quasi-isomorphism: the left hand side computes $\mathcal{H}_*(A,A)$ and the induced map is then an isomorphism. Because the canonical map

$$\Omega^1_Q = Q \otimes_Q \Omega^1_Q \longrightarrow A \otimes_Q \Omega^1_Q$$

is a quasi-isomorphism, it follows that $\Omega^1_Q \longrightarrow \Omega^1_A$ is a quasi-isomorphism. In view of Lemma 2.5, we see that the map $\text{HKR}_A$ is a quasi-isomorphism.

Before moving on, we would like to highlight the following three points that will be revisited when we develop the general operadic formalism for HKR theorems:

1. For $q : Q \longrightarrow A$ a cofibrant resolution of $A$ a smooth algebra the map $q : \Omega^1_Q \longrightarrow \Omega^1_A$ is a quasi-isomorphism. This is intrinsic to the category of commutative algebras and thus independent of the map of operads $\text{Ass} \longrightarrow \text{Com}$.

2. Showing that for any cofibrant algebra $Q$ the map $\text{Def}_*(f^* Q, f^* Q) \longrightarrow \Omega^*_Q = S^*_c(\text{Def}_*(Q))$ is a quasi-isomorphism is independent of smoothness, and depends on the map of operads $f : \text{Ass} \longrightarrow \text{Com}$. In particular, we can take “affine” algebras as test algebras in this step, which we called “cobar algebras” above.

3. The way the two previous points are put together is by noting that, since the functor $S^c$ preserves quasi-isomorphisms, we have that $\Omega^*_Q : \Omega^*_Q \longrightarrow \Omega^*_A$ is a quasi-isomorphism. Here it is the only step where we use the universal enveloping algebra functor associated to $\text{Lie} \longrightarrow \text{Ass}$ preserves quasi-isomorphisms, by the classical PBW theorem.
3 The HKR theorem for general operadic algebras

In this section we generalise the classical notions from the previous section to algebras over operads, and prove Theorem 3.22, generalising the classical Theorem 2.4 to smooth algebras for morphisms of operads satisfying a natural homological condition.

Some conventions for this section.

- We fix once and for all a morphism of non-dg Koszul operads $\mathcal{P} \rightarrow \mathcal{Q}$ which we assume comes from a map of quadratic data. All (co)operads will be homologically graded with zero differential and by $A$ we will always denote a $\mathcal{Q}$-algebra.
- For any operad $\mathcal{O}$, we denote by $\mathcal{Q}: \mathcal{O}-\text{Alg} \rightarrow \mathcal{O}-\text{Alg}$ a fixed choice of cofibrant replacement functor for $\mathcal{O}$-algebras.
- When context allows, we will usually simply write $\mathcal{Q}$ for a cofibrant replacement of some algebra $A'$ which will be clear from context.
- In particular, we will sometimes need to use the composition $U\mathcal{Q}$ where $U = U_\mathcal{O}: \mathcal{O}-\text{Alg} \rightarrow \text{Ass}-\text{Alg}$ is the associative universal envelope functor, in which case we will usually to write this $U\mathcal{Q}$ when the algebra we applied it to is clear from context.

3.1 Deformation complexes and (co)homology

In this section we introduce the formalism of (co)tangent homology and cohomology for algebras over an operad. We refer the reader to the article [41] of J. Millès for a thorough and comprehensive study of this theory, and point to the Appendix, where some useful recollections on algebra over operads and their operadic modules is given. The reader can consult Appendix A.2 for details on derivations and associative universal envelopes of algebras over operads.

Definition 3.1 Let $A$ be a $\mathcal{Q}$-algebra. We define the tangent complex of $A$ with values in an operadic $A$-module $M$ by

$$\text{Def}^*(A,M) = \text{Der}(QA,M) = \text{hom}_{U\mathcal{Q}A}(\Omega^1_{QA},N).$$

Note that we dropped the subscript $\mathcal{Q}$, which will be clear from context. Observe that this is defined up to natural quasi-isomorphism, and is well-defined in the derived category of complexes.

The morphism $f: \mathcal{P} \rightarrow \mathcal{Q}$ induces a restriction functor $f^*: \mathcal{Q}-\text{Alg} \rightarrow \mathcal{P}-\text{Alg}$ that assigns $A$ to the $\mathcal{P}$-algebra $f^*A$ with the same underlying object as $A$ along with the $\mathcal{P}$-algebra structure given by the composition $\mathcal{P} \rightarrow \mathcal{Q} \rightarrow \text{End}_A$. Observe that we can take $\mathcal{Q}(f^*A)$ as a cofibrant replacement $Q(f^*A) \rightarrow f^*(QA)$ of the $\mathcal{P}$-algebra $f^*(QA)$. In this way, we obtain a natural map

$$\text{Def}^*(f,M) : \text{Def}^*(A,M) \rightarrow \text{Def}^*(f^*A,f^*M)$$

for every operadic $A$-module $M$.

Definition 3.2 The cohomology of $\text{Def}^*(A,M)$ is, by definition, the tangent cohomology of $A$ with values in $M$, and we will write it $\mathcal{H}^*_\mathcal{Q}(A,M)$. 

Remark 3.3 When $A$ is an associative algebra, $\mathcal{H}^s(A, M)$ differs from the classical Hochschild cohomology groups $\operatorname{HH}^s(A, N)$ of $A$ only in that we do not quotient out by inner derivations in degree zero and we discard the 0th classical Hochschild cohomology group of $A$ with values in $N$ so that

$$\mathcal{H}^s(A, M) = \begin{cases} 
\operatorname{HH}^{s+1}(A, M) & \text{if } s \geq 1, \\
\operatorname{Der}(A, M) & \text{if } s = 0.
\end{cases}$$

In a similar fashion to tangent cohomology, we define the cotangent complex.

Definition 3.4 The cotangent homology of $A$ with coefficients in $M$ through the cotangent complex of $A$ which is obtained as

$$\operatorname{Def}_s(A, M) = \Omega^1_{QA} \otimes_{UQA} M$$

and write it $\mathcal{H}^Q_s(A, M)$.

Note that from Proposition A.4, in case we take $f : \text{Ass} \rightarrow \text{Com}$, $A$ a commutative algebra and choose $\Omega^*_B A$ for the cofibrant resolution for the associative algebra $A$, we have that

$$\operatorname{Def}_s(f^* A, f^* A) = s^{-1} C_s(A, A)^+.$$

It is useful to note there are universal coefficients for these homology theories [18]. Indeed, writing the functors as compositions, where we write $U$ for the associative enveloping algebra $UA$ to lighten the notation

$$\operatorname{Def}^s(A, M) = \operatorname{hom}_U(\operatorname{Def}_s(A, U), M),$$

$$\operatorname{Def}_s(A, M) = \operatorname{Def}_s(A, U) \otimes_U M$$

we obtain two universal coefficient spectral sequences,

$$E^2_{s, t} = \operatorname{Tor}_s^U(-, \mathcal{H}^Q_s(A, U)) \Rightarrow \mathcal{H}^Q_{s+t}(X, -),$$

$$E^2_{s, t} = \operatorname{Ext}_U^s(\mathcal{H}^Q_t(X, U), -) \Rightarrow \mathcal{H}^{s+t}_Q(A, -),$$

that explain the relation between (co)homology theories given by Ext and Tor functors and the operadic theories. For example, in case we do this for the associative operad, we observe that

$$\mathcal{H}^{Ass}_t(A, UA) = 0 \quad \text{for } t \geq 1,$$

which shows Hochschild (co)homology of associative algebras is given by Tor and Ext functors. When $A$ is commutative, the cotangent homology groups, usually known as the André–Quillen homology groups $\mathcal{H}^\text{Com}_s(A, UA) = \mathcal{H}^\text{Com}_s(A, A)$ are in general non-zero in higher degrees, so there are obstructions to this comparison.

3.2 Smooth algebras

In analogy with the characterization of smoothness for commutative algebras of Theorem 2.3, we introduce the following definition.
Definition 3.5 The $\mathcal{Q}$-algebra $A$ is smooth if for every operadic $A$-module $M$, 

$$\mathcal{H}^1_\mathcal{Q}(A,M) = 0.$$ 

or, what is the same, if $\mathcal{H}^0_\mathcal{Q}(A,-)$ is an exact functor in operadic $A$-modules.

Although our focus lies on non-dg algebras, it is useful to remark that the definition is in general not invariant under quasi-isomorphisms. Indeed, suppose that $q : A \longrightarrow A'$ is a quasi-isomorphism of dg $\mathcal{Q}$-algebras and let us assume first that $A'$ is smooth, and let $M$ be an operadic $A$-module. There is a map 

$$\mathcal{H}^1_\mathcal{Q}(A,M) \longrightarrow \mathcal{H}^1_\mathcal{Q}(A',\psi_!(M)) = 0$$ 

but, unless we assume that $q_!$ is well-behaved (that is, flatness assumptions on $q$), there is no reason to expect this to be an isomorphism. However, if $A$ is smooth then any cofibrant replacement of $A$ is one of $A'$, so that in this case $A$ smooth implies $A'$ smooth. It is immediate that every free (i.e. affine) $\mathcal{Q}$-algebra is smooth.

Let us now consider a related condition: we say that $A$ is quasi-smooth if for some —and hence, every— cofibrant replacement $p : QA \longrightarrow A$, the induced map on Kahler differential forms 

$$\Omega^1_p : p_!\Omega^1_QA \longrightarrow \Omega^1_A$$ 

is a quasi-isomorphism of operadic $A$-modules. Before relating the notions of smoothness and quasi-smoothness, we record the following lemma:

Lemma 3.6 Let $q : X \longrightarrow Y$ be a map of complexes of operadic $A$-modules, and suppose that for every operadic $A$-module $M$ the map 

$$q^* : \text{hom}_A(Y,M) \longrightarrow \text{hom}_A(X,M)$$ 

is a quasi-isomorphism. Then $q$ is a quasi-isomorphism.

Proof. Let us take $J$ an injective cogenerator of the category of left $UA$-modules. Then the $p$th cohomology group of $\text{hom}_A(X,J)$ identifies with 

$$F_p(X) := \text{hom}_A(H_p(X),J)$$ 

because $J$ is injective. Since $J$ is also a cogenerator, the collection of functors $\{F_p\}_{p \in \mathbb{Z}}$ detects quasi-isomorphisms, which gives what we wanted. 

Remark. Observe that the reverse implications is not true. Indeed, let us consider the commutative algebra $A = k[x]$, the trivial $A$-module $M = Y = k$ and the complex $X : A \longrightarrow A$ where the differential is given by multiplication by $x$. Then the quotient map $q : X \longrightarrow Y$ is a quasi-isomorphism, but the induced map $\text{hom}_A(Y,M) \longrightarrow \text{hom}_A(X,M)$ is not: the right hand side computes $\text{Ext}^1_A(k,M)$, and this may not always be concentrated in degree 0.
With this lemma at hand, we can prove the following proposition. It is interesting to compare it with Corollary 7.3.5 in [27]. While we make a statement about the behaviour of the induced morphism
\[ \Omega^1_p : p_! \Omega^1_B \longrightarrow \Omega^1_A \]
when \( f \) is an acyclic fibration onto a smooth algebra, that corollary makes a statement about the behaviour of that map when \( f \) is an acyclic cofibration; in both cases the conclusion is that the map induced is a quasi-isomorphism.

**Proposition 3.7** Every smooth \( \Omega \)-algebra is quasi-smooth.

**Proof.** Suppose that \( A \) is smooth, and let \( Q \longrightarrow A \) be a cofibrant replacement, let us show that the map \( p_! \Omega^1_Q \longrightarrow \Omega^1_A \) is a quasi-isomorphism of operadic \( A \)-modules. By the previous two lemmas, it suffices to show that for every operadic \( A \)-module \( M \), the induced map
\[ p^* : \text{hom}_A(\Omega^1_A, M) \longrightarrow \text{hom}_A(p_! \Omega^1_Q, M) \]
is a quasi-isomorphism. By adjunction, the codomain is naturally isomorphic to
\[ \text{hom}_{UQ}(\Omega^1_Q, p^* M) \]
so we obtain \( p^* \) identifies naturally with the map \( \text{hom}_A(\Omega^1_A, M) \longrightarrow \text{hom}_Q(\Omega^1_Q, M) \) representing the pullback along \( p \)
\[ p^* : \text{Der}(A, M) \longrightarrow \text{Der}(Q, p^* M). \]
Since \( A \) is smooth and the right hand side computes \( \mathcal{H}^* \) of \( A \), it follows that this map is a quasi-isomorphism: it induces the identity of \( \mathcal{H}^0(A, M) = \text{Der}(A, M) \). We conclude that \( \Omega^1_p \) is a quasi-isomorphism, which means that \( A \) is quasi-smooth, as we wanted. \( \square \)

**Remark 3.8** It is important to observe that the notion of quasi-smoothness may be quite weak. For example, every unital associative algebra is quasi-smooth, owing to the fact that the module of associative Kähler preserves quasi-isomorphisms. Indeed, in this case this functor fits into an exact sequence
\[ 0 \longrightarrow \Omega^1_A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0 \]
and the second and third functor clearly preserve quasi-isomorphisms.

### 3.3 Left Koszul morphisms

In this section we collect some facts about (left) Koszul morphisms between weight graded operads, which we introduce. We refer the reader to the excellent monograph [42, Section 2.5] for the case of algebras. We say a symmetric sequence \( X \) is **diagonally pure** if for each \( p \in \mathbb{N} \) the component \( X_p \) of homological degree \( p \) is concentrated in weight \( p \). With this at hand, let us introduce the kind of left dg \( \mathcal{P} \)-modules that interest us.
Lemma 3.11 Let $f : \mathcal{P} \to \mathcal{Q}$ be a map of left $\mathcal{P}$-modules and suppose that $\mathcal{F} = (\mathcal{P} \circ \mathcal{Y}, d)$ is a quasi-free complex mapping onto $\mathcal{Q}$ and that $\mathcal{Q}$ is a resolution of $\mathcal{N}$. Then there exists a map of left $\mathcal{P}$-modules $\mathcal{F} \to \mathcal{Q}$ extending $f$, and any two such choices are homotopic.

It is easy to see that any diagonally pure quasi-free resolution $(\mathcal{P} \circ \mathcal{Y}, d)$ is minimal. Indeed, the resulting complex is of the form $(\mathcal{Y}, d)$. Since $d$ preserves the weight degree but lowers the homological degree, the differential $\tilde{d}$ does too and, since $\mathcal{Y}$ is diagonally pure, $\tilde{d}$ vanishes.

Definition 3.10 We define $\text{Tor}^\mathcal{P}(k, \mathcal{Q})$ as the homology of the dg module $k \circ_\mathcal{P} \mathcal{F}$ where $\mathcal{F}$ is any quasi-free resolution of $\mathcal{Q}$ in left $\mathcal{P}$-modules. For each $i, j \in \mathcal{P}$ we write $\text{Tor}^\mathcal{P}_i(k, \mathcal{Q})_{(j)}$ for the component of Tor in homological degree $i$ and weight degree $j$.

It is useful to note that this is well defined, since the category of left $\mathcal{P}$-modules admits a model structure in which the fibrations are the arity-wise surjections, the weak equivalences are the quasi-isomorphisms, and the quasi-free left modules are included in the class of cofibrant objects. It is important that we are working over a field of characteristic zero, so that $\mathcal{P}$ is $\Sigma$-cofibrant.

Lemma 3.11 Let $f : \mathcal{M} \to \mathcal{N}$ be a map of left $\mathcal{P}$-modules and suppose that $\mathcal{F} = (\mathcal{P} \circ \mathcal{Y}, d)$ is a quasi-free complex mapping onto $\mathcal{M}$ and that $\mathcal{R}$ is a resolution of $\mathcal{N}$. Then there exists a map of left $\mathcal{P}$-modules $\mathcal{F} \to \mathcal{R}$ extending $f$, and any two such choices are homotopic.

Lemma 3.12 The map $f$ is left Koszul if and only if $\text{Tor}^\mathcal{P}(k, \mathcal{Q})$ is concentrated on the diagonal.

Proof. Let us show we can construct minimal quasi-free resolutions $\mathcal{F} = (\mathcal{P} \circ \mathcal{X}, d) \to \mathcal{Q}$. To do this, let us consider an equivariant section $\sigma$ of the projection $\mathcal{Q} \to k \circ_\mathcal{P} \mathcal{Q}$, which exists since we work over a field of characteristic zero, and let $\mathcal{X}_0 = \sigma(k \circ_\mathcal{P} \mathcal{Q})$, so we have an epimorphism

$$f_0 : \mathcal{P} \circ \mathcal{X}_0 \to \mathcal{Q} = \mathcal{X}_{-1}.$$

The kernel $\mathcal{K}_0$ of this map is a left $\mathcal{P}$-module, so we may repeat this and take $\mathcal{X}_1$ a minimal generating set for $\mathcal{K}_0$ obtained from an equivariant section of the projection $\mathcal{K}_0 \to k \circ_\mathcal{P} \mathcal{K}_0$, along with $f_1 : \mathcal{P} \circ \mathcal{X}_1 \to \mathcal{K}_0$. Extend the construction above to $\mathcal{F}_1 = \mathcal{P} \circ (\mathcal{X}_0 \oplus s \mathcal{X}_1)$ where the differential is the unique map $\mathcal{X}_0 \oplus s \mathcal{X}_1 \to \mathcal{F}_1$ that vanishes on $\mathcal{X}_0$ and maps $s \mathcal{X}_1$ onto $\mathcal{K}_0$. In this way, $\mathcal{H}_0(\mathcal{F}_1)$ is isomorphic to $\mathcal{Q}$ through the map $f_0$. We can now continue this process by adjoining generators in homological degree 2 to obtain $\mathcal{F}_2$ with $\mathcal{H}_1(\mathcal{F}_2) = 0$ and $\mathcal{H}_0(\mathcal{F}_2)$ isomorphic to $\mathcal{Q}$. Continuing, in the limit, we obtain the desired resolution.

Since the resolution is minimal, we see that $\mathcal{X}$ is isomorphic to $\text{Tor}^\mathcal{P}(k, \mathcal{Q})$, so $\mathcal{X}$ must be concentrated in the diagonal. Conversely, it is clear that if we have a diagonally pure resolution, then $\text{Tor}^\mathcal{P}(k, \mathcal{Q})$ is concentrated on the diagonal.

Let us recall the following from [16].

Definition 3.13 We say a morphism of operads $f : \mathcal{P} \to \mathcal{Q}$ enjoys the PBW property if there is an endofunctor $\mathcal{X} : \mathcal{C} \to \mathcal{C}$ on the category underlying that of $\mathcal{P}$-algebras so that for each $\mathcal{P}$-algebra $A$ there is a natural isomorphism

$$f_!(A)^\# \to \mathcal{X}(A^\#).$$
The main result of [16], if we take monads there to be algebraic operads, is the following:

**Theorem 3.14** The morphism of operads $f : \mathcal{P} \to \mathcal{Q}$ satisfies the PBW property if and only if it makes $\mathcal{Q}$ into a free right module over $\mathcal{P}$. In this case, the functor $X$ is a basis for $\mathcal{Q}$ as a right $\mathcal{P}$-module.

Our first main theorem shows the PBW property above is related, by Koszul duality, to the notion of left Koszul morphisms, at least when $\mathcal{P}$ and $\mathcal{Q}$ are Koszul operads. Note this is an extension of [42, Corollary 5.9] to algebraic operads.

**Theorem 3.15** (Duality) A map between Koszul operads is left Koszul if and only if its Koszul dual map satisfies the PBW property.

To do this, we just need two technical lemmas, beginning with the following simple homological criterion for freeness, which we recall from Proposition 4.1 in [16], for example. We phrase it in a slightly different way than we did there:

**Lemma 3.16** A right $\mathcal{P}$-module is $\mathcal{M}$ is free if and only if for every $i, j \in \mathbb{N}$, the group $\text{Tor}^\mathcal{P}_{j-i}(\mathcal{M}, k)^{(i)}$ vanishes unless $i = j$.

Note we are simply saying that the homology of $\mathcal{M} \circ \mathcal{P} k$ is concentrated in degree 0. The second lemma relates the derived functors $k \circ \mathcal{P} \mathcal{Q}$ and $\mathcal{P}! \circ \mathcal{Q}! k$ in case $\mathcal{P}$ and $\mathcal{Q}$ are Koszul operads. We point the reader to Theorem 5.8 in [42] which proves the result for associative algebras.

**Lemma 3.17** Let $f : \mathcal{P} \to \mathcal{Q}$ be a morphism of Koszul operads, and let $f^! : \mathcal{Q}^! \to \mathcal{P}^!$ be its dual morphism. For each $j, i \in \mathbb{N}$ we have a natural isomorphism:

$$\text{Tor}^\mathcal{Q}_{j-i}(\mathcal{P}^!, k)^{(i)} \to \text{Tor}^\mathcal{P}_{j}(k, \mathcal{Q})^*(j).$$

**Proof.** It suffices to note that, in the category of right $\mathcal{P}$-modules, $k$ admits a resolution $\mathcal{P}^! \circ \mathcal{P} \to k$ while, in the category of left $\mathcal{Q}^!$-modules, $k$ admits a resolution $\mathcal{Q}^! \circ \mathcal{Q} \to k$. Applying the functor $- \circ \mathcal{P} \mathcal{Q}$ in the first case and the functor $\mathcal{P}^! \circ \mathcal{Q}^! -$ in the second case, we obtain two complexes $\mathcal{P}^! \circ \mathcal{Q}$ and $\mathcal{P}^! \circ \mathcal{Q}$ that are related by the duality described in the statement of the lemma.

**Proof of Theorem 3.15.** The previous three lemmas immediately imply the result.

### 3.4 The HKR morphism

In this section we construct, for each map $f : \mathcal{P} \to \mathcal{Q}$ and each $\mathcal{Q}$-algebra $A$, an operadic analogue of the classical HKR map. This map relates the deformation complex of the $\mathcal{P}$-algebra $f^* A$ to a certain space of ‘differential forms’ on $A$ depending functorially, as in the classical setting, on $\Omega^1_A$.

**Remark 3.18** We cannot avoid making the point that, while the morphisms of operads in [16, 34] enjoying the PBW property involve a statement about the pushforward functor $f_!$ on $\mathcal{P}$-algebras, the morphisms of operads we are interested in involve a statement about the pullback functor $f^*$ on $\mathcal{Q}$-algebras. As mentioned in the introduction, this follows the ‘mantra’ promoted in [45].
To begin, let us take a quasi-free resolution $\mathcal{F} = (P \circ Y, d)$ of $\Omega$ in left $\mathcal{P}$-modules. Let us recall from Lemma A.3 that if $Q = (\Omega(V), d)$ is a cofibrant resolution of $A$, then $\Omega^1_Q \otimes_U Q$ is canonically isomorphic to $V \otimes \Omega(V)$, while $\Omega^1_{Q,A}$ is canonically isomorphic to $V \otimes A$. We will be interested in applying the functor $k \circ P \mathcal{F} = Y$ to the space $\Omega^1_Q \otimes_U Q$, relative to the algebra $Q$, as the following definition explains.

**Definition 3.19** We define $\Omega^*_Q$, the space of differential forms on $A$ associated to $F$, as the chain complex $Y(V) \otimes_Q V$. Its differential is the one induced from $Q$ and $\mathcal{F}$.

The following proposition shows that to each resolution we may associate an ‘HKR map’.

**Proposition 3.20** There exists a map of chain complexes

$$\text{HKR}_A : \text{Def}^*_A(f^*A) \longrightarrow \Omega^*_Q,$$

for every choice of resolution $\mathcal{F}$. Moreover, if $Q \longrightarrow A$ is a cofibrant resolution of $A$ in the category of $\Omega$-algebras, there is a commutative diagram

$$\begin{array}{ccc}
\text{Def}^*_A(f^*Q) & \longrightarrow & \text{Def}^*_A(f^*A) \\
\downarrow & & \downarrow \\
\Omega^*_Q & \longrightarrow & \Omega^*_Q.
\end{array}$$

Observe that, unlike the case where $A$ is a commutative algebra, the source and the target of the HKR map do not admit natural operadic $A$-module structures. We call $\text{HKR}_A$ the Hochschild–Kostant–Rosenberg map associated to $f$ and the algebra $A$. Observe, moreover, that $\text{HKR}_A$ manifestly depends on the resolution $\mathcal{F} = (P \circ Y, d)$ and on the cofibrant resolution $Q$ of $A$. This is not a problem for us, since the map it induces on homology does not. With this at hand, we can define the HKR property:

**Definition 3.21** We say that $f$ satisfies the HKR property if the map HKR is a quasi-isomorphism for every smooth $\Omega$-algebra.

**Proof Proposition 3.20.** Let $A$ be any $\Omega$-algebra and let us pick a cofibrant resolution of $A$ of the form $Q = (\Omega(V), d)$. By the HKR property of $f$, the non-dg $\mathcal{P}$-algebra $(\Omega(V), 0)$ admits a cofibrant resolution of the form $(\mathcal{P} \circ Y(V), d_V) \longrightarrow \Omega(V)$ and hence perturbing this we obtain a resolution

$$Z = (\mathcal{P} \circ Y(V), d_V + \delta) \longrightarrow (\Omega(V), d)$$

of the $\mathcal{P}$-algebra $(\Omega(V), d)$. If we use this resolution to compute the cotangent complex of the $\mathcal{P}$-algebra $Q = (\Omega(V), d)$, we obtain a complex of the form

$$\text{Def}^*_A(f^*A) = \Omega^1_Z \otimes_U Q = (Y(V) \otimes \Omega(V), \delta_1)$$
and, tautologically, the cotangent complex of the $\mathcal{Q}$-algebra $A$ may be computed through the complex

$$\text{Def}_*(A) = \Omega^1_{\mathcal{Q}} \otimes U \mathcal{Q} = (V \otimes \mathcal{Q}(V), \delta'_1).$$

The fact all of the constructions and isomorphisms above are natural, with the exception of the perturbation process, means that, in fact, the complex $(V \otimes \mathcal{Q}(V), \delta'_1)$ is obtained from the complex $(V \otimes \mathcal{Q}(V), \delta_1)$ by the endofunctor $\mathcal{Y}$ relative to the algebra $\mathcal{Q}(A)$. The commutativity of the diagram is then immediate, although we cannot promote $\Omega^*_{\mathcal{F}^* \mathcal{A}}$ to a bona-fide functor and hence $\text{HKR}_{\mathcal{A}}$ to a natural transformation.

With this at hand, our second main result for maps satisfying the HKR property is the following ‘operadic HKR theorem’, which we now prove with a series of lemmas. Its immediate application is the corollary that follows it, which we will use heavily later on.

**Theorem 3.22** Every left Koszul map between Koszul operads has the HKR property.

**Corollary 3.23** (PBW criterion) Every map between Koszul operads whose dual has the PBW property satisfies the HKR property.

**Proof.** This follows immediately from Theorem 3.15.

**Remark 3.24** It is natural to wonder whether a converse to this last corollary exists. The condition that the resolution $\mathcal{F}$ be diagonally pure makes a certain spectral sequence collapse and gives our result, modulo the computations and various lemmas that we have made use of. In a generic case, one should expect an “HKR spectral sequence” to exist coming from the resolution $\mathcal{F}$ and, in favourable cases, one may obtain an HKR theorem without requiring that $\mathcal{F}$ be diagonally pure. It would certainly be interesting to have an example of this behaviour.

The proof of Theorem 3.22 relies on the following fundamental lemma relating the two different constructions arising from the twisting morphisms $\phi$ and $\psi$ associated to the Koszul operads $\mathcal{P}$ and $\mathcal{Q}$, respectively.

**Lemma 3.25** Let $\mathcal{P}^i$ and $\mathcal{Q}^i$ be the Koszul dual cooperads to $\mathcal{P}$ and $\mathcal{Q}$. For the commutative diagram of maps of (co)operads and twisting morphisms

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{f} & \mathcal{Q} \\
\phi \uparrow & & \uparrow \psi \\
\mathcal{P}^i & \xrightarrow{g} & \mathcal{Q}^i,
\end{array}$$

there is a natural isomorphism of functors $\mathcal{B}_\phi f^* = g^i \mathcal{B}_\psi : \Omega^* \mathcal{P} \text{-Alg} \rightarrow \mathcal{P}^i \text{-Cog}$, where $f^*$ denotes the restriction of scalars functor and $g^i$ denotes the coinduction functor.

**Proof.** Ignoring the additional differentials produced by the bar construction, $\mathcal{B}_\psi$ produces the cofree conilpotent $\mathcal{Q}^i$-coalgebra functor on the underlying chain complex and $g^i$ is the right adjoint of the corestriction of scalars functor $g_*$. Since the composition of right adjoints is a right adjoint,
we conclude that up to bar-differentials both $B_\phi f^*$ and $g^! B_\psi$ correspond to the cofree conilpotent $\Omega^1$-coalgebra on the underlying space. The commutativity of the diagram above guarantees that both differentials are the same.

**Corollary 3.26** In the conditions of the previous lemma, if the coinduction functor $g^!$ satisfies the PBW property, there is a quasi-isomorphism of functors

$$f^* \Omega_\psi \simeq \Omega_\phi g^!: \Omega^1 \text{-Cog} \to \mathcal{P}\text{-Alg}.$$  

**Proof.** Our hypothesis on $g^!$ implies it preserves quasi-isomorphisms [34, Corollary 1.1]. This implies that there are quasi-isomorphisms of functors

$$\Omega_\phi g^! \simeq \Omega_\phi B_\psi \Omega_\psi = \Omega_\phi f^* \Omega_\psi \simeq f^* \Omega_\psi.$$

This is what we wanted. \qed

Recall from the discussion after Proposition A.2 that the operadic $A$-module $\Omega^1_A$ of Kähler differentials on $A$ is the quotient of the free operadic $A$-module generated by symbols $da$ for $a \in A$ subject to a generalized Leibniz rule.

**Lemma 3.27** The map HKR$_A$ of Proposition 3.20 is a quasi-isomorphism for $\Omega$-algebras obtained as a cobar construction.

**Proof.** For a $\Omega$-algebra of the form $A = \Omega_\psi(V)$, with $V$ a $\Omega^1$-coalgebra, let us compute the quasi-isomorphism type of $\text{Def}_*(f^*A)$. Taking the cofibrant resolution $Q = \Omega_\phi B_\phi f^* A$ of $A$ in the category of $\mathcal{P}$-algebras given by the bar-cobar resolution, we have that:

$$\text{Def}_*(f^*A) = (\Omega^1_Q \otimes U_Q Q, d)$$

by Definition 3.4

$$= (B_\phi f^* A \otimes Q, d')$$

by Lemma A.3

$$= (f^* A \otimes g^! B_\psi \Omega_\psi(V), d')$$

by Lemma 3.25

$$\simeq (f^* A \otimes g^!(V), d')$$

since $g^!$ is PBW.

Here $d'$ denotes the only non-internal differential which is transported along the isomorphism of graded vector spaces provided by Lemma A.3.

On the other hand, since $A$ is quasi-free, $\Omega^1_A = (U_Q(A) \otimes V, d')$, endowed with an additional transferred differential. It follows that as an operadic $A$-module,

$$\Omega^*_{\mathcal{F}, A} = (A \otimes g^!(V), d') ,$$

so in particular as chain complexes $\Omega^*_{\mathcal{F}_A}$ and $\text{Der}_*(f^*A)$ are quasi-isomorphic. It remains to see that this quasi-isomorphism gives an quasi-inverse to HKR$_A$. Keeping track of the quasi-isomorphisms above, one can see that filtering $\text{Def}_*(f^*A)$ by the appropriate word lengths we recover the quasi-inverse at the level of the associated graded complexes, which reduces our claim to that of free $\Omega$-algebras (with zero differential). This is what we wanted. \qed
Proposition 3.28  Suppose that A is a smooth $\mathcal{Q}$-algebra and that the functor $g^!$ preserves quasi-isomorphisms. Then $\Omega^*_{\mathcal{F},A}$ is a complex with homology $g^!\mathcal{H}_0(A,A)$ concentrated in degree zero.

Proof. This follows immediately since $\text{Def}^*(f^*Q,f^*A) = Q \otimes U \Omega^1_Q$ is a complex with homology concentrated in degree 0, where it equals $\mathcal{H}_0(A,A) = A \otimes U \Omega^1_A$, while $g^!\mathcal{H}_0(A,A) = A \otimes g^!(\Omega^1_A)$, so all we need to conclude is the fact $g^!$ preserves quasi-isomorphisms.

Proof of Theorem 3.22. Since the algebra $A$ is smooth, we know by Proposition 3.7 that the morphism $p: \Omega^1_Q \rightarrow \Omega^1_A$ is a quasi-isomorphism, and hence so is the map

$$Q \otimes_U Q \Omega^1_Q \rightarrow A \otimes_U A \Omega^1_A.$$ 

By Proposition 3.28, the resulting map $\Omega^*_{\mathcal{F},Q} \rightarrow \Omega^*_{\mathcal{F},A}$ is a quasi-isomorphism. We have shown in Lemma 3.27 that HKR$_{QA}$ is a quasi-isomorphism (for $QA$ is cofibrant) and since $A$ is smooth, the commutativity of the following diagram

$$\text{Def}^*(f^*Q,f^*Q) \xrightarrow{\text{HKR}_Q} \Omega^*_{\mathcal{F},Q}$$

$$\downarrow \sim$$

$$\text{Def}^*(f^*A,f^*A) \xrightarrow{\text{HKR}_A} \Omega^*_{\mathcal{F},A}$$

shows that HKR$_A$ is a quasi-isomorphism. This is what we wanted. □

3.5 The case of cohomology

As before, let us take a quasi-free resolution $(\mathcal{P} \circ \mathcal{Y}, d)$ of $\mathcal{Q}$ where $\mathcal{Y}$ is a diagonal endofunctor and, for the $\mathcal{Q}$-algebra $A$, let $(\mathcal{Q}(V), d)$ be a quasi-free resolution. Recall that in this case the underlying graded vector space to $\text{Def}^*_Q(A)$ is given by $\text{hom}(V, \mathcal{Q}(V))$.

Definition 3.29  Let $A$ be a $\mathcal{Q}$-algebra. We define Poly$^*(A)$, the \textit{poly-vector fields on A relative to $f$}, to be the chain complex

$$\text{Poly}^*(A) := (\text{hom}(\mathcal{Y}(V), \mathcal{Q}(V)), \delta).$$

The dual result for tangent cohomology of smooth $\mathcal{Q}$-algebras is the following. The proof is quite similar to the case of cotangent homology, so we only sketch the details. We will make use of the ‘smaller complex’ Poly$^*(Q,A) := \text{hom}(\mathcal{Y}(V), A)$. The non-internal part of the differential makes use of the map $(\mathcal{Q}(V), d) \rightarrow A$.

Theorem 3.30  If $f$ is left Koszul then for every smooth $\mathcal{Q}$-algebra $A$ the map

$$\text{HKR}^A : \text{Def}^*(f^*A) \rightarrow \text{Poly}^*(A)$$

is a quasi-isomorphism.

Lemma 3.31  The conclusion of Theorem 3.30 holds for cobar algebras.
Proof. For a $\Omega$-algebra of the form $A = \Omega_\Psi(V)$, with $V$ a $\Omega^1$-coalgebra, let us compute the quasi-isomorphism type of $\text{Der}^*(f^*A)$. Taking the cofibrant resolution $Q$ given by the bar-cobar resolution $\Omega f_\phi B f^*A$ of $f^*A$ in the category of $\mathcal{P}$-algebras, we have that:

\[
\begin{align*}
\text{Def}^*_P(f^*A) &= \text{hom}_Q(\Omega^1,vQ) \\
&= \text{hom}(B f^*A, Q) \\
&= \text{hom}(g^! B f^*A(v), Q) \\
&\sim \text{hom}(g^!(V), Q) \\
&\sim \text{hom}(g^!(V), Q) \quad \text{by Lemma 3.25}
\end{align*}
\]

Proceeding as in the case of homology, we obtain a quasi-inverse to the HKR map.

Proof of Theorem 3.30. If $A$ is a smooth algebra, then for any cofibrant replacement $p: Q \sim A$ there are induced quasi-isomorphisms

\[
\text{Der}(Q,Q) \longrightarrow \text{Der}(Q,A) \leftarrow \text{Der}(A,A)
\]

since $\text{hom}_{UQ}(\Omega^1, -)$ is exact. Furthermore, similarly to Proposition 3.7 one can show that $\text{Poly}^*(Q) \to \text{Poly}^*(Q,A)$ and $\text{Poly}^*(A) \to \text{Poly}^*(Q,A)$ are quasi-isomorphisms. The result follows from the commutativity of the diagram

\[
\begin{array}{c}
\text{Def}^*(f^*Q, f^*Q) \xrightarrow{\text{HKR}_Q} \text{Poly}^*(Q) \\
\downarrow \sim \\
\text{Def}^*(f^*Q, f^*A) \xrightarrow{\sim} \text{Poly}^*(Q,A) \\
\downarrow \sim \\
\text{Def}^*(f^*A, f^*A) \xrightarrow{\text{HKR}_A} \text{Poly}^*(A)
\end{array}
\]

where we now use coefficients to be able to draw the zig-zag of quasi-isomorphisms.

Remark 3.32 The classical cohomological version of the HKR theorem establishes not only that the Hochschild cohomology and the space of poly-vector fields of a smooth commutative algebra are isomorphic as chain complexes, but that they are isomorphic Lie algebras. In the operadic setting, tangent cohomology is also a Lie algebra via the bracket defined by the commutator of derivations. Unless the endofunctor $\mathcal{Y}$ carries some extra structure, it is not clear a priori how to endow $\text{Poly}^*(A)$ with a Lie algebra structure that makes our map an isomorphism of Lie algebras. However, we point the reader to Theorem 4.10 below where $\mathcal{Y}$ can be taken to be an operad itself, and where we explain how in the classical case we, do recover the Lie algebra structure on poly-vector fields.
4 Computations and examples

4.1 Revisiting the classical HKR theorem

Let us show that our formalism recovers the classical HKR theorem exactly.

**Proposition 4.1** The morphism $\text{Ass} \to \text{Com}$ enjoys the HKR property, and the induced map

$$\text{HKR}_A : \text{Def}_e(f^*A) \to \Omega^*_A$$

coincides, up to a suspension, with the classical HKR quasi-isomorphism.

**Proof.** We offer two points of view:

1. We can produce a resolution of the form $(\text{Ass} \circ \text{Lie}^\dagger, d)$ coming from the functorial Koszul resolution on free commutative algebras $g : T(S_c(V[-1])[1]) \to S(V)$, which is manifestly diagonally pure, since the homological degree in $S_c(V[-1])[1] = \text{Lie}^\dagger(V)$ coincides with the weight degree.

2. The Koszul dual morphism is PBW, that is, $\text{Ass}$ is a free right Lie-module, so Theorem 3.15 implies the result. Moreover, the lemma preceding it shows that we may take $Y = \text{Lie}^\dagger$ as in the previous item; we need only pay attention to the shift in homological degree.

We conclude, in particular, that $\text{Tor}^{\text{Ass}}(k, \text{Com}) \simeq \text{Lie}^\dagger$ as weight graded dg $\Sigma$-modules, so we may take $V \mapsto \text{Hom}(V, \Omega^*_A) = S_c(A[V[-1]]_1)$ as the functor witnessing the classical HKR property. The HKR map for a commutative algebra $A$ induces an isomorphism

$$s^{-1}\text{HH}_e(A, A) = \text{Hom}_e(A, A) \to \Omega^*_A = S_c(\Omega^*_A[-1])[1].$$

It differs from the classical HKR isomorphism map by a desuspension and in that we do not have, in degree zero, the identity map of $A = \text{HH}_0(A)$ onto $A$. Otherwise, our formalism recovers the HKR map exactly.

4.2 New HKR theorems

**Permutative algebras.** A permutative algebra [5] is an associative algebra $A$ such that for every $x, y$ and $z \in A$, we have that $x(yz) = x(zy)$. Permutative algebras are algebras over a binary quadratic operad, denoted $\text{Perm}$ which is the linearisation of a set operad $\text{Perm}_0$. The Koszul dual operad of $\text{Perm}$ is the operad $\text{PreLie}$ controlling pre-Lie algebras. Both these algebraic structures play an important role in the study of operadic deformation theory [3, 14, 15].

Clearly, every commutative algebra is a permutative algebra via the same product. Since a permutative product is in particular associative, there is a factorisation of the map of operads $f : \text{Ass} \to \text{Com}$ via the permutative operad: $\text{Ass} \to \text{Perm} \to \text{Com}$. 
**Proposition 4.2** The map $\psi : \text{Perm} \rightarrow \text{Com}$ enjoys the HKR property, with generating sequence $\text{RT} \neq 1$ that assigns a set $I$ to the set of rooted trees with vertices labeled by $I$ for which no vertex has exactly one child.

**Proof.** The Koszul dual map to $\psi$ is the anti-symmetrisation map $\phi : \text{Lie} \rightarrow \text{PreLie}$. In [16] this map was shown to satisfy the PBW property. Moreover, in [11], it was show that the generators of PreLie as a right Lie-module are given by the functor $\text{RT} \neq 1$ as in the statement of the proposition. The result follows from Corollary 3.23. □

Theorems 3.22 and 3.30 allow us to compute the permutative (co)tangent (co)homology of commutative algebras.

**Corollary 4.3** Let $A$ be a smooth commutative algebra.

- The cotangent homology $\mathcal{H}^*(\psi^* A)$ is isomorphic to the algebra of “tree-wise” differential forms $\text{RT} \neq 1 (\Omega^*_A)$ over the classical differential forms of $A$.
- Dually, the tangent cohomology $\mathcal{H}^*(\psi^* A)$ is isomorphic to the algebra of “tree-wise” poly-vector fields $\text{RT}^\vee (\text{Poly}^*(A))$. □

**Corollary 4.4** For every smooth commutative algebra $A$ there is a quasi-isomorphism

$$\text{Def}^\circ_{\text{Perm}}(A) \longrightarrow \text{RT}^\vee \neq 1 (\text{Poly}^*(A)).$$

Moreover, the natural map $\text{Def}^\circ_{\text{Perm}}(A) \longrightarrow \text{Def}^\circ_{\text{Ass}}(A)$ induces, in homology, the natural map

$$\text{RT}^\vee \neq 1 (\text{Poly}^*(A)) \longrightarrow \text{Poly}^*(A)$$

given by the augmentation $\text{RT}^\vee \neq 1 \longrightarrow k$. □

*Enriched pre-Lie algebras of Dotsenko and Foissy.* In [12] the authors define a functor that assigns to every Hopf cooperad $\mathcal{C}$ an operad $\text{PreLie}_\mathcal{C}$ of $\mathcal{C}$-enriched pre-Lie algebras. The example we are interested in is the following, where this functor recovers the operad of pre-Lie algebras and the operad of braces algebras. Such braces algebras and related structures, conceived originally in [31], appeared in the literature in several opportunities [21, 22, 35], and are relevant in deformation theory [14], for example.

1. If $\mathcal{C} = u\text{Com}^*$ is the Hopf cooperad of unital commutative coalgebras, one obtains the operad $\text{PreLie}$ governing pre-Lie algebras.
2. If $\mathcal{C} = u\text{Ass}^*$ is the Hopf cooperad of unital associative coalgebras, one obtains the operad $\text{Br}$ governing brace algebras.
3. The unit map $u\text{Com}^* \longrightarrow u\text{Ass}^*$ gives the map $g : \text{PreLie} \longrightarrow \text{Br}$ constructed in [8].

By Proposition 2 in that article, every connected Hopf cooperad $\mathcal{C}$ admits a structure of associative algebra for the Cauchy product in the category symmetric sequences —what are usually called twisted associative algebras—, in such a way that every morphism of Hopf cooperads $\mathcal{C} \longrightarrow \mathcal{C}'$ induces a morphism of twisted associative algebras.
Definition 4.5 (Proof of Theorem 1 in [12]) Given a species \( \mathcal{X} \), there is a species of enriched trees, which we write \( \mathcal{T}_R \), where each vertex is decorated by an element of \( \mathcal{C}' \) with the condition that every vertex of maximal depth is decorated by an element of \( \mathcal{X} \). Similarly, \( \mathcal{T}_L \) is the species of \( \mathcal{C}' \)-enriched trees, with the condition that the root vertex is decorated by an element of \( \mathcal{X} \).

The main result of [12] is as follows.

Theorem. Let \( \varphi : \mathcal{C} \rightarrow \mathcal{C}' \) be a morphism of connected Hopf cooperads and let us consider \( \mathcal{C}' \) as a \( \mathcal{C} \)-bimodule by viewing \( \varphi \) as a map of twisted associative algebras. Then:

1. if \( \mathcal{C}' \) is left \( \mathcal{C} \)-free with generators \( \mathcal{X} \) then the operad \( \text{PreLie}_{\mathcal{C}'} \) is free as a left \( \text{PreLie}_{\mathcal{C}'} \)-module with generators \( \mathcal{T}_L \) and,
2. if \( \mathcal{C}' \) is right \( \mathcal{C} \)-free with generators \( \mathcal{X} \) then the operad \( \text{PreLie}_{\mathcal{C}'} \) is free as a right \( \text{PreLie}_{\mathcal{C}'} \)-module with generators \( \mathcal{T}_R \).

An immediate consequence of this result is the following, since the map of twisted associative algebras \( u_{\text{Com}}^* \rightarrow u_{\text{Ass}}^* \) is both left and right free.

Corollary 4.6 (Theorem 2 in [12]) The brace operad \( \text{Br} \) is free as a left and as a right \( \text{PreLie} \)-module.

From this, we obtain the following HKR theorem for smooth brace algebras.

Theorem 4.7 The map \( g : \text{PreLie} \rightarrow \text{Br} \) satisfies the HKR property: for every smooth brace algebra \( A \) there exists a natural quasi-isomorphism

\[ \text{HKR}_A : \text{Def}_\ast (g^\ast A) \rightarrow \Omega_{\mathcal{T}_A}^\ast \]

where \( \Omega_{\mathcal{T}_A}^\ast = \mathcal{T}_R(\Omega_1^\ast) \) where \( \mathcal{T}_R \) is the endofunctor of rooted trees with vertices of maximal depth decorated by Lie words.

Diassociative algebras. Diassociative algebras were introduced by J.-L. Loday in [37]. A diassociative algebra [39, Section 13.6] consists of a vector space \( V \) along with two associative operations

\[ \vdash : V \otimes V \rightarrow V \quad \text{and} \quad \dashv : V \otimes V \rightarrow V \]

satisfying the following set of three quadratic relations:

\[ (x_1 \vdash x_2) \dashv x_3 = x_1 \vdash (x_2 \vdash x_3), \quad (x_1 \vdash x_2) \vdash x_3 = x_1 \vdash (x_2 \vdash x_3), \quad (x_1 \dashv x_2) \vdash x_3 = x_1 \dashv (x_2 \vdash x_3). \]

Any permutative algebra gives rise to a diassociative algebra by defining both products to be the permutative product, so that we have a map \( \text{Dias} \rightarrow \text{Perm} \), whose Koszul dual is the map \( \text{PreLie} \rightarrow \text{Dend} \). In [16] the authors proved that this morphism is PBW and, since it is known that \( \text{Dend} \) is a left free \( \text{Ass} \)-module with basis the operad of braces \( \text{Br} \), we can use Corollary 4.6 and our main theorem to obtain the following result:
**Theorem 4.8** For every smooth permutative algebra $A$ there is a quasi-isomorphism

$$\text{HKR}_A : \text{Def}^\text{Dias}_*(A) \longrightarrow \Omega^*_A$$

where the endofunctor $\mathcal{Y}$ is given by $\text{Ass} \circ \mathcal{T}_R$ and $\mathcal{T}_R$ is the endofunctor of Theorem 4.7.

---

**The work of J. Griffin.** Let us now connect our formalism with the one developed by J. Griffin. Motivated by the Hodge decomposition of Hochschild cohomology [1,19], Griffin [26] considered the problem —like we do— of computing the cohomology of a pull-back algebra $f^*A$ under a morphism of operads $f : P \longrightarrow Q$. Since his motivation is slightly different from ours, there is no mention of HKR-type theorems in his paper, nor of smooth algebras. However, one can find the following result in *ibidem*, which relates the Quillen homology of a $Q$-algebra $A$ to that of its pull-back, which can be seen as a first approximation to the problem of computing the cohomology of $f^*A$, and which contains already a clear link between his and our formalism; see Theorem 3.7-(II) in [26].

**Theorem 4.9** Let $f : P \longrightarrow Q$ be a map of Koszul operads and let $g : Q^i \longrightarrow P^i$ be its Koszul dual map. Suppose that $P^i$ is a free right $Q^i$-comodule with basis $X$. Then for every $Q$-algebra $A$ there is an isomorphism

$$B_P(f^*A) \longrightarrow X \circ B_Q(A).$$

Moreover, Griffin goes on to consider the case of maps of Koszul operads $P \longrightarrow Q$ where $Q$ is obtained from $P'$ and another operad $P$ by a filtered distributive law [13] as originally defined by V. Dotsenko in [10]; see [26, Theorems 5.15 and 5.18]. The following result offers a complementary technique to compute the tangent cohomology of a $P$-algebra coming from a smooth $Q$-algebra under the projection $f : P \longrightarrow Q$.

**Theorem 4.10** (Filtered HKR theorem) Suppose $P$ is obtained from $Q$ and $R$ by a filtered distributive law, so that $P$ is isomorphic to $Q \circ R$ as a right $R$-module. For every smooth $Q$-algebra $A$ the cotangent homology of $f^*A$ is given by the endofunctor

$$A \longrightarrow R^i(\Omega^*_A)$$

of ”$R^i$-enriched differential forms” on $A$.

**Proof.** Since we are working over a field of characteristic zero, Theorem 5.4 in [13] guarantees that $P^i$ is a free right $Q^i$-module with generators $R^i$, so the claim follows.

---

**4.3 The operadic butterfly of J.-L. Loday**

Let us recall from [38] that we can arrange certain nine operads into a “butterfly” diagram of morphisms, as in the figure above. We record those maps which we know satisfy the PBW property and which we know satisfy the HKR property. Most of the claims follow immediately by duality (Theorem 3.15) from the results obtained in [16], or by the following simple remark:
Figure 1: The operadic butterfly.

**Remark 4.11** Note that if $f: P \to Q$ is PBW, then we must have $\dim_k P(n) \leq \dim_k Q(n)$ for each $n \in \mathbb{N}$. In particular, since $\dim_k \text{Dias}(2) > \dim_k \text{Ass}(2)$, $\dim_k \text{Leib}(2) > \dim_k \text{Lie}(2)$, and since $\dim_k \text{PreLie}(3) > \dim_k \text{Ass}(3)$, the respective maps in Figure 1 are not PBW.

It would be interesting to determine if the remaining arrows enjoy the HKR or the PBW property or if, perhaps, they enjoy none of the two.

**Remark 4.12** It is well known [6] that the map of operads $\text{Lie} \to \text{PreLie}$ makes its codomain a free left module. However, the generators exhibiting PreLie as a left free Lie-module are not concentrated in weight zero, so that, as expected, the map from Perm onto Com is not PBW. In fact, in general, the extra weight degree we have considered means a map $f: P \to Q$ that is left free will not be left Koszul unless it is the identity, which shows that it is crucial to replace the ‘left free’ condition to a left Koszul condition.

## A Recollections on operads

### A.1 Operads and their algebras and modules

Let us fix a reduced symmetric operad $P$ and write $P$-$\text{Alg}$ for the category of dg $P$-algebras. The operad $P$, viewed as a monad, gives the left adjoint

$$P : \Sigma \text{dgMod} \to P$-$\text{Alg}$$

$$\# : P$-$\text{Alg} \to \Sigma \text{dgMod}.$$
Fix a dg \( \mathcal{P} \)-algebra \( A \) as before. An \textit{operadic A-module} is a dg \( \Sigma \)-module \( M \) along with an action \( \gamma_M : \mathcal{P} \circ (A, M) \longrightarrow M \) so that

\[
\gamma_M(1 \circ (\gamma_A, \gamma_M)) = \gamma_M(\gamma \circ (1, 1)).
\]

Here \( \mathcal{P} \circ (A, M) \) is the submodule of \( \mathcal{P}(A \oplus M) \) which is linear in \( M \).

It is useful to note that if \( \mathcal{P} = A_\Sigma \) and if \( A \) is an \( \mathcal{P} \)-algebra or, what is the same, an associative algebra, then an operadic \( A \)-module is the same as an \( A \)-bimodule and \textit{not} a left (or right) \( A \)-module. Similarly, the operadic modules for commutative algebras are the symmetric bimodules.

In fact, there is a functor

\[ U_\mathcal{P} : \mathcal{P} \text{-Alg} \longrightarrow \text{Ass-Alg}, \]

the last being the category of dga algebras, so that the category of operadic \( A \)-modules is isomorphic to the category of left \( U_\mathcal{P}(A) \)-modules of the associative algebra \( U_\mathcal{P}(A) \).

**Definition A.1** We call \( U_\mathcal{P}(A) \) the \textit{associative enveloping algebra of} \( A \).

Concretely, \( U_\mathcal{P}(A) \) is spanned by trees with one leaf pointed by the only element in \( k \) under the relation that identifies the corolla with root \( \mu \circ_i \nu \) with the corolla with root \( \mu \) and \( \nu \) acting on the leaves \( i, i+1, \ldots \), and we will write a generic element by

\[ u(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n) \]

where \( u \) is an operation of \( \mathcal{P} \) and the empty slot corresponds to the leaf marked by \( k \). The algebra structure is defined by concatenation through the pointed leaf and the root through the partial composition \( \circ_i \) of \( \mathcal{P} \). We refer the reader to [33] for a useful reinterpretation of \( U_\mathcal{P} \) through the language of 2-colored operads.

As useful examples, we note that in the case of associative and Lie algebras, we recover the usual notion of enveloping algebra: for an associative algebra \( A \) we have that \( U_{A_\Sigma}(A) = A \otimes A^{\text{op}} \), for a Lie algebra \( L \) we have that \( U_{\text{Lie}}(L) = U(L) \); note in both cases we are considering non-unital algebras and non-unital actions.

Given a map of \( \mathcal{P} \)-algebras \( f : B \longrightarrow A \), we obtain two maps

\[ f^* : A \text{Mod} \longrightarrow B \text{Mod} \quad \text{and} \quad f_! : B \text{Mod} \longrightarrow A \text{Mod} \]

corresponding respectively to the restriction and extension of scalars, and a map

\[ U_\mathcal{P}(f) : U_\mathcal{P}(B) \longrightarrow U_\mathcal{P}(A). \]

Then the previous two adjoint functors are simply the usual functors of restriction and extension for \( U_\mathcal{P}(f) \). We can also describe the free modules as follows. If \( X \) is a dg \( \Sigma \)-module, we have a coequalizer diagram

\[ \mathcal{P}(\mathcal{P}(A), X) \longrightarrow \mathcal{P}(A, X) \longrightarrow A \circ \mathcal{P} X \]
where the arrows are \( \gamma(1,1) \) and \( 1(\gamma_A,1) \) and \( A \circ_{\mathcal{P}} X \) is the free operadic \( A \)-module on \( X \), so that

\[
A \circ_{\mathcal{P}} - : \Sigma \mathfrak{dgMod} \longrightarrow \mathcal{A} \mathfrak{Mod}
\]
is left adjoint to the forgetful functor

\[
\# : \mathcal{A} \mathfrak{Mod} \longrightarrow \Sigma \mathfrak{dgMod}.
\]

Graphically, generators of \( A \circ_{\mathcal{P}} X \) correspond to corollas with their root labeled by an operation of \( \mathcal{P} \), all whose leaves are labeled by elements of \( A \) except for one, which is labeled by an element of \( X \), and we impose the relations for each \( i, l, n \in \mathbb{N} \), each pair of operations \( \mu, \nu \in \mathcal{P} \) with \( \mu \) of arity \( n \) and each \( n \)-tuple \((a_1, \ldots, a_n)\) of elements of \( A \),

\[
\mu(a_1, \ldots, a_i, \nu(a_{i+1}, \ldots, a_{i+l}), a_{i+l+1}, \ldots, a_n, x) = (\mu \circ \nu)(a_1, \ldots, a_n, x).
\]

In case \( A \) or \( \mathcal{P} \) are graded, signs will appear owing to the Koszul sign rule.

### A.2 Derivations and Kähler differentials

As before, let us fix an operad \( \mathcal{P} \), and let us also fix a \( \mathcal{P} \)-algebra \( A \). If \( M \) is an operadic \( A \)-module then a \( \mathcal{P} \)-derivation of \( M \) is a linear map \( d : A \longrightarrow \mathcal{M} \) such that \( \gamma_M(1 \circ_d \varepsilon) = \varepsilon_A \). For a fixed choice \( f : B \longrightarrow A \) of a map \( \mathcal{P} \)-algebras, we say \( d \) is \( B \)-linear whenever it vanishes on the image of \( f \).

Following [27], we write \( \text{Der}_B(A, M) \) for the complex of such derivations, which defines a functor

\[
\text{Der}_B(A, -) : \mathcal{A} \mathfrak{Mod} \longrightarrow \mathfrak{k} \mathfrak{Ch}.
\]

In particular, if \( u : A \longrightarrow U \) is a map of \( \mathcal{P} \)-algebras, then \( U \) is an operadic \( A \)-module and we can consider \( \text{Der}_B(A, U) \) the complex of \( B \)-linear derivations \( A \longrightarrow U \). We refer the reader to [39, §12.3.19] for a proof of the following:

**Proposition A.2** The functor \( \text{Der}_B(A, -) \) is representable. \( \square \)

We call the representing module the module of relative Kähler differentials and write it \( \Omega^1_{A|B} \).

Explicitly, \( \Omega^1_A \) is the coequalizer of the diagram

\[
A \circ_{\mathcal{P}} d\mathcal{P}(A) \longrightarrow A \circ_{\mathcal{P}} dA \longrightarrow \Omega^1_A
\]

so that \( \Omega^1_A \) is the free operadic \( A \)-module on a copy \( dA \) of \( A \) where we additionally impose the relations that, for each \( i, l, n \in \mathbb{N} \), each pair of operations \( \mu, \nu \in \mathcal{P} \) with \( \mu \) of arity \( n \) and each \( n \)-tuple \((a_1, \ldots, a_n)\) of elements of \( A \),

\[
\mu(a_1, \ldots, a_i, d', a_{i+l+1}, \ldots, a_n) = \sum_{t=1}^l (\mu \circ_i \nu)(a_1, \ldots, da_{i+t}, \ldots, a_n)
\]
The arrows are as follows: the uppermost arrow is induced from the map
\[ 1(1, \gamma_A) : \mathcal{P}(A, d\mathcal{P}(A)) \to \mathcal{P}(A, dA), \]
while the lowermost arrow is induced from the following three maps:

1. The arrow \( \mathcal{P}(A, d\mathcal{P}(A)) \to \mathcal{P}(A, \mathcal{P}(A, dA)) \) induced from the infinitesimal composite \( 1 \circ d : d\mathcal{P}(A) \to \mathcal{P}(A, dA) \) obtained from the isomorphism \( d : A \to dA \),
2. The arrow \( \mathcal{P}(A, \mathcal{P}(A, dA)) \to (\mathcal{P} \circ (1)) \mathcal{P}(A, dA) \) which is an inclusion and
3. The arrow \( \gamma(A_1, 1, 1) \).

The module of relative Kähler differentials \( \Omega^1_{A/B} \) is defined similarly, with the extra relation that \( dB = 0 \). It is functorial in both arguments in the following way. If we have a pair of morphisms \( B \overset{f}{\to} A \overset{g}{\to} C \) of \( \mathcal{P} \)-algebras we can consider any \( A \)-linear derivation of \( A \) as a \( B \)-linear derivation, so we get a morphism
\[ \Omega^1_{C/B} : \mathcal{P} \Omega^1_{C/A} \to \Omega^1_{C/B} \]
representing the restriction. Similarly, any \( B \)-linear derivation \( g^*d : A \to f^*M \) so we obtain a morphism
\[ \Omega^1_{g/B} : g^*\Omega^1_{A/B} \to \Omega^1_{C/B}. \]

The following lemma describes Kähler differentials and derivations of free algebras. In particular, it follows the corresponding complexes of derivations and of differentials of \( \mathcal{P} \)-algebras of the form \( (\mathcal{P}(V), d) \) are simple, and correspond to “nc-vector fields” \( X : V \to \mathcal{P}(V) \) determined on the coordinates \( v \in V \) by some vector field \( X : \partial_v \mapsto X(\partial_v) \), and to “nc-differential forms” \( f(v)dv \) where \( f(v) \in Y \) is a function on the coordinates.

**Lemma A.3** Let \( A = \mathcal{P}(V) \) be the free \( \mathcal{P} \)-algebra on \( V \). Write \( i : V \to \mathcal{P}(V) \) for the canonical inclusion. Then \( \Omega^1_A \) is canonically isomorphic to the free operadic \( X \)-module generated by \( V \), and we have isomorphisms of complexes
\[ i^* : \text{Der}(A) \to \text{hom}(V, A), \quad i_* : A \otimes V \to A \otimes_{UX} \Omega^1_A \]
that assign a derivation \( f : A \to A \) to its restriction \( fi \) and \( x \otimes v \) to the class of \( xdv \).

**Proof.** Since \( X \) is free, any derivation \( f : X \to X \) is determined by its restriction to \( V \), and \( i^* \) is a bijection. From this and the Yoneda lemma it follows that \( \Omega^1_X \) is the free left \( UX \)-module generated by \( V \), and hence that the canonical map \( X \otimes V \to X \otimes_{UX} \Omega^1_X \) is an isomorphism.

In particular, if we consider a commutative algebra \( A \) and the bar-cobar resolution \( Y = \Omega BA \), we get the following:

**Lemma A.4** There is a natural isomorphism \( A \otimes_{UX} \Omega^1_Y \to A \otimes s^{-1}BA = s^{-1}C_*(A, A) \) where the right hand side is the cyclic Hochschild complex computing Hochschild homology of \( A \) away from degree 0.
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