Non-tangential, radial and stochastic asymptotic properties of harmonic functions on trees *†

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Abstract

For a harmonic function on a tree with random walk whose transition probabilities are bounded between two constants in $(0, 1/2)$, it is known that the radial and stochastic properties of convergence, boundedness and finiteness of energy are all a.s. equivalent. We prove here that the analogous non-tangential properties are a.e. equivalent to the above ones.

We are interested in the comparison between some non-tangential asymptotic properties of harmonic functions on a tree and the corresponding radial properties, using analogous stochastic ones. We proved in a previous work [6], under a reasonable uniformity hypothesis, the almost sure equivalence between different radial and stochastic properties: convergence, boundedness and finiteness of the energy. The probabilistic-geometric methods, adapted from those we used in the setting of manifolds of negative curvature [5], were flexible and presumed to extend to the non-tangential case for trees.

A recent article [2] shows by combinatorial methods the equivalence of the three non-tangential corresponding properties in the particular case of homogeneous trees. It seems to be time to show explicitly that our methods give in a swift way the non-tangential results for general trees satisfying the uniformity hypothesis above.

We use our previous results to compare the non-tangential notions with the radial and stochastic ones: we prove on one hand that the stochastic convergence implies the non-tangential convergence in the section 3 and on the other hand that the non-tangential boundedness implies almost surely the finiteness of the non-tangential energy in the section 4. The notations are fixed in the section 1 and our main result is stated in the section 2.

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1 Setting

Let us briefly fix the notations (for details see [6]). We consider a tree \((S, A)\) i.e. a non-oriented, locally finite, connected and simply connected graph with vertices in \(S\) and edges in \(A\). We will use the usual notions of path, distance and geodesic path and note \(x \sim y\) iff \((x, y) \in A\).

We also consider a transient random walk \((X_n)_n\) on \(S\) such that the transition probability \(p(x, y) > 0\) iff \(x \sim y\). Denote by \(P_x\) the distribution of the walk starting from \(x\) and by \(p_n(x, y)\) the probability \(P_x[X_n = y]\) of reaching \(y\) from \(x\) in \(n\) steps.

The Green function \(G(x, y) = \sum_{n=0}^{\infty} p_n(x, y)\) is finite by transience. Denote by \(H(x, y)\) the probability of reaching \(y\) starting from \(x\). If \(z\) is on the geodesic path \([x, y]\), the simple connectivity implies

\[
H(x, y) = H(x, z)H(z, y) \quad \text{and} \quad G(x, y) = H(x, z)G(z, y). 
\]

(1)

If \(U \subset S\), the Green function of \(U\), defined on \(U \times U\), is the expectation of the number of times the walk starting from \(x\) hits \(y\) before exiting \(U\).

The Laplacian of a function \(f\) on \(S\) is \(\Delta f(x) = E_x[f(X_1)] - f(x)\). The function \(f\) is harmonic if \(\Delta f = 0\).

Let \(u\) be a fixed harmonic function. The stochastic energy of \(u\) is \(J^*(u) = \sum_{k=0}^{\infty} (\Delta u^2)(X_k)\) (non-negative terms). The events \(L^{**}\), \(N^{**}\) and \(J^{**}\) are defined respectively by the convergence of \((u(X_n))_n\), its boundedness and the finiteness of the stochastic energy. The Martingale theorem implies \(J^{**} \subset L^{**}\) \((P_x\text{-almost sure inclusion}) [6]\). It is known since P. Cartier [3] that geometric and Martin compactifications agree and the random walk converges almost surely to a point of the boundary \(\partial S\). The exit law starting from \(x\) is the harmonic measure \(\mu_x\) and \(\mu = (\mu_x)_x\) is a family of equivalent measures. Conditioning by Doob’s method of \(h\)-processes gives probabilities \(P^*_x\) (ending at \(\theta\)). Asymptotic events verify 0–1 law and we define sets of events \(L^* = \{\theta \in \partial S | P^*_x(L^{**}) = 1\}\), \(N^* = \{\theta \in \partial S | P^*_x(N^{**}) = 1\}\), \(J^* = \{\theta \in \partial S | P^*_x(J^{**}) = 1\}\), which determine stochastic notions of convergence, boundedness and finiteness of the energy at \(\theta \in \partial S\). For \(\theta \in L^*, \lim_{n \to \infty} u(X_n) = P^*_x\text{-a.s.}\) constant (independent from \(x\)) and called the stochastic limit at \(\theta\).

Fix a base point \(o\). For \(\theta \in \partial S\), \(\gamma_\theta\) is the geodesic ray from \(o\) to \(\theta\) and for \(c \in \mathbb{N}\), \(\Gamma^c_\theta = \{y \in S | d(y, \gamma_\theta) \leq c\}\) is a non-tangential tube. Let \(u\) be a harmonic function. For \(c \in \mathbb{N}\), its \(c\)-non-tangential energy at \(\theta\) is \(J^c_\theta(u) = \sum_{y \in \Gamma^c_\theta} \Delta u^2(y)\) and its radial energy at \(\theta\) is \(J^0_\theta(u) = J^0_\theta(u) = \sum_{k=0}^{\infty} \Delta u^2(\gamma_\theta(k))\). There is radial convergence, boundedness or finiteness of the energy depending wether \((u(\gamma_\theta(n)))_n\) converges, is bounded or has finite radial energy. There is non-tangential convergence of \(u\) at \(\theta\) if for all \(c \in \mathbb{N}\), \(u(y)\) has a limit when \(y\) goes to \(\theta\) staying in \(\Gamma^c_\theta\). There is non-tangential boundedness (resp. finiteness of the energy) if for all \(c \in \mathbb{N}\), \(u\) is bounded on \(\Gamma^c_\theta\) (resp. \(J^c_\theta(u) < +\infty\)).
2 Main result

We now suppose \((\mathcal{H})\): \(\exists \varepsilon > 0, \exists \eta > 0, \forall x \sim y, \varepsilon \leq p(x, y) \leq \frac{1}{2} - \eta\), a discrete analogue of the pinched curvature for manifolds. It also forces at least three neighbors for each vertex, and ensures transience. We proved in \([6]\):

**Theorem 2.1** For a harmonic function \(u\) on a tree with random walk satisfying \((\mathcal{H})\), the notions of radial convergence, radial boundedness, radial finiteness of the energy, stochastic convergence, stochastic boundedness, stochastic finiteness of the energy, are \(\mu\)-almost equivalent.

We prove here the following theorem:

**Theorem 2.2** Under the same hypotheses, the notions of non-tangential convergence, non-tangential boundedness and non-tangential finiteness of the energy are \(\mu\)-almost equivalent to the notions above.

Considering the trivial implications, it is sufficient to prove that stochastic convergence implies non-tangential convergence and non-tangential boundedness implies almost surely non-tangential finiteness of the energy.

3 Stochastic implies NT convergence

The first implication needs the following lemma due to A. Ancona in a general setting \([1]\), but easily proved here by simple connectivity:

**Lemma 3.1** If \((x_n)_n\) is a sequence converging non-tangentially to \(\theta \in \partial S\), the walk hits \(P_\theta^\mu\)-a.s. infinitely many \(x_n\).

Let us see how this lemma helps. Assume that the harmonic function \(u\) has a stochastic limit \(l \in \mathbb{R}\) at \(\theta\) but does not converge non-tangentially towards \(l\) at \(\theta\). There exists \(\delta > 0\) and a sequence \((x_n)_n\) converging non-tangentially to \(\theta\) such that \(|u(x_n) - l| \geq \delta\) for all \(n\). As the random walk \((X_k)_k\) hits \(P_\theta^\mu\)-a.s. infinitely many \(x_n\) by the lemma, one can extract a subsequence \((X_{k_j})_j\) such that \(|u(X_{k_j}) - l| \geq \delta\) for all \(j\). Hence, \(P_\theta^\mu\)-almost surely, the function \(u\) does not converge towards \(l\) along \((X_{k_j})_j\) which leads to a contradiction.

Let us now prove the lemma. Recall that the principle of the method of Doob’s \(h\)-processes is to consider a new Markov chain defined by \(p^\theta(x, y) = \frac{K_\theta(y)}{K_\theta(x)} p(x, y)\) where the Martin kernel \(K_\theta(x)\) is defined as \(\lim_{y \to \theta} \frac{G(x, y)}{G(y, \theta)}\) (see for example \([1]\)). This formula leads to analogous formulae for the \(p_\theta^\mu\) and the associated functions \(H^\theta\) and \(G^\theta\). Consider for a fixed \(n\) the projection \(y_n\) of \(x_n\) on the geodesic ray \(\gamma_\theta\) (see \([1]\)). As the random walk starting from \(o\) and conditioned to end at \(\theta\) hits almost surely \(y_n\) due to the tree structure, the strong Markov property gives \(H^\theta(o, x_n) = H^\theta(y_n, x_n) = \frac{K_\theta(x_n)}{K_\theta(y_n)} H(y_n, x_n)\). By definition of the Martin kernel, \(\frac{K_\theta(x_n)}{K_\theta(y_n)} = \lim_{y \to \theta} \frac{G(x_n, y)}{G(y, \theta)}\) and \(G(x_n, y) = H(x_n, y_n)G(y_n, y)\) as soon as \(y_n \in [x_n, y]\), so \(H^\theta(o, x_n) = H(x_n, y_n)H(y_n, x_n)\). The distance between \(x_n\) and
$y_n$ is bounded as $(x_n)_n$ converges non-tangentially to $\theta$, hence the last product is bounded from below by a constant $C > 0$ using $(H)$. By Fatou’s lemma, the probability conditioned to end at $\theta$ of hitting infinitely many $x_n$ is not smaller than $C$ and the asymptotic 0-1 law ensures that it equals 1, which completes the lemma’s proof.

## 4 NT boundedness implies finite NT energy

Denoting $N_c = \{\theta \in \partial S | \sup_{x \in \Gamma^\theta} |u| < +\infty\}$ and $J_c = \{\theta \in \partial S | J^\theta(u) < +\infty\}$, we will show that for all $c \in \mathbb{N}$, $N_{c+1}^N \subset J_c$, which will give the wanted result by monotonous intersection. Let us write $N_{c+1}^N = \bigcup_{N \in \mathbb{N}} N_{c+1}^N$, where

$$N_{c+1}^N = \left\{ \theta \in \partial S \mid \sup_{x \in \Gamma^\theta_{c+1}} |u| \leq N \right\}.$$

By countability it is sufficient to prove that for all $N$, $N_{c+1}^N \subset J_c$. Let us fix $N \in \mathbb{N}$. Denote $\Gamma = \bigcup_{\theta \in N_{c+1}^N} \Gamma_c^\theta$ and $\tau$ the exit time from $\Gamma$. As

$$M_n = u^2(X_n) - \sum_{k=0}^{n-1} \Delta u^2(X_k)$$

is a martingale (see [6]), Doob’s stopping time theorem for the bounded exit time $\tau \wedge n$ gives $E_\theta [M_{\tau \wedge n}] = E_\theta [M_0] = u^2(\theta) \geq 0$, hence

$$E_\theta \left[ \sum_{k=0}^{\tau \wedge n-1} \Delta u^2(X_k) \right] \leq E_\theta \left[ u^2(X_{\tau \wedge n}) \right].$$

As $X_{\tau \wedge n}$ is at distance at most 1 from $\Gamma$, it lies in a tube $\Gamma^\theta_{c+1}$ where $\theta \in N_{c+1}^N$ and $|u(X_{\tau \wedge n})| \leq N$. When $n$ goes to $\infty$, monotonous convergence ($\Delta u^2 \geq 0$) and the desintegration formula (see [6]) give then, for $\mu$-almost all $\theta \in \partial S$,

$$E_\theta^\theta \left[ \sum_{k=0}^{\tau-1} \Delta u^2(X_k) \right] < +\infty.$$

Let us use a conditioned version of formula 2 from [6], which will be proved later:

**Lemma 4.1** For a function $\varphi \geq 0$ on $\Gamma$ and $\tau$ the exit time of $\Gamma$,

$$E_\theta^\theta \left[ \sum_{k=0}^{\tau-1} \varphi(X_k) \right] = \sum_{y \in \Gamma} \varphi(y) G_\Gamma(\theta, y) K_\theta(y).$$
This lemma implies that for $\mu$-almost all $\theta \in \partial S$, $\sum_{y \in \Gamma} \Delta u^2(y) G_{\Gamma}(o, y) K_{\theta}(y)$ is finite. In order to get an energy, we will show that $G_{\Gamma}(o, y) K_{\theta}(y)$ is bounded from below using the two following lemmas. The first one is due to A. Ancona [1] but has a very simple proof in the present context of trees. The second one enables comparison between $G_{\Gamma}$ and $G$.

**Lemma 4.2** For all $c \in \mathbb{N}$, $\exists \alpha > 0$, $\forall \theta \in \partial S$, $\forall y \in \Gamma_{c}^\theta$, $G(o, y) K_{\theta}(y) \geq \alpha$.

**Lemma 4.3** For $U \subset S$ containing $\Gamma_{c}^\theta$ and $\tau$ the exit time of $U$,

$$\lim_{y \in \Gamma_{c}^\theta, y \rightarrow \theta} \frac{G_{U}(o, y)}{G(o, y)} = P_{o}[\tau = +\infty].$$

By lemma 4.2 for $\mu$-almost all $\theta \in \mathcal{N}_{c+1}^N$,

$$\sum_{y \in \Gamma_{c}^\theta} \Delta u^2(y) \frac{G_{\Gamma}(o, y)}{G(o, y)} < +\infty.$$

If we show that for $\mu$-almost all $\theta \in \mathcal{N}_{c+1}^N$, $P_{o}[\tau = +\infty] > 0$, lemma 4.3 gives $\mathcal{N}_{c+1}^N \subset \mathcal{J}_{c}$. The proof of that fact is the same as in the analogous radial proof [6] which completes the theorem’s proof.

Let us now prove the lemmas. Concerning lemma 4.1 using Fubini,

$$E_{o}^{\theta} \left[ \sum_{k=0}^{\tau-1} \varphi(X_k) \right] = \sum_{k=0}^{\infty} E_{o}^{\theta} \left[ \varphi(X_k) 1_{(k<\tau)} \right].$$

The random variable $\varphi(X_k) 1_{(k<\tau)}$ being measurable with respect to the $\sigma$-algebra generated by $(X_i)_{i \leq k}$ (see [6]) and using formula 2 from [6], the expectation above equals

$$\sum_{k=0}^{\infty} E_{o} \left[ \varphi(X_k) 1_{(k<\tau)} K_{\theta}(X_k) \right] = E_{o} \left[ \sum_{k=0}^{\infty} \varphi(X_k) 1_{(k<\tau)} K_{\theta}(X_k) \right]$$

$$= \sum_{y \in \Gamma} \varphi(y) G_{\Gamma}(o, y) K_{\theta}(y),$$

which finishes the proof of lemma 4.1.

Let us prove lemma 4.2. Denote $\pi(y)$ the projection of $y$ on $\gamma_{\theta}$ (see [Q]) and remark that for $z \in (\pi(y), \theta)$, $G(o, z) = H(o, \pi(y)) G(\pi(y), z)$ and $G(y, z) = H(y, \pi(y)) G(\pi(y), z)$ by formula [11]. Hence $G(y, z) / G(o, z) = H(y, \pi(y)) / H(o, \pi(y))$ does not depend anymore on $z$ and its limit when $z$ goes to $\theta$ is then $K_{\theta}(y) = H(y, \pi(y)) / H(o, \pi(y))$. By formula [11],

$$G(o, y) K_{\theta}(y) = H(y, \pi(y)) \frac{G(o, y)}{H(o, \pi(y))} = H(y, \pi(y)) H(\pi(y), y) G(y, y).$$
But \( G(y, y) \geq p_2(y, y) \geq 3\varepsilon^2 \) and \( H(y, \pi(y))H(\pi(y), y) \geq \varepsilon^2 \) by (H) and \( d(y, \pi(y)) \leq c \), which finishes the proof of lemma 4.2.

Let us prove lemma 4.3:

\[
G_U(o, y) = G(o, y) - E_o[G(X_\tau, y)1_{(\tau < +\infty)}]
\]

\[
= G(o, y) \left(1 - E_o \left[ \frac{G(X_\tau, y)}{G(o, y)} 1_{(\tau < +\infty)} \right] \right)
\]

and by definition of Martin’s kernel, if we could switch the limit and expectation, by a conditioning formula \[6\],

\[
\lim_{y \in \Gamma^o, \gamma \to \theta} \frac{G_U(o, y)}{G(o, y)} = 1 - E_o[K_\theta(X_\tau)1_{(\tau < +\infty)}] = P^o_\theta[\tau = +\infty].
\]

We now justify that inversion by Lebesgue’s theorem. The idea is to bound, when \( \tau \) is finite, \( \frac{G(X_\tau, y)}{G(o, y)} \) by a multiple of \( K_\theta(X_\tau) \). We compare for that purpose \( G(X_\tau, y) \) with \( K_\theta(X_\tau) \). Denote again by \( \pi \) the projection function on \( \gamma_\theta \). We distinguish two cases

If \( \pi(X_\tau) \in [o, \pi(y)] \), \( \frac{G(X_\tau, y)}{K_\theta(X_\tau)} = \frac{G(o, y)}{K_\theta(o)} = G(o, y) \), by formula \[1\] and the remark that this formula also implies by definition of \( K_\theta \) and by taking the limit that \( K_\theta(X_\tau) = H(X_\tau, \pi(X_\tau))K_\theta(\pi(X_\tau)) \) and \( K_\theta(o) = H(o, \pi(X_\tau))K_\theta(\pi(X_\tau)) \).

If \( \pi(X_\tau) \not\in [o, \pi(y)] \), again \( \frac{G(X_\tau, y)}{K_\theta(X_\tau)} = \frac{G(o, y)}{K_\theta(o)} \). We also have, by definition and formula \[1\] \( K_\theta(\pi(X_\tau)) = (H(o, \pi(X_\tau)))^{-1} \), hence the quotient above equals \( H(o, \pi(X_\tau))G(\pi(X_\tau), y) = H(o, \pi(y))H(\pi(y), \pi(X_\tau))G(\pi(X_\tau), y) \). We know that \( G \) is bounded (see \[7\] \[9\]) and \( H \) is a probability, so it just remains to compare \( H(o, \pi(y)) \) with \( G(o, y) \). But \( \frac{H(o, \pi(y))}{G(o, y)} = (G(\pi(y), y))^{-1} \) and \( \frac{1}{3\varepsilon^2} \) is bounded by \( \frac{1}{3\varepsilon^2} \).

Merging the two cases gives a constant \( \beta \) such that \( \frac{G(X_\tau, y)}{K_\theta(X_\tau)} \leq \beta G(o, y) \), which enables to use Lebesgue’s theorem and completes the proof of lemma 4.3.

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