OPERADS AND CATEGORIES IN REPRESENTATION STABILITY THEORY

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Abstract. We make precise the relation between operads and some of the categories that have appeared in representation stability theory. We give new examples of such categories and show that these categories are quasi-Gröbner. We also show that several operadic categories of Batanin–Markl that encode generalized operads are quasi-Gröbner.

INTRODUCTION

Several quasi-Gröbner categories that have appeared in representation stability theory are related to operads or operad-like structures. The simplest examples include the category $\mathbf{OI}$ of ordered finite injections and the category $\mathbf{FS}^{op}$, the opposite of the category of finite surjections. These categories arise in a similar, but not exactly the same, way from operads: the category $\mathbf{OI}$ is related to the operad $\mathbf{As}$ of semigroups, while the category $\mathbf{FS}^{op}$ is related to the operad $\mathbf{Com}$ of commutative semigroups. At first glance this relation may seem superficial, however properties of an operad can be used in the study of modules over the corresponding category. For example the Koszulity property of the operad $\mathbf{Com}$ has been used in the study of modules over $\mathbf{FS}^{op}$ in [Tos19].

We will use six construction of categories in this work. These appear in pairs connected by discrete opfibrations. Discrete opfibrations relate Gröbner and quasi-Gröbner properties of their étale and base categories as described in Section 1.

The six constructions of categories from an operad $P$ are: the PROP of $P$; the category $\mathcal{C}(P)$; the universal enveloping category $\mathcal{U}_{Op}(P)$ (SGK94); the twisted arrow category $\mathcal{Tw}_{Op}(P)$ (Bur22, Hon20); the enveloping category $\mathcal{U}(A)$ of a $P$-algebra $A$, where often $A$ is the terminal $P$-algebra; and the twisted arrow category $\mathcal{Tw}(A)$ of a $P$-algebra $A$. All these constructions arise from essentially the same idea. In Section 2 we give definitions of these constructions, show how familiar categories fit into this picture, and prove quasi-Gröbner property in several new cases, including several examples of operadic categories of Batanin–Markl ([BM15]), and the category $\mathbf{ncCS}$ that might serve instead of the category $\mathbf{FS}^{op}$ as the indexing category of modules arising from modular operads and analogous to the module considered in [Tos19].

In Section 3 we study categories of the form $\mathcal{C}(P)$ with $P$ an operad encoding a generalized operad. These categories are related to, but different from the categories of graphs that appeared in representation stability theory. The Gröbner category $\mathcal{PT}$ from [Bar15] is the full subcategory of $\mathcal{C}(pOp)$ on trees without input leaves, where $pOp$ is the operad of planar operads. The category $\mathcal{G}_{op}$ from [PR19] is related to the category $\mathcal{C}(mOp_{(g,n)})$, where $mOp_{(g,n)}$ is the operad of genus-graded modular operads, however the latter category admits morphisms that change the genus of
the underlying graph. In Section 3 we show that most of these categories $\mathcal{C}(P)$ are quasi-Gröbner.

Initially there was expectation that the property of being quasi-Gröbner for a category $\mathcal{C}(P)$ can be deduced from familiar properties of an operad $P$. For example, the operad $pOp$ admits quadratic Gröbner basis with generators $X$ such that if one orders the elements $p$ of $pOp$ via the order on their canonical representatives (the smallest preimages of $p$ in the free operad $F(X)$), then this order is compatible with operadic composition, which in turn implies property (G1) for the category $\mathcal{C}(pOp)$, and moreover suggests a way to define a lingual structure on $\mathcal{C}(pOp)$. However, it seems that in general familiar properties of operads are not relevant to the proofs of the quasi-Gröbner property of categories $\mathcal{C}(P)$.

**Prerequirements.** We will use [SS17] as the reference for the basic notions of representation stability theory, of which we will use only the properties (F), (S), (G1) and (G2). We will assume that the reader is familiar with the definition and basic examples of operads. We only consider (coloured) operads in the category of sets.

1. **Key lemmas**

**Lemma 1.1.** Let $G : \mathcal{C} \to \mathcal{D}$ be a discrete opfibration surjective on objects. If $\mathcal{C}$ is quasi-Gröbner, then $\mathcal{D}$ is quasi-Gröbner.

**Proof.** Discrete opfibrations that are surjective on objects satisfy property (F), with the identity morphisms being the morphisms $f_1$ that ensure property (F). □

**Lemma 1.2.** Let $G : \mathcal{C} \to \mathcal{D}$ be a discrete opfibration with finite fibers, surjective on objects. If $\mathcal{C}$ is quasi-Gröbner, then $\mathcal{D}$ is quasi-Gröbner.

**Proof.** The morphisms $f_1 : x \to G(y_i)$ that ensure property (F) are the identity morphisms, with $y_i$ being all the objects over $x$. □

**Lemma 1.3.** Let $\mathcal{D}$ be a directed category that satisfies (G1) and let $G : \mathcal{C} \to \mathcal{D}$ be a faithful functor. Then $\mathcal{C}$ satisfies (G1).

**Proof.** Let $c$ be an object of $\mathcal{C}$ and $\prec$ be the well-order on $G(c)/\mathcal{D}$ that ensures the property (G1). For every morphism $h$ from $G(c)$ in $\mathcal{D}$ choose any well-order $\prec_h$ on the set $G^{-1}(c, h)$ of morphisms $f$ from $c$ such that $G(f) = h$. Define the order $\prec'$ on $\mathcal{C}$ so that $f \prec' f'$ if $G(f) \prec G(f')$ or if $G(f) = G(f')$ and $f \prec_{G(f)} f'$. This is a well-order.

Let $f$ and $f'$ be two morphisms from $c$ with the same target, and $g$ be a morphism from this target. If $f \prec' f'$, then by faithfulness $G(f) \neq G(f')$, and thus $G(f) \prec G(f')$. By admissibility of $\prec$ we have $G(g \circ f) \prec G(g \circ f')$, and thus $g \circ f \prec' g \circ f'$. The order $\prec'$ is admissible. □

**Lemma 1.4.** Let $G : \mathcal{C} \to \mathcal{D}$ be a discrete opfibration. If $\mathcal{D}$ is Gröbner, then $\mathcal{C}$ is Gröbner.

**Proof.** Any slice category $x/\mathcal{C}$ is isomorphic to the slice category $x/\mathcal{D}$. If $\mathcal{D}$ satisfies property (G2) then $\mathcal{C}$ satisfies property (G2). Discrete opfibrations are faithful, and property (G1) follows from Lemma 1.3. □

**Lemma 1.5.** Let $G : \mathcal{A} \to \mathcal{B}$ be a discrete opfibration. If $\mathcal{B}$ is quasi-Gröbner, then $\mathcal{A}$ is quasi-Gröbner.

**Proof.** Since $\mathcal{B}$ is quasi-Gröbner, there is essentially surjective functor $\Phi : \mathcal{D} \to \mathcal{B}$ that satisfies property (F), where $\mathcal{D}$ is Gröbner. Let $\mathcal{C}$ be the pullback of $\Phi : \mathcal{D} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{B}$, and $\Phi' : \mathcal{C} \to \mathcal{A}$ be the functor in the pullback square. Discrete
opfibrations are stable under pullbacks, thus \( \mathcal{C} \rightarrow \mathcal{D} \) is a discrete opfibration, and \( \mathcal{C} \) is Gröbner.

Recall that pullbacks of categories are formed by taking fibered products of sets: the objects (and the morphisms) in \( \mathcal{C} \) are the pairs of objects (respectively the pairs of morphisms) in \( \mathcal{A} \) and \( \mathcal{D} \) that have the same image in \( \mathcal{B} \), and the functors \( \mathcal{C} \rightarrow \mathcal{A} \) and \( \mathcal{C} \rightarrow \mathcal{D} \) are the projection maps.

To show that \( \Phi' : \mathcal{C} \rightarrow \mathcal{A} \) is essentially surjective, let \( A \) be an object in \( \mathcal{A} \). Since \( \Phi : \mathcal{D} \rightarrow \mathcal{B} \) is essentially surjective, there is an object \( D \) in \( \mathcal{D} \) such that there is an isomorphism \( G(A) \rightarrow \Phi(D) \) in \( \mathcal{B} \). Let \( A \rightarrow A' \) be the lift of this isomorphism to \( \mathcal{A} \). Lifts of isomorphisms are isomorphisms, thus \( A' \) is isomorphic to \( A \). The pair \( (A', D) \) is an object of \( \mathcal{C} \), and its image under \( \Psi' \) is isomorphic to \( A \). Thus \( \Phi' \) is essentially surjective.

Next we prove the property (F) for \( \Phi' \). Let \( A \) be an object in \( \mathcal{A} \). The property (F) of \( \Phi \) with respect to \( G(A) \) gives objects \( D_i \) in \( \mathcal{D} \) and morphisms \( f_i : G(A) \rightarrow \Phi(D_i) \) in \( \mathcal{B} \). Let \( f'_i : A \rightarrow A_i \) be the lifts of \( f_i \) to \( \mathcal{A} \). The pairs \( (A_i, D_i) \) are objects of \( \mathcal{C} \). We will show that \( f'_i \) and \( (A_i, D_i) \) ensure the property (F) of \( \Phi' \) with respect to \( A \).

Let \( (A', D') \) be an object of \( \mathcal{C} \), and \( A \rightarrow A' = \Phi'((A', D')) \) be a morphism in \( \mathcal{A} \). We have to find a nice factorization of \( A \rightarrow A' \) through some \( f'_i \). Let \( D_i \rightarrow D' \) be the morphism in \( \mathcal{D} \) given by the property (F) of \( \Phi \) with respect to \( G(A \rightarrow A') \) and \( D' \), i.e. such that \( G(A \rightarrow A') = \Phi(D_i \rightarrow D') \circ f_i \). Let \( (A_i, D_1) \rightarrow (X, D') \) be the lift of \( D_i \rightarrow D' \) to \( \mathcal{C} \). Opfibration property for \( \mathcal{A} \rightarrow \mathcal{B} \) ensures that \( X = A' \) and that \( (A \rightarrow A') = \Psi'((A_i, D_1) \rightarrow (A', D') \circ f'_i) \).

\[\text{□}\]

2. Categories from operads

2.1. The general construction. All categories in this work stem from the following two constructions, described respectively in [GK94] and [Bur22].

**Definition 2.1.** Let \( Q \) be a \( C \)-coloured operad and \( A \) be a \( Q \)-algebra. The universal enveloping category \( U_Q(A) \) of \( A \) has the colours of \( Q \) as its object. A morphism in \( U_Q(A) \) is an equivalence class of expressions \( p(c_1, a_2, \ldots, a_n) \) where \( p \in Q(c_1, \ldots, c_n; a_0) \) for some \( c_i \in C \) and \( a_i \in A(c_i) \). The equivalence is generated by relations \( (p \circ q)(c_1, a_2, \ldots, a_{n+m-1}) \sim p(c_1, a_2, \ldots, a_{i-1}, q(a_{i+1}), \ldots, a_{i+m}) \), where \( 2 \leq i \leq n \) and \( q \) is composable with \( p \), and by relations \( p^\sigma(c_1, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) \sim p(c_1, a_2, \ldots, a_n) \) for all permutations \( \sigma \in S_n \) that preserve \( 1 \). The source of a morphism \( p(c_1, a_2, \ldots, a_n) \) is \( c_1 \), and the target is \( c_0 \), where \( p \) is in \( Q(c_1, \ldots, c_n; c_0) \).

**Definition 2.2.** Let \( Q \) be a \( C \)-coloured operad and \( A \) be a \( Q \)-algebra. The twisted arrow category \( Tw_Q(A) \) of \( A \) has the elements of \( A \) as its objects. A morphism in \( Tw_Q(A) \) is an equivalence class of expressions \( p(a_1, a_2, \ldots, a_n) \) where \( p \in Q(c_1, \ldots, c_n; a_0) \) for some \( c_i \in C \) and \( a_i \in A(c_i) \). The equivalence is generated by relations \( (p \circ q)(a_1, a_2, \ldots, a_{n+m-1}) \sim p(a_1, a_2, \ldots, a_{i-1}, q(a_{i+1}), \ldots, a_{i+m}) \), where \( 2 \leq i \leq n \) and \( q \) is composable with \( p \), and by relations \( p^\sigma(a_1, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) \sim p(a_1, a_2, \ldots, a_n) \) for all permutations \( \sigma \in S_n \) that preserve \( 1 \). The source of a morphism \( p(a_1, a_2, \ldots, a_n) \) is \( a_1 \), and the target is \( p(a_1, a_2, \ldots, a_n) \) as seen as an element of \( A(c_0) \).

In both cases the composition of morphisms is computed via operadic substitution. The functor \( G : Tw_Q(A) \rightarrow U_Q(A) \) that sends an element of \( A \) to the corresponding colour of \( Q \) is a discrete opfibration. If \( A \) is the terminal \( Q \)-algebra then \( Tw_Q(A) = U_Q(A) \).

2.2. Cobordisms. An example of the construction \( Tw \) is the category \( CS \) whose objects are connected orientable surfaces with indexed boundaries and morphisms are cobordisms between boundaries. The subcategory \( ncCS \) of \( CS \) may serve
instead of $\text{FS}^{\text{op}}$ as the indexing category of modules related to modular operads, e.g. of the module $H_i(M_{g,n})$ considered in [Tos19].

**Proposition 2.3.** Let $\text{mOp}$ be the operad whose algebras are modular operads, and $\text{uCom}_m$ be the modular envelope of the terminal cyclic operad. Let $\text{Cob}$ be the category with finite sequences of circles as objects and orientable cobordisms between disjoint unions of indexed circles as morphisms. The category $\text{U}_m \text{Op}(\text{uCom}_m)$ is isomorphic to the subcategory $\text{Cob}'$ of $\text{Cob}$ that contains all the morphisms of $\text{Cob}$ except the non-trivial morphisms from the empty sequence $\emptyset$. The category $\text{Tw}_m \text{Op}(\text{uCom}_m)$ is isomorphic to the category $\text{CS}$, the full subcategory of the slice category $\emptyset/\text{Cob}$ on non-empty connected cobordisms from the empty set. The projection functor $G : \text{CS} \rightarrow \text{Cob}'$ is a discrete opfibration surjective on objects.

**Proof.** For the definition of $\text{mOp}$ and for the proof see Section 1.1 and Proposition 2.33 in [Bur22]. □

**Proposition 2.4.** The category $\text{CS}$, and thus the category $\text{Cob}'$, are not quasi-Gröbner.

**Proof.** Let $f_i$ be the sequence of morphisms in $\text{CS}$ from the hemisphere to the closed surface of genus $i$. Let $M$ be the module over $\text{CS}$ equal to $\mathbb{Q}$ on the hemisphere and on the closed surfaces, and to 0 elsewhere, with maps $\mathbb{Q} \rightarrow \mathbb{Q}$ being the identity maps. This module is finitely generated (by the hemisphere). The submodule $N$ of $M$ that is equal to 0 on the hemisphere and to $\mathbb{Q}$ on the closed surfaces is not finitely generated. □

The proof shows that any reasonable quasi-Gröbner subcategory of $\text{CS}$ consists of morphisms $f$ such that the intersection of the target of $f$ and of the boundary of any connected component of the cobordism $f$ is non-trivial. In particular, a reasonable quasi-Gröbner subcategory of $\text{CS}$ cannot encode the module structure maps $M_{g,n} \rightarrow M_{g+1,n-2}$.

**Definition 2.5.** The categories $\text{ncCob}'$ and $\text{ncCS}$ are the subcategories of $\text{Cob}'$ and $\text{CS}$ on all morphisms such that the target boundary of each connected component of the corresponding cobordism is non-trivial.

To prove that these categories are quasi-Gröbner we use the following category.

**Definition 2.6.** The category $\text{gOS}$ of graded ordered surjections has the sets $\underline{n} = \{1, \ldots, n\}$ as objects for $n \geq 0$. A morphism $f : \underline{n} \rightarrow \underline{m}$ in $\text{gOS}$ is a surjective map $f : \underline{n} \rightarrow \underline{m}$ such that $\min f^{-1}(i) < \min f^{-1}(j)$ for all $i < j$ in $\underline{m}$, together with a map $g_f : \underline{m} \rightarrow \mathbb{N}$ called grading. Composition $h \circ f$ of $f : \underline{n} \rightarrow \underline{m}$ and $h : \underline{m} \rightarrow \underline{k}$ is given by the composition of set-maps and by the grading $g_{h \circ f}(i) = g_h(i) + \sum_{j \in h^{-1}(i)} g_f(j)$.

**Proposition 2.7.** The category $\text{gOS}^{\text{op}}$ is Gröbner.

**Proof.** The category $\text{gOS}^{\text{op}}$ is directed. Let $U : \text{gOS}^{\text{op}} \rightarrow \text{OS}^{\text{op}}$ be the forgetful functor and $\prec$ be the admissible order on $\text{OS}^{\text{op}}$. Define the ordering on the slice categories of $\text{gOS}^{\text{op}}$ so that $f \prec h$ if $U(f) \prec U(h)$ or $U(f) = U(h)$ and for some $i$ we have $g_f(j) = g_h(j)$ for all $j < i$, and $g_f(i) < g_h(i)$. This is an admissible order, and property (G1) holds.

To show that property (G2) holds take a sequence of morphisms $f_1, f_2, \ldots$ in $\text{gOS}^{\text{op}}$ with the same source. Since (G2) holds for $\text{OS}^{\text{op}}$ we can take an infinite subsequence $f_{i_1}, f_{i_2}, \ldots$ of this sequence so that $U(f_{i_1}), U(f_{i_2}), \ldots$ is non-decreasing. Then we can further take subsequences of $f_{i_1}, f_{i_2}, \ldots$ so that the value of the grading on the first element of the source is non-decreasing, then the value of the grading
on the second element is non-decreasing, and so on until the value on the last element of the source is non-decreasing. This gives an infinite increasing subsequence in the order $\leq$, i.e. property (G2) holds. □

**Proposition 2.8.** The categories $\text{ncCS}$ and $\text{ncCob}$ are quasi-Gröbner.

**Proof.** The functor $\text{ncCS} \to \text{ncCob}$ is a discrete opfibration, thus if $\text{ncCob}$ is quasi-Gröbner, then $\text{ncCS}$ is quasi-Gröbner.

Let $\Phi : \text{gOS}^{op} \to \text{ncCob}$ be the functor that sends the opposite of $f : n \to m$ to the cobordism that for all $i$ connects the $i$-th circle with the circles in $f^{-1}(i)$ by the surface of genus $g_f(i)$. The functor $\Phi$ is surjective on objects. Let $x$ be an object in $\text{ncCob}$ and let $f_i$ be all the morphisms in $\text{ncCob}$ from $x$ such that for each connected component of a cobordism $f_i$ its target boundary is a circle. Any morphism from $x$ factors uniquely as a composition of some $f_i$ with a morphism in the image of $\Phi$. The morphisms $f_i$ ensure that $\Phi$ satisfies property (F). □

### 2.3. The four constructions.

A $C$-colored operad $P$ can be seen as an algebra over the operad $s\text{Op}_C$ whose algebras are $C$-coloured operads. This operad is described in [BM07, 1.5.6]. This leads to the following constructions.

**Definition 2.9.** Let $P$ be an operad. The twisted arrow category $\text{Tw}(P) = \text{Tw}_{s\text{Op}}(P)$ of $P$ ([Bar22] [Hoa20]) has the operations of $P$ as its objects. Morphisms are represented by planar rooted trees with half-edges. A morphism $f$ in $\text{Tw}(P)$ from an operation $p$ of arity $n$ to an operation of arity $m$ corresponds to unique tree of height 3 (or height 2 if $n = 0$), with $m$ input leaves, with exactly one middle vertex, which is marked by the source $p$ of $f$, with the remaining vertices marked by some operations $q_0, q_1, \ldots, q_n$ from $P$, with the number of input edges of each vertex equal to the arity of the operation that marks this vertex, with half-edges coloured by the corresponding colours of operations so that two halves of the same edge have the same colour, with the lower vertex connected to the middle vertex by its first input edge, with leaves indexed from 1 to $m$ so that for each vertex the indices of leaves above it increase in planar order. The target of a morphism is computed by the evaluation of the corresponding tree. Composition $f \circ g$ of trees $f$ and $g$ is computed by first grafting for all $i$ the $i$-th upper vertex of $f$ into the leaf of $g$ indexed by $i$, grafting the root of $g$ to the first input edge of the lower vertex of $f$, which produces a tree of height 5, and then evaluating the maximal subtrees of this tree that do not contain the middle vertex. See Figure [II] for example.
Definition 2.10. The universal enveloping category $U(P) = U_{sOp}(P)$ of a $C$-coloured operad $P$ has tuples $(c_0, \ldots, c_n)$ of elements of $C$ as objects. Its morphisms correspond to the same trees as above, except the middle vertex is not marked by an operation, and in a morphism from $(c_0, \ldots, c_n)$ for all $i$ the colour of the $i$-th edge adjacent to the middle vertex is $c_i$, where the 0-th edge is the output edge.

The category $U(P)$ first appears in [PRE14] as the opposite of the category of pointed operators of $P$, i.e. as the category $(\Gamma^+_P)^{op}$.

Definition 2.11. The category $C(P)$ from [BDBW18] that we will denote by $C(P)$ is the wide subcategory of $Tw(P)$ on morphisms such that the lower vertex is marked by an identity operation. These morphisms can be seen as trees of height 2 with the root vertex marked by the source.

Definition 2.12. The category $PROP(P)$ of an operad $P$ is the wide subcategory of $U(P)$ on morphisms such that the lower vertex is marked by an identity operation. It is (the opposite of) the PROP obtained in the usual way from an operad $P$.

These categories form the following commutative diagram.

$$
\begin{array}{c}
C(P) \\ \downarrow \\
PROP(P) \\
\end{array}
\quad
\begin{array}{c}
Tw(P) \\ \downarrow \\
U(P) \\
\end{array}
$$

Lemma 2.13. The functors $G : C(P) \to PROP(P)$ and $G : Tw(P) \to U(P)$ are discrete opfibrations.

Corollary 2.14. For any set-operad $P$ if $U(P)$ is quasi-Gr"obner then $Tw(P)$ is quasi-Gr"obner. If $PROP(P)$ is quasi-Gr"obner then $C(P)$ is quasi-Gr"obner. If $Tw(P) \to U(P)$ is surjective on objects, then the opposite implications hold.

In general one should not expect the categories $Tw(P)$ and $U(P)$ to be quasi-Gr"obner even for reasonable operads $P$.

Proposition 2.15. The twisted arrow category $Tw(sOp)$ of the operad of single-coloured operads, equivalently the Moerdijk–Weiss category $\Omega$, is not quasi-Gr"obner.

Proof. For $i \geq 3$ let operations $p_i$ be the trees with two vertices, with the root vertex having 2 input edges, with the second vertex attached to the first input edge of the root vertex, with the second vertex having $i$ input edges. Let $M$ be the $\mathbb{Q}$-module over $Tw(sOp)$ generated by $id_2$, an object of $Tw(sOp)$. Up to automorphisms, there is only one morphism $id_2 \to p_i$ for all $i$, and there is no operation $q$ with morphisms $q \to p_i$ and $q \to p_j$ for $i \neq j$. Let $N$ be the submodule of $M$ such that $N(p) = M(p)$ if $p$ has at least 3 leaves and $N(p) = 0$ otherwise. This module is not finitely generated: $N(p_i) \neq 0$ for all $i$, and these cannot be generated by a finite number of objects. Thus $\Omega$ is not quasi-Gr"obner. \hfill $\square$

Similar reasoning shows that the twisted arrow categories (and thus the universal enveloping categories) of the operads that encode planar, cyclic, modular operads, PROPs, wheeled PROPs, properads, and similar operad-like structures are not quasi-Gr"obner.

2.4. Classical examples. The table below gives categories $Tw(P)$, $U(P)$, $C(P)$ and $PROP(P)$ (or in some cases the skeletons of these categories) for the operads $uAs$ of monoids, $As$ of semigroups, $uCom$ of commutative monoids, and $Com$ of commutative semigroups.
Here the category $R_{>0}$ is the wide subcategory of $Tw(P)$ that consists of morphisms such that all the vertices in the corresponding trees, except possibly the source vertex, have non-zero arity. The category $\Delta$ is the simplex category, $\mathrm{OI}$ is the category of order preserving injections, $\Delta_{ep}$ is the interval category, i.e. the wide subcategory of $\Delta$ on endpoint-preserving maps, $\mathrm{FA}(as)$ is the category of non-commutative sets, $\mathrm{FS}(as)$ is the subcategory of surjections in $\mathrm{FA}(as)$, $\mathrm{FA}$ is the category of finite sets, $\mathrm{FS}$ is the category of surjections in $\mathrm{FA}$. The subscripts denote the corresponding subcategories on sets that have $(\pm)$ at least one element, $(\pm\pm)$ at least two elements, $(\ast)$ a marked element that is preserved by the maps, $(\ast\ast)$ at least one element in addition to the marked element, while $(\ast p)$ denotes the wide subcategory of endpoint-preserving maps. Notice also that $\Delta_{ep}$ is equivalent to $\Delta^{op}$.

**Lemma 2.16.** Let $P$ be an operad such that the set $P(\cdot;c)$ is finite for all colours $c$ of $P$, and let $\mathcal{C}$ be the category $Tw(P)$, $\mathcal{U}(P)$, $\mathcal{C}(P)$, or $PROP(P)$. Let $R_{>0}$ be the wide subcategory of $\mathcal{C}$ on morphisms represented by trees with non-source vertices marked by operations of non-zero arity. If $R_{>0}$ is quasi-Gröbner, then $\mathcal{C}$ is quasi-Gröbner.

**Proof.** The inclusion $R_{>0} \to \mathcal{C}$ is essentially surjective. For an object $p$ of $\mathcal{C}$ let $f_i$ be all the morphisms from $p$ represented by trees whose non-source vertices are marked either by identity operations or by operations of arity 0, with leaves permuted trivially. These morphisms ensure the property (F) for the inclusion $R_{>0} \to \mathcal{C}$.  

**Proposition 2.17.** The categories in the table above are quasi-Gröbner.

**Proof.** Since the category $\mathrm{OI}$ is Gröbner, the categories $\mathrm{OI}^+_+$ and $\mathrm{OI}^{\ast\ast}$ are Gröbner. Lemma 2.16 implies that $Tw(uAs)$ is quasi-Gröbner. The inclusions $\mathrm{OI}^{\ast}_{ep} \to \mathrm{OI}^+_+$ and $\mathrm{OI}^{\ast\ast}_{ep} \to \mathrm{OI}^{\ast\ast}$ satisfy property (S), therefore $\mathrm{OI}^{\ast}_{ep}$ and $\mathrm{OI}^{\ast\ast}_{ep}$ are Gröbner, and by Lemma 2.16 the category $\Delta_{ep}$ is quasi-Gröbner. The discrete opfibrations $Tw(uAs) \to \mathcal{U}(uAs)$ and $\mathcal{C}(uAs) \to PROP(uAs)$ are surjective on objects, which implies that $\mathrm{FA}^+_+(as)^{op}$ and $\mathrm{FA}(as)^{op}$ are quasi-Gröbner. There is a discrete opfibration from $R_{>0} \subset Tw(uAs)$ to $\mathcal{U}(As)$ that is surjective on objects, and its restriction $\mathcal{C}_{uAs} \cap R_{>0} \to PROP(As)$ is also a discrete opfibration surjective on objects, which implies that $\mathrm{FS}^+_+(as)^{op}$ and $\mathrm{FS}(as)^{op}$ are quasi-Gröbner.

The categories $\mathrm{FA}^{op}$ and $\mathrm{FS}^{op}$ are known to be quasi-Gröbner. Since $\mathrm{FS}^{op}$ is quasi-Gröbner, $\mathrm{FS}^+_+(as)^{op}$ is quasi-Gröbner. Recall that the category $\mathrm{OS}^{op}$, the opposite of the category of ordered finite surjections, is Gröbner. Consider its objects as the sets $\{0, \ldots, n\}$, with 0 as the marked element that is preserved by the maps. There is a functor $\mathrm{OS}^{op} \to \mathrm{FS}^{op}$ that satisfies property (F) and is surjective on objects, thus $\mathrm{FS}^{op}$ is quasi-Gröbner, and thus $\mathrm{FS}^+_+(as)^{op}$ is quasi-Gröbner. Lemma 2.16 implies that $\mathrm{FA}^+_+(as)^{op}$ is quasi-Gröbner.

This leads to another example of quasi-Gröbner category. This example might be new, though it is fairly obvious. It comes from the general construction discussed in the beginning.

**Proposition 2.18.** Connes cyclic category $\Lambda$ and its subcategory of cyclic injections are quasi-Gröbner.
Proof. There is discrete opfibration \( \Lambda \simeq \text{Tw}_{c\text{Op}}(A \mathcal{S}) \to \mathcal{U}_{c\text{Op}}(A \mathcal{S}) \simeq \text{FA}(\mathcal{S})^{op} \), where \( c\text{Op} \) is the operad whose algebras are cyclic operads, and \( A \mathcal{S} \) is the cyclic operad of monoids, see [Bur22 Proposition 2.31 and 2.32]. Its restriction to the category of injections (or to the opposite of the category of surjections, which is the same category by the self-duality \( \Lambda \simeq \Lambda^{op} \)) is a discrete opfibration over \( \text{FS}(\mathcal{S})^{op} \). \( \square \)

2.5. Operadic categories. The opposites of categories \( \mathcal{C}(P) \) are operadic categories ([BM15]), and many operadic categories arise from operads in this way. Opposites of operadic categories are often quasi-Gröbner.

**Definition 2.19.** Let \( S \) be a semigroup. Let \( N(S) \) be the presheaf over \( \text{OI}_{++} \) such that \( N(S)([n]) = S^n \), with presheaf maps defined in the same way as in the nerve construction for monoids. Further take the restriction of \( N(S) \) to \( \text{OI}_{++}^{ep} \).

The category \( \text{OI}_{++}^{ep}/N(S) \) has finite non-empty sequences of elements \( s_i \) of \( S \) as objects. Morphisms correspond to substitutions of elements \( s_i \) by sequences \( s_{i_1}, \ldots, s_{i_k}, \) such that \( s_{i_1} \cdots s_{i_k} = s_i \).

The category \( \text{OI}_{++}^{ep}/N(S) \) is the opposite of the operadic category described in [Moz22 A.1]. In the definition of the latter the subcategory of surjections of \( \Delta \) is used instead of \( \text{OI}_{++}^{ep} \), but these are equivalent.

The categories \( \text{OI}_{++}^{ep}/N(S) \) and \( \text{OI}_{++}^{ep}/N(S) \) can be seen as the categories \( \text{Tw}(P) \) and \( \mathcal{C}(P) \) where \( P \) is the Baez–Dolan plus construction of the semigroup \( S \) as an algebra over the operad \( A \mathcal{S} \) of semigroups.

**Proposition 2.20.** Let \( S \) be a semigroup. If \( S \) is not finite, then \( \text{OI}_{++}/N(S) \) is not quasi-Gröbner.

**Proof.** The proof is analogous to that of Proposition 2.15: the role of operations \( p_i \) is played by sequences \( (s_1, s_i) \) for pairwise different \( s_i \) in \( S \), and the role of morphisms \( id_2 \to p_i \) is played by inclusions \( (s_1) \to (s_1, s_i) \). \( \square \)

**Proposition 2.21.** Let \( S \) be a semigroup such that for any element \( s \) in \( S \) the number of decompositions of \( s \) into a product of non-identity elements of \( S \) is finite. Then the category \( \text{OI}_{++}^{ep}/S \) is Gröbner.

**Proof.** The projection \( \text{OI}_{++}^{ep}/S \to \text{OI}_{++}^{ep} \) is faithful. By Lemma 1.3 \( \text{OI}_{++}^{ep}/S \) satisfies property (G1).

If \( S \) does not have the identity element, then the slice categories are finite, and the property (G2) holds. Assume that \( S \) has the identity element \( e \). The condition on \( S \) implies that \( e \) cannot be decomposed into non-trivial product of elements of \( S \). To prove property (G2) let \( x = (s_1, \ldots, s_n) \) be an object in \( \text{OI}_{++}^{ep}/S \) and let \( f_i, i \in \mathbb{N} \), be a sequence of morphisms from \( x \). These morphisms correspond to decompositions of elements \( s_j \) into products of elements of \( S \). Since the number of these decompositions, up to multiplication by \( e \), is finite, there is a subsequence of \( f_i \) such that for each \( j \) the morphisms decompose the element \( s_j \) in the same way up to multiplication by \( e \). We can further choose a subsequence such that for each \( j \) the number of elements \( e \) between any two adjacent non-trivial elements in the decomposition of \( s_j \), and also the number of elements \( e \) before the first non-trivial element and the number of elements \( e \) after the last non-trivial element, is non-decreasing. This gives a non-decreasing subsequence of morphisms. \( \square \)

**Proposition 2.22.** Let \( S \) be the group \( \mathbb{Z}/2 \). The category \( \text{OI}_{++}^{ep}/S \) is Gröbner.

**Proof.** Again the projection \( \text{OI}_{++}^{ep}/S \to \text{OI}_{++}^{ep} \) is faithful, which implies property (G1).
Let $x = (s_1, \ldots, s_n)$ be an object in $\text{OI}_{+\text{op}}/S$ and let $f_i, i \in \mathbb{N}$, be a sequence of morphisms from $x$. We can choose a subsequence $f_i$ such that for all $j$ the number of symbols 1 in the subsequence $(s'_1, \ldots, s'_{j+1})$ that replaces $s_j$ is either stable or increases, and if it is stable, we can further choose subsequence such that the numbers of symbols 0 to the left of the first, in between of $l$-th and $(l + 1)$-th for all $l$, and to the right of the last symbol 1 are non-decreasing. Take any $f_i$ in the subsequence. Let $h$ be the morphism from the target of $f_i$ that, for all $j$ such that the number of symbols 1 in $(s'_{j}, \ldots, s'_{j+1})$ increases with $i$, replaces the symbols 0 in subsequence $(s'_1, \ldots, s'_{j+1})$ by subsequence $(1, 1)$. There is some $N$ such that there is a morphism from $h \circ f_j$ to morphism $f_i$ in the subsequence for all $l > N$, i.e. $f_j \leq f_i$.

Example 2.23. Consider the opposite of the category $\mathcal{C}$ from Example 3. This category is obtained from the commutative monoid $\mathbb{R}_{\geq 0}$ as follows. Let $N(\mathbb{R}_{\geq 0})$ be the nerve of $\mathbb{R}_{\geq 0}$, seen as a presheaf over the Segal’s category $\mathcal{FA}_{\text{op}}$, and consider the restriction of $N(\mathbb{R}_{\geq 0})$ to the subcategory $\mathcal{FA}_{\text{op}}$ of active morphisms. Then $\mathcal{C}_{\text{op}}$ is the full subcategory of $\mathcal{FA}_{\text{op}}/N(\mathbb{R}_{\geq 0})$ on objects $(s_1, \ldots, s_n)$ such that $\sum_i s_i \leq 1$.

Let $M$ be the module generated by $(1)$ and $N$ be the submodule of $M$ such that $N((s_1, \ldots, s_n)) = 0$ if all $s_i$ except one equal to 0, and $N(x) = M(x)$ on the remaining objects $x$. There are morphisms $(1) \rightarrow (p, 1 - p)$ for all $p \in [0, 1]$, such that $N((p, 1 - p)) \neq 0$. The module $N$ is not finitely generated, and the category $\mathcal{C}_{\text{op}}$ is not quasi-Gröbner.

3. Graph-based categories

Next we consider the categories $\mathcal{C}(P)$ and their subcategories for the operads $P$ described in [Bur22, Section 1.1]. These operads are the operads that encode planar, symmetric, cyclic, modular and genus-graded modular operads. Operations of these operads, and thus objects in the corresponding categories $\mathcal{C}(P)$, are graphs with half-edges endowed with additional structure. To avoid confusion, the operations of these operads whose underlying graphs are trees will be called operadic trees, while the trees that correspond to morphisms in categories $\mathcal{C}(P)$ will be called 2-level trees.

Just like in the cases above, we will be more interested in the subcategories $\mathcal{C}(P_{\neq 0})$ of categories $\mathcal{C}(P)$, where $P_{\neq 0}$ is the suboperad of an operad $P$ consisting of all its operations of non-zero arity. By Lemma 2.10 if $\mathcal{C}(P_{\neq 0})$ is quasi-Gröbner then $\mathcal{C}(P)$ is quasi-Gröbner. It will be convenient to have a concrete description of these categories.

Definition 3.1. A graph with half edges is a finite sets $V$ of vertices, a finite set $H$ of half-edges, an involution $\text{inv}$ on $H$, the adjacency map $t : H \rightarrow V$. A fixed point of the involution is called a leaf. A two-element orbit of the involution is called edge. The set $V$ together with the set of edges can be seen as the usual graph, where an edge $\{h_1, h_2\}$ connects the vertices $t(h_1)$ and $t(h_2)$.

Definition 3.2. An operadic graph is a graph with half edges $(V, H, \text{inv}, t)$ together with an order on the leaves, an order on each of the sets $t^{-1}(v)$ of half-edges adjacent to a vertex $v$, and an order on vertices. The orders on half-edges and on leaves will often be given by bijection with sets $\{0, \ldots, n\}$, while the order on vertices will be given by bijection with sets $\{1, \ldots, n\}$.

Proposition 3.3. The category $\mathcal{C}((m\text{Op}_{(g,n)} )_{\neq 0})$ has the following concrete description. Its objects are operadic graphs endowed with a genus map $g : V \rightarrow \mathbb{N}$. A morphism $f : p \rightarrow q$ given by a 2-level tree with upper vertices $q_i$ corresponds to
embedding of operadic graphs $q_l$ into the vertices of $p$. For all $l$ the number of leaves of the operadic graph $q_l$ coincides with the number of half-edges adjacent to the $l$-th vertex of $p$, and the genus of the $l$-th vertex of $p$ is equal to $(\sum_{v \in q_l} g(v) + g(q_l))$, where $g(v)$ is the genus of the vertex $v$ of $q_l$ and $g(q_l)$ is the genus of the graph of $q_l$. If the $i$-th half-edge of the $j$-th vertex is connected to the $k$-th half-edge of the $l$-th vertex in $p$, then $f$ connects the $i$-th leaf of $q_j$ to the $k$-th leaf of $q_l$. If the $i$-th half-edge of the $j$-th vertex is the $k$-th leaf of $p$, then the $i$-th leaf of $q_j$ becomes the $k$-th leaf of $q$. Additionally $f$ determines the order on vertices of $q$ according to the indices of leaves of the 2-level tree of $f$. The functor $C((mOp_{(g,n)} {\neq} 0)) \to FS^{op}$ sends an operadic graph with $n$ vertices to $\{1, \ldots, n\}$ and a morphism $f : p \to q$ that inserts $q_l$ into the $l$-th vertex of $p$ to the opposite of the map $h$ such that $h^{-1}(l)$ is the set of indices of vertices of the subgraph $q_l$ of $q$.

Instead of categories $C(P)$ one may want to consider the wide subcategories $Active(P)$ of categories $Tw(P)$ on active morphisms, i.e. morphisms such that the lower vertices in the corresponding 3-level trees have arity 1 and are marked by invertible operations. Notice though that if an operad $P$ is such that for any colour $c$ of $P$ the number of invertible operations in $P$ with input colour $c$ is finite, then the inclusion of $C(P)$ into $Active(P)$ is essentially surjective and satisfies property (F), and thus if $C(P)$ is quasi-Gröbner then $Active(P)$ is quasi-Gröbner. This finiteness condition holds for the operads that we consider.

Let $pOp$ be the operad of planar operads. The category $C(pOp)$ is equivalent to the subcategory of active morphisms of the planar version of the Moerdijk–Weiss dendroidal category $\Omega_{pl}$. The objects of $C(pOp_{pl})$ are the operadic graphs that are planar rooted trees with half-edges, with vertices indexed from 1 to $n$, with leaves indexed in planar order. A morphism $f : p \to q$ in $C(pOp_{pl})$ embeds an operadic tree $q_j$ into the $j$-th vertex $v_j$ of the operadic tree $p$ for all $j$, and indexes the vertices of $q$ so that $v \prec v'$ in $q$ if $v$ and $v'$ belong to the same $q_j$ and $v \prec v'$ as vertices of $q_j$ or if $v \in q_i, v' \in q_j$ and $v_i \prec v_j$ as the vertices of $p$.

**Proposition 3.4.** The category $C(pOp_{pl})$ is quasi-Gröbner.

**Proof.** For any operad $P$ there is a functor $F : C(P) \to FS^{op}$ that sends an operation $p$ of arity $n$ to $\{1, \ldots, n\}$ and the morphism represented by a 2-level tree $h$ to the opposite of the set-map $f$ such that $f^{-1}(i)$ consists of indices of leaves above the $i$-th upper vertex of the 2-level tree $h$.

The functor $F : C(pOp_{pl}) \to FS^{op}$ is faithful. Indeed, a morphism $h : p \to q$ in $C(pOp_{pl})$ substitutes certain operadic trees $q_j$ into the vertices of $p$, and assigns order to the vertices of $q$. Such a morphism $h$ is determined by the operation $q$ and by the partition of the planar tree of $q$ into subtrees $q_j$, and this partition is determined by the indices of the vertices of the subtrees $q_j$ of $q$, i.e. it is determined by $F(h)$.

Let $D$ be the full subcategory of $C(pOp_{pl})$ on operadic trees with vertices ordered in clockwise depth-first search order starting from the root. The inclusion $D \to C(pOp_{pl})$, an equivalence of categories, is essentially surjective and satisfies property (F). The image of the restriction of $F : C(pOp_{pl}) \to FS^{op}$ to $D$ lies in $OS^{op}$. Since $F : D \to OS^{op}$ is faithful and $OS^{op}$ satisfies property (G1), by Lemma 1 the category $D$ satisfies property (G1).

It remains to prove that $D$ satisfies property (G2), and this is done in the lemma below. The main difficulty is posed by the operadic trees that contain vertices of arity 0 (i.e. vertices without input edges). If one is interested only in the full subcategory $D_+$ of $C(pOp_{pl})$ on trees without vertices arity 0, then one may proceed as follows. Let $f_i$ be a sequence of morphisms in $D_+$ from the same object $p$. Let $q_{ji}$ be the operadic tree that is substituted into the $j$-th vertex of $p$ under the morphism
f_. For any fixed j the trees q_ji have the same number of input leaves, and we can choose a subsequence of f_i such that the trees q_ji are homeomorphic to each other and differ only by the number of vertices of degree 2. We can further choose a subsequence such that the corresponding numbers of vertices of degree 2 between any two adjacent vertices of degree not equal to 2, and also the numbers of vertices of degree 2 between leaves and vertices of degree not equal to 2, are non-decreasing. Doing this for each j, we get a non-decreasing sequence of morphisms f_i. □

Lemma 3.5. The property (G2) for D is equivalent to the relative Kruskal’s tree theorem from [Bar15], i.e. to the property (G2) for the category PT.

Proof. There is a functor G : PT → D that sends a planar rooted tree T to the same tree (without input leaves and with the root leaf added to the root vertex) and sends a morphism f : T → T’ to the morphism that for each vertex v of T embeds into v the maximal subtree of T’ that contains only the vertices above or equal to f(v) and, for all the vertices w that are the children of v, does not contain the vertices f(w), yet contains the half-edges directly below the vertices f(w).

There is a faithful functor F : D → PT: on objects it adds a vertex of arity 0 to each input leaf, on morphisms it sends f : p → q to the map of planar trees that sends a vertex v of p to the lowest vertex in the subtree of q that is embedded into v by f, and sends the vertices above the leaves of p to the corresponding vertices above the leaves of q.

The composition F ∘ G is the identity functor. The functor G is fully faithful. This allows to view PT as a full subcategory of D. In particular, if D satisfies property (G2), then PT satisfies property (G2).

For any object p in D let F’ : p/PT → F(p)/PT be the functor induced by F. The functor F’ is fully faithful. Thus if PT satisfies (G2), then D satisfies (G2). □

Proposition 3.6. The category C(sOp) is quasi-Gröbner.

Proof. Notice that the inclusion C(pOp) → C(sOp) induced by the inclusion of operads pOp → sOp is not essentially surjective, since, unlike the case of the category Tw(sOp), the same planar trees with different permutation on leaves are in general not isomorphic to each other.

Let D’ be the subcategory of C(sOp) with objects being operadic trees with any permutation on leaves and with permutation on vertices given by the clockwise depth-first search order starting from the root; and with morphisms being such that the upper vertices in the 2-level trees that represent these morphisms belong to pOp, i.e. the corresponding operations q_j are operadic trees with trivial permutation on leaves. The inclusion D’ → C(sOp) is essentially surjective and satisfies property (F): any morphism p → q in C(sOp) is a composition of an isomorphism that permutes input edges of vertices of the operadic tree p (and permutes the order on vertices) and of a morphism from D’.

Let D be the Gröbner category from Proposition 3.4. The functor D’ → D that forgets the indices of leaves of trees is a discrete opfibration. Thus D’ is Gröbner. □

Proposition 3.7. The inclusion C(sOp) → C(cOp) induced by the inclusion of operads sOp → cOp is an equivalence of categories.

Proof. Any object p in C(cOp) is isomorphic to an object of C(sOp) via the morphism that substitutes into the vertices of p operadic trees q_j that are corollas with cyclically permutated leaves. For any object p in C(sOp) if the operadic trees q_j are such that the substitution of these trees into p is an object of C(sOp), then the operadic trees q_j are in sOp. □
The next two categories are related to the operadic category \textit{Gr} from Section 3.

**Proposition 3.8.** The category \( \mathcal{C}(\text{mOp}_{\neq 0}) \) is not quasi-Gröbner.

**Proof.** Observe that there are morphisms \( f_i : id_1 \to p_i \) from the corolla with two leaves, ordered trivially, to the graphs on two vertices, of genus \( i \), without loops, and with the 0-th leaf adjacent to the first vertex and the 1-st leaf adjacent to the second vertex. Let \( M \) be the module generated by \( id_1 \) and let \( N \) be the maximal submodule of \( M \) that is trivial on the graphs of genus 0 and on the graphs with only one vertex. The module \( N \) is non-trivial over objects \( p_i \), and thus is not finitely generated. \( \blacklozenge \)

**Proposition 3.9.** The category \( \mathcal{C}((\text{mOp}_{(g,n)})_{\neq 0}) \) is quasi-Gröbner.

**Proof.** Denote \( \mathcal{C}((\text{mOp}_{(g,n)})_{\neq 0}) \) by \( \mathcal{C} \). Notice that morphisms in \( \mathcal{C} \) preserve the number of leaves of operadic graphs. Let \( \mathcal{C}_0 \), \( \mathcal{C}_1 \) and \( \mathcal{C}' \) be the full subcategories of \( \mathcal{C} \) on operadic graphs without leaves, with exactly one leaf, and with at least one leaf respectively. There is a functor \( \mathcal{C}_1 \to \mathcal{C}_0 \) that removes the leaf. This functor is full and surjective on objects. If \( \mathcal{C}' \) is quasi-Gröbner, then \( \mathcal{C}_1 \) is quasi-Gröbner, and then \( \mathcal{C}_0 \) is quasi-Gröbner, which implies that \( \mathcal{C} \), as the disjoint union of \( \mathcal{C}' \) and \( \mathcal{C}_0 \), is quasi-Gröbner.

As in the previous proofs, to prove that \( \mathcal{C}' \) is quasi-Gröbner we take a subcategory \( \mathcal{Z} \) of \( \mathcal{C}' \) such that \( \mathcal{Z} \) does not have non-trivial endomorphisms and the image of \( \mathcal{Z} \) under the functor \( \mathcal{C}' \to \mathcal{F}\mathcal{S}^{op} \) lies in \( \mathcal{O}\mathcal{S}^{op} \). The objects of the subcategory \( \mathcal{Z} \) are operadic graphs with at least one leaf, with the first vertex being the vertex adjacent to the 0-th leaf, and with the 0-th half-edge of the first vertex being the 0-th leaf of the graph, with vertices ordered in clockwise depth-first search order, and such that the edges that are traversed by the clockwise depth-first search that starts from the first vertex contain exactly one 0-th half-edge and one non-zero half-edge of adjacent vertices. The morphisms \( f : p \to q \) of \( \mathcal{Z} \) are the morphisms of \( \mathcal{C}' \) such that for all \( l \) the order on the leaves of the operadic graph \( q_l \) that is inserted by \( f \) into the vertices of \( p \) is the order in which the clockwise depth-first search over \( q_l \) that starts from the 0-th leaf traverses the leaves of \( q_l \). This condition on the morphisms implies that the 0-th leaf of \( q_l \) is the 0-th half-edge of its vertex and that the clockwise depth-first search over \( q \) that starts from the first vertex traverses (the smallest vertices of) the subgraphs \( q_l \) sequentially from \( q_1 \) to \( q_n \). The latter property implies that the image of \( \mathcal{Z} \) under \( \mathcal{C}' \to \mathcal{F}\mathcal{S}^{op} \) lies in \( \mathcal{O}\mathcal{S}^{op} \). Endomorphisms in \( \mathcal{C}' \) insert trees \( q_l \) into vertices, and the condition on the morphisms in \( \mathcal{Z} \) implies that these trees are identity operations, i.e. endomorphisms in \( \mathcal{Z} \) are trivial.

The inclusion \( \mathcal{Z} \to \mathcal{C}' \) is essentially surjective. To prove that this inclusion satisfies property (F) we will use the following observation. Let \( p \) be an object of \( \mathcal{Z} \), and let \( i : p \to r \) be an isomorphism in \( \mathcal{C}' \) that for some \( j \) substitutes into the \( j \)-th vertex of \( p \) an operadic tree \( r_j \) that has one vertex and whose 0-th leaf is the 0-th half-edge of the first vertex of \( r_j \), substitutes the identity operations into the remaining vertices of \( p_i \), and does not permute the indices of vertices. Then there is exactly one isomorphism \( i' : r \to p' \) in \( \mathcal{C}' \) with \( p' \) in \( \mathcal{Z} \) that substitutes the identity operations into the first \( j \) vertices of \( r \), for all \( l > j \) substitutes trees with one vertex and with cyclic permutation on leaves into the remaining vertices of \( r \), and does not permute the indices of the first \( j \) vertices of \( r \).

Let \( f : p \to q \) be a morphism in \( \mathcal{C}' \) between objects in \( \mathcal{Z} \). Let \( q_l \) be the operadic graph that is inserted into the \( l \)-th vertex of \( p \) by \( f \). Since the 0-th leaf of \( q_l \) is the 0-th half-edge of its vertex, the 0-th leaf of \( q_l \) is the 0-th half-edge of its vertex. The operation \( q_l \) can be represented as the composition \( x_1 \circ q_1 q'_1 \), where \( x_1 \) is a tree with one vertex with the 0-th leaf being the 0-th half-edge of its vertex, and with \( q'_1 \) such
that the order on the leaves is the order in which the clockwise depth-first search traverses the leaves of \(q_1\). The substitution of the operation \(x_1\) into the first vertex of \(p\) gives an isomorphism from \(p\) in \(C'\). As explained in the previous paragraph, this isomorphism can be extended to an isomorphism \(i_1 : p \rightarrow p_1\) with \(p_1\) in \(Z\).

This gives decomposition of \(f\) as \(f_1 \circ i_1\). If \(p\) has \(n\) vertices, doing the same for \(f_{i-1}\) and the \(i\)-th vertex of \(p_{i-1}\), with \(i\) ranging from 2 to \(n\), gives the isomorphism \(i = i_{n-1} \circ \cdots \circ i_1 : p \rightarrow p_n\) such that \(f = f_n \circ i\). The morphism \(f_n\) substitutes operations \(q_i\) into vertices of \(p_n\), and the orders on leaves of \(q_i\) are such that \(f_n\) is in \(Z\). This shows that the inclusion \(Z \rightarrow C'\) satisfies property (F). It remains to prove that \(Z\) is quasi-Gröbner.

The image of the functor \(Z \rightarrow \mathbf{FS}^{op}\) lies in \(\mathbf{OS}^{op}\), however this functor is not faithful, e.g. there are two morphisms from a graph with one vertex of degree 1 and one loop to the graph with one vertex of degree 0 with two loops, and these morphisms are mapped to the same map in \(\mathbf{FS}^{op}\). We will construct an essentially surjective discrete fibration \(\mathcal{G} \rightarrow \mathcal{D}\) with finite fibers. The composition \(\mathcal{G} \rightarrow \mathcal{D} \rightarrow \mathbf{OS}^{op}\) will be faithful, which implies that \(\mathcal{G}\) satisfies property (G1). By Lemma 1.2 to prove that \(Z\) is quasi-Gröbner it will suffice to prove that \(\mathcal{G}\) satisfies property (G2).

An object of \(\mathcal{G}\) is an object \(p\) of \(Z\) endowed with a colour map \(col : H \rightarrow \mathbb{N}\) from the half-edges of \(p\) that satisfies the following:

1. For any edge of \(p\) its two half-edges have the same colour, which will be called the colour of the edge.
2. Let \(p'\) be the operadic graph obtained from \(p\) by repeated removal of all the vertices of arity 0 and genus 0 together with the adjacent edges (i.e. we remove both half-edges of the adjacent edge), until no vertex of arity 0 and genus 0 remains, except possibly the last vertex. Then the removed edges have colour 0 and the remaining edges have colour different from 0.
3. If a path in \(p'\) from a vertex \(v\) to a vertex \(w\) consists only of vertices of degree 2 and genus 0, then the edges in this path have the same colour. Let \(p''\) be the graph obtained from \(p'\) by replacing such maximal paths with edges that have the same colours as the edges in the paths that they replace.
4. Half-edges of \(p''\) that belong to different edges have different colours.
5. For all \(h\) in \(H\) we have \(col(h) \leq g(v) + g(p) + l(p)\), where \(g(v)\) is the genus of a vertex \(v\), \(g(p)\) is the genus of the graph of \(p\), and \(l(p)\) is the number of leaves of \(p\).

The morphisms are required to preserve colours, i.e. for a morphism \(f : p \rightarrow q\) that inserts \(q_i\) into the \(l\)-th vertex of \(p\) the leaves of \(q_i\) (seen as the half-edges of \(q_i\)) have the same colours as the half-edges of \(p\) to which these leaves correspond under \(f\). These conditions imply that the forgetful functor \(\mathcal{G} \rightarrow Z\) is a discrete fibration surjective on objects and that \(\mathcal{G} \rightarrow \mathbf{OS}^{op}\) is faithful, which implies that \(\mathcal{G}\) satisfies property (G1).

It remains to prove property (G2) for \(\mathcal{G}\). Let \(f_j\) be a sequence of morphisms from \(p\) in \(\mathcal{G}\), with \(f_i\) inserting \(q_i\) into the \(l\)-th vertex of \(p\). Let \(q'_i\) be the coloured operadic graphs obtained from the graphs \(q_i\), by repeated removal of all the vertices of arity 0 and genus 0 until no such vertices remain (except possibly the last vertex), and by replacing the maximal paths whose vertices have degree 2 with edges, and let \(f'_i\) be the corresponding morphisms from \(p\). Since the genus of the \(l\)-th vertex of \(p\) is equal to \((\sum_{v \in q_i} g(v) + g(q_{li}))\) and the number of half-edges of the \(l\)-th vertex of \(p\) is equal to the number of leaves of \(q_{li}\), the total number of possible coloured operadic graphs \(q'_{li}\) is finite, and there is a subsequence \(f'_i\) of the sequence \(f'_i\) that consists of the same morphism \(f : p \rightarrow q'\) repeated infinitely often. Let \(f_j\) be the
corresponding subsequence of \( f_i \). Observe that there are morphisms \( g_j \) such that 
\[ f_j = g_j \circ f, \]
where \( g_j \) insert planar operadic trees into the vertices of \( q' \). If \( g_{j_1} \leq g_{j_2} \), then \( f_{j_1} \leq f_{j_2} \). Let \( r \) be an object of the subcategory \( D \) of \( C(s\text{Op}_{\neq 0}) \) (described in Proposition \ref{prop:subcategory}) that has the same number of vertices as \( q' \) and such that the \( l \)-th vertex of \( r \) has the same arity as the \( l \)-th vertex of \( q' \) for all \( l \). To morphisms \( g_j \) correspond morphisms \( g'_j \) from \( r \) in \( D \) such that \( g_j \) and \( g'_j \) insert the same planar trees in their vertices. Since \( D \) satisfies property (G2), there are some \( j_1 < j_2 \) such that \( g'_{j_1} \leq g'_{j_2} \), which implies that \( g_{j_1} \leq g_{j_2} \), and \( f_{j_1} \leq f_{j_2} \). \( \square \)

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