FRACTIONAL RELAXATION
AND FRACTIONAL OSCILLATION MODELS
IN VOLVING ERDÉLYI-KOBER INTEGRALS

Moreno Concezzi, 1 Roberto Garra, 2 Renato Spigler, 1

Abstract

We consider fractional relaxation and fractional oscillation equations involving Erdélyi–Kober integrals. In terms of Riemann–Liouville integrals, the equations we analyze can be understood as equations with time-varying coefficients. Replacing Riemann–Liouville integrals with Erdélyi–Kober-type integrals in certain fractional oscillation models, we obtain some more general integro-differential equations. The corresponding Cauchy-type problems can be solved numerically, and, in some cases analytically, in terms of Saigo–Kilbas Mittag–Leffler functions. The numerical results are obtained by a treatment similar to that developed by K. Diethelm and N.J. Ford to solve the Bagley–Torvik equation. Novel results about the numerical approach to the fractional damped oscillator equation with time-varying coefficients are also presented.

MSC 2010: Primary 34C26; 26A33; 65L05; Secondary 326A33; 26A48; 34A08; 33E12

Key Words and Phrases: anomalous relaxation; fractional relaxation; fractional oscillations; Erdélyi–Kober integrals

1. Introduction

In this paper, we consider fractional relaxation and fractional oscillation models with time-varying coefficients, involving Erdélyi–Kober-type integrals. In the recent years, a number of papers have been devoted to the analysis of fractional relaxation and oscillations as well as to their applications, see, e.g., [14, 24] and references therein. For a probabilistic approach to fractional relaxation equations, we refer to [2]. Non-Debye relaxation...
phenomena in dielectrics are often modeled by means of fractional differential equations that generalize the classical relaxation equation. In this framework, Mittag–Leffler-type functions play a relevant role to describe anomalous relaxation, including special cases such as the Cole–Cole [4], the Davidson–Cole [6], and the Havriliak–Negami [10, 12] models (see also [17] for a short review about analytical representations of relaxation functions in non-Debye processes). In recent theoretical investigations [3, 11], the authors studied fractional relaxation models with time-varying coefficients resorting to different approaches. In the first part of this paper, we consider a further approach in this direction, studying a new fractional model involving Erdélyi–Kober-type integrals. In this framework, we provide both analytical and numerical results.

In the second part of the paper, we consider a fractional generalization of the damped oscillator equation with time-dependent elasticity involving Erdélyi–Kober-type integrals. Fractional generalizations of the damped oscillator equation have been subject of recent research, especially in the framework of fractional mechanics; we refer, in particular, to the recent books [21, 26] and references therein. Moreover, we observe that a connection between fractional diffusion equations involving Erdélyi–Kober-type operators and the density law of the generalized grey Brownian motion has been recently studied in [28]. This fact stresses the role of Erdélyi–Kober integral operators in the future developments of fractional calculus.

As far as we know, only few works concerning analysis as well as numerical treatment of fractional damped oscillators with time-dependent coefficients can be found in the literature. We will show that some generalizations of the classical equations of mechanics by means of Erdélyi–Kober-type integrals imply the analysis of integro-differential equations with time-varying coefficients involving Caputo-type fractional derivatives. This topic seems to be promising and interesting both, for numerical and analytical studies of fractional differential equations with variable coefficients, and their applications in mechanics.

2. Fractional relaxation models involving Erdélyi–Kober-type integrals

A reasonable generalization of the classical relaxation model can be obtained replacing the ordinary time derivative with the the Caputo fractional derivative, defined by (see [24])

\[ D_t^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - s)^{m-1-\alpha} \frac{d^m}{ds^m} f(s) \, ds, \quad (1.1) \]
where $\alpha > 0$, $m = \lceil \alpha \rceil$, and $f \in AC^m[a,b]$. Therefore, the so-called fractional relaxation equation is simply given by

$$ D_t^{1-\alpha} u(t) = -\lambda u(t), \quad \lambda > 0, \quad t \geq 0, \quad \alpha \in [0,1). \quad (1.2) $$

In this case, the fractional model describes an anomalous relaxation process with a Mittag–Leffler decay, that is with an asymptotically power law behavior.

Since for absolutely continuous functions, we have

$$ J_t^{1-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad (1.4) $$

is the Riemann-Liouville fractional integral of order $\beta > 0$ (see, e.g., [19]), we can write (1.2), in the equivalent integral form,

$$ u(t) - u(0) = -\lambda J_t^{1-\alpha} u(t). \quad (1.5) $$

In this section we consider a generalization of the fractional relaxation equation in the integral form (1.5), where the Riemann–Liouville integral is replaced by the Erdélyi–Kober-type fractional integral (see for example [27, 31, 32])

$$ I_{m}^{\eta,\alpha} f = \frac{t^{-m\alpha-\max}}{\Gamma(\alpha)} \int_0^t (t^m - s^m)^{-\alpha-1} s^{mn} f(s) \, ds, \quad \eta \geq 0, \quad m > 0. \quad (1.6) $$

By this, we will be able to obtain some general results concerning anomalous relaxation with time-dependent coefficients and hereditary effects, including both, classical (ordinary) and fractional relaxation models as special cases.

More general Erdélyi–Kober operators, with $\beta > 0$ instead of $m$ in (1.6), have been studied, and the corresponding “Erdélyi–Kober fractional derivatives” were introduced in [29], and more recently analyzed, e.g., in [23]. Applications of such operators in the framework of fractional mechanics should be the subject of further research.

For simplicity, in (1.6) we will assume $m = 1$, being $\alpha \in (0,1)$. In fact, in this section we consider purely relaxation phenomena. Moreover, we assume that the relaxation coefficient there follows a power-law behavior, i.e., $\lambda(t) = \lambda t^{1-2\alpha}$. We are interested to understand the interplay between dynamical coefficients and Erdélyi–Kober-type integrals in generalized mechanical models. Therefore, our first model is based on the
following generalization of equation (1.5)

\[ u(t) - u_0 = -\frac{\lambda}{t^{2\alpha - 1}} I^{1-\alpha}_1 u(t) \]

\[ = -\frac{\lambda t^{-\alpha-\eta}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^\eta u(s) \, ds \]

\[ = -\lambda t^{\alpha-\eta} J^{1-\alpha}_1 (t^\eta u(t)). \] (1.7)

We observe that, for \( \eta = 0 \) and \( \alpha = 0 \) we recover the classical relaxation model. Hence, the effect of introducing Erdélyi–Kober-type integrals in the fractional relaxation equation is essentially that of providing power law time-varying coefficients. We will show that this integral equation can be treated in a similar way as that considered in [3].

Equation (1.7) can be handled in two equivalent ways. The first one is based on the solution of the following system of coupled integro-differential equations

\[
\begin{cases}
  u(t) - u_0 = -\lambda t^{-\eta-\alpha} D_1^\alpha g(t) \\
  D_t g = t^\eta u(t),
\end{cases}
\] (1.8)

which is clearly equivalent to (1.7) (here \( D_t g \equiv D_1^t := dg/dt \) and \( g(0) = 0 \)). This approach is interesting in view of a numerical integration of equation (1.7), and is suggested by a method developed by Diethelm and Ford in [8, 9], to solve numerically the well-known Bagley–Torvik equation [1].

The second approach, based on simple analytical manipulations, allows to reduce equation (1.7) to that considered recently in [3]. Indeed, applying the first order derivative to both sides of (1.7), we obtain

\[ \frac{d}{dt} \left[ u(t) + \lambda t^{-\eta-\alpha} J^{1-\alpha}_1 (t^\eta u(t)) \right] = 0. \] (1.9)

We will study the case

\[ t^{\eta+\alpha} u(t) + \lambda J^{1-\alpha}_1 (t^\eta u(t)) = 0. \] (1.10)

Let us recall that

\[ D_t^{1-\alpha} f_t^{1-\alpha} f(t) = f(t), \] (1.11)

and define \( r(t) := t^{\eta+\alpha} u(t) \). Then, taking the Caputo derivative of order \( 1 - \alpha \) of both sides of (1.10), we obtain the equation

\[ D_t^{1-\alpha} r(t) = -\lambda t^{-\alpha} r(t). \] (1.12)

Such equation has recently been studied by Capelas de Oliveira et al. in [3], and its explicit solution can be written in terms of the so-called Saigo–Kilbas Mittag–Leffler function, for \( \alpha \in (0, 1/2) \) (see the recent monograph [13] for more details). In particular, here we recall the following useful result (see, e.g., [19], pp. 232–233).
Lemma 1.1. The function
\[ E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}(-\lambda t^{\alpha+\beta}) = 1 + \sum_{k=1}^{\infty} (-\lambda)^{k} t^{k(\alpha+\beta)} \prod_{j=0}^{k-1} \frac{\Gamma \left( \alpha \left( j + j \frac{\beta}{\alpha} + \frac{\beta}{\alpha} \right) + 1 \right)}{\Gamma \left( \alpha \left( j + j \frac{\beta}{\alpha} + \frac{\beta}{\alpha} + 1 \right) + 1 \right)} . \]

solves the following fractional Cauchy problem:
\[
\begin{cases}
D^\alpha_t y(t) = -\lambda t^\beta y(t), & t \geq 0, \; \alpha \in (0,1], \; -\alpha < \beta \leq 1 - \alpha, \\
y(0) = 1.
\end{cases}
\]

(1.14)

We recall that, according to Kilbas et al. [19] (Remark 4.8, p. 233), uniqueness of the solution to the Cauchy problem in (1.14) has been established only for \( \beta \geq 0 \).

In view of Lemma 1.1, a solution to equation (1.12), for \( \alpha \in (0,1/2) \), is given by
\[ r(t) = E_{1-\alpha,1-\frac{\alpha}{1-\alpha},1-\frac{\alpha}{1-\alpha}}(-\lambda t^{1-2\alpha}) , \]

(1.15)

and going back to the original function, we have that
\[ u(t) = \frac{E_{1-\alpha,1-\frac{\alpha}{1-\alpha},1-\frac{\alpha}{1-\alpha}}(-\lambda t^{1-2\alpha})}{t^{\alpha+\eta}} . \]

(1.16)

We observe that, for \( \alpha = \eta = 0 \), equation (1.16) becomes
\[ u(t) = E_{1,1,0}(-\lambda t) = e^{-\lambda t} , \]

(1.17)

which coincides with the solution of the classical relaxation equation with initial condition \( u(0) = 1 \).

The study of the conditions for the complete monotonicity of a solution to the fractional Cauchy problem in (1.14) is a central topic for the physical sense of the anomalous relaxation models, since it ensures that in isolated systems the energy decays monotonically (see [15] for a complete discussion about the physical realizability of the viscoelastic models).

In [3], the authors have discussed the complete monotonicity of the so-called Saigo–Kilbas Mittag–Leffler function (1.13). In particular, they proved that the solution (1.13) is completely monotonic provided that \( \alpha \in (0,1] \) and \( \beta \in (-\alpha,1-\alpha] \). In our case, these conditions correspond to the assumption that \( \alpha \in (0,1/2) \), that ensures the complete monotonicity of the solution. We refer to [25] and references therein for a recent discussion concerning the relevance of completely monotonic functions in dielectrics.
1.1. Numerical results

In this section, we present some numerical results concerning the solution of the fractional Cauchy problem in (1.14). In Fig. 2, we show the numerical solutions to such problem, for different values of $\alpha \in (0, 1]$, and varying the parameter $\beta$, with $-\alpha < \beta \leq 1 - \alpha$.

The decay of the response function, for a fixed $\alpha$, is faster than exponential for short times, and power-law-like asymptotically, as pointed out also in [3]. The decay is clearly faster for lower values of $\alpha$.

In Fig.1, we observe that, for a fixed value of $\alpha$ ($\alpha = 0.9$) for short times the solution to equation (1.13) decays faster with respect to the classical exponential relaxation decay.

On the other hand, for a fixed value of $\alpha$, the role played by the parameter $\beta$ is clear: increasing $\beta$ the solution decays faster (see Fig.2). In order to understand the physical meaning of the fractional relaxation model described by the Cauchy problem in (1.14), we observe that, for $\alpha = 1$, it reduces to

$$\begin{cases}
D_t y(t) = -\lambda t^\beta y(t), & t \geq 0, \beta > -1 \\
y(0) = 1,
\end{cases}$$

(1.18)

that is to a relaxation equation with the time-dependent coefficient $\lambda(t) := \lambda t^\beta$, whose solution is given by

$$y(t) = \exp \left\{ -\lambda \frac{t^{\beta+1}}{\beta+1} \right\}.$$

From this special case, we can understand the role of the parameter $\beta$ in the solution, as confirmed by the numerical results.

In Figs 1 and 2, we plotted the numerical solution of (1.14) for several values of the parameters. The numerical method we used is an adaptive improvement of the predictor-corrector method earlier introduced in [7], and developed in [5]. Starting from an arbitrary discretization step, $h$, we chose locally a step size inversely proportional to the size of the (classical) derivative of the solution being computed, so that, at the $i$-th step, the time step

$$h_i := \frac{c_i h}{|u(t_i) - u(t_{i-1})|},$$

$i = 1, 2, \ldots, N$, $t_i = \sum_{j=1}^i h_j$, $t_0 = 0$, is used. Here $N$ is the number of the integration nodes, and

$$c_i = \frac{u'(t_{i-2}) - u'(t_{i-3})}{u'_\alpha(t_{i-1}) - u'(t_{i-2})},$$

where $u'$ denotes the (classical) first derivative of $u'$, see [5].
Figure 1. Numerical solution of (1.14) for $\alpha = 1$ and $\alpha = 0.9$, $\lambda = 1$, $\beta = 0.5$, and initial discretization step $h = 0.001$, computed for $t \in [0, 5]$.

In Fig. 3, we show the convergence rate for the numerical solution of (1.14), plotting $\log |u'/u|$ vs $\beta$. The graph exhibits a linear negative relation between the values of $\beta$ and the rate of convergence defined as before.

2. Damped fractional oscillator involving Erdélyi–Kober-type integrals

In this section, we consider a fractional generalization of the damped oscillator equation, involving Erdélyi–Kober-type integrals. There are many investigations nowadays about fractional damped oscillators involving Caputo derivatives, in the framework of the fractional mechanics (see for example [10, 21, 22] and references therein). Here we consider a special form of the fractional damped equation with time-varying coefficients, containing Erdélyi–Kober-type integrals, and we assume the elastic term to be time dependent. Hereafter, $D_t$ will denote the first order ordinary time derivative. A Caputo-type fractional generalization of the damped oscillator equation is thus given by

$$D_t^\alpha D_t x(t) + \lambda D_t x(t) + k x(t) = 0,$$

(2.1)

where $\lambda > 0$ represents the viscosity coefficient and $k > 0$ is the elasticity constant. Since $^C D_t^\alpha x(t) = J_t^{1-\alpha} D_t x(t)$, (2.1) can also be written as

$$J_t^{1-\alpha} (D_t^2 x)(t) + \lambda D_t x(t) + k x(t) = 0.$$  (2.2)

Our purpose is to consider a further, more general model of damped oscillator with memory effects and time-varying coefficients, replacing in (2.2) the Riemann–Liouville integral with the Erdélyi-Kober-type integral

$$\left( I_t^{\eta,1-\alpha} u \right)(t) = \frac{t^{\alpha-1-\eta}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^\eta u(s) \, ds.$$  (2.3)

Moreover, we will assume that the elasticity term depends on time, and, in particular, $k(t) = k_0/t$, where we set $k_0 := \lambda(\eta + 1 - \alpha) > 0$. We
Figure 2. Numerical solution of (1.14) for $\alpha = 0.9$, $\lambda = 1$, initial discretization step $h = 0.001$ and several values of $\beta$, computed for $t \in [0, 5]$. 
thus obtain the following fractional damped oscillator equation with time-varying coefficients,

\[
\left[D_\alpha^\eta D_t^\nu + \lambda t^{\nu+1-\alpha} D_t + k_0 t^{\eta-\alpha}\right] u(t) = 0.
\]  

(2.4)

Note that, for \( \eta = 0 \) and \( \alpha = 1 \), (2.4) reduces to the classical damped oscillator equation with time-dependent elastic term. At best of our knowledge no papers can be found in the literature concerning fractional damped oscillator equations with time-varying coefficients. Looking at (2.4), we can see that, replacing the Riemann-Liouville integral with the Erdélyi-Kober integral, power law time-variable viscosity and elasticity coefficients appear.

The physical meaning of having such a combination (in \( k_0 \)) of the original constants, \( \alpha \) and \( \eta \), is that elasticity, damping, and viscosity effects turn out to be coupled, that is, we may expect that damping ratio and (local) oscillation frequency (if any) are linked. However, we will see from the numerical simulations that oscillatory effects strongly depend on the order \( \alpha \) of the fractional derivatives involved in the model equation.

We now show how this choice is useful from the mathematical point of view. Equation (2.4) can be rewritten, decoupling it into the two auxiliary
coupled equations
\[
\begin{aligned}
D_\alpha^\eta g(t) &= -\lambda t^{\eta + 1 - \alpha} u(t), \\
t^{\eta} D_t u(t) &= D_t g(t).
\end{aligned}
\] (2.5)

This can be shown just by taking the fractional derivative of order \(\alpha\) of both sides of the second equation of the system (2.5), and replacing there \(D_\alpha^\eta g\) as given by the first equation, recalling that \(k_0 = \lambda (\eta + 1 - \alpha)\). We obtain
\[
D_\alpha^\eta D_t g(t) = D_t D_\alpha^\eta g(t) = -\lambda D_t (t^{\eta + 1 - \alpha} u(t)) = D_\alpha^\eta (t^\eta D_t u(t)),
\] (2.6)

where we exploited the fact that, being \(g(0) = 0\), ordinary and fractional derivatives commute. Moreover, in this case the Riemann–Liouville and the Caputo fractional derivatives of \(g(t)\) coincide (see, e.g., [29, pp. 73-74]). This seems to be an interesting result, since equation (2.4) is a fractional differential equation with time-varying coefficients, classically related to fractionally damped oscillation.

The previous equation can be written as a “hybrid” system, in the sense that it involves both, classical and fractional derivatives. Here, the range of the parameters is \(0 < \alpha \leq 1\), \(\lambda > 0\), and \(\eta \geq 0\). Writing \(v(t) := D_t u(t)\), we obtain
\[
\begin{aligned}
D_t u(t) &= v(t) \\
D_\alpha^\eta (t^\eta v(t)) &= -\lambda t^\delta v - \mu t^\gamma u,
\end{aligned}
\] (2.7)

where we set, for short, \(\delta := \eta + 1 - \alpha\), \(\mu := \lambda (\eta + 1 - \alpha)\), and \(\gamma := \eta - \alpha\), and assume as initial conditions \(u(0) = v(0) = 1\). Setting \(w(t) := t^\eta v(t)\), we have
\[
\begin{aligned}
D_t u(t) &= t^{-\eta} w(t) \\
D_\alpha^\eta w(t) &= -\lambda t^{\delta - \eta} w(t) - \mu t^\gamma u(t),
\end{aligned}
\] (2.8)

This approach, adopted to solve numerically (2.4), is suggested again by the method of Diethelm and Ford in [8], extended according to the adaptive pattern developed in [5]), to solve the celebrated Bagley–Torvik equation [1],
\[
A D_\alpha^\eta y + B C D_\alpha^{3/2} y + Cy = f(t),
\] (2.9)

where \(A \neq 0\), \(B\), and \(C\) are real constants. We recall that in [8] equation (2.9) was rewritten as
\[
\begin{aligned}
D_\alpha^{1/2} y_1 &= y_2 \\
D_\alpha^{1/2} y_2 &= y_3 \\
D_\alpha^{1/2} y_3 &= y_4 \\
D_\alpha^{1/2} y_4 &= A^{-1} [-Cy_1 - By_4 + f(t)],
\end{aligned}
\] (2.10)

with the initial conditions \(y_1(0) = y_0\), \(y_2(0) = 0\), \(y_3 = y_0\), \(y_4(0) = 0\).

However, using Gr" uwald-Letnikov discretization, we face the problem of
having unevenly spaced nodes, which problem can be solved resorting to some interpolation. The latter can be merely linear, since the additional error introduced is of the second order, while the order of the method of [8] is of order \(3/2\). Quadratic interpolation was also tried, which introduces third-order (hence negligible) errors, at the price of some little complications.

### 3. Numerical results

In this section, we present some results for the numerical solution of (2.8). We first discuss the role played by the parameters \(\alpha, \lambda,\) and \(\eta\) entering equation (2.8). The fractional order of the derivative has an intrinsic damping effect on oscillatory phenomena, as discussed for example in [33]. In order to understand the role of the parameters \(\lambda\) and \(\eta\), we observe that for \(\alpha = 1\), (2.8) becomes

\[
\left[ D_t^2 + \left( \frac{\eta}{t} + \lambda \right) D_t + \frac{\lambda \eta}{t} \right] u(t) = 0, \tag{3.1}
\]

where we recall that we assumed \(\lambda > 0\) and \(\eta \geq 0\). This is a damped oscillator with time-dependent viscosity and elasticity terms. We remark that in our fractional model, time-variable viscosity, damping, and elasticity terms are coupled. This implies that damping and (instantaneous) oscillation frequency are linked. When \(\eta = 0\) and \(\alpha = 1\), (2.8) reduces to the classical relaxation equation with constant coefficients.

Equation (3.1) can actually be solved explicitly, see, e.g., [18, 30, 34]. A special solution is given by

\[
u(t) = e^{-\lambda t} \left\{ c_1 + c_2 \int_0^t v^{-\eta} e^{\lambda v} dv \right\},
\]

c_1 and c_2 being two arbitrary constants. If \(c_2 \neq 0\), we can see that

\[
u(t) \sim c_2 \frac{1}{t^{\eta}}, \text{ as } t \to +\infty,
\]

for \(\eta > 0\), being \(\lambda > 0\), that is, \(u(t)\) decays monotonically as \(t\) gets large.

The oscillatory behavior of the general solution can be seen proceeding as follows. Setting \(u := p(t)v\) with \(p(t) = t^{-\eta/2} e^{-\lambda t/2}\), \(v\) will satisfy the equation

\[
v'' + q(t)v = 0, \tag{3.2}
\]

where

\[
q(t) := \frac{\eta(2 - \eta)}{4t^2} + \frac{\lambda \eta}{2t} - \frac{\lambda^2}{4}. \tag{3.3}
\]
Now, as is well known, where \( q(t) \geq k^2 > 0 \), for any arbitrary but fixed constant \( k \), the solution is oscillatory in the interval, \((t_-(k), t_+(k))\), where

\[
t_{\pm} = \frac{\lambda \eta \pm \sqrt{2\eta \lambda^2 + 4k^2 \eta (2-\eta)}}{\lambda^2 + 4k^2}.
\]

For small \( k \)'s we have

\[
t_+(k) - t_-(k) \approx 2\sqrt{\frac{2\eta}{\lambda}} \left[ 1 - \frac{k^2}{\lambda^2} (2 + \eta) \right],
\]

and hence about \( t_+(k) - t_-(k) \approx 2\sqrt{2\eta}/\lambda \). By the Sturm’s comparison theorem, the distance between consecutive zeros of \( u(t) \) is less than \( \pi/k \). In fact, by such a theorem, between any two consecutive zeros of any nontrivial solution to \( u'' + k^2 u = 0 \), there exists at least one zero of the solution to (3.2)-(3.3). Since \( t_+(k) - t_-(k) \) (for small \( k \)'s) decreases roughly according to \( \sqrt{\eta}/\lambda \), we infer that several oscillations occur in the interval \((t_-, t_+)\).

On the other hand, the numerical results below show the “physical” role of the fractional order of derivatives, \( \alpha \), which is to introduce a rather strong oscillatory behavior in the solutions, for some special combinations of the parameters \( \lambda \) and \( \eta \).

Generally speaking, varying the parameters \( \lambda \), \( \eta \), and \( \alpha \), in analogy with the well-known solutions of the classical damped oscillator equation, the solutions can be either \textit{overdamped} (i.e., they may have a purely monotonic decay), or \textit{underdamped} (i.e., they may exhibit an oscillatory decay).

In our numerical solutions, we observe both behaviors, according to the physical parameters. In Fig. 4 we plotted the numerical solution of (2.8), for several values of \( \lambda \), keeping fixed the other coefficients. We observed an underdamped-like oscillatory decay. In Fig. 3 we consider the solutions for several values of \( \alpha \) and \( \lambda \), for a fixed value of \( \eta \). In this case, for a fixed \( \alpha \), changing \( \lambda \), the behavior of the solution switches from an overdamped-like monotonic decay to an oscillatory decay pattern, for increasing values of \( \lambda \), as expected. A similar less pronounced switch between different behaviors can be observed in Fig. 3, varying \( \eta \) and \( \alpha \).

In Fig. 6 we plotted \(|u'/u|\) vs \( t \), for \( \lambda = 1, 2, 3 \), correspondingly to the numerical solution of (2.8).

In Fig. 7 we plotted \(|u'/u|\) vs \( t \), for \( \lambda = 5 \) and large \( t \), correspondingly to the numerical solution of (2.8).

**Acknowledgments**

This work was carried out within the framework of the Italian GNFM-INdAM.
Figure 4. Numerical solution of (2.8) for $\alpha = 1.0, \eta = 0.5, \mu = 0.5, \beta = 0.5, \gamma = -0.5$, and different values of $\lambda$, initial discretization step $h = 0.1$, computed for $t \in [0, 40]$. 
Figure 5. Numerical solution of (2.8) for $\alpha = 1.0$, $\eta = 0.5$, $\mu = 0.5$, $\beta = 0.5$, $\gamma = -0.5$, and different values of $\lambda$, initial discretization step $h = 0.1$, computed for $t \in [0, 10]$.

Figure 6. Numerical solution of (2.8) plotted as $|u'/u|$ for $\alpha = 1.0$, $\eta = 0.5$, $\mu = 0.5$, $\beta = 0.5$, $\gamma = -0.5$, and several values of $\lambda$, initial discretization step $h = 0.1$, computed for $t \in [2, 10]$. 
Figure 7. Numerical solution of (2.8), plotted as $|u'/u|$, for $\alpha = 0.9$, $\lambda = 5$, $\eta = 0.5$, $\mu = 0.6$, $\beta = 0.6$, $\gamma = -0.4$, the initial discretization step $h = 0.1$, computed for $t \in [0, 150]$.
Figure 8. Numerical solution of (2.8), plotted for several values of $\alpha \in (0, 1]$ and $\lambda$, with fixed $\eta = 0.5$, initial discretization step $h = 0.001$, computed for $t \in [0, 5]$. 
Figure 9. Numerical solution of (2.8), plotted for different values of $\alpha \in (0,1]$ and $\eta$, with fixed $\lambda = 10$, initial discretization step $h = 0.001$, computed for $t \in [0, 5]$. 


References

[1] R.L. Bagley and P.J. Torvik, Fractional calculus a different approach to the analysis of viscoelastically damped structures, *AIAA J.*, 21 (1983), no. 5, 741-748.

[2] L. Beghin, Fractional relaxation equations and Brownian crossing probabilities of a random boundary, *Advances in Applied Probability*, 44:479–505, (2012).

[3] E. Capelas de Oliveira, F. Mainardi, and J. Vaz Jr., Fractional models of anomalous relaxation based on the Kilbas and Saigo function, *Meccanica*, 10.1007/s11012-014-9930-0, (2014).

[4] K.S. Cole, R.H. Cole, Dispersion and absorption in dielectrics. I. Alternating current characteristics, *J. Chem. Phys.* 9 (1941), 341–351.

[5] M. Concezzi and R. Spigler, Some analytical and numerical properties of the Mittag-Leffler functions, *Fract. Calc. Appl. Anal.*, 18, No. 1 (2015), 64-94; DOI: 10.1515/fca-2015-0006.

[6] D.W. Davidson, R.H. Cole, Dielectrics relaxation in glycerol, propylene glycol, and n-propanol, *J. Chem. Phys.* 19 (1951), 1484–1490.

[7] K. Diethelm and A.D. Freed, “The Frac PECE subroutine for the numerical solution of differential equations of fractional order”, in: S. Heinzel, T. Plesser (Eds.), Forschung und Wissenschaftliches Rechnen 1998, Gessellschaft fur Wissenschaftliche Datenverarbeitung, Göttingen, 1999, pp. 57-71.

[8] K. Diethelm and N.J. Ford, Numerical solution of the Bagley-Torvik equation, *BIT* 42, No. 3 (2002), 490-507.

[9] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer, New York, 2004.

[10] J.-S. Duan, Z. Wang, and S.-Z. Fu, The zeros of the solutions of the fractional oscillation equation, *Fract Calc. Appl. Anal.*, 17, No. 1, (2014), 10–22, DOI: 10.2478/s13540-014-0152-x

[11] R. Garra, A. Giusti, F. Mainardi, and G. Pagnini, Fractional relaxation with time-varying coefficient, *Fract. Calc. Appl. Anal.* 17, No. 2 (2014), 424–439; DOI: 10.2478/s13540-014-0178-0.

[12] R. Garrappa, G. Maione, M. Popolizio, Time-domain simulation for fractional relaxation of Havriliak-Negami type, Proceedings of the 2014 International Conference on Fractional Differentiation and Its Applications (ICFDA), Catania, Italy, June 23-25, 2014; DOI: http://dx.doi.org/10.1109/ICFDA.2014.6967399.

[13] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, *Mittag-Leffler functions. Related Topics and Applications*, Springer Monographs in Mathematics, Berlin, (2014)
[14] R. Gorenflo and F. Mainardi, Fractional calculus, integral and differential equations of fractional order, In: A. Carpinteri and F. Mainardi (Editors), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien (1997), pp. 223-276. [E-print http://arxiv.org/abs/0805.3823]
[15] A. Hanyga, Physically acceptable viscoelastic models, in Hutter, K. and Wang, Y. (Editors), Trends in Applications of Mathematics to Mechanics, Shaker Verlag GmbH, Aachen. 12 pp., (2005)
[16] S.J. Havriliak, S. Negami, A complex plane representation of dielectric and mechanical relaxation processes in some polymers, Polymer, 8 (1967) 161–210.
[17] H. Hilfer, Analytical representations for relaxation functions of glasses, J Non-Cryst Solids, 305:122–126, (2002)
[18] E. Kamke, Differentialgleichungen: Lösungsmethoden und Lösungen, 1, Gewöhnliche Differentialgleichungen, B.G. Teubner, Leipzig, 1977.
[19] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies 204, Elsevier, Amsterdam 2006.
[20] V. Kiryakova, Generalized Fractional Calculus and Applications. Longman Scientific & Technical and J. Wiley, New York (1994).
[21] M. Klime, On solutions of linear fractional differential equations of a variational type, The Publishing Office of Czestochowa University of Technology, Czestochowa 2009.
[22] M. Li, S.C. Lim, C. Cattani, and M. Scalia, Characteristic Roots of a Class of Fractional Oscillators, Advances in High Energy Physics, Vol. 2013 (2013), Article ID 853925, 7 pages.
[23] Y. Luchko, J. Trujillo, Caputo-type modification of the Erdélyi-Kober fractional derivative, Fract. Calc. Appl. Anal. 10, No 3 (2007), 249–267.
[24] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College Press, London, (2010).
[25] F. Mainardi and R. Garrappa, On complete monotonicity of the Prabhakar function and non-Debye relaxation in dielectrics, J. Comput. Phys., (2015), in press, DOI:10.1016/j.jcp.2014.08.006
[26] A.B. Malinowska and D.F. M. Torres, Introduction to the Fractional Calculus of Variations, Imperial College Press, London and World Scientific Publishing, Singapore, 2012.
[27] A.C. McBride, A theory of fractional integration for generalized functions, SIAM J. Math. Anal. 6(3):583–599, (1975).
[28] G. Pagnini, Erdélyi-Kober fractional diffusion, Fract. Calc. Appl. Anal., 15, No. 1 (2012), 117-127; DOI: 10.2478/s13540-012-0008-1.
[29] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[30] A.D. Polyanin and V.F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd Ed., Chapman & Hall/CRC, Boca Raton, 2003.

[31] I.N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*, North-Holland Publ., Amsterdam (1966).

[32] I.N. Sneddon, *The use in mathematical physics of Erdélyi–Kober operators and some of their generalizations*, In: *Lect. Notes Math.* 457, Springer–Verlag, New York (1975), 37–79.

[33] A. Tofighi, The intrinsic damping of the fractional oscillator, *Physica A*, 329(1–2), (2003), 29–34

[34] F.G. Tricomi, *Funzioni Ipergeometriche Confluenti* (Italian), Ed. Cremonese, Roma, 1954.

1 Department of Mathematics and Physics
Roma Tre University
1, Largo S. Leonardo Murialdo
00146 Rome, ITALY
e-mail: concezzi@mat.uniroma3.it
e-mail: spigler@mat.uniroma3.it

2 Department of Statistical Sciences
“Sapienza” University of Rome
5, P.le Aldo Moro
00185 Rome, ITALY
e-mail: roberto.garra@sba.uniroma1.it

Received: MONTH DD, YYYY