Kropina spaces of constant curvature II.  
(long version)

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**Introduction.** In 1991, Matsumoto considered the necessary and sufficient conditions for a Kropina space to be of constant curvature and obtained

**Theorem M**([3], [4]) An n-dimensional Kropina space is of constant curvature $K$, if and only if, putting $r_{ij} = \frac{(b_i + b_j)}{2}$ and $s_{ij} = \frac{(b_i - b_j)}{2}$, we have

1. $r_{ij} = Mb^2a_{ij}$
2. $U_i = u_i + Mb_i$ is a gradient vector field,
3. $w_{ij} = s_{ij}s^j$ is written in the form
4. $s_{ij; k}$ is written in the form
5. the curvature tensor $R_{hijk}$ of the associated Riemannian space has the form

where the symbol $A_{ij}$ denotes interchanges of indices $i$ and $j$ and subtraction,

In [7], we characterized a Kropina space by some Riemannian space $(M, h)$ and a vector field $W$ of constant length 1 on it, and rewrite Theorem M using $h$ and $W$. But, a few years after we noticed that we should replace (3) in Theorem M with the condition

which is equivalent to the equality in Lemma 2 of [3]. Then, it follows that the condition 4 of Theorem 2 in [7] should be omitted. Therefore, we consider the same problem in another way.

In this paper, we rewrite the condition for a Kropina space to be of constant curvature with a Riemannian metric $h$ and a vector filed $W$ and obtain the necessary and sufficient conditions for a Kropina space to be of constant curvature.

In the first section, we will show again that a Kropina space is characterized by some Riemannian metric $h$ and a vector filed $W$ of constant length 1 on it, and in the second section, we will rewrite the coefficients of the geodesic spray in a Kropina space by the Riemannian metric $h$ and the vector field $W$.

In Finsler geometry, the necessary and sufficient condition for a Kropina space to be of scalar curvature is well-known ([2]). We rewrite the condition by $h$ and $W$, and using the relation of $h$ and $W$ we obtain the necessary and sufficient conditions for a Kropina space to be of constant curvature by straightforward calculations. Our main result is Theorem 4. This Theorem 4 is the corrected version of Theorem 2 in [7]. Therefore, this paper is the corrected version of [7].
1 The characterization of a Kropina metric.

Let \((M, \alpha)\) be an \(n (\geq 2)\)-dimensional differential manifold endowed with a Riemannian metric \(\alpha\). A Kropina space \((M, \alpha^2/\beta)\) is a Finsler space whose fundamental function is given by \(F = \alpha^2/\beta\), where \(\alpha = \sqrt{a_{ij}(x)y^iy^j}\) and \(\beta = b_i(x)y^i\). Even though Kropina spaces can be studied in more general case ([4]), in this paper, we suppose that the matrix \((a_{ij})\) is positive definite.

Let us remark that for a Kropina space \((M, \alpha^2/\beta)\) the Kropina metric \(F = \alpha^2/\beta\) can be rewritten as follows:

\[
\frac{\alpha^2}{F^2} - \frac{\beta}{F} + \frac{b^2}{4} = \frac{b^2}{4},
\]

where \(b^2 = a^{ij}b_i b_j\) and \((a^{ij}) = (a_{ij})^{-1}\). Let \(\kappa(x)\) be a function of \((x^i)\) alone. Multiplying the above equation by \(e^{\kappa(x)}\), we have

\[
e^{\kappa(x)}a_{ij} \frac{y^i y^j}{F} - e^{\kappa(x)}a_{ij} \frac{y^i}{F} b^j + \frac{1}{4} e^{\kappa(x)}a_{ij} b^i b^j = \frac{e^{\kappa(x)}b^2}{4}.,
\]

Defining a new Riemannian metric \(h = \sqrt{h_{ij}(x)y^iy^j}\) and a vector field \(W = W^i(\partial/\partial x^i)\) on \(M\) by

\[
h_{ij} = e^{\kappa(x)}a_{ij} \quad \text{and} \quad 2W_i = e^{\kappa(x)}b_i,
\]

where \(W_i = h_{ij}W^j\), the equation (1.1) reduces to

\[
\frac{|y|}{F} - W = |W|.
\]

In the above equation, the symbol \(|\cdot|\) denotes the length of a vector with respect to the Riemannian metric \(h\).

We notice that the equation \(|W| = 1\) holds if and only if the function \(\kappa(x)\) satisfies the condition

\[
e^{\kappa(x)}b^2 = 4.
\]

Suppose that the function \(\kappa(x)\) satisfies (1.3), then we have \(|W| = 1\) and

\[
\frac{|y|}{F} - W = 1.
\]

Therefore, in each tangent space \(T_y M\), the indicatrix of the Kropina metric necessarily goes through the origin.

Conversely, consider a Riemannian space \((M, h)\), where \(h = \sqrt{h_{ij}(x)y^iy^j}\), and a unit vector field \(W = W^i(\partial/\partial x^i)\) on it. Then, we consider the metric \(F\) characterized by (1.4). Solving for \(F\) from (1.4), we get

\[
F = \frac{|y|^2}{\{\sqrt{2h(y, W)}\}^2}.
\]

Comparing the above equality with a Kropina metric \(F = \alpha^2/\beta\), we obtain (1.2) and from the assumption \(|W| = 1\) we get (1.3).

Summarizing the above discussion, we obtain
Theorem 1. Let \((M, \alpha)\) be an \(n(\geq 2)\)-dimensional Riemannian space with a Riemannian metric \(\alpha = \sqrt{a_{ij}(x)y^iy^j}\). For a Kropina space \((M, F = \alpha^2/\beta)\), where \(\beta = b_i(x)y^i\), we define a new Riemannian metric \(h = \sqrt{h_{ij}(x)y^iy^j}\) and a vector field \(W = W^i(\partial/\partial x^i)\) of constant length 1 by (1.2) and (1.3). Then, the Kropina metric \(F\) satisfies the equation (1.4).

Conversely, suppose that \(h = \sqrt{h_{ij}(x)y^iy^j}\) is a Riemannian metric and \(W = W^i(\partial/\partial x^i)\) is a vector field of constant length 1 on \((M, h)\). Consider the metric \(F\) defined by (1.4). Then, defining \(a_{ij}(x) := e^{-\kappa(x)}h_{ij}(x)\) and \(b_i(x) := 2e^{-\kappa(x)}W_i\) by (1.2) using some function \(\kappa(x)\) of \((x)\) alone, we get \(F = \alpha^2/\beta\) and it follows the function \(\kappa(x)\) satisfies (1.3).

\section{The coefficients of the geodesic spray in a Kropina space.}

From the theory of Riemannian spaces, we have the following theorem:

**Theorem A** ([6]) Let \((M, g)\) and \((M, g^* = e^\rho g)\), where \(g = \sqrt{g_{ij}(x)y^iy^j}\) and \(g^* = \sqrt{g^*_{ij}(x)y^iy^j}\) respectively, be two \(n\)-dimensional Riemannian spaces which are conformal each other. Furthermore, let \(\gamma^i_jk\) and \(\gamma^{*i}_jk\) be the coefficients of Levi-Civita connection of \((M, g)\) and \((M, g^*)\), respectively. Then, we have

\[ g^*_{ij} = e^{2\rho}g_{ij}, \quad g^{*ij} = e^{-2\rho}g^{ij}, \quad \gamma^{*i}_jk = \gamma^i_jk + \rho_j\delta^i_k + \rho_k\delta^i_j - \rho^i g_{jk}, \]

where \(\rho_i = \partial\rho/\partial x^i\) and \(\rho^i = g^{ij}\rho_j\).

From (1.2), we have \(h_{ij} = e^\kappa a_{ij}\). Applying Theorem A, we get

\[ h_{ij}^* = \gamma_{ij}^* + \frac{1}{2}\kappa_{ij}^* + \frac{1}{2}\kappa_{k}^*\delta_{ij}^* - \frac{1}{2}\kappa^* a_{jk}, \]

where \(h_{ij}^*\) and \(\gamma_{ij}^*\) are the coefficients of Levi-Civita connection of \((M, h)\) and \((M, \alpha)\) respectively, \(\kappa_i = \partial\kappa/\partial x^i\) and \(\kappa^i = a^i_j\kappa_j\). Transvecting (2.1) by \(y^j y^k\), we get

\[ h_{00}^* = \alpha_{00}^* + \kappa_0 y^j - \frac{1}{2}h_{00}^* \sigma^j, \]

where \(\sigma^j = h^{ij}\kappa_j\).

We denote the covariant derivative in the Riemannian space \((M, \alpha)\) by \((\cdot)_{\alpha}\) and put as follows:

\[ s_{ij} := \frac{b_{ij} - b_{ji}}{2}, \quad r_{ij} := \frac{b_{ij} + b_{ji}}{2}, \quad s_j := b^i s_{ij}, \quad r_j := b^i r_{ij}. \]

In [1], the authors have shown that the coefficients \(G^i\) of the geodesic spray in a Finsler space \((M, F = \alpha\phi(\beta/\alpha))\) are given by

\[ 2G^i = \alpha_{00}^* + 2\alpha s_i^0 + 2\Theta(r_{00} - 2\alpha s_0^0)\left(\frac{y^i}{\alpha} + \frac{\omega'}{\omega - s\omega}\right), \]

where

\[ \omega := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{\omega - s\omega'}{2(1 + s\omega + (b^2 - s^2)\omega')}\]
For a Kropina space, we have $\phi(s) = 1/s$, hence we obtain

$$\phi' = -\frac{1}{s^2}, \quad \omega = -\frac{1}{2s}, \quad \omega' = \frac{1}{2s^2}$$

and

$$\frac{\omega'}{\omega - s\omega'} = -\frac{1}{2s}, \quad r_{00} - 2\alpha\omega s_0 = r_{00} + F s_0, \quad 1 + s\omega + (b^2 - s^2)\omega' = \frac{b^2}{2s^2}, \quad \Theta = -\frac{s}{b^2}.$$

Substituting the above equalities in (2.3) and using (2.2), we get

$$2G^i = h_{\gamma^0^i}^j - \kappa_0 y^i + \frac{1}{2} h_{00}^j - F s^i_0 - \frac{1}{b^2}(r_{00} + F s_0)(\frac{2}{F} y^i - b^i).$$

From Theorem 1, for a Kropina space $(M, \alpha^2/\beta)$, a new Riemannian metric $h = \sqrt{h_{ij}(x)y^i y^j}$ and a vector field $W = W^i(\partial/\partial x^i)$ are defined by (1.2) and (1.3). So, the vector field $W$ satisfies the condition $|W| = 1$ and we have $F = h_{00}/2W_0$.

Therefore, we get

(2.4) \quad 2G^i = h_{\gamma^0^i}^j + 2\Phi^i,$n

where

(2.5) \quad 2\Phi^i := -\kappa_0 y^i + \frac{1}{2} h_{00}^j - \frac{h_{00}}{2W_0} s^i_0 - \frac{1}{b^2}(r_{00} + \frac{h_{00}s_0}{2W_0})(\frac{4W_0}{h_{00}} y^i - b^i).$

Using (2.1), we have

$$b_{i;j} = 2e^{-\kappa W^i_{||j}} + e^{-\kappa}\left(\kappa_i W_j - \kappa_j W_i - W_r^r h_{ij}\right),$$

where the notation $(||)$ stands for the $h$-covariant derivative in the Riemannian space $(M, h)$.

**Remark 1** We can introduce a Finsler connection $\Gamma^* = (h_{\gamma^i^j}^k(x), N^j_i := h_{\gamma^i^j}^k(x) y^k, C^i_{jk})$ associated with the linear connection $h_{\gamma^i^j}^k(x)$ of the Riemannian space $(M, h)$. The $h$-covariant derivative are defined as follows ([2]):

For a vector field $W^i(x)$ on $M$,

$$1, \quad W^i(x)_{||j} = \frac{\partial W^i}{\partial x^j} - \frac{\partial W^i}{\partial y^s} N^s_j + h_{\gamma^i^j}^s W^s = \frac{\partial W^i}{\partial x^j} + h_{\gamma^i^j}^s W^s.$$

For a reference vector $y^i$,

$$2, \quad y^i_{||j} = \frac{\partial y^i}{\partial x^j} - \frac{\partial y^i}{\partial y^s} N^s_j + h_{\gamma^i^j}^s y^s = -N^i_j + N^i_j = 0.$$
We put
\[ R_{ij} := \frac{W_{i||j} + W_{j||i}}{2}, \quad S_{ij} := \frac{W_{i||j} - W_{j||i}}{2}, \quad R^i_j := h^{ir}R_{rj}, \quad S^i_j := h^{ir}S_{rj}. \]

It follows
\[ r_{ij} = 2e^{-\kappa} \left( R_{ij} - \frac{1}{2} W_r \kappa^r h_{ij} \right), \quad s_{ij} = 2e^{-\kappa} \left( S_{ij} + \frac{\kappa_i W_j - \kappa_j W_i}{2} \right). \]

Furthermore, we get
\[
\begin{align*}
    s^i_j &= 2S^i_j + \kappa^i W_j - \kappa_j W^i, \\
    s^i_0 &= 2S^i_0 + W_0 \kappa^i - \kappa_0 W^i, \\
    s_i &= 2e^{-\kappa} \left( 2S_i + W_i \kappa^r W_0 - \kappa_i \right), \\
    s_0 &= 2e^{-\kappa} \left( 2S_0 + W_r \kappa^r W_0 - \kappa_0 \right), \\
    r_{00} &= 2e^{-\kappa} \left( R_{00} - \frac{1}{2} W_r \kappa^r h_{00} \right), \\
    b^i &= a^{ir}b_r = e^\kappa h^{ir} \frac{2W_r}{e^\kappa} = 2W^i.
\end{align*}
\]

Substituting the above equality in (2.5), we have
\[ (2.6) \quad 2\Phi^i = \frac{h_{00}}{W_0} (S_0 W^i - S^i_0) + (R_{00} W^i - 2S_0 y^i) - \frac{2W_0}{h_{00}} R_{00} y^i. \]

Multiplying now the above equality by $2h_{00}W_0$, we get
\[ 4h_{00}W_0 \Phi^i = 2(h_{00})^2 (S_0 W^i - S^i_0) + 2h_{00}W_0 (R_{00} W^i - 2S_0 y^i) - 4(W_0)^2 R_{00} y^i \]

and by putting
\[ A^i_{(1)} := 2(S_0 W^i - S^i_0), \quad A^i_{(2)} := 2(R_{00} W^i - 2S_0 y^i), \quad A^i_{(3)} := -4R_{00} y^i, \]

it follows
\[ (2.7) \quad 4h_{00}W_0 \Phi^i = (h_{00})^2 A^i_{(1)} + h_{00}W_0 A^i_{(2)} + (W_0)^2 A^i_{(3)}. \]

### 3 The necessary and sufficient conditions for a Kropina space to be of constant curvature.

In this section, we consider a Kropina space $(M, F = \alpha^2/\beta)$ of constant curvature $K$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and $\beta = b_i(x) y^i$. Furthermore, we suppose that the matrix $(a_{ij})$ is always positive definite and that the dimension $n$ is more than or equal two. Hence, it follows that $\alpha^2$ is not divisible by $\beta$. This is an important relation and is equivalent to that $h_{00}$ is not divisible by $W_0$. Using these, we will obtain the necessary and sufficient conditions for a Kropina space to be of constant curvature.

#### 3.1 The curvature tensor of a Kropina space.

Let $R^{i}_{j\ kl}$ be the $h$-curvature tensor of Cartan connection in Finsler space. The Berwald’s spray curvature tensor is
\[ (3.1) \quad (b)R^{i}_{j\ kl} = A_{(kl)} \left\{ \frac{\partial G^i_j}{\partial x^k} + G^r_j G^i_r \right\}. \]
It is well-known that the equality \( R^i_{0kl} = h\gamma^i_{0j} + 2\Phi^i_j \) holds good ([2]).

From \( 2G^i = h\gamma^i_{0j} + 2\Phi^i_j \), it follows
\[
G^i_j = h\gamma^i_{0j} + \Phi^i_j \quad \text{and} \quad G^i_{jk} = h\gamma^i_{jk} + \Phi^i_{jk}.
\]

Substituting the above equalities in (3.2), we get
\[
R^i_{jkl} = hR^i_{jkl} + A_{(kl)}\left\{ \Phi^i_{j|k} + \Phi^i_{r|k} \Phi^r_i \right\}.
\]

The following Proposition is well-known ([2]):

**Proposition 1.** The necessary and sufficient condition for a Finsler space \((M, F)\) to be of scalar curvature \(K\) is that the equality
\[
R^i_{00} = KF^2(\delta^i_l - l^i l_l),
\]
where \(l^i = g^i/F\) and \(l_l = \partial F/\partial g^i\), holds.

If the equality (3.2) holds good and \(K\) is constant, then the Finsler space is called of constant curvature \(K\).

For a Kropina space of constant curvature \(K\), the equality (3.2) holds and \(K\) is constant.

Since \(F = \alpha^2/\beta\) is written as \(F = h_{00}/(2W_0)\), we have
\[
\delta^i_l - l^i l_l = \delta^i_l - \frac{2W_0 h_{0l} - h_{00} W_l}{h_{00} W_0} g^i.
\]

Using the curvature we obtained above, we have
\[
R^i_{00} = hR^i_{00} + 2\Phi^i_{|l} - \Phi^i_{l|0} + 2 \Phi^r \Phi_{r|l} - \Phi_{r|l} \Phi^i_r.
\]

Substituting the above equalities in (3.2), we get
\[
(3.3) KF^2\left(\delta^i_l - \frac{2W_0 h_{0l} - h_{00} W_l}{h_{00} W_0} g^i\right) = hR^i_{00} + 2\Phi^i_{|l} - \Phi^i_{l|0} + 2 \Phi^r \Phi_{r|l} - \Phi_{r|l} \Phi^i_r.
\]

### 3.2 Rewriting the equation (3.3) using \(h_{00}\) and \(W_0\).

(1) The calculations for \(\Phi^i_{|l}\).

First, applying the \(h\)-covariant derivative \(|l|\) to (2.7), it follows
\[
4h_{00} W_{0|l} \Phi^i + 4h_{00} W_0 \Phi^i_{|l} = (h_{00})^2 A^i_{(1)|l} + h_{00} W_0 A^i_{(2)|l} + h_{00} W_0 A^i_{(3)|l} + (W_0)^2 A^i_{(3)|l} + 2W_0 W_0 A^i_{(3)}
\]

and using again (2.7), we get
\[
4h_{00} (W_0)^2 \Phi^i_{|l} = (h_{00})^2 W_0 A^i_{(1)|l} - (h_{00})^2 A^i_{(1)} W_0|l + h_{00} (W_0)^2 A^i_{(2)|l} + (W_0)^3 A^i_{(3)|l} + (W_0)^2 A^i_{(3)} W_0|l.
\]

Putting now
\[
B^i_{(1)|l} = A^i_{(1)|l}, \quad B^i_{(2)|l} = -A^i_{(1)} W_0|l, \quad B^i_{(22)|l} = A^i_{(2)|l}, \quad B^i_{(3)|l} = A^i_{(3)|l}, \quad B^i_{(4)|l} = A^i_{(3)} W_0|l,
\]
we have
\[(3.4) \quad 4h_{00}(W_0)^2\Phi^i_{|\mu} = (h_{00})^2W_0B^i_{(1)\mu} + (h_{00})^2B^i_{(21)\mu} + h_{00}(W_0)^2B^i_{(22)\mu} + (W_0)^3B^i_{(3)\mu} + (W_0)^2B^i_{(4)\mu}.
\]

(2) The calculations for $\Phi^i_{\mu}$.
Secondly, derivating (2.7) by $y^j$, we get
\[
8h_{00}W_0\Phi^i + 4h_{00}W_1\Phi^i + 4h_{00}W_0\Phi^i_{\mu} = (h_{00})^2A^i_{(1)\mu} + h_{00}W_0A^i_{(2)\mu} + h_{00}(4h_{00}A^i_{(1)} + W_1A^i_{(2)}) + (W_0)^2A^i_{(3)\mu} + 2W_0(W_1A^i_{(3)} + h_{00}A^i_{(2)}),
\]
where the notation $(\mu)$ stands for the derivation by $y^\mu$.
Using the above equality and (2.7), we obtain
\[(3.5) \quad 4(h_{00})^2(W_0)^2\Phi^i_{\mu} = (h_{00})^3W_0C^i_{(0)\mu} + (h_{00})^3C^i_{(11)\mu} + (h_{00})^3(W_0)^2C^i_{(12)\mu} + (h_{00})^2W_0C^i_{(21)\mu} + h_{00}(W_0)^3C^i_{(22)\mu} + h_{00}(W_0)^2C^i_{(3)\mu} + (W_0)^3C^i_{(4)\mu},
\]
where
\[
\begin{align*}
C^i_{(0)\mu} &:= A^i_{(1)\mu} = 2(S_0W^i - S^i_0)W_1, & C^i_{(11)\mu} &:= -A^i_{(1)}W_1 = -2(S_0W^i - S^i_0)W_1, \\
C^i_{(12)\mu} &:= A^i_{(2)\mu} = 4(R_0W^i - S_0y^i - S^i_0\delta^l_1), & C^i_{(21)\mu} &:= 2A^i_{(1)}h_{0\mu} = 4(S_0W^i - S^i_0)h_{0\mu}, \\
C^i_{(22)\mu} &:= A^i_{(3)\mu} = -4(2R_0y^i + R_0\delta^l_1), & C^i_{(3)\mu} &:= W_1A^i_{(3)} = -4R_0W_1y^i, \\
C^i_{(4)\mu} &:= -2A^i_{(3)}h_{0\mu} = 8R_0h_{0\mu}y^i.
\end{align*}
\]

(3) The calculations for $\Phi^i_{\mu\nu}\Phi^i_{\nu}$.
Applying the $h$-covariant derivative $|\nu$ to (3.5), we get
\[
8(h_{00})^2W_0W_0|0\Phi^i + 4(h_{00})^2(W_0)^2\Phi^i_{\mu|0} = (h_{00})^3W_0C^i_{(0)|0} + (h_{00})^3(W_0)|0C^i_{(0)\mu|0} + (h_{00})^3(W_0)^2C^i_{(11)\mu|0} + (h_{00})^2W_0|0C^i_{(12)\mu|0} + (h_{00})^2W_0(W_0)|0C^i_{(21)\mu|0} + h_{00}(W_0)^3C^i_{(22)\mu|0} + h_{00}(W_0)^2(3W_0|0C^i_{(22)\mu|0} + C^i_{(3)\mu|0}) + 2h_{00}W_0W_0|0C^i_{(3)\mu|0} + (W_0)^3C^i_{(4)\mu|0} + 3(W_0)^2W_0|0C^i_{(4)\mu|0}.
\]
Using the above equality and (3.5), we obtain
\[(3.6) \quad 4(h_{00})^2(W_0)^2\Phi^i_{\mu|0} = (h_{00})^3(W_0)^2D^i_{(1)|0} + (h_{00})^3W_0D^i_{(21)|0} + (h_{00})^3D^i_{(31)|0} + (h_{00})^3W_0D^i_{(22)|0} + (h_{00})^3W_0D^i_{(32)|0} + (h_{00})^2W_0D^i_{(41)|0} + (h_{00})^2W_0D^i_{(42)|0} + (W_0)^4D^i_{(5)|0} + (W_0)^3D^i_{(6)|0},
\]
where
\[
\begin{align*}
D^i_{(1)|0} &= C^i_{(0)|0}, & D^i_{(21)|0} &= C^i_{(11)|0} - W_0|0C^i_{(0)|0}, & D^i_{(31)|0} &= -2W_0|0C^i_{(11)|0}, \\
D^i_{(22)|0} &= C^i_{(12)|0}, & D^i_{(32)|0} &= C^i_{(21)|0}, & D^i_{(41)|0} &= -W_0|0C^i_{(21)|0}, \\
D^i_{(33)|0} &= C^i_{(22)|0}, & D^i_{(42)|0} &= W_0|0C^i_{(22)|0} + C^i_{(3)|0|0}, & D^i_{(5)|0} &= C^i_{(4)\mu|0}, & D^i_{(6)|0} &= W_0|0C^i_{(4)\mu|0}.
\end{align*}
\]

(4) The calculations for $\Phi^i\Phi^i_{\tau}$.
Using (3.5), we have

\[
(3.7) \quad 16(h_{00})^4(W_0)^3 \Phi_{r} \Phi^i_{r}
\]

\[
= (h_{00})^6(W_0)^2 E^{(01)l} + (h_{00})^6 W_0 E^{(11)l} + (h_{00})^6 E^{(21)l} + (h_{00})^5(W_0)^3 E^{(12)l}
\]

\[
+ (h_{00})^5(W_0)^2 E^{(22)l} + (h_{00})^5 W_0 E^{(31)l} + (h_{00})^4(W_0)^4 E^{(23)l} + (h_{00})^4(W_0)^3 E^{(32)l}
\]

\[
+ (h_{00})^4(W_0)^2 E^{(41)l} + (h_{00})^3(W_0)^5 E^{(33)l} + (h_{00})^3(W_0)^4 E^{(42)l} + (h_{00})^3(W_0)^3 E^{(51)l}
\]

\[
+ (h_{00})^2(W_0)^6 E^{(43)l} + (h_{00})^2(W_0)^5 E^{(52)l} + (h_{00})^2(W_0)^4 E^{(61)l} + h_{00}(W_0)^6 E^{(62)l}
\]

\[
+ h_{00}(W_0)^5 E^{(7)l} + (W_0)^6 E^{(8)l},
\]

where

\[
E^{(0)l} := C^{(0)}_r C^{(0)r}_l, \quad E^{(1)l} := C^{(1)}_r C^{(1)r}_l, \quad E^{(2)l} := C^{(2)}_r C^{(2)r}_l,
\]

\[
E^{(3)l} := C^{(3)}_r C^{(3)r}_l + C^{(0)}_r C^{(0)r}_l + C^{(2)}_r C^{(2)r}_l + C^{(1)}_r C^{(1)r}_l, \quad E^{(4)l} := C^{(4)}_r C^{(4)r}_l + C^{(3)}_r C^{(3)r}_l + C^{(1)}_r C^{(1)r}_l + C^{(2)}_r C^{(2)r}_l,
\]

\[
E^{(5)l} := C^{(5)}_r C^{(5)r}_l + C^{(4)}_r C^{(4)r}_l + C^{(3)}_r C^{(3)r}_l + C^{(2)}_r C^{(2)r}_l + C^{(1)}_r C^{(1)r}_l + C^{(0)}_r C^{(0)r}_l,
\]

\[
E^{(6)l} := C^{(6)}_r C^{(6)r}_l + C^{(5)}_r C^{(5)r}_l + C^{(4)}_r C^{(4)r}_l + C^{(3)}_r C^{(3)r}_l + C^{(2)}_r C^{(2)r}_l + C^{(1)}_r C^{(1)r}_l + C^{(0)}_r C^{(0)r}_l,
\]

\[
E^{(7)l} := C^{(7)}_r C^{(7)r}_l + C^{(6)}_r C^{(6)r}_l + C^{(5)}_r C^{(5)r}_l + C^{(4)}_r C^{(4)r}_l,
\]

\[
E^{(8)l} := C^{(8)}_r C^{(8)r}_l.
\]

(5) The calculations for \( \Phi_{r} \Phi^i_{r} \).

Derivating (3.5) by \( y_r \), we get

\[
16 h_{00} \Phi_{r} + 8 (h_{00})^2 W_0 \Phi_{r} + 4 (h_{00})^2 W_0 \Phi_{r} + 4 (h_{00})^2 (W_0)^2 \Phi_{r} + 4
\]

\[
= 6 (h_{00})^2 h_{00} W_0 C^{(0)}_r + (h_{00})^3 W_0 C^{(0)}_r + (h_{00})^3 W_0 C^{(0)}_r + (h_{00})^3 W_0 C^{(0)}_r
\]

\[
+ 6 (h_{00})^2 h_{00} W_0 C^{(1)}_r + (h_{00})^3 W_0 C^{(1)}_r + (h_{00})^3 W_0 C^{(1)}_r
\]

\[
+ 4 (h_{00})^2 W_0 C^{(2)}_r + 2 (h_{00})^2 W_0 C^{(2)}_r + (h_{00})^2 W_0 C^{(2)}_r + (h_{00})^2 W_0 C^{(2)}_r
\]

\[
+ 4 (h_{00})^2 W_0 C^{(3)}_r + (h_{00})^2 W_0 C^{(3)}_r + (h_{00})^2 W_0 C^{(3)}_r + (h_{00})^2 W_0 C^{(3)}_r
\]

\[
+ 2 (h_{00})^2 W_0 C^{(4)}_r + 3 h_{00} (W_0)^2 W_0 C^{(4)}_r + h_{00} (W_0)^2 C^{(4)}_r
\]

\[
+ 2 (h_{00})^2 W_0 C^{(5)}_r + 2 h_{00} W_0 W_0 C^{(5)}_r + h_{00} (W_0)^2 C^{(5)}_r
\]

\[
+ 3 (W_0)^2 W_0 C^{(6)}_r + (W_0)^3 C^{(6)}_r.
\]

Using the above equality, (3.5) and \( C^{(0)}_r = 0 \), we have

\[
4 (h_{00})^3 (W_0)^3 \Phi_{r} \Phi^i_{r}
\]

\[
= (h_{00})^4 W_0 H_{(0)rl} + (h_{00})^4 H^{(0)}_{(1)rl} + (h_{00})^3 (W_0)^3 H^{(0)}_{(2)rl} + (h_{00})^3 (W_0)^2 H^{(0)}_{(2)rl}
\]

\[
+ (h_{00})^3 W_0 H^{(1)}_{(1)rl} + (h_{00})^2 (W_0)^4 H^{(1)}_{(3)rl} + (h_{00})^2 (W_0)^3 H^{(1)}_{(3)rl}
\]

\[
+ h_{00} (W_0)^4 H^{(2)}_{(2)rl} + h_{00} (W_0)^3 H^{(3)}_{(2)rl} + (W_0)^4 H^{(3)}_{(2)rl},
\]

where

\[
H^{(0)}_{(0)rl} = C^{(1)}_{(1)rl} - W_0 C^{(0)}_{(0)rl}, \quad H^{(1)}_{(1)rl} = -2 W_0 C^{(1)}_{(1)rl}, \quad H^{(2)}_{(2)rl} = C^{(1)}_{(1)rl},
\]
\[ H_{(12)r}^i = 2h_0 C_{(0)(0)}^i + C_{(21)r}^i, \quad H_{(21)r}^i = -W_r C_{(21)}^i + 2h_0 C_{(11)r}^i, \]
\[ H_{(13)r}^i = C_{(22)r}^i, \quad H_{(22)r}^i = W_r C_{(22)}^i + C_{(31)r}^i, \quad H_{(31)r}^i = C_{(41)r}^i - 2h_0 C_{(22)r}^i, \]
\[ H_{(41)r}^i = W_r C_{(41)}^i - 2h_0 C_{(31)r}^i, \quad H_{(51)r}^i = -4h_0 C_{(41)}^i. \]

Using the above equality and (2.7), we get

\[(3.8)16(h_{00})^4(W_0)^4\Phi_r^i\Phi_i^j = (h_{00})^6W_0 J_{(11)r}^i + (h_{00})^6J_{(21)r}^i + (h_{00})^5(W_0)^3J_{(12)r}^i + (h_{00})^5(W_0)^2J_{(22)r}^i + (h_{00})^5J_{(31)r}^i + (h_{00})^4(W_0)^4J_{(23)r}^i + (h_{00})^4(W_0)^3J_{(32)r}^i + (h_{00})^4(W_0)^2J_{(41)r}^i + (h_{00})^4(W_0)^3J_{(51)r}^i + (h_{00})^4(W_0)^2J_{(52)r}^i + (h_{00})^4(W_0)^3J_{(61)r}^i + (h_{00})^4(W_0)^2J_{(62)r}^i + (h_{00})^4(W_0)^3J_{(71)r}^i + (h_{00})^4(W_0)^2J_{(72)r}^i,
\]

where

\[
\begin{align*}
J_{(11)r}^i &= A_{(1)}^r H_{(01)r}^i, \quad J_{(21)r}^i = A_{(1)}^r H_{(11)r}^i, \quad J_{(12)r}^i = A_{(1)}^r H_{(02)r}^i, \\
J_{(22)r}^i &= A_{(1)}^r H_{(12)r}^i + A_{(2)}^r H_{(01)r}^i, \quad J_{(31)r}^i = A_{(1)}^r H_{(21)r}^i + A_{(2)}^r H_{(11)r}, \\
J_{(23)r}^i &= A_{(1)}^r H_{(13)r}^i + A_{(2)}^r H_{(02)r}^i, \quad J_{(32)r}^i = A_{(1)}^r H_{(22)r}^i + A_{(2)}^r H_{(12)r} + A_{(3)}^r H_{(01)r}^i, \\
J_{(41)r}^i &= A_{(2)}^r H_{(21)r}^i + A_{(3)}^r H_{(11)r}^i, \quad J_{(33)r}^i = A_{(2)}^r H_{(23)r}^i + A_{(3)}^r H_{(13)r}^i, \\
J_{(42)r}^i &= A_{(1)}^r H_{(33)r}^i + A_{(2)}^r H_{(22)r}^i + A_{(3)}^r H_{(12)r}^i, \\
J_{(51)r}^i &= A_{(1)}^r H_{(41)r}^i + A_{(3)}^r H_{(21)r}^i, \quad J_{(43)r}^i = A_{(3)}^r H_{(31)r}^i, \\
J_{(52)r}^i &= A_{(2)}^r H_{(33)r}^i + A_{(3)}^r H_{(23)r}^i, \quad J_{(61)r}^i = A_{(2)}^r H_{(41)r}^i + A_{(3)}^r H_{(21)r}, \\
J_{(62)r}^i &= A_{(3)}^r H_{(33)r}^i, \quad J_{(71)r}^i = A_{(2)}^r H_{(42)r}^i + A_{(3)}^r H_{(32)r}^i, \quad J_{(72)r}^i = A_{(3)}^r H_{(51)r}^i.
\end{align*}
\]

(6) The main relation.

Multiplying (3.3) by 16(h_{00})^6(W_0)^4 and using F^2 = (h_{00})^2/\{4(W_0)^2\}, we have the equality

\[
4K(h_{00})^6(W_0)^2 h^i - 8K(h_{00})^5(W_0)^2 h_0 g^i + 4K(h_{00})^6 W_0 W_0 g^i.
\]

Substituting (3.4), (3.5), (3.6), (3.7) and (3.8) in the above equality, we get

\[
4K(h_{00})^6(W_0)^2 \delta^i - 8K(h_{00})^5(W_0)^2 h_0 g^i + 4K(h_{00})^6 W_0 W_0 g^i = \]
\[
- (h_{00})^6(2J_{(11)r}^i - E_{(11)r}^i) + (h_{00})^6 W_0(2J_{(21)r}^i - E_{(11)r}^i),
\]

\[
+ (h_{00})^5(2J_{(21)r}^i - E_{(21)r}^i) + (h_{00})^5(W_0)^3(8B_{(1)r}^i - 4D_{(1)r}^i + 2J_{(21)r}^i - E_{(12)r}^i)
\]

\[
+ (h_{00})^5(W_0)^2(-4D_{(21)r}^i + 8B_{(21)r}^i + 2J_{(22)r}^i - E_{(22)r}^i),
\]

\[
+ (h_{00})^5 W_0(2J_{(31)r}^i - 4D_{(31)r}^i - E_{(31)r}^i),
\]

\[
+ (h_{00})^4(W_0)^4(16h_0 R_{(0)r}^i + 8B_{(22)r}^i - 4D_{(22)r}^i + 2J_{(23)r}^i - E_{(23)r}^i),
\]

\[
+ (h_{00})^4(W_0)^3(2J_{(32)r}^i - 4D_{(32)r}^i - E_{(32)r}^i) + (h_{00})^4(W_0)^2(2J_{(41)r}^i - E_{(41)r}^i - 4D_{(41)r}^i)
\]

\[
+ (h_{00})^4(W_0)^5(8B_{(3)r}^i - 4D_{(33)r}^i + 2J_{(33)r}^i - E_{(33)r}^i),
\]

\[
+ (h_{00})^4(W_0)^4(2J_{(42)r}^i + 8B_{(42)r}^i - 4D_{(42)r}^i - E_{(42)r}^i),
\]

\[
+ (h_{00})^4(W_0)^3(2J_{(51)r}^i - E_{(51)r}^i + (h_{00})^2(W_0)^2(2J_{(43)r}^i - E_{(43)r}^i)
\]

\[
+ (h_{00})^2(W_0)^5(2J_{(52)r}^i - 4D_{(52)r}^i - E_{(52)r}^i) + (h_{00})^2(W_0)^4(2J_{(61)r}^i - 4D_{(61)r}^i - E_{(61)r}^i)
\]

\[
+ h_0(W_0)^6(2J_{(62)r}^i - E_{(62)r}^i) + h_0(W_0)^5(2J_{(71)r}^i - E_{(71)r}^i) + (W_0)^6(2J_{(8)r}^i - E_{(8)r}^i).
\]
Taking into account the equalities $J_{(21)i}^l = 0$ and $E_{(21)i}^l = 0$, we have

\begin{equation}
(3.9) \quad (h_{00})^4 P_{(5)i}^l + (h_{00})^2 Q_{(9)i}^l + (W_0)^4 R_{(9)i}^l = 0,
\end{equation}

where

\begin{align*}
P_{(5)i}^l & = (h_{00})^2 W_0 ( - E_{(0)i}^l - 4K \delta_i^l ) \\
& \quad + (h_{00})^2 (2J_{(11)i}^l - E_{(11)i}^l - 4KW_i y^i) \\
& \quad + (W_0)^2 (8B_{(1)l}^i - 4D_{(1)i}^l + 2J_{(12)i}^l - E_{(12)i}^l) \\
& \quad + (W_0) ( - 4D_{(21)i}^l + 8B_{(21)i}^l + 2J_{(22)i}^l - E_{(22)i}^l + 8K h_{00} y^i) \\
& \quad + (W_0)^3 (16B_{(22)i}^l - 4D_{(22)i}^l + 2J_{(23)i}^l - E_{(23)i}^l) \\
& \quad + (W_0)^2 (2J_{(32)i}^l - 4D_{(32)i}^l - E_{(32)i}) \\
Q_{(9)i}^l & = (h_{00})^3 (2J_{(31)i}^l - 4D_{(31)i}^l - E_{(31)i}) \\
& \quad + (h_{00})^2 W_0 (2J_{(41)i}^l - E_{(41)i}^l - 4D_{(41)i}^l) \\
& \quad + h_{00} (W_0)^3 (8B_{(3)l}^i - 4D_{(33)i}^l + 2J_{(33)i}^l - E_{(33)i}^l) \\
& \quad + h_{00} (W_0)^3 (2J_{(42)i}^l + 8B_{(41)i}^l - 4D_{(42)i}^l - E_{(42)i}^l) \\
& \quad + (W_0) ( - 4D_{(51)i}^l + 2J_{(51)i}^l - E_{(51)i}) \\
& \quad + (W_0)^3 (2J_{(43)i}^l - E_{(43)i}) \\
& \quad + (W_0)^4 (2J_{(52)i}^l - 4D_{(5)i}^l - E_{(52)i}) \\
& \quad + (W_0)^3 (2J_{(61)i}^l - 4D_{(6)i}^l - E_{(61)i}) \\
R_{(9)i}^l & = h_{00} W_0 (2J_{(62)i}^l - E_{(62)i}^l) \\
& \quad + (W_0)^2 (2J_{(7)i}^l - E_{(7)i}^l) \\
& \quad + (W_0)^2 (2J_{(8)i}^l - E_{(8)i}^l).
\end{align*}

We call $P_{(5)i}^l$, $Q_{(9)i}^l$ and $R_{(9)i}^l$ the curvature part, the vanishing part and the Killing part, respectively.

**Proposition 2** The necessary and sufficient condition for a Kropina space $(M,F = \alpha^2 / \beta = h_{00}/2W_0)$ to be of constant curvature $K$ is that (3.9) holds good.

### 3.3 The Killing part.

We consider the Killing part $R_{(9)i}^l$ and obtain the conclusion that the vector field $W$ is Killing.

First, we have

\begin{align*}
J_{(62)i}^l & = - 64R_{00}(2R_{00} h_{0i} + h_{00} R_{0i}) y^i - 32(R_{00})^2 h_{00} \delta_i^l, \\
E_{(62)i}^l & = - 64R_{00} h_{00} R_{0i} y^i - 128(R_{00})^2 h_{00} y^i, \\
J_{(7)i}^l & = - 32R_{00} \{ (3R_{00} W_0 - 4S_0 h_{00}) h_{0i} + h_{00} R_{00} W_1 \} y^i, \\
E_{(7)i}^l & = - 32(R_{00})^2 (h_{00} W_i y^i + W_0 h_{00} y^i), \\
2J_{(8)i}^l - E_{(8)i}^l & = 192(R_{00})^2 h_{00} h_{00} y^i.
\end{align*}

Using the above equalities, we get

\begin{equation}
R_{(9)i}^l = - 32h_{00} R_{00} \left( W_0 (2R_{00} h_{00} \delta_i^l + 2h_{00} R_{00} y^i + 7R_{00} h_{00} y^i) - 8S_0 h_{00} h_{00} y^i + R_{00} h_{00} W_1 y^i \right).
\end{equation}
Substituting the above equality in (3.9) and dividing it by \( W_0 h_{00} \), we get

\[
(3.10) \quad (h_{00})^3 P_{(5)i}^i + h_{00} Q_{(9)i}^i + 8S_0 h_{00} h_{00} y_i' + 7 R_{00} h_{00} W_i y_i' = 0.
\]

**Lemma 1**  
In the equation (3.10), it follows that \( R_{00} \) is divisible by \( h_{00} \).

**(Proof.)** Suppose that \( R_{00} \) is not divisible by \( h_{00} \) and since \((h_{ij})\) is positive definite, \((R_{00})^2\) is not divisible by \( h_{00} \).

Taking into account that \( P_{(5)i}^i \) and \( Q_{(9)i}^i \) are homogeneous polynomials of \( y^i \) and that \((W_0)^2\) is not divisible by \( h_{00} \), it follows that the equation

\[
h_{00} y_i' = h_{00} \eta_i',
\]

where \( \eta_i'(x) \) is a function of \((x^i)\) alone, holds good. Transvecting the above equation by \( W_i \), we get

\[
W_0 y_i' = h_{00} \eta_i'(x) W_i.
\]

Since \( h_{00} \) is not divisible by \( W_0 \), the above equation is impossible. Q.E.D.

Therefore, it follows that \( R_{00} \) is divisible by \( h_{00} \) and the following equation holds:

\[
R_{00} = c(x) h_{00},
\]

where \( c(x) \) is a function of \((x^i)\) alone. Derivating the above equation by \( y^i \) and \( y^j \), we get

\[
(3.11) \quad W_{i||j} + W_{j||i} = 2c(x) h_{ij}.
\]

Transvecting (3.11) by \( W^i W^j \), we get \( W_{i||j} W^i W^j = c(x) h_{ij} W^i W^j \) and using \( h_{ij} W^i W^j = |W|^2 = 1 \) and \( W_{i||j} W^i = 0 \), we obtain \( c(x) = 0 \). Therefore, it follows that the equality

\[
(3.12) \quad R_{ij} = 0
\]

holds good. Hence, we have that \( W \) is a Killing vector field. Therefore, we can state

**Lemma 2**  
If a Kropina space \((M, \alpha^2/\beta)\) is of constant curvature \( K \), then
(1) \( W(x) \) is a Killing vector field,
and then
(2) \( \text{Killing part } R_{(9)i}^i = 0 \).

Using (3.12), the equation (3.10) reduces to

\[
(3.13) \quad (h_{00})^2 P_{(5)i}^i + Q_{(9)i}^i = 0
\]

and we have the following equalities:

\[
(3.14) \quad W_{i||j} = S_{ij}, \quad S_{j} = W_{i||j} W^i = 0, \quad W_{0||j} = S_{0j}, \quad W_{i||0} = S_{i0}, \quad W_{0||0} = 0.
\]
3.4 The vanishing part.

In this subsection, we will show that the equality $Q^{i}_{(9)t} = 0$ holds from the conclusion $\mathbb{R}_{00} = 0$ in the previous subsection.

Using (3.12) and (3.14), the $A's$, $B's$, $C's$, $D's$, $E's$, $H's$ and $J's$ reduce to

\[
A^{i}_{(1)t} = -2W^{i}_{|0}|t, \quad B^{i}_{(1)t} = -2W^{i}_{|0}|t, \quad B^{i}_{(21)t} = 2W^{i}_{|0}W_{0}|t, \quad C^{i}_{(0)t} = -2W^{i}_{|t}, \\
C^{i}_{(11)t} = 2W^{i}_{|0}W_{t}, \quad C^{i}_{(21)t} = -4W^{i}_{|0}h_{0t}, \quad D^{i}_{(1)t} = -2W^{i}_{|t|0}, \\
D^{i}_{(21)t} = 2W^{i}_{|0}W_{t} + 2W^{i}_{|0}W_{0}, \quad D^{i}_{(32)t} = -4W^{i}_{|0}h_{0t}, \quad E^{i}_{(0)t} = 4W^{i}_{|t}W^{r}_{|t}, \\
E^{i}_{(1)t} = -4W^{i}_{|t}W^{r}_{|0}W_{t}, \quad E^{i}_{(22)t} = 8W^{i}_{|0}W_{0}|t + 8W^{i}_{|r}W^{r}_{|0}h_{0t}, \\
H^{i}_{(01)rt} = 2W^{i}_{|r}W_{t} + 2W^{i}_{|0}W_{t}, \quad H^{i}_{(11)rt} = -4W^{i}_{|t}W^{r}_{|t}, \\
H^{i}_{(12)r4} = -4(h_{0r}W^{i}_{|l} + W^{i}_{|r}h_{0l} + W^{i}_{|0}h_{rl}), \quad H^{i}_{(21)r4} = 4(W_{r}W^{i}_{|0}h_{0l} + h_{0r}W^{i}_{|0}W_{t}), \\
J^{i}_{(1)t} = -4W^{i}_{|t}W^{r}_{|0}W_{t}, \quad J^{i}_{(22)t} = 8W^{i}_{|r}W^{r}_{|0}h_{0t} + 8W^{i}_{|0}W_{t}|0, \\
\]

and the others are zero. Using the above equalities, we get

\[
2J^{i}_{(31)t} - 4D^{i}_{(31)t} - E^{i}_{(31)t} = 0, \quad 2J^{i}_{(41)t} - E^{i}_{(41)t} - 4D^{i}_{(41)t} = 0, \\
8B^{i}_{(3)} - 4D^{i}_{(33)t} + 2J^{i}_{(33)t} - E^{i}_{(33)t} = 0, \quad 2J^{i}_{(42)t} + 4B^{i}_{(4)} - 4D^{i}_{(42)t} - E^{i}_{(42)t} = 0, \\
2J^{i}_{(51)t} - E^{i}_{(51)t} = 0, \quad 2J^{i}_{(43)t} - E^{i}_{(43)t} = 0, \\
2J^{i}_{(52)t} - 4D^{i}_{(52)t} - E^{i}_{(52)t} = 0, \quad 2J^{i}_{(61)t} - 4D^{i}_{(61)t} - E^{i}_{(61)t} = 0.
\]

Therefore, from Lemma 2, it follows

**Lemma 3** If a Kropina space $(\mathbb{M}, \alpha^{2}/\beta)$ is of constant curvature $K$, then

1. vanishing part $Q^{i}_{(9)t} = 0$,

and then

2. curvature part $P^{i}_{(5)t} = 0$.

3.5 The curvature part.

In this subsection, we will show that Lemma 3 implies that $(\mathbb{M}, h)$ is a Riemannian space of constant curvature $K$.

Using the results given at the beginning of the previous subsection, we have

\[
-E^{i}_{(0)t} - 4K\delta^{i}_{t} = -4W^{i}_{|r}W^{r}_{|t} - 4K\delta^{i}_{t}, \\
2J^{i}_{(11)t} - E^{i}_{(11)t} - 4KW^{i}_{l}y^{l} = -4W^{i}_{|r}W^{r}_{|0}W_{t} - 4KW^{i}_{l}y^{l}, \\
8B^{i}_{(1)} - 4D^{i}_{(11)t} + 2J^{i}_{(12)t} - E^{i}_{(12)t} = -16W^{i}_{|0}|t + 8W^{i}_{|0}|0, \\
-4D^{i}_{(21)t} + 8B^{i}_{(21)t} + 2J^{i}_{(22)t} - E^{i}_{(22)t} + 8Kho_{y^{i}} = -8W^{i}_{|0}|0}W_{t} + 8W^{i}_{|r}W^{r}_{|0}h_{0t} + 8Kho_{y^{i}}, \\
16h_{0}^{i} + 8B^{i}_{(22)t} - 4D^{i}_{(22)t} + 2J^{i}_{(23)t} - E^{i}_{(23)t} = 16h_{0}^{i}, \\
2J^{i}_{(32)t} - 4D^{i}_{(32)t} - E^{i}_{(32)t} = 16W^{i}_{|0}h_{0t}.
\]

Therefore, from (2) of Lemma 3 and the above equalities we have

\[
(3.15) - \frac{1}{4}P^{i}_{(5)t} = (h_{00})^{2}W_{0}(W^{i}_{|r}W^{r}_{|t} + K\delta^{i}_{t}) + (h_{00})^{2}(W^{i}_{|r}W^{r}_{|0}W_{t} + KW_{l}y^{l}) \\
+ 2h_{00}(W_{0})^{2}(2W^{i}_{|0}|t - W^{i}_{|0}|0) + 2h_{00}W_{0}(W^{i}_{|0}|0W_{t} - W^{i}_{|r}W^{r}_{|0}h_{0t} - Kho_{y^{i}}) \\
- 4(W_{0})^{3} h_{0}^{i} - 4(W_{0})^{2}W^{i}_{|0}|0}h_{0t} = 0,
\]
First, we consider the term \((h_{00})^2(W_i\|_i W^r\|_0 W_l + K W_l y^i)\) which does not contain \(W_0\). Taking into account that \((h_{00})^2\) is not divisible by \(W_0\), the following equality must hold good

\[ (3.16) \quad W^i\|_i W^r\|_0 W_l + K W_l y^i = W_0 c_l^i(x), \]

where \(c_l^i(x)\) are functions of \((x^i)\) alone, must hold. Transvecting \((3.16)\) by \(W^l\) and dividing it by \(W_0\), we get

\[ W^i\|_i W^r\|_0 + K y^i = c_l^i(x) y^l. \]

Derivating it by \(y^l\), we have

\[ W^i\|_i W^r\|_l + K \delta^i_l = c_l^i(x). \]

Substituting the above equality in \((3.16)\), we get

\[ W^i\|_i W^r\|_0 W_l + K W_l y^i = W_0(W^i\|_i W^r\|_l + K \delta^i_l). \]

Transvecting the above equality by \(W^l\), we get

\[ W^i\|_i W^r\|_0 + K y^i = K W_0 W^i, \]

where we have used \(W^i\|_i W^r\|_l W^l = 0\). Derivating the above equality by \(y^l\), we get

\[ (3.17) \quad W^i\|_i W^r\|_l = K W_l W^i - K \delta^i_l. \]

Substituting \((3.17)\) in \((3.15)\) and dividing it by \(2W_0\), it follows

\[ (3.18) \quad K(h_{00})^2W_l W^i + h_{00}W_0(2W^i\|_l W_l - W^i\|_l W^l) + h_{00}(W^i\|_l W_l - K W_0 W^i h_{0l}) - 2(W_0)^2 \cdot h^0 R^i_{0l} - 2W_0 W^i\|_l h_{0l} = 0. \]

Transvecting the above equality by \(h_{0i}\), we get

\[ (3.19) \quad W_{0\|i\|0} = K(h_{00}W_i - W_0 h_{0i}). \]

Using \(W_{i\|i} + W_{j\|i} = 0\) and \((3.19)\), we have

\[ (3.20) \quad W_{l\|0\|0} = -K(h_{00}W_l - W_0 h_{0l}). \]

Derivating \((3.19)\) by \(y^i\), we have

\[ W^i\|_l W^l_{\|0} + W_{0\|l\|i} = K(2h_{0i}W_l - W_i h_{0l} - W_0 h_{il}). \]

From the above equality, we have

\[ (3.21) \quad W_{i\|0\|l} = -W_{i\|l\|0} - K(2h_{0i}W_i - W_l h_{0i} - W_0 h_{li}). \]

Using \((3.20)\) and \((3.21)\), the equality \((3.18)\) reduces to

\[ 3h_{00}(K W_l y^i - K h_{0l} W^i - W^i\|_l W_l) - 2W_0(h^0 R^i_{0l} + K y^i h_{0l} - K h_{00} \delta^i_l) = 0. \]

Since \(h_{00}\) is not divisible by \(W_0\), it follows that the equality

\[ (3.22) \quad h^0 R^i_{0l} + K h_{0l} y^i - K h_{00} \delta^i_l = h_{00} d^i_l(x), \]
where \( d^i_t(x) \) are functions of \((x^i)\) alone, must hold. Transvecting the above equality by \( y^j \), we get \( d^i_t(x)y^j = 0 \). Derivating the above equality by \( y^l \), we get \( d^i_t(x) = 0 \). Substituting it in (3.22), we get

\[
(3.23) \quad hR^i_{0\ell} + Kh_{0l}y^l - Kh_{00}\delta^i_l = 0.
\]

We can rewrite the above equality as

\[
(3.23) \quad hR^i_{0\ell} = K(h_{00}\delta^i_l - h_{0l}y^l).
\]

Derivating (3.23) by \( y^j \) and \( y^k \), we get

\[
(3.24) \quad hR^i_{jk} + hR^i_{kl} = K(2h_{jk}\delta^i_l - h_{jl}\delta^i_k - h_{kl}\delta^i_j),
\]

and interchanging \( j \) and \( l \), we obtain

\[
(3.25) \quad hR^i_{lj} + hR^i_{lj} = K(2h_{lk}\delta^i_j - h_{lj}\delta^i_k - h_{jk}\delta^i_l).
\]

Subtracting (3.25) from (3.24), we get

\[
(3.25) \quad hR^i_{jk} + 2hR^i_{kj} - hR^i_{kj} = 3K(h_{jk}\delta^i_l - h_{kl}\delta^i_j).
\]

Since the left-hand side of the above equality can be changed as follows:

\[
(3.25) \quad hR^i_{jk} + 2hR^i_{kj} - hR^i_{kj} = 2hR^i_{jk} - hR^i_{jk} - hR^i_{kj} = 2hR^i_{jk} + hR^i_{kj} = 3hR^i_{jk},
\]

we obtain

\[
(3.25) \quad hR^i_{jk} = K(h_{jk}\delta^i_l - h_{kl}\delta^i_j).
\]

This means that the Riemannian space \((M, h)\) is of constant curvature \( K \).

Therefore, we obtain

**Theorem 2** Let \( M \) be an \((\geq 2)\)-dimensional Riemannian manifold. Put

\[
\alpha = \sqrt{a_{ij}(x)y^iy^j} \quad \text{and} \quad \beta = b_i(x)y^i.
\]

Let \((M, \alpha^2/\beta)\) be a Kropina space and define a new Riemannian metric \( h = \sqrt{h_{ij}(x)y^iy^j} \) and a vector field \( W \) with \(|W| = 1\) by (1.2) and (1.3).

If the Kropina space \((M, \alpha^2/\beta)\) is of constant curvature \( K \), then the vector field \( W \) is a Killing one and the Riemannian space \((M, h)\) is of constant curvature \( K \).

### 3.6 The converse of Theorem 2.

Let \((M, \alpha^2/\beta)\) be a Kropina space and define a new Riemannian metric \( h = \sqrt{h_{ij}(x)y^iy^j} \) and a vector field \( W \) with \(|W| = 1\) by (1.2) and (1.3). Suppose that the vector field \( W \) is a Killing one and that the Riemannian space \((M, h)\) is of constant curvature \( K \). To prove that the Kropina space \((M, \alpha^2/\beta)\) is of constant curvature \( K \), we have only to show that the equality (3.9) holds.

Since the vector field \( W \) is a Killing one, we have \( R_{00} = 0 \). Taking into account (2) of Lemma 2 and (1) of Lemma 3, the Killing part \( R \) and the vanishing part \( Q \) vanishes respectively.

Therefore, we have only to show that the curvature part \( P^i_{(5|l)} \) vanishes. At the rest of this subsection, we will prove it. The curvature part \( P^i_{(5|l)} \) is defined by (3.15).
First, we give the following Lemma 4:

**Lemma 4**  
For a Killing vector field $W = W^i(\partial/\partial x^i)$ of constant length $|W| = 1$, the equality

$$(3.26) 
W_{i||j||k} = W_r^r R_k^{r\ ij}$$

holds good.

**(Proof.)** From the Ricci’s formula, it follows

$$(3.27) 
W_{i||j||k} - W_{i||k||j} = -W_r^r R_i^{r\ jk}.$$  

On the other hand, since $W$ is a Killing vector field, we have

$$W_{i||j} + W_{j||i} = 0.$$  

Applying the $h$-covariant derivative $||k$ to the above equality, we get

$$W_{i||j||k} + W_{j||i||k} = 0.$$  

Replacing $i, j, k$ by $j, k, i$ and $k, i, j$ respectively, we obtain the following equalities:

$$W_{j||k||i} + W_{k||j||i} = 0,$$

$$W_{k||i||j} + W_{i||k||j} = 0.$$  

Subtracting the second equality from the first equality and adding the third equality to it, we get

$$(3.28) 
W_{i||j||k} + W_{i||k||j} - W_r^r R_j^{r\ ik} - W_r^r R_k^{r\ ij} = 0.$$  

From (3.27) and (3.28), we get

$$2W_{i||j||k} = -W_r^r R_j^{r\ ik} + W_r^r R_j^{r\ ij} + W_r^r R_k^{r\ ij}.$$  

Using the formula

$$h R_i^{r\ jk} + h R_j^{r\ ki} + h R_k^{r\ ij} = 0,$$  

we have

$$2W_{i||j||k} = W_r^r R_j^{r\ ki} + W_r^r R_k^{r\ ij} + W_r^r R_j^{r\ ij} + W_r^r R_k^{r\ ij} = 2W_r^r R_k^{r\ ij}$$

that is, (3.26) holds good. Q.E.D.

From the assumption that the Riemannian space $(M, h)$ is of constant curvature, we have

$$(3.29) 
h R_k^{r\ ji} = K(h_{kj}\delta_i^r - h_{ki}\delta_j^r).$$  

Using the above equality and (3.26), we get

$$(3.30) 
W_i^{r||j||k} = K(\delta_i^r W_j - h_{kj} W^r)$$
and from here and \( y^i_{||j} = 0 \) (See, Remark 1), it follows
\[
W^i_{||0} = K(\delta^i_l W_0 - h_{l0} W^i), \quad W^i_{|0|} = K(y^i W_0 - h_{00} W^i), \quad W^i_{||i} = K(y^i W_i - h_{i0} W^i).
\]

(3.31)

From (3.29), we have
\[
h R^i_{0l} = K(h_{00} \delta^i_l - h_{0l} y^i)
\]
and applying the \( h \)-covariant derivative \( ||i \) to the equality \( |W|^2 = W_r W^r h^{rs} = 1 \), we get
\[
W_{||r} W^r = -W_{|r} W^r = 0.
\]

Furthermore, applying the \( h \)-covariant derivative \( ||l \) to the above equality, we obtain
\[
W_{||l} W^r_{||l} + W_{|l|l} W^r = 0.
\]

From the above equality and (3.30), we have
\[
W_{||l} W^r_{||l} = -W_{|l|l} W^r = K(h_{il} W_l - h_{li} W_i) W^r = K(W_l W_i - h_{li}).
\]

(3.33)

Substituting the equalities (3.31)-(3.33) in (3.15), we can easily recognize the curvature part \( P^i_{5l} = 0 \). Therefore, (3.9) holds good. Hence, from Proposition 2, we get

**Theorem 3** Let \( M \) be an \( n \geq 2 \)-dimensional Riemannian space. Put \( \alpha = \sqrt{a_{ij}(x)y^i y^j} \) and \( \beta = b_i(x)y^i \). Let \( (M, \alpha^2/\beta) \) be a Kropina space and define a new Riemannian metric \( h = \sqrt{h_{ij}(x)y^i y^j} \) and a vector field \( W = W^i(\partial/\partial x^i) \) of constant length \( |W| = 1 \) by (1.2) and (1.3).

If the vector field \( W = W^i(\partial/\partial x^i) \) is a Killing one and the Riemannian space \( (M, h) \) is of constant curvature \( K \), the Kropina space \( (M, \alpha^2/\beta) \) is of constant curvature \( K \).

From Theorem 2 and Theorem 3, we have

**Theorem 4** Let \( (M, \alpha^2/\beta) \) be an \( n \geq 2 \)-dimensional Kropina space, where \( \alpha^2 = a_{ij}(x)y^i y^j, \beta = b_i(x)y^i \) and the matrix \( (a_{ij}) \) is positive definite. For the Kropina space, we define a new Riemannian metric \( h = \sqrt{h_{ij}(x)y^i y^j} \) and a vector field \( W = W^i(\partial/\partial x^i) \) of constant length \( |W| = 1 \) on \( M \) by (1.2) and (1.3).

Then, the Kropina space \( (M, \alpha^2/\beta) \) is of constant curvature \( K \) if and only if the following conditions hold:

1. \( W_{||j} + W_{j||} = 0 \), that is, \( W = W^i(\partial/\partial x^i) \) is a Killing vector field.
2. The Riemannian space \( (M, h) \) is of constant curvature \( K \).

Let \( (M, F = \alpha^2/\beta) \) be an \( n \geq 2 \)-dimensional Kropina space. From Theorem 1, for this Kropina metric \( F = \alpha^2/\beta \), we can define a Riemannian metric \( h = \sqrt{h_{ij}(x)y^i y^j} \) and a vector field \( W = W^i(\partial/\partial x^i) \) of constant length 1 on \( M \) by (1.2) and (1.3). We suppose that the vector field \( W \) is a Killing one. Then, we have \( R_{00} = 0 \). From this assumption, we get the second equation of (3.14), that is, \( S_0 = 0 \). Substituting \( R_{00} = 0 \), \( S_0 = 0 \) and \( F = h_{00}/(2W_0) \) in (2.6), we obtain the equation \( \Phi^i = -FS^i_0 \). Substituting this in (2.4),
we get

**Theorem 5** Let \((M, F = \alpha^2/\beta)\) be an \(n(\geq 2)\)-dimensional Kropina space.

We define a Riemannian metric \(h = \sqrt{h_{ij}(x)y^i y^j}\) and a vector field \(W = W^i(\partial/\partial x^i)\) of constant length \(|W| = 1\) on \((M, h)\) by (1.2) and (1.3).

Suppose that the vector field \(W\) is a Killing one, then the coefficients \(G^i\) of the geodesic spray of the Kropina space \((M, \alpha^2/\beta)\) is written as follows:

\[
2G^i = h_{\gamma^0}^i - 2FS_0^i,
\]

where \(h_{\gamma^0}^i\) are Christoffel symbols of the Riemannian space \((M, h)\).

**Remark 2** The geodesic spray of the Randers space \((M, \alpha + \beta)\) is given in the subsection 2.3 ([5] p.5). Comparing to this, the geodesic spray of the Kropina space \((M, F = \alpha^2/\beta)\) is in a very simple form (3.33).

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