CONTROLLABILITY OF A BASIC COCHLEA MODEL

SCOTT W. HANSEN
Department of Mathematics
Iowa State University
Ames, IA 50011, USA

Abstract. Two variations of a basic model for a cochlea are described which consist of a basilar membrane coupled with a linear potential fluid. The basilar membrane is modeled as an array of oscillators which may or may not include longitudinal elasticity. The fluid is assumed to be a linear potential fluid described by Laplace’s equation in a domain that surrounds the basilar membrane. The problem of controllability of the system is considered with control active on a portion of the basilar membrane. Approximate controllability is proved for both models and moreover lack of exact controllability is shown to hold when longitudinal stiffness is not included.

1. Introduction. The cochlea is the sensory organ inside the inner ear that converts mechanical vibrations transmitted through the middle ear from the ear drum to nerve impulses corresponding to specific frequencies. The history of cochlea modeling goes back at least to von Helmholtz (1863) [5], who proposed the “piano string” theory, wherein the cochlea was considered to be a sequence of resonators that each resonate at a specific frequency. Later experiments by Békésy [1] from the 1940’s into the 1960’s indicated the presence of traveling wave phenomena which played a key role in describing where along the length of the cochlea, the most resonance occurs. Békésy was awarded the Nobel Prize 1961 for his “traveling wave theory” and related work on the cochlea. Various mathematical models for the cochlea have been proposed since the 1950’s which incorporate the fluid-elastic coupling of the cochlear fluid and the basilar membrane; see e.g., Ranke [16], Lighthill, [9], Neely [13]. For general history of cochlea modeling see Luce [10], [11].

The goal of this paper is to analyze the controllability structure of the cochlea. We consider here two variations of perhaps the most basic of type of model for a cochlea that includes the fluid structure coupling. The motion of the cochlear fluid is assumed to be governed by a potential function that satisfies the two-dimensional Laplace’s equation. The flexible portions of the fluid boundary consist of the basilar membrane (BM), the oval window (OW) and round window (RW). For simplicity we model the OW and RW as springs and BM is modeled as and a continuous array of springs, which could include longitudinal elasticity. Such assumptions are generally well-accepted, (see Neely, [13]) at least as a first approximation and are used in many papers, e.g., [18], [13], [7], [8], [6] We do not consider here active

2010 Mathematics Subject Classification. Primary: 93B05, 93C20; Secondary: 74F10, 74K15.

Key words and phrases. Controllability, cochlea model, distributed control, fluid structure interaction, potential fluid.

The research was supported in part by the Institute for Mathematics for its Applications during Spring of 2016. This research is also supported by NSF grant DMS-1312952.
tuning effect due e.g., to inner and outer hair cell sensing and actuation; see e.g., Rhode [17], Neely and Kim [14] and [11].

Roughly, we wish to determine the extent to which controlling the displacements over a small portion of the BM and possibly also on the RW and OW influences the motion of the entire basilar membrane. A better understanding of the controllability structure of a cochlea could be helpful in improving the design of hearing aids and cochlear implants.

While there exists a fairly large volume of literature on control of fluid-elastic structures, and also a large literature on analysis of cochlear mechanics, there seems to be very few articles that consider either mathematical well-posedness or controllability. We mention the thesis of Chepkwony [2], who proved a simpler version of the approximate controllability result Theorem 4.1 in the case where the basilar membrane does not include stiffness and also investigated boundary control by the multiplier method in the case of small membrane elasticity. Some related models that are partly applicable to the cochlea include Hansen and Lyashenko [4] and Hansen [3] where exact boundary controllability results for beams and membranes with potential fluid coupling.

We begin by describing the model and proving well-posedness in appropriate function spaces. Secondly, we prove that approximate controllability holds with control active on an arbitrarily small open subset of the BM. In the case without longitudinal elasticity, we show that when control is active on a proper subinterval of the BM, exact controllability does not hold.

In terms of controllability, the presence of the potential fluid creates a technical difficulty since the model becomes nonlocal. Thus approximate controllability can not be obtained from unique continuation of the zero solution from the elastic component of the system. Instead we deduce approximate controllability from the unique continuation property of the elliptic potential equation that governs the fluid.

This paper is organized as follows: In section 2 we formulate the cochlea model in the polygonal geometry described by diagram Figure 1. In section 3 we prove well-posedness of the model. In section 4 we prove the main controllability results and comment on possible geometrical generalizations.

2. Cochlea model. The precise geometrical assumptions that we require are described later in Remark 1, but for the purpose of explaining the model, we consider the two-dimensional polygonal geometry for the cochlear cavity indicated in Figure 1. The domain of the cochlear fluid $\Omega$ is assumed to have a boundary $\Gamma = \partial \Omega$ consisting of a fixed boundary $\Gamma_f$ and a flexible boundary $\Gamma_O \cup \Gamma_R \cup \Gamma_B$ corresponding to the oval window (OW), round window (RW) and basilar membrane (BM), respectively. The lengths of the OW and RW are $\ell_O$ and $\ell_R$, respectively. By scaling we can assume that the BM has length $\pi$. The BM is the union of lower and upper portions of the BM: $\Gamma_B = \Gamma_B^- \cup \Gamma_B^+$. We consider the BM to have a positive thickness $h$ and view the end of the BM as a part of the fixed boundary $\Gamma_f$.

In terms of the mechanics of the cochlea, sound waves cause vibrations of the ear drum which are transmitted through the middle ear to the stapes, which is attached to the OW on the $\Gamma_O$ portion of the boundary of the cochlear cavity. Since the fluid is essentially incompressible, inward motions at the OW are matched by outward motions at the RW on $\Gamma_R$, creating a pressure difference in the upper and lower
chambers of the cochlea, which drives the motion of the BM. It is worth pointing out that in an actual cochlea there is a hole, called the helicotrema in the (2 dimensional) BM that allows fluid to flow between the upper and lower chambers. Therefore in this sense the gap between right end of the BM and $\Gamma_f$ would seem to be a necessary feature of the model, although it not considered in many models.

![Figure 1. The domain of the cochlear fluid.](image_url)

We use $(x, y)$ with the $x$-axis through the centerline of the BM, and the $y$-axis coinciding with $\Gamma_O$ and $\Gamma_R$ in Figure 1. The transverse displacement of the BM is given by $w(x, t), 0 < x < \pi, t > 0$, where $x$ is the horizontal coordinate and $y$ is the vertical coordinate. The displacement of the OW and RW are described by $\eta(y, t)$ on $\Gamma_O, t > 0$ and $\xi(y, t)$ on $\Gamma_R, t > 0$, respectively.

The vibrations of the membranes on the OW and RW are modeled as spring-mass systems. For simplicity we assume the displacements are constant with respect to $y$ over their respective domains, i.e., $\eta(y, t) = \eta(t)$ and $\xi(y, t) = \xi(t)$. (Perhaps a more physically plausible family of motions would be of the form $\eta(y, t) = \eta(t)\nu(y)$ and $\xi(y, t) = \xi(t)\zeta(y)$, but this leads to essentially the same model.)

The fluid is assumed to have a constant density $\rho$ and a velocity potential satisfying Laplace’s equation. More precisely, the fluid velocity is given by $\nabla \phi$, where $\phi$ satisfies $\Delta \phi = 0$ on $\Omega$. Matching velocities on $\Gamma = \partial \Omega$ requires that $\phi$ and its outward normal derivative $\phi_n$ satisfy

$$
\begin{align*}
\Delta \phi &= 0 \quad \text{in } \Omega \\
\phi_n &= 0 \quad \text{on } \Gamma_f \\
\phi_n &= -\phi_y = -w_t \quad \text{on } \Gamma_B^+ \\
\phi_n &= \phi_y = w_t \quad \text{on } \Gamma_B^- \\
\phi_n &= -\phi_x = \eta_t \quad \text{on } \Gamma_O \\
\phi_n &= -\phi_x = \xi_t \quad \text{on } \Gamma_R
\end{align*}
$$

(1)

In (1), $\Omega$ is assumed to be a fixed domain so that the velocities are matched on the equilibrium positions of the flexible boundary. Thus we do not consider nonlinearities due to the free boundary problem that results from matching velocities on
the deformed boundary. On the other hand, the scale of deformations are on the order of nanometers and are well within the linear realm. Furthermore we do not consider the effects of viscosity of the fluid. This assumption is supported partially by an analysis in [6].

Since (1) is a pure Neumann problem, the motions of the OW and RW are constrained to have average 0. Hence

$$\xi = -\frac{\ell_O}{\ell_R} \eta. \quad (2)$$

The incompressibility constraint does not however restrict the possible motions of the BM since a displacement in the upper chamber is balanced by an opposite displacement in the lower chamber.

After eliminating $\xi$, (1) becomes

$$\begin{align*}
\Delta \phi &= 0 \quad \text{in } \Omega \\
\phi_n &= 0 \quad \text{on } \Gamma_f \\
\phi_n &= -w_t \quad \text{on } \Gamma_B^+ \\
\phi_n &= w_t \quad \text{on } \Gamma_B^- \\
\phi_n &= \eta_t \quad \text{on } \Gamma_O \\
\phi_n &= -\frac{\ell_O}{\ell_R} \eta_t \quad \text{on } \Gamma_R
\end{align*} \quad (3)$$

The BM is modeled as a continuous array of springs with linear mass density $m_0(x)$ and stiffness density $k_0(x)$, $x \in (0, \pi)$. If longitudinal elasticity is included, in addition to the spring coupling, the BM has a variable, strictly positive stress coefficient $\sigma(x)$ ($x \in (0, \pi)$). Typically, the stiffness of the BM decreases by several orders of magnitude over the length of the cochlea and hence variable stiffness coefficients $\mu_0(x)$, $\sigma(x)$ is considered physically essential to the model.

For simplicity, we assume that the spring constants for the OW and RW have respective constant mass densities $m_1 = m_1(y)$ and $m_2 = m_2(y)$ and constant stiffness densities $k_1 = k_1(y)$ and $k_2 = k_2(y)$ on their domains, either $\Gamma_O$ or $\Gamma_R$.

The energy $E(t)$ is the sum of the kinetic $K(t)$ and potential $P(t)$ energies where

$$K = \frac{1}{2} \int_0^\pi \rho |\nabla \phi|^2 \, d\Omega + \int_0^\pi \frac{1}{2} m_0 w_t^2 \, dx + \int_{\Gamma_O} \frac{1}{2} m_1 \eta_t^2 \, dy + \int_{\Gamma_R} \frac{1}{2} m_2 \xi_t^2 \, dy$$

and

$$P = \frac{1}{2} \int_0^\pi (k_0 w^2 + \sigma(x) (w_x)^2) \, dx + \int_{\Gamma_O} \frac{1}{2} k_1 \eta^2 \, dy + \int_{\Gamma_R} \frac{1}{2} k_2 \xi^2 \, dy.$$

We assume the BM is subject to external applied force $F_0(x, t)$, $0 < x < \pi$, while the oval window is subject to external applied constant force density $F_1(t)$ on $\Gamma_O$. The work integral is given by

$$W(t) = \int_0^\pi F_0 w \, dx + \int_0^{\ell_O} F_1 \eta \, dy.$$

We form the action integral of the Lagrangian $L = \int_0^T (K + W - P) \, dt$, where using the displacement assumptions and the constraint (2),

$$L = \int_0^T \left\{ \frac{1}{2} \int_\Omega \rho |\nabla \phi|^2 \, d\Omega + \frac{1}{2} \int_0^\pi (m_0(x) w_t^2 - k_0(x) w^2 - \sigma(x) w_x^2) \, dx \\
+ \frac{\ell_O}{2} (m_1 \eta_t^2 - k_1 \eta^2) + \frac{\ell_R^2}{\ell_R} (m_2 \eta_2^2 - k_2 \eta^2) + \int_0^\pi F_0 w \, dx + \ell_O F_1 \eta \right\} \, dt.$$
In much of this paper we will need to distinguish between the cases \( \sigma \equiv 0 \) and \( \sigma(x) \geq \sigma_0 > 0 \), which have very different mathematical and control theoretic properties.

If \( \sigma = 0 \) no boundary conditions need to be specified.

In the case \( \sigma > 0 \) boundary conditions need to be specified at the ends \( x = 0 \) and \( x = \pi \) of the BM, and various boundary conditions could be considered. The geometry of Figure 1 seems to suggest boundary conditions of the form

\[
\begin{align*}
  w(0, t) &= 0, & w_x(\pi) &= 0 \quad (t > 0).
\end{align*}
\]

On the other hand, since there should be some flexibility of the structure supporting the BM, it might be more realistic to replace the Dirichlet condition in (4) by a Robin type condition \( \alpha w(0, t) + w_x(0, t) = 0 \). However, to fix ideas, for now assume the boundary conditions in (4) hold if \( \sigma > 0 \).

The equations of motion can be obtained from the principle of virtual work. That is, the first variation, with respect to the class of admissible variations satisfying (1) and (2), of \( L \) is set to zero. One obtains

\[
\begin{align*}
  m_0(x) \ddot{w} - (\sigma w_x)_x + k_0(x) w + [\rho \phi_t]_{\Gamma_R} &= F_0, \quad x \in (0, \pi), \quad t > 0 \quad (5) \\
  \tilde{m}_1 \ddot{\eta} + \tilde{k}_1 \eta + \rho [\phi_t]_{\Gamma_O,R} &= F_1, \quad t > 0 \quad (6)
\end{align*}
\]

where \( \phi \) satisfies (1) and where we have used

\[
\tilde{m}_1 = (m_1 + \frac{\ell_O}{\ell_R} m_2), \quad \tilde{k}_1 = (k_1 + \frac{\ell_O}{\ell_R} k_2)
\]

The natural energy space \( E \) for the system is

\[
E = (w, \eta, w_t, \eta_t, \phi) \in H^{1,\sigma}(\Omega) \times \mathbb{R} \times L^2(\Gamma_B) \times \mathbb{R} \times H^1(\Omega).
\]

Since the energy of the fluid is completely determined by the Neumann data on \( \Gamma \), the fluid can be eliminated as a state variable. To this end we define the Neumann to Dirichlet map, \( \Lambda_0 \) as follows:

\[
y = \Lambda_0 f \iff \begin{cases} 
\Delta \phi &= 0 \text{ on } \Omega \\
\phi_n &= f \text{ on } \Gamma \\
y &= \phi |_{\Gamma} \text{ on } \Gamma.
\end{cases}
\]

Then \( \Lambda_0 : \tilde{H}^{-1/2}(\Gamma) = \{ z \in H^{-1/2}(\Gamma) : <z, 1> = 0 \} \to H^{1/2}(\Gamma) + C \), where \( H^{1/2}(\Gamma) + C \) denotes the quotient space of \( H^{1/2}(\Gamma) \) functions identified up to an additive constant.

Since \( \Omega \) is a Lipschitz domain, it is well-known (see [12]) that

\[
\Lambda_0 : \tilde{L}^2(\Gamma) \to H^{1}(\Gamma) + C \quad \text{continuously.}
\]
(Here and later we use $\tilde{M}$ to indicate the subspace of $M$ orthogonal to constants and $M + C$ for the quotient space where elements of $M$ are identified if they differ by a constant.) Furthermore, it is shown in [4] that $\Lambda_0$ is self-adjoint in the sense that

$$\int_{\Gamma} g \Lambda_0 f \, d\Gamma = \int_{\Gamma} f \Lambda_0 g \, d\Gamma \quad \forall f, g \in \tilde{L}^2(\Gamma).$$  \hfill (9)

For $g \in L^2(0, \pi)$, $\alpha \in \mathbb{R}$ define

$$\tilde{g} = \begin{cases} 0 & \text{on } \Gamma_f \cup \Gamma_O \cup \Gamma_R \\ -g & \text{on } \Gamma_B^+ \\ g & \text{on } \Gamma_B^- \end{cases}, \quad \tilde{\alpha} = \begin{cases} 0 & \text{on } \Gamma_f \cup \Gamma_B^+ \cup \Gamma_B^- \\ \alpha & \text{on } \Gamma_O \\ -\frac{\alpha}{\ell}\tilde{\alpha} & \text{on } \Gamma_R. \end{cases} \hfill (10)$$

Due to the structure in (10) the functions $\tilde{g}$ and $\tilde{\alpha}$ belong to $\tilde{L}^2(\Gamma)$. Define

$$\Lambda : L^2(0, \pi) \to H^1(0, \pi) : \Lambda g = \Lambda_0 \tilde{g} \mid_{\Gamma_B^0} - \Lambda_0 \tilde{g} \mid_{\Gamma_B^+}$$

$$S : \mathbb{R} \to H^1(0, \pi) : S\alpha = (\Lambda_0 \tilde{\alpha} \mid_{\Gamma_B^0} - \Lambda_0 \tilde{\alpha} \mid_{\Gamma_B^+})$$

$$T : L^2(0, L) \to \mathbb{R} : Tg = \ell_O \Lambda_0 \tilde{g} \mid_{\Gamma_O, R}$$

$$H : \mathbb{R} \to \mathbb{R} : H\alpha = \ell_O \Lambda_0 \tilde{\alpha} \mid_{\Gamma_O, R}.$$  

The cochlea system (5), (6), (3) can now be written:

$$(m_0 + \rho \Lambda)w_{tt} - (\sigma w_x)_x + \rho S\eta_{tt} + k_0 w = F_0 \quad \text{on } (0, \pi) \times \mathbb{R}^+$$

$$(\hat{m}_1 + \rho H)\eta_{tt} + \rho \tilde{T}w_{tt} + k_1 \eta = F_1 \quad \text{on } \mathbb{R}^+. \hfill (11)$$

We consider initial conditions of the form

$$(w, \eta, w_t, \eta_t)|_{t=0} = (w^0, \eta^0, w^1, \eta^1). \hfill (12)$$

3. Existence and uniqueness of solutions.

**Proposition 1.** $R = \begin{pmatrix} \Lambda & S \\ T & H \end{pmatrix}$ is positive, compact and self-adjoint on $X = L^2(0, \pi) \times \mathbb{R}$. Furthermore $R : X \to H^1(0, \pi) \times \mathbb{R}$ continuously.

**Proof.** From the definition of $\Lambda$ and (8) we have

$$\| \Lambda g \|_{H^1(0, \pi)} = \| \Lambda_0 \tilde{g} \mid_{\Gamma_B^0} - \Lambda_0 \tilde{g} \mid_{\Gamma_B^+} \|_{H^1(0, \pi)}$$

$$\leq 2 \| \Lambda_0 \tilde{g} \|_{H^1(0, \pi)}$$

$$\leq C \| \tilde{g} \|_{\tilde{L}^2(0, \pi)} = 2C \| g \|_{L^2(0, \pi)}.$$  

Thus $\Lambda$ has the required continuity property. Similar arguments establish the required continuity estimates for operators $S$, $T$, $H$. The compactness of $R$ on $X$ follows.

To prove self-adjointness, we show that $\Lambda$ and $H$ are self-adjoint and $S = T^*$. If $f, g \in L^2(0, \pi),$

$$\langle \Lambda f, g \rangle = \int_0^\pi (\Lambda_0 \tilde{f} \mid_{\Gamma_B^0} - \Lambda_0 \tilde{f} \mid_{\Gamma_B^+}) g \, dx = \int_{\Gamma_B^-} (\Lambda_0 \tilde{f}) \tilde{g} \, dx + \int_{\Gamma_B^+} (\Lambda_0 \tilde{f}) \tilde{g} \, dx$$

$$= \int_{\Gamma^-} \tilde{g} \Lambda_0 \tilde{f} \, d\Gamma = \int_{\Gamma} \tilde{f} \Lambda_0 \tilde{g} \, d\Gamma \quad \text{(using (9))}$$

$$= \int_{\Gamma_B^-} f \Lambda_0 \tilde{g} \, dx + \int_{\Gamma_B^+} -f \Lambda_0 \tilde{g} \, dx = \int_0^\pi f(\Lambda_0 \tilde{g} \mid_{\Gamma_B^0} - \Lambda_0 \tilde{g} \mid_{\Gamma_B^+}) \, dx$$

$$= \langle f, \Lambda g \rangle.$$
Thus $\Lambda$ is self-adjoint.

The previous calculation also shows that for $f \in L^2(0, \pi)$,

$$< \Lambda f, f > = \int_\Gamma \hat{f} \Lambda_0 \hat{f} \, d\Gamma. \tag{13}$$

Let $\phi$ be a solution of

$$\Delta \phi = 0 \quad \text{on } \Omega, \quad \phi_n = \hat{f} \quad \text{on } \Gamma. \tag{14}$$

Then $\Lambda_0 \hat{f} = \phi|\Gamma$ and applying the divergence theorem,

$$\int_\Gamma \hat{f} \Lambda_0 \hat{f} \, d\Gamma = \int_\gamma (\phi_n |\Gamma) (\phi |\Gamma) \, d\Gamma = \int_\Omega |\nabla \phi|^2 \, d\Omega. \tag{15}$$

This combined with (13) shows that $\Lambda$ is nonnegative. However, if $\nabla \phi$ vanishes on $\Omega$ then since $\Omega$ is simply connected, $\phi$ is constant. It then follows that $\hat{f}$ is constant and has average zero. Therefore $\hat{f} = 0$ on $\Gamma$. Thus $\Lambda$ is positive.

Next consider

$$\Delta \phi = 0 \quad \text{on } \Omega, \quad \phi_n = \tilde{\eta} \quad \text{on } \Gamma,$$

where $\tilde{\eta}$ is a constant function.

To see that $H$ is self-adjoint note that

$$< H \eta, v > = < \ell_O [\Lambda_0 \tilde{\eta}] |_{\Gamma, O,R}, v >$$

$$= \ell_O \left( \left( \frac{1}{\ell_O} \int_{\Gamma, O} \Lambda_0 \tilde{\eta} \, ds \right) v - \left( \frac{1}{\ell_R} \int_{\Gamma, R} \Lambda_0 \tilde{\eta} \, ds \right) \right) v$$

$$= \ell_O \left( \frac{1}{\ell_O} \int_{\Gamma, O} v |\Gamma, O \Lambda_0 \tilde{\eta} \, ds - \frac{1}{\ell_R} \int_{\Gamma, R} v |\Gamma, R \Lambda_0 \tilde{\eta} \, ds \right)$$

$$= \ell_O \left( \frac{1}{\ell_O} \int_{\Gamma, O} \tilde{v} \Lambda_0 \tilde{\eta} \, ds - \frac{1}{\ell_R} \int_{\Gamma, R} \tilde{v} \Lambda_0 \tilde{\eta} \, ds \right)$$

$$= \int_\Gamma \tilde{v} \Lambda_0 \tilde{\eta} \, ds, \tag{17}$$

where $\tilde{v}$ is defined in terms of $v$ in the same way as $\tilde{\eta}$ was from $\eta$, using (10). Self-adjointness and positivity follows from (9) and (15).

Next, recall $S : \mathbb{R} \to H^1(0, \pi)$ is defined by $S \eta = (\Lambda_0 \tilde{\eta} |_{\Gamma, -} - \Lambda_0 \tilde{\eta} |_{\Gamma, +})$ where $\tilde{\eta}$ is defined by (16) and $T : L^2(0, \pi) \to \mathbb{R}$ by

$$Tg = \ell_O [\Lambda_0 \tilde{g}] |_{\Gamma, O,R} = \ell_O \left( \frac{1}{\ell_O} \int_{\Gamma, O} (\Lambda_0 \tilde{g} |\Gamma, O) \, dy - \frac{1}{\ell_R} \int_{\Gamma, R} (\Lambda_0 \tilde{g} |\Gamma, R) \, dy \right),$$

where $\tilde{g}$ is defined from $g$ by (10). Calculations similar to the one leading to (17) reveal that

$$< Tg, \eta > = \int_\Gamma \tilde{\eta} \Lambda_0 \tilde{g} \, ds$$

and

$$< g, S \eta > = \int_\Gamma \tilde{g} \Lambda_0 \tilde{\eta} \, ds.$$ 

Hence using (9) it follows that $T^* = S$. This completes the proof.

In matrix form the initial value problem (11), (12) is:

$$MV_{tt} + \rho RV_{tt} + AV = F, \quad V(0) = V^0, \quad V'(0) = V^1 \tag{18}$$
where
\[ M = \begin{pmatrix} m_0 & 0 \\ 0 & \tilde{m}_1 \end{pmatrix}, \quad R = \begin{pmatrix} \Lambda & S \\ T & H \end{pmatrix}, \quad A = \begin{pmatrix} k_0 - \frac{\partial}{\partial x}(\sigma \frac{\partial}{\partial x}) & 0 \\ 0 & \tilde{k}_1 \end{pmatrix}, \]
(19)
\[ F = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}, \quad V = \begin{pmatrix} w \\ \eta \end{pmatrix}, \quad V^0 = \begin{pmatrix} w^0 \\ \eta^0 \end{pmatrix}, \quad V^1 = \begin{pmatrix} w^1 \\ \eta^1 \end{pmatrix}. \]
(20)

Let
\[ X = X^0 = L_2(0, \pi) \times \mathbb{R}, \quad X^1 = \mathcal{H}^{1,\sigma} \times \mathbb{R}, \quad X^2 = \mathcal{H}^{2,\sigma} \times \mathbb{R}. \]

Define the energy inner product \( \langle \cdot, \cdot \rangle_e \) on \( X^1 \times X^0 \) by
\[ \langle \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \rangle_e = a \left( \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} , \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \right) + b \left( \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} , \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \right) \]
(21)
where
\[ a(V, \hat{V}) = a(\{\varphi, \alpha\}, \{\hat{\varphi}, \hat{\alpha}\}) = \int_0^\pi \sigma(x)\varphi_x\hat{\varphi}_x + k_0\varphi\hat{\varphi} \, dx + \tilde{k}_1\hat{\alpha}\hat{\alpha} \quad \forall V, \hat{V} \in X^1, \]
\[ b(V, \hat{V}) = \left( (M + \rho R)V, \hat{V} \right)_{X^0 \times X^0}, \quad \forall V, \hat{V} \in X^0. \]

Since \( a(\cdot, \cdot) \) is a continuous, symmetric, positive definite form on \( X^1 \), by the Lax-Milgram theorem, \( A \) extends to a continuous mapping \( X^1 \rightarrow X^{-1} \), where \( X^{-1} \) is the dual of \( X^1 \) with respect to \( X^0 \).

We assume that the mass distribution \( m_0 \) and spring stiffness distribution \( k_0 \) are strictly positive and bounded on \([0, \pi]\). We have for \( U \) and \( V \) in \( X^1 \)
\[ b((M + \rho R)^{-1}AU, V) = b((M + \rho R)^{-1}AU, V) = a(U, V) = b(U, (M + \rho R)^{-1}AV). \]
And hence
\[ \hat{A} = (M + \rho R)^{-1}A \]
is strictly positive, and self-adjoint relative to \( b(\cdot, \cdot) \). If \( \sigma = 0 \) then \( \hat{A} \) is also bounded. In first order form the system (18) becomes:
\[ \frac{d}{dt} \begin{pmatrix} V \\ V_t \end{pmatrix} = \mathcal{A} \begin{pmatrix} V \\ V_t \end{pmatrix} + \begin{pmatrix} 0 \\ (M + \rho R)^{-1}F \end{pmatrix}, \quad \begin{pmatrix} V(0) \\ V_t(0) \end{pmatrix} = \begin{pmatrix} V^0 \\ V^1 \end{pmatrix} \]
(22)
where
\[ \mathcal{A} = \begin{pmatrix} 0 & I \\ -\hat{A} & 0 \end{pmatrix}. \]

It is easy to show that \( \mathcal{A} : \mathcal{D}(\mathcal{A}) = X^2 \times X^1 \rightarrow X^1 \times X^0 \) is skew-adjoint with respect to the energy inner product. Hence by Stone’s theorem, (e.g., see Pazy, [15]) \( \mathcal{A} \) is the generator of a unitary group. Moreover in the case \( \sigma = 0 \), the generator is bounded, which implies (see e.g., [15]) that the group is uniformly continuous. Thus we have the following:

**Theorem 3.1.** Assume that \( m_0 \) and \( k_0 \) are bounded and strictly positive on \([0, \pi]\).
Then \( \mathcal{A} \) is an infinitesimal generator of a strongly continuous group of unitary operators on \( X^1 \times X^0 \). Given the control \( F \in L_2(0, \tau; X^0) \), and initial data \( V^0 \in X^1 \), \( V^1 \in X^0 \) the solution \( V \) of initial value problem
\[ (M + \rho R)V_{tt} + AV = F \quad \text{with} \quad V(0) = V^0, \quad V_t(0) = V^1, \]
(23)
satisfies
\[ V \in C([0, \tau]; X^1) \cap C^1([0, \tau]; X^0). \]
Moreover, if \( \sigma \equiv 0 \) then the semigroup is uniformly continuous.
4. Approximate controllability. Below, we show that a distributed control that is active on an arbitrary open subset of the BM is enough to obtain approximate controllability. It is not necessary to have control active on the OW or RW.

**Theorem 4.1.** Let \( \omega \) be an open subset of \((0, \pi)\). Given \( \{V^0, V^1\}, \{W^0, W^1\} \in X^1 \times X^0, T > 0, \) and \( \epsilon > 0, \) there exists \( F = \begin{pmatrix} F_0 \\ 0 \end{pmatrix} \) with \( F_0 \in L^2((0,T) \times \omega) \) for which the solution at time \( T \) of (18) satisfies
\[
\| (V(T), V_z(T)) - (W^0, W^1) \|_{X^1 \times X^0} < \epsilon.
\]

**Proof.** Let \( \Phi_t : L^2(\omega \times (0,t)) \rightarrow X^1 \times X^0 \) denote the mapping from the control \( F_0 \) to the solution of (23) with \( V^0 = V^1 = 0 \). Then the solution of (23) at time \( T \) is
\[
\begin{pmatrix} V(T) \\ V_z(T) \end{pmatrix} = e^{TA} \begin{pmatrix} V^0 \\ V^1 \end{pmatrix} + \Phi_T F_0.
\]

Consequently to prove Theorem 4.1 it is enough to prove that \( \Phi_T \) has dense range. Equivalently we need to show \( \Phi_T^* Z = 0 \) is one to one. Putting \( \Phi_T^* Z = 0 \) is equivalent to the following “observed problem”:
\[
m_0 w_{tt} + \rho \Lambda w_{tt} - (\sigma w_x)_x + \rho S_\eta_{tt} + k_0 w = 0 \quad \text{on } (0, \pi) \times [0,T] \tag{24}
\]
\[
m_1 \eta_{tt} + \rho \Gamma w_{tt} + \rho H_\eta_{tt} + \hat{k}_1 \eta = 0 \quad \text{on } [0,T] \tag{25}
\]
\[
w_t = 0 \quad \text{on } \omega \times [0,T] \tag{26}
\]
\[
(w, \eta, w_t, \eta_t) = Z = (z^0, z^1, z^2) \quad t = 0, \tag{27}
\]

where we have used the unitary group property of Theorem 3.1 in assigning data at time 0, rather than \( T \) in (27). We wish to show that \( Z = 0 \).

First consider the case \( \sigma \equiv 0 \). Since the generator of the unitary group in Theorem 3.1 is bounded, the solutions in Theorem 3.1 with \( F \equiv 0 \) are \( C^\infty \) in time. Hence the equations (24)–(26) may be differentiated in time. Thus we have \( w_{tt} = 0 \) on \( \omega \times [0,T] \). Using this fact and (26) in the differentiated form of (24), (26) we obtain
\[
\rho \Lambda w_{ttt} + \rho S_\eta_{ttt} = 0 \quad \text{on } \omega \times [0,T] \tag{28}
\]
\[
m_1 \eta_{ttt} + \rho \Gamma w_{ttt} + \rho H_\eta_{ttt} + \hat{k}_1 \eta_t = 0 \quad \text{on } [0,T]. \tag{29}
\]

Recall that \( \Delta f = y \) means
\[
\begin{align*}
\Delta \phi &= 0 \quad \text{in } \Omega \\
\phi|_{\Gamma^+_B - \phi|_{\Gamma^-_B} &= y \\
\partial_n \phi &= \begin{cases} f & \text{on } \Gamma^+_B \\
-f & \text{on } \Gamma^-_B \end{cases} \\
\partial_n \phi &= 0 \quad \text{on } \Gamma_O \cup \Gamma_f \cup \Gamma_R 
\end{align*}
\]

Also, \( S\alpha = g \) means
\[
\begin{align*}
\Delta \phi &= 0 \quad \text{in } \Omega \\
\phi|_{\Gamma^+_B - \phi|_{\Gamma^-_B} &= g \\
\partial_n \phi &= \tilde{\alpha} \quad \text{on } \Gamma_O \cup \Gamma_R \\
\partial_n \phi &= 0 \quad \text{on } \Gamma \backslash \{ \Gamma_O \cup \Gamma_R \} \\
\tilde{\alpha} &= \begin{cases} \alpha & \text{on } \Gamma^+_O \\
-\frac{\partial \alpha}{\partial n} & \text{on } \Gamma^-_R \\
0 & \text{otherwise} \end{cases}
\end{align*}
\]

Thus, the solution of (24)–(26) at \( T \) is given by
\[
\begin{align*}
\Delta \phi &= 0 \quad \text{in } \Omega \\
\phi|_{\Gamma^+_B - \phi|_{\Gamma^-_B} &= y \\
\partial_n \phi &= \begin{cases} f & \text{on } \Gamma^+_B \\
-f & \text{on } \Gamma^-_B \end{cases} \\
\partial_n \phi &= 0 \quad \text{on } \Gamma_O \cup \Gamma_f \cup \Gamma_R 
\end{align*}
\]
By superposition, we have that $z = \Lambda f + S \alpha$ means
\[ \Delta \psi = 0 \quad \text{in } \Omega \]
\[ \psi|_{\Gamma_B^+} - \psi|_{\Gamma_B^-} = z \]
\[ \partial_n \psi = \begin{cases} \tilde{\alpha} & \text{on } \Gamma_O \cup \Gamma_R \\tilde{f} & \text{on } \Gamma_B^- \cup \Gamma_B^+ \end{cases} \quad \text{on } \Gamma_f \]
(31)

Applying this to equation (28) yields
\[ \Delta \psi = 0 \quad \text{in } \Omega \]
\[ \psi|_{\Gamma_B^+} - \psi|_{\Gamma_B^-} = \begin{cases} 0 & x \in \omega \\bar{m}_0 w_{tt} + k_0 w_t & x \in (0, \pi) \setminus \omega \end{cases} \]
\[ \partial_n \psi = \begin{cases} \tilde{\alpha} = \tilde{\eta}_{tt} & \text{on } \Gamma_O \cup \Gamma_R \\tilde{f} = \tilde{w}_{tt} & \text{on } \Gamma_B^- \cup \Gamma_B^+ \end{cases} \quad \text{on } \Gamma_f \]
(32)

Since $w_{ttt}$ is even with respect to $y$, we have
\[ \Delta \psi = 0 \quad \text{in } \Omega \]
\[ \psi|_{\omega \cap (\Gamma_B^+)} - \psi|_{\omega \cap (\Gamma_B^-)} = 0 \]
\[ \partial_n \psi = 0 \quad \text{on } \omega \]

The upper chamber of the cochlea (see Fig. 1) is described by cartesian coordinates $(x,y)$ with $y > h/2$ and $y < -h/2$ for the lower chamber. Let $D$ denote the open rectangle $0 < x < \pi$, $h/2 < y < \min (h/2 + \ell_O, h/2 + \ell_R)$. For $(x,y) \in D$ define
\[ \zeta(x,y) = \frac{\psi(x,y) - \psi(x,-y)}{2} \]

We have
\[ \Delta \zeta = 0 \quad \text{in } D \]
\[ \zeta = 0 \quad \text{on } \omega \]
\[ \partial_n \zeta + \bar{w}_y(x,y) + \bar{w}_x(x,-y) = 0 \quad \text{on } \omega \]
(34)

The problem is overdetermined. Thus $\zeta \equiv 0$ in $D$. Thus $\psi$ in (31) is even with respect to $y$. Since the region $D$ has at least part of $\Gamma_R$ and $\Gamma_O$ for its boundary and their normal vectors point in the same half plane, they can not be opposite signs. It follows from (33) that $\eta_{ttt} \equiv 0$. Similarly, as the normal vectors to $\Gamma_B^+$ and $\Gamma_B^-$ point in opposite directions, (and using the definition of $\tilde{f} = w_{ttt}$ in (29)), $w_{ttt}$ must also vanish on $(0,\pi)$ for all $t$. Using this information, the differentiated dual system (24), (25), (26) reduces to
\[ k_0 w_t = 0 \quad \text{on } (0,\pi) \times [0,T] \]
\[ k_1 \eta_t = 0 \quad \text{on } [0,T]. \]

Under our positivity assumptions for $k_0 k_1$, we obtain
\[ w_t = 0 \quad \text{on } (0,\pi) \times [0,T] \]
\[ \eta_t = 0 \quad \text{on } [0,T]. \]

This together with (24), (25) result in
\[ w = 0 \quad \text{on } (0,\pi) \times [0,T] \]
\[ \eta = 0 \quad \text{on } [0,T]. \]
Thus \( Z = 0 \) as required.

Next consider the case \( \sigma > 0 \). Let \( A \) denote the semigroup generator in (22). By continuity of the solution map in Theorem 3.1 and density of \( D(A^2) \) in \( X^1 \times X^0 \), it is enough to show that any solution to (24)–(27) with the initial data \( Z \) in (27) belonging to \( D(A^2) \) must be the trivial solution. Therefore let \( Z \in D(A^2) \). The corresponding solution is in \( C^1([0,T];D(A)) \) and hence the system (24)–(27) maybe be differentiated with respect to time. Since \( \sigma \) is time independent, we again obtain (28), (29). This again leads to the conclusion that \( w_{ttt} \) and \( \eta_{ttt} \) vanish on \((0, \pi) \times (0, T)\). This means that after differentiating (24)-(26) one is left with

\[
- (\sigma w_{tt})_x + k_0 w_t = 0 \quad \text{on} \quad (0, \pi) \times [0, T] \\
\hat{k}_1 \eta_t = 0 \quad \text{on} \quad [0, T] \\
w_t = 0 \quad \text{on} \quad \omega \times [0, T].
\]

Since \( \{w_t, \eta_t, w_{tt}, \eta_{tt}\} \in D(A) \), in particular \( w_t \in H^2_1 \) (see (7)) and satisfies the differential equation above, this means \( w_t \equiv 0 \) on \((0, \pi) \times [0, T]\) and \( \eta_t \equiv 0 \) on \([0, T]\). Then again we can use (24), (25) to deduce \(- (\sigma(x)w_x)_x + k_0(x)w = 0 \) on \([0, \pi] \times [0, T]\). Now the boundary conditions imply \( w \) vanishes and hence, \( Z = 0 \). This completes the proof.

\[ \square \]

4.1. Lack of exact controllability when \( \sigma = 0 \). We mention that the system (23) with \( F = \begin{pmatrix} F_0 \\ 0 \end{pmatrix} \) would be exactly controllable in time \( T \) if given any \( \{V^0, V^1\}, \{W^0, W^1\} \in X^1 \times X^0 \), there exists \( F_0 \in L^2((0,T) \times \omega) \) for which the solution at time \( T \) of (18) satisfies \( (V(T), V_t(T)) = (W^0, W^1) \).

\textbf{Proposition 2.} Assume \( \sigma = 0 \) and the control region \( \omega \) has a nontrivial complement \( \omega^c \) on \((0, \pi)\). Then the system in Theorem 3.1 is not exactly controllable.

\textbf{Proof.} To explain the idea of the proof more directly, we consider the problem without the effects of the oval and round window. In this context, the system (24)–(27) becomes

\[
m_0 w_{tt} + \rho Aw_{tt} + k_0 w = 0 \quad \text{on} \quad (0, \pi) \times [0, T] \\
(w, w_t) = (z^0, z^1) := Z \quad t = 0.
\]

(35) (36)

For exact controllability in time \( T \) to hold, the observed system (35), (36) needs to satisfy for some \( K > 0 \)

\[
K \int_0^T \int_\omega |w_t|^2 \, dx \, dt \geq \|Z\|_{X \times X}^2 \quad Z \in X \times X.
\]

(37)

(Here, \( X = L^2(0, \pi) \)). To show no such \( K \) exists, consider the sequence of initial data for (36) of the form

\[
Z^n = (0, \sin(n x) \chi_{\omega^c}),
\]

where \( \chi_{\omega^c} \) is the indicator function for \( \omega^c \). Then \( \|Z^n\|_{X \times X}^2 \rightarrow |\omega^c|/2 \) as \( n \rightarrow \infty \). Let \( y^n \) denote the solution to

\[
m_0 y_{tt} + k_0 y = 0 \quad \text{on} \quad (0, \pi) \times [0, T] \\
(y, y_t) = Z = (z^0, z^1) \quad t = 0.
\]

(38) (39)

Then

\[
y^n = \left( \sqrt{\frac{m_0}{k_0}} \sin \sqrt{\frac{k_0}{m_0}} t \right) (\sin n x) \chi_{\omega^c},
\]
and $u^n = y^n - w^n$ satisfies

\begin{align}
  m_0 u^n_{tt} + \rho \Lambda u^n_t + k_0 u^n &= \rho \Lambda y^n_{tt} \quad \text{on } (0, \pi) \times [0, T] \tag{40} \\
  (u^n, u^n_t) &= (0, 0) \quad t = 0. \tag{41}
\end{align}

In the equivalent variational form (40) becomes

\[
  c(u^n_{tt}, \varphi) + (k_0 u^n, \varphi) = \rho(\Lambda y^n_{tt}, \varphi) = \rho(y^n_{tt}, \Lambda \varphi)
\]

However since since $\sin nx$ tends weakly to zero in $X$ as $n \to \infty$, the right hand side above also goes to zero uniformly on $[0, T]$. It follows that $\|u^n\|_{X}$ and $\|u^n_t\|_{X}$ also tend to 0 on $[0, T]$. Consequently $\int_0^T \int_0^\pi |u^n|^2 \, dx \, dt \to 0$ so that (37) can not hold.

The same argument can be applied to the general system of Theorem 3.1.

**Remark 1.** Our main results remain valid for domains more general than indicated in Figure 1. More precisely, the geometric assumptions on the fluid domain $\Omega$ that we have utilized are the following:

- $\Omega$ is simply connected. If the BM divides the domain in two separate fluid regions a different model results; see [3].
- $\Omega$ is assumed to have a Lipschitz boundary. This requirement imposes a small thickness $h > 0$ in the BM. If $h = 0$, $\Omega$ becomes a non-Lipschitz domain and the regularity of Proposition 1 is not guaranteed to hold from classical elliptic theory, e.g., [12]. On the other hand, the right angles in Fig. 1 are not necessary, in fact $\partial \Omega$ can be smooth.
- **Symmetry assumptions:** In the proof of the approximate controllability we have used: (i) A portion of the domain of the fluid adjacent to the BM, RW, OW should be symmetric in the transverse direction with respect to the BM. Furthermore, (ii) $\Gamma_f^+$ and $\Gamma_f^-$ are flat and parallel portions of $\partial \Omega$, with exterior normal vectors in opposite directions.

**Remark 2.** Here, we do not answer the question of whether of exact controllability holds in the case $\sigma > 0$, but can give an affirmative answer on a closely related (but much simpler) boundary control problem. Instead of the full system (11), (12), consider the following simplified system without the effects due to the OW and RW, i.e., $\Gamma_f$ now includes $\Gamma_O$ and $\Gamma_R$. In the constant coefficient case, where $m_0$ is 1, $\rho$, $\sigma$ and $k_0$ are constant, and Neumann control is applied at the left end the system becomes:

\begin{align}
  (I + \rho \Lambda)w_{tt} - \sigma w_{xx} + k_0 w &= 0 \quad \text{on } (0, \pi) \times \mathbb{R}^+ \\
  w_x(0, t) = u(t), \quad w_x(\pi, t) = 0, \quad t > 0 \\
  w(x, 0) = w^0, \quad w_t(x, 0) = w^1 \quad \text{on } (0, \pi). \tag{42}
\end{align}

It was shown in [4] that the system (42) becomes exactly controllable if the density $\rho$ is sufficiently small, i.e., there exists $T > 2\pi$ such that given initial data $(w^0, w^1) \in L^2(0, \pi) \times H^1(0, \pi)$, there exists $f \in L^2(0, T)$ for which $(w, w_t)|_{t=T} = (0, 0)$.

**Acknowledgments.** Much of this paper was written while the author was visiting the Institute for Mathematics and its Applications during his visit from January to May of 2016. The author would like to gratefully acknowledge their hospitality.
REFERENCES

[1] G. von Békésy, *Experiments in Hearing*, McGraw-Hill Inc., New York, 1960.
[2] Isaak Chepkwony, *Analysis and Control Theory of some Cochlea Models*, Ph.D. thesis, Department of Mathematics, Iowa State University, Ames, IA, 2006.
[3] S. W. Hansen, Exact controllability of an elastic membrane coupled with a potential fluid, *Int. J. Appl. Math. Comput. Sci.*, 11 (2001), 1231–1248.
[4] S. W. Hansen and A. Lyashenko, Exact controllability of a beam in an incompressible inviscid fluid, *Disc. Cont. Dyn. Syst.*, 3 (1997), 59–78.
[5] H. L. F. von Helmholtz, On the sensations of tone as a physiological basis for the theory of music, (Translation by A. J. Ellis of *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik*: Verlag von Fr. Vieweg u. Sohn. 4th ed., 1877; originally published 1863) Dover, New York, 1954.
[6] J. B. Keller and J. C. Neu, Asymptotic analysis of a viscous cochlear model *J. Acoust. Soc. Amer.*, 77 (1985), 2107–2110.
[7] R. J. Leveque, C. S. Peskin and P. D. Lax, Solution of a two-dimensional cochlea model using transform techniques, *SIAM J. Applied Math.*, 45 (1988), 450—464.
[8] R. J. Leveque, C. S. Peskin and P. D. Lax, Solution of a two-dimensional cochlea model with fluid viscosity, *SIAM J. Applied Math.*, 48 (1988), 191—213.
[9] J. Lighthill, Energy flow in the cochlea, *J. Fluid Mech.*, 106 (1981), 149–213.
[10] R. D. Luce, *Sound and Hearing. A Conceptual Introduction*, Lawrence Erlbaum Assoc. Inc., Publishers, Hillsdale, New Jersey, 1993.
[11] G. A. Manley and R. R. Fay, *Active Processes and Otoacoustic Emissions in Hearing*, Springer Science & Business Media 30, 2007.
[12] J. Necas, *Les Methodes Directes en theorie des equations Elliptiques*. Paris: Masson, 1967.
[13] S. T. Neely, Mathematical modeling of cochlear mechanics, *J. Acoust. Soc. Am.*, 78 (1985), 345–352.
[14] S. T. Neely and D. O. Kim, An active cochlear model showing sharp tuning and high sensitivity, *Hearing Research*, 9 (1983) 123–130.
[15] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
[16] O. F. Ranke, Theory of operation of cochlear: A contribution to the hydrodynamics of the cochlear, *J. Acoust. Soc. Am.*, 22 (1950), 772–777.
[17] W. S. Rhode, Observations of the vibration of the Basilar Membrane in squirrel monkeys using the Mössbauer technique, *Journal of the Acoustical Society of America*, 49 (1971), 1218–1231.
[18] J. Xin, Dispersive instability and its minimization in time-domain computation of steady-state responses of cochlea models, *J. Acoust. Soc. Am.*, 115 (2004), 2173–2177.

Received August 2016; revised September 2016.

E-mail address: shansen@iastate.edu