Information Causality and Noisy Computations

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We reformulate the information causality in a more general framework by adopting the results of signal propagation and computation in a noisy circuit. In our framework, the information causality leads to a broad class of Tsirelson inequalities. This fact allows us to subject information causality to experimental scrutiny. A no-go theorem for reliable nonlocal computation is also derived. Information causality prevents any physical circuit from performing reliable computations.

I. INTRODUCTION

As a physical theory, quantum mechanics has been extremely successful in describing the microscopic physics. Nevertheless, its current framework is incapable of explaining the nature of quantum entanglement. Attempts to remedy this situation have been made by reconstructing quantum mechanics in terms of physical principles. These physical principles should be able to yield or constrain the non-local correlation implied by the quantum entanglement. One such candidate is the principle of space-time causality. This principle will constrain the possible non-local correlation such that any physical theory must be no-signaling [1], i.e., signal cannot be send in the way of violating causality. However, a broad class of no-signaling theories other than quantum mechanics exist. Certain features, usually thought of as specifically quantum, are common for many of these theories [2, 3]. Clearly, no-signaling is insufficient as a principle to single out quantum mechanics.

Some of these theories are allowed to have more non-local correlation than quantum mechanics [2–6]. Specifically, the non-local correlation in these theories can violate Bell-type inequalities by more than Tsirelson’s bound [1, 7]. From this perspective, we should search for a physical principle as follows. The principle can single out Tsirelson’s bound as a limitation on the extent of the allowed correlation for a physical theory. With the advent of quantum information science, some principles of information theoretic flavor have been proposed. These proposed candidates set the constraints on the physically realizable correlations. In this Letter, we focus on a promising candidate — the information causality. Information causality states that, in a bipartite code protocol prepared with any physically local or non-local resources, the accessible information gain cannot exceed the amount of classical communication. In [8] information causality is demonstrated by a generic task similar to random access codes (RAC) and oblivious transfer. In this task, a database of \(k\) bits is prepared: 
\[
\vec{a} := (a_0, a_1, \ldots, a_{k-1}),
\]
where each \(a_i\) is a random variable, which is only known by the first party, Alice. A second, distant party, Bob, is given a random variable \(b \in (0, \ldots, k - 1)\) along with a bit \(\alpha\) send by Alice. With the bit \(\alpha\) and the pre-shared correlation with Alice, Bob’s task is to optimally guess the bit \(a_b\). Then, according to information causality the quantity \(I\) has an upper bound
\[
I = \sum_{i=0}^{k-1} I(a_i; \beta|b = i) \leq 1. \tag{1}
\]
Here \(I(a_i; \beta|b = i)\) is the Shannon mutual information between \(a_i\) and Bob’s guessing bit \(\beta\) under the condition \(b = i\). Classically, \(I\) can reach 1 once \(\alpha = a_i\) and \(I(a_i; a_j) = \delta_{ij}\) (i.e., the Kronecker delta).

To perform the RAC task, Alice and Bob can use (earlier prepared and distributed) correlations among either classical or quantum systems. These no-signaling correlation resources can be simulated by the no-signaling box (NS-box). The NS-box correlates the inputs and outputs of Alice and Bob in an imperfect way subjected to the probabilistic noise. The noise of NS-box is intrinsically inherited from the underlying physical theory such as quantum mechanics. The quantity \(I\) in (1) is unavoidably affected by the intrinsic noise of NS-box. In this framework, the signal decay theorem in [9, 10] for a noisy circuit is exploited to yield a tight bound for \(I(a_b; \beta|b)\) in terms of noise of NS-box. According to information causality, the tight bound should also obey the upper bound in (1). By expressing the tight bound in terms of correlation functions between Alice’s and Bob’s measurement outcomes, this then yields our main result — a broad class of multi-setting Tsirelson-type inequalities. As a result, we can then subject the physical principle of information causality to scrutiny by experimentally verifying or falsifying the generalized Tsirelson’s bounds.

Without classical communication, the RAC can be regarded as nonlocal computation. Therein, distant Alice and Bob compute a general Boolean function without knowing the other’s input. Here, NS-box can be regarded as a noisy gate for non-local computation [11]. Noise of the gate is closely related to the reliability of non-local computation. The computational noise of the gate is related to the intrinsic reliability of the physically realized NS-box. In this aspect, we can tackle a fundamental question on noisy computation with its nonlocal version.
As raised by von Neumann [12], this question is originally stated as follows. Could physical circuits of finite size perform the reliable noisy non-local computation of any Boolean function? Based on constraint by the information causality for any physical circuit, we will see that the answer is negative in non-local computation.

The paper is organized as follows. In the next section we derive the Tsirelson-type inequalities from the information causality by using the theorem of signal propagation. In the Section III we discuss the implication of information causality on the nonlocal quantum computation and yield a no-go theorem for reliable nonlocal quantum computation. Finally we briefly conclude our paper in section IV. Moreover, in the Appendix we give the details of verifying our newly-derived Tsirelson-type inequalities by using the method of semidefinite programing.

II. TSIRELSON-TYPE INEQUALITIES

We start by reformulating the NS-box as a noisy distributed gate for nonlocal computation. The NS-box is initially distributed between two distant parties, Alice and Bob. Locally, Alice and Bob input bit strings \(\tilde{x}\) and \(\tilde{y}\), respectively, into half of the box, which then outputs bits \(A_\tilde{x}\) and \(B_{\tilde{y}}\), respectively. The lengths of the bit strings can be chosen by design. Our NS-box is further characterized by the conditional joint probabilities \(\Pr[A_\tilde{x} + B_{\tilde{y}} = f(\tilde{x}, \tilde{y}) | \tilde{x}, \tilde{y}]\). Therein, \(f(\tilde{x}, \tilde{y})\) is the task function. Notably, if the NS-box is physically realizable, these joint probabilities must fulfill the no-signaling conditions.

In the RAC protocol, the chosen task function \(f(\tilde{x}, \tilde{y})\) depends on how we encode Alice’s database \(\tilde{a}\) and Bob’s given random variable \(b\) into \(\tilde{x}\) and \(\tilde{y}\), respectively. From now on, we will implicitly use the following protocol. Firstly, Alice encodes her database \(\tilde{a}\) into the \((k-1)\)-bit string \(\tilde{x} := (x_1, \cdots, x_{k-1})\) by \(x_i = a_0 + a_i\). Alice’s half of NS-box then produces an outcome \(A_\tilde{x}\). At the same time, Bob encodes the given \(b\) in to \((k-1)\)-bit string \(\tilde{y} := (y_1, \cdots, y_{k-1})\) by \(y_i = \delta_{b,i}\) for \(b \neq 0\), and \(\tilde{y} = \tilde{0}\) for \(b = 0\). Bob’s half of NS-box then produces an outcome \(B_{\tilde{y}}\). Secondly, Alice sends Bob a bit \(\alpha = a_0 + A_\tilde{x}\). The optimal strategy for Bob’s task is to output a guess bit \(\beta = \alpha + B_{\tilde{y}}\). As a result, Bob can decode Alice’s bit \(a_0\) successfully whenever \(A_\tilde{x} + B_{\tilde{y}} = \tilde{x} \cdot \tilde{y} \pmod{2}\). Most of the calculations in this Letter are modulo-2 defined.

In quantum mechanics, Alice’s and Bob’s outcomes can be produced by performing the corresponding measurement of \(2^{k-1}\) and \(k\) settings, respectively. For the above protocol, the success probability of Bob’s task in guessing Alice’s bit \(a_0\) is related to the one for noisy computation as follows

\[
\Pr[\beta = a_0 | b] = \frac{1}{N_\tilde{x}} \sum_{\tilde{x}} \Pr[A_\tilde{x} + B_{\tilde{y}} = f(\tilde{x}, \tilde{y}) | \tilde{x}, \tilde{y}],
\]

where \(N_\tilde{x}\) is the cardinality of the input space spanned by the encoding \(\{\tilde{x}\}\). By defining the correlation functions between Alice’s and Bob’s measurement outcomes as \(C_{\tilde{x}, \tilde{y}} := \sum_{A_\tilde{x}=0,1} \sum_{B_{\tilde{y}}=0,1} (-1)^{A_\tilde{x} + B_{\tilde{y}}} \Pr[A_\tilde{x}, B_{\tilde{y}} | \tilde{x}, \tilde{y}]\), we find

\[
\xi_{\tilde{y}} = \frac{1}{N_\tilde{x}} \sum_{\tilde{x}} (-1)^{f(\tilde{x}, \tilde{y})} C_{\tilde{x}, \tilde{y}},
\]

where the coding noise parameter is defined as \(\xi_{\tilde{y}} := 2 \Pr[\beta = a_0 | b] - 1\). The sub-index \(\tilde{y}\) of \(\xi_{\tilde{y}}\) is understood to be equivalent to Bob’s given parameter \(b\) via encoding.

One of the main results of this paper is a broad class of Tsirelson’s bound implied by information causality, i.e.,

\[
|\sum_{(\tilde{y})} \xi_{\tilde{y}}| = \frac{1}{N_\tilde{x}} \sum_{(\tilde{x}), (\tilde{y})} (-1)^{f(\tilde{x}, \tilde{y})} C_{\tilde{x}, \tilde{y}} \leq \sqrt{k}.
\]

For \(k = 2\), it is easy to check that (4) is the Tsirelson’s bound \(|C_{0,0} + C_{0,1} + C_{1,0} - C_{1,1}| \leq 2\sqrt{2} [7]\). For the case of \(k > 2\) with \(f(\tilde{x}, \tilde{y}) = \tilde{x} \cdot \tilde{y}\), we have verified (4) to be the Tsirelson’s bound in quantum mechanics by using the semidefinite programing [14]. Please see Appendix for more detailed discussions.

Indeed, later we will see that information causality will render (4). This implies that information causality can be tested by experimental verification or refutation via the measurement of the correlation functions of a quantum system.

In order to arrive the Tsirelson’s bound (4) from the information causality constraint (1), we need to relate \(I(a_0, \beta | b)\) to \(\xi_{\tilde{y}}\). It turns out that this can be done by using the following signal decay theorem on the signal propagation [9, 10].

**Theorem 1:** Let \(X, Y\) and \(Z\) be Boolean random variables. Consider a cascade of two communication channels: \(X \rightarrow Y \rightarrow Z\). \(X\) and \(Y\) are the input and the output of the first channel, respectively. Let \(Y\) in turn be the input of a cascading binary symmetric channel \(C_\epsilon\) with a noise parameter \(\epsilon\), i.e.,

\[
C_\epsilon = \begin{pmatrix}
\frac{1}{2} (1 + \epsilon) & \frac{1}{2} (1 - \epsilon) \\
\frac{1}{2} (1 - \epsilon) & \frac{1}{2} (1 + \epsilon)
\end{pmatrix}.
\]

Let \(Z\) be the output of \(C_\epsilon\), i.e., \(Z = Y\) with the bit-flipping probability \(\frac{1}{2} (1 - \epsilon)\)

\[
\frac{I(X; Z)}{I(X; Y)} \leq \epsilon^2.
\]

A special case arises if the first channel is noiseless or trivial, i.e., \(I(X; Y = X) = 1\) such that \(I(X; Z) \leq \epsilon^2\). Note also that regardless of the properties of the second
channel, there is a data processing inequality $I(X; Z) \leq I(X; Y)$.

We apply this theorem to our RAC protocol as follows. Because Alice’s database $a_0, a_1, \cdots, a_{k-1}$ are random variables and independent of each other, so that all the $a_j$’s with $j \neq i$ can be fixed without disturbing $I(a_i : \beta | b)$. Let $X = a_1, Y = a_0 + f(\bar{x}, \bar{y})$, and $Z = \beta$. Here $Y$ is Bob’s ideal answer and hence $I(X; Y) = 1$. The coding noise $\epsilon$ for our protocol is $\xi_{\bar{y}}$, then according to the Theorem 1, we have

$$I(a_i : \beta | b = i) \leq \epsilon_i^2. \quad (6)$$

Therefore, the information causality in Eq. (1) yields

$$I \leq \sum_{\{\bar{y}\}} \xi_{\bar{y}}^2 \leq 1. \quad (7)$$

In [8], similar inequalities are derived to avoid the divergence of $I$, which justifies the information causality. However, such trouble does not exist in our reformulation because of the tight bound of Theorem 1. With the help of (3) the second inequality in (7) becomes a quadratic Tsirelson-type inequality for the correlation function $C_{\bar{x}, \bar{y}}$. Moreover, using the Cauchy-Schwarz inequality, we can obtain $|\sum_{\{\bar{y}\}} \xi_{\bar{y}}| \leq \sqrt{k}$, which results in the linear Tsirelson inequality of Eq. (4).

III. NOISY NONLOCAL COMPUTATION

In the previous discussion we have considered the information causality using a single nonlocal NS-box. Instead, we can treat the NS-box as a non-local gate for performing the nonlocal computation, i.e., computing the function $f(\bar{x}, \bar{y})$ [11]. Unlike using the same gate for the RAC, no classical communication between Alice and Bob is required to perform the nonlocal computation. In details, Alice’s and Bob’s local outputs are $A_{\bar{x}}$ and $B_{\bar{y}}$, respectively. The computation is successful if $A_{\bar{x}} + B_{\bar{y}} = f(\bar{x}, \bar{y})$. The computational noise parameter is defined as

$$\epsilon_{\bar{x}, \bar{y}} := 2 \Pr[A_{\bar{x}} + B_{\bar{y}} = f(\bar{x}, \bar{y}) | \bar{x}, \bar{y}] - 1. \quad (8)$$

From (8) and (3) the computational noise of the gate is related to its coding noise by

$$\xi_{\bar{y}} = \frac{1}{N_{\bar{x}}} \sum_{\{\bar{y}\}} \epsilon_{\bar{x}, \bar{y}}. \quad (9)$$

Basically, computational errors inherently come from the gate noise. Information causality constraints the noisy extent of the NS-box as a gate. From this perspective, information causality is deeply connected with nonlocal computation.

Furthermore, we can combine the NS-box gates to form a more complicated circuit without worrying about the coding protocol. Then the total task function for the whole circuit will be a complicated function, i.e., a composite of task functions of all NS-boxes. We can then try to answer the following fundamental question: could a noiseless (nonlocal) computation be simulated using a noisy nonlocal physical resource?

Specifically, we consider the so-called $(n, k, l)$-circuit, $G$, formed by cascading layers of noisy gates into a circuit in the form of a directed, acyclic tree (see Fig. 1). On the top of $G$, there are $n$ inputs to the NS-boxes — the leaves; at the bottom there is only one NS-box — the root. The longest path from the leaves to the root is called the depth of the circuit, denoted by $l$. The maximum input number of a gate in $G$ is $k$. Note that, in [8] $G$ comprises $k = 2$ gates and is exploited to compress $n$ bits of $\bar{x}$ into one bit $A_{\bar{x}}$. However, there is no restriction on the task function for each NS-box, as long as the final circuit is a consistent acyclic tree diagram.

We then use the circuit $G$ to perform the following nonlocal computation. Alice’s $n$-bit database $\vec{a} = (a_0, a_1, \cdots, a_{n-1})$ is given to the leaves of $G$, and a conditional input $b \in \{0, 1, \cdots, n - 1\}$ is given to the distant Bob. The previous encoding $\vec{a} \rightarrow \vec{x}$ and $b \rightarrow \bar{y}$ for the RAC protocol, is also exploited here. Alice’s output is properly encoded and then fed into the NS-box at the next layer, again with Bob’s conditional input. The same procedure is performed recursively until reaching the root, with its output as the answer to the total task function at the root.

Alternatively, Bob’s decoding gates can be thought to be noise free, and the computational noise is only due to Alice’s encoding gates, and vice versa. This makes it easier to understand the above procedure of noisy computation. Now we can consider the information flow of $G$.

Theorem 2: For a noisy, local circuit $G$ with an ar-
where $\Delta := 1 + \delta$ with a probability $1 - \epsilon$ output asymptotically becomes random because $\Delta \to \infty$. In summary, this implies that information causality prevents any physically realizable $(n, k, l)$-circuit from achieving reliable computations of excessively complicated functions, i.e., with either too many inputs or lengthy steps needed.

The above result applies only when classical communication between Alice and Bob is disallowed. Under such circumstances, the noise of the gate is intrinsically constrained by the underlying physical theory. Otherwise, the classical communication can be exploited to improve the reliability of the gates so that the no-go result could be lifted.

IV. CONCLUSION

We show how information causality leads to Tsirelson bounds in a much easier way. A series of new Tsirelson bounds are then derived. These bounds provide some playground to test the information causality by experiments, as done before to test the Bell inequality. Moreover, deep ramifications concerning non-local quantum computation are also found and discussed. Especially, the no-go theorem for the reliable nonlocal quantum computation deserves more study to clarify its physical implication.

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Appendix

In this appendix, we write down the detail of getting the Tsirelson-type inequalities derived from IC, and also check these inequalities directly by semidefinite programming (SDP).

We review the the RAC protocol as follows. Alice has a database of $k$ bits $a_0, a_1, \ldots, a_{k-1}$ where $a_i \in \{0, 1\}$ is the random variable $\forall i \in (0, \ldots, k - 1)$. The distant Bob is given a random variable $b \in (0, \ldots, k - 1)$ and a bit $\alpha$ sent by Alice. Bob’s task is to guess $a_\alpha$. Here we will consider the RAC protocol with different settings. Case (a) is proposed in the main text. In case (b), Alice’s and Bob’s settings are modified. In the following, Alice’s input is denoted by an $N$-bit string $\vec{x} = x_1 \ldots x_N$. Let $N = 1 + \sum_{i=1}^{2^{i-1}} x_i$, $1 \leq x \leq 2^N$. Bob’s input is denoted by $N$- bit string $\vec{y} = y_1 \ldots y_N$.

- Case (a)

Here $N = k - 1$, and $x_i = a_0 + a_i \forall i \in (1, \ldots, k - 1)$. $y_i = \delta_{i,b} \forall i \in (1, \ldots, k - 1)$, if $b \neq 0$, $\vec{y} = \vec{0}$ if $b = 0$.

Let $y = 1 + \sum_{i=1}^{N} i y_i$, $1 \leq y \leq k$. In this case, the
The optimal solution of dual function is bounded under some vector $y$ subjected to certain conditions associated with Tomita inequality from information causality following the procedure in the main text is

$$\sum_{\{x\}, \{y\}} (-1)^{\|x\|} C_{x,y} \leq 2^{k-1} \sqrt{k}. \quad (10)$$

**Case (b)**

Here $N = k$, $x_i = a_{i-1}$, and $y_i = \delta_{i,b+1} \forall i \in \{1, \ldots, k\}$. Let $y = \sum_{i=1}^{N} t y_i$, $1 \leq y \leq k$. Then, the Tseirelson-type inequality from information causality is

$$\sum_{\{x\}, \{y\}} (-1)^{\|x\|} C_{x,y} \leq 2^k \sqrt{k}. \quad (11)$$

We now briefly introduce the semidefinite programming [14]. SDP is the problem of optimizing a linear function subjected to certain conditions associated with a positive semidefinite matrix $X$, i.e., $v^T X v \geq 0$, for $v \in \mathbb{C}^n$, and is denoted by $X \succeq 0$. It can be formulated as the standard primal problem as follows. Given the $n \times n$ symmetric matrices $C$ and $D_q$’s with $q = 1, \ldots, m$, we like to optimize the $n \times n$ positive semidefinite matrix $X \succeq 0$ such that we can achieve the following:

$$\begin{align*}
\text{minimize} & \quad \text{Trace}(C^T X) \\
\text{subject to} & \quad \text{Trace}(D_q^T X) = b_q, \quad q = 1, \ldots, m.
\end{align*} \quad (12)$$

Corresponding to the above primal problem, we can obtain a dual problem via a Lagrange approach [17]. The Lagrange duality can be understood as the following. If the primal problem is

$$\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{s.t.} & \quad f_q(x) \leq 0, \quad q \in 1 \ldots m. \quad (13a) \\
& \quad h_q(x) = 0, \quad q \in 1 \ldots p, \quad (13b)
\end{align*}$$

the Lagrange function can be defined as

$$L(x, \lambda, \nu) = f_0(x) + \sum_{q=1}^{m} \lambda_q f_q(x) + \sum_{q=1}^{p} \nu_q h_q(x). \quad (14)$$

where $\lambda_1, \ldots, \lambda_m$, and $\nu_1, \ldots, \nu_p$ are Lagrange multipliers respectively. Due to the problem and (14), the minima of $f_0$ is bounded by (14) under the constraints when $\lambda_1, \ldots, \lambda_m \geq 0$.

$$\inf_{x} f_0 \geq \inf_{x} L(x, \lambda, \nu).$$

Then the Lagrange dual function is obtained.

$$g(\lambda, \nu) = \inf_{x} L(x, \lambda, \nu).$$

$$g(\lambda, \nu) \leq p \text{ is the optimal solution of } f_0(x) \text{), for } \lambda_1, \ldots, \lambda_m \geq 0 \text{ and arbitrary } \nu_1, \ldots, \nu_p. \text{ The dual problem is defined.}$$

$$\begin{align*}
\text{maximize} & \quad g(\lambda, \nu) \\
\text{s.t.} & \quad \lambda_q \geq 0. \quad (q \in \{1 \ldots m\})
\end{align*} \quad (15)$$

We can use the same method to define the dual problem for SDP. From the primal problem of SDP (12), we can write down the dual function by using minimax inequality [18].

$$\begin{align*}
\inf_{X \succeq 0} \text{Trace}(C^T X) = \inf_{X \succeq 0} \text{Trace}(C^T X) + \sum_{q=1}^{m} y_q(b_q - \text{Trace}(D_q^T X)) \\
= \inf_{X \succeq 0} \sup_{y} \sum_{q=1}^{m} y_q(b_q) + \text{Trace}((C^T - \sum_{q=1}^{m} y_q D_q^T)X) \\
\geq \sup_{y} \inf_{X \succeq 0} \sum_{q=1}^{m} y_q(b_q) + \text{Trace}((C - \sum_{q=1}^{m} y_q D_q)^T X) \\
= \sup_{y} \inf_{X \succeq 0} \sum_{q=1}^{m} y_q(b_q) + \text{Trace}((C - \sum_{q=1}^{m} y_q D_q)^T X). \quad (16)
\end{align*}$$

The optimal solution of dual function is bounded under some vector $y$ [20].

$$\sup_{y} \inf_{X \succeq 0} \sum_{q=1}^{m} y_q(b_q) + \text{Trace}((C - \sum_{q=1}^{m} y_q D_q)^T X) = \sup_{y} \sum_{q=1}^{m} y_q(b_q) \quad \text{when } C - \sum_{q=1}^{m} y_q D_q \succeq 0 \quad -\infty \quad \text{otherwise.} \quad (17)$$
The corresponding dual problem is

\[
\text{maximize} \quad \sum_{q=1}^{m} y_q (b_q) \\
\text{s.t.} \quad S = C - \sum_{q=1}^{m} y_q D_q \geq 0.
\] (18a, 18b)

If the feasible solutions for the primal problem and the dual problem attain their minimal and maximal values denoted as \( p' \) and \( d' \) respectively, then \( p' \geq d' \), which is called the duality gap. This implies that the optimal solution of primal problem is bounded by dual problem. This then leads to the following: Both the primal and the dual problems attain their optimal solutions when the duality gap vanishes, i.e., \( d' = p' \).

We now use SDP to check the Tsirelson-type bound. To cast the above problem of finding the Tsirelson’s bound in the context of quantum mechanics, we need to use Tsirelson’s theorem [15]. It says that for any quantum state \(| \Psi \rangle \in A \otimes B \) shared by two observers Alice and Bob with their measurement outcomes being \( A_x \in [-1, 1] \) and \( B_y \in [-1, 1] \), respectively. The correlation function can be expressed by the inner product of two real unit vectors \( \alpha_x, \beta_y \in \mathbb{R}^t \). Therein, \( t \) and \( v \) are the numbers of Alice’s and Bob’s measurement settings, respectively. In detail, \( C_{x,y} \) used in (10) or (11), the Tsirelson’s theorem guarantees that we have \( C_{x,y} = \alpha_x \cdot \beta_y \). Then, we can cast the problem of finding the Tsirelson bound in (10) or (11) into the following form of optimal problem for SDP:

\[
\text{maximize} \quad | \sum_{\{\overline{x}, \overline{y}\}} (-1)^{\overline{x} \cdot \overline{y}} \alpha_x \cdot \beta_y | \\
\text{s.t.} \quad \| \alpha_x \| = \| \beta_y \| = 1, \quad \forall x, y. (19a, 19b)
\]

Then, the associated dual problem is

\[
\text{minimize} \quad \sum_{q=1}^{m} y_q \\
\text{s.t.} \quad S = \sum_{q=1}^{m} y_q D_q - C \geq 0. (20a, 20b)
\]

We now will turn the problem (19) into the primal problem (12) by constructing the matrices \( X, C, \) and \( A_i \)'s from the unit vectors \( \alpha_x \) and \( \beta_y \). Following the way in [16], the mapping is as follows. Define the matrix \( P \) whose columns are vectors \( (\alpha_1, ..., \alpha_t, \beta_1, ..., \beta_v) \). Then the SDSP matrix \( X \) is given by \( P^T P \), which can be put into the following block form

\[
X = \begin{pmatrix} E & F \\ G & H \end{pmatrix}
\]

where the matrix elements of each block are \( E_{ij} = \alpha_i \cdot \alpha_j, F_{ib} = \alpha_i \cdot \beta_b, G_{aj} = \beta_a \cdot \alpha_j \) and \( H_{ab} = \beta_a \cdot \beta_b \) with \( i, j = 1, \ldots, t \) \( (t = 2^N) \) and \( a, b = 1, \ldots, v \) \( (v = k) \). Note that \( F \) and \( G \) are used in (19), and instead \( E \) and \( H \) are used in (19b). Therefore, we can write down the matrices \( C = D_q \)'s accordingly so that the problem (19) is equivalent to the primal problem (12). It is easy to see that \( C \) is a matrix with only non-vanishing off-diagonal block of matrix elements given by \((-1)^{x \cdot y}\)’s and \( D_q \)'s are the diagonal matrices with \( (D_q)_{st} = \delta_{s, q} \delta_{k, q} \). We omit their detailed forms here.

We take \( k = 2 \) and \( k = 3 \) in case(a) for example.

- **k=2**

Here \( \overline{x} = x_1 \) and \( \overline{y} = y_1 \). According Eq. (10), we want to maximize \(| C_{0,0} + C_{0,1} + C_{1,0} - C_{1,1} | \). Using the Tsirelson theorem, it is equivalent to maximizing \( \alpha_1 \cdot \beta_1 + \alpha_2 \cdot \beta_2 + \alpha_3 \cdot \beta_3 - \alpha_2 \cdot \beta_1 - \alpha_2 \cdot \beta_3 \). Such Tsirelson bound has been showed by Wehner [16] using SDP. We just show the numerical result. For more details, please see [16]. After using SeDuMi program [19] to solve SDP, the optimal for both primal and dual problem is 2.8284. It is consistent with the Tsirelson bound [7] \((2\sqrt{2})\) for the case two settings per site.

- **k=3**

Here \( \overline{x} = x_1 x_2 \) and \( \overline{y} = y_1 y_2 \). Notably, \( \overline{y} \in \{00, 10, 01\} \). The problem which we want to solve is

\[
\text{maximize} \quad | C_{00,00} + C_{00,10} + C_{00,01} + C_{01,00} + C_{01,10} - C_{01,01} \\
+ C_{10,00} - C_{10,10} + C_{10,01} + C_{11,00} - C_{11,10} - C_{11,01} | \\
= \text{maximize} \quad \alpha_1 \cdot \beta_1 + \alpha_1 \cdot \beta_2 + \alpha_1 \cdot \beta_3 + \alpha_2 \cdot \beta_1 + \alpha_3 \cdot \beta_1 + \alpha_3 \cdot \beta_2 + \alpha_3 \cdot \beta_3 + \alpha_2 \cdot \beta_1 - \alpha_2 \cdot \beta_3 + \alpha_2 \cdot \beta_2 - \alpha_4 \cdot \beta_2 - \alpha_4 \cdot \beta_3.
\] (21)

The \( X \) matrix for primal problem is \( X = S^T S \) where the columns of \( S \) correspond the unit vec-
tors \((\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3)\).

\[
X = \begin{pmatrix}
\alpha_1 \cdot \alpha_1 & \alpha_1 \cdot \alpha_2 & \alpha_1 \cdot \alpha_3 & \alpha_1 \cdot \alpha_4 & \alpha_1 \cdot \beta_1 & \alpha_1 \cdot \beta_2 & \alpha_1 \cdot \beta_3 \\
\alpha_2 \cdot \alpha_1 & \alpha_2 \cdot \alpha_2 & \alpha_2 \cdot \alpha_3 & \alpha_2 \cdot \alpha_4 & \alpha_2 \cdot \beta_1 & \alpha_2 \cdot \beta_2 & \alpha_2 \cdot \beta_3 \\
\alpha_3 \cdot \alpha_1 & \alpha_3 \cdot \alpha_2 & \alpha_3 \cdot \alpha_3 & \alpha_3 \cdot \alpha_4 & \alpha_3 \cdot \beta_1 & \alpha_3 \cdot \beta_2 & \alpha_3 \cdot \beta_3 \\
\alpha_4 \cdot \alpha_1 & \alpha_4 \cdot \alpha_2 & \alpha_4 \cdot \alpha_3 & \alpha_4 \cdot \alpha_4 & \alpha_4 \cdot \beta_1 & \alpha_4 \cdot \beta_2 & \alpha_4 \cdot \beta_3 \\
\beta_1 \cdot \alpha_1 & \beta_1 \cdot \alpha_2 & \beta_1 \cdot \alpha_3 & \beta_1 \cdot \alpha_4 & \beta_1 \cdot \beta_1 & \beta_1 \cdot \beta_2 & \beta_1 \cdot \beta_3 \\
\beta_2 \cdot \alpha_1 & \beta_2 \cdot \alpha_2 & \beta_2 \cdot \alpha_3 & \beta_2 \cdot \alpha_4 & \beta_2 \cdot \beta_1 & \beta_2 \cdot \beta_2 & \beta_2 \cdot \beta_3 \\
\beta_3 \cdot \alpha_1 & \beta_3 \cdot \alpha_2 & \beta_3 \cdot \alpha_3 & \beta_3 \cdot \alpha_4 & \beta_3 \cdot \beta_1 & \beta_3 \cdot \beta_2 & \beta_3 \cdot \beta_3
\end{pmatrix}
\]

According to (21), the matrix \(C\) is defined

\[
C = \frac{1}{2} \times \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1
\end{pmatrix}
\]

The norm of the vectors \((\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3)\)

\[
X = \begin{pmatrix}
1.0000 & 0.3333 & 0.3333 & -0.3333 & 0.5774 & 0.5774 & 0.5774 \\
0.3333 & 1.0000 & -0.3333 & 0.3333 & 0.5774 & -0.5774 & 0.5774 \\
0.3333 & -0.3333 & 1.0000 & 0.3333 & 0.5774 & 0.5774 & -0.5774 \\
-0.3333 & 0.3333 & 0.3333 & 1.0000 & 0.5774 & -0.5774 & -0.5774 \\
0.5774 & 0.5774 & 0.5774 & 0.5774 & 1.0000 & 0.0000 & 0.0000 \\
0.5774 & -0.5774 & 0.5774 & -0.5774 & 0.0000 & 1.0000 & -0.0000 \\
0.5774 & 0.5774 & -0.5774 & -0.5774 & 0.0000 & -0.0000 & 1.0000
\end{pmatrix}
\]

\(X\) satisfying the constraint that \(X\) is SDSP with non-negative eigenvalues [20].

- **For the case \(k = 3\) to \(k = 8\)**

After setting up the SDP for finding the Tsirelson bound, we still use the package named SeDuMi to solve it for both case (a) and (b) with any value of \(k\). The result agrees extremely well with the bound obtained from information causality up to \(O(10^{-4})\). To be more concrete, the numerical results are shown below: for case (a) up to \(k = 8\), we have

| \(k\) | 3   | 4   | 5   | 6   | 7   | 8   |
|------|-----|-----|-----|-----|-----|-----|
| SDP  | 6.9282 | 16.0000 | 35.7771 | 78.3837 | 169.3281 | 362.0387 |

This agrees extremely well with the RHS of (10).

Similarly, for case (b) up to \(k = 8\), we have

| \(k\) | 3   | 4   | 5   | 6   | 7   | 8   |
|------|-----|-----|-----|-----|-----|-----|
| SDP  | 13.8564 | 32.0000 | 71.5542 | 156.7673 | 338.6562 | 724.0773 |

It again agrees extremely well with (11). Therefore, based on our numerical simulation, information causality indeed singles out the Tsirelson bound of a physical theory such as quantum mechanics.

[1] S. Popescu and D. Rohrlich, Found. Phys. **24**, 379 (1994).
[2] Ll. Masanes, A. Acín, and N. Gisin, Phys. Rev. A 73,
[3] J. Barrett, Phys. Rev. A 75, 032304 (2007).
[4] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Phys. Rev. A 71, 022101 (2005).
[5] J. Barrett, L. Hardy, and A. Kent, Phys. Rev. Lett. 95, 010503 (2005).
[6] V. Scarani, N. Gisin, N. Brunner, L. Masanes, S. Pino, and A. Acín, Phys. Rev. A 74, 042339 (2006).
[7] B. S. Tsirelson Lett. Math. Phys. 4, 93 (1980).
[8] M. Pawlowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Zukowski, Nature, 461, 1101 (2009).
[9] W. Evans and L. J. Schulman, Proceedings of the 34th Annual Symposium on Foundations of Computer Science, 594 (1993).
[10] W. Evans and L. J. Schulman, IEEE Trans. Inf. Theory, 45 2367 (1999).
[11] N. Linden, S. Popescu, A. J. Short, and Andreas Winter, Phys. Rev. Lett. 99, 180502 (2007).
[12] J. von Neumann, in Automata Studies, C. E. Shannon and J. McCarthy, Eds. Princeton, NJ: Princeton Univ. Press, 1956, pp. 43.
[13] N. Pippenger, IEEE Transactions on Information Theory, 34(2):194-197, March 1988.
[14] L. Vandenberghe and S. Boyd, SIAM Review 38, 1 (1996).
[15] B. Tsirelson, Hadronic J. Suppl. 8, 329 (1993).
[16] S. Wehner, Phys. Rev. A 73, 022110 (2006).
[17] Boyd, Stephen and Vandenberghe, Lieven (2004). Convex Optimization. Cambridge University Press.
[18] http://homepages.cwi.nl/~monique/ow-seminar-sdp/files/ow_intro.pdf
[19] J. Sturm and AdvOL, http://sedumi.mcmaster.ca.
[20] C. Helmberg. Semidefinite programming for combinatorial optimization. Technical Report ZIB-Report ZR-00-34, Konrad-Zuse-Zentrum Berlin, 2000.