Structure theorems for AP rings

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Abstract

In “New Proofs of the structure theorems for Witt Rings”, the first author shows how the standard ring-theoretic results on the Witt ring can be deduced in a quick and elementary way from the fact that the Witt ring of a field is integral and from the specific nature of the explicit annihilating polynomials he provides. We will show in the present article that the same structure results hold for larger classes of commutative rings and not only for Witt rings. We will construct annihilating polynomials for these rings.

1 Introduction

Using his theory of multiplicative forms, Pfister obtained in 1966 several structure theorems for Witt rings of quadratic forms over fields. The proofs of these results were later simplified and Harrison, Leicht and Lorenz added different results concerning the ideal theory of these rings. In 1987, the first author produced polynomials annihilating all non-singular quadratic forms of a given dimension in the Witt ring. He showed that the structure results also follow from the fact that the Witt ring is integral and the specific form of the polynomials. The main goal of this article is to obtain these structure theorems for more general classes of rings.

In the second section we will look at commutative rings $R$ additively generated by a subset of this ring $R$. The elements of this subset being zeros of a polynomial $q(X)$ with integer coefficients. In these rings $R$, which we will call annihilating polynomial rings or AP rings for short, one can define a length map $\ell : R \to \mathbb{N}$ and construct polynomials $p_n(X) \in \mathbb{Z}[X]$ in such a way that all elements of length $n$ are annihilated by $p_n(X)$. We will give many examples of AP rings.

In the third section we will obtain structure results for all AP rings and in the fourth section we will look at AP rings with generating polynomial $q(X) = X^{2^n} - 1$. We will show a close relationship between the signatures and the prime ideals in these rings and assuming an extra ‘admissible’ condition on AP rings we will be able to produce the complete classification of the spectrum, in analogy to the work of Harrison, Leicht and Lorenz on the spectrum of Witt rings. Finally, looking at the specific nature of the prime ideals in these AP rings, we will obtain an analogue of Pfister’s local-global principle.

In the last section we will construct the annihilating polynomials for different choices of $q(X)$. 

1
2 Annihilating polynomial rings

2.1 Definition

Let \( q(X) \) be a monic polynomial with integer coefficients. Let \( R \) be a commutative ring and \( S \) a subset, such that

(R1) \( R \) is additively generated by the subset \( S \subseteq R \) and

(R2) \( q(s) = 0 \) for all \( s \in S \).

A commutative ring \( R \) satisfying the conditions (R1) and (R2) will be called an annihilating polynomial ring or AP ring for short. \( S \) will be called a generating set and \( q(X) \) a generating polynomial for the AP ring \( R \).

The condition (R1) enables us to introduce a notion of length. Given an element \( r \in R \) we denote by \( \ell_S(r) \) the least number \( l \) such that it is possible to write \( r \) as a sum of \( l \) elements of \( S \), i.e. \( r = \sum_{i=1}^{l} \epsilon_i a_i \) with \( \epsilon_i = \pm 1, a_i \in S \); we call \( \ell_S(r) \) the length of \( r \) (with respect to \( S \)). In this way, we have defined a map \( \ell_S : R \rightarrow \mathbb{N} \) which we shall call the length map of \( R \) relative to \( S \).

We will write \( \ell(r) \) for short if the choice of the generating set \( S \) for the ring \( R \) is clear.

Recall this general proposition for commutative rings.

Let \( q_1(X), q_2(X), \ldots, q_n(X) \) be monic polynomials with integer coefficients and such that none of the \( q_i(X) \) has a multiple root in the complex numbers \( \mathbb{C} \). Write \( R_i \) for the set of roots of \( q_i(X) \) in \( \mathbb{C} \) and \( T_n \) for the set of all complex numbers \( \sigma \) expressible in the form \( \sigma = \sum_{i=1}^{n} \epsilon_i \sigma_i \) where \( \epsilon_i = \pm 1, \sigma_i \in R_i \) for each \( i \). We write \( p_n(X) = \prod_{\sigma \in T_n} (X - \sigma) \) which is a monic polynomial with integer coefficients, without multiple roots. We have the following

**Proposition 2.1.** [Lewis] Let \( R \) be a commutative ring. Let \( x_1, x_2, \ldots, x_n \) be elements of \( R \) such that \( q_i(x_i) = 0 \) for \( i = 1, 2, \ldots, n \) and let \( x = \sum_{i=1}^{n} \epsilon_i x_i \) where \( \epsilon_i = \pm 1 \). Then \( p_n(x) = 0 \) in \( R \).

**Proof.** See [3]. \( \square \)

Let \( R \) be an AP ring with generating polynomial \( q(X) \). Let \( T_1 \) denote the set of roots of \( q(X) \) in \( \mathbb{C} \) and \( T_n \) the set of complex numbers expresible in the form \( \sigma = \sum_{i=1}^{n} \epsilon_i \sigma_i \) where \( \epsilon_i = \pm 1, \sigma_i \in T_1 \). Put \( p_n(X) = \prod_{\sigma \in T_n} (X - \sigma) \). We have:

**Corollary 2.2.** Let \( R \) be an AP ring. Every element of length \( n \) is annihilated by \( p_n(X) \) in \( R \). In particular, \( R \) is integral.

**Proof.** Let \( r \) be an element of length \( n \) in the AP ring \( R \) with generating set \( S \) and generating polynomial \( q(X) \). Then there exist elements \( a_1, a_2, \ldots, a_n \in S \) such that \( r = \sum_{i=1}^{n} \epsilon_i a_i, \epsilon_i = \pm 1 \) and \( q(a_1) = q(a_2) = \ldots = q(a_n) = 0 \). From the previous proposition it follows that \( p_n(r) = 0 \). \( \square \)
Before giving examples of AP rings, let us give an example of constructing an annihilating polynomial.

Let $R$ be an AP ring with generating polynomial $q(X) = X^2 - 1$. The roots of the generating polynomial are $-1$ and $1$. The possible values for the sum of $n$ elements out of $\{-1, 1\}$ lie in $\{-n, -n+2, \ldots, n-2, n\}$. The annihilating polynomial for an element of length $n$ is thus the $n$-th Lewis polynomial

$$p_n(X) = (X-n)(X-(n-2))\ldots(X+(n-2))(X+n).$$

Further examples of constructing the annihilating polynomial for a given AP ring will be given in section 5.

2.2 Examples of AP rings

(i) $\mathbb{Z}$ is an AP ring. It is additively generated by $S = \{-1, 1\}$ and we can take the generating polynomial $q(X) = X^2 - 1$.

(ii) A product $\prod \mathbb{Z}$ of finitely many copies of $\mathbb{Z}$ is an AP ring. It is generated by the elements $S = \{\pm e_i\}$ where $e_i$ has a 1 in the $i$-th place and zero elsewhere, and we can take $q(X) = X^3 - X$ as a generating polynomial.

(iii) Let $G$ be an abelian group of finite exponent $n$. The group ring $\mathbb{Z}[G]$ is an AP ring with generating set $S = G$. Since every element has finite order $n$, we can take $q(X) = X^n - 1$ as the generating polynomial.

(iv) Let $K$ be an ideal in the AP ring $R$. Then $R/K$ is a AP ring as well, with generating set $S/K$. If $q(X)$ is a generating polynomial for the AP ring $R$, then $q(X)$ is a generating polynomial for $R/K$ as well. Examples are $\mathbb{Z}_n$, abstract Witt rings (see [10]),

(v) Witt-Grothendieck $\hat{W}(F)$ and Witt rings $W(F)$ of quadratic forms over fields $F$ (see [6] or [11] for further details on quadratic forms and Witt rings).

Let $F$ be a field of characteristic not 2. Consider the group ring $\mathbb{Z}[F^*/F^{*2}]$ and the canonical ring homomorphism

$$\pi : \mathbb{Z}[F^*/F^{*2}] \rightarrow \hat{W}(F)$$

defined by $\alpha \mapsto \langle \alpha \rangle$ for $\alpha \in F^*/F^{*2}$. $\pi$ is surjective since every bilinear space is an orthogonal sum of 1-dimensional ones and therefore

$$\hat{W}(F) \cong \mathbb{Z}[F^*/F^{*2}]/\ker(\pi).$$

The fact that $\hat{W}(F)$ is an AP ring follows from examples (iii) and (iv) and we can take $q(X) = X^2 - 1$.

$W(F)$ is the quotient ring of $\hat{W}(F)$ by the ideal generated by the hyperbolic spaces $\{n[H] \mid n \in \mathbb{Z}\}$. By example (iv), $W(F)$ is an AP ring with generating polynomial $q(X) = X^2 - 1$. 

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Let $F$ be a field of characteristic not $2$. Denote by $\langle \alpha \rangle_n$ the class $\alpha F^*/F^*2^n$ in $F^*/F^*2^n$ and consider the ideal $R_n(F)$ of $\mathbb{Z}[F^*/F^*2^n]$ generated by $\langle 1 \rangle_n + \langle -1 \rangle_n$ and $(\sum_{i=0}^{2^n-1} \langle \alpha^i \rangle_n)(1 - (1 + \alpha)_n)$ for all $\alpha \in F^*$.

The ring $W_n(F) := \mathbb{Z}[F^*/F^*2^n]/R_n(F)$ is called the Witt ring of level $n$ of $F$ and is an AP ring with generating polynomial $q(X) = X^{2^n} - 1$.

The Burnside ring of a finite group (see [13] for further details on Burnside rings). Let $G$ be a finite group. The set of isomorphism classes of finite $G$-sets form a commutative associative semi-ring $\Omega^+(G)$ with unit under disjoint union and cartesian product. The Grothendieck ring $\Omega(G)$ constructed from this semi-ring is called the Burnside ring of $G$.

Additively, $\Omega(G)$ is the free abelian group on isomorphism classes of transitive $G$-sets. Equivalently, an additive $\mathbb{Z}$-basis for $\Omega(G)$ is given by the $[G/H]$ in $\Omega(G)$ where $(H)$ runs through the set $C(G)$ of conjugacy classes of subgroups of $G$.

There exist an injective ring homomorphism

$$
\varphi : \Omega(G) \longrightarrow \prod_{(H) \in C(G)} \mathbb{Z}
$$

induced by

$$
T \mapsto \langle |T^H| \mid (H) \in C(G) \rangle
$$

where $|T^H|$ denotes the number of elements of the $G$-set $T$ fixed under $H$.

Each generator $y = [G/H]$ of $\Omega(G)$ maps to an element $(n_1, n_2, \ldots, n_k)$ ($k = |C(G)|$) and this is annihilated in $\prod_{i=1}^k \mathbb{Z}$ by the polynomial $q_y(X) = \prod_{i=1}^k (X - n_i)$. By the injectivity of the above ring homomorphism $\varphi$, $q_y(X)$ annihilates $y$ in $\Omega(G)$. Let $q(X) = \prod q_y(X)$, the product taken over $S$, the finite set of generators $y$ of $\Omega(G)$. Then $\Omega(G)$ is an AP ring with generating polynomial $q(X)$.

The Witt ring of a central simple algebra In [9] the first author and Tignol define this ring as a quotient of a group ring of abelian groups of exponent two. They generalize the notion of Witt ring of a skew field done earlier by Craven and Sladek (see [2] and [12]). All these Witt rings are AP rings with generating polynomial $q(X) = X^2 - 1$. Note that this immediately implies that the Lewis polynomials annihilate the elements of these rings.

In the next section we will obtain structure results for these AP rings using the fact that these rings are integral and the specific nature of the annihilating polynomials.
3 Structure theorems for AP rings

We will start this section with investigating the spectrum of AP rings. Let us fix the following notations first.

\[ \text{Nil} R \] is the nilradical of \( R \), i.e. set of nilpotent elements of \( R \), \( \text{Nil} R = \cap P \) where \( P \) runs through all the prime ideals of \( R \).

\[ R_t \] is the torsion subgroup of the additive group of \( R \).

\[ \text{Spec} R \] is the set of prime ideals of \( R \).

\[ \text{Max} R \] and \( \text{Min} R \) are the subsets of \( \text{Spec} R \) consisting of the maximal ideals and minimal prime ideals of \( R \) respectively.

**Lemma 3.1.** Let \( R \) be an AP ring. Then

(i) A prime ideal \( P \) of \( R \) is maximal if and only if there is a rational prime \( p \) with \( p \cdot 1_R \in P \).

(ii) If \( R_t \subset \text{Nil} R \) then a prime ideal \( P \) of \( R \) is minimal if and only if \( P \cap \mathbb{Z} = 0 \), maximal otherwise, and every maximal ideal of \( R \) properly contains a minimal prime ideal.

(iii) \( \text{Nil} R \subset R_t \).

**Proof.**

(i),(ii) Since every AP ring \( R \) is integral over \( \mathbb{Z} \), this follows from [5, Lemma 2.5.].

(iii) Let \( r \in \text{Nil} R \) and let \( k \in \mathbb{N} \) be such that \( r^k = 0 \) and \( r^{k-1} \neq 0 \). Since \( R \) is an AP ring, \( r \) is annihilated by some \( p_n(X) \). Write \( r^n + a_{n-1}r^{n-1} + \ldots + a_1r + a_0 = 0 \) where \( a_i \in \mathbb{Z} \). Suppose that \( a_i \) is the non-zero number of lowest index (\( i = 0 \) or \( i = 1 \) since \( p_n(X) \) has no multiple roots (see 2.1)).

Multiplying the above equation with \( r^{k-1} \) \( (k - i - 1 \geq 0, \text{since } k \geq 2) \) yields \( a_ir^{k-1} = 0 \). Using this and multiplying the equation with \( a_ir^{k-1-2} \) we obtain \( a_i^2r^{k-2} = 0 \). Repeating this process will give \( a_i^{k-1}r = 0 \), i.e. \( r \in R_t \).

\[ \square \]

Let \( R \) be an AP ring with generating polynomial \( q(X) \) and a multiplicatively closed generating set \( S \). Suppose \( R \) is of the form \( \bigoplus s \in S \mathbb{Z}s \). Let \( \chi \) be a (monoid-)morphism of \( S \) into \( \mathbb{C} \),

\[ \chi : S \rightarrow \mathbb{C}, \]

mapping 1 to 1. Remark that for every \( s \in S \), \( \chi(s) \) will be a root of \( q(X) \).

Let \( K \) be the field generated over \( \mathbb{Q} \) by all the roots of \( q(X) \) (i.e. by all the \( \chi(s) \) and let \( \mathbb{C} \) be the integral closure of \( \mathbb{Z} \) in \( K \). Every morphism \( \chi \) of \( S \) to \( \mathbb{C} \) extends to a ring homomorphism

\[ \phi_\chi : R \rightarrow \mathbb{C} \]
in the obvious way by defining

\[ \phi_{\chi}(\sum a_{i}s_{i}) = \sum a_{i}\chi(s_{i}) \quad \text{for all} \quad a_{i} \in \mathbb{Z}, \: s_{i} \in S. \]

The spectrum of \( R = \bigoplus_{s \in S} \mathbb{Z}s \) is completely determined by the following lemma.

**Lemma 3.2.** The minimal prime ideals of \( R = \bigoplus_{s \in S} \mathbb{Z}s \) are the kernels \( P_{\chi} \) of the morphisms \( \phi_{\chi} : R \rightarrow C \). The maximal ideals of \( \mathbb{Z}[S] \) are of the form \( M_{\chi,p} = \phi_{\chi}^{-1}(p) \) where \( p \) is a non-zero (and thus maximal) prime ideal of \( C \).

**Proof.** The proof of [5, Lemma 3.1.] extends in an obvious way from abelian torsion groups \( G \) to sets \( S \) and the assertion follows. \( \square \)

A signature \( \sigma \) of \( R \) is a (necessarily surjective) ring homomorphism \( \sigma : R \rightarrow \mathbb{Z} \). We call a prime ideal \( \mathcal{P} \) a signature ideal of \( R \) if \( R/\mathcal{P} \) is isomorphic to \( \mathbb{Z} \). We will call \( X(R) = \{ \mathcal{P} | \mathcal{P} \text{ a signature ideal of } R \} \) the space of signatures of \( R \).

**Proposition 3.3.** Let \( R = \bigoplus_{s \in S} \mathbb{Z}s \) be the AP ring generated by \( q(X) \). If \( q(X) \) has all its roots in \( \mathbb{Z} \) then

\[ \text{Spec}(R) = \text{Min}(R) \cup \text{Max}(R) \]

where

\[ \text{Min}(R) = X(R) \]

and

\[ \text{Max}(R) = \{ \mathcal{P} + pR \mid \mathcal{P} \in X(R), \: \text{and } p \text{ a rational prime} \}. \]

**Proof.** \( q(X) \) has all its roots in \( \mathbb{Z} \), so \( \chi(s) \in \mathbb{Z} \) for all \( s \in S \). Using the above notation we get that \( C = \mathbb{Z} \) and \( K = \mathbb{Q} \). All the minimal prime ideals, as kernels of homomorphisms of \( R = \bigoplus_{s \in S} \mathbb{Z}s \) to \( \mathbb{Z} \) are the signature ideals of \( R \). Since all the maximal ideals of \( \mathbb{Z} \) are of the form \( p\mathbb{Z} \), the maximal ideals of \( R \) are of the form \( \mathcal{P} + pR \), where \( \mathcal{P} \in X(R) \). \( \square \)

The Burnside ring \( \Omega(G) \) of a finite group \( G \) satisfies the conditions of the previous proposition (see example (viii)). For a subgroup \( U \leq G \) and a \( G \)-set \( S \) the map \( S \mapsto \#S^{U} \) (i.e. the number of elements in \( S \) invariant under \( U \)) extends to a ring homomorphism \( \phi_{U} : \Omega(G) \rightarrow \mathbb{Z} \). Define for \( p \) being 0 or a prime number the prime ideal \( \mathfrak{p}_{U,p} = \{ x \in \Omega(G) | \#x^{U} \equiv 0 \mod 2 \} \). Since \( X(\Omega(G)) = \{ \ker(\phi_{U}) \mid U \leq G \} \) we can retrieve Dress’ result on the description of the spectrum of the Burnside ring \( \Omega(G) \):

**Proposition 3.4 (Dress).** (see [8, Proposition 1.]) One has \( \mathfrak{p}_{U,p} \subset \mathfrak{p}_{V,q} \) if and only if \( p = q \) and \( \mathfrak{p}_{U,p} = \mathfrak{p}_{V,q} \) or \( p = 0, q \neq 0 \) and \( \mathfrak{p}_{V,q} \subset \mathfrak{p}_{U,p} \). Especially, \( \mathfrak{p}_{U,p} \) is minimal, respectively maximal, if and only if \( p = 0 \), respectively \( p \neq 0 \).
4 AP rings with generating polynomial $X^{2^k} - 1$

We will now take a closer look at AP rings with generating polynomial $q(X) = X^{2^k} - 1$ and a group $S = G$ as a set of generators. We will obtain some additional structure theorems.

Consider $I := (1 - a | a \in G)$. We will call this ideal the \textit{fundamental ideal of $R$}.

Observe that for any AP ring $R$, $|R/I| \leq 2$, since every element is a sum of an even or an odd number of elements of $G \cup -G$.

**Proposition 4.1.** Let $R$ be an AP ring with generating polynomial $X^{2^k} - 1$ and a group $S = G$ as a set of generators. The following conditions are equivalent:

(i) $R$ does not have odd characteristic.

(ii) $|R/I| = 2$.

(iii) $\sum_{i=1}^{m} \varepsilon_i a_i = 0$, $a_i \in G \Rightarrow m \in 2\mathbb{Z}$.

\textbf{Proof.}

(iii) $\Rightarrow$ (ii): Suppose $|R/I| = 1$, then $1 \in I$ and we have $1 + \sum_i \varepsilon_i(a_i - 1) = 0$, contradicting (iii).

(ii) $\Rightarrow$ (i): Suppose $R$ has odd characteristic $k$ and let $r \in R$. Then either $r \in I$ or $r = r + k \in I$ implying $|R/I| = 1$, a contradiction.

(i) $\Rightarrow$ (iii): Assume $r = \sum_{i=1}^{m} a_i = 0$ with $m$ odd. Since $p_m(r) = 0$ this implies that $p_m(0) = 0$ in $R$ contradicting the fact that $R$ has even characteristic. See also lemma [52].

\[ \square \]

Any AP ring $R$ which satisfies the conditions of the previous proposition will be called \textit{admissible}.

These admissibility conditions are not always satisfied, e.g. $\mathbb{Z}/n\mathbb{Z}$ is an AP ring for all $n \in \mathbb{N}$, but is only admissible if $n$ is even.

Consider the Arason-Pfister property $AP(k)$

If $r = a_1 + a_2 + \ldots + a_n \in I^k$, with $a_i \in G \cup -G$, $n < 2^k$, then $r = 0$.

We have the following

**Proposition 4.2.** An AP ring is admissible if and only if $AP(1)$ holds.

Let $R$ be an admissible AP ring. The unique isomorphism $R/I \cong \mathbb{Z}/2\mathbb{Z}$ induces a homomorphism

\[ \varphi : R \to \mathbb{Z}/2\mathbb{Z} \]

\[ r \mapsto \varphi(r) \equiv \ell(r) \mod 2, \forall r \in R. \]
Proposition 4.3. Let \( R \) be an admissible AP ring. Then the fundamental ideal \( I \) is the only prime ideal of index 2 in \( R \).

**Proof.** Suppose \( P \) is a prime ideal of index 2. For every generating element \( a \) we have \( 0 = (1+a^{2^k-1})(1-a^{2^k-1}) \in P \). Since \( 2 \in P \), this implies that \( 1-a^{2^k-1} \in P \). Using this observation we obtain finally that \( 1-a \in P \). Since \( 1-a \) are exactly the generators for \( I \), we get \( I \subseteq P \) and thus \( I = P \) by maximality of \( I \).

From now on we look at the special case \( k = 1 \), i.e. where \( R \) is an AP ring with generating polynomial \( q(X) = X^2 - 1 \).

Remark 4.4. AP rings satisfying these conditions are for example Witt rings of fields, Witt rings of central simple algebras and abstract Witt rings.

Proposition 4.5. Let \( R \) be an AP ring with generating polynomial \( q(X) = X^2 - 1 \). If \( P \) is a prime ideal of \( R \), then either \( P \) is of finite index which is a prime number, or \( R/P \) is isomorphic to \( \mathbb{Z} \).

**Proof.** Let \( P \) be a prime ideal in \( R \). Since \( R/P \) is an integral domain and \( (a-1)(a+1) = 0 \) for all \( a \in G \), we have that \( a = 1 + P \) or \( a = -1 + P \) for all \( a \in G \). Since \( R \) is additively generated by \( G \), we have that \( R/P \) is cyclic generated by \( 1 + P \). If the characteristic of the integral domain \( R/P \) is \( p \), then \( P \) is of finite index \( p \) in \( R \). Otherwise, the characteristic of \( R/P \) is 0 and so \( R/P \) is isomorphic to \( \mathbb{Z} \).

Proposition 4.6. Let \( R \) be an AP ring with generating polynomial \( q(X) = X^2 - 1 \). There is a canonical one-to-one correspondence between signature ideals in \( R \) and the different signatures of \( R \).

**Proof.** Let \( \sigma \) be a signature of \( R \). It is clear that \( \ker(\sigma) \) is a signature ideal in \( R \). Conversely, suppose that \( P \) is a signature ideal of \( R \). The isomorphism \( R/P \cong \mathbb{Z} \) induces an unique homomorphism \( R \to \mathbb{Z} \) with kernel \( P \).

Let \( \mathcal{P} \) be a signature ideal of \( R \) and consider the obvious prime ideals of finite index \( p \), namely \( \mathcal{P} + pR \). We will show that these ideals are the only prime ideals of finite index in \( R \). Let us first recall the following result:

**Lemma 4.7.** Let \( R \) be an AP ring with generating polynomial \( q(X) = X^2 - 1 \) and assume \( X(R) \neq \emptyset \). Then \( \mathcal{P} + pR \) is the unique prime ideal of finite index \( p \), containing the signature ideal \( \mathcal{P} \).

Moreover, these ideals are the only ideals of finite index \( p \) different from 2 since:

**Lemma 4.8.** Let \( R \) be an AP ring with generating polynomial \( q(X) = X^2 - 1 \) and assume \( X(R) \neq \emptyset \). Let \( \mathcal{Q} \) be a prime ideal of finite index \( p \neq 2 \). Then \( \mathcal{Q} = \mathcal{P} + pR \) for some signature ideal \( \mathcal{P} \in X(R) \).
Proof. We will construct a signature $\sigma : R \rightarrow \mathbb{Z}$ such that $P = \ker(\sigma)$ and $P \subset \mathbb{Q}$. The above lemma will then complete the proof.

Let $\mathcal{M}$ be the minimal prime ideal such that $\mathcal{M} \subset \mathbb{Q}$. Since $a^2 = 1$, for all $a \in G$, we have $a - 1 \in \mathcal{M}$ or $a + 1 \in \mathcal{M}$. Define $\sigma_\mathcal{M} : G \rightarrow \{-1, +1\}$ for all $a \in G$ by

$$\sigma_\mathcal{M}(a) = \begin{cases} -1, & \text{if } a + 1 \in \mathcal{M} \\ +1, & \text{if } a - 1 \in \mathcal{M}. \end{cases} \quad (1)$$

Define $\sigma : R \rightarrow \mathbb{Z}$ for all $r = \sum_{i=1}^{k} \varepsilon_i a_i \in R$, where $\varepsilon_i = \pm 1$, $a_i \in G$ by

$$\sigma(r) = \sum_{i=1}^{k} \varepsilon_i \sigma_\mathcal{M}(a_i).$$

We claim that $\sigma$ is a well-defined signature of $R$. Proof of claim:

Suppose $r = \sum_{i=1}^{k} \varepsilon_i a_i = 0 \in R$. Then

$$\sigma(r) = \sum_{i=1}^{k} \varepsilon_i \sigma_\mathcal{M}(a_i) = \sum_{i=1}^{k} \varepsilon_i (a_i - \sigma_\mathcal{M}(a_i)) \in \mathcal{M} \cap \mathbb{Z}.$$  

Since the AP ring $R$ is integral over $\mathbb{Z}$, any minimal prime ideal of $R$ lies over $\mathbb{Z}$, i.e. $\mathcal{M} \cap \mathbb{Z} = \{0\}$. So $\sigma(r) = 0$ and $\sigma$ is well-defined. Further, it is obvious that $\sigma(1) = 1$ and $\sigma(r_1 + r_2) = \sigma(r_1) + \sigma(r_2)$, for all $r_1, r_2 \in R$, from the definition of $\sigma$. To show that $\sigma(r_1 r_2) = \sigma(r_1) \sigma(r_2)$, for all $r_1, r_2 \in R$, observe that it is sufficient to show that $\sigma_\mathcal{M}(ab) = \sigma_\mathcal{M}(a) \sigma_\mathcal{M}(b)$, for all $a, b \in G$.

$$(a - \sigma_\mathcal{M}(a))(b - \sigma_\mathcal{M}(b)) = ab - \sigma_\mathcal{M}(a)b - \sigma_\mathcal{M}(b)a + \sigma_\mathcal{M}(a)\sigma_\mathcal{M}(b) \in \mathcal{M}.$$ Since $\sigma_\mathcal{M}(a)b - \sigma_\mathcal{M}(a)\sigma_\mathcal{M}(b) \in \mathcal{M}$ and $\sigma_\mathcal{M}(b)a - \sigma_\mathcal{M}(b)\sigma_\mathcal{M}(a) \in \mathcal{M}$, we have $ab - \sigma_\mathcal{M}(a)\sigma_\mathcal{M}(b) \in \mathcal{M}$, i.e. $\sigma_\mathcal{M}(a)\sigma_\mathcal{M}(b) = \sigma_\mathcal{M}(ab)$.

Finally, consider the signature ideal $P = \{r \in R | \sigma(r) = 0\}$.

For $r = \sum_{i=1}^{k} \varepsilon_i a_i \in P$, we have $\sum_{i=1}^{k} \varepsilon_i \sigma_\mathcal{M}(a_i) = 0$. This implies (that $k$ is even and) that one half of the $\varepsilon_i \sigma_\mathcal{M}(a_i)$ equals $-1$ and the other half equals $+1$. Without loss of generality we can assume that $\varepsilon_i \sigma_\mathcal{M}(a_i) = -1$ for $1 \leq i \leq k/2$ and $\varepsilon_i \sigma_\mathcal{M}(a_i) = +1$ for $k/2 < i \leq k$. We can rewrite

$$r = \sum_{i=1}^{k} \varepsilon_i a_i$$

$$= \sum_{i=1}^{k} \varepsilon_i (a_i - \sigma_\mathcal{M}(a_i)) \in \mathcal{M}$$

So $P \subset \mathcal{M} \subset \mathbb{Q}$.
This brings us to the complete classification of the prime ideal spectrum of an admissible AP ring $R$, denoted $\text{Spec}(R)$:

**Proposition 4.9.**

Let $R$ be an admissible AP ring with generating polynomial $q(X) = X^2 - 1$.

If $X(R) = \emptyset$, then $\text{Spec}(R) = \{I\}$.

Otherwise $\text{Spec}(R) = \text{Min}(R) \cup \text{Max}(R)$ where $\text{Min}(R) = X(R)$ and $\text{Max}(R) = \{I\} \cup \{P + pR \mid P \in X(R), p \text{ odd prime}\}$.

In particular, $R$ is a local ring if and only if $X(R) = \emptyset$.

The admissible AP ring $R$ is a local ring if, and only if, $I$ is the only prime ideal in $R$. Otherwise a prime ideal of $R$ is either a minimal prime ideal or a maximal ideal. The minimal prime ideals are the ideals in $\text{Min}(R)$. The maximal ideals are the ones given by $\text{Max}(R)$.

We will determine the following objects, which are very useful to determine the structure of a ring $R$:

- $\text{Nil}(R)$, the nil radical of $R$, consisting of all nilpotent elements of $R$,
- $\text{Jac}(R)$, the Jacobson radical of $R$, the intersection of all maximal ideals of $R$,
- $R^\times$, the units (or invertible elements) of $R$,
- $\text{Zd}(R)$, the zerodivisors in $R$ (including the zero element),
- the idempotents in $R$ and
- $R_t$, the torsion elements of $R$.

We will make a distinction between two cases, namely $X(R) = \emptyset$ and $X(R) \neq \emptyset$. From now on, $R$ will denote an admissible AP ring.

### 4.1 $X(R) = \emptyset$

In this case $I$ is the only prime (and thus the maximal) ideal in $R$, which makes $R$ a local ring.

Since $\text{Nil}(R)$ is the intersection of all prime ideals in $R$, we get $\text{Nil}(R) = I$. Obviously, $\text{Jac}(R) = I$, as it is the only maximal ideal.

In a local ring $R$, with unique maximal ideal $I$, the multiplicative group of invertible elements consists of the elements in $R \setminus I$.

For a commutative ring $R$, the set of zerodivisors, $\text{Zd}(R)$, is the union of a certain set of prime ideals in $R$. Given a zerodivisor $z \in \text{Zd}(R)$, the prime ideal $P$ containing $z$ is the prime ideal $P$, maximal in the sense that $(R \setminus \text{Zd}(R)) \cap P = \emptyset$. 
That $\text{Zd}(R) = I$ follows from the observation that $I = \text{Nil}(R) \subseteq \text{Zd}(R)$ and that $R \setminus I = R^e \subseteq R \setminus \text{Zd}(R)$.

The only idempotents are the trivial ones, i.e. 0 and 1. This follows from the observation that for $e \in R$ one has $e \in I$ or $e - 1 \in I$. Since $\text{Zd}(R) = I$, we have $e(e - 1) = 0$ implying $e - 1 = 0$ or $e = 0$ respectively. So $R$ is a 'connected' ring.

Since $I = \text{Nil}(R)$, we have $2 = 1 + 1 \in \text{Nil}(R)$. So there exists a natural number $k$, such that $2^k = 0$ in $R$. This implies that $2^k r = 0$, $\forall r \in R$. So $R_t = R$.

These results can be summarized in the following

**Proposition 4.10.** Let $R$ be an admissible AP ring with generating polynomial $q(X) = X^2 - 1$ and assume that $X(R) = \emptyset$. Then all elements in $R$ are torsion elements. Moreover, for an element $r \in R$, the following conditions are equivalent:

(i) $r \in I$,

(ii) $r$ is nilpotent,

(iii) $r$ is a zerodivisor,

(iv) $r$ is not invertible,

(v) $r$ belongs to every prime ideal in $R$.

### 4.2 \(X(R) \neq \emptyset\)

In this case $I$ is not the only (maximal) prime ideal in $R$ and $R$ is not a local ring.

For every prime $p$ and every signature ideal $\mathcal{P} \in X(R)$, $\mathcal{P} + pR$ is another maximal ideal.

For a given signature ideal $\mathcal{P}$ the intersection, ranging over all primes $p$, of $\mathcal{P} + pR$ is just $\mathcal{P}$. This implies that

$$\text{Jac}(R) = \bigcap_{\mathcal{P} \in X(R), p \text{ prime}} \mathcal{P} + pR = \bigcap_{\mathcal{P} \in X(R)} \mathcal{P} = \text{Nil}(R).$$

First remark that $R_t \subseteq \text{Nil}(R) = \text{Jac}(R)$. Suppose that $r \in R_t$ and $r \notin \mathcal{P} + pR$ for some $\mathcal{P} \in X(R)$, $p$ prime. Consider the ideal $J = (\mathcal{P} + pR + rR)$. Since $\mathcal{P} + pR$ is maximal, $J = R$. So, $1 = s + rt$, for some $s \in \mathcal{P} + pR$, $t \in R$. Let $k \in \mathbb{N}$ such that $kr = 0$, then we have $k = ks \in \mathcal{P} + pR$. Since $\mathcal{P} + pR$ has finite index $p$, this implies $p|k$, and so $r \in \mathcal{P} + pR$, a contradiction.
Every nilpotent element is even \((l(r^k) \equiv l(r)^k \mod 2)\). Suppose that \(r \in \text{Nil}(R)\) and \(l(r) = n, n\) even. Then \(p_n(r) = 0\) i.e.

\[
r^{n+1} + c_{n-1}r^{n-1} + \ldots + c_3r^3 + c_1r = 0, \text{ where } c_1 \neq 0.
\]

If \(r^k = 0\), \(r^{k-1} \neq 0\) then multiplying the equation by \(r^{k-3}\) yields \(c_1r^{k-2} = 0\). Multiplying the equation by \(c_1r^{k-5}\) then yields \(c_1^2r^{k-4} = 0\). Repeating this process will give \(c_1^l r = 0\) for some \(l \in \mathbb{N}\). So \(r \in R_t\).

This concludes that

\[
R_t = \text{Nil}(R) = \text{Jac}(R) = \bigcap_{\mathcal{P} \in X(R)} \mathcal{P}.
\]

The unit group \(R^\times = \{r \in R \mid \sigma(r) = \pm 1, \text{ for all signatures } \sigma \text{ of } R\}\). This can be seen as follows:

If \(r \in R^\times\) then there exists an \(s \in R\) such that \(rs = 1\). This implies that \(\sigma(r)\sigma(s) = 1\) and since \(\sigma(r) \in \mathbb{Z}\) we have \(\sigma(r) = \pm 1\). On the other hand, suppose that \(\sigma(r) = \pm 1\) for all signatures \(\sigma\) of \(R\). Then we have \(\sigma(r^2-1) = 0\) for all \(\sigma\). So, \(r^2 - 1 \in \bigcap_{\mathcal{P}} \mathcal{P} = \text{Nil}(R)\). There exists a \(k \in \mathbb{N}\) such that \((r^2 - 1)^k = 0\). Now \((r^2 - 1)\) nilpotent implies \(r^2\) invertible, i.e. \(r \in R^\times\).

Recall that for a commutative ring \(R\), the set of zerodivisors, \(\text{Zd}(R)\), is the union of a certain set of prime ideals in \(R\). We will first show that \(\bigcup_{\mathcal{P} \in X(R)} \mathcal{P} \subseteq \text{Zd}(R)\). Suppose that \(r \in \mathcal{P} \subseteq X(R)\). Then \(p_n(r) = 0\) for some \(n\) even. This implies that \(r[(r-n) \ldots (r-2)(r+2) \ldots (r+n)] = 0\). Since \(\sigma((r-n) \ldots (r-2)(r+2) \ldots (r+n)) \neq 0\) we have \(r \in \text{Zd}(R)\).

Denote by \(R_{t,p}\) the subset of \(R_t\) consisting of the \(p\)-torsion elements of \(R\). If \(R_{t,p} \neq \{0\}\), then \(p \in \mathcal{P} + pR\) is a zerodivisor, implying that \(\mathcal{P} + pR \subseteq \text{Zd}(R)\).

Since all the signature ideals \(\mathcal{P} \subseteq \mathcal{P} + pR\), we have \(\text{Zd}(R) = \bigcup_{\mathcal{P} \in X(R)} \mathcal{P}\) when \(R_t = \{0\}\) and \(\text{Zd}(R) = \bigcup_{\mathcal{P} \in X(R), \text{p prime}} \mathcal{P} + pR\) for all \(p\) prime such that \(R_{t,p} \neq \{0\}\). Since \(\mathcal{P} \subseteq I = \mathcal{P}_2\) for all \(\mathcal{P} \in X(R)\), we have \(\text{Zd}(R) \subseteq I\) and the same arguments hold as in the case \(X(R) = \emptyset\) to prove that \(R\) is a ‘connected’ ring, i.e. 0 and 1 are the only idempotents.

These results can be summarized in the following propositions:

**Proposition 4.11.** Let \(R\) be an admissible AP ring with generating polynomial \(q(X) = X^2 - 1\) and assume that \(X(R) \neq \emptyset\).

Then for an element \(r \in R\), the following conditions are equivalent:

(i) \(r\) is a torsion element,

(ii) \(r\) is nilpotent,

(iii) \(r\) belongs to every prime ideal in \(R\),

(iv) \(r\) belongs to every signature ideal in \(R\).

The equivalence (i) \(\Leftrightarrow\) (iv) is in fact Pfister’s local-global principle.

The set of zerodivisors is completely described by the following
Proposition 4.12. Let $R$ be an admissible AP ring with generating polynomial $q(X) = X^2 - 1$ and assume that $X(R) \neq \emptyset$.

If $R$ is torsion-free then $Zd(R) = \bigcup_{P \in X(R)} P$, the union of all signature ideals in $R$.

Otherwise, $Zd(R) = \bigcup_{P \in X(R), p}(P + pR)$, for $p$ prime such that $R$ has non-zero $p$-torsion.

5 Constructing annihilating polynomials

We will construct annihilating polynomials for AP rings for different choices of the generating polynomial $q(X)$.

5.1 $q(X) = X^2 - 1$

This is the well-known case described in [8]. The roots of the generating polynomial are $-1$ and $1$. The possible values for the sum of $n$ elements out of $\{-1, 1\}$ lie in $\{-n, -(n-2), \ldots, n-2, n\}$. The annihilating polynomial for an element of length $n$ is thus the $n$-th Lewis polynomial

$$p_n(X) = (X - n)(X - (n-2)) \ldots (X + (n-2))(X + n).$$

5.2 $q(X) = X^4 - 1$

Write $R_1 = \{-1, 1, -i, i\}$ for the set of roots of $q(x) = x^4 - 1$ in $\mathbb{C}$. Denote by $R_j$ the subset of the complex numbers $\mathbb{C}$ consisting of sums of $j$ elements of $R_1$. Since $0 \in R_2$ we have $R_n \subset R_{n+2}$ for all $n \in \mathbb{N}^*$. Consider $D_n = R_n \setminus R_{n-2}$ for $n > 2$. Put $D_1 = R_1$ and $D_2 = R_2$.

Now define the monic integer polynomial $t_n(X) \in \mathbb{Z}[X]$ as follows:

$$t_n(x) = (x^n - n^4) \prod_{\substack{a + b = n \\ a, b \in \mathbb{N}^*}} (x^4 - 2(a^2 - b^2)x^2 + (a^2 + b^2)^2)$$

$$t_1(x) = x^4 - 1$$
$$t_2(x) = (x^4 - 16)(x^4 + 4)$$
$$t_3(x) = (x^4 - 6x^2 + 25)(x^4 + 6x^2 + 25)$$
$$t_4(x) = (X^4 - 256)(x^4 - 16X^2 + 100)(x^4 + 64)(x^4 + 16X^2 + 100)$$

Lemma 5.1. The polynomial $t_n(x)$ has the property that $t_n(z) = 0$ for all $z \in D_n$.

Proof. For $n > 1$, this follows from the observation that

$$D_n = \{a + bi \in \mathbb{Z}[i] : |a| + |b| = n\}.$$
Now we are able to construct the annihilating polynomial $p_n(x)$. Since $R_n \subset R_{n+2}$ and $0 \in R_{2n}$ for all $n \in \mathbb{N}^*$ it follows that:

For $n$ even $p_n(x) = t_n(X)t_{n-2}(X) \ldots t_2(X)X$.

For $n$ odd $p_n(x) = t_n(X)t_{n-2}(X) \ldots t_1(X)$.

A few examples:

$p_1(x) = x^4 - 1$
$p_2(x) = x(x^4 - 16)(x^4 + 4)$
$p_3(x) = (x^4 - 1)(x^4 - 6x^2 + 25)(x^4 + 6x^2 + 25)$
$p_4(x) = x(x^4 - 16)(x^4 + 4)(x^4 - 256)(x^4 - 16X^2 + 100)(x^4 + 64)(x^4 + 16X^2 + 100)$

Elements in the Witt ring of level 2 of dimension $n$ will be annihilated by $p_n(X)$.

**5.3 $q(X) = X^{2^k} - 1$**

Let $R$ be an AP ring with generating polynomial $q(X) = X^{2^k} - 1$. The set of roots of $q(X)$ is generated by a primitive $2^k$-th root of unity $\zeta$. In general it is very difficult to find an explicit expression for the annihilating polynomial $p_n(X)$ but we will obtain some properties. We prove the following

**Lemma 5.2.** Let $R$ be an AP ring with generating polynomial $q(X) = X^{2^k} - 1$ and let $p_n(X)$ be the annihilating polynomial. Then $p_n(0)$ is odd whenever $n$ is odd.

**Proof.** Let $R_1 = \{\zeta, \zeta^2, \ldots, \zeta^{2^k}\}$ be the set of roots of $q(X) = X^{2^k} - 1$ in $\mathbb{C}$ and let $R_i$ be the subset of the complex numbers $\mathbb{C}$ consisting of sums of $i$ elements of $R_1$. Note that

$$R_n = \{\sum_{i=1}^{2^k} a_i \zeta^i \mid a_i \in \mathbb{N}^* \text{ and } \sum_{i=1}^{2^k} a_i = n\}.$$ 

Put $P := \prod_{\sigma \in R_n} \sigma$.

Since $\zeta^i = -(1 - \zeta)(1 + \zeta + \ldots + \zeta^{i-1}) + 1$ it follows that

$$P = n^l + (1 - \zeta)m(\zeta)$$

for some complex function $m$ and $l = \#R_n$. Then

$$P^{2^k} = n^l 2^k + 2m'(\zeta)$$

for some rational valued function $m'$ (since $P \in \mathbb{Z}$). It follows that $P$ is odd whenever $n$ is odd. The result now follows, since $p_n(0) \mid P$. $\square$

The following lemma will provide us an upper bound on the degree of the annihilating polynomials.

**Lemma 5.3.** Let $R$ be an AP ring with generating polynomial $q(X) = X^{2^k} - 1$ and let $p_n(X)$ be the annihilating polynomial. Then $\deg p_n(X) \leq 2^{n-1}(2^k - 1) + 1$. 

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Proof. To determine the degree of the annihilating polynomial \( p_n(X) \) we have to calculate the number of different elements in the set

\[ R_n = \{ \sum_{i=1}^{2^k} a_i \zeta^i \mid a_i \in \mathbb{N}^* \text{ and } \sum_{i=1}^{2^k} a_i = n \} . \]

We write

\[ \sum_{i=1}^{2^k} a_i \zeta^i = \sum_{i=1}^{2^{k-1}} (a_i - a'_i) \zeta^i \]

where \( a'_i = a_{2^{k-1}+i} \)

\[ = \sum_{i=1}^{2^{k-1}} b_i \zeta^i \]

where \( b_i > 0 \)

\[ + \sum_{i=1}^{2^{k-1}} b'_i \zeta^i \]

where \( b'_i < 0 \)

\[ + 0. \]

The number of elements in (1) and (2) are for symmetry reasons the same and it follows that \( \#R_n = 2 \times \#R_{n-1} - 1 \).

Since \( \#R_1 = 2^k \), we obtain that \( \deg p_n(X) \leq \#R_n = 2^{n-1}(2^k - 1) + 1 \).

An example of an AP ring with generating polynomial \( X^{2^k} - 1 \) is a Witt rings of level \( k \) as defined in [4].

5.4 \( q(X) = X^2 - 2^k X \)

Write \( R_1 = \{ 0, 2^k \} \) for the set of roots of \( q(x) = x^2 - 2^k X \) in \( \mathbb{C} \). Denote by \( R_n \) the subset of the complex numbers \( \mathbb{C} \) consisting of sums of \( n \) elements of \( R_1 \).

So \( R_n = \{ 0, 2^k, 2 \cdot 2^k, \ldots, n \cdot 2^k \} \) and \( p_n(X) \) is given by

\[ p_n(X) = X(X - 2^k)(X - 2 \cdot 2^k) \ldots (X - n \cdot 2^k) . \]

An example of an AP ring with generating polynomial \( X^2 - 2^k X \) is for example the subring of the Witt ring additively generated by all \( k \)-fold Pfister forms of this Witt ring. In this setting, \( p_n(X) \) will annihilate all sums of \( n \) \( k \)-fold Pfister forms.

5.5 Table of Marks

The table of marks of a finite group \( G \), as introduced by Burnside in his classic theory of groups of finite order [1], is a matrix whose rows and columns are labeled by the conjugacy classes of subgroups of \( G \) and where for two subgroups \( H_1 \) and \( H_2 \) the \((H_1, H_2)\)-entry is the number of fixed points of \( H_2 \) in the transitive action of \( G \) on the cosets of \( H_1 \) in \( G \). This table of marks is sometimes called a Burnside matrix. If \( n_1, n_2, \ldots, n_k \in \mathbb{N} \) are the different entries in the table of marks of a group \( G \), then, by example 2.2.(viii), the polynomial \( q(X) = \prod_{i=1}^k (X - n_i) \) is a generating polynomial for the Burnside ring \( \Omega(G) \).
Example 5.4. The table of marks of the alternating group $A_5$ is given by

|     | $e$ | $C_2$ | $C_3$ | $V_4$ | $C_5$ | $S_3$ | $D_{10}$ | $A_4$ | $A_5$ |
|-----|-----|-------|-------|-------|-------|-------|----------|-------|-------|
| $e$ | 60  | 0     | 0     | 0     | 0     | 0     | 0        | 0     | 0     |
| $C_2$ | 30  | 2     | 0     | 0     | 0     | 0     | 0        | 0     | 0     |
| $C_3$ | 20  | 0     | 2     | 0     | 0     | 0     | 0        | 0     | 0     |
| $V_4$ | 15  | 3     | 0     | 3     | 0     | 0     | 0        | 0     | 0     |
| $C_5$ | 12  | 0     | 0     | 2     | 0     | 0     | 0        | 0     | 0     |
| $S_3$ | 10  | 2     | 1     | 0     | 1     | 0     | 0        | 0     | 0     |
| $D_{10}$ | 6   | 2     | 0     | 1     | 0     | 1     | 0        | 0     | 0     |
| $A_4$ | 5   | 1     | 2     | 1     | 0     | 0     | 0        | 1     | 0     |
| $A_5$ | 1   | 1     | 1     | 1     | 1     | 1     | 1        | 1     | 1     |

The generating polynomial for the AP ring $\Omega(A_5)$ is thus given by

$$q(X) = (X - 60)(X - 30)(X - 20)(X - 15)(X - 10)(X - 6)(X - 5)(X - 3)(X - 2)(X - 1)X.$$ 

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