A FLOW METHOD FOR A GENERALIZATION OF \( L_p \) CHRISTOFFEL-MINKOWSKI PROBLEM

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Abstract. In this paper, a generalization of the \( L_p \)-Christoffel-Minkowski problem is studied. We consider an anisotropic curvature flow and derive the long-time existence of the flow. Then under some initial data, we obtain the existence of smooth solutions to this problem for \( c = 1 \).

1. Introduction

Recently, as important developments of the classical Brunn-Minkowski theory in convex geometry, the dual Brunn-Minkowski theory is developing rapidly. Dual curvature measures and their associated variational formulas was firstly introduced by Huang, Lutwak, Yang and Zhang in their recent groundbreaking work [8]. Then the dual curvature measures were extended into the \( L_p \) case in [13] and then studied in [1, 4, 9].

In this paper, we consider the following fully nonlinear equation, which is a generalization of \( L_p \)-Christoffel-Minkowski problem

\[
\frac{h^{1-p}}{(h^2 + |\nabla h|^2)^{\frac{n-2}{2}}} \sigma_k(x)f(x) = c \text{ on } S^{n-1}
\]

for some positive constant \( c \). Here \( f \) is a given positive and smooth function on the unit sphere \( S^{n-1} \) and \( h \) is the support function defined on \( S^{n-1} \). \( \sigma_k(x,t) \) is the \( k \)-th elementary symmetric function for principal curvature radii and \( \nabla \) is the Levi-Civita connection on \( S^{n-1} \).

Equation (1) is just the smooth case of \( L_p \) dual Minkowski problem when \( k = n - 1 \).

When \( q = n, k = n - 1 \), Eq. (1) reduces to the \( L_p \)-Minkowski problem, which has been extensively studied, see e.g. Schneider’ book [14]. For \( q = n, 1 \leq k < n - 1 \), Eq. (1) is known as the \( L_p \)-Christoffel-Minkowski problem and is the classical Christoffel-Minkowski problem for \( p = 1 \). Under a sufficient condition on the prescribed function, existence of solution for the classical Christoffel-Minkowski problem was given in [5].
The $L_p$-Christoffel-Minkowski problem is related to the problem of prescribing $k$-th $p$-area measures. Hu, Ma & Shen in [7] proved the existence of convex solutions to the $L_p$-Christoffel-Minkowski problem for $p \geq k + 1$ under appropriate conditions. Using the methods of geometric flows, Ivaki in [10] and then Sheng & Yi in [15] also gave the existence of smooth convex solutions to the $L_p$-Christoffel-Minkowski problem for $p \geq k + 1$. In case $1 < p < k + 1$, Guan & Xia in [6] established the existence of convex body with prescribed $k$-th even $p$-area measures. In [3], the authors considered a generalized $L_p$-Christoffel-Minkowski problem and gave the the existence of smooth solutions by curvature flow method. To the best of our knowledge, there is no other existence result about Eq. (1).

The existence of smooth solutions for Eq. (1) is concerned in this paper. In general, there is no variational structure, We use a flow method involving $\sigma_k$, support function and radial function to give the existence of smooth solutions for Eq. (1) wich $c = 1$.

Let $M_0$ be a smooth, closed, strictly convex hypersurface in the Euclidean space $\mathbb{R}^n$, which encloses the origin and is given by a smooth embedding $X_0 : S^{n-1} \to \mathbb{R}^n$. Consider a family of closed hypersurfaces $\{M_t\}$ with $M_t = X(S^{n-1}, t)$, where $X : S^{n-1} \times [0, T) \to \mathbb{R}^n$ is a smooth map satisfying the following initial value problem:

\begin{equation}
\frac{\partial X}{\partial t}(x, t) = f(\nu)\sigma_k(x, t)[X, \nu]^{2-p}|X|^{q-n} - \eta(t)X,
\end{equation}

$X(x, 0) = X_0(x)$.

Here $\nu$ is the unit outer normal vector of $M_t$ at the point $X(x, t)$. $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$, $\eta$ is a scalar function to be specified later, and $T$ is the maximal time for which the solution exists. We use $\{e_{ij}\}, 1 \leq i, j \leq n - 1$ and $\nabla$ for the standard metric and Levi-Civita connection of $S^{n-1}$ respectively. Principal radii of curvature are the eigenvalues of the matrix

$$b_{ij} := \nabla_i \nabla_j h + e_{ij} h$$

with respect to $\{e_{ij}\}$. $\sigma_k(x, t)$ is the $k$-th elementary symmetric function for principal curvature radii of $M_t$ at $X(x, t)$ and $k$ is an integer with $1 \leq k < n - 1$. In this paper, $\sigma_k$ is normalized so that $\sigma_k(1, \ldots, 1) = 1$.

Geometric flows with speed of symmetric polynomial of the principal curvature radii of the hypersurface have been extensively studied, see e.g. [16].

On the other hand, anisotropic curvature flows provide alternative methods to prove the existences of elliptic PDEs arising from convex geometry, see e.g. [2, 4, 10, 12, 15].

The scalar function $\eta(t)$ in (2) is usually used to keep $M_t$ normalized in a certain sense, see for examples [4, 10, 15]. In this paper, $\eta$ is given by

\begin{equation}
\eta(t) = \frac{\int_{S^{n-1}} \rho^{q-n} h \sigma_k \, dx}{\int_{S^{n-1}} \frac{1}{f(x)} h^p \, dx},
\end{equation}

where $h(\cdot, t)$ and $\rho(u, t)$ are the support function and radial function of the convex hypersurface $M_t$.

To obtain the long-time existence of flow (2), we need some constraints on $f$. 

\(\)
(A): Let $s$ be the arc-length parameter and $f$ be a smooth function on $S^{n-1}$. On every great circle, $f$ satisfies

$$f_{ss} - \frac{1}{k+1} \left( k + \frac{q-n}{q-n-k-1} + \frac{p-2}{p+k-1} \right) f_s^2 f^{-1} + (p+k-1-q+n) f > 0.$$ 

The main results of this paper are stated as follows.

**Theorem 1.** Assume $M_0$ is a smooth, closed and strictly convex hypersurface in $\mathbb{R}^n$. Suppose $k$ is an integer with $1 \leq k < n - 1$ and $k+1 < q-n < p-k-1$. Suppose $f$ is a smooth positive function on $S^{n-1}$ satisfying (A). Then flow (2) has a unique smooth solution $M_t$ for all time $t > 0$. Moreover, when $t \to \infty$, a subsequence of $M_t$ converges in $C^\infty$ to a smooth, closed, strictly convex hypersurface.

For the proof of Theorem 1, we will see it is enough to obtain the uniform positive upper and lower bounds for support functions of $\{M_t\}$ under condition $0 \leq q-n < p-k-1$. And the stronger condition $k+1 < q-n < p-k-1$ is only used to derive the uniform bound of principal curvature, see Lemma 5 for details.

**Corollary 1.** Under the assumptions of Theorem 1, there exists a smooth solution to equation (1) with $c = 1$.

This paper is organized as follows. In section 2, we give some basic knowledge about the flow (2) and evolution equations of some geometric quantities. In section 3, the long-time existence of flow (2) will be obtained. First, we obtain the uniform positive upper and lower bounds for support functions of $\{M_t\}$. Based on the bounds of support functions, we obtain the uniform bounds of principal curvatures by constructing proper auxiliary functions. The long-time existence of flow (2) then follows by standard arguments. In section 4, under some special initial condition, we prove that a subsequence of $\{M_t\}$ converges to a smooth solution to equation (1) with $c = 1$, completing the proofs of Corollary 1.

### 2. Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space, and $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$. Assume $M$ is a smooth closed strictly convex hypersurface in $\mathbb{R}^n$. Without loss of generality, we may assume that $M$ encloses the origin. The support function $h$ of $M$ is defined as

$$h(x) := \max_{y \in M} \langle y, x \rangle, \quad \forall x \in S^{n-1},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.

Denote the Gauss map of $M$ by $\nu_M$. Then $M$ can be parametrized by the inverse Gauss map $X : S^{n-1} \to M$ with $X(x) = \nu^{-1}_M(x)$. The support function $h$ of $M$ can be computed by

$$h(x) = \langle x, X(x) \rangle, \quad x \in S^{n-1}. $$
Note that $x$ is just the unit outer normal of $M$ at $X(x)$. Differentiating (4), we have
\[ \nabla_i h = \langle \nabla_i x, X(x) \rangle + \langle x, \nabla_i X(x) \rangle. \]
Since $\nabla_i X(x)$ is tangent to $M$ at $X(x)$, we have
\[ \nabla_i h = \langle \nabla_i x, X(x) \rangle. \]
It follows that
\[ (5) \quad X(x) = \nabla h + hx. \]

By differentiating (4) twice, the second fundamental form $A_{ij}$ of $M$ can be computed in terms of the support function, see for example [16],
\[ (6) \quad A_{ij} = \nabla_{ij} h + he_{ij}, \]
where $\nabla_{ij} = \nabla_i \nabla_j$ denotes the second order covariant derivative with respect to $e_{ij}$. The induced metric matrix $g_{ij}$ of $M$ can be derived by Weingarten’s formula,
\[ (7) \quad e_{ij} = \langle \nabla_i x, \nabla_j x \rangle = A_{im} A_{lj} g^{ml}. \]
The principal radii of curvature are the eigenvalues of the matrix $b_{ij} = A_{ik} g_{kj}$. When considering a smooth local orthonormal frame on $S^{n-1}$, by virtue of (6) and (7), we have
\[ (8) \quad b_{ij} = A_{ij} = \nabla_{ij} h + h\delta_{ij}. \]
We will use $b^{ij}$ to denote the inverse matrix of $b_{ij}$.

From the evolution equation of $X(x,t)$ in flow (2), we derive the evolution equation of the corresponding support function $h(x,t)$:
\[ (9) \quad \frac{\partial h(x,t)}{\partial t} = f(x) \sigma_k(x,t) h^{2-p} \rho^{q-n} - \eta(t) h(x,t). \]

The radial function $\rho$ of $M$ is given by
\[ \rho(u) := \max \{ \lambda > 0 : \lambda u \in M \}, \quad \forall \ u \in S^{n-1}. \]
Note that $\rho(u)u \in M$.

From (5), $u$ and $x$ are related by
\[ (10) \quad \rho(u)u = \nabla h(x) + h(x)x \]
and
\[ \rho^2 = |\nabla h|^2 + h^2. \]
Let $x = x(u,t)$, by (10), we have
\[ \log \rho(u,t) = \log h(x,t) - \log \langle x, u \rangle. \]
Differentiating the above identity, we have
\[
\frac{1}{\rho(u,t)} \frac{\partial \rho(u,t)}{\partial t} = \frac{1}{h(x,t)} \left( \nabla h \cdot \frac{\partial x(u,t)}{\partial t} + \frac{\partial h(x,t)}{\partial t} \right) - \frac{u}{\langle x,u \rangle} \cdot \frac{\partial x(u,t)}{\partial t}
\]
\[
= \frac{1}{h(x,t)} \frac{\partial h(x,t)}{\partial t} + \frac{1}{h(x,t)} [\nabla h - \rho(u,t)u] \cdot \frac{\partial x(u,t)}{\partial t} - u \langle x, u \rangle \cdot \frac{\partial x(u,t)}{\partial t}
\]
\[
= \frac{1}{h(x,t)} \frac{\partial h(x,t)}{\partial t}.
\]

The evolution equation of radial function then follows from (9),
\[
\frac{\partial \rho}{\partial t}(u,t) = f(x) \sigma_k(u,t) h^{1-p} \rho^{q-n-1} - \eta(t) \rho(u,t),
\]
where \(\sigma_k(u,t)\) denotes the fundamental symmetric function of principal radii at \(\rho(u,t)u \in M_t\) and \(f\) takes value at the unit normal vector \(x(u,t)\).

In the rest of the paper, we take a local orthonormal frame \(\{e_1, \ldots, e_{n-1}\}\) on \(S^{n-1}\) such that the standard metric on \(S^{n-1}\) is \(\{\delta_{ij}\}\). Double indices always mean to sum from 1 to \(n-1\). We denote partial derivatives \(\frac{\partial \sigma_k}{\partial b_{ij}}\) and \(\frac{\partial^2 \sigma_k}{\partial b_{ab} \partial b_{mn}}\) by \(\sigma_{ij}^{k}\) and \(\sigma_{k}^{ab, mn}\) respectively. For convenience, we also write
\[
N = f(x) h^{2-p} \rho^{q-n},
\]
\[
F = N \sigma_k.
\]

By the flow equation (2), we can derive evolution equations of some geometric quantities.

**Lemma 1.** The following evolution equations hold along the flow (2).
\[
\partial_t b_{ij} - N \sigma_k^{ab} \nabla_{ab} b_{ij}
\]
\[
= (k+1) N \sigma_k \delta_{ij} - N \sigma_k^{ab} \delta_{ij} b_{ab} + N (\sigma_k^{ja} b_{ja} - \sigma_k^{ja} b_{ia})
\]
\[
+ N \sigma_k^{ab, mn} \nabla_j b_{ab} \nabla_i b_{mn} + (\sigma_k \nabla_i N + \nabla_j \sigma_k \nabla_i N + \nabla_i \sigma_k \nabla_j N - b_{ij} \eta(t))
\]
\[
\partial_t b^{ij} - N \sigma_k^{ab} \nabla_{ab} b^{ij}
\]
\[
= -(k+1) N \sigma_k^{ja} b^{ja} + N \sigma_k^{ab} \delta_{ij} b^{ij} - N b^{ja} (\sigma_k^{ra} b_{rb} - \sigma_k^{rb} b_{ra})
\]
\[
- N b^{il} b^{js} (\sigma_k^{ab, mn} + 2 \sigma_k^{am} b^{nb}) \nabla_l b_{ab} \nabla_s b_{mn}
\]
\[
- b^{ja} b^{ib} (\sigma_k \nabla_{ab} N + \nabla_b \sigma_k \nabla_{a} N + \nabla_a \sigma_k \nabla_{b} N) + \eta(t) b^{ij}.
\]

For the specific computations, one can refer to Lemma 2.3 in [10].

3. **The long-time existence of the flow**

In this section, we will give a priori estimates about support functions and curvatures to obtain the long-time existence of flow (2) under assumptions of Theorem 1.
In the rest of this paper, we assume that $M_0$ is a smooth, closed, strictly convex hypersurface in $\mathbb{R}^n$ and $h : S^{n-1} \times [0,T) \to \mathbb{R}$ is a smooth solution to the Eq. (9) with the initial $h(\cdot, 0)$ the support function of $M_0$. Here $T$ is the maximal time for which the solution exists. Let $M_t$ be the convex hypersurface determined by $h(\cdot, t)$, and $\rho(\cdot, t)$ be the corresponding radial function.

We first give the uniform positive upper and lower bounds of $h(\cdot, t)$ and $\rho(\cdot, t)$ for $t \in [0, T)$.

**Lemma 2.** Let $h$ be a smooth solution of (9) on $S^{n-1} \times [0,T)$ and $f$ be a positive, smooth function on $S^{n-1}$. $p > 1$ and $0 \leq q - n < p - k - 1$. Then

\begin{equation}
\frac{1}{C} \leq h(x, t) \leq C, \tag{12}
\end{equation}

\begin{equation}
\frac{1}{C} \leq \rho(u, t) \leq C, \tag{13}
\end{equation}

where $C$ is a positive constant independent of $t$.

**Proof.** Let $J(t) = \int_{S^{n-1}} h^{p-1} \frac{1}{f(x)} \, dx$. We claim that $J(t)$ is unchanged along the flow (2). It is because

\[ J'(t) = \int_{S^{n-1}} ph^{p-1} \frac{1}{f(x)} \partial_t h \, dx \]
\[ = p \int_{S^{n-1}} h^{p-1} \frac{1}{f(x)} (\sigma_k h^{2-p} \rho^{q-n} f(x) - \eta(t) h) \, dx \]
\[ = 0. \]

For each $t \in [0, T)$, suppose that the maximum of radial function $\rho(\cdot, t)$ is attained at some $u_t \in S^{n-1}$. Let

\[ R_t = \max_{u \in S^{n-1}} \rho(u, t) = \rho(u_t, t) \]

for some $u_t \in S^{n-1}$. By the definition of support function, we have

\[ h(x, t) \geq R_t \langle x, u_t \rangle, \quad \forall x \in S^{n-1}. \]

Denote the hemisphere containing $u_t$ by $S_{u_t}^+ = \{ x \in S^{n-1} : \langle x, u_t \rangle > 0 \}$. Since $p > 1$, we have

\[ J(0) = J(t) \geq \int_{S_{u_t}^+} h^{p-1} \frac{1}{f(x)} \, dx \]
\[ \geq \int_{S_{u_t}^+} R_t^p \langle x, u_t \rangle^{p} \frac{1}{f(x)} \, dx \geq \frac{1}{f_{\text{max}}} \int_{S_{u_t}^+} R_t^p \langle x, u_t \rangle^{p} \, dx \]
\[ = \frac{1}{f_{\text{max}}} \int_{S^+} R_t^p x_1^p \, dx, \]

where $S^+ = \{ x \in S^{n-1} : x_1 > 0 \}$. 

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Denote \( S_1 = \{ x \in \mathbb{S}^{n-1} : x_1 \geq 1/2 \} \), then
\[
J(0) \geq \frac{1}{f_{\max}} \int_{S_1} R_t^p \left( \frac{1}{2} \right)^p \, dx = \frac{1}{f_{\max}} R_t^p \left( \frac{1}{2} \right)^p |S_1|,
\]
which implies that \( R_t \) is uniformly bounded from above.

Now, we estimate the lower bound of \( h \). First we explain that \( \eta(t) \) is bounded from above. Since mixed volumes are monotonic increasing, see [14, page 282], we have for each \( t \in [0, T) \),
\[
(14) \quad h_{\min}^{k+1}(t) \leq \frac{\int_{S^{n-1}} h \sigma_k \, dx}{\omega_{n-1}} \leq h_{\max}^{k+1}(t),
\]
here \( h_{\min}(t) = \min_{x \in S^{n-1}} h(x, t) \) and \( h_{\max}(t) = \max_{x \in S^{n-1}} h(x, t) \).

Now we recall the definition of \( \eta(t) \) and notice that \( J(t) \) is unchanged along the flow (2). From the upper bound of \( h_{\max}(t) \) and (14), we have
\[
\eta(t) \leq c_3
\]
for some positive constant \( c_3 \) independent of \( t \).

Suppose the minimum of \( h \) is attained at a point \( (y_t, t) \). It follows that
\[
\sigma_k(y_t, t) \geq h_{\min}^k(t).
\]
Then in the sense of the lim inf of difference quotient, we have
\[
\frac{\partial h_{\min}(t)}{\partial t} \geq f_{\min} h_{\min}^{k+2-p+q-n} - c_3 h_{\min}.
\]
The right hand of the above inequality is positive for \( h_{\min}(t) \) small enough if \( k + 1 - p + q - n < 0 \). The lower bound of \( h_{\min}(t) \) follows from the maximum principle.

**Remark 1.** When \( \eta = 1 \), \( J(t) \) is not unchange along the flow any more. The upper bound of \( h \) can be estimated as follows.

Suppose the maximum of \( h(x, t) \) is attained at a point \( (x_t, t) \). At \( (x_t, t) \),
\[
\sigma_k(x_t, t) \leq h_{\max}^k(t).
\]
Hence, we have
\[
\frac{\partial h_{\max}(t)}{\partial t} \leq f_{\max} h_{\max}^{k+2-p+q-n} - h_{\max}.
\]
The right hand of the above inequality becomes negative for \( h_{\max}(t) \) large enough if \( k + 1 - p + q - n < 0 \). The upper bound of \( h_{\max}(t) \) follows. The lower bound \( h(x, t) \) can be obtained by using the same method for the case \( \eta(t) \neq 1 \).

By the equality \( \rho^2 = h^2 + |\nabla h|^2 \), we can obtain the gradient estimate of support function from Lemma [2]
Corollary 2. Under the assumptions of Lemma 2, we have
\[ |\nabla h(x,t)| \leq C, \quad \forall (x,t) \in S^{n-1} \times [0,T), \]
where \( C \) is a positive constant depending only on constants in Lemma 2.

The uniform bounds of \( \eta(t) \) can be derived from Lemma 2.

Lemma 3. Under the assumptions of Lemma 2, \( \eta(t) \) is uniformly bounded above and below from zero.

Proof. From the proof of Lemma 2, we know \( \eta(t) \) has positive upper bound and
\[
h_{\text{min}}^{k+1}(t) \leq \frac{\int_{S^{n-1}} h \sigma_k \, dx}{\omega_{n-1}} \leq h_{\text{max}}^{k+1}(t).
\]
Lemma 2 also gives
\[
\int_{S^{n-1}} \rho^{q-n} h \sigma_k \, dx \geq c \int_{S^{n-1}} h \sigma_k \, dx,
\]
here \( c \) is a constant depending on the bounds of \( h(x,t) \). This together with the uniform of \( h(x,t) \) implies that \( \eta(t) \) is bounded from below. \( \square \)

To obtain the long-time existence of the flow (2), we need to establish the uniform bounds on principal curvatures. By Lemma 2, for any \( t \in [0,T) \), \( h(\cdot,t) \) always ranges within a bounded interval \( I' = [1/C, C] \), where \( C \) is the constant in Lemma 2. First, we give the estimates of \( \sigma_k \).

Lemma 4. Under the assumptions of Lemma 2
\[
\frac{1}{C} \leq \sigma_k(x,t) \leq C, \quad \forall (x,t) \in S^{n-1} \times [0,T),
\]
where \( C \) is a positive constant independent of \( t \).

Proof. Recall that \( F = h^{2-p} \rho^{q-n} \sigma_k f(x) \). According to Lemma 2 uniform bounds of \( \sigma_k \) will follow from those of \( F \).

First, we compute the evolution equation of \( F \). Since
\[
\partial_t \sigma_k = \sigma_k^{ij} \partial_i (\nabla_{ij} h + \delta_{ij} h)
\]
\[
= \sigma_k^{ij} \nabla_{ij} (\partial_i h) + \sigma_k^{ij} \delta_{ij} \partial_i h
\]
\[
= \sigma_k^{ij} \nabla_{ij} F - \eta(t) \sigma_k^{ij} \nabla_{ij} h + \sigma_k^{ij} \delta_{ij} F - \eta(t) \sigma_k^{ij} \delta_{ij} h
\]
\[
= \sigma_k^{ij} \nabla_{ij} F + \sigma_k^{ij} \delta_{ij} F - k \sigma_k \eta(t).
\]
The above equality and (11) gives that
\[
\partial_t F = (2-p) f(x) h^{1-p} \rho^{q-n} \sigma_k \partial_i h + (q-n) f(x) h^{2-p} \rho^{q-n-1} \sigma_k \partial_i \rho + f(x) h^{2-p} \rho^{q-n} \partial_i \sigma_k
\]
\[
= (q-n+2-p) \frac{F^2}{h} - (q-n+k+2-p) \eta(t) F + NF \sigma_k^{ij} \delta_{ij} + N \sigma_k^{ij} \nabla_{ij} F.
\]
Next, we consider the evolution equation of \( \frac{F}{h} = f(x)h^{1-p} \rho^q \sigma_k \).

\[
\partial_t \left( \frac{F}{h} \right) - N \sigma_k^{ij} \nabla_{ij} \left( \frac{F}{h} \right) \\
= \frac{1}{h} (\partial_t F - N \sigma_k^{ij} \nabla_{ij} F) - \frac{F}{h^2} (\partial_t h - N \sigma_k^{ij} \nabla_{ij} h) + 2 \frac{N}{h} \sigma_k^{ij} \nabla_i \left( \frac{F}{h} \right) \nabla_j h \\
= (q - n + k + 1 - p) \left( \frac{F}{h} \right)^2 - (q - n + k + 1 - p) \eta(t) \left( \frac{F}{h} \right) \\
+ 2 \frac{N}{h} \sigma_k^{ij} \nabla_i \left( \frac{F}{h} \right) \nabla_j h.
\]

(15)

Since \( \eta(t) \) is uniformly bounded and \( q - n + k + 1 - p < 0 \), we have \( \frac{F}{h} \) is bounded from below and above. The uniform bounds on \( \sigma_k \) follow. \( \square \)

Now we can derive the upper bounds of principal curvatures \( \kappa_i(x, t) \) of \( M_t \) for \( i = 1, \cdots, n - 1 \).

**Lemma 5.** Under the assumptions of Theorem 4, we have

\[ \kappa_i \leq C, \quad \forall (x, t) \in \mathbb{S}^{n-1} \times [0, T), \]

where \( C \) is a positive constant independent of \( t \).

**Proof.** By rotation, we assume that the maximal eigenvalue of \( \frac{b_{ij}}{h} \) at \( t \) is attained at point \( x_i \) in the direction of the unit vector \( e_1 \in T_{x_i} \mathbb{S}^{n-1} \). We also choose orthonormal vector field such that \( b_{ij} \) is diagonal.

By the evolution equation of \( b_{ij} \) in Lemma 1, we have

\[
\partial_t \left( \frac{b_{11}}{h} \right) - N \sigma_k^{ij} \nabla_{ij} \left( \frac{b_{11}}{h} \right) \\
= \frac{2}{h} N \eta(t) \sigma_k^{ij} \nabla_i \frac{b_{11}}{h} \nabla_j + \frac{N}{h^2} b_{11} \sigma_k^{ij} \nabla_{ij} \frac{b_{11}}{h} - (k + 1) \frac{N}{h} \sigma_k (b_{11})^2 + \frac{N}{h} \sigma_k \delta_{ij} \frac{b_{11}}{h} \\
- \frac{N}{h} (b_{11})^2 (\sigma_k^{ij, mn} + 2 \sigma_k^{im} b_{nj}) \nabla_1 b_{ij} \nabla_1 b_{mn} \\
- \frac{1}{h} (b_{11})^2 (\nabla_1 N \sigma_k + 2 \nabla_1 \sigma_k \nabla_1 N) - \frac{b_{11}}{h^2} N \sigma_k + \frac{2 b_{11}}{h} \eta(t) \\
= \frac{2}{h} N \sigma_k^{ij} \nabla_i \frac{b_{11}}{h} \nabla_j - (k + 1) \frac{N}{h} \sigma_k (b_{11})^2 - \frac{N}{h^2} (b_{11})^2 (\sigma_k^{ij, mn} + 2 \sigma_k^{im} b_{nj}) \nabla_1 b_{ij} \nabla_1 b_{mn} \\
- \frac{1}{h} (b_{11})^2 (\nabla_1 N \sigma_k + 2 \nabla_1 \sigma_k \nabla_1 N) + (k - 1) \frac{b_{11}}{h^2} N \sigma_k + \frac{2 b_{11}}{h} \eta(t).
\]

According to inverse concavity of \( (\sigma_k)^{\frac{1}{k}} \), we obtain by [16]

\[
(\sigma_k^{ij, mn} + 2 \sigma_k^{im} b_{nj}) \nabla_1 b_{ij} \nabla_1 b_{mn} \geq \frac{k + 1}{k} \left( \nabla_1 \sigma_k \right)^2.
\]
On the other hand, by Schwartz inequality, the following inequality holds

\[ 2|\nabla_1 \sigma_k \nabla_1 N| \leq \frac{k + 1}{k} \frac{N(\nabla_1 \sigma_k)^2}{\sigma_k} + \frac{k}{k + 1} \frac{\sigma_k(\nabla_1 N)^2}{N}. \]

Hence, we have at \((x_t, t)\)

\[ \partial_t \frac{b_{11}}{h} \leq -\frac{(b_{11})^2}{h} \sigma_k \left[ \nabla_{11} N - \frac{k}{k + 1} \frac{(\nabla_1 N)^2}{N} + (k + 1)N + (1 - k) \frac{Nb_{11}}{h} \right] + \frac{2b_{11}}{h} \eta(t). \]

Let \(s\) be the arc-length of the great circle passing through \(x_t\) with the unit tangent vector \(e_1\). Notice that

\[ \nabla_{11} N = \frac{k}{k + 1} \frac{(\nabla_1 N)^2}{N} + (k + 1)N = (k + 1)N \frac{k}{k + 1} \left( N^{\frac{1}{k + 1}} + (N^{\frac{1}{k + 1}})_{ss} \right). \]

It is easy to compute

\[ N_s = f_s h^{2-p} \rho_1^{q-n} + (2 - p) f h^{1-p} \rho_1^{q-n} h_s + (q - n) f h^{2-p} \rho_1^{q-n} (h h_s + h_{ss} h_s), \]
\[ N_{ss} = f_{ss} h^{2-p} \rho_1^{q-n} + 2(2 - p) f h^{1-p} \rho_1^{q-n} f_s h_s + 2(q - n) f h^{2-p} \rho_1^{q-n} f_s h_s b_{ss} + 2(q - n)(2 - p) f h^{1-p} \rho_1^{q-n} h_s^2 b_{ss} + (2 - p)(1 - p) f h^{1-p} \rho_1^{q-n} h_s^2 + (q - n)(q - n - 2) f h^{2-p} \rho_1^{q-n} b_{ss}^2 h_s^2 + (2 - p) f h^{1-p} \rho_1^{q-n} h_s^2 + (q - n) f h^{2-p} \rho_1^{q-n} (b_{ss}^2 - h b_{ss} + h_s^2 + h_{ss} h_m). \]
We have by direct computations

\[ 1 + N^{-\frac{1}{2+p}} \left( N^{\frac{1}{2+p}} \right)^{ss} = 1 + \frac{1}{k+1} N^{-1} N_{ss} - \frac{k}{(k+1)^2} N^{-2} N^2_s \]

\[ = 1 + \frac{1}{k+1} f_{ss} f^{-1} + \frac{1}{k+1} 2(2-p) h^{-1} f^{-1} f_s h_s + \frac{1}{k+1} 2(q-n) f^{-1} \rho^{-2} f_s h_s b_{ss} \]

\[ + \frac{1}{k+1} 2(q-n)(2-p) h^{-1} \rho^{-2} b_{ss}^2 h_s^2 + \frac{1}{k+1} (2-p)(1-p) h^{-2} h_s^2 \]

\[ + \frac{1}{k+1} (q-n)(q-n-2) \rho^{-4} b_{ss}^2 h_s^2 + \frac{1}{k+1} (2-p) h^{-1} h_s \]

\[ + \frac{1}{k+1} (q-n) \rho^{-2} (b_{ss}^2 - h_b h_s + h_s^2 + h_{ssm} h_m) \]

\[ = \frac{k}{(k+1)^2} f_s f^{-2} - \frac{k}{(k+1)^2} (2-p) h^{-2} h_s^2 - \frac{k}{(k+1)^2} (q-n) \rho^{-2} b_{ss}^2 h_s^2 \]

\[ = \frac{k}{(k+1)^2} 2(2-p) f^{-1} h^{-1} f_s h_s - \frac{k}{(k+1)^2} 2(q-n) f^{-1} \rho^{-2} f_s h_s b_{ss} \]

\[ = \frac{k}{(k+1)^2} 2(2-p)(q-n) h^{-1} \rho^{-2} h_s^{-2} b_{ss}. \]

\[ = 1 + \frac{1}{k+1} f_{ss} f^{-1} - \frac{k}{(k+1)^2} f_s f^{-2} + \frac{2-p}{k+1} h^{-1} h_s \]

\[ + \frac{2(p-k)}{(k+1)^2} h^{-1} f^{-1} f_s h_s + \frac{2(q-n)}{(k+1)^2} f^{-1} \rho^{-2} f_s h_s b_{ss} \]

\[ + \frac{2}{(k+1)^2}(2-p)(q-n) h^{-1} \rho^{-2} h_s^2 b_{ss} + \frac{1}{(k+1)^2} (p-2)(p+k-1) h^{-2} h_s^2 \]

\[ + \frac{q-n}{(k+1)^2}(q-n-2k-2) \rho^{-4} b_{ss}^2 h_s^2 + \frac{1}{k+1} (q-n) \rho^{-2} (b_{ss}^2 - h_b h_s + h_s^2 + h_{ssm} h_m). \]

At \((x_t, t)\), we have

\[ 0 = \nabla_m b^{11} = \nabla_m b^{11} = \frac{b^{11}}{h} - \frac{b^{11}}{h^2} h_m. \]

Then

\[ h_{ssm} h_m = (b_{ss} - h)_m h_m = b_{ssm} h_m, \]

\[ = - \frac{b_{ss}}{h} h_m - h_m^2. \]
From this we get

\[ b_{ss}^2 - hb_{ss} + h_s^2 + h_{ssm}h_m \]

= \[ b_{ss}^2 - hb_{ss} + h_s^2 - \frac{b_{ss}}{h}h_m^2 - h_m^2 \]

= \[ b_{ss}^2 + h_s^2 - \frac{b_{ss}}{h} \rho^2 - \rho^2 + h^2. \]

Now, we have

\[
1 + N^{-\frac{k}{k+1}} \left( N^{\frac{1}{k+1}} \right)_{ss} \\
= 1 + \frac{1}{k+1} f_{ss} f^{-1} + \frac{k}{(k+1)^2} f_{ss}^2 f^{-2} + \frac{2 - \rho}{k+1} h^{-1} h_{ss} \\
+ \frac{2(2 - p)}{(k+1)^2} h^{-1} f_{ss} h_s + \frac{2(q - n)}{(k+1)^2} f_{s} h_{ss} h_s \\
+ \frac{q - n}{(k+1)^2} (q - n - 2k - 2) \rho^{-2} h_s^2 h_s^2 \\
- \frac{1}{k+1} (q - n) + \frac{1}{k+1} (q - n) \rho^{-2} \left( b_{ss}^2 - \frac{b_{ss}}{h} \rho^2 + h_s^2 + h^2 \right). 
\]

Since \( 1 > \rho^{-2} h_s^2 \) and \( q > n + k + 1 \), we have

\[
\frac{2(q - n)}{(k+1)^2} f_{s} h_{ss} h_s + \frac{q - n}{(k+1)^2} (q - n - 2k - 2) \rho^{-2} h_s^2 h_s^2 + \frac{1}{k+1} (q - n) \rho^{-2} b_{ss}^2 \\
> \frac{2(q - n)}{(k+1)^2} f_{s} h_{ss} h_s + \frac{q - n}{(k+1)^2} (q - n - k - 1) \rho^{-2} b_{ss}^2 h_s^2 \\
= \frac{q - n}{(k+1)^2} \left( (q - n - k - 1) \frac{1}{2} \rho^{-2} h_s h_{ss} + f_{s} f^{-1} \frac{1}{(q - n - k - 1)^2} \right)^2 \\
- \frac{q - n}{(k+1)^2} \frac{1}{q - n - k - 1} f_{s}^2 f^{-2}. 
\]

We also have

\[
\frac{2(2 - p)}{(k+1)^2} h^{-1} f_{ss} f^{-1} h_s + \frac{1}{(k+1)^2} (p - 2)(p + k - 1) h^{-2} h_s^2 \\
= \frac{(p - 2)}{(k+1)^2} \left( (p + k - 1) \frac{1}{2} h^{-1} h_s - \frac{1}{(p + k - 1)^2} f_{s} f^{-1} \right)^2 \\
- \frac{(p - 2)}{(k+1)^2} \frac{1}{p + k - 1} f_{s}^2 f^{-2}. 
\]
Noticing that $p \geq 2$, we have
\[
1 + N^{-\frac{1}{4+p}} \left( N^{\frac{1}{4+p}} \right)_{ss} \geq \frac{p + n - q + k - 1}{k + 1} + \frac{2 - p - q + n}{k + 1} h^{-1} b_{ss} + \frac{2}{(k + 1)^2} (2 - p)(q - n) \rho^{-2} h^{-1} b_{ss} h_s^2
\]
\[
+ \frac{1}{k + 1} f_{ss} f^{-1} \left( \frac{k}{(k + 1)^2} + \frac{1}{(k + 1)^2} q - n - k - 1 \right) f_s^2 f^{-2}
\]
\[
= \frac{2 - p - q + n}{k + 1} h^{-1} b_{ss} + \frac{2}{(k + 1)^2} (2 - p)(q - n) \rho^{-2} h^{-1} b_{ss} h_s^2
\]
\[
+ \frac{1}{k + 1} f^{-1} \left( f_{ss} - \frac{1}{k + 1} \left( k + \frac{q - n}{q - n - k - 1} + \frac{p - 2}{p - k - 1} \right) f_s^2 f^{-1} + (p + k - 1 - q + n) f \right)
\]
Since $f$ satisfies (A), we have
\[
\partial_t \frac{b^{11}}{h} \leq -\left( \frac{b^{11}}{h} \right)^2 N \sigma_k (c_f h - c_0 b_{11}) + \frac{2 b^{11}}{h} \eta(t).
\]
Here $c_f$ is a positive constant depending on $f$ and $c_0$ is a positive constant depending on the uniform bounds of $h$ and $|\nabla h|$. By the uniform bounds on $h$, $f$, $\eta$ and $\sigma_k$, we conclude
\[
\partial_t \frac{b^{11}}{h} \leq -c_1 \left( \frac{b^{11}}{h} \right)^2 + c_2 \frac{b^{11}}{h}.
\]
Here $c_1$ and $c_2$ are positive constants independent of $t$. The maximum principle then gives the upper bound of $b^{11}$ and the result follows. \qed

**proof of Theorem 1.** Combining Lemma 4 and Lemma 5, we see that the principal curvatures of $M_t$ has uniform positive upper and lower bounds. This together with Lemma 2 and Corollary 2 implies that the evolution equation (9) is uniformly parabolic on any finite time interval. Thus, the result of [11] and the standard parabolic theory show that the smooth solution of (9) exists for all time. And by these estimates again, a subsequence of $M_t$ converges in $C^\infty$ to a positive, smooth, strictly convex hypersurface $M_\infty$ in $\mathbb{R}^n$. \qed

In general, the problem (9) does not have any variational structure, we can not expect the convergence of the flow to a solution for all initial hypersurfaces. In the next section, we will choose an special to obtain the existence of smooth solutions for equation (9) with $c = 1$.

### 4. Existence of solution

Since $q - n + k + 1 - p < 0$, we can choose initial hypersurface satisfying $(\frac{F}{h} - 1) M_0 > 0$. We have proved that flow (2) exists for all time when $\eta = 1$. We will use flow (9) with $\eta = 1$ to obtain the convergence.
By (15), we have
\[
\partial_t \left( \frac{F - 1}{h} \right) - N \sigma_k^{ij} \nabla_{ij} \left( \frac{F}{h} - 1 \right) \nabla_j \left( \frac{F}{h} - 1 \right).
\]
From the assumption about the initial data, the positivity of \( \frac{F}{h} - 1 \) is preserved along the flow and
\[
\partial_t h = F - h > 0
\]
holds for all time. Since \( h \) is positive and bounded from above and below, we have
\[
C \geq h(x, t) - h(x, 0) = \int_0^{+\infty} (F - h) dt.
\]
This implies that there exists a subsequence of times \( t_j \to \infty \) such that
\[
F(t_j) - h(t_j) \to 0 \text{ as } t_j \to \infty.
\]
By passing to the limit, we obtain
\[
\tilde{h}^{1-p} \rho^{p-n} \tilde{\sigma}_k(x) f(x) = 1 \text{ on } \mathbb{S}^{n-1},
\]
where \( \tilde{\sigma}_k, \tilde{h} \) and \( \tilde{\rho} \) are the \( k \)-th elementary symmetric function for principal curvature radii, the support function and radial function of \( M_\infty \). The proof of Corollary 1 is completed.

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