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Dimension of ergodic measures projected onto self-similar sets with overlaps

Thomas Jordan and Ariel Rapaport

Abstract

For self-similar sets on $\mathbb{R}$ satisfying the exponential separation condition we show that the dimension of natural projections of shift invariant ergodic measures is equal to $\min\{1, \frac{h}{\chi}\}$, where $h$ and $\chi$ are the entropy and Lyapunov exponent, respectively. The proof relies on Shmerkin’s recent result on the $L^q$ dimension of self-similar measures. We also use the same method to give results on convolutions and orthogonal projections of ergodic measures projected onto self-similar sets.

1. Introduction and statement of results

The dimension of self-similar measures on the line has been the subject of much attention going back over 40 years, since [8]. While the dimension of self-similar measures is well understood when the open set condition is satisfied, it has been a long-standing problem to see how the dimension behaves when the condition is not satisfied. Hochman, in [5], made significant progress by showing that the dimension of self-similar measures can be found as long as an exponential separation condition is satisfied, which is a much weaker condition than the open set condition.

Self-similar measures can be thought of as the projection of Bernoulli measures from a shift space to the self-similar set. So it is also possible to consider the question of what happens when general ergodic measures are projected. In the non-overlapping case it is possible to easily adapt the standard proof to obtain that the dimension is given by the ratio of the entropy to the Lyapunov exponent, a result which can also be seen in several other settings, for example, [9].

In the overlapping case it is easy to see that the ratio of entropy with Lyapunov exponent is always an upper bound (see [4, Theorem 2.8; 13, Section 3], where in addition it is shown that such measures are exact dimensional). In [13, Theorem 7.2] this is also shown to be a lower bound almost everywhere for certain families satisfying a transversality condition. However, the techniques used by Hochman in the exponential separation case for self-similar measures do not apply, since they rely on the convolution structure of self-similar measures. More precisely when the self-similar measure is homogeneous, that is when all of the contractions are the same, it is possible to represent it as a convolution of an arbitrarily small copy of itself with some other measure $\nu$ on $\mathbb{R}$. Outside of the homogeneous case, it is possible to obtain such a representation by taking $\nu$ to be a measure on the affine group of $\mathbb{R}$.

Fortunately it turns out that the result of Shmerkin [12], on the $L^q$ dimension of self-similar measures for $q > 1$, can be used to give the dimension of the projection of arbitrary ergodic...
measures. The ideas used involve an analysis of numbers of intersections of cylinders, which are similar to the ideas introduced by Rams in [11]. In addition, similar ideas combined with other results from [12] can be used to give a different proof of a result of Hochman and Shmerkin on the dimension of convolutions of times $n$ and times $m$ invariant measures. In particular the result in [7, Theorem 1.3] on the convolution of times $n$, times $m$ invariant measures is a special case of Theorem 3.1 in this paper. In Section 4 we show how the same ideas can be used to give a result on the orthogonal projections of ergodic measures supported on self-similar sets in the plane.

**Notation**

Before stating our main result we need to state our setting formally and fix the notation we will be using. In what follows the base of the log and exp functions is always 2, so that $\exp(a) = 2^a$ for $a \in \mathbb{R}$. This means our definitions of entropy and Lyapunov exponent are slightly different to usual, where the usual exponential and logarithm are used, but fits in more with the use of entropy dimension used in [5, 12].

Let $\Lambda$ be a finite non-empty set, and for each $\lambda \in \Lambda$ fix $0 < |r_{\lambda}| < 1$ and $a_{\lambda} \in \mathbb{R}$. Let

$$\Phi = \{ \phi_{\lambda}(x) = r_{\lambda}x + a_{\lambda} \}_{\lambda \in \Lambda}$$

be the associated self-similar iterated function system (IFS) on $\mathbb{R}$. Let $K$ be the attractor of $\Phi$, that is, $K$ is the unique non-empty compact subset of $\mathbb{R}$ with

$$K = \bigcup_{\lambda \in \Lambda} \phi_{\lambda}(K).$$

Write $\Omega = \Lambda^\mathbb{N}$ and let $\sigma : \Omega \to \Omega$ be the left shift. Given $n \geq 1$ and $\lambda_1 \cdots \lambda_n = w \in \Lambda^n$ write $[w] \subset \Omega$ for the cylinder set corresponding to $w$, $r_w$ for $r_{\lambda_1} \cdots r_{\lambda_n}$, and $\phi_w$ for $\phi_{\lambda_1} \circ \cdots \circ \phi_{\lambda_n}$. For $(\omega_k)_{k \geq 0} = \omega \in \Omega$ set $\omega|_n = \omega_0 \cdots \omega_{n-1} \in \Lambda^n$. Let $\Pi : \Omega \to K$ be the coding map for $\Phi$, that is,

$$\Pi \omega = \lim_{n \to \infty} \phi_{\omega|_n}(0) \text{ for } \omega \in \Omega.$$

We will always assume that our system satisfies an exponential separation condition introduced by Hochman in [5]. We define the distance between two affine maps $g_i(x) = r_i x + a_i$ on $\mathbb{R}$ as

$$d(g_1, g_2) = \begin{cases} |a_1 - a_2| & \text{if } r_1 = r_2 \\ \infty & \text{if } r_1 \neq r_2. \end{cases}$$

It is easy to see that the following definition is equivalent to the one given in [12, Section 6.4].

**Definition 1.** We say that the IFS $\Phi$ has exponential separation if there exist $c > 0$ and an increasing sequence $\{ n_j \}_{j \geq 1} \subset \mathbb{N}$ such that

$$d(\phi_{w_1}, \phi_{w_2}) \geq c^{n_j}$$

for all $j \geq 1$ and $w_1, w_2 \in \Lambda^{n_j}$ with $w_1 \neq w_2$.

This condition is satisfied for instance if $\{ r_{\lambda} \}_{\lambda \in \Lambda}$ and $\{ a_{\lambda} \}_{\lambda \in \Lambda}$ are all algebraic numbers and the maps in $\Phi$ generate a free semigroup. Additionally, in [5, Theorem 1.8; 6, Theorem 1.10] Hochman has shown that in quite general parametrized families of self-similar iterated function systems, the exponential separation condition holds outside of a set of parameters of packing and Hausdorff co-dimension at least 1.
For $\delta > 0$ and $x \in \mathbb{R}$ write $B(x, \delta)$ for the interval $[x - \delta, x + \delta]$. A Borel probability measure $\theta$ on $\mathbb{R}$ is said to be exact dimensional if there exists a number $s \geq 0$ with
\[
\lim_{\delta \downarrow 0} \frac{\log \theta(B(x, \delta))}{\log \delta} = s \text{ for } \theta\text{-a.e. } x \in \mathbb{R},
\]
in which case we write $\dim \theta = s$.

Given a Borel probability measure $\mu$ on $\Omega$ we write $\Pi \mu$ for the push-forward of $\mu$ by $\Pi$. Assuming $\mu$ is $\sigma$-invariant and ergodic, it follows from [4, Theorem 2.8] that $\Pi \mu$ is exact dimensional. We write $h_\mu$ for the entropy of $\mu$ and $\chi_\mu$ for its Lyapunov exponent with respect to $\{r_\lambda\}_{\lambda \in \Lambda}$, that is,
\[
\chi_\mu = \sum_{\lambda \in \Lambda} \mu[\lambda] \log |r_\lambda|.
\]

**Main result and structure of the paper**

**Theorem 1.1.** Suppose that $\Phi$ has exponential separation, and let $\mu$ be a $\sigma$-invariant and ergodic probability measure on $\Omega$. Then,
\[
\dim \Pi \mu = \min \left\{ 1, \frac{h_\mu}{\chi_\mu} \right\}.
\]

The proof of Theorem 1.1 is given in the next section. We first construct suitable self-similar measures, and apply Shmerkin’s results on the $L^q$ dimension to these measures. We then show that these results, together with the connection between the self-similar and ergodic measures, yield that the dimension can only drop by an amount which can be made arbitrarily small. For a full definition of $L^q$ dimensions of a measure we refer the reader to [12, Section 1.3]. The key result we will be using connects $L^q$ dimensions to bounds on the local dimension and is [12, Lemma 1.7].

In the rest of the paper we state some other applications of this method to convolutions of ergodic measures and to orthogonal projections of ergodic measures on the plane.

### 2. Proof of Theorem 1.1

Fix a $\sigma$-invariant and ergodic measure $\mu$ on $\Omega$, and write $h$ for $h_\mu$ and $\chi$ for $\chi_\mu$. We start with the construction of suitable Bernoulli measures. Let $\beta = \min \{1, \frac{h}{\chi}\}$, and in order to obtain a contradiction assume that $\dim \Pi \mu < \beta$. In particular, we have $h > 0$. Let $0 < \epsilon < \beta - \dim \Pi \mu$ be small in a manner depending on $\Phi$ and $\mu$, let $\delta > 0$ be small with respect to $\epsilon$, and let $m > 1$ be large with respect to $\delta$.

Write,
\[
W = \left\{ w \in \Lambda^m : 2^{-m(h+\delta)} \leq \mu[w] \leq 2^{-m(h-\delta)} \text{ and } |r_w| \geq 2^m(\chi-\delta) \right\}.
\]

By combining Egorov’s theorem with the Shannon–Macmillan–Breiman theorem and the ergodic theorem (applied to the function $\omega \mapsto \log r_{\omega 0}$), it can be seen that by taking $m$ sufficiently large we can obtain that
\[
\mu(\bigcup_{w \in W}[w]) > 1 - \delta. \tag{2.1}
\]

For $w \in \Lambda^m$ set
\[
p_w = \begin{cases} 
\mu[w] \cdot c & \text{if } w \in W \\
2^{-mc^{-1}} \cdot c & \text{otherwise}
\end{cases},
\]
where \( c > 0 \) is chosen so that \( \sum_{w \in \Lambda^m} p_w = 1 \). By (2.1) and by assuming that \( \epsilon^{-1} > \log |\Lambda| \) it follows that \( 1/2 \leq c \leq 2 \). Write \( p = (p_w)_{w \in \Lambda^m} \) and let \( \nu \) be the measure on \( \Omega \) with
\[
\nu[w_1 \cdots w_l] = p_{w_1} \cdots p_{w_l} \quad \text{for each } w_1, \ldots, w_l \in \Lambda^m.
\]
We now relate the expected behaviour of the \( L^q \) dimension of \( \Pi \nu \) to the expected dimension of \( \Pi \mu \). Write \( q \) for \( \delta^{-1} \) and let \( \tau > 0 \) be the unique solution to
\[
\sum_{w \in \Lambda^m} p_w^q |r_w|^{-\tau} = 1.
\]

**Lemma 2.1.** By taking \( \epsilon \) and \( \delta \) to be small enough, and \( m \) to be large enough, we may assume that
\[
\frac{\tau}{q-1} \geq \frac{h}{-\chi} - O(\delta).
\]

**Proof.** Write \( \rho_1 = \min_{\Lambda \subseteq \Lambda} |r_{\lambda}|, \rho_2 = \max_{\Lambda \subseteq \Lambda} |r_{\lambda}|, \|\ell\|_q = \sum_{w \in \Lambda^m} p_w^q, \) and \( \|\ell\|_{\infty} = \max_{w \in \Lambda^m} p_w \). Then
\[
0 \geq \log \left( \sum_{w \in \Lambda^m} p_w^q \rho_2^{-m\tau} \right) = \log \|\ell\|_q - m\tau \log \rho_2.
\]

We may assume \( \delta < h \), hence
\[
\|\ell\|_q \geq \|\ell\|_{\infty} \geq 2^{-mg(h+\delta)} \geq 2^{-2mh},
\]
and so
\[
\tau \leq \frac{\log \|\ell\|_q}{m \log \rho_2} \leq \frac{-2qh}{\log \rho_2}.
\]

From this and by the definitions of \( W \) and \( p \),
\[
1 = \sum_{w \in \Lambda^m} p_w^q \cdot |r_w|^{-\tau}
\leq \sum_{w \in \Lambda^m} p_w^q \cdot 2^{m\tau(\delta-\chi)} + \sum_{w \in \Lambda^m \setminus W} c^q 2^{-m\epsilon^{-1}} \cdot \rho_1^{-m\tau}
\leq 2^{m\tau(\delta-\chi)} \|\ell\|_q + \sum_{w \in \Lambda^m \setminus W} \exp \left( q \left( 1 - m\epsilon^{-1} + 2mh \frac{\log \rho_1}{\log \rho_2} \right) \right)
\leq 2^{m\tau(\delta-\chi)} \|\ell\|_q + \exp \left( m \log |\Lambda| + q \left( 1 - m\epsilon^{-1} + 2mh \frac{\log \rho_1}{\log \rho_2} \right) \right).
\]

By choosing \( \epsilon \) small enough in a manner depending on \( \Phi \) and \( \mu \) we may clearly assume that
\[
m \log |\Lambda| + q \left( 1 - m\epsilon^{-1} + 2mh \frac{\log \rho_1}{\log \rho_2} \right) < -1.
\]

Hence
\[
1/2 \leq 2^{m\tau(\delta-\chi)} \|\ell\|_q,
\]
and so
\[
\tau \geq \frac{-1 - \log \|\ell\|_q}{m(\delta - \chi)}.
\]
We also have
\[ \|p\|_q^q \leq \|p\|_\infty \sum_{w \in \Lambda^m} p_w \]
\[ \leq c^{q-1} \exp(-m(h - \delta)(q - 1)) \]
\[ \leq \exp((q - 1)(1 - m(h - \delta))). \]

Hence by assuming that \( m \) is large enough with respect to \( \delta \),
\[ \frac{\tau}{q - 1} \geq \frac{h - \delta}{\delta - \chi} - \delta, \]
which completes the proof of the lemma. \( \square \)

To apply Shmerkin’s result we will need the following lemma. Its proof is a simple consequence of the fact that \( \Phi \) has exponential separation, and is therefore omitted.

**Lemma 2.2.** The IFS \( \{\varphi_w\}_{w \in \Lambda^m} \) has exponential separation.

We can now use Shmerkin’s result on the \( L^q \) dimension of self-similar measures with exponential separation. Fix some \( 0 < \alpha < \min\{\tau/(q - 1), 1\} \).

**Lemma 2.3.** There exists \( \eta_0 > 0 \), which depends on all previous parameters, such that
\[ \Pi_\sigma^j \nu(B(x, \eta)) \leq \eta^{(1-\delta)\alpha} \text{ for all } 0 \leq j < m, 0 < \eta \leq \eta_0 \text{ and } x \in \mathbb{R}. \] (2.3)

**Proof.** By Lemma 2.2 the IFS \( \{\varphi_w\}_{w \in \Lambda^m} \) has exponential separation. Thus from \([12, \text{Theorem 6.6}]\) it follows that the \( L^q \) dimension of \( \Pi_\nu \) is equal to \( \min\{\tau/(q - 1), 1\} \). Write
\[ \alpha' = \frac{1}{2}\left(\alpha + \min\left\{\frac{\tau}{q - 1}, 1\right\}\right), \]
then by \([12, \text{Lemma 1.7}]\) and \( q = \delta^{-1} \) it follows that there exists \( \eta_1 > 0 \) with
\[ \Pi_\nu(B(x, \eta)) \leq \eta^{(1-\delta)\alpha'} \text{ for all } 0 < \eta \leq \eta_1 \text{ and } x \in \mathbb{R}. \]

Let \( \eta_0 > 0 \) be small with respect to \( \eta_1, m, |\Lambda| \) and \( \alpha' - \alpha \). Given a Borel set \( E \subset \Omega \) write \( \nu|_E \) for the restriction of \( \nu \) to \( E \). For every \( 0 \leq j < m, 0 < \eta \leq \eta_0, x \in \mathbb{R}, \) and \( u \in \Lambda^j \),
\[ \Pi_\sigma^j(\nu|_u)(B(x, \eta)) = \nu\{\omega \in [u] : \Pi_\sigma^j \omega \in B(x, \eta)\} \]
\[ = \nu\{\omega \in [u] : \varphi_u^{-1} \Pi \omega \in B(x, \eta)\} \]
\[ = \nu\{\omega \in [u] : \Pi \omega \in B(\varphi u x, \eta r_u)\} \]
\[ \leq \Pi_\nu(B(\varphi u x, \eta r_u)) \leq \eta^{(1-\delta)\alpha'}. \]

Hence,
\[ \Pi_\sigma^j \nu(B(x, \eta)) = \sum_{u \in \Lambda^j} \Pi_\sigma^j(\nu|_u)(B(x, \eta)) \leq |\Lambda|^m \eta^{(1-\delta)\alpha'} < \eta^{(1-\delta)\alpha}, \]
which completes the proof of the lemma. \( \square \)
We now need to relate the behaviour of the Bernoulli measure \( \nu \) and our original ergodic measure \( \mu \). Define \( f : \Omega \rightarrow \mathbb{R} \) by

\[
f(\omega) = -\frac{1}{m} 1_{\mathcal{W}(\omega|m)} \log \mu[\omega|m],
\]
for all \( \omega \in \Omega \). By the definition of \( f \) and \( \mathcal{W} \) we have that \( \int f \, d\mu \leq h + \delta \). Let \( N \geq 1 \) be large with respect to all previous parameters. Let \( \Omega_0 \) be the set of all \( \omega \in \Omega \) such that for every \( n \geq N \),

1. \( \mu[\omega|n,m] < 2^{-nm(h-\delta)} \);
2. \( |r_{\omega|m}| < 2^{nm(\chi+\delta)} \);
3. \( \frac{1}{nm} \sum_{k=0}^{nm-1} f(\sigma^k \omega) + \frac{1}{enm} \sum_{k=0}^{nm-1} 1_{\{\sigma^k \omega\}|m} \notin \mathcal{W} \} \leq h + 2\delta(1+\epsilon^{-1}) \).

By \( \int f \, d\mu \leq h + \delta \) and (2.1), and since \( \mu \) is ergodic, we may assume that \( \mu(\Omega_0) > 1/2 \). Note that the fact that \( \mu \) is ergodic for \( \sigma \) does not necessarily imply that \( \mu \) is ergodic for \( \sigma^m \), the following lemma allows us to take care of this.

**Lemma 2.4.** There exists a global constant \( c_1 > 1 \) such that for every \( \omega \in \Omega_0 \) and \( n \geq N \),

\[
-\frac{1}{nm} \log \sigma^j \nu[\omega|n,m] \leq h + c_1 \delta / \epsilon \text{ for some } 0 \leq j < m.
\]  

**Proof.** Let \( \omega \in \Omega_0 \) and \( n \geq N \), then by partitioning (3) into \( m \) sums we can see there must exist \( 0 \leq j < m \) such that

\[
\frac{1}{n} \sum_{k=1}^{n-1} f(\sigma^{km-j} \omega) + \frac{1}{en} \sum_{k=1}^{n-1} 1_{\{\sigma^{km-j} \omega\}|m} \notin \mathcal{W} \} \leq h + 2\delta(1+\epsilon^{-1}).
\]  

By the definition of \( \nu \),

\[
\sigma^j \nu[\omega|n,m] = \nu(\sigma^{-j}[\omega|m-j]) \cdot \left( \prod_{k=1}^{n-1} \nu[(\sigma^{km-j} \omega)|m] \right) \cdot \nu[(\sigma^{nm-j} \omega)|j].
\]

Since \( p_w \geq c 2^{-nc^{-1}} \) for every \( w \in \Lambda^m \) we may assume that \( N \) is sufficiently large so that

\[
-\frac{1}{nm} \log \nu(\sigma^{-j}[\omega|m-j]) - \frac{1}{nm} \log \nu[(\sigma^{nm-j} \omega)|j] \leq \delta / 2.
\]

From this, (2.5), (2.6) and \( c \geq 1/2 \), we now get

\[
-\frac{1}{nm} \log \sigma^j \nu[\omega|n,m] \leq -\frac{1}{nm} \sum_{k=1}^{n-1} \log \nu[(\sigma^{km-j} \omega)|m] \leq \frac{\delta}{2}
\]

\[
\leq -\frac{1}{nm} \sum_{k=1}^{n-1} \log (p_{(\sigma^{km-j} \omega)|m}/c) + \delta
\]

\[
= \frac{1}{n} \sum_{k=1}^{n-1} f(\sigma^{km-j} \omega) + \frac{1}{en} \sum_{k=1}^{n-1} 1_{\{\sigma^{km-j} \omega\}|m} \notin \mathcal{W} \} \leq h + 3\delta(1+\epsilon^{-1}),
\]

which completes the proof of the lemma. \( \square \)
We are now ready to complete the proof of the theorem. For a Borel set $E \subset \Omega$ write $\mu_0(E) = \frac{\mu(E \cap \Omega)}{\mu(\Omega)}$. Since $\Pi \mu_0 \ll \Pi \mu$, it follows by [10, Theorem 2.12] that for $\Pi \mu_0$-a.e. $x \in \mathbb{R}$ the limit

$$\lim_{\eta \downarrow 0} \frac{\Pi \mu_0(B(x, \eta))}{\Pi \mu(B(x, \eta))}$$

exists, and it is positive and finite. Thus, since $\Pi \mu$ is exact dimensional, the same goes for $\Pi \mu_0$ with

$$\dim \Pi \mu_0 = \dim \Pi \mu < \beta - \epsilon.$$

Let $n \geq N$ and $x \in \mathbb{R}$ be with

$$\log \frac{\Pi \mu_0(B(x, 2^{nm}\chi))}{nm\chi} < \beta - \epsilon.$$

Write

$$U = \{ w \in \Lambda^m : [w] \cap \Pi^{-1}(B(x, 2^{nm}\chi)) \neq \emptyset \text{ and } \mu_0[w] > 0 \}.$$ 

Since $\mu(\Omega_0) > 1/2$,

$$2^{nm(\beta - \epsilon)} < \Pi \mu_0(B(x, 2^{nm}\chi)) \leq \sum_{w \in U} \mu_0[w] \leq 2 \sum_{w \in U} \mu[w]. \quad (2.7)$$

For each $w \in U$ we have $\mu_0[w] > 0$, hence $\Omega_0 \cap [w] \neq \emptyset$, and so $\mu[w] < 2^{-nm(h - \delta)}$. From this and (2.7) we get

$$2^{nm(\beta - \epsilon)} < 2^{1-nm(h - \delta)} \cdot |U|.$$ 

For $0 \leq j < m$ write

$$U_j = \{ w \in U : \sigma^j \nu[w] \geq \exp(-nm(h + c_1\delta/\epsilon)) \}.$$ 

From (2.4) and $n \geq N$, and since $\Omega_0 \cap [w] \neq \emptyset$ for each $w \in U$, it follows that $U = \cup_{j=0}^{m-1} U_j$. Hence there exists $0 \leq j < m$ with

$$|U_j| \geq |U|/m > 2^{nm(\beta - \epsilon)} \cdot 2^{nm(h - \delta)} \cdot \frac{1}{2m}. \quad (2.8)$$

Without loss of generality we may assume that $\text{diam}(K) \leq 1$. Given $w \in U_j$ we have $\Pi[w] \cap B(x, 2^{nm}\chi) \neq \emptyset$. Since $\Omega_0 \cap [w] \neq \emptyset$,

$$\text{diam}(\Pi[w]) = \text{diam}(\varphi_w(K)) \leq |r_w| < 2^{nm(\chi + \delta)},$$

which implies $[w] \subset \Pi^{-1}(B(x, 2^{nm(\chi + 2\delta)}))$. Hence, by the definition of $U_j$,

$$\Pi \sigma^j \nu(B(x, 2^{nm(\chi + 2\delta)})) \geq \sigma^j \nu(\cup_{w \in U_j} [w]) \geq |U_j| \cdot \exp(-nm(h + c_1\delta/\epsilon)).$$

From this and (2.8),

$$\Pi \sigma^j \nu(B(x, 2^{nm(\chi + 2\delta)})) \geq \frac{1}{2m} \exp(nm(\chi(\beta - \epsilon) - O(\delta/\epsilon))).$$

On the other hand, by (2.3) and by assuming that $n$ is large enough,

$$\Pi \sigma^j \nu(B(x, 2^{nm(\chi + 2\delta)})) \leq \exp(nm(\chi + 2\delta)(1 - \delta)\alpha).$$

Hence

$$\frac{1}{2m} \exp(nm(\chi(\beta - \epsilon) - O(\delta/\epsilon))) \leq \exp(nm(\chi + 2\delta)(1 - \delta)\alpha),$$

and so by taking logarithm on both sides, dividing by $nm\chi$, and letting $n$ tend to $\infty$, we get

$$\beta - \epsilon + O(\delta/\epsilon) \geq (1 + 2\delta/\chi)(1 - \delta)\alpha.$$
Now by (2.2) and since this holds for every $0 \leq \alpha < \min\{\frac{1}{\delta}, 1\}$,
\[
\beta - \epsilon + O(\delta/\epsilon) \geq (1 + 2\delta/\chi)(1 - \delta) \min\left\{\frac{h}{\chi} - O(\delta), 1\right\}.
\] (2.9)

Recall that $\delta$ is arbitrarily small with respect to $\epsilon$ and that $\beta = \min\{1, \frac{h}{\chi}\}$. Hence (2.9) gives a contradiction, and so we must have $\dim \Pi \mu \geq \beta$. Since it always holds that $\dim \Pi \mu \leq \beta$ (see [4, Theorem 2.8; 13, Section 3] for details of how to prove this), this completes the proof of Theorem 1.1.

3. Convolutions of ergodic measures

In this section we show how to use the ideas from the proof of Theorem 1.1 to prove a result on the convolution of ergodic measures.

For $i = 1, 2$ let $\Phi_i = \{\varphi_{\lambda, i}(x) = r_i x + a_{\lambda, i}\}_{\lambda \in \Lambda_i}$ be a homogeneous self-similar IFS on $\mathbb{R}$, write $\Omega_i = \Lambda_i^0$, let $\Pi_i : \Omega_i \rightarrow \mathbb{R}$ be the coding map for $\Phi_i$, let $\sigma_i : \Omega_i \rightarrow \Omega_i$ be the left shift, let $\mu_i$ be a $\sigma_i$-invariant and ergodic probability measure on $\Omega_i$, and write $h_i$ for the entropy of $\mu_i$. We also write $\theta$ for the convolution $\Pi_1 \mu_1 \star \Pi_2 \mu_2$.

Recall that in Section 1 a distance $d$ was defined between affine maps from $\mathbb{R}$ to $\mathbb{R}$. We say that $\Phi_1, \Phi_2$ are jointly exponentially separated if there exist $c > 0$ and an increasing sequence $\{n_j\}_{j \geq 1} \subset \mathbb{N}$ such that
\[
d(\varphi_{w_1, i}, \varphi_{w_2, i}) \geq c^{n_j} \quad \text{for } i = 1, 2, j \geq 1 \text{ and } w_1, w_2 \in \Lambda_i^{n_j} \text{ with } w_1 \neq w_2.
\]

**Theorem 3.1.** Suppose that $\log r_1 / \log r_2 \notin \mathbb{Q}$ and that $\Phi_1, \Phi_2$ are jointly exponentially separated. Then $\theta$ is exact dimensional and
\[
\dim \theta = \min\left\{1, \frac{h_1}{-\log r_1} + \frac{h_2}{-\log r_2}\right\}.
\]

In the case of self-similar measures the theorem follows almost directly from [12, Theorem 7.2], which is the main ingredient of our proof. In [7, Theorem 1.3] the above result is shown for systems $\Phi_i$ of the form
\[
\{\varphi_{\lambda, i}(x) = x/n_i + \lambda t_i/n_i\}_{\lambda=0}^{n_i-1},
\]
where $t_1, t_2 > 0$ are real and $n_1, n_2$ are positive integers with $\log n_1 / \log n_2 \notin \mathbb{Q}$. Such systems are clearly jointly exponentially separated (in fact they satisfy the more restrictive open set condition).

**Preparations for the proof of Theorem 3.1**

Given a Borel probability measure $\zeta$ on $\mathbb{R}$ write $\dim_H \zeta$ and $\dim_P^* \zeta$ for its lower Hausdorff and upper packing dimensions. That is,
\[
\dim_H \zeta = \sup\left\{s \geq 0 : \liminf_{\eta \downarrow 0} \frac{\log \zeta(B(x, \eta))}{\log \eta} \geq s \text{ for } \zeta\text{-a.e. } x \in \mathbb{R}\right\}
\]

and
\[
\dim_P^* \zeta = \inf\left\{s \geq 0 : \limsup_{\eta \downarrow 0} \frac{\log \zeta(B(x, \eta))}{\log \eta} \leq s \text{ for } \zeta\text{-a.e. } x \in \mathbb{R}\right\}.
\]
Clearly \( \dim_H \zeta \leq \dim^*_P \zeta \), and \( \zeta \) has exact dimension \( s \) if and only if \( s = \dim_H \zeta = \dim^*_P \zeta \). Given a Borel set \( E \subset \mathbb{R} \) denote its Hausdorff dimension by \( \dim_H E \). It is well known that

\[
\dim_H \zeta = \inf \{ \dim_H E : E \subset \mathbb{R} \text{ is Borel and } \zeta(E) > 0 \}. \tag{3.1}
\]

For further details on these notions see [2, Section 10].

Recall that the total variation distance between Borel probability measures \( \zeta_1, \zeta_2 \) on \( \mathbb{R} \) is defined by

\[
d_{TV}(\zeta_1, \zeta_2) = \sup \{|\zeta_1(E) - \zeta_2(E)| : E \subset \mathbb{R} \text{ is Borel}\}.
\]

**Lemma 3.1.** The function which takes a probability measure \( \zeta \) on \( \mathbb{R} \) to \( \dim_H \zeta \) is upper semicontinuous with respect to the total variation distance.

**Proof.** Let \( \zeta \) be a probability measure on \( \mathbb{R} \) and let \( s > \dim_H \zeta \). By (3.1) there exists a Borel set \( E \subset \mathbb{R} \) with \( \zeta(E) > 0 \) and \( \dim_H E < s \). Now suppose that \( \xi \) is another probability measure on \( \mathbb{R} \) with \( d_{TV}(\zeta, \xi) < \zeta(E) \). Then

\[
\zeta(E) > \zeta(E) - d_{TV}(\zeta, \xi) > 0,
\]

and so by (3.1),

\[
\dim_H \xi \leq \dim_H E < s.
\]

This completes the proof of the lemma. \( \square \)

**Proof of Theorem 3.1**

We let

\[
\beta = \min \left\{ 1, \frac{h_1}{-\log r_1} + \frac{h_2}{-\log r_2} \right\}.
\]

By Theorem 1.1 it follows that \( \Pi_i \mu_i \) has exact dimension \( \min\{1, \frac{h_i}{-\log r_i}\} \) for \( i = 1, 2 \). Thus, it is easy to see that \( \Pi_1 \mu_1 \times \Pi_2 \mu_2 \) has exact dimension,

\[
\min \left\{ 1, \frac{h_1}{-\log r_1} \right\} + \min \left\{ 1, \frac{h_2}{-\log r_2} \right\}.
\]

Now since \( \theta \) is a linear projection of \( \Pi_1 \mu_1 \times \Pi_2 \mu_2 \),

\[
\dim^*_P \theta \leq \min \{1, \dim(\Pi_1 \mu_1 \times \Pi_2 \mu_2)\} = \beta.
\]

Thus it suffices to prove that \( \dim_H \theta \geq \beta \). Assume by contradiction that \( \dim_H \theta < \beta \). Let \( 0 < \epsilon < \beta - \dim_H \theta \) be small in a manner depending on \( \Phi_i \) and \( \mu_i \), let \( \delta > 0 \) be small with respect to \( \epsilon \), and let \( m \geq 1 \) be large with respect to \( \delta \).

For \( i = 1, 2 \) write

\[
\mathcal{W}_i = \left\{ w \in \Lambda_i^m : 2^{-m(h_i + \delta)} \leq \mu_i[w] \leq 2^{-m(h_i - \delta)} \right\}.
\]

We may assume that

\[
\mu_i(\cup_{w \in \mathcal{W}_i}[w]) > 1 - \delta. \tag{3.2}
\]

For \( w \in \Lambda_i^m \) set

\[
p_{w,i} = \begin{cases} 
\mu_i[w] \cdot c_i & \text{if } w \in \mathcal{W}_i \\
2^{-mc_i} \cdot c_i & \text{otherwise},
\end{cases}
\]
where \( c_i > 0 \) is chosen so that \( \sum_{w \in \Lambda_m^n} p_{w,i} = 1 \). By (3.2) it follows that \( 1/2 \leq c_i \leq 2 \). Write \( p_i = (p_{w,i})_{w \in \Lambda_m^m} \) and let \( \nu_i \) be the measure on \( \Omega_i \) with
\[
\nu_i[w_1 \cdots w_l] = p_{w_1,i} \cdots p_{w_l,i} \text{ for each } w_1, \ldots, w_l \in \Lambda_m^m.
\]
For \( t > 0 \) and \( x \in \mathbb{R} \) set \( S_t x = tx \) and \( \xi_t = \Pi_1 \nu_1 + S_t \Pi_2 \nu_2 \). Write \( q \) for \( \delta^{-1} \). Given a Borel probability measure \( \zeta \) on \( \mathbb{R} \) denote by \( D(\zeta, q) \) the \( L^q \) dimension of \( \zeta \).

**Lemma 3.2.** There exists a constant \( c_1 \geq 1 \), which depends only on \( r_1, r_2 \), such that
\[
D(\xi_t, q) > \beta - c_1 \delta \text{ for all } t > 0.
\]

**Proof.** For \( i = 1, 2 \) we have
\[
\| p_i \|_q^q \leq \| p_i \|_\infty^{q-1} \sum_{w \in \Lambda_m^m} p_{w,i} \leq \exp(\frac{\log \| p_i \|_q^q}{(q-1) \log r_i}) = \exp(\frac{\log \| p_i \|_\infty}{(q-1) \log r_i}).
\]
From this and [12, Theorem 6.2],
\[
D(\Pi_1 \nu_i, q) = \min \left\{ 1, \frac{\log \| p_i \|_q^q}{(q-1) \log r_i} \right\} \geq \min \left\{ 1, \frac{h_i - \delta}{-\log r_i} \right\}.
\]
From the fact that \( \Phi_i \) are jointly exponentially separated it follows easily that the systems \( \{ \varphi_{w,i} \}_{w \in \Lambda_m^m} \) are also jointly exponentially separated. From this and the assumption \( \log r_1 / \log r_2 \notin \mathbb{Q} \), by [12, Theorem 7.2], and since \( D(S_t \Pi_2 \nu_2, q) = D(\Pi_2 \nu_2, q) \) for \( t > 0 \), we get
\[
D(\xi_t, q) = \min \{ 1, D(\Pi_1 \nu_1, q) + D(S_t \Pi_2 \nu_2, q) \}
\]
\[
\geq \min \left\{ 1, \frac{h_1}{-\log r_1} + \frac{h_2}{-\log r_2} \right\} - O_{r_1, r_2}(\delta),
\]
which completes the proof of the lemma. \( \square \)

Fix some \( 0 < \alpha < \beta - c_1 \delta \).

**Lemma 3.3.** There exists \( \eta_0 > 0 \), which depends on all previous parameters, such that for every \( 0 \leq j_1, j_2 < m \),
\[
\Pi_1 \sigma_{j_1}^{i_j} \nu_1 \ast \Pi_2 \sigma_{j_2}^{i_j} \nu_2 (B(x, \eta)) \leq \eta^{(1-\alpha)\eta} \text{ for all } 0 < \eta \leq \eta_0 \text{ and } x \in \mathbb{R}.
\]

**Proof.** Write
\[
T = \left\{ r_1^{j_1} r_2^{-j_2} : 0 \leq j_1, j_2 < m \right\}.
\]
By [12, Lemma 1.7], (3.3), and \( q = \delta^{-1} \), it follows that there exists \( \eta_1 > 0 \) with
\[
\xi_t(B(x, \eta)) \leq \eta^{(1-\delta)(\beta - c_1 \delta)} \text{ for all } t \in T, \ 0 < \eta \leq \eta_1 \text{ and } x \in \mathbb{R}.
\]

Let \( \eta_0 > 0 \) be small with respect to \( \eta_1 \) and all previous parameters. Let \( 0 \leq j_1, j_2 < m \), \( 0 < \eta \leq \eta_0 \), \( x \in \mathbb{R} \), \( u_1 \in \Lambda_1^{j_1} \), and \( u_2 \in \Lambda_2^{j_2} \). Write \( b = \varphi_{u_1,1} \circ \varphi_{u_2,2}^{-1}(0) \), then
\[
\Pi_1 \sigma_{j_1}^{i_j}(\nu_1|_{[u_1]}) \ast \Pi_2 \sigma_{j_2}^{i_j}(\nu_2|_{[u_2]})(B(x, \eta))
\]
\[
= \nu_1 \times \nu_2 \left\{ (\omega_1, \omega_2) \in [u_1] \times [u_2] : \Pi_1 \sigma_{j_1}^{i_j} \omega_1 + \Pi_2 \sigma_{j_2}^{i_j} \omega_2 \in B(x, \eta) \right\}
\]
\[
= \nu_1 \times \nu_2 \left\{ (\omega_1, \omega_2) \in [u_1] \times [u_2] : \varphi_{u_1,1}^{-1} \Pi_1 \omega_1 + \varphi_{u_2,2}^{-1} \Pi_2 \omega_2 \in B(x, \eta) \right\}
\]
\[
\leq \nu_1 \times \nu_2\left\{ (\omega_1, \omega_2) : \Pi_1 \omega_1 + S_{r_1 i - 2} \Pi_2 \omega_2 \in B(\varphi_{u_1,1} x - b, r_1^{2j} \eta) \right\}
= \xi_{r_1 i - 2} (B(\varphi_{u_1,1} x - b, r_1^{2j} \eta)) \leq \eta^{(1-\delta)(\beta - c_1 \delta)}.
\]

Hence,
\[
\Pi_1 \sigma_1^{2j} \nu_1 \ast \Pi_2 \sigma_2^{2j} \nu_2 (B(x, \eta)) = \sum_{u_1 \in \Lambda_1^{i1}} \sum_{u_2 \in \Lambda_1^{i2}} \Pi_1 \sigma_1^{2j} (\nu_1 \mid u_1) \ast \Pi_2 \sigma_2^{2j} (\nu_2 \mid u_2) (B(x, \eta))
\leq |\Lambda_1|^m |\Lambda_2|^m \eta^{(1-\delta)(\beta - c_1 \delta)} \leq \eta^{(1-\delta)\alpha},
\]
which completes the proof of the lemma. \(\square\)

For \(i = 1, 2\) and \(\omega \in \Omega_i\) set
\[
f_i(\omega) = -\frac{1}{m} W_i(\omega) \log \mu_i[\omega]\]
then \(\int f_i \, d\mu_i \leq h_i + \delta\). Let \(N \geq 1\) be large with respect to all previous parameters. For \(n \geq 1\)
write \(n_i = \left\lceil \frac{n}{-\log r_i} \right\rceil\). Let \(\Omega_{0,i}\)
be the set of all \(\omega \in \Omega_i\) such that for every \(n \geq N\),
\begin{itemize}
\item \(\mu_i[\omega]_{n,m} \leq 2^{-n_i m(h_i - \delta)}\);
\item \(\frac{1}{n_i m} \sum_{k=0}^{n_i m - 1} f_i(\sigma_k \omega) + \frac{1}{r_i m} \sum_{k=0}^{n_i m - 1} \{\sigma_k \omega \mid \omega \notin W_i\} \leq h_i + 2\delta(1 + \epsilon^{-1})\).
\end{itemize}

By (3.2), the fact that \(\int f_i \, d\mu_i \leq h_i + \delta\), Egorov’s theorem and the ergodicity of \(\mu_i\), we may assume that \(\mu_i(\Omega_{0,i}) > 1 - O(\delta)\).

**Lemma 3.4.** There exists a global constant \(c_2 > 1\) such that for \(i = 1, 2\), \(\omega \in \Omega_{0,i}\), and \(n \geq N\),
\[
-\frac{1}{n_i m} \log \sigma_i^{2j} \nu_i[\omega]_{n,m} \leq h_i + c_2 \delta / \epsilon \text{ for some } 0 \leq j < m.
\]

**Proof.** The proof uses exactly the same method as the proof of Lemma 2.4. \(\square\)

Let us resume with the proof of the theorem. For \(i = 1, 2\) and a Borel set \(E \subset \Omega_i\)
write \(\mu_{0,i}(E) = \mu_i(E \mid \Omega_{0,i})\), and write \(\theta_0\) for \(\Pi_1 \mu_{0,1} \ast \Pi_2 \mu_{0,2}\). By Lemma 3.1 the function which takes
a probability measure \(\zeta\) on \(\mathbb{R}\) to \(\text{dim}_H \zeta\) is upper semi-continuous with respect to the total
variation distance. From \(\mu_i(\Omega_{0,i}) > 1 - O(\delta)\) it follows that the total variation distance between
\(\theta\) and \(\theta_0\) is \(O(\delta)\). Thus we may assume that
\[
\text{dim}_H \theta_0 \leq \text{dim}_H \theta + \epsilon/2 < \beta - \epsilon/2.
\]

Let \(n \geq N\) and \(x \in \mathbb{R}\) be with
\[
\frac{\log \theta_0(B(x, 2^{-nm})}{-nm} < \beta - \epsilon/2.
\]

Let \(g : \Omega_1 \times \Omega_2 \to \mathbb{R}\) be with \(g(\omega_1, \omega_2) = \Pi_1 \omega_1 + \Pi_2 \omega_2\), then \(\theta_0 = g(\mu_{0,1} \times \mu_{0,2})\). Let \(U\) be the
set of all pairs of words \((w_1, w_2) \in \Lambda_{n+1}^{nm} \times \Lambda_{n+2}^{nm}\) such that
\[
([w_1] \times [w_2]) \cap g^{-1}(B(x, 2^{-nm})) \neq \emptyset,
\]
and \(\mu_{0,i}[w_i] > 0\) for \(i = 1, 2\).
Since \( \mu_1(\Omega_{0,i}) > 1 - O(\delta) > 1/2, \)
\[
2^{-nm(\beta-\epsilon/2)} < \theta_0(B(x,2^{-nm})) = g(\mu_{0,1} \times \mu_{0,2})(B(x,2^{-nm}))
\leq \sum_{(w_1,w_2) \in U} \mu_{0,1}[w_1]\mu_{0,2}[w_2] \leq 4 \sum_{(w_1,w_2) \in U} \mu_1[w_1]\mu_2[w_2]. \tag{3.6}
\]
For each \((w_1,w_2) \in U\) we have \(\mu_{0,i}[w_i] > 0\) for \(i = 1,2\), hence \(\Omega_{0,i} \cap [w_i] \neq \emptyset\), and so \(\mu_i[w_i] < 2^{-n,m(h,\delta)}\). From this and (3.6) we get
\[
2^{-nm(\beta-\epsilon/2)} < \exp(2 - n_1 mh_1 - n_2 mh_2 + \delta m(n_1 + n_2)) \cdot |U|.
\]
For \(0 \leq j_1, j_2 < m\) write
\[
U_{j_1,j_2} = \left\{ (w_1,w_2) \in U : \sigma_1^{j_1} \nu_1[w_i] \geq \exp(-n_i m(h_i + c_2 \delta/\epsilon)) \text{ for } i = 1,2 \right\}.
\]
From (3.5) and \(n \geq N\), and since \(\Omega_{0,i} \cap [w_i] \neq \emptyset\) for \(i = 1,2\) and \((w_1,w_2) \in U\), it follows that \(U = \bigcup_{j_1,j_2=0}^{m-1} U_{j_1,j_2}\). Hence there exist \(0 \leq j_1, j_2 < m\) with
\[
|U_{j_1,j_2}| \geq |U|/m^2 > \frac{1}{4m^2} \exp(n_1 mh_1 + n_2 mh_2 - nm(\beta-\epsilon/2) - \delta m(n_1 + n_2)). \tag{3.7}
\]
For \(i = 1,2\) let \(K_i\) be the attractor of \(\Phi_i\). Without loss of generality we may assume that \(\text{dim}(K_i) \leq 1\). Given \((w_1,w_2) \in U_{j_1,j_2}\) we have
\[
g([w_1] \times [w_2]) \cap B(x,2^{-nm}) \neq \emptyset.
\]
Also, since \(n_i = \lceil -\frac{n}{\log r_i} \rceil\),
\[
\text{diam}(g([w_1] \times [w_2])) = \text{diam}(\Pi_1[w_1]) + \text{diam}(\Pi_2[w_2])
\leq \text{diam}(\varphi_{w_1,1}(K_1)) + \text{diam}(\varphi_{w_2,2}(K_2)) \leq r_1^{n_1 m} + r_2^{n_2 m} < 2^{1-nm},
\]
which implies that
\[
[w_1] \times [w_2] \subset g^{-1}(B(x,2^{-nm})).
\]
Hence, by the definition of \(U_{j_1,j_2}\),
\[
g(\sigma_1^{j_1} \nu_1 \times \sigma_2^{j_2} \nu_2)(B(x,2^{-nm})) \geq \sigma_1^{j_1} \nu_1 \times \sigma_2^{j_2} \nu_2(\cup_{(w_1,w_2) \in U_{j_1,j_2}} [w_1] \times [w_2])
\geq |U_{j_1,j_2}| \cdot \exp(-n_1 mh_1 - n_2 mh_2 - (n_1 + n_2)m c_2 \delta/\epsilon).
\]
From this and (3.7),
\[
g(\sigma_1^{j_1} \nu_1 \times \sigma_2^{j_2} \nu_2)(B(x,2^{-nm})) \geq \frac{1}{4m^2} \exp(-nm(\beta-\epsilon/2 + O_{r_1,r_2}(\delta/\epsilon))).
\]
On the other hand, by (3.4) and by assuming that \(n\) is large enough,
\[
g(\sigma_1^{j_1} \nu_1 \times \sigma_2^{j_2} \nu_2)(B(x,2^{-nm})) \leq \exp((2-nm)(1-\delta)\alpha).
\]
Hence
\[
\frac{1}{4m^2} \exp(-nm(\beta-\epsilon/2 + O_{r_1,r_2}(\delta/\epsilon))) \leq \exp((2-nm)(1-\delta)\alpha),
\]
and so by taking logarithm on both sides, dividing by \(-nm\), and letting \(n\) tend to \(\infty\), we get
\[
\beta - \epsilon/2 + O_{r_1,r_2}(\delta/\epsilon) \geq (1-\delta)\alpha.
\]
Since this holds for every \(0 < \alpha < \beta - c_1 \delta\),
\[
\beta - \epsilon/2 + O_{r_1,r_2}(\delta/\epsilon) \geq (1-\delta)(\beta - c_1 \delta). \tag{3.8}
\]
Now recall that \(\delta\) is arbitrarily small with respect to \(\epsilon\), and so (3.8) gives a contradiction. Thus we must have \(\text{dim}_H \theta \geq \beta\), which completes the proof of the theorem.
4. Orthogonal projections of ergodic measures

In this section we show how to use the ideas above in order to prove a result on the orthogonal projections of ergodic measures. As in previous sections, the main ingredient in the proof is a result from [12].

Let \( U \) be a \( 2 \times 2 \) orthogonal matrix with \( U^n \neq \text{Id} \) for all \( n \geq 1 \) and let \( 0 < r < 1 \). Let \( \Phi = \{ \varphi_\lambda(x) = rUx + a_\lambda \}_{\lambda \in \Lambda} \) be a self-similar IFS on \( \mathbb{R}^2 \). Suppose that \( \Phi \) satisfies the open set condition. Let \( S^1 \) be the unit circle of \( \mathbb{R}^2 \).

For \( z \in S^1 \) and \( y \in \mathbb{R}^2 \) write \( P_z y = \langle z, y \rangle \). Write \( \Omega = \Lambda^\infty \), let \( \sigma : \Omega \to \Omega \) be the left shift, and let \( \Pi : \Omega \to K \) be the coding map for \( \Phi \).

**Theorem 4.1.** Let \( \mu \) be a \( \sigma \)-invariant and ergodic measure on \( \Omega \). Write \( h \) for the entropy of \( \mu \). Then for every \( z \in S^1 \) the measure \( P_z \Pi \mu \) is exact dimensional and

\[
\dim P_z \Pi \mu = \min \left\{ 1, \frac{h}{-\log r} \right\}.
\]

In [7, Theorem 1.6] the above result is shown for self-similar measures and it is shown for Gibbs measures in [1]. The methods used in [1, 7] do not seem to adapt to general ergodic measure. However the results in [1] do work for Gibbs measures on self-conformal sets as well as on self-similar sets. We do not know how to extend our results to the setting of self-conformal sets.

**Sketch of the proof of Theorem 4.1**

The proof is almost identical to the ones given for Theorems 1.1 and 3.1, thus we only provide a short sketch.

Let \( \beta = \min \{ 1, \frac{h}{-\log r} \} \), then it suffices to show that \( \dim_H P_z \Pi \mu \geq \beta \) for all \( z \in S^1 \). Assume by contradiction that there exists \( z \in S^1 \) with \( \dim_H P_z \Pi \mu < \beta \). Let \( 0 < \epsilon < \beta - \dim_H P_z \Pi \mu \) be small in a manner depending on \( \Phi \) and \( \mu \), let \( \delta > 0 \) be small with respect to \( \epsilon \), and let \( m \geq 1 \) be large with respect to \( \delta \).

Next we construct a Bernoulli measure \( \nu \) which corresponds to \( \mu \) as in the proof of Theorem 3.1. Namely, write

\[
W = \left\{ w \in \Lambda^m : 2^{-m(h+\delta)} \leq \mu[w] \leq 2^{-m(h-\delta)} \right\},
\]

for \( w \in \Lambda^m \) set

\[
p_w = \begin{cases} 
\mu[w] \cdot c & \text{if } w \in W \\
2^{-m\epsilon^{-1}} \cdot c & \text{otherwise}
\end{cases}
\]

(where \( 1/2 \leq c \leq 2 \) is a normalizing constant), and let \( \nu \) be the measure on \( \Omega \) with,

\[
\nu[w_1 \cdots w_l] = p_{w_1} \cdots p_{w_l} \text{ for each } w_1, \ldots, w_l \in \Lambda^m.
\]

Write \( q \) for \( \delta^{-1} \), and recall that given a Borel probability measure \( \zeta \) on \( \mathbb{R} \) its \( L^q \) dimension is denoted by \( D(\zeta, q) \).

**Lemma 4.1.** There exists a constant \( c_1 \geq 1 \), which depends only on \( r \), such that

\[
D(P_z \Pi \nu, q) > \beta - c_1 \delta \text{ for all } v \in S^1.
\] (4.1)

**Proof.** We have

\[
\|p\|_q^q \leq \|p\|_{\infty}^{\delta^{-1}} \sum_{w \in \Lambda^m} p_w \leq \exp(-m(h-\delta)(q-1)).
\]
From this and [12, Theorem 8.2] it follows that for all $v \in S^1$,

$$D(P_v \Pi \nu, q) = \min \left\{ 1, \frac{\log \| p \|_q}{(q - 1) \log r^m} \right\} \geq \min \left\{ 1, \frac{h - \delta}{-\log r} \right\},$$

which completes the proof of the lemma. \[\square\]

Fix some $0 < \alpha < \beta - c_1 \delta$.

**Lemma 4.2.** There exists $\eta_0 > 0$, which depends on all previous parameters, such that for every $0 \leq j < m$,

$$P_z \Pi \sigma^j \nu(B(x, \eta)) \leq \eta^{(1-\delta)\alpha} \text{ for all } 0 < \eta \leq \eta_0 \text{ and } x \in \mathbb{R}. \quad (4.2)$$

**Proof.** Write

$$T = \{ U^j z : 0 \leq j < m \}.$$

By [12, Lemma 1.7], (4.1), and $q = \delta^{-1}$, it follows that there exists $\eta_1 > 0$ with

$$P_z \Pi \nu(B(x, \eta)) \leq \eta^{(1-\delta)(\beta - c_1 \delta)} \text{ for all } v \in T, 0 < \eta \leq \eta_1 \text{ and } x \in \mathbb{R}.$$

Let $\eta_0 > 0$ be small with respect to $\eta_1$ and all previous parameters. Let $0 \leq j < m, 0 < \eta \leq \eta_0, x \in \mathbb{R}$, and $u \in \Lambda^j$. Write $b = \langle z, U^{-j} \varphi_u(0) \rangle$, then

$$P_z \Pi \sigma^j (\nu|_u)(B(x, \eta)) = \nu\{ \omega \in [u] : P_z \Pi \sigma^j \omega \in B(x, \eta) \}$$

$$= \nu\{ \omega \in [u] : P_z \varphi_u^{-1} \Pi \omega \in B(x, \eta) \}$$

$$= \nu\{ \omega \in [u] : P_{U^j} \Pi \omega \in B(x + b, r^j \eta) \}$$

$$\leq P_{U^j} \Pi \nu(B(x + b, r^j \eta)) \leq \eta^{(1-\delta)(\beta - c_1 \delta)}.$$

Hence,

$$P_z \Pi \sigma^j \nu(B(x, \eta)) = \sum_{u \in \Lambda^j} P_z \Pi \sigma^j (\nu|_u)(B(x, \eta)) \leq |\Lambda|^m \eta^{(1-\delta)(\beta - c_1 \delta)} < \eta^{(1-\delta)\alpha},$$

which completes the proof of the lemma. \[\square\]

After this point the argument proceeds exactly as in the proofs of Theorems 1.1 and 3.1, and we can complete the proof of Theorem 4.1.

5. Applications and remarks

For a self-similar set the similarity dimension $s$ is defined to be the unique solution of $\sum_{\lambda \in \Lambda} r_{\lambda}^s = 1$. In the case where the similarity dimension is less than or equal to 1 there is a more straightforward proof for Theorem 1.1, where $\nu$ can simply be taken to be the self-similar measure with weight $r_{\lambda}^s$ for each $\lambda \in \Lambda$. In fact with this assumption Theorem 1.1 can be extended to show no dimension drop for non-invariant measures and sets, where the dimension on the symbolic space is defined to be compatible with the self-similar set.

We can also give a general bound on the dimension of an ergodic measure $\mu$, projected to a self-similar set, in terms of $L^q$ dimensions. As above we let $\nu$ be the self-similar measure with weight $r_{\lambda}^s$ for each $\lambda \in \Lambda$, where $s$ is the similarity dimension. If for $q > 1$ we let $D \Pi \nu(q)$ denote
the $L^q$ dimension of $\Pi \nu$ and $\alpha_{\min} = \lim_{q \to \infty} D_{\Pi \nu}(q)$, then for any $0 < \alpha < \alpha_{\min}$ there exists $C > 0$ such that

$$\Pi \nu(B(x, r)) \leq C r^\alpha \text{ for all } x \in \mathbb{R} \text{ and } r > 0.$$ 

Now by using some of the ideas appearing in the proofs above, it can be shown that

$$\dim \Pi \mu \geq \frac{h(\mu)}{-\chi_\mu} - (s - \alpha_{\min}).$$

This can be applied in situations where exponential separation is not satisfied but the $L^q$ spectrum is known, for examples of this see [3]. In the case where $s > 1$ it may be possible to adapt the methods given earlier to produce better methods, but this will be very dependent on the specific system.

If we have a diagonal self-affine system in the plane, satisfying suitable separation conditions, then we can combine our Theorem 1.1 with [4, Theorem 2.11] to show that the dimension of any ergodic measure will be the Lyapunov dimension (the Lyapunov dimension is the natural generalization of the entropy divided by Lyapunov exponent formula for ergodic measures projected on self-affine systems).

To give the full details of this let $\Lambda$ be a finite non-empty set and for each $\lambda \in \Lambda$ let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\varphi(x, y) = (a_\lambda x + s_\lambda, b_\lambda y + t_\lambda)$ where $0 < |a_\lambda|, |b_\lambda| < 1$ and $s_\lambda, t_\lambda \in \mathbb{R}$. In this setting there exists a unique non-empty compact set $K$ such that $K = \cup_{\lambda \in \Lambda} \varphi(K)$. Let $\Omega = \Lambda^\mathbb{N}$ and denote by $\Pi : \Omega \to K$ the natural projection to the self-affine set. We assume that this map is finite to one. In particular this is satisfied when the strong separation condition holds, in which case $\Pi$ is injective.

Let $\pi : \Omega \to \mathbb{R}$ denote the projection to the self-similar set given by $\{a_\lambda x + s_\lambda\}_{\lambda \in \Lambda}$ and $\pi_y : \Omega \to \mathbb{R}$ denote the projection to the self-similar set given by $\{b_\lambda y + t_\lambda\}_{\lambda \in \Lambda}$. For a fixed ergodic measure $\mu$ on $\Omega$ we let $\chi_x(\mu)$, $\chi_y(\mu) > 0$ denote the Lyapunov exponents with respect to the respective self-similar systems. We also let $\chi_1(\mu) = \min\{\chi_x(\mu), \chi_y(\mu)\}$ and $\chi_2(\mu) = \max\{\chi_x(\mu), \chi_y(\mu)\}$, and let $\pi_1$ be the projection which corresponds to the smaller Lyapunov exponent $\chi_1(\mu)$.

In this setting [4, Theorem 2.11] gives that

$$\dim_H \Pi \mu = \frac{h(\mu)}{-\chi_1(\mu)} + \frac{h(\mu) - h(\mu)}{-\chi_2(\mu)}.$$ 

Here $h(\mu)$ is the usual entropy and $h(\mu)$ is the projected entropy which satisfies $\dim(\pi_1 \mu, h_1(\mu) = h(\mu)$ (see [4, Theorem 2.8]). Now suppose that the direction corresponding to the smaller Lyapunov exponent satisfies exponential separation. Then Theorem 1.1 gives that if $\chi_1(\mu) \geq h(\mu)$ then $h(\mu) = h(\mu)$ and so

$$\dim_H \Pi \mu = \frac{h(\mu)}{-\chi_1(\mu)}.$$ 

On the other hand if $\chi_1(\mu) < h(\mu)$ then Theorem 1.1 gives that $\chi_1(\mu) = h(\mu)$ and so

$$\dim_H \Pi \mu = 1 + \frac{h(\mu) - \chi_1(\mu)}{-\chi_2(\mu)}.$$ 

This means that whenever the self-similar set corresponding to the smaller Lyapunov exponent satisfies exponential separation,

$$\dim_H \pi \mu = \min\left\{\frac{h(\mu)}{-\chi_1(\mu)}, 1 + \frac{h(\mu) - \chi_1(\mu)}{-\chi_2(\mu)}\right\}$$

which is what we required.

Note that the requirement of $\Pi : \Omega \to X$ being finite to 1 is used to show that the projected entropy in [4, Theorem 2.11] is the same as the usual entropy. It should be possible to weaken
this assumption considerably. For example to a suitable exponential separation condition for the diagonal self-affine set.

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