THREE, FOUR AND FIVE-DIMENSIONAL FULLERENES

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Abstract

We explore some generalizations of fullerenes $F_v$ (simple polyhedra with $v$ vertices and only 5- and 6-gonal faces) seen as $(d-1)$-dimensional simple manifolds (preferably, spherical or polytopal) with only 5- and 6-gonal 2-faces. First, finite and planar (infinite) 3-fullerenes are described. Three infinite families of spherical 4-fullerenes are presented in Sections 4-6. The Construction A gives 4-polytopes by suitable insertion of fullerenes $F_{30}(D_{5h})$ into glued 120-cells. The Construction B gives 3-spheres by growing dodecahedra and barrels $F_{24}$ around of given fullerene. The Construction C gives 4-fullerenes from special decoration of given 4-fullerene, which add facets $F_{20}, F_{24}, F_{26}$ and $F_{28}(T_d)$ only. Some 5-fullerenes are obtained, by a variation of gluing of two regular tilings 5333 of hyperbolic 4-space or of their suitable quotients.

1 Introduction

We define here $d$-fullerene as a $(d-1)$-dimensional simple (i.e. $d$-valent) manifold (on any surface), such that any 2-face is 5- or 6-gon. We are specially interested in the $d$-fullerenes, which are spherical, i.e. homeomorphic to the $(d-1)$-sphere, and, moreover, polytopal, i.e. convex. So, the dual of a $d$-fullerene is a $(d-1)$-dimensional simplicial manifold, such that any $(d-3)$-face is adjacent to five or six $(d-2)$-simplexes.

We will use the following notation: $F_v(G)$ denotes a fullerene, i.e simple polyhedron with only 5- and 6-gonal faces, having $v$ vertices and the group of symmetry $G$. Those polyhedra are important in Organic Chemistry; see, for example, a monograph [FoMa95]. In particular, the regular dodecahedron $F_{20}(I_h)$ and the “hexagonal barrel” (unique $F_{24}$) will be also denoted by $Do$ and $B_6$, respectively.

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All regular (i.e. such that the group of automorphisms is transitive on $i$-faces for any $i$) partitions, which are $d$-fullerenes, are (in classical notation of [Cox73]):

for $d = 3$: 53 (the smallest fullerene $F_{20}$) and 63 (the tiling of Euclidean plane by regular hexagons; it can be seen as the fullerene $F_{\infty}$);
for $d = 4$: 533 (the 120-cell) and 633;
for $d = 5$: 5333 (the tiling of hyperbolic 4-space by 120-cell).

All 4- and 5-fullerenes below are constructed from five above regular partitions. But, perhaps, some of those constructions can be applied also to regular star-polytopes $\tilde{S}_3^3$, the great stellated dodecahedron, and $\tilde{S}_2^3$; they cover 2-, 3-sphere, respectively, 7, 191 times and isomorphic, i.e. topologically equivalent, to the dodecahedron and the 120-cell, respectively. All other, i.e. not simple, regular partitions with only 5- or 6-gonal 2-faces, are: hyperbolic 5m, 6m (for any $m \geq 4$); 534, 5334; 535, 5335 and 634, 635; 536. The angle between faces of dodecahedral facet is $90^\circ$, $72^\circ$, $60^\circ$ for 534, 535, 536, respectively.

The purpose of this note is to expose a zoo of examples of $d$-fullerenes for $d = 3, 4, 5$; we not considered embeddability of those $(d-1)$-manifolds.

2 3-fullerenes

Consider first 3-fullerenes. It turns out, that all finite 3-fullerenes are:

(i) spherical fullerenes, i.e. usual fullerene polyhedra on the sphere $S^2$;
(ii) toroidal and Klein bottal polyhexes, i.e. 3-valent partitions of the torus $T^2$ or the Klein bottle $K^2$ into hexagons;
(iii) elliptic fullerenes, i.e. 3-valent partitions of the real projective (i.e. elliptic) plane $P^2$ into six 5-gons and some $p_6$ hexagons.

In fact, the Euler characteristic $v - e + p_5 + p_6$ of any finite 3-fullere
is $\frac{24}{6} \geq 0$, because $3v = 2e = 5p_5 + 6p_6$. On the other hand, it is $2 - 2g$ for any orientable and $2 - g$ for any non-orientable surface of genus $g$. So, the only possible surfaces are the $S^2, T^2, P^2, K^2$ with $g = 0, 1, 1, 2$ and $p_5 = 12, 0, 6, 0$, respectively; $v = \frac{5p_5}{3} + 2p_6$. Toroidal and Klein bottal polyhexes are classified, for example, in [Neg85]; the smallest (polyhedral) ones are the Heawood graph (dual $K_7$) and dual $K_{3,3,3}$, respectively. The elliptic fullerenes are exactly folded centrally-symmetric spherical fullerenes, i.e. their $i$-faces $(i = 0, 1, 2)$ are pairs of its antipodal $i$-faces. The smallest one is the Petersen graph (dual $K_6$) on $P^2$, i.e. folded Do. Actually, the centrally-symmetric fullerenes are exactly those of symmetry $C_i, C_{2h}, D_{2h}, D_{6h}, D_{3d}, D_{5d}, T_h, I_h$; clearly, 4 divides $v$. So, there are 30 such fullerenes with $v \leq 60$: the $F_{20}$, the $F_{32}(D_{3d})$ (one of two $F_{32}$ with 6-ring of hexagons separating two identical blocks of six
pentagons), the $F_{36}(D_{6h})$ (elongated $F_{24}$) and 3,3,3,6,9 of them with 40,44,48,52,56,60 vertices, respectively. There are 107 such fullerenes with isolated pentagons and $v \leq 140$.

All compact $d$-fullerenes are finite. As an example of infinite 3-fullerene, let us introduce planar fullerenes, i.e. 3-valent partitions of the plane into (combinatorial) 6-gons and $p_5$ 5-gons. It turns out, that such partitions have $p_5 \leq 6$. For $p_5 = 0, 1$ such 3-fullerene is unique, for any $2 \leq p_5 \leq 6$ there is an infinity of them.

In fact, above criterion follows from a theorem of Alexandrov (see, for example page 92 of [Ale58]), that any complete (i.e. without boundary) metric of non-negative curvature on the plane, can be realized as a metric of convex surface in 3-space. Consider planar metric, with respect of which all faces of our planar fullerene became regular polygons. Its curvature will be zero on interior points (of faces and edges) and non-negative on vertices. But the convex surface will be at most a half of the sphere and so we are done. Apropos, the same argument implies that a partition of the plane into hexagons and $p_4$ quadrangles (respectively, $p_3$ triangles) has $p_4 \leq 3$ (respectively, $p_3 \leq 2$).

## 3 4-fullerenes

First, by analogy with above planar fullerenes, consider space fullerenes, i.e. 4-valent tilings of 3-space by some usual fullerenes. Examples of those infinite 4-fullerenes are tilings used (in [Wel84] on pp. 74, 136-139, 659-664) for description of clathrate structures of some ice-like or silicate compounds. In particular, they are space-fillings: a) by $D_0$ and $B_6$ in proportion 1:3; b) by slightly distorted $D_0$ and the $F_{28}(T_d)$ in proportion 2:1; c) by $D_0$, $B_6$ and the $F_{26}$. Those and other space fullerenes, whose cells have isolated hexagons occur also as crystalline t.c.p. (tetrahedrally close-packing phases) of metallic alloys. Following simple construction gives new space fullerene, having, as tiles, $D_0$, $B_6$ and the $F_{36}(D_{5h})$ (elongated $B_6$). Take infinite 4-fullerenes, given in the end of Section 5 below, and glue, by hexagons, an infinite pile of those parallel regions, in order to fill the 3-space. Elongated $B_6$ will come from pairs of glued $B_6$, as in Construction A, the $F_{36}(D_{5h})$ (elongated $D_0$) came from $D_0$.

For any finite 4-fullerene, denote by $(v, e, p, q)$ its $f$-vector, giving the number of its $i$-faces for $i = 0, 1, 2, 3$. The Euler characteristic, i.e. the number $v - e + p - q = 0$ for any finite closed 3-manifold. Also a $i$-face of simple 3-manifold is the intersection of exactly 4-$i$ ($i+1$)-faces; so, $e = 2v$, $p = v + q$ and the number $p_5$ of pentagonal 2-faces is $6q$.

All 3-faces of $d$-fullerenes are 3-fullerenes; we want, moreover, those
fullerenes to be close relatives of $Do$. Besides of $B_6$ (the unique next to $Do$ by the number of vertices), two other fullerenes with isolated hexagons, the unique $F_{26}$ and unique $F_{28}(T_d)$, are also candidates. The duals of those three polyhedra are known in Chemistry (under the name Frank-Kasper polyhedra $Z_{14}, Z_{15}, Z_{16}$) and in Physics, where they appear as disclinations (rotational defects) with respect to the vertex figure of the local icosahedral order. The fullerene $F_{30}(D_{5h})$ (elongated $Do$) will also appear below, in the Construction A, as well as the $F_{32}(D_{3h})$, in Construction D; the $F_{36}(D_{6d})$ appeared above in this Section.

Some relevant facts and analogies for 4-fullerenes are:

(i) It is well-known (see, for example, [Bok95]) that the boundary of the 120-cell is the unique simple equifacetted 3-sphere with (combinatorial) facet $F_{20}$. But [She66] has shown that every 4-polytope can be approximated arbitrary closely (in the Hausdorff distance) by a polytope whose facets are projective images of the dodecahedron. Remind also that the 120-cell is the universal polytope in the sense that any regular $\leq 4$-polytope, including star-polytopes, can be inscribed (vertices into vertices) into it ([Cox73], page 269). In Chapter 10 of [FTo64], it is conjectured that 120-cell is isoperimetrically best (i.e. it has the least volume among the 4-polytopes of unit in-radius, having 120 cells) and it is proved that it is locally the best. See also [Con67], [Miy90] for some operations on the 120-cell.

(ii) Pasini [Pas98] proved non-existence of 4-dimensional football, i.e. equifacetted 4-fullerene with (combinatorial) facet $F_{60}(I_h)$. Clearly, any equifacetted spherical 4-fullerene with $q$ (combinatorial) facets $F_n$, has $v = \frac{4n}{3}$ vertices. Perhaps, 120-cell is unique such 4-fullerene.

(iii) There exists (non-simple, of course) a tiling of Euclidean 3-space by (116 types of polyhedra isomorphic to) the $Do$; the question about tiling of 4-space by 4-polytopes isomorphic to the 120-cell, is open (see [Sch84]). $Do$, $B_6$ (actually, each pentagonal $i$-sided prism $B_i$, $i \geq 5$, see Section 5 below) tiles alone the hyperbolic 3-space $H^3$ with vertex valency 6; also 535 is 12-valent tiling of $H^3$ by $Do$.

(iv) We can show, using Theorem 6 from [DSt97], that the skeleton of the dual of any 4-fullerene does not embed isometrically (up to a scale) in any cubic lattice; the 120-cell also does not embed ([DGr97]) this way.

The Table below presents three families of spherical 4-fullerenes, constructed in this note.

The columns 2, 3 of the Table give the number of vertices and 2-faces; the number of edges is $2V$, clearly. The next five columns give the number of corresponding fullerenes among cells of a 4-fullerene; here $F_{20}, F_{24}$ are $Do, B_6$ while $F_{28}, F_{30}$ are the unique fullerenes with symmetry $T_d, D_{5h}$, respectively, and given number of vertices. The last column $F'$ gives: the
number of 3-cells $F$, when it is a fullerene in the Construction B, and the number of 3-cells in $F$, when it is a 4-fullerene in the construction C. The symbols $v, p = (p_5, p_6), q$ denote the number of vertices, of 2-faces and (for the Construction C) of cells of $F$. In particular, $C_1(120 - \text{cell})$ has, as cells, $2p_5 + q = 1560$ dodecahedra and $v = 600$ fullerenes $F_{28}$.

Table. $f$-vectors of some finite 4-dimensional fullerenes

|       | $V$  | $P$    | $F_{20}$ | $F_{24}$ | $F_{28}$ | $F_{30}$ | $F'$ |
|-------|------|--------|----------|----------|----------|----------|------|
| $A_i$ | 120- | $600$  | $720$    | $120$    | $0$      | $0$      | $0$  |
|       | 560$i$ + 40 | $666i + 54$ | $94i + 26$ | $0$      | $0$      | $12i - 12$ |     |
| $B(F)$ | $30v$ | $\frac{11v}{2} + 10$ | $\frac{v}{2} + 48$ | $2v - 40$ | $0$      | $0$      | $2$  |
| $C_1(F)$ | $20v$ | $20v + 3p$ | $2p_5$   | $2p_6$   | $v$      | $0$      | $q$  |

4 Construction of polytopal 4-fullerenes $A_i$

It will be a 4-dimensional analogue of the following simple construction of the $i$-layered dodecahedron $F_{20+10i}$; see Figure above for such $F_{30}$. Stellate a face $t$ of $Do$ (i.e. extend face-planes of its 5 neighbors until
their intersection; so we got a 5-pyramid on the face). Then do a projective transformation, sending the apex of 5-pyramid to infinity so that the 5-pyramid became right regular 5-prism. The image of our regular dodecahedron will be inscribed in the continuation of above 5-prism. The face t became larger and its opposite became smaller, but they both remain regular 5-gons; all other faces became irregular. Only one of the six 5-axes of symmetry of Do will remain. Take the mirror reflection of such modified dodecahedron on the face-plane of t. Two such dodecahedra glued by the “large” regular face, obtained from t, form the convex 3-polytope $F_{30}(D_{5h})$. It has exactly two regular 5-gonal faces: “small” ones from both modified dodecahedra. On each of them we can continue the same procedure and get general $i$-layered dodecahedron $F_{20+10i}$ with symmetry $D_{5d}$ for even $i > 0$ and $D_{5h}$ for odd $i$. (This tube is the dual of 2-capped pile of $i$ 5-anti-prisms.)

Apply same procedure to the 120-cell in 4-space. Stellate a dodecahedral face t until we get a pyramid on it. By a projective transformation, sending the apex to infinity, it will be transformed into a right prism, having $Do$ as a base. The 120-cell will be modified: t became larger, its opposite became smaller, but both remain regular; all other dodecahedral cells became irregular. Take the mirror reflection of the modified 120-cells on the 3-space, containing t; we get (from two modified 120-cells, glued by the “large” regular dodecahedron) the convex 4-polytope $A_1(120-cell)$. It has exactly two faces $Do$, “small” $Do$’s of two modified 120-cells, the other dodecahedra are irregular. The continuation of this procedure on each of “small” $Do$ gives the 4-polytope $A_i(120-cell)$. See its $f$-vector in the Table above; exactly $30i - 30$ its 2-faces are hexagons.

We can apply the construction A to any non-exposed dodecahedral cell of a $A_i(120-cell)$, i.e. having only dodecahedral neighbors: we obtain 3-spheres, but now there is no guarantee of convexity. When operation A is applied to several non-exposed dodecahedra, no two of them should have the same dodecahedral neighbor. In order to enumerate such possibilities, the solution of the following extremal problem will be of interest (we give it in dual form for the 600-cell): estimate the maximal number of vertices in the 600-cell with all pairwise distances (in the skeleton, having diameter 5) at least 3. It is at most 9, clearly, and at least 6: take 3 suitable vertices on each of two 10-gons (among all 12), which lie in two orthogonal planes.

In Chapter 4 (Sections 1.7 and 1.8) of [SaM97], four 4-fullerenes, having each 144 dodecahedral cells and $12k$ cells $B_6$ for $k = 2, 3, 4, 6$, are constructed from the 600-cell, by inverting the Hopf map of the 3-sphere on the 2-sphere. Also in Chapter 7 (Sections 2.7, 3 and 4.2) of [SaM97] crystal agregats are given, which can be used to construct 4-fullerenes.
5 Construction B of spherical 4-fullerenes

Fix a 3-fullerene $F$ with $v$ vertices, $p = \frac{3v}{2} + 2$ faces and $e = \frac{3v}{2}$ edges. From an interior point $o$ of $F$ take on the ray through each vertex $b$ a point $b'$ with distance $d(o, b') = d(o, b) + 1$. Put on each face of $F$ dodecahedra $Do$ on 5-gons and barrels $B_6$ on 6-gons, so that their lateral sides coincide. (Always in this construction $Do$ and $B_6$ are combinatorial.) We got 1-corona: $F$ itself and $p$ polyhedra of the 1-st floor. The surface of 1-corona consists of $p$ 1-anti-faces, i.e. opposite ones to the faces of $F$ and others, which are organized in $v$ 3-hedral triples of 5-gons with the central vertex $b'$ (for each of $v$ vertices $b$ of $F$). Put $v$ new $Do$ into those $v$ 3-hedral angles, one $Do$ for each. We got a 2-corona with the 2-nd floor, consisting of $v$ dodecahedra. Each of them is adjacent to 1-corona in 3 faces (of its 3-hedral angle) and to 3 neighbors on the 2-nd floor; so that the 6 remaining faces are free. Each of the 12 5-gonal (or $p - 12$ hexagonal) 1-anti-faces is incident to 5 (or 6, respectively) dodecahedra of the 2-nd floor. Those 5 (or 6) 5-gons form a half-dodecahedron (or a half-barrel, respectively). Add for each of them the second half in order to obtain $p$ new polyhedra; they form 3-rd floor. We got 3-corona. The surface of 3-corona consists of $p$ 2-anti-faces (i.e. the faces, opposite to 1-anti-faces) and $e$ quadruples, i.e. two edge-adjacent 5-gons and two other 5-gons, edge-adjacent to the first two via each vertex of the edge of their adjacency. First two 5-gons are from the surface of 2-corona, two others are from the surface of the 3-corona. Take now two copies of 3-corona. (Remind, that each i-corona is a 3-ball in 3-space.) Now we will join them in 4-space, putting between them $e$ new dodecahedra, which will form the 4-th floor for each copy. Also the corresponding 2-anti-faces of them will coincide. Each $Do$ of the 4-th floor is incident to each copy of 3-corona by a quadruple and to four neighbors on the 4-th floor.

Clearly, $B(Do)$ is the 120-cell itself and $B(B_6)$ consists only of two (combinatorial types of) fullerenes $Do$ and $B_6$.

In fact, the construction B can be similarly applied to any simple 3-polytope with, say, $v$ vertices and any given $p$-vector $(p_3, p_4, ...)$, where $p_i$ is the number of $i$-gonal faces for any $i \geq 3$. The above construction will give simple 3-sphere with $30v$ vertices, $60v$ edges, $\frac{71v}{2} + 10$ 2-faces (including $5p_i$ $i$-faces for each $i$, except 5) and $\frac{11v}{2} + 10$ cells, including two original 3-polytopes, $4p_i$ $i$-gonal barrels $B_i$ ($p_i$ on both 1-st and both 3-rd floors) and $\frac{7v}{2}$ dodecahedra ($v$ on each of both 2-nd floors and $\frac{3v}{2}$, i.e. the number $e$ of edges, on the common 4-th floor). Note that $B_5$ is $Do$ and so, $4p_5 + \frac{7v}{2}$ is the number of all dodecahedra.

It looks difficult to decide when the construction B leads to a 4-
polytope (i.e. a convex 3-sphere) even when applied to such polyhedra as the regular tetrahedron, the cube or the barrel $B_3$. $B_3$ is a cube with two opposite vertices truncated; $B_4$ is the dual of 2-capped 4-anti-prism (one of all 8 polyhedra whose faces are regular triangles). $B_3$ is called in [BoS95] *Dürer octahedron*; among constructions, given in [BoS95], there are two simple equifacetted 3-manifolds: one with 10 facets $B_3$ and one with 26 facets $B_4$ (the first one is a non-polytopal 3-sphere).

Construction B can be applied also to any simple partition of Euclidean or hyperbolic plane. In particular, taking, as original $F$, the “graphite” $63 = F_\infty$ (Euclidean plane, partitioned by regular hexagons), one get a region of 3-space, bounded by two parallel planes and filled by several layers of $Do$ and $B_6$.

### 6 Construction C of 4-fullerenes $C_j(F)$

We give this construction in dual terms of general simplicial 3-manifold; it was inspired by [SaM85]. Apply following four operations to the simplicial 3-manifold $F^*$, which is the dual to given simple 3-manifold $F$:

1) Transform each edge into a 4-path of three edges, by adding two new vertices on each edge, and subdivide each tetrahedron, using the new edges, into four tetrahedra and one truncated tetrahedron.

2) By projecting all faces from an interior point, subdivide each truncated tetrahedron into four tetrahedra and four 6-pyramids.

3) Glue each two 6-pyramids having a common base into a 6-bipyramid (cf. two 4-pyramids, glued into the octahedron in transition to the f.c.c. lattice $A_3$).

4) Subdivide each 6-bipyramid into six tetrahedra with common edge, linking its apexes.

Denote the obtained simplicial complex by $C_1(F)$; iterating the above procedure $j$ times produces $C_j(F)$. If $F$ has $v$ vertices, $p$ 2-faces (including $p_{5}$ 5-gonal and $p_{6}$ 6-gonal ones), $q$ cells, then $C_1(F)$ has $20v + 3p$ 2-faces (including $2v + 3p_{6}$ hexagons) and only following cells: all cells of $F$ plus $2p_{5}$ dodecahedra, $2p_{6}$ hexagonal barrels $B_6$ and $v$ fullerenes $F_{28}(T_d)$. So, if $F$ is a 4-fullerene (for example, the 120-cell or one obtained by the above constructions A or B), then any $C_j(F)$ is also 4-fullerene.

If the original simplicial manifold $F^*$ is spherical, then its $j$-th simplicial subdivision, described above, is also spherical. But the question of preserving the convexity is difficult. Operations 1), 2), 4) could be arranged in order to preserve it. (For example, chosen interior points of the tetrahedra should be moved “out” within 4-th dimension in order to get edges between neighbors, then suitable two points *around* of
each edge should be found and so on.) But the operation 3) can destroy the convexity. Moreover, four above topological operations can be seen separately, which is not the case of their metrical counterparts.

The dualization of another decoration of 600-cell, given in [Mo83] and Section 3.4 of [SaM85], produces from it another infinite family $D_j$, $j \geq 0$, of spherical 4-fullerenees, having $61^j \times 600$ vertices and, as cells, only dodecahedra, $B_6$ and the fullerene $F_{32}(D_{3h})$. Similarly to the construction $C_j(F)$, one can generalize it to a construction (say, $D_j(F)$), which gives an infinite family of 4-fullerenees (having, besides of cells of $F$, only cells $B_6$ and $F_{32}(D_{3h})$) starting from a 4-fullerene $F$. A mixed construction (choosing suitably operation C or D on each step) gives asymptotically non-periodic 4-fullerenees, having, besides of cells of $F$, only cells $B_6$, $F_{28}(T_d)$ and $F_{32}(D_{3h})$.

Both decorations of 600-cell, leading to $C_j(120-cell)$ and $D_j(120-cell)$, are given in [SaM85] as two examples of inflation method, which can be used in order to get other spherical $d$-fullerenees. Roughly, it consists of finding out, in a simplicial $d$-polytope $F^*$, a suitable “large” $(d-1)$-simplex, containing an integer number of “small” (fundamental) simplices; this number was 20, 61 for Constructions $C_1(F)$, $D_1(F)$ with $F$ being 120-cell. The decoration of $F^*$ comes by “barycentric homothety” (suitable projection of the “large” simplex on the “small”one) as the orbit of new points under the symmetry group.

## 7 Quotient d-fullerenees and polyhexes

The terms of this Section are, for example, from [Thu97], [Rag72].

The Poincaré dodecahedral space ([Poi04]) is, actually, a 4-fullerene with $f$-vector $(v, e, p, c) = (5, 10, 6 = p_5, 1)$. It comes from Do by gluing of its opposite faces with $1/10$ right-handed rotation. The Poincaré fullerene is not spherical, but it is locally spherical (i.e. of constant positive curvature); it has same Betti groups as $S^3$, but its fundamental group is not trivial (it has order 120). Actually, the Poincaré fullerene is the quotient 3-manifold of $S^3$ by the binary icosahedral group $I_h$ of order 120; so its $f$-vector is just $1/120$ of that one of 120-cell.

Following three compact quotients 3-manifolds have only 5-gonal and (only for the third one, called Lbell space) 6-gonal 2-faces, but they are not 4-fullerenees: (i) Seifert-Weber hyperbolic space with $f$-vector $(1, 6, 6 = p_5, 1)$, which is the quotient of 535 by the torsion-free group, having $Do$ as the fundamental domain; (ii) the quotient of 534 (or of 6-valent tiling of $H^3$ by $B_6$) by a torsion-free group, having eight right-angled $Do$ (or $B_6$) as the fundamental domain, with $f$-vector $(20, 60, 48 =$
\( p_5, 8 \) (or \((24, 72, 48 + 8 = p_5 + p_6, 8)\)).

In general, a theorem of Selberg (see Lemma 8 in [Sel60], [Bor63], 6.11-6.16 in [Rag72]) gives, that if a discrete group of motions of a symmetric space has compact fundamental domain, then it has a torsion-free normal subgroup of finite index. So, a quotient of a \( d \)-fullerene by a such group of symmetry, is a finite \( d \)-fullerene.

This method gives, for example, compact 4-fullerenes as quotients of the space fullerenes, given in the beginning of Section 3, and, on cylinder or half-cylinder, of 4-fullerenes, given in the end of Section 5.

Consider now polyhexes, i.e. \( d \)-fullerenes, having only 6-gonal 2-faces. The regular tilings \( 63 = F_{\infty} \) and \( 633 \) are examples of such infinite simply connected 3- and 4-fullerenes. All quotient surfaces of the Euclidean plane \( R^2 \) by discontinuous and fixed point free group of isometries, are: torus \( T^2 \), cylinder, its twist (Mbius surface) and Klein bottle \( K^2 \); the group is generated, respectively, by two translations, a translation, a glide reflection and by a translation and glide reflection. So, those four types of quotients of 63 are polyhexes; they are not simply connected.

Smallest quotient polyhexes on \( T^2 \) realize graphs \( C_6, K_{3,3} \) (two embeddings), 3-cube with \( p_6 = 1, 3, 4 \), respectively. But such polyhexes, the set of all faces, edges and vertices of which form a topological complex (i.e. no loops and double edges in the graph and the intersection of any two faces is an edge, a vertex or empty), exist if and only if \( p_6 \geq 7 \). On the other hand, the “greatest” polyhex 633 (the convex hull of the vertices of 63, realized on a horosphere, i.e. on a sphere with center at infinity) is an example of not cocompact (i.e. with not compact fundamental domain), but cofinite (i.e. of finite volume) 4-fullerene.

8 5-fullerenes

The regular tiling 5333 of hyperbolic 4-space by 120-cells is an infinite 5-fullerene: all its 2-faces are 5-gons.

The following is an infinite family of 5-fullerenes, having both 5-gonal and 6-gonal 2-faces. Take two copies of the tilings 5333 and glue them in some pairs of corresponding 120-cells. Delete now from the manifold the interiors of those 120-cells. For each of them, any corresponding pair (from both 5333) of neighboring 120-cells glue in a 4-polytope \( A_1 \), described in the Section 4. If the tilings are glued in only one 120-cell, the 4-manifold is the direct product of the 3-sphere and the Euclidean line; so it is simply connected.

Now, using Selberg theorem, as in previous Section, we get a finite 5-fullerene as quotient of 5333 by its symmetry group. It is a compact 4-
manifold, partitioned into a finite number of 120-cell’s. The same gluing as above, can be applied to it, in order to produce new 5-fullerenes.

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