Activity spectrum from waiting-time distribution

Mauro Politi$^{a,1}$ and Enrico Scalas$^{b,1}$

$^a$Department of Physics, Università degli Studi di Milano, via Celoria 16, 20133 Milano, Italy

$^b$Department of Advanced Science and Technology, Università degli Studi del Piemonte Orientale, 15100 Alessandria, Italy

Abstract

In high frequency financial data not only returns but also waiting times between trades are random variables. In this work, we analyze the spectra of the waiting-time processes for tick-by-tick trades. The numerical problem, strictly related with the real inversion of Laplace transforms, is analyzed by using Tikhonov’s regularization method. We also analyze these spectra by a rough method using a comb of Dirac’s delta functions.

Key words: Econophysics; Exponential distribution; Inverse problems
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1 Introduction

It has been previously shown that waiting times between orders as well as trades do not follow an exponential distribution [1,2,3]. This phenomenon can be explained by variable activity during the trading day, leading to a suitable mixture of exponential distributions in order to describe the distribution of durations [4]. In this paper, we study the activity spectrum, by numerically inverting the empirical survival function. The paper is organized as follows: In section 2, we give the basic theoretical background and we present two methods to derive the activity spectrum. In section 3, the methods are applied to real financial data. Finally, section 4 contains our conclusions.

1 Corresponding authors
E-mail: mauro.politi@unimi.it, enrico.scalas@mfn.unipmn.it
2 Theory

In tick-by-tick financial data, the waiting time (duration), $\tau$, between two consecutive trades is a random variable. Let us call $\psi(\tau)$ the probability density of durations. If we suppose that the duration process is a mixture of exponential processes, we can write:

$$
\psi(\tau) = \int_0^\infty g(\lambda)\lambda e^{-\lambda \tau} d\lambda,
$$

where $g(\lambda)$ is the spectrum of activity satisfying

$$
\int_0^\infty g(\lambda) d\lambda = 1.
$$

A similar equation can be written for the survival function (the complementary cumulative distribution function) $\Psi(\tau) = 1 - \int_0^\tau \psi(\tau') d\tau'$:

$$
\Psi(\tau) = \int_0^\infty g(\lambda)e^{-\lambda \tau} d\lambda.
$$

From eq. (3), the activity spectrum can be seen as the solution of a Fredholm problem of the first kind:

$$
\Psi(\tau) = \int_0^\infty g(\lambda) K(\lambda, \tau) d\lambda,
$$

with the kernel $K$ equal to

$$
K(\lambda, \tau) = e^{-\lambda \tau}.
$$

$\Psi(\tau)$ is indeed easily accessible from empirical data; however the problem becomes the real inversion of a Laplace transform for discrete and noisy real data [5].

We can rewrite our linear problem, defining a matrix

$$
K = \{k_{ij}\} \quad k_{ij} = e^{-h_{ij}},
$$

for $i, j = 1, ..., \tau_{max}$, where $\tau_{max}$ is the largest waiting time; the value of the $h$-parameter has to be chosen with the aim of covering all the spectrum. In fact, we can think of the index $j$ as the waiting time, whereas $\lambda_i = hi$ are values of $\lambda$ in which we want to determine the unknown function $g(\lambda)$. The equation 3 then becomes a matrix equation:

$$
\Psi = K g,
$$
where $\Psi$ is the vector of $\Psi$ values and $g$ is the unknown vector of activities $g(\lambda_i)$. The problem is ill-conditioned, in fact the ratio between the maximum and minimum elements in the matrix $K$ is equal to $e^{h(\tau_{max}^2-1)}$ and the ratio between the maximum and minimum empirical values of $\Psi$ is equal to the number of duration data points. For the data set described below, $\tau_{max} = 196$ s and the number of durations is 55559.

2.1 Tikhonov’s Method

Several techniques are available in applied mathematics to solve ill-conditioned linear systems. One of the most powerful and commonly used is Tikhonov’s regularization method [6,7,8].

We can think of the solution of a linear system as the minimum of the functional

$$L[g] = ||Kg - \Psi||^2,$$

and the key idea of the method is to introduce a regularization parameter, a positive real number $\mu$, and a regularization matrix (often the identity matrix $I$) such that the functional becomes

$$\hat{L}_\mu[g] = ||K\hat{g}_\mu - \Psi||^2 + \mu||\hat{g}_\mu||^2 \quad \mu > 0.$$  

Some theorems are available for error estimation [6,7,8].

The procedure to find the minimum of $\hat{L}_\mu[x]$ can be reduced to the problem of determining an inverse matrix for an optimal value of $\mu$. In fact, one can show that

$$\hat{g}_\mu = (K^TK + \mu I)^{-1}K^T\Psi,$$

where $K^T$ is the transpose of $K$. In order to find the optimal value of $\mu$ it is usually possible to apply the Generalized Cross Validation technique [9] or the L-curve method [10], that are less subjected to ill-conditioning of the matrix, but in our case, the matrix is too ill-conditioned even for these methods. To circumvent this difficulty, we have used Tikhonov’s method for a large number of different $\mu$ values and we have compared the rebuilt survival function

$$\hat{\Psi}_\mu = K\hat{g}_\mu$$

with the empirical one by means of the Kolmogorov-Smirnov test. The best result will be the best fit of the empirical data.
2.2 The method of Dirac’s delta comb

In this case, we assume the spectrum to be a comb of Dirac’s delta functions:

\[ g(\lambda) = \sum_{i=1}^{M} a_i \delta(\lambda - \lambda_i), \]  

(12)

where \( M \) is a suitable number of time intervals of constant activity, \( \lambda_i \), in which the trading period has been divided and \( a_i \) are suitable weights such that \( \sum_{i=1}^{M} a_i = 1 \). As a consequence, the survival function becomes:

\[ \Psi(\tau) = \sum_{i=1}^{M} a_i e^{-\lambda_i \tau}. \]  

(13)

We use the following procedure to estimate the parameters \( a_i \) and \( \lambda_i \). Let us fix a time-window, \( \Delta T \), and let us consider the minimum number \( N_j \) of waiting times for which the sum

\[ T_j = \sum_{i=1}^{N_j} \tau_i \]  

(14)

is larger than \( \Delta T \). Then a term is added in eq. (12) with the following parameters:

\[ \lambda_j = N_j / T_j \quad a_j = N_j / N. \]  

(15)

where \( N \) is the total number of data. The new interval starts when \( \tau_{N_j} \) occurs. In this way the normalization

\[ \sum_{i=1}^{M} a_i = 1 \]  

(16)

arises naturally. With this method, the value \( M \) is unknown at the beginning. Again, we test the rebuilt survival function with the Kolmogorov-Smirnov test for different values of \( \Delta T \) in order to find the optimal size. We used this simple method also to estimate the parameter \( h \) in the matrix (6).

3 Results

In order to calibrate the methods, we have first applied them to synthetic data sets extracted from an exponential or a Mittag-Leffler distribution \([11,12]\). Here, however, we only report results on activity spectrum estimates for real market data.

As in ref.\([2]\), we have considered NYSE General Electric tick-by-tick data of October 1999. After filtering the data, 55559 waiting times were recorded,
Fig. 1. Survival function for GE OCT 99 data. The solid line represents an exponential fit with $\lambda = \frac{1}{\tau_0}$.

Fig. 2. General Electric data analysis. KS probability as a function of the time interval.

with a mean $\tau_0 \simeq 8.85s$. The empirical survival function is shown in figure Fig.1.

The matrix in eq. (6) has a free parameter: $h$. It defines the range of $\lambda$ in which we can evaluate the function $g(\lambda)$. In our case, we fix $h$ to be 0.0015 based on a preliminary analysis using the Dirac’s delta comb method. The size of the matrix is $\tau_{max} \times \tau_{max} = 196 \times 196$. Therefore, our spectrum ranges from 0 to

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Fig. 3. Dotted line: empirical survival function. Solid line: survival function built by means of the time-splitting method with $\Delta T = 1500$ s.

Fig. 4. KS probability vs $\mu$ for the Tikhonov method. We can see that the optimal value of $\mu$ is around 0.654

$\sim 2.5$ times $\lambda_0 = 1/\tau_0$.

As for the time-split method, we found that the optimal value for $\Delta T$ is around 1500s (see Fig. 2). In Fig. 3, we present the rebuilt survival function.

As for Tikhonov’s method, in Fig. 4, we present the goodness of fit based on the KS test as a function of the parameter $\mu$. Using this criterion, the optimal
value is $\mu \simeq 0.654$. In Fig. 5, we can see the optimal reconstruction of the empirical survival function.

4 Summary and conclusions

By assuming that the survival function of intertrade durations can be written as a mixture of exponential distributions, we have proposed two methods to reconstruct the activity spectrum. The first method is based on Tikhonov’s regularization. The second method uses an ansatz of intervals of constant activity. In this paper, we have not given any rigorous convergence proof and the methods outlined above were just heuristic.

The code used for this paper is available from [13] or it can be obtained from the authors. Unfortunately, for copyright reasons, we cannot publish the market data-set, but we can provide a full synthetic data-set based on the Mittag-Leffler function. More details about the methods will be available in a forthcoming paper as well as in the PhD thesis by Mauro Politi.

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References

[1] R. Engle and J. Russell, *Autoregressive conditional duration: A new model for irregularly spaced transaction data*, Econometrica **66**, 1127–1162, 1998.

[2] E. Scalas, R. Gorenflo, H. Luckock, F. Mainardi, M. Mantelli, and M. Raberto *Anomalous waiting times in high-frequency financial data*, Quantitative Finance, **4**, 695–702, 2004.

[3] E. Scalas, T. Kaizoji, M. Kirchler, J. Huber, and A. Tedeschi, *Waiting times between orders and trades in double auction markets*, Physica A, **366**, 463–471, 2006.

[4] E. Scalas, *Mixtures of compound Poisson processes as models of tick-by-tick financial data*, [http://arxiv.org/abs/physics/0608217](http://arxiv.org/abs/physics/0608217).

[5] V. V. Kryzhyi *Numerical inversion of the Laplace transform: analysis via regularized analytic continuation*, Inverse Problems **22**, 579-597, 2006.

[6] A. N. Tikhonov and V. Y. Arsenin, *Solutions of ill-posed problems*. Washington, D.C.: V.H. Winston and Sons, 1977.

[7] B. Hofmann, *Regularization for Applied Inverse and Ill-Posed Problems*, B.G. Teubner Leipzig, Teubner-Texte zur Mathematik, Bd. 85, Leipzig, 1986.

[8] C. W. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg, Braunschweig, 1993.

[9] G. Wahba, *Practical Approximate Solutions to Linear Operator Equations when the data are Noisy*, SIAM Journal on Numerical Analysis, **14**, 651–667, 1977.

[10] M. Hanke, *Limitations of the L-curve method in ill-posed problems*, BIT, **36**, 287–301, 1996.

[11] A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*. Springer Verlag, Wien and New York, pp. 223-276, 1997.

[12] T. J. Kozubowski and S. T. Ratchev *Univariate Geometric Stable Laws*. Journal of Computational Analysis and Applications, **1**, 177-217, 1999.

[13] [http://www.mfn.unipmn.it~scalas/colloquium06.html](http://www.mfn.unipmn.it~scalas/colloquium06.html)