Approximating Maximum Independent Set for Rectangles in the Plane

Joseph S. B. Mitchell*

July 7, 2021

Abstract

We give a polynomial-time constant-factor approximation algorithm for maximum independent set for (axis-aligned) rectangles in the plane. Using a polynomial-time algorithm, the best approximation factor previously known is $O(\log \log n)$. The results are based on a new form of recursive partitioning in the plane, in which faces that are constant-complexity and orthogonally convex are recursively partitioned into a constant number of such faces.

1 Introduction

Given a set $\mathcal{R} = \{R_1, \ldots, R_n\}$ of $n$ axis-aligned rectangles in the plane, the maximum independent set of rectangles (MISR) problem seeks a maximum-cardinality subset, $\mathcal{R}^* \subseteq \mathcal{R}$, of rectangles that are independent, meaning that any two rectangles of $\mathcal{R}^*$ are interior-disjoint.

The MISR is an NP-hard (13, 19) special case of the fundamental Maximum Independent Set (MIS) optimization problem, in which, for a given input graph, one seeks a maximum-cardinality subset of its vertices such that no two vertices are joined by an edge. The MIS is an optimization problem that is very difficult to approximate, in its general form; the best polynomial-time approximation algorithm is an $O(n/\log n)$-approximation [6], and there is no polynomial-time $n^{1-\delta}$-approximation algorithm, for any fixed $\delta > 0$, unless P=NP [25].

Because the MIS is such a challenging problem in its general setting, the MIS in special settings, particularly geometric settings, has been extensively studied. For string graphs (arcwise-connected sets), an $n^\varepsilon$-approximation algorithm is known [14]. For outerstring graphs, a polynomial-time exact algorithm is known [20], provided a geometric representation of the graph is given. For geometric objects that are “fat” (e.g., disks, squares), a Polynomial-Time Approximation Scheme (PTAS) is known [12]. Fatness is critical in many geometric approximation settings. The MISR is one of the most natural geometric settings for MIS in which the objects are not assumed to be fat – rectangles can have arbitrary aspect ratios. The MISR arises naturally in various applications, including resource allocation, data mining, and map labeling.

For the MISR, the celebrated results of Adamaszek, Har-Peled, and Wiese [1, 2, 3, 18] give a quasipolynomial-time approximation scheme, yielding a $(1-\varepsilon)$-approximation in time $n^{\text{poly}(\log n/\varepsilon)}$. Chuzhoy and Ene [9] give an improved, but still not polynomial, running time: they present a $(1-\varepsilon)$-approximation algorithm with a running time of $n^{O((\log \log n/\varepsilon)^*)}$. Recently, Grandoni, Kratsch, and Wiese [17] give a parameterized approximation scheme, which, for given $\varepsilon > 0$ and integer $k > 0$, takes time $f(k, \varepsilon)n^{g(\varepsilon)}$ and either outputs a solution of size $\geq k/(1+\varepsilon)$ or determines that the optimum solution has size $< k$.

Turning now to polynomial-time approximation algorithms for MISR, early algorithms achieved an approximation ratio of $O(\log n)$ [4, 21, 23] and $[\log_k n]$, for any fixed $k$ [6]. In a significant breakthrough, the $O(\log n)$ approximation bound was improved to $O(\log \log n)$ [8, 7], which has remained the best approximation factor during the last decade. Indeed, it has been a well known open problem to improve on

*Stony Brook University, Stony Brook, NY 11794-3600, joseph.mitchell@stonybrook.edu
this approximation factor: “even for the axis-parallel rectangles case currently has no constant factor approximation algorithm in polynomial time” [1], and “obtaining a PTAS, or even an efficient constant-factor approximation remains elusive for now” [9].

Our Contribution

We give a polynomial-time $O(1)$-approximation algorithm for MISR, improving substantially on the prior $O(\log \log n)$ approximation factor for a polynomial-time algorithm. The algorithm is based on a novel means of recursively decomposing the plane, into constant-complexity (orthogonally convex) “corner-clipped rectangles” (CCR), such that any set of disjoint (axis-aligned) rectangles has a constant-fraction subset that “nearly respects” the decomposition, in a manner made precise below. The fact that the CCRs are of constant complexity, and the partitioning recursively splits a CCR into a constant number of CCRs, allow us to give an algorithm, based on dynamic programming, that computes such a subdivision, while maximizing the size of the subset of $R$ of rectangles that are not penetrated (in their interiors) by the cuts. Together, the structure theorem and the dynamic program, imply the result.

Two key ideas are used in obtaining our result: (1) instead of a recursive partition based on a binary tree, partitioning each problem into two subproblems, we utilize a $K$-ary partition, with constant $K \leq 3$; (2) instead of subproblems based on rectangles (arising from axis-parallel cuts), we allow more general cuts, which are constant-complexity planar graphs having edges defined by axis-parallel segments, with resulting regions being orthogonally convex (CCR) faces of constant complexity. Proof of our structural theorem relies on a key idea to use maximal expansions of an input set of rectangles, enabling us to classify rectangles of an optimal solution in terms of their “nesting” properties with respect to abutting maximal rectangles. This sets up a charging scheme that can be applied to a subset of at least half of the maximal expansions of rectangles in an optimal solution: those portions of cuts that cross these rectangles can be accounted for by means of a charging scheme, allowing us to argue that a constant fraction subset of original rectangles exists that (nearly) respects a recursive partitioning into CCRs. These properties then enable a dynamic programming algorithm to achieve a constant factor approximation in polynomial time.

2 Preliminaries

Throughout this paper, when we refer to a rectangle we mean an axis-aligned, closed rectangle in the plane. We let $[x, x'] \times [y, y']$ denote the rectangle whose projection on the $x$-axis (resp., $y$-axis) is the (closed) interval $[x, x']$ (resp., $[y, y']$). We often speak of the sides (or edges) of a rectangle; we consider each rectangle to have four sides (top, bottom, left, right), each of which is a closed (horizontal or vertical) line segment. Two nonparallel sides of a rectangle meet at a common endpoint, one of the corners of the rectangle, often distinguished as being northeast, northwest, southwest, or southeast. When we speak of rectangles being disjoint, we mean that their interiors are disjoint. We say that two rectangles abut (or are in contact) if their interiors are disjoint, but there is an edge of one rectangle that has a nonzero-length overlap with an edge of the other rectangle. We say that a horizontal/vertical line segment $\sigma$ penetrates a rectangle $R \in \mathcal{R}$ if $\sigma$ intersects the interior of $R$. We say that a horizontal/vertical line segment $\sigma$ crosses another horizontal/vertical segment $\sigma'$ if the two segments intersect in a point that is interior to both segments; similarly, we say that $\sigma$ crosses a rectangle $R$ if the intersection $\sigma \cap R$ is a subsegment of $\sigma$ that is contained in the interior of $\sigma$.

The input to our problem is a set $\mathcal{R}$ of $n$ arbitrarily overlapping (axis-aligned) rectangles in the plane. Without loss of generality, we can assume that the $x$-coordinates and $y$-coordinates that define the rectangles $\mathcal{R}$ are distinct and are integral. Let $x_1, x_2, \ldots, x_{2n}$, with $x_1 < x_2 < \cdots < x_{2n}$, be the sorted $x$-coordinates of the left/right sides of the $n$ input rectangles $\mathcal{R}$; similarly, let $y_1 < y_2 < \cdots < y_{2n}$ be the $y$-coordinates of the top/bottom sides of the input. We let $BB(\mathcal{R}) = [x_1, x_{2n}] \times [y_1, y_{2n}]$ denote the axis-aligned bounding box (minimal enclosing rectangle) of the rectangles $\mathcal{R}$. The arrangement of the horizontal/vertical lines ($x = x_i$, $y = y_j$) that pass through the sides of the rectangles $\mathcal{R}$ determine a grid $\mathcal{G}$, with faces (rectangular grid cells), edges (line segments along the defining lines), and vertices (at points where the defining lines cross).
The nesting relationship among maximal rectangles. Consider a maximal rectangle, another rectangle of its interior, and each being maximal within the set of its four corners of containing exactly one of the boundaries. Following this process: Within the bounding box of another maximal rectangle, or with the boundary of another maximal rectangle, we extend each rectangle of the corresponding edge of the rectangles containing the corresponding rectangle. This results in a new set of independent rectangles within the bounding box of another maximal rectangle, or with the boundary of another maximal rectangle, obtained from a given rectangle by subtracting from it a set of up to four interior-disjoint rectangles, with each rectangle containing exactly one of the four corners of it in its interior. In an earlier version of this paper \[22\], we utilized the full generality of a CCR, having potentially all four corners clipped (resulting in a CCR with up to 12 sides). In this paper we employ a simplified case analysis, and we restrict ourselves to CCRs with at most one corner clipped, so our CCRs are “L-shaped” orthogonal polygons with 6 sides, 1 reflex vertex (of internal angle 270 degrees), and 5 convex vertices (each with internal angle 90 degrees). Refer to Figure 1. From now on, when we say “CCR” we mean a rectangle or an L-shaped CCR, with a single clipped corner.

Corner-Clipped Rectangles (CCRs). An orthogonal polygon is a simple polygon\[^1\] whose boundary consists of a finite set of axis-parallel segments. A corner-clipped rectangle is a special orthogonal polygon obtained from a given rectangle by subtracting from it a set of up to four interior-disjoint rectangles, with each rectangle containing exactly one of the four corners of it in its interior.

In the maximum independent set of rectangles (MISR) problem on we seek a subset, of maximum cardinality, such that the rectangles in are independent in that they have pairwise disjoint interiors. Our main result is a polynomial-time algorithm to compute an independent subset of of cardinality \(\Omega(k^*)\).

Maximal rectangles. Let \(I\) be an independent set of rectangles within the bounding box \(BB(\mathcal{R})\). We say that the set \(I\) is maximal within \(BB(\mathcal{R})\) if each rectangle \(R \in I\) has all four of its sides in nonzero-length contact with a side of another rectangle of \(I\), or with the boundary of \(BB(\mathcal{R})\). An arbitrary independent subset \(I = \{R_1, \ldots, R_k\} \subseteq R\) of input rectangles \(R_i\) can be converted into an independent set of rectangles, \(R'_i \supseteq R_i\), that are maximal within \(B\), in any of a number of ways, “expanding” each \(R_i\) appropriately. (Note that the super-rectangles, \(R'_i\) need not be elements of the input set \(R\).) To be specific, we define \(I' = \{R'_1, \ldots, R'_k\}\) to be the set of maximal expansions, \(R'_i \supseteq R_i\), of the rectangles \(R_i\), obtained from the following process: Within the bounding box \(B\), we extend each rectangle \(R_i\) upwards (in the +y direction) until it contacts another rectangle of \(I\) or contacts the top edge of \(BB(\mathcal{R})\). We then extend these rectangles leftwards, then downwards, and then rightwards, in a similar fashion. This results in a new set \(I'\) of independent rectangles within \(BB(\mathcal{R})\), with each rectangle \(R'_i\) containing the corresponding rectangle \(R_i\), and each being maximal within the set \(I'\) of rectangles: each \(R'_i\) is in contact, on all four sides, either with another rectangle of \(I'\), or with the boundary of \(BB(\mathcal{R})\). Note too that the sides of each \(R'_i\) lie on the grid \(G\) of horizontal/vertical lines through sides of the input rectangles \(R_i\); the coordinates defining the rectangles \(I'\) are a subset of the set of coordinates of the input rectangles \(R\). See Figure 2.

The nesting relationship among maximal rectangles. Consider a maximal rectangle, \(R'_i \in I'\). By definition of maximality, each of the four sides of \(R'_i\) is in contact with the boundary of \(BB(\mathcal{R})\) or with the boundary of another maximal rectangle, \(R'_j\). If \(R'_i\) and \(R'_j\) are in contact and the corresponding edge of \(R'_i\) lies within the interior of the corresponding edge of \(R'_j\), then we say that \(R'_i\) is nested on that side where

\[^1\]We consider rectangles and polygons to be closed regions that include their boundary and their interior.
Figure 2: Example of the maximal expansions $R'_i$ (in light red) of a set $I$ of independent rectangles $R_i$ (in magenta).

it contacts $R'_j$. For example, if $R'_i$ is nested on its top side, where it contacts the bottom side of $R'_j$, then the left side of $R'_j$ is strictly left of the left side of $R'_i$, and the right side of $R'_j$ is strictly right of the right side of $R'_i$. For a rectangle $R'_i$ that is in contact with the boundary of $BB(R)$, we say that $R'_i$ is nested on a side that is contained in the interior of a boundary edge of the bounding box $BB(R)$. We say that $R'_i$ is nested vertically if it is nested on its top side or its bottom side (or both); similarly, we say that $R'_i$ is nested horizontally if it is nested on its left side or its right side (or both). Refer to Figure 3. A simple observation that comes from the definition, and the fact that rectangles in $I'$ do not overlap, is the following:

Observation 1. For a set $I'$ of independent rectangles that are maximal within $BB(R)$, a rectangle $R'_i \in I'$ cannot be nested both vertically and horizontally.

Proof. Consider a rectangle $R'_i \in I'$ and assume that it is nested vertically, with its top side being a horizontal segment interior to the bottom side of $R'_j$. If $R'_i$ were also nested horizontally, abutting one of the dashed rectangles (of $I'$) to the left/right of $R'_i$, then we get a contradiction to the interior disjointness of rectangles, as a red point would be interior to two rectangles of $I'$.

Figure 3: Proof of Observation 1. If $R'_i$ is nested vertically, with its top side being contained in the interior of the bottom side of $R'_j$, then the red points, which are slightly shifted top corners of $R'_i$, must be interior to $R'_j$. However, if $R'_i$ were nested horizontally on its right side, then a red point would be interior to the abutting rectangle to the right (and a similar statement holds if $R'_i$ were to be nested on its left side.) This a contradiction to the interior disjointness of the rectangles of $I'$.

Proof. Consider a rectangle $R'_i \in I'$ and assume that it is nested vertically, with its top side being a horizontal segment interior to the bottom side of a rectangle $R'_j \in I'$ that abuts $R'_i$ to its north. Let $c_{NE} = (x_{max}, y_{max})$ be the northeast corner of $R'_i$; then, we know from the nesting relationship with $R'_j$, and the assumed integrality of coordinates defining the input rectangles, that the shifted corner point $(x_{max} + 1/2, y_{max} + 1/2)$ is interior to $R'_j$. Similarly, the shifted corner point $(x_{min} - 1/2, y_{max} + 1/2)$, just northwest of the northeast corner, $c_{NW}$, of $R'_i$ is interior to $R'_j$. However, if $R'_i$ were to be nested horizontally on its right side, then $(x_{max} + 1/2, y_{max} + 1/2)$ must be interior to the abutting rectangle to the right (and a similar statement holds if $R'_i$ were to be nested on its left side.) This a contradiction to the interior disjointness of the rectangles of $I'$.

CCR Partitions. We define a recursive partitioning of a rectangle $B \subseteq BB(R)$ as follows. At any stage of the decomposition, we have a partitioning of $B$ into polygonal faces (CCR faces), each of which is a CCR
(meaning a rectangle or an L-shaped CCR). (Initially, there is a single rectangular face, $B$.) A set, $\chi$, of axis-parallel line segments (called horizontal cut segments and vertical cut segments) on the grid $G$ that partitions a CCR face $f$ into two or more CCR faces is said to be a CCR-cut of $f$. We often use $\sigma$ to refer to a vertical cut segment of $\chi$, and we will consider $\sigma$ to be a maximal segment that is contained in the union of grid edges that make up $\chi$. A CCR-cut $\chi$ is $K$-ary if it partitions $f$ into at most $K$ CCR faces. Necessarily, a $K$-ary CCR-cut $\chi$ consists of $O(K)$ horizontal/vertical cut segments, since each of the resulting $K$ subfaces are (constant-complexity, orthogonal) CCR faces. By recursively making $K$-ary CCR-cuts, we obtain a $K$-ary hierarchical partitioning of $B$ into CCR faces. We refer to the resulting (finite) recursive partitioning as a $K$-ary CCR-partition of $B$. Associated with a CCR-partition is a partition tree, whose nodes correspond to the CCR faces, at each stage of the recursive partitioning, and whose edges connect a node for a face $f$ to each of the nodes corresponding to the subfaces of $f$ into which $f$ is partitioned by a $K$-ary CCR-cut. The root of the tree corresponds to $B$; at each level of the tree, a subset of the nodes at that level have their corresponding CCR faces partitioned by a CCR-cut, into at most $K$ subfaces that are children of the respective nodes. The CCR faces that correspond to leaves of the partition tree are called leaf faces. We say that a CCR-partition is perfect with respect to a set of rectangles if none of the rectangles have their interiors penetrated by the CCR-cuts of the CCR-partition, and each leaf face of the CCR-partition contains exactly one rectangle. (This terminology parallels the related notion of a “perfect BSP”, mentioned below.) We say that a CCR-partition is nearly perfect with respect to a set of rectangles if for every CCR-cut in the hierarchical partitioning each of the $O(1)$ cut segments of the CCR-cut penetrates at most 2 rectangles of the set, and each leaf face of the CCR-partition contains at most one rectangle. (Having this property will enable a dynamic programming algorithm to optimize the size of a set of rectangles, over all nearly perfect partitions of the set, since there is only a constant amount of information to store with each subproblem defined by a CCR face in the CCR-partition.)

Comparison with BSPs. A binary space partition (BSP) is a recursive partitioning of a set of objects using hyperplanes (lines in 2D); a BSP is said to be perfect if none of the objects are cut by the hyperplanes defining the partitioning. An orthogonal BSP uses cuts that are orthogonal to the coordinate axes. There are sets of axis-aligned rectangular objects for which there is no perfect orthogonal BSP; see Figure. A CCR-partition is a generalization of an orthogonal binary space partition (BSP), allowing more general shape faces (and cuts) in the recursive partitioning, and allowing more than 2 (but still a constant number of) children of internal nodes in the partition tree.

Conjecture 1. [Pach-Tardos] For any set of $n$ interior-disjoint axis-aligned rectangles in the plane, there exists a subset of size $\Omega(n)$ that has a perfect orthogonal BSP.
Figure 5: Left: A set of (gray) rectangles and a BSP that respects it. Middle: A set of (gray) rectangles that does not have a perfect BSP (but does have a perfect CCR-partition). Right: A perfect BSP partition of 3/4 of the rectangles in the middle figure (with 1/4 of the rectangles, in white, removed).

3 The Structure Theorem

Our main structure theorem states that for any set of \( k \) interior disjoint rectangles in the plane there is a constant fraction size subset with respect to which there exists a nearly perfect CCR-partition. We state the theorem here and we utilize it for the algorithm in the next section; the proof of the structure theorem is given in Section 5.

While Conjecture 1 remains open, and if true would imply a constant-factor approximation for MISR (via dynamic programming), our structural result suffices for obtaining a constant factor approximation (Theorem 4.1), which is based on optimizing over recursively defined CCR-partitions using dynamic programming.

**Theorem 3.1.** For any set \( I = \{R_1, \ldots, R_k\} \) of \( k \) interior disjoint (axis-aligned) rectangles in the plane within a bounding box \( B \), there exists a \( K \)-ary CCR-partition of the bounding box \( B \), with \( K \leq 3 \), recursively cutting \( B \) into rectangles and (L-shaped) corner-clipped rectangles (CCRs), such that the CCR-partition is nearly perfect with respect to a subset of \( I \) of size \( \Omega(k) \).

4 The Algorithm

The algorithm utilizes dynamic programming. A subproblem \( S \) of the DP is specified by giving (1) a CCR \( Q \) whose edges lie on the grid \( G \) defined by the set of vertical/horizontal lines through the sides of the input rectangles, and (2) a set \( I_S \subseteq R \) of \( O(1) \) input rectangles (these are “special” rectangles that we explicitly specify) that are penetrated by some of the edges of \( Q \), at most 2 per edge. Since \( Q \) is either a rectangle or an L-shaped CCR, \( Q \) has at most 6 edges/vertices, and we know that \( |I_S| \leq 12 \) (a more careful count that exploits the structure of the cuts used in our CCR-partitions yields \( |I_S| \leq 6 \)) and that there are only a polynomial number of possible CCRs.

For a subproblem \( S = (Q, I_S) \), let \( R(S) \subseteq R \) denote the subset of the input rectangles that lie within \( Q \) and are pairwise disjoint from the specified rectangles \( I_S \). Let \( f(S) \) denote the cardinality of a maximum-cardinality subset, \( I^*(S) \), of \( R(S) \) that is independent and for which there exists a CCR-partition of \( Q \) that is nearly perfect with respect to \( I^*(S) \).

The objective for the subproblem \( S \) specified by \( (Q, I_S) \) is to compute a CCR-partitioning of \( Q \) to maximize the cardinality of the subset of \( R(S) \) with respect to which there is a CCR-partitioning of \( Q \) that is nearly perfect.

The optimization is done by iterating over all candidate \( K \)-ary (\( K \leq 3 \)) CCR-cuts of \( Q \), together with choices of special rectangles associated with the vertical cut segments of a CCR-cut, resulting in the following recursion:

\[
f(S) = \begin{cases} 
0 & \text{if } R(S) = \emptyset, \\
\max_{X \in \gamma(S), I_X} \left( f(S_1) + \cdots + f(S_K) + |I_X| \right) & \text{otherwise}, 
\end{cases}
\]
where the optimization is over the set, $\gamma(S)$ of eligible CCR-cuts within the grid $\mathcal{G}$ that partition $Q$ into $K$ CCRs ($2 \leq K \leq 3$) and over the choices of the set $I_h \subseteq I$ of $O(1)$ special rectangles, with at most 2 rectangles of $I_h$ penetrated by each cut segment of $\chi$. The cut $\chi \in \gamma(S)$, together with the choice of $I_h$, results in a set of $K$ subproblems $S_1, \ldots, S_K$, each specified by a CCR, together with a set of specified special rectangles (a subset of the $O(1)$ rectangles of $I_h$). Since the eligible cuts are of constant complexity within the grid of $O(n)$ vertical/horizontal lines, $|\gamma(S)|$ is of polynomial size. Also, there are a polynomial number of choices for the set $I_h$, since each set is of constant size. Thus, the algorithm runs in polynomial time.

The desired solution is given by evaluating $f(S_0)$, on the subproblem $S_0 = (BB(R), R)$ defined by the full bounding box, $BB(R)$, of the input set $R$ of rectangles.

The proof of correctness is based on mathematical induction on the number of input rectangles of $R$ that are contained in subproblem $S$. The base case is determined by subproblems $S$ for which there is no rectangle of $R$ inside the CCR; we correctly set $f(S) = 0$ for such subproblems. With the induction hypothesis that $f(S)$ is correctly computed for subproblems $S$ containing at most $k$ rectangles of $R$, we see that, for a subproblem $S$ having $k + 1$ rectangles of $R$, since we optimize over all CCR-cuts partitioning $Q$ into at most 3 CCRs, respecting $I_S$, and over all choices of $O(1)$ special rectangles penetrated by each considered CCR-cut, with each of the values $f(S_i)$ being correctly computed (by the induction hypothesis), we obtain a correct value $f(S)$. Since the algorithm runs in polynomial time, and our structure theorem shows that there exists an independent subset of $R$ of size at least $\Omega(k^*)$ for which there is a CCR-partition respecting this subset, we obtain our main result:

**Theorem 4.1.** There is a polynomial-time $O(1)$-approximation algorithm for maximum independent set for a set of axis-aligned rectangles in the plane.

**Remark.** While efficiency is not the focus of our attention here, other than to establish polynomial-time, we comment briefly on the overall running time of the dynamic program. A naive upper bound comes from the fact that there are at most $O(n^6)$ L-shaped CCRs defined on the set of vertical/horizontal lines through sides of the input rectangles, and there are at most $O(n^6)$ specified rectangles $I_S$ in any subproblem. The most complex cut $\chi$ among the cases in the proof of Lemma 5.2 is seen to have 3 horizontal cut segments and 1 vertical cut segment, with at most 2 associated specified rectangles along the vertical cut segment and at most additional specified rectangle along each of the horizontal cut segments; thus, there are at most $O(n^9)$ choices for $(\chi, I_h)$. Thus, the optimization is over at most $O(n^9)$ choices, for each of at most $O(n^{12})$ subproblems. The overall time bound of $O(n^{21})$ is surely an over-estimate.

## 5 Proof of the Structure Theorem

In this section we prove the main structure theorem, Theorem 3.1, which states that for any set $I = \{R_1, \ldots, R_k\}$ of $k$ interior disjoint (axis-aligned) rectangles in the plane within a bounding box $B$, there exists a $K$-ary CCR-partition of the bounding box $B$, with $K \leq 3$, recursively cutting $B$ into corner-clipped rectangles (CCRs), such that the CCR-partition is nearly perfect with respect to a subset of $I$ of size $\Omega(k)$ (specifically, we show size at least $k/4$).

First, we consider the set $I'$ of maximal expansions, as described previously, of the $k$ input rectangles $I = \{R_1, \ldots, R_k\}$ within $B$. Recall that, appealing to Observation 1, the set $I'$ is the disjoint union of subsets: $I' = I_h \cup I_v \cup I_o$, where $I_h$ is the subset of “red” rectangles that are nested horizontally, $I_v$ is the subset of “blue” rectangles that are nested vertically, and $I_o$ is the subset of “gray” rectangles that are not nested on any of their four sides. Refer to Figure 4. It cannot be that both $|I_h|$ and $|I_v|$ are greater than $k/2$; thus, assume, without loss of generality, that $|I_h| \leq k/2$. Then, there are $|I_v| + |I_o| \geq k/2$ “non-red” rectangles.

---

2Throughout the discussion going forward in this section, the assumption that the number of red (maximal) rectangles is at most $k/2$ will permeate, leading to apparent “bias” in places where we construct “fence” segments that are horizontal, require constraints on vertical cut segments in Lemma 5.2, focus on charging off rectangles that are crossed by vertical cut segments, while horizontal cut segments never cross rectangles of $I'$, etc; in all of these places, there is a symmetric statement corresponding to the case in which, instead, the number of blue (maximal) rectangles is at most $k/2$, and there are at least $k/2$ “non-blue” rectangles.
We now describe a process by which a subset of $\Omega(k)$ of the rectangles of $I'$ is chosen, and a CCR-partition, $\Pi$, is constructed that is nearly perfect with respect to this subset, and, thus, by the following claim, is nearly perfect with respect to the corresponding subset of (sub-)rectangles $I$.

Claim 5.1. If a CCR-partition is nearly perfect with respect to a set $A' \subset I'$ of maximal rectangles, then it is nearly perfect with respect to the corresponding set $A \subset I$ of input rectangles.

Proof. This follows immediately from the fact that, for each $R' \in I'$, the (unique) corresponding $R \in I$ is a subrectangle of $R'$.

Initially, all rectangles of $I'$ are active. During the process of recursively partitioning $BB(\mathcal{R})$ into a CCR-partition, a subset of the rectangles of $I'$ will be discarded and removed from active status. In the end, the set of remaining active rectangles of $I'$ have the property that the CCR-partition is nearly perfect with respect to the remaining active rectangles. We show (Claim 5.8), via a charging scheme, that at most a constant fraction ($3/4$) of the rectangles of $I'$ are discarded, implying that the final set of active rectangles of $I'$ has cardinality at least $k/4 = |I'|/4$.

At each stage in the recursive partitioning, a CCR $Q$ that contains more than one rectangle of $I'$ (and thus more than one rectangle of $I$) is partitioned via a cut $\chi$, within the grid $\mathcal{G}$, into at most $K = 3$ CCR faces, via a set of $O(1)$ horizontal and vertical cut segments. Details of how we determine these CCR-cuts are presented in the statement and proof of Lemma 5.2 below, after we introduce the notion of “fences”.

A property of the cuts $\chi$ we construct is that no rectangle of $I'$ is crossed by a horizontal cut segment (Lemma 5.2 (iii), together with the property that the horizontal segments (“fences”) that serve as input to the lemma are defined to have the property that they do not cross any of the rectangles of $I'$); horizontal cut segments can follow the boundary of a rectangle of $I'$, and might penetrate up to two rectangles of $I'$ (namely, the rectangle(s) that contain each of the two endpoints of the segment), but cannot cross any such rectangle.

A vertical cut segment, $\sigma$, of $\chi$ can penetrate and also cross some rectangles of $I'$: we discard those rectangles that are crossed by such a cut segment $\sigma$. On any vertical cut segment at most two rectangles (the rectangle(s) containing each of the two endpoints of $\sigma$) can be penetrated but not crossed; these rectangles will not be discarded and are the source of the use of “nearly perfect” cuts in our method. The ability to find such cuts $\chi$ is assured by Lemma 5.2. Those rectangles that are crossed by a vertical cut segment and discarded must be accounted for, to assure that a significant fraction remain in the end. This is where a charging scheme is utilized, which we describe below, after we introduce fences.

Fences. An important aspect of the method we will describe for construction of a CCR-partition that respects a large subset of $I'$ is the enforcement of constraints that prevent the cuts we produce from crossing rectangles of $I'$ without being able to “pay” for the crossing, through our charging scheme (below). We achieve this by establishing certain fences, which are horizontal line segments, on the grid $\mathcal{G}$, each “anchored” on a left/right side of the current CCR face, $Q$. The fences become constraints on CCR-cuts of $Q$, as we require that a CCR-cut $\chi$ have no vertical cut segments that cross any fence (and all horizontal cut segments will be contained within the set of fence segments); refer to Lemma 5.2 below.

For each of the 4 corners, $c_{NE}, c_{NW}, c_{SW}, c_{SE}$, of a rectangle $R \in I'$, we let the corresponding shifted corner (denoted $c'_{NE}, c'_{NW}, c'_{SW}, c'_{SE}$) be the point interior to $R$, shifted towards the interior of $R$ by $1/2$ in both $x$- and $y$-coordinates (recall that the defining coordinates of the input rectangles are assumed, without loss of generality, to be integers); for example, if $c_{NE} = (x, y)$, then $c'_{NE} = (x - 1/2, y - 1/2)$. (There is nothing special about the number $1/2$; any real number strictly between 0 and 1 could be used to define shifted corners for the rectangles, which have integer coordinates.)

We say that a corner, $c$, of a rectangle $R \in I'$, is exposed to the right within $Q$ if a horizontal rightwards ray, $\rho$, with endpoint at its corresponding shifted corner, $c'$, does not penetrate any rectangle of $I'$ (other than $R$) that lies fully within $Q$ before reaching the right boundary of $Q$. If a corner, $c$, of rectangle $R$ is not exposed to the right within $Q$, then the rightwards ray $\rho$ penetrates at least one rectangle that lies fully within $Q$ before reaching the right boundary of $Q$: the first such rectangle, $R'$, that is penetrated by $\rho$ after $\rho$ exits $R$ is called the right-neighbor of $R$ at $c$. We similarly define the notion of a corner $c$ being exposed to
the left within $Q$ and the notion of a left-neighbor of $R$ at $c$ (in the case that the corner $c$ is not exposed to the left). Note that, from the definition, the northeast corner of $R$ is exposed to the right (resp., left) if and only if the northwest corner of $R$ is exposed to the right (resp., left); a similar statement holds for southeast and southwest corners. Thus, we say that the top side of $R$ is exposed to the left (resp., right) if the northeast and northwest corners of $R$ are exposed to the left (resp., right). If the top side of $R$ is not exposed to the left (resp., right), then we speak of the top-left-neighbor of $R$ (resp., top-right-neighbor of $R$), which is the left-neighbor (resp., right-neighbor) of $R$ at a top corner of $R$. Similarly, we speak of a bottom side of $R$ being exposed to the left/right, and, if not exposed, having a bottom-left-neighbor or bottom-right-neighbor. We use the term left-neighbor of a rectangle to refer to a top-left-neighbor or a bottom-left-neighbor; we similarly use the term right-neighbor.

When a CCR $Q$ is partitioned by a CCR-cut, $\chi$, a vertical cut segment, $\sigma$, of $\chi$ may cause some top/bottom sides of other rectangles of $I'$ within the new CCR faces (subfaces of $Q$ on either side of $\sigma$) to become exposed to the left or to the right within their respective new CCR faces: a leftwards/rightwards ray (from a shifted corner of some rectangle) that previously penetrated a rectangle of $I'$ (within $Q$) now reaches $\sigma$, a new vertical boundary of a subface, before the penetration. When a vertical cut segment $\sigma$ of a CCR-cut causes the top/bottom side of a rectangle $R \in I'$ that lies fully within a subface to become exposed to its left (resp., right) within its new CCR face, we establish a fence (horizontal line segment) that contains the top/bottom side of $R$ and extends to the left (resp., right) side of the corresponding right-neighbor (resp., left-neighbor) of the rectangle $R$.

Figure 6: Examples showing the establishment of fences after a cut along vertical segment $\sigma$ causes the northwest corner (left and right cases) or the southwest corner (middle case) of $R$ to become exposed to the left. The newly established fence segment is shown in red, extending from $\sigma$ to the right side of $R_r$, the top-right-neighbor (left and right cases) or bottom-right-neighbor (middle case).

Note that, by its definition, a fence segment does not cross any rectangle of $I'$; it may penetrate a (left/right-neighbor) rectangle of $R \in I'$ near one of the two ends of the fence segment. (Since an input rectangle can lie within the interior of its corresponding maximal rectangle in $I'$, this means that the fence segment may cross an input rectangle that is contained within one of the at most two penetrated maximal rectangles of $I'$.) Refer to Figure 7, we use the convention that fences are shown in red (resp., blue) if anchored on their left (resp., right) endpoint. (This color choice of red/blue for fences has nothing to do with the “red” and “blue” color terminology we had above for rectangles in the set $I_h$ and $I_v$.) As cuts are made during the process, we establish fences in order to maintain the following invariant during the course of the recursive partitioning:

---

In an earlier version of this paper [22], the fence segment extended only to the right (or left) side of the newly exposed rectangle $R$, rather than continuing to the right (resp., left) side of the corresponding right-neighbor (resp., left-neighbor) of $R$. By extending the fence through the left/right-neighbor, we are able to ensure (Claim 5.6) that no rectangle has both its northwest and northeast corners charged (or both the southwest and southeast corners), as the extended fence ensures that the left/right-neighbor is not crossed by a cut.
For each rectangle $R \in I'$ that lies fully within a CCR $Q$, if the top/bottom side of $R$ is exposed to the left (resp., right) within $Q$, then there is a (horizontal) fence segment $\alpha$ (resp., $\beta$) that includes the top/bottom side of $R$ and extends leftwards to the left side of $Q$ and rightwards to the right side of the top/bottom-right-neighbor of $R$ (resp., extends rightwards to the right side of $Q$ and leftwards to the left side of the top/bottom-left-neighbor of $R$).

We say that a rectangle $R \in I'$, with $R$ fully contained within a CCR $Q$, is grounded if its top or bottom side is contained within the top or bottom boundary of $Q$ or within a fence segment within $Q$, or if $R$ is penetrated (but not crossed) by a fence segment that extends from one vertical side of $R$ to the other. Lemma 5.2 guarantees that the CCR-cuts we utilize do not have any vertical cut segments $\sigma$ that cross any fences and, therefore, do not cross any grounded rectangles (since a vertical segment that crosses a rectangle necessarily crosses its top and bottom sides and any horizontal segment that penetrates the rectangle, extending from one vertical side to the other). It can be, however, that a vertical cut segment $\sigma$ penetrates (without crossing) a grounded rectangle, e.g., if the segment $\sigma$ terminates on a fence or on

Figure 7: Example showing the establishment of fences after a cut. In this figure, the CCR is a rectangle, the maximal rectangles are shown, with red ones (nested to the left or to the right) shown in light red, and all non-red rectangles shown in gray. A vertical cut segment $\sigma$ (in green) is shown. The cut results in the top/bottom sides of some rectangles becoming exposed; we show the respective shifted corners as blue/red hollow dots, with dark green arrows showing the exposure to the cut $\sigma$, as well as showing the additional penetrated neighboring rectangle, which defines the extent of the fence segment. Rectangles that are crossed by the cut are marked with an “X”: (Note that the first and last rectangles intersected by $\sigma$ are penetrated but not crossed by $\sigma$.) All of the newly established fences, anchored on $\sigma$, are shown in red (if the left endpoint is anchored on $\sigma$) or in blue (if the right endpoint is anchored on $\sigma$).
a top/bottom side of \(Q\); however, a vertical cut segment \(\sigma\) can penetrate, without crossing, at most two grounded rectangles, one containing the top endpoint of \(\sigma\) and one containing the bottom endpoint of \(\sigma\). (This is the source of the “2” in the definition of a nearly perfect cut.) Each rectangle \(R \in I'\) is a maximal expansion of an original input rectangle of \(I\); since the original rectangle is a subrectangle of \(R\), it can happen that a vertical cut segment \(\sigma\) crosses the original rectangle (or it could completely miss intersecting it) associated with a grounded rectangle \(R \in I'\), while not crossing, but only penetrating, \(R\).

Lemma 5.2 below shows that the recursive partitioning process is feasible, showing that for any CCR face \(Q\) and any set of (horizontal) fences anchored on the left/right sides of \(Q\), it is always possible to find a CCR-cut with the properties claimed in the lemma; in particular, the cut should “respect” the given set of fences, in that no vertical cut segment of the cut crosses a fence segment. We initialize the set of fences in order that the Fence Invariant holds at the beginning of the process: for each rectangle \(R \in I'\) whose top/bottom side is exposed to the left within \(B\), we establish a fence along its top/bottom side, anchored on the left side of \(B\), extending rightwards to the right side of the (top/bottom)-right-neighbor of \(R\). We do similarly for rectangles whose top/bottom side is exposed to the right within \(B\). See Figure 8. (If the top/bottom side of \(R\) is contained within the top/bottom side of \(B\), we need not establish a fence (though we do show them in the figure), since the fence segment would be contained in the top/bottom side of \(B\).) If there exists a rectangle \(R \in I'\) whose top/bottom side is exposed both to the left and to the right within \(B\), then we simply cut \(B\) with a horizontal cut segment along the top/bottom side of \(R\), from the left side of \(B\) to the right side of \(B\), since this horizontal segment penetrates no rectangles of \(I'\).

![Figure 8: The initial fences, shown as bold red/blue segments, for the shown set of maximal rectangles within a rectangle B.](image)

We appeal to the following technical lemma, whose proof is based on a simple enumeration of cases, given in the appendix.

**Lemma 5.2.** Let \(Q\) be a CCR whose edges lie on a grid lines of \(G\). Let \(\{\alpha_1, \ldots, \alpha_{k_\alpha}\}\) be a set of \(k_\alpha \geq 0\) “red” horizontal anchored (grid) segments within \(Q\), that are anchored with left endpoints on the left side of \(Q\), and let \(\{\beta_1, \ldots, \beta_{k_\beta}\}\) be a set of \(k_\beta \geq 0\) “blue” horizontal anchored (grid) segments within \(Q\), that are anchored with right endpoints on the right side of \(Q\). Then, assuming that \(Q\) contains at least two grid cells (faces of \(G\)), there exists a CCR-cut \(\chi\) with the following properties:

(i) \(\chi\) partitions \(Q\) into \(O(1)\) (at most 3) CCR faces;

(ii) \(\chi\) is comprised of \(O(1)\) horizontal/vertical segments on the grid \(G\), with endpoints on the grid;

(iii) horizontal cut segments of \(\chi\) are a subset of the given red/blue anchored segments;

(iv) vertical cut segments of \(\chi\) do not cross any of the given red/blue anchored segments;

(v) there is at most 1 vertical cut segment of \(\chi\).
The lemma shows that a CCR, \( Q \), with anchored (grid) segments has a CCR-cut, on the grid, partitioning \( Q \) into at most 3 CCRs (CCR faces in the grid). Applying this recursively (and finitely, due to the finiteness of the grid), until there is at most one rectangle of \( I' \) within each CCR face, we will be able to conclude, based on the charging scheme below, the proof of the claimed structure theorem.

**Remark.** There is an equivalent “rotated” version of Lemma 5.2 that applies to a set of vertical (fence) segments anchored on the top/bottom sides of \( Q \), with vertical cut segments lying along the anchored segments, and at most 1 horizontal cut segment not crossing any anchored (vertical) segments. The rotated version applies in our proof of the structure theorem in the case that the non-blue maximal rectangles have cardinality at least \( k/2 \).

**Charging Scheme.** Consider a vertical cut segment, \( \sigma \), that is part of a CCR-cut \( \chi \) of a CCR \( Q \), given by Lemma 5.2. The segment \( \sigma \) penetrates some (possibly empty) subset, \( I'_\sigma \), of rectangles of \( I' \) that lie within \( Q \); the intersection of \( \sigma \) with the rectangles \( I'_\sigma \) is a set of (interior-disjoint) subsegments of \( \sigma \), at most two of which share an endpoint with \( \sigma \) and are thus not contained within the interior of \( \sigma \). Thus, at most two of the rectangles of \( I'_\sigma \) are not crossed by \( \sigma \).

**Claim 5.3.** If a rectangle of \( I'_\sigma \) is crossed by \( \sigma \), then neither its top nor its bottom is exposed to the left or right within \( Q \).

**Proof.** This is immediate from the Fence Invariant, and the fact (Lemma 5.2 part (iv)) that vertical cut segments do not cross fences.

Our charging scheme assigns (“charges off”) each non-red rectangle \( R \) that is crossed by a vertical cut segment \( \sigma \) to two corners of rectangles of \( I' \) (one corner to the left of \( R \), one corner to the right of \( R \)), each receiving 1/2 unit of charge. If the rectangle \( R \) is red, we do not attempt to charge it off, as our charging scheme is based on charging to corners of rectangles that lie to the left/right of \( R \) (within the vertical extent of \( R \)), and a red rectangle is, by definition, nesting on its left or on its right (or both), implying that we will not have available the corners we need to be able to charge. Instead, our accounting scheme below will take advantage of the fact that the number of non-red rectangles is at least \( k/2 \).

Consider a non-red rectangle \( R \in I' \) within \( Q \) that is crossed by a vertical cut segment \( \sigma \). By property (v) of Lemma 5.2 there are no other vertical cut segments of \( \chi \). We charge the crossing of \( R \) to 2 corners, each receiving 1/2 unit of charge to “pay” for the crossing of \( R \): we charge 1/2 to a corner of a rectangle of \( I' \) that lies to the right of \( R \) (within the vertical extent of \( R \)), and 1/2 to a corner of a rectangle of \( I' \) that lies to the left of \( R \) (within the vertical extent of \( R \)).

We describe and illustrate the assignment of charge to a corner that lies to the right of \( R \): the assignment to a corner that lies to the left of \( R \) is done symmetrically. Specifically, let \( c \) be the northeast corner of \( R \), and let \( c' \) be the corresponding shifted corner (slightly to the southwest of \( c \)). By Claim 5.3 \( c \) is not exposed to the right within \( Q \); thus, the rightwards horizontal ray from \( c' \) must penetrate at least one rectangle of \( I' \) that is fully within \( Q \) before reaching the right boundary of \( Q \). Let \( R_r \) be the first (leftmost) rectangle that is penetrated (\( R_r \) is the top-right-neighbor of \( R \)). Let \( y^+ \) and \( y^- \) (resp., \( y^+_r \) and \( y^-_r \)) be the top and bottom y-coordinates of rectangle \( R \) (resp., \( R_r \)). Similarly, let \( x^+, x^-, x^+_r \), and \( x^-_r \) denote the x-coordinates of the left/right sides of \( R \) and \( R_r \), so that \( R = [x^-, x^+] \times [y^-, y^+] \) and \( R_r = [x^-_r, x^+_r] \times [y^-_r, y^+_r] \). By definition of \( R_r \), we know that \( y^+_r \geq y^+ \) and that \( x^+ \leq x^-_r \). Refer to Figure 9 where we illustrate the charging scheme (for charging to the right of \( R \)), enumerating the 8 potential cases, depending if (i) \( y^+ = y^+_r \) or \( y^+ < y^+_r \), (ii) \( y^- > y^-_r \) or \( y^- \leq y^-_r \), (iii) \( x^+ = x^-_r \) or \( x^+ < x^-_r \). We itemize the 8 cases below:

1. \([y^+ = y^+_r, y^- > y^-_r, \text{and } x^+ = x^-_r]\) We charge the northwest corner, \( c_r \), of rectangle \( R_r \).

After the cut along \( \sigma \), rectangle \( R_r \) will have its top side exposed to the left, so, to maintain the Fence Invariant, we establish a horizontal fence segment (shown in red), anchored on \( \sigma \), passing through the top side of \( R_r \), and extending rightwards to the right side of the top-right-neighbor, \( R'_r \), of \( R_r \). (We know that the top side of \( R_r \) is not exposed to the right side of \( Q \), since, if it were exposed, then, by the Fence Invariant, at the moment of first exposure there would be a fence extending along the top side of \( R_r \), and leftwards through \( R \) (the top-left-neighbor of \( R_r \)), to the left side of \( R \), implying that
Figure 9: Case analysis for the charging scheme when a non-red rectangle $R$ is crossed by a vertical portion (shown dashed), $\sigma$, of a cut $\chi$. The charged corner is labelled $c_r$, for the possible cases (1), (3), (5), (6). The fences established after the cut are shown in red. Here, we only show the case of charging to the right of the crossed rectangle $R$, assigning a charge of $1/2$ to an appropriately selected corner; symmetrically, we also charge $1/2$ to a corner that is to the left of $R$. 

this fence segment is crossed by $\sigma$, a contradiction.) This fence will become exposed to the left (at a point along $\sigma$) after the cut along $\sigma$. In the figure, we show the case in which $R'_r$ is abutting $R_r$; the case in which there is a gap between $R_r$ and its top-right-neighbor $R'_r$ is similar.

(2) $[y^+ = y_r^+, y^- > y_r^-, \text{ and } x^+ < x_r^-.]$ In this case, there is a rectangular region (shown in gray in the figure) separating the right side of $R$ from the left side of $R_r$. This gray rectangle must intersect the interior of at least one rectangle of $I'$; otherwise, $R$ is not maximal, as it can be extended to the right, until it abuts $R_r$. Thus, there must be some rectangle in $I'$ whose interior overlaps with the interior of the gray rectangle; let $W \in I'$ be such a rectangle whose top side has the maximum $y$-coordinate among all such rectangles. Then, we get a contradiction as follows: If the top side of $W$ lies at or above the top sides of $R$ and $R_r$, then the rightwards ray from $c'$ penetrates $W$ before penetrating $R_r$, contradicting the definition of $R_r$; on the other hand, if the top side of $W$ lies below the rightwards
ray from \( c' \), then we get a contradiction to the maximality of \( W \), since it can be extended upwards, to the top of the gray rectangle. Thus, this case cannot happen.

(3) \( [y^+ = y^+_r, \ y^- \leq y^-_r, \text{ and } x^+ = x^-_r.] \) This is handled exactly as in case (1): we charge the northwest corner \( c_r \) and establish the fence segment (shown in red), through the top side of \( R_r \), extending to the right side of the top-right-neighbor, \( R'_r \).

(4) \( [y^+ = y^+_r, \ y^- \leq y^-_r, \text{ and } x^+ < x^-_r.] \) Exactly as in case (2), this case cannot happen.

(5) \( [y^+ < y^+_r, \ y^- \leq y^-_r, \text{ and } x^+ = x^-_r.] \) We charge the southwest corner, \( c_r \), of rectangle \( R_r \). After the cut along \( \sigma \), rectangle \( R_r \) will have its bottom side exposed to the left, so, to maintain the Fence Invariant, we establish a horizontal fence segment (shown in red), anchored on \( \sigma \), passing through the bottom side of \( R_r \), and extending rightwards to the right side of the bottom-right-neighbor, \( R'_r \), of \( R_r \). (We know that the bottom side of \( R_r \) is not exposed to the right side of \( Q \), since, if it were exposed, then, by the Fence Invariant, at the moment of first exposure there would be a fence extending along the bottom side of \( R_r \), and leftwards through \( R \) (the bottom-left-neighbor of \( R_r \)), to the left side of \( R \), implying that this fence segment is crossed by \( \sigma \), a contradiction.)

(6) \( [y^+ < y^+_r, \ y^- \leq y^-_r, \text{ and } x^+ < x^-_r.] \) We charge the southwest corner, \( c_r \), of rectangle \( R_r \). Consider the gray rectangle, \( [x^+, x^-] \times [y^-, y^-] \); we claim that there can be no rectangle \( W \in I' \) whose interior overlaps with this gray rectangle, for, if such a rectangle existed, then one such rectangle, \( W \), that maximizes the \( y \)-coordinate of the top side fails to be maximal. Thus, no rectangle fully within the subface to the right of \( \sigma \) is penetrated by the leftwards ray from \( c'_r \), before it reaches \( \sigma \), on the left side of the subface. Thus, after the cut along \( \sigma \), rectangle \( R_r \) will have its bottom side exposed to the left. In order to maintain the Fence Invariant, we establish a horizontal fence segment (shown in red), anchored on \( \sigma \), passing through the bottom side of \( R_r \), and extending rightwards to the right side of the bottom-right-neighbor, \( R'_r \), of \( R_r \). (As in case (5), we know that there must be a bottom-right-neighbor, since the bottom side of \( R_r \) cannot have been exposed to the right, as this would imply the existence of a fence across \( R \), contradicting the fact that the cut segment \( \sigma \) crosses no fence segment.) led to a fence that

(7) \( [y^+ < y^+_r, \ y^- > y^-_r, \text{ and } x^+ = x^-_r.] \) We get a contradiction to the fact that \( R \) is non-red, since the inequalities \( y^+ < y^+_r \) and \( y^- > y^-_r \) say that \( R \) is nesting on its right. Thus, this case cannot happen.

(8) \( [y^+ < y^+_r, \ y^- > y^-_r, \text{ and } x^+ < x^-_r.] \) By reasoning exactly as in case (2), this case cannot happen.

Summarizing the above case enumeration of the charging scheme, we state the following facts:

**Claim 5.4.** A corner of a rectangle is charged at most once (with a charge of 1/2).

**Proof.** When a corner \( c_r \) is charged, as illustrated in the case enumeration of Figure 2 (cases (1), (3), (5), (6)), the corner \( c_r \) becomes exposed to the left and a horizontal fence segment is established (according to the Fence Invariant), extending from the segment \( \sigma \) along the top/bottom edge of the rectangle whose corner \( c_r \) is being charged, and rightwards to the right side of the top/bottom-right-neighbor of that rectangle. Once exposed to the left, a corner remains exposed to the left in the CCR face containing it, for the remainder of the process of recursive partitioning.

**Claim 5.5.** If a corner, \( c_r \), of rectangle \( R_r \in I' \) is charged, then \( R_r \) has not previously been crossed by a vertical cut segment, is not currently (as part of \( \chi \)) crossed by a vertical cut segment, and will not be crossed by vertical cut segments later in the recursive CCR-partition.

**Proof.** In all cases (1), (3), (5) and (6), if \( R_r \) had previously been crossed by a vertical cut segment, \( \sigma' \), then the northeast corner \( c \) of \( R \) would have become exposed to the right (since \( R_r \) is the top-right-neighbor of \( R \)), and a fence segment would have been established through \( c \), along the top side of \( R \) (extending leftwards to the left side of the top-left-neighbor of \( R \)); this fence prevents \( R \) from being crossed, since a vertical cut segment never crosses a fence segment.
There is no other vertical cut segment of $\chi$ (recall property (v) of Lemma 5.2), besides $\sigma$; thus, $R_r$ is not crossed by a vertical cut segment of $\chi$.

Later in the recursive CCR-partition, the rectangle $R_r$ cannot be crossed by a vertical cut segment, since such a crossing would imply a crossing of the fence we established through the top or bottom side of $R_r$ (and this is ruled out by property (iv) of Lemma 5.2). Indeed, we establish fences in order to enforce that once a rectangle has a corner charged, it cannot be crossed by later cuts in the recursive partitioning.

**Claim 5.6.** At most two corners of a rectangle are charged: it is never the case that both a northwest and a northeast corner of the same rectangle is charged (and a similar statement holds for southwest and southeast corners).

**Proof.** If, as in cases (1) and (3), the northwest corner, $c_r$, of a rectangle $R_r \in I'$ is charged for the crossing of a rectangle $R$, then a fence is established along the top side of $R_r$, extending rightwards to the right side of the top-right-neighbor, $R'_r$, of $R_r$. This fence prevents $R'_r$ from being crossed later by a vertical cut segment; thus, the northeast corner of $R_r$ cannot be charged, since, if the northeast corner were charged for the crossing of some rectangle $R'$ to its right, then $R'$ must be the top-right-neighbor of $R_r$ (just as we saw in cases (1) and (3) that when the northwest corner $c_r$ is charged for the crossing of $R$ by $\sigma$, the rectangle $R$ must be the top-left-neighbor of $R_r$), and we know that the top-right-neighbor of $R_r$ cannot be crossed by a vertical cut segment, since it is protected by the previously established fence. A similar statement holds for cases (5) and (6), when the southwest corner of $R_r$ is charged for the crossing of $R$: it cannot be that the southeast corner is also charged, as the bottom-right-neighbor of $R_r$ is protected by the fence segment established.

The set of red rectangles can be partitioned into two sets: those that are cut (let $h_0$ be the number) and those that are not cut (let $h_0$ be the number). Similarly, non-red rectangles can be partitioned into two sets: those that are cut (let $m_0$ be the number) and those that are not cut (let $m_0$ be the number). By Claim 5.5 if any rectangle is charged, then it is never crossed by a cut; thus, there are at most $h_0 + m_0$ rectangles that are charged, and we know from Claim 5.4 that the amount of charge received by any of these rectangles is at most 1 (the crossing of a rectangle $R$ by $\sigma$ resulted in a charge of 1/2 to a northwest corner and 1/2 to a southeast corner). We conclude that

**Claim 5.7.** The total sum of all charges is at most $h_0 + m_0$, and thus $m_\chi \leq h_0 + m_0$.

**Proof.** By Claim 5.6, at most two corners of a rectangle is charged (each with a charge of 1/2), and by Claim 5.5, each is charged at most once. Since, by specification of the charging scheme, there is exactly one unit of charge per crossed non-red rectangle, we get that $m_\chi \leq h_0 + m_0$.

In the recursive partitioning, we discard every one of the $h_0 + m_\chi$ rectangles that are crossed by vertical cut segments; what remains in the end are the $h_0 + m_0$ rectangles that were not crossed and therefore not discarded. (These remaining rectangles either appear as isolated rectangles within a leaf face of the CCR-partition, or appear as special rectangles, which are penetrated but not crossed.) Now, Claim 5.7 together with the fact that $m_0 + m_\chi \geq k/2$ (by assumption), imply that

$$h_0 + m_0 \geq m_\chi \geq (k/2) - m_0,$$

from which we see that

$$2h_0 + 4m_0 \geq k,$$

and thus,

$$4(h_0 + m_0) \geq k.$$

This implies that

$$h_0 + m_0 \geq k/4,$$

which proves our main claim (with a 4-approximation).

**Claim 5.8.** The total number, $h_0 + m_0$, of rectangles that are not crossed by vertical cut segments (and thus that survive) in the recursive partitioning is $h_0 + m_0 \geq k/4$. 

15
Improving the Approximation Factor

While our main goal in this paper is simply to get an $O(1)$-approximation algorithm that runs in polynomial time, we briefly discuss how our methods yield an improvement of the approximation factor 4 described above, if some additional care is taken in the charging scheme.

Consider the cases ((1), (3), (5) and (6)), where we described the charging of 1/2 to a corner $c_r$ (northwest or southwest of $R_r$, the top-right-neighbor of $R$), to account for half of the cost of $\sigma$ crossing $R$ (the other half being charged, by a symmetric argument, to a corner to the left of $R$). We now impose a preference to charge a northwest corner, if possible, of a rectangle for which $R$ is its top-left-neighbor. In cases (1) and (3), $c_r$ is already a northwest corner. In cases (5) and (6), provided that the bottom of $R_r$ is not aligned with the bottom of $R$ (i.e., provided that $y^- > y^-$), there must be (by maximality) at least one rectangle abutting $R_r$ from below, so we let $\bar{R}_r$ be that rectangle abutting $R_r$ with leftmost left side (so that $R$ is the top-left-neighbor of $\bar{R}_r$); instead of the southwest corner of $R_r$, we can charge the northwest corner of $\bar{R}_r$, and the top of $\bar{R}_r$ will become exposed after the cut along $\sigma$, so we establish a fence, anchored on $\sigma$, containing the top side of $\bar{R}_r$, and extending rightwards to the right side of the top-right-neighbor of $\bar{R}_r$. In cases (5) and (6), if $y^- = y^-$, then we go ahead and charge the southwest corner of $\bar{R}_r$, as we do not have the option to charge the northwest corner of $\bar{R}_r$, which abuts $R$ from below, since the interior of $\bar{R}_r$ lies strictly below the bottom side of $R$ (implying that $R$ is not the top-left-neighbor of $\bar{R}_r$).

Now, consider a rectangle $R_r$ for which both of its left corners (northwest and southwest) is charged (an amount 1/2 each). As the above argument shows, this happened because the northwest corner of $R_r$ was charged (in case (1) or (3)), since the top side of $R_r$ is aligned with the top of a crossed (by $\sigma$) rectangle $R_1$, and its bottom side was also aligned with the bottom side of a crossed rectangle $R_2$. (Further, from maximality, we see that the right sides of $R_1$ and $R_2$ are abutting the left side of $R_r$.)

There are two cases: (A) $R_r$ is red (in which case it must be nesting on its right, since it is not nesting on its left); and (B) $R_r$ is non-red. In case (A), we keep the full charge of 1/2 assigned to each of the two left corners of $R_r$; this is accounted for in the term $h_0$ of the total charge, since $R_r$ is then one of the $h_0$ red rectangles that is not crossed. In case (B), we know (by the same cases analysis, cases (1)-(8), which showed the existence of a chargeable corner to the right of $R$) that there must be a chargeable corner, $c_{rr}$, on the left side of a rectangle $R_{rr}$ that is the top-right-neighbor of $R_r$. We reallocate the two charges (of 1/2 each) to the left corners of $R_r$: instead, we assign charges of 1/3 to each of the two left corners of $R_r$, and assign 1/3 to the corner $c_{rr}$. To enable this charging scheme, we must modify slightly our specification of fences, so that a fence segment extends not just to the right side of the (immediate) top/bottom-right-neighbor of a rectangle whose top/bottom becomes exposed to the left, but so that the fence extension continues through the next rectangle penetrated to the right. This extended fence prevents $R_{rr}$ from having more than two of its corners charged (each with 1/3), as it will not be possible to charge both northwest and northeast (or to charge both southwest and southeast) corners, just as we saw in the proof of Claim 5.5. With this extended fence, now penetrating two rectangles (one of which may be crossed) instead of one, our notion of “nearly perfect” must allow for the possibility of at most 3 (still $O(1)$) rectangles penetrated per edge of a CCR subproblem. This, in turn, implies an increased (polynomial) running time in the dynamic program.

This modification of the charging yields an improved approximation factor: Since each of the $h_0$ uncrossed red rectangles has at most two corners charged, each with 1/2, and each of the $m_0$ uncrossed non-red rectangles has at most a total charge of 2/3 (arising from either a single corner charge of 1/2, or two corners charged, each with 1/3), we get that the total charge, $m_\chi$, is at most $h_0 + (2/3)m_0$, which shows, using $m_0 + m_\chi \geq k/2$

$$h_0 + (2/3)m_0 \geq m_\chi \geq (k/2) - m_0,$$

from which we get

$$2h_0 + (10/3)m_0 \geq k,$$

implying that

$$h_0 + m_0 \geq 3k/10,$$

yielding a 10/3-approximation.
This method of offloading charge to a next “layer” of neighboring rectangles can be continued to additional layers, yielding better and better factors, approaching a factor of 3. For example, if $R_{rr}$ has both of its left corners charged (each with an offloaded charge of 1/3), as a result of there being two rectangles ($R_{r1}$ and $R_{r2}$) each having both of their left corners charged (due to a sequence, $R_1$, $R_2$, $R_3$, $R_4$, of consecutive rectangles being crossed by $\sigma$), then, if $R_{rr}$ is non-red, we offload to the corner of a right-neighbor $R_{rrr}$, while extending fences into the next neighboring rectangle; this results in a factor 22/7 (the 4 charges of 1/2 get distributed as charges of 2/7 to each of the two left corners of $R_{r1}$ and $R_{r2}$, the two left corners of $R_{rr}$, and one left corner of $R_{rrr}$, so that each non-red uncrossed rectangle receives charge at most 4/7). The running time goes up, though, with each application of this method, as there become more special rectangles needed to specify within the state of the dynamic program, as we allow fences to penetrate more neighboring layers. It appears that this approach to obtaining an improvement is “stuck” at a factor 3 + $\epsilon$, so that additional ideas may be needed to improve the charging scheme further.

6 Conclusion

Our main result is a first polynomial-time constant-factor approximation algorithm for maximum independent set for axis-aligned rectangles in the plane. Our goal here was to devise new methods that achieve a constant factor; can the methods be developed further to yield an improved constant approximation factor?

In the time since the original version [22] appeared, where an initial approximation factor of 10 was given, Galvez et al [15] obtained an improved factor of 6 via a novel variant of methods from [22] (using a fairly simple, less technical argument), and a further improvement to factor 4 (using more involved technical arguments). The same authors have a further improvement to factor 3 [16], which requires considerably more technical effort and new ideas.

More ambitiously, is there a PTAS? Our approach seems to give up at least a factor of 2 in its method of handling the nestedness issue, in order to guarantee enough rectangles can be charged to nearby rectangle corners. Can the running time of a constant-factor approximation algorithm be improved significantly? Can the Pach-Tardos Conjecture [1] be resolved? (This would yield a constant-factor approximation algorithm with an improved running time.)

Appendix

Proof of Lemma 5.2

Proof. The proof is by a simple enumeration of cases, each of which is illustrated in Figure 11.

In the case in which $k_\alpha = 0$ or $k_\beta = 0$ (there are no fence segments, or only fence segments anchored on one side (left or right) of $Q$), a cut consisting of a single vertical chord (within the grid $G$) suffices to partition $Q$ into two subfaces, satisfying the claim. So, assume now that $k_\alpha, k_\beta \geq 1$.

In the case itemization below, we first consider the case (case (1)) in which $Q$ is a rectangle and then consider the cases (cases (2) through (6)) in which $Q$ is an L-shaped CCR, which, without loss of generality, has its northeast corner clipped, as shown in the figures. Cases (2) through (6) are distinguished based on how a particular rectangle, $C$, shown shaded gray in Figure 11, has its top/bottom sides among the fences and the top/bottom sides of $Q$. The rectangle $C$ is one face (cell) in the vertical decomposition (also known as the trapezoidal decomposition [10]) of $Q$ with respect to the fence segments: the vertical decomposition is a decomposition of $Q$ into rectangular faces (cells) induced by the fence segments, together with vertical segments through endpoints of fences (and the single reflex vertex, $v$, of $Q$) that are maximal within $Q$ without crossing any fence segment. Refer to Figure 10. Specifically, we let $C$ be the unique rectangular face containing the slightly shifted point $v'$ (shown as a small black dot in the figures), which is just northwest of the integral reflex vertex $v = (v_x, v_y)$, say, at $v' = (v_x - 1/2, v_y + 1/2)$.

We partition into cases according to the top/bottom sides of the rectangle $C$: the bottom of $C$ might lie on the bottom side, $b$, of $Q$, or on a blue fence $\beta^-$, or on a red fence $\alpha$; the top of $C$ might lie on the top side, $t$, of $Q$, or on a blue fence $\beta^+$ anchored on the right side $r^+$ at a point strictly above the reflex vertex $v$. In the case in which $k_\alpha = 0$ or $k_\beta = 0$, a single vertical chord is sufficient to partition $Q$ into two subfaces, satisfying the claim.
Figure 10: The vertical decomposition of an L-shaped CCR $Q$ with respect to red and blue fence segments. The case shown is a CCR with the northeast corner clipped; the unique reflex vertex is $v$, and the slightly shifted, to the northwest, point $v'$ is shown as a small dot. The cell $C$ is defined to be the unique (rectangular) face containing $v'$. Two examples are shown, both the the top and bottom sides of $C$ contained in blue fence segments $\beta^+$ (anchored on $r^+$) and $\beta^-$ (which we consider to be anchored on the top endpoint of $r$, or equivalently the bottom endpoint of $r^+$).

$v$. (The top side of $C$ cannot lie on a fence that is not anchored on the upper right side, $r^+$, of $Q$, since $C$ is a face in the vertical decomposition that contains $v'$, and thus abuts the right side segment $r^+$ of $Q$.)

We have the following cases:

1. [\(Q\) is a rectangle] Let $\beta$ be the blue fence with leftmost left endpoint.
   - (a) [The rightmost red endpoint is left of the left endpoint of $\beta$] Then, a vertical chord through the left endpoint of $\beta$ does not cross any fence, so we cut along this vertical chord, and along $\beta$, obtaining 3 subfaces, each a rectangle.
   - (b) [Otherwise] A vertical ray (e.g., upwards) from the left endpoint of $\beta$ meets a red fence, $\alpha$ before reaching the boundary of $Q$. We make a “Z” cut along $\beta$, a vertical segment to $\alpha$, and a portion of $\alpha$, obtaining two subfaces, each a CCR.

2. [The top of $C$ lies on $t$, the bottom on $b$] We make a straight vertical cut along the right side of $C$, obtaining 2 rectangle subfaces.

3. [The top of $C$ lies on $t$, the bottom on blue fence $\beta$] Since $C$ contains $v'$, the fence $\beta$ must be anchored on the right side, $r$, of $Q$, at or below the top endpoint of $r$ (if anchored at the top endpoint of $r$, we can equivalently consider $\beta$ to be anchored at reflex vertex $v$). We make an “L” cut along the left side of $C$ and a portion of $\beta$. The left side of $C$ may have been determined by the left endpoint of $\beta$ (case (3)(a) in the figure), or it may have been determined by the right endpoint of a red fence (case (3)(b) in the figure), in which case our cut includes the red fence segment as well. Assuming (as shown in the figure) $\beta$ is anchored on a point of the right side $r$ below vertex $v$, we obtain 2 subfaces that are L-shaped CCRs, and one rectangular subface (in case (3)(b)); if $\beta$ is anchored at reflex vertex $v$, the figures would be similar, but there would be 1 subface that is an L-shaped CCR, and one or two (in case (3)(b)) rectangular subfaces.

4. [The top of $C$ lies on blue fence $\beta$, the bottom on $b$] This case is like (3), and we obtain 2 subfaces that are CCRs, and possibly one rectangular subface (in case (4)(b)). (An alternative cut is to make a single vertical cut along the right side of $C$.)

5. [The top of $C$ lies on blue fence $\beta^+$, the bottom on blue fence $\beta^-$] The left side of $C$ is a vertical segment with endpoints on the blue fences $\beta^+$ and $\beta^-$. Informally, consider sweeping this segment leftwards, keeping it of maximal length, while not crossing any fence segment: the upper endpoint moves along a “staircase” going leftwards and making jumps upwards, while the lower endpoint moves along a staircase going leftwards and making jumps downwards, until one of five events (enumerated
Figure 11: The cases in the proof of Lemma 5.2. The rectangular cell $C$ is shown shaded gray; it is the cell of the vertical decomposition that contains $v'$, the slightly shifted (to the northwest) reflex vertex $v$ of $Q$, shown as a small black dot. Cuts are shown in thick green. The fences are shown in blue if their right endpoint lies on the right boundary (sides $r^+$ or $r$) of $Q$, and are referred to using the letter beta (“$\beta$, $\beta^-$, $\beta^+$”); fences with left endpoint on the left boundary (side $\ell$) of $Q$ are shown in red and are referred to using a letter alpha (“$\alpha$”). Note that in cases (3) and (5) the blue fence segment is shown anchored at a point interior to the right side, $r$; it could be anchored at the top endpoint of $r$, or equivalently at the reflex vertex $v$. Only those fence segments that are needed to identify the subcases are shown; there may be numerous other red/blue fences.
as cases (a) through (e) below) takes place. More formally, starting at (rectangular) cell \( C \) in the vertical decomposition, we follow a path leftwards in the dual graph of the decomposition, from cell to a unique left neighboring cell, until we come to a cell, \( C' \) (which is possibly \( C \) itself), that has 2 (or more) left neighbors (because its left side is determined by the right endpoint of a red fence), or until we reach a cell whose top and bottom sides are not both on blue fences. We have the following subcases depending on the classification of the cell \( C' \):

(a) [The top of \( C' \) is on the top side, \( t \), of \( Q \)] We obtain 2 subfaces that are L-shaped CCRs, and one rectangular subface.
(b) [The bottom of \( C' \) is on the bottom side, \( b \), of \( Q \)] We obtain 2 subfaces that are L-shaped CCRs, and one rectangular subface.
(c) [The top of \( C' \) is on a red fence segment \( \alpha \)] We obtain 3 subfaces that are L-shaped CCRs.
(d) [The bottom of \( C' \) is on a red fence segment \( \alpha \)] We obtain 3 subfaces that are L-shaped CCRs.
(e) [The left side of \( C' \) has a right endpoint of a red fence \( \alpha \)] We obtain 3 subfaces that are L-shaped CCRs.

In Figure 10, one example (on the left) has \( C' = C_2 \) (case (d), with the bottom of \( C' \) on a red fence), and one example (on the right) has \( C' = C_3 \) (case (e), with the left side of \( C' \) determined by the right endpoint of a red fence).

(6) [The top of \( C \) is on a blue fence \( \beta \), the bottom of \( C \) is on a red fence \( \alpha \)] We obtain 2 subfaces: one rectangle and one L-shaped CCR.

In all cases, we see that there is a single vertical cut segment, at most 3 subfaces, and all portions of the cuts lie on the grid, with horizontal portions along fence segments. This concludes the proof of the lemma.

Acknowledgements

I thank Mathieu Mari for his input on an earlier draft [22]. I thank Arindam Khan and the other coauthors of [15] for discussions and for sharing their preprint [16].

This work was partially supported by the National Science Foundation (CCF-2007275), the US-Israel Binational Science Foundation (BSF project 2016116), Sandia National Labs, and DARPA (Lagrange).

References

[1] Anna Adamaszek, Sariel Har-Peled, and Andreas Wiese. Approximation schemes for independent set and sparse subsets of polygons. *Journal of the ACM (JACM)*, 66(4):29, 2019.

[2] Anna Adamaszek and Andreas Wiese. Approximation schemes for maximum weight independent set of rectangles. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 400–409. IEEE, 2013.

[3] Anna Adamaszek and Andreas Wiese. A QPTAS for maximum weight independent set of polygons with polylogarithmically many vertices. In *Proceedings 25th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 645–656. SIAM, 2014.

[4] Pankaj K Agarwal, Marc van Kreveld, and Subhash Suri. Label placement by maximum independent set in rectangles. *Computational Geometry: Theory and Applications*, 3(11):209–218, 1998.

[5] Piotr Berman, Bhaskar DasGupta, Shanmugavelayutham Muthukrishnan, and Suneeta Ramaswami. Improved approximation algorithms for rectangle tiling and packing. In *Proceedings 12th Annual ACM-SIAM Symposium on Discrete Algorithms*, volume 7, pages 427–436, 2001.
[6] Ravi Boppana and Magnús M. Halldórsson. Approximating maximum independent sets by excluding subgraphs. *BIT Numerical Mathematics*, 32(2):180–196, 1992.

[7] Parinya Chalermsook. Coloring and maximum independent set of rectangles. In Leslie Ann Goldberg, Klaus Jansen, R. Ravi, and José D. P. Rolim, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 123–134, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.

[8] Parinya Chalermsook and Julia Chuzhoy. Maximum independent set of rectangles. In *Proceedings 20th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 892–901. Society for Industrial and Applied Mathematics, 2009.

[9] Julia Chuzhoy and Alina Ene. On approximating maximum independent set of rectangles. In 2016 *IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 820–829. IEEE, 2016.

[10] Mark de Berg, Otfried Cheong, Marc J. van Kreveld, and Mark H. Overmars. *Computational Geometry: Algorithms and Applications*, 3rd Edition. Springer, 2008.

[11] Mark De Berg, Marko M de Groot, and Mark H Overmars. Perfect binary space partitions. *Computational geometry*, 7(1-2):81–91, 1997.

[12] Thomas Erlebach, Klaus Jansen, and Eike Seidel. Polynomial-time approximation schemes for geometric intersection graphs. *SIAM Journal on Computing*, 34(6):1302–1323, 2005.

[13] Robert J Fowler, Michael S Paterson, and Steven L Tanimoto. Optimal packing and covering in the plane are np-complete. *Information processing letters*, 12(3):133–137, 1981.

[14] Jacob Fox and János Pach. Computing the independence number of intersection graphs. In *Proceedings 22nd Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1161–1165. SIAM, 2011.

[15] Waldo Galvez, Arindam Khan, Mathieu Mari, Tobias Momke, Madhusudhan Reddy, and Andreas Wiese. A 4-approximation algorithm for maximum independent set of rectangles. *arXiv preprint arXiv:2106.00623*, June, 2021.

[16] Waldo Galvez, Arindam Khan, Mathieu Mari, Tobias Momke, Madhusudhan Reddy, and Andreas Wiese. Personal communication, July, 2021.

[17] Fabrizio Grandoni, Stefan Kratsch, and Andreas Wiese. Parameterized approximation schemes for independent set of rectangles and geometric knapsack. *arXiv preprint arXiv:1906.10982*, 2019.

[18] Sariel Har-Peled. Quasi-polynomial time approximation scheme for sparse subsets of polygons. In *Proceedings Annual Symposium on Computational Geometry*, pages 120–129, 2014.

[19] Hiroshi Imai and Takao Asano. Finding the connected components and a maximum clique of an intersection graph of rectangles in the plane. *Journal of Algorithms*, 4(4):310–323, 1983.

[20] J. Mark Keil, Joseph S.B. Mitchell, Dinabandhu Pradhan, and Martin Vatshelle. An algorithm for the maximum weight independent set problem on outerstring graphs. *Computational Geometry: Theory and Applications*, 60:19–25, 2017. Special issue devoted to selected papers from CCCG 2015. URL: http://www.sciencedirect.com/science/article/pii/S092577211630044X

[21] Sanjeev Khanna, S. Muthukrishnan, and Paterson Mike. On approximating rectangle tiling and packing. In *Proceedings 9th Annual ACM-SIAM Symposium on Discrete Algorithms*, volume 95, page 384. SIAM, 1998.

[22] Joseph S. B. Mitchell. Approximating maximum independent set for rectangles in the plane. *arXiv preprint arXiv:2101.00326*, January, 2021.
[23] Frank Nielsen. Fast stabbing of boxes in high dimensions. *Theoretical Computer Science*, 246(1-2):53–72, 2000.

[24] János Pach and Gábor Tardos. Cutting glass. *Discrete & Computational Geometry*, 24(2-3):481–496, 2000.

[25] David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3:103–128, 2007.