True Gravity in Atmospheric Ekman Layer Dynamics

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Abstract True gravity is a three-dimensional vector field, \( \mathbf{g}(\lambda, \varphi, z) = i g_\lambda + j g_\varphi + k g_z \), with \((\lambda, \varphi, z)\) the (longitude, latitude, height) and \((i, j, k)\) the corresponding unit vectors. The longitudinal-latitudinal component of the true gravity, \( g_\lambda = i g_\lambda \), is neglected completely in meteorology through using the standard gravity \((-g_0,0,0)\) or the effective gravity \([-g(\varphi)\mathbf{K}]\). Here, \( \mathbf{K} \) (or \( \mathbf{K} \)) is normal to the Earth spherical (or ellipsoidal) surface. Such simplification of \( \mathbf{g}(\lambda, \varphi, z) \) has never been challenged. This study uses the classical atmospheric Ekman layer dynamics as an example to illustrate the importance of \( \mathbf{g}_\lambda \). The standard gravity \((-g_0\mathbf{K})\) is replaced by the true gravity \( \mathbf{g} \) in the classical atmospheric Ekman layer equation with a constant eddy viscosity \((\zeta)\) and height-dependent-only density \(\rho(z)\) represented by an e-folding stratification. New formulas for the Ekman spiral and Ekman pumping are obtained. The second derivative of the gravity disturbance \((T)\), \(\hat{c}^2 \mathbf{g} \mathbf{K} \hat{c}^2\), causes the Ekman pumping in addition to the geostrophic vorticity \((\zeta)\). With \(\hat{c}^2 \mathbf{g} \mathbf{K} \hat{c}^2\) from the EIGEN-6C4 static gravity model, and \(\zeta\) calculated from July sea level pressure \((p)\) data from the Comprehensive Ocean-Atmosphere Data Set, the global mean strength of the Ekman pumping over the world oceans is 3.69 cm s\(^{-1}\) due to \( \mathbf{g}_\lambda \), which is much larger than 0.33 cm s\(^{-1}\) due to the geostrophic vorticity. It implies the urgency to use the true gravity \( \mathbf{g}(\lambda, \varphi, z) \) into atmospheric GCM and weather forecast although the results are obtained from specific density field and gravity model.

Plain Language Summary Meteorologists use the spherical (or ellipsoidal) surfaces represented by latitude \((\varphi)\) and longitude \((\lambda)\) as the horizontal and the direction normal to them represented by height \((z)\) as the vertical. It is not correct since the vertical direction is represented by the true gravity \( \mathbf{g}(\lambda, \varphi, z) \), and the horizontal surfaces are the equipotential surfaces of \( \mathbf{g}(\lambda, \varphi, z) \) such as the geoid surface which is nearest to the Earth spherical (or ellipsoidal) surface \((z=0)\). In the \((\lambda, \varphi, z)\) coordinates, the horizontal vector \( \mathbf{g}(\lambda, \varphi, z) \) has latitudinal and longitudinal components, which are neglected completely in meteorology. This study uses the atmospheric Ekman layer dynamics and the true gravity data from the EIGEN-6C4 static gravity model as an example to show the importance of using the true gravity \( \mathbf{g}(\lambda, \varphi, z) \) in the atmospheric dynamics, weather forecast, and climate change.

1. Introduction

Meteorologists usually use the Earth-fixed polar spherical coordinate system with \((\lambda, \varphi, z)\) representing the longitude, latitude, and spherical normal (or height) with \((i, j, k)\) the corresponding unit vectors. The unit vector \( k \) is perpendicular to the spherical surface (upward positive), and does not represent the vertical direction since the Earth true gravity \( \mathbf{g}(\lambda, \varphi, z) = g_\lambda i + g_\varphi j + g_z k \) represents the vertical. However, this important body force \( \mathbf{g}(\lambda, \varphi, z) \) has been highly simplified in meteorology.

When the Earth is assumed as a uniform sphere with no rotation, the gravity exerted on an air particle from this spherical Earth is a single-directional vector

\[
\mathbf{g}_S = -g_0 \mathbf{k}, \quad g_0 = 9.81 \text{ m s}^{-2}
\]

where \( g_0 \) is the standard gravity. The corresponding standard gravity potential \((E_0)\) is given by

\[
E_0(z) = -g_0 \cdot \mathbf{g}_S = \frac{dE_0}{dz} \mathbf{k}
\]

The spherical Earth becomes ellipsoidal when the rotation is considered. The gravity exerted on an air particle from this ellipsoidal Earth is also a single-directional vector

\[
\mathbf{g}_E = -g(\varphi) \mathbf{K}
\]
where $K$ is the unit vector perpendicular to the ellipsoidal surface (upward positive); $g_e$ is the sum of the gravitational and centrifugal accelerations, called the effective gravity vector (Vallis, 2006) or the normal gravity vector (Vaniček & Krakiwsky, 1986), and also without latitudinal-longitudinal component in the oblate spherical coordinates. Its intensity $g(\varphi)$ is determined analytically. For example, the World Geodetic System 1984 uses the Somigliana equation (National Geospatial-Intelligence Agency, 1984) to represent $g(\varphi)$

$$
    g(\varphi) = g_e \left[ \frac{1 + \kappa \sin^2 \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}} \right] e^2 = \frac{a^2 - b^2}{a^2} \kappa = \frac{b g_p - a g_e}{a g_e}
$$

(4)

where $(a, b)$ are the equatorial and polar semi-axes; $a$ is used for the Earth radius, $R = a = 6.3781364 \times 10^6$ m; $b = 6.3567523 \times 10^6$ m; $e$ is the spheroid’s eccentricity; $g_e = 9.780$ m/s$^2$, is the gravity at the equator; and $g_p = 9.832$ m/s$^2$ is the gravity at the poles. The corresponding effective gravity potential $(E)$ is given by

$$
    E(z, \varphi) = -g(\varphi) z; g_E = \frac{\partial E}{\partial z} K
$$

(5a)

which is usually associated with the oblate spheroid coordinate system $(\lambda, \varphi_{ob}, z_{ob})$, which shares the same longitude $(\lambda)$ with the polar spherical coordinate system but has different latitude $(\varphi_{ob})$ and height $(z_{ob})$ coordinates (Gill, 1982).

The effective gravity vector $g_e$ has no $(\lambda, \varphi_{ob})$ components in the oblate spherical coordinates, but has latitudinal and radial $(\varphi, z)$ components in the polar spherical coordinates. To keep the dynamical balances involving the Coriolis force and pressure that determine the large-scale horizontal flow, it is conventionally to use the direction of $g_e$ as in the direction of the unit vector $k$, and then for all geometric purposes to regard the surfaces of constant $E(z, \varphi)$ as if they were true sphere, that is,

$$
    g_e \approx -g(\varphi) k
$$

(5b)

which leads to zero latitudinal component of $g_e$ (Vallis, 2006). Such a treatment has been well accepted by the meteorological community. The dynamic equation is represented in the polar spherical coordinates with the effective gravity represented by Equation 5a. The geometric difference between the oblate and polar spherical coordinates is around 0.17% (Gill, 1982). However, the dynamical error may be larger (Vallis, 2006).

With rotation and non-uniform mass distribution (i.e., considering the true Earth), the gravity exerted on an air particle is a three-dimensional vector field,

$$
    g(\lambda, \varphi, z) = -g(\varphi) k + \delta g(\lambda, \varphi, z)
$$

(6)

where $\delta g$ is the gravity disturbance (Hackney & Featherstone, 2003); and $g(\lambda, \varphi, z)$ is the true gravity. The true gravity $g(\lambda, \varphi, z)$ has never been recognized in meteorology since meteorologists are only familiar with the standard and effective gravities. The corresponding true gravity potential $(V)$ is the sum of the effective gravity potential $(E)$ and the disturbing gravity potential $(T)$,

$$
    V = E + T = -g(\varphi) z + T, \delta g = \nabla T(\lambda, \varphi, z)
$$

(7)

where

$$
    \nabla T = i \frac{1}{R \cos \varphi} \frac{\partial}{\partial \lambda} + j \frac{1}{R} \frac{\partial}{\partial \varphi} + k \frac{\partial}{\partial z}
$$

(8)

is the three-dimensional vector differential operator; and $R = 6.3781364 \times 10^6$ m, is the Earth radius. Substitution of Equation 7 into Equation 6 leads to

$$
    g(\lambda, \varphi, z) = g_h + g_k k, g_h = \nabla T, g_z = \frac{\partial T}{\partial z} - g(\varphi)
$$

(9)

where $\nabla = i \frac{\partial}{R \cos \varphi \partial \lambda} + j \frac{\partial}{R \partial \varphi}$ is the two-dimensional vector differential operator; $g_h$ is the latitudinal-longitudinal gravity vector; and $g_k k$ is the component in the direction of $k$. Obviously, $g_h$ is not the difference between $g_e$ and $g_k$, or the latitudinal component of $g_e$. Since the deviation of the deflected-vertical component of the gravity $(g_z)$ to the constant $(-g_0)$ is around four orders of magnitude smaller than $g_0$, which leads to

$$
    g_z \approx -g_0
$$

(10)

Substitution of Equation 10 into Equation 9 leads to
\[ g(\lambda, \phi, z) \approx g_h - g_k \]  

The spherical expansion of the disturbing static gravity potential \( T \) in the polar spherical coordinates outside the Earth masses is given by (Kostelecký et al., 2015)

\[
T(\lambda, \phi, z) = \frac{GM}{(R + z)} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left( \frac{R}{R + z} \right)^l \left[ (C_{l,m} - C_{l,m}^e) \cos m\lambda + S_{l,m} \sin m\lambda \right] P_{l,m}(\sin \phi),
\]

where \( M = 5.9736 \times 10^{24} \text{ kg} \), is the mass of the Earth; \( G = 6.674 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2} \), is the gravitational constant; \((C_{l,m}^e, S_{l,m})\) are the harmonic geopotential coefficients with \(C_{l,m}^e\) belonging to the reference ellipsoid; and \(P_{l,m}(\sin \phi)\) are the Legendre associated functions with \((l, m)\) the degree and order of the harmonic expansion. According to Equation 12, the ratio between \( T(\lambda, \phi, z) \) to \( T(\lambda, \phi, 0) \) through the troposphere can be roughly estimated by

\[
\frac{T(\lambda, \phi, z)}{T(\lambda, \phi, 0)} \approx \frac{R}{(R + z)} \approx 1, \quad H \geq z \geq 0
\]

where \( H = 10.4 \text{ km} \), is the height of the troposphere. Since \( R \) is the Earth radius and more than three orders of magnitude larger than \( H \). This leads to the thin layer approximation that the disturbing static gravity potential \( T(\lambda, \phi, z) \) does not change with \( z \) in the whole troposphere

\[
T(\lambda, \phi, z) \approx T(\lambda, \phi, 0), \quad H \geq z \geq 0
\]

which makes

\[
\nabla^2 T(\lambda, \phi, z) \approx \nabla^2 T(\lambda, \phi, 0);
\]

The true gravity potential satisfies the Laplace equation outside the Earth surface (Vaniček & Krakiwsky, 1986),

\[
\nabla^2 V + \frac{\partial^2 V}{\partial z^2} = 0
\]

Substitution of Equation 7 into Equation 16 leads to the Laplace equation for the disturbing gravity potential

\[
\nabla^2 T + \frac{\partial^2 T}{\partial z^2} = 0
\]

Use of the second formula in Equation 9 gives

\[
\nabla \cdot g_h = -\frac{\partial^2 T}{\partial z^2} \approx -\frac{\partial^2 T}{\partial z^2}_{|z=0}
\]

where the thin-layer approximation Equation 14 for the troposphere is used.

The unit vector \( k \) (or \( K \)) does not represent the vertical direction since the Earth true gravity \( g(\lambda, \phi, z) \) represents the vertical. We may call the direction of \( k \) the deflected-vertical. The angle between \( -k \) and \( g \) is the vertical deflection. The spherical (or ellipsoidal) surfaces are not the horizontal surfaces since the equipotential surfaces of \( g(\lambda, \phi, z) \) such as the geoid surface represent the horizontal surfaces. We may call the spherical (or ellipsoidal) surfaces the deflected-horizontal surfaces.

The turbulent mixing in atmospheric planetary boundary layer is treated as a diffusion process similar to molecular diffusion, with an eddy viscosity \( K \), which is many orders of magnitude larger than the molecular viscosity. The turbulent mixing generates ageostrophic wind (called the Ekman spiral), decaying by an e-folding over a height as the wind vector rotate to the right (left) in the northern (southern) hemisphere through one radian (Ekman, 1905). Along with the Ekman spiral, several important processes such as Ekman pumping can be identified.

As in other atmospheric dynamics, the Ekman theory was established using the standard gravity \((-g_k k)\) (Holton, 2004; Pedlosky, 1987), rather than the true gravity \(g(\lambda, \phi, z)\). The longitudinal-latitude component of the true gravity, \(g_\phi (=g_k \hat{j} + g_h \hat{i})\), is neglected completely. Use of the standard gravity \((-g_k k)\) instead of the true gravity \(g\) is based on the comparison that the strength of the deflected-vertical component \(|g_h|\) is 5–6 orders of magnitude larger than the strength of the deflected-horizontal gravity \(|g_h|\). This comparison
is not correct because such a huge difference in magnitude between the components in \( k \) and in \((i,j)\) also occurs in the pressure gradient force in large-scale atmospheric dynamics. But, the pressure gradient force in \((i,j)\) is never neglected against the pressure gradient force in \( k \). Thus, the feasibility of using the standard gravity \((-g_k)\) in meteorology needs to be investigated. The Ekman dynamics provides a theoretical framework for such a study.

The rest of the study is outlined as follows. Section 2 presents the dynamic equation with the true gravity for atmospheric Ekman layer. Section 3 shows the new Ekman layer solution and new formula for the Ekman pumping due to the use of the true gravity. Section 4 describes the two datasets: (a) the second derivative of the disturbing static gravity potential \((\partial^2 T_{z}/\partial^2 z)\) from the EIGEN-6C4 model (Kostelecký et al., 2015), and (b) the climatological July mean sea level pressure \((p)\) data from the Comprehensive Ocean-Atmosphere Data Set (COADS) (Slutz et al., 1985). Section 5 presents the global Ekman pumping velocity due to \(g_h\) and due to the geostrophic vorticity calculated from the sea level pressure. Section 6 shows the feasibility of using the \((\lambda, \phi, z)\) coordinates. Section 7 presents the conclusions.

2. Dynamic Equation With the True Gravity

Steady-state large-scale atmospheric dynamic equation with the Boussinesq approximation (replacement of density \( \rho \) by a constant \( \rho_0 \) except \( \rho \) being multiplied by the gravity and incompressibility) is given by (Chu, 2021)

\[
\rho_0 \left[ 2\Omega \times \mathbf{U} \right] = -\nabla p + \rho \mathbf{g} + \rho \mathbf{F} \quad (19a)
\]

\[
\nabla \cdot \mathbf{U} = 0 \quad (19b)
\]

if the pressure gradient force, true gravity \( g(\lambda, \phi, z) \), and friction are the only real forces. Here, \( \Omega = \Omega \left[ \cos \phi + k \sin \phi \right] \) is the Earth rotation vector with \( \Omega = 2\pi/(86,164 \text{ s}) \) the Earth rotation rate; \( \rho \) is the density; \( \rho_0 = 1.225 \text{ kg/m}^3 \), is the characteristic density near the ocean surface; \( \mathbf{U} = (u, v) \), is the deflected-horizontal velocity vector; \( w \) is the deflected-vertical velocity; \( p \) is the pressure; and \( \mathbf{F} \) is the turbulent diffusive force due to the vertical shear represented by

\[
\mathbf{F} = \frac{\partial}{\partial z} \left( K \frac{\partial \mathbf{U}}{\partial z} \right) \quad (20)
\]

Let \( \mathbf{U}_g \) be the geostrophic wind

\[
\rho_0 \left[ 2\Omega \times \mathbf{U}_g \right] = -\nabla p \quad (21)
\]

Substitution of Equation 21 into Equation 19a leads to the dynamic equation for the Ekman layer

\[
\rho_0 \left[ 2\Omega \times (\mathbf{U} - \mathbf{U}_g) \right] = \rho \mathbf{g}_h + \rho \mathbf{F}, \quad (22)
\]

where \( \mathbf{g}_h \) is independent on \( z \) in the troposphere (see Equation 18).

Baroclinicity (i.e., non-zero latitudinal or longitudinal density gradient) and spatially varying eddy viscosity \( K \) affect the Ekman layer dynamics (Chu, 2015; Sun & Sun, 2020). To limit the study on the effect of \( \mathbf{g}_h \), the eddy viscosity \( K \) is assumed constant and the density varies in the \( z \)-direction only, that is, the geostrophic wind does not depend on \( z \),

\[
\partial \mathbf{U}_g / \partial z = 0.
\]

Furthermore, a special density stratification is selected for this study as the e-folding decreasing with height

\[
\frac{\rho}{\rho_0} = s(z), s(z) = \exp \left( -\frac{z}{H} \right), \quad H = 10.4 \text{ km} \quad (23)
\]

where \( H \) is the height of the troposphere. Substitution of Equation 23 into Equation 22 leads to

\[
2\Omega \times (\mathbf{U} - \mathbf{U}_g) = s(z) \mathbf{g}_h + K \frac{\partial^2 \mathbf{U}}{\partial z^2} \quad (24)
\]
With the complex variables, the deflected-horizontal gravity ($g_h$), Ekman velocity ($U$), and geostrophic wind ($U_g$) are defined by

$$G_h = g_x + ig_y, U = u_x + iv_x, U_g = u_y + iv_y, i = \sqrt{-1}$$

(25)

Equation 24 is converted into

$$\frac{\partial^2 U}{\partial z^2} - i\frac{f}{K}(U - U_g) = -\frac{s(z)}{K} G_h.$$  

(26)

Substitution of Equation 23 into Equation 26 leads to

$$\frac{\partial^2 U}{\partial z^2} - i\frac{f}{K}(U - U_g) = -\frac{G_h}{K} \exp\left(-\frac{z}{H}\right)$$

(27)

The Ekman velocity $U$ satisfies the upper boundary condition,

$$U \rightarrow U_g \text{ as } z \rightarrow \infty$$

(28)

and the surface boundary condition

$$U = 0 \text{ as } z = 0$$

(29)

3. Ekman Layer Solution

Equation 27 with the boundary conditions Equations 28 and 29 has the exact solution

$$U(z) = U_g - \left[U_g - \Gamma\left(\delta^2 + 2i\right)G_h\right] \exp\left(-i(z + H)\right)$$

(30)

$$\frac{D_E}{\pi H} = \sqrt{\frac{2Kf}{f}}$$

(31)

Here, $D_E$ is the Ekman layer depth; and $\delta$ is the ratio between the Ekman layer depth ($D_E$) and the height of troposphere ($H$). Converting the Ekman layer spiral (Equation 30) into the vector form

$$u = u_y - \left[u_y - \Gamma\left(\delta^2 + 2g_\phi\right)\right] \cos\left(\frac{\pi z}{D_E}\right) + \left[v_x - \Gamma\left(\delta^2 + 2g_\phi\right)\right] \sin\left(\frac{\pi z}{D_E}\right) e^{-\pi z / D_E}$$

(32)

$$v = v_y - \left[-u_y - \Gamma\left(\delta^2 + 2g_\phi\right)\right] \sin\left(\frac{\pi z}{D_E}\right) + \left[v_x - \Gamma\left(\delta^2 + 2g_\phi\right)\right] \cos\left(\frac{\pi z}{D_E}\right) e^{-\pi z / D_E}$$

The eddy viscosity $K$ is taken as a constant ($K = 5 \text{ m}^2\text{s}^{-1}$) in the atmospheric planetary boundary layer (Holton, 2004). The parameter $\delta$ is estimated by

$$\delta = \sqrt{\frac{2Kf}{fH}} = \frac{0.0252}{\sqrt{\sin\varphi}}, \text{ for } K = 5 \text{ m}^2\text{s}^{-1}, H = 10.4 \text{ km}, \Omega = \frac{2\pi}{86164 \text{s}}$$

(33)

where the parameter $\delta$ varies from 0.0854 at $\varphi = 5^\circ$ (N or S) to 0.0252 at $\varphi = 90^\circ$ (N or S). The range of $\pi\delta$ and the maximum values of $\delta^2$ (at $\varphi = 5^\circ$S or N) are estimated by

$$0.07917 \leq n\delta \leq 0.26283, \delta^2 \leq 0.7286 \times 10^{-2}$$

(34)

It is reasonable to neglect terms with $\delta^2$ in Equation 32. The Ekman profile (Equation 32) is simplified by

$$u = u_y - \left[u_y + \frac{g_\phi}{f}\right] \cos\left(\frac{\pi z}{D_E}\right) + \left[v_x + \frac{g_\phi}{f}\right] \sin\left(\frac{\pi z}{D_E}\right) e^{-\pi z / D_E} + \frac{g_\phi}{f} e^{-iH}$$

(35)
where the parameter $\Gamma$ is simplified as $\Gamma = 1/(2f)$. Substitution of Equation 35 into the continuity (Equation 19b) and integration with respect to $z$ from $z = 0$ to $z = D_E$ leads to

$$\dot{w}(D_E) = \frac{D_E}{R} \left( \frac{1}{\cos \phi} \frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \phi} \right) dz$$

(36)

Substitution of Equation 35 into Equation 36 gives the Ekman pumping velocity

$$\dot{w}(D_E) = \frac{D_E}{2\pi f} \zeta_e - \frac{D_E}{2\pi f} \nabla \cdot \mathbf{g}_h$$

(37)

where the following approximations in the definite integration (Equation 36) are used

$$e^{-z} = 0.04321 << 1, e^{-D_E} \approx 1$$

(38)

and

$$\zeta_e = k \cdot \nabla \times \mathbf{U}_e$$

(39)

is the geostrophic vorticity.

Equation 37 clearly shows that $\nabla \cdot \mathbf{g}_h$ causes the Ekman pumping in addition to the geostrophic vorticity $\zeta_e$.

Substitution of Equation 18 into Equation 37 leads to

$$\dot{w}(D_E) = \frac{D_E}{2\pi f} \zeta_e + \frac{D_E}{2\pi f} \frac{\partial^2 T}{\partial z^2}$$

(40)

where the second term in the righthand side

$$\dot{w}_g(D_E) = \frac{D_E}{2\pi f} \frac{\partial^2 T}{\partial z^2}$$

(41)

is the Ekman pumping due to the true gravity; and the first term

$$\dot{w}_g(D_E) = \frac{D_E}{2\pi f} \frac{\partial^2 T}{\partial z^2} \left( \nabla^2 p - \frac{\cot \phi}{R^2} \frac{\partial p}{\partial \phi} \right)$$

(42)

is due to the geostrophic vorticity. Here, Equations 21 and 39 are used. A nondimensional C-number is defined by

$$C = \frac{O\left|\dot{w}_g(D_L)\right|}{O\left|\dot{w}_e(D_L)\right|}$$

(43)

to represent the relative importance of the true gravity versus the geostrophic vorticity on the Ekman pumping. When the order of magnitude of $|\dot{w}_g(D_L)|$ is non-negligible in comparison to the order of magnitude of $|\dot{w}_e(D_L)|$, the true gravity should be used in the atmospheric Ekman dynamics.

4. Data Sources

Two openly available datasets were used to calculate $w_g(D_E)$ and $w_e(D_E)$. The first one is the global second derivative of the disturbing static gravity potential $\partial^2 T/\partial z^2$ data from the static gravity model EIGEN-6C4 (Kostelecký et al., 2015) at http://icgem.gfz-potsdam.de/home. The EIGEN-6C4 was developed jointly by the GFZ Potsdam and GRGS Toulouse up to degree and order 2,190, on $1^\circ \times 1^\circ$ grids (Figure 1), with $-603.6$ Eotvos as the minimum and 642.8 Eotvos as the maximum ($1$ Eotvos = $10^{-9}$ s$^{-2}$). The second one is the global climatological July mean sea level pressure ($p$) data on $2^\circ \times 2^\circ$ grids (Figure 2) from the Comprehensive Ocean-Atmosphere Data Set (COADS) (Slutz et al., 1985) at https://iridl.ldeo.columbia.edu/SOURCES/. COADS/.

5. Global Ekman Pumping Velocity due to $\partial^2 T/\partial z^2$

With the openly available EIGEN-6C4 $\partial^2 T/\partial z^2$ and COADS sea level pressure ($p$) data, the global Ekman pumping velocity due to $\mathbf{g}_h$, $|\dot{w}_g(D_E)|$ is calculated using Equation 41 (Figure 3a), and due to geostrophic vorticity $\zeta_e|\dot{w}_g(D_E)|$ is calculated using Equation 42 (Figure 4a). The equatorial region ($5^\circ$S–$5^\circ$N) is not included since the geostrophic balance does not exist there. The histogram of $\dot{w}_g(D_E)$ shows the Gaussian
type distribution (Figure 3b), and the histogram of \( |\omega_{tg}(D_{E})| \) (Figure 3c) shows the near Gamma distribution with the mean of 3.69 cm s\(^{-1}\), and standard deviation of 10.26 cm s\(^{-1}\). Similarly, the histogram of \( \omega_{gv}(D_{E}) \) shows the Gaussian type distribution (Figure 4b), and the histogram of \( |\omega_{gv}(D_{E})| \) (Figure 4c) shows the near Gamma distribution too with the mean of 0.33 cm s\(^{-1}\), and standard deviation of 0.44 cm s\(^{-1}\). If the global mean value is treated as the order of magnitude, the C-number can be estimated by

\[
C \approx \frac{O\left[\omega_{tg}(D_{E})\right]}{O\left[\omega_{gv}(D_{E})\right]} \frac{\text{mean}\left[\omega_{tg}(D_{E})\right]}{\text{mean}\left[\omega_{gv}(D_{E})\right]} = \frac{3.69 \text{ cm/s}}{0.33 \text{ cm/s}} = 11.18
\]

which is surprisingly large. Note that the EIGEN-6C4 is not the only one static gravity models available in the geodetic community. The high C-number (11.18) obtained here is only for a specific gravity model EIGEN-6C4 and a specific atmospheric density field (Equation 23). With other gravity models and atmospheric density fields, the C-number will vary. However, it clearly shows that \( g_{h} \) cannot be neglected in the Ekman layer dynamics.

6. True-Vertical Coordinate Versus Deflected-Vertical Coordinate

As mentioned in the Introduction section, the true vertical direction \( e_{3} \) (upward positive) is with the true gravity \( g \).

\[
g(\lambda, \varphi, z) = -g(\lambda, \varphi, z)e_{3}(\lambda, \varphi, z).
\]

The true horizontal surfaces are the equipotential surfaces of the true gravity \( V(\lambda, \varphi, z) \). The geoid is one of them. On a true horizontal surface, the orthogonal unit vectors are represented by \( e_{1}(\lambda, \varphi, z), e_{2}(\lambda, \varphi, z) \), but not \( (i, j) \). With such a true-vertical coordinate, the true gravity \( g \) has the vertical component only with
no true-horizontal component. This treatment seems attractive to meteorologists. However, it is not feasible at all since the unit vectors $[\mathbf{e}_1(\lambda, \phi, z), \mathbf{e}_2(\lambda, \phi, z), \mathbf{e}_3(\lambda, \phi, z)]$ vary at each point inside the troposphere, and it is almost impossible to convert any atmospheric model (theoretical or numerical) with the standard gravity ($-g_0 \mathbf{k}$) into the model with the true gravity $g$ using the reference coordinates with the unit vectors $[\mathbf{e}_1(\lambda, \phi, z), \mathbf{e}_2(\lambda, \phi, z), \mathbf{e}_3(\lambda, \phi, z)]$. The alternative treatment is to keep the deflected-vertical direction $\mathbf{k}$ and deflected-horizontal surface ($i, j$). The unit vectors ($i, j, k$) are independent on ($\lambda, \phi, z$). It is easy to replace the standard gravity ($-g_0 \mathbf{k}$) by the true gravity $g (= g_h - g_0 \mathbf{k})$ into any atmospheric models.

7. Conclusions

The Earth gravity has three forms: standard gravity ($-g_0 \mathbf{k}$), effective gravity [$-g(\phi) \mathbf{K}$], and true gravity $g(\lambda, \phi, z)$. Meteorologists only use the standard gravity (most often) and effective gravity, but not the true gravity. The deflected-vertical (i.e., normal to the Earth spherical/ellipsoidal surface) is treated as the “vertical,” the deflected-horizontal (i.e., the Earth spherical/ellipsoidal surfaces) is taken as the “horizontal.” The true gravity $g(\lambda, \phi, z)$ has latitudinal-longitudinal (i.e., deflected-horizontal) component $g_h$, which is not the latitudinal component of the effective gravity and neglected completely in the dynamic equation.

This study demonstrates the importance of $g_h$ in the Ekman layer dynamics. With the constant eddy viscosity $K$ and the e-folding type height-decreasing density, new equation for the atmospheric Ekman layer

Figure 2. Climatological July mean (a) sea level pressure $p$ (unit: hPa), obtained from the Comprehensive Ocean-Atmosphere Data Set at https://iridl.ldeo. columbia.edu/SOURCES/.COADS/. Negative (positive) geostrophic vorticity ($\zeta_g$) associated with high (low) pressure leads to the downward (upward) Ekman pumping velocity.
dynamics was derived including both geostrophic vorticity $\zeta_g$ and $g_h$. The two openly available datasets of $\frac{\partial^2 T}{\partial z^2}$ from the static gravity model Eigen-6C4 and climatological July mean sea level pressure ($p$) over the global oceans were used to obtain the mean of $|w_t(G_x)|$ as 3.69 cm s$^{-1}$, and the mean of $|w_v(G_y)|$ as 0.33 cm s$^{-1}$. It clearly shows the importance of $g_h$ in the atmospheric Ekman layer dynamics.

Note that the results in this study is only for the specially selected density field represented by the e-folding stratification not for the density in the real atmosphere and with the one particular gravity model (i.e., EIGEN-6C4). Thus, an atmospheric general circulation model is needed to thoroughly investigate the effects.
of $g_c$ in atmospheric dynamics and weather forecast using the true gravity data $g(\lambda, \varphi, z)$ from various gravity models.

Finally, the **geopotential** is the negation of the gravity potential. The existing definition of the geopotential $[-E(z) = g_c z]$ or $[-E(z) = g(\varphi) z]$ in meteorology needs to be replaced by $[-V = g(\varphi) z - T]$. It may drastically impact atmospheric dynamics, weather forecast, and climate change.

**Figure 4.** (a) Ekman pumping velocity (cm s$^{-1}$) due to the geostrophic vorticity, $w_{gv}(D_p)$ over the global oceans, calculated with the Comprehensive Ocean-Atmosphere Data Set climatological July mean sea level pressure ($p$) data, (b) histogram of $w_{gv}(D_p)$, and histogram of $|w_{gv}(D_p)|$.
Data Availability Statement

Data from the EIGEN-6C4 model at the website: http://icgem.gfz-potsdam.de/home, and the COADS for the climatological July mean sea level pressure data at the website: https://iridl.ldeo.columbia.edu/SOURCES/.COADS/.

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