KOPPELMAN FORMULAS ON FLAG MANIFOLDS

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ABSTRACT. We construct Koppelman formulas on manifolds of flags in \( \mathbb{C}^N \) for forms with values in any holomorphic line bundle as well as in the tautological vector bundles and their duals. As an application we obtain new explicit proofs of some vanishing theorems of the Bott-Borel-Weil type by solving the corresponding \( \bar{\partial} \)-equation. We also construct reproducing kernels for harmonic \((p, q)\)-forms in the case of Grassmannians.

1. INTRODUCTION

The classical Koppelman formula is an integral formula of the kind

\[
\varphi(z) = \int_{\partial\Omega} K \wedge \varphi + \int_{\Omega} K \wedge \bar{\partial} \varphi + \bar{\partial} z \int_{\Omega} K \wedge \varphi + \int_{\Omega} P \wedge \varphi,
\]

which represents differential forms on a smoothly bounded domain \( \Omega \subseteq \mathbb{C}^n \). Here \( K \) is an integrable differential form on \( X \times X \), and \( P \) is a smooth one. The formula (1) is equivalent to the equation of currents

\[
\bar{\partial} K = [\Delta] - P,
\]

where \( [\Delta] \) is the \((n, n)\)-current given by integration over the diagonal \( \Delta \subseteq \mathbb{C}^n \times \mathbb{C}^n \).

In this paper we consider the case of the complex manifold \( X \) of all flags \( V_1 \subseteq \cdots \subseteq V_k \) of linear subspaces of \( \mathbb{C}^N \) of a given type, and construct integral formulas for smooth differential forms with values in certain holomorphic vector bundles \( V \to X \). To be more precise we construct integral kernels \( K \) and \( P \), integrable and smooth respectively, so that the formula (2) holds for any smoothly bounded domain \( \Omega \subseteq X \) and smooth \((p, q)\)-form on \( \Omega \) with values in the holomorphic line bundle \( L \to X \) (or in any of the tautological vector bundles). Here \( K \) is an integrable section of the bundle \( V \boxtimes V^* \otimes \mathbb{T}_\mathbb{C}^N \to X \times X \), and \( P \) is a smooth section of the same bundle. In this setting the current \( [\Delta] \) is given by the tensor product of the integration current associated to \( \Delta \subseteq X \times X \) and the identity section of \( (V \boxtimes V^*)|_\Delta \).

We apply the formula (1) to prove certain vanishing theorems for Dolbeault cohomology groups with values in line bundles. Indeed, if we let \( \Omega \) be equal to \( X \), the boundary term on the right hand side of the formula vanishes. Therefore \( \bar{\partial} \)-closed forms \( \varphi \) are represented by the third plus the fourth term, and hence the third terms provides a solution to the \( \bar{\partial} \)-equation if one can prove that the fourth term vanishes.

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The approach of this paper is a straightforward generalization of the one in [3] which deals with Grassmannians. That paper in turn is an application of (a slight generalization of) the method developed in [2] for constructing integral formulas on complex manifold. The key ingredient is to have a manifold $X$ admitting the Diagonal Property, i.e., that $X \times X$ admits a holomorphic vector bundle $E \to X \times X$ of rank equal to the dimension of $X$, and a holomorphic section $\eta$ of $E$ which vanishes to the first order on the diagonal $\Delta \subseteq X \times X$ and is nonzero elsewhere. We will therefore be very brief and refer to [3] and [2] for the framework in which our constructions take place, as well as for a historical background on integral formulas.

The paper is organized as follows. In Section 2 we recall some preliminaries on flag manifolds and related vector bundles. In Section 3 we show that $X$ admits the Diagonal property, and use this to give a Koppelman formula for scalar valued differential forms. In Section 4 we construct weights which allow us to give integral formulas for forms with values in vector bundles. Section 5 is concerned with applications. We prove vanishing theorems for Dolbeault cohomology groups, and, in the special case of Grassmannians, we also identify the fourth term on the right hand side in (1) with the reproducing kernel for the space of harmonic $(p, q)$-forms.

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2. Preliminaries

2.1. Flag manifolds. A flag $F$ of type $d := (d_1, \ldots, d_k)$ is a $k$-tuple $F = (V_1, \ldots, V_k)$ of subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k$ of $\mathbb{C}^N$, where $\dim V_i = d_i$, and $V_k = \mathbb{C}^N$. We fix the flag $F_0 = (V_1^0, \ldots, V_k^0)$, with $V_i^0 = \text{Span}_{\mathbb{C}}\{e_1, \ldots, e_{d_i}\}$, where $\{e_1, \ldots, e_N\}$ is the standard basis for $\mathbb{C}^N$.

Let $X$ be the set of all flags of subspaces of type $(d_1, \ldots, d_k)$ of $\mathbb{C}^N$. The group $G := GL(N, \mathbb{C})$ then acts transitively on $X$, and the stabilizer of the reference point $(V_1^0, \ldots, V_k^0)$ is

$$P := \left\{ \left( \begin{array}{ccc} A_1 & * & * \\ 0 & \ddots & * \\ 0 & \cdots & A_k \end{array} \right) \in GL(N, \mathbb{C}) \right\}$$

where $A_1$ is a block of size $d_1 \times d_1$, and $A_i$ is a block of size $(d_i - d_{i-1}) \times (d_i - d_{i-1})$ for $i = 2, \ldots, k$.

On the level of Lie algebras, we have the decomposition

$$g = n \oplus p,$$

with

$$p := \left\{ \left( \begin{array}{ccc} A_1 & * & * \\ 0 & \ddots & * \\ 0 & \cdots & A_k \end{array} \right) \in M_{NN}(\mathbb{C}) \right\},$$

$$n := \left\{ \left( \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ B_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ B_{(k-1)1} & \cdots & B_{(k-1)(k-1)} & 0 \end{array} \right) \in M_{NN}(\mathbb{C}) \right\},$$
where $B_{i1}$ is a block of size $(d_{i+1} - d_i) \times d_1$, and $B_{ij}$ is a block of size $(d_{i+1} - d_i) \times (d_j - d_{j-1})$, $j = 2, \ldots, k - 1$. From this, we see that $X$ is a complex manifold of dimension

$$n := \dim \mathbb{C} \mathbb{N} = \sum_{j=1}^{k-1} (N - d_j)(d_j - d_{j-1}) = \sum_{j=1}^{k-1} d_j(d_{j+1} - d_j).$$

Here we use the convention that $d_0 = 0$.

2.2. **Vector bundles over $X$.** For each $i = 1, \ldots, k$, we have a tautological holomorphic vector bundle

$$H_i := \{(F, v) \in X \times \mathbb{C}^N \mid v \in V_i\},$$
and a corresponding quotient bundle $F_i := X \times \mathbb{C}^N / H_i$. We let $q_i : X \times \mathbb{C}^N \to F_i$ denote the holomorphic projection map, and

$$Q_i : F_i \to F_{i+1}, \quad v + H_i \mapsto v + H_{i+1}$$

be the natural quotient map.

Note that the restriction of the Euclidean metric on $\mathbb{C}^N$ equips each $H_i$ with a Hermitian metric which is invariant under the action of the group $U(N)$. Moreover, we can identify the quotient bundle $F_i$ with the orthogonal complement, $H_i^\perp$, of $H_i$. We let $\varphi_i : F_i \to H_i^\perp$ denote this identification, the inverse of which is the restriction of $q_i$ to $H_i^\perp$. These identifications allow us to define $U(N)$-invariant Hermitian metrics on the $F_i$. In the sequel, we will be concerned with the Cartesian product $X \times X$. We will denote points in the first component by $z$, and points in the second component by $\zeta$. We let $H_{i,z} \to X \times X$ denote the pullback bundle of $H_i$ with respect to the projection of $X \times X$ onto the first component. Analogously, we define $H_{i,\zeta}$ and the corresponding constructions for the $F_i$. Finally, we let $q_{i,z} : X \times X \times \mathbb{C}^N \to F_{i,z}$ denote the natural quotient morphism.

2.3. **The Picard group of $X$.** Since $X$ can be described as quotient of $G$ by a parabolic subgroup, it follows that any holomorphic line bundle over $X$ is homogeneous under $G$, i.e., corresponds to a holomorphic character $\chi : P \to \mathbb{C}^\times$ (cf. [6]). It is well known that the group of holomorphic characters of $P$ is generated by the characters $\chi_1, \ldots, \chi_k$, where

$$\chi_i(P) = \det A_1 \cdots \det A_i,$$

where $P = LU$ is the Levi decomposition of $P$ with

$$L := \left\{ \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & A_k \end{pmatrix} \in GL(N, \mathbb{C}) \right\},$$

$$U := \left\{ \begin{pmatrix} I_{d_1} & * & * \\ 0 & \ddots & * \\ 0 & \cdots & I_{(d_k - d_{k-1})} \end{pmatrix} \in GL(N, \mathbb{C}) \right\}.$$

The corresponding homogeneous line bundle

$$L_i := G \times \mathbb{C}$$
is equivalent to \( \det H_i := \wedge^d_i H_i \) by the morphism of \( G \)-homogeneous line bundles given by
\[
[(g,v)] \mapsto g.v,
\]
where the \( G \)-action on the right hand side is the linear action on \( \wedge^d_i \mathbb{C}^N \) induced from the standard action on \( \mathbb{C}^N \). The dual line bundles \( L_i^{-1} := L_i^* \) are ample, and we have isomorphisms \( \wedge^d_i \mathbb{C}^N \to H^0(X, L_i^{-1}) \) given by
\[
v \mapsto s_v,
\]
where \( s_v(z).u := \langle u, v \rangle, \quad v \in \wedge^d_i \mathbb{C}^N, \quad u \in \wedge^d_i H_i \).

where the pairing \( \langle \cdot, \cdot \rangle \) is given by the natural inner product on \( \wedge^d_i \mathbb{C}^N \) induced from the one on \( \mathbb{C}^N \).

3. The Diagonal Property

Consider the holomorphic vector bundle \( \sum_{i=1}^{k-1} F_{i,z} \otimes H_i^* \), which has rank \( \sum_{i=1}^{k-1} d_j (N-d_j) \). We define a holomorphic section \( \eta \) of this bundle by
\[
\eta(z,\zeta) := \sum_{i=1}^{k-1} \eta_i(z,\zeta),
\]
\[
\eta_i(z,\zeta) v := q_{i,z}(v), \quad v \in H_i \subset \mathbb{C}^N.
\]
Then \( \eta \) defines the diagonal \( \Delta \subset X \times X \), i.e., \( \eta \) vanishes to first order along \( \Delta \) and is non-zero elsewhere. Moreover, \( \eta \) is a section of the subbundle \( E := \ker \Phi \), where
\[
\Phi : \sum_{i=1}^{k-1} \text{Hom}((H_i)^* \otimes (F_i) \otimes) \rightarrow \sum_{i=1}^{k-2} \text{Hom}((H_i)^* \otimes (F_{i+1}) \otimes),
\]
\[
\Phi(\sum_{i=1}^{k-1} f_i) := \sum_{i=1}^{k-2} f_{i+1} \circ Q_i \circ f_i.
\]

The map \( \Phi \) is surjective, and it follows that \( E \) is of rank \( n (= \dim X) \). We thus have

**Proposition 1.** The section \( \eta \) is a holomorphic section of the rank-\( n \) bundle \( E \) and \( \eta \) defines the diagonal \( \Delta \subset X \times X \).

**Remark 2.** The construction of \( \eta \) occurred to us rather soon as a generalization of the construction for Grassmannians (cf. [3]). However, we did not realize that it defines a section of a subbundle of the right rank, and therefore dismissed it. It was pointed out to us by Prof. Pragacz that it actually is a section of \( E \). We are very grateful to him for this.

Equip \( E \) with the \( U(N) \)-invariant Hermitian metric (here \( U(N) \) is identified with the diagonal subgroup of \( U(N) \times U(N) \) \( \langle \cdot, \cdot \rangle_E \)) induced from the Euclidean metric on \( \mathbb{C}^N \) via the tautological vector bundles \( H_i \) and their quotients \( F_i \). Let \( D_E \) be the Chern connection with respect to this metric, and let \( \Theta_E \) be its curvature, and let \( \tilde{D}_E \) and \( \tilde{\Theta}_E \) denote their natural images as sections of the vector bundle
\[
G_E := \Lambda[T^* X \times X \otimes E \oplus E^*] \rightarrow X \times X.
\]
Let $\delta_\eta$ be mapping $\wedge E^* \to \wedge E^*$ defined by interior multiplication by $\eta$, and let $\sigma$ be the section of $E^*$ given by forming the inner product with the normalization of $\eta$, i.e.,
\[ \sigma(z, \zeta)(v) := \frac{(v, \eta(z, \zeta))_E}{|\eta(z)|^2_E}. \]
Then $\delta_\eta \sigma = 1$ on $X \times X \setminus \Delta$. Moreover, put $u = \frac{\sigma}{\nabla_\sigma} = \sum_{k=1}^n \sigma \wedge (\partial \sigma)^{k-1}$, which is a section of $G_E$. Finally, let $\int_E : G_E \to T^*(X \times X)$ be the mapping which takes a point to the coefficient of the term with degree $n$ in both $E$ and $E^*$. We then have the following Koppelman formula for scalar valued differential forms, see Theorem 3.4 in [2].

**Theorem 3.** Define $K$ and $P$ by
\[
K(z, \zeta) := \int_E u \wedge \left( \frac{D_E \eta}{2\pi i} + \frac{i \bar{\Theta}_E}{2\pi} \right)_n, \quad P(z, \zeta) := \int_E \left( \frac{D_E \eta}{2\pi i} + \frac{i \bar{\Theta}_E}{2\pi} \right)_n,
\]
where $(\cdot)_n = (\cdot)^n/n!$. Then $K$ is integrable and $P$ is smooth, and moreover, $K$ and $P$ satisfy the equation of currents
\[ \partial K = [\Delta] - P. \]
Notice that for degree reasons we have $P = \int_E (i \bar{\Theta}_E/2\pi)_n = \det(i \Theta/2\pi) = c_n(E)$ so that $P$ is the $n$th Chern form of $E$.

4. **Weighted integral formulas**

We recall the definition of a weight for a holomorphic vector bundle $V \to X$. We first put $G_{E,V} := \text{Hom}(V_\zeta, V_\zeta) \otimes G_E$, and define the operator
\[ \nabla_\eta := \delta_\eta - \bar{\partial} \]
on the space of smooth sections of $G_{E,V}$.

**Definition 4.** Let $g = g_{0,0} + \ldots + g_{n,n}$ be a smooth section of $G_{E,V}$, where $g_{k,k}$ is a section of $\Lambda^k E^* \wedge T^*_{0,1}(X \times X)$. Then $g$ is called a weight for $V$ if $\nabla_\eta g = 0$ and $g_{0,0} |_\Delta = \text{Id} \in \text{Hom}(V_\zeta, V_\zeta) |_\Delta$.

Now, following [2], p. 54, one easily shows that
\[
K_g(z, \zeta) := \int_E g \wedge u \wedge \left( \frac{D_E \eta}{2\pi i} + \frac{i \bar{\Theta}_E}{2\pi} \right)_n, \quad P_g(z, \zeta) := \int_E g \wedge \left( \frac{D_E \eta}{2\pi i} + \frac{i \bar{\Theta}_E}{2\pi} \right)_n
\]
satisfy (4).

4.1. **Weights for the tautological vector bundles.** To begin with, we define a section $\gamma_{0,i}$ of the bundle $H_i \otimes H_i^*$ by
\[ \gamma_{0,i}(z, \zeta)v := \pi_{H,i,z} v, \quad v \in H_i \zeta. \]
Next, we define $\gamma_{1,i} \in \Gamma(X \times X, (H_i \otimes H_i^*) \otimes E^* \otimes T^*_{0,1})$. Let $\xi$ and $v$ be (germs of) smooth sections of $E$ and $H_i \zeta$ respectively, and let $\xi$ be decomposed as $\xi = \xi_1 + \cdots + \xi_k$, where $\xi_i \in (F_i)_\zeta \otimes (H_i^*)_\zeta$. Then we let
\[ \gamma_{1,i}(\xi \otimes v) := -\pi_{H,i}(\bar{\partial} \varphi_i(\xi_i(v))). \]
As a linear operator $\Gamma(X \times X, E \otimes H_i \zeta) \to \Gamma(X \times X, H_i \zeta \otimes T^*_{0,1})$ is $C^\infty(X \times X)$-linear (cf. [3]), and hence defines a section of the appropriate bundle.
Proposition 5. The section \( G_i = G_i(z, \zeta) := \gamma_{0,i} + \gamma_{1,i} \) is a weight for the tautological vector bundle \( H_i \).

Proof. In view of the identity \( \delta_\eta \gamma_{1,i} = \delta_\eta \gamma_{1,i} \), the statement follows immediately from the proof of the analogous statement for the Grassmannian of \( d_i \)-dimensional subspaces of \( \mathbb{C}^N \) (cf. [3]). \( \square \)

4.2. Weights for line bundles. We recall from [3] that we can form exterior powers, tensor powers, and duals of weights. The following proposition is an immediate consequence of this construction.

Proposition 6. Define the section \( g_i := \wedge^d G_i \). Then we have

a) \( g_i \) is a weight for the line bundle \( L_i \),

b) the section \( g_i^{-1} := g_i^* \) is a weight for \( L_i^* \),

c) the section \( g_i^{m_1} \otimes g_i^{m_2} \) is a weight for the line bundle \( L \) with factorization \( L = L_1^{m_1} \otimes \cdots \otimes L^{m_k} \), into integer powers of the generators \( L_i \).

5. Applications

5.1. Vanishing theorems. In this section we prove a vanishing theorem of the Bott-Borel-Weil type for Dolbeault cohomology. We reduce the problem to the case of Grassmannians by fiber bundle techniques. In the Grassmannian case the corresponding vanishing theorems are proved explicitly by solving the \( \bar{\partial} \)-equation using Koppelman formulas.

Theorem 7. For \( i = 1, \ldots, k \), the cohomology group \( H^q(X, L_i) \) is trivial for \( r \leq 0 \) and \( q > 0 \).

Proof. Let \( Y := \text{Gr}(d_i, N) \) be the Grassmannian of \( d_i \)-dimensional subspaces of \( \mathbb{C}^n \), and let \( \tilde{L}_i \rightarrow Y \) be the determinant of the tautological vector bundle over \( Y \). Let \( \tau : X \rightarrow Y \) be the natural projection \( (V_1, \ldots, V_k) \mapsto V_i \). Then \( \tau \) defines a holomorphic fiber bundle with fiber

\[
\tau^{-1}(W) = X_{W}^1 \times X_{W}^2,
\]

where \( X_{W}^1 \) is the manifold of all flags \( V_1 \subset \cdots \subset V_{i-1} \subset W \) with \( \dim V_j = d_j \), for \( j = 1, \ldots, i-1 \), and \( X_{W}^2 \) is the manifold of all flags \( W \subset V_{i+1} \subset \cdots \subset V_k = \mathbb{C}^N \), with \( \dim V_j = d_j, j = i+1, \ldots, k \). Moreover, \( \tau^* \tilde{L}_i = L_i \). Let \( \{ U_j \} \) be an open cover of \( Y \) such that both the bundles \( \tilde{L}_i \rightarrow Y \) and \( X \rightarrow Y \) are locally trivial over each \( U_j \), and such that \( U_j \) is biholomorphically equivalent to an open ball in \( \mathbb{C}^m \) (where \( m=\dim Y \)). Since \( \tau \) has compact and connected fibers, the identity

\[
H^0(\tau^{-1}(O), L_i^r) \cong H^0(O, \tilde{L}_i^r)
\]

holds for any sufficiently small open ball \( O \subseteq Y \), and \( r \in \mathbb{Z} \). Let \( \mathcal{F}(r) \) denote the sheaf of germs of holomorphic sections of \( L_i^r \), and let \( \tilde{\mathcal{F}}(r) \) denote the sheaf of germs of holomorphic sections of \( \tilde{L}_i^r \). From the isomorphism (6) it follows that we have the isomorphism

\[
\tau_* \mathcal{F}(r) \cong \tilde{\mathcal{F}}(r),
\]

of sheaves over \( Y \). Here \( \tau_* \) is the direct image of \( \mathcal{F}(r) \). Recall that that the \( q \)th direct image of \( \mathcal{F}(r) \), \( R^q \tau_* \mathcal{F}(r) \), is the sheafification of the presheaf \( O \mapsto \)
with respect to the Laplacian $d\bar{d}$.

**Proof.** To begin with, observe that the curvature tensor $U$ from [4, Exercise B2, p.227], that a associated to projection onto the space of harmonic forms in the special case of Grassmannians. Let the projection onto the space of harmonic $(p,q)$-forms, i.e., the cohomology of the trivial line bundle. We can now prove by induction on $n = \dim X$ that $H^q(X, L_i^r) = 0$ for $r \leq 0$. If $n = 1$, then $X = P^1(\mathbb{C})$, and the statement is proved by explicitly by solving the $\partial$-equation using a Koppelman formula in [3, Thm. 12]. For the induction step, assume that the claim is proved for all manifolds of flags of dimension strictly smaller than $n$. In particular, $H^q(X^1_W, O) = 0$, and $H^q(X^2_W, O) = 0$ for $q > 0$ (cf. [1, Ch. IX, Cor. 5.22]). In [3, Thm. 12], the claim that the isomorphism

$$H^q(X, L_i^r) \cong H^q(Y, \tilde{L}_i^r)$$

(cf. [1, p.234, Cor. 13.9]). In [3, Thm. 12], the claim that $H^q(Y, \tilde{L}_i^r) = 0$ for $q > 0$ and $r \leq 0$ is proved by solving the $\tilde{\partial}$-equation explicitly using a Koppelman formula. The statement therefore follows from the induction hypothesis.

5.2. Harmonic forms. In this section we will construct reproducing kernels for harmonic forms in the special case of Grassmannians. Let therefore $X := G(m, N)$ be the Grassmannian of $m$-dimensional subspaces of $\mathbb{C}^N$, and let $A^{p,q}(X)$ denote the space of $(p,q)$-forms on $X$.

We recall the Hodge-decomposition

$$A^{p,q}(X) = \mathcal{H}^{p,q}(X) \oplus \bar{\partial}A^{p,q-1}(X) \oplus \bar{\partial}^*A^{p,q+1}(X)$$

(8)

into an orthogonal sum with respect to the $L^2$-inner product

$$\langle \varphi, \psi \rangle := \int_X \varphi \wedge * \psi,$$

where $*$ denotes the (antilinear) Hodge operator defined by a $U(N)$-invariant Hermitian metric on $T(X)$. Here $\mathcal{H}^{p,q}(X)$ denotes the harmonic $(p,q)$-forms, i.e., the kernel of the operator $\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} : A^{p,q}(X) \to A^{p,q}(X)$.

**Theorem 8.** The operator $A^{p,q}(X) \to \mathcal{H}^{p,q}(X)$, $\varphi \mapsto \int_X \varphi \wedge P$ is the orthogonal projection onto the space of harmonic $(p,q)$-forms.

**Proof.** To begin with, observe that the curvature tensor $\Theta$ of $E = F_z \otimes H^2_2$ is associated to $U(N)$-invariant metrics on the respective bundles, and is therefore $U(N) \times U(N)$-invariant. Since $X$ is a Riemannian symmetric space, it follows, e.g., from [4, Exercise B2, p.227], that a $U(N)$-invariant $(p,q)$-form on $X$ is harmonic with respect to the Laplacian $dd^* + d^*d$. But since $X$ is Kähler we have

$$\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} = \frac{1}{2}(dd^* + d^* d)$$
and we see that $U(N)$-invariant $(p,q)$-forms indeed are harmonic. It follows that $z \mapsto P(z, \zeta), \zeta \mapsto P(z, \zeta)$, and $\int \varphi \wedge P$ are harmonic for any $\varphi \in A^{p,q}(X)$.

Secondly, if $\varphi$ is harmonic, the Koppelman formula yields

$$\varphi = \bar{\partial} \int \varphi \wedge K + \int \varphi \wedge P.$$  

Since the second term on the right hand side is harmonic, it follows from (8) that $\bar{\partial} \int \varphi \wedge K = 0$, i.e., that $\varphi = \int \varphi \wedge P$.

Assume now that $\varphi = \bar{\partial} \psi$ for some $\psi \in A^{p,q-1}(X)$. Then, since $\zeta \mapsto P(z, \zeta)$ is $\bar{\partial}$-closed,

$$\int \bar{\partial} \psi \wedge P = \int \bar{\partial} (\psi \wedge P) = \int d(\psi \wedge P) = 0.$$  

Finally, if $\varphi = \bar{\partial}^* \psi$ for some $\psi \in A^{p,q+1}(X)$,

$$\int \bar{\partial}^* \varphi \wedge P = \pm \langle \bar{\partial}^* \psi, * P \rangle = \pm \langle \psi, \bar{\partial}^* P \rangle = \pm \langle \psi, * \bar{\partial}^* P \rangle = 0$$

since $\zeta \mapsto P(z, \zeta)$ is $\bar{\partial}^*$-closed. This proves the theorem. \hfill \square

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