A Note on the Set of After-measurement States
in Generalized Quantum Measurement

I.D. Ivanovic
Physics Department, Carleton University
Ottawa ON. Canada
igor@physics.carleton.ca

Abstract The sets of after-measurement states for standard and generalized quantum measurements are compared. It is shown that for a SIC-POVM generalized measurement, the ratio of the volume of the set of after-measurement states and the volume of the simplex generated by individual outcomes quickly tends to zero with increase of the number of dimensions. The volumes used are based on the Hilbert-Schmidt norm. Some consequences on actual realizations, having finite collections of systems are discussed.

States and Standard Measurements. Few well known facts about standard measurements may be useful. Let the system be described in a d-dimensional complex Hilbert space \( H \). A pure state is a projector onto a normalized vector \( |\Psi\rangle, \langle \Psi|\Psi\rangle = 1 \), and its state is a ray projector \( P = |\Psi\rangle\langle \Psi| \). Generally, a state is an operator on \( H \) satisfying \( W \geq 0, \text{tr}(W) = 1 \). A state can be expressed as a weighted sum of its eigen-projectors \( W = \sum_k r_k P_k \) where \( \sum_k r_k = 1 \), and \( P_k P_r = \delta_{kr} P_k \). The set of all states over \( H \) will be denoted by \( \mathcal{V}_W = \{ W | W \geq 0, \text{tr}(W) = 1 \} \). An orthogonal ray-resolution of the identity (ORRI) over \( H \) is a set of projectors \( \{ P_k \} \) satisfying

\[
\sum_k P_k = I_H, \quad P_k P_r = \delta_{kr} P_k, \quad \text{tr}(P_k) = 1
\]

The convex set of all convex combinations of an ORRI \( \{ P_k \} \), \( \text{conv}(\{ P_k \}) \) is a commutative simplex, identical to the classical, discrete probability simplex, having pure states \( P_k \) as extremal points and normalized identity \( W_o = \frac{1}{d} I_H \) as its baricenter. The set of all states, \( \mathcal{V}_W \), can be obtained by applying all unitary transformations to an initial, commutative, simplex. The point common to all simplices is \( W_o \).

The most natural way to look at \( \mathcal{V}_W \) is as a convex set in the space of Hermitian operators over \( H \) using Hilbert-Schmidt distance. A standard measurement, defined by complete, nondegenerate observable \( A \), having
ORRI \( \{P_k\} \) is represented by a change of state

\[
W_{am} = \sum_k P_k W_{pm} P_k = \sum_k \text{tr}(WP_k) P_k
\]

where \( p(a_k) = \text{tr}(W_{pm} P_k) \) and \( \langle A \rangle = \sum_k p(a_k) a_k = \text{tr}(AW_{pm}) = \text{tr}(AW_{am}) \). Here \( W_{pm} \) is a pre-measurement state and \( W_{am} \) is the after-measurement state of the system or to be more precise of an infinite ensemble of systems.

Formally, \( W_{am} \) is an orthogonal projection of \( W_{pm} \) onto the simplex defined by the \( \{P_k\} \). Again, the easiest way to visualize this is to deduct the \( W_o \) from all states, working in the hyperplane \( \text{tr}(A) = 1 \), then the simplex of commuting states defined by \( \{P_k\} \) is in e.g. two dimensions is a segment of length \( \sqrt{2} \). The midpoint is \( W_o \). In three dimensions a commutative simplex is equilateral triangle, edge \( \sqrt{2} \), the baricenter is, as always, \( W_o \).

Once an ORRI \( \{P_k\} \) is given one may identify the three set of states:

i) set of \( \text{tr} = 1 \) linear combinations of \( P_k \)'s,

\[
V(\{P_k\}) = \{W|W = \sum_k a_k P_k, \sum a_k = 1, W \geq 0 \},
\]

ii) set of all convex combinations of \( P_k \)'s,

\[
\text{conv}(\{P_k\}) = \{W|W = \sum_k a_k P_k, \ a_k \geq 0, \ \sum a_k = 1 \}
\]

iii) the set of all possible aftermeasurement states

\[
V_{am}(\{P_k\}) = \{W|W = \sum_k P_k W_{pm} P_k, W_{pm} \in V_W \}.
\]

All three sets are identical

\[
V(\{P_k\}) = \text{conv}(\{P_k\}) = V_{am}(\{P_k\}).
\]

Furthermore, this type of measurement, corresponding to an ORRI, can be selective e.g. when the systems are 'tagged' and states with outcome \( a_k \) are selected. The other type is non-selective when \( W_{am} \) is all we know about the state. ORRI measurements are repeatable i.e. immediately after e.g. \( a_k \) is observed on a system, another measurement of observable A should give the same result and a consequence it that it is also repeatable on the ensemble i.e. \( \sum_k P_k W_{am} P_k = W_{am} \). Obviously, almost no measurements satisfy these conditions but as a paradigm it mirrors our ideas of distinguishability into orthogonality.
Generalized Quantum Measurement In a generalized quantum measurement (GM) the resolution of the identity is as a rule a nonorthogonal one (NRI) and it is formally given by

$$W_{am} = \sum_k A_k W_{pm} A_k^\dagger$$

where $\sum_k A_k^\dagger A_k = I_d$ [1] but it is possible that $\sum_k A_k A_k^\dagger \neq I_d$. The possibility for a GM to displace $W_o$ indicates that there is a part of it which is a preparation, not simply a measurement. To make things simple we will consider only non-orthogonal ray resolutions (NRRI), which will be ‘stripped’ of their unitary part. Namely, using the polar decomposition $A_k = U_k Q_k$ only ray-projector factor $Q_k$ will be kept. The subset of GM we will consider is then

$$W_{am} = \sum_k c_k Q_k W_{pm} Q_k, \quad \sum_k c_k Q_k = I, \quad tr(Q_k) = 1.$$ 

where $Q_k$s are a linearly independent set. In this way, a GM is always a contraction having $W_o$ as one of the fixed points.

Realizations of GM , come as a rule, come from a Naimark-like constructions, either by expanding the space, making $H = H_S$ a subspace of a larger space, $H = H_S \oplus H_A$, or by making $H$ a factor space of $H = H_S \otimes H_A$.

In the first case[2], the original space $H_S$ is enlarged so that the NRRI $\{Q_k\}$ is a projection of an ORRI from the enlarged space i.e. $c_k Q_k = P_S P_k P_S$ where $\{P_k\}$ is an ORRI from the enlarged space $H$ and $P_S$ is the projector onto original space $H$. One must notice that the measurement should be made with $\{P_r\}$s on a state from $V_W$ and then projected or rotated back into $V_W$.

A more frequent situation is when an ancila is attached to the system. In this case, an ORI performed on the ancila, after a unitary transformation is performed on $H_S \otimes H_A$, results in an NRI measurement on the system. The most straightforward construction is given in [3].

The system in state $W_{pm}$ is attached to the ancila in a specified state e.g. $P_o^A$. A unitary transformation is then applied to the $W_{pm} \otimes P_o^A$ resulting in $U_{S\otimes A} W_{pm} \otimes P_o^A U_{S\otimes A}^\dagger$ such that

$$W_{am} = tr_A(\sum_k (I \otimes P_k^A)(U_{S\otimes A} W_{pm} \otimes P_o^A U_{S\otimes A}^\dagger)(I \otimes P_k^A)) = \sum_k c_k tr(W_{pm} Q_k) Q_k$$

(2)
One should notice that the measurement of $I_S \otimes \{P_k^A\}$ on the ancila serves only to tag the systems in $H$ while the state of the system is already the one given by eq.(2). So one needs a classical communication between the ancila and the system to identify individual systems and their states. Strictly speaking, no actual measurement is performed on the system, what happened is an unitary transformation and a 'distant' selection [4]. The state of the system, after the unitary transformation is already

$$\text{tr}_A(U_S \otimes A W_{pm} \otimes P^A U_{S \otimes A}^\dagger) = \text{tr}_A(\sum_k (I_S \otimes P^A_k)(U_S \otimes A W_{pm} \otimes P^A_k U_{S \otimes A}^\dagger)(I_S \otimes P^A_k))$$

**What is a measurement result in a GM?** In the case of a GM based on an NRRI $\{Q_k\}$, satisfying $\sum_k c_k Q_k = I, tr Q_k = 1$, NRRI defines three sets of states:

1) set of all $tr = 1$ linear combinations of $\{Q_k\}$

$$V(\{Q_k\}) = \{W | W = \sum a_k Q_k \geq 0, \sum a_k = 1, a_k - real\}$$

2) noncommutative simplex

$$\text{conv}(\{Q_k\}) = \{W | W = \sum a_k Q_k \geq 0, a_k \geq 0, \sum a_k = 1\}$$

and

3) the set of all possible after-measurement states

$$V_{am}(\{Q_k\}) = \{W | W = \sum c_k Q_k W_{pm} Q_k, W_{pm} \in V_W\}$$

It is easy to see that

$$V(\{Q_k\}) \supset \text{conv}(\{Q_k\}) \supset V_{am}(\{Q_k\})$$

The first inclusion is obvious, the second follows if one performs a measurement on one of the extremal points from $\text{conv}(\{Q_k\})$, e.g. $W_{pm} = Q_{k_o}$. The state after the measurement is

$$W_{am} = \sum c_k tr(Q_k Q_{k_o}) Q_k = c_{k_o} Q_{k_o} + \sum_{k \neq k_o} c_k tr(Q_{k_o} Q_k) Q_k$$

In order for $Q_{k_o}$ to remain an extremal point of $\text{conv}(\{Q_k\})$, $c_{k_o}$ must be 1 and $tr(Q_k Q_{k_o}) = 0$. 

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Therefore, due to nonorthogonality between the ray-projectors from \( \{Q_k\} \), and in this case the lack of repeatability, the map of at least some of the extremal points of \( \text{conv}(\{Q_k\}) \) can not remain extremal points, otherwise this NRRI would be an ORRI.

As commented in [5], if the result of a measurement on certain number of identically prepared systems is still outside of \( V_{am}(\{Q_k\}) \), one should continue with measurements till the after-measurement state touches the boundary of \( V_{am}(\{Q_k\}) \) or goes into \( V_{am}(\{Q_k\}) \). Should one continue with measurement or stop at the boundary? This, of course, has no bearing on an infinite ensemble, but it may affect any actual realization.

An interesting situation may occur in the following situation. Assume that we know nothing about \( W_{pm} \), while the resulting \( W_{am} \), after certain finite number of observations, is still outside of \( V_{am}(\{Q_k\}) \): one may be forced to change the expected values of a subset of states for the second part of the ensemble, knowing that the final result should belong to \( V_{am}(\{Q_k\}) \), or to be prepared to say that quantum mechanical description is incomplete. Furthermore an observer on S may communicate the results to ancila A, making the future results for an ORI on the ancila also more predictable.

Finally, what is actually measured? In principle, one can calculate the expected values of all observables which are a linear combinations of \( \{Q_k\} \); also, depending on the span of projectors, a position of a pre-measurement state is reduced to a better defined subset of \( V_W \).

**SIC-POVM** If an NRRI is symmetric-informationally complete SIC-POVM [6] i.e. if \( d^2 \) projectors \( \{Q_k\} \) satisfy

\[
\frac{1}{d} \sum_k Q_k = I_H ; \quad \text{tr}(Q_kQ_r) = \frac{d\delta_{kr} + 1}{(d+1)}
\]

one can make some more specific conclusions.

First, \( \{Q_k\} \) spans the operator space and \( V(\{Q_k\}) \supset V_W \). This means that any pre-measurement state may be written as \( W_{pm} = \sum_k a_k Q_k \). The after-measurement state is then

\[
W_{am} = \sum_{k,r} a_k c_r \text{tr}(Q_kQ_r)Q_r = \frac{1}{(d+1)} I + \frac{1}{(d+1)} W_{pm} = \frac{d}{(d+1)} W_o + \frac{1}{(d+1)} W_{pm}
\]
So, in this measurement all states are contracted (in the $tr(A) = 1$ hyperplane) by a factor of $\frac{1}{(d+1)}$ (cf. [7]). First thing that one may observe is that all after-measurement states must be nonsingular. One can say that unless all events from $\{Q_k\}$ occur the state is definitely not allowed as a result.

Furthermore, the set of states ”shrinks”, but the original shape of $V_W^{(d)}$ is preserved. A possible problem is that we do not have a simple characterization or parameterization of the set of states, so even if an after-measurement state is inside the sphere of radius $\frac{1}{(d+1)} \sqrt{\frac{(d-1)}{d}}$, it may not be an image of a state, rather, one would have to ”stretch” the state to its original size to establish was it actually a state or not.

Finally, the set of admissible after-measurement states shrinks really quickly with increased $d$. Due to the fact that all three sets

$$V_W \supset conv(\{Q_k\}) \supset V_{am}(\{Q_k\})$$

have the same dimensions, one can compare their volumes.

The volume of the $conv(\{Q_k\})$ in the hyperplane $tr(A) = 1$, which is a $d^2 - 1$ dimensional simplex of edge $\left(\frac{2d}{d+1}\right)^{1/2}$ is

$$\mathcal{V}(conv\{Q_k\}) = \left(\frac{2d}{d+1}\right)^{\frac{d^2-1}{2}} \frac{d}{(d^2-1)!} \frac{(d^2-1)^2}{2^d}.$$ 

The volume of states is, cf. [8],

$$\mathcal{V}(V_W) = \sqrt{d} (2\pi)^d \frac{d(d-1)}{2} \frac{\Gamma(1) \ldots \Gamma(d)}{\Gamma(d^2)}$$

and the volume of the after-measurement states (results) for a SIC-POVM $\{Q_k\}$ is then

$$\mathcal{V}(V_{am}(\{Q_k\})) = \frac{\mathcal{V}(V_W)}{(d+1)^{(d^2-1)}}$$

As a result, almost immediately, even for small $d$’s

$$\frac{\mathcal{V}(conv(\{Q_k\}))}{\mathcal{V}(V_W)} \to 0$$
but also
\[ \frac{\mathcal{V}(V_{am})}{\mathcal{V}(\text{conv}\{Q_k\})} \to 0 \]

Again, for infinite ensembles this is unimportant, but for any actual realization it probably is.

One should also notice that in a state reconstruction, tomography, or state determination, when it is made using ORIs a similar but not as drastic situation may occur. E.g. first ORI measurement fixes a set of admissible pre-measurement states, the result of the following measurements must fit into it. The simplest situation would be if the result of e.g. measurement of spin 1/2 component \( S_z \) gives distribution \{1 − a, a\}. If the next measurement is e.g. of \( S_x \) than as long as the result is outside \{1/2 + b, 1/2 − b\} where \(-\sqrt{a(1−a)} \leq b \leq \sqrt{a(1−a)}\), the result of the state determination is actually not a state.

To conclude with, generalized measurements are indeed generalization of standard ORI measurements, but when they are not ORIs or combinations of ORIs they are mostly either clever state determinations or distant state preparations. It is indeed very difficult to change a well established name, as generalized measurement is, but more specifications may be necessary.

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