PALEY-WIENER THEOREM FOR THE WEINSTEIN TRANSFORM AND APPLICATIONS

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Abstract. In this paper our aim is to establish the Paley-Wiener Theorems for the Weinstein Transform. Furthermore, some applications are presents, in particular some properties for the generalized translation operator associated with the Weinstein operator are proved.

keywords: Weinstein transform, Paley-Wiener Theorem.

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1. Introduction

Paley-Wiener theorem is any theorem that relates decay properties of a function or distribution at infinity with analyticity of its Fourier transform [7]. For example, the classical Paley-Wiener theorem (see [2]) stated that states that a necessary and sufficient condition for a square-integrable function $\varphi$ to be extendable to an entire function in the complex plane with an exponential type bound

$$|\varphi(z)| \leq Ce^{Az}$$

if $\varphi$ is band-limited, i.e. the Fourier transform $\hat{\varphi}$ has compact support. Higher dimensional extensions of the Paley-Wiener theorem have been studied. There are plenty of Paley-Wiener type theorems since there are many kinds of bound for decay rates of functions and many types of characterizations of smoothness. In this regard a wide number of papers have been devoted to the extension of the theory on many other transforms and different classes of functions, for example, the Mellin transform [10], Hankel transform [4, 11], Jacobi transform [12] and Clifford-Fourier transform [8, 13].

Since the Weinstein transform are natural generalizations of the Fourier transforms, it is natural to ask whether such a representation for entire functions is possible in this case also. The aim of this paper is to obtain an analogue of the Paley-Wiener theorem for Weinstein transforms. As applications of this results, some properties for the generalized translation operator associated with the Weinstein operator are established.

2. Harmonic analysis Associated with the Weinstein Operator

In order to set up basic and standard notation we briefly overview the Weinstein operator and related harmonic analysis. Main references are [1] [2].

In the following we denote by

- $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0, \infty)$.
- $x = (x_1, \ldots, x_d, x_{d+1}) = (x', x_{d+1})$. 

• \(-x = (-x', x_{d+1})\).
• \(C_*(\mathbb{R}^{d+1})\), the space of continuous functions on \(\mathbb{R}^{d+1}\), even with respect to the last variable.
• \(S_*(\mathbb{R}^{d+1})\), the space of the \(C^\infty\) functions, even with respect to the last variable, and rapidly decreasing together with their derivatives.
• \(L^p_\alpha(\mathbb{R}^{d+1})\), \(1 \leq p \leq \infty\), the space of measurable functions \(f\) on \(\mathbb{R}^{d+1}\) such that
\[
\|f\|_{\alpha,p} = \left(\int_{\mathbb{R}^{d+1}} |f(x)|^p \, d\mu_\alpha(x)\right)^{1/p} < \infty, \quad p \in [1, \infty),
\]
\[
\|f\|_{\alpha,\infty} = \operatorname{ess sup}_{x \in \mathbb{R}^{d+1}} |f(x)| < \infty,
\]
where
\[
d\mu_\alpha(x) = x^{2\alpha+1}_d \, dx = x^{2\alpha+1}_d \, dx_1 ... dx_{d+1}.
\]
• \(P^{d+1}_{*,L}\) the set of homogeneous polynomials on \(\mathbb{R}^{d+1}\) of degree \(l\), even with respect to the last variable.
• \(A_\alpha(\mathbb{R}^{d+1}) = \{ \varphi \in L^1_\alpha(\mathbb{R}^{d+1}); F_{W,\alpha} \varphi \in L^1_\alpha(\mathbb{R}^{d+1}) \}\) the Wiener algebra space.
• \(S^d_\alpha = \{ x \in \mathbb{R}^{d+1}; \|x\| = 1 \} \).

We consider the Weinstein operator \(\Delta_{W,\alpha}^d\) defined on \(\mathbb{R}^{d+1}\) by
\[
\Delta_{W,\alpha}^d = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha, \quad \alpha > -1/2,
\]
where \(\Delta_d\) is the Laplacian operator for the \(d\) first variables and \(L_\alpha\) is the Bessel operator for the last variable defined on \((0, \infty)\) by
\[
L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}}.
\]

The Weinstein operator \(\Delta_{W,\alpha}^d\) have remarkable applications in different branches of mathematics. For instance, they play a role in Fluid Mechanics [3].

2.1. The eigenfunction of the Weinstein operator. For all \(\lambda = (\lambda_1, ..., \lambda_{d+1}) \in \mathbb{C}^{d+1}\), the system
\[
\begin{aligned}
\frac{\partial^2 u}{\partial x_j^2}(x) &= -\lambda_j^2 u(x), & \text{if } 1 \leq j \leq d \\
L_\alpha u(x) &= -\lambda_{d+1}^2 u(x), \\
u(0) &= 1, \quad \frac{\partial u}{\partial x_j}(0) = 0, \quad \text{and } \frac{\partial^2 u}{\partial x_{d+1}^2}(0) = -i\lambda_j, & \text{if } 1 \leq j \leq d,
\end{aligned}
\]
has a unique solution on \(\mathbb{R}^{d+1}\), denoted by \(\Lambda_{\alpha}^d(\lambda, \cdot)\), and given by
\[
\Lambda_{\alpha}^d(\lambda, x) = e^{-i <x', \lambda'>} j_\alpha(x_{d+1} \lambda_{d+1})
\]
where \(x = (x', x_{d+1})\), \(\lambda = (\lambda', \lambda_{d+1})\) and \(j_\alpha\) is the normalized Bessel function of index \(\alpha\) defined by
\[
j_\alpha(x) = \Gamma(\alpha + 1) \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!(\alpha + k + 1)}.
\]
The function \((\lambda, x) \mapsto \Lambda^d_\alpha(\lambda, x)\) has a unique extension to \(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}\), and satisfied the following properties:

**Proposition 1.** i). For all \((\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}\) we have

\[
(5) \quad \Lambda^d_\alpha(\lambda, x) = \Lambda^d_\alpha(x, \lambda)
\]

ii). For all \((\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}\) we have

\[
(6) \quad \Lambda^d_\alpha(\lambda, -x) = \Lambda^d_\alpha(-\lambda, x)
\]

iii). For all \((\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}\) we get

\[
(7) \quad \Lambda^d_\alpha(\lambda, 0) = 1.
\]

vi). For all \(\nu \in \mathbb{N}^{d+1}\), \(x \in \mathbb{R}^{d+1}\) and \(\lambda \in \mathbb{C}^{d+1}\) we have

\[
(8) \quad \left| D^\nu_\lambda \Lambda^d_\alpha(\lambda, x) \right| \leq \|x\|^{|\nu|} e^{\|x\|\|\lambda\|}
\]

where \(D^\nu_\lambda = \partial^\nu / (\partial \lambda_1^{\nu_1} \ldots \partial \lambda_{d+1}^{\nu_{d+1}})\) and \(|\nu| = \nu_1 + \ldots + \nu_{d+1}\). In particular, for all \((\lambda, x) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}\), we have

\[
(9) \quad \left| \Lambda^d_\alpha(\lambda, x) \right| \leq 1.
\]

2.2. The Weinstein transform.

**Definition 1.** The Weinstein transform is given for \(\varphi \in L^1_{\alpha}(\mathbb{R}^{d+1}_+\) by

\[
(10) \quad \mathcal{F}_W,\alpha(\varphi)(\lambda) = \int_{\mathbb{R}^{d+1}_+} \varphi(x) \Lambda^d_\alpha(\lambda, x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}^{d+1}_+,
\]

where \(\mu_\alpha\) is the measure on \(\mathbb{R}^{d+1}_+\) given by the relation [7].

Some basic properties of this transform are as follows. For the proofs, we refer [1, 2].

**Proposition 2.** (1) For all \(\varphi \in L^1_{\alpha}(\mathbb{R}^{d+1}_+\), the function \(\mathcal{F}_W,\alpha(\varphi)\) is continuous on \(\mathbb{R}^{d+1}_+\) and we have

\[
(11) \quad \|\mathcal{F}_W,\alpha \varphi\|_{\alpha,\infty} \leq \|\varphi\|_{\alpha,1}.
\]

(2) The Weinstein transform is a topological isomorphism from \(\mathcal{S}_+(\mathbb{R}^{d+1}_+)\) onto itself. The inverse transform is given by

\[
(12) \quad \mathcal{F}^{-1}_{W,\alpha} \varphi(\lambda) = C_{\alpha, d} \mathcal{F}_W,\alpha \varphi(-\lambda), \quad \text{for all } \lambda \in \mathbb{R}^{d+1}_+,
\]

where

\[
(13) \quad C_{\alpha, d} = \frac{1}{(2\pi)^{d/2} \Gamma(\alpha + 1)}.
\]

(3) Parseval formula: For all \(\varphi, \phi \in \mathcal{S}_+(\mathbb{R}^{d+1}_+)\), we have

\[
(14) \quad \int_{\mathbb{R}^{d+1}_+} \varphi(x) \overline{\phi(x)} d\mu_\alpha(x) = C_{\alpha, d} \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_W,\alpha(\varphi)(x) \overline{\mathcal{F}_W,\alpha(\phi)(x)} d\mu_\alpha(x)
\]
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(4) Plancherel formula: For all $\varphi \in \mathcal{S}_\ast (\mathbb{R}^{d+1}_+)$, we have

$$
\int_{\mathbb{R}^{d+1}_+} |\varphi(x)|^2 d\mu_\alpha(x) = C_{\alpha,d} \int_{\mathbb{R}^{d+1}_+} |\mathcal{F}_{W,\alpha}\varphi(x)|^2 d\mu_\alpha(x)
$$

(5) Inversion formula: If $\varphi \in A_\alpha (\mathbb{R}^{d+1}_+)$, then

$$
\varphi(\lambda) = C_{\alpha,d} \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_{W,\alpha}\varphi(x)\Lambda^d_\alpha (\lambda, x) d\mu_\alpha(x), \ a.e. \ \lambda \in \mathbb{R}^{d+1}_+
$$

2.3. The translation operator associated with the Weinstein operator.

**Definition 2.** The translation operator $\tau^\alpha_x$, $x \in \mathbb{R}^{d+1}_+$ associated with the Weinstein operator $\Delta^d_\alpha$, is defined for a continuous function $\varphi$ on $\mathbb{R}^{d+1}_+$ which is even with respect to the last variable and for all $y \in \mathbb{R}^{d+1}_+$ by

$$
\tau^\alpha_x \varphi(y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^\pi \varphi \left( x' + y, \sqrt{x_{d+1}^2 + y_{d+1}^2 + 2x_{d+1}y_{d+1} \cos \theta} \right) (\sin \theta)^{2\alpha} d\theta.
$$

By using the Weinstein kernel, we can also define a generalized translation, for a function $\varphi \in \mathcal{S}_\ast (\mathbb{R}^{d+1}_+)$ and $y \in \mathbb{R}^{d+1}_+$ the generalized translation $\tau^\alpha_x \varphi$ is defined by the following relation

$$
\mathcal{F}_{W,\alpha}(\tau^\alpha_x \varphi)(y) = \Lambda^d_\alpha (x, y) \mathcal{F}_{W,\alpha}(\varphi)(y)
$$

The following proposition summarizes some properties of the Weinstein translation operator.

**Proposition 3.** The translation operator $\tau^\alpha_x$, $x \in \mathbb{R}^{d+1}_+$ satisfies the following properties:

i). For $\varphi \in C_\ast (\mathbb{R}^{d+1}_+)$, we have for all $x, y \in \mathbb{R}^{d+1}_+$

$$
\tau^\alpha_x \varphi(y) = \tau^\alpha_y \varphi(x) \quad \text{and} \quad \tau^\alpha_0 \varphi = \varphi.
$$

ii). Let $\varphi \in L^1_\alpha (\mathbb{R}^{d+1}_+)$, $1 \leq p \leq \infty$ and $x \in \mathbb{R}^{d+1}_+$. Then $\tau^\alpha_x \varphi$ belongs to $L^p_\alpha (\mathbb{R}^{d+1}_+)$ and we have

$$
\| \tau^\alpha_x \varphi \|_{\alpha,p} \leq \| \varphi \|_{\alpha,p}
$$

Note that the $A_\alpha (\mathbb{R}^{d+1}_+)$ is contained in the intersection of $L^1_\alpha (\mathbb{R}^{d+1}_+)$ and $L^\infty_\alpha (\mathbb{R}^{d+1}_+)$ and hence is a subspace of $L^2_\alpha (\mathbb{R}^{d+1}_+)$. For $\varphi \in A_\alpha (\mathbb{R}^{d+1}_+)$ we have

$$
\tau^\alpha_x \varphi(y) = C_{\alpha,d} \int_{\mathbb{R}^{d+1}_+} \Lambda^d_\alpha (x, z) \Lambda^d_\alpha (-y, z) \mathcal{F}_{W,\alpha} \varphi(z) d\mu_\alpha(z).
$$

By using the generalized translation, we define the generalized convolution product $\varphi \ast_W \psi$ of the functions $\varphi, \psi \in L^1_\alpha (\mathbb{R}^{d+1}_+)$ as follows

$$
\varphi \ast_W \psi(x) = \int_{\mathbb{R}^{d+1}_+} \tau^\alpha_x \varphi(-y) \psi(y) d\mu_\alpha(y).
$$

This convolution is commutative and associative, and it satisfies the following properties:
Proposition 4. i). For all $\varphi, \psi \in L^1_\alpha(\mathbb{R}^{d+1})$, (resp. $\varphi, \psi \in S_*(\mathbb{R}^{d+1})$), then $\varphi \ast W \psi \in L^1_\alpha(\mathbb{R}^{d+1})$, (resp. $\varphi \ast W \psi \in S_*(\mathbb{R}^{d+1})$) and we have
\begin{equation}
F_{W,\alpha}(\varphi \ast W \psi) = F_{W,\alpha}(\varphi)F_{W,\alpha}(\psi)
\end{equation}

ii). Let $p, q, r \in [1, \infty]$, such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $\varphi \in L^p_\alpha(\mathbb{R}^{d+1})$ and $\psi \in L^q_\alpha(\mathbb{R}^{d+1})$ the function $\varphi \ast W \psi$ belongs to $L^r_\alpha(\mathbb{R}^{d+1})$ and we have
\begin{equation}
\|\varphi \ast W \psi\|_{\alpha,r} \leq \|\varphi\|_{\alpha,p} \|\psi\|_{\alpha,q}.
\end{equation}

2.4. Heat functions related to the Weinstein operator. The generalized heat kernel $E^\alpha_t(x), x \in \mathbb{R}^{d+1}, t > 0$ associated with the Weinstein operator $\Delta^d_{W,\alpha}$ is given by
\begin{equation}
E^\alpha_t(x) = \frac{2}{\pi^\frac{d}{2} \Gamma(\alpha + 1)} \frac{1}{(4t)^{\alpha+1} + \frac{d}{2}} e^{-\|y\|^2/4t},
\end{equation}
which is a solution of the generalized heat equation:
\begin{equation}
\frac{\partial}{\partial t}E^\alpha_t(x) - \Delta^d_{W,\alpha}E^\alpha_t(x) = 0,
\end{equation}
and satisfied the following properties (see [6]):

Proposition 5. i). For $x \in \mathbb{R}^{d+1}, t > 0$, the function $\tau^\alpha_x E^\alpha_t$ is nonnegative.

ii). For $t > 0$, we have
\begin{equation}
\|E^\alpha_t\|_{\alpha,1} = 1.
\end{equation}

iii). For $x \in \mathbb{R}^{d+1}, t > 0$, we have
\begin{equation}
\int_{\mathbb{R}^{d+1}} \tau^\alpha_x E^\alpha_t(y) d\mu_\alpha(y) = 1.
\end{equation}

iv). For $t > 0$, we have
\begin{equation}
F_{W,\alpha}E^\alpha_t(x) = e^{-t\|x\|^2}
\end{equation}

3. Paley-Wiener Theorem for the Weinstein Transform

In this section we prove a sharp Paley-Wiener theorem for the Weinstein transform and study its consequences. We suppose that $d \geq 1$ and $\lambda > 0$. For a non-negative integer $l$, we put
\begin{equation}
\mathcal{H}^\alpha_l = \{ P \in \mathcal{P}_*; P \text{ is homogeneous such that } \Delta_\alpha P = 0 \},
\end{equation}
which is called the space of generalized spherical harmonics of degree $l$. We fix a $P_l \in \mathcal{H}^\alpha_l$ and define the Weinstein coefficients of $\varphi \in S(\mathbb{R}^{d+1})$ in the angular variable by
\begin{equation}
\varphi_{l,\alpha}(\lambda) = \int_{S^d} \varphi(\lambda t)P_l(t) d\sigma_\alpha(t),
\end{equation}
with $d\sigma_\alpha(t) = t^{2\alpha+1} d\sigma_{d+1}(t)$. Then the Weinstein spherical harmonic coefficients of $\varphi \in S(\mathbb{R}^{d+1})$ are given by
\begin{equation}
\Phi_{l,\alpha}(\lambda) = \lambda^{-1} \int_{S^d} F_{W,\alpha}(\varphi)(\lambda, t) P_l(t) d\sigma_\alpha(t)
\end{equation}
Theorem 1. Let \( \varphi \in S_\ast(\mathbb{R}_+^{d+1}) \) and \( R \) be a positive number. Then \( \varphi \) is supported in \( \{x; \|x\| < R\} \) if and only if the Weinstein spherical harmonic coefficients of \( \varphi \) extends to an entire function of \( \lambda \in \mathbb{C} \) satisfying the estimate
\[
|\Phi_{l,\alpha}(\lambda)| \leq c_{l,\alpha} e^{R\|\Im(\lambda)\|},
\]

Proof. In [5] (see proof of Proposition 5), the authors proved that
\[
\Phi_{l,\alpha}(\lambda) = C_{l,\alpha} \int_0^\infty \varphi_{l,\alpha}(r) j_{\alpha + \frac{d}{2} + l}(\lambda r) r^{2\alpha + 2l + d + 1} dr,
\]
where \( \varphi_{l,\alpha}(r) \) as defined in (26). Thus, \( \Phi_{l,\alpha} \) is the Hankel transform of order \( \alpha + \frac{d}{2} + l \) of the function \( \varphi_{l,\alpha}(l) \). So, Paley-Wiener Theorem for the Hankel transform (see [4]) and (26) completes the proof. \( \blacksquare \)

Theorem 2. A \( \varphi \in S_\ast(\mathbb{R}_+^{d+1}) \) be supported in \( \{x; \|x\| < R\} \) if and only if the function \( \mathcal{F}_{W,\alpha}\varphi \) extends to an entire function of \( y \in \mathbb{C}^{d+1} \) which satisfies
\[
|\mathcal{F}_{W,\alpha}\varphi(y)| \leq c e^{R\|\Im(y)\|}, \tag{28}
\]

Proof. Since the Weinstein kernel \( \Lambda_d^\alpha(x,y) \) is an entire function in \( y \in \mathbb{C}^{d+1} \) and satisfied
\[
|\Lambda_d^\alpha(x,y)| \leq e\|x\|\|\Im(y)\|,
\]
On the other hand, by the definition of the Weinstein transform we have
\[
|\mathcal{F}_{W,\alpha}\varphi(y)| \leq c e^{R\|\Im(y)\|},
\]
where \( c = \|\varphi\|_{1,\alpha} \).

Conversely, assume that the function \( \mathcal{F}_{W,\alpha}\varphi \) is an entire function of \( y \in \mathbb{C}^{d+1} \) and satisfied (28), thus implies that the function
\[
\Phi_{l,\alpha}(\lambda) = \lambda^{-1} \int_{S^d_+} \mathcal{F}_{W,\alpha}(\varphi)(\lambda, t) P_l(t) d\sigma(t), \ \lambda \in \mathbb{C}
\]
is an entire function of exponential type \( R \), from which the converse follows from the Theorem [1]. \( \blacksquare \)

Corollary 1. Let \( \varphi \in S_\ast(\mathbb{R}_+^{d+1}) \) be supported in \( \{x; \|x\| < R\} \). Then \( \tau_y^{\alpha}\varphi \) is supported in \( \{x; \|x\| < R + \|y\|\} \).

Proof. Let \( \psi(x) = \tau_y^{\alpha}\varphi(x) \). Then \( \mathcal{F}_{W,\alpha}\psi(z) = \Lambda_d^\alpha(y, z) \mathcal{F}_{W,\alpha}\varphi(z) \). Thus
\[
|\mathcal{F}_{W,\alpha}\psi(z)| = \left| \Lambda_d^\alpha(y, z) \mathcal{F}_{W,\alpha}\varphi(z) \right| \leq c e^{(R + \|y\|)\|\Im(z)\|},
\]
i.e \( \mathcal{F}_{W,\alpha}\psi(z) \) extends to \( \mathbb{C}^{d+1} \) as an entire function of type \( R + \|y\| \). Consequently, from the previous Theorem we conclude that \( \tau_y^{\alpha}\varphi \) is supported in \( \{x; \|x\| < R + \|y\|\} \). \( \blacksquare \)
Corollary 2. Let $\varphi \in \mathcal{S}_*(\mathbb{R}^{d+1}_+)$. If $\varphi$ is supported in $\{x; \|x\| < R\}$, then the following inequality

$$
\|\tau_x^\alpha \varphi - \varphi\|_{\alpha,p} \leq \tilde{C}_{\alpha,d} \|x\| (R + \|x\|)^{\frac{d+2\alpha+2}{p}}
$$

holds for all $1 \leq p \leq \infty$.

Proof. Inversion formula $(16)$ and $(18)$ yields that

$$
\tau_x^\alpha \varphi(y) - \varphi(y) = C_{\alpha,d} \int_{\mathbb{R}^{d+1}_+} \left( \Lambda^d_{\alpha}(x,\xi) - 1 \right) \Lambda^d_{\alpha}(-y,\xi) F_{W,\alpha} \varphi(\xi) d\mu_{\alpha}(\xi).
$$

Combining $(8)$ and $(30)$ and using the mean value theorem we get

$$
\left| \tau_x^\alpha \varphi(y) - \varphi(y) \right| \leq C_{\alpha,d} \|x\| \int_{\mathbb{R}^{d+1}_+} \|\xi\| |F_{W,\alpha} \varphi(\xi)| d\mu_{\alpha}(\xi).
$$

As $\varphi$ is supported in $\{y; \|y\| < R\}$ and $\tau_x^\alpha \varphi$ is supported in $\{y; \|y\| < R + \|x\|\}$, we can restrict the integration domain above to $\{y; \|y\| < R + \|x\|\}$. A short calculation gives the desired result. \hfill ■

Corollary 3. If $\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)$, then

$$
\lim_{t \to 0} \|\varphi \ast W E_t^\alpha - \varphi\|_{\alpha,1} = 0.
$$

Proof. From the fact $\tau_x^\alpha E_t^\alpha \geq 0$ and $\tau_x^\alpha \varphi(y) = \tau_y^\alpha \varphi(x)$ we get

$$
\|\varphi \ast W E_t^\alpha\|_{\alpha,1} \leq \int_{\mathbb{R}^{d+1}_+} \int_{\mathbb{R}^{d+1}_+} |\varphi(y)| |\tau_x^\alpha E_t^\alpha(y) d\mu_{\alpha}(y) d\mu_{\alpha}(x)|.
$$

By Fubini’s Theorem and the equality $(24)$ we obtain

$$
\|\varphi \ast W E_t^\alpha\|_{\alpha,1} \leq \|\varphi\|_{\alpha,1}.
$$

For a given $\epsilon > 0$, we choose $\psi \in \mathcal{S}_*(\mathbb{R}^{d+1}_+)$ such that $\|\varphi - \psi\|_{\alpha,1} < \epsilon/3$. Now, we can write $\varphi \ast W E_t^\alpha - \varphi$ in the following form

$$
\varphi \ast W E_t^\alpha - \varphi = [(f - g) \ast W E_t^\alpha] + [\psi - \varphi] + [\psi \ast W E_t^\alpha - \psi].
$$

So, the triangle inequality and $(32)$ leads to

$$
\|\varphi \ast W E_t^\alpha - \varphi\|_{\alpha,1} \leq \frac{2\epsilon}{3} + \|\psi \ast W E_t^\alpha - \psi\|_{\alpha,1}.
$$

On the other hand, by using $(23)$ we have

$$
\psi \ast W E_t^\alpha(y) - \psi(x) = \int_{\mathbb{R}^{d+1}_+} (\tau_y^\alpha \psi(-x) - \psi(x)) E_t^\alpha(y) d\mu_{\alpha}(y)
$$

$$
= \int_{\mathbb{R}^{d+1}_+} (\tau_y^\alpha \psi(x) - \psi(x)) E_t^\alpha(y) d\mu_{\alpha}(y)
$$

$$
= \int_{\mathbb{R}^{d+1}_+} (\tau_y^\alpha \psi(x) - \psi(x)) E_t^\alpha(y) d\mu_{\alpha}(y).
$$
Thus implies then
\[ \| \psi * \mathcal{E}_\alpha \|_{\alpha,1} \leq \int_{\mathbb{R}^d_+} \| \tau_\alpha \psi - \psi \|_{\alpha,1} \mathcal{E}_\alpha(y) d\mu_\alpha(y). \]

If \( \psi \) is supported in \( \{ x; \| x \| < R \} \), by Corollary 2 we get
\[ \| \psi * \mathcal{E}_\alpha \|_{\alpha,1} \leq \tilde{C}_{\alpha,d} \int_{\mathbb{R}^d_+} \| y \| (R + \| y \|)^{d+2\alpha+2} \mathcal{E}_\alpha(y) d\mu_\alpha(y) \]
\[ = \frac{2\tilde{C}_{\alpha,d}}{\pi^d \Gamma(\alpha + 1)(4t)^{\alpha+1/d+2}} \int_{\mathbb{R}^d_+} \| y \| (R + \| y \|)^{d+2\alpha+2} e^{-\| y \|^2/4t} d\mu_\alpha(y) \]
\[ = \frac{4\tilde{C}_{\alpha,d} \sqrt{t}}{\pi^d \Gamma(\alpha + 1)} \int_{\mathbb{R}^d_+} \| y \| \left( R + 2\sqrt{t} \| y \| \right)^{d+2\alpha+2} e^{-\| y \|^2} d\mu_\alpha(y), \]
which can be made smaller than \( \epsilon/3 \) by choosing \( \epsilon \) small. So, the proof of Corollary 3 is completes. \( \blacksquare \)

As a consequence of the Corollary 3 we reobtain the inversion formula (16).

**Corollary 4.** For \( \varphi \in \mathcal{A}_\alpha(\mathbb{R}^d_+) \), then for almost \( x \in \mathbb{R}^d_+ \)
\[ \varphi(x) = \mathcal{C}_{\alpha,d} \int_{\mathbb{R}^d_+} \mathcal{F}_{W,\alpha} \varphi(y) \Lambda_\alpha(x,-y) d\mu_\alpha(y) \] (33)

**Proof.** Let \( \varphi \in \mathcal{S}(\mathbb{R}^d_+) \). Using (14), (17) and (25) we have
\[ \varphi * \mathcal{E}_\alpha(x) = \int_{\mathbb{R}^d_+} \tau_\alpha \mathcal{E}_\alpha(y) \varphi(y) d\mu_\alpha(y) \]
\[ = \mathcal{C}_{\alpha,d} \int_{\mathbb{R}^d_+} \mathcal{F}_{\alpha,W} \left( \tau_\alpha \mathcal{E}_\alpha(\cdot) \right)(-y) \mathcal{F}_{\alpha,W} \varphi(y) d\mu_\alpha(y) \]
\[ = \mathcal{C}_{\alpha,d} \int_{\mathbb{R}^d_+} e^{-t\| y \|^2} \Lambda_\alpha(x,-y) \mathcal{F}_{\alpha,W} \varphi(y) d\mu_\alpha(y). \]
This extends to \( \varphi \in L_\alpha^1(\mathbb{R}^d_+) \), since the convolution operator extends to \( L_\alpha^1(\mathbb{R}^d_+) \), as a bounded operator by the inequality (32). Letting \( t \to 0^+ \), applying Corollary 3 to the left-hand side and the dominant convergence theorem to the right-hand side, we see that the inversion formula follows almost everywhere. \( \blacksquare \)

**Corollary 5.** Suppose that \( U \subseteq \mathbb{R}^d_+ \) is open. Suppose, further that \( x_1, x_2, \ldots, x_N \in \mathbb{R}^d_+ \)
are pairwise distinct and \( z_1, z_2, \ldots, z_N \in \mathbb{C} \). If \( \sum_{k=1}^N z_k \Lambda_\alpha^d(x_k, \xi) = 0 \) for all \( \xi \in U \), then \( z_k = 0 \) for all \( k \in \{ 1, 2, \ldots, N \} \).

**Proof.** By successive analytic continuation in each coordinate we can derive that the assumption actually means that \( \sum_{k=1}^N z_k \Lambda_\alpha^d(x_k, \xi) = 0 \) for all \( \xi \in \mathbb{R}^d_+ \). Let \( f \in \mathcal{S}_\alpha(\mathbb{R}^d_+) \)
be a test function. Then
\[ 0 = \sum_{k=1}^N z_k \Lambda_\alpha^d(x_k, \xi) \mathcal{F}_{W,\alpha} f(\xi) = \mathcal{F}_{W,\alpha} \left( \sum_{k=1}^N z_k \tau_\alpha \mathcal{F}_{W,\alpha} f(\xi) \right)(\xi), \]
for all $\xi \in \mathbb{R}^{d+1}_+$. Since $\tau_{x_k}^\alpha f \in \mathcal{S}(\mathbb{R}^{d+1}_+)$ and The Weinstein transform $\mathcal{F}_{W,\alpha}$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^{d+1}_+)$ onto itself, we have

\[
\sum_{k=1}^{N} z_k \tau_{x_k}^\alpha f(\xi) = 0,
\]

for all $\xi \in \mathbb{R}^{d+1}_+$. Now take $f$ to be compactly supported and support contained in the ball around zero with radius $\epsilon$ then for $\tau_{x_j}^\alpha f$ is compactly supported and support contained in the ball around zero with radius $\epsilon + \|x_k\|$ for all $k \in \{1, \ldots, N\}$, by means of Corollary 1.

Now, suppose that $\epsilon < \min_{j \neq k} \|x_k\| - \|x_j\|$ thus implies that $\tau_{x_k}^\alpha f(x_j) = 0$, for all $j \neq k$, and $\tau_{x_k}^\alpha f(x_k) \neq 0$.

So, by again using (34) we obtain that $z_k \tau_{x_k}^\alpha f(x_k) = 0$ for all $k \in \{1, \ldots, N\}$ and consequently $z_k = 0$.

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