ON THE COLUMN-ROW PROPERTY OF OPERATOR SPACES

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Abstract. In this article, we study the following question asked by Michael Hartz in a recent paper [Har20]: which operator spaces satisfy the column-row property? We provide a complete classification of the column-row property in the cases of $C^*$-algebras (denoted by $A$) and non-commutative $L_p$-spaces over semifinite von Neumann algebras $M$. In particular, we prove that both $A$ and $L_p(M)$ for $1 \leq p \neq 2 \leq \infty$, have the column-row property if and only if both $A$ and $M$ are subhomogeneous. Moreover, if the column-row constant for $A$ is 1, then it has to be abelian. En route we study several other relevant properties of operator spaces that are related to the column-row property. We verify their existence and non-existence for various natural examples of operator spaces.

1. Introduction

In [Tre04], while studying the Corona problem for Dirichlet spaces on the unit disc $\mathbb{D}$, the author discovered an important property involving the corresponding multiplier algebra, which in recent times, is known as the column-row property. The column-row property has emerged as an important tool in extending classical results on Hardy spaces to complete Nevanlinna-Pick (cnp) spaces. In a remarkable recent work [Har20], Hartz showed that every cnp space has the column-row property with the corresponding constant as 1. The notion of column-row property has led to a plethora of important results for cnp spaces, to name a few: (a) factorization for weak-product spaces; (b) interpolating sequences; (c) Corona problem etc. (see [AHMR212, CH19, Har20, Tre04]). Motivated by the inherent operator space structure of multiplier algebras and the immense application of this property, Hartz asked the following question in [Har20]:

Question. Which operator spaces satisfy the column-row property?

Our aim in this article is to initiate a study for this question in the general setting of operator spaces, by looking at several examples. For the theory of operator spaces and related important results we refer to the excellent monographs [ER00, Pau02, Pis03]. Let us begin the description of column-row property by considering a concrete operator space $E \subseteq B(H)$. For a sequence $e := (e_1, \ldots, e_n)$ in $E$, we define

$$C_e := \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} : H \to H^n.$$
as the following column operator by
\[ C_e(\zeta) := \begin{bmatrix} e_1(\zeta) \\ \vdots \\ e_n(\zeta) \end{bmatrix} \in H^n. \]

Similarly, let us define the corresponding row operator
\[ R_e := [e_1 \cdots e_n] : H^n \to H \]
by
\[ R_e( \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{bmatrix} ) := \sum_{j=1}^n e_j(\zeta_j). \]

Definition 1.1. A concrete operator space \( E \subseteq B(H) \) is said to have the column-row property (in short CRP) if there exists a constant \( C > 0 \) such that for any finite sequence \( e \) in \( E \) with \( \|C_e\|_{H \to H^n} \leq 1 \) we have \( \|R_e\|_{H^n \to H} \leq C \).

In this article, we begin our study of CRP by looking at \( C^* \)-algebras. One of our most definite results in this article is the following:

Theorem 1.2. Let \( A \) be a \( C^* \)-algebra. Then \( A \) has CRP if and only if \( A \) is subhomogeneous. Furthermore, one can take the constant \( C = 1 \) in Definition 1.1 if and only if \( A \) is abelian.

We give two proofs of the above theorem. However, the heart of the proofs are essentially the same and depends on the structure theory of von Neumann algebras. One more essential tool is the theory of operator space tensor products which plays also a crucial role in the whole paper.

Note that Hartz proved a stronger result by showing that cnp spaces satisfy the column-matrix property [Har20, Corollary 3.6]. Motivated by these results, we study the following properties, which are closely related to CRP:

(i) Transpose property (TP);
(ii) Column-Matrix property (CMP).

Both these properties are defined in Section 3. Note that one can also study the so called Row-Column property (RCP) and Row-Matrix property (RMP). It is an obvious observation that CRP with TP implies RCP. But in most cases, CRP is very much asymmetrical. For instance, multiplier algebras of the Drury-Arveson space and Dirichlet space satisfy CRP but do not satisfy RCP and thus, in particular TP (see [AHMR211, Subsection 4.2]). However, it follows from the definition that an operator space \( E \) has CRP or CMP if and only if the
opposite operator space $E^{op}$ has RCP or RMP, respectively. Hence it is enough to study CRP or CMP only. In the context of $C^*$-algebras we prove that CRP coincides with TP.

Apart from Theorem 1.2, we have studied these properties for many naturally occurring operator spaces. Let us briefly mention some of the positive results for the interest of the reader:

- $MIN(E), MAX(E)$ and $L_p(\Omega)$ for $1 \leq p \leq \infty$, have both CRP and TP with constant 1, where $MIN(E)$ and $MAX(E)$ denote the so called minimal and maximal quantization of a Banach space $E$. However, we were only able to show that $MIN(E)$ has CMP with constant 1.
- The operator Hilbert space over the indexing set $I$, denoted by $OH(I)$, has all the three properties namely, CRP, TP and CMP with constant 1, which further implies that $S_2$ (the class of Hilbert-Schimdt operators on $\ell_2$) has all the three properties as well with constant 1.
- The column Hilbert space $C$ has the column-row property with constant 1.

For the negative results, we observe that the row Hilbert space $R$ does not have the column-row property. We also study the column-row property for the Shatten-$p$ classes, denoted by $S_p$. In the non-commutative situation, we could show that $S_p$ does not have the column-row property for $1 \leq p \neq 2 \leq \infty$. The proof of this fact relies on estimates of norms of certain rows and columns consisting of elements of $S_p$ and are of independent interest. Also we have been able to completely characterize when $L_p(M)$ has CRP, where $M$ is a semifinite von Neumann algebra.

**Theorem 1.3.** Let $M$ be a semifinite von Neumann algebra. Let $1 \leq p \neq 2 \leq \infty$. Then $L_p(M)$ has CRP iff $M$ is subhomogeneous.

We also study the completely bounded version of the above concepts. It was shown in [Har20, Section 5.1] that cnb spaces do not have completely bounded CRP with constant 1. However, it turns out that no non-trivial operator space can have completely bounded CRP.

**Theorem 1.4.** Let $E$ be an operator space which is not equal to the zero vector space. Then $E$ does not have the completely bounded CRP or completely bounded CMP.

Let us now briefly discuss the manner in which the rest of the article has been organised. Section 2 contains the definition of the several operator spaces in our study along with definitions of the various other properties of our interest. In section 3 we have focussed our study on $C^*$-algebras and proved Theorem 1.2. Finally, in section 4 we discuss these properties for other operator spaces.

2. Preliminaries.

In this section, we recall the definitions of the several examples of operator spaces essential for our study. We begin by observing that CRP can also be defined using the matricial norm structure. For an abstract operator space $(E, \|\cdot\|_{M_n(E)})_{n \geq 1}$ and any sequence $e$ in $E$, let us
define
\[ \tilde{C}_e := \begin{bmatrix}
  e_1 & 0 & \ldots & 0 \\
  e_2 & 0 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  e_n & 0 & \ldots & 0
\end{bmatrix} \in M_n(E) \]
and
\[ \tilde{R}_e := \begin{bmatrix}
  e_1 & e_2 & \ldots & e_n \\
  0 & 0 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & 0
\end{bmatrix} \in M_n(E). \]

We say that \( E \) has CRP if and only if there exists a constant \( C > 0 \) such that for all finite sequences \( e \subseteq E \) with \( \| \tilde{C}_e \|_{M_n(E)} \leq 1 \), we have \( \| \tilde{R}_e \|_{M_n(E)} \leq C \). One can immediately see that this definition is equivalent to Definition 1.1 by Ruan’s characterization [Rua88, Theorem 3.1].

Let \( E \) be an operator space. Given \( x = [x_{ij}]_{i,j=1}^n \in M_n(E) \) we denote the linear transpose map by \( t(x) := [x_{ji}]_{i,j=1}^n \). The transpose maps taking columns to rows or rows to columns are also denoted by \( t \). Sometimes we also denote it by \( t_n \) to specify the dimension of the spaces under consideration.

**Definition 2.1.** An operator space \( E \) is said to have the transpose property (in short TP) if there exists a constant \( C > 0 \) such that for all \( x \in M_n(E) \), we have \( \| t(x) \|_{M_n(E)} \leq C \| x \|_{M_n(E)} \).

It is a trivial observation that if an operator space \( E \) has TP, then for all \( x \in M_n(E) \), \( \| x \|_{M_n(E)} \leq C \| t(x) \|_{M_n(E)} \). Moreover, if \( E \) has TP then \( E \) has CRP.

**Definition 2.2.** Let \( E \) be an operator space. We say \( E \) has the column-matrix property (in short CMP) if there exists a constant \( C > 0 \) such that for all \( [x_{ij}]_{i,j=1}^n \in M_n(E) \), we have
\[
\| [x_{ij}]_{i,j=1}^n \|_{M_n(E)} \leq C \| M_n(E) \|
\]
Recall that a linear map \( u : E \to F \) between two operator spaces is called \textit{completely bounded} if \( \sup_{n \geq 1} \| id_{E_n} \otimes u \|_{M_n(E) \to M_n(F)} < \infty \), where \( id_E : E \to E \) is the identity map for any vector space \( E \). In this case, one denotes \( \| u \|_{cb} := \sup_{n \geq 1} \| id_{E_n} \otimes u \|_{M_n(E) \to M_n(F)} \). The map \( u \) is called a complete isometry if \( id_{E_n} \otimes u \) is an isometry for all \( n \geq 1 \). We denote by \( I_n \) to be the identity matrix in \( M_n \).

\textbf{Definition 2.3.} Let \( E \) be an operator space. The \textit{opposite} operator space, \( E^{\text{op}} \) is defined to be the same space as \( E \), but with the following matricial norm: \( \| [x_{ij}]_{i,j=1}^n \|_{M_n(E^{\text{op}})} := \| [x_{ji}]_{i,j=1}^n \|_{M_n(E)} \), for all \( [x_{ij}]_{i,j=1}^n \in M_n(E^{\text{op}}) \).

It is clear that \( E \) has TP if and only if \( \| id_E \|_{cb,E \to E^{\text{op}}} < \infty \). We refer [Pis03] for more details on opposite operator spaces.

If \( A \) is a \( C^* \) algebra then let us fix a faithful \(*\)-representation of \( A \) on \( B(H) \). Then \( M_{m,n}(A) \) can be identified as a vector subspace of \( B(H^n, H^m) \). The canonical operator space structure on \( A \) is obtained by borrowing the norm from \( B(H^n, H^m) \), and we will be using this canonical structure in the sequel. We recommend the readers [ER00] and [Pis03] for more details on this concept.

An operator space \( E \) is called \( n \)-minimal if there exists a compact Hausdorff space \( K \) for which \( E \) has a completely isometric embedding into \( C(K, M_n) \).

\textbf{Definition 2.4.} A \( C^* \)-algebra \( A \) is called \( n \)-subhomogeneous if all the irreducible representations of \( A \) have dimensions at most \( n \). A \( C^* \)-algebra is called subhomogeneous if it is \( n \)-subhomogeneous for some \( n \in \mathbb{N} \).

Note that for an operator space \( E \) we have a notion of operator subspace and the dual operator space (see [ER00]). We refer [ER00] Chapter 7 and Chapter 8 and [Pis03] for various facts on the operator space projective and injective tensor product. We will also need various other examples of operator spaces, namely minimal and maximal quantization, operator Hilbert space, row and column Hilbert spaces and non-commutative \( L_p \)-spaces. Each one of these concepts are well known and recalled before their proofs (see [ER00] and [Pis03] for more details).

\section{Column-Row property for \( C^* \) algebras.}

In this section, we focus on proving our results for \( C^* \)-algebras. It was shown in [Roy05] that a \( C^* \)-algebra \( A \) is \( n \)-minimal if and only if \( A \) is \( n \)-subhomogeneous. Therefore, we conclude that a \( C^* \)-algebra \( A \) is \( n \)-subhomogeneous if and only if \( \| id_A \|_{cb,A \to A^{op}} \leq n \). Thus, we get that

\textbf{Proposition 3.1.} Let \( A \) be a \( C^* \)-algebra. Then \( A \) has TP if and only if \( A \) is subhomogeneous.

Let us now recall the properties of the bidual of a \( C^* \) algebra (see [Pis20]). It is well known that \( A^{**} \) becomes a von Neumann algebra by extending the algebra operations of \( A \), known as the universal enveloping von Neumann algebra of \( A \). The following well-known fact is important for the sequel.
Fact: $B(\ell_2)$ does not have the CRP since for any $n \in \mathbb{N}$ we have

$$\|[E_{11}, \ldots , E_{1n}]\| = \sqrt{n}; \quad \|[E_{11}, \ldots , E_{1n}]^*\| = 1,$$

where $E_{ij}$ are the elementary matrices for $1 \leq i, j \leq n$.

**Lemma 3.2.** Let $E$ be an operator space and $F \subseteq E$ be an operator subspace. Then the following is true:

(i) $E$ has CRP (or TP) then so does $F$.

(ii) $E$ has CRP, CMP or TP then $E^{op}$ has RCP, RMP or TP respectively with the same constant.

(iii) $E$ has CRP if and only if there exists a constant $C > 0$ such that for all $n \geq 1$,

$$\|[t_n \otimes id_E]_{M_{n,1} \otimes E \to M_{1,n} \otimes E} \leq C.$$

(iv) $E$ has CRP (resp. TP) if and only if $E^{**}$ has CRP (resp. TP) with the same constant.

**Proof.** The proofs of (i) and (ii) are straightforward and therefore, not included here.

(iii) To prove (iii) we notice from [ER00, Corollary 8.1.3] that

$$M_{n,1}(E) = M_{n,1} \widehat{\otimes} E, \quad \text{and } M_{1,n}(E) = M_{1,n} \widehat{\otimes} E,$$

where $\widehat{\otimes}$ denotes the operator space injective tensor product of operator spaces. Moreover, under the above identification we also have $(t_n \otimes id_E)([x_1, \ldots , x_n]^*) = [x_1, \ldots , x_n]$ for all $x_i \in E$, $1 \leq i \leq n$. This proves part (iii) of the lemma.

(iv) Note that if $E^{**}$ has CRP, then $E$ has CRP by part (i) of the lemma as the natural inclusion of $E$ inside $E^{**}$ is a completely isometry [ER00, Proposition 3.2.1].

Now assume that $E$ has CRP. Equivalently, by part (ii) of the lemma, there exists a constant $C > 0$ such that for all $n \geq 1$, $\|[t_n \otimes id_E]_{M_{1,n} \otimes E \to M_{1,n} \otimes E} \leq C$. Therefore, for all $n \geq 1$, we get

$$\|[t_n \otimes id_E]^{**}_{(M_{n,1} \otimes E)^{**} \to (M_{1,n} \otimes E)^{**}} \leq C. \quad (3.1)$$

Note that we have the completely isometric identifications

$$M_{n,1}(E^{**}) = M_{n,1} \widehat{\otimes} E^{**} = (M_{1,n} \widehat{\otimes} E^*)^* = (M_{n,1}^* \widehat{\otimes} E^*)^* = (M_{1,n} \widehat{\otimes} E)^{**}.$$

Here we have used the duality between the projective and injective tensor product of operator spaces (see [Pis03, Theorem 4.1]) and the fact that $M_{n,1}^* = M_{1,n}$ completely isometrically (see [Pis03, Exercise 2.3.5]). Similarly, we get $M_{1,n}(E^{**}) = M_{1,n} \widehat{\otimes} E^{**}$ completely isometrically. Moreover, under these identifications $(t_n \otimes id_E)^{**}$ becomes $t_n \otimes id_{E^{**}}$. Therefore, from (3.1), we obtain that

$$\|t_n \otimes id_{E^{**}}\|_{M_{n,1} \otimes E^{**} \to M_{1,n} \otimes E^{**}} \leq C. \quad (3.2)$$

The proof follows from part (ii) of the lemma.

This completes the proof of the lemma.

**Remark 3.1.** Note that for $C^*$-algebras, part (iv) of the above lemma can be proved without using operator space tensor product. This follows from the simple fact that any $C^*$-algebra $A$ is $w^*$-dense in the universal enveloping von Neumann algebra $A^{**}$ (see [Pis20]).
We are now ready to prove Theorem 1.2. The outline of the following proof was suggested by Michael Hartz in a communication through email.

**Proof of Theorem 1.2.** Let $A$ has CRP with constant $C$. Assume that dimension of $A$ is infinity. If $A^{**}$ contains an isomorphic copy of $M_n$, we must have

$$\sqrt{n} = \|[E_{11} \ldots E_{1n}]\| \leq C\|[E_{11} \ldots E_{1n}]^\ast\| = C.$$  

It follows that $A^{**}$ cannot contain a type $I_n$ factor for $n > C^2$. This implies that in the type decomposition of $A^{**}$ [Tak79 Chapter V, Theorem 1.9] there is no type $II$ and type $III$ summand. Otherwise for each $n \in \mathbb{N}$, $A^{**}$ will contain a copy of $M_n$. This follows from [Tak79 Chapter V, Proposition 1.22] and [Tak79 Chapter V, Proposition 1.35]. Therefore, $A^{**}$ is of the form $\oplus M_j$ where $M_j$ is a von Neumann algebra of type $I_j$ and $j \leq C^2$. This proves that $A$ is subhomogeneous (see [Bla06 IV.1.4.6 Proposition]).

Now, suppose that $C = 1$. If $A$ is non-abelian, then by the decomposition of von Neumann algebra $A^{**}$ into type $I$, $II_1$, $II_\infty$ and $III$, it follows that $M_n \subseteq A^{**}$ for some $n > 1$, which further implies that

$$1 < \sqrt{2} = \|[E_{11}, E_{12}]\| \leq C\|[E_{11}, E_{12}]^\ast\| = C,$$

which is a contradiction. This completes the proof of the theorem. \qed

In view of Theorem 1.2 and Proposition 3.1 we have the following corollary.

**Corollary 3.3.** Let $A$ be a $C^*$-algebra. Then $A$ has CRP if and only if $A$ has TP.

We have an alternative approach to proving Theorem 1.2 using the Haagerup tensor product in the following manner:

**Proof of Theorem 1.2:** Recall that the Haagerup tensor product norm on the algebraic tensor product is defined by

$$\|z\|_h := \inf \left\{ \| \sum_{i=1}^n x_i x_i^\ast \|^\frac{1}{2} \| \sum_{i=1}^n y_i^\ast y_i \|^\frac{1}{2} : z = \sum_{i=1}^n x_i \otimes y_i \right\} \quad (z \in A \otimes A).$$

Let us consider the linear map $\text{inv} : A \otimes A \rightarrow A \otimes A$, which is defined on the elementary tensors by $\text{inv}(x \otimes y) := x^\ast \otimes y^\ast$. We claim that this map can be extended continuously on $A \otimes_h A := \overline{A \otimes A}^{\|\cdot\|_h}$. For proving this claim, let us note that by CRP there exists a positive constant $C$ such that $\| \sum_{i=1}^n x_i x_i^\ast \| \leq C\| \sum_{i=1}^n x_i^\ast x_i \|$ and $\| \sum_{i=1}^n y_i^\ast y_i \| \leq C\| \sum_{i=1}^n y_i y_i^\ast \|$ for all finite sequences $(x_i)_{i=1}^n, (y_i)_{i=1}^n$ in $A$. Therefore, we obtain that

$$\|z\|_h \leq C^2 \inf \left\{ \| \sum_{i=1}^n x_i^\ast x_i \|^{\frac{1}{2}} \| \sum_{i=1}^n y_i y_i^\ast \|^{\frac{1}{2}} : z = \sum_{i=1}^n x_i \otimes y_i \right\} = C^2 \|\text{inv}(z)\|.$$  

Note that $\text{inv} : A \otimes A \rightarrow A \otimes A$ is an involution. Thus, by replacing $z$ with $\text{inv}(z)$ we obtain that $\|\text{inv}(z)\|_h \leq C^2\|z\|_h$ for all $z \in A \otimes A$. This establishes the claim and thus, the result follows from [Kum01 Part (iii) Theorem 2.2.]. If $C = 1$, then $A$ is abelian follows from [Kum01 Theorem 2.3]. This completes the proof. \qed
4. Column-Row property and column-matrix property for some well-known Operator spaces

It is well-known that given any Banach space $E$, there are two natural operator space structures on $E$, namely the $MIN(E)$ and $MAX(E)$ operator spaces. We refer [Pis03] and [ER00] Section 3.3, Page 47 for more on these spaces.

**Proposition 4.1.** Let $E$ be a Banach space. Then both $MIN(E)$ and $MAX(E)$ have TP with constant 1.

**Proof.** Let $[x_{ij}]_{i,j=1}^n \in M_n(MIN(E))$. Then by definition we get

$$\| [x_{ij}]_{i,j=1}^n \|_{M_n(MIN(E))} = \sup \left\{ \| \Phi(x_{ij}) \|_{i,j=1}^n : \Phi : E \to \mathbb{C}, \| \Phi \| \leq 1 \right\}.$$  

Note that as the transposition map is an isometry on matrix algebras, we have

$$\sup \left\{ \| \Phi(x_{ij}) \|_{i,j=1}^n : \Phi : E \to \mathbb{C}, \| \Phi \| \leq 1 \right\} = \sup \left\{ \| \Phi(x_{ji}) \|_{i,j=1}^n : \Phi : E \to \mathbb{C}, \| \Phi \| \leq 1 \right\}.$$  

This shows that $\| [x_{ij}]_{i,j=1}^n \|_{M_n(MIN(E))} = \| [x_{ji}]_{i,j=1}^n \|_{M_n(MIN(E))}$, i.e. $MIN(E)$ has TP.

Let us now deal with the case on the maximal structure. Note that by definition we have,

$$\| [x_{ij}]_{i,j=1}^n \|_{M_n(MAX(E))} = \sup \left\{ \| \Phi(x_{ij}) \|_{i,j=1}^n : \Phi : E \to M_r \text{ for some } r \geq 1, \| \Phi \| \leq 1 \right\}.$$  

Since transposition is an isometry on matrix algebras, we have $[x_{ij}]_{i,j=1}^n \in M_n(MAX(E))$ and

$$\sup \left\{ \| \Phi(x_{ij}) \|_{i,j=1}^n : \Phi : E \to M_r \text{ for some } r \geq 1, \| \Phi \| \leq 1 \right\} = \sup \left\{ \| \Phi(x_{ji}) \|_{i,j=1}^n : \Phi : E \to M_r \text{ for some } r \geq 1, \| \Phi \| \leq 1 \right\}.$$  

The last equality follows from the fact that given any $\Phi : E \to M_r$ with $\| \Phi \| \leq 1$, the map $x \mapsto \Phi(x)^t$ is again contractive. Thus, $\| [x_{ij}]_{i,j=1}^n \|_{M_n(MAX(E))} = \| [x_{ji}]_{i,j=1}^n \|_{M_n(MAX(E))}$, which implies that $MAX(E)$ has TP with constant 1. This completes the proof. □

It is known that Pisier’s self dual operator Hilbert space [Pis96] has TP (see [Fid99]). For the sake of completeness, let us provide a proof of the above fact.

**Proposition 4.2.** Given an index set $I$, let $OH(I)$ be the corresponding operator Hilbert space. Then $OH(I)$ has TP with constant 1.

**Proof.** Let $x \in M_n(OH(I))$ be of the form $x = \sum_{k \in I} a_k \otimes e_k$, where $(e_k)_{k \in I}$ be the orthonormal basis of $OH(I)$ and $a_k \in M_n$ and $a_k = 0$ for all but finitely many $k$. Then by [Pis96] Theorem 1.1] we have that $\| x \|_{M_n(OH(I))} = \| \sum_{k \in I} a_k \otimes \overline{a_k} \|_{B(\ell_2^I \otimes \ell_2^I)}$. Note that $x^t = \sum a_k^t \otimes e_k$. Therefore, we have

$$\| x^t \|_{M_n(OH(I))} = \| \sum_{k \in I} a_k^t \otimes \overline{a_k} \|_{B(\ell_2^I \otimes \ell_2^I)} = \sum_{k \in I} (a_k^t \otimes \overline{a_k}) \|_{B(\ell_2^I \otimes \ell_2^I)} = \| \sum_{k \in I} a_k \otimes \overline{a_k} \|_{B(\ell_2^I \otimes \ell_2^I)} = \| x \|_{M_n(OH(I))}.$$
By density, we conclude that $OH(I)$ has TP with constant $1$.\qed

**Lemma 4.3.** Let $a_i \in OH(I)$ for $1 \leq i \leq n$. Then

$$
\| [a_1 \ldots a_n] \|_{M_n,1(OH(I))} = \left( \sum_{i,j=1}^{n} |\langle a_i, a_j \rangle|^2 \right)^{\frac{1}{2}}.
$$

**Proof.** For any $[x_{ij}]_{i,j=1}^{n} \in M_n(S_2)$, we have the following fact from [Pis03, Exercise 7.5]

$$
\| [x_{ij}]_{i,j=1}^{n} \|_{M_n(S_2)} = \| \left[ \langle x_{ij}, x_{kl} \rangle \right]_{k,l=1}^{n} \|_{M_n^2}^{\frac{1}{2}}.
$$

We use this to calculate the norm $\| [x_{ij}]_{i,j=1}^{n} \|_{M_n(S_2)}$, where $x_{ij} = 0$ for $j \geq 2$ and $x_{i1} = a_i$ for $1 \leq i, j \leq n$. Thus, for any $i \in \{1, \ldots, n\}$,

$$
[x_{i1}, x_{kl}]_{k,l=1}^{n} = \left[ \begin{array}{c}
\langle x_{i1}, x_{11} \rangle & \ldots & 0 \\
\langle x_{i1}, x_{21} \rangle & \ldots & 0 \\
\vdots & \ddots & \vdots \\
\langle x_{i1}, x_{n1} \rangle & \ldots & 0
\end{array} \right] = \left[ \begin{array}{c}
\langle a_i, a_1 \rangle & \ldots & 0 \\
\langle a_i, a_2 \rangle & \ldots & 0 \\
\vdots & \ddots & \vdots \\
\langle a_i, a_n \rangle & \ldots & 0
\end{array} \right].
$$

For the sake of computation, we shall denote the latter matrix by $C_i$ and

$$
\tilde{C} := \left[ \begin{array}{cccc}
C_1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
C_n & 0 & \ldots & 0
\end{array} \right]_{n \times n} \left[ \begin{array}{cccc}
\langle a_1, a_1 \rangle & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\langle a_n, a_n \rangle & \ldots & 0
\end{array} \right]_{n^2 \times n^2}.
$$

Thus, from the definition of $[x_{ij}]_{i,j=1}^{n}$, we get $[[x_{ij}, x_{kl}]_{k,l=1}^{n}]_{i,j=1}^{n} = \tilde{C}$. Hence

$$
\| [[x_{ij}, x_{kl}]_{k,l=1}^{n}]_{i,j=1}^{n} \|_{M_n^2} = \| \tilde{C} \|_{M_n^2} = \left( \sum_{i,j=1}^{n} |\langle a_i, a_j \rangle|^2 \right)^{\frac{1}{2}}.
$$

The last equality follows from the fact that the kernel of $\tilde{C}^* \tilde{C}$ has dimension $n^2 - 1$ and the only non-zero eigen value is $\sum_{i,j=1}^{n} |\langle a_i, a_j \rangle|^2$. This completes the proof.\qed

Moreover, we have the following property as well.

**Proposition 4.4.** $OH(I)$ has CMP with constant $1$.

**Proof.** Note that by the description of the norm of $OH(I)$ we have the formula

$$
\| [x_{ij}]_{i,j=1}^{n} \|_{M_n(OH(I))} = \| [[x_{ij}, x_{kl}]_{k,l=1}^{n}]_{i,j=1}^{n} \|_{M_n^2}^{\frac{1}{2}}.
$$
From Lemma 4.3 we get \( \|a_1 \ldots a_n\|_{M_n(OH(I))} = (\sum_{i,j=1}^n |a_i, a_j|^2)^{\frac{1}{2}} \), and thus we get the trivial inequality \( \|x_{ij}\|_{M_n(OH(I))} \leq (\sum_{i,j=1}^n \sum_{k,l=1}^n |\langle x_{ij}, x_{kl} \rangle|^2)^{\frac{1}{2}} \). This completes the proof.

We will now establish our results for Schatten-\( p \) classes. Let us begin with a brief overview of these operator spaces. Given a von Neumann algebra \((\mathcal{M}, \tau)\) with normal faithful semifinite trace \(\tau\), let \(L_p(\mathcal{M}, \tau)\) be the corresponding non-commutative \(L_p\)-space for \(0 < p < \infty\). One denotes \(L_\infty(\mathcal{M}) = \mathcal{M}\). When \((\mathcal{M}, \tau) = (B(\ell_2), Tr)\) with \(Tr\) being the natural trace on \(B(\ell_2)\), the corresponding non-commutative \(L_p\)-spaces are called the Schatten-\( p \) classes and are denoted by \(S_p\) for \(1 \leq p < \infty\). The space \(S_\infty\) is the space of all compact operators on \(\ell_2\). These spaces are denoted by \(S_p^n\) when \(\ell_2\) is replaced by \(\ell_2^n\) for \(1 \leq p \leq \infty\). Note that \(M_n(\mathcal{M})\) is again a von Neumann algebra equipped with the canonical tensor trace. Equipped with the canonical operator space structures \((\mathcal{M}*, \mathcal{M})\) becomes an interpolation couple (see [Pis03, Section 2.7 and Chapter 7] and [Fid99]), where \(\mathcal{M}^*\) is the predual of \(\mathcal{M}\). We identify the predual \(\mathcal{M}^*\) with \(L_1(\mathcal{M})\) via the map \(\phi : L_1(\mathcal{M}) \to \mathcal{M}^*\) as \(\phi(y)(x) := \tau(xy)\) for all \(y \in L_1(\mathcal{M})\) and \(x \in \mathcal{M}\). Moreover, \(L_1(\mathcal{M})\) has a natural operator spaces structure induced by \(\mathcal{M}^*\) (see [Pis03, Page 139]). We have the following description of the operator space structure of \(L_p(\mathcal{M}, \tau)\) from the convention established in [Pis03]. For \(x \in M_n(L_p(\mathcal{M}, \tau))\), we have for all \(1 \leq p \leq \infty\)

\[
\|x\|_{M_n(L_p(\mathcal{M}, \tau))} = \sup\{\|AXB\|_{L_p(M_n(\mathcal{M}))} : \|A\|_{S_p^n} \leq 1, \|B\|_{S_p^n} \leq 1\}.
\]

Here \(AXB\) is the usual product of matrices. Note that for \(p = 2\) the operator space structure on \(L_2(\mathcal{M}, \tau)\) agrees with the operator space structure of operator Hilbert space [Fid99]. It follows from [Pis98, Lemma 1.7] that for a map \(u : L_p(\mathcal{M}) \to L_p(\mathcal{N})\) we have \(\|u\|_{cb,L_p(\mathcal{M})\to L_p(\mathcal{N})} = \|id_{S_p^n} \otimes u\|_{L_p(M_n(\mathcal{M}))\to L_p(M_n(\mathcal{N}))}\). Moreover, \(u\) is a complete isometry iff \(id_{S_p^n} \otimes u\) is an isometry for all \(n \geq 1\).

**Remark 4.1.** Note that the norm on \(M_n(L_p(\mathcal{M}))\) is very different from \(L_p(M_n(\mathcal{M}))\) as can be seen from our computations in the proof of Theorem 4.7.

In the sequel, we shall frequently use the following block matrices for proving our results.

\[
A_n := \begin{bmatrix}
E_{11} & E_{12} & \ldots & E_{1n} \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{bmatrix}_{n \times n}, \quad B_n := \begin{bmatrix}
E_{11} & E_{21} & \ldots & E_{n1} \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{bmatrix}_{n \times n}.
\]

Now let us record a straightforward observation that will be useful in the sequel.

**Lemma 4.5.** Let \(1 \leq p < \infty\). Then

\[
\|A_n\|_{S_p^n} = \sqrt{n} \text{ and } \|B_n\|_{S_p^n} = n^{\frac{1}{p}}.
\]
Proof. Note that
\[
\left\| (A_n^* A_n)^{\frac{1}{2}} \right\|_{S_p^2} = \left\| \left( \sum_{j=1}^{n} E_{1j} E_{1j}^* \right)^{\frac{1}{2}} \right\|_{S_p^n} = \sqrt{n}.
\]
For the second matrix, notice that \(\|\cdot\|_p\) is unitary invariant. Therefore, by changing rows and columns, we'll have a block matrix with \(I_n\) at the \((1,1)\) position and zeros elsewhere. Moreover, one has \(\|I_n\|_p = n^\frac{1}{p}\). This completes the proof of the lemma. \(\square\)

**Lemma 4.6.** Let \(y = [y_{kl}]_{k,l=1}^{n} \in M_n(S_n^\infty)\) such that \(\|y\|_{M_n(S_n^\infty)} \leq 1\). Then for all \(a \in S_n^2\) with \(\|a\|_2 \leq 1\), we have
\[
\sum_{l=1}^{n} \left| \sum_{j,k=1}^{n} a_{jk} y_{kl} \right|^2 \leq 1,
\]
where \(y_{kl} = [y_{ij}]_{i,j=1}^{n}\).

Proof. Note that \(z = [z_{lk}]_{l,k=1}^{n}\) has norm \(\leq 1\), where \(z_{lk} = y_{kl}^*\). For all \(1 \leq k \leq n\), consider \(v_k := (v_{1k}, \ldots, v_{nk}) \in \ell^2\) such that \(\|v\|_{\ell^2} \leq 1\), where \(v := (v_1, \ldots, v_n)\). Note that we must have \(\|vz\|_{\ell^2}^2 \leq 1\) as well. Hence we get
\[
(4.2) \quad \sum_{l=1}^{n} \left\| \sum_{k=1}^{n} z_{lk} v_k \right\|^2 \leq 1.
\]
Note that \(z_{lk} v_k = (\sum_{j=1}^{n} z_{ij} v_{jk})_{i=1}^{n}\), where \(z_{lk} = [z_{ij}]_{i,j=1}^{n}\). Therefore, we get \(\sum_{k=1}^{n} z_{lk} v_k = (\sum_{k=1}^{n} \sum_{j=1}^{n} z_{ij} v_{jk})_{i=1}^{n}\). Thus, form equation (4.2) we obtain that
\[
\sum_{l=1}^{n} \sum_{i=1}^{n} \left| \sum_{k=1}^{n} \sum_{j=1}^{n} z_{ij} v_{jk} \right|^2 \leq 1.
\]
Fixing \(i = 1\) in the above inequality we get
\[
\sum_{l=1}^{n} \left| \sum_{k=1}^{n} \sum_{j=1}^{n} z_{lk} v_{jk} \right|^2 \leq 1.
\]
Rewriting the above inequality we get
\[
\sum_{l=1}^{n} \left| \sum_{k=1}^{n} \sum_{j=1}^{n} y_{ij} v_{jk} \right|^2 \leq 1.
\]
Putting \(v_{jk} = a_{jk}\) for \(1 \leq j, k \leq n\) in the above, we obtain the desired inequality. \(\square\)

**Lemma 4.7.** Let \(1 \leq p \neq 2 \leq \infty\). Then \(S_p\) does not have CRP.

Proof. Note that the case \(p = 2\) has been dealt in Proposition 4.2. Let us fix \(2 < p < \infty\). By [Pis98 Theorem 3.4], it is enough to find the column-row constant for \(S_p^n\). Let us consider
\[
(4.3) \quad A_n \in M_n(S_p^n).
\]
Therefore, from the above inequality and Lemma 4.5, we obtain that
\[ 1 \leq n^{-\frac{1}{2p}} \| A_n \|_{M_n(S_p)} \geq \| aA_n b \|_{L_p(M_n(B(\ell_2)))}. \]
Therefore, from the above inequality and Lemma 4.5, we obtain that
\[ \| A_n \|_{M_n(S_p)} \geq n^{-\frac{1}{2p}} \| A_n \|_{S_p} = n^{\frac{1}{2} - \frac{1}{2p}} = n^{\frac{1}{2p}}. \]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Now we shall estimate \( \| t(A_n) \|_{M_n(S_p)} \), for which let us begin by calculating \( \| t(A_n) \|_{M_n(S_p^n)} \) and then applying the method of complex interpolation. Note that the operator space structure of \( S_p^n \) agrees with the operator Hilbert space structure \( \text{Fid99} \). Therefore, by [Pis03 Theorem 7.1] and Lemma 4.3, we get
\[
A_n = (\sum_{i=1}^{n} \sum_{j=1}^{n} |E_{11}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |E_{1i}|^2)^{\frac{1}{2}} = n^{\frac{1}{2}}.
\]
Therefore, we obtain
\[
\| t(A_n) \|_{M_n(S_p^n)} = \| E_{11} \ 0 \ldots \ 0 \|_{M_n(S_p^n)} = n^{\frac{1}{2}}.
\]
We know that
\[
\| t(A_n) \|_{M_n(B(\ell_2^n))} = 1.
\]
Therefore, by complex interpolation we get \( \| t(A_n) \|_{M_n(S_2)} \leq n^{\frac{1}{2p}} \). Now notice that
\[
\frac{\| A_n \|}{\| t(A_n) \|} \geq \frac{n^{\frac{1}{2p}}}{n^{\frac{1}{2p}}} = n^{\frac{p-2}{2p}} \rightarrow \infty,
\]
as \( n \rightarrow \infty \) since \( p > 2 \). Thus, \( S_p \) does not have CRP for \( 2 < p < \infty \). Now we will study the case for \( p \in [1,2) \), for which let us consider
\[
B_n \in M_n(S_p^n).
\]
Then by choosing \( a \) and \( b \) as in condition (4.4) in the formula (4.1), we have the estimate
\[
\| B_n \|_{M_n(S_p^n)} \geq \| aB_n b \|_{L_p(M_n(B(\ell_2^n)))} = n^{\frac{1}{2p}}.
\]
Now let us estimate \( \| t(B_n) \|_{M_n(S_p^n)} \). To do this we use the method of complex interpolation again. It is easy to see that \( \| t(B_n) \|_{M_n(S_p^n)} = \| A_n \|_{M_n(S_p^n)} \). For the sake of computation, let us denote the latter block matrix by \( z := (z_{ij})_{i,j=1}^{n} \) that is, \( z_{1i} = E_{1i} \) for \( 1 \leq i \leq n \) and \( z_{ij} = 0 \) for
Remark 4.2. In view of [LZ221, Lemma 5.3] and Lemma 3.2, we have the estimate
\[ \| t_n \otimes \text{id}_{S_p} \|_{M_{n,1} \otimes S_p \to M_{1,n} \otimes S_p} \leq n^{\frac{|p-1|}{p}}. \]
Hence we have
\[ n^{\frac{p-2}{p}} \leq \| t_n \otimes id_{S_p^p} \|_{M_{n,1} \otimes S_p^p \rightarrow M_{1,n} \otimes S_p^p} \leq n^{\frac{p-2}{p}}. \]

**Proposition 4.8.** For an operator space \( E \), \( \text{MIN}(E) \) has the CMP with constant 1.

**Proof.** Let \([f_{ij}]_{i,j=1}^{n} \in M_n(C(K))\). Then we have \( \| [f_{ij}(s)] \|_{op} \leq \left( \sum_{i,j=1}^{n} |f_{ij}(s)|^2 \right)^{1/2} \). Taking supremum over \( s \in K \) we obtain that \( C(K) \) has the column-matrix property. Now we know that \( \text{MIN}(E) \) embeds completely isometrically into \( C(K) \) for some compact Hausdorff topological space \( K \) \cite[Proposition 3.3.1]{ER00}. This completes the proof. \( \square \)

**Proposition 4.9.** Let \( \Omega \) be a \( \sigma \)-finite measure space. Then \( L_p(\Omega) \) has TP with constant 1 for all \( 1 \leq p \leq \infty \).

**Proof.** At first, let us observe that both the maps \( Id : L_\infty(\Omega) \rightarrow L_\infty(\Omega)^{op} \) and \( Id : L_1(\Omega) \rightarrow L_1(\Omega)^{op} \) are complete isometries. This is due to Proposition 4.1 and the fact that the natural operator space structure on \( L_\infty(\Omega) \) agrees with the MIN \cite[Proposition 3.3.1]{ER00} and the natural operator space structure of \( L_1(\Omega) \) is induced by \( \text{MAX}(L_\infty^*) \) via the inclusion (see \cite[Chapter 2]{Pis98} and \cite[Page 51]{ER00}). Hence, the result follows by Riesz-Thorin complex interpolation. \( \square \)

**Proof of Theorem 1.3.** Let \( \mathcal{M} \) be subhomogeneous. Then it follows from \cite[Lemma 4.4]{LZ22} that \( L_p(\mathcal{M}) \) has TP. For the converse, if possible assume that \( \mathcal{M} \) is not subhomogeneous. Then by \cite[Lemma 2.1]{LZ22} it follows that for all \( n \geq 1 \) there is a complete isometry from \( S_p^n \) into \( L_p(\mathcal{M}) \). Hence by Lemma 4.7 \( L_p(\mathcal{M}) \) cannot have CRP for \( p \neq 2 \). \( \square \)

**Corollary 4.10.** Let \( \mathcal{M} \) is a semifinite von Neumann algebra. Let \( 1 \leq p \neq 2 \leq \infty \). Then \( L_p(\mathcal{M}) \) has CRP iff \( L_p(\mathcal{M}) \) has TP.

The next set of operator spaces which we shall investigate are the row and column Hilbert spaces denoted by \( R \) and \( C \), respectively.

**Proposition 4.11.** \( C \) has CRP with constant 1 but \( R \) does not have CRP.

**Proof.** We refer the reader to \cite[Section 3.4, Page 54]{ER00}, for the matrix norms of \( R \) and \( C \) spaces. The matrix norm for row operator spaces is \( \| [x_{jk}] \|_R := \left\| \left( \sum_{i=1}^{n} x_{ki}, x_{ji} \right)_{jk} \right\|_{M_n}^{1/2} \). It follows that \( \| [x_1, \ldots, x_n] \| = \left( \sum_{i=1}^{n} \| x_i \|^2 \right)^{1/2} \) and \( \| [x_1, \ldots, x_n]^t \| = \left\| \left( x_1, x_1, \ldots, x_1, x_n \right) \right\|_{M_n}^{1/2} \). In particular, if we choose orthonormal vectors, then
\[ \| [e_1, \ldots, e_n] \| = \sqrt{n}, \quad \| [e_1, \ldots, e_n]^t \| = 1. \]

This implies that the row operator space does not admit CRP. Now we shall investigate column operator spaces. Let us note that the matrix norm is \( \| [x_{jk}] \|_C := \left\| \left( \sum_{i=1}^{n} x_{ik}, x_{ij} \right)_{jk} \right\|_{M_n}^{1/2} \) and
Thus, \( \| [x_1, \ldots, x_n]^t \| = (\sum_{i=1}^{n} \| x_i \|^2)^{\frac{1}{2}} \). Furthermore,

\[
\| [x_1, \ldots, x_n] \| = \left\| \left[ \| x_1 \|^2, \langle x_2, x_1 \rangle, \ldots, \langle x_n, x_1 \rangle \right] \right\|_{\frac{1}{2}} = (\| x_1 \|^4 + \sum_{i=2}^{n} |\langle x_i, x_1 \rangle|^2)^{\frac{1}{2}} \\
\leq \| x_1 \| \left( \sum_{i=1}^{n} \| x_i \|^2 \right)^{\frac{1}{2}}.
\]

Without the loss of generality, if we choose \( x_1 \) to be an unit vector then

\[
\| [x_1, \ldots, x_n] \| \leq \left( \sum_{i=1}^{n} \| x_i \|^2 \right)^{\frac{1}{2}} = \| [x_1, \ldots, x_n]^t \|.
\]

This proves that \( C \) has CRP. \( \square \)

We shall now study the completely bounded versions of the notions CRP and CMP. To state our result we introduce the following definition.

**Definition 4.12.** Let \( E \) be an operator space. \( E \) is said to satisfy the completely bounded CRP if there exists a constant \( C > 0 \) such that for all \( n \geq 1 \), we have

\[
\| [x_1 \ldots x_n]^t \|_{cb, M_{n, 1}(E) \to M_{1, n}(E)} < C.
\]

The notion of completely bounded CMP is defined in a similar manner. We now show that there is no non-trivial operator space with the completely bounded CRP.

**Proof of Theorem** \( \square \)

Let \( e \in E \) be such that \( \| e \| = 1 \). Then it follows from the property of the operator space injective tensor product \( [ER00, \text{Page } 142] \) that the map \( i_n : M_{n, 1} \to M_{n, 1} \otimes E \) given by \( x \mapsto x \otimes e \) is a complete isometry. Also \( t_n(x \otimes e) = t_n(x) \otimes e \) for all \( x \in M_{n, 1} \). Note that \( i_d M_N \otimes t_n (x_{ij} \otimes e)_{i, j=1}^{N} = [t_n(x_{ij})]_{i, j=1}^{N} \otimes e \). Therefore, from the fact that the operator space injective tensor product is associative \( [ER00] \), we have that

\[
\| t_n \|_{cb, M_{n, 1} \otimes E \to M_{1, n} \otimes E} \geq \sup_{N \geq 1, |x_{ij}|_{i, j=1}^{N} \neq 0} \frac{\| t_n(x_{ij}) \|_{i, j=1}^{N}}{\| x_{ij} \|_{i, j=1}^{N}} = \sqrt{n}.
\]

The last estimate in the above inequality follows from \( [Pis03, \text{Page } 22] \). This shows that \( E \) does not have completely bounded CRP. The fact that \( E \) does not have completely bounded CMP follows easily from this. This completes the proof of the theorem. \( \square \)

The following proposition is interesting in its own right. It shows that the completely bounded norm of \( t_n \otimes id_E \) is equal to the completely bounded norm of certain operator from column to matrices.

**Proposition 4.13.** Let \( E \) be an operator space. Then for all \( n \geq 1 \)

\[
\| t_n \otimes id_E \|_{cb, M_{n, 1} \otimes E \to M_{1, n} \otimes E} = \| [x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn}]^t \|_{cb, M_{n, 2, 1}(E) \to M_n(E)}.
\]
Proof. It is a straightforward observation that the right hand side dominates the left hand side. Also we may assume that the left hand side is finite. Now for proving the converse, suppose there exists $C > 0$ such that

$$
\| t_n \otimes id_E \|_{cb,M_{1,n} \otimes E \rightarrow M_{1,n} \otimes E} < C
$$

where $t_n : M_{n,1} \rightarrow M_{1,n}$ is the usual transpose map. Therefore, for all $n \geq 1$ we get

$$
\| id_{M_{n,1}} \otimes (t_n \otimes id_E) \|_{cb,M_{n,1} \otimes M_{n,1} \otimes E \rightarrow M_{n,1} \otimes M_{1,n} \otimes E} < C.
$$

Note that we have the identification $M_{n,1} \otimes M_{1,n} \cong M_n$ completely isometrically via the map, $e_i^t \otimes e_j \mapsto e_{ij}$ (see [ER00, Section 9.3]) and $M_n \otimes E \cong M_n(E)$ (see [ER00, Corollary 8.1.3]) via the canonical identification. Using these identifications it is easy to observe that

$$
id_{M_{n,1}} \otimes (t_n \otimes id_E)([x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn}]^t) = [x_{ij}]_{i,j=1}^n.
$$

The proof is completed by taking infimum over all $C$. \qed

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