Higher Spin Interacting Quantum Field Theory and Higher Order Conformal Invariant Lagrangians

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PhD Thesis

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Abstract

This thesis includes several original results. All of them are already published or submitted for publication. The thesis is based on articles \[51\], \[12\], \[33\], \[34\], \[35\], \[36\], \[30\], \[37\] and reproduces the results of \[50\] for completeness.

I present here the short summary of main results:

The ultraviolet singular structure of the bulk-to-bulk propagators for higher spin gauge fields in AdS\(_4\) space is analyzed in details. One loop mass renormalization is studied on a simple example.

The conformal invariant Lagrangian with the \(k\)-th power of Laplacian for the hierarchy of conformally coupled scalars with increasing scaling dimensions connected with the \(k\)-th Euler density is rederived using the Fefferman-Graham ambient space approach. The corresponding gauged ambient metric, Fefferman-Graham expansion and extended Penrose-Brown-Henneaux transformations are proposed and analyzed.

Linearized gauge invariant interactions of scalar and general higher even spin fields in the AdS\(_D\) space are obtained. A generalized Weyl transformation is proposed and the corresponding Weyl invariant action for cubic coupling of a scalar to a spin \(\ell\) field is constructed.

Using Noether’s procedure several cubic interactions between different HS gauge fields are derived, including cubic selfinteraction of even spin gauge fields in a flat background. Then the main result - the complete off-shell gauge invariant Lagrangian for the trilinear interactions of Higher Spin Fields with arbitrary spins \(s_1, s_2, s_3\) in a flat background is presented. All possibilities with different numbers of derivatives are discussed. Restrictions on the number of derivatives are obtained. For any possible number of derivatives this interaction is uniquely fixed by gauge invariance up to partial integration and field redefinition.

Finally an off-shell generating function for all cubic interactions of Higher Spin gauge fields is presented. It is written in a compact way, and turns out to have a remarkable structure.
"The good lord is subtle, but he is not malicious."
Albert Einstein
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Higher Spin gauge field theory is one of the most important and puzzling problems in modern quantum field theory. The first attempts to deal with a quantum theory of high spin particles date back to late 1930-s and 1940-s, with the works of Dirac [1], Fierz and Pauli [2], Rarita and Schwinger [3] and others. One of the most important concepts in quantum field theory is Poincaré symmetry. In the classical papers [4] and [5] the irreducible representations of Poincaré group were classified. These irreducible representations are characterized by two quantities - mass and spin. After discovery of nonabelian gauge theory by Yang and Mills [6] the role of gauge symmetries in quantum field theory was acknowledged. It became clear that the invariance of the Lagrangian with respect to local symmetry is the cornerstone of any field theory.

There are higher rank tensor representations of the Poincaré group [4, 5], which are not yet associated with any physical field theory. It is clear that the higher rank tensor representations of the Poincaré group, the Higher Spin fields, are gauge fields that one has to introduce gauging higher derivative symmetries of the action.

It is natural to assume that Higher Spin gauge field interactions are much weaker than gravitational ones, that’s why we don’t see any evidence for these interactions. They should be important only in a very high energy regime. An interesting speculative application of Higher Spin gauge fields might be also it’s connection to Dark Matter and/or Dark Energy.

Nowadays the best candidate for quantum gravity is String Theory. In the spectrum of the String Theory there are excitations with any high spin, therefore String Theory gives another motivation for investigations of Higher Spin gauge fields to take place.

A new motivation for investigating Higher Spin gauge field theories arose during the last decade after discovering the holographic duality between the $O(N)$ sigma model in $d = 3$ space and Higher Spin gauge field theory living in the space $AdS_4$ [7], which is an interesting special case of AdS/CFT conjecture [8]. This case of holography is especially important by the existence of two conformal points.
of the boundary theory and the possibility to describe them by the same HSF
gauge theory with the help of spontaneous breaking of higher spin gauge symmetry
and mass generation by a corresponding Higgs mechanism ([9]-[13] and references
therein).

After the nonabelian gauge theory of vector fields [6] appeared, there were
numerous attempts to construct a gauge theory with a gauge group which mixes
space-time and internal symmetries in a nontrivial way. These attempts resulted
in several no-go theorems. The most general result was obtained by Coleman and
Mandula in [14]. They have proved a theorem on the impossibility of combining
space-time and internal symmetries in any but a trivial way, which holds not
only for Lie groups but is also applicable to infinite-parameter groups. The
moral of this theorem is that if the assumptions of the theorem hold, there can’t
be Higher Spin charges. Therefore Higher Spin fields, if existing, don’t participate
in interactions. In order to have an interacting theory of Higher Spin fields one
has to loosen some of the assumptions of the Coleman-Mandula theorem. Then
it was shown that it is possible to overcome this theorem introducing graded
Lie algebras [15, 16, 17], which give rise to the supersymmetric theories. It was
shown in [18] that the only possible algebras that mix space-time and internal
symmetries are graded Lie algebras, which in addition to the standard generators
of Poincaré algebra include also supersymmetry generators with spin one-half.

Two other no-go theorems were formulated by Steven Weinberg and Edward
Witten in 1980 [19]. One of them rules out electrically charged fields with spin
$s > 1/2$, the other theorem forbids theories with a Lorentz covariant energy-
momentum tensor which include fields with spin $s > 1$. These theorems don’t
apply to gauge theories though, therefore the search for Higher Spin gauge theo-
ries wasn’t proved to be meaningless.

Despite all the no-go results (see also [20]), using the Lagrangian formulation
of Higher Spin theories by Singh and Haagen [21, 22], the consistent Lagrangian
description for free Higher Spin gauge fields both in flat space and in constantly
curved backgrounds was given by Fronsdal in [23, 25] for bosonic fields and by
Fang and Fronsdal in [24, 26] for fermionic Higher Spin fields. Fronsdal’s theory
of Higher Spin gauge fields includes some new features. There are constraints
on Higher Spin gauge fields and the gauge parameters. In order to have a gauge
invariant field equation of motion that is linear in the Higher Spin gauge field
and of second order in the derivatives, the gauge parameter should satisfy the
tracelessness condition. This is so-called Fronsdal’s first constraint. In order to
have a gauge invariant kinetic Lagrangian for free Higher Spin gauge fields, the
field itself should be double traceless. This is Fronsdal’s second constraint. On-
shell gauge symmetry allows to gauge away all nonphysical components of the field
and to obtain a traceless transversal tensor field with a simple Klein-Gordon-like
equation of motion. For the spin $s$ field we get two possible helicities: $\pm s$. The
possible deformation of the gauge algebra of Fronsdal’s Lagrangian, which should
lead to the Higher Spin interacting gauge theory is considered as a challenge
already 30 years. Fronsdal’s theory of Higher Spin gauge fields is a natural
generalization of linearized gravity, and is called also metric-like formulation of
Higher Spin gauge fields.

The generalization of Christoffel symbol and Riemann curvature of linearized
gravity for Higher Spin cases was given by deWitt and Freedman in [27]. There
are \( s - 1 \) Christoffel symbols for Higher Spin gauge fields, with different numbers
of derivatives (from 1 to \( s - 1 \)). They are all linear in the field and transform
under gauge transformations in a simple way. The curvature of the Higher Spin
field, also linear in the field, is invariant with respect to gauge transformations.
Interesting properties of Higher Spin field curvatures are discussed in [28]. The
full nonlinear form for the deWitt-Freedman curvature and Christoffel symbols
(if any) is still unknown.

Despite the fact that consistent equations of motion for Higher Spin gauge
fields are known over twenty years [29], the question of existence of Lagrangian for
interacting Higher Spin gauge fields is still open. The subject of special interest is
a minimal selfinteraction of even spin gauge fields, where one can naively expect
the existence of an Einstein-Hilbert type nonlinear action for any single even spin
gauge field. Although there are known restrictions on Higher Spin theories in flat
space-time, the recent development [30] has shown that there is a local higher
derivative cubic interaction Lagrangian for gauge fields with any higher spins in
flat space-time of any dimensions. This shifts the no-go theorems to the quartic
power of fields in interaction Lagrangians, where one can expect the final battle
for the existence of local (or nonlocal) Lagrangians for interacting HS gauge field
theory in flat space.

Gauge symmetry, which is a redundancy of non-physical degrees of freedom in
the Lagrangian, is the main principle which helps to choose the right Lagrangian
for the given theory. To quote C. N. Yang, gauge symmetry dictates the form
of the interaction. In this thesis we will show that for Higher Spin gauge fields
interactions are uniquely determined by gauge symmetry.

The free Lagrangian for Higher Spin gauge fields both in flat space and in
constantly curved backgrounds (dS and AdS) are known over thirty years [23,
24, 25, 26]. In contrast to free theory, attempts to construct Lagrangians for
interacting theories haven’t been successful yet beyond the cubic vertices. In this
thesis we are going to discuss only trilinear interactions of Higher Spin gauge
fields.

Our recent results [30], [32]-[37] on Higher Spin gauge field cubic interactions
in flat space, which certainly reproduce the flat limit of the famous Fradkin-
Vasiliev vertex for higher spin coupling to gravity [38], show that all interactions
of higher spin gauge fields with any spins \( s_1, s_2, s_3 \) both in flat space and in dS

\[^*\]This quotation along with a very beautiful review of the history of gauge symmetries you
can find in [31].
or AdS are unique. This was already proven for some low spin cases of both the Fradkin-Vasiliev vertex for $2, s, s$ and the nonabelian vertex for $1, s, s$ in [39].

The first important step towards cubic interactions in Higher Spin gauge field theory in covariant formulation was done in 1984 by Berends, Burgers and van Dam [40]. They constructed a cubic selfinteraction Lagrangian for spin three gauge fields and proved impossibility of extension to higher orders. Their arguments are based on gauge algebra, which does not close for a single spin three nonabelian field. The authors give an optimistic hope that it will be possible to extend this Lagrangian to higher orders if one takes into account corrections from interactions with gauge fields with spins higher than three. A recent discussion on this subject appeared from Bekaert, Boulanger and Leclerq [41]. They show the impossibility to close this non-abelian (spin 3) algebra taking into account corrections from interactions of other fields with spins higher (or lower) than three.

The first successful result on Higher Spin gravitational interactions was derived in the already mentioned work by Fradkin and Vasiliev [38], where a cubic coupling of Higher Spin gauge fields to linearized gravity was constructed in the constantly curved background. The interesting property of this Lagrangian is its non-analyticity in the cosmological constant, therefore excluding a flat space limit. However it was shown already in [39] that after rescaling of Higher Spin gauge fields one can observe a flat limit for the Fradkin-Vasiliev interactions. In our approach the spin $s$ gauge field has scaling dimension $[\text{length}]^{s-2}$, and the Fradkin-Vasiliev vertex has a flat limit with $2s-2$ derivatives (minimal possible number) in the $2-s-s$ interaction which has the same scaling dimension as the Einstein-Hilbert Lagrangian terms. As it was shown by Metsaev in [42] using a light cone gauge approach, there are three different couplings to linearized gravity with different numbers of derivatives for any higher spin $s$ field, and in general $\min\{s_1, s_2, s_3\} + 1$ different possibilities with different numbers of derivatives for the $s_1-s_2-s_3$ interaction. All these interactions were derived in a covariant off-shell formulation in [30], and I am going to discuss them in this Thesis.

For some important results on higher spin cubic interactions see [43]-[47] and references therein. For recent reviews see [48].

The only Section in this Thesis that is not directly connected to Higher Spin theory is Section 4.1 where I give brief introduction independently.

All the Chapters in this Thesis are more or less independent, in some cases there are even differences in conventions, therefore I give notations and conventions independently where needed.

The formalism which I use in this Thesis is developed in Chapter 2 where I present also well known results in the free theory of Higher Spin fields: the Fronsdal Lagrangian, the deWitt-Freedman curvatures and Christoffel symbols for HS gauge fields and the Bianchi identities that connect them as well as some

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†The cubic interaction Lagrangian is unique up to partial integration and field redefinition.
new connections between these quantities following from their Bianchi identities.

In Chapter 3 the ultraviolet singular structure of the bulk-to-bulk propagators for higher spin gauge fields in $AdS_4$ space is analyzed in details. One loop mass renormalization corresponding to interactions with the Higgs scalar are studied. This mass renormalization is finite and connected with the anomalous dimensions of those currents in the corresponding boundary $CFT_3$ that cease to be conserved when the interaction is switched on. In particular it is proportional to $\ell - 2$ for a spin $\ell$ field.

In Chapter 4 the hierarchy of conformally coupled scalars with increasing scaling dimensions $\Delta_k = k - d/2$, $k = 1, 2, 3, \ldots$ connected with the $k$-th Euler density in the corresponding space-time dimensions $d \geq 2k$ is proposed. The corresponding conformal invariant Lagrangian with the $k$-th power of Laplacian for the already known cases $k = 1, 2$ is reviewed, and the subsequent case of $k = 3$ is completely constructed and analyzed. The same hierarchy is redervied using the Fefferman-Graham $d + 2$ dimensional ambient space approach. The corresponding mysterious "holographic" structure of these operators is clarified. We explore also the $d + 2$ dimensional ambient space origin of the Ricci gauging procedure proposed by A. Iorio, L. O’Raifeartaigh, I. Sachs and C. Wiesendanger as another method of constructing the Weyl invariant Lagrangians. The corresponding gauged ambient metric, Fefferman-Graham expansion and extended Penrose-Brown-Henneaux transformations are proposed and analyzed.

Then another generalization of conformal coupling of the scalar to the gravity is considered. The explicit form of linearized gauge invariant interactions of scalar and general higher even spin fields in the $AdS_D$ space is obtained. In the case of general spin $\ell$ a generalized ‘Weyl’ transformation is proposed and the corresponding ‘Weyl’ invariant action is constructed. In both cases the invariant actions of the interacting higher even spin gauge field and the scalar field include the whole tower of invariant actions for couplings of the same scalar with all gauge fields of smaller even spin.

In section 5.1 of Chapter 5 several trilinear interactions of higher spin fields involving two equal ($s = s_1 = s_2$) and one higher even ($s_3 > s$) spin are presented. Interactions are constructed on the Lagrangian level using Noether’s procedure together with the corresponding next to free level fields of the gauge transformations. In certain cases when the number of derivatives in the transformation is $2s - 1$ the interactions lead to the currents constructed from the generalization of the gravitational Bell-Robinson tensors. In section 5.2 of Chapter 5 using Noether’s procedure the complete cubic selfinteraction for the case of spin $s = 4$ in a flat background is presented and the cubic selfinteraction for general spin $s$ with $s$ derivatives in the same background is discussed. The leading term of the latter interaction together with the leading gauge transformation of first field order are presented. In section 5.3 of Chapter 5 the complete solution for the trilinear interactions of arbitrary spins $s_1, s_2, s_3$ in a flat background is presented, the possibility to enlarge this construction to higher order interactions in the
gauge field is discussed. Finally the expansion of a general spin $s$ gauge transformation into powers of the field and the related closure of the gauge algebra in the general case are discussed.

In the recent paper [49] by Sagnotti and Taronna the authors proposed an on-shell generating function for the general HS cubic interaction presented in [30] from a massless limit of String Theory. In the final Chapter 6 of this thesis I am going to present an off-shell extension of that generating function, which can surprisingly be enhanced with a beautiful Grassmann structure, the string origin of which is not clear yet.

In this Thesis I am not going to address many important directions in Higher Spin gauge field theories like Mixed Symmetry Higher Spin fields (see [87]-[102] and references therein), Unconstrained Higher Spins ([89], [90], [103] and references therein), Unfolded formulation of Higher Spin dynamics ([104]-[110] and references therein), Vasiliev equations ([29], [128] and references therein), BRST approach to Higher Spins ([110]-[114]), Light Cone gauge formulation ([12], [116]-[117] and references therein), some AdS/CFT aspects ([10], [115] and references therein), some String-inspired constructions ([118]-[122] and references therein), partially massless Higher Spin fields ([123]-[127] and references therein).

Another weakness of this Thesis is the lack of a group-theoretical description: the study of the non-abelian gauge algebra which stands behind the interactions presented in this Thesis as well as "Higher Spin geometry" interpretations (if any) will follow in the future. The interactions we discuss in this Thesis don’t include fermionic half-integer spin fields, which is another important direction to generalize these results.
Chapter 2

Free Higher Spin Gauge fields

2.1 Technical setup and important relations in free HSF theory

We work with Higher Spin Gauge Fields in Fronsdal (metric-like) formulation [23, 25], in which the spin \( s \) field is double traceless fully symmetric \( s \)-th rank tensor. The most elegant and convenient way of handling symmetric tensors such as \( h^{(s)}_{\mu_1\mu_2...\mu_s}(z) \) is by contracting it with the \( s' \)th tensorial power of a vector \( a^\mu \) of the tangential space at the base point \( z \):

\[
h^{(s)}(z; a) = \sum_{\mu_i} (\prod_{i=1}^{s} a^{\mu_i}) h^{(s)}_{\mu_1\mu_2...\mu_s}(z).
\]  

(2.1.1)

In this way we obtain a homogeneous polynomial in the vector \( a^\mu \) of degree \( s \). In this formalism the symmetrized gradient, trace and divergence are

- \( \text{Grad} : h^{(s)}(z; a) \Rightarrow \text{Grad} h^{(s+1)}(z; a) = (a\nabla)h^{(s)}(z; a), \)  

(2.1.2)

- \( \text{Tr} : h^{(s)}(z; a) \Rightarrow \text{Tr} h^{(s-2)}(z; a) = \frac{1}{s(s-1)} \Box_a h^{(s)}(z; a), \)  

(2.1.3)

- \( \text{Div} : h^{(s)}(z; a) \Rightarrow \text{Div} h^{(s-1)}(z; a) = \frac{1}{s} (\nabla a) h^{(s)}(z; a). \)  

(2.1.4)

Next we introduce the notation \( *_a, *_b \) for a contraction in the symmetric spaces of indices \( a \) or \( b \)

\[
*_a = \frac{1}{(s!)^2} \prod_{i=1}^{s} \left( \partial_a^{\mu_i} \partial_a^{\mu_i} \right).
\]  

(2.1.5)

To manipulate reshuffling of different sets of indices we employ other differentials with respect to \( a \) and \( b \), e.g. \( (a\partial_b) \) or \( (b\partial_a) \). Then we see that operators \( (a\partial_b), a^2, b^2 \)

*To distinguish easily between "a" and "z" spaces we introduce for space-time derivatives \( \partial_a \) the notation \( \nabla_\mu. \)
are dual (or adjoint) to \((b\partial_a), \square_a, \square_b\) with respect to the "star" product of tensors with two sets of symmetrized indices \((2.1.5)\):

\[
\frac{1}{n}(a\partial_b) f^{(m-1,n)}(a, b) *_{a,b} g^{(m,n-1)}(a, b) = f^{(m-1,n)}(a, b) *_{a,b} \frac{1}{m}(b\partial_a) g^{(m,n-1)}(a, b), \quad (2.1.6)
\]

\[
a^2 f^{(m-2,n)}(a, b) *_{a,b} g^{(m,n)}(a, b) = f^{(m-2,n)}(a, b) *_{a,b} \frac{1}{m(m-1)} \square_a g^{(m,n)}(a, b)
\]

(an analogous equation for \(b^2\)) \(\quad (2.1.7)\)

In the same fashion gradients and divergences are dual with respect to the full scalar product in the space \((z; a, b)\)

\[
(a\nabla) f^{(m-1,n)}(z; a, b) *_{a,b} g^{(m,n)}(z; a, b) = -f^{(m-1,n)}(z; a, b) *_{a,b} \frac{1}{m}(\nabla \partial_a) g^{(m,n)}(z; a, b)
\]

(an analogous equation for \((b\nabla)\)) \(\quad (2.1.8)\)

We will use the deWit-Freedman curvature and Christoffel symbols \([27, 85]\). The n-th deWit-Freedman-Christoffel symbol is

\[
\Gamma^{(s)}_{(n)}(z; b, a) \equiv \Gamma^{(s)}_{(n)p_1 \ldots p_s \mu_1 \ldots \mu_t} b^{p_1} \ldots b^{p_s} a^{\mu_1} \ldots a^{\mu_t} = [(b\nabla) - \frac{1}{n}(a\nabla)(b\partial_a)] \Gamma^{(s)}_{(n-1)}(z; b, a), \quad (2.1.9)
\]

or in another way

\[
\Gamma^{(s)}_{(n)}(z; b, a) = \left( \prod_{k=1}^{s} [(b\nabla) - \frac{1}{k}(a\nabla)(b\partial_a)] \right) h^{(s)}(z; a). \quad (2.1.10)
\]

We contracted them with the degree \(s\) tensorial power of one tangential vector \(a^{\mu}\) in the first set of \(s\) indices and with a similar tensorial power of another tangential vector \(b^{\nu}\) in its second set. The deWit-Freedman curvature and n-th Christoffel symbol are then written as

\[
\Gamma^{(s)}(z; b, a) = \Gamma^{(s)}(z; b, \lambda a) = \Gamma^{(s)}(z; \lambda b, a) = \lambda^s \Gamma^{(s)}(z; b, a), \quad (2.1.11)
\]

\[
\Gamma^{(s)}_{(n)}(z; b, a) = \Gamma^{(s)}_{(n)}(z; b, \lambda a) = \lambda^n \Gamma^{(s)}_{(n)}(z; b, a), \quad (2.1.12)
\]

\[
\Gamma^{(s)}_{(n)}(z; \lambda b, a) = \lambda^n \Gamma^{(s)}_{(n)}(z; b, a), \quad (2.1.13)
\]

\[
\Gamma^{(s)}(z; b, a) = \Gamma^{(s)}_{(n)}(z; b, a)|_{n=s}. \quad (2.1.14)
\]

Now one can prove that \([27, 86]\):

\[
(a\partial_b) \Gamma^{(s)}(z; a, b) = (b\partial_a) \Gamma^{(s)}(z; a, b) = 0. \quad (2.1.15)
\]

These "primary Bianchi identities" are manifestations of the hidden antisymmetry.
Using the following commutation relations
\[
[(b\partial_a)^n, (a\nabla)^m] = \sum_{k=1}^{\min\{n,m\}} k! \binom{n}{k} \binom{m}{k} (b\nabla)^{n-k}(a\partial_a)^{m-k}, (2.1.16)
\]
\[
\Box_b(b\nabla)^i = i(i-1)(b\nabla)^{i-2}\Box,
\]
\[
\partial^b_a(b\nabla)^i\partial^b_a(b\partial_a)^j = ij(b\nabla)^{i-1}(b\partial_a)^{j-1}(\nabla\partial_a),
\]
\[
\Box_b(b\partial_a)^j = j(j-1)(b\partial_a)^{j-2}\Box_a,
\]
and mathematical induction we can prove that
\[
\Gamma_{(n)}^{(s)}(z; b, a) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} (b\nabla)^{n-k}(a\nabla)^k(b\partial_a)^k h^{(s)}(z; a). (2.1.20)
\]

The gauge variation of a spin $s$ field is
\[
\delta h^{(s)}(z; a) = s(a\nabla)\epsilon^{(s-1)}(z; a), (2.1.21)
\]
with traceless gauge parameter
\[
\Box_a\epsilon^{(s-1)}(z; a) = 0, (2.1.22)
\]
for the double traceless gauge field
\[
\Box_a^2 h^{(s)}(z; a) = 0. (2.1.23)
\]
The gauge variation of n-th Christoffel symbol is
\[
\delta \Gamma_{(n)}^{(s)}(z; b, a) = \frac{(-1)^n}{n!} (a\nabla)^{n+1}(b\partial_a)^n \epsilon^{(s-1)}(z; a), (2.1.24)
\]
putting here $n = s$ we obtain gauge invariance for the curvature
\[
\delta \Gamma_{(s)}^{(s)}(z; b, a) = 0. (2.1.25)
\]
Tracelessness of the gauge parameter (2.1.22) implies that b-traces of all Christoffel symbols are gauge invariant
\[
\Box_b \delta \Gamma_{(n)}^{(s)}(z; b, a) = \frac{(-1)^n}{(n-2)!} (a\nabla)^{n+1}(b\partial_a)^{n-2} \Box_a \epsilon^{(s-1)}(z; a) = 0. (2.1.26)
\]
Thus for the second order gauge invariant field equation we can use the trace of the second Christoffel symbol, the so called Fronsdal tensor:
\[
F^{(s)}(z; a) = \frac{1}{2} \Box_b \Gamma_{(2)}^{(s)}(z; b, a)
\]
\[
= \Box h^{(s)}(z; a) - (a\nabla)(\nabla\partial_a)h^{(s)}(z; a) + \frac{1}{2}(a\nabla)^2 \Box_a h^{(s)}(z; a). (2.1.27)
\]
Using equation (2.1.20) for Christoffel symbols and after long calculations we obtain the following expression

\[ \square_b \Gamma^{(s)}(z; b, a) = \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} (n - k)(n - k - 1)(b \nabla)^{n-k-2}(a \nabla)^k (b \partial_a)^k \mathcal{F}^{(s)}(z; a). \]  

(2.1.28)

We have expressed the b-trace of any \( \Gamma^{(s)}_{(n)} \) through the Fronsdal tensor or the b-trace of the second Christoffel symbol, which is the same (2.1.27), which means that there are only two nontrivial gauge invariant objects–Fronsdal tensor and deWit-Freedman curvature of higher spin gauge field. But this is not the whole story. Using mathematical induction and (2.1.16)-(2.1.19) again we can show that

\[ n - 1 \prod_{k=3}^{n} [(b \nabla) - \frac{1}{k} (a \nabla)(b \partial_a)] \mathcal{F}^{(s)}(z; a). \]  

(2.1.29)

In particular for the trace of the curvature we can write

\[ \square_b \Gamma^{(s)}(z; b, a) = s(s - 1) \mathcal{U}(a, b, 3, s) \mathcal{F}^{(s)}(z; a), \]  

where we introduced an operator mapping the Fronsdal tensor on the trace of the curvature

\[ \mathcal{U}(a, b, 3, s) = \prod_{k=3}^{s} [(b \nabla) - \frac{1}{k} (a \nabla)(b \partial_a)]. \]  

(2.1.30)

(2.1.31)

Now let us consider this curvature in more detail. First we have the symmetry under exchange of \( a \) and \( b \)

\[ \Gamma^{(s)}(z; a, b) = (-1)^s \Gamma^{(s)}(z; b, a). \]  

(2.1.32)

Therefore the operation ”a-trace” can be defined by (2.1.30) with exchange of \( a \) and \( b \) at the end. The mixed trace of the curvature can be expressed through the \( a \) or \( b \) traces using ”primary Bianchi identities” (2.1.15)

\[ (\partial_a \partial_b) \Gamma^{(s)}(z; b, a) = -\frac{1}{2} (b \partial_a) \square_b \Gamma^{(s)}(z; b, a) = -\frac{1}{2} (a \partial_b) \square_a \Gamma^{(s)}(z; b, a). \]  

(2.1.33)

The next interesting properties of the higher spin curvature and corresponding Ricci tensors are so called generalized secondary or differential Bianchi identities.

\[ \text{Which means that all b-traces of all Christoffel symbols are zero On-shell when Fronsdal equation } \mathcal{F}^{(s)}(z; a) = 0 \text{ holds.} \]
CHAPTER 2. FREE HIGHER SPIN GAUGE FIELDS

We can formulate these identities in our notation in the following compressed form ([... ] denotes antisymmetrization)

\[ \frac{\partial}{\partial a^\mu} \frac{\partial}{\partial b^\nu} \nabla_\lambda \Gamma^{(s)}(z; a, b) = 0. \]  \hfill (2.1.34)

This relation can be checked directly from representation (2.1.20). Then contracting with \( a^\mu \) and \( b^\nu \) we get a symmetrized form of (2.1.34)

\[ s \nabla_\mu \Gamma^{(s)}(z; a, b) = (a \nabla) \partial_\mu \Gamma^{(s)}(z; a, b) + (b \nabla) \partial_\mu \Gamma^{(s)}(z; a, b). \]  \hfill (2.1.35)

Now we can contract (2.1.35) with \( a \partial_\mu \) and using (2.1.33) obtain a connection between the divergence and the trace of the curvature

\[ (s - 1)(\nabla \partial_\mu) \Gamma^{(s)}(z; a, b) = [(b \nabla) - \frac{1}{2} (a \nabla)(b \partial_\nu)] \Box_b \Gamma^{(s)}(z; a, b). \]  \hfill (2.1.36)

These two identities with a similar identity for the Fronsdal tensor

\[ (\nabla \partial_\mu) F^{(s)}(z; a) = \frac{1}{2} (a \nabla) \Box_a F^{(s)}(z; a), \]  \hfill (2.1.37)

play an important role for the construction of the interaction Lagrangian. To complete the free field information we present here Fronsdal’s Lagrangian in terms of our quantities:

\[ \mathcal{L}_0(h^{(s)}(a)) = -\frac{1}{2} h^{(s)}(a) \ast_a F^{(s)}(a) + \frac{1}{8 s(s - 1)} \Box_a h^{(s)}(a) \ast_a \Box_a F^{(s)}(a). \]  \hfill (2.1.38)

To obtain the equation of motion we vary (2.1.38) and obtain

\[ \delta \mathcal{L}_0(h^{(s)}(a)) = -(F^{(s)}(a) - \frac{a^2}{4} \Box_a F^{(s)}(a)) \ast_a \delta h^{(s)}(a). \]  \hfill (2.1.39)

Zero order gauge invariance can be checked easily by substitution of (2.1.21) into this variation and use of the duality relation (2.1.8) and identity (2.1.37) taking into account tracelessness of the gauge parameter (2.1.22).
Chapter 3

Propagator of HSF and One-Loop Diagram

3.1 General setup for higher spin propagators

Here we would like to give all the conventions about $AdS_{d+1}$ metric. We work in Euclidian $AdS_{d+1}$ with the following metric, curvature and covariant derivatives:

$$ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu = \frac{L^2}{(z_0)^2} \delta_{\mu\nu}dz^\mu dz^\nu, \quad \sqrt{g} = \frac{L^{d+1}}{(z_0)^{d+1}},$$

$$[\nabla_\mu, \nabla_\nu]V_\lambda^\rho = R_{\mu\nu\lambda}^\sigma V_\sigma^\rho - R_{\mu\nu\sigma}^\rho V_\lambda^\sigma,$$

$$R_{\mu\nu\lambda}^\rho = -\frac{1}{(z_0)^2} \left( \delta_{\mu\rho} \delta_{\nu\lambda} - \delta_{\nu\rho} \delta_{\mu\lambda} \right) = -\frac{1}{L^2} \left( g_{\mu\lambda}(z) \delta_\rho^\nu - g_{\nu\lambda}(z) \delta_\rho^\mu \right),$$

$$R_{\mu\nu} = -\frac{d}{(z_0)^2} \delta_{\mu\nu} = -\frac{d}{L^2} g_{\mu\nu}(z), \quad R = -\frac{d(d+1)}{L^2}.$$  \hspace{1cm} (3.1.2)

For simplicity we will from now on put $L = 1$ during all calculations keeping in mind that we can always restore the $AdS$ radius from dimensional consideration.

Then we can write the starting point of the investigation of higher spin gauge field propagators, namely Fronsdal’s equation of motion (we introduce here the “geometric AdS mass” $\mu_\ell^2$) [25] for the double traceless spin $\ell$ field, $\Box_a \Box_a h^{(\ell)} = 0$:

$$\mathcal{F}(h^{(\ell)}(z;a)) = \left[ \square - \mu_\ell^2 \right] h^{(\ell)}(z;a) - a^2 \Box_a h^{(\ell)}(z;a) \right.$$  

$$- \left( a\nabla \right) \left[ \nabla^\mu \frac{\partial}{\partial a^\mu} h^{(\ell)} - \frac{1}{2} (a\nabla) \Box_a h^{(\ell)}(z;a) \right] = 0, \quad (3.1.1)$$

$$\mu_\ell^2 = \left( \ell^2 + \ell(d-5) - 2(d-2) \right), \quad (3.1.2)$$

*We will always try to keep general $d$ in all possible formulas admitting of course that for $AdS_4$ theory it should be set to 3 at the end.
The most important property of this equation is higher spin gauge invariance with the traceless parameter $e^{(l-1)}(z;a)$,
\begin{align}
\delta h^{(l)}(z;a) &= (a\nabla)e^{(l-1)}(z;a), \\
\Box_a e^{(l-1)}(z;a) &= 0, \\
\delta F(h^{(l)}(z;a)) &= 0.
\end{align}

The most natural gauge fixing condition for Fronsdal’s equation is the so called traceless de Donder gauge
\begin{align}
D^{(l-1)}(h^{(l)}) &= \nabla^\mu \frac{\partial}{\partial a^\mu} h^{(l)} - \frac{1}{2} (a\nabla)\Box_a h^{(l)} = 0,
\end{align}

In this gauge Fronsdal’s equation simplifies to
\begin{align}
\mathcal{F}^{dD}(h^{(l)}) &= [\Box - \mu_\epsilon^2]h^{(l)} - a^2\Box_a h^{(l)} = 0.
\end{align}

Note that any deviation in (3.1.7) of $\mu_\epsilon^2$ from (3.1.2) leads to a massive higher spin field.

We can write now Fronsdal’s gauge invariant action in the concise form
\begin{align}
S^\epsilon &= \frac{1}{2} \int \sqrt{g}d^{d+1}z \left\{ \psi^{(l)}(z;a) \ast_a \mathcal{F}(h^{(l)}(z;a)) - \frac{1}{4s(s-1)}\Box_a h^{(l)}(z;a) \ast_a \Box_a \mathcal{F}(h^{(l)}(z;a)) \right\}, (3.1.8)
\end{align}

This action is gauge invariant due to the "Bianchi" identity (2.1.37).

Next we can write our double traceless field $h^{(l)}(z;a)$ as a set of the two traceless spin $\ell$ and $\ell - 2$ fields $\psi^{(l)}(z;a)$ and $\theta^{(l-2)}(z;a)$
\begin{align}
 h^{(l)}(z;a) &= \psi^{(l)} + \frac{a^2}{2\alpha_0} \theta^{(l-2)}(z;a), \\
 \Box_a h^{(l)} &= \theta^{(l-2)}, \quad \Box_a \psi^{(l)} = \Box_a \theta^{(l-2)} = 0, \\
 \alpha_0 &= d + 2\ell - 3.
\end{align}

Applying the de Donder gauge condition we see that the fields $\psi^{(l)}(z;a)$ and $\theta^{(l-2)}(z;a)$ completely separate in the action (3.1.8)
\begin{align}
S^\epsilon &= \frac{1}{2} \int \sqrt{g}d^{d+1}z \left\{ \psi^{(l)}(z;a) \ast [\Box - \mu_\epsilon^2] \psi^{(l)}(z;a) \\
&\quad - \frac{\alpha_0 - 2}{4\alpha_0} \theta^{(l-2)}(z;a) \ast [\Box - \mu_\epsilon^2] \theta^{(l-2)}(z;a) \right\}, (3.1.12)
\end{align}

with the following diagonal field equations and de Donder gauge condition connecting $\psi^{(l)}$ and $\theta^{(l-2)}$
\begin{align}
\nabla^\mu \frac{\partial}{\partial a^\mu} \psi^{(l)} &= \frac{\alpha_0 - 2}{2\alpha_0} (a\nabla)\theta^{(l-2)} - \frac{a^2}{2\alpha_0} \nabla^\mu \frac{\partial}{\partial a^\mu} \theta^{(l-2)}, \\
(\Box + \ell) \psi^{(l)} &= \Delta_\ell (\Delta_\ell - d) \psi^{(l)}, \\
(\Box + \ell - 2) \theta^{(l-2)} &= \Delta_\theta (\Delta_\theta - d) \theta^{(l-2)}, \\
\Delta_\ell &= d + \ell - 2, \quad \Delta_\theta = d/2 + 1/2 \sqrt{(\alpha_0 - 1)(\alpha_0 + 7)}.
\end{align}
So we realize that only in the de Donder gauge we have a diagonal equation of motion for the physical $\psi^{(\ell)}$ components but this component is not transversal due to the presence of $\theta^{(\ell-2)}$. This is the most convenient gauge for the quantization and construction of bulk-to-bulk propagators and for the investigation of $AdS_4/CFT_3$ correspondence in the case of the critical conformal $O(N)$ boundary sigma model. We also mentioned that in the boundary limit only the traceless mode $\psi^{(\ell)}$ survives but the nonphysical trace mode $\theta^{(\ell-2)}$ can create a Goldstone mode and enters the bulk tree dynamics and the loops.

The negative sign of the $\theta$ part in the action (3.1.8) suggests to quantize this higher spin field with a formalism of Gupta-Bleuler type, so that a state with $n$ quanta of $\theta$ has norm squared of signature $(-1)^n$ yielding a pseudo Hilbert space. Applying de Donder’s constraint (3.1.14) on field operators, the “physical” Hilbert space is the kernel of the annihilation operator part of this constraint inside the pseudo Hilbert space. Finally zero norm states are projected out. In the context of this work it is only relevant that the two-point function of $\theta$ satisfies
\[
\langle \theta^{(\ell-2)}(z_1; a), \theta^{(\ell-2)}(z_2; c) \rangle \leq 0 \quad (3.1.18)
\]
as a distribution.

### 3.2 Propagators in de Donder’s gauge

On AdS space which is a constant curvature space the geodesic distance $\eta$ enters all invariant expressions of the relative distance of two points. The standard variable $\zeta = \cosh \eta$ can be expressed by Poincaré coordinates as
\[
\zeta(z_1, z_2) = \frac{(z_1^0)^2 + (z_2^0)^2 + (\bar{z}_1 - \bar{z}_2)^2}{2z_1^0 z_2^0} = 1 + \frac{(z_1 - z_2)^\mu (z_1 - z_2)^\nu \delta_{\mu\nu}}{2z_1^0 z_2^0}. \quad (3.2.1)
\]
The propagators are bitensorial quantities which are presented in the algebraic basis of homogeneous functions of $I_1, I_2, I_3, I_4$
\[
I_1(a, c) := (a \partial_1)(c \partial_2)\zeta(z_1, z_2), \quad (3.2.2)
\]
\[
I_2(a, c) := (a \partial_1)\zeta(z_1, z_2)(c \partial_2)\zeta(z_1, z_2), \quad (3.2.3)
\]
\[
I_3(a, c) := a_1^2 I_2^2 + c_2^2 I_1^2, \quad (3.2.4)
\]
\[
I_4 := a_1^2 c_2^2, \quad (3.2.5)
\]
\[
I_{1a} := (a \partial_1)\zeta(z_1, z_2), \quad I_{2c} := (c \partial_2)\zeta(z_1, z_2), \quad (3.2.6)
\]
\[
(a \partial_1) = a^\mu \frac{\partial}{\partial z_1^\mu}, \quad (c \partial_2) = c^\mu \frac{\partial}{\partial z_2^\mu}, \quad (3.2.7)
\]
\[
a_1^2 = g_{\mu\nu}(z_1)a^\mu a^\nu, \quad c_2^2 = g_{\mu\nu}(z_2)c^\mu c^\nu. \quad (3.2.8)
\]
of degree $\ell$, the spin of the field. All important formulas for this "advanced technology" of working with higher spin field theory in AdS space one can find...
in Appendix A. We are interested only in that part of the propagator expansion which neglects traces. So it is a map from a space of $\ell + 1$ functions $\{F_k(\zeta)\}_{k=0}^\ell$ to a space of bitensors parameterized by $I_1$ and $I_2$ only, namely

$$
\Psi^{(\ell)}[F_k] = \sum_{k=0}^\ell F_k(\zeta) I_1^{\ell-k} I_2^k, \quad (3.2.9)
$$

$$(\square + \ell) \Psi^{(\ell)}[F_k] = \Delta_\ell(\Delta_\ell - d)\Psi^{(\ell)}[F_k] + O(a_1^2; c_2^2). \quad (3.2.10)
$$

In the variable $\zeta$ the analytic properties of QFT n-point functions are conveniently described. In particular the two-point functions or propagators are analytic in the $\zeta$ plane with singularities at $\zeta = \pm 1$ and at $\zeta = \infty$, which in most cases are logarithmic branch points. Analyticity is therefore meant in general on infinite covering planes. All AdS field theories are symmetric under the exchange $\zeta \rightarrow -\zeta$.

Another variable used often is $u = \zeta - 1$, the “chordal distance”, more precisely one half the square of the chordal distance. The series expansions for two-point functions in $u$ converge in a radius 2, whereas the series expansions in powers of $\zeta^{-1}$ converge for $|\zeta| > 1$. These analytic properties remind us of Legendre functions. Indeed if propagator functions can be identified as Gaussian hypergeometric functions, these are Legendre functions and the ”quadratic transformations” can be applied. Using formulas from Appendix A we can show that in de Donder’s gauge the propagator satisfy the following set of differential equations for the functions $F_k(\zeta)$ or correspondingly $\Phi_k(u)$ of (3.2.9) following from equation (3.1.15)

$$(\zeta^2 - 1)F''_k + (d + 1 + 4k)\zeta F'_k + X_k F_k + 2\zeta(k + 1)^2F_{k+1} + 2(\ell - k + 1)F'_{k-1} = 0, \quad (3.2.11)$$

$$X_k = k(d + 2\ell - k) + 2\ell - (\ell - 2)(\ell + d - 2). \quad (3.2.12)$$

The ”dimension” of the higher spin field $\Delta_\ell = \ell + d - 2$ has been inserted. Moreover we use $F_{-1} = F_{\ell+1} = 0$. The dimension of the AdS space is $d + 1$, we interpolate analytically in $d$ if this is technically required. Our issue is to solve these equations by expansion in powers of $\zeta^{-1}$ or $u$. This leads to matrix recursion equations which necessitate some linear algebra operations.

As an ansatz for the series expansion of $F_k(\zeta)$ at $\zeta = \infty$ we use

$$F_k(\zeta) = \zeta^{-\alpha-k} \sum_{n=0}^{\infty} c_{kn} \zeta^{-2n}. \quad (3.2.13)$$

Denote $\xi = \alpha + 2n$. Then a two term recursion of the form

$$D_n \left( \begin{array}{c} c_{0m} \\ c_{1m} \\ \vdots \\ c_{\ell,n} \end{array} \right) = C_{n-1} \left( \begin{array}{c} c_{0,n-1} \\ c_{1,n-1} \\ \vdots \\ c_{\ell,n-1} \end{array} \right), \quad (3.2.14)$$
results with the two matrices
\[ C_{n-1} = \text{diag}\{ (\xi - 1)(\xi - 2), \xi(\xi - 1), \ldots (\xi + \ell - 1)(\xi + \ell - 2) \}. \quad (3.2.15) \]
and the entries of the matrix \( D_n \)
\[ (D_n)_{k,k-1} = -2(\ell - k + 1)(\xi + k - 1), \quad (3.2.16) \]
\[ (D_n)_{k,k} = \xi^2 - \xi(d + 2k) - 4k^2 + 2\ell(k + 1) - (\ell - 2)(\ell + d - 2), \quad (3.2.17) \]
\[ (D_n)_{k,k+1} = 2(k + 1)^2. \quad (3.2.18) \]
The determinant of \( D_0 \) is a polynomial of degree \( 2(\ell + 1) \) of the variable \( \alpha \) with roots which we identify with the "roots" of the differential equation system. For arbitrary \( \ell \) we have
\[ \det D_0 = [(\alpha + \ell - 2)(\alpha + 2 - \ell - d)][(\alpha + \ell - 2)(\alpha - \ell - d)] \times \prod_{n=0}^{\ell-2} [\alpha^2 - (d + 4 + 2n)\alpha - ((\ell - 2)d + (\ell + n)^2 - (n + 2)(3n + 4))]. \quad (3.2.19) \]
Each square bracket represents one eigenvalue of \( D_0 \) and contributes two roots. The quadratic factors lead in almost all cases to two irrational roots that are neither degenerate among themselves nor with the other roots, but there are exceptions which have two integer roots e.g. for \( d = 3 : (\ell, n) \in \{(4,1), (6,4), (9,2), (9,5), (11,8), (15,8)\ldots\} \). Two roots are said to be degenerate, if their difference is an integer. For the case of expansions in powers of \( \zeta^2 \) as in (3.2.13), this integer must be even. In such case the solution with the bigger root enters the other one with a \( \log \zeta \) factor.

The following roots are of particular (physical) importance
\[ \alpha_p = \ell + d - 2, \quad (3.2.20) \]
\[ \alpha_c = \ell + d. \quad (3.2.21) \]
We call the first root \( \alpha_p \) "principal" because it has the value of the dimension \( \Delta \) of the field which enters the field equation in the form \( \Delta(\Delta - d) \). The second root is a "companion" of it, since they appear for all \( \ell \) as such pair (see (3.2.19)). It is degenerate with the principal root and the solution of it enters the principal solution with a \( \log \zeta \) factor on the next to leading power in the expansion. The bigger ones of the two roots in the exceptional cases quoted above are also bigger than the principal root \( \ell + 1 \) (for the same \( \ell \)) but their distance to it are odd numbers except for the case \( (\ell, n) = (15,8) \), where the distance to \( \ell + 1 \) is sixteen and the \( \log \zeta \) term appears at a very high power.

For the principal root the equation for the eigenvector of \( D_0 \)
\[ D_0(\alpha_p) \begin{pmatrix} c_{00}^{(\alpha_p)} \\ c_{10}^{(\alpha_p)} \\ \vdots \\ c_{l0}^{(\alpha_p)} \end{pmatrix} = 0, \quad (3.2.22) \]
can be solved for each \( \ell \). We find

\[
  c_{k,0}^{(\alpha_p)} = (-1)^k \binom{\ell}{k},
\]

which is easy to prove by using the general expression for the rows of the matrix \( D_n \) as given in (3.2.16) - (3.2.18). The consequence of this result is that the leading term of \( \Psi^{(\ell)}[F_k(\alpha_p)] \) at \( \zeta = \infty \) is the well known expression \( \zeta^{-\Delta}(I_1 - \zeta^{-1}I_2)^\ell \). Already at next order in \( \zeta^{-2} \) log-terms appear.

For the companion root \( \alpha_c \) the eigenvector for \( D_0 \) can be derived by a little bit more algebra for any \( \ell \)

\[
  c_{k,0}^{(\alpha_c)} = (-1)^k \left( \binom{\ell}{k} + (d + 2\ell - 2) \binom{\ell - 1}{k - 1} \right).
\]

The actual construction of a solution for the pair of roots starts with the bigger one, \( \alpha_c \). Its solution takes the form

\[
  F_k(\zeta; \alpha_c) = \zeta^{-\Delta-2} \sum_{n=0}^{\infty} \zeta^{-2n} \sum_{s=0}^{\ell} \Pi_n(\alpha_c)_{k,s} c_{s,0}^{(\alpha_c)},
\]

where we used

\[
  H_n(\alpha_c) = D_n(\alpha_c)^{-1}C_{n-1}(\alpha_c) = H_1(\alpha_c + 2(n - 1)), \quad (3.2.26)
\]

\[
  \Pi_n(\alpha_c) = \Pi^{n-1}_{r=0} H_1(\alpha_c + 2r). \quad (3.2.27)
\]

and the left arrow denotes ordering of the product with increasing \( r \) from right to left. In this context we note that if a nonsingular matrix \( S(\alpha) \) would exist, so that \( H_1 \) could be diagonalized by

\[
  H_1(\alpha) = S^{-1}(\alpha + 2)\Delta(\alpha)S(\alpha), \quad (3.2.28)
\]

then \( F_k(\zeta; \alpha) \) would be a generalized hypergeometric function.

Having constructed the solution for the companion root we turn to the principal root. We recognize that \( D_n(\alpha_p) \) can be spectrally decomposed in the following fashion

\[
  D_n \chi_i = \lambda_i \chi_i, \quad (3.2.29)
\]

\[
  D_n^T \psi_i = \lambda_i \psi_i, \quad (3.2.30)
\]

\[
  D_n = \sum_{i=0}^{l} \lambda_i \chi_i \otimes \psi_i^T, \quad (3.2.31)
\]

\[
  \psi_i^T \chi_j = \delta_{ij} \quad (3.2.32)
\]
Denote further
\[ \rho^T = \psi^T C_{n-1}. \]  
(3.2.33)

All these quantities can be determined as functions of \( \xi \), and it is easily verified that (3.2.28) is not fulfilled.

One of the eigenvalues of \( D_1(\alpha_p) \) vanishes, we denote it \( \lambda_0 \), so that \( D_1(\alpha_p) \) cannot be inverted. We perform a deformation of our differential equation system replacing \( \alpha_p \) only in \( \lambda_0 \) and in the prefactor \( \zeta^{-\alpha_p} \) by \( \alpha_p + \epsilon \). All other eigenvalues and the eigenvectors remain unchanged. Then we continue the whole procedure known from the companion root, all \( H_n \) will remain singularity free. At the end we subtract a certain multiple \( \gamma \) of \( (\epsilon^{-1} + \mu)\Psi^{(\ell)}[F_k(\alpha_c)] \) so that the limit \( \epsilon \to 0 \) can be performed and the log-terms appearing are \( -\gamma \log \zeta \Psi^{(\ell)}[F_k(\alpha_c)] \).

The additional parameter \( \mu \) is in principle arbitrary showing that the principal solution containing a log factor is a coset with respect to adding the companion solution. This parameter can, however, be normalized in a standard fashion by requiring that the \((l+1)\)-tupel of coefficients \( c_{k,n}^{(\alpha_p)} \) where at level \( n \) the log term appears first, is orthogonal to the eigenvector \( \psi_0 \) belonging to the deformed eigenvalue. We close this discussion with the remark that on the boundary of AdS space i.e. \( \zeta = \infty \) any linear combination
\[ \Psi^{(\ell)}[F_k(\alpha_p)] + A\Psi^{(\ell)}[F_k(\alpha_c)] \]  
(3.2.34)
is indistinguishable from the pure principal solution. Thus the boundary constraint fixes only the whole coset and not any representative of it.

In order to render the expansions of \( F_k \) around \( \zeta = 1(u = 0) \) a visually different expression, we shall denote them \( \Phi_k \). The expansions are
\[ \Phi_k(u) = u^\alpha \sum_{n=0}^{\infty} a_{k,n} u^n. \]  
(3.2.35)

Again we obtain matrix recursion relations
\[
A_n \begin{pmatrix} a_{0,n} \\ a_{1,n} \\ \vdots \\ a_{\ell,n} \end{pmatrix} + B_{n-1} \begin{pmatrix} a_{0,n-1} \\ a_{1,n-1} \\ \vdots \\ a_{\ell,n-1} \end{pmatrix} + E \begin{pmatrix} a_{0,n-2} \\ a_{1,n-2} \\ \vdots \\ a_{\ell,n-2} \end{pmatrix} = 0. \]  
(3.2.36)

We define
\[ \xi = \alpha + n, \]  
(3.2.37)

and obtain the matrices
\[
(A_n)_{k,k} = \xi(2\xi + d + 4k - 1), \]  
(3.2.38)
\[
(A_n)_{k,k-1} = 2\xi(\ell - k + 1), \]  
(3.2.39)
\[
(B_{n-1})_{k,k} = (\xi - 1)(\xi + d + 4k - 1) + X_k, \]  
(3.2.40)
\[
(B_{n-1})_{k,k+1} = 2(k + 1)^2 = (E)_{k,k+1}. \]  
(3.2.41)
Here we used the shorthand (see (3.2.12))

\[ X_k(\lambda) = k(2\lambda + 2\ell - k + 1) + 2\ell - (\ell - 2)(2\lambda + \ell - 1), \]

(3.2.42)

and \( d = 2\lambda + 1 \) has been introduced. Therefore \( A_n \) is of lower triangular shape with eigenvalues \( \xi(2\xi + d + 4k - 1) \). The root system is therefore

- \( \ell + 1 \) times the root zero;
- the \( \ell + 1 \) roots \( \alpha_m = -\lambda - 2m, \quad 0 \leq m \leq \ell \).

Both sets are degenerate among themselves, and if \( d \) is odd, the second set is degenerate with respect to the first one. The first set produces regular solutions, the second set produces poles if \( d \) is odd, which it is in the case of present interest. Nevertheless we will regard \( d \) as a free real parameter in order to handle the degeneracy with the regular cases. The solution for \( \alpha_0 \) in combination with any regular solution has the appropriate singular behaviour at \( u = 0 \) needed for a propagator, namely applying Fronsdal’s differential operator the correct delta function is created.

Any solution is obtained by requiring

\[ A_0 \begin{pmatrix} a_{0,0} \\ a_{1,0} \\ \vdots \\ a_{\ell,0} \end{pmatrix} = 0. \]

(3.2.43)

This requirement is solved for the regular solutions \( \Phi_k^{(r)}(u) \) (for which \( A_0 = 0 \) and the solution is trivial) by

\[ a_k^{(r)} = \delta_{k,r}. \]

(3.2.44)

For any such solution \( r \) we obtain next

\[ a_{k,1}^{(r)} = -(A_1^{-1} B_0)_{k,r} \]

\[ = -(A_1^{-1})_{k,r} (B_0)_{rr} - (A_1^{-1})_{k,r-1} (B_0)_{r-1,r}, \]

(3.2.45)

where we insert

\[ (A_1)_{r,r} = d + 4r + 1, \]

(3.2.46)

\[ (A_1)_{r,r-1} = 2(\ell - r + 1), \]

(3.2.47)

\[ (B_0)_{r,r} = X_r, \]

(3.2.48)

\[ (B_0)_{r-1,r} = 2r^2, \]

(3.2.49)
and obtain

\[(A^{-1}_1)_{k,r} = (-2)^{k-r} \prod_{s=r+1}^{k} (\ell - s + 1) \left[ \prod_{s=r}^{k} (d + 4s + 1) \right]^{-1}
\text{ (for } k > r), \quad (3.2.50)\]

\[(A^{-1}_1)_{r,r} = (d + 4r + 1)^{-1}, \quad (3.2.51)\]

\[(A^{-1}_1)_{k,r} = 0 \quad (\text{for } k < r). \quad (3.2.52)\]

There is no sign of any singularity caused by the degeneracy. Finally we get

\[a^{(r)}_{k,1} = -X_r (A^{-1}_1)_{k,r} - 2r^2 (A^{-1}_1)_{k,r-1}, \quad (3.2.53)\]

which vanishes for \( r > k + 1 \).

We turn now to the nonanalytic solutions \( \Phi_k(u, \alpha_m) \) with roots \( \alpha_m = -\lambda - 2m \) and concentrate on the case \( m = 0 \) because this is the perturbative Green function for the Fronsdal differential operator. At the beginning we assume \( \lambda \notin \mathbb{Z} \) in order to avoid the degeneracy with the regular solutions. In this case we have

\[(A_0)_{k,k} = -4\lambda k, \quad (3.2.54)\]

\[(A_0)_{k,k-1} = -2\lambda (\ell - k + 1), \quad (3.2.55)\]

and the equation

\[\sum_r (A_0)_{k,r} c^{(\alpha_0)}_{r,0} = 0 \quad (3.2.56)\]

is solved by

\[c^{(\alpha_0)}_{k,0} = \left( -\frac{1}{2} \right)^k \binom{\ell}{k}. \quad (3.2.57)\]

Next we treat the \( A_1 \) matrix

\[(A_1)_{k,k} = 2(1 - \lambda)N_k, \quad N_k = 2k + 1, \quad (3.2.58)\]

\[(A_1)_{k,k-1} = 2(1 - \lambda)(\ell - k + 1), \quad (3.2.59)\]

\[(A^{-1}_1)_{k,r} = [2(1 - \lambda)]^{-1} \beta_{k,r}, \text{ for } k \geq r \text{ and zero else,} \quad (3.2.60)\]

\[\beta_{k,r} = (-\ell)_{k-r} \prod_{s=r}^{k} N_s]^{-1}. \quad (3.2.61)\]

The \( B_0 \) matrix is

\[(B_0)_{k,k} = -\lambda(\lambda + 4k + 1) + X_k := Z_k(\lambda), \quad (3.2.62)\]

\[(B_0)_{k,k+1} = 2(k + 1)^2. \quad (3.2.63)\]

The matrix \( E \) is still not needed for \( n = 1 \).
We define the matrix
\[(H_1)_{k,r} = -(A_1^{-1}B_0)_{k,r} = [2(\lambda - 1)]^{-1}\{\beta_{k,r}(B_0)_{r,r} + \beta_{k,r-1}(B_0)_{r-1,r}\}, \tag{3.2.64}\]
and obtain for the coefficients \(c^{(\alpha_0)}_{k,1}\)
\[c^{(\alpha_0)}_{k,1} = \sum_{r=0}^{k+1} (H_1)_{k,r} \left(-\frac{1}{2}\right)^r \binom{\ell}{r}. \tag{3.2.65}\]
All these coefficients inherit a pole in \(\lambda\) at the value 1.

This pole does not appear in one eigenvalue only as in the \(\zeta = \infty\) case. This is due to the fact that for \(\lambda = 1\) there exist \(\ell + 1\) degenerate regular solutions and therefore the pole appears in all \(\ell + 1\) eigenvalues simultaneously. It is straightforward to calculate the residues of all matrix elements of \(H_1\) and to derive the expressions
\[\rho_k = \sum_{r=0}^{k+1} \text{res}(H_1)_{k,r} \left(-\frac{1}{2}\right)^r \binom{\ell}{r}. \tag{3.2.66}\]
Then we subtract from this solution at \(n = 1\) the regular solution
\[(\lambda - 1)^{-1}\left[\sum_{r=0}^{\ell} \rho_r \Phi^{(r)}(u)\right], \tag{3.2.67}\]
obtaining in the limit the log term of \(\Psi^{(\ell)}[\Phi_k(u, \alpha_0)]\)
\[-\log u \left[\sum_{r=0}^{\ell} \rho_r \Phi^{(r)}(u)\right]. \tag{3.2.68}\]
We mention that the leading term of \(\Psi^{(\ell)}[\Phi_k(u, \alpha_0)]\) is
\[u^{-1}(I_1 - \frac{1}{2}I_2)^\ell. \tag{3.2.69}\]

The situation with the Green function type solution is the same as with the solution which is constrained by the AdS boundary condition: The UV constraint is satisfied by a coset, namely any linear combination of regular solutions can be added to the solution \(\Psi^{(\ell)}[\Phi(\alpha_0)]\). In turn this may also be used to normalize the solutions \(\Phi_k(\alpha_0)\). We can namely require that on the level \(n = 1\) on which \(\log u\) appears first, all the coefficients \(c^{(\alpha_0)}_{k,1}\) are made to vanish by appropriate subtraction of regular solutions.
CHAPTER 3. PROPAGATOR OF HSF AND ONE-LOOP DIAGRAM

3.3 Propagators in Feynman’s gauge

In this section we consider the higher spin gauge field propagators analyzed in the previous section and in [75], [76], [77], in an approach developed originally for the spin $\ell = 0, 1, 2$ only in [78], [79], [80], but now generalized for all $\ell$ with a slight modification of arguments. Namely we consider our propagator working directly in the space of conserved currents

$$h^{(\ell)}(z_1; a) = \int \sqrt{g} d^{d+1}z_2 K^{(\ell)}(z_1, a; z_2, c) * c J^{(\ell)}(z_2, c), \quad (3.3.1)$$

where

$$K^{(\ell)}(z_1, a; z_2, c) = \Psi^{(\ell)}[F_k(u(z_1; z_2))] + \text{traces}. \quad (3.3.2)$$

Taking into account the conservation properties of the current $J^{(\ell)}(z_2, c)$ we can formulate the ansatz following from (3.1.15)

$$[\Box + \ell - \Delta_{\ell}(\Delta_{\ell} - d)]\Psi^{(\ell)}[F_k(u(z_1; z_2))] = -I_1^{\ell}\delta_{d+1}(z_1; z_2) + \text{traces} + \left(c\nabla_2 \left(I_{1a}\Psi^{(\ell-1)}[\Lambda_k(u(z_1; z_2))]\right)\right). \quad (3.3.3)$$

This means that applying the gauge fixed equation of motion at the first argument of the bilocal propagator we get zero (or more precisely a delta function in the coincident points) due to a gauge transformation at the second argument.

Here we should make some comments on the delta function in curved $AdS$ space. Our notation in (3.3.3) means

$$\delta_{(d+1)}(z_1; z_2) = \frac{\delta_{(d+1)}(z_1 - z_2)}{\sqrt{g}(z)}, \quad \int \delta_{(d+1)}(z_1 - z_2) d^{d+1}z_1 = 1. \quad (3.3.4)$$

In the polar coordinate system defined in Appendix A the invariant measure (for $d = 3$) is

$$\sqrt{g} d^4z = u(u + 2)dud\Omega_3. \quad (3.3.5)$$

Therefore we can define

$$\frac{\delta_{(4)}(z - z_{pole})}{\sqrt{g}(z)} = \frac{\delta(u)}{u(u + 2)\Omega_3} = -\frac{\delta_{(1)}(u)}{(u + 2)\Omega_3}, \quad (3.3.6)$$

$$u\delta_{(1)}(u) = -\delta(u).$$

This $u$-dependence of the measure leads to the idea that short distance singularities in $D = d + 1 = 4$ dimensional $AdS$ space should start from $\frac{1}{u^2}$ not from $\frac{1}{u}$.

Then using the gradient map (A.34), (A.35) we can derive

$$(c\nabla_2 \left(I_{1a}\Psi^{(\ell-1)}[\Lambda_k(u)]\right) = \Psi^{(\ell)}[\Lambda'_{k-1}(u) + (k + 1)\Lambda_k(u)], \quad \Lambda_\ell = 0 \quad (3.3.7)$$
Combining this with the Laplacian map \( (A.31)-(A.33) \) and \( (3.3.1) \) we obtain the following set of \( \ell + 1 \) equations for \( z_1 \neq z_2 \) (unlike the case \( (3.2.11) \) we do not insert the value of \( \Delta_\ell \) here)

\[
u(u + 2)F''_k + (d + 1 + 4k)(u + 1)F'_k + 2(\ell - k + 1)F'_{k-1} + 2(u + 1)(k + 1)^2F_{k+1} \\
+ [2\ell + k(d + 2\ell - k)]F_k - \Delta_\ell(\Delta_\ell - d)F_k = \Lambda'_{k-1} + (k + 1)\Lambda_k.
\]

To analyze this system we write the \( k = 0, 1 \) and \( \ell - 1, \ell \) cases explicitly

\[
u(u + 2)F''_0 + (d + 1)(u + 1)F'_0 - \Delta_\ell(\Delta_\ell - d)F_0 + 2(u + 1)F_1 \\
+ 2\ell F_0 = \Lambda_0,
\]

\[O(F''_0, F'_1, F_1, F_2) + 2\ell F'_0 = \Lambda'_0 + 2\Lambda_1,
\]

\[
\vdots
\]

\[
u(u + 2)F''_{\ell-1} + (d + 1 + 4\ell)(u + 1)F'_{\ell-1} + [\ell^2 + \ell(d + 2\ell - \Delta_\ell(\Delta_\ell - d))]F_{\ell} \\
+ 2F'_{\ell-1} = \Lambda'_{\ell-1},
\]

and we see that this system for \( 2\ell + 1 \) functions is separable. One solution is obtained if we put

\[
F_k = 0, \quad k = 1, 2, \ldots, \ell,
\]

\[
\Lambda_k = 0, \quad k = 1, 2, \ldots, \ell - 1,
\]

and submit \( F_0(u) \) to the Gaussian hypergeometric equation

\[
u(u + 2)F''_0(u) + (d + 1)(u + 1)F'_0(u) - \Delta_\ell(\Delta_\ell - d)F_0(u) = 0,
\]

supplemented with a noncontradictory solution for the remaining gauge parameter \( \Lambda_0(u) \)

\[
\Lambda_0(u) = 2\ell F_0(u).
\]

So we prove that with an appropriate choice of the gauge freedom we can obtain the propagator in Feynman’s gauge in the form

\[
K^{(0)}(z_1, a; z_2, c) = I'_1F_0(u) + \text{traces},
\]

where \( F_0(u) \) is the solution of the equation for the scalar field with dimension \( \Delta_\ell \) \( (3.3.15) \) \[78\]. The solution of this equation is well known and can be written in two different forms \[80, 81\]. The first form is \( (\zeta = u + 1) \)

\[
F_0(\zeta) = C(\ell, d)2^{\Delta_\ell}\zeta^{-\Delta_\ell}F_1\left(\frac{\Delta_\ell}{2}, \frac{\Delta_\ell + 1}{2}, \Delta_\ell - \frac{d}{2} + 1; \frac{1}{\zeta^2}\right).
\]

This form is suitable for an investigation of the infrared behaviour. We see immediately that near the boundary limit we have

\[
F_0(\zeta) \sim \zeta^{-\Delta_\ell}|_{d=3} = \zeta^{-(\ell+1)}, \quad \text{if} \quad \zeta \to \infty,
\]
which is just wanted for AdS/CFT correspondence. Indeed comparing \( \Delta_\ell \) and \( \Delta_\theta \) in (3.1.15)-(3.1.17) we see that the propagator of the nonphysical mode \( \theta \) falls off in the boundary limit faster than the propagator for the physical mode \( \psi \), as it should be.

But for us the second form of this expression obtained after a quadratic transformation of the hypergeometric function listed in the Appendix B (B.1) is more interesting

\[
F_0(u) = C(\ell, d) \left( \frac{2}{u} \right)^{\Delta_\ell} \binom{\Delta_\ell}{\Delta_\ell - \frac{d}{2} + \frac{1}{2}, 2\Delta_\ell - d + 1; -\frac{2}{u}}. \tag{3.3.20}
\]

The normalization constant \( C(\ell, d) \) is chosen to obtain the \( \delta \) function on the right hand side of (3.3.3)

\[
C(\ell, d) = \frac{\Gamma(\Delta_\ell)\Gamma(\Delta_\ell - \frac{d}{2} + \frac{1}{2})}{(4\pi)^{(d+1)/2}\Gamma(2\Delta_\ell - d + 1)} \bigg|_{d=3} = \frac{\ell!(\ell - 1)!}{16\pi^2(2\ell - 1)!}. \tag{3.3.21}
\]

To investigate the ultraviolet limit of (3.3.20) we have to use the second formula (B.2) of Appendix B and take carefully the limit \( d \to 3 \) to obtain

\[
\left( \frac{2}{u} \right)^{\Delta_\ell} \binom{\Delta_\ell, \Delta_\ell - \frac{d}{2} + \frac{1}{2}, 2\Delta_\ell - d + 1; -\frac{2}{u}} \bigg|_{d=3} = \frac{(2\ell - 1)!}{(\ell - 1)!} \left\{ \frac{2}{\ell!u} \right. \\
+ \frac{1}{(\ell - 2)!} \sum_{n=0}^{\ell - 2} \frac{(\ell + 1)_n(2 - \ell)_n}{n!(n + 1)!} \left[ \Upsilon_{\ell, n} + \log \left( \frac{u}{2} \right) \left( -\frac{u}{2} \right)^n \right]\left\}, \tag{3.3.22}
\]

where the rational number \( \Upsilon_{\ell, n} \) is expressed by the \( \psi \) functions

\[
\Upsilon_{\ell, n} = \psi(\ell + n + 1) + \psi(\ell - n - 1) - \psi(n + 1) - \psi(n + 2). \tag{3.3.23}
\]

So we see now that in the ultraviolet limit we get

\[
F_0(u) \bigg|_{d=3} \approx -\frac{1}{8\pi^2 u} + O(1, u, \log u, u \log u, \ldots). \tag{3.3.24}
\]

This main singular term in the propagator of the scalar field with dimension \( \Delta_\ell \) does not depend on the field dimension and behaves always like \( \frac{1}{8\pi^2 u} \). For example we have the same singularity in the propagator of the conformally coupled scalar in \( AdS_4 \) (see [82])

\[
\Sigma[u(z_1, z_2)] = \frac{1}{8\pi^2} \left( \frac{1}{u} \pm \frac{1}{u + 2} \right), \tag{3.3.25}
\]

\[
(\Box + 2)\Sigma[u(z_1, z_2)] = -\delta(4)(z_1; z_2). \tag{3.3.26}
\]

So we observe some universality in the UV behaviour of higher spin propagators in Feynman’s gauge:

\textit{For any spin} \( \ell \) \textit{the main term of the propagator has the form} \( \frac{1}{8\pi^2 u} \).

Comparing with (3.2.69) we deduce that in de Donder gauge we have the same picture because
• $I_1(a, c; u) \to a^\mu c_\nu$ if $u \to 0$.
• $I_2(a, c; u) = I_3(a, c; u) \to 0$ if $u \to 0$.
• $I_4(a, c; u) \to a^2 c^2$ if $u \to 0$.

So finally we can formulate the following statement:

*The higher spin propagator in Feynman’s gauge is simplest and most convenient for the calculation of any Feynman diagram. Just we have to couple it with conserved currents to make sure that we preserve gauge invariance. The UV-behaviour of the propagator is universal and described by (3.3.24).*

### 3.4 Spin $\ell$, $\ell - 2$ and scalar interaction and mass renormalization

In this section we will discuss some interaction between two neighboring higher spin gauge fields and a scalar containing two derivatives. On this linearized level of understanding the higher spin gauge invariance it is possible to construct an interaction of the gauge field contracted with the conserved current formed from gauge fields of the nearest different spin ($\ell \pm 2$), conformally coupled in the $AdS_{d+1}$ background with the scalar $\sigma(z)$

$$\Box \sigma(z) + \frac{d^2 - 1}{4}\sigma(z) = 0,$$

and two derivatives

$$S_{int} = \frac{g_\ell}{\sqrt{N}} \int \sqrt{g} d^4 z h^{(\ell)}(z; a) J^{(\ell)}[h^{(\ell \pm 2)}(z; a), \sigma(z)].$$

Here we introduce an unknown coupling parameter $g_\ell$ normalized as $O(\frac{1}{\sqrt{N}})$ as it follows from $AdS_4/CFT_3$ correspondence. The conservation condition following from the gauge transformation for $h^{(\ell)}(z)$ (3.1.3) with the traceless parameter $\epsilon^{(\ell-1)}$ is

$$\nabla^\mu \frac{\partial}{\partial a_\mu} J^{(\ell)}[h^{(\ell \pm 2)}(z; a), \sigma(z)] = O(a^2).$$

This equation could be used to construct all possible currents with properties mentioned above.

Returning to the equation (3.4.3) we note first that the operator

$$\mathcal{D}^{(1)} = (a \nabla) - \frac{1}{2} a^2 \nabla^\mu \frac{\partial}{\partial a^\mu}$$

is dual to the de Donder gauge operator

$$\mathcal{D}^{(-1)} = \nabla^\mu \frac{\partial}{\partial a^\mu} - \frac{1}{2} (a \nabla) \Box_a,$$
with respect to the full scalar product, and second that this operator commutes with the divergence in the following way (see (A.37) and (A.38))

\[ \nabla^\mu \partial_{a\mu} \vec{D}^{(1)} j^{(\ell-1)}(z, a) = [\Box - (\ell - 1)(d + \ell - 2)] j^{(\ell-1)}(z, a) + O(a^2). \] (3.4.6)

Then taking into account (A.39) we can see that with the natural choice \( j^{(\ell-1)}(z, a) = (a \nabla)[h^{(\ell-2)}(z; a)\sigma(z)] \) one can obtain \( (\mu_{\ell-2}^2 \) is the AdS mass of the trace part of \( h^{(\ell)} \) (3.1.13))

\[ \nabla^\mu \partial_{a\mu} \vec{D}^{(1)}(a \nabla)[h^{(\ell-2)}(z; a)\sigma(z)] = (a \nabla)[\Box - \mu_{\ell-2}^2] (h^{(\ell-2)}(z; a)\sigma(z)) + O(a^2). \] (3.4.7)

This can be integrated easily and we restore conserved current from \( \square \) in the following form

\[ J^{(\ell)}[h^{(\ell-2)}, \sigma] = \vec{D}^{(1)}(a \nabla)[h^{(\ell-2)}(z; a)\sigma(z)] - \frac{a^2}{2} [\Box - \mu_{\ell-2}^2] (h^{(\ell-2)}(z; a)\sigma(z)) + O(a^4). \] (3.4.8)

Note that all \( O(a^4) \) terms are unimportant due to the double tracelessness of \( h^{(\ell)}(z; a) \). At this point we will apply for simplicity de Donder’s gauge condition to all types of gauge fields. Then using free equations of motion only for the fields \( h^{(\ell-2)} \) and \( \sigma \) that form the conserved current, and neglecting the first part due to de Donder’s gauge condition for the gauge field \( h^{(\ell)} \), we obtain the following effective current

\[ J^{(\ell)}[h^{(\ell-2)}, \sigma] = -\frac{a^2}{2} \left[ 2 \nabla^\mu (\nabla_\mu h^{(\ell-2)} \sigma) + \left( \frac{1 - d^2}{4} - \mu_{\ell-2}^2 - \mu_{\ell-2}^2 \right) h^{(\ell-2)} \sigma \right]. \] (3.4.9)

Note that this interaction vanishes if we require a free equation of motion for the field \( h^{(\ell)} \) coupled to the conserved current.

The next step of our consideration is the construction of the conserved current \( J^{(\ell)}[h^{(\ell+2)}, \sigma] \) which is dual to the former one, where the gauge field inside the current has a spin higher than the gauge field coupled with the current. Exploring in a similar way the conservation condition (3.4.3) and using divergence instead of gradient on stage (3.3.7) and formula (A.36) instead of (A.39) we obtain the following solution

\[ J^{(\ell)}[h^{(\ell+2)}, \sigma] = \vec{D}^{(1)}(\nabla \partial_a) [\theta^{(\ell)}(z; a)\sigma(z)] - [\Box - \mu_{\ell}^2] (\theta^{(\ell)}(z; a)\sigma(z)), \] (3.4.10)

where

\[ \theta^{(\ell)}(z; a) = \Box_a h^{(\ell+2)}(z; a). \] (3.4.11)

Inserting in (3.4.10) \( \ell - 2 \) instead of \( \ell \) and using the equation of motion for the fields inside the current and de Donder’ gauge condition for the external gauge
field, we obtain the effective current

\[ \mathcal{J}^{(\ell-2)}[h^{(\ell)}], \sigma] = -2 \nabla^\mu (\nabla_\mu \theta^{(\ell-2)} \sigma) + \left( \frac{1 - d^2}{4} - \mu^2_{(\ell-2)} - \mu_{\sigma(\ell-2)}^2 \right) \theta^{(\ell-2)} \sigma. \]

(3.4.12)

Comparing with (3.4.9) we see that in both cases we have the same effective interaction between the physical mode \( \psi^{(\ell-2)} \), the trace mode \( \theta^{(\ell-2)} \) and the scalar, and the conserved current (3.4.12) up to an overall normalization is dual to the conserved current (3.4.9) due to the equations of motion for the fields forming the currents in each cases. Thus we prove that one can use Feynman’s gauge for propagators coupled to these two currents and can turn now to investigate some loop diagram for a study of mass renormalization or quantum mass generation phenomena. Actually we considered all interactions of two neighbouring higher spin fields with a Higgs scalar that are minimal with respect to the number of derivatives, and which can generate mass in a loop. Though from (3.4.6) follows that we can introduce in principle many other \( j^{(\ell-1)}(z; a) \), all of them will contain more derivatives in front of the quantized fields and will generate counterterms that are not suitable for finite mass renormalization.

So we see that only one reasonable one loop diagram can be constructed from the interactions considered in this section. It is a loop formed by the scalar \( \sigma \) and the nonphysical trace mode \( \theta^{(\ell)} \) and with physical but off-shell external lines \( \psi^{(\ell)} \). Actually we would like to calculate the following quadratic part of the effective action

\[ \frac{g^2}{N} \int \sqrt{g} \, d^4 z_1 \int \sqrt{g} \, d^4 z_2 \mathcal{F}^{(\ell)}(\psi^{(\ell)}(z_1; a)) \ast_{a} \mathcal{F}^{(\ell)}(\psi^{(\ell)}(z_2; c)) \Theta^{(\ell)}[u(z_1, z_2); a, c] \ast_{c} \Theta^{(\ell)}[u(z_1, z_2); a, c] \mathcal{F}(\psi^{(\ell)}(z_2; c)). \]

(3.4.13)

where \( \mathcal{F}^{(\ell)}[h^{(\ell+2)}, \sigma; z_2; c] \) is presented in (3.4.10). Performing a partial integration and taking into account tracelessness of \( \theta^{(\ell)} \) we get the following expression

\[ \frac{g^2}{N} \int \sqrt{g} \, d^4 z_1 \int \sqrt{g} \, d^4 z_2 \mathcal{F}(\psi^{(\ell)}(z_1; a)) \ast_{a} \Sigma[u(z_1, z_2)] \Theta^{(\ell)}[u(z_1, z_2); a, c] \ast_{c} \mathcal{F}(\psi^{(\ell)}(z_2; c)). \]

(3.4.14)

Here \( \mathcal{F}(\psi^{(\ell)}) \) is the traceless part of Fronsdal’s operator, \( \Sigma[u] \) is the scalar propagator (3.3.25) and

\[ \Theta^{(\ell)}[u(z_1, z_2); a, c] = \langle \theta^{(\ell)}(z_1; a), \theta^{(\ell)}(z_2; c) \rangle \]

(3.4.15)

is a trace part of the \( h^{(\ell+2)} \) propagator. We want to understand the singular part of this loop.

¿From now on we follow a technique developed in [82] and [83] where the trace anomaly of the scalar mode in external higher spin field was successfully calculated from a one loop diagram. First we can use an AdS transformation to fix the point \( z_1 \) as a pole for the coordinate system \( z_2 \). Then the integration measure can be expressed through the chordal distance \( u \) as it is explained in
the Appendix A. The singularity of the product of the scalar and the higher spin propagators is relevant if it is at least $1/u^2$ because one power of $u$ is compensated by the integration measure (see (A.11) and explanation hereafter). Then from the relative coefficient between the $\psi$ and $\theta$ modes in (3.1.12) evaluated for spin $\ell + 2$ and $d = 3$, from (3.3.24) and an additional sign from the indefinite metric (3.1.18) we deduce

$$
\langle \psi^{(\ell)}(z_1; a), \theta^{(\ell)}(z_2; c) \rangle = -\frac{4(\ell + 2)}{\ell + 1} \frac{1}{8\pi^2 u} + O(u, \log u, \ldots),
$$

(3.4.16)

Multiplying hereafter with the scalar propagator we get the unique singular term of the loop function

$$
\{ \Sigma[u] \Theta^{(\ell)}[u] \}_{\text{sing}} = -\frac{(\ell + 2)}{16\pi^4(\ell + 1)} I_1^\ell \frac{1}{u^2}.
$$

(3.4.17)

Using a standard formula of analytic dimensional regularization in $AdS$ space (see [82] and [83])

$$
\left[ \frac{1}{u^{n-\epsilon}} \right]_{\text{sing}} = -\frac{1}{\epsilon} \frac{(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(u),
$$

(3.4.18)

for our distribution with $n = 2$ and the definition of the delta function (3.3.6), we obtain

$$
\{ \Sigma[u] \Theta^{(\ell)}[u] \}_{\text{sing}} = -\frac{\Omega_3(\ell + 2)}{8\pi^4(\ell + 1)} (a^\mu c_\mu) \frac{1}{\epsilon} \delta(4)(z_1; z_2).
$$

(3.4.19)

Before inserting this expression in (3.4.14) we have to be sure that we preserved gauge invariance during regularization. At this stage it means that we have to preserve the conservation condition (3.4.3) for the current as a Ward identity for the correlator in (3.4.13). Then taking into account that after partial integration we got gauge invariant Fronsdal’s operators instead of external lines we deduce that we just should write these external lines as a gauge invariant object during dimensional regularization or in other words for $d = 3 - \epsilon$. From the formula for the geometric $AdS$ mass $\mu_\ell^2$ (3.1.2) we see that

$$
\mathcal{F}^{d=3-\epsilon}(h^{(\ell)}(z; a)) = \mathcal{F}^{d=3}(h^{(\ell)}(z; a)) + \epsilon(\ell - 2)h^{(\ell)}(z; a).
$$

(3.4.20)

Then inserting this and (3.4.19) in (3.4.14) we obtain immediately as local singularity of our diagram

$$
-\frac{1}{\epsilon} \frac{g_5^2 \Omega_3(\ell + 2)}{8N\pi^4(\ell + 1)} \int \sqrt{g} d^4 z \mathcal{F}(\psi^{(\ell)}(z; a)) *_a \mathcal{F}(\psi^{(\ell)}(z; a)),
$$

(3.4.21)

supplemented with the additional finite local term

$$
-\frac{g_5^2 \Omega_3(\ell - 2)(\ell + 2)}{4N\pi^4(\ell + 1)} \int \sqrt{g} d^4 z \psi^{(\ell)}(z; a) *_a \mathcal{F}(\psi^{(\ell)}(z; a)).
$$

(3.4.22)
The first singular term can be dropped adding the same singular local and gauge invariant counterterm to the effective action as (3.4.21) but with opposite sign. The second finite local part is not gauge invariant itself and cannot be absorbed by adding the local finite invariant counterterm but can be absorbed by finite renormalization of the mass term. Indeed let us add an additional finite local counterterm proportional to

\[
\int \sqrt{g} d^4 z [\mathcal{F}(\psi^{(\ell)}) - \delta m^2_\ell \psi^{(\ell)}] *_a [\mathcal{F}(\psi^{(\ell)}) - \delta m^2_\ell \psi^{(\ell)}].
\]

(3.4.23)

Then we see that if

\[
\delta m^2_\ell = \frac{g^2_i \Omega_3 (\ell - 2)(\ell + 2)}{8N\pi^4 (\ell + 1)} = \frac{g^2_i (\ell - 2)(\ell + 2)}{4N\pi^2 (\ell + 1)},
\]

(3.4.24)

we will cancel (3.4.22) without any additional term in the given order of perturbation theory ($O(1/N)$), absorbing the additional $O(1)$ finite local $\mathcal{F}^2$ term in the infinite singular gauge invariant counterterm, since an additional finite renormalization supplementing the infinite one fixes the renormalization scheme. We see that our mass renormalization implies a soft symmetry breaking because we got only a finite mass generation. In other words all our infinite counterterms are gauge invariant.

So we see that we got mass renormalization as it was expected from $AdS_4/CFT_3$ correspondence and formulated in terms of boundary $CFT$ theory in [9]. In principle we can compare this mass with the answer obtained in [9] from anomalous dimensions of higher spin currents in the $O(N)$ sigma model

\[
\delta m^2_\ell = \frac{1}{N} \frac{16(\ell - 2)}{3\pi^2}.
\]

(3.4.25)

We got the same interesting $(\ell - 2)$ factor protecting the spin 2 graviton field, that corresponds to the boundary energy momentum tensor, from renormalization and found a prediction for the coupling $g_\ell$. But we will not compare them at this stage because it is not the full solution of the problem. We did not include in our consideration the all other possible interactions and the corresponding one loop diagrams. It is also interesting to compare this UV approach with another IR ansatz including the St"uckelberg and Goldstone mechanism which was considered in [11].
Chapter 4

Conformal invariant Lagrangians

4.1 Conformal invariant powers of the laplacian, FG ambient metric and Ricci gauging

The problem of constructing conformally invariant Lagrangians or differential operators in various dimensions and for various fields has quite a long history. This problem attracts attention primarily because it is always a nontrivial task to construct local conformal or Weyl invariants in higher dimensions \([52, 53, 54]\). The \(AdS/CFT\) correspondence \([8]\) increased interest in this old problem as well as returned the attention to the seminal mathematical paper by Fefferman and Graham (FG) on conformal invariants \([55]\). In this section we discuss conformal coupling of a scalar field with gravity in different dimensions which has been a subject of interest in quantum field theory in curved spacetimes \([56]\). In recent years it has attracted special attention in the context of new developments in the area of \(AdS/CFT\) \([8]\) correspondence, and in investigations of higher order and higher spin gravitating systems in general \([57]\). Conformally invariant field theories in higher dimensions are particularly interesting because they present a universal tool for investigations of their quantum properties, such as conformal or trace anomalies \([58]\). Another important property of conformally invariant theories in arbitrary dimensions is, that the method of dimensional regularization can be employed as a conformally invariant regularization in higher dimensions for the construction of anomalous effective actions \([59]\). Note also that in connection with higher spin gauge field interactions with a scalar field, this coupling and Weyl invariance itself, can be generalized (see next section and \([32, 33]\)). Our goal in this section is to establish the connections between different ways of construction of the local conformal invariant Lagrangians or differential operators in \(d\) dimensions \([50, 60]\) and the FG \(d + 2\) dimensional ambient Ricci flat space method \([55]\).

In this work we propose a hierarchy of such couplings of gravity to scalar fields with increasing scaling dimensions parameterized by a natural number \(k\), and
living in all space-time dimensions \( d \geq 2k \). Actually this hierarchy corresponds to the conformally invariant \( k \)-th power of the Laplacian acting on a scalar field with conformal dimension \( \Delta_{(k)} = k - d/2 \), in spacetime dimensions \( d \geq 2k \). From the other hand we propose the connection between this hierarchy and the \( k \)-th Euler density \( E_{(k)} \) lifted to spacetime dimensions greater than \( 2k \). For completeness, we verify this proposal in the well known text book case of \( k = 1 \) \[56\]. We then turn to the known case in \( d = 4 \) \[61, 62\], and the fourth order conformally covariant operator in dimension \( d \geq 4 \) obtained in \[63, 64\] long ago, which provides us with a further check of our proposal, now involving the second Euler density \( E_{(2)} \). In the subsection 4.1.4 we perform the new calculation of the locally Weyl invariant third power of the Laplacian in spacetime dimensions \( d \geq 6 \), or in another words we construct a conformally invariant action for the scalar with conformal dimension \( 3 - d/2 \) coupled with gravity. In all three cases we have found the corresponding Euler density \( E_{(k)} \) as part of the invariant action, proportional to the first order of \( \Delta_{(k)} \), and without derivatives. Taking into account the rather technical character of this section we devote a substantial subsection, subsection 4.1.1 with a more or less complete technical setup and all the formulas which we have used in our calculations.

The main FG idea consists in the confidence that the lower dimensional diffeomorphisms and local conformal invariants can be obtained from corresponding reparametrization invariant counterparts in the higher dimensional space where \( d \) dimensional conformal invariance is realized as a part of \( d + 2 \) dimensional diffeomorphisms (we review the FG method in subsection 4.1.2). On the other hand the FG expansion is connected with \( AdS_{d+1}/CFT_d \) correspondence and plays a crucial role in derivation of the holographic anomalies in different dimensions \[65\]. This point forced us in subsection 4.1.5 to derive again, using the FG ambient space method, the hierarchy of conformally invariant powers of the Laplacian (or invariant Lagrangian) in spacetime dimensions \( d \geq 2k \) acting on a scalar field obtained in subsections 4.1.3 4.1.4 by the direct Noether procedure, whose conformal dimension is \( \Delta_{(k)} = k - d/2 \). This ambient space derivation unveiled the remarkable and mysterious feature of these differential invariants namely the appearance of the \( 2k \) dimensional holographic anomaly in the \( k \)-th member of this hierarchy \[50\] (recent mathematical development in the holographic formalism for conformally invariant operators is considered in \[66\]).

Then we propose also (subsection 4.1.6) an extended or gauged FG \( d + 2 \) dimensional space to establish a connection between the FG expansion and another interesting method of constructing the Weyl invariant Lagrangians obtained in \[60\] by A. Iorio, L. O’Raifeartaigh, I. Sachs and C. Wiesendanger and named “Ricci gauging”. The magic and universality of the \( d + 2 \) dimensional FG method is defined by the existence of so-called Penrose-Brown-Henneaux (PBH) diffeomorphisms \[67\] considered in details for usual FG metric in \[59\] and \[68\]. In subsection 4.1.6 we consider the new PBH transformation for gauged ambient spaces to explore some properties of the FG expansion in the presence of the
CHAPTER 4. CONFORMAL INVARIANT LAGRANGIANS

Weyl gauge field and the holographic origin of the Ricci gauging.

4.1.1 Notations and Conventions

We work in a $d$-dimensional curved space with metric $g_{\mu\nu}$ and use the following conventions for covariant derivatives and curvatures:

\[ \nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\sigma} V^\sigma - \Gamma^\sigma_{\mu\lambda} V^\rho, \]  
\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}), \]  
\[ [\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\mu\sigma\nu\lambda} V^\sigma - R^\sigma_{\mu\lambda\nu\rho} V^\rho, \]  
\[ R^\rho_{\mu\lambda\nu} = \partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\lambda\nu} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda}, \]  
\[ R_{\mu\lambda} = R^{\rho}_{\mu\rho\lambda}, \quad R = R^\rho_{\rho\rho}. \]  

The corresponding local conformal transformations (Weyl rescalings)

\[ \delta g_{\mu\nu} = 2(\sigma(x)) g_{\mu\nu}, \quad \delta g^{\mu\nu} = -2(\sigma(x)) g^{\mu\nu}, \]  
\[ \delta \Gamma^\lambda_{\mu\nu} = \partial_\mu \sigma \delta^\lambda_{\nu} + \partial_\nu \sigma \delta^\lambda_{\mu} - g_{\mu\nu} \partial^\lambda \sigma, \]  
\[ \delta R^\rho_{\mu\lambda\nu} = \nabla_\mu \partial_\lambda \sigma \delta^\rho_{\nu} - \nabla_\nu \partial_\lambda \sigma \delta^\rho_{\mu} + g_{\mu\lambda} \nabla_\nu \partial^\rho \sigma - g_{\nu\lambda} \nabla_\mu \partial^\rho \sigma, \]  
\[ \delta R_{\mu\lambda} = (d-2) \nabla_\mu \partial_\lambda \sigma + g_{\mu\lambda} \Box \sigma, \]  
\[ \delta R = -2\sigma R + 2(d-1) \Box \sigma, \]  

are first order in the infinitesimal local scaling parameter $\sigma$.

We then introduce the Weyl ($W$) and Schouten ($K$) tensors, as well as the scalar $J$

\[ R_{\mu\nu} = (d-2) K_{\mu\nu} + g_{\mu\nu} J, \quad J = \frac{1}{2(d-1)} R, \]  
\[ W^\rho_{\mu\nu\lambda} = R^\rho_{\mu\nu\lambda} - K_{\mu\lambda} \delta^\rho_{\nu} + K_{\nu\lambda} \delta^\rho_{\mu} - K^\rho_{\nu\mu} g_{\lambda\nu} + K^\rho_{\mu\nu} g_{\lambda\nu}, \]  
\[ \delta K^\rho_{\mu\nu} = \nabla_\mu \partial_\nu \sigma, \]  
\[ \delta J = -2\sigma J + \Box \sigma, \]  
\[ \delta W^\rho_{\mu\nu\lambda} = 0, \]  

which are more convenient because their conformal transformations are "diagonal".

To describe the Bianchi identity with these tensors, we have to introduce the so-called Cotton tensor

\[ C_{\mu\nu\lambda} = \nabla_\mu K_{\nu\lambda} - \nabla_\nu K_{\mu\lambda}, \]  
\[ \delta C_{\mu\nu\lambda} = -\partial_\alpha \sigma W^\alpha_{\mu\nu\lambda}, \quad C_{\mu\nu\lambda} = 0. \]
All important properties of these tensors following from the Bianchi identity can then be listed as

\begin{align*}
\nabla_{[\alpha} W_{\mu\nu]\lambda}^\rho &= g_{\lambda[\alpha} C_{\mu\nu]}^\rho - \delta_{[\alpha}^\rho C_{\mu\nu]\lambda} , \\
\nabla_{\alpha} W_{\mu\nu\lambda}^\alpha &= (3 - d) C_{\mu\nu\lambda} , \\
\nabla^\mu K_{\mu\nu} &= \partial_\nu J , \\
C_{\mu\nu}^\nu &= 0 , \\
\nabla^\lambda C_{\mu\nu\lambda} &= 0 .
\end{align*}

Finally we introduce the last important conformal object in the above listed hierarchy, namely the symmetric and traceless Bach tensor

\begin{equation}
B_{\mu\nu} = \nabla^\lambda C_{\lambda\mu\nu} + K_{\alpha\nu}^\lambda W_{\lambda\mu\nu}^\alpha ,
\end{equation}

whose conformal transformation and divergence are expressed in terms of the Cotton and the Schouten tensors as follows

\begin{align*}
\delta B_{\mu\nu} &= -2\sigma B_{\mu\nu} + (d - 4)\nabla^\lambda \sigma (C_{\lambda\mu\nu} + C_{\lambda\nu\mu}) , \\
\nabla^\mu B_{\mu\nu} &= (d - 4) C_{\alpha\nu\beta} K_{\alpha\beta}^\nu .
\end{align*}

Note that only in four dimensions is the Bach tensor conformally covariant and divergenceless.

This basis of $B, C, K, J, W$ tensors we will use in subsections 4.1.3, 4.1.4 to construct directly a hierarchy of conformally invariant Lagrangians or differential operators originating from powers of the Laplacian in spacetime dimensions $d \geq 2k$, describing the nonminimal coupling of gravity with a scalar field whose conformal dimension is $\Delta_{(k)} = k - d/2$. Finally for any scalar $f^\Delta(x)$ with arbitrary scaling dimension $\Delta$ we can easily derive the following important relations

\begin{align*}
\delta \left( \nabla_\mu \partial_\nu f^\Delta \right) &= \Delta \sigma \nabla_\mu \partial_\nu f^\Delta + \Delta f^\Delta \nabla_\mu \partial_\nu \sigma + (\Delta - 1) \partial_\mu (\sigma \partial_\nu f^\Delta) + g_{\mu\nu} \partial^\lambda \sigma \partial_\lambda f^\Delta , \\
\delta \left( \Box f^\Delta \right) &= (\Delta - 2) \sigma \Box f^\Delta + \Delta f^\Delta \Box \sigma + (d + 2\Delta - 2) \partial^\lambda \sigma \partial_\lambda f^\Delta
\end{align*}

by using the transformation (4.1.7) for Christoffel symbols.

### 4.1.2 Ambient metric and Fefferman-Graham expansion

In this section we review the FG ambient space method for constructing local conformal invariants\[55\]. We define the $d + 2$ dimensional ambient space with

\begin{align*}
\delta W_{ijk}^m &= 0 , \\
\delta K_{ij} &= \nabla_i \partial_j \sigma , \\
\delta J &= -2\sigma J + \Box \sigma , \\
\delta C_{ijk} &= -\partial_m \sigma W_{ijk}^m , \\
\delta B_{ij} &= -2\sigma B_{ij} + (d - 4) \nabla_k \sigma (C_{kij} + C_{kji}) ,
\end{align*}

and it is all one needs to construct any conformally invariant object in arbitrary dimensions\[50,53\].

---

*This basis of $B, C, K, J, W$ tensors forms a closed system with respect to local conformal (or Weyl) transformations of the boundary metric $\delta g_{ij}(x) = 2\sigma(x)g_{ij}(x)$.
the set of coordinates \( \{ x^\mu \} = \{ t, \rho, x^i; i = 1, 2, \ldots d \} \) and the following Ricci flat metric
\[
d s^2_A = g^A_{\mu\nu}(t, \rho, x^i)dx^\mu dx^\nu = \frac{t^2}{\ell^2}h_{ij}(x, \rho)dx^i dx^j - \rho dt^2 - t dtd\rho , \tag{4.1.27}
\]
where
\[
h_{ij}(x, \rho) = g_{ij}(x) + \rho h^{(1)}_{ij}(x) + \rho^2 h^{(2)}_{ij}(x) + \ldots \ldots \tag{4.1.28}
\]
is the well known FG expansion with an arbitrary boundary value of the metric \( g_{ij}(x) = h_{ij}(x, \rho)|_{\rho=0} \) and a set of the higher \( \rho \) derivatives
\[
\frac{n!}{\rho^n}h_{ij}^{(n)}(x) = \partial^n \rho^n h_{ij}(x, \rho)|_{\rho=0}
\]
fixed by the Ricci flatness condition in ambient space
\[
R^A_{\mu\nu} = 0 . \tag{4.1.29}
\]
This condition produces the following set of equations
\[
R^A_{it} = R^A_{ti} = R^A_{tt} \equiv 0 , \tag{4.1.30}
\]
\[
R^A_{\rho\rho} = \frac{1}{2} \left[ h_{kl}^{(h)} h_{kl}^{(h)} - \frac{1}{2} h_{ij}^{(h)} h_{jk}^{(h)} h_{li}^{(h)} \right] = 0 , \tag{4.1.31}
\]
\[
R^A_{i\rho} = \frac{1}{2} \left[ \nabla^{(h)} i h_{kl}^{(h)} - \nabla^{(h)} k h_{il}^{(h)} \right] = 0 , \tag{4.1.32}
\]
\[
\ell^2 R^A_{ij} = \ell^2 R_{ij}[h] - (d-2)h_{ij}' - h_{kl}^{(h)} h_{kl}^{(h)} h_{ij}' + \rho \left[ 2h_{ij}' - 2h_{il}' h_{jm}' + h_{kl}^{(h)} h_{kl}^{(h)} h_{ij}' \right] = 0 , \tag{4.1.33}
\]
where \( \ldots \ldots = \partial_{\rho} \ldots \ldots \) and \( \nabla^{(h)} i , R_{ij}[h] \) are covariant derivative and Ricci tensor of the metric \( h_{ij}(x, \rho) \), respectively. It was shown in \([55]\) that this system of equations is equivalent to the \( d+1 \) dimensional Einstein’s equations with negative cosmological constant (see \([68]\) for details). This can be easily seen from the following consideration

- The \( AdS_{d+1} \) bulk can be found in \( d+2 \) dimensional ambient space as a \( d+1 \) dimensional surface defined as
\[
t^2 \rho = \ell^2 , \quad \rho > 0 . \tag{4.1.34}
\]
on which the metric (4.1.27) induces the standard Poincaré metric for coordinates \( \{ x^a \} = \{ \rho, x^i \} \)
\[
d s^2_{Bulk} = g_{ab}^{Bulk}(x, \rho)dx^a dx^b = \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{1}{\rho}h_{ij}(x, \rho)dx^i dx^j . \tag{4.1.35}
\]
- The corresponding bulk Ricci tensor is related to the nonzero components of the ambient Ricci tensor in the way
\[
R^A_{ab} = R^{Bulk}_{ab} + \frac{d}{\ell^2}g^{Bulk}_{ab} , \tag{4.1.36}
\]
and condition (4.1.29) leads to the negative constant curvature

\[ R_{ab} = 0 \Rightarrow R^{Bulk}_{ab} = R^{Bulk}_{ab} g^{ab}_{Bulk} = -\frac{d(d+1)}{\ell^2}. \] (4.1.37)

Therefore (4.1.29) leads, as in the case of \( AdS_{d+1}/CFT_d \) correspondence [65], to the same solutions for \( h^{(n)}_{ij}(x) \) in the FG expansion (4.1.28) in terms of covariant objects constructed from the boundary value \( g_{ij}(x) \)

\[
\begin{align*}
  h^{(1)}_{ij}(x) &= \ell^2 K_{ij}, & h^{(1)} &= g^{ij}(x) h^{(1)}_{ij}(x) = \ell^2 J, \\
  h^{(2)}_{ij}(x) &= \frac{\ell^4}{4} \left\{ \frac{B_{ij}}{d-4} + K_{ij}^{m} K_{mij} \right\}, & h^{(2)} &= g^{ij}(x) h^{(2)}_{ij}(x) = \frac{\ell^4}{4} K_{ij}^{m} K_{mij}, \\
  h^{(3)} &= g^{ij}(x) h^{(3)}_{ij}(x) = \frac{\ell^6}{6(d-4)} K_{ij} B_{ij},
\end{align*}
\] (4.1.38) (4.1.39) (4.1.40)

The connection of the Fefferman-Graham ambient metric expansion and \( AdS/CFT \) correspondence was investigated and developed by many authors. We do not pretend here to present an exhaustive list of citations in this field and just quote a number of articles important for us in this area [59], [65], [68]. For us the most important result of [55] is the elegant method of constructing conformal invariants (covariants) in \( d \) dimensions from reparametrization invariant (covariant) combinations of the curvature and it’s covariant derivatives in \( d + 2 \) dimensional ambient space equipped with a Ricci flat metric (4.1.27) by truncation to the \( d \) dimensional boundary at \( \rho = 0 \) and \( t = const. \). In the simplest case of a Riemannian curvature tensor this prescription gives for nonvanishing components (see [68] for detailed derivation)

\[
\begin{align*}
  R^{A}_{ijk} \big|_{\rho=0} &= W_{ijk} \big|^{t}, & (4.1.41) \\
  R^{A}_{ijk} \big|_{\rho=0} &= t C_{ijk}, & (4.1.42) \\
  R^{A}_{\rho ij} \big|_{\rho=0} &= \frac{t \ell^2}{2} B_{ij} \big|_{\rho=0}. & (4.1.43)
\end{align*}
\]

Using this the authors derived in [55] the first nontrivial invariant obtained from \( (\nabla^{A}_{m} R^{A}_{ijkl})^{2} \) and discussed in details in [53]. In the same article Fefferman and Graham predicted that usual Laplacian in ambient \( d+2 \) dimensional space should produce conformal invariant second order differential operator in dimension \( d \), which is the first representative in the hierarchy of conformal operators for scalar fields constructed here in subsections 4.1.3, 4.1.4.

### 4.1.3 Hierarchies of conformal scalars and Euler densities

In this section we introduce the hierarchy of scalar fields \( \varphi(k) \), where \( k = 1, 2, 3, \ldots \) with the corresponding scaling dimensions and infinitesimal conformal
transformations

\[ \Delta_{(k)} = k - d/2, \quad (4.1.44) \]
\[ \delta \varphi_{(k)} : = \Delta_{(k)} \sigma \varphi_{(k)}. \quad (4.1.45) \]

Each of these exist in the spacetime dimensions \( d \geq 2k \), and with the minimal dimension vanishing, \( \Delta_{(k)} = 0 \) when \( d = 2k \).

Let us now introduce the hierarchy of the Euler densities

\[ E_{(k)} := \frac{1}{2k(d - 2k)!} \delta_{\alpha_1 \ldots \alpha_{d-2k} \mu_1 \mu_2 \ldots \mu_{2k-1} \mu_{2k}} R_{\mu_1 \mu_2} \ldots R_{\mu_{2k-1} \mu_{2k}}. \quad (4.1.46) \]

This set of objects exist as Lagrangians in space time dimensions \( d \geq 2k \), but for the minimal case \( d = 2k \), \( E_k \) is a total divergence such that its integral is a topological invariant, the Euler characteristic. In these dimensions \( E_k \) trivialize as Lagrangians but describe the topological part of the trace anomaly in the corresponding even space-time dimension \( 2k \).

The idea of this section is the following observation: There should be a one to one correspondence between the conformally coupled scalars \( \varphi_{(k)} \) and the Euler densities \( E_{(k)} \).

Our first step in proving this is to start from the action of the well known non minimal conformally coupled scalar in the space-time dimension \( d \) and with conformal dimension \( \Delta_1 = 1 - d/2 \)

\[ S_{(1)} = \frac{1}{2} \int d^d x \sqrt{g} \left\{ g^{\mu \nu} \partial_\mu \varphi_{(1)} \partial_\nu \varphi_{(1)} - \frac{d - 2}{4(d - 1)} R \varphi_{(1)}^2 \right\}. \quad (4.1.47) \]

We first see that the second term without derivatives and proportional to the scaling dimension can be written in the form \(-\frac{d-2}{4(d-1)} R = \Delta_{(1)} J\). After that the proof of the conformal invariance of the action (4.1.47) becomes trivial: We write (4.1.26) for \( \Delta = \Delta_{(1)} \) and use (4.1.14), from which it follows that \( \delta \left[ \sqrt{g} \varphi_{(1)} \left( -\Delta_{(1)} J + \varphi_{(1)} \right) \right] = 0 \). We next evaluate (4.1.46) for \( k = 1 \)

\[ E_{(1)} = 2(d - 1) J. \quad (4.1.48) \]

Finally we see that (4.1.47) can be rewritten in the form

\[ S_{(1)} = \frac{1}{2} \int d^d x \sqrt{g} \left\{ -\varphi_{(1)} \Box \varphi_{(1)} + \frac{\Delta_{(1)}}{2(d - 1)} E_{(1)} \varphi_{(1)}^2 \right\}. \quad (4.1.49) \]

We now see that derivative independent part of the conformally invariant action is proportional to the scaling dimension times the first Euler density. Note again that both objects degenerate in minimal dimension \( d = 2 \) where the Laplacian

\[ \Box \]
itself is conformally invariant and the Euler density describes the topological invariant, which is the two dimensional trace anomaly.

The next step in our considerations is the $k = 2$ case. Again this higher derivative action (or 4-th order conformal invariant operator) is known since many years \cite{61, 62} for dimension 4 as well as for general $d$ \cite{63, 64}. All this is presented in \cite{53} where many of the invariant objects are considered. In our work, we rederived this Lagrangian just applying the Noether procedure to the local conformal variation of the following suitable object

$$S^0_{(2)} = \frac{1}{2} \int d^4x \sqrt{g} \left( \hat{D}_{(2)} \varphi_{(2)} \right)^2,$$

(4.1.50)

whose Weyl transformation includes only the first derivatives of the parameter. In (4.1.50) and thereafter, we use the notation

$$\hat{D}_{(k)} := \Box - \Delta_{(k)} J, \quad k = 1, 2, 3, \ldots,$$

(4.1.51)

$$\delta \left( \hat{D}_{(k)} \varphi_{(k)} \right) = (\Delta_{(k)} - 2) \hat{D}_{(k)} \varphi_{(k)} + 2(k - 1) \partial^\mu \sigma \partial_\mu \varphi_{(k)} - 2 \partial_\mu \partial_\sigma \varphi_{(k)} + 2 \partial^\mu \partial_\nu \varphi_{(k)} = \hat{D}_{(k)},$$

(4.1.52)

$$\hat{D}_{(k) \mu \nu} := \nabla_\mu \partial_\nu - \Delta_{(k)} K_{\mu \nu}, \quad g^{\mu \nu} \hat{D}_{(k) \mu \nu} = \hat{D}_{(k)} \varphi_{(k)},$$

(4.1.53)

$$\delta \left( \hat{D}_{(k) \mu \nu} \varphi_{(k)} \right) = \Delta_{(k)} \sigma \hat{D}_{(k) \mu \nu} \varphi_{(k)} + (\Delta_{(k)} - 1) \partial (\mu \nu \sigma \partial_\lambda \varphi_{(k)} + g_{\mu \nu} \partial^\lambda \sigma \partial_\lambda \varphi_{(k)}).$$

(4.1.54)

Performing the functional integration of the Weyl variation of the (4.1.50) is now just a matter of some partial integration, elimination of the second derivatives of $\sigma$ using (4.1.13), (4.1.14) and cancelation of terms linear in $\partial \sigma$ using the Bianchi identity (4.1.20). It should be noted here that all these types of calculations could instead be performed using the powerful method proposed in \cite{60}. Here we presented only the direct Noether procedure because that will be more suitable for us in the next section. After all these manipulations we arrive at the following action

$$S^1_{(2)} = \frac{1}{2} \int d^4x \sqrt{g} \left\{ \left( \hat{D}_{(2)} \varphi_{(2)} \right)^2 + 4 K^{\mu \nu} \partial_\mu \varphi_{(2)} \partial_\nu \varphi_{(2)} - 2 J \partial_\mu \varphi_{(2)} \partial_\mu \varphi_{(2)} + 2 \Delta_{(2)} (K^2 - J^2) \varphi_{(2)}^2 \right\}$$

(4.1.55)

Then after some work we can evaluate $E_{(2)}$ using (4.1.46) and (4.1.12) to be

$$E_{(2)} = W^2 - 4(d - 3)(d - 2) \left( K^2 - J^2 \right).$$

(4.1.56)

We see that the $\varphi_{(2)}^2$ term in (4.1.55) which is linear in $\Delta_{(2)}$, is proportional to the Weyl tensor independent part of the Euler density. The other term without derivatives is proportional to $\Delta_{(2)}^2$. This noninvariant part of the four dimensional trace anomaly arises in $AdS/CFT$ \cite{65} and carries the name "holographic", and corresponds to the maximally supersymmetric gauge theory on the boundary of $AdS_4$. 

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The combination
\[-\frac{1}{2} \int d^d x \sqrt{g} \left\{ \frac{\Delta}{2(d - 3)(d - 2)} W^2 \phi^2 \right\}, \tag{4.1.57}\]
on the other hand is also conformally invariant and can be added to \(4.1.55\) at no cost. This leads us to our final result
\[S_E^{(2)} = \frac{1}{2} \int d^d x \sqrt{g} \left\{ \phi^2 \Delta J \phi^2 - \frac{\Delta}{2(d - 3)(d - 2)} E^2 \phi^2 \right\}, \tag{4.1.58}\]
confirming our main observation in the \(k = 2\) case.

4.1.4 The \(\Delta_3 = 3 - d/2\) case

To confirm our main observation, verified for \(k = 1, 2\) above, and present it as an assertion for general \(k\), we need to carry out this verification in the next nontrivial case of \(k = 3\). This is the content of the present subsection, which consists of the explicit calculation of the conformally invariant action analogous to \(4.1.49\) and \(4.1.58\) for \(k = 1, 2\). In this case we will follow again the same strategy.

Taking into account that \(\hat{D}^{(3)} \phi (3)\) scales as an object with the dimension \(\Delta^{(1)} = \Delta^{(3)} - 2\) we start from the following initial Lagrangian
\[S^0^{(3)} = -\frac{1}{2} \int d^d x \sqrt{g} \left\{ \hat{D}^{(3)} \phi (3) \left( \hat{D}^{(3)} + 2J \right) \hat{D}^{(3)} \phi (3) \right\}, \tag{4.1.59}\]
with the more or less simple Weyl variation
\[\delta S^0^{(3)} = - \int d^d x \sqrt{g} \left\{ 4 \hat{D}^{(3)} \phi (3) \left( \Delta^{(3)} \phi (3) \partial^\sigma \partial_\sigma J + 4(\Delta^{(3)} - 2) K^{\mu \nu} \partial_\mu \sigma \partial_\nu \phi (3) \right) \right. \]
\[- \left. 2 \hat{D}^{(3)} \phi (3) \left( \hat{D}^{(3)} \phi (3) \partial^\sigma J - 4 \hat{D}^{(3)} \phi (3) K^{\mu \nu} \partial_\mu \phi (3) \partial_\nu J - 2 \Delta^{(3)} \delta (K^2) \phi (3) \right) \right\}. \tag{4.1.60}\]
The second line in \(4.1.60\) can be integrated adding to the \(S^0^{(3)}\) the following terms
\[S_1^{(3)} = - \int d^d x \sqrt{g} \left\{ 2(\hat{D}^{(3)} \phi (3))^2 J - 8 \hat{D}^{(3)} \phi (3) \hat{D}^{(3)} \phi (3) K^{\mu \nu} \right. \]
\[- 4 \hat{D}^{(3)} \phi (3) \partial^\sigma \phi (3) \partial^\lambda J - 4 \Delta^{(3)} \hat{D}^{(3)} \phi (3) K^2 \phi (3) \right\}. \tag{4.1.61}\]
Writing the variation of the \(S^{0+1}\) is rather more complicated. First we should separate the Laplacians from \(\Delta^{(3)} J\) in the terms with \(\hat{D}^{(3)} \phi (3)\), then, performing some partial integrations we redistribute derivatives and separate the terms
CHAPTER 4. CONFORMAL INVARIANT LAGRANGIANS

\[ \partial_\mu \varphi_2(3) \partial_\nu \varphi_2(3) \] and \[ \varphi_2^2(3) \] that are irreducible under partial integration. After some manipulations, using (4.1.16) and Bianchi identities, we obtain

\[ \delta S_0 \left( \partial_\mu \varphi_2(3) \partial_\nu \varphi_2(3) \right) + \delta S_1 \left( \partial_\mu \varphi_2(3) \partial_\nu \varphi_2(3) \right) = -\delta S_2 \left( \partial_\mu \varphi_2(3) \partial_\nu \varphi_2(3) \right) - \delta S_\Delta \left( \partial_\mu \varphi_2(3) \partial_\nu \varphi_2(3) \right) \]

where

\[ S_2 \left( \partial_\mu \varphi_2(3) \partial_\nu \varphi_2(3) \right) = \int d^4x \sqrt{|g|} \left\{ 24K^\mu_\nu \partial_\mu \varphi_2(3) \partial_\nu \varphi_2(3) + 24\Delta_\varphi \partial_\lambda \varphi_2(3) \left( \partial_\mu \varphi_2(3) \right) \right\} \]

(4.1.62)

Now to cancel the second line in (4.1.62) with the Cotton tensor we have to turn to the Bach tensor transformation (4.1.23). It is easy to see that the following combination

\[ S_B \left( \partial_\mu \varphi_2(3) \partial_\nu \varphi_2(3) \right) = -\frac{8}{d-4} \int d^4x \sqrt{|g|} \left\{ B^\mu_{\lambda \nu} \partial_\mu \varphi_2(3) \partial_\nu \varphi_2(3) + \Delta_\varphi B^\mu_\nu \varphi_2(3) \right\} \]

(4.1.65)

make our action completely locally conformal invariant. It follows that the required locally Weyl invariant action for the \( k = 3 \) case is

\[ S_3 = \sum_{i=0}^2 S_i(3) + S_\Delta(3) + S_B(3) \]  

(4.1.66)

Now we analyze the linear on \( \Delta_\varphi \varphi_2^2(3) \) part of (4.1.66):

\[ 4\Delta_\varphi \int d^4x \sqrt{|g|} \left\{ J^3 - 3K^2J + 2K^3 - \frac{2}{d-4} B^\mu_\nu \varphi_2(3) \right\} \varphi_2^2(3) \]

(4.1.67)

We see again that this part coincides with the so-called "holographic" anomaly [65] in 6 dimensions written in general spacetime dimension \( d \) (see also [68] for the role of the Bach tensor in holography). The main property of the holographic anomaly is that it is a special combination of the Euler density with the other three Weyl invariants [69] which reduce the topological part of the anomaly to the expression (4.1.67) (see [70] for the correct separation), which is zero for the Ricci flat metric.

But this is for the anomaly itself in \( d = 6 \). Here we are concerned with the invariant Lagrangian and presence of the scalar field and the integral make our considerations easier. To get the invariant action with the whole third Euler density, we have to perform some more work, and find that there is another invariant action with the maximum of four derivatives. This action can be obtained, using the same Noether procedure, to render the following initial term

\[ S_W^0 = \frac{8}{(d-3)(d-4)} \int d^4x \sqrt{|g|} W^{\mu\nu_\alpha_\beta} \tilde{D}_{\mu_\nu} \varphi_2(3) \tilde{D}_{\alpha_\beta} \varphi_2(3) \]

(4.1.68)
invariant. After some lengthy but straightforward calculation we arrive at the following locally conformal invariant action.

\[ S_W = S^B_W - S^0_W - S^\Delta_W, \quad \delta S_W = 0, \quad (4.1.69) \]

where

\[ S^1_W = \int d^4x \sqrt{g} \left\{ 16W^{\mu\nu\alpha\beta}K_{\alpha\beta} \frac{1}{(d-3)} + 3W^2g^{\mu\nu} - 12W^{2\mu\nu} \right\} \partial_{\mu}\varphi(3) \partial_{\nu}\varphi(3), \quad (4.1.70) \]

\[ S^\Delta_W = \Delta(3) \int d^4x \sqrt{g} \left\{ \frac{12W^{\mu\nu\alpha\beta}K_{\mu\nu}K_{\alpha\beta}}{(d-3)} + \frac{3W^2J - 12W^{2\mu\nu}K_{\mu\nu}}{(d-3)(d-4)} \right\} \varphi^2(3), \quad (4.1.71) \]

To derive this we used the Bianchi identity (4.1.18) contracted with the Weyl tensor. This leads to the following relation

\[ \frac{1}{2} \partial_\alpha W^2 - 2\nabla_\mu W^2_\mu = 2(d-4)C_{\lambda\rho} W^{\lambda\rho}, \quad (4.1.72) \]

which generates the terms quadratic in the Weyl tensor in (4.1.69)-(4.1.71). Therefore the existence of the invariant (4.1.69) allows us to replace the Bach tensor dependent term \( S^B_W \) in (4.1.66) with \( W \) dependent terms and obtain

\[ S^A_W = \sum_{i=0}^{2} S^i_W + S^0_W + S^1_W + S^\Delta_W + S^\Delta_W. \quad (4.1.73) \]

Then we see that all terms proportional to \( \Delta_3\varphi^2(3) \) are accumulated in the last two terms of (4.1.73)

\[ S^\Delta_W + S^\Delta_W = \frac{3\Delta_3}{(d-5)(d-4)(d-3)} \int d^4x \sqrt{g} A\varphi^2(3), \quad (4.1.74) \]

where

\[ A = (d-5)[W^2J - 4W^{2\mu\nu}K_{\mu\nu}] + 4(d-5)(d-4)W^{\mu\nu\alpha\beta}K_{\mu\nu}K_{\alpha\beta} \]
\[ + \frac{4}{3}(d-5)(d-4)(d-3)[J^3 - 3K^2J + 2K^3]. \quad (4.1.75) \]

We can now insert (4.1.12) in (4.1.46) for \( k = 3 \) and get

\[ E_{(3)} = \frac{16}{3} W^3 \frac{32}{3} W^3 + 8A, \quad (4.1.76) \]

\[ W^3 = W^{\alpha\beta\mu\nu} W^{\lambda\rho}_{\mu\nu} W^{\lambda\rho}_{\alpha\beta}, \quad W^3 = W^{\alpha\mu\nu} W^{\lambda\mu\rho}_{\alpha\beta}. \quad (4.1.77) \]

In the same way as in the \( k = 2 \) case we can add these two additional invariant actions with the appropriate coefficients:

\[ S_W = \frac{1}{2} \int d^4x \sqrt{g} \frac{4\Delta_3(W^3 + 2W^3)}{(d-5)(d-4)(d-3)} \varphi^2(3), \quad (4.1.78) \]
and restore the Euler density containing Lagrangian
\[ S_{E}^{(3)} = S_{A}^{(3)} + S_{W}^{3} = \frac{1}{2} \int d^{d}x \sqrt{g} \left\{ -\varphi^{(2)} \Box^{2} \varphi^{(2)} + \cdots + \frac{3 \Delta_{3}}{4(d-5)(d-4)(d-3)} E^{(3)} \varphi^{2(3)} \right\}, \tag{4.1.79} \]
where we put \( \cdots \) instead of the other terms with derivatives, or terms proportional to \( \Delta_{3}^{2} \) and \( \Delta_{3}^{3} \). These terms can be readily read off (4.1.59), (4.1.61), (4.1.68) and (4.1.70).

We have proved our assertion concerning the connection between the hierarchy of conformally coupled scalars with the dimensions \( \Delta_{k} \) and Euler densities \( E_{(k)} \) for the \( k = 1, 2, 3 \), and have constructed the conformal coupling of the third scalar with gravity in dimensions \( d \geq 6 \). This action in spacetime dimension \( d = 6 \) or equivalently for \( \Delta_{3}(3) = 0 \) degenerates to a conformal invariant operator for dimension 0 scalars obtained in [71, 72] from cohomological considerations of the effective action.

### 4.1.5 Hierarchies of conformal invariant powers of Laplacian from ambient space

In subsection 4.1.3 we introduced the hierarchy of scalar fields \( \varphi_{(k)} \), where \( k = 1, 2, 3, \ldots \) with the corresponding scaling dimensions \( \Delta_{(k)} = k - d/2 \) and infinitesimal conformal transformations (4.1.45). Each of these exists in the spacetime dimensions \( d \geq 2k \), and with the minimal vanishing dimension, \( \Delta_{(k)} = 0 \) when \( d = 2k \) and couples with gravity in the conformally invariant way through the hierarchy of the conformally invariant \( k \)-th power of the Laplacian
\[ \hat{\mathcal{L}}_{(k)} = \Box^{k} + \cdots + \Delta_{(k)} a_{(k)}. \tag{4.1.80} \]

The interesting point of this consideration was the appearance [50] of the so-called holographic anomaly \( a_{(k)} \) [65], namely the derivative independent part of the conformally invariant \( k \)-th power of Laplacian is the scaling dimension times the holographic anomaly in dimension \( d = 2k \) written in general spacetime dimension \( d \).

In this subsection we will explain this remarkable property of the above hierarchy, namely that one obtains conformal invariant operators from the \( k \)-th power of the Laplace-Beltrami operator constructed from the ambient metric which acts on the \( d+2 \) dimensional scalar field and from using the FG holographic expansion (4.1.28). So we concentrate on
\[ (\Box^{A})^{k} f(x, t, \rho), \tag{4.1.81} \]

\(^{3}\)We use the notation \( \Box^{A} \) for the Laplacian in ambient space. The \( \Box_{h} \) is the Laplacian constructed from \( h_{ij}(x, \rho) \) and a simple \( \Box \) corresponds to the boundary metric \( g_{ij}(x) \).
where
\[
\Box_{A} = \frac{\ell^2}{t^2} \Box + \frac{4 \rho}{t^2} \partial^2 \rho - \frac{4}{t} \partial_t \partial_{\rho} + h^{ij} h_{ij}' \left( \frac{2 \rho}{\ell^2} \partial_{\rho} - \frac{1}{t} \partial_t \right) - \frac{2(d-2)}{t^2} \partial_{\rho}. \tag{4.1.82}
\]

For doing that first of all we have to understand the right truncation for the \(d + 2\) dimensional scalar \(f(x,t,\rho)\) to the \(d\) dimensional scalar \(\varphi_k(x)\). Taking into account that we do not want to consider AdS/CFT behaviour for the scalar field we can take it \(\rho\) independent. Then from simple scaling arguments we arrive at the following ansatz
\[
f(x,t,\rho) = t^{\Delta(k)} \varphi(k)(x). \tag{4.1.83}
\]

Then we see that (4.1.82) reduces to
\[
\Box_{A} \left[ t^{\Delta(k)} \varphi(k)(x) \right] = \ell^2 t^{\Delta(k)-2} \left[ \Box_t \varphi(k)(x) - \frac{\Delta(k)}{\ell^2} h^{ij} h_{ij}' \varphi(k)(x) \right], \tag{4.1.84}
\]
so that inserting \(k = 1\) and using (4.1.89) we obtain
\[
\Box_{A} \left[ t^{\Delta(1)} \varphi(1)(x) \right] \big|_{\rho=0} = \ell^2 t^{-d/2} \left( \Box - \Delta(1) J \right) \varphi(1)(x), \tag{4.1.85}
\]
where we recognize in the brackets the well known conformal Laplacian
\[
\hat{\Box}_{(1)} = \Box - \Delta(1) J = \Box + \frac{(d-2)}{4(d-1)} R. \tag{4.1.86}
\]

The next step in our ambient space considerations is the \(k = 2\) case. First we rewrite the last term in (4.1.82) in the \(\Delta(k)\) dependent form
\[
-2 \frac{d-2}{t^2} \partial_{\rho} = \frac{4 \Delta(k) - 4(k-1)}{t^2} \partial_{\rho}. \tag{4.1.87}
\]

Inserting (4.1.84) in (4.1.82) and expanding in \(\rho\) we obtain
\[
\Box_{A} \left[ t^{\Delta(k)} \varphi(k)(x) \right] = \ell^4 t^{\Delta(k)-4} f_{(k)}(\rho, x) = \ell^4 t^{\Delta(k)-4} \left\{ \left( \Box - \frac{\Delta(k)}{\ell^2} h^{(1)} \right)^2 + \frac{2}{\ell^2} h^{(1)} \Box \right.
\]
\[
- \frac{4(3-k)}{\ell^2} \left[ h^{(1)ij} \nabla_i \partial_j + \frac{1}{2} (\nabla^n h^{(1)}) \partial_n \right] + \frac{2}{\ell^4} \Delta(k) \left[ (3-k) h^{(1)ij} h_{ij}^{(1)} - h^{(1)2} \right]
\]
\[
+ \frac{\rho \Delta(k)}{\ell^4} \left( 8 h^{(1)ij} h_{ij}^{(2)} - 4 h^{(1)ij} h_{jn}^{(1)} h^{(1)n}_{i} + 3 h^{(1)ij} h_{ij}^{(1)} \right)
\]
\[
+ \rho O(\nabla) + \rho O(3-k) + \rho O(\Delta^{2}(k)) + O(\rho^2) \right\} \varphi(k)(x), \tag{4.1.88}
\]
where we use the following relations
\[
\nabla_j h^{(1)ij} = \nabla_i h^{(1)}, \quad h^{(2)} = \frac{1}{4} h^{(1)ij} h_{ij}^{(1)}, \tag{4.1.89}
\]
\[
\nabla_j h^{(2)ij} + \frac{1}{2} \nabla_i h^{(2)} = \frac{1}{2} h^{(1)jn} \nabla_j h_{ni}^{(1)} + \frac{1}{4} h_{ij}^{(1)} \nabla^j h^{(1)}, \tag{4.1.90}
\]
\[
h^{(3)} = \frac{2}{3} h^{(1)ij} h_{ij}^{(2)} - \frac{1}{6} h^{(1)ij} h_{jn}^{(1)} h^{(1)n}_{i}, \tag{4.1.91}
\]
obtained from $\rho$ expansion of (4.1.31) and (4.1.32). Now inserting in (4.1.88) $k = 2$ and $\rho = 0$ and using (4.1.38) we obtain
\[\Box^2 A \left[ t^{\Delta(k)} \varphi(k)(x) \right] |_{\rho=0} = \ell^4 t^{\Delta(k)-4} \mathcal{L}(2) \varphi(k)(x), \quad (4.1.92)\]
\[\mathcal{L}(2) = (\Box - \Delta(2))J^2 - 4 \nabla_i K^{ij} \partial_j + 2 \nabla^i J \partial_i + 2 \Delta(2) (K^2 - J^2). \quad (4.1.93)\]

Again this fourth order higher derivative conformal invariant operator is known since many years [61, 62] for dimension 4 as well as for general $d$ [63, 64]. This operator was rederived in [50] and here in subsection 4.1.3 as a kinetic operator for the second Lagrangian of the hierarchy of conformally coupled scalars by simply applying the Noether procedure.

Now we can evaluate the general expression for Euler densities
\[E(k) := \frac{1}{2k(d-2)!} \delta_{i_1 \ldots i_{d-2k} j_1 \ldots j_{2k-1} j_{2k}} R^{k_1 k_2}_{j_1 j_2} \ldots R^{k_{2k-1} k_{2k}}_{j_{2k-1} j_{2k}}. \quad (4.1.94)\] for $k = 2$ and obtain
\[2 \Delta(2) (K^2 - J^2) = -\frac{\Delta(2)}{2(d-3)(d-2)} (E(2) - W^2) \quad (4.1.95)\]

So we see that the last term in (4.1.93), which is linear in $\Delta(2)$, is proportional to the Weyl tensor independent part of the Euler density. Thus we recognize as $a(k)$ of (4.1.80) for both the $k = 1, 2$ cases (4.1.85), (4.1.92)
\[a(1) = -\frac{1}{\ell^2} h^{(1)} = -\frac{1}{2(d-1)} E(1), \quad (4.1.96)\]
\[a(2) = 2(h^{(1)ij} h^{(1)}_{ij} - h^{(1)2}) = -\frac{1}{2(d-3)(d-2)} (E(2) - W^2). \quad (4.1.97)\]

The "holographic" trace anomaly arises in $AdS/CFT$ [65] and corresponds to the maximally supersymmetric gauge theories on the boundary of $AdS_3$ and $AdS_5$. To check our statement as an assertion for general $k$, we need to carry out this verification in the next nontrivial case of $k = 3$ obtained in subsection 4.1.4 by the Noether procedure [50] (the sixth order conformally invariant operator in $d = 6$ was obtained in [71] from cohomological consideration). We performed the full calculation inserting (4.1.88) with $k = 3$ in (4.1.82) and have found full agreement with the formula (4.1.73) ((56) of [50]). Here, to avoid cumbersome formulas, we will trace only the derivative independent term linear in $\Delta(3)$. First of all we see from (4.1.82) and (4.1.87) the relation
\[\Box A \ell^4 t^{\Delta(k)-4} f(k)(\rho, x) = \ell^6 t^{\Delta(k)-6} \left[ \Box + (4 - \Delta(k))h^{(1)} + 4(5 - k) \partial_\rho + O(\rho) \right] f(k)(\rho, x). \quad (4.1.98)\]

Then it is easy to see that the relevant terms in (4.1.88) are only two derivative free expressions with the $\ell^{-4}$ in front. Now because both derivative free terms in
(4.1.88) are already with a $\Delta_{(k)}$ factor, the operator (4.1.98) contributes only as $4h^{(1)} + 8\partial_\rho$ if $k = 3$ and we have to just multiply the derivative free part of the second line in (4.1.88) (it is just $-\frac{2\Delta_{(k)}}{\ell^6} h^{(1)2}$ for $k=3$) by $4h^{(1)}$ and add it to the third line of (4.1.88) with factor 8 instead of the $\rho$. So finally we have

$$\Box^2 [t^{\Delta_{(3)}} \varphi_{(k)}(x)] |_{\rho=0} = \ell^6 t^{\Delta_{(3)}} \hat{L}_{(3)} \varphi_{(3)}(x) = \ell^6 t^{\Delta_{(3)}} \hat{L}_{(3)} \varphi_{(3)}(x) = \ell^6 t^{\Delta_{(3)}} \hat{L}_{(3)} \varphi_{(3)}(x) - \frac{3\Delta_{(3)}}{\ell^6} \left[ 8h^{(1)ij}h^{(2)}_{ij} - 4h^{(1)ij}h^{(1)k}h_i - 3h^{(1)ij}h^{(1)}_{ij} - h^{(1)3} \right] \varphi_{(3)}(x). \quad (4.1.99)$$

Now using again (4.1.38) and (4.1.39) we see that

$$a_{(3)} = -8 \left[ J^3 - 3K_i^j K_i^j J + 2K_i^j K_{jn} K_n^j - \frac{2}{d-4} K_i^j B_{ij} \right]. \quad (4.1.100)$$

We see again that this part coincides with the so called "holographic" anomaly [65] in 6 dimensions written in general spacetime dimension $d$ (see also [68]). The important property of the holographic anomaly is that it is a special combination of the Euler density with three other Weyl invariants [69], [70] which reduce the topological part of the anomaly to the expression (4.1.100), which is zero for the Ricci flat metric (see [73] for recent results on purely algebraic considerations of the general structure of the Weyl anomaly in arbitrary $d$).

### 4.1.6 The ambient space, PBH diffeomorphisms and Ricci gauging

In this subsection we consider an ambient space origin of another method of construction of $d$ dimensional local conformal invariants. This is the so-called Ricci gauging proposed by A. Iorio, L. O’Raifeartaigh, I. Sachs and C. Wiesendanger in [60]. Ricci gauging is very effective when we start from a scale invariant matter field Lagrangian and want to generalize it to a local Weyl or conformal invariant Lagrangian. The prescription developed in [60] consists of two steps

1. First of all we have to perform Weyl gauging by introduction of the corresponding Weyl gauge field $A_i(x)$. For the scalar field it looks like

$$\partial_i \varphi_{(k)}(x) \rightarrow D_i \varphi_{(k)}(x) = (\partial_i - \Delta_{(k)} A_i(x)) \varphi_{(k)}(x), \quad (4.1.101)$$

$$\delta A_i(x) = \partial_i \sigma(x), \quad \delta D_i \varphi_{(k)}(x) = \Delta_{(k)} D_i \varphi_{(k)}(x), \quad (4.1.102)$$

with the additional "pure gauge" conditions $\nabla_i A_j = \nabla_j A_i$ for elimination of the self invariant combinations of $A_i$ constructed from the field strength $F_{ij} = \partial_i A_j$.

2. After Weyl gauging the actions with a conformally invariant flat space limit (scale invariant) contain the field $A_i$ only in the combinations
\[ \Omega_{ij}[A] = \nabla_i A_j(x) + A_i A_j - \frac{g_{ij}}{2} g^{kl} A_k A_l, \quad \delta \Omega_{ij}[A] = \nabla_i \sigma(x), \quad (4.1.103) \]

\[ \Omega[A] = g^{ik} \nabla_i A_k(x) + \frac{d-2}{2} g^{kl} A_k A_l, \quad \delta \Omega[A] = \Box \sigma(x), \quad (4.1.104) \]

and therefore can be replaced by

\[ K_{ij} = \Omega_{ij}[A] \quad \text{and} \quad J = \Omega[A]. \quad (4.1.105) \]

The authors of [60] called this procedure Ricci gauging.

To understand this Ricci gauging on the level of \( d + 2 \) dimensional gauged ambient space of Fefferman and Graham we turn first to the idea of PBH diffeomorphisms [67] of the higher dimensional spaces, which reduce to conformal transformations on the lower dimensional boundary or embedded subspace. Actually the PBH transformations can be defined as higher dimensional diffeomorphisms which leave the form of the higher dimensional metric invariant. The PBH transformations for the bulk metric (4.1.35) are constructed and analyzed in [59] and [74]. For the \( d + 2 \) dimensional ambient metric (4.1.27) PBH diffeomorphisms are considered in [68]. The existence of such a transformations is another reason why the reparametrization invariant powers of the Laplacian in ambient space reduce to the Weyl invariant operators in \( d \) dimensional space as considered in the previous section. Following [68] we define PBH transformations of (4.1.27) as diffeomorphisms (Lie derivative along the vector \( \zeta^\mu(t, \rho, x) \))

\[ \delta g^A_{\mu\nu}(x^\rho) = \mathcal{L}_{\zeta(t, \rho, x)} g^A_{\mu\nu}(t, \rho, x) = \zeta^\lambda(t, \rho, x) \partial_{\lambda} g^A_{\mu\nu}(t, \rho, x) \]

\[ + g^A_{\mu\lambda}(t, \rho, x) \partial_{\nu} \zeta^\lambda(t, \rho, x) + g^A_{\nu\lambda}(t, \rho, x) \partial_{\mu} \zeta^\lambda(t, \rho, x), \quad (4.1.106) \]

satisfying the conditions

\[ \delta g^A_{tt}(t, \rho, x) = \delta g^A_{t\rho}(t, \rho, x) = \delta g^A_{\rho\rho}(t, \rho, x) = \delta g^A_{ti}(t, \rho, x) = 0. \quad (4.1.107) \]

The corresponding infinitesimal PBH transformations are [59], [68]

\[ \zeta^i(t, \rho, x) = t \sigma(x), \quad (4.1.108) \]

\[ \zeta^\rho(t, \rho, x) = -2 \rho \sigma(x), \quad (4.1.109) \]

\[ \zeta^i(t, \rho, x) = \zeta^i(0, x), \quad h_{ij}(\rho, x) \partial_\rho \zeta^i(\rho, x) = \frac{\rho^2}{2} \partial_\rho \sigma(x), \quad (4.1.110) \]

\[ \delta h_{ij}(\rho, x) = 2 \sigma(x)(1 - \rho \partial_\rho) h_{ij}(\rho, x) + \mathcal{L}_{\zeta(\rho, x)} h_{ij}(\rho, x). \quad (4.1.111) \]

We see that PBH transformations depend on two free parameters \( \sigma(x) \) and \( \zeta^i(x) = \zeta^i(0, x) \) corresponding to the local Weyl and local diffeomorphisms of the boundary metric \( g_{ij}(x) = h_{ij}(0, x) \). All other terms \( n! \zeta^{(n)i}(x) = \frac{\partial^n}{\partial \rho^n} \zeta(\rho, x)|_{\rho=0} \) of the \( \rho \) expansion of the \( \zeta^i(\rho, x) \) are expressed through \( \sigma(x) \) according to the relation (4.1.110). This dependence fixes the special unhomogeneous forms of
the Weyl transformations of the FG coefficients, which is in full agreement with
the direct solution (4.1.37)-(4.1.39) of the corresponding equations (4.1.29) or
(4.1.37) (see [59] for details).

To include the Weyl gauge field $A_i(x)$ in this game and find an ambient space
description of the Ricci gauging we introduce a generalized $d + 2$ dimensional
gauged ambient space with the following metric

$$ds^2_{GA} = \frac{t^2}{\ell^2} \left[ h_{ij}(\rho, x) + \rho \ell^2 A_i(x) A_j(x) \right] dx^i dx^j - \rho dt^2 - t \left[ dt + t A_i(x) dx^i \right] d\rho.$$  

(4.1.112)

Then we consider corresponding $d + 2$ dimensional diffeomorphisms conserving
the form of (4.1.112)

$$\delta g_{tt}(t, \rho, x) = \delta g_{tp}(t, \rho, x) = \delta g_{\rho\rho}(t, \rho, x) = \delta g_{ti}(t, \rho, x) = 0,$$  

(4.1.113)

and giving for $A_i(x)$ a gauge transformation with the Weyl parameter $\sigma(x)$
(4.1.102). The corresponding solution gives for new PBH transform ations

$$\zeta^t(t, \rho, x) = t \sigma(x),$$  

(4.1.114)

$$\zeta^\rho(t, \rho, x) = -2\rho \sigma(x),$$  

(4.1.115)

$$\zeta^i(t, \rho, x) = \zeta^i(x),$$  

(4.1.116)

$$\delta h_{ij}(\rho, x) = 2\sigma(x)(1 - \rho \partial_\rho) h_{ij}(\rho, x) + \mathcal{L}_{\zeta(x)} h_{ij}(\rho, x),$$  

(4.1.117)

$$\delta A_i(x) = \partial_i \sigma(x) + \mathcal{L}_{\zeta(x)} A_i(x).$$  

(4.1.118)

Comparing with (4.1.108)-(4.1.111) we see that we were lucky with the ansatz
(4.1.112) to restore the Weyl part of the PBH transformation with the proper
gauge transformation for $A_i(x)$. The only difference that we have here is the $\rho$-
independence of the bulk diffeomorphisms $\zeta^i(x)$ and correspondingly the absence
of the condition (4.1.110). It is a price for the additional gauge field transformation
(4.1.118). However, this difference is very essential for the FG expansion.

Putting $\zeta^i(x) = 0$ we get from (4.1.117) for pure Weyl transformations of the FG
coefficients $n! h_{ij}^{(n)}(x)$ only the homogeneous parts

$$\delta g_{ij}(x) = 2\sigma(x) g_{ij}(x),$$  

(4.1.119)

$$\delta h_{ij}^{(1)}(x) = 0,$$  

(4.1.120)

$$\delta h_{ij}^{(2)}(x) = -2\sigma(x) h_{ij}^{(2)}(x).$$  

(4.1.121)

So it seems really as a Weyl gauged version of the FG expansion. For making the
final check of this assertion we turn now to the Ricci flatness condition for the
gauged ambient metric (4.1.112). Inverting the metric (4.1.112) we obtain

$$
\begin{pmatrix}
\ell^2 A^2 & -2\gamma & -\ell^2 \frac{A^i}{\ell^2} A^j \\
-2\gamma & \frac{4\gamma}{\ell^2} & \frac{2\ell^2 A^i}{\ell^2} \frac{A^j}{\ell^2} \\
-\ell^2 A^i & \frac{2\ell^2 A^i}{\ell^2} & \frac{\ell^2}{\ell^2} h^{ij}
\end{pmatrix},
\quad (4.1.122)
$$
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where

\[
\gamma = 1 + \rho \ell^2 A^2, \quad A^2(\rho, x) = h^{nm}(\rho, x)A_n(x)A_m(x), \quad (4.1.123)
\]

\[
A^i(\rho, x) = h^{ik}(\rho, x)A_k(x). \quad (4.1.124)
\]

Then the calculation of the Christoffel symbols and Ricci tensor became straightforward if we admit the condition \( F_{ij} = 0 \). After a long calculation we see that the first four equations

\[
R^G_{\ell \ell} = R^G_{\rho \rho} = R^G_{\ell t} \equiv 0,
\]

\[
R^G_{\rho \rho} = \frac{1}{2} \left[ h^{kl} h''_{kl} - \frac{1}{2} h^i j k l h'_{kl} h'_{ij} \right] = 0,
\]

are the same as in the usual ambient space. But the last two undergo a change

\[
R^G_{ij} = \frac{1}{2} \left[ \nabla_i (h'_{kl}) - \nabla_k (h'_{il}) \right] + \frac{1}{2} h^{kl} h'_{kl} A_i + \frac{d-2}{2} h'_{il} h^{kl} A_l - \rho h''_{ik} h'_{kl} A_l = 0
\]

\[
\ell^2 R^G_{ij} = \ell^2 R_{ij}[h] - (d-2) h'_{ij} - \gamma h^{kl} h'_{kl} h_{ij} + \rho \gamma \left[ 2 h''_{ij} - 2 h'_{il} h^{lm} h'_{mj} + h^{kl} h'_{kl} h'_{ij} \right]
\]

\[
- (d-2) (\nabla_i (h_{kl}) + A_i A_l - A^2 h_{ij}) - h_{ij} \nabla_k (h_{kl}) A^k
\]

\[
+ \rho [h^{kl} h'_{kl} \nabla_i (h_{kl}) A_l - (d-4) A^2 h'_{kl} - 2 A^k (h'_{ik} A_l + h'_{kj} A_l)]
\]

\[
- h^{kl} (h'_{ki} \nabla_i (h_{kl}) A_l + h'_{kj} \nabla_k (h_{kl}) A_i) + \nabla_k (h'_{ij} A^k) + 2 \rho h_{ik} A^k h'_{kl} A_l = 0. \quad (4.1.128)
\]

Then inserting in \( (4.1.128) \) \( \rho = 0 \) we obtain instead of \( (4.1.38) \) the following solution for the first coefficient of the FG expansion

\[
\frac{1}{\ell^2} h^{(1)}_{ij}(x) = K_{ij} - \nabla_i A_j - A_i A_j + \frac{1}{2} g_{ij} A_k A_l g^{kl}
\]

\[
= K_{ij} - \Omega_{ij} [A], \quad (4.1.129)
\]

\[
\frac{1}{\ell^2} h^{(1)}(x) = J - \Omega [A], \quad (4.1.130)
\]

So we see that \( (4.1.129) \) is Weyl invariant which is in agreement with the PBH transformation \( (4.1.120) \). On the other hand we see that Ricci gauging leads to a trivialization of the Fefferman-Graham expansion. Indeed the Ricci gauging condition \( (4.1.105) \) means

\[
h^{(1)}_{ij} \equiv 0. \quad (4.1.131)
\]

Moreover because equations \( (4.1.126)-(4.1.128) \) express recursively each next \( h^{(n)}_{ij} \) through the nonzero powers of previous ones we can conclude that all higher \( h^{(n)}_{ij} \) coefficients of the FG expansion are trivialized after imposing the Ricci gauging condition. The final conclusion which we can make now is the following: The FG expansion for a gauged ambient metric \( (4.1.112) \) can be obtained from the usual
expansion for (4.1.27) by the Weyl gauging. For example we can easily guess the next coefficient

\[ h^{(2)}_{ij}(x) = \frac{\ell^4}{4} \left\{ \frac{\tilde{B}_{ij}}{d - 4} + (K^m_i - \Omega^m_i[A])(K_{mj} - \Omega_{mj}[A]) \right\} , \quad (4.1.132) \]

where

\[ \tilde{B}_{ij} = B_{ij} - (d - 4)A^k(C_{kij} + C_{kji}) - (d - 4)A_iW_{kij} \quad (4.1.133) \]

is the Weyl gauged Bach tensor.

4.2 Conformal invariant interaction of a scalar field with the higher spin field in AdS

4.2.1 The cases of spin two and spin four

We work in Euclidian AdS with the following metric, curvature and covariant derivatives:

\[ ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu = \frac{L^2}{(z^0)^2} \delta_{\mu\nu}dz^\mu dz^\nu, \quad \sqrt{-g} = \frac{L^D}{(z^0)^D} , \]
\[ [\nabla_\mu, \nabla_\nu]V^\rho_\lambda = R^\rho_{\mu\lambda\sigma}V^\sigma_\sigma - R^\rho_{\mu\nu\sigma}V^\sigma_\lambda , \]
\[ R^\rho_{\mu\nu\lambda} = \frac{1}{(z^0)^2} (\delta^\rho_{\mu\lambda} - \delta^\rho_{\nu\lambda}) - \frac{1}{L^2} \left( g_{\mu\lambda}(z)\delta^\rho_\nu - g_{\nu\lambda}(z)\delta^\rho_\mu \right) , \]
\[ R_{\mu\nu} = -\frac{D - 1}{(z^0)^2} \delta_{\mu\nu} = -\frac{D - 1}{L^2} g_{\mu\nu}(z) , \quad R = -\frac{(D - 1)D}{L^2} . \]

In [32] the authors constructed gauge and generalized Weyl invariant actions for spin two and four gauge fields interacting with a scalar field. Here we review these results in the form suitable for a generalization to arbitrary higher even spin fields. We work with double traceless higher spin fields in Fronsdal’s formulation [23], [25] where the free field equation of motion for the higher spin $\ell$ field $h_{\mu_1...\mu_\ell}$ reads

\[ F_{\mu_1...\mu_\ell} = \Box h_{\mu_1...\mu_\ell} - \ell \nabla_{(\mu_1} \nabla^\rho h_{\mu_2...\mu_\ell)\rho} + \frac{\ell(\ell - 1)}{2} \nabla_{(\mu_1} \nabla_{\mu_2} h_{\mu_3...\mu_\ell)\rho}^\rho \]
\[ + \frac{\ell^2 + \ell(D - 6) - 2(D - 3)}{L^2} h_{\mu_1...\mu_\ell} + \frac{\ell(\ell - 1)}{L^2} g_{(\mu_1\mu_2} h_{\mu_3...\mu_\ell)\rho}^\rho = 0 \quad (4.2.1) \]

This equation is invariant under gauge transformation.

\*We denote symmetrization of indices by round brackets.
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\[ \delta h_{\mu_1 \ldots \mu_\ell} = \ell \nabla (\epsilon_{\mu_1 \ldots \mu_\ell}) = \nabla_{\mu_1} \epsilon_{\mu_2 \ldots \mu_\ell} + \text{c.p.} \tag{4.2.2} \]

where

\[ h_{\mu_1 \ldots \mu_{\ell-4} \rho \sigma} = 0, \tag{4.2.3} \]
\[ \epsilon_{\mu_1 \ldots \mu_{\ell-3} \rho} = 0. \tag{4.2.4} \]

The trace of Fronsdal’s tensor reads as

\[ r^{(\ell)}_{\mu_1 \ldots \mu_{\ell-2}} = -\frac{1}{2} Tr F(h^{(\ell)}) = \nabla_{\alpha} \nabla_{\beta} h^{(\ell)\alpha\beta\mu_1 \ldots \mu_{\ell-2}} - \Box h^{(\ell)\alpha\mu_1 \ldots \mu_{\ell-2}} \]
\[ - \frac{\ell - 2}{2} \nabla^{(\mu_1} \nabla_{\alpha} h^{(\ell)\mu_2 \ldots \mu_{\ell-2)}\alpha\beta} - \frac{(\ell - 1)(D + \ell - 3)}{L^2} h^{(\ell)\alpha\mu_1 \ldots \mu_{\ell-2}}. \tag{4.2.5} \]

For the case \( \ell = 2 \) one can see [32] that a Weyl invariant action is

\[ S_{WI}(\phi, h^{(2)}) = S_0(\phi) + S_1^{\Psi}(\phi, h^{(2)}) + S_1^r(\phi, h^{(2)}). \tag{4.2.6} \]

where

\[ S_0(\phi) = \frac{1}{2} \int d^Dz \sqrt{-g} [\nabla_\mu \phi \nabla^\mu \phi + \frac{D(D - 2)}{4L^2} \phi^2], \tag{4.2.7} \]
\[ S_1^{\Psi}(\phi, h^{(2)}) = \frac{1}{2} \int d^Dz \sqrt{-gh^{(2)\mu\nu}} \Psi^{(2)}_{\mu\nu}(\phi), \tag{4.2.8} \]
\[ \Psi^{(2)}_{\mu\nu}(\phi) = -\nabla_\mu \phi \nabla_\nu \phi + \frac{D-2}{4L^2} (\nabla_\lambda \phi \nabla^\lambda \phi + \frac{D(D - 2)}{4L^2} \phi^2), \tag{4.2.9} \]
\[ S_1^r(\phi, h^{(2)}) = \frac{1}{8} \frac{D-2}{D-1} \int d^Dz \sqrt{-g} r^{(2)}(h^{(2)}) \phi^2, \tag{4.2.10} \]
\[ r^{(2)}(h^{(2)}) = \nabla_\mu \nabla_\nu h^{(2)\mu\nu} - \Box h^{(2)\mu}_{\mu} - \frac{D-1}{L^2} h^{(2)\mu}_{\mu} \tag{4.2.11} \]

which is of course the linearized form of (4.1.47) and is invariant with respect to the gauge and Weyl transformations \( \Delta \)

\[ \delta^1_{\epsilon} \phi = \epsilon^\mu(z) \nabla_\mu \phi; \quad \delta^0_{\epsilon} h^{(2)}_{\mu\nu} = 2 \nabla_{(\mu \epsilon_\nu)}, \tag{4.2.12} \]
\[ \delta^1_{\Delta} \phi(z) = \Delta \sigma(z) \phi(z); \quad \delta^0_{\sigma} h^{(2)}_{\mu\nu} = 2 \sigma(z) g_{\mu\nu}. \tag{4.2.13} \]
\[ \Delta = 1 - \frac{D}{2} \tag{4.2.14} \]

\( \Delta \) is so-called conformal weight of the scalar and gets fixed by conformal invariance condition.
Now we turn to the case $\ell = 4$. In [32] the authors started from the action (4.2.7) and applied Noether’s procedure using the following higher spin ‘reparametrization’ of the scalar field with a traceless third rank symmetric tensor parameter

$$
\delta_1^\ell \phi(z) = \epsilon^{\mu\lambda}(z) \nabla_\mu \nabla_\lambda \phi(z), \quad \epsilon^\alpha_{\alpha\mu} = 0. \tag{4.2.15}
$$

The variation of (4.2.7) is

$$
\delta_1^\ell S_0(\phi) = \int d^Dz \sqrt{-g} \left\{ -\nabla^{(\alpha} \epsilon^{\mu\lambda)} \nabla_\mu \nabla_\alpha \phi \nabla_\lambda \phi + \epsilon^{\mu\nu}_{(1)} \frac{1}{2} \nabla_\mu \nabla_\alpha \phi \nabla_\nu \nabla_\alpha \phi \\
+ \frac{D(D + 2)}{8L^2} \nabla_\mu \phi \nabla_\nu \phi \right\} - \nabla^{(\mu} \epsilon^{\nu)} \phi \nabla_\mu \phi \nabla_\nu \phi + \frac{g_{\mu\nu}}{2} \left( \nabla_\lambda \phi \nabla_\lambda \phi + \frac{D(D - 2)}{4L^2} \phi^2 \right) \\
+ \left( \nabla^2 \phi - \frac{D(D - 2)}{4L^2} \phi \right) \nabla_\mu \left( \epsilon^{\mu\nu}_{(1)} \nabla_\nu \phi \right). \tag{4.2.17}
$$

We see immediately that the first two lines of (4.2.17) produce interactions with the spin four and two currents. From the other hand the last line in (4.2.17) is proportional to the equation of motion following from $S_0(\phi)$ and therefore can be absorbed after gauging by the trace of the spin four gauge field $(2\epsilon^{\mu\nu}_{(1)} \rightarrow h^{(4)\alpha\mu\nu})$

performing the following field redefinition of $\phi$

$$
\phi \rightarrow \phi + \frac{1}{2} \nabla_\mu \left( h^{(4)\alpha\mu\nu} \nabla_\nu \phi \right). \tag{4.2.18}
$$

Such a type of field redefinition is a standard correction of Noether’s procedure and means that we always can drop from the cubic part of the action terms proportional to the equation of motion following from the quadratic part of the initial action.

So finally we see that the action

$$
S^{\text{GI}}(\phi, h^{(2)}, h^{(4)}) = S_0(\phi) + S_1^{\Psi(2)}(\phi, h^{(2)}) + S_1^{\Psi(4)}(\phi, h^{(4)}), \tag{4.2.19}
$$

where $S_0(\phi)$, $S_1^{\Psi(2)}(\phi, h^{(2)})$ are defined in (4.2.7)-(4.2.9) and

$$
S_1^{\Psi(4)}(\phi, h^{(4)}) = \frac{1}{4} \int d^Dz \sqrt{-gh^{(4)\mu\nu\alpha\beta}} \Psi^{(4)}_{\mu\nu\alpha\beta}(\phi) \tag{4.2.20}
$$

and

$$
\Psi^{(4)}_{\mu\nu\alpha\beta}(\phi) = \nabla_{(\mu} \nabla_\nu \phi \nabla_\alpha \nabla_\beta \phi - g_{\mu\nu} [\nabla_\alpha \nabla_\gamma \phi \nabla_\beta \phi] \nabla_\gamma \phi + \frac{D(D + 2)}{4L^2} \nabla_\alpha \phi \nabla_\beta \phi, \tag{4.2.21}
$$

From now on we will never make a difference between a variation of the Lagrangians or the actions discarding all total derivative terms and admitting partial integration if necessary. For compactness we introduce shortened notations for divergences of the tensorial symmetry parameters

$$
\epsilon^{\mu\nu...}_{(1)} = \nabla_\lambda \epsilon^{\lambda\mu\nu...}, \quad \epsilon^{\mu...}_{(2)} = \nabla_\nu \nabla_\lambda \epsilon^{\nu\lambda\mu...}, \ldots \tag{4.2.16}
$$
is invariant with respect to the gauge transformations of the spin four field with an additional spin two field gauge transformation inspired by the second divergence of the spin four gauge parameter.

Thus the linearized action for a scalar field interacting with the spin two and four gauge field

Thus we introduced a gauge invariant interaction of the scalar with the spin four field

We write the generalized Weyl transformation law for the spin four case as in

where we introduced a generalized "conformal" weight $\Delta^4$.*

We have to mention that our $\Delta^4$ here differs from $\tilde{\Delta}$ in [32] because of field redefinition (4.2.18) which is the reason why $S^r(4)$ from [32] turned into (4.2.21). When we make field redefinition, we add to the Lagrangian terms which are not Weyl invariant, and in order to restore Weyl invariance we have to change the coefficient $\Delta_4$.††
4.2.2 Gauge invariant interaction for the spin \( \ell \) case

Here we generalize our construction to the general spin \( \ell \) case. Again following [32] we apply the following gauge transformation

\[
\delta^a_\epsilon \phi(z) = \epsilon^{\mu_1 \mu_2 \ldots \mu_{\ell-1}}(z) \nabla_{\mu_1} \nabla_{\mu_2} \ldots \nabla_{\mu_{\ell-1}} \phi(z), \tag{4.2.34}
\]

\[
\delta^a_\epsilon h^{(\ell)}_{\alpha \mu_1 \ldots \mu_{\ell}} = \epsilon^{(\ell)}_{(e_{-1})} \nabla^{(\ell)}(\mu_1 \mu_2 \ldots \mu_{\ell-1}), \quad \delta^a_\epsilon h^{(\ell)}_{\alpha \mu_1 \ldots \mu_{\ell-2}} = 2 \epsilon^{(1)}_{(e_{-2})}, \tag{4.2.35}
\]

\[
\epsilon_{\alpha \mu_3 \ldots \mu_{\ell-1}} = 0 \tag{4.2.36}
\]

to the action [4.2.7] and obtain the following starting variation for Noether’s procedure

\[
\delta^1_\epsilon S_0(\phi) = \int d^Dz \sqrt{-g} \{ \nabla^\alpha \epsilon^{\mu_1 \ldots \mu_{\ell-1}} \nabla_\alpha \phi \nabla_{\mu_1} \ldots \nabla_{\mu_{\ell-1}} \phi + \epsilon^{\mu_1 \ldots \mu_{\ell-1}} \nabla_\alpha \phi \nabla^\alpha \nabla_{\mu_1} \ldots \nabla_{\mu_{\ell-1}} \phi + \frac{D(D-2)}{4L^2} \epsilon^{\mu_1 \ldots \mu_{\ell-1}} \phi \nabla_{\mu_1} \ldots \nabla_{\mu_{\ell-1}} \phi \}. \tag{4.2.37}
\]

Using the following notations

\[
T(n,k) = \nabla^\alpha \epsilon^{\mu_1 \ldots \mu_{n-1}} \nabla_{\mu_1} \ldots \nabla_{\mu_{k-1}} \nabla_\alpha \phi \nabla_{\mu_k} \ldots \nabla_{\mu_{n-1}} \phi, \tag{4.2.38}
\]

\[
M(n,k) = \epsilon^{\mu_1 \ldots \mu_{n-1}} \nabla_{\mu_1} \ldots \nabla_{\mu_k} \nabla_\alpha \phi \nabla_{\mu_{k+1}} \ldots \nabla_{\mu_n} \nabla^\alpha \phi, \tag{4.2.39}
\]

\[
N(n,k) = \epsilon^{\mu_1 \ldots \mu_{n-1}} \nabla_{\mu_1} \ldots \nabla_{\mu_k} \nabla_{\mu_{k+1}} \ldots \nabla_{\mu_n} \phi, \tag{4.2.40}
\]

and commutation relation (D.1) from Appendix D we rewrite (4.2.37) in the form

\[
\delta^1_\epsilon S_0(\phi) = \int d^Dz \sqrt{-g} \{ T(\ell, 1) + M(\ell - 1, 0) + \frac{(\ell - 1)(\ell - 2)}{2L^2} N(\ell - 1, 1) + \frac{D(D-2)}{4L^2} N(\ell - 1, 0) \}. \tag{4.2.41}
\]

Then using relations between \( T(m,n), M(m,n) \) and \( N(m,n) \) from Appendix C and after some algebra we 'diagonalize’ (4.2.41)

\[
\delta^1_\epsilon S_0(\phi) = \sum_{m=1}^{\ell} (-1)^m \left( \frac{\ell - m - 1}{m - 1} \right) \int d^Dz \sqrt{-g} \{ -T(2m, m) + \frac{1}{2} M(2m - 2, m - 1) + \frac{(D + 2m - 2)(D + 2m - 4)}{8L^2} N(2m - 2, m - 1) - \frac{m - 1}{\ell - 2m + 1} \epsilon^{\mu_1 \ldots \mu_{2m-2}}(\nabla_{\mu_1} \ldots \nabla_{\mu_{m-1}}[\nabla^2 \phi - \frac{D(D-2)}{4L^2} \phi] \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-2}} \phi) \} \tag{4.2.42}
\]

Further performing a final symmetrization in (4.2.42), we obtain the following elegant expression

\[
\delta^1_\epsilon S_0(\phi) = \int d^Dz \sqrt{-g} \left\{ \sum_{m=1}^{\ell} \left( \frac{\ell - m - 1}{m - 1} \right) \left[ -\nabla^{(\mu_2m, \epsilon^{\mu_1 \ldots \mu_{2m-1}}(\epsilon_{(e_{-2m+1})} \phi) \psi^{(2m)})} \mu_{3 \ldots \mu_{2m}} \right] \right. \\
+ \left[ \nabla^2 \phi - \frac{D(D-2)}{4L^2} \phi \right] \sum_{m=2}^{\ell} \left( \frac{\ell - m - 1}{m - 2} \right) \nabla_{\mu_1} \ldots \nabla_{\mu_{m-1}} \left( \epsilon_{(e_{-2m+1})}^{\mu_1 \ldots \mu_{2m-2}}(\epsilon_{(e_{-2m+1})}^{\mu_1 \ldots \mu_{2m-2}} \phi) \phi \right) \right\}. \tag{4.2.43}
\]
where

\[
\Psi_{\mu_1 \ldots \mu_{2m}}^{(2m)} = (-1)^m \{ \nabla_{\mu_1} \ldots \nabla_{\mu_m} \phi \nabla_{\mu_{m+1}} \ldots \nabla_{\mu_{2m}} \phi \\
- \frac{m}{2} g_{\mu_0 \mu_{2m-1}} g^{\alpha \beta} \nabla_{(\mu_1} \ldots \nabla_{\mu_{m-1}} \nabla_\alpha \phi \nabla_{(\mu_m} \ldots \nabla_{\mu_{2m-2}} \nabla_\beta \phi \\
- \frac{m(D + 2m - 2)(D + 2m - 4)}{8L^2} g_{\mu_0 \mu_{2m-1}} g_{\mu_{m-1} \mu_{2m-2}} \phi \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-2}} \phi \} \tag{4.2.44}
\]

and we admitted symmetrization for the set \( \mu_1, \ldots \mu_{2m} \) of indices. So we see that miraculously the coefficients in (4.2.44) don’t depend on \( \ell \) ! All \( \ell \)-dependence is concentrated in the second line of (4.2.43) proportional to the equation of motion for the action (4.2.43). This part like in the spin four case can be removed by an appropriate field redefinition (see (4.2.49), (4.2.50), (D.6))

\[
\phi \rightarrow \phi + \sum_{m=2}^{\ell} \frac{m - 1}{2(\ell - 2m + 1)} \nabla_{\mu_1} \ldots \nabla_{\mu_{m-1}} (h_{(2m)}^{(2m)} \phi_{(2m)} \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-2}} \phi) \tag{4.2.45}
\]

and we can drop these terms from our consideration. Thus we arrive at the following spin \( \ell \) gauge invariant action

\[
S^{(\ell)}(\phi, h^{(2)}, h^{(4)}, \ldots, h^{(\ell)}) = S_0(\phi) + \sum_{m=1}^{\ell} S^{(2m)}_1(\phi, h^{(2m)}) \tag{4.2.46}
\]

where

\[
S^{(2m)}_1(\phi, h^{(2m)}) = \frac{1}{2m} \int d^D z \sqrt{-g} h^{(2m)\mu_1 \ldots \mu_{2m}} \Psi_{\mu_1 \ldots \mu_{2m}}^{(2m)} \\
= \frac{(-1)^m}{2m} \int d^D z \sqrt{-g} \{ h^{(2m)\mu_1 \ldots \mu_{2m}} \nabla_{\mu_1} \ldots \nabla_{\mu_m} \phi \nabla_{\mu_{m+1}} \ldots \nabla_{\mu_{2m}} \phi \\
- \frac{m}{2} h_{\alpha \mu_1 \ldots \mu_{2m-2}}^{(2m)} \nabla_{(\mu_1} \ldots \nabla_{\mu_{m-1}} \nabla_\mu \phi \nabla_{(\mu_m} \ldots \nabla_{\mu_{2m-2}} \nabla_\mu \phi \\
- \frac{m(D + 2m - 2)(D + 2m - 4)}{8L^2} h_{\alpha \mu_1 \ldots \mu_{2m-2}}^{(2m)} \nabla_{\mu_1} \ldots \nabla_{\mu_{m-1}} \phi \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-2}} \phi \} \tag{4.2.47}
\]

and the final form of the improved gauge transformations

\[
\delta_\epsilon \phi(z) = \epsilon^{\mu_1 \mu_2 \ldots \mu_{\ell-1}}(z) \nabla_{\mu_1} \nabla_{\mu_2} \ldots \nabla_{\mu_{\ell-1}} \phi(z), \tag{4.2.48}
\]

\[
\delta_\epsilon h_{\alpha \mu_1 \ldots \mu_{2m-2}}^{(2m)} = 2m \nabla_{(\mu_2} \epsilon^{(2m)\mu_1 \ldots \mu_{2m-1})}, \quad \delta_\epsilon h_{\alpha \mu_1 \ldots \mu_{2m-2}}^{(2m)} = 2 \epsilon^{(2m)\mu_1 \ldots \mu_{2m-1}}, \tag{4.2.49}
\]

\[
\epsilon^{(2m)\mu_1 \ldots \mu_{2m-1}} = \binom{\ell - m - 1}{m - 1} \epsilon^{\mu_1 \ldots \mu_{2m-1}} \tag{4.2.50}
\]

Now we can insert \( m = \frac{\ell}{2} \) into (4.2.44) and compare our general expression for \( S^{(\ell)}_1(\phi, h^{(\ell)}) \) with the already known cases of spin two (the energy momentum
tensor for the scalar field) \((4.2.9)\) and spin four \((4.2.21)\). We can easily see that for these cases \(S_{1}^{(\ell-2,4)}(\phi, h^{(\ell)})\) exactly reproduces \((4.2.9)\) and \((4.2.21)\) respectively. So we found the gauge invariant action for a general spin \(\ell\) gauge field coupled to a scalar and this action has the following property:

*The gauge invariant action* \(S^{GI}(\phi, h^{(2)}, h^{(4)}, ..., h^{(\ell)})\) *for a spin \(\ell\) gauge field coupled to a scalar includes gauge invariant actions of the tower of all smaller even spin gauge fields coupled to the same scalar in an analogous way.*

Note that this statement holds true only if we think of an even number of divergencies applied to the gauge parameter as a possible redefinition of gauge parameter of smaller even spin gauge fields, in that case this amazing hierarchy of all smaller even spin currents appear. Another possibility is to regard divergencies of the gauge parameter as gauge transformation for divergencies of the trace of the spin \(\ell\) field and make an appropriate field redefinition. In that case we don’t need to introduce smaller spin currents, but the field redefinition will be of another form. The current of spin \(\ell\) is the same in both approaches, it is unique, and in the flat space limit reproduces currents constructed in \([3, 10]\) and \([45]\) applying a partial integration and field redefinition. The interesting point is that this symmetric form of currents is unique, and the natural generalization of the energy-momentum tensor of the scalar field \((4.2.9)\).

### 4.2.3 Weyl invariant action for a higher spin field coupled to a scalar

In this section we introduce generalized Weyl transformations for higher spin fields and derive a Weyl invariant action for a higher spin field coupled to a scalar field. Following \([32]\) we write the generalized Weyl transformation for the even spin \(l\) field in the form

\[
\delta_{0}^{\ell} h^{(\ell)\mu_{1}...\mu_{\ell}} = \ell (\ell - 1) \sigma^{\mu_{1}...\mu_{\ell-2}} g_{\mu_{\ell-1}\mu_{\ell}}, \quad (4.2.51)
\]

\[
\delta_{0}^{\alpha} h^{(\ell)\alpha\mu_{1}...\mu_{\ell-2}} = 2 (D + 2\ell - 4) \sigma^{\mu_{1}...\mu_{\ell-2}}, \quad (4.2.52)
\]

\[
\delta_{1}^{\phi} = \Delta_{\ell} \sigma^{\mu_{1}...\mu_{\ell-2}} \nabla_{\mu_{1}}... \nabla_{\mu_{\ell-2}} \phi. \quad (4.2.53)
\]

Then we assume that the Weyl invariant action for a spin \(\ell\) field should be accompanied with similar Weyl invariant actions for smaller spin gauge fields and therefore can be constructed from \((4.2.46)\) adding \(\frac{\ell}{2}\) additional terms

\[
S^{WI}(\phi, h^{(2)}, h^{(4)}, ..., h^{(\ell)}) = S^{GI}(\phi, h^{(2)}, ..., h^{(\ell)}) + \sum_{m=1}^{\ell/2} S_{1}^{(2m)}(\phi, h^{(2m)}), \quad (4.2.54)
\]

where each \(S_{1}^{(2m)}\) is gauge invariant itself. In the case of spin two we had only the linearized Ricci scalar (see \((4.2.10)\)) and for the spin four case we had two terms constructed from the spin four generalization of the Ricci scalar (see \((4.2.26)\)).
Now we will see that the generalization of the Ricci scalar for a higher spin field
namely the trace of Fronsdal’s operator \([4.2.5]\) (see \([25,32]\)) is the only gauge
invariant combination of two derivatives and a higher spin field which we need
to construct the Weyl invariant action \([4.2.44]\) starting from \([4.2.46]\). We will
use the following strategy for solving our problem: We apply transformation
\([4.2.51]-[4.2.53]\) to \([4.2.46]\) and try to compensate it with the variation of

\[
\sum_{m=1}^{\ell/2} S_1^{(2m)}(\phi, h^{(2m)}), \text{ where } \\
S_1^{(\ell)}(\phi, h^{(2)}, ..., h^{(\ell)}) = \\
\frac{1}{2} \sum_{m=0}^{\ell/2-1} \xi_m \int d^D z \sqrt{-g} \nabla_{\mu_{2m+1}} ... \nabla_{\mu_{\ell-2}} r^{(\ell)}_{\mu_1 ... \mu_{\ell-2}} \nabla_{\mu_1} ... \nabla_{\mu_m} \phi \nabla_{\mu_{m+1}} ... \nabla_{\mu_{2m}} \phi \\
\]  

(4.2.55)

introducing necessarily gauge and Weyl transformations for lower spin gauge fields

\[
\delta_\sigma h^{(2m)\mu_1 ... \mu_{2m}} = 2m(2m - 1) C_\ell^m \sigma^{(\mu_1 ... \mu_{2m-2}} g^{\mu_{2m-1} \mu_{2m})}, \quad m = 1, ..., \ell/2, (4.2.56) \\
C_\ell^{\ell/2} = 1. \\
(4.2.57)
\]

In other words we solve the equation

\[
\delta_\sigma^{W1}(\phi, h^{(2)}, ..., h^{(\ell)}) = \delta_\sigma S_0 + \sum_{s=1}^{\ell/2} \delta_\sigma^{S_0^{(2s)}} + \sum_{s=1}^{\ell/2} \delta_\sigma^{S_1^{(2s)}} = 0 (4.2.58)
\]

which consists of a system of \(\ell + 1\) equations for \((\ell/2 + 1)(\ell/2 + 2)/2\) dependent
variables

\[
\Delta_\ell, \\
C_\ell^m, \quad m = 1, 2, ..., \ell/2, \\
\xi_n^s, \quad n = 0, 1, ..., s - 1; \; s = 1, ..., \ell/2. \\
(4.2.59-4.2.61)
\]

but when we find \(\xi_{\ell/2-k}\) we also find \(\xi_{s-k}\) for any \(s \geq k\). In other words we find
a whole diagonal of this triangle matrix

\[
\begin{pmatrix}
C_\ell^1 & C_\ell^2 & \cdots & C_\ell^{\ell/2-1} & C_\ell^{\ell/2} & \Delta_\ell \\
\xi_1 & \xi_1 & \xi_1 & \xi_1 & \xi_1 & \xi_1 \\
\xi_{\ell-2} & \xi_{\ell-2} & \xi_{\ell-2} & \xi_{\ell-2} & \xi_{\ell-2} \\
\xi_4 & \xi_4 & \xi_4 & \xi_4 & \xi_4 \\
\xi_2 & \xi_2 & \xi_2 & \xi_2 & \xi_2 \\
\end{pmatrix} \\
(4.2.62)
\]
which helps us to solve the whole system. We have two equations for any column of this matrix besides the last, for which we have one equation for \( \Delta \). We start from the last column and go to the left. When we take any column and two equations for that column of variables, we have only two variables to find if we already solved all columns to the right of that one (it is easy to see that the first two rows of (4.2.62) are all we need to find out. The second row gives the solution for any spin \( \ell \). \( \xi \)-s in lower rows are just particular case and can be determined by putting concrete spin value in a general solution, which means that the independent variables are only first two rows of the (4.2.62) and the number of variables in these two rows is \( \ell + 1 \), just as much as equations we have.

This right-to-left method can be used only due to the fact that we solve system for general spin case. This is a deductive method which we use. Another approach is an inductive method - one could solve equations for concrete cases of spin 2,4,6... and obtaining all rows lower than second, and therefore whole Weyl invariant Lagrangian for lower spins, solve first two rows. Of course this is impossible for general spin \( \ell \). That means that our system has a unique solution. Placing all complicated Weyl variations of (4.2.58) into the Appendix E, we present here the resulting system of equations for the unknown variables (4.2.59)-(4.2.61):

\[
\Delta_\ell = 1 - \frac{D}{\ell/2} \tag{4.2.63}
\]

\[
\frac{(-1)^{\ell/2}}{2}(\Delta_\ell - \frac{\ell - 2}{2}) - (D + 2\ell - 5)\xi^{\ell/2-1}_\ell = 0 \tag{4.2.64}
\]

\[
(-1)^m C^m_\ell + \sum_{s=m+1}^{\ell/2} mC_s^m \xi^m_{2s} = 0, \quad (m = 1, ..., \ell/2 - 1) \tag{4.2.65}
\]

\[
\frac{(-1)^{m-1}}{2}(m - 1)C^m_\ell - C^m_\ell (D + 4m - 5)\xi^{m-1}_{2m} + \frac{1}{2} \sum_{s=m+1}^{\ell/2} C_s^m [-m(m - 1)\xi^m_{2s} - (2s - 2m + 2)(D + 2s + 2m - 5)\xi^{m-1}_{2s}] = 0
\]

\[
(m = 1, ..., \ell/2 - 1) \tag{4.2.66}
\]

The solution of this system is universal \( \Delta_\ell = \Delta = 1 - \frac{D}{\ell/2} \) and

\[
\xi^m_\ell = \frac{(-1)^m}{2^{\ell-2m}(\ell/2)} \left( \frac{\ell/2}{m} \right) \frac{(D/2 + m - 1)\ell/2-m}{(D+\ell-1/2 + m - 1)\ell/2-m} \tag{4.2.67}
\]

\[
\xi^m_\ell = \frac{(-1)^m}{2^{\ell-2m}(\ell/2)} \left( \frac{\ell/2}{m} \right) \frac{(D/2 + m - 1)\ell/2-m}{(D+\ell-1/2 + m - 1)\ell/2-m} \tag{4.2.68}
\]

\[
C^m_\ell = \frac{(-1)^{\ell/2-m}}{2^{\ell-2m}(\ell/2)} \left( \frac{\ell/2-1}{m-1} \right) \frac{(D/2 + m - 1)\ell/2-m}{(D+\ell-1/2 + 2m)\ell/2-m} \tag{4.2.69}
\]

These expressions completely fix (4.2.55) and therefore the full Weyl invariant
action (4.2.54), and also determine the transformation law for the whole tower of higher spin gauge fields (4.2.56).\(^{1}\)

\(^{1}\)It is easy to see from formula (E.4) that we get also a redefinition of the gauge parameters for all lower even spin fields which in the spin 4 case coincides with formula (4.2.32).
Chapter 5

Cubic Interactions of HSF

5.1 Off-Shell construction of some Higher Spin gauge field cubic interactions

5.1.1 Exercises on spin one field couplings with the higher spin gauge fields

We start this section constructing the well known interaction of the electromagnetic field \( A_\mu \) in flat \( D \) dimensional space-time with the linearized spin two field. Hereby we illustrate how Noether’s procedure regulates the relation between gauge symmetries of different spin fields. The standard free Lagrangian of the electromagnetic field is

\[
L_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \partial_\mu A_\nu \partial_\mu A_\nu + \frac{1}{2} (\partial A)^2, \tag{5.1.1}
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial A = \partial_\mu A^\mu. \tag{5.1.2}
\]

To construct the interaction we propose a possible form for the action of the spin two linearized gauge symmetry

\[
\delta^{\text{lin}}_{\varepsilon} h^{(2)\mu\nu}(x) = 2 \partial^{[\mu} \varepsilon^{\nu]}(x) = \partial^\mu \varepsilon^\nu(x) + \partial^\nu \varepsilon^\mu(x), \tag{5.1.3}
\]

on the spin one gauge field \( A_\mu(x) \). Then Noether’s procedure fixes this coupling (1-1-2 interaction) of the electromagnetic field with linearized gravity correcting when necessary the proposed transformation.

We start from the following general ansatz for a gauge variation of \( A_\mu \) with respect to a spin 2 gauge transformation with vector parameter \( \varepsilon^\rho \)

\[
\delta^1_{\varepsilon} A_\mu = -\varepsilon^\rho \partial_\rho A_\mu + C \varepsilon^\rho \partial_\rho A_\mu. \tag{5.1.4}
\]

Then we apply this variation (5.1.4) to (5.1.1) and after some algebra neglecting
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total derivatives we obtain

\[ \delta^1 L_0 = \partial^\mu \varepsilon^\nu \partial_\mu A_\rho \partial_\nu A^\rho - \frac{1}{2} \varepsilon^{(1)} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \varepsilon^{(1)} (\partial A)^2 + C \partial^\mu \varepsilon^\nu \partial_\mu A_\rho \partial^\rho A_\nu - 1 \varepsilon^{(1)} (\partial A)^2 \]

Then we have to compensate (or integrate) this variation using the gauge variation of the spin 2 field \((5.1.3)\) and its trace \(\delta^0 h^{(2)\mu}_\mu = 2 \varepsilon^{(1)}\). We see immediately that the last line in \((5.1.5)\) is irrelevant but can be dropped by choice of the free constant \(C = 1\). With this choice we have instead of \((5.1.4)\)

\[ \delta^1 A_\mu = -\varepsilon^\rho \partial_\rho A_\mu + \varepsilon^\rho \partial_\mu A_\rho = \varepsilon^\rho F\rho\mu, \]

so that our spin two transformation now is manifestly gauge invariant with respect to the spin one gauge invariance

\[ \delta^0_\alpha A_\mu = \partial_\mu \sigma, \]

and our spin one gauge invariant free action \((5.1.1)\) keeps this property also after spin two gauge variation. Namely \((5.1.5)\) now can be written as

\[ \delta^1 L_0 = \partial^\mu \varepsilon^\nu F\mu\nu F^\nu\rho - \frac{1}{4} \varepsilon^{(1)} F\mu\nu F^{\mu\nu}. \]

This variation can be compensated introducing the following 2-1-1 interaction

\[ L_1 (A_\mu, h^{(2)\mu\nu}) = \frac{1}{2} h^{(2)\mu\nu} \Psi^{(2)}_{\mu\nu}, \]

where

\[ \Psi^{(2)}_{\mu\nu} = -F_{\mu\rho} F^\rho\nu + \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}, \]

is the well known energy-momentum tensor for the electromagnetic field.

Thus we solved Noether’s equation

\[ \delta^1 L_0 (A_\mu) + \delta^0_\alpha L_1 (A_\mu, h^{(2)\mu\nu}) = 0 \]

in this approximation completely, defining a first order transformation and interaction term at the same time. Finally note that the corrected Noether’s procedure spin two transformation of the spin one field \((5.1.6)\) can be written as a combination of the usual reparametrization for the contravariant vector \(A_\mu(x)\) (non

*Using the same conventions as in previous Chapter (see \((4.2.16)\)).
invariant with respect to (5.1.7)) and spin one gauge transformation with the special field dependent choice of the parameter \( \sigma(x) = \varepsilon^\rho(x)A_\rho(x) \)

\[
\delta^1_\varepsilon A_\mu = \varepsilon^\rho F^\rho_\mu = -\varepsilon^\rho \partial_\rho A_\mu - \partial_\mu \varepsilon^\rho A_\rho + \partial_\mu (\varepsilon^\rho(x)A_\rho(x)), \quad (5.1.12)
\]

A symmetry algebra of these transformations can be understood from commutator

\[
[\delta^1_\varepsilon, \delta^1_\varepsilon] A_\mu(x) = \delta^1_{[\varepsilon, \varepsilon]} A_\mu(x) + \partial_\mu \left( \varepsilon^\rho(x) F^\rho_\mu(x) \right) \quad (5.1.13)
\]

\[
[\eta, \varepsilon]^\lambda = \eta^\rho \partial_\rho \varepsilon^\lambda - \varepsilon^\rho \partial_\rho \eta^\lambda \quad (5.1.14)
\]

So we see that algebra of transformations (5.1.12) close on field dependent gauge transformation (5.1.7).

Now we turn to the first nontrivial case of the vector field interaction with a spin four gauge field with the following zero order spin four gauge variation

\[
\delta^0_\eta \epsilon^{h\rho\lambda\sigma} = 4 \partial_\mu \epsilon^{\rho\lambda\sigma}, \quad \delta^0_\eta \epsilon^\rho_\lambda \sigma = 2 \epsilon^{\lambda\sigma} \quad (5.1.15)
\]

where we have a symmetric and traceless gauge parameter \( \epsilon^{\mu\nu\lambda\sigma} \) to construct a gauge variation for \( A_\mu \). According to the previous lesson we start from a spin one gauge invariant ansatz for the spin four transformation of \( A_\mu \)

\[
\delta^1_\varepsilon A_\mu = \epsilon^{\rho\lambda\sigma} \partial_\rho F^\rho_\mu. \quad (5.1.16)
\]

Thus we have now the following variation of \( \mathcal{L}_0 \)

\[
\delta^1_\varepsilon \mathcal{L}_0 = \delta^1_\varepsilon (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}) = (\delta^1_\varepsilon A_\nu) \partial_\mu F^{\mu\nu} = -\partial_\mu (\epsilon^{\rho\lambda\sigma} \partial_\rho \partial_\lambda F^{\rho\sigma}) F^{\mu\nu}. \quad (5.1.17)
\]

After some algebra, again neglecting total derivatives and using the Bianchi identity for \( F_{\mu\nu} \)

\[
\partial_\mu F^{\rho\lambda} + \partial_\nu F^{\rho\lambda} + \partial_\lambda F_{\mu\nu} = 0, \quad (5.1.18)
\]

and taking into account the important relation

\[
-\partial^\mu \epsilon^{\rho\lambda\sigma} \partial_\rho F^{\nu}_{\mu} \partial_\lambda F^{\sigma}_{\nu} = -\partial_\mu (\epsilon^{\rho\lambda\sigma} \partial_\rho F^{\nu}_{\mu} \partial_\lambda F^{\sigma}_{\nu}) + \frac{1}{4} \epsilon^{\lambda\sigma}_{(1)} \partial^{\nu \rho} F^{\mu\lambda} \partial^\rho F^{\nu\sigma} - \frac{1}{2} \epsilon^{\lambda\sigma}_{(1)} \partial_\mu F^{\nu}_{\rho} \partial_\rho F^{\nu\sigma}, \quad (5.1.19)
\]

we arrive at the following form of the variation convenient for our analysis

\[
\delta^1_\varepsilon \mathcal{L}_0 = \delta^1_\varepsilon (\epsilon^{\rho\lambda\sigma} \partial_\rho F^{\nu}_{\mu} \partial_\lambda F^{\sigma}_{\nu}) + \frac{1}{4} \epsilon^{\lambda\sigma}_{(1)} \partial^{\nu \rho} F^{\mu\lambda} \partial^\rho F^{\nu\sigma} + \frac{1}{4} \epsilon^{\lambda\sigma}_{(1)} \partial_\lambda F^{\nu}_{\rho} \partial_\rho F^{\nu\sigma} - \frac{1}{2} \partial^\rho \epsilon^{\mu\rho\lambda\sigma} \partial_\lambda F^{\rho\sigma}_{\mu} F^{\mu\nu} - \partial_\lambda (\epsilon_{(1)}^{\lambda\sigma} F^{\nu}_{\rho}) \partial_\nu F^{\mu\nu} - \frac{1}{4} \epsilon_{(3)}^{\nu\lambda\sigma} \partial_\lambda F^{\rho\sigma}_{\nu} F^{\rho\mu} \quad (5.1.20)
\]
Returning to the gauge variation of the spin four field (5.1.15) we notice that all terms in the first line of (5.1.20) and the first two terms in the second line can be integrated to the interaction terms. The last term in the second line is proportional to the free field equations but is not integrable, so we can cancel this term only by changing the initial variation of $A_\mu$ (5.1.16). The modified form of (5.1.16) is

$$\delta_1 A_\mu = \epsilon^{\rho\lambda\sigma} \partial_\rho \partial_\lambda F_{\mu\sigma} + \frac{1}{2} \partial_\rho \epsilon_{\mu\lambda\sigma} \partial^\lambda F^{\sigma\rho}. \quad (5.1.21)$$

Therefore

$$\mathcal{L}_1 = \frac{1}{4} h^{(4)\mu\nu\rho\lambda} \partial_\rho F^{\mu\nu}_\lambda F^{\lambda\sigma}_\rho - \frac{1}{8} h^{(4)\rho\lambda\sigma} \partial^\rho F_{\mu\lambda} \partial^\mu F_{\nu\sigma} - \frac{1}{8} h^{(4)\rho\lambda\sigma} \partial_\rho F_{\mu\nu} \partial_\sigma F^{\mu\nu}$$

$$+ \partial_\lambda (\frac{1}{2} h^{(4)\rho\lambda\sigma} \partial_\rho F^{\mu\nu}_\sigma) \partial_\beta F^{\nu\mu}_\beta + \frac{1}{8} h^{(4)\rho\lambda\sigma} \partial^\rho F_{\mu\lambda} \partial^\mu F_{\nu\sigma}$$

$$- \frac{1}{2} h^{(2)\mu\nu} F_{\mu\sigma} F^\sigma_\nu + \frac{1}{8} h^{(2)\rho} F_{\mu\nu} F^{\mu\nu}. \quad (5.1.22)$$

But the two terms in the second line are proportional to the equation of motion for the initial Lagrangian (5.1.1), hence they are not physical and can be removed by the following field redefinition

$$A_\mu \rightarrow A_\mu - \partial_\lambda (h^{(2)\alpha\lambda\sigma} F_{\mu\sigma}) - \frac{1}{4} h^{(2)\alpha\mu\sigma} \partial_\beta F^{\beta\sigma}. \quad (5.1.23)$$

So we can drop the second line of (5.1.22).

Another novelty in comparison with the previous case is the third line of (5.1.20). Comparing with (5.1.8) we see that we can integrate these two terms introducing an additional spin two field coupling and compensate the first and third line introducing the following linearized Lagrangian for the coupling of the electromagnetic field to the spin four and spin two fields

$$\mathcal{L}_1(A_\mu, h^{(2)\mu\nu}, h^{(4)\mu\alpha\beta}) = \frac{1}{4} h^{(4)\mu\nu\rho\lambda} \Psi^{(4)}_{\mu\nu\rho\lambda} + \frac{1}{2} h^{(2)\mu\nu} \Psi^{(2)}_{\mu\nu}, \quad (5.1.24)$$

where the current $\Psi^{(2)}_{\mu\nu}$ is the same energy-momentum tensor (5.1.9) and

$$\Psi^{(4)}_{\mu\nu\rho\lambda} = \partial_\rho (F^{\mu\nu}_\lambda \partial_\beta F_{\nu\mu}_\beta) - \frac{1}{2} g_{(\mu\nu} \partial^\lambda F_{\alpha\sigma \theta^\sigma F_{\beta\lambda})_{\rho}} - \frac{1}{2} g_{(\mu\nu} \partial_\alpha F^{\sigma\rho} \partial_\beta F_{\sigma\rho}). \quad (5.1.25)$$

The whole lagrangian

$$\mathcal{L}_0(A_\mu) + \mathcal{L}_1(A_\mu, h^{(2)\mu\nu}, h^{(4)\mu\alpha\beta}), \quad (5.1.26)$$

is invariant with respect to the spin one gauge transformations and the following higher spin transformations

$$\delta^1 A_\mu = \epsilon^{\rho\lambda\sigma} \partial_\rho \partial_\lambda F_{\mu\sigma} + \frac{1}{2} \partial_\rho \epsilon_{\mu\lambda\sigma} \partial^\lambda F^{\sigma\rho},$$

$$\delta^0 h^{(4)\mu\nu\rho\lambda} = 4 \partial_\rho (\epsilon^{\mu\nu\rho\lambda})^\beta, \quad \delta^0 h^{(2)\mu\nu} = 2 \epsilon^{(2)}_{\mu\nu}, \quad \delta^0 h^{(2)\mu} = 2 \epsilon_{(3)}. \quad (5.1.27)$$
Therefore we proved that like the previously investigated scalar–higher spin coupling case (previous Chapter), the interaction with the spin four gauge field leads to the additional interaction with the lower even spin two field. Following [34] we review here also vector field coupling to the general HS field. We start from following gauge variation

$$\delta^1_\ell A_\mu = \epsilon^1_\ell \mu_1 \cdots \mu_{l-1} \nabla_{\mu_1} \cdots \nabla_{\mu_{l-1}} F_{\mu_{l-1} \mu}.$$  

(5.1.28)

\[ \hat{L}_0 = \sum_{m=1}^{\ell/2} \left( \frac{\ell - m - 1}{m - 1} \right) \nabla^{\mu_1} \cdots \nabla^{\mu_{m-1}} (\nabla^{\nu_{m+1}} \cdots \nabla^{\nu_{l-2m}} (\nabla^{\nu_{l-2m-1}} \nabla_{\mu_1} \cdots \nabla_{\mu_{m-1}} F_{\mu_{m-1}} F_{\mu_m}) \nabla_{\alpha} F^{\alpha \mu} + \frac{\ell - m - 1}{2m} \nabla^{\mu_1} \cdots \nabla^{\mu_{m-1}} (\epsilon^{\mu_{l+1} \cdots \mu_{l+2m-3}} (\epsilon^{\nu_{l+1} \cdots \nu_{l+2m-3}} \nabla_{\mu_1} \cdots \nabla_{\mu_{m-1}} F_{\nu_{m-1}} F_{\nu_{m-2}}) \nabla_{\alpha} F^{\alpha \mu} \right) \]

\[ = \sum_{m=1}^{\ell/2} \left( \frac{\ell - m - 1}{m - 1} \right) \nabla^{\mu_1} \cdots \nabla^{\mu_{m-1}} (\nabla^{\nu_{m+1}} \cdots \nabla^{\nu_{l-2m}} (\nabla^{\nu_{l-2m-1}} \nabla_{\mu_1} \cdots \nabla_{\mu_{m-1}} F_{\nu_{m-1}} F_{\nu_{m-2}}) \nabla_{\alpha} F^{\alpha \mu} \right) \]

where

$$\Psi_{\mu_1 \cdots \mu_{2m}}(A_\mu) = (-1)^m (-\nabla^{\mu_1} \cdots \nabla^{\mu_{m-1}} F_{\mu_m} \nabla^{\nu_{m+1}} \cdots \nabla^{\nu_{l-2m}} (\nabla^{\nu_{l-2m-1}} \nabla_{\mu_1} \cdots \nabla_{\mu_{m-1}} F_{\nu_{m-1}} F_{\nu_{m-2}}) \nabla_{\alpha} F^{\alpha \mu} + \frac{m - 1}{2} g_{\mu_1 \mu_2} \nabla_{\mu_1} \cdots \nabla_{\mu_m} \nabla_{\mu_{m+1}} \cdots \nabla_{\mu_{m+2}} \cdots \nabla_{\mu_{m-1}} F_{\mu_{m-1}} F_{\mu_{m-2}} + \frac{m}{4} g_{\mu_1 \mu_2} \nabla_{\mu_1} \cdots \nabla_{\mu_{m+1}} F^{\rho \sigma} \nabla_{\mu_{m+2}} \cdots \nabla_{\mu_{2m}} F^{\rho \sigma})$$

(5.1.30)

and we admitted symmetrization for the set $\mu_1 \cdots \mu_{2m}$ of indices. This means that when we change our initial variation $\delta^1_\ell A_\mu$ to

$$\delta^1_\ell A_\mu = \epsilon^1_\ell \mu_1 \cdots \mu_{l-1} \nabla_{\mu_1} \cdots \nabla_{\mu_{l-1}} F_{\mu_{l-1} \mu}$$

$$- \sum_{m=1}^{\ell/2} \left( \frac{\ell - m - 1}{m - 1} \right) \nabla^{\mu_1} \cdots \nabla^{\mu_{m-1}} (\nabla^{\nu_{m+1}} \cdots \nabla^{\nu_{l-2m}} (\nabla^{\nu_{l-2m-1}} \nabla_{\mu_1} \cdots \nabla_{\mu_{m-1}} F_{\nu_{m-1}} F_{\nu_{m-2}}) \nabla_{\alpha} F^{\alpha \mu} \right) \]

(5.1.31)

and also take into account appropriate field redefinition

$$A_\mu = A_\mu + \sum_{m=1}^{\ell/2} \left( \frac{\ell - m - 1}{m - 1} \right) \frac{m - 1}{2m} \nabla^{\mu_1} \cdots \nabla^{\mu_{m-1}} (\epsilon^{\mu_{l+1} \cdots \mu_{l+2m-3}} (\epsilon^{\nu_{l+1} \cdots \nu_{l+2m-3}} \nabla_{\mu_1} \cdots \nabla_{\mu_{m-1}} F_{\nu_{m-1}} F_{\nu_{m-2}}) \nabla_{\alpha} F^{\alpha \mu} \right) \]

(5.1.32)
we can see that the gauge invariant Lagrangian for interaction of electromagnetic field with the higher even spin \( \ell \) field is

\[
\mathcal{L}_1(A_\mu, h^{(2)}, h^{(4)}, ..., h^{(\ell)}) = \sum_{m=1}^{\ell/2} \frac{1}{2m} h^{(2m)\mu_1...\mu_{2m}} \Psi^{(2m)}_{\mu_1...\mu_{2m}} (A_\mu). \tag{5.1.33}
\]

This result is similar to the scalar case investigated in the section 4.2. The same tower of even spin gauge fields appear when we construct gauge invariant interaction with higher spin fields. The generalization to the non-Abelian charged vector (Yang-Mills) fields is trivial. In scalar case we went further and constructed Weyl invariant lagrangian. The Weyl invariance can’t be generalized for spin one case. That is the price for spin one manifest gauge invariance (in all interactions vector field is represented by it’s curvature \( F_{\mu\nu} \)). Here we wanted to mention that \( \text{AdS}_D \) corrections to (5.1.30) have following basic properties. As in the scalar case there are no \( 1/L^4 \) or higher corrections. The \( 1/L^2 \) term is proportional to \( \ell - 2 \). As a result, for 1-1-2 interaction we don’t have any difference between interaction in the flat space and \( \text{AdS} \).

### 5.1.2 Generalization to the 2-2-4 and 2-2-6 interactions

In this section we turn to the spin two field as a lower spin field in the construction of the higher spin gauge invariant interactions with spin 4 and spin 6 gauge potentials. And again we want to keep manifest the lower spin two gauge invariance.

So proceeding similarly as in the previous section we start from the free spin two Pauli-Fierz Lagrangian \( [2] \)

\[
\mathcal{L}_0(h^{(2)}_{\mu\nu}) = \frac{1}{2} \partial_\mu h^{(2)}_{\alpha\beta} \partial^\mu h^{(2)\alpha\beta} - \partial_\alpha h^{(2)\alpha\beta} \partial_\beta h^{(2)\mu} + \partial_\mu h^{(2)\alpha} \partial_\beta h^{(2)\beta} - \frac{1}{2} \partial_\mu h^{(2)\alpha} \partial^\mu h^{(2)\beta}, \tag{5.1.34}
\]

and try to solve the following Noether’s equation

\[
\delta_1 \mathcal{L}_0(h^{(2)}_{\mu\nu}) + \delta^0_1 \mathcal{L}_1(h^{(4)}_{\mu\nu}, h^{(4)\alpha\beta\gamma\delta}) = 0. \tag{5.1.35}
\]

For this purpose we introduce the following starting ansatz for the spin four transformation of the spin two field

\[
\delta^1_\epsilon h^{(2)}_{\mu\nu} = \epsilon^{\rho\lambda\alpha} \partial_\rho \Gamma_{\lambda\sigma,\mu\nu}, \tag{5.1.36}
\]

where \( \Gamma_{\lambda\sigma,\mu\nu} \) is the spin two gauge invariant symmetrized linearized Riemann curvature

\[
\Gamma_{\alpha\beta,\mu\nu} = \frac{1}{2} (R_{\alpha\mu,\beta\nu} + R_{\beta\mu,\alpha\nu}), \tag{5.1.37}
\]

\[
\Gamma_{(\alpha\beta,\mu)\nu} = 0, \tag{5.1.38}
\]
introduced by de Witt and Freedman for higher spin gauge fields together with
the higher spin generalization of the Christoffel symbols [27]. This symmetrized
curvature is more convenient for the construction of an interaction with symmet-
ric tensors. The corresponding Ricci tensor (Fronsdal operator for higher spin
generalization) and scalar can be defined in the usual manner using traces
\begin{align}
\mathcal{F}_{\mu\nu} &= \Gamma_{\mu\nu,\lambda}^\lambda = \Box h_{\mu\nu}^{(2)} - 2\partial_{(\mu}\partial^{\alpha}h_{\nu)}^{(2)\alpha} + \partial_{\mu}\partial_{\nu}h_{\alpha}^{(2)\alpha}, \\
\mathcal{F} &= \mathcal{F}_\mu^\mu = 2(\Box h_{\mu}^{(2)\mu} - \partial_{\mu}\partial_{\nu}h_{\mu\nu}^{(2)}).
\end{align}
In terms of these objects the Bianchi identities can be written as
\begin{align}
\partial_\lambda \Gamma_{\mu\nu,\alpha\beta} &= \partial_{(\mu}\Gamma_{\nu)\lambda,\alpha\beta} + \partial_{(\alpha}\Gamma_{\beta)\lambda,\mu\nu}, \\
\partial_\lambda \mathcal{F}_{\alpha\beta} &= \partial^\mu \Gamma_{\mu\lambda,\alpha\beta} + \partial_{(\alpha}\mathcal{F}_{\beta)\lambda}, \\
\partial^\lambda \mathcal{F}_{\lambda\mu} &= \frac{1}{2} \partial_\mu \mathcal{F}_\alpha^\alpha.
\end{align}
Then a variation of (5.1.34) with respect to (5.1.36) is
\begin{align}
\delta^1 \mathcal{L}_0(h_{\mu\nu}^{(2)}) &= \frac{\delta \mathcal{L}_0}{\delta h_{\mu\nu}^{(2)}} \delta^1 h_{\mu\nu}^{(2)} = -(\mathcal{F}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \mathcal{F}) e^{\rho\lambda\sigma} \partial_\rho \Gamma_{\lambda\sigma,\mu\nu}.
\end{align}
To integrate it and solve the equation (5.1.35) we submit to the following strategy:
1) First we perform a partial integration and use the Bianchi identity (5.1.42)
to lift the variation to a curvature square term.
2) Then we make a partial integration again and rearrange indices using
(5.1.38) and (5.1.41) to extract an integrable part.
3) Symmetrizing expressions in this way we classify terms as
- integrable
- integrable and subjected to field redefinition (proportional to the free field
equation of motion)
- non integrable but reducible by deformation of the initial ansatz for the
gauge transformation (again proportional to the free field equation of motion)

Then if no other terms remain we can construct our interaction together with
the corrected first order transformation. Following this strategy after some fight
with formulas we obtain the following expression
\begin{align}
\delta^1 \mathcal{L}_0(h_{\mu\nu}^{(2)}) &= \frac{\partial^\alpha \Gamma_{\mu\nu,\alpha\beta}}{\partial h_{\mu\nu}^{(2)}} \frac{\delta \mathcal{L}_0}{\delta h_{\alpha\beta}^{(2)}} + \partial^\rho \epsilon^{\mu\nu}_{\alpha\beta} \Gamma_{\beta\rho,\mu\nu} \frac{\delta \mathcal{L}_0}{\delta h_{\alpha\beta}^{(2)}} + \epsilon^{\mu\nu}_{(1)} \Gamma_{\mu\nu,\alpha\beta} \frac{\delta \mathcal{L}_0}{\delta h_{\alpha\beta}^{(2)}} + \frac{\delta \mathcal{L}_0}{\delta h_{\alpha\beta}^{(2)}}.
\end{align}
where
\[
\Psi_{(\Gamma)\alpha\beta\mu\nu}^{(4)} = \Gamma_{(\alpha\beta,\rho\sigma\Gamma_{\mu\nu})} - \frac{2}{3} g_{(\alpha\beta\Gamma_{\mu\nu})}^{\rho\sigma\lambda} \Gamma_{\rho\sigma\lambda},
\]
(5.1.46)
\[
\Psi_{(F)\alpha\beta\mu\nu}^{(4)} = F_{(\alpha\beta,\Gamma_{\mu\nu})} - g_{(\alpha\beta\Gamma_{\mu\nu})}^{\rho\sigma\lambda} \frac{\delta L_0}{\delta h_{(2,\alpha\beta,\Gamma_{\mu\nu})}} + g_{(\alpha\beta\Gamma_{\mu\nu})}^{\rho\sigma\lambda} \frac{\delta L_0}{\delta h_{(2,\mu\nu)}} F_{\nu},
\]
(5.1.47)
\[
\frac{\delta L_0}{\delta h_{(2)\alpha\beta}} = -F_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta},
\]
(5.1.48)

So we see immediately that in (5.1.45) only the last term of the second line is not integrable but proportional to the equation of motion and can be dropped by the correction to the initial gauge transformation (5.1.36). On the other hand taking into account (5.1.15) and (5.1.46)-(5.1.48) we can compensate \(\Psi_{(F)}^{(4)}\) and the first term in the second line of (5.1.45) by the following field redefinition

\[
h_{(2)\mu\nu} \rightarrow h_{(2)\mu\nu} - \frac{1}{2} h_{(4)\alpha\lambda\sigma} \Gamma_{\lambda\sigma,\mu\nu} - \frac{1}{4} h_{(4)\alpha\mu\nu} F_{\alpha\lambda} + \frac{1}{4} h_{(4)\alpha\lambda}^{\alpha(\mu} F_{\nu)}\lambda.
\]
(5.1.49)

Thus after field redefinition we arrive at the 4-2-2 gauge invariant interaction
\[
L_1(h_{(2)\mu\nu}, h_{(2)\alpha\beta\mu\nu}) = \frac{1}{4} h_{(4)\alpha\beta\mu\nu} \Psi_{(\Gamma)\alpha\beta\mu\nu}(h_{(2)\mu\nu}),
\]
(5.1.50)

with the following gauge transformations
\[
\delta_1 e^\rho_{\lambda\sigma} \partial_\rho \Gamma_{\lambda\sigma,\mu\nu} - \partial_\rho e^{\lambda\sigma(\mu} \Gamma_{\nu)}^{\rho\lambda\sigma},
\]
(5.1.51)
\[
\delta_0 e^{(4)\rho\lambda\sigma} = 4 \partial^{(\mu} e^{\rho\lambda\sigma)}, \quad \delta_0 h_{(4)\rho\lambda\sigma} = 2 e^{(1)\lambda\sigma}.
\]
(5.1.52)

Now in possession of knowledge about the 2-2-4 interaction we start to construct the most nontrivial interaction in this section between spin 2 and spin 6 gauge fields. We would like to check the appearance of the 2-2-4 coupling during the construction of 2-2-6 which we expect from the analogy with the scalar case considered in [32, 33] and the 1-1-4 case considered in the previous subsection. To proceed we have to solve the following Noether’s equation
\[
\delta_1^1 L_0(h_{(2)\mu\nu}^{(2)}) + \delta_1^0 L_1(h_{(2)\mu\nu}^{(2)}, h_{(2)\alpha\beta\lambda\rho\sigma\delta}^{(6)}) = 0,
\]
(5.1.53)

with a starting ansatz for the spin 6 first order gauge transformation for the spin 2 field:
\[
\delta_1^1 h_{(2)\mu\nu}^{(2)}(x) = e^{\alpha(\beta\rho\lambda\sigma}(x) \partial_\alpha \partial_\beta \partial_\rho \Gamma_{\lambda\sigma,\mu\nu}(x),
\]
(5.1.54)

and the standard zero order gauge transformation for the spin 6 gauge field
\[
\delta_0^0 h_{(4)\mu\alpha\beta\rho} = 6 \partial^{(\mu} e^{\rho\alpha\beta\sigma)}, \quad \delta_0^0 h_{(4)\rho\alpha\beta\rho} = 2 e^{(1)\alpha\beta\rho}.
\]
(5.1.55)
CHAPTER 5. CUBIC INTERACTIONS OF HSF

First of all we have to transform the variation

$$\delta^1 \mathcal{L}_0(h^{(2)}_{\mu\nu}) = -(\mathcal{F}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \mathcal{F}) \epsilon^{\alpha\beta\mu\rho} \partial_{\alpha} \partial_{\beta} \partial_{\rho} \Gamma_{\lambda\sigma\mu\nu},$$

(5.1.57)

into a form convenient for integration. Following the same strategy as before in the 2-2-4 case, using many times partial integration and Bianchi identities (5.1.38), (5.1.41)-(5.1.43), we obtain after tedious but straightforward calculations

$$\delta^1 \mathcal{L}_0(h^{(2)}_{\mu\nu}) = \delta_{\omega} (\epsilon^{\alpha\beta\mu\rho}) \Psi^{(6)}_{(\Gamma)\alpha\beta\mu\nu \rho} - \delta_{\omega} (\epsilon^{\alpha\beta\mu\nu}) \Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu} + \frac{4}{3} \rho^{\mu\nu\lambda\sigma} \partial_\lambda \partial_\sigma \Gamma_{\rho\beta\mu\nu} - \frac{1}{3} \rho^{\mu\nu\lambda\sigma} \partial_\sigma \Gamma_{\rho\lambda\mu\nu} \delta \mathcal{L}_0 \delta \mathcal{L}_0 \delta h^{(2)}_{\mu\nu},$$

(5.1.58)

where

$$\Psi^{(6)}_{(\Gamma)\alpha\beta\mu\nu \rho} = \partial_{\omega} \Gamma_{\beta\mu\nu \rho}, \sigma \delta - g_{\omega \beta \mu} \partial_\rho \Gamma_{\lambda \sigma \mu \nu}, \rho \sigma \delta,$$

(5.1.59)

$$\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu} = \Gamma_{\alpha \beta \mu \nu}, \rho \sigma \rho \sigma \delta - \frac{2}{3} g_{\omega \beta \mu} \Gamma_{\rho \sigma \lambda \nu}, \rho \sigma \lambda,$$

(5.1.60)

and $R^{\mu\nu}_{\text{int}}(\Gamma, \mathcal{F}) \frac{\delta \mathcal{L}_0}{\delta h^{(2)}_{\mu\nu}}$ are remaining integrable terms proportional to the equation of motion. Indeed the symmetric tensor $R^{\mu\nu}_{\text{int}}(\Gamma, \mathcal{F})$ is expressed through the only integrable combinations of derivatives of gauge parameter

$$R^{\mu\nu}_{\text{int}}(\Gamma, \mathcal{F}, \epsilon) = \epsilon_{(1)}^{\alpha\beta\mu\nu} \partial_\alpha \partial_\beta \Gamma_{\lambda\delta} \mu\nu - \frac{1}{3} \partial_\lambda \epsilon_{(1)}^{\alpha\beta\mu\nu} \partial_\lambda \Gamma_{\lambda\beta\delta} + \partial_\lambda \left[ \partial_{\omega} \epsilon_{(1)}^{\alpha\beta\mu\nu} \partial_\omega \mathcal{F}_{\beta} \right]$$

$$- \frac{2}{3} \partial_\lambda \left[ \epsilon_{(1)}^{\alpha\beta\mu\nu} \partial_\omega \mathcal{F} \right] + \frac{1}{6} \epsilon_{(1)}^{\alpha\beta\mu\nu} \partial_\alpha \partial_\beta \mathcal{F} + \partial_{\omega} \epsilon_{(2)}^{\alpha\beta\mu\nu} \mathcal{F}_{\alpha\beta} + \frac{5}{3} \partial_{\omega} \epsilon_{(2)}^{\alpha\beta\mu\nu} \partial_\lambda \mathcal{F}_{\alpha\beta}$$

$$- \frac{5}{3} \partial_\lambda \left[ \epsilon_{(1)}^{\alpha\beta\mu\nu} \partial_\omega \mathcal{F} \right] + \frac{1}{6} \delta_{(1)}^{\alpha\beta\mu\nu} \mathcal{F}_{\alpha\beta} + \frac{1}{6} \partial\partial\epsilon_{(1)}^{\alpha\beta\mu\nu} \partial_\lambda \mathcal{F}_{\alpha\beta} - \frac{1}{2} \epsilon_{(3)}^{(\mu)} \mathcal{F}_{\alpha}.$$

(5.1.61)

Substituting into this expression $\partial_{\omega} \epsilon_{(1)}^{\alpha\beta\mu\nu}$ with $\frac{1}{6} \delta_{(1)}^{\alpha\beta\mu\nu}$, $\partial_{\omega} \epsilon_{(2)}^{\alpha\beta\mu\nu}$ with $\frac{1}{4} \delta_{(2)}^{\alpha\beta\mu\nu}$, and correspondingly $2 \epsilon_{(1)}^{\alpha\beta\mu\nu}$ and $2 \epsilon_{(2)}^{\alpha\beta\mu\nu}$ with their traces, we define a field redefinition for $h^{(2)}_{\mu\nu}$

$$h^{(2)}_{\mu\nu} \rightarrow h^{(2)}_{\mu\nu} + R^{\mu\nu}_{\text{int}}(\Gamma, \mathcal{F}, h^{(6)}, h^{(4)}),$$

(5.1.62)

using which we can drop the third line in (5.1.58). The second line in (5.1.58) can be cancelled by the following deformation of the initial ansatz for the transformation (5.1.54)

$$\delta^1 h^{(2)}_{\alpha\beta} = \epsilon^{\mu\nu\rho\lambda} \partial_\mu \partial_\nu \partial_\rho \Gamma_{\lambda\sigma,\alpha\beta} - \frac{4}{3} \partial_{\omega} \epsilon^{\mu\nu\lambda\sigma} \partial_\lambda \partial_\sigma \Gamma_{\rho\beta\mu\nu} + \frac{1}{3} \partial_{\omega} \partial_\lambda \epsilon_{\alpha\beta}^{\mu\nu\sigma} \partial_\sigma \Gamma_{\rho\lambda\mu\nu}.$$
Thus we arrive at the promised result that the 2-2-6 interaction automatically includes also the 2-2-4 interaction constructed above, and the corresponding trilinear interaction Lagrangian is

\[
\mathcal{L}_1(h^{(2)}, h^{(4)}, h^{(6)}) = -\frac{1}{6} h^{(6)\alpha\beta\gamma\lambda\rho} \Psi^{(6)}_{\Gamma} \partial_\alpha \Gamma_{\beta\gamma\lambda\rho} + \frac{1}{4} h^{(4)\alpha\beta\mu\nu} \Psi^{(4)}_{\Gamma} \partial_{\alpha} \Gamma_{\beta\mu\nu} \\
= -\frac{1}{6} h^{(6)\alpha\beta\gamma\lambda\rho} \partial_\alpha \Gamma_{\beta\gamma\lambda\rho} \sigma\delta \partial_\sigma \Gamma_{\lambda\rho,\sigma\delta} + \frac{1}{6} h^{(6)\alpha\beta\gamma\lambda\rho} \partial_\alpha \Gamma_{\beta\gamma\lambda\rho,\sigma\delta} \partial_\sigma \Gamma_{\lambda\rho,\sigma\delta} \\
+ \frac{1}{12} h^{(6)\alpha\beta\gamma\lambda\rho} \partial_\alpha \Gamma_{\beta\gamma\lambda\rho,\sigma\delta} \partial_\sigma \Gamma_{\lambda\rho,\sigma\delta} + \frac{1}{4} h^{(4)\alpha\beta\mu\nu} \Gamma_{\alpha\beta,\rho\sigma} \Gamma_{\mu\nu,\rho\sigma} - \frac{1}{6} h^{(4)\alpha\beta\mu\nu} \Gamma_{\mu,\rho,\sigma,\lambda} \Gamma_{\nu,\rho,\sigma,\lambda}. \quad (5.1.64)
\]

This formula together with the corrected gauge transformation (5.1.63) solves completely Noether’s equation (5.1.53).

### 5.1.3 2s-s-s interaction Lagrangian

Now we turn to the generalization of the Noether procedure of the 2-2-4 case to the general s-s-2s interaction construction. So we must propose a first order variation of the spin s field with respect to a spin 2s gauge transformation. Remembering that Fronsdal’s higher spin gauge potential is double traceless, we must make sure that the same holds for the variation. Expanding the general variation in powers of \(a\)

\[
\delta h^{(s)}(a) = \delta h^{(s)}_{(1)}(a) + a^2 \delta h^{(s-2)}(a) + (a^2)^2 \delta h^{(s-4)}(a) + \ldots, \quad (5.1.65)
\]

we see that the double tracelessness condition \(\Box^2 a \delta h^{(s)}(a) = 0\) expresses the third and higher terms of the expansion (5.1.65) through the first two free parameters \(\delta h^{(s)}_{(1)}(a)\) and \(\delta h^{(s-2)}(a)\). From the other hand Fronsdal’s tensor is double traceless by definition and therefore all these \(O(a^4)\) terms are unimportant because they do not contribute to (2.1.39). This leaves us freedom in the choice of \(\delta h^{(s-2)}(a)\). Substituting (5.1.65) in (2.1.39) we discover that the following choice of \(\delta h^{(s-2)}(a)\)

\[
\delta h^{(s-2)}(a) = \frac{1}{2(D + 2s - 2)} \Box a \delta h^{(s)}_{(1)}(a), \quad (5.1.66)
\]

reduces our variation (2.1.39) to

\[
\delta_{(1)} \mathcal{L}_0(h^{(s)}(a)) = -\mathcal{F}^{(s)}(a) *_a \delta h^{(s)}_{(1)}(a). \quad (5.1.67)
\]

\(^1\)For completeness we present here the solution for \(\delta h^{(s-2)}(a)\) following from the double tracelessness condition

\[
\delta h^{(s-2)}(a) = -\frac{1}{8\alpha_1\alpha_2} \left[ \Box^2 a \delta h^{(s)}_{(1)}(a) + 4\alpha_1 \Box a \delta h^{(s-2)}(a) \right],
\]

\(\alpha_k = D + 2s - (4 + 2k), \quad k \in \{1, 2\}.\)
Then we propose the following spin 2s transformation of the spin s potential
\[ \delta h^{(s)}_{(1)}(a) = \tilde{U}(b, a, 2, s) e^{2s-1}(z; b) *_b \Gamma^{(s)}(z; b, a), \quad (5.1.68) \]
where
\[ \tilde{U}(b, a, 2, s) = \frac{(-1)^s}{(s - 1)!} \prod_{k=2}^{s} \left[ (\nabla \partial_b) - \frac{1}{k} (a \partial_b)(\nabla \partial_a) \right], \quad (5.1.69) \]
is operator dual to
\[ [(b \nabla) \frac{1}{2} (a \nabla)(b \partial_a)] U(b, a, 3, s) = \prod_{k=2}^{s} [(b \nabla) \frac{1}{2} (a \nabla)(b \partial_a)], \quad (5.1.70) \]
with respect to the \(*_{a,b}\) contraction product. Taking into account \((2.1.30)\) and Bianchi identities \((2.1.36)\) we get
\[ \delta(1) \mathcal{L}_0(h^{(s)}(a)) = \varepsilon^{2s-1}(z; b) *_b \Gamma^{(s)}(z; b, a) *_a [(b \nabla) \frac{1}{2} (a \nabla)(b \partial_a)] U(b, a, 3, s) \mathcal{F}^{(s)}(z; a) \]
\[ = \varepsilon^{2s-1}(z; b) *_b \Gamma^{(s)}(z; b, a) *_a \frac{1}{s(s - 1)} [(b \nabla) \frac{1}{2} (a \nabla)(b \partial_a)] \Box_b \Gamma^{(s)}(z; b, a) \]
\[ = \varepsilon^{2s-1}(z; b) *_b \Gamma^{(s)}(z; b, a) *_a \frac{1}{s} (\nabla \partial_b) \Gamma^{(s)}(z; b, a) \]
\[ = -(b \nabla) \varepsilon^{2s-1}(b) *_b \Gamma^{(s)}(b, a) *_a \Gamma^{(s)}(b, a) - \varepsilon^{2s-1}(b) *_b \nabla_\mu \Gamma^{(s)}(b, a) *_a \frac{1}{s} \partial_\mu \Gamma^{(s)}(b, a). \quad (5.1.71) \]

Then using a secondary Bianchi identity \((2.1.35)\) and a primary one \((2.1.15)\) one can show that
\[ -\varepsilon^{2s-1}(b) *_b \nabla_\mu \Gamma^{(s)}(b, a) *_a \frac{1}{s} \partial_\mu \Gamma^{(s)}(b, a) \]
\[ = \frac{1}{2s(s + 1)(2s - 1)} (\nabla \partial_b) \varepsilon^{2s-1}(b) *_b \partial_\mu \Gamma^{(s)}(b, a) *_a \partial_\mu \Gamma^{(s)}(b, a). \quad (5.1.72) \]

Putting all together we see that the integrated first order interaction Lagrangian
(with generalized Bell-Robinson current \([40]\))
\[ \mathcal{L}_1(h^{(s)}(a), h^{(2s)}(b)) = \frac{1}{2s} h^{(2s)}(z; b) *_b \Psi^{(2s)}(\Gamma)(z; b), \quad (5.1.73) \]
\[ \Psi^{(2s)}(\Gamma)(z; b) = \Gamma^{(s)}(b, a) *_a \Gamma^{(s)}(b, a) - \frac{b^2}{2(s + 1)} \partial_\mu \Gamma^{(s)}(b, a) *_a \partial_\mu \Gamma^{(s)}(b, a). \quad (5.1.74) \]

supplemented with transformation \((5.1.68)\) for \(h^{(s)}(a)\) and the standard zero order for \(h^{(2s)}(a)\)
\[ \delta_0 h^{(2s)}(z; b) = 2s(b \nabla) e^{2s-1}(z; b), \quad (5.1.75) \]
\[ \delta_0 \Box_b h^{(2s)}(z; b) = 4s(\nabla \partial_b) e^{2s-1}(z; b), \quad (5.1.76) \]
completely solves Noether’s equation

\[ \delta_1 \mathcal{L}_0(h^{(s)}(a)) + \delta_0 \mathcal{L}_1(h^{(s)}(a), h^{(2s)}(b)) = 0. \]  

(5.1.77)

Note that here just as in the 2-2-4 case we did not obtain an interaction with lower spins because all derivatives included in the ansatz were used for the lifting to the second curvature.

5.2 Cubic selfinteraction for Higher Spin gauge fields

5.2.1 Higher spin gauge field selfinteraction: The Noether’s procedure

Here we present again Fronsdal’s Lagrangian

\[ \mathcal{L}_0(h^{(s)}(a)) = -\frac{1}{2} h^{(s)}(a) \star_a \mathcal{F}^{(s)}(a) + \frac{1}{8s(s-1)} \Box_a h^{(s)}(a) \star_a \Box_a \mathcal{F}^{(s)}(a), \]  

(5.2.1)

where \( \mathcal{F}^{(s)}(z; a) \) is the so-called Fronsdal tensor

\[ \mathcal{F}^{(s)}(z; a) = \Box h^{(s)}(z; a) - s(a \nabla)D^{(s-1)}(z; a), \]  

(5.2.2)

and \( D^{(s-1)}(z; a) \) is the so-called de Donder tensor or traceless divergence of the higher spin gauge field

\[ D^{(s-1)}(z; a) = \text{Div} h^{(s-1)}(z; a) - \frac{s - 1}{2} (a \nabla) \text{Tr} h^{(s-2)}(z; a), \]  

(5.2.3)

\[ \Box_a D^{(s-1)}(z; a) = 0. \]  

(5.2.4)

The initial gauge variation of a spin \( s \) field that is of field order zero is

\[ \delta_{(0)} h^{(s)}(z; a) = s(a \nabla) \epsilon^{(s-1)}(z; a), \]  

(5.2.5)

with the traceless gauge parameter

\[ \Box_a \epsilon^{(s-1)}(z; a) = 0, \]  

(5.2.6)

for the by definition double traceless gauge field

\[ \Box^2_a h^{(s)}(z; a) = 0. \]  

(5.2.7)

Therefore on this level we can see from (5.2.5) and (5.2.6) that a correct generalization of the Lorentz gauge condition in the case of \( s > 2 \) could be only the so-called de Donder gauge condition

\[ D^{(s-1)}(z; a) = 0. \]  

(5.2.8)
The equation of motion following from (5.2.1) is

\[ \delta L_0 (h^{(s)}(a)) = - (\mathcal{F}^{(s)}(a) - \frac{a^2}{4} \Box_a \mathcal{F}^{(s)}(a)) \ast_a \delta h^{(s)}(a), \tag{5.2.9} \]

and zero order gauge invariance (when \( \delta h^{(s)}(a) = \delta^{(0)} h^{(s)}(a) \)) can be checked by substitution of (5.2.5) into this variation and use of the duality relation (2.1.8) and identity (2.1.37) taking into account tracelessness of the gauge parameter (5.2.6).

Now we turn to the formulation of Noether’s general procedure for constructing the spin \( s \) cubic selfinteraction. Similar to [40] Noether’s equation in this case looks like

\[ \delta(1) L_0 (h^{(s)}(a)) + \delta_0 L_1 (h^{(s)}(a)) = 0, \tag{5.2.10} \]

where \( L_1 (h^{(s)}(a)) \) is a cubic interaction Lagrangian and \( \delta(1) h^{(s)}(a) \) is a gauge transformation that is of first order in the gauge field. Actually equation (5.2.10) just expresses in the cubic order on the field the general gauge invariance

\[ \delta L (h^{(s)}(a)) = \frac{\delta \mathcal{L}(h^{(s)}(a))}{\delta h^{(s)}(a)} \ast_a \delta h^{(s)}(a) = 0, \tag{5.2.11} \]

where

\begin{align*}
\mathcal{L}(h^{(s)}(a)) &= L_0(h^{(s)}(a)) + L_1(h^{(s)}(a)) + \ldots, \tag{5.2.12} \\
\delta h^{(s)}(a) &= \delta^{(0)} h^{(s)}(a) + \delta(1) h^{(s)}(a) + \ldots. \tag{5.2.13}
\end{align*}

Combining (5.2.9) and (5.2.10) we obtain the following functional Noether’s equation

\[ \delta^{(0)} L_1 (h^{(s)}(a)) = (\mathcal{F}^{(s)}(a) - \frac{a^2}{4} \Box_a \mathcal{F}^{(s)}(a)) \ast_a \delta(1) h^{(s)}(a), \tag{5.2.14} \]

and we would like to present in this section the solution of the latter equation for the case \( s = 4 \) and propose a generalization for any even \( s \).

First we investigate a first order variation of the spin \( s \) gauge transformation. Remembering that Fronsdal’s higher spin gauge potential has scaling dimension \( \Delta_s = s - 2 \) (zero for the \( s = 2 \) graviton case) and ascribing the same dimensions to the free part of the Lagrangian that is quadratic in the fields and derivatives \( L_0(h^{(s)}(a)) \) and to the interaction \( L_1(h^{(s)}(a)) \) cubic in the fields, we arrive at the idea that the number of derivatives in the interaction should be \( s \). This type of interacting theories will behave in the same way as gravity. Then we can easily conclude from (5.2.10) that the number of derivatives in the first order variation \( \delta(1) h^{(s)}(a) \) should be \( s - 1 \). For \( s = 2 \) this consideration is of course in full agreement with the linearized expansion of the Einstein-Hilbert action.

\[^{\text{†}}\text{From now on we will admit integration everywhere where it is necessary (we work with a Lagrangian as with an action) and therefore we will neglect all } d \text{ dimensional space-time total derivatives when making a partial integration.}\]
The next observation is connected with double tracelessness of Frønsdal’s higher spin gauge potential. This means that we must make sure that the same holds for the variation. Expanding the general variation in powers of $a^2$

$$\delta_{(1)}h^{(s)}(a) = \delta_{(1)}\tilde{h}^{(s)}(a) + a^2\delta_{(1)}h^{(s-2)}(a) + (a^2)^2\delta h^{(s-4)}(a) + \ldots,$$  \hspace{1cm} (5.2.15)

we see that the double tracelessness condition $\Box^2_\alpha \delta h^{(s)}(a) = 0$ expresses the third and higher terms of the expansion (5.2.15) through the first two free parameters $\delta_{(1)}h^{(s)}(a)$ and $\delta_{(1)}h^{(s-2)}(a)$. From the other hand Frønsdal’s tensor (and the r.h.s of (5.2.14)) is double traceless by definition and therefore all these $O(a^4)$ terms are unimportant because they do not contribute to (5.2.14). This leaves us freedom in the choice of initial $\delta_{(1)}h^{(s-2)}(a)$. Using this freedom we can shift the initial first order variation in the following way

$$\delta_{(1)}h^{(s)}(a) \Rightarrow \delta_{(1)}h^{(s)}(a) + \frac{a^2}{2(D + 2s - 2)} \Box_\alpha \delta h^{(s)}_{(1)}(a),$$  \hspace{1cm} (5.2.16)

and discover that (5.2.14) reduces to

$$\delta_{(0)}L_1(h^{(s)}(a)) = F^{(s)}(a) \ast_{\alpha} \delta h^{(s)}_{(1)}(a).$$  \hspace{1cm} (5.2.17)

Now to solve this equation we can formulate the following strategy:

1) First we can start from any cubic ansatz with $s$ derivatives $L_1(h^{(s)}(a))$ suitable in respect to the zero order variation (5.2.5) and variate it inserting in the l.h.s. of (5.2.17).

2) Then we make a partial integration and rearrange indices to extract an integrable part due to terms proportional to Frønsdal’s tensor $F^{(s)}(a)$ (or $TrF^{(s)}(a)$) in agreement with the r.h.s. of (5.2.17).

3) Symmetrizing expressions in this way we classify terms as

- integrable
- integrable and subjected to field redefinition (proportional to Frønsdal’s tensor)
- non integrable but reducible by deformation of the initial ansatz for the gauge transformation (again proportional to Frønsdal’s tensor)

\footnote{For completeness we present here the solution for $\delta h^{(s-4)}(a)$ following from the double tracelessness condition

$$\delta h^{(s-4)}(a) = -\frac{1}{8\alpha_1\alpha_2} \left[ \Box^2_\alpha \delta h^{(s)}_{(1)}(a) + 4\alpha_1 \Box_\alpha \delta h^{(s-2)}(a) \right],$$  \hspace{1cm} \alpha_k = D + 2s - (4 + 2k), \hspace{0.5cm} k \in \{1, 2\}.}
CHAPTER 5. CUBIC INTERACTIONS OF HSF

Then if no other terms remain we can construct our interaction together with the corrected first order transformation. Following this strategy we will consider the \( s = 2 \) and \( s = 4 \) cases in the next subsections in detail. The exact and unique results after field redefinition and partial integration that are presented in the next two subsections are in full agreement with the prediction for general even spin \( s \). To formulate this prediction let us first introduce a classification of cubic monoms with \( s \) derivatives. We will call leading terms all those monoms without traces and divergences or equivalently without \( \tilde{h}^{(s-2)} = Tr : h^{(s)} \) and \( D^{(s-1)} \), where the derivatives are contracted only with gauge fields and not with other derivatives. This type of terms is interesting because any partial integration will map such term to the terms of the same type and create one additional term with a divergence, which we can map to \( D \) dependent and trace dependent terms.

Another important point of this class of monoms is that inside of this class we have the following important term involving the linearized Freedman-de Witt gauge invariant curvature [27, 85]

\[
\mathcal{L}_{1}^{initial}(h^{(s)}(a)) = \frac{1}{2s}h^{(s)}(b) \ast_{b} \Gamma^{(s)}(b, a) *_{a} h^{(s)}(a), \quad (5.2.18)
\]

\[
\Gamma^{(s)}(z; b, a) = \sum_{k=0}^{s} \frac{(-1)^{k}}{k!} (b\nabla)^{s-k}(a\nabla)^{k}(b\partial_{a})^{k}h^{(s)}(z; a). \quad (5.2.19)
\]

This term we can use (and we used it in the case \( s=4 \)) as an initial ansatz for the solution of \( (5.2.17) \). Using \( (2.1.25) \) and \( (2.1.36) \) we see that

\[
\delta_{(0)}\mathcal{L}_{1}^{initial}(h^{(s)}(a)) = -\epsilon^{(s-1)}(z; b)(b\nabla)h^{(s)}(a) *_{a} \ast_{b} \Gamma^{(s)}(b, a) + O(F^{(s)}). \quad (5.2.20)
\]

It is easy to see from \( (5.2.19) \) that after variation in the r.h.s. of \( (5.2.20) \) we get \( s+1 \) monoms linear on the gauge parameter \( \epsilon^{(s-1)}(z; b) \) and quadratic in the gauge field, where some of them contain two factors \( (b\nabla) \) of contracted derivatives. These terms we can separate as next level terms including the de Donder tensor \( D^{(s-1)}(z; b) \). To prove this statement we note first that due to partial integration there is the following simple formula:

\[
F(z)\nabla_{\mu}G(z)\nabla^{\mu}H(z) = \frac{1}{2}(\square F(z)G(z)H(z) - F(z)\square G(z)H(z) - F(z)G(z)\square H(z)). \quad (5.2.21)
\]

The objects \( F(z), G(z), H(z) \) in our case are proportional to \( h^{(s)}(z; a) \) or \( \epsilon^{(s-1)}(z; a) \). Then using the definition of Fronsdal’s operator \( (5.2.22) \) and from \( (5.2.3) \) and \( (5.2.5) \) follows the transformation rule

\[
\delta_{(0)}D^{(s-1)}(z; a) = \square \epsilon^{(s-1)}(z; a). \quad (5.2.22)
\]

This implies that we can classify all terms with contracted derivatives (i.e. terms with Laplacians) as monoms containing \( D^{(s-1)}(z; a) \) or \( \delta_{(0)}D^{(s-1)}(z; a) \) which therefore vanish in the de Donder gauge. Actually according to the r.h.s of
we can during functional integration always replace any $\Box h^{(s)}(a)$ with $\mathcal{F}^{(s)}(a) + s(a\nabla)D^{(s-1)}(a)$ obtaining a contribution to $\delta_{(1)}$ and shifting this monom to the next level class comprising one more order of the de Donder tensor.

Operating in this way we can integrate Noether’s equation (5.2.17) (or equivalently express the r.h.s. of (5.2.20) as $-\delta_{(0)}\mathcal{L}_{cubic}^{(1)}(h^{(s)}) + O(\mathcal{F}^{(s)})$) using the initial ansatz (5.2.20) step by step: integrating first the leading terms without any de Donder tensor or trace, then integrate terms involving only traces but not $D^{(s-1)}(z; a)$. That is the solution in de Donder gauge. After that we can continue the integration and obtain terms linear on $D^{(s-1)}(z; a)$, quadratic and so on. The procedure will be closed when we obtain a sufficient number of $D^{(s-1)}(z; a)$ to stop the production of terms with contracted derivatives and therefore the production of new level terms coming from formula (5.2.21).

Collecting the leading terms and rearranging by partial integration derivatives in a cyclic way so that each derivative acting on a tensor gauge field is contracted with the preceding tensor we finally come to the following prediction for the leading terms of the interaction for a general spin $s$ gauge field:

$$\mathcal{L}_{\text{leading}}^{(1)}(h^{(s)}(z)) = \frac{1}{3s(s!)^3} \sum_{\alpha+\beta+\gamma=s}^s \left(\begin{array}{c} s \\ \alpha, \beta, \gamma \end{array}\right) \int_{z_1, z_2, z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3)$$

$$\left[(\nabla_1\partial_\alpha)^\gamma(\nabla_2\partial_\alpha)^\beta(\nabla_3\partial_\alpha)^\gamma(\partial_\alpha\partial_\beta)^\alpha(\partial_\beta\partial_\gamma)^\beta(\partial_\gamma\partial_\alpha)^\gamma\right] h(\alpha; z_1)h(\beta; z_2)h(\gamma; z_3), \quad (5.2.23)$$

where the relative coefficients between monoms are trinomial coefficients:

$$\left(\begin{array}{c} s \\ \alpha, \beta, \gamma \end{array}\right) = \frac{s!}{\alpha!\beta!\gamma!}, \quad s = \alpha + \beta + \gamma. \quad (5.2.24)$$

Correspondingly the leading term of the first order gauge transformation should be

$$\delta_{\text{leading}}^{(1)}h^{(s)}(c; z) = \frac{1}{s!(s-1)!} \sum_{\alpha+\beta+\gamma=s} (-1)^\beta \left(\begin{array}{c} s-1 \\ \alpha-1, \beta, \gamma \end{array}\right) \int_{z_1, z_2} \delta(z - z_1)\delta(z - z_2)$$

$$\left[(c\nabla_1)^\gamma(c\nabla_2)^\beta(c\nabla_3)^\gamma(\partial_\alpha\partial_\beta)^\alpha(\partial_\beta\partial_\gamma)^\beta(\partial_\gamma\partial_\alpha)^\gamma\right] \epsilon(\alpha; z_1)h(\beta; z_2). \quad (5.2.25)$$

Splitting the trinomial into two binomials we can rewrite this expression in a more elegant way

$$\delta_{\text{leading}}^{(1)}h^{(s)}(c; z) = \frac{1}{s!} \sum_{k=0}^{s-1} k! \left(\begin{array}{c} s-1 \\ k \end{array}\right) \gamma^{(k)}_{(s-1)}(c, b; a)^{*a,b}, (a\nabla)^{s-k-1}(c\partial_\alpha)^{s-k}h^{(s)}(b) \quad (5.2.26)$$

where

$$\gamma^{(k)}_{(s-1)}(c, b; a) = \frac{k!}{(s-1)!} \sum_{i=0}^k (-1)^i \left(\begin{array}{c} k \\ i \end{array}\right) (c\nabla)^{k-i}(b\nabla)^i(c\partial_\alpha)^{i} \left[(a\partial_\alpha)^{s-1-k}\epsilon^{(s-1)}(b)\right]. \quad (5.2.27)$$
Comparing with (5.2.19) we see that
\[ \gamma_{(s-1)}^{(k)}(c, b, a) = \Gamma^{(k)}(c, b; h_{(a)}^{(k)}(b)), \] (5.2.28)
where
\[ h_{a}^{(k)}(b) = \frac{k!}{(s-1)!} \left[ (a\partial_{b})^{s-1-k} e^{(s-1)}(b) \right], \] (5.2.29)
and therefore the \( \gamma_{(s-1)}^{(k)}(c, b; a) \) coefficients inherit in the \( c, b \) index spaces all properties of the corresponding spin \( k \) curvature described in details in Section 2.1. In the next two sections we show for the \( s = 2, 4 \) cases that fixing the leading terms by partial integration and field redefinition leads to the unique solution of Noether’s equation (5.2.17).

5.2.2 Cubic selfinteraction and Noether’s procedure, the spin two example

Using our general basis for the spin 2 case
\[ h_{\mu\nu}, \] (5.2.30)
\[ D_\mu = (\nabla h)_\mu - \frac{1}{2} \nabla_\mu h, \] (5.2.31)
\[ h = h_\mu^\mu, \] (5.2.32)
we can rewrite the free Fronsdal (linearized Einstein-Hilbert gravity) Lagrangian for the spin two gauge field in the following way:
\[ L_0 = -\frac{1}{2} h^{\mu\nu}(\Box h_{\mu\nu} - 2\nabla_{(\mu}D_{\nu)}) + \frac{1}{4} h(\Box h - 2(\nabla D)), \] (5.2.33)
\[ (\nabla D) = \nabla_\mu D_\mu. \] (5.2.34)
This action is invariant with respect to the zero order gauge transformation
\[ \delta_{(0)} h_{\mu\nu} = 2\nabla_{(\mu} e_{\nu)}. \] (5.2.35)

According to our strategy described in the previous section we obtain the following cubic interaction Lagrangian
\[ L_1(h^{(2)}) = \frac{1}{2} h^{\alpha\beta}\nabla_\alpha \nabla_\beta h_{\mu\nu} h^{\mu\nu} + h^{\alpha\mu}\nabla_\alpha h^{\beta\nu}\nabla_\beta h_{\mu\nu} \]
\[-\frac{1}{4}(\nabla D)h_{\mu\nu} h^{\mu\nu} -\frac{1}{2} h^{\mu\nu}\nabla_\mu h D_\nu, \] (5.2.36)
supplemented with the Lie derivative form of the first order transformation law
\[ \delta_{(1)} h_{\mu\nu} = e^\rho \nabla_\rho h_{\mu\nu} + 2\nabla_{(\mu} e^\rho h_{\nu)\rho}, \] (5.2.37)
and the following field redefinition leading to this minimized form of Lagrangian (5.2.36)

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{1}{4}(hh_{\mu\nu} - 2h^\rho_{\mu}h_{\nu\rho} - \frac{1}{2(D-2)}h^2g_{\mu\nu}), \]  

(5.2.38)

Note that the interaction Lagrangian in de Donder gauge

\[ D_\mu = 0, \]  

(5.2.39)

reduces to the first two leading terms of (5.2.36). This minimized form of the leading terms is equivalent to the expansion up to cubic terms of the Einstein-Hilbert action (see [40]) after partial integration and field redefinition, and is in full agreement with (5.2.23) for \( s = 2 \).

To see the same for the first order transformation law (5.2.37) and (5.2.26) we note that the second term in the (5.2.37) can be written in the form involving the vector curvature \( \gamma^{(1)}_{\mu\nu} = 2\nabla_{[\mu}\varepsilon_{\nu]} \) and the additional field redefinition

\[ (\nabla_{(\mu}\varepsilon^\rho - \nabla^\rho\varepsilon_{(\mu})h_{\nu)\rho} + (\nabla_{(\mu}\varepsilon^\rho + \nabla^\rho\varepsilon_{(\mu})h_{\nu)\rho} = (\nabla_{(\mu}\varepsilon^\rho - \nabla^\rho\varepsilon_{(\mu})h_{\nu)\rho} + \frac{1}{2}\delta_\varepsilon(h_{(\mu}^\rho h_{\nu)\rho}). \]  

(5.2.40)

Consequently the first order gauge variation becomes

\[ \delta^{(1)}h_{\mu\nu} = \varepsilon^\rho\nabla_\rho h_{\mu\nu} + \gamma^{(1)}_{(\mu}^\rho h_{\nu)\rho}, \]  

(5.2.41)

and the field redefinition (5.2.38) reduces to

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{1}{4}(hh_{\mu\nu} - \frac{1}{2(D-2)}h^2g_{\mu\nu}). \]  

(5.2.42)

5.2.3 The cubic selfinteraction for spin four

We start this nontrivial case by introducing the free Fronsdal's Lagrangian for the spin four gauge field \( h_{\alpha\beta\gamma\delta} \)

\[ \mathcal{L}_0(h^{(4)}) = -\frac{1}{2}h^{\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta} + \frac{3}{2}\bar{F}_{\alpha\beta}, \]  

(5.2.43)

\[ F_{\alpha\beta\gamma\delta} = \Box h_{\alpha\beta\gamma\delta} - 4\nabla_{(\alpha}D_{\beta\gamma\delta)}, \]  

(5.2.44)

\[ \bar{F}_{\alpha\beta} = F_{\gamma\alpha\beta} = \Box h_{\alpha\beta} - 2(\nabla D)_{\alpha\beta}, \]  

(5.2.45)

which is invariant under

\[ \delta_0(h_{\alpha\beta\gamma\delta}) = 4\nabla_{(\alpha}\varepsilon_{\beta\gamma\delta)}, \]  

(5.2.46)
where we defined the de Donder tensor and the trace of the gauge field by

\[ D_{\alpha\beta\gamma} = (\nabla h)_{\alpha\beta\gamma} - \frac{3}{2} \nabla (\alpha h_{\beta\gamma}), \]

\[ h_{\beta\gamma} = h_{\beta\gamma\alpha}, \]

\[ D_{\alpha\beta} = 0, \quad \bar{h}_{\beta} = 0. \]

(5.2.47) (5.2.48) (5.2.49)

The spin four case is much more complicated than the spin two case and includes all difficulties and complexities of a general spin s interaction but remains inside the domain of problems which one can handle analytically. To apply our strategy and integrate the corresponding Noether’s equation completely we have to introduce the following table to classify terms and levels of the interaction Lagrangian.

| $D$ | $h$ | 0 | 1 | 2 |
|-----|-----|---|---|---|
| 0   | $hhh$ | $Dhh$ | $DDh$ |
| 1   | $\bar{hhh}$ | $\bar{h}Dh$ | $\bar{h}DD$ |
| 2   | $\bar{hh}h$ | $\bar{hh}D$ |
| 3   | $\bar{h}h\bar{h}$ |

(5.2.50)

This table introduces some "coordinate system" for classification of our interaction

\[ \mathcal{L}_1 = \sum_{i,j=0,1,2,3} \mathcal{L}_{ij}^{\text{int}}(h^{(4)}), \]

(5.2.51)

where

\[ \mathcal{L}_{ij}^{\text{int}}(h^{(4)}) \sim \nabla^{4-i}(D)^i(\bar{h}^{(4)})^j(h^{(4)})^{3-j-i}. \]

(5.2.52)

In this notation the leading term described in the second section is $\mathcal{L}_{00}^{\text{int}}(h^{(4)})$. On the other hand the first column of table (5.2.50) is nothing else but the interaction
Lagrangian in de Donder gauge $D_{\alpha\gamma} = 0$ and can be expressed as a sum

$$\mathcal{L}_{\text{int}}^{\text{dD}}(h^{(4)}) = \sum_{j=0}^{3} \mathcal{L}_{0j}^{\text{int}}(h^{(4)}).$$  \hfill (5.2.53)

Integrating Noether's equation step by step (cell by cell in means of (5.2.50)) starting from the initial curvature ansatz (5.2.18), we obtain after very long and tedious calculations the following cubic interaction Lagrangian:

$$\mathcal{L}_{00}^{\text{int}}(h^{(4)}) = \frac{1}{8} h^{\alpha\beta\gamma\delta} h^{\mu\nu\lambda\rho} \Gamma_{\alpha\beta\gamma\delta,\mu\nu\lambda\rho}^{(4)} - \nabla_{\mu} h_{\alpha\beta\gamma\delta} \nabla^{\alpha} h^{\gamma\nu\lambda\rho} \nabla^{\delta} h_{\mu\nu\lambda\rho} + \frac{3}{4} \nabla_{\mu} h_{\alpha\beta\gamma\delta} \nabla^{\alpha} h^{\gamma\delta\lambda\rho} \nabla^{\beta} h_{\mu\nu\lambda\rho} + 3 \nabla_{\mu} \nabla_{\nu} h_{\alpha\beta\gamma\delta} h^{\alpha\nu\lambda\rho} \nabla^{\beta} \nabla^{\gamma} h_{\mu\nu\lambda\rho},$$  \hfill (5.2.54)

$$\mathcal{L}_{01}^{\text{int}}(h^{(4)}) = -\frac{3}{2} h_{\alpha\beta\gamma\delta} \nabla^{\alpha} \nabla^{\beta} h^{\gamma\nu\lambda\rho} \nabla^{\delta} \nabla_{\nu} \bar{h}_{\lambda\rho} - 3 h_{\alpha\beta\gamma\delta} h_{\mu\nu\lambda\rho} \nabla^{\delta} \nabla^{\beta} \nabla^{\gamma} \nabla^{\lambda} \bar{h}_{\mu\nu},$$  \hfill (5.2.55)

$$\mathcal{L}_{02}^{\text{int}}(h^{(4)}) = -\frac{3}{2} \nabla_{\mu} \nabla_{\nu} h_{\alpha\beta\gamma\delta} \nabla^{\alpha} \bar{h}_{\lambda\rho} \nabla^{\beta} \nabla^{\gamma} \nabla^{\delta} \bar{h}_{\mu\nu} + \frac{3}{4} h_{\alpha\beta\gamma\delta} \nabla^{\alpha} \nabla^{\mu} \bar{h}_{\beta\nu} \nabla^{\gamma} \nabla^{\delta} \bar{h}_{\mu\nu} - \frac{3}{4} \nabla_{\mu} \nabla_{\nu} h_{\alpha\beta\gamma\delta} \nabla^{\alpha} \bar{h}_{\lambda\rho} \nabla^{\beta} \nabla^{\gamma} \nabla^{\delta} \bar{h}_{\mu\nu},$$  \hfill (5.2.56)

$$\mathcal{L}_{03}^{\text{int}}(h^{(4)}) = 3 \nabla_{\mu} \nabla_{\nu} \bar{h}_{\alpha\beta} \nabla^{\alpha} h^{\mu\lambda} \nabla^{\beta} \bar{h}_{\nu\lambda} - \frac{3}{4} \nabla_{\mu} \bar{h}_{\nu\alpha} \nabla^{\mu} \bar{h}_{\nu\lambda}(\nabla_{\nu} \bar{h}),$$  \hfill (5.2.57)

$$\mathcal{L}_{10}^{\text{int}}(h^{(4)}) = 3 \nabla_{\alpha} \nabla_{\nu} D_{\lambda\rho\beta} h^{\alpha\beta\gamma\delta} \nabla_{\gamma} h_{\nu\lambda\rho\delta} + \frac{3}{2} \nabla^{\alpha} D_{\alpha\beta\lambda} \nabla^{\mu} h^{\alpha\beta\gamma\delta} \nabla^{\lambda} h_{\mu\nu\gamma\delta} - 2 \nabla^{\delta} D_{\nu\lambda\rho\beta} \nabla^{\nu} h_{\alpha\beta\gamma\delta} \nabla^{\lambda} h_{\mu\nu\gamma\delta},$$  \hfill (5.2.58)
\[ \mathcal{L}_{11}^{\text{int}}(h^{(4)}) = \]
\[ - \frac{1}{2} \bar{h}^{\gamma \delta} \nabla_{\gamma} h_{\mu \nu \lambda \rho} \nabla_{\delta} \nabla^{\mu} D^{\nu \lambda \rho} + \frac{1}{2} \bar{h}^{\gamma \delta} \nabla_{\gamma} h_{\mu \nu \lambda \rho} \nabla^{\mu} D^{\nu \lambda \rho} \]
\[ + \frac{3}{4} \nabla^{\mu} \bar{h}^{\gamma \delta} h_{\mu \nu \lambda \rho} \nabla_{\gamma} D_{\delta}^{\nu \lambda \rho} - \frac{3}{4} \nabla_{\mu} \bar{h}^{\gamma \delta} h_{\mu \nu \lambda \rho} \nabla_{\gamma} D_{\delta}^{\nu \lambda \rho} \]
\[ + \frac{9}{4} \nabla_{\mu} \bar{h}^{\gamma \delta} \nabla_{\rho} h_{\gamma \delta \nu \lambda} \nabla^{\lambda} D^{\mu \nu \rho} + \frac{3}{4} \bar{h}^{\gamma \delta} \nabla_{\rho} h_{\gamma \mu \nu \lambda} \nabla^{\mu} D_{\delta}^{\nu \lambda \rho} \]
\[ + \frac{3}{4} \bar{h}^{\gamma \delta} \nabla_{\rho} h_{\gamma \mu \nu \lambda} \nabla^{\mu} D_{\delta}^{\nu \lambda \rho} - \frac{3}{4} \bar{h}^{\gamma \delta} \nabla_{\rho} h_{\gamma \mu \nu \lambda} \nabla^{\mu} D_{\delta}^{\nu \lambda \rho} \]
\[ - \frac{3}{4} (\nabla D)^{\gamma \delta} \nabla_{\gamma} h_{\mu \nu \rho} - \frac{3}{4} D^{\mu \nu \rho} (\nabla D)^{\gamma \delta} \nabla_{\gamma} h_{\mu \nu \rho} + \frac{1}{4} \bar{h}^{\gamma \delta} \nabla D^{\mu \nu \rho} \nabla_{\gamma} h_{\mu \nu \rho} \]
\[ + 6 \nabla^{\mu} \nabla_{\nu} (\nabla D)^{\gamma \delta} h_{\gamma \lambda} h_{\delta \mu \nu \lambda} + \frac{1}{4} \bar{h}^{\gamma \delta} \nabla D^{\mu \nu \rho} D_{\mu} h_{\gamma \delta \mu \nu \rho} \]
\[ - \frac{3}{8} \bar{h}^{\gamma \delta} \nabla D^{\mu \nu \rho} \nabla_{\mu} h_{\gamma \delta \mu \nu \rho}, \quad (5.2.59) \]

\[ \mathcal{L}_{12}^{\text{int}}(h^{(4)}) = \]
\[ \frac{3}{4} D^{\mu \nu \rho} (\nabla \bar{h})^{\delta} \nabla_{\delta} \nabla_{\mu} h_{\nu \rho} - \frac{9}{8} \nabla_{\gamma} \nabla_{\delta} D^{\mu \nu \rho} \nabla^{\gamma \delta} \nabla_{\mu} h_{\nu \rho} \]
\[ - 3 D^{\mu \nu \rho} \bar{h}^{\gamma \delta} \nabla_{\gamma} \nabla_{\nu} \nabla_{\rho} h_{\delta \mu} - 3 \nabla_{\mu} D_{\nu \rho \gamma} \nabla^{\nu \rho} \bar{h}^{\gamma \delta} \nabla_{\delta} h_{\mu \nu \rho} \]
\[ - \frac{3}{2} (\nabla D)^{\gamma \delta} \nabla_{\gamma} \bar{h}_{\mu \nu \rho} \nabla_{\delta} h_{\mu \nu} - \frac{3}{8} (\nabla D)^{\gamma \delta} \bar{h}_{\mu \nu \rho} \nabla_{\gamma} \nabla_{\delta} h_{\mu \nu} \]
\[ + \frac{3}{2} \nabla^{\mu} (\nabla D)^{\gamma \delta} \nabla_{\gamma} \bar{h}_{\mu \nu \rho} \nabla_{\delta} h_{\mu} + \frac{3}{2} \nabla^{\mu} (\nabla D)^{\gamma \delta} \nabla_{\gamma} \bar{h}_{\mu \nu \rho} \nabla_{\delta} h_{\mu} \]
\[ + \frac{9}{4} (\nabla D)^{\gamma \delta} \nabla_{\gamma} \bar{h}_{\mu \nu \rho} \nabla_{\delta} h_{\mu} + \frac{3}{2} \nabla^{\mu} (\nabla D)^{\gamma \delta} \nabla_{\gamma} \bar{h}_{\mu \nu \rho} \nabla_{\delta} h_{\mu} \]
\[ + \frac{3}{8} \nabla^{\mu} \nabla^{\nu} (\nabla D)^{\gamma \delta} \bar{h}_{\gamma \delta \mu \nu \rho} + \frac{3}{8} \nabla^{\mu} (\nabla D)^{\gamma \delta} \bar{h}_{\gamma \delta \mu \nu \rho} \]
\[ - \frac{3}{2} \nabla^{\mu} (\nabla D)^{\gamma \delta} \bar{h}_{\gamma \delta \mu \nu \rho} \nabla_{\mu} h_{\delta \mu \nu \rho}, \quad (5.2.60) \]

\[ \mathcal{L}_{20}^{\text{int}}(h^{(4)}) = \]
\[ 3 D^{\gamma \delta} \nabla_{\mu} \nabla_{\nu} h_{\rho \alpha \beta \gamma} - \frac{9}{4} D^{\gamma \delta} \nabla_{\mu} h_{\beta \gamma \rho \alpha} + \frac{1}{2} (\nabla D)^{\gamma \delta} D^{\mu \nu \rho} \nabla_{\gamma} h_{\delta \mu \nu \rho}, \quad (5.2.61) \]

\[ \mathcal{L}_{21}^{\text{int}}(h^{(4)}) = \]
\[ - \frac{3}{4} \bar{h}^{\gamma \delta} \nabla_{\gamma} D^{\mu \nu \rho} \nabla_{\delta} D_{\mu \nu \rho} + \frac{1}{8} (\nabla D \bar{h}^{\gamma \delta}) D^{\mu \nu \rho} D_{\mu \nu \rho} \]
\[ + \frac{3}{4} \nabla_{\gamma} D^{\mu \nu \rho} \nabla_{\delta} D_{\mu \nu \rho} + \frac{9}{4} \bar{h}^{\gamma \delta} \nabla_{\gamma} D^{\mu \nu \rho} \nabla_{\delta} D_{\mu \nu \rho} \]
\[ + 3 h^{\gamma \delta} \nabla^{\mu \nu \rho} D_{\gamma \delta \mu \nu \rho} + \frac{3}{4} \nabla^{\nu \rho} \bar{h}^{\gamma \delta} D_{\gamma \delta \mu \nu \rho} + \frac{3}{4} \nabla^{\nu \rho} \bar{h}^{\gamma \delta} D_{\gamma \delta \mu \nu \rho} \]
\[ - \frac{3}{4} \bar{h}^{\gamma \delta} (\nabla D)^{\mu \nu \rho} \nabla_{\gamma} D_{\mu \nu \rho} + 6 \bar{h}^{\gamma \delta} \nabla^{\mu \nu \rho} (\nabla D)^{\gamma \delta} D_{\mu \nu \rho} \]
\[ + 3 h^{\gamma \delta} (\nabla D)^{\gamma \delta} (\nabla D)^{\mu \nu \rho}, \quad (5.2.62) \]
Collecting factors coming with Fronsdal’s equation of motion (Fronsdal’s tensor) in Noether’s equation we obtain next to the free term $\delta(0)h$ of the gauge transformation law for the spin four field the linear term

$$
\delta(1)h_{\alpha\beta\gamma\delta} = \epsilon^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\nu \nabla_\rho h_{\alpha\beta\gamma\delta} + 3(\nabla_\alpha \epsilon^{\mu\nu\rho\sigma} - \nabla_\rho \epsilon^{\alpha\mu\nu\sigma}) \nabla_\mu h_{\beta\gamma\delta}^\rho + 3(\nabla_\alpha \nabla_\beta \epsilon_{\nu\rho}^{\mu} - 2\nabla_\alpha \nabla_\nu \epsilon_{\beta\rho}^{\mu} + \nabla_\nu \nabla_\rho \epsilon_{\alpha\beta}^{\mu}) \nabla_\mu h_{\gamma\delta}^\nu + 3\nabla_\alpha \nabla_\beta \epsilon_{\nu\rho}^{\mu} \nabla_\mu h_{\gamma\delta}^\nu + 3\nabla_\alpha \nabla_\beta \epsilon_{\nu\rho}^{\mu} \nabla_\mu h_{\gamma\delta}^\nu + 3\nabla_\alpha \nabla_\beta \epsilon_{\nu\rho}^{\mu} \nabla_\mu h_{\gamma\delta}^\nu + (\text{trace terms } O(g_{\alpha\beta})),
$$

(5.2.63)

where we assumed symmetrization of the indices $\alpha, \beta, \gamma, \delta$ and the spin four field redefinition $h_{\alpha\beta\gamma\delta} \rightarrow h_{\alpha\beta\gamma\delta} - \frac{9}{8} \nabla_\mu h_{\alpha\beta\gamma\delta} + \frac{1}{4} \nabla_\mu (\nabla_\alpha h_{\beta\gamma\delta}) - \frac{3}{4} \nabla_\mu \left[ (\nabla_\alpha h_{\beta\gamma\delta}) \right] + 1 \nabla_\mu \nabla_\nu h_{\alpha\beta\gamma\delta} + \nabla_\nu (\nabla_\alpha h_{\beta\gamma\delta}) - \frac{3}{2} \nabla_\mu \epsilon_{\alpha\beta\gamma\delta} \nabla_\nu h_{\beta\gamma\delta} - \frac{3}{2} \nabla_\mu \epsilon_{\alpha\beta\gamma\delta} \nabla_\nu h_{\beta\gamma\delta} + \frac{1}{4} \nabla_\mu (\nabla_\alpha D_{\beta\gamma\delta}) - \frac{3}{2} h_{\alpha\beta\gamma\delta} D_{\gamma\delta}$.

(5.2.64)

where symmetrization over the indices $\alpha, \beta, \gamma, \delta$ is also understood.

Finally note that we did not obtain an $L^{\text{inter}}_{30} \sim (D)^3$ part of interaction (that’s why we didn’t draw corresponding cell in the first row of (5.2.50)) because we started the leading part $L^{\text{inter}}_{00}$ (5.2.51) from the curvature term and fixed in this way partial integration freedom. After that as it was mentioned above all other terms of interaction could be constructed in a unique way up to some field redefinition. This particular way of derivative rearrangement (including partial integration of all other level terms) does not lead to a $(D)^3$ term as opposed to other ways of rearranging the derivatives by means of the partial integration freedom. On the other hand if we rearrange the derivatives as described in the subsection 5.2.1 we get leading part of the interaction $L^{\text{inter}}_{00}$ in complete agreement with our prediction (5.2.23) for $s = 4$. The same is true for the transformation law (5.2.63) and (5.2.20).
5.3 General cubic interaction $s_1 - s_2 - s_3$

5.3.1 Notations

Here we present Fronsdal’s Lagrangian:

\[ L_0(h^{(s)}(a)) = -\frac{1}{2} h^{(s)}(a) \ast_a F^{(s)}(a) + \frac{1}{8s(s-1)} \Box_a h^{(s)}(a) \ast_a \Box \mathcal{F}^{(s)}(a), \]  

where $\mathcal{F}^{(s)}(z; a)$ is the Fronsdal tensor

\[ \mathcal{F}^{(s)}(z; a) = \Box h^{(s)}(z; a) - s(a \nabla) D^{(s-1)}(z; a), \]  

and $D^{(s-1)}(z; a)$ is the deDonder tensor or traceless divergence of the higher spin gauge field

\[ D^{(s-1)}(z; a) = Div h^{(s-1)}(z; a) - \frac{s-1}{2}(a \nabla) Tr h^{(s-2)}(z; a), \]

\[ \Box_a D^{(s-1)}(z; a) = 0. \]  

The initial gauge variation of order zeroth in the spin $s$ field is

\[ \delta_{(0)} h^{(s)}(z; a) = s(a \nabla) \epsilon^{(s-1)}(z; a), \]  

with the traceless gauge parameter for the double traceless gauge field

\[ \Box_a \epsilon^{(s-1)}(z; a) = 0, \]

\[ \Box_a^2 h^{(s)}(z; a) = 0. \]  

Therefore at this point we can see from (5.3.1) and (5.3.2) that the de Donder gauge condition $5.2.8$ is a correct generalization of the Lorentz gauge condition in the case of spin $s > 2$. Finally we note that in deDonder gauge (5.2.8) $\mathcal{F}^{(s)}(z; a) = \Box h^{(s)}(z; a)$ and the field $h^{(s)}$ decouples from it’s trace in Fronsdal’s Lagrangian (5.3.1).

5.3.2 Noether’s theorem in leading order: Trinomial coefficients

We consider three potentials $h^{(s_1)}(z_1; a), h^{(s_2)}(z_2; b), h^{(s_3)}(z_3; c)$ whose spins $s_i$ are assumed to be ordered

\[ s_1 \geq s_2 \geq s_3. \]  

For the interaction we make the cyclic ansatz

\[ L_i^{(0,0)}(h^{(s_1)}(a), h^{(s_2)}(b), h^{(s_3)}(c)) = \sum_{n_1, n_2, n_3} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \int dz_1 dz_2 dz_3 \delta(z_3 - z_1) \delta(z_2 - z_1) \]

\[ \hat{T}(Q_{12}, Q_{23}, Q_{31}|n_1, n_2, n_3) h^{(s_1)}(z_1; a) h^{(s_2)}(z_2; b) h^{(s_3)}(z_3; c), \]
where
\[ \hat{T}(Q_{12}, Q_{23}, Q_{31}|n_1, n_2, n_3) = (\partial_a \partial_b)^{Q_{12}}(\partial_b \partial_c)^{Q_{23}}(\partial_c \partial_a)^{Q_{31}}(\partial_a \nabla_2)^{n_1}(\partial_b \nabla_3)^{n_2}(\partial_c \nabla_1)^{n_3}, \]
(5.3.3)

and the notation \((0, 0)\) as a superscript means that it is an ansatz for terms without \(\text{Div}h^{s_i-1}) = (\nabla_i \partial_a)h^{(s_i)}(a_i)\) and \(\text{Tr}h^{s_i-2}) = \frac{1}{s_i(s_i-1)} \partial_a h^{(s_i)}(a_i)\). Denoting the number of derivatives by \(\Delta\) we have
\[ n_1 + n_2 + n_3 = \Delta. \]
(5.3.4)

As balance equations we have
\[ n_1 + Q_{12} + Q_{31} = s_1, \]
\[ n_2 + Q_{23} + Q_{12} = s_2, \]
\[ n_3 + Q_{31} + Q_{23} = s_3. \]
(5.3.5)

These equations are solved by
\[ Q_{12} = n_3 - \nu_3, \]
\[ Q_{23} = n_1 - \nu_1, \]
\[ Q_{31} = n_2 - \nu_2. \]
(5.3.6)

Since the l.h.s. cannot be negative, we have
\[ n_i \geq \nu_i. \]
(5.3.7)

The \(\nu_i\) are determined to be
\[ \nu_i = \frac{1}{2}(\Delta + s_i - s_j - s_k), \quad i, j, k \text{ are all different.} \]
(5.3.8)

These \(\nu_i\) must also be nonnegative, since otherwise the natural limitation of the \(n_i\) to nonnegative values would imply a boundary value problem which has only a trivial solution. It follows that the minimally possible \(\Delta\) is expressed by Metsaev’s (see [12] equ. (5.11)-(5.13)) formula (using the ordering of the \(s_i\)).
\[ \Delta_{\text{min}} = \max [s_i + s_j - s_k] = s_1 + s_2 - s_3. \]
(5.3.9)

For example
\[ \Delta_{\text{min}} = 6 \quad \text{for} \quad s_1 = s_2 = 4, s_3 = 2. \]
(5.3.10)

This value and the ordering of the \(s_i\) implies for the \(\nu_i\)
\[ \nu_1 = s_1 - s_3, \]
\[ \nu_2 = s_2 - s_3, \]
\[ \nu_3 = 0. \]
(5.3.11)
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From this result and the experience with the cubic selfinteraction for \( s = 4 \) we can guess that the coefficient \( C \) in the ansatz is a trinomial

\[
C_{s_1, s_2, s_3}^{n_1, n_2, n_3} = \text{const} \left( \frac{s_3}{n_1 - s_1 + s_3, n_2 - s_2 + s_3, n_3} \right),
\]

which entails

\[
\sum_{ij} Q_{ij} = \Delta - \sum_i \nu_i,
\]

\[
\sum_i \nu_i = 3/2\Delta - 1/2 \sum_i s_i,
\]

and the expression \( \Delta_{\min} \) for \( \Delta_{\min} \).

For the proof of this equation \( (5.3.12) \) we use Noether’s theorem to derive recursion relations which are then solved. By variation w.r.t. \( h \) we obtain three currents whose divergences must vanish on shell. We need only do the explicit variation once:

\[
J^{(3)}(z_3; c) = \sum C_{s_1, s_2, s_3}^{n_1, n_2, n_3} \int dz_1 dz_2 \delta(z_3 - z_1)\delta(z_3 - z_2)
\]

\[
(\partial a \partial b)Q^{12}(b \partial c)Q^{23}(c \partial)Q^{31}(\partial a \partial)_{n_1}Q(\partial b \partial)_{n_2}Q(c \partial)_{n_3}h^{(s_1)}(z_1; a)h^{(s_2)}(z_2; b),
\]

having the divergence

\[
(\partial c \partial)J^{(3)}(z_3; c) = \sum C_{s_1, s_2, s_3}^{n_1, n_2, n_3}
\]

\[
\{n_3(\nabla_1 \nabla_3)(\partial a \partial b)Q^{12}(b \partial c)Q^{23}(c \partial)Q^{31}(\partial a \partial)_{n_1}Q(\partial b \partial)_{n_2}Q(c \partial)_{n_3}^{-1}
\]

\[
+Q^{23}(\partial a \partial b)Q^{12}(b \partial c)Q^{31}(\partial a \partial)_{n_1}Q(\partial b \partial)_{n_2}Q(c \partial)_{n_3}^{n_2+1}(c \partial)_{n_3}^{-1}
\]

\[
+Q^{31}(\partial a \partial b)Q^{12}(b \partial c)Q^{23}(c \partial)Q^{31}(\partial a \partial)_{n_1}Q(\partial b \partial)_{n_2}Q(c \partial)_{n_3}^{n_2+1}(c \partial)_{n_3}^{-1}\}
\]

\[
h^{(s_1)}(z_1; a)h^{(s_2)}(z_2; b) \mid z_1 = z_2 = z_3.
\]

This divergence (and the corresponding divergences of the currents \( J^{(1,2)} \)) must vanish on shell.

We shall develop now a recursive algorithm. First we study the terms not containing any deDonder expression \( D^{(s_i-1)}, i = 1, 2, 3 \):

\[
D^{(s_i-1)} = \frac{1}{s_i}[(\partial a_i \nabla_i) - 1/2(a_i \nabla_i)\square a_i]h^{(s_i)}(z_i; a_i), \quad a_i = a, b, c.
\]

We use that

\[
(\nabla_1 \nabla_3) = 1/2[\square_2 - \square_1 - \square_3],
\]

and

\[
\square_i h^{(s_i)}(z_i; a_i) = \mathcal{F}^{(s_i)}(z_i; a_i) + s_i(a_i \nabla_i)D^{(s_i-1)},
\]

\[
\square_i \epsilon^{(s_i-1)}(z_i; a_i) = \delta_i^{(0)}D^{(s_i-1)},
\]
where $\mathcal{F}(s_i)(z_i; a_i)$ is Fronsdal’s gauge invariant equation of motion and can be dropped on shell. So the $n_3$-term of (5.3.15) does not contribute to the leading order terms. On the other hand the $Q_{23}$-term is purely leading order. The $Q_{31}$-term contains

$$ (\partial_a \nabla_3) = -(\partial_a \nabla_2) - (\partial_a \nabla_1). \quad (5.3.20) $$

Only the first term yields a leading order contribution, the next one is a divergence term.

In the leading order terms we renumber the powers $n_1 \to n_1 + 1$ in the $Q_{23}$-term and $n_2 \to n_2 + 1$ in the leading order $Q_{31}$ term. We get

$$ [(n_1 + 1 - \nu_1)C_{n_1,n_2,n_3}^{s_1,s_2,s_3} - (n_2 + 1 - \nu_2)C_{n_1,n_2+1,n_3}^{s_1,s_2,s_3}] (\partial_b \partial_c)^{n_3-\nu_3} (\partial_c \partial_a)^{n_2-\nu_2} (\partial_a \nabla_2)^{n_1+1} (\partial_b \nabla_3)^{n_2+1} (c \nabla_1)^{n_3} = 0. \quad (5.3.21) $$

It follows that the factor in the square bracket must vanish. Two analogous relations follow from the two other currents. The solution of these three recursion relations is

$$ C_{n_1,n_2,n_3}^{s_1,s_2,s_3} = \text{const} \left( \sum n_i - \sum \nu_i, \ n_1 - \nu_1, n_2 - \nu_2, n_3 - \nu_3 \right), \quad (5.3.22) $$

which is equivalent to (5.3.12) for $\Delta = \Delta_{\text{min}}$ and therefore $\nu_3 = 0$, and describes also all other $\Delta > \Delta_{\text{min}}$ cases. Comparison with (5.3.13) proves that in the $\Delta_{\text{min}}$ case we can present the trinomial coefficient also as

$$ C_{Q_{12},Q_{23},Q_{31}}^{s_1,s_2,s_3} = \text{const} \left( \sum \frac{s_{\text{min}}}{Q_{12}, Q_{23}, Q_{31}} \right). \quad (5.3.23) $$

We see that the number of contractions between indices of our three fields $Q_{12}, Q_{23}, Q_{31}$ define coefficients in our interaction completely.

Finally we want to make a remark concerning the case where two or all three of these fields are equal. Then we get only two or one current whose divergences vanish on shell. But in this case we have a symmetry which restores the result (5.3.12), (5.3.21) and shows that this is correct in all cases.

### 5.3.3 Cubic interactions for arbitrary spins: Complete solution of the Noether’s procedure

To derive the next terms of interaction containing one deDonder expression we turn to the Lagrangian formulation of the task and solve Noether’s equation

$$ \sum_{i=1}^{3} \delta_i^{(1)} \mathcal{L}_i^0(h^{(s_i)}(a)) + \sum_{i=1}^{3} \delta_i^{(0)} \mathcal{L}_i(h^{(s_i)}(a), h^{(s_2)}(b), h^{(s_3)}(c)) = 0. \quad (5.3.24) $$
where
\[
\delta_i^{(0)} h^{(s_i)} (a_i) = s_i (a_i \nabla_i) \epsilon^{s_i-1}(z_i; a_i) \tag{5.3.25}
\]
\[
\mathcal{L}_i^{(0)} (h^{(s_i)} (a)) = - \frac{1}{2} h^{(s_i)} (a_i) *_{a_i} \mathcal{F}^{(s_i)} (a_i) + \frac{1}{8s_i(s_i - 1)} \Box_{a_i} h^{(s_i)} (a_i) *_{a_i} \Box_{a_i} \mathcal{F}^{(s_i)} (a_i) \tag{5.3.26}
\]

Shifting \( \delta_i^{(1)} \) by a trace term in the same way as in previous section we obtain the following functional equation:

\[
\sum_{i=1}^{3} \delta_i^{(0)} \mathcal{L}_i (h^{(s_1)} (a), h^{(s_2)} (b), h^{(s_3)} (c)) = 0 + O(\mathcal{F}^{(s_i)} (a_i)). \tag{5.3.27}
\]

We solve this equation starting from the ansatz [5.3.25, 5.3.3] and integrating level by level in means of its dependence on deDonder tensors and traces of higher spin gauge fields.

Actually we have to solve the following equation:

\[
C_{\{n_i\}}^{[s_i]} \hat{T}(Q_{ij}|n_i) [(a \nabla_i)\epsilon^{(s_i-1)}h^{(s_2)}h^{(s_3)} + h^{(s_1)}(b \nabla_2)\epsilon^{(s_2-1)}h^{(s_3)} + h^{(s_1)}h^{(s_2)}(c \nabla_3)\epsilon^{(s_3-1)}] = 0 + O(\mathcal{F}^{(s_i)} (a_i), D^{(s_i-1)} (a_i), \Box_{a_i} h^{(s_i)} (a_i)). \tag{5.3.28}
\]

Taking into account that due to [5.3.3]

\[
\hat{T}(Q_{ij}|n_i)(a_i \nabla_i)\epsilon^{(s_i-1)}(a_i) = [\hat{T}(Q_{ij}|n_i), (a_i \nabla_i)]\epsilon^{(s_i-1)}(a_i), \tag{5.3.29}
\]

we see that all necessary information for the recursion can be found calculating these commutators

\[
[\hat{T}(Q_{ij}|n_i), (a \nabla_1)] = Q_{12} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1|n_1, n_2, n_3 + 1)
- Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31}|n_1, n_2 + 1, n_3)
+ n_1 \hat{T}(Q_{12}, Q_{23}, Q_{31}|n_1 - 1, n_2, n_3)(\nabla_1 \nabla_2)
- Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31}|n_1, n_2, n_3)(\partial_1 \nabla_2), \tag{5.3.30}
\]

\[
[\hat{T}(Q_{ij}|n_i), (b \nabla_2)] = Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31}|n_1 + 1, n_2, n_3)
- Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31}|n_1, n_2, n_3 + 1)
+ n_2 \hat{T}(Q_{12}, Q_{23}, Q_{31}|n_1, n_2 - 1, n_3)(\nabla_2 \nabla_3)
- Q_{23} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31}|n_1, n_2, n_3)(\partial_2 \nabla_3), \tag{5.3.31}
\]

\[
[\hat{T}(Q_{ij}|n_i), (c \nabla_3)] = Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31}|n_1, n_2 + 1, n_3)
- Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1|n_1 + 1, n_2, n_3)
+ n_3 \hat{T}(Q_{12}, Q_{23}, Q_{31}|n_1, n_2, n_3 - 1)(\nabla_3 \nabla_1)
- Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31}|n_1 - 1, n_2, n_3)(\partial_3 \nabla_1), \tag{5.3.32}
\]
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where we used relations like (5.3.18) and (5.3.20). In these commutators we can use also the following identities

\[
\nabla_1 \nabla_2 = \frac{1}{2} (\square_1 - \square_2 - \square_1), \\
\nabla_2 \nabla_3 = \frac{1}{2} (\square_2 - \square_1 - \square_3), \\
\nabla_3 \nabla_1 = \frac{1}{2} (\square_3 - \square_2 - \square_1). \quad (5.3.33)
\]

Now we see immediately from the first two lines of (5.3.30)-(5.3.32) that these contribute to (5.3.27) as leading order terms and yield the same equations for the \( C_{n,i}^{s} \) coefficients as (5.3.21)

\[
(Q_{31} + 1)C_{n_1,n_2+1,n_3}^{s_1,s_2,s_3} - (Q_{12} + 1)C_{n_1,n_2,n_3+1}^{s_1,s_2,s_3} = 0, \quad (5.3.34) \\
(Q_{12} + 1)C_{n_1,n_2+1,n_3}^{s_1,s_2,s_3} - (Q_{23} + 1)C_{n_1+1,n_2,n_3}^{s_1,s_2,s_3} = 0, \quad (5.3.35) \\
(Q_{23} + 1)C_{n_1+1,n_2,n_3}^{s_1,s_2,s_3} - (Q_{31} + 1)C_{n_1,n_2+1,n_3}^{s_1,s_2,s_3} = 0. \quad (5.3.36)
\]

with the solution (5.3.22) or (5.3.23).

To find the full interaction we follow the same strategy as in the case of \( s = 4 \) selfinteraction and introduce the following classification for the higher order interaction terms in \( D \) and \( \bar{h} = Tr h \):

\[
\mathcal{L}_I = \sum_{i,j=0,1,2,3} \mathcal{L}_I^{(i,j)}(h^{(s_i)}), \quad (5.3.37)
\]

where

\[
\mathcal{L}_I^{(i,j)}(h^{(s)}) \sim \nabla^{s-i}(D)^j(\bar{h}(s)^j)(h^{(s)})^{3-j-i}. \quad (5.3.38)
\]

In this notation the leading term described in the second section is \( \mathcal{L}_I^{(0,0)}(h^{(s)}) \).

To integrate Noether’s equation next to the leading term we have to insert in (5.3.27) the last two lines of (5.3.30)-(5.3.32) and use two important relations (5.3.18), (5.3.19). Thus we arrive at the following \( O(D) \) solution:

\[
\mathcal{L}_I^{(1,0)} = \sum_{n_1} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z) + \left[ \frac{s_1 n_1}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij}) n_1 - 1, n_2, n_3 D^{(s_1-1)} h^{(s_2)} h^{(s_3)} \\
+ \frac{s_2 n_2}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij}) n_1, n_2 - 1, n_3 h^{(s_1)} h^{(s_2)} D^{(s_2-1)} h^{(s_3)} \\
+ \frac{s_3 n_3}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij}) n_1, n_2, n_3 - 1 h^{(s_1)} h^{(s_2)} D^{(s_3-1)} \right]. \quad (5.3.39)
\]

The detailed proof of this formula can be found in the Appendix F where we describe also derivations of all other terms.
The next $O(D^2)$ and $O(D^3)$ level Lagrangians are

\[
L^{(2,0)}_I = \sum_{n_i} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_3 - z) \delta(z_2 - z)
\]

\[
\begin{align*}
&+ \frac{s_3 n_3 s_1 n_1}{2} C^{s_1, s_2, s_3}_{n_1, n_2, n_3} \hat{T}(Q_{ij} | n_1 - 1, n_2, n_3 - 1) D^{(s_1 - 1)} h^{(s_2)} D^{(s_3 - 1)} \\
&+ \frac{s_1 n_1 s_2 n_2}{2} C^{s_1, s_2, s_3}_{n_1, n_2, n_3} \hat{T}(Q_{ij} | n_1 - 1, n_2 - 1, n_3) D^{(s_1 - 1)} D^{(s_2 - 1)} h^{(s_3)} \\
&+ \frac{s_2 n_2 s_3 n_3}{2} C^{s_1, s_2, s_3}_{n_1, n_2, n_3} \hat{T}(Q_{ij} | n_1, n_2 - 1, n_3 - 1) h^{(s_1)} D^{(s_2 - 1)} D^{(s_3 - 1)}
\end{align*}
\]

(5.3.40)

and

\[
L^{(3,0)}_I = \sum_{n_i} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_3 - z) \delta(z_2 - z)
\]

\[
\begin{align*}
&+ \frac{s_3 n_3 s_2 n_2 s_1 n_1}{2} C^{s_1, s_2, s_3}_{n_1, n_2, n_3} \hat{T}(Q_{ij} | n_1 - 1, n_2 - 1, n_3 - 1) D^{(s_1 - 1)} D^{(s_2 - 1)} D^{(s_3 - 1)}
\end{align*}
\]

(5.3.41)

The remaining terms in the Lagrangian contain at least one trace:

\[
L^{(0,1)}_I = L^{(0,2)}_I = 0,
\]

(5.3.42)

\[
L^{(0,3)}_I = \sum_{n_i} C^{s_1, s_2, s_3}_{n_1, n_2, n_3} \frac{Q_{12} Q_{23} Q_{31}}{8} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z)
\]

\[
\left[ \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} - 1 | n_1, n_2, n_3) \Box_a h^{(s_1)} \Box_b h^{(s_2)} \Box_c h^{(s_3)} \right],
\]

(5.3.43)

\[
L^{(1,1)}_I = \sum_{n_i} C^{s_1, s_2, s_3}_{n_1, n_2, n_3} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z)
\]

\[
\begin{align*}
&+ \frac{s_1 Q_{12} n_2}{4} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} - 1 | n_1, n_2 - 1, n_3) D^{(s_1 - 1)} \Box_b h^{(s_2)} h^{(s_3)} \\
&+ \frac{s_2 Q_{23} n_3}{4} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} - 1 | n_1, n_2, n_3 - 1) h^{(s_1)} D^{(s_2 - 1)} \Box_c h^{(s_3)} \\
&+ \frac{s_3 Q_{31} n_1}{4} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} - 1 | n_1 - 1, n_2, n_3) \Box_a h^{(s_1)} h^{(s_2)} D^{(s_3 - 1)}
\end{align*}
\]

(5.3.44)

\[
L^{(1,2)}_I = \sum_{n_i} C^{s_1, s_2, s_3}_{n_1, n_2, n_3} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z)
\]

\[
\begin{align*}
&+ \frac{s_1 Q_{12} Q_{23} n_3}{8} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} - 1 | n_1, n_2, n_3 - 1) D^{(s_1 - 1)} \Box_b h^{(s_2)} \Box_c h^{(s_3)} \\
&+ \frac{s_2 Q_{23} Q_{31} n_1}{8} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} - 1 | n_1 - 1, n_2, n_3) \Box_a h^{(s_1)} D^{(s_2 - 1)} \Box_c h^{(s_3)} \\
&+ \frac{s_3 Q_{31} Q_{12} n_2}{8} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} - 1 | n_1, n_2 - 1, n_3) \Box_a h^{(s_1)} \Box_b h^{(s_2)} D^{(s_3 - 1)}
\end{align*}
\]

(5.3.45)
$L_I^{(2,1)} = \sum_{n_1} C^{n_1, s_2, s_3}_{n_1, n_2, n_3} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z)$

\[+ \frac{s_2 s_3 Q_{31} n_1 n_2}{4} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1 - 1, n_2 - 1, n_3) \Box_a \hat{h}^{(s_1)} D^{(s_2-1)} D^{(s_3-1)}
\[+ \frac{s_1 s_2 Q_{23} n_3 n_1}{4} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1 - 1, n_2, n_3 - 1) D^{(s_1-1)} D^{(s_2-1)} \Box_c \hat{h}^{(s_3)}
\[+ \frac{s_3 s_1 Q_{12} n_2 n_3}{4} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3 - 1) D^{(s_1-1)} \Box_b \hat{h}^{(s_2)} D^{(s_3-1)} \right].
\]

(5.3.46)

So we integrated all cells of the following classification table corresponding to (5.3.37)

| $\hat{h}$ | 0 | 1 | 2 | 3 |
|-----------|---|---|---|---|
| 0         | $hhh$ | $Dh\hat{h}$ | $DD\hat{h}$ | $DDD$ |
| 1         | 0 | $\bar{h}D\hat{h}$ | $\bar{h}D\hat{D}$ |
| 2         | 0 | $\bar{h}\hat{h}D$ |
| 3         | $\bar{h}\bar{h}\hat{h}$ |

(5.3.47)

and proved that after fixing the freedom of partial integration in the leading term (i.e. our cyclic ansatz) all other terms of interaction can be integrated in a unique way when we avoid additional partial integration during recursions.

Summarizing we see that the interaction Lagrangian in deDonder gauge $D^{(s-1)}(z; a) = 0$ can be expressed as a sum

$L_I^{dD}(h^{(s)}) = \sum_{j=0}^{3} L_I^{(0,j)}(h^{(s)}).
\]

(5.3.48)

and it is nothing else than the first column of this table. Therefore (5.3.42) means that in deDonder gauge the traces of the HS fields decouple from the fields as they do in the free Lagrangian.
5.3.4 Discussion: Towards gauge transformations as open Lie algebras

If all spins in the cubic interaction are equal $s$, we can derive the first order gauge transformation of $h^{(s)}$ from the r. h. s. of Noether’s equation \((5.3.27)\) taken off shell

\[
\begin{align*}
& [O(\mathcal{F}) \quad \text{part of} \quad \delta^{0}_{\epsilon(s-1)} \mathcal{L}_I] \\
= & \sum_{n_1} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \int dz_1 dz_2 dz_3 \delta (z_1 - z) \delta (z_2 - z) \delta (z_3 - z) \\
+ & \frac{s_1 n_1}{2} \hat{T}(Q_{12} - 1, n_2, n_3) \epsilon^{(s_1-1)} \mathcal{F}^{(s_2)} h^{(s_3)} \\
+ & \frac{s_1 Q_{12} n_2}{4} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3) \epsilon^{(s_1-1)} \Box_b \mathcal{F}^{(s_2)} h^{(s_3)} \\
+ & \frac{s_3 s_1 Q_{12} n_2 n_3}{4} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3 - 1) \epsilon^{(s_1-1)} \Box_b \mathcal{F}^{(s_2)} D^{(s_3-1)} \\
+ & \frac{s_1 Q_{12} Q_{23} n_2}{8} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} | n_1, n_2, n_3 - 1) \epsilon^{(s_1-1)} \Box_b \mathcal{F}^{(s_2)} \Box_c h^{(s_3)} \\
+ & \frac{s_1 n_1}{2} \hat{T}(Q_{12} - 1, n_2, n_3) \epsilon^{(s_1-1)} h^{(s_2)} \mathcal{F}^{(s_3)} \\
+ & \frac{s_1 Q_{12} n_2}{4} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3) \epsilon^{(s_1-1)} \Box_b h^{(s_2)} \mathcal{F}^{(s_3)} \\
+ & \frac{s_1 s_2 n_1 n_2}{4} \hat{T}(Q_{12}, Q_{23}, Q_{31} | n_1 - 1, n_2 - 1, n_3) \epsilon^{(s_1-1)} D^{(s_2-1)} \mathcal{F}^{(s_3)} \\
+ & \frac{s_1 Q_{12} Q_{23} n_2}{8} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} | n_1, n_2, n_3 - 1) \epsilon^{(s_1-1)} \Box_b h^{(s_2)} \Box_c \mathcal{F}^{(s_3)} \bigg].
\end{align*}
\tag{5.3.49}
\]

If we assume moreover that the gauge transformations form a Lie algebra of power series in some "coupling constant” $g$, we can following along the ideas of Berends, Burger and Van Dam in their classical paper \cite{46} derive conclusions on the higher order interactions. We sum up simple results:

The arguments of these authors to show that such power series algebra does not exist for $s = 3$, cannot be generalized to even spins.

For a given gauge function $\epsilon^{(s-1)}(z; a)$ the gauge transformation is a substitution (classically) with expansion

\[
h \rightarrow h + \delta_\epsilon h = h + \nabla \epsilon + \sum_{n \geq 1} g^n \Theta_n (h, h, \ldots h; \epsilon), \tag{5.3.50}
\]

with $\Theta_n$ depending on $\epsilon$ linearly and on $h$ in the n’th power. Moreover we assume that the commutator of two such transformations is given by

\[
[\delta_\epsilon, \delta_\eta] h = \delta_{C(h; \epsilon, \eta)} h, \tag{5.3.51}
\]
with the expansion
\[ C(h; \epsilon, \eta) = g \sum_{n \geq 0} g^n C_n(h, h, \ldots h; \epsilon, \eta), \] (5.3.52)

where each \( C_n \) depends on \( \epsilon \) and \( \eta \) linearly and on \( h \) in the \( n \)’th power. As substitutions gauge transformations are associative and their infinitesimals must satisfy the Jacobi identity. At order \( g^2 \) this is e.g.
\[ \sum_{\eta, \epsilon, \zeta} \{ C_1(\nabla \zeta; \eta, \epsilon) + C_0(C_0(\eta, \epsilon), \zeta) \} = 0. \] (5.3.53)

The commutator can also be expanded
\[ [\delta_\eta, \delta_\epsilon] = g(\Theta_1(\nabla \epsilon; \eta) - \Theta_1(\nabla \eta; \epsilon)) + g^2\{[\Theta_1(\Theta_1(h; \epsilon); \eta) - \Theta_1(\Theta_1(h; \eta); \epsilon)] + [\Theta_2(\nabla \epsilon, h; \eta) - \Theta_2(\nabla \eta, h; \epsilon)] + [\Theta_2(h, \nabla \epsilon; \eta) - \Theta_2(h, \nabla \eta; \epsilon)]\} + O(g^3). \] (5.3.54)

Inserting this expansion into the definition of the functions \( C_n \) we obtain
\[ \nabla C_0(\eta, \epsilon) = \Theta_1(\nabla \epsilon; \eta) - \Theta_1(\nabla \eta; \epsilon), \] (5.3.55)
\[ \nabla C_1(h; \eta, \epsilon) = \Theta_1(\Theta_1(h; \epsilon); \eta) - \Theta_1(\Theta_1(h; \eta); \epsilon) + \Theta_2(\nabla \epsilon, h; \eta) - \Theta_2(\nabla \eta, h; \epsilon) + \Theta_2(h, \nabla \epsilon; \eta) - \Theta_2(h, \nabla \eta; \epsilon). \] (5.3.56)

Assume that \( \Theta_1(h; \epsilon) \) has been extracted from (5.3.49) for the case of equal spins \( s \). Then the order of derivations in \( \Theta_1 \) is
\[ \nabla \Theta_1(h; \epsilon) = s - 1. \] (5.3.57)

Inserting this result into (5.3.53), (5.3.55) we obtain the number of derivations in \( C_0, C_1 \) as
\[ \nabla C_0(\eta, \epsilon) = s - 1 \quad \text{and} \quad \nabla C_1(h; \eta, \epsilon) = 2s - 3. \] (5.3.58)

This implies
\[ \nabla \Theta_2(h, h; \epsilon) = 2s - 3. \] (5.3.59)

Consequently the quartic interaction must contain \( 2s - 2 \) derivatives. The argument can be continued to still higher interactions: For \( n \)’th order interactions the number is \( (s - 2)(n - 2) + 2 \). This result is equivalent to introduction of a scale \( L \) and dimensions in the following way
\[ [h] = L^{s-2}, \quad [\nabla] = \frac{1}{L}, \] (5.3.60)
with a dimensionless coupling constant $g$, so that each term in the power series has the same dimension. Note that in the case of $\Delta = \Delta_{\text{min}}$ we obtained in the previous subsections of this section and in previous section the same dimensions for cubic selfinteractions and a free Fronsdal’s action.

In [46] the argument was presented that for spin $s = 3$ a Lie algebra of gauge transformations in the form of power series does not exist, the problem starting with the second power. The argument was based on the term

$$(\partial_a \nabla_2)^{s-1} e^{(s-1)}(z_1; a) h^{(s)}(z_2; b),$$

which exists in $\Theta_1$. Such term is present in fact for any spin, as can be inspected from (5.3.49). Namely, in the fifth term of the square bracket of (5.3.49) (this is the unique localization) we get such expression for $n_1 = s, n_2 = n_3 = 0$. In equation (5.3.56) in the first line we have thus $2s - 2$ derivatives acting on the field $h$ in either term. In no other terms of (5.3.56) such expression appears. Therefore they must cancel inside this line and they do cancel indeed for even spin only. There is in this case no obstruction of the power series algebra by these arguments. A deeper investigation of such algebras will follow in the future.
Chapter 6

Generating function of HSF cubic interactions

6.1 Generating function for the Free Lagrangian of all higher spin gauge fields

We introduce a generating function for HS gauge fields by

\[ \Phi(z; a) = \sum_{s=0}^{\infty} \frac{1}{s!} h^{(s)}(z; a) \]  

(6.1.1)

where we assume that the spin \( s \) field has scaling dimension \( s - 2 \), the \( a_i \) vectors have dimension \( -1 \), and therefore all terms in the generating function for higher spin gauge fields (6.1.1) have the same dimension \(-2\).

A zeroth order gauge transformation for this field reads as

\[ \delta_\Lambda^0 \Phi(z; a) = (a \nabla) \Lambda(z; a), \]  

(6.1.2)

\[ \delta_\Lambda^0 D_a \Phi(z; a) = \Box \Lambda(z; a), \]  

(6.1.3)

\[ \delta_\Lambda^0 \Box_a \Phi(z; a) = 2(\nabla \partial_a) \Lambda(z; a). \]  

(6.1.4)

where

\[ \Lambda(z; a) = \sum_{s=1}^{\infty} \frac{1}{(s-1)!} \epsilon^{(s-1)}(z; a), \]  

(6.1.5)

is the generating function of the gauge parameters and is dimensionless.\[ \star \]

Fronsdal’s constraint for the gauge parameter reads as

\[ \Box_a \Lambda(z; a) = 0, \]  

(6.1.6)

\[ \star \text{The gauge parameter for spin } s \text{ field } \epsilon^{(s-1)} \text{ has scaling dimension } s - 1, \text{ therefore after contraction with } s - 1 \text{ a-s becomes dimensionless.} \]
CHAPTER 6. GENERATING FUNCTION OF HSF CUBIC INTERACTIONS

For a spin $s$ field gauge variation we get as expected
\[ \delta_0 h^{(s)}(z; a) = s (a \nabla) e^{(s-1)}(z; a), \] (6.1.7)

The second Fronsdal constraint of the gauge field reads in these notations
\[ \Box^2_a \Phi(z; a) = 0, \] (6.1.8)

We introduced the "de Donder" operator
\[ D_{a_i} = (\partial_{a_i} \nabla_i) - \frac{1}{2} (a_i \nabla_i) \Box_{a_i}, \] (6.1.9)

This operator is "linear" in $\partial_{a_i}$.

Here we write the quadratic Lagrangian for free higher spin gauge fields in general form using the generating function for HS fields (6.1.1). First we introduce Fronsdal’s operator
\[ \mathcal{F}_{a_i} = \Box_i - (a_i \nabla_i)(\nabla_i \partial_i) + \frac{1}{2} (a_i \nabla_i)^2 \Box_{a_i}, \] (6.1.10)

or with the help of (6.1.9)
\[ \mathcal{F}_{a_i} = \Box_i - (a_i \nabla_i)D_{a_i}. \] (6.1.11)

The operator of the equation of motion can be written in the form
\[ \mathcal{G}_{a_i} = \mathcal{F}_{a_i} - \frac{a_i^2}{4} \Box_{a_i} \mathcal{F}_{a_i}, \] (6.1.12)

Now we can write the free Lagrangian for all gauge fields of any spin in a symmetric elegant form
\[ \mathcal{L}^{free}(z) \rightleftarrows \frac{\kappa}{2} \exp[\lambda^2 \partial_{a_1} \partial_{a_2}] \int_{z_1 z_2} \delta(z_1 - z) \delta(z_2 - z) \\
\left\{ (\nabla_1 \nabla_2) - \lambda^2 D_{a_1} D_{a_2} - \frac{\lambda^4}{4} (\nabla_1 \nabla_2) \Box_{a_1} \Box_{a_2} \right\} \Phi(z_1; a_1) \Phi(z_2; a_2) \mid_{a_1 = a_2 = 0} \] (6.1.13)

where $\lambda$ has scaling dimension $-1$, therefore $\lambda^2$ compensates the dimension of the operator in the exponent. We will see that all relative coupling constants of HS interactions can be expressed as powers of $\lambda$. The parameter $\kappa$ is a constant which makes the action dimensionless (analogous to Einstein’s constant and simply connected with the latter). It has scaling dimension $6 - d$, where $d$ is the space-time dimension. For Einstein’s constant $\kappa_E$ we get
\[ \kappa_E^{-2} = \kappa \lambda^4 \] (6.1.14)
CHAPTER 6. GENERATING FUNCTION OF HSF CUBIC INTERACTIONS

It is now obvious that in the free Lagrangian there is no mixing between gauge fields of different spin. It can also be written in such forms

\[
\mathcal{L}_{\text{free}}(z) = -\frac{1}{2} \exp[\lambda^2 \partial_a \partial_{a_2}] \int_{z_1} \delta(z_1 - z)(G_{a_1})\Phi(z_1; a_1)\Phi(z; a_2) \big|_{a_1 = a_2 = 0} = -\frac{1}{2} \exp[\lambda^2 \partial_a \partial_{a_2}] \int_{z_2} \delta(z_2 - z)(G_{a_2})\Phi(z; a_1)\Phi(z_2; a_2) \big|_{a_1 = a_2 = 0}
\]

These expressions reproduce Fronsdal’s Lagrangians for all gauge fields with any spin.

6.2 Generating Function for Cubic Interactions

We are going to present a very beautiful and compact form of all HS gauge field interactions derived in the previous chapter. First we rewrite the leading term of a general trilinear interaction of higher spin gauge fields with any spins \(s_1, s_2, s_3\)

\[
\mathcal{L}^{\text{leading}}(h^{(s_1)}(z), h^{(s_2)}(z), h^{(s_3)}(z))
= \sum_{\alpha+\beta+\gamma=n} \frac{1}{\alpha!\beta!\gamma!} \int_{z_1, z_2, z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3) 
\]

\[
[(\nabla_1 \partial_c)^{s_3-n+\gamma}(\nabla_2 \partial_a)^{s_1-n+\alpha}(\nabla_3 \partial_b)^{s_2-n+\beta} (\partial_a \partial_{b})^{\gamma} (\partial_b \partial_{c})^{\alpha} (\partial_c \partial_{a})^{\beta}]
\]

\[
h^{(s_1)}(a; z_1)h^{(s_2)}(b; z_2)h^{(s_3)}(c; z_3),
\]

where the number of derivatives is

\[
\Delta = s_1 + s_2 + s_3 - 2n,
\]

\[
0 \leq n \leq \min(s_1, s_2, s_3)
\]

As we see, the minimal and maximal possible numbers of derivatives are

\[
\Delta_{\text{min}} = s_1 + s_2 + s_3 - 2\min(s_1, s_2, s_3),
\]

\[
\Delta_{\text{max}} = s_1 + s_2 + s_3.
\]

The case of \(\Delta_{\text{min}}\) is important also because only in that case the interaction \((6.2.1)\) has the same dimension as the lowest spin field free Lagrangian.

These interactions trivialize only if we have two equal spin values and the third value is odd. This we call the \(\ell - s - s\) case, where \(\ell\) is odd. In that case we should have a multiplet of spin \(s\) fields, with at least two charges to couple to the spin \(\ell\) field. As example consider an odd spin self-interaction. In the case of \(\ell - \ell - \ell\) odd spin self interaction, the number of possible charges in the multiplet should be at least 3.
The same Lagrangian can be written in the following way (due to a constant normalization factor $2^A$)

$$L^{leading}(h^{(s_1)}(z), h^{(s_2)}(z), h^{(s_3)}(z))$$

$$= \sum_{\alpha + \beta + \gamma = n} \frac{1}{\alpha!\beta!\gamma!} \int_{z_1, z_2, z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3)$$

$$\left[ (\nabla_{12} \partial_1)^{s_1 - \alpha} (\nabla_{23} \partial_2)^{s_2 - \beta} (\nabla_{31} \partial_3)^{s_3 - \gamma} (\partial_a \partial_b)^{\alpha} (\partial_b \partial_c)^{\beta} (\partial_c \partial_a)^{\gamma} \right]$$

$$h^{(s_1)}(a; z_1)h^{(s_2)}(b; z_2)h^{(s_3)}(c; z_3), \quad (6.2.6)$$

where

$$\nabla_{12} = \nabla_1 - \nabla_2, \quad (6.2.7)$$

$$\nabla_{23} = \nabla_2 - \nabla_3, \quad (6.2.8)$$

$$\nabla_{31} = \nabla_3 - \nabla_1. \quad (6.2.9)$$

Now we can see that the following expression is a generating function for the leading term of all interactions of HS gauge fields.

$$A^{00} = \int_{z_1, z_2, z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3) \exp W$$

$$\Phi_1(z_1; a_1 + \frac{1}{2} \nabla_{23})\Phi_2(z_2; a_2 + \frac{1}{2} \nabla_{31})\Phi_3(z_3; a_3 + \frac{1}{2} \nabla_{12}) \mid_{a_1 = a_2 = a_3 = 0} \quad (6.2.10)$$

with

$$W = \frac{\lambda^2}{2} [(\partial_{a_1} \partial_{a_2})(\partial_{a_3} \nabla_{12}) + (\partial_{a_2} \partial_{a_3})(\partial_{a_1} \nabla_{23}) + (\partial_{a_3} \partial_{a_1})(\partial_{a_2} \nabla_{31})] \quad (6.2.11)$$

This can be written in another form

$$A^{00}(\Phi(z)) = \int_{z_1, z_2, z_3} \delta(z - z_{1,2,3}) \exp \tilde{W} \times \Phi(z_1; a_1)\Phi(z_2; a_2)\Phi(z_3; a_3) \mid_{a_1 = a_2 = a_3 = 0} \quad (6.2.12)$$

where

$$\tilde{W} = \frac{\lambda^2}{2} [(\partial_{a_1} \partial_{a_2})(\partial_{a_3} \nabla_{12}) + (\partial_{a_2} \partial_{a_3})(\partial_{a_1} \nabla_{23}) + (\partial_{a_3} \partial_{a_1})(\partial_{a_2} \nabla_{31})]$$

$$+ \frac{1}{2} [(\partial_{a_1} \nabla_{12}) + (\partial_{a_2} \nabla_{23}) + (\partial_{a_3} \nabla_{31})], \quad (6.2.13)$$

$$\int_{z_1, z_2, z_3} \delta(z - z_{1,2,3}) = \int_{z_1, z_2, z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3) \quad (6.2.14)$$

for brevity. Furthermore we will always assume this integration with delta functions, without writing it explicitly. The operator in the second row of (6.2.13) is a dimensionless operator, therefore it does not need any dimensional constant multiplier.

Now we can derive all other terms in the Lagrangian using the following important relation

$$[\exp \tilde{W}, A] = \exp \tilde{W} [\tilde{W}, A] + \exp \tilde{W} [\tilde{W}, [\tilde{W}, A]] + \exp \tilde{W} [\tilde{W}, [\tilde{W}, [\tilde{W}, A]]] + \ldots \quad (6.2.15)$$
for any operator $A$. And therefore

$$[\exp \hat{W}, (a_1 \nabla_1)] = \exp \hat{W} [\hat{W}, (a_1 \nabla_1)],$$

(6.2.16)

$$[\exp \hat{W}, (a_2 \nabla_2)] = \exp \hat{W} [\hat{W}, (a_2 \nabla_2)],$$

(6.2.17)

$$[\exp \hat{W}, (a_3 \nabla_3)] = \exp \hat{W} [\hat{W}, (a_3 \nabla_3)].$$

(6.2.18)

The following commutators will be used many times while deriving trace and divergence terms

$$[\hat{W}, (a_1 \nabla_1)] = -\frac{\lambda^2}{4} [(\partial_{a_2} \nabla_2)(\partial_{a_3} \nabla_1) + (\partial_{a_3} \nabla_3)(\partial_{a_2} \nabla_1)] + \frac{1}{2} \gamma \gamma (\partial_{a_2} \partial_{a_3} + 1) \nabla_1 \nabla_2 \nabla_3, \quad (6.2.19)$$

$$[\hat{W}, (a_2 \nabla_2)] = -\frac{\lambda^2}{4} [(\partial_{a_3} \nabla_3)(\partial_{a_1} \nabla_2) + (\partial_{a_1} \nabla_1)(\partial_{a_3} \nabla_2)] + \frac{1}{2} \gamma \gamma (\partial_{a_1} \partial_{a_3} + 1) \nabla_2 \nabla_3 \nabla_1, \quad (6.2.20)$$

$$[\hat{W}, (a_3 \nabla_3)] = -\frac{\lambda^2}{4} [(\partial_{a_1} \nabla_1)(\partial_{a_2} \nabla_3) + (\partial_{a_2} \nabla_2)(\partial_{a_1} \nabla_3)] + \frac{1}{2} \gamma \gamma (\partial_{a_1} \partial_{a_2} + 1) \nabla_3 \nabla_1 \nabla_2. \quad (6.2.21)$$

Note that

$$\nabla_1 \nabla_2 \nabla_3 = \nabla_1 - \nabla_2,$$  

(6.2.22)

$$\nabla_2 \nabla_3 \nabla_1 = \nabla_2 - \nabla_3,$$  

(6.2.23)

$$\nabla_3 \nabla_1 \nabla_2 = \nabla_3 - \nabla_1,$$  

(6.2.24)

which is obvious because

$$\nabla_1 + \nabla_2 + \nabla_3 = 0. \quad (6.2.25)$$

We are working with the same type of diagram as in previous chapter.

\[
\begin{array}{|c|c|c|c|c|}
\hline
D_{a_1} & 0 & 1 & 2 & 3 \\
\hline
\square_{a_1} & A^{00} & A^{10} & A^{20} & A^{30} \\
\hline
0 & & & & \\
\hline
1 & A^{01} & A^{11} & A^{21} \\
\hline
2 & A^{02} & A^{12} \\
\hline
3 & A^{03} \\
\hline
\end{array}
\]

\( (6.2.26) \)

\footnote{We always understand partial integrations to be performed, working with a Lagrangian as with an action.}
CHAPTER 6. GENERATING FUNCTION OF HSF CUBIC INTERACTIONS

Now we take a gauge variation of \( \mathcal{A}^{00} \), and find generating functions for all other terms in the cubic Lagrangian. A simple but elegant structure is exhibited by the first row of the diagram

\[
\mathcal{A}^{10}(\Phi(z)) = \mathcal{A}^{20}(\Phi(\ z\ )) = 0, \tag{6.2.27}
\]

\[
\mathcal{A}^{20}(\Phi(z)) = \frac{1}{4} \exp \dot{W} \{ +[\lambda^2(\partial_a \partial_a) + 1][\lambda^2(\partial_a \partial_a) + 1]D_{a_2}D_{a_1} \\
+ [\lambda^2(\partial_a \partial_a) + 1][\lambda^2(\partial_a \partial_a) + 1]D_{a_2}D_{a_2} \\
+ [\lambda^2(\partial_a \partial_a) + 1][\lambda^2(\partial_a \partial_a) + 1]D_{a_2}D_{a_3} \} \\
\Phi(z_1; a_1)\Phi(z_2; a_2)\Phi(z_3; a_3) \big|_{a_1=a_2=a_3=0} \tag{6.2.28}
\]

Other terms are

\[
\mathcal{A}^{01}(\Phi(z)) = 0, \tag{6.2.29}
\]

\[
\mathcal{A}^{11}(\Phi(z)) = \frac{\lambda^2}{16} \exp \dot{W} \{ +[\lambda^2(\partial_a \partial_a) + 1](\partial_a \cdot \nabla)D_{a_2} \\
- [\lambda^2(\partial_a \partial_a) + 1](\partial_a \cdot \nabla)D_{a_1} \\
+ [\lambda^2(\partial_a \partial_a) + 1](\partial_a \cdot \nabla)D_{a_3} \\
- [\lambda^2(\partial_a \partial_a) + 1](\partial_a \cdot \nabla)D_{a_2} \\
+ [\lambda^2(\partial_a \partial_a) + 1](\partial_a \cdot \nabla)D_{a_2} \\
- [\lambda^2(\partial_a \partial_a) + 1](\partial_a \cdot \nabla)D_{a_3} \} \\
\Phi(z_1; a_1)\Phi(z_2; a_2)\Phi(z_3; a_3) \big|_{a_1=a_2=a_3=0} \tag{6.2.30}
\]

and so on.

All these expressions can be written in a very elegant form. First we introduce Grassmann variables by

\[
\eta_{a_1}, \bar{\eta}_{a_1}, \eta_{a_2}, \bar{\eta}_{a_2}, \eta_{a_3}, \bar{\eta}_{a_3}. \tag{6.2.31}
\]

Then we change expressions in the formula (6.2.12) in a following way

\[
(\partial_a \partial_a) \rightarrow (\partial_a \partial_a) + \frac{1}{4} \eta_{a_2} \bar{\eta}_{a_2} \Box_{a_2} + \frac{1}{4} \eta_{a_2} \bar{\eta}_{a_2} \Box_{a_1}, \tag{6.2.32}
\]

\[
(\partial_a \partial_a) \rightarrow (\partial_a \partial_a) + \frac{1}{4} \eta_{a_2} \bar{\eta}_{a_2} \Box_{a_3} + \frac{1}{4} \eta_{a_3} \bar{\eta}_{a_3} \Box_{a_2}, \tag{6.2.33}
\]

\[
(\partial_a \partial_a) \rightarrow (\partial_a \partial_a) + \frac{1}{4} \eta_{a_3} \bar{\eta}_{a_3} \Box_{a_2} + \frac{1}{4} \eta_{a_3} \bar{\eta}_{a_3} \Box_{a_1}, \tag{6.2.34}
\]

\[
(\partial_a \nabla_{a_2}) \rightarrow (\partial_a \nabla_{a_2}) + \eta_{a_1} \bar{\eta}_{a_1} D_{a_2} - \eta_{a_1} \bar{\eta}_{a_1} D_{a_3}, \tag{6.2.35}
\]

\[
(\partial_a \nabla_{a_2}) \rightarrow (\partial_a \nabla_{a_2}) + \eta_{a_2} \bar{\eta}_{a_2} D_{a_3} - \eta_{a_2} \bar{\eta}_{a_2} D_{a_1}, \tag{6.2.36}
\]

\[
(\partial_a \nabla_{a_2}) \rightarrow (\partial_a \nabla_{a_2}) + \eta_{a_3} \bar{\eta}_{a_3} D_{a_2} - \eta_{a_3} \bar{\eta}_{a_3} D_{a_2}. \tag{6.2.37}
\]
and can write
\[ \mathcal{A}(\Phi(z)) = \int d\eta_{a_1} d\bar{\eta}_{a_1} d\eta_{a_2} d\bar{\eta}_{a_2} d\eta_{a_3} d\bar{\eta}_{a_3} (1 + \eta_{a_1} \bar{\eta}_{a_1})(1 + \eta_{a_2} \bar{\eta}_{a_2})(1 + \eta_{a_3} \bar{\eta}_{a_3}) \]
\[ \exp \hat{W} \Phi(z_1; a_1) \Phi(z_2; a_2) \Phi(z_3; a_3) |_{a_1 = a_2 = a_3 = 0}. \quad (6.2.38) \]

where
\[ \hat{W} = \frac{1}{2} [1 + \lambda^2 (\partial_{a_1} \partial_{a_2} + \frac{1}{4} \eta_{a_1} \bar{\eta}_{a_2} \Box_{a_2} + \frac{1}{4} \eta_{a_2} \bar{\eta}_{a_1} \Box_{a_1})][\partial_{a_3} \nabla_{12} + \eta_{a_3} \bar{\eta}_{a_1} D_{a_1} - \eta_{a_3} \bar{\eta}_{a_2} D_{a_2}] \]
\[ + \frac{1}{2} [1 + \lambda^2 (\partial_{a_2} \partial_{a_3} + \frac{1}{4} \eta_{a_2} \bar{\eta}_{a_3} \Box_{a_3} + \frac{1}{4} \eta_{a_3} \bar{\eta}_{a_2} \Box_{a_2})][\partial_{a_1} \nabla_{23} + \eta_{a_1} \bar{\eta}_{a_2} D_{a_2} - \eta_{a_1} \bar{\eta}_{a_3} D_{a_3}] \]
\[ + \frac{1}{2} [1 + \lambda^2 (\partial_{a_3} \partial_{a_1} + \frac{1}{4} \eta_{a_3} \bar{\eta}_{a_1} \Box_{a_1} + \frac{1}{4} \eta_{a_1} \bar{\eta}_{a_3} \Box_{a_3})][\partial_{a_2} \nabla_{31} + \eta_{a_2} \bar{\eta}_{a_3} D_{a_3} - \eta_{a_2} \bar{\eta}_{a_1} D_{a_1}] \quad (6.2.39) \]

This operator generates all terms in the cubic interaction of any three HS fields with any possible number of derivatives \( \Delta \) in the range \( \Delta_{\min} \leq \Delta \leq \Delta_{\max} \).

Another possible form of the \( \hat{W} \) operator is

\[ \hat{W} = [1 + \lambda^2 (\partial_{a_1} \partial_{a_2} + \frac{1}{2} \eta_{a_1} \bar{\eta}_{a_2} \Box_{a_2})][(\partial_{a_3} \nabla_1) + \frac{1}{2} \eta_{a_3} \bar{\eta}_{a_1} D_{a_1} - \frac{1}{2} \eta_{a_3} \bar{\eta}_{a_2} D_{a_2} + \frac{1}{2} \eta_{a_3} \bar{\eta}_{a_2} D_{a_3}] \]
\[ + [1 + \lambda^2 (\partial_{a_2} \partial_{a_3} + \frac{1}{2} \eta_{a_2} \bar{\eta}_{a_3} \Box_{a_3})][(\partial_{a_1} \nabla_2) + \frac{1}{2} \eta_{a_1} \bar{\eta}_{a_2} D_{a_2} - \frac{1}{2} \eta_{a_1} \bar{\eta}_{a_3} D_{a_3} + \frac{1}{2} \eta_{a_1} \bar{\eta}_{a_3} D_{a_1}] \]
\[ + [1 + \lambda^2 (\partial_{a_3} \partial_{a_1} + \frac{1}{2} \eta_{a_3} \bar{\eta}_{a_1} \Box_{a_1})][(\partial_{a_2} \nabla_3) + \frac{1}{2} \eta_{a_2} \bar{\eta}_{a_3} D_{a_3} - \frac{1}{2} \eta_{a_2} \bar{\eta}_{a_1} D_{a_1} + \frac{1}{2} \eta_{a_2} \bar{\eta}_{a_1} D_{a_2}] \quad (6.2.40) \]

This case generates the Lagrangian derived in previous chapter. The leading term of that Lagrangian is \( \mathcal{L} \). Two operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) generate two Lagrangians that differ from each other just by partial integration and field redefinition. All interactions of HS gauge fields with any number of derivatives are unique and are generated by both operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \).

In the case of \( \mathcal{L}_1 \) we have

| \( \Box_{a_1} \) | 0 | 1 | 2 | 3 |
|----------------|----|----|----|----|
| 0              | 0  | 0  | 0  | 0  |
| 1              | 0  | A^{11} | A^{21} |
| 2              | A^{02} | A^{12} |
| 3              | A^{03} |

(6.2.41)
In the case of (6.2.40) we have

\[
\begin{array}{c|cccc}
\triangle D_{a_i} & 0 & 1 & 2 & 3 \\
\hline
0 & A^{00} & A^{10} & A^{20} & A^{30} \\
1 & 0 & A^{11} & A^{21} &  \\
2 & 0 & A^{12} &  &  \\
3 &  & A^{03} &  &  \\
\end{array}
\]  

(6.2.42)

Both forms of the same cubic Lagrangian are very useful for further investigations.
Chapter 7

Conclusions and outlook

In this Thesis I present first of all recent results in the theory of Higher Spin gauge field interactions. In addition a powerful method of constructing higher order conformal invariants is presented.

The cubic interaction problem in Higher Spin gauge field theory is solved in the most general case covering all possibilities. The other beauty of the result is the compactness and clearness of final formulas, that can be derived from a Generating Function, which reproduces all possible nontrivial interactions between massless Higher Spin gauge fields in a flat background spacetime. However, it is not trivial to continue this construction to higher orders on the gauge field. The solution of the main problem of existence of a local (or nonlocal) Lagrangian for Higher Spin gauge fields is very close now: the problem of the quartic interaction is an urgent and real task which could shed light on the solution in all orders.

These results can be used also to test AdS/CFT correspondence between critical O(N) sigma model and Higher Spin gauge field theory on AdS (after deformation to AdS from an even dimensional flat space).

Another important point here is that the interactions in flat space-time derived in this Thesis are independent of the space-time dimensions. At last we note that the structure of the Generating Function for these interactions leads to new connections with the massless regime of String Theory.
Appendix A

The Euclidean $\text{AdS}_{d+1}$ metric

$$ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu = \frac{1}{(z^0)^2} \delta_{\mu\nu} dz^\mu dz^\nu \quad (A.1)$$

can be realized as an induced metric for the hypersphere defined by the following embedding procedure in $d + 2$ dimensional Minkowski space

$$X^A X^B \eta_{AB} = -X_{-1}^2 + X_0^2 + \sum_{i=1}^{d} X_i^2 = -1, \quad (A.2)$$

$$X_{-1}(z) = \frac{1}{2} \left( \frac{1}{z_0} + \frac{z_0^2 + \sum_{i=1}^{d} z_i^2}{z_0} \right), \quad (A.3)$$

$$X_0(z) = \frac{1}{2} \left( \frac{1}{z_0} - \frac{z_0^2 + \sum_{i=1}^{d} z_i^2}{z_0} \right), \quad (A.4)$$

$$X_i(z) = \frac{z_i}{z_0}. \quad (A.5)$$

Using these embedding rules we can identify the variable $\zeta(z, w)$ as an $SO(1, d+1)$ invariant scalar product

$$- X^A(z) Y^B(w) \eta_{AB} = \frac{1}{2z_0w_0} \left( 2z_0w_0 + \sum_{\mu=0}^{d} (z - w)^2_\mu \right) = \zeta = u + 1, \quad (A.6)$$

and therefore can be realized by $\cosh$ of a hyperbolic angle. Indeed we can introduce another embedding

$$X_{-1}(\eta, \omega_\mu) = \cosh \eta, \quad (A.7)$$

$$X_\mu(\eta, \omega_\mu) = \sinh \eta \omega_\mu, \quad \sum_{\mu=0}^{d} \omega_\mu^2 = 1, \quad (A.8)$$

$$ds^2 = d\eta^2 + \sinh^2 \eta d\Omega_d. \quad (A.9)$$

In these coordinates the chordal distance $u$ between an arbitrary point $X^A(\eta, \Omega_\mu)$ and the pole of the hypersphere $Y^A(\eta = 0, \omega_\mu)$ is simply

$$\zeta = -X^A Y^B \eta_{AB} = \cosh \eta. \quad (A.10)$$
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Therefore the invariant measure is expressed as

\[
\sqrt{g}d\eta d\Omega_d = (\sinh \eta)^d d\eta d\Omega_d = [u(u + 2)]^{d-1} du d\Omega_d. \tag{A.11}
\]

So we see that the integration measure for \( d = 3 \) (\( D = d + 1 = 4 \)) will cancel one order of \( u^{-n} \) in short distance singularities and we have to count the singularities starting from \( u^{-2} \) which is "logarithmically divergent" in standard QFT terminology.

In this article we use the following rules and relations for \( u(z, z'), I_{1a}, I_{2c} \) and the bitensorial basis \( \{ I_i \}_i \):

\( \Box u = (d + 1)(u + 1), \quad \nabla_\mu \partial_\nu u = g_{\mu\nu}(u + 1), \quad g^{\mu\nu} \partial_\mu u \partial_\nu u = u(u + 2), \tag{A.12} \)

\( \partial_\mu \partial_\nu u \nabla_\mu u = (u + 1) \partial_\nu u, \quad \partial_\mu \partial_\nu u \nabla_\nu u = g_{\mu\nu} + \partial_\mu u \partial_\nu u, \tag{A.13} \)

\( \nabla_\mu \partial_\nu u \nabla_\nu u = \partial_\nu u \partial_\nu u, \quad \nabla_\mu \partial_\nu u = g_{\mu\nu}, \tag{A.14} \)

\( \frac{\partial}{\partial a_\mu} I_{1a} \frac{\partial}{\partial a_\mu} I_{1a} = u(u + 2), \quad \frac{\partial}{\partial a_\mu} I_1 \frac{\partial}{\partial a_\mu} I_1 = (u + 1) I_{2c}, \tag{A.15} \)

\( \frac{\partial}{\partial a_\mu} I_1 \frac{\partial}{\partial a_\mu} I_1 = c_2^2 + I_{2c}^2, \quad \frac{\partial}{\partial a_\mu} I_2 \frac{\partial}{\partial a_\mu} I_2 = (u + 1) I_{2c}^2, \quad \Box I_4 = 2(d + 1) c_2^2, \tag{A.16} \)

\( \frac{\partial}{\partial a_\mu} I_2 \frac{\partial}{\partial a_\mu} I_2 = u(u + 2) I_{2c}^2, \quad \Box I_3 = 2(d + 1) I_{2c}^2 + 2c_2^2 u(u + 2), \tag{A.17} \)

\( \nabla_\mu \frac{\partial}{\partial a_\mu} I_1 = (d + 1) I_{2c}, \quad \nabla_\mu \frac{\partial}{\partial a_\mu} I_2 = (d + 2)(u + 1) I_{2c}, \quad \nabla_\mu I_1 \partial_\mu u = I_2, \tag{A.18} \)

\( \nabla_\mu \frac{\partial}{\partial a_\mu} I_3 = 4 I_1 I_{2c} + (d + 2)(u + 1) c_2^2 I_{1a}, \quad \nabla_\mu I_2 \partial_\mu u = 2(u + 1) I_2, \tag{A.19} \)

\( \frac{\partial}{\partial a_\mu} I_1 \partial_\mu u = (u + 1) I_{2c}, \quad \frac{\partial}{\partial a_\mu} I_2 \partial_\mu u = u(u + 2) I_{2c}, \quad \frac{\partial}{\partial a_\mu} I_1 \nabla_\mu I_1 = I_1 I_{2c}, \tag{A.20} \)

\( \frac{\partial}{\partial a_\mu} I_2 \nabla_\mu I_2 = 2(u + 1) I_{2c} I_2, \quad \nabla_\mu I_1 \nabla_\mu I_1 = a_1^2 I_{2c}^2, \quad \Box I_1 = I_1, \tag{A.21} \)

\( \nabla_\mu I_1 \nabla_\mu I_2 = I_{2c} (u + 1) I_1 + (u + 1) I_{2c}^2, \quad \Box I_2 = (d + 2) I_2 + 2(u + 1) I_1, \tag{A.22} \)

\( \nabla_\mu I_2 \nabla_\mu I_2 = I_{2c}^2 + 2(u + 1) I_1 I_2 + a_1^2 I_{2c}^2 (u + 1)^2 + c_2^2 I_{1a}^2, \tag{A.23} \)

\( a_1^2 \nabla_\mu I_{1a} = \alpha^2(u + 1), \quad a_1^2 \nabla_\mu I_{2c} = I_1, \quad a_1^2 \nabla_\mu I_1 = a^2 I_{2c}, \quad a_1^2 \nabla_\mu I_2 = a^2(u + 1) I_{2c} + I_{1a} I_1, \tag{A.24} \)

Using these relations we can derive (\( F'_k := \frac{\partial}{\partial a} F_k(u) \))

- Divergence map

\[
\nabla_\mu \frac{\partial}{\partial a_\mu} \Psi^\ell[F] = I_{2c} \Psi^{\ell-1}[\text{Div}_\ell F] + O(c_2^2), \tag{A.27}
\]

\[
(D_{\text{Div}} F)_k = (\ell - k)(u + 1) F'_k + (k + 1) u(u + 2) F'_k \tag{A.28}
\]

\[+(\ell - k)(\ell + d + k + 1)(u + 1) F'_{k+1}(A.28)\]
CHAPTER 7. CONCLUSIONS AND OUTLOOK

• Trace map

\[ \Box_\ell \Psi^\ell [F] = I_2^2 \Psi^{\ell - 2}[\text{Tr}_\ell F] + O(c_2^2), \]  
(A.29)

\[ (\text{Tr}_\ell F)_k = (\ell - k)(\ell - k - 1)F_k + 2(k + 1)(\ell - k - 1)(u + 1)F_{k+1} + (k + 2)(k + 1)u(u + 2)F_{k+2}. \]  
(A.30)

• Laplacian map

\[ \Box_1 \Psi^\ell [F] = \Psi^\ell [\text{Lap}_\ell F] + O(a_1^2, c_2^2), \]  
(A.31)

\[ (\text{Lap}_\ell F)_k = u(u + 2)F''_k + (d + 1 + 4k)(u + 1)F'_k + [\ell + k(d + 2\ell - k)]F_k \]  
\[ + 2(u + 1)(k + 1)^2F_{k+1} + 2(\ell - k + 1)F'_{k-1}, \]  
(A.32)

\[ \Box F_k(u) = u(u + 2)F''_k + (d + 1)(u + 1)F'_k. \]  
(A.33)

• Gradient map

\[ (a \cdot \nabla)_1 \Psi^\ell [F] = I_{1a} \Psi^\ell [\text{Grad}_\ell F] + O(a_1^2), \]  
(A.34)

\[ (\text{Grad}_\ell F)_k = F''_k + (k + 1)F_{k+1}. \]  
(A.35)

At the end we present all important commutation relations working in the space of symmetric rank \( n \) tensors

\[ [[\nabla \partial_a, \Box] f^{(n)}(z, a)] = 2(a \nabla)\Box a - (d + 2n - 2)(\nabla \partial_a) f^{(n)}(z, a); \]  
(A.36)

\[ [[\nabla \partial_a, (a \nabla)] f^{(n)}(z, a)] = \Box f^{(n)}(z, a) + [\nabla_\mu, (a \nabla)] \partial^\mu_a f^{(n)}(z, a); \]  
(A.37)

\[ [\nabla_\mu, (a \nabla)] \partial^\mu_a f^{(n)}(z, a) = [a^2 \Box a - n(d + n - 1)] f^{(n)}(z, a); \]  
(A.38)

\[ [\Box, (a \nabla)] f^{(n)}(z, a) = [2a^2(\nabla \partial_a) - (d + 2n)(a \nabla)] f^{(n)}(z, a); \]  
(A.39)

\[ a^2 a f^{(n)}(z, a)] = 2(d + 2n + 1) f^{(n)}(z, a) + a^2 \Box a f^{(n)}(z, a). \]  
(A.40)
Appendix B

These two useful hypergeometric identities we learned from the book of H. Bateman and A. Erdelyi “Higher transcendental functions” V.1, McGraw-Hill Book company Inc. 1953.

\[ _2F_1(a, b, 2b; z) = \left( \frac{1 - z}{2} \right)^{-a} _2F_1 \left( \frac{a + 1}{2}, b + \frac{1}{2}; \left( \frac{z}{2 - z} \right)^2 \right), \quad (B.1) \]

\[ _2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)} (-z)^{-a} _2F_1(a, 1 - c + a, 1 - b + a; z^{-1}) \]
\[ + \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)} (-z)^{-b} _2F_1(a, 1 - c + b, 1 - a + b; z^{-1}). (B.2) \]
Appendix C

Here we present the basic relations between different T-s, M-s and N-s which we use in section 4.2.

\[
T(n, k) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} T(n - i, k + m - i), \quad (C.1)
\]

\[
T(n, k) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} T(n - i, k - m), \quad (C.2)
\]

and the same for M and N. There is another important relation

\[
T(n, k) = -M(n - 1, k) - \frac{k(k - 1)}{2L^2} N(n - 1, k - 1)
- \left[ \frac{(n - k - 1)(2D + n - k - 4)}{2L^2} + \frac{D(D - 2)}{4L^2} \right] N(n - 1, k)
- \epsilon_{(\ell-n)} \nabla_{\mu_1} \cdots \nabla_{\mu_k} \phi \nabla_{\mu_{k+1}} \cdots \nabla_{\mu_{n-1}} \left( \Box - \frac{D(D - 2)}{4L^2} \right) \phi, \quad (C.3)
\]

and the 'symmetrization' relations

\[
M(2k + 1, k) = M(2k + 1, k + 1) = -\frac{1}{2} M(2k, k), \quad (C.4)
\]

\[
N(2k + 1, k) = N(2k + 1, k + 1) = -\frac{1}{2} N(2k, k), \quad (C.5)
\]

\[
T(2m, m) = \nabla^{(\alpha} \epsilon_{(\ell-2m)}^{\mu_1 \cdots \mu_{2m-1})} \nabla_{\mu_1} \cdots \nabla_{\mu_{m-1}} \nabla_{\alpha} \phi \nabla_{\mu_m} \cdots \nabla_{\mu_{2m-1}} \phi
+ \frac{(m - 1)(m - 2)}{6L^2} N(2m - 2, m - 1) - \frac{(m - 1)(m - 2)}{12L^2} N(2m - 4, m - 2), \quad (C.6)
\]

\[
M(2m - 2, m - 1) = \epsilon_{(\ell-2m+1)}^{\mu_1 \cdots \mu_{2m-2}} \nabla_{(\mu_1} \cdots \nabla_{\mu_{m-1}} \nabla_{\alpha)} \phi \nabla^{(\mu_m} \cdots \nabla^{\mu_{2m-2}} \nabla^{\alpha)} \phi
+ \frac{(m - 1)(m - 2)}{3L^2} N(2m - 2, m - 1, m - 1) - \frac{(m - 1)(m - 2)}{6L^2} N(2m - 4, m - 2) \quad (C.7)
\]

We must mention here that these relations are satisfied up to full derivatives and therefore admit integration.
Appendix D

We use the following commutation relations in $AdS_D$

\[
\epsilon^{\mu_1...\mu_{\ell-1}}[\nabla^\mu, \nabla_{\mu_1}...\nabla_{\mu_k}]\phi = \frac{k(k-1)}{2L^2} \epsilon^{\mu\mu_2...\mu_{\ell-1}}\nabla_{\mu_2}...\nabla_{\mu_k}\phi, \quad (D.1)
\]

\[
[\nabla_{\mu_1}...\nabla_{\mu_k}, \nabla^\mu]\epsilon^{\mu_1...\mu_{\ell-1}} = \frac{2k(D+\ell-2) - k(k+1)}{2L^2} \epsilon^{\mu_{\mu_k+1}...\mu_{\ell-1}}, \quad (D.2)
\]

\[
\epsilon^{\mu_1...\mu_{\ell-1}}[\nabla_\mu, \nabla_{\mu_1}...\nabla_{\mu_k}]\nabla^\mu \phi = \frac{k(2D+k-3)}{2L^2} \epsilon^{\mu_1\mu_2...\mu_{\ell-1}}\nabla_{\mu_1}...\nabla_{\mu_k}\phi, \quad (D.3)
\]

\[
\epsilon^{\mu_1...\mu_{\ell-1}}[\nabla^2, \nabla_{\mu_1}...\nabla_{\mu_k}]\phi = \frac{k(D+k-2)}{L^2} \epsilon^{\mu_1\mu_2...\mu_{\ell-1}}\nabla_{\mu_1}...\nabla_{\mu_k}\phi, \quad (D.4)
\]

where $\epsilon^{\mu_1...\mu_{\ell-1}}$ is the symmetric and traceless tensor. Finally we list all necessary binomial identities

\[
\sum_{k=0}^{n-m}(-1)^k\binom{n}{k} = (-1)^{n-m}\binom{n-1}{m-1}, \quad \sum_{k=0}^{n-m}(-1)^k\binom{n}{m+k} = \binom{n-1}{m-1}, \quad (D.5)
\]

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \binom{\ell-m-1}{m-2} = \frac{m-1}{\ell-2m+1}\binom{\ell-m-1}{m-1}. \quad (D.6)
\]
Appendix E

Here we present all Weyl variations necessary for the derivation of (4.2.63)-(4.2.66)

\[
\delta_0 S_0 = \Delta_\ell \int d^D z \sqrt{-g} \left\{ \sum_{m=1}^{\ell-1} \binom{\ell - m - 2}{m - 1} \nabla^{(\mu_2^m \sigma^{\mu_1 \ldots \mu_{2m-1}})} \Psi(2m)_{\mu_1 \ldots \mu_{2m}} 
+ \sum_{m=1}^{\ell-1} \frac{(-1)^{m-1}}{2} \binom{\ell - m - 3}{m - 1} \Box \sigma^{\mu_1 \ldots \mu_{2m}}_{(\ell - 2m - 2)} \nabla_{\mu_1} \ldots \nabla_{\mu_m} \phi \nabla_{\mu_{m+1}} \ldots \nabla_{\mu_{2m}} \phi 
+ \sum_{m=1}^{\ell-1} (-1)^m \binom{\ell - m - 3}{m - 1} \sigma^{\mu_1 \ldots \mu_{2m}}_{(\ell - 2m - 2)} \nabla_{\mu_1} \ldots \nabla_{\mu_m} \nabla_\alpha \phi \nabla_{\mu_{m+1}} \ldots \nabla_{\mu_{2m}} \nabla_\alpha \phi 
+ O\left(\frac{1}{L^2}\right) \right\}.
\]

(E.1)

We don’t have to calculate \( O\left(\frac{1}{L^2}\right) \) terms because they can be fixed from flat space considerations and gauge invariance of Fronsdal’s operator in AdS. The first term in (E.1) can be cancelled by an additional gauge transformation of all gauge fields with spin less than \( \ell \). To cancel other terms we calculate the variation of \( \sum_{m=1}^{\ell/2} S_1^{\Psi(2m)}(\phi, h^{(2m)}) \):

\[
\delta_0 S_1^{\Psi(2m)}(\phi, h^{(2)}, \ldots, h^{(2m)}) 
= C_{\ell}^m \int d^D z \sqrt{-g} \left\{ -(m - 1) \left[ \nabla^{(\mu_{2m-2} \sigma^{\mu_1 \ldots \mu_{2m-3}})}_{(\ell - 2m + 1)} \Psi(2m-2)_{\mu_1 \ldots \mu_{2m-2}} 
+ \frac{(-1)^{m}}{2} \Box \sigma^{\mu_1 \ldots \mu_{2m-2}}_{(\ell - 2m)} \nabla_{\mu_1} \ldots \nabla_{\mu_{m-1}} \phi \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-2}} \phi \right] 
+ (-1)^m \left( 1 - \frac{D}{2} \right) \sigma^{\mu_1 \ldots \mu_{2m-2}}_{(\ell - 2m)} \nabla_{\mu_1} \ldots \nabla_{\mu_{m-1}} \nabla_\alpha \phi \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-2}} \nabla_\alpha \phi \right\}.
\]

(E.2)
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and the variation of $\sum_{m=1}^{\ell/2} S_1^{(2m)}(\phi, h^{(2m)})$:

$$
\delta^0_\sigma S_1^{(\ell)} = \frac{1}{2} \sum_{m=1}^{\ell-1} \int d^D z \sqrt{-g} \left\{ [2m(2m-1)\xi^m_\ell - 2(m-1)(D + 4m - 8)\xi^{m-1}_\ell] \times
\right.
$$

$$
\times \nabla^{(\mu_2m-2}\sigma^{\mu_1...\mu_{2m-3}}(\ell-2m+1)}_\mu^{(2m-2)} \Psi^{(2m-2)} \}
$$

$$
- \frac{1}{2} \int d^D z \sqrt{-g} \left\{ (\ell - 2)(D + 2\ell - 8)\xi^{\ell/2-1}_\ell \nabla^{(\mu_\ell-2}\sigma^{\mu_1...\mu_{\ell-3}}(\ell-1)}_\mu^{(\ell-2)} \right\}
$$

$$
+ \frac{1}{2} \sum_{m=1}^{\ell-1} \xi^m_\ell \int d^D z \sqrt{-g} \left\{ 2m(1 - \frac{D}{2})\sigma^{\mu_1...\mu_{2m-2}}_\ell \nabla_{\mu_1}...\nabla_{\mu_{m-1}} \nabla_\alpha \phi \nabla_{\mu_m}...\nabla_{\mu_{2m-2}} \nabla_\alpha \phi \right\}
$$

$$
+ \frac{1}{2} \sum_{m=1}^{\ell-1} \int d^D z \sqrt{-g} \left\{ -m(m - 1)\xi^m_\ell - (\ell - 2m + 2)(D + \ell + 2m - 5)\xi^{m-1}_\ell \times
\right.
$$

$$
\times \Box^{(\mu_2m-2} \nabla_{\mu_1}...\nabla_{\mu_{m-1}} \phi \nabla_{\mu_m}...\nabla_{\mu_{2m-2}} \phi \right\}
$$

$$
- \frac{1}{2} \int d^D z \sqrt{-g} \left\{ 2(D + 2\ell - 5)\xi^{\ell/2-1}_\ell \Box^{(\mu_1...\mu_{\ell-2}} \nabla_{\mu_1}...\nabla_{\mu_{\ell/2-1}} \phi \nabla_{\mu_{\ell/2}}...\nabla_{\mu_{\ell-2}} \phi \right\}
$$

$$
+ O\left(\frac{1}{L^2}\right) \quad (E.3)
$$
\[ \delta_\sigma S^{W1}(\phi, h^{(2)}, \ldots, h^{(\ell)}) = \delta_\sigma S_0 + \sum_{s=1}^{\ell/2} \delta_\sigma S_1 + \sum_{s=1}^{\ell/2} \delta_\sigma S_{1}^{(s)} \]

\[ = \sum_{m=1}^{\ell/2} \int d^Dz \sqrt{-g} \left\{ \left( \frac{\ell - m - 2}{m - 2} \right) \Delta_\ell - mC_{\ell}^{m+1} \left[ 1 + (D + 4m - 4)\xi_{2m+2}^{m} \right] \right\} \]

\[ + \frac{1}{2} \sum_{s=m+2}^{\ell/2} \left\{ [2m + 2)(2m + 1)\xi_{2s+1}^{m+1} - 2m(D + 4m - 4)\xi_{2s}^{m}] \right\} \times \]

\[ \times \nabla^{(\mu_2m) \sigma_{(\ell - 2m - 1)}^{\mu_1 \ldots \mu_2m - 1}} \Psi^{(2m)}_{\mu_1 \ldots \mu_2m} \]

\[ + \int d^Dz \sqrt{-g} \left\{ \frac{(-1)^{\ell/2}}{2} \left( \Delta_\ell - \frac{\ell - 2}{2} \right) - (D + 2\ell - 5)\xi_{\ell}^{\ell/2 - 1} \right\} \times \]

\[ \times \Box \sigma_{(\ell - 2m)}^{\mu_1 \ldots \mu_{\ell - 2} \mu_1 \ldots \mu_{\ell - 2}} \phi \nabla_{\mu_1} \ldots \nabla_{\mu_{\ell - 2}} \phi \]

\[ + \sum_{m=1}^{\ell/2} \int d^Dz \sqrt{-g} \left\{ \frac{(-1)^m}{2} \left[ \left( \frac{\ell - m - 2}{m - 2} \right) \Delta_\ell - (m - 1)C_{\ell}^{m} - C_{\ell}^{m}(D + 4m - 5)\xi_{2m+1}^{m-1} \right] \right\} \times \]

\[ \times \Box \sigma_{(\ell - 2m)}^{\mu_1 \ldots \mu_{\ell - 2} \mu_1 \ldots \mu_{\ell - 2}} \phi \nabla_{\mu_1} \ldots \nabla_{\mu_{\ell - 2}} \phi \]

\[ + \int d^Dz \sqrt{-g} \{ (-1)^{\ell/2}(1 - \frac{D}{2} - \Delta_\ell) \sigma_{(\ell - 2m)}^{\mu_1 \ldots \mu_{\ell - 2} \mu_1 \ldots \mu_{\ell - 2}} \nabla_{\mu_1} \ldots \nabla_{\mu_{\ell - 2}} \nabla^{\alpha} \phi \}
\]

\[ + \sum_{m=1}^{\ell/2} \int d^Dz \sqrt{-g} \{ (-1)^{m-1} \left( \frac{\ell - m - 2}{m - 2} \right) \Delta_\ell + (-1)^{m}(1 - \frac{D}{2})C_{\ell}^{m} \}
\]

\[ + \left( 1 - \frac{D}{2} \right) \sum_{s=m+1}^{\ell/2} mC_{s}^{\ell} \xi_{2s}^{m} \sigma_{(\ell - 2m)}^{\mu_1 \ldots \mu_{\ell - 2} \mu_1 \ldots \mu_{\ell - 2}} \nabla_{\mu_1} \ldots \nabla_{\mu_{\ell - 2}} \nabla^{\alpha} \phi \] (E.4)

From this expression we can derive our system of equations \([4.2.63]-[4.2.66]\).
Appendix F
Proof of $\mathcal{L}_I^{(\hat{i},\hat{j})}$

The expression for $\mathcal{L}_I^{(1,0)}$ (5.3.39) is right since the following remaining group of terms vanishes:

\begin{align*}
&+\frac{s_1 n_1 s_3}{2} C_{(n_i)}^{(s_1)} [\hat{T}(Q_{ij}|n_1 - 1, n_2, n_3), (c \nabla_3)] (\epsilon^{(s_1-1)} h^{(s_2)} D^{(s_3-1)} + D^{(s_1-1)} h^{(s_2)} \epsilon^{(s_3-1)}) \quad (F.1) \\
&+\frac{s_1 n_1 s_2}{2} C_{(n_i)}^{(s_1)} [\hat{T}(Q_{ij}|n_1 - 1, n_2, n_3), (b \nabla_2)] (D^{(s_1-1)} \epsilon^{(s_2-1)} h^{(s_3)} - \epsilon^{(s_1-1)} D^{(s_2-1)} h^{(s_3)}) \quad (F.2) \\
&+\frac{s_2 n_2 s_1}{2} C_{(n_i)}^{(s_1)} [\hat{T}(Q_{ij}|n_1, n_2 - 1, n_3), (a \nabla_1)] (D^{(s_1-1)} \epsilon^{(s_2-1)} h^{(s_3)} + \epsilon^{(s_1-1)} D^{(s_2-1)} h^{(s_3)}) \quad (F.3) \\
&+\frac{s_2 n_2 s_3}{2} C_{(n_i)}^{(s_1)} [\hat{T}(Q_{ij}|n_1, n_2 - 1, n_3), (c \nabla_3)] (\epsilon^{(s_1)} h^{(s_2)} D^{(s_3-1)} - h^{(s_1)} \epsilon^{(s_2-1)} D^{(s_3-1)}) \quad (F.4) \\
&+\frac{s_3 n_3 s_2}{2} C_{(n_i)}^{(s_1)} [\hat{T}(Q_{ij}|n_1, n_2, n_3 - 1), (b \nabla_2)] (h^{(s_1)} D^{(s_2-1)} \epsilon^{(s_3-1)} + h^{(s_1)} \epsilon^{(s_2-1)} D^{(s_3-1)}) \quad (F.5) \\
&+\frac{s_3 n_3 s_1}{2} C_{(n_i)}^{(s_1)} [\hat{T}(Q_{ij}|n_1, n_2, n_3 - 1), (a \nabla_1)] (\epsilon^{(s_1-1)} h^{(s_2)} D^{(s_3-1)} - D^{(s_1-1)} h^{(s_2)} \epsilon^{(s_3-1)}) \quad (F.6) \\
&-s_1 Q_{12} s_2 C_{(n_i)}^{(s_1)} [\hat{T}(Q_{12} - 1, Q_{23}, Q_{31}|n_i)] \epsilon^{(s_1-1)} D^{(s_2-1)} h^{(s_3)} \quad (F.7) \\
&-s_2 Q_{23} s_3 C_{(n_i)}^{(s_1)} [\hat{T}(Q_{12}, Q_{23} - 1, Q_{31}|n_i)] h^{(s_1)} \epsilon^{(s_2-1)} D^{(s_3-1)} \quad (F.8) \\
&-s_3 Q_{31} s_1 C_{(n_i)}^{(s_1)} [\hat{T}(Q_{12}, Q_{23}, Q_{31} - 1|n_i)] D^{(s_1-1)} h^{(s_2)} \epsilon^{(s_3-1)}. \quad (F.9)
\end{align*}

Indeed calculating commutators in the leading order and using relation (5.3.35) we see that

\begin{align*}
(F.1) + (F.6) \\
&= s_1 s_2 (Q_{23} + 1) C_{n_1+1,n_2,n_3}^{s_1} \hat{T}(Q_{12}, Q_{23}, Q_{31}|n_1, n_2 + 1, n_3) D^{(s_1-1)} h^{(s_2)} \epsilon^{(s_3-1)}, \quad (F.10)
\end{align*}

which exactly cancels (F.9) after a corresponding shift of $n_2$ and using relation (5.3.36). In a similar way we can prove cancelation of the other two sets of three lines.

To prove formulas for $\mathcal{L}_I^{(2,0)}$ and $\mathcal{L}_I^{(3,0)}$ we should manage the commutators of
T operators with a, b, c, gradients in the following expression

\[
\frac{s_1 s_2 s_3}{2} C_{(n_1)}^{(s_i)} \left[ \begin{array}{l}
n_1 n_3 \hat{T}(Q_{ij} | n_1 - 1, n_2, n_3 - 1), (b \nabla_2) \right] \left( D^{(s_1-1)} D^{(s_2-1)} \epsilon^{(s_3-1)} + D^{(s_1-1)} \epsilon^{(s_2-1)} D^{(s_3-1)} \right) \\
n_2 n_3 \hat{T}(Q_{ij} | n_1, n_2 - 1, n_3 - 1), (a \nabla_1) \right] \left( D^{(s_1-1)} \epsilon^{(s_2-1)} D^{(s_3-1)} + \epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)} \right) \\
n_1 n_2 \hat{T}(Q_{ij} | n_1 - 1, n_2 - 1, n_3), (c \nabla_3) \right] \left( \epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)} + D^{(s_1-1)} D^{(s_2-1)} \epsilon^{(s_3-1)} \right)
\end{array} \right]
\]

\[
- n_3 Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2, n_3 - 1) \left( \epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)} - \epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)} \right) \\
- n_2 Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1, n_2 - 1, n_3) \left( D^{(s_1-1)} D^{(s_2-1)} \epsilon^{(s_3-1)} - D^{(s_1-1)} \epsilon^{(s_2-1)} D^{(s_3-1)} \right) \\
- n_1 Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1 - 1, n_2, n_3) \left( \epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)} - \epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)} \right)
\]

(F.11)

and use again (F.3.34)-(F.3.36) to show that (F.11) is zero.

The remaining terms are:

\[
\frac{1}{2} C_{(n_1)}^{(s_i)} \left[ \left( \epsilon^{(s_1-1)} \hat{h}^{(s_2-2)} \hat{h}^{(s_3)} \right) \right]
\]

\[
- n_3 Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2, n_3 - 1), (b \nabla_2) \right] \left( D^{(s_1-1)} \epsilon^{(s_2-2)} D^{(s_3-1)} \right) \\
- n_2 Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1, n_2 - 1, n_3) \left( \epsilon^{(s_1-1)} \epsilon^{(s_2-1)} \hat{h}^{(s_3-2)} \right) \\
- n_1 Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1 - 1, n_2, n_3) \left( \epsilon^{(s_1-1)} + \epsilon^{(s_2-1)} \hat{h}^{(s_3-2)} \right)
\]

(F.12)

and

\[
\frac{s_1 s_2 s_3}{4} C_{(n_1)}^{(s_i)} \left[ \left( \delta \hat{h}^{(s_2-2)} D^{(s_3-1)} \epsilon^{(s_3-1)} + 2(\nabla D)^{(s_1-1)} \hat{h}^{(s_2)} \epsilon^{(s_3-1)} \right) \right]
\]

\[
- s_3 s_1 (s_1 - 1) n_1 C_{(n_1)}^{(s_i)} Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1 - 1, n_2, n_3) \\
- s_1 s_2 (s_2 - 1) n_2 C_{(n_1)}^{(s_i)} Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3) \\
- s_2 s_3 (s_3 - 1) n_3 C_{(n_1)}^{(s_i)} Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1, n_2, n_3 - 1)
\]

(F.13)

\[
- \frac{s_3 s_1 (s_1 - 1) n_1}{4} C_{(n_1)}^{(s_i)} Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1 - 1, n_2, n_3) \\
- \frac{s_1 s_2 (s_2 - 1) n_2}{4} C_{(n_1)}^{(s_i)} Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3) \\
- \frac{s_2 s_3 (s_3 - 1) n_3}{4} C_{(n_1)}^{(s_i)} Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1, n_2, n_3 - 1)
\]

(F.14)
The last $DD\bar{h}$ terms coming from our calculation are:

\[
\frac{s_1 s_2 s_3}{4} C_{(n_1)}^{(s_i)} \left[ 
- (s_3 - 1)n_1 n_3 Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1 - 1, n_2, n_3 - 1) \\
( D^{(s_1 - 1)} D^{(s_2 - 1)} \delta \bar{h}^{(s_3 - 2)} + 2 D^{(s_1 - 1)} \epsilon^{(s_2 - 1)} (\nabla D)^{(s_3 - 2)} ) \\
- (s_2 - 1)n_2 n_3 Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3 - 1) \\
( D^{(s_1 - 1)} \delta \bar{h}^{(s_2 - 2)} D^{(s_3 - 1)} + 2 \epsilon^{(s_1 - 1)} (\nabla D)^{(s_2 - 2)} D^{(s_3 - 1)} ) \\
- (s_1 - 1)n_1 n_2 Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1 - 1, n_2 - 1, n_3) \\
( \delta \bar{h}^{(s_1 - 2)} D^{(s_2 - 1)} D^{(s_3 - 1)} + 2 (\nabla D)^{(s_3 - 2)} D^{(s_2 - 1)} \epsilon^{(s_1 - 1)} ) \right].
\] (F.15)

These terms can be used in the same fashion for proving the remaining part of $\mathcal{L}_{i,j}^{(i,j)}$ to contain traces.
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