RIESZ MEANS AND BILINEAR RIESZ MEANS ON MÉTIVIER GROUPS

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ABSTRACT. In this paper, we investigate the $L^p$-boundedness of the Riesz means and the $L^{p_1} \times L^{p_2} \rightarrow L^p$ boundedness of the bilinear Riesz means on Métivier groups. Métivier groups are generalization of Heisenberg groups and general H-type groups. Because general Métivier groups only satisfy the non-degeneracy condition and have high-dimensional centre, we have to use different methods and techniques from those on Heisenberg groups and H-type groups.

1. Introduction

Métivier groups, introduced by Métivier [9] in his study of analytic hypoellipticity, are two-step nilpotent Lie groups satisfying a non-degeneracy condition. They are also characterized by the property that the quotients with respect to the hyperplanes contained in the centre are general Heisenberg groups. H-type groups, introduced by Kaplan [5], are typical examples of Métivier groups. Heisenberg groups are the simplest Métivier groups. Casarino and Ciatti [2] investigated the spectral resolution of the sub-Laplacian on Métivier groups, and proved a Stein-Tomas restriction theorem in terms of the mixed norms. So, we can use the spectral decomposition of the sub-Laplacian to define the Riesz means and the bilinear Riesz means on Métivier groups. Mauceri [8] and Müller [11] obtained the same results on the $L^p$-boundedness of the Riesz means on Heisenberg groups by using different methods. We proved the $L^{p_1} \times L^{p_2} \rightarrow L^p$ boundedness of the bilinear Riesz means on Heisenberg groups [9, 6]. To extend these results to general Métivier groups, we notice that there are two essential difficulties: one is that Heisenberg groups have one dimensional centre but the centre dimension of Métivier groups is in general bigger than one; the other is that on H-type groups, the non-degeneracy condition becomes a better orthogonality condition, which is not true on general Métivier groups. Therefore, in this paper, we shall introduce some different techniques from those used in [8, 9] and [11, 6] to obtain the $L^p$-boundedness of the Riesz means and $L^{p_1} \times L^{p_2} \rightarrow L^p$ boundedness of the bilinear Riesz means on Métivier groups.

This paper is organized as follows. In Section 2, we recall the spectral decomposition of the sub-Laplacian on Métivier groups and define the Riesz means and bilinear Riesz means. In Section 3, we give the pointwise estimates for the kernel of the Riesz means and the kernel of the bilinear Riesz means. In Section 4, we present the $L^p$-boundedness of the Riesz means. In the rest of this paper, we study the $L^{p_1} \times L^{p_2} \rightarrow L^p$ boundedness of the bilinear Riesz means, which for the case of $1 \leq p_1, p_2 \leq 2$ in Section 5 and for some particular cases in Section 6. In Section 7, we outline the bilinear interpolation method and obtain the results in other cases.

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2. Preliminaries

We first recall Métivier groups. Let $\mathfrak{g}$ be a real finite dimensional two step nilpotent Lie algebra, equipped with an inner product $\langle \cdot, \cdot \rangle$. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$, $[\mathfrak{g}_1, \mathfrak{g}_2] = [\mathfrak{g}_2, \mathfrak{g}_2] = 0$ and $\dim \mathfrak{g}_1 = d$, $\dim \mathfrak{g}_2 = m$. Let $\{X_1, X_2, \cdots, X_d\}$ be an orthonormal basis of $\mathfrak{g}_1$, $\{U_1, U_2, \cdots, U_m\}$ be an orthonormal basis of $\mathfrak{g}_2$.

Define for $\mu \in \mathfrak{g}_2$ the skew symmetric form $\omega_\mu$ on $\mathfrak{g}_1$ by

$$\omega_\mu(X, Y) = \mu(\langle X, Y \rangle)$$

and the corresponding metric

$$(J_{\mu})_{jk} = \omega_\mu(X_j, X_k).$$

We say that the corresponding connected, simply connected Lie group $G$ of $\mathfrak{g}$ is a Métivier group (M-type group) if $\omega_\mu$ is non-degenerate for all $\mu \neq 0$. Under this non-degenerate hypothesis, $J_\mu$ is a $d \times d$ skew symmetric, invertible metric for all $\mu \neq 0$. It follows that $d = 2n$ is even. Since the exponential mapping is a bijection, we shall parametrise the element $g = \exp \left( \sum_{i=1}^{2n} x_i X_i + \sum_{j=1}^{m} u_j U_j \right) \in G$ by $(x, u)$ where $x = (x_1, \cdots, x_{2n}) \in \mathbb{R}^{2n}$, $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$. By the Baker-Campbell-Hausdorff formula, the group multiplication is given by

$$(x, u) \cdot (y, v) = (x + y, u + v + \frac{1}{2} [x, y]),$$

where $[x, y] = \langle x, J_1 y \rangle, \langle x, J_2 y \rangle, \cdots, \langle x, J_m y \rangle \rangle \in \mathbb{R}^m$, $J_k$ is $2n \times 2n$ skew-symmetric, non-degenerate metric for any $k = 1, \cdots, m$. The identity element of $G$ is $(0, 0)$ and the inversion of $(x, u)$ is denoted by $(x, u)^{-1} = (-x, -u)$.

A homogeneous structure on $G$ is obtained by defining the dilations $\delta_t(x, u) = (tx, t^2 u)$, $t > 0$. The homogeneous dimension of $G$ is

$$Q = 2n + 2m.$$

The Haar measure on $G$ coincides with the Lebesgue measure on $\mathfrak{g}$ denoted by $dxdu$. It is easy to verify that the Jacobian determinant of the dilations $\delta_t$, $t > 0$ is constant, equal to $t^Q$. Define a homogeneous norm of degree one under the dilations $\{\delta_t, t > 0\}$ on $G$ by

$$|\omega| = |\langle x, u \rangle| = \left( \frac{1}{16} |x|^4 + |u|^2 \right)^{\frac{1}{4}}, \quad \omega = (x, u) \in G.$$

This norm satisfies the triangle inequality $|\omega \cdot \omega'| \leq |\omega| + |\omega'|$ and leads to a left-invariant distance $d(\omega, \omega') = |\omega^{-1} : \omega'|$.

For any $f, g \in L^1(G)$, their convolution is defined by

$$f * g(x, u) = \int_G f((x, u) \cdot (y, v)^{-1})g(y, v)dydv.$$

The $\mu$-twisted convolution of two suitable functions or distributions on $\mathfrak{g}_1$ is defined by

$$\phi \times_{\mu} \psi(x) = \int_{\mathfrak{g}_1} \phi(x - y)\psi(y)e^{\mu(\langle x, y \rangle)}d_{\mu}y,$$

where $d_{\mu}y = \sqrt{\det J_\mu}dy$. Given $f \in L^1(G)$, we define the Fourier transform with respect to the central variables by

$$f^\mu(x) = \int_{\mathfrak{g}_2} f(x, u)e^{\mu(u)}du.$$
It is easy to verify that

\[(f \ast g)^\mu = f^\mu \ast g^\mu.\]

For any \(X \in g_1\), we obtain a left-invariant vector field \(X\) on \(G\) defined by

\[Xf(g) = \frac{d}{dt} f(g \cdot (tX))|_{t=0}, \quad f \in C^\infty(G), \ g \in G.\]

Then, the orthonormal basis of \(g_1, \{X_1, X_2, \cdots, X_{2n}\}\), is associated to the left-variant vector field:

\[X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{m} \langle x, J_k e_j \rangle \frac{\partial}{\partial u_k}, \quad e_j = (0, \cdots, 1_j, 0, \cdots, 0_{2n})^T, \ j = 1, 2, \cdots, 2n.\]

The sub-Laplacian on \(G\) is the left-invariant hypoelliptic operator defined by

\[\mathcal{L} = - \sum_{j=1}^{2n} X_j^2 = -\Delta_x - \sum_{k=1}^{m} \langle x^T J_k, \nabla_x \rangle \frac{\partial}{\partial u_k} - \frac{1}{4} \sum_{k,l=1}^{n} \langle x^T J_k, x^T J_l \rangle \frac{\partial^2}{\partial u_k \partial u_l},\]

where

\[\Delta_x = \sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2}, \quad \nabla_x = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_{2n}} \right)^T.\]

For any \(f \in \mathcal{S}(G)\) and \(j = 1, \cdots, n\), the Fourier transform of \(X_j f\) with respect ot the central variables is given by

\[(X_j f)^\mu(x) = \int_{g_2} \frac{\partial f}{\partial x_j}(x, u) e^{iu(x)} du + \frac{1}{2} \sum_{k=1}^{m} \langle x, J_k e_j \rangle \int_{g_2} \frac{\partial f}{\partial u_k}(x, u) e^{iu(x)} du\]

\[= \left( \frac{\partial}{\partial x_j} - \frac{i}{2} \langle x, J_\mu e_j \rangle \right) f^\mu(x).\]

Setting

\[X_j^\mu = \frac{\partial}{\partial x_j} - \frac{i}{2} \langle x, J_\mu e_j \rangle,\]

we have that

\[(X_j f)^\mu = X_j^\mu f^\mu.\]

Define

\[\Delta^\mu = - \sum_{j=1}^{2n} \left( X_j^\mu \right)^2.\]

It follows that

\[(\mathcal{L} f)^\mu = \Delta^\mu f^\mu.\]

Next, we let \(S^{m-1} = \{\eta \in g_2^* : |\eta| = 1\}\) denote the unit sphere of \(\mathbb{R}^m \cong g_2^*\). Since that for any fixed \(\eta \in S^{m-1}\), the corresponding metrix \(J_\eta\) is skew symmetric and non-degenerate, then there exists a \(2n \times 2n\) invertible matrix \(A_\eta\) such that

\[J_\eta = A_\eta^T J_{2n} A_\eta\]

where \(J_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0 \end{pmatrix}\). Clearly, \((\det A_\eta)^2 = \det J_\eta\). Since \(\det J_\eta\), which is a polynomial function of the components of \(\eta\), never vanishes on \(S^{m-1}\), then there exists a positive constant \(K\) such that for any \(\eta \in S^{m-1}\)

\[
\frac{1}{K} \leq |\det A_\eta|^2 = |\det J_\eta| \leq K.
\]
Let \((Z_1, \ldots, Z_{2n}) = (X_1, \ldots, X_{2n})A_\eta^{-1}\). \{Z_1, \ldots, Z_{2n}\} is an orthonormal basis of \(g_1\) such that the corresponding metric of \(\omega_\eta\) is \(J_{2n}\), i.e.

\[ \omega_\eta(Z_j, Z_k) = (J_{2n})_{jk}. \]

The new coordinates of the element in \(g_1\) in term of the basis \(\{Z_1, \ldots, Z_{2n}\}\) is \(z = A_\eta x\). For any \(\lambda > 0\), we have that

\[ (Z_j f)^\lambda_\eta (z) = \left( \frac{\partial}{\partial z_j} - \frac{i\lambda}{2} \langle z, J_{2n} e_j \rangle \right) f^{\lambda_\eta}(z). \]

Set

\[ Z_j^{\lambda_\eta} = \frac{\partial}{\partial z_j} - \frac{i\lambda}{2} \langle z, J_{2n} e_j \rangle. \]

More explicitly,

\[ Z_j^{\lambda_\eta} = \frac{\partial}{\partial z_j} + \frac{i\lambda}{2} z_{n+j}, \quad Z_j^{\lambda_\eta} = \frac{\partial}{\partial z_{j+n}} - \frac{i\lambda}{2} z_j, \quad j = 1, \ldots, n. \]

It follows that

\[ L^{\lambda_\eta} = -\sum_{j=1}^{2n} \left( Z_j^{\lambda_\eta} \right)^2 = -\sum_{j=1}^{2n} \frac{\partial^2}{\partial z_j^2} - i\lambda \sum_{j=1}^n \left( z_{n+j} \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_{n+j}} \right) + \frac{\lambda^2}{4} \sum_{j=1}^{2n} z_j^2 \]

\[ = -\Delta z - i\lambda \sum_{j=1}^n \left( z_{n+j} \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_{n+j}} \right) + \frac{\lambda^2}{4} |z|^2, \]

and

\[ (L f)^\lambda_\eta = L^{\lambda_\eta} f^{\lambda_\eta}. \]

Notice that \(L^{\lambda_\eta}\) coincides with the the \(\lambda\)-scaled special Hermite operator \(L^\lambda\) on \(g_1\). The \(\lambda\)-twisted convolution of \(f, g \in \mathcal{S}(g_1)\) in terms of the coordinates \(z\) is given by

\[ \phi \times_\lambda \psi(z) = \int_{g_1} \phi(z - w) \psi(w) e^{\frac{i\lambda}{2} \phi(w)} dw. \]

Next, we give the spectral decomposition of the sub-Laplacian \(L\) on \(G\). We have to recall some facts about the special Hermite expansion. The Hermite polynomials \(H_k\) on \(\mathbb{R}\) is defined by

\[ H_k(t) = (-1)^k \frac{d^k}{dt^k} \left( e^{-t^2} \right) e^{t^2}, \quad k = 0, 1, 2, \ldots. \]

Let

\[ h_k(t) = \left( \frac{2^k k! \sqrt{\pi}}{\sqrt{2}} \right) ^{-1} H_k(t) e^{-\frac{1}{2} t^2}, \quad k = 0, 1, 2, \ldots. \]

For any multiindex \(\alpha\), the Hermite functions \(\Phi_\alpha\) on \(\mathbb{R}^n\) is defined by

\[ \Phi_\alpha(y) = \prod_{j=1}^n h_{\alpha_j}(y_j), \quad y = (y_1, \ldots, y_n) \in \mathbb{R}^n. \]

For each pair of multiindices \(\alpha, \beta\), the special Hermite function \(\Phi_{\alpha, \beta}\) on \(\mathbb{C}^n\) is given by

\[ \Phi_{\alpha, \beta}(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \Phi_\alpha \left( \xi + \frac{y}{2} \right) \Phi_\beta \left( \xi - \frac{y}{2} \right) d\xi, \quad z = x + iy \in \mathbb{C}^n. \]
The special Hermite functions form a complete orthonormal system for $L^2(\mathbb{C}^n)$. Since that $\mathfrak{g}_1 \simeq \mathbb{C}^n$, then for any $g \in L^2(\mathfrak{g}_1)$ we have the special Hermite expansion

$$g(z) = \sum_{\alpha \in \mathbb{N}^m} \sum_{\beta \in \mathbb{N}^n} (f, \Phi_{\alpha\beta}) \Phi_{\alpha\beta}(z).$$

We define the Laguerre functions on $\mathbb{C}^n$ by

$$\varphi_k(z) = L^n_k \left( \frac{1}{2} |z|^2 \right) e^{-\frac{i}{2} |z|^2}, \quad k = 0, 1, 2, \ldots,$$

where

$$L^n_k(x) = \frac{1}{k!} \frac{d^k}{dx^k} \left( e^{-x} x^{k+\nu} \right) e^{x} x^{-\nu}, \quad k = 0, 1, 2, \ldots, \quad \nu > -1,$$

is the Laguerre polynomial on $\mathbb{R}$ of type $\nu$ and degree $k$. It follows that

$$\sum_{|\alpha| = k} |\alpha| \Phi_{\alpha\alpha}(z) = (2\pi)^{-\frac{n}{2}} \varphi_k(z),$$

and the special Hermite expansion can be written as

$$g(z) = (2\pi)^{-\frac{n}{2}} \sum_{k=0}^{\infty} g \times_{\lambda=1} \varphi_k(z).$$

Set $\varphi_k^\lambda(z) = \varphi_k(\sqrt{\lambda}z)$ for any $\lambda > 0$. By a dilation argument, we have the expansion

$$g(z) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \sum_{k=0}^{\infty} g \times_\lambda \varphi_k^\lambda(z),$$

where $g \times_\lambda \varphi_k^\lambda$ is the eigenfunction of the operator $L^\lambda$ with the eigenvalue $\lambda(2k + n)$.

Let $g_\eta = g \circ A_\eta^{-1}$. (2.2) yields that

$$g(x) = g_\eta(A_\eta x) = g_\eta(z) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \sum_{k=0}^{\infty} g_\eta \times_\lambda \varphi_k^\lambda(z) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \sum_{k=0}^{\infty} \left( g_\eta \times_\lambda \varphi_k^\lambda \right) \circ A_\eta(x).$$

So, we can get the following results:

**Proposition 1.** [2] Let $\eta \in S^{m-1}$ and $\lambda \in \mathbb{R}$. For any $g \in \mathcal{S}(\mathfrak{g}_1)$, $\Pi_k^\eta g = (g_\eta \times_\lambda \varphi_k^\lambda) \circ A_\eta$ is the eigenfunction of the operator $\Delta_\lambda^\eta$ with the eigenvalue $\lambda(2k + n)$, and $e^{-i\lambda\eta(u)} \Pi_k^\eta g = e^{-i\lambda\eta(u)} (g_\eta \times_\lambda \varphi_k^\lambda) \circ A_\eta$ is the eigenfunction of the sub-Laplacian $\mathcal{L}$ on $\mathcal{G}$ with the eigenvalue $\lambda(2k + n)$.

**Proof.** Since that $(X_1^\eta, \ldots, X_{2n}^\eta) = (Z_1^\eta, \ldots, Z_{2n}^\eta) A_\eta$, then

$$\Delta_\lambda^\eta (g \circ A_\eta) = \left( L^\lambda g \right) \circ A_\eta.$$ 

Notice that $\Pi_k^\eta g = g_\eta \times_\lambda \varphi_k^\lambda$ is the eigenfunction of $L^\lambda$ with the eigenvalue $\lambda(2k + n)$. We have

$$\Delta_\lambda^\eta \left( (g_\eta \times_\lambda \varphi_k^\lambda) \circ A_\eta \right) = \lambda(2k + n) (g_\eta \times_\lambda \varphi_k^\lambda) \circ A_\eta = \lambda(2k + n) \left( g_\eta \times_\lambda \varphi_k^\lambda \right) \circ A_\eta.$$

This implies $(g_\eta \times_\lambda \varphi_k^\lambda) \circ A_\eta$ is the eigenfunction of the operator $\Delta_\lambda^\eta$ with the eigenvalue $\lambda(2k + n)$.

Since that for any $f \in \mathcal{S}(\mathcal{G})$, $(\mathcal{L} f)^\eta = \Delta_\lambda^\eta f^\eta$, then we have that

$$\left( \mathcal{L} \left( e^{-i\lambda\eta(u)} \Pi_k^\eta g \right) \right)^\eta = \Delta_\lambda^\eta \left( e^{-i\lambda\eta(u)} \Pi_k^\eta g \right)^\eta = \left( e^{-i\lambda\eta(u)} \right)^\eta \Delta_\lambda^\eta \left( \Pi_k^\eta g \right).$$
The spectral decomposition of the sub-Laplacian $L$ produces that
\[ L(e^{-i\lambda\eta(u)}\Pi_k^\eta g) = \lambda(2k + n) \left( e^{-i\lambda\eta(u)}\Pi_k^\eta g \right)^{\lambda\eta}. \]
This yields that $L \left( e^{-i\lambda\eta(u)}\Pi_k^\eta g \right) = \lambda(2k + n) e^{-i\lambda\eta(u)}\Pi_k^\eta g$, and we can conclude that $e^{-i\lambda\eta(u)}\Pi_k^\eta g$ is the eigenfunction of $L$ with the eigenvalue $\lambda(2k + n)$.

Define $e_k^{\lambda\eta}(x, u) = e^{-i\lambda\eta(u)} \left( \varphi_k^\lambda \circ A_\eta \right)(x) |\det A_\eta|$. We can verify that
\[ (f * e_k^{\lambda\eta})(x, u) = e^{-i\lambda\eta(u)} \left( (f^{\lambda\eta} \times \varphi_k^\lambda) \circ A_\eta \right)(x). \]

The Proposition [1] tells that $f * e_k^{\lambda\eta}$ is the eigenfunction of $L$ with the eigenvalue $\lambda(2k + n)$. Let $\tilde{e}_k^{\lambda\eta} = e_k^{\lambda + \pi m \eta}$. Clearly, $f * \tilde{e}_k^{\lambda\eta}$ is the eigenfunction of $L$ with the eigenvalue $\lambda$. The inversion formula of the Fourier transform and (2.3) imply that for any $f \in \mathcal{S}(\mathbb{G})$,
\[
 f(x, u) = \frac{1}{(2\pi)^m} \int_0^\infty \int_{S^{m-1}} f^{\lambda\eta}(x) e^{-i\lambda\eta(u)} \lambda^{m-1} d\sigma(\eta) d\lambda
 = \frac{1}{(2\pi)^m} \int_0^\infty \int_{S^{m-1}} \left( \frac{\lambda}{2\pi} \right)^n \sum_{k=0}^\infty e^{-i\lambda\eta(u)} \left( (f^{\lambda\eta} \times \varphi_k^\lambda) \circ A_\eta \right)(x) \lambda^{m-1} d\sigma(\eta) d\lambda
 = \frac{1}{(2\pi)^m} \int_0^\infty \int_{S^{m-1}} \left( \frac{\lambda}{2\pi} \right)^n \sum_{k=0}^\infty (f * e_k^{\lambda\eta})(x, u) \lambda^{m-1} d\sigma(\eta) d\lambda
 = \int_0^\infty \left( \sum_{k=0}^\infty \left( \frac{\lambda}{2\pi(2k + n)} \right)^{n+m} \int_{S^{m-1}} f * \tilde{e}_k^{\lambda\eta}(x, u) d\sigma(\eta) \right) d\lambda.
\]

Here $d\sigma(\eta)$ denotes the induced Lebesgue measure on the unit sphere $S^{m-1}$. Define
\[ P_\lambda f(x, u) = \sum_{k=0}^\infty \left( \frac{\lambda}{2\pi(2k + n)} \right)^{n+m} \int_{S^{m-1}} f * \tilde{e}_k^{\lambda\eta}(x, u) d\sigma(\eta). \]

$P_\lambda f$ is the eigenfunction of the sub-Laplaceian $L$ with the eigenvalue $\lambda$. Then, we have that
\[ f = \int_0^\infty P_\lambda f d\lambda. \]
The spectral decomposition of the sub-Laplaceian $L$ on $\mathbb{G}$ is given by
\[ Lf = \int_0^\infty \lambda P_\lambda f d\lambda. \]

We now define the Riesz means associated to the sub-Laplacian $L$ for $f \in \mathcal{S}(\mathbb{G})$ by
\[ S_\delta_R f = \int_0^\infty \left( 1 - \frac{\lambda}{R} \right)^\delta P_\lambda f d\lambda. \]
Since that $P_\lambda$ is the convolution operator, we have
\[
 S_\delta_R f(x, u) = \int_{\mathbb{G}} f((x, u) \cdot (y, v)^{-1}) S_\delta_R (y, v) dy dv,
\]
where the kernel is given by
\[
 S_\delta_R (x, u) = \int_0^\infty \left( 1 - \frac{\lambda}{R} \right)^\delta \left( \sum_{k=0}^\infty \left( \frac{\lambda}{2\pi(2k + n)} \right)^{n+m} \int_{S^{m-1}} \tilde{e}_k^{\lambda\eta}(x, u) d\sigma(\eta) \right) d\lambda.
\]
If $\delta > 0$, the bilinear Riesz means $S_\delta$ are bounded. Let $S_{1,2}$ be the kernel of the Riesz means $S_{1}$. Because of this, we see that the $L^p$-boundedness of $S_{1,2}$ can be deduced from the $L^p$-boundedness of $S_\delta$. We write $S_\delta = S_{1,2}$ and it is sufficient to consider the bounds of $S_\delta$.

The bilinear Riesz means associated to $S_{1,2}$ are defined by

$$S_{1,2}^\alpha(f, g)(x, u) = \int_{\mathcal{G}} f((x, u) \cdot (x_1, u_1)^{-1}) g((x, u) \cdot (x_2, u_2)^{-1}) \times S_{1,2}^\alpha((x_1, u_1), (x_2, u_2)) dx_1 du_1 dx_2 du_2,$$

where the kernel is given by

$$S_{1,2}^\alpha((x_1, u_1), (x_2, u_2)) = \frac{1}{(2\pi)^2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_0^\infty \int_0^\infty \left(1 - \frac{(2k + n)\lambda_1 + (2l + n)\lambda_2}{R}\right)^\alpha \times \int_{\mathcal{G}} e^{\lambda_1\eta_1(x_1, u_1)} d\sigma(\eta_1)$$

$$\times \int_{\mathcal{G}} e^{\lambda_2\eta_2(x_2, u_2)} d\sigma(\eta_2) \lambda_1^{n+m-1} \lambda_2^{n+m-1} d\lambda_1 d\lambda_2$$

$$= \frac{1}{(2\pi)^2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_{\mathcal{G}} \int_{\mathcal{G}} \left(1 - \frac{(2k + n)\lambda_1 + (2l + n)\lambda_2}{R}\right)^\alpha$$

$$\times e^{\lambda_1\eta_1(x_1, u_1)} e^{\lambda_2\eta_2(x_2, u_2)} |\mu_1|^n |\mu_2|^n d\mu_1 d\mu_2.$$

Note that

$$S_{1,2}^\alpha((x_1, u_1), (x_2, u_2)) = R^2 S_1^\alpha \left( (\sqrt{R}x_1, Ru_1), (\sqrt{R}x_2, Ru_2) \right).$$

By a dilation argument, the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of $S_{1,2}^\alpha$ is deduced from the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of $S_1^\alpha$ as $1/p = 1/p_1 + 1/p_2$. Thus, we will concentrate on the operator $S_1^\alpha$ and write $S_\alpha = S_1^\alpha$.

### 3. Pointwise Estimate for the Kernel

In this section, we investigate the pointwise estimates for the kernel of the Riesz means $S_\delta$ and the bilinear Riesz means $S_\alpha$.

**Theorem 1.** Let $S_\delta(x, u)$ be the kernel of the Riesz means $S_\delta$. Assume that $N$ is a positive integer. If $\delta > 2N - 1$, then

$$|S_\delta(x, u)| \leq C_N \left(1 + \left(\left|A_{\frac{\mu}{|\mu|}}x, u\right|\right)^{-2N}.$$

for any $\omega = (x, u) \in \mathcal{G}$ and $\mu \in \mathbb{R}^n \setminus \{0\}$. 
Proof. Since that
\[ e_k^\delta(x,u) = e^{-i\mu(u)} \left( \varphi_k^{[\mu]} \circ A_{\frac{\mu}{m}} \right)(x) \left| \det A_{\frac{\mu}{m}} \right|, \]
we have
\[ S^\delta(x,u) = \frac{1}{(2\pi)^{n+m}} \sum_{k=0}^\infty \int_{\mathbb{R}^m} (1 - (2k + n) |\mu|)^\frac{\delta}{2} e_k^\delta(x,u) |\mu|^n d\mu. \]
Let \( z = A_{\frac{\mu}{m}} x \) and
\[ F(x,u) = \frac{1}{(2\pi)^{n+m}} \sum_{k=0}^\infty \int_{\mathbb{R}^m} e^{-i\mu(u)} (1 - (2k + n) |\mu|)^\delta \varphi_k^{[\mu]}(z) \left| \det A_{\frac{\mu}{m}} \right| |\mu|^n d\mu. \]
\( F(z,u) \) is a radial function with respect to \( z \). We set \( r = |z| \) and define
\[ R_k(\mu, F) = \frac{2(1-n)k!}{(k+n-1)!} \int_0^\infty F^\mu(r) \varphi_k^{[\mu]}(r) r^{2n-1} dr, \]
where
\[ F^\mu(r) = \int_{\mathbb{R}^m} F(r,u)e^{i\mu(u)} du. \]
Using the inversion formula of the Fourier transform and the Laguerre expansion of the radial function (see [15]), we have that
\[ F(z,u) = \sum_{k=0}^\infty \int_{\mathbb{R}^m} e^{-i\mu(u)} R_k(\mu, F) \varphi_k^{[\mu]}(z) |\mu|^n d\mu. \]
Notice that
\[ \| \varphi_k \|_\infty = c_n \frac{(k+n-1)!}{k!}. \]
So, if we can show that
\[ \sum_{k=0}^\infty \int_{\mathbb{R}^m} |R_k(\mu, F)| \frac{(k+n-1)!}{k!} |\mu|^n d\mu < \infty \]
then \( F \) is bounded. Similarly, if we can show that for any \( N \in \mathbb{N}^+ \), there exists a constant \( C_N > 0 \) such that for each \( j = 1, 2, \cdots, m, \)
\[ \sum_{k=0}^\infty \int_{\mathbb{R}^m} R_k \left( \mu, \left( iu_j - \frac{\mu_j}{|\mu|} \frac{1}{4} |z|^2 \right)^N F \right) \left| \frac{(k+n-1)!}{k!} |\mu|^n d\mu < C_N, \]
then
\[ \sum_{k=0}^\infty \left( \left| \left( \sup_j |u_j|^2 + \sup_j \frac{\mu_j^2}{|\mu|^2} \frac{1}{4} |z|^4 \right)^N F \right)^2 \leq C_N^2, \]
namely,
\[ |F(z,u)| \leq C_N (1 + |(z,u)|)^{-2N}. \]
This yields that for any \( \omega = (x,u) \in G \) and \( \mu \in \mathbb{R}^m \setminus \{0\} , \)
\[ |S^\delta(x,u)| = |F(A_{\frac{\mu}{m}} x,u)| \leq C_N \left( 1 + \left| \left( A_{\frac{\mu}{m}} x,u \right) \right| \right)^{-2N}. \]
Thus, to obtain Theorem 1 it suffices to prove (3.4). Fixing \( j = 1, \cdots, m \). We first let \( N = 1 \) and calculate \( R_k \left( \mu, \left( iu_j - \frac{\mu_j}{|\mu|} \frac{1}{4} r^2 \right) F \right) \). From (3.2), we know that

\[
R_k(\mu, iu_j F) = \frac{2^{(1-n)c!}}{(k+n-1)!} \int_0^\infty (iu_j F)^\mu(r) \varphi_k^{(\mu)}(r) r^{2n-1} dr.
\]

Since that

\[
(iu_j F)^\mu(r) = \frac{\partial}{\partial \mu_j} F^\mu(r),
\]

then

\[
R_k(\mu, iu_j F) = \frac{\partial}{\partial \mu_j} R_k(\mu, F) - \frac{2^{(1-n)c!}}{(k+n-1)!} \int_0^\infty F^\mu(r) \frac{\partial}{\partial \mu_j} \varphi_k^{(\mu)}(r) r^{2n-1} dr.
\]

Noticing

\[
\frac{\partial}{\partial \mu_j} \varphi_k^{(\mu)}(r) = \frac{1}{2} r^2 \frac{\mu_j}{|\mu|} (L_k^{n-1})' \left( \frac{1}{2} |\mu| r^2 \right) e^{-\frac{1}{4} |\mu| r^2} - \frac{1}{2} r^2 \frac{\mu_j}{|\mu|} L_k^{n-1} \left( \frac{1}{2} |\mu| r^2 \right) e^{-\frac{1}{4} |\mu| r^2},
\]

and using the recursion formula (see [15])

\[
r \frac{d}{dr} L_k^{n-1}(r) = kL_k^{n-1}(r) - (k+n-1)L_k^{n-1}_{k-1}(r),
\]

we have that

\[
(3.5) \quad \frac{\partial}{\partial \mu_j} \varphi_k^{(\mu)}(r) = \frac{\mu_j}{|\mu|} \left( |\mu|^{-1} k \varphi_k^{(\mu)}(r) - |\mu|^{-1} (k+n-1) \varphi_{k-1}^{(\mu)}(r) - \frac{1}{4} r^2 \varphi_k^{(\mu)}(r) \right).
\]

Then,

\[
R_k \left( \mu, \left( iu_j - \frac{\mu_j}{|\mu|} \frac{1}{4} r^2 \right) F \right) = \frac{\partial}{\partial \mu_j} R_k(\mu, F) - \frac{\mu_j}{|\mu|} \left( \frac{k}{|\mu|} R_k(\mu, F) - \frac{k}{|\mu|} R_{k-1}(\mu, F) \right).
\]

(3.1) and (3.3) yield that

\[
R_k(\mu, F) = (1 - (2k+n) |\mu|)^\delta \left| \det A_{\mu} \right|.
\]

We set \( \sigma = (2k+n) |\mu| \) and \( \psi(\sigma) = (1-\sigma)^\delta \). Then, \( R_k \left( \mu, \left( iu_j - \frac{\mu_j}{|\mu|} \frac{1}{4} r^2 \right) F \right) \) can be rewritten as

\[
R_k \left( \mu, \left( iu_j - \frac{\mu_j}{|\mu|} \frac{1}{4} r^2 \right) F \right) = \frac{\partial}{\partial \mu_j} \left( \left| \det A_{\mu} \right| \psi(\sigma) \right) - \left| \det A_{\mu} \right| \mu_j \left( \frac{k}{|\mu|} \psi(\sigma) - \frac{k}{|\mu|} \psi(\sigma - 2 |\mu|) \right)
\]

\[
= \frac{\partial}{\partial \mu_j} \left( \left| \det A_{\mu} \right| \psi(\sigma) \right) - \left| \det A_{\mu} \right| k \frac{\mu_j}{|\mu|^2} \frac{\partial}{\partial \mu} \left( (2k+n) |\mu| \right)
\]

\[
+ \left| \det A_{\mu} \right| k \frac{\mu_j}{|\mu|^2} \left( \frac{\partial}{\partial \mu} \left( (2k+n) |\mu| \right) - \psi((2k+n) |\mu|) - \psi((2k+2n) |\mu|) \right).
\]

Using Taylor expansion, we get that

\[
\frac{\partial}{\partial \mu} \left( (2k+n) |\mu| \right) - \psi((2k+n) |\mu|) + \psi((2k-2+n) |\mu|)
\]

\[
= 4 |\mu|^2 \int_{k-1}^{k} (s+1-k) \psi''((2s+n) |\mu|) ds.
\]
The proof of Theorem 1 is completed.

Proof. We take Corollary 1. Let

\[ \frac{1}{k} | \frac{\partial}{\partial k} \psi((2k + n) | \mu|) - k \psi((2k + n) | \mu|) + k \psi((2k - 2 + n) | \mu|) \bigg| \det A_{\frac{\mu}{|\mu|^2}} | | \mu|^n d\mu \]

\[ = 4k \int_{k-1}^{k} (s + 1 - k) \bigg( \int_{\mathbb{R}^m} |\mu_j| (1 - (2s + 2n) |\mu|)^{\delta - 2} \bigg| \det A_{\frac{\mu}{|\mu|^2}} | | \mu|^n d\mu \bigg) ds \]

\[ \leq C \int_{k-1}^{k} (2s + 2n)^{-n - m} ds \]

\[ \leq C(2k + n)^{-n - m} \]

if \((1 - |\mu|)^{\delta - 2}\) is integrable on \(\mathbb{R}^m\). At the same time, we see that

\[ \frac{\partial}{\partial \mu_j} \left( \left| \det A_{\frac{\mu}{|\mu|^2}} \right| \psi(\sigma) \right) - \left| \det A_{\frac{\mu}{|\mu|^2}} \right| k \frac{\mu_j}{|\mu|^2} \frac{\partial}{\partial k} \psi((2k + n) | \mu|) \bigg| \mu|^n d\mu \]

\[ = \int_{\mathbb{R}^m} \psi((2k + n) \lambda) \left( \int_{\mathbb{R}^{m-1}} \left| \det A_{\eta} \right| d\eta \right) \lambda^{n+m-1} d\lambda \]

\[ \leq (2k + n)^{-n - m}. \]

An iteration of the process shows that \(R_k \left( \mu, \left( iu_j - \frac{\mu_j}{|\mu|^2} \right) \right) F \) also satisfies

\[ \int_{\mathbb{R}^m} \left| R_k \left( \mu, \left( iu_j - \frac{\mu_j}{|\mu|^2} \right) \right) F \right| \bigg| \mu|^n d\mu \leq C_1(2k + n)^{-n - m} \]

provided that \((1 - |\mu|)^{\delta - 2}\) is integrable on \(\mathbb{R}^m\). Thus, when \(l = N\) and \(\delta > 2N - 1\) we have

\[ \int_{\mathbb{R}^m} \left| R_k \left( \mu, \left( iu_j - \frac{\mu_j}{|\mu|^2} \right) \right)^{N} F \right| \bigg| \mu|^n d\mu \leq C_N(2k + n)^{-n - m}, \]

which implies that

\[ \sum_{k=0}^{\infty} \int_{\mathbb{R}^m} \left| R_k \left( \mu, \left( iu_j - \frac{\mu_j}{|\mu|^2} |z|^2 \right)^{N} F \right) \right| \frac{(k + n - 1)!}{k!} | | \mu|^n d\mu < \infty. \]

The proof of Theorem 1 is completed. \(\square\)

**Corollary 1.** Let \(1 \leq p \leq \infty\). If \(\delta > Q + 1\), then \(S^*\) is bounded from \(L^p(G)\) into \(L^p(G)\).

**Proof.** We take \(N = \frac{Q}{2} + 1\). If \(\delta > Q + 1\), we have \(\delta > 2N - 1\). Then, Theorem 1 is available. By Hölder’s inequality and Young’s inequality, we conclude that

\[ \| S^* f \|_p \leq \| f \|_p \int_{G} \left( 1 + \left| \left( A_{\frac{\mu}{|\mu|^2}} x, u \right) \right| \right)^{-2N} dx du \]

\[ \leq \| f \|_p \left| \det A_{\eta} \right|^{-1} \int_{G} \left( 1 + \left( |x, u| \right) \right)^{-2N} dx du \]
Clearly, the proof is completed.

Next, we show the pointwise estimate of the kernel of the bilinear Riesz means $S^\alpha$.

**Theorem 2.** For any $N \in \mathbb{N}^+$, if $\alpha > 4N - 1$, then for any $\omega_1 = (x_1, u_1), \omega_2 = (x_2, u_2) \in G$ and any $\mu_1, \mu_2 \in \mathbb{R}^m \setminus \{0\}$,

$$|S^\alpha((x_1, u_1), (x_2, u_2))| \leq C_N \left(1 + \left|\left(A_{\mu_1^{(1)}} x_1, u_1\right)\right|^{-2N} \left(1 + \left|\left(A_{\mu_2^{(2)}} x_2, u_2\right)\right|^{-2N} \right).$$

**Proof.** Choose a nonnegative function $\varphi \in C_0^\infty(\frac{1}{2}, 1)$ satisfying $\sum_{i=0}^{\infty} \varphi(2^is) = 1$, $s > 0$. For each $j \geq 0$, we set function

$$\varphi_j^\alpha(s, t) = (1 - s - t)^\alpha \varphi(2^j (1 - s - t)),$$

and define bilinear operator

$$T^\alpha_j(f, g) = \int_0^\infty \int_0^\infty \varphi_j^\alpha(\lambda_1, \lambda_2) P_{\lambda_1} f P_{\lambda_2} g d\lambda_1 d\lambda_2.$$

It is easy to see that

$$T^\alpha_j(f, g)(\omega) = \int_G \int_G f(\omega_1^{-1}) g(\omega_2^{-1}) K^\alpha_j(\omega_1, \omega_2) d\omega_1 d\omega_2,$$

where the kernel $K^\alpha_j$ is given by

$$K^\alpha_j((x_1, u_1), (x_2, u_2)) = \frac{1}{(2\pi)^Q} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \varphi_j^\alpha((2k + n) |\mu_1|, (2l + n) |\mu_2|) e_{k}^{\mu_1} (x_1, u_1) e_{l}^{\mu_2} (x_2, u_2) |\mu_1|^n |\mu_2|^n d\mu_1 d\mu_2 \times \varphi_k^{\mu_1} \left(A_{\mu_1^{(1)}} x_1\right) \varphi_l^{\mu_2} \left(A_{\mu_2^{(2)}} x_2\right) \det A_{\mu_1^{(1)}} |\mu_1|^n \det A_{\mu_2^{(2)}} |\mu_2|^n d\mu_1 d\mu_2.$$

Clearly,

$$S^\alpha(\omega_1, \omega_2) = \sum_{j=0}^{\infty} K^\alpha_j(\omega_1, \omega_2).$$

Fixing $j \geq 0$. We first estimate $K^\alpha_j$. Let $z_1 = A_{\mu_1^{(1)}} x_1$, $z_2 = A_{\mu_2^{(2)}} x_2$ and

$$F((z_1, u_1), (z_2, u_2)) = \frac{1}{(2\pi)^Q} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{-i\mu_1 u_1} e^{-i\mu_2 u_2} \varphi_j^\alpha((2k + n) |\mu_1|, (2l + n) |\mu_2|) \times \varphi_k^{\mu_1} (z_1) \varphi_l^{\mu_2} (z_2) \det A_{\mu_1^{(1)}} |\mu_1|^n \det A_{\mu_2^{(2)}} |\mu_2|^n d\mu_1 d\mu_2.$$

Then, $F((z_1, u_1), (z_2, u_2))$ is bi-radial with respect to $z_1$ and $z_2$. We set $r_1 = |z_1|$, $r_2 = |z_2|$ and define

$$R_k(l, \mu_1, \mu_2, F) = \frac{2^{(1-n)k}!}{(k + n - 1)! (l + n - 1)!} \int_0^\infty F^{\mu_1, \mu_2}(r_1, r_2) \varphi_k^{\mu_1} (r_1) \varphi_l^{\mu_2} (r_2) r_1^{2n-1} r_2^{2n-1} dr_1 dr_2,$$
where
\[ F_{\mu_1, \mu_2}(r_1, r_2) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{i\mu_1(u_1)} e^{i\mu_2(u_2)} F((r_1, u_1), (r_2, u_2)) du_1 du_2. \]

Using the inversion formula of the Fourier transform and the Laguerre expansion of the radial function (see [15]), the function \( F \) can be written as
\[
F((z_1, u_1), (z_2, u_2)) = \frac{1}{(2\pi)^{2q}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{-i(\mu_1(u_1) + \mu_2(u_2))} F_{\mu_1, \mu_2}(z_1, z_2) |\mu_1|^n |\mu_2|^n d\mu_1 d\mu_2
\]
\[
= \frac{1}{(2\pi)^{2q}} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{-i(\mu_1(u_1) + \mu_2(u_2))} R_{k,l}(\mu_1, \mu_2, F) \varphi_k^l(z_1) \varphi_l^k(z_2) |\mu_1|^n |\mu_2|^n d\mu_1 d\mu_2.
\]

Hence, if we can show that
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |R_{k,l}(\mu_1, \mu_2, F)|(k + n - 1)! (l + n - 1)! d\mu_1 d\mu_2 < \infty
\]
then \( F \) is bounded. Let \( u_1 \) be the \( p \)-component of \( u_1 \), \( u_2 \) be the \( q \)-component of \( u_2 \). Similarly, if we can show that for any fixed \( j \geq 0 \) and \( N \in \mathbb{N}^+ \), there exists a constant \( C_{j,N} \) such that for any \( p, q = 1, 2, \ldots, m \),
\[
(3.7) \quad \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |R_{k,l}(\mu_1, \mu_2, 1)| \left| \frac{(k + n - 1)! (l + n - 1)!}{k! l!} |\mu_1|^n |\mu_2|^n d\mu_1 d\mu_2 \leq C_{j,N}, \right.
\]
then
\[
\left| (z_1, u_1)^{4N} (z_2, u_2)^{4N} F((z_1, u_1), (z_2, u_2)) \right|
\]
\[
= \left( \sup_p \left| u_1^{(p)} \right|^2 + \sup_p \left| \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{1}{16} |z_1|^4 \right| \right)^N \left( \sup_q \left| u_2^{(q)} \right|^2 + \sup_q \left| \frac{\mu_2^{(q)}}{|\mu_2|^2} \frac{1}{16} |z_2|^4 \right| \right)^N \left| F((z_1, u_1), (z_2, u_2)) \right|
\]
\[
= \sup_p \left( \left| u_1^{(p)} \right|^2 + \left| \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{1}{16} |z_1|^4 \right| \right)^N \sup_q \left( \left| u_2^{(q)} \right|^2 + \left| \frac{\mu_2^{(q)}}{|\mu_2|^2} \frac{1}{16} |z_2|^4 \right| \right)^N \left| F((z_1, u_1), (z_2, u_2)) \right|^2
\]
\[
\leq C_{j,N}^2.
\]

This implies that for any \( \mu, \mu_2 \in \mathbb{R}^m \setminus \{0\}, \)
\[
|K_j^2 ((x_1, u_1), (x_2, u_2)) | = \left| F \left( \left( A_{\mu_1^2}, x_1, u_1 \right), \left( A_{\mu_2^2}, x_2, u_2 \right) \right) \right|
\]
\[
\leq C_{j,N} \left( 1 + \left| \left( A_{\mu_1^2}, x_1, u_1 \right) \right|^{-2} \right)^N \left( 1 + \left| \left( A_{\mu_2^2}, x_2, u_2 \right) \right|^{-2} \right)^N.
\]
We now prove (3.7). Fixing $p, q = 1, 2, \cdots, m$. We first calculate
\[ R_{k,l} \left( u_1, u_2, \left( \frac{u_1}{|\mu_1|} - \frac{1}{4} |z_1|^2 \right) \left( \frac{u_2}{|\mu_2|} - \frac{1}{4} |z_2|^2 \right) F \right). \]

From (3.6), we know that
\[
R_{k,l}(\mu_1, \mu_2, iu_1(q), iu_2(q) F) = \frac{2^{(1-n)k}}{(k+n-1)!} \int_0^\infty \int_0^\infty \left( \frac{iu_1(q)}{\mu_1} \cdot \frac{iu_2(q)}{\mu_2} F \right)^{\mu_1+\mu_2} (r_1, r_2) \times \phi_k^{\mu_1}(r_1) \phi_l^{\mu_2}(r_2) r_1^{n-1} r_2^{n-1} dr_1 dr_2.
\]

Since that
\[
\left( \frac{i u_1(q)}{\mu_1} \cdot \frac{i u_2(q)}{\mu_2} F \right)^{\mu_1+\mu_2} (r_1, r_2) = \frac{\partial}{\partial \mu_1(q)} \frac{\partial}{\partial \mu_2(q)} F^{\mu_1+\mu_2}(r_1, r_2),
\]
we have
\[
R_{k,l}(\mu_1, \mu_2, iu_1(q), iu_2(q) F) = \frac{\partial}{\partial \mu_1(q)} \frac{\partial}{\partial \mu_2(q)} R_{k,l}(\mu_1, \mu_2, F)
\]
\[
\frac{2^{(1-n)k}}{(k+n-1)!} \frac{2^{(1-n)!}}{(l+n-1)!} \int_0^\infty \int_0^\infty \frac{\partial}{\partial \mu_1(q)} F^{\mu_1+\mu_2}(r_1, r_2) \frac{\partial}{\partial \mu_2(q)} \left( \phi_k^{\mu_1}(r_1) \phi_l^{\mu_2}(r_2) r_1^{n-1} r_2^{n-1} \right) dr_1 dr_2
\]
\[
\frac{2^{(1-n)k}}{(k+n-1)!} \frac{2^{(1-n)!}}{(l+n-1)!} \int_0^\infty \int_0^\infty \frac{\partial}{\partial \mu_1(q)} F^{\mu_1+\mu_2}(r_1, r_2) \frac{\partial}{\partial \mu_1(q)} \left( \phi_k^{\mu_1}(r_1) \phi_l^{\mu_2}(r_2) r_1^{n-1} r_2^{n-1} \right) dr_1 dr_2
\]
\[
\frac{2^{(1-n)k}}{(k+n-1)!} \frac{2^{(1-n)!}}{(l+n-1)!} \int_0^\infty \int_0^\infty F^{\mu_1+\mu_2}(r_1, r_2) \frac{\partial}{\partial \mu_1(q)} \frac{\partial}{\partial \mu_2(q)} \left( \phi_k^{\mu_1}(r_1) \phi_l^{\mu_2}(r_2) r_1^{n-1} r_2^{n-1} \right) dr_1 dr_2.
\]

(3.5) tells
\[
\frac{\partial}{\partial \mu_1(q)} \phi_k^{\mu_1}(r_1) = \frac{\mu_1(p)}{|\mu_1|} \left( |\mu_1|^{-1} k \phi_k^{\mu_1}(r_1) - |\mu_1|^{-1} (k+n-1) \phi_{k-1}^{\mu_1}(r_1) - \frac{1}{4} \frac{r_1^2}{r_2} \phi_k^{\mu_1}(r_1) \right),
\]
\[
\frac{\partial}{\partial \mu_2(q)} \phi_l^{\mu_2}(r) = \frac{\mu_1(q)}{|\mu_2|} \left( |\mu_2|^{-1} l \phi_l^{\mu_2}(r) - |\mu_2|^{-1} (l+n-1) \phi_{l-1}^{\mu_2}(r) - \frac{1}{4} \frac{r_1^2}{r_2} \phi_l^{\mu_2}(r) \right),
\]
then, it follows that
\[
R_{k,l}(\mu_1, \mu_2, iu_1(q), iu_2(q) F) = \frac{\partial}{\partial \mu_1(q)} \frac{\partial}{\partial \mu_2(q)} R_{k,l}(\mu_1, \mu_2, F) - \frac{\mu_1(p)}{|\mu_1|} \frac{\mu_2(q)}{|\mu_2|} R_{k,l}(\mu_1, \mu_2, F) \frac{1}{4} |z_1|^2 \frac{1}{4} |z_2|^2 F
\]
\[
+ \frac{\mu_1(p)}{|\mu_1|} \frac{\partial}{\partial \mu_2(q)} R_{k,l}(\mu_1, \mu_2, F) + \frac{\mu_2(q)}{|\mu_2|} \frac{\partial}{\partial \mu_1(p)} R_{k,l}(\mu_1, \mu_2, F)
\]
\[
- \frac{\mu_1(p)}{|\mu_1|} \frac{\partial}{\partial \mu_2(q)} \left( R_{k,l}(\mu_1, \mu_2, F) - R_{k,l-1}(\mu_1, \mu_2, F) \right)
\]
\[
- \frac{\mu_1(p)}{|\mu_1|} \frac{\partial}{\partial \mu_2(q)} \left( R_{k,l}(\mu_1, \mu_2, F) - R_{k,l-1}(\mu_1, \mu_2, F) \right).
\]
\[
+ k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} (R_{k,l,-1}(\mu_1, \mu_2, F) - R_{k,l}(\mu_1, \mu_2, F)) \\
+ k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} (R_{k,-l}(\mu_1, \mu_2, F) - R_{k,l}(\mu_1, \mu_2, F)) \\
+ k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} \left( R_{k,l}(\mu_1, \mu_2, \frac{1}{4} |z_2|^2 F) - R_{k,-l}(\mu_1, \mu_2, \frac{1}{4} |z_2|^2 F) \right) \\
+ k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} \left( R_{k,l}(\mu_1, \mu_2, \frac{1}{4} |z_1|^2 F) - R_{k,-l}(\mu_1, \mu_2, \frac{1}{4} |z_1|^2 F) \right).
\]

In the same way, we get that

\[
R_{k,l}(\mu_1, \mu_2, -iu_1^{(p)} \cdot \frac{\mu_2^{(q)}}{|\mu_2|^2} \frac{1}{4} |z_2|^2 F)
\]

\[
= - \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\partial}{\partial \mu_1^{(q)}} R_{k,l}(\mu_1, \mu_2, \frac{1}{4} |z_1|^2 F) - \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} R_{k,l}(\mu_1, \mu_2, \frac{1}{4} |z_1|^2 \frac{1}{4} |z_2|^2 F) \\
+ k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} \left( R_{k,l}(\mu_1, \mu_2, \frac{1}{4} |z_2|^2 F) - R_{k,-l}(\mu_1, \mu_2, \frac{1}{4} |z_2|^2 F) \right),
\]

and

\[
R_{k,l}(\mu_1, \mu_2, -iu_2^{(p)} \cdot \frac{\mu_1^{(q)}}{|\mu_1|^2} \frac{1}{4} |z_1|^2 F)
\]

\[
= - \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\partial}{\partial \mu_1^{(q)}} R_{k,l}(\mu_1, \mu_2, \frac{1}{4} |z_1|^2 F) - \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} R_{k,l}(\mu_1, \mu_2, \frac{1}{4} |z_1|^2 \frac{1}{4} |z_2|^2 F) \\
+ k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} \left( R_{k,l}(\mu_1, \mu_2, \frac{1}{4} |z_1|^2 F) - R_{k,-l}(\mu_1, \mu_2, \frac{1}{4} |z_1|^2 F) \right).
\]

Combining (3.9), (3.10) and (3.11), we have that

\[
R_{k,l} \left( \mu_1, \mu_2, \left( iu_1^{(p)} - \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{1}{4} |z_1|^2 \right) \left( iu_2^{(p)} - \frac{\mu_2^{(q)}}{|\mu_2|^2} \frac{1}{4} |z_2|^2 \right) F \right)
\]

\[
= R_{k,l} \left( \mu_1, \mu_2, iu_1^{(p)} \cdot iu_2^{(q)} F \right) + R_{k,l} \left( \mu_1, \mu_2, -iu_1^{(p)} \mu_2^{(q)} \frac{1}{|\mu_2|^2} |z_2|^2 F \right) \\
+ R_{k,l} \left( \mu_1, \mu_2, -iu_2^{(p)} \mu_1^{(q)} \frac{1}{|\mu_1|^2} |z_1|^2 F \right) + R_{k,\ell} \left( \mu_1, \mu_2, \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{1}{4} |z_1|^2 \frac{\mu_2^{(q)}}{|\mu_2|^2} \frac{1}{4} |z_2|^2 F \right)
\]

\[
= \frac{\partial}{\partial \mu_1^{(p)}} \frac{\partial}{\partial \mu_2^{(q)}} R_{k,l}(\mu_1, \mu_2, F) - 2 \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} R_{k,l}(\mu_1, \mu_2, \frac{1}{4} |z_1|^2 \frac{1}{4} |z_2|^2 F)
\]

\[
- l \frac{\mu_2^{(q)}}{|\mu_2|^2} \frac{\partial}{\partial \mu_1^{(p)}} \left( R_{k,l}(\mu_1, \mu_2, F) - R_{k,-l}(\mu_1, \mu_2, F) \right) - k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\partial}{\partial \mu_2^{(q)}} \left( R_{k,l}(\mu_1, \mu_2, F) - R_{k,-l}(\mu_1, \mu_2, F) \right)
\]

\[
+ k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} \left( R_{k,l-1}(\mu_1, \mu_2, F) - R_{k,l}(\mu_1, \mu_2, F) + R_{k,-l}(\mu_1, \mu_2, F) - R_{k,-l-1}(\mu_1, \mu_2, F) \right).
\]
\[ +2k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} \left( R_{k,l}(\mu_1, \mu_2, 1, z_2^2 F) - R_{k-1,l}(\mu_1, \mu_2, 1, z_2^2 F) \right) \\
+2k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} \left( R_{k,l}(\mu_1, \mu_2, 1, z_1^2 F) - R_{k-1,l}(\mu_1, \mu_2, 1, z_1^2 F) \right). \]

Let \( \sigma = (2k + n) |\mu_1| + (2l + n) |\mu_2| \) and
\[ \psi(\sigma) = (1 - \sigma)^2 \varphi(2'(1 - \sigma)). \]

Notice that
\[ R_{k,l}(\mu_1, \mu_2, F) = \psi(\sigma) \left| \det A_{\mu_1^{(p)}} \right| \left| \det A_{\mu_2^{(q)}} \right|. \]

Then,
\[ R_{k,l} \left( \mu_1, \mu_2, \left( iu_1^{(p)} - \frac{\mu_1^{(p)}}{|\mu_1|^2} |z_1|^2 \right) \left( iu_2^{(q)} - \frac{\mu_2^{(q)}}{|\mu_2|^2} |z_2|^2 \right) F \right) = \left| \det A_{\mu_1^{(p)}} \right| \left| \det A_{\mu_2^{(q)}} \right| \left( \frac{\partial}{\partial \mu_1^{(p)}} \frac{\partial}{\partial \mu_2^{(q)}} \psi(\sigma) \right) \]
\[ -l \frac{\mu_2^{(q)}}{|\mu_2|^2} \frac{\partial}{\partial \mu_1^{(p)}} (\psi(\sigma) - \psi(\sigma - 2 |\mu_2|)) - k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\partial}{\partial \mu_2^{(q)}} (\psi(\sigma) - \psi(\sigma - 2 |\mu_1|)) \]
\[ -k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} (\psi(\sigma) - \psi(\sigma - 2 |\mu_1|)) + C_2 \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} (\psi(\sigma) - \psi(\sigma - 2 |\mu_2|)) - C_3 \psi(\sigma) \]
\[ + \frac{\partial}{\partial \mu_1^{(p)}} \frac{\partial}{\partial \mu_2^{(q)}} \left| \det A_{\mu_1^{(p)}} \right| \left| \det A_{\mu_2^{(q)}} \right| (\psi(\sigma) - \psi(\sigma - 2 |\mu_2|)) \]
\[ -k \frac{\mu_1^{(p)}}{|\mu_1|^2} \left| \det A_{\mu_1^{(p)}} \right| \left| \det A_{\mu_2^{(q)}} \right| (\psi(\sigma) - \psi(\sigma - 2 |\mu_1|)). \]

We set
\[ \psi_{p,q}^3(\sigma) = \left| \det A_{\mu_1^{(p)}} \right| \left| \det A_{\mu_2^{(q)}} \right| \left( \frac{\partial}{\partial \mu_1^{(p)}} \frac{\partial}{\partial \mu_2^{(q)}} \psi(\sigma) \right) \]
\[ -l \frac{\mu_2^{(q)}}{|\mu_2|^2} \frac{\partial}{\partial \mu_1^{(p)}} (\psi(\sigma) - \psi(\sigma - 2 |\mu_2|)) - k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\partial}{\partial \mu_2^{(q)}} (\psi(\sigma) - \psi(\sigma - 2 |\mu_1|)) \]
\[ -k \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} (\psi(\sigma) - \psi(\sigma - 2 |\mu_1|)) + C_2 \frac{\mu_1^{(p)}}{|\mu_1|^2} \frac{\mu_2^{(q)}}{|\mu_2|^2} (\psi(\sigma) - \psi(\sigma - 2 |\mu_2|)) - C_3 \psi(\sigma) \].
To verify (3.12), we rewrite

\[
\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \psi((2k+n)|\mu_1|,(2l+n)|\mu_2|)|\mu_1|^n|\mu_2|^n \, d\mu_1 d\mu_2 \leq C 2^{-j \alpha} 2^{-j} (2k+n)^{-n-m} (2l+n)^{-n-m}.
\]

Clearly, \(\psi_{p,q}^1(\sigma)\) also satisfies the first property. We claim that it satisfies (3.12)

\[
\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \psi_{p,q}^1((2k+n)|\mu_1|,(2l+n)|\mu_2|)|\mu_1|^n|\mu_2|^n \, d\mu_1 d\mu_2 \leq C 2^{-j(\alpha-4)} 2^{-j} (2k+n)^{-n-m} (2l+n)^{-n-m}.
\]

To verify (3.12), we rewrite \(\psi_{p,q}^1(\sigma)\) as

\[
\psi_{p,q}^1(\sigma) = \frac{n \mu_1 \mu_2}{2 |\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial k} \frac{\partial}{\partial l} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
- \frac{n \mu_1 \mu_2}{2 |\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial l} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|) - \psi((2k-2+n)|\mu_1| + (2l+2+n)|\mu_2|)
\]

\[
+ \frac{n \mu_1 \mu_2}{2 |\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial k} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
+ \frac{n \mu_1 \mu_2}{2 |\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial l} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
+ 2kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial k} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
- 2kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial l} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|) - \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
+ 2kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial k} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
- 2kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial l} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|) - \psi((2k-2+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
- kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial k} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
+ kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial l} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
+ kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial k} (2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
- kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial l} (2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
- kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial k} (2k-2+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
- kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial l} (2k-2+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
+ kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial k} (2k-2+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
- kl \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial l} (2k-2+n)|\mu_1| + (2l+n)|\mu_2|)
\]

\[
-C_1 k \frac{\mu_1 \mu_2}{|\mu_1|^2 |\mu_2|^2} \frac{\partial}{\partial k} \psi((2k+n)|\mu_1| + (2l+n)|\mu_2|)
\]
Using Taylor expansion, we have that

\[ \psi'(k+n) |\mu_1| + (2l + n) |\mu_2| \]  

\[ - \frac{n^2}{4} - 2k \left( \frac{1}{|\mu_1|^2 |\mu_2|^2} \right) \psi''(k+n) |\mu_1| + (2l + n) |\mu_2| \]

\[ + C_1 k \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \psi''(k+n) |\mu_1| + (2l + n) |\mu_2| \]

\[ + C_1 k \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \psi''(k+n) |\mu_1| + (2l + n) |\mu_2| \]

\[ - C_3 \psi((2k+n) |\mu_1| + (2l + n) |\mu_2|). \]

Using Taylor expansion, we have that

\[ \psi_{p,q}^1(\sigma) \]

\[ = - k\frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 16 |\mu_1|^2 |\mu_2|^2 \int_{k-l}^{k} (t+1-k)(s+1-l) \psi^{(4)}((2t+n) |\mu_1| + (2s+n) |\mu_2|) ds dt \right) \]

\[ + \frac{n k}{2} \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 8 |\mu_1|^2 |\mu_2|^2 \int_{k-l}^{k} (t+1-k) \psi^{(3)}((2t+n) |\mu_1| + (2s+n) |\mu_2|) ds dt \right) \]

\[ + \frac{n l}{2} \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 8 |\mu_1|^2 |\mu_2|^2 \int_{k-l}^{k} (s+1-l) \psi^{(3)}((2k+n) |\mu_1| + (2s+n) |\mu_2|) ds \right) \]

\[ + 2k l \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 8 |\mu_1|^2 |\mu_2|^2 \int_{k-l}^{k} (s+1-l) \psi^{(3)}((2k+n) |\mu_1| + (2s+n) |\mu_2|) ds \right) \]

\[ + 2k l \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 8 |\mu_1|^2 |\mu_2|^2 \int_{k-l}^{k} (t+1-k) \psi^{(3)}((2t+n) |\mu_1| + (2l + n) |\mu_2|) ds \right) \]

\[ - C_1 k \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 4 |\mu_1|^2 \int_{k-l}^{k} (t+1-k) \psi^{(2)}((2t+n) |\mu_1| + (2l + n) |\mu_2|) ds \right) \]

\[ - C_2 l \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 4 |\mu_1|^2 \int_{k-l}^{k} (s+1-l) \psi^{(2)}((2k+n) |\mu_1| + (2s+n) |\mu_2|) ds \right) \]

\[ + \left( \frac{n^2}{4} - 2k l \right) \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 4 |\mu_1|^2 |\mu_2|^2 \psi^{(2)}((2k+n) |\mu_1| + (2l+n) |\mu_2|) \right) \]

\[ + C_1 k \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 4 |\mu_1|^2 |\mu_2|^2 \psi^{(2)}((2k+n) |\mu_1| + (2l+n) |\mu_2|) \right) \]

\[ + C_2 l \frac{\mu_1}{|\mu_1|^2 |\mu_2|^2} \left( 4 |\mu_1|^2 |\mu_2|^2 \psi^{(2)}((2k+n) |\mu_1| + (2l+n) |\mu_2|) \right) \]

\[ - C_3 \psi((2k+n) |\mu_1| + (2l+n) |\mu_2|). \]

It follows that

\[ \int_{R^n} \int_{R^n} |\psi_{p,q}^1(\sigma)| |\mu_1|^n |\mu_2|^n d\mu_1 d\mu_2 \leq c_1 \sigma^{2-j} \int_{k-l}^{k} (2t+n)^{-n-m-1}(2s+n)^{-n-m-1} dt ds \]
An iteration of the process shows that for any 
\[ N \]
\[ p, q \]
\[ (3.13) \]
which yields that
\[ 1 \]
\[ 18 \]
\[ MIN WANG^{1} \]
\[ HUA ZHU^{1} \]
This proves that \[ \psi_{p,q}^{(1)}(\sigma) \] has the property \[ (3.12) \]. At the same time, applying the same way in Theorem \[ 1 \] we can obtain that
\[
\int_{\mathbb{R}^{m}} \int_{\mathbb{R}^{m}} \left| \frac{\partial}{\partial \mu_{1}^{(p)}} \frac{\partial}{\partial \mu_{2}^{(q)}} \right| \left| \det A_{\mu_{1}^{(p)}} \right| \left| \det A_{\mu_{2}^{(q)}} \right| - \frac{t^{(p)}}{\mu_{1}^{(p)}} \frac{\partial}{\partial \mu_{1}^{(p)}} \left( \left| \det A_{\mu_{1}^{(p)}} \right| \left| \det A_{\mu_{2}^{(q)}} \right| (\psi(\sigma) - \psi(\sigma - 2 |\mu_{1}|)) \right) \right| |\mu_{1}|^{n} |\mu_{2}|^{n} \ d\mu_{1} d\mu_{2}
\]
\[ \leq C^{2-j(a-2)} 2^{-j} (2k + n)^{-n-m} (2l + n)^{-n-m}. \]

It follows that
\[
\int_{\mathbb{R}^{m}} \int_{\mathbb{R}^{m}} R_{k,l} \left( |\mu_{1}^{(p)}| \frac{1}{|\mu_{1}|} |z_{1}|^{2} \right) \left( |\mu_{2}^{(q)}| \frac{1}{|\mu_{2}|} |z_{2}|^{2} \right) F \right| |\mu_{1}|^{n} |\mu_{2}|^{n} \ d\mu_{1} d\mu_{2}
\]
\[ \leq C^{2-j(a-2)} 2^{-j} (2k + n)^{-n-m} (2l + n)^{-n-m}. \]

An iteration of the process shows that for any \[ N \in \mathbb{N}^{+}, \]
\[
\int_{\mathbb{R}^{m}} \int_{\mathbb{R}^{m}} R_{k,l} \left( |\mu_{1}^{(p)}| \frac{1}{|\mu_{1}|} |z_{1}|^{2} \right) \left( |\mu_{2}^{(q)}| \frac{1}{|\mu_{2}|} |z_{2}|^{2} \right) F \right| |\mu_{1}|^{n} |\mu_{2}|^{n} \ d\mu_{1} d\mu_{2}
\]
\[ \leq C^{2-j(a-4N)} 2^{-j} (2k + n)^{-n-m} (2l + n)^{-n-m}. \]

Thus, for any \[ p, q = 1, 2, \ldots, m, \] we have
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_{\mathbb{R}^{m}} \int_{\mathbb{R}^{m}} R_{k,l} \left( |\mu_{1}^{(p)}| \frac{1}{|\mu_{1}|} |z_{1}|^{2} \right) \left( |\mu_{2}^{(q)}| \frac{1}{|\mu_{2}|} |z_{2}|^{2} \right) F \right| \frac{(k + n - 1)! (l + n - 1)!}{k! l!} |\mu_{1}|^{n} |\mu_{2}|^{n} \ d\mu_{1} d\mu_{2} \leq C^{2-j(a-4N+1)},
\]
which yields that
\[ (3.13) \]
\[ |K_{j}^{a} ((x_{1}, u_{1}), (x_{2}, u_{2}))| \leq C_{N}^{2-j(a-4N+1)} \left( 1 + \left| A_{\mu_{1}^{(p)}} x_{1}, u_{1} \right| \right)^{-2N} \left( 1 + \left| A_{\mu_{2}^{(q)}} x_{2}, u_{2} \right| \right)^{-2N}. \]
Consequently, for any fixed \((x_1, u_1), (x_2, u_2) \in \mathbb{G}\) and \(\mu_1, \mu_2 \in \mathbb{R}^m \setminus \{0\}\), whenever \(\alpha > 4N - 1\), we have

\[
|S^\alpha((x_1, u_1), (x_2, u_2))| \leq C_N \sum_{j=0}^{\infty} C_N 2^{-j(\alpha-4N+1)} \left(1 + \left| \left( A_{\mu_1} x_1, u_1 \right) \right| \right)^{-2N} \left(1 + \left| \left( A_{\mu_2} x_2, u_2 \right) \right| \right)^{-2N}.
\]

The proof of Theorem 2 is completed.

\[\square\]

**Corollary 2.** Let \(1 \leq p_1, p_2 \leq \infty\) and \(1/p = 1/p_1 + 1/p_2\). If \(\alpha > 2Q + 3\), then \(S^\alpha\) is bounded from \(L^{p_1}(\mathbb{G}) \times L^{p_2}(\mathbb{G})\) into \(L^{p}(\mathbb{G})\).

**Proof.** We take positive integer \(N = \frac{Q}{2} + 1\). If \(\alpha > 2Q + 3\), we have \(\alpha > 4N - 1\). So, Theorem 2 is available. Then, applying the Hölder’s inequality, the Young’s inequality and changing variables, we conclude that

\[
\|S^\alpha(f, g)\|_p \leq C \|f\|_{p_1} \|g\|_{p_2} \int_{\mathbb{G}} \left(1 + \left| \left( A_{\mu_1} x_1, u_1 \right) \right| \right)^{-2N} dx \int_{\mathbb{G}} \left(1 + \left| \left( A_{\mu_2} x_2, u_2 \right) \right| \right)^{-2N} dx_2 du_2.
\]

The proof is completed.

\[\square\]

Note that the index in Corollary 1 and Corollary 2 are very high. Thus, in the rest part of this paper, we shall use different methods to give lower indices.

4. **Boundedness of the Riesz means \(S^\delta\)**

Let \(\mathbb{G}\) be a M-type group with Lie algebra \(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2\) where \(\mathfrak{g}_2\) is the center of \(\mathfrak{g}\). As above, we assume that \(\dim \mathfrak{g}_1 = 2n\) and \(\dim \mathfrak{g}_2 = m\). The mixed norm on \(\mathbb{G}\) is defined by

\[
\|f\|_{L^p(\mathfrak{g}_2)L^q(\mathfrak{g}_1)} = \left( \left( \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} |f(x, u)|^r \, du \, dx \right)^{\frac{q}{r}} \right)^{\frac{1}{q}}, \quad 1 \leq p, r \leq \infty.
\]

Casarino and Ciatti [2] proved the following Stein-Tomas restriction theorem in terms of the mixed norms:

**Lemma 1.** [2] Let \(\mathbb{G}\) be a M-type group with Lie algebra \(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2\) where \(\mathfrak{g}_2\) is the center of \(\mathfrak{g}\). Let \(\dim \mathfrak{g}_1 = 2n\) and \(\dim \mathfrak{g}_2 = m\).\(\) Then, for all \(1 \leq p \leq 2 \leq q \leq \infty\), \(1 \leq r \leq \frac{2m+2}{m+3}\) and all \(f \in \mathcal{S}(\mathbb{G})\), we have that

\[
\|P_\lambda f\|_{L^r(\mathfrak{g}_2)L^q(\mathfrak{g}_1)} \leq C\lambda^{\left(\frac{2}{r}-1\right)+n\left(\frac{1}{p}-\frac{1}{q}\right)^{-1}} \|f\|_{L^r(\mathfrak{g}_2)L^q(\mathfrak{g}_1)}.
\]

Particularly, when \(p = r\) and \(q = q^r\), we have the following result:
Theorem 3. Let $G$ be a $M$-type group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_2$ is the center of $\mathfrak{g}$. Let $\dim \mathfrak{g}_1 = 2n$ and $\dim \mathfrak{g}_2 = m$. Then, for all $1 \leq p \leq \frac{2m+2}{m+3}$ and all $f \in \mathcal{S}(G)$, we have that

$$\|P_\lambda f\|_{L^p(G)} \leq C_\lambda Q(\frac{1}{p} - \frac{1}{2})^{-1} \|f\|_{L^p(G)}.$$ 

Applying Theorem 3, we can obtain the $L^p$-boundedness of the Riesz means $S^\delta$.

Theorem 4. Let $G$ be a $M$-type group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_2$ is the center of $\mathfrak{g}$. Let $\dim \mathfrak{g}_1 = 2n$ and $\dim \mathfrak{g}_2 = m$. Suppose that $1 \leq p \leq \frac{2m+2}{m+3}$. If $\delta > Q\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}$, then $S^\delta$ is bounded on $L^p(G)$.

Proof. We take a partition of unity $\sum_{\omega} \varphi(2^j s) = 1$ where $\varphi \in C^\infty_0(\mathbb{R})$ is a nonnegative function, and write $\varphi_j^\delta(s) = (1 - s)^\delta_\omega \varphi(2^j (1 - s))$ for each $j \geq 0$. Define

$$T_j^\delta f = \int_0^\infty \int_0^\infty \varphi_j^\delta(\lambda) P_\lambda f d\lambda.$$ 

It is obvious that

$$S^\delta = \sum_{j=0}^\infty T_j^\delta.$$ 

Theorem 4 will be proved once we show that when $\delta > Q\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}$, there exists an $\varepsilon > 0$ such that for each $j \geq 0$,

$$(4.1) \quad \left\|T_j^\delta\right\|_{L^p \to L^p} \leq 2^{-\varepsilon j}.$$

In order to prove (4.1), we define $B_j = \{\omega : |\omega| \leq 2^{j(1+\gamma)}\} \subseteq G$ and split the kernel $s_j^\delta(\omega)$ of $T_j^\delta$ into two parts

$$s_j^\delta(\omega) = s_j^1(\omega) + s_j^2(\omega),$$

where $s_j^1(\omega) = s_j^\delta \chi_{B_j}(\omega)$, $s_j^2(\omega) = s_j^\delta \chi_{G\setminus B_j}(\omega)$. Here $\chi_A$ stands for the characteristic function of set $A$ and $\gamma > 0$ is to be fixed. Clearly, (4.1) is the consequence of the estimates

$$\left\|f \ast s_j^l\right\|_{L^p} \leq C 2^{-\varepsilon j} \|f\|_p, \quad l = 1, 2.$$

We first consider the convolution with $s_j^2(\omega)$. Set $R_\lambda^1(\omega)$ to be the kernel of the Riesz means $\int_0^1 (1 - \frac{t}{\lambda})^1 P dt$. Then,

$$\lambda \to R_\lambda^0(\omega)$$

is a function of bounded variation, and the kernel of each $T_j^\delta$ can be written as

$$s_j^\delta(\omega) = \int_0^\infty \varphi_j^\delta(\lambda) \frac{\partial}{\partial \lambda} R_\lambda^0(\omega) d\lambda.$$ 

Integrating by parts and using the identity

$$\frac{\partial}{\partial \lambda}\left(\lambda^N R_\lambda^N(\omega)\right) = N \lambda^{N-1} R_\lambda^{N-1}(\omega),$$

where $N$ is a positive integer, we get the relation

$$s_j^\delta(\omega) = c_N \int \left(\frac{\partial^2}{\partial \lambda^2} \varphi_j^\delta(\lambda)\right) \lambda^{2N+1} R_\lambda^{2N+1}(\omega) d\lambda.$$ 

Theorem 4 tells that

$$(4.3) \quad \sup_{\lambda \in [0, 1]} \lambda^{2N+1} R_\lambda^{2N+1}(\omega) \leq \left(1 + \left|\left(\frac{A_{\tilde{m}} x}{u}\right)\right|^{-2N}\right)^{-\frac{1}{2}}.$$
Note that

\[ |\phi^2(\lambda)| \leq 2^{j(2N+2)} \]

imply that

\[ |s^2_j(\omega)| \leq C2^{j(2N+2)} \left( 1 + \left| \left( A_{\mu_{\omega_{\omega}}} x, u \right) \right|^2 \right)^{-2N}. \]

By changing variables, it follows that

\[ \int_G |s^2_j(\omega)| \, d\omega = \int_{|\omega| > 2^j(1+\gamma)} |s^2_j(\omega)| \, d\omega \]
\[ \leq C2^{j(2N+2)} \int_{|\omega| \geq 2^j(1+\gamma)} \left( 1 + \left| \left( A_{\mu_{\omega_{\omega}}} x, u \right) \right|^2 \right)^{-2N} \, dzdu \]
\[ \leq C2^{j(2N+2)} \text{det} A_{\mu_{\omega_{\omega}}}^{-1} \int_{|\omega| \geq 2^j(1+\gamma)} \left( 1 + |(z, u)| \right)^{-2N} \, dzdu. \]

Because there exists positive constant \( K, H \) such that

\[ \sqrt{K} \leq \left| \text{det} A_{\mu_{\omega_{\omega}}} \right|^{-1} \leq \frac{1}{\sqrt{K}} \quad \text{and} \quad \frac{1}{H} \leq \left| A_{\mu_{\omega_{\omega}}}^{-1} \right| \leq H |z| \]

for any \( \mu \in \mathbb{R}^m \setminus \{0\} \), so we have that

\[ \int_G |s^2_j(\omega)| \, d\omega \leq C2^{j(2N+2)} \int_{|\omega| \geq 2^j(1+\gamma)} \left( 1 + |(z, u)| \right)^{-2N} \, dzdu \]
\[ \leq C2^{j(2N+2)} \int_{|\omega| \geq 2^j(1+\gamma)} \left( 1 + |\omega| \right)^{-2N} \, d\omega \]
\[ \leq C2^{j(2N+2)2j(1+\gamma)(-2N+Q)}. \]

Choosing \( N \) large enough such that

\[ 2N\gamma > Q(1 + \gamma) + 2, \]

it follows that

\[ \int_G |s^2_j(\omega)| \, d\omega \leq C2^{-\varepsilon j} \]

for some \( \varepsilon > 0 \). By Young’s inequality, we conclude for any \( 1 \leq p \leq \infty \),

\[
(4.4) \quad \|f \ast s^2_j\|_p \leq C2^{-\varepsilon j} \|f\|_p.
\]

Next, we consider the convolution with \( s^1_j \). For any \( \xi \in \mathbb{G} \) and \( R > 0 \), we set \( B_j(\xi, R) = \{\omega : |\xi^{-1}\omega| \leq R2^{j(1+\gamma)}\} \subseteq \mathbb{G} \) and split the function \( f \) into three parts: \( f = f_1 + f_2 + f_3 \) where \( f_1 = f_{\chi_{B_j(\xi, R)}} \), \( f_2 = f_{\chi_{B_j(\xi, R)} \setminus B_j(\xi, \frac{\omega}{4})} \) and \( f_3 = f_{\chi_{G \setminus B_j(\xi, \frac{\omega}{4})}} \). Assume that \( |\xi^{-1}\omega| \leq \frac{3}{4}2^{j(1+\gamma)} \). Since that \( f_3 \) is supported on \( \mathbb{G} \setminus B_j(\xi, \frac{\omega}{4}) \), then \( f_3 \neq 0 \) leads to

\[ |\xi^{-1} \cdot \omega \omega' \cdot \omega^{-1}| \geq \frac{5}{4}2^{j(1+\gamma)}. \]

It follows that

\[ |\omega'| \geq 2^{j(1+\gamma)}. \]

Note that \( s^1_j \) is supported on \( B_j \). Hence, \( f_3 \ast s^1_j = 0 \). Since \( f_2 \) is supported on \( B_j(\xi, \frac{5}{4}) \setminus B_j(\xi, \frac{3}{4}) \), then \( f_2 \neq 0 \) leads to

\[ |\omega'| > \frac{1}{2}2^{j(1+\gamma)}. \]
Repeating the proof of (4.4), we can get that
\[(4.5) \quad \|f_2 * s^1_j\|_{L^p(B_j(\xi, \frac{\lambda}{4}))} \leq C 2^{-\varepsilon j} \|f_2\|_{L^p} \leq C 2^{-\varepsilon j} \|f\|_{L^p(B_j(\xi, \frac{\lambda}{4}))}.\]
Taking the $L^p$ norm with respect to $\xi$ on the both side of (4.5), it follows that
\[
\left( \int_G \int_{B_j(\xi, \frac{\lambda}{4})} |f_2 * s^1_j(\omega)|^p d\omega d\xi \right)^{\frac{1}{p}} \leq C 2^{-\varepsilon j} \left( \int_G \int_{B_j(\xi, \frac{\lambda}{4})} |f(\omega)|^p d\omega d\xi \right)^{\frac{1}{p}}.
\]
Changing the variable and exchanging the order of integration, the left side
\[
\left( \int_G \int_{B_j(\xi, \frac{\lambda}{4})} |f_2 * s^1_j(\omega)|^p d\omega d\xi \right)^{\frac{1}{p}} = \left( \int_{|\omega| \leq \frac{\lambda}{2} 2^{j(1+\gamma)}} \int_G \int_{B_j(\xi, \frac{\lambda}{4})} |f_2 * s^1_j(\xi\omega)|^p d\omega d\xi \right)^{\frac{1}{p}}
\]
\[
= \left( \int_{|\omega| \leq \frac{\lambda}{2} 2^{j(1+\gamma)}} \int_G |f_2 * s^1_j(\xi\omega)|^p d\xi d\omega \right)^{\frac{1}{p}}
\]
\[
= \left( \frac{1}{4} 2^{j(1+\gamma)} \right)^{\frac{p}{q}} \|f_2 * s^1_j\|_p.
\]
In the same way, the right side
\[
C 2^{-\varepsilon j} \left( \int_{|\omega| \leq \frac{\lambda}{2} 2^{j(1+\gamma)}} \int_G |f(\omega)|^p d\omega d\xi \right)^{\frac{1}{p}} = C 2^{-\varepsilon j} \left( \frac{5}{4} 2^{j(1+\gamma)} \right)^{\frac{p}{q}} \|f\|_p.
\]
Hence, we have that
\[(4.6) \quad \|f_2 * s^1_j\|_p \leq C 2^{-\varepsilon j} \|f\|_p.
\]
Now it remains to estimate $f_1 * s^1_j$. Since that $f_1 \neq 0$ implies
\[
|\xi^{-1} \cdot \omega\omega^{-1}| \leq \frac{3}{4} 2^{j(1+\gamma)},
\]
it follows that
\[
|\omega'| \leq 2^{j(1+\gamma)}.
\]
So, we have
\[(4.7) \quad f_1 * s^1_j(\omega) = f_1 * s^1_j(\omega) \quad \text{for any} \quad \omega \in B_j(\xi, \frac{1}{4}).
\]
Note that
\[(4.8) \quad f * s^\delta_j = T^\delta_j f = \int_{1-2^{-j+1}}^{1-2^{-j-1}} \varphi^\delta_j(\lambda) P_{\lambda} f d\lambda.
\]
Then,
\[
\left\| f * s^\delta_j \right\|_2^2 = \left\langle T^\delta_j f, T^\delta_j f \right\rangle = \left\langle \left( T^\delta_j \right)^2 f, f \right\rangle
\]
\[
\leq \left\| \left( T^\delta_j \right)^2 f \right\|_{\varphi'} \|f\|_p.
\]
Since
\[
\left( T^\delta_j \right)^2 f = \int_{1-2^{-j+1}}^{1-2^{-j-1}} \left( \varphi^\delta_j(\lambda) \right)^2 P_{\lambda} f d\lambda,
\]
using Theorem 3, we get that
\[
\| \left( T_j^\alpha \right)^2 f \|_{p'} \leq \int_{1-2^{-j-1}}^{1-2^{-j+1}} \left( \varphi_j^\delta (\lambda) \right)^2 \| P_\lambda f \|_{p'} d\lambda \\
\leq C \int_{1-2^{-j-1}}^{1-2^{-j+1}} \left( \varphi_j^\delta (\lambda) \right)^2 \lambda^{Q \left( \frac{1}{p} - \frac{1}{2} \right) - 1} \| f \|_{p'} d\lambda \\
\leq C 2^{-2j\delta - j} \| f \|_p.
\]
So,
\[
\| f * s_j^\delta \|_p^2 \leq C 2^{-2j\delta - j} \| f \|_p^2.
\]
This estimate, together with (4.7) and Hölder’s inequality yield that
\[
\| f_1 * s_j^\delta \|_{L^p(B_j(\xi, \frac{1}{4}))} = \| f_1 * s_j^\delta \|_{L^p(B_j(\xi, \frac{1}{4}))} \\
\leq 2^{jQ(1+\gamma) \left( \frac{1}{p} - \frac{1}{2} \right)} \| f_1 * s_j^\delta \|_2 \\
\leq C 2^{-j\delta - \frac{2}{p} + 2jQ(1+\gamma) \left( \frac{1}{p} - \frac{1}{2} \right)} \| f_1 \|_p \\
\leq C 2^{-j\delta - \frac{2}{p} + 2jQ(1+\gamma) \left( \frac{1}{p} - \frac{1}{2} \right)} \| f \|_{L^p(B_j(\xi, \frac{1}{4}))}.
\]
Taking the $L^p$ norm with respect to $\xi$, we have
\[
\| f_1 * s_j^\delta \|_{L^p} \leq C 2^{-j\delta - \frac{2}{p} + 2jQ(1+\gamma) \left( \frac{1}{p} - \frac{1}{2} \right)} \| f \|_{L^p}.
\]
Thus, when $\delta > Q \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2}$, we can choose $\gamma > 0$ such that
\[
\delta > Q(1+\gamma) \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2},
\]
which implies that there exists an $\varepsilon > 0$ such that
\[
\| f_1 * s_j^\delta \|_{L^p} \leq C 2^{-\varepsilon j} \| f \|_{L^p}.
\]
The proof of Theorem 4 is completed. \hfill \Box

5. Boundedness of $S^\alpha$ for $1 \leq p_1, p_2 \leq 2$

From this section, we begin to investigate the $L^{p_1} \times L^{p_2} \rightarrow L^p$ boundedness of the bilinear Riesz means $S^\alpha$. We first consider the case of $1 \leq p_1, p_2 \leq 2$.

**Lemma 2.** Suppose $m \in L^\infty(\mathbb{R})$. Define operator $T_m f = \int_a^b m(\lambda) P_\lambda f d\lambda$ with $0 \leq a < b$. Then, for any $1 \leq p \leq 2$, we have
\[
\| T_m f \|_2 \leq C \| m \|_{\infty} \left( (b-a) b^{n+m-1} \right) \left( \frac{1}{p} - \frac{1}{2} \right) \| f \|_p.
\]

**Proof.** Since that
\[
P_\lambda f = \sum_{k=0}^\infty \frac{\lambda^{n+m-1}}{(2\pi(2k+n))^{n+m}} \int_{\mathbb{R}^{n-m}} f * \tilde{e}_k^\lambda (x, u) d\sigma(\eta),
\]
the operator $T_m$ can be written as
\[
T_m f = \int_a^b m(\lambda) P_\lambda f d\lambda = f * G_m,
\]
where the kernel $G_m$ is given by $G_m(x, u)$.
By interpolation with the trivial estimate

Applying the Plancherel theorem in the variable $u$ and using Young’s inequality, it follows that

Changing variables in the integral and using the estimate

where

Applying the Plancherel theorem in the variable $u$ and the orthogonality of $\varphi_k^{[\mu]}$, we obtain that

Changing variables in the integral and using the estimate

we get that

Thus,

Using Young’s inequality, it follows that

By interpolation with the trivial estimate

$$
\|T_m f\|_2 \leq \|m\|_\infty \|f\|_1 \leq C \left( (b-a)^{n+m-\frac{1}{2}} \right) \|m\|_\infty \|f\|_1,
$$
we conclude that
\[ \|T_m f\|_2 \leq C \left( (b-a) b^{n+m-1} \right)^{\frac{1}{p} - \frac{1}{2}} \|m\|_\infty \|f\|_p. \]
The proof is completed. \(\square\)

**Theorem 5.** Suppose that \(1 \leq p_1, p_2 \leq 2\) and \(1/p = 1/p_1 + 1/p_2\). If \(\alpha > Q \left( \frac{1}{p} - 1 \right)\), then \(S^\alpha\) is bounded from \(L^{p_1}(G) \times L^{p_2}(G)\) into \(L^p(G)\).

**Proof.** In the proof of Theorem 2, we have defined the operator
\[ T_j^\alpha(f, g) = \int_0^\infty \int_0^\infty \varphi_j^\alpha(\lambda_1, \lambda_2) P_{\lambda_1} f P_{\lambda_2} g d\lambda_1 d\lambda_2, \]
and showed that
\[ S^\alpha = \sum_{j=0}^\infty T_j^\alpha. \]
So, Theorem 5 would follow if we can show that when \(\alpha > Q \left( \frac{1}{p} - 1 \right)\), there exists an \(\varepsilon > 0\) such that for each \(j \geq 0\),
\[ \|T_j^\alpha\|_{L^{p_1} \times L^{p_2} \to L^p} \leq 2^{-\varepsilon j}. \]
Fixing \(j \geq 0\). To prove (5.2), we define \(B_j = \{ \omega : |\omega| \leq 2^j(1+\gamma) \} \subseteq G\) and split the kernel of \(T_j^\alpha\), denoted by \(K_j^\alpha\), into four parts:
\[ K_j^\alpha = K_j^1 + K_j^2 + K_j^3 + K_j^4, \]
where
\[ K_j^1(\omega_1, \omega_2) = K_j^\alpha(\omega_1, \omega_2) \chi_{B_j}(\omega_1) \chi_{B_j}(\omega_2), \]
\[ K_j^2(\omega_1, \omega_2) = K_j^\alpha(\omega_1, \omega_2) \chi_{B_j}(\omega_1) \chi_{B_j^c}(\omega_2), \]
\[ K_j^3(\omega_1, \omega_2) = K_j^\alpha(\omega_1, \omega_2) \chi_{B_j^c}(\omega_1) \chi_{B_j}(\omega_2), \]
\[ K_j^4(\omega_1, \omega_2) = K_j^\alpha(\omega_1, \omega_2) \chi_{B_j^c}(\omega_1) \chi_{B_j^c}(\omega_2). \]
Let \(T_j^l\) be the bilinear operator with kernel \(K_j^l, l = 1, 2, 3, 4\). Then, (5.2) would be the consequence of the estimates
\[ \|T_j^l\|_{L^{p_1} \times L^{p_2} \to L^p} \leq 2^{-\varepsilon j}, \quad l = 1, 2, 3, 4. \]
First consider \(T_j^4\). \[3.13\] tells that for any \(N \in \mathbb{N}^+\) and \(\omega_1 = (x_1, u_1), \omega_2 = (x_2, u_2) \in G\),
\[ |K_j^\alpha((x_1, u_1), (x_2, u_2))| \leq C_N 2^{-j\left(\alpha - 4N + 1\right)} \left(1 + \left|\left(A_{\frac{\mu_1}{|B_j|}} x_1, u_1\right)\right|\right)^{-2N} \left(1 + \left|\left(A_{\frac{\mu_2}{|B_j^c|}} x_2, u_2\right)\right|\right)^{-2N}. \]
Applying Hölder’s inequality, Young’s inequality and changing variables, we can easily get that
\[ \|T_j^4(f, g)\|_p \leq C_N 2^{j\cdot 4N} \|f\|_{p_1} \|g\|_{p_2} \left( \int_{|x_1, u_1| \geq 2^j(1+\gamma)} \left(1 + \left|\left(A_{\frac{\mu_1}{|B_j|}} x_1, u_1\right)\right|\right)^{-2N} dx_1 du_1 \right) \times \left( \int_{|x_2, u_2| \geq 2^j(1+\gamma)} \left(1 + \left|\left(A_{\frac{\mu_2}{|B_j^c|}} x_2, u_2\right)\right|\right)^{-2N} dx_2 du_2 \right). \]
\[
\begin{align*}
\leq \quad & C_N 2^{J+4N} \|f\|_{p_1} \|g\|_{p_2} \left( \int_{1}(z_1, u_1) \right)^{-2N} dz_1 du_1 \\
\quad \times \quad & \left( \int_{1}(z_2, u_2) \right)^{-2N} dz_2 du_2 \\
\leq \quad & C_N 2^{J+4N} \|f\|_{p_1} \|g\|_{p_2} \left( \int_{1}(z_1, u_1) \right)^{-2N} dz_1 du_1 \\
\quad \times \quad & \left( \int_{1}(z_2, u_2) \right)^{-2N} dz_2 du_2 \\
\leq \quad & C_N 2^{J+4N} \|f\|_{p_1} \|g\|_{p_2} \left( \int_{1}(\omega_1) \right)^{-2N} d\omega_1 \\
\quad \times \quad & \left( \int_{1}(\omega_2) \right)^{-2N} d\omega_2 \\
\leq \quad & C_N 2^{J+4N} \|f\|_{p_1} \|g\|_{p_2} \left( \int_{1}(\omega_1) \right)^{-2N} d\omega_1 \\
\quad \times \quad & \left( \int_{1}(\omega_2) \right)^{-2N} d\omega_2 \\
\leq \quad & C_N 2^{J+4N} 2^{(1+\gamma)(-4N+2Q)} \|f\|_{p_1} \|g\|_{p_2}.
\end{align*}
\]

Choosing \(N\) large enough such that
\[
4N\gamma > 2Q(1 + \gamma),
\]
we have
\[
(5.3) \quad \|T^J\|_{L^p \times L^p \to L^p} \leq 2^{-\varepsilon j}
\]
for some \(\varepsilon > 0\).

Consider the estimate of \(T^J\). As the proof of Theorem 4, the kernel \(K_j^\alpha\) can be rewritten as
\[
K_j^\alpha(\omega_1, \omega_2) = c_N \int_0^\infty \int_0^\infty \varphi^\alpha_j(\lambda_1, \lambda_2) \frac{\partial}{\partial \lambda_1} R_{\lambda_1}(\omega_1) G_{\lambda_2}(\omega_2) d\lambda_1 d\lambda_2,
\]
where
\[
G_{\lambda_2}(\omega_2) = G_{\lambda_2}(x, u) = \sum_{k=0}^{\infty} \frac{\lambda_2^{n+m-1}}{(2\pi(2k+n+1))^{n+m}} \int_{Sm-1} \hat{e}^{\lambda_2^2}(x, u) d\eta(\eta_2).
\]

Intergration by parts and using the identity (4.2), we get that
\[
K_j^\alpha(\omega_1, \omega_2) = c_N \int_0^1 \int_0^1 \left( \partial_{\lambda_1}^{2N+2} \varphi_j(\lambda_1, \lambda_2) \right) \lambda_1^{2N+1} R_{\lambda_1}(\omega_1) G_{\lambda_2}(\omega_2) d\lambda_1 d\lambda_2.
\]
So,
\[
|K_j^\alpha(\omega_1, \omega_2)|
\leq C \int_0^1 \left| \lambda_1^{2N+1} R_{\lambda_1}^{2N+1}(\omega_1) \chi_{B_j}(\omega_1) \right| \left| \int_0^1 \left( \partial_{\lambda_1}^{2N+2} \varphi_j(\lambda_1, \lambda_2) \right) G_{\lambda_2}(\omega_2) \chi_{B_j}(\omega_2) d\lambda_2 \right| d\lambda_1
\leq C \sup_{\lambda_1 \in [0, 1]} \left| \lambda_1^{2N+1} R_{\lambda_1}^{2N+1}(\omega_1) \chi_{B_j}(\omega_1) \right| \int_0^1 \left| \int_0^1 \left( \partial_{\lambda_2}^{2N+2} \varphi_j(\lambda_1, \lambda_2) \right) G_{\lambda_2}(\omega_2) \chi_{B_j}(\omega_2) d\lambda_2 \right| d\lambda_1.
\]
It follows that
\[
\|T^3_j(f,g)\|_p \leq C \|f\|_{p_1} \|g\|_{p_2} \int_{|\omega_1| \geq 2^{j(1+\gamma)}} \sup_{\lambda_1 \in [0,1]} \lambda_1^{2N+1} R_{\lambda_1}^{2N+1}(\omega_1) \, d\omega_1 \times \int_{|\omega_2| \leq 2^{j(1+\gamma)}} \left( \|\partial^{2N+2}_2 \varphi_j^\alpha(\lambda_1, \lambda_2)\| G_{\lambda_2}(\omega_2) \, d\omega_2 \right) \, d\lambda_1 \, d\omega_2.
\]
Applying the Cauchy-Schwartz’s inequality and Lemma 2, we get that
\[
\int_{|\omega_2| \leq 2^{j(1+\gamma)}} \left( \|\partial^{2N+2}_2 \varphi_j^\alpha(\lambda_1, \lambda_2)\| G_{\lambda_2}(\omega_2) \, d\omega_2 \right) \leq C^2(1+\gamma) \frac{2}{2} \leq C^2(1+\gamma) \frac{2}{2} 2^{j(2N+2)}.
\]
On the other hand, (4.3) implies that
\[
\int_{|\omega_1| \geq 2^{j(1+\gamma)}} \sup_{\lambda_1 \in [0,1]} \lambda_1^{2N+1} R_{\lambda_1}^{2N+1}(\omega_1) \, d\omega_1 \leq C \int_{(x_1, u_1) \geq 2^{j(1+\gamma)}} \left( 1 + \left| \left( A_{\omega_1} x_1, u_1 \right) \right| \right)^{-2N} \, dx_1 \, du_1 \leq C \int_{|\omega_1| \geq 2^{j(1+\gamma)}} (1 + |\omega_1|)^{-2N} \, d\omega_1 \leq C 2^{j(1+\gamma)(-2N+Q)}.
\]
Thus,
\[
\|T^3_j(f,g)\|_p \leq C \|f\|_{p_1} \|g\|_{p_2} 2^{j(1+\gamma)\frac{Q}{2}} 2^{j(2N+2)} 2^{j(1+\gamma)(-2N+Q)}.
\]
Choose \(N\) large enough such that
\[
2N\gamma \geq \frac{3}{2} Q(1+\gamma) + 2,
\]
we have
(5.4) \[
\|T^3_j\|_{L^p \times L^p \to L^p} \leq C 2^{-\varepsilon j}
\]
for some \(\varepsilon > 0\). Obviously, (5.4) also holds for \(T^2_j\).

Now, it remains to estimate \(T^1_j\). \(T^1_j\) is denoted by
\[
T^1_j(f,g)(\omega) = \int_{G} \int_{G} f(\omega_1^{-1}) g(\omega_2^{-1}) K_j^1(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2.
\]
We assume that \(\omega \in B_j(\xi, \frac{1}{4})\) and split the functions \(f, g\) into three parts respectively: \(f = f_1 + f_2 + f_3, g = g_1 + g_2 + g_3\), where
\[
\begin{align*}
f_1 &= f \chi_{B_j(\xi, \frac{1}{4})}, & g_1 &= g \chi_{B_j(\xi, \frac{1}{4})}, \\
f_2 &= f \chi_{B_j(\xi, \frac{1}{4}) \setminus B_j(\xi, \frac{3}{4})}, & g_2 &= g \chi_{B_j(\xi, \frac{1}{4}) \setminus B_j(\xi, \frac{3}{4})}, \\
f_3 &= f \chi_{G \setminus B_j(\xi, \frac{1}{4})}, & g_3 &= g \chi_{G \setminus B_j(\xi, \frac{1}{4})}.
\end{align*}
\]
Based on this decomposition, we notice that if \( f_3 \neq 0 \) or \( g_3 \neq 0 \),
\[
|\xi^{-1} \cdot \omega\omega^{-1}| \geq \frac{5}{4} 2^{j(1+\gamma)} \quad \text{or} \quad |\xi^{-1} \cdot \omega\omega^{-1}| \geq \frac{5}{4} 2^{j(1+\gamma)}.
\]
Since \( |\xi^{-1}\omega| \leq \frac{1}{4} 2^{j(1+\gamma)} \), by the triangle inequality, we get that
\[
|\omega_1| \geq 2^{j(1+\gamma)} \quad \text{or} \quad |\omega_2| \geq 2^{j(1+\gamma)}.
\]
Because the kernel \( K_j^1 \) is supported on \( B_j \times B_j \), we have \( T_j^1(f_3,g) = 0 \) and \( T_j^1(f,g_3) = 0 \). Similarly, \( f_2 \neq 0 \) and \( g_2 \neq 0 \) yield that
\[
|\omega_1| \geq \frac{1}{2} 2^{j(1+\gamma)} \quad \text{and} \quad |\omega_2| \geq \frac{1}{2} 2^{j(1+\gamma)}.
\]
So, we can repeat the proof of (5.3) to obtain that
\[
\|T_j^1(f_2,g_2)\|_{L_p(B_j(\xi,\frac{1}{4}))} \leq C 2^{-\varepsilon j} \|f_2\|_{p_1} \|g_2\|_{p_2}
\]
(5.5)
Taking the \( L^p \) norm with respect to \( \xi \), as the proof of Theorem 4, we get that
\[
\|T_j^1(f_2,g_2)\|_p \leq C 2^{-\varepsilon j} \|f\|_{p_1} \|g\|_{p_2}.
\]
(5.6)
If \( f_1 \neq 0 \) and \( g_2 \neq 0 \), we have
\[
|\omega_1| \leq 2^{j(1+\gamma)} \quad \text{and} \quad |\omega_2| \geq \frac{1}{2} 2^{j(1+\gamma)}.
\]
Repeating the proof of (5.4), we can conclude that
\[
\|T_j^1(f_1,g_2)\|_{L_p(B_j(\xi,\frac{1}{4}))} \leq C 2^{-\varepsilon j} \|f_1\|_{p_1} \|g_2\|_{p_2}
\]
\[
\leq C 2^{-\varepsilon j} \|f\|_{L^1(B_j(\xi,\frac{1}{4}))} \|g\|_{L^2(B_j(\xi,\frac{1}{4}))}.
\]
(5.7)
Obviously, (5.7) also holds for \( T_j^1(f_2,g_1) \). Finally, we consider \( T_j^1(f_1,g_1) \). Because \( f_1, g_1 \neq 0 \) implies that
\[
|\omega_1| \leq 2^{j(1+\gamma)} \quad \text{and} \quad |\omega_1| \leq 2^{j(1+\gamma)},
\]
so
\[
T_j^1(f_1,g_1)(\omega) = T_j^\alpha(f_1,g_1)(\omega)
\]
(5.8)
for any \( \omega \in B_j(\xi,\frac{1}{4}) \). Notice that \( T_j^\alpha \) can be written as
\[
T_j^\alpha(f,g) = \int_0^\infty \int_0^\infty \varphi_j^\alpha(\lambda_1,\lambda_2) P_{\lambda_1} f P_{\lambda_2} g d\lambda_1 d\lambda_2
\]
\[
= C \int_{[-1,1]^2} \varphi_j^\alpha(|\lambda_1|,|\lambda_2|) P_{\lambda_1} f P_{\lambda_2} g d\lambda_1 d\lambda_2.
\]
Since that for any fixed \( s \in [-1,1] \), the function
\[
t \to \varphi_j^\alpha(|s|,|t|)
\]
is supported in \([-1,1]\) and vanishes at endpoints \( \pm 1 \), we can expand this function in Fourier series by considering a periodic extension on \( \mathbb{R} \) of period 2, i.e.
\[
\varphi_j^\alpha(|s|,|t|) = \sum_{k \in \mathbb{Z}} \gamma_j^{\alpha}(s) e^{i\pi k t},
\]
Thus, whenever \( \alpha > Q \)
\( \epsilon > 0 \) such that
\[
\sup_{s \in [-1, 1]} |\gamma_{j,k}^\alpha(s)|(1 + |k|)^{1+\delta} \leq C2^{-j(\alpha-\delta)}.
\]
Then, \( T_j^\alpha \) can be expressed by
\[
T_j^\alpha(f, g) = C \int_{[-1, 1]^2} \varphi_j^\alpha(|\lambda_1|, |\lambda_2|) P_{\lambda_1} f P_{\lambda_2} g d\lambda_1 d\lambda_2
\]
\[
= C \sum_{k \in \mathbb{Z}} \int_{[-1, 1]^2} \gamma_{j,k}^\alpha(\lambda_1)e^{i\pi k \lambda_2} P_{\lambda_1} f P_{\lambda_2} g d\lambda_1 d\lambda_2
\]
\[
= C \sum_{k \in \mathbb{Z}} \int_{-1}^1 \gamma_{j,k}^\alpha(\lambda_1) P_{\lambda_1} f d\lambda_1 \int_{-1}^1 e^{i\pi k \lambda_2} P_{\lambda_2} g d\lambda_2.
\]
Applying the Cauchy-Schwartz’s inequality and Lemma 2, we have
\[
\|T_j^\alpha(f, g)\|_1 \leq C \sum_{k \in \mathbb{Z}} \left\| \int_{-1}^1 \gamma_{j,k}^\alpha(\lambda_1) P_{\lambda_1} f d\lambda_1 \right\|_2 \left\| \int_{-1}^1 e^{i\pi k \lambda_2} P_{\lambda_2} g d\lambda_2 \right\|_2
\]
\[
\leq C \sum_{k \in \mathbb{Z}} (1 + |k|)^{-1-\delta} \left( \sup_{s \in [-1, 1]} |\gamma_{j,k}^\alpha(s)|(1 + |k|)^{1+\delta} \right) \|f\|_{p_1} \|g\|_{p_2}
\]
(5.9)
\[
\leq C2^{-j(\alpha-\delta)} \|f\|_{p_1} \|g\|_{p_2}.
\]
Using the Hölder’s inequality and (5.8), we have
\[
\|T_j^1(f_1, g_1)\|_{L^p(B_j(\xi, \frac{1}{4}))} \leq 2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)} \|T_j^1(f_1, g_1)\|_{L^1(B_j(\xi, \frac{1}{4}))}
\]
\[
= 2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)} \|T_j^0(f_1, g_1)\|_{L^1(B_j(\xi, \frac{1}{4}))}
\]
\[
\leq C2^{-j(\alpha-\delta)2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)}} \|f_1\|_{p_1} \|g_1\|_{p_2}
\]
\[
\leq C2^{-j(\alpha-\delta)2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)}} \|f\|_{L^p(B_j(\xi, \frac{3}{4}))} \|g\|_{L^{p_2}(B_j(\xi, \frac{3}{4}))}.
\]
Taking the \( L^p \) norm with respect to \( \xi \), it follows that
\[
(5.10) \quad \|T_j^1(f_1, g_1)\|_p \leq C2^{-j(\alpha-\delta)2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)}} \|f\|_{p_1} \|g\|_{p_2}.
\]
Combining (5.6), (5.7) and (5.10), we can conclude that
\[
\|T_j^1(f, g)\|_p \leq C2^{-j(\alpha-\delta)2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)}} \|f\|_{p_1} \|g\|_{p_2}.
\]
Thus, whenever \( \alpha > Q\left(\frac{1}{p} - 1\right) \), we can choose suitable \( \gamma, \delta > 0 \) such that
\[
\alpha > Q(1 + \gamma)\left(\frac{1}{p} - 1\right) + \delta,
\]
which means that there exists an \( \varepsilon > 0 \) such that
\[
\|T_j^1\|_{L^1 \times L^{p_2} \to L^p} \leq 2^{-\varepsilon j}.
\]
The proof of Theorem 5 is completed.

6. BOUNDEDNESS OF $S^\alpha$ FOR PARTICULAR POINTS

In this section, we investigate the boundedness of $S^\alpha$ for some specific triples of points $(p_1, p_2, p)$. 

6.1. The point $(1, \infty, 1)$.

**Theorem 6.** If $\alpha > \frac{Q}{2}$, then $S^\alpha$ is bounded from $L^1(\mathbb{G}) \times L^\infty(\mathbb{G})$ to $L^1(\mathbb{G})$.

**Proof.** We keep the notations in Section 5. Note that (5.3), (5.4), (5.6) and (5.7) hold for any $\alpha > 0$. So, the proof of Theorem 5 is valid except from the estimate of $T_j^1(f_1, g_1)$. According to (5.9), we know that for any $0 < \delta < \alpha$,

$$\|T_j^\alpha(f, g)\|_1 \leq C2^{-j(\alpha-\delta)} \|f\|_1 \|g\|_2.$$ 

Using the Hölder’s inequality, we have

$$\|T_j^1(f_1, g_1)\|_{L^1(B_j(\xi, \frac{1}{4}))} = \|T_j^\alpha(f_1, g_1)\|_{L^1(B_j(\xi, \frac{1}{4}))} \leq C2^{-j(\alpha-\delta)} \|f\|_{L^1(B_j(\xi, \frac{1}{4}))} \|g\|_{L^2(B_j(\xi, \frac{1}{4}))} \leq C2^{-j(\alpha-\delta)}2^{j(\gamma+\frac{2}{Q})} \|f\|_{L^1(B_j(\xi, \frac{1}{4}))} \|g\|_{L^\infty(B_j(\xi, \frac{1}{4}))}.$$ 

Taking the $L^p$ norm with respect to $\xi$ yields that

$$\|T_j^1(f_1, g_1)\|_1 \leq C2^{-j(\alpha-\delta)}2^{j(\gamma+\frac{2}{Q})} \|f\|_1 \|g\|_\infty.$$ 

Thus, whenever $\alpha > \frac{Q}{2}$, we can choose $\gamma, \delta > 0$ such that $\alpha > \frac{(1+\gamma)}{2}Q + \delta$, which means that there exists $\varepsilon > 0$ such that

$$\|T_j^1(f_1, g_1)\|_1 \leq C2^{-\varepsilon j} \|f\|_1 \|g\|_\infty.$$ 

$\square$

6.2. The point $(\infty, \infty, \infty)$.

**Theorem 7.** If $\alpha > Q - \frac{1}{2}$, then $S^\alpha$ is bounded from $L^\infty(\mathbb{G}) \times L^\infty(\mathbb{G})$ into $L^\infty(\mathbb{G})$.

**Proof.** We still keep the notation in last Section. To proof this Theorem, it suffices to estimate $T_j^\alpha(f_1, g_1)$. Notice that

$$P_\lambda f(x, u) = \sum_{k=0}^\infty \frac{\lambda^{n+m-1}}{(2\pi(2k+n))^{n+m}} \int_{\mathbb{R}^{m-1}} f \ast e_k^{\lambda\eta}(x, u)d\sigma(\eta),$$

and

$$f \ast e_k^{\lambda\eta}(x, u) = e^{-i\lambda\eta(u)} \left( (f^{\lambda\eta} \times_\lambda \varphi_k) \circ A_\eta \right)(x).$$

Then, $T_j^\alpha(f, g)$ can be written as

$$T_j^\alpha(f, g)(x, u) = \frac{1}{(2\pi)^q} \sum_{k=0}^\infty \sum_{l=0}^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_j^\alpha((2k+n)\lambda_1, (2l+n)\lambda_2) \int_{\mathbb{R}^{m-1}} f \ast e_k^{\lambda\eta_1}(x, u)d\sigma(\eta_1) \times \int_{\mathbb{R}^{m-1}} f \ast e_l^{\lambda\eta_2}(x, u)d\sigma(\eta_2) \lambda_1^{n+m-1} \lambda_2^{n+m-1} d\lambda_1 d\lambda_2$$

$$= \frac{1}{(2\pi)^q} \sum_{k=0}^\infty \sum_{l=0}^\infty \int_0^\infty \int_0^\infty \varphi_j^\alpha((2k+n)\lambda_1, (2l+n)\lambda_2) \int_{\mathbb{R}^{m-1}} e^{-i\lambda_1\eta_1(u)} \left( (f^{\lambda\eta_1} \times_\lambda \varphi_k) \circ A_{\eta_1} \right)(x)d\sigma(\eta_1) \right)$$
Using (5.1), we get that for any $0 < \delta < m$,

\[
\|T_\delta^n (f, g)\|_\infty \leq C \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\varphi_j^n((2k + n) | \mu_1 |, (2l + n) | \mu_2)|
\]

\[
\times \left\| \left( \int_{\mathbb{R}^m} \varphi_k^{n} (x) d\mu_1(x) \right) \left( \int_{\mathbb{R}^m} \varphi_l^{n} (y) d\mu_2(y) \right) \left( \int_{\mathbb{R}^m} f_{\mu_1}^{n-1} (x) d\mu_1(x) \right) \left( \int_{\mathbb{R}^m} f_{\mu_2}^{n-1} (y) d\mu_2(y) \right) \right\|_\infty \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| f_{\mu_1}^{n-1} \|_{2} \| f_{\mu_2}^{n-1} \|_{2} \| d\mu_1 d\mu_2
\]

\[
\leq C \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left\| \varphi_j^n((2k + n) | \mu_1 |, (2l + n) | \mu_2)| \right\|_\infty \\left\| \left( \int_{\mathbb{R}^m} \varphi_k^{n} (x) d\mu_1(x) \right) \left( \int_{\mathbb{R}^m} \varphi_l^{n} (y) d\mu_2(y) \right) \left( \int_{\mathbb{R}^m} f_{\mu_1}^{n-1} (x) d\mu_1(x) \right) \left( \int_{\mathbb{R}^m} f_{\mu_2}^{n-1} (y) d\mu_2(y) \right) \right\|_\infty \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| f_{\mu_1}^{n-1} \|_{2} \| f_{\mu_2}^{n-1} \|_{2} \| d\mu_1 d\mu_2
\]

\[
\leq C \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left\| \varphi_j^n((2k + n) | \mu_1 |, (2l + n) | \mu_2)| \right\|_\infty \\left\| \left( \int_{\mathbb{R}^m} \varphi_k^{n} (x) d\mu_1(x) \right) \left( \int_{\mathbb{R}^m} \varphi_l^{n} (y) d\mu_2(y) \right) \left( \int_{\mathbb{R}^m} f_{\mu_1}^{n-1} (x) d\mu_1(x) \right) \left( \int_{\mathbb{R}^m} f_{\mu_2}^{n-1} (y) d\mu_2(y) \right) \right\|_\infty \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| f_{\mu_1}^{n-1} \|_{2} \| f_{\mu_2}^{n-1} \|_{2} \| d\mu_1 d\mu_2
\]

\[
\leq C \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left( \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left\| \varphi_j^n((2k + n) | \mu_1 |, (2l + n) | \mu_2)| \right\|_\infty \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| f_{\mu_1}^{n-1} \|_{2} \| f_{\mu_2}^{n-1} \|_{2} \right) \right)^{\frac{1}{2}} \| d\mu_1 d\mu_2 \right)^{\frac{1}{2}}
\]

\[
\leq C \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} k^{n+\frac{3}{2}} \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left| \varphi_j^n((2k + n) | \mu_1 |, (2l + n) | \mu_2)| \right|^2 \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| d\mu_1 d\mu_2 \right)^{\frac{1}{2}}
\]

\[
\leq C \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} k^{n+\frac{3}{2}} \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left| \varphi_j^n((2k + n) | \mu_1 |, (2l + n) | \mu_2)| \right|^2 \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| d\mu_1 d\mu_2 \right)^{\frac{1}{2}}
\]

\[
\leq C \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} k^{n+\frac{3}{2}} \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left| \varphi_j^n((2k + n) | \mu_1 |, (2l + n) | \mu_2)| \right|^2 \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| d\mu_1 d\mu_2 \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_{\|\mu_1\| \leq 1} \int_{\|\mu_2\| \leq 1} \left\| f_{\mu_1}^{n-1} \right\|_{2}^2 \left\| g_{\mu_2}^{n-1} \right\|_{2}^2 \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| d\mu_1 d\mu_2 \right)^{\frac{1}{2}}
\]

\[
\leq C \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} k^{n+\frac{3}{2}} \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left| \varphi_j^n((2k + n) | \mu_1 |, (2l + n) | \mu_2)| \right|^2 \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| d\mu_1 d\mu_2 \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_{\|\mu_1\| \leq 1} \int_{\|\mu_2\| \leq 1} \left\| f_{\mu_1}^{n-1} \right\|_{2}^2 \left\| g_{\mu_2}^{n-1} \right\|_{2}^2 \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| d\mu_1 d\mu_2 \right)^{\frac{1}{2}}
\]

\[
\leq C \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} k^{n+\frac{3}{2}} \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left| \varphi_j^n((2k + n) | \mu_1 |, (2l + n) | \mu_2)| \right|^2 \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| d\mu_1 d\mu_2 \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_{\|\mu_1\| \leq 1} \int_{\|\mu_2\| \leq 1} \left\| f_{\mu_1}^{n-1} \right\|_{2}^2 \left\| g_{\mu_2}^{n-1} \right\|_{2}^2 \| \mu_1^{n+\frac{3}{2}} \|_{2} \| \mu_2^{n+\frac{3}{2}} \|_{2} \| d\mu_1 d\mu_2 \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \sum_{l=0}^{\infty} k^{-\frac{m+l+1}{2}} \right) \left( \sum_{k=0}^{\infty} l^{-\frac{m+k+1}{2}} \right) \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |\varphi_j^\alpha (|\mu_1|, |\mu_2|)|^2 |\mu_1|^{n+\delta} |\mu_2|^{n+\delta} \, d\mu_1 d\mu_2 \right)^{\frac{1}{2}} \\
\times \left( \int_{|\mu_1| \leq 1} \left\| f_{\mu_1}^{\mu_1} \left| \mu_1 \right|^{-\delta} \, d\mu_1 \right\|^2 \left\| f_{\mu_2}^{\mu_2} \left| \mu_2 \right|^{-\delta} \, d\mu_2 \right\|^2 \right)^{\frac{1}{2}} \\
\leq C 2^{-j(\alpha + \frac{1}{2})} \left( \int_{|\mu_1| \leq 1} \left\| f_{\mu_1}^{\mu_1} \left| \mu_1 \right|^{-\delta} \, d\mu_1 \right\|^2 \left\| f_{\mu_2}^{\mu_2} \left| \mu_2 \right|^{-\delta} \, d\mu_2 \right\|^2 \right)^{\frac{1}{2}}.
\]

Because

\[
T_j^1 (f_1, g_1)(\omega) = T_j^\alpha (f_1, g_1)(\omega) \quad \text{for any } \omega \in B_j (\xi, \frac{1}{4}),
\]

we have

\[
(6.1) \quad \left\| T_j^1 (f_1, g_1) \right\|_{L^\infty (B_j (\xi, \frac{1}{4}))} = \left\| T_j^\alpha (f_1, g_1) \right\|_{L^\infty (B_j (\xi, \frac{1}{4}))} \leq C 2^{-j(\alpha + \frac{1}{2})} \left( \int_{|\mu_1| \leq 1} \left\| f_{\mu_1}^{\mu_1} \left| \mu_1 \right|^{-\delta} \, d\mu_1 \right\|^2 \left\| f_{\mu_2}^{\mu_2} \left| \mu_2 \right|^{-\delta} \, d\mu_2 \right\|^2 \right)^{\frac{1}{2}}.
\]

Considering the integral about \( \mu_1 \), we notice that

\[
\left\| f_{\mu_1}^{\mu_1} \right\|^2 \leq \det A_{\frac{\mu_1}{|\mu_1|}} \left\| f_{\mu_1}^{\mu_1} \right\|_{L^\infty (B_j (\xi, \frac{1}{4}))} \leq C 2^{j(\gamma + \mu)} \left\| f_{\mu_1}^{\mu_1} \right\|_{L^\infty (B_j (\xi, \frac{1}{4}))}.
\]

So, we can get that

\[
\int_{|\mu_1| \leq 1} \left\| f_{\mu_1}^{\mu_1} \left| \mu_1 \right|^{-\delta} \, d\mu_1 \right\|^2 \leq C 2^{j(\gamma + \mu)} \left\| f_{\mu_1}^{\mu_1} \right\|_{L^\infty (B_j (\xi, \frac{1}{4}))}.
\]

From (6.1) and the above estimates, we have

\[
\left\| T_j^1 (f_1, g_1) \right\|_{L^\infty (B_j (\xi, \frac{1}{4}))} \leq C 2^{-j(\alpha + \frac{1}{2})} \left\| f_{\mu_1}^{\mu_1} \right\|_{L^\infty (B_j (\xi, \frac{1}{4}))} \left\| g_{\mu_2}^{\mu_2} \right\|_{L^\infty (B_j (\xi, \frac{1}{4}))}.
\]

It follows that

\[
\left\| T_j^1 (f_1, g_1) \right\|_{L^\infty} \leq C 2^{-j(\alpha + \frac{1}{2})} \left\| f_{\mu_1}^{\mu_1} \right\|_{L^\infty} \left\| g_{\mu_2}^{\mu_2} \right\|_{L^\infty}.
\]
Therefore, whenever $\alpha > Q - \frac{1}{2}$, we can choose $\gamma, \delta > 0$ such that $\alpha > (1 + \gamma)(Q + 2\delta) - \frac{1}{2}$, which implies there exists an $\varepsilon > 0$ such that

$$
\|T_j^\alpha(f_1, g_1)\|_{L^\infty} \leq C2^{-\varepsilon j} \|f\|_{L^\infty} \|g\|_{L^\infty}.
$$

The proof of Theorem \[\ast\] is completed.

6.3. The point \((2, \infty, 2)\).

**Theorem 8.** If $\alpha > \frac{Q-1}{2}$, then $S^\alpha$ is bounded from $L^2(\mathbb{G}) \times L^\infty(\mathbb{G})$ to $L^2(\mathbb{G})$.

**Proof.** As above, it suffices to estimate $T_j^\alpha(f_1, g_1)$. We write $T_j^\alpha(f, g)$ as

$$
T_j^\alpha(f, g)(x, u) = \frac{1}{(2\pi)^Q} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{-i(u\cdot\mu_2)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \varphi_j^\alpha((2k + n) |\mu_1|, (2l + n) |\mu_2|) \times \left( f_{\mu_1, \mu_2} \varphi_{\mu_1, \mu_2} \right) \left( g_{\mu_1, \mu_2} \varphi_{\mu_1, \mu_2} \right) \left( A_{\mu_1, \mu_2} x \right) |\mu_1^n| |\mu_2^n| d\mu_1 d\mu_2
$$

Then, applying the Plancherel theorem in variable $u$, the Minkowski’s inequality, the orthogonality of the special Hermite projections and \[\ast\], we get that

$$
\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} |T_j^\alpha(f, g)(x, u)|^2 \, dx du = \frac{1}{(2\pi)^Q} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{-i(u\cdot\mu_2)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \varphi_j^\alpha((2k + n) |\mu_1|, (2l + n) |\mu_2|) \times \left( f_{\mu_1, \mu_2} \varphi_{\mu_1, \mu_2} \right) \left( g_{\mu_1, \mu_2} \varphi_{\mu_1, \mu_2} \right) \left( A_{\mu_1, \mu_2} x \right) |\mu_1^n| |\mu_2^n| d\mu_2 d\mu_1
$$

$$
\leq \frac{C}{(2\pi)^Q} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \varphi_j^\alpha((2k + n) |\mu_1 - \mu_2|, (2l + n) |\mu_2|) \times \left( f_{\mu_1, \mu_2} \varphi_{\mu_1, \mu_2} \right) \left( g_{\mu_1, \mu_2} \varphi_{\mu_1, \mu_2} \right) \left( A_{\mu_1, \mu_2} x \right) |\mu_1 - \mu_2| |\mu_2^n| d\mu_2 d\mu_1
$$

$$
\leq C \left( \frac{Q}{2\pi} \right)^2 \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \varphi_j^\alpha((2k + n) |\mu_1 - \mu_2|, (2l + n) |\mu_2|) \times \left( f_{\mu_1, \mu_2} \varphi_{\mu_1, \mu_2} \right) \left( g_{\mu_1, \mu_2} \varphi_{\mu_1, \mu_2} \right) \left( A_{\mu_1, \mu_2} x \right) |\mu_1 - \mu_2| |\mu_2^n| d\mu_2 d\mu_1.
$$
Applying (2.4), the Plancherel theorem in variable \( u \) and the orthogonality of the special Hermite projections, we obtain that

\[
\|f\|_2^2 = \frac{1}{(2\pi)^Q} \int_{R^m} \int_{R^m} e^{-i\mu_1 (u)} \sum_{k=0}^{\infty} \left( \left( f^{|\mu_1|}_{|\mu_1|} \times |\mu_1| \right) \circ A_{|\mu_1|} \right)(x) |\mu_1|^n d\mu_1 \int_{R^m} e^{-i\mu_1 (u)} \sum_{k=0}^{\infty} \left( \left( f^{|\mu_1|}_{|\mu_1|} \times |\mu_1| \right) \circ A_{|\mu_1|} \right)(x) |\mu_1|^n d\mu_1 dx
\]
Thus, whenever

\[ \rho > 0 \text{ such that \ } \alpha > \rho \]

At the same time, (6.2) tells that

Taking the \( \| \cdot \|_{L^1(\mathbb{R}_+)} \)

Thus, we can obtain the intermediate boundedness of \( S \) norm with respect to \( \xi \), we get that

\[
\int_{|\mu_2|\leq 1} \left\| g_1(x) \mu_2 \right\|_2^2 |\mu_2|^{-\delta} d\mu_2 \leq C2^{j(1+\gamma)(Q+2\delta)} \|g\|_{L^\infty(B_j(\xi,\frac{\rho}{4}))}^2.
\]

Thus, the proof of Theorem 8 is completed.

\[ \square \]

7. BILINEAR INTERPOLATION

Because \( p_1 \) and \( p_2 \) are symmetric, we have obtained, in two sections above, the boundedness of the bilinear Riesz means \( S^\alpha \) at some specific triples of points \((p_1, p_2, p)\) like

\[
(1, 1, \frac{2}{3}), (2, 2, 1), (\infty, \infty, \infty), (1, 2, \frac{2}{3}), (2, 1, \frac{2}{3}), (1, \infty, 1), (\infty, 1, 1), (2, \infty, 2), (\infty, 2, 2).
\]

We can obtain the intermediate boundedness of \( S^\alpha \) by using of the bilinear interpolation via complex method adapted to the setting of analytic families or real method in [3]. Bernicot et al. [1] described how to make use of real method. We outline this argument for reader’s convenience.

Consider a spherical decomposition of \( S^\alpha \) as

\[
S^\alpha = \sum_{j=0}^\infty 2^{-j\alpha}T_{j,\alpha},
\]

where

\[
T_{j,\alpha}(f, g) = \int_0^\infty \int_0^\infty \varphi_{j,\alpha}(\lambda_1, \lambda_2) P_{\lambda_1}f P_{\lambda_2}g d\mu(\lambda_1)d\mu(\lambda_2)
\]

and

\[
 \varphi_{j,\alpha}(s, t) = 2^{j\alpha} (1 - s - t)^{\alpha} (1 - s - t). \]

In the preceding sections, we have actually obtained the estimates of the form

\[
(7.1) \quad \|T_{j,\alpha}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq C2^{j\alpha(p_1, p_2)}
\]
at some triples of points \((p_1, p_2, p)\). Since \(\alpha(p_1, p_2)\) only depends on the point \((p_1, p_2, p)\), (7.1) also holds for any other \(T_{j, \alpha'}\), i.e.,

\[
\|T_{j, \alpha'}\|_{L^p_1 \times L^p_2 \to L^p} \leq C2^{j\alpha(p_1, p_2)}.
\]

So, fixing \(j\) and \(\alpha'\) and applying bilinear real interpolation theorem on \(T_{j, \alpha'}\), we can conclude that if the point \((p_1, p_2, p)\) satisfies

\[
\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p}\right) = (1 - \theta) \left(\frac{1}{p_1^0}, \frac{1}{p_2^0}, \frac{1}{p^0}\right) + \theta \left(\frac{1}{p_1^1}, \frac{1}{p_2^1}, \frac{1}{p^1}\right)
\]

for some \(\theta \in (0, 1)\) and \((p_1^0, p_2^0, p^0), (p_1^1, p_2^1, p^1)\), we have that

\[
\|T_{j, \alpha'}\|_{L^p_1 \times L^p_2 \to L^p} \leq C2^{j(\alpha(p_1, p_2) + \theta \alpha(p_1^1, p_2^1))}.
\]

Define \(\alpha(p_1, p_2) = (1 - \theta)\alpha(p_1^0, p_2^0) + \theta \alpha(p_1^1, p_2^1)\) and let \(\alpha' = \alpha\). It follows that

\[
\|S^\alpha\|_{L^p_1 \times L^p_2 \to L^p} \leq \sum_{j=0}^{\infty} 2^{-j\alpha} \|T_{j, \alpha}\|_{L^p_1 \times L^p_2 \to L^p} \leq C \sum_{j=0}^{\infty} 2^{-j\alpha} 2^{j\alpha(p_1, p_2)}.
\]

Thus, when \(\alpha > \alpha(p_1, p_2)\), we have \(\|S^\alpha\|_{L^p_1 \times L^p_2 \to L^p} \leq C\), i.e., \(S^\alpha\) is bounded from \(L^p_1(G) \times L^p_1(G)\) to \(L^p(G)\).

Based on the above argument, we can get the full results on the \(L^p_1 \times L^p_2 \to L^p\) boundedness of the operator \(S^\alpha\).

**Theorem 9.** Let \(1 \leq p_1, p_2 \leq \infty\) and \(1/p = 1/p_1 + 1/p_2\).

1. (region I) For \(2 \leq p_1, p_2 \leq \infty\) and \(p \geq 2\), if \(\alpha > Q \left(1 - \frac{1}{p}\right) - \frac{1}{2}\), then \(S^\alpha\) is bounded from \(L^p_1(G) \times L^p_1(G)\) to \(L^p(G)\).

2. (region II) For \(2 \leq p_1, p_2 \leq \infty\) and \(1 \leq p \leq 2\), if \(\alpha > (Q - 1) \left(1 - \frac{1}{p}\right)\), then \(S^\alpha\) is bounded from \(L^p_1(G) \times L^p_1(G)\) to \(L^p(G)\).

3. (region III) For \(1 \leq p_1 \leq 2 \leq p_2 \leq \infty\) and \(p \geq 1\), if \(\alpha > Q \left(\frac{1}{2} - \frac{1}{p_2}\right) - \left(1 - \frac{1}{p}\right)\), then \(S^\alpha\) is bounded from \(L^p_1(G) \times L^p_1(G)\) to \(L^p(G)\); For \(1 \leq p_2 \leq 2 \leq p_1 \leq \infty\) and \(p \geq 1\), if \(\alpha > Q \left(\frac{1}{2} - \frac{1}{p_1}\right) - \left(1 - \frac{1}{p}\right)\), then \(S^\alpha\) is bounded from \(L^p_1(G) \times L^p_1(G)\) to \(L^p(G)\).
(4) (region IV) For $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$ and $p \leq 1$, if $\alpha > Q \left( \frac{1}{p_1} - \frac{1}{2} \right)$, then $S^\alpha$ is bounded from $L^{p_1}(G) \times L^{p_1}(G)$ to $L^p(G)$; For $1 \leq p_2 \leq 2 \leq p_1 \leq \infty$ and $p \leq 1$, if $\alpha > Q \left( \frac{1}{p_2} - \frac{1}{2} \right)$, then $S^\alpha$ is bounded from $L^{p_1}(G) \times L^{p_1}(G)$ to $L^p(G)$.

(5) (region V) For $1 \leq p_1, p_2 \leq 2$, if $\alpha > Q \left( \frac{1}{p} - 1 \right)$, then $S^\alpha$ is bounded from $L^{p_1}(G) \times L^{p_1}(G)$ to $L^p(G)$.

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