An invariance group for a linear combination of two Saalschützian $4\text{F}_3(1)$ hypergeometric series

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Abstract

We explore a function $L(\vec{x}) = L(a, b, c, d; e; f, g)$ which is a linear combination of two Saalschützian $4\text{F}_3(1)$ hypergeometric series. We demonstrate a fundamental two-term relation satisfied by the $L$ function and show that the fundamental two-term relation implies that the Coxeter group $W(D_5)$, which has 1920 elements, is an invariance group for $L(\vec{x})$. The invariance relations for $L(\vec{x})$ are classified into six types based on a double coset decomposition of the invariance group. The fundamental two-term relation is shown to generalize classical results about hypergeometric series. We derive Thomae’s identity for $3\text{F}_2(1)$ series, Bailey’s identity for terminating Saalschützian $4\text{F}_3(1)$ series, and Barnes’ second lemma as consequences of the fundamental two-term relation.

1 Introduction

Invariance groups for hypergeometric series have been studied extensively in the past. A hypergeometric series is trivially invariant under permutations of its numerator and denominator parameters thus giving us an invariance group isomorphic to the cross product of two symmetric groups. The existence of nontrivial two-term relations and their combined use with the trivial relations leads to larger invariance groups that have been the subject of study over the last twenty-five years by Beyer et al. [5], Srinivasa Rao et al. [12], and others.

The series of type $3\text{F}_2(1)$ have been studied since the nineteenth century. In 1879 Thomae [17] obtained a number of two-term relations for $3\text{F}_2(1)$ series. One of those relations is known today as Thomae’s identity (see [2, p. 14]). Thomae’s identity was later rediscovered (with an explicit proof provided) by Ramanujan (see [8, p. 104]). In 1923 Whipple [20] re-visited Thomae’s work and introduced a more convenient notation, in terms of his Whipple parameters, that indexed the two-term relations found by Thomae. In a recent paper Krattenthaler and Rivoal [10] described other families of two-term relations for $3\text{F}_2(1)$ series that are not consequences of the identities found by Thomae.

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A two-term relation for terminating Saalschützian \( _4F_3(1) \) series, based on identities relating very-well-poised \( _7F_6(1) \) series to terminating Saalschützian \( _4F_3(1) \) series, was given by Whipple [21, Eq. (10.11)] in 1925. The same two-term relation appeared later in Bailey’s monograph [2, p. 56] and is often referred to today as Bailey’s identity.

The first mention of an invariance group for hypergeometric series seems to be due to Hardy. In [8, p. 111] it is implied that the symmetric group \( S_5 \) is an invariance group for the \( _3F_2(1) \) series. In 1987 Beyer et al. [5] rediscovered that Thomae’s identity combined with the trivial invariances under permutations of the numerator and denominator parameters implies that \( S_5 \) is an invariance group for the \( _3F_2(1) \) series. Beyer et al. also showed in the same paper [5] that Bailey’s identity combined with the trivial invariances implies that the symmetric group \( S_6 \) is an invariance group for the terminating Saalschützian \( _4F_3(1) \) series.

The goal of this paper is to extend the results stated above to Saalschützian \( _4F_3(1) \) series. We examine a function \( L(a, b, c, d; e; f, g) \) (see (2.2) for the definition) which is a linear combination of two Saalschützian \( _4F_3(1) \) series. This particular linear combination of two Saalschützian \( _4F_3(1) \) series appears in [14] in the evaluation of the Mellin transform of a spherical principal series \( GL(4, \mathbb{R}) \) Whittaker function.

In Section 3 we derive a fundamental two-term relation (see (3.4)) satisfied by \( L(a, b, c, d; e; f, g) \). The fundamental two-term relation (3.4) is derived through a Barnes integral representation of \( L(a, b, c, d; e; f, g) \) and generalizes both Thomae’s and Bailey’s identities in the sense that the latter two identities can be obtained as limiting cases of our fundamental two-term relation (see Section 5).

In Section 4 we show that the two-term relation (3.4) combined with the trivial invariances of \( L(a, b, c, d; e; f, g) \) under permutations of \( a, b, c, d \) and interchanging \( f, g \) implies that the function \( L(a, b, c, d; e; f, g) \) has an invariance group \( G_L \) isomorphic to the Coxeter group \( W(D_5) \), which is of order 1920. (See [9] for general information on Coxeter groups.) The invariance group \( G_L \) is given as a matrix group of transformations of the affine hyperplane

\[
V = \{(a, b, c, d, e, f, g)^T \in \mathbb{C}^7 : e + f + g - a - b - c - d = 1\}\tag{1.1}
\]

The 1920 invariances of the \( L \) function that follow from the invariance group \( G_L \) are classified into six types based on a double coset decomposition of \( G_L \) with respect to its subgroup \( \Sigma \) consisting of all the permutation matrices in \( G_L \).

To the best of the author’s knowledge, using such a double coset decomposition is a new way of describing all the relations induced by an invariance group and does not have an analog in the literature before.

Some consequences of the fundamental two-term relation (3.4) are shown in Section 5. In particular, as already mentioned, we show that Thomae’s and Bailey’s identities follow as limiting cases of (3.4). We also show that Barnes’ second lemma (see [4] or [2, p. 42]) follows as a special case of (3.4) when we take \( d = g \).
Versions of the $L$ function (in terms of very-well-poised $\gamma F_b(1)$ series, see (2.3)) were examined in the past by Bailey [1], Whipple [22], and Raynal [13]. Bailey obtained two-term relations that were later re-visited by Whipple and Raynal. However, there is no mention of an underlying invariance group.

A basic hypergeometric series analog of the $L$ function (in terms of $\phi_7$ series) was studied by Van der Jeugt and Srinivasa Rao [19]. The authors establish an invariance group isomorphic to $W(D_5)$ for the $\phi_7$ series, but do not classify all two-term relations, or consider how they could imply results about lower-order series.

Very recently Formichella et al. [7] explored a function $K(a; b, c, d; e, f, g)$ which is a different linear combination of two Saalschützian $\phi_3(1)$ series from the function $L(a, b, c, d; e, f, g)$. The linear combination of two Saalschützian $\phi_3(1)$ series studied by Formichella et al. appears in the theory of archimedean zeta integrals for automorphic $L$ functions (see [15, 16]). The function $K(a; b, c, d; e, f, g)$ behaves very differently from $L(a, b, c, d; e, f, g)$. Formichella et al. obtain in [7] a two-term relation satisfied by $K(a; b, c, d; e, f, g)$ and show that their two-term relation implies that the symmetric group $S_6$ is an invariance group for $K(a; b, c, d; e, f, g)$. In a future work by the author of the present paper and by Green and Stade, the connection between the $K$ and the $L$ functions will be studied.

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2 Hypergeometric series and Barnes integrals

The hypergeometric series of type $\phi_{p+1} F_p$ is the power series in the complex variable $z$ defined by

$$
\phi_{p+1} F_p \left[ \begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{p+1})_n}{n!(b_1)_n (b_2)_n \cdots (b_p)_n} z^n,
$$

(2.1)

where $p$ is a positive integer, the numerator parameters $a_1, a_2, \ldots, a_{p+1}$ and the denominator parameters $b_1, b_2, \ldots, b_p$ are complex numbers, and the rising factorial $(a)_n$ is given by

$$(a)_n = \begin{cases} a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} & n > 0, \\ 1 & n = 0. \end{cases}
$$

The series in (2.1) converges absolutely if $|z| < 1$. When $|z| = 1$, the series converges absolutely if Re($\sum_{i=1}^{p} b_i - \sum_{i=1}^{p+1} a_i$) $> 0$ (see [2] p. 8). We assume that no denominator parameter is a negative integer or zero. If a numerator parameter is a negative integer or zero, the series has only finitely many nonzero terms and is said to terminate.
When \( z = 1 \), the series is said to be of unit argument and of type \( {}_{p+1}F_p(1) \). If \( \sum_{i=1}^{p} b_i = \sum_{i=1}^{p+1} a_i + 1 \), the series is called Saalschützian. If \( 1 + a_1 = b_1 + a_2 = \ldots = b_p + a_{p+1} \), the series is called well-poised. A well-poised series that satisfies \( a_2 = 1 + \frac{1}{2} a_1 \) is called very-well-poised.

Our main object of study in this paper will be the function \( L(a, b, c, d; e; f, g) \) defined by

\[
L(a, b, c, d; e; f, g) = \frac{4F_3[a, b, c, d; 1]}{\sin \pi e \Gamma(e) \Gamma(f) \Gamma(g) \Gamma(1 + a - e) \Gamma(1 + b - e) \Gamma(1 + c - e) \Gamma(1 + d - e) - 4F_3[1 + a - e, 1 + b - e, 1 + c - e, 1 + d - e; 1]}
\]

\[
= \frac{\Gamma(1 + d + g - e) \pi \Gamma(g) \Gamma(1 + g - e) \Gamma(f - d) \Gamma(1 + a + d - e) \Gamma(1 + b + d - e) \Gamma(1 + c + d - e)}{\Gamma(1 + b + d - e) \Gamma(1 + g - e) \Gamma(1 + c + d - e) - \frac{1}{2} \Gamma(d + g - e), g - a, g - b, g - c, d, 1 + d - e;}
\]

\[
\cdot \tau F_6 \left[ \frac{1}{2} (d + g - e), 1 + a + d - e, 1 + b + d - e, 1 + c + d - e, 1 + g - e, g; 1 \right],
\]

provided that \( \text{Re}(f - d) > 0 \). Therefore our results on the \( L \) function can also be interpreted in terms of the very-well-poised \( \tau F_6(1) \) series given in \((2.3)\).

Fundamental to the derivation of a nontrivial two-term relation for the \( L \) function will be the notion of a Barnes integral, which is a contour integral of the form

\[
\int_{C} \prod_{i=1}^{n} \Gamma^{\varepsilon_i}(a_i + t) \prod_{j=1}^{m} \Gamma^{\varepsilon_j}(b_j - t) \, dt,
\]

where \( n, m \in \mathbb{Z}^+: \varepsilon_i, \varepsilon_j = \pm 1 \); and \( a_i, b_j, t \in \mathbb{C} \). The path of integration is the imaginary axis, indented if necessary, so that any poles of \( \prod_{i=1}^{n} \Gamma^{\varepsilon_i}(a_i + t) \) are to the left of the contour and any poles of \( \prod_{j=1}^{m} \Gamma^{\varepsilon_j}(b_j - t) \) are to the right of the contour. This path of integration always exists, provided that, for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), we have \( a_i + b_j \notin \mathbb{Z} \) whenever \( \varepsilon_i = \varepsilon_j = 1 \).

From now on, when we write an integral of the form \((2.4)\), we will always mean a Barnes integral with a path of integration as just described.
A Barnes integral can often be evaluated in terms of hypergeometric series using the Residue Theorem, provided that we can establish the necessary convergence arguments. This is the approach we take in the next section. We will make use of the extension of Stirling’s formula to the complex numbers (see [18, Section 4.42] or [23, Section 13.6]):

\[ \Gamma(a + z) = \sqrt{2\pi} e^{-\frac{z^2}{4}} \frac{z^{a-\frac{1}{2}}}{(1 + O(1/|z|))} \text{ uniformly as } |z| \to \infty, \quad (2.5) \]

provided that \(-\pi + \delta \leq \text{arg}(z) \leq \pi - \delta, \, \delta \in (0, \pi)\).

When applying the Residue Theorem, we will use the fact that the gamma function has simple poles at \(t = -n, n = 0, 1, 2, \ldots\), with

\[ \text{Res}_{t=-n} \Gamma(t) = \frac{(-1)^n}{n!}. \quad (2.6) \]

When simplifying expressions involving gamma functions, the reflection formula for the gamma function will often be used:

\[ \Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin \pi t}. \quad (2.7) \]

Finally, we will use a result about Barnes integrals known as Barnes’ lemma (see [3] or [2, p. 6]):

**Lemma 2.1 (Barnes’ lemma).** If \(a, \beta, \gamma, \delta \in \mathbb{C}\), we have

\[
\frac{1}{2\pi i} \int_t \frac{\Gamma(a + t)\Gamma(\beta + t)\Gamma(\gamma - t)\Gamma(\delta - t)}{\Gamma(\alpha + \beta + \gamma + \delta)} \, dt = \frac{\Gamma(a + \gamma)\Gamma(\alpha + \delta)\Gamma(\beta + \gamma)\Gamma(\beta + \delta)}{\Gamma(\alpha + \beta + \gamma + \delta)}, \quad (2.8)
\]

provided that none of \(a + \gamma, a + \delta, \beta + \gamma\) and \(\beta + \delta\) is an integer.

### 3 Fundamental two-term relation

In this section we show that the function \(L(a, b, c, d; e; f, g)\) defined in [22] can be represented as a Barnes integral. The Barnes integral representation will then be used to derive a fundamental two-term relation satisfied by the \(L\) function.

**Proposition 3.1.**

\[
L(a, b, c, d; e; f, g) = \frac{1}{\pi \Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1 + a - e)\Gamma(1 + b - e)\Gamma(1 + c - e)\Gamma(1 + d - e)} \cdot \frac{1}{2\pi i} \int_t \frac{\Gamma(a + t)\Gamma(\beta + t)\Gamma(\gamma + t)\Gamma(\delta + t)\Gamma(1 + e - t)\Gamma(1 + d - e)}{\Gamma(\beta + t)\Gamma(\gamma + t)} \, dt. \quad (3.1)
\]

In the proof of Proposition 3.1, we will need the following statement.
Lemma 3.2. For every \( \varepsilon > 0 \), there is a constant \( K = K(\varepsilon) \), such that if \( \text{dist}(z, Z) \geq \varepsilon \), then
\[
|\sin \pi z| \geq Ke^{\pi|\text{Im}(z)|}.
\]
(3.2)

Proof. Let \( z = x + iy \). We have
\[
\sin \pi z = \frac{1}{2i} \left( e^{i\pi(x+iy)} - e^{-i\pi(x+iy)} \right) = \sin \pi x \cosh \pi y + i \cos \pi x \sinh \pi y.
\]

Since \( |\sinh \pi y| \leq \cosh \pi y \), it follows that \( |\sin \pi y| \leq |\pi z| \leq \cosh \pi y \).

We may assume that \( \varepsilon \in (0, 1) \). If \( \text{dist}(z, Z) \geq \varepsilon \), then at least one of the following two statements holds:
(a) \( \text{dist}(x, Z) \geq \varepsilon/2 \).
(b) \( |y| \geq \varepsilon/2 \).

If (a) holds, then
\[
|\sin \pi z| \geq |\sin \pi x| \cosh \pi y \geq \sin(\pi\varepsilon/2) \cosh \pi y \geq \frac{1}{2} \sin(\pi\varepsilon/2)e^{\pi|y|}.
\]

If (b) holds, then
\[
|\sin \pi z| \geq \sinh \pi |y| = \frac{1}{2} e^{\pi|y|}(1 - e^{-2\pi|y|}) \geq \frac{1}{2}(1 - e^{-\pi\varepsilon})e^{\pi|y|}.
\]

Thus (3.2) holds with \( K = \frac{1}{2} \min\{\sin(\pi\varepsilon/2), 1 - e^{-\pi\varepsilon}\} \).

Proof of Proposition 3.1. Let
\[
I \left[ a, b, c, d; e; f, g \right] = \frac{1}{2\pi i} \int \frac{\Gamma(a + t)\Gamma(b + t)\Gamma(c + t)\Gamma(d + t)\Gamma(1 - e - t)\Gamma(-t)}{\Gamma(f + t)\Gamma(g + t)} dt.
\]
(3.3)

For \( N \geq 1 \), let \( C_N \) be the semicircle of radius \( \rho_N \) on the right side of the imaginary axis and center at the origin, chosen in such a way that \( \rho_N \to \infty \) as \( N \to \infty \) and
\[
\varepsilon := \inf_N \text{dist}(C_N, Z \cup (Z - e)) > 0.
\]

The formula (2.7) gives
\[
G(t) := \frac{\Gamma(a + t)\Gamma(b + t)\Gamma(c + t)\Gamma(d + t)\Gamma(1 - e - t)\Gamma(-t)}{\Gamma(f + t)\Gamma(g + t)}
= \frac{-\pi^2\Gamma(a + t)\Gamma(b + t)\Gamma(c + t)\Gamma(d + t)}{\Gamma(f + t)\Gamma(g + t)\Gamma(e + t)\Gamma(1 + t)\sin \pi t \sin \pi(e + t)}.
\]

By Stirling’s formula (2.5),
\[
\frac{\Gamma(a + t)\Gamma(b + t)\Gamma(c + t)\Gamma(d + t)}{\Gamma(f + t)\Gamma(g + t)\Gamma(e + t)\Gamma(1 + t)} \sim t^{a+b+c+d-e-f-g-1} = t^{-2}.
\]


By Lemma 3.2 there exists a constant $K = K(\varepsilon)$ such that

$$\frac{1}{\sin \pi t \sin \pi (e + t)} \leq \frac{1}{K^2} \text{ if } t \in C_N, \ N = 1, 2, \ldots.$$

Therefore we obtain by the above estimates that there is a constant $\tilde{K} > 0$ such that

$$|G(t)| \leq \frac{\tilde{K}}{\rho^2 N} \text{ as } N \to \infty.$$

Thus

$$\int_{C_N} G(t) \, dt \leq \frac{\tilde{K}}{\rho^2 N} \pi \rho N \to 0 \text{ as } N \to \infty,$$

which implies

$$\int_{C_N} G(t) \, dt \to 0 \text{ as } N \to \infty.$$

It follows that the integral given by $I \left[ a, b, c, d; e, f, g \right]$ is equal to the sum of the residues of the poles of $\Gamma(1 - e - t)$ and $\Gamma(-t)$. Adding up the residues and making use of (2.7), we obtain

$$I \left[ a, b, c, d; e, f, g \right] = \frac{\pi \Gamma(a) \Gamma(b) \Gamma(c) \Gamma(d)}{\sin \pi e \Gamma(e) \Gamma(f) \Gamma(g)} I_{4F3} \left[ a, b, c, d; e, f, g; 1 \right] - \frac{\pi \Gamma(1 + a - e) \Gamma(1 + b - e) \Gamma(1 + c - e) \Gamma(1 + d - e)}{\sin \pi e \Gamma(1 + f - e) \Gamma(1 + g - e) \Gamma(2 - e)} I_{4F3} \left[ 1 + a - e, 1 + b - e, 1 + c - e, 1 + d - e; 1 + f - e, 1 + g - e, 2 - e; 1 \right],$$

from which the result follows.

The fundamental two-term relation satisfied by $L(a, b, c, d; e, f, g)$ is given in the next proposition.

**Proposition 3.3.**

$$L \left[ a, b, c, d; e, f, g \right] = L \left[ a, b, g - c, g - d; 1 + a + b - f; 1 + a + b - e, g \right]. \quad (3.4)$$

**Proof.** Let $I \left[ a, b, c, d; e, f, g \right]$ be as given in (3.3). As a first step, we will prove that

$$I \left[ a, b, c, d; e, f, g \right] \frac{\Gamma(c) \Gamma(d) \Gamma(1 + a - e) \Gamma(1 + b - e)}{\Gamma(f - a) \Gamma(f - b) \Gamma(g - c) \Gamma(g - d)}.$$

$$= \frac{\Gamma(1 + a - e) \Gamma(1 + b - e)}{\Gamma(1 + f - e) \Gamma(1 + g - e) \Gamma(2 - e)} I_{4F3} \left[ 1 + a - e, 1 + b - e, 1 + c - e, 1 + d - e; 1 + f - e, 1 + g - e, 2 - e; 1 \right],$$

\[ (3.5) \]
By Barnes’ lemma,

\[
\frac{\Gamma(a + t)\Gamma(b + t)}{\Gamma(f + t)} = \frac{1}{2\pi i} \frac{\Gamma(f - a)\Gamma(f - b)}{\Gamma(f - a - b + u)\Gamma(a - u)\Gamma(b - u)} \int_u \Gamma(t + u)\Gamma(f - a - b + u)\Gamma(a - u)\Gamma(b - u) \, du
\]

and

\[
\frac{\Gamma(c + t)\Gamma(d + t)}{\Gamma(g + t)} = \frac{1}{2\pi i} \frac{\Gamma(g - c)\Gamma(g - d)}{\Gamma(g - c - d + v)\Gamma(c - v)\Gamma(d - v)} \int_v \Gamma(t + v)\Gamma(g - c - d + v)\Gamma(c - v)\Gamma(d - v) \, dv.
\]

We re-write the integral for \( I_{[a, b, c, d; e; f, g]} \) by substituting for the above expressions, changing the order of integration, so that we integrate with respect to \( t \) first, and then applying Barnes’ Lemma again to the integral with respect to \( t \). We obtain

\[
I_{[a, b, c, d; e; f, g]} = \frac{\Gamma(c)\Gamma(d)(1 + a - e)(1 + b - e)}{-4\pi^2 \Gamma(c)\Gamma(d)\Gamma(1 + a - e)\Gamma(1 + b - e)\Gamma(f - a)\Gamma(f - b)\Gamma(g - c)\Gamma(g - d)} \int_u \Gamma(f - a - b + u)\Gamma(a - u)\Gamma(b - u)\Gamma(1 - e + u) \cdot \left( \int_v \frac{\Gamma(g - c - d + v)\Gamma(c - v)\Gamma(d - v)\Gamma(1 - e + v)}{\Gamma(1 - e + u + v)} \, dv \right) du. \tag{3.6}
\]

After the substitution \( v \mapsto c + d - f + v \) in the inside integral, it is easily checked (using the Saalschützian condition \( e + f + g - a - b - c - d = 1 \)) that the right-hand side of (3.6) is invariant under the transformation

\[
(a, b, c, d; e; f, g) \mapsto (a, b, g - c, g - d; 1 + a + b - f; 1 + a + b - e, g),
\]

which proves (3.5). The result in the proposition now follows immediately from (3.5) upon writing the two \( L \) functions in (3.4) in terms of their Barnes integral representations (3.1).
4 Invariance group

In the previous section we showed that the function $L(a, b, c, d; e; f, g)$ satisfies the two-term relation (3.4). If we define

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 1 \\
0 & 0 & -1 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix} \in GL(7, \mathbb{C}),$$

(4.1)

then (3.4) can be expressed as $L(\vec{x}) = L(A\vec{x})$.

If $\sigma \in S_7$, we will identify $\sigma$ with the matrix in $GL(7, \mathbb{C})$ that permutes the standard basis $\{e_1, e_2, \ldots, e_7\}$ of the complex vector space $\mathbb{C}^7$ according to the permutation $\sigma$. For example,

$$(123) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Let

$$G_L = \langle (12), (23), (34), (67), A \rangle \leq GL(7, \mathbb{C}).$$

(4.2)

The two-term relation (3.4) along with the trivial invariances of the function $L(a, b, c, d; e; f, g)$ under permutations of $a, b, c, d$ and interchanging $f, g$ implies that $G_L$ is an invariance group for $L(a, b, c, d; e; f, g)$, i.e. $L(\vec{x}) = L(\alpha\vec{x})$ for every $\alpha \in G_L$.

The goal of this section is to find the isomorphism type of the group $G_L$ and further to describe the two-term relations for the $L$ function in terms of a double coset decomposition of $G_L$ with respect to its subgroup $\Sigma$ defined as follows:

$$\Sigma = \langle (12), (23), (34), (67) \rangle.$$

(4.3)

The group $\Sigma$ is a subgroup of $G_L$ consisting of permutation matrices. It is clear that $\Sigma \cong S_4 \times S_3$ and so $|\Sigma| = 48$. We note that if $\sigma \in \Sigma, \alpha \in G_L$, the multiplication $\sigma\alpha$ permutes the rows of $\alpha$, and the multiplication $\alpha\sigma$ permutes the columns of $\alpha$. A double coset of $\Sigma$ in $G_L$ is a set of the form

$$\Sigma_0\Sigma = \{\sigma\alpha \tau : \sigma, \tau \in \Sigma\},$$

for some $\alpha \in G_L$.

(4.4)

The distinct double cosets of the form (4.4) partition the group $G_L$ and give us a double coset decomposition of $G_L$ with respect to $\Sigma$. (See [6] p. 119 for more on double cosets.)
In Theorem 4.1 below we show that the group $G_L$ is isomorphic to the Coxeter group $W(D_5)$, which is of order 1920. In Theorem 4.2 we show that the subgroup $\Sigma$ is the largest permutation subgroup of $G_L$ and obtain a double coset decomposition of $G_L$ with respect to $\Sigma$. We list a representative for each of the six double cosets obtained and give the six invariance relations induced by those representatives (see (4.6)–(4.11)). The six invariance relations (4.6)–(4.11) listed are all the “different” types of invariance relations in the sense that every other invariance relation can be obtained by permuting the first four entries and permuting the last two entries on the right-hand side of a listed invariance relation (which corresponds to permuting the rows of the accompanying matrix), and by permuting $a, b, c, d$ and permuting $f, g$ on the right-hand side of a listed invariance relation (which corresponds to permuting the columns of the accompanying matrix).

**Theorem 4.1.** The group $G_L$ is isomorphic to the Coxeter group $W(D_5)$, which is of order 1920.

**Proof.** The Dynkin diagram of the Coxeter group $W(D_5)$ is given by the graph with vertices labeled 1, 2, 3, 4, 1', where $i, j \in \{1, 2, 3, 4\}$ are connected by an edge if and only if $|i - j| = 1$, and 1' is connected to 2 only. The presentation of $W(D_5)$ is given by

$$W(D_5) = \langle s_1, s_2, s_3, s_4, s_{1'} : (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ii} = 1$ for all $i$; and for $i$ and $j$ distinct, $m_{ij} = 3$ if $i$ and $j$ are connected by an edge, and $m_{ij} = 2$ otherwise. It is well-known that the order of $W(D_5)$ is $2^4 \cdot 5! = 1920$ (see [9, Section 2.11]).

Consider the elements of $G_L$ given by

$$a_1 = (34), a_2 = (23), a_3 = (34)a, a_4 = (67), a_{1'} = (12).$$

(4.5)

It is clear $G_L = \langle a_i : i \in \{1, 2, 3, 4, 1'\} \rangle$. A direct computation shows that

$$(a_ia_j)^{m_{ij}} = 1, \text{ for all } i, j \in \{1, 2, 3, 4, 1'\}.$$

Therefore if we define $\varphi(s_i) = a_i$ for every $i \in \{1, 2, 3, 4, 1'\}$, $\varphi$ extends (uniquely) to a surjective homomorphism from $W(D_5)$ onto $G_L$ (see [6, Section 1.6]). Since $W(D_5)$ is a finite group, if we show that $G_L$ and $W(D_5)$ have the same order, it will follow that $\varphi$ is an isomorphism and the theorem will be proved.

Since $\varphi$ is a surjective homomorphism, the First Isomorphism Theorem for groups (see [6, p. 98]) implies that $|G_L| = |\text{Im}(\varphi)|$ must divide $|W(D_5)| = 1920$. Therefore if we show that $|G_L| > 960 = \frac{1920}{2}$, then necessarily $|G_L| = 1920$. We will obtain an estimate on the order of $G_L$ by computing the sizes of the double cosets $\Sigma A \Sigma$ and $\Sigma((123)(67)a)^2 \Sigma$ of $\Sigma$ in $G_L$, where $\Sigma$ is as given in (4.3).
The matrix $A$ is given by

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & -1 & -1 & 1 & 0 & 1 \\
0 & 0 & -1 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.$$

We see that all the rows of $A$ are distinct as sequences. Therefore multiplying $A$ on the left by $\sigma$, for $\sigma \in \Sigma$, will give us 48 matrices in $G_L$ that belong to the double coset $\Sigma A \Sigma$. We note that the products $\sigma A$, for $\sigma \in \Sigma$, amount to obtaining all possible permutations of the first four rows of $A$ and all possible permutations of the last two rows of $A$. By considering products of the form $A\sigma$, for $\sigma \in \Sigma$, we can permute the first four columns of $A$ and the last two columns of $A$ in every possible way. If we first permute columns of $A$ that are different as multisets, and then permute the rows of the resulting matrix in all 48 different ways, we obtain 48 new elements in $G_L$ that belong to the double coset $\Sigma A \Sigma$. Now, the first and second columns of $A$ are the same as multisets and so are the third and the fourth columns. Thus we permute the first four columns in $\frac{4!}{2!2!} = 6$ different ways. The sixth and seventh columns of $A$ are different as multisets and so we permute them in 2 different ways. In total, we permute the columns of $A$ in $6 \cdot 2 = 12$ different ways and then we permute the rows of each of the resulting matrices in all 48 possible ways to obtain that the number of matrices that belong to the double coset $\Sigma A \Sigma$ is $12 \cdot 48$.

Next we consider the matrix

$$A_1 = ((123)(67)A)^2 = \begin{pmatrix}
0 & -1 & -1 & -1 & 0 & 1 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & -1 & -2 & -1 & 1 & 1 & 1 \\
0 & 0 & -1 & -1 & 0 & 1 & 1 \\
0 & -1 & -1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}.$$

We see that $A_1$ contains an entry of $-2$, which is not the case with $A$, implying that the double cosets $\Sigma A_1 \Sigma$ and $\Sigma A \Sigma$ are distinct. All the rows of $A_1$ are distinct as sequences. The first, second and third columns of $A_1$ are different as multisets and the fourth column represents the same multiset as the second column. The sixth and seventh columns of $A_1$ are the same as multisets. Thus we permute the columns of $A_1$ in $\frac{4!}{2!} = 12$ different ways and then we permute the rows of each of the resulting matrices in all 48 possible ways to obtain that the number of matrices that belong to the double coset $\Sigma A_1 \Sigma$ is $12 \cdot 48$.

Considering the number of matrices that belong to the double cosets $\Sigma A \Sigma$ and $\Sigma A_1 \Sigma$, we see that the group $G_L$ contains at least $12 \cdot 48 + 12 \cdot 48 > 960$ elements. Therefore $|G_L| = |W(D_5)|$ and the theorem is proved.
As stated before Theorem 4.1, we are interested in the complete double coset decomposition of $G_L$ with respect to $\Sigma$ since this will classify all the invariance relations for the function $L(a, b, c, d; e, f, g)$ in a convenient way. We use the same technique as in the proof of Theorem 4.1 given by permuting columns that are different as multisets and then permuting the rows of the resulting matrices in every possible way. We obtain that there are six double cosets of $\Sigma$ in $G_L$. Representative matrices for the double cosets are $I_7$, $A$, $((123)(67)A)^2$, $((123)(67)A)^3$, $((123)(67)A)^4$. The corresponding double coset sizes are $1 \cdot 48$, $12 \cdot 48$, $12 \cdot 48$, $12 \cdot 48$, $2 \cdot 48$, $1 \cdot 48$. Furthermore, the representative matrices are all seen to have different entries (as, for example, we determined for the matrices $A$ and $((123)(67)A)^2$ in the proof of Theorem 4.1) so that $\Sigma$ must indeed be the largest permutation subgroup of $G_L$. Each representative matrix gives rise to an invariance relation. Theorem 4.2 summarizes the result.

**Theorem 4.2.** Let $\Sigma$ be as defined in (4.3). Then $\Sigma$ consists of all the permutation matrices in $G_L$. There are six double cosets in the double coset decomposition of $G_L$ with respect to $\Sigma$. Representative matrices for the double cosets are $I_7$, $A$, $((123)(67)A)^2$, $((123)(67)A)^3$, $((123)(67)A)^4$ and the corresponding double coset sizes are $1 \cdot 48$, $12 \cdot 48$, $12 \cdot 48$, $12 \cdot 48$, $2 \cdot 48$, $1 \cdot 48$. The corresponding invariances of the $L$ function are given by

\[
L \left[ a, b, c, d; e, f, g \right] = L \left[ a, b, c, d; e, f, g \right],
\]

(4.6)

\[
L \left[ a, b, c, d; e, f, g \right] = L \left[ a, b, g - c, g - d; 1 + a + b - f; 1 + a + b - e, g \right],
\]

(4.7)

\[
L \left[ a, b, c, d; e, f, g \right] = L \left[ 1 + a - c, 1 + a + b - e, 1 + a + d - e; g - c, g - a, f - c \right],
\]

(4.8)

\[
L \left[ a, b, c, d; e, f, g \right] = L \left[ 1 + d - e, 1 + a - e, g - c, g - b; 1 + g - b - c, 1 + a + d - e, 1 + g - e \right],
\]

(4.9)

\[
L \left[ a, b, c, d; e, f, g \right] = L \left[ g - a, g - b, g - c, g - d; 1 + g - f; 1 + g - e, g \right],
\]

(4.10)

\[
L \left[ a, b, c, d; e, f, g \right] = L \left[ 1 + c - e, 1 + d - e, 1 + a - e, 1 + b - e; 2 - e, 1 + g - e, 1 + f - e \right].
\]

(4.11)

5 Applications of the fundamental two-term relation

In this final section we prove some consequences of the fundamental two-term relation given in Proposition 3.3. As a first step, we write the two $L$ functions in (3.4) in terms of their definitions as linear combinations of two $4F_3(1)$ series.
We obtain

\[
\begin{align*}
\frac{4F_3 \left[ a, b, c, d; e, f, g; 1 \right]}{\sin \pi e} \Gamma(f) & \Gamma(g) (1 + a - e) \Gamma(1 + b - e) \Gamma(1 + c - e) \Gamma(1 + d - e) \\
- \frac{4F_3 \left[ 1 + a - e, 1 + b - e, 1 + d - e; 1 \right]}{1 + f - e, 1 + g - e, 1 + d - e; 1} = \\
\frac{4F_3 \left[ a, b, g - c, g - d; 1 + a + b - f, 1 + a + b - e, g; 1 \right]}{\sin \pi e} \\
\Gamma(b) & \Gamma(c) \Gamma(d) \Gamma(1 + f - e) \Gamma(1 + g - e) \Gamma(1 + d - e)
\end{align*}
\]

Using the fact that \( \lim_{n \to \infty} (1 + a + b + c + d - f - g) \) depend on \( a \). In equation (5.1) we let \( |a| \to \infty \). Using Stirling’s formula (2.5) and the conditions (5.2), we obtain

\[
\begin{align*}
3F_2\left[ b, c, d; f, g; 1 \right] = \\
\frac{\Gamma(f) \Gamma(g) (f + g - b - c - d)}{3F_2\left[ b, g - c, g - d; f + g - c - d, g; 1 \right]} \\
\frac{\Gamma(f + g - c - d) \Gamma(g) \Gamma(f - b)}{\Gamma(f + g - b - d) \Gamma(f + g - b - c)}.
\end{align*}
\]

We note that the conditions (5.2) are needed for the absolute convergence of the two \( 3F_2(1) \) series in (5.3). Applying (5.3) twice yields Thomae’s identity

\[
\begin{align*}
3F_2\left[ b, c, d; f, g; 1 \right] = \\
\frac{\Gamma(f) \Gamma(g) (f + g - b - c - d)}{3F_2\left[ f - b, g - b, f + g - b - c; f + g - b - d, f + g - b - c; 1 \right]} \\
\frac{\Gamma(f + g - b - d) \Gamma(f + g - b - c)}{\Gamma(b) \Gamma(f + g - b - d) \Gamma(f + g - b - c)}.
\end{align*}
\]

In fact, applying (5.4) twice gives (5.3), so that (5.3) and (5.4) are equivalent.

Next in equation (5.1) we let \( a \to -n \), where \( n \) is a nonnegative integer. Using the fact that \( \lim_{e \to -n} \frac{\Gamma(e)}{\Gamma(n)} = 0 \) and then formula (2.7) to simplify the result, we obtain Bailey’s identity

\[
\begin{align*}
4F_3\left[ -n, b, c, d; e, f, g; 1 \right] = \\
\frac{(e - b)_n (f - b)_n}{(e)_n (f)_n} 4F_3\left[ -n, b, g - c, g - d; 1 - n + b - f, 1 - n + b - e, g; 1 \right],
\end{align*}
\]

\[
(5.5)
\]
which holds provided that \( e + f + g - b - c - d + n = 1 \).

Thomae’s and Bailey’s identities have been shown in \( [7] \) in a similar way to be limiting cases of a fundamental two-term relation satisfied by the function \( K(a; b, c, d; e, f, g) \).

As a final application, in the fundamental two-term relation \( (3.4) \) we let \( d = g \). We express the left-hand side as a Barnes integral according to Proposition \( (3.1) \) and we write the right-hand side in terms of two \( 4F3(1) \) series according to the definition of the \( L \) function. The condition \( d = g \) causes one of the terms on the right-hand side to go to zero and the \( 4F3(1) \) series in the other term to be trivially equal to one. If we simplify the result further using \( (2.7) \), we obtain

\[
\frac{1}{2\pi i} \int_t \frac{\Gamma(a + t)\Gamma(b + t)\Gamma(c + t)\Gamma(1 - e - t)\Gamma(-t)}{\Gamma(f + t)} dt = \frac{\Gamma(a)\Gamma(b)\Gamma(c)(1 + a - e)\Gamma(1 + b - e)\Gamma(1 + c - e)}{\Gamma(f - a)\Gamma(f - b)\Gamma(f - c)},
\]

which holds provided that \( e + f - a - b - c = 1 \). The equation \( (5.6) \) is precisely the statement of Barnes’ second lemma.

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