Wave equations on space-times of low regularity:
Existence results and regularity theory in the framework of
generalized function algebras

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Abstract

We present recent developments concerning Lorentzian geometry in algebras of generalized functions. These have, in particular, raised a new interest in refined regularity theory for the wave equation on singular space-times.

Keywords: wave equation, generalised Lorentzian geometry, algebras of generalized functions.

MSC 2000: 35L05, 58J45.

1 Introduction

Generalized function algebras have proved themselves a valuable tool for non-linear analysis of partial differential equations while maintaining consistency with Schwartz’ distributional framework ([9, 18] and the references therein). Their essential feature is they admit the space of smooth functions as a faithful subalgebra, which, in view of the Schwartz impossibility result [19], is the optimal possibility.

More recent scientific work also focuses on developing a geometric theory of generalized functions ([5, 12] et al.). From the very beginning of this line of research, one had in mind applications to fields such as general relativity. The foundations of generalised pseudo-Riemannian geometry were laid in [14], with further work investigating flows of generalized vector fields [13], and generalised connections on principal bundles [15].

The work of Vickers and Wilson [20] on the wave-equation on conical space-times has initiated research on existence and uniqueness results for the initial value problem of the wave-equation for a wider class of singular space-times. A first generalization of their result concerned static-space times [16] and, just recently, a quite general existence and uniqueness result has been established, thus dropping the condition of staticity [8]. In the course of this work, it became imperative to study generalized Lorentzian geometry [17], cf. below.

The program of this article is to report on recent developments in Lorentzian geometry in a generalised framework and to present results on the wave-equation on singular space-times. Finally, we suggest that regularity information for the wave-equation can be retrieved from refined energy estimates. This study is motivated by concrete examples, cf. [7].

For notation in the field, we refer to the standard reference [9].

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2 Lorentzian geometry in algebras of generalized functions

Loosely speaking, a singular pseudo-Riemannian metric is modelled as a net of smooth metrics, 
\[(g_\varepsilon) \pmod{\mathcal{N}(T_0^2(M))},\]
where \(g_\varepsilon\) is a pseudo-Riemannian metric of fixed index for sufficiently small smoothing parameter \(\varepsilon\). As generalized functions are not determined pointwise, but only by evaluation on generalized points, one can characterize candidates for generalized pseudo-Riemannian metrics in terms of bilinear forms on the finite dimensional module \(\tilde{\mathbb{R}}^n\) (for more details, see [14], Proposition 2.1 and [17]).

**Theorem 2.1.** Let \(g \in \mathcal{G}^0_2(X)\). The following are equivalent:

1. For each chart \((V_\alpha, \psi_\alpha)\) and each \(\tilde{x} \in (\psi_\alpha(V_\alpha))\), the map \(g_\alpha(\tilde{x}) : \tilde{\mathbb{R}}^n \times \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}\) is symmetric and non-degenerate.

2. \(g : \mathcal{G}^0_1(X) \times \mathcal{G}^0_1(X) \to \mathcal{G}(X)\) is symmetric and \(\det(g)\) is invertible in \(\mathcal{G}((\psi_\alpha(V_\alpha)))\).

3. \(\det g\) is invertible in \(\mathcal{G}((\psi_\alpha(V_\alpha)))\) and for each relatively compact open set \(V \subset X\) there exists a representative \((g_\varepsilon)\varepsilon\) of \(g\) and \(\varepsilon_0 > 0\) such that \(g_\varepsilon\big|_V\) is a smooth pseudo-Riemannian metric for all \(\varepsilon < \varepsilon_0\).

A generalized pseudo Riemannian metric then is defined as a non-degenerate symmetric \(g \in \mathcal{G}^0_1(X)\) with locally constant index. We were motivated by item (2) to continue the research in the direction of elaborating a concept of causality from the viewpoint of bilinear forms on \(\tilde{\mathbb{R}}^n\) on the basis of positivity concepts revisited in the introduction. The first step is, based on a basic perturbation theory, to introduce a well defined index associated to a symmetric matrix \(A \in M(n \times n, \tilde{\mathbb{R}})\): First, “ordered eigenvalues” of \(A\) are defined, \(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\). Let \(j \geq 0\) such that \(\lambda_1 << 0, \ldots, \lambda_j << 0, \lambda_{j+1} >> 0, \ldots, \lambda_n >> 0\), then \(j\) is called the index of \(A\). (This index need not exist.) Finally, the index of a symmetric bilinear form \(b\) is then defined as the index of \(A := b(e_i, e_j), e_i\) determining the canonical basis of \(\tilde{\mathbb{R}}^n\). A symmetric bilinear form \(b\) is called

- positive definite, if its index is 0.
- Lorentzian, if its index is 1.

Causality is introduced in the following way: Let \(b\) be Lorentzian. A vector \(v \in \tilde{\mathbb{R}}^n\) is called

- time-like, if \(b(v, v) << 0\),
- null, if \(b(v, v) = 0\),
- space-like, if \(b(v, v) >> 0\)

Unlike the standard theory, this concept is not a trichotomy (i.e., there are vectors which are neither time-like, null or space-like), since \(<<\) does not give rise to a total ordering \(<<=\). Essential in the following natural basic considerations is the characterization of free vectors:

**Theorem 2.2.** Let \(v\) be an element of \(\mathbb{R}^n\). The following are equivalent:

1. For any positive definite symmetric bilinear form \(h\) on \(\tilde{\mathbb{R}}^n\) we have 
   \[h(v, v) >> 0\]

2. The coefficients of \(v\) with respect to some (hence any) basis span \(\tilde{\mathbb{R}}\).

3. \(v\) is free.
4. For each representative \((v_\varepsilon)_{\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)\) of \(v\) there exists some \(\varepsilon_0 \in I\) such that for each \(\varepsilon < \varepsilon_0\) we have \(v_\varepsilon \neq 0\) in \(\mathbb{R}^n\).

Further consequences concern, for instance, characterizations of symmetric positive forms, also, direct summands, the algebraic structure of \(\mathbb{R}^n\) (being not semi-simple), etc. In order to establish so-called dominant energy conditions for a class of super-energy tensors, it was necessary to establish the inverse Cauchy-Schwarz inequality for time-like vectors in a Lorentzian space-time. We refer the interested reader to the foundational paper \([17]\).

3 The wave-equation

The results of the preceding section have been elaborated in close relation with the project of generalising the paper of Vickers and Wilson \([20]\) for the wave equation on conical space-times. Only the new framework allows to formulate our recent result, as proved in \([8]\). We give here a brief overview. The initial value problem under consideration corresponds to the scalar wave equation,

\[
\Box_g u = f \\
u_\Sigma = v \\
\tilde{\xi} u_\Sigma = w,
\]

where \(\Box_g\) is the d’Alembertian induced by a generalized Lorentzian metric \(g\) on a smooth manifold \(M\). Here \(\xi\) is a \(C^\infty\) vector field, and \(\tilde{\xi}\) the corresponding unit field; further, \(\Sigma\) is the space-like (in the generalized sense) initial surface. The main theorem in our joint paper with R. Steinbauer \([8]\) establishes local existence of scalar solutions to \((3.1)\), under the following regularity assumptions (cf. also the conclusion section of that paper)

1. For all \(K\) compact in \(M\), for all orders of derivative \(k \in \mathbb{N}_0\) and all \(k\)-tuples of vector fields \(\eta_1, \ldots, \eta_k \in \mathcal{X}(M)\) and for any representative \((g_\varepsilon)_{\varepsilon}\) we have:

- \(\sup_{p \in K} \| \mathcal{L}_{\eta_1} \ldots \mathcal{L}_{\eta_k} g_\varepsilon \|_m = O(\varepsilon^{-k})\) \((\varepsilon \to 0)\);
- \(\sup_{p \in K} \| \mathcal{L}_{\eta_1} \ldots \mathcal{L}_{\eta_k} g_\varepsilon^{-1} \|_m = O(\varepsilon^{-k})\) \((\varepsilon \to 0)\).

2. For all \(K\) compact in \(M\), we have

\[
\sup_{p \in K} \| \nabla^\varepsilon \xi \|_m = O(\log(\varepsilon)), \quad (\varepsilon \to 0),
\]

where \(\nabla^\varepsilon\) denotes the covariant derivative with respect to the Lorentzian metric \(g_\varepsilon\).

3. a causality assumption, to guarantee smooth solutions.

Different arguments suggest that the regularity estimates derived in \([8]\) might be improved substantially by refined energy estimates based on the adaption of suitable Sobolev embedding theorems (see, e.g., \([1, 10]\)) to our setting. In order to apply these results, we require better control on the asymptotic growth of Sobolev-constants corresponding to the regularizing sequences of Riemannian metrics induced on \(t = \text{constant}\) slices of our Lorentzian space-time. This, in turn, requires control on the injectivity radius of the induced Riemannian metrics, which is possible if we have control on the diameter, curvature and volume \([4]\). Investigations on this topic are underway.

3.1 Regularity issues 1: Wave equations for Lorentzian metrics of H"older regularity

The class of metrics in the preceding section can be considered as generalized Hölder-Zygmund regular of order \(s = 0\) (cf. \([11]\) and, for an adaption to the geometric setting, cf. \([7]\)). In the
latter reference, we have investigated a two dimensional space-time \((\mathbb{R}^2, g)\), with \(g\) a Lorentzian metric of classic Hölder regularity \(C^{0,\alpha}\), \(\forall \alpha > 0\). The latter is designed in such a way that the principal symbol of its induced d’Alembertian \(\Box_g\) is precisely the strictly hyperbolic operator used by Colombini & Spagnolo [6] (who present an example of a wave equation with Hölder continuous coefficients and smooth data, not admitting distributional solutions).

In [7] we bring the metric, by means of a coordinate transformation \(\Phi\) of low regularity \(C^{1,\alpha}\) \((\alpha < 1)\), into a form that is evidently conformally flat. This suggests that solutions \(\bar{u}\) for the wave-equation in flat space-time can then be pulled back via \(\Phi\) to prospective solutions \(u\) of \(\Box_g u = 0\). This procedure, however, raises questions concerning the solution concept, since the coordinate transform is not \(C^2\), the minimal regularity required for the standard analysis to hold. Nevertheless, for regularisations (that is, \(C^\infty\) mappings \((\Phi_\varepsilon)\)) of \(\Phi\), the above is rigorous and we can show that the smooth pulled back solutions \((u_\varepsilon)\) even give rise to generalized solutions, that is, elements of the special algebra \(\mathcal{G}\). So we have, by means of a geometric argument, found solutions of a non-trivial strictly hyperbolic differential equation with coefficients of regularity beyond Lipschitz.

### 3.2 Regularity issues 2: Collapsing Riemannian manifolds

An alternative direction for research concerning wave equations on manifolds with singular metrics involves situations where the induced Riemannian metric on three-dimensional hypersurfaces of the Lorentzian manifold “collapses” as \(\varepsilon \to 0\). In particular, in the case of Cheeger-Gromov collapse [2, 3], one constructs nets of Riemannian metrics, \(h_\varepsilon\), with the property that, as \(\varepsilon \to 0\), the sectional curvature of \(h_\varepsilon\) is uniformly bounded, but the injectivity radius of \(h_\varepsilon\) converges uniformly to zero. In this case, the volume of the three-manifold converges to zero as \(\varepsilon \to 0\), but the curvature remains bounded. Taking, for example, a four-dimensional Lorentzian metric of the form \(g_\varepsilon = -dt^2 + h_\varepsilon\), we would have asymptotic conditions

\[
\begin{align*}
\text{dvol}_{g_\varepsilon} &= O(\varepsilon), \\
\text{Curvature} &= O(1)
\end{align*}
\]

as \(\varepsilon \to 0\).

These asymptotic conditions are quite different from those investigated in [8, 20] and, as such, will lead to (generalised) solutions of the wave equation that obey quite different asymptotic conditions from those that are standard in Colombeau theory. On the other hand, Cheeger-Gromov collapse is of considerable interest from a differential geometric viewpoint, and is a field where ideas from the field of generalised functions may, perhaps, be usefully applied.

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