An Optimal EDG Method for Distributed Control of Convection Diffusion PDEs

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Abstract

We propose an embedded discontinuous Galerkin (EDG) method to approximate the solution of a distributed control problem governed by convection diffusion PDEs, and obtain optimal a priori error estimates for the state, dual state, their fluxes, and the control. Moreover, we prove the optimize-then-discretize (OD) and discretize-then-optimize (DO) approaches coincide. Numerical results confirm our theoretical results.

1 Introduction

We study the following distributed optimal control problem:

\[
\min J(u) = \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \gamma \|u\|_{L^2(\Omega)}^2, \quad \gamma > 0,
\]

subject to

\[
-\Delta y + \beta \cdot \nabla y = f + u \quad \text{in } \Omega,
\]

\[
y = g \quad \text{on } \partial \Omega,
\]

where \(\Omega \subset \mathbb{R}^d \ (d \geq 2)\) is a Lipschitz polyhedral domain with boundary \(\Gamma = \partial \Omega\), \(f \in L^2(\Omega)\), \(g \in C^0(\partial \Omega)\), and the vector field \(\beta\) satisfies

\[
\nabla \cdot \beta \leq 0. \quad (3)
\]

Optimal control problems for convection diffusion equations have been extensively studied using many different finite element methods, such as standard finite elements [11, 13], mixed finite elements [13, 35, 39], discontinuous Galerkin (DG) methods [16, 21, 33, 34, 36, 40, 41] and hybrid discontinuous Galerkin (HDG) methods [17, 18]. HDG methods were first introduced by Cockburn et al. in [4] for second order elliptic problem, and then they have been applied to many other problems [2, 3, 5, 7, 8, 23–26, 32]. HDG methods keep the advantages of DG methods, but

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have a lower number of globally coupled degrees of freedom compared to mixed methods and DG methods. However, the degrees of freedom for HDG methods is still larger compared to standard finite element methods. Embedded discontinuous Galerkin (EDG) methods were first proposed in [15], and then analyzed in [6]. EDG methods are obtained from the HDG methods by forcing the numerical trace space to be continuous. This simple change significantly reduce the number of degrees of freedom and make EDG methods competitive for flow problems [27] and many other applications [9, 10, 19, 27, 29].

In [38], we utilized an EDG method for a distributed optimal control problems for the Poisson equation. We obtained optimal convergence rates for the state, dual state and the control, but suboptimal convergence rates for their fluxes. This suboptimal flux convergence rate for the Poisson equation is a limitation of the EDG method with equal order polynomial degrees for all variables [6]. However, Zhang, Xie, and Zhang recently proposed a new EDG method and proved optimal convergence rates for all variables for the Poisson equation [37]. This EDG new method is obtained by simply using a lower degree finite element space for the flux. In this work, we use this new EDG method to approximate the solution of the above convection diffusion distributed optimal control problem, and in Section 3 we prove optimal convergence rates for all variables.

There are two main approaches to compute the numerical solution of PDE constrained optimal control problems: the optimize-then-discretize (OD) and discretize-then-optimize (DO) approaches. In the OD approach, one first derives the first-order necessary optimality conditions, then discretizes the optimality system, and then solves the resulting discrete system by utilizing efficient iterative solvers [31]. In the DO approach, one first discretizes the PDE optimization problem to obtain a finite dimensional optimization problem, which is then solved by existing optimization algorithms, such as [1, 28]. The discretization methods for which these two approaches coincide are called commutative. Intuitively, the DO approach is more straightforward in practice; however, not all discretization schemes are commutative. In the non-commutative case, the DO approach may result in badly behaved numerical results; see, e.g., [20, 22]. Therefore, devising commutative numerical methods is very important. In Section 2, we prove the EDG method studied here is commutative for the convection diffusion distributed control problem. Moreover, we provide numerical examples to confirm our theoretical results in Section 4.

## 2 EDG scheme for the optimal control problem

### 2.1 Notation

Throughout the paper we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on $\Omega$ with norm $\| \cdot \|_{m,p,\Omega}$ and seminorm $| \cdot |_{m,p,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with norm $\| \cdot \|_{m,\Omega}$ and seminorm $| \cdot |_{m,\Omega}$. Specifically, $H_0^1(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial \Omega \}$. We denote the $L^2$-inner products on $L^2(\Omega)$ and $L^2(\Gamma)$ by

$$
(v, w) = \int_{\Omega} vw \quad \forall v, w \in L^2(\Omega),
$$

$$
\langle v, w \rangle = \int_{\Gamma} vw \quad \forall v, w \in L^2(\Gamma).
$$

Define the space $H(\text{div}, \Omega)$ as

$$
H(\text{div},\Omega) = \{ v \in [L^2(\Omega)]^d, \nabla \cdot v \in L^2(\Omega) \}.
$$

Let $T_h$ be a collection of disjoint elements that partition $\Omega$. We denote by $\partial T_h$ the set $\{ \partial K : K \in T_h \}$. For an element $K$ of the collection $T_h$, let $e = \partial K \cap \Gamma$ denote the boundary face of $K$ if
the $d - 1$ Lebesgue measure of $e$ is non-zero. For two elements $K^+$ and $K^-$ of the collection $\mathcal{T}_h$, let 
$e = \partial K^+ \cap \partial K^-$ denote the interior face between $K^+$ and $K^-$ if the $d - 1$ Lebesgue measure of $e$ is non-zero. Let $\varepsilon_h^0$ and $\varepsilon_h^\partial$ denote the set of interior and boundary faces, respectively. We denote by $\varepsilon_h$ the union of $\varepsilon_h^0$ and $\varepsilon_h^\partial$. We finally introduce

$$(w,v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w,v)_K, \quad \langle \zeta, \rho \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \langle \zeta, \rho \rangle_{\partial K}. \tag{7}$$

Let $\mathcal{P}^k(D)$ denote the set of polynomials of degree at most $k \geq 0$ on a domain $D$. We introduce the discontinuous finite element spaces

$$V_h := \{v \in [L^2(\Omega)]^d : v|_K \in [\mathcal{P}^k(K)]^d, \forall K \in \mathcal{T}_h\}, \tag{4}$$

$$W_h := \{w \in L^2(\Omega) : w|_K \in \mathcal{P}^{k+1}(K), \forall K \in \mathcal{T}_h\}, \tag{5}$$

$$M_h := \{\mu \in L^2(\varepsilon_h) : \mu|_e \in \mathcal{P}^{k+1}(e), \forall e \in \varepsilon_h\}. \tag{6}$$

Define $M_h(o)$ and $M_h(\partial)$ in the same way as $M_h$, but with $\varepsilon_h^0$ and $\varepsilon_h^\partial$ replacing $\varepsilon_h$. Note that $M_h$ consists of functions which are continuous inside the faces (or edges) $e \in \varepsilon_h$ and discontinuous at their borders. In addition, for any function $w \in W_h$ we use $\nabla w$ to denote the piecewise gradient on each element $K \in \mathcal{T}_h$. A similar convention applies to the divergence $\nabla \cdot \mathbf{r}$ for all $\mathbf{r} \in V_h$.

For EDG methods, we only change the space of numerical traces $M_h$, which is discontinuous, into a continuous space $\widetilde{M}_h$ as follows:

$$\widetilde{M}_h := M_h \cap C^0(\varepsilon_h). \tag{7}$$

The spaces $\widetilde{M}_h(o)$ and $\widetilde{M}_h(\partial)$ are defined in the same way as $M_h(o)$ and $M_h(\partial)$.

Recall we assume the Dirichlet boundary data $g$ is continuous. Let $\mathcal{I}_h$ be an interpolation operator, so that $\mathcal{I}_h g$ is a continuous interpolation of $g$ on $\varepsilon_h^0$.

Again, in most of the EDG works in the literature the polynomial degree is equal for the three spaces $V_h$, $W_h$, and $\widetilde{M}_h$. We lower the polynomial degree for the flux space $V_h$ as in the recent work [37].

### 2.2 Optimize-then-Discretize

First, we consider the optimize-then-discretize (OD) approach: we use the EDG method to discretize the optimality system for the convection diffusion control problem.

It is well known that the optimal control problem [1]–[2] is equivalent to the optimality system

$$-\Delta y + \beta \cdot \nabla y = f + u \quad \text{in } \Omega, \tag{8a}$$

$$y = g \quad \text{on } \partial \Omega, \tag{8b}$$

$$-\Delta z - \nabla \cdot (\beta z) = y - y_d \quad \text{in } \Omega, \tag{8c}$$

$$z = 0 \quad \text{on } \partial \Omega, \tag{8d}$$

$$z + \gamma u = 0 \quad \text{in } \Omega. \tag{8e}$$

For $\mathbf{q} = -\nabla y$ and $\mathbf{p} = -\nabla z$, the mixed weak form of the optimality system [8a]–[8e] is given by

$$(\mathbf{q}, \mathbf{r}) - (y, \nabla \cdot \mathbf{r}) + (y, \mathbf{r} \cdot \mathbf{n}) = 0, \tag{9a}$$

$$(\nabla \cdot (\mathbf{q} + \beta y), w) - (y\nabla \cdot \beta, w) = (f + u, w), \tag{9b}$$

$$(\mathbf{p}, \mathbf{r}) - (z, \nabla \cdot \mathbf{r}) = 0, \tag{9c}$$

$$(\nabla \cdot (\mathbf{p} - \beta z), w) = (y - y_d, w), \tag{9d}$$

$$(z + \gamma u, v) = 0. \tag{9e}$$
for all \((r, w, v) \in H(\text{div}, \Omega) \times L^2(\Omega) \times L^2(\Omega)\).

To approximate the solution of this system, the EDG method seeks approximate fluxes \(q_h, p_h \in V_h\), states \(y_h, z_h \in W_h\), interior element boundary traces \(\tilde{y}_h^0, \tilde{z}_h^0 \in \tilde{M}_h(\partial)\), and control \(u_h \in W_h\) satisfying

\[
(q_h, r_1)_{T_h} - (y_h, \nabla \cdot r_1)_{T_h} + \langle \tilde{y}_h^0, r_1 \cdot n \rangle_{\partial T_h \setminus e_h^\partial} = -\langle I_h g, r_1 \cdot n \rangle_{e_h^\partial},
\]

\[
-(q_h + \beta y_h, \nabla w_1)_{T_h} - (y_h \nabla \cdot \beta, w_1)_{T_h} + \langle \tilde{y}_h^0, w_1 \rangle_{\partial T_h \setminus e_h^\partial} - (u_h, w_1)_{T_h} = -(\beta \cdot n I_h g, w_1)_{e_h^\partial}
\]

\[
+ (f, w_1)_{T_h},
\]

(10a)

(10b)

(10c)

for all \((r_1, w_1) \in V_h \times W_h\),

\[
(p_h, r_2)_{T_h} - (z_h, \nabla \cdot r_2)_{T_h} + \langle \tilde{z}_h^0, r_2 \cdot n \rangle_{\partial T_h \setminus e_h^\partial} = 0,
\]

\[
-(p_h - \beta z_h, \nabla w_2)_{T_h} + \langle \tilde{p}_h, w_2 \rangle_{\partial T_h}
\]

\[
- (\beta \cdot n \tilde{z}_h^0, w_2)_{\partial T_h \setminus e_h^\partial} - (y_h, w_2)_{T_h} = -(y_d, w_2)_{T_h},
\]

(10d)

(10e)

for all \((r_2, w_2) \in V_h \times W_h\),

\[
\langle \tilde{q}_h \cdot n + \beta \cdot n \tilde{y}_h^0, \mu_1 \rangle_{\partial T_h \setminus e_h^\partial} = 0,
\]

\[
\langle \tilde{p}_h \cdot n - \beta \cdot n \tilde{z}_h^0, \mu_2 \rangle_{\partial T_h \setminus e_h^\partial} = 0,
\]

(10f)

(10g)

for all \(\mu_1, \mu_2 \in \tilde{M}_h(\partial)\), and the optimality condition

\[
(z_h + \gamma u_h, w_3)_{T_h} = 0,
\]

(10h)

for all \(w_3 \in W_h\).

The numerical traces on \(\partial T_h\) are defined as

\[
\tilde{q}_h \cdot n = q_h \cdot n + h^{-1}(y_h - \tilde{y}_h^0) + \tau_1(y_h - \tilde{y}_h^0) \quad \text{on} \ \partial T_h \setminus e_h^\partial,
\]

\[
\tilde{p}_h \cdot n = p_h \cdot n + h^{-1}(y_h - \tilde{y}_h^0) + \tau_1(y_h - \tilde{y}_h^0) \quad \text{on} \ \varepsilon_h^\partial,
\]

\[
\tilde{p}_h \cdot n = p_h \cdot n + h^{-1}(z_h - \tilde{z}_h^0) + \tau_2(z_h - \tilde{z}_h^0) \quad \text{on} \ \partial T_h \setminus e_h^\partial.
\]

(10i)

(10j)

(10k)

where \(\tau_1\) and \(\tau_2\) are positive stabilization functions defined on \(\partial T_h\). We show below that the OD and DO approaches coincide if \(\tau_2 = \tau_1 - \beta \cdot n\). The implementation of the OD approach is very similar to the HDG method in [18], and hence is omitted here.

### 2.3 Discretize-then-Optimize

Now we derive the optimality conditions for the discretize-then-optimize (DO) approach when the optimal control problem is discretized by the EDG method. Therefore, we solve

\[
\min_{u_h \in W_h} \frac{1}{2} \|y_h - y_d\|_{T_h}^2 + \frac{\gamma}{2} \|u_h\|_{T_h}^2, \quad \gamma > 0,
\]

(11)
subject to the discrete state equations
\begin{align}
(q_h, r_1)_{\mathcal{T}_h} - (y_h, \nabla \cdot r_1)_{\mathcal{T}_h} + (\tilde{y}_h^0, r_1 \cdot n)_{\partial \mathcal{T}_h \setminus \gamma_h^0} &= -\langle I_h g, r_1 \cdot n \rangle_{\mathcal{T}_h}, \\
-(q_h + \beta y_h, \nabla w_1)_{\mathcal{T}_h} - (y_h \nabla \cdot \beta, w_1)_{\mathcal{T}_h} + \langle q_h, n, w_1 \rangle_{\partial \mathcal{T}_h} \\
+((h^{-1} + \tau_1) y_h, w_1)_{\partial \mathcal{T}_h} + \langle \beta \cdot n - (h^{-1} + \tau_1) y_h^0, w_1 \rangle_{\partial \mathcal{T}_h \setminus \gamma_h^0} \\
- (u_h, w_1)_{\mathcal{T}_h} &= -\langle \beta \cdot n - (h^{-1} + \tau_1) I_h g, w_1 \rangle_{\partial \mathcal{T}_h \setminus \gamma_h^0} + (f, w_1)_{\mathcal{T}_h}, \\
(q_h, n) + (h^{-1} + \tau_1) (y_h - \tilde{y}_h^0), \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \gamma_h^0} &= 0,
\end{align}
for any \((r_1, w_1, \mu_1) \in V_h \times W_h \times \widetilde{M}_h(o)\).

The discretized Lagrangian functional is defined by
\begin{align}
\mathcal{L}_h(q_h, y_h, \tilde{y}_h^0; p_h, z_h, \tilde{z}_h^0) &= \frac{1}{2} \left| y_h - y_d \right|^2_{\mathcal{T}_h} + \frac{\gamma}{2} \left| u_h \right|^2_{\mathcal{T}_h} \\
+ (q_h, \beta y_h, \nabla z_h)_{\mathcal{T}_h} + (y_h \nabla \cdot \beta, z_h)_{\mathcal{T}_h} - \langle q_h, n, z_h \rangle_{\partial \mathcal{T}_h} \\
- ((h^{-1} + \tau_1) y_h, z_h^0)_{\partial \mathcal{T}_h} - \langle (\beta \cdot n - h^{-1} - \tau_1) y_h^0, z_h \rangle_{\partial \mathcal{T}_h \setminus \gamma_h^0} \\
+ (u_h, z_h)_{\mathcal{T}_h} - \langle (\beta \cdot n - h^{-1} - \tau_1) I_h g, z_h \rangle_{\partial \mathcal{T}_h \setminus \gamma_h^0} + (f, z_h)_{\mathcal{T}_h}, \\
+ \langle q_h, n + (h^{-1} + \tau_1) (y_h - \tilde{y}_h^0), \tilde{z}_h^0 \rangle_{\partial \mathcal{T}_h \setminus \gamma_h^0}.
\end{align}

Since the constraint PDE is linear and the cost functional is convex, the necessary and sufficient optimality conditions can be obtained by setting the partial Fréchet-derivatives of \((15)\) with respect to the flux \(q_h\), state \(y_h\), numerical trace \(\tilde{y}_h^0\) and control \(u_h\) equal to zero. Thus, we obtain the system consisting of the discrete adjoint equations
\begin{align}
\frac{\partial \mathcal{L}_h}{\partial q_h} r_2 &= (p_h, r_2)_{\mathcal{T}_h} + (\nabla z_h, r_2)_{\mathcal{T}_h} - (z_h, r_2 \cdot n)_{\partial \mathcal{T}_h} + (\tilde{z}_h^0, r_2 \cdot n)_{\partial \mathcal{T}_h \setminus \gamma_h^0} \\
&= (p_h, r_2)_{\mathcal{T}_h} - (z_h, \nabla \cdot r_2)_{\mathcal{T}_h} + (\tilde{z}_h^0, r_2 \cdot n)_{\partial \mathcal{T}_h \setminus \gamma_h^0} = 0,
\end{align}
\begin{align}
\frac{\partial \mathcal{L}_h}{\partial y_h} w_2 &= - (\nabla \cdot p_h, w_2)_{\mathcal{T}_h} + (\beta \nabla z_h, w_2)_{\mathcal{T}_h} + (z_h \nabla \cdot \beta, w_2)_{\mathcal{T}_h} \\
- ((h^{-1} + \tau_1) z_h, w_2)_{\partial \mathcal{T}_h} - ((h^{-1} + \tau_1) \tilde{z}_h^0, w_2)_{\partial \mathcal{T}_h \setminus \gamma_h^0} + (y_h - y_d, w_2)_{\mathcal{T}_h} \\
= (p_h - \beta z_h, \nabla w_2)_{\mathcal{T}_h} - (p_h \cdot n + (h^{-1} + \tau_1 - \beta \cdot n) z_h, w_2)_{\partial \mathcal{T}_h} \\
+ ((h^{-1} + \tau_1) \tilde{z}_h^0, w_2)_{\partial \mathcal{T}_h \setminus \gamma_h^0} + (y_h - y_d, w_2)_{\mathcal{T}_h} = 0,
\end{align}
\begin{align}
\frac{\partial \mathcal{L}_h}{\partial \tilde{y}_h^0} \mu_2 &= (p_h \cdot n - (\beta \cdot n - h^{-1} - \tau_1) z_h - (h^{-1} + \tau_1) \tilde{z}_h^0, \mu_2)_{\partial \mathcal{T}_h \setminus \gamma_h^0} = 0.
\end{align}

Furthermore, we obtain the same optimality condition \((10)\) as in the OD approach:
\begin{align}
\frac{\partial \mathcal{L}_h}{\partial u_h} w_3 &= (\gamma u_h + z_h, w_3)_{\mathcal{T}_h} = 0.
\end{align}

In the OD approach, if the stabilization functions \(\tau_1\) and \(\tau_2\) satisfy
\begin{align}
\tau_2 = \tau_1 - \beta \cdot n,
\end{align}
then by comparing the above discrete adjoint equations with \((10)\) we obtain identical discrete systems; therefore, the two approaches coincide in this case, i.e., OD = DO.
2.4 Implementation of DO

In the DO approach, we need to deal with a large optimization problem (11) and (12)-(14) since the EDG method generates three variables: the flux $q_h$, the scalar variable $y_h$, and the numerical trace $\tilde{y}_h$. Fortunately, we can reduce the large scale problem into a smaller problem using the local solver for the EDG method.

2.4.1 Matrix equations

Assume $V_h = \text{span}\{\varphi_i\}_{i=1}^{N_1}$, $W_h = \text{span}\{\phi_i\}_{i=1}^{N_2}$, and $\tilde{M}_h(o) = \text{span}\{\psi_i\}_{i=1}^{N_3}$. Then

$$
q_h = \sum_{j=1}^{N_1} \alpha_j \varphi_j, \quad y_h = \sum_{j=1}^{N_2} \beta_j \phi_j, \quad \tilde{y}_h = \sum_{j=1}^{N_3} \gamma_j \psi_j, \quad u_h = \sum_{j=1}^{N_2} \zeta_j \phi_j.
$$

(17)

Substitute (17) into (11)-(14) to give the following finite dimensional optimization problem:

$$
\min_{\zeta \in \mathbb{R}^{N_2}} \frac{1}{2} \beta^T A_6 \beta - b_1^T \beta + \frac{1}{2} \zeta^T A_6 \zeta
$$

(18a)

subject to

$$
\begin{bmatrix}
A_1 & -A_2 & A_3 & 0 \\
A_2^T & A_4 & A_5 & -A_6 \\
A_3^T & A_7 & -A_8 & 0
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
\gamma \\
\zeta
\end{bmatrix}
= \begin{bmatrix}
-b_2 \\
-b_3 - b_4 \\
0
\end{bmatrix},
$$

(18b)

where $\alpha, \beta, \gamma, \zeta$ are the coefficient vectors for $q_h, y_h, \tilde{y}_h, u_h$, respectively, and

$$
A_1 = [(\varphi_j, \varphi_i)_{\mathcal{T}_h}], \quad A_2 = [(\phi_j, \nabla \cdot \varphi_i)_{\mathcal{T}_h}], \quad A_3 = [(\psi_j, \varphi_i \cdot n)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h}],
$$

$$
A_4 = -[(\phi_j, \nabla \cdot (\beta \cdot \phi_i))_{\mathcal{T}_h}] + [(h^{-1} + \tau_1) \phi_j, \phi_i)_{\partial \mathcal{T}_h}],
$$

$$
A_5 = [(\beta \cdot n - h^{-1} - \tau_1) \psi_j, \phi_i)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h}], \quad A_6 = [(\phi_j, \phi_i)_{\mathcal{T}_h}],
$$

$$
A_7 = [(h^{-1} + \tau_1) \phi_j, \psi_i)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h}], \quad A_8 = [(h^{-1} + \tau_1) \psi_j, \psi_i)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h}],
$$

$$
b_1 = [(y_d, \phi_i)_{\mathcal{T}_h}], \quad b_2 = [(I_h g, r_1 \cdot n)_{\mathcal{E}_h}], \quad b_3 = [(f, \phi_i)_{\mathcal{T}_h}],
$$

$$
b_4 = [(\beta \cdot n - h^{-1} - \tau_1) g, \phi_i)_{\mathcal{E}_h}].
$$

Due to the discontinuous nature of the approximation spaces $V_h$ and $W_h$, the first two equations of (18b) can be used to eliminate both $\alpha$ and $\beta$ in an element-by-element fashion. As a consequence, we can write system (18b) as

$$
\begin{align*}
\alpha &= G_1 \gamma + G_2 \zeta + H_1, \\
\beta &= G_3 \gamma + G_4 \zeta + H_2, \\
G_5 \gamma + G_6 \zeta &= H_3.
\end{align*}
$$

(19)

We provide details on the element-by-element construction of the coefficient matrices $G_1, \ldots, G_6$ and $H_1, H_2, H_3$ in the appendix.

Substituting (19) into (18) gives the reduced optimization problem

$$
\min_{\zeta \in \mathbb{R}^{N_2}} \frac{1}{2} \left[ \gamma^T \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} \gamma \\ \zeta \end{bmatrix} + \begin{bmatrix} b_5^T \\ b_6^T \end{bmatrix} \begin{bmatrix} \gamma \\ \zeta \end{bmatrix} \right],
$$

(20a)
subject to
\[
\begin{bmatrix} G_5 & G_6 \end{bmatrix} \begin{bmatrix} \gamma \\ \zeta \end{bmatrix} = H_3.
\] (20b)

where
\[
B_1 = G_3^T A_6 G_3, \quad B_2 = G_3^T A_6 G_4, \quad B_3 = G_4^T A_6 G_3, \quad B_4 = G_4^T A_6 G_4 + A_6,
\]
\[
b_5 = G_3^T (A_6 H_2 - b_1), \quad b_6 = G_4^T (A_6 H_2 - b_1).
\]

Remark 1. In the DO approach, we need to solve the optimization problem (20); there are many existing optimization algorithms \cite{14} that can efficiently solve this problem.

3 Error Analysis

Next, we provide a convergence analysis of the above EDG method for the optimal control problem. Throughout this section, we assume \( \beta \in [W^{1,\infty}(\Omega)]^d \), \( \Omega \) is a bounded convex polyhedral domain, the solution is smooth enough, and \( h \leq 1 \).

3.1 Main result

For our theoretical results, we require the stabilization functions \( \tau_1 \) and \( \tau_2 \) are chosen to satisfy

(A1) \( \tau_2 = \tau_1 - \beta \cdot n \).

(A2) For any \( K \in T_h \), \( \min (\tau_1 - \frac{1}{2} \beta \cdot n)_{\partial K} > 0 \).

We note that (A1) and (A2) imply

\[
\min (\tau_2 + \frac{1}{2} \beta \cdot n)_{\partial K} > 0 \quad \text{for any } K \in T_h.
\] (21)

Furthermore, (A1) implies the OD and DO approaches yield equivalent results; therefore, all of our convergence analysis is for the OD approach.

Theorem 1. We have

\[
\left\| q - q_h \right\|_{T_h} \lesssim h^{k+1} (|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}),
\]
\[
\left\| p - p_h \right\|_{T_h} \lesssim h^{k+1} (|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}),
\]
\[
\left\| y - y_h \right\|_{T_h} \lesssim h^{k+2} (|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}),
\]
\[
\left\| z - z_h \right\|_{T_h} \lesssim h^{k+2} (|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}),
\]
\[
\left\| u - u_h \right\|_{T_h} \lesssim h^{k+2} (|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}).
\]

3.2 Preliminary material

Next, we introduce the standard \( L^2 \)-orthogonal projection operators \( \Pi_V \) and \( \Pi_W \) as follows:

\[
(\Pi_V q, r)_K = (q, r)_K \quad \forall r \in [P_k(K)]^d, \quad (22a)
\]
\[
(\Pi_W y, w)_K = (y, w)_K \quad \forall w \in P_{k+1}(K). \quad (22b)
\]
We use the following well-known bounds:

\[ \|q - \Pi_V q\|_{T_h} \leq C h^{k+1} \|q\|_{k+1,\Omega}, \quad \|y - \Pi_W y\|_{T_h} \leq C h^{k+2} \|y\|_{k+2,\Omega}, \]  
\[ \|y - \Pi_W y\|_{\partial T_h} \leq C h^{k+\frac{3}{2}} \|y\|_{k+2,\Omega}, \quad \|q - \Pi_V q\|_{\partial T_h} \leq C h^{k+\frac{3}{2}} \|q\|_{k+1,\Omega}, \]  
\[ \|y - I_h y\|_{\partial T_h} \leq C h^{k+\frac{3}{2}} \|y\|_{k+2,\Omega}, \quad \|w\|_{\partial T_h} \leq C h^{-\frac{1}{2}} \|w\|_{T_h}, \forall w \in W_h, \]  

where \( I_h \) is the continuous interpolation operator introduced earlier.

We define the following EDG operators \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \).

\[
\mathcal{B}_1(q_h, y_h, \tilde{y}_h; r_1, w_1, \mu_1) = (q_h, r_1)_{T_h} - (y_h, \nabla \cdot r_1)_{T_h} + \langle \tilde{y}_h, r_1 \cdot n \rangle_{\partial T_h \setminus \varepsilon_h^0} + \langle q_h + \beta y_h, \nabla w_1 \rangle_{T_h} - (\nabla \cdot \beta y_h, w_1)_{T_h} \]

\[ + \langle q_h \cdot n + (h^{-1} + \tau_1)y_h, w_1 \rangle_{\partial T_h} + ((\beta \cdot n - h^{-1} - \tau_1)\tilde{y}_h, w_1)_{\partial T_h \setminus \varepsilon_h^0} \]

\[ - \langle q_h \cdot n + \beta \cdot n \tilde{y}_h + (h^{-1} + \tau_1)(y_h - \tilde{y}_h), \mu_1 \rangle_{\partial T_h \setminus \varepsilon_h^0}, \]  

\[ \mathcal{B}_2(p_h, z_h, \tilde{z}_h; r_2, w_2, \mu_2) = (p_h, r_2)_{T_h} - (z_h, \nabla \cdot r_2)_{T_h} + \langle \tilde{z}_h, r_2 \cdot n \rangle_{\partial T_h \setminus \varepsilon_h^0} - (p_h - \beta z_h, \nabla w_2)_{T_h} \]

\[ + \langle p_h \cdot n + (h^{-1} + \tau_2)z_h, w_2 \rangle_{\partial T_h} - ((\beta \cdot n + h^{-1} + \tau_2)\tilde{z}_h, w_2)_{\partial T_h \setminus \varepsilon_h^0} \]

\[ - \langle p_h \cdot n - \beta \cdot n \tilde{z}_h + (h^{-1} + \tau_2)(z_h - \tilde{z}_h), \mu_2 \rangle_{\partial T_h \setminus \varepsilon_h^0}. \]  

By the definition of \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \), we can rewrite the EDG formulation of the optimality system (10) as follows: find \((q_h, p_h, y_h, z_h, u_h, \tilde{y}_h, \tilde{z}_h) \in V_h \times V_h \times W_h \times W_h \times \tilde{M}_h(o) \times \tilde{M}_h(o)\) such that

\[
\mathcal{B}_1(q_h, y_h, \tilde{y}_h; r_1, w_1, \mu_1) = (f + u_h, w_1)_{T_h} - \langle I_h g, (\beta \cdot n - \tau_1 - h^{-1})w_1 + r_1 \cdot n \rangle_{\varepsilon_h^0}, \]

\[ \mathcal{B}_2(p_h, z_h, \tilde{z}_h; r_2, w_2, \mu_2) = (y_h - y_d, w_2)_{T_h}, \]

\[ (z_h + \gamma u_h, w_3)_{T_h} = 0, \]

for all \((r_1, r_2, w_1, w_2, w_3, \mu_1, \mu_2) \in V_h \times V_h \times W_h \times W_h \times \tilde{M}_h(o) \times \tilde{M}_h(o)\).

Next, we present two fundamental properties of the operators \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \), and show the EDG equations (26) have a unique solution. The proofs of these results are similar to proofs in [17, 18] and are omitted. We note that condition (A1) is used in the proof of Lemma 2 which is fundamental to the error analysis in this work.

**Lemma 1.** For any \((v_h, w_h, \mu_h) \in V_h \times W_h \times \tilde{M}_h\), we have

\[
\mathcal{B}_1(v_h, w_h, \mu_h; v_h, w_h, \mu_h) = (v_h, v_h)_{T_h} + \langle (h^{-1} + \tau_1 - \frac{1}{2} \beta \cdot n)(w_h - \mu_h), w_h - \mu_h \rangle_{\partial T_h \setminus \varepsilon_h^0} \]

\[ - \frac{1}{2} \langle \nabla \cdot \beta w_h, w_h \rangle_{T_h} + \langle (h^{-1} + \tau_1 - \frac{1}{2} \beta \cdot n) w_h, \mu_h \rangle_{\varepsilon_h^0}, \]

\[
\mathcal{B}_2(v_h, w_h, \mu_h; v_h, w_h, \mu_h) = (v_h, v_h)_{T_h} + \langle (h^{-1} + \tau_2 + \frac{1}{2} \beta \cdot n)(w_h - \mu_h), w_h - \mu_h \rangle_{\partial T_h \setminus \varepsilon_h^0} \]

\[ - \frac{1}{2} \langle \nabla \cdot \beta w_h, w_h \rangle_{T_h} + \langle (h^{-1} + \tau_2 + \frac{1}{2} \beta \cdot n) w_h, \mu_h \rangle_{\varepsilon_h^0}. \]
Lemma 2. The EDG operators satisfy
\[ \mathcal{B}_1(q_h, y_h, \tilde{y}_h^o; p_h, -z_h, -\tilde{z}_h^o) + \mathcal{B}_2(p_h, z_h, \tilde{z}_h^o; -q_h, y_h, \tilde{y}_h^o) = 0. \]

Proposition 1. There exists a unique solution of the EDG equations \([26]\).

3.3 Proof of Main Result
To prove the convergence result, we split the proof into six steps. We first consider the following auxiliary problem: find
\[ (q_h(u), p_h(u), y_h(u), z_h(u), \tilde{y}_h^o(u), \tilde{z}_h^o(u)) \in V_h \times V_h \times W_h \times \tilde{W}_h(o) \times \tilde{W}_h(o) \]
such that
\[ \begin{align*}
\mathcal{B}_1(q_h(u), y_h(u), \tilde{y}_h^o(u); r_1, w_1, \mu_1) &= (f + u, w_1)_{T_h} - \langle I_h g, (\beta \cdot n - \tau_1 - h^{-1}) w_1 + r_1 \cdot n \rangle_{\epsilon_h^o}, \\
\mathcal{B}_2(p_h(u), z_h(u), \tilde{z}_h^o(u); r_2, w_2, \mu_2) &= (y_h(u) - y_d, w_2)_{T_h},
\end{align*} \tag{27a,b} \]
for all \((r_1, r_2, w_1, w_2, \mu_1, \mu_2) \in V_h \times V_h \times W_h \times \tilde{W}_h(o) \times \tilde{W}_h(o).

In Steps 1-3, we focus on the primary variables, i.e., the state \(y\) and the flux \(q\), and we use the following notation:
\[ \begin{align*}
\delta q &= q - \Pi_V q, & \varepsilon^q_h &= \Pi_V q - q_h(u), \\
\delta y &= y - \Pi_W y, & \varepsilon^y_h &= \Pi_W y - y_h(u), \\
\delta \tilde{y} &= y - I_h y, & \varepsilon^\tilde{y}_h &= I_h y - \tilde{y}_h(u), \\
\tilde{\delta}_1 &= \delta q \cdot n + \beta \cdot n \delta \tilde{y} + (\tau_1 + h^{-1})(\delta y - \delta \tilde{y}), & \varepsilon^\tilde{\delta}_1 &= \Pi_T \tilde{\delta}_1 + (\delta_1, \mu_1)_{\partial T_h \setminus \epsilon_h^o}.
\end{align*} \tag{28} \]
where \(\tilde{y}_h(u) = \tilde{y}_h^o(u)\) on \(\varepsilon^\tilde{y}_h\) and \(\tilde{y}_h(u) = I_h g\) on \(\varepsilon^\tilde{y}_h\), which implies \(\varepsilon^\tilde{\delta}_1 = 0\) on \(\varepsilon^\tilde{\delta}_1\).

3.3.1 Step 1: The error equation for part 1 of the auxiliary problem \([27a]\).

Lemma 3. We have the following error equation
\[ \mathcal{B}_1(\varepsilon^q_h, \varepsilon^y_h, \varepsilon^\tilde{y}_h; r_1, w_1, \mu_1) = -\langle \tilde{\delta}_1, r_1 \cdot n \rangle_{\partial T_h} + (\beta \delta^y, \nabla w_1)_{T_h} + (\nabla \cdot \beta \delta^y, w_1)_{T_h} - (\delta_1, w_1)_{\partial T_h} + (\delta_1, \mu_1)_{\partial T_h \setminus \epsilon_h^o}. \tag{29} \]

Proof. By definition of the operator \(\mathcal{B}_1\) in \([24]\), we have
\[ \begin{align*}
\mathcal{B}_1(\Pi_V q, \Pi_W y, \Pi_T y; r_1, w_1, \mu_1) &= (\Pi_V q, r_1)_{T_h} - (\Pi_W y, \nabla \cdot r_1)_{T_h} + \langle I_h y, r_1 \cdot n \rangle_{\partial T_h \setminus \epsilon_h^o} \\
&- (\Pi_V q + \beta \Pi_W y, \nabla w_1)_{T_h} - (\nabla \cdot \beta \Pi y, w_1)_{T_h} \\
&+ (\Pi_V q - \tau_1 + h^{-1}) \Pi W y, w_1)_{\partial T_h} + \langle (\beta \cdot n - \tau_1 - h^{-1}) I_h y, w_1 \rangle_{\partial T_h \setminus \epsilon_h^o} \\
&- (\Pi_V q - \beta \cdot n I_h y + (\tau_1 + h^{-1})(\Pi W y - I_h y), \mu_1)_{\partial T_h \setminus \epsilon_h^o}.
\end{align*} \]
Using properties of the $L^2$-orthogonal projection operators \( (22) \) gives
\[
\mathcal{B}_1(\Pi_V \mathbf{q}, \Pi_W y, I_h y; r_1, w_1, \mu_1)
= (q, r_1)_{\mathcal{T}_h} - (y, \nabla \cdot r_1)_{\mathcal{T}_h} + \langle y, r_1 \cdot n \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0} - \langle \delta \tilde{y}, r_1 \cdot n \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
- (q + \beta y, \nabla w_1)_{\mathcal{T}_h} + (\beta \delta y, \nabla w_1)_{\mathcal{T}_h} - (\nabla \cdot \beta y, w_1)_{\mathcal{T}_h} + (\nabla \cdot \delta \tilde{y}, w_1)_{\mathcal{T}_h}
+ (q \cdot n + (\tau_1 + h^{-1}) y, w_1)_{\partial \mathcal{T}_h} - \langle \delta q \cdot n \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
+ \langle (\beta \cdot n - \tau_1 - h^{-1}) y, w_1 \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0} - \langle (\beta \cdot n - \tau_1 - h^{-1}) \delta \tilde{y}, w_1 \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
- (q \cdot n + \beta \cdot n y, \mu_1)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
+ \langle \delta q \cdot n + \beta \cdot n \delta \tilde{y} \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}.
\]
Note that the exact solution \( \mathbf{q} \) and \( y \) satisfies
\[
(q, r_1)_{\mathcal{T}_h} - (y, \nabla \cdot r_1)_{\mathcal{T}_h} + \langle y, r_1 \cdot n \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0} = -(q, r_1 \cdot n)_{\mathcal{E}_h^0},
-(q + \beta y, \nabla w_1)_{\mathcal{T}_h} - (\nabla \cdot \beta y, w_1)_{\mathcal{T}_h} + \langle q \cdot n + \beta \cdot n y, w_1 \rangle_{\partial \mathcal{T}_h} = (f + u, w_1)_{\mathcal{T}_h},
-\langle q \cdot n + \beta \cdot n y, \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0} = 0,
\]
for all \((r_1, w_1, \mu_1) \in V_h \times W_h \times \widetilde{M}_h(o)\). Therefore, we have
\[
\mathcal{B}_1(\Pi_V \mathbf{q}, \Pi_W y, I_h y; r_1, w_1, \mu_1)
= -(g, r_1 \cdot n)_{\mathcal{E}_h^0} - \langle \delta \tilde{y}, r_1 \cdot n \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0} + (\beta \delta y, \nabla w_1)_{\mathcal{T}_h}
+ (\nabla \cdot \beta \delta y, w_1)_{\mathcal{T}_h} + (f + u, w_1)_{\mathcal{T}_h} - \langle \delta q \cdot n \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
+ \langle (\beta \cdot n - \tau_1 - h^{-1}) y, w_1 \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0} - \langle (\beta \cdot n - \tau_1 - h^{-1}) \delta \tilde{y}, w_1 \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}
+ \langle \delta q \cdot n + \beta \cdot n \delta \tilde{y} \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}.
\]
Finally, subtracting \((27a)\) from the above equation completes the proof. \( \square \)

### 3.3.2 Step 2: Estimate for \( \varepsilon_h^q \) by an energy argument.

First, we give an auxiliary result that is very similar to a result from [30]. The proof is also very similar, and is omitted.

**Lemma 4.** We have
\[
\| \nabla \varepsilon_h^q \|_{\mathcal{T}_h} \lesssim \| \varepsilon_h^q \|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \| \varepsilon_h^\tilde{y} \|_{\partial \mathcal{T}_h}.
\]
(30)

**Lemma 5.** We have
\[
\| \varepsilon_h^q \|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \| \varepsilon_h^\tilde{y} \|_{\partial \mathcal{T}_h} \lesssim h^{k+1}(\| \mathbf{q} \|_{k+1} + \| y \|_{k+2}).
\]
(31)

**Proof.** Taking \((r_1, w_1, \mu_1) = (\varepsilon_h^q, \varepsilon_h^q, \varepsilon_h^\tilde{y})\) in \((29)\) in Lemma 3 gives
\[
\mathcal{B}_1(\varepsilon_h^q, \varepsilon_h^q, \varepsilon_h^\tilde{y}; \varepsilon_h^q, \varepsilon_h^q, \varepsilon_h^\tilde{y})
= -(\delta \tilde{y}, \varepsilon_h^q \cdot n)_{\partial \mathcal{T}_h} + (\beta \delta y, \nabla \varepsilon_h^q)_{\mathcal{T}_h}
+ (\nabla \cdot \beta \delta y, w_1)_{\mathcal{T}_h} - \langle \delta \tilde{y}, \varepsilon_h^q \cdot n \rangle_{\partial \mathcal{T}_h}
= T_1 + T_2 + T_3 + T_4,
\]
where we used $\varepsilon_h^2 = 0$ on $\varepsilon_h^0$. We estimate $T_i$, for $i = 1, 2, 3, 4$, as follows. First,

$$T_1 \leq Ch^{-1}\|\hat{\delta}_y\|^2_{\partial T_h} + \frac{1}{4}\|\varepsilon_h^q\|^2_{T_h},$$

where we used trace and inverse inequalities. For the second term $T_2$, by Lemma 4 we have

$$T_2 \leq C\|\delta^y\|^2_{T_h} + \frac{1}{4h}\|\varepsilon_h^q\|^2_{T_h} + \frac{1}{4h}\|\varepsilon_h^y - \hat{\delta}_y\|^2_{\partial T_h}.$$

For the third term $T_3$, we have

$$T_3 \leq C\|\delta^y\|^2_{T_h} + \frac{1}{2}\|(-\nabla \cdot \beta)\frac{1}{2}\varepsilon_h^y\|^2_{T_h}.$$

For the last term $T_4$,

$$T_4 \leq Ch\|\hat{\delta}_1\|^2_{\partial T_h} + \frac{1}{4h}\|\varepsilon_h^y - \hat{\delta}_y\|_{\partial T_h}.$$

Sum all the estimates for $\{T_i\}_{i=1}^4$ to obtain

$$\|\varepsilon_h^q\|^2_{T_h} + h^{-1}\|\varepsilon_h^y - \hat{\delta}_y\|^2_{\partial T_h} \leq h^{-1}\|\hat{\delta}_y\|^2_{\partial T_h} + \|\delta^y\|^2_{T_h} + h\|\hat{\delta}_1\|^2_{\partial T_h} \leq h^{2(k+1)}(|q|_{k+1}^2 + |y|_{k+2}^2).$$

\[\square\]

### 3.3.3 Step 3: Estimate for $\varepsilon_h^y$ by a duality argument.

Next, we introduce the dual problem for any given $\Theta$ in $L^2(\Omega)$:

$$\begin{align*}
\Phi + \nabla \Psi &= 0 \quad \text{in} \ \Omega, \\
\nabla \cdot (\Phi - \beta \Psi) &= \Theta \quad \text{in} \ \Omega, \\
\Psi &= 0 \quad \text{on} \ \partial \Omega.
\end{align*}$$

Since the domain $\Omega$ is convex, we have the following regularity estimate

$$\|\Phi\|_{1,\Omega} + \|\Psi\|_{2,\Omega} \leq C_{reg} \|\Theta\|_{\Omega}.$$  

(33)

We use the following notation below:

$$\hat{\delta} = \Phi - \Pi_V \Phi, \quad \hat{\delta}^y = \Psi - \Pi_W \Psi, \quad \hat{\delta}_y = \Psi - \mathcal{I}_h \Psi.$$  

(34)

Lemma 6. We have

$$\|\varepsilon_h^y\|_{T_h} \lesssim h^{k+2}(|q|_{k+1} + |y|_{k+2}).$$  

(35)

Proof. First we take $(r_1, w_1, \mu_1) = (\Pi_V \Phi, -\Pi_W \Psi, -\mathcal{I}_h \Psi)$ in equation [29] to get

$$\begin{align*}
\mathcal{B}_1(\varepsilon_h^q, \varepsilon_h^y, \Pi_V \Phi, -\Pi_W \Psi, -\mathcal{I}_h \Psi) &= (\varepsilon_h^q, \Pi_V \Phi)_{T_h} - (\varepsilon_h^y, \nabla \cdot \Pi_V \Phi)_{T_h} + (\varepsilon_h^y, \Pi_V \Phi \cdot \mathcal{n})_{\partial T_h \setminus \varepsilon_h^h} \\
&\quad + (\varepsilon_h^q + \beta \varepsilon_h^y, \nabla \Pi_W \Psi)_{T_h} + (\nabla \cdot \beta \varepsilon_h^y, \Pi_W \Psi)_{T_h} \\
&\quad - (\beta \mathcal{n} - h^{-1} + \tau_1)\varepsilon_h^y, \Pi_W \Psi)_{\partial T_h} \\
&\quad - ((\beta \cdot \mathcal{n} - h^{-1} + \tau_1)\varepsilon_h^y, \Pi_W \Psi)_{\partial T_h \setminus \varepsilon_h^h} \\
&\quad + (\varepsilon_h^q \cdot \mathcal{n} + \beta \mathcal{n} \varepsilon_h^y + (h^{-1} + \tau_1)(\varepsilon_h^y - \hat{\delta}_y), \mathcal{I}_h \Psi)_{\partial T_h \setminus \varepsilon_h^h}.
\end{align*}$$
Moreover, we have

\[-(\varepsilon_h^y, \nabla \cdot \Pi_V \Phi)_{\partial \Omega_h} = (\nabla \varepsilon_h^y, \Phi)_{\partial \Omega_h} - (\varepsilon_h^y, \Pi_V \Phi \cdot n)_{\partial \Omega_h} \]
\[= -(\varepsilon_h^y, \nabla \cdot \Phi)_{\partial \Omega_h} + (\varepsilon_h^y, \delta \Phi \cdot n)_{\partial \Omega_h}, \]
\[(\varepsilon_h^q, \nabla \Pi_W \Psi)_{\partial \Omega_h} = - (\nabla \cdot \varepsilon_h^q, \Psi)_{\partial \Omega_h} + (\varepsilon_h^q, \nabla \cdot (\Pi_W \Psi))_{\partial \Omega_h} \]
\[= (\varepsilon_h^q, \nabla \Psi)_{\partial \Omega_h} - (\varepsilon_h^q, \nabla \Phi)_{\partial \Omega_h}, \]
\[(\beta \varepsilon_h^y, \nabla \Pi_W \Psi)_{\partial \Omega_h} + (\nabla \cdot \beta \varepsilon_h^y, \Pi_W \Psi)_{\partial \Omega_h} = (\varepsilon_h^y, \nabla \cdot (\beta \Pi_W \Psi))_{\partial \Omega_h} \]
\[= (\varepsilon_h^y, \nabla \cdot (\beta \Psi))_{\partial \Omega_h} - (\varepsilon_h^y, \nabla \cdot (\beta \delta \Phi))_{\partial \Omega_h} \]
\[= (\varepsilon_h^y, \nabla \cdot (\beta \Psi))_{\partial \Omega_h} + (\beta \cdot (\nabla \varepsilon_h^y), \delta \Psi)_{\partial \Omega_h} \]
\[= (\beta \cdot \nabla \varepsilon_h^y, \delta \Psi)_{\partial \Omega_h}. \]

Together with the dual problem \cite{22}, using \( \Theta = -\varepsilon_h^y \), we have

\[
B(\varepsilon_h^q, \varepsilon_h^y, \varepsilon_h^q; \Pi_V \Phi, -\Pi_W \Psi, -I_h \Psi) \]
\[= (\varepsilon_h^q, \Phi)_{\partial \Omega_h} - (\varepsilon_h^y, \nabla \cdot \Phi)_{\partial \Omega_h} + (\varepsilon_h^y, \delta \Phi \cdot n)_{\partial \Omega_h} \]
\[+ (\varepsilon_h^q, \nabla \Psi)_{\partial \Omega_h} - (\varepsilon_h^q, \nabla \Phi)_{\partial \Omega_h} + (\varepsilon_h^y, \nabla \cdot (\beta \Psi))_{\partial \Omega_h} \]
\[+ (\beta \cdot (\nabla \varepsilon_h^y), \delta \Psi)_{\partial \Omega_h} - (\beta \cdot \nabla \varepsilon_h^y, \delta \Psi)_{\partial \Omega_h} \]
\[= (\varepsilon_h^q, \varepsilon_h^y)_{\partial \Omega_h} + (\varepsilon_h^y - \varepsilon_h^q, \delta \Phi \cdot n)_{\partial \Omega_h} - (\varepsilon_h^q, \nabla \Phi)_{\partial \Omega_h} \]
\[+ (\beta \cdot (\nabla \varepsilon_h^y), \delta \Psi)_{\partial \Omega_h} - (\beta \cdot n \varepsilon_h^y, \delta \Psi)_{\partial \Omega_h} \]
\[+ ((\tau_1 + h^{-1})(\varepsilon_h^y - \varepsilon_h^q) + (\beta \cdot \nabla \varepsilon_h^y, \delta \Psi)_{\partial \Omega_h} \]
\[+ (\beta \cdot (\nabla \varepsilon_h^y), \delta \Psi)_{\partial \Omega_h}. \]

Here, we used that \( (\varepsilon_h^q, \Phi \cdot n)_{\partial \Omega_h} = 0, \Psi = \varepsilon_h^q = 0 \) on \( \varepsilon_h^q \), and

\[ (\beta \cdot n \varepsilon_h^y, \delta \Psi)_{\partial \Omega_h} = 0, \]

since \( \varepsilon_h^q \) is single-valued on interior faces and \( \varepsilon_h^q = 0 \) on boundary faces. On the other hand, from equation \cite{29},

\[
B(\varepsilon_h^q, \varepsilon_h^y, \varepsilon_h^q; \Pi_V \Phi, -\Pi_W \Psi, -I_h \Psi) \]
\[= -(\delta \Psi, (\Pi_V \Phi \cdot \nabla \Pi_W \Psi)_{\partial \Omega_h} - (\nabla \cdot \beta \Psi, \nabla \Pi_W \Psi)_{\partial \Omega_h} \]
\[+ (\delta \Psi, \Pi_W \Psi)_{\partial \Omega_h} \]
\[= : \sum_{i=1}^9 T_i. \]
We estimate each terms separately. For the first term

\[ T_1 \leq \|\delta^n\|_{\partial \Gamma_h} \|\delta^\Phi\|_{\partial \Gamma_h} \lesssim h^{\frac{1}{2}} \|\delta^n\|_{\partial \Gamma_h} \|\Phi\|_{1,\Omega} \lesssim h^{\frac{1}{2}} \|\delta^n\|_{\partial \Gamma_h} \|\varepsilon^n\|_{\Gamma_h}. \]

For the second term,

\[ T_2 \lesssim h^{\frac{3}{2}} \|\delta_1\|_{\partial \Gamma_h} \|\Psi\|_{2,\Omega} \lesssim h^{\frac{3}{2}} \|\delta_1\|_{\partial \Gamma_h} \|\varepsilon^n\|_{\Gamma_h}. \]

For the third term \( T_3 \),

\[ T_3 \lesssim \|\beta\|_{0,\infty,\Omega} \|\delta^n\|_{\Gamma_h} (\|\nabla \delta^n\|_{\Gamma_h} + \|\nabla \Psi\|_{\Omega}) \lesssim \|\delta^n\|_{\Gamma_h} (\|\Psi\|_{2,\Omega} + \|\Psi\|_{1,\Omega}) \lesssim \|\delta^n\|_{\Gamma_h} \|\varepsilon^n\|_{\Gamma_h}. \]

For \( T_4 \),

\[ T_4 \lesssim \|\beta\|_{1,\infty,\Omega} \|\delta^n\|_{\Gamma_h} \|\Pi W \Psi\|_{\Gamma_h} \lesssim \|\delta^n\|_{\Gamma_h} \|\varepsilon^n\|_{\Gamma_h}. \]

For \( T_5 \),

\[ T_5 \lesssim \|\varepsilon^n - \varepsilon^n_{\partial \Gamma_h} \|_{\partial \Gamma_h} \|\delta^\Phi\|_{\partial \Gamma_h} \lesssim h^{\frac{1}{2}} \|\varepsilon^n - \varepsilon^n_{\partial \Gamma_h} \|_{\partial \Gamma_h} \|\Phi\|_{1,\Omega} \lesssim h^{\frac{1}{2}} \|\varepsilon^n - \varepsilon^n_{\partial \Gamma_h} \|_{\partial \Gamma_h} \|\varepsilon^n\|_{\Gamma_h}. \]

For \( T_6, T_7, \) and \( T_9 \), following the same idea for \( T_5 \), we have

\[ T_6 \lesssim h \|\varepsilon^n_{\partial \Gamma_h} \|_{\Gamma_h} \|\varepsilon^n\|_{\Gamma_h}, \]
\[ T_7 \lesssim h^{\frac{3}{2}} \|\varepsilon^n_{\partial \Gamma_h} \|_{\partial \Gamma_h} \|\varepsilon^n\|_{\Gamma_h}, \]
\[ T_9 \lesssim \|\beta\|_{0,\infty,\Omega} h^{\frac{3}{2}} \|\varepsilon^n - \varepsilon^n_{\partial \Gamma_h} \|_{\partial \Gamma_h} \|\varepsilon^n\|_{\Gamma_h}. \]

And by Lemma 4 we have

\[ T_8 \lesssim \|\beta\|_{0,\infty,\Omega} h \|\nabla \varepsilon^n\|_{\Gamma_h} \|\Psi\|_{1} \lesssim h (\|\varepsilon^n\|_{\Gamma_h} + h^{-\frac{1}{2}} \|\varepsilon^n_{\partial \Gamma_h} \|_{\partial \Gamma_h}) \|\varepsilon^n\|_{\Gamma_h}. \]

Therefore, summing the estimates and using the bounds (23) and Lemma 5 gives the result.

The triangle inequality yields optimal convergence rates for \( \|q - q_h(u)\|_{\Gamma_h} \) and \( \|y - y_h(u)\|_{\Gamma_h} \):

**Lemma 7.** We have

\begin{align*}
\|q - q_h(u)\|_{\Gamma_h} &\leq \|\delta^n\|_{\Gamma_h} + \|\varepsilon^n_{\partial \Gamma_h} \|_{\Gamma_h} \lesssim h^{k+1}(\|q\|_{k+1} + |y|_{k+2}), \quad (36a) \\
\|y - y_h(u)\|_{\Gamma_h} &\leq \|\delta^n\|_{\Gamma_h} + \|\varepsilon^n_{\partial \Gamma_h} \|_{\Gamma_h} \lesssim h^{k+2}(\|q\|_{k+1} + |y|_{k+2}). \quad (36b)
\end{align*}
3.3.4 Step 4: The error equation for part 2 of the auxiliary problem (27b).

Next, we bound the error between the solution of the dual convection diffusion equation (3c)-(3d) for \( z \) and the auxiliary HDG equation (27b). 

First we define
\[
\delta^p = p - \Pi_V p, \quad \delta^2 = z - \Pi_W z, \quad \hat{\delta} = z - I_h z, \quad \hat{\delta}^2 = z - I_h z - \hat{z}_h(u),
\]
(37)
where \( \hat{z}_h(u) = \hat{z}_h(u) \) on \( \varepsilon_h^\theta \) and \( \hat{z}_h(u) = 0 \) on \( \varepsilon_h^\theta \). This gives \( \hat{\varepsilon}_h^\theta = 0 \) on \( \varepsilon_h^\theta \).

Following the same idea with Lemma 3, we have the following error equation:

**Lemma 8.** We have
\[
\mathcal{B}_2(\varepsilon_h^p, \varepsilon_h^\theta, \varepsilon_h^\hat{\delta}; r_2, w_2, \mu_2) \\
= -\langle \delta^2, r_2 \cdot n \rangle_{\partial T_h} - (\beta \delta^2, \nabla w_2)_{T_h} \\
- (\hat{\delta}_2, w_2)_{\partial T_h} + (\hat{\delta}_2, \mu_2)_{\partial T_h} + (y_h(u) - y, w_2)_{T_h},
\]
(38)

3.3.5 Step 5: Estimates for \( \varepsilon_h^\theta \) and \( \varepsilon_h^\hat{\delta} \) by an energy and duality argument.

First, it is easy to see that Lemma 4 still holds for \( \varepsilon_h^\theta, \varepsilon_h^\hat{\delta} \), and \( \varepsilon_h^p \).

**Lemma 9.** We have
\[
\| \nabla \varepsilon_h^\theta \|_{T_h} \leq C(\| \varepsilon_h^\theta \|_{T_h} + h^{-\frac{1}{2}}\| \varepsilon_h^\theta - \varepsilon_h^\hat{\delta} \|_{\partial T_h}).
\]
(39)

Also, to estimate \( \varepsilon_h^p \) we need the following discrete Poincaré inequality that is very similar to a result from [30]. The proof is essentially the same, and is omitted.

**Lemma 10.** We have
\[
\| \varepsilon_h^\theta \|_{T_h} \leq C(\| \nabla \varepsilon_h^\theta \|_{T_h} + h^{-\frac{1}{2}}\| \varepsilon_h^\theta - \varepsilon_h^\hat{\delta} \|_{\partial T_h}).
\]
(40)

**Lemma 11.** We have
\[
\| \varepsilon_h^p \|_{T_h} + h^{-\frac{1}{2}}\| \varepsilon_h^\theta - \varepsilon_h^\hat{\delta} \|_{\partial T_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}),
\]
(41)
\[
\| \varepsilon_h^\theta \|_{T_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}).
\]
(42)

**Proof.** Since \( \varepsilon_h^\hat{\delta} = 0 \) on \( \varepsilon_h^\theta \), the energy identity for \( \mathcal{B}_2 \) in Lemma 1 gives
\[
\mathcal{B}_2(\varepsilon_h^p, \varepsilon_h^\theta, \varepsilon_h^\hat{\delta}; \varepsilon_h^\hat{\delta}, \varepsilon_h^\hat{\delta}) \\
= \| \varepsilon_h^p \|_{T_h}^2 + \| (h^{-1} + \tau_2 + \frac{1}{2}\beta \cdot n) \varepsilon_h^\theta \|_{\partial T_h}^2 + \frac{1}{2}\| (-\nabla \beta) \varepsilon_h^\theta \|_{T_h}^2.
\]

Take \( (r_2, w_2, \mu_2) = (\varepsilon_h^p, \varepsilon_h^\theta, \varepsilon_h^\hat{\delta}) \) in the error equation (38) to obtain
\[
\| \varepsilon_h^p \|_{T_h}^2 + \| (h^{-1} + \tau_2 + \frac{1}{2}\beta \cdot n) \varepsilon_h^\theta \|_{\partial T_h}^2 + \frac{1}{2}\| (-\nabla \beta) \varepsilon_h^\theta \|_{T_h}^2 \\
= -\langle \delta^2, \varepsilon_h^\theta \rangle_{\partial T_h} - (\beta \delta^2, \nabla \varepsilon_h^\theta)_{T_h} \\
- (\hat{\delta}_2, \varepsilon_h^\theta - \varepsilon_h^\hat{\delta})_{\partial T_h} + (y_h(u) - y, \varepsilon_h^\hat{\delta})_{T_h} \\
=: T_1 + T_2 + T_3 + T_4.
\]
By the same argument as in the proof of [Lemma 5] apply (39) and (40) to get

\[ T_1 \lesssim h^{-\frac{1}{2}} \| \delta^\circ \| \sigma_{T_h} \| \varepsilon_P \|_{T_h}, \]
\[ T_2 \lesssim \| \beta \|_{0, \infty, \Omega} \| \delta^\circ \|_{T_h} \| \nabla \varepsilon_h \|_{T_h}, \]
\[ T_3 \lesssim \| \beta \|_{0, \infty, \Omega} \| \delta^\circ \|_{T_h} \| \varepsilon_P \|_{T_h} + h^{-\frac{1}{2}} \| \varepsilon_h^\circ - \varepsilon_h \|_{\sigma_{T_h}}, \]
\[ T_4 \lesssim \| y - y_h(u) \|_{T_h} \| \varepsilon_h \|_{T_h} \]
\[ \lesssim \| y - y_h(u) \|_{T_h} (\| \varepsilon_P \|_{T_h} + h^{-\frac{1}{2}} \| \varepsilon_h^\circ - \varepsilon_h \|_{\sigma_{T_h}}). \]

Finally, applying (23) and [Lemma 7] yields (41). Together with (41) and (40), we can obtain (42). □

3.3.6 Step 6: Estimate for $\varepsilon_h^\circ$ by a duality argument.

Next, we introduce the dual problem for any given $\Theta$ in $L^2(\Omega)$:

\[ \Phi + \nabla \Psi = 0 \quad \text{in } \Omega, \]
\[ \nabla \cdot \Phi - \beta \cdot \nabla \Psi = \Theta \quad \text{in } \Omega, \]
\[ \Psi = 0 \quad \text{on } \partial \Omega. \]  \hspace{1cm} (43)

Since the domain $\Omega$ is convex, we have the following regularity estimate

\[ \| \Phi \|_{1, \Omega} + \| \Psi \|_{2, \Omega} \leq C_{\text{reg}} \| \Theta \|_{\Omega}. \]  \hspace{1cm} (44)

Lemma 12. We have

\[ \| \varepsilon_h^\circ \|_{T_h} \lesssim h^{k+2}(\| q \|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}). \]  \hspace{1cm} (45a)

Proof. Consider the dual problem (43), and let $\Theta = \varepsilon_h^\circ$. Take $(r_2, w_2, \mu_2) = (\Pi_V \Phi, -\Pi_W \Psi, -I_h \Psi)$ in (38) in Lemma 8. Since $\Psi = 0$ on $\varepsilon_h^\circ$ we have

\[ \mathcal{B}_2(\varepsilon_P, \varepsilon_h^\circ, \varepsilon_h^\circ; \Pi_V \Phi, -\Pi_W \Psi, -I_h \Psi) \]
\[ = (\varepsilon_P^\circ, \Pi_V \Phi)_{T_h} - (\varepsilon_h^\circ, \nabla \cdot \Pi_V \Phi)_{T_h} + (\varepsilon_h^\circ, \Pi_V \Phi \cdot n)_{\sigma_{T_h}} \]
\[ + (\varepsilon_h^\circ - \beta \varepsilon_h^\circ, \nabla \Pi_W \Psi)_{T_h} - (\varepsilon_h^\circ n - \beta \cdot n \varepsilon_h^\circ + \tau_2(\varepsilon_h^\circ + \delta^\circ) - I_h \Psi)_{\sigma_{T_h}}. \]

Moreover, we have

\[ - (\varepsilon_h^\circ, \nabla \cdot \Pi_V \Phi)_{\sigma_{T_h}} = (\nabla \varepsilon_h^\circ, \Phi)_{T_h} - (\varepsilon_h^\circ, \Pi_V \Phi \cdot n)_{\sigma_{T_h}} \]
\[ = - (\varepsilon_h^\circ, \nabla \Phi)_{T_h} + (\varepsilon_h^\circ, \delta^\circ \cdot n)_{\sigma_{T_h}}. \]

\[ (\varepsilon_h^\circ, \nabla \Pi_W \Psi)_{T_h} = - (\nabla \cdot \varepsilon_h^\circ, \Psi)_{T_h} + (\varepsilon_h^\circ n, \Pi_W \Psi)_{\sigma_{T_h}} \]
\[ = (\varepsilon_h^\circ, \nabla \Psi)_{T_h} - (\varepsilon_h^\circ n, \delta^\circ \Psi)_{\sigma_{T_h}}. \]

\[ - (\beta \varepsilon_h^\circ, \nabla \Pi_W \Psi)_{T_h} = - (\beta \varepsilon_h^\circ, \nabla \delta^\circ)_{T_h} + (\beta \varepsilon_h^\circ, \nabla \Psi)_{T_h} \]
\[ = - (\beta \cdot \nabla \varepsilon_h^\circ, \delta^\circ)_{\sigma_{T_h}} + (\nabla \cdot \beta \varepsilon_h^\circ, \delta^\circ \Psi)_{T_h} \]
\[ + (\beta \cdot \nabla \varepsilon_h^\circ, \delta^\circ \Psi)_{T_h} + (\beta \varepsilon_h^\circ, \nabla \Psi)_{T_h}. \]
Then we have

$$
\mathcal{B}_2(\varepsilon^p_h, \varepsilon^\xi_h, \varepsilon^\xi_{\nabla}; \Pi_V \Phi, -\Pi_W \Psi, -I_h \Psi)
= (\varepsilon^p_h, \Phi)_{\partial T_h} - (\varepsilon^\xi_h, \nabla \cdot \Phi)_{\partial T_h} + (\delta^\Phi \cdot \nabla \Phi)_{\partial T_h} + (\varepsilon^\xi_h, \Phi \cdot \nabla \Phi)_{\partial T_h}
+ (\varepsilon^p_h, \Psi)_{\partial T_h} - (\varepsilon^\xi_h, \delta^\Phi)_{\partial T_h} - (\beta \cdot \nabla \varepsilon^\xi_h, \delta^\Phi)_{\partial T_h}
+ (\varepsilon^\xi_h, \Phi)_{\partial T_h} + (\beta \cdot \nabla \varepsilon^\xi_h, \delta^\Phi)_{\partial T_h}
+ (\beta \cdot \nabla \varepsilon^\xi_h, \delta^\Phi)_{\partial T_h} + \langle \tau_2 (\varepsilon^\xi_h - \varepsilon^\xi_{\nabla}), \delta^\Phi - \delta^\xi \rangle_{\partial T_h}
= (\varepsilon^\xi_h, \varepsilon^\xi_{\nabla})_{\partial T_h} + \langle \varepsilon^\xi_h - \varepsilon^\xi_{\nabla}, \delta^\Phi \cdot n \rangle_{\partial T_h} + (\varepsilon^p_h, \nabla \Phi)_{\partial T_h} + (\beta \cdot \nabla \varepsilon^\xi_h, \delta^\Phi)_{\partial T_h}
+ (\beta \cdot \nabla \varepsilon^\xi_h, \delta^\Phi)_{\partial T_h} + \langle (\tau_2 + h^{-1}) (\varepsilon^\xi_h - \varepsilon^\xi_{\nabla}), \delta^\Phi - \delta^\xi \rangle_{\partial T_h}.
$$

Here, we used $\langle \varepsilon^\xi_h, \Phi \cdot n \rangle_{\partial T_h} = 0$, which holds since $\varepsilon^\xi_h$ is single-valued function on interior edges and $\varepsilon^\xi_{\nabla} = 0$ on $\partial \Omega$. We also used $\langle \beta \cdot \nabla \varepsilon^\xi_h, \delta^\Phi \rangle_{\partial T_h} = 0$, which is derived similarly.

On the other hand, by Lemma 8

$$
\mathcal{B}_2(\varepsilon^p_h, \varepsilon^\xi_h, \varepsilon^\xi_{\nabla}; \Pi_V \Phi, -\Pi_W \Psi, -I_h \Psi)
= -(\delta^\xi, \Pi_V \Phi \cdot n)_{\partial T_h} + (\beta \delta^\xi, \nabla \Pi_V \Psi)_{\partial T_h}
+ (\delta^\xi, \Pi_V \Phi - I_h \Psi)_{\partial T_h} - (y_h(u) - y, \Pi_V \Psi)_{\partial T_h}.
$$

Comparing the above two equalities gives

$$
||\varepsilon^\xi_h||^2_{T_h} = -\langle \varepsilon^\xi_h - \varepsilon^\xi_{\nabla}, \delta^\Phi \cdot n + (\tau_2 + h^{-1}) (\delta^\Phi - \delta^\xi) - \beta \cdot n \delta^\Phi \rangle_{\partial T_h}
+ \langle \varepsilon^p_h \cdot \nabla \Phi \rangle_{\partial T_h} + (\beta \cdot \nabla \varepsilon^\xi_h, \delta^\Phi)_{\partial T_h} + (\nabla \cdot \beta \varepsilon^\xi_h, \delta^\Phi)_{\partial T_h}
- \langle \delta^\xi, \delta^\Phi \cdot n \rangle_{\partial T_h} + (\beta \delta^\xi, \nabla \Pi_V \Psi)_{\partial T_h}
+ \langle \delta^\xi, \Pi_V \Phi - I_h \Psi \rangle_{\partial T_h} - (y_h(u) - y, \Pi_V \Psi)_{\partial T_h}
= \sum_{i=1}^{8} R_i.
$$

For the terms $R_1$-$R_4$, Lemma 11 gives

$$
R_1 = -\langle \varepsilon^\xi_h - \varepsilon^\xi_{\nabla}, \delta^\Phi \cdot n + (\tau_2 + h^{-1}) (\delta^\Phi - \delta^\xi) \rangle_{\partial T_h}
\lesssim h^{\frac{\frac{3}{2}}{2}} \| (\tau_2 + h^{-1} + \beta \cdot n) \frac{3}{2} (\varepsilon^\xi_h - \varepsilon^\xi_{\nabla}) \|_{\partial T_h} (\| \Phi \|_{1, \Omega} + \| \Psi \|_{1, \Omega})
\lesssim h^{\frac{\frac{3}{2}}{2}} \| (\tau_2 + h^{-1} + \beta \cdot n) \frac{3}{2} (\varepsilon^\xi_h - \varepsilon^\xi_{\nabla}) \|_{\partial T_h} \varepsilon^\xi_h \|_{T_h},
$$

$$
R_2 \lesssim h^{\frac{\frac{3}{2}}{2}} \| \varepsilon^p_h \|_{\partial T_h} \| \Psi \|_{2, \Omega} \lesssim h^{\frac{\frac{3}{2}}{2}} \| \varepsilon^p_h \|_{T_h} \varepsilon^\xi_h \|_{T_h},
\lesssim h^{\frac{\frac{3}{2}}{2}} \| \Psi \|_{1, \Omega},
R_3 \lesssim h \| \beta \|_{0, \infty, \Omega} h^{\frac{3}{2}} \| \varepsilon^\xi_h \|_{T_h} \| \Psi \|_{1, \Omega},
R_4 \lesssim h \| (-\nabla \cdot \beta) \frac{3}{2} \varepsilon^\xi_h \|_{T_h} \| \Psi \|_{1, \Omega} \lesssim h \| (-\nabla \cdot \beta) \frac{3}{2} \varepsilon^\xi_h \|_{T_h} \varepsilon^\xi_h \|_{T_h}.
$$

For $R_5$, we have

$$
R_5 \lesssim h^{\frac{\frac{3}{2}}{2}} \| \delta^\xi \|_{\partial T_h} \varepsilon^\xi_h \|_{T_h}.
$$
For the terms \( R_6 \) and \( R_8 \), we use the triangle inequality, the regularity estimate (33), and the assumption \( h \leq 1 \) to give

\[
R_6 \lesssim \| \beta \|_{0, \infty, \Omega} \| \delta \|_{\tau_h} (\| \nabla \delta \|_{\tau_h} + \| \Psi \|_{\tau_h}) \lesssim \| \beta \|_{0, \infty, \Omega} \| \delta \|_{\tau_h} \| \varepsilon_h \|_{\tau_h},
\]
\[
R_8 \lesssim \| y_h(u) - y \|_{\tau_h} \| \varepsilon_h \|_{\tau_h}.
\]

For the term \( R_7 \),

\[
R_7 \lesssim h^2 \| \delta p \cdot n \| + (\tau_1 + h^{-1})(\| \delta \cdot \|_{\partial \tau_h} \| \Psi \|_{2, \Omega}) \lesssim h^2 (\| \delta p \|_{\partial \tau_h} + \| \delta \|_{\tau_h} + \| \delta \cdot \|_{\partial \tau_h}) \| \varepsilon_h \|_{\tau_h}.
\]

Summing \( R_1 \) to \( R_8 \), together with (23), (39), (41), and (42) gives

\[
\| \varepsilon_h \|_{\tau_h} \lesssim h^{k+2} (\| q \|_{k+1} + \| y \|_{k+2} + \| p \|_{k+1} + \| z \|_{k+2}).
\]

The triangle inequality gives optimal convergence rates for \( \| p - p_h(u) \|_{\tau_h} \) and \( \| z - z_h(u) \|_{\tau_h} \):

**Lemma 13.**

\[
\| p - p_h(u) \|_{\tau_h} \leq \| \delta p \|_{\tau_h} + \| p \|_{\tau_h}
\]
\[
\lesssim h^{k+1} (\| q \|_{k+1} + \| y \|_{k+2} + \| p \|_{k+1} + \| z \|_{k+2}),
\]

\[
\| z - z_h(u) \|_{\tau_h} \leq \| \delta z \|_{\tau_h} + \| z \|_{\tau_h}
\]
\[
\lesssim h^{k+2} (\| q \|_{k+1} + \| y \|_{k+2} + \| p \|_{k+1} + \| z \|_{k+2}).
\]

**3.3.7 Step 7: Estimates for \( u - u_h \|_{\tau_h}, \| y - y_h \|_{\tau_h}, \text{ and } \| z - z_h \|_{\tau_h}.**

Next, we bound the error between the solutions of the auxiliary problem and the EDG discretization of the optimality system (36). We use these error bounds and the error bounds in Lemma 7 and Lemma 13 to obtain the main result.

The proofs in Steps 7 and 8 are similar to proofs in our earlier work [18]; we include the proofs here to make the final steps self-contained.

For the remaining steps, we denote

\[
\zeta_q = q_h(u) - q_h, \quad \zeta_y = y_h(u) - y_h, \quad \zeta_{\bar{y}} = \bar{y}_h(u) - \bar{y}_h,
\]
\[
\zeta_p = p_h(u) - p_h, \quad \zeta_z = z_h(u) - z_h, \quad \zeta_{\bar{z}} = \bar{z}_h(u) - \bar{z}_h.
\]

Subtracting the auxiliary problem and the EDG problem gives the following error equations

\[
\mathcal{B}_1(\zeta_q, \zeta_y, \zeta_{\bar{y}}; r_1, w_1, \mu_1) = (u - u_h, w_1)_{\tau_h},
\]
\[
\mathcal{B}_2(\zeta_p, \zeta_z, \zeta_{\bar{z}}; r_2, w_2, \mu_2) = -(\zeta_y, w_2)_{\tau_h}.
\]

**Lemma 14.** We have

\[
\gamma \| u - u_h \|_{\tau_h}^2 + \| y_h(u) - y_h \|_{\tau_h}^2
\]
\[
= (z_h + \gamma u_h, u - u_h)_{\tau_h} - (z_h(u) + \gamma u, u - u_h)_{\tau_h}.
\]
Proof. First, we have
\[
(z_h + \gamma u_h, u - u_h)_{\mathcal{T}_h} - (z_h(u) + \gamma u, u - u_h)_{\mathcal{T}_h} = -\langle \zeta_z, u - u_h \rangle_{\mathcal{T}_h} + \gamma \| u - u_h \|_{\mathcal{T}_h}^2.
\]
Next, Lemma 2 gives
\[
\mathcal{B}_1(\zeta_q, \zeta_y, \zeta_y; \zeta_p, -\zeta_z, -\zeta \hat{z}) + \mathcal{B}_2(\zeta_p, \zeta_z, \zeta \hat{z}; -\zeta_q, \zeta_y, \zeta \hat{y}) = 0.
\]
On the other hand, using the definition of $\mathcal{B}_1$ and $\mathcal{B}_2$ gives
\[
\mathcal{B}_1(\zeta_q, \zeta_y, \zeta_y; \zeta_p, -\zeta_z, -\zeta \hat{z}) + \mathcal{B}_2(\zeta_p, \zeta_z, \zeta \hat{z}; -\zeta_q, \zeta_y, \zeta \hat{y}) = -\langle u - u_h, \zeta_z \rangle_{\mathcal{T}_h} - \| \zeta_y \|_{\mathcal{T}_h}^2.
\]
Comparing the above two equalities gives
\[
-(u - u_h, \zeta_z)_{\mathcal{T}_h} = \| \zeta_y \|_{\mathcal{T}_h}^2.
\]
This completes the proof. □

Theorem 2. We have
\begin{align*}
\| u - u_h \|_{\mathcal{T}_h} &\lesssim h^{k+2}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}), \\
\| y - y_h \|_{\mathcal{T}_h} &\lesssim h^{k+2}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}), \\
\| z - z_h \|_{\mathcal{T}_h} &\lesssim h^{k+2}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}).
\end{align*}

Proof. Recalling the continuous and discretized optimality conditions (8e) and (26c) gives
\[
\gamma \| u - u_h \|_{\mathcal{T}_h} + \| \zeta_y \|_{\mathcal{T}_h} = (z_h + \gamma u_h, u - u_h)_{\mathcal{T}_h} - (z_h(u) + \gamma u, u - u_h)_{\mathcal{T}_h}
\]
\[
= -(z_h(u) - z, u - u_h)_{\mathcal{T}_h} \leq \| z_h(u) - z \|_{\mathcal{T}_h} \| u - u_h \|_{\mathcal{T}_h} \leq \frac{1}{2\gamma} \| z_h(u) - z \|_{\mathcal{T}_h}^2 + \frac{\gamma}{2} \| u - u_h \|_{\mathcal{T}_h}^2.
\]
By Lemma 13 we have
\[
\| u - u_h \|_{\mathcal{T}_h} + \| \zeta_y \|_{\mathcal{T}_h} \lesssim h^{k+2}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}).
\]
Then, by the triangle inequality and Lemma 7 we obtain
\[
\| y - y_h \|_{\mathcal{T}_h} \lesssim h^{k+2}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}).
\]
Finally, since $z = \gamma u$ and $z_h = \gamma u_h$ we have
\[
\| z - z_h \|_{\mathcal{T}_h} \lesssim h^{k+2}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}).
\]
□
3.3.8 Step 8: Estimate for $\|q - q_h\|_{\mathcal{T}_h}$ and $\|p - p_h\|_{\mathcal{T}_h}$.

**Lemma 15.** We have

$$
\|q\|_{\mathcal{T}_h} \lesssim h^{k+2}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}), \tag{52a}
$$

$$
\|p\|_{\mathcal{T}_h} \lesssim h^{k+2}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}). \tag{52b}
$$

**Proof.** By Lemma 1, the error equation (48a), and the estimate (51) we have

$$
\|q\|_{\mathcal{T}_h}^2 \lesssim B_1(\zeta_q; \zeta_q; \zeta_y; \zeta_z; \zeta_y; \zeta_y) = (u - u_h; \zeta_y)_{\mathcal{T}_h}
\leq \|u - u_h\|_{\mathcal{T}_h}\|\zeta_y\|_{\mathcal{T}_h}
\lesssim h^{2k+4}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2})^2.
$$

Similarly, by Lemma 1, the error equation (48b), Lemma 13 and Theorem 2 we have

$$
\|p\|_{\mathcal{T}_h}^2 \lesssim B_2(\zeta_p; \zeta_z; \zeta_z; \zeta_p; \zeta_z; \zeta_z) = -(\zeta_y; \zeta_y)_{\mathcal{T}_h}
\leq \|\zeta_y\|_{\mathcal{T}_h}\|\zeta_z\|_{\mathcal{T}_h}
\leq \|\zeta_y\|_{\mathcal{T}_h}(\|z_h - z\|_{\mathcal{T}_h} + \|z - z_h\|_{\mathcal{T}_h})
\lesssim h^{2k+4}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2})^2.
$$

The above lemma along with the triangle inequality, Lemma 7 and Lemma 13 complete the proof of the main result:

**Theorem 3.** We have

$$
\|q - q_h\|_{\mathcal{T}_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}), \tag{53a}
$$

$$
\|p - p_h\|_{\mathcal{T}_h} \lesssim h^{k+1}(|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}). \tag{53b}
$$

4 **Numerical Experiments**

In this section, we present two numerical examples to confirm our theoretical results. We consider the problems on a square domain $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. For the two examples, we take $\gamma = 1$, $\tau_1 = 1$, $\beta = [x_2, x_1]$ and the exact state $y(x_1, x_2) = \sin(\pi x_1)$. We used the optimize-then-discretize (OD) approach in Example 1 and the discretize-then-optimize (DO) approach in Example 2. In these examples, the data $f$, $g$, and $y_d$ is generated from the optimality system after we specified the exact dual state $z(x_1, x_2) = \sin(\pi x_1) \sin(4\pi x_2)$.

Numerical results for $k = 0$ and $k = 1$ for the two approaches are shown in Table 1. The observed convergence rates and numerical results exactly match the theoretical results.

**Example 1.** For the OD approach, we set the stabilization parameter $\tau_2$ using (A1); hence, conditions (A1)-(A2) are satisfied. We obtain optimal convergence rates for all variables for $k = 0$ and $k = 1$ in Table 1 and Table 2 respectively. This matches our theoretical results.

**Example 2.** For the DO approach, we used the same data as in Example 1. From the tables we can see that the numerical results are exactly the same with the OD approach, which confirms our theoretical results.
| $h/\sqrt{2}$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
|---|---|---|---|---|---|
| $\|q - q_h\|_{0,\Omega}$ | 2.8775E-01 | 1.4501E-01 | 7.2649E-02 | 3.6342E-02 | 1.8173E-02 |
| order | - | 0.98861 | 0.99716 | 0.99929 | 0.99982 |
| $\|p - p_h\|_{0,\Omega}$ | 2.1036E-01 | 1.0341E-01 | 5.1480E-02 | 2.5712E-02 | 1.2852E-02 |
| order | - | 1.0244 | 1.0063 | 1.0016 | 1.0004 |
| $\|y - y_h\|_{0,\Omega}$ | 1.1842E-02 | 3.2095E-03 | 8.4824E-04 | 2.1887E-04 | 5.5641E-05 |
| order | - | 1.8834 | 1.9198 | 1.9544 | 1.9759 |
| $\|z - z_h\|_{0,\Omega}$ | 1.8304E-02 | 5.3420E-03 | 1.4422E-03 | 3.7460E-04 | 9.5451E-05 |
| order | - | 1.7767 | 1.8891 | 1.9449 | 1.9725 |

Table 1: Example 1 Errors for the state $y$, adjoint state $z$, and the fluxes $q$ and $p$ when $k = 0$ with the OD approach.

| $h/\sqrt{2}$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
|---|---|---|---|---|---|
| $\|q - q_h\|_{0,\Omega}$ | 1.8365E-02 | 4.9165E-03 | 1.2726E-03 | 3.2189E-04 | 8.0742E-05 |
| order | - | 1.9012 | 1.9498 | 1.9831 | 1.9952 |
| $\|p - p_h\|_{0,\Omega}$ | 1.6649E-02 | 5.6050E-03 | 1.5952E-03 | 4.1463E-04 | 1.0475E-04 |
| order | - | 1.5707 | 1.8129 | 1.9439 | 1.9848 |
| $\|y - y_h\|_{0,\Omega}$ | 1.3524E-03 | 1.8347E-04 | 2.3956E-05 | 3.0691E-06 | 3.8882E-07 |
| order | - | 2.9849 | 2.9371 | 2.9645 | 2.9807 |
| $\|z - z_h\|_{0,\Omega}$ | 3.2125E-03 | 4.2489E-04 | 5.4721E-05 | 6.9745E-06 | 8.8190E-07 |
| order | - | 2.9186 | 2.9569 | 2.9719 | 2.9834 |

Table 2: Example 1 Errors for the state $y$, adjoint state $z$, and the fluxes $q$ and $p$ when $k = 1$ with the OD approach.

| $h/\sqrt{2}$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
|---|---|---|---|---|---|
| $\|q - q_h\|_{0,\Omega}$ | 2.8775E-01 | 1.4501E-01 | 7.2649E-02 | 3.6342E-02 | 1.8173E-02 |
| order | - | 0.98861 | 0.99716 | 0.99929 | 0.99982 |
| $\|p - p_h\|_{0,\Omega}$ | 2.1036E-01 | 1.0341E-01 | 5.1480E-02 | 2.5712E-02 | 1.2852E-02 |
| order | - | 1.0244 | 1.0063 | 1.0016 | 1.0004 |
| $\|y - y_h\|_{0,\Omega}$ | 1.1842E-02 | 3.2095E-03 | 8.4824E-04 | 2.1887E-04 | 5.5641E-05 |
| order | - | 1.8834 | 1.9198 | 1.9544 | 1.9759 |
| $\|z - z_h\|_{0,\Omega}$ | 1.8304E-02 | 5.3420E-03 | 1.4422E-03 | 3.7460E-04 | 9.5451E-05 |
| order | - | 1.7767 | 1.8891 | 1.9449 | 1.9725 |

Table 3: Example 2 Errors for the state $y$, adjoint state $z$, and the fluxes $q$ and $p$ when $k = 0$ with the DO approach.
\[
\frac{h}{\sqrt{2}} \quad 1/8 \quad 1/16 \quad 1/32 \quad 1/64 \quad 1/128
\]
\[
\|q - q_h\|_{0, \Omega} \quad 1.8365E-02 \quad 4.9165E-03 \quad 1.2726E-03 \quad 3.2189E-04 \quad 8.0742E-05
\]
\[
\text{order} \quad - \quad 1.9012 \quad 1.9498 \quad 1.9831 \quad 1.9952
\]
\[
\|p - p_h\|_{0, \Omega} \quad 1.6649E-02 \quad 5.6050E-03 \quad 1.5952E-03 \quad 4.1463E-04 \quad 1.0475E-04
\]
\[
\text{order} \quad - \quad 1.5707 \quad 1.8129 \quad 1.9439 \quad 1.9848
\]
\[
\|y - y_h\|_{0, \Omega} \quad 1.3524E-03 \quad 1.8347E-04 \quad 2.3956E-05 \quad 3.0691E-06 \quad 3.8882E-07
\]
\[
\text{order} \quad - \quad 2.8819 \quad 2.9371 \quad 2.9645 \quad 2.9807
\]
\[
\|z - z_h\|_{0, \Omega} \quad 3.2125E-03 \quad 4.2489E-04 \quad 5.4721E-05 \quad 6.9745E-06 \quad 8.8190E-07
\]
\[
\text{order} \quad - \quad 2.9186 \quad 2.9569 \quad 2.9719 \quad 2.9834
\]

Table 4: Example 2 Errors for the state \(y\), adjoint state \(z\), and the fluxes \(q\) and \(p\) when \(k = 1\) with the DO approach.

5 Conclusions

We considered a recently proposed EDG method to approximate the solution of an optimal distributed control problems for an elliptic convection diffusion equation. We showed the optimize-then-discretize and discretize-then-optimize approaches coincide, and proved optimal a priori error estimates for the control, state, dual state, and their fluxes. EDG methods are known to be competitive for convection dominated problems; therefore, this new EDG method has potential for optimal control problems involving such PDEs.

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6 Appendix

By simple algebraic operations in equation (18b), we obtain the following formulas for \(G_1, G_2, G_3, G_4, H_1, \) and \(H_2\) in (19):

\[
G_1 = -A_1^{-1} A_2 (A_4 + A_2^T A_1^{-1} A_2)^{-1} (A_5 - A_2^T A_1^{-1} A_3) - A_1^{-1} A_3,
\]

\[
G_2 = A_1^{-1} A_2 (A_4 + A_2^T A_1^{-1} A_2)^{-1} A_6,
\]

\[
G_3 = -(A_4 + A_2^T A_1^{-1} A_2)^{-1} (A_5 - A_2^T A_1^{-1} A_3),
\]

\[
G_4 = (A_4 + A_2^T A_1^{-1} A_2)^{-1} A_6,
\]

\[
H_1 = A_1^{-1} A_2 (A_4 + A_2^T A_1^{-1} A_2)^{-1} (b_3 - b_4 + A_2^T A_1^{-1} b_2) - A_1^{-1} b_2,
\]

\[
H_2 = (A_4 + A_2^T A_1^{-1} A_2)^{-1} (b_3 - b_4 + A_2^T A_1^{-1} b_2).
\]

In general, forming these quantities is impractical; however, for the EDG method described in this work these matrices can be easily computed. We briefly sketch this process below.

Since the spaces \(V_h\) and \(W_h\) consist of discontinuous polynomials, some of the system matrices are block diagonal and each block is small and symmetric positive definite (SSPD). The inverse of
such a matrix is another matrix of the same type, and the inverse is easily computed by inverting each small block. Furthermore, the inverse of each small block can be computed in parallel.

It can be checked that $A_1$ is a SSPD block diagonal matrix, and therefore $A_1^{-1}$ is easily computed and is also a SSPD block diagonal matrix. Therefore, $G_1, G_2, G_3, G_4, H_1,$ and $H_2$ are easily computed since $A_4 + A_2^T A_1^{-1} A_2$ is also a SSPD block diagonal matrix. Also, once these quantities are computed, $G_5, G_6,$ and $H_3$ in (19) are also easy to compute using (18b).

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