On covariant expansion of the gravitational Stückelberg trick

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A new approach to expanding the “Stückelberized” fiducial metric in a covariant manner is developed. The idea is to consider the curved 4-dimensional space as a codimension-one hypersurface embedded in a 5-dimensional Minkowski bulk, in which the 5-dimensional Goldstone modes can be defined as usual. After solving one constraint among the five 5-dimensional Goldstone modes and projecting onto the 4-dimensional hypersurface, we are able to express the “Stückelberized” fiducial metric in terms of the 4-dimensional Goldstone modes as well as 4-dimensional curvature quantities. We also compared the results with expressions got using the Riemann Normal Coordinates (RNC) in Gao et al [Phys. Rev. D90, 124073 (2014)] and find that, after a simple field redefinition, results got in two approaches exactly coincide.

I. INTRODUCTION

Attempts to explain the primordial and late time acceleration of the cosmic background through the use of theories beyond general relativity (GR) with a cosmological constant (see [1, 2] for reviews and [3] for a short introduction). Such theories typically contain additional degrees of freedom to the two tensor modes of GR. Instead of introducing extra fields by hand, one approach to these new degrees of freedom is to construct effective theories with less gauge redundancies comparing with GR. This can be achieved most straightforwardly by introducing a fiducial metric $f_{\mu\nu}$, which does not change under the coordinates transformation and thus explicitly breaks the general covariance.

If $f_{\mu\nu}$ is degenerate and has only one nonvanishing timelike eigenvector, one gets the effective field theory (EFT) of inflation [4, 5] and recent so-called “theories beyond Horndeski” [6, 7], where the time diffeomorphism is broken and thus generally an additional scalar mode arises [8–10]. Spatial symmetries may be broken by considering $f_{\mu\nu}$ with nonvanishing spacelike eigenvectors. If the number of nonvanishing spacelike eigenvectors is equal to the spatial dimension, one arrives at some sort of massive gravity theories (typically Lorentz-violating, e.g. [11, 12]). A nondegenerate $f_{\mu\nu}$ breaks all spacetime symmetries, through which a Lorentz-invariant massive gravity can be constructed [13] (see [14, 15] for reviews). In this note, we concentrate on the case of a nondegenerate $f_{\mu\nu}$.

The fiducial metric $\tilde{f}_{\mu\nu}$, whose existence breaks the general covariance explicitly, can always be thought of as the “gauge-fixed” version of some covariant tensor field. This is just the idea of gravitational Stückelberg trick, which dates from [16, 17] in the study of open string theory. We may promote the fiducial metric $\tilde{f}_{\mu\nu}$ as

$$\tilde{f}_{\mu\nu} \rightarrow f_{\mu\nu}(\phi) \equiv \tilde{f}_{ab}(\phi(x)) \frac{\partial \phi^a(x)}{\partial x^\mu} \frac{\partial \phi^b(x)}{\partial x^\nu},$$

(1)

where the “Stückelberized” fiducial metric $f_{\mu\nu}$ transforms as a tensor as long as each of the four (we are working with 4-dimensional spacetime) Stückelberg fields $\{\phi^a\}$ transform as scalars under a general coordinates transformation of spacetime. The fixed $\tilde{f}_{\mu\nu}$ is recovered by choosing the so-called “unitary gauge” with $\phi^a \rightarrow \delta^a_\mu x^\mu$. In practice, we may expand the Stückelberg fields around the unitary gauge and concentrate on the behavior of their fluctuations [18]

$$\phi^a - \tilde{f}^a_{\mu\nu} x^\mu \equiv -\delta^a_\mu \hat{x}^\mu. \tag{2}$$

When the fiducial metric $\tilde{f}_{\mu\nu}$ is flat, it has been well-known that in the so-called decoupling limit (some limit of energy scales where the interactions among different types of degrees of freedom get simplified), $\pi_\mu$ defined in (2) behaves as a spacetime vector. In this case, we can fix a gauge in which the helicity-1 and helicity-0 parts of the graviton are encoded in $\pi_\mu$ [19]. It is just in this way that the Boulware-Deser ghost [20] can be seen most transparently [21, 22]. This argument, however, cannot be simply applied to a general fiducial metric $f_{\mu\nu}$. First, naively plugging (2) into (1) would inevitably yield noncovariant expressions [23, 24]. More seriously, as was well explained in [23, 25, 26], $\pi_\mu$ defined in (2) is not a vector and does not capture the helicity-1 and helicity-0 modes of the graviton correctly, either when going beyond the decoupling limit or when the fiducial metric has curvature.

This problem was systematically solved in [24] by employing the the Riemann Normal Coordinates (RNC), where a covariant formulation of the Stückelberg expansion with a general fiducial metric was developed. A decoupling limit analysis similar to the case of a flat fiducial metric was consistently performed in [24], where the helicity modes can be characterized correctly [24]. On the other hand, when dealing with the de Sitter fiducial metric, an alternative approach to the covariant Stückelberg expansion was developed in [23]. The idea is to embed the $d$-dimensional de Sitter space into a $(d + 1)$-dimensional Minkowski background, in which the Goldstone modes can be identified as in (2). Then by projecting to the $d$-dimensional de Sitter space, (1) can be expanded in terms of the correct helicity modes in $d$ dimensions, in a covariant man-

1 See also [23, 22] for related progresses on the Stückelberg analysis and decoupling limit of massive gravity around a general background.
The purpose of this note, is to develop this technique further, and more systematically, by considering a general fiducial metric.

This note is organized as follows. In Sec II we briefly review the main results in [24] on the covariant Stückelberg expansion based on RNC. In Sec III we establish the basic formalism of embedding the 4-dimensional curved space into a 5-dimensional Minkowski one, and determine the “covariant” Goldstone modes in the 4 dimensions. In Sec IV we derive the covariant expansions for the fiducial metric in terms of 4-dimensional quantities, and compare them with the corresponding results in [24]. Finally we briefly summarize in Sec V.

II. COVARIANT EXPANSION BASED ON THE RIEMANN NORMAL COORDINATES

In [24], the Riemann Normal Coordinates (RNC) was employed to derive covariant expressions for the Stückelberg expansion in the presence of a general fiducial metric. The idea is to consider a one-parameter family of diffeomorphisms of the spacetime with parameter $\lambda$:

$$\phi_\lambda : p \mapsto \phi_\lambda(p),$$

where $p$ is a given spacetime point. The Stückelberg fields at point $p$ are defined as the coordinate values of its image $\phi_\lambda(p)$ with $\lambda = -1$:

$$\phi^\mu|_p \equiv x^\mu|_{\phi^{-1}(p)}. \quad (4)$$

The “covariant” Goldstone modes $\pi^\mu$ are thus defined as the standard RNC’s of the image point $\phi^{-1}(p)$, i.e., the tangent vector of the geodesic at point $p$ connecting $p$ and its image $\phi^{-1}(p)$:

$$\dot{\phi}^\mu|_p = x^\mu - \pi^\mu - \frac{1}{2} \bar{\Gamma}_{\nu\rho}^\mu \pi^\nu \pi^\rho + \frac{1}{6} \left( \partial_{\nu} \bar{\Gamma}^\rho_{\nu\rho} - 2 \bar{\Gamma}^{\rho}_{\nu\rho} \bar{\Gamma}^\lambda_{\nu\rho} \right) \pi^\nu \pi^\rho \pi^\sigma + \cdots, \quad (5)$$

where $x^\mu|_p \equiv \dot{\phi}^\mu|_p$, and $\bar{\Gamma}_{\nu\rho}^\mu$ is the Christoffel symbol associated with $\bar{f}_{\mu\nu}$. Comparing with [2], (4) also implies a nonlinear relation between $\bar{\pi}_\mu$ and $\pi_\mu$.

By plugging (5) into (1) and carefully dealing with the Christoffel symbols and their derivatives, it is possible to expand the “Stückelbergized” fiducial metric $f_{\mu\nu}$ covariantly in terms of $\bar{\pi}_\mu$, $\bar{f}_{\mu\nu\rho}$ as well as their covariant derivatives (with respect to $\bar{f}_{\mu\nu}$). An equivalent and more convenient approach, is to evaluate (see [24] for details)

$$f_{\mu\nu} = e^{-\phi^{\nu}} \bar{f}_{\mu\nu} \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \tilde{L}^n f_{\mu\nu}, \quad (6)$$

$$= \sum_{n=0}^{\infty} f^{(n)\text{RNC}}_{\mu\nu},$$

where $\tilde{L}^n f_{\mu\nu}$ is the $n$-th order Lie derivative of $\bar{f}_{\mu\nu}$ with respect to $\pi^\mu$. Straightforward calculations yield[3]

$$f^{(1)\text{RNC}}_{\mu\nu} = -2 \bar{\nabla}_\mu \bar{\pi}_\nu, \quad (7)$$

$$f^{(2)\text{RNC}}_{\mu\nu} = \bar{\nabla}_\mu \bar{\pi}_\rho \bar{\nabla}_\nu \pi^\rho - \bar{R}_{\mu\nu\rho\sigma} \pi^\rho \pi^\sigma, \quad (8)$$

$$f^{(3)\text{RNC}}_{\mu\nu} = \frac{1}{3} \pi^\alpha \pi^\beta \left( \pi^\rho \nabla^\mu \bar{R}^{\alpha\beta}_{\rho\sigma\lambda} - 4 \bar{R}_{\mu\rho\sigma\lambda} \bar{R}_{\nu\alpha\beta\lambda} - 4 \bar{R}_{\mu\rho\sigma\lambda} \bar{R}_{\nu\alpha\beta\lambda} \cdot \cdot \cdot \right), \quad (9)$$

$$f^{(4)\text{RNC}}_{\mu\nu} = -\frac{1}{12} \left( \bar{R}^{\alpha\beta\rho\sigma} \bar{R}_{\nu\alpha\beta\rho\sigma} - 4 \bar{R}_{\mu\rho\sigma\lambda} \bar{R}_{\nu\alpha\beta\lambda} + 1 \bar{\nabla}_\rho \bar{R}^{\alpha\beta\rho\sigma} \pi^\alpha \pi^\beta \pi^\rho \pi^\sigma + \bar{\nabla}_\rho \bar{R}^{\alpha\beta\rho\sigma} \pi^\alpha \pi^\beta \pi^\rho \pi^\sigma \right). \quad (10)$$

Please note in deriving (8)-(10), $\pi^\nu \nabla^\mu \pi^\mu = 0$ is used, since $\pi^\mu$ is the tangent vector of geodesics.

III. STÜCKELBERG BY EMBEDDING

A different approach to the covariant Stückelberg expansion was introduced in [23] in the study of massive gravity on de Sitter background. This approach is based on the observation that the $d$-dimensional de Sitter space can be embedded into a $(d+1)$-dimensional Minkowski one, in which the Goldstone modes can be identified easily as in (2) and the Stückelberg expansion of the fiducial metric can be performed as usual. Then the $d$-dimensional quantities can be got, in a covariant manner, by simply projecting the $(d+1)$-dimensional ones onto the $d$-dimensional de Sitter space.

In general, to embed an arbitrary $d$-dimensional space into a $(d+1)$-dimensional Minkowski one is not always possible. While as the first attempt, in this note, we restrict ourselves to the subclass of 4 dimensional metrics which can be embedded (at least locally) into a 5 dimensional flat bulk. In this case, the corresponding Stückelbergized fiducial metric is given by

$$f_{\mu\nu} = \eta_{MN} \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu}, \quad (11)$$

where $\{X^M\}$ with $M = 0, 1, 2, 3, 4$ are Cartesian coordinates of the 5-dimensional Minkowski space. As we shall see, in the 5-dimensional flat bulk, the unitary gauge and the decomposition of the Stückelberg fields can be performed in a standard manner. All the subtleties are thus in the projection from 5 dimensions to 4 dimensions.

It is convenient to introduce another set of coordinates $\{x^a\}$ for the 5-dimensional Minkowski bulk such that its metric takes the form

$$ds^2 = h_{ab} dx^a dx^b, \quad (12)$$

with

$$h_{ab}(x) = \eta_{MN} \frac{\partial X^M(x)}{\partial x^a} \frac{\partial X^N(x)}{\partial x^b}. \quad (13)$$

Note in [23], $f^{(n)\text{RNC}}_{\mu\nu}$ were evaluated only up to the cubic order in $\pi^\mu$. Here we also evaluate $f^{(4)\text{RNC}}_{\mu\nu}$ for late convenience.

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2 This is in fact a standard approach in order to define perturbations “covariantly”, as in the well-known background field method [31]. For massive gravity, RNC was also suggested in [23] (footnote 5).

3 Note in [23], $f^{(n)\text{RNC}}_{\mu\nu}$ were evaluated only up to the cubic order in $\pi^\mu$. Here we also evaluate $f^{(4)\text{RNC}}_{\mu\nu}$ for late convenience.
Note we also have $b^{ab} \partial X^M(x) \partial X^N(x) = \eta^{MN}$.

The embedding of the 4-dimensional hypersurface into the 5-dimensional bulk can be parametrized by a constraint among coordinates $\{X^M\}$:

$$\Phi \left( X^M \right) = 0,$$

which is a scalar function under diffeomorphism and global Lorentz transformation of $\{X^M\}$. Since the hypersurface is codimensional one, its normal vector (with normalization $\eta^{MN} n_M n_N = 1$) is thus given by

$$n_M = N \partial_M \Phi,$$

with $\partial_M = \partial \Phi / \partial X^M$ and

$$N = \frac{1}{\sqrt{\eta^{MN} \partial_M \Phi \partial_N \Phi}}$$

Here the sign of $n_M$ is chosen to be compatible with the fact that $n_M$ is spacelike, such that the induced metric on the 4-dimensional hypersurface

$$h_{MN} = \eta_{MN} - n_M n_N,$$

is Lorentzian.

### A. Unitary gauge

Up to now, the formalism is quite general. We can now fix a gauge (unitary gauge) by choosing a specified mapping between the two set of coordinates:

$$x^a \mapsto X^M \equiv \bar{X}^M(x),$$

which corresponds to a special choice of $\{x^a\}$-coordinates adapted to the embedding, i.e.,

$$\{x^a\} = \{x^\mu, y\}, \quad \text{with } \mu = 0, 1, 2, 3,$$

such that the metric of 5-dimensional bulk in this particular $\{x^a\}$-coordinates takes the form

$$ds^2 = \bar{h}_{ab} dx^a dx^b \equiv \bar{f}_{\mu \nu} dx^\mu dx^\nu + 2 N_\mu dx^\mu dy + (N^2 + N_\mu N^\mu) dy^2,$$

where $\bar{f}_{\mu \nu}$ is just the fixed induced metric on the hypersurface, which we treat as being given beforehand. In $\bar{h}_{ab}$ we write $f_{\mu \nu}$ in order to emphasize it is the expression in the unitary gauge. It is always possible to choose $y$-coordinate to be normal to the hypersurface so that $\bar{h}_{\mu y} = N_\mu = 0$. Here we keep $N_\mu \neq 0$ for generality, while as we shall see later, all the contributions from $N_\mu$ drop out in the final expressions. At this point, keep in mind that $N$ and $N_\mu$ must be determined by $\bar{f}_{\mu \nu}$ in order to make sure that $\bar{h}_{ab}$ is indeed describing a flat space.

In the following, we use

$$e^M_a = \frac{\partial \bar{X}^M(x)}{\partial x^a},$$

for short. In this unitary gauge, the normal vector to the hypersurface is given by, in $\{X^M\}$-coordinate:

$$n^M = \frac{1}{N} (e^M_y - N^M),$$

and in $\{x^a\}$-coordinates:

$$n_a \equiv \{0, \cdots, 0, n_y\} = \{0, \cdots, 0, N\}.$$

The induced metric on the hypersurface $\bar{X}$ thus becomes

$$h_{ab} = h_{ab} - n_a n_b.$$
In \( \{ x^a \} \)-coordinate, the constraint (31) becomes
\[
\sum_{n=1} (-1)^n \Phi_{a_1 \cdots a_n} x^{a_1} \cdots x^{a_n} = 0, \tag{32}
\]
where
\[
\Phi_{a_1 \cdots a_n} = e^{M_1} \cdots e^{M_n} \frac{\partial^n \Phi}{\partial X^{M_1} \cdots \partial X^{M_n}} = \nabla_{a_1} \cdots \nabla_{a_n} \Phi. \tag{33}
\]
Note \( \Phi_{a_1 \cdots a_n} \) is symmetric since the 5-dimensional bulk is essentially flat.

By definition,
\[
\nabla_a \Phi = \frac{1}{N} n_a, \tag{34}
\]
and a further derivative yields
\[
\nabla_a \nabla_b \Phi = -\frac{1}{N} \left( n_a n_b \rho - 2n_a n_b - K_{ab} \right), \tag{35}
\]
with shorthands
\[
\rho = \xi_5 \ln N, \quad a_a = -D_a \ln N. \tag{36}
\]
In the above, \( D_a \) is the tangent covariant derivative (compatible with \( h_{ab} \) in (24)), \( K_{ab} \equiv D_b n_a \) is the extrinsic curvature. The third derivative is decomposed to \( h_{ab} \) as (7, 10)
\[
\nabla_a \nabla_b \nabla_c \Phi = \frac{1}{N} \left( n_a n_b n_c \omega + 2n_a n_b n_c \lambda_c \right.
\]
\[
\left. +n_a \zeta_{bc} + \lambda_a n_b n_c + 2\zeta_{a(bn_c)} + \chi_{abc} \right), \tag{37}
\]
with
\[
\omega = \rho^2 - \xi_5 \rho - 2a_a a_a, \tag{38}
\]
\[
\lambda_a = \xi_5 a_a - 2\rho a_a - 2K_a' a_c', \tag{39}
\]
\[
\zeta_{ab} = 2a_a a_b - \rho K_{ab} + \xi_5 K_{ab} - 2K_{ac'} K_a'^b, \tag{40}
\]
\[
\chi_{abc} = a_a K_{bc} + 2K_{a(bc)} + D_a K_{bc}, \tag{41}
\]
where \( \rho \) and \( a_a \) are defined in (37). The full decomposition of \( \nabla_a \nabla_b \nabla_c \nabla_d \Phi \) is rather involved and we prefer not to present it in this note. For our purpose to solve \( \pi^\perp \) up to the fourth order in \( \pi^\mu \), only the purely tangent part of \( \nabla_a \nabla_b \nabla_c \nabla_d \Phi \) is needed, which reads
\[
\nabla_a \nabla_b \nabla_c \nabla_d \Phi \supset \frac{1}{N} \chi_{abcd}, \tag{42}
\]
with (37)
\[
\chi_{abcd} = K_{ab} \zeta_{cd} + K_{ac} \zeta_{bd} + K_{ad} \zeta_{bc} + a_a \chi_{bcd} + D_a \chi_{bcd}, \tag{43}
\]
where \( \zeta_{ab} \) etc. are defined in (40)-(42).

Supposing that \( \pi^\perp \) can be solved in terms of \( \pi^\mu \) perturbatively as
\[
\pi^\perp = \sum_{n=1} \pi^\perp_{(n)}, \tag{44}
\]
where \( \pi^\perp_{(n)} \sim O((\pi^\mu)^n) \). Plugging (34), (35) and (37) into (42) and using the definition for \( \pi^\perp \) and \( \pi^\mu \), after some manipulations we have
\[
\pi^\perp = \frac{1}{2} \frac{\pi^\mu \pi^\nu K_{\mu\nu}}{N^{1/2}} - \frac{1}{6} \pi^\mu \pi^\nu \pi^\rho \nabla_\rho \tilde{K}_{\mu\nu} \tag{45}
\]
\[
+ \frac{1}{24} \pi^\mu \pi^\nu \pi^\rho \pi^\sigma \left( 3 \tilde{K}_{\mu\nu} \tilde{K}_\rho \tilde{K}_\sigma + \nabla_\nu \tilde{K}_\rho \nabla_\sigma \right) \tag{46}
\]
\[
+ O \left( (\pi^\mu)^{5/2} \right). \tag{47}
\]
(46) explicitly depends on the extrinsic curvature \( \tilde{K}_{\mu\nu} \) and its derivatives. While from the 4-dimensional point of view, the fiducial metric should not “know” anything about the embedding. As we shall see in the next section, in the final expressions for the Stückelbergized fiducial metric \( f_{\mu\nu} \), all the dependence on the extrinsic curvature gets suppressed after using the Gauss relation and a simple field redefinition.

IV. COVARIANT EXPANSION OF THE FIDUCIAL METRIC

We are now ready to expand the “Stückelbergized” fiducial metric given by
\[
f_{\mu\nu} = h_{\mu\nu} \tag{48}
\]
with \( h_{ab} \) given in (13), in terms of \( \pi^\mu \) as well as 4-dimensional curvature quantities. Expanding around the unitary gauge by plugging (45) into (13), we get
\[
f_{\mu\nu} = \tilde{f}_{\mu\nu} - 2\eta_{MN} \bar{D}_\mu \bar{X}^M \bar{D}_\nu \bar{X}^N + \eta_{MN} \bar{D}_\mu \bar{X}^M \bar{D}_\nu \bar{X}^N, \tag{49}
\]
which is exact. Note in deriving (48), we used \( \eta_{MN} \bar{D}_\mu \bar{X}^M \bar{D}_\nu \bar{X}^N \equiv \tilde{f}_{\mu\nu} \). Our purpose is to rewrite (48) in a covariant manner in terms of 4 dimensional quantities.

For the second term in (48), it is easy to show that (see (13))
\[
\eta_{MN} \bar{D}_\mu \bar{X}^M \bar{D}_\nu \bar{X}^N = e^M_{\mu} e^N_{\nu} \partial_N \pi_M \tag{50}
\]
\[
\equiv \nabla_\nu \pi_M + \tilde{K}_{\mu\nu} \pi^\perp_M, \tag{51}
\]
where $\pi$ is defined in (29). For the last term in (48), first we have

$$
\eta_{MN}\partial_\mu \pi^M \partial_\nu \pi^N
$$

(50)

and at the quadratic order,

$$
f^{(2)\text{Ebd}}_{\mu\nu} = \nabla_\mu \pi^\rho \nabla_\nu \pi^\rho - (K_{\mu\rho} K_{\nu\sigma} - K_{\mu\sigma} K_{\nu\rho}) \pi^\rho \pi^\sigma,
$$

(56)

where in the last step we used the Gauss relation

$$
K_{\mu\rho} K_{\nu\sigma} - K_{\mu\sigma} K_{\nu\rho} = \tilde{R}_{\mu\nu\rho\sigma},
$$

(57)

since the 5 dimensional bulk is flat. Note $f^{(1)\text{Ebd}}_{\mu\nu}$ in (55) and $f^{(2)\text{Ebd}}_{\mu\nu}$ in (56) exactly coincides with the results got in the RNC approach $f^{(1)\text{RNC}}_{\mu\nu}$ in (7) and $f^{(2)\text{RNC}}_{\mu\nu}$ in (8) respectively.

At the cubic order,

$$
f^{(3)\text{Ebd}}_{\mu\nu} = f^{(3)\text{RNC}}_{\mu\nu} - \frac{1}{3} \nabla_{(\mu} (\pi^\alpha \pi^\beta \pi^\sigma K_{\nu)\alpha\beta}),
$$

(59)

where $f^{(3)\text{RNC}}_{\mu\nu}$ is given in (9). Please note in deriving (59), we never use $\pi^\nu \nabla_\mu \pi^\mu = 0$, which is the crucial assumption in the RNC approach (see Sec II and (24) for details). The second term on the right-hand-side of (59), may be absorbed by a field redefinition

$$
\pi_{\mu} \rightarrow \tilde{\pi}_{\mu} \equiv \pi_{\mu} + \Delta^{(3)}_{\mu} + \mathcal{O} \left( (\pi_{\mu})^2 \right),
$$

(60)

with

$$
\Delta^{(3)}_{\mu} = \frac{1}{6} \pi^\alpha \pi^\beta \pi^\sigma K_{\mu\alpha\beta}.
$$

(61)

This can be seen easily since $f^{(1)\text{RNC}}_{\mu\nu}(\Delta^{(3)}_{\mu})$ will exactly reproduce the second term on the right-hand-side of (59).

Similarly, at the quartic order, first we got the expression for $f^{(4)\text{Ebd}}_{\mu\nu}$ depending on the extrinsic curvature

$$
f^{(4)\text{Ebd}}_{\mu\nu} = \frac{1}{12} \pi^\alpha \pi^\beta \pi^\rho \pi^\sigma \left[ 3 K_{\alpha\beta} (K_{\mu\rho} K_{\nu\sigma} K_{\lambda\kappa} - K_{\mu\lambda} K_{\nu\rho} K_{\sigma\kappa}) + K_{\mu\nu} \nabla_\alpha \nabla_\beta K_{\rho\sigma} - 3 \nabla_\mu K_{\alpha\beta} \nabla_\nu K_{\rho\sigma} - 4 K_{\sigma(\mu} \nabla_{\nu)} \nabla_\alpha K_{\beta\rho}) - \frac{1}{3} \pi^\alpha \pi^\beta \pi^\rho \nabla_\mu (\pi^\sigma K_{\nu\beta\rho} - 3 \nabla_\nu K_{\alpha\beta} K_{\rho\sigma} - 3 K_{\nu(\mu} \nabla_{\sigma) K_{\beta\rho}}) \right.
$$

$$
+ \pi^\alpha \pi^\beta \nabla_\mu \nabla_\nu \nabla_\rho \nabla_\sigma K_{\alpha\beta\rho\sigma}.
$$

(62)
Using the Gauss relation \( \pi \) again, after some manipulations, \( f^{(4)RNC}_{\mu\nu} \) can be recast as

\[
f^{(4)Ebd}_{\mu\nu} = f^{(4)RNC}_{\mu\nu} + 2 \nabla_{(\mu} \pi^{\rho} \nabla_{\nu)} \Delta_{(3)}^{(3)} - 2 \tilde{R}_{(\mu} \pi^{\rho}_{\nu)} \Delta_{(3)}^{(3)} \pi_{\rho} - 2 \nabla_{(\mu} \Delta_{(3)}^{(4)} \),
\]

where \( f^{(4)RNC}_{\mu\nu} \) is given in (10), \( \Delta_{(3)}^{(3)} \) is defined in (61), and

\[
\Delta_{(4)}^{(4)} = - \frac{1}{24} \pi_{\alpha} \pi_{\beta} \pi^{\rho} \pi^{\sigma} (\tilde{K}_{\alpha\beta} \nabla_{\mu} \tilde{K}_{\sigma\rho} + 2 \tilde{K}_{\sigma\rho} \nabla_{\alpha} \tilde{K}_{\beta\mu}) .
\]

It is interesting to note that although \( f^{(4)Ebd}_{\mu\nu} \) itself does not coincide with \( f^{(4)RNC}_{\mu\nu} \), their difference can be absorbed by the following field redefinition

\[
\pi_{\mu} \rightarrow \tilde{\pi}_{\mu} = \pi_{\mu} + \Delta_{(3)}^{(3)} + \Delta_{(4)}^{(4)} + \mathcal{O} ( (\pi_{\mu})^{5}) ,
\]

which is also consistent with (65). That is, we have

\[
f^{Ebd}_{\mu\nu} (\tilde{\pi}_{\rho}) = f^{RNC}_{\mu\nu} (\tilde{\pi}_{\rho}) ,
\]

where \( \tilde{\pi}_{\mu} \) is given in (65) up to the fourth order.

\section{V. Conclusion}

The problem of covariant formulation for the Stückelberg analysis with a non-flat fiducial metric (or around a general background) has been known for some time. In [24] a covariant Stückelberg expansion was developed based on the Riemann normal coordinates. In this note we explore an alternative approach by considering the 4-dimensional curved space being a hypersurface embedded in a 5-dimensional Minkowski background, in which the Goldstone modes and the Stückelberg expansion can be performed in the standard manner. After eliminating one Goldstone modes through the constraint (26) and then projecting onto the 4-dimensional hypersurface, we are able to expand the Stückelbergized fiducial metric \( (11) \) in terms of 4-dimensional Goldstone modes \( \pi_{\mu} \) as well as 4-dimensional geometric quantities, which are given in (53), (60), (59) and (63) respectively, up to the fourth order in \( \pi_{\mu} \). Strikingly, the two approaches (RNC and embedding), although quite different from each other, give exactly coincide results after a simple field redefinition (65).

We expect the formalism developed in this note, may shed some light on the Stückelberg analysis as well as the decoupling limit of the massive gravity on a general background. There are some questions left to be answered. First it is important to find a geometric meaning for the field redefinition (65), which implies that the correct “covariant” Goldstone modes from the 4-dimensional point of view are actually nonlinear functions \( \pi_{\mu} \) (instead of \( \pi_{\mu} \) themselves). Moreover, as being emphasized before, higher codimensions may be needed in order to embed a general curved space into a flat background. It is thus interesting to generalize the formalism in this note to such a case.

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\section*{Appendix A: Projection of \( \nabla_{a} A_{b} \)}

Let us consider a codimension-one hypersurface in an arbitrary background with metric \( g_{ab} \). The normal vector \( n_{a} \) is normalized as \( n_{a} n^{a} = +1 \), which is spacelike. The induced metric on the hypersurface is \( h_{ab} = g_{ab} - n_{a} n_{b} \). For an arbitrary vector field \( A_{a} \), we may write

\[
A_{a} \equiv n_{a} A_{\perp} + A_{a \parallel} ,
\]

with

\[
A_{\perp} = n^{a} A_{a} , \quad A_{a \parallel} = h_{ab}^{a} A_{ab} .
\]

It is easy to show that

\[
h_{a}^{a} h_{b}^{b} \nabla_{a} A_{b} = D_{a} A_{\parallel} + K_{ab} A_{\perp} ,
\]

\[
h_{a}^{a} n^{b} \nabla_{a} A_{b} = D_{a} A_{\perp} - K_{ab} A_{\parallel} ,
\]

where \( D_{a} \) is the covariant derivative compatible with \( h_{ab} \), \( K_{ab} \equiv D_{a} n_{b} \) is the extrinsic curvature.

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