Spin-structures and proper group actions

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Abstract

We generalise Atiyah and Hirzebruch’s vanishing theorem for actions by compact groups on compact Spin-manifolds to possibly non-compact groups acting properly and cocompactly on possibly non-compact Spin-manifolds. As corollaries, we obtain some vanishing results for ̂A-type genera.

Introduction

In 1970, Atiyah and Hirzebruch [2] proved the following remarkable result.

Theorem 1. Let N be a compact, connected even dimensional manifold and K be a compact connected Lie group acting smoothly and non-trivially on N. Suppose also that N has a K-invariant Spin structure. Then the equivariant index of the Dirac operator on N vanishes in the representation ring of K,

\[ \text{index}_K(\partial N) = 0 \in R(K). \]

In particular, \( \int_N \hat{A}(N) = 0. \)

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Their result then inspired many, especially Witten \cite{cite13} who studied two-dimensional quantum field theories and the index of the Dirac operator on free loop space $\mathcal{L}N$, relating it to the rigidity of certain Dirac-type operators on $N$ and the elliptic genus, which was proved in \cite{cite4,cite11}.

Our goal in this note is to extend the theorem to the non-compact setting. More precisely, let $M$ be a complete Riemannian manifold, on which a connected Lie group $G$ acts properly and isometrically. Suppose $M/G$ is compact. Suppose $M$ has a $G$-equivariant Spin-structure. Let

$$\text{index}_G(\emptyset_M) \in K_\bullet(C^*_r G)$$

be the equivariant index of the associated Spin-Dirac operator. Here $K_\bullet(C^*_r G)$ is the $K$-theory of the reduced group $C^*$-algebra of $G$, and $\text{index}_G$ denotes the analytic assembly map used in the Baum–Connes conjecture \cite{cite3,cite9}. If $G$ is compact, then $K_\bullet(C^*_r G) = R(G)$, and the analytic assembly map is the usual equivariant index. Atiyah and Hirzebruch’s result generalises as follows.

**Theorem 2.** If there is a point in $M$ whose stabiliser in $G$ is not a maximal compact subgroup of $G$, then

$$\text{index}_G(\emptyset_M) = 0,$$

if $G$ has a property\footnote{See the text leading up to Lemma 5 for the precise formulation of this property.} that one of its double covers always has.

From this theorem, we will deduce vanishing of characteristic classes related to the $\hat{A}$ class in Corollary 7 as well as an application in Corollary 8. In Corollary 9, we give an equivalent statement of Theorem 2 that does not use $C^*_r G$ or the analytic assembly map.

There are many group actions that satisfy the hypotheses of Theorem 2. Indeed, let $K < G$ be a maximal compact subgroup, and suppose that $G$ has the property mentioned in Theorem 2. Then if $K$ acts on a compact Spin-manifold $N$ as in Theorem 1, then Theorem 2 applies to the action by $G$ on the fibred product $G \times_K N$, as we will see. If $K = S^1$, then it is proved in the theorem in Section 2.3 in \cite{cite2} that any compact oriented manifold $X$ with $\int_X \hat{A}(X) = 0$ has the property that $mX$ (for some $m \in \mathbb{N}$) is oriented cobordant to a compact Spin manifold $N$ which has a non-trivial
$S^1$-action on each of its components. Then the action by $K$ on $N$ satisfies the hypotheses of Theorem 1 so that the to the action by $G$ on $G \times_K N$ satisfies the conditions of Theorem 2.

Note that if $N$ is a compact Spin-manifold with the trivial $K$-action, the action by $G$ on $G/K \times N$ does not satisfy the hypotheses of Theorem 2: all stabilisers are conjugate to $K$ and hence maximal compact subgroups.

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1 Preliminaries

We begin by recalling the smooth version of Abels’ slice theorem for proper group actions. Let $M$ be a smooth manifold, and let $G$ be a connected Lie group acting properly on $M$. Let $K < G$ be maximal compact.

**Theorem 3** (p. 2 of [1]). There is a smooth, $K$-invariant submanifold $N \subset M$, such that the map $[g, n] \mapsto gn$ is a $G$-equivariant diffeomorphism

\[ G \times_K N \cong M \] (1.1)

*Here the left hand side is the quotient of $G \times N$ by the action by $K$ given by*

\[ k \cdot (g, n) = (gk^{-1}, kn), \]

*for $k \in K$, $g \in G$ and $n \in N$.*

We call (1.1) an associated Abels fibration of $M$, as it is a fibre bundle over $G/K$ with fibre $N$. From now on, fix $N$ as in Theorem 3.

The fixed point set $N^K$ of the action by $K$ on $N$ is related to the action by $G$ on $M$ in the following way.

**Lemma 4.** One has

\[ M_{(K)} = G \cdot N^K \cong G/K \times N^K, \]

*where $M_{(K)}$ is the set of points in $M$ with stabilisers conjugate to $K$.*
Proof. Let \( m \in M_{(K)} \) and write \( m = [g, n] \) for \( g \in G \) and \( n \in N \), under the correspondence (1.1). Then \( G_m = gK_n g^{-1} \). So \( G_m \) is conjugate to \( K \) if and only if \( K_n \) is. Since \( K_n < K \), it is conjugate to \( K \) precisely if it equals \( K \). \( \square \)

Now fix a \( K \)-invariant inner product on the Lie algebra \( g \) of \( G \), and let \( p \subset g \) be the orthogonal complement to the Lie algebra \( t \) of \( K \). Suppose \( \text{Ad} : K \to \text{SO}(p) \) lifts to

\[
\tilde{\text{Ad}} : K \to \text{Spin}(p).
\]

This is always possible if one replaces \( G \) by a double cover. Indeed, consider the diagram

\[
\begin{array}{ccc}
\tilde{K} & \xrightarrow{\tilde{\text{Ad}}} & \text{Spin}(p) \\
\pi_K & & \pi \\
\downarrow & & \downarrow 2:1 \\
K & \xrightarrow{\text{Ad}} & \text{SO}(p),
\end{array}
\]

where

\[
\tilde{K} := \{ (k, a) \in K \times \text{Spin}(p); \text{Ad}(k) = \pi(a) \};
\]

\[
\pi_K(k, a) := k;
\]

\[
\tilde{\text{Ad}}(k, a) := a,
\]

for \( k \in K \) and \( a \in \text{Spin}(p) \). Then for all \( k \in K \),

\[
\pi_K^{-1}(k) \cong \pi^{-1}(\text{Ad}(k)) \cong \mathbb{Z}_2,
\]

so \( \pi_K \) is a double covering map. Since \( G/K \) is contractible, \( \tilde{K} \) is the maximal compact subgroup of a double cover of \( G \).

Suppose \( M \) has a \( G \)-equivariant Spin-structure \( P_M \to M \). In Section 3.2 of [6] and Section 3.2 of [8], an induction procedure of equivariant Spin\(^c\)-structures from \( N \) to \( M \) is described, which we will denote by \( \text{Ind}_N^M \) here. We will use the fact that any \( G \)-equivariant Spin-structure on \( M \) can be obtained via this induction procedure. (See also Proposition 3.10 in [8].)

**Lemma 5.** There is a \( K \)-equivariant Spin-structure \( P_N \to N \) such that

\[
P_M = \text{Ind}_N^M (P_N).
\]
Proof. Let $\mathfrak{p}_N \to N$ be the trivial vector bundle $N \times p \to N$, with the diagonal $K$-action. It has the $K$-equivariant Spin-structure

$$N \times \text{Spin}(p) \to N$$

on $\mathfrak{p}_N$, where $K$ acts diagonally on $N \times \text{Spin}(p)$ via the lift (1.2) of the adjoint action by $K$ on $p$. Since

$$TM = G \times_K (TN \oplus \mathfrak{p}_N)$$

(see Proposition 2.1 and Lemma 2.2 in [6]), the restriction $P_{M|N}$ is a $K$-equivariant Spin-structure on

$$TM|_N = TN \times \mathfrak{p}_N.$$ 

In the proof of the two out of three lemma in Section 3.1 of [10], a Spin$^c$-structure $P_N$ on $TN$ is constructed given the above data. In this case, this is a $K$-equivariant Spin structure on $TN$. In Lemma 3.9 of [8], it is shown that

$$P_M = \text{Ind}^M_N(P_N).$$ 

\[ \square \]

2 Proof of Theorem 2

Suppose $M/G$ is compact.

The quantisation commutes with induction techniques of [6, 7], suitably adapted to the Spin-setting, allow us to deduce our main result from Atiyah and Hirzebruch’s Theorem 1. This involves the Dirac induction map

$$\text{D-Ind}_K^G : R(K) \to K_*(C^*_rG),$$

which is an isomorphism for almost connected Lie groups, cf. [5]. We will use the fact that it relates the equivariant indices of the Spin-Dirac operators $\mathfrak{g}_N$ on $N$ and $\mathfrak{g}_M$ on $M$, associated to the Spin-structures $P_N$ and $P_M$, respectively, to each other. (See also Theorem 5.7 in [8].)

Proposition 6. One has

$$\text{D-Ind}_K^G(\text{index}_K(\mathfrak{g}_N)) = \text{index}_G(\mathfrak{g}_M) \in K_*(C^*_rG).$$
Proof. Let $K^*_K(N)$ and $K^*_G(M)$ be the equivariant $K$-homology groups \[3\] of $N$ and $M$, respectively. In Theorem 4.6 in \[6\] and Theorem 4.5 in \[7\], a map

$$K\text{-Ind}^G_K : K^*_K(N) \rightarrow K^*_G(M)$$

is constructed, such that the following diagram commutes:

$$
\begin{array}{c}
K^*_G(M) \xrightarrow{\text{index}_G} K^*_*(C^*_r G) \\
\downarrow \text{K-Ind}^G_K \\
K^*_K(N) \xrightarrow{\text{index}_K} D\text{-Ind}^G_K
\end{array}
$$

In Section 6 of \[6\], it is shown that the $K$-homology class of a Spin$^c$-Dirac operator on $N$, associated to a connection $\nabla^N$ on the determinant line bundle of a Spin$^c$-structure, is mapped to the class of a Spin$^c$-Dirac operator on $M$ associated to a connection $\nabla^M$ induced by $\nabla^N$ on the determinant line bundle of the induced Spin$^c$-structure, by the map $K\text{-Ind}^G_K$. In the Spin-setting, both connections $\nabla^N$ and $\nabla^M$ are trivial connections on trivial line bundles. Hence one gets

$$K\text{-Ind}^G_K(\partial / N) = [\partial / M],$$

and the result follows. \[ \square \]

This allows us to deduce Theorem 2 from Theorem 1.

Proof of Theorem 2. In the setting of Theorem 2, let $N \subset M$ be as in Theorem 3. Consider a $K$-equivariant Spin-structure on $N$ as in Lemma 5. By Proposition 6, we have

$$\text{index}_G(\partial_M) = D\text{-Ind}^G_K(\text{index}_K(\partial_N)).$$

The stabiliser of a point $m \in M$ is a maximal compact subgroup of $G$ if and only if $m \in M_{(K)}$. Hence, by Lemma 4, the condition on the stabilisers of the action by $G$ on $M$ is equivalent to the action by $K$ on $N$ being nontrivial. So Theorem 1 implies that

$$\text{index}_K(\partial_N) = 0,$$

and the result follows. \[ \square \]
3 Consequences

Let $c \in C^\infty_c(M)$ be a cutoff function, that is a non-negative function satisfying
\[ \int_M c(g^{-1}m) \, dg = 1 \]
for all $m \in M$. Let $\tau : C^*_r G \to \mathbb{C}$ be the von Neumann trace determined by
\[ \tau(R(f)^*R(f)) = \int_G |f(g)|^2 \, dg, \]
for $f \in L^1(G) \cap L^2(G)$, where $R$ denotes the right regular representation. This induces a morphism $\tau_* : K_*(C^*_r G) \to \mathbb{R}$. The following fact follows immediately from Theorem 2 and Theorem 6.12 in [12].

**Corollary 7.** Under the hypotheses of Theorem 2, one has
\[ 0 = \tau_* (\text{index}_G(\emptyset_M)) = \int_M c \cdot \hat{A}(M). \]

We note that the right hand side of (3.1) is independent of the choice of cutoff function $c$, cf. [12].

As an application of Corollary 7, one has the following generalisation of the second theorem in Section 2.2 of [2].

**Corollary 8.** Let $M$ be a complete, connected, oriented Riemannian manifold with $w_2(M) = 0$ and suppose that $\int_M c \cdot \hat{A}(M) \neq 0$. Then any closed subgroup $G$ (in the compact–open topology) of the group of all orientation preserving isometries of $M$ is a discrete group, if there is a point in $M$ whose stabiliser in $G$ is not a maximal compact subgroup of $G$.

**Proof.** In this setting, the Myer–Steenrod theorem implies that $G$ is a Lie group. The action on $M$ by the identity component $G_0$ of $G$ satisfies the conditions of Theorem 2. So if $G_0$ is nontrivial, then $\int_M c \cdot \hat{A}(M) = 0$ by Corollary 7. \qed

Let $G$ be a connected Lie group acting properly on a manifold $M$. Then by Abels’ Theorem 3, there is a proper equivariant projection map.
$p: M \to G/K$, where $K$ is a maximal compact subgroup of $G$. The map $p_*$ induced on $K$-homology relates the equivariant indices on $M$ and $G/K$ by the diagram

$$
\begin{array}{c}
K^G_*(M) \\
\downarrow p_* \\
K^G_*(G/K)
\end{array}
\xrightarrow{\text{index}_G} \xleftarrow{\text{index}_G} K_*(C^*_r G).
$$

It was shown in [5], Theorem 1.1, that the equivariant index on $G/K$ defines an isomorphism $K^G_*(G/K) \cong K_*(C^*_r G)$. Using this, we deduce an equivalent statement of Theorem 2 that does not use $C^*_r G$ or the equivariant index.

**Corollary 9.** Under the hypotheses of Theorem 2, one has

$$p_*[\varnothing_M] = 0 \in K^G_*(G/K).$$

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