DU VAL CURVES AND THE POINTED BRILL-NOETHER THEOREM

GAVRIL FARKAS AND NICOLA TARASCA

Abstract. We show that a general curve in an explicit class of what we call Du Val pointed curves satisfies the Brill-Noether Theorem for pointed curves. Furthermore, we prove that a generic pencil of Du Val pointed curves is disjoint from all Brill-Noether divisors on the universal curve. This provides explicit examples of smooth pointed curves of arbitrary genus defined over $\mathbb{Q}$ which are Brill-Noether general. A similar result is proved for 2-pointed curves as well using explicit curves on elliptic ruled surfaces.

The pointed Brill-Noether Theorem concerns the study of linear series on a general pointed algebraic curve $[C, p]$ with prescribed ramification at the marked point $p$. Recall that for a point $p \in C$ and a linear series $\ell = (L, V) \in G^d_d(C)$, one denotes by

$$\alpha^\ell(p) : 0 \leq \alpha_0^\ell(p) \leq \ldots \leq \alpha_r^\ell(p) \leq d - r$$

the ramification sequence of $\ell$ at $p$. One says that $p \in C$ is a ramification point of $\ell$ if $\alpha_r^\ell(p) > 0$. For instance, the ramification points of the canonical linear series are precisely the Weierstrass points of $C$. The total number of ramification points of $\ell$, counted with appropriate multiplicities, is given by the Plücker formula, see for instance [EH1] Proposition 1.1. Fixing a Schubert index $\alpha : 0 \leq \alpha_0 \leq \ldots \leq \alpha_r \leq d - r$, one can ask when a general pointed curve $[C, p]$ of genus $g$ carries a linear series $\ell \in G^d_d(C)$ with ramification sequence $\alpha^\ell(p) \geq \alpha$. The locus $G^r_d(C, p, \alpha)$ of linear series on $C$ satisfying this condition is a generalized determinantal variety of expected dimension

$$\rho(g, r, d, \alpha) := \rho(g, r, d) - w(\alpha),$$

where $\rho(g, r, d) := g - (r + 1)(g - d + r)$ and $w(\alpha) := \alpha_0 + \ldots + \alpha_r$ is the weight of $\alpha$. It is proved in [EH2] Theorem 1.1 that for a general pointed curve $[C, p] \in \mathcal{M}_{g,1}$, each component of $G^r_d(C, p, \alpha)$, if nonempty, has dimension precisely $\rho(g, r, d, \alpha)$. Moreover, [EH2] Proposition 1.2 establishes that $G^r_d(C, p, \alpha) \neq \emptyset$ if and only if

$$\sum_{i=0}^r \max\{\alpha_i + g - d + r, 0\} \leq g.$$
Since curves in the polarization class of a $K3$ surface have no obvious distinguished marked points, it is far from clear how to extend the results of [Laz] to the case of pointed curves. In [ABFS], an explicit specialization of Lazarsfeld’s curves emerging from the paper [ABS] is worked out. It is shown that suitably general singular plane curves of degree $3g$ having multiplicity $g$ at eight points in $\mathbb{P}^2$ and multiplicity $g-1$ at a further ninth point verify the Brill-Noether-Petri Theorem. Such curves, which belong to the closure in $\overline{\mathcal{M}}_g$ of the locus of curves lying on $K3$ surfaces, are called Du Val curves of genus $g$.

One aim of this paper is to show that the Du Val curves introduced in [ABFS] lead to Brill-Noether general smooth pointed curves of any genus defined over $\mathbb{Q}$. The essential observation is that, unlike curves on general $K3$ surfaces, Du Val curves have a distinguished marked point with respect to which a pointed Brill-Noether Theorem can be established.

We begin by recalling the setting of [ABFS]. Let $S'$ be the blow-up of $\mathbb{P}^2$ at nine points $p_1, \ldots, p_9$ which are general in the sense of [ABFS] (see also Section 1 for the precise definition). Let $E_1, \ldots, E_9$ be the exceptional curves on $S'$. We denote by $J' \in |-K_{S'}|$ the unique smooth plane cubic passing through $p_1, \ldots, p_9$ and consider the linear system on $S'$

$$L_g := |3g\ell - gE_1 - \cdots - gE_8 - (g-1)E_9|,$$

where $\ell \in \text{Pic}(S')$ is the proper transform of a line in $\mathbb{P}^2$. The main result of [ABFS] is that a general curve $C' \in L_g$ verifies the Brill-Noether-Petri Theorem. For each $g \geq 1$, the points $p_1, \ldots, p_9$ determine a 10-th point $p_{10}^{(g)}$ which is the base point of $L_g$. In fact, $p_{10} \in C' \cdot J'$, for every $C' \in L_g$. The point $p_{10}$ is determined by the relation

$$p_{10} = p_{10}^{(g)} = -gp_1 - \cdots - gp_8 - (g-1)p_9 \in J',$$

with respect to the group law of the elliptic curve. Under the genericity assumptions on the points $p_1, \ldots, p_9$ we started with, the points $p_{10}^{(g)}$ are distinct from one another, as well as from $p_1, \ldots, p_9$, see also Proposition 1. As in [ABFS], we set $S := \text{Bl}_{p_{10}}(S)$ and, by slight abuse of notation, we denote by $E_1, \ldots, E_{10}$ the corresponding exceptional curves. If $C$ is the strict transform of $C'$, then $|\mathcal{O}_S(C)|$ is a base point free linear system of curves of genus $g$ having a section induced by $E_{10}$ (note that $C \cdot E_{10} = 1$).

A pointed Du Val curve is a smooth pointed curve $[C, p] \in \overline{\mathcal{C}}_g$, where $C \subset S$ is as above and $\{p\} = C \cdot E_{10}$. Before stating our main results, we recall that for a linear system $\ell \in G_d^r(C)$ and points $p_1, \ldots, p_n \in C$, the pointed Brill-Noether number is defined as

$$\rho(\ell, p_1, \ldots, p_n) := \rho(g, r, d) - w(\alpha^\ell(p_1)) - \cdots - w(\alpha^\ell(p_n)).$$

**Theorem 1.** A general pointed Du Val curve $[C, p]$ verifies the pointed Brill-Noether Theorem, that is, $\dim G_{d}^r(C, p, \alpha) = \rho(g, r, d, \alpha)$, when $G_{d}^r(C, p, \alpha) \neq \emptyset$. In particular, for every linear system $\ell$ on $C$, one has $\rho(\ell, p) \geq 0$.

Since the points $p_1, \ldots, p_9$ can be chosen to have rational coefficients, $p = p_{10}^{(g)} \in \mathbb{P}^2(\mathbb{Q})$ and then $[C, p]$ is also defined over $\mathbb{Q}$. Hence, paralleling [ABFS] Corollary 1.3, our Theorem 1 provides examples of Brill-Noether general pointed curves of arbitrary genus $g$ defined over $\mathbb{Q}$.
If \( W_g \) denotes the locus of Weierstrass points in \( C_g \) (known to be an irreducible divisor on the universal curve), by direct calculation we show that the image of the family \( j : \mathbb{P}^1 \to \mathcal{C}_g \) induced by a Lefschetz pencil of Du Val curves on \( S \) satisfies 
\[
j(\mathbb{P}^1) \cap W_g = \emptyset,
\]
that is, for every pointed Du Val curve \([C,p]\), the marked point \( p \) is not a Weierstrass point of \( C \). As we point out in Corollary 1, this implies that \( j(\mathbb{P}^1) \) is disjoint from all pointed Brill-Noether divisors on \( \mathcal{C}_g \). We refer to Section 1 for detailed background on pointed Brill-Noether divisors on \( \mathcal{C}_g \).

0.1. **Brill-Noether general 2-pointed curves on elliptic ruled surfaces.** The Brill-Noether problem can be formulated for \( n \)-pointed curves \([C, p_1, \ldots, p_n]\) and concerns the variety of linear series \( \ell \in G^d_r(C) \) having prescribed ramification \( \alpha_i(\ell) \geq \alpha^i \) for \( i = 1, \ldots, n \), given in terms of fixed Schubert indices \( \alpha^1, \ldots, \alpha^n \). In Section 2, using decomposable elliptic ruled surfaces, we exhibit for the first time examples of smooth 2-pointed curves of arbitrary genus verifying the 2-pointed Brill-Noether Theorem. The construction is inspired by a very nice note of Treibich [Treb].

We start with an elliptic curve \( J \) and a non-torsion line bundle \( \eta \in \text{Pic}^0(J) \). The decomposable ruled surface 
\[
\phi : Y := \mathbb{P}(\mathcal{O}_J \oplus \eta) \to J
\]
is endowed with two disjoint sections \( J_0 \) and \( J_1 \) respectively. Pick a point \( r \in J \) and denote by \( f := \phi^{-1}(r) \) the corresponding ruling of \( Y \). We denote by \( s = s(g) \in J \) the point determined by the equation \( \mathcal{O}_J(s-r) = \eta^\otimes g \). The linear system \( |gJ_0 + f| \) consists of curves of genus \( g \) and has two base points, namely 
\[
\{p\} := \phi^{-1}(r) \cdot J_1 \quad \text{and} \quad \{q\} := \phi^{-1}(s) \cdot J_0,
\]
respectively. We establish the following result:

**Theorem 2.** The 2-pointed curve \([C, p, q] \in \mathcal{M}_{g,2}\), where \( C \in |gJ_0 + f| \) is a general element and \( p \) and \( q \) are as above, verifies the 2-pointed Brill-Noether Theorem. In particular, for every linear series \( \ell \in G^d_r(C) \) the inequality \( \rho(\ell, p, q) \geq 0 \) holds.

A Brill-Noether general 2-pointed curve supports a Brill-Noether general 1-pointed curve obtained by dropping either marked point. In particular, both 1-pointed curves \([C, p]\) and \([C, q]\) in the statement of Theorem 2 verify the 1-pointed Brill-Noether Theorem as well. For details, we refer to Section 2. The proofs of both Theorems 1 and 2 are intimately related, and rely on a canonical degeneration within the corresponding linear system on the surface to a singular curve with an elliptic tail. This leads to an inductive argument in the genus, which ultimately proves the desired Brill-Noether type theorems.

Arguably, for many applications, the curves constructed in Theorem 2 are the simplest known examples of Brill-Noether general smooth curves of arbitrary genus. They combine two desirable features: (i) The canonical elliptic tail degeneration in \( |gJ_0 + f| \) provides a system of Brill-Noether general curves of any genus on the surface \( Y \), which invites inductive proofs and reduction to genus 1 curves and Schubert calculus problems in the spirit of limit linear series, and (ii) The general curve in \( |gJ_0 + f| \) being smooth, one need not build-up the degeneration set-up typical for limit linear series applications. A vivid
instance of their use is the recent proof in [FK1] of the Prym-Green Conjecture concerning the naturality of the resolution of a paracanonical curve $\varphi_{K\otimes\eta} : C \hookrightarrow \mathbb{P}^{g-2}$, where $C$ is a general curve of odd genus and $\eta$ is an $\ell$-torsion line bundles on $C$. The conjecture is proven for odd $g$ and arbitrary $\ell$ using precisely the curves constructed in Theorem 2. For a proof of the Prym-Green Conjecture using special $K3$ surfaces instead — but only in the range $\ell \geq \sqrt{\frac{g+2}{2}}$ — see [FK2].

Finally, we show in Theorem 4 that Brill-Noether general one-pointed smooth curves can be constructed also on indecomposable elliptic ruled surfaces.

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1. Pointed Du Val curves and Weierstrass points

We assume familiarity with the theory of limit linear series in the sense of [EH1]. We need a few facts concerning divisor classes on the universal curve $\mathcal{C}_g := \mathcal{M}_g$. The rational Picard group $\text{Pic}(\mathcal{C}_g)$ is generated by the Hodge class $\lambda$, the relative cotangent class $\psi$, the boundary divisor class $\delta_{\text{irr}} := [\Delta_{\text{irr}}]$ of irreducible pointed stable curves of genus $g$ and by the classes $\delta_i := [\Delta_i]$, where for each $i = 1, \ldots, g-1$, the boundary divisor $\Delta_i \subset \mathcal{C}_g$ corresponds to a transverse union of two smooth curves of genus $i$ and $g-i$ respectively, meeting in one point, the marked points lying on the genus $i$ component. If $\pi : \mathcal{C}_g \rightarrow \mathcal{M}_g$ is the morphism forgetting the marked point, the boundary divisors on $\mathcal{M}_g$ and those on $\mathcal{C}_g$ are related by the following formulas:

$$\pi^*(\delta_{\text{irr}}) = \delta_{\text{irr}}, \quad \pi^*(\delta_i) = \delta_i + \delta_{g-i}, \quad \text{for } 1 \leq i < \frac{g}{2}, \quad \text{and } \quad \pi^*(\delta_{\frac{g}{2}}) = \delta_{\frac{g}{2}}, \quad \text{for } g \text{ even.}$$

If $\alpha : 0 \leq \alpha_0 \leq \ldots \leq \alpha_r \leq d-r$ is a Schubert index of type $(r,d)$, we introduce the complementary Schubert index $\alpha^c : 0 \leq d-r - \alpha_r \leq \ldots \leq d-r - \alpha_0 \leq d-r$. When $\alpha_i = 0$ for $i = 0, \ldots, r$, we say that $\alpha$ is the trivial Schubert index. We recall the definition of pointed Brill-Noether divisors on $\mathcal{C}_g$. Fix integers $r, d \geq 1$ and a Schubert index $\alpha : 0 \leq \alpha_0 \leq \ldots \leq \alpha_r \leq d-r$ such that the expected dimension of the locus of linear series of type $g^r_d$ on a curve of genus $g$ with prescribed ramification $\alpha$ at a given point equals $-1$. In other words,

$$\rho(g, r, d, \alpha) := g - (r+1)(g-d+r) - w(\alpha) = -1.$$ 

Let $\mathcal{C}^r_{g,d}(\alpha) := \{[C, p] \in \mathcal{C}_g : G^r_d(C, p, \alpha) \not= \emptyset\}$ be the corresponding pointed Brill-Noether locus. For instance,

$$\mathcal{W}_g := \mathcal{C}^{g-2}_{g,2g-2}(0, \ldots, 0, 1) = \{[C, p] \in \mathcal{C}_g : H^0(C, \omega_C(-gp)) \not= 0\}$$

is the divisor of Weierstrass points. Since $\mathcal{W}_g$ can be parametrized by the Hurwitz space of $g$-fold covers of $\mathbb{P}^1$ having a point of total ramification, it follows from [Arb] Theorem 2.5 that $\mathcal{W}_g$ is an irreducible divisor. If $\rho(g, r, d) = -1$, then $\mathcal{C}^r_{g,d}(0, \ldots, 0)$ is the pull-back to $\mathcal{C}_g$ of the Brill-Noether divisor $\mathcal{M}^r_{g,d}$ consisting of curves carrying a $g^r_d$. 

Cukierman [Cuk] computed the class of the closure $\overline{W}_g$ of the Weierstrass divisor in $\overline{C}_g$:

$$(2) \quad \overline{W}_g = -\lambda + \binom{g+1}{2} \psi - \sum_{i=1}^{g-1} \binom{g-i+1}{2} \delta_i \in \text{Pic}(\overline{C}_g).$$

We also recall [EH2] that the class of the pull-back to $\overline{C}_g$ of the Brill-Noether divisors $\overline{M}_{g,d}$ is given by the formula

$$(3) \quad [\overline{C}_{g,d}(0, \ldots, 0)] = c_{g,d,r} \cdot \overline{B}N_g,$$

where $c_{g,d,r} \in \mathbb{Q}_{>0}$ and

$$\overline{B}N_g := (g+3)\lambda - \frac{g+1}{6} \delta_{\text{irr}} - \sum_{i=1}^{g-1} i(g-i) \delta_i \in \text{Pic}(\overline{C}_g).$$

Remarkably, the pointed Brill-Noether divisors only span a 2-dimensional cone in $\text{Pic}(\overline{C}_g)$. It is shown in [EH3] Theorem 1.2 that $\overline{C}_{g,d}(\alpha)$ is a proper subvariety of $\overline{C}_g$, having a unique divisorial component. The class of this component, which we shall denote by $[\overline{C}_{g,d}(\alpha)] \in CH^1(\overline{C}_g)$, can be written as a linear combination

$$[\overline{C}_{g,d}(\alpha)] = \mu \cdot [\overline{W}_g] + \nu \cdot \overline{B}N_g,$$

for non-negative rational constants $\mu$ and $\nu$, which are determined in [FT].

**Definition 1.** We say that a pointed curve $[C, p] \in \overline{C}_g$ is Brill-Noether general, if for every choice of integers $r, d$ and a corresponding Schubert index $\alpha$ of type $(r, d)$, we have

$$\dim G^r_d(C, p, \alpha) = \rho(g, r, d, \alpha) \quad \text{or} \quad G^r_d(C, p, \alpha) = \emptyset.$$

In particular, for every linear series $\ell \in G^r_d(C)$, the inequality $\rho(\ell, p) \geq 0$ holds.

If $[C, p]$ is a Brill-Noether general pointed curve, by letting $\alpha$ be the trivial Schubert index, we obtain that $C$ is a Brill-Noether general (unpointed) curve.

**Lemma 1.** A pointed curve $[C, p] \in \overline{C}_g$ carries no linear series $\ell$ with $\rho(\ell, p) < 0$ if and only if it does not belong to any locus $\overline{C}_{g,d}(\alpha)$, where $\rho(g, r, d, \alpha) = -1$.

**Proof.** One implication being obvious, assume first there exists a linear series $\ell \in G^r_d(C)$ with $w(\alpha^\ell(p)) > \rho(g, r, d) \geq 0$. Then we can find a Schubert index

$$\alpha' : 0 \leq \alpha'_0 \leq \ldots \leq \alpha'_r \leq d - r$$

with $w(\alpha') = \rho(g, r, d) + 1 \leq w(\alpha^\ell(p))$, such that $\alpha' \leq \alpha^\ell(p)$ (lexicographically). Hence $\rho(g, r, d, \alpha') = -1$ and $[C, p] \in C_{g,d}(\alpha')$. Finally, assume we are in the case when there exists a linear series $\ell \in G^r_d(C)$ with $\rho(g, r, d) < -1$. Then we can find $d' > d$ and a Schubert index $\alpha' : 0 \leq \alpha'_0 \leq \ldots \leq \alpha'_{d'} \leq d'-r$ with $\alpha'_0 \leq d' - d$ and $w(\alpha') = \rho(g, r, d') + 1$. Hence $[C, p] \in C_{g,d'}(\alpha')$, which finishes the proof. $\square$

We now turn to Du Val surfaces. In what follows, we denote by $\equiv$ linear equivalence of divisors on varieties. Following [ABFS] Proposition 2.3, we recall that a set of nine distinct
points $p_1, \ldots, p_9$ in $\mathbf{P}^2$ is said to be general if on the blown-up plane $S' := \text{Bl}_{(p_1, \ldots, p_9)}(\mathbf{P}^2)$, every effective divisor

$$D' = d\ell - \nu_1 E_1 - \cdots - \nu_9 E_9$$

with $\nu_i \geq 0$ and satisfying $D \cdot J' = 0$ is necessarily a multiple of $J'$. In particular, if $p_1, \ldots, p_9$ are general points, then the sum $p_1 + \cdots + p_9 \in J'$ is not torsion.

**Remark 1.** Examples of sets of nine general points in $\mathbf{P}^2(\mathbb{Q})$ are easy to produce, if one starts with a concrete elliptic curve defined over $\mathbb{Q}$. For instance, it is shown in [ABFS] that the following points lying on the elliptic curve $E : y^2 = x^3 + 17$ are general: $p_1 = (-2, 3)$, $p_2 = (-1, -4)$, $p_3 = (2, 5)$, $p_4 = (4, 9)$, $p_5 = (52, 375)$, $p_6 = (5234, 37866)$, $p_7 = (8, -23)$, $p_8 = (43, 282)$, and $p_9 = \left(\frac{1}{7}, -\frac{33}{7}\right)$.

Recall the definition (1) of the points $p^{(g)}_{10} \in J'$, where $g \geq 1$.

**Proposition 1.** Assume that the points $p_1, \ldots, p_9$ are general. Then for $k = 2, \ldots, g$, the difference $p^{(k)}_{10} - p^{(k-1)}_{10} \in \text{Pic}^0(J')$ is not torsion.

**Proof.** Using (1), we obtain that $p^{(k-1)}_{10} - p^{(k)}_{10} = p_1 + \cdots + p_9$ (with respect to the group law of $J'$), for each $k \geq 2$. As pointed out, this is not a torsion point on $J'$.

We now introduce the pointed Du Val pencil in $\overline{\mathcal{C}}_g$, which is a lift under the forgetful map $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ of the pencil of unpointed curves introduced in Section 4 of [ABFS]. Recall that $S := \text{Bl}_{p^{(g)}_{10}}(S')$ and we denote by $L_g$ the proper transform of the linear system on $S'$ denoted by the same symbol in the Introduction. The linear system of Du Val curves of genus $g - 1$ on $S$, that is,

$$\Lambda_{g-1} := |3(g - 1)\ell - (g - 1)E_1 - \cdots - (g - 1)E_8 - (g - 2)E_9|$$

appears as a hyperplane in the $g$-dimensional linear system $L_g$. It consists precisely of the curves of the form $D + J \in L_g$, where $J \subset S$ denotes the proper transform of $J'$ and $D \in \Lambda_{g-1}$. Since $J \equiv 3\ell - E_1 - \cdots - E_{10}$, note that $D \cdot J = 1$.

We now choose a Lefschetz pencil in $L_g$, which has $2g - 2 = C^2$ base points. Let $X := \text{Bl}_{2g-2}(S)$ be the blow-up of $S$ at those points. Since $C \cdot E_{10} = 1$ for $C \in L_g$, the corresponding fibration $f : X \to \mathbf{P}^1$ has a section induced by the proper transform of $E_{10}$ on $X$. This induces a pencil in the universal curve

$$j : \mathbf{P}^1 \to \overline{\mathcal{C}}_g.$$ 

In what follows it will be convenient to use the notation $C_1 \cup_p C_2$, for a stable curve consisting of two irreducible components $C_1$ and $C_2$ respectively, meeting transversally at a point $p$.

**Proposition 2.** The intersection numbers of the pointed Du Val pencil with the generators of $\text{Pic}(\overline{\mathcal{C}}_g)$ are as follows:

$$j^*(\lambda) = g, \quad j^*(\psi) = 1, \quad j^*(\delta_{1r}) = 6(g + 1), \quad j^*(\delta_1) = 1, \quad j^*(\delta_i) = 0 \quad \text{for} \quad i = 2, \ldots, g - 1.$$  

**Proof.** One has $j^*(\psi) = -E^2_{10} = 1$. The restrictions of the classes $\lambda, \delta_{1r}, \delta_2, \ldots, \delta_{g-2}$ follow from [ABFS] Theorem 4.1 and are copied here for the sake of completeness. There
exists precisely one element of the pencil $f$ of the type $D + J$, for some $D \in \Lambda_{g-1}$. Since $E_{10} \cdot J = 1$ while $E_{10} \cdot D = 0$, the marked point lies on the elliptic component of this singular element. The corresponding pointed stable curve is $[D \cup_{p_{10}} (g-1) J', p_{10}^{(g)}] \in \overline{\mathcal{C}}_g$. Hence $j^*(\delta_1) = 1$, and since $\pi^*(\delta_1) = \delta_1 + \delta_{g-1}$, it follows that $j^*(\delta_{g-1}) = 0$. □

By direct computation, using (2) and (3), it follows that the pencil $j(P^1) \subset \overline{\mathcal{C}}_g$ has intersection number zero with the Brill-Noether class $\mathcal{BN}_g$ as well as with the Weierstrass divisor $\overline{W}_g$, that is,

$$j^*(\mathcal{BN}_g) = (g + 3)g - \frac{g + 1}{6}(6g + 6) - (g - 1) = 0, \quad \text{and}$$

$$j^*(\overline{W}_g) = -g + \left(\frac{g + 1}{2}\right) - \left(\frac{g}{2}\right) = 0.$$

Since the class of any pointed Brill-Noether divisor lies in the cone spanned by these classes [EH3] Theorem 1.2, it follows that the intersection number of $j(P^1)$ with the closure of any pointed Brill-Noether divisor is zero as well.

We are now in a position to complete the proof of our main result.

Proof of Theorem 1. We shall establish by induction on $g$ that the general member of the Du Val pencil satisfies the pointed Brill-Noether Theorem. For $g = 1$, we have that $[C, p] \in \mathcal{C}_1$ and it is well-known that each smooth pointed elliptic curve is Brill-Noether general, see e.g. [EH2] Theorem 1.1. Assuming the statement for Du Val curves of genus $g - 1$, suppose by contradiction that there exist $r, d \geq 1$ and a Schubert index $\alpha$ such that $\dim G^r_d(C, p, \alpha) > \rho(g, r, d, \alpha)$, for each $C \in L_g$, where $\{p\} = C \cap E_{10}$.

Let $j : P^1 \to \overline{\mathcal{C}}_g$ be a Lefschetz pencil of Du Val curves on $S$. As explained in Proposition 2, the pencil contains a unique elliptic tail degeneration $[D \cup_{p_{10}} (g-1) J', p_{10}^{(g)}]$, where $D$ is an element of $\Lambda_{g-1}$. Then the variety

$$\overline{\mathcal{C}}^r_d(D \cup J', p_{10}^{(g)}, \alpha)$$

of limit linear series $\ell = (\ell_D, \ell_J') \in G^r_d(D) \times G^r_d(J')$ on $D \cup_{p_{10}} (g-1) J'$ satisfying the ramification condition $\alpha^\ell(p_{10}^{(g)}) \geq \alpha$ is of dimension at least $\rho(g, r, d, \alpha) + 1$. Note that $[D, p_{10}^{(g-1)}]$ can be assumed to be a general Du Val curve of genus $g - 1$, for every curve from $\Lambda_{g-1}$ appears as an elliptic tail degeneration in a genus $g$ Du Val pencil.

Let $\ell$ be a general point of an irreducible component $Z$ of $\overline{G}^r_d(D \cup J', p_{10}^{(g)}, \alpha)$ of maximal dimension, and set $\beta := \alpha^\ell(p_{10}^{(g-1)})$. By the additivity of the Brill-Noether number with respect to marked points, we write

$$\rho(g, r, d, \alpha) = \rho(\ell, p_{10}^{(g)}) \geq \rho(\ell_D, p_{10}^{(g-1)}) + \rho(\ell_J', p_{10}^{(g)} \cup p_{10}^{(g-1)}).$$

By the construction in [EH1] Theorem 3.3 of the variety of limit linear series, $Z$ is birational to an irreducible component of the product

$$G^r_d(D, p_{10}^{(g-1)}, \beta) \times G^r_d(J', (p_{10}^{(g-1)}, \beta^c), (p_{10}^{(g)}, \alpha)).$$
By assumption, each component of $G_d'(D, p_{10}^{(g-1)}, \beta)$ has dimension $\rho(g - 1, r, d, \beta)$.

Moving to $J'$, first observe that $\rho(J', p_{10}^{(g)}, p_{10}^{(g-1)}) \geq 0$. Indeed, assuming otherwise, we denote by $(a_0, \ldots, a_r)$ and $(b_0, \ldots, b_r)$ the vanishing sequences of $\ell_{J'}$ at the points $p_{10}^{(g-1)}$ and $p_{10}^{(g)}$ respectively, and obtain that there exist indices $0 \leq i < j \leq r$ such that

$$a_i + b_{r-i} = a_j + b_{r-j} = d.$$  

In particular, the underlying line bundle of the linear series $\ell_{J'}$ corresponds to the divisors $a_i \cdot p_{10}^{(g-1)} + b_{r-i} \cdot p_{10}^{(g)} \equiv a_j \cdot p_{10}^{(g-1)} + b_{r-j} \cdot p_{10}^{(g)}$, from which it follows that $p_{10}^{(g-1)} - p_{10}^{(g)}$ is a torsion class in $\text{Pic}^0(J')$, which contradicts Proposition 1.

Furthermore, it implicitly follows from [EHI], and it is spelled-out explicitly in [Oss] Lemma 2.1, that every 2-pointed elliptic curve $[E, x, y] \in \mathcal{M}_{1,2}$, where the difference $O_E(x - y)$ is not a torsion class, is Brill-Noether general. This follows from the observation that for a line bundle $L \in \text{Pic}^d(E)$ which is not given by a divisor on $E$ supported only at $x$ and $y$, the flags in $H^0(E, L) \cong \mathbb{C}^d$ given by the vanishing of sections at $x$ and $y$ respectively, are transversal. In particular, Schubert cycles in $G(r + 1, H^0(E, L))$ defined in terms of these flags intersect in the expected dimension. Applying this fact to the case at hand, we find

$$\dim G_d'(J', (p_{10}^{(g-1)}, \beta^c), (p_{10}^{(g)}, \alpha)) = \rho(1, r, d, \beta^c, \alpha) := \rho(1, r, d) - w(\beta^c) - w(\alpha).$$

Putting all together, we obtain that

$$\rho(g, r, d, \alpha) < \dim Z = \dim G_d'(D, p_{10}^{(g-1)}, \beta) + \dim G_d'(J', (p_{10}^{(g-1)}, \beta^c), (p_{10}^{(g)}, \alpha))$$

$$= \rho(g - 1, r, d, \beta) + \rho(1, r, d, \beta^c, \alpha) \leq \rho(g, r, d, \alpha),$$

which is a contradiction. Therefore, the singular pointed curve $(D \cup J', p_{10}^{(g)})$ is Brill-Noether general.\hfill \Box

**Corollary 1.** The image of a Du Val pencil $j : \mathbb{P}^1 \to \overline{\mathcal{C}}_g$ is disjoint from all pointed Brill-Noether divisors $\overline{\mathcal{C}}_{g,d}(\alpha)$.

**Proof.** As noted in Proposition 2, we have $j(\mathbb{P}^1) \cdot \overline{\mathcal{C}}_{g,d}(\alpha) = 0$. Either $j(\mathbb{P}^1) \cap \overline{\mathcal{C}}_{g,d}(\alpha) = \emptyset$, or else, $j(\mathbb{P}^1) \subset \overline{\mathcal{C}}_{g,d}(\alpha)$. The proof of Theorem 1 rules out the second possibility.\hfill \Box
In general it is not known whether $C_{r,g,d}(\alpha)$ is pure of codimension 1. However, when this happens, for instance in the case of the Weierstrass divisor $W_g$, Corollary 1 shows that every pointed Du Val curve is Brill-Noether general with respect to linear series of that type.

1.1. Towards the effective cone of $\overline{C}_g$. The Slope Conjecture [HM] on effective divisors on $\overline{M}_g$ used to predict that the Brill-Noether divisors $\overline{M}_{g,d}$ of curves with a linear series $g$ where $\rho(g, r, d) = -1$ are extremal. Via Lazarsfeld’s result [Laz], an equivalent formulation of the Slope Conjecture is that the rational curve $R \subset \overline{M}_g$ induced by a Lefschetz pencil of genus $g$ curves on a general polarized $K3$ surface $(X, H)$, with $H^2 = 2g - 2$ is nef, that is, it intersects every effective divisor on $\overline{M}_g$ non-negatively. Note that the intersection numbers of $R$ with the generators of Pic$(\overline{M}_g)$ are given as follows, see for instance [FP]:

$$R \cdot \lambda = g + 1, \quad R \cdot \delta_{\text{irr}} = 6g + 18 \quad \text{and} \quad R \cdot \delta_i = 0, \quad \text{for } i = 1, \ldots, \left\lfloor \frac{g}{2} \right\rfloor.$$  

Although the Slope Conjecture is false for high $g$, see [FP] and [Far], it is known to hold for $g \leq 9$ and $g = 11$. The statement played an important role in Mukai’s work on alternative birational models of $\overline{M}_g$ for $g = 7, 8, 9$ and has guided the search for geometric divisors $D$ on $\overline{M}_g$ having small slope, that is, necessarily contain the locus in $\overline{M}_g$ of curves that lie on $K3$ surfaces.

It is an interesting question to find an adequate definition of the notion of slope for effective divisors on the universal curve and an analogue of the Slope Conjecture on $\overline{C}_g$.

Problem 1. For what values of $g$ is the Du Val pencil $j : P^1 \to \overline{C}_g$ nef, that is, $j^*(D) \geq 0$, for every effective divisor $D$ on $\overline{C}_g$? For which $g$ does this inequality hold for all effective divisors $D$ on $\overline{C}_g$ such that $\pi(D) = \overline{M}_g$?

In light of Corollary 1, a closely related question is whether the Weierstrass divisor $\overline{W}_g$ is extremal in the effective cone Eff$(\overline{C}_g)$. The hypothesis that $\overline{W}_g$ is extremal has recently received further credence due to [Pol]. Note that for the pull-backs to $\overline{C}_g$ of the effective divisors on $\overline{M}_{g_1+10}$ constructed in [Far], Problem 1 has a negative answer. For instance, when $g = 10$, the divisor in question is

$$Z_{10} := \{ [C, p] \in \mathcal{C}_{10} : C \text{ lies on a } K3 \text{ surface} \},$$

and $[Z_{10}] = 7\lambda - 5\delta_{\text{irr}} - \delta_1 - \delta_0 - 12\delta_2 - 12\delta_8 - \cdots \in \text{Pic}(\mathcal{C}_{10})$, see [FP] Theorem 1.6. By applying Proposition 2, we compute $j^*(\mathbb{Z}_{10}) = -1 < 0$. We are unaware of any example of an effective divisor $D$ on $\overline{C}_g$ that is not a pull-back of an effective divisor from $\overline{M}_g$ and which satisfies $j^*(D) < 0$.

2. Brill-Noether general two-pointed curves via elliptic surfaces

In this section we construct explicit smooth 2-pointed curves of arbitrary genus verifying the Brill-Noether Theorem. Given a smooth curve $C$, distinct points $p, q \in C$ and two Schubert indices

$$\alpha : 0 \leq \alpha_0 \leq \ldots \leq \alpha_r \leq d - r \quad \text{and} \quad \beta : 0 \leq \beta_0 \leq \ldots \leq \beta_r \leq d - r,$$
we consider the variety $G^\rho_d(C, (p, \alpha), (q, \beta))$ of linear series $\ell \in G^\rho_d(C)$ verifying ramification conditions at two points:

$$\alpha^\ell(p) \geq \alpha \quad \text{and} \quad \alpha^\ell(q) \geq \beta.$$ 

We say that $[C, p, q]$ satisfies the 2-pointed Brill-Noether Theorem if for any $\alpha$ and $\beta$,

$$\dim G^\rho_d(C, (p, \alpha), (q, \beta)) = \rho(g, r, d, \alpha, \beta) := \rho(g, r, d) - w(\alpha) - w(\beta),$$

unless $G^\rho_d(C, (p, \alpha), (q, \beta)) = \emptyset$. Eisenbud and Harris [EH2] Theorem 1.1 established the 2-pointed Brill-Noether Theorem for general 2-pointed curves by use of degeneration. As in the case of 1-pointed curves, up to now no explicit example of a smooth Brill-Noether general 2-pointed curve has been known. We construct such curves using decomposable elliptic ruled surfaces.

We start with an elliptic curve $J$ and consider a non-torsion line bundle $\eta \in \text{Pic}^0(J)$. Let

$$\phi : Y := \mathbb{P}(\mathcal{O}_J \oplus \eta) \to J$$

be the ruled surface corresponding to a decomposable rank 2 vector bundle. We denote by $J_0$ and $J_1$ the disjoint sections of $Y$ such that

$$N_{J_0/Y} = \eta \quad \text{and} \quad N_{J_1/Y} = \eta^\vee.$$ 

In particular, $J_0^2 = J_1^2 = 0$. Observe that $J_0 \equiv J_0 - \phi^*(\eta)$. We fix a point $r \in J$ and let $f = f_r := \phi^{-1}(r)$ be the corresponding ruling. For each $g \geq 1$, we denote by $s = s^{(g)}$ the point on the base elliptic curve $J$ determined by

$$\mathcal{O}_J(s^{(g)} - r) = \eta^\otimes g.$$ 

Since $\eta$ is not a torsion class, we have $s^{(g)} \neq r$, for all $g \geq 1$. Furthermore, the difference $s^{(g)} - s^{(g-1)} \in \text{Pic}^0(J)$ is not a torsion class. As explained in the Introduction, we set

$$\{p\} = J_1 \cdot f_r \quad \text{and} \quad \{q^{(g)}\} := J_0 \cdot f_{s^{(g)}}.$$ 

**Lemma 2.** We have that $h^0(Y, \mathcal{O}_Y(gJ_0 + f_r)) = g + 1$. The general point of the linear system $|gJ_0 + f_r|$ is a smooth curve of genus $g$ passing through the points $p$ and $q^{(g)}$.

**Proof.** By direct calculation, using Riemann-Roch, we find that

$$h^0(Y, \mathcal{O}_Y(gJ_0 + f_r)) = h^0(\mathcal{O}_J(r) \otimes \text{Sym}^g(\mathcal{O}_J \oplus \eta)) = \deg(\mathcal{O}_J(r) \otimes \text{Sym}^g(\mathcal{O}_J \oplus \eta)) = g + 1.$$ 

Furthermore, since $K_Y \equiv -2J_0 + \omega^*(\eta) \equiv -2J_1 + \phi^*(\eta^\vee)$, from the adjunction formula we obtain that a smooth curve $C \in |gJ_0 + f_r|$ has genus $g$.

From [FGP] Proposition 11, since $\eta$ is non-torsion, the base points of $|gJ_0 + f_r|$ lie on $J_0 + J_1 = [-K_Y]$. Since $\mathcal{O}_{J_1}(gJ_0 + f_r) = \mathcal{O}_{J_1}(p)$, the point $p$ must lie in the base locus of $|gJ_0 + f_r|$. Finally, since $\mathcal{O}_{J_0}(gJ_0 + f_r) = \eta^\otimes g \otimes \mathcal{O}_{J_0}(f_r) = \mathcal{O}_{J_0}(q^{(g)})$, it follows that $q^{(g)}$ belongs to each curve $C \in |gJ_0 + f_r|$. Hence, the base locus of $|gJ_0 + f_r|$ consists of the points $p$ and $q^{(g)}$. \qed
Therefore, on each curve from the linear system \(|gJ_0 + f|\) we can single out two marked points, \(p\) and \(q = q^{(g)}\). These are precisely the points for which the Brill-Noether Theorem will be proved.

**Theorem 3.** The 2-pointed curve \([C, p, q] \in \mathcal{M}_{g,2}\), where \(C \in |gJ_0 + f|\) is general and \(p\) and \(q := q^{(g)}\) are as above, verifies the 2-pointed Brill-Noether Theorem, that is,

\[
\dim G^r_d \left( C, (p, \alpha), (q, \beta) \right) = \rho(g, r, d, \alpha, \beta) \quad \text{or} \quad G^r_d \left( C, (p, \alpha), (q, \beta) \right) = \emptyset,
\]

for all Schubert indices \(\alpha\) and \(\beta\).

**Proof.** Assume by contradiction that for a 2-pointed curve \([C, p, q^{(g)}]\), where \(C \in |gJ_0 + f|\) is a general element, the Brill-Noether Theorem fails for certain Schubert indices \(\alpha\) and \(\beta\), that is, there exists a component of \(G^r_d \left( C, (p, \alpha), (q, \beta) \right)\) whose dimension exceeds \(\rho(g, r, d, \alpha, \beta)\). Then, similarly to the proof of Theorem 1, we consider a specialization of \(C\) to the sublinear system \(\{J_0\} + \{(g - 1)J_0 + f\} \cong \mathbb{P}^{g-1}\), which appears as a hyperplane in \(|gJ_0 + f| \cong \mathbb{P}^g\). The 2-pointed curve corresponding to the general element of this subsystem is a curve of the form

\[
[D \cup J_0, p \in D, q^{(g)} \in J_0] \in \mathcal{M}_{g,2},
\]

where \(D \in \{(g - 1)J_0 + f\}\) is a smooth curve of genus \(g - 1\) passing through \(p\) and the point \(q^{(g-1)} \in J_0 \cdot f_{s^{(g-1)}}\). Note that \(D \cap J_0 = \{q^{(g-1)}\}\). Observe moreover that under the isomorphism \(\phi = \phi_{|J_0} : J_0 \cong J\), we have

\[
q^{(g)} - q^{(g-1)} = \phi^* (s^{(g)}) - \phi^* (s^{(g-1)}) = \phi^* (\eta) \in \text{Pic}^0(J_0),
\]

that is, the difference \(q^{(g)} - q^{(g-1)}\) is not torsion on \(J_0\).

The proof now follows by induction. By semicontinuity, the variety of limit linear series \(\ell\) of type \(g^r_d\) on \(D \cup J_0\) verifying the ramification conditions \(\alpha^\ell(p) \geq \alpha\) and \(\alpha^\ell(q^{(g)}) \geq \beta\) must have a component \(Z\) of dimension strictly greater than \(\rho(g, r, d, \alpha, \beta)\). Denote by \(\ell = (\ell_D, \ell_{J_0})\) a general point of \(Z\). We may assume that \(\ell\) is a refined limit linear series. Set \(\gamma := \alpha^\ell_{\mathcal{M}}(q^{(g-1)})\). Then \(Z\) is birationally isomorphic to the product

\[
G^r_d \left( D, (p, \alpha), (q^{(g-1)}, \gamma) \right) \times G^r_d \left( J_0, (q^{(g-1)}, \gamma^c), (q^{(g)}, \beta) \right).
\]

By induction on the genus, we may assume that \([D, p, q^{(g-1)}] \in \mathcal{M}_{g-1,2}\) satisfies the 2-pointed Brill-Noether Theorem, in particular

\[
\dim G^r_d \left( D, (p, \alpha), (q^{(g-1)}, \gamma) \right) = \rho(g - 1, r, d, \alpha, \gamma).
\]

Since \(q^{(g)} - q^{(g-1)} \in \text{Pic}^0(J_0)\) is not torsion, as we have observed \([J_0, q^{(g-1)}, q^{(g)}] \in \mathcal{M}_{1,2}\) is a Brill-Noether general 2-pointed curve, hence

\[
\dim G^r_d \left( J_0, (q^{(g-1)}, \gamma^c), (q^{(g)}, \beta) \right) = \rho(1, r, d, \gamma^c, \beta).
\]

Using the additivity of the Brill-Noether number, we have

\[
\dim Z = \rho(g - 1, r, d, \alpha, \gamma) + \rho(1, r, d, \gamma^c, \beta) = \rho(g, r, d, \alpha, \beta),
\]

a contradiction. \(\square\)
Remark 2. Since a Brill-Noether general \( n \)-pointed curve supports a Brill-Noether general \( m \)-pointed curve for all \( m < n \) obtained by dropping \( n - m \) of the marked points, it follows that the curve \( C \in |gJ_0 + f_r| \) satisfies the (unpointed) Brill-Noether Theorem as well.

Remark 3. The Du Val curves considered in [ABFS] and in Section 1 of this paper are known to lie in the closure in \( \mathcal{M}_g \) of the locus of curves of genus \( g \) lying on a K3 surface. Algebraic surfaces \( \mathcal{S} \subset \mathbb{P}^g \) having canonical hyperplane sections have been classified by Epema [Epe]. All such surfaces are potentially limits in \( \mathbb{P}^g \) of smooth polarized K3 surfaces of degree \( 2g - 2 \). A criterion for when such surfaces smooth to K3 surfaces is given in [ABS] Corollary 26. Du Val surfaces, as well as the decomposable elliptic ruled surfaces considered in Theorem 2, are minimal models of corresponding instances of such objects, see [Epe], as well as [ABS] Proposition 29. It is natural to ask whether there are explicit examples of Brill-Noether general pointed curves, other than those which are limits of curves on K3 surfaces.

2.1. Brill-Noether general pointed curves on indecomposable elliptic ruled surfaces. Here we show how Brill-Noether general pointed smooth curves can be constructed also on indecomposable elliptic ruled surfaces. We fix again an elliptic curve \( J \) and denote by \( E \) the unique indecomposable vector bundle on \( J \) defined by the exact sequence

\[
0 \to \mathcal{O}_J \to \mathcal{E} \to \mathcal{O}_J \to 0.
\]

Let \( \varphi: X' := \mathbb{P}(\mathcal{E}) \to J \) be the induced ruled surface. We fix a point \( r \in J \) and set \( f := \varphi^{-1}(r) \), therefore \( f^2 = 0 \). Let \( J_0 \subset X' \) be the unique section of \( \varphi \) having \( \mathcal{N}_{J_0/X'} = \mathcal{O}_{J_0} \) and set \( \{q\} := J_0 \cdot f_r \). In a way similar to the proof of Lemma 2, one can show that the general element of the linear system \( |gJ_0 + f| \) is a curve of genus \( g \) passing through the point \( q \).

Each curve \( C \in |gJ_0 + f| \) has a distinguished marked point, namely \( q \in C \cdot J_0 \). In [Tre], Treibich considers curves in the linear system \( |gJ_0 + f| \) and sketches an argument using Fay’s trisecant formula for showing that a general curve \( C \in |gJ_0 + f| \) is Brill-Noether general. Reasoning in a way very similar to the proof of Theorem 3, we prove the stronger fact that the general curve \( [C, q] \) satisfies the pointed Brill-Noether Theorem.

Since the linear system \( |gJ_0 + f| \) cuts out on a general curve \( C \in |gJ_0 + f| \) the linear system \( |\omega_C| + 2g \), it follows that \( |gJ_0 + f| \) has a further base point \( q' \), infinitely near to \( q \).\(^1\) We denote by \( \varepsilon: X := \text{Bl}_{q,q'}(X') \to X' \) the blow-up of \( X' \) at \( q \) and \( q' \) and by \( E \), respectively \( E' \), the corresponding exceptional divisors. We keep denoting by \( J_0 \) and \( f \) the strict transforms of the curves denoted by the same symbols on \( X' \). Finally, let \( C \subset X \) be the strict transform of a curve in the linear system \( |gJ_0 + f| \). Then \( C \cdot E' = 1 \) and \( C^2 = 2g - 2 \). A Lefschetz pencil in this linear system induces a family of pointed curves

\[
\iota: \mathbb{P}^1 \to \overline{\mathcal{C}}_g.
\]

Proposition 3. The numerical features of the pencil \( \iota: \mathbb{P}^1 \to \overline{\mathcal{C}}_g \) are as follows:

\[
\iota^*(\lambda) = g - 1, \quad \iota^*(\psi) = 1, \quad \iota^*(\delta_{ir}) = 6(g - 1), \quad \iota^*(\delta_1) = 1, \quad \iota^*(\delta_{g-1}) = 1,
\]

and \( \iota^*(\delta_i) = 0 \) for \( i = 2, \ldots, g - 2 \).

\(^1\)Added in February 2023. We are grateful to Edoardo Sernesi for pointing out to us this fact, which was overlooked in the published version of this paper.
Proof. We blow-up $X$ at the $2g-2$ base points of a Lefschetz pencil in $|C|$ and denote by $h: \tilde{X} \to \mathbb{P}^1$ the induced fibration. Clearly $h^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ and $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^1(J, \mathcal{O}_J)$ is 1-dimensional, therefore $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. Accordingly, 

$$i^*(\lambda) = \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) + g - 1 = g - 1.$$ 

By the Noether formula, the total number of singular fibres in the pencil $i$ is given by 

$$t^*(\delta) = c_2(\tilde{X}) + 4g - 4 = 12\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) - K_X^2 + 4g - 4 = 6g - 4.$$ 

In the pencil $i$ there exists a unique curve from the linear system $|(g-1)J_0 + f|$ with $J_0$, which is viewed as a hyperplane inside $|gJ_0 + f|$. This singular curve is of the type 

$$t = [D \cup E \cup J_0, \tilde{q} := E \cdot E'] \in \overline{\mathcal{C}}_g,$$ 

where $D \in |(g-1)J_0 + f|$ is a smooth curve of genus $g - 1$ with $D \cap J_0 = \emptyset$ (on $X$). Forgetting the marked point $\tilde{q}$, the stable model of this curve is $[D \cup q J_0] \in \overline{\mathcal{M}}_g$. The point $t$ lies on both boundary divisors $\Delta_1$ and $\Delta_{g-1}$, which implies $i^*(\delta_1) = i^*(\delta_{g-1}) = 1$, therefore $i^*(\delta_{irr}) = 6(g - 1)$. 

Corollary 2. The numerical features of the pencil $i := \pi \circ t : \mathbb{P}^1 \to \overline{\mathcal{M}}_g$ obtained by forgetting the marked point, are given by: 

$$i^*(\lambda) = g - 1, \ i^*(\delta_{irr}) = 6(g - 1), \ i^*(\delta_1) = 2, \ i^*(\delta_i) = 0, \text{ for } i = 2, \ldots, \left\lfloor \frac{g}{2} \right\rfloor.$$ 

Proof. The only thing which has to be observed is that $i^*(\delta_1) = i^*(\delta_1) + i^*(\delta_{g-1}) = 2$. 

Using Proposition 3 it is now immediate to check that the pencil $i$, just like the Du Val pencil, satisfies the relations 

$$i^*(\mathcal{B}N_g) = 0 \text{ and } i^*(|\overline{\mathcal{M}}_g|) = 0.$$ 

Theorem 4. The general pointed curve $[C, q]$, where $C \in |gJ_0 + f|$ and $\{q\} = J_0 \cdot f$, verifies the pointed Brill-Noether Theorem. 

Proof. The proof proceeds by induction on $g$ in a way mirroring the proofs of Theorems 1 and 3. Assume by contradiction that the pointed Brill-Noether Theorem fails for every smooth curve $[C, q]$, where $C \in |gJ_0 + f|$. By choosing a Lefschetz pencil $i$ in $|gJ_0 + f|$ as above, the same conclusion holds for the degenerate pointed curve $t = [D \cup E \cup J_0, \tilde{q}]$. That is, the variety of limit linear series $\ell$ of type $g_d^r$ on $D \cup E \cup J_0$ such that $a^g(\tilde{q}) \geq \alpha$ has a component $Z$ of dimension strictly greater than $\rho(g, r, d, \alpha)$, for some $r, d,$ and $\alpha$. For $\ell = (\ell_D, \ell_E, \ell_J)$ a general point of $Z$, let $\gamma_D := \alpha^g_\ell(D \cdot E)$ and $\gamma_J := \alpha^g_\ell(J_0 \cdot E)$. Then $Z$ is birationally isomorphic to 

$$G^r_d(D, (D \cdot E, \gamma_D)) \times G^r_d(E, (E \cdot D, \gamma_E), (E \cdot J_0, \gamma_J), (\tilde{q}, \alpha)) \times G^r_d(J_0, (J_0 \cdot E, \gamma_J)).$$ 

Both the 3-pointed rational curve $[E, E \cdot D, E \cdot J_0, \tilde{q}]$, as well as the 1-pointed elliptic curve $[J_0, J_0 \cdot E] \in \mathcal{M}_{1, 1}$ verify the pointed Brill-Noether Theorem. By induction the same can be assumed for $[D, D \cdot E] \in \mathcal{C}_{g-1}$. It follows that 

$$\dim Z = \rho(g - 1, r, d, \gamma_D) + \rho(0, r, d, \gamma_D, \gamma_J, \alpha) + \rho(1, r, d, \gamma_J) = \rho(g, r, d, \alpha),$$ 

a contradiction.
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