ON THE COMPACTIFICATIONS OF SPACES WITH APPLICATIONS

A. TARIZADEH AND M. R. REZAEE HURI

Abstract. In this paper, new advances on the compactifications of topological spaces, in particular on the Alexandroff and Stone-
Čech compactifications have been made. Specially, it is proved that the minimal spectrum of the direct product of a family of integral domains and also the maximal spectrum of the direct product of a family of local rings both indexed by a set X are the Stone-Čech compactification of the discrete space X. These results improve all of the previous constructions of the Stone-Čech compactification of a discrete space. Some applications of this study are given. Specially and surprisingly, it is shown that the Stone-Čech compactification of an arbitrary topological space X is extracted from the Stone-Čech compactification of the discrete space X by passing to a its appropriate quotient. The Alexandroff compactification of a discrete space is also constructed by a new and interesting approach.

1. Introduction

Compactification is one of the main topics which is investigated in this paper from new points of view. Among various compactifications, the Stone-Čech compactification of a discrete space X is particularly important. One of the main reasons of its importance is that it admits a semigroup structure whenever X is a (commutative) semigroup, and this semigroup structure has vast and interesting applications in diverse fields of mathematics specially in combinatorial number theory, Ramsey theory, topological dynamics and Ergodic theory. Perhaps as another main reason for the importance of this compactification is its vital role in proving Theorem 5.4.

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An accessible concrete description of this compactification often remains elusive. For instance the semigroup $\beta N$, the Stone-Čech compactification of the natural numbers, is amazingly complicated and there are some unanswered questions about its structure. For example, whether or not $\beta N$ contains any elements of finite order which are not idempotent still remains a challenging open problem; please consider [5] and [14] and their rich bibliography for further studies.

Classically, the Stone-Čech compactification of a discrete space is usually constructed via the ultrafilters on that space. In this paper, following a suggestion of Pierre Deligne and then generalizing his idea, we find two new and very interesting ways to construct this compactification using only the standard methods of commutative algebra. In fact in Theorem 4.5 we prove that the minimal spectrum of the direct product of a family of integral domains indexed by a set $X$ is the Stone-Čech compactification of the discrete space $X$. In Theorem 6.4 it is also shown that the maximal spectrum of the direct product of a family of local rings indexed by $X$ is the Stone-Čech compactification of the discrete space $X$. These results improve all of the former constructions of the Stone-Čech compactification of a discrete space, and also show that this compactification is independent of choosing of integral domains and local rings. In particular, we get that $\beta X = \text{Spec} \mathcal{P}(X)$ here $\beta X$ denotes the Stone-Čech compactification of the discrete space $X$, and the classical approach is also recovered, see Remark 4.10. Then various applications of these results are given. In particular, these results allow us to understand the number of prime ideals of the infinite direct products of integral domains and local rings more precisely. Also surprisingly, the Stone-Čech compactification of an arbitrary topological space $X$ is obtained from the Stone-Čech compactification of the discrete space $X$ by passing to a its certain quotient, see Theorem 5.4.

We have also found a new way to build the Alexandroff (one-point) compactification of a discrete space, see Corollary 3.5. This result tells us that for any set $X$ then $\alpha X = \text{Spec} \mathcal{R}$ here $\alpha X$ denotes the Alexandroff compactification of the discrete space $X$ and $\mathcal{R}$ is the subring of $\mathcal{P}(X)$ consisting of all subsets of $X$ which are either finite or cofinite (i.e., its complement is finite). Then in Theorem 7.2, we show that every totally disconnected compactification of a discrete space $X$ is precisely of the form $\text{Spec} \mathcal{R}'$ where the ring $\mathcal{R}'$ satisfies in the extensions of rings $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{P}(X)$. (This result is also proved in [3] Theorems 2.2 and 2.3] by another approach using Boolean algebras). Therefore
all of the totally disconnected compactifications of a discrete space are in the scope of the Zariski topology. In particular, up to isomorphisms, this class forms a set and the extensions of the corresponding rings puts a partial ordering over this set in a way that the Alexandroff compactification is the minimal one and the Stone-Čech compactification is the maximal one. If $\tilde{X}$ is an arbitrary compactification of a discrete space $X$ then $\pi_0(\tilde{X})$, the space of connected components of $\tilde{X}$, is canonically isomorphic to $\text{Spec} \left( \text{Clop}(\tilde{X}) \right)$ and the ring $\text{Clop}(\tilde{X})$ satisfies in the extensions of rings $\mathcal{R} \subseteq \text{Clop}(\tilde{X}) \subseteq \mathcal{P}(X)$. In summary, all of the compactifications of a discrete space are characterized in more precise and certain way.

Extremally disconnected spaces possess interesting properties and they form a spectacular class of topological spaces. For instance, it is well known that in the category of compact spaces, the projective objects are precisely the extremally disconnected spaces. Using this result and Corollary 4.9, then we obtain two interesting results on extremally disconnected spaces, see Theorems 8.1 and 8.2.

It is well known that if the discrete space $X$ is also a (commutative) semigroup then its operation can be extended uniquely to an operation on $\beta X$ which forms a semigroup structure as well, see [5, Theorems 4.1 and 4.4]. This result opens new horizons to explore the basic and also sophisticated properties of the semigroup $\beta X$, though some of them have been done in the literature over the years (see [5] and its bibliography), but there is a pressing need for new constructions to aid the development and the understanding the algebraic structure of this semigroup specially $\beta \mathbb{N}$ more deeply. We have made very little contributions to this subject but the results are general and including Theorems 9.1, 9.2 and 9.3. Indeed, in Theorem 9.1 we reformulate this important result into a more standard form and then it is proven by a new approach. Then in Theorems 9.2 and 9.3 various aspects of the semigroup $\beta X$ are investigated, specially it is shown that this semigroup structure is actually functorial.

In summary, this paper has revolutionized our insight and understanding to the compactifications of topological spaces specially to the Stone-Čech compactification.
2. Preliminaries

In this section we recall some material which is needed in the sequel.

In this paper, all rings are commutative. It is well known that a ring $R$ is absolutely flat (or, von-Neumann regular) if and only if for each $f \in R$ there exists some $g \in R$ such that $f = f^2 g$.

If $f$ is a member of a ring $R$ then $D(f) = \{ p \in \text{Spec}(R) : f \notin p \}$ and $V(f) = \text{Spec}(R) \setminus D(f)$.

If $\varphi : A \to B$ is a morphism of rings then the induced map $\text{Spec}(B) \to \text{Spec}(A)$ is denoted by $\text{Spec}(\varphi)$ or by $\varphi^*$.

If $X$ is a set then its power set $\mathcal{P}(X)$ together with the symmetric difference $A + B = (A \cup B) \setminus (A \cap B)$ as the addition and the intersection $A \cdot B = A \cap B$ as the multiplication form a commutative ring whose zero and unit are respectively the empty set and the whole set $X$. The ring $\mathcal{P}(X)$ is called the power set ring of $X$. If $f : X \to Y$ is a function then the map $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by $A \leadsto f^{-1}(A)$ is a morphism of rings. In fact, the assignments $X \leadsto \mathcal{P}(X)$ and $f \leadsto \mathcal{P}(f)$ form a faithful contravariant functor from the category of sets to the category of Boolean rings. We call it the power set functor.

By $\text{Fin}(X)$ we mean the set of all finite subsets of $X$. It is an ideal of $\mathcal{P}(X)$.

A ring is called a Boolean ring if each element is an idempotent. Power set rings are typical examples of Boolean rings. It is easy to see that every Boolean ring is a commutative ring, and in a Boolean ring every prime ideal is a maximal ideal.

If $X$ is a topological space then by $\text{Clop}(X)$ we mean the set of clopen (both open and closed) subsets of $X$. Then $\text{Clop}(X)$ is a subring of $\mathcal{P}(X)$. If $f : X \to Y$ is a continuous map of topological spaces then the map $\text{Clop}(f) : \text{Clop}(Y) \to \text{Clop}(X)$ given by $A \leadsto f^{-1}(A)$ is a morphism of rings.

If $\text{Fun}(X, K)$ is the set of all functions from a set $X$ to a ring $K$, then this set with the usual addition and multiplication of functions is a commutative ring. This ring is canonically isomorphic to $\prod_{x \in X} K$. 
Similarly above, if \( f : X \to Y \) is a function then the induced map \( \text{Fun}(f) : \text{Fun}(Y, K) \to \text{Fun}(X, K) \) given by \( g \mapsto g \circ f \) is a morphism of rings. So \( \text{Fun}(-, K) \) is a faithful contravariant functor from the category of sets to category of commutative rings. It is easy to see that \( f \) is injective if and only if \( \text{Fun}(f) \) is surjective. Also, \( f \) is surjective if and only if \( \text{Fun}(f) \) is injective.

By a compact space we mean a quasi-compact and Hausdorff topological space.

**Definition 2.1.** By a compactification of a topological space \( X \) we mean a compact space \( \tilde{X} \) together with a continuous open embedding \( \eta : X \to \tilde{X} \) such that \( \eta(X) \) is a dense subspace of \( \tilde{X} \). If moreover \( \tilde{X} \setminus \eta(X) \) consisting only a single point then \( \tilde{X} \) is called the one-point or the Alexandroff compactification of \( X \) and it is denoted by \( \alpha X \), and this single point is often called the point at infinity.

**Definition 2.2.** The Stone-\( Č\)ech compactification of a topological space \( X \) is the pair \( (\beta X, \eta) \) where \( \beta X \) is a compact space and \( \eta : X \to \beta X \) is a continuous map such that the following universal property holds. For each such pair \( (Y, \varphi) \), i.e. \( Y \) is a compact space and \( \varphi : X \to Y \) is a continuous map, then there exists a unique continuous map \( \tilde{\varphi} : \beta X \to Y \) such that \( \varphi = \tilde{\varphi} \circ \eta \).

Let \( R \) be a ring. The set of minimal prime ideals of \( R \) is denoted by \( \text{Min}(R) \) and the set of maximal ideals of \( R \) is denoted by \( \text{Max}(R) \). Note that the minimal spectrum \( \text{Min}(R) \) is not necessarily Zariski quasi-compact. The Jacobson radical of \( R \) is denoted by \( J \).

There exists a unique topology on \( \text{Spec}(R) \) such that the collection of \( V(I) \) with \( I \) is a finitely generated ideal of \( R \) forms a basis for its opens. This topology is called the flat topology. For more details please see \([10]\).

Recall that a subspace \( Y \) of a topological space \( X \) is called a retraction (or, retract) of \( X \) if there exists a continuous function \( \gamma : X \to Y \) such that \( \gamma(y) = y \) for all \( y \in Y \). As an example of retractions, it is well known that a commutative ring \( R \) is a Gelfand ring (i.e., every prime ideal of \( R \) is contained in a unique maximal ideal of \( R \)) if and only if \( \text{Max}(R) \) is the Zariski retraction of \( \text{Spec}(R) \), see \([2]\). Theorem
A ring $R$ is called a mp-ring if every prime ideal of $R$ contains a unique minimal prime ideal of $R$. Clearly a ring $R$ is a mp-ring if and only if for every distinct minimal primes $p$ and $q$ of $R$ then $p + q = R$.

3. Alexandroff compactification

Let $\mathcal{R}$ be the set of all subsets of a set $X$ which are either finite or cofinite (i.e. its complement is finite). Then clearly $\mathcal{R}$ is a subring of $\mathcal{P}(X)$. In the following result the maximal ideals of $\mathcal{R}$ are characterized.

Recall that if $x \in X$ then $m_x = \mathcal{P}(X \setminus \{x\})$ is a maximal ideal of $\mathcal{P}(X)$.

**Theorem 3.1.** Let $X$ be an infinite set. Then the maximal ideals of $\mathcal{R}$ are precisely $\text{Fin}(X)$ or of the form $m_x \cap \mathcal{R}$ where $x \in X$.

**Proof.** First we have to show that $\text{Fin}(X)$ is a maximal ideal of $\mathcal{R}$. Clearly $\text{Fin}(X) \neq \mathcal{R}$ since $X$ is infinite. If there exists an ideal $I$ of $\mathcal{R}$ strictly containing $\text{Fin}(X)$ then we may choose some $A \in I$ such that $A \notin \text{Fin}(X)$. It follows that $A^c \in \text{Fin}(X)$ and so $1 = A + A^c \in I$. Hence, $\text{Fin}(X)$ is a maximal ideal of $\mathcal{R}$. Conversely, let $M$ be a maximal ideal of $\mathcal{R}$ such that $M \neq m_x \cap \mathcal{R}$ for all $x \in X$. It follows that $A_x := X \setminus \{x\} \in \mathcal{R} \setminus M$ for all $x \in X$. But $\{x\}.A_x = 0 \in M$. Therefore $\{x\} \in M$ for all $x \in X$. This yields that $\text{Fin}(X) \subseteq M$ and so $\text{Fin}(X) = M$. □

**Remark 3.2.** Let $X$ be an infinite set. Here we give a second proof to show that $\text{Fin}(X)$ is a maximal ideal of $\mathcal{R}$. There exists a maximal ideal $M$ of $\mathcal{P}(X)$ such that $\text{Fin}(X) \subseteq M$ since $\text{Fin}(X) \neq \mathcal{P}(X)$. We have then $\text{Fin}(X) = M \cap \mathcal{R}$. Thus $\text{Fin}(X)$ is a maximal ideal of $\mathcal{R}$.

**Lemma 3.3.** If $p$ and $q$ are distinct minimal prime ideals of a ring $R$ then there exist $f \in R \setminus p$ and $g \in R \setminus q$ such that $fg = 0$.

**Proof.** It suffices to show that $0 \in S = (R \setminus p)(R \setminus q)$. If not, then there exists a prime ideal $P$ of $R$ such that $P \cap S = \emptyset$. This yields that
\[ p = P = q \] which is a contradiction. \( \square \)

The following result is well known, we prove it by a new and short approach.

**Corollary 3.4.** For any ring \( R \) then the space \( \text{Spec}(R) \) is Hausdorff if and only if every prime ideal of \( R \) is a maximal ideal.

**Proof.** The implication “\( \Rightarrow \)” holds more generally, because every point of a Hausdorff space is a closed point. The converse implies from Lemma 3.3 \( \square \)

The following result puts to the compactification a purely algebraic insight. This new point of view culminates in Theorems 4.5, 5.4, 6.4 and 7.2.

**Corollary 3.5.** If \( X \) is an infinite set then \( \text{Spec}(R) \) is the Alexandroff compactification of the discrete space \( X \).

**Proof.** By Corollary 3.4, the space \( \text{Spec}(R) \) is Hausdorff and so it is compact. The map \( \eta : X \to \text{Spec}(R) \) given by \( x \mapsto m_x \cap R \) is an open embedding. Because by Theorem 3.1 \( D({x}) = \{m_x \cap R\} \) for all \( x \in X \). Now if \( A \) is a subset of \( X \) then \( \eta(A) = \bigcup_{x \in A} D({x}) \). If \( U \) is an open neighborhood of \( \text{Fin}(X) \) in \( \text{Spec}(R) \) then \( U^c \) is a finite set. Hence, \( \eta(X) \) is a dense subspace of \( \text{Spec}(R) \). \( \square \)

Clearly \( \text{Fin}(X) \) is the point at infinity of the Alexandroff compactification of the infinite discrete space \( X \).

4. **Minimal spectrum as Stone-Cech compactification**

The following result generalizes [2] p. 460.

**Proposition 4.1.** Consider the canonical ring map \( \pi : R \to S^{-1}R \) where \( S \) is a multiplicative subset of a ring \( R \), and let \( f \in R \). Then \( f \in \bigcap_{p \in \text{Im} \pi^*} p \) if and only if there exists some \( g \in S \) such that \( fg \) is nilpotent.
Proof. If \( f \in \bigcap_{p \in \text{Im} \pi^*} p \) then:

\[
f/1 \in \bigcap_{p \in \text{Im} \pi^*} S^{-1}p = \bigcap_{q \in \text{Spec}(S^{-1}R)} q = \sqrt{0}.
\]

Thus there exist a natural number \( n \geq 1 \) and some \( g \in S \) such that \( f^n g = 0 \). Hence, \( fg \) is nilpotent. The reverse implication is easy. \( \Box \)

Corollary 4.2. ([4, Lemma 1.1] and [6, Lemma 3.1]) Let \( p \) be a prime ideal of a ring \( R \). Then \( p \) is a minimal prime ideal of \( R \) if and only if for each \( f \in p \) there exists some \( g \in R \setminus p \) such that \( fg \) is nilpotent.

Proof. It is an immediate consequence of Proposition 4.1. \( \Box \)

Theorem 4.3. Let \( R \) be a ring. Then the Zariski topology over \( \text{Min}(R) \) is finer than the flat topology. Moreover, these two topologies over \( \text{Min}(R) \) are the same if and only if \( \text{Min}(R) \) is Zariski compact.

Proof. Let \( f \in R \). If \( p \in W = \text{Min}(R) \cap V(f) \) then by Corollary 4.2 there exists some \( g \in R \setminus p \) such that \( fg \) is nilpotent. This yields that \( p \in \text{Min}(R) \cap D(g) \subset W \). Therefore \( W \) is a Zariski open of \( \text{Min}(R) \). Hence the Zariski topology over \( \text{Min}(R) \) is finer than the flat topology. The set \( \text{Min}(R) \) is Zariski Hausdorff, see Lemma 3.3. It is also easy to see that \( \text{Min}(R) \) is flat quasi-compact. Therefore if these two topologies over \( \text{Min}(R) \) are the same then \( \text{Min}(R) \) is Zariski compact. Conversely, suppose \( \text{Min}(R) \) is Zariski compact. In the above we observed that \( U = \text{Min}(R) \cap D(f) \) is a Zariski clopen of \( \text{Min}(R) \). Every closed subspace of a quasi-compact space is quasi-compact. Thus there exist a finitely many elements \( g_1, \ldots, g_n \in R \) such that \( U^c = \text{Min}(R) \setminus U = \bigcup_{i=1}^n \text{Min}(R) \cap D(f_i) \). It follows that \( U = \text{Min}(R) \cap V(I) \) where \( I = (g_1, \ldots, g_n) \) is a finitely generated ideal of \( R \). Thus \( U \) is a flat open of \( \text{Min}(R) \). \( \Box \)

Throughout this paper, \( \Lambda = \prod_{x \in X} R_x \) where each \( R_x \) is an integral domain. For each \( f = (f_x) \in \Lambda \), the set \( \text{Supp}(f) = \{x \in X : f_x \neq 0\} \) is simply denoted by \( S(f) \). Clearly \( \text{S}(fg) = \text{S}(f) \cap \text{S}(g) \) for all \( f, g \in \Lambda \).
Corollary 4.4. The space $\text{Min}(\Lambda)$ is compact.

**First proof.** By Theorem 4.3 it suffices to show that for each $f \in \Lambda$ then $U = \text{Min}(\Lambda) \cap D(f)$ is a flat open of $\text{Min}(\Lambda)$. Consider the sequence $e = (e_x) \in \Lambda$ where $e_x$ is either 0 or 1, according as $x \in S(f)$ or $x \notin S(f)$. Then clearly $ef = 0$ and $g = ge$ for all $g \in \text{Ann}(f)$. Hence $\text{Ann}(f)$ is generated by the sequence $e$. Now let $p \in \text{Min}(\Lambda) \cap V(e)$. If $f \in p$ then by Corollary 4.2 there exists some $h \in \Lambda \setminus p$ such that $fh$ is nilpotent. But $\Lambda$ is a reduced ring. Hence $h \in \text{Ann}(f)$. Thus $h = he \in p$. But this is a contradiction. This shows that $U = \text{Min}(\Lambda) \cap V(e)$ is a flat open of $\text{Min}(\Lambda)$.

**Second proof.** The ideal $(f)$ as $\Lambda$–module is isomorphic to $(1 - e)$. Thus every principal ideal of $\Lambda$ is a projective module. The ring $\Lambda$ is also a reduced mp-ring because it is easy to see that a ring $R$ is a reduced mp-ring if and only if $\text{Ann}(f) + \text{Ann}(g) = \text{Ann}(fg)$ for all $f, g \in R$. It is also well known that the minimal spectrum of a reduced mp-ring $R$ is Zariski compact if and only if every principal ideal of $R$ is a projective $R$–module, see [13, Proposition 3.4]. Therefore the space $\text{Min}(\Lambda)$ is compact. □

For each $x \in X$ then $p_x := \ker \pi_x$ is a minimal prime ideal of $\Lambda$ and it is generated by the sequence $1 - \Delta_x$ where $\pi_x : \Lambda \to R_x$ is the canonical projection, $\Delta_x = (\delta_{x,y})_{y \in X}$ and $\delta_{x,y}$ is the Kronecker delta.

Theorem 4.5. The space $\text{Min}(\Lambda)$ together with the canonical map $\eta : X \to \text{Min}(\Lambda)$ given by $x \mapsto p_x$ is the Stone-Čech compactification of the discrete space $X$.

**Proof.** By Corollary 4.3 the space $\text{Min}(\Lambda)$ is compact. It remains to check out the universal property of the Stone-Čech compactification. Let $Y$ be a compact topological space and $\varphi : X \to Y$ a function. We shall find a continuous function $\tilde{\varphi} : \text{Min}(\Lambda) \to Y$ such that $\varphi = \tilde{\varphi} \circ \eta$ and then we show that such function is unique. If $p \in \text{Min}(\Lambda)$ then the subsets $S(f)$ with $f \in \Lambda \setminus p$ have the finite intersection property. It follows that the subsets $\varphi(S(f))$ and so their closures $\overline{\varphi(S(f))}$ with $f \in \Lambda \setminus p$ have the finite intersection property. This yields that $\bigcap_{f \in \Lambda \setminus p} \varphi(S(f)) \neq \emptyset$ because $Y$ is quasi-compact. We claim that this intersection has exactly one point. If $y$ and $y'$ are two distinct points of the intersection then there exist disjoint opens $U$ and $V$ in $Y$ such
that \( y \in U \) and \( y' \in V \). Then consider the sequence \( f \in \Lambda \) where \( f_x \) is either 0 or 1, according as \( x \in \varphi^{-1}(U) \) or \( x \notin \varphi^{-1}(U) \). Then we have either \( f \in p \) or \( 1 - f \in p \) since \( f \) is an idempotent. If \( f \in p \) then \( \varphi^{-1}(V) \cap S(1 - f) \neq \emptyset \). So we may choose some \( x \) in this intersection. Thus \( x \notin \varphi^{-1}(U) \), hence \( f_x = 1 \). But this is a contradiction since \( x \in S(1 - f) \). If \( 1 - f \in p \) then \( \varphi^{-1}(U) \cap S(f) \neq \emptyset \), but this is again a contradiction. Hence, there exists a unique point \( y_p \in Y \) such that \( \bigcap_{f \in \Lambda \setminus p} \varphi\left(S(f)\right) = \{y_p\} \). This establishes the claim. Then we define the map \( \tilde{\varphi} : \text{Min}(\Lambda) \rightarrow Y \) as \( p \leadsto y_p \). It is easy to see that \( \varphi(x) \in \bigcap_{f \in \Lambda \setminus p_x} \varphi\left(S(f)\right) \) for all \( x \in X \). Therefore \( \varphi = \tilde{\varphi} \circ \eta \). Now we show that \( \tilde{\varphi} \) is continuous. Let \( U \) be an open of \( Y \) and let \( p \in (\tilde{\varphi})^{-1}(U) \). There exists an open neighborhood \( V \) of \( y_p \) such that \( \overline{V} \subseteq U \), because it is well known that every compact space is a normal space. Let \( h \in \Lambda \) be a sequence which is defined as \( h_x = 1 \) or \( h_x = 0 \), according as \( x \in \varphi^{-1}(V) \) or \( x \notin \varphi^{-1}(V) \). Then \( p \in D(h) \), since if \( h \in p \) then \( 1 - h \notin p \) and so \( \varphi^{-1}(V) \cap S(1 - h) \neq \emptyset \), which is impossible. To conclude the continuity of \( \tilde{\varphi} \) we show that \( \text{Min}(\Lambda) \cap D(h) \subseteq (\tilde{\varphi})^{-1}(U) \).

Suppose there exists some \( q \in \text{Min}(\Lambda) \cap D(h) \) such that \( y_q \notin U \). Thus \( y_q \in W := Y \setminus \overline{V} \). It follows that \( W \cap \varphi\left(S(h)\right) \neq \emptyset \). But this is impossible since \( S(h) = \varphi^{-1}(V) \) and so \( W \cap \varphi\left(S(h)\right) \subseteq W \cap V = \emptyset \). Therefore \( \tilde{\varphi} \) is continuous. If \( \text{Min}(\Lambda) \cap D(f) \) is non-empty then \( f \neq 0 \) and so there exists some \( x \in X \) such that \( p_x \in D(f) \). This shows that \( \eta(X) \) is a dense subspace of \( \text{Min}(\Lambda) \), hence the uniqueness of \( \tilde{\varphi} \) is deduced from the basic fact that if two continuous maps into a Hausdorff space agree on a dense subspace of the domain, they are equal. \( \square \)

**Remark 4.6.** The space \( \text{Min}(\Lambda) \) is the compactification of the discrete space \( X \) in the sense of Definition 2.1. Because the map \( \eta : X \rightarrow \text{Min}(\Lambda) \) given by \( x \leadsto p_x \) is an open embedding since \( \{p_x\} = \text{Min}(\Lambda) \cap D(\Delta_x) \) for all \( x \in X \).

**Lemma 4.7.** If each \( R_x \) is a field then every prime ideal of \( \Lambda \) is a maximal ideal.

**Proof.** Let \( p \) be a prime ideal of \( \Lambda \) and \( f \in \Lambda \setminus p \). Then consider the sequence \( q = (g_x) \in \Lambda \) where \( g_x \) is 1 or \( 1/f(x) \), according as \( f_x = 0 \) or \( f_x \neq 0 \). Then it is obvious that \( f(1 - fg) = 0 \in p \). This yields
that $1 - fg \in \mathfrak{p}$. Therefore $\Lambda/\mathfrak{p}$ is a field. As a second proof, the assertion is also deduced from the fact that $\Lambda$ is an absolutely flat ring. □

**Corollary 4.8.** The space $\text{Spec}(\Lambda)$ together with the canonical map $\eta : X \rightarrow \text{Spec}(\Lambda)$ is the Stone-Čech compactification of the discrete space $X$ if and only if each $R_x$ is a field.

**Proof.** If each $R_x$ is a field then the assertion is deduced from Theorem 4.5 and Lemma 4.7. Conversely, if $\mathfrak{m}$ is a maximal ideal of $R_x$ then $\pi^{-1}_x(\mathfrak{m}) = \pi^{-1}_x(0)$ because $\text{Spec}(\Lambda)$ is Hausdorff and so every prime ideal of $\Lambda$ is a maximal ideal. But $\pi_x$ is surjective and so the induced map $\pi^*_x$ is injective. Therefore the zero ideal of $R_x$ is a maximal ideal and so it is a field. □

**Corollary 4.9.** The space $\text{Spec}(P(X))$ together with the canonical map $\eta : X \rightarrow \text{Spec}(P(X))$ given by $x \sim m_x = P(X \setminus \{x\})$ is the Stone-Čech compactification of the discrete space $X$.

**Proof.** The map $P(X) \rightarrow \prod_{x \in X} \mathbb{Z}_2$ given by $A \sim \chi_A$ is an isomorphism of rings where $\chi_A$ is the characteristic function of $A$ and $\mathbb{Z}_2 = \{0, 1\}$. Then apply Corollary 4.8. □

**Remark 4.10.** Now the classical approach to construct the Stone-Čech compactification of a discrete space $X$ is easily recovered. Indeed, if $X$ is a set then one can easily check that the map $M \sim P(X) \setminus M = \{A \in P(X) : A^c \in M\}$ is a homeomorphism from $\text{Spec}(P(X))$ onto $\mathcal{F}(X)$, the space of ultrafilters on $X$ equipped with the Stone topology. Recall that the collection of $d(A) = \{F \in \mathcal{F}(X) : A \in F\}$ with $A \in P(X)$ forms a basis for the opens of the Stone topology. Indeed, this homeomorphism builds a bridge between the theory of ultrafilters and commutative algebra. Using this identification, then one can easily convert all of the theory of ultrafilters, Boolean algebras, lattice theory and other related fields to the Grothendieck’s style of mathematics in commutative algebra which is more standard than those theories.

**Proposition 4.11.** If $X$ is a Hausdorff topological space and $S$ is a dense subspace of $X$ then the cardinality of $X$ is at most the cardinality of $P(P(S))$. 

Proof. The map \( \varphi : X \to \mathcal{P}(\mathcal{P}(S)) \) which sends each point \( x \in X \) into the set of all \( S \cap D \) is injective. Recall that a (closed) subset \( D \) of \( X \) is called a closed domain of \( X \) if \( D \) is the closure of its interior. To see the injectivity of \( \varphi \), suppose \( \varphi(x) = \varphi(y) \). If \( x \neq y \) then we may choose disjoint opens \( U \) and \( V \) of \( X \) such that \( x \in U \) and \( y \in V \). But \( D := \overline{U} \) is a closed domain of \( X \), and so \( S \cap D \in \varphi(x) \). Thus there exists a closed domain \( D' \) of \( X \) containing \( y \) such that \( S \cap D = S \cap D' \). This yields that \( D = D' \) since \( S \cap D \) is dense in every closed domain \( D \). It follows that \( y \in \overline{U} \). But this is a contradiction and we win. \( \Box \)

As a consequence of Proposition 4.11 and the generalized continuum hypothesis, we get that if \( X \) is a set with the cardinality \( \kappa \) and \( \tilde{X} \) is a compactification of the discrete space \( X \) then \( |\tilde{X}| \in \{\kappa, 2^\kappa, 2^{2^\kappa}\} \).

**Corollary 4.12.** If \( X \) and \( Y \) are two sets then we have the following canonical bijections:

\[
\text{Mor}_{\text{Set}}(X, \beta Y) \simeq \text{Mor}_{\text{Top}}(\beta X, \beta Y) \simeq \text{Mor}_{\text{Ring}}(\mathcal{P}(Y), \mathcal{P}(X)).
\]

**Proof.** The first bijection is implied form Corollary 4.9, and the second bijection is an immediate consequence of [12, Theorem 5.6]. \( \Box \)

5. The Stone–Čech compactification of an arbitrary space

It is well known that every topological space \( X \) admits the Stone–Čech compactification. In this section, we give a completely new and more natural way to prove it. Indeed, this compactification is obtained from the Stone–Čech compactification of the discrete space \( X \) by passing to a its certain quotient. To see this, first we prove some new results.
which are so interesting in their own right. The work [7] was the main motivation in emerging the ideas of this section.

Let $X$ be a topological space, $x \in X$ and $M$ a maximal ideal of $\mathcal{P}(X)$. Then we say that $M$ is Zariski convergent to the point $x$ if $U$ is an open subset of $X$ containing $x$ then $M \in \mathcal{D}(U)$.

**Lemma 5.1.** Let $X$ be a set. If $M$ is a maximal ideal of $\mathcal{P}(X)$ then $\mathcal{P}(\eta)^*(M)$ is Zariski convergent to the point $M \in \beta X = \text{Spec } \mathcal{P}(X)$.

**Proof.** Let $U$ be an open of $\beta X$ such that $M \in U$. If $U \in \mathcal{P}(\eta)^*(M)$ then $\eta^{-1}(U) \subseteq M$. But there exists some $A \in \mathcal{P}(X)$ such that $M \in \mathcal{D}(A) \subseteq U$. If $x \in A$ then $\eta(x) = m_x \in \mathcal{D}(A)$ and so $x \in \eta^{-1}(U)$. This shows that $A \subseteq \eta^{-1}(U)$. Thus $A \in M$. But this is a contradiction. □

**Lemma 5.2.** Let $X$ be a topological space and let $A$ be a subset of $X$ with the property that $\mathcal{D}(A)$ contains every maximal ideal of $\mathcal{P}(X)$ which is Zariski convergent to a point of $A$. Then $A$ is an open subset of $X$.

**Proof.** Take $x \in A$ and let $S$ be the set of all open subsets of $X$ which are containing $x$. Then by the hypothesis, the ideal of $\mathcal{P}(X)$ generated by $A$ and the elements $U^c = X \setminus U$ with $U \in S$ is the whole ring. Thus we may find a finite number $U_1, \ldots, U_n$ of elements of $S$ such that $X = A \cup (\bigcup_{i=1}^n U_i^c)$. It follows that $x \in \bigcap_{i=1}^n U_i \subseteq A$. Hence, $A$ is an open of $X$. □

Note that the converse of the above lemma holds trivially.

Let $\varphi : X \rightarrow Y$ be a continuous map of topological spaces. If a maximal ideal $M$ of $\mathcal{P}(X)$ is Zariski convergent to some point $x \in X$ then clearly $\mathcal{P}(\varphi)^*(M)$ is Zariski convergent to $\varphi(x)$. In the following result we establish its converse.

**Corollary 5.3.** Let $\varphi : X \rightarrow Y$ be a function between topological spaces with the property that $\mathcal{P}(\varphi)^*(M)$ is Zariski convergent to $\varphi(x)$ whenever a maximal ideal $M$ of $\mathcal{P}(X)$ is Zariski convergent to some point $x \in X$. Then $\varphi$ is continuous.
Proof. It is easily deduced from Lemma 5.2. □

Theorem 5.4. Every topological space $X$ admits the Stone–Čech compactification.

Proof. Consider the equivalence relation $\sim$ on $\beta X = \text{Spec} \mathcal{P}(X)$ defined as $M \sim N$ if $\varphi : X \to Y$ is a continuous function to a compact space $Y$ then $\tilde{\varphi}(M) = \tilde{\varphi}(N)$ where $\tilde{\varphi} : \beta X \to Y$ is the unique continuous function such that $\varphi = \tilde{\varphi} \circ \eta$, see the proof of Theorem 4.5. Now to prove that the pair $(X', \pi \circ \eta)$ is the Stone–Čech compactification of the space $X$ it suffices to show that $\pi \circ \eta : X \to X'$ is continuous where $\pi : \beta X \to X' = \beta X/\sim$ is the canonical map and $X'$ is equipped with the quotient topology. To prove the continuity of $\pi \circ \eta$, by Corollary 5.3 it will be enough to show that if a maximal ideal $M$ of $\mathcal{P}(X)$ is Zariski convergent to some point $x \in X$ then $\mathcal{P}(\pi \circ \eta)^*(M)$ is Zariski convergent to the point $(\pi \circ \eta)(x)$. We have $\mathcal{P}(\pi \circ \eta)^*(M) = \mathcal{P}(\pi)^*(\mathcal{P}(\eta)^*(M))$. By Lemma 5.1 $N := \mathcal{P}(\eta)^*(M)$ is Zariski convergent to the point $M \in \beta X$. Thus $\mathcal{P}(\pi)^*(N)$ is Zariski convergent to the point $\pi(M)$ since $\pi$ is continuous. Then we show that $M \sim m_x$. Because take $A \in \mathcal{P}(X) \setminus M$ and let $V$ be an open of a compact space $Y$ such that $\varphi(x) \in V$ where $\varphi : X \to Y$ is a continuous map. Then $\varphi^{-1}(V) \notin M$. Note that $S(A) = A$. Now if $V \cap \varphi(A) = \emptyset$ then $A \in M$, a contradiction. Hence, $\varphi(x) \in \varphi(S(A))$. Thus by the definition of $\tilde{\varphi}$, see the proof of Theorem 4.5, we get that $\varphi(x) = \tilde{\varphi}(M)$ and so $M \sim m_x$. Therefore $\mathcal{P}(\pi \circ \eta)^*(M)$ is Zariski convergent to the point $\pi(M) = (\pi \circ \eta)(x)$. Note that during to verify the universal property of the Stone–Čech compactification for the pair $(X', \pi \circ \eta)$, the uniqueness is deduced from the fact that $(\pi \circ \eta)(X)$ is a dense subspace of $X'$. □

It seems that the notion “Zariski convergent” still has the potential that can be developed further to apply for other purposes.

6. Maximal spectrum as Stone–Čech compactification

Let $R$ be a ring and $f \in R$. If $m \in U = \text{Max}(R) \cap D(f)$ then there exist some $g \in m$ and $h \in R$ such that $1 = fh + g$. This yields that $m \in \text{Max}(R) \cap V(g) \subseteq U$. Thus $U$ is a flat open of $\text{Max}(R)$. Therefore the flat topology over $\text{Max}(R)$ is finer than the Zariski topology.
Proposition 6.1. For a ring $R$ the following are equivalent.
(i) $R/\mathfrak{J}$ is absolutely flat.
(ii) The Zariski and flat topologies over $\text{Max}(R)$ are the same.
(iii) $\text{Max}(R)$ is flat compact.

Proof. (i) $\Rightarrow$ (ii) : If $f \in R$ then there exists some $g \in R$ such that $f(1-fg) \in \mathfrak{J}$. It follows that $\text{Max}(R) \cap V(f) = \text{Max}(R) \cap D(1-fg)$.
(ii) $\Rightarrow$ (iii) : The subset $\text{Max}(R)$ is Zariski quasi-compact and flat Hausdorff. (iii) $\Rightarrow$ (i) : See [11, Theorem 4.5]. □

Lemma 6.2. Let $R$ be a ring such that $R/\mathfrak{J}$ is absolutely flat. Then the clopens of $\text{Max}(R)$ are precisely of the form $\text{Max}(R) \cap V(f)$ where $f \in R$.

Proof. By Proposition 6.1, the Zariski and flat topologies over $\text{Max}(R)$ are the same. If $f \in R$ then we observed that $\text{Max}(R) \cap V(f)$ is a clopen of $\text{Max}(R)$. Conversely, let $U$ be a clopen of $\text{Max}(R)$. It is easy to see that every closed subspace of a quasi-compact space is quasi-compact. Hence, we may write $U = \bigcup_{k=1}^{n} \text{Max}(R) \cap V(I_k)$ where each $I_k$ is a (finitely generated) ideal of $R$. This yields that $U = \text{Max}(R) \setminus U = \text{Max}(R) \cap V(J)$ where $J$ is a (finitely generated) ideal of $R$. It follows that $I + J = R$. Thus there exist some $f \in I$ and $g \in J$ such that $f + g = 1$. This implies that $U = \text{Max}(R) \cap V(f)$. □

Throughout this paper, $\Gamma = \prod_{x \in X} R_x$ where each $R_x$ is a local ring with the maximal ideal $m_x$. For each $x \in X$ then $\mathcal{M}_x := \pi_x^{-1}(m_x)$ is a maximal ideal of $\Gamma$ because the ring map $\Gamma/\mathcal{M}_x \to R_x/m_x$ induced by the canonical projection $\pi_x : \Gamma \to R_x$ is an isomorphism.

If $f = (f_x) \in \Gamma$ then we define $\Omega(f) = \{x \in X : f_x \notin m_x\}$. It is obvious that $f$ is invertible in $\Gamma$ if and only if $\Omega(f) = X$. It is also easy to see that $\Omega(fg) = \Omega(f) \cap \Omega(g)$ for all $f, g \in \Gamma$.

Lemma 6.3. Let $f \in \Gamma$. Then $\Omega(f) = \emptyset$ if and only if $f \in \mathfrak{J}$.

Proof. If $\Omega(f) = \emptyset$ then $f_x \in m_x$ for all $x$. This yields that $\Omega(1 + fg) = X$ for all $g \in \Gamma$. Thus $f \in \mathfrak{J}$. Conversely, if $f \in \mathfrak{J}$
then \( f \in \mathcal{M}_x \) for all \( x \). So \( \Omega(f) \) is empty. \( \Box \)

**Theorem 6.4.** The space \( \text{Max}(\Gamma) \) together with the canonical map \( \eta : X \to \text{Max}(\Gamma) \) given by \( x \mapsto \mathcal{M}_x \) is the Stone-Čech compactification of the discrete space \( X \).

**Proof.** If \( f \in \Gamma \) then consider the sequence \( g = (g_x) \in \Gamma \) such that \( g_x \) is either 0 or \( f_x^{-1} \), according as \( f_x \in m_x \) or \( f_x \notin m_x \). Then \( \Omega(1 + fh(1 - fg)) = X \) for all \( h \in \Gamma \). Hence, \( f(1 - fg) \in \mathcal{J} \). Thus \( \Gamma/\mathcal{J} \) is absolutely flat. Therefore by Proposition 6.1, the space \( \text{Max}(\Gamma) \) is compact. Then we verify the universal property of the Stone-Čech compactification. Let \( Y \) be a compact topological space and \( \varphi : X \to Y \) a function. If \( M \in \text{Max}(\Gamma) \) then by Lemma 6.3 the subsets \( \Omega(f) \) with \( f \in \Gamma \setminus M \) have the finite intersection property. Thus by a similar argument as applied in the proof of Theorem 4.5 there exists a unique point \( y_M \in Y \) such that \( \bigcap_{f \in \Gamma \setminus M} \varphi(\Omega(f)) = \{y_M\} \). Then we define the map \( \tilde{\varphi} : \text{Max}(\Gamma) \to Y \) as \( M \mapsto y_M \). Again exactly like the proof of Theorem 4.5 it is shown that \( \varphi = \tilde{\varphi} \circ \eta \) and \( \tilde{\varphi} \) is continuous. Finally, to prove the uniqueness of \( \tilde{\varphi} \) it suffices to show that \( \eta(X) \) is a dense subspace of \( \text{Max}(\Gamma) \). The space \( \text{Max}(\Gamma) \) is totally disconnected, see [11, Proposition 4.4]. It is well known that in a compact totally disconnected space, the collection of clopens is a basis for the opens. Using this and Lemma 6.2 then the collection of \( \text{Max}(\Gamma) \cap V(f) \) with \( f \in \Gamma \) forms a basis for the opens of \( \text{Max}(\Gamma) \). Now if \( \text{Max}(\Gamma) \cap V(f) \) is non-empty then \( \Omega(f) \neq X \). Hence there exists some \( x \in X \) such that \( \mathcal{M}_x \in \text{Max}(\Gamma) \cap V(f) \). Therefore \( \eta(X) \) is a dense subspace of \( \text{Max}(\Gamma) \). \( \Box \)

**Remark 6.5.** The canonical map \( \eta : X \to \text{Max}(\Gamma) \) given by \( x \mapsto \mathcal{M}_x \) is an open embedding. In fact, \( \{\mathcal{M}_x\} = \text{Max}(\Gamma) \cap D(\Delta_x) \) for all \( x \in X \). To see this let \( M \in \text{Max}(\Gamma) \cap D(\Delta_x) \) and \( f \in M \). If \( f \notin \mathcal{M}_x \) then \( \Omega(1 - \Delta_x + \Delta_x f) = X \) and so \( 1 - \Delta_x + \Delta_x f \) is invertible in the ring \( \Gamma \). But this is a contradiction because \( 1 - \Delta_x + \Delta_x f \in M \). Therefore \( M \subseteq \mathcal{M}_x \) and so \( M = \mathcal{M}_x \).

**Corollary 6.6.** There exists a unique homeomorphism \( \text{Min}(\Lambda) \simeq \text{Max}(\Gamma) \) such that \( p_x \) is mapped into \( \mathcal{M}_x \) for all \( x \in X \).
Proof. It is deduced from the universal property of the Stone-ćešech compactification by taking into account Theorems \[4.5\] and \[6.4\]. □

Corollary 6.7. Let \( R \) be a ring and let \( X \) be a subset of Spec\( (R) \). Then the following spaces are canonically homeomorphic.
(i) Min\( (\prod_{p \in X} R/p) \).
(ii) Spec\( (\prod_{p \in X} \kappa(p)) \).
(iii) Max\( (\prod_{p \in X} R_p) \).

Proof. It is an immediate consequence of Corollary \[6.6\]. □

Corollary 6.8. If \( X \) is an infinite set with the cardinality \( \kappa \) then \(|\text{Min}(\Lambda)| = |\text{Max}(\Gamma)| = |\text{Spec} \mathcal{P}(X)| = 2^{2^\kappa} \).

Proof. It follows from Corollaries \[6.6\] and \[6.6\] and the fact that the cardinality of the Stone-ćešech compactification of the infinite discrete space \( X \) is equal to \( 2^{2^\kappa} \), to see the proof of this fact please consider \[[5, Theorem 3.58] \) or \[[14, Theorem on page 71] \). □

Corollary 6.9. Let \( X \) and \( Y \) be two sets with the cardinalities \( \kappa \) and \( \lambda \), respectively. Then the cardinality of the set of ring maps \( \mathcal{P}(Y) \to \mathcal{P}(X) \) is either \( \lambda^\kappa \) or \( 2^{\kappa 2^\lambda} \), according as \( Y \) is finite or infinite.

Proof. It is deduced from Corollaries \[4.12\] and \[6.8\]. □

If \( \lambda \) is an infinite cardinal then \( \kappa 2^\lambda = \max\{\kappa, 2^\lambda\} \). To see this apply Cantor’s theorem and the fact that if \( \kappa \) is an infinite cardinal then \( \kappa^\kappa = \kappa \).

Corollary 6.10. If \( X \) is a set with the cardinality \( \kappa \), then the cardinality of the set of ring maps \( \mathcal{P}(X) \to \mathcal{P}(X) \) is either \( \kappa^\kappa \) or \( 2^{2^\kappa} \), according as \( X \) is finite or infinite.

Proof. It is an immediate consequence of Corollary \[6.9\]. □

The ring \( \Gamma \) is a clean ring and so it can be shown that the collection of Max\( (\Gamma) \cap V(f) \) with \( f \in R \) an idempotent forms a basis for the opens
Every clean ring is a Gelfand ring. Thus we may consider the retraction map \( \gamma : \text{Spec}(\Gamma) \to \text{Max}(\Gamma) \) which sends each prime ideal \( p \) of \( \Gamma \) into the unique maximal ideal containing \( p \). This map is continuous, see [2, Theorem 1.2]. In the following result, \( K_x = R_x/\mathfrak{m}_x \) is the residue field of the local ring \( R_x \) and \( \pi : \Gamma \to \Gamma' = \prod_{x \in X} K_x \) is the canonical ring map.

**Corollary 6.11.** The map \( \gamma \circ \pi^*: \text{Spec}(\Gamma') \to \text{Max}(\Gamma) \) is a homeomorphism.

**Proof.** It is deduced from the universal property of the Stone-Čech compactification by taking into account Corollary 4.8 and Theorem 6.4. \( \square \)

### 7. Totally disconnected compactifications

In this section it is shown that every totally disconnected compactification of a discrete space \( X \) is precisely of the form \( \text{Spec}(\mathcal{R}') \) where the ring \( \mathcal{R}' \) satisfies in the extensions of rings \( \mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{P}(X) \) and \( \mathcal{R} \) is the ring of finite or cofinite subsets of \( X \), see §3.

**Lemma 7.1.** Let \( f : X \to Y \) be a continuous map of topological spaces such that \( f(X) \) is a dense subspace of \( Y \). Then the induced map \( \text{Clop}(f) : \text{Clop}(Y) \to \text{Clop}(X) \) is an injective morphism of rings.

**First proof.** Let \( A \) be a clopen of \( Y \) such that \( f^{-1}(A) = \emptyset \). If \( A \) is non-empty then \( A \cap f(X) \) is non-empty. But this is a contradiction.

**Second proof.** Let \( D_1 \) and \( D_2 \) be two clopens of \( Y \) such that \( f^{-1}(D_1) = f^{-1}(D_2) \). Suppose there exists some \( y \in D_1 \) such that \( y \notin D_2 \). It follows that \( (D_1 \cap D_2^c) \cap f(X) \neq \emptyset \). Hence there exists some \( x \in X \) such that \( f(x) \in D_1 \cap D_2^c \). But this is a contradiction. Therefore \( D_1 = D_2 \). \( \square \)

**Theorem 7.2.** Every totally disconnected compactification of a discrete space \( X \) is precisely of the form \( \text{Spec}(\mathcal{R}') \) where the ring \( \mathcal{R}' \) satisfies in the extensions of rings \( \mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{P}(X) \).

**Proof.** It is easy to see that for any such ring \( \mathcal{R}' \) then \( \text{Spec}(\mathcal{R}') \) together with the canonical open embedding \( \eta : X \to \text{Spec}(\mathcal{R}') \) which
sends each point $x \in X$ into $m_x \cap R'$ is a totally disconnected compactification of the discrete space $X$. Conversely, let $(\tilde{X}, \eta)$ be a totally disconnected compactification of a discrete space $X$. By [12, Corollary 5.4], the space $\tilde{X}$ is homeomorphic to $\text{Spec}(R)$ where $R = \text{Clop}(\tilde{X})$. By Lemma 7.1, the induced map $\text{Clop}(\eta) : R \to \text{Clop}(X) = \mathcal{P}(X)$ is an injective ring map. So the ring $R$ is isomorphic to $R'$, the image of $\text{Clop}(\eta)$. It remains to show that $R \subseteq R'$. Take $A \in R$. If $A$ is finite then $D := \eta(A) = \bigcup_{x \in A} \{\eta(x)\}$ is a closed subset of $\tilde{X}$ and so $D \in \text{Clop}(\tilde{X})$. Therefore $A = \eta^{-1}(D) \in \mathcal{R}'$. But if $A$ is cofinite then the above argument shows that $A^c \in \mathcal{R}'$, and so $A = 1 - A^c \in \mathcal{R}'$. □

Remark 7.3. If $(\tilde{X}, \eta)$ is an arbitrary compactification of a discrete space $X$ then by [12, Theorem 5.2], the space of connected components $\pi_0(\tilde{X})$ is homeomorphic to $\text{Spec}(\mathcal{R}')$ where $\mathcal{R}' = \text{Clop}(\tilde{X})$. Also $\mathcal{R}'$, via the ring map $\text{Clop}(\eta)$, can be viewed as a subring of $\mathcal{P}(X)$ and containing $\mathcal{R}$. Note that there are compactifications of a discrete space which are not totally disconnected.

8. Extremally disconnected spaces

Recall that a topological space $X$ is called extremally (or, extremely) disconnected if the closure of every open of $X$ is again an open of $X$. This notion is symmetric that is, a space $X$ is extremally disconnected if and only if the interior of every closed of $X$ is again a closed of $X$. Every irreducible space is extremally disconnected since in such space every non-empty open is dense. In particular, if $R$ is an integral domain then $\text{Spec}(R)$ is extremally disconnected.

It is well known that in the category of Hausdorff spaces, epimorphisms are precisely the maps with dense image. By a similar argument, it is shown that in the category of compact spaces, epimorphisms are precisely the surjective maps. Therefore in the category of compact spaces, the projective objects are precisely the extremally disconnected spaces, because it is well known that a compact space $X$ is extremally disconnected if and only if every continuous and surjective map $f : Y \to X$ with $Y$ compact admits a continuous section that is, a continuous map $g : X \to Y$ such that $f \circ g$ is the identity, see [11, Tag 08YN] or [3, Theorem 2.5], also see [8]. Now using this, then we obtain the following results.
Theorem 8.1. If $X$ is a compact and extremally disconnected space then $X$ is a retraction of $\text{Spec} \mathcal{P}(X)$.

Proof. There exists a (unique) continuous map $f : \text{Spec} \mathcal{P}(X) \to X$ such that $f(m_x) = x$ for all $x \in X$. Thus there exists a continuous map $g : X \to \text{Spec} \mathcal{P}(X)$ such that $f \circ g$ is the identity. Hence $g$ induces a homeomorphism $h : X \to Y$ onto its image. Clearly $(h \circ f)(y) = y$ for all $y \in Y$. Hence, $X \simeq Y$ is a retraction of $\text{Spec} \mathcal{P}(X)$. □

Note that in the above proof, $\eta(X) \subseteq Y$ if and only if $Y = \text{Spec} \mathcal{P}(X)$.

Theorem 8.2. For any set $X$ then the space $\text{Spec} \mathcal{P}(X)$ is extremally disconnected.

Proof. Let $f : Z \to \text{Spec} \mathcal{P}(X)$ be a continuous surjective map with $Z$ a compact space. By the axiom of choice, there exists a function $\sigma : X \to Z$ such that $f(\sigma(x)) = \eta(x)$ for all $x \in X$. Thus there exists a (unique) continuous function $\tilde{\sigma} : \text{Spec} \mathcal{P}(X) \to Z$ such that $\sigma = \tilde{\sigma} \circ \eta$. But $f \circ \tilde{\sigma}$ and the identity map agree on $\eta(X)$, hence $f \circ \tilde{\sigma}$ is the identity. Therefore $\text{Spec} \mathcal{P}(X)$ is extremally disconnected. □

9. Semigroup structure on $\beta X$

In the remaining of this paper, $\beta X = \text{Spec} \mathcal{P}(X)$ together with the canonical map $\eta : X \to \beta X$ denotes the Stone-Čech compactification of the discrete space $X$. If $f : X \to Y$ is a function then by Corollary 4.9 there exists a unique continuous function $\beta f : \beta X \to \beta Y$ such that $(\beta f)(m_x) = m_{f(x)}$ for all $x \in X$. This yields that $\beta f = \mathcal{P}(f)^*$. In particular, if $f : X \to Y$ is injective then $\beta f : \beta X \to \beta Y$ is as well.

Let $(S, \ast)$ be a semigroup which $S$ is simultaneously a topological space. If the operation $\ast : S \times S \to S$ is continuous (here $S \times S$ is equipped with the product topology) then $(S, \ast)$ is called a topological semigroup. But it may happen that the operation $\ast$ is not necessarily continuous. This leads us to a weaker notion. The pair $(S, \ast)$ is called a left topological semigroup if the operation $\ast$ is left semi-continuous that is, for each $p \in S$ then the map $\ell_p : S \to S$ given by $x \mapsto p \ast x$ is continuous. The right topological semigroup is defined dually. Obviously every topological semigroup is both right topological and left
topological semigroup. We have then the following interesting result.

**Theorem 9.1.** The operation of every commutative semigroup \((X,\cdot)\) can be extended uniquely to an operation \(\ast\) on \(\beta X\) such that: \((\beta X,\ast)\) is a left topological semigroup, the canonical map \(\eta : X \to \beta X\) is a morphism of semigroups and \(m_x \ast M = M \ast m_x\) for all \(M \in \beta X\) and \(x \in X\). If moreover \(e\) is the identity of \(X\) then \(m_e\) is the identity of \(\beta X\).

**Proof.** If \(x \in X\) then by Theorem 4.8 there exists a unique continuous function \(\varphi_x : \beta X \to \beta X\) such that \(\varphi_x(m_y) = m_{x,y}\) for all \(y \in X\). For a fixed \(M \in \beta X\), again by Theorem 4.8 there exists a unique continuous map \(\theta_M : \beta X \to \beta X\) such that \(\theta_M(m_x) = \varphi_x(M)\) for all \(x \in X\). Now we define the operation \(\ast\) on \(\beta X\) as \(M \ast N = \theta_M(N)\). Then we show this operation is associative. To prove this it suffices to show that \(\theta_M \circ \theta_N = \theta_L\) for every \(M, N \in \beta X\) with \(L = \theta_M(N)\). To see this it will be enough to show that \(\theta_M \circ \varphi_x = \varphi_x \circ \theta_M\) for all \(M \in \beta X\) and \(x \in X\). But to see the latter it suffices to show that \(\theta_M \circ \varphi_x\) and \(\varphi_x \circ \theta_M\) agree on \(\eta(X)\), (recall that if two continuous maps into a Hausdorff space agree on a dense subspace of the domain, they are equal). This reduces to show that \(\varphi_x \circ \varphi_y = \varphi_{x,y}\) for all \(x, y \in X\). Finally, to see this it suffices to show that \((\varphi_x \circ \varphi_y)(m_z) = \varphi_{x,y}(m_z)\) for all \(z \in X\). But the latter obviously holds since the operation of \(X\) is associative. Clearly \(\ell_M = \theta_M\) for all \(M \in \beta X\). Hence, \((\beta X,\ast)\) is a left topological semigroup. The map \(\eta\) is a morphism of semigroups since \(\varphi_x = \theta_{m_x}\) for all \(x \in X\). This also yields that \(m_x \ast M = M \ast m_x\) for all \(M \in \beta X\) and \(x \in X\). To see the uniqueness of \(\ast\), suppose there is another operation \(\ast'\) on \(\beta X\) such that \((\beta X, \ast')\) is a left topological semigroup, the canonical map \(\eta : X \to (\beta X, \ast')\) is a morphism of semigroups and \(m_x \ast' M = M \ast' m_x\) for all \(M \in \beta X\) and \(x \in X\). Then clearly for each \(x \in X\), the maps \(\ell_{m_x}\) and \(\ell'_{m_x}\) agree on \(\eta(X)\), hence they are equal. It follows that for each \(M \in \beta X\), then \(\ell_M\) and \(\ell'_M\) agree on \(\eta(X)\), hence they are equal. The latter implies that \(\ast = \ast'\). Finally, if \(e\) is the identity element of \(X\) then \(\varphi_e\) is the identity map. It follows that \(m_e\) is the identity element of \(\beta X\). □

Note that the operation \(\ast\) of Theorem 9.1 is not necessarily commutative. Hence, we may define a new operation \(\times\) on \(\beta X\) as \(M \times N := \theta_N(M) = N \ast M\). Then it is easy to see that \((\beta X, \times)\) is a right topological semigroup. Therefore we may consider \(\beta X\) as left topological or right topological semigroup, depending on the preferred construction,
but for $X$ infinite never both.

In the proof of Theorem 9.1 we have $\varphi_x = \mathcal{P}(f_x)^*$ for all $x \in X$ where the function $f_x : X \to X$ defined by $f_x(y) = x.y$. By [12, Theorem 5.6], there exists a (unique) morphism of rings $h_M : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $\theta_M = \text{Spec}(h_M)$. The following result provides the rule of this morphism.

**Theorem 9.2.** Let $(X, \cdot)$ be a commutative semigroup and $M$ a maximal ideal of $\mathcal{P}(X)$. Then the map $\zeta_M : \mathcal{P}(X) \to \mathcal{P}(X)$ given by $A \mapsto \{x \in X : f_x^{-1}(A) \notin M\}$ is a morphism of rings and $\theta_M = \text{Spec}(\zeta_M)$.

**Proof.** It is not hard to see that the map $\zeta_M$ is actually a morphism of rings. To see $\theta_M = \text{Spec}(\zeta_M)$ it suffices to show that $\theta_M(m_x) = \zeta_M^{-1}(m_x)$ for all $x \in X$. □

The category whose objects are the left topological monoids and whose morphisms are the continuous morphisms of monoids is called the category of left topological monoids.

**Theorem 9.3.** The assignments $X \mapsto \beta X$ and $h \mapsto \beta f$ form a faithful covariant functor from the category of commutative monoids to the category of left topological monoids.

**Proof.** By the universal property of the Stone-Čech compactification, it is a functor provided that we could prove that if $f : X \to Y$ is a morphism of commutative monoids then $\beta f : \beta X \to \beta Y$ is a morphism of monoids. Clearly $\beta f$ preserves the identities. It remains to show that $(\beta f) \circ \theta_M = \theta_{M'} \circ (\beta f)$ for all $M \in \text{Spec} \mathcal{P}(X)$ with $M' = (\beta f)(M)$, for the notations see the proof of Theorem 9.1. To see this it suffices to show that these functions agree on $\eta(X)$. To see the latter it will be enough to show that $(\beta f) \circ \varphi_x = \varphi_{f(x)} \circ (\beta f)$ for all $x \in X$. But clearly these maps agree on $\eta(X)$, hence they are equal. □

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Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.
E-mail address: ebulfedz1978@gmail.com, mohammadrezae.rezaee@yahoo.com