INFINITE DIMENSIONAL GEOMETRY OF $M_1 = \text{Diff}_+ (S^1)/\text{PSL}(2, \mathbb{R})$ AND $q_R$-CONFORMAL SYMMETRIES. II. GEOMETRIC QUANTIZATION AND HIDDEN SYMMETRIES OF VERMA MODULES OVER VIRASORO ALGEBRA.

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This article being a continuation of the first part [1] is addressed to specialists in representation theory, infinite dimensional geometry, quantum algebra, mathematical physics and informatics of interactive systems. The interrelations of the infinite dimensional geometry of the homogeneous Kähler manifold $M = \text{Diff}_+ (S^1)/S^1$ (see e.g. [2] and numerous refs wherein), its quotient $M_1 = \text{Diff}_+ (S^1)/\text{PSL}(2, \mathbb{R})$ and the $q_R$-conformal symmetries are discussed. Considering $q_R$-conformal symmetries in the Verma modules as Nomizu operators [1] one receives a connection in a bundle over $M_1$ with fibers isomorphic to Verma modules over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. An interpretation of this connection in terms of geometric quantization is proposed. Simultaneously, another interpretation of $q_R$-conformal symmetries based on a geometric construction of the Verma modules over Virasoro algebra by the orbit method [3] as hidden symmetries is formulated. Such hidden symmetries are local over $M_1$, i.e. commute with the natural action of the commutative ring $\mathcal{O}(M_1)$ of all polynomial germs of holomorphic functions on $M_1$ in the Verma modules over Virasoro algebra.

1. Preliminary definitions

1.1. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and Verma modules over it. Lobachevski-Berezin $C^*$-algebra and $q_R$-conformal hidden symmetries in the Verma modules over $\mathfrak{sl}(2, \mathbb{C})$. Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is a three-dimensional space of $2 \times 2$ complex matrices with zero trace supplied by the standard commutator $[X, Y] = XY - YX$, where the right hand side multiplication is the standard matrix multiplication. In the basis

$$
\begin{align*}
  l_{-1} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, &
  l_0 &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, &
  l_1 &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},
\end{align*}
$$

the commutation relations are of the form:

$$
[l_i, l_j] = (i - j)l_{i+j} \quad (i, j = -1, 0, 1).
$$

Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is $\mathbb{Z}$-graded: $\deg(l_i) = -\text{ad}(l_0)l_i = i$, where $\text{ad}(X)$ is the adjoint action operator in the Lie algebra: $\text{ad}(X)Y = [X, Y]$. Therefore, $\mathbb{Z}$-graded
modules over $\mathfrak{sl}(2, \mathbb{C})$ are $l_0$-diagonal. A vector $v$ in a $\mathbb{Z}$-graded module over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is called extremal iff $l_1 v = 0$ and the linear span of vectors $l^n_{-1} v$ ($n \in \mathbb{Z}_+$) coincides with the module itself (i.e. $v$ is a cyclic vector). A $\mathbb{Z}$-graded module with an extremal vector (in this case it is defined up to a multiplier) is called extremal [4]. An extremal module is called the Verma module [5] iff the action of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ on the Verma module with an extremal vector (in this case it is defined up to a multiplier) is called extremal [4]. An extremal weight of the Verma module is the number defined for all complex numbers $h$ and are pairwise nonisomorphic. Below we shall consider the Verma modules with real extremal weights only.

The Verma module $V_h$ over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ with the extremal weight $h$ may be realized in the space $\mathbb{C}[z]$ of polynomials of a complex variable $z$. The formulas for the generators of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ are of the form:

$$l_{-1} = z, \quad l_0 = z\partial_z + h, \quad l_1 = z\partial_z^2 + 2h\partial_z,$$

here $\partial_z = \frac{d}{dz}$.

The Verma module is nondegenerate (i.e. does not contain any proper submodule) iff $h \neq -\frac{n}{2}$ ($n \in \mathbb{Z}_+$). The Verma module $V_h$ is called unitarizable (or Hermitian) iff it admits a structure of the pre-Hilbert space such that $l^n_i = l_i^*$. The completion of the unitarizable Verma module will be denoted by $V_h^{\text{Hilb}}$. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ acts in $V_h^{\text{Hilb}}$ by the unbounded operators. Also it is rather useful to consider the formal Verma modules $V_h^{\text{form}}$, which are realized in the space $\mathbb{C}[[z]]$ of formal power series of a complex variable $z$, whereas the formulas for generators of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ coincide with ones above. Note that $V_h \subseteq V_h^{\text{Hilb}} \subseteq V_h^{\text{form}}$ and modules $V_h, V_h^{\text{Hilb}}, V_h^{\text{form}}$ form the Gelfand triple or the Dirac equipment of the Hilbert space $V_h^{\text{Hilb}}$. An action of the real form of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ generated by the anti-Hermitean operators $i\partial_0, l_1 - l_{-1}$ and $i(l_1 + l_{-1})$ in the Hilbert space $V_h^{\text{Hilb}}$ by the unbounded operators is exponentiated to a unitary representation of the corresponding simply connected Lie group.

In the nonunitarizable Verma module over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ there exists the unique (up to a scalar multiple) indefinite sesquilinear form $(\cdot, \cdot)$ such that $(l_i v_1, v_2) = (v_1, l_{-i} v_2)$ for any two vectors $v_1$ and $v_2$ from the Verma module. If this sesquilinear form is nondegenerate (in this case the Verma module is nondegenerate) then it has a signature $(n, \infty)$, where $n$ is finite, and therefore, there is defined a Pontryagin completion [6] of the Verma module. The corresponding module in which the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ acts by the unbounded operators will be denoted by $V_h^{\text{Pontr}}$. The following chain of inclusions holds: $V_h \subseteq V_h^{\text{Pontr}} \subseteq V_h^{\text{form}}$. An action of the real form of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ generated by the anti-Hermitean (with respect to the nondegenerate indefinite sesquilinear form $(\cdot, \cdot)$) operators $i\partial_0, l_1 - l_{-1}$ and $i(l_1 + l_{-1})$ in the Pontryagin space $V_h^{\text{Pontr}}$ by the unbounded operators is exponentiated to a pseudounitary representation of the corresponding simply connected Lie group.

Let us describe some hidden symmetries in the Verma modules over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

**Proposition 1** [7]. In the nondegenerate Verma module $V_h$ over the Lie algebra
\( \mathfrak{sl}(2, \mathbb{C}) \) there are uniquely defined the operators \( D \) and \( F \) such that
\[
[D, l_{-1}] = 1, \quad [D, l_0] = D, \quad [D, l_1] = D^2,
\]
\[
[l_{-1}, F] = 1, \quad [l_0, F] = F, \quad [l_1, F] = F^2.
\]

If the Verma modules are realized in the space \( \mathbb{C}[z] \) of polynomials of a complex variable \( z \) then
\[
D = \partial_z, \quad F = z\frac{1}{\xi+2\hbar},
\]
where \( \xi = z\partial_z \). The operators \( F \) and \( D \) obey the following relations:
\[
[FD, DF] = 0, \quad [D, F] = q_R(1 - DF)(1 - FD),
\]
where \( q_R = \frac{1}{2\hbar - 1} \). In the unitarizable Verma module \( (q_R \neq 0) \) the operators \( F \) and \( D \) are bounded and \( F^* = D, D^* = F \).

The algebra generated by the variables \( t \) and \( t^* \) with the relations \([tt^*, t^*t] = 0 \) and \([t, t^*] = q_R(1 - tt^*)(1 - t^*t)\) being the Berezin quantization of the Lobachevskii plane realized in the unit complex disc (the Poincaré realization) \([8]\) is called the Lobachevskii–Berezin algebra. Proposition 1 allows to consider the Lobachevskii–Berezin algebra as a \( C^* \)-algebra.

**Proposition 2** [7]. In the nongenerate Verma module \( V_h \) over the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) there are uniquely defined the operators \( L_n \) \((n \in \mathbb{Z})\) such that
\[
[l_i, L_n] = (i - n)L_{i+n} \quad (i = -1, 0, 1; \ n \in \mathbb{Z}),
\]
moreover, \( L_i = l_i \) \((i = -1, 0, 1)\). If the Verma modules are realized in the space \( \mathbb{C}[z] \) of polynomials of a complex variable \( z \) then
\[
L_k = (xi + (k + 1)\hbar)\partial_z^k \quad (k \geq 0), \quad L_{-k} = z^k \frac{\xi+(k+1)\hbar}{(\xi+2\hbar)\cdots(\xi+2\hbar+k-1)} \quad (k \geq 1),
\]
where \( \xi = z\partial_z \). The operators \( L_n \) obey the following relations:
\[
[L_n, L_m] = (n - m)L_{n+m}, \text{ if } n, m \geq -1 \text{ or } n, m \leq 1.
\]

In the unitarizable Verma module the operators \( L_n \) are unbounded and \( L_i^* = L_{-i} \).

The operators \( L_n \) are called the \( q_R \)-conformal symmetries. They may be symbolically represented in the form:
\[
L_n = D^{nh}L_0D^{n(1-h)}, \quad L_{-n} = F^{n(1-h)}L_0F^{nh}.
\]

To supply the symbolical recording by a sense one should use the general commutation relations
\[
[L_n, f(D)] = (-D)^{n+1}f'(D) \quad (n \geq -1), \quad [L_{-n}, f(F)] = F^{n+1}f'(D) \quad (n \geq -1)
\]
for \( n = 0 \).

The commutation relations for the operators \( D, F \) and the generators of \( q_R \)-conformal symmetries with the generators of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) mean that the families \( J_k \) and \( L_k \) \((k \in \mathbb{Z})\), where \( J_i = D^i, \ J_{-i} = F^i \ (i \in \mathbb{Z}_+)\), are families of tensor operators [6,9] for the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \).
1.2. Infinite dimensional \( \mathbb{Z} \)-graded Lie algebras: the Witt algebra \( \mathfrak{w}^\mathbb{C} \) of all Laurent polynomial vector fields on a circle and the Virasoro algebra \( \mathfrak{vir}^\mathbb{C} \), its one-dimensional nontrivial central extension. Group \( \text{Diff}_+(\mathbb{S}^1) \) of diffeomorphisms of a circle \( \mathbb{S}^1 \) and the Virasoro-Bott group \( \text{Vir} \). Flag manifold \( M = \text{Diff}_+(\mathbb{S}^1)/\mathbb{S}^1 \) of the Virasoro-Bott group and infinite dimensional Kähler manifold \( M_1 = \text{Diff}_+(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R}) \). The Lie algebra \( \text{Vect}(\mathbb{S}^1) \) is realized in the space of \( C^\infty \)-smooth vector fields \( v(t)\partial_t \) on a circle \( \mathbb{S}^1 \simeq \mathbb{R}/2\pi \mathbb{Z} \) with the commutator

\[
[v_1(t)\partial_t, v_2(t)\partial_t] = (v_1(t)v_2'(t) - v_1'(t)v_2(t))\partial_t.
\]

In the basis

\[
s_n = \sin(nt)\partial_t, \quad c_n = \cos(nt)\partial_t, \quad h = \partial_t
\]

the commutation relations have the form:

\[
[\mathfrak{s}_n, \mathfrak{s}_m] = \frac{1}{2}((m-n)s_{n+m} + \text{sgn}(n-m)(n+m)s_{|n-m|}), \\
[\mathfrak{c}_n, \mathfrak{c}_m] = \frac{1}{2}((m-n)s_{n+m} + \text{sgn}(n-m)(n+m)s_{|n-m|}), \\
[\mathfrak{s}_n, \mathfrak{c}_m] = \frac{1}{2}((m-n)c_{n+m} - (m+n)c_{|n-m|}) - n\delta_{nm}h, \\
[h, \mathfrak{s}_n] = nc_n, \quad [h, \mathfrak{c}_n] = -ns_n.
\]

Let us denote by \( \text{Vect}^\mathbb{C}(\mathbb{S}^1) \) the complexification of the Lie algebra \( \text{Vect}(\mathbb{S}^1) \). In the basis \( \mathfrak{e}_n = i e^{ikt}\partial_t \) the commutation relations in the Lie algebra \( \text{Li} \text{ Vect}^\mathbb{C}(\mathbb{S}^1) \) have the form:

\[
[e_j, e_k] = (j - k)e_{j+k}.
\]

It is rather convenient to consider an imbedding of the circle \( \mathbb{S}^1 \) into the complex plane \( \mathbb{C} \) with the coordinate \( z \), so that \( z = e^{it} \) on the circle and the elements of the basis \( \mathfrak{e}_k \ (k \in \mathbb{Z}) \) are represented by the Laurent polynomial vector fields:

\[
\mathfrak{e}_k = z^{k+1}\partial_z.
\]

The \( \mathbb{Z} \)-graded Lie algebra generated by the Laurent polynomial vector fields (i.e. by the finite linear combinations of elements of the basis \( \mathfrak{e}_k \)) is called the Witt algebra and is denoted by \( \mathfrak{w}^\mathbb{C} \). The Witt algebra \( \mathfrak{w}^\mathbb{C} \) is the complexification of the subalgebra \( \mathfrak{w} \) of the algebra \( \text{Vect}(\mathbb{S}^1) \) generated by the trigonometric polynomial vector fields on a circle \( \mathbb{S}^1 \), i.e. by the finite linear combinations of elements of the basis \( \mathfrak{s}_n, \mathfrak{c}_n \) and \( h \).

The Lie algebra \( \text{Vect}(\mathbb{S}^1) \) admits a nontrivial one-dimensional central extension defined by the Gelfand-Fuchs 2-cocycle [10]:

\[
c(v_1(t)\partial_t, v_2(t)\partial_t) = \int_0^{2\pi} (v_1'(t)v_2''(t) - v_1''(t)v_2'(t)) \ dt.
\]

This extension being continued to the complexification \( \text{Vect}^\mathbb{C}(\mathbb{S}^1) \) of the Lie algebra \( \text{Vect}(\mathbb{S}^1) \) and reduced to the subalgebra \( \mathfrak{w}^\mathbb{C} \) defines a central one-dimensional extension of the Witt algebra, which is called the Virasoro algebra and is denoted by \( \mathfrak{vir}^\mathbb{C} \). The Virasoro algebra is generated by the elements \( \mathfrak{e}_k \ (k \in \mathbb{Z}) \) and the central element \( c \) with the commutation relations:

\[
[e_j, e_k] = (j - k)e_{j+k} + \frac{j^3 - j}{12}c\delta(j + k).
\]
and is the complexification of a central extension $\text{vir}$ of the Lie algebra $\mathfrak{w}$. Because

$$\int_0^{2\pi} (v_1(t)v'_2(t) - v_2(t)v'_1(t)) \, dt$$

is trivial, the cocycle, which defines the Virasoro algebra, is indeed equivalent to the Gelfand-Fuchs 2-cocycle above and is known under the same name. The modified Gelfand-Fuchs cocycle is $\mathfrak{sl}(2,\mathbb{C})$–invariant, where $\mathfrak{sl}(2,\mathbb{C})$ is a subalgebra of $\text{vir}^\mathbb{C}$ generated by $e_{-1}$, $e_0$ and $e_1$, so it is handier in practice. Below we shall use the modified version of Gelfand-Fuchs 2-cocycle under this name only. In the irreducible representation the central element $c$ of the Virasoro algebra is mapped to a scalar operator, which is proportional to the identity operator with a coefficient $c$ called the central charge.

Let $\text{Diff}(S^1)$ denote the group of all diffeomorphisms of the circle $S^1$). The group manifold $\text{Diff}(S^1)$ splits into two connected components, the subgroup $\text{Diff}_+(S^1)$ and the coset $\text{Diff}_-(S^1)$. The diffeomorphisms in $\text{Diff}_+(S^1)$ preserve the orientation on the circle $S^1$ and those in $\text{Diff}_-(S^1)$ reverse it. The Lie algebra of $\text{Diff}_+(S^1)$ can be identified with $\text{Vect}(S^1)$.

The infinite-dimensional group $\text{Vir}$ corresponding to the Virasoro algebra $\text{vir}$ (more precisely, to the central extension of the Lie algebra $\text{Vect}(S^1)$ defined by the Gelfand-Fuchs cocycle, whereas the Virasoro algebra $\text{vir}$ is an extension of the real form $\mathfrak{w}$ of the Witt algebra $\mathfrak{w}^\mathbb{C}$) is a central extension of the group $\text{Diff}(S^1)$. The corresponding 2-cocycle was calculated by R.Bott [11]:

$$c(g_1, g_2) = \int \log(g'_1 \circ g_2) \log(g'_2).$$

The group $\text{Vir}$ is called the Virasoro-Bott group.

The flag manifold $M$ of the Virasoro-Bott group is a homogeneous space with transformation group $\text{Diff}_+(S^1)$ and isotropy group $S^1$). There exist several different realizations of this manifold [12-15].

**Algebraic realization.** The space $M$ can be realized as a conjugacy class in the group $\text{Diff}_+(S^1)$ or in the Virasoro-Bott group $\text{Vir}$.

**Probabilistic realization.** Let $P$ be the space of real probability measures $\mu = u(t) \, dt$ with smooth positive density $u(t)$ on $S^1$. The group $\text{Diff}_+(S^1)$ naturally acts on $P$ by the formula

$$g : u(t) \, dt \mapsto u(g^{-1}(t)) \, dg^{-1}(t).$$

The action is transitive and the stabilizer of the point $(2\pi)^{-1} \, dt$ is isomorphic to $S^1$, therefore, $P$ can be identified with $M$.

**Orbital realization.** The space $M$ can be considered as an orbit of the coadjoint representation of $\text{Diff}_+(S^1)$ or $\text{Vir}$. Namely, the elements of the dual space $\text{vir}^*$ of the Virasoro algebra $\text{vir}$ are identified with the pairs $(p(t) \, dt^2, b)$; the coadjoint action of $\text{Vir}$ has the form

$$K(g)(p, b) = (gp - bS(g), b),$$

where

$$S(g) = \frac{g'''}{g'} - \frac{3}{2} \left( \frac{g''}{g'} \right)^2.$$
is the Schwarzian (the Schwarz derivative) and $gp$ denotes the natural action of $g$ on the quadratic differential $p$. The orbit of the point $(a \cdot dt^2, b)$ coincides with $M$ if $a/b \neq -n^2/2, n = 1, 2, 3, \cdots$. Therefore, a family $\omega_{a,b}$ of symplectic structures (Kirillov forms) is defined on $M$ (cf.\[16\]).

Analytic realization. Let us consider the space $S$ of univalent functions $f(z)$ on the unit disk $D_+$ such that $f(0) = 0$, $f'(0) = 1$ and $f'(e^{it}) \neq 0$ [17-19] (ordinarily the least condition is omitted; so below we shall consider a proper subclass of a conventional class $S$ under the same notation). The Taylor coefficients $c_1, c_2, c_3, \cdots$ in the expansion

$$f(z) = z + c_1 z^2 + c_2 z^3 + \cdots + c_n z^{n+1} + \cdots$$

form a coordinate system on $S$. The class $S$ can be naturally identified with $M$ via the Kirillov construction [14]. The Lie algebra $\text{Vect}(S^1)$ acts on $S$ by the formulas

$$L_v f(z) = -if^2(z) \oint \left( \frac{w f'(w)}{f(w)} \right)^2 \frac{v(w)}{f(w) - f(z)} \frac{dw}{w}.$$ 

The Kirillov construction supply $M$ by the complex structure. The symplectic structure $\omega_{a,b}$ coupled with the complex structure determines a Kähler metric $w_{a,b}$ on $M$. More detailed information on the infinite-dimensional geometry of the flag manifold $M$ is contained in [2] (see also refs wherein). Note only that the curvature tensors of the Kähler connections on $M$ were calculated in [15].

The subgroup $S^1$ is contained in each of the subgroups $H_k$, $k = 1, 2, 3, \cdots$, generated by the generators $i e_0, e_k - e_{-k}$ and $i(e_k + e_{-k})$. The subgroup $H_k$ is isomorphic to the $k$-folded covering of the group $H_1 = \text{PSL}(2, \mathbb{R})$ and acts on $S^1$ by the formulas

$$z \mapsto \left( \frac{az + b}{bz + a} \right)^{1/k} = \alpha z \left( \frac{1 + b z^{-k}}{1 + b z^k} \right)^{1/k},$$

where $\alpha = (a/|a|)^{2/k}$ is an univalued function on $H_k$.

The homogeneous space $M_k = \text{Diff}_+(S^1)/H_k$ is a symplectic manifold and can be identified with the orbit with $a/b = -k^2/2$ in the coadjoint representation of the Virasoro-Bott group. The manifold $M_k$ is a quotient of $M$ by the $\text{Diff}_+(S^1)$-invariant foliation $\mathcal{F}_k$ generated by $s_k$ and $c_k$ (considered as elements of a reductive basis on $M$ [15]). Note that though $M$ is a reductive space the manifolds $M_k$ are not reductive. The foliations $\mathcal{F}_k$ are complex foliations, which were discovered by V.Yu.Ovsienko and O.D.Ovsienko. The foliation $\mathcal{F}_1$ is holomorphic whereas other foliations are not holomorphic. Hence, the almost complex structure on $M_1$ is integrable and, therefore, $M_1 = \text{Diff}_+(S^1)/\text{PSL}(2, \mathbb{R})$ is a homogeneous Kähler manifold. This is not true for $k \geq 2$. The main characteristics of the Kähler metrics on $M_1$ were calculated in [15] (really, there were calculated their back-liftings to $M$). In particular, these Kähler metrics are einsteinian.

Let us consider any point $f(z)$ of $M$. A fiber of the first Ovsienko foliation $\mathcal{F}_1$ consists of points

$$f_b(z) = \frac{f(z)}{1 - b f(z)},$$

where $b \in \mathbb{C} \setminus [f(D_+)]^{-1}$. Therefore, $M_1$ may be identified with the class $S^{(1)}$ of the univalent functions $f(z)$ such that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'(e^{it}) \neq 0$.\]
The variational formulas for an action of $\text{Vect}(\mathbb{S}^1)$ on $S^{(1)}$ are of the form

$$L_v^{(1)} f(z) = L_v f(z) - [L_v f]_2 f^2(z),$$

where

$$[L_v f]_2 = \frac{1}{2} (L_v f)''(0) = i \int \frac{w(f'(w))^2 v(w) dw}{f^3(w)}.$$

1.3. Verma modules over the Virasoro algebra and their construction by the orbit method [3]. The algebras $w^C$ and $\text{vir}^C$ as a $\mathbb{Z}$-graded algebras possess Verma modules over them, which was studied by many authors (see e.g. [20-22] and also [23,24]). Namely, put $\text{vir}_+^C = \text{span}(e_k, k \geq 0)$, $\chi_{h,c}$ be a character of $\text{vir}_+^C$ defined as

$$\chi_{h,c}(e_k) = 0, \text{ if } k > 0, \quad \chi_{h,c}(e_0) = h, \quad \chi_{h,c}(c) = c.$$

Verma module $V_{h,c}$ is a $\text{vir}^C$–module induced from the character $\chi_{h,c}$ of the subalgebra $\text{vir}_+^C$. Otherwords,

$$V_{h,c} = U(\text{vir}^C) \otimes_{U(\text{vir}_+^C)} V_{\chi_{h,c}},$$

where $V_\chi$ is a $\text{vir}_+^C$–module defined by a character $\chi$. $U(\text{vir}^C)$ and $U(\text{vir}_+^C)$ are the universal enveloping algebras of the Lie algebras $\text{vir}^C$ and $\text{vir}_+^C$. The Verma module $V_{h,c}$ is $\mathbb{Z}$–graded $\text{vir}^C$–module; if $h$ and $c$ are real (what will be supposed below) there is defined the unique up to a multiple invariant Hermitian form in $V_{h,c}$. The Verma module is unitarizable if and only if the Hermitian form is positive definite; let us denote by $D_n(h,c)$ the determinant of this form in the $n$-th homogeneous component of $V_{h,c}$ in the basis $e_1^{k_1} \cdots e_j^{k_j} v$, $k_j \geq 0$ ($v$ is the extremal vector, i.e. $e_0 v = hv$, $e_{-m} v = 0$), then as it was shown by V.G.Kac, B.L.Feigin and D.B.Fuchs

$$D_n(h,c) = A \prod_{0 < \alpha \leq \beta} \Phi_{\alpha,\beta}^{p(n-\alpha \beta)},$$

where

$$\Phi_{\alpha,\beta}(h,c) = (h + \frac{c-13}{24} (\beta^2 - 1) + \frac{\alpha \beta - 1}{2}) (h + \frac{c-13}{24} (\alpha^2 - 1) + \frac{\alpha \beta - 1}{2}) + \frac{\alpha^2 - \beta^2}{16},$$

$$\Phi_{\alpha,\alpha}(h,c) = h + \frac{c-13}{24} (\alpha^2 - 1).$$

If for any $\alpha$, $\beta$ $\Phi_{\alpha,\beta}(h,c) \neq 0$, then the module $V_{h,c}$ is irreducible and is not contained in any other Verma module; if there exists exactly one pair $(\alpha, \beta)$ such that $\Phi_{\alpha,\beta}(h,c) = 0$ then there three possibilities may be realized: 1) $\alpha \beta < 0$, then $V_{h,c}$ may be imbedded into the Verma module $V_{h+\alpha \beta, c}$, 2) $\alpha \beta > 0$, then $V_{h,c}$ contains a submodule $V_{h+\alpha \beta, c}$, 3) either $\alpha = 0$ or $\beta = 0$, then $V_{h,c}$ is irreducible and is not a submodule of another Verma module. If there exists two pairs $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ such that $\Phi_{\alpha_1, \beta_1}(h,c) = 0$, then there exists an infinite number of pairs $(\alpha, \beta)$, which possess such property; this situation is realized if

$$c_{1,2} = 1 - \frac{6((\alpha_1 \pm \alpha_2) - (\beta_1 \pm \beta_2))^2}{(\alpha_1 \pm \alpha_2)(\beta_1 \pm \beta_2)}$$

and

$$h_{1,2} = \frac{(\alpha_2 \beta_1 - \alpha_1 \beta_2)^2 - ((\alpha_1 \pm \alpha_2) - (\beta_1 \pm \beta_2))^2}{4(\alpha_1 \pm \alpha_2)(\beta_1 \pm \beta_2)}.$$
In this case the structure of Verma modules is described by the Feigin–Fuchs theory.

The Verma module $V_{h,c}$ is unitarizable if $h > 0$, $c > 1$; the Verma module $V_{h,c}$ contains an unitarizable quotient if (a) $h > 0$, $c > 1$; (b) $c = 1 - \frac{6}{p(p+1)}$, $h = \frac{(\alpha p - \beta (p+1))^2 - 1}{4p(p+1)}$, $\alpha, \beta, p \in \mathbb{Z}$; $p \geq 2$; $1 \leq \alpha \leq p$, $1 \leq \beta \leq p - 1$.

Let us now describe a geometric way of the construction of the Verma modules over the Virasoro algebra based on the orbit method [3]. It uses the following facts:

1. to any $\text{Diff}_+(S^1)$–invariant Kähler metric $w_{h,c}$ (new parameters $h$ and $c$ are related to $a$ and $b$ as $a = h - \frac{c}{12}$, $b = \frac{c}{12}$ and $h = a + b$, $c = 12b$) on the space $M = \text{Diff}_+(S^1)/S^1$ one can assign a linear holomorphic bundle $E_{h,c}$ over $M$ with the following properties:

   a) $E_{h,c}$ is the Hermitian bundle, with metric $\exp(-U_{h,c})d\lambda d\bar{\lambda}$, where $\lambda$ is the coordinate in the fiber and $K_{h,c} = \exp(U_{h,c})$ is the Bergman kernel function, i.e. the exponential of the Kähler potential of the metric $w_{h,c}$ (the Kähler potentials for $w_{h,c}$ were calculated in [15,3]),

   b) the algebra $\text{vir}^C$ holomorphically acts in the prescribed bundle by covariant derivatives with respect to the Hermitian connection with curvature form $2\pi i \omega_{h,c}$ (here $\omega_{h,c}$ is the symplectic structure corresponded to $w_{h,c}$);

2. let $\mathcal{O}(E_{h,c})$ be the space of all polynomial germs (in some natural trivialization) of sections of the bundle $E_{h,c}$ in the coordinates $c_1, \ldots, c_k, \ldots$. The action of $\text{vir}^C$ in its $\mathbb{Z}$-graded module $\mathcal{O}(E_{h,c})$ $(\deg(c_k) = k)$ is defined by the formulas

\[
L_p = \partial_p + \sum_{k \geq 1} (k + 1) c_k \partial_{k+p} \quad (p > 0), \quad L_0 = \sum_{k \geq 1} k c_k \partial_k + h,
\]

\[
L_{-1} = \sum_{k \geq 1} ((k + 2)c_{k+1} - 2c_1 c_k) \partial_k + 2hc_1,
\]

\[
L_{-2} = \sum_{k \geq 1} ((k + 3)c_{k+2} - (4c_2 - c_1^2)c_k - b_k(c_1, \ldots, c_{k+2})) \partial_k
\]

\[
+ h(4c_2 - c_1^2) + \frac{c}{2} (c_2 - c_1^2),
\]

\[
L_{-n} = \frac{1}{(n-2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 0),
\]

where $\partial_k = \frac{\partial}{\partial c_k}$ and $b_k$ are the Laurent coefficients of the function $1/(zf(z))$.

Let us choose the basis $e^{a_1, \ldots, a_n} = c_1^{a_1} \ldots c_n^{a_n}$ in $\mathcal{O}(E_{h,c})$ and let $\mathcal{O}^*(E_{h,c})$ be the space of all linear functionals $p$ on $\mathcal{O}(E_{h,c})$ such that the relation $p(x) \neq 0$ implies $\deg(x) \leq N$. The space $\mathcal{O}^*(E_{h,c})$ is called the Fock space of the pair $(M, E_{h,c})$ and is denoted by $F(E_{h,c})$. The Verma module $V_{h,c}$ over $\text{vir}^C$ is realized in the Fock space $F(E_{h,c})$. The formal Verma module $V_{h,c}^{\text{form}}$ may be obtained if one omits a condition on the functionals $p$. If we choose the basis $e_{a_1, \ldots, a_n} =: c_1^{a_1} \ldots c_n^{a_n}$ in $F(E_{h,c})$ such that

\[
\langle e_{a_1, \ldots, a_n}, e^{b_1, \ldots, b_m} \rangle = a_1! \ldots a_n! \delta_n^m \delta_{a_1}^{b_1} \ldots \delta_{a_n}^{b_n},
\]
then the action of the Virasoro algebra in this basis is given by the formulas

\[
L_{-p} = c_p + \sum_{k \geq 1} c_{k+p} \partial_k \quad (p > 0), \quad L_0 = \sum_{k \geq 1} kc_k \partial_k + h,
\]

\[
L_1 = \sum_{k \geq 1} c_k ((k + 2) \partial_{k+1} - 2 \partial_1 \partial_k) + 2h \partial_1,
\]

\[
L_2 = \sum_{k \geq 1} c_k ((k + 3) \partial_{k+2} - (4 \partial_2 - \partial_1^2) \partial_k - b_k (\partial_1, \ldots, \partial_{k+2}))
\]

\[
+ h(4 \partial_2 - \partial_1^2) + \frac{c}{2} (\partial_2 - \partial_1^2),
\]

\[
L_n = \frac{(-1)^n}{(n-2)!} \text{ad}^{n-2} L_1 \cdot L_2 \quad (n > 2).
\]

The action of \( \text{vir}^C \) is exponentiated to the projective action of Vir in a certain subspace of \( V^\text{form}_{h,c} \). The Bergman kernel function expanded by \( c_k \) and \( \tilde{c}_k \) coincides with the invariant Hermitian form in the Verma module \( V_{h,c} \) written in the basis \( e_{a_1, \ldots, a_n} \). If the Verma module is unitarizable then the action of \( \text{vir}^C \) is exponentiated to the unitary action of the universal covering \( \tilde{\text{Vir}} \) of the Virasoro-Bott group in the space \( V_{h,c}^{\text{Hilb}} \). In the nonunitarizable case the situation is slightly more complicated [23].

The construction above may be applied to the manifold \( M_1 \) instead of \( M \). The Kähler metrics \( w_c \) on \( M_1 \) are just the reduction of degenerate metrics \( w_{h,c} \) on \( M \) with \( h = 0 \). So the bundle \( E_c \) over \( M_1 \) related to the metric \( w_c \) is a restriction of \( E_{0,c} \) over \( M \) on \( M_1 \) (here \( M_1 \) is imbed into \( M \) as a subclass \( S^{(1)} \) of \( S \)). The action of \( \text{vir}^C \) in the space \( \mathcal{O}(E_c) \) is defined by the formulas

\[
L_p = \partial_p + \sum_{k \geq 0} (k + 1) c_k \partial_{k+p} \quad (p \geq 2),
\]

\[
L_1 = \sum_{k \geq 2} (k + 1) c_k \partial_{k+1} - \Gamma,
\]

\[
L_0 = \sum_{k \geq 2} c_k \partial_k,
\]

\[
L_1 = \sum_{k \geq 2} (k + 2) c_{k+1} \partial_k - 3c_2 \Gamma,
\]

\[
L_2 = \sum_{k \geq 2} ((k + 3) c_{k+2} - 4c_2 c_k - b_k (0, c_2, \ldots, c_{k+2})) \partial_k + \frac{c}{2} c_2 - 5c_3 \Gamma,
\]

\[
L_{-n} = \frac{1}{(n-2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 0),
\]

where

\[
\Gamma = 2 \sum_{k \geq 2} c_k \partial_{k+1} + \sum_{i,j \geq 2} c_i c_j \partial_{i+j+1}.
\]

The Fock space \( F(E_c) \) is a \( \mathbb{Z} \)-graded \( \text{vir}^C \)-module in which the module \( W_c \) (a
quotient of $V_{0,c}$) is realized. The Virasoro algebra $\text{vir}^C$ acts in $W_c$ as

\[
L_{-p} = c_p + \sum_{k \geq 2} (k + 1)c_{k+p}\partial_k \quad (p \geq 2),
\]

\[
L_{-1} = \sum_{k \geq 2} (k + 1)c_{k+1}\partial_k - \Gamma^*,
\]

\[
L_0 = \sum_{k \geq 2} c_k\partial_k,
\]

\[
L_1 = \sum_{k \geq 2} (k + 2)c_k\partial_{k+1} - 3\Gamma^*\partial_2,
\]

\[
L_2 = \sum_{k \geq 2} c_k((k + 3)\partial_{k+2} - 4\partial_2\partial_k - b_k(0, \partial_2, \ldots, \partial_{k+2})) + \frac{c}{2}\partial_2 - 5\Gamma^*\partial_3,
\]

\[
L_n = \frac{(-1)^n}{(n-2)!} \text{ad}^{n-2} L_1 \cdot L_2 \quad (n > 2).
\]

where

\[
\Gamma^* = 2 \sum_{k \geq 2} c_{k+1}\partial_k + \sum_{i,j \geq 2} c_{i+j+1}\partial_i\partial_j.
\]

2. Geometric quantization and hidden symmetries of Verma modules over the Virasoro algebra

2.1. Verma modules over the Virasoro algebra and geometric quantization. Let us consider an arbitrary (topologically trivial for simplicity) Kähler manifold $N$ and a linear bundle $E$ over it. There exists a canonical imbedding of the manifold $N$ into the projectivization $\mathbb{P}(F(E))$ of the Fock space $F(E)$ of the bundle $E$. If $E$ is the Hermitian bundle and the scalar products in fibers is defined via the Bergman kernel function (the exponent of a Kähler potential) of the Kähler metric $w$ on $N$ then $F(E)$ is a pseudo-Hilbert space and the imbedding $N \hookrightarrow \mathbb{P}(F(E))$ is isometric. The bundle $E$ over $N$ may be restored as a restriction $\tau|_N$ of the canonical bundle $\tau$ over $\mathbb{P}(F(E))$ onto the submanifold $N$. Note that $\tau$ possess a natural Hermitian connection, which restriction onto $E = \tau|_N$ is just the prequantization connection in $E$ (the Hermitian connection is called a prequantization one if its curvature is equal to $2\pi\omega$, where the symplectic form $\omega$ is the imaginary part of the Kähler metric $w$ [25,26]).

If the Kähler manifold $N$ is not topologically trivial then one is able to consider it locally and then to globalize all constructions. At least, any isometric imbedding of a Kähler manifold $N$ into a projective space $\mathbb{P}(V)$ supplies $N$ by a prequantization bundle $E = \tau|_N$ (a Hermitian bundle with a prequantization connection) over it.

A natural generalization of projective spaces are Grassmannians. It is rather interesting to generalize the geometric quantization schemes exposed above onto imbeddings into Grassmannians. Indeed, let us consider a Kähler manifold $N$ and its isometric imbedding into the Grassmannian $\text{Gr}(V; V_0)$ of all subspaces of the Hilbert (Euclidean) space $V$, which are isometric to its subspace $V_0$. One is able to consider a canonical bundle $\tau$ over $\text{Gr}(V; V_0)$ with fibers isometric to $V_0$ and a canonical Hermitian connection $\nabla$ in it and then to restrict $\tau$ and $\nabla$ onto the submanifold $N$. Thus, one obtains a Hermitian bundle $\hat{E}$ over $N$ and a Hermitian
connection $\nabla$ in it. In general, $\nabla$ is not flat and has a non-trivial (and nonscalar) curvature.

In particular, one may consider the Kähler manifold $N$ with a prequantization linear bundle $E$ and put $V = F(E)$. Let us also consider a holomorphic bundle $N \mapsto N_1$. For any fiber $p$ of this bundle let us put $V_0 = F(p) \subseteq V$. Then, one receives an imbedding of $N_1$ into $\text{Gr}(V; V_0)$ and the construction above supplies $N_1$ by the Hermitian bundle $\hat{E}$ over it with fibers isometric to $V_0$ and Hermitian connection $\nabla$ in it.

One may specify the homogeneous case $N = G/H$, where $G$ is a real reductive Lie group and $H$ is its certain subgroup, $E$ is a prequantization bundle over $N$. The Fock space $F(E)$ realizes a Verma module over $g^C$ [27]. Let also $G_0$ be a subgroup of $G$ such that $H \subseteq G_0$, and $G_0/H \mapsto G/H \mapsto G/G_0$ is a holomorphic bundle. Then one obtains an imbedding $G/G_0 \mapsto \text{Gr}(V; V_0)$, where $V = F(E)$ and $V_0 = F(E|_{G_0/H})$, the Hermitian bundle $\hat{E}$ over $G/G_0$ and Hermitian connection $\nabla$ in it. The situation is straightforwardly generalized to infinite dimensional case (e.g. $G$ is the Virasoro-Bott group).

In the homogeneous case the bundle $\hat{E}$ possesses a natural action of the Lie algebra $g^C$ by operators $L_X (X \in g^C)$. The covariant derivatives $\nabla_X$ along the elements of $g^C$ are not closed. In the physical terms their commutators realize the so-called nonscalar Schwinger terms [28]. The operators $A_X = L_X - \nabla_X$ are fiberwise and are called the Nomizu operators [29] of the connection $\nabla$.

The Fock space $F(\hat{E})$ of the bundle $\hat{E}$ over $G/G_0$ is naturally identified with the Fock space $F(E)$ of the bundle $E$ over $G/H$ and, therefore, realizes a Verma module over $g^C$. Hence, the covariant derivatives $\nabla_X$ and the Nomizu operators $A_X$ act in the Verma module.

Specifying the situation for the Virasoro-Bott group one obtains the following proposition.

**Theorem 1.** The Verma module $V_{h,c}$ over the Virasoro algebra may be realized in the Fock space $F(\hat{E}_{h,c})$ of the vector bundle $\hat{E}_h$ over $M_1 = \text{Diff}_+(S^1)/\text{PSL}(2, \mathbb{R})$ with fibers isomorphic to the Verma module $V_h$ over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. The Nomizu operators $A_X (X \in \text{vir}^C)$ of the Hermitian connection $\nabla$ in $\hat{E}$ being considered in any fixed fiber coincide with the $q_R$-conformal symmetries in $V_h$.

The theorem follows from the fact that the Nomizu operators $A_X$ are tensor ones in $V_h$ with respect to $\mathfrak{sl}(2, \mathbb{C})$ and, therefore, coincide with $q_R$-conformal symmetries up to a multiple. However, for $X \in \mathfrak{sl}(2, \mathbb{C}) A_X$ are just the generators of $\mathfrak{sl}(2, \mathbb{C})$, hence, the multiple is equal to 1 \hfill \square

This theorem is parallel to the main theorem of [1]. Below we shall discuss the action of the operators $\nabla_X$ and $A_X (X \in \text{vir}^C)$ in the Verma modules over the Virasoro algebra.

**Remark 1.** $\hat{E}_{h,c} = \hat{E}_{h,0} \otimes E_c$, where $E_c$ is a linear bundle over $M_1$. This representation is valid for the connections $\nabla$ also.

### 2.2. Nomizu operators $A_X$ as hidden symmetries in the Verma modules over the Virasoro algebra.

Let us realize the Verma modules $V_{h,c}$ over the Virasoro algebra $\text{vir}^C$ in the Fock spaces $F(E_{h,c})$ of the Hermitian bundles $E_{h,c}$. Then the commutative algebra $\mathcal{O}(M_1)$ of all polynomial germs of holomorphic functions on $M_1$ naturally acts in $F(E_{h,c})$ as a subalgebra of $\mathcal{O}(M)$. 

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Theorem 2. There exists a unique family of tensor operators in the Verma module $V_{h,c}$ over the Virasoro algebra $\text{vir}^C$, which transform as the adjoint representation of $\text{vir}^C$, and which are local over $M_1$, i.e. commute with the action of $\mathcal{O}(M_1)$ in the module $V_{h,c}$. These tensor operators coincide with the Nomizu operators $A_X$ up to a multiple.

Note that tensor operators from a family form the same family being multiplied on a number.

A proof of the theorem 2 is just the same as of theorem 1.

Remark 2. 

$[L_X, A_Y] = A_{[X,Y]}$

for any $X, Y \in \text{vir}^C$.

2.3. Setting hidden symmetries free.

Definition 1 [30].

A. A linear space $v$ is called a Lie composite iff there are fixed its subspaces $v_1, \ldots, v_n$ (dim $v_i > 1$) supplied by the compatible structures of Lie algebras. Compatibility means that the structures of the Lie algebras induced in $v_i \cap v_j$ from $v_i$ and $v_j$ are the same. The Lie composite is called dense iff $v_1 \cup \ldots \cup v_n = v$ (here $\cup$ denotes the sum of linear spaces). The Lie composite is called connected iff for all $i$ and $j$ there exists a sequence $k_1, \ldots, k_m$ ($k_1 = i$, $k_m = j$) such that $v_{k_i} \cap v_{k_{i+1}} \neq \emptyset$.

B. A representation of the Lie composite $v$ in the space $H$ is the linear mapping $T : v \mapsto \text{End}(H)$ such that $T|_v$ is a representation of the Lie algebra $v_i$ for all $i$.

C. Let $g$ be a Lie algebra. A linear mapping $T : g \mapsto \text{End}(H)$ is called the composed representation of $g$ in the linear space $H$ iff there exists a set $g_1, \ldots, g_n$ of the Lie subalgebras of $g$, which form a dense connected composite and $T$ is its representation.

Reducibility and irreducibility of representations of the Lie composites are defined in the same manner as for Lie algebras. The set of representations of the fixed Lie composite is closed under the tensor product and, therefore, may be supplied by the structure of tensor category.

Let us consider two subalgebras $p_\pm$ of the Witt algebra $w^C$ generated by $e_i$ with $i \geq -1$ and $i \leq 1$; note that $p_+ \cap p_- = \mathfrak{sl}(2, \mathbb{C})$. The triple $(w^C; p_+, p_-)$ is a dense connected Lie composite.

Proposition 3 [30]. The $q_R$-conformal symmetries in the Verma module $V_h$ over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ form a representation of the Lie composite $(w^C; p_+, p_-)$ and, hence, the composed representation of the Witt algebra $w^C$.

Theorems 1,2 and Proposition 3 imply Theorem 3.

Theorem 3. The operators $\nabla_X$ ($X \in \text{vir}^C$) in the Verma module $V_{h,c}$ over the Virasoro algebra form a representation of the Lie composite $(w^C; p_+, p_-)$ and, hence, the composed representation of the Witt algebra $w^C$.

Remark 3. 

$[L_X, \nabla_Y] = \nabla_{[X,Y]}$

for any $X, Y \in \text{vir}^C$. 

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It is naturally to consider the operators $\nabla_X$ as hidden symmetries in the Verma modules $V_{h,c}$ over the Virasoro algebra and to try to set them free [31]. To do it one needs in some generalization of the procedure of the setting hidden symmetries free (as well as of definition of hidden symmetries) in view of the infinite dimensionality of the family $\nabla_X$.

**Definition 2.**

A. Let $\mathfrak{g}$ be a $\mathbb{Z}$–graded Lie algebra and $A$ be a $\mathbb{Z}$–graded associative algebra such that $\mathfrak{g}$ is a $\mathbb{Z}$–graded subalgebra of $\text{Der}(A)$; a linear subspace $V$ of $A$ is called a space of hidden symmetries iff (1) $V$ is a $\mathfrak{g}$–submodule of $A$, (2) the Weyl symmetrization defines mappings $W_\pm: S^\cdot(V_\pm) \to A$, and $W_0: S^\cdot(V_0) \to A$, where $V_\pm$ and $V_0$ are the subspaces of $V$ of elements of positive, negative and zero degree, respectively (here $S^\cdot(H)$ is the symmetric algebra over a linear space $H$) such that the products $W_+(A)W_0(B)W_-(C)$ $(A \in S^\cdot(V_+), B \in S^\cdot(V_0), C \in S^\cdot(V_-))$ are dense in $A$. The elements of $V$ are called hidden symmetries with respect to $\mathfrak{g}$). A $\mathbb{Z}$–graded associative algebra $F$ such that $\mathfrak{g} \subset \text{Der}(F)$ is called an algebra of the set free hidden symmetries iff (1) $F$ is generated by $V$, (2) there exists a $\mathfrak{g}$–equivariant epimorphism of algebras $F \to A$, (3) the Weyl symmetrization define mappings $W_\pm: S^\cdot(V_\pm) \to F$ and $W_0: S^\cdot(V_0) \to F$ so that their product $W_+W_0W_-$ defines the isomorphism of $\mathfrak{g}$–modules $S^\cdot(V_+) \otimes S^\cdot(V_0) \otimes S^\cdot(V_-)$ of $\mathfrak{g}$–modules.

Let $\mathfrak{g}$ be a Lie algebra, $V$ be a certain $\mathfrak{g}$–module, $\mathcal{A}_s$ be a family of associative algebras, parametrized by $s \in \mathcal{S}$ such that $\mathfrak{g} \subset \text{Der}(\mathcal{A}_s), \pi_s: V \to \mathcal{A}_s$ be a family of $\mathfrak{g}$–equivariant imbeddings such that $\pi_s(V)$ is a space of hidden symmetries in $\mathcal{A}_s$ with respect to $\mathfrak{g}$ for a generic $s$ from $\mathcal{S}$. An associative algebra $F$ is called an algebra of the $\mathcal{A}_s, s \in \mathcal{S}$–universally set free hidden symmetries iff $F$ is an algebra of the set free hidden symmetries corresponding to $V \simeq \pi_s(V)$ for generic $\mathcal{A}_s (s \in \mathcal{S})$.

The definitions are a natural $\mathbb{Z}$–graded generalizations of ones from [31].

The following theorem, which unravel the algebraic structure of operators $\nabla_X$, holds.

**Theorem 4.** The operators $\nabla_X$ in the Verma modules $V_{h,c}$ over the Virasoro algebra $\text{vir}_C$ form a family of hidden symmetries. For any $c$ there exists a unique algebra $F_c$, which set these hidden symmetries free $\text{End}(V_{h,c})$–universally ($h \in \mathbb{R}$). The algebras $F_c$ are the quotients of the central extension $F$ of the algebra $F_0$ by the ideal generated by $c - c \cdot 1$, where $c$ is the central element.

**Remark 4.** The theorem means that the algebra $F$ may be represented in the model $M_c$ of the Verma modules $V_{h,c}$ over the Virasoro algebra [27] for any $c$.

The theorem 4 easily follows from the dimension counting.

3. **Conclusions**

Thus, some natural hidden symmetries in the Verma modules over the Virasoro algebra are constructed in terms of geometric quantization. Their differential geometric meaning is established and their expression via $q_R$–conformal symmetries in the Verma modules over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is found. The analysis and the unraveling of the algebraic structure of these families of hidden symmetries are performed.
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