ASYMPTOTICS FOR THE HIGHER-ORDER DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

Pavel I. Naumkin
Centro de Ciencias Matemáticas, UNAM Campus Morelia, AP 61-3 (Xangari)
Morelia CP 58089, Michoacán, MEXICO

Isahi Sánchez-Suárez∗
Universidad Politécnica de Uruapan
CP 60210 Uruapan Michoacán, MEXICO

(Communicated by Robert M. Strain)

Abstract. We study the Cauchy problem for the derivative higher-order nonlinear Schrödinger equation
\[
\begin{aligned}
  &i\partial_t v + \frac{a}{2} \partial_x^2 v - \frac{b}{4} \partial_x^4 v = (\partial_x v)^2, \quad t > 1, \quad x \in \mathbb{R}, \\
  &v(1, x) = v_0(x), \quad x \in \mathbb{R},
\end{aligned}
\]
where \( a, b > 0 \). Our aim is to prove global existence and calculate the large time asymptotics of solutions. We develop the factorization techniques originated in papers [13, 10, 12]. Also we follow the method of papers [9, 11] to transform the quadratic nonlinearity to critical cubic nonlinearities similarly to the normal forms of Shatah [18].

1. Introduction. We consider the Cauchy problem for the one dimensional derivative higher-order nonlinear Schrödinger equation
\[
\begin{aligned}
  &i\partial_t v + \frac{a}{2} \partial_x^2 v - \frac{b}{4} \partial_x^4 v = (\partial_x v)^2, \quad t > 1, \quad x \in \mathbb{R}, \\
  &v(1, x) = v_0(x), \quad x \in \mathbb{R},
\end{aligned}
\]
where \( a, b > 0 \). We take the initial time at \( t = 1 \) for the convenience of the forthcoming calculations. Equation (1.1) occurs in the study of deep water wave dynamics [7], solitary waves [15, 16], vortex filaments [8], and so on. To explain the significance of the quadratic nonlinearity \((\partial_x v)^2\) in the model we refer the papers [4, 17].

Changing \( u = (i\partial_x) v \), we rewrite equation (1.1) as
\[
\begin{aligned}
  &i\partial_t u + \frac{a}{2} \partial_x^2 u - \frac{b}{4} \partial_x^4 u = (\partial_x u)^2, \quad t > 1, \quad x \in \mathbb{R}, \\
  &u(1, x) = u_0(x), \quad x \in \mathbb{R},
\end{aligned}
\]
where \( u_0 = (i\partial_x) v_0, \quad S = \partial_x (i\partial_x)^{-1}, \quad (i\partial_x) = \sqrt{1 - \partial_x^2}. \)

2020 Mathematics Subject Classification. Primary: 35B40; Secondary: 35Q35.
Key words and phrases. Asymptotics of solutions, derivative nonlinear Schrödinger equation, higher order, factorization techniques, modified scattering.

The work of P. I. N. is partially supported by CONACYT project 283698 and PAPIIT project IN103221.

* Corresponding author.
The global in time existence of small solutions to the Cauchy problem (1.2) is a difficult problem, since comparing the time decay rate of the linear part of equation (1.2) (see [1]) with that for the right-hand side of (1.2), we conclude that the quadratic nonlinearity should be considered as subcritical in the one dimensional case. To overcome this obstacle we follow the method of paper [9, 11], where we use the method similar to the normal forms of Shatah (see [18]) to transform the quadratic nonlinearity to critical cubic nonlinearities involving bilinear operators. Thus we will show below that the nonlinearity in equation (1.2) due to it’s oscillating structure acts as a critical one for large time. Another difficulty in our problem comes from the estimates involving the operator $J = x + a i \partial_x - b i \partial_x^3$, which plays a crucial role in the large time asymptotics of solutions. To overcome this difficulty we develop the factorization technique originated in papers [13, 10, 12].

We introduce Notations and Function Spaces. $L^p = \{ \phi \in S'; \| \phi \|_{L^p} < \infty \}$ is the usual Lebesgue space with norm $Vert \phi \vert_{L^p} = (\int_{\mathbb{R}} \phi(x)^p dx)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\| \phi \|_{L^\infty} = \sup_{x \in \mathbb{R}} |\phi(x)|$ for $p = \infty$. The weighted Sobolev space is $H^{m,s}_p = \{ \phi \in S'; \| \phi \|_{H^{m,s}_p} = \| \langle x \rangle^s (i \partial_x)^m \phi \|_{L^p} < \infty \}$, with $m, s \in \mathbb{R}, 1 \leq p \leq \infty, \langle x \rangle = \sqrt{1 + x^2}, (i \partial_x) = \sqrt{1 - \partial_x^2}$. Below $F$ stands for the Fourier transform $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \phi(x) dx$, and $F^{-1}$ is the inverse Fourier transformation $F^{-1} \phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi \xi} \phi(\xi) d\xi$. We also use the notations $H_2^{m,s} = H_2^{m,0}, H_3^m = H_3^{m,0}$. Let $C(I; B)$ be the space of continuous functions from the time interval $I$ to a Banach space $B$.

Now we state the results of this paper. Denote the dilation operator $D_t(\phi(x) = t^{-\frac{3}{2}} \phi(\frac{x}{t}))$, the scaling operator $(B\phi)(x) = \phi(\mu(x)),\mu(x)$ is defined by the equation $\lambda(t, \eta) = e^{ie\theta(\eta)}$, $\Theta(\eta) = \eta \Lambda'(\eta) - \Lambda(\eta)$, $\Lambda(\xi) = \frac{4\xi^4}{\lambda'(\xi)\lambda(\xi)\lambda(\xi)} 2\lambda(\xi)$, $\mu(x)$ is defined by the equation $\lambda'(\mu) = x$.

**Theorem 1.1.** Let the initial data $u_0 \in H^2 \cap H^{1,1}$ and the norm $\| u_0 \|_{H^2 \cap H^{1,1}}$ be sufficiently small. Then there exists a unique global solution $u$ of the Cauchy problem (1.2) such that $u \in C \left( [1, \infty); H^2 \cap H^{1,1} \right)$. Moreover there exists a unique final state $W_+ \in L^\infty$ such that the following asymptotics

$$u(t) = D_t BWMW_+ \exp \left(-i\lambda(\xi) |W_+|^2 \log t \right) + O \left(t^{-\frac{3}{2} - \delta} \right)$$

is valid as $t \to \infty$ uniformly with respect to $x \in \mathbb{R}, \delta > 0$.

The rest of our paper is organized as follows. In Section 2 we transform equation (1.2) following the method of paper [9, 11] similarly to the normal forms of Shatah. Section 3 is devoted to the Factorization Techniques for the transformed equation. Preliminary estimates related to solutions of the linear problem are obtained in Section 4. Section 5 and Section 6 deal with the estimates for the bilinear operators $\mathcal{Z}$ and $\mathcal{G}$, respectively. In Section 7 we estimate the nonlinearity in equation (3.4). Finally we prove Theorem 1.1 in Section 9 by using a priori estimates for the solutions in the norm

$$\sup_{t \in [1, T]} \left( t^{-\gamma_2} \| u(t) \|_{H^2} + t^{-\gamma} \| \mathcal{Z} u(t) \|_{H^1} \right) + \sup_{t \in [1, T]} \left( \| \xi \right. \hat{\psi} \left( t \right) \left. \|_{L^\infty}\right),$$

obtained in Section 8, where $\gamma > 3\gamma_2 > 0$ are small, $w(t) = \mathcal{U}(t) F^{-1} \hat{\psi}$, and $\hat{\psi}$ is defined by equation (2.2) below.

2. **Transformation of equation.** Define the new dependent variable $\hat{\varphi} = \mathcal{F} \mathcal{U}(-t)$ $w$, where the free evolution group $\mathcal{U}(t) = F^{-1} e^{-i\lambda(\xi)t} F$ and the symbol $\Lambda(\xi) =$
Using the equation $\psi_t = \mathcal{L}\psi$ with $\mathcal{L} = i\partial_t + \frac{a}{2} \partial_x^2 - \frac{b}{4} \partial_x^4$, applying the Fourier transformation to equation (1.2), using formulas

$$
\mathcal{F}(uv) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\eta)\hat{v}(\xi-\eta)d\eta,
$$

$\mathcal{F}\pi = \tilde{u}(\xi)$, we get $i\partial_t\hat{\varphi} = K_0 \left( \hat{\varphi}, \hat{\varphi} \right)$, where the symmetric bilinear operator

$$
K_0 \left( \hat{\varphi}, \hat{\varphi} \right) = \mathcal{F} \mathcal{U} (-t) \left( \left( \partial_x \mathcal{U} (t) \mathcal{F}^{-1} \hat{\varphi} \right) \partial_x \mathcal{U} (t) \mathcal{F}^{-1} \hat{\varphi} \right)
$$

$$
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega (\xi, \eta)} \mathcal{F}^{-1} \hat{\varphi} (t, \eta - \xi) f(t, -\eta) \eta d\eta
$$

and $\omega (\xi, \eta) = \Lambda (\xi) + \Lambda (\eta - \xi) + \Lambda (\eta)$.

Using the identity $e^{i\omega (\xi, \eta)} = H(t, \xi, \eta) \partial_t \left( te^{i\omega (\xi, \eta)} \right)$, where $H(t, \xi, \eta) = (1 + it\omega (\xi, \eta))^{-1}$, we obtain

$$
i\partial_t \left( K_1 \left( \hat{\varphi}, \hat{\varphi} \right) \right) = -K_0 \left( \hat{\varphi}, \hat{\varphi} \right) - 2K_1 \left( i\hat{\varphi}, \hat{\varphi} \right) + \frac{1}{t} K_2 \left( \hat{\varphi}, \hat{\varphi} \right),
$$

(2.1)

where we define the symmetric bilinear operators

$$
K_j \left( \hat{\varphi}, \hat{\varphi} \right) = \int_{\mathbb{R}} e^{i\xi (\xi, \eta)} f(t, \eta - \xi) f(t, -\eta) H_j(t, \xi, \eta) d\eta
$$

with $H_1(t, \xi, \eta) = \frac{\xi}{\sqrt{2\pi}} \frac{(\xi, \eta)}{(\xi - \eta, \eta)} H(t, \xi, \eta)$ and $H_2(t, \xi, \eta) = \frac{\xi}{\sqrt{2\pi}} \frac{(\xi, \eta)}{(\xi - \eta, \eta)} it\partial_t H(t, \xi, \eta)$.

Taking identity (2.1) with $\psi = \varphi$ and multiplying by $\{\xi\} = 1 - \xi^{-1}$, we get

$$
i\partial_t \left( K_3 \left( \hat{\varphi}, \hat{\varphi} \right) \right) = -\{\xi\} K_0 \left( \hat{\varphi}, \hat{\varphi} \right) - 2K_3 \left( i\hat{\varphi}, \hat{\varphi} \right) + \frac{1}{t} K_5 \left( \hat{\varphi}, \hat{\varphi} \right),
$$

where $K_3 = \{\xi\} K_1$ and $K_5 = \{\xi\} K_2$. Also multiplying identity (2.1) by $\{\xi\}^{-1}$ we find

$$
i\partial_t \left( K_4 \left( \hat{\psi}, \hat{\psi} \right) \right) = -\{\xi\}^{-1} K_0 \left( \hat{\psi}, \hat{\psi} \right) - 2K_4 \left( i\hat{\psi}, \hat{\psi} \right) + \frac{1}{t} K_6 \left( \hat{\psi}, \hat{\psi} \right),
$$

where $K_4 = \{\xi\}^{-1} K_1$ and $K_6 = \{\xi\}^{-1} K_2$. Then since $\{\xi\} + \{\xi\}^{-1} = 1$, we have

$$
K_0 \left( \hat{\varphi}, \hat{\varphi} \right) = -i\partial_t \left( K_3 \left( \hat{\varphi}, \hat{\varphi} \right) + K_4 \left( \hat{\psi}, \hat{\psi} \right) \right) + \{\xi\}^{-1} \left( K_0 \left( \hat{\varphi}, \hat{\varphi} \right) - K_0 \left( \hat{\psi}, \hat{\psi} \right) \right)
$$

$$
- 2K_4 \left( i\hat{\varphi}, \hat{\varphi} \right) - 2K_4 \left( i\hat{\psi}, \hat{\psi} \right) + \frac{1}{t} \left( K_5 \left( \hat{\varphi}, \hat{\varphi} \right) + K_6 \left( \hat{\psi}, \hat{\psi} \right) \right).
$$

Using the equation $i\partial_t\hat{\varphi} = K_0 \left( \hat{\varphi}, \hat{\varphi} \right)$, we get

$$
K_0 \left( \hat{\varphi}, \hat{\varphi} \right) = -i\partial_t \left( K_3 \left( \hat{\varphi}, \hat{\varphi} \right) + K_4 \left( \hat{\psi}, \hat{\psi} \right) \right) + \{\xi\}^{-1} \left( K_0 \left( \hat{\varphi}, \hat{\varphi} \right) - K_0 \left( \hat{\psi}, \hat{\psi} \right) \right)
$$

$$
- 2K_3 \left( K_0 \left( \hat{\varphi}, \hat{\varphi} \right), \hat{\varphi} \right) - 2K_4 \left( i\hat{\psi}, \hat{\psi} \right) + \frac{1}{t} \left( K_5 \left( \hat{\varphi}, \hat{\varphi} \right) + K_6 \left( \hat{\psi}, \hat{\psi} \right) \right).
$$

And choosing a new dependent variable $\hat{\psi}$ such that

$$
\hat{\psi} = \hat{\varphi} + K_3 \left( \hat{\varphi}, \hat{\varphi} \right) + K_4 \left( \hat{\psi}, \hat{\psi} \right),
$$

(2.2)

we obtain the following equation

$$
i\partial_t\hat{\psi} = \{\xi\}^{-1} \left( K_0 \left( \hat{\varphi}, \hat{\varphi} \right) - K_0 \left( \hat{\psi}, \hat{\psi} \right) \right)
$$

$$
- 2K_3 \left( K_0 \left( \hat{\varphi}, \hat{\varphi} \right), \hat{\varphi} \right) - 2K_4 \left( i\hat{\psi}, \hat{\psi} \right) + \frac{1}{t} \left( K_5 \left( \hat{\varphi}, \hat{\varphi} \right) + K_6 \left( \hat{\psi}, \hat{\psi} \right) \right).$$
We next return to the $x$-representation. Define $w(t) = \mathcal{U}(t) F^{-1} \hat{\psi}$, $u(t) = \mathcal{U}(t) F^{-1} \hat{\varphi}$, i.e. substitute $\hat{\psi} = \mathcal{FU}(t) \psi$, $\hat{\varphi} = \mathcal{FU}(t) \phi$.

Since
\[
\mathcal{U}(t) F^{-1} (\xi)^{-1} \mathcal{K}_0 \left( \frac{\mathcal{S}\psi}{\mathcal{S}\varphi} \right) = (\mathcal{S}u)^2
\]
and $\mathcal{U}(t) F^{-1} (\xi)^{-1} \mathcal{K}_0(\tilde{\psi}, \tilde{\varphi}) = (\mathcal{S}w)^2$, applying $\mathcal{U}(t) \mathcal{F}^{-1}$ to the above equation, and using
\[
\mathcal{U}(t) F^{-1} \mathcal{L} \hat{\psi} = \mathcal{L} \mathcal{U}(t) F^{-1} \hat{\psi} = \mathcal{L} w,
\]
we get
\[
\mathcal{L} w = (\mathcal{S} u)^2 - (\mathcal{S} w)^2 - 2Z_3 \left( \langle \mathcal{L} w, (\mathcal{S} u)^2 \rangle, u \right) - 2Z_4 (\mathcal{L} w, w) + \frac{1}{t} \left( Z_5 (u, u) + Z_6 (w, w) \right),
\]
where we denote the symmetric bilinear operators
\[
Z_j (\phi_1, \phi_2) = \mathcal{U}(t) \mathcal{F}^{-1} \mathcal{K}_j \left( \mathcal{FU}(-t) \phi_1, \mathcal{FU}(-t) \phi_2 \right).
\]

Next we represent
\[
Z_j (u, w) = \mathcal{U}(t) \mathcal{F}^{-1} \mathcal{K}_j \left( \mathcal{FU}(-t) u, \mathcal{FU}(-t) w \right)
\]
\[
= \mathcal{F}^{-1} \int e^{-it\mathcal{A}(\xi)} \int e^{it\omega(t, \eta)} e^{it\mathcal{A}(\eta, \xi)} \tilde{u}(t, \eta - \xi) \tilde{w}(t, \eta) \mathcal{H}_j (t, \xi, \eta) d\eta
\]
\[
= \frac{1}{t} \int \tilde{g}_j (t, \xi - \eta, \eta) \tilde{u}(t, \eta - \xi) \tilde{w}(t, \eta) d\eta,
\]
where $\tilde{g}_j (t, \xi, \eta) = \mathcal{H}_j (t, \xi + \eta, \eta)$. Also we have $\tilde{g}_3 (t, \xi, \eta) = \{ \xi + \eta \} \tilde{g}_1 (t, \xi, \eta)$, $\tilde{g}_4 (t, \xi, \eta) = \{ \xi + \eta \} \tilde{g}_2 (t, \xi, \eta)$, $\tilde{g}_6 (t, \xi, \eta) = \{ \xi + \eta \} \tilde{g}_2 (t, \xi, \eta)$. We find
\[
Z_j (u, w) = \mathcal{F}^{-1} \int \tilde{g}_j (t, \xi - \eta, \eta) \tilde{u}(t, \eta - \xi) \tilde{w}(t, \eta) d\eta
\]
\[
= \int \tilde{g}_j (t, y, z) u(t, x - y) w(t, x - z) dydz
\]
with the kernel $g_j (t, y, z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \tilde{g}_j (t, \xi, \eta) e^{i\eta y + i\xi z} d\xi d\eta$.

From the equation $\hat{\psi} = \hat{\varphi} + \mathcal{K}_3(\hat{\varphi}, \hat{\varphi}) + \mathcal{K}_4(\hat{\psi}, \hat{\psi})$ it follows the relation $w = u + Z_3 (u, u) + Z_4 (w, w)$. Then we find $Z_j (u, u) = Z_j (w, w) + R_{j1} + R_{j2}$, where $R_{j1} = -2Z_j (w, Z_3 (u, u) + Z_4 (w, w))$, $R_{j2} = Z_j (Z_3 (u, u) + Z_4 (w, w), Z_3 (u, u) + Z_4 (w, w))$.

Using $Z_5 (w, w) + Z_6 (w, w) = Z_2 (w, w)$, we write
\[
\frac{1}{t} \left( Z_5 (u, u) + Z_6 (w, w) \right) = \frac{1}{t} Z_2 (w, w) + \frac{1}{t} \left( R_{51} + R_{52} \right).
\]

Also applying $Z_3 (w, w) + Z_4 (w, w) = Z_1 (w, w)$, we get
\[
Z_3 (u, u) + Z_4 (w, w) = Z_1 (w, w) + R_{31} + R_{32}.
\]
Hence we have $u = w - Z_1 (w, w) - R_{31} - R_{32}$. Next we transform
\[
(\mathcal{S} u)^2 - (\mathcal{S} w)^2 = -2SwSZ_1 (w, w) + 4 (\mathcal{S} w) \left( \mathcal{S} Z_3 (w, Z_1 (w, w)) \right)
\]
\[
+ (\mathcal{S} Z_1 (w, w))^2 + r_1,
\]
where the remainder contains the fifth and higher order terms

\[ r_1 = 4 (\mathcal{S} w) \left( \mathcal{S} \mathcal{Z}_3 (w, R_{31} + R_{32}) \right) - 2 (\mathcal{S} w) \mathcal{S} R_{32} + 2 \mathcal{S} \mathcal{Z}_1 (w, w) \left( \mathcal{S} R_{31} + \mathcal{S} R_{32} \right) \]

\[ + (\mathcal{S} R_{31} + \mathcal{S} R_{32})^2. \]

Also denoting \( \mathcal{Z}_8 (u, w) = \mathcal{Z}_3 \left( (i \partial_x) u, w \right) \), we get

\[ \mathcal{Z}_3 \left( (i \partial_x) (\mathcal{S} u)^2, u \right) = \mathcal{Z}_8 \left( (\mathcal{S} w)^2, w \right) - 2 \mathcal{Z}_8 \left( (\mathcal{S} w \mathcal{Z}_1 (w, w)), w \right) \]

\[ - \mathcal{Z}_8 \left( (\mathcal{S} w)^2, Z_1 (w, w) \right) + r_2, \]

where

\[ r_2 = 2 \mathcal{Z}_8 \left( \mathcal{S} w \mathcal{Z}_1 (w, w), Z_1 (w, w) \right) - \mathcal{Z}_8 \left( (\mathcal{S} w)^2 - 2 \mathcal{S} w \mathcal{Z}_1 (w, w), R_{31} + R_{32} \right) \]

\[ + \mathcal{Z}_8 \left( 4 (\mathcal{S} w) \left( \mathcal{S} \mathcal{Z}_3 (w, Z_1 (w, w)) \right) + \left( \mathcal{S} \mathcal{Z}_1 (w, w) \right)^2 \right) \]

\[ + \mathcal{Z}_8 \left( r_1, w - Z_1 (w, w) - R_{31} - R_{32} \right). \]

Thus we obtain

\[ \mathcal{L} w = -2 (\mathcal{S} w \mathcal{Z}_3 (w, w)) - 2 \mathcal{Z}_8 \left( (\mathcal{S} w)^2, w \right) \]

\[ + 4 (\mathcal{S} w) \left( \mathcal{S} \mathcal{Z}_3 (w, Z_1 (w, w)) \right) + 4 \mathcal{Z}_8 \left( \mathcal{S} w \mathcal{Z}_3 (w, w), w \right) \]

\[ + \left( \mathcal{Z}_1 (w, w) \right)^2 - 2 \mathcal{Z}_8 \left( (\mathcal{S} w)^2, Z_1 (w, w) \right) + \frac{1}{\mathcal{L}} \mathcal{Z}_2 (w, w) - 2 \mathcal{Z}_4 (\mathcal{L} w, w) + r_3, \]

where \( \mathcal{Z}_7 (u, w) = \mathcal{S} \mathcal{Z}_1 (u, w), r_3 = \frac{1}{\mathcal{L}} (R_{31} + R_{32}) + r_1 - 2 r_2 \). Define the operator \( \mathcal{Y}_w \phi = 2 \mathcal{Z}_4 (w, \phi) \), then we can write

\[ (1 + \mathcal{Y}_w) \mathcal{L} w = -2 \mathcal{S} w \mathcal{Z}_7 (w, w) - 2 \mathcal{Z}_8 \left( (\mathcal{S} w)^2, w \right) + \left( \mathcal{Z}_7 (w, w) \right)^2 \]

\[ + 4 \mathcal{Z}_8 \left( \mathcal{S} w \mathcal{Z}_7 (w, w), w \right) + 2 \mathcal{Z}_8 \left( (\mathcal{S} w)^2, Z_1 (w, w) \right) + \frac{1}{\mathcal{L}} \mathcal{Z}_2 (w, w) + r_3. \]

Using the identities \((1 + \mathcal{Y}_w)^{-1} = 1 - \mathcal{Y}_w + (1 + \mathcal{Y}_w)^{-1} \mathcal{Y}_w^2\) for the cubic terms and \((1 + \mathcal{Y}_w)^{-1} = 1 - (1 + \mathcal{Y}_w)^{-1} \mathcal{Y}_w\) for all the other summands, we obtain

\[ \mathcal{L} w = -2 \mathcal{S} w \mathcal{Z}_7 (w, w) - 2 \mathcal{Z}_8 \left( (\mathcal{S} w)^2, w \right) + \mathcal{Z}_8 \left( \mathcal{S} w \mathcal{Z}_7 (w, w), w \right) \]

\[ + 4 \mathcal{Z}_4 \left( w, \mathcal{Z}_8 \left( (\mathcal{S} w)^2, w \right) \right) + 4 \mathcal{S} w \left( \mathcal{S} \mathcal{Z}_3 (w, Z_1 (w, w)) \right) \]

\[ + \left( \mathcal{Z}_7 (w, w) \right)^2 - 2 \mathcal{Z}_8 \left( (\mathcal{S} w)^2, Z_1 (w, w) \right) + \frac{1}{\mathcal{L}} \mathcal{Z}_2 (w, w) + r_4, \]

where \( \mathcal{Z}_8 (u, w) = 4 \mathcal{Z}_4 (u, w) + 4 \mathcal{Z}_8 (u, w) \), and the remainder term

\[ r_4 = (1 + \mathcal{Y}_w)^{-1} r_3 - 2 (1 + \mathcal{Y}_w)^{-1} \mathcal{Y}_w \left( \mathcal{S} w \mathcal{Z}_7 (w, w) \right) - 2 (1 + \mathcal{Y}_w)^{-1} \mathcal{Y}_w \left( \mathcal{S} w \mathcal{Z}_1 (w, w) \right) \]

\[ - 4 (1 + \mathcal{Y}_w)^{-1} \mathcal{Y}_w \mathcal{Z}_8 \left( \mathcal{S} w \mathcal{Z}_7 (w, w), w \right) - (1 + \mathcal{Y}_w)^{-1} \mathcal{Y}_w \left( \mathcal{Z}_7 (w, w) \right)^2 \]

\[ - 4 (1 + \mathcal{Y}_w)^{-1} \mathcal{Y}_w \left( \mathcal{S} w \left( \mathcal{S} \mathcal{Z}_3 (w, Z_1 (w, w)) \right) \right) \]

\[ + 2 (1 + \mathcal{Y}_w)^{-1} \mathcal{Y}_w \mathcal{Z}_8 \left( (\mathcal{S} w)^2, Z_1 (w, w) \right) - \frac{1}{\mathcal{L}} (1 + \mathcal{Y}_w)^{-1} \mathcal{Y}_w \mathcal{Z}_2 (w, w) = \sum_{j=1}^8 r_{4j}. \]

This is our target equation.
3. Factorization techniques. We have for the free evolution group
\[ U(t) F^{-1} \phi = F^{-1} e^{-it\Lambda(\xi)} \phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it(\xi - \Lambda(\xi))} \phi(\xi) d\xi, \]
where \( \Lambda(\xi) = \frac{a}{2} \xi^2 + \frac{b}{4} \xi^4 \). Consider the stationary point \( \mu(x) \) defined by the equation \( \Lambda'(\mu) = x \). Since \( \Lambda''(\xi) = a + 3b\xi^2 > 0, \) then \( \Lambda'(\xi) = a\xi + b\xi^3 \) is monotonous. Hence there exists a unique stationary point \( \mu(x) = \frac{r}{|x|} ((\frac{1}{2b}|x| + \sqrt{\frac{1}{27} b^3 + \frac{1}{49} x^2})^\frac{1}{3} - \frac{a}{3b} (\frac{1}{2b}|x| + \sqrt{\frac{1}{27} b^3 + \frac{1}{49} x^2})^{-\frac{1}{3}} \) such that \( \Lambda'(\mu(x)) = x \) for all \( x \in \mathbb{R} \). Then we write
\[ U(t) F^{-1} \phi = D_t \sqrt{\frac{t}{2\pi}} e^{it\Theta(\mu(x))} \int_{\mathbb{R}} e^{-it(\Lambda(\xi) - \Lambda(\mu(x))) - x(\xi - \mu(x))} \phi(\xi) d\xi = D_t BM Q \phi, \]
where the dilation operator \( D_t \phi(x) = t^{-\frac{3}{2}} \phi \left( \frac{x}{t} \right) \), the scaling operator \( (B \phi)(x) = \phi(\mu(x)) \), the multiplication factor \( M(t, \eta) = e^{it\Theta(\eta)} \), \( \Theta(\eta) = \eta \Lambda'(\eta) - \Lambda(\eta) \), the phase function \( S(\xi, \eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi - \eta) \) and the defect operator
\[ Q(t) \phi = \int_{\mathbb{R}} e^{-itS(\xi, \eta)} \phi(\xi) d\xi. \]
Also we define the adjoint defect operator
\[ Q^*(t) \phi = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{it\Lambda(\xi) - it\xi x} e^{it\Theta(\mu(x))} \phi(\mu(x)) dx = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi, \eta)} \phi(\eta) \Lambda''(\eta) d\eta. \]
Thus we have the representation for the free evolution group \( U(t) F^{-1} = D_t BM Q \) and for the inverse evolution group \( F U(-t) = Q^* MB^{-1} D_t^t \) with the inverse scaling operator \( (B^{-1}\phi)(\eta) = \phi(\Lambda'(\eta)) \) and the inverse dilation operator \( D_t^{-1} \phi(x) = t^{-\frac{3}{2}} \phi(xt) \).

Define the operators \( A_k = \frac{1}{\Lambda''(\eta)} M^k \partial_\eta M^k, \ k = 0, 1, \) such that \( A_1 = i\eta + A_0 \). We have \( i\xi Q^* = Q^* A_1 \). Since
\[ S(\xi, \eta) + k\Theta(\eta) = \Omega_{k+1}(\xi) + (1 + k) S \left( \frac{\xi}{1 + k}, \eta \right), \]
where \( \Omega_{k+1} = \Lambda(\xi) - (k + 1) \Lambda \left( \frac{\xi}{k+1} \right) \) for \( k \neq -1 \), by the definition of the operator \( Q^*(t) \) we obtain
\[ Q^*(t) M^k \phi = e^{it\Omega_{k+1}} D_{k+1} Q^* ((k + 1) t) \phi. \]
Finally we mention that the operator \( J = U(t) x U(-t) = x - t\Lambda'(-i\partial_x) \), with \( \Lambda'(-i\partial_x) = F^{-1} \Lambda'(\xi) F \), plays a crucial role in the large time asymptotic estimates. Note that \( J \) commutes with \( L = i\partial_t + \frac{9}{2} \partial_x^2 - \frac{1}{4} \partial_x^4 \), i.e. \( [J, L] = 0 \).

We next apply the factorization formulas to equation (2.5) to deduce equation for \( \hat{\psi} = F U(-t) w(t) \). Since \( F U(-t) L = i\partial_t F U(-t) \), applying the operator \( F U(-t) = Q^* MB^{-1} D_t^t \) to equation (2.5) we get
\[ i\partial_t \hat{\psi} = \sum_{j=1}^9 \Phi_j, \]
where
\[ \Phi_1 = -2Q^* MB^{-1} D_t^t \left( \hat{S} w \hat{Z}(w, w) \right), \]
\[ \Phi_2 = -2Q M B^{-1} D_t^{-1} Z_8 \left( (Sw)^2, w \right), \]
\[ \Phi_3 = Q M B^{-1} D_t^{-1} Z_9 \left( Sw Z_7 (w, w), w \right), \]
\[ \Phi_4 = 4Q M B^{-1} D_t^{-1} Z_4 \left( w, Z_8 \left( (Sw)^2, w \right) \right), \]
\[ \Phi_5 = Q M B^{-1} D_t^{-1} \left( Z_7 (w, w) \right)^2, \]
\[ \Phi_6 = 4Q M B^{-1} D_t^{-1} \left( Sw \left( Z_10 (w, Z_1 (w, w)) \right) \right), \]
\[ \Phi_7 = 2Q M B^{-1} D_t^{-1} Z_8 \left( (Sw)^2, Z_1 (w, w) \right), \]
\[ \Phi_8 = \frac{1}{t} Q M B^{-1} D_t^{-1} Z_2 (w, w), \]
\[ \Phi_9 = Q M B^{-1} D_t^{-1} r_4, \]

with \( Z_{10} = S Z_3 \). We substitute \( \phi_j = D_t BM^{\alpha_1} W_j \) into \( B^{-1} D_t^{-1} Z_j (\phi_1, \phi_2) \), with \( \alpha_1, \alpha_2 \in \mathbb{R} \), then by formula (2.4) we get
\[ B^{-1} D_t^{-1} Z_j (\phi_1, \phi_2) \]
\[ = B^{-1} D_t^{-1} \left( D_t BM^{\alpha_1} W_1, D_t BM^{\alpha_2} W_2 \right) \]
\[ = B^{-1} D_t^{-1} \int_{\mathbb{R}^2} g_j (t, y, z) \left( D_t BM^{\alpha_1} W_1 \right) (t, x - y) \left( D_t BM^{\alpha_2} W_2 \right) (t, x - z) dy dz \]
\[ = t^{-\frac{1}{2}} B^{-1} \int_{\mathbb{R}^2} g_j (t, xt - y, xt - z) \left( BM^{-\alpha_1} W_1 \right) \left( t, \frac{y}{t} \right) \times \left( BM^{-\alpha_2} W_2 \right) \left( t, \frac{z}{t} \right) dy dz \]
\[ = t^{-\frac{1}{2}} B^{-1} \int_{\mathbb{R}^2} g_j (t, yt - x, xt - z) \left( BM^{-\alpha_1} W_1 \right) (t, y) \left( BM^{-\alpha_2} W_2 \right) (t, z) dy dz. \]

Next changing \( y = \Lambda' (\eta) \), \( z = \Lambda' (\zeta) \), we find
\[ B^{-1} D_t^{-1} Z_j (\phi_1, \phi_2) \]
\[ = t^{\frac{1}{2}} \int_{\mathbb{R}^2} g_j (t, t \Lambda' (\xi) - yt, t \Lambda' (\xi) - zt) \left( M^{-\alpha_1} W_1 \right) (t, \mu (y)) \left( M^{-\alpha_2} W_2 \right) (t, \mu (z)) dy dz \]
\[ = t^{\frac{1}{2}} \int_{\mathbb{R}^2} g_j (t, t (\Lambda' (\xi) - \Lambda' (\eta)), t (\Lambda' (\xi) - \Lambda' (\zeta))) \left( M^{-\alpha_1} W_1 \right) (t, \eta) \times \left( M^{-\alpha_2} W_2 \right) (t, \zeta) \Lambda'' (\eta) \Lambda'' (\zeta) d\eta d\zeta \]
\[ = t^{\frac{1}{2}} \int_{\mathbb{R}^2} e^{iP_{\alpha_1, \alpha_2} (\xi, \eta, \zeta)} g_j (t, t (\Lambda' (\xi) - \Lambda' (\eta)), t (\Lambda' (\xi) - \Lambda' (\zeta))) \times \left( M^{-\alpha_2} W_2 \right) (t, \zeta) \Lambda'' (\eta) \Lambda'' (\zeta) d\eta d\zeta \]
\[ = t^{\frac{1}{2}} \int_{\mathbb{R}^2} e^{iP_{\alpha_1, \alpha_2} (\xi, \eta, \zeta)} g_j (t, t (\Lambda' (\xi) - \Lambda' (\eta)), t (\Lambda' (\xi) - \Lambda' (\zeta))) \Lambda'' (\eta) \Lambda'' (\zeta) \Lambda'' (\xi) \Lambda'' (\eta) d\eta d\zeta \]
\[ = t^{\frac{1}{2}} M^{-\alpha_1 + \alpha_2} \int_{\mathbb{R}^2} h_{j, \alpha_1, \alpha_2} (t, \xi, \eta, \zeta) \Lambda'' (\eta) \Lambda'' (\zeta) d\eta d\zeta \]
\[ = t^{\frac{1}{2}} M^{-\alpha_1 + \alpha_2} \mathcal{G}_{j, \alpha_1, \alpha_2} \left( W_1, W_2 \right), \]

where the operator
\[ \mathcal{G}_{j, \alpha_1, \alpha_2} (\phi_1, \phi_2) = t^2 \int_{\mathbb{R}^2} h_{j, \alpha_1, \alpha_2} (t, \xi, \eta, \zeta) \phi_1 (t, \eta) \phi_2 (t, \zeta) d\eta d\zeta, \]

the kernel
\[ h_{j, \alpha_1, \alpha_2} (t, \xi, \eta, \zeta) = e^{iP_{\alpha_1, \alpha_2} (\xi, \eta, \zeta)} g_j (t, t (\Lambda' (\xi) - \Lambda' (\eta)), t (\Lambda' (\xi) - \Lambda' (\zeta))) \Lambda'' (\eta) \Lambda'' (\zeta) \Lambda'' (\xi) \Lambda'' (\eta) \Lambda'' (\zeta) \Lambda'' (\xi) \Lambda'' (\eta) d\eta d\zeta \]
and
\[ P_{\alpha_1, \alpha_2} (\xi, \eta, \zeta) = (\alpha_1 + \alpha_2) \Theta (\xi) - \alpha_1 \Theta (\eta) - \alpha_2 \Theta (\zeta), \]
\[\alpha_1, \alpha_2 \in \mathbb{R}. \text{ Thus we get} \]
\[B^{-1}D^1_t Z_j (D_t BM^{\alpha_1} W_1, D_t BM^{\alpha_2} W_2) = t^{-\frac{1}{2}} M^{-(\alpha_1 + \alpha_2)} \Phi_{j, \alpha_1, \alpha_2} (W_1, W_2). (3.3)\]

Define \( q = Q \hat{\psi}, q_S = Q S \hat{\psi}, S = i \xi (\xi^{-1}, \text{ so that } w = D_t BM q, Sw = D_t BM q_S. \) We transform equation (3.2). Using formulas (3.3) and (3.1) with \( k = 3, -4, \) we find

\[
\begin{align*}
\Phi_1 &= -2t^{-1} Q^* q_S \hat{G}_{7,1,1} (\xi, \eta), \\
\Phi_2 &= -2t^{-1} Q^* \hat{G}_{8, -2, -1} (q, \eta), \\
\Phi_3 &= t^{-\frac{3}{2}} e^{it\eta^{-2}D_{-2} Q^* (-2t) \hat{G}_{9, 1, 1} (q_S \hat{G}_{7,1,1} (\xi, \eta), \eta),} \\
\Phi_4 &= 4t^{-\frac{1}{2}} e^{it\eta^{-2}D_{-2} Q^* (-2t) \hat{G}_{4, 1, 1} (\eta_{S, -2, 1} (q^2_S, \eta), \eta),} \\
\Phi_5 &= t^{-\frac{3}{2}} e^{it\eta^{-2}D_{4} Q^* (4t) \hat{G}_{7,1,1} (\eta, \eta),} \\
\Phi_6 &= 4t^{-\frac{1}{2}} e^{it\eta^{-2}D_{-2} Q^* (-2t) \hat{G}_{10,1,1} (\eta, \eta),} \\
\Phi_7 &= 2t^{-\frac{1}{2}} e^{it\eta^{-2}D_{-2} Q^* (4t) \hat{G}_{8, -2, -2} (q^2_S, \hat{G}_{1,1,1} (\xi, \eta), \eta),} \\
\Phi_8 &= t^{-\frac{3}{2}} e^{it\eta^{-2}D_{-2} Q^* (-2t) \hat{G}_{2,1,1} (\xi, \eta),} \\
\text{where } \Omega_{-2} &= \Lambda(\xi) + 2\Lambda (\xi), \Omega_4 = \Lambda(\xi) - 4\Lambda (\xi).
\end{align*}
\]

Thus we obtain
\[
i \partial_t \hat{\psi} = t^{-1} \Psi_1 + t^{-\frac{3}{2}} e^{it\eta^{-2} \Psi_2 + t^{-\frac{1}{2}} e^{it\eta^{-2} \Psi_3 + \Phi_9, (3.4)}
\]

where
\[
\begin{align*}
\Psi_1 &= -2Q^* q_S \hat{G}_{7,1,1} (\xi, \eta) - 2Q^* \hat{G}_{8, -2, -1} (q^2_S, \eta), \\
\Psi_2 &= D_{-2} Q^* (-2t) \hat{G}_{9, 1, 1} (q_S \hat{G}_{7,1,1} (\xi, \eta), \eta) \\
&\quad + 4D_{-2} Q^* (-2t) \hat{G}_{10,1,1} (\eta, \hat{G}_{1,1,1} (\xi, \eta), \eta) \\
&\quad + D_{-2} Q^* (-2t) \hat{G}_{2,1,1} (\xi, \eta) + 4D_{-2} Q^* (-2t) \hat{G}_{4, 1, 1} (\eta_{S, -2, 1} (q^2_S, \eta), \eta) \\
&= \Psi_{21} + \Psi_{22} + \Psi_{23} + \Psi_{24}
\end{align*}
\]

and
\[
\Psi_3 = D_{4} Q^* (4t) \left( \hat{G}_{7,1,1} (\eta, \eta), \right)^2 - 2G_{8, -2, -2} (q^2_S, \hat{G}_{1,1,1} (\xi, \eta)) = \Psi_{31} + \Psi_{32}.
\]

We use this equation to estimate the norms \( \| \langle \xi \rangle \hat{\psi} \|_{L^\infty} \) and \( \| \langle \xi \rangle \partial_\xi \hat{\psi} \|_{L^2}. \)

4. Preliminaries.

4.1. Estimates for \( Q. \) Denote the kernel \( A(t, \eta) = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi, \eta)} d\xi. \) As in paper [12] we find the representation
\[
A(t, \eta) = \frac{1}{\sqrt{i \Lambda^p (\eta)}} + O \left( \frac{1}{\sqrt{T}} \right)
\]
for all \( t > 0, \eta \in \mathbb{R}, \) where \( 1 \leq p < \infty. \) Also as in paper [12] we obtain the following result.

**Lemma 4.1.** Let \( \delta \in [0, 2]. \) Then the estimate
\[
\left\| \langle \eta \rangle \hat{\psi} \right\|_{L^\infty} \leq C t^{-\frac{1}{2}} \left\| \langle \xi \rangle \partial_\xi \hat{\psi} \right\|_{L^2}
\]
is valid for all \( t \geq 1. \)
As a consequence of Lemma 4.1 we have the estimate for \( j = 0, 1, 2, 3 \)
\[
\| \langle \eta \rangle^{\min(\frac{3}{2}, 1)} Q \xi \phi \|_{L^\infty} \leq C \left( \| \langle \xi \rangle \phi \|_{L^\infty} + t^{-\frac{1}{2}} \| \langle \xi \rangle \phi \|_{L^2} \right).
\]

There are many papers devoted to the \( L^2 \) - estimates of pseudodifferential operators (see, e.g. [2, 5, 6, 14]). Consider the following pseudodifferential operator \( a(t, x, D) \phi = \int e^{ix\xi} a(t, x, \xi) \hat{\phi}(\xi) d\xi \). We will use the following result due to [14].

**Lemma 4.2.** Let the symbol \( a(t, x, \xi) \) be such that
\[
\sup_{x, \xi, t, \geq 1} |\partial^k_x \partial^l_{\xi} a(t, x, \xi)| \leq C
\]
for \( k, l = 0, 1 \). Then \( \| a(t, x, D) \phi \|_{L^2} \leq C \| \phi \|_{L^2} \) for all \( t \geq 1 \).

Next we find the \( L^2 \) - estimate for the weighted defect operator
\[
\mathcal{V}_h \phi = \sqrt{\frac{t}{2 \pi}} \int e^{-iS(\xi, \eta)} h(\xi, \eta) \phi(\xi) d\xi.
\]

**Lemma 4.3.** Let \( h(\xi, \eta) \) satisfy the estimate
\[
\sup_{\xi, x \in \mathbb{R}, t \geq 1} |\partial^k_x \partial^l_{\xi} h(\xi, \mu (xt^{-1}))| \leq C
\]
for \( k, l = 0, 1 \). Then the inequality \( \| \sqrt{\mathcal{V}^* \mathcal{V}_h} \phi \|_{L^2} \leq C \| \phi \|_{L^2} \) is true for \( t \geq 1 \).

**Proof.** Changing \( \eta = \mu (x) \) we get
\[
\mathcal{V}_h \phi = \frac{1}{\sqrt{2 \pi}} MB^{-1} D_t^{-1} \int e^{ix_x} h(\xi, \mu (xt^{-1})) e^{-itA(\xi)} \phi(\xi) d\xi
\]
\[
= \frac{1}{\sqrt{2 \pi}} MB^{-1} D_t^{-1} a(t, x, D) F^{-1} e^{-itA} \phi,
\]
where we introduce the pseudodifferential operator
\[
a(t, x, D) \phi = \int e^{ix_x} a(t, x, \xi) \hat{\phi}(\xi) d\xi
\]
with a symbol \( a(t, x, \xi) = h(\xi, \mu (xt^{-1})) \). In view of conditions of the lemma, we obtain the following estimate
\[
|\partial^k_x \partial^l_{\xi} a(t, x, \xi)| \leq C |\partial^k_x \partial^l_{\xi} h(\xi, \mu (xt^{-1}))| \leq C
\]
for all \( \xi, x \in \mathbb{R}, t \geq 1, k, l = 0, 1 \). Therefore, applying Lemma 4.2, we find
\[
\| a(t, x, D) \phi \|_{L^2} \leq C \| \phi \|_{L^2}
\]
for \( t \geq 1 \). Also using equalities \( \| \sqrt{\mathcal{V}^* \mathcal{V}_h} \phi \|_{L^2} = \| \phi \|_{L^2} \) and \( \| D_t^{-1} \phi \|_{L^2} = \| \phi \|_{L^2} \), we get
\[
\| \sqrt{\mathcal{V}^* \mathcal{V}_h} \phi \|_{L^2} \leq C \| \sqrt{\mathcal{V}^* \mathcal{V}_h} D_t^{-1} a(t, x, D) F^{-1} e^{-itA} \phi \|_{L^2} \leq C \| \phi \|_{L^2}.
\]

Lemma 4.3 is proved.

Next we find the \( L^2 \) - estimate for the derivative of the defect operator \( \mathcal{Q} \).

**Lemma 4.4.** Let \( \delta \in [0, 2] \). Then the estimate
\[
\| \sqrt{\mathcal{V}^* \mathcal{V}} \langle \eta \rangle^{-\delta} \partial_\eta \mathcal{Q} \phi \|_{L^2} \leq C \| \langle \xi \rangle^{-\delta} \phi \|_{L^2} + C \| \langle \xi \rangle^{-\delta} \phi \|_{L^2}
\]
is true for all \( t > 0 \).
Proof. Integration by parts yields
\[ \langle \eta \rangle^{-\delta} \partial_\eta Q\phi = - \langle \eta \rangle^{-\delta} \sqrt{\frac{t}{2\pi}} \int \frac{d\xi}{2\pi} e^{-itS(\xi,\eta)} \left( \partial_\xi \partial_\eta S(\xi,\eta) - \partial_\eta \partial_\xi S(\xi,\eta) \right) \phi(\xi) d\xi \]
\[ - \langle \eta \rangle^{-\delta} \sqrt{\frac{t}{2\pi}} \int \frac{d\xi}{2\pi} e^{-itS(\xi,\eta)} \partial_\xi \partial_\eta S(\xi,\eta) \phi_\xi(\xi) d\xi. \]
Since \( \partial_\eta S(\xi,\eta) = -\Lambda'(\eta)(\xi-\eta) \) and \( \partial_\xi S(\xi,\eta) = \Lambda'(\xi) - \Lambda'(\eta) \), we get
\[ \langle \eta \rangle^{-\delta} \partial_\eta Q\phi = I_1 + I_2 + I_3, \]
where
\[ I_1 = \sqrt{\frac{t}{2\pi}} \int e^{-itS(\xi,\eta)} \psi_1(\xi,\eta) \langle \xi \rangle^{-\delta} \phi(\xi) d\xi, \]
with
\[ \psi_1(\xi,\eta) = \langle \xi \rangle^\delta \langle \eta \rangle^{-\delta} \frac{b(\nu + 3b)}{(\nu + \xi + \eta^2)^2} \]
\[ I_2 = - \langle \eta \rangle^{-\delta} Q\frac{a}{\nu + \xi + \eta^2} \phi_\xi(\xi), \]
and
\[ I_3 = - \sqrt{\frac{t}{2\pi}} \int e^{-itS(\xi,\eta)} \psi_2(\xi,\eta) \langle \xi \rangle^{-\delta} \phi_\xi(\xi) d\xi, \]
where
\[ \psi_2(\xi,\eta) = \langle \xi \rangle^\delta \langle \eta \rangle^{-\delta} \frac{b\eta(2a\nu - a\xi + 3b\nu^2)}{(a + b(\xi^2 + \eta^2))}(a + b\xi^2)^2. \]
To apply Lemma 4.3 we check the condition
\[ \sup_{\xi,\eta \in \mathbb{R}, t \geq 1} \left| \partial_{\xi}^k \partial_{\mu}^l \delta_j(\xi, \mu(\nu t^{-1})) \right| \leq C \]
for \( k, l = 0, 1 \), and \( j = 1, 2 \), if \( \delta \in [0, 2] \). Then by Lemma 4.3 we get \( \|\sqrt{\Lambda^2}I_1\|_{L^2} \leq C\|\langle \xi \rangle^{-\delta} \phi\|_{L^2} \) and
\[ \|\sqrt{\Lambda^2}I_2\|_{L^2} + \|\sqrt{\Lambda^2}I_3\|_{L^2} \leq C \|\langle \xi \rangle^{-\delta} \phi\|_{L^2}. \]
Thus the estimate of the lemma is valid for all \( t > 0 \), if \( \delta \in [0, 2] \). Lemma 4.4 is proved.

As a consequence of Lemma 4.4 we have
\[ \|\sqrt{\Lambda^2} \langle \eta \rangle^{\min(2,3-j)} tA_0 Q\xi^j \phi\|_{L^2} \leq C \|\langle \xi \rangle \phi\|_{L^\infty} + \|\langle \xi \rangle \phi_\xi\|_{L^2} \]
for \( j = 0, 1, 2, 3 \). Next we find the \( L^2 \) - estimate for the derivative \( \partial_1 \) of the defect operator \( Q \).

**Lemma 4.5.** The estimate \( \|\sqrt{\Lambda^2} \langle \eta \rangle^{-1} tQ_\epsilon \phi\|_{L^2} \leq C(\|\langle \xi \rangle \phi\|_{L^\infty} + \|\langle \xi \rangle \phi_\xi\|_{L^2}) \) is true for all \( t > 0 \).

**Proof.** Note that \( S(\xi,\eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi-\eta), \Lambda(\xi) = \frac{\nu}{2}\xi^2 + \frac{\nu}{2}\xi^4 \). Then using the identity \( A_1 = (A_0 + i\eta) \), we get
\[ A_1^4 - 4(\nu \eta)^3 A_1 + 3(\nu \eta)^4 = A_0 A_1^3 + i\eta A_0 A_1^2 + (\nu \eta)^2 A_0 A_1 - 3(\nu \eta)^3 A_0. \]
Hence
\[ Q_\epsilon \phi = \frac{1}{2t} Q\phi + \frac{ai}{2} \left( A_1^2 - 2i\eta A_1 + (\nu \eta)^2 \right) Q\phi - \frac{ib}{4} \left( A_1^4 - 4(\nu \eta)^3 A_1 + 3(\nu \eta)^4 \right) Q\phi. \]
Applying the estimate of Lemma 4.4 we obtain
\[ \left\| \frac{1}{t} \mathcal{A}^\eta(t \xi) -1 t \mathcal{Q}_t \phi \right\|_{L^2} \leq \sum_{j=0}^{3} \left\| \mathcal{A}^{\eta}(\eta)^{2-j} \mathcal{A}_0 \mathcal{Q} \xi_j \phi \right\|_{L^2} \leq C \left( \| \phi \|_{L^\infty} + \| \phi \|_{L^2} \right). \]

Lemma 4.5 is proved. \( \square \)

4.2. Estimates for \( \mathcal{Q}^* \). Denote the conjugate kernel
\[ \mathcal{A}^* (t, \xi) = \sqrt{\frac{i}{2\pi}} \int_{\mathbb{R}} e^{iS(\xi, \eta)} \Lambda^\eta(\eta) \, d\eta. \] In paper [12] we proved the estimate
\[ \mathcal{A}^* (t, \xi) = \sqrt{\mathcal{A}^{\eta}(\xi)} + O \left( t^{1/p - 1/2} (\xi)^{3-1/2} \right) \]
for all \( t > 0, \xi \in \mathbb{R} \), where \( 1 < p < \infty \). Also we have the estimate \( |\mathcal{A}^* (t, \xi)| \leq C |\xi| \)
for all \( t \geq 1, \xi \in \mathbb{R} \). Next as in paper [12] we obtain the estimate of the operator \( \mathcal{Q}^* \)
in the norm \( L^\infty \).

Lemma 4.6. The estimate
\[ \| \mathcal{Q}^* \phi - \mathcal{A}^* (t, \xi) \phi (\xi) \|_{L^\infty} \leq C |t|^{-1/4} \left\| \mathcal{A}^{\eta}(\eta)^{3/2} t \mathcal{A}_0 \phi \right\|_{L^2} \]
is valid for all \( |t| \geq 1 \).

We now estimate the derivative \( \partial_\xi \mathcal{Q}^* \).

Lemma 4.7. Let \( \alpha \neq 0 \). Then the estimate
\[ \| \partial_\xi \mathcal{D}_1 \mathcal{Q}^* (at) \phi \|_{L^2} \leq C \sum_{j=0,1,2} \left\| \mathcal{A}^{\eta}(\eta)^{2-j} t \mathcal{A}_0 \mathcal{A}^j \phi \right\|_{L^2} \]
is true for all \( |t| \geq 1 \).

Proof. Changing \( at = t' \) we reduce the proof to the case of \( \alpha = 1 \). Since \( \partial_\xi S(\xi, \eta) = \Lambda' (\xi) - \Lambda' (\eta) \) and \( i \xi \mathcal{Q}^* = \mathcal{Q}^* A_1 \), we have
\[ \partial_\xi \mathcal{Q}^* \phi = it \Lambda' (\xi) \mathcal{Q}^* \phi - it \mathcal{Q}^* \Lambda' (\eta) = t \mathcal{Q}^* (A_1 - i \eta) \phi - t \mathcal{Q}^* \left( A_1^3 - (i \eta)^3 \right) \phi. \]
Using the identity \( A_1 = i \eta + \mathcal{A}_0 \), we get \( A_1^3 - (i \eta)^3 = \mathcal{A}_0 \mathcal{A}_1^2 + i \eta \mathcal{A}_0 \mathcal{A}_1 - \eta^2 \mathcal{A}_0. \)
Therefore
\[ \partial_\xi \mathcal{Q}^* \phi = t \mathcal{Q}^* \mathcal{A}_0 \phi - t \mathcal{Q}^* \mathcal{A}_0 \mathcal{A}_1^2 \phi - t \mathcal{Q}^* i \eta \mathcal{A}_0 \mathcal{A}_1 \phi + t \mathcal{Q}^* \eta^2 \mathcal{A}_0 \phi. \]
Applying \( \| \mathcal{Q}^* \phi \|_{L^2} = \| \mathcal{A}^{\eta} \phi \|_{L^2} \), we find the estimate of the lemma. Lemma 4.7 is proved. \( \square \)

We now estimate the derivative \( \partial_t \mathcal{Q}^* \).

Lemma 4.8. The estimate
\[ \| t \partial_t \mathcal{D}_1 \mathcal{Q}^* (at) \phi \|_{L^2} \leq C \left\| \sqrt{\mathcal{A}^{\eta}} \phi \right\|_{L^2} + C \sum_{j=0}^{3} \left\| \sqrt{\mathcal{A}^{\eta}(\eta)^{3-j}} t \mathcal{A}_0 \mathcal{A}^j \phi \right\|_{L^2} \]
is true for all \( |t| \geq 1 \).
Proof. As above we consider the case of $\alpha = 1$. Since

$$S(\xi, \eta) = -\frac{a}{2} (i\eta)^2 - 2i\eta \xi + (i\xi)^2 + \frac{b}{4} (3(i\eta)^4 + (i\xi)^4 - 4(i\eta)^3 (i\xi)),$$

using the identities $i\xi Q^* = Q^*A_1$, $A_1 = i\eta + A_0$, $A_k\eta^j = \eta^j A_k + \frac{y^j}{\xi\eta}$, we get

$$(i\eta)^2 - 2i\eta A_1 + A_1^2 = -i\eta A_0 + A_0 A_1,$$

$$3(i\eta)^4 + A_1^4 = 4A_1(i\eta)^3 = A_0 A_1^3 + i\eta A_0 A_1^2 - \eta^2 A_0 A_1 + 3i\eta^2 A_0 + \frac{12i\eta^2}{tA^\alpha(\eta)}.$$

Hence

$$Q_1^* \phi = \frac{1}{2t} Q^* \phi + \frac{a_1}{2} Q^* (i\eta A_0 - A_0 A_1) \phi + \frac{b_1}{4} Q^* (A_0 A_1^3 + i\eta A_0 A_1^2 - \eta^2 A_0 A_1 + 3i\eta^2 A_0 + \frac{12i\eta^2}{tA^\alpha(\eta)}) \phi.$$

Applying the identity $\|Q^* \phi\|_{L^2} = \|\sqrt{A^\alpha(\eta)}\phi\|_{L^2}$, we find

$$\| (\xi)^{-1} \partial_t Q^* \phi \|_{L^2} \leq C t^{-1} \| \sqrt{A^\alpha(\eta)} \phi \|_{L^2} + C \sum_{j=0}^3 \| \sqrt{A^\alpha(\eta)}^3 - j A_0 A_1^j \phi \|_{L^2}.$$

Lemma 4.8 is proved.

\[\Box\]

5. Estimates for bilinear operators.

5.1. Estimate for the kernel $g$. Consider the estimate for the kernel $g(t, y, z) = t \int_{\mathbb{R}^2} \hat{g}(t, \xi, \eta) e^{i(y\xi + iz\eta)} d\xi d\eta$.

\textbf{Lemma 5.1.} Assume that for all $\xi, \eta \in \mathbb{R}$, $t \geq 1$

$$\left| \partial^j_\xi \partial^l_\eta \hat{g}(t, \xi, \eta) \right| \leq \frac{C (|\xi| + |\eta|)^{N-j-l}}{1 + t \left( \xi^2 \eta^2 + \eta^2 \langle \eta \rangle^2 \right)}$$

if $j, l = 0, 1, \ldots, N$, $N = 2, 3$, and

$$\left| \partial^j_\xi \partial^l_\eta \hat{g}(t, \xi, \eta) \right| \leq \frac{C t^{j+l-N}}{1 + t \left( \xi^2 \eta^2 + \eta^2 \langle \eta \rangle^2 \right)}$$

if $j, l = N + 1, \ldots, N + K$, $K \geq 1$. Then the estimate

$$|g(t, y, z)| \leq C \frac{1}{(\{y\} + \{z\}) (y + z)^{N-\frac{\gamma_1}{2}}} \left( \left\langle \frac{y}{\sqrt{t}} \right\rangle + \left\langle \frac{z}{\sqrt{t}} \right\rangle \right)^K$$

is true for all $y, z \in \mathbb{R}$, $t \geq 1$, where $\gamma_1 > 0$ is small.

\textbf{Proof.} Define the cut of function $\phi_1(\xi, \eta) \in C^\infty(\mathbb{R}^2)$ such that $\phi_1(\xi, \eta) = 1$ for $|\xi| + |\eta| \leq 1$ and $\phi_1(\xi, \eta) = 0$ for $|\xi| + |\eta| \geq 2$. Also $\phi_2(\xi, \eta) = 1 - \phi_1(\xi, \eta)$. Then we represent

$$g(t, y, z) = t \int_{\mathbb{R}^2} \hat{g}(t, \xi, \eta) e^{i y \xi + iz \eta} \phi_1 \left( \frac{\xi}{\delta}, \frac{\eta}{\delta} \right) d\xi d\eta$$

$$+ t \int_{\mathbb{R}^2} \hat{g}(t, \xi, \eta) e^{i y \xi + iz \eta} \phi_2 \left( \frac{\xi}{\delta}, \frac{\eta}{\delta} \right) d\xi d\eta = I_1 + I_2.$$
Let \( |y| \geq |z| \). If \( |y| \leq 1 \), we estimate the first integral as follows for \( \delta > \frac{1}{2} \):

\[
|I_1| \leq C \int_{\mathbb{R}^2} \left| \partial_{\xi} \left( \frac{\xi}{\delta} \right) \right| \xi^2 (\xi^2 + \eta^2)^{1/2} d\xi d\eta \leq C \int_0^1 dr + C \int_1^{2\delta} dr \leq C \delta.
\]

In \( I_2 \) we integrate \( N \) times by parts with respect to \( \xi \):

\[
|I_2| = \frac{C t}{|y|} \int_{\mathbb{R}^2} \left| \partial_{\xi}^N \left( \frac{\xi}{\delta} \right) \right| \xi^2 (\xi^2 + \eta^2)^{1/2} d\xi d\eta \leq \frac{C}{|y|^N} \int_\delta^\infty \frac{r dr}{r^2 + r^4} \leq \frac{C}{|y|^{N - \frac{1}{2} + K}}.
\]

We choose \( \delta = \frac{1}{|y| + \frac{1}{t} \cdot t} \). Then we get \( |g(t,y,z)| \leq C(|y| + \{z\})^{-1} \) for \( |y| + \{z\} \leq 1 \).

In the domain \( 1 \leq |y| \leq \sqrt{t} \) we integrate \( N - 1 \) times by parts with respect to \( \xi \) and one time using the identity \( e^{iy_\xi} = (1 + i \xi) \cdot e^{iy_\xi} \):

\[
|g(t,y,z)| = \frac{C t}{|y|^{N-1}} \int_{\mathbb{R}^2} \left| \partial_{\xi} \left( (1 + i \xi) \cdot e^{iy_\xi} \right) \right| d\xi d\eta \leq \frac{C t}{|y|^{N-1}} \int_{\mathbb{R}^2} (1 + |y|) \left( 1 + t (\xi^2 + \eta^2 + \xi^4 + \eta^4) \right) \frac{d\xi d\eta}{|y|^{N - \frac{1}{2} + K}} \leq \frac{C}{(\langle y \rangle + \langle z \rangle)^{N - \frac{1}{2} + K}}.
\]

Lemma 5.1 is proved.

\[\square\]

5.2. Estimates for operators \( Z \). In the next lemma we estimate the operator

\[
Z(u,w) = \int_{\mathbb{R}^2} g(t,y,z) u(t,x-y)w(t,x-z) dydz.
\]

**Lemma 5.2.** Let the estimate of Lemma 5.1 be fulfilled for the kernel \( g(t,y,z) \). Then the estimates

\[
\|Z(u,w)\|_{L^p} \leq C t^{\gamma_1 - \frac{3}{2}} \|u\|_{L^p} \|w\|_{L^\infty}
\]

and

\[
\|Z(u,w)\|_{L^1} \leq C t^{\gamma_2 - \frac{3}{2}} \|u\|_{L^2} \|w\|_{L^2}
\]

are true for \( 1 \leq p \leq \infty \), where \( \gamma_1 > 0 \) is small.

**Proof.** We find by the Young inequality

\[
\|Z(u,w)\|_{L^p} \leq \left\| \int_{\mathbb{R}^2} \left| u(t,x-y) \right| \left| w(t,x-z) \right| dydz \right\|_{L^p}
\]

Then

\[
\|Z(u,w)\|_{L^p} \leq C t^{\gamma_1 - \frac{3}{2}} \|u\|_{L^p} \|w\|_{L^\infty}
\]

and

\[
\|Z(u,w)\|_{L^1} \leq C t^{\gamma_2 - \frac{3}{2}} \|u\|_{L^2} \|w\|_{L^2}
\]

are true for \( 1 \leq p \leq \infty \), where \( \gamma_1 > 0 \) is small.

**Proof.** We find by the Young inequality

\[
\|Z(u,w)\|_{L^p} \leq C t^{\gamma_1 - \frac{3}{2}} \|u\|_{L^p} \|w\|_{L^\infty}
\]

and

\[
\|Z(u,w)\|_{L^1} \leq C t^{\gamma_2 - \frac{3}{2}} \|u\|_{L^2} \|w\|_{L^2}
\]

are true for \( 1 \leq p \leq \infty \), where \( \gamma_1 > 0 \) is small.
Let the estimate of Lemma 5.1 be fulfilled for

Estimates for operators

6.

Lemma 5.2 is proved. □

Also we have

and

P

(•)

and

Lemma 5.3 is proved. □

Proof. Applying the operator \( J \), we get \( JZ(u, w) = xZ(u, u) + ati∂zZ(u, u) - bti∂3zZ(u, u) \). We have by Lemma 5.2

\[
\|xZ(u, w)\|_{L^2} \leq C \|Z(Ju, w)\|_{L^2} + \left\| \int_{\mathbb{R}^2} yg(t, y, z) u(t, x-y)w(t, x-z)dydz \right\|_{L^2} \\
+ Ct \left\| \int_{\mathbb{R}^2} \partial_y g(t, y, z) \left( a + b\partial_y^2 \right) u(t, x-y)w(t, x-z)dydz \right\|_{L^2} \\
\leq C|t| \left\| u\right\|_{L^\infty} \|Ju\|_{L^2} + C\|u\|_{L^\infty} \|w\|_{H^2},
\]

\[
\|t\partial_z Z(u, w)\|_{L^2} \leq Ct \left\| \int_{\mathbb{R}^2} \partial_y g(t, y, z) u(t, x-y)w(t, x-z)dydz \right\|_{L^2} \\
\leq C\|u\|_{L^\infty} \|w\|_{L^2},
\]

and

\[
\|t\partial^3_z Z(u, w)\|_{L^2} \leq Ct \left\| \int_{\mathbb{R}^2} \partial_y \partial_z g(t, y, z) \partial_t^2 \left( u(t, x-y)w(t, x-z) \right) dydz \right\|_{L^2} \\
\leq C\|u\|_{L^\infty} \|w\|_{H^2} + C\|w\|_{L^\infty} \|u\|_{H^2}.
\]

Lemma 5.3 is proved. □

6. Estimates for operators \( \mathcal{G} \). In this section we consider the estimates of the operator

\[
\mathcal{G}(\phi, f) = t^2 \int_{\mathbb{R}^2} h(t, \xi, \eta, \zeta) \phi(\eta) f(\zeta) d\eta d\zeta,
\]

where the kernel

\[
h(t, \xi, \eta, \zeta) = e^{itP(\xi, \eta, \zeta)} g(t, t (\Lambda(\xi) - \Lambda(\eta)), t (\Lambda'(\xi) - \Lambda'(\eta))) \Lambda''(\eta) \Lambda''(\zeta)
\]

and \( P(\xi, \eta, \zeta) = (\alpha_1 + \alpha_2) \Theta(\xi) - \alpha_1 \Theta(\eta) - \alpha_2 \Theta(\zeta). \) Changing \( \xi = \mu(x), \eta = \mu(x-y), \zeta = \mu(x-z), \) we get

\[
\mathcal{G}(\phi, f) = t^2 \mathcal{B}^{-1} \int_{\mathbb{R}^2} \tilde{h}(t, x, y, z) \phi(\mu(x-y)) f(\mu(x-z)) dydz,
\]
where the kernel $\tilde{h}(t, x, y, z) = e^{itP(\mu(x), \mu(x-y), \mu(x-z))} g(t, y, z)$. We have the identity

$$A_{-\alpha_1} A_{-\alpha_2} e^{itP(\mu(x), \mu(x-y), \mu(x-z))} = e^{itP(\mu(x), \mu(x-y), \mu(x-z))] (A_{-\alpha_1} + A_{-\alpha_2}),$$

where the operator $A_\alpha = \frac{1}{i\lambda(y)} M^a \partial_y M^a$, $\alpha \in \mathbb{R}$. Hence we get

$$A_{-\alpha_1} A_{-\alpha_2} G_{j, \alpha_1, \alpha_2} (\phi, f) = G_{j, \alpha_1, \alpha_2} (A_{-\alpha_1} \phi, f) + G_{j, \alpha_1, \alpha_2} (\phi, A_{-\alpha_2} f), \quad (6.1)$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$.

### 6.1. Estimates in $L^2$.

**Lemma 6.1.** Let the estimate of Lemma 5.1 be fulfilled for $g(t, y, z)$. Then the estimate

$$\left\| \sqrt{\Lambda^\theta} (\xi) \beta G(\phi, f) \right\|_{L^2} \leq C t^{2-\gamma} \left\| (\xi) \beta_1 \phi \right\|_{L^\infty} \left\| \sqrt{\Lambda^\theta} (\xi) \beta_2 f \right\|_{L^2}$$

is true, where $\beta \geq 0, \beta_1, \beta_2 \in \mathbb{R}$ are such that $\beta_1 + \beta_2 = \beta$ and $K \geq 2^n$.

**Proof.** Using estimate $\langle \mu(x) \rangle \leq \langle \mu(y) \rangle \langle \mu(x - y) \rangle \leq C \langle y \rangle^{\frac{3}{2}} \langle \mu(x - y) \rangle$ and estimate of Lemma 5.1 we find

$$\langle \xi \rangle^\beta |G(\phi, f)| = t^2 B^{-1} \int_{\mathbb{R}^2} \langle \mu(x) \rangle^\beta \left| \left\| \left( \langle x \rangle^{\beta_1} \langle y \rangle^{\beta_2} \langle z \rangle^{\beta_2} \right) \left( \langle y \rangle^{\beta_1} + \langle z \rangle^{\beta_2} \right) \right| \langle \mu(x - y) \rangle \langle \mu(x - z) \rangle \right\|_{L^\infty} \left\| \sqrt{\Lambda^\theta} (\xi) \beta_2 f \right\|_{L^2}$$

We use the equality $\left\| \sqrt{\Lambda^\theta} B^{-1} \phi \right\|_{L^2} = \left\| \phi \right\|_{L^2}$, then we obtain

$$\left\| \sqrt{\Lambda^\theta} (\xi) \beta G(\phi, f) \right\|_{L^2} \leq C \left\| \langle \mu \rangle^{\beta_1} B \phi \right\|_{L^\infty} \left\| \langle \mu \rangle^{\beta_2} B f \right\|_{L^2}$$

Next we estimate the derivative $\partial_\xi G$.

**Lemma 6.2.** Let the estimate of Lemma 5.1 be fulfilled for $g(t, y, z)$. Then the estimate

$$\left\| \sqrt{\Lambda^\theta} (\eta) \beta \partial_\xi G(\phi, f) \right\|_{L^2} \leq C t^{3-\gamma} \left\| (\eta) \beta_1 f \right\|_{L^\infty} \left\| \sqrt{\Lambda^\theta} (\theta) \beta_2 \phi \right\|_{L^2}$$

is true for all $t \geq 1$, where $\beta \geq 0, \beta_1, \beta_2 \in \mathbb{R}$ are such that $\beta_1 + \beta_2 = \beta$ and $K \geq 2^n$.\[\square\]
Proof. Since \( (\eta)^{\beta} t A_0 = B^{-1} (\mu (x))^{\beta} \partial_x \) we get
\[
(\eta)^{\beta} t A_0 G (\phi, f) = t^2 B^{-1} (\mu (x))^{\beta} \int_{\mathbb{R}^2} \tilde{h} (t, x, y, z) \phi (\mu (x - y)) \partial_x f (\mu (x - z)) \, dydz
\]
\[
+ t^2 B^{-1} (\mu (x))^{\beta} \int_{\mathbb{R}^2} \tilde{h} (t, x, y, z) \partial_x \phi (\mu (x - y)) \, f (\mu (x - z)) \, dydz
\]
\[
+ t^2 B^{-1} (\mu (x))^{\beta} \int_{\mathbb{R}^2} (\partial_x \tilde{h} (t, x, y, z)) \phi (\mu (x - y)) \, f (\mu (x - z)) \, dydz
\]
\[
\equiv I_1 + I_2 + I_3.
\]
In view of the equality \( \| \sqrt{N} B^{-1} \phi \|_{L^2} = \| \phi \|_{L^2} \) and estimate of Lemma 5.1 we get
\[
\left\| \sqrt{N} I_1 \right\|_{L^2} \leq C t^2 \left\| \int_{\mathbb{R}^2} \langle \mu (x) \rangle^{\beta} |\phi (\mu (x - y))| |\partial_x f (\mu (x - z))| \, dydz \right\|_{L^2}
\]
\[
\leq C \left\| \langle \mu \rangle^{\beta_1} B \phi \right\|_{L^\infty} \left\| \langle \mu \rangle^{\beta_2} \partial_x B f \right\|_{L^2}
\]
\[
\times \int_{\mathbb{R}^2} \langle \mu (y) \rangle^{\beta_1} \langle \mu (z) \rangle^{\beta_2} t^2 \, dydz
\]
\[
\leq C t^{\gamma \frac{3N}{2}} \left\| \langle \eta \rangle^{\beta_1} f \right\|_{L^\infty} \left\| \sqrt{N} \langle \eta \rangle^{\beta_2} t A_0 f \right\|_{L^2}.
\]
Finally using equalities \( \Theta (\eta) = \eta \Lambda' (\eta) - \Lambda (\eta) \), \( \Theta' (\eta) = \eta \Lambda'' (\eta) \), \( \mu^{\prime} (x) = \frac{1}{\Lambda'' (\mu (x))} \), \( \partial_x \Theta (\mu (x)) = \mu^{\prime} (x) \Theta' (\mu (x)) = \mu (x) \), we find
\[
\partial_x P (\mu (x), \mu (x - y), \mu (x - z)) = \alpha_1 (\mu (x) - \mu (x - y)) + \alpha_2 (\mu (x) - \mu (x - z)) = O (|y| + |z|).
\]
Hence we obtain
\[
\left\| \sqrt{N} I_2 \right\|_{L^2} \leq C t^{\gamma \frac{3N}{2}} \left\| \langle \eta \rangle^{\beta_1} f \right\|_{L^\infty} \left\| \sqrt{N} \langle \eta \rangle^{\beta_2} t A_0 f \right\|_{L^2}.
\]
Lemma 6.2 is proved. \( \square \)

Next we estimate the derivative \( \partial_t G (\phi, f) \).
Lemma 6.3. Let the estimate of Lemma 5.1 be fulfilled for \( g(t,y,z), t\partial_t g(t,y,z) \) and \((y\partial_y + z\partial_z) g(t,y,z)\). Then integrating by parts we get

\[
\left\| t \phi f \right\|_{L^2} 
\leq C \int_{t}^{t+2N} \left\| \nabla^\alpha \phi \right\|_{L^\infty} \left( \left\| \phi \right\|_{L^\infty} + \left\| \nabla^\alpha \phi \right\|_{L^\infty} \right) \left( \left\| \phi \right\|_{L^\infty} + \left\| \nabla^\alpha \phi \right\|_{L^\infty} \right)
\]

is true for all \( t \geq 1 \).

Proof. We have

\[
t (t g(t,y,z)) = 2t^2 B^{-1} \int_{\mathbb{R}^2} \tilde{h}(t,x,y,z) \phi(\mu(x-y)) f(\mu(x-z)) \, dydz
\]

\[
+ \int_{\mathbb{R}^2} \tilde{h}(t,x,y,z) \phi(\mu(x-y)) f(\mu(x-z)) \, dydz
\]

\[
\equiv I_1 + I_2.
\]

The integral \( I_1 \) is estimated as above. Consider the second integral. Using the identity \( i t \alpha_1 \mu(x-y) e^{-i t \alpha_1 \Theta(\mu(x-y))} = \partial_y e^{-i t \alpha_1 \Theta(\mu(x-y))} \), we find \( t \partial_t \tilde{h}(t,x,y,z) = \tilde{h}_1(t,x,y,z) + \partial_y \tilde{h}_2(t,x,y,z) + \partial_z \tilde{h}_3(t,x,y,z) \), where

\[
\tilde{h}_1(t,x,y,z) = i t \alpha_1 (\mu(x) - \mu(x-y))^2 \left( \frac{a}{2} + \frac{3b}{4} (\mu(x))^2 + (\mu(x-y))^2 \right) \tilde{h}(t,x,y,z)
\]

\[
+ i t \alpha_2 (\mu(x) - \mu(x-z))^2 \left( \frac{a}{2} + \frac{3b}{4} (\mu(x))^2 + (\mu(x-z))^2 \right) \tilde{h}(t,x,y,z)
\]

\[
+ e^{i t \alpha_1 \Theta(\mu(x-y))} t \partial_t g(t,yt,zt)
\]

\[
- e^{i t \alpha_1 \Theta(\mu(x-z))} \partial_y (2 (\mu(x) - \mu(x-y))
\]

\[
\times \left( \frac{a}{2} + \frac{3b}{4} (\mu(x))^2 + (\mu(x-y))^2 \right) \tilde{h}(t,x,y,z)
\]

\[
- e^{i t \alpha_1 \Theta(\mu(x-z))} \partial_z (2 (\mu(x) - \mu(x-z))
\]

\[
\times \left( \frac{a}{2} + \frac{3b}{4} (\mu(x))^2 + (\mu(x-z))^2 \right) \tilde{h}(t,x,y,z)
\]

\[
\tilde{h}_2(t,x,y,z) = 2 (\mu(x) - \mu(x-y)) \left( \frac{a}{2} + \frac{3b}{4} (\mu(x))^2 + (\mu(x-y))^2 \right) \tilde{h}(t,x,y,z)
\]

\[
\tilde{h}_3(t,x,y,z) = 2 (\mu(x) - \mu(x-z)) \left( \frac{a}{2} + \frac{3b}{4} (\mu(x))^2 + (\mu(x-z))^2 \right) \tilde{h}(t,x,y,z)
\]

Then integrating by parts we get \( I_2 = I_{21} + I_{22} + I_{23} \), where

\[
I_{21} = t^2 B^{-1} \int_{\mathbb{R}^2} \tilde{h}_1(t,x,y,z) \phi(\mu(x-y)) f(\mu(x-z)) \, dydz,
\]

\[
I_{22} = -t^2 B^{-1} \int_{\mathbb{R}^2} \tilde{h}_2(t,x,y,z) \partial_y \phi(\mu(x-y)) f(\mu(x-z)) \, dydz,
\]

\[
I_{23} = -t^2 B^{-1} \int_{\mathbb{R}^2} \tilde{h}_3(t,x,y,z) \partial_z \phi(\mu(x-y)) f(\mu(x-z)) \, dydz,
\]
\[ I_{23} = -t^2 B^{-1} \int_{\mathbb{R}^2} \tilde{h}_3(t, x, y, z) \phi(\mu(x - y)) \partial_z f(\mu(x - z)) \, dydz. \]

Since \( \mu(x) - \mu(x - y) = O(y) \) we find the estimates

\[
\begin{aligned}
&\left| \tilde{h}_1(t, x, y, z) \right| \\
\leq & \frac{C \left( 1 + t (\langle |y| + |z| \rangle) \right) \left( \langle |y| \rangle \langle \mu(x - y) \rangle^2 + \langle z \rangle^2 \langle \mu(x - z) \rangle^2 \right)}{(\{yt\} + \{zt\}) (\langle yt \rangle + \langle zt \rangle)^{N-\gamma_1} (\langle y\sqrt{t} \rangle + \langle z\sqrt{t} \rangle)^K}, \\
&\left| \tilde{h}_2(t, x, y, z) \right| \\
\leq & \frac{C |y| \langle \mu(x - y) \rangle^2}{(\{yt\} + \{zt\}) (\langle yt \rangle + \langle zt \rangle)^{N-\gamma_1} (\langle y\sqrt{t} \rangle + \langle z\sqrt{t} \rangle)^K}, \\
&\left| \tilde{h}_3(t, x, y, z) \right| \\
\leq & \frac{C |z| \langle \mu(x - z) \rangle^2}{(\{yt\} + \{zt\}) (\langle yt \rangle + \langle zt \rangle)^{N-\gamma_1} (\langle y\sqrt{t} \rangle + \langle z\sqrt{t} \rangle)^K}.
\end{aligned}
\]

Therefore we obtain

\[
\begin{aligned}
\left\| \sqrt{N^7} I_{21} \right\|_{L^2} \leq C \int_{\mathbb{R}^2} \left\| B f \right\|_{L^\infty} \left\| \langle \mu(x) \rangle^2 B \phi \right\|_{L^2} t^2 \left( 1 + t (\langle |y| + |z| \rangle) \right) \langle y \rangle^2 \, dydz \\
+ C \int_{\mathbb{R}^2} \left\| B \phi \right\|_{L^\infty} \left\| \langle \mu(x) \rangle^2 B f \right\|_{L^2} t^2 \left( 1 + t (\langle |y| + |z| \rangle) \right) \langle z \rangle^2 \, dydz \\
\leq Ct^{\gamma_1 - \frac{3N}{2}} \left\| f \right\|_{L^\infty} \left\| \sqrt{N^7} \langle \xi \rangle^2 \phi \right\|_{L^2} + Ct^{\gamma_1 - \frac{3N}{2}} \left\| \phi \right\|_{L^\infty} \left\| \sqrt{N^7} \langle \xi \rangle^2 f \right\|_{L^2}.
\end{aligned}
\]

And using \( \langle \eta \rangle^2 t A_0 = B^{-1} \langle \mu(x) \rangle^2 \partial_z \) we get

\[
\begin{aligned}
\left\| \sqrt{N^7} I_{22} \right\|_{L^2} \leq C \int_{\mathbb{R}^2} \left\| B f \right\|_{L^\infty} \left\| \langle \mu(x) \rangle^2 B \partial_z \phi \right\|_{L^2} t^2 \langle y \rangle \langle y \rangle^2 \, dydz \\
\leq Ct^{\gamma_1 - \frac{3N}{2}} \left\| \langle y \rangle \phi \right\|_{L^\infty} \left\| \sqrt{N^7} \langle \xi \rangle^2 \left( t A_0 \phi \right) \right\|_{L^2}
\end{aligned}
\]

and

\[
\begin{aligned}
\left\| \sqrt{N^7} I_{23} \right\|_{L^2} \leq C \int_{\mathbb{R}^2} \left\| B \phi \right\|_{L^\infty} \left\| \langle \mu(x) \rangle^2 B \partial_z f \right\|_{L^2} t^2 \langle z \rangle \langle z \rangle^2 \, dydz \\
\leq Ct^{\gamma_1 - \frac{3N}{2}} \left\| \langle \phi \rangle_{L^\infty} \right\| \left\| \sqrt{N^7} \langle \xi \rangle^2 \left( t A_0 f \right) \right\|_{L^2}.
\end{aligned}
\]

Lemma 6.3 is proved.

\[ \Box \]

6.2. Estimates in \( L^\infty \). First we estimate \( G(\phi, f) \) in \( L^\infty \).

**Lemma 6.4.** Let the estimate of Lemma 5.1 be fulfilled for \( g(t, y, z) \). Then the estimate \( \| (\xi)^{\beta} G(\phi, f) \|_{L^\infty} \leq Ct^{\gamma_1 - \frac{3N}{2}} \| \xi \|_{L^\infty} \| (\xi)^{\beta} \phi \|_{L^\infty} \| (\xi)^{\beta} f \|_{L^\infty} \) is true, where \( \beta \geq 0, \beta_1, \beta_2 \in \mathbb{R} \) are such that \( \beta_1 + \beta_2 = \beta \) and \( K \geq 3 |\beta_1| + |\beta_2| \).
Proof. Using estimate of Lemma 5.1 we find
\[
\|\langle \xi \rangle^{\beta} G(\phi, f)\|_{L^\infty} \leq C t^2 \|\mathcal{B}^{-1} (\mu(x))^{\beta} \int_{\mathbb{R}^2} |g(t, yt, zt) - \phi(\mu(x-y))| |f(\mu(x-z))| dydz \|_{L^\infty}
\]
\[
\leq C \|\langle \xi \rangle^{\beta_1} \phi\|_{L^\infty} \|\langle \xi \rangle^{\beta_2} f\|_{L^\infty}
\times \left| \left \langle \mu(y) \right \rangle^{\beta_1} (\mu(z))^{\beta_2} t^2 dydz \right|^{1/2}
\leq C t^{\gamma_1 - \gamma_2} \left| \left \langle \xi \right \rangle^{\beta_1} \phi\right|_{L^\infty} \left| \left \langle \xi \right \rangle^{\beta_2} f\right|_{L^\infty}.
\]
Lemma 6.4 is proved.

Next we obtain the asymptotics of $G(\phi, f)$.

**Lemma 6.5.** Let the estimate of Lemma 5.1 be fulfilled for $g(t, y, z)$ with $N = 3$. Then the asymptotic representation
\[
G(\phi, f) = \sqrt{2\pi} \Re \left \langle \phi(\xi) f(\xi) \right \rangle \hat{g}(t, -\alpha_1 \xi, -\alpha_2 \xi) + O \left( t^{\gamma_1 - \gamma_2} \|\phi\|_{L^\infty} \|f\|_{L^\infty} \right)
\]
\[
+ O \left( t^{-1/2} (\|\partial_\xi \phi\|_{L^2} \|f\|_{L^\infty} + \|\partial_\xi f\|_{L^2} \|\phi\|_{L^\infty}) \right)
\]
is true for all $t \geq 1$, $\xi \in \mathbb{R}$.

**Proof.** We represent \(G(\phi, f)\) as
\[
G(\phi, f) = t^2 \int_{\mathbb{R}^2} h(t, \xi, \eta, \zeta) (\phi(\eta) f(\zeta) - \phi(\xi) f(\xi)) \, d\eta d\zeta
\]
\[
+ \phi(\xi) f(\xi) t^2 \int_{\mathbb{R}^2} h(t, \xi, \eta, \zeta) \, d\eta d\zeta = I_1 + I_2.
\]
The first summand $I_1$ is a remainder
\[
|I_1| \leq C \|\partial_\xi \phi\|_{L^2} \|f\|_{L^\infty} t^2 \int_{\mathbb{R}^2} |h(t, \xi, \eta, \zeta)| |\xi - \eta|^{1/2} \, d\eta d\zeta
\]
\[
+ C \|\partial_\xi f\|_{L^2} \|\phi\|_{L^\infty} t^2 \int_{\mathbb{R}^2} |h(t, \xi, \eta, \zeta)| |\xi - \zeta|^{1/2} \, d\eta d\zeta.
\]
By the estimate of Lemma 5.1, changing $\xi = \mu(x)$, $\eta = \mu(x-y)$, $\zeta = \mu(x-z)$, we get
\[
t^2 \int_{\mathbb{R}^2} |h(t, \xi, \eta)| \left| |\xi - \eta|^{1/2} + |\xi - \zeta|^{1/2} \right| \, d\eta d\zeta
\]
\[
\leq C t^2 \int_{\mathbb{R}^2} |g(t, t(y), t(z))| \left( |\mu(x) - \mu(x-y)|^{1/2} + |\mu(x) - \mu(x-z)|^{1/2} \right) \, dy dz
\]
\[
\leq C \int_{\mathbb{R}^2} \left( |t(y)| + |t(z)| \right) \left( |t(y)| + |t(z)| \right)^{N-\gamma_1} \left( |y\sqrt{T} + z\sqrt{T}| \right)^R \, dy dz \leq C t^{\gamma_1 - 1/4}.
\]
Hence the first term is estimated as
\[
|I_1| \leq C t^{\gamma_1 - 1/4} \left( \|\partial_\xi \phi\|_{L^2} \|f\|_{L^\infty} + \|\partial_\xi f\|_{L^2} \|\phi\|_{L^\infty} \right)
\]
Since $\partial_x \Theta(\mu(x)) = \Theta'(\mu(x)) \mu'(x) = \mu(x)$, by the Taylor Theorem we have
\[
P(\mu(x), \mu(x-y), \mu(x-z)) = (\alpha_1 y + \alpha_2 z) \mu(x) + O(\gamma^2 + z^2).
\]
Then changing as above \( \xi = \mu(x) \), \( \eta = \mu(x - y) \), \( \zeta = \mu(x - z) \) we find
\[
\begin{align*}
t^2 \int_{\mathbb{R}^2} h(t, \xi, \eta, \zeta) \, d\eta d\zeta = & \quad t^2 \mathcal{B}^{-1} \int_{\mathbb{R}^2} e^{i \mu(x) \omega_{1y + \omega_{2z}}} \left( e^{it \mathcal{O}(\omega^2 + z^2)} - 1 \right) g(t, yt, zt) \, dydz \\
& + t^2 \mathcal{B}^{-1} \int_{\mathbb{R}^2} e^{i \mu(x) \omega_{1y + \omega_{2z}}} g(t, yt, zt) \, dydz = I_3 + I_4.
\end{align*}
\]

Here \( I_3 \) is a remainder
\[
|I_3| \leq C t^3 \int_{\mathbb{R}^2} \frac{(y^2 + z^2) \, dydz}{\langle ty \rangle \langle tz \rangle (\langle ty \rangle + \langle tz \rangle)^{3\zeta_1} (\langle y \rangle + \langle z \rangle)^{\zeta_1}} \leq Ct^{\zeta_1 - \frac{1}{2}}.
\]
And the last summand \( I_4 \) is the Fourier transform
\[
I_4 = \int_{\mathbb{R}^2} e^{i \xi (\omega_{1y + \omega_{2z})}} g(t, y, z) \, dydz = \int_{\mathbb{R}^2} e^{i \xi (\omega_{1y + \omega_{2z})}} dydz \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}^2} \hat{g}(t, \zeta, \eta) e^{iy \zeta + iz \eta} d\zeta d\eta = (2\pi)^{\frac{1}{2}} \hat{g}(t, -\alpha_1 \xi, -\alpha_2 \xi).
\]

Lemma 6.5 is proved. \( \square \)

Application of Lemma 6.5 for the case of \( \mathcal{G}_{7.1,1}(\phi, f) \) and \( \mathcal{G}_{3,-2,1}(\phi, f) \) yields.

**Corollary 1.** The following asymptotics
\[
\mathcal{G}_{7.1,1}(\phi, f) = \frac{2i \xi^3 \phi(\xi) f(\xi)}{\langle \xi \rangle^2 (\Lambda(2\xi) + 2\Lambda(\xi))} + O \left( t^{\zeta_1 - \frac{1}{2}} \| \phi \|_{L^\infty} \| f \|_{L^\infty} \right)
\]
\[
+ O \left( t^{-\frac{1}{2}} (\| \partial_t \phi \|_{L^2} \| f \|_{L^\infty} + \| \partial_\xi f \|_{L^2} \| \phi \|_{L^\infty}) \right)
\]

and
\[
\mathcal{G}_{8,-2,1}(\phi, f) = \frac{2 \xi^2 \phi(\xi) f(\xi)}{\Lambda(2\xi) + 2\Lambda(\xi)} + O \left( t^{\zeta_1 - \frac{1}{2}} \| \phi \|_{L^\infty} \| f \|_{L^\infty} \right)
\]
\[
+ O \left( t^{-\frac{1}{2}} (\| \partial_t \phi \|_{L^2} \| f \|_{L^\infty} + \| \partial_\xi f \|_{L^2} \| \phi \|_{L^\infty}) \right)
\]

are true for all \( t \geq 1, \xi \in \mathbb{R} \).

**Proof.** Remind that in the case of \( \mathcal{G}_{7.1,1}(\phi, f) \) we have
\[
\hat{g}_7(t, \xi, \eta) = \frac{t \xi \eta (\xi + \eta)}{\sqrt{2\pi} \langle \xi \rangle \langle \eta \rangle (1 + it (\Lambda(\xi + \eta) + (\lambda(\xi) + \Lambda(\eta)))},
\]
\( \alpha_1 = \alpha_2 = 1 \). Then using Lemma 6.5 we find
\[
\begin{align*}
\sqrt{2\pi} \hat{g}_7(t, -\xi, -\eta) &= - \frac{2t \xi^3}{\langle \xi \rangle^2 (1 + it (\Lambda(2\xi) + 2\Lambda(\xi)))} \\
&= \frac{2t \xi^3}{\langle \xi \rangle^2 (\Lambda(2\xi) + 2\Lambda(\xi))} - \frac{2i \xi^3}{\langle \xi \rangle^2 (\Lambda(2\xi) + 2\Lambda(\xi))} (\Lambda(2\xi) + 2\Lambda(\xi)) \\
&= \frac{2t \xi^3}{\langle \xi \rangle^2 (\Lambda(2\xi) + 2\Lambda(\xi))} + O \left( t^{-\frac{1}{2}} \right).
\end{align*}
\]
In the case of $G_{3, -2, 1} (\phi, f)$ we have
\[
\hat{g}_s (t, \xi, \eta) = \frac{it \xi \eta \{ \xi + \eta \} \{ \xi + \eta \}}{\sqrt{2 \pi} \langle \eta \rangle (1 + it (\Lambda (\xi + \eta) + \Lambda (\xi + \eta)))},
\]
\[\alpha_1 = -2, \alpha_2 = 1.\] Then Lemma 6.5 yields
\[
\sqrt{2 \pi} \hat{g}_s (t, 2 \xi, -\xi) = \frac{2it \xi^2 \{ \xi \}}{1 + it (\Lambda (2 \xi) + 2 \Lambda (\xi))} = \frac{2 \xi^2 \{ \xi \}}{\Lambda (2 \xi) + 2 \Lambda (\xi)} + O \left( t^{-\frac{1}{2}} \right).
\]
Corollary 1 is proved.

7. Estimates for the nonlinearity. Define the norm
\[
\| \psi \|_Y = \| \langle \xi \rangle \hat{\psi} \|_{L^\infty} + t^{-\tau_2} \| \langle \xi \rangle^2 \hat{\psi} \|_{L^2} + t^{-\gamma} \| \langle \xi \rangle \partial_\xi \hat{\psi} \|_{L^2},
\]
where $\gamma > 3\gamma_2 > 3\gamma_1 > 0$ are small. In the next lemma we establish relation of different norms with $\| \hat{\psi} \|_Y$.

Lemma 7.1. Assume that the norm $\| \hat{\psi} \|_Y$ is small. Then the following estimates
\[
\| u \|_{H_{\infty}} + \| w \|_{H_{\infty}} \leq Ct^{-\frac{1}{2}} \| \hat{\psi} \|_Y
\]
and
\[
\| \langle \xi \rangle \hat{\varphi} \|_{L^\infty} + t^{-\tau_2} \| \langle \xi \rangle^2 \hat{\varphi} \|_{L^2} + t^{-\frac{1}{2}} \gamma \| \langle \xi \rangle \partial_\xi \hat{\varphi} \|_{L^2} \leq C \| \hat{\psi} \|_Y
\]
are true.

Proof. By the factorization formula $w (t) = U (t) F^{-1} \hat{\psi} = D_1 B M Q \hat{\psi}$ and Lemma 4.1 with $\delta = 1$ we get the estimate
\[
\| w \|_{H_{\infty}} \leq Ct^{-\frac{1}{2}} \| \hat{\psi} \|_Y + Ct^{-\frac{1}{2}} \| \langle \xi \rangle \partial_\xi \hat{\psi} \|_{L^2} \leq Ct^{-\frac{1}{2}} \| \hat{\psi} \|_Y.
\]
Then by the relation $u = w - Z_3 (u, u) - Z_4 (w, w)$ and using Lemma 5.2 we find
\[
\| u \|_{H_{\infty}} \leq \| w \|_{H_{\infty}} + \| Z_3 (u, u) \|_{H_{\infty}} + \| Z_4 (w, w) \|_{H_{\infty}} \leq \| w \|_{H_{\infty}} + Ct^{\gamma_1} \| u \|_{H_{\infty}^2} + Ct^{\gamma_1} \| w \|_{H_{\infty}^2}.
\]
Since the norm $\| \hat{\psi} \|_Y$ is small we obtain $\| u \|_{H_{\infty}} \leq Ct^{-\frac{1}{2}} \| \hat{\psi} \|_Y$. Next we have
\[
\| K_1 (\phi, f) \|_{L^\infty} \leq Ct \int e^{i\omega (t, \eta - \xi)} \hat{\phi} (t, \eta - \xi) \hat{f} (t, -\eta) \| \langle \xi \rangle \eta \|_Y H (t, \xi, \eta) \| \| \leq C \| \hat{\phi} \|_{L^\infty} \| \hat{f} \|_{L^\infty} \int e^{i\omega (t, \eta)} \| \langle \xi \rangle \eta \|_Y H (t, \xi, \eta) \| \| \leq C \| \hat{\phi} \|_{L^\infty} \| \hat{f} \|_{L^\infty}.
\]
We use the relation $\hat{\psi} = \hat{\varphi} + K_3 (\overline{\varphi}, \overline{\varphi}, \overline{\psi}) + K_4 (\overline{\psi}, \overline{\varphi}, \overline{\varphi})$, then
\[
\| \langle \xi \rangle \hat{\varphi} \|_{L^\infty} \leq \| \langle \xi \rangle \hat{\varphi} \|_{L^\infty} + \| \langle \xi \rangle \{ \xi \} K_1 (\overline{\varphi}, \overline{\varphi}) \|_{L^\infty} + \| \langle \xi \rangle K_1 (\overline{\psi}, \overline{\varphi}) \|_{L^\infty} \leq \| \langle \xi \rangle \hat{\varphi} \|_{L^\infty} + \| \langle \xi \rangle \hat{\varphi} \|_{L^\infty}^2 + \| \langle \xi \rangle \hat{\varphi} \|_{L^\infty}^2.
\]
Since the norm $\| \hat{\varphi} \|_Y$ is small we obtain $\| \langle \xi \rangle \hat{\varphi} \|_{L^\infty} \leq C \| \hat{\varphi} \|_{L^\infty}$. To estimate the norm $\| \langle \xi \rangle^2 \hat{\varphi} \|_{L^2} = \| u \|_{H^2}$ we use the relation $u = w - Z_3 (u, u) - Z_4 (w, w)$. Application of Lemma 5.2 yields
\[
\| Z_3 (u, u) \|_{H^2} \leq Ct^{\gamma_1} \| u \|_{H^2} \| u \|_{L^\infty}.
and \( \| \mathcal{Z}_4 (w, w) \|_{H^2} \leq C t^{\gamma_1} \| w \|_{H^2} \| w \|_{L^\infty} \). Hence

\[
\| u \|_{H^2} \leq \| u \|_{H^2} + \| \mathcal{Z}_3 (u, u) \|_{H^2} + \| \mathcal{Z}_4 (w, w) \|_{H^2}
\]

\[
\leq \| u \|_{H^2} + C t^{\gamma_1} \| u \|_{H^2} \| u \|_{L^\infty} + C t^{\gamma_1} \| w \|_{H^2} \| u \|_{L^\infty} \leq C \| \hat{\psi} \|_Y.
\]

Finally we estimate the norm \( \| \langle \xi \rangle \partial_\xi \hat{\varphi} \|_{L^2} = \| J u \|_{H^1} \). Application of Lemma 5.3 yields

\[
\| J u \|_{H^1} \leq \| J w \|_{H^1} + \| J \mathcal{Z}_3 (u, u) \|_{H^1} + \| J \mathcal{Z}_4 (w, w) \|_{H^1}
\]

\[
\leq C t^{\gamma} \| \hat{\varphi} \|_Y + C t^{\gamma_1} \| u \|_{H^1} \| J u \|_{H^1} + C t \| u \|_{H^1} \| \hat{\varphi} \|_{L^2}
\]

\[
+ C t^{\gamma_1} \| w \|_{H^1} \| J w \|_{H^1} + C t \| u \|_{H^1} \| J u \|_{H^1} \| w \|_{H^2} \leq C t^{\gamma + \gamma_1} \| \hat{\varphi} \|_Y.
\]

Lemma 7.1 is proved. \( \square \)

7.1. Estimates in \( L^\infty \). In the next lemma we calculate the asymptotic representation for the nonlinear term \( \Psi_1 \) in equation (3.4). Denote \( \varepsilon = \| \hat{\psi} \|_Y, \lambda (\xi) = 4 \xi (\xi (\lambda (2 \xi) + 2 \lambda (\xi))) \).

**Lemma 7.2.** The asymptotic representation \( \Psi_1 = \lambda (\xi) |\hat{\psi}|^2 \hat{\varphi} + O (t^{3 - \frac{1}{4}} \varepsilon^3) \) is true for all \( t \geq 1, \xi \in \mathbb{R} \).

**Proof.** By Lemma 4.6

\[
- 2 \mathcal{Q}^* \eta \mathcal{Z}_7,1,1 (\eta, \eta)
\]

\[
= - 2 A^* (t, \xi) \eta \mathcal{Z}_7,1,1 (\eta, \eta) + C t^{\gamma} \left\| \sqrt{\lambda^\prime} (\eta) \right\|_{L^2}
\]

and

\[
- 2 \mathcal{Q}^* G_{8, -2,1} (q_S, \eta)
\]

\[
= - 2 A^* (t, \xi) G_{8, -2,1} (q_S, \eta) + C t^{\gamma} \left\| \sqrt{\lambda^\prime} (\eta) \right\|_{L^2}
\]

also \( A^* (t, \xi) = \sqrt{i \lambda^\prime (\xi)} + O (t^{\frac{3}{4} - \frac{1}{2}} \xi^3 + \frac{1}{2}) \). By Lemma 4.4 with \( \delta = 0 \) we have

\[
\left\| \sqrt{\lambda^\prime} \partial_\eta q \right\|_{L^2} + \left\| \sqrt{\lambda^\prime} \partial_\eta q S \right\|_{L^2} \leq C \varepsilon.
\]

In view of Lemma 4.1 with \( \delta = 0 \) we find \( \| q \|_{L^\infty} + \| q S \|_{L^\infty} \leq C \varepsilon \). Via Lemma 6.2 with \( N = 3, \beta = 2 \), we get

\[
\left\| \sqrt{\lambda^\prime} \partial_\eta G_{7,1,1} (\eta, \eta) \right\|_{L^2} \leq C t^{\gamma} \| q \|_{L^\infty} \left\| \sqrt{\lambda^\prime} (\eta) \right\|_{L^2} + C \| q \|_{L^\infty} \left\| \sqrt{\lambda^\prime} \partial_\eta q \right\|_{L^2} \leq C \varepsilon^3 t^{2 \gamma}
\]

and

\[
\left\| \sqrt{\lambda^\prime} \partial_\eta G_{8, -2,1} (q_S, \eta) \right\|_{L^2}
\]

\[
\leq C t^{\gamma} \| q S \|_{L^\infty} \left\| \sqrt{\lambda^\prime} (\eta) \right\|_{L^2} + C \| q S \|_{L^\infty} \left\| \sqrt{\lambda^\prime} \partial_\eta q \right\|_{L^2}
\]

\[
+ C \| q \|_{L^\infty} \| q S \|_{L^\infty} \left\| \sqrt{\lambda^\prime} \partial_\eta q S \right\|_{L^2} \leq C \varepsilon^3 t^{2 \gamma}.
\]

Hence we can estimate the remainders

\[
\left\| \sqrt{\lambda^\prime} (\eta) \right\|_{L^\infty} \left\| \sqrt{\lambda^\prime} \partial_\eta q \right\|_{L^2} + C \left\| (\eta) \frac{1}{2} \right\|_{L^2} \left\| \sqrt{\lambda^\prime} \partial_\eta q S \right\|_{L^2} + C \| q \|_{L^\infty} \left\| \sqrt{\lambda^\prime} \partial_\eta q S \right\|_{L^2} \leq C \varepsilon^3 t^{2 \gamma}.
\]
\[ \leq C\varepsilon^3 t^{2\gamma}. \]

In the same manner
\[
\left\| \sqrt{\Lambda'} \langle \eta \rangle^{-\frac{1}{2}} \partial_n (G_{8,-2,1} (q_S^2, \eta)) \right\|_{L^2} \leq C\varepsilon^3 t^{2\gamma}.
\]

By virtue of Corollary 1
\[ G_{7,1,1} (\eta, \eta) = \frac{2i\xi^3 \eta^2}{\Lambda (2\xi) + 2\Lambda (\xi)} + O \left( t^{\gamma - \frac{1}{4} \varepsilon^2} \right) \]
and \[ G_{8,-2,1}(q_S^2, \eta) = \frac{2\varepsilon^2 (\xi) q_S^2 \eta}{\Lambda (2\xi) + 2\Lambda (\xi)} + O(t^{\gamma - \frac{1}{4} \varepsilon^3}). \]
Hence
\[ \Psi_1 = 4\xi^2 \sqrt{i\Lambda'' (\xi)} \frac{i\xi^2 q_S^2 - \{ \xi \} q_S^2 \eta}{\Lambda (2\xi) + 2\Lambda (\xi)} + O \left( t^{\gamma - \frac{1}{4} \varepsilon^3} \right). \]

By Lemma 4.1 with \( \delta = 0 \) we have \( q = \frac{\hat{\psi}}{\sqrt{i\Lambda'' (\xi)}} + O(\varepsilon t^{-\frac{1}{4}}) \) and \( q_S = \frac{i\xi \hat{\psi}}{\langle \xi \rangle \sqrt{i\Lambda'' (\xi)}} + O(\varepsilon t^{-\frac{1}{4}}). \) Therefore
\[
i\xi^2 q_S^2 - \{ \xi \} q_S^2 \eta = \frac{\xi^2}{\langle \xi \rangle^2 \sqrt{i\Lambda'' (\xi)} \Lambda'' (\xi)} \frac{\hat{\psi}}{\bar{\psi}} + O \left( t^{\gamma - \frac{1}{4} \varepsilon^3} \right).
\]

Lemma 7.2 is proved.

7.2. Estimates of \( \Psi_2, \Psi_3 \) and \( \Phi_9 \) in \( L^\infty \). We next show that \( \Psi_2, \Psi_3 \) and \( \Phi_9 \) in equation (3.4) are the remainders.

Lemma 7.3. The estimates \( \| \Psi_2 \|_{L^\infty} + \| \Psi_3 \|_{L^\infty} \leq C\varepsilon^2 t^{\frac{1}{4} + 3\gamma} \), \( \| \Phi_9 \|_{L^\infty} \leq C\varepsilon^3 t^{4\gamma - \frac{3}{2}} \) are true for all \( t \geq 1 \).

Proof. We use estimate of Lemma 4.6
\[
\left\| \Psi_2 \right\|_{L^\infty} \leq C \left\| \langle \xi \rangle^2 G_{9,1,1} (q_S G_{7,1,1} (\eta, \eta), \eta) \right\|_{L^\infty} + C \left\| t^{-\frac{1}{4}} \sqrt{\Lambda'} \langle \eta \rangle^\frac{1}{2} t \mathcal{A}_0 G_{9,1,1} (q_S G_{7,1,1} (\eta, \eta), \eta) \right\|_{L^2}.
\]

In view of Lemma 4.1 with \( \delta = 0 \) we find
\[
\left\| \langle \eta \rangle q \right\|_{L^\infty} + \left\| \langle \eta \rangle q_S \right\|_{L^\infty} \leq C \left\| \hat{\psi} \right\|_{L^\infty} + C t^{-\frac{1}{4}} \left\| \partial_x \hat{\psi} \right\|_{L^2} \leq C \varepsilon.
\]

By Lemma 6.4
\[
\left\| \langle \xi \rangle^2 G (\phi, f) \right\|_{L^\infty} \leq C t^{\gamma_1 \frac{3}{2} \gamma_2} \left\| \langle \xi \rangle^\frac{3}{2} \phi \right\|_{L^\infty} + C t^{\gamma_1} \left\| \langle \xi \rangle^\frac{3}{2} f \right\|_{L^\infty}.
\]

Hence
\[
\left\| \langle \xi \rangle^2 G_{9,1,1} (q_S G_{7,1,1} (\eta, \eta), \eta) \right\|_{L^\infty} 
\leq C t^{\frac{3}{2} \gamma_1} \left\| \langle \eta \rangle q \right\|_{L^\infty} \left\| \langle \eta \rangle^2 G_{7,1,1} (\eta, \eta) \right\|_{L^\infty} \leq C t^{\gamma_1} \left\| \langle \eta \rangle q \right\|_{L^\infty}^3 \left\| q_S \right\|_{L^\infty} \leq C t^{3 \varepsilon}.
\]

By Lemma 6.4 we find
\[
\left\| \langle \eta \rangle^\frac{3}{2} G_{7,1,1} (\eta, \eta) \right\|_{L^\infty} \leq C \left\| \langle \eta \rangle q \right\|_{L^\infty}^3 \leq C \varepsilon^2.
\]

By Lemma 4.4 with \( \delta = 2 \) we get
\[
\left\| \sqrt{\Lambda'} t \mathcal{A}_0 q \right\|_{L^2} \leq C \left\| \langle \xi \rangle^{-\frac{1}{2}} \partial_x \hat{\psi} \right\|_{L^2} + C \left\| \langle \xi \rangle^{-\frac{1}{2}} \hat{\psi} \right\|_{L^2} \leq C \varepsilon \gamma.
\]
Hence by Lemma 6.2
\[
\left\| \sqrt{\lambda} \langle \eta \rangle^{1/2} t A_0 \mathcal{G}_{9,1,1} (q_s \mathcal{G}_{7,1,1} (\eta, \overline{q}), \overline{q}) \right\|_{L^2} \\
\leq C t^{\gamma + 1/2} \left\| q_s \right\|_{L^\infty} \left\| \langle \eta \rangle^{1/2} \mathcal{G}_{7,1,1} (\eta, \overline{q}) \right\|_{L^\infty} \left\| \sqrt{\lambda} q_s \right\|_{L^2} \\
+ C t^{\gamma} \left\| q \right\|_{L^\infty} \left\| \langle \eta \rangle^{1/2} \mathcal{G}_{7,1,1} (\eta, \overline{q}) \right\|_{L^\infty} \left\| \sqrt{\lambda} t A_0 q_s \right\|_{L^2} \\
+ C t^{\gamma} \right\| q_s \right\|_{L^\infty} \left\| \langle \eta \rangle^{1/2} \mathcal{G}_{7,1,1} (\eta, \overline{q}) \right\|_{L^\infty} \left\| \sqrt{\lambda} t A_0 q \right\|_{L^2} \\
+ C t^{2\gamma} \left\| \langle \eta \rangle q \right\|_{L^2}^2 \left\| q_s \right\|_{L^\infty} \left\| \sqrt{\lambda} \right\|_{L^2} \leq C t^{2\gamma + 1/2} \varepsilon^4.
\]
Therefore \( \left\| \Psi_{21} \right\|_{L^\infty} \leq C t^{2\gamma + 1/2} \varepsilon^4 \). In the same manner we estimate \( \Psi_{22}, \Psi_{23} \) and \( \Psi_{24} \). Thus we obtain \( \left\| \Psi_2 \right\|_{L^\infty} \leq C \varepsilon^2 t^{\frac{1}{4} + \gamma} \).

Next we consider \( \Psi_3 \). In view of estimate of Lemma 4.6 we find
\[
\left\| \Psi_{31} \right\|_{L^\infty} \\
\leq C \left\| \langle \xi \rangle^{2} \mathcal{G}_{7,1,1} (\eta, \overline{q}) \right\|_{L^\infty}^2 + C \left\| t \right\|^{-1} \left\| \langle \eta \rangle^{1/2} \mathcal{G}_{7,1,1} (\eta, \overline{q}) \right\|_{L^\infty} \left\| \sqrt{\lambda} t A_0 \mathcal{G}_{7,1,1} (\eta, \overline{q}) \right\|_{L^2}.
\]
By Lemma 4.1 with \( \delta = 0 \) we find \( \left\| \langle \eta \rangle q \right\|_{L^\infty} \leq C \varepsilon \). Applying Lemma 4.4 we get
\[
\left\| \langle \eta \rangle^{1/2} \mathcal{G}_{7,1,1} (\eta, \overline{q}) \right\|_{L^\infty} \leq C t^{\gamma + 1/2} \left\| \langle \xi \rangle \right\|_{L^\infty}^2 \leq C t^{\gamma}.\]
By Lemma 4.4 with \( \delta = 0 \) we get
\[
\left\| \sqrt{\lambda} t A_0 q \right\|_{L^2} \leq C \left\| \partial_\xi \hat{\psi} \right\|_{L^2} + C \left\| \hat{\psi} \right\|_{L^2} \leq C t\gamma.
\]
Hence by Lemma 6.2
\[
\left\| \sqrt{\lambda} t A_0 \mathcal{G}_{7,1,1} (\eta, \overline{q}) \right\|_{L^2} \\
\leq C t^{\gamma + 1/2} \left\| q \right\|_{L^\infty} \left\| \sqrt{\lambda} q \right\|_{L^\infty} \left\| \sqrt{\lambda} t A_0 q \right\|_{L^2} \leq C t^{2\gamma + 1/2} \varepsilon^4.
\]
Therefore \( \left\| \Psi_{31} \right\|_{L^\infty} \leq C t^{2\gamma + 1/2} \varepsilon^4 \). In the same manner we estimate \( \Psi_{32} \). Hence we obtain \( \left\| \Psi_3 \right\|_{L^\infty} \leq C \varepsilon^2 t^{\frac{1}{4} + \gamma} \). Next we estimate \( \Phi_9 \). We have
\[
\left\| \Phi_9 \right\|_{L^\infty} = \left\| Q^{-1} B^{-1} D_t^{-1} r_4 \right\|_{L^\infty} \leq C t^{1/2} \left\| \langle \eta \rangle^2 B^{-1} D_t^{-1} r_4 \right\|_{L^1} \leq C t^{1/2} \left\| D_t^{-1} r_4 \right\|_{L^1} \leq C \left\| r_4 \right\|_{L^1}.
\]
By Lemma 5.2
\[
\left\| (1 + \mathcal{Y}_w) \phi \right\|_{L^1} \leq \left\| \phi \right\|_{L^1} + C \left\| \mathcal{Z}_4 (w, \phi) \right\|_{L^1} \leq (1 + C \left\| w \right\|_{L^\infty}) \left\| \phi \right\|_{L^1} \leq (1 + C \varepsilon) \left\| \phi \right\|_{L^1}.
\]
Hence \( \left\| (1 + \mathcal{Y}_w)^{-1} \phi \right\|_{L^1} \leq C \left\| \phi \right\|_{L^1} \). Also
\[
\left\| \mathcal{Y}_w \phi \right\|_{L^1} = 2 \left\| \mathcal{Z}_4 (w, \phi) \right\|_{L^1} \leq C \left\| w \right\|_{L^\infty} \left\| \phi \right\|_{L^1} \leq C \varepsilon t^{-1/2} \left\| \phi \right\|_{L^1}.
\]
Consider \( r_{41} \). We have
\[
\left\| r_{41} \right\|_{L^1} = \left\| (1 + \mathcal{Y}_w)^{-1} r_3 \right\|_{L^1} \leq C \left\| r_3 \right\|_{L^1}.
and decay. So we use the oscillation to represent

Thus we obtain

Thus, we obtain

Then

and

Thus we obtain

Thus we get \( \| \Phi_\gamma \|_{L^\infty} \leq \sum_{j=1}^{8} \| r_{4j} \|_{L^1} \leq C \varepsilon^{3} 4^{j-\frac{1}{2}} \). Lemma 7.3 is proved. \( \square \)

7.3. Derivative \( \partial_\xi \). Derivation of equation (3.4) yields

The last two terms on the right-hand side of the above equation have a slow time decay. So we use the oscillation to represent

\( i t \Omega_{k+1}^{-\frac{1}{2}} e^{i t \Omega_{k+1} \Phi} = i \partial_\xi \left( \Omega_{k+1}^{-\frac{1}{2}} t^{\frac{1}{2}} H_{k+1} e^{i t \Omega_{k+1} \Phi} \right) - i \Omega_{k+1}^{-t} e^{i t \Omega_{k+1} \Phi} \partial_\xi (H_{k+1} \Phi) \)

with \( H_{k+1} = \left( \frac{1}{2} + it \Omega_{k+1} \right)^{-1} \). Then we find

\[
\begin{align*}
\partial_\xi \left( \Omega_{k+1}^{-\frac{1}{2}} t^{\frac{1}{2}} H_{k+1} e^{i t \Omega_{k+1} \Phi} \right) &= t^{-\frac{1}{2}} \partial_\xi \phi + t^{-\frac{1}{2}} e^{i t \Omega_{k+1} \Phi} \partial_\xi \phi + t^{-\frac{1}{2}} e^{i t \Omega_{k+1} \Phi} \partial_\xi \phi + (\xi) \partial_\xi \phi \\
\end{align*}
\]
Using Lemma 4.4 we find

\[ -i\Omega'_{-2} t^\gamma e^{i\Omega_2 \gamma} \partial_t (H_{-2} \Psi_2) - i\Omega' t^{\frac{3}{2}} e^{i\Omega_1 \gamma} \partial_t (H_4 \Psi_3). \]  

We need to estimate the right-hand side of equation (7.1) in \( L^2 \) with a weight \( \langle \xi \rangle \).

**Lemma 7.4.** The estimate

\[ \| \langle \xi \rangle \partial_\xi \Psi_1 \|_{L^2} + t^{-\frac{3}{2}} \| \langle \xi \rangle \partial_\xi \Psi_2 \|_{L^2} + t^{-\frac{1}{2}} \| \langle \xi \rangle \partial_\xi \Psi_3 \|_{L^2} + t^1 \| \langle t \partial_\xi \Psi_2 \rangle \|_{L^2} + \| t \partial_\xi \Psi_3 \|_{L^2} \leq C \varepsilon t^\gamma \]

is true for all \( t \geq 1 \).

**Proof.** Consider the term \( \partial_\xi \Psi_1 \). We observe that the operators \( \mathcal{G}_{7,1,1} \) and \( \mathcal{G}_{8,-2,1} \) satisfy Lemma 6.2 with \( N = 3 \). This fact enables us to obtain optimal time rate for \( \partial_\xi \Psi_1 \). By Lemma 4.7 we have

\[ \| \langle \xi \rangle \partial_\xi Q^* \psi S_{7,1,1} (\vec{q}, \vec{q}) \|_{L^2} \leq C \sum_{j=0,1,2,3} \| \sqrt{\Lambda''} \langle \eta \rangle^{2-j} t A_0 A_1^{j+k} \psi S_{7,1,1} (\vec{q}, \vec{q}) \|_{L^2}. \]

Using Lemma 4.4 we find

\[ \| \sqrt{\Lambda''} \langle \eta \rangle^{2-j} t A_0 A_1^{j+k} \psi S_{7,1,1} (\vec{q}, \vec{q}) \|_{L^2} \leq C \varepsilon t^\gamma \]

for \( j = 0, 1, 2, 3 \). Also by virtue of Lemma 4.1 we have

\[ \| \langle \eta \rangle^{2-j} A_1^j \psi S_{7,1,1} (\vec{q}, \vec{q}) \|_{L^2} \leq C \varepsilon \]

for \( j = 0, 1, 2, 3 \). Therefore applying Lemma 6.2 with \( N = 3 \) we get

\[ \| \sqrt{\Lambda''} \langle \eta \rangle^{2} t A_0 A_1 \psi S_{7,1,1} (\vec{q}, \vec{q}) \|_{L^2} \leq C \| \langle \eta \rangle \|_{L^2} + \| \langle \eta \rangle \|_{L^2} \]

for \( j = 0, 1, 2, 3 \). Hence as above

\[ \| \sqrt{\Lambda''} t A_0 A_1 \psi S_{7,1,1} (\vec{q}, \vec{q}) \|_{L^2} \leq C \| \langle \eta \rangle \|_{L^2} + \| \langle \eta \rangle \|_{L^2} \]

In the same manner we estimate the other terms. Therefore we find

\[ \| \langle \xi \rangle \partial_\xi Q^* \psi S_{7,1,1} (\vec{q}, \vec{q}) \|_{L^2} \leq C \varepsilon t^\gamma. \]

Similarly we consider the term \( \langle \xi \rangle \partial_\xi Q^* \mathcal{G}_{8,-2,1} (\vec{q}_S, \vec{q}) \). Hence we get

\[ \| \langle \xi \rangle \partial_\xi \Psi_2 \|_{L^2} \leq C \varepsilon t^\gamma. \]

Next we estimate \( \partial_\xi \Psi_3 \). As above by Lemma 4.7 we have

\[ \| \langle \xi \rangle \partial_\xi Q^* (-2t) \mathcal{G}_{0,1,1} \psi S_{7,1,1} (\vec{q}, \vec{q}) \|_{L^2} \]
\[ \leq C \sum_{j=0,1,2,k=0,1} \left\| \sqrt{N^j} \langle \eta \rangle^{2-j} tA_0 A_{j-2}^{l+k} g_{9,1.1} (q_s g_{7,1.1} (\bar{q}, \bar{q}), \bar{q}) \right\|_{L^2}. \]

By Lemma 6.2 we find
\[ \leq C t^{\frac{3}{2} + \frac{1}{2}} \left\| \langle \eta \rangle^{2} q_s g_{7,1.1} (\bar{q}, \bar{q}) \right\|_{L^\infty} \left\| \sqrt{N^j q} \right\|_{L^2} + C t^{\frac{3}{2}} \left\| \langle \eta \rangle^{2} q_s g_{7,1.1} (\bar{q}, \bar{q}) \right\|_{L^\infty} \left\| \sqrt{N^j q_t A_0 q} \right\|_{L^2} + C t^{\frac{3}{2}} \left\| \langle \eta \rangle q \right\|_{L^\infty} \left\| \langle \eta \rangle q_s \right\|_{L^\infty} \left\| \sqrt{N^j q_t A_0 g_{7,1.1} (\bar{q}, \bar{q})} \right\|_{L^2} \leq C \varepsilon^{4} t^{\gamma + \frac{1}{2}}. \]

By identity (6.1) we obtain
\[ A_{-2} g_{9,1.1} (q_s g_{7,1.1} (\bar{q}, \bar{q}), \bar{q}) = g_{9,1.1} ((A_t q_s) g_{7,1.1} (\bar{q}, \bar{q}), \bar{q}) + 2 g_{9,1.1} (q_s g_{7,1.1} (A_t q_s), \bar{q}) + g_{9,1.1} (q_s g_{7,1.1} (q_s, A_t q_s)). \]

Hence as above
\[ A_{-2} \left\| \sqrt{N^j} q_t A_0 A_{-2} g_{8,-2,1} (q_s, \bar{q}) \right\|_{L^2} \leq \left\| \sqrt{N^j} q_t A_0 g_{9,1.1} ((A_t q_s) g_{7,1.1} (\bar{q}, \bar{q}), \bar{q}) \right\|_{L^2} + 2 \left\| \sqrt{N^j} q_t A_0 g_{9,1.1} (q_s g_{7,1.1} (A_t q_s), \bar{q}) \right\|_{L^2} + \left\| \sqrt{N^j} q_t A_0 g_{9,1.1} (q_s g_{7,1.1} (\bar{q}, \bar{q}), A_t q_s) \right\|_{L^2} \leq C \varepsilon^{4} q^{2} t^{\gamma + \frac{1}{2}}. \]

In the same manner we consider the other terms to get
\[ \left\| (\xi) \partial_t \xi (q_s g_{7,1.1} (\bar{q}, \bar{q}), \bar{q}) \right\|_{L^2} \leq C \varepsilon^{4} t^{\gamma + \frac{1}{2}}. \]

Thus we find \[ \left\| (\xi) \partial_t \Psi_2 \right\|_{L^2} \leq C \varepsilon^{4} t^{\gamma + \frac{1}{2}}. \] Next we consider \( \partial_t \Psi_3 \). By Lemma 4.7 we have
\[ \left\| (\xi) \partial_t \xi (q_s g_{7,1.1} (\bar{q}, \bar{q}), \bar{q}) \right\|_{L^2} \leq C \sum_{j=0,1,2,k=0,1} \left\| \sqrt{N^j} (\eta)^{2-j} tA_0 A_{j-2}^{l+k} (g_{7,1.1} (\bar{q}, \bar{q}))^2 \right\|_{L^2}. \]

Then as above by Lemma 6.2, Lemma 4.4 and Lemma 4.1 we have
\[ \left\| \sqrt{N^j} (\eta)^{2} tA_0 \left( g_{7,1.1} (\bar{q}, \bar{q}) \right)^2 \right\|_{L^2} \leq C t^{\frac{3}{2}} \left\| \langle \eta \rangle q \right\|_{L^\infty} \left\| \sqrt{N^j q} \right\|_{L^2} + C \left\| \langle \eta \rangle q \right\|_{L^\infty} \left\| \sqrt{N^j tA_0 q} \right\|_{L^2} \leq C \varepsilon^{4} t^{\gamma + \frac{1}{2}}. \]

Next by identity (6.1) we obtain
\[ A_4 \left( g_{7,1.1} (\bar{q}, \bar{q}) \right)^2 = 2 \left( g_{7,1.1} (\bar{q}, \bar{q}) \right) \left( g_{7,1.1} (\bar{q}, A_t q_s) \right) \]

Hence as above
\[ \left\| \sqrt{N^j} \eta_t A_0 A_4 (g_{7,1.1} (\bar{q}, \bar{q}))^2 \right\|_{L^2} \]
\[
\leq C t^{3} \left\| \langle \eta \rangle \right\|_{L^{\infty}} \| A_{1}q \|_{L^{\infty}} \left\| q \right\|_{L^{\infty}} \left\| \sqrt{\nabla q} \right\|_{L^{2}} + C \left\| \langle \eta \rangle \right\|_{L^{\infty}} \| A_{1}q \|_{L^{\infty}} \left\| \sqrt{\nabla^2 \langle \eta \rangle^{j_{2}}} t A_{0}q \right\|_{L^{2}} + C \left\| \langle \eta \rangle \right\|_{L^{\infty}} \left\| q \right\|_{L^{2}} \left\| \sqrt{\nabla^2 A_{1}q} \right\|_{L^{2}} + C \left\| \langle \eta \rangle \right\|_{L^{\infty}} \left\| q \right\|_{L^{2}} \left\| \sqrt{\nabla t A_{0}q} \right\|_{L^{2}} + C \left\| \langle \eta \rangle \right\|_{L^{\infty}} \left\| q \right\|_{L^{2}} \left\| \sqrt{\nabla t A_{0}A_{1}q} \right\|_{L^{2}} \leq C \varepsilon^{4} t^{3/2}.
\]

The other terms in \( \partial_{\xi} \Psi_{3} \) are considered in the same way. So we get
\[
\left\| \langle \xi \rangle \partial_{\xi} \Psi_{3} \right\|_{L^{2}} \leq C \varepsilon^{4} t^{3/2}.
\]

Consider \( \partial_{\xi} \Phi_{9} \). We have
\[
\left\| \langle \xi \rangle \partial_{\xi} \Phi_{9} \right\|_{L^{2}} = \| \mathcal{F}U (-t) \mathcal{J} r_{4} \|_{L^{2}} = \| \mathcal{J} r_{4} \|_{\mathcal{H}^{4}}.
\]

Then applying Lemma 5.3 we find
\[
\left\| \mathcal{J} (1 + \gamma w)^{-1} \phi \right\|_{\mathcal{H}^{4}} \leq \| \mathcal{J} \phi \|_{\mathcal{H}^{4}} + \sum_{j=1}^{\infty} \| \mathcal{J} \gamma^{j} \phi \|_{\mathcal{H}^{4}} \leq \| \mathcal{J} \phi \|_{\mathcal{H}^{4}} + C \varepsilon t^{1/2} \| \phi \|_{L^{\infty}} + C \varepsilon^{4} t^{3/2} \| \phi \|_{L^{2}}.
\]

Therefore
\[
\| \mathcal{J} r_{41} \|_{\mathcal{H}^{4}} \leq \frac{1}{t} \| \mathcal{J} r_{41} \|_{\mathcal{H}^{4}} + C \varepsilon t^{1/2} \| R_{51} \|_{L^{\infty}} + C \varepsilon t^{-1} \| R_{51} \|_{L^{2}} + \frac{1}{t} \| \mathcal{J} r_{52} \|_{\mathcal{H}^{4}} + C \varepsilon t^{1/2} \| R_{52} \|_{L^{\infty}} + C \varepsilon t^{-1} \| R_{52} \|_{L^{2}} + \| \mathcal{J} r_{1} \|_{\mathcal{H}^{4}} + C \varepsilon t^{1/2} \| r_{11} \|_{L^{\infty}} + C \varepsilon t^{1/2} \| r_{11} \|_{L^{2}} + \| \mathcal{J} r_{2} \|_{\mathcal{H}^{4}} + C \varepsilon t^{1/2} \| r_{22} \|_{L^{\infty}} + C \varepsilon t^{1/2} \| r_{22} \|_{L^{2}}.
\]

Applying Lemma 5.3 we have \( \| \mathcal{J} r_{41} \|_{\mathcal{H}^{4}} \leq C \varepsilon^{4} t^{1/2} \). Similarly we get
\[
\| \mathcal{J} r_{42} \|_{\mathcal{H}^{4}} \leq C \varepsilon^{4} t^{1/2} \| \gamma w \left( \overline{w} \overline{Z}(w, w) \right) \|_{L^{\infty}} + C \varepsilon t^{1/2} \| \gamma w \left( \overline{w} \overline{Z}(w, w) \right) \|_{L^{2}} + C \varepsilon t^{1/2} \| \gamma w \left( \overline{w} \overline{Z}(w, w) \right) \|_{L^{2}} \leq C \varepsilon^{4} t^{1/2} \| \mathcal{J} r_{42} \|_{\mathcal{H}^{4}} \leq C \varepsilon^{4} t^{3/2}.
\]

The other terms are estimated in the same manner. So we have \( \| \mathcal{J} r_{41} \|_{\mathcal{H}^{4}} \leq C \varepsilon^{4} t^{3/2} \). Finally we estimate the time derivatives \( \partial_{\xi} \Psi_{2} \) and \( \partial_{\xi} \Psi_{3} \). By Lemma 4.8 we find
\[
\| \mathcal{D}_{-2} \partial_{t} \mathcal{Q}^{+} (-2t) \mathcal{G}_{9,1,1} (q s \mathcal{G}_{7,1,1} (\overline{\eta}, \overline{\eta}), \overline{\eta}) \|_{L^{2}} \leq C \left\| \sqrt{\nabla^2} \mathcal{G}_{9,1,1} (q s \mathcal{G}_{7,1,1} (\overline{\eta}, \overline{\eta}), \overline{\eta}) \right\|_{L^{2}} + C \sum_{j=0}^{3} \left\| \sqrt{\nabla^2} (\eta)^{j} t A_{0}A_{1}^{j}_{1} \mathcal{G}_{9,1,1} (q s \mathcal{G}_{7,1,1} (\overline{\eta}, \overline{\eta}), \overline{\eta}) \right\|_{L^{2}}.
\]

Using Lemma 4.1, Lemma 4.4 and Lemma 6.3 we get
\[
\| \sqrt{\nabla^2} (\partial_{t} \mathcal{G}_{9,1,1}, q s \mathcal{G}_{7,1,1} (\overline{\eta}, \overline{\eta}), \overline{\eta}) \|_{L^{2}} \leq C t^{3/2} \left\| q \right\|_{L^{2}} \left\| \sqrt{\nabla^2} (\eta)^{2} (q s \mathcal{G}_{7,1,1} (\overline{\eta}, \overline{\eta}) \right\|_{L^{2}} + C t^{3/2} \left\| (q s \mathcal{G}_{7,1,1} (\overline{\eta}, \overline{\eta})) \right\|_{L^{2}} \left\| \sqrt{\nabla^2} (\eta)^{2} q \right\|_{L^{2}}
\]
8. A priori estimates. First we state the local existence of solutions to the Cauchy problem (1.2) in the functional space $H^2 \cap H^{1,1}$ (see [3] for the proof.)

**Theorem 8.1.** Assume that the initial data $u_0 \in H^2 \cap H^{1,1}$. Then there exists a time $T > 1$ which depends on the norm $\|u_0\|_{H^2 \cap H^{1,1}}$ such that the Cauchy problem (1.2) has a unique solution $U(t) \in C ([1, T]; H^2 \cap H^{1,1})$. Existence time $T$ is large when the norm $\|u_0\|_{H^2 \cap H^{1,1}}$ is small.

To prove a global result, we define the following norm

$$\|w\|_{X_T} = \sup_{t \in [1, T]} \left( t^{-\gamma_2} \|w(t)\|_{H^2} + t^{-\gamma} \|J w(t)\|_{H^{1,1}} \right) + \sup_{t \in [1, T]} \left( \|\partial_\xi \hat{\psi}(t)\|_{L^\infty} \right),$$

where $\gamma > 3\gamma_2 > 0$ are small. As a consequence of Lemma 7.1 we have $\|w(t)\|_{H^\infty} \leq C t^{-\frac{1}{2}}, \|\partial_\xi \hat{\psi}(t)\|_{L^\infty} \leq C t^{\gamma},$ with $\epsilon = \|w\|_{X_T}$.

**Lemma 8.2.** Let the initial data $u_0 \in H^2 \cap H^{1,1}$ have a small norm $\|u_0\|_{H^2 \cap H^{1,1}}$. Then the estimate $\|w\|_{X_T} < C \epsilon$ is true for all $T \geq 1$.

**Proof.** Arguing by the contradiction, we can find a maximal time interval $T > 1$ such that $\|w\|_{X_T} \leq C \epsilon$. First we prove a priori estimate in $L^\infty$ norm. In the domain $|\xi| \geq (t)^{\gamma}$ we use the Sobolev embedding inequality

$$\|\langle \xi \rangle \hat{\psi}\|_{L^\infty(|\xi| \geq (t)^{\gamma})} \leq C \|\langle \xi \rangle \hat{\psi}\|_{L^2(|\xi| \geq (t)^{\gamma})} \|\partial_\xi \langle \xi \rangle \hat{\psi}\|_{L^2(|\xi| \geq (t)^{\gamma})} \leq C t^{-\frac{1}{2}} \|\langle \xi \rangle \hat{\psi}\|_{L^2(|\xi| \geq (t)^{\gamma})} \|\partial_\xi \langle \xi \rangle \hat{\psi}\|_{L^2(|\xi| \geq (t)^{\gamma})} < C \epsilon \langle t \rangle^{-\frac{5}{2} + \frac{3}{2} \gamma} < C \epsilon.$$

By Lemma 4.5 we obtain

$$\|\sqrt{A^\gamma} \langle \xi \rangle^{-1} t \partial_\xi q\|_{L^2} \leq C \epsilon + \|\sqrt{A^\gamma} \langle \xi \rangle^{-1} Q t \partial_\xi \hat{v}\|_{L^2} \leq C \epsilon.$$

Hence

$$\|D_{-2} Q^* (-2t) \mathcal{G}_{0,1,1} (t \partial_\xi q \mathcal{G}_{0,1,1} (\xi, \eta), \eta)\|_{L^2} \leq \left\|\sqrt{A^\gamma} \mathcal{G}_{0,1,1} (t \partial_\xi q \mathcal{G}_{0,1,1} (\xi, \eta), \eta)\right\|_{L^2} \leq C t^{\gamma} \|\langle \eta \rangle q\|_{L^\infty} \|\mathcal{G}_{0,1,1} (\xi, \eta)\|_{L^\infty} \right\|\sqrt{A^\gamma} \langle \xi \rangle^{-1} t \partial_\xi q\|_{L^2} \leq C \epsilon t^{\gamma}.$$

In the same manner we estimate the other terms. Hence we find $\|t \partial_\xi \Psi_2\|_{L^2} + \|t \partial_\xi \Psi_3\|_{L^2} \leq C \epsilon$. Lemma 7.4 is proved. \qed
if $\nu > 3\gamma$. Therefore we need to estimate the function $\langle \xi \rangle \hat{\psi} (t, \xi)$ in the domain $|\xi| \leq \langle t \rangle^\nu$. By equation (3.4) in view of Lemma 7.2 and Lemma 7.3 we get

$$i\partial_t \langle \xi \rangle \hat{\psi} = t^{-1} \lambda (\xi) \left| \hat{\psi} \right|^2 \langle \xi \rangle \hat{\psi} + O \left( \varepsilon^2 \langle \xi \rangle t^{3\gamma - \frac{2}{\nu}} \right).$$

Multiplying the above equation by $\langle \xi \rangle \hat{\psi}(t, \xi)$, and taking the imaginary part, we get

$$\frac{d}{dt} \left| \langle \xi \rangle \hat{\psi}(t, \xi) \right| \leq C\varepsilon^2 t^{\nu+3\gamma - \frac{2}{\nu}}$$

in the domain $|\xi| \leq \langle t \rangle^\nu$. Define $t_1$ such that $\langle t_1 \rangle^\nu = |\xi|$, then integrating in time from $t_1$ to $t$ we find

$$\left| \langle \xi \rangle \hat{\psi}(t_1, \xi) \right| \leq \left| \langle \xi \rangle \hat{\psi}(t_1, \xi) \right| + C\varepsilon^2 \int_{t_1}^t t^{\nu+3\gamma - \frac{2}{\nu}} dt < C\varepsilon.$$

Thus we obtain $\left| \langle \xi \rangle \hat{\psi}(t) \right|_{L^\infty} < C\varepsilon$. Next by equation (2.3) we get

$$Lw = \left( S u \right)^2 - (Sw)^2 - 2Z_8 \left( \left( S u \right)^2, u \right) - 2Z_4 (Lw, w) + \frac{1}{t} (Z_5 (u, u) + Z_6 (w, w)).$$

We use the relation $u = w - Z_4 (u, u) - Z_4 (w, w)$, then

$$\left\| \left( S u \right)^2 - (Sw)^2 \right\|_{H^2} \leq C \left\| (Su + Sw) (SZ_3 (u, u) + SZ_4 (w, w)) \right\|_{H^2} \leq C \left( \|u\|_{H^2} + \|w\|_{H^2} \right) \left( \|u\|_{L^\infty}^2 + \|w\|_{L^\infty}^2 \right) + C \left( \|u\|_{L^\infty} + \|w\|_{L^\infty} \right) \left( \|u\|_{H^2} + \|w\|_{L^\infty} \|w\|_{H^2} \right) \leq C\varepsilon^3 t^{\gamma_2 - 1}.$$

Also we have the estimates

$$Z_8 \left( \left( S u \right)^2, u \right) \leq \|u\|_{L^\infty}^2 \|u\|_{H^2} \leq C\varepsilon^3 t^{\gamma_2 - 1},$$

$$Z_5 (u, u) \leq \|u\|_{L^\infty}^2 \|u\|_{H^1} \leq C\varepsilon^3 t^{\gamma_2 - 1},$$

$$Z_6 (w, w) \leq \|w\|_{L^\infty}^2 \|w\|_{H^1} \leq C\varepsilon^3 t^{\gamma_2 - 1}$$

and

$$Z_4 (Lw, w) \leq \|w\|_{L^\infty} \|Lw\|_{H^1} \leq C\varepsilon t^{-\frac{1}{2}} \|Lw\|_{H^1}.$$

Then by equation (2.5) we get $\|Lw\|_{H^1} \leq C\varepsilon^3 t^{\gamma_2 - 1} + C\varepsilon t^{-\frac{1}{2}} \|Lw\|_{H^1}$. Hence $\|Lw\|_{H^1} \leq C\varepsilon^3 t^{\gamma_2 - 1}$. Therefore $\|Z_4 (Lw, w)\|_{H^2} \leq C\varepsilon^3 t^{\gamma_2 - 1}$. Now by equation $\frac{d}{dt} \|w\|_{H^2} \leq C\varepsilon t^{\gamma_2}$. Integrating in time, we get $\|w\|_{H^2} < C\varepsilon t^{\gamma_2}$.

We need to estimate $\|Jw\|_{H^1} = \left\| \langle \xi \rangle \partial_\xi \hat{\psi} \right\|_{L^2}$. By equation (7.1) and Lemma 7.4 we find

$$\frac{d}{dt} \left\| \frac{Y}{Y} \right\|_{L^2} \leq t^{-1} \left\| \langle \xi \rangle \partial_\xi \Psi_1 \right\|_{L^2} + t^{-\frac{2}{\nu}} \left\| \langle \xi \rangle \partial_\xi \Psi_2 \right\|_{L^2} + t^{-\frac{2}{2}} \left\| \langle \xi \rangle \partial_\xi \Psi_3 \right\|_{L^2} + \left\| \langle \xi \rangle \partial_\xi \Phi_0 \right\|_{L^2}$$

with

$$Y = \langle \xi \rangle \left( \partial_\xi \hat{\psi} - \Omega_2 t^{\frac{1}{2}} \partial_\xi \partial_t \hat{\psi} - \Omega_4 t^{\frac{1}{2}} \partial_t \hat{\psi} \right).$$

The integration in time yields $\|Y\|_{L^2} \leq C\varepsilon t^{\gamma}$. Hence $\left\| \langle \xi \rangle \partial_\xi \hat{\psi} \right\|_{L^2} \leq C\varepsilon t^{\gamma}$. Thus we obtain $\|w\|_{X_\nu} < C\varepsilon$, which yields a desired contradiction. Lemma 8.2 is proved. \qed
9. Proof of Theorem 1.1. By Lemma 8.2 we see that a priori estimate of $\|u\|_{X_T}$ ≤ $C\varepsilon$ is true for all $T > 0$. Therefore the global existence of solutions of the Cauchy problem (1.2) satisfying estimate $\|u\|_{X_\infty} \leq C\varepsilon$ follows by a standard continuation argument by the local existence Theorem 8.1. Thus we have the global in time existence of small solutions to the Cauchy problem (1.2).

Now we turn to the proof of the asymptotic formula for the solutions $u$ of the Cauchy problem (1.2). As in the proof of Lemma 8.2 by equation (3.4) in view of Lemma 7.2 and Lemma 7.3 we get

$$i\partial_t \tilde{\psi} = t^{-1}\lambda(\xi) \left|\tilde{\psi}\right|^2 \tilde{\psi} + O\left(\varepsilon^2 t^{3\gamma - \frac{3}{2}}\right),$$

where $\lambda(\xi) = \frac{4\varepsilon^4}{\chi'(\xi)(\chi(2\xi) + 2\chi'(\xi))}$. To exclude the resonant term we make a change

$$v = \tilde{\psi} \exp\left(i\lambda(\xi) \int_1^t \left|\tilde{\psi}(\tau, \xi)\right|^2 \frac{d\tau}{\tau}\right),$$

then we get $v_t = O(\varepsilon^3 t^{3\gamma - \frac{3}{2}})$. Integrating in time and taking the $L^\infty$-norm, we obtain

$$\|v_t(t) - v_t(s)\|_{L^\infty} \leq C\varepsilon \int_s^t \tau^{3\gamma - \frac{3}{2}} d\tau \leq C\varepsilon s^{3\gamma - \frac{3}{4}}$$

for any $t > s > 0$. Therefore there exists a unique final state $v_+ \in L^\infty$ such that $\|v(t) - v_+\|_{L^\infty} \leq C\varepsilon t^{3\gamma - \frac{3}{4}}$ for all $t > 0$. Since $\int_1^t |\tilde{\psi}(\tau, \xi)|^2 d\tau = |v(\tau, \xi)|^2$, we write

$$\int_1^t \left|\tilde{\psi}(\tau, \xi)\right|^2 \frac{d\tau}{\tau} = \int_1^t |v(\tau, \xi)|^2 \frac{d\tau}{\tau}$$

and denote $\Phi(t) = \int_1^t |v(\tau, \xi)|^2 \frac{d\tau}{\tau} - |v_+|^2 \log t$. We study the asymptotics in time of the remainder term $\Phi(t)$. We have

$$\Phi(t) - \Phi(s) = \int_s^t \left(\|v(t)\|^2 - |v(t)|^2\right) \frac{d\tau}{\tau} + \left(\|v(t)\|^2 - |v_+|^2\right) \log \frac{t}{s}$$

and $\|\Phi(t) - \Phi(s)\|_{L^\infty} \leq C\varepsilon^2 s^{3\gamma - \frac{3}{4}}$ for any $t > s > 0$. Hence there exists a unique real-valued function $\Phi_+$ such that $\Phi_+ \in L^\infty$ and $\|\Phi(t) - \Phi_+\|_{L^\infty} \leq C\varepsilon^2 t^{3\gamma - \frac{3}{4}}$. Therefore we obtain

$$\int_1^t \left|\tilde{\psi}(\tau, \xi)\right|^2 \frac{d\tau}{\tau} = \int_1^t |v(\tau, \xi)|^2 \frac{d\tau}{\tau} = \Phi(t) - |v_+|^2 \log t + O\left(\varepsilon^2 t^{3\gamma - \frac{3}{4}}\right)$$

for all $t > 0$. Then we find that

$$\|\exp\left(i\lambda(\xi) \int_1^t \left|\tilde{\psi}(\tau, \xi)\right|^2 \frac{d\tau}{\tau}\right) - \exp\left(i\lambda(\xi) |v_+|^2 \log t + i \frac{i}{\chi'(\xi)} \Phi_+\right)\|_{L^\infty} \leq Ct^{3\gamma - \frac{3}{4}}$$

for all $t > 0$. Thus we get the large time asymptotics

$$\|\tilde{\psi}(t, \xi) - v_+ \exp\left(-i\lambda(\xi) \int_1^t \left|\tilde{\psi}(\tau, \xi)\right|^2 \frac{d\tau}{\tau}\right)\|_{L^\infty} = \|v(t) - v_+\|_{L^\infty} \leq Ct^{3\gamma - \frac{3}{4}}$$

and

$$\|v_+ \exp\left(-i\lambda(\xi) \int_1^t \left|\tilde{\psi}(\tau, \xi)\right|^2 \frac{d\tau}{\tau}\right) - v_+ \exp\left(-i\lambda(\xi) \left|\tilde{\psi}_+\right|^2 \log t + \Phi_+\right)\|_{L^\infty} \leq Ct^{3\gamma - \frac{3}{4}}.$$
Therefore we obtain the estimate
\[ \left\| \hat{\psi}(t, \xi) - W_+ \exp \left( -i\lambda(\xi) |W_+|^2 \log t \right) \right\|_{L^\infty} \leq C t^{3\gamma - \frac{1}{4}} \]
with \( W_+(\xi) = v_+(\xi) e^{-i\lambda(\xi) \Phi_+}(\xi) \). Note that \( W_+ \in L^\infty \).

Using the factorization formula \( u(t) = D_t B M \hat{\phi} \) and Lemma 4.1 with \( \delta = 0 \) we find
\[ u(t) = D_t B M W_+ \exp \left( -i\lambda(\xi) |W_+|^2 \log t \right) + O \left( t^{-\frac{1}{2} - \gamma} \right). \]
This completes the proof of the asymptotics. Theorem 1.1 is proved.

**Acknowledgments.** We are grateful to unknown referees for many useful suggestions and comments.

**REFERENCES**

[1] M. Ben-Artzi, H. Koch and J. C. Saut, Dispersion estimates for fourth order Schrödinger equations, *C. R. Math. Acad. Sci.*, **330** (2000), 87–92.

[2] A. P. Calderon and R. Vaillancourt, A class of bounded pseudo-differential operators, *Proc. Nat. Acad. Sci.*, **69** (1972), 1185–1187.

[3] Th. Cazenave, *Semilinear Schrödinger equations*, Courant Institute of Mathematical Sciences, American Mathematical Society, Providence, RI, 2003.

[4] S. Cohn, Resonance and long time existence for the quadratic semilinear Schrödinger equation, *Commun. Pure Appl. Math.*, **45** (1992), 973–1001.

[5] R. R. Coifman and Y. Meyer, Au dela des operateurs pseudo-differentiels, Societe Mathematique de France, Paris, 1978.

[6] H. O. Cordes, On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators, *J. Funct. Anal.*, **18** (1975), 115–131.

[7] K. B. Dysthe, Note on a modification to the nonlinear Schrödinger equation for application to deep water waves, *Proc. R. Soc. Lond. Ser. A*, **369** (1979), 105–114.

[8] Y. Fukumoto and H. K. Mofatt, Motion and expansion of a viscous vortex ring: I. A higher-order asymptotic formula for the velocity, *J. Fluid. Mech.*, **417** (2000), 1–45.

[9] N. Hayashi and P. I. Naumkin, A quadratic nonlinear Schrödinger equation in one space dimension, *J. Differ. Equ.*, **186** (2002), 165–185.

[10] N. Hayashi and P. I. Naumkin, The initial value problem for the cubic nonlinear Klein-Gordon equation, *Z. Angew. Math. Phys.*, **59** (2008), 1002–1028.

[11] N. Hayashi and P. I. Naumkin, Asymptotic behavior for a quadratic nonlinear Schrödinger equation, *Electron. J. Differential Equations*, **15** 2008, 38 pp.

[12] N. Hayashi and P.I. Naumkin, On the inhomogeneous fourth-order nonlinear Schrödinger equation, *J. Math. Phys.*, **56** (2015), 25 pp.

[13] N. Hayashi and T. Ozawa, Scattering theory in the weighted \( L^2(R^n) \) spaces for some Schrödinger equations, *Ann. I. H. P. (Phys. Théor.)*, **48** (1988), 17-37.

[14] I. L. Hwang, The \( L^2 \)-boundedness of pseudodifferential operators, *Trans. Amer. Math. Soc.*, **302** (1987), 55–76.

[15] V. L. Karpman, Stabilization of soliton instabilities by high-order dispersion: fourth order nonlinear Schrödinger-type equations, *Phys. Rev. E*, **53** (1996), 1336–1339.

[16] V. L. Karpman and A. G. Shagalov, Stability of soliton described by nonlinear Schrödinger-type equations with high-order dispersion, *Physica D*, **144** (2000), 194–210.

[17] T. Ozawa, Remarks on quadratic nonlinear Schrödinger equations, *Funkcial. Ekvac.*, **38** (1995), 217–232.

[18] J. Shatah, Normal forms and quadratic nonlinear Klein-Gordon equations, *Commun. Pure Appl. Math.*, **38** (1985), 685–696.

Received September 2020; revised January 2021.

E-mail address: pavelnimatemor.unam.mx
E-mail address: isahi_ss@hotmail.com