Abstract

It is shown that a locally compact second countable group $G$ has the Haagerup property if and only if there exists a sharply weak mixing $0$-type measure preserving free $G$-action $T = (T_g)_{g \in G}$ on an infinite $\sigma$-finite standard measure space $(X, \mu)$ admitting an exhausting $T$-Følner sequence (i.e. a sequence $(A_n)_{n=1}^\infty$ of measured subsets of finite measure such that $A_1 \subset A_2 \subset \cdots$, $\bigcup_{n=1}^\infty A_n = X$ and $\lim_{n \to \infty} \sup_{g \in K} \frac{\mu(T_g A_n \triangle A_n)}{\mu(A_n)} = 0$ for each compact $K \subset G$). It is also shown that a pair of groups $H \subset G$ has property (T) if and only if there is a $\mu$-preserving $G$-action $S$ on $X$ admitting an $S$-Følner sequence and such that $S \upharpoonright H$ is weakly mixing. These refine some recent results by Delabie-Jolissaint-Zumbrunnen and Jolissaint.

1 Introduction

Throughout this paper $G$ is a non-compact locally compact second countable group. It has the Haagerup property if there exists a weakly continuous unitary representation $V$ of $G$ in a separable Hilbert space $\mathcal{H}$ such that $\lim_{g \to \infty} V(g) = 0$ in the weak operator topology and

\[(*) \text{ for each } \epsilon > 0 \text{ and every compact subset } K \subset G, \text{ there is a unit vector } \xi \in \mathcal{H} \text{ such that } \sup_{g \in K} \|V(g)\xi - \xi\| < \epsilon.\]
The amenable groups, $SO(n, 1)$ and $SU(n, 1)$ for each $n \geq 2$, the free groups, the Coxeter groups have the Haagerup property [Ch–Va]. The class of discrete countable Haagerup groups is closed under free products and wreath products [CoStVa]. For more information about the Haagerup property we refer to [Ch–Va]. There is a purely dynamical description of this property: $G$ is Haagerup if and only if there exists a mixing non-strongly ergodic probability preserving free $G$-action [Ch–Va, Theorem 2.2.2] (see §2 for the definitions). Recently, an infinite measure preserving counterpart of this result was discovered in [DeJoZu]:

**Theorem A.** $G$ has the Haagerup property if and only if there is a 0-type measure preserving $G$-action $T = (T_g)_{g \in G}$ on an infinite $\sigma$-finite measure space $(X, \mathcal{B}, \mu)$ admitting a sequence of non-negative unit vectors $(\xi_n)_{n=1}^{\infty}$ in $L^2(X, \mu)$ such that $\lim_{n \to \infty} \sup_{g \in K} \langle \xi_n \circ T_g, \xi_n \rangle = 1$ for each compact $K \subset G$.

We recall that $T$ is called of 0-type if $\lim_{g \to \infty} \mu(T_g A \cap B) = 0$ for all subsets $A, B \in \mathcal{B}$ of finite measure. In this paper we provide a much shorter alternative proof of Theorem A which is grounded on the Moore-Hill concept of restricted infinite products of probability measures [Hi].

We note that the 0-type for infinite measure preserving systems is a natural counterpart of the mixing for probability preserving systems. However unlike mixing, the 0-type is not a “strong” asymptotic property. It implies neither weak mixing nor ergodicity. Moreover, the totally dissipative actions are all of 0-type. In view of that the description in Theorem A does not look sharp from the ergodic theory point of view. Our first main result in this work is the following finer ergodic criterion of the Haagerup property.

**Theorem B.** The following are equivalent.

(i) $G$ has the Haagerup property.

(ii) There exists a sharply weak mixing (conservative) 0-type measure preserving free $G$-action $T$ on an infinite $\sigma$-finite standard measure space admitting an exhausting $T$-Følner sequence of subsets.

(iii) There exists a sharply weak mixing (conservative) 0-type measure preserving free $G$-action $T$ on an infinite $\sigma$-finite standard measure space

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1Conservativeness, ergodicity, weak mixing and sharp weak mixing are not spectral invariants of the underlying dynamical systems. Hence the principal difference of Theorem B from Theorem A is that it provides non-spectral ergodic characterization of the Haagerup property.
Haagerup property, Kazhdan pairs and infinite ergodic actions

$$(X, \mathcal{B}, \mu)$$ admitting a $T$-Følner sequence $(A_n)_{n=1}^{\infty}$ such that $\mu(A_n) = 1$ for all $n \in \mathbb{N}$.

We say that $(A_n)_{n=1}^{\infty}$ is $T$-Følner if $\mu(A_n) < \infty$ and

$$\sup_{g \in K} \frac{\mu(A_n \triangle T_g A_n)}{\mu(A_n)} \to 0 \quad \text{as } n \to \infty$$

for each compact subset $K \subset G$. If $A_1 \subset A_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} A_n = X$, we say that $(A_n)_{n=1}^{\infty}$ is exhausting. We note that sharp weak mixing (see §2 for the definition) implies ergodicity and weak mixing. To prove (the non-trivial part of) Theorem B, we apply the Moore-Hill construction [Hi] to the mixing non-strongly ergodic $G$-action from [Ch–Va, Theorem 2.2.2] (cf. the construction of $II_\infty$ ergodic Poisson suspensions of countable amenable groups from [DaKo]). Then we observe that the action $T$ that we obtain is IDPFT (see §2 and [DaLe], where such actions were introduced). Hence, by the properties of IDPFT systems, $T$ is sharply weak mixing whenever we show that it is conservative. To show the conservativeness of $T$ it remains to choose the parameters of the Moore-Hill construction in an appropriate way.

As a corollary from Theorem B, we obtain one more dynamical characterization of the Haagerup property in terms of Poisson actions.

**Corollary C.** $G$ has the Haagerup property if and only if there exists a mixing (probability preserving) Poisson $G$-action that is not strongly ergodic.

Our next purpose is to obtain a “parallel” characterization of property (T) which is a reciprocal to the Haagerup property. We recall [Jo1, Definition 1.1] that given a non-compact closed subgroup $H$ of $G$, the pair $H \subset G$ has property (T) if for each unitary representation $V$ of $G$ satisfying $[\ast]$, there is a unit vector which is invariant under $V(h)$ for every $h \in H$. The property (T) for a single group $G$ corresponds to the (T)-property for the pair $G \subset G$. Using the techniques developed for proving Theorem B we obtain an ergodic (non-spectral) characterization of Kazhdan pairs that refines a spectral characterization from [Jo2].

**Theorem D.** (i) Assume that a pair $H \subset G$ has property (T). Let $S = (S_g)_{g \in G}$ be a measure preserving $G$-action on a $\sigma$-finite infinite standard measure space $(Y, \mathcal{C}, \nu)$, such that $S \upharpoonright H := (S_h)_{h \in H}$ has no invariant subsets of positive finite measure. Then this action admits no $S$-Følner sequences.
(ii) If a pair \( H \subset G \) does not have property (T) then there is a measure preserving \( G \)-action \( S \) on a \( \sigma \)-finite infinite measure space which has an exhausting \( S \)-Følner sequence and such that \( S \upharpoonright H \) is weakly mixing.

Let us say that \( S \upharpoonright H \) is of **weak 0-type** if there is a subsequence \( h_n \to \infty \) in \( H \) such that \( \lim_{n \to \infty} \nu(S_{h_n} A \cap B) = 0 \) for all subsets \( A, B \in \mathcal{C} \) of finite measure. Then replacing the “has no invariant subsets of positive finite measure” in (i) with a stronger “is of weak 0-type”, and the “weakly mixing” in (ii) with a weaker “of weak 0-type” we obtain exactly [Jo2, Theorem 1.5].

**Corollary E.** A pair \( H \subset G \) has property (T) if and only if every (probability preserving) Poisson \( G \)-action with weakly mixing \( H \)-subaction is strongly ergodic. The same is also true with “ergodic” in place of “weakly mixing”.

The outline of the paper is as follows. In Section 2 we state all necessary definitions related to the basic dynamical concepts of group actions both in the nonsingular and and finite measure preserving cases, restricted infinite powers of probability measures, IDPFT actions and Poisson actions. In Section 3 we prove Theorems B and Corollary C. Section 4 is devoted to the proof of Theorems D and Corollary E.

## 2 Definitions and preliminaries

**Nonsingular and measure preserving \( G \)-actions**

Nonsingular actions appear in the proof of Theorem B. We remind several basic concepts related to them (see [Aa], [ScWa], [GlWe2], [DaKo]).

**Definition 2.1.** Let \( S = (S_g)_{g \in G} \) be a nonsingular \( G \)-action on a standard probability space \((Z, \mathcal{F}, \kappa)\).

(i) \( S \) is called **totally dissipative** if the partition of \( Z \) into the \( S \)-orbits is measurable and the \( S \)-stabilizer of a.e. point is compact, i.e. there is a measurable subset of \( Z \) which meets a.e. \( S \)-orbit exactly once, and for a.e. \( z \in Z \), the subgroup \( \{ g \in G \mid S_g z = z \} \) is compact in \( G \).

(ii) \( S \) is called **conservative** if there is no \( S \)-invariant subset \( A \subset Z \) of positive measure such that the restriction of \( S \) to \( A \) is totally dissipative.

(iii) There is a unique (mod 0) partition of \( X \) into two invariant subsets \( \mathcal{D}(S) \) and \( \mathcal{C}(S) \) such that \( S \upharpoonright \mathcal{D}(S) \) is totally dissipative and \( S \upharpoonright \mathcal{D}(S) \) is conservative. We call \( \mathcal{D}(S) \) and \( \mathcal{C}(S) \) the **dissipative** and **conservative part** of \( S \) respectively.
(iv) $S$ is called \textit{ergodic} if each measurable $S$-invariant subset of $Z$ is either \(\mu\)-null or \(\mu\)-conull.

(v) $S$ is called \textit{weakly mixing} if for each ergodic probability preserving $G$-action $R = (R_g)_{g \in G}$, the product $G$-action $(S_g \times R_g)_{g \in G}$ is ergodic.

(vi) $S$ is called \textit{sharply weak mixing} \cite{DaLe} if $S$ is conservative, ergodic and for each ergodic conservative nonsingular $G$-action $R = (R_g)_{g \in G}$, the product $G$-action $(S_g \times R_g)_{g \in G}$ is either ergodic or totally dissipative.

We also remind some concepts related to finite measure preserving actions.

\textbf{Definition 2.2.} Suppose that $\kappa(Z) = 1$ and $\kappa \circ S_g = \kappa$ for all $g \in G$.

(i) $S$ is called \textit{mixing} if $\lim_{g \to \infty} \kappa(S_g A \cap B) = \kappa(A)\kappa(B)$ for all $A, B \in \mathcal{F}$.

(ii) A sequence of Borel subsets $(A_n)_{n=1}^\infty$ in $X$ of strictly positive measure is called \textit{T-asymptotically invariant} if for each compact subset $K \subset G$, we have that $\sup_{g \in K} \kappa(A_n \triangle T_g A_n) \to 0$ as $n \to \infty$.

(iii) $T$ is called \textit{strongly ergodic} if each $T$-asymptotically invariant sequence $(A_n)_{n=1}^\infty$ is trivial, i.e. $\lim_{n \to \infty} \kappa(A_n)(1 - \kappa(A_n)) = 0$.

We now state a corollary from the Schmidt-Walters theorem \cite[Theorem 2.3]{ScWa} as it appeared in \cite[Theorem 7.3]{ArIsMa}. For a detailed proof of a sharper result, we refer to \cite[Theorem 7.14]{ArIsMa}.

\textbf{Lemma 2.3.} Let $S = (S_g)_{g \in G}$ be a mixing measure preserving action on a standard probability space $(Y, \mathcal{C}, \nu)$ and let $R = (R_g)_{g \in G}$ be a conservative nonsingular $G$-action on a standard probability space $(Y, \mathcal{C}, \nu)$. If

$$F : Y \times Z \to \mathbb{C}$$

is an $(S \times R)$-invariant Borel function then there exists a Borel $R$-invariant function $f : Z \to \mathbb{C}$ such that $F(y, z) = f(z)$ at a.e. $(y, z) \in Y \times Z$.

\textbf{Corollary 2.4.} Let $S = (S_g)_{g \in G}$ be a mixing measure preserving action on a standard probability space $(Y, \mathcal{C}, \nu)$ and let $R = (R_g)_{g \in G}$ be a nonsingular $G$-action on a standard probability space $(Z, \mathcal{D}, \kappa)$. Then

$$\mathcal{D}(S \times R) = Y \times \mathcal{D}(R) \quad \text{and} \quad \mathcal{C}(S \times R) = Y \times \mathcal{C}(R).$$
Restricted infinite powers of probability measures

Let \((Y, \mathcal{C}, \gamma)\) be a standard non-atomic probability space. Fix a sequence \(B := (B_n)_{n=1}^\infty\) of subsets from \(\mathcal{C}\) of positive measure. Let \((X, \mathcal{B}) := (Y, \mathcal{C})^\otimes\). For each \(n \in \mathbb{N}\), we set \(B^n := Y^n \times B_{n+1} \times B_{n+2} \times \cdots \in \mathcal{B}\). Then

\[
B^1 \subset B^2 \subset \cdots.
\]

We define a measure \(\gamma^B\) on \((X, \mathcal{B})\) by the following sequence of restrictions (see [Hi] for details):

\[
\gamma^B | B^n := \frac{\gamma}{\gamma(B_1)} \otimes \cdots \otimes \frac{\gamma}{\gamma(B_n)} \otimes \frac{\gamma | B_{n+1}}{\gamma(B_{n+1})} \otimes \frac{\gamma | B_{n+2}}{\gamma(B_{n+2})} \otimes \cdots.
\]

Since the restrictions are compatible, \(\gamma^B\) is well defined. We note that \(\gamma^B\) is supported on the subset \(\bigcup_{n=1}^\infty B^n \subset Y\) and \(\gamma^B(B^n) = \prod_{j=1}^n \gamma(B_j)^{-1}\) for each \(n\). Hence, \(\gamma^B\) is \(\sigma\)-finite. It is infinite if and only if \(\prod_{n=1}^\infty \gamma(B_n) = 0\).

**Definition 2.5.** We call \(\gamma^B\) the restricted infinite power of \(\gamma\) with respect to \(B\).

Given a \(\gamma\)-preserving Borel bijection \(T\) of \(Y\), we let \(T := \bigotimes_{n=1}^\infty T\) and \(TB := (TB_n)_{n=1}^\infty\). A straightforward verification shows that \(\gamma^B \circ T^{-1} = \gamma^{TB}\).

**Proposition 2.6.** If \(\sum_{n=1}^\infty \frac{\gamma(B_n \triangle TB_n)}{\gamma(B_n)} < \infty\) then \(T\) preserves \(\gamma^B\).

**Proof.** For each \(n \in \mathbb{N}\) and an arbitrary Borel subset \(A \subset Y^n\), we let \(A' := A \times B_{n+1} \times B_{n+2} \times \cdots \in \mathcal{B}\). Then for every \(m > n\),

\[
(\gamma^{TB} | (TB)^m)(A') = \frac{\gamma^{\otimes n}(A)}{\gamma(B_1) \cdots \gamma(B_n)} \prod_{j>m} \frac{\gamma(B_j \cap TB_j)}{\gamma(B_j)}.
\]

Passing to the limit as \(m \to \infty\), we obtain that

\[
\gamma^{TB}(A') = \lim_{m \to \infty} (\gamma^{TB} | (TB)^m)(A') = \frac{\gamma^{\otimes n}(A)}{\gamma(B_1) \cdots \gamma(B_n)} = \gamma^B(A').
\]

Hence \(\gamma^B \circ T^{-1} = \gamma^{TB} = \gamma^B\), as desired. \(\Box\)

We note that under the condition of Proposition 2.6,

\[
\gamma^B(TB^n \cap B^n) = \prod_{j=1}^n \gamma(B_j)^{-1} \prod_{j>n} \frac{\gamma(B_j \cap TB_j)}{\gamma(B_j)}.
\]

Hence

\[
\lim_{n \to \infty} \frac{\gamma^B(TB^n \cap B^n)}{\gamma^B(B^n)} = 1.
\]
We note that the Hilbert space $L^2(X, \gamma^B)$ is the infinite tensor product of the sequence of Hilbert spaces $(L^2(Y, \gamma(B_n)^{-1}\gamma))^\infty_{n=1}$ along the stabilizing sequence $(1_{B_n})^\infty_{n=1}$ of unit vectors (see [Gu] for the definition). In particular, we see that the linear subspace 

$$\bigcup_{n=1}^{\infty} \left\{ f \otimes \bigotimes_{j>n} 1_{B_j} \mid f \in L^2(Y^n, \gamma^\otimes_n) \right\}$$

is dense in $L^2(X, \gamma^B)$. Let $U_T$ and $U_T$ stand for the Koopman operators associated to $T$ and $T$ in $L^2(Y, \gamma)$ and $L^2(X, \gamma^B)$ respectively. The following proposition is verified straightforwardly.

**Lemma 2.7.** For each $n \in \mathbb{N}$ and arbitrary functions $f, r \in L^2(Y^n, \gamma^\otimes_n)$, we have that

$$U_T \left( f \otimes \bigotimes_{j>n} 1_{B_j} \right) = \lim_{m \to \infty} (U_T)^{\otimes n} f \otimes \left( \bigotimes_{j=n+1}^{m} 1_{TB_j} \right) \otimes \bigotimes_{j>m} 1_{B_j}$$

and

$$\langle U_T \left( f \otimes \bigotimes_{j>n} 1_{B_j} \right), r \otimes \bigotimes_{j>n} 1_{B_j} \rangle = \frac{\langle (U_T)^{\otimes n} f, r \rangle \prod_{j=1}^{n} \gamma(B_j)}{\prod_{j>n} \gamma(TB_j \cap B_j) \gamma(B_j)}.$$

The above limit is considered in the strong topology in $L^2(X, \gamma^B)$.

**IDPFT-actions**

IDPFT-actions were introduced in [DaLe] in the case, where $G = \mathbb{Z}$. In [DaKa], IDPFT actions of arbitrary discrete countable groups were studied. In this paper we consider IDPFT-actions for arbitrary locally compact second countable groups.

**Definition 2.8.** Let $T_n = (T_n(g))_{g \in G}$ be an ergodic measure preserving $G$-action on a standard probability space $(Y_n, \mathcal{C}_n, \nu_n)$, let $\mu_n$ be a probability measure on $\mathcal{C}_n$ and let $\mu_n \sim \nu_n$ for each $n \in \mathbb{N}$. We put $(X, \mathcal{B}, \mu) := \bigotimes_{n=1}^{\infty} (Y_n, \mathcal{C}_n, \mu_n)$, $T(g) := \bigotimes_{n=1}^{\infty} T_n(g)$ and $T := (T(g))_{g \in G}$. If $\mu \circ T(g) \sim \mu$ for each $g \in G$ then the nonsingular dynamical system $(X, \mathcal{B}, \mu, T)$ is called an infinite direct product of finite types (IDPFT).

We will need the following fact, extending [DaLe] Proposition 2.3] from $\mathbb{Z}$-actions to arbitrary $G$-actions.

**Proposition 2.9.** Let $(X, \mathcal{B}, \mu, T)$ be an IDPFT system as in Definition 2.8. If $T_n$ is mixing for each $n \in \mathbb{N}$ then $T$ is either totally dissipative or conservative. If $T$ is conservative then $T$ is sharply weak mixing.
Proof. It follows from Corollary 2.4 that
\[ \mathcal{D}(T) = Y_1 \times \cdots \times Y_n \times \mathcal{D} \left( \bigotimes_{j > n} T_j \right) \mod 0 \]
for each \( n > 0 \). By Kolmogorov’s 0-1 law that either \( \mu(\mathcal{D}(T)) = 0 \) and hence \( T \) is conservative or \( \mu(\mathcal{D}(T)) = 1 \) and hence \( T \) is totally dissipative. Thus, the first claim is proved. We do not provide a proof for the second claim because it is an almost verbal repetition of the proof of [DaLe, Proposition 2.3]: just replace the reference to [DaLe, Theorem B] there with a reference to Lemma 2.3. Of course, \( \mu \) is not concentrated on a single orbit. \( \square \)

**Poisson suspensions** (see [LaPe] and [Ro] for details)

Let \((X, \mathcal{B})\) be a standard Borel space and let \( \mu \) be an infinite \( \sigma \)-finite nonatomic measure on \( X \). Let \( X^* \) be the set of \( \mathbb{Z}_+ \)-valued (\( \sigma \)-finite) measures on \( X \). For each subset \( A \in \mathcal{B} \) with \( 0 < \mu(A) < \infty \), we define a mapping \( N_A : X^* \to \mathbb{R} \cup \{+\infty\} \) by setting \( N_A(\omega) := \omega(A) \). Let \( \mathcal{B}^* \) stand for the smallest \( \sigma \)-algebra on \( X^* \) such that the mappings \( N_A \) are all \( \mathcal{B}^* \)-measurable. There is a unique probability measure \( \mu^* \) on \((X^*, \mathcal{B}^*, \mu)\) satisfying the following two conditions:

- the measure \( \mu^* \circ N_A^{-1} \) is the Poisson distribution with parameter \( \mu(A) \) for each \( A \in \mathcal{B} \) with \( \infty > \mu(A) > 0 \),

- given a finite family \( A_1, \ldots, A_q \) of mutually disjoint subsets \( A_1, \ldots, A_q \) of \( X \) of finite positive measure, the corresponding random variables \( N_{A_1}, \ldots, N_{A_q} \) defined on the space \((X^*, \mathcal{B}^*, \mu^*)\) are independent.

Then \((X^*, \mathcal{B}^*, \mu^*)\) is a Lebesgue space. The mapping \( N_A \) is finite \( \mu^* \)-almost everywhere for each \( A \in \mathcal{B} \) with \( \infty > \mu(A) > 0 \). Moreover, for \( \mu^* \)-a.e. \( \omega \), there exist countably many points \( x_j \in X, \ j \in \mathbb{N} \), such that \( \omega = \sum_{j \in \mathbb{N}} \delta_{x_j} \).

For each \( \mu \)-preserving \( G \)-action \( T = (T_g)_{g \in G} \), we define a \( G \)-action \( T^* = (T_g^*)_{g \in G} \) on \((X^*, \mathcal{B}^*, \mu^*)\) by setting
\[ T_g^* \omega := \omega \circ T_g^{-1} \quad \text{for all } \omega \in X^* \text{ and } g \in G. \]

Then \( T^* \) preserves \( \mu^* \).

**Definition 2.10.** The dynamical system \((X^*, \mathcal{B}^*, \mu^*, T^*)\) is called the Poisson suspension of \((X, \mathcal{B}, \mu, T)\). A probability preserving \( G \)-action is called Poisson if it is isomorphic to a Poisson suspension of some infinite \( \sigma \)-finite measure preserving \( G \)-action.
Unitary representations of $G$ and Koopman representations of measure preserving actions

Let $V = (V(g))_{g \in G}$ be a weakly continuous unitary representation of $G$ in a separable Hilbert space $H$. We will always assume that $V$ is a complexification of an orthogonal representation of $G$ in a real Hilbert space.

**Definition 2.11.** $V$ is called *weakly mixing* if $V$ has no nontrivial finite dimensional invariant subspaces.

The Fock space $F(H)$ over $H$ is the orthogonal sum $\bigoplus_{n=0}^{\infty} H \otimes_n$, where $H \otimes_n$ is the $n$-th symmetric tensor power of $H$ when $n > 0$ and $H \otimes_0 := \mathbb{C}$. By $\exp V = (\exp V(g))_{g \in G}$ we denote the corresponding unitary representation of $G$ in $F(H)$, i.e. $\exp V(g) := \bigoplus_{n=0}^{\infty} V(g) \otimes_n$ for each $g \in G$ (see [Gu] for details).

Let $T = (T_g)_{g \in G}$ be a measure preserving $G$-action on a $\sigma$-finite nonatomic standard measure space $(X, \mathcal{B}, \mu)$. Denote by $U_T = (U_T(g))_{g \in G}$ the associated (weakly continuous) unitary Koopman representation of $G$ in $L^2(X, \mu)$:

$$U_T(g) f := f \circ T_{g}^{-1}, \quad \text{for all } g \in G.$$  

If $\mu(X) < \infty$, we let

$$L^2_0(X, \mu) := L^2(X, \mu) \ominus \mathbb{C} = \left\{ f \in L^2(X, \mu) \mid \int_X f \, d\mu = 0 \right\}.$$  

We will need the following fact.

**Fact 2.12.** Let $V$ and $(X, \mathcal{B}, \mu, T)$ be as above in this subsection.

(i) If $\mu(X) < \infty$ then $T$ is weakly mixing if and only if $U_T \mid L^2_0(X, \mu)$ is weakly mixing [BeRo].

(ii) $V$ is weakly mixing if and only if $(\exp V) \mid (F(H) \ominus \mathbb{C})$ is weakly mixing [GlWe1, Theorem A3].

(iii) If $\mu(X) = \infty$ then $U_T^*$ is canonically unitarily equivalent to $\exp U_T$ [Ro].

(iv) If $(Y, \mathcal{C}, \nu, S)$ denote the Gaussian dynamical system ($G$-action) associated with $V$ then $U_S$ is canonically unitarily equivalent to $\exp V$ [Gu].

(v) $V$ is weakly mixing if and only if there is a sequence $g_n \to \infty$ in $G$ such that $V(g_n) \to 0$ weakly as $n \to \infty$ [BeRo, Corollary 1.6, Theorem 1.9].
3 The Haagerup property

In this section we prove Theorem B (and hence Theorem A) and Corollary C. Prior to this we state a folklore proposition.

Proposition 3.1. Let $S = (S_g)_{g \in G}$ be a measure preserving $G$-action on an infinite $\sigma$-finite standard measure space $(Z, \mathcal{B}, \kappa)$. Let $\mathcal{B}_0 \subset \mathcal{B}$ stand for the ring of subsets of finite measure. Let $Z_1 \subset Z_2 \subset \cdots$ be a sequence of subsets from $\mathcal{B}_0$ such that

- $\bigcup_{n=1}^\infty Z_n = Z$ and
- for each $n > 0$, there is a finite partition $\mathcal{P}_n$ of $Z_n$ into subsets of equal measure such that the sequence $(\mathcal{P}_n)_{n=1}^\infty$ approximates $(\mathcal{B}, \kappa)$, i.e. for each $B \in \mathcal{B}_0$ and $\epsilon > 0$, there is $N > 0$ such that if $n > N$ then there exists a $\mathcal{P}_n$-measurable subset $B_n$ with $\kappa(B \triangle B_n) < \epsilon$.

(i) If for each compact subset $K \subset G$, an integer $n > 0$ and a $\mathcal{P}_n$-atom $A$, there exist a finite family $g_1, \ldots, g_l \in G \setminus K$ and mutually disjoint subsets $A_1, \ldots, A_l$ of $A$ such that $\bigcup_{i=1}^l S_{g_i}A_i \subset A$ and $\kappa(\bigcup_{i=1}^l A_i) > 0.1\kappa(A)$ then $S$ is conservative.

(ii) If for each $n > 0$ and every pair of $\mathcal{P}_n$-atoms $A$ and $B$, there exist a finite family $g_1, \ldots, g_l \in G$ and mutually disjoint subsets $A_1, \ldots, A_l$ of $A$ such that $\bigcup_{i=1}^l S_{g_i}A_i \subset B$ and $\kappa(\bigcup_{i=1}^l A_i) > 0.1\kappa(A)$ then $S$ is ergodic.

Idea of the proof. (i) Take a subset $A \subset Z$ of positive finite measure and a compact subset $K \subset G$. Our purpose is to find $g \in G \setminus K$ such that $\mu(A \cap S_gA) > 0$. For that, we select $n > 0$ and a $\mathcal{P}_n$-measurable subset $B_n$ such that $\mu(A \triangle B_n) \leq 0.001\mu(A)$. Then there is a $\mathcal{P}_n$-atom $B \subset B_n$ such that $\mu(A \cap B) > 0.999\mu(B)$. Utilizing the condition of (i), we can find a subset $C \subset B$ and $g \in G \setminus K$ such that $\mu(A \cap C) > 0.9\mu(C)$ and $\mu(A \cap S_gC) > 0.9\mu(C)$. This yields that $\mu(A \cap S_gA) > \mu((A \cap C) \cap (S_gA \cap C)) > 0$.

(ii) is proved in a similar way.

Proof of Theorem B. The implications (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i) are trivial.

We now prove (i)$\Rightarrow$(ii). Let $G$ have the Haagerup property. By [Ch–Va, Theorem 2.2.2], there is a mixing measure preserving free $G$-action $T = (T_g)_{g \in G}$ on a standard probability space $(Y, \mathcal{C}, \gamma)$ and a $T$-asymptotically invariant sequence $B := (B_n)_{n=1}^\infty$ such that $\gamma(B_n) = 0.5$ for each $n \in \mathbb{N}$. 

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Passing to a subsequence, if necessary, we may (and will) assume that for each compact subset \( K \subset G \),

\[
(3.1) \quad \sum_{n=1}^{\infty} \sup_{g \in K} \gamma(T_g B_n \triangle B_n) < +\infty.
\]

We now let \((X, \mathcal{F}) := (Y, \mathcal{C}) \otimes \mathbb{N}\). Endow this standard Borel space with the \((\sigma\text{-finite})\) restricted infinite power \(\gamma^B\) of \(\gamma\) with respect to \(B\). Since \(\prod_{n \in \mathbb{N}} \gamma(B_n) = 0\), it follows that \(\gamma^B(X) = \infty\). For \(g \in G\), let \(T_g := T_g^\infty\). In view of (3.1), it follows from Proposition 2.6 that \(\gamma^B \circ T_g = \gamma^B\). Thus, \(T := (T_g)_{g \in G}\) is a measure preserving \(G\)-action on \((X, \mathcal{F}, \gamma^B)\). Since the map \(X \ni x = (y_n)_{n=1}^{\infty} \to y_1 \in Y\) intertwines \(T\) with \(T\) and \(T\) is free, \(T\) is free too. For each \(n \in \mathbb{N}\), define a subset \(B_n \subset X\) in the same way as in §1. It follows from (2.1) that the sequence \((B_n)_{n \in \mathbb{N}}\) is \(T\)-Følner. Moreover, it is exhausting and \(\gamma^B(B_n) = 2^n\) for each \(n\). To show that \(T\) is of 0-type, we first note that since \(T\) is mixing then for each pair of integers \(n < m\),

\[
\lim_{g \to \infty} \prod_{j=n}^{m} \frac{\gamma(T_g B_j \cap B_j)}{\gamma(B_j)} = \prod_{j=n}^{m} \gamma(B_j) = 2^{-m+n-1}.
\]

Hence given two functions \(f, r \in L^2(Y, \gamma^\otimes n)\), we deduce from Lemma 2.7 that

\[
\lim_{g \to \infty} \left\langle U_T \left( f \otimes \bigotimes_{j > n} 1_{B_j} \right), r \otimes \bigotimes_{j > n} 1_{B_j} \right\rangle = 0.
\]

This implies that \(T\) is of 0-type\(^2\).

Thus, it remains to show that \(T\) is sharply weak mixing. To this purpose, we first prove that upon a replacement of \(\gamma^B\) with an equivalent probability measure, \(T\) is an IDPFT. For that, we choose a sequence of reals \((\epsilon_n)_{n=1}^{\infty}\) such that \(0 < \epsilon_n < 1\) for each \(n\) and \(\sum_{n=1}^{\infty} \epsilon_n < \infty\). Then we define, for each \(n \in \mathbb{N}\), a Borel function \(\phi_n : Y \to \mathbb{R}_+\) by setting

\[
\phi_n := 2\epsilon_n 1_{Y \setminus B_n} + 2(1 - \epsilon_n) 1_{B_n}.
\]

Since \(\gamma(B_n) = \frac{1}{2}\), a simple verification yields that \(\int_Y \phi_n d\gamma = 1\). Denote by \(\mu_n\) the probability measure on \(Y\) such that \(\mu_n \sim \gamma\) and \(\frac{d\mu_n}{d\gamma} := \phi_n\). We now recall that given two probability measures \(\alpha\) and \(\beta\) on \((Y, \mathcal{C})\) such that \(\alpha \prec \gamma\) and \(\beta \prec \gamma\), the squared Hellinger distance between \(\alpha\) and \(\beta\) is

\[
H^2(\alpha, \beta) := \frac{1}{2} \int_Y \left( \sqrt{\frac{d\alpha}{d\gamma}} - \sqrt{\frac{d\beta}{d\gamma}} \right)^2 d\gamma.
\]

\(^2\)It is worthy to note that at this point we have proved completely Theorem A.
A straightforward computation yields that
\[ H^2\left( \frac{1}{\gamma(B_n)} \gamma \mid B_n, \mu_n \right) = \frac{1}{2} \int_Y (\sqrt{2} \cdot 1_{B_n} - \sqrt{\phi_n})^2 d\gamma = \frac{1}{2} \left( (\sqrt{1 - \epsilon_n} - 1)^2 + \epsilon_n \right) \]
and hence
\[ \sum_{n=1}^{\infty} H^2\left( \frac{1}{\gamma(B_n)} \gamma \mid B_n, \mu_n \right) < +\infty. \] Therefore, by [Hi, Theorems 3.9, 3.6], \( \gamma^B \sim \bigotimes_{n=1}^{\infty} \mu_n \). Hence \( T \) is an IDPFT, as claimed. We now deduce from Proposition 2.9 that \( T \) is either sharply weak mixing or totally dissipative. Therefore, to complete the proof of (i) \( \Rightarrow \) (ii), it would be enough to show that \( T \) is conservative. Unfortunately, we can not prove this fact. However we observe that if one replaces \( B \) with an arbitrary infinite subsequence \( B' \) and associates a \( G \)-action \( T' \) with \( B' \) in the same way as we associated \( T \) with \( B \) then \( T' \) possesses the same properties that we established for \( T \); it is free, 0-type, IDPFT and it admits an exhausting \( T' \)-Følner sequence. Therefore, to complete the proof of (i) \( \Rightarrow \) (ii), it suffices (in view of Proposition 2.9) to select a subsequence \( B' \) of \( B \) such that the corresponding \( G \)-action \( T' \) is conservative.

Let \( (\mathcal{P}_n)_{n=1}^{\infty} \) be a sequence of finite partitions of \( Y \) into Borel subsets of equal measure such that the sequence \( (\mathcal{P}_n)_{n=1}^{\infty} \) approximates \( (\mathcal{C}, \gamma) \). Fix a sequence \( (K_n)_{n=1}^{\infty} \) of compact subsets of \( G \) such that \( K_1 \subset K_2 \subset \cdots \) and \( \bigcup_{n=1}^{\infty} K_n = G \). Since the \( G \)-action \( (T_g^g)_{g \in G} \) on \( Y^n \) preserves the probability measure \( \gamma^\otimes n \), this action is conservative. Hence for each subset \( A \subset Y^n \), there is \( g \in G \setminus K_n \) such that \( \gamma^\otimes n(T_g^g A \cap A) > 0 \). Moreover, if we fix a countable dense subgroup \( G' \) of \( G \) then we can additionally claim that \( g \in G' \). Applying a standard exhaustion argument, we construct a sequence \( (A_m)_{m=1}^{\infty} \) of mutually disjoint subsets of \( A \) and a sequence \( (g_m)_{m=1}^{\infty} \) of elements of \( G' \) such that \( A = \bigcup_{m=1}^{\infty} A_m = \bigcup_{m=1}^{\infty} T_{g_m}^g A_m \) and \( g_m \in G' \setminus K_n \) for each \( m \in \mathbb{N} \). Hence there is \( M > 0 \) such that
\[
\left( \bigcup_{m=1}^{M} A_m \right) \cup \left( \bigcup_{m=1}^{M} T_{g_m}^g A_m \right) \subset A \quad \text{and} \quad \gamma^\otimes n \left( \bigcup_{m=1}^{M} A_m \right) = \gamma^\otimes n \left( \bigcup_{m=1}^{M} T_{g_m}^g A_m \right) > 0.5 \gamma^\otimes n(A).
\]

Then for every \( n > 0 \), we apply this argument to each atom \( P \) of the partition \( (\mathcal{P}_n)_{n=1}^{\infty} \) of \( Y^n \) to determine a finite subset \( F_n \subset G \setminus K_n \) and a family of measured subsets \( (P_f)_{f \in F_n} \) of \( P \) satisfying the following conditions:

- \( P_f \cap P_h = \emptyset \) and \( (T_f)^{\otimes n} P_f \cap (T_h)^{\otimes n} P_h = \emptyset \) if \( f \neq h \),

- \( \bigcup_{f \in F_n} (T_f)^{\otimes n} P_f \subset P \) and
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serving free notation, we let $F$. Then $P(T_\gamma B_{k,m} \cap B_{k,m}) = 0.5 \gamma \otimes (\gamma(B_{k,m}))$ for each $n$.

This is possible because $B$ is $T$-asymptotically invariant. For simplicity of notation, we let $F_n := F_{k_n}$, $B_n := B_{k_n}$, $B' := (B_n')_{n=1}^\infty$,

$$(B')^n := Y^n \times B_{n+1}' \times B_{n+2}' \cdots ,$$

$P_n := P_{k_n}$ and $P_f := P_f \in P_n$ if $f \in F_n$. Then $(P_n')_{n=1}^\infty$ is a sequence of finite partitions of $Y$ into Borel subsets of equal measure and the sequence $(P_n')_{n=1}^\infty$ approximates $(\gamma, \gamma_B)$. We set

$P_n^* := \{P' \times B_{n+1} ' \times B_{n+2} ' \times \cdots \mid P' \in (P_n')_{n=1}^\infty\}.$

Then $P_n^*$ is a finite partition of $(B')^n$ into subsets of equal measure for each $n \in \mathbb{N}$. Moreover, the sequence $(P_n^*)_{n=1}^\infty$ approximates $(\mathcal{B}, \gamma_B)$.

Take an atom $P^* \in P_n^*$ for some $n \in \mathbb{N}$. Then $P^* = P' \times B_{n+1} ' \times B_{n+2} ' \times \cdots$ for an atom $P' \in (P_n')_{n=1}^\infty$. We now set for each $f \in F_n$,

$$P_f^* := P_f' \times (B_{n+1}' \cap T_{f-1} B_{n+1}') \times (B_{n+2}' \cap T_{f-1} B_{n+2}') \times \cdots .$$

Then $(P_f^*)_{f \in F_n}$ are mutually disjoint Borel subsets of $P^*$ and $(T_f P_f^*)_{f \in F_n}$ are also mutually disjoint Borel subsets of $P^*$. Moreover,

$$\gamma_B\left(\bigcup_{f \in F_n} P_f^*\right) = \sum_{f \in F_n} \frac{\gamma \otimes (P_f')}{{\prod_{j=1}^n \gamma(B_j')}} \frac{\gamma(T_f B_k' \cap B_k')}{\gamma(B_k')} > \frac{1}{2} \sum_{f \in F_n} \frac{\gamma \otimes (P_f')}{{\prod_{j=1}^n \gamma(B_j')}} $$

(3.2)

It follows from this and Proposition 3.1(i) that $T'$ is conservative, as desired.

To prove (i) $\Rightarrow$ (iii), we first note that there exists a mixing measure preserving free $G$-action $T$ on a standard probability space $(Y, \mathcal{C}, \gamma)$ that admits, for each $m \in \mathbb{N}$, a $T$-asymptotically invariant sequence $(A_{n,m})_{n=1}^\infty$ with
\[ \gamma(A_{n,m}) = 2^{-m} \text{ for all } n, m \in \mathbb{N}. \]

Utilizing this action \( T \) and the sequence \( B := (A_{n,1})_{n=1}^{\infty} \) we construct the dynamical system \((X, \mathcal{B}, \gamma^{A'}, T')\) as above. The standard diagonalization argument applied to \((A_{n,m})_{n=1}^{\infty}\) yields a \(T\)-Følner sequence \((A_n)_{n=1}^{\infty}\) such that \(\gamma(A_n) = 2^{-n}\) for each \(n \in \mathbb{N}\). We now set \(A^n := Y^{n-1} \times A_n \times B'_{n+1} \times B'_{n+2} \times \cdots \subset X\). Then

\[
\gamma^{B'}(A^n) = \frac{\gamma(A_n)}{\prod_{j=1}^{n} \gamma(B'_j)} = 1 \quad \text{and} \\
\gamma^{B'}(T'_g A^n \cap A^n) = \left( \prod_{j=1}^{n-1} \gamma(B'_j) \right) \frac{\gamma(A_n \cap T_g A_n)}{\gamma(B'_n)} \prod_{j>n} \frac{\gamma(B'_j \cap T_g B'_j)}{\gamma(B'_j)}
\]

Since \(\sup_{g \in K} \prod_{j>n} \frac{\gamma(B'_j \cap T_g B'_j)}{\gamma(B'_j)} \to 1\) for each compact subset \(K \subset G\) in view of (3.1), we obtain that \(\sup_{g \in K} \gamma^{B'}(T'_g A^n \cap A^n) \to 1\) as \(n \to \infty\), as desired.

Proof of Corollary C. The “if” part follows from [Ch–Va, Theorem 2.2.2]. We prove the “only if” part. Let \(G\) has the Haagerup property. By Theorem B, there exists a sharply weak mixing 0-type measure preserving free \(G\)-action \(T\) on an infinite \(\sigma\)-finite standard measure space \((X, \mathcal{B}, \mu)\) admitting a \(T\)-Følner sequence \((A_n)_{n=1}^{\infty}\) such that \(\mu(A_n) = 1\) for all \(n \in \mathbb{N}\). Denote by \((X^*, \mathcal{B}^*, \mu^*, T^*)\) the Poisson suspension of \((X, \mathcal{B}, \mu, T)\). For each \(n \in \mathbb{N}\), we set \([A_n]_0 := \{\omega \in X^* \mid \omega(A_n) = 0\}\). Then

\[
\mu^*([A_n]_0) = e^{-\mu(A_n)} = e^{-1}
\]

and

\[
\mu^*(T^*_g [A_n]_0 \cap [A_n]_0) = \mu^*([T_g A_n \cup A_n]_0) = e^{-\mu(T_g A_n \cup A_n)}.
\]

Since for each compact subset \(K \subset G\),

\[
\sup_{g \in K} |\mu(T_g A_n \cup A_n) - \mu(A_n)| \to 0
\]

3Indeed, let a mixing action \(S = (S_g)_{g \in G}\) on a standard probability space \((Z, \kappa)\) have an asymptotically invariant sequence \((A_n)_{n=1}^{\infty}\) with \(\kappa(A_n) = \frac{1}{n}\) as in [Ch–Va] Theorem 2.2.2]. Consider an infinite product \(G\)-action \(R = ((S_g)_{g \in G})\). Of course, \(R\) is mixing. For each \(m > 1\), let \(A_{n,m} := (A_n)^R \times Y \times Y \times \cdots \subset Y^N\). The sequence \((A_{n,m})_{n=1}^{\infty}\) is \(R\)-asymptotically invariant and \(\kappa^R(A_{n,m}) = 2^{-m}\).
we obtain that
\[ \sup_{g \in K} |\mu^*(T_g^* [A_n]_0 \cap [A_n]_0) - \mu^*([A_n]_0)| \to 0 \]
as \( n \to \infty \). Thus, the sequence \([A_n]_0\) is nontrivial and \( T^* \)-asymptotically invariant. Thus \( T^* \) is not strongly ergodic. Since \( T \) is of 0-type, it follows from Fact 1.12(iii) that 
\[ U_T^*(g) \rvert_{L^2_0(X^*, \mu^*)} \to 0 \]
weakly as \( g \to \infty \). Hence \( T^* \) is mixing. \( \square \)

4 Pairs of groups with Kazhdan property (T)

In this section we prove Theorem D and Corollary E. Prior to this we note that if \( S \) is a weakly mixing measure preserving \( G \)-action on an infinite \( \sigma \)-finite standard measure space \((Y, \mathcal{F}, \nu)\) then the Koopman representation \( U_S \) is weakly mixing. Indeed, suppose that \( U_S \) contains a finite dimensional subspace. Then there is a unitary representation \( V \) of \( G \) in a finite dimensional Hilbert space \( H \) and a nontrivial mapping \( F : Y \to H \) such that \( F(S_g y) = V(g) F(y) \) for each \( g \in G \) at a.e. \( y \in Y \). By [GlWe2, Theorem 1.1], \( F \) is constant a.e. Since \( \nu \) is infinite and the mapping \( Y \ni y \mapsto \langle F(y), h \rangle \) belongs to \( L^2(Y, \nu) \) for each \( h \in H \), it follows that \( F = 0 \). Therefore \( U_S \) is weakly mixing, as claimed. It now follows from Fact 2.12(v) that \( S \) is of weak 0-type. Therefore Theorem D implies [Jo2, Theorem 1.5].

**Proof of Theorem D.** (i) If there is an \( S \)-Følner sequence then \((*)\) holds for the Koopman representation \( U_S \) of \( G \). Hence the restriction of \( U_S \) to \( H \) has a nontrivial invariant vector. Hence the action \( S \rvert H \) has an invariant subset of positive finite measure. This contradicts to the condition of (i).

(ii) Let \( H \subset G \) do not have property (T). The beginning of our argument is a slight modification of the proof of [Jo2, Theorem 1.5]. There exists a conditionally negative definite function \( \psi : G \to \mathbb{R}_+ \) which is unbounded on \( H \) [Jo1, Theorem 1.2(a4')]. By the Schoenberg theorem, for each \( t > 0 \), the function \( \phi_t := e^{-t^{-1} \psi} : G \to \mathbb{R}_+ \) is positive definite. Hence the GNS-construction yields a triplet \((V_t, \mathcal{H}_t, \xi_t)\), consisting of a separable Hilbert space \( \mathcal{H}_t \), a unitary representation \( V_t \) of \( G \) in \( \mathcal{H}_t \) and a \( V_t \)-cyclic unit vector \( \xi_t \in \mathcal{H}_t \) such that \( \langle V_t(g) \xi_t, \xi_t \rangle = \phi_t(g) \) for each \( g \in G \). Since \( \psi \) is unbounded on \( H \), there is a sequence \((h_k)_{k=1}^\infty \) of elements in \( H \) such that \( \psi(h_k) \to +\infty \). As was shown in the proof of [Jo1, Lemma 2.1], \( \psi(g_1 h_k g_2) \to +\infty \) as \( k \to \infty \) for all \( g_1, g_2 \in G \). Since \( \xi_t \) is \( V_t(G) \)-cyclic, it follows that \( V_t(h_k) \to 0 \) weakly.

\(^4\)This is a formula on the first page of the paper.
as \( k \to \infty \) for each \( t > 0 \). Thus, the restriction of \( V_t \) to \( H \) is weakly mixing by Fact 2.12(v). Since \( \phi_t \) takes only real values, \( V_t \) is the complexification of an orthogonal representation of \( G \). We will now argue as in the case (a) of the proof of the main result from [CoWe]. Let

\[
\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n \quad \text{and} \quad V := \bigoplus_{n=1}^{\infty} V_n.
\]

Of course, the restriction of \( V \) to \( H \) is weakly mixing. Moreover, \( V \) is also the complexification of an orthogonal representation of \( G \). Denote by \( T = (T_g)_{g \in G} \) the corresponding Gaussian measure preserving \( G \)-action on a standard probability space \((Y, \mathcal{C}, \gamma)\). By Fact 2.12(iv), the associated Koopman representation \( U_T \) of \( G \) is unitarily equivalent to \( \exp V \). Since \( \exp(V \upharpoonright H) = (\exp V) \upharpoonright H \), it follows from Fact 2.12(ii) that the unitary representation \( (U_T(h))_{h \in H} \) is weakly mixing on \( L^2_0(Y, \gamma) \). Hence \( (T_h)_{h \in H} \) is weakly mixing by Fact 2.12(i). We recall that \( \mathcal{H} \) is a subspace of \( \mathcal{F}(\mathcal{H}) \) which is canonically isomorphic to \( L^2(Y, \gamma) \). Since \( \mathcal{H}_n \) is a subspace of \( \mathcal{H} \), we obtain that \( \xi_n \in L^2(Y, \gamma) \), \( \xi_n \) is a centered Gaussian variable on \( Y \) and

\[
\langle U_T(g)\xi_n, \xi_n \rangle = \langle V_n(g)\xi_n, \xi_n \rangle = \phi_n(g) = e^{-\psi(g)/n} \quad \text{for each} \quad g \in G.
\]

Let \( B_n := \{ y \in Y \mid \xi_n(y) > 0 \} \). Denote by \( \vartheta_n \) the distribution of the random variable \( \xi_n \). Then \( \gamma(B_n) = \vartheta_n((0, +\infty)) = 0.5 \) by the symmetry of \( \vartheta_n \). As in the proof of the main result from [CoWe], we obtain that

\[
\gamma(T_n B_n \Delta B_n) = \frac{\arccos(U_T(g)\xi_n, \xi_n)}{\pi} = \frac{\arccos e^{-\psi(g)/n}}{\pi}.
\]

This yields that the sequence \( \textbf{B} := (B_n)_{n=1}^{\infty} \) is \( T \)-asymptotically invariant. Moreover, without loss of generality we may (and will) assume that (3.1) holds for \( \textbf{B} \).

Consider now the dynamical system \((X, \mathcal{B}, \gamma^B, \textbf{T})\) as in the proof of Theorem B. Then \( \gamma^B(X) = \infty \) and \( \textbf{T} \) preserves \( \gamma^B \) by Proposition 2.6. As was shown in the proof of Theorem B, the sequence \((\textbf{B}^n)_{n \in \mathbb{N}} \) (defined there) is \( \textbf{T} \)-Følner and exhausting.

Reasoning as in the proof of Theorem B, we have only to choose a subsequence \( \textbf{B}' \) in \( \textbf{B} \) in such a way that the restriction of \( \textbf{T}' \) (we recall again that \( \textbf{T}' \) is determined by \( \textbf{B}' \)) to \( H \) is weakly mixing. Let \((\mathcal{P}_n)_{n=1}^{\infty} \) denote the same sequence of finite partitions as in the proof of Theorem B. Since \((T_h)_{h \in H} \) is weakly mixing, the \( H \)-action \((T_h^\otimes 2n)_{h \in H} \) on the probability space \((X^{2n}, \gamma^\otimes 2n) \) is ergodic. Hence for every \( n > 0 \), there is a subset \( F_n \subset H \) such that for every two atoms \( P, Q \) of the partition \((\mathcal{P}_n)^\otimes 2n \) of \( B_n \times B_n \), there is a family of measurable subsets \((P_f)_{f \in F_n} \) of \( P \) such that
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\[ P_f \cap P_h = \emptyset \text{ and } (T_f)^{\otimes 2n} P_f \cap (T_f)^{\otimes 2n} P_f = \emptyset \text{ if } f \neq h, \]

\[ \bigcup_{f \in F} (T_f)^{\otimes 2n} P_f \subset Q \text{ and } \]

\[ \gamma^{\otimes 2n}(\bigcup_{f \in F_n} P_f) > 0.5 \gamma^{\otimes 2n}(P). \]

We now select a sequence \( k_1 < k_2 < \ldots \) of integers in such a way that

\[ \max_{f \in F_n} \prod_{m > n} \frac{\gamma(T_f B_{km} \cap B_{km})}{\gamma(B_{km})} > 0.5 \quad \text{for each } n. \]

As in the proof of Theorem B, we let \( F'_n := F_{k_n}, B'_n := B_{k_n}, B' = (B'_n)_{n=1}^{\infty}, (B')^n := Y^n \times B'_{n+1} \times B'_{n+2} \times \cdots, \mathcal{P}'_n := \mathcal{P}_{k_n} \) and \( P'_f := P_f \in \mathcal{P}'_n \) if \( f \in F'_n \).

We also set

\[ P^* := P' \times (B'_{n+1} \times B'_{n+2} \times \cdots)^\otimes \subset (B')^n \times (B')^n, \]

\[ Q^* := Q' \times (B'_{n+1} \times B'_{n+2} \times \cdots)^\otimes \subset (B')^n \times (B')^n \quad \text{and} \]

\[ P^*_f := P'_f \times ((B'_{n+1} \cap T^{-1}_f B'_{n+1}) \times (B'_{n+2} \cap T^{-1}_f B'_{n+2}) \times \cdots)^\otimes \]

for all atoms \( P', Q' \in (\mathcal{P}'_n)^{\otimes 2n} \). Then \( \mathcal{P}'_n := (P^*)_{P' \in (\mathcal{P}'_n)^{\otimes 2n}} \) is a finite partition of \( (B')^n \times (B')^n \) into subsets of equal measure and the sequence \((\mathcal{P}'_n)_{n=1}^{\infty}\) approximates \((\mathcal{B} \otimes \mathcal{B}, \gamma^{B'} \otimes \gamma^{B'}).\) We also note that \( (P^*_f)_{f \in F_n} \) are mutually disjoint subsets of \( P^* \) and \((T_f \times T_f) P^*_f \) are mutually disjoint subsets of \( Q^*. \)

Arguing as in 3.2, we obtain that

\[ (\gamma^{B'} \otimes \gamma^{B'})(\bigcup_{f \in F_n} P^*_f) > \frac{1}{8}(\gamma^{B'} \otimes \gamma^{B'})(P^*). \]

It follows from that and Proposition 3.1(ii) that the \( H \)-action \((T'_g \times T'_g)_{g \in H}\) is ergodic. Hence \( T' \mid H \) is weakly mixing by [GlWe2 Theorem 1.1]. \( \square \)

Corollary E follows from Theorem D in the same way as Corollary C follows from Theorem B.

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