Poisson Source Localization on the Plane.
Cusp Case.

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Abstract

This work is devoted to the problem of estimation of the localization of Poisson source. The observations are inhomogeneous Poisson processes registered by the $k \geq 3$ detectors on the plane. We study the asymptotic properties of the Bayes estimators in the asymptotic of large intensities. It is supposed that the intensity functions of the signals arriving in the detectors have cusp-type singularity. We show the consistency, limit distributions and the convergence of moments of these estimators.

Key words: Inhomogeneous Poisson process, Poisson source, sensors, Bayes estimators, cusp-type singularity.

1 Introduction

Suppose that we have $k \geq 3$ detectors at the points $D_j, j = 1, \ldots, k$ with the coordinates $\vartheta_j = (x_j, y_j), j = 1, \ldots, k$ on the plane and a source of emission of Poisson signals at the point $D_0$ with coordinates $\vartheta_0 = (x_0, y_0)$. We consider the problem of estimation of the position $\vartheta_0 = (x_0, y_0)$ by the observations $X = (X_1, \ldots, X_k)$ of Poisson signals $X_j = (X_j(t), 0 \leq t \leq T)$ received by detectors [7].

An example of such model is given on the Fig. 1.
This is our third work devoted to this problem of identification of localization of the source (see the Introduction in the work [4] where we give the review of the engineering literature on this subject).

The intensity function $\lambda_{j,n}(\vartheta_0, t)$ of the Poisson process received by the $j$-th detector taken in this work and in [1], [4] is of the form

$$\lambda_{j,n}(\vartheta_0, t) = n\lambda_j(t - \tau_j) \mathbb{1}_{t \geq \tau_j} + n\lambda_0, \quad 0 \leq t \leq T. \quad (1)$$

Here $\tau_j = \tau_j(\vartheta_0)$ is the instant of arriving of the Poisson signal at the $j$-th detector, which is calculated by the formula $\tau_j(\vartheta_0) = \nu^{-1}\|\vartheta_j - \vartheta_0\|$, where $\nu > 0$ is the known rate of propagation of the signal and $\|\cdot\|$ is Euclidean norm on the plane. The Poisson signals are received in the presence of Poisson noise of the known intensity $n\lambda_0 > 0$. The exact calculation of the error of estimation $E_{\vartheta_0}\|\hat{\vartheta} - \vartheta_0\|^2$ ($\hat{\vartheta}$ is some estimator) in this essentially non linear statistical problem is very difficult problem. Moreover the most interesting are the situations where the errors of estimation are small. To obtain small errors and have possibility to calculate it we have to consider one or another type of asymptotics. That is why we introduce the large parameter $n$ in the intensity function (1) and study the errors of estimation in the asymptotics $n \to \infty$. This means that the signal and noise are sufficiently large and the estimators $\hat{\vartheta} = \hat{\vartheta}_n$ take values not too far from the true value: $E_{\vartheta_0}\|\hat{\vartheta} - \vartheta_0\|^2 = o(1)$. Recall that the similar mathematical model can be used in the problem of GPS-localization on the plane. In this case we have $k$ emitters of the Poisson signals and an object which receives these signals. The positions of the emitters are known and the problem is in the estimation of the position of the object by the observations of the signals. The intensity functions of the received Poisson signals depend on the distance between the emitters and the object and the receiver has to defined its position by these observations (see, e.g. [12]).

The goal of the works [1], [4] and of this one is to evaluate the errors $E_{\vartheta_0}\|\hat{\vartheta} - \vartheta_0\|^2$ and $E_{\vartheta_0}\|\tilde{\vartheta} - \vartheta_0\|^2$, where $\hat{\vartheta}_n$ is the maximum likelihood estimator
MLE) and \( \hat{\theta}_n \) is the Bayes estimator (BE) with the quadratic loss function. The difference between these three works is in the conditions of regularity of the functions \( \lambda_j(\cdot) \) and as a consequence of it the rates of convergence of the errors are different. Let us remind this class of models and errors of estimation with the help of the Poisson process with intensity function

\[
\lambda_n(\vartheta, t) = 2n \left| \frac{t - \vartheta}{\delta} \right|^\kappa \mathbb{I}_{[0 \leq t - \vartheta \leq \delta]} + 2n \mathbb{I}_{[t \geq \vartheta + \delta]} + n, \quad 0 \leq t \leq T.
\]

Here the unknown parameter \( \vartheta \) is one-dimensional, \( \vartheta \in (\alpha, \beta) \subset [0, T] \). Choosing the different values of \( \kappa \) we obtain statistical problems of different regularity. The examples of such intensities are given on the Fig. 2, where we put \( n = 1 \).

Figure 2: Intensity functions of different regularity: a) \( \kappa = \frac{5}{8} \), b) \( \kappa = \frac{1}{2} \), c) \( \kappa = \frac{1}{8} \), d) \( \kappa = 0 \), e) \( \kappa = -\frac{3}{8} \).

The cases a) and b) correspond to the regular (smooth, LAN) case. In the case c) we have cusp-type singularity. The case d) corresponds to change-point model of observations and the case e) is explosion-type singularity. The
rates of convergence of errors in these cases are

\[ a) \quad \mathbb{E} \| \hat{\vartheta}_n - \vartheta_0 \|^2 \approx \frac{C}{n}, \]
\[ b) \quad \mathbb{E} \| \hat{\vartheta}_n - \vartheta_0 \|^2 \approx \frac{C}{n \ln n}, \]
\[ c) \quad \mathbb{E} \| \hat{\vartheta}_n - \vartheta_0 \|^2 \approx \frac{C}{n^{2^{m+1}}}, \]
\[ d) \quad \mathbb{E} \| \hat{\vartheta}_n - \vartheta_0 \|^2 \approx \frac{C}{n^2}, \]
\[ e) \quad \mathbb{E} \| \hat{\vartheta}_n - \vartheta_0 \|^2 \approx \frac{C}{n^{2^{m+1}}} \]

For the case a) see [8], the case b) was considered in [1], for the case c) see [2], for the case d) see [9] and the case e) was studied in [3].

We have to note that the study of MLE and BE in all these cases was done with the help of some general results concerning the behavior of estimators developed by Ibragimov and Khasminskii [5]. Their method is based on the study of the normalized likelihood ratio random fields, which we remind below in this section.

We have to note that the study of MLE and BE in all these cases was done with the help of some general results concerning the behavior of estimators developed by Ibragimov and Khasminskii [5]. Their method is based on the study of the normalized likelihood ratio random fields, which we remind below in this section.

We have k independent observations of inhomogeneous Poisson processes \( X^n = (X_1, \ldots, X_k) \) with intensities \( \lambda_j(\vartheta_0) \) depending on \( \tau_j(\vartheta_0) \). We suppose that the position of the source \( \vartheta_0 \in \Theta \) is unknown and we have to estimate \( \vartheta_0 \) by the observations \( X^n \). Here \( \Theta \subset \mathbb{R}^2 \) is a convex bounded and open set.

It seems that the mathematical study of this class of models was not yet sufficiently developed. The statistical models of inhomogeneous Poisson process with intensity functions having discontinuities along some curves depending on unknown parameters were considered in [10], Sections 5.2 and 5.3. Statistical inference for point processes can be found in the works [6], [13] and [14].

Let us recall the definitions of the MLE and BE. The functions \( \lambda_j(\cdot) \) are bounded and the constant \( \lambda > 0 \) therefore the measures induced by the processes \( X_j \) in the space of their realizations are equivalent [11]. The likelihood ratio function \( L(\vartheta, X^n) \) is

\[
\ln L(\vartheta, X^n) = \sum_{j=1}^{k} \int_{\tau_j}^{T} \ln \left( 1 + \frac{\lambda_j(t - \tau_j)}{\lambda_0} \right) dX_j(t) - n \sum_{j=1}^{k} \int_{\tau_j}^{T} \lambda_j(t - \tau_j) dt.
\]

Of course, \( \tau_j = \tau_j(\vartheta) \) and the observations \( X^n = (X_1, \ldots, X_n) \), where \( X^n_j = (X_j(t), 0 \leq t \leq T), j = 1, \ldots, k \) are counting processes from \( k \) detectors. The maximum likelihood estimator (MLE) \( \hat{\vartheta}_n \) and Bayesian estimator (BE) \( \tilde{\vartheta}_n \) for the quadratic loss function are defined by the “usual” relations

\[
L(\hat{\vartheta}_n, X^n) = \sup_{\vartheta \in \Theta} L(\vartheta, X^n)
\]

(2)
and
\[ \tilde{\theta}_n = \frac{\int_{\Theta} \tilde{\theta} p(\tilde{\theta}) L(\tilde{\theta}, X^n) d\tilde{\theta}}{\int_{\Theta} p(\tilde{\theta}) L(\tilde{\theta}, X^n) d\tilde{\theta}}. \tag{3} \]

Here \( p(\tilde{\theta}), \tilde{\theta} \in \Theta \) is the prior density. We suppose that it is positive, continuous function on \( \Theta \). In this work we study the BE only. The case of MLE for this model of observations (two-dimensional cusp) will be considered later.

## 2 Main result

Suppose that there exists a source of Poisson signals at some point \( \vartheta_0 = (x_0, y_0) \in \Theta \subset \mathbb{R}^2 \) and \( k \geq 3 \) sensors (detectors) on the same plane located at the points \( \vartheta_j = (x_j, y_j), j = 1, \ldots, k \). The source was activated at the (known) instant \( t = 0 \) and the signals from the source (inhomogeneous Poisson processes) are registered by all \( k \) detectors. The signal arrives at the \( j \)-th detector at the instant \( \tau_j \). Of course, \( \tau_j = \tau_j(\vartheta_0) \) is the time necessary for the signal to arrive in the \( j \)-th detector defined by the relation
\[ \tau_j(\vartheta_0) = \frac{1}{\nu} \| \vartheta_j - \vartheta_0 \|, \]
where \( \nu > 0 \) is the known speed of propagation of the signal and \( \| \cdot \| \) is the Euclidean norm (distance) in \( \mathbb{R}^2 \).

The intensity function of the Poisson process \( X^n_j = (X_j(t), 0 \leq t \leq T) \) registered by the \( j \)-th detector is
\[ \lambda_{j,n}(\vartheta_0, t) = nS_j(t - \tau_j(\vartheta_0)) + n\lambda_0, \quad 0 \leq t \leq T, \tag{4} \]
where \( nS_j(t - \tau_j(\vartheta_0)) \) is the intensity function of the signal and \( n\lambda_0 > 0 \) is the intensity of the noise. We suppose that the function \( S_j(\cdot) \) of the signal can be presented as follows
\[ S_j(t - \tau_j) = \lambda_j(t - \tau_j) \left| \frac{t - \tau_j}{\delta} \right|^{\kappa} I_{\{0 \leq t - \tau_j \leq \delta\}} + \lambda_j(t - \tau_j) I_{\{t - \tau_j > \delta\}}. \tag{5} \]

Here \( \delta > 0 \) is some small parameter. This means that the signal is strongly increasing function on the interval \( [\tau_j, \tau_j + \delta] \) and non differentiable at the point \( t = \tau_j \). For simplicity of the exposition we suppose that the noise level in all detectors is the same.

Introduce the notations: \( \varphi_n = n^{-\frac{1}{k+1}} \) and for \( j = 1, \ldots, k \)
\[ \tau_j(\vartheta_0 + \nu \varphi_n u) = \tau_j(\vartheta_0) - \varphi_n(m_j, u) + \| u \|^2 \mathcal{O}(\varphi_n^2), \tag{6} \]
\[ m_j = \left( \frac{x_j - x_0}{\rho_j}, \frac{y_j - y_0}{\rho_j} \right), \quad \rho_j = \| \vartheta_j - \vartheta_0 \|, \quad \| m_j \| = 1, \]
\[ \alpha_j = \inf_{\vartheta \in \Theta} \tau_j(\vartheta), \quad \beta_j = \sup_{\vartheta \in \Theta} \tau_j(\vartheta), \quad \mathcal{T}_j = [\alpha_j, \beta_j]. \]
Conditions $\mathcal{C}$.

$\mathcal{C}_1$. The set $\Theta$ is open, convex, bounded and such that $0 < \alpha_j < \beta_j < T$.

$\mathcal{C}_2$. The source can not be in the detector, i.e., $\vartheta_0 \neq \vartheta_j$.

$\mathcal{C}_3$. The parameters $\kappa \in (0, \frac{1}{2})$ and $\delta \in (0, T)$.

$\mathcal{C}_4$. The functions $\lambda_j(t) > 0$ have continuous derivatives $\lambda_j'(\cdot)$.

$\mathcal{C}_5$. There is at least three detectors which are not on the same line.

By the condition $\mathcal{C}_1$ we have $\min_j \rho_j > 0$. This condition is quite restrictive because if we take as $\Theta$ the region including $\vartheta_0$ and all $\vartheta_j$ we have to suppose that there exists $\varepsilon > 0$ such that the discs $\mathbb{C}_j = \{ \vartheta_0 : \| \vartheta_j - \vartheta_0 \| \leq \varepsilon \}$ are excluded from $\Theta$, but in this case the set $\Theta$ is no more convex. Note that it is possible to modify the proof in such a way that the consistency and convergence to the limit distribution are uniform on compacts $\mathbb{K} \subset \Theta$ which do not include the positions of the detectors $\vartheta_j$. Another point, when we do the re-normalization $\vartheta = \vartheta_0 + \nu \varphi_n u$ with $u \in \mathbb{U}_n = \{ u : \vartheta_0 + \nu \varphi_n u \in \Theta \}$ we have to exclude the values $u$ which correspond to $\vartheta \in \mathbb{C}_j$. To avoid such problems we extend the normalized likelihood ratio random field to include these values $u$, but the true value $\vartheta_0$ is always separated from $\vartheta_j$.

Introduce the notations: $\lambda_j = \lambda_j (0)$,

$$
\mathbb{B}_j = \{ u : \langle m_j, u \rangle < 0 \}, \quad \mathbb{B}_j^c = \{ u : \langle m_j, u \rangle \geq 0 \}, \quad \gamma_j = \frac{\lambda_j}{\delta^\kappa \lambda_0},
$$

$$
J_j (u) = J_{j,-} (u) \mathbb{I}_{\{u \in \mathbb{B}_j\}} + J_{j,+} (u) \mathbb{I}_{\{u \in \mathbb{B}_j^c\}}, \quad u \in \mathbb{R}^2,
$$

$$
J_{j,-} (u) = \gamma_j \int_0^\infty \left[ |s + \langle m_j, u \rangle|^\kappa \mathbb{I}_{\{s < - \langle m_j, u \rangle \}} - |s|^\kappa \right] dW_j (s),
$$

$$
J_{j,+} (u) = \gamma_j \int_{\{s > - \langle m_j, u \rangle \}} \left[ |s + \langle m_j, u \rangle|^\kappa - |s|^\kappa \mathbb{I}_{\{s > 0 \}} \right] dW_j (s),
$$

$$
R_j (u) = R_{j,-} \mathbb{I}_{\{u \in \mathbb{B}_j\}} + R_{j,+} \mathbb{I}_{\{u \in \mathbb{B}_j^c\}}, \quad u \in \mathbb{R}^2,
$$

$$
R_{j,-} = \gamma_j^2 \int_0^\infty \left[ |s - 1|^\kappa \mathbb{I}_{\{s < 1 \}} - |s|^\kappa \right]^2 ds,
$$

$$
R_{j,+} = \gamma_j^2 \int_{-1}^\infty \left[ |s + 1|^\kappa - |s|^\kappa \mathbb{I}_{\{s > 0 \}} \right]^2 ds.
$$

Here $W_j (\cdot), j = 1, \ldots, k$ are independent Wiener processes. The limit likelihood ratio field is

$$
Z (u) = \exp \left\{ \sum_{j=1}^k \left[ J_j (u) - \frac{|\langle m_j, u \rangle|^{2\kappa+1}}{2} R_j (u) \right] \right\}, \quad u \in \mathbb{R}^2.
$$
Note that this is a product of \( k \) independent random fields

\[
Z(u) = \prod_{j=1}^{k} Z_j(u), \quad Z_j(u) = \exp \left\{ J_j(u) - \frac{|\langle m_j, u \rangle|^{2\kappa+1}}{2} R_j(u) \right\}.
\]

Introduce as well the random vector \( \tilde{\zeta} \), which has the same distribution as the limit of the normalized BE

\[
\tilde{\zeta} = \nu \int_{\mathbb{R}^2} u Z(u) \, du = \nu \int_{\mathbb{R}^2} Z(u) \, du.
\]

Remark, that if all detectors are on the same line, then the consistent identification is impossible because the same signals come from the symmetric with respect to this line possible locations of the source.

We have the following minimax lower bound on the mean square errors of all estimators \( \tilde{\vartheta}_n \): Let the conditions \( C \) be fulfilled then for any \( \vartheta_0 \in \Theta \)

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{\Vert \vartheta - \vartheta_0 \Vert \leq \delta} \ n^{\frac{2}{2\kappa+1}} \mathbb{E}_{\vartheta_0} \left\Vert \tilde{\vartheta}_n - \vartheta \right\Vert^2 \geq \mathbb{E}_{\vartheta_0} \left\Vert \zeta \right\Vert^2.
\]

For the proof see, e.g., [5], Theorem 2.12.1.

We call the estimator \( \tilde{\vartheta}_n \) asymptotically efficient, if for all \( \vartheta_0 \in \Theta \) we have the equality

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{\Vert \vartheta - \vartheta_0 \Vert \leq \delta} \ n^{\frac{2}{2\kappa+1}} \mathbb{E}_{\vartheta_0} \left\Vert \tilde{\vartheta}_n - \vartheta \right\Vert^2 = \mathbb{E}_{\vartheta_0} \left\Vert \tilde{\zeta} \right\Vert^2.
\]

**Theorem 1** Let the conditions \( \mathcal{R} \) be fulfilled then the BE \( \tilde{\vartheta}_n \) is uniformly consistent, converges in distribution

\[
 n^{\frac{1}{2\kappa+1}} (\tilde{\vartheta}_n - \vartheta_0) \Rightarrow \tilde{\zeta},
\]

for any \( p > 0 \)

\[
\lim_{n \to \infty} \ n^{\frac{p}{2\kappa+1}} \mathbb{E}_{\vartheta_0} \left\Vert \tilde{\vartheta}_n - \vartheta_0 \right\Vert^p = \mathbb{E}_{\vartheta_0} \left\Vert \zeta \right\Vert^p,
\]

and BE is asymptotically efficient.

**Proof.** The properties of estimators mentioned in this theorem we verify with the help of approach developed by Ibragimov and Khasminskii [5]. The similar method was used in the preceding our works [1] and [4]. For the
convenience of understanding we remind it here once more. Introduce the normalized likelihood ratio random field

$$Z_n(u) = \frac{L(\vartheta_0 + \nu \varphi_n u, X^n)}{L(\vartheta_0, X^n)}, \quad u \in U_n = \{u : \vartheta_0 + \nu \varphi_n u \in \Theta\}$$

where the normalizing function $\varphi_n = n^{-\frac{1}{2k+1}}$.

Suppose that we already proved the convergence

$$Z_n(\cdot) \implies Z(\cdot).$$

Then the limit distribution of the BE can be obtained as follows (see [5]). Below we change the variables $\vartheta = \vartheta_u = \vartheta_0 + \nu \varphi_n u$.

$$\tilde{\vartheta}_n = \frac{\int_{\Theta} \theta p(\theta) L(\theta, X^T) \, d\theta}{\int_{\Theta} p(\theta) L(\theta, X^T) \, d\theta} = \vartheta_0 + \nu \varphi_n \frac{\int_{U_n} up(\vartheta_u) L(\theta_u, X^T) \, du}{\int_{U_n} p(\vartheta_u) L(\theta_u, X^T) \, du}.$$

Hence

$$\varphi_n^{-1} (\tilde{\vartheta}_n - \vartheta_0) = \nu \frac{\int_{U_n} up(\vartheta_u) Z_n(u) \, du}{\int_{U_n} p(\vartheta_u) Z_n(u) \, du} \implies \nu \frac{\int_{\mathbb{R}^2} u Z(u) \, du}{\int_{\mathbb{R}^2} Z(u) \, du} = \tilde{\zeta}.$$

Recall that $p(\vartheta_u) \to p(\vartheta_0) > 0$.

The properties of the $Z_n(u)$ required in the Theorem 1.10.2 [5] are checked in the three lemmas below. Remind that this approach to the study of the properties of these estimators was applied in [8], [10]. Here we use some obtained there inequalities.

**Lemma 1** Let the conditions $\mathcal{C}$ be fulfilled, then the finite dimensional distributions of the random field $Z_n(u), u \in U_n$ converge to the finite dimensional distributions of the limit random field $Z(u), u \in \mathbb{R}^2$ and this convergence is uniform on compacts $K \in \Theta$.

**Proof.** Let us denote $d\pi_{j,n}(t) = dX_j(t) - n[S_j(t - \tau_j(\vartheta_0)) + \lambda_0] \, dt$ and put
\[ \vartheta_u = \vartheta_0 + \nu \varphi_n u, \; \tau_j = \tau_j (\vartheta_0). \]

Then we can write

\[
\begin{align*}
\ln Z_n (u) &= \sum_{j=1}^{k} \int_0^T \ln \left( \frac{S_j (t - \tau_j (\vartheta_u)) + \lambda_0}{S_j (t - \tau_j) + \lambda_0} \right) d\pi_{j,n} (t) \\
&\quad - n \sum_{j=1}^{k} \int_0^T \left[ \frac{S_j (t - \tau_j (\vartheta_u)) + \lambda_0}{S_j (t - \tau_j) + \lambda_0} - 1 \right] \ln \left( \frac{S_j (t - \tau_j (\vartheta_u)) + \lambda_0}{S_j (t - \tau_j) + \lambda_0} \right) d\tau_j \\
&\quad - \ln \left( \frac{S_j (t - \tau_j (\vartheta_u)) + \lambda_0}{S_j (t - \tau_j) + \lambda_0} \right) [S_j (t - \tau_j) + \lambda_0] d\tau_j \\
&= \sum_{j=1}^{k} \int_0^T F_j (t, \vartheta_u) d\pi_{j,n} (t) - n \sum_{j=1}^{k} \int_0^T G_j (t, \vartheta_u) d\tau_j
\end{align*}
\]

with obvious notation.

Let \( u \in \mathbb{B}_j \). Then \( \tau_j (\vartheta_u) > \tau_j \). Following the same arguments as that given in \([2]\), we obtain the asymptotic (\( n \to \infty \)) relations:

\[
\begin{align*}
J_{j,n} (u) &= \int_0^T F_j (t, \vartheta_u) d\pi_{j,n} (t) = \int_{\tau_j}^{\tau_j + \delta} F_j (t, \vartheta_u) d\pi_{j,n} (t) (1 + o (1)) \\
I_{j,n} (u) &= n \int_0^T G_j (t, \vartheta_u) d\tau_j = n \int_{\tau_j}^{\tau_j + \delta} G_j (t, \vartheta_u) d\tau_j (1 + o (1))
\end{align*}
\]

For \( t \in [\tau_j, \tau_j - \varphi_n (m_j, u)] \) as \( \varphi_n \to 0 \) we obtain the expansions

\[
\begin{align*}
\lambda_j (t - \tau_j (\vartheta_u)) &= \lambda_j (0) + (t - \tau_j (\vartheta_u)) \lambda_j' (0) (1 + o (1)) = \lambda_j + o (1) \\
\lambda_j (t - \tau_j (\vartheta_u)) &= \lambda_j (t - \tau_j) + \varphi_n (m_j, u) \lambda_j' (t - \tau_j) + O (\varphi_n^2) \|u\|^2, \\
\left| \frac{t - \tau_j (\vartheta_u)}{\delta} \right|^\kappa &= \delta^{-\kappa} \left| t - \tau_j + \varphi_n (m_j, u) + O (\varphi_n^2) \right|^\kappa \\
&= \delta^{-\kappa} \left| t - \tau_j + \varphi_n (m_j, u) \right|^\kappa + O (\varphi_n^{2\kappa})
\end{align*}
\]

Here we used the inequality \( |a + b|^\kappa \leq |a|^\kappa + |b|^\kappa \).

Further, for \( \tau_j \leq t \leq \tau_j - \varphi_n (m_j, u) \) and \( \|u\| < L \) we can write

\[
\begin{align*}
\ln \left( \frac{S_j (t - \tau_j (\vartheta_u)) + \lambda_0}{S_j (t - \tau_j) + \lambda_0} \right) &= \ln \left( \frac{\lambda_0}{\lambda_j (t - \tau_j) \left| \frac{t - \tau_j}{\delta} \right|^\kappa + \lambda_0} \right) \\
&= - \ln \left( 1 + \frac{\lambda_j}{\lambda_0} \left| \frac{t - \tau_j}{\delta} \right|^\kappa \right) (1 + O (\varphi_n)) \\
&= - \frac{\lambda_j}{\lambda_0} \left| \frac{t - \tau_j}{\delta} \right|^\kappa (1 + O (\varphi_n^{2\kappa}))
\end{align*}
\]
For \( t \in [\tau_j - \varphi_n(m_j, u), \delta] \) the similar relations are

\[
\ln \left( \frac{S_j(t - \tau_j(\vartheta_u) + \lambda_0)}{S_j(t - \tau_j) + \lambda_0} \right) = \ln \left( \frac{\lambda_j (t - \tau_j(\vartheta_u)) \left| \frac{t - \tau_j(\vartheta_u)}{\delta} \right|^\kappa + \lambda_0}{\lambda_j (t - \tau_j) \left| \frac{t - \tau_j}{\delta} \right|^\kappa + \lambda_0} \right) = \ln \left( 1 + \frac{\lambda_j (t - \tau_j(\vartheta_u)) \left| \frac{t - \tau_j(\vartheta_u)}{\delta} \right|^\kappa - \lambda_j (t - \tau_j) \left| \frac{t - \tau_j}{\delta} \right|^\kappa}{\lambda_j (t - \tau_j) \left| \frac{t - \tau_j}{\delta} \right|^\kappa + \lambda_0} \right)
\]

therefore

\[
\mathbb{E}_{\vartheta_0} (J_{j,n}(u))^2 = \int_0^T F_j(t, \vartheta_u)^2 \lambda_{j,n}(\vartheta_0, t) \, dt = \frac{\lambda_j^2 n}{\lambda_0} \int_{\tau_j - \varphi_n(m_j, u)}^{\tau_j} \left| \frac{t - \tau_j}{\delta} \right|^{2\kappa} \, dt + o(1)
\]

\[
+ \frac{\lambda_j^2 n}{\lambda_0 \delta^{2\kappa}} \varphi_n^{2\kappa+1} \int_0^{\varphi_n(m_j, u)} |s|^{2\kappa} \, ds
\]

\[
+ \frac{\lambda_j^2 n}{\lambda_0 \delta^{2\kappa}} \varphi_n^{2\kappa+1} \int_{\varphi_n(m_j, u)}^{\varphi_n} \left[ |s + \langle m_j, u \rangle|^\kappa - |s|^\kappa \right]^2 \, ds + o(1)
\]

\[
= \gamma_j^2 \langle m_j, u \rangle^{2\kappa+1} \int_0^1 |v|^{2\kappa} \, dv + \gamma_j^2 \langle m_j, u \rangle^{2\kappa+1} \int_1^{\varphi_n(m_j, u)} \left[ |v - 1|^\kappa - |v|^\kappa \right]^2 \, dv + o(1)
\]

\[
= \gamma_j^2 \langle m_j, u \rangle^{2\kappa+1} \int_0^{\varphi_n(m_j, u)} \left[ |v - 1|^\kappa \mathbb{I}_{v \geq 1} - |v|^\kappa \right]^2 \, dv + o(1)
\]

where we changed the variables \( t = \tau_j + s\varphi_n \) and \( s = -\langle m_j, u \rangle \). Recall that \( n\varphi_n^{2\kappa+1} = \nu^{2\kappa+1} \) and \( \gamma_j^2 = \lambda_j^2 \nu^{2\kappa+1} \lambda_0^{-1} \delta^{-2\kappa} \). Hence for \( u \in \mathbb{B}_- \) we obtain the following limit

\[
R_n = \int_0^{\varphi_n(m_j, u)} \left[ |v - 1|^\kappa \mathbb{I}_{v \geq 1} - |v|^\kappa \right]^2 \, dv
\]
$$\longrightarrow \int_0^\infty \left[ |v-1|^\kappa \mathbb{I}_{\{v\geq 1\}} - |v|^\kappa \right]^2 dv = R_{j,-}.$$  

These arguments allow us to write the representation

$$J_{j,n}(u) = \gamma_j \int_0^{\tau_j} \left[ |s + \langle m_j, u \rangle|^\kappa \mathbb{I}_{\{s \geq -\langle m_j, u \rangle\}} - |s|^\kappa \right] dW_{j,n}(s) + o(1).$$  

Here

$$W_{j,n}(s) = \frac{1}{\sqrt{\lambda_0 n \varphi_n}} \left[ X_j(\tau_j + s \varphi_n) - X_j(\tau_j) - \int_{\tau_j}^{\tau_j + s \varphi_n} \lambda_{j,n}(\vartheta_0, v) dv \right],$$

$$E_{\vartheta_0} W_{j,n}(s)^2 = \frac{n}{\lambda_0 n \varphi_n} \int_{\tau_j}^{\tau_j + s \varphi_n} \lambda_{j,n}(\vartheta_0, v) dv = s + o(1),$$

$$E_{\vartheta_0} W_{j,n}(s) = 0, \quad E_{\vartheta_0} W_{j,n}(s_1) W_{j,n}(s_2) = s_1 \wedge s_2 + o(1).$$

The standard central limit theorem provides us the corresponding convergence of stochastic integrals. For any $u_1, \ldots, u_M \in B_j$ we have the joint asymptotic normality of the stochastic integrals

$$Y_{j,n} \equiv \left( J_{j,n}(u_1), \ldots, J_{j,n}(u_M) \right) \Longrightarrow Y_j \equiv \left( J_j(u_1), \ldots, J_j(u_M) \right),$$

where

$$J_j(u) = \gamma_j \int_0^\infty \left[ |s + \langle m_j, u \rangle|^\kappa \mathbb{I}_{\{s \geq -\langle m_j, u \rangle\}} - |s|^\kappa \right] dW_j(s).$$

Moreover, the similar arguments give us the convergence

$$Y_n \equiv \left( Y_{1,n}, \ldots, Y_{k,n} \right) \Longrightarrow Y \equiv \left( Y_1, \ldots, Y_k \right) \quad (7)$$

Consider now the values $u \in B_j^c$. Then $\tau_j(\vartheta_0) \leq \tau_j(\vartheta_0)$ or asymptotically $\tau_j(\vartheta_0) - \varphi_n \langle m_j, u \rangle + O(\varphi_n^2) \leq \tau_j(\vartheta_0)$. The similar arguments allow us to verify the convergence (7) with the limit process

$$J_j(u) = \gamma_j \int_{-\langle m_j, u \rangle}^{\infty} \left[ |s + \langle m_j, u \rangle|^\kappa \mathbb{I}_{\{s \leq \langle m_j, u \rangle\}} + \left( |s + \langle m_j, u \rangle|^\kappa - |s|^\kappa \right) \right] dW(s).$$

Therefore we have the convergence of finite-dimensional distributions of the stochastic integrals.
For the ordinary integral $I_{j,n}(u)$ we have the similar representation ($u \in \mathbb{B}_t$, $G_{j,t} = G_j(t, u)$)

$$I_{j,n}(u) = n \int_{0}^{\tau_j} G_{j,t} \, dt + n \int_{\tau_j}^{\tau_j(\vartheta_u)} G_{j,t} \, dt + n \int_{\tau_j(\vartheta_u)}^{\tau_j + \delta} G_{j,t} \, dt$$

$$+ n \int_{\tau_j + \delta}^{\tau_j(\vartheta_u) + \delta} G_{j,t} \, dt + n \int_{\tau_j(\vartheta_u) + \delta}^{T} G_{j,t} \, dt$$

$$= n \int_{\tau_j}^{\tau_j(\vartheta_u)} G_{j,t} \, dt + n \int_{\tau_j(\vartheta_u)}^{\tau_j + \delta} G_{j,t} \, dt + n \int_{\tau_j + \delta}^{\tau_j(\vartheta_u) + \delta} G_{j,t} \, dt + n \int_{\tau_j(\vartheta_u) + \delta}^{\tau_j + \delta} G_{j,t} \, dt + o(1)$$

For $t \in [0, \tau_j]$ we have $G_j(t, u) = 0$ and for $t \in [\tau_j + \delta, T]$ the function $G_j(t, u)$ has continuous bounded derivative and we can write

$$n \int_{\tau_j(\vartheta_u) + \delta}^{T} G_{j,t} \, dt \leq C n \varphi_n^2 \|u\|^2 = o(1).$$

Consider the case $t \in [\tau_j, \tau_j(\vartheta_u)]$. Using expansion $\ln(1 + x) = x - \frac{x^2}{2} + O(x^3)$ we can write

$$\frac{\lambda_{j,n}(\vartheta_u, t)}{\lambda_{j,n}(\vartheta_0, t)} - 1 - \ln \left( \frac{\lambda_{j,n}(\vartheta_u, t)}{\lambda_{j,n}(\vartheta_0, t)} \right) = \frac{\lambda_0}{\lambda_j(t - \tau_j) \left| \frac{t - \tau_j}{\delta} \right|^\kappa + \lambda_0}$$

$$- 1 - \ln \left( \frac{\lambda_0}{\lambda_j(t - \tau_j) \left| \frac{t - \tau_j}{\delta} \right|^\kappa + \lambda_0} \right) = \frac{\lambda_0^2}{2\lambda_0^2 \delta^2 \kappa} |t - \tau_j|^{2\kappa} (1 + o(1)).$$

For $t \in [\tau_j(\vartheta_u), \tau_j + \delta]$ we have

$$\frac{\lambda_{j,n}(\vartheta_u, t)}{\lambda_{j,n}(\vartheta_0, t)} - 1 - \ln \left( \frac{\lambda_{j,n}(\vartheta_u, t)}{\lambda_{j,n}(\vartheta_0, t)} \right) = \frac{\lambda_j(t - \tau_j(\vartheta_u)) \left| \frac{t - \tau_j(\vartheta_u)}{\delta} \right|^\kappa + \lambda_0}{\lambda_j(t - \tau_j) \left| \frac{t - \tau_j}{\delta} \right|^\kappa + \lambda_0}$$

$$- 1 - \ln \left( \frac{\lambda_j(t - \tau_j(\vartheta_u)) \left| \frac{t - \tau_j(\vartheta_u)}{\delta} \right|^\kappa + \lambda_0}{\lambda_j(t - \tau_j) \left| \frac{t - \tau_j}{\delta} \right|^\kappa + \lambda_0} \right)$$

$$= \frac{\lambda_0^2}{2\lambda_0^2 \delta^2 \kappa} \left( |t - \tau_j(\vartheta_u)|^{2\kappa} - |t - \tau_j|^\kappa \right)^2 (1 + o(1)).$$

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These relations allow us to write
\[ I_{j,n} = \frac{n\lambda_j^2}{2\lambda_0^2} \int_{\tau_j}^{\tau_j(\varphi_u)} |t - \tau_j|^{2\kappa} \, dt \]
\[ + \frac{n\lambda_j^2}{2\lambda_0^2} \int_{\tau_j(\varphi_u)}^{\tau_j (\varphi_u) + \delta} \left( |t - \tau_j (\varphi_u)|^{2\kappa} - |t - \tau_j|^{2\kappa} \right) \, dt + o(1) \]
\[ = \frac{n\rho_j^{2\kappa+1}\lambda_j^2}{2\lambda_0^2} \int_0^{(m_j,u)} |s|^{2\kappa} \, ds \]
\[ + \frac{n\lambda_j^2 \rho_j^{2\kappa+1}}{2\lambda_0^2} \int_0^{\delta \rho_j (\varphi_u) + \tau_j(\varphi_u)} \left( |s + \langle m_j, u \rangle|^\kappa - |s|^\kappa \right)^2 \, ds + o(1) \]
\[ = \frac{\lambda_j^2 |\langle m_j, u \rangle|^{2\kappa+1}}{2\lambda_0^2} \int_0^1 |s|^{2\kappa} \, ds \]
\[ + \frac{n\lambda_j^2 \rho_j^{2\kappa+1}}{2\lambda_0^2} \int_1^{\tau_j - \tau_j(\varphi_u) + \delta \rho_j (\varphi_u)} \left( |v - 1|^\kappa - |v|^\kappa \right)^2 \, ds + o(1) \]
\[ \to \frac{\gamma_j^2}{2} \int_0^\infty |s|^{2\kappa} \, ds + \left( |s + \langle m_j, u \rangle|^\kappa - |s|^\kappa \right)^2 \, ds. \]

Note that all convergences mentioned above are uniform on compacts \( K \subset \Theta. \)

**Lemma 2** Let the condition \( R_2 \) be fulfilled, then there exists a constant \( C > 0, \) which does not depend on \( n \) such that for any \( R > 0 \)
\[ \sup_{\varphi_u \in \Theta} \sup_{\|u_1 - u_2\| \leq R} \|u_1 - u_2\|^{-2\kappa-1} \mathbf{E}_{\varphi_u} \left| Z_n^{1/2} (u_1) - Z_n^{1/2} (u_2) \right|^2 \leq C (1 + R). \]

**Proof.** We have the estimate (see, e.g. [10])
\[ \mathbf{E}_{\varphi_u} \left| Z_n^{1/2} (u_1) - Z_n^{1/2} (u_2) \right|^2 \leq \sum_{j=1}^k \int_0^T \left[ \frac{\lambda_{j,n} \left( \varphi_{u_2}, t \right) - \lambda_{j,n} \left( \varphi_{u_1}, t \right)}{\sqrt{\lambda_{j,n} \left( \varphi_{u_2}, t \right) + \lambda_{j,n} \left( \varphi_{u_1}, t \right)}} \right]^2 \, dt \]
\[ = \sum_{j=1}^k \int_0^T \frac{n^2 \left[ S_j \left( t - \tau_j \left( \varphi_{u_2} \right) \right) - S_j \left( t - \tau_j \left( \varphi_{u_1} \right) \right) \right]^2}{\left[ \sqrt{\lambda_{j,n} \left( \varphi_{u_2}, t \right) + \lambda_{j,n} \left( \varphi_{u_1}, t \right)} \right]^2} \, dt \]
\[ \leq \frac{n}{4\lambda_0} \sum_{j=1}^k \int_0^T \left[ S_j \left( t - \tau_j \left( \varphi_{u_2} \right) \right) - S_j \left( t - \tau_j \left( \varphi_{u_1} \right) \right) \right]^2 \, dt, \]
where we used the estimate $\lambda_{j,n}(\vartheta_u, t) \geq n\lambda_0$. Suppose that $\tau_j(\vartheta_{u_1}) < \tau_j(\vartheta_{u_2})$ and denote $\Delta_t = \sqrt{n} [S_j(t - \tau_j(\vartheta_{u_2})) - S_j(t - \tau_j(\vartheta_{u_1}))]$. Then

$$
\int_0^T \Delta_t^2 dt = \int_0^{\tau_j(\vartheta_{u_1})} \Delta_t^2 dt + \int_{\tau_j(\vartheta_{u_1})}^{\tau_j(\vartheta_{u_2})} \Delta_t^2 dt + \int_{\tau_j(\vartheta_{u_2})}^T \Delta_t^2 dt
$$

$$
= \int_{\tau_j(\vartheta_{u_1})}^{\tau_j(\vartheta_{u_2})} \Delta_t^2 dt + \int_{\tau_j(\vartheta_{u_2})}^T \Delta_t^2 dt.
$$

Remark that the function $\Delta_t = 0$ on the interval $[0, \tau_j(\vartheta_{u_1})]$ and $\Delta_t = nS_j(\vartheta_{u_1}, t)$ on the interval $[\tau_j(\vartheta_{u_1}), \tau_j(\vartheta_{u_2})]$. Therefore

$$
\int_{\tau_j(\vartheta_{u_1})}^{\tau_j(\vartheta_{u_2})} \Delta_t^2 dt \leq Cn \int_{\tau_j(\vartheta_{u_1})}^{\tau_j(\vartheta_{u_2})} \left| \frac{t - \tau_j(\vartheta_{u_1})}{\delta} \right|^{2\kappa} dt \leq Cn \left| \frac{\tau_j(\vartheta_{u_2}) - \tau_j(\vartheta_{u_1})}{\delta} \right|^{2\kappa + 1}
$$

$$
\leq Cn \varphi_n^{2\kappa + 1} \|u_2 - u_1\|^{2\kappa + 1} = C \|u_2 - u_1\|^{2\kappa + 1}.
$$

Further

$$
\int_{\tau_j(\vartheta_{u_2})}^T \Delta_t^2 dt = \int_{\tau_j(\vartheta_{u_2})}^{\tau_j(\vartheta_{u_2}) + \delta} \Delta_t^2 dt + \int_{\tau_j(\vartheta_{u_2}) + \delta}^{\tau_j(\vartheta_{u_2}) + \delta} \Delta_t^2 dt + \int_{\tau_j(\vartheta_{u_2}) + \delta}^T \Delta_t^2 dt. \quad (8)
$$

Using the estimate

$$
|\lambda_j (t - \tau_j(\vartheta_{u_2})) - \lambda_j (t - \tau_j(\vartheta_{u_1}))|^2 \leq C \varphi_n^2 \|u_2 - u_1\|^2
$$

we obtain for the first integral

$$
\int_{\tau_j(\vartheta_{u_2})}^{\tau_j(\vartheta_{u_2}) + \delta} \Delta_t^2 dt = n \int_{\tau_j(\vartheta_{u_2})}^{\tau_j(\vartheta_{u_2}) + \delta} \left[ \lambda_j (t - \tau_j(\vartheta_{u_2})) \left| \frac{t - \tau_j(\vartheta_{u_2})}{\delta} \right|^\kappa - \lambda_j (t - \tau_j(\vartheta_{u_1})) \left| \frac{t - \tau_j(\vartheta_{u_1})}{\delta} \right|^\kappa \right]^2 dt
$$

$$
\leq Cn \varphi_n^2 \|u_2 - u_1\|^2 + Cn \int_{\tau_j(\vartheta_{u_2})}^{\tau_j(\vartheta_{u_2}) + \delta} \left[ |t - \tau_j(\vartheta_{u_2})|^\kappa - |t - \tau_j(\vartheta_{u_1})|^\kappa \right]^2 dt
$$

$$
\leq C \varphi_n^{1-2\kappa} \|u_2 - u_1\|^2
$$

$$
+ Cn \varphi_n^{2\kappa + 1} \int_0^{\tau_j(\vartheta_{u_2}) - \tau_j(\vartheta_{u_1}) + \delta} \left[ |s|^\kappa - |s - \frac{\tau_j(\vartheta_{u_1}) - \tau_j(\vartheta_{u_2})}{\varphi_n}|^\kappa \right]^2 ds
$$

$$
\leq C \varphi_n^{1-2\kappa} \|u_2 - u_1\|^2 + C \|u_2 - u_1\|^{2\kappa + 1}.
$$
where we used the relations
\[ |\tau_j(\vartheta_{u_1}) - \tau_j(\vartheta_{u_2})| + \langle m_j, u_1 \rangle - \langle m_j, u_2 \rangle \leq C\varphi_n \|u_2 - u_1\|^2, \]
\[ \int_0^\infty \left[ |s| - |s - \langle m_j, u_1 - u_2 \rangle| \right]^2 ds \leq \|u_2 - u_1\|^{2\kappa + 1} \int_0^\infty \left[ |v| - |v - \langle m_j, e \rangle| \right]^2 dv \leq C \|u_2 - u_1\|^{2\kappa + 1}. \]

Here we set \( s = v \|u_2 - u_1\| \) and \( e = \|u_2 - u_1\|^{-1}(u_2 - u_1). \)

As the function \( S(t) \) has a bounded derivative \( S'(t) \) on the interval \([\tau_j(\vartheta_{u_2}) + \delta, T]\) we can write
\[ \int_{\tau_j(\vartheta_{u_2}) + \delta}^T \Delta_t^2 dt \leq Cn\varphi_n^2 \|u_2 - u_1\|^2 \leq C (1 + R) \|u_2 - u_1\|^{2\kappa + 1}. \]

The other cases can be estimated by a similar way.

**Lemma 3** Let the conditions \( \mathcal{C} \) be fulfilled, then there exists a constant \( \kappa > 0 \), which does not depend on \( n \) such that
\[ \sup_{\vartheta_0\in\Theta} E_{\vartheta_0}Z_{j,n}^{\frac{1}{2}}(u) \leq e^{-\kappa\|u\|^2/\varphi_n}. \] (9)

**Proof.** Let us denote \( \theta_u = \vartheta_0 + \nu\varphi_n u \) and put
\[ Z_{j,n}(u) = \exp \left\{ \int_0^T \ln \left( \frac{\lambda_{j,n}(\theta_u, t)}{\lambda_{j,n}(\vartheta_0, t)} \right) dX_j(t) - \int_0^T [\lambda_{j,n}(\theta_u, t) - \lambda_{j,n}(\vartheta_0, t)] dt \right\}. \]

By Lemma 2.2 in [10] we can write
\[ E_{\vartheta_0}Z_{j,n}^{\frac{1}{2}}(u) = \exp \left\{ -\frac{1}{2} \int_0^T \left[ \sqrt{\lambda_{j,n}(\theta_u, t)} - \sqrt{\lambda_{j,n}(\vartheta_0, t)} \right]^2 dt \right\}. \]

Hence
\[ E_{\vartheta_0}Z_{n}^{\frac{1}{2}}(u) = \prod_{j=1}^k E_{\vartheta_0}Z_{j,n}^{\frac{1}{2}}(u) \]
\[ = \exp \left\{ -\frac{1}{2} \sum_{j=1}^k \int_0^T \left[ \sqrt{\lambda_{j,n}(\theta_u, t)} - \sqrt{\lambda_{j,n}(\vartheta_0, t)} \right]^2 dt \right\}. \] (10)
First for simplicity of calculation we write

\[
\int_0^T \left[ \sqrt{\lambda_{j,n}(\vartheta, t)} - \sqrt{\lambda_{j,n}(\vartheta_0, t)} \right]^2 dt \\
= \int_0^T \frac{[\lambda_{j,n}(\vartheta, t) - \lambda_{j,n}(\vartheta_0, t)]^2}{\sqrt{\lambda_{j,n}(\vartheta, t)} + \sqrt{\lambda_{j,n}(\vartheta_0, t)}} dt \\
\geq c_j \int_0^T [S_j(t - \tau_j(\vartheta)) - S_j(t - \tau_j)]^2 dt, \quad (11)
\]

where \( c_j = (4\lambda_M)^{-1} > 0 \) and \( \lambda_M = \lambda_0 + \max_{t \in \mathcal{T}_j} S_j(t) \). Therefore it is sufficient to study the integral

\[
I_j(\vartheta) = \int_0^T [S_j(t - \tau_j(\vartheta)) - S_j(t - \tau_j)]^2 dt \\
= \int_{\tau_j(\vartheta) \land \tau_j} [S_j(t - \tau_j(\vartheta)) - S_j(t - \tau_j)]^2 dt.
\]

We evaluate these integrals on two sets \( \mathcal{A} = \{ \vartheta : \|\vartheta - \vartheta_0\| \leq h \} \) and \( \mathcal{A}^c \). Here \( h > 0 \) is some small number. Recall that we denoted \( \tau_j = \tau_j(\vartheta_0) \).

Let \( \vartheta \in \mathcal{A} \cap \mathcal{B} \), where \( \mathcal{B} = \{ \vartheta \in \mathcal{A} : \tau_j(\vartheta) > \tau_j(\vartheta_0) \} \). Moreover \( \tau_j(\vartheta) - \tau_j(\vartheta_0) < \delta \). Then

\[
I_j(\vartheta) \geq \int_{\tau_j}^{\tau_j(\vartheta)} S_j(t - \tau_j)^2 dt + \int_{\tau_j}^{\tau_j(\vartheta) + \delta} [S_j(t - \tau_j(\vartheta)) - S_j(t - \tau_j)]^2 dt \\
= \int_0^{\tau_j(\vartheta) - \tau_j} S_j(s)^2 ds + \int_0^{\tau_j(\vartheta) + \delta - \tau_j} [S_j(s) - S_j(s - \Delta_t)]^2 ds.
\]

where \( \Delta_t(\tau_j) = \tau_j - \tau_j(\vartheta) \). Further (below \( \lambda_m = \min_{t \in \mathcal{T}_j} \lambda_j(t) > 0 \))

\[
\int_0^{\tau_j(\vartheta) - \tau_j} \lambda_j(s)^2 \left( \frac{s}{\delta} \right)^{2\kappa} ds \geq \frac{\lambda_m^2}{\delta^{2\kappa}} \int_0^{\tau_j(\vartheta) - \tau_j} s^{2\kappa} ds = \frac{\lambda_m^2 |\tau_j(\vartheta) - \tau_j|^{2\kappa+1}}{\delta^{2\kappa} (2\kappa + 1)}.
\]

Recall that

\[
\tau_j(\vartheta) - \tau_j = \langle m_j, \vartheta - \vartheta_0 \rangle + O(h^2) = \langle m_j, e \rangle \|\vartheta - \vartheta_0\| + O(h^2),
\]

where the unit vector \( e = (\vartheta - \vartheta_0) \|\vartheta - \vartheta_0\|^{-1} \). Therefore

\[
\int_0^{\tau_j(\vartheta) - \tau_j} \lambda_j(s)^2 \left( \frac{s}{\delta} \right)^{2\kappa} ds \geq \frac{\lambda_m^2 |\langle m_j, e \rangle|^{2\kappa+1}}{\delta^{2\kappa} (2\kappa + 1)} \|\vartheta - \vartheta_0\|^{2\kappa+1} (1 + o(\|\vartheta - \vartheta_0\|)).
\]
and we can take such \( h \) that
\[
\int_{0}^{\tau_j(\vartheta) - \tau_j} \lambda_j (s)^2 \left( \frac{s}{\delta} \right)^{2\kappa} \, ds \geq \frac{\lambda_m^2}{2\delta^{2\kappa} (2\kappa + 1)} \| \vartheta - \vartheta_0 \|^{2\kappa + 1}.
\]

For the second integral we have \( (\delta_* = \tau_j - \tau_j (\vartheta) + \delta > 0) \)
\[
\int_{0}^{\delta_*} [S_j (s) - S_j (s - \Delta (\tau_j))]^2 \, ds
= \frac{1}{\delta^{2\kappa}} \int_{0}^{\delta_*} \left[ \lambda_j (s) s^{\kappa} - \lambda_j (s - \Delta (\tau_j)) |s - \Delta (\tau_j)|^{\kappa} \right]^2 \, ds
\geq \frac{\lambda_m^2}{\delta^{2\kappa}} \int_{0}^{\delta_*} \left[ s^{\kappa} - |s - \Delta (\tau_j)|^{\kappa} \right]^2 \, ds - C \| \vartheta - \vartheta_0 \|^2
\geq \frac{\lambda_m^2}{\delta^{2\kappa}} \int_{0}^{\delta_*} \left[ v^{\kappa} - |v - 1|^{\kappa} \right]^2 \, dv \Delta (\tau_j)^{2\kappa + 1} - C \| \vartheta - \vartheta_0 \|^{2\kappa + 1}
- C \| \vartheta - \vartheta_0 \|^2,
\]
where we used the relation \( \lambda_j (s - \Delta (\tau_j)) = \lambda_j (s) + O (\Delta (\tau_j)) \) and set \( s = v \Delta (\tau_j) \).

These estimates from below of the integral allow us to write
\[
\sum_{j=1}^{k} I_j (\vartheta) \geq \gamma \sum_{j=1}^{k} |\langle m_j, e \rangle|^{2\kappa + 1} \| \vartheta - \vartheta_0 \|^{2\kappa + 1} - C \| \vartheta - \vartheta_0 \|^2.
\]
As \( k \geq 3 \) we have
\[
Q (e) = \sum_{j=1}^{k} |\langle m_j, e \rangle|^{2\kappa + 1}, \quad \inf_{\| e \| = 1} Q (e) = q_1 > 0.
\]
Indeed, if \( q_1 = 0 \), then there exists a vector \( e_* \) such that \( Q (e_*) = 0 \) and this vector is orthogonal to all \( m_j, j = 1, \ldots, k \). Of course, this is impossible. Therefore we can take such sufficiently small \( h \) that for \( \vartheta \in \mathcal{A} \cap \mathcal{B} \) we obtain the estimate
\[
\sum_{j=1}^{k} \int_{0}^{T} |S_j (t - \tau_j (\vartheta)) - S_j (t - \tau_j (\vartheta_0))|^2 \, dt \geq \gamma_1 \| \vartheta - \vartheta_0 \|^{2\kappa + 1} \quad (12)
\]
with some positive \( \gamma_1 \). For the other values of \( \vartheta \in \mathcal{A} \) we have the similar estimates.
Let us consider these integrals for the values \( \vartheta \in \mathbb{A}^c \). According to (11) we have to study the function

\[ g(h) = \inf_{\vartheta_0 \in \Theta} \inf_{\| \vartheta - \vartheta_0 \| > h} \frac{1}{k} \sum_{j=1}^{k} \int_{0}^{T} \left[ S_j(t - \tau_j(\vartheta)) - S_j(t - \tau_j(\vartheta_0)) \right]^2 dt, \]

and show that \( g(h) > 0 \).

Suppose that \( g(h) = 0 \), then this implies that there exists at least one point \( \vartheta^* \in \Theta \) such that \( \| \vartheta^* - \vartheta_0 \| \geq h \) and for all \( j = 1, \ldots, k \) we have

\[ \int_{0}^{T} \left[ S_j(t - \tau_j(\vartheta^*)) - S_j(t - \tau_j(\vartheta_0)) \right]^2 dt = 0. \]

Let \( \tau_j(\vartheta^*) > \tau_j \). Then for all \( t \in [\tau_j, \tau_j(\vartheta^*)] \) we have

\[ \lambda_j(t - \tau_j)|t - \tau_j|^\kappa = 0 \]

and for \( t \in [\tau_j(\vartheta^*), \tau_j + \delta] \)

\[ \lambda_j(t - \tau_j(\vartheta^*))|t - \tau_j(\vartheta^*)|^\kappa = \lambda_j(t - \tau_j)|t - \tau_j|^\kappa. \]

Of course we can have these two equalities if and only if \( \tau_j(\vartheta^*) = \tau_j(\vartheta_0) \) for all \( j = 1, \ldots, k \). Recall that \( \lambda_j(t) \) are strictly positive functions. From the geometry of the model it follows that it is impossible to have two different points such that the distances from these points and \( k \geq 3 \) detectors coincide.

Therefore for \( \vartheta \in \mathbb{A}^c \)

\[ \sum_{j=1}^{k} \int_{0}^{T} \left[ S_j(t - \tau_j(\vartheta)) - S_j(t - \tau_j(\vartheta_0)) \right]^2 dt \geq g(h) \]

\[ \geq \frac{g(h) \| \vartheta - \vartheta_0 \|^{2\kappa + 1}}{D^{2\kappa + 1}} \geq \gamma_2 \| \vartheta - \vartheta_0 \|^{2\kappa + 1}, \tag{13} \]

where \( D = \sup_{\vartheta_1, \vartheta_2 \in \Theta} \| \vartheta_1 - \vartheta_2 \| \).

From the estimates (12) and (13) it follows that if we put \( \vartheta = \vartheta_0 + \nu \varphi_n u \), then

\[ \sum_{j=1}^{k} \int_{0}^{T} \left[ \sqrt{\lambda_{j,n}(\vartheta, t)} - \sqrt{\lambda_{j,n}(\vartheta_0, t)} \right]^2 dt \geq \gamma n \| \vartheta - \vartheta_0 \|^{2\kappa + 1} \]

\[ = \gamma n^{2\kappa + 1} \| u \|^{2\kappa + 1}. \]

This estimate and (10) prove (9).

The properties of the likelihood ratio field \( Z_n(\cdot) \) established in the lemmas 1-3 are sufficient conditions for the Theorem 1.10.2 in [5]. Therefore the Theorem 1 is proved.
3 Discussion

There are several problems which naturally arise for this model of observations. Note that the properties of the MLE $\hat{\vartheta}_n$ can be studied too. This requires a special modification of Lemma 2 to verify the tightness of the corresponding family of measures.

In this work we supposed that the source starts emission at the instant $t = 0$. It is interesting to consider the more general statement with unknown start of the emission. The limit distributions of the MLE and BE are unknown and it will be interesting to have some pictures obtained by numerical simulations for the densities of these vectors. The numerical simulations can provide us the values $\mathbb{E}_{\vartheta_0} \| \hat{\zeta} \|^2$ and $\mathbb{E}_{\vartheta_0} \| \tilde{\zeta} \|^2$ of the limit variances of these estimators.

Note that it is possible to construct a consistent estimator of $\vartheta_0$ in two steps as it was proposed in [1]. First we estimate $k$ moments $\tau = (\tau_1, \ldots, \tau_k)$ of arriving signals in detectors, say, $\tilde{\tau}_{1,n}, \ldots, \tilde{\tau}_{k,n}$. Recall that

$$\tilde{\xi}_{j,n} = \kappa + 1 (\tilde{\tau}_{j,n} - \tau_j) \implies \tilde{\xi}_j, \quad j = 1, \ldots, k$$

where $\tilde{\xi}_j$ are independent random variables. Hence we have

$$\nu^2 \tilde{\tau}_{j,n} = \nu^2 \tau_j^2 + 2 \varphi_n \tau_j \tilde{\xi}_{j,n} + \varphi_n^2 \tilde{\xi}^2_{j,n} = \rho_j^2 + 2 \nu \rho_j \tilde{\xi}_{j,n} \varphi_n + O (\varphi_n^2).$$

Then we write the equations

$$(x_j - x_0^*)^2 + (y_j - y_0^*)^2 = \nu^2 \tilde{\tau}_{j,n} = \rho_j^2 + 2 \nu \rho_j \tilde{\xi}_{j,n} \varphi_n + O (\varphi_n^2), \quad j = 1, \ldots, k,$$

and obtain the least squares estimator $\hat{\vartheta}_n^*$, which is consistent and has the same rate of convergence as the BE $\tilde{\vartheta}_n$ studied in this work. See details in the Section 3, [1].

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