ON THE POSITIVE FIXED POINTS OF QUARTIC OPERATORS

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Abstract

In this paper, we consider the operator of degree fourth (quartic) on $\mathbb{R}^2$. We show that the existence of positive fixed point of quartic operators and they can have up to five positive fixed points. Besides, obtained the theorems for the number of positive fixed points of quartic operators. Furthermore, gives sufficient conditions for the number of positive fixed points of quartic operators and at the end gives examples to satisfy conditions. Investigated quartic operators had arisen when examining translation-invariant Gibbs measures for models in [7] on the Cayley tree of order $k = 4$. In the paragraph 5 given model, which the study translation-invariant Gibbs measure reduced the study the fixed points of a quartic operator.

1 Quatric operators on $\mathbb{R}^2$

We introduce

$$R_+^2 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}, \quad R_+^2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}. $$

We consider the following operator $Q$ on the cone $R_+^2$:

$$Q(x, y) = \left( \sum_{i=0}^{4} C^i_4 a_i x^{4-i} y^i, \sum_{i=0}^{4} C^i_4 b_i x^{4-i} y^i \right), \tag{1}$$

where $C^i_4 = \frac{4!}{(4-i)!i!}$ (binomial coefficient), $a_i > 0$ and $b_i > 0$ for all $i \in \{0, 1, 2, 3, 4\}$. This operator is composed of two polynomials of the fourth degree in two variables, therefore we call the operator $Q$ a quartic. Clearly, an arbitrary non-trivial positive fixed point $(x_0, y_0) \in R_+^2$ of the quartic operator (QO) $Q$ is a strictly positive, i.e. $(x_0, y_0) \in R_+^2$. Denote by $N_{\mathbb{R}_+^2}(Q)$ the number of fixed points of the QO $Q$ that belongs to $R_+^2$.
Lemma 1. If \( \omega = (x_0, y_0) \in \mathbb{R}_+^2 \) is a fixed point of the QO \( Q \), then \( \xi_0 = \frac{\omega_0}{x_0} \) is root of the algebraic equation
\[
 a_4 \xi_5 + (4a_3 - b_4) \xi_4 + (6a_2 - 4b_3) \xi_3 + (4a_1 - 6b_2)\xi^2 - (a_0 - 4b_1)\xi - b_0 = 0.
\] (2)

Proof. Let the point \( \omega = (x_0, y_0) \in \mathbb{R}_+^2 \) be a fixed point of QO \( Q \). Then
\[
 \sum_{i=0}^{4} C_i^{j} a_4^i x_0^4 - i y_0^i = x_0, \sum_{i=0}^{4} C_i^{j} b_4^i x_0^4 - i y_0^i = y_0.
\]
From the notation \( \frac{\omega_0}{x_0} = \xi_0 \) we obtain
\[
 x_0^4 \left( \sum_{i=0}^{4} C_i^{j} a_4^i \xi_0^i \right) = x_0, \ x_0^4 \left( \sum_{i=0}^{4} C_i^{j} b_4^i \xi_0^i \right) = \xi_0 x_0.
\]
Hence, we have
\[
 \frac{1}{\xi_0} = \frac{1}{\sum_{i=0}^{4} C_i^{j} a_4^i \xi_0^i}.
\]
By the last equality we get
\[
 a_4 \xi_5^5 + (4a_3 - b_4) \xi_4^4 + (6a_2 - 4b_3) \xi_3^3 + (4a_1 - 6b_2)\xi^2 - (a_0 - 4b_1)\xi - b_0 = 0.
\]
Lemma 1 is proved. \( \Box \)

Lemma 2. If \( \xi_0 \) is a positive root of the algebraic equation (2), then the point \( \omega_0 = (x_0, \xi_0 x_0) \in \mathbb{R}_+^2 \) and \( \xi_0 \) is a fixed point of the QO \( Q \), where
\[
 x_0 = \frac{1}{\sqrt[4]{\sum_{i=0}^{4} C_i^{j} a_4^i \xi_0^i}}.
\] (3)

Proof. Let \( \xi_0 > 0 \) and \( \xi_0 \) is a root of the equation (2). Put \( y_0 = \xi_0 x_0 \), where \( x_0 \) is a given by the equality (3) and \( \omega_0 = (x_0, \xi_0 x_0) \). From the equality \( y_0 = \xi_0 x_0 \) we have
\[
 \sum_{i=0}^{4} C_i^{j} a_4^i x_0^4 - i y_0^i = \sum_{i=0}^{4} C_i^{j} a_4^i x_0^4 - i \xi_0 x_0^i = x_0^4 \left( \sum_{i=0}^{4} C_i^{j} a_4^i \xi_0^i \right) = \frac{1}{\sqrt[4]{\sum_{i=0}^{4} C_i^{j} a_4^i \xi_0^i}},
\]
i.e.
\[
 \sum_{i=0}^{4} C_i^{j} a_4^i x_0^4 - i y_0^i = x_0.
\]
On the other hand
\[
 a_4 \xi_5^5 + (4a_3 - b_4) \xi_4^4 + (6a_2 - 4b_3) \xi_3^3 + (4a_1 - 6b_2)\xi^2 - (a_0 - 4b_1)\xi - b_0 = 0.
\]
Then we get
\[
 b_0 + 4b_1 \xi_0 + 6b_2 \xi_0^2 + 4b_3 \xi_0^3 + b_4 \xi_0^4 = a_0 \xi_0 + 4a_1 \xi_0^2 + 6a_2 \xi_0^3 + 4a_3 \xi_0^4 + a_4 \xi_0^5 = \xi_0 \left( a_0 + 4a_1 \xi_0 + 6a_2 \xi_0^2 + 4a_3 \xi_0^3 + a_4 \xi_0^4 \right)
\]
From the last equality we have
\[
 \frac{\xi_0}{\sqrt[4]{\sum_{i=0}^{4} C_i^{j} a_4^i \xi_0^i}} = \frac{4}{\sum_{i=0}^{4} C_i^{j} b_4^i \xi_0^i} = x_0^4 \left( \sum_{i=0}^{4} C_i^{j} b_4^i \xi_0^i \right) = \sum_{i=0}^{4} C_i^{j} b_4^i x_0^4 - i y_0^i = y_0.
\]
We put
\[ \mu_0 = a_4, \mu_1 = 4a_3 - b_4, \mu_2 = 6a_2 - 4b_3, \mu_3 = 4a_1 - 6b_2, \mu_4 = a_0 - 4b_1, \mu_5 = b_0 \]
and define the polynomial \( P_5(\xi) \) of degree 5:
\[
P_5(\xi) = \mu_0 \xi^5 + \mu_1 \xi^4 + \mu_2 \xi^3 + \mu_3 \xi^2 + \mu_4 \xi - \mu_5.
\]

By the Lemmas 1, 2 the following corollary is correct.

**Corollary 1.** The number of positive fixed points of the QO \( Q \) equal to the number of positive roots of the polynomial \( P_5(\xi) \).

**Lemma 3.** The polynomial \( P_5(\xi) \) has at least one positive root.

**Proof.** Clearly, that \( P_5(0) = -\mu_5 < 0 \) and \( P_5(+\infty) = +\infty \). It means that there exists \( c > 0 \) such that \( P_4(c) = 0 \).

**Proposition 1.** The polynomial \( P_5(\xi) \) can have up to five positive roots.

**Proof.** We have the following table for the number of sign changes in the sequence of the coefficients of the polynomial \( P_5(\xi) \) (Tab. 1). By using this table and the Descartes rule, we can conclude that the polynomial \( P_5(\xi) \) can have up to five the positive roots (see [1], pp. 27-29).

| \( P_5(\xi) \) | \( \mu_0 \) | \( \mu_1 \) | \( \mu_2 \) | \( \mu_3 \) | \( \mu_4 \) | \( \mu_5 \) | the number of sign changes |
|---|---|---|---|---|---|---|---|
| 1 | + | - | + | - | + | - | 5 |

Table 1.

\[ \square \]

## 2 Sufficient conditions for number of positive fixed points of the QO \( Q \)

In this section we study a number of positive fixed points of the quartic operator \( Q \).

We use the following designations
\[
Q = \left(\frac{a}{3}\right)^3 + \left(\frac{b}{2}\right)^2,
\]
where
\[
a = -\frac{p^2}{12} - r, \quad b = -\frac{p^3}{108} + \frac{pr}{3} - \frac{q^2}{8},
\]
and
\[
p = \frac{15\mu_0\mu_2 - 6\mu_4^2}{25\mu_0^2}, \quad q = \frac{50\mu_2^2\mu_3 + 8\mu_4^3 - 30\mu_0\mu_4\mu_2}{125\mu_0^3},
\]
\[
r = \frac{15\mu_0\mu_2^2\mu_2 - 50\mu_2^2\mu_1\mu_3 - 3\mu_4^3 - 125\mu_0^3\mu_4}{625\mu_0^3}.
\]

When \( Q < 0 \), we define following number
\[
z_0 = 2\sqrt{-\frac{a}{3} \cos \left(\frac{2\pi}{3} + \frac{\alpha}{3}\right)} - \frac{p}{3},
\]
where
\[
\cos \alpha = -\frac{b}{2} \left(\frac{3}{a}\right)^{\frac{3}{2}}, \quad \alpha \in [0, \pi].
\]
We also introduce
\[
\xi_{1,2}^{ext} = \frac{1}{2} \left( \sqrt{2z_0 + 4 \left( \frac{p}{2} + z_0 + \frac{q}{2 \sqrt{2z_0}} \right)} - \frac{\mu_1}{5\mu_0} \right),
\]
\[
\xi_{3,4}^{ext} = \frac{1}{2} \left( -\sqrt{2z_0 + 4 \left( \frac{p}{2} + z_0 - \frac{q}{2 \sqrt{2z_0}} \right)} - \frac{\mu_1}{5\mu_0} \right).
\]

Let us note that if \( z_0 > 0 \) then the numbers \( \xi_{1,2}^{ext}, \xi_{3,4}^{ext} \) are real.

For \( z_0 > 0 \), denote
\[
\xi_{\min} = \min \{ \xi_{1,2}^{ext}, \xi_{3,4}^{ext} \}, \quad \xi_{\max} = \max \{ \xi_{1,2}^{ext}, \xi_{3,4}^{ext} \}.
\]

**Theorem 1.** Let \( Q < 0, z_0 > 0, \xi_{\max} < 0 \). Then the QO has a unique fixed point in \( \mathbb{R}^2_+ \), i.e., \( N_{f^{fix}}^{Q} (Q) = 1 \).

**Proof.** By the Corollary 1 quite enough to find the positive roots of the polynomial \( P_5(\xi) \).

Let \( Q < 0 \). Then by the Cardano’s formula cubic equation \( \eta^3 + a\eta + b = 0 \) has three real roots. They have the following form [2]:
\[
\eta_k = 2 \sqrt{\frac{a}{3}} \cos \left( \frac{\alpha + 2\pi(k - 2)}{3} \right), \quad k = 1, 2, 3,
\]
where
\[
\cos \alpha = -\frac{2}{3} \left( -\frac{a}{3} \right)^{\frac{3}{2}}, \quad \alpha \in [0, \pi].
\]

We have \( \eta_3 < \eta_1 < \eta_2 \). The number \( z_0 = \eta_3 - \frac{5}{3}(z_0 > 0) \) is the least root of the following cubic equation:
\[
z^3 + pz^2 + \frac{q^2}{4} - z - \frac{q^2}{8} = 0.
\]

On the other hand if all roots are strictly positive of the equation [5], then all roots \( \omega_j, \quad (j = 1, \ldots, 4) \) of the equation
\[
\omega^4 + p\omega^3 + q\omega + r = 0
\]
are real [3]. By the Ferrari’s method the numbers \( \xi_{j}^{ext} = \omega_j - \frac{\mu_1}{5\mu_0} (j = 1, \ldots, 4) \) are roots of the equation \( P_5^2(\xi) = 0 \), i.e.
\[
5\mu_0 \xi^4 + 4\mu_0 \xi^3 + 3\mu_0 \xi^2 + 2\mu_0 \xi + \mu_0 = 0.
\]
Thus, the numbers \( \xi_{j}^{ext} (j = 1, \ldots, 4) \) are the local extreme points of the function \( P_5(\xi) \). Besides \( \xi_{\max} = \max_{j \in \{1, \ldots, 4\}} \xi_{j}^{ext} < 0 \) and \( P_5(\infty) = +\infty \). Therefore the function \( P_5(\xi) \) is an increasing on the set \( (\xi_{\max}, +\infty) \). Consequently, by the inequality \( P_5(0) < 0 \) the polynomial \( P_5(\xi) \) has a unique positive root.

**Theorem 2.** Let \( Q < 0, z_0 > 0, \xi_{\min} > 0 \). If the polynomial \( P_5(\xi) \) satisfies one of the following conditions
(a) \( P_5(\xi_{\min}) = 0 \),
(b) \( P_5(\xi_{\max}) = 0 \),
then the QO has at least two fixed points in \( \mathbb{R}^2_+ \), i.e., \( N_{f^{fix}}^{Q} (Q) \geq 2 \).

**Proof.** Let \( Q < 0, z_0 > 0, \xi_{\min} > 0 \). Then all four roots of the equation [1] is real and they strictly positive.

And so the function \( P_5(\xi) \) is an increasing on the set \( (-\infty, \xi_{\min}) \cup (\xi_{\max}, +\infty) \).

(a) \( P_5(\xi_{\min}) = 0 \). Then \( \max_{\xi \in (-\infty, \xi_{\min})} P_5(\xi) = P_5(\xi_{\min}) = 0 \) and the number \( \xi_1 = \xi_{\min} \) is the root of the polynomial \( P_5(\xi) \). Besides we know, that the function \( P_5(\xi) \) is an increasing on the set \( (\xi_{\max}, +\infty) \). Therefore exist \( \xi_2 \in (\xi_{\max}, +\infty), \) where \( P_5(\xi_2) = 0 \). And so the polynomial \( P_5(\xi) \) has at least two strictly positive roots.

(b) \( P_5(\xi_{\max}) = 0 \). Then \( \min_{\xi \in [\xi_{\max}, +\infty]} P_5(\xi) = P_5(\xi_{\max}) = 0 \) and the number \( \xi_1 = \xi_{\max} \) is the root of the polynomial \( P_5(\xi) \). The function \( P_5(\xi) \) is an increasing on the set \( [\xi_{\max}, +\infty) \), so exist \( \xi_0 \neq \xi_{\max} \) which the function \( P_5(\xi) \) is decreasing on the set \( (\xi_0, \xi_{\max}) \). Also we have \( P_5(0) < 0 \) and \( \max_{\xi \in \{\xi_0, \xi_{\max}\}} P_5(\xi) = P_5(\xi_0) > 0 \). Therefore the polynomial \( P_5(\xi) \) has another unique positive root \( \xi_2 \) on the set \( (0, \xi_{\min}) \).

**Theorem 3.** Let \( Q < 0, z_0 > 0, \xi_{\min} > 0 \). If for the polynomial \( P_5(\xi) \) following condition

\[
\]
(c) $P_5(\xi_{\text{min}}) > 0$, $P_5(\xi_{\text{max}}) < 0$

is satisfies, then the QO $Q$ has at least three fixed points in $\mathbb{R}^2_+$, i.e., $N_{\text{fix}}^\{(Q)\} \geq 3$.

Proof. Let $Q < 0$, $z_0 > 0$, $\xi_{\text{min}} > 0$. Then all four roots of the equation 3 is real and they strictly positive. And so the function $P_5(\xi)$ is an increasing on the set $(-\infty, \xi_{\text{min}}) \cup (\xi_{\text{max}}, +\infty)$.

(c) Let $P_5(\xi_{\text{min}}) > 0$, $P_5(\xi_{\text{max}}) < 0$. We have known that $P_5(0) < 0$ and $P_5(+\infty) = +\infty$. It indicates that the polynomial $P_5(\xi)$ has at least three positive roots, such that $\xi_1 \in (0, \xi_{\text{min}})$, $\xi_2 \in (\xi_{\text{min}}, \xi_{\text{max}})$ and $\xi_3 \in (\xi_{\text{max}}, +\infty)$.

In the next, we assume that $Q < 0$, $z_0 > 0$. Then we can define following set of real numbers (set of extremal points of the function $P_5(\xi)$):

$$\mathfrak{A} = \{\xi_{\text{ext}}^1, \xi_{\text{ext}}^2, \xi_{\text{ext}}^3, \xi_{\text{ext}}^4\}.$$

Points of the set of $\mathfrak{A}$ are real and they are different. Therefore we can have denote following

$$\lambda_1 = \xi_{\text{min}}, \quad \lambda_2 = \min (\mathfrak{A}\\{\xi_{\text{min}}\}), \quad \lambda_3 = \max (\mathfrak{A}\\{\xi_{\text{max}}\}), \quad \lambda_4 = \xi_{\text{max}}.$$

Consequently, we have the following relation

$$\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4.$$

It follows the function $P_5(\xi)$ is an increasing (decreasing) on the set $(-\infty, \lambda_1) \cup (\lambda_2, \lambda_3) \cup (\lambda_4, +\infty)$ ($(\lambda_1, \lambda_2) \cup (\lambda_3, \lambda_4)$). The function $P_5(\xi)$ has a local maximum value at the points $\lambda_1, \lambda_3$ and local minimum values at the points $\lambda_2$ and $\lambda_4$. Using these property and by the corollary 1, we can give the following table:

| On the number of the positive fixed points of the quartic operators, when $Q < 0$, $z_0 > 0$, $\lambda_1 > 0$. | Sufficient condition: $P_5(\lambda_1) < 0$, $P_5(\lambda_3) < 0$ | $N_{\text{fix}}^\{(Q)\}$ |
|---|---|---|
| 1. | $P_5(\lambda_1) < 0$, $P_5(\lambda_3) < 0$ | 1 |
| 2. | $P_5(\lambda_1) = 0$, $P_5(\lambda_3) < 0$ | 2 |
| 3. | $P_5(\lambda_1) < 0$, $P_5(\lambda_3) = 0$ | |
| 4. | $P_5(\lambda_2) > 0$, $P_5(\lambda_4) = 0$ | |
| 5. | $P_5(\lambda_2) = 0$, $P_5(\lambda_4) > 0$ | 3 |
| 6. | $P_5(\lambda_2) > 0$, $P_5(\lambda_4) = 0$ | 4 |
| 7. | $P_5(\lambda_1) = 0$, $P_5(\lambda_4) = 0$ | |
| 8. | $P_5(\lambda_1) = 0$, $P_5(\lambda_3) = 0$ | |
| 9. | $P_5(\lambda_2) = 0$, $P_5(\lambda_4) = 0$ | |
| 10. | $P_5(\lambda_2) > 0$, $P_5(\lambda_4) < 0$ | |
| 11. | $P_5(\lambda_1) < 0$, $P_5(\lambda_3) > 0$, $P_5(\lambda_4) < 0$ | |
| 12. | $P_5(\lambda_1) > 0$, $P_5(\lambda_2) < 0$, $P_5(\lambda_4) > 0$ | |
| 13. | $P_5(\lambda_1) = 0$, $P_5(\lambda_3) > 0$, $P_5(\lambda_4) < 0$ | |
| 14. | $P_5(\lambda_1) > 0$, $P_5(\lambda_3) = 0$ | |
| 15. | $P_5(\lambda_2) = 0$, $P_5(\lambda_4) < 0$ | |
| 16. | $P_5(\lambda_1) > 0$, $P_5(\lambda_2) < 0$, $P_5(\lambda_4) = 0$ | 5 |
| 17. | $P_5(\lambda_1) > 0$, $P_5(\lambda_2) < 0$, $P_5(\lambda_3) > 0$, $P_5(\lambda_4) < 0$ | |

Table 2.

3 Examples

In this section we give the concrete examples, which the QO $Q$ has the three and the five positive fixed points.

Example 1. We consider an example for QO $Q$, which satisfies the 7th condition in the table 2. We define QO $Q$ by the equality:

$$Q(x, y) = (52x^4 + 8x^3y + 42x^2y^2 + 4xy^3 + y^4, 16x^4 + 4x^3y + 72x^2y^2 + 5xy^3 + 14y^4).$$
Then for the QO \( Q \) we have
\[
a_0 = 52, \ a_1 = 2, \ a_2 = 7, \ a_3 = 1, \ a_4 = 1, \ b_0 = 16, \ b_1 = 1, \ b_2 = 12, \ b_3 = \frac{5}{4}, \ b_4 = 1.
\]
Therefore we get
\[
\mu_0 = a_4 = 1, \ \mu_1 = 4a_3 - b_4 = -10, \ \mu_2 = 6a_2 - 4b_3 = 37, \ \mu_3 = 4a_1 - 6b_2 = -64,
\]
\[
\mu_4 = a_0 - 4b_1 = 52, \ \mu_5 = b_0 = 16.
\]
For the QO \( Q \) the polynomial \( P_5(\xi) \) has the following form
\[
P_5(\xi) = \xi^5 - 10\xi^4 + 37\xi^3 - 64\xi^2 + 52\xi - 16.
\]
Then we have
\[
Q = -\frac{7}{12500}, \quad z_0 = \frac{1}{20} (13 - \sqrt{105}),
\]
\[
\xi_1^{\text{ext}} = 1, \quad \xi_2^{\text{ext}} = 2, \quad \xi_3^{\text{ext}} = \frac{1}{10} (25 - \sqrt{105}), \quad \xi_4^{\text{ext}} = \frac{1}{10} (25 + \sqrt{105}).
\]
By the denotes we have
\[
\lambda_1 = 1, \quad \lambda_2 = \frac{1}{10} (25 - \sqrt{105}), \quad \lambda_3 = 2, \quad \lambda_4 = \frac{1}{10} (25 + \sqrt{105}).
\]
So \( Q < 0, \quad z_0 > 0, \quad \lambda_1 > 0, \quad P_5(\lambda_1) = P_5(1) = 0, \quad P_5(\lambda_3) = P_5(2) = 0. \)

By the 7th condition in the table 2, the QO \( Q \) has three positive fixed points.

It is easy to verify that the polynomial \( P_5(\xi) \) has three positive roots and they have the following form:
\[
\xi_{1,2} = 1, \quad \xi_{3,4} = 2, \quad \xi_5 = 4.
\]

By the Corollary 1 the QO \( Q \) has three positive fixed points.

**Example 2.** Now, we consider the following QO \( Q \):
\[
Q(x, y) = \left( 31x^4 + \frac{1}{2}x^3y + \frac{43}{3}x^2y^2 + \frac{1}{4}xy^3 + \frac{1}{5}y^4, 10x^4 + x^3y + 31x^2y^2 + \frac{2}{3}xy^3 + 3y^4 \right).
\]

Then
\[
P_5(\xi) = \frac{1}{5} \xi^5 - \frac{11}{40} \xi^4 + \frac{41}{3} \xi^3 - \frac{61}{2} \xi^2 + 30\xi - 106.
\]

Also
\[
a = -\frac{7}{3}, \quad b = -\frac{20}{27}, \quad Q = -\frac{1}{3},
\]
and
\[
z_0 = \frac{1}{8}.
\]

Consequently,
\[
\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3, \quad \lambda_4 = 5.
\]

The following relations are true:
\[
Q = -\frac{1}{3} < 0, \quad z_0 = \frac{1}{8} > 0, \quad \lambda_1 = 1 > 0,
\]
\[
P_5(\lambda_1) = P_5(1) = \frac{37}{60} > 0, \quad P_5(\lambda_2) = P_5(2) = -\frac{4}{15} < 0,
\]
\[
P_5(\lambda_3) = P_5(3) = \frac{7}{20} > 0, \quad P_5(\lambda_4) = P_5(5) = -\frac{95}{12} < 0,
\]

By the 17th condition in the table 2, the QO \( Q \) has five positive fixed points.
4 Application

In this section, we show that the number of fixed points of the operator $\mathbb{Q}$ corresponds to a non-trivial positive fixed point of the Hammerstein integral operator, which plays an important role in the theory of Gibbs measures.

A Cayley tree $\Gamma^k = (V, L)$ of order $k \geq 1$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k+1$ edges incident to each vertex. Here $V$ is the set of vertices and $L$ that of edges.

Consider models where the spin takes values in the set $[0, 1]$. We introduce the Hamiltonian $H_{\mathbb{Q}}$ on $C_{+}[0, 1]$ defined by:

$$(R_k f)(t) = \left( \frac{1}{0} \int K(t, u)f(u)du \right)^k, \quad k \in \mathbb{N}$$

where $K(t, u) = \exp(J/\beta u)$, $f(t) > 0, t, u \in [0, 1]$.

In [1,2] was shown that a translation-invariant Gibbs measure of the model $H_{\mathbb{Q}}$ corresponds to a solution $f(t) \in C_{+}[0, 1]$ of the following equation:

$$(R_k f)(t) = f(t). \quad (10)$$

For every $k \geq 2$ we define the Hammerstein integral operator $H_k$ acting in $C_{+}[0, 1]$ as following:

$$(H_k f)(t) = \frac{1}{0} \int K(t, u)f^k(u)du.$$

**Lemma 4.** Let $k \geq 2$. The equation (10) has a non-trivial positive solution iff the Hammerstein integral operator has a non-trivial positive fixed point and

$$N^f_{\mathbb{Q}}(R_k) = N^f_{\mathbb{Q}}(H_k),$$

where $N^f_{\mathbb{Q}}(T)$ is a number of non-trivial positive fixed points of the operator $T$.

**Proof.** At first, we notice that the equation (10) has at least one solution in $C_{+}[0, 1]$ (see Theorem 3.5 in [3]).

**Necessity.** Let $k \geq 2$ and $f(t) \in C_{+}[0, 1]$ be a solution of the (10), then $f(0) = 1$. Define the following function

$$g(t) = \frac{1}{0} \int K(0, u)f(u)du$$

where $0 < \int K(0, u)f(u)du < 1$. From $f(0) = 1$, $g$ corresponds to exactly one function $f$. Then we have

$$(H_k g)(t) = g(t).$$
Sufficiency. Let $k \geq 2$ and $g = g(t) \in C^0_t[0,1]$ be a fixed point of the operator $H_k$. It is clear that $\int_0^1 K(0,u)g^k(u)du = g(0) > 0$. Define non-trivial positive continuous function

$$f(t) = \left(\frac{g(t)}{g(0)}\right)^k.$$  \hspace{1cm} (12)

Because of $g$ is a fixed point of the operator $H_k$, the last equality is well-defined, i.e., each function $f$ corresponds to exactly one function $g$.

Then

$$(R_k f)(t) = \left(\frac{1}{0} K(t,u)f(u)du\right)^k \left(\frac{1}{0} K(0,u)g^k(u)du\right)^k = \left(\frac{g(t)}{g(0)}\right)^k = f(t).$$

It is easy to check that the relations (11) and (12) establish a one-to-one correspondence between $F_\alpha = \{f(t) \in C^0_t[0,1] : R_k f = f\}$ and $G_\alpha = \{g(t) \in C^0_t[0,1] : H_k g = g\}$.

Consequently $N^{fix}_+(R_k) = N^{fix}_+(H_k)$.

This completes the proof. $\square$

From Lemma 4 we can conclude that the number of the non-trivial positive fixed points of the operator $H_k$ is equal to the number of translation-invariant Gibbs measures of the model $H$. \hspace{1cm} (9)

Let functions $\phi_1(t)$, $\varphi_2(t)$ and $\phi_3(t)$, $\phi_4(t)$ belong to $C^\infty_t[0,1]$. We consider the Hamiltonian $H_{14}$ on the Cayley tree of order $k = 4$ with the function of potential

$$\xi_{1,4} = \frac{1}{J\beta} \ln \left(\phi_1(t)\varphi_2(u) + \phi_2(t)\varphi_2(u)\right).$$  \hspace{1cm} (13)

In a similar manner, we can prove that for the number non-trivial positive fixed points of the integral operator $H_{14}$ the following equality holds, as in the \hspace{1cm} (11) (see Lemma 7):

$$N^{fix}_+(H_{14}) = N^{fix}_+(Q).$$

By the last equality we can conclude that number of nontrivial positive fixed points of the quartic operator $Q$ \hspace{1cm} (11) on the $\mathbb{R}^2$ is equal to number of translation-invariants Gibbs measures for the model $H$ \hspace{1cm} (11) on the Cayley tree of order four.

**Open problem.** It has not been constructed yet the exact function of potential $\xi_{1,4}$ \hspace{1cm} (13) for satisfy the conditions of the Theorems 2-3. This requires constructing non-trivial positive continuous functions $\varphi_1(t)$, $\varphi_2(t)$ and $\phi_3(t)$, $\phi_4(t)$ in such a way that the corresponding coefficients

$$a_i = \int_0^1 \varphi_1(u)\phi_3^{k-i}(u)\phi_4^{i}(u)du, \quad b_i = \int_0^1 \varphi_2(u)\phi_3^{k-i}(u)\phi_4^{i}(u)du, \quad i = 0,\ldots,4$$

of the nonlinear operator $Q$ need satisfy the conditions of the Theorems 2-3 (as in the above examples 1-2).

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