Geometrical bounds of the irreversibility in classical and open quantum systems

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We derive geometrical bounds on the irreversibility for both classical and open quantum systems that satisfy the detailed balance conditions. Using the information geometry, we prove that the irreversible entropy production is bounded from below by a Wasserstein-like distance between the initial and final states, thus generalizing the Clausius inequality. The Wasserstein-like metric can be regarded as a discrete-state generalization of the Wasserstein metric, which plays an important role in the optimal transport theory. Notably, the derived bounds closely resemble classical and quantum speed limits, implying that the minimum time required to transform a system state is constrained by the associated entropy production.

Introduction.— Irreversibility, which is quantified by entropy production, is a fundamental concept in classical and quantum thermodynamics [1–3]. Most macroscopic phenomena in nature are irreversible, despite that physical processes at the microscopic level are time symmetric in general. According to the second law of thermodynamics, a system undergoing an irreversible process is associated with a positive production of entropy on average, ∆S_{tot} ≥ 0. This bound can be saturated only when operations are performed in the infinite-time quasistatic limit. However, real processes must be completed in a finite time, thus being accompanied by a certain dissipation. Having a tighter lower bound on entropy production not only deepens our understanding of how much heat needs to be dissipated but also provides insights into building quantum technologies such as quantum computation [4] and quantum heat engines [5].

In recent years, there has been a great effort to characterize dissipation of thermodynamic processes via means of information geometry [6–16]. The relation of the irreversible entropy production and the distance between thermodynamic states was well established for classical systems near equilibrium [17, 18]. Moreover, a lower bound on dissipation in terms of the Wasserstein distance [19] has been derived for nonequilibrium systems governed by Langevin equations [20–22]. For a closed driven quantum system, Ref. [23] showed that the entropy production is bounded from below by the Bures length between the final state and the corresponding equilibrium state. Following a similar approach, Ref. [24] determined a geometrical upper bound for equilibration processes of open quantum systems. With the help of information geometry, other important relations, such as speed limits [25–28] and a trade-off between efficiency and power of microscopic heat engines [29], have been successfully derived.

In this Letter, we enlarge the family of these universal relations by investigating classical and open quantum systems that satisfy the detailed balance conditions. Examples include equilibration processes, which have received a considerable interest in the field of nonequilibrium physics [30–32]. Specifically, we derive geometrical lower bounds on the irreversible entropy production for both classical and open quantum systems described by master equations. The spaces of discrete distributions and quantum states are treated as Riemannian manifolds, on which the time evolution of the system state is described by a smooth curve. By defining a Wasserstein-like metric, we prove that the entropy production is bounded from below by the square of the geodesic distance between the initial and final states divided by the process time. The derived bounds are stronger than the conventional inequality of the second law, thus can be considered as generalizations of the Clausius inequality. The equality is attained only when the path described by the system dynamics is a geodesic. The Wasserstein-like metric is a discrete-state generalization of the Wasserstein metric, which measures the distance between two distributions and is widely used in optimal transport problems [19]. Interestingly, the obtained inequalities can be interpreted as speed limits [33–37] for classical and quantum systems, establishing a trade-off relation between the speed of the state transformation and dissipation cost. We numerically illustrate the results on three systems: classical two- and three-level models and an equilibration process of a two-level atom.

Riemannian geometry.— We briefly describe some concepts of the Riemannian geometry, which are used in this study. Let M be a smooth Riemannian manifold equipped with a metric g_p on the tangent space at each point p ∈ M. Note that such metrics can be infinitely defined as long as they satisfy the linearity, symmetry, and positive-definite conditions. For example, for classical discrete-state systems, M can be the collection of discrete distributions p = [p_1, . . ., p_N]^\top, where p_n ≥ 0 and \sum_{n=1}^{N} p_n = 1. In the quantum case, M can be the space of density operators ρ, which are positive and have unit trace, ρ > 0 and tr{ρ} = 1. Given a smooth curve \{γ(t)\}_{t∈S^b} on the manifold, its length ℓ(γ) can be defined in terms of γ(t), among which the dot denotes...
the time derivative. Then, the geodesic distance between two points \( p \) and \( q \) can be defined as the minimum length over all smooth curves \( \gamma \) connecting these points, \( d(p, q) = \inf_{\gamma} \{ \ell(\gamma) \} \). Throughout this Letter, we use notation \( \langle \cdot, \cdot \rangle \) for the scalar inner product, i.e., \( \langle x, y \rangle = x^\top y \) for the classical case and \( \langle X, Y \rangle = \text{tr} \{ X^\top Y \} \) for the quantum case.

**Bounds for classical systems.**— First, we consider a discrete-state system during a time period \( \tau \). The system is in contact with a heat bath at the inverse temperature \( \beta = 1/T \). The stochastic transitions between states occur due to the interaction with the heat bath. The dynamics obey a time-continuous Markov jump process and are described by the master equation,

\[
\dot{p}_n(t) = \sum_{m \neq n} [R_{nm}(t)p_m(t) - R_{mn}(t)p_n(t)],
\]

where \( p_n(t) \) is the probability to find the system in state \( n \) at time \( t \) and \( R_{nm}(t) \) is the transition rate from state \( n \) to state \( m \) (\( 1 \leq n \neq m \leq N \)), which can be time-dependent. We assume that the transition rates satisfy the detailed balance conditions, \( R_{nm}(t)e^{-\beta E_n(t)} = R_{mn}(t)e^{-\beta E_m(t)} \) for all \( m \neq n \), where \( E_n(t) \) is the instantaneous energy of state \( n \) at time \( t \). When the transition rates are time-independent, the system always relaxes to a unique equilibrium state after a sufficiently long time, irrespective of the initial state. Herein, we define the instantaneous equilibrium state \( p^{eq}(t) \) as \( p^{eq}_n(t) \propto e^{-\beta E_n(t)} \).

According to the framework of stochastic thermodynamics [1], the irreversible entropy production \( \Delta S_{tot} \) is quantified via changes in the system’s Shannon entropy and the heat flow dissipated into the environment. Specifically, \( \Delta S_{tot} = \int_0^\tau \sigma_{tot}(t) dt \), where \( \sigma_{tot}(t) = \sigma(t) + \sigma_m(t) \) is the total entropy production rate and \( \sigma(t) = \sum_{mn} R_{nm}p_m \ln(p_n/p_m) \) and \( \sigma_m(t) = \sum_{mn} R_{mn}p_n \ln(R_{mn}/R_{nm}) \) are the rates of system and medium entropy productions, respectively. Using the detailed balance conditions, the entropy production rate can be explicitly calculated as \( \sigma_{tot}(t) = \{ \langle f(t), \dot{p}(t) \rangle \} \), where \( \langle f(t), \dot{p}(t) \rangle = -\nabla_p D(p(t)\parallel p^{eq}(t)) \) is the vector of thermodynamic forces. Here, \( D(p\parallel q) = \sum p \ln(p/q) \) is the Kullback–Leibner (KL) divergence between distributions \( p \) and \( q \), and \( \nabla_p := [\partial p_1, \ldots, \partial p_N]^\top \) denotes the gradient with respect to \( p \). The second law of thermodynamics, \( \Delta S_{tot} \geq 0 \), can be obtained from the positivity of the entropy production rate \( \sigma_{tot}(t) \). In what follows, we will derive sharper lower bounds of \( \Delta S_{tot} \) in terms of geometrical distances between the initial state \( p(0) \) and the final state \( p(\tau) \).

The master equation [Eq. (1)] can be rewritten in an alternative way as \[ K_p(t) := \sum_{n \neq m} R_{nm}(t)p_m(t)\Phi\left( \frac{p_n(t)}{p_m(t)} \right)E_{nm} \]

where \( K_p(t) \) is a symmetric positive semi-definite matrix, given by

\[
\ell_c(\gamma) := \tau \int_0^\tau \langle f(t), \dot{p}(t) \rangle dt.
\]

The distance between two points \( p_0 \) and \( p_\tau \) is then reads

\[
W_c(p_0, p_\tau) := \inf_{\gamma} \ell_c(\gamma),
\]

where the infimum is taken over all smooth curves \( \{ \gamma(t) \} \) that connect \( p_0 \) and \( p_\tau \) on the manifold. It is evident that \( W_c \) is a measure of distance between points. The quadratic term \( \langle f, K_p f \rangle \) in the integral is known as the dissipation function [39] and closely related to the thermodynamic divergence of a path [17]. In the context of optimal transport theory, \( W_c \) can be regarded as a Wasserstein-like distance, which is an extension of Benamou–Brenier flow formulation of the original \( L^2 \)-Wasserstein distance for discrete spaces [19, 42]. In practice, \( W_c \) can be numerically calculated using the geodesic equations, whose solution determines the shortest path between two points. Since \( \sigma_{tot}(t) = \{ \langle f(t), \dot{p}(t) \rangle \} = \{ \langle f(t), K_p(t)f(t) \rangle \}, \sqrt{\tau \Delta S_{tot}} \) is the thermodynamic length of the path described by the system dynamics. As the first main result, we obtain the following bound:

\[
\Delta S_{tot} \geq \frac{W_c(p(0), p(\tau))^2}{\tau}.
\]

Inequality (5) provides a stronger bound than the Clausius inequality of the second laws and is valid as long as the transition rates satisfy the detailed balance conditions. It indicates that the entropy production is bounded from below by the Wasserstein-like distance between the initial and final distributions. Another physical implication is that the geometrical bound imposes a constraint on the space of distributions that are accessible in a limited time from the initial state using a fixed dissipation budget. From a geometrical point of view, Eq. (5) can be considered a discrete-state generalization of the relation between dissipation and the Wasserstein distance.
which has been studied for continuous-state Langevin dynamics [20, 22].

The distance \( W_c \) can be further bounded from below by the total variation distance, \( d_1(p_0, p_\tau) = \sum_{n=1}^N |p_{0n} - p_{\tau n}| \), as [38]

\[
W_c(p_0, p_\tau)^2 \geq \frac{d_1(p_0, p_\tau)^2}{2A_1},
\]

where \( A_1 := \tau^{-1} \int_0^\tau \sum_{mn} R_{mn}(t) \gamma_n(t) dt \) is the average dynamical activity along the geodesic path \( \{ \gamma(t) \} \) and characterizes the time scale of the system. The dynamical activity indicates the time-symmetric changes in the system and plays important roles in studies of nonequilibrium phenomena [43]. From Eqs. (5) and (6), classical speed limits of the state transformation can be obtained

\[
\tau \geq \frac{W_c(p_0, p_\tau)^2}{\Delta S_{tot}} \geq \frac{d_1(p_0, p_\tau)^2}{2\Delta S_{tot}A_1}.
\]  

These inequalities imply a trade-off relation between the time needed to transform the system state and physical quantities such as entropy production and dynamical activity, i.e., fast transformation necessitates high dissipation and freeness. The last bound in inequality (7) is analogous to but distinct from a bound derived in Ref. [37], where \( A_1 \) is replaced by the average dynamical activity along the path described by the time evolution of the system.

Bounds for open quantum systems.— Next, we consider an open quantum system which is weakly coupled to a heat bath at the inverse temperature \( \beta \). The time evolution of the density operator \( \rho(t) \) of the system is described by the Lindblad master equation [44, 45],

\[
\dot{\rho} = \mathcal{L}(\rho) := -i[H(t), \rho] + \mathcal{D}(\rho),
\]

where \( \mathcal{L} \) is the Lindblad operator, \( H(t) \) is the Hamiltonian, and \( \mathcal{D}(\rho) \) is the dissipator given by

\[
\mathcal{D}(\rho) := \sum_{\mu, \nu} \alpha_{\mu}(\omega) \left[ 2L_{\mu}(\omega)\rho L_{\mu}(\omega) - \{L_{\mu}(\omega) L_{\mu}(\omega), \rho\} \right].
\]

Here, \( [A, B] = AB + BA \) is the anti-commutator and \( L_{\mu}(\omega) \) is a jump operator that satisfies \( L_{\mu}^\dagger(\omega) = L_{\mu}(\omega) \) and \( \{L_{\mu}(\omega), H\} = \omega L_{\mu}(\omega) \). Note that jump operators and coupling coefficients can be time-dependent: however, the time notation is omitted for simplicity. We assume the detailed balance conditions \( \alpha_{\mu}(\omega) e^{i\omega} = \alpha_{\mu}(-\omega) \) and the system is ergodic [46] (i.e., \( \{L_{\mu}(\omega), X\} = 0 \) for all \( \mu, \omega \) if and only if \( \omega \) is proportional to the identity operator). These assumptions are sufficient conditions for the Gibbs state \( \rho^eq(t) := e^{-\beta H(t)} / Z_\beta(t) \) to be the instantaneous stationary state of the Lindblad master equation, i.e., \( \mathcal{L}[\rho^eq(t)] = 0 \) [47, 48]. Here, \( Z_\beta(t) := \text{tr} \{e^{-\beta H(t)}\} \) is the partition function.

The irreversible entropy production during the time period \( \tau \) is \( \Delta S_{tot} = \int_0^\tau \sigma_{tot}(t) dt \), where \( \sigma_{tot}(t) = \dot{S} + \beta \dot{Q} \) is the entropy production rate. Here, \( \dot{S} = -\text{tr} \{\dot{\rho}(t) \ln \rho(t)\} \) denotes the von Neumann entropy flux of the system and \( \dot{Q} = -\text{tr} \{H(t) \dot{\rho}(t)\} \) denotes the heat flux dissipated from the system to the bath. The entropy production rate can be rewritten as \( \sigma_{tot}(t) = -\langle \ln \rho(t) - \ln \rho^eq(t), \dot{\rho}(t) \rangle \) and is nonnegative due to the monicity of the relative entropy under a complete positive trace preserve map, from which the Clausius inequality \( \Delta S_{tot} \geq 0 \) can be obtained.

We construct an operator \( K_\rho \) such that the Lindblad master equation [Eq. (8)] can be alternatively expressed as \( \dot{\rho} = K_\rho(\rho) - \ln \rho + \ln \rho^eq \) [38]. For arbitrary density operator \( \rho \), we define a tilted operator \( \{\tilde{\rho}\rho(A) := e^{-\theta/2} \int_0 e^{i\theta} \rho A e^{-\theta} ds, \theta \text{ is a real number.} \}

Using this operator, \( K_\rho \) can be explicitly constructed as \( K_\rho(\nu) := i\beta^{-1}[\nu, \rho] + O_\rho(\nu) \), where \( O_\rho(\nu) \) is a self-adjoint positive operator given by \( O_\rho(\nu) := \sum_{\mu, \omega} e^{-\beta \omega} 2\alpha_{\mu}(\omega) [L_{\mu}(\omega), [\rho, \beta \omega ([L_{\mu}(\omega), \nu])] \), which can be interpreted as a quantum analogue of the Onsager operator. For arbitrary smooth curve \( \{\gamma(t)\} \), there exists a unique vector field of traceless self-adjoint operators \( \{\nu(t)\} \) such that \( \dot{\gamma}(t) = K_\rho[\nu(t)] \) for all \( t \). Analogous to the classical case, one can define a metric such that the gradient flow of the entropy production equals to the flow associated to the system dynamics [49, 50]. Specifically, we define the metric \( g_{\nu}(\gamma, \gamma) = \langle \nu, \gamma, \gamma \rangle \), which is always nonnegative because \( \langle \nu, K_\rho(\nu) \rangle \geq 0 \). Based on this quantum metric, the thermodynamic length of a path \( \{\gamma(t)\}_{0 \leq t \leq \tau} \) can be measured as

\[
\ell_q(\gamma)^2 := \tau \int_0^{\tau} \langle \nu, K_\rho(\nu) \rangle dt.
\]

The distance between two states \( \rho_0 \) and \( \rho_\tau \) is then defined as \( W_q(\rho_0, \rho_\tau) = \inf_{\nu} \{\ell_q(\gamma)\} \), where the infimum taken over smooth curves with end points \( \gamma(0) = \rho_0 \) and \( \gamma(\tau) = \rho_\tau \). In the classical limit, \( W_q \) reduces to \( W_c \), thus \( W_q \) is a natural quantum analogue of the classical Wasserstein-like distance. Setting \( \phi := -\langle \ln \rho - \ln \rho^eq \rangle + \text{tr} \{\ln \rho - \ln \rho^eq\} \), \( \phi \) is a traceless self-adjoint operator and satisfies \( \dot{\rho} = K_\rho(\phi) \). Then, the entropy production rate can be expressed as \( \sigma_{tot}(t) = \langle \phi, \dot{\rho} \rangle = \langle \phi, K_\rho(\phi) \rangle \). Consequently, a geometrical bound of the entropy production can be obtained as the second main result,

\[
\Delta S_{tot} \geq \frac{W_q(\rho_0, \rho_\tau)^2}{\tau}.
\]

Inequality (11) provides a sharper bound than the conventional second laws of thermodynamics. Moreover, it can also be interpreted as a quantum speed limit, providing a lower bound of the time required to transform the system state in terms of dissipation and the geometrical distance between states. Because it is difficult to explicitly compute the distance \( W_q \) in general, we provide a lower bound of \( W_q \) in terms of the trace-like (or quantum total variation) distance \( d_2(\rho_0, \rho_\tau) = \sum_{n=1}^N |a_n - b_n| \), where \( a_1 \leq a_2 \leq \cdots \leq a_N \) and \( b_1 \leq b_2 \leq \cdots \leq b_N \) are
Increasing eigenvalues of \( \rho_0 \) and \( \rho_f \), respectively. Specifically, we prove that \( \mathcal{W}_c(\rho_0, \rho_f) \geq d_2(\rho_0, \rho_f)^2/4 \mathcal{A}_2 \), where \( \mathcal{A}_2 := \tau^{-1} \int_0^\infty \sum_{\mu, \omega} \alpha_{\mu}(\omega) \omega |L_{\mu}(\omega)|^2 dt \) characterizes the average time scale of the quantum system and \(|\mathcal{A}_2|_\infty\) denotes the spectral norm of the operator \( \mathcal{A}_2 \). Consequently, the entropy production is also bounded from below by the trace-like distance between the initial and final states,

\[
\Delta S_{\text{tot}} \geq \frac{d_2(\rho_0, \rho_f)^2}{4 \mathcal{A}_2}. \tag{12}
\]

For equilibration processes (i.e., the Hamiltonian and jump operators are time-independent), the entropy production can be bounded by the distance of the average change in the energy, \( d_3(\rho_0, \rho_f) = \text{tr} \{ H(\rho_0 - \rho_f) \} \), as \( |\mathcal{A}_2|_\infty \) [38]

\[
\Delta S_{\text{tot}} \geq \frac{d_3(\rho_0, \rho_f)^2}{\tau \mathcal{A}_3}, \tag{13}
\]

where \( \mathcal{A}_3 := \sum_{\mu, \omega} \alpha_{\mu}(\omega) \omega^2 |L_{\mu}(\omega)|^2 \). Inequalities (12) and (13) provide lower bounds not only on the entropy production, but also on the equilibration time, which is an essential quantity in quantum state preparation [51] and offers insights into understanding of thermalization [30]. As an application, one can approximately estimate the equilibration time without the need of solving the Lindblad master equation, which may be time-consuming in the weak coupling limit.

Examples.— First, we illustrate the derived bound in Eq. (5) on a time-dependent two-level system and the equilibration process of a three-level system. For the two-level system, instantaneous energies of states 1 and 2 are \( \mathcal{E}_1(t) = \beta^{-1} \ln[(1-a+b(t+1)/\tau)/(a-bt/\tau)] \) and \( \mathcal{E}_2(t) = 0 \), respectively, and \( 0 < b < a < 1 \) are positive constants. The transition rates are \( R_{12}(t) = 1, R_{21}(t) = e^{i \mathcal{E}_1(t)} \).

The probability distribution and the entropy production can be analytically calculated, \( p_1(t) = a - bt/\tau \) and \( \Delta S_{\text{tot}} = b^{-1} \int_0^t \ln[(1-a+b(t+1)/\tau)/(1-a+bt/\tau)] dt \).

We plot the entropy production and the Wasserstein-like distance in Fig. 1(a) for varying time \( \tau \). As can be seen, the entropy production is tightly bounded from below by the distance \( \mathcal{W}_c \) for all times, which numerically verifies Eq. (5). Regarding the three-level system, the transition rates are time-independent and equal to \( R_{mn} = w_{mn} e^{\beta(\mathcal{E}_n - \mathcal{E}_m)/2} \text{sech}[\beta(\mathcal{E}_n - \mathcal{E}_m)/2] \), where \( w_{mn} = w_{nm} \) are nonnegative constants. It is evident that the detailed balance conditions, \( R_{mn} p_m^\text{eq} = R_{nm} p_n^\text{eq} \), are satisfied. Under these conditions, Ref. [52] proved that the entropy production in equilibration processes is bounded from below by an information-theoretical quantity of the initial and final states, \( \Delta S_{\text{tot}} \geq D[\rho(0)||\rho(\tau)] \). We fix the transition rates and plot the entropy production, the Wasserstein-like distance, and the KL divergence as functions of the time \( \tau \) in Fig. 1(b). As illustrated, the distance term \( \mathcal{W}_c^2/\tau \) and the KL divergence always lie below the entropy production \( \Delta S_{\text{tot}} \). The Wasserstein-like distance is tight in the short-time regime, while the KL divergence is saturated in the long-time limit. Therefore, these two bounds complementarily characterize the irreversibility of equilibration processes.

Next, we verify the derived bounds in Eqs. (12) and (13) using a model of a two-level atom interacting with a thermal bath of photons, which has been widely studied in the literature [53]. The dynamics are described by the Lindblad master equation

\[
\dot{\rho} = -i[H, \rho] + \alpha \tilde{n}(\omega)(2\sigma_+ \rho \sigma_- - \{ \sigma_+ \sigma_- \}) + \alpha \tilde{n}(\omega) + 1)(2\sigma_- \rho \sigma_+ - \{ \sigma_+ \sigma_- \}),
\]

where \( H = \omega \sigma_z/2, \sigma_z = (\sigma_x + i \sigma_y)/2, \alpha \) is a positive damping rate, and \( \tilde{n}(\omega) = (e^{\beta \omega} - 1)^{-1} \) is the bosonic occupation number in thermal equilibrium. The den-
sity operator $\rho(t)$ in this system is exactly solvable [54] and the entropy production can be explicitly evaluated as $\Delta S_{\text{tot}} = S[\rho(0)||\rho^n] - S[\rho(\tau)||\rho^n]$, where $S[\rho_1||\rho_2] := \text{tr}\{\rho_1(\ln \rho_1 - \ln \rho_2)\}$ is the relative entropy of $\rho_1$ with respect to $\rho_2$. If the reverse triangle inequality for the relative entropy $S[\rho(0)||\rho^n] \geq S[\rho(0)||\rho(\tau)] + S[\rho(\tau)||\rho^n]$ holds, one can further bound dissipation by quantum Fisher information and Wigner–Yanase metrics [24]. However, this inequality does not hold in general. A simple counterexample can be found when $\alpha_\mu(\omega) \to 0$. In this vanishing coupling limit, the entropy production vanishes because the relative entropy is invariant under unitary transform. On the other hand, $S[\rho(0)||\rho(\tau)]$ is always positive, thus $\Delta S_{\text{tot}} < S[\rho(0)||\rho(\tau)]$. We vary the time $\tau$ and plot the entropy production, the derived lower bounds, and the relative entropy in Fig. 1(c). As can be seen, $\Delta S_{\text{tot}}$ is always bounded from below by $d_2(\rho(0), \rho(\tau))^2/4\tau A_2$ and $d_2(\rho(0), \rho(\tau))^2/\tau A_3$, while the inequality $\Delta S_{\text{tot}} \geq S[\rho(0)||\rho(\tau)]$ is clearly violated.

Conclusions.— In the present Letter, we have derived geometrical bounds of the irreversibility for both classical and open quantum systems. These bounds are stronger than the conventional second laws of thermodynamics and can be interpreted as speed limits. Exploring analogous bounds in systems where the detailed balance conditions are broken would be a promising direction.

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Supplemental Material for
“Geometrical bounds of the irreversibility in classical and open quantum systems”
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This supplementary material describes the calculations introduced in the main text. Equation and figure numbers are prefixed with S [e.g., Eq. (S1) or Fig. S1]. Numbers without this prefix [e.g., Eq. (1) or Fig. 1] refer to items in the main text.

We denote by $\mathcal{H}$ the complex Hilbert space with dimension $N > 0$, by $\mathcal{L}(\mathcal{H})$ the set of linear operators, and by $\mathcal{S}(\mathcal{H})$ the set of self-adjoint operators. The inner product $\langle \cdot, \cdot \rangle$ is defined as $\langle x, y \rangle = x^* y$ for $x, y \in \mathbb{R}^{N \times 1}$ (classical case) and $\langle X, Y \rangle = \text{tr} \{X^\dagger Y\}$ for $X, Y \in \mathcal{L}(\mathcal{H})$ (quantum case).

S1. CLASSICAL MARKOV JUMP PROCESSES

A. Alternative expression of the classical master equation

We show that the master equation $\dot{\mathbf{v}} = \mathbf{K}_p \mathbf{f}$, where $\mathbf{R} = [R_{mn}]$ with $R_{mn} = -\sum_{m \neq n} R_{mn}$, $\mathbf{K}_p = \sum_{1 \leq n < m \leq N} R_{nm} p_{n}^{\text{eq}} \Phi \left( \frac{p_n^{\text{eq}}}{p_m^{\text{eq}}}, \frac{p_m}{p_n^{\text{eq}}} \right) \mathbf{E}_{nm}$, and $\mathbf{f} = -\nabla_p D(p\|p^{\text{eq}})$. Here, $\nabla_p = [\partial_p, \ldots, \partial_{p_N}]^\top$. Specifically, we need to show that

$$
(K_p \mathbf{f})_n = \sum_{m \neq n} [R_{nm} p_m - R_{mn} p_n]
$$

(S1)

holds for all $n$. Indeed, using relations $f_n = -(\ln p_n - \ln p_n^{\text{eq}} - 1)$ and $R_{nm} p_m^{\text{eq}} = R_{mn} p_n^{\text{eq}}$, Eq. (S1) can be verified as follows:

$$
(K_p \mathbf{f})_n = \sum_{m \neq n} R_{nm} p_m^{\text{eq}} \Phi \left( \frac{p_n^{\text{eq}}}{p_m^{\text{eq}}}, \frac{p_m}{p_n^{\text{eq}}} \right) (\mathbf{E}_{nm} \mathbf{f})_n
$$

(S2a)

$$
= \sum_{m \neq n} R_{nm} p_m^{\text{eq}} \Phi \left( \frac{p_n^{\text{eq}}}{p_m^{\text{eq}}}, \frac{p_m}{p_n^{\text{eq}}} \right) (f_n - f_m)
$$

(S2b)

$$
= \sum_{m \neq n} R_{nm} p_m^{\text{eq}} \frac{p_n^{\text{eq}}}{p_m^{\text{eq}}} - p_m^{\text{eq}} / p_m
ln p_n - ln p_n^{\text{eq}} - ln p_m + ln p_m^{\text{eq}} (ln p_m - ln p_m^{\text{eq}} - ln p_n + ln p_n^{\text{eq}})
$$

(S2c)

$$
= \sum_{m \neq n} R_{nm} p_m^{\text{eq}} \left( \frac{p_m}{p_m^{\text{eq}}} - \frac{p_n}{p_n^{\text{eq}}} \right)
$$

(S2d)

$$
= \sum_{m \neq n} [R_{nm} p_m - R_{mn} p_n]
$$

(S2e)

B. Properties of the matrix $K_p$

The matrix $K_p$ is symmetric and positive semi-definite, and has the following properties.

Lemma 1. For arbitrary distribution $\mathbf{p}$ satisfying $p_n > 0$ for all $n$, $\text{ker}(K_p) = \{ \mathbf{v} \in \mathbb{R}^{N \times 1} \mid \mathbf{v} \propto \mathbf{1} \}$.

Proof. Since the system is ergodic, there exists a set of $N - 1$ unordered pairs, $\mathcal{E} = \{(i,j) \mid R_{ij} \neq 0\}$, such that for arbitrary $n \neq m$, there is a path $n = i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_k = m$ and $(i_l, i_{l+1}) \in \mathcal{E}$ for all $0 \leq l \leq k - 1$. Assuming $\mathbf{v} \in \text{ker}(K_p)$ then

$$
0 = \langle \mathbf{v}, K_p \mathbf{v} \rangle = \sum_{m \neq n} R_{nm} p_m^{\text{eq}} \Phi \left( \frac{p_n^{\text{eq}}}{p_m^{\text{eq}}}, \frac{p_m}{p_n^{\text{eq}}} \right) (v_m - v_n)^2.
$$

(S3)

This means that $v_i - v_j = 0$ for all $(i,j) \in \mathcal{E}$, or equivalently, $\mathbf{v} \propto \mathbf{1}$.

□
Lemma 2. There exists a vector $v$ that satisfies the relation $\dot{p} = K_p v$. Such vector is unique if the condition $(1, v) = 0$ is considered.

Proof. For any $v$ satisfying $K_p v = 0$ [i.e., $v \in \ker(K_p)$], then $v \propto 1 \rightarrow v^T \dot{p} = 0$, or equivalently, $\dot{p} \in \ker(K_p)$. According the Fredholm alternative, the equation $\dot{p} = K_p v$ always has a nonzero solution $v$. Assuming there exist two solutions $v_1$ and $v_2$ such that $(1, v_1) = (1, v_2) = 0$, then $K_p(v_1 - v_2) = 0 \Rightarrow v_1 - v_2 = c1$ for some $c \in \mathbb{R}$. Moreover, $(1, v_1 - v_2) = 0 \Rightarrow Nc = 0 \Rightarrow c = 0$, which proves the uniqueness of $v$. \hfill \Box

C. Geodesic equations of the Wasserstein-like distance

Here we derive geodesic equations, which determine the geodesic path between two distributions $p_0$ and $p_\tau$. We consider the following functional, which is minimized with the geodesic path,

$$J[p(t)] = \int_0^\tau \langle v(t), K_p v(t) \rangle dt,$$

where $v(t)$ is related to $p(t)$ via the equation $\dot{p}(t) = K_p v(t)$. Consider an arbitrary perturbation path $\{q(t)\}_{0 \leq t \leq \tau}$ that satisfies $q(0) = q(\tau) = 0$ and $\sum_n q_n(t) = 0$ for all $0 \leq t \leq \tau$. Because the functional $J[\gamma(t)]$ is minimized when $\gamma = p + \epsilon q$, the function $\Theta(\epsilon) = J[p(t) + \epsilon q(t)]$ has a minimum at $\epsilon = 0$ and thus $\Theta(0) = 0$. The functional evaluated at $\gamma = p + \epsilon q$ can be written as

$$J[p(t) + \epsilon q(t)] = \int_0^\tau \langle \vartheta(t), K_{p+\epsilon q} \vartheta(t) \rangle dt,$$

where $\vartheta(t)$ is determined via the relation $\dot{p}(t) + \epsilon \dot{q}(t) = K_{p+\epsilon q} \vartheta(t)$. From Eq. (S5), we have

$$0 = \Theta'(0) = \int_0^\tau \left[ (\partial_v \vartheta(t), K_p v(t)) + \langle v(t), \partial_v K_{p+\epsilon q} v(t) \rangle + \langle v(t), K_p \partial_v \vartheta(t) \rangle \right] dt.$$

Hereafter, the notation of evaluation at $\epsilon = 0$ is omitted for the sake of conciseness. The first and third terms in Eq. (S6) are equal due to the symmetry of $K_p$, $(\partial_v \vartheta(t), K_p v(t)) = \langle v(t), K_p \partial_v \vartheta(t) \rangle$. Taking the partial derivative of both sides of equation $\dot{p}(t) + \epsilon \dot{q}(t) = K_{p+\epsilon q} \vartheta(t)$ with respect to $\epsilon$ and evaluating at $\epsilon = 0$, we obtain

$$\dot{q}(t) = \partial_v K_{p+\epsilon q} v(t) + K_p \partial_v \vartheta(t) \Rightarrow \langle v(t), \partial_v K_{p+\epsilon q} v(t) \rangle = \langle v(t), \dot{q}(t) \rangle - \langle v(t), \partial_v K_{p+\epsilon q} v(t) \rangle.$$

From Eqs. (S6) and (S7), we have

$$0 = \int_0^\tau [2\langle v(t), \dot{q}(t) \rangle - \langle v(t), \partial_v K_{p+\epsilon q} v(t) \rangle] dt = - \int_0^\tau [2\langle \dot{v}(t), q(t) \rangle + \langle v(t), \partial_v K_{p+\epsilon q} v(t) \rangle] dt.$$

Since

$$\partial_v K_{p+\epsilon q} = \sum_{1 \leq n < m \leq N} R_{nm} p_{nm} q_{nm} \frac{\partial}{\partial v} \Phi \left( \frac{p_n + \epsilon q_n}{p_n}, \frac{p_m + \epsilon q_m}{p_m} \right) E_{nm},$$

we have

$$\langle v(t), \partial_v K_{p+\epsilon q} v(t) \rangle = \sum_{n,m} R_{nm} [v_m(t) - v_n(t)]^2 \Psi \left( \frac{p_n(t)}{p_n}, \frac{p_m(t)}{p_m} \right) q_n(t) = \langle r(t), q(t) \rangle,$$

where $\Psi(x, y) = [x - \Phi(x, y)]/[x \ln x - y \ln y]$ and $r_n(t) := \sum_m R_{nm} [v_m(t) - v_n(t)]^2 \Psi \left( \frac{p_n(t)}{p_n}, \frac{p_m(t)}{p_m} \right)$. From Eqs. (S8) and (S10), we have

$$\int_0^\tau (2\dot{v}(t) + r(t), q(t)) dt = 0$$

Because $\{q(t)\}$ is an arbitrary perturbation path, the term in the inner product must be zero, i.e., $\dot{v}(t) + r(t)/2 = 0$. Finally, we obtain the geodesic equations, which determine the shortest path between states $p_0$ and $p_\tau$,

$$\begin{cases}
\dot{p}(t) - K_p v(t) = 0, \\
\dot{v}(t) + \frac{1}{2} r(t) = 0,
\end{cases}$$

with the boundary conditions $p(0) = p_0$ and $p(\tau) = p_\tau$. 

\hfill \Box
D. Lower bound of the Wasserstein-like distance in terms of the total variation distance

Here we derive the lower bound of the Wasserstein distance in terms of the total variation distance, \( d_1(p, q) = \sum_n |p_n - q_n| \). The distance \( d_1(p, q) \) can also be expressed in a variational form as

\[
d_1(p, q) = \max_{\|w\|_\infty \leq 1} \{ w^\top (p - q) \} = \max_{\|w\|_\infty \leq 1} \{ w, p - q \},
\]

where the maximum is taken over all real vectors \( w = [w_1, \ldots, w_N]^\top \) and \( \|w\|_\infty := \max_n |w_n| \). The equality is attained when \( w_n = \text{sign}(p_n - q_n) \), where \( \text{sign}(x) \) is the sign function of \( x \), defined as \( \text{sign}(x) = 1 \) for \( x \geq 0 \) and \(-1\) otherwise. From the definition of the Wasserstein-like distance, given a fixed positive number \( \delta > 0 \), there exists a smooth curve \( p(t) \) with end points \( p_0 \) and \( p_\tau \) such that

\[
\tau \int_0^\tau \langle v, K_p v \rangle dt \leq W_c(p_0, p_\tau)^2 + \delta.
\]

Here, \( v(t) \in \mathbb{R}^{N \times 1} \) is determined via the equation \( \dot{p}(t) = K_p v(t) \). For arbitrary vector \( w \) with \( \|w\|_\infty \leq 1 \), we have

\[
\langle w, p_\tau - p_0 \rangle = \int_0^\tau \langle w, K_p v \rangle dt \\
\leq \left( \int_0^\tau \langle w, K_p w \rangle dt \right)^{1/2} \left( \int_0^\tau \langle v, K_p v \rangle dt \right)^{1/2} \\
\leq \left( \tau^{-1} \int_0^\tau \langle w, K_p w \rangle dt \right)^{1/2} (W_c(p_0, p_\tau)^2 + \delta)^{1/2}.
\]

(S15c)

Now, we further bound the first term in Eq. (S15c). Using inequalities \( \Phi(x, y) \leq (x + y)/2 \) and \( (w_n - w_m)^2 \leq 4 \), we obtain

\[
\langle w, K_p w \rangle = \sum_{m > n} R_{nm} F_{m}^e \Phi \left( \frac{p_n}{p_n^\text{eq}}, \frac{p_m}{p_m^\text{eq}} \right) \langle w, E_{nm} w \rangle \\
= \sum_{m > n} R_{nm} F_{m}^e \Phi \left( \frac{p_n}{p_n^\text{eq}}, \frac{p_m}{p_m^\text{eq}} \right) (w_n - w_m)^2 \\
\leq 2 \sum_{m > n} R_{nm} F_{m}^e \left( \frac{p_n}{p_n^\text{eq}} + \frac{p_m}{p_m^\text{eq}} \right) \\
= 2 \sum_{m > n} [R_{nm} p_m + R_{mn} p_n].
\]

(S16a)

Consequently,

\[
W_c(p_0, p_\tau)^2 + \delta \geq \frac{\langle w, p_0 - p_\tau \rangle^2}{2\tau^{-1} \int_0^\tau \sum_{m > n} [R_{nm}(t)p_m(t) + R_{mn}(t)p_n(t)] dt}.
\]

(S17)

Taking the maximum over all \( w \) and the limit \( \delta \to 0 \), we obtain

\[
W_c(p_0, p_\tau)^2 \geq \frac{d_1(p_0, p_\tau)^2}{2A_1},
\]

(S18)

where \( A_1 := \tau^{-1} \int_0^\tau \sum_{m > n} R_{mn}(t) \gamma_n(t) dt \) is the average dynamical activity along the geodesic path \( \{ \gamma(t) \}_{0 \leq t \leq \tau} \).

S2. OPEN QUANTUM SYSTEMS

A. Alternative expression of the Lindblad master equation

Here we show that the Lindblad master equation can be written as

\[
\dot{\rho} = \mathcal{K}_\rho (-\ln \rho + \ln \rho^{\text{eq}}),
\]

(S19)

where \( \mathcal{K}_\rho : \nu \mapsto i\beta^{-1}[\nu, \rho] + \mathcal{O}_\rho(\nu) \) and \( \mathcal{O}_\rho \) is an operator defined by

\[
\mathcal{O}_\rho(\nu) := \sum_{\mu, \omega} e^{-\beta \omega/2} \alpha_\mu(\omega) [L_\mu(\omega), [\rho, L^\dagger_\mu(\omega), \nu]].
\]

(S20)
For a density operator $\rho = \sum_n r_n |v_n\rangle\langle v_n|$, where $\sum_n r_n = 1$ and $\{ |v_n\rangle \}_n$ are orthonormal eigenvectors, the titled operator can also be expressed as

$$[\rho]_\theta(A) = e^{-\theta/2} \int_0^1 e^{\theta \rho s} A e^{(1-s)\rho} ds = \sum_{n,m} \Phi(\theta r_n, \theta r_m) (v_n|A|v_m)\langle v_n|v_m\rangle.$$

(S21)

Here, $\Phi(x, y)$ is the logarithmic mean of positive numbers $x$ and $y$, given by $\Phi(x, y) = (x-y)/[\ln(x) - \ln(y)]$ for $x \neq y$ and $\Phi(x, x) = x$. Since $\rho^\omega = e^{-\beta H} | Z_{\beta} \rangle$ and $[\ln \rho, \rho] = 0$, we have $i\beta^{-1}[-\ln \rho + \ln \rho^\omega, \rho] = -i[H, \rho]$. Thus, we only need to show that

$$O_\rho(-\ln \rho + \ln \rho^\omega) = \sum_{\mu, \omega} \alpha_\mu(\omega) \left[ 2L_\mu(\omega) \rho L_\mu^\dagger(\omega) - \{ L_\mu(\omega) L_\mu(\omega), \rho \} \right].$$

(S22)

First, we show that $[\rho]_\theta([A, \ln \rho] - \theta A) = e^{-\theta/2} A \rho - e^{\theta/2} \rho A$ for arbitrary operator $A \in \mathcal{L}(\mathcal{H})$ and $\theta \in \mathbb{R}$. Indeed, one can transform as follows:

$$[\rho]_\theta([A, \ln \rho] - \theta A) = e^{-\theta/2} \int_0^1 e^{\theta s} e^{s \ln \rho} (A \ln \rho - \ln \rho A - \theta A) e^{(1-s)\ln \rho} ds$$

(S23a)

$$= -e^{-\theta/2} \int_0^1 e^{\theta s} e^{s \ln \rho} (\ln \rho + \theta \mathbb{1}) e^{(1-s)\ln \rho} + e^{\theta s} e^{s \ln \rho} A (-\ln \rho) e^{(1-s)\ln \rho} ds$$

(S23b)

$$= e^{-\theta/2} (A e^{\ln \rho} - e^{\ln \rho + \theta \mathbb{1}} A)$$

(S23c)

$$= e^{-\theta/2} (A - \theta e^{\ln \rho} + e^{\ln \rho + \theta \mathbb{1}} A).$$

(S23d)

Next, using the relation $[L_\mu(\omega), H] = -\omega L_\mu^\dagger(\omega)$, one immediately obtains

$$[\rho]_{\beta\omega} ([L_\mu(\omega), -\ln \rho + \ln \rho^\omega]) = [\rho]_{\beta\omega} ([L_\mu^\dagger(\omega), -\ln \rho - \beta H])$$

(S24a)

$$= [\rho]_{\beta\omega} ([L_\mu(\omega), \ln \rho] + \beta [L_\mu^\dagger(\omega), H])$$

(S24b)

$$= [\rho]_{\beta\omega} ([L_\mu(\omega), \ln \rho] - \beta \omega L_\mu^\dagger(\omega))$$

(S24c)

$$= e^{\beta\omega / 2} \rho L_\mu^\dagger(\omega) - e^{-\beta\omega / 2} L_\mu^\dagger(\omega) \rho.$$

(S24d)

Consequently, noticing that $L_\mu^\dagger(\omega) = L_\mu(-\omega)$ and $\alpha_\mu(\omega) = e^{\beta\omega / 2} \alpha_\mu(-\omega)$, one can verify Eq. (S22) as follows:

$$O_\rho(-\ln \rho + \ln \rho^\omega)$$

(S25a)

$$= \sum_{\mu, \omega} \alpha_\mu(\omega) [L_\mu(\omega), [\rho]_{\beta\omega} ([L_\mu^\dagger(\omega), -\ln \rho + \ln \rho^\omega])]$$

(S25b)

$$= \sum_{\mu, \omega} e^{-\beta\omega / 2} \alpha_\mu(\omega) [L_\mu(\omega), e^{\beta\omega / 2} \rho L_\mu^\dagger(\omega) - e^{-\beta\omega / 2} L_\mu^\dagger(\omega) \rho]$$

(S25c)

$$= \sum_{\mu, \omega} \alpha_\mu(\omega) [e^{-\beta\omega} L_\mu(\omega) L_\mu^\dagger(\omega) + L_\mu(\omega) \rho L_\mu^\dagger(\omega) + e^{-\beta\omega} L_\mu^\dagger(\omega) \rho L_\mu^\dagger(\omega) - \rho L_\mu^\dagger(\omega) L_\mu^\dagger(\omega)]$$

(S25d)

$$= \sum_{\mu, \omega} \{ \alpha_\mu(\omega) \{ L_\mu(\omega) \rho L_\mu^\dagger(\omega) - L_\mu^\dagger(\omega) \rho L_\mu(\omega) \} + \alpha_\mu(-\omega) \{ L_\mu(-\omega) \rho L_\mu^\dagger(-\omega) - L_\mu^\dagger(-\omega) \rho L_\mu(-\omega) \} \}$$

(S25e)

$$= \sum_{\mu, \omega} \alpha_\mu(\omega) [2L_\mu(\omega) \rho L_\mu^\dagger(\omega) - \{ L_\mu(\omega) L_\mu(\omega), \rho \}].$$

(S25f)

**B. Properties of the quantum Wasserstein-like metric**

Here we provide several properties of the defined metric in the main text.

**Lemma 3.** The product $\langle \xi, O_\rho(\cdot) \rangle$ satisfies the conjugate-symmetry condition, $\langle \xi, O_\rho(\nu) \rangle = \langle \nu, O_\rho(\xi) \rangle^*$, for all operators $\nu$ and $\xi$. Here, $^*$ denotes the complex conjugate.

**Proof.** For arbitrary operator $A \in \mathcal{L}(\mathcal{H})$ and $\theta \in \mathbb{R}$, we have

$$\langle \xi, [A, \rho]_{\theta}([A^\dagger, \nu]) \rangle = \text{tr} \{ \xi^\dagger [A, \rho]_{\theta}([A^\dagger, \nu]) \}$$

(S26a)
= \text{tr} \left\{ [\xi^\dagger, A][\rho]_\theta([A^\dagger, \nu]) \right\} 
= \sum_{n,m} \Phi(e^{-\beta/2}r_n,e^{\beta/2}r_m)(v_n[A^\dagger,\xi])v_m(\xi,A)v_n). 
\tag{S26b}
\end{align}

Here, we used Eq. (S21) in the last equation [Eq. (S26c)]. By swapping \( \xi \) and \( \nu \), one obtains
\begin{align}
\langle \nu, [A, [\rho]_\theta([A^\dagger, \xi]) \rangle \rangle^* = \sum_{n,m} \Phi(e^{-\beta/2}r_n,e^{\beta/2}r_m)(v_n[A^\dagger,\xi])v_m(\nu,A)v_n^* 
= \sum_{n,m} \Phi(e^{-\beta/2}r_n,e^{\beta/2}r_m)(v_n[A^\dagger,\xi])v_m(\nu,A)v_n 
= \langle \xi, [A, [\rho]_\theta([A^\dagger, \nu]) \rangle \rangle.
\tag{S27}
\end{align}

Since \( O_\rho(\nu) = \sum_{\mu,\omega} e^{-\beta\omega^2/2} \alpha_\mu(\omega)[L_\mu(\omega),[\rho]_{\beta,\omega}([L_\mu^\dagger(\omega), \nu])] \), Eq. (S27) implies that
\begin{align}
\langle \nu, O_\rho(\xi) \rangle^* = \langle \xi, O_\rho(\nu) \rangle.
\tag{S28}
\end{align}

From Eq. (S26c), one can see that
\begin{align}
\langle \xi, [A, [\rho]_\theta([A^\dagger, \xi]) \rangle \rangle = \sum_{n,m} \Phi(e^{-\beta/2}r_n,e^{\beta/2}r_m)(v_n[A^\dagger,\xi])v_m(\xi,A)v_n)^2 \geq 0.
\tag{S29}
\end{align}

Therefore, \( \langle \xi, O_\rho(\xi) \rangle \geq 0 \) for arbitrary operator \( \xi \). The equality is obtained only when \( [L_\mu^\dagger(\omega), \xi] = 0 \) for all \( \mu \) and \( \omega \). When \( \xi \) is a self-adjoint operator, i.e., \( \xi^\dagger = \xi \), we have \( \langle \xi, K_\rho(\xi) \rangle = \langle \xi, O_\rho(\xi) \rangle \geq 0 \).

**Proposition 4.** An self-adjoint operator \( \nu \) satisfies \( K_\rho(\nu) = 0 \) if and only if \( \nu \) is spanned by \( I_N \).

**Proof.** Since \( K_\rho(I_N) = 0 \), we only need to show that if \( K_\rho(\nu) = 0 \) then \( \nu \) is spanned by \( I_N \). Note that \( \langle \nu, K_\rho(\nu) \rangle = \langle \nu, O_\rho(\nu) \rangle = 0 \), which occurs only when \( [L_\mu^\dagger(\omega), \nu] = 0 \) for all \( \mu \) and \( \omega \). Since the dynamics of the quantum system are ergodic, this implies that \( \nu \) is spanned by \( I_N \).

**Proposition 5.** \( K_\rho(\nu) \) is a traceless self-adjoint operator for all \( \nu \in \mathcal{F}(H) \).

**Proof.** Since
\begin{align}
K_\rho(\nu) = i\beta^{-1}[\nu, \rho] + O_\rho(\nu) = i\beta^{-1}[\nu, \rho] + \sum_{\mu,\omega} e^{-\beta\omega^2/2} \alpha_\mu(\omega)[L_\mu(\omega),[\rho]_{\beta,\omega}([L_\mu^\dagger(\omega), \nu])] \tag{S30}
\end{align}
is a linear combination of commutators, \( \text{tr} \left\{ K_\rho(\nu) \right\} = 0 \) is immediately derived. Note that \( (i\beta^{-1}[\nu, \rho])^\dagger = i\beta^{-1}[\nu, \rho] \), we only need to show that \( O_\rho(\nu) \) is self-adjoint. Using relations \( [\rho]_\theta(A)^\dagger = [\rho]_{-\theta}(A^\dagger) \), \( [A,B]^\dagger = [B^\dagger,A^\dagger] \), \( e^{-\beta\omega^2/2} \alpha_\mu(\omega) = e^{\beta\omega^2/2} \alpha_\mu(-\omega) \), and \( L_\mu^\dagger(\omega) = L_\mu(-\omega) \), we can prove that \( O_\rho(\nu) \) is self-adjoint as follows:
\begin{align}
O_\rho(\nu)^\dagger = \sum_{\mu,\omega} e^{-\beta\omega^2/2} \alpha_\mu(\omega)[L_\mu(\omega),[\rho]_{\beta,\omega}([L_\mu^\dagger(\omega), \nu])]^\dagger 
= \sum_{\mu,\omega} e^{-\beta\omega^2/2} \alpha_\mu(\omega[[\rho]_{\beta,\omega}([L_\mu^\dagger(\omega), \nu])]^\dagger, L_\mu(\omega)] 
= \sum_{\mu,\omega} e^{\beta\omega^2/2} \alpha_\mu(-\omega)[L_\mu(-\omega),[\rho]_{-\beta,\omega}([L_\mu^\dagger(-\omega), \nu])] 
= O_\rho(\nu). \tag{S31}
\end{align}

**Lemma 6.** For arbitrary density operator \( \rho \) and traceless self-adjoint operator \( \vartheta \), there exists a unique traceless self-adjoint operator \( \nu \) such that \( \vartheta = K_\rho(\nu) \).
Proof. Let $B = \{\chi_{j,k}\}_{1 \leq j,k \leq N}$ denote the set of generalized Gell-Mann matrices, which span the space of operators of the complex Hilbert space $\mathcal{H}$. Specifically, $\chi_{j,k}$ can be expressed as follows:

$$
\chi_{j,k} = \begin{cases} 
E_{k,j} + E_{j,k}, & \text{if } j < k \\
i(E_{k,j} - E_{j,k}), & \text{if } j > k \\
\sqrt{\frac{2}{j(j+1)}} \left( \sum_{l=1}^{j} E_{l,l} - jE_{j+1,j+1} \right), & \text{if } j = k < N \\
N^{-1}I_N, & \text{if } j = k = N
\end{cases}
$$

(S36)

Here, $E_{j,k}$ denote the matrix with 1 in the $jk$-th entry and 0 elsewhere. By this construction, each $\chi_{j,k}$ is a Hermitian matrix and $\{\chi_{j,k}\} = \delta_{jN} \delta_{kN}$ for all $(j,k) \neq (N,N)$. It is convenient to define a set $\mathcal{B} := B \setminus \{\chi_{N,N}\}$. For arbitrary traceless self-adjoint operator $A$, there exists real coefficients $c_{j,k} \in \mathbb{R}$ such that $A = \sum_{j,k} c_{j,k} \chi_{j,k}$. Taking the trace of two sides of the equation, we obtain $0 = \text{tr} \left\{ A \right\} = \sum_{j,k} c_{j,k} \text{tr} \{ \chi_{j,k} \} = c_{N,N}$. This implies that $A$ can be expressed as a linear combination of matrices in $\mathcal{B}$ with all real coefficients.

According to Propositions 4 and 5, it is obvious that $K_\rho(\chi_{j,k})$ is a nonzero traceless self-adjoint operator for all $(j,k) \neq (N,N)$. We show that $\{K_\rho(\chi_{j,k})\}_{(j,k) \neq (N,N)}$ is an independent set, i.e., $\sum_{(j,k) \neq (N,N)} c_{j,k} K_\rho(\chi_{j,k}) = 0$ only when $c_{j,k} = 0$ for all $j,k$. Indeed, from the linearity of $K_\rho$, we have

$$
0 = \sum_{(j,k) \neq (N,N)} c_{j,k} K_\rho(\chi_{j,k}) = K_\rho \left( \sum_{(j,k) \neq (N,N)} c_{j,k} \chi_{j,k} \right) = 0.
$$

(S37)

Due to Propositions 4, $\sum_{(j,k) \neq (N,N)} c_{j,k} \chi_{j,k}$ must be spanned by $I_N = \chi_{N,N}$, i.e., $\sum_{(j,k) \neq (N,N)} c_{j,k} \chi_{j,k} = -c_{N,N} \chi_{N,N}$ for some $c_{N,N}$. This is equivalent to $\sum_{1 \leq j,k \leq N} c_{j,k} \chi_{j,k} = 0$. Since $\mathcal{B}$ is a basis of $\mathcal{H}$, this happens only when $c_{j,k} = 0$ for all $j,k$. Note that $\{K_\rho(\chi_{j,k})\}_{(j,k) \neq (N,N)}$ has $N^2 - 1$ elements, therefore one can add another matrix $\phi$ to form a new basis of $\mathcal{H}$. Then, $I_N$ can be expressed in terms of elements of the new basis as

$$
I_N = z\phi + \sum_{(j,k) \neq (N,N)} c_{j,k} K_\rho(\chi_{j,k}).
$$

(S38)

Taking the trace of two sides of Eq. (S38), we have $N = z \text{tr} \{ \phi \}$, which indicates that $z \neq 0$. Therefore, $\phi$ can be expressed in terms of $I_N$ and $\{K_\rho(\chi_{j,k})\}_{(j,k) \neq (N,N)}$ as

$$
\phi = z^{-1} \left[ I_N - \sum_{(j,k) \neq (N,N)} c_{j,k} K_\rho(\chi_{j,k}) \right].
$$

(S39)

Equation (S39) implies that an arbitrary matrix can be expressed as a linear combination of elements in the following set:

$$
\mathcal{S} := \{I_N\} \cup \{K_\rho(\chi_{j,k})\}_{(j,k) \neq (N,N)}.
$$

(S40)

Equivalently, $\mathcal{S}$ is a basis of $\mathcal{H}$. Consequently, due to the fact that $K_\rho(\chi_{j,k})$ is traceless and self-adjoint, arbitrary traceless self-adjoint operator $\vartheta$ can be expressed in terms of $\{K_\rho(\chi_{j,k})\}_{(j,k) \neq (N,N)}$ with real coefficients $c_{j,k}$ as

$$
\vartheta = \sum_{(j,k) \neq (N,N)} c_{j,k} K_\rho(\chi_{j,k}) = K_\rho \left( \sum_{(j,k) \neq (N,N)} c_{j,k} \chi_{j,k} \right).
$$

(S41)

Defining $\nu := \sum_{(j,k) \neq (N,N)} c_{j,k} \chi_{j,k}$, which is a traceless self-adjoint operator, one readily obtains $\vartheta = K_\rho(\nu)$. Finally, we prove the uniqueness of $\nu$. Assuming that there exist two traceless self-adjoint operators $\nu_1$ and $\nu_2$ such that $\vartheta = K_\rho(\nu_1) = K_\rho(\nu_2)$, then $K_\rho(\nu_1 - \nu_2) = 0$. Applying the result in Proposition 4, $\nu_1 - \nu_2 = zI_N$ for some $z \in \mathbb{C}$. Thus, $zN = \text{tr} \{ zI_N \} = \text{tr} \{ \nu_1 - \nu_2 \} = 0 \Rightarrow z = 0$. This implies the uniqueness of $\nu$.

Lemma 7. Given an arbitrary traceless self-adjoint operator $\nu$, the equality $\langle \nu + zI_N, K_\rho(\nu + zI_N) \rangle = \langle \nu, K_\rho(\nu) \rangle$ holds for arbitrary number $\lambda \in \mathbb{C}$.

Proof. Since $K_\rho(\nu + zI_N) = K_\rho(\nu) + K_\rho(zI_N) = K_\rho(\nu)$, we have

$$
\langle \nu + zI_N, K_\rho(\nu + zI_N) \rangle = \langle \nu + zI_N, K_\rho(\nu) \rangle = \langle \nu, zI_N, K_\rho(\nu) \rangle
$$

(S42a)

$$
= \langle \nu, K_\rho(\nu) \rangle + \langle zI_N, K_\rho(\nu) \rangle
$$

(S42b)

$$
= \langle \nu, K_\rho(\nu) \rangle + z^* \text{tr} \{ K_\rho(\nu) \}
$$

(S42c)

$$
= \langle \nu, K_\rho(\nu) \rangle.
$$

(S42d)

Here we used the traceless property of $K_\rho$, which is obtained in Proposition 5. 

□
C. Lower bound of the quantum Wasserstein-like distance in terms of the trace-like distance

Here we derive the lower bound of the quantum Wasserstein-like distance \( \mathcal{W}_q(\rho_0, \rho_\tau) \) in terms of the trace-like distance. From the definition of the quantum Wasserstein-like distance, given a fixed positive number \( \delta > 0 \), there exists a smooth curve \( \rho(t) \) with end points \( \rho_0 \) and \( \rho_\tau \) such that

\[
\tau \int_0^\tau \langle \nu, \mathcal{K}_\rho(\nu) \rangle dt \leq \mathcal{W}_q(\rho_0, \rho_\tau)^2 + \delta. \tag{S43}
\]

Here, \( \nu(t) \in \mathcal{D}(\mathcal{H}) \) is a traceless self-adjoint operator that satisfies \( \rho(t) = \mathcal{K}_\rho(\nu(t)) \). Let \( \rho(t) = \sum_n \epsilon_n(t)|\epsilon_n(t)\rangle \langle \epsilon_n(t)| \) be the spectral decomposition with orthogonal basis \( |\epsilon_n(t)\rangle \langle \epsilon_n(t)| \) = \( \delta_{nm} \), then we define the self-adjoint operator \( \phi(t) := \sum_n c_n|\epsilon_n(t)\rangle \langle \epsilon_n(t)| \), where \( c_n \leq 1 \) are constants which will be determined later. It is evident that \( \phi(t) \) commutes with \( \rho(t) \), i.e., \( [\phi, \rho] = 0 \). Now, using relations \( \rho = i\beta^{-1}[\nu, \rho] + \mathcal{O}_\rho(\nu) \) and \( \langle \phi, [\nu, \rho] \rangle = 0 \), we have

\[
\sum_n c_n(\epsilon_n(\tau) - \epsilon_n(0)) = \text{tr} \left\{ \int_0^\tau \phi(t) \rho(t) dt \right\} \tag{S44a}
\]

\[
= \int_0^\tau \langle \phi, i\beta^{-1}[\nu, \rho] + \mathcal{O}_\rho(\nu) \rangle dt \tag{S44b}
\]

\[
= \int_0^\tau \langle \phi, \mathcal{O}_\rho(\nu) \rangle dt \tag{S44c}
\]

\[
\leq \left( \int_0^\tau \langle \phi, \mathcal{O}_\rho(\phi) \rangle dt \right)^{1/2} \left( \int_0^\tau \langle \nu, \mathcal{O}_\rho(\nu) \rangle dt \right)^{1/2} \tag{S44d}
\]

\[
\leq \left( \tau^{-1} \int_0^\tau \langle \phi, \mathcal{O}_\rho(\phi) \rangle dt \right)^{1/2} (\mathcal{W}_q(\rho_0, \rho_\tau)^2 + \delta)^{1/2}. \tag{S44e}
\]

The first term in the last inequality (S44e) can be rewritten as

\[
\langle \phi, \mathcal{O}_\rho(\phi) \rangle = \sum_{\mu, \omega} e^{-\beta \omega/2} \alpha_\mu(\omega) \langle \phi, [L_\mu(\omega), [\rho, \beta_\omega([L_\mu(\omega), \phi])]) \rangle \tag{S45a}
\]

\[
= \sum_{\mu, \omega} e^{-\beta \omega/2} \alpha_\mu(\omega) \text{tr} \left\{ [\phi, L_\mu(\omega)] [\rho, \beta_\omega([L_\mu(\omega), \phi])] \right\} \tag{S45b}
\]

\[
= \sum_{\mu, \omega} e^{-\beta \omega/2} \alpha_\mu(\omega) [L_\mu(\omega), [\rho, \beta_\omega([L_\mu(\omega), \phi])]) \tag{S45c}
\]

Before processing further, we prove the following result.

**Proposition 8.** For arbitrary operator \( X \), real number \( \theta \), and density operator \( \rho \),

\[
\langle X, [\rho]_\theta(X) \rangle \leq \frac{1}{2} (e^{\theta/2} + e^{-\theta/2}) \| X \|_\infty^2 \tag{S46}
\]

holds, where \( \| X \|_\infty \) denotes the spectral norm of the operator \( X \).

**Proof.** Using Eq. (S21), we have

\[
\langle X, [\rho]_\theta(X) \rangle = \sum_{n,m} \Phi(e^{\theta/2}r_n, e^{-\theta/2}r_m) \langle v_n | X | v_m \rangle \langle v_m | X^\dagger | v_n \rangle. \tag{S47}
\]

Applying the inequality \( \Phi(x, y) \leq (x + y)/2 \) and using the relation \( \sum_n |v_n\rangle \langle v_n| = \mathbb{I}_N \), we obtain

\[
\langle X, [\rho]_\theta(X) \rangle \leq \frac{1}{2} \sum_{n,m} \left( e^{\theta/2}r_n + e^{-\theta/2}r_m \right) \langle v_n | X | v_m \rangle \langle v_m | X^\dagger | v_n \rangle \tag{S48a}
\]

\[
= \frac{1}{2} \sum_{n,m} e^{\theta/2}r_n \langle v_n | X | v_m \rangle \langle v_m | X^\dagger | v_n \rangle + \frac{1}{2} \sum_{m,n} e^{-\theta/2}r_m \langle v_m | X^\dagger | v_n \rangle \langle v_n | X | v_m \rangle \tag{S48b}
\]

\[
= \frac{1}{2} \sum_{n} e^{\theta/2}r_n \| X \|_\infty^2 + \frac{1}{2} \sum_{m} e^{-\theta/2}r_m \| X \|_\infty^2 \tag{S48c}
\]

\[
\leq \frac{1}{2} \sum_{n} e^{\theta/2}r_n \| X \|_\infty^2 \tag{S48d}
\]

\[
= \frac{1}{2} (e^{\theta/2} + e^{-\theta/2}) \| X \|_\infty^2 . \tag{S48e}
\]

Here, we used the fact that \( \langle v_n | X^\dagger | v_n \rangle \leq \| X \|_\infty^2 \) in Eq. (S48d) and \( \sum_n r_n = 1 \) in Eq. (S48e). □
Now, going back to our problem, applying the Proposition 8 with $X = [L^\dagger_{\mu}(\omega), \phi]$ and $\theta = \beta \omega$, one obtains
\[
\langle [L^\dagger_{\mu}(\omega), \phi], [\rho]_{\beta \omega}([L^\dagger_{\mu}(\omega), \phi]) \rangle \leq \frac{1}{2} (e^{-\beta \omega/2} + e^{\beta \omega/2}) ||[L^\dagger_{\mu}(\omega), \phi]]\|_\infty \leq 2(e^{-\beta \omega/2} + e^{\beta \omega/2}) ||L^\dagger_{\mu}(\omega)\|_\infty. \tag{S49}
\]
Here, we used inequalities $||A, B||_\infty \leq ||AB||_\infty + ||BA||_\infty$ and $||AB||_\infty \leq ||A||_\infty ||B||_\infty$ for all $A, B \in \mathcal{L}(\mathcal{H})$. Consequently,
\[
\langle \phi, \mathcal{O}_\rho(\phi) \rangle \leq 2 \sum_{\mu, \omega} e^{-\beta \omega/2} \alpha_{\mu}(\omega) (e^{-\beta \omega/2} + e^{\beta \omega/2}) ||L^\dagger_{\mu}(\omega)\|_\infty^2 = 4 \sum_{\mu, \omega} \alpha_{\mu}(\omega) ||L^\dagger_{\mu}(\omega)\|_\infty^2. \tag{S50}
\]
From Eqs. (S44e) and (S50), the following inequality is readily obtained
\[
\mathcal{W}_q(\rho_0, \rho_t)^2 + \delta \geq \frac{(\sum_n \epsilon_n(\tau) - \epsilon_n(0))^2}{4\tau^{-1} \int_0^\tau \sum_{\mu, \omega} \alpha_{\mu}(\omega)||L^\dagger_{\mu}(\omega)||_\infty^2 dt}. \tag{S51}
\]
Setting $c_n = \text{sign}[\epsilon_n(\tau) - \epsilon_n(0)]$ and taking the limit $\delta \to 0$ in Eq. (S51), a lower bound of the quantum Wasserstein-like distance is obtained
\[
\mathcal{W}_q(\rho_0, \rho_t) \geq \frac{\sum_n |\epsilon_n(\tau) - \epsilon_n(0)|}{2\sqrt{\tau^{-1} \int_0^\tau \sum_{\mu, \omega} \alpha_{\mu}(\omega)||L^\dagger_{\mu}(\omega)||_\infty^2 dt}}. \tag{S52}
\]
From Eq. (S52), we wish to bound the Wasserstein-like distance by the trace-like distance $d_2(\rho_0, \rho_t) = \sum_{n=0}^N |a_n - b_n|$, where $a_1 \leq a_2 \leq \cdots \leq a_N$ and $b_1 \leq b_2 \leq \cdots \leq b_N$ are increasing eigenvalues of $\rho_0$ and $\rho_t$. Given two arrays of real numbers, $\{x_n\}$ and $\{y_n\}$, one can prove that
\[
\sum_n |x_n - y_n| \geq \sum_n |x_n - y_{p(n)}|, \tag{S53}
\]
where $\{p(n)\}$ is a permutation of $\{n\}$ such that $y_{p(n)} \geq y_{p(m)}$ if $x_n \geq x_m$. Therefore, $\sum_n |\epsilon_n(\tau) - \epsilon_n(0)| \geq d_2(\rho_0, \rho_t)$. Consequently, one can obtain the bound in terms of the trace-like distance as
\[
\mathcal{W}_q(\rho_0, \rho_t) \geq \frac{d_2(\rho_0, \rho_t)}{2\sqrt{\tau^{-1} \int_0^\tau \sum_{\mu, \omega} \alpha_{\mu}(\omega)||L^\dagger_{\mu}(\omega)||_\infty^2 dt}}. \tag{S54}
\]

D. Lower bound of the entropy production in terms of the distance of the average change in energy

Here we derive the lower bound of the entropy production $\Delta S_{\text{tot}}$ in terms of the distance $d_3(\rho_0, \rho_t) = |\text{tr}\{H(\rho_0 - \rho_t)\}|$. The Lindblad master equation can be expressed as $\dot{\rho}(t) = -i[H, \rho(t)] + \mathcal{O}_\rho(\phi(t))$, where $\phi(t) := -\text{tr}(\rho(t) + \rho^q)$. Using relations $\text{tr}\{H[H, \rho]\} = 0$ and $\Delta S_{\text{tot}} = \int_0^\tau \langle \phi, \mathcal{O}_\rho(\phi)\rangle dt$, we obtain
\[
|\text{tr}\{H(\rho_0 - \rho_t)\}| = |\text{tr}\{H \int_0^\tau \dot{\rho}(t) dt\}| \tag{S55a}
\]
\[
= |\int_0^\tau \text{tr}\{H, \mathcal{O}_\rho(\phi)\} dt| \tag{S55b}
\]
\[
\leq \left( \int_0^\tau \langle H, \mathcal{O}_\rho(\phi)\rangle dt \right)^{1/2} \left( \int_0^\tau \langle \phi, \mathcal{O}_\rho(\phi)\rangle dt \right)^{1/2} \leq \left( \int_0^\tau \langle H, \mathcal{O}_\rho(\phi)\rangle dt \right)^{1/2} \sqrt{\Delta S_{\text{tot}}}. \tag{S55c}
\]

The first term in the last equation [(S55d)] can be rewritten as
\[
\langle H, \mathcal{O}_\rho(\phi) \rangle = \sum_{\mu, \omega} e^{-\beta \omega/2} \alpha_{\mu}(\omega) \langle H, [L^\dagger_{\mu}(\omega), [\rho]_{\beta \omega}([L^\dagger_{\mu}(\omega), H])] \rangle \tag{S56a}
\]
\[
= \sum_{\mu, \omega} e^{-\beta \omega/2} \alpha_{\mu}(\omega) \text{tr} \{ [H, L^\dagger_{\mu}(\omega)][\rho]_{\beta \omega}([L^\dagger_{\mu}(\omega), H]) \} \tag{S56b}
\]
\[
= \sum_{\mu, \omega} e^{-\beta \omega/2} \alpha_{\mu}(\omega) \langle [L^\dagger_{\mu}(\omega), H], [\rho]_{\beta \omega}([L^\dagger_{\mu}(\omega), H]) \rangle. \tag{S56c}
\]
Applying the Proposition 8 with $X = [L^\dagger_\mu(\omega), H]$ and $\theta = \beta \omega$, one obtains

$$\langle [L^\dagger_\mu(\omega), H], [\rho, \beta \omega([L^\dagger_\mu(\omega), H])] \rangle \leq \frac{1}{2} (e^{-\beta \omega/2} + e^{\beta \omega/2}) \| [L^\dagger_\mu(\omega), H] \|_\infty^2 = \frac{1}{2} (e^{-\beta \omega/2} + e^{\beta \omega/2}) \omega^2 \| L^\dagger_\mu(\omega) \|_\infty^2. \quad (S57)$$

Consequently,

$$\langle H, \rho(H) \rangle \leq \frac{1}{2} \sum_{\mu, \omega} e^{-\beta \omega/2} \alpha_\mu(\omega) (e^{-\beta \omega/2} + e^{\beta \omega/2}) \omega^2 \| L^\dagger_\mu(\omega) \|_\infty^2 = \sum_{\mu, \omega} \alpha_\mu(\omega) \omega^2 \| L^\dagger_\mu(\omega) \|_\infty^2. \quad (S58)$$

From Eqs. (S55d) and (S58), the following inequality is readily obtained

$$\Delta S_{\text{tot}} \geq \frac{d_3(\rho_0, \rho_{\tau})^2}{\tau \sum_{\mu, \omega} \alpha_\mu(\omega) \omega^2 \| L^\dagger_\mu(\omega) \|_\infty^2}. \quad (S59)$$