MINIMAL SURFACES IN $S^3$ FOLIATED BY CIRCLES

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Abstract. We deal with minimal surfaces in the unit sphere $S^3$, which are one-parameter families of circles. Minimal surfaces in $\mathbb{R}^3$ foliated by circles were first investigated by Riemann, and a hundred years later Lawson constructed examples of such surfaces in $S^3$. We prove that in $S^3$ there are only two types of minimal surfaces foliated by circles, crossing the principal lines at a constant angle. The first type surfaces are foliated by great circles, which are bisectrices of the principal lines, and we show that these minimal surfaces are the well-known examples of Lawson. The second type surfaces, which are new in the literature, are families of small circles, and the circles are principal lines. We give a constructive formula for these surfaces. An application to the theory of minimal foliated semi-symmetric hypersurfaces in $\mathbb{R}^4$ is given.

1. Introduction

In the present paper we deal with minimal surfaces in the unit sphere $S^3$ in the four-dimensional Euclidean space $\mathbb{R}^4$, equipped with the standard Euclidean metric $\langle \cdot, \cdot \rangle$. A surface $M^2$ in $S^3$ is given by a unit vector-valued function $l(u,v)$ in $\mathbb{R}^4$, defined in a domain $D \subset \mathbb{R}^2$, i.e.

$$l(u,v) = (l^1(u,v), l^2(u,v), l^3(u,v), l^4(u,v)),$$

where $\langle l(u,v), l(u,v) \rangle = 1$, $(u,v) \in D$. Since our considerations are local, we assume that the parameters $(u,v)$ are isothermal (conformal) ones, which means that $\langle l_u, l_u \rangle = \langle l_v, l_v \rangle; \langle l_u, l_v \rangle = 0$.

The minimal surfaces in $S^3$ are determined by the solutions $l = l(u,v)$ of the following system of partial differential equations

$$\Delta l + |\nabla l|^2 l = 0;$$

$$\langle l_u, l_u \rangle = \langle l_v, l_v \rangle; \quad \langle l_u, l_v \rangle = 0; \quad \langle l, l \rangle = 1,$$

where $\nabla$ and $\Delta$ denote the gradient and the laplacian operators, respectively, computed with respect to the Euclidean metric in $\mathbb{R}^4$.

System (1.1) is the Euler - Lagrange system of harmonic maps, which has been intensively investigated in the last decades by variational methods (see for example [5], [6], [7], [8], [14] and the references there).

Our aim is to find the minimal surfaces in $S^3$, that satisfy a certain geometric property: locally they are one-parameter families of circles. The variational methods cannot be applied for studying the geometric structure of the minimal surfaces, that is why we use a different method, which is based rather on the differential geometry of surfaces in $\mathbb{R}^4$ than on PDE methods.

It is well known that the only minimal rotatitional surface in $\mathbb{R}^3$ is the catenoid, which is a surface fibred by circles in parallel planes. Other minimal surfaces in $\mathbb{R}^3$, which are foliated by circles in parallel planes, were discovered by Riemann [13]. They are usually referred in the literature as Riemann examples. Ennepe [2] proved that catenoids and Riemann

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examples are the only minimal surfaces in $\mathbb{R}^3$ foliated by circles. A surface in $\mathbb{R}^3$, which is determined by a smooth one-parameter family of circles is also called a cyclic surface. Cyclic surfaces of constant mean curvature and cyclic surfaces of constant Gauss curvature in $\mathbb{R}^3$ are described in [12] and [11].

Our idea to find the minimal surfaces in $S^3$, which are one-parameter family of circles, is motivated by what happens for cyclic minimal surfaces in $\mathbb{R}^3$.

A well known example of a minimal surface in $S^3$ is the Clifford torus (the standard flat torus), which consists of two orthogonal families of circles. Generalizations of the Clifford torus are the so called Lawson tori [10]. They have two orthogonal families of parametric lines, one of them consists of circles, and the other one consists of curves with constant Frenet curvatures in $\mathbb{R}^4$.

The circles on the Lawson torus cross the principal lines at an angle $\frac{\pi}{4}$. In this paper we find all minimal surfaces in $S^3$, which are one-parameter families of circles, crossing the principal lines at a constant angle. We call these surfaces generalized tori. In Theorem 2.2 we prove that there are only two types of generalized tori in $S^3$: the first one is characterized by the condition that the circles are bisectrices of the principal lines (we call these surfaces generalized tori of first type), and the second one is characterized by the condition that the circles are principal lines (we call them generalized tori of second type). We show that all generalized tori of first type are Lawson tori (Theorem 2.3). In Theorem 2.4 we give a constructive formula for the generalized tori of second type.

In Section 3 we point out the relation between the theory of minimal surfaces in $S^3$ and the theory of minimal foliated semi-symmetric hypersurfaces in $\mathbb{R}^4$. Each minimal surface in $S^3$ generates a minimal foliated semi-symmetric hypersurface in $\mathbb{R}^4$ according to a special construction given in [4]. We illustrate how this construction can be applied to two examples of minimal surfaces in $S^3$ for obtaining the first and the second type helicoids, which are special minimal foliated semi-symmetric hypersurfaces. We also apply the construction to a class of generalized tori of second type, which are new surfaces in the literature, and thus we obtain new minimal foliated semi-symmetric hypersurfaces in $\mathbb{R}^4$.

2. Generalized tori in $S^3$

Let $M^2 : l = l(u, v), (u, v) \in \mathcal{D}$ be a surface, parameterized by isothermal parameters, and lying on the unit sphere $S^3$ in $\mathbb{R}^4$, i.e. the vector-valued function $l(u, v)$ satisfies the equalities:

\begin{equation}
\langle l_u, l_u \rangle = \langle l_v, l_v \rangle = E(u, v); \quad \langle l_u, l_v \rangle = 0; \quad \langle l, l \rangle = 1.
\end{equation}

Since $l, l_u, l_v$ are mutually orthogonal, there exists a unique (up to a sign) unit vector field $n(u, v)$, such that $\{l, l_u, l_v, n\}$ form an orthogonal basis in $\mathbb{R}^4$. Differentiating the equalities (2.1), we get the following derivative formulas:

\begin{align*}
l_{uu} &= \frac{E_u}{2E} l_u - \frac{E_v}{2E} l_v - E l + a_{11} n; \\
l_{uv} &= \frac{E_u}{2E} l_u + \frac{E_v}{2E} l_v + a_{12} n; \\
l_{vv} &= -\frac{E_u}{2E} l_u + \frac{E_v}{2E} l_v - E l + a_{22} n,
\end{align*}

where $a_{ij}(u, v), i, j = 1, 2$ are functions defined in $\mathcal{D}$. Hence,

\begin{equation}
l_{uu} + l_{vv} + 2E l = (a_{11} + a_{22}) n.
\end{equation}
$M^2$ is a minimal surfaces in $S^3$ if and only if $a_{11} + a_{22} = 0$. Equality (2.2) implies that $M^2$ is minimal if and only if $l_{uu} + l_{vv} + 2El = 0$.

Consequently, the problem of finding the minimal surfaces in $S^3$ is equivalent to the solvability of the following system

$$\Delta l(u,v) + 2E(u,v)l(u,v) = 0;$$

$$\langle l_u, l_u \rangle = \langle l_v, l_v \rangle = E(u,v); \quad \langle l_u, l_v \rangle = 0; \quad \langle l, l \rangle = 1$$

(2.3)

with an appropriate $C^\infty$ smooth scalar function $E(u,v) > 0$ in a small neighbourhood $\mathcal{D}$ of the origin.

Let us note that according to the theorem of Hélein (see [5], p. 346) the solutions of system (1.1) (and hence of (2.3)) are $C^\infty$ smooth, because $\mathcal{D}$ is a two-dimensional domain.

Now let $M^2 : l = l(u,v)$ be a minimal surface in $S^3$. Then the derivative formulas of $M^2$ are as follows:

$$l_{uu} = \frac{E_u}{2E}l_u - \frac{E_v}{2E}l_v - El + an;$$

$$l_{uv} = \frac{E_u}{2E}l_u + \frac{E_v}{2E}l_v + bn;$$

$$l_{vv} = -\frac{E_u}{2E}l_u + \frac{E_v}{2E}l_v - El - an,$$

(2.4)

where $a(u,v)$ and $b(u,v)$ are functions in $\mathcal{D}$. The derivatives $n_u$ and $n_v$ of $n(u,v)$ satisfy

$$n_u = -\frac{a}{E}l_u - \frac{b}{E}l_v; \quad n_v = -\frac{b}{E}l_u + \frac{a}{E}l_v.$$  

(2.5)

Using the Gauss and Codazzi equations (or equivalently the identities of the mixed third derivatives of $l(u,v)$ and mixed second derivatives of $n(u,v)$) from (2.4) and (2.5) we get that the functions $a(u,v)$ and $b(u,v)$ are harmonic functions, satisfying the Cauchy - Riemann conditions:

$$b_u(u,v) = a_v(u,v); \quad b_v(u,v) = -a_u(u,v).$$

The Gauss and Codazzi equations for $M^2$ also imply the identity

$$a^2 + b^2 = \frac{1}{2}\Delta E - \frac{E_u^2 + E_v^2}{2E} + E^2.$$  

(2.6)

As it is well known the Gauss curvature $K$ of $M^2$ is given by

$$K = \frac{E_u^2 + E_v^2}{2E^2} - \frac{\Delta E}{2E^2} = -\frac{1}{2E}\Delta \ln E.$$  

(2.7)

Hence, equalities (2.6) and (2.7) imply that the Gauss curvature $K$ is expressed by the functions $a$ and $b$ as follows:

$$K = 1 - \frac{a^2 + b^2}{E^2}.$$  

(2.8)

The simplest case, in which problem (2.3) can be solved completely, is the case $K = \text{const}$. In [9] it is proved that: If $M^2$ is a minimal surface in $S^3$ of constant Gauss curvature $K$, then either $K = 1$ and $M^2$ is totally geodesic, or $K = 0$ and $M^2$ is an open piece of the Clifford torus.

From (2.8) it follows that the case $M^2$ is totally geodesic in $S^3$ (i.e. $M^2$ is a sphere with radius 1) corresponds to $a = b = 0$. Further on we shall consider only the case $(a,b) \neq (0,0)$.

Let us recall that the Clifford torus is a surface in $\mathbb{R}^4$, parameterized as follows:

$$\mathcal{M} : l(u,v) = (\cos u \cos v; \cos u \sin v; \sin u \cos v; \sin u \sin v).$$
A direct computation shows that \( l(u, v) \) satisfies the equality \( l_{uu} + l_{vv} + 2l = 0 \), and \( \langle l_u, l_v \rangle = \langle l_v, l_v \rangle = 1 \); \( \langle l_u, l_u \rangle = 0 \); \( \langle l, l \rangle = 1 \). The parametric lines \( u = \text{const} \) and \( v = \text{const} \) of \( \mathcal{M} \) are circles.

Another example of minimal surfaces in \( S^3 \), which is a generalization of the Clifford torus, is given in [10]. H. B. Lawson proved that every ”ruled” minimal surface in \( S^3 \) is an open submanifold of one of the surfaces \( \mathcal{M}_\alpha \) given by

\[
\mathcal{M}_\alpha: l(x, y) = (\cos x \cos \alpha y; \cos x \sin \alpha y; \sin x \cos y; \sin x \sin y)
\]

for some constant \( \alpha > 0 \). Here ”ruled” means one-parameter family of great circles in \( S^3 \). The surface \( \mathcal{M}_1 \) is the Clifford torus, and it is the only surface \( \mathcal{M}_\alpha \) with constant Gauss curvature (see [9]). We shall call the surfaces \( \mathcal{M}_\alpha \) (\( \alpha \neq 1 \)) Lawson tori.

The tangent space of \( \mathcal{M}_\alpha \) is spanned by the vector fields

\[
l_x(x, y) = (-\sin x \cos \alpha y; -\sin x \sin \alpha y; \cos x \cos y; \cos x \sin y), \\
l_y(x, y) = (-\alpha \cos x \sin \alpha y; \alpha \cos x \cos \alpha y; -\sin x \sin y; \sin x \cos y);
\]

and the coefficients \( E, F, G \) of the first fundamental form of \( \mathcal{M}_\alpha \) are given by \( E = 1; \ F = 0; \ G = G(x) = \alpha^2 \cos^2 x + \sin^2 x \).

Using (2.9) and (2.10) we find the unit normal vector field \( n(u, v) \) of \( \mathcal{M}_\alpha \) orthogonal to \( \{l, l_u, l_v\} \):

\[
n(x, y) = \frac{1}{\sqrt{\alpha^2 \cos^2 x + \sin^2 x}}(\sin x \sin \alpha y; -\sin x \cos \alpha y; -\alpha \cos x \sin y; \alpha \cos x \cos y).
\]

A direct computation shows that the Gauss curvature of \( \mathcal{M}_\alpha \) is given by

\[
K = 1 - \frac{\alpha^2}{(\alpha^2 \cos^2 x + \sin^2 x)^2}
\]

and obviously \( K \neq \text{const} \) when \( \alpha \neq 1 \).

Let us consider the Lawson torus \( \mathcal{M}_\alpha \) for \( \alpha \neq 1 \). In such case the parametric lines \( y = y_0 = \text{const} \) of \( \mathcal{M}_\alpha \) are circles, while the parametric lines \( x = x_0 = \text{const} \) are curves in \( \mathbb{R}^4 \) with constant Frenet curvatures.

The parametrization (2.9) of \( \mathcal{M}_\alpha \) is not an isothermal one. We shall find isothermal parameters of \( \mathcal{M}_\alpha \), which are also principal parameters of the surface. Let us consider the following change of the parameters:

\[
\pi = \int_0^x \frac{1}{\sqrt{G(\tau)}} d\tau; \quad \pi = v.
\]

Then we obtain

\[
\overline{E} = \overline{G} = G(x(\pi)); \quad \overline{F} = 0,
\]

i.e. the parameters \( (\overline{u}, \overline{v}) \) are isothermal. A direct computation shows that the vector-function \( \overline{l}(\overline{u}, \overline{v}) \) satisfies the system:

\[
\overline{l}_{u\overline{u}} = \frac{\overline{E}}{2 \overline{E}} \overline{E} l - \overline{E} l; \quad \overline{l}_{v\overline{v}} = \frac{\overline{E}}{2 \overline{E}} l + \alpha n; \quad \overline{l}_{u\overline{v}} = -\frac{\overline{E}}{2 \overline{E}} l - \overline{E} l.
\]

Hence, for the Lawson torus \( \mathcal{M}_\alpha \) the functions \( a \) and \( b \) in formulas (2.4) are \( a = 0, \ b = \alpha = \text{const} \), and \( \overline{E} = \overline{E}(\pi) \). The circles on \( \mathcal{M}_\alpha \) are the parametric \( \pi \)-lines.

If we change the isothermal parameters \( (\overline{u}, \overline{v}) \) with isothermal parameters \( (u, v) \) in the following way

\[
u = \frac{\sqrt{2}}{2} (\overline{u} + \overline{v}); \quad v = \frac{\sqrt{2}}{2} (\overline{u} - \overline{v}),
\]
then \( \tilde{E}(u,v) = (l_u, l_u) = (l_v, l_v) = \tilde{G}(u,v) \), and \( l(u,v) \) satisfies (2.4) with \( a = \alpha, b = 0 \), and \( \tilde{E} \) instead of \( E \).

With respect to \((u,v)\) the surface \( M_\alpha \) is parameterized by principal lines, i.e. the shape operator, corresponding to the normal vector field \( n(u,v) \), is in diagonal form. The circles on \( M_\alpha \) are bisectrices of the principal lines.

Now we shall find all minimal surfaces in \( S^3 \), which are one-parameter families of circles, crossing the principal lines under a constant angle. We call these surfaces generalized tori in \( S^3 \). They generalize the Lawson tori.

**Proposition 2.1.** Let \( M^2 \) be a minimal surface in \( S^3 \) with non-constant Gauss curvature. Then \( M^2 \) can locally be parameterized by principal lines, and the new parameters are isothermal ones.

**Proof.** Let \( M^2 : l = l(u,v), (u,v) \in \mathcal{D} \) be a minimal surface in \( S^3 \), parameterized by isothermal parameters. Then the derivative formulas (2.4) of \( M^2 \) hold good, and the functions \( a(u,v) \) and \( b(u,v) \) are harmonic functions, satisfying the Cauchy - Riemann conditions. In case of \( b(u,v) \equiv 0 \) the parameters \((u,v)\) are principal ones. Let \( b(u_0, v_0) \neq 0, (u_0, v_0) \in \mathcal{D} \).

Then there exists \( \mathcal{D}_0 \subset \mathcal{D} \) such that \( b(u,v) \neq 0 \) for all \((u,v) \in \mathcal{D}_0 \). We shall prove that there exist isothermal parameters \((x,y)\) such that \( \tilde{b}(x,y) = (l_{xy}, n) = 0 \). If
\[
\begin{align*}
x &= x(u,v); & y &= y(u,v)
\end{align*}
\]
is a holomorphic change of the parameters \((x(u,v)\) and \(y(u,v)\) satisfy the Cauchy - Riemann conditions), then
\[
\tilde{b}(x,y) = 2au_x u_y - b(u_x^2 - u_y^2).
\]
Hence,
\[
\tilde{b} = 0 \quad \iff \quad b \left( \frac{u_x}{u_y} \right)^2 - 2a \left( \frac{u_x}{u_y} \right) - b = 0.
\]

From the inverse change of the parameters, using the Cauchy - Riemann conditions, we have \( x_u = \frac{u_x}{u_x^2 + u_y^2}, x_v = -\frac{u_y}{u_x^2 + u_y^2} \), and hence we obtain that \( \tilde{b} = 0 \) if and only if
\[
b \left( \frac{x_u}{x_v} \right)^2 + 2a \left( \frac{x_u}{x_v} \right) - b = 0,
\]
i.e. \( \frac{x_u}{x_v} = \frac{a \pm \sqrt{a^2 + b^2}}{b} \). We denote \( \beta(u,v) = \frac{-a + \sqrt{a^2 + b^2}}{b}, \gamma(u,v) = \frac{-a - \sqrt{a^2 + b^2}}{b} \).

Now, let us consider the equations
\[
(2.11) \quad \frac{dv}{du} = \beta(u,v); \quad \frac{dv}{du} = \gamma(u,v).
\]
For each point \((u_0,v_0) \in \mathcal{D}_0 \) there exists \( \mathcal{D}_1 \subset \mathcal{D}_0 \) and functions \( \Phi(u,v) \neq 0, \Psi(u,v) \neq 0 \) in \( \mathcal{D}_1 \), such that the integral curves of the first equation in (2.11) are given by \( \Phi(u,v) = \text{const} \), while the integral curves of the second equation in (2.11) are \( \Psi(u,v) = \text{const} \). Hence,
\[
(2.12) \quad \Phi_u = -\beta \Phi_v, \quad \Psi_u = -\gamma \Psi_v.
\]

We consider the following smooth change of the parameters:
\[
(2.13) \quad x = \Phi(u,v), \quad y = \Psi(u,v); \quad (u,v) \in \mathcal{D}_1.
\]
When \((u,v)\) runs in \( \mathcal{D}_1 \) the parameters \((x,y)\) describe a domain \( \mathcal{D} \subset \mathbb{R}^2 \). Now \( x_u = \Phi_u; \quad x_v = \Phi_v; \quad y_u = \Psi_u; \quad y_v = \Psi_v \). Using that \( \beta \gamma = -1 \) and equalities (2.12), we get \( (l_x, l_y) = 0 \), i.e. the parametrization \((x,y)\) is orthogonal one. We shall prove that this parametrization is isothermal. With respect to the new parameters the coefficients of the first fundamental
form are \( E = \langle l_x, l_x \rangle, \quad F = \langle l_x, l_y \rangle = 0, \quad \text{and} \quad G = \langle l_y, l_y \rangle. \) Then for the surface \( M^2 \) the following derivative formulas hold:

\[
\begin{align*}
    l_{xx} &= \frac{E_x}{2E} l_x - \frac{E_y}{2E} l_y - E l + a n; \\
    l_{xy} &= \frac{E_y}{2E} l_x + \frac{G_x}{2G} l_y; \\
    l_{yx} &= \frac{G_x}{2G} l_x + \frac{E_y}{2E} l_y - G l - a n,
\end{align*}
\]

(2.14)

\[
    n_x = -\frac{a}{E} l_x; \quad n_y = \frac{a}{G} l_y.
\]

(2.15)

Taking into account the second fundamental form of \( M^2 \) as a surface in \( \mathbb{R}^4 \), from (2.14) we calculate that the Gauss curvature \( K \) is given by

\[
    K = 1 - \frac{a^2}{EG}.
\]

(2.16)

Using that \( n_{xy} = n_{yx}, \quad l_{xxy} = l_{xyx}, \quad l_{xyy} = l_{yyx} \), from (2.14) and (2.15) we obtain

\[
\begin{align*}
    a_x &= 0; \quad a_y = 0; \quad (E - G)E_y = 0; \quad (E - G)G_x = 0; \\
    \frac{E_{yy}}{2E} + \frac{G_{xx}}{2G} - \frac{3E_y^2}{4E^2} - \frac{G_x^2}{4G^2} + \frac{E_xG_x - E_yG_y}{4EG} + E - \frac{a^2}{G} = 0; \\
    \frac{E_{yy}}{2E} + \frac{G_{xx}}{2G} - \frac{3G_y^2}{4E^2} - \frac{E_y^2}{4E^2} + \frac{E_xG_x - E_yG_y}{4EG} + G - \frac{a^2}{E} = 0.
\end{align*}
\]

(2.17)

(2.18)

If we assume that \( E(x_0, y_0) - G(x_0, y_0) \neq 0 \) at some point \((x_0, y_0) \in \overline{D}\), and hence \( E - G \neq 0 \) in a neighbourhood \( \overline{D}_0 \subset \overline{D} \) of \((x_0, y_0)\), then from (2.17) we get \( E_y = G_x = 0 \) in \( \overline{D}_0 \). Now equalities (2.16), (2.17), and (2.18) imply that \( K = 0 \) in \( \overline{D}_0 \), which contradicts the assumption in the theorem. Hence, \( E - G \equiv 0 \) in \( \overline{D}_0 \). Consequently, the parameters \((x, y)\), defined by change (2.13), are principal ones. \( \square \)

Now let \( M^2 : l = l(u, v) \) be a minimal surface in \( S^3 \), parameterized locally by isothermal principal parameters, i.e. \( b = 0 \). Using that \( b_u = a_v, \quad b_v = -a_u \), we get \( a = \text{const} \). Without loss of generality we assume that \( a = 1 \) (if \( a \neq 1 \), we multiply the parameters by \( \sqrt{|a|} \)). Hence, the derivative formulas (2.4) and (2.5) hold with \( a = 1, \quad b = 0 \).

**Theorem 2.2.** Let \( M^2 \) be a minimal surface in \( S^3 \) with non-constant Gauss curvature. If on \( M^2 \) there exists a family of circles, crossing the principal lines under a constant angle \( \theta \), then the circles are either principal lines \( (\theta = 0 \) or \( \theta = \frac{\pi}{2} \)) or bisectrices of the principal lines \( (\theta = \frac{\pi}{4} \) or \( \theta = \frac{3\pi}{4} \)).

**Proof.** Let \( M^2 : l = l(u, v) \) be parameterized locally by principal parameters. Suppose that on \( M^2 \) there exists a family of circles, crossing the principal lines under a constant angle \( \theta \). Let

\[
\begin{align*}
    x &= \cos \theta u + \sin \theta v; \\
    y &= -\sin \theta u + \cos \theta v; \quad \theta = \text{const}, \quad \theta \in [0; 2\pi).
\end{align*}
\]

(2.19)

Then from (2.19) we get

\[
\begin{align*}
    E_x &= \cos \theta E_u + \sin \theta E_v; \\
    E_y &= -\sin \theta E_u + \cos \theta E_v.
\end{align*}
\]

(2.20)
Using (2.4) with $a = 1$, $b = 0$, and (2.20) we calculate
\[ l_{xx} = \frac{E_x}{2E} l_x - \frac{E_y}{2E} l_y - E l + \cos 2\theta n; \]
\[ l_{xy} = \frac{E_x}{2E} l_x + \frac{E_y}{2E} l_y - \sin 2\theta n; \]
\[ l_{yy} = -\frac{E_x}{2E} l_x + \frac{E_y}{2E} l_y - E l - \cos 2\theta n. \]

Let us denote $a = \cos 2\theta$, $b = \sin 2\theta$ $(a, b - \text{const})$.

Now we shall consider an arbitrary $x$-line $c : l(x) = l(x, y_0)$. The curve $c$ is a circle if and only if its Frenet curvatures are $\kappa = \text{const}$; $\tau = \sigma = 0$. Using (2.21) we calculate the tangent vector $t_c$ and the principal normal vector $n_c$ of $c$:
\[ t_c = \frac{l_x}{\sqrt{E}}; \quad n_c = \frac{1}{\kappa} \left( -\frac{E_y}{2E^2} l_y - l + \frac{a}{E} n \right), \]
where $\kappa^2 = \frac{E_y^2}{4E^3} + 1 + \frac{a^2}{E^2}$. The derivatives of $t_c$ and $n_c$ are:
\[ t'_c = \kappa n_c; \]
\[ n'_c = -\kappa t_c - \left[ \left( \frac{1}{\kappa} \right)' \frac{E_y}{2E^2} + \frac{1}{\kappa} \left( \left( \frac{E_y}{2E^2} \right)' \frac{E_x E_y}{4E^3} + \frac{a b}{E^2} \right) \right] \frac{l_y}{\sqrt{E}} \]
\[ - \left( \frac{1}{\kappa} \right)' \frac{l}{\sqrt{E}} + \left[ \left( \frac{1}{\kappa} \right)' \frac{a}{E} + \frac{1}{\kappa} \left( -b \frac{E_y}{2E^2} + \left( \frac{a}{E} \right)' \right) \right] \frac{n}{\sqrt{E}}. \]

From (2.22) it follows that $c$ is a circle if and only if
\[ \kappa = \text{const}; \]
\[ \left( \frac{E_y}{2E^2} \right)' \frac{E_x}{4E^3} + \frac{a b}{E^2} = 0; \]
\[ \left( \frac{a}{E} \right)' - b \frac{E_y}{2E^2} = 0, \]
which is equivalent to
\[ 2a E_x + b E_y = 0; \]
\[ 2E E_{xy} - 3E_x E_y + 4ab E = 0. \]

Analogously, the $y$-lines are circles if and only if
\[ b E_x - 2a E_y = 0; \]
\[ 2E E_{xy} - 3E_x E_y - 4ab E = 0. \]

From (2.20) we calculate
\[ E_{xy} = -\sin \theta \cos \theta E_{uu} + \cos 2\theta E_{uv} + \sin \theta \cos \theta E_{uw}. \]

From the first equality of (2.23), using (2.20), we get $\cos^3 \theta E_u - \sin^3 \theta E_v = 0$. All solutions of this equation are given by
\[ E = \varphi (\sin^3 \theta u + \cos^3 \theta v) \]
for arbitrary smooth function $\varphi$. Hence, $E_u = \sin^3 \theta \varphi'$; $E_v = \cos^3 \theta \varphi'$; $E_{uu} = \sin^6 \theta \varphi''$; $E_{vv} = \cos^6 \theta \varphi''$; $E_{uv} = \cos^6 \theta \varphi''$. From the second equality of (2.23), using (2.25) and
(2.26) we obtain \( \sin 2\theta \cos 2\theta (\varphi \varphi'' - \frac{3}{2} \varphi'^2 - 4\varphi) = 0 \). Consequently, the \( x \)-lines are circles if and only if
\[
E = \varphi (\sin^3 \theta u + \cos^3 \theta v);
\]
(2.27)
\[
\sin 2\theta \cos 2\theta (\varphi \varphi'' - \frac{3}{2} \varphi'^2 - 4\varphi) = 0.
\]

Analogously, using (2.24) we obtain that the \( y \)-lines are circles if and only if
\[
E = \varphi (\cos^3 \theta u - \sin^3 \theta v);
\]
(2.28)
\[
\sin 2\theta \cos 2\theta (\varphi \varphi'' - \frac{3}{2} \varphi'^2 - 4\varphi) = 0.
\]

Thus the condition the \( x \)-lines (or \( y \)-lines) to be circles leads to the following cases:

I. \( \sin 2\theta = 0 \), i.e. \( \theta = 0 \) or \( \theta = \frac{\pi}{2} \).

This case corresponds to \( x = u; \ y = v \) or \( x = v; \ y = -u \). From (2.27) and (2.28) we obtain that the \( u \)-lines are circles if and only if \( E = \varphi (v) \), and the \( v \)-lines are circles if and only if \( E = \varphi (u) \). In this case one of the families of principal lines is a family of circles.

II. \( \cos 2\theta = 0 \), i.e. \( \theta = \frac{\pi}{4} \) or \( \theta = \frac{3\pi}{4} \).

This case corresponds to \( x = \frac{\sqrt{2}}{2} (u+v) \); \( y = \frac{\sqrt{2}}{2} (-u+v) \) or \( x = \frac{\sqrt{2}}{2} (-u+v) \); \( y = -\frac{\sqrt{2}}{2} (u+v) \).

From (2.27) and (2.28) we obtain that the \( x \)-lines are circles if and only if \( E = \varphi (x) \), and the \( y \)-lines are circles if and only if \( E = \varphi (y) \). In this case one of the families of bisectrices of the principal lines is a family of circles.

III. \( \varphi \varphi'' - \frac{3}{2} \varphi'^2 - 4\varphi = 0 \), and \( \sin 2\theta \cos 2\theta \neq 0 \).

We shall prove that this case is not possible. Since \( E > 0 \) then \( E = \varphi (\tau) = e^{z(\tau)} \) for some function \( z = z(\tau) \), \( \tau = \sin^3 \theta u + \cos^3 \theta v \) (or \( \tau = \cos^3 \theta u - \sin^3 \theta v \)). Moreover, \( E \neq const \), i.e. \( z'(\tau) \neq 0 \). The equality \( \varphi \varphi'' - \frac{3}{2} \varphi'^2 - 4\varphi = 0 \) implies
\[
(2.29)
\]
\[
z'' - \frac{1}{2} z'^2 - 4e^{-z} = 0.
\]

On the other hand, using identity (2.6), we obtain
\[
(2.30)
\]
\[
(\cos^6 \theta + \sin^6 \theta) z'' + 4 \sinh z = 0.
\]

Let us denote \( \lambda = \cos^6 \theta + \sin^6 \theta = const \). Multiplying (2.30) by \( z' \) and integrating we get
\[
\frac{\lambda}{2} z'^2 (\tau) + 4 \cosh z(\tau) = \frac{\lambda}{2} z'^2 (0) + 4 \cosh z(0).
\]

Equalities (2.29) and (2.30) imply
\[
4 \sinh z + \frac{\lambda}{2} z'^2 + 4\lambda e^{-z} = 0.
\]

Using the last two equalities we obtain
\[
4(1 - \lambda) e^{-z} = \frac{\lambda}{2} z'^2 (0) + 4 \cosh z(0) = const.
\]

Since \( 1 - \lambda = 3 \sin^2 \theta \cos^2 \theta \neq 0 \), we get \( e^{-z(\tau)} = const \), i.e. \( z(\tau) = const \), which contradicts the condition \( z'(\tau) \neq 0 \), i.e. \( E \neq const \). \( \square \)

From Theorem 2.2 it follows that there are only two types of generalized tori in \( S^3 \): the first one is characterized by the condition that one of the families of bisectrices of the principal
lines is a family of circles (we shall call such surfaces generalized tori of first type), and the second one is characterized by the condition that one of the families of principal lines is a family of circles (we shall call these surfaces generalized tori of second type).

**Theorem 2.3.** Let $M^2$ be a generalized torus of first type with non-constant Gauss curvature. Then $M^2$ is a Lawson torus $\mathcal{M}_\alpha$ for some $\alpha > 0$, $\alpha \neq 1$.

**Proof.** Let $M^2$ be a generalized torus of first type with non-constant Gauss curvature. In this case the derivative formulas (2.4) hold, with $a = 0$; $b = 1$, and $E = E(u)$ (or $E = E(v)$). We shall consider only the first case, i.e. $E = E(u), E_u \neq 0$. The second one can be investigated analogously. The derivative formulas in this case look like:

\begin{align*}
l_{uu} &= \frac{E_u}{2E} l_u - E l; \quad n_u = -\frac{1}{E} l_v; \\
l_{uv} &= \frac{E_u}{2E} l_v + n; \quad n_v = -\frac{1}{E} l_u; \\
l_{vv} &= -\frac{E_u}{2E} l_u - E l;
\end{align*}

(2.31)

We shall prove that the parametric $u$-lines are great circles. Let $c : c(u) = l(u, v_0), v_0 = \text{const}$ be an arbitrary $u$-line. From (2.31) it follows that $\dot{c} = l_u$, and the tangent vector $t_c$ of $c$ is $t_c = c' = \frac{\dot{c}}{\sqrt{E}}$. We calculate $t_c' = \frac{t_c}{\sqrt{E}} = \frac{1}{\sqrt{E}} \left( \frac{l_{uu}}{\sqrt{E}} - \frac{E_u}{2E \sqrt{E}} l_u \right) = -l$. Hence, the curvature $\kappa$ of $c$ is $\kappa = 1$, i.e. $c$ is a great circle. Consequently, $M^2$ is a one-parameter family of great circles. According to [10, Prop. 7.2] $M^2$ is on open submanifold of $\mathcal{M}_\alpha$ for some $\alpha > 0$, $\alpha \neq 1$. □

Now we shall consider a generalized torus of second type with non-constant Gauss curvature. In this case the derivative formulas (2.4) hold, with $a = 1$; $b = 0$, and $E = E(u)$ (or $E = E(v)$). We consider the case $E = E(u), E_u \neq 0$. In this case the parametric $v$-lines are circles. We shall prove that the different $v$-lines are circles with different radius.

Now the derivative formulas are as follows:

\begin{align*}
l_{uu} &= \frac{E_u}{2E} l_u - E l + n; \quad n_u = -\frac{1}{E} l_v; \\
l_{uv} &= \frac{E_u}{2E} l_v; \quad n_v = \frac{1}{E} l_u; \\
l_{vv} &= -\frac{E_u}{2E} l_u - E l - n;
\end{align*}

(2.32)

Let $c : c(v) = l(u_0, v), u_0 = \text{const}$ be an arbitrary $v$-line. As in the proof of Theorem 2.3 from (2.32) we calculate the curvature $\kappa(u_0)$ of $c$: $\kappa(u_0) = \sqrt{1 + \frac{E_u^2(u_0)}{4E^3(u_0)} + \frac{1}{E^2(u_0)}}$.

For different values of the constant $u_0$ the curvatures $\kappa(u_0)$ are different. We note that $\kappa(u_0) > 1$, i.e. the circles are not great ones. Therefore, the generalized tori of second type are different from the Lawson tori.
Since \( E(u) > 0 \), we write \( E(u) \) in the form \( E(u) = e^{z(u)}, \ z(u) \neq \text{const} \). Then the system (2.32) is rewritten in the following form

\[
\begin{align*}
  l_{uu} - \frac{z'(u)}{2} l_u + e^{z(u)} l - n &= 0; \\
  n_u + e^{-z(u)} l_u &= 0; \\
  l_{uv} - \frac{z'(u)}{2} l_v &= 0; \\
  n_v - e^{-z(u)} l_v &= 0; \\
  l_{vv} + \frac{z'(u)}{2} l_u + e^{z(u)} l + n &= 0.
\end{align*}
\]

(2.33)

We look for classical solutions of the above system for the vector-valued functions \( l(u, v), \ l_u(u, v), \ l_v(u, v) \) and \( n(u, v) \) in a neighbourhood of the origin under the following initial conditions:

\[
\begin{align*}
  l(0, 0) &= e_1; \\
  l_u(0, 0) &= e^{\frac{s}{2}} e_2 = \sqrt{E(0)} e_2; \\
  l_v(0, 0) &= e^{\frac{s}{2}} e_3 = \sqrt{E(0)} e_3; \\
  n(0, 0) &= e_4,
\end{align*}
\]

(2.34)

where \( \{e_1, e_2, e_3, e_4\} \) is the standard orthonormal basis in \( \mathbb{R}^4 \), and \( s = \text{const} = z(0) = \ln E(0) \).

Since the function \( E(u) \) satisfies identity (2.6) with \( a = 1, \ b = 0 \), it follows that \( z(u) \) is the solution of the following ordinary differential equation

\[
z''(u) + 4 \sinh z(u) = 0; \quad z(0) = s; \quad z'(0) = 2t,
\]

(2.35)

where \( s \) and \( t \) are arbitrary constants.

In order to find explicitly \( z(u) \) we note that the identity (2.6) holds for an arbitrary minimal surface in \( S^3 \), parameterized by isothermal parameters. So, let us consider again the Lawson torus \( M_{\alpha} \), defined by (2.9). We change the parameters \( (x, y) \) by new parameters \( (u, v) \) in the following way

\[
\begin{align*}
  u &= h(x) = \sqrt{\alpha} \int_0^x \frac{1}{\sqrt{\alpha^2 \cos^2 \tau + \sin^2 \tau}} \, d\tau; \\
  v &= \sqrt{\alpha} y,
\end{align*}
\]

and denote by \( h^{-1} \) the inverse function of \( h \). Then we obtain an isothermal parametrization of \( M_{\alpha} \), and the function

\[
E(u) = e^{z(u)} = \frac{\alpha^2 \cos^2 h^{-1}(u) + \sin^2 h^{-1}(u)}{\alpha}
\]

satisfies (2.6) with \( a = 0; \ b = 1 \). Hence the function

\[
z(u) = \ln \frac{\alpha^2 \cos^2 h^{-1}(u) + \sin^2 h^{-1}(u)}{\alpha}
\]

(2.36)

is a solution of (2.35) with \( t = 0 \) and \( s = \ln \alpha, \ \alpha > 0, \ \alpha \neq 1 \). (It can be calculated directly that the function \( z(u) \), defined by (2.36) satisfies (2.35) with \( z(0) = \ln \alpha, \ \alpha > 0, \ \alpha \neq 1 \) and \( z'(0) = 0 \).)

We will prove that every solution \( \tilde{z}(u) \) of (2.35) with arbitrary \( t \) and \( s \) can be obtained from (2.36) by the formula \( \tilde{z}(u) = z(u + u_0) \) for a suitable choice of constants \( u_0 \) and \( \alpha \). Since (2.35) is an autonomous equation and \( z(u) \) is its solution, then it follows that \( \tilde{z}(u) = z(u + u_0) \) is also a solution of this equation. Therefore we have to check only the initial conditions. Let \( x_0 = h^{-1}(u_0) \). Simple calculations give us the equalities

\[
\begin{align*}
  \tilde{z}(0) &= z(u_0) = \ln \frac{\alpha^2 \cos^2 x_0 + \sin^2 x_0}{\alpha} = s; \\
  \tilde{z}'(0) &= z'(u_0) = \frac{(1 - \alpha^2) \sin 2x_0}{\sqrt{\alpha} \sqrt{\alpha^2 \cos^2 x_0 + \sin^2 x_0}} = 2t,
\end{align*}
\]
which imply
\[
\sin 2x_0 = \frac{2\alpha t e^{\frac{s}{2}}}{1 - \alpha^2}; \quad \cos 2x_0 = \frac{1 + \alpha^2 - 2\alpha e^s}{1 - \alpha^2}.
\]
Using that \( \sin^2 2x_0 + \cos^2 2x_0 = 1 \), we get the following equation for \( \alpha \):
\[
e^s \alpha^2 - (1 + e^{2s} + t^2 e^s)\alpha + e^s = 0,
\]
whose positive solutions are
\[
\alpha = \frac{1 + e^{2s} + t^2 e^s \pm \sqrt{(1 + e^{2s} + t^2 e^s)^2 - 4e^{2s}}}{2e^s}.
\]
For the above choice of \( \alpha \) and \( u_0 = h(x_0) = \frac{1}{2} \arctan \frac{2\alpha t e^{\frac{s}{2}}}{1 + \alpha^2 - 2\alpha e^s} \) the initial conditions are satisfied.

It is curious to mention that all solutions of (2.35) are periodic ones with period \( \omega = h(\pi) \).

In order to simplify system (2.33) we change the vector-function \( L(u, v) \) with vector-function \( L(u, v) \) determined by
\[
(2.37) \quad l(u, v) = e^{\frac{z(u)}{2}} L(u, v).
\]
We get the system
\[
L_{uu}(u, v) + \frac{z'(u)}{2} L_u(u, v) + e^{-z(u)} L(u, v) - e^{-\frac{z(u)}{2}} n = 0;
\]
\[
L_{uv}(u, v) = 0;
\]
\[
L_{vv}(u, v) + \frac{z'(u)}{2} L_v(u, v) + \left[ \left( \frac{z'(u)}{2} \right)^2 + e^{z(u)} \right] L(u, v) + e^{-\frac{z(u)}{2}} n = 0;
\]
\[
n_u + e^{-\frac{z(u)}{2}} \left( L_u + \frac{z'(u)}{2} L \right) = 0;
\]
\[
n_v - e^{-\frac{z(u)}{2}} L_v = 0.
\]
The initial conditions (2.34) for \( l(u, v) \) imply the following initial conditions for \( L(u, v) \):
\[
L(0, 0) = e^{-\frac{s}{2}} e_1; \quad L_u(0, 0) = e_3; \quad L_v(0, 0) = -t e^{-\frac{s}{2}} e_1 + e_2; \quad n(0, 0) = e_4.
\]
From the second equality in (2.38) it follows that \( L(u, v) = f(u) + g(v) \), where \( f(u) \) and \( g(v) \) are vector-functions, satisfying the system
\[
f''(u) + \frac{z'(u)}{2} f'(u) + e^{-z(u)} (f(u) + g(v)) - e^{-\frac{z(u)}{2}} n = 0;
\]
\[
g''(v) + \frac{z'(u)}{2} f'(u) + \left[ \left( \frac{z'(u)}{2} \right)^2 + e^{z(u)} \right] (f(u) + g(v)) + e^{-\frac{z(u)}{2}} n = 0;
\]
\[
n_u + e^{-\frac{z(u)}{2}} \left( f'(u) + \frac{z'(u)}{2} (f(u) + g(v)) \right) = 0;
\]
\[
n_v - e^{-\frac{z(u)}{2}} g'(v) = 0;
\]
\[
f'(0) = -t e^{-\frac{s}{2}} e_1 + e_2; \quad g'(0) = e_3; \quad f(0) + g(0) = e^{-\frac{s}{2}} e_1.
\]
Without loss of generality we assume that \( g(0) = 0; \ f(0) = e^{-\frac{s}{2}} e_1 \).
Let us fix \( u = 0 \) in the fourth equality of (2.39). Then after integration we get 
\[
n(0, v) = e^{-\frac{s}{v}}g(v) + e_4.
\]

Now using the second equality of (2.39) we obtain that \( g(v) \) satisfies the initial problem 
\[
g''(v) + (t^2 + 2 \cosh s)g(v) = -e^{\frac{s}{v}}e_1 - te_2 - e^{-\frac{s}{v}}e_4;
\]
\[
g(0) = 0; \quad g'(0) = e_3.
\]

Simple computations give us 
\[
g(v) = \frac{1}{t^2 + 2 \cosh s} (\cos \sqrt{t^2 + 2 \cosh s} v - 1)(e^{\frac{s}{v}}e_1 + te_2 + e^{-\frac{s}{v}}e_4) + \frac{1}{\sqrt{t^2 + 2 \cosh s}} \sin \sqrt{t^2 + 2 \cosh s} v e_3.
\]

(2.40)

Now, multiplying (2.35) with \( z'(u) \) and integrating from 0 to \( u \), we get that \( z'(u) \) satisfies the equality 
\[
(z'(u))^2 + 8 \cosh z(u) = 4t^2 + 8 \cosh s.
\]

Using the first and the second equality of (2.39), (2.40) and (2.41), setting \( v = 0 \) we obtain that \( f(u) \) satisfies the initial problem:
\[
f''(u) + z'(u)f'(u) + (t^2 + 2 \cosh s)f(u) = e^{\frac{s}{u}}e_1 + te_2 + e^{-\frac{s}{u}}e_4;
\]
\[
f(0) = e^{-\frac{s}{u}}e_1; \quad f'(0) = -te^{-\frac{s}{u}}e_1 + e_2.
\]

Therefore the solution \( f(u) \) of the above system can be written in the form 
\[
f(u) = p(u) + \frac{1}{t^2 + 2 \cosh s} (e^{\frac{s}{u}}e_1 + te_2 + e^{-\frac{s}{u}}e_4),
\]

where \( p(u) \) is the unique solution of the following linear homogenous system:
\[
p''(u) + z'(u)p'(u) + (t^2 + 2 \cosh s)p(u) = 0;
\]
\[
p(0) = \frac{1}{t^2 + 2 \cosh s} [e^{-\frac{s}{u}}(t^2 + e^{-s})e_1 - te_2 - e^{-\frac{s}{u}}e_4]; \quad p'(0) = -te^{-\frac{s}{u}}e_1 + e_2.
\]

Now, using (2.37) we obtain that the solution \( l(u, v) \) of problem (2.33), (2.34) is given by 
\[
l(u, v) = e^{\frac{z(u)}{u}} p(u) + \frac{e^{\frac{z(u)}{u}}}{t^2 + 2 \cosh s} \cos \sqrt{t^2 + 2 \cosh s} v (e^{\frac{s}{u}}e_1 + te_2 + e^{-\frac{s}{u}}e_4) + \frac{e^{\frac{z(u)}{u}}}{\sqrt{t^2 + 2 \cosh s}} \sin \sqrt{t^2 + 2 \cosh s} v e_3.
\]

If we denote \( \beta = \sqrt{t^2 + 2 \cosh s} \) \( (\beta = \text{const}) \), then \( l(u, v) \) is rewritten in the form:
\[
l(u, v) = e^{\frac{z(u)}{u}} p(u) + \frac{e^{\frac{z(u)}{u}}}{\beta^2} [\cos \beta v (e^{\frac{s}{u}}e_1 + te_2 + e^{-\frac{s}{u}}e_4) + \beta \sin \beta v e_3].
\]

We remind that the function \( z(u) \) is given explicitly by (2.36).

Thus we prove the following result:

**Theorem 2.4.** Let \( M^2 : l = l(u, v) \) be a generalized torus of second type with non-constant Gauss curvature. Then
\[
l(u, v) = e^{\frac{z(u)}{u}} p(u) + \frac{e^{\frac{z(u)}{u}}}{\beta^2} [\cos \beta v (e^{\frac{s}{u}}e_1 + te_2 + e^{-\frac{s}{u}}e_4) + \beta \sin \beta v e_3],
\]
where \( \beta = \sqrt{t^2 + 2 \cosh s} \) for arbitrary constants \( s \) and \( t \), the scalar function \( z(u) \) is the solution of the equation
\[
z'' + 4 \sinh z(u) = 0; \quad z(0) = s; \quad z'(0) = 2t,
\]
and is given explicitly by (2.36), \( p(u) \) is a vector function, which is a solution of the system
\[
p''(u) + z'(u)p'(u) + \beta^2 p(u) = 0;
\]
and \( e_1, e_2, e_3, e_4 \) is the standard orthonormal basis in \( \mathbb{R}^4 \).

3. Application to the theory of minimal foliated semi-symmetric hypersurfaces

In this section we shall relate the theory of minimal surfaces in \( S^3 \) with the theory of minimal foliated semi-symmetric hypersurfaces in \( \mathbb{R}^4 \).

For an \( n \)-dimensional Riemannian manifold \( (M^n, g) \) we denote by \( T_pM^n \) the tangent space to \( M^n \) at a point \( p \in M^n \) and by \( \mathfrak{X}M^n \) - the algebra of all vector fields on \( M^n \). The associated Levi-Civita connection of the metric \( g \) is denoted by \( \nabla \), the Riemannian curvature tensor \( R \) is defined by
\[
R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}; \quad X, Y \in \mathfrak{X}M^n.
\]

A semi-symmetric space is a Riemannian manifold \( (M^n, g) \), whose curvature tensor \( R \) satisfies the identity
\[
R(X, Y) \cdot R = 0
\]
for all vector fields \( X, Y \in \mathfrak{X}M^n \). (Here \( R(X, Y) \) acts as a derivation on \( R \)).

According to the classification of Z. Szabó [13] the main class of semi-symmetric spaces is the class of all Riemannian manifolds foliated by Euclidean leaves of codimension two.

The foliated semi-symmetric hypersurfaces in Euclidean space \( \mathbb{E}^{n+1} \) are the hypersurfaces of type number two, i.e. hypersurfaces whose rank of the second fundamental form is equal to two everywhere. They are characterized by a second fundamental form
\[
h = \nu_1 \eta_1 \otimes \eta_1 + \nu_2 \eta_2 \otimes \eta_2, \quad \nu_1 \nu_2 \neq 0,
\]
where \( \eta_1 \) and \( \eta_2 \) are unit one-forms, \( \nu_1 \) and \( \nu_2 \) are functions on the hypersurface \( M^n \). The Euclidean leaves of the foliation are the integral submanifolds of the distribution \( \Delta_0 \), determined by the one-forms \( \eta_1 \) and \( \eta_2 \), i.e. \( \Delta_0(p) = \{ X \in T_pM^n \mid \eta_1(X) = 0, \eta_2(X) = 0 \} \), \( p \in M^n \). A special class of foliated semi-symmetric hypersurfaces is the class of ruled hypersurfaces.

A hypersurface \( M^n \) of type number two is minimal if \( \nu_1 + \nu_2 = 0 \).

The foliated semi-symmetric hypersurfaces in \( \mathbb{E}^{n+1} \) are characterized in [3] by the following

**Theorem 3.1.** A hypersurface \( M^n \) in Euclidean space \( \mathbb{E}^{n+1} \) is locally a foliated semi-symmetric hypersurface if and only if it is the envelope of a two-parameter family of hyperplanes in \( \mathbb{E}^{n+1} \).

Using the characterization of a foliated semi-symmetric hypersurface as the envelope of a two-parameter family of hyperplanes, each such hypersurface is determined by a pair of a unit vector-valued function \( l(u, v) \) and a scalar function \( r(u, v) \), defined in a domain \( D \subset \mathbb{R}^2 \).

Since the vector fields \( l_u \) and \( l_v \) are linearly independent, then the vector-valued function \( l(u, v) \) determines a two-dimensional surface \( M^2 : l = l(u, v) \), \( (u, v) \in D \) in \( \mathbb{E}^{n+1} \). Without loss of generality it can be assumed that the surface \( M^2 \) is parameterized locally by isothermal
parameters, i.e. \( E = G, \ F = 0 \). Then the generated foliated semi-symmetric hypersurface \( M^n \) is given in [4] by

\[
X(u, v, w^\alpha) = rl + \frac{r_u}{E} l_u + \frac{r_v}{E} l_v + w^\alpha b_\alpha, \quad \alpha = 1, \ldots, n - 2,
\]

where \((u, v) \in D, \ w^\alpha \in \mathbb{R}, \ \alpha = 1, \ldots, n - 2, \) and \( b_1(u, v), \ldots, b_{n-2}(u, v), \) \((u, v) \in D \) are \( n - 2 \) mutually orthogonal unit vectors, orthogonal to \( \text{span}\{l, l_u, l_v\} \).

The minimal foliated semi-symmetric hypersurfaces in \( \mathbb{E}^{n+1} \) are characterized analytically in [4] by the following

**Theorem 3.2.** Let \( M^n \) be a hypersurface in \( \mathbb{E}^{n+1} \) which is the envelope of a two-parameter family of hyperplanes, determined by a unit vector-valued function \( l(u, v) \), represented by isothermal parameters, and a scalar function \( r(u, v) \). Then \( M^n \) is minimal if and only if \( l(u, v) \) and \( r(u, v) \) satisfy the equalities

\[
\Delta l(u, v) + 2E(u, v) l(u, v) = 0;
\]

\[
\Delta r(u, v) + 2E(u, v) r(u, v) = 0.
\]

Hence, the minimal foliated semi-symmetric hypersurfaces in \( \mathbb{R}^4 \) are generated by the solutions of system (2.3), i.e. by the minimal surfaces in \( S^3 \).

Now we shall construct examples of minimal foliated semi-symmetric hypersurfaces in \( \mathbb{R}^4 \), which are generated by some minimal surfaces in \( S^3 \).

The simplest example of a minimal surface in \( S^3 \) is the sphere \( S^2 = S^3 \cap \mathbb{R}^3 \). We assume that \( \mathbb{R}^3 \) is the subspace of \( \mathbb{R}^4 \) orthogonal to \( e_4 \), i.e. \( \mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\} \). An isothermal parametrization of \( S^2 \) is given by

\[
S^2 : l(u, v) = \frac{1}{\cosh u} (\cos v; \sin v; \sinh u; 0).
\]

A direct computation shows that \( E = \langle l_u, l_u \rangle = \langle l_v, l_v \rangle = \frac{1}{\cosh^2 u}, \ F = \langle l_u, l_v \rangle = 0 \) and obviously \( l(u, v) \) satisfies the equality

\[
\Delta l(u, v) + \frac{2}{\cosh^2 u} l(u, v) = 0.
\]

The normal vector field \( n(u, v) \) of \( S^2 \) is \( n = e_4 = (0; 0; 0; 1) \). According to Theorem 3.2 the corresponding differential equation for the scalar function \( r(u, v) \) is

\[
\Delta r(u, v) + \frac{2}{\cosh^2 u} r(u, v) = 0.
\]

Every solution \( r(u, v) \) of (3.2) together with the sphere \( S^2 : l = l(u, v) \) generate a minimal foliated semi-symmetric hypersurface in \( \mathbb{R}^4 \) according to formula (3.1).

One solution of (3.2) is

\[
r(u, v) = \left( v + \frac{\pi}{2} \right) \tanh u.
\]

Now let us consider the minimal foliated semi-symmetric hypersurface \( M^3 \) generated by \( l(u, v) \) and this solution \( r(u, v) \). Calculating \( r_u, \ r_v, \ l_u, \ l_v \) and applying formula (3.1), we obtain

\[
M^3 : X(u, v, w) = \left( - \sinh u \ \sin v; \ \sinh u \ \cos v; \ v + \frac{\pi}{2}; \ w \right).
\]

After the following change of the parameters \( u^1 = \sinh u; \ t = v + \frac{\pi}{2} \), we obtain the hypersurface

\[
M^3 : X(u^1, t, w) = u^1 (\cos t \ e_1 + \sin t \ e_2) + t e_3 + w e_4.
\]
The hypersurface $M^3$, whose radius-vector $X = X(u^1, t, w)$ is determined by (3.3), is the so-called \textit{generalized helicoidal ruled hypersurface}, obtained by G. Aumann (see \cite{1}, Theorem 4)). It is called also a \textit{first type helicoid} in $\mathbb{R}^4$. It is a generalization of the \textit{right helicoid} in $\mathbb{R}^3$.

The next well-known example of a minimal surface in $S^3$ is the Clifford torus

$$M : l(u, v) = (\cos u \cos v; \cos u \sin v; \sin u \cos v; \sin u \sin v).$$

The normal vector field $n(u, v)$ of $M$ is

$$n(u, v) = (\sin u \sin v; -\sin u \cos v; -\cos u \sin v; \cos u \cos v).$$

Since $E = \langle l_u, l_u \rangle = \langle l_v, l_v \rangle = 1$, then according to Theorem \[3.2\] the corresponding equation for the scalar function $r(u, v)$ is

$$\Delta r(u, v) + 2r(u, v) = 0. \tag{3.4}$$

If we take the trivial solution $r(u, v) = 0$ of (3.4), we get the following minimal foliated semi-symmetric hypersurface

$$M^3 : X(u, v, w) = w n(u, v) = w(\sin u \sin v; -\sin u \cos v; -\cos u \sin v; \cos u \cos v).$$

After the following change of the parameters $u^1 = -w \sin u; u^2 = w \cos u; t = v + \frac{\pi}{2}$, we obtain the hypersurface

$$M^3 : X(u^1, u^2, t) = u^1(\cos t e_1 + \sin t e_2) + u^2(\cos t e_3 + \sin t e_4). \tag{3.5}$$

The hypersurface $M^3$, whose radius-vector $X = X(u^1, u^2, t)$ is determined by (3.5), is the minimal ruled hypersurface obtained by G. Aumann in \cite{1} Theorem 1. This hypersurface is known as a \textit{second type helicoid}. The first and the second type helicoids are the only minimal ruled hypersurfaces in $\mathbb{R}^4$ (see \cite{1}).

Thus we showed that the first type helicoid is generated by the sphere $S^2$ in $S^3$, while the second type helicoid is generated by the Clifford torus.

Our scheme of constructing minimal foliated semi-symmetric hypersurfaces in $\mathbb{R}^4$ can be applied to each minimal surface $M^2 : l = l(u, v)$ in $S^3$ and each solution of the corresponding differential equation for the scalar function $r(u, v)$.

We shall illustrate how this construction can be applied to the generalized torus of second type, given in Theorem \[2.1\] in the special case when $t = 0, s = \ln \alpha, \alpha > 0, \alpha \neq 1$. Since the calculations are too long and complicated, we give only a short sketch of the construction. In this case the solution $l(u, v)$ is defined by

$$l(u, v) = f(u) \left[ p(u) + \frac{\alpha}{\alpha^2 + 1} \cos \beta v \left( \sqrt{\alpha} e_1 + \frac{1}{\sqrt{\alpha}} e_4 \right) + \sqrt{\frac{\alpha}{\alpha^2 + 1}} \sin \beta v e_3 \right],$$

where $f(u) = \sqrt{\frac{\alpha^2 \cos^2 h^{-1}(u) + \sin^2 h^{-1}(u)}{\alpha}}, h^{-1}(u)$ is the inverse function of $h$ given on page \[10\], $p(u)$ is the solution of system (2.42), and $\beta = \sqrt{\frac{\alpha^2 + 1}{\alpha}}$. As a solution of the corresponding differential equation for $r(u, v)$ we choose $r(u, v) = f(u) \sqrt{\frac{\alpha}{\alpha^2 + 1}} \sin \beta v$. We calculate $l_u(u, v), l_v(u, v)$, and using (2.33) and (2.40) we find the vector-valued function
\[ n(u, v) = \frac{1 - \alpha^2}{\alpha(\alpha^2 + 1)}(e_1 - \alpha e_4) + \frac{\alpha}{(\alpha^2 + 1)f(u)} \cos \beta v \left( \sqrt{\alpha e_1 + \frac{1}{\sqrt{\alpha}}}e_4 \right) \]

\[ + \sqrt{\frac{\alpha}{\alpha^2 + 1} f(u)} \sin \beta v \frac{1}{e_3} - \frac{p(u)}{f(u)} - \frac{1 - \alpha^2}{\alpha} \int_0^u \frac{\sin h^{-1}(s)}{f^2(s)} p(s) ds. \]

Applying formula (3.1) we obtain the following minimal foliated semi-symmetric hypersurface \( M^3 \):

\[ X(u, v, w) = \frac{1 - \alpha^2}{2\sqrt{\alpha(\alpha^2 + 1)}} \frac{\sin 2h^{-1}(u)}{f(u)} \sin \beta v p'(u) \]

\[ + \sqrt{\frac{\alpha}{\alpha^2 + 1} f(u)} \left( \frac{\alpha^2 \sin^2 h^{-1}(u) + \sin^2 h^{-1}(u)}{\alpha^2 f^2(u)} \right) \sin \beta v p(u) \]

\[ - \left( \sqrt{\frac{\alpha}{\alpha^2 + 1}} \right)^3 \frac{1}{2f^2(u)} \sin 2\beta v \left( \sqrt{\alpha e_1 + \frac{1}{\sqrt{\alpha}}}e_4 \right) \]

\[ + \left( 1 - \frac{\alpha}{(\alpha^2 + 1)f^2(u)} \sin^2 \beta v \right) e_3 + w n(u, v). \]

Unfortunately, in this case we cannot write by means of elementary functions the hypersurface \( M^3 \) as in the previous examples, because we cannot find an explicit solution \( p(u) \) of linear system (2.42).

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MINIMAL SURFACES IN $S^3$ FOLIATED BY CIRCLES

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