ERDŐS–KO–RADO AND HILTON–MILNER THEOREMS FOR TWO-FORMS

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ABSTRACT. In this short note we show that both generalizations of celebrated Erdős–Ko–Rado theorem and Hilton–Milner theorem to the setting of exterior algebra in the simplest non-trivial case of two-forms follow from the folklore puzzle about possible arrangements of an intersecting family of lines.

1. Introduction

We recall that a family of sets is called intersecting if any two sets of the family have a non-empty intersection. We assume that $n$ is strictly greater than 4 throughout the note. The celebrated Erdős–Ko–Rado theorem [EKR61] states

Suppose that $r \leq n/2$. If $F$ is an intersecting family of $r$-element subsets of $\{1, \ldots, n\}$, then $|F| \leq \binom{n-1}{r-1}$. If more strongly $r < n/2$, then the equality $|F| = \binom{n-1}{r-1}$ holds only if all the sets in $F$ share a common element.

Hilton and Milner [HM67] obtained the following stability extension of this result:

Suppose that $r < n/2$. If $F$ is an intersecting family of $r$-element subsets of $\{1, \ldots, n\}$ such that there is no common element for all sets of $F$, then $|F| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.

Except for the case $r = 3$, there is a unique up to renaming extremal intersecting family in the Hilton–Milner result (see [FF86]).

Recently, the Erdős–Ko–Rado theorem was extended to the exterior algebra (see [SW21, Theorem 2.3] and [Woo20, Theorem 1.4]), where instead of intersecting families of sets, the authors used self-annihilating subspaces of the space of $r$-forms over $\mathbb{R}^n$ or $\mathbb{C}^n$. We will say that a subspace $W$ of $\Lambda^r(\mathbb{R}^n)$ is self-annihilating if $w_1 \wedge w_2 = 0$ for any two $r$-forms $w_1$ and $w_2$ of $W$. The result may be formulated as follows.

Suppose that $r < n/2$. If $W$ is a self-annihilating subspace of $\Lambda^r(\mathbb{R}^n)$, then $\dim W \leq \binom{n-1}{r-1}$.

This result implies the inequality of the Erdős–Ko–Rado theorem (consider the linear hull of forms $e_{i_1} \wedge \cdots \wedge e_{i_r}$ for $\{i_1, \ldots, i_r\} \in F$). However, neither the characterization of extremal configurations, nor an extension of the Hilton–Milner theorem were obtained in the setting of exterior algebra. The reasonable conjecture is that $\dim W = \binom{n-1}{r-1}$ if and only if all $r$-forms of $W$ are of the form $a \wedge v$ for some fixed $a \in \mathbb{R}^n$.

In this note we show that in the simplest non-trivial case of $r = 2$ both the characterization of extremal configurations in the Erdős–Ko–Rado theorem and the extension of the Hilton–Milner theorem follows from the folklore fact, which we state without proof:

Folklore lemma. Let $L$ be a set of lines in $\mathbb{R}P^{n-1}$ such that any two of them intersect. Then either all lines pass through one point, or all lines lie in a two-dimensional subspace.

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Our small contribution is the following theorem.

**Theorem 1.** Suppose $n \geq 5$, and $W$ is a self-annihilating subspace of $\Lambda^2(\mathbb{R}^n)$. Then

1. $\dim W \leq n - 1$, and the equality holds if and only if all 2-forms of $W$ are of the form $a \wedge v$ for some fixed $a \in \mathbb{R}^n$.

2. if there is no $a \in \mathbb{R}^n$ such that any 2-form of $W$ is of the form $a \wedge v$, then $\dim W \leq 3$.

More strongly, $\dim W = 3$ in this case if and only if $W$ is the linear hull of forms $x_1 \wedge x_2, x_2 \wedge x_3, x_3 \wedge x_1$ for some linearly independent $x_1, x_2, x_3 \in \mathbb{R}^n$.

**Proof.** Any two-form can be written in the standard form $e_1 \wedge e_2 + \cdots + e_{2k+1} \wedge e_{2k+2}$ in some basis [DS08, Theorem 1.1]. Thus, if a two-form $w$ satisfies $w \wedge w = 0$, then it’s decomposable, that is, $w = v_1 \wedge v_2$ for some $v_1, v_2 \in \mathbb{R}^n$. Consequently, all elements of $W$ are decomposable. That is, they correspond to two-dimensional subspaces of $\mathbb{R}^n$ or, equivalently, to lines in $\mathbb{R}P^{n-1}$.

By Folklore lemma, there are two cases:

1. All the lines pass through one point. Then there are at most $n - 1$ linearly independent of them, which easily yields (1).

2. All the lines belong to some two-dimensional subspace. Then there are at most 3 of them that can be linearly independent, and we have (2).

\qed

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