General Calculation Method of Normal Stress on Cross Section of Combined Tension (Compression) Bending Deformation

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Abstract: Tension (compression) bending combined deformation is a commonly used combined deformation form in engineering. The current material mechanics textbooks use the superposition method to calculate the stress on the cross section of the combined deformed member, whose characteristics are simple and easy to grasp. However, the premise of superposition principle is that the internal force, stress, strain and displacement have a linear relationship with external force. If the above linear relationship cannot be guaranteed, the superposition principle cannot be applied. In this case, the stress-strain relationship must be used for calculation. Based on the assumption of plane and unidirectional force, in this paper, rods with arbitrary cross-sectional shape under arbitrary load were taken as the research object, and the general calculation methods of the stress on the cross-section during the combined deformation of tension and bending were studied. This paper aimed at expanding students' thinking mode and improving students' learning ability in the process of material mechanics teaching.

1. Introduction
In the teaching of material mechanics, the combination of two or more basic deformations is called combined deformation. The current material mechanics textbook[1-4] uses the superposition method to calculate the stress on the cross section of the combined deformation member, that is, the external force is simplified or decomposed according to the external force characteristics of the basic deformation, and then the internal force and stress caused by each basic deformation are calculated; and finally, the stress results under each basic deformation are superimposed to obtain the stress calculation under the combined deformation[1]. In the teaching process, to facilitate the application of the superposition method, common combination deformations are divided into bending and bending combinations (oblique bending), tension (compression) bending combination, bending and torsion combination and tension (compression) bending and torsion combination, etc. Among them, the combined deformation of tension (compression) and bending is common in engineering. For example, the frame of the small press shown in Figure 1(a) produces both tensile and bending deformations, and the beam of the small crane shown in Figure 1(b) produces both compression and bending deformations.
As long as students have mastered the knowledge of basic deformation, it is relatively simple to use the superposition method to calculate the combined deformation stress. However, the premise of superposition principle is that the internal force, stress, strain and displacement have a linear relationship with external force. When the above linear relationship cannot be guaranteed, the superposition principle cannot be applied [1]. In this case, the relationship between the internal force, stress and deformation and the external force can only be obtained by means of the stress-strain relationship. To improve students’ learning ability and broaden their thinking mode, it is necessary to introduce to students the stress calculation and analysis methods under the most general stress conditions during the teaching process. This paper takes the combined deformation of tension (compression) and bending as an example, based on the assumption of plane and unidirectional force, derives the calculation of normal stress at any point on the cross section of any shape in the most general case, and discusses the calculation results. After completing the teaching content of material mechanics at the same time, the analysis method can be used as an extended content to explain, and make some necessary supplements to the results of material mechanics.

2. Calculation of internal force
Figure 2 shows a straight rod with an arbitrary shape in cross-section. The axis of the rod is the x-axis, and any two mutually perpendicular centroid axes in the cross-section are the y-axis and the z-axis, which are without loss of generality. This paper assumes that the members bear the axial uniform load \( q_x(x) \), the transverse uniform load \( q_y(x) \) and \( q_z(x) \) simultaneously, and it is stipulated that the load concentration consistent with the coordinate axis direction is positive, and the rod will undergo a combination of tension and bending deformation [5].

Figure 2 Combination of stretch and bending deformation rods
To obtain the internal force on any cross section of the member, using the section method, a micro segment is cut from the bar in Figure 2 with two cross-sections 1-1 and 2-2 separated by $dx$, and its force projection in the $xy$ plane and the $xz$ plane is shown in Figure 3. The balance equation of the micro-section can be listed:

$$\begin{aligned}
F_N + dF_N + q_x(x)dx - F_N &= 0 \\
F_{Sy} + dF_{Sy} + q_y(x)dx - F_{Sy} &= 0 \\
F_{Sz} + dF_{Sz} + q_z(x)dx - F_{Sz} &= 0 \\
M_x + dM_x + \frac{1}{2}q_y(x)dx \cdot dx - F_{Sy}dx - M_x &= 0 \\
M_y + dM_y + \frac{1}{2}q_z(x)dx \cdot dx - F_{Sz}dx - M_y &= 0
\end{aligned}$$

(1)

Where, $F_N$ is the axial force, $F_{Sy}$ and $F_{Sz}$ are the shear forces along the $y$-axis and $z$-axis respectively, and $M_y$ and $M_z$ are the bending moments on the $y$-axis and $z$-axis respectively, simplifying the above equations and omiting high-order infinite small quantities, obtaining

$$\begin{aligned}
\frac{dF_N}{dx} &= -q_x(x) \\
\frac{dF_{Sy}}{dx} &= -q_y(x) \\
\frac{dF_{Sz}}{dx} &= -q_z(x) \\
\frac{dM_x}{dx} &= F_{Sy} \\
\frac{dM_y}{dx} &= F_{Sz}
\end{aligned}$$

(2)

It can be seen from equation (2) that, no matter how complicated the external force it receives, the internal forces on the cross-section of the combined deformation remain independent of each other. Therefore, when calculating the internal force of the combined deformation, the external force can be decomposed into the component forces along the $x$-axis, $y$-axis, and $z$-axis, and then calculated for each internal force.

3. Calculation of deformation
Taking the micro-sections as shown in Figure 4, $u$, $v$, and $w$ are used to represent the displacements of the cross-section centroid along the $x$, $y$, and $z$ axes, respectively. Under the action of axial force, shear force and bending moment, the cross section, in addition to translation $u$, $v$ and $w$ along the $x$, $y$ and $z$ axes along the centroid, there also are rotations about the $y$ and $z$ axes. Therefore, the displacement $u_1$ of any point $M$ along the $x$-direction on the cross section is not only related to the displacement $u$ of the cross-section centroid along the $x$-axis, but also related to the rotation of the cross-section around the $y$ and $z$ axes. In the case of small deformation, the angles of rotation of the section around the $y$ and $z$ axes are $\theta_3$ and $\theta_5$, respectively (as shown in Figure 4). Suppose the coordinates of any point $M$ on the cross section are $y$ and $z$, the displacement of point $M$ along the $x$-axis caused by the translation of the section along the $x$ axis is $u$, and the displacements of the $M$ point along the $x$-axis caused by the corners of the cross section $\theta_3$ and $\theta_5$ are $-rac{dy}{dx}$ and $-rac{dz}{dx}$, respectively. Therefore, the displacement of point $M$ along the $x$-axis is:

$$u_1 = u - y \frac{dy}{dx} - z \frac{dz}{dx} \quad (3)$$

Figure 5 Deformation of any longitudinal line segment

In Figure 5, taking a longitudinal fiber $MN$ with coordinates $(y, z)$, if the displacement of point $M$ in the $x$ direction is $u_1$, the displacement of point $N$ should be $u_1 + \frac{\partial u}{\partial x} dx$. Based on this, the strain of the longitudinal fiber $MN$ can be obtained as:

$$\varepsilon_x = \frac{u_1 - \frac{\partial u}{\partial x} dx - u_1}{dx} = \frac{\partial u_1}{\partial x} \quad (4)$$

Substituting equation (3) into equation (4), the normal strain at any point under the combined deformation of tension (compression) and bending can be obtained:

$$\varepsilon_x = \frac{du}{dx} - y \frac{d^2 v}{dx^2} - z \frac{d^2 w}{dx^2} \quad (5)$$

The first term on the right side of the above equation represents the normal strain caused by axial tension (compression), and the second and third terms represent the normal strain caused by bending in the $xy$ and $xz$ planes, respectively.

From equation (5), the linear strain at any point of the combined deformation of tension and bending is not only related to the axial deformation, but also related to the degree of bending (curvature) of the rod in each direction. From equation (5), it can be seen that the linear strain of the cross section is liner distribution when the combination of tension and bending is deformed.

4. Calculation of stress

Assuming that each longitudinal fiber is in a unidirectional force state when the rod undergoes combined deformation, Hooke's law can be used to obtain the normal stress at any $M (y, z)$ point of the cross section during combined deformation:

$$\sigma_x = E \varepsilon_x = E \left( \frac{du}{dx} - y \frac{d^2 v}{dx^2} - z \frac{d^2 w}{dx^2} \right) \quad (6)$$

The internal force corresponding to the normal stress $\sigma_x$ on the cross section of the beam is:
Substituting equation (6) into equation (7), it can be obtained that:

\[
\begin{align*}
F_{Nx} &= \int_A \sigma_x \, dA \\
M_x &= \int_A y \sigma_x \, dA \\
M_y &= \int_A z \sigma_x \, dA
\end{align*}
\]  

(7)

In the equation, \( S_z = \int_A y \, dA \) and \( S_y = \int_A z \, dA \) are the static moments of the \( z \)-axis and \( y \)-axis of the cross section, \( A = \int_A dA \) is the area of the cross section, \( I_y = \int_A y^2 \, dA \) and \( I_z = \int_A z^2 \, dA \) is the moment of inertia across the \( z \)-axis and \( y \)-axis, and \( I_{yz} = \int_A yz \, dA \) is the product of inertia across the \( y \)-axis and \( z \)-axis. Since the \( y \)-axis and \( z \)-axis are the centroid axes of the cross section, so \( S_z = \int_A y \, dA = 0 \) and \( S_y = \int_A z \, dA = 0 \), then the equation (8) can be simplified as:

\[
\begin{align*}
F_{Nx} &= E A \frac{du}{dx} \\
M_x &= -EI_y \frac{d^2 v}{dx^2} - EI_{yz} \frac{d^2 w}{dx^2} \\
M_y &= -EI_y \frac{d^2 v}{dx^2} - EI_{yz} \frac{d^2 w}{dx^2}
\end{align*}
\]  

(9)

It can be solved by equation (9):

\[
\begin{align*}
\frac{du}{dx} &= \frac{F_{Nx}}{EA} \\
\frac{d^2 v}{dx^2} &= - \frac{M_z - M_y I_{yz}}{E (I_y - I_{yz})} \\
\frac{d^2 w}{dx^2} &= - \frac{M_y - M_z I_{yz}}{E (I_y - I_{yz})}
\end{align*}
\]  

(10)

Substituting equation (10) into equation (6), the general calculation equation for normal stress under combined tension (compression) and bending deformation is obtained:

\[
\sigma_x = \frac{F_{Nx}}{A} + \frac{M_y I_y - M_z I_{yz}}{I_y - I_{yz}} y + \frac{M_z I_z - M_y I_{yz}}{I_z - I_{yz}} z
\]  

(11)

Equations (10) and (11) are general expressions for the calculation of bar deformation and stress under combined tension (compression) and bending deformation, which are suitable for combined tension (compression) and bending deformation under arbitrary cross-sections and load forms.

The first term of equation (10) and the first term on the right side of equation (11) represent the axial deformation and stress generated by the rod under the action of axial force \( F_{Nx} \), which are independent of the bending moments \( M_z \) and \( M_y \). Therefore, for the combined deformation of tension (compression) and bending. It can also calculate the deformation and stress when the tensile (compression) deformation and the bending deformation act separately, and then superimpose them. If the rod is only subjected to axial external force, the equations (10) and (11) are the same as the deformation and stress calculation equations during uniaxial tension (compression)\(^{[1-4]}\).

The cross-section of the rod used in engineering generally has at least one axis of symmetry. Supposing the \( y \)-axis is the axis of symmetry, and the \( z \)-axis is the main inertia of the centroid of the cross-section, that is, \( I_z = 0 \), the equations (10) and (11) can be simplified to:

\[
\begin{align*}
\frac{du}{dx} &= \frac{F_{Nx}}{EA} \\
\frac{d^2 v}{dx^2} &= - \frac{M_z}{EI_z} \\
\frac{d^2 w}{dx^2} &= - \frac{M_y}{EI_y}
\end{align*}
\]  

(12)
\[ \sigma_x = \frac{F_N}{A} + \frac{M_{Yx}}{I_z} + \frac{M_{Xz}}{I_y} \]  \tag{13}

In this case, the deformation caused by the combined deformation is only related to the internal force in this direction, and the stress at any point under the combined deformation is equal to the algebraic sum of the stress generated by each internal force at that point, that is, the stress superimposition\[^6\text{-}\text{7}].\]

5. Conclusion

Based on the plane assumption and unidirectional force assumption in material mechanics, this paper derives the general equation (Equation 11) for calculating the normal stress on the cross section of a combined tension and bending member. This equation is applicable to the calculation of any bending stress at any section. Based on this equation, the calculation of deformation and stress in unidirectional tension and compression and pure bending can also be discussed. Introducing this analysis method into the teaching of material mechanics, the deformation of bars can be explained from general to special, which is more convenient for students to understand the content and analysis method of the textbook. Besides, through the analysis process from general to special, students have a deeper understanding of the formula of stress calculation under simple deformation in the mechanics of materials to know what it is and why it is.

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