Structure Identifiability of an NDS With LFT Parametrized Subsystems

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Abstract—Requirements on subsystems have been made clear in this article for a linear time invariant networked dynamic system (NDS), under which subsystem interconnections can be estimated from external output measurements. In this NDS, subsystems may have distinctive dynamics, and subsystem interconnections are arbitrary. It is assumed that system matrices of each subsystem depend on its (pseudo) first principle parameters (FPP) through a linear fractional transformation. It has been proven that if in each subsystem, the transfer function matrix (TFM) from its internal inputs to its external outputs is of full normal column rank, while the TFM from its external inputs to its internal outputs is of full normal row rank, then the structure of the NDS is identifiable. Moreover, under some particular situations like there is no direct information transmission from an internal input to an internal output in each subsystem, a necessary and sufficient condition is established for NDS structure identifiability. A matrix valued polynomial rank-based equivalent condition is further derived, which depends affinely on subsystem (pseudo) FPPs and can be independently verified for each subsystem. From this condition, some necessary conditions are obtained for both subsystem dynamics and its (pseudo) FPPs, using the Kronecker canonical form of a matrix pencil.

Index Terms—First principle parameter (FPP), Kronecker canonical form (KCF), large scale system, linear fractional transformation (LFT), matrix pencil, networked dynamic system (NDS), structure identifiability.

I. INTRODUCTION

Networked dynamic systems (NDS) have been attracting research attention for a long time, which are some times also called large-scale systems, composite systems, especially in the 60s of last century [15], [20], [27]. With technology developments, especially those in communications and computers, the scale of an NDS becomes larger and larger. Moreover, some new issues also arise, such as attack prevention, random communication delay/failure, etc. On the other hand, some classic problems including revealing the structure of an NDS from measurements, computationally efficient conditions for NDS controllability/observability verifications, etc., still remain challenging [3], [4], [9], [13], [14], [16], [24]. Among these, an essential issue is NDS identification which is widely realized as the basis for developing effective methods in NDS analysis and synthesis [4], [19], [27].

Particularly, in order to monitor the behaviors of an NDS or to improve its performances, it is usually required to understand the dynamics of its subsystems, as well as their interactions. While in some applications both of them are known from the NDS working principles and/or constructions, there are also various situations in which both of them or one of them must be estimated from experiment data. For example, in an NDS with wireless communications, some subsystem interactions may fail to work due to unpredictable communication congestion; in an NDS constituted from several mobile robots, information exchange among these robots may vary with changing environments; and in a gene regulation network, a direct interaction is usually hard and/or too expensive to measure; etc. In these applications, it is invaluable to understand subsystem interactions from measured experiment data [4], [14], [19], [20], [27].

In NDS dynamics description, the adopted approaches can be briefly divided into two categories. One of them treats each measured variable as a node, while transfer functions among these variables as edges, which has been used in many researches on NDS analysis and synthesis. Examples include [4], [7], and [22]. The other approach treats each subsystem as a node, while interactions among subsystems as edges. This approach has also been widely adopted, for example, [4], [5], [14], [15], [20], [21], and [27]. In [4], characteristics of these two descriptions were investigated from a structural informativity aspect, comparing with the extensively adopted transfer function matrix (TFM) description and the state space model. An important observation there is that, while all these four descriptions can describe the same input/output behavior of a system, each one may characterize a different notion of system structure. In addition, no matter which of the aforementioned two approaches is adopted in NDS dynamics descriptions, further efforts are still required for developing efficient methods that estimate the associated model from experiment data. The latter has also been observed in [21], [22], and [27].

In particular, several recent studies make it clear that there is no guarantee that NDS subsystem interactions can always be estimated from experiment data. For example, it is shown in [4] that even if the TFM of an NDS can be perfectly estimated, there are still possibilities that its subsystem interactions cannot be identified. More specifically, the subsystem structure description is in general more structurally informative than a TFM description. To clarify situations under which an NDS structure can be identified, some eigenvector-based conditions are derived in [12] for an NDS with descriptor subsystems and diffusive subsystem.
coupling, so that variations of its subsystem interactions can be detected. Waarde et al. [21] studied topology identifiability when subsystems of an NDS are coupled through their outputs. It is proven that an NDS is topologically identifiable only when the constant kernel of a TFM, which is completely determined by subsystem dynamics, is equal to a zero vector. It has also been shown there that this condition becomes also sufficiently under some special situations.

In this article, we investigate requirements on the NDS model adopted in [26]–[28] such that its subsystem interactions can be identified from experiment data. In this NDS, each subsystem is permitted to have distinctive dynamics, and subsystem interactions are directed. In addition, the system matrices of each subsystem depend on its (pseudo) first principle parameters (FPP) through a linear fractional transformation (LFT). This NDS model includes those adopted in [4], [12], and [21] as special cases, and may be considered as the most general one among linear time invariant (LTI) NDS models. It is proven that the structure of this NDS model is identifiable, if two TFMs associated with each of its subsystems independently, are, respectively, of full normal column rank (FNCR) and full normal row rank (FNRR). Based on this result, it is further shown that this FNCR/FNRR condition is, respectively, equivalent to the FNCR of two matrix-valued polynomials (MVP) that depends affinely on the (pseudo) FPPs of each subsystem. Moreover, a necessary and sufficient condition is established for NDS structure identifiability, under the following two situations. The first is that in each subsystem, there is no direct information transmission from an internal input to an internal output. The second is that the TFM from its internal inputs to its external outputs can be expressed as the multiplication of an FNCR TFM and the TFM from its internal inputs to its internal outputs, while the TFM from its external inputs to its internal outputs can be expressed as the multiplication of that TFM and an FNRR TFM. This condition can be verified for each tuple of subsystems independently, and its computational complexity increases only quadratically with the number of subsystems in an NDS.

The rest of this article is organized as follows. At first, in Section II, problem descriptions are given, together with the NDS model adopted in this article and some preliminary results. NDS structure identifiability is studied in Section III, in which some necessary and sufficient conditions on the TFMs of a subsystem are derived. Section IV investigates relations between NDS structure identifiability and subsystem (pseudo) FPPs. A numerical example is given in Section V to illustrate applicability of these theoretical results. Finally, Section VI concludes this article.

The following notation and symbols are adopted. \( C \) and \( C^m \) stand, respectively, for the set of complex numbers and the \( m \) dimensional complex Euclidean space. \( \text{det}(\cdot) \) represents the determinant of a square matrix, \( \|\cdot\|_2 \) the (right) null space of a matrix, \(^{-1}\) the matrix whose columns form a base of the (right) null space of a matrix, while \( A \otimes B \) the Kronecker product of two matrices. \( \text{diag}\{X_{i=1}^{L_i}\} \) denotes a block diagonal matrix with its \( i \)th diagonal block being \( X_i \), while \( \text{col}\{X_{i=1}^{L_i}\} \) the vector/matrix stacked by \( X_{i=1}^{L_i} \) with its \( i \)th row block/vector/matrix being \( X_i \), and \( \text{vec}(X) \) the vector stacked by the columns of the matrix \( X \). \( L_m, 0_m, \) and \( 0_m \times n \) represent, respectively, the \( m \) dimensional identity matrix, the \( m \) dimensional zero column vector and the \( m \times n \) dimensional zero matrix. The subscript is usually omitted if it does not lead to confusions. The superscript \( T \) is used to denote the transpose of a matrix/vector. When a constant matrix has its rank equal to the number of its rows/columns, it is called full row/column rank. Moreover, a matrix valued function is called full normal row/column rank, if its maximum rank is equal to the number of its rows/columns.

II. PROBLEM DESCRIPTION AND SOME PRELIMINARIES

In a real world NDS, its subsystems may have distinctive dynamics. For an LTI NDS, a model is suggested in [26]–[28] to describe relations among subsystem inputs, outputs, and its (pseudo) FPPs. More specifically, for an NDS \( \Sigma \) consisting of \( N \) subsystems, the following model is utilized to describe the dynamics of its \( i \)th subsystem \( \Sigma_i \)

\[
\delta(x(t,i)) = [Ax_i(xv) Bx_i(xv)] [x(t,i)] \\
z(t,i) = [Ax_i(xv) Bx_i(xv)] [x(t,i)] \\
y(t,i) = [Cx_i(xv) Dx_i(xv)] [u(t,i)] \\
\]

(1)

Moreover, interactions among NDS subsystems are described by the following equation:

\[
v(t) = \Phi x(t) \]

(2)

in which \( z(t) \) and \( v(t) \) are assembly expressions, respectively, for the NDS internal output and input vectors. That is, \( z(t) = \text{col}\{z(t,i)\}_{i=1}^{N} \) and \( v(t) = \text{col}\{v(t,i)\}_{i=1}^{N} \). In addition, the system matrices \( Ax_i(xv) \), etc., of the subsystem \( \Sigma_i \) are assumed to depend on its (pseudo) FPPs through the following LFT:

\[
[\begin{bmatrix} Ax_i(xv) & Bx_i(xv) \\ Ax_i(xv) & Dx_i(xv) \end{bmatrix} = [\begin{bmatrix} A_{0}^{x} & A_{0}^{y} & A_{0}^{z} \\ A_{0}^{x} & A_{0}^{y} & A_{0}^{z} \end{bmatrix} [\begin{bmatrix} P^{x} & P^{y} & P^{z} \\ P^{x} & P^{y} & P^{z} \end{bmatrix} [\begin{bmatrix} C_{0}^{x} & C_{0}^{y} & C_{0}^{z} \\ C_{0}^{x} & C_{0}^{y} & C_{0}^{z} \end{bmatrix} \]
\]

(3)
sent to other subsystems and signals gotten from other subsystems. In addition, the matrix $\Phi$ describes influences among different NDS subsystems, and is called subsystem connection matrix (SCM). If each subsystem is regarded as a node and each nonzero element of its SCM as an edge, a digraph can be associated with an NDS, which is usually called its structure or topology [4]. More specifically, we have the following definition.

**Definition 1:** The structure of the NDS $\Sigma$ is a digraph, in which each node denotes a subsystem, while a directed and weighted edge represents a nonzero element of the SCM $\Phi$. In particular, for each $1 \leq i, j \leq N$, each $1 \leq \alpha \leq m_{x_i}$ and each $1 \leq \beta \leq m_{x_j}$, when the element of the SCM $\Phi$ is not equal to zero that is in its $(\sum_{p=0}^{i-1} m_{yp} + \alpha)th$ row and $(\sum_{q=0}^{j-1} m_{xq} + \beta)th$ column, then there is a directed edge from Node $j$ to Node $i$ with its weight equal to that nonzero element, indicating direct influences from the $j$th subsystem $\Sigma_j$ to the $i$th subsystem $\Sigma_i$.

The above model reflects the well known fact that in a real world plant, elements in its system matrices are usually not algebraically independent of each other, and some of them can even not be tuned in system designs. A more detailed discussion can be found in [26]–[28] on its application motivations, which can also be seen from the artificial example of the following Section V, in which the system matrices of each subsystem are determined completely by three pseudo FPPs.

To have a concise presentation, the dependence of a system matrix of the subsystem $\Sigma_i$ on its parameter matrix $P(i)$ is usually not explicitly expressed, except when this omission may cause some significant confusions.

Substituting (2) into (1), the aforementioned NDS model takes completely the same form as the generalized state space model suggested in [4], except that all the associated matrices are restricted to be block diagonal or the product of a block diagonal matrix and the SCM $\Phi$. This may mean that the model adopted in this article is more flexible and more structurally informative than that model of [4]. In addition, when the internal and external output vectors $z(t,i)$ and $y(t,i)$ are identically equal for each $1 \leq i \leq N$, and/or the SCM $\Phi$ is restricted to be symmetric, etc., this NDS model reduces to those adopted in [5], [12], and [21].

To clarify dependence of the NDS $\Sigma$, the external output vector of its $i$th subsystem $\Sigma_i$, etc., on the NDS SCM $\Phi$, they are sometimes also written as $\Sigma(\Phi)$, $y(t,i,\Phi)$, etc. In addition, the following assumptions are adopted throughout this article.

- **A.1)** Every NDS subsystem, that is, $\Sigma_i$ with $i \in \{1,2,\ldots,N\}$, is well posed. This is equivalent to that the matrix $\left|I - \Phi \text{diag}(A_{x_i}(i))\right|$ is invertible.

- **A.2)** The NDS $\Sigma$ itself is well posed. This is equivalent to that the matrix $I - \Phi \text{diag}(A_{x}(i))$ is invertible.

Recall that the well-posedness of a system means that its states respond solely to each pair of their initial values and external inputs, which implies that Assumptions A.1) and A.2) are necessary for a system to properly work [11], [20], [25], [27]. All these two assumptions should therefore, be met by a practical system, and are hence not quite restrictive.

**Definition 2:** The structure of the NDS of (1) and (2) is identifiable, if for an arbitrary initial state vector $\text{col}\{x(0,i)|i=1^{\text{N}}\}$ and any two distinct SCMs $\Phi_1$ and $\Phi_2$ satisfying Assumption A.2), there exists at least one external input time series $\text{col}\{u(t,i)|i=1^{\text{N}}\}$ such that the external output time series $\text{col}\{y(t,i,\Phi_1)|i=1^{\text{N}}\}$ of the NDS $\Sigma(\Phi_1)$ is different from the external output time series $\text{col}\{y(t,i,\Phi_2)|i=1^{\text{N}}\}$ of the NDS $\Sigma(\Phi_2)$. Otherwise, the structure of this NDS is called unidentifiable.

From this definition, it is clear that if the structure of an NDS is not identifiable, then its subsystem interactions cannot be determined through only experiments and the subsequent estimations, no matter what probing signals are used to stimulate the NDS, how long an experiment data length is, and what estimation algorithm is adopted. In other words, for this NDS, experiments only are not informative enough to distinguish its structure. This means that the structure identifiability defined above is a property held by an NDS.

The objectives of this article are to develop a condition for verifying NDS structure identifiability that is both attractive from a computation aspect and scalable with its subsystem number, as well as to derive requirements on each subsystem from which an NDS can be constructed whose structure can be estimated from experiment data.

Identifiability is an important issue in system identification, and has been attracting extensive attention for a long time in various fields [1], [22], [27]. While diverse definitions have been given and many verification methods have been developed, this terminology traditionally and basically means parameter identifiability, and is sometimes also called a priori/structural identifiability, etc. The former reflects the fact that identifiability is an essential prerequisite on a system model before estimation, while the latter means that identifiability is usually a generic property of a system model (rather than identifiability of the structure of a system). In general, identifiability verification is mathematically hard and still remains challenging, even for an LTI system [1], [21], [22].

When a subsystem interaction is regarded as a system parameter, the structure identifiability of Definition 2 becomes the traditional parameter identifiability. More precisely, it has been shown through straightforward algebraic manipulations in [26] that the system matrices of the whole NDS $\Sigma$ can be expressed as an LFT of the SCM $\Phi$. Limited to the author’s knowledge, however, there are still not any published works dealing with parameter identifiability of this kind of systems. The most closely related work appears to be [18], in which system matrices of an LTI plant are assumed to depend on its parameters through some polynomials. Noting that a matrix polynomial can always be converted into an LFT [25], it seems safe to declare that the system model adopted there is a special case of that used in this article. On the other hand, while some conditions have been derived in [18] when system’s impulse responses or noise-free input/output data are available, their computation costs increase double-exponentially with the number of system states, which may greatly restrict their applicability to NDS analysis and synthesis, noting that scalability is essential for a criterion to work properly with an NDS, especially when it has a large number of subsystems.

As argued in [4], [14], [15], [19]–[21], [27], the structure information is important in NDS analysis and synthesis. To reflect this aspect and be consistent with [4], [15], [24], and [27], the terminology “structure identifiability” is adopted in this article, which is also called topology identifiability in [21] and network identifiability in [19] and [22]. In addition, note that in the analysis and synthesis of a large-scale NDS, initial states of the NDS are usually not completely known, and the
data length of an experiment is often not very large. These two factors have also been taken into account in the above definition.

To develop a computationally feasible condition for verifying NDS structure identifiability, the following results are introduced [6], [8], [28].

**Lemma 1:** Divide a matrix $A$ as $A = [A_1^T, A_2^T, A_3^T]^T$, and assume that $A_1$ is not of full column rank (FCR). Then the matrix $A$ is of FCR, if and only if the matrix $A_2A_3^T$ is of FCR.

**Lemma 2:** Assume that $A_i[i = 1, i = 1, j = 1]$ and $B_j[j = 1, j = 1, i = 1]$ are some matrices having compatible dimensions, and the matrix $[A_1^{[1]} A_2^{[2]} ... A_n^{[m]}]$ is of FCR. Then the matrix

$$\begin{bmatrix}
\text{diag} \{A_1^{[1]}, A_2^{[1]}, A_3^{[1]}) \} & \ldots & \text{diag} \{A_1^{[m]}, A_2^{[m]}, A_3^{[m]}) \}
\end{bmatrix}
$$

is of FCR, if and only if the following matrix has this property:

$$\begin{bmatrix}
\text{diag} \{A_1^{[1]}, A_3^{[1]}) \} & \ldots & \text{diag} \{A_1^{[m]}, A_3^{[m]}) \}
\end{bmatrix}
$$

The following definitions and results are well known on matrix pencils, which can be found in many published works including [2] and [10].

**Definition 3:** Let $G$ and $H$ be two arbitrary $m \times n$ dimensional real matrices. A MVP $\Psi(\lambda) = \lambda G + H$ is called a matrix pencil.

1) This matrix pencil is called regular, whenever $m = n$ and there exists a $\lambda \in \mathbb{C}$, such that det$(\Psi(\lambda)) \neq 0$.

2) If both the matrices $G$ and $H$ are invertible, then this matrix pencil is called strictly regular.

3) If there exist two nonsingular real matrices $U$ and $V$, such that $\Psi(\lambda) = U \overline{\Psi}(\lambda) V$ is satisfied at each $\lambda \in \mathbb{C}$ by two matrix pencils $\Psi(\lambda)$ and $\overline{\Psi}(\lambda)$, then these two matrix pencils are said to be strictly equivalent.

The following symbols are adopted throughout this article. For an arbitrary nonnegative integer $m$, the symbol $H_m(\lambda)$ stands for an $m \times m$ dimensional strictly regular matrix pencil, while the symbols $K_m(\lambda), N_m(\lambda), L_m(\lambda)$, and $J_m(\lambda)$, respectively, for matrix pencils having the following definitions:

$$K_m(\lambda) = \lambda I_m + \begin{bmatrix}
0 & I_{m-1}
\end{bmatrix}, \quad N_m(\lambda) = \lambda \begin{bmatrix}
0 & I_m
\end{bmatrix} + I_m$$

$$L_m(\lambda) = \begin{bmatrix}
K_m(\lambda)
0
1
\end{bmatrix}, \quad J_m(\lambda) = \begin{bmatrix}
K_m^T(\lambda)
0
1
\end{bmatrix}.$$  

**Lemma 3:** For any matrix pencil $\Psi(\lambda)$, there are some unique nonnegative integers $\xi_H, \zeta_K, \xi_L, \xi_N, \xi_J, \xi_L(j), \xi_N(j), \xi_J(j)$, and $\xi_\Lambda(\lambda)$, as well as some unique positive integers $\xi_H(\lambda), \xi_K(\lambda), \xi_L(\lambda), \xi_N(\lambda), \xi_J(\lambda)$, and $\xi_\Lambda(\lambda)$, such that $\Psi(\lambda)$ is strictly equivalent to the block diagonal matrix pencil $\overline{\Psi}(\lambda)$ defined as

$$\overline{\Psi}(\lambda) = \begin{bmatrix}
H_{\xi_H}(\lambda), & K_{\xi_K}(\lambda), & L_{\xi_L}(\lambda), & N_{\xi_N}(\lambda), & J_{\xi_J}(\lambda)
\end{bmatrix}.$$  

The following results are obtained in [28], which explicitly characterizes the null spaces of the matrix pencils $H(\lambda), K(\lambda), N(\lambda), L(\lambda),$ and $J(\lambda)$. This characterization is helpful in clarifying requirements on a subsystem, with which an NDS can be constructed with its structure identifiable.

**Lemma 4:** Let $m$ be an arbitrary nonnegative integer. Then the matrix pencils defined, respectively, in (4) and (5) have the following null spaces.

1) $H_m(\lambda)$ is not of full rank (FR) only at $m$ isolated complex values of the variable $\lambda$. All these values are not equal to zero.

2) $N_m(\lambda)$ is always of FR.

3) $J_m(\lambda)$ is always of FCR.

4) $K_m(\lambda)$ is singular only at $\lambda = 0$, and $K_m^*(0) = \text{col}\{1, 0, m-1\}$.

5) $L_m(\lambda)$ is not of FCR at any complex $\lambda$, and $L_m^+(\lambda) = \text{col}\{1, -1\}.$

**III. NDS Structure Identifiability**

To establish conditions on a subsystem such that the structure of an NDS constituted from it is identifiable, for each subsystem $\Sigma_i$ of the NDS $\Sigma$, in which $i = 1, 2, \ldots, N$, define TFMs $G_{zu}(\lambda, i), G_{zv}(\lambda, i), G_{yu}(\lambda, i)$, and $G_{yv}(\lambda, i)$, respectively, as

$$\begin{bmatrix}
G_{yu}(\lambda, i) & G_{yv}(\lambda, i)
G_{zu}(\lambda, i) & G_{zv}(\lambda, i)
\end{bmatrix} = \begin{bmatrix}
D_{u}(i) & C_{v}(i)
D_{z}(i) & B_{v}(i)
\end{bmatrix} + \begin{bmatrix}
C_{x}(i)
A_{x}(i)
\end{bmatrix}$$

$$\times \begin{bmatrix}
[\lambda I_m - A_{xx}(i)]^{-1} B_{x}(i)
A_{x}(i)
\end{bmatrix}$$

in which $\lambda$ stands for the Laplace transformation variable $s$ when the NDS $\Sigma$ is of continuous time, and for the $Z$ transformation variable $z$ when the NDS $\Sigma$ is of discrete time. Moreover, define block diagonal TFMs $G_{s, \#}(\lambda)$ with $* = z$ or $y$ and $\# = u$ or $v$ as $G_{s, \#}(\lambda) = \text{diag}(G_{s, \#}(\lambda, i))_{i=1}^N$.

Note that the well-posedness of the NDS $\Sigma$ is equivalent to that the matrix $I_m - A_{zx} \Phi$ is invertible, in which $A_{zx} = \text{diag}(A_{zx}(i))_{i=1}^N$ and $m_0 = \sum_{i=1}^N m_{z_0}$ [26], [28]. On the other hand, define matrices $A_{xx}, A_{zx},$ and $A_{xx}$, respectively, as $A_{xx} = \text{diag}(A_{xx}(i))_{i=1}^N$, $A_{zx} = \text{diag}(A_{zx}(i))_{i=1}^N$, and $A_{xx} = \text{diag}(A_{xx}(i))_{i=1}^N$. Moreover, denote $\sum_{i=1}^N m_{z_0}$ by $m_0$. Then when the NDS $\Sigma$ satisfies Assumption A.2, from the block diagonal structure of the TFM $G_{s, xx}(\lambda)$, we have that

$$I_m - A_{zx} \Phi$$

$$= \left\{I_m - A_{zx} \Phi \right\}^{-1} \left(I_m - A_{zx} \Phi \right)^{-1} \times$$

$$A_{xx} \left[I_m - A_{xx} A_{xx}^{-1} A_{zx} \Phi \right].$$

(7)
Hence, from the well-known determinant equality \( \det(I - AB) = \det(I - BA) \) [6], [8], we have that
\[
\det\{ I_{m_x} - G_{zv}(\lambda)\Phi \}
\]
\[
= \det(I_{m_x} - A_{zv}) \times \det(I_{m_x} - A_{zv})^{-1} A_{zv} \quad \lambda I_{m_x} - A_{xx} \quad A_{xx}^{-1} A_{xv} \Phi \}
\]
\[
= \det(I_{m_x} - A_{zv}) \times \det(I_{m_x} - A_{zv}) \times \det(\lambda I_{m_x} - A_{xx}) \times \det(I_{m_x} - A_{zv}) \Phi \}
\]
\[
= \det^{-1}(\lambda I_{m_x} - A_{xx}) \times \det(I_{m_x} - A_{zv}) \Phi \}
\]
\[
\times \det(\lambda I_{m_x} - A_{xx}) - [A_{xx} + A_{xv} \Phi (I_{m_x} - A_{zv})^{-1} A_{zx}] \}
\]
(8)

Recall that all the matrices and TFMs in the above equation are of a finite dimension. This means that when the NDS \( \Sigma \) is well posed, \( \det(I_{m_x} - G_{zv}(\lambda)\Phi) \) is not identically equal to zero. Hence, its inverse is well defined. On the basis of these observations, it can be straightforwardly shown that the TFM from the external input \( \text{col}\{u(t,i)\in\mathbb{N}\} \) of the NDS \( \Sigma(\Phi) \) to its external output \( \text{col}\{y(t,i)\in\mathbb{N}\} \), denote it by \( H(\lambda, \Phi) \), can be expressed as
\[
H(\lambda, \Phi) = G_{yu}(\lambda) + G_{yv}(\lambda) \Phi \quad I_{m_x} - G_{zv}(\lambda)\Phi \quad -G_{zu}(\lambda).
\]
(9)

Moreover, the following results can be established for the structure identifiability of the NDS \( \Sigma \), while their proof is deferred to the Appendix.

**Theorem 1:** Assume that the NDS \( \Sigma \) is well posed. Then its structure is identifiable, if and only if for each SCM pair \( \Phi_1 \) and \( \Phi_2 \) satisfying \( \Phi_1 \neq \Phi_2 \), \( H(\lambda, \Phi_2) \neq H(\lambda, \Phi_1) \) at every \( \lambda \in C \).

This theorem makes it clear that the structure identifiability studied in this article is equivalent to that of [4], [5], and [21], in which the structure of an NDS is called identifiable if any two different SCMs lead to different TFMs of the whole system, or lead to distinguishable time domain responses with a particular probing signal under distinctive initial conditions.

The necessity and sufficiency of the above condition are to some extent clear from an application viewpoint. Particularly, when there are two sets of subsystem interactions that lead to the same external outputs for each external stimulus, it is not out of imagination that these two subsystem interaction sets result in the same TFM of the whole NDS from its external inputs to its external outputs. On the other hand, if two distinctive SCMs lead to the same NDS TFM, the external outputs of the corresponding NDSs are usually hard to be distinguished under the same external stimulations.

On the basis of these results, as well as properties of an LFT, a computationally feasible condition is derived for the structure identifiability of the NDS \( \Sigma \).

**Theorem 2:** Assume that the NDS \( \Sigma \) satisfies Assumptions A.1) and A.2). If for each \( i = 1, 2, \ldots, N \), the TFM \( G_{yv}(\lambda, i) \) is of FNCR, while the TFM \( G_{zu}(\lambda, i) \) is of FNRR, then the structure of this NDS is identifiable.

The proof of the above theorem is given in the Appendix.

From this theorem, it is clear that structure identifiability of an NDS can be completely determined by the dynamics of its individual subsystems. This is quite attractive in NDS constructions including subsystem dynamics selection, external input/output position determination, etc., as well as experiment designs for NDS identification.

Theorem 2 also makes it clear that through adding external outputs and/or external inputs, it is possible to change the structure of an NDS from being unidentifiable to being identifiable. Noting that adding an external input/output usually increases information in experiment data about an NDS structure, this observation is consistent well with intuitions in actual applications.

Note that the conditions of Theorem 2 depend only on the TFMs \( G_{yv}(\lambda, i) \) and \( G_{zu}(\lambda, i) \). It is straightforward to extend them to an NDS whose subsystem dynamics are described in a descriptor form. Compared with the results of [12], the conditions are only sufficient. On the other hand, there are less restrictions on subsystem dimensions and interactions, and the conditions can be verified with each subsystem independently. More specifically, the condition of [12] is both necessary and sufficient. Each subsystem, however, is required to be square and subsystems are asked to be connected through their external outputs by a positive semidefinite SCM. In addition, the condition there depends on the SCM \( \Phi \) which may in general not be very competitive in large-scale NDS analysis and synthesis.

When subsystem parameters are known, the associated subsystem matrices are completely determined and therefore, the corresponding TFMs \( G_{yv}(\lambda, i) \in C_{m_{yi} \times m_{zi}} \) and \( G_{zu}(\lambda, i) \in C_{m_{yi} \times m_{ui}} \). Under such a condition, the situation of Theorem 2 can be simply verified using their Smith–McMillan forms, etc.

More precisely, let \( r_{yi}^{[i]} \) and \( r_{zi}^{[i]} \) with \( i = 1, 2, \ldots, N \), stand, respectively, for the maximum ranks of the TFMs \( G_{yv}(\lambda, i) \) and \( G_{zu}(\lambda, i) \) when \( \lambda \) varies over the set \( C \). Then it is obvious from the dimensions of these TFMs that \( 0 \leq r_{yi}^{[i]} \leq \max(m_{yi}, m_{yi}) \) and \( 0 \leq r_{zi}^{[i]} \leq \max(m_{zi}, m_{ui}) \). Moreover, their Smith–McMillan forms can be, respectively, written as
\[
G_{yv}(\lambda, i) = U_{yv}(\lambda, i) \begin{bmatrix} \text{diag}(\alpha_{yv}^{[i]}(\lambda, i))_{j=1}^{r_{yi}^{[i]}} & 0 \\ 0 & 0 \end{bmatrix} V_{yv}(\lambda, i)
\]
(10)
\[
G_{zu}(\lambda, i) = U_{zu}(\lambda, i) \begin{bmatrix} \text{diag}(\alpha_{zu}^{[i]}(\lambda, i))_{j=1}^{r_{zi}^{[i]}} & 0 \\ 0 & 0 \end{bmatrix} V_{zu}(\lambda, i)
\]
(11)
in which the zero matrices in general have different dimensions, while \( U_{yv}(\lambda, i) \), \( U_{zu}(\lambda, i) \), \( V_{yv}(\lambda, i) \), and \( V_{zu}(\lambda, i) \) are, respectively, \( m_{yi} \times m_{yi} \), \( m_{zi} \times m_{zi} \), \( m_{yi} \times m_{yi} \), and \( m_{ui} \times m_{ui} \) dimensional unimodular MVPs. \( \alpha_{yv}^{[i]}(\lambda, i) \in C_{r_{yi}^{[i]}} \) and \( \alpha_{zu}^{[i]}(\lambda, i) \in C_{r_{zi}^{[i]}} \) are real coefficient rational functions that are not identically equal to zero and have a finite degree.

As argued in the proof of Theorem 2, the TFM \( G_{yv}(\lambda, i) \) is of FNCR, if and only if \( r_{yi}^{[i]} = m_{yi} \). On the other hand, the TFM \( G_{zu}(\lambda, i) \) is of FNRR, if and only if \( r_{zi}^{[i]} = m_{ui} \). Note that the dimensions of a subsystem in an NDS are usually not very large. This means that the Smith–McMillan forms of the aforementioned TFMs can be easily obtained in general, and therefore, the conditions of Theorem 2 can be checked without any significant computation difficulties.

While the above theorem gives a condition for the NDS structure identifiability which can be easily verified, it is only sufficient and is in general not necessary, which is illustrated by the numerical example of Section V. In addition, this condition may not be satisfied easily in some applications, as it requires that in each NDS subsystem, the number of its external outputs is not smaller than that of its internal inputs, while the number
of its external inputs is not smaller than that of its internal outputs. However, it currently still appears mathematically difficult to establish a necessary and sufficient condition that is computationally feasible for the NDS described by (1) and (2). Nevertheless, for some particular kinds of NDSs, applicable results have been obtained.

For this purpose, partition the unimodular MVPs $V_{zy}(\lambda, i)$ and $U_{zu}(\lambda, i)$ of (10) and (11), respectively, as

$$V_{zy}(\lambda, i) = \begin{bmatrix} V_{zy}^{[1]}(\lambda, i) \\ V_{zy}^{[2]}(\lambda, i) \end{bmatrix}, U_{zu}(\lambda, i) = \begin{bmatrix} U_{zu}^{[1]}(\lambda, i) \\ U_{zu}^{[2]}(\lambda, i) \end{bmatrix}$$

in which the sub-MVP $V_{zy}^{[1]}(\lambda, i)$ has $r_{zy}^{[1]}$ rows, the sub-MVP $U_{zu}^{[1]}(\lambda, i)$ has $r_{zu}^{[1]}$ columns, while the other sub-MVPs have compatible dimensions. Moreover, denote the highest degrees of the MVP $V_{zy}^{[1]}(\lambda, i)$ and the MVP $U_{zu}^{[1]}(\lambda, i)$, respectively, by $d_{zy}^{[1]}(i)$ and $d_{zu}^{[1]}(i)$. Then these two MVPVs can be rewritten as

$$V_{zy}^{[1]}(\lambda, i) = \sum_{j=0}^{d_{zy}^{[1]}(i)} V_{zy}^{[1]}(i, j) \lambda^j, \quad U_{zu}^{[1]}(\lambda, i) = \sum_{j=0}^{d_{zu}^{[1]}(i)} U_{zu}^{[1]}(i, j) \lambda^j$$

in which $V_{zy}^{[1]}(i, j)$ and $U_{zu}^{[1]}(i, j)$ are, respectively, $r_{zy} \times m_{z}$ and $m_{z} \times r_{zu}$ dimensional real matrices. Furthermore, for each $i, j = 1, 2, \ldots, N$, denote the following matrix by $\Xi_{zy}^{(i, j)}$:

$$\Xi_{zy}^{(i, j)} = \left\{ \begin{array}{l} \min\{k|k + d_{zy}^{[1]}(i)\} \\ \sum_{s=\max(0, k-d_{zy}^{[1]}(i))}^{d_{zy}^{[1]}(i)} U_{zu}^{[1]}(i, k-s) \otimes V_{zy}^{[1]}(j, s) \end{array} \right\}_{k=0}^{d_{zy}^{[1]}(i)}$$

With these symbols, the following results are obtained for the structure identifiability of the NDS of (1) and (2), while their proof is postponed to the Appendix.

**Theorem 3**: Assume that each subsystem of the NDS $\Sigma$ is well posed. Moreover, assume that its subsystem TFM $G_{zy}(\lambda, i)$ is identical to zero for each $i = 1, 2, \ldots, N$. Then the structure of this NDS is identifiable, if and only if for every $i, j = 1, 2, \ldots, N$, the matrix $\Xi_{zy}^{(i, j)}$ is of FCR.

Note that the condition of this theorem can be checked for each tuple of subsystems independently, its computational complexity increases only quadratically with the subsystem number, and is therefore, scalable for an NDS with its subsystem number. This is attractive in large-scale NDS analysis and synthesis, in which the scalability of a condition is essential from computational considerations. On the other hand, the proof of the above theorem also reveals that in addition to using the Smith–McMillan forms of the TFM $G_{zy}(\lambda, i)$ and $G_{zu}(\lambda, i)$, similar results can be obtained through their left and right coprime matrix polynomial descriptions, which have also been widely adopted in system analysis and synthesis [11], [25]. The details are omitted due to their obviousness.

The assumption that $G_{zy}(\lambda, i) \equiv 0$ for every $1 \leq i \leq N$, means that in each subsystem, there does not exist any direct information transmission from an internal input to an internal output, which is quite restrictive, and may not be easily satisfied in actual applications. This condition, however, does not mean that in the NDS $\Sigma$, each subsystem is isolated from other subsystems. This can be seen from its TFM given by (9). On the other hand, the derivations of Theorem 3 are quite helpful in getting conditions for structure identifiability of another class of NDSs which includes those of [4], [5], [12], and [21] as a special case. More precisely, in addition to the situation of Theorem 3, a similar necessary and sufficient condition can also be established using similar arguments for NDS structure identifiability, provided that the TFM from the internal inputs to the external outputs can be expressed as an FCR TFM multiplied by the TFM from its internal inputs to its internal outputs, while the TFM from the external inputs to the internal outputs can be expressed as that TFM multiplied by an FNRR TFM.

In particular, for each $i, j \in \{1, 2, \ldots, N\}$, define a matrix $\Xi_{zy}^{(i, j)}$ as its counterpart $\Xi_{zy}^{(i, j)}$, using the Smith–McMillan form of the TFM $G_{zy}(\lambda, i)$ and that of the TFM $G_{zu}(\lambda, j)$, then through similar arguments as those in the proof of Theorem 3, a necessary and sufficient condition is obtained under the aforementioned situation for the structure identifiability of the NDS $\Sigma$ of (1) and (2).

**Corollary 1**: Assume that each subsystem of the NDS $\Sigma$, as well as the NDS itself, is well posed. Moreover, assume that there exist an FNC TFM $G_{zy}(\lambda)$ and an FNRR TFM $G_{zu}(\lambda)$, such that

$$G_{zy}(\lambda) = G_{zy}(\lambda)G_{zu}(\lambda), \quad G_{zu}(\lambda) = G_{zu}(\lambda)G_{zu}(\lambda).$$

Then the structure of this NDS is identifiable, if and only if for every $i, j = 1, 2, \ldots, N$, the matrix $\Xi_{zy}^{(i, j)}$ is of FCR.

**Proof**: Let $\Phi_1$ and $\Phi_2$ be two arbitrary SCMs satisfying the well-posedness assumption. Substitute (13) into (41). Then from (9), we have that

$$H(\lambda, \Phi_1) - H(\lambda, \Phi_2)$$

$$= \overline{G}_{zy}(\lambda)G_{zu}(\lambda) \left\{ I_{m_{z}} - \Phi_2 G_{zu}(\lambda) \right\}^{-1} (\Phi_1 - \Phi_2)$$

$$\times \left\{ I_{m_{z}} - G_{zu}(\lambda)\Phi_1 \right\}^{-1} G_{zu}(\lambda) \overline{G}_{zu}(\lambda)$$

$$= \overline{G}_{zy}(\lambda) \left\{ I_{m_{z}} - G_{zu}(\lambda)\Phi_2 \right\}^{-1} G_{zu}(\lambda) (\Phi_1 - \Phi_2) G_{zu}(\lambda)$$

$$\times \left\{ I_{m_{z}} - \Phi_1 G_{zu}(\lambda) \right\}^{-1} \overline{G}_{zu}(\lambda).$$

Now assume that the TFM $G_{zy}(\lambda)$ and $G_{zu}(\lambda)$ are, respectively, of FNC and FNRR. Noting that both the TFM $\left\{ I_{m_{z}} - \Phi_1 G_{zu}(\lambda) \right\}^{-1}$ and $\left\{ I_{m_{z}} - G_{zu}(\lambda)\Phi_2 \right\}^{-1}$ are invertible, it can be straightforwardly declared that the TFM $G_{zy}(\lambda) \left\{ I_{m_{z}} - G_{zu}(\lambda)\Phi_2 \right\}^{-1}$ and $\left\{ I_{m_{z}} - \Phi_1 G_{zu}(\lambda) \right\}^{-1} G_{zu}(\lambda)$ are also, respectively, of FNC and FNRR.

From these conditions, it can be proven using similar arguments as those in the proof of Theorem 2 that $H(\lambda, \Phi_1) - H(\lambda, \Phi_2) \equiv 0$, if and only if $G_{zu}(\lambda)(\Phi_1 - \Phi_2) G_{zu}(\lambda) \equiv 0$.

The proof can now be completed by the same token as those in proving Theorem 3.

**Note** That the matrix $\Xi_{zy}^{(i, j)}$ of Corollary 1 shares the same structure with the matrix $\Xi_{zy}^{(i, j)}$ of Theorem 3. It is obvious that the condition of Corollary 1 has the same computational advantages as that of Theorem 3. That is, its computational costs increase in general quadratically with the NDS subsystem number.

In some real world problems, there may have some apriori structure information about subsystem connections. For example, some elements of the SCM $\Phi$ must be fixed to be zero, which means that some subsystem internal outputs are prohibited to directly affect some subsystem internal inputs, etc. The conditions of Theorem 3 and Corollary 1 can be easily modified to these situations. In particular, it is straightforward from (56) that under the aforementioned condition, the only required modifications
are to remove the corresponding column(s) from the matrix \( \mathbf{E}_\Sigma(i,j) \) or the matrix \( \mathbf{E}_\Sigma^H(i,j) \). In addition, the right null spaces of these two matrices give a complete description of the SCMs that cannot be identified from experimental data. The details are omitted due to space considerations.

While the hypothesis adopted in Theorem 3 and Corollary 1, that is, \( G_{\mathbf{g}v}(\lambda, i) \equiv 0 \) for each \( i = 1,2,\ldots,N \), \( G_{\mathbf{g}v}(\lambda) = \mathbf{G}_{\mathbf{g}v}(\lambda) \mathbf{G}_{\mathbf{g}v}(\lambda) \) and \( \mathbf{G}_{\mathbf{g}u}(\lambda) = \mathbf{G}_{\mathbf{g}v}(\lambda) \mathbf{G}_{\mathbf{g}u}(\lambda) \) with \( \mathbf{G}_{\mathbf{g}v}(\lambda) \) and \( \mathbf{G}_{\mathbf{g}u}(\lambda) \) are, respectively, of FNCR and FNRR, are quite restrictive, the corresponding NDS model still includes those adopted in [4], [5], [12], and [21] as a special case, noting that the models adopted in these works require that all subsystem outputs are measured, no matter it is an internal output or an external output. This requirement is equivalent to that in (1), \( z(t,i) \) is identically equal to \( y(t,i) \) for each \( 1 \leq i \leq N \) and at each time instant \( t \). On the other hand, the conditions of [5] and [4] are based on the TFM of the NDS, which may not be very helpful in system selection and parameter tuning. While Theorems 2 and 3, as well as Corollary 1, as can be equivalently expressed as (17) under the condition that each initial state of this subsystem is equal to zero, the following equalities are obtained:

\[
\begin{align*}
\mathbf{r}(\lambda, i) &= P(i)w(\lambda, i) \\
\mathbf{x}(\lambda, i) &= \begin{bmatrix} A_{\mathbf{xx}}^{[0]}(i) & A_{\mathbf{xx}}^{[0]}(i) & B_{\mathbf{xx}}^{[0]}(i) \\
A_{\mathbf{xx}}^{[0]}(i) & G(i) & F_u(i) \\
A_{\mathbf{xx}}^{[0]}(i) & A_{\mathbf{xx}}^{[0]}(i) & D_u^{[0]}(i) \\
C_{\mathbf{xx}}^{[0]}(i) & H_y(i) & C_{\mathbf{xx}}^{[0]}(i) & D_u^{[0]}(i) \\
\end{bmatrix} \mathbf{x}(\lambda, i) \\
&+ \begin{bmatrix} F_x(i) \\
H_x(i) \\
-A_{\mathbf{xx}}^{[0]}(i)^{-1} \\
\end{bmatrix} \lambda I_{m_x},
\end{align*}
\]

Define TFM \( H_{\mathbf{s}1}(\lambda,i) \) with \( * = \mathbf{w}, \mathbf{z}, \mathbf{y}, \mathbf{v}, \) or \( \mathbf{u} \), as follows:

\[
\begin{align*}
\mathbf{H}_{\mathbf{w}1}(\lambda,i) &= \mathbf{H}_{\mathbf{w}v}(\lambda,i) \mathbf{H}_{\mathbf{w}u}(\lambda,i) \\
\mathbf{H}_{\mathbf{z}1}(\lambda,i) &= \mathbf{H}_{\mathbf{z}v}(\lambda,i) \mathbf{H}_{\mathbf{z}u}(\lambda,i) \\
\mathbf{H}_{\mathbf{y}1}(\lambda,i) &= \mathbf{H}_{\mathbf{y}v}(\lambda,i) \mathbf{H}_{\mathbf{y}u}(\lambda,i) \\
\mathbf{H}_{\mathbf{v}1}(\lambda,i) &= \mathbf{H}_{\mathbf{v}w}(\lambda,i) \mathbf{H}_{\mathbf{v}u}(\lambda,i) \\
\mathbf{H}_{\mathbf{u}1}(\lambda,i) &= \mathbf{H}_{\mathbf{u}w}(\lambda,i) \mathbf{H}_{\mathbf{u}v}(\lambda,i)
\end{align*}
\]

Then straightforward matrix operations prove that the relations among \( w(\lambda,i), z(\lambda,i), y(\lambda,i), \) etc., which are given by (19), can be equivalently expressed as

\[
\begin{align*}
\mathbf{w}(\lambda,i) &= \begin{bmatrix} \mathbf{H}_{\mathbf{w}1}(\lambda,i) & \mathbf{H}_{\mathbf{w}v}(\lambda,i) & \mathbf{H}_{\mathbf{w}u}(\lambda,i) \end{bmatrix} \mathbf{r}(\lambda,i) \\
\mathbf{z}(\lambda,i) &= \begin{bmatrix} \mathbf{H}_{\mathbf{z}1}(\lambda,i) & \mathbf{H}_{\mathbf{z}v}(\lambda,i) & \mathbf{H}_{\mathbf{z}u}(\lambda,i) \end{bmatrix} \mathbf{v}(\lambda,i) \\
\mathbf{y}(\lambda,i) &= \begin{bmatrix} \mathbf{H}_{\mathbf{y}1}(\lambda,i) & \mathbf{H}_{\mathbf{y}v}(\lambda,i) & \mathbf{H}_{\mathbf{y}u}(\lambda,i) \end{bmatrix} \mathbf{u}(\lambda,i)
\end{align*}
\]
Let $m_{wi}$ stand for the dimension of the auxiliary signal vector $w(t, i)$. With similar arguments as those of (7) and (8), it can be proven that if this subsystem is well posed, then the TFMs $I_{m_{wi}} - H_{\text{wr}}(\lambda, i)P(i)$ is not identically equal to zero. That is, its inverse is well defined. Combining (18) and (21) together, direct algebraic manipulations show that the TFMs $G_{\Sigma\Sigma}(\lambda, i)$, $G_{zu}(\lambda, i)$, $G_{yy}(\lambda, i)$, and $G_{yu}(\lambda, i)$ of the previous section, can also be expressed as

$$
\begin{bmatrix}
G_{\Sigma\Sigma}(\lambda, i) & G_{zu}(\lambda, i) \\
G_{yy}(\lambda, i) & G_{yu}(\lambda, i)
\end{bmatrix}
= 
\begin{bmatrix}
H_{\Sigma\Sigma}(\lambda, i) & H_{zu}(\lambda, i) \\
H_{yy}(\lambda, i) & H_{yu}(\lambda, i)
\end{bmatrix}
\times \begin{bmatrix}
H_{\Sigma\Sigma}(\lambda, i) & H_{zu}(\lambda, i) \\
H_{yy}(\lambda, i) & H_{yu}(\lambda, i)
\end{bmatrix}^{-1}.
$$

(22)

That is, in the $i$th subsystem $\Sigma_i$, all the TFMs including those from its external/internal inputs to its external/internal outputs, can be expressed as an LFT of the matrix constituted from its (pseudo) FPPs.

From this LFT expression, the following results are established for the TFMs $G_{yy}(\lambda, i)$ of FNCR.

**Theorem 4:** Assume that the $i$th subsystem $\Sigma_i$ is well posed. Then its TFM $G_{yy}(\lambda, i)$ is of FNCR, if and only if there exists a $\lambda \in C$ such that the matrix pencil $M(\lambda, i)$ defined as follows is of FCR:

$$
M(\lambda, i) = \begin{bmatrix}
\lambda I_{m_{wi}} - A_{xx}^{[0]}(i) & -A_{xy}^{[0]}(i) & -H_x(i) \\
C_0^{[0]}(i) & C_0^{[0]}(i) & H_y(i) \\
P(i)F_x(i) & P(i)F_y(i) & P(i)G(i) - I_{m_{pi}}
\end{bmatrix}
$$

(23)

in which $m_{pi}$ stands for the number of the rows in the matrix $P(i)$ constructed from the (pseudo) FPPs of this subsystem.

The proof of the above theorem is provided in the Appendix.

The matrix pencil $M(\lambda, i)$ in the above theorem has a form very similar to the matrix pencil $\lambda M(\lambda)$ of [20] and [28] which is used for controllability/observability verification of an NDS. The conditions, however, are completely different. More precisely, in NDS controllability/observability verifications, the matrix pencil $M(\lambda)$ is required to be FCR at each $\lambda \in C$. But the above theorem only asks for the existence of one particular $\lambda \in C$, at which the matrix pencil $M(\lambda, i)$ is of FCR. On the other hand, some techniques of [26] and [28] can be borrowed here to deal with NDS structure identifiability.

When the matrix $[C_0^{[0]}(i) C_0^{[0]}(i) H_y(i)]$ is of FCR, it is obvious that at each $\lambda \in C$, the matrix pencil $M(\lambda, i)$ is of FCR. That is, the TFM $G_{yy}(\lambda, i)$ is certainly of FNCR. Therefore, in the remaining of this section, we only investigate the situation in which this matrix is column rank deficient. In this case, its right null space has nonzero elements and $[C_0^{[0]}(i) C_0^{[0]}(i) H_y(i)]$ is not a zero vector.

Partition the matrix $[C_0^{[0]}(i) C_0^{[0]}(i) H_y(i)]$ as

$$
[C_0^{[0]}(i) C_0^{[0]}(i) H_y(i)] = \text{col} \{N_x(i), N_y(i), N_w(i)\}
$$

(24)

in which the submatrices $N_x(i)$, $N_y(i)$, and $N_w(i)$, respectively, have $m_{xi}$, $m_{yi}$, and $m_{pi}$ rows. Then according to Lemma 1, the matrix pencil $M(\lambda, i)$ is of FCR at a particular $\lambda \in C$, if and only if the following matrix pencil $\bar{M}(\lambda, i)$ is of FCR:

$$
\bar{M}(\lambda, i) = \begin{bmatrix}
\lambda I_{m_{wi}} - A_{xx}^{[0]}(i) & -A_{xy}^{[0]}(i) & -H_x(i) \\
C_0^{[0]}(i) & C_0^{[0]}(i) & H_y(i) \\
P(i)F_x(i) & P(i)F_y(i) & P(i)G(i) - I_{m_{pi}}
\end{bmatrix}
$$

(25)

To verify whether or not the matrix pencil $\bar{M}(\lambda, i)$ is of FNCR, the KCF of the matrix pencil $\lambda N_x(i) - [A_{xx}^{[0]}(i)N_x(i) + A_{xy}^{[0]}(i)N_y(i) + H_x(i)N_w(i)]$ is utilized.

From Lemma 3, there exist two invertible real matrices $U(i)$ and $V(i)$, some unique nonnegative integers $\xi_1^{[i]}, \xi_2^{[i]}, \xi_3^{[i]}, \xi_4^{[i]}, \xi_5^{[i]}, \xi_6^{[i]}, \xi_7^{[i]}, \xi_8^{[i]}, \xi_9^{[i]}, \xi_{10}^{[i]}, \xi_{11}^{[i]}$, and $\xi_1^{[i]}(j) = \xi_2^{[i]}(j) = \xi_5^{[i]}(j) = \xi_6^{[i]}(j) = 1$, as well as some unique positive integers $\xi_1^{[i]}(j) = \xi_3^{[i]}(j) = \xi_4^{[i]}(j) = \xi_7^{[i]}(j) = \xi_8^{[i]}(j) = \xi_9^{[i]}(j) = \xi_{10}^{[i]}(j) = \xi_{11}^{[i]}(j) = 1$.

(26)

in which

$$
K(\lambda, i) = \text{diag} \left\{ L_{\xi_1^{[i]}(j)}(\lambda), L_{\xi_2^{[i]}(j)}(\lambda), L_{\xi_3^{[i]}(j)}(\lambda), L_{\xi_4^{[i]}(j)}(\lambda), L_{\xi_5^{[i]}(j)}(\lambda), L_{\xi_6^{[i]}(j)}(\lambda), L_{\xi_7^{[i]}(j)}(\lambda), L_{\xi_8^{[i]}(j)}(\lambda), L_{\xi_9^{[i]}(j)}(\lambda), L_{\xi_{10}^{[i]}(j)}(\lambda), L_{\xi_{11}^{[i]}(j)}(\lambda) \right\}
$$

(27)

From this KCF and Lemma 2, the following results are obtained, while their proof is deferred to the Appendix.

**Corollary 2:** Define matrices $\Theta(i)$ and $\Pi(i)$, respectively, as

$$
\Theta(i) = [F_x(i)N_x(i) + F_y(i)N_y(i) + G(i)N_w(i)]V^{-1}(i), \ m(i)$$

$$
\Pi(i) = N_w(i)V^{-1}(i), \ m(i)
$$

in which $m(i) = \xi_1^{[i]} + \sum_{j=1}^{\xi_{11}^{[i]}} \xi_1^{[i]}(j)$, while $V^{-1}(i), m(i)$ is the submatrix of the inverse of the matrix $V(i)$ consisting of its first $m(i)$ columns. Then the matrix pencil $\bar{M}(\lambda, i)$ is of FNCR, if and only if the following matrix pencil $\bar{M}(\lambda, i)$ is:

$$
\bar{M}(\lambda, i) = \text{diag} \left\{ L_{\xi_1^{[i]}(j)}(\lambda), L_{\xi_2^{[i]}(j)}(\lambda), L_{\xi_3^{[i]}(j)}(\lambda), L_{\xi_4^{[i]}(j)}(\lambda), L_{\xi_5^{[i]}(j)}(\lambda), L_{\xi_6^{[i]}(j)}(\lambda), L_{\xi_7^{[i]}(j)}(\lambda), L_{\xi_8^{[i]}(j)}(\lambda), L_{\xi_9^{[i]}(j)}(\lambda), L_{\xi_{10}^{[i]}(j)}(\lambda), L_{\xi_{11}^{[i]}(j)}(\lambda) \right\}
$$

(28)

Compared with the matrix pencil $\bar{M}(\lambda, i)$, the matrix pencil $\bar{M}(\lambda, i)$ usually has much less columns. This means that the condition of Corollary 2 is in general much more computationally attractive than that of Theorem 4. On the other hand, from the proof of the above corollary, it is also clear that if $\xi_1^{[i]} = 0$, that is, if there does not exist a matrix pencil in the form of $L_1(\lambda)$ in the KCF of the matrix pencil $\lambda N_x(i) - [A_{xx}^{[0]}(i)N_x(i) + A_{xy}^{[0]}(i)N_y(i) + H_x(i)N_w(i)]$, then the matrix pencil $\bar{M}(\lambda, i)$, and therefore, the TFM $G_{yy}(\lambda, i)$, is certainly of FNCR.

To establish a more direct and computationally attractive condition on subsystem dynamics and (pseudo) FPPs, partition the matrix $\Theta(i)$ and the matrix $\Pi(i)$, respectively, as

$$
\Theta(i) = \begin{bmatrix}
\Theta_1(i) & \Theta_2(i) & \cdots & \Theta_{\xi_{11}^{[i]}}(i)
\end{bmatrix}
$$

(29)

$$
\Pi(i) = \begin{bmatrix}
\Pi_1(i) & \Pi_2(i) & \cdots & \Pi_{\xi_{11}^{[i]}}(i)
\end{bmatrix}
$$

(30)
Here, for each \( j = 1, 2, \ldots, \xi_L^{[i]} \), both the submatrix \( \Theta_j(i) \) and the submatrix \( \Pi_j(i) \) have \( \xi_L^{[i]}(j) + 1 \) columns. Define a positive integer \( \xi_L^{[i]} \) as

\[
\xi_L^{[i]} = \max_{j \in \{1, 2, \ldots, \xi_L^{[i]}\}} \xi_L^{[i]}(j). \tag{31}
\]

Moreover, for every \( j \) belonging to the set \( \{1, 2, \ldots, \xi_L^{[i]}\} \), define a matrix \( \Theta_j(i) \) and a matrix \( \Pi_j(i) \), respectively, through

\[
\Theta_j(i) = [\Theta_j(i) \ 0], \quad \Pi_j(i) = [\Pi_j(i) \ 0] \tag{32}
\]

so that all of them have \( \xi_L^{[i]} + 1 \) columns.

Using the above symbols, on the basis of the structure of the null space of a matrix pencil with the form \( L_x(\lambda) \), the following conditions are derived for the TFM \( G_{yz}(\lambda, i) \) to be FNCR. These conditions are computationally more attractive, give more direct requirements on subsystem dynamics and (pseudo) FPPs, and, therefore, may be more insightful in selecting subsystem dynamics and parameters.

**Theorem 5:** Define MVPs \( \Theta(\lambda, i) \) and \( \Pi(\lambda, i) \), respectively, as

\[
\Theta(\lambda, i) = \begin{bmatrix} \Theta(\lambda, i) \col \{ \lambda^k \xi_L^{[i]}(1) \}_{k=0} \cdots \Theta(\lambda, i) \col \{ \lambda^k \xi_L^{[i]}(\xi_L^{[i]}) \}_{k=0} \end{bmatrix},
\]

\[
\Pi(\lambda, i) = \begin{bmatrix} \Pi(\lambda, i) \col \{ \lambda^k \xi_L^{[i]}(1) \}_{k=0} \cdots \Pi(\lambda, i) \col \{ \lambda^k \xi_L^{[i]}(\xi_L^{[i]}) \}_{k=0} \end{bmatrix}.
\]

Moreover, define a matrix \( \Gamma(i) \) as

\[
\Gamma(i) = \left[ (\Theta(\lambda, i) \otimes I) \text{vec}(P(i)) - \text{vec}(\Pi(\lambda, i)) \cdots (\Theta(\lambda, i) \otimes I) \text{vec}(P(i)) - \text{vec}(\Pi(\lambda, i)) \right].
\]

Then

1) the TFM \( G_{yz}(\lambda, i) \) is of FNCR, if and only if the MVP \( P(i) \Theta(\lambda, i) - \Pi(\lambda, i) \) is;

2) the TFM \( G_{yz}(\lambda, i) \) is of FNCR, only if the matrix \( \Gamma(i) \) is of FCR.

The proof of this theorem is also deferred to the Appendix. Using similar arguments as those between (69) to (73) in the proof of Corollary 2, it can be proven that the MVP \( P(i) \Theta(\lambda, i) - \Pi(\lambda, i) \) is of FNCR, if and only if its Smith form has the following structure:

\[
U(\lambda, i) \begin{bmatrix} \text{diag} \left\{ \alpha^{[j]}(\lambda, i) \xi_L^{[i]}(j) \right\}_{j=1} \end{bmatrix} V(\lambda, i)
\]

in which both \( U(\lambda, i) \) and \( V(\lambda, i) \) are unimodular MVPs with a compatible dimension, while \( \alpha^{[j]}(\lambda, i) \xi_L^{[i]}(j) \) are some nonzero and real coefficient polynomials with a finite degree. The latter can be verified through various standard methods developed in system analysis and synthesis [11], [25].

Note that

\[
P(i) \Theta(\lambda, i) - \Pi(\lambda, i) = [P(i) - I] \begin{bmatrix} \Theta(\lambda, i) \ \Pi(\lambda, i) \end{bmatrix}. \tag{33}
\]

It is obvious that the MVP \( P(i) \Theta(\lambda, i) - \Pi(\lambda, i) \) is of FNCR, only if the MVP \( \text{col}(\Theta(\lambda, i), \Pi(\lambda, i)) \) is. As the latter is independent of the subsystem parameters, it gives conditions on the dynamics of a subsystem, such that an NDS can be constituted from it with the NDS structure identifiable.

**V. Numerical Example**

To illustrate theoretical results obtained in the previous sections, an artificial NDS is constructed in this section which has two subsystems and each of them consists of 3 capacitors, 6 operation amplifiers, 2 varistors, and 18 resistors, as shown in Fig. 1. To make presentation concise, it is assumed that in each subsystem, all the capacitors take the same value \( C_i \), while all the resistors take the same value \( R_i \). In addition, the maximum values of the varistors, that is, \( R_{i+1} \) and \( R_{i+2} \), are assumed to be \( R_i \). Hence, the SCM \( \Phi \) of this artificial NDS is a \( 2 \times 4 \) dimensional real matrix.

Define pseudo FPPs \( T_i, k_{11}, \text{and } k_{12} \) of the \( i \)th subsystem \( \Sigma_i \) with \( i = 1, 2 \), respectively, as

\[
T_i = R_i C_i, \quad k_{11} = \frac{R_{i+1}}{R_i}, \quad k_{12} = \frac{R_{i+2}}{R_i}.
\]

To investigate influences of the varistors \( R_{i+1} \) and \( R_{i+2} \) on NDS structure identifiability, assume that the time constant \( T_i \) is prescribed for each subsystem.

Take the voltage of each capacitor as a state of a subsystem, while the voltage of the leftmost capacitor as its output. Under the above assumptions, the system matrices of each subsystem can be written as follows:

\[
A_{xx}^{[0]}(i) = -\frac{1}{T_i} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_{xx}^{[0]}(i) = -\frac{1}{T_i} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
B_{x}^{[0]}(i) = -\frac{1}{T_i} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{x}^{[0]}(i) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
A_{x}^{[0]}(i) = 0_2, \quad B_{x}^{[0]}(i) = 0_{2 \times 2}
\]
\( C_X^{[0]}(i) = [1 \ 0 \ 0], \quad C_Y^{[0]}(i) = 0, \quad D_u^{[0]}(i) = [0 \ 0] \)

\( H_X(i) = -\frac{1}{T_i} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_Y(i) = \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}, \quad H_Z(i) = [0 \ 0] \)

\( F_X(i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_Y(i) = 0, \quad F_u(i) = 0_{2 \times 2}, G(i) = I_2. \)

Moreover,

\[ P(i) = \begin{bmatrix} \frac{k_1}{k_1+1} & 0 \\ 0 & \frac{k_1}{k_1+2} \end{bmatrix}. \]

From these system matrices, direct algebraic manipulations show that for each \( i = 1, 2 \), \( G_{XV}(\lambda, i) \) is a nonzero \( 1 \times 1 \) dimensional transfer function, while the TFM \( G_{ZU}(\lambda, i) \) has the following expression:

\[ G_{ZU}(\lambda, i) = \frac{T_i(2k_{i1} - 1)}{(\lambda T_i + 1)(\lambda T_i + k_1 + 3)} \]

Note that the time constant \( T_1 \) always takes a positive value. It is clear that \( \det \{ G_{ZU}(\lambda, i) \} = 0 \), if and only if \( k_{i1} = 0.5 \).

These observations mean that for each \( i = 1, 2 \), the TFM \( G_{XV}(\lambda, i) \) is always of FNCR, while the TFM \( G_{ZU}(\lambda, i) \) is not of FNRR when and only when \( k_{i1} \) takes the value 0.5.

As relations have already become clear between the pseudo FPPs \( k_{11}/k_{21} \) and the required properties that the TFM \( G_{XV}(\lambda, i) \) is of FNCR/the TFM \( G_{ZU}(\lambda, i) \) is of FNRR, results of Theorem 5, etc., are no longer required for this example.

It can therefore be declared from Theorem 2 that, if each of the NDS subsystems \( \Sigma_1 \) and \( \Sigma_2 \) has its pseudo FPP \( k_{11}/k_{21} \) taking a value different from 0.5, then the structure of this artificial NDS is identifiable.

To investigate the influences of the pseudo FPPs \( k_{11} \) and \( k_{21} \) on the structure identifiability of the artificial NDS, fix the other pseudo FPPs as \( T_1 = T_2 = 1, k_{12} = 0.4, \) and \( k_{22} = 0.9. \) Moreover, the pseudo FPPs \( k_{11} \) and \( k_{21} \) are assumed to take the same value \( k_1 \) that varies between 0 and 1. For each value of \( k_1 \), the following quantity \( d_{\text{id}}^{[F]} \) is adopted as a frequency domain measure for the distance of an NDS to the set of NDSs with an unidentifiable structure.

\[ d_{\text{id}}^{[F]} = \inf_{\Phi_1, \Phi_2 \neq \Phi_1} \| H(\lambda, \Phi_2) - H(\lambda, \Phi_1) \|_{\infty} \]

in which \( \| \cdot \|_{\infty} \) stands for the \( L_\infty \) norm of a TFM, while \( \sigma(\cdot) \) is the maximum singular value of a matrix. Moreover, the TFMs \( H(\lambda, \Phi_1) \) and \( H(\lambda, \Phi_2) \) have a definition of (9).

Note that the \( L_\infty \) norm of a TFM is defined as the supremum of the maximum singular value of its frequency response, which is well known to be an induced norm \([25],[27]\). On the other hand, the maximum singular value of a matrix is also an induced norm that is extensively adopted in matrix difference measurements \([6],[8]\). The above quantity appears reasonable from an application viewpoint. Another option is to use the Frobenius norm of a matrix in the definition of the above quantity \( d_{\text{id}}^{[F]} \). Our simulation results, however, show that the conclusions are almost the same.

While from an application viewpoint, the above quantity may be a good candidate in measuring the distance of an NDS to the set of NDSs with an unidentifiable structure, there are currently still some mathematical difficulties to calculate it analytically. It is therefore, estimated in this section through numerical simulations.

Specifically, for a prescribed value of the parameter \( k_1, 10^4 \) SCM \( \Phi_{1s} \) are randomly and independently generated. In addition, for each randomly generated SCM \( \Phi_{1} \), \( 2.0000 \times 10^5 \) SCM \( \Phi_{2s} \) are generated, also randomly and independently. Furthermore, for each generated SCM pair \( \Phi_1 \) and \( \Phi_2 \), the \( L_\infty \)-norm of the TFM \( H(\lambda, \Phi_2) - H(\lambda, \Phi_1) \) is divided by the maximum singular value of the matrix \( \Phi_2 - \Phi_1 \). Finally, the minimum value of this division over all the generated SCM \( (\Phi_1, \Phi_2) \) pairs is used as an estimate for the above frequency domain measure \( d_{\text{id}}^{[F]} \) on the distance of an NDS to the set of NDSs with an unidentifiable structure, while the SCM pair \( \Phi_1 \) and \( \Phi_2 \) associated with this minimum value are recorded, respectively, as \( \Phi_{[1]} \) and \( \Phi_{[2]} \).

In the aforementioned SCM generation, each element of the SCM is produced independently, according to the continuous uniform distribution over the interval \((-1, 1)\).

On the other hand, it is inviable from an application viewpoint to see the maximum difference in the time domain between the outputs of two artificial NDSs that are closest to each other in some sense and associated with different SCMs, with sampled input/output data of a finite length. However, exhausting all the possible probing signals are computationally prohibitive. Instead, a pseudo random binary signal (PRBS) is adopted as testing signals, which has been extensively utilized in traditional system identifications and is widely recognized as an informative probing signal that is persistently exciting \([7],[19]\). Note that it is observed in \([23]\) that all the input/output trajectories of a system can be expressed as a linear combination of the input/output trajectory of the system stimulated by a persistently exciting signal. This probing signal selection appears reasonable.

Particularly, for each value of \( k_1 \), stimulate simultaneously the NDS \( \Sigma_{\Phi_{[1]}} \) and the NDS \( \Sigma_{\Phi_{[2]}} \) with four independent PRBSs that take values from the set \([-1, 1]\). Recall that \( \Phi_{[1]} \) and \( \Phi_{[2]} \) stand for the SCM pair among all the randomly generated samples, that make the cost function \( \| H(\lambda, \Phi_2) - H(\lambda, \Phi_1) \|_{\text{min}} \) achieve its minimum value. That is, these two NDSs are the closest ones among all the sampled NDS pairs.

The differences \( e(t)_{|t=0}^M \) between the outputs of these two artificial NDSs are measured by their extended \( l_2 \)-norm \( \| e(t) \|_2 \) defined as \( \| e(t) \|_2 = \sqrt{\sum_{t=0}^{M-1} e^T(t)e(t)} \), in which \( e(t) = y(t, \Phi_{[1]}) - y(t, \Phi_{[2]}) \) stands for the output difference at the \( t \)th sampling instant, while \( M \) is the number of all sampled time instants. With this information available, the quantity \( d_{\text{id}}^{[T]} \) is calculated which is defined as

\[ d_{\text{id}}^{[T]} = \frac{\| e(t) \|_2}{M\sigma(\Phi_{[2]} - \Phi_{[1]})} \]

This quantity is referred here to as an estimate of a time domain measure for the distance of an NDS to the set of NDSs with an unidentifiable structure.

For each prescribed value of \( k_1 \), let \( A(k_1) \) stand for the block diagonal matrix constituted from the state transition matrices of the two associate NDSs \( \Sigma_{\Phi_{[1]}} \) and \( \Sigma_{\Phi_{[2]}} \), while \( \rho_{\text{max}}(A(k_1)) \) and \( \rho_{\text{max}}(A(k_1)) \), respectively, the maximum and minimum of the absolute value of its eigenvalues. In actual
simulations, an NDS is sampled with a constant period set to \( \rho_{\text{max}}(A(k_{11})) \), while the sample number to the maximum of \( 2.0000 \times 10^4 \) and \( 100 \times \rho_{\text{max}}(A(k_{11})) \). This simulation setting is adopted to get sufficient information about the required output differences without too much computation costs.

The computation results are listed in Table I for both the frequency and the time domain distances. For reference, \( \Sigma \) is not very close to zero, which is equivalent to that the SCM is in fact a nonlinear transformation. As the multiplication of this TFM and an FNRR TFM. This condition is necessary and sufficient that the (pseudo) FPPs of its subsystems make an MVP have an FNCR/FNRR which depends affinely on these (pseudo) FPPs. Moreover, a matrix rank-based necessary and sufficient condition is established for NDS structure identifiability under the following two situations. The first is that in each subsystem, no direct information transmission exists from an internal input to an internal output. The second is that, the TFM from its internal inputs to its internal outputs, while the TFM from its internal inputs to its external outputs can be expressed as the multiplication of an FNCR TFM and the TFM from its internal inputs to its external outputs can be expressed as the multiplication of this TFM and an FNRR TFM. This condition can be independently verified with each tuple of subsystems, and is scalable for large-scale NDSs.

Table I

| \( k_1 \) | \( d_{\text{scm}} \) | \( d_{\text{id}}^{[F]} \) | \( d_{\text{id}}^{[T]} \) |
|---|---|---|---|
| 0.0500 | 1.6138 | 5.7399 \times 10^{-2} | 7.0091 \times 10^{-5} |
| 0.1000 | 1.2643 | 5.2766 \times 10^{-2} | 7.1263 \times 10^{-5} |
| 0.1500 | 0.7771 \times 10^{-1} | 4.0053 \times 10^{-2} | 5.3975 \times 10^{-5} |
| 0.2000 | 1.2356 | 4.0338 \times 10^{-2} | 5.2044 \times 10^{-5} |
| 0.2500 | 1.9906 | 3.4435 \times 10^{-2} | 3.9401 \times 10^{-5} |
| 0.3000 | 2.0254 | 3.2411 \times 10^{-2} | 4.0897 \times 10^{-5} |
| 0.3500 | 1.2953 | 2.5649 \times 10^{-2} | 3.3994 \times 10^{-5} |
| 0.4000 | 2.1483 | 1.9829 \times 10^{-2} | 2.8705 \times 10^{-5} |
| 0.4500 | 1.8778 | 1.4281 \times 10^{-2} | 1.8896 \times 10^{-5} |
| 0.5000 | 1.7358 | 1.2580 \times 10^{-2} | 1.5758 \times 10^{-5} |
| 0.5500 | 1.7297 | 1.1353 \times 10^{-2} | 1.6384 \times 10^{-5} |
| 0.6000 | 2.2571 | 1.4394 \times 10^{-2} | 2.0664 \times 10^{-5} |
| 0.6500 | 1.3653 | 2.0464 \times 10^{-2} | 2.6208 \times 10^{-5} |
| 0.7000 | 1.8584 | 2.2381 \times 10^{-2} | 3.0936 \times 10^{-5} |
| 0.7500 | 1.7534 | 2.9100 \times 10^{-2} | 3.6359 \times 10^{-5} |
| 0.8000 | 2.1193 | 3.1190 \times 10^{-2} | 4.4965 \times 10^{-5} |
| 0.8500 | 1.2331 | 3.0762 \times 10^{-2} | 5.0705 \times 10^{-5} |
| 0.9000 | 1.4441 | 3.4386 \times 10^{-2} | 5.1064 \times 10^{-5} |
| 0.9500 | 1.8608 | 3.8835 \times 10^{-2} | 4.9123 \times 10^{-5} |

\( d_{\text{id}}^{[T]} \) are, respectively, normalized by a factor \( 10^2 \) and a factor \( 10^1 \) to improve figure readability.

Clearly, both the estimated \( d_{\text{id}}^{[F]} \) and the estimated \( d_{\text{id}}^{[T]} \) increase almost monotonically with the magnitude of the deviation of the pseudo FPPs \( k_{11} \) and \( k_{21} \) from 0.5000. On the other hand, their curves have very similar variation patterns. This may suggest that these two indices are consistent well with each other, and may be good candidates for a metric in measuring distance of an NDS to the set of NDSs with an unidentifiable structure.

VI. CONCLUSION

In this article, we have investigated conditions on a subsystem such that an LTI NDS constructed from it has an identifiable structure. Except well-posedness, there are neither any other restrictions on subsystem dynamics, nor any other restrictions on subsystem connections. It is proven that the structure of an LTI NDS is identifiable, if the TFMs of its subsystems meet some rank conditions. Based on this result, it has been further shown that in order to guarantee the satisfaction of this condition, it is necessary and sufficient that the (pseudo) FPPs of its subsystems make an MVP have an FNCR/FNRR which depends affinely on these (pseudo) FPPs. Moreover, a matrix rank-based necessary and sufficient condition is established for NDS structure identifiability under the following two situations. The first is that in each subsystem, no direct information transmission exists from an internal input to an internal output. The second is that, the TFM from its internal inputs to its external outputs can be expressed as the multiplication of an FNCR TFM and the TFM from its internal inputs to its external outputs, while the TFM from its external inputs to its internal outputs can be expressed as the multiplication of this TFM and an FNRR TFM. This condition can be independently verified with each tuple of subsystems, and is scalable for large-scale NDSs.

From these results, it is conjectured that rather than the particular value of subsystem (pseudo) FPPs, it is the connections among subsystem states, internal/external inputs/outputs and (pseudo) FPPs that determine the structure identifiability of an NDS. That is, structure identifiability of an NDS is also a generic property, which is similar to its controllability and observability, as well as NDS identifiability with a prescribed.

Fig. 2. Normalized estimate of the distance of the NDS to the set of NDSs with an unidentifiable structure. ♦: frequency domain distance. ◦: time domain distance.
structure. This is an interesting topic under investigations, from which graph-based conditions on NDS structure identifiability can be expected, that are quite attractive in the analysis and synthesis of a large-scale NDS. In addition, further efforts are required to get a computationally scalable necessary and sufficient condition removing the assumptions like that on direct internal input–output information delivery. It is also interesting to develop a more appropriate metric measuring the distance of an NDS to the set of NDSs with an unidentifiable structure, taking into account both application significance and computational feasibility. This is an important issue, which appears closely related to the hyper-ribbon structure of an NDS associated data space, and the existence of sloppy parameter variations in its SCM, which have been extensively observed when a model has a large number of parameters to be estimated [17].

**APPENDIX**

**PROOF OF SOME TECHNICAL RESULTS**

**Proof of Theorem 1:** For each \( i \in \{1, 2, \ldots, N\} \), let \( x(0, i) \) denote the initial value of the state vector of the \( i \)th subsystem \( \Sigma_i \). Take the Laplace transformation on both sides of (1) when \( \delta(\cdot) \) is the derivative of a function with respect to time, and the \( Z \) transformation when \( \delta(\cdot) \) represents a forward time shift operation. Moreover, let \( * \) represent the associated signal after the transformation, in which \( * = x, u, v, y, z \). Then according to the properties of the Laplace/\( Z \) transformation, we have the following relations

\[
\begin{align*}
\hat{x}(\lambda, i) &- x(0, i) \quad z(\lambda, i) \\
\hat{y}(\lambda, i) &= \left[ A_{xx}(i) A_{xv}(i) B_{x}(i) \right. \\
&\left. \left. A_{xz}(i) A_{zv}(i) B_{z}(i) \right] \hat{x}(\lambda, i) + \left[ C_{x}(i) C_{v}(i) D_{u}(i) \right] \hat{u}(\lambda, i). \quad (34)
\end{align*}
\]

For each \( \# = x, v, z \) or \( \# = \) \( z \), define a vector \( \#(\lambda) \) as \( \#(\lambda) = \{\#(\lambda, i)\}_{i=1}^{N} \). Moreover, denote the vector \( \text{col}(x(0, i)_{i=1}^{N}) \) by \( x(0) \). Furthermore, define a matrix \( D_u \) as \( D_u = \text{diag}(D_{u}(i)_{i=1}^{N}) \). In addition, define matrices \( A_{x\#}, B_{\#}, \) and \( C_{\#} \) with \( \# = x, y, v \) or \( \# = z \), respectively, as \( A_{x\#} = \text{diag}(A_{x\#}(i)_{i=1}^{N}) \), \( B_{\#} = \text{diag}(B_{\#}(i)_{i=1}^{N}) \), and \( C_{\#} = \text{diag}(C_{\#}(i)_{i=1}^{N}) \). With these symbols, relations among all the transformed signals of all the subsystems in the NDS \( \Sigma \), which is given by (34), can be compactly represented by

\[
\begin{align*}
\hat{x}(\lambda, i) &- x(0, i) \quad z(\lambda, i) \\
\hat{y}(\lambda, i) &= \left[ A_{xx} A_{xv} B_{x} \right. \\
&\left. \left. A_{xz} A_{zv} B_{z} \right] \hat{x}(\lambda, i) + \left[ C_{x} C_{v} D_{u} \right] \hat{u}(\lambda, i). \quad (35)
\end{align*}
\]

Denote \( \sum_{k=1}^{N} m_{uk} \) by \( m_{u} \). From (35), as well as the definitions of the TFMs \( G_{ux}(\lambda), G_{xv}(\lambda), G_{yu}(\lambda), \) and \( G_{yv}(\lambda) \), direct algebraic manipulations show that

\[
\begin{align*}
\hat{z}(\lambda) &= \left[ G_{ux}(\lambda) G_{xu}(\lambda) \right. \\
&\left. \left. G_{yv}(\lambda) G_{yu}(\lambda) \right] \hat{v}(\lambda) + \left[ A_{xx} C_{x} \right] \hat{x}(\lambda, i) \quad (36)
\end{align*}
\]

On the other hand, from (2), we have that the following relation exists between the transformed internal input/output vectors of the NDS \( \Sigma \):

\[
\begin{align*}
v(\lambda) &= \Phi_x z(\lambda). \quad (37)
\end{align*}
\]

Combining (36) and (37) together, and recalling that the inverse of the TFM \( I_{m_u} - G_{uv}(\lambda) \) is well defined when the NDS \( \Sigma \) is well posed, we immediately have that

\[
y(\lambda, \Phi) = G(\lambda, \Phi) x(0) + H(\lambda, \Phi) u(\lambda). \quad (38)
\]

Here, in order to clarify the dependence of NDS outputs on its SCM \( \Phi \), the vector valued function \( y(\lambda, \Phi) \) is replaced by \( y(\lambda, \Phi) \). In addition

\[
G(\lambda, \Phi) = \left\{ C_{x} + G_{yv}(\lambda) \Phi \left[ I_{m_u} - G_{uv}(\lambda) \Phi \right]^{-1} A_{xz} \right\} \times (\lambda I_{m_u} - A_{xx})^{-1}. \]

Let \( \Phi_1 \) and \( \Phi_2 \) be two arbitrary different SCMs in the NDS \( \Sigma \). Then with the same but arbitrary initial state vector \( x(0) \) and the same also arbitrary inputs \( u(\lambda) \), the difference between the associated outputs can be expressed as

\[
y(\lambda, \Phi_1) - y(\lambda, \Phi_2) = \left[ G(\lambda, \Phi_1) - G(\lambda, \Phi_2) \right] x(0) + [H(\lambda, \Phi_1) - H(\lambda, \Phi_2)] u(\lambda). \quad (39)
\]

Assume that there exist two different SCMs \( \Phi_1 \) and \( \Phi_2 \) for the NDS \( \Sigma \), such that \( H(\lambda, \Phi_1) = H(\lambda, \Phi_2) \) for every \( \lambda \in C \). The above equation implies that if all the initial states of the NDS \( \Sigma \) are equal to zero, then for these two SCMs, we have that for each input series, the following equality holds:

\[
y(\lambda, \Phi_1) = y(\lambda, \Phi_2) \quad \forall \lambda \in C. \quad (40)
\]

Recall that both the Laplace transformation and the \( Z \)-transformation are bijective mappings [11], [25], [27]. It is obvious that for these two SCMs \( \Phi_1 \) and \( \Phi_2 \), the associated outputs of the NDS \( \Sigma (\Phi_1) \) and the NDS \( \Sigma (\Phi_2) \) are always the same, no matter what input signals are used to stimulate them and how long the outputs are measured, provided that its initial states are all equal to zero. This implies that the structure of this NDS \( \Sigma \) is not identifiable.

On the contrary, assume that for any two arbitrary different SCMs \( \Phi_1 \) and \( \Phi_2 \) of the NDS \( \Sigma \), there exist some \( \lambda \in C \) at which \( H(\lambda, \Phi_1) \neq H(\lambda, \Phi_2) \). Then according to (39), if \( G(\lambda, \Phi_1) - G(\lambda, \Phi_2)] x(0) \neq 0 \), then the input \( u(t)_{t=0}^{\infty} \) satisfying \( u(\lambda) \equiv 0 \), that is, the zero inputs, leads to \( y(\lambda, \Phi_1) \neq y(\lambda, \Phi_2) \). In other words, the outputs of the NDS \( \Sigma \) associated, respectively, with the SCMs \( \Phi_1 \) and \( \Phi_2 \) are different.

On the other hand, for an initial state vector \( x(0) \) satisfying \( [G(\lambda, \Phi_1) - G(\lambda, \Phi_2)] x(0) = 0 \) for every \( \lambda \in C \), let \( u(\lambda) = \epsilon_j \) in which \( j \) is an element of the set consisting of the column indices of the TFM \( H(\lambda, \Phi_1) - H(\lambda, \Phi_2) \) with which the corresponding column is not consistently equal to zero, while \( \epsilon_j \) is the \( j \)th standard basis of the \( m_u \) dimensional Euclidean space \( C^{m_u} \) in which \( m_u = \sum_{k=1}^{N} m_{uk} \). From (39), it is obvious that this input satisfies \( y(\lambda, \Phi_1) \neq y(\lambda, \Phi_2) \). That is, there exists at least one input time series, such that the outputs of the NDS \( \Sigma (\Phi_1) \) and the NDS \( \Sigma (\Phi_2) \) are not equal to each other at every time instant.

The above arguments mean that if the aforementioned condition is satisfied, then the structure of the NDS \( \Sigma \) is identifiable. This completes the proof.

**Proof of Theorem 2:** Let \( \Phi_1 \) and \( \Phi_2 \) be two arbitrary SCMs satisfying the well-posedness assumption. Then both the TFM \( I_{m_u} - G_{uv}(\lambda) \Phi_1 \) and the TFM \( I_{m_u} - G_{uv}(\lambda) \Phi_2 \) are invertible. This implies that both the TFM \( H(\lambda, \Phi_1) \) and the TFM \( H(\lambda, \Phi_2) \) are well defined. From the definitions of the TFM \( H(\lambda, \Phi) \), we have that

\[
H(\lambda, \Phi_1) = H(\lambda, \Phi_2) = G_{yv}(\lambda) \left\{ \Phi_1 \left[ I_{m_u} - G_{uv}(\lambda) \Phi_1 \right]^{-1} \right\},
\]
in which $\Delta(\lambda) = [I_{m_\tau} - \Phi_2 G_{zu}(\lambda)]^{-1}(\Phi_1 - \Phi_2) \times [I_{m_\tau} - G_{zu}(\lambda)]\Phi_1^{-1} G_{zu}(\lambda)$

Note that $\det \{I_{m_\tau} - \Phi_2 G_{zu}(\lambda)\} = \det \{I_{m_\tau} - G_{zu}(\lambda)\} \Phi_1$.

This means that the TFM $I_{m_\tau} - \Phi_2 G_{zu}(\lambda)$ is invertible for almost all $\lambda \in \mathbb{C}$. These imply that if $\Phi_1 = \Phi_2$, then $\Delta(\lambda) = 0$ for all the $\lambda \in \mathbb{C}$.

On the contrary, assume that $\Delta(\lambda) = 0$ for almost all the $\lambda \in \mathbb{C}$. As both the TFM $I_{m_\tau} - \Phi_2 G_{zu}(\lambda)$ and the TFM $I_{m_\tau} - G_{zu}(\lambda)\Phi_1$ are invertible, it can be proven using arguments similar to those between the following (42) and (47), that there certainly exists at least one $\lambda_0 \in \mathbb{C}$, such that $\Delta(\lambda_0) = 0$, while $I_{m_\tau} - \Phi_2 G_{zu}(\lambda_0)$ and $I_{m_\tau} - G_{zu}(\lambda_0)\Phi_1$ are invertible. From the definition of the TFM $\Delta(\lambda)$, this means that $\Phi_1 = \Phi_2$.

The above arguments mean that $\Phi_1 = \Phi_2$ if and only if $\Delta(\lambda) = 0$ for all the $\lambda \in \mathbb{C}$.

On the other hand, according to the Smith–McMillan form of a TFM, it can be declared that there exist a nonnegative integer $m_\tau$ not greater than $m_\tau$, an $m_\tau \times m_\tau$ dimensional unimodular matrix $U_{zu}(\lambda)$, an $m_\tau \times m_\tau$ dimensional unimodular matrix $V_{zu}(\lambda)$, as well as nonzero and real coefficient rational functions $\alpha^{[i]}_{zu}(\lambda)|_{i=1}^{m_\tau}$, with all of them having a finite degree, such that

$$G_{zu}(\lambda) = U_{zu}(\lambda) \begin{bmatrix} \text{diag} \{\alpha^{[i]}_{zu}(\lambda)|_{i=1}^{m_\tau}\} & 0 \\ 0 & V_{zu}(\lambda) \end{bmatrix}. \quad (42)$$

Here, the dimensions of the zero matrices are in general different. They are not clearly indicated for brevity.

Divide the unimodular matrix $U_{zu}(\lambda)$ as $U_{zu}(\lambda) = [U_{zu,1}^1(\lambda) U_{zu,2}^1(\lambda)]$ with $U_{zu,1}^1(\lambda)$ having $m_\tau$ columns. Then from (42), we have that

$$G_{zu}(\lambda) = [U_{zu,1}^1(\lambda) \begin{bmatrix} \text{diag} \{\alpha^{[i]}_{zu}(\lambda)|_{i=1}^{m_\tau}\} & 0 \\ 0 & V_{zu}(\lambda) \end{bmatrix}] U_{zu,2}^1(\lambda). \quad (43)$$

As $U_{zu}(\lambda)$ is an unimodular matrix, there exists another unimodular matrix $U_{zu,2}^1(\lambda)$, such that

$$U_{zu,2}^1(\lambda) U_{zu,2}^1(\lambda) = I_{m_\tau}. \quad (44)$$

Partition the unimodular matrix $U_{zu}(\lambda)$ as $U_{zu}(\lambda) = \cup \{U_{zu,1}^i(\lambda), U_{zu,2}^i(\lambda)\}$ with $U_{zu,1}^i(\lambda)$ having $m_\tau$ rows. It can then be declared from (44) that

$$U_{zu,2}^i(\lambda) U_{zu,2}^i(\lambda) = 0. \quad (45)$$

Construct a polynomial vector $\zeta(\lambda)$ as

$$\zeta(\lambda) = \zeta(\lambda)|_{i=1}^{m_\tau} \text{diag} \{\alpha^{[i]}_{zu}(\lambda)|_{i=1}^{m_\tau}\} \quad (46)$$

in which $\zeta(\lambda)$ is an arbitrary $m_\tau - m_\tau$ dimensional row polynomial with real coefficients that does not make the associated polynomial vector $\zeta(\lambda)$ being equal to zero at each $\lambda \in \mathbb{C}$. The existence of this polynomial vector is guaranteed by the fact that the MVP $U_{zu,2}^i(\lambda)$ is unimodular, which means that the sub-MVP $U_{zu,2}^i(\lambda)$ is of FRR at each complex $\lambda$. Substitute this $\zeta(\lambda)$ into (43). It is immediate from (45) that

$$\zeta(\lambda) G_{zu}(\lambda) = \zeta(\lambda) U_{zu,2}^1(\lambda) \begin{bmatrix} \text{diag} \{\alpha^{[i]}_{zu}(\lambda)|_{i=1}^{m_\tau}\} & 0 \\ 0 & V_{zu}(\lambda) \end{bmatrix} \equiv 0. \quad (47)$$

The above arguments show that if the integer $m_\tau$ is smaller than $m_\tau$, then the TFM $G_{zu}(\lambda)$ is row rank deficient at every $\lambda \in \mathbb{C}$, and is therefore, not of FNRR.

Assume now that for each $i \in \{1, 2, \ldots, N\}$, the TFM $G_{zu}(\lambda, i)$ is of FNRR, while the TFM $G_{zu}(\lambda, i)$ is of FNCR. From the block diagonal structure of the TFM $G_{zu}(\lambda)$ and $G_{zu}(\lambda)$, it can be directly declared that the TFM $G_{zu}(\lambda)$ is of FNRR, while the TFM $G_{zu}(\lambda)$ is of FNCR.

From these observations and (47), it is clear that there exist an $m_\tau \times m_\tau$ dimensional unimodular matrix $U_{zu}(\lambda)$, an $m_\tau \times m_\tau$ dimensional unimodular matrix $V_{zu}(\lambda)$, as well as nonzero and real coefficient rational functions $\alpha^{[i]}_{zu}(\lambda)|_{i=1}^{m_\tau}$ with all of them having a finite degree, such that

$$G_{zu}(\lambda) = U_{zu}(\lambda) \begin{bmatrix} \text{diag} \{\alpha^{[i]}_{zu}(\lambda)|_{i=1}^{m_\tau}\} & 0 \\ 0 & V_{zu}(\lambda) \end{bmatrix}. \quad (48)$$

In addition, noting that a TFM is of FNCR if and only if its transpose is of FNRR. This implies that there also exist an $m_\tau \times m_\tau$ dimensional unimodular matrix $U_{zu}(\lambda)$, an $m_\tau \times m_\tau$ dimensional unimodular matrix $V_{zu}(\lambda)$, as well as nonzero and real coefficient rational functions $\alpha^{[i]}_{zu}(\lambda)|_{i=1}^{m_\tau}$ with all of them having a finite degree, such that

$$G_{zu}(\lambda) = U_{zu}(\lambda) \begin{bmatrix} \text{diag} \{\alpha^{[i]}_{zu}(\lambda)|_{i=1}^{m_\tau}\} & 0 \\ 0 & V_{zu}(\lambda) \end{bmatrix}. \quad (49)$$

Equations (48) and (49) mean that the TFM $G_{zu}(\lambda)$ is right invertible for almost all $\lambda \in \mathbb{C}$, while the TFM $G_{zu}(\lambda)$ is left invertible for almost every $\lambda \in \mathbb{C}$.

More precisely, define sets $\Lambda_{zu}$ and $\Lambda_{zu}$, respectively, as

$$\Lambda_{zu} = \bigcup_{i=1}^{m_{\tau}} \{\lambda \mid \alpha^{[i]}_{zu}(\lambda) = 0, \lambda \in \mathbb{C} \}$$

$$\Lambda_{zu} = \bigcup_{i=1}^{m_{\tau}} \{\lambda \mid \alpha^{[i]}_{zu}(\lambda) = 0, \lambda \in \mathbb{C} \}.$$
in which $\Phi_1(i,j)$ and $\Phi_2(i,j)$ are $m_{\psi_1} \times m_{\psi_2}$ dimensional real submatrices. Moreover, denote $\Phi_1(i,j)$ and $\Phi_2(i,j)$ with $i, j = 1, 2, \ldots, N$, by $\Delta(i,j)$ for brevity. Then from the consistent block diagonal structure of the TFMs $G_{\psi_1}(\lambda)$ and $G_{\psi_2}(\lambda)$, it is immediate that $H(\lambda, \Phi_1) - H(\lambda, \Phi_2) \equiv 0$ if and only if for every $i, j = 1, 2, \ldots, N$

$$G_{\psi_1}(\lambda, \lambda) \Delta(i,j) G_{\psi_2}(\lambda, \lambda) \equiv 0. \quad (52)$$

Substitute (10) and (11) into the above equation. Noting that both the MVPs $U_{\psi_1}(\lambda, i)$ and $V_{\psi_2}(\lambda, i)$ are unimodular, as well as the rational functions $\alpha_{\psi_1}^{[j]}(\lambda, i) \eta_{\psi_1}^{[j]}$ and $\alpha_{\psi_1}^{[j]}(\lambda, i) \eta_{\psi_1}^{[j]}$ are not identically equal to zero and have a finite degree, it can be straightforwardly shown that (52) is satisfied, if and only if

$$V_{\psi_1}^{[i]}(\lambda, i) \Delta(i,j) U_{\psi_2}^{[j]}(\lambda, j) \equiv 0. \quad (53)$$

In addition, from (12), we have that

$$V_{\psi_1}^{[i]}(\lambda, i) \Delta(i,j) U_{\psi_2}^{[j]}(\lambda, j) = \left( \sum_{p=0}^{d_{\psi_1}^{[i][j]}} V_{\psi_1}^{[i]}(i,p) \lambda^p \right) \Delta(i,j) \left( \sum_{q=0}^{d_{\psi_2}^{[j][j]}} U_{\psi_2}^{[j]}(j, q) \lambda^q \right)$$

$$= \sum_{p=0}^{d_{\psi_1}^{[i][j]}} \sum_{q=0}^{d_{\psi_2}^{[j][j]}} V_{\psi_1}^{[i]}(i,p) \Delta(i,j) U_{\psi_2}^{[j]}(j, q) \lambda^{p+q}$$

$$= \sum_{k=0}^{d_{\psi_1}^{[i][j]} + d_{\psi_2}^{[j][j]}} \min(k, d_{\psi_1}^{[i][j]}) \sum_{s = \max(0, k - d_{\psi_2}^{[j][j]})} V_{\psi_1}^{[i]}(i, k-s) \Delta(i,j) U_{\psi_2}^{[j]}(j, s) \lambda^k. \quad (54)$$

Therefore, (53) is satisfied, if and only if for each $k = 0, 1, \ldots, d_{\psi_1}^{[i][j]} + d_{\psi_2}^{[j][j]}$

$$\sum_{s = \max(0, k - d_{\psi_2}^{[j][j]})} V_{\psi_1}^{[i]}(i, k-s) \Delta(i,j) U_{\psi_2}^{[j]}(j, s) = 0 \quad (55)$$

which is equivalent to

$$\text{vec} \left( \sum_{s = \max(0, k - d_{\psi_2}^{[j][j]})} V_{\psi_1}^{[i]}(i, s) \Delta(i,j) U_{\psi_2}^{[j]}(j, k-s) \right) = \left[ \min(k, d_{\psi_1}^{[i][j]}) \sum_{s = \max(0, k - d_{\psi_2}^{[j][j]})} U_{\psi_2}^{[j]}(j, k-s) \otimes V_{\psi_1}^{[i]}(i, s) \right] \text{vec}(\Delta(i,j)) = 0. \quad (56)$$

Assume now that the matrix $E_{\psi_2}^{[i][j]}(\lambda, i)$ is of FCR. Then (53)–(56) mean that (52) has a unique solution $\Delta(i,j) = 0$, and vice versa. It can therefore be declared from Theorem 1 that the condition that the matrix $E_{\psi_2}^{[i][j]}(\lambda, i)$ is of FCR for each $i, j = 1, 2, \ldots, N$, is both necessary and sufficient for the structure of the NDS $\Sigma$ being identifiable.

This completes the proof.

**Proof of Theorem 4:** From (22), we have that

$$G_{\psi_1}(\lambda, i) = H_{\psi_1}(\lambda, i) + H_{\psi_2}(\lambda, i) P(i) \times [I_{m_{\psi_1}} - H_{\psi_2}(\lambda, i) P(i)]^{-1} H_{\psi_2}(\lambda, i).$$

For a particular $\lambda \in C$, assume that there is a vector $\alpha$ satisfying $G_{\psi_1}(\lambda, i) \alpha = 0$. Define a vector $\beta$ as

$$\beta = P(i) \left[ I_{m_{\psi_1}} - H_{\psi_2}(\lambda, i) P(i) \right]^{-1} H_{\psi_2}(\lambda, i) \alpha. \quad (58)$$

Obviously, the vector $\beta$ can also be expressed as

$$\beta = [I_{m_{\psi_1}} - P(i) H_{\psi_2}(\lambda, i)]^{-1} P(i) H_{\psi_2}(\lambda, i) \alpha. \quad (59)$$

Hence, the vectors $\alpha$ and $\beta$ satisfy

$$\begin{bmatrix} H_{\psi_1}(\lambda, i) \\
P(i) H_{\psi_2}(\lambda, i) \end{bmatrix} \begin{bmatrix} \alpha \\
\beta \end{bmatrix} = 0. \quad (60)$$

On the other hand, from (20), it can be straightforwardly proven that

$$\begin{bmatrix} H_{\psi_1}(\lambda, i) \\
\beta \end{bmatrix} = \begin{bmatrix} 0 \\
-I_{m_{\psi_1}} \end{bmatrix} + [P(i) H_{\psi_2}(\lambda, i)]^{-1} \begin{bmatrix} C_{x_1}^{[0]}(\lambda, i) \\
F_x(i) \end{bmatrix} \begin{bmatrix} \alpha \\
\beta \end{bmatrix}.$$ (61)

in which the zero matrices in general have different dimensions. Define a vector $\xi$ as

$$\xi = \left( \lambda I_{m_{\psi_1}} - A_{xx}^{[0]}(\lambda, i) \right)^{-1} A_{xx}^{[0]}(\lambda, i) H_x(i) \alpha \beta. \quad (62)$$

Then we have that

$$\lambda I_{m_{\psi_1}} - A_{xx}^{[0]}(\lambda, i) - A_{xx}^{[0]}(\lambda, i) H_x(i) \alpha \beta = 0. \quad (63)$$

Moreover, from (60) and (61), as well as the definition of the vector $\xi$, direct matrix manipulations show that

$$\begin{bmatrix} C_{x_1}^{[0]}(\lambda, i) \\
C_{x_1}^{[0]}(\lambda, i) \\
H_x(i) \end{bmatrix} \begin{bmatrix} \alpha \\
\beta \end{bmatrix} = 0. \quad (64)$$

Combining (63) and (64) together, the definition of the matrix pencil $M(\lambda, i)$ leads immediately to the following equality:

$$M(\lambda, i) \text{col}\{\xi, \alpha, \beta\} = 0. \quad (65)$$

Assume now that the TFM $G_{\psi_1}(\lambda, i)$ is not of FCR. Then for an arbitrary $\lambda \in C$, there exists a nonzero vector $\alpha$ satisfying $G_{\psi_1}(\lambda, i) \alpha = 0$. The above arguments show that under such a situation, the corresponding vector $\text{col}\{\xi, \alpha, \beta\}$ with its subvectors $\beta$ and $\xi$ being defined, respectively, by (58) and (62), is also nonzero and satisfies (65). This means that the matrix pencil $M(\lambda, i)$ is not of FCR, also.

On the contrary, assume that the matrix pencil $M(\lambda, i)$ is not of FNC. Then for each $\lambda \in C$, there exists at least one nonzero vector $\xi$ such that $M(\lambda, i) \xi = 0$. Partition this vector $\xi$ as

$$\xi = \text{col}\{\xi, \alpha, \beta\} \quad (66)$$

with the subvectors $\xi$, $\alpha$, and $\beta$ having, respectively, $m_{\xi_1}$, $m_{\psi_1}$, and $m_{\psi_1}$ elements. On the basis of (65), direct algebraic manipulations show that the subvector $\alpha$ must not be a zero vector and satisfies $G_{\psi_1}(\lambda, i) \alpha = 0$. Hence, the TFM $G_{\psi_1}(\lambda, i)$ is also not of FNC.

This completes the proof.

**Proof of Corollary 2:** Substitute the KCF of (26) into (25), the following equality is obtained:

$$\bar{M}(\lambda, i) = \text{diag}\{U(i), I_{m_{\psi_1}}\} \bar{M}(\lambda, i) V(i) \quad (67)$$
in which
\[[\tilde{M}(\lambda, i) = \begin{bmatrix}
K(\lambda, i) & P(\lambda) \{F_0(i)N_y(i) + F_1(i)N_y(i) + G(i)N_w(i)\}V^{-1}(i) - N_w(i)V^{-1}(i)
\end{bmatrix}.\tag{68}\]

Note that both the matrix $U(i)$ and the matrix $V(i)$ are invertible and independent of the complex variable $\lambda$. It is obvious that the matrix pencil $M(\lambda, i)$ is of FNCR, if and only if the matrix pencil $\tilde{M}(\lambda, i)$ is. As the matrix pencil $F(i)$ is in fact the submatrix of the matrix pencil $\tilde{M}(\lambda, i)$ constituted from its first $m(i)$ columns, this means that the matrix pencil $\tilde{M}(\lambda, i)$ is of FNCR, only if the matrix pencil $\tilde{M}(\lambda, i)$ is.

On the contrary, assume that the matrix pencil $\tilde{M}(\lambda, i)$ is of FNCR. Then there exists at least one $\lambda_0 \in C$, such that for an arbitrary $m(i)$ dimensional nonzero complex vector $\xi$, the matrix $\tilde{M}(\lambda_0, i)$ satisfies $\tilde{M}(\lambda_0, i)\xi \neq 0$.

On the other hand, according to the Smith form of an MVP, there exist a nonnegative integer $m(i)$, an $(\sum_{j=1}^{s[i]} \xi_k(j) + m_{pi}) \times (\sum_{j=1}^{s[i]} \xi_k(j) + m_{pi})$ dimensional unimodular matrix $\tilde{U}(\lambda, i)$, an $m(i) \times m(i)$ dimensional unimodular matrix $\tilde{V}(\lambda, i)$, as well as nonzero and real coefficient polynomials $\tilde{\xi}^[(\lambda)](\xi)^{m(i)}$ with a finite degree, such that
\[
\tilde{M}(\lambda, i) = \tilde{U}(\lambda, i) \begin{bmatrix}
\text{diag} \left\{ \tilde{\xi}^[(\lambda)](\xi)^{m(i)} \right\} & 0 \\
0 & 0
\end{bmatrix} \tilde{V}(\lambda, i),\tag{69}\]
in which the zero matrices may not have the same dimension.

Assume that $m(i) < m(i)$. Divide the unimodular matrix $\tilde{V}(\lambda, i)$ as $\tilde{V}(\lambda, i) = \text{col}\{\tilde{V}_1(\lambda, i), \tilde{V}_2(\lambda, i)\}$ with $\tilde{V}(\lambda, i)$ having $m(i)$ rows. Then the sub-MVP $\tilde{V}_2(\lambda, i)$ is not empty. Moreover, from (69), we have that
\[
\tilde{M}(\lambda, i) = \tilde{U}(\lambda, i) \begin{bmatrix}
\text{diag} \left\{ \tilde{\xi}^[(\lambda)](\xi)^{m(i)} \right\} & 0 \\
0 & 0
\end{bmatrix} \tilde{V}_1(\lambda, i).\tag{70}\]

Note that $\tilde{V}(\lambda, i)$ is an unimodular matrix. There exists another unimodular matrix $\tilde{V}^{[\xi]}(\lambda, i)$ satisfying
\[
\tilde{V}(\lambda, i)\tilde{V}^{[\xi]}(\lambda, i) = \text{Im}(m(i)).\tag{71}\]
Partition the unimodular matrix $\tilde{V}^{[\xi]}(\lambda, i)$ as $\tilde{V}^{[\xi]}(\lambda, i) = [\tilde{V}^{[\xi]}_1(\lambda, i) \quad \tilde{V}^{[\xi]}_2(\lambda, i)]$ with $\tilde{V}^{[\xi]}_2(\lambda, i)$ having $m(i)$ rows of the $\tilde{M}(\lambda, i)$ columns. It can then be declared from (71) that the sub-MVP $\tilde{V}^{[\xi]}_2(\lambda, i)$ is of FCR at every $\lambda \in C$, and
\[
\tilde{V}_1(\lambda, i)\tilde{V}^{[\xi]}_2(\lambda, i) = 0.\tag{72}\]

Combining (69)–(72) together, we have that for an arbitrary $\lambda \in C$ and an arbitrary complex vector $\xi$ with an appropriate dimension
\[
\tilde{M}(\lambda, i)\tilde{V}^{[\xi]}(\lambda, i)\xi = \tilde{U}(\lambda, i) \begin{bmatrix}
\text{diag} \left\{ \tilde{\xi}^[(\lambda)](\xi)^{m(i)} \right\} & 0 \\
0 & 0
\end{bmatrix} \tilde{V}_1(\lambda, i)\tilde{V}^{[\xi]}_2(\lambda, i)\xi = 0.\tag{73}\]

This is a contradiction with the assumption that the matrix pencil $\tilde{M}(\lambda, i)$ is of FNCR. Hence, $m(i) = m(i)$. This means that when the matrix pencil $\tilde{M}(\lambda, i)$ is of FNCR, it is column rank deficient only at a finite number of $\lambda \in C$. Particularly, let $\Lambda_1(i)$ denote the set of the complex numbers at which the matrix pencil $\tilde{M}(\lambda, i)$ is column rank deficient. Then
\[
\Lambda_1(i) = \bigcup_{j=1}^{s[i]} \left\{ \lambda \mid \tilde{\xi}^[(\lambda)](\xi) = 0, \ \lambda \in C \right\}.\tag{74}\]

Let $\Lambda_2(i)$ denote the set of the complex numbers at which the matrix pencil $H_{j\xi_k(j)}(\lambda)$ is singular. Then from Lemma 4, this set also consists of only finitely many elements. On the other hand, Lemma 4 also reveals that the matrix pencils $K_{j\xi_k(j)}(\lambda)$ with $j = 1, 2, \ldots, \xi_k^{[i]}$ are not of FCR only at $\lambda = 0$, while all the matrix pencils $N_{j\xi_k^{[i]}(\lambda)}$ with $j = 1, 2, \ldots, \xi_k^{[i]}$ and $J_{j\xi_k^{[i]}(\lambda)}$ with $j = 1, 2, \ldots, \xi_k^{[i]}$ are of FCR at each $\lambda \in C$.

The above arguments show that if the matrix pencil $\tilde{M}(\lambda, i)$ is of FNCR, then for each
\[
\lambda \in C \setminus \{ \Lambda_1(i) \cup \Lambda_2(i) \cup \{0\} \}
\]
all the matrix pencils $H_{j\xi_k(j)}(\lambda)$, $K_{j\xi_k(j)}(\lambda)$, $N_{j\xi_k^{[i]}(\lambda)}$, and $J_{j\xi_k^{[i]}(\lambda)}$, as well as the matrix pencil $\tilde{M}(\lambda, i)$, are of FCR. As both the set $\Lambda_1(i)$ and the set $\Lambda_2(i)$ have only finitely many elements, the set $C \setminus \{ \Lambda_1(i) \cup \Lambda_2(i) \cup \{0\} \}$ is not empty. Hence, the existence of the desirable $\lambda$ is guaranteed.

From (67) and Lemma 2, as well as the block diagonal structure of the matrix pencil $K(\lambda, i)$, it can be further declared that at every $\lambda$ satisfying (75), the matrix pencil $\tilde{M}(\lambda, i)$ is of FCR, also.

This completes the proof. \hfill $\Diamond$

Proof of Theorem 5: Note that the requirement that the matrix pencil $M(\lambda, i)$ is of FNCR is equivalent to that the matrix pencil $\tilde{M}(\lambda, i)$ is of FNCR. On the other hand, from Lemma 4, we know that for each $j = 1, 2, \ldots, \xi_k^{[i]}$ and for an arbitrary $\lambda \in C$
\[
\text{null}(L_{\xi_k^{[i]}(\lambda)}(-\lambda)) = \begin{bmatrix} a_1 \lambda^{[i]}_{k=1} \\ a_2 \lambda^{[i]}_{k=1} \\ \vdots \\ a_c \lambda^{[i]}_{k=1} \end{bmatrix}, \quad a_j \in C.\tag{76}\]

Hence, for an arbitrary $\alpha$ satisfying $\text{diag}(L_{\xi_k^{[i]}(\lambda)}(\alpha)) = 0$, there certainly exist some complex numbers, denote them by $a_1, a_2, \ldots, a_{\xi_k^{[i]}}$, such that
\[
\alpha = \text{col}\left\{ a_1 \lambda^{[i]}_{k=1} \right\}, \quad a_j \in C.\tag{77}\]

On the contrary, direct matrix multiplications show that every vector $\alpha$ having an expression of (77) belongs to the null space of the matrix $\text{diag}(L_{\xi_k^{[i]}(\lambda)}(\alpha))_{j=1}^{\xi_k^{[i]}}$.

Denote the vector $\text{col}\{ a_j \lambda^{[i]}_{k=1} \}$ by $a$. From the above observations, it is straightforward to prove that the matrix pencil $\tilde{M}(\lambda, i)$ is of FNCR, if and only if there exists a $\lambda \in C$, such that
\[
[P(\lambda)\Theta(i) - \Pi(i)]\text{col}\left\{ a_1 \lambda^{[i]}_{k=1} \right\} = [P(\lambda)\Theta(i) - \Pi(\lambda, i)]a \neq 0.\tag{78}\]
for arbitrary complex numbers $a_1, a_2, \cdots$ and $a_{-1}$ that are not simultaneously equal to zero, which is equivalent to that $a \neq 0$. Therefore, the last inequality of (78) exactly means that the MNP $P(i)\Theta(i) - \Pi(i)$ is of FNCR.

On the other hand, from the definition of the integer $e[i]$, as well as those of the matrices $\Sigma_j(i)_{j=1}^{[i]}$ and $\Pi_j(i)_{j=1}^{[i]}$, it is obvious that

$$\begin{align*}
\lbrack P(i)\Theta(i) - \Pi(i) \rbrack & \col \begin{bmatrix} a_j \begin{bmatrix} 1, \lambda \end{bmatrix}^{[i]}_{k=1} \end{bmatrix}_{j=1}^{e[i]} \\
\end{align*}$$

It can therefore be declared that if the matrix pencil $\tilde{M}(\lambda, i)$ is of FNCR, then for every $a \neq 0$

$$\begin{align*}
\sum_{j=1}^{[i]} a_j \lbrack P(i)\Sigma_j(i) - \Pi_j(i) \rbrack \neq 0. \quad (80)
\end{align*}$$

Note that, vec$\left( \sum_{j=1}^{[i]} a_j [ P(i)\Sigma_j(i) - \Pi_j(i) ] \right) = \Gamma(i)a$. The inequality of (80) implies that the matrix $\Gamma(i)$ is of FCR. This completes the proof.

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