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On some group properties of heat and mass transfer equations

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Abstract. Heat and mass transfer equations with variable transport coefficients are under study. The forms of unknown thermal conductivity, diffusion and Dufour coefficients are found by means of Lie group theory. It is shown that arbitrary elements have the power-law, logarithmic and exponential dependencies on temperature and concentration.

1. Introduction
The object of the study is the evolution equations describing heat and mass transfer in binary mixtures of liquids and gases. These equations in their classical form were known many years ago [1] and they were widely used in mathematical modelling. They are treated by means of numerical, qualitative and asymptotic analysis up to now. The Cauchy and boundary value problems for such equations are imposed and solved in a wide variety of statements. Many of these problems are included in the classical textbooks on mathematical physics.

In recent 20 years mathematical models of heat and mass transfer became more complex [2]. It is caused by high requirements to accuracy for describing physical processes and by the rapid development of computers as well. The classical heat and mass transfer models take into account two main processes. These are thermal conductivity and diffusion. Moreover, equation for temperature may not be connected with equation for mass distribution. In fact, the heat and mass transfer is more complex phenomena. Convection and cross effects influence essentially. Furthermore, for some cases it is impossible to avoid dependence of transport coefficients on temperature and concentration. Modern development of experimental equipment and computers allow taking into account collateral effects mentioned above more accurately.

We pay attention to study of heat and mass transfer equations taking into consideration Soret and Dufour effects. The Soret effect (or thermal diffusion) is the molecular transport of mass caused by the thermal gradient in a fluid mixture while the Dufour effect (or diffusive thermal conductivity) is the heat flux caused by mass concentration gradient in a fluid mixture. These effect are often called reciprocal ones. From the mathematical point of view temperature and concentration equations with these effects became essentially nonlinear and they can not be considered separately.

For the investigation of qualitative properties of these equations we need a powerful mathematical method. We use Lie symmetry analysis as one of the universal tool for the study of differential equations of arbitrary form. The basic principals of this theory and many examples of its application are described in books [3] and [4]. The number of papers devoted to analysis of
simple and complex temperature and diffusion equations by means of Lie symmetry technique is growing permanently. Starting with the example of nonlinear thermal conductivity equation in book [3] and results of the preprint [5] many successive papers deal with the mathematical models with variable transport coefficients. Various classes of diffusion–convection–reaction equations with different types of coefficient dependence on unknown function are investigated in [6]–[8] and in the works cited therein. Papers [9] and [10] should be mention as the results of analysis of the group properties and exact solutions of nonlinear heat transfer with delay.

It is necessary to note that there are not so many papers devoted to the study of systems of heat and mass transfer equations. Nevertheless, the group properties of nonlinear reaction–diffusion equations with constant coefficients are studied in [11]. Group classification problem for constant and variable transport coefficients is solved for general convection equations with five arbitrary parameters in [12] and [13] respectively. The results of symmetry analysis of the heat and mass transfer equations taking into account the Soret effect without convection terms are presented in [14].

In this paper we study symmetry properties of heat and mass transfer equations which can not be considered separately. They are connected with each other by means of cross-effects. We find the forms of arbitrary transport coefficients and calculate admissible transformations of dependent and independent variables. These transformations allow to construct new exact solutions of the governing equations. The paper is organized as follows. In Section 2 we describe the mathematical model of mentioned above process. In Section 3 Lie symmetry analysis of the governing equations is carried out. The determining and classifying equations are derived. They are analyzed in detail in Section 4. The results of group classification are gathered into a table for the reader’s convenience.

2. Governing equations

Due to taking into account Soret and Dufour effect the thermal and diffusive fluxes have the sophisticated forms

\[ J_c = -\rho(D\nabla C + D^\theta \nabla T) \quad \text{and} \quad J_q = -(\kappa \nabla T + D^F \nabla C) \]

respectively [1].

Here thermal diffusion \((D^\theta)\) and Dufour \((D^F)\) coefficients are presented in the most generalized forms as the functions of temperature \(T\) and concentration of the lighter component \(C\), \(\rho\) is the density of the mixture. According to heat and mass transfer theory the mathematical model of this process can be written in the form

\[ \frac{\partial T}{\partial t} = \text{div} \left( \kappa(T, C) \nabla T + D^F(T, C) \nabla C \right), \quad (1) \]

\[ \frac{\partial C}{\partial t} = \text{div} \left( D(T, C) \nabla C + D^\theta(T, C) \nabla T \right). \quad (2) \]

Here \(t\) is the time, \(\nabla = (\partial_{x^1}, \partial_{x^2}, \partial_{x^3})\), \((x^1, x^2, x^3)\) is the coordinate vector. The thermal conductivity \(\kappa(T, C)\) and diffusion coefficient \(D(T, C)\) are positive and non-zero. The Dufour coefficient \(D^F\) is positive and the thermal diffusion coefficient \(D^\theta\) has an arbitrary sign.

Dependence of transport coefficients on temperature and concentration and the cross effects make the governing equations new and interesting for analysis. On the basis of information about symmetry properties of equations (1) and (2) the heat and mass transfer processes can be studied more thoroughly.
3. Derivation of determining and classifying equations

The classical method for group analysis is properly described in well-known monographs of L.Ovsiannikov [3] and P.Olver [4]. Here we apply this technique step by step omitting cumbersome formulas given in the above mentioned books. We search for an admissible generator in the form

\[ X = \xi^0 \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i} + \xi^T \frac{\partial}{\partial T} + \eta^1 \frac{\partial}{\partial C} + \eta^2 \frac{\partial}{\partial T}, \]

where coefficients \( \xi^i, \ i = 0, \ldots, 3, \) and \( \eta^j, \ j = 1, 2, \) depend on dependent \( T, C \) and independent \( t, x^i, \ i = 1, 2, 3, \) variables.

We split the resulting equations with respect to the mixed partial derivatives of the third order and obtain that

\[ \xi^0 = \xi^0(t), \quad \xi^i = \xi^i(t, x^i), \quad i = 1, 2, 3. \]

Furthermore the coefficients at the mixed partial derivatives of the second order satisfy the Killing equations

\[ \xi^2_{x^i} + \xi^1_{x^2} = 0, \quad \xi^3_{x^1} + \xi^1_{x^3} = 0, \quad \xi^3_{x^2} + \xi^2_{x^3} = 0. \]  \hfill (3)

Splitting with respect to derivatives \( T_{x^i x^j}, \) and \( C_{x^i x^j}, \ i = 1, 2, 3, \) gives the following equations

\[ \xi^1_{x^1} - \xi^2_{x^2} = 0, \quad \xi^1_{x^1} - \xi^3_{x^3} = 0, \quad \xi^2_{x^2} - \xi^3_{x^3} = 0. \]  \hfill (4)

We also obtain that

\[ \eta^1_{TT} = \eta^1_{TC} = \eta^1_{CC} = 0, \quad \eta^2_{TT} = \eta^2_{TC} = \eta^2_{CC} = 0 \]  \hfill (5)

if the following inequalities are fulfilled

\[ D(D + \kappa) - D^\theta D^F \neq 0, \quad D^F \neq 0. \]

Equations (3)–(5) are determining and give preliminary information about the coefficients of the generator \( X. \) The functions \( \xi^i, \ i = 0, \ldots, 3, \) and \( \eta^i, \ j = 1, 2, \) are connected with the parameters of the governing equations through the following equalities

\[ -\xi^i_t = D\left(2\eta^2_{C x^i} - \sum_{j=1}^{3} \xi^j_{x^j x^i}\right) + 2D^\theta \eta^1_{C x^i} + \left(\frac{\partial D}{\partial T} + \frac{\partial D^\theta}{\partial C}\right)\eta^1_{x^i} + 2\frac{\partial D}{\partial C} \eta^2_{x^i}, \]  \hfill (6)

\[ -\xi^i_t = \kappa\left(2\eta^1_{T x^i} - \sum_{j=1}^{3} \xi^j_{x^j x^i}\right) + 2D^F \eta^2_{T x^i} + \left(\frac{\partial \kappa}{\partial C} + \frac{\partial D^F}{\partial T}\right)\eta^2_{x^i} + 2\frac{\partial \kappa}{\partial T} \eta^1_{x^i}, \]  \hfill (7)

\[ 2D\eta^2_{T x^i} + 2D^\theta \eta^1_{T x^i} + D^\theta \sum_{j=1}^{3} \xi^j_{x^j x^i} + \left(\frac{\partial D}{\partial T} + \frac{\partial D^\theta}{\partial C}\right)\eta^1_{x^i} + 2\frac{\partial D^\theta}{\partial T} \eta^2_{x^i} = 0, \]  \hfill (8)

\[ 2\kappa \eta^1_{C x^i} + 2D^F \eta^2_{C x^i} + D^F \sum_{j=1}^{3} \xi^j_{x^j x^i} + \left(\frac{\partial \kappa}{\partial C} + \frac{\partial D^F}{\partial T}\right)\eta^1_{x^i} + 2\frac{\partial D^F}{\partial T} \eta^2_{x^i} = 0, \]  \hfill (9)

\[ \kappa(2\xi^1_{x^i} - \xi^0_t) + D^\theta \eta^1_{C} - D^F \eta^2_T = \eta^1 \frac{\partial \kappa}{\partial T} + \eta^2 \frac{\partial \kappa}{\partial C}. \]  \hfill (10)
and classifying equations (6) – (15). It results 
transfer coefficients we should vary the arbitrary elements and split with respect to them in the 

In order to find the common part of Lie symmetries for equations (1) and (2) with arbitrary 
i

There are 18 classifying equations in the above system because \( i = 1, 2, 3 \) in equations (6)–(9). 
In order to find the common part of Lie symmetries for equations (1) and (2) with arbitrary 
transfer coefficients we should vary the arbitrary elements and split with respect to them in the 
classifying equations (6) – (15). It results 

\[
\eta^1 = 0, \quad \eta^2 = 0
\]

and 

\[
\begin{align*}
\xi^0 &= c_0 t + c_1, \quad \xi^1 = c_2 x^1 + c_3 x^2 + c_4 x^3 + c_5, \\
\xi^2 &= -c_3 x^1 + c_2 x^2 + c_6 x^3 + c_7, \quad \xi^3 = -c_4 x^1 - c_6 x^2 + c_1 x^3 + c_8,
\end{align*}
\]

where \( c_i, i = 0, \ldots, 8 \), are arbitrary constants. Consequently, assuming that one of constant 
c\( i \) is equal to unit and the other constants are zero we obtain eight generators of the basic Lie 
algebra. They have the following form 

\[
L^0 = \{ \partial_t, \partial_{x^i}, 2t\partial_t + \sum_{i=1}^3 x^i \partial_{x^i}, x^j \partial_{x^i} - x^i \partial_{x^j}, \ i, j = 1, 2, 3, \ i \neq j. \}
\]

We have three types of transformations in the basic Lie group generated by the basic algebra 
\( L^0 \). These are transition with respect to time and spatial variables, dilatation of independent 
variables and rotations. Such transformations are typical for many equations in fluid mechanics. 
It is interested to note that there is no generator depending on arbitrary functions in the basic 
group.

Further we make some assumptions. Firstly, we suppose that the thermal diffusion coefficient 
\( D^\theta \) vanishes for simplification of the problem. Such statement of the problem is justified at study 
of gas mixtures in which the Dufour effect is more essential then in liquid ones. Secondly, we 
consider the classified functions depending on two variables \( T \) and \( C \). The cases of dependencies 
on \( T \) or \( C \) separately will be studied in further works. Thereby we solve the group classification 
problem with respect to the functions \( D(T, C), \kappa(T, C) \) and \( D^F(T, C) \).
4. Analysis of the classifying equations

In this section we solve the group classification problem for equations (1) and (2) at \(D^0 \equiv 0\). We should find all possible forms of functions \(D\), \(\kappa\) and \(D^F\) and extensions of the basic Lie algebra \(L^0\).

It is necessary to note that equations (1) and (2) do not change their differential forms under following linear transformations

\[ T' = AT + H, \quad C' = AC + G, \tag{18} \]

where \(A \neq 0\), \(H\) and \(G\) are constant. All of the classified functions are found with respect of transformations (18).

Under the assumption \(D^0 = 0\) equation (12) takes the form

\[ (\kappa - D)\eta_2^2 = 0. \]

If \(\kappa = D\) we subtract equation (10) from equation (11) and obtain \(\eta_2^2 = 0\) also due to \(D^F \neq 0\). That is why we may consider the case \(\eta_2^2 = 0\) only. Then three equations (8) transform into

\[ \frac{\partial D}{\partial T}\eta_i^2 = 0, \quad i = 1, 2, 3. \]

Since we consider the case when \(D\) depends on \(T\) and \(C\) essentially then \(\eta^2\) does not depend on \(x^i\), \(i = 1, 2, 3\), and \(t\) due to equation (15). We should find other coordinates of the generator \(X\). We differentiate equation (11) with respect to \(x^i\), \(i = 1, 2, 3\), and subtract from equations (6) respectively. Further manipulations with obtained equalities lead to vanishing of all third order derivatives of \(\xi^1\), \(\xi^2\), \(\xi^3\) with respect to \(x^1\), \(x^2\) and \(x^3\). Using equations (3) and (4) we find out that the coordinates of the generator \(X\) satisfy equations (16) and (17). Corresponding to (5) the coefficients \(\eta^2\) and \(\eta^1\) are linear with respect to \(T\) and \(C\). They are calculated from equation (6):

\[ \eta^1 = c_9 T + c_{10} C + c_{11}, \quad \eta^2 = c_{12} C + c_{13}. \]

System of classifying equations (6)–(15) reduces to three equations

\[ \kappa(c_0 - 2c_2) + (c_9 T + c_{10} C + c_{11})\frac{\partial \kappa}{\partial T} + (c_{12} C + c_{13})\frac{\partial \kappa}{\partial C} = 0, \]

\[ D(c_0 - 2c_2) + (c_9 T + c_{10} C + c_{11})\frac{\partial D}{\partial T} + (c_{12} C + c_{13})\frac{\partial D}{\partial C} = 0, \tag{19} \]

\[ D^F(c_0 - 2c_2 + c_{12} - c_9) + (c_9 T + c_{10} C + c_{11})\frac{\partial D^F}{\partial T} + (c_{12} C + c_{13})\frac{\partial D^F}{\partial C} + (\kappa - D)c_{10} = 0. \]

For solution of the group classification problem we should consider equations (19) in an general form

\[ A_i F_i + (A_2 T + A_3 C + A_4)\frac{\partial F_i}{\partial T} + (A_5 C + A_6)\frac{\partial F_i}{\partial C} = 0, \quad i = 1, 2, \]

\[ F_3(A_1 + A_5 - A_2) + (A_2 T + A_3 C + A_4)\frac{\partial F_3}{\partial T} + (A_5 C + A_6)\frac{\partial F_3}{\partial C} + (F_1 - F_2)A_3 = 0, \]

where \(F_1 = \kappa\), \(F_2 = D\) and \(F_3 = D^F\).
The solution of the equations considered depends on the coefficients $A_i$, $i = 1, \ldots, 6$. These coefficients are not equal to zero simultaneously. The basic role belongs to $A_2$ and $A_6$. If they are equal to each other or equal to zero we have different forms for functions $F_i$, $i = 1, 2, 3$. In this way we should study the following cases.

1. Let $A_2 \neq A_5$.

1.1 If $A_2 \neq 0$ and $A_5 \neq 0$ then we set $A_4 = A_6 = 0$ using transformation (18).  
1.2 If $A_2 = 0$ and $A_5 \neq 0$ then we set $A_6 = 0$ using transformation (18).  
1.3 If $A_2 \neq 0$, $A_5 = 0$ then we set $A_4 = 0$ using transformation (18).  
1.3.1 $A_6 \neq 0$.  
1.3.2 $A_6 = 0$.

Cases 1.3.1 and 1.3.2 give different forms for classified functions.  
2. Another case is $A_2 = A_5$ then we have three different variants.  
2.1 If $A_2 \neq 0$ then we can set $A_4 = A_6 = 0$ using transformation (18).  
2.2 If $A_2 = 0$ and $A_6 \neq 0$ we can not use transformation (18).  
2.3 If $A_2 = 0$ and $A_6 = 0$ then we can set $A_4 = 0$ if $A_3 \neq 0$ using transformation (18).

As an example one of this cases is described here in detail. We consider case 1.3.1. After integrating equations for $F_i$, $i = 1, 2, 3$, we have

$$F_i = f_i(w')e^{-A_1 C/A_6}, \quad i = 1, 2,$$

$$F_3 = \left(\frac{A_3}{A_2} e^{- \frac{A_2 C}{A_6} (f_1(w') - f_2(w')) + f_3(w')}\right) e^{(\frac{-A_1}{A_6} + \frac{A_2}{A_6}) C},$$

where $w' = (T + A_3 C/A_2 + A_2 A_4 + A_3 A_6)e^{-A_2 C/A_6}$ is the argument of arbitrary functions $f_1$, $f_2$ and $f_3$. We chose $A = 1$ and $H = A_2 A_4 + A_3 A_6$ in transformation (18) for variable $T$ and obtain simpler form for argument $w'$

$$w = \left(T + \frac{A_3}{A_2} C\right)e^{- \frac{A_2 C}{A_6}}.$$  

Using notation

$$\frac{A_3}{A_2} = \alpha, \quad -\frac{A_2}{A_6} = \varepsilon, \quad -\frac{A_1}{A_6} = \gamma$$

we have the unknown functions in the form

$$\kappa = f_1(w)e^{\gamma C}, \quad D = f_2(w)e^{\gamma C}, \quad D^F = (\alpha\varepsilon e^{C}(f_1(w) - f_2(w)) + f_3(w))e^{(\gamma - \varepsilon) C}.$$

Further we should substitute these forms of classified functions into equations (19) and calculate constants $c_i$, $i = 0, 2, 9 - 13$. Due to functions $f_i$, $i = 1, 2, 3$, are arbitrary we can split resulting equations and obtain

$$c_0 = 2c_2 - \gamma c_{13}, \quad c_9 = -\varepsilon c_{13}, \quad c_{10} = -\alpha \varepsilon c_{13}, \quad c_{11} = -\alpha c_{13}, \quad c_{12} = 0.$$  

Thereby all constants are eliminated by means of $c_{13}$. When $c_{13} = 1$ and other constants are zero in equations (16) and (17) we have the following additional generator

$$-\gamma t \partial_t + \partial_C - \alpha \varepsilon C \partial_T - \varepsilon T \partial_T - \alpha \partial_T$$

which extents the basic Lie group.

Repeating the same procedure for the other six cases we get the exhausted result of group classification. Classified functions and admissible generators are gathered in Table 1, where the
constants $\alpha$, $\beta$, $\gamma$, $\varepsilon$ are arbitrary ones such that $\varepsilon \neq 0$, $\delta \neq 0$, $-1$. The functions $f_i = f_i(w)$, $i = 1, 2, 3$, are arbitrary smooth non-zero functions. There is their argument $w$ in the last column of Table 1. The eight generators of the basic Lie algebra $L^0$ are included into the third column of Table 1 for all presented cases. The following generators are involved to extensions of the Lie basic algebra $L^0$:

$$Z = t\partial_t, \quad T^1 = T\partial_T, \quad T^3 = \partial_T, \quad C^1 = C\partial_C, \quad C^2 = T\partial_C, \quad C^3 = \partial_C.$$  

As one can see the generators from the third column of Table 1 are the linear combinations of generators of time, temperature and concentration dilatation extent $L^0$ as well as temperature and concentration transition. There is the generator $T^2$ in this set also. It occurs due to taking into account Dufour effect in the governing equations. This generator displays relations between diffusion and thermal conductivity processes.

**Table 1.** Results of group classification for essential dependence of classified functions on $T$ and $C$.

| $\kappa$, $D$ | $D^F$ | generators | $w$ |
|----------------|--------|------------|-----|
| $e^{T/C} f_i$ | $e^{T/C}(f_2 - f_1) + f_3$ | $\gamma Z - T^2$ | $C$ |
| $e^{C} f_i$ | $e^C(\alpha(T_f - f_1) + f_3)$ | $\gamma Z - \alpha T^2 + \beta T^3$ | $T + \alpha C^2/2 - \beta C$ |
| $e^{C} f_i$ | $(\alpha e^C f_1 - f_2) + (T/\alpha C^2 + \beta T^3)$ | $\gamma Z - \alpha T^2 + \beta T^3$ | $T + \alpha C^2/2 - \beta C$ |
| $C^i f_i$ | $C^i(\alpha(T_f - f_1) + f_3)$ | $\gamma Z - \alpha T^2 + \beta T^3$ | $T + \alpha C^2/2 - \beta C$ |
| $C^i f_i$ | $\alpha C^i(f_1 - f_2) + C^{\gamma - 1} f_3$ | $\gamma Z + (\alpha + \delta T^1) + \delta T^1$ | $(T + \alpha C) C^\delta$ |
| $C^i f_i$ | $C^i(\beta f_1 - f_2) \ln C + f_3$ | $\gamma Z - \alpha T^2 - T^1$ | $T/C + \beta \ln C$ |

| $(T + \alpha C) f_i$ | $(T + \alpha C) \varepsilon (\alpha f_1 - f_2) + f_3(T + \alpha C)$ | $\varepsilon Z - \alpha T^2 - T^1$ | $C$ |

5. Conclusion
The equations describing heat and mass transfer in a binary mixture with thermodiffusion and diffusive thermal conduction are considered. Group classification with respect to three physical parameters is performed. The transformations left unchanged the structure of the governing equations are used for simplification of the form of classified functions. It should be noted that on the basis of group classification the governing equations can be studied more thoroughly with the use of the symmetry properties incorporated in these equations. New exact solutions and generalizations of the known solutions can be obtained.

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