Correlations for pairs of periodic trajectories for open billiards

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Abstract

In this paper we prove two asymptotic estimates for pairs of closed trajectories for open billiards similar to those established by Pollicott and Sharp (2006 *Invent. Math.* 163 10–24) for closed geodesics on negatively curved compact surfaces. The first of these estimates holds for general open billiards in any dimension. The more intricate second estimate is established for open billiards satisfying the so-called Dolgopyat type estimates. This class of billiards includes all open billiards in the plane and open billiards in $\mathbb{R}^N$ ($N \geq 3$) satisfying some additional conditions.

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1. Introduction

In [PoS2] Pollicott and Sharp prove some interesting asymptotic estimates concerning the distribution of lengths $\lambda(\gamma)$ of closed geodesics $\gamma$ on a compact surface $V$ of negative curvature. Given a finite symmetric set $S$ of generators of the fundamental group $\pi_1(V)$, for each closed geodesic $\gamma$ on $V$ let $|\gamma|$ denote the minimal number of elements of $S$ needed to write down an element of $\pi_1(V)$ conjugate to $\gamma$. Given real numbers $a, b$, the first asymptotic estimate in [PoS2] concerns the number $\pi(n, [a, b])$ of all pairs $(\gamma, \gamma')$ of closed geodesics with $|\gamma|, |\gamma'| \leq n$ and $a \leq \lambda(\gamma) - \lambda(\gamma') \leq b$, and has the same form as that in theorem 1 below except that the constant $h_0$ that appears in [PoS2] depends on the set of generators $S$. The second asymptotic in [PoS2] is much more delicate and involves a family of intervals $I_n = [z + \epsilon_n a, z + \epsilon_n b], z \in \mathbb{R}$, where $\epsilon_n \to 0$ subexponentially (see the formula in theorem 2 below which has the same form as the one in [PoS2]). In the proof of this a crucial role is
played by Dolgopyat’s estimates [D] which apply to any Anosov flow on a compact surface (and also to some Anosov flows on higher dimensional compact manifolds), and in particular to geodesic flows on compact surfaces of negative curvature.

In this paper we prove similar asymptotic estimates for the billiard flow in the exterior of several strictly convex bodies in $\mathbb{R}^N (N \geq 2)$ having smooth boundaries and satisfying the so-called no-eclipse condition (H) defined below. In this case there is a natural coding for the closed trajectories using configurations (admissible sequences) of convex bodies and the constant $h_0 > 0$ that appears in the asymptotic formulae is just the topological entropy of the billiard ball map from boundary to boundary, and $h_0$ depends only on the number of obstacles (see section 4). The first asymptotic estimate (theorem 1) holds for any open billiard in any dimension. As in [PoS2], the second asymptotic (theorem 2) relies heavily on Dolgopyat type (see section 4). The first asymptotic estimate (theorem 1) holds for any open billiard in any

**Corollary:**

Correlations for periods of periodic orbits have been studied earlier in the physical literature. It appears [Aal] was the first paper in this area, where some conjectures were made and numerical results for three chaotic systems were described. We refer to [PoS2] for other references concerning problems and results on correlations for periodic orbits in the physical and mathematical literature.

We now proceed to state precisely the results in this paper.

Let $K$ be a subset of $\mathbb{R}^N (N \geq 2)$ of the form $K = K_1 \cup K_2 \cup \ldots \cup K_{k_0}$, where $K_i$ are compact strictly convex disjoint domains in $\mathbb{R}^N$ with $C^r (r \geq 3)$ boundaries $\Gamma_i = \partial K_i$ and $k_0 \geq 3$. Set $\Omega = \mathbb{R}^N \setminus K$. Throughout this paper we assume that $K$ satisfies the following (no-eclipse) condition:

\[
\text{(H)} \quad \text{for every pair } K_i, K_j \text{ of different connected components of } K \text{ the convex hull of } K_i \cup K_j \text{ has no common points with any other connected component of } K.
\]

With this condition, the billiard flow $\phi_t$ defined on the cosphere bundle $S^* (\Omega)$ in the standard way is called an open billiard flow. It has singularities; however, its restriction to the non-wandering set $\Lambda$ has only simple discontinuities at reflection points. Moreover, $\Lambda$ is compact, $\phi_t$ is hyperbolic and transitive on $\Lambda$, and it follows from [St1] that $\phi_t$ is non-lattice and therefore by a result of Bowen [B], it is topologically weak-mixing on $\Lambda$.

Given a periodic billiard trajectory (ray) $\gamma$ in $\Omega$, denote by $d_\gamma$ the period (return time) of $\gamma$ and by $T_\gamma$ the primitive period (length) of $\gamma$. For any configuration $j = (j_1, j_2, \ldots, j_m)$ with $j_\mu \in \{1, \ldots, k_0\}$, $j_\mu \neq j_{\mu+1}$ and $|j| = m$, there exists a unique periodic reflecting ray $\gamma$ with reflecting points on $\Gamma_{j_1, \ldots, \Gamma_{j_m}}$ following the configuration $j$ (see [I1, PS1]) and we set $|\gamma| = |j| = m$. We denote by $m_\gamma$ the number of reflections of $\gamma$ and by $P_\gamma$ the linear Poincaré map related to $\gamma$ (see [PS1]). For $a < b$ consider

\[
\pi(n, [a, b]) = \#(\gamma, \gamma') : |\gamma|, |\gamma'| \leq n, a \leq T_\gamma - T_{\gamma'} \leq b
\]

and denote by $h_0 > 0$ the topological entropy of the billiard ball map related to $\phi_t$ which coincides with the topological entropy of the shift map $\sigma$ on a naturally defined symbol space $\Sigma_\Lambda$ (see section 2.2 for the notation). This implies $h_0 = \log \lambda$, where $\lambda > 1$ is the maximal positive simple eigenvalue of the matrix $A$, so $h_0$ depends only on the number $k_0$ of connected
components of $K$. Then we have
\[
\#\{y : |y| \leq n\} \sim \frac{\epsilon_n^h}{e^{h_0} - 1} n^\beta, \quad n \to +\infty.
\]

Our first result is the following theorem.

**Theorem 1.** There exists $\beta > 0$ such that for any $a < b$ we have
\[
\pi(n, [a, b]) \sim \frac{(b - a)\epsilon_n^h}{(2\pi)^{1/2} \beta (e^{h_0} - 1)^2} n^{5/2}, \quad n \to +\infty.
\]

This estimate is derived from lemma 1 in section 4 and the argument from the proof of the first result in [PoS2].

To obtain a more precise result we use the Dolgopyat type estimates (3.14) from section 3.3. Consider a sequence of intervals $I_n(z) = [z + \epsilon_n a, z + \epsilon_n b]$, where $\epsilon_n \to 0$. We say that $\epsilon_n$ goes to 0 subexponentially if $\limsup_{n \to +\infty} |\log \epsilon_n|/n = 0$. Using the notation of sections 2 and 3, the second result in this paper is the following analogue of theorem 2 in [PoS2].

**Theorem 2.** Assume that the estimates (3.14) hold for the Ruelle operator $L_{irr-h_0}$, $|f| \geq l_0 > 1$, and that the strong stable and the strong unstable laminations $\{W^s_k(x)\}_{k \in \Lambda}$ and $\{W^u_k(x)\}_{k \in \Lambda}$ are Lipschitz in $x \in \Lambda$. Then there exists $\beta > 0$ such that for any $a < b$ and for every sequence $\epsilon_n$ going to 0 subexponentially, we have
\[
\lim_{n \to +\infty} \sup_{z \in \mathbb{R}} \frac{\beta h^{5/2} n}{\epsilon_n e^{2h_0}} \pi(n, I_n(z)) - \frac{(b - a)\epsilon_n^h}{(2\pi)^{1/2} \beta (e^{h_0} - 1)^2} e^{-z^2/2h^2} = 0.
\]

In particular, for any fixed $z \in \mathbb{R}$ we get
\[
\pi(n, I_n(z)) \sim \frac{(b - a)\epsilon_n^h}{(2\pi)^{1/2} \beta (e^{h_0} - 1)^2} e^{2h_0} n^{5/2}, \quad n \to +\infty.
\]

Note that the assumptions of theorem 2 are satisfied for $N = 2$ without any additional geometric conditions. The central point in the proof of theorem 2 is lemma 4 in section 4, where the Dolgopyat type estimates are used. We should note that lemma 4 deals with Ruelle transfer operators defined by means of the symbolic coding $\Sigma_A^+$ using the connected components $K_j$ of $K$ (see section 2 for the notation). This coding is very natural and most convenient for a variety of problems where the open billiard flow is involved. However, $\Sigma_A^+$ does not have the properties of a coding by means of a Markov family and therefore Dolgopyat type estimates do not automatically apply to it. One needs to make a transition from one coding to the other, and this is done in sections 2 and 3. As a result of this one identifies a class of functions on the symbolic model $\Sigma_A^+$ to which the estimates (3.14) can be applied. Unfortunately, this class of functions is not of the form $F_\theta(\Sigma_A^+)$ for some $\theta > 0$, and it is very doubtful that Dolgopyat type estimates would apply to all functions in $F_\theta(\Sigma_A^+)$. This rather serious difficulty appears also in the analysis of the approximation of the cut-off resolvent of the Dirichlet Laplacian in [PS2], where we deal with phases and amplitudes related to the connected components $K_j$ of $K$ (and not to Markov sections $R_j$) to build an approximation. To overcome it, we use a link between the Ruelle operators $L_{-s f \hat{g}}$ and $L_{-s r \hat{g}}$ corresponding to each other via the transition from one coding to the other. The dynamical zeta function related to the billiard flow is independent of the choice of coding and by using thermodynamic formalism (see, e.g. [PP]) one can show that the eigenvalues of the operators $L_{-s f \hat{g}}$ and $L_{-s r \hat{g}}$ coincide with their multiplicities. However, this does not imply similar estimates for the norms of the iterations of these operators. To get such estimates we use the explicit link between $L_{-s f \hat{g}}$ and $L_{-s r \hat{g}}$ established in proposition 4 for a special class of functions. From the analysis in section 4 we need to have estimates for the operators $L_{-s f \hat{g}}$. 

...
Proposition 1. \( \partial K_i \) is using the boundary components \( \tilde{\eta} \) and this corresponds to the case when \( g = -h_0 \). It is important to note that we may code the periodic rays using a Markov family; however, different Markov families will lead to rather different symbolic models, possibly with different number of symbols. The coding using the connected components of \( K \) is very natural and has a clear geometrical and physical meaning.

In the proofs of theorems 1 and 2 (see section 4) we use the analytic arguments in [PoS2] with minor changes, so we omit the details. Most of the arguments below concern the natural symbolic model for the open billiard flow. On the other hand, our result in section 3 shows that there exists a class of functions for which the Dolgopyat estimates hold for the Ruelle operator related to the coding with obstacles and this has been applied in [PS2] in the analysis of the analytic continuation of the cut-off resolvent of the Laplacian.

In section 5 we discuss some open problems related to correlations of pairs of periodic trajectories when \( \epsilon_n \) goes to 0 faster than \( e^{-h_0 n} \) and their relationship with some separation conditions.

2. Symbolic codings

In this section we compare the Ruelle transfer operators related to two different codings of the billiard flow \( \phi_t \) on \( \Lambda \)—the first of these is related to a Markov family for \( \Lambda \), while the second is using the boundary components \( \partial K_i \).

Fix a large ball \( B_0 \) containing \( K \) in its interior. For any \( x \in \Gamma = \partial K \) we will denote by \( v(x) \) the outward unit normal to \( \Gamma \) at \( x \).

It follows from lemma 3.1 in [II] that there exist \( \delta_1 > 0 \) and \( 0 < d_0 < \frac{1}{2} \min_{\nu, x} \text{dist}(K_i, K_j) \) such that for any \( i = 1, \ldots, k_0, x \in \Gamma_i \) and \( \xi \in S^{n-1} \) with \( 0 \leq (\xi, v(x)) \leq \delta_1 \) and \( \eta = \xi - 2(\xi, v(x))v(x) \), at least one of the rays \( \{x + t\xi : t \geq 0\} \), \( \{x + t\eta : t \leq 0\} \) has no common points with \( \cup_{j \neq i} B(K_j, d_0) \), where

\[
B(A, d_0) = \{y + w : y \in A, w \in \mathbb{R}^N, \|w\| \leq d_0\}.
\]

Let \( z_0 = (x_0, u_0) \in S^*(\Omega) \). Denote by

\[
X_1(z_0), X_2(z_0), \ldots, X_m(z_0), \ldots
\]

the successive reflection points (if any) of the forward trajectory

\[
\gamma_+(z_0) = \{\text{pr}_1(\phi_t(z_0)) : 0 \leq t\}.
\]

Similarly, we will denote by \( \gamma_-(z_0) \) the backward trajectory determined by \( z_0 \) and by

\[
\ldots, X_{-m}(z_0), \ldots, X_{-1}(z_0), X_0(z_0)
\]

its backward reflection points (if any). If \( \gamma(z_0) = \gamma_+(z_0) \cup \gamma_-(z_0) \) is bounded (i.e. it has infinitely many reflection points both forwards and backwards), we will say that it has an itinerary \( \eta = (\eta_j)_{j=-\infty}^{\infty} \) (or that it follows the configuration \( \eta \)) if \( X_j(z_0) \in \partial K_{\eta_j} \) for all \( j \in \mathbb{Z} \). We will say that the itinerary \( \eta \) is admissible if \( \eta_j \neq \eta_{j+1} \) for all \( j \).

The following is a consequence of the hyperbolicity of the billiard flow in the exterior of \( K \) and can be derived from the works of Sinai on general dispersing billiards ([Si1, Si2] and from Ikawa’s papers on open billiards ([II]; see also [Bu])). In this particular form it can be found in [Sj] (see also chapter 10 in [PS1]).

**Proposition 1.** There exist global constants \( C > 0 \) and \( \alpha \in (0, 1) \) such that for any admissible configuration \( \iota = (i_0, i_1, i_2, \ldots, i_m) \) and any two billiard trajectories in \( \Omega \) with successive reflection points \( x_0, x_1, \ldots, x_m \) and \( y_0, y_1, \ldots, y_m \), both following the configuration \( \iota \), we have

\[
\|x_j - y_j\| \leq C (\alpha^j + \alpha^{m-j}), \quad 0 \leq j \leq m.
\]
Moreover, C and α can be chosen so that if \((x_0, (x_1 - x_0)/\|x_1 - x_0\|)\) and \((y_0, (y_1 - y_0)/\|y_1 - y_0\|)\) belong to the same unstable manifold of the billiard flow, then
\[
\|x_j - y_j\| \leq C\alpha^{m-j}, \quad 0 \leq j \leq m.
\]

As a consequence of this one obtains that there is an one-to-one correspondence between the bounded (in both directions) billiard trajectories in \(\Omega\) and the set of admissible itineraries \(\eta\). In particular, this implies that the intersections of the non-wandering set \(\Lambda\) with cross-sections to the billiard flow \(\phi_t\) are Cantor sets.

For \(x \in \Lambda\) and a sufficiently small \(\epsilon > 0\) let
\[
W^s_\epsilon(x) = \{y \in S^s(\Omega) : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0, d(\phi_t(x), \phi_t(y)) \to \infty, 0 \}\]
\[
W^u_\epsilon(x) = \{y \in S^u(\Omega) : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \leq 0, d(\phi_t(x), \phi_t(y)) \to \infty, 0 \}\]
be the (strong) stable and unstable manifolds of size \(\epsilon\), and let \(E^u(x) = T_1W^u_\epsilon(x)\) and \(E^s(x) = T_1W^s_\epsilon(x)\). For any \(A \subset S^s(\Omega)\) and \(I \subset \mathbb{R}\) denote \(\phi_I(A) = \{\phi_t(y) : y \in A, t \in I\}\).

It follows from the hyperbolicity of \(\Lambda\) (cf e.g. [KH]) that if \(\epsilon > 0\) is sufficiently small, there exists \(\delta > 0\) such that if \(x, y \in \Lambda\) and \(d(x, y) < \delta\), then \(W^s_\epsilon(x)\) and \(\phi_{[-\epsilon,\epsilon]}(W^u_\epsilon(y))\) intersect at exactly one point \([x, y] \in \Lambda\). That is, \([x, y] \in W^s_\epsilon(x)\) and there exists a unique \(t \in [-\epsilon, \epsilon]\) such that \(\phi_t([x, y]) \in W^u_\epsilon(y)\). Setting \(\Delta(x, y) = t\), one defines the so-called **temporal distance function** \(\Delta(x, y)\). For \(x, y \in \Lambda\) with \(d(x, y) < \delta\), define
\[
\pi_t \circ \phi_I : W \to \phi_{[-\epsilon, \epsilon]}(W^u_\epsilon(y))\]

Thus, for a fixed \(y \in \Lambda\), \(\pi_t \circ \phi_I : W \to \phi_{[-\epsilon, \epsilon]}(W^u_\epsilon(y))\) is the projection along local stable manifolds defined on a small open neighbourhood \(W\) of \(y\) in \(\Lambda\).

### 2.1. Coding via a Markov family

Given \(E \subset \Lambda\) we will denote by \(\operatorname{Int}_A(E)\) and \(\partial \Lambda E\) the interior and the boundary of the subset \(E\) in the topology of \(\Lambda\), and by \(\operatorname{diam}(E)\) the diameter of \(E\). We will say that \(E\) is an **admissible subset** of \(W^u_\epsilon(z) \cap \Lambda (z \in \Lambda)\) if \(E\) coincides with the closure of its interior in \(W^u_\epsilon(z) \cap \Lambda\). Admissible subsets of \(W^u_\epsilon(z) \cap \Lambda\) are defined similarly. Following [D], a subset \(R\) of \(\Lambda\) will be called a **rectangle** if it has the form
\[
R = [U, S] = \{[x, y] : x \in U, y \in S\},
\]
where \(U\) and \(S\) are admissible subsets of \(W^u_\epsilon(z) \cap \Lambda\) and \(W^s_\epsilon(z) \cap \Lambda\), respectively, for some \(z \in \Lambda\). For such \(R\), given \(\xi = [x, y] \in R\), we will denote \(W^u_\epsilon(\xi) = \{[x', y'] : x' \in U\}\) and \(W^s_\epsilon(\xi) = \{[x, y'] : y' \in S\} \subset W^u_\epsilon(x)\).

Let \(R = \{R_i\}_{i=1}^k\) be a family of rectangles with \(R_i = [U_i, S_i], U_i \subset W^u_\epsilon(z) \cap \Lambda\) and \(S_i \subset W^s_\epsilon(z) \cap \Lambda\), respectively, for some \(z_i \in \Lambda\). Set
\[
R = \bigcup_{i=1}^k R_i.
\]

The family \(R\) is called **complete** if there exists \(T > 0\) such that for every \(x \in \Lambda\), \(\phi_T(x) \in R\) for some \(t \in (0, T]\). The **Poincaré map** \(\mathcal{P} : R \to R\) related to a complete family \(R\) is defined by \(\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R\), where \(\tau(x) > 0\) is the smallest positive time with \(\phi_{\tau(x)}(x) \in R\). The function \(\tau\) is called the first return time associated with \(R\). Note that \(\tau\) is constant on each of the set \(W^u_\epsilon(x), x \in R_i\). A complete family \(R = \{R_i\}_{i=1}^k\) of rectangles in \(\Lambda\) is called a **Markov family** of size \(\chi > 0\) for the flow \(\phi\) if \(\operatorname{diam}(R_i) < \chi\) for all \(i\) and:

(a) for any \(i \neq j\) and any \(x \in \operatorname{Int}_A(R_i) \cap \mathcal{P}^{-1}(\operatorname{Int}_A(R_j))\) we have
\[
\mathcal{P}(\operatorname{Int}_A(W^u_\epsilon(R_i))) \subset \operatorname{Int}_A(W^u_\epsilon(\mathcal{P}(x))), \quad \mathcal{P}(\operatorname{Int}_A(W^u_\epsilon(R_i))) \supset \operatorname{Int}_A(W^u_\epsilon(\mathcal{P}(x)));
\]
(b) for any \(i \neq j\) at least one of the sets \(R_i \cap \phi_{[0,\chi]}(R_j)\) and \(R_j \cap \phi_{[0,\chi]}(R_i)\) is empty.
Replacing \( \theta_j \) for all integers \( i \), we may assume that the Markov family \( \mathcal{R} \) is a Cantor set, we may assume that the Markov family \( \mathcal{R} \) is chosen in such a way that
\[(c) \text{ for any } i = 1, \ldots, k \text{ we have } \partial_{\delta_i} R_i = \emptyset.\]

Finally, partitioning every \( R_i \) into finitely many smaller rectangles, cutting \( R_i \) along some unstable leaves, and removing some rectangles from the family formed in this way, we may assume that
\[(d) \text{ for every } x \in R \text{ the billiard trajectory of } x \text{ from } x \text{ to } \mathcal{P}(x) \text{ makes exactly one reflection.}\]

From now on we will assume that \( \mathcal{R} = \{R_i\}_{i=1}^k \) is a fixed Markov family for \( \phi_1 \) of size \( \chi < \epsilon_0/2 \) satisfying the above conditions (a)–(d). Set
\[U = \bigcup_{i=1}^k U_i.\]

The shift map \( \tilde{\sigma} : U \rightarrow U \) is given by \( \tilde{\sigma} = \pi(U) \circ \mathcal{P} \), where \( \pi(U) : R \rightarrow U \) is the projection along stable leaves.

Let \( A = (A_{ij})_{i,j=1}^k \) be the matrix given by \( A_{ij} = 1 \) if \( \mathcal{P}(\text{Int}_A(R_i)) \cap \text{Int}_A(R_j) \neq \emptyset \) and \( A_{ij} = 0 \) otherwise. Consider the symbol space
\[\Sigma_A = \{(i_j)_{j=-\infty}^{\infty} : 1 \leq i_j \leq k, A_{ij} = 1 \text{ for all } j \},\]

with the product topology and the shift map \( \sigma : \Sigma_A \rightarrow \Sigma_A \) given by \( \sigma((i_j)) = ((i'_j)) \), where \( i'_j = i_{j+1} \) for all \( j \). As in [B] one defines a natural map
\[\Psi : \Sigma_A \rightarrow R.\]

Namely, given any \( i = (i_j)_{j=-\infty}^{\infty} \in \Sigma_A \) there is exactly one point \( x \in R_i \) such that \( \mathcal{P}^j(x) \in R_i \) for all integers \( j \). We then set \( \Psi((i_j)) = x \). One checks that \( \Psi \circ \sigma = \mathcal{P} \circ \Psi \) on \( R \). It follows from condition (c) above that the map \( \Psi \) is a bijection. Moreover, \( \Psi \) is Lipschitz if \( \Sigma_A \) is considered with the metric \( d_{\theta} \) for some appropriately chosen \( \theta \in (0, 1) \), where \( d_{\theta}(\xi, \eta) = 0 \) if \( \xi = \eta \) and \( d_{\theta}(\xi, \eta) = \theta^{|i|} \) if \( |i| < m \) and \( m \) is maximal with this property.

Replacing \( \theta \) by an appropriate \( \rho \in (\theta, 1) \), makes \( \Psi^{-1} \) a Lipschitz map.

In a similar way one deals with the one-sided subshift of finite type
\[\Sigma_A^+ = \{(i_j)_{j=0}^{\infty} : 1 \leq i_j \leq k, A_{ij} = 1 \text{ for all } j \geq 0 \},\]

where the shift map \( \sigma : \Sigma_A^+ \rightarrow \Sigma_A^+ \) is defined in a similar way: \( \sigma((i_j)) = ((i'_j)) \), where \( i'_j = i_{j+1} \) for all \( j \geq 0 \). The metric \( d_{\theta} \) on \( \Sigma_A^+ \) is defined as above. One checks that there exists a unique map \( \psi : \Sigma_A^+ \rightarrow U \) such that \( \psi \circ \pi = \pi(U) \circ \Psi \), where \( \pi : \Sigma_A \rightarrow \Sigma_A^+ \) is the natural projection.

Note that the roof function \( r : \Sigma_A \rightarrow [0, \infty) \) defined by \( r(\xi) = \tau(\Psi(\xi)) \) depends only on the forward coordinates of \( \xi \in \Sigma_A \). Indeed, if \( \xi_i = \eta_i \), where \( \xi_i = (\xi^i_j)_{j=0}^{\infty} \), then for \( x = \Psi(\xi) \) and \( y = \Psi(\eta) \) we have \( x, y \in R_i \) for \( i = \xi_0 = \eta_0 \) and \( \mathcal{P}^i(x) \) and \( \mathcal{P}^i(y) \) belong to the same \( R_i \) for all \( j \geq 0 \). This implies that \( x \) and \( y \) belong to the same local stable fibre in \( R_i \) and therefore \( \tau(x) = \tau(y) \). Thus, \( r(\xi) = r(\eta) \). So, we can define a roof function \( r : \Sigma_A^+ \rightarrow [0, \infty) \) such that \( r \circ \pi = \tau \circ \Psi \).

Setting \( r_n(\xi) = r(\xi) + r(\sigma(\xi)) + \cdots + r(\sigma^{n-1}(\xi)) \) for any integer \( n \geq 1 \) and any \( \xi \in \Sigma_A^+ \), we have the following proposition.
Proposition 2. There exists a bijection $\psi : \Sigma_+ \rightarrow U$ such that $\psi \circ \sigma = \tilde{\sigma} \circ \psi$ and a function $r : \Sigma_+ \rightarrow [0, \infty)$ such that for any integer $n \geq 1$ and any $\xi \in \Sigma_+$ we have $r_n(\xi) = t_n(\psi(\xi))$, i.e. this is the length of the billiard trajectory in $\Omega$ determined by $\psi(\xi)$ from $\psi(\xi)$ to the $n$th intersection with a rectangle from $\mathbb{R}$.

Let $B(\Sigma_+)$ be the space of bounded functions $g : \Sigma_+ \rightarrow \mathbb{C}$ with its standard norm $\|g\|_0 = \sup_{\xi \in \Sigma_+} |g(\xi)|$. Given a function $g \in B(\Sigma_+)$, the Ruelle transfer operator $L_g : B(\Sigma_+) \rightarrow B(\Sigma_+)$ is defined by

$$(L_g h)(\xi) = \sum_{\sigma(\eta) = \xi} e^{\rho(h)} h(\eta).$$

Let $F_\varphi(\Sigma_+)$ denote the space of $d_0$-Lipschitz functions $g : \Sigma_+ \rightarrow \mathbb{C}$ with the norm $\|f\|_0 = \|f\|_0 + \|f\|_\sigma$, where

$$\|f\|_\sigma = \sup \left\{ \frac{|f(\xi) - f(\eta)|}{d_0(\xi, \eta)} : \xi, \eta \in \Sigma_+, \xi \neq \eta \right\}.$$ 

If $g \in F_\varphi(\Sigma_+)$, then $L_g$ preserves the space $F_\varphi(\Sigma_+)$. 

2.2. Coding using boundary components

Here we assume that $K$ is as in section 1. Denote by $A$ the $\kappa_0 \times \kappa_0$ matrix with entries $A(i, j) = 1$ if $i \neq j$ and $A(i, i) = 0$ for all $i$, and set

$\Sigma_A = \{\ldots, \eta_{-m}, \ldots, \eta_{-1}, \eta_0, \eta_1, \ldots, \eta_m, \ldots : 1 \leq \eta_j \leq \kappa_0, \eta_j \in \mathbb{N}, \eta_j \neq \eta_{j+1} \text{ for all } j \in \mathbb{Z}, \}$

$\Sigma_A^+ = \{\ldots, \eta_{-m}, \ldots, \eta_{-1}, \eta_0, \eta_1, 1 \leq \eta_j \leq \kappa_0, \eta_j \in \mathbb{N}, \eta_j \neq \eta_{j+1} \text{ for all } j \in \mathbb{Z} \}$

The shift operators $\sigma : \Sigma_A \rightarrow \Sigma_A$ and $\sigma : \Sigma_A^+ \rightarrow \Sigma_A^+$ are defined as before.

We will now define two important functions $f$ and $g$ on $\Sigma_A$. For the second one in particular we need some preliminary information.

A phase function on an open set $U$ in $\mathbb{R}^N$ is a smooth ($C'$) function $\varphi : U \rightarrow \mathbb{R}$ such that $\|
abla \varphi\| = 1$ everywhere in $U$. For $x \in U$ the level surface

$$C_\varphi(x) = \{y \in U : \varphi(y) = \varphi(x)\}$$

has a unit normal field $\pm \nabla \varphi(y)$.

The phase function $\varphi$ defined on $U$ is said to satisfy condition $(P)$ on $\Gamma_j$ if:

(i) the normal curvatures of $C_\varphi$ with respect to the normal field $-\nabla \varphi$ are non-negative at every point of $C_\varphi$;

(ii) $U^i(\varphi) = \{y + t \nabla \varphi(y) : t \geq 0, y \in U\} \supset \cup_{i \neq j} K_i$.

A natural extension of $\varphi$ on $U^i(\varphi)$ is obtained by setting $\varphi(y + t \nabla \varphi(y)) = \varphi(y) + t$ for $t \geq 0$ and $y \in U$.

For any $\delta > 0$ and $V \subset \Omega$ denote by $S^*_\delta(V)$ the set of those $(x, u) \in S^*(\Omega)$ such that $x \in V$ and there exist $y \in \Gamma$ and $t \geq 0$ with $y + tu = x$, $y + su \in \mathbb{R}^N \setminus K$ for all $s \in (0, t)$ and $(u, v(y)) \geq \delta$.

We keep the same notation as for $\Sigma_A$ and $\Sigma_A^+$; it will be clear from the context which of these we mean in each particular case.
Note that condition (H) implies the existence of a constant \( \delta_0 > 0 \) depending only on the obstacle \( K \) such that any \((x, u) \in S^*(\Omega)\) whose backward and forward billiard trajectories both have common points with \( K \). This defines a function \( g(\xi) = \|P_0(\xi) - P_1(\xi)\|\) and that \( g(\xi) \) is the local unstable manifold of \( x_0 \).

Proposition 3. There exists a constant \( \epsilon_0 > 0 \) such that for any \( z_0 = (x_0, u_0) \in S^*_0(\Omega \cap B_0) \) whose backward trajectory \( \gamma_-(z_0) \) has an infinite number of reflection points \( X_j = X_j(z_0) \) (\( j \leq 0 \)) and \( n \in \Sigma_A \) is its itinerary, the following hold:

(a) There exists a smooth \((C^1)\) phase function \( \psi = \psi_0 \) satisfying condition \((P)\) on \( \mathcal{U} = B(x_0, \epsilon_0) \cap \Omega \) such that \( \psi(x_0) = 0 \), \( u_0 = \nabla\psi(x_0) \), and such that for any \( x \in C_\psi(x_0) \cap \mathcal{U}^*(\psi) \) the billiard trajectory \( \gamma_-(x, \nabla\psi(x)) \) has an itinerary \( n \) and therefore \( d(\phi_t(x, \nabla\psi(x)), \phi_t(\epsilon_0)) \to 0 \) as \( t \to -\infty \). That is,

\[
W_{\text{loc}}^u(z_0) = \{(x, \nabla\psi(x)) : x \in C_\psi(x_0) \cap \mathcal{U}^*(\psi)\}
\]

is the local unstable manifold of \( z_0 \).

(b) If \((y, v) \in S^*(\Omega \cap B_0)\) is such that \( y \in C_\psi(x_0) \) and \( \gamma_-(y, v) \) has the same itinerary \( n \), then \( v = \nabla\psi(y) \), i.e. \((y, v) \in W_{\text{loc}}^u(z_0)\).

Note that \( W_{\text{loc}}^u(z_0) \) coincides locally near \( z_0 \) with \( W_{\text{loc}}^u(z_0) \) defined above.

In what follows we will use the notation \( C_n(z_0) = C_\psi(x_0) \). Denote by \( G_\eta(z_0) \) the Gauss curvature of \( C_\psi(z_0) \) at \( x_0 \).

Given \( \xi \in \Sigma_A \), let

\[
\ldots, P_{-2}(\xi), P_{-1}(\xi), P_0(\xi), P_1(\xi), P_2(\xi), \ldots
\]

be the successive reflection points of the unique billiard trajectory in the exterior of \( K \) such that \( P_j(\xi) \in K_\xi \) for all \( j \in \mathbb{Z} \). Set

\[
f(\xi) = \|P_0(\xi) - P_1(\xi)\|
\]

and define the map

\[
\Phi : \Sigma_A \to \Lambda_{\partial K} = \Lambda \cap S^*_0(\partial K)
\]

by \( \Phi(\xi) = (P_0(\xi), (P_1(\xi) - P_0(\xi))/\|P_1(\xi) - P_0(\xi)\|) \). Then \( \Phi \) is a bijection such that \( \Phi \circ \sigma = B \circ \Phi \), where \( B : \Lambda_{\partial K} \to \Lambda_{\partial K} \) is the billiard ball map.

Next, set \( \xi = (\ldots, \xi_{-m}, \xi_{-m+1}, \ldots, \xi_{-1}, \xi_0) \in \Sigma_A \) and choose an arbitrary point \( x_0 \) on the segment \([P_0(\xi), P_1(\xi)]\) such that \( z_0 = (x_0, u_0) \in S^*_0(\Omega) \), where \( u_0 = (P_1(\xi) - P_0(\xi))/\|P_1(\xi) - P_0(\xi)\| \). Let \( C_{\xi}(z_0) = C_\psi(x_0) \) for some phase function \( \psi \). Setting \( t_\cdot = \|x_0 - P_0(\xi)\| \) and \( t_\xi = \|x_0 - P_1(\xi)\| \), consider the surfaces

\[
C_{\xi}^{-}(P_0(\xi)) = \{x - t_\cdot \nabla\psi(x) : x \in C_{\xi}(z_0)\}, \quad C_{\xi}^{+}(P_1(\xi)) = \{x + t_\cdot \nabla\psi(x) : x \in C_{\xi}(z_0)\}.
\]

Clearly \( C_{\xi}^{-}(P_0(\xi)) \) is the surface passing through \( P_0(\xi) \) obtained by shifting \( C_{\xi}(z_0) \) in free space \( \mathbb{R}^N \), \( t_\cdot \) units backwards along its normal field, while \( C_{\xi}^{+}(P_1(\xi)) \) is the surface passing through \( P_1(\xi) \) obtained by shifting \( C_{\xi}(z_0) \) in free space \( \mathbb{R}^N \), \( t_\cdot \) units forwards along its normal field. Let \( G_{\xi}^{-}(P_0(\xi)) \) and \( G_{\xi}^{+}(P_1(\xi)) \) be the Gauss curvatures of \( C_{\xi}^{-}(P_0(\xi)) \) at \( P_0(\xi) \) and that of \( C_{\xi}^{+}(P_1(\xi)) \) at \( P_1(\xi) \), respectively. Set

\[
g(\xi) = \frac{1}{N - 1} \ln \frac{G_{\xi}^{+}(P_1(\xi))}{G_{\xi}^{-}(P_0(\xi))}.
\]

This defines a function \( g : \Sigma_A \to \mathbb{R} \).
Choosing appropriately $\theta \in (0, 1)$, we can define as in section 2.1 the space $F_\theta(\Sigma_A)$ and we get $f, g \in F_\theta(\Sigma_A)$ (see, e.g., [II]). By Sinai’s lemma (see, e.g., [PP]), there exist functions $\hat{f}, \hat{g} \in F_\theta(\Sigma_A)$ depending on future coordinates only and $\chi_f, \chi_g \in F_\theta(\Sigma_A)$ such that

$$f(\xi) = \tilde{f}(\xi) + \chi_f(\sigma^i \xi), \quad g(\xi) = \tilde{g}(\xi) + \chi_g(\sigma^i \xi)$$

for all $\xi \in \Sigma_A$. As in the proof of Sinai’s lemma, for any $k = 1, \ldots, \kappa_0$ choose and fix an arbitrary sequence $\eta^{(k)} = (\ldots, \eta_{-m}^{(k)}, \ldots, \eta_{-1}^{(k)}, \eta_0^{(k)}) \in \Sigma_A$ with $\eta_0^{(k)} = k$. Then for any $\xi \in \Sigma_A$ (or $\xi \in \Sigma_A$) set

$$e(\xi) = (\ldots, \eta_{-m}^{(k)}, \ldots, \eta_{-1}^{(k)}, \eta_0^{(k)} = \xi_0, \xi_1, \ldots, \xi_m, \ldots) \in \Sigma_A.$$ 

Then we have

$$\chi_f(\xi) = \sum_{n=0}^{\infty} [f(\sigma^n(\xi)) - f(\sigma^n e(\xi))].$$

A similar formula holds for $\chi_g$.

As in section 2.1, given any function $V \in B(\Sigma_A^*)$, the Ruelle transfer operator $L_V : B(\Sigma_A^*) \rightarrow B(\Sigma_A^*)$ is defined by

$$(L_V W)(\xi) = \sum_{\sigma(\eta) = \xi} e^{V(\eta)} W(\eta).$$

2.3. Another coding related to the Markov family

Here we define another coding which uses the symbolic model $\Sigma_A^*$. We will then define representatives $\hat{f}$ and $\hat{g}$ of the functions $f$ and $g$ on $\Sigma_A^*$ and consider the corresponding Ruelle operators $L_{\hat{f}}, L_{\hat{g}}$.

Let $V_0$ be the set of those $(p, u) \in S^*(\Omega)$ such that $p = q + t u$ and $(p, u) = \phi_t(q, u)$ for some $(q, u) \in S_{\delta K}^*(\Omega)$ with $|u, v(q)| > 0$ and some $t \geq 0$. Clearly $V_0$ is an open subset of $S^*(\Omega)$ containing $\Lambda$. Setting $\omega(p, u) = (q, u)$, we get a smooth map $\omega : V_0 \rightarrow S_{\delta K}^*(\Omega)$.

Consider the bijection $S = \Phi^{-1} \circ \omega \circ \Psi : \Sigma_A \rightarrow \Sigma_A$. It induces a bijection $S : \Sigma_A^* \rightarrow \Sigma_A^*$. Moreover, $S \circ \sigma = \sigma \circ S$.

Define the functions $f', g' : \Sigma_A \rightarrow \mathbb{R}$ by $f'(i) = f(S(i))$ and $g'(i) = g(S(i))$.

Next, repeating a part of the previous subsection, for any $i = 1, \ldots, k$ choose

$$\hat{\nu}(i) = (\ldots, j_{-m}^{(i)}, \ldots, j_0^{(i)}),$$

such that $(j_0^{(i)}, i) \in \Sigma_A^-$. It is convenient to make this choice in such a way that $\hat{\nu}(i)$ corresponds to the local unstable manifold $U_i \subset \Lambda \cap W^u(\xi_i)$ (see the beginning of section 2.1), i.e. the backward itinerary of every $z \in U_i$ coincides with $\hat{\nu}(i)$.

Now for any $\hat{i} = (i_0, i_1, \ldots) \in \Sigma_A^*$ (or $\hat{i} \in \Sigma_A$) set

$$\hat{e}(i) = (\hat{\nu}(i_0), i_0, i_1, \ldots) \in \Sigma_A.$$ 

According to the choice of $\hat{\nu}(i)$, we have $\Psi(\hat{e}(i)) = \psi(i) \in U_{\hat{i}}$. (Note that without the above special choice we would only have that $\Psi(\hat{e}(i))$ and $\psi(i) \in U_{\hat{i}}$ lie on the same stable leaf in $R_{\hat{i}}$.)

Next, define

$$\hat{f}_f(\xi) = \sum_{n=0}^{\infty} \left[ f'(\sigma^n(\hat{\xi})) - f'(\sigma^n \hat{e}(\hat{\xi})) \right], \quad \hat{\xi} \in \Sigma_A.$$
We will now describe a natural relationship between the operators $\tau_n(x)$ and $\lambda(x)$ on a manifold. Thus, we have a bijection and dependency on future coordinates only. Then $\lambda(x)$ depends on future coordinates only, so they can be regarded as functions on $\Sigma_A^+$. Moreover, $\lambda(x) = t = \hat{\lambda}(i)$. This defines a function $\lambda : R \rightarrow [0, \infty)$ so that $\lambda \circ \Psi = \hat{\lambda}$.

Before continuing, note that

$$\hat{\lambda}(\hat{\gamma}(i)) = \hat{\lambda}(i) - \hat{\gamma}(i), \quad \hat{i} \in \Sigma_A.$$  

Indeed, given $\hat{i} = (\ldots, i_0, i_1, \ldots) \in \Sigma_A$, let $x = \Psi(\hat{i}) \in R_0$ and $y = \Psi(\hat{\gamma}(i)) \in R_0$. Since $\hat{i}$ and $\hat{\gamma}(i)$ have the same forward coordinates, it follows that $x$ and $y$ lie on the same local stable manifold. Thus, $\tau^m(x) = \tau^m(y)$ and so $r_m(i) = r_m(\hat{\gamma}(i))$ for all integers $n \geq 1$. Moreover, the $m$th reflection points of the billiard trajectories determined by $x$ and $y$ are Const-$\theta^m$-close for

$$\hat{\lambda}(\hat{\gamma}(i)) = \hat{\lambda}(i) - \hat{\gamma}(i), \quad \hat{i} \in \Sigma_A.$$  

Indeed, given $\hat{i} = (\ldots, i_0, i_1, \ldots) \in \Sigma_A$, let $x = \Psi(\hat{i}) \in R_0$ and $y = \Psi(\hat{\gamma}(i)) \in R_0$. Since $\hat{i}$ and $\hat{\gamma}(i)$ have the same forward coordinates, it follows that $x$ and $y$ lie on the same local stable manifold. Thus, $\tau^m(x) = \tau^m(y)$ and so $r_m(i) = r_m(\hat{\gamma}(i))$ for all integers $n \geq 1$. Moreover, the $m$th reflection points of the billiard trajectories determined by $x$ and $y$ are Const-$\theta^m$-close for
some global constants Const > 0 and \( \theta \in (0, 1) \). Thus, \(|\hat{\lambda}(\sigma^m(i)) - \hat{\lambda}(\sigma^m(\hat{e}(i)))| \leq \text{Const} \theta^m\), too (possibly with a different global constant \( \text{Const} > 0 \)).

Set \( t = \hat{\lambda}(i) \) and \( t' = \hat{\lambda}(\hat{e}(i)) \). Given an integer \( m \geq 1 \), consider

\[
A_m = \sum_{n=0}^{m} \left[ f'(\sigma^n(i)) - f'(\sigma^n(\hat{e}(i))) \right].
\]

Then \( \hat{\chi}_f(i) = \lim_{m \to \infty} A_m \). Moreover,

\[
A_m = [r_m(i) + t - \hat{\lambda}(\sigma^m(i))] - [r_m(\hat{e}(i)) + t' - \hat{\lambda}(\sigma^m(\hat{e}(i)))]
\]

and letting \( m \to \infty \), we obtain (3.3).

Next, for any \( i \in \Sigma_A \) and any \( m \geq 1 \) we have

\[
f_m(i) = f'(i) + f'(\sigma i) + \cdots + f'(\sigma^{m-1}i)
\]

\[
= [f(i) + \hat{\chi}_f(i)] + [f(\sigma i) + \hat{\chi}_f(\sigma i)] + \cdots + [f(\sigma^{m-1}i) + \hat{\chi}_f(\sigma^{m-1}i)]
\]

\[
= \hat{f}_m(i) + \hat{\chi}_f(i).
\]

Since \( r_m(i) = f_m(i) + \hat{\lambda}(\sigma^m i) - \hat{\lambda}(i) \), combining the above and (3.3), it follows that

\[
r_m(i) = \hat{f}_m(i) + \hat{\lambda}(\sigma^m i) - \hat{\lambda}(i) - \hat{\lambda}(\sigma^m(\hat{e}(i))) - \hat{\lambda}(\hat{e}(i)).
\]

Thus,

\[
r_m(i) = \hat{f}_m(i) + \hat{\lambda}(\sigma^m i) - \hat{\lambda}(\hat{e}(i)), \quad i \in \Sigma_A, \quad m \geq 1. \tag{3.4}
\]

Note that \( \hat{f}_m(i) = \hat{f}_m(\hat{e}(i)) \), since \( \hat{f} \) depends on future coordinates only.

Now we will find a relationship between the powers of the operators \( \mathcal{L}_{-s r^k \hat{g}} \) and \( \mathcal{L}_{-s f^k \hat{g}} \).

Given \( s \in \mathbb{C} \), consider the function \( h_s : U \to \mathbb{R} \) defined by

\[
h_s(x) = e^{-s \hat{\lambda}(x)}.
\]

It gives rise to a function \( \hat{h}_s : \Sigma_A \to \mathbb{R} \) defined by

\[
\hat{h}_s(i) = h_s(\psi(i)) = h_s(\Psi(\hat{e}(i))) = e^{-s \hat{\lambda}(\hat{e}(i))}, \quad i \in \Sigma_A. \tag{3.5}
\]

(See the note after the definition of \( \hat{e}(i) \) in section 2.3.)

It now follows from (3.4) and (3.5) that for any function \( V : \Sigma_A \to \mathbb{C} \), any \( s \in \mathbb{C} \) and any integer \( n \geq 1 \) we have

\[
\mathcal{L}_{-s r^k \hat{g}}^{n+1} \left( \hat{h}_s \cdot V \right)(i) = \sum_{\sigma^m i \in \Sigma_A} e^{-s r_m(i) + \hat{\lambda}(\sigma^m(i))} e^{-s \hat{\lambda}(\hat{e}(i))} V(k)
\]

\[
= \sum_{\sigma^m i \in \Sigma_A} e^{-s f_m(i) + \hat{\lambda}(\sigma^m(i)) - \hat{\lambda}(\hat{e}(i))} V(k)
\]

\[
= \sum_{\sigma^m i \in \Sigma_A} e^{-s \hat{f}_m(i) - \hat{\lambda}(\hat{e}(i))} V(k)
\]

\[
= e^{-s \hat{\lambda}(\hat{e}(i))} \sum_{\sigma^m i \in \Sigma_A} e^{-s \hat{f}_m(i) + \hat{\lambda}(\sigma^m(i))} V(k) = \hat{h}_s(i) \cdot \left( \mathcal{L}_{-s f^k \hat{g}}^{n+1} V \right)(i).
\]

Thus,

\[
\frac{1}{\hat{h}_s} \cdot \mathcal{L}_{-s r^k \hat{g}}^{n+1} \left( \hat{h}_s \cdot V \right) = \mathcal{L}_{-s f^k \hat{g}}^n V. \tag{3.6}
\]
3.2. Relationship between $L_{-t_f \hat{z}}$ and $L_{-t_f \hat{z}'}$

Here we will use arguments similar to those in section 3.1 to find a relationship between the operators $L^n_{-t_f \hat{z}}$ and $L^n_{-t_f \hat{z}'}$.

For any $p = 1, 2, \ldots, k_0$ fix an arbitrary point $y_p \in \Lambda \cap S^\star_{\delta_0}(\Omega)$ such that $\eta^{(p)} \in \Sigma_A^-$ corresponds to the unstable manifold $W^u(y_p)$, i.e. the backward itinerary of every $z \in W^u(y_p) \cap V_0$ coincides with $\eta^{(p)}$. Define $\kappa : \Sigma_A^+ \to \mathbb{R}$ by

$$\kappa(i) = [\Psi(\hat{e}(i)), y_{\|\delta|}, \xi = S(i)] \in \Sigma^+_A.$$

We will now prove that

$$\hat{\chi}_f(i) - \chi_f(S(i)) = \lambda(\kappa(i)) - \hat{\lambda}(\hat{e}(i)), \quad \hat{i} \in \Sigma_A.$$

It is important that the right-hand side of (3.7) depends only on the future coordinates of $\hat{i}$. Indeed, given $\hat{i} = (\ldots; i_0, i_1, \ldots) \in \Sigma_A$, set $x = \Psi(\mathbf{e}(\hat{i})) \in U_1, \xi = S(i), \xi_0 = p$ and $y = \kappa(i) = [x, y_p]$. Note that $y \in W^s(x)$, and so the forward billiard trajectories of $x$ and $y$ converge. On the other hand, $\phi_t(y) \in W^u(y_p)$ for some $t \in \mathbb{R}$ (in fact, $t = \Delta(x, y_p)$; see section 2 for the definition of $\Delta$), so the billiard trajectory determined by $y$ has backward itinerary $\eta^{(p)} \in \Sigma_A^+$ and forward itinerary $(\xi_0, \xi_1, \ldots) \in \Sigma_A^+$, i.e. this is the trajectory determined by $e(\xi)$.

Since $y \in W^s(x)$, the $j$th reflection points $p_j$ and $q_j$ ($j \geq 0$) of the billiard trajectories determined by $x$ and $y$, respectively, are Const $\theta_j$-close for some global constants $\text{Const} > 0$ and $\theta \in (0, 1)$.

Set $\hat{t} = \hat{\lambda}(\hat{e}(i)) = \lambda(x)$ and $t^\prime = \lambda(\kappa(i)) = \lambda(\hat{e}(i))$. Given an integer $m \geq 1$, consider

$$A_m = \sum_{n=0}^{m-1} \left[ f'(\sigma^n(\hat{i})) - f'(\sigma^n(\hat{e}(i))) \right] - \sum_{n=0}^{m-1} \left[ f(\sigma^n(\xi)) - f(\sigma^n(e(\xi))) \right].$$

Then $\hat{\chi}_f(i) - \chi_f(S(i)) = \lim_{m \to \infty} A_m$. Since $f'(\sigma^n(\hat{i})) = f(\sigma^n(\xi))$, we have

$$A_m = -\sum_{n=0}^{m-1} f'(\sigma^n(\hat{e}(i))) + \sum_{n=0}^{m-1} f(\sigma^n(e(\xi))).$$

The first sum in this expression is the length of the billiard trajectory determined by $x$ from $p_{m+1}$ till $p_m$, while the second is the length of the billiard trajectory determined by $y$ from $q_m$ till $q_{m+1}$. Since $\|q_m - p_m\| \leq \text{Const} \theta^m$ and $y \in W^s(x)$, it now follows that $A_m = -t + t^\prime + O(\theta^m)$, and letting $m \to \infty$ proves (3.7).

In a similar way, using the definition of the function $g$, one derives that

$$\hat{\chi}_g(i) - \chi_g(\hat{i}) = \frac{1}{N-1} \ln G_{\hat{e}(i)}(P_0(\hat{e}(i))) - \frac{1}{N-1} \ln G_{\hat{e}(i)}(P_0(e(\xi))), \quad \hat{i} \in \Sigma_A.$$

(3.8)

Note that $G_{\hat{e}(i)}(P_0(\hat{e}(i)))$ is the Gauss curvature of a shift of $U_{\delta_0}$ backwards along the corresponding billiard trajectory at $P_0(\hat{e}(i))$ (so this is uniquely determined by the forward coordinates of $\hat{i}$), while $G_{\hat{e}(i)}(P_0(e(\xi)))$ is the Gauss curvature of the shift of $W^u(y_{\|\delta|})$ at $P_0(e(\xi))$ (so again this is uniquely determined by the forward coordinates of $\hat{i}$).

Given $\hat{i} \in \Sigma_A$, set $\xi = S(\hat{i})$. Then (3.7) implies

$$\hat{f}_m(i) - \hat{f}_m(\hat{i}) = \left[ f'_m(i) - \hat{f}'_m(\hat{i}) + \hat{f}_m(\sigma^m(\hat{i})) - [f_m(\hat{i}) - \chi_f(\hat{i}) + \chi_f(\sigma^m(\xi))] \right] = -[\hat{f}_m(\hat{i}) - \hat{f}_m(\sigma^m(\hat{i})) + \chi_f(\hat{i}) - \chi_f(\sigma^m(\xi))].$$

(3.9)
Similarly,
\[
\tilde{g}_m(i) - \tilde{g}_m(\xi) = -[\tilde{x}_g(i) - \tilde{x}_g(\sigma^m i)] + [x_g(\xi) - x_g(\sigma^m \xi)] .
\] (3.10)

Given \( s \in \mathbb{C} \), consider the functions \( d_s : U \rightarrow \mathbb{R} \) and \( \hat{d}_s : \Sigma_A^+ \rightarrow \mathbb{R} \) defined by
\[
d_s(\Psi(i)) = e^{s[\tilde{x}_g(i) - \tilde{x}_g(\xi)]} \frac{\sum_{\sigma^n \xi = \xi} e^{-s\tilde{x}_g(i) + \tilde{x}_g(\sigma^n \xi)}}{G(i, \xi, \mathbb{R})}, \quad \hat{d}_s(i) = \hat{d}_s(\xi) = \mathcal{S}(i) .
\] (3.11)

Note that by (3.7) and (3.8),
\[
\hat{d}_s(i) = e^{s[\tilde{x}_g(i) - x_g(\xi)]} - e^{s[\tilde{x}_g(i) - x_g(\xi)]}, \quad \xi = S(i).
\] (3.12)

Now for any function \( V : \Sigma_A^+ \rightarrow \mathbb{R} \), any \( s \in \mathbb{C} \) and any integer \( n \geq 1 \), setting \( \xi = S(i) \) and \( \zeta = S(\xi) \) and using (3.9) and (3.10), we get
\[
L^n_{f + \xi \tilde{g}} \left( (\hat{d}_s \circ S^{-1}) \cdot (V \circ S^{-1}) \right)(\xi) = \sum_{\sigma^n \xi = \xi} e^{-s\tilde{x}_g(i) + \tilde{x}_g(\sigma^n \xi)} \hat{d}_s(\xi) V(\xi)
\[
= \sum_{\sigma^n \xi = \xi} e^{-s\tilde{x}_g(i) + \tilde{x}_g(\sigma^n \xi)} \hat{d}_s(\xi) \cdot \mathcal{S}(i).
\]

Combining the latter with (3.6) yields
\[
L^n_{f + \xi \tilde{g}} \left( (\hat{d}_s \circ S^{-1}) \cdot (V \circ S^{-1}) \right)(\xi) = \hat{d}_s(i) \cdot \mathcal{L}^n_{f + \xi \tilde{g}}(\mathcal{S}) \cdot (\mathcal{S}(i)).
\]

Setting \( u = (\hat{d}_s \circ S^{-1}) \cdot (V \circ S^{-1}) \in C(\Sigma_A^+) \) in the above yields the following proposition.

**Proposition 4.** For any \( u \in C(\Sigma_A^+) \) we have
\[
L^n_{f + \xi \tilde{g}}(u)(\xi) = \hat{d}_s(i) \cdot \mathcal{L}^n_{f + \xi \tilde{g}}(\mathcal{S}) \cdot (\mathcal{S}(i)), \quad \hat{d}_s(i) \cdot (\mathcal{S}(i)).
\] (3.13)

In particular, the eigenvalues of \( \mathcal{L}_{f + \xi \tilde{g}} \) and \( \mathcal{L}_{-f + \xi \tilde{g}} \) coincide with their multiplicities.

### 3.3. Dolgopyat type estimates

To obtain Dolgopyat type estimates for the left-hand side of (3.13) we can use the Dolgopyat type estimates for \( \mathcal{L}_{f + \xi \tilde{g}} \), provided \( u \circ S \) is determined by a Lipschitz function on \( R \) (with respect to the distance on \( R \)) determined by the standard metric in \( S^*(\Omega) \). To do so we also need to show that the functions \( d_s \) and \( h_s \) given by (3.11) and (3.5) are Lipschitz on \( U \).

**Lemma 1.** Assume that the strong stable and the strong unstable laminations \( \{W^s(x)\}_{x \in \Lambda} \) and \( \{W^u(x)\}_{x \in \Lambda} \) are Lipschitz in \( x \in \Lambda \). Then the functions \( d_s \) and \( h_s \) are Lipschitz on \( U \).

**Proof.** Consider the function \( d_s \) on \( U \); for some fixed \( i \), \( k = 1, \ldots, k \). Given \( i \), there exists \( \tilde{p} = 1, \ldots, k_0 \) such that \( pr_i(\omega(U)) \subset \Gamma_{\tilde{p}} \). Fix an arbitrary \( \tilde{z} \in \Lambda \cap \Sigma_A^-(\Omega) \) with a backward itinerary \( \eta(\tilde{p}) \in \Sigma_A^- \) and \( pr_i(\omega(\tilde{z})) \in \Gamma_{\tilde{p}} \) and an arbitrary \( \tilde{y} \in U_i \) with backward itinerary \( \gamma(\tilde{y}) \in \Sigma_A^- \). Consider the surface \( C = C^{\eta(\tilde{p})}(\tilde{z}) \) (see section 2.2 for the definition).
Note that for any $i = (i_0, i_1, \ldots) \in \Sigma_+^*$ with $i_0 = \bar{t}$ and any $y \in U_{\bar{t}}$, locally the surface $C_{\xi(i)}(y)$ coincides with $pr_1(\phi_t(U))$ for some $t = t(y)$ which is a Lipschitz function of $y \in U_{\bar{t}}$ (in fact $t(y)$ extends smoothly to a neighbourhood of $U_{\bar{t}}$ in the local unstable manifold containing it). Thus, $G_{\xi(i)}(y)$ is a Lipschitz function of $y \in U_{\bar{t}}$.

Similarly, for $\xi = (\xi_0, \xi_1, \ldots) \in \Sigma_+^*$ with $\xi_0 = \bar{p}$ and $z \in \Lambda$ with $pr_1(\omega(z)) \in \Gamma_{\bar{p}}$, $G_{\xi(i)}(z)$ is a Lipschitz function of $z$.

Next, let $x \in U_{\bar{t}}$. Then $x = \psi(i)$ for some $i = (i_0, i_1, \ldots) \in \Sigma_+^*$ with $i_0 = \bar{t}$, and by the choice of $\hat{e}(i)$ (see section 2.3), we have $x = \Psi(\hat{e}(i)) \in U_{\bar{t}}$. Thus, $\hat{\lambda}(\hat{e}(i)) = \lambda(x)$, and the definition of $\kappa$ gives $\kappa(i) = [x, y_{\bar{p}}]$. Moreover, $x = \Psi(\hat{e}(i))$ shows that $P_0(\hat{e}(i)) = pr_1(\omega(x))$.

Finally, note that $P_0(e(\xi)) = pr_1(\omega([x, y_{\bar{p}}]))$. Indeed, the point $z = [x, y_{\bar{p}}]$ lies on $W^s(x)$, so its forward itinerary in the model $\Sigma_+^*$ is the same as that of $x = \psi(i)$, i.e. it is $i$. Thus, the forward itinerary of $z$ in the model $\Sigma_+^*$ is $S(i) = \xi$. On the other hand, $\phi_i(z) \in W^s(y_{\bar{p}})$ for some (small) $s \in \mathbb{R}$, so $z$ has the same backward itinerary in the model $\Sigma_+$ as $y_{\bar{p}}$, i.e. it is $\eta(p)$. Thus, $z$ lies on the trajectory determined by $e(\xi) = (\eta^{\bar{p}}; \xi) \in \Sigma_+$, and therefore $P_0(e(\xi)) = pr_1(\omega(z)) = pr_1(\omega([x, y_{\bar{p}}]))$.

It now follows from the above and (3.11) that

$$\ln d_i(x) = s[\lambda([x, y_{\bar{p}}]) - \lambda(x)] - \frac{1}{N-1} \ln \frac{G_{\xi(i)}(pr_1(\omega(x)))}{G_{\xi(i)}(pr_1(\omega([x, y_{\bar{p}}])))}.$$ 

Since $\lambda : \mathbb{R} \to [0, \infty)$ is Lipschitz (it is actually smooth on a neighbourhood of $\Lambda$ in $S^*(\Omega)$) and $[\cdot, \cdot]$ is uniformly Lipschitz, it follows that $d_i(x)$ is Lipschitz with $\text{Lip}(d_i) \leq \text{Const} |s|$ when $\text{Re}(s) \leq \text{Const}$.

The same argument applies to the function $h_i$.

Denote by $C^{Lip}(U)$ the space of Lipschitz functions $v : U \to \mathbb{C}$ for such $v$ let $\text{Lip}(v)$ denote the Lipschitz constant of $v$, and for $t \in \mathbb{R}$, $|t| \geq 1$, define

$$\|v\|_{\text{Lip}, t} = \|v\|_0 + \frac{\text{Lip}(v)}{|t|}, \quad \|v\|_0 = \sup_{s \in \mathbb{C}} |v(s)|.$$ 

Let $P(F)$ denote the topological pressure of $F$ defined by

$$P(F) = \sup_{\mu \in M_2} \left[ h(\mu) + \int_{\Sigma_+} F \, d\mu \right],$$ 

where $M_2$ is the set of all probability measures on $\Sigma_+^*$ invariant with respect to $\hat{\sigma}$ and $h(\mu)$ is the measure theoretic entropy of $\hat{\sigma}$ with respect to $\mu$.

In the next section we will need an estimate for the iterations $L_{-\delta F - h_0}$. Below we deal with the more general situation when $\hat{g}$ and $\hat{h}$ are not necessarily constant functions. In particular, we study the case when $g$ is defined as in section 2.2 by Gauss curvatures at reflection points. This analysis is motivated by applications related to the dynamical zeta function (see [PS2]). The case $g = h_0$ is covered by the same argument.

We will apply Dolgoypat type estimates ([D]) established in the case of open billiard flows in [St2] for $N = 2$ and in [St3] for $N \geq 3$ under certain assumptions. We are now going to state these assumptions in detail.
The following pinching condition\(^4\) is one of the assumptions needed below:

\[
\frac{1}{C} \| e^{\alpha t} u \| \leq \| \phi_t(x) \cdot u \| \leq C e^{\beta t} \| u \|, \quad u \in E^u(x), \quad t > 0,
\]

for some constants \( \alpha, \beta > 0 \) depending on \( x \) but independent of \( u \) with \( \alpha \leq \alpha_x \leq \beta_x \) and \( 2\alpha_x - \beta_x \geq \alpha \) for all \( x \in \Lambda \).

Note that when \( N = 2 \) this condition is always satisfied. For \( N \geq 3 \), (P) follows from certain estimates on the eccentricity of the connected components \( K_j \) of \( K \). According to general regularity results [PSW], (P) implies that \( W^u(x) \) and \( W^s(x) \) are Lipschitz in \( x \in \Lambda \).

Next, consider the following non-flatness condition:

(NF): For every \( x \in \Lambda \) there exists \( \epsilon_x > 0 \) such that there is no \( C^1 \) submanifold \( X \) of \( W^u(x) \) of positive codimension with \( \Lambda \cap W^u(x) \subset X \).

Clearly this condition is always satisfied if \( N = 2 \), while for \( N \geq 3 \) it is at least generic. In the proof of the main result in [St3] this condition plays a technical role, and one would expect that a future refinement of the proof would remove it.

Next, we need some definitions from [St3]. Given \( \xi \in \Lambda \), let \( \exp^\xi : E^\xi(z) \to W^\xi(z) \) and \( \exp^\xi_\tau : E^\xi(z) \to W^\xi(z) \) be the corresponding exponential maps. A vector \( b \in E^u(z) \setminus \{0\} \) is called tangent to \( \Lambda \) at \( z \) if there exist infinite sequences \( \{t^{(m)}\} \subset E^u(z) \) and \( \{t_m\} \subset \mathbb{R} \setminus \{0\} \) such that \( \exp^\xi(t_m, t^{(m)}) \in \Lambda \cap W^u(z) \) for all \( m \), \( t^{(m)} \to b \) and \( t_m \to 0 \) as \( m \to \infty \). It is easy to see that a vector \( b \in E^u(z) \setminus \{0\} \) is tangent to \( \Lambda \) at \( z \) if there exists a \( C^1 \) curve \( z(t) \) \( 0 < t \leq a \) in \( W^u(z) \) for some \( a > 0 \) with \( z(0) = z \), \( \dot{z}(0) = b \), and \( z(t) \in \Lambda \) for arbitrarily small \( t > 0 \).

In a similar way one defines tangent vectors to \( \Lambda \) in \( E^s(z) \).

Denote by \( da \) the standard symplectic form on \( T^*(\mathbb{R}^N) = \mathbb{R}^N \times \mathbb{R}^N \). The following condition says that \( da \) is in some sense non-degenerate on the ‘tangent space’ of \( \Lambda \) near some of its points:

(ND): There exist \( z_0 \in \Lambda \), \( \epsilon > 0 \) and \( \mu_0 > 0 \) such that for any \( \bar{z} \in \Lambda \cap W^u(z_0) \) and any unit vector \( b \in E^u(\bar{z}) \) tangent to \( \Lambda \) at \( \bar{z} \) there exist \( \tilde{z} \in \Lambda \cap W^u(z_0) \) arbitrarily close to \( \bar{z} \) and a unit vector \( a \in E^s(\tilde{z}) \) tangent to \( \Lambda \) at \( \tilde{z} \) with \( |d\alpha(a, b)| \geq \mu_0 \).

Clearly when \( N = 2 \) this condition is always satisfied. In fact, it seems very likely (and there is some evidence supporting it) that this condition is always satisfied for open billiard flows.

Given a real-valued function \( q \in \mathcal{F}_\phi(\Sigma^s_A) \), there exists a unique number \( s(q) \in \mathbb{R} \) such that \( P(-s(q)r + g) = 0 \). The following is an immediate consequence of the main result in [St3].

**Theorem 3.** Assume that the billiard flow \( \phi \) over \( \Lambda \) satisfies conditions (P), (NF) and (ND). Let \( q \in \mathcal{F}_\phi(\Sigma^s_A) \) be a real-valued function such that \( q \circ \psi^{-1} \in C^{l-p}(U) \). Then for every \( a > 0 \) there exist constants \( \sigma(g) < s(q) \), \( C = C(a) > 0 \) and \( 0 < \rho < 1 \) such that for \( s = \tau + \rho t \) with \( \sigma(g) \leq \tau, \ |\tau| \leq a, \ |t| \geq 1 \) and \( n = \rho |\log|t|| + l \), \( p \in \mathbb{N}, \ 0 \leq l \leq |\log|t|| - 1 \), for every function \( v : \Sigma^s_A \to C \) with \( v \circ \psi^{-1} \in C^{l-p}(U) \) we have

\[
\| (L^\tau_{-tr+q} v) \circ \psi^{-1} \|_{L^p, J} \leq C \rho^{|\log|t||} \| v \|_{L^p, J} \| v \circ \psi^{-1} \|_{L^p, J}.
\] (3.14)

As mentioned above, conditions (P), (NF) and (ND) are always satisfied for \( N = 2 \), so (3.14) holds for \( N = 2 \) without any additional assumptions.

\(^4\) It appears that in the proof of the Dolgopyat type estimates in the case of open billiard flows (and some geodesic flows), one should be able to replace condition (P) by just assuming Lipschitzness of the stable and unstable laminations—this will be the subject of some future work.
Now combining (3.13) and theorem 3, we obtain estimates for the iterations $L^n_{-\tau} u$, provided $v = u \circ S$ is determined by a Lipschitz function on $R$.

To relate the quantities $P(-\tau r + \hat{g})$ and $P(-\tau \hat{f} + \hat{g})$, consider

$$T_n = \sum_{\sigma i = L} e^{-\tau d_{\sigma i}(v) + \hat{g}(v)} = \sum_{\sigma i = L} e^{-\tau \hat{f}_i - \hat{g}(v)},$$

where $d_{\sigma i}(v)$ is the length of the periodic trajectory $\gamma(i)$ determined by $i \in \Sigma^s$. Since $S \circ \sigma = \sigma \circ S$, we get

$$g_{\sigma i}(j) = g_\sigma(Si) = \hat{g}_\sigma(Si).$$

Setting $Si = v \in \Sigma^s$, we have $\sigma^nv = v$ and the periodic trajectory $\gamma(v)$ determined by $v$ has length $d_{\sigma i}(v) = d_{\sigma j}(v)$. Thus

$$T_n = \sum_{\sigma^nv = v} e^{-\tau d_{\sigma i}(v) + \hat{g}_\sigma(v)} = \sum_{\sigma^nv = v} e^{-\tau \hat{f}_i(v) + \hat{g}_\sigma(v)} = T_n.$$

On the other hand, it follows from a general property of the topological pressure that

$$P(-\tau r + \hat{g}) = \lim_{n \to \infty} \frac{1}{n} \log T_n,$$

(see, e.g. [R] or theorem 20.3.7 in [KH] from which this property can be derived). Similarly,

$$P(-\tau \hat{f} + \hat{g}) = \lim_{n \to \infty} \frac{1}{n} \log T_n,$$

so we get

$$P(-\tau r + \hat{g}) = P(-\tau \hat{f} + \hat{g}).$$

(3.15)

Introduce the number $s_0 \in \mathbb{R}$ such that $P(-s_0 \tau + \hat{g}) = P(-s_0 \hat{f} + \hat{g}) = 0$.

As a consequence of (3.15), theorem 3, lemma 1 and proposition 4 we obtain the following theorem.

Theorem 4. Assume that the billiard flow $\phi_0$ over $\Lambda$ satisfies conditions (P), (NF) and (ND). Then for any $a > 0$ there exist constants $\sigma_0 < s_0$, $C' = C'(a) > 0$ and $0 < \rho < 1$ so that for any $s = \tau + i t \in \mathbb{C}$ with $\tau \geq \sigma_0$, $|\tau| \leq a$, $|t| \geq 1$, any integer $n = p[\log |t|] + l$, $p \in \mathbb{N}$, $0 \leq l \leq [\log |t|] - 1$, and any function $u : \Sigma^s \to \mathbb{R}$ such that $u \circ S \circ \psi^{-1} \in C^{Lip}(U)$ we have

$$\|L^n_{-\tau} u \circ S \circ \psi^{-1}\|_{Lip,t} \leq C' \rho^{p[\log |t|]} \|P(-\tau \hat{f} + \hat{g})\|_u \circ S \circ \psi^{-1}\|_{Lip,t}.$$  

(3.16)

Proof. Using theorem 3 with $q = \hat{g}$, we find constants $\sigma_0$, $C$ and $\rho$ satisfying (3.14) with $q = \hat{g}$. Given $a > 0$, let $s = \tau + i t \in \mathbb{C}$ be such that $\tau \geq \sigma_0$, $|\tau| \leq a$ and $|t| \geq 1$, and set $n = p[\log |t|] + l$, $p \in \mathbb{N}$, $0 \leq l \leq [\log |t|] - 1$.

Consider an arbitrary function $u \in C(\Sigma^s)$ such that $v = u \circ S \circ \psi^{-1} \in C^{Lip}(U)$. It follows from lemma 1 (and its proof) that there exists a constant $\text{Const} > 0$, depending on $a$, such that for $s \in \mathbb{C}$ with $|\text{Re}(s)| \leq a$ we have $\text{Lip}(h_i) \leq \text{Const} |t|$ and $\text{Lip}(d_i) \leq \text{Const} |t|$. From the definitions of these functions we also have $\|h_i\|_0 \leq \text{Const}$, $\|d_i\|_0 \leq \text{Const}$, $\|d_i\|_0 \leq \text{Const}$ and $\|h_i/d_i\|_0 \leq \text{Const}$. This implies $\text{Lip}(h_i/d_i) \leq \text{Const} |t|$ and $\text{Lip}(d_i/h_i) \leq \text{Const} |t|$. Hence

$$\text{Lip}((h_i/d_i) v) \leq \|h_i/d_i\|_0 \text{Lip}(v) + \|v\|_0 \text{Lip}(h_i/d_i) \leq \text{Const} \text{Lip}(v) + |t| \|v\|_0.$$
Thus,
\[ \| (h_s/d_s) v \|_{\text{Lip},t} = \| (h_s/d_s) v \|_0 + \frac{\text{Lip}((h_s/d_s) v)}{|t|} \]
\[ \leq \text{Const} \left( \| v \|_0 + \frac{\text{Lip}(v)}{|t|} \right) = \text{Const} \| v \|_{\text{Lip},t}. \]

Similarly,
\[ \left\| \frac{d_s}{h_s} L_{-r+\hat{g}}^n \left( \frac{h_s}{d_s} v \right) \right\|_{\text{Lip},t} \leq \text{Const} \left\| L_{-r+\hat{g}}^n \left( \frac{h_s}{d_s} v \right) \right\|_{\text{Lip},t}. \]

Using the above, (3.14), (3.13) and (3.15) we get
\[ \left\| (L_{-r+\hat{g}}^n \circ S \circ \psi^{-1}) \right\|_{\text{Lip},t} = \left\| \frac{d_s}{h_s} L_{-r+\hat{g}}^n \left( \frac{h_s}{d_s} v \right) \right\|_{\text{Lip},t} \leq \text{Const} \left( P(\sigma P) \right) \| (h_s/d_s) v \|_{\text{Lip},t} \]
\[ \leq \text{Const} \rho^{\text{Lip}(|t|)} L_{r+\hat{g}}^n \| (h_s/d_s) v \|_{\text{Lip},t}. \]

This proves (3.16). \qed

4. Proofs of theorems 1 and 2

Consider the space \( \Sigma_A \) and the function \( f : \Sigma_A \rightarrow \mathbb{R}^+ \) introduced in section 2.2. Let \( M_\sigma \) be the space of all probability measures on \( \Sigma_A \) invariant with respect to \( \sigma \). For a continuous function \( G : \Sigma_A \rightarrow \mathbb{R} \), the pressure \( P(G) \) is defined by
\[ P(G) = \sup \left[ h(\mu) + \int G \, d\mu : \mu \in M_\sigma \right], \]
where \( h(\mu) \) is the measure theoretic entropy of \( \sigma \) with respect to \( \mu \). The measure of maximal entropy \( \mu_0 \) for \( \sigma \) is determined by
\[ h(\mu_0) = \sup \{ h(\mu) : \mu \in M_\sigma \} = P(0) = h_0, \]
where \( h_0 > 0 \) being the topological entropy of \( \sigma \). Since \( \sigma \) is conjugated to the billiard ball map \( B : \Lambda_{\alpha K} \rightarrow \Lambda_{\alpha K} \), introduced in section 2, \( h_0 \) coincides with the topological entropy of \( B \).

On the other hand, the matrix \( A \) related to the symbolic codings with obstacles has a unique maximal simple eigenvalues \( \lambda > 1 \) and \( h_0 = \log \lambda \). Note that if we consider, as in section 2, the function \( \tilde{f} \in F_\sigma(\Sigma_A) \) depending only on future coordinates, then \( P(\tilde{f}) = P(f) \).

Next, consider the space \( \Sigma_A^\tau = \Sigma_A \times \Sigma_A \) and the shift operator \( \sigma(x, y) = (\sigma x, \sigma y) \). Note that \( \Sigma_A^\tau \) is a subshift of finite type with matrix
\[ \tilde{A}((i, j), (i', j')) = A(i, j)A(i', j'). \]

Given a function \( G : \Sigma_A \rightarrow \mathbb{R} \), we define the pressure \( P(G) \) by
\[ P(G) = \sup \left[ h(m) + \int G \, dm : m \in M_\tilde{\sigma} \right], \]
where \( M_\tilde{\sigma} \) being the space of probability measures on \( \Sigma_A^\tau \) invariant with respect to \( \tilde{\sigma} \). As in [PoS2], the measure of maximal entropy \( m_0 \) for \( \tilde{\sigma} \) is equal to \( \mu_0 \times \mu_0 \) and the topological entropy of \( \tilde{\sigma} \) is \( 2h_0 \).

Consider the function \( F(x, y) = \tilde{f}(x) - \tilde{f}(y), x, y \in \Sigma_A^\tau \), and note that \( F(sR)|_{s=0} = 2h_0 \),
\[ \frac{d}{ds} F(sR)|_{s=0} = \int F(x, y) \, dm_0(x, y) = \int \tilde{f}(x) \, d\mu_0(x) - \int \tilde{f}(y) \, d\mu_0(y) = 0. \]
Here we used the fact that
\[
\frac{d}{ds}P(s f)\big|_{s=0} = \int \tilde{f}(x) \, d\mu_0.
\]

**Lemma 2.** There do not exist constants \(a > 0\) and \(c \in \mathbb{R}\) and continuous functions \(\psi: \Sigma_A \to \mathbb{R}\) and \(M: \Sigma_A \to a\mathbb{Z}\) such that
\[
F = \psi \circ \bar{\sigma} - \psi + M + c.
\]

**Proof.** Let \(d = \text{dist}(K_1, K_2) > 0\). Assume that the above equation holds for some \(a > 0\), \(c \in \mathbb{R}\) and continuous functions \(\psi: \Sigma_A \to \mathbb{R}\) and \(M: \Sigma_A \to a\mathbb{Z}\). Fix a point \(y \in \Sigma_A\) with \(\sigma^2y = y\) corresponding to a periodic trajectory with 2 reflection points and length 2\(d\). Then for every \(n = 2k \geq 2\) and \(\sigma^n x = x\), we get
\[
F_n(x, y) = \tilde{f}_{2k}(x) - \tilde{f}_{2k}(y) = M_{2k}(x, y) + 2kc
\]
so
\[
\tilde{f}_{2k}(x) = M_{2k}(x, y) + 2kq, \quad \forall k \in \mathbb{N}
\]
with \(q = c+d\). Now we will exploit the construction in lemma 5.2 in [St1], where configurations 
\[
\alpha_k = \{1, 2, 1, 2, ..., 1, 2, 3, 1\}, \quad k \in \mathbb{N}
\]
with \(4k + 2\) terms have been considered. Let \(T_k\) be the periods of the primitive periodic rays following the configurations \(\alpha_k\). It was shown in [St1] that \(T_k\) satisfy the estimate
\[
T_{k-1} + 4d < T_k < T_{k-1} + 4d + C\delta^{2k-4}
\]
with some global constants \(0 < \delta < 1, \ C > 0\) independent of \(k\). Then
\[
T_k = a z_k + (4k + 2)q, \quad T_{k-1} = a z_{k-1} + (4k - 2)q, \quad 2d = a z_0 + 2q
\]
with some integers \(z_0, z_{k-1}, z_k \in \mathbb{Z}\). This implies
\[
z_{k-1} + 2z_0 < z_k < z_{k-1} + 2z_0 + C\delta^{2k-4}.
\]
Letting \(k \to \infty\), we obtain a contradiction. \(\square\)

An application of lemma 2 shows (see [PP]) that there exists \(\beta > 0\) such that
\[
\frac{d^2}{ds^2}P(s F)\big|_{s=0} = \beta^2 > 0.
\]

Repeating the proof of lemmas 1.4 and 1.5 in [PoS2], we obtain the following lemma.

**Lemma 3.** The function \(t \to e^{itF}\) has a Taylor expansion
\[
e^{itF} = e^{2b_0} \left(1 - \frac{\beta^2 t^2}{2} + O(|t|^3)\right)
\]
with \(O(|t|^3)\) uniform on bounded intervals. Moreover, there exists a change of coordinates \(v = v(t)\) such that for \(|t| \leq \epsilon\) we have \(e^{itF} = e^{2b_0}(1 - v^2)\).

We say that \(x \in \Sigma_A^+\) is a prime point if \(\sigma^n x = x\) for some \(n \geq 2\) and there are no integers \(m < n\) with the property \(\sigma^m x = x\). Next, as in [PoS2], given a continuous non-negative function \(\chi: \mathbb{R} \to \mathbb{R}\) with compact support, we introduce the function
\[
\rho_N(\chi) = \sum_{\substack{\nu, \nu' \in P, \\ \nu' < N}} \chi(T_{\nu'} - T_{\nu}).
\]
Then
\[ \rho_N(\chi) = \sum_{n,m=1}^{N} \sum_{|\gamma| = m} \sum_{|\gamma'| = n} \chi(T_\gamma - T_{\gamma'}) = \Xi_N(\chi) + O\left(\|\chi\|_\infty (\log N)^2 e^{3h_0 N/2}\right) \]
with
\[ \Xi_N(\chi) = \sum_{|\gamma| = m} \sum_{|\gamma'| = n} \frac{1}{nm} \sum_{\sigma, \tau \text{ prime points}} \chi(T_\gamma - T_{\gamma'}) \]
\[ = \sum_{|\gamma| = m} \sum_{|\gamma'| = n} \frac{1}{nm} \sum_{\sigma, \tau \text{ prime points}} \chi(\tilde{f}_n(x) - \tilde{f}_m(y)). \]

Next the proof of theorem 1 follows without any change that of theorem 1 in [PoS2] and we omit the details.

**Proof of theorem 2.** The crucial point is an estimate for the iterations of the Ruelle operator
\[ L_{i\tau f - h_0} w(\xi) = \sum_{\sigma \eta = \xi} e^{i\tilde{f}(\eta) - h_0} w(\eta) = e^{-h_0} \sum_{\sigma \eta = \xi} e^{i\tilde{f}(\eta)} w(\eta). \]
It was shown in section 3 that \( P(-\tau r - h_0) = P(-\tau \tilde{f} - h_0) \). This shows that \( P(-\tau r - h_0) \mid_{r=0} = h_0 \), so for the Ruelle operator \( L_{-\tau r - h_0} \) and \( s = i\tau \), \(|\tau| \geq 1\) we can apply the estimates (3.14) with \( v = 1 \). Next, the Ruelle operator \( L_{i\tau f - h_0} \) is conjugated to \( L_{i\tau \tilde{f} - h_0} \) by (3.13), hence the estimates (3.14) can be applied to \( L_{i\tau \tilde{f} - h_0} \). More precisely, there exist \( 0 < \rho_1 < 1 \) and \( C > 0 \) so that for \( n = p[\log |\tau|] + l, \ 0 \leq l \leq [\log |\tau|] - 1 \) we have
\[ \|L_{i\tau \tilde{f} - h_0}^n\|_\infty \leq C^{1/2} \rho_1^{p[\log |\tau|]}, \quad n \geq 1. \]
This yields
\[ \|L_{i\tau f - h_0}^n\|_\infty \leq C^{1/2} e^{h_0 n} \rho_1^{p[\log |\tau|]/2}, \quad n \geq 1, \]
with \( 0 < \rho = \rho_1^{1/2} < 1 \).

Consider the transfer operator
\[ (L_{i\tau F} v)(x, y) = \sum_{\tilde{b}(x, y) = \tilde{b}(x, y)} e^{i\tilde{F}(x', y')} v(x', y'). \]
Since
\[ (L_{i\tau F}^n)(x_0, y_0) = \sum_{\sigma^* \tilde{b} = \tilde{b}_0} e^{i\tilde{F}_n(x_0, y_0)} = \sum_{\sigma^* \tilde{b} = \tilde{b}_0} e^{i\tilde{f}_n(x_0, y_0)} = \sum_{\sigma^* \tilde{b} = \tilde{b}_0} e^{-i\tilde{f}_n(y_0)} \]
\[ = L_{i\tau f}^n(1(x_0)) L_{-i\tau f}^n(1(y_0)), \]
it follows that
\[ \|L_{i\tau f}^n\|_\infty \leq C e^{2h_0 n} \rho^{p[\log |\tau|]} \leq C e^{2h_0 n} \rho^{-\log |\tau|} \leq C e^{2h_0 n} \min\{\rho^n |\tau|^{1/2}, 1\} \]
(4.1)
with \( \alpha = |\log \rho| \). Thus we obtain the following lemma.

**Lemma 4.** There exists \( C > 0, \ 0 < \rho < 1 \) and \( \alpha > 0 \) such that for \( |\tau| \geq 1 > 0 \) we have
\[ \|L_{i\tau f}^n\|_\infty \leq C e^{2h_0 n} \min\{\rho^n |\tau|^{\alpha}, 1\} \].
Next, consider the function
\[ S_N(t) = \sum_{n,m=1}^{N} e^{i(f_n(x) - f_m(y))}. \]

Let \( \chi \in C^\infty_0(\mathbb{R}) \) be a non-negative function and let \( \hat{\chi} \) be the Fourier transform of \( \chi \). Given a sequence \( \epsilon_n = O(e^{-\eta n}) \), denote
\[ A_2(N, z) = \left| \int_{|t| \geq \beta \sqrt{N}} e^{\frac{i}{\beta \sqrt{N}} \left[ e^{-2\beta \epsilon_n N} \left( \frac{t}{\beta \sqrt{N}} \right) \hat{\chi} \left( \frac{\epsilon_n t}{\beta \sqrt{N}} \right) \right]} \right|. \]

As in [PoS2], applying lemma 4 we obtain the following lemma.

**Lemma 5.** For sufficiently small \( \eta > 0 \) and for \( N \to +\infty \) we have \( \sup_{z \in \mathbb{R}} A_2(N, z) = 0 \).

The rest of the proof of theorem 2 closely follows arguments in [PoS2] and we omit the details.

By the same arguments we get the following proposition.

**Proposition 5.** Let \( \epsilon_n \) be a sequence such that \( \epsilon_n = O(e^{-\eta n}) \), where \( \eta > 0 \) is a sufficiently small constant. Then for
\[ \omega(n, I_n(z)) = \{|\gamma, \gamma'| : |\gamma| = |\gamma'| = n, z + \epsilon_n a \leq T_\gamma - T_{\gamma'} \leq z + \epsilon_n b| \}
we have
\[ \lim_{n \to +\infty} \sup_{z \in \mathbb{R}} \left| \frac{n^{5/2}}{\epsilon_n e^{2\beta n}} \omega(n, I_n(z)) - \frac{b - a}{(2\pi)^{1/2} e^{-\epsilon_n^2/2\beta^2 n}} \right| = 0. \]

## 5. Separation condition for the lengths of the primitive periodic rays

In this section we discuss some open problems related to the distribution of the lengths of primitive periodic rays. Let \( \mathcal{P} \) be the set of all primitive periodic rays in \( \Omega \). Let \( \Xi \) be the set of the lengths of rays \( \gamma \in \mathcal{P} \) and let \( \Sigma \) be the set of periods of all periodic rays in \( \Omega \). It is known that
\[ \#\{ \gamma \in \mathcal{P} : T_\gamma \leq x \} = \frac{e^{hx}}{h^x} (1 + o(1)), \quad x \to +\infty, \quad (5.1) \]
where
\[ h = \sup_{\mu \in M_\mu} \frac{h(m)}{\int f \ d\mu} > 0 \]
is the topological entropy of the open billiard flow in \( \Omega \). In particular, if we have Dolgopyat type estimates for the Ruelle operator \( L_{-t} \) for \( s = \tau + it \), \( |t| \geq 1 \) and \( h - \epsilon \leq \tau \leq h \), we can obtain a sharper estimate than (5.1)(see [PoS1], [St2], [St3])
\[ \#\{ \gamma \in \mathcal{P} : T_\gamma \leq x \} = li(e^{hx}) + O(e^{\epsilon x}), \quad x \to +\infty, \]
where \( li(x) = \int_2^x dt / \log t \) and \( 0 < c < h \). It was proved in [PS1, chapter 3], that for generic obstacles \( K \) the periods of the primitive rays are rationally independent, that is
\[ \gamma \neq \delta \Rightarrow \frac{T_\gamma}{T_\delta} \notin \mathbb{Q}, \quad \forall \gamma, \forall \delta \in \mathcal{P}. \]

Thus for generic obstacles we have \( T_\gamma \neq T_\delta \) for any distinct elements \( \gamma \) and \( \delta \) of \( \mathcal{P} \).
In what follows we assume that the latter property is satisfied. Let $T_\gamma$, $\gamma \in \mathcal{P}$, be ordered as a sequence

$$T_1 < T_1 < \cdots < T_n < \cdots$$

Introduce the intervals

$$J(\gamma, \delta) = [T_\gamma - e^{-\delta T_\gamma}, T_\gamma + e^{-\delta T_\gamma}], \quad \gamma \in \mathcal{P}, \quad \delta > 0.$$ 

Obviously, the number of pairs $(T_\gamma, T_\delta)$ lying in $J(\gamma, \delta)$ decreases with $\delta$.

**Definition 1.** We say that the obstacle $K$ satisfies separation condition $(S)$ if there exists $\delta > h > 0$ such that

$$J(\gamma, \delta) \cap \Pi = T_\gamma, \quad \forall \gamma \in \mathcal{P}. \tag{5.2}$$

We say that $K$ satisfies condition $(S_2)$ if there exists $\delta > h > 0$ such that

$$\# \{\gamma \in \mathcal{P} : T_\gamma \leq x, J(\gamma, \delta) \cap \Pi = T_\gamma \} \sim e^{\frac{x}{\delta}}, \quad x \to +\infty. \tag{5.3}$$

Condition $(S)$ was introduced in [P2]; however, we are not aware of any geometric conditions on $K$ that would imply $(S)$. The same can be said about $(S_2)$. It is important to note that proposition 5 cannot be applied with the sequence $\epsilon_n = e^{-\delta T_\gamma}$, $\delta > h$. On the other hand, it was shown in [P2] that under condition $(S)$ there exists a global constant $A_0 > 0$ such that

$$\# \{d_\gamma \in \Xi : d_\gamma \in J(\gamma, \delta)\} \leq A_0 T_\gamma, \quad \forall \gamma \in \mathcal{P}.$$ 

Thus, the distribution of all periods in $\Xi$ is such that we have no clustering of periods with big density in $J(\gamma, \delta)$. It is clear that condition $(S_2)$ is much weaker than $(S)$.

Let $\chi(t) \in C_0^\infty(-1, 1)$ be a positive function such that $\chi(t) = 1$ for $|t| \leq \epsilon_0 < 1/2$ with Fourier transform $\hat{\chi}(\xi) \geq 0$, $\forall \xi \in \mathbb{R}$. Consider the function $\varphi_j(t) = \chi(e^{\delta T_\gamma}(t - T_j))$. For the proof of the so-called Modified Lax–Phillips Conjecture (MLPC) for the Dirichlet problem in $\mathbb{R} \times \Omega$ (see [12]) it is necessary to find $\delta > h$ and a sequence $j \to +\infty$ so that

$$\left| \sum_{\gamma} \left( -1 \right)^{\gamma} / T_\gamma \right| \sim e^{\frac{1}{2} \eta T_\gamma}, \tag{5.4}$$

with $\eta_0 > 0$, $\eta > 0$ independent of $j$. Here $P_\gamma$ is the linear Poincaré map related to $\gamma$ introduced in section 1. It is easy to see that under condition $(S_2)$ we can arrange (5.4) for a suitable sequence $j \to +\infty$, so (MLPC) follows from $(S_2)$.

Indeed, assuming $(S_2)$, consider the number of iterated periodic rays $\mu \notin \mathcal{P}$ with lengths $d_\mu \in J(\gamma, \delta)$, provided $\gamma \in \mathcal{P}$, $T_\gamma \leq x$. If $d_\mu = k T_\gamma$, $k \geq 2$, then $2 \leq k \leq \frac{x+1}{d_\mu}$, where $d_\mu = 2 \min_{x \neq j} \mathrm{dist}(K_j, K_j)$. Thus, using (5.1), the number of such non-primitive periodic rays $\mu$ for $x \geq M_0$ large enough is not greater than

$$\frac{C}{h(x+1)} \sum_{k=2}^{x+1/d_\mu} k e^{-\frac{x}{x+1}} \leq C e^{\frac{x}{x+1}} \left( 1 + e^{-\frac{x}{x+1}} \right) \sum_{k=3}^{x+1/d_\mu} \frac{k}{2} \leq \frac{1}{2} e^{\frac{x}{x+1}}.$$

Consequently, taking into account (5.3), it is possible to find an infinite number of intervals $J(\gamma_j, \delta)$ with $\gamma_j \in \mathcal{P}$ such that $T_{\gamma_j} \to +\infty$ and

$$J(\gamma_j, \delta) \cap \Xi = T_{\gamma_j}.$$ 

For such rays the sum on the left-hand side of (5.4) is reduced to the term $T_{\gamma_j} / \det(I - P_{\gamma_j})^{1/2}$ and inequality (5.4) follows from the estimate

$$\det(I - P_{\gamma_j}) \leq e^{T_{\gamma_j}}.$$
with some global constant \( c > 0 \) (see, for instance, appendix A.1 in [P1]). By the argument of Ikawa [I2] we conclude that (5.4) implies (MLPC).

The above analysis shows that \((S_2)\) can be replaced by a weaker condition in order to satisfy (5.4). Let \(P_e\) (respectively \(P_o\)) be the set of primitive periodic rays with an even (respectively odd) number of reflections and let \(\Pi_e\) (respectively \(\Pi_o\)) be the set of periods of \(\gamma \in P_e\) (respectively \(\gamma \in P_o\)). We have (see [G, X1, X2]) the following analogue of (5.1):

\[
\#\{\gamma \in P_e : T_\gamma \leq x\} \sim e^{\frac{\delta x}{2h}}, \quad x \to +\infty,
\]

\[
\#\{\gamma \in P_o : T_\gamma \leq x\} \sim e^{\frac{\delta x}{2h}}, \quad x \to +\infty.
\]

**Definition 2.** We say the obstacle \(K\) satisfies condition \((S_3)\) if there exists \(\delta > h\) such that

\[
\#\{\gamma \in P_e : T_\gamma \leq x, \ J(\gamma, \delta) \cap \Pi_o = \emptyset\} \sim e^{\frac{\delta x}{2h}}, \quad x \to +\infty. \quad (5.5)
\]

Under condition \((S_3)\) in every interval \(J(\gamma, \delta), \ \gamma \in P_e\), we may have an arbitrary clustering of periods in \(\Pi_e\) and this leads to a sum of terms with positive signs in (5.4). Assuming \((S_3)\) fulfilled, it is not hard to deduce that there exists a sequence of \(\gamma_j \in P_e, \ T_{\gamma_j} \to +\infty, \) such that

\[
J(\gamma_j, \delta) \cap \Xi \subset (\Pi_e \cup 2\Pi), \quad \forall j \in \mathbb{N},
\]

and the latter implies (MLPC). We leave the details to the reader.

It is an interesting open problem to investigate whether conditions \((S_2), \ (S_3)\) or some other similar condition are fulfilled.

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