An ISS self-triggered implementation of linear controllers. *

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Abstract

Nowadays control systems are mostly implemented on digital platforms and, increasingly, over shared communication networks. Reducing resources (processor utilization, network bandwidth, etc.) in such implementations increases the potential to run more applications on the same hardware. We present a self-triggered implementation of linear controllers that reduces the amount of controller updates necessary to retain stability of the closed-loop system. Furthermore, we show that the proposed self-triggered implementation is robust against additive disturbances and provide explicit guarantees of performance. The proposed technique exhibits an inherent trade-off between computation and potential savings on actuation.

1 Introduction

The majority of control systems are nowadays implemented on digital platforms equipped with microprocessors capable of running real-time operating systems. This creates the possibility of sharing the computational resources between control and other kinds of applications thus reducing the deployment costs of complex control systems. Many control systems are also implemented over shared communication media making it necessary to share the communication medium. The concept of self-triggered control was introduced by Velasco et al in [1] to take advantage of the possibility (or necessity) of sharing resources. The key idea of self-triggered control is to compute, based on the current state measurements, the next instant of time at which the control law is to be recomputed. In between updates of the controller the control signal is held constant, and the appropriate generation of the update times guarantees the stability of the

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closed-loop system. Under a periodic implementation, the control law is executed every $T$ units of time, regardless of the current state of the plant. Hence this period $T$ has to be chosen in order to guarantee stability under all possible operating conditions. On the other hand, under self-triggered implementations the time between updates is a function of the state, and thus less control executions are expected. On the other hand, the intervals of time in which no attention is devoted to the plant pose a new concern regarding the robustness of self-triggered implementations.

The contribution of this paper is to describe a self-triggered implementation for linear systems, in which the times between controller updates are as large as possible so as to enforce desired levels of performance subject to the computational limitations of the digital platform. By increasing the available computational resources, the performance guarantees improve while the number of controller executions is reduced. Hence, the proposed technique reduces the actuation requirements (and communication, in networked systems) in exchange for computation. Furthermore, we also show that the proposed self-triggered implementation results in an exponentially input-to-state stable closed-loop system with corresponding gains depending on the available computational resources. A preliminary version of these results appeared in the conference papers [2] and [3].

Several self-triggered implementations have been proposed in the last years, both for linear [4] and non-linear [5] plants. The latter when applied to linear systems degenerates into a periodic implementation, while [4] makes use of very conservative approximations. In contrast with those two techniques the approach followed in the present work provides large inter-execution times for linear systems by not requiring a continuous decay of the Lyapunov function in use, much in the spirit of [6]. Computing exactly the maximum allowable inter-execution times guaranteeing stability requires the solution of transcendental equations for which closed form expressions do not exist. Our proposal computes approximations of these maximum allowable inter-execution times while providing stability guarantees. The idea advocated in this paper, trading communication/actuation for computation, was already explored in [7]. However, their approach is aimed at loosely coupled distributed systems, where local actuation takes place continuously and communication between subsystems is reduced by means of state estimators. In the analysis of robustness of the proposed implementation the authors were influenced by the approach followed in [8] and [9]. Finally, the notion of input-to-state stability [10] is fundamental in the approach followed in the present paper.

2 Notation

We denote by $\mathbb{R}^+$ the positive real numbers. We also use $\mathbb{R}^+_0 = \mathbb{R}^+ \cup \{0\}$. The usual Euclidean ($l_2$) vector norm is represented by $\| \cdot \|$. When applied to a matrix $| \cdot |$ denotes the $l_2$ induced matrix norm. A matrix $P \in \mathbb{R}^{n \times n}$ is said to be positive definite, denoted $P > 0$, whenever $x^T P x > 0$ for all $x \neq 0$, $x \in \mathbb{R}^n$. 


and a matrix $A$ is said to be Hurwitz when all its eigenvalues have strictly negative real part. We denote by $I$ the identity matrix. By $\lambda_\min(P), \lambda_\max(P)$ we denote the minimum and maximum eigenvalues of $P$ respectively. A function $\gamma : [0, \infty[ \to \mathbb{R}^+_0$, is of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, $\gamma(0) = 0$ and $\gamma(s) \to \infty$ as $s \to \infty$. Given an essentially bounded function $\delta : \mathbb{R}^+_0 \to \mathbb{R}^m$ we denote by $\| \delta \|_\infty$ its $L_\infty$ norm, i.e., $\| \delta \|_\infty = (\text{ess sup}_{t \in \mathbb{R}^+_0} \{ |\delta(t)| \}) < \infty$. We consider linear systems described by the differential equation $\frac{d}{dt} \xi = A\xi + B\chi + \delta$ with inputs $\chi : \mathbb{R}^+_0 \to \mathbb{R}^p$ and $\delta : \mathbb{R}^+_0 \to \mathbb{R}^p$ essentially bounded piecewise continuous functions of time. The input $\chi$ will be used to denote controlled inputs, while $\delta$ will denote disturbances. We refer to such systems as control systems. Solutions of a control system with initial condition $x$ and inputs $\chi$ and $\delta$, denoted by $\xi_{x,\chi,\delta}$, satisfy: $\xi_{x,\chi,\delta}(0) = x$ and $\frac{d}{dt} \xi_{x,\chi,\delta}(t) = A\xi_{x,\chi,\delta}(t) + B\chi(t) + \delta(t)$ for almost all $t \in \mathbb{R}^+_0$. The notation will be relaxed by dropping the subindex when it does not contribute to the clarity of exposition. A linear feedback law for a linear control system is a map $u = Kx$; we will sometimes refer to such a law as a controller for the system.

**Definition 2.1** (Lyapunov function). A function $V : \mathbb{R}^m \to \mathbb{R}^+_0$, is said to be a Lyapunov function for a linear system $\dot{\xi} = A\xi$ if for every $x \in \mathbb{R}^m$; $\alpha(|x|) \leq V(x) \leq \beta(|x|)$ for some $\alpha, \beta \in \mathcal{K}_\infty$, and there exists $\lambda \in \mathbb{R}^+$ such that for every $x \in \mathbb{R}^m$:

$$\frac{\partial V}{\partial x} Ax \leq -\lambda V(x).$$

We will refer to $\lambda$ as the rate of decay of the Lyapunov function. In what follows we will consider functions of the form $V(x) = (x^TPx)^+$, in which case $V$ is a Lyapunov function for system $\dot{\xi} = A\xi$ if and only if $P > 0$ and $A^TP + PA \leq -2\lambda I$ for some $\lambda \in \mathbb{R}^+$, the rate of decay.

**Definition 2.2** (EISS). A control system $\dot{\xi} = A\xi + \delta$ is said to be exponentially input-to-state stable (EISS) if there exists $\lambda \in \mathbb{R}^+, \sigma \in \mathbb{R}^+$ and $\gamma \in \mathcal{K}_\infty$ such that for any $t \in \mathbb{R}^+_0$ and for all $x \in \mathbb{R}^m$:

$$|\xi_{x,\delta}(t)| \leq \sigma |x|e^{-\lambda t} + \gamma(\| \delta \|_\infty).$$

We shall refer to $(\beta, \gamma)$, where $\beta(s, t) = s\sigma e^{-\lambda t}$, as the EISS gains of the EISS estimate. If no disturbance is present, i.e., $\delta = 0$, an EISS system is said to be globally exponentially stable (GES).

### 3 A self-triggered implementation for stabilizing linear controllers.

Consider the sampled-data system:

$$\dot{\xi}(t) = A\xi(t) + B\chi(t) + \delta(t) \quad (1)$$

$$\chi(t) = K\xi(t_k), \ t \in [t_k, t_{k+1}] \quad (2)$$
where \( \{t_k\}_{k \in \mathbb{N}} \) is a divergent sequence of update times for the controller, and \( A+BK \) is Hurwitz. The signal \( \delta \) can be used to describe measurement disturbances, actuation disturbances, unmodeled dynamics, or other sources of uncertainty as described in [9].

A self-triggered implementation of the linear stabilizing controller (2) for the plant (1) is given by a map \( \Gamma : \mathbb{R}^m \rightarrow \mathbb{R}^+ \) determining the controller update time \( t_{k+1} \) as a function of the state \( \xi(t_k) \) at the time \( t_k \), i.e., \( t_{k+1} = t_k + \Gamma(\xi(t_k)) \). If we denote by \( \tau_k \) the inter-execution times \( \tau_k = t_{k+1} - t_k \), we have \( \tau_k = \Gamma(\xi(t_k)) \). Once the map \( \Gamma \) is defined, the expression closed-loop system refers to the sampled-data system (1) and (2) with the update times \( t_k \) defined by \( t_{k+1} = t_k + \Gamma(\xi(t_k)) \).

The problem we solve in this paper is the following:

**Problem 3.1.** Given a linear system (1) and a linear stabilizing controller (2), construct a self-triggered implementation \( \Gamma : \mathbb{R}^m \rightarrow \mathbb{R}^+ \) of (2) that renders EISS the closed-loop system defined by (1), (2), while enlarging the inter-execution times.

In order to formally define the self-triggered implementation proposed in this paper, we need to introduce two maps:

- \( h_c \), a continuous-time output map and
- \( h_d \), a discrete-time version of \( h_c \).

Let \( V \) be a Lyapunov function of the form

\[
V(x) = (x^TPx)^2
\]

for \( \xi = (A+BK)\xi \), with rate of decay \( \lambda_o \). The output map \( h_c : \mathbb{R}^m \times \mathbb{R}^+_0 \rightarrow \mathbb{R}^+_0 \) is defined by:

\[
h_c(x, t) := V(\xi_x(t)) - V(x)e^{-\lambda t}
\]

for some \( 0 < \lambda < \lambda_o \). Note that by enforcing:

\[
h_c(\xi_x(t_k), t) \leq 0, \quad \forall t \in [0, \tau_k] \quad \forall k \in \mathbb{N}
\]

the closed-loop system satisfies:

\[
V(\xi_x(t)) \leq V(x)e^{-\lambda t}, \quad \forall t \in \mathbb{R}^+_0 \quad \forall x \in \mathbb{R}^m
\]

which implies exponential stability of the closed-loop system in the absence of disturbances, i.e., when \( \delta(t) = 0 \) for all \( t \in \mathbb{R}^+_0 \).

Our objective is to construct a self-triggered implementation enforcing (4). Since no digital implementation can check (4) for all \( t \in [t_k, t_{k+1}] \), we consider instead the following discrete-time version of (4) based on a sampling time \( \Delta \in \mathbb{R}^+ \):

\[
h_d(\xi_x(t_k), n) := h_c(\xi_x(t_k), n\Delta) \leq 0 \quad \forall n \in \left[0, \left\lfloor \frac{\tau_k}{\Delta} \right\rfloor \right],
\]

and for all \( k \in \mathbb{N} \). This results in the following self-triggered implementation, first introduced by the authors in [2], where we use \( N_{\text{min}} := \left\lfloor \frac{\tau_{\text{min}}}{\Delta} \right\rfloor \), \( N_{\text{max}} := \left\lceil \frac{\tau_{\text{max}}}{\Delta} \right\rceil \), and \( \tau_{\text{min}} \) and \( \tau_{\text{max}} \) are design parameters. A similar approach was followed in [11] in the context of event-triggered control.
Definition 3.2. The map $\Gamma_d : \mathbb{R}^n \to \mathbb{R}^+$ defined by:

$$\Gamma_d(x) := \max\{\tau_{\min}, n_k \Delta\}$$

$$n_k := \max_{n \in \mathbb{N}} \{n \leq N_{\max}|h_d(x, s) \leq 0, s = 0, \ldots, n\}$$

prescribes a self-triggered implementation of the linear stabilizing controller (2) for plant (1).

Note that the role of $\tau_{\min}$ and $\tau_{\max}$ is to enforce explicit lower and upper bounds, respectively, for the inter-execution times of the controller. The upper bound enforces robustness of the implementation and limits the computational complexity.

Remark 3.3. Linearity of (1) and (2) enables us to compute $h_d$ as a quadratic function of $\xi(t_k)$. Moreover, through a Veronese embedding we can implement the self-triggered policy described in Definition 3.2 so that its computation has space complexity $q^{m(m+1)/2}$ and time complexity $q + (2q + 1)m(m+1)/2$ where $q := N_{\max} - N_{\min}$. For reasons of space we omit these details. They can be found in [2].

4 Main results

The proofs of all the results reported in this section can be found in the Appendix. The following functions will be used to define EISS-gains:

$$\rho_P := \left(\frac{\lambda_M(P)}{\lambda_m(P)}\right)^{1/2}, \quad \gamma_{\mu,T}(s) := s\frac{\lambda_M(P)}{\lambda_m(P)} \int_0^T |e^{Ar}| dr.$$  

We start by establishing a result explaining how the design parameter $\tau_{\min}$ should be chosen. The function $\Gamma_d$ can be seen as a discrete-time version of the function $\Gamma_c : \mathbb{R}^m \to \mathbb{R}^+_0$ defined by:

$$\Gamma_c(x) := \max_{\tau \in \mathbb{R}_0^+} \{\tau \leq \tau_{\max}|h_c(x, s) \leq 0, \forall s \in [0, \tau]\}. \quad (5)$$

If we use $\Gamma_c$ to define an ideal self-triggered implementation, the resulting inter-execution times are no smaller than $\tau_{\min}^*$ which can be computed as detailed in the next result.

Lemma 4.1. The inter-execution times generated by the self-triggered implementation in (5) are lower bounded by:

$$\tau_{\min}^* = \min\{\tau \in \mathbb{R}^+: \det M(\tau) = 0\} \quad (6)$$

where:

$$M(\tau) := C(e^{F_T \tau} C_T PC e^{F \tau} - C_T PC e^{-\lambda \tau}) C_T,$$

$$F := \begin{bmatrix} A + BK & BK \\ -A - BK & -BK \end{bmatrix}, \quad C := [I \ 0].$$
The computation of $\tau^*_{\min}$ described in Lemma 4.1 can be regarded as a formal procedure to find a sampling period for periodic implementations (also known as maximum allowable time interval or MATI). It should be contrasted with the frequently used ad-hoc rules of thumb [12], [13] (which do not provide stability guarantees). Moreover, an analysis similar to the one in the proof of this lemma can also be applied, mutatis mutandis, to other Lyapunov-based triggering conditions, like the ones appearing in [14] and [4]. Notice that the self-triggered approach always renders times no smaller than the periodic implementation, since under a periodic implementation the controller needs to be executed every $\tau^*_{\min}$ (in order to guarantee performance under all possible operating points).

The second and main result establishes EISS of the proposed self-triggered implementation and details how the design parameters $\tau_{\min}, \tau_{\max}, \Delta$, and $\lambda$ affect the EISS-gains.

**Theorem 4.2.** If $\tau_{\min} \leq \tau^*_{\min}$, the self-triggered implementation in Definition 3.2 renders the closed-loop system EISS with gains $(\beta, \gamma)$ given by:

\[
\beta(s, t) := \rho P g(\Delta, N_{\max}) e^{-\lambda t} s,
\]

\[
\gamma(s) := \gamma_{P,N_{\max}} \Delta(s) \left( \frac{\lambda_{\min}(P) g(\Delta, N_{\max})}{1 - e^{-\lambda \tau_{\min}}} + \gamma_{I,N_{\max}} \Delta(s) \right)
\]

where:

\[
g(\Delta, N_{\max}) := \rho P \left( e^{\frac{(s+2\lambda)\mu \Delta}{\mu - \rho}} + e^{2\lambda(N_{\max} - 1)\Delta} \left( e^{\frac{(s+2\lambda)\mu \Delta}{\mu - \rho}} - e^{\frac{2\lambda \mu \Delta}{\mu - \rho}} \right)^{\frac{1}{2}} \right),
\]

\[
\rho := \lambda_{M}(G), \quad \mu := \lambda_{m}(G),
\]

\[
G := \begin{bmatrix}
P_{\frac{1}{2}} A P_{\frac{1}{2}} + (P_{\frac{1}{2}} A P_{\frac{1}{2}})^T & P_{\frac{1}{2}} B K P_{\frac{1}{2}}^T \\
(P_{\frac{1}{2}} B K P_{\frac{1}{2}}^T)^T & 0
\end{bmatrix}.
\]

Note that while $\tau_{\min}$ is constrained by $\tau^*_{\min}, \tau_{\max}$ can be freely chosen. However, by enlarging $\tau_{\max}$ (and thus $N_{\max}$) we are degrading the EISS-gains. It is also worth noting that by enlarging $\tau_{\max}$ one can allow longer inter-execution times, and compensate the performance loss by decreasing $\Delta$, at the cost of performing more computations.

Let us define the maximum exact inter-execution time from $x$ as $\tau^*(x) := \min\{\Gamma_{c}(x), \tau_{\max}\}$, where the upper bound is required to obtain robustness against disturbances. The third and final result states that the proposed self-triggered implementation is optimal in the sense that it generates the longest possible inter-execution times given enough computational resources. Hence, by enlarging the inter-execution times we are effectively trading actuation for computation. The proof of the following proposition follows from the proof of Theorem 4.2.

**Proposition 4.3.** The inter-execution times provided by the self-triggered implementation in Definition 3.2 are bounded from below as follows:

\[
\Gamma_{d}(x) \geq \tau^*(x) - \Delta, \ \forall \ x \in \mathbb{R}^m.
\]
Note that even if $\Gamma_d(x) \geq \tau^*(x)$ the performance guarantees provided in Theorem 4.2 still hold.

**Remark 4.4.** When implementing self-triggered policies on digital platforms several issues related to real-time scheduling need to be addressed. For a discussion of some of these issues we refer the readers to [15]. Here, we describe the minimal computational requirements for the proposed self-triggered implementation under the absence of other tasks. There are three main sources of delays: measurement, computation, and actuation. Since the computation delays dominate the measurement and actuation delays, we focus on the former. The computation of $\Gamma_d$ is divided in two steps: a preprocessing step performed once by execution, and a running step performed $n$ times when computing $h_d(x,n)$. The preprocessing step has time complexity $(m^2 + m)/2$ and the running step has time complexity $m^2 + m$. If we denote by $\tau_c$ the time it takes to execute an instruction in a given digital platform, the self-triggered implementation can be executed if:

$$\frac{3}{2}(m^2 + m)\tau_c \leq \tau_{\text{min}}, \quad (m^2 + m)\tau_c \leq \Delta.$$

The first inequality imposes a minimum processing speed for the digital platform while the second equality establishes a lower bound for the choice of $\Delta$.

We refer the interested reader to [2] and [3] for numerical examples illustrating the proposed technique and the guarantees it provides.

## 5 Conclusions

This paper described a self-triggered implementation of stabilizing feedback control laws for linear systems. The proposed technique guarantees exponential input-to-state stability of the closed-loop system with respect to additive disturbances. Furthermore, the proposed self-triggered implementation allows the tuning of the resulting performance and complexity through the selection of the parameters $\Delta$, $\lambda$, $\tau_{\text{min}}$ and $\tau_{\text{max}}$. The performance guarantees can be improved and the inter-execution times enlarged by increasing the computational complexity of the implementation.

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6 Appendix: Proofs

Proof of Lemma 4.1. It can be verified that \( h_c \) satisfies \( h_c(x, 0) = 0 \) and \( \left. \frac{\partial h_c}{\partial t} \right|_{t=0} h_c(x, t) < 0 \), \( \forall x \in \mathbb{R}^m \), which, by continuity of \( h_c \), implies the existence of some \( \tau^*_{\min}(x) > 0 \) such that \( \Gamma_c(x) \geq \tau^*_{\min}(x) \). Let us define the variables \( \eta(t) = \xi(t) - \xi(t_k), t \in [t_k, t_{k+1}] \) and \( \zeta = [\xi^T, \eta^T]^T \). Note that at the controller update times \( \eta(t_k) = 0 \).

Under this new notation, system (1) with controller (2), in the absence of disturbances, can be rewritten as \( \dot{\xi}(t) = F\zeta(t) \) with solution \( \zeta(t) = e^{Ft}y \), \( y = [x^T, 0]^T \). Let us denote by \( \hat{h}_c \) the map \( \hat{h}_c(y, t) = V(C\zeta(t)) - V(Cy)e^{-\lambda t} \).

While it is not possible to find \( \Gamma_c \) in closed form, we can find its minimum value by means of the Implicit Function Theorem. Differentiating \( \phi(x) = \hat{h}_c(y, \Gamma_c(x)) = 0 \) with respect to the initial condition \( x \) we obtain:

\[
\frac{d\phi}{dx} = \left. \frac{\partial h_c}{\partial t} \right|_{t=\Gamma_c(x)} \frac{d\Gamma_c}{dx} + \frac{\partial h_c}{\partial y} \frac{dy}{dx} = 0.
\]

The extrema of the map \( \Gamma_c \) are defined by the following equation:

\[
\frac{d\Gamma_c}{dx} = -\left( \frac{\partial h_c}{\partial t} \bigg|_{t=\Gamma_c(x)} \right)^{-1} \left( \frac{\partial h_c}{\partial y} \right) \frac{dy}{dx} = 0.
\]

Hence, the extrema of \( \Gamma_c \) satisfy either \( \left. \frac{\partial h_c}{\partial y} \right|_{t=\Gamma_c(x)} dy \big|_{x, t} = 0 \) for some \( t \in \mathbb{R}^+ \) or \( \frac{d\Gamma_c}{dx}(x, t) = 0 \). The latter case corresponds to situations in which for some \( x \) the map \( \hat{h}_c \) reaches zero exactly at an extremum, and thus can be disregarded as violations of the condition \( h_c(t_k, t) \leq 0 \). Combining \( \frac{\partial h_c}{\partial y} \frac{dy}{dx}(\tau, x) = 0 \) into matrix form we obtain:

\[
M(\tau)x = 0. \tag{7}
\]

The solution to this equation provides all extrema of the map \( \Gamma_c(x) \) that incur a violation of \( h_c(x, t) \leq 0 \). Thus, the minimum \( \tau \) satisfying (7) corresponds to the smallest time at which \( h_c(x, \tau) = 0 \), \( \left. \frac{\partial h_c}{\partial t} \bigg|_{t=\tau} \right) > 0 \) can occur. Since the left hand side of (7) is linear in \( x \), it is sufficient to check when the matrix has a nontrivial nullspace. Hence the equality (6). \( \square \)

We introduce now a Lemma that will be used in the proof of Theorem 4.2.

Lemma 6.1. Consider system (1) and a positive definite function \( V(x) = (x^T P x)^\frac{1}{2}, P > 0 \). For any given \( 0 \leq T < \infty \) the following bound holds:

\[
V(\xi_{x\chi}^\delta(t)) \leq V(\xi_{x\chi_0}(t)) + \gamma_{P, T}(\|\delta\|_{\infty}), \forall t \in [0, T].
\]

Proof. Applying the triangular inequality and using Lipschitz continuity of \( V \) we have:

\[
V(\xi_{x\chi}^\delta(t)) = |V(\xi_{x\chi_0}(t)) + V(\xi_{x\chi}^\delta(t)) - V(\xi_{x\chi_0}(t))| \leq V(\xi_{x\chi_0}(t)) + \frac{\lambda_{\delta}(P)}{\lambda_{m}(P)}|\xi_{x\chi}^\delta(t) - \xi_{x\chi_0}(t)|.
\]
Integrating the dynamics of $\xi$ and after applying H"older’s inequality one can conclude that:

$$|\xi_{x\chi}(t) - \xi_{x\chi}(0)| \leq \int_0^t |e^{Ar}|dr\|\delta\|_{\infty}.$$  

And thus for all $t \in [0, T]$:

$$V(\xi_{x\chi}(t)) \leq V(\xi_{x\chi}(0)) + \frac{\mu_1(P)}{\lambda_2(P)} \int_0^T |e^{Ar}|dr\|\delta\|_{\infty}.$$  

\textbf{Proof of Theorem 4.2.} We start by proving that in the absence of disturbances the following bound holds:

$$|\xi_{x\chi}(t_k + \tau)| \leq g(\Delta, N_{\text{max}})|\xi_{x\chi}(t_k)|e^{-\lambda \tau}, \ \forall \tau \geq 0. \quad (8)$$  

Let $W(x) = x^TPx$ and use $W(t)$ to denote $W(\xi_{x\chi}(t))$, with $\xi$ determined by (1), (2), and $\tau_k = \Gamma_d(\xi(t_k))$. By explicitly computing $\dot{W}(t)$ one obtains:

$$\dot{W}(t) = \left[(P^T\ddot{\xi}(t))T (P^T\ddot{\xi}(t_k))T\right] G \left[(P^T\ddot{\xi}(t))T (P^T\ddot{\xi}(t_k))T\right] T,$$

for $t \in [t_k, t_{k+1}]$, and thus the following bounds hold:

$$\mu (W(t) + W(t_k)) \leq \dot{W}(t) \leq \rho (W(t) + W(t_k)),$$

for $t \in [t_k, t_{k+1}]$. After integration, one can bound the trajectories of $W(t)$, when $t + s$ belongs to the interval $[t_k, t_{k+1}]$, as:

$$W(t + s) \leq e^{\rho s}W(t) + W(t_k)(e^{\rho s} - 1),$$

$$W(t + s) \geq e^{\mu s}W(t) + W(t_k)(e^{\mu s} - 1).$$

Let us denote $t_k + n\Delta$ by $r_n$ for succinctness of the expressions that follow. An upper bound for $W(t)$ valid for $t \in [r_n, r_{n+1}]$ is then provided by:

$$\left\{ \begin{array}{ll}
\begin{align*}
\dot{W}(t_{n+1}) & = e^{\rho s}W(t_n) + W(t_k)(e^{\rho s} - 1), \\
\dot{W}(t) & = e^{\mu s}W(t_k)(e^{\mu (s - \Delta)} - 1) + \frac{\mu \Delta}{\mu - \rho}
\end{align*}
\end{array}\right. \quad s \in [0, s^*].$$

The maximum for the bound of $W(t_{n+1} + s)$ for $s \in (0, \Delta^*)$ is attained at the point at which the two branches of the bound meet, i.e. at $s = s^*$, as the first branch is monotonically increasing in $s$, and the second branch monotonically decreasing. The point $s^*$ can be computed as:

$$s^* = \frac{1}{\rho - \mu} \log \left(\frac{W(r_{n+1}) + W(t_k)}{W(r_n) + W(t_k)}\right) + \frac{\mu \Delta}{\mu - \rho}.$$

and thus $W(r_n, W(s^*))$ can be bounded as:

$$e^{\frac{\mu \Delta}{\mu - \rho}} \left((W(r_n) + W(t_k))^{\frac{\mu}{\mu - \rho}} (W(r_{n+1}) + W(t_k))^{\frac{\rho}{\mu - \rho}}\right).$$
which is monotonically increasing on $W(r_n)$, $W(r_{n+1})$, and $W(t_k)$. If $S(t) = W(t_k)e^{-2\lambda(t-t_k)}$, we have:

$$V(t_k) = W(t_k)e^{-2\lambda(t-t_k)}M(f(t_{\text{failow}})) \leq -S(t_k) +
\begin{multline}
e^{\mu t_k \Delta} \left((S(r_n) + S(t_k)) - (S(r_{n+1}) + S(t_k))\right)
\end{multline}
$$

where we used the fact that, if $t_{\text{min}} \leq t_{\text{min}}^*$, $\Gamma_d$ enforces (in the absence of disturbances) $W(r_n) \leq S(r_n)$ for all $n \in \mathbb{N}$, $n \leq n_k$. From the previous expression we can obtain $W(r_n + s^*) \leq g(\Delta, n)S(r_n + s^*)$ where:

$$g(\Delta, n) = -e^{2\lambda n^2} + e^{2\lambda n^2(1 + e^{2\lambda n^2})} = 2\lambda_n^2 + 2\lambda_n^2 e^{2\lambda n^2} + e^{2\lambda n^2 - s^*}$$

The value of $s^*$ can be further bounded to obtain a simpler expression:

$$s^* \leq \frac{\mu \Delta}{\mu - \rho}.$$

Using this bound for $s^*$ and letting $n$ take its maximum possible value $n = N_{\text{max}} - 1$, the following chain of inequalities holds:

$$\rho P \tilde{g}(\Delta, n) \leq \rho P \tilde{g}(\Delta, N_{\text{max}} - 1) \leq g(\Delta, N_{\text{max}})$$

for all $n \in [0, N_{\text{max}}]$, which leads to the bound:

$$W(t) \leq \rho P^{-1} g(\Delta, N_{\text{max}}) S^\frac{1}{2}(t).$$

Note that (9) does not depend on $t_k$ or $n$. Finally, apply the bounds:

$$\lambda_n^2(P)|x| \leq V(x) = \sqrt{x^T P x} \leq \lambda_{\max}^2(P)|x|.$$  

(10)

to obtain (8). From Lemma 6.1 and the condition enforced by the self-triggered implementation we have:

$$V(\xi(t_{k+1})) \leq V(\xi(t_k)) e^{-\lambda_k} + \gamma P(\|\delta\|_\infty).$$

Iterating the previous equation it follows:

$$V(\xi(t_k)) \leq e^{-\lambda_k} V(\xi(t_0)) + \gamma P(\|\delta\|_\infty) \sum_{i=0}^{k-1} e^{-\lambda_{\min} i} \leq e^{-\lambda_k} V(\xi(t_0)) + \gamma P(\|\delta\|_\infty) \frac{1}{1 - e^{-\lambda_{\min}}}.$$

Assuming, without loss of generality, that $t_0 = 0$, the following bound also holds:

$$|\xi_k(t_k)| \leq \rho P |x| e^{-\lambda_k + \lambda_{\min}^2(P)} \frac{\gamma P(\|\delta\|_\infty)}{1 - e^{-\lambda_{\min}}}.$$  

(11)
where we used (10). From (8) and Lemma 6.1 one obtains:

\[ |\xi_x(t_k + \tau)| \leq g(\Delta, N_{\max})|\xi_x(t_k)|e^{-\lambda \tau} + \gamma I(\|\delta\|_{\infty}), \]  

(12)

for all \( \tau \in [0, N_{\max}\Delta] \). Combining (11) and (12) results in:

\[ |\xi_x(t_k + \tau)| \leq g(\Delta, N_{\max})\rho P|x|e^{-\lambda(t_k + \tau)} \\
+ e^{-\lambda \tau}\gamma P(\|\delta\|_{\infty}) \frac{\lambda_m^+(P)g(\Delta, N_{\max})}{1 - e^{-\lambda \tau}} \\
+ \gamma I(\|\delta\|_{\infty}), \]

and after denoting \( t_k + \tau \) by \( t \) we can further bound:

\[ |\xi_x(t)| \leq g(\Delta, N_{\max})\rho P|x|e^{-\lambda t} \\
+ \gamma P(\|\delta\|_{\infty}) \frac{\lambda_m^+(P)g(\Delta, N_{\max})}{1 - e^{-\lambda \tau}} + \gamma I(\|\delta\|_{\infty}), \]

which is independent of \( k \) and concludes the proof. \( \square \)