In analogy with the classical complexity class Total Functional NP (TFNP), we introduce the complexity class of Total Functional QMA (TFQMA). In this complexity class one is given a family of quantum circuits $Q_n$ that take as input a classical string $x$ of length $n$ and a quantum state $|\psi\rangle$ on $\text{poly}(n)$ qubits, for some polynomial $\text{poly}(n)$, such that for all $x$ there exists at least one witness, i.e. a state $|\psi\rangle$ such that $Q_n(x, |\psi\rangle) = 1$ with probability $\geq 2/3$. The functional problem is then, given $Q_n$ and $x$, find a $|\psi\rangle$ such that $Q_n(x, |\psi\rangle) = 1$ with probability $\geq 2/3$. The complexity of this class lies between the functional analogs of BQP and QMA, denoted FBQP and FQMA respectively. We show that TFQMA can equivalently be defined as the functional analog of QMA $\cap$ coQMA. We provide examples of problems that lie in TFQMA, coming from areas such as the complexity of $k$-local Hamiltonians and public key quantum money. In the context of black-box groups, we note that Group Non-Membership, which was known to belong to QMA, in fact belongs to TFQMA. We also provide an oracle with respect to which we have a separation between FBQP and TFQMA. In the conclusion we discuss the relation between TFQMA, public key quantum money, and the complexity of quantum states.

1 Introduction

Classical complexity classes are generally defined as consisting of decision problems. But functional analogs of these classes can also be defined. The functional analog of NP is denoted FNP (Functional NP). As a simple example, the functional analog of the travelling salesman problem is the following: given a weighted graph and a length $\ell$, either output a circuit with length less than $\ell$, or output NO if such a circuit does not exist. The functional analog of P, denoted FP, is the subset of FNP for which the output can be computed in polynomial time.

Total functional NP (TFNP), introduced in [1] and which lies between FP and FNP, is the subset of FNP for which it can be shown that the NO outcome never occurs. As an example, factoring (given an integer $n$, output the prime factors of $n$) lies in TFNP since for all $n$ a (unique) set of prime factors exists, and it can be verified in polynomial time that the factorisation is correct. TFNP can also be defined as the functional analog of NP $\cap$ coNP. It can be shown that TFNP is strictly included in FNP except if NP = coNP [1].

TFNP contains many natural and important problems, including factoring, local search problems[2, 3, 4], computational versions of Brower’s fixed point theorem[5], finding Nash equilibrium[6, 7], etc. Although there probably do not exist complete problems for TFNP, there are many syntactically defined subclasses of TFNP that contain complete problems, and for which some of the above natural problems could be shown to be complete. For recent work in this direction, see [8].

The quantum analog of NP is QMA [9]. QMA has been extensively studied, and contains a rich set of complete problems, see e.g. [10]. These complete problems are all promise problems. For instance the most famous one, the $k$-local Hamiltonian problem, involves a promise that the ground state energy is either less than $a$ or greater than $b$, with $b - a = 1/\text{poly}(n)$, for some polynomial $\text{poly}(n)$, and the problem is to determine which is the case.

The functional analog of QMA, which we de-
note by FQMA can be defined as follows: 

**FQMA.** Let $Q_n$ be a family of uniform quantum circuits of size $\text{poly}(n)$ that take as input a classical bit string $x \in \{0,1\}^n$ of length $n$, and a quantum state $|\psi\rangle \otimes |0\rangle^\otimes k$ where $|\psi\rangle$ is a state of $\text{poly}(n)$ qubits, and $|0\rangle^\otimes k$ are $k = \text{poly}(n)$ ancilla qubits in the standard basis state, and that outputs a classical bit. The functional computational problem is: for all $n$, given $x \in \{0,1\}^n$, either output $|\psi\rangle$ such that $\Pr[Q_n(x,|\psi\rangle) = 1] \geq 2/3$; or output $|\text{NO}\rangle$ if for all $|\psi\rangle$, $\Pr[Q_n(x,|\psi\rangle) = 1] \leq 1/3$; where we have the promise that only these two cases occur.

The functional class $\text{FBQP}$ is the subset of FQMA such that the output of the FQMA problem can be computed in polynomial time on a quantum computer.

Here we introduce the functional class $\text{TFQMA}$ (Total functional QMA), as the subset of FQMA such that only the YES answer of the FQMA problem occurs, i.e. such that for all $n$, given $x \in \{0,1\}^n$, there exists a $|\psi\rangle$ such that $\Pr[Q_n(x,|\psi\rangle) = 1] \geq 2/3$. The functional computational problem is then to output such a $|\psi\rangle$.

The most interesting problems in TFQMA are not promise problems, rather they have a structure such that one can prove that only the YES answer occurs.

The main aim of the present paper is to show that TFQMA is an interesting and rich complexity class. To this end, after giving precise definitions of the above complexity classes in Section 2, we provide several examples of problems that belong to TFQMA. These problems are related to problems previously studied in quantum complexity, such as commuting quantum $k$-SAT, commuting $k$-local Hamiltonian, the Quantum Lovász Local Lemma (QLLL) [11], public key quantum money based on knots [12]. We show how these problems can be adapted to fit into the TFQMA framework. In the next section (Section 4) we consider relativized problems. In the context of black-box groups, we show that Group Non-Membership, which was known to belong to QMA [13], in fact belongs to TFQMA. We also exhibit problems based on the Quantum Fourier Transform (QFT) and provide a simple oracle with respect to which there is a separation between FBQP and TFQMA. In Section 5 we go back to the formal definition of TFQMA, and show that with a natural definition of equality between functional computational classes, we can identify $\text{TFQMA} = F(\text{QMA} \cap \text{coQMA})$, i.e. it is the functional analog of $\text{QMA} \cap \text{coQMA}$. Finally, in Section 6 we discuss how TFQMA is related to public key quantum money and to the complexity of quantum states, introduce additional functional computational classes that are of potential interest (when there is a unique witness, when there is a gap between the acceptance and rejection probabilities; functional analogs of CQMA and MA), and present a list of open questions.

2 Preliminary Definitions

**Definition 1. Quantum Verification Procedure.** A quantum verification procedure is a uniform family of quantum circuits $Q = \{Q_n\}$ that take as input a classical bit string $x$ of length $|x| = n$, a quantum state $|\psi\rangle$ of $m = \text{poly}(n)$ qubits, and $k = \text{poly}(n)$ ancilla qubits in the standard basis state $|0\rangle^\otimes k$, for some polynomial poly$(n)$, and which outputs 0 or 1 obtained by running the circuit on its input and measuring the first qubit of the resulting state in the standard basis.

We denote by $Q_n(x,|\psi\rangle) \in \{0,1\}$ the outcome of the verification procedure. We interpret outcome 1 as accept and the outcome 0 as reject.

**Definition 2. QMA.** The class $\text{QMA}$ is the set of languages $L \subseteq \{0,1\}^*$ such that there exists a quantum verification procedure $P$ for which the following holds:

\[
\forall x \in L, \exists |\psi\rangle s.t. \Pr[P(x,|\psi\rangle) = 1] \geq 2/3 \tag{1}
\]
\[
\forall x \notin L, \forall |\psi\rangle, \Pr[P(x,|\psi\rangle) = 1] \leq 1/3 \tag{2}
\]

If $x \in L$ we say that any $|\psi\rangle$ satisfying Eq. (1) is a witness for $x$. We call a quantum verification procedure for which Eqs. (1) and (2) hold a quantum verification procedure for $L$.

**Definition 3. coQMA.** The class $\text{coQMA}$ is the set of languages $L \subseteq \{0,1\}^*$ such that there exists a quantum verification procedure $P'$ for which the following holds:

\[
\forall x \in L, \forall |\psi\rangle, \Pr[P'_n(x,|\psi\rangle) = 1] \leq 1/3 \tag{3}
\]
\[
\forall x \notin L, \exists |\psi\rangle s.t. \Pr[P'_n(x,|\psi\rangle) = 1] \geq 2/3 \tag{4}
\]

**Definition 4. Functional QMA (FQMA).** Given a quantum verification procedure $P = \{Q_n\}$ for $L \in \text{QMA}$, the functional computational problem $\Pi_Q \in \text{FQMA}$ is, for all $x \in
Some additional cases of commuting 2-local Hamiltonian problem is in interactions; approximating the ground state is planar and nearly Euclidean, the pla-qubits, the 3-local Hamiltonian where the systems are qutrits and the interaction graph is planar, and the pla-qubit mapping, in which case there is a basis state with the same eigenvalues $h=(h_1,...,h_n)$ (a collision).

Existence. By the pigeonhole principle. There exists a basis of joint eigenstates. This basis has $2^n$ orthogonal states. To each basis state is associated a bit string $h=(h_1,...,h_n)$. Since the number of bit strings is equal to the number of basis states, either there is a 1–to–1 mapping, in which case there is a basis state with $h=(0,0,...,0)$, or there is at least one collision, i.e. at least two orthogonal basis states that have associated the same bit string $h$. 

3 Problems in TFQMA

In this section we provide examples of problems in TFQMA.

3.1 Eigenstates of commuting $k$-local Hamiltonian

3.1.1 Background

A $k$-local Hamiltonian is a Hermitian matrix $H=\sum_{a=1}^{A} H_a$ operating on the Hilbert space of $n$ $d$-dimensional particles (qudits), where each term $H_a$ (sometimes called constraint) is a Hermitian operator that acts non trivially on at most $k$ particles.

The $k$-local Hamiltonian problem is to determine whether the ground state of $H$ has energy $\leq a$ or $\geq b$, with $b-a \geq 1/\text{poly}(n)$, for some polynomial $\text{poly}(n)$, with the promise that only one of these cases occurs. The $k$-local Hamiltonian problem is QMA complete even when $k=2$.

The commuting $k$-local Hamiltonian is the case where the operators $H_a$ commute.

It was shown by Bravyi and Vyalyi that the commuting 2-local Hamiltonian problem is in NP. Some additional cases of commuting $k$-local Hamiltonian problems also in NP are: the 3-local Hamiltonian where the systems are qubits, the 3-local Hamiltonian where the systems are qutrits and the interaction graph is planar, and nearly Euclidean, the planar square lattice of qubits with plaquette-wise interactions; approximating the ground state energy when the interaction graph is a locally expanding graph. The complexity of the commuting $k$-local Hamiltonian problem in the general case is unknown.

A particularly interesting case is when each $H_a$ is a projector, i.e. it has only 0,1 eigenvalues. The $k$-local Hamiltonian problem in this case reduces to the questions whether $H$ has a frustration free eigenstate, i.e. an eigenstate with eigenvalue 0. This is known as quantum $k$-SAT, and was introduced in [18] where it was shown that quantum 2-SAT is in P and quantum $k$-SAT for $k \geq 4$ is QMA$_1$ complete (where QMA$_1$ is the subset of QMA in which the accepting probability in the case of YES instances is 1). It was later shown that quantum 3-SAT is also QMA$_1$ complete.

3.1.2 Frustration-Free or Degenerate Eigenspace of commuting quantum $k$-SAT

Here we consider the commuting $k$-local Hamiltonian with all constraints projectors. In this case there exists a basis of the Hilbert space where each basis state is also an eigenstate of all the constraints $H_a$. For such a basis state $|\psi\rangle$, we denote $h_a$ its eigenvalues: $H_a|\psi\rangle=h_a|\psi\rangle$, and set $h=(h_1,...,h_A)$. Observe that $h$ is a bit string and that given such an eigenstate, one can efficiently determine $h$ by measuring each $H_a$ in succession (the order is immaterial since the $H_a$ commute).
Verification. Measure all the $H_a$ on each state, if $h = (0,0,...,0)$ accept. Otherwise check that all the eigenvalues $h = (h_1,...,h_n)$ are equal. Check that the states are pairwise orthogonal using the SWAP test. (The SWAP test was introduced in [20].)

The existence argument is based on the pigeonhole principle, and therefore the problem has a form very similar to the problems in the Polynomial Pigeonhole Principle (PPP) class introduced in [5]. It has the following classical analog: given a $k$-SAT formula with $n$ variables and $n$ clauses, either find a satisfying assignment, or find two assignments such that the clauses all have the same value. Note that if one or more of the constraints $H_a = I$ is the identity operator, then there is no frustration free state, and the only possible output is a collision.

3.1.3 Almost degenerate states of commuting $k$-local Hamiltonian.

We now modify the above problem to the case where the local terms in the Hamiltonian are not projectors, but arbitrary local Hermitian operators.

Problem 2. Almost degenerate eigenspace of commuting $k$-local Hamiltonian. Given a commuting $k$-local Hamiltonian acting on $n$ qubits with $A$ terms $H = \sum_{a=1}^{A} H_a$, with the local terms bounded by $0 \leq H_a \leq \alpha I$, for some $\alpha > 0$, output several copies of two orthogonal states which are eigenstates of the $H_a$: for all $a$, $H_a|\psi_1\rangle = h_{1a}^a|\psi_1\rangle$ and $H_a|\psi_2\rangle = h_{2a}^a|\psi_2\rangle$, with almost identical energies $|E_1^a - E_2^a| \leq \Lambda \alpha 2^{-n}$ where $E_i^a = \sum_a h_{ia}^a$ for $i = 1, 2$.

Existence. There exists a basis of joint eigenstates. This basis has $2^n$ orthogonal states. To each basis state is associated a string $h = (h_1,...,h_A)$ of eigenvalues of the $H_a$ and an energy $E = \sum_a h_a$. The energies lie in the range $0 \leq E \leq \Lambda \alpha$. Hence by the pigeonhole principle, there are at least two energies that differ by at most $\Lambda \alpha 2^{-n}$.

Verification. Measure all the $H_a$ on each state to obtain the eigenvalues $h^1 = (h_1^1, h_2^1,...,h_A^1)$ and $h^2 = (h_1^2, h_2^2,...,h_A^2)$. Compute the energies $E^1 = \sum_a h_{1a}^a$ and $E^2 = \sum_a h_{2a}^a$. If $h^1 \neq h^2$ and if $|E^1 - E^2| \leq \Lambda \alpha 2^{-n}$ then accept. If $h^1 = h^2$ and if a SWAP test shows that the pairs of eigenstates are orthogonal, then accept. Otherwise reject.

3.1.4 Multiple copies of eigenstates of commuting $k$-local Hamiltonian.

The quantum no–cloning principle suggests another type of problem, namely producing several copies of a state that has certain properties. In the present case the required property is that the state be an eigenstate of the $H_a$’s.

Note that finding a single random eigenstate of the $H_a$’s is easy: take any pure state and measure all the $H_a$ operators on the state. To create two identical copies, we can try the following procedure: start with the maximally entangled state $|\phi^+\rangle = 2^{-n/2}\sum_{i=0}^{2^n-1} |i\rangle_1|i\rangle_2$ (which can be efficiently produced). Now measure the $H_a$’s on the first system. Denote by $h = (h_1,...,h_n)$ the measured eigenvalues. If the corresponding eigenspace is one-dimensional, the state after the measurement is $|\psi_h\rangle_1|\psi_h\rangle_2$, where $|\psi_h\rangle$ denotes the complex conjugate of the state $|\psi\rangle$ in the standard basis. (If the corresponding eigenspace is degenerate with degeneracy $J$, the state after the measurement is $J^{-1/2}\sum_{j=1}^{J} |\psi_{hj}\rangle_1|\psi_{hj}\rangle_2$ where $|\psi_{hj}\rangle$ is an orthonormal basis of the eigenspace with eigenvalues $h$). Thus if the $H_a$’s are real in the standard basis, we can efficiently create two identical eigenstates. But we do not know an efficient procedure to create two identical eigenstates when the $H_a$’s are complex, nor do we know of an efficient procedure to create three identical eigenstates when the $H_a$ are real.

These remarks lead to the following problem:

Problem 3. Multiple copies of eigenstates of commuting $k$-local Hamiltonian. Given a commuting $k$-local Hamiltonian acting on $n$ qubits with $A$ terms, output 2 states with the same eigenvalues $h = (h_1,...,h_A)$ if the $H_a$ are complex, or output 3 states with the same eigenvalues $h$ if the $H_a$ are real.

Existence. Trivial.

Verification. Measure all the $H_a$ on each state. Accept if the outcomes $h = (h_1,...,h_A)$ are equal.

3.2 Quantum Lovász Local Lemma

The Quantum Lovász Local Lemma (QLLL) introduced in [11] provides conditions under which the quantum $k$-SAT problem is satisfiable.
As an example we give the following result taken from [11]: Let \( \{\Pi_1, \ldots, \Pi_m\} \) be a \( k\)-QSAT instance where all projectors have rank at most \( r \). If every qubit appears in at most \( D = 2k/(e \cdot r \cdot k) \) projectors, then the problem is satisfiable. For our purposes we will call the hypotheses of this statement the QLLL conditions.

A Constructive Quantum Lovász Local Lemma provides conditions under which the frustration free state can be efficiently constructed by a quantum algorithm, i.e. is in FBQP. Initial results used commutativity of the constraints [21, 22]. But this condition was dropped in [23] which provides a constructive algorithm under a uniform gap constraint: for any subset \( S \) of the constraints, the gap of \( H_S = \sum_{i \in S} \Pi_i \) must be greater than \( \epsilon = 1/\text{poly}(m) \), for some polynomial \( \text{poly}(n) \), where the gap is the difference between the two smallest eigenvalues of \( H_S \), and \( \epsilon \) is independent of \( S \). Note that there is no known quantum algorithm that can check whether the uniform gap constraint is satisfied.

It is not known how the constructive algorithm of [23] works when the uniform gap condition does not hold. It may be that it always outputs a state close to the ground state. It may also be that it sometimes outputs a state far from the ground state. If the latter is true, then this gives rise to an interesting problem in TFQMA.

**Problem 4. Approximate ground state under QLLL conditions.** Given a quantum \( k\)-SAT problem involving \( n \) qubits that satisfies the QLLL conditions, output a state with energy less then \( \epsilon = 1/\text{poly}(n) \).

**Existence.** By the QLLL conditions, the Hamiltonian has a frustration free state, i.e. a state with energy 0.

**Verification.** Exponentiation of a Hamiltonian that is a sum of local terms can be done efficiently [24]. Hence using the phase estimation algorithm, one can estimate the ground state energy in time \( O(1/\text{poly}(\epsilon)) = O(\text{poly}(n)) \).

We note that the analogous classical problem is in FBPP (the functional analog of BPP), as there exist efficient randomized classical algorithms to find a satisfying assignment [25, 26].

### 3.3 Quantum money based on knots.

As we discuss in the conclusion, there is a close connection between problems in TFQMA and public key quantum money [32]. Here we show how the scheme of [12] in which the quantum money consists of coherent superposition of (representations of) knots induces a problem in TFQMA.

We first recall that any knot can be represented by a grid diagram \( G \), which itself can be encoded by two disjoint permutations \( \Pi_X \) and \( \Pi_O \) of \( D \) elements. We denote by \( (G) = (\Pi_X, \Pi_O) \) a quantum encoding of such a grid diagram. The one-variate Alexander polynomial \( A(G) \) can be efficiently computed from the representation of a knot \( G \) [33]. For a positive real number \( x \) we denote by \([x] \) the smallest integer which is at least \( x \), and we set \( [x] = [x - 1/2] \).

**Problem 5. Uniform superpositions over knots.** Given a grid diagram \( G \) of dimension \( D = d(G) \) for a knot, and \( A = A(G) \) the Alexander polynomial of this knot, output the following superposition over grid diagrams \( G' \) of dimension between 2 and \( 2D \) with the same Alexander polynomial:

\[
|\$_{D,A} \rangle = \frac{1}{\sqrt{N}} \sum_{G' : d(G') \leq 2D, A(G') = A} \sqrt{q(d(G'))} |G' \rangle,
\]

where \( N \) is a normalisation factor, and \( q(d') \) is the following quasi-Gaussian distribution over grid diagram dimensions between 2 and \( 2D \):

\[
q(d') = \left[ \frac{y(d')}{y_{\text{min}}} \right], \quad \text{where} \quad y(d') = \frac{1}{d' \left[ \frac{d'}{D} \right]^{2}} \exp \left( -(d' - D)^2 / 2D \right), \quad \text{for} \quad 2 \leq d' \leq 2D,
\]

and \( y_{\text{min}} \) is the minimum value of \( y(d') \).

**Existence.** Trivial.

**Verification.** Use the verification procedure described in [12], which for completeness we recall briefly. Denote by \( |\phi \rangle \) the state that must be verified.

1. Verify that \( |\phi \rangle \) is a superposition of basis vectors that validly encode grid diagrams. If this is the case then move on to step 2, otherwise reject.

2. Measure the Alexander polynomial on \( |\phi \rangle \). If this is measured to be \( A(G) \) then continue on to step 3. Otherwise, reject. (Because the distribution
$q(d')$ is strongly peaked around $D$, this step will accept with high probability on a valid state).

4. Apply the Markov chain verification algorithm described in [12]. If $|\phi\rangle$ passes this step, accept the state. Otherwise, reject. This is the crucial step that checks that the state is a coherent superposition of knots which can be mapped one into the other by elementary grid moves, that is elementary moves that map a knot onto an equivalent knot.

Roughly speaking, the Markov chain verification algorithm of [12] is based on Markov matrices $P_s$ that are such that they leave the states Eq. (6) invariant, but will transform a knot $|G\rangle$ into another knot through a local grid move, with $s$ labeling the local grid moves. One then prepares the uniform superposition over possible moves $|S^+\rangle = S^{-1/2} \sum_{s=1}^{S} |s\rangle$ where $S$ is the number of possible Markov matrices, and checks that the state $|\phi\rangle|S^+\rangle$ is invariant under the transformation $V = \sum_s P_s |s\rangle\langle s|$. (There is an additional technical complication which is that one wants to keep the weights $\sqrt{q(d(G'))}$ invariant. We refer to [12] for how this is implemented through the addition of an additional register, and a slightly more complicated Markov walk).

In analogy with Problem 3 we can of course also request multiple copies of the same knot state.

**Problem 6. Multiple copies of knot states.** Fix $D$. Output $|S_{D,A}\rangle^{\otimes 3}$, i.e. output 3 identical knot states given by Eq. (6).

In Problem 6 we ask for 3 copies since the knot states are real, see comment in Section 3.1.4.

**Existence.** Trivial.

**Verification.** First carry out SWAP tests to check that the 3 states are identical. Reject if these tests fail.

Then carry out the verification procedure of Problem 5. Reject also if the verification algorithms fails. Reject if the Alexander polynomials measured in step 2 are not all identical.

4 Relativized Problems

Here we give problems in which the quantum computer has access to an oracle. The complexity is counted as the complexity of the quantum algorithm, including the number of calls to the oracle which each count as one computational step.

4.1 Finding a marked state

We give a very simple oracle, which is the basis of Grover’s algorithm [34, 35] with respect to which we have a separation between FBQP and TFQMA. See [36, 28] for previous use in separating complexity classes. Consider the oracle $A$ that marks the state $|\psi\rangle$: $U|\psi\rangle = -|\psi\rangle$ and $U|\phi\rangle = |\phi\rangle$ for all $|\phi\rangle$ such that $\langle \phi | \psi \rangle = 0$.

**Problem 7. Finding a marked state.** Given access to oracle $A$, output the marked state $|\psi\rangle$.

**Existence.** Trivial.

**Verification.** Given state $|\chi\rangle$, prepare the state $(|0\rangle + |1\rangle)|\chi\rangle$, then conditional on the first qubit being 1 act with $U$ on the second register, finally measure the first qubit in the $|\pm\rangle = |0\rangle \pm |1\rangle$ basis. If $|\chi\rangle = |\psi\rangle$, then one will obtain outcome $-$, while if $|\chi\rangle \perp |\psi\rangle$, one will obtain outcome $+$. It is well known that finding such a marked state among $n$ states requires $\Theta(d^{1/2})$ queries to the oracle. The lower bound follows from arguments in [36], and the upper bound is given by Grover’s algorithm.

4.2 Group Non–Membership

Watrous [13] showed in the context of black-box groups that Group Non–Membership is in QMA. It follows from the construction that it is also in TFQMA. We recall the key results from [13].

We use Babai and Szemerédi’s model of black-box groups with unique encoding [29], adapted to the quantum context. In this model we know how to multiply elements of the group, but we dont know anything else about the group. More precisely, each element $x \in G$ is represented by a randomly chosen label $l(x)$. We have access to a group oracle $B$ to perform the group operations. Suppose that the state of the quantum computer is

$$|\psi\rangle = \sum_{x,y,z} \psi_{xyz}|l(x)|l(y)|z\rangle,$$  \hspace{1cm} (7)
Problem 8. Group (Non) Membership. 

If we know the representation of the unit element $|l(e)⟩$, then the oracle can be used to compute the inverse of an element (by inputing $|l(x)⟩|l(e)⟩$), and matrix multiplication (by first computing $x^{-1}$, and then inputing $|l(x^{-1})⟩|l(y)⟩$).

Suppose you receive as input elements $g_1, ..., g_n$ and $h$ of $G$. Denote by $H = \langle g_1, ..., g_n \rangle$ the subgroup of $G$ generated by $g_1, ..., g_n$. Group (Non) Membership is the question: is $h \in H$ or $h \notin H$? If $h \in H$ then there exists a succinct classical certificate [29]. If $h \notin H$ then there exists a succinct quantum certificate [13] (note that [28] provides evidence that the certificate when $h \notin H$ could be classical, in which case Group (Non) Membership would be in TFCQMA$^B$.

Existence and Verification. See [29] and [13].

The key part of the proof in [13] is exhibiting a quantum algorithm that accepts on state $|ψ_H⟩ = \frac{1}{|H|^{1/2}} \sum_{x \in H} |l(x)⟩$, and which rejects on states orthogonal to $|ψ_H⟩$. Hence we can rephrase the above problem as follows:

Problem 9. Uniform superposition over subgroup. Given access to oracle $B$, and given elements $g_1, ..., g_n$ and $h$ of $G$, either output the classical certificate showing that $h \in H$, or output the quantum certificate showing that $h \notin H$.

Existence. Trivial.

Verification. See [13].

4.3 Problems based on QFT

We consider here problems in which the verification procedure is based on the efficiency of the Quantum Fourier Transform and the phase estimation algorithm [37, 38, 39].

We consider a unitary $U$, acting on $n$ qubits that can be efficiently exponentiated. More precisely suppose that we have access to an oracle $C$ which, given input $(k, |ψ⟩)$ where $k \in \{0, ..., m\}$ is classical, outputs

$$C(k, |ψ⟩) = (k, U^k|ψ⟩), \quad k \in \{0, ..., m\},$$

where we are interested below in the case where $m$ is exponentially large (for instance $m = 2^n$).

Unitaries that can be efficiently exponentiated were studied in [40] in the context of the time energy uncertainty. The only explicit example we are aware of where $U$ can be efficiently exponentiated but cannot be efficiently diagonalised is when $U$ is the time evolution of a commuting $k$-local Hamiltonian: $U = \exp(iH)$ with $H = \sum_a H_a$, where $H_a$ is $k$-local and the $H_a$ all commute. As a consequence the verification procedures given below can be used in place of the ones used in Problems 1, 2 and 3. However, if additional classes of unitaries that can be efficiently exponentiated but cannot be efficiently diagonalized are discovered, then this provides new TFQMA problems, which justifies interest in the present approach.

We denote by $φ \in [0, 1]$ and $|ψ_{φα}\rangle$ the eigenvalues and eigenstates of $U$:

$$U|ψ_{φα}\rangle = e^{i2πφ}|ψ_{φα}\rangle,$$

$$⟨ψ_{φ',α'}|ψ_{φα}\rangle = δ_{α',α}δ φ' φ,$$

where $α$ labels orthogonal states with the same eigenvalue. We denote by $d(φ, φ') = \min\{||φ - φ'||, 1 - ||φ - φ'||\}$ the distance on the unit circle between the angles $2πφ$ and $2πφ'$.

Problem 10. Almost degenerate eigenspace of $U$. Given access to the oracle $C$ of Eq. (9) with $m = 2^n$, output $N = O(1)$ copies of the state $|ψ_{φα}\rangle|ψ_{φ'α'}\rangle$, where $(φ', α') \neq (φ, α)$ and $d(φ, φ') \leq 2^{-n}$. That is one must output orthogonal states which are almost degenerate (have almost the same eigenvalue).

Existence. Since $U$ acts on $n$ qubits, it has $2^n$ eigenstates, which form an orthonormal basis of the Hilbert space with eigenvalues in $[0, 1]$. By the pigeonhole principle, there must be at least $2$ eigenstates with eigenvalues $φ, φ'$ satisfying $d(φ, φ') \leq 2^{-n}$.

Verification.

Step 1: Randomly choose one of the $N$ pairs.

On each member of the selected pair carry out the phase estimation algorithm, obtaining two estimates $\hat{φ}$ and $\hat{φ}'$. Reject if $d(\hat{φ}, \hat{φ}') > 5/2^n$. 

Step 2: Carry out the SWAP test on the remaining $N - 1$ pairs. Reject if the fraction of failed SWAP’s is too far from 1/2, i.e. is not consistent with the two states in each pair being orthogonal.

It is easy to show that Step 1 will accept with probability at least $2/3$ on a pair of states $|\psi_{\phi_0}\rangle|\psi_{\phi'}\rangle$ with $d(\phi, \phi') \leq 2^{-n}$. Indeed in [39] it is shown that the phase estimation algorithm acting on an eigenstate $|\psi_{\phi_0}\rangle$ produces an estimated eigenvalue $\hat{\phi}$ with error bounded by

$$\Pr\left[ d(\phi, \hat{\phi}) > \frac{k}{2^n} \right] < \frac{1}{2k - 1}. \quad (11)$$

Using the triangle inequality we have

$$d(\hat{\phi}, \hat{\phi'}) \leq d(\phi, \hat{\phi}) + d(\phi, \phi') + d(\phi', \hat{\phi'}). \quad (12)$$

Hence if $d(\hat{\phi}, \hat{\phi'}) > 5/2^n$ and $d(\phi, \phi') \leq 2^{-n}$, then either $d(\phi, \hat{\phi}) > 2/2^n$ or $d(\phi', \hat{\phi'}) > 2/2^n$. From Eq. (11) the probability of at least one of the later events occurring is less than 1/3, hence $Pr[d(\hat{\phi}, \hat{\phi'})] > 5/2^n < 1/3$.

We can also consider the problem of outputting several identical eigenstates.

**Problem 11.** Provide several identical eigenstate. Given access to the oracle $C$ of Eq. (9), output three identical states with the same eigenvalues, i.e. output the state $|\psi_{\phi_0}\rangle^\otimes 3$ for some $(\phi, \alpha)$.

**Existence.** Trivial.

**Verification.** Step 1: Carry out SWAP tests to check that the states are identical. If one of the SWAP test fails, reject.

Step 2: Carry out the phase estimation algorithm on the three states yielding outcome $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\phi}_3$. Accept if $d(\hat{\phi}_i, \hat{\phi}_j) < 5/2^n$ for the three pairs $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$, otherwise reject.

On states of the form $|\psi_{\phi_0}\rangle^\otimes 3$ this verification procedure will accept with probability greater than $(8/9)^3 \approx 0.70$. Indeed Step 1 will always accept and does not affect the state. Considering Step 2, the triangle inequality implies that if $d(\hat{\phi}_i, \phi) < k/2^n$ for $i = 1, 2, 3$, then $d(\hat{\phi}_i - \phi) < 2k/2^n$ for the three pairs $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$. Using Eq. (11), the former event occurs with probability at least $(1 - 1/(2k - 1))^3$. Setting $k = 5$ yields the above bound.

Problems 10 and 11 are expected to be hard because outputting an eigenstate of $U$ with a specified eigenvalue is expected to be hard in general. It is instructive however to consider variants of the problem that are easy. For instance outputting a random eigenstate of $U$ and the corresponding eigenvalue (up to precision $2^{-n}$) is easy: choose an arbitrary initial state and run the phase estimation algorithm. The output of the algorithm will be an approximate eigenvalue $\hat{\phi}$, and the state after running the algorithm will be a superposition of eigenstates with eigenvalues close to $\hat{\phi}$. And if one carries out this procedure on one half of a maximally entangled state, one obtains a superposition of eigenstate times their complex conjugate (see remark in Section 3.1.4) This is why we request 3 copies in Problem 11.

Note also that if we have additional information on the structure of $U$, constructing eigenstates may become easy. For instance suppose that there is a set of orthogonal states on which $U$ acts like $U|\chi_j\rangle = |\chi_{j+1}\rangle$, where $j = 0, \ldots, N - 1$, where we identify $|\chi_N\rangle = |\chi_0\rangle$, and suppose that we can efficiently implement the transformation $V$ which transforms the computational basis state $|j\rangle$ into $|\chi_j\rangle$: $V(j)|0\rangle = |0\rangle|\chi_j\rangle$. Then acting with $V$ on the state $N^{-1/2}\sum_{j=0}^N e^{2\pi jk/N}|j\rangle|0\rangle$ will yield an eigenstate of $U$ with eigenvalue $e^{2\pi k/N}$.

5 On Completeness

5.1 Motivation

Reference [1] contains two important results on the complexity of TFNP. First, it identifies TFNP with the functional analog of $NP \cap coNP$: "TFNP = F(NP\cap coNP)" (we put quotes because as discussed below, equality between functional classes must be treated with some precaution). Second, it shows that if there is a FNP complete problem that lies in TFNP, then $NP = coNP$.

Here we show that an analog of the first result holds: "TFQMA = F(QMA \cap coQMA)" (again with quotes). We then discuss why the second results does not seem to generalize.

5.2 TFQMA = F(QMA \cap coQMA)

We begin by defining QMA \cap coQMA and F(QMA \cap coQMA).

**Definition 6.** QMA \cap coQMA. The class QMA \cap coQMA is the set of languages $L \subseteq \{0, 1\}^*$ that belong both to QMA and to coQMA, i.e. such that there exists two quantum verification
procedures $Q$ and $Q'$ such that for all $x \in L$, Eqs. (1) and (3) hold, and for all $x \notin L$, Eqs. (2) and (4) hold.

**Definition 7.** Functional QMA∩coQMA ($F(QMA \cap coQMA)$). Given two quantum verification procedure $Q = \{Q_n\}$ and $Q' = \{Q'_n\}$ for $L \in QMA \cap coQMA$, such that for all $x \in L$, Eqs. (1) and (3) hold, and for all $x \notin L$, Eqs. (2) and (4) hold, the functional computational problem $\Pi_{QQ'} \in F(QMA \cap coQMA)$ is, for all $x \in L$, output $|\psi\rangle_1 \otimes |\text{NO}\rangle_2$ where $|\psi\rangle$ satisfies Eq. (1), and for all $x \notin L$, output $|\text{NO}\rangle_1 \otimes |\psi'\rangle_2$ where $|\psi'\rangle$ satisfies Eq. (4). That is the output of the functional computational problem $\Pi_{QQ'}$ is the tensor product of the outputs of the functional computational problems $\Pi_Q$ and $\Pi_{Q'}$.

The difficulty we encounter is that the output of the functional problems in $F(QMA \cap coQMA)$ do not have the same format as the output of the functional problems in FQMA. Since the format used should be irrelevant, we define an equivalence between functional computational problems.

**Definition 8.** Equivalence of functional computational problems under reformatting. Functional computational problems $\Pi_Q$ and $\Pi_{Q'}$ are equivalent if there exists 1) a uniform family of unitary transformations $U = \{U_n\}$ that act on $\text{poly}(n)$ qubits, for some polynomial $\text{poly}(n)$, 2) bijections $F(x)$ between $\Pi_Q(x)$ and $\Pi_{Q'}(x)$, and 3) functions $k(n), l(n)$, such that for all $x \in \{0,1\}^*$ with $n = |x|$, for all $|\psi\rangle \in \Pi_Q(x)$,

$$U_n (|\psi\rangle \otimes |\text{NO}\rangle^k) = \left(|F(\psi)\rangle \otimes |\text{NO}\rangle^l\right). \quad (13)$$

We denote this equivalence as $\Pi_Q \equiv_U \Pi_{Q'}$.

One easily checks that this is an equivalence relation. The inclusion of the factors $|\text{NO}\rangle^k$ and $|\text{NO}\rangle^l$ allows for the use of ancilla in the reformatting. The fact that these factors appear on both sides of Eq. (13) ensures transitivity of the relation.

We can now define inclusion of functional computational classes.

**Definition 9.** Inclusion of functional computational classes. Functional class $F$ is included up to reformatting in functional class $F'$ if there exists a uniform family of unitary transformations $U = \{U_n\}$ that act on $\text{poly}(n)$ qubits, for some polynomial $\text{poly}(n)$, such that for each computational problem $\Pi_Q \in F$, there exists a computational problem $\Pi_{Q'} \in F'$ such that $\Pi_{Q'}(x) \equiv_U \Pi_Q(x)$ where the family of unitary transformation $U$ is fixed, i.e. independent of $Q$. We denote this as $F \subseteq_U F'$.

**Definition 10.** Equality (up to reformatting) of functional computational classes. Functional class $F$ is equal up to reformatting in functional class $F'$ if $F \subseteq_U F'$ and $F' \subseteq_U F$. We denote this as $F =_U F'$.

**Theorem 1.** Functional class TFQMA and $F(QMA \cap coQMA)$ are equal up to reformatting: $\text{TFQMA} =_U F(QMA \cap coQMA)$.

**Proof.** For one direction, consider an arbitrary computational problem $\Pi_Q \in \text{TFQMA}$ associated to a family of circuits $Q = \{Q_n\}$. The set of possible outputs of this computational problem $\Pi_Q(x)$ is the set of witnesses for $Q = \{Q_n\}$.

Consider now the two quantum verification procedures given by the following families of circuits $\hat{Q} = \{Q_n\}$ and $\hat{Q}' = \{Q'_n = 0\}$ (i.e. circuits $Q'_n$ always output 0). This pair of quantum verification procedures belongs to $\text{QMA} \cap \text{coQMA}$ with language $L = \{0,1\}^*$.

According to Definition 7 the output of the functional computational problem $\Pi_{\hat{Q}\hat{Q}'}$ is $\Pi_{\hat{Q}\hat{Q}'}(x) = \{|\psi\rangle_1 \otimes |\text{NO}\rangle_2\}$ where $|\psi\rangle_1$ is witness for $x$.

Therefore there is a simple unitary transformation, independent of $x$, that maps all states of $\Pi_Q(x)$ to states of $\Pi_{\hat{Q}\hat{Q}'}(x)$: append one ancilla qubit in state $|\text{NO}\rangle_2$ and carry out the transformation $|\text{NO}\rangle_2 \rightarrow |\text{NO}\rangle_2$. Hence according to Definition 9, $\text{TFQMA} \subseteq_U F(QMA \cap coQMA)$.

For the other direction, consider an arbitrary computational problem $\Pi_{QQ'} \in F(QMA \cap coQMA)$ associated to the two quantum verification procedures $Q = \{Q_n\}$ and $Q' = \{Q'_n\}$, and the corresponding language $L$.

Consider the following quantum verification procedure $\mathcal{Q} = \{\mathcal{Q}_n\}$ that takes as input a classical string of length $x$ and a quantum state of $\text{poly}(n)$ qubits, for some polynomial $\text{poly}(n)$, and which on input $(x, |\text{NO}\rangle \otimes |\psi\rangle)$ runs the circuit $\mathcal{Q}_n$ on inputs $(x, |\psi\rangle)$, i.e. computes $Q_n(x, |\psi\rangle)$, and on input $(x, |\text{NO}\rangle \otimes |\psi\rangle)$ runs the circuit $\mathcal{Q}'_n$ on inputs $(x, |\psi\rangle)$, i.e. computes $Q'_n(x, |\psi\rangle)$. That is the quantum verification procedure measures the first qubit of the witness, and if the outcome is
0 implements circuit $Q_n$, and if the outcome is 1 implements circuit $Q'_{n'}$.

The set of possible outputs of the functional computational problem $\Pi_Q$ is therefore: for all $x \in L$, $\Pi_Q(x) = \{0\} \otimes |\psi\rangle$ where $|\psi\rangle$ is a witness for $Q_n$ on input $x$; and for all $x \notin L$, $\Pi_Q(x) = \{1\} \otimes |\psi'\rangle$ where $|\psi'\rangle$ is a witness for $Q'_{n'}$ on input $x$.

Therefore there is a simple unitary transformation independent of $x$ that maps all states in $\Pi_Q(x)$ into states in $\Pi_{QQ'}$ (namely the transformation that implements $|0\rangle|\varphi\rangle \rightarrow |\varphi\rangle_1|\text{NO}\rangle_2$ and $|1\rangle|\varphi\rangle \rightarrow |\text{NO}\rangle_1|\varphi\rangle_2$ for all $|\varphi\rangle$). Hence according to Definition 9, $F(QMA \cap \text{coQMA}) \subseteq L \subseteq \text{TFQMA}$. 

\[\]

5.3 Other Complexity Results

In reference [1] it is shown that any problem in FNP that reduces to a problem in TFNP is in $F(NP = \text{coNP})$. This implies that if an FNP complete problem reduces to a problem in TFNP, then NP = coNP. We have tried without success to prove an analog result for TFQMA. We sketch here why a naive attempt to copy of the argument in [1] does not work. First we need a notion of reduction.

Definition 11. Reduction of FQMA problems. Problem $\Pi_Q$ in FQMA with circuits $Q = \{Q_n\}$ for language $L \in \text{QMA}$ reduces to problem $\Pi_{Q'}$ in FQMA with circuits $Q' = \{Q'_{n'}\}$ for language $L'$ if there exists a polynomial $\text{poly}(n)$ and a uniform family of classical circuits $f_n : \{0,1\}^n \rightarrow \{0,1\}^{n'}$ with $n' = \text{poly}(n)$ and a uniform family of quantum circuits $R = \{R_{n'}\}$ that take as input a classical bit string $x'$ of length $|x'| = n'$, a quantum state $|\psi'\rangle$ of $m' = \text{poly}(n')$ qubits, and $k' = \text{poly}(n')$ ancilla qubits in the standard basis state $|0^{k'}\rangle = |0\rangle^\otimes k'$, and which output a state of $m = \text{poly}(n)$ qubits, such that for all $x$ for all $|\varphi\rangle \in \Pi_{Q'}(x')$ where $x' = f_n(x)$, we have $R_{n'}(x', |\varphi\rangle \otimes |0^{k'}\rangle) \in \Pi_Q(x)$.

Suppose now that a problem $\Pi_Q$ in FQMA reduces to a problem $\Pi_{Q'}$ in TFQMA. Denote by $Q_n$ and $Q'_{n'}$ the corresponding families of quantum circuits, and denote by $f_n$ and $Q_n$ the classical and quantum circuits used in Definition 11. The aim is to show that there is a quantum verification procedure $\hat{Q} = \{\hat{Q}_n\}$ that satisfies Eqs. (3) and (4). The naive attempt is as follows: on input $(x, |\varphi\rangle)$ the quantum verification procedure does the following: 1) Compute $x' = f_n(x)$; 2) Check that $|\varphi\rangle$ is a witness for $Q'_{n'}(x')$; 3) Check that $R_{n'}(x', |\varphi\rangle) = |\text{NO}\rangle$. If both steps 2 and 3 succeed, then accept; otherwise reject.

If $x \notin L$, then there exists a state $|\varphi\rangle$ such that both steps 2 and 3 succeed (namely the witness for $Q'_{n'}(x')$). Hence condition of Eq. (4) is satisfied.

The problem arises when $x \in L$. In this case we want that for all states $|\varphi\rangle$, either step 2 or step 3 fails. The problem is that when $x \in L$, there may exist states that are almost witnesses for $Q'_{n'}(x')$, i.e. states for which $Q'_{n'}(x')$ accepts with probability $2/3 - \epsilon$ for arbitrarily small $\epsilon$, and thus for which step 2 almost always accepts. However, for such states we have no control over $R_{n'}(x', |\varphi\rangle)$, in particular $R_{n'}(x', |\varphi\rangle)$ may be equal to $|\text{NO}\rangle$ with high probability. On such states the above procedure will accept with high probability, whereas it should reject.

6 Discussion

6.1 Public key quantum money

Public key quantum money was introduced in [32]. We recall the requirements for a public key quantum money scheme:

1. Efficient Generation. There exists a polynomial-time quantum algorithm that produces both a quantum money state $|\$\rangle$ and an associated serial number $x \in \{0,1\}^n$.

2. Efficient Check. There is a quantum verification procedure with uniform family of circuits $\{Q_n\}$ which accepts with probability greater than $2/3$ on input $(x, |\$\rangle)$. The quantum verification procedure is public, i.e. anyone with access to a quantum computer can run the verification algorithm.

Note that the acceptance probability of a quantum verification procedure can be brought exponentially close to 1 [9], and this can be done without increasing the witness size and without significantly distorting the witness [41, 42]. Thus the verification step can be repeated exponentially many times on the same quantum money state.

3. No Forging. Given one piece of quantum money $(x, |\$\rangle)$, it is hard to generate a quantum state $|\psi\rangle$ on $2n$ qubits such that each
part of $|\psi\rangle$ (along with the original serial number $x$) passes the verification algorithm.

There are close links between public key quantum money schemes and TFQMA. Indeed, quantum money schemes are necessarily based on explicit problems, in the same way that the interesting TFQMA problems are explicit (as opposed to e.g. QMA complete problems which are promise problems). Furthermore, condition 2 is based on a quantum verification procedure, with the condition that for any serial number $x$, there exists a valid quantum money state $|\psi_x\rangle$, which is essentially the same as the condition that differentiates TFQMA from FQMA (namely for all inputs $x$ there exists at least one witness). Finally condition 3 is related (but stronger) than the hardness requirement that a problem in TFQMA is not in FBQP, i.e. a witness cannot be efficiently produced on a quantum computer.

Condition 1 does not have an analog in the definition of TFQMA. For instance in [12] a probabilistic scheme is used to generate money states and serial numbers $(x, |S_x\rangle)$. However it is interesting to note that the Problems introduced in Section 3 can sometimes be modified to produce random pairs $(x, |S_x\rangle)$. (For instance as a quantum money analog of Problem 3, take a random state $|\varphi\rangle$ and measure the operators $H_a$ on $|\varphi\rangle$ to obtain a pair $(h, |\psi_h\rangle)$ where $h = (a_1, ..., a_A)$ are the eigenvalues of $H_a$ and $|\psi_h\rangle$ is an eigenstate of the $H_a$: $H_a|\psi_h\rangle = h_a|\psi_h\rangle$. Then $h$ would be the classical certificate, and $|\psi_h\rangle$ the corresponding quantum money state). Note that an additional condition that a quantum money scheme should have is that randomly generated pairs $(x, |S_x\rangle)$ should be hard to forge on average, which is stronger than the requirement that a TFQMA problem not lie in FBQP.

Finally we note another difference, namely that in the case of quantum money, all strings $x$ need not be valid serial numbers. For instance in the case of knot states [12] not all polynomials are valid Alexander polynomials (which is why in Problem 5 we used as classical input a knot diagram – from which the Alexander polynomial can be efficiently computed – rather than a polynomial).

The above shows that there is a close relation between TFQMA and public key quantum money. One can therefore expect that progress on both questions will be closely linked.

6.2 Testing Quantum States

The introduction of quantum complexity classes has also shed a new light on the problem of understanding how complex are different quantum states. Indeed most quantum states of $n$ qubits are extremely complex, as they require $O(2^n)$ parameters to describe (one possibility is to give the coefficients of all the $2^n$ basis states up to logarithmic precision). But a subset of states can be described much more synthetically. One can think of this as a form of Kolmogorov complexity: given a state $|\psi\rangle$ of $n$ qubits, how many classical bits $x$ are required to specify $|\psi\rangle$ (either exactly, or to high precision).

Thus the class FBQP suggests a natural way to quantify the complexity of a quantum state by the amount of resources that are required to build the state, i.e. by the size of the smallest quantum circuit that will produce the desired state as output.

The complexity classes TFQMA and FQMA can be thought of as alternative, more general ways to quantify the complexity of (families of) quantum states. Indeed, $\Pi_Q(x)$ is the subspace of accepting states for circuit $Q_n$ and input $x$. Thus the pair $(Q_n, x)$ provides a succinct description of $\Pi_Q(x)$. In the Kolmogorov sense introduced above the complexity of the different definitions is the same ($O(n)$), but the definitions of the subspaces are less and less explicit as we go from FBQP to TFQMA to FQMA, and in this sense the complexity of the accepting subspaces increases as we go from one class to the other.

In order to refine the above argument, it is useful to consider an operational interpretation. Suppose that one is given some states which one suspects are YES instances of some TFQMA or FQMA problem. (Such states could possibly be preexisting in the universe somewhere, see for instance the discussion in [27] chapter 14). Maybe a bit more realistically, it could be that such states can be created by small depth large width quantum circuits. Then using the large size of Avogadro’s number (the number of atoms in a mole, i.e. in a few grams), namely $N_A = 6 \cdot 10^{23}$, one could possibly create rather large instances of such states by running a small depth quantum circuit on $N_A$ atoms. Even more realistically such states could have been produced by a random procedure, as in the preparation of quantum money). We would like to test whether
the state is a YES instance of the TFQMA or FQMA problem. In the YES instance, there are states (the witnesses) with acceptance probability $\geq 2/3$, and these will be accepted by our test. But there may also exist states with acceptance probability $2/3 - \epsilon$ for arbitrary small $\epsilon$. What do we do with those states? An operational procedure will not be able to distinguish them from genuine witnesses. One possibility is to consider a gapped version of QMA, as follows:

**Definition 12. Gapped Verification Procedure.** A quantum verification procedure is gapped if for all $x$ the witness Hilbert space decomposes into a direct sum of two spaces $V^+(x) \oplus V^-(x)$ such that

\begin{align}
\forall |\psi\rangle \in V^+(x), \Pr[Q_n(x, |\psi\rangle)] &= 1 \geq a \quad (14)
\forall |\psi\rangle \in V^-(x), \Pr[Q_n(x, |\psi\rangle)] &= 1 \leq b \quad (15)
\end{align}

where $a - b \geq 1/poly(n)$, for some polynomial $\mathit{poly}(n)$. If $V^+(x)$ is non empty, we call any state in $V^+(x)$ a witness for $x$.

In this definition one can also consider the spectrum of the POVM element $M_1 = Q_n(x)\Pi_1Q_n(x)$ with $\Pi_1$ the projector on the first qubit being $|1\rangle$, corresponding to the circuit accepting. The gapped condition is equivalent to the eigenvalues of $M_1$ being either $\geq a$ or $\leq b$. The corresponding eigenvectors of $M_1$ are a basis of $V^+(x)$ and $V^-(x)$ respectively.

Using this definition, one can introduce, the classes gapQMA, cogapQMA, FgapQMA, TFgapQMA by replacing quantum verification procedures by gapped quantum verification procedures in Definitions 2, 3, 4, 5.

For these gapped classes the above problem does not arise. Indeed, using the procedure of [41, 42] one can without loss of generality take $a = 1 - \exp(-\mathit{poly}(n))$ and $b = \exp(-\mathit{poly}(n))$, whereupon one can operationally distinguish with exponentially small probability of mistake whether a given state is a witness (belongs to $V^+(x)$) or is not a witness.

Note that Problems 1, 2, 3 belong to TFgapQMA. It is not clear whether the other problems we have introduced belong to TFgapQMA. It would be interesting to find other problems in TFgapQMA.

### 6.3 Open questions.

We have provided several examples of problems belonging to TFQMA, showing that it is an noteworthy complexity class. We sketch here some interesting open questions.

Can one find additional problems in TFQMA? Note that in the classical case there are many problems that belong to TFNP, including some problems of real practical importance, such as local search problems and finding Nash equilibria. Are there problems of real practical importance in TFQMA?

One can define restrictions of the classes FQMA and TFQMA. One restriction are the classes FgapQMA and TFgapQMA mentioned above. Another possibility is to require that there is a unique witness, i.e., that the witness Hilbert space is one-dimensional. The corresponding classes could be denoted UniqueFQMA, UniqueTFQMA, UniqueFgapQMA, UniqueTFgapQMA.

When the witness is classical, the class QMA becomes CQMA. When in addition the verifier is classical, one obtains the classical classe MA. In analogy with the definitions of Section 5, one can define the functional problems associated to these classes FCQMA and FMA, and the corresponding total functions TFCQMA and TFMA. In all these cases one can introduce gapped versions, and unique versions of the functional classes.

Can one find (interesting) examples of problems in any of the above classes? (Note that [28] provides evidence that the certificate for Group Non–Membership could be classical, in which case Problem 8 would be in TFCQMA$^B$).

In the case of TFNP, there exist a number of syntactically defined subclasses, such as Polynomial Local Search (PLS), Polynomial Parity Argument (PPA), Polynomial Parity Argument on a Directed Graph (PPAD), Polynomial Pigeonhole Principle (PPP), etc..., each with some complete problems. Are there syntactically defined subclasses of TFQMA? If these syntactically defined subclasses of TFQMA exist, do they have natural complete problems? Do the syntactically defined subclasses of TFNP (such as PLS, PPA, PPAD, PPP, etc...) have quantum analogs? Note that Problems 1, 2, 10 are based on the pigeonhole principle which is also at the basis of class PPP. These problems may fit into a quantum analog of PPP. Problems 3, 6 which are based on the im-
possibility of quantum cloning most likely belong to a new quantum class.

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