PROOF OF A LORENTZ AND LEVI-CIVITA THESIS

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ABSTRACT. A formal proof of the thesis by Lorentz and Levi-Civita that the left-hand side of Einstein field equations represents the real energy-momentum-stress tensor of the gravitational field.

Summary – 1. Introduction. Aim of the paper. – 2. Mathematical preliminaries. – 3. Proof that the left-hand side of the Einstein field equations gives the true energy-momentum-stress tensor of the gravitational field. – 4. A fundamental consequence. – Appendix: On the pseudo energy-tensor.

PACS 04.20 – General relativity.

1. – As it has been remarked [1], if I is the action integral of any field (of any tensorial nature) – say \( \varphi(x) \), \( x \equiv (x^0, x^1, x^2, x^3) \) – acting in a pseudo-Riemannian spacetime, and we perform the variation of I – say \( \delta g I \) – generated by the variation \( \delta g_{jk} \), \( (j, k = 0, 1, 2, 3) \), of the metric tensor \( g_{jk}(x) \) (possibly interacting with \( \varphi(x) \)),

\[
\delta g I = \int_D (\ldots)^{jk} \delta g_{jk} \sqrt{-g} \, d^4x ,
\]

– where \( D \) is a fixed spacetime domain – , the expression \( (\ldots)^{jk} \) is a symmetrical tensor, which represents the energy-momentum-stress tensor of \( \varphi(x) \). This statement has been verified for various fields [1]. And its general validity can be intuitively understood bearing in mind that I is an action integral, with the Lagrange density of \( \varphi(x) \) as integrand.

We shall prove that the above statement holds also if \( \varphi(x) \equiv g_{jk}(x) \), thus corroborating a famous (and debated!) thesis by Lorentz [2] and Levi-Civita [3] – see also Pauli [4] (and the references therein).

The essential merit of the following demonstration is its independence of the Einstein field equations (and of the Bianchi relations).

2. – Let \( \sqrt{-g} \, S [g_{jk}(x), g_{jk,m}(x), g_{jk,mn}(x), \ldots] \) be a generic scalar density which is a function of the metric \( g_{jk}(x) \) and of a finite number of its ordinary derivatives [5]. We do not assume that \( \sqrt{-g} \, S \) is a Lagrange density, and therefore the integral
\[ J = \int_D S \sqrt{-g} \, d^4x \]

is not an action integral. We have:

\[ \delta_g J = \int_D \frac{\delta(S \sqrt{-g})}{\delta g_{jk}} \delta g_{jk} \, d^4x \]

the variational derivative \( \delta(S \sqrt{-g})/\delta g_{jk} \) is equal to

\[ \frac{\partial}{\partial x^m} \left( \frac{\partial(S \sqrt{-g})}{\partial x^m} \right) - \frac{\partial^2}{\partial x^m \partial x^n} \left( \frac{\partial(S \sqrt{-g})}{\partial g_{jm,n}} \right) \]

putting \( \delta(S \sqrt{-g})/\delta g_{jk} := P^{jk} \sqrt{-g} \), we can write:

\[ \delta_g J = \int_D P^{jk} \sqrt{-g} \delta g_{jk} \, d^4x \]

Let us now consider the particular \( \delta g_{jk} \) — say \( \delta^* g_{jk} \) — which is generated by an infinitesimal change of the co-ordinates \( x \):

\[ x'^j = x^j + \varepsilon^j(x) \]

we assume that \( \varepsilon^j(x) \) is zero on the bounding surface \( \partial D \). The corresponding variation of \( J \) — say \( \delta^* J \) — will be equal to zero, because \( J \) is an invariant.

We have:

\[ g_{mn}(x) = \frac{\partial x'^j}{\partial x^m} \frac{\partial x'^k}{\partial x^n} g'_{jk}(x') \]

and we consider the \( \delta^* g_{jk} \) for fixed values of the coordinates, i.e.:

\[ \delta^* g_{jk} := g'_{jk}(x') - g_{jk}(x') = g'_{jk}(x') - g_{jk}(x) - g_{jk,s}(x) \varepsilon^s \]

It follows immediately from eqs. (3), (6), (6') that

\[ \delta^* g_{mn} = -g_{mn,j} \varepsilon^j - g_{mj} \varepsilon^j_{,n} - g_{nj} \varepsilon^j_{,m} \]

from eq. (3) we get:

\[ \delta^*_g J = \int_D P^{mn} \sqrt{-g} \delta^* g_{mn} \, d^4x = \]

\[ = \int_D P^{mn} \left( -g_m; \varepsilon^j_{,n} - g_{nj} \varepsilon^j_{,m} - g_{mn,j}; \varepsilon^j \right) \sqrt{-g} \, d^4x = \]

\[ = \int_D \left[ 2 \left( P^n_j \sqrt{-g},_n - g_{mn,j} P^{mn} \sqrt{-g} \right) \right] \varepsilon^j \, d^4x = \]

\[ = 2 \int_D P^n_j;_m \varepsilon^j \sqrt{-g} \, d^4x = 0 \]
if the colon denotes a covariant derivative; in the last passage we use the following property of any symmetrical tensor $S^{mn}$:

$$(8') \quad S^{m}_{j;m} \sqrt{-g} = (S^{n}_{j} \sqrt{-g})_{,m} - \frac{1}{2} g_{mn,j} S^{mn} \sqrt{-g}.$$ 

Accordingly:

$$(9) \quad P_{j;m}^{m} = 0 \quad (j = 0, 1, 2, 3).$$

3. The result (9) has a mere mathematical interest. It becomes physically significant when $J$ is the action integral, say $A$, given by

$$(10) \quad A = \int_{D} R \sqrt{-g} \, d^{4}x,$$

where $R = R^{jk}g_{jk}$ is the Ricci scalar. We shall not use the fact that the $g_{jk}$’s are (a priori) independent variables, because we do not wish to deduce from the action $A$ the Einstein field equations.

Standard procedures (see, e.g., Hilbert’s method in Appendix, 3)) tell us that

$$(11) \quad \delta_{g}A = \int_{D} \left( R^{jk} - \frac{1}{2} g^{jk} R \right) \sqrt{-g} \, \delta g_{jk} \, d^{4}x ;$$

the analogue of eq.$(8)$ is:

$$(12) \quad \delta^{*}_{g}A = 2 \int_{D} \left( R^{k}_{j} - \frac{1}{2} \delta^{j}_{k} R \right) \varepsilon^{j} \sqrt{-g} \, d^{4}x = 0 ,$$

from which:

$$(13) \quad \left( R^{jk} - \frac{1}{2} g^{jk} R \right)_{;k} = 0 \quad (j = 0, 1, 2, 3).$$

Thus, quite independently of the field equations, we see that the symmetrical tensor $R^{jk} - (1/2)g^{jk}R$ satisfies four conservation equations. Of course, eqs. (13) are identically satisfied by virtue of Bianchi relations, but the above method – which is essentially due to the conceptions of Emmy Noether (6) – evidences the conservative property of $R^{jk} - (1/2)g^{jk}R$, and attributes it the nature of an energy-momentum-stress tensor. Properly speaking, $[R^{jk} - (1/2)g^{jk}R]/\kappa$, if $\kappa$ is the Newton-Einstein gravitational constant, represents the Einsteinian energy tensor, as it was emphasized by Lorentz (2) and Levi-Civita (3). And the fact that this tensor is a function only of the potential $g^{jk}$ implies that it is the unique energy-momentum-stress tensor of the gravitational field.
4. The fact that $[R^{jk} - (1/2)g^{jk}R]/\kappa$ is the true energy-momentum-stress tensor of the gravitational field has a very important consequence: the mathematical undulatory solutions of the equations $R^{jk} - (1/2)g^{jk}R = 0 = R^{jk}$ are quite devoid of physical meaning, because they do not transport energy, momentum, stress. This was the first demonstration of the physical non-existence of the gravitational waves. Quite different demonstrations have been given in recent years, see e.g. [7], and references therein.

In his fundamental memoir [3], Levi-Civita proved also the nature of mere mathematical fiction (Eddington [8]) of the well-known pseudo energy-tensor of the metric field $g_{jk}$. –

A useful discussion with Dr. T. Marsico is gratefully acknowledged.

**APPENDIX**

α) The full illogicality of the notion of pseudo energy-tensor can be seen also in the following way. The usual definition of this pseudo tensor is:

\[ \sqrt{-g} t^m_n \stackrel{\text{DEF}}{=} \frac{\partial (L \sqrt{-g})}{\partial g_{jk,n}} g_{jk,m} - \delta_n^m L \sqrt{-g} \]  

the function $L$:

\[ L \equiv g^{mn} (\Gamma^s_{mn} \Gamma^r_{sr} - \Gamma^r_{ms} \Gamma^s_{nr}) \]

yields the Lagrangean field equations:

\[ \frac{\partial (L \sqrt{-g})}{\partial g_{jk}} - \frac{\partial}{\partial x^n} \left[ \frac{\partial (L \sqrt{-g})}{\partial g_{jk,n}} \right] = 0 \]

Now, the left-hand side of (A.3) is **not** equal to

\[ - \left( R^{jk} - \frac{1}{2} g^{jk}R \right) \sqrt{-g} \]

as it is commonly affirmed. Indeed:

i) A non tensor entity cannot be equal to a tensor density –  

ii) The above affirmed equality has its origin in a “negligence”: in the customary variational deduction of the Einstein field equations the variation of $\int_D R \sqrt{-g} \, d^4x$ is “reduced” to the variation of $\int_D L \sqrt{-g} \, d^4x$.  

But in his “reduction” two perfect differentials in the integrand have been omitted, because on the boundary $\partial D$ the variations of the $g_{jk}$ and of their first derivatives are zero (by assumption): this omission destroys the tensor-density character of the initial expressions. –

β) It is likely that the pseudo energy-tensor would not have been invented if the authors had followed Hilbert’s procedure [9]. This Author started from the fact that (with our previous notations) the explicit evaluation of
the variational derivative $\delta (R \sqrt{-g}) / \delta g^{mn}$ gives the following Lagrangean expressions:

$$(A.5) \quad \frac{\partial (R \sqrt{-g})}{\partial g^{mn}} - \frac{\partial}{\partial x^k} \left[ \frac{\partial (R \sqrt{-g})}{\partial g^{mn}_{,k}} \right] + \frac{\partial^2}{\partial x^k \partial x^l} \left[ \frac{\partial (R \sqrt{-g})}{\partial g^{mn}_{,kl}} \right] ;$$

Hilbert wrote: “... spezialiere man zunächst das Koordinatensystem so, daß für den betrachteten Weltpunkt die $g^{mn}_s$ samtlich verschwinden.”. I.e., he chose a local coordinate-system for which the first derivatives of $g^{mn}$ are equal to zero. Thus, only the first term of (A.5) gives a non-zero contribution, and we have that (A.5) is equal to

$$(A.6) \quad \sqrt{-g} \left( R_{mn} - \frac{1}{2} g_{mn} R \right) .$$

There is no room in this procedure for false (pseudo) tensor entities.

References

[1] W. Pauli, *Teoria della Relatività* (Boringhieri, Torino) 1958, sect. 55. See also: V. Fock, *The Theory of Space, Time and Gravitation*, Second Revised Edition (Pergamon Press, Oxford, etc) 1964, sects. 31*, 48, 60; A. Loinger, *Nuovo Cimento*, 110A (1997) 341.

[2] H.A. Lorentz, *Amst. Versl.*, 25 (1916) 468; (this memoir is written in Dutch – an English translation would be desirable).

[3] T. Levi-Civita, *Rend. Acc. Lincei*, 26 (1917) 381; an English translation in *arXiv:physics/9906004* (June 2nd, 1999). See also: *Idem*, *ibid.*, 11 (s.6*) (1930) 3 and 113.

[4] See Pauli [1], sects. 23 and 61.

[5] E. Schrödinger, *Space-Time Structure* (Cambridge University Press, Cambridge) 1960, Chapt. XI; P.A.M. Dirac, *General Theory of Relativity* (J. Wiley and Sons, New York, etc) 1975, sect.30.

[6] E. Noether, *Gött Nachr.*, (1918) 235 (“Invariante Variationsprobleme”).

[7] A. Loinger and T. Marsico, *arXiv:1006.3844* [physics.gen-ph] 19 Jun 2010.

[8] A. S. Eddington, *The Mathematical Theory of Relativity*, Second Edition (Cambridge University Press, Cambridge) 1960, p.148. See also H. Bauer, *Phys. Z.*, 19 (1918) 163.

[9] D. Hilbert, *Gött Nachr.*: Erste Mitteilung, vorgelegt am 20. Nov. 1915; zweite Mitteilung, vorgelegt am 23. Dez. 1916 – *Math. Annalen*, 92 (1924) 1.

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