SYMBOLIC POWERS, SET-THEORETIC COMPLETE INTERSECTION AND CERTAIN INVARIANTS

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Abstract. In this survey article we give a brief history of symbolic powers and its connection with the interesting problem of set-theoretic complete intersection. We also state a few problems and conjectures. Recently, in connection to symbolic powers is the containment problem. We list a few interesting results and related problems on the resurgence, Waldschmidt constant and Castelnuovo-Mumford regularity.

1. Introduction

Let $A$ be a Noetherian ring and $I$ an ideal in $A$ with no embedded components. Then the ideal $I^{(n)} := A \cap (\cap_{p \in Ass(A/I)} I^n A_p)$ is the $n$-th symbolic power of $I$. The study of $n$-th symbolic power has been important for the last few decades mainly because of its connection with algebraic geometry. In recent years, it has become even more active area of research mainly because several interesting associated invariants. We list some of the interesting questions and open problems.

To understand the connection with algebraic geometry, let $k$ be a field and let $\mathbb{A}^d$ (or $\mathbb{A}^d_k$) denote the set of all $d$-tuples $a = (a_1, \ldots, a_d)$ where $a_i \in k$ for all $i = 1, \ldots, d$. The set $\mathbb{A}^d$ is called the affine $d$-space of dimension $d$ over $k$. We say that a subset $Y$ in $\mathbb{A}^d$ is a zero set if it is the set of common zeros of a collection of polynomials $f_1, \ldots, f_m \in R := k[X_1, \ldots, X_d]$ and we denote it by $Y = Z(f_1, \ldots, f_m)$. We can define a topology on $\mathbb{A}^n$ by defining the closed sets to be the zero sets. If $I = (f_1, \ldots, f_m)$, then $Y = Z(I)$. To every subset of $Y \subset \mathbb{A}^n$ we can define the ideal of $I(Y) := \{ f \in k[X_1, \ldots, X_n] | f(P) = 0 \text{ for all } P \in Y \}$. An irreducible closed subset $Y$ of $\mathbb{A}^n$ called an affine algebraic set.

Let $Y$ be an algebraic set. We say that $Y$ is defined set-theoretically by $n$ elements if there exists $n$ elements $f_1, \ldots, f_n \in R$ such that $I(Y) = \sqrt{(f_1, \ldots, f_n)}$. In 1882, Kronecker showed that $I$ can be set
theoretically defined by \( d + 1 \) equations \([44]\). Later, this result was improved by Storch \([61]\) and by Eisenbud and Evans \([25]\). It follows from their work that if \( \mathbb{k} \) is algebraically closed and \( I \) is an homogeneous ideal, then \( I \) can be defined set-theoretically by \( d \) elements. Hence it was of interest to know which ideals \( I \) could be defined set-theoretically by \( d - 1 \) elements. If \( I \) is locally complete intersection of pure dimension one, then \( I \) can be defined set-theoretically by \( d - 1 \) elements \([27], [7], [50], [65]\). In 1992, Lyubeznick showed that if \( V \) is an algebraic set in \( \mathbb{A}^d_{\mathbb{k}} \) and \( \text{char}(\mathbb{k}) = p > 0 \), then \( V \) can be defined set-theoretically by \( d - 1 \) elements \([46]\).

In 1978, Cowsik and Nori proved a remarkable result. They showed that if \( \text{char}(\mathbb{k}) = p > 0 \), then any affine curve is a set-theoretic complete intersection \([12, \text{Theorem 1}]\). If \( \text{char}(\mathbb{k}) = 0 \), then one of the best known results in \( \text{char}(\mathbb{k}) = 0 \) is the result of Herzog which was later also proved by Bresinsky \([8]\). He showed that all monomial curves in \( \mathbb{A}^3 \) are set-theoretic complete intersection.

In 1981 Cowsik proved an interesting result which connects commutative algebra and algebraic geometry:

**Theorem 1.1.** \([11]\) Let \((R, \mathfrak{m})\) a Noetherian local ring and \( \mathfrak{p} \neq \mathfrak{m} \) a prime ideal. If the symbolic Rees algebra \( R_s(\mathfrak{p}) := \oplus_{n \geq 0} \mathfrak{p}^n \) is Noetherian, then \( \mathfrak{p} \) can be defined set-theoretically by \( d - 1 \) elements.

However, the converse need not be true. Cowsik’s result motivated several researchers to investigate the Noetherian property of the symbolic Rees algebra. In 1987, Huneke gave necessary and sufficient conditions for \( R_s(\mathfrak{p}) \) to be Noetherian when \( \dim R = 3 \) \([41]\). Huneke’s result was generalised in 1991 for \( \dim R \geq 3 \) by Morales \([51]\). All these results paved a new way to study the famous problem on set theoretic complete intersection. In section 2 we discuss some of these problems.

We also have the famous result due to Zariski \([68]\) and Nagata \([52]\) which states that if \( \mathbb{k} \) is an algebraically closed field, then the \( n \)-th symbolic power of a given prime ideal consists of the elements that vanish up to order \( n \) on the corresponding variety \([24]\). This result has been generalised to perfect fields \( \mathbb{k} \) and radical ideals \([17, \text{Proposition 2.14, Exercise 2.15}]\).

In general, symbolic powers of ideals are hard to compute. Hence recently, associated to symbolic powers of ideals Bocci and Harbourne introduced a quantity called the resurgence \([4]\). In section 3 we will state the recent developments on this quantity and related invariants like the Waldschmidt constant and Castelnuovo-Mumford regularity.
2. Set-theoretic complete intersection and symbolic Rees algebra

2.1. Set-theoretic complete intersection in $\mathbb{A}^n$ and $\mathbb{P}^n$.
Throughout this section $R = \mathbb{k}[X_1, \ldots, X_d]$ where $X_1, \ldots, X_d$ are variables. We say that a radical ideal $I \subset R$ is a set theoretic complete intersection of there exists $h = \text{ht}(I)$ elements such that $I = \sqrt{(f_1, \ldots, f_h)}$. Let $a := (a_1, \ldots, a_d)$ be positive integers such that $\gcd(a_1, \ldots, a_d) = 1$. Let $C(a) := \{(a_1, t^{a_2}, \ldots, t^{a_d}) | t \in \mathbb{k}\}$ be a curve in $\mathbb{A}^n$. If $\phi : R \rightarrow \mathbb{k}[T]$ is the homomorphism given by $\phi(X_i) = T^{a_i}$ for all $i = 1, \ldots, d$. Then $p(C(a)) := \ker(\phi)$ is the prime ideal defining the curve $C(a)$. In other words, $I(C(a)) = p(C(a))$. We say that $C(a)$ can be defined set-theoretically by $d-1$ elements if there exists $d-1$ elements $f_1, \ldots, f_{d-1} \in p$ such that $p = \sqrt{(f_1, \ldots, f_{d-1})}$.

Let $d = 3$, then we have the interesting result:

**Theorem 2.1.** Let $\gcd(a_1, a_2, a_3) = 1$.

1. [35] Then one of the following is true:
   a. $p(C(a))$ is a complete intersection.
   b. There exists integers $\alpha_i, \beta_i, \gamma_i; i = 1, 2$ such that $p(C(a))$ is generated by $2 \times 2$ minors of the matrix $\begin{pmatrix} X_1^{\alpha_1} & X_2^{\beta_1} & X_3^{\gamma_1} \\ X_2^{\alpha_2} & X_3^{\beta_2} & X_1^{\gamma_2} \end{pmatrix}$.

2. [Herzog (Unpublished work)] and [8] $p(C(a))$ is a set-theoretic complete intersection.

Such a result is not known for $\mathbb{A}^d$, $d \geq 4$. The first result in higher dimension was given by Bresinsky where he showed that certain Gorenstein curves in $\mathbb{A}^4$ are set-theoretic complete intersection [9]. In 1990, Patil proved the following result [54]:

**Theorem 2.2.** Let $a_1, a_2, \ldots, a_{d-2}$ be an arithmetic sequence. Then $p(C(a))$ is a set-theoretic complete intersection.

In 1980, Valla showed that certain determinantal ideals were set-theoretic complete intersection [66]. As a consequence they prove the following:

**Theorem 2.3.** [66] Example 3.3] Let $q, m$ be positive integers with $\gcd(q, m) = 1$. Put $a_i = 2q + 1 + (i - 1)m$, for $i = 1, 2, 3$. Then $p(C(a))$ is a set-theoretic complete intersection.

In the past forty years several researchers have given interesting examples of affine varieties which are set-theoretic complete intersection. However, the following question is still open.
**Question 2.4.** Let $k$ be a field of characteristic zero and $d \geq 4$. Is every curve $C(a) \subseteq A^d$ a set theoretic complete intersection?

We would like to bring to the attention a paper of Moh where he considered the set-theoretic complete intersection problem of analytic space curves over an algebraically closed field. Let $k[[X,Y,Z]]$ and $k[[T]]$ be power series rings and $\phi : k[[X,Y,Z]] \to k[[T]]$ be given by $\phi(X) = T^a + \cdots$, $\phi(Y) = T^b + \cdots$ and $\phi(Z) = T^c + \cdots$. Let $p = \ker(\phi)$. Such curves are called Moh curves. Moh showed that if $(a-2)b < c$, then $p$ is a set-theoretic complete intersection [48]. In [41] Huneke showed that the symbolic Rees algebra of the Moh curve parameterized by $(t^6, t^7 + t^{10}, t^8)$ is Noetherian. However, it is not easy to describe the defining ideal of a Moh curve. The following question is still open:

**Question 2.5.** Let $\phi : k[[X,Y,Z]] \to k[[T]]$ be given by $\phi(X) = T^6 + T^{31}$, $\phi(Y) = T^8$ and $\phi(Z) = T^{10}$. Let $p = \ker(\phi)$. Is $p$ a set-theoretic complete intersection?

We now focus our attention on curves in the projective space $\mathbb{P}_k^n$. It is a long standing question whether every connected subvariety in $\mathbb{P}_k^n$ is a set-theoretic complete intersection [36]. The answer is not known even for curves in $\mathbb{P}^3$. We list a few results in this direction.

Let $a := (a_1, \ldots, a_d)$ be integers such that gcd$(a_1, \ldots, a_d) = 1$ and $0 = a_0 < a_1 < a_2 \cdots < a_d$. Put $S = k[X_0, X_1, \ldots, X_d]$ and let $\psi : S \to k[T, U]$ be the homomorphism given by $\psi(X_i) = T^{a_i} - a_i U^{a_i}$ for all $i = 0, \ldots, d$. Then $p(C(a)) := \ker(\psi)$ is the prime ideal defining ideal of the curve $C(a)$. In other words, $I(C(a)) = p(C(a))$.

One of the most simplest and interesting example was given by Hartshorne.

**Theorem 2.6.** [37] Let $k$ be a field of positive characteristic $p$ and $a = (1, a-1, a)$, where $a \geq 4$. Then $p(C(a))$ is a set-theoretic complete intersection.

Hartshorne’s result was generalised by Ferrand [27]. Robbiano and Valla studied the curve $C(a)$ for $a = (1, 3, 4)$ [56]. In 1991, Moh generalised the work of Hartshorne and Ferrand [49].

An interesting result in $\mathbb{P}^3$ is:

**Theorem 2.7.** [56], [62] Let $C$ be a curve in $\mathbb{P}^3$. Let $I(C) \subset S = k[X_0, X_1, X_2, X_3]$ be the ideal of the $C$. If $S/I(C)$ is Cohen-Macaulay, then $I(C)$ is a set-theoretic complete intersection.

A considerable amount of research has been done in this area. It is impossible to list all of them there. The list of references is not
exhaustive. For a good collection of articles on set-theoretic complete intersection one can read [33]. An interesting survey on this topic was also given by Lyubneznik [45].

2.2. Symbolic Rees Algebra.

The work of Cowsik, Huneke and Morales motivated several researchers to work on the symbolic powers of a prime ideal and the symbolic Rees algebra $R_s(p) = \oplus_{n \geq 0} p^{(n)}$.

If $(R, m)$ a Noetherian local ring of dimension $d$ and $p$ a prime ideal of height $d - 1$. Some of the interesting questions on the symbolic Rees algebra are: (1) Is it Noetherian? (2) Is it Cohen-Macaulay? (3) Is it Gorenstein? An answer to question (1) would imply that $p$ is a set-theoretic complete intersection, by Cowsik’s result.

If $\phi : \mathbb{k}[[X_1, \ldots, X_d]] \to \mathbb{k}[[T]]$ is the homomorphism given by $\phi(X_i) = T^a_i$ for all $i = 1, \ldots, d$, then $p(\mathfrak{a}) := \ker(\phi)$.

In 1982, Huneke gave examples prime ideals $p$ in $\mathbb{k}[[X_1, X_2, X_3]]$ whose symbolic Rees algebra $R_s(p)$ is Noetherian [40]. In fact he showed the following result.

**Theorem 2.8.** $R_s(p(\mathfrak{a}))$ is Noetherian in the following cases:

1. $\mathfrak{a} = (2t + 1, 2r + s, s + r + rs)$, where either $s \leq r$ or $s > r$ and $t = 1$.
2. $\mathfrak{a} = (s + 2, 2r + 1, s + 1 + rs)$, $2 \leq r \leq s$.
3. $\mathfrak{a} = (t + s + 1, tr + t + 1, rs + r + s)$, $r \leq t$ and $s \geq 1$.

The first result which gave necessary and sufficient conditions for $R_s(p)$ to be Noetherian was by Huneke [41, Theorem 2.1]. This result was later generalised by Morales [51, Theorem 2.1]. A consequence of their result is:

**Theorem 2.9.** Let $(R, m)$ be a regular local ring of dimension $d$ and $p$ a prime ideal of height $d - 1$. Then $R_s(p)$ is Noetherian if and only if there exists $x \in m \setminus p$ and elements $f_i \in p^{(k_i)}$, $i = 1, \ldots, f_{d-1}$, such that

$$\ell \left( \frac{R}{(p, f_1, \ldots, f_{d-1}, x)} \right) = \ell \left( \frac{R}{(p, x)} \right) k_1 \cdots k_{d-1}.$$

Using this criteria several researchers have produced examples of monomial curves $p \in \mathbb{k}[[X_1, X_2, X_3]]$ such that the symbolic Rees algebra $R_s(p)$ is Noetherian. We cite a few examples here [59, 13, 30, 60, 31, 42, 29, 58].

One interesting question is: If $R$ is a Noetherian ring, and $p$ is a prime ideal, is the symbolic Rees algebra $R_s(p)$ Noetherian? Rees provided an example which implies that this question does not have a positive answer [55]. Later Cowsik conjectured that if $R$ is a regular local ring
and \( p \) is a prime ideal, then \( R_\mathfrak{s}(\mathfrak{p}) \) is Noetherian. In 1990, Roberts gave a counter example to Cowsik’s conjecture \[57\]. In 1994, Goto, Nishida and Watanabe gave an infinite class of monomial curves whose symbolic Rees algebra \( R_\mathfrak{s}(\mathfrak{p}) \) is Noetherian if the characteristic of \( k \) is \( p \), but if the characteristic of \( k \) is zero, then \( R_\mathfrak{s}(\mathfrak{p}) \) is not Noetherian \[32\].

We end this section stating a problem which is still open:

**Problem 2.10.** Let \( k \) be an algebraically closed field. Can one classify all monomial curves in \( \mathbb{A}^3_k \) for which \( R_\mathfrak{s}(\mathfrak{p}) \) is Noetherian?

An interesting paper in this direction is \[13\].

### 3. Resurgence, Waldschmidt Constant and Regularity

One of the reasons the symbolic Rees algebra is hard to analyse is because it is not easy to describe the symbolic powers even for curves in \( \mathbb{A}^3_k \). Hence, one would like to compare the symbolic powers and ordinary powers of an ideal. If \( I \) is an ideal in a Noetherian ring \( R \), then from the definition of symbolic powers it follows that \( I^n \subseteq I^{(n)} \) and in fact for any proper ideal nonzero ideal \( I \), \( I^r \subseteq I^{(n)} \) holds if and only if \( r \geq n \). A challenging problem to determine for which positive integers \( n \) and \( r \) the containment \( I^{(n)} \subseteq I^r \) holds true. In \[64\] Swanson compared the symbolic powers and ordinary powers of several ideals.

For the rest of this section we will assume that \( S = k[X_0, \ldots, X_d] \) and \( I \) is an homogenous ideal in \( S \). Hence in 2001, Ein, Lazarsfeld and Smith proved a very interesting result. It follows from their result

**Theorem 3.1.** \[23\]. Let \( I \subset S \) be a proper ideal. If \( h \) is the largest height of an associated prime of \( I \), then \( I^{(hn)} \subseteq I^n \) for all \( n \geq 0 \).

In 2002, Hochster and Huneke proved a stronger result \[39\]. It follows from the above results that if \( d = \dim R \), then \( I^{(n)} \subseteq I^r \) for \( n \geq (d - 1)r \). In this direction, Harbourne raised the following: conjecture in:

**Conjecture 3.2.** \[1\] Conjecture 8.4.2] For any homogeneous ideal \( 0 \neq I \subset S \), \( I^{(n)} \subseteq I^r \) if \( n \geq rd - (d - 1) \).

In the same paper they remark that from the methods in \[39\] there is enough evidence for this conjecture to be true at least when the characteristic of \( k \) is \( p \) and \( r = p^t \) for \( t > 0 \) (see \[1\] Example 8.4.4]. This has led to the study of the least integer \( n \) for which \( I^{(n)} \subseteq I^r \) holds for a given ideal \( I \) and for an integer \( r \). To answer this question
C. Bocci and B. Harbourne defined an asymptotic quantity \[ 5 \] called resurgence which is defined as

\[
\rho(I) := \sup\{m/r \mid I^{(m)} \not\subseteq I^r\}.
\]

Hence if \( m > \rho(I)r \), then \( I^{(m)} \subseteq I^r \).

In general resurgence is not easy to compute. Hence it is useful to give bounds. From the results in [23] it follows that \( \rho(I) \leq d - 1 \).

Another interesting invariant is the Waldschmidt constant. This constant was introduced by Waldschmidt in [67]. Let \( I \) be an homogenous ideal and let \( \alpha(I) \) denote the least degree of a homogeneous generator of \( I \). Then we have the famous conjecture due to Nagata:

**Conjecture 3.3.** [53] Let \( V \) be a finite set of \( n \) points in \( \mathbb{P}^2_{\mathbb{C}} \) and \( I(V) \) be the corresponding homogenous ideal in \( \mathbb{C}[X_0, X_1, X_2] \). Then \( \alpha(I^{(m)}) \geq m\sqrt{n} \).

This conjecture is still open in general. It is know only in a few cases. Define

\[
\gamma(I) := \lim_{n \to \infty} \frac{\alpha(I^{(n)})}{n}.
\]

\( \gamma(I) \) is called the Waldschmidt constant ([67], [4]). Since \( \alpha \) is subadditive, i.e., \( \alpha(I^{(n+m)}) \leq \alpha(I^{(n)}) + \alpha(I^{(m)}) \), it follows that \( \gamma(I) \) exists ([67], [3]). Moreover, there is a lower bound for \( \rho(I) \). It follows from [5, Lemma 2.3.2]:

**Lemma 3.4.** Let \( 0 \neq I \subset S \) be a homogenous ideal. Then \( \gamma(I) \geq 1 \) and

\[
1 \leq \frac{\alpha(I)}{\gamma(I)} \leq \rho(I).
\]

Related to the Waldschmidt constant is the following conjecture:

**Conjecture 3.5** (Chudnovsky). Let \( V \) be a set of points in \( \mathbb{P}^d \) and \( I(V) \) be the corresponding homogenous ideal in \( S \). Then

\[
\gamma(I) \geq \frac{\alpha(I) + d - 1}{d}.
\]

Recently, Chudnovsky’s conjecture has attracted the attention of researchers ([22], [28], [17]).

For any homogenous ideal \( I \) we can define the Castelnuovo-Mumford regularity as follows. Let \( M \) be a finitely generated graded \( S \)-module. Let

\[
F_\bullet : 0 \to F_r \to F_{r-1} \to \cdots \to F_1 \to F_0 \to 0
\]
be a minimal free resolution of $M$ where $F_i = \bigoplus_j S[-j]^{b_{ij}}$. Put $b_i(M) = \max\{j \mid b_{ij} \neq 0\}$. Then $\text{reg}(M) = \max_i\{b_i(M) - i\}$.

Bounds on the Castelnuovo-Mumford regularity has been of interest. As the list is long we state only a few results. In 1997, Swanson proved that if $I$ is a homogenous ideal, then there exists an integer $r$ such that $\text{reg}(I^m) \leq mr$ for any $m$ [63]. Later, it was proved that asymptotically $\text{reg}(I^m)$ is a linear function of $m$ by [43], [15].

The behaviour of Castelnuovo-Mumford regularity of symbolic powers is not easy to predict. From a result of Cutkosky, Herzog and Trung, it follows that if $I$ is an ideal of points in a projective space and the symbolic Rees algebra $R_s(I) = \bigoplus_{n \geq 0} I^{(n)}$ is Noetherian, then $\text{reg}(I^{(n)})$ is a quasi-polynomial ([16, Theorem 4.3]). Moreover, $\lim_{n \to \infty} \left( \frac{\text{reg}(I^{(n)})}{n} \right)$ exists and can even be irrational [14]. For a nice survey article on Castelnuovo-Mumford regularity see [10].

Bocci and Harbourne showed showed that if $I$ is a zero dimensional subscheme in a projective space, then $\alpha(I)/\gamma(I) \leq \rho(I) \leq \text{reg}(I)/\gamma(I)$ [5, Theorem 1.2.1]. Hence, if $\alpha(I) = \text{reg}(I)$, then $\rho(I) = \alpha(I)/\gamma(I)$. Later, Harbourne and Huneke raised the following Conjecture:

**Conjecture 3.6.** [35, Conjecture 2.1] Let $I$ be an ideal of fat points in $S$ and $\mathfrak{m} = (X_0, \ldots, X_d)$. Then $I^{(nd)} \subseteq \mathfrak{m}^{(d-1)} I^n$ holds true for all $I$ and $n$.

In the same paper they showed that the conjecture is true for fat point ideals arising as symbolic powers of radical ideals generated in a single degree in $\mathbb{P}^2$. Recently, there has been a renewed interest on the Waldschmidt constant mainly due to the containment problem. In [2] the Waldschmidt constant for square free monomial ideals was computed. In fact they showed that if $\gamma(I)$ can be expressed as the value to a certain linear program arising from the structure of the associated primes of $I$. The Waldschmidt constant has also been computed for Stanley-Risner ideals [3].

The resurgence and the Waldschmidt constant has been studied in a few cases: for certain general points in $\mathbb{P}^2$ [4], smooth subschemes [34], fat linear subspaces [20], special point configurations [21] and monomial ideals [6].

We now briefly state our results on resurgence, Waldschmidt constant and Castelnuovo-Mumford regularity. Putting weights on monomial curves $\mathcal{C}(\underline{a})$ in $\mathbb{A}^d$, we can consider them as weighted points in a weighted projective space $\mathbb{P}_k(\underline{a})$. Hence the bounds for resurgence in [5] hold true. For $\underline{a} = (3, 3 + m, 3 + 2m)$ these invariants have been
computed in \[19\]. For \( q \geq 1 \), and \( \gcd(2q + 1, m) = 1 \) these invariants have been computed \( \underline{a} = (2q + 1, 2q + 1 + m, 2q + 1 + 2m) \), these invariants have been computed in \[18\]. In these cases the generators of the symbolic powers of \( \mathfrak{p}(\mathcal{C}(\underline{a})) \) has been computed. In \[20\], for monomial curves \( \mathfrak{p}(\mathcal{C}(\underline{a})) \) in \( \mathbb{P}^3 \), where \( \underline{a} = (m, 2m, 2q + 1 + 2m) \), \( q, m \) are positive integers and \( \gcd(2q + 1, m) = 1 \), these invariants have been computed.

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