Existence of similar point configurations in thin subsets of $\mathbb{R}^d$

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Abstract. We prove the existence of similar and multi-similar point configurations (or simplexes) in sets of fractional Hausdorff measure in Euclidean space. These results can be viewed as variants, for thin sets, of theorems for sets of positive density in $\mathbb{R}^d$ due to Furstenberg, Katznelson and Weiss [11], Bourgain [2] and Ziegler [16]. Let $d \geq 2$ and $E \subset \mathbb{R}^d$ be a compact set. For $k \geq 1$, define

$$\Delta_k(E) = \left\{ (|x^1 - x^2|, \ldots, |x^i - x^j|, \ldots, |x^k - x^{k+1}|) : \{x^i\}_{i=1}^{k+1} \subset E \right\} \subset \mathbb{R}^{k(k+1)/2},$$

the $(k+1)$-point configuration set of $E$. For $k \leq d$, this is (up to permutations) the set of congruences of $(k+1)$-point configurations in $E$; for $k > d$, it is the edge-length set of $(k+1)$-graphs whose vertices are in $E$. Previous works by a number of authors have found values $s_{k,d} < d$ so that if the Hausdorff dimension of $E$ satisfies $\dim_H(E) > s_{k,d}$, then $\Delta_k(E)$ has positive Lebesgue measure. In this paper we study more refined properties of $\Delta_k(E)$, namely the existence of (exactly) similar or multi–similar configurations. For $r \in \mathbb{R}$, $r > 0$, let

$$\Delta'_k(E) := \{r\vec{t} \in \Delta_k(E) : r\vec{t} \in \Delta_k(E)\} \subset \Delta_k(E).$$

We show that for a natural measure $\nu_k$ on $\Delta_k(E)$, $\dim_H(E) > s_{k,d}$ and all $r \in \mathbb{R}_+$, one has $\nu_k (\Delta'_k(E)) > 0$. Thus, there exist many pairs, $\{x^1, x^2, \ldots, x^{k+1}\}$ and $\{y^1, y^2, \ldots, y^{k+1}\}$, in $E$ which are similar by the scaling factor $r$. We also show there are many pairs $(r_1, r_2)$ with $\nu_k (\Delta'_{k1}(E) \cap \Delta'_{k2}(E)) > 0$, i.e., there exist triply-similar configurations in $E$. Further extensions yield existence of multi–similar configurations of any multiplicity.

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1. Introduction

Furstenberg, Katznelson and Weiss [11] proved that if \( A \subset \mathbb{R}^2 \) has positive upper Lebesgue density and \( A_\delta \) denotes its \( \delta \)-neighborhood, then, given vectors \( u, v \) in \( \mathbb{R}^2 \), there exists \( r_0 \) such that, for all \( r > r_0 \) and any \( \delta > 0 \), there exists \( \{x, y, z\} \subset A_\delta \) forming a triangle congruent to \( \{0, ru, rv\} \), i.e., similar to \( \{0, u, v\} \) via scaling factor \( r \). Under the same assumptions, Bourgain [2] proved in \( \mathbb{R}^d \) that if \( u^1, \ldots, u^k \in \mathbb{R}^d \), \( k \leq d \), there exists \( r_0 \) such that, for all \( r > r_0 \) and any \( \delta > 0 \), there exists \( \{x^1, x^2, \ldots, x^{k+1}\} \subset A_\delta \) forming a simplex similar to \( \{0, u^1, \ldots, u^k\} \) via scaling \( r \). Moreover, Bourgain showed that if \( k < d \) and the simplex is non-degenerate, i.e., of positive \( k \)-dimensional volume, then the conclusion holds with \( \delta = 0 \), i.e., one can take \( \{x^1, x^2, \ldots, x^{k+1}\} \subset A \). He further showed that the conclusion does not in general hold when \( \delta = 0 \), at least for \( k = 2 = d \), using arithmetic progressions, i.e., degenerate triangles. The most general result of this type is due to T. Ziegler [16]:

**Theorem 1.1.** Let \( d \geq 2, k \geq 2 \). Suppose \( A \subset \mathbb{R}^d \) is of positive upper Lebesgue density, and let \( A_\delta \) denote the \( \delta \)-neighborhood of \( A \). Let \( V = \{0, u^1, v^1, \ldots, v^k\} \subset \mathbb{R}^d \). Then there exists \( r_0 > 0 \) such that, for all \( r > r_0 \) and any \( \delta > 0 \), there exists \( \{x^1, \ldots, x^{k+1}\} \subset A_\delta \) similar to \( \{0, u^1, \ldots, u^k\} \) via scaling \( r \).

The purpose of this paper is to prove variants of these results for compact sets of Hausdorff dimension \( < d \), sometimes referred to as thin sets. In this context, it is impossible to obtain conclusions nearly as sweeping as those in Theorem 1.1. For one thing, in compact sets the distances are necessarily bounded by the diameter of the set. A more fundamental problem is that it is known, for instance, that there exist compact subsets of \( \mathbb{R}^2 \) of full Hausdorff dimension that do not contain vertices of any equilateral triangle (Falconer [10]). Nevertheless, we are able to prove that if the Hausdorff dimension of a compact set is sufficiently large, then given any \( r > 0 \) there exist many pairs of point configurations consisting of \( (k + 1) \) points of this set that are similar up to a translation and scaling by \( r \). We also give extensions from similarities to multi-similarities, where there exist configurations in the set \( E \) similar to each other via multiple scaling parameters. Moreover, as in [2], we obtain the existence of exactly similar configurations, i.e., all of whose vertices are in \( E \), not just in a \( \delta \)-neighborhood of \( E \).

We need the following definition. Recall that a thin set \( E \) supports a Frostman measure; see [14] for the definition of Frostman measures and their basic properties.

**Definition 1.2.** Let \( d \geq 2, 1 \leq k \leq d \), \( E \subset \mathbb{R}^d \) a compact set, and \( \mu \) be a Frostman measure on \( E \). For \( x^1, \ldots, x^{k+1} \in \mathbb{R}^d \), let \( \vec{v}_{k,d}(x^1, \ldots, x^{k+1}) = \mathbb{R}^{k(k+1)/2} \) be the vector with entries \(|x^i - x^j|, 1 \leq i < j \leq k + 1\) listed in the lexicographic order. Denote points of \( \mathbb{R}^{k(k+1)/2} \) by \( \tilde{t} = (t^j) \).
(i) Define a measure \( \nu_k \) on \( \mathbb{R}^{k(k+1)/2} \) induced by the measure \( \mu \) on \( E \) by the relation, for \( f(\vec{t}) \in C_0(\mathbb{R}^{k(k+1)/2}) \),

\[
(1.1) \quad \int_{\mathbb{R}^{k(k+1)/2}} f(\vec{t}) \, d\nu_k(\vec{t}) := \int \cdots \int f(\vec{v}_{k,d}(x^1, \ldots, x^{k+1})) \, d\mu(x^1) \cdots d\mu(x^{k+1}).
\]

(ii) Define the \( k \)-simplex set or \( (k+1) \)-point configuration set of \( E \),

\[
(1.2) \quad \Delta_k(E) := \{ \vec{v}_{k,d}(x^1, \ldots, x^{k+1}) : x^j \in E \} \subset \mathbb{R}^{k(k+1)/2}.
\]

Note that the \( k \)-simplex will necessarily be degenerate if \( k > d \).

The measure \( \nu_k \) has total mass \( \leq \mu(E)^{k+1} \), and is supported on \( \Delta_k(E) \). For \( k \leq d \), \( \Delta_k(E) \) can be considered, modulo the symmetric group \( S_{k+1} \) acting on the \( x^i \), as the set of congruence classes of \( (k+1) \)-point configurations in \( E \), or equivalently the set of (possibly degenerate) \( k \)-simplexes in \( \mathbb{R}^d \) spanned by points of \( E \). The action of the finite group \( S_{k+1} \) is irrelevant for our results, which are expressed in terms of certain sets of configurations having positive measure. (The situation when \( k > d \) is discussed below.)

The study of the Lebesgue measure of the distance set \( \Delta_1(E) \) for thin sets was begun in 1986 by Falconer [9]. He proved that if \( \dim_H(E) > \frac{d+1}{2} \), then the Lebesgue measure of \( \Delta_1(E) \) is positive. Bourgain [3] improved Falconer’s exponent in the plane to \( \frac{13}{9} \) in 1994, and Wolff [15] further improved it to \( \frac{4}{3} \) in 1999. In 2004, Erdo\'gan [8] improved the exponent in dimensions \( d \geq 3 \) to \( \frac{d}{2} + \frac{1}{3} \), and this is where things stood for a long time until recent improvements. Recently, Du, Guth, Ou, Wang, Wilson and Zhang [5] proved that the Lebesgue measure of \( \Delta_1(E) \) is positive if \( \dim_H(E) > \frac{d}{2} \) for \( d = 3 \) and \( \dim_H(E) > \frac{d}{2} + \frac{1}{4} + \frac{d+1}{4(2d+1)(d-1)} \) for \( d \geq 4 \); this was improved to \( \dim_H(E) > \frac{d^2}{2d-1} \) for \( d \geq 4 \) by Du and Zhang [6]. Even more recently, Guth, Iosevich, Ou and Wang [13] proved that if a planar set satisfies \( \dim_H(E) > \frac{5}{4} \), then the Lebesgue measure of \( \Delta_1(E) \) is positive.

For \( 1 < k \leq d \), the best results known are due to Erdo\'gan, Hart and Iosevich [7] and Greenleaf, Iosevich, Liu and Palsson [12]. The former proved that \( L^{k(k+1)/2}(\Delta_k(E)) > 0 \) if \( \dim_H(E) > \frac{d+k+1}{2} \) and the latter obtained the threshold \( \frac{d^k+1}{k+1} \), improved to \( \frac{8}{5} \) for \( k = d = 2 \).

We note that all of these results, except for [13], are proven by establishing that the measure \( \nu_k \) defined by (1.1) has a density in \( L^2(\mathbb{R}^{k(k+1)/2}) \).
Definition 1.3. Let $d \geq 2$, $1 \leq k$. The $L^2$-threshold for the $k$-simplex problem (or $(k+1)$-point configuration problem) in $\mathbb{R}^d$ is

$$s_{k,d} := \inf \left\{ s : \text{dim}_H(E) > s \implies \int_{\Delta_k(E)} \nu_k^2(\vec{t}) d\vec{t} < \infty \right\},$$

where $E$ runs over all compact sets $E \subset \mathbb{R}^d$.

The case of $k > d$ needs to be treated somewhat differently, since in that range the set $\Delta_k(E) \subset \mathbb{R}^{k(k+1)/2}$ has lower dimension than $\mathbb{R}^{k(k+1)/2}$ and so cannot have positive Lebesgue measure, regardless of the Hausdorff dimension of $E$. This stems from the fact that, when $k > d$, specifying the $(k+1)/2$ pairwise distances between $k+1$ points in $\mathbb{R}^d$ gives an over-determined system: knowing only some of the distances determines the rest. Thus, although $\Delta_k(E)$ still makes sense, the setup has to be modified. In Chatzikonstantinou, Iosevich, Mkrtchyan and Pakianathan [4] it was shown that for $k > d$ the set of congruence classes of $(k+1)$-tuples of elements of $E$ can be naturally viewed as a subset of $\mathbb{R}^{m}$; if $m := d(k+1) - \binom{d+1}{2}$ appropriately chosen distances are specified, then the other distances are determined, up to finitely many possibilities. Let $P$ be such a collection of $m$ edges. In the terminology of [4], $P$ is a maximally independent (in $\mathbb{R}^d$) subset of the edges of the complete graph on $k+1$ vertices. Extend the definition of $\vec{v}_{k,d}$ to the case $k > d$ by setting $\vec{v}_{k,d}(x^1, \ldots, x^{k+1}) = (|x^i - x^j|_{i,j} \in P) \in \mathbb{R}^m$, where the entries in the range are ordered lexicographically. Using this, we can define a measure $\nu_k$ on $\mathbb{R}^m$ and a set $\Delta_k(E) \subset \mathbb{R}^m$ similarly to (1.1) and (1.2). Note that $\nu_k$ and $\Delta_k(E)$ will depend on the choice of $P$, but for our purposes this is irrelevant, so we will fix a particular $P$ once and for all.

While $\vec{v}_{k,d}(x^1, \ldots, x^{k+1})$ doesn’t determine the congruence class of $(x^1, \ldots, x^{k+1})$ uniquely, it identifies it up to a finite number of possibilities. The number of these possibilities is bounded above by a constant $u_{d,k}$, depending only on $d$ and $k$. In this sense, congruence classes of $(k+1)$-tuples of elements of a compact set $E$ in $\mathbb{R}^d$ for $k > d \geq 2$ can be naturally viewed as a subset of $\mathbb{R}^m$. It was shown in [4] that if $k > d$ and the Hausdorff dimension of $E$ is greater than $d - \frac{1}{k+1}$, then the $m$-dimensional Lebesgue measure of the set of congruence classes of $(k+1)$-point configurations with endpoints in $E$ is positive, and, as with most of the results for $k \leq d$, this was shown by first establishing that the measure $\nu_k$ defined by (1.1) has a density in $L^2(\mathbb{R}^m)$.

We now turn to the results of this paper. With the theorems of Furstenberg-Katznelson-Weiss, Bourgain and Ziegler in mind, obtaining more refined structural information about $\Delta_k(E)$ is of natural interest. In our setting, the questions need
to reflect the fact that, since $E$ is compact, all the pairwise distances are bounded. We are going to prove that if $\dim_{H}(E) > s_{k,d}$, then among the $k$-simplexes of $E$, all possible similarity scaling factors occur, and do so with positive $\nu_{k}$-measure. Furthermore, we show that multi-similarities of arbitrarily high multiplicity occur as well. Thus, this holds for the values of $\dim_{H}(E)$ in all of the positive results referred to above, with the possible exception of [13].

To make this more precise, for $r \in \mathbb{R}_{+} := (0, \infty)$ let
\begin{equation}
\Delta_{k}^{r}(E) := \{ \vec{t} \in \Delta_{k}(E) : r \vec{t} \in \Delta_{k}(E) \} \subset \Delta_{k}(E),
\end{equation}
the set of all $k$-simplexes $\vec{t}$ in $E$ for which there is also a simplex in $E$ similar to $\vec{t}$ via the scaling factor $r$. Interchanging the roles of the two simplexes in such a pair, $\{\vec{t}, r\vec{t}\} \subset \Delta_{k}(E)$, note for later use that
\begin{equation}
\Delta_{k}^{1/r}(E) = \Delta_{k}(E).
\end{equation}

One can not only look for similar pairs $\{\vec{t}, r\vec{t}\} \subset \Delta_{k}(E)$, but more generally for similarities of higher multiplicity.

**Definition 1.4.** A collection $\{\vec{t}, r_{1}\vec{t}, \ldots, r_{n-1}\vec{t}\} \subset \Delta_{k}(E)$, with $\{1, r_{1}, \ldots, r_{n-1}\}$ pairwise distinct, is an $n$-similarity of $k$-simplexes in $E$, also referred to as a multi-similarity of multiplicity $n$.

Our main results are the following. All are obtained under the assumptions that $d \geq 2; 1 \leq k \leq d; E \subset \mathbb{R}^{d}$ is compact, $\mu$ is a Frostman measure on $E$ and $\nu_{k}$ is the measure induced by $\mu$ as in Def. 1.2.

**Theorem 1.5.** Let $d \geq 2, 1 \leq k$ and $E \subset \mathbb{R}^{d}$ compact. Suppose that $\dim_{H}(E) > s_{k,d}$, the $L^{2}$-threshold for the $k$-simplex problem. Then there is a uniform lower bound
\[\nu_{k}(\Delta_{k}^{r}(E)) \geq C(k, E) > 0 \text{ for all } r > 0.\]

With the same notation and assumptions as in Thm. 1.5, we also have:

**Theorem 1.6.** Suppose that $\dim_{H}(E) > s_{k,d}$. Then there exist distinct $r_{1}, r_{2} > 0$, with $\nu_{k}(\Delta_{k}^{r_{1}}(E) \cap \Delta_{k}^{r_{2}}(E)) > 0$. In fact, for any partition $\mathbb{R}_{+} = \bigsqcup_{\alpha \in A} R_{\alpha}$ with each $R_{\alpha} \neq \emptyset$ and countable, there exist distinct $\alpha_{1}, \alpha_{2} \in A$ and $r_{1} \in R_{\alpha_{1}}, r_{2} \in R_{\alpha_{2}}$, such that $\nu_{k}(\Delta_{k}^{r_{1}}(E) \cap \Delta_{k}^{r_{2}}(E)) > 0$.

**Theorem 1.7.** Suppose that $\dim_{H}(E) > s_{k,d}$. Then for all $n \in \mathbb{N}$, there exists an \[M = M(n, k, E) \in \mathbb{N}\] such that for any distinct $r_{1}, \ldots, r_{M} \in \mathbb{R}_{+}$, there exist distinct $r_{i_{1}}, \ldots, r_{i_{n}}$ such that
\[\nu_{k}(\Delta_{k}^{r_{i_{1}}}(E) \cap \Delta_{k}^{r_{i_{2}}}(E) \cap \cdots \cap \Delta_{k}^{r_{i_{n}}}(E)) > 0.\]
Remark 1.8. More explicitly, Thm. 1.5 says that, given any \( r > 0 \), there exist \((k + 1)\)-point configurations \( \{x^1, x^2, \ldots, x^{k+1}\} \) and \( \{y^1, y^2, \ldots, y^{k+1}\} \) in \( E \) which are similar via the scaling factor \( r \), i.e., there exists a translation \( \tau \in \mathbb{R}^d \) and a rotation \( \theta \in O_d(\mathbb{R}) \) such that \( y^j = r \theta (x^j + \tau) \), \( 1 \leq j \leq k + 1 \). In the language of Def. 1.4, among the \((k + 1)\)-point configurations or \( k \)-simplexes in \( E \), there exist (many) similarities of multiplicity 2.

In fact, by Thm. 1.6 there exist triple-similarities in \( E \), i.e., pairs \( r_1, r_2 \) of scalings and triples of \((k + 1)\)-point configurations, \( \{x^j\}, \{y^j\}, \{z^j\} \) in \( E \) such that \( y^j = r_1 \theta_1 (x^j + \tau_1) \), \( z^j = r_2 \theta_2 (x^j + \tau_2) \) for appropriate \( \theta_1, \theta_2 \in O_d(\mathbb{R}) \) and \( \tau_1, \tau_2 \in \mathbb{R}^d \), and furthermore that \( r_1, r_2 \) can be arranged to lie in different subsets of a partition of \( \mathbb{R}_+ \) as stated. For example, decomposing \( \mathbb{R}_+ \) into the multiplicative cosets of \( \mathbb{Q}_+ \), there exist similarities of multiplicity 3 with \( r_2/r_1 \) irrational; similarly, replacing the positive rationals with the positive algebraic numbers, there exist such with \( r_2/r_1 \) transcendental.

Finally, Thm. 1.7 shows that there exist multi-similar \((k + 1)\)-point configurations in \( E \) of arbitrarily high multiplicity, and that the scaling factors can be chosen to come from an arbitrary set of distinct elements of \( \mathbb{R}_+ \), as long as that set has large enough cardinality relative to the desired similarity multiplicity.

Remark 1.9. Denoting \( x = (x^1, \ldots, x^{k+1}) \), when \( k > d \) the fact that \( \vec{v}_{k,d}(x) = \vec{v}_{k,d}(y) \) does not imply that \( x \) and \( y \) are congruent and hence \( \vec{v}_{k,d}(x) = r \vec{v}_{k,d}(y) \) doesn’t imply \( x \) and \( y \) are similar. However, the conclusions of Remark 1.8 still hold: Recall that \( \vec{v}_{k,d}(x) \) determines the congruence type of \( x \) up to at most \( u_{d,k} \) choices. Using Thm. 1.7 with \( nu_{d,k} \) instead of \( n \) we see that \( \exists x, x_{i_1}, \ldots, x_{nu_{d,k}} \) and \( r_{i_1}, \ldots, r_{nu_{d,k}} \) such that \( \vec{v}_{k,d}(x), \vec{v}_{k,d}(x_{i_1}), \ldots, \vec{v}_{k,d}(x_{nu_{d,k}}) \) are all congruent. It follows that \( x, x_{i_1}, \ldots, x_{nu_{d,k}} \) all fall within at most \( u_{d,k} \) congruence classes and thus, by the pigeon hole principle, at least \( n + 1 \) of them must be in the same congruence class. This argument, naturally, applies to the conclusions of Remark 1.8 for the other Theorems as well.

2. Proofs of Theorems 1.6 and 1.7

We start by showing that Thms. 1.6 and 1.7 follow from Thm. 1.5 by measure-theoretic arguments. To prove Thm. 1.6, let \( \mathbb{R}_+ = \bigsqcup_{\alpha \in A} R_\alpha \) be a partition of \( \mathbb{R}_+ \) into a (necessarily uncountable) collection of nonempty countable subsets. From the definition (1.3), it follows that each \( \Delta_\xi(E) \) is \( \nu_k \)-measurable. Hence, if for each
$\alpha \in A$, with slight abuse of notation we define the set
\[ \Delta_k^\alpha(E) := \bigcup_{r \in R_\alpha} \Delta_k^r(E), \]
then, being a countable union of measurable sets, each $\Delta_k^\alpha(E)$ is $\nu_k$-measurable. Furthermore, combining $R_\alpha \neq \emptyset$, the monotonicity of $\nu_k$ and Thm. 1.5, one sees that each $\nu_k(\Delta_k^\alpha(E)) > 0$. However, $\nu_k(\Delta_k(E)) \leq \mu(E)^{k+1} < \infty$, and no finite (or even $\sigma$-finite) measure space can be the pairwise disjoint union of an uncountable collection of measurable subsets of positive measure. Thus, there must exist $\alpha_1 \neq \alpha_2$ such that $\Delta_k^{\alpha_1}(E) \cap \Delta_k^{\alpha_2}(E) \neq \emptyset$; it follows that there are $r_j \in R_{\alpha_j}$, $j = 1, 2$, such that $\Delta_k^{r_j}(E) \cap \Delta_k^{r_2}(E) \neq \emptyset$. For the full claim of Thm. 1.6, that there exist distinct $\alpha_1, \alpha_2 \in A$ and $r_1 \in R_{\alpha_1}, r_2 \in R_{\alpha_2}$, such that $\nu_k(\Delta_k^{r_1}(E) \cap \Delta_k^{r_2}(E)) > 0$, first make a choice of one representative from each of the $R_\alpha$, choose an arbitrary countably infinite subset of these, and then apply Thm. 1.7.

For the proof of Thm. 1.7, we use the uniform lower bound from Thm. 1.5, $\nu_k(\Delta_k^r(E)) \geq C(E, k) > 0$, $\forall r \in \mathbb{R}$, combined with $\nu_k(\Delta_k(E)) < \infty$. Thm. 1.7 then follows from the following measure-theoretic pigeon-hole principle, which might be of independent interest and whose proof is deferred to the Appendix, Sec. 6.

**Lemma 2.1.** Let $\mathcal{X} = (X, \mathcal{M}, \sigma)$ be a finite measure space. For $0 < c < \sigma(X)$, let $\mathcal{M}_c = \{A \in \mathcal{M} : \sigma(A) \geq c\}$. Then, for every $n \in \mathbb{N}$, there exists an $N = N(\mathcal{X}, c, n) \in \mathbb{N}$ such that for any collection $\{A_1, \ldots, A_N\} \subset \mathcal{M}_c$ of cardinality $N$, there is a subcollection $\{A_{i_1}, \ldots, A_{i_n}\}$ of cardinality $n$ such that $\sigma(A_{i_1} \cap \cdots \cap A_{i_n}) > 0$ and hence $A_{i_1} \cap \cdots \cap A_{i_n} \neq \emptyset$.

### 3. Proof of Theorem 1.5

To keep the exposition simple, we first prove Thm. 1.5 in the case $k \leq d$. In Sections 3 and 4 we will assume $k \leq d$. In the case $k > d$ the arguments are very similar. Essentially, since $\tilde{v}_{k,d}(x^1, \ldots, x^{k+1})$ determines the congruence type of $(x^1, \ldots, x^{k+1})$ up to at most $u_{d,k}$ choices, the constant $u_{k,d}$ will appear throughout the proof. However, since the results here are up to multiplicative constants, this doesn’t play any essential role.

For $\epsilon > 0$, define an approximation of $\nu_k$ on $\mathbb{R}^{k(k+1)/2}$ by
\begin{equation}
\nu_k^{\ell}(\tilde{t}) = \int \cdots \int \prod_{1 \leq i < j \leq k+1} \sigma_{ij}^\epsilon(x^i - x^j) \prod_{l=1}^{k+1} d\mu(x^l),
\end{equation}
where $\sigma_t$ is the normalized surface measure on the sphere of radius $t$ in $\mathbb{R}^d$ and $\sigma_t^\epsilon(x) := \sigma_t * \rho_\epsilon(x)$, with $\rho \in C_0^\infty(\mathbb{R}^d)$, $\rho \geq 0$, supp($\rho$) $\subset \{|t| < 1\}$, $\int \rho = 1$ and
\[ \rho_\varepsilon(x) = \varepsilon^{-d} \rho(\varepsilon^{-1} x) \]. Then each \( \nu_k^\varepsilon \in C_0^\infty \) and \( \nu_k^\varepsilon \to \nu_k \) weak* as \( \varepsilon \to 0 \). Thus,

\[ \nu_k(\Delta_k^\varepsilon(E)) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{k(k+1)/2}} \nu_k^\varepsilon(r\vec{t}) d\nu_k(\vec{t}) \].

By (1.1), for \( \varepsilon \) fixed,

\[ \int_{\mathbb{R}^{k(k+1)/2}} \nu_k^\varepsilon(r\vec{t}) d\nu_k(\vec{t}) = \int \nu_k^\varepsilon(r(x_1 - x^2), \ldots, r(x_k - x^{k+1})) d\mu(x_1) \ldots d\mu(x^{k+1}) \].

Using the definition in (3.1), we see that this is (3.2)

\[ \approx \varepsilon^{-(\frac{k+1}{2})} \int \cdots \int_{\|x_i - x^j - r|y_i - y^j|\| < \varepsilon; 1 \leq i < j \leq k+1} d\mu(x_1) \ldots d\mu(x^{k+1}) d\mu(y_1) \ldots d\mu(y^{k+1}) \].

Here, and throughout, we write \( X \lesssim Y \) (resp. \( X \approx Y \)) if there exist constants \( 0 < c < C \), depending only on \( k, E \) and the choice of \( \rho \) (and thus implicitly on \( d \)), such that \( X \lesssim CY \) (resp., \( cY \leq X \leq CY \)).

For each rotation \( \theta \in O_d(\mathbb{R}) \), define a measure \( \lambda_{r,\theta} \) on \( \mathbb{R}^d \) by

\[ \int f(z) d\lambda_{r,\theta}(z) = \int \int f(u - r\theta v) d\mu(u)d\mu(v), f \in C_0(\mathbb{R}^d). \]

This is the push-forward of \( \mu \times \mu \) under the map \( (u, v) \to u - r\theta v \), has total mass \( ||\lambda_{r,\theta}|| = \mu(E)^2 \), and is supported in \( E - r\theta E \). The following shows that, for \( E \) of the Hausdorff dimensions in question, for a.e. \( \theta \), \( \lambda_{r,\theta} \) is absolutely continuous with a density in \( L^{k+1}(\mathbb{R}^d) \), which we denote by \( \lambda_{r,\theta}(\cdot) \). Let \( d\theta \) denote the Haar probability measure on \( O_d(\mathbb{R}) \).

**Proposition 3.1.** With the notation above, \( \lim \inf_{\varepsilon \to 0} \) of the expression on the right hand side of (3.2) is

\[ \approx \int \int \lambda_{r,\theta}^{k+1}(z) dz d\theta. \]

By definition, the quantity on the right hand side of (3.3) is finite if \( \dim_H(E) > s_{k,d} \), the \( L^2 \)-threshold for the \( k \)-simplex problem. Prop. 3.1 was proved in [12] in the case \( r = 1 \); the proof in the general case is similar, but we supply it in the next section for the sake of completeness.
Continuing with the proof of Thm. 1.5, by Hölder we have
\[
(3.4) \quad \mu(E)^2 = \int \int \lambda_{r,\theta}(z) \cdot 1 \, dz \, d\theta \\
\leq \left( \int \int \lambda_{r,\theta}^{k+1}(z) \, dz \, d\theta \right)^{\frac{1}{k+1}} \times \left( \int \int_{\text{supp}(\lambda_{r,\theta}) \times \text{Os}_d(\mathbb{R})} 1^{\frac{1}{k+1}} \, dz \, d\theta \right)^{\frac{k}{k+1}}.
\]
Since supp(\lambda_{r,\theta}), being contained in E - r\theta E, has Lebesgue measure \( \lesssim (1 + r^d)\mu(E) \), we divide both sides of (3.4) by the second factor on the right hand side and raise to the \( k+1 \) power to obtain
\[
\mu(E)^{k+1}(1 + r^d)^{-(k+1)} \lesssim \int \int \lambda_{r,\theta}^{k+1}(z) \, dz \, d\theta.
\]
Combining this with Prop. 3.1, we conclude that, for \( \dim_H(E) > s_{k,d} \) and \( 0 < r \leq 1 \),
\[
(3.5) \quad \liminf_{\varepsilon \to 0} \int \nu_k^r(r\bar{t}) d\nu_k^r(t) \approx \int \int \lambda_{r,\theta}^{k+1}(z) \, dz \, d\theta \gtrsim 1.
\]
It follows that \( \liminf_{\varepsilon \to 0} \nu_k(\{ \bar{t} : r\bar{t} \in \Delta_{k,\varepsilon}(E) \}) \gtrsim 1 \), where \( \Delta_{k,\varepsilon}(E) \) is the \( \varepsilon \)-neighborhood of \( \Delta_k(E) \). Since the sets \( \{ \bar{t} : r\bar{t} \in \Delta_{k,\varepsilon}(E) \} \) are nested as \( \varepsilon \to 0 \), we conclude that, for \( 0 < r \leq 1 \),
\[
(3.6) \quad \nu_k(\{ \bar{t} : r\bar{t} \in \Delta_k(E) \}) \gtrsim 1.
\]
However, by (1.4), \( \nu_k(\{ \bar{t} : r\bar{t} \in \Delta_k(E) \}) = \nu_k(\{ \bar{t} : r^{-1}\bar{t} \in \Delta_k(E) \}) \), so (3.6) holds for \( 1 \leq r < \infty \) as well. This completes the proof of Thm. 1.5, up to the verification of Prop. 3.1.

4. Proof of Proposition 3.1

We will follow closely the argument in [12, Sec. 2]. It will be convenient to denote an ordered \( (k+1) \)-tuple \( (x^1, \ldots, x^{k+1}) \) of elements of \( \mathbb{R}^d \) by \( x \). If the corresponding set \( \{x^1, \ldots, x^{k+1}\} \) is a nondegenerate simplex (i.e., affinely independent), then
\[
\pi(x) := \text{span}\{x^2 - x^1, \ldots, x^{k+1} - x^1\}
\]
is a \( k \)-dimensional linear subspace of \( \mathbb{R}^d \). \( \Delta(x) \) will denote the (unoriented) simplex generated by \( \{x^1, \ldots, x^{k+1}\} \), i.e., the closed convex hull, which is contained in the affine plane \( x^1 + \pi(x) \). Both \( \pi(x) \) and \( \Delta(x) \) are independent of the order of the \( x^j \). If \( \{x^1, \ldots, x^{k+1}\} \) is similar to \( \{y^1, \ldots, y^{k+1}\} \) by a scaling factor \( r \), then, up to permutation of \( y^1, \ldots, y^{k+1} \), there exists a \( \theta \in \text{O}(d) \) such that \( x^j - x^1 = r\theta(y^j - y^1), 2 \leq j \leq k+1 \), which is equivalent with \( x^j - x^1 = r\theta(y^j - y^i), 1 \leq i < j \leq k+1 \), and \( \Delta(x) = (x^1 - r\theta y^1) + r\theta \Delta(y) \).
The group $O(d)$ acts on the Grassmanians $G(k, d)$ and $G(d-k, d)$ of $k$ (resp., $d-k$) dimensional linear subspaces of $\mathbb{R}^d$, and if $x$ is similar to $y$, one has $\pi(x) = \theta \pi(y)$ and $\pi(x)^\perp = \theta (\pi(y)^\perp)$. The set of $\theta \in O(d)$ fixing $\pi(x)$ is a conjugate of $O(d-k) \subset O(d)$, and we refer to this as the stabilizer of $x$, denoted $\text{Stab}(x)$.

For $x, y$ similar, let $\tilde{\theta} \in O(d)$ be such that it transforms $y$ to $x$. I.e. we have $\pi(x) = \tilde{\theta} \pi(y)$ and $x^i - x^j = r \theta \omega^i(y^i - y^j)$ for all $\omega \in \text{Stab}(y)$. For each $y$, take a cover of $O(d)/\text{Stab}(y)$ by balls of radius $\epsilon$ (with respect to some Riemannian metric) with finite overlap. Since the dimension of $O(d)/\text{Stab}(y)$ is that of $O(d)/O(d-k)$, namely

$$\frac{d(d-1)}{2} - \frac{(d-k)(d-k-1)}{2} = kd - \frac{k(k+1)}{2},$$

one needs $N(\epsilon) \lesssim \epsilon^{-(kd-k(k+1))}$ balls to cover it. In these balls, choose sample points, $\tilde{\theta}_m(y)\text{, }1 \leq m \leq N(\epsilon)$.

One sees that

$$\{ (x, y) : ||x^i - x^j| - r|y^i - y^j|| \leq \epsilon, \ 1 \leq i < j \leq k+1 \}$$

$$\subseteq \bigcup_{m=1}^{N(\epsilon)} \{(x, y) : \left| (x^i - x^j) - r \tilde{\theta}_m(y) \omega(y^i - y^j) \right| \lesssim \epsilon, \forall 1 \leq i < j \leq k+1, \omega \in \text{Stab}(y) \}.$$

Thus, the right hand side of (3.2) is bounded above by

$$\sum_{m=1}^{N(\epsilon)} \mu^{2(k+1)} \{(x, y) : \left| (x^i - x^j) - r \tilde{\theta}_m(y) \omega(y^i - y^j) \right| \lesssim \epsilon, \forall 1 \leq i < j \leq k+1, \omega \in \text{Stab}(y) \}.$$  \hspace{1cm} (4.1)

When picking the $N(\epsilon)$ balls, if each point of $O(d)/\text{Stab}(y)$ is covered by at most $p = p(d)$ of the balls, then the quantity above becomes a lower bound when multiplied by $1/p$. Thus, the right hand side of (3.2) is $\approx$ to (4.1), which can also be written as

$$\sum_{m=1}^{N(\epsilon)} e^{-k d - \frac{k(k+1)}{2}} \mu^{2(k+1)} \{(x, y) : \left| (x^i - r \tilde{\theta}_m(y) \omega y^i) - (x^j - r \tilde{\theta}_m(y) \omega y^j) \right| \lesssim \epsilon, \forall 1 \leq i < j \leq k+1, \omega \in \text{Stab}(y) \}.$$  \hspace{1cm} (4.2)
Since this holds for any choice of sample points \( \tilde{\theta}_m(y) \), we can pick these points such that they minimize (up to a factor of 1/2, say) the quantity

\[
\mu^{2(k+1)} \{(x, y) : |(x^i - r\tilde{\theta}_m(y)\omega y^j) - (x^j - r\tilde{\theta}_m(y)\omega y^j)| \leq \epsilon, \\
\quad \forall 1 \leq i < j \leq k + 1, \omega \in \text{Stab}(y)\}.
\]

The \( N(\epsilon) \) preimages, under the natural projection from \( \mathbb{O}(d) \), of the balls used to cover \( \mathbb{O}(d)/\text{Stab}(y) \) are \( \epsilon \)-tubular neighborhoods of the preimages of the sample points \( \tilde{\theta}_m(y) \), which we denote \( T_1^\epsilon, \ldots, T_{N(\epsilon)}^\epsilon \). Since \( \dim(\mathbb{O}(d)/\text{Stab}(y)) = kd - \frac{k(k+1)}{2} \), each \( T_m^\epsilon \) has measure \( \sim \epsilon^{kd-\frac{k(k+1)}{2}} \) with respect to the Haar measure \( d\theta \). Since the infimum over a set is less than or equal to the average over the set, it follows that

\[
\mu^{2(k+1)} \left\{ (x, y) : \left| (x^i - r\tilde{\theta}_m(y)\omega y^j) - (x^j - r\tilde{\theta}_m(y)\omega y^j) \right| \leq \epsilon, \\
\quad \forall 1 \leq i < j \leq k + 1, \omega \in \text{Stab}(y) \right\} 
\]

\[
\approx \frac{1}{\epsilon^{kd-\frac{k(k+1)}{2}}} \int_{T_m^\epsilon} \mu^{2(k+1)} \left\{ (x, y) : \left| (x^i - r\theta y^j) - (x^j - r\theta y^j) \right| \leq \epsilon, 1 \leq i < j \leq k+1 \} d\theta.
\]

The quantity in (4.2) is thus

\[
\approx \epsilon^{-kd} \sum_{m=1}^{N(\epsilon)} \int_{T_m^\epsilon} \mu^{2(k+1)} \left\{ (x, y) : \left| (x^i - r\theta y^j) - (x^j - r\theta y^j) \right| \leq \epsilon, 1 \leq i < j \leq k + 1 \} d\theta,
\]

which, since the \( \{T_m^\epsilon\} \) have finite overlap, is

\[
\approx \epsilon^{-kd} \int \mu^{2(k+1)} \left\{ (x, y) : \left| (x^i - r\theta y^j) - (x^j - r\theta y^j) \right| \leq \epsilon, 1 \leq i < j \leq k + 1 \} d\theta,
\]

and taking the \( \text{lim inf} \), we obtain something \( \approx \) the expression (3.3). This completes the proof of Proposition 3.1, and thus Thms. 1.5, 1.6, and 1.7.

5. Open question

The following is a natural question pertaining to the subject matter of Thm. 1.5:

- In [1] it was shown that if \( E \) is a compact subset of \( \mathbb{R}^d \), of Hausdorff dimension greater than \( \frac{d+1}{2} \), then there exists a non-empty open interval \( I \) such that, for any \( t \in I \), there exist \( x^1, x^2, \ldots, x^{k+1} \in E \) such that \( |x^{j+1} - x^j| = t \), \( 1 \leq j \leq k \). In view of Thm. 1.5, it seems reasonable to ask: given any \( r > 0 \), do there exist \( x, y, z \in E \) such that \( |x - z| = r|x - y|? \) This can be regarded as a pinned version of the case \( k = 1 \) of Thm. 1.5, in the sense that the
endpoint $x$ is common to both segments whose length is being compared. Similar questions can be raised when $k > 1$.

6. Appendix: A measure-theoretic Pigeon Hole Principle

Unable to find Lemma 2.1 in the literature, and believing that it should be useful for other problems, we prove it here. Without loss of generality the total measure $\sigma(X)$ can be normalized to be equal to 1, so for the proof we restate the result as

**Lemma 6.1.** Let $\mathcal{X} = (X, \mathcal{M}, \sigma)$ be a probability space. For $0 < c < 1$, let $\mathcal{M}_c = \{ A \in \mathcal{M} : \sigma(A) \geq c \}$. Then, for every $n \in \mathbb{N}$, there exists an $N = N(\mathcal{X}, c, n) \in \mathbb{N}$ such that, for any collection $\{ A_1, \ldots, A_N \} \subset \mathcal{M}_c$ of cardinality at least $N$, there is a subcollection $\{ A_{i_1}, \ldots, A_{i_n} \}$ of cardinality $n$ such that $\sigma(A_{i_1} \cap \cdots \cap A_{i_n}) > 0$ and hence $A_{i_1} \cap \cdots \cap A_{i_n} \neq \emptyset$.

To start the proof, first we establish the following claim, which is a quantitative strengthening of the statement for $n = 2$:

**Claim 6.2.** Let $\mathcal{X} = (X, \mathcal{M}, \sigma)$ be a probability space. Then for any $0 < c < 1$ there exists $P_c \in \mathbb{N}$ such that for any $N > P_c$, if $\{ A_1, \ldots, A_N \} \subset \mathcal{M}_c$, then there exist distinct $i, j \leq N$ such that $\sigma(A_i \cap A_j) \geq c^3/3$.

**Proof.** Suppose not. Let $S \subset (0,1)$ be the set of all $c \in (0,1)$ such that the statement of the claim is false, and suppose $c \in S$. Then for every $N \in \mathbb{N}$ there exists a subset $\{ A_1, \ldots, A_{2N} \} \subset \mathcal{M}_c$ such that $\sigma(A_i) \geq c$ for all $i$ but $\sigma(A_i \cap A_j) < c^3/3$ for all $i \neq j$. Consider the sets $A_{2i-1} \cup A_{2i}$, $i = 1, \ldots, N$. We have

$$
\sigma(A_{2i-1} \cup A_{2i}) = \sigma(A_{2i-1}) + \sigma(A_{2i}) - \sigma(A_{2i-1} \cap A_{2i}) > c + c - \frac{c^3}{3} = 2c - \frac{c^3}{3}.
$$

Since $\sigma(X) = 1 \geq \sigma(A_{2i-1} \cup A_{2i})$, this implies $1 > 2c - \frac{c^3}{3}$. In particular, since $c < 1$, we have $c \lesssim 0.52 < 3/5$; hence $[3/5, 1) \cap S = \emptyset$.

Moreover,

$$
\sigma( (A_{2i-1} \cup A_{2i}) \cap (A_{2j-1} \cup A_{2j}) ) = \sigma( (A_{2i-1} \cap A_{2j}) \cup (A_{2i-1} \cap A_{2j-1}) \cup (A_{2i} \cap A_{2j}) \cup (A_{2i} \cap A_{2j-1}) ) \< c^3 \leq \sigma( A_{2i-1} \cap A_{2j} ) + \sigma(A_{2i-1} \cap A_{2j-1}) + \sigma(A_{2i} \cap A_{2j}) + \sigma(A_{2i} \cap A_{2j-1}) } \leq \frac{(2c - c^3/3)^3}{3} \text{ since } 0 < c < 1.
$$

Thus, there exist $N$ sets, namely $A_1 \cup A_2, \ldots, A_{2N-1} \cup A_{2N}$, such that each has measure at least $f(c) := 2c - \frac{c^3}{3}$ but all pairwise intersections have measure less than $\frac{f(c)^3}{3}$.
Thus, we have shown that if \( c \in S \) then \( f(c) \in S \) as well. However if \( 0 < c < 1 \), then there exists \( k \in \mathbb{N} \) such that \( f^k(c) > 3/5 \) and is thus \( \notin S \) (where \( f^k \) denotes \( f \) composed with itself \( k \) times). It follows that \( S \) must be empty. \( \square \)

We use Claim 6.2 as a building block for the proof of Lemma 6.1, which is by induction on \( n \). If \( n = 1 \), then we can take \( N = 1 \), since any \( A_{i_1} \in \mathcal{M}_c \) satisfies the statement. If \( n = 2 \) then any \( N \geq \lceil 1/c \rceil \) suffices, since there cannot be more than \( 1/c \) pairwise disjoint sets of measure \( \geq c \) each; alternatively, one may simply invoke Claim 6.2.

Now suppose that the conclusion of Lemma 6.1 holds for some \( n, n \geq 2 \). Set \( N = 2N(\mathcal{X}, c^3/3, n) + P_c \), and suppose \( \{A_1, \ldots, A_N\} \subset \mathcal{M}_c \) is a collection of cardinality \( N \). Since \( N > P_c \), by Claim 6.2 there exist distinct \( i, j \leq N \) such that \( \sigma(A_i \cap A_j) > c^3/3 \). Let \( B_1 = A_i \cap A_j \). Removing \( A_i \) and \( A_j \) from the collection we still have \( N - 2 > P_c \) sets, so can find another pair whose intersection has measure at least \( c^3/3 \). Repeating this procedure \( N(\mathcal{X}, c^3/3, n) \) times, one finds sets \( B_1, \ldots, B_N(\mathcal{X}, c^3/3, n) \in \mathcal{M}_{c^3/3} \). By the induction hypothesis there exist \( 0 < i_1 < i_2 < \cdots < i_n \leq N(\mathcal{X}, c^3/3, n) \) such that \( \sigma(B_{i_1} \cap \cdots \cap B_{i_n}) > 0 \). Since \( B_{i_1} \cap \cdots \cap B_{i_n} \) is the intersection of \( 2n \) distinct sets from the collection \( \{A_1, \ldots, A_N\} \), the intersection of any \( n + 1 \) of those \( 2n \) will have positive measure, completing the induction step. \( \square \)

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