Hilbert scheme of rational curves on a generic quintic threefold

B. Wang
(汪 镭)
Aug, 2018

Abstract

Let $X_0$ be a generic quintic threefold in projective space $\mathbb{P}^4$ over the complex numbers. For a fixed natural number $d$, let $R_d(X_0)$ be the open sub-scheme of the Hilbert scheme, parameterizing irreducible rational curves of degree $d$ on $X_0$. In this paper, we show that

1) $R_d(X_0)$ is smooth and of expected dimension,
2) Combining the Calabi-Yau condition on $X_0$, we further show that it consists of immersed rational curves.
3) Parts (1) and (2) imply a statement of Clemens’ conjecture: if $C_0 \in R_d(X_0)$ and $c_0 : \mathbb{P}^1 \to C_0$ is the normalization, the normal sheaf is isomorphic to the vector bundle

\[ N_{c_0/X_0} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1). \]

Contents

1 Introduction
  1.1 Statements ............................................. 2
  1.2 Outline of the proof .................................... 2

Key words: Generic quintic 3-fold, Hilbert scheme, Jacobian matrix, Clemens’ conjecture

2000 Mathematics subject classification: 14J70, 14J30, 14C05, 14B12
1 Introduction

We work over the complex numbers, \( \mathbb{C} \), and the Euclidean topology. A generic (or general) point is referred to a point in a complement of a proper complex analytic subset of an irreducible analytic (or projective) space. The following is the statement of the result.

1.1 Statements

Theorem 1.1.
Let \( X_0 \) be a generic quintic threefold in \( \mathbb{P}^4 \) over \( \mathbb{C} \). Let \( R_d(X_0) \) be the open sub-scheme of the Hilbert scheme, parameterizing irreducible rational curves of degree \( d \) on \( X_0 \). Then

1. \( R_d(X_0) \) is smooth, and of expected dimension

\[
\left( \dim(X_0) + 2 - \deg(X_0) \right)d + \dim(X_0) - 3 = 0
\]

2. \( R_d(X_0) \) consists of immersed rational curves. Precisely, if

\[
c_0 : \mathbb{P}^1 \to C_0
\]

is the normalization of \( C_0 \in R_d(X_0) \), then \( c_0 \) is an immersion, i.e.
the differential of \( c_0 \) is injective everywhere and \( c_0 \) is an isomorphism to its image on an open set.

For the same \( X_0 \), but smooth embedding \( c_0 \), Clemens, in [2], proposed

\[
N_{c_0/X_0} \simeq \mathcal{O}_\mathbb{P}^1(-1) \oplus \mathcal{O}_\mathbb{P}^1(-1). \tag{1.1}
\]

Notice that in deformation theory, part (1) in Theorem 1.1 implies

\[
H^1(N_{c_0/X_0}) = 0.
\]

Now using part (2) and \( X_0 \) being Calabi-Yau, we obtain that the normal sheaf \( N_{c_0/X_0} \) must be a vector bundle decomposed as

\[
N_{c_0/X_0} \simeq \mathcal{O}_\mathbb{P}^1(-k - 2) \oplus \mathcal{O}_\mathbb{P}^1(k), \tag{1.2}
\]

with \( k \geq -1 \). Applying the Serre duality with the fact \( H^1(N_{c_0/X_0}) = 0 \), we obtain that \( k = -1 \). Therefore Theorem 1.1 implies

**Corollary 1.2.** The statement of Clemens’ conjecture – formula (1.1) is correct for all rational curves on \( X_0 \).

**Remark** The example of Vainsencher ([5]) shows the immersed, singular rational curves exist on \( X_0 \). So our corollary proves a statement of the amended Clemens’ conjecture ([3]).

### 1.2 Outline of the proof

Our approach is in the local deformation theory with the focus on algebra, more specifically, the linear algebra and the differential algebra. In past, the deformation idea is geometric and is well-known (for instance see [1]). But while it worked in special cases, it also taught us that in general cases, any kind of deformation of \( X_0 \) will encounter the serious hurdle in geometry. In this paper, we’ll present a method that realizes the new idea in algebra:

1. use the down-to-earth approach to set-up the space of rational curves.
2. then switch the focus to the first order deformation, and use linear algebra and differential algebra to reformulate and calculate a Jacobian matrix.

Let’s start the problem in geometry.
1.2.1 Geometric setting

This is the down-to-earth approach in affine coordinates, that avoids modulo $GL(2)$ group action of $\mathbb{P}^1$. Let

$$M = (H^0(\mathcal{O}_{\mathbb{P}^1}(d)))^\oplus 5 \simeq \mathbb{C}^{5d+5},$$

be the space of all 5 tuples of homogeneous polynomials in two variables of degree $d$. The open set $M_d$ of $M$ represents (but is not equal to)

$$\{c \in \text{Hom}_{\text{bir}}(\mathbb{P}^1, \mathbb{P}^4) : \deg(c(\mathbb{P}^1)) = d\}.$$

Let $f_0 \in H^0(\mathcal{O}_{\mathbb{P}^4}(5))$ and $X_0 = \text{div}(f_0)$ be the 3-fold associated to the quintic section. For a generic quintic 3-fold $X_0$, let $I_{X_0}$ be an irreducible component of the incidence scheme

$$\{c \in M_d : c^*(f_0) = 0\} \subset M_d,$$

where we use the same $c$ to denote the regular map $\mathbb{P}^1 \to \mathbb{P}^4$ associated to the point $c \in M_d$. Then we’ll prove

**Proposition 1.3.**

$I_{X_0}$ is smooth and $\dim(I_{X_0}) = 4$.

It is well understood that Proposition 1.3 is just the variant version of the main part of Theorem 1.1 ([1], [4]). But the style of the formulation prepares for a change. We switch the focus to another scheme, which will lead us to a different algebra later on. We call it the projected incidence scheme $I_L$. Let

$$S = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(5)))$$

be the space of quintic 3-folds of $\mathbb{P}^4$. Let $\mathbb{P} \subset S$ be a 2-dimensional plane and $\mathbb{L} \subset S$ be an analytic open subset. We define $I_L$ to be an irreducible component of the scheme

$$\{c \in M_d : c^*(f) = 0, [f] \in \mathbb{L}\} \subset M_d,$$

where $[f]$ is a point in $\mathbb{L}$ representing the quintic $\text{div}(f)$. To prove Proposition 1.3, it suffices to prove the same property in the local complex geometry for $I_L$. 

4
Proposition 1.4. Assume $\mathbb{L}$ satisfies the “pencil condition” that requires all components $I_\mathbb{L}$ are disjoint. Then the following assertions are true.

1. For such a sufficiently small subset $\mathbb{L}$ of $\mathbb{P}$, containing an $S$-generic point, if all components $I_\mathbb{L}$ are smooth and of dimension 6, then Proposition 1.3 is true.

2. There exist such a plane $\mathbb{P}$ and its open set $\mathbb{L}$ containing an $S$-generic $f_0$ such that all components $I_\mathbb{L}$ are smooth and of dimension 6, or equivalently a Jacobian matrix of each $I_\mathbb{L}$ has full rank.

Remark

A more technical description of pencil condition is: for the generic $[f_0] \in \mathbb{L}$, a generic rational curve from each component $I_{X_0}$ (with $X_0 = \text{div}(f_0)$) does not lie in any quintic of $\mathbb{L}$ other than $X_0$. Because the incidence relation is linear in $S$, the pencil condition on $\mathbb{L}$ is also a condition on $\mathbb{P}$. The “pencil condition” is a necessary condition for $I_\mathbb{L}$ to be smooth.

1.2.2 Algebraic setting–the Jacobian matrix

Theorem 1.1 follows from Proposition 1.3 which follows from the part (2) of Proposition 1.4. In the standard geometry, each $I_{X_0}$ has no difference from its corresponding $I_\mathbb{L}$. But we found that their elementary algebra are different. Thus we change the focus to algebra and try to prove that

a Jacobian matrix $J_\mathbb{L}$ of $I_\mathbb{L}$ has full rank. \[ (1.4) \]

The algebra in (1.4) comes from the affine coordinates of $M$. Let’s express $I_\mathbb{L}$ in affine coordinates of $M$. Let $\mathbb{P}$ be spanned by three non-collinear quintic polynomials $f_0, f_1, f_2$ whose generic open set $\mathbb{L}$ satisfies the “pencil condition” and contains an $S$-generic quintic. Next we define global functions. Choose generic $5d + 1$ points $t_i \in \mathbb{P}^1$ (generic in $\text{Sym}^{5d+1}(\mathbb{P}^1)$), and let

$$ t = (t_1, \cdots, t_{5d+1}) \in \text{Sym}^{5d+1}(\mathbb{P}^1). $$

In the rest of the paper, we’ll also use the following conventions in affine coordinates:

---

1The same algebraic approach to $I_\mathbb{L}$ fails for $\text{dim}(\mathbb{L}) = 0, 1$, and is inaccessible (to us) for $\text{dim}(\mathbb{L}) \geq 3$.  

5
(a) $t_i$ or $t$ denotes a complex number in an affine open set $\mathbb{C} \subset \mathbb{P}^1$,
(b) for the restriction of $c \in M_d$ to an affine set $\mathbb{C} \subset \mathbb{P}^1$, $c(t)$ denotes the image
\[
\mathbb{C} \overset{c}{\rightarrow} \mathbb{C}^5 \\
t \mapsto c(t),
\]
(c) a quintic $f$ is a homogeneous polynomial of degree 5 in 5 variables, i.e. $f \in \mathcal{O}(\mathbb{C}^5)$.

We should note that in these affine coordinates, the incidence relation $c^*(f) = 0$ can be expressed as the composition $f(c(t)) = 0$ for all $t \in \mathbb{C}$. With these affine expressions, $I_L$ for an generic and sufficiently small open set $L$ (satisfying the pencil condition) is defined by $5d - 1$ polynomials
\[
\begin{vmatrix}
  f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\
  f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\
  f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2))
\end{vmatrix} \tag{1.5}
\]
for $i = 3, \cdots, 5d + 1$ and variable $c \in M$, where $| \cdot |$ denotes the determinant of a matrix. This is because that the incidence scheme
\[
\{(c, f) : f(c(t)) = 0, \text{ fixed, distinct } t_1, \cdots, t_{5d+1} \in \mathbb{C}^1\}
\]
implies that $I_L$ is defined by
\[
\begin{vmatrix}
  f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\
  f_2(c(t_j)) & f_1(c(t_j)) & f_0(c(t_j)) \\
  f_2(c(t_k)) & f_1(c(t_k)) & f_0(c(t_k))
\end{vmatrix} \tag{1.6}
\]
for $1 \leq i, j, k \leq 5d + 1$. Because of pencil condition,
\[
\begin{pmatrix}
  f_2(c(t_1)), f_1(c(t_1)), f_0(c(t_1)) \\
  f_2(c(t_2)), f_1(c(t_2)), f_0(c(t_2))
\end{pmatrix}
\]
are linearly independent for all $c$ in $I_L$ for a sufficiently small $L$ (by the genericity of $t_1, t_2$). This implies the polynomials (1.5) and (1.6) define the same scheme around $I_L$. Let $C_M$ be a system of affine coordinates for $M$, which determines an isomorphism
\[
M \simeq \mathbb{C}^{5d+5}.
\]
Then the set of polynomials in (1.5) gives a rise to a holomorphic map
\[ M \simeq \mathbb{C}^{5d+5} \rightarrow \mathbb{C}^{5d-1}, \]
whose differential map is represented by a \((5d - 1) \times (5d + 5)\) matrix \(J_L\). In this paper, we simply call \(J_L\) the Jacobian matrix of the scheme \(I_L\), which is uniquely defined with the affine coordinates above (We are only concerned with local \(J_L\) around \(I_L\)).

### 1.2.3 Jacobian matrices in differential algebra

We introduce differential algebra to dig deep into the Jacobian matrix \(J_L\). We consider differential 1-forms \(\phi_i\) on \(M\) for \(i = 3, \cdots, 5d + 1\):

\[
\phi_i = d \begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix}
\]

(1.8)

where \(d\) is the differential on the regular functions of \(M\). Using these forms, we define a \((5d - 1)\)-form,

\[
\omega(\mathbb{L}, t) = \wedge_{i=3}^{5d+1} \phi_i \in H^0(\Omega^{5d-1}_M)
\]

(1.9)

The \(\omega(\mathbb{L}, t)\) is the wedge product of all the row vectors of the Jacobian matrix \(J_L\). Thus the non-vanishing of it is equivalent to the non-degeneracy of \(J_L\) (of full rank).

**Proposition 1.5.** For a generic \(P\), there exists an open set \(L\) of the plane \(P\) consisting of \(S\)-generic quintics and satisfying the pencil condition such that the form \(\omega(\mathbb{L}, t)\) is nowhere zero on \(I_L\). Therefore \(J_L\) has full rank.

To attack \(\omega(\mathbb{L}, t)\), we appeal to differential algebra. We first expand the determinant (1.5), then apply the product rule (from calculus). This yields an expression for \(\phi_i\).

\[
\phi_i = \begin{vmatrix} f_1(c(t_1)) & f_0(c(t_1)) \\ f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} df_2(c(t_i)) + \begin{vmatrix} f_0(c(t_1)) & f_2(c(t_1)) \\ f_0(c(t_2)) & f_2(c(t_2)) \end{vmatrix} df_1(c(t_i)) + \begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{vmatrix} df_0(c(t_i)) + \sum_{l=0, j=1}^{l=\frac{2}{2}, j=2} h_{ij}^l(c) df_l(c(t_j)),
\]

(1.10)
where $h_{ij}^k(c)$ are polynomials in $c$. We let $c_g$ be a generic point of $\mathbb{L}$. We should notice that each $\phi_i$ at $c_g$ is a linear combination of seven exact 1-forms,

$$
\left. \frac{df_3(c(t_i))}{c_g}, \text{ and } \left. \frac{df_i(c(t_j))}{c_g} \right|
$$

where $l = 0, 1, 2$, $j = 1, 2$,

$$
f_3 = \delta_2 f_2 + \delta_1 f_1 + \delta_0 f_0.
$$

for

$$
\delta_2 = \begin{bmatrix}
  f_1(c_g(t_1)) & f_0(c_g(t_1)) \\
  f_1(c_g(t_2)) & f_0(c_g(t_2))
\end{bmatrix}, \delta_1 = \begin{bmatrix}
  f_0(c_g(t_1)) & f_2(c_g(t_1)) \\
  f_0(c_g(t_2)) & f_2(c_g(t_2))
\end{bmatrix}, \delta_0 = \begin{bmatrix}
  f_2(c_g(t_1)) & f_1(c_g(t_1)) \\
  f_2(c_g(t_2)) & f_1(c_g(t_2))
\end{bmatrix}.
$$

Among these seven 1-forms, six of them, $\left. \frac{df_l(c(t_j))}{c_g} \right|$ are shared by all $\phi_i$. Then in differential algebra, the non-vanishing of $\omega(\mathbb{L}, t)$ evaluated at a point $c_g$ is implied by the linear independence of $5d + 5 = 5d - 1 + 6$ differential 1-forms

$$
\left. \frac{df_3(c(t_k))}{c_g}, \left. \frac{df_l(c(t_j))}{c_g} \right|, \left. \frac{df_i(c(t_j))}{c_g} \right|
$$

in the cotangent space $T^*_c M$, for $k = 3, \cdots, 5d + 1$, $l = 0, 1, 2$, $j = 1, 2$. Notice $5d + 5$ is exactly the dimension of $M$. Thus we can have the final switch back to a square matrix (different from the Jacobian matrix $J_L$),

**Proposition 1.6.** The form $\omega(\mathbb{L}, t)$ is non-vanishing at $c_g$ if the square matrix

$$
\mathcal{A} = \left. \frac{\partial \left( f_3(c(t_3)), \cdots, f_3(c(t_{5d+1})), f_0(c(t_1)), \cdots, f_2(c(t_2)) \right)}{\partial C_M} \right|_{c_g}
$$

is non-degenerate, where the row vectors of $\mathcal{A}$ represent the differential 1-forms in (1.12). To emphasize the dependence, $\mathcal{A}$ is also expressed as

$$
\mathcal{A}(C_M, f_0, f_1, f_2, t).
$$
1.2.4 The hidden key to the proof

So far we only established the setting, but nothing is substantial. Next we should see that the key to the proof hides inside of the differential algebra of (1.10). To analyze it, we observe (1.10) carefully. First take a pause to note that we are permitted to fix the generic \( f_0 \) and manipulate \( f_1, f_2 \) freely to achieve a non-vanishing \( \omega(L, t) \). Among four terms in (1.10), the summation \( \sum(\cdot) \) and the two other terms with multiples of differentials \( df_0(c(t_i)) \) (1.14) may be accessible by a free choice of \( f_1, f_2 \). So the only inaccessible term is

\[
\left| \begin{array}{cc}
 f_2(c(t_1)) & f_1(c(t_1)) \\
 f_2(c(t_2)) & f_1(c(t_2)) \\
\end{array} \right| df_0(c(t_1))
\]

This is due to the genericity of \( f_0 \). But now we’ll let it vanish by choosing \( t_1, t_2 \) in

\[
\left| \begin{array}{cc}
 f_2(c(t_1)) & f_1(c(t_1)) \\
 f_2(c(t_2)) & f_1(c(t_2)) \\
\end{array} \right| .
\]

So to simplify the hidden algebra in \( \omega(L, t) \), we first fix a generic \( c_g \) in \( I_L \), then arrange

\[
\left| \begin{array}{cc}
 f_2(c_g(t_1)) & f_1(c_g(t_1)) \\
 f_2(c_g(t_2)) & f_1(c_g(t_2)) \\
\end{array} \right| = 0, i.e. \delta_0 = 0.
\]

In this way we reduce the \( \phi_i \) at \( c_g \) to

\[
\phi_i|_{c_g} = \left( df_3(c(t_i)) + \sum_{l=2, j=2}^{l=0, j=1} h_{ij}^l(c) df_l(c(t_j)) \right)|_{c_g}, \tag{1.17}
\]

where the troubled term \( f_3 \) is reduced to \( f_3 = \delta_1 f_1 + \delta_2 f_2 \) which has no constraints. The formula (1.17) showed the calculation of \( A \) boils down to the calculation of differentials \( df_3(c(t_i)) \) at \( c_g \), which can be simplified by specializations.

So key to the proof in the algebra (1.10) indicates that it is sufficient to freely specialize the first order deformation (\( f_1 \) and \( f_2 \)) of the generic quintic

\[\text{Proposition 1.3 exactly addressed this lone (without a multiple) differential } df_0(c(t_i)) \text{ of generic } f_0 \text{ for all } i. \text{ Actually in local deformation theory, the differential } df_0(c(t_i)) \text{ is the only hurdle for Clemens’ conjecture.}\]
This is contrary to that in [1] where the whole quintic 3-fold \( f_0 \) must be deformed to achieve the same deformation result.

This key idea, in terms of the algebra of \( A \), is simply the reduction of the matrix. It can be achieved by two manipulations: (a) make (1.15) vanish, (b) specialize \( f_1, f_2 \). So to unfold the key idea, we must realize the specializations, to which the most of paper is devoted. For this we choose a specific specialization \( f_1, f_2 \) (we'll not be surprised to see different kinds of specialization). Let

\[
f_0 = \text{generic}, f_1 = z_0 \cdots z_2 q, f_2 = z_0 \cdots z_5 \quad (1.18)
\]

where \( z_0, \cdots, z_4 \) are generic homogeneous coordinates of \( \mathbb{P}^4 \), and \( q \) is a quadratic, homogeneous polynomial in \( z_0, \cdots, z_4 \). Applying a particular type of coordinates, called “quasi-polar coordinates” \( C'_{\mathcal{M}} \) (section 3.1) associated to the special \( f_1, f_2 \), we can break the Jacobian matrix

\[
\mathcal{A}(C'_{\mathcal{M}}, f_0, f_1, f_2, t')|_{c_g} \xrightarrow{\text{row equiv.}} \begin{pmatrix} D & 0 \\ 0 & \text{Jac}(C'_{\mathcal{M}}, c_g) \end{pmatrix} \quad (1.19)
\]

where \( D \) is a non-degenerate diagonal matrix of size \((5d - 2) \times (5d - 2)\) and \( \text{Jac}(C'_{\mathcal{M}}, c_g) \) is a \( 7 \times 7 \) matrix. Next we study the remaining \( 7 \times 7 \) matrix, \( \text{Jac}(C'_{\mathcal{M}}, c_g) \). It can be reduced more by specializing \( t', c_g, \text{and } q \). Finally we piece them together as a matrix \( \mathcal{A} \), and deform \( C_M, f_0, f_1, f_2, t \)

to a generic position. Lastly \( c_g \) can be changed to all rational curves because the number of the rational curves on \( X_0 \) is finite.

The following arrows give a guide for multiple switches along the logic line (backwards)

\[
R_d(X_0) \Rightarrow I_{X_0} \Rightarrow I_L \xrightarrow{\text{from geometry}} J_L \Rightarrow \omega(\mathbb{L}, t) \Rightarrow \mathcal{A} \quad (1.20)
\]

In the past, the incidence schemes \( I_{X_0} \) and \( I_S \) (over the total space \( S \)) are well studied. But after \( I_L \) (in (1.20)), our ideas of the proof parted ways with the past work in this field.
1.3 Technical notations and assumptions

In this subsection, we collect all technical definitions and assumptions used in this paper. Some of them may already be defined before, but we’ll repeat them more precisely.

Notations:
(1) $S$ denotes the space all quintics, i.e. $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5)))$. Let $[f]$ denote the image of $f$ under the map $H^0(\mathcal{O}_{\mathbf{P}^4}(5)) \backslash \{0\} \rightarrow S$.

(2) Let $M$ be $$(H^0(\mathcal{O}_{\mathbf{P}^1}(d)))^\oplus 5 \cong \mathbb{C}^5d + 5$$ and $M_d$ be the subset that parametrizes all birational-to-its-image maps $\mathbf{P}^1 \rightarrow \mathbf{P}^4$ whose push-forward cycles have degree $d$.

(3) Throughout the paper, for $c \in M_d$ the same letter $c$ also denotes its projectivization in $\mathbf{P}(M_d)$, which is regarded a regular map $c : \mathbf{P}^1 \rightarrow \mathbf{P}^4$.

Let $c^*(\sigma)$ denote the pull-back section of section $\sigma$ of some bundle over $\mathbf{P}^4$. The bundles will not always be specified, but they are apparent in the context. If $f \in H^0(\mathcal{O}_{\mathbf{P}^1}(5))$, $c^*(f)$ is also denote by $f(c)$, the symbol from the composition in affine coordinates. If $Y$ is quasi-affine scheme, $\mathcal{O}(Y)$ denotes the ring of regular functions on $Y$.

(4) Let $\alpha \in T_c M$, and $g : M \rightarrow H^0(\mathcal{O}_{\mathbf{P}^1}(r))$. 

11
be a regular map. Then the image $g_*(\alpha)$ of $\alpha$ under the differential map at $c_0$ is denoted by the symbol of partial derivatives

$$\frac{\partial g(c_0(t))}{\partial \alpha} \in T_{g(c_0)}(H^0(\mathcal{O}_{\mathbb{P}^1}(r))) = H^0(\mathcal{O}_{\mathbb{P}^1}(r)).$$  \hspace{1cm} (1.21)

(use the identification $T_{g(c_0)}(H^0(\mathcal{O}_{\mathbb{P}^1}(r))) = H^0(\mathcal{O}_{\mathbb{P}^1}(r))$).

(5) If $Y$ is a scheme, $|Y|$ denotes the induced reduced scheme of $Y$.

(6) (a) If $f \in H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \setminus \{0\}$ is a quintic polynomial other than $f_0$, we denote the direction of the line through two points $[f], [f_0]$ in the projective space, $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(5)))$ by $\overrightarrow{f}$. So

$$\overrightarrow{f} \in T_{[f_0]} \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(5))).$$

(b) Note that the vector $\overrightarrow{f}$ is well-defined up-to a non-zero multiple. In case when $c_0$ can deform to all quintics to the first order, i.e. the map in the formula (2.1) below is surjective, this naturally gives a section $<\overrightarrow{f}>$ of the bundle $c_0^*(T_{\mathbb{P}^4})$ (may not be unique), to each deformation $\overrightarrow{f}$ of the quintic $f_0$. This is easily can be understood as the direction of the moving $c_0$ in the deformation $(\overrightarrow{f}, <\overrightarrow{f}>)$ of the pair $(c_0, f_0)$.

(7) Let $Pr : M_d \times S \to S$ be the projection map.

Let $\Gamma$ be the union of the open sets of all irreducible components of the incidence scheme

$$\{(c, [f]) \subset M_d \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(5))) : c^*(f) = 0\}$$

dominating $S$ such that fibres of the map on each component

$$\Gamma_i \to Pr(\Gamma_i)$$

$$\cap \hspace{1cm} \cap$$

$$\Gamma \hspace{1cm} Pr(\Gamma)$$

are equal dimensional. It was shown by Katz ([4]) that there is such a $\Gamma$ with fibre dimension 4.

(8) Let $f_0, f_1, f_2 \in H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \setminus \{0\}$ be non-collinear and $f_0$ be generic. Let

$$\mathbb{P} = \text{span}([f_0], [f_1], [f_2])$$
and let $\mathbb{L}$ be an open subset of $\mathbb{P}$, containing $[f_0]$ and satisfying the pencil condition

(9) Let $\Gamma$ be that in (7).

Let

$$\Gamma_L \subset \Gamma \cap (M \times \mathbb{L})$$

be an irreducible component, and

$$\Gamma_{f_0},$$

be an irreducible component of

$$P(\Gamma \cap (M \times \{[f_0]\}))$$

where $P$ is the projection $M \times S \rightarrow M$. Let $I_L = P(\Gamma_L)$ for a component $\Gamma_L$. In general, these notations are extended to any subset $B$ of $Pr(\Gamma) \subset S$, so there are $I_B$ and $\Gamma_B$ surjective to $B$. When $B$ consists of one point $[f_0]$, $I_B$ is also written as $I_{f_0}$ or $I_{\text{div}(f_0)}$. This notation then is consistent with the notation $I_X$ used in section 1.2. Note that $P : \Gamma_L \rightarrow I_L$ is an isomorphism due to the pencil condition.

Combing with (7), we obtain two projections restricted to $\Gamma$,

\[
\begin{array}{c}
\Gamma \\
\downarrow \text{Pr} \\
S \\
\downarrow \text{P} \\
M.
\end{array}
\]

The goal is to know the properties of $Pr$. But we obtain the information through attacking $P$.

(10) The term we use the most is “Jacobian matrix”. Let $\Delta^n \subset \mathbb{C}^n$ be an analytic open set with coordinates $x_1, \ldots, x_n$. Let $f_1, \ldots, f_m$ be holomorphic functions on $\Delta^n$. For any positive integers $m' \leq m$, $n' \leq n$ and a point $p \in \Delta^n$, we define

\[
\frac{\partial(f_1, f_2, \ldots, f_{m'})}{\partial(x_1, x_2, \ldots, x_{n'})}\bigg|_p := \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_{n'}} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_{n'}} \\
\vdots & \vdots & & \vdots \\
\frac{\partial f_{m'}}{\partial x_1} & \frac{\partial f_{m'}}{\partial x_2} & \cdots & \frac{\partial f_{m'}}{\partial x_{n'}}
\end{pmatrix}\bigg|_p.
\]
to be the Jacobian matrix of functions $f_1, \ldots, f_{m'}$ in $x_1, \ldots, x_{n'}$. So it is a Jacobian matrix with a particular $C^{\infty}$ map between Euclidean spaces. But in context we’ll skip the descriptions of Euclidean spaces and $C^{\infty}$ maps.

This definition coincides with the one used before.

The rest of paper is organized as follows.

In section 2, we give a proof of part (1), Proposition 1.4. It shows some equivalence of two different incidence schemes, $I_{X_0}$ and $I_L$. Thus we can shift the focus from $I_{X_0}$ to $I_L$. These are all in the first order. In section 3, we construct specializations to calculate differential form $\omega(L, t)$, which represents the Jacobian matrix of $I_L$. It leads the proof of Propositions 1.3, 1.4. Theorem 1.1 is just an invariant expression of them.

2 Equivalence of the incidence schemes, $I_{X_0}$ and $I_L$

2.1 First order deformations of the pair

Let’s start the problem in its first order. It’ll lead to some relevant equivalence of $I_{X_0}$ and $I_L$.

Lemma 2.1. If $(c_0, [f_0]) \in \Gamma$ is a generic point, then the projection

$$T_{(c_0, [f_0])}\Gamma \to T_{[f_0]} S$$

(2.1)

is surjective.

Proof. Let $|\Gamma| \subset \Gamma$ be the reduced scheme of the scheme $\Gamma$. In a neighborhood of a generic point $(c_0, [f_0]) \in |\Gamma|$, the projection is a smooth map. By the assumption, the projection

$$|\Gamma| \to S$$

(2.2)

is dominant. Thus

$$T_{(c_0, [f_0])}|\Gamma| \to T_{[f_0]} S$$

(2.3)

is surjective. This proves the lemma. □
To elaborate (6), section 1.3, we apply this lemma to obtain that for any \( \alpha \in T_{[f_0]}S \), there is a section denoted by

\[ < \alpha > \in H^0(c_0^*(T_{P^4})) \]

such that

\[ (\alpha, < \alpha >) \]

is tangent to the universal hypersurface

\[ X = \{(x, [f]) \in P^4 \times S : x \in \text{div}(f)\}. \]

Note that \(< \alpha >\) is unique up to a section in \( H^0(c_0^*(T_{X_0})) \). But we will always fix \(< \alpha >\) as in introduction.

### 2.2 The incidence schemes

There are two kinds of incidence schemes \( I_{X_0} \) and \( I_L \). In this subsection, we show certain equivalence between them. This is meant to shift our focus from \( I_{X_0} \) to \( I_L \). More specifically we’ll accomplish two goals:

(a) the dimension of the Zariski tangent space of the incidence scheme \( I_{X_0} \) will force \( c_0 \) to be an immersion.

(b) If \( L \) satisfies the pencil condition we can reduce the problem to that over the projected incidence scheme \( I_L \).

**Lemma 2.2.** Let \([f_0] \in S\) be a generic point, \( L_2 \subset S \) an open set of the pencil containing \( f_0 \) and another quintic \( f_2 \). Assume they determine the components \( I_{f_0}, I_{L_2} \) satisfying

\[ I_{f_0} \subset I_{L_2}, c_0^*(f_2) \neq 0 \text{ for generic } c_0 \in I_{f_0}. \quad (2.4) \]

Then

(a)

\[ \frac{T_{c_0}I_{f_0}}{\ker} \cong H^0(c_0^*(T_{X_0})). \quad (2.5) \]

where \( \ker \) is a line in \( T_{c_0}I_{f_0} \).
(b) \[ \dim(T_{(c_0, [f_0])} \Gamma L_2) = \dim(T_{c_0} I_{f_0}) + 1, \] (2.6)

and furthermore

\[ \dim(T_{c_0} I_{L_2})) = \dim(T_{c_0} I_{f_0}) + 1, \] (2.7)

(c) If \( \dim(T_{c_0} P(\Gamma L_2)) = 5 \), then

(1) \( c_0 \) is an immersion,

(2) and

\[ H^1(N_{c_0}/X_0) = 0. \] (2.8)

**Proof.** (a). Let \( a_i(c, f), i = 0, \cdots, 5d \) be the coefficients of polynomial \( f(c(t)) \) in parameter \( t \). Then the scheme

\[ \Gamma \]

in \( M \times \mathbb{P}^4 \) is defined by homogeneous polynomials

\[ a_i(c, f) = 0, i = 0, \cdots, 5d, \text{ locally around } (c_0, [f_0]). \]

Let \( \alpha \in T_{c_0} M \). The equations on \( \alpha \)

\[ \frac{\partial a_i(c_0, f_0)}{\partial \alpha} = 0, \text{ all } i \] (2.9)

by the definition, are necessary and sufficient conditions for \( \alpha \) to lie in

\[ T_{(c_0, [f_0])} I_{f_0}. \]

On the other hand there is an evaluation map \( e \):

\[ M \times \mathbb{P}^1 \to \mathbb{P}^4 \]

\[ (c, t) \to c(t) \] (2.10)

The differential map \( e_* \) gives a morphism \( e_*: \)

\[ T_{c_0} M \stackrel{e_*}{\to} H^0(c_0^*(\mathbb{P}^1)) \]

\[ \alpha \to e_*(\alpha) \] (2.11)

(Note \( c_0 \) is birational to its image. Thus \( c_0^*(\mathbb{P}^1) \) exists). Suppose there is an \( \alpha \) such that \( e_*(\alpha) = 0 \). We may assume \( c_0 \) is a map

\[ \mathbb{C}^1 \to \mathbb{C}^5 \setminus \{0\}. \]
Since $c_0$ is birational to its image, there is a Zariski open set

$$U_{P^1} \subset \mathbb{C}^1 \subset P^1$$

and an open set

$$V \subset c_0(P^1) \subset \mathbb{C}^5 \setminus \{0\}$$

such that $c_0|_{U_{P^1}}$ is an isomorphism

$$U_{P^1} \to V.$$  

Due to the equation $e_\ast(\alpha) = 0$, on $T_Uc_{P^1}$

$$(\alpha_0(t), \cdots, \alpha_4(t)) = \lambda(t)c_0(t)$$

on $V$ (at each point $(c_0(t), \cdots, c_4(t))$ of $V$) where $\lambda(t)$ lies in $O(U_{P^1})$. Because $(\alpha_0(t), \cdots, \alpha_4(t))$ is parallel to $c_0(t)$ at all points $t \in P^1$, $\lambda(t)$ can be extended to $P^1$. Hence $\lambda(t)$ is in $H^0(O_{P^1})$. So it is a constant (independent of $t$). Therefore $\alpha \in \mathbb{C}^{5d+5}$ is parallel to

$$c_0 \neq 0 \in \mathbb{C}^{5d+5}.$$  

This shows that

$$dim(\ker(e_m)) = 1.$$  

By the dimension count, $e_m$ must be surjective.

For any $\alpha \in H^0(c_0^\ast(T_{P^1}))$, $\alpha \in H^0(c_0^\ast(T_{X_0}))$ if and only if

$$\frac{\partial f_0(c_0(t))}{\partial \alpha} = 0,$$  

(2.12)

for generic $t \in P^1$. Notice equations (2.9) and (2.12) are exactly the same. Therefore $e_m$ induces an isomorphism

$$\frac{T_{c_0}^{\text{int}}}{\ker(e_m)} \xrightarrow{e_m} H^0(c_0^\ast(T_{X_0}))$$  

(2.13)

This proves part (a).

(b). Let

$$< f_2 >= M \in e_m^{-1}(< f_2 >)$$  

(2.14)

be an inverse of the section $< f_2 >$ in the map (2.11). Since

$$\frac{\partial f_0(c_0(t))}{\partial < f_2 >} = -c_0^\ast(f_2) \neq 0,$$  

(2.15)
by part (a), $< \vec{f}_2 >_M$ does not lie in $T_c I_{f_0}$. Hence

$$T_c P(\Gamma_{L_2}) = T_c I_{f_0} + \mathbb{C} < \vec{f}_2 >_M$$

(2.16)

has dimension $\dim(T_c I_{f_0}) + 1$, where $P$ is the projection

$$P : M \times S \to M$$

Notice

$$T_c P(\Gamma_{L_2}) \simeq T_{(c_0,f_0)} \Gamma_{L_2}.$$  

Hence

$$\dim(T_{(c_0,f_0)} \Gamma_{L_2}) = \dim(T_c I_{L_2}) = \dim(T_c I_{f_0}) + 1.$$  

(2.17)

(c) If $\dim(T_c P(\Gamma_{L_2}))=5$, then by part (a), (b),

$$\dim(H^0(c_0^*(T_{X_0}))) = 3.$$  

(2.18)

Now we consider it from a different point of view. Because $c_0$ is a birational map to its image, there are finitely many points $t_i \in \mathbb{P}^1$ where the differential map

$$(c_0)_* : T_{t_i} \mathbb{P}^1 \to T_{c_0(t_i)} \mathbb{P}^4$$

(2.19)

is a zero map. Assume its vanishing order at $t_i$ is $m_i$. Let

$$m = \sum_i m_i.$$  

(2.20)

Let $s(t) \in H^0(\mathcal{O}_{\mathbb{P}^1}(m))$ such that

$$\text{div}(s(t)) = \sum_i m_i t_i.$$  

The sheaf morphism $(c_0)_*$ is injective and induces a composed morphism $\xi_s$ of sheaves

$$T_{\mathbb{P}^1}(\rightarrow) c_0^*(T_{X_0}) \xrightarrow{\pi'} c_0^*(T_{X_0}) \otimes \mathcal{O}_{\mathbb{P}^1}(-m).$$  

(2.21)

It is easy to see that the induced bundle morphism $\xi_s$ is injective. Let

$$N_m = \frac{c_0^*(T_{X_0}) \otimes \mathcal{O}_{\mathbb{P}^1}(-m)}{\xi_s(T_{\mathbb{P}^1})}.$$  

(2.22)
Then
\[ \dim(H^0(N_m)) = \dim(H^0(c_0^*(T_{X_0}) \otimes O_{P^1}(-m))) - 3. \] (2.23)

On the other hand, three dimensional automorphism group of \( P^1 \) gives a rise to a 3-dimensional subspace \( Au \) of
\[ H^0(c_0^*(T_{X_0})). \]

By (2.18), \( Au = H^0(c_0^*(T_{X_0})) \). Over each point \( t \in P^1 \), \( Au \) spans a one dimensional subspace. Hence
\[ c_0^*(T_{X_0}) = O_{P^1}(2) \oplus O_{P^1}(-k_1) \oplus O_{P^1}(-k_2), \] (2.24)
where \( k_1, k_2 \) are some positive integers. This implies that
\[ \dim(H^0(c_0^*(T_{X_0}) \otimes O_{P^1}(-m))) = \dim(H^0(O_{P^1}(2 - m))). \] (2.25)

Then
\[ \dim(H^0(c_0^*(T_{X_0}) \otimes O_{P^1}(-m))) = 3 - m. \] (2.26)

Since \( \dim(H^0(N_m)) \geq 0 \), by the formula (2.23), \(-m \geq 0 \). By the definition of \( m \), \( m = 0 \). Hence \( c_0 \) is an immersion.

Next we prove (2). Notice that \((c_0)_*(T_{P^1})\) is a sub-bundle generated by global sections. It must be the \( O_{P^1}(2) \) summand in (2.24) because \( k_1, k_2 \) are positive. Therefore
\[ N_{c_0/X_0} \simeq O_{P^1}(-k_1) \oplus O_{P^1}(-k_2). \] (2.27)

Since \( \deg(c_0^*(T_{X_0})) = 0 \), \( k_1 = k_2 = 1 \).

Therefore
\[ H^1(N_{c_0/X_0}) = 0. \] (2.28)

Now we can describe the case for \( P \). Recall \( \mathbb{L} \) is an open set of \( P \) spanned by \( f_0, f_1, f_2 \), where \( f_0 \) is generic. Let
\[ \mathbb{L}_2 \subset \mathbb{L} \]
be a pencil containing the generic \([f_0] \).
Lemma 2.3. Let $[f_0], [f_1], [f_2]$ be non-collinear quintic hypersurfaces and $f_0$ is generic. Let $\mathbb{L}_2$ be an open set of the pencil span($[f_0], [f_2]$) as in Lemma 2.2. Also assume that an open set $\mathbb{L}$ of the span of $[f_0], [f_1], [f_2]$ satisfies the pencil condition. and $f_0 \in \mathbb{L}_2 \subset \mathbb{L}$ as before. We choose components

$I_{f_0} \subset I_{\mathbb{L}_2} \subset I_{\mathbb{L}}$,

and let $c_0 \in I_{f_0}$. Then

$$\dim(T_{c_0} I_{\mathbb{L}}) = \dim(T_{c_0} I_{\mathbb{L}_2}) + 1. \quad (2.29)$$

Proof. The formula (2.16) asserts

$$T_{c_0} I_{\mathbb{L}_2} \simeq T_{c_0} I_{f_0} \oplus \mathbb{C} \prec f_2 \succ_M. \quad (2.30)$$

Note

$$T_{c_0} I_{\mathbb{L}} \simeq T_{c_0} I_{f_0} \oplus \mathbb{C} \prec f_2 \succ_M + \mathbb{C} \prec f_1 \succ_M. \quad (2.31)$$

Thus it suffices to show the section $\prec f_1 \succ_M$ does not lie in

$$T_{c_0} I_{f_0} \oplus \mathbb{C} \prec f_2 \succ_M.$$

By the notation (6) in section 1.3, this is equivalent to

$$f_1(c_0(t)) + \epsilon_2 f_2(c_0(t)) \neq 0$$

for all complex numbers $\epsilon_2$. By the “pencil condition”, the lemma is proved. \qed

Lemmas 2.2, 2.3 imply

Corollary 2.4. The part (1), Proposition 1.4 is correct.

So in the rest paper, we concentrate on part (2) of Proposition 1.4.
3 Projected incidence scheme $I_L$

In this section, all neighborhoods and the word “local” are in the sense of Euclidean topology. The topic of this section is the computation of the differential from

$$\omega(I, t)$$

through multiple block Jacobian matrices of $A$ under specializations. This mainly corresponds to the “specialization” step in the key to the proof in section 1.2.4. Specifically there are only two methods in the calculations of matrices (they must be specialized first):

1. For a Jacobian matrix of large size, we construct special coordinates – quasi-polar coordinates, to reduces it to a diagonal matrix.
2. For a Jacobian matrix of smaller size, use the free choice of $f_1, f_2$ and other parameters to show the linear independence of row vectors.

We organize them in three subsections

3.1 Quasi-polar coordinates

We introduce local analytic coordinates of the affine space $M_d$, that will simplify expressions of differentials on $M$.

**Definition 3.1.** (polar coordinates) Let $a_0 \in H^0(\mathcal{O}_\mathbb{P}^1(d))$ be a non-zero element satisfying the zeros are distinct. Then there is a Euclidean neighborhood $U \subset H^0(\mathcal{O}_\mathbb{P}^1(d))$ of $a_0$, which has analytic coordinates $r, w_1, \ldots, w_d$ ($r \neq 0$) such that all elements $a \in U$ has an expression

$$a = r \prod_{j=1}^{d} (t - w_j).$$

(3.1)
We call \( \{r, w_1, \cdots, w_d\} \) the polar coordinates of \( H^0(\mathcal{O}_\mathbb{P}: (d)) \) at \( a_0 \).

Next we fix the notations of polar coordinates in the \( M_d \). Let non-zero 
\( c = (c^0, \cdots, c^4) \) with
\[
c^i \in H^0(\mathcal{O}_\mathbb{P}: (d)), i = 0, \cdots, 4
\]
be a varied point of \( M_d \) in a small analytic neighborhood. We assume the equations \( c^i(t) = 0, i \leq 4 \), always have \( 5d \) distinct zeros
\[
\theta^i_j, \text{ for } i \leq 4, j \leq d
\]
We have polar coordinates for \( M \) (around some points). We denote them by
\[
 r_i, \theta^i_j, \quad j = 1, \cdots, d, i = 0, \cdots, 4 \tag{3.2}
\]
with \( r_i \neq 0 \) satisfying
\[
c^i(t) = r_i \prod_{j=1}^{d} (t - \theta^i_j). \tag{3.3}
\]
The values of the center point \( c_g \) of the neighborhood are denoted by
\[
 \hat{r}_i, \hat{\theta}^i_j \quad \text{for } i = 0, \cdots, 4, j = 1, \cdots, d.
\]

Next we define quasi-polar coordinates that are associated to the special quintics we are going to choose later. They are the polar coordinates for \( M \) with a deformation for last two components \( c^3, c^4 \). Let \( q \) be a homogeneous quadratic polynomial in variables \( z_0, \cdots, z_4 \). Let
\[
h(c, t) = \delta_1 q(c(t)) + \delta_2 c^3(t)c^4(t). \tag{3.4}
\]
for \( c \in M \), where \( \delta_i, i = 1, 2 \) are two complex numbers, generic in \( \mathbb{C}^2 \). Assume for \( c \) in a small analytic neighborhood, \( h(c, t) = 0 \) has \( 2d \) distinct zeros. Let \( \gamma_1, \cdots, \gamma_{2d} \) be the zeros of \( h(c, t) = 0 \). Similar to the polar coordinates, we let
\[
h(c, t) = R \prod_{k=1}^{2d} (t - \gamma_k), R \neq 0
\]
It is clear that
\[ R = \delta_1 q(r_0, r_1, r_2, r_3, r_4) + \delta_2 r_3 r_4, \]
and \( \gamma_k \) are analytic functions of \( c \).

( Notice \( R \) is the value of \( h(c, t) \) at \( t = \infty \), the coefficient of the highest order.). Let the coordinates values at the center point be \( \hat{R}, \hat{\gamma}_k \).

**Proposition 3.2.** Let \((\delta_1, \delta_2)\) and \( q \) be generic. Let \( U_{c_g} \subset M \) be an analytic neighborhood of a center point \( c_g \) as above.

Let
\[ \varphi : U_{c_g} \rightarrow \mathbb{C}^{5d+5} \] (3.5)
be a regular map that is defined by
\[ (\theta_0^1, \cdots, \theta_d^1, r_0, r_1, r_2, r_3, r_4) \]
\[ \downarrow \varphi \]
\[ (\theta_0^1, \cdots, \theta_d^1, r_0, r_1, r_2, r_3, r_4, \gamma_1, \cdots, \gamma_{2d}). \] (3.6)

Then \( \varphi \) is an isomorphism to its image.

**Proof.** It suffices to prove the complex differential of \( \varphi \) at \( c_g \) is an isomorphism for a specific \( q, \delta_i \). So we assume that
\[ \delta_1 = 0, \delta_2 = 1. \]

Then \( h(c, t) = c^3(t)c^4(t) \). Hence \( \gamma_k, k = 1, \cdots, 2d \) are just
\[ \theta_i^j, i = 3, 4, j = 1, \cdots, d. \]

So \( \varphi \) is the identity map. We complete the proof. \( \square \)

**Definition 3.3.** By Proposition 3.2, for generic \((\delta_1, \delta_2), q\),
\[ \theta_0^1, \cdots, \theta_d^1, r_0, r_1, r_2, r_3, r_4, \gamma_1, \cdots, \gamma_{2d} \] (3.7)
are local analytic coordinates of \( M \) around \( c_g \), and \( c_g \) corresponds to the coordinate values with \( \circ \) accent. We denote the system of coordinates by
\[ C'_M \]
and will be called quasi-polar coordinates.
3.2 The largest block matrix

The specifically tailored quasi-polar coordinates above will automatically imply that the largest block in (1.19) is a diagonal matrix.

Let’s define this matrix. Choose a generic homogeneous coordinates \([z_0, \cdots, z_4]\) for \(\mathbb{P}^4\). Let

\[
f_3 = z_0z_1z_2(\delta_1 q + \delta_2 z_3 z_4).
\]

be a quintic polynomial, where \((\delta_1, \delta_2), q\) are generic. (Later we will choose \(f_1 = z_0 \cdots z_4, f_2 = z_0 z_1 z_2 q\) for the specialization). Let \(c_g \in M_d\) such that

\[
f_3(c_g(t)) \neq 0
\]

Recall we have denoted the zeros of \(c_g^j(t) = 0\) by \(\tilde{\theta}_i^j\) and zeros of

\[
(\delta_1 q + \delta_2 z_3 z_4)|_{c_g(t)} = 0
\]

for varied \(c\) by \(\gamma_k, k = 1, \cdots, 2d\). We assume \(\tilde{\theta}_i^j, i = 0, \cdots, 4, j = 1, \cdots, d\) are distinct, and \(\gamma_k, k = 1, \cdots, 2d\) are also distinct. For the simplicity we denote 5d complex numbers

\[
\tilde{\theta}_1^0, \cdots, \tilde{\theta}_2^d, \tilde{\gamma}_1^0, \cdots, \tilde{\gamma}_2^d
\]

by

\[
\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_{5d}.
\]

So \(\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_{5d}\) are zeros of \(f_3(c_g(t)) = 0\).

Lemma 3.4. Recall in Definition 3.3,

\[
\theta_0^1, \cdots, \theta_2^d, \gamma_1^0, \cdots, \gamma_2^d, r_0, \cdots, r_4
\]

are analytic coordinates of \(M\) around the point \(c_g\).

Then

(a) the Jacobian matrix

\[
J(c_g) = \frac{\partial(f_3(c_g(\tilde{t}_1)), \cdots, f_3(c_g(\tilde{t}_{5d}))}{\partial(\theta_1^0, \cdots, \theta_2^d, \gamma_1^0, \cdots, \gamma_2^d)}
\]
is equal to a diagonal matrix $D$ whose diagonal entries are

$$\frac{\partial f_3(c_g(\tilde{t}_1))}{\partial \theta_1^0}, \ldots, \frac{\partial f_3(c_g(\tilde{t}_{3d}))}{\partial \theta_d^2}, \frac{\partial f_3(c_g(\tilde{t}_{3d+1}))}{\partial \gamma_1}, \ldots, \frac{\partial f_3(c_g(\tilde{t}_{5d}))}{\partial \gamma_{2d}}$$ (3.11)

which are all non-zeros.

(b) For $i = 1, \ldots, 5d, l = 0, \ldots, 4$

$$\frac{\partial f_3(c_g(\tilde{t}_i))}{\partial r_l} = 0.$$

Proof. Note $\hat{\theta}_j, i = 0, \ldots, n, j = 1, \ldots, d$ are distinct and $\hat{\gamma}_k, k = 0, \ldots, 2d$ are also distinct. Thus the coordinates in Definition 3.3 exist. Applying $C_M'$ coordinates to $f_3(c(t))$, we have

$$f_3(c(t)) = r_0r_1r_2R \prod_{i=0,j=1,k=1}^{i=2,j=d,k=2d} (t - \theta_j^i)(t - \gamma_k).$$ (3.12)

Notice right hand side of (3.12) is in analytic coordinates $C_M'$, and $R$ is a polynomial in variables $r_1, \ldots, r_4$. Both parts of Lemma 3.4 follow from the expression (3.12). We complete the proof. \qed

### 3.3 Non vanishing of the differential form

At last we go back to the beginning to construct $\mathbb{L}$, then $\omega(\mathbb{L}, t)$. All ingredients in the hidden key will be carefully examined, and remaining terms in the specialization will be calculated one-by-one.

Proof. of Proposition 1.5: We compute the non-vanishing $\omega(\mathbb{L}, t)$ in two steps: (1) reduce it to the square matrix $\mathcal{A}$, (2) break the $\mathcal{A}$ and calculate two of its block matrices.

Step 1: We’ll use a generic $f_0$. It suffices to prove it for special choices of $f_1, f_2$ and distinct $t_1, \ldots, t_{5d+1}$. So let $z_0, \ldots, z_4$ be general homogeneous coordinates of $\mathbb{P}^4$. Let

$$f_2 = z_0z_1z_2z_3z_4.$$
Let
\[ f_1 = z_0 z_1 z_2 q, \]
where \( q \) is a generic quadratic homogeneous polynomial in \( z_0, \cdots, z_4 \). Such a choice of quintics satisfies the pencil condition in Proposition 1.4. The pencil condition for a neighborhood \( L \) of \([f_0]\) is equivalent to saying that any non-zero vector \( \vec{g} \) of the plane spanned by \( \vec{f}_1, \vec{f}_2 \) is sent to a section
\[ < \vec{g} > \not\in H^0(T_{C_0/X_0}). \tag{3.13} \]
Because the coordinates in \( f_1, f_2 \) are generic, the pencil condition, i.e. (3.13) is satisfied. Let \( c_g \in I_L \) be a generic point in \( I_L \) where \( L \) is an sufficiently small analytic open set of the plane spanned by \( f_0, f_1, f_2 \). By the genericity of \( f_0 \), we may assume
\[ c_g = (c_g^0, \cdots, c_g^4) \]
satisfies that \( c_g^i \neq 0 \) for all \( i \) and equations
\[ c_g^i(t) = 0, \; i = 0, \cdots, 4 \]
have \( 5d \) distinct zeros \( \theta_j^i \in \mathbb{P}^1 \). (it is quite important to notice that \( c_g \) does not lie in any individual \( f_0, f_1, f_2 \), but it does lie in a linear combination of them). To calculate the Jacobian matrix \( J_L \), we need to choose auxiliary data: defining equations of \( I_L \) and local coordinates of \( M \). To have defining equations, we choose \( 5d + 1 \) distinct points \( t_i \) on \( C \subset \mathbb{P}^1 \), denoted by \( t' = (t_1, \cdots, t_{5d+1}) \).

(1) \( t_{5d+1} \) is free and \( t_1, t_2 \) are general satisfying
\[ \begin{vmatrix} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{vmatrix} = 0, \tag{3.14} \]

(2) \( t_3, \cdots, t_{5d} \) are the \( 5d - 2 \) complex numbers
\[ \theta_j^i, \gamma_k, \quad (i, j) \neq (0, 1), (1, 1) \]
\[ 1 \leq k \leq 2d, 0 \leq i \leq 2, 1 \leq j \leq d. \]
satisfying that \( \gamma_k \) are the zeros of

\[
\delta_1 q(c_g(t)) + \delta_2 z_3 z_4|_{c_g(t)} = 0,
\]

(3.15)

with

\[
\delta_1 = \begin{vmatrix}
  f_0(c_g(t_1)) & f_2(c_g(t_1)) \\
  f_0(c_g(t_2)) & f_2(c_g(t_2))
\end{vmatrix},
\]

\[
\delta_2 = \begin{vmatrix}
  f_1(c_g(t_1)) & f_0(c_g(t_1)) \\
  f_1(c_g(t_2)) & f_0(c_g(t_2))
\end{vmatrix}.
\]

(3.16)

and \( \theta_j \) are all zeros of

\[
c_0^g(t)c_1^g(t)c_2^g(t) = 0,
\]

but excluding \( \theta_0^g, \theta_1^g \). Hence \( t_3, \cdots, t_{5d} \) are just all zeros of

\[
\delta_1 f_1(c_g(t)) + \delta_2 f_2(c_g(t)) = 0.
\]

(3.17)

but excluding two zeros \( \theta_0^g, \theta_1^g \). They are distinct because \( c_g \) is generic in \( I_L \).

We claim that

\[
(\delta_1, \delta_2)
\]

(3.18)

is generic in \( \mathbb{C}^2 \). Proof of the claim: The curve \( c_g \) lies in \( I_L \), but does not lie in

\[
P(\Gamma_{span(f_0, f_1)}), \text{ and } P(\Gamma_{span(f_0, f_2)}),
\]

where \( P \) is the projection \( M \times S \rightarrow M \). Hence vectors

\[
\{ (f_0(c_g(t)), f_1(c_g(t))) \}_{t \in \mathbb{P}^1}
\]

span \( \mathbb{C}^2 \). This implies

\[
\delta_2 = \begin{vmatrix}
  f_1(c_g(t_1)) & f_0(c_g(t_1)) \\
  f_1(c_g(t_2)) & f_0(c_g(t_2))
\end{vmatrix}
\]

is generic in \( \mathbb{C} \). Similarly \( \delta_1 \) is generic in \( \mathbb{C} \). By the generacity of the coordinates (that determine \( f_1, f_2 \)), \( (\delta_1, \delta_2) \in \mathbb{C}^2 \) is generic.

These \( 5d + 1 \) points give specific generators of the scheme \( I_L \):

\[
f(c(t_1)), \cdots, f(c(t_{5d+1})).
\]

27
Then we recall the formulation of differential algebra in the introduction as follows. As in the introduction, we obtain the $5d-2$ form $\omega(\mathbb{L}, t')$ (which is a collection of all maximal minors of the Jacobian matrix). As in (1.10), we expand 1-forms $\phi_i, i = 3, \cdots, 5d + 1$ to obtain that

$$
\phi_i = \begin{vmatrix} f_1(c(t_1)) & f_0(c(t_1)) \\ f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} df_2(c(t_i)) + \begin{vmatrix} f_0(c(t_1)) & f_2(c(t_1)) \\ f_0(c(t_2)) & f_2(c(t_2)) \end{vmatrix} df_1(c(t_i)) \\
+ \begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{vmatrix} df_0(c(t_i)) + \sum_{l=0, j=1}^{l=2, j=2} h_{ij}^i(c_g) df_l(c(t_j))
$$

(3.19)

By the assumption for $t_1, t_2$, 

$$
\begin{vmatrix} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{vmatrix} = 0.
$$

We obtain

$$
\phi_i|_{c_g} = \delta_1 df_1(c(t_i)) + \delta_2 df_2(c(t_i)) + \sum_{l=0, j=1}^{l=2, j=2} h_{ij}^i(c_g) df_l(c(t_j))
$$

(3.20)

$$
= df_3(c(t_i)) + \sum_{l=0, j=1}^{l=2, j=2} h_{ij}^i(c_g) df_l(c(t_j))
$$

where

$$
f_3 = \delta_1 f_1 + \delta_2 f_2.
$$

Notice $(\delta_1, \delta_2)$ is generic.

The expression says that non-vanishing of $\omega(\mathbb{L}, t')$ is the linear independence of the $5d + 5$ differential 1-forms,

$$
\begin{align*}
&df_3(c(t_1)), \cdots, df_3(c(t_{5d+1})) \\
&df_0(c(t_1)), df_1(c(t_1)), df_2(c(t_1)), \\
&df_0(c(t_2)), df_1(c(t_2)), df_2(c(t_2))
\end{align*}
$$

(3.21)

in the cotangent space $(T_{c_g}M)^*$, where 6 functions

$$
\begin{align*}
f_0(c(t_1)), f_1(c(t_1)), f_2(c(t_1)), \\
f_0(c(t_2)), f_1(c(t_2)), f_2(c(t_2))
\end{align*}
$$

come form the linear combinations

$$
\sum_{l=0, j=1}^{l=2, j=2} h_{ij}^i(c_g) df_l(c(t_j))
$$

28
in (3.20). This is clearly equivalent to the non-degeneracy of Jacobian matrix
\[
\mathcal{A}(C_M, f_0, f_1, f_2, t') = \frac{\partial \left( f_3(c(t_3)), \ldots, f_3(c(t_{5d+1})), f_0(c(t_1)), \ldots, f_2(c(t_2)) \right)}{\partial C_M},
\]
(3.22)
at \, c_g, where \, C_M \, is any analytic coordinates' chart of \, M \, around \, c_g.

Then it suffices to prove Proposition 1.6.

The \,(5d + 5) \times (5d + 5) \, matrix
\[
\mathcal{A}(C_M, f_0, f_1, f_2, t') \tag{3.23}
\]
is non-degenerate at generic point \, c_g \, of \, I_L.

Step 2:

Proof. of Proposition 1.6: With the above choices, \,(\delta_1, \delta_2), q \, are all generic. So we can choose the quasi-polar coordinates \, C'_M \, defined in Definition 3.3 to be the local coordinates around \, c_g \in I_L. We divide \, \mathcal{A} \, to block matrices in the following
\[
\begin{pmatrix}
\mathcal{A}_{11} & \mathcal{A}_{12} \\
\mathcal{A}_{21} & \mathcal{A}_{22}
\end{pmatrix}
\tag{3.24}
\]
where \, \mathcal{A}_{ij} \, are the Jacobian matrices:

(a)
\[
\mathcal{A}_{11} = \frac{\partial (f_3(c(t_3)), f_3(c(t_4)), \ldots, f_3(c(t_{5d})))}{\partial (\theta_2^0, \ldots, \theta_1^1, \ldots, \theta_2^2, \gamma_1, \ldots, \gamma_{2d})}, (\cdot := \text{omit}) \tag{3.25}
\]
(b)
\[
\mathcal{A}_{12} = \frac{\partial (f_3(c(t_3)), f_3(c(t_4)), \ldots, f_3(c(t_{5d})))}{\partial (\theta_1^0, \theta_1^1, r_0, r_1, r_2, r_3, R)} \tag{3.26}
\]
(c)
\[
\mathcal{A}_{21} = \frac{\partial (f_3(c(t_{5d+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2)))}{\partial (\theta_2^0, \ldots, \theta_1^1, \ldots, \theta_2^2, \gamma_1, \ldots, \gamma_{2d})} \tag{3.27}
\]
(d) \[
\mathcal{A}_{22} = \frac{\partial(f_3(c(t_{5d+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2)))}{\partial(\theta^0_1, \theta^1_1, r_0, r_1, r_2, r_3, R)}.
\]
(\(\mathcal{A}_{22}\) is a 7 \(\times\) 7 matrix.)

Applying the preparation in sections 3.1, 3.2, we can found that \(\mathcal{A}_{11}|_{c_g}\) is a non-zero diagonal matrix and

\[
\mathcal{A}_{12}|_{c_g} = 0.
\]
(Using Lemma 3.4). Therefore it suffices to show

\[
\text{det}(\mathcal{A}_{22}|_{c_g}) \neq 0.
\]

Notice \(t_{5d+1}\) is generic on \(\mathbb{P}^1\). The genericity of \(q\) makes curve in \(\mathbb{C}^7\),

\[
\left(\frac{\partial f_3(c(t))}{\partial \theta^0_1}, \frac{\partial f_3(c(t))}{\partial \theta^1_1}, \frac{\partial f_3(c(t))}{\partial r_0}, \ldots, \frac{\partial f_3(c(t))}{\partial r_4}\right)
\]

span the entire space \(\mathbb{C}^7\). This means the first row vector of

\[
\mathcal{A}_{22}|_{c_g}
\]
is linearly independent of other 6 row vectors. Hence it suffices for us to show the 6 \(\times\) 6 Jacobian matrix

\[
\mathcal{B}(c_g) = \frac{\partial(f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2)))}{\partial(\theta^0_1, \theta^1_1, r_1, r_2, r_3, r_4)}|_{c_g}
\]
is non degenerate (the column of partial derivatives with respect to \(r_0\) is eliminated). To attack \(\mathcal{B}(c_g)\), we continue to specialize. This time we change the point \(c_g\). To show \(\mathcal{B}(c_g)\) is non-degenerate, it suffices to show it is non-degenerate for a special \(c'_g \in I_L\). To do that, we let \(\mathbb{L}_2\) be an open set of pencil through \(f_0, f_2\). Then \(I_L\) must contain a component \(I_{L_2}\) where \(q\) is generic.

Let \(c'_g\) be a generic point of \(I_{L_2}\) (\(c'_g\) lies in a lower dimensional subvariety \(I_{L_2}\), but it is still in \(M_d\) because \(f_0\) is generic in \(S\)). Because \(q\) is generic with respect to 1st, 2nd, 5th and 6th rows and \(c'_g(t_1), c'_g(t_2)\) are distinct, two middle rows of the matrix \(\mathcal{B}(c_g)\),

\[
\left(\frac{\partial f_1(c(t_1))}{\partial \theta^0_1}, \frac{\partial f_1(c(t_1))}{\partial \theta^0_1}, \frac{\partial f_1(c(t_1))}{\partial r_1}, \ldots, \frac{\partial f_1(c(t_1))}{\partial r_4}\right)|_{c_g}
\]

\[
\left(\frac{\partial f_1(c(t_2))}{\partial \theta^0_1}, \frac{\partial f_1(c(t_2))}{\partial \theta^0_1}, \frac{\partial f_1(c(t_2))}{\partial r_1}, \ldots, \frac{\partial f_1(c(t_2))}{\partial r_4}\right)|_{c_g}
\]

(3.32)
in $\mathbb{C}^6$ must be linearly independent of 1st, 2nd, 5th and 6th rows (because $q$ can vary freely as $c'_g$ stays fixed). Then we reduce the non-degeneracy of $\mathcal{B}(c'_g)$ to the non-degeneracy of $4 \times 4$ matrix

$$Jac(f_0, c'_g) = \frac{\partial \big(f_2(c(t_1)), f_2(c(t_2)), f_0(c(t_1)), f_0(c(t_2))\big)}{\partial (\theta_1^0, r_2, r_3, r_4)} |_{c'_g}. \quad (3.33)$$

Finally we write down the matrix $Jac(f_0, c'_g)$,

$$Jac(f_0, c'_g) = \frac{1}{\lambda} \begin{pmatrix}
\frac{t_1 - \theta_1^0}{\partial f_0(c(t_1))} & 1 & 1 & 1 \\
\frac{t_2 - \theta_1^0}{\partial f_0(c(t_2))} & 1 & 1 & 1 \\
\frac{t_1 - \theta_1^0}{\partial f_0(c(t_1))} & 1 & 1 & 1 \\
\frac{t_2 - \theta_1^0}{\partial f_0(c(t_2))} & 1 & 1 & 1
\end{pmatrix}, \quad (3.34)$$

where $\lambda$ is a non-zero complex number. We further compute to have

$$Jac(f_0, c'_g) = \lambda \frac{1}{\lambda} \begin{pmatrix}
1 & 1 & 1 & 1 \\
\frac{2 \partial f_0(c'_g(t_1))}{\partial (\theta_1^0)} & \frac{2 \partial f_0(c'_g(t_2))}{\partial (\theta_1^0)} & \frac{2 \partial f_0(c'_g(t_1))}{\partial (\theta_1^0)} & \frac{2 \partial f_0(c'_g(t_2))}{\partial (\theta_1^0)}
\end{pmatrix}, \quad (3.35)$$

where $\theta_1^0$ is one of complex roots of $c'_g(t) = 0$. Since $t_1, t_2$ are only required to satisfy one equation (3.14), by the genericity of $q$, we may assume $(t_1, t_2) \in \mathbb{C}^2$ is generic. To show the non-degeneracy of $Jac(f_0, c'_g)$, we consider the Jacobian, i.e. the determinant of the Jacobian matrix,

$$J(f) = \begin{vmatrix}
\frac{\partial f_1}{\partial \theta_1^0} |_{c'_g(t_1)} & \frac{\partial f_5}{\partial \theta_1^0} |_{c'_g(t_1)} & \frac{1}{\lambda} \\
\frac{\partial f_1}{\partial \theta_1^0} |_{c'_g(t_2)} & \frac{\partial f_5}{\partial \theta_1^0} |_{c'_g(t_2)} & \frac{1}{\lambda} \\
\frac{\partial f_1}{\partial \theta_1^0} |_{c'_g(t_1)} & \frac{\partial f_5}{\partial \theta_1^0} |_{c'_g(t_1)} & \frac{1}{\lambda} \\
\frac{\partial f_1}{\partial \theta_1^0} |_{c'_g(t_2)} & \frac{\partial f_5}{\partial \theta_1^0} |_{c'_g(t_2)} & \frac{1}{\lambda}
\end{vmatrix}.$$

where the image $C'_g = c'_g(\mathbb{P}^1)$ lies in $\text{div}(f)$. We calculate

$$J(f) = \begin{vmatrix}
f_1 |_{c'_g(t_1)} & f_5 |_{c'_g(t_1)} & 1 \\
f_1 |_{c'_g(t_2)} & f_5 |_{c'_g(t_2)} & 1 \\
f_1 |_{c'_g(t_1)} & f_5 |_{c'_g(t_1)} & 1 \\
f_1 |_{c'_g(t_2)} & f_5 |_{c'_g(t_2)} & 1
\end{vmatrix}. \quad (3.36)$$
where
\[ f_4 = z_2 \partial f / \partial z_2 - z_4 \partial f / \partial z_4 \]
\[ f_5 = z_3 \partial f / \partial z_3 - z_4 \partial f / \partial z_4 \]
are two quintics. Hence \( J(f) = 0 \) implies
\[
\left. \begin{array}{c}
(f_4, f_5) \\
\end{array} \right|_{c'_1(t_1)}
\]
\[
\left. \begin{array}{c}
(f_4, f_5) \\
\end{array} \right|_{c'_2(t_2)}
\]
are linearly dependent. Since \((t_1, t_2) \in \mathbb{C}^2\) is generic, then there exist two complex numbers \( \epsilon_1, \epsilon_2 \) that are not all zero such that
\[
(c'_g)^*(\epsilon_1 f_4 + \epsilon_2 f_5) = 0.
\]
Hence
\[
\eta = (0, 0, \epsilon_1 c'_2, \epsilon_2 c'_3, -\epsilon_1 c'_4 - \epsilon_2 c'_4)
\]
in \( T_{c'_g} M \) gives a holomorphic section of the normal sheaf \( N_{\text{div}(f)}/c'_g \), where \( c'_i \) is the \( i \)-th component of \( c'_g \). Notice two vanishing components of \( \eta \) can be moved to a general position (because the coordinates of \( \mathbb{P}^4 \) is generic), while the normal sheaf \( N_{\text{div}(f)}/c'_g \) containing \( \eta \) is fixed (independent of coordinates of \( \mathbb{P}^4 \)). This contradiction implies \( J(f) \neq 0 \). Hence \( J(f_0) \neq 0 \). Then the determinant of \( \text{Jac}(f_0, c'_g) \) can’t be zero. We complete the proof of Proposition 1.6, therefore of Proposition 1.5.

As indicated in the introduction, Proposition 1.5 implies part (2) of Proposition 1.4. Then the section 2 showed this is sufficient for theorem 1.1.
References

[1] H. Clemens, Homological equivalence, modulo algebraic equivalence, is not finitely generated, Publ. Math IHES 58 (1983), pp 19-38

[2] ———, Curves on higher-dimensional complex projective manifolds, Proc. International Cong.Math., Berkeley 1986, pp. 634–640.

[3] D. Cox and S. Katz, Mirror symmetry and algebraic geometry, Math. survey and monographs, AMS (68)1999

[4] S. Katz, On the finiteness of rational curves on quintic threefolds, Comp.Math., (60)1986, pp. 151-162

[5] I. Vainsencher, Enumeration of n-fold tangent hypersurfaces to a surface, J. Algebraic geometry 4(1995), pp. 503-526.

Department of Mathematics, Rhode Island college, Providence, RI 02908
E-mail address: binwang64319@gmail.com