Approximating Node-Weighted $k$-MST on Planar Graphs

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Abstract. We study the problem of finding a minimum weight connected subgraph spanning at least $k$ vertices on planar, node-weighted graphs. We give a $(4 + \varepsilon)$-approximation algorithm for this problem. In the process, we use the recent LMP primal-dual 3-approximation for the node-weighted prize-collecting Steiner tree problem [4] and the Lagrangian relaxation [6]. In particular, we improve the procedure of picking additional vertices given by Sadeghian [17] by taking a constant number of recursive steps and utilizing the limited guessing procedure of Arora and Karakostas [1]. We argue that our approach can be interpreted as a generalization of a result by Kőnemann et al. [12]. Together with a result by Mestre [14] this implies that our bound is essentially best possible among algorithms that utilize an LMP algorithm for the Lagrangian relaxation as a black box.

1 Introduction

We consider the node-weighted variant of the fundamental, well-studied $k$-MST problem. Given a graph $G = (V, E)$ with non-negative node weights $w: V \to \mathbb{R}_+$ and a positive integer $k$, we consider a problem of finding a minimum cost connected subgraph of $G$ spanning $k$ vertices. In analogy to the edge-weighted case, we call this problem node-weighted $k$-MST (NW-$k$-MST) because the solution can be assumed to be a tree.

It was already observed that this problem is $\Omega(\log n)$-hard to approximate [10]. However, the problem becomes easier, when we restrict $G$ to be a planar graph. The focus of this work is to provide an approximation algorithm with small factor for this case.

1.1 Related work

Edge-weighted $k$-MST The standard, edge-weighted $k$-MST problem was thoroughly studied. In the sequence of papers [8, 19] the 2-approximation algorithm for prize-collecting Steiner tree problem [10] was used to finally obtain
the 2-approximation algorithm for k-MST. These results were to some extent explained by Chudak et. al [6] in terms of Lagrangian Relaxation. In particular they were able to get a 5-approximation algorithm in this framework, known mostly from Jain and Vazirani’s work on the k-median problem [11]. In these algorithms, the Lagrangian multiplier preserving (LMP) property plays a crucial role. The LMP property is also satisfied by the Goemans-Williamson algorithm for the prize-collecting Steiner tree problem (PC-ST).

**Node-weighted k-MST** The problem NW-k-MST was already studied in the more general quota setting, where each node has also associated some profit, and the goal is to find the minimum cost connected set of vertices having at least some total profit \( \Pi \). In particular, an \( O(\log n) \)-approximation was given in [10] for this problem. However, this result was based on their invalid \( O(\log n) \)-approximation for NW-PC-ST. Recently, Bateni et. al [2] proposed a correct algorithm for NW-PC-ST, but without LMP guarantee. The result on the Quota problem was finally restored by Könoann et al. [13] where an LMP algorithm was developed. In the related master thesis [17], Sadeghian gives also an alternative way of picking vertices \(^4\) in the reduction for the quota problem. In these results, the constant lost in the process was not optimized.

**Node-weighted planar Steiner problems** Recently, the planar variants of Steiner problems received increased attention. In particular, Demaine et. al [7] obtained a 6-approximation for the node-weighted Steiner forest problem. The factor was further improved to 3 by Moldenhauer [15]. Both results rely on the moat-growing algorithm similar to that of Goemans and Williamson. Currently the best result for this problem is the 2.4 approximation by Berman and Yaroslavtsev [3] where they use a different oracle for determining violated sets.

More general network design problems on planar graphs where also studied by Chekuri et. al [5]. Finally, the result of Moldenhauer was generalized to the prize-collecting variant by Byrka et. al [4], establishing an LMP 3-approximation for NW-PC-ST on planar graphs. We note that our result highly relies on this last algorithm.

**Partial cover** Below, we argue that our problem on arbitrary graphs generalizes the partial cover problem. In this problem we are given a set cover instance along with a positive integer \( k \). The objective is to cover at least \( k \) ground elements by a family of sets of minimum cost. In the prize-collecting version of the problem every element has a penalty and the objective is to minimize the sum of costs of the chosen sets and the penalties of the elements that are not covered. Könoann et al. [12] describe a unified framework for partial cover. They show how to obtain an approximation algorithm for a class \( \mathcal{I} \) of partial cover instances if there is an

\(^4\) by picking vertices we mean augmenting the smaller solution with some vertices of larger solution. This is an important ingredient for the Lagrangian Relaxation technique
\( r \)-approximate LMP algorithm for the corresponding prize-collecting version. In particular, their result implies a \( (\frac{4}{3} + \varepsilon) r \)-approximation algorithm for the class \( T \). Mestre \[14\] shows that no algorithm that uses an LMP algorithm as a black box can obtain a ratio better than \( \frac{4}{3} r \) so these results are essentially optimal.

1.2 Our result

We give a polynomial-time \( (4 + \varepsilon) \)-approximation algorithm for the NW-\( k \)-MST problem on planar graphs. Our results extend to an algorithm for the quota node-weighted Steiner tree problem on planar graphs with the same factor.

The main technique we use is the Lagrangian relaxation framework (as mentioned in related works above) where two solutions — one with less and the other with more than \( k \) nodes — are combined to obtain a feasible tree. In order to optimize the constant we employ additional techniques and ideas.

First of all, we use limited guessing similar to that of Arora and Karakostas \[1\] where they improve Garg’s 3 approximation for edge-weighted \( k \)-MST to \( 2 + \varepsilon \). This additional guessing allows them to pay \( \varepsilon \cdot \text{OPT} \) instead of \( \text{OPT} \) for connecting a single set of vertices to the rest of the solution. We provide a node-weighted variant of this idea and also use it more extensively, because we have to buy multiple (but still a constant number of) such connections.

Secondly, we extend the procedure of picking additional vertices of Sadeghian \[17\]. He finds some cost-effective subset of vertices, which is two times larger than needed. We show that by picking vertices in certain order and applying recursion a constant number of times, we are able to pick almost exactly the number of nodes that is needed. Although, the number of components of this set might be arbitrary, we need to buy only a constant number of connections to restore connectivity.

Finally, we show that the cheaper of this combined solution and the solution having more than \( k \) nodes is a \( 4 + \varepsilon \) approximation.

1.3 Organization of the paper

We start by describing a way of guessing a skeleton of the optimum solution in Section 2.

Next, we abstract from Lagrangian Relaxation and move straight to the description of picking additional vertices in Section 3 where we establish our result.

Then in Section 4 the framework is described as well as the details of using the LMP 3-approximation for NW-PC-ST so that the produced trees contain guessed skeleton.

2 Pruning the Instance

First, we assume that we know \( \text{OPT} \) up to a factor \( 1 + \varepsilon \) by using standard guessing techniques \[8\]. A node \( v \) is called \( \varepsilon \)-distant to a node set \( U \subseteq V \) if there
exists a path $P$ in $G$ from $v$ to a node $u \in U$ of node weight $c(V(P) \setminus \{u\}) \leq \varepsilon \cdot \text{OPT}$.

**Lemma 1.** Consider an optimum solution $T$ and an $\varepsilon > 0$. Then there exists a set $W \subseteq V(T)$ of size at most $1/\varepsilon$ such that each node in $T$ is $\varepsilon$-distant to $W \cup \{r\}$.

**Proof.** Consider $T$ as a tree rooted at $r$. For any node $u$ in this tree let $T_u$ denote the subtree hanging from $u$. A subtree $T_u$ is called *good* if for any node in $T_u$ the node weight (including the weight of the end nodes) of the unique path from this node to $u$ within $T_u$ is at most $\varepsilon \cdot \text{OPT}$.

We traverse $T$ in a bottom-up fashion starting with the leaves. We maintain the invariant (by removing subtrees) that for all nodes $u$ visited so far and still being in $T$, the subtree $T_u$ is good. To this end, when we encounter a node $u$ such that $T_u$ is good we just continue with the traversal. If $T_u$ is bad, however, then there must be a path $P$ within $T_u$ ending in $u$ of node weight $c(P) \geq \varepsilon \cdot \text{OPT}$. We include $u$ into $W$ and assign $P$ as a *witness* to $u$. Because of our invariant for all (if any) children $v$ of $u$ we have that $T_v$ is good. This means in particular that for all nodes $z$ in $T_u$ the node weight (excluding the weight of $u$) of the path from $z$ to $u$ is at most $\varepsilon \cdot \text{OPT}$. Finally, remove $T_u$ from $T$ and continue with the traversal. We stop when we reach the root $r$ at which point we remove the remaining tree (for the sake of analysis).

First, note that the set $W$ has cardinality at most $1/\varepsilon$ because we assigned to each node in $W$ a witness path of weight at least $\varepsilon \cdot \text{OPT}$ and because the witness paths are pairwise node-disjoint. Second, observe that whenever we removed a node $z$ from $T$ as part of a subtree $T_u$, the node weight (excluding the weight of $u$) of the path from $z$ to $u$ was at most $\varepsilon \cdot \text{OPT}$. Hence, for every node in $T$ there exists such a path to a node in $W \cup \{r\}$ at the end of the tree traversal since every node was removed.

In the sequel, we will call such a set $W$ whose existence is provided by the above lemma an $\varepsilon$-*skeleton*.

In a pre-preprocessing, we iterate over all $n^{O(1/\varepsilon)}$ many sets $W \subseteq V$ with $|W| \leq 1/\varepsilon$ and thereby guessing the $\varepsilon$-skeleton $W$ whose existence is guaranteed by the above lemma. Moreover, we prune all nodes $u$ from the instance that are not $\varepsilon$-distant to $W \cup \{r\}$.

### 3 The $(4 + \varepsilon)$-Approximation Algorithm

In Chapter 3 of [17] Sadeghian describe a $O(\log n)$ approximation for Node-Weighted Quota Steiner Tree Problem. His result is established using a framework of [6], repeated also in [16] where a primal-dual LMP approximation algorithm for the prize-collecting Steiner tree problem can be used along with the Lagrangian relaxation method to obtain an approximation algorithm for the quota version of the problem. Sadeghian loses some large constant factor in
We now show, that carefully injecting the LMP 3-approximation algorithm for NW-PC-ST on planar graphs given in [4] into his analysis yields a \((4 + \varepsilon)\)-approximation. However, in the process, we need a more efficient way to pick additional vertices. We show that it possible to pick a cheap set of these vertices. Although it will not be connected, it will have the constant number of components which we can connect cheaply via \(\varepsilon\)-skeleton.

The analysis relies on the framework which guarantees us the following lemma.

**Lemma 2.** We can produce trees \(T_1\) and \(T_2\) containing all the vertices \(W\) from the \(\varepsilon\)-skeleton and root \(r\) of sizes \(|T_1| \leq k \leq |T_2|\), and such that for \(\alpha_1 + \alpha_2 = 1\) with \(\alpha_1 |T_1| + \alpha_2 |T_2| = k\) we have that

\[
\alpha_1 c(T_1) + \alpha_2 c(T_2) \leq (3 + \varepsilon)OPT
\]

The construction of these trees \(T_1\) and \(T_2\) and the proof of above lemma is described in the Section 4.

Let now \(q = k - |T_1|\) be the number of vertices that are missing from the tree \(T_1\). We will now show, that these vertices can be picked from \(T_2 \setminus T_1\) without paying too much.

**Lemma 3.** It is possible to find a (not necessarily connected) set \(S\) of at least \(q\) vertices in \(T_2 \setminus T_1\) of cost at most

\[
(1 + \varepsilon)\alpha_2 c(T_2)
\]

which can be connected to \(T_1\) by buying additional \(\mathcal{O}(\log(1/\varepsilon^2))\) many \(\varepsilon\)-distant vertices (where \(\varepsilon_2\) is any constant).

**Proof.** Here, we substantially extend the analysis in [17]. Consider a graph \(T'_2\) constructed from \(T_2\) by identifying all vertices from \(T_1 \cap T_2\) to a single vertex \(r'\). Define the cost of this vertex \(r'\) to 0 (we will buy \(T_1\) anyway). From now on, whenever we count the cardinality of some subset \(S\) of vertices in \(T'_2\), we do not count vertex \(r'\).

**Definition 4.** The subset of vertices \(S\) is cost-effective if \(\frac{c(S)}{|S|} \leq \frac{c(T'_2)}{|T'_2|}\).

**Fact 5.** If cost-effective set \(S\) has size \((1 + \varepsilon_2)q\) then its cost is at most \((1 + \varepsilon_2)\alpha_2 c(T_2)\)

**Proof.**

\[
c(S) \leq |S| \frac{c(T'_2)}{|T'_2|} \leq (1 + \varepsilon_2)q \frac{c(T_2)}{|T'_2| - |T_1|} \leq (1 + \varepsilon_2)\alpha_2 c(T_2)
\]

where we used the fact that \(\alpha_2 = \frac{k - |T_1|}{|T_2| - |T_1|}\). \(\square\)
So now, our goal is to find a cost-effective set \( S \) in \( T'_2 \) of size only slightly larger than \( q \). First, we start with the procedure for picking at most \( 2q \) vertices as in [17]. Initialize graph \( H \) with any spanning tree of \( T'_2 \). Observe that \( H \) is cost-effective. Consider any edge \( e \) of \( H \). Let \( X \) and \( Y \) be the two components that would be created after removing the edge \( e \) from \( H \). At least one of these two components must be cost-effective. For any cost-effective component from this two, say \( X \), do the following. If \( X \) has enough vertices, i.e. \( |X| \geq q \), remove \( Y \) from \( H \) and continue. Otherwise, identify vertices of \( X \) to a single super vertex and set its cost to the sum of all vertices in \( X \). Also, define the super-cardinality of this new super vertex to \( |X| \).

It can be seen that after repeating this procedure as many times as possible, the graph \( H \) will be a star graph with super-cardinality at least \( q \). Let \( p \) be the number of leaves of \( H \). The case when \( p \leq 1 \) is easy and will be dealt separately. Assume now that \( p \geq 2 \). It is easy to see that the central vertex of the star graph \( H \), call it \( c \), is not a super vertex. Moreover, every leaf \( v \) must be cost-effective (otherwise either we would remove \( v \), or \( H \) would consist of two nodes). Observe also, that the super-cardinality of each leaf is at most \( q \). Hence adding leaves to \( S \) one by one, would eventually lead to the set \( S \) with super-cardinality at most \( 2q \) (and at least \( q \)). Finally, \( S \) could be connected to \( T'_1 \) by a single path from vertex \( c \).

We now modify this procedure of adding leaves. First, consider them in the order of decreasing super-cardinalities. To this end, let \( v_1, v_2, \ldots, v_p \) be leaves of \( H \) and \( s_1 \geq s_2 \geq \cdots \geq s_p \) be the corresponding super-cardinalities. Find the smallest \( i \) such that \( \sum_{j=1}^{i} s_j + s_{i+1} \geq q \). If \( s_{i+1} = 1 \), then the desired set \( S \) consist of all vertices in \( v_1, v_2, \ldots, v_{i+1} \) and it has exactly \( q \) vertices. Otherwise, add the first \( i \) leaves to the set \( S \). Let \( t = \sum_{j=1}^{i} s_j \) be the number of vertices added to \( S \). Now, instead of adding to \( S \) all vertices in the super vertex \( s_{i+1} \), we expand this super vertex back to the original graph and repeat above process with new number of vertices to pick equal to \( q' = q - t \). Observe that, because of sorting we have that \( t \geq \frac{1}{2}q \), which also implies that \( q' \leq \frac{1}{2}q \). This process is repeated recursively \( l \) times—where \( l \) is a parameter— but in the last call we take the last leaf completely.

Let now \( q_1, q_2, \ldots, q_l \) be the numbers of vertices to pick in respective recursive calls (note that \( q_1 = q \) and \( q_j \leq \frac{1}{2}q_{j-1} \)). The total number of picked vertices is then at most \( q + 2q' \leq (1 + 2^{l+2})q \). Therefore, to find the desired set \( S \) of at most \( (1+\varepsilon)2q \) vertices, we need only a constant number of recursive calls — parameter \( l \) is only \( O(\log(1/\varepsilon_2)) \). Moreover all the vertices of \( S \) can be connected to \( T'_1 \) by buying paths from the central nodes of all the \( l \) star graphs that appeared in the process. This finishes the proof. \( \square \)

To construct a feasible solution, take the set \( S \) guaranteed by above lemma and connect it to \( T'_1 \) by the \( O(\log(1/\varepsilon_2)) \) shortest paths to the \( \varepsilon \)-skeleton. Denote this solution by \( \text{SOL}_1 \). Let also \( \text{SOL}_2 \) be the entire tree \( T'_2 \).

**Lemma 6.** The cost of the cheaper of the two solutions \( \text{SOL}_1 \) and \( \text{SOL}_2 \) is at most \( (4 + \varepsilon)\text{OPT} \).
Proof. Let $\alpha = \alpha_2$ and $\beta = \frac{\epsilon(T_2)}{OPT}$. Using this notation we can bound the cost of these two solutions by

$$c(SOL_1) \leq c(T_1) + (1 + \epsilon_2)\alpha c(T_2) + \epsilon \cdot O(\log(1/\epsilon_2)) \cdot OPT$$

$$\leq (3(1 + \epsilon) + \alpha \beta) \cdot OPT + \epsilon \cdot O(\log(1/\epsilon_2)) \cdot OPT$$

$$c(SOL_2) = c(T_2) \leq \frac{3(1 + \epsilon) - (1 - \alpha)\beta}{\alpha} \cdot OPT$$

It can be now easily verified that the cheaper of the two solutions will not cost more than $(4 + \epsilon)OPT$. For the exact derivation of this claim, we refer the reader to the proof of Lemma 6 in the work on Partial Covering [12], with $r = 3$.

We note that arbitrary $\epsilon$ in the lemma statement, can be guaranteed by the appropriate choices for the number of repetitions $l$ and the epsilon used in the $\epsilon$-skeleton. \hfill \Box

4 Lagrangian Relaxation and Moat Growing on Planar Graphs

In this section we prove Lemma 2. The proof utilizes Lagrangian Relaxation and follows framework similar to [6].

We start with the LP relaxation for the NW-$k$-MST problem. However, we have to ensure that the feasible solutions contain all guessed vertices $W$ of the $\epsilon$-skeleton. This LP is as follows:

\begin{align*}
\min & \quad \sum_{v \in V \setminus \{r\}} x_v c_v \\
\text{s.t.} & \quad \sum_{v \in F(S)} x_v + \sum_{\substack{X : S \subseteq X \\cap W = \emptyset}} z_X \geq 1 \quad \forall S \subseteq V \setminus \{r\} \\
& \quad x_v + \sum_{\substack{X : v \in X \\cap W = \emptyset}} z_X \geq 1 \quad \forall v \in V \setminus \{r\} \\
& \quad \sum_{X \subseteq V \setminus \{r\}} |X| z_X \leq n - k \quad (1) \\
& \quad x_v \geq 0 \quad \forall v \in V \setminus \{r\} \\
& \quad z_X \geq 0 \quad \forall X \subseteq V \setminus \{r\}
\end{align*}

Now we move constraint (1) to the objective function obtaining the following Lagrangian Relaxation:
\[
\begin{align*}
\min & \quad \sum_{v \in V \setminus \{r\}} x_v c_v + \lambda \left( \sum_{X \subseteq V \setminus \{r\}} |X| z_X - (n - k) \right) \\
\text{s.t.} & \quad \sum_{v \in T(S)} x_v + \sum_{X : S \subseteq X, X \cap W = \emptyset} z_X \geq 1 \quad \forall S \subseteq V \setminus \{r\} \\
& \quad x_v + \sum_{X : v \in X, X \cap W = \emptyset} z_X \geq 1 \quad \forall v \in V \setminus \{r\} \\
& \quad x_v \geq 0 \quad \forall v \in V \setminus \{r\} \\
& \quad z_X \geq 0 \quad \forall X \subseteq V \setminus \{r\}
\end{align*}
\]

The above LP (ignoring the constant \(-\lambda(n - k)\) term in the objective function) is exactly the LP for the node-weighted prize-collecting Steiner tree (NW-PC-ST) in which the penalty of each vertex in \(V'\) is equal to the parameter \(\lambda\) with a slight modification that the subset of vertices \(W\) is required to be in the solution.

Let \(V' = V \setminus W\). Consider now, the dual of the LR(\(\lambda\)):

\[
\begin{align*}
\max & \quad \sum_{S \subseteq V \setminus \{r\}} y_S + \sum_{v \in V \setminus \{r\}} p_v - \lambda(n - k) \quad (DLR(\lambda)) \\
\text{s.t.} & \quad \sum_{S : v \in F(S)} y_S + p_v \leq c_v \quad \forall v \in V \setminus \{r\} \\
& \quad \sum_{X \subseteq S} y_X + \sum_{v \in S} p_v \leq \lambda |S| \quad \forall S \subseteq V' \setminus \{r\} \\
& \quad y_S \geq 0 \quad \forall S \subseteq V \setminus \{r\}
\end{align*}
\]

Now, the slightly modified primal-dual LMP 3 approximation for (NW-PC-ST) given in [4] can be used with penalties \(\lambda\) to produce the tree \(T^\lambda\) and the dual solution \((y^\lambda, p^\lambda)\) such that

\[
c(T^\lambda) + 3\lambda(n - |T^\lambda|) \leq 3 \left( \sum_{S \subseteq V \setminus \{r\}} y^\lambda_S + \sum_{v \in V \setminus \{r\}} p^\lambda_v \right) \quad (2)
\]

where \(T^\lambda\) contains all vertices of \(W\). The description of this algorithm is deferred to Subsection 4.1. Let us now see how we can use it to finish the proof of Lemma 2. We proceed essentially as in [17] and [8]. By subtracting \(3\lambda(n - k)\) from both sides of inequality (2) and simplifying the notation so that DS =
\[ \sum_{S \subseteq V \setminus \{r\}} y^\lambda_S + \sum_{v \in V \setminus \{r\}} p^\lambda_v \] denotes the value of a dual solution we have that
\[ c(T^\lambda) + 3\lambda(|T^\lambda| - k) \leq 3(DS_\lambda - \lambda(n - k)) \leq 3 \cdot DLR(\lambda) \leq 3 \cdot OPT \]

Observe that for \( \lambda = 0 \) the algorithm could output a tree with at least \( k \) vertices. In this case the resulting tree is a 3-approximation so we do not need the merging procedure described in Section 3. Otherwise, for some large \( \lambda \), e.g. the maximum cost of a vertex, the resulting tree would contain all the vertices. Therefore, we do the binary search for \( \lambda \) such that \( |T^\lambda| \) is close to \( k \). In a lucky event \( |T^\lambda| = k \) and then we don’t need the merging procedure described in Section 3. Otherwise, we would obtain \( \lambda_1 \) and \( \lambda_2 \) such that \( |T^{\lambda_1}| < k < |T^{\lambda_2}| \). By making enough steps of the binary search we can ensure that \( \lambda_2 - \lambda_1 \leq \frac{\varepsilon \cdot OPT}{3n} \). Let these trees be \( T_1 \) and \( T_2 \). Now, by setting \( \alpha_1 = \frac{|T_2| - k}{|T_2| - |T_1|} \) and \( \alpha_2 = \frac{k - |T_2|}{|T_2| - |T_1|} \) and using inequality (2) twice we have that
\[ \alpha_1 c(T_1) + \alpha_2 c(T_2) \leq 3 \left( (\alpha_1 DS_1 + \alpha_2 DS_2 - \alpha_1 \lambda_1(n - |T_1|)) - \alpha_2 \lambda_2(n - |T_2|) \right) \leq 3 \left( (\alpha_1 DS_1 + \alpha_2 DS_2 - \lambda_2(n - k) + (\lambda_2 - \lambda_1)(n - |T_1|)) \right) \leq 3 \left( OPT + (\lambda_2 - \lambda_1)n \right) \leq (3 + \varepsilon) \cdot OPT \]

where we used the fact that the convex combination of \( DS_1 \) and \( DS_2 \) is a feasible solution for \( DLR(\lambda_2) \).

### 4.1 Moat Growing

In this subsection we describe the slight technical modification needed in the primal-dual algorithm for NW-PC-ST problem on planar graphs given in [4]. We also give the short description of the resulting algorithm for completeness. Observe, that there are two differences in the LPs used.

First, we have additional constraints and corresponding dual variables \( p_v \). This is due to the fact, that in our setting all vertices can have both nonzero penalty and cost, while in the previous setting the reduction step was employed so that each vertex is a terminal with some penalty and zero cost or a Steiner vertex with zero penalty. However, this reduction step is equivalent to setting \( p_v \) to minimum of cost and penalty and defining the reduced costs and reduced penalties. This does not influence the approximation factor, nor the LMP guarantee. See also Section 2.1 of Sadeghian [17] for details.

Second modification comes from the fact that we have to include some guessed vertices \( W \) in the solution. However, it is enough to treat these vertices in the same way as terminals.

Now, we give the description of the algorithm. First, we do the mentioned reduction of eliminating \( p_v \) variables. This would make some vertices terminal and the other Steiner vertices. We also add all the guessed vertices to the set of terminals and set their penalty to infinite.
The algorithm maintains a set of moats, i.e. the family of disjoint sets of vertices. In each step, these moats can be viewed as the components of the graph induced by bought nodes. Each moat has an associated potential equal to the total penalty of vertices inside this moat minus the sum of the dual variables for all the subsets of this moat. The moat with the positive potential is active, with exception that the moat containing the root is always inactive.

The algorithm raises simultaneously all the active moats. For the growth of the moat we pay with its potential. We can have two events.

In the first event, some vertex goes tight, i.e. the inequality in the dual program becomes inequality for this vertex. In this case we buy this vertex and merge all the neighboring moats, setting the potential accordingly to the sum of all previous moats. We declare this new moat inactive whenever it contains a root vertex.

In the second event, some moat goes tight, i.e. the inequality in the dual program becomes tight for some set of vertices. This corresponds to the situation when the potential of this moat drops to zero. In this case we declare this moat inactive and we mark all the previously unmarked terminals inside it as marked with the current time. Observe that in the dual we do not have these inequalities for sets containing guessed vertices $W$. This means, that all the vertices of $W$ will be connected to the root vertex.

We repeat this process until we do not have any active moats. Then we start pruning phase. We consider all the bought vertices in the reverse order of buying. We delete a vertex $v$ if the removal of $v$ would not disconnect any unmarked terminal or any terminal marked with time greater than the time of buying the vertex $v$. We return the pruned set of bought vertices as the solution.

The result in [4] implies that the above algorithm run with initial penalty $\lambda$ for all vertices in $V'$ returns a tree $T^\lambda$ satisfying the inequality (2).

5 Conclusions

The $4 + \varepsilon$ approximation factor was obtained for the NW-$k$-MST problem on planar graphs. In the process we used the Lagrangian Relaxation technique. Our work can be interpreted as a generalization of a work on Partial Cover [12]. The result by Mestre [14] implies that our factor is essentially best possible using the underlying LMP algorithm for the NW-PC-ST as a black-box. It shows that one would have to exploit internals of the algorithm or planarity to beat our factor.

We raise the open question of improving the factor to a constant better than 4. We would expect the result matching the factor of 3 of the LMP algorithm. This would require a careful analysis of the inner-workings of the moat growing algorithm. This work was made for the edge-weighted variant by Garg [30]. However, the node-weighted variant seems to be more difficult. The changes in the solutions by increasing initial potentials of vertices can be much larger. In particular, one can observe situations of node-flips in which two potentially distant vertices exchange their presence in the solution. Also, in contrast to edge-weighted variant, a single node can be adjacent to any number of moats and not
only two. This in turn causes the large difference in two trees produced by the
algorithm. In particular, the OLD vertices as described in Garg [8] can form any
number of connected components which may be difficult to connect.

6 Acknowledgments

We would like to thank Zachary Friggstad for initial discussions on the problem.
The authors were supported by the NCN grant number 2015/18/E/ST6/00456.

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