Geometric phase and quantum phase transition in an inhomogeneous periodic XY spin-1/2 model

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The notion of geometric phase has been recently introduced to analyze the quantum phase transitions of many-body systems from the geometrical perspective. In this work, we study the geometric phase of the ground state for an inhomogeneous period-two anisotropic XY model in a transverse field. This model encompasses a group of familiar spin models as its special cases and shows a richer critical behavior. The exact solution is obtained by mapping on a fermionic system through the Jordan-Wigner transformation and constructing the relevant canonical transformation to realize the diagonalization of the Hamiltonian coupled in the k-space. The results show that there may exist more than one quantum phase transition point at some parameter regions and these transition points correspond to the divergence or extremum properties of the Berry curvature.

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I. INTRODUCTION

Since the existence of the adiabatic geometric phase that was first revealed in pioneer work of Berry [1], the concepts of geometric phases have been extensively generalized along many directions [2–4], and now the applications of geometric phases can be found in various physical fields [5–9]. Recently, the close relation between geometric phases and quantum phase transitions (QPTs) has been revealed gradually [10–12] and increasing interest has been drawn to the role of geometric phases in detecting QPTs for many-body systems [13]. Essentially, quantum phase transitions happen at zero temperature and cannot give a complete description of properties of the ground state in many-body systems. In the past few years, a lot of efforts have been devoted to the study of QPTs from other perspectives, such as quantum order or topological order [14], entanglement measures [15, 16], and quantum fidelity based on the concept of quantum information [17, 20]. Generally, in the vicinity of QPTs, the changes in the ground state driven by external parameters of the Hamiltonian will lead to an energy-level crossing or avoided energy-level crossing between the ground state and the excited state [21], and the features of level structures can be captured by the geometric phase of the ground state because the features of energy-level crossing or avoided crossing correspond to the divergence or extremum property of the Berry curvature. From the geometrical perspective, the geometric phase is a reflection of the global curvature in the parameter space of the Hamiltonian.

In the present work, we shall use the geometric phase of ground state to detect the QPTs for a period-two inhomogeneous anisotropic XY spin-1/2 chain in a transverse field, in which the nearest-neighbor interactions and the degree of anisotropy will take alternating parameters between the neighbor sites [22–27]. To the best of our knowledge, the previous studies of geometric phase as a witness of QPTs mainly concerned homogeneous spin chains for simplicity [10–12]. On the other hand, inhomogeneous systems will exhibit rich phase diagrams and it would be interesting to investigate whether the geometric phase is able to characterize the quantum phase transition in these more complicate systems. So far, many methods have been introduced to investigate the inhomogeneous spin chains in different limited conditions [22–28], however an explicit expression of the ground state, which is necessary for the derivation of the geometric phase, is still lacking. In our scheme, by mapping the spin Hamiltonian on a fermionic system through the Jordan-Wigner transformation and the Fourier transformation, we derive a general canonical transformation to realize the diagonalization of the fermionic system Hamiltonian coupled in the k-space and construct the exact expression of the ground state. Our results show that there exist more than one critical points at some parameter region and the critical points correspond to the divergence or extremum property of the Berry curvature of the Hamiltonian parameter space.

II. MODEL

The system under consideration is an inhomogeneous periodic anisotropic XY spin-1/2 chain [22–27], which consists of N cells with two sites in each cell, and in an external magnetic field. Its Hamiltonian is given by

\[ \mathcal{H} = \sum_{l=1}^{N} \left[ J_{a}(1 + \gamma_{a}) S_{l,a}^{x} S_{l+1,b}^{x} + J_{a}(1 - \gamma_{a}) S_{l,a}^{y} S_{l+1,b}^{y} \right. \\
+ \left. J_{b}(1 + \gamma_{b}) S_{l,b}^{x} S_{l+1,a}^{x} + J_{b}(1 - \gamma_{b}) S_{l,b}^{y} S_{l+1,a}^{y} \right] + h(S_{l,a}^{z} + S_{l,b}^{z}). \] (1)
where $S_{\alpha}^m(\alpha = x, y, z; \ m = a, b)$ are the local spin operators, $J_{\alpha}$ is the exchange coupling, $\gamma_m$ is the anisotropy in the in-plane interaction on the $m$ site in the $l$th cell and $h$ is the external field. We assume periodic boundary conditions and choose $N$ to be odd for convenience. This model encompasses a group of other well-known spin models as its special cases [23], such as quantum Ising model in a transverse field for $\gamma_a = \gamma_b = 1$ and $J_a = J_b$, the transverse field XX model for $\gamma_a = \gamma_b = 0$ and $J_a = J_b$, and the uniform transverse field anisotropic XY model for $\gamma_a = \gamma_b$ and $J_a = J_b$.

In order to obtain the geometric phase in this system, we consider rotating this model by applying a rotation of angle $\varphi$ about the $z$ axis to each spin [10], i.e., $D_z(\varphi) = \prod_{l=1}^{N} \exp(i\varphi(S_{l,a}^z + S_{l,b}^z))$, and then we have $\mathcal{H}(\varphi) = D_z(\varphi) \mathcal{H} D_z(\varphi^*)$, in which $D_z(\varphi)$ is the relevant rotation operator and we have set $h = 1$ for simplicity. It can be verified that $\mathcal{H}(0) = \mathcal{H}(\pi)$, and $\mathcal{H}(\varphi)$ is $\pi$ periodic in $\varphi$ because the quadratic form about the $x$ and $y$ axes appears symmetric in Eq. (1). Considering the unitarity of the rotation operator $U(\varphi)$, the critical behavior and energy spectrum of the family of Hamiltonians parametrized by $\varphi$ are obviously $\varphi$ independent. The spin Hamiltonian can be mapped exactly on a spinless fermion model through the Jordan-Wigner transformation

$$S_{l,a}^+ = \exp \left[ i\frac{\pi}{2} \sum_{l=1}^{N} \sum_{m' = a, b} C_{l,m'}^a C_{l,m'a}^\dagger \right] C_{l,a}^\dagger,$$

$$S_{l,b}^+ = \exp \left[ i\frac{\pi}{2} \sum_{l=1}^{N} \sum_{m' = a, b} C_{l,m'}^b C_{l,m'b}^\dagger + C_{l,a}^\dagger C_{l,a} \right] C_{l,b}^\dagger,$$

where $S_{l,a}^\pm = S_{l,m}^* \pm i S_{l,m}^z$ are the spin ladder operators and $c_{l,a}'$ and $c_{l,a}$ are the fermion creation and annihilation operators. The original Hamiltonian $\mathcal{H}(\varphi)$ is transformed into

$$\mathcal{H}(\varphi) = - \sum_{l=1}^{N} \left( \frac{J_a}{2} C_{l,a}^\dagger C_{l,a} + \frac{J_b}{2} C_{l,b}^\dagger C_{l,b} \right) + \left( \frac{J_a}{2} e^{-i\varphi} C_{l+1,a}^\dagger C_{l,a}^\dagger + H.C. \right) + \left( \frac{J_b}{2} e^{-i\varphi} C_{l+1,b}^\dagger C_{l,b}^\dagger + H.C. \right) + h(C_{l,a}^\dagger C_{l,a} + C_{l,b}^\dagger C_{l,b} - 1).$$

In the fermion case, the periodic boundary conditions $S_{N+1}^m(\alpha = x, y, z; \ m = a, b)$ on the spin degrees of freedom imply that $C_{N+1,m}^\dagger = \exp[i\varphi] \sum_{l=1}^{N} \sum_{m' = a, b} C_{l,m'}^a C_{l,m'}^\dagger C_{l,m}^\dagger$ in which $\sum_{l=1}^{N} \sum_{m' = a, b} C_{l,m'}^a C_{l,m'}^\dagger C_{l,m}^\dagger$ is just the total fermion number $N_F$. Thus the boundary conditions on the fermionic system are $C_{N+1,m}^a = e^{i\pi N_F} C_{1,m}^a$, and the fermionic system will obey periodic or antiperiodic conditions depending on whether $N_F$ is even or odd [30]. However, the differences between the two boundary conditions are negligible in the thermodynamic limit where the second-order QPTs occur [11, 29]. Without loss of generality, we assume the periodic boundary condition on the fermionic system, which means that $N_F$ is always even and $C_{N+1,m} = C_{1,m}$. This periodic boundary condition enables us to introduce a Fourier transformation,

$$c_{l,a} = \frac{1}{\sqrt{N}} \sum_k e^{ikR_{l,a}} a_k,$$

$$c_{l,b} = \frac{1}{\sqrt{N}} \sum_k e^{ik(R_{l,a}+\alpha)} b_k$$

(4)

to the Hamiltonian $H(\varphi)$, in which $k = (2\pi/2aN)n$ and $n = -\frac{N-1}{2}, -\frac{N-1}{2} + 1, \ldots, \frac{N-1}{2}$. Here $R_{l,a}(R_{l,b} = R_{l,a} + \alpha)$ is defined as the coordinate of site $a$ ($b$) on the $l$th cell in the one-dimensional lattice with the lattice parameter $2a$. Hence, the Hamiltonian $\mathcal{H}(\varphi)$ transformed into the momentum space is given by

$$H_{\varphi} = - \sum_k \left( h(a_k^\dagger a_k + b_k^\dagger b_k) - 1 \right) + \left( \frac{J_a}{2} e^{i\varphi} a_k^\dagger a_k + \frac{J_b}{2} e^{-i\varphi} a_{-k}^\dagger a_{-k} \right) + H.C. \right) + \left( \frac{J_a}{2} e^{i\varphi} a_k^\dagger a_k + \frac{J_b}{2} e^{-i\varphi} a_{-k}^\dagger a_{-k} \right) + H.C. \right) \right) \right).$$

(5)

This Hamiltonian has a quadratic form in fermion operators and can be exactly diagonalized. We note that the Hamiltonian $H_{\varphi}$ can be expressed as $H_{\varphi} = \sum_k (\Gamma_k^a M_k^a \Gamma_k + h)$ with matrix $\Gamma_k^a = (a_k^\dagger, a_{-k}, b_k^\dagger, b_{-k})$ and $M_k^a$ is a $4 \times 4$ Hermitian matrix. Therefore, we can always find a unitary transformation matrix $U$ which can be inserted in the Hamiltonian as $H_{\varphi} = \sum_k (\Gamma_k^a U_k^a M_k^a U_k^a \Gamma_k + h)$ and then transform the matrix $M_k^a$ into a diagonal matrix $U_k M_k U_k^a$. That is to say, the term $U_k \Gamma_k^a$ is equivalent to introducing the following canonical transformation and define a set of new operators, i.e.,

$$\gamma_k = \frac{1}{\sqrt{2}} (e^{2i\varphi} \cos \frac{\gamma_k}{2} a_k + e^{i\delta_k} e^{-i\varphi} \sin \frac{\gamma_k}{2} a_k^\dagger),$$

$$\eta_k = \frac{1}{\sqrt{2}} (e^{2i\varphi} \sin \frac{\gamma_k}{2} a_k + e^{i\delta_k} e^{-i\varphi} \cos \frac{\gamma_k}{2} a_k^\dagger),$$

$$\mu_k = \frac{1}{\sqrt{2}} (e^{2i\varphi} \cos \frac{\delta_k}{2} a_k - e^{i\varphi} e^{-i\varphi} \sin \frac{\delta_k}{2} a_k^\dagger),$$

$$\nu_k = \frac{1}{\sqrt{2}} (e^{2i\varphi} \sin \frac{\delta_k}{2} a_k + e^{i\delta_k} e^{-i\varphi} \cos \frac{\delta_k}{2} a_k^\dagger).$$

(6)
where
\[ \delta_k = \arg (J_a e^{ika} + J_b e^{-ika}) , \]
\[ \sigma_k = \arg (J_a \gamma_a e^{ika} - J_b \gamma_b e^{-ika}) , \]
\[ \zeta_k = \sqrt{J_a^2 + J_b^2 + 2 J_a J_b \cos 2ka} , \]
\[ \xi_k = \sqrt{J_a^2 \gamma_a^2 + J_b^2 \gamma_b^2 - 2 J_a J_b \gamma_a \gamma_b \cos 2ka} , \]
and
\[ \cos \theta_k = \frac{h - \xi_k}{\sqrt{(h - \xi_k)^2 + (\xi_k)^2}} , \]
\[ \cos \beta_k = \frac{h + \xi_k}{\sqrt{(h + \xi_k)^2 + (\xi_k)^2}} . \]

Using the set of quasiparticle operators \( \gamma_k, \eta_k, \mu_k \) and \( \nu_k \), we can write the Hamiltonian \( H_\phi \) in the explicit diagonal form as
\[ H_\phi = \sum_{q=\gamma,\eta,\mu,\nu} \sum_k \Lambda_{q,k} \left( \frac{q_k^* q_k - \frac{1}{2}}{2} \right) , \]
where \( \Lambda_{q,k} (q = \gamma, \eta, \mu, \nu) \) are the eigenvalues of the Hamiltonian matrix \( M_k \). Now, they are the relevant quasiparticle energy spectra as follows:

\[ \Lambda_{\gamma,k} = -\frac{1}{2} \left( h - \frac{\zeta_k}{2} \right) - \frac{1}{2} \sqrt{(h - \frac{\zeta_k}{2})^2 + (\frac{\xi_k}{2})^2} , \]
\[ \Lambda_{\eta,k} = -\frac{1}{2} \left( h + \frac{\zeta_k}{2} \right) + \frac{1}{2} \sqrt{(h + \frac{\zeta_k}{2})^2 + (\frac{\xi_k}{2})^2} , \]
\[ \Lambda_{\mu,k} = -\frac{1}{2} \left( h + \frac{\zeta_k}{2} \right) - \frac{1}{2} \sqrt{(h + \frac{\zeta_k}{2})^2 + (\frac{\xi_k}{2})^2} , \]
\[ \Lambda_{\nu,k} = -\frac{1}{2} \left( h - \frac{\zeta_k}{2} \right) + \frac{1}{2} \sqrt{(h - \frac{\zeta_k}{2})^2 + (\frac{\xi_k}{2})^2} . \]

Furthermore, it can be verified that the general canonical transformation Eq. (13) can be reduced to the familiar Bogoliubov transformation in the case of the uniform anisotropic XY model.

### III. GEOMETRIC PHASE AND QUANTUM PHASE TRANSITION

Now, let us focus on the geometric phase of the ground state. We have introduced the family of Hamiltonians parameterized by \( \varphi \), and this family of Hamiltonians \( H(\varphi) \) can be described as a result of adiabatic rotation of the physical system. The geometric phase of the ground state will be accumulated when the system finish a cyclic evolution, corresponding to varying the angle \( \varphi \) from 0 to \( \pi \) \([\mathcal{H}(\varphi) \) is \( \pi \) periodic in \( \varphi \).

The Hamiltonian \( H_\phi \) in Eq. (9) has been diagonalized in the set of quasiparticle number operators, which allows us to determine all the eigenvalues and eigenvectors. We note that the energy spectra \( \Lambda_{\eta,k} \geq 0, \Lambda_{\nu,k} \geq 0 \) and \( \Lambda_{\gamma,k} \leq 0, \Lambda_{\mu,k} \leq 0 \). The ground state, denoted as \( |g(\varphi)\rangle \), corresponds to the state with the lowest energy, which consists of state with no \( \eta \) and \( \nu \) fermions occupied but with \( \gamma \) and \( \mu \) fermions occupied. Explicitly, the ground state can be constructed as follows

\[ |g(\varphi)\rangle = C^{-\frac{1}{2}} \prod_{k>0} \left( \gamma_{-k}^\dagger \mu_{-k}^\dagger \mu_{-k} \eta_{-k} \eta_{-k} \nu_{-k} \nu_{-k} \right) |0\rangle_a \otimes |0\rangle_b , \]

where \( C^{-1/2} \) is the normalized factor and \( |0\rangle_a \) and \( |0\rangle_b \) are the vacuum states of the sublattices \( a \) and \( b \), respectively. It is easy to check that \( \eta_k |g(\varphi)\rangle = 0, \nu_k |g(\varphi)\rangle = 0 \) and \( \gamma_{-k}^\dagger |g(\varphi)\rangle = 0, \mu_{-k}^\dagger |g(\varphi)\rangle = 0 \) for all \( k \). The corresponding ground-state energy \( E_g \) is

\[ E_g = \sum_k (\Lambda_{\gamma,k} + \Lambda_{\mu,k} + h) . \]

The geometric phase of the ground state, denoted \( \mathcal{B}_g \), is given by

\[ \mathcal{B}_g = \int_0^\pi \left\langle g(\varphi) \right| i \frac{\partial}{\partial \varphi} \left| g(\varphi) \right\rangle d\varphi . \]

Substituting Eq. (11) into Eq. (13), we have

\[ \mathcal{B}_g = \frac{1}{2C} \int_0^\pi \sum_{k>0} \left( \nu_k^\dagger \nu_{-k}^\dagger \eta_k \eta_{-k} \mu_{-k}^\dagger \mu_k \gamma_{-k} \right) |0\rangle_a \otimes |0\rangle_b d\varphi . \]

The factor of \( \frac{1}{2} \) before the normalized factor \( C^{-1} \) is due to the repeated calculations about the \( k \) and \(-k\) operators. Straightforward calculation is tedious. Nevertheless the result can be derived concisely from the following consideration. We note that for each term of \( \gamma_k \frac{\partial}{\partial \varphi} \gamma_k^\dagger \) and \( \gamma_{-k} \frac{\partial}{\partial \varphi} \gamma_{-k}^\dagger \) in the integrand yield the same results of \( -2i \cos^2 \frac{\theta_k}{2} \). In the same way, the terms of \( \mu_k \frac{\partial}{\partial \varphi} \mu_k^\dagger \) and \( \mu_{-k} \frac{\partial}{\partial \varphi} \mu_{-k}^\dagger \) yield the results of \( -2i \cos^2 \frac{\beta_k}{2} \), the terms of \( \eta_k \frac{\partial}{\partial \varphi} \eta_{-k}^\dagger \) and \( \eta_{-k} \frac{\partial}{\partial \varphi} \eta_k^\dagger \) yield the results of \( -2i \cos^2 \frac{\beta_k}{2} \), and the terms of \( \nu_k \frac{\partial}{\partial \varphi} \nu_{-k}^\dagger \) and \( \nu_{-k} \frac{\partial}{\partial \varphi} \nu_k^\dagger \) yield the results of \( -2i \cos^2 \frac{\beta_k}{2} \). Finally, the overall result is

\[ \mathcal{B}_g = \frac{i}{2} \int_0^\pi \sum_{k>0} 8(-i \cos^2 \frac{\theta_k}{2} - i \cos^2 \frac{\beta_k}{2}) d\varphi \]
\[ = 2\pi \left[ (N - 1) + \sum_{k>0} (\cos \theta_k + \cos \beta_k) \right] \]
\[ = 2\pi \sum_{k>0} (\cos \theta_k + \cos \beta_k) . \]

To study the quantum criticality, we are interested in the properties under the thermodynamic limit when the
parameter regions for the inhomogeneous periodic systems. An interesting thing is that there may exist two quantum phase transition points in some cases of inhomogeneous periodic systems. We can coexist in some parameter regions. In order to further understand the QPTs of this inhomogeneous periodic system, we present numerical results for the derivative of its geometric phase density as a function of different parameters (\(J_a, J_b, \gamma_a, \gamma_b\)) in the Hamiltonian.

In Fig. 1, we plot it as a function of \(\alpha = J_b/J_a\) and \(h\) with fixed parameters \(J_a = 1\) and \(\gamma_a = \gamma_b = 1\), which describes an inhomogeneous periodic Ising model in a transverse field \(h\). As shown in Fig. 1, the peak of curves for \(\partial_h \beta_g(\alpha, h)\) becomes sharp with the increasing of the lattice size \(N\). A notable feature is that the divergence of the curve in the thermodynamic limit only exists in the parameter region of \(J_b/J_a = 1\) and \(\gamma_a = \gamma_b = 1\), which correspond to the case of the uniform quantum Ising model, while in the other parameters regions, the curves only show extremum points.

In Fig. 2, we illustrate the derivative of the geometric phase of the ground state in various cases of inhomogeneous periodic systems. An interesting thing is that there may exist two quantum phase transition points in some parameter regions for the inhomogeneous period-two systems. The number of transition points and the corresponding divergence or extremum properties of curves are dependent on the parameters of the Hamiltonian, which is quite different from those of the quantum Ising model and anisotropic XY model in a transverse field. As shown in Figs. 2(a) and 2(c), the derivatives of the geometric phase only display the extremum instead of the divergence properties even under the thermodynamic limit condition. On the other hand, in Figs. 2(b) and 2(d), the extremum and divergence properties can coexist in some parameter regions. In order to further understand the divergence or extremum property of \(\partial_h \beta_g(\alpha, h)\), we choose a section of Fig. 2(c) plotted in Fig. 3, in which the Hamiltonian parameter takes \(J_a = 1\), \(J_b = 2\), \(\gamma_a = 0.05\) and \(\gamma_b = 0.08\). In this case, the transition point of QPT is characterized by the extremum point.

As shown in Fig. 3, there is no real divergence even in the thermodynamic limit but it tends to two extremum points with the increasing of the lattice size \(N\). The transition points in the thermodynamic limit can also be obtained by the finite-size analysis of positions of extremum points for different size systems. Our results show that the position of the first extremum point approaches the first QPT point \(h_c1\) in a way of \(h_m = h_c1(1 - \text{const}N^{-1.004})\) with the transition point \(h_c1 = 0.559908\) and the second one approaches as \(h_m = h_c2(1 - \text{const}N^{-1.017})\) with the transition point \(h_c2 = 1.47561\).

FIG. 1: (color online) (a) The derivatives of the geometric phase \(d\beta_g/dh\) for an inhomogeneous periodic Ising model in a transverse field \(h\), as a function of the Hamiltonian parameters \(\alpha = J_b/J_a\) and \(h\), in which \((J_a = 1, \gamma_a = \gamma_b = 1)\). The curves correspond to different lattice sizes \(N = 51\); (b) \(N = 101\); (c) \(N = 501\); (d) \(N \to \infty\).

In this case, we introduce the notation of the geometric phase density as \(\beta_g = \lim_{N \to \infty} \beta_g/N\), thus, we have

\[
\beta_g = \lim_{N \to \infty} \frac{2\pi}{N} \sum_{k>0} (\cos \theta_k + \cos \beta_k) = \int_0^\pi (\cos \theta_k + \cos \beta_k) dk .
\] (16)

In this case, the summation \(\frac{2\pi}{N} \sum_{k>0}\) has been replaced by the integral \(\int_0^\pi dk\) with \(\lim_{N \to \infty} \frac{2\pi}{N} \sum_{k>0}\). To better understand the QPTs of this inhomogeneous periodic model and how the geometric phase of the ground state is used as a witness to detect them, we present numerical results for the derivative of its geometric phase \(\partial_h \beta_g\) as a function of different parameters \((J_a, J_b, \gamma_a, \gamma_b)\) in the Hamiltonian.

FIG. 2: (color online) (a) The derivatives of the geometric phase \(d\beta_g/dh\) as a function of \(\gamma = \gamma_a = \gamma_b\) and \(h\) with the fixed parameters \(J_a = 1\), \(J_b = 0.5\) and lattice sizes \(N \to \infty\); (b) \(d\beta_g/dh\) as a function of \(J_b\) and \(h\) with the fixed parameters \(J_a = 1\), \(\gamma_a = 0.2\), \(\gamma_b = 0.4\); (c) \(d\beta_g/dh\) as a function of \(\gamma_a\) and \(h\) with the fixed parameters \(J_a = 1\), \(J_b = 2\), \(\gamma_a = 0.05\); (d) \(d\beta_g/dh\) as a function of \(J_b\) and \(h\) with the fixed parameters \(J_a = 1\), \(\gamma_a = 0.2\), \(\gamma_b = 0.1\).
FIG. 3: (color online) (a) The derivatives $d\beta_g/ dh$ for the inhomogeneous periodic $XY$ model ($J_x = 1, J_y = 2, \gamma_a = 0.05, \gamma_b = 0.08$) as a function of the transverse field $h$. The curves correspond to different lattice sizes $N = 51, 101, 501, \infty$. (b) and (c) The positions of the first extremum point changes and tends as $N^{-1.004}$ towards the first QPT point $h = 0.559908$. The positions of the second extremum changes and tends as $N^{-1.037}$ towards the second QPT point $h = 1.47561$.

IV. DISCUSSIONS AND CONCLUSIONS

As shown above, the geometry phase can be used as a detector for the more complicated QPTs in the inhomogeneous system. This is because there exists an intrinsic connection between the geometric phase and the energy level structure. Furthermore, similar connection is also reflected in the fidelity. The relation between the fidelity and Berry phase has been unveiled in terms of geometric tensors. The intrinsic relationship between the fidelity and the characterization of QPTs has also been studied in Ref. 19. For a general Hamiltonian of the quantum many-body system undergoing QPTs given by

$$H(\lambda) = H_0 + \lambda H_1 ,$$

where $H_1$ is supposed to be the driving term with $\lambda$ as the control parameter, the second derivative of the ground-state energy can be expressed as

$$\frac{\partial^2}{\partial \lambda^2} E_g(\lambda) = \sum_{n \neq g} \frac{2 |\langle n(\lambda) | H_1 | g(\lambda) \rangle|^2}{E_g(\lambda) - E_n(\lambda)} .$$

Here $E_g$ is the ground-state energy, $n(\lambda)$ are the eigenstates of $H(\lambda)$, and $g(\lambda)$ is the ground state. On the other hand, the geometric phase of the system can be obtained by introducing another parameter $R$ to the Hamiltonian Eq. (17), i.e.,

$$H(R, \lambda) = H_0(R) + \lambda H_1(R) .$$

which is generated by a unitary transformation $H(R, \lambda) = \mathcal{U}(R) H(\lambda) \mathcal{U}^\dagger(R)$. Here, $\mathcal{U}(R)$ is unitary and satisfies $[\mathcal{U}(R), H(\lambda)] \neq 0$ to ensure the nontrivial transformation. Obviously, such a transformation keeps the energy-level structures invariant and the critical behavior of the system is thus $R$ independent. The eigenvalues are only characterized by the parameter $\lambda$. On the other hand, we note that the geometric phase adiabatically undergoing a closed path $C_{R,\lambda}$ in the $R$ space can be expressed as

$$\beta_g(C_{R,\lambda}) = - \int_{S(C_{R,\lambda})} \Omega_g(R, \lambda) \cdot dS ,$$

where $\Omega_g(R, \lambda)$ is the Berry curvature given by

$$\Omega_g(R, \lambda) = \frac{i}{\hbar} \sum_{n \neq g} \frac{\langle g(R, \lambda) | \nabla_R | n(R, \lambda) \rangle \times \langle n(R, \lambda) | \nabla_R | g(R, \lambda) \rangle}{(E_n(\lambda) - E_g(\lambda))^2} .$$

From the expressions of Eqs. (18) and (21), it is not hard to find that for both of them the singularities may come from the vanishing energy gap in the thermodynamic limit. In the inhomogeneous $XY$ model, we find that a gapless excitation occurs only when $\Lambda_{nk} \rightarrow 0$ or $\Lambda_{nk} \rightarrow 0$, which demands $\xi_k \rightarrow 0$. Hence, this condition can be achieved only in the thermodynamic limit $N \rightarrow \infty$ and for the appropriate parameters of the Hamiltonian, i.e., $J_x \gamma_a = J_y \gamma_b$. Apart from these special cases, there exist no solution for $\Lambda_{nk} = 0$ or $\Lambda_{nk} = 0$ and a non-zero energy gap opened. Consequently, the Berry curvature in the thermodynamic limit only develops extremum points instead of divergence.

In summary, we present an exact diagonalization approach for an inhomogeneous periodic anisotropic $XY$ model in a transverse field. By introducing a general canonical transformation, we construct an explicit expression for the ground state, and based on this, we study the geometric phase of the ground state and QPTs for this model. Different from the Ising chain and anisotropic $XY$ chain in a transverse field, the inhomogeneous periodic spin chains exhibit a richer behavior of QPTs. Our results show that there may exist more than one phase transition point at some parameter regions. In the language of geometric phase, detecting the QPTs of a many-body system driven by the external parameter $\lambda$ is equivalent to finding a path $C_{R,\lambda}$ in the parameter space of the Hamiltonian, in which the Berry curvature comes to the divergence or extremum points.

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