Safe Control and Learning Using the Generalized Action Governor

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ABSTRACT
This article introduces a general framework for safe control and learning based on the generalized action governor (AG). The AG is a supervisory scheme for augmenting a nominal closed-loop system with the ability of strictly handling prescribed safety constraints. In the first part of this article, we present a generalized AG methodology and analyze its key properties in a general setting. Then, we introduce tailored AG design approaches derived from the generalized methodology for linear and discrete systems. Afterward, we discuss the application of the generalized AG to facilitate safe online learning, which aims at safely evolving control parameters using real-time data to enhance control performance in uncertain systems. We present two safe learning algorithms based on, respectively, reinforcement learning and data-driven Koopman operator-based control integrated with the generalized AG to exemplify this application. Finally, we illustrate the developments with a numerical example.

KEYWORDS
safe autonomy, safety-critical systems, supervisory control, safe reinforcement learning, data-driven control

1. Introduction

Safety is a major concern in the development and operation of autonomous systems. Many safety specifications can be expressed as time-domain constraints on system state and control variables. Control methods that can explicitly and strictly handle constraints include model predictive control (MPC; Camacho & Alba, 2013; Borrelli, Bemporad, & Morari, 2017), the control barrier function (CBF; Tee, Ge, & Tay, 2009; Ames, Xu, Grizzle, & Tabuada, 2016), reference governors (RGs; Garone, Di Cairano, & Kolmanovsky, 2017; Kolmanovsky & Li, 2022), and reachability-based methods (Schürmann, Kochdumper, & Althoff, 2018; Herbert et al., 2021), etc. The action governor (AG) is a more recently proposed supervisory scheme that augments a nominal closed-loop system with the capability of handling constraints (N. Li et al., 2020;
It is placed after the nominal controller and supervises the control actions generated by the controller, adjusting unsafe actions to safe ones. The AG is designed based on discrete-time models and methods that exploit set-based computations, which distinguish it from the CBF which is most frequently based on continuous-time models (Tee et al., 2009; Ames et al., 2016) [although discrete-time formulations exist, as in Zeng, Zhang, and Sreenath (2021); Xiong, Zhai, Tavakoli, and Xia (2022)] and does not use set-based computations. The AG was first proposed for discrete-time linear models in N. Li et al. (2020) and then extended to uncertain piecewise-affine models in Y. Li et al. (2021a).

The first part of this article aims to generalize the AG methodology. Instead of focusing on system models that take a specific form such as linear or piecewise-affine as in N. Li et al. (2020); Y. Li et al. (2021a), we make minimal assumptions on the model and describe the offline design principles as well as online operation procedures of the AG in such a general setting. Then, we present tailored design procedures and algorithms for linear systems and discrete systems with finite state and action spaces. Note that the procedure and algorithm for linear systems presented in this article are different from those of N. Li et al. (2020) – the ones presented in this article are derived based on the generalized AG methodology developed in this article, which are computationally efficient and scalable but restricted to convex constraints, while those of N. Li et al. (2020) are not. Moreover, we present a novel procedure and algorithm that handle discrete systems.

As many autonomous systems operate in a priori uncertain or changing conditions, online learning capability that evolves control parameters using real-time data to adapt to operating conditions is highly desirable for these systems. The learning must be performed in a safe manner, i.e., constraints must be satisfied during the learning process. However, most conventional learning approaches, including most reinforcement learning (RL) algorithms (Sutton & Barto, 2018), do not have the ability to strictly respect constraints during learning. This has been a major impediment to using these approaches for online learning. To overcome this obstacle, a learning algorithm may be integrated with a constraint handling scheme to realize safe learning (Hsu, Hu, & Fisac, 2023). For instance, safe RL using MPC is proposed in Zanon and Gros (2020); S. Li and Bastani (2020); Wabersich and Zeilinger (2021) and using CBF in Cheng, Orosz, Murray, and Burdick (2019); Choi, Castaneda, Tomlin, and Sreenath (2020); Marvi and Kiumarsi (2021). Integrating Q-learning and an AG to realize safe Q-learning has also been proposed in our previous conference paper (Y. Li et al., 2021b). In the second part of this article, we extend the discussion on safe online learning using the AG. We first briefly review the integration of Q-learning and the AG for safe Q-learning, and then propose a new safe learning strategy based on data-driven Koopman control. Koopman operator-based control, where control is determined based on a data-driven Koopman linear model of a nonlinear system, has been attracting extensive attention (Korda & Mezić, 2018; Bruder, Fu, Gillespie, Remy, & Vasudevan, 2020). In this article, we propose to integrate it with the AG toward an alternative safe learning strategy to safe RL. To the best of our knowledge, such a safe learning strategy has not been proposed before. Therefore, the contributions of this article are:

(1) We present a generalized AG methodology. Unlike previous AG design approaches that only apply to linear or piecewise-affine models, the methodology presented in this article is independent of the form of the model and hence applies to significantly more general systems. Moreover, unlike the majority of existing constraint handling schemes including the CBF and previous AG ap-
proaches that rely on positive invariance of a safe set to ensure recursive safety (Tee et al., 2009; Ames et al., 2016; N. Li et al., 2020; Y. Li et al., 2021a), the generalized AG methodology presented in this article relaxes the reliance to returnability of a safe set. The relaxation from positive invariance to returnability enables more flexible design of the safe set, leading to the following three consequences: i) For systems for which positively invariant safe sets are not easy to construct, it may be possible/easier to compute a returnable safe set (Gilbert & Kolmanovsky, 2002; Kolmanovsky & Li, 2022). ii) For some systems a positively invariant safe set may have a rather complex representation, e.g., involve many inequality conditions, which entails high memory requirement for storing it and high computational cost of solving the online optimization problem that uses it. In contrast, a returnable safe set may have a less complex representation, hence facilitating its storage and the associated online computations (Kolmanovsky & Li, 2022). iii) For discrete systems, this relaxation enables designing a safe set using the algorithm proposed in Section 3.2 of this article.

(2) Induced by the generalized AG methodology, we present tailored AG design approaches for linear and discrete systems with bounded uncertainties. For linear systems, we show that the maximum output admissible set (MOAS), which was studied and used mainly in the RG framework (Garone et al., 2017; Kolmanovsky & Li, 2022), can be used to define the safe set for the AG to handle constraints. This leads to a new, computationally efficient and scalable AG design approach for linear systems with convex constraints, which is different from the previous approach of N. Li et al. (2020). For discrete systems with finite state and action spaces, we present an algorithm for computing the safe set, which is also new.

(3) We discuss the application of the generalized AG to support safe online learning. In addition to a review of safe RL via an effective integration of Q-learning and the AG (with details not given in our conference paper Y. Li et al., 2021b), we also propose a new safe learning strategy, which is based on the integration of data-driven Koopman operator-based control and the AG. An example is reported illustrating the effectiveness of this new safe learning strategy in improving control performance using online data for an uncertain system.

This article is organized as follows: In Section 2, we introduce the problem treated in this article, including the basic models and assumptions. In Section 3, we present the generalized AG methodology, analyze its key properties, and derive tailored approaches for linear and discrete systems. In Section 4, we discuss the application of the generalized AG for safe online learning. We use an example to illustrate the developments in Section 5 and conclude the paper in Section 6.

2. Models and Assumptions

In this article, we consider systems the dynamics of which can be represented by the following discrete-time model:

$$x(t+1) = f(x(t), u(t), w(t))$$  \hspace{1cm} (1)

where $t \in \mathbb{Z}_+$ denotes the discrete time; $x(t) \in \mathcal{X}$ represents the system state at time $t$, taking values in the state space $\mathcal{X}$; $u(t) \in \mathcal{U}$ represents the control action at time $t$, taking values in the action space $\mathcal{U}$; $w(t) \in \mathcal{W}$ represents an uncertainty such as an
unmeasured external disturbance, which is assumed to take values in a known bounded set $W$; and $f : X \times U \times W \rightarrow X$ is the state transition function. At this time, we make no further assumptions on the spaces $X$, $U$, and $W$ and on the function $f$. For instance, $X$, $U$, and $W$ can be continuous or discrete spaces, and $f$ can be nonlinear.

This article deals with safety-critical applications where it is assumed that the system during operation must satisfy the following constraints on state and control variables:

$$\left(x(t), u(t)\right) \in C, \quad \forall t \in \mathbb{Z}_+$$

where $C$ is a subset of $X \times U$. A constraint on state $x(t)$ can represent a process variable bound or a collision avoidance requirement; a constraint on control action $u(t)$ can represent an actuator power or range limit, which may be state-dependent.

Our goal is to develop a supervisory control scheme, termed the generalized action governor (AG), to ensure the system’s adherence to the safety condition (2). We assume there is at hand a nominal control policy, $\pi_0 : X \times V \rightarrow U$, that is,

$$u_0(t) = \pi_0(x(t), v(t))$$

where $v(t) \in V$ represents a reference signal, which typically corresponds to the current control objective (e.g., a setpoint for tracking) and is passed to the policy from a higher-level planner or a human operator. We write the closed-loop system under the nominal policy $\pi_0$ as

$$x(t + 1) = f_{\pi_0}(x(t), v(t), w(t)) = f(x(t), \pi_0(x(t), v(t)), w(t))$$

and write its response at $t$ for a given initial condition $x(0) = x$, constant reference $v(\tau) \equiv v$, and disturbance sequence $w(\cdot) = \{w(\tau)\}_{\tau=0}^{\infty}$ taking values in $W$ as $\phi_{\pi_0}(t \mid x, v, w(\cdot))$. This nominal closed-loop system shall have some stability-type response characteristics [to ensure the existence of nonempty safe and returnable set as defined in (5) and (6)]. For instance, for a constant $v(t) \equiv v \in V$, the system state $x(t)$ converges to the steady state corresponding to $v$, i.e., $x(t) \approx x_v(v)$ as $t \to \infty$. Note that we do not require the nominal policy $\pi_0$ to have the ability to handle the safety constraints in (2) for all $v \in V$ and do not assume it is an optimal policy. Therefore, for many systems, such a policy is easier to design than one that achieves both stability and constraint handling. For instance, for linear systems $\pi_0$ may be designed based on PID or linear quadratic regulator (LQR).

We now consider a compact set $\Pi_{\pi_0}$ of pairs $(x, v)$ that satisfies the following two properties:

i (Safety). For all $(x, v) \in \Pi_{\pi_0}$ and $w(\cdot)$ in $W$,

$$\left(\phi_{\pi_0}(t \mid x, v, w(\cdot)), \pi_0(\phi_{\pi_0}(t \mid x, v, w(\cdot)), v)\right) \in C$$

for all $t \in \mathbb{Z}_+$.

ii (Returnability). For all $(x, v) \in \Pi_{\pi_0}$ and $w(\cdot)$ in $W$, there exists $\hat{t} = \hat{t}(x, v, w(\cdot)) > 0$ such that

$$\left(\phi_{\pi_0}(\hat{t} \mid x, v, w(\cdot)), v\right) \in \Pi_{\pi_0}.$$
The returnability of \( \Pi_{\pi_0} \) means that state trajectories of the nominal closed-loop system (4) for constant \( v \) beginning in \( \Pi_{\pi_0} \) eventually return to \( \Pi_{\pi_0} \). If for all \( (x, v) \in \Pi_{\pi_0} \) and \( w(\cdot) \) taking values in \( \mathcal{W} \) the time \( \hat{t} = \hat{t}(x, v, w(\cdot)) \) in (6) is equal to 1, then \( \Pi_{\pi_0} \) is \textit{positively invariant}. Thus, positively invariant sets belong to the larger class of returnable sets. Given a nominal control policy \( \pi_0 \), a variety of tools exist for computing/estimating a set \( \Pi_{\pi_0} \) that satisfies the two properties – \textit{safety} and \textit{returnability} (or the stronger property \textit{positive invariance}). In Section 3, we introduce two examples, one for linear systems and the other for discrete nonlinear systems.

3. Generalized Action Governor

The generalized AG enforces (2) by adjusting actions according to the following algorithm:

\[
\begin{align*}
    (7) & \quad u(t) = \begin{cases} 
        \hat{u}(t) & \text{if (8) is feasible} \\
        \pi_0(x(t), \hat{v}(t)) & \text{otherwise}
    \end{cases} \\
\end{align*}
\]

where \( \hat{u}(t) \) is determined according to

\[
\begin{align*}
    (8a) & \quad \hat{u}(t) \in \arg\min_{u \in U} \text{dist}_x(x(t), u_1(t), u) \\
    \text{s.t.} & \quad (x(t), u) \in \mathcal{C} \\
    & \quad f(x(t), u, w) \in \text{proj}_x(\Pi_{\pi_0}), \forall w \in \mathcal{W}
\end{align*}
\]

and \( \hat{v}(t) \) is determined according to: If \( x(t) \in \text{proj}_x(\Pi_{\pi_0}) \), then

\[
\begin{align*}
    (9a) & \quad \hat{v}(t) \in \arg\min_{v \in V} \text{dist}_x(x(t), u_1(t), \pi_0(x(t), v)) \\
    \text{s.t.} & \quad (x(t), v) \in \Pi_{\pi_0}
\end{align*}
\]

and \( \hat{v}(t) = \hat{v}(t-1) \) otherwise.

In (8) and (9), \( u_1(t) \) is the action before adjustment. It does not have to be the action from the nominal policy \( \pi_0 \), but can be from another policy \( \pi_1 \) that is currently being used or an exploration action in order to update \( \pi_1 \) through learning. The function \( \text{dist}_x(\cdot, \cdot) \) measures the distance between \( u_1(t) \) and \( u \) (or \( \pi_0(x(t), v) \)) in order to minimize the adjustment. A typical choice is \( \text{dist}_x(u_1, u) = \|u_1 - u\| \), where \( \| \cdot \| \) represents a norm, while the subscript \( x \) indicates that such a distance function can be state-dependent in general. The function \( \text{proj}_x \) projects the set \( \Pi_{\pi_0} \) onto the state space \( \mathcal{X} \). The principal idea of the AG algorithm (7)-(9) is as follows: At each \( t \in \mathbb{Z}_+ \), if there exist actions \( u \) that satisfy the constraint (2) at the present time [according to (8b)] and ensure the satisfaction of (2) for future times to be possible [according to (8c)], then we let \( u(t) \) be the one of such actions that is closest to the action before adjustment [according to (8a)]. If such actions do not exist, then we use \( \pi_0 \) as a backup plan and let \( u(t) = \pi_0(x(t), \hat{v}(t)) \). In the latter case, if \( x(t) \in \text{proj}_x(\Pi_{\pi_0}) \) (which is equivalent to (9) being feasible), then we select the reference signal \( \hat{v}(t) \) to be the one that minimizes the action adjustment [according to (9a)] while satisfying (9b); we let \( \hat{v}(t) = \hat{v}(t-1) \) otherwise.

The generalized AG algorithm has the following properties:
Proposition 1 (All-Time Safety). If (8) is feasible at the initial time \( t = 0 \), then the system trajectory \((x(t), u(t))\) under the AG supervision satisfies the constraints in (2) for all \( t \in \mathbb{Z}_+ \).

Proof. Let \( \tau \in \mathbb{Z}_+ \) be arbitrary. Because (8) is feasible at the initial time 0, there must exist a time \( \tau' \in [0, \tau] \) that is the last time where (8) is feasible. According to (7) and (8b), \((x(\tau'), u(\tau')) = (x(\tau'), \hat{u}(\tau')) \in C\). Then, if \( \tau' = \tau \), we are done. If \( \tau' \prec \tau \), according to (7) and (8c), it must hold that \( x(\tau' + 1) = f(x(\tau'), u(\tau'), w(\tau')) = f(x(\tau'), \hat{u}(\tau'), w(\tau')) \in \text{proj}_{Z}(\Pi_{\tau_0}) \). Then, (9) must be feasible at \( \tau' + 1 \), and accordingly, there must exist a time \( \tau'' \in [\tau' + 1, \tau] \) that is the last time where (9) is feasible. According to (9b), \((x(\tau''), \hat{v}(\tau'')) \in \Pi_{\tau_0} \). Note that according to the definition of \( \tau'' \) above, (8) is infeasible over \([\tau' + 1, \tau]\), and according to the definition of \( \tau'' \) above, (9) is infeasible over \([\tau'' + 1, \tau]\). Then, according to (7) and (9), \( u(t) = \pi_0(x(t), \hat{v}(t)) \) and \( \hat{v}(t) = \hat{v}(\tau'') \) for all \( t \in [\tau'', \tau] \). Consequently, \( x(\tau) = \phi_{\tau_0}(\tau - \tau'', x(\tau''), \hat{v}(\tau''), w(\cdot)) \) and \( u(\tau) = \pi_0(x(\tau), \hat{v}(\tau'')) \). Then, according to \((x(\tau''), \hat{v}(\tau'')) \in \Pi_{\tau_0}\) and the safety property of \( \Pi_{\tau_0} \) in (5), we have \((x(\tau), u(\tau)) \in C\). Therefore, we have shown that in both cases, \( \tau' = \tau \) and \( \tau' \prec \tau \), \((x(\tau), u(\tau)) \in C\). Since \( \tau \in \mathbb{Z}_+ \) is arbitrary, we have shown that (2) is satisfied for all times. \( \square \)

Proposition 2 (Eventual Feasibility). If (8) or (9) is feasible at some time \( \tau \), then there exists a future time \( \tau' > \tau \) such that (8) or (9) is feasible at \( \tau' \).

Proof. If (8) is feasible at \( \tau \), according to (7) and (8c), it must hold that \( x(\tau + 1) = f(x(\tau), u(\tau), w(\tau)) = f(x(\tau), \hat{u}(\tau), w(\tau)) \in \text{proj}_{Z}(\Pi_{\tau_0}) \). Then, (9) must be feasible at \( \tau' = \tau + 1 \). For the case where (8) is infeasible while (9) is feasible at \( \tau \), according to (9b), \((x(\tau), \hat{v}(\tau)) \in \Pi_{\tau_0} \). Let us assume that neither (8) nor (9) will be feasible for any future time \( t > \tau \). In this case, according to (7) and (9), \( u(t) = \pi_0(x(t), \hat{v}(t)) \) and \( \hat{v}(t) = \hat{v}(\tau) \) for all \( t \geq \tau \). Consequently, \( x(t) = \phi_{\tau_0}(t - \tau, x(\tau), \hat{v}(\tau), w(\cdot)) \) and \( u(t) = \pi_0(x(t), \hat{v}(\tau)) \). However, according to \((x(\tau), \hat{v}(\tau)) \in \Pi_{\tau_0}\) and the returnability property of \( \Pi_{\tau_0} \) in (6), there must exist \( \hat{t} > \tau \) such that \((x(\hat{t}), \hat{v}(\tau)) = \phi_{\tau_0}(\hat{t} - \tau, x(\tau), \hat{v}(\tau), w(\cdot)), \hat{v}(\tau)) \in \Pi_{\tau_0} \). That is, (9) will be feasible at \( \hat{t} \). This contradicts the assumption of infeasibility of (8) and (9) for all \( t > \tau \) made above. Therefore, we have shown the necessary existence of a future time \( \tau' > \tau \) such that (8) or (9) is feasible at \( \tau' \). \( \square \)

It can be seen from the proof that the returnability of \( \Pi_{\tau_0} \) plays a key role in the eventual feasibility result of Proposition 2.

While the AG algorithm (7)-(9) and its properties stated in Propositions 1 and 2 apply to general systems, in what follows we present two design approaches of the AG tailored for linear systems and discrete systems.

3.1. Generalized action governor for linear systems

Suppose (1) is linear with an additive uncertainty, i.e.,

\[
x(t + 1) = f(x(t), u(t), w(t)) = Ax(t) + Bu(t) + Ew(t)
\]

where \( A, B \) and \( E \) are matrices of consistent dimensions, \( w(t) \) takes values in a compact polyhedral set \( \mathcal{W} = \{w : Gw \leq g\} \), and the constraints in (2) can be expressed as...
linear inequalities, i.e.,

\[ (x(t), u(t)) \in \mathcal{C} \iff y(t) = Cx(t) + Du(t), \ y(t) \in \mathcal{Y} = \{ y : Hy \leq h \}. \quad (11) \]

In this case, one can consider a linear nominal policy (3),

\[ u_0(t) = \pi_0(x(t), v(t)) = Kx(t) + Lv(t) \quad (12) \]

which leads to the following closed-loop system

\[
\begin{aligned}
x(t+1) &= f_{\pi_0}(x(t), v(t), w(t)) = \tilde{A}x(t) + \tilde{B}v(t) + \tilde{E}w(t) \\
y(t) &= \tilde{C}x(t) + \tilde{D}v(t) \quad (13)
\end{aligned}
\]

where \( \tilde{A} = A + BK \), \( \tilde{B} = BL \), \( \tilde{C} = C + DK \), and \( \tilde{D} = DL \). The nominal policy (12) can be designed flexibly, but needs to ensure that the closed-loop matrix \( \tilde{A} \) is Schur, i.e., all eigenvalues of \( \tilde{A} \) are strictly inside the unit disk of the complex plane. As will be shown below, this nominal policy is useful for offline computing a set \( \Pi_{\pi_0} \) that has the desired safety and returnability properties; meanwhile, this nominal policy is never directly used for online control. In particular, one can consider a set \( \Pi_{\pi_0} \) that is defined as follows:

\[ \Pi_{\pi_0} = \tilde{O}_{\infty} = \left( \bigcap_{t=0}^{t'} \mathcal{O}_t \right) \cap (\mathcal{X} \times \Omega') \quad (14) \]

where

\[
\begin{align*}
\mathcal{O}_t &= \left\{ (x, v) : \tilde{C}\tilde{A}^t x + (\tilde{C}(I - \tilde{A}^t)(I - \tilde{A})^{-1}\tilde{B} + \tilde{D})v \in \mathcal{Y}_t \right\} \\
\mathcal{Y}_t &= \mathcal{Y} \sim \tilde{C} \left( \bigoplus_{k=0}^{t-1} \tilde{A}^k \mathcal{E} \mathcal{W} \right) \\
\Omega' &= \left\{ v : (\tilde{C}(I - \tilde{A})^{-1}\tilde{B} + \tilde{D})v \in (1 - \varepsilon)\mathcal{Y}_t \right\}
\end{align*}
\]

with \( 0 < \varepsilon \ll 1 \). In (15), \( \oplus \) denotes the Minkowski sum operator and \( \sim \) denotes the Pontryagin difference operator between sets (Kolmanovsky & Gilbert, 1998). Under mild assumptions, for all \( t' \) sufficiently large, the set \( \tilde{O}_{\infty} \) defined in (14), called the MOAS, becomes independent of \( t' \), and it is safe and positively invariant (hence returnable; Kolmanovsky & Gilbert, 1998), i.e.,

\[ (x(t), v(t)) \in \tilde{O}_{\infty} \implies (\phi_{\pi_0}(1 | x(t), v(t), w(\cdot)), v(t)) \in \tilde{O}_{\infty}, \forall w(t) \in \mathcal{W} \quad (16) \]

where \( \phi_{\pi_0}(1 | x(t), v(t), w(\cdot)) \) is defined below (4). Under the positive invariance of \( \Pi_{\pi_0} = \tilde{O}_{\infty} \), the AG algorithm (7)-(9) reduces to

\[
\begin{align*}
u(t) &\in \arg\min_{u \in \mathcal{U}} \text{dist}_{x(t)}(u_1(t), u) \quad (17a) \\
\text{s.t.} \quad &Cx(t) + Du \in \mathcal{Y} \quad (17b) \\
&Ax(t) + Bu \in \text{proj}_{E}(\tilde{O}_{\infty}) \sim EW. \quad (17c)
\end{align*}
\]
This reduction is due to the following result:

**Proposition 3 (Recursive Feasibility).** Under the positive invariance of \( \bar{\mathcal{O}}_\infty \) in (16), if (17) is feasible at \( t \), it is necessarily feasible at \( t+1 \).

**Proof.** If (17) is feasible at \( t \), according to (17c), it must hold that \( x(t+1) = Ax(t) + Bu(t) + Ew(t) \in \text{proj}_f(\bar{\mathcal{O}}_\infty) \). Thus, there must exist \( v \) such that \( (x(t+1), v) \in \bar{\mathcal{O}}_\infty \). Now consider \( u = \pi_0(x(t+1), v) = Kx(t+1) + Lv \). Because \( (x(t+1), v) \in \bar{\mathcal{O}}_\infty \subset \mathcal{O}_0 \), according to the definition of \( \mathcal{O}_0 \) in (15), \( u \) must satisfy \( Cx(t+1) = Cx(t+1) + D(Kx(t+1) + Lv) = (C + DK)x(t+1) + DLv = \tilde{C}x(t+1) + DLv \in \mathcal{Y} \). Furthermore, the positive invariance (16) implies that \( \phi_{\pi_0}(1 | x(t+1), v, w(\cdot)) = (Ax(t+1) + Bv + Ew, v) \in \bar{\mathcal{O}}_\infty \) for all \( w \in \mathcal{W} \). Thus, \( \tilde{AX}(t+1) + Bv + Ew = Ax(t+1) + B(Kx(t+1) + Lv) + Ew = Ax(t+1) + Bu + Ew \in \text{proj}_f(\bar{\mathcal{O}}_\infty) \) for all \( w \in \mathcal{W} \), which is equivalent to \( Ax(t+1) + Bu \in \text{proj}_f(\bar{\mathcal{O}}_\infty) \approx \mathcal{E} \mathcal{W} \). Therefore, we have shown that \( u = Kx(t+1) + Lv \) satisfies both constraints (17b) and (17c) and is a feasible solution at \( t+1 \). This proves the necessary feasibility of (17) at \( t+1 \). \( \square \)

Note that the computation of \( \bar{\mathcal{O}}_\infty \) for linear systems is simple and scalable (Kolmanovsky & Gilbert, 1998). Therefore, the approach in this subsection represents a computationally efficient and scalable method for designing the AG for linear systems, and it is different from the approach of N. Li et al. (2020).

### 3.2. Generalized action governor for discrete systems

The second special case we consider is where the spaces \( \mathcal{X}, \mathcal{U}, \mathcal{W} \) and \( \mathcal{V} \) are all finite, and the transition function \( f \) is a mapping between finite spaces. In this case, the system (1) is referred to as a discrete system. This case is of interest due to several reasons: First, when the uncertainty \( w(t) \) takes values according to a probability distribution over \( \mathcal{W} \), the system (1) admits a Markov decision process (MDP) representation. The MDP is a fundamental model for sequential decision-making and is widely used in many disciplines including robotics and manufacturing (Thrun, Burgard, & Fox, 2005). An MDP model also admits a graph representation, which further broadens the range of its applications. This means the techniques presented in this subsection for discrete systems can be used to develop safety supervisors for MDP models in these application areas. Second, via appropriate discretization, a general nonlinear system can be approximated by a discrete system on finite spaces, and many control synthesis techniques such as dynamic programming are based on such a discrete approximation (Puterman, 2014). Accordingly, the techniques presented in this subsection for discrete systems can be used to treat general nonlinear systems in an approximate manner.

When (1) is discrete, Algorithm 1 can be used to compute a set \( \Pi_{\pi_0} \) that is safe and returnable. The algorithm relies on an initial set of pairs \( (x, v) \), denoted as \( \Pi^{0}_{\pi_0} \), that is safe and positively invariant under \( \pi_0 \), i.e., \( (x, v) \in \Pi^{0}_{\pi_0} \) implies

\[
\begin{align}
(i) & \quad (x, \pi_0(x, v)) \in \mathcal{C} \\
(ii) & \quad (f_{\pi_0}(x, v, w), v) \in \Pi^{0}_{\pi_0}, \forall w \in \mathcal{W}.
\end{align}
\]

While there may exist many sets that satisfy the above properties, one of them may be particularly easy to compute – that is the set of safe steady-state pairs. Recall that the nominal policy \( \pi_0 \) is assumed to stabilize the system in terms of making \( x(t) \approx x_0(v) \) as \( t \to \infty \), where \( x_0(v) \) denotes the steady state corresponding to \( v \). In the disturbance
free case, \( W = \{0\} \), \( \Pi^0_{\pi_0} \) is the set of \((x, u)(v), v)\) that satisfies \((x, u)(v), \pi_0(x, u)(v), v)\) \( \in C \). When there is disturbance, \( W \neq \{0\} \), states in the vicinity of \( x, u)(v) \) may need to be checked for safety and be included in order to ensure positive invariance of \( \Pi^0_{\pi_0} \). Such a set of safe steady-state pairs is easy to compute when the steady-state map \( x, u)(v) \) is known; otherwise it may be constructed using data from steady-state experiments.

### Algorithm 1 Computation of \( \Pi_{\pi_0} \) for discrete systems

1. Initialize the datasets \( D^+_x \) as \( \Pi^0_{\pi_0} \), \( D^-_x \) as empty, \( D^\text{remain}_x \) as \((X \times V) \setminus \Pi^0_{\pi_0} \), and the counter \( k \) as 0;
2. while \( D^\text{remain}_x \) is nonempty and \( k < k_{\text{max}} \) do
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. 
14. Set \( \Pi_{\pi_0} \) as any set satisfying \( \Pi^0_{\pi_0} \subset \Pi_{\pi_0} \subset D^+_x \).

In Algorithm 1, \( D^+_x \) represents a set of state-reference pairs that will safely and eventually enter \( \Pi^0_{\pi_0} \), regardless of disturbance (and stay in \( \Pi^0_{\pi_0} \) after that by the positive invariance of \( \Pi^0_{\pi_0} \)). \( D^-_x \) represents a set of state-reference pairs that will eventually violate the safety requirement \((x, u) \in C \) under some disturbance realization, and \( D^\text{remain}_x \) is the set of pairs that remain to be verified for safety and returnability. In particular, if a pair \((x, v)\) is safe (checked in Step 4) and will enter \( D^+_x \) for all possible disturbance values (checked in Step 5), then this pair is added to \( D^+_x \) (because once entering \( D^+_x \), it is in the course of safely and eventually entering \( \Pi^0_{\pi_0} \)); if a pair \((x, v)\) is not safe or will enter \( D^-_x \) under some disturbance value (checked in Step 7), then this pair is added to \( D^-_x \) (because once entering \( D^-_x \), it is in the course of violating \((x, u) \in C \) at some future time). If a pair \((x, v)\) is determined as either safe and eventually entering \( \Pi^0_{\pi_0} \) or unsafe, it is removed from the remaining set \( D^\text{remain}_x \). The algorithm terminates when \( D^\text{remain}_x \) becomes empty (i.e., all state-reference pairs have been determined as either safe and eventually entering \( \Pi^0_{\pi_0} \) or unsafe) or when the maximum number of iterations \( k_{\text{max}} \) is reached. The desired set \( \Pi_{\pi_0} \) is selected as any set that contains \( \Pi^0_{\pi_0} \) and is contained in \( D^+_x \). Note that since any \((x, v) \in D^+_x \) will safely and eventually enter \( \Pi^0_{\pi_0} \), any such selection of \( \Pi_{\pi_0} \) is safe and returnable. While \( \Pi_{\pi_0} = D^+_x \) is a trivial valid selection, other selections may be preferred in certain cases, e.g., for storage simplicity. We also note that although the initial set \( \Pi^0_{\pi_0} \) is already safe and returnable (since it is safe and positively invariant), it may be small, e.g., when it is the set of safe steady-state pairs. Algorithm 1 enlarges the set and hence enables the AG to have a higher flexibility for selecting safe actions.
4. Safe Online Learning

The AG can be used to enable safe online learning (i.e., learning while satisfying prescribed safety specifications). A diagram illustrating the proposed safe learning architecture is given in Fig. 1. A learning algorithm evolves the control policy to improve its performance or adapt to changes in system parameters or operating conditions. In order to learn, *excitation signals* (or, *exploratory actions*) must be applied (Ladosz, Weng, Kim, & Oh, 2022), which may cause the system behavior to violate prescribed safety specifications, even if the original control policy was designed to respect these specifications. In this case, an AG can be used to supervise the learning process – it is placed between the control policy and the system and adjusts the actions that potentially lead to safety violations to safe actions. Various learning algorithms can be integrated with the AG according to this architecture to realize safe online learning. For illustration, we present two examples that involve different learning algorithms in what follows.

**Figure 1.** Safe online learning architecture.

### 4.1. Safe Q-learning

Q-learning is a classical RL algorithm and serves as the foundation of many more advanced RL algorithms. The readers are referred to Watkins and Dayan (1992) and Sutton and Barto (2018) for reviews of the Q-learning algorithm and its variants. In order for Q-learning to converge, the algorithm must explore different actions at each state. An effective exploration strategy is called $\varepsilon$-greedy (Whitehead & Ballard, 1991), where

$$
    u(t) \in \begin{cases} 
        \arg\max_u \tilde{Q}(x(t), u) & \text{with prob. } = 1 - \varepsilon \\
        \text{random action in } \mathcal{U} & \text{with prob. } = \varepsilon
    \end{cases}
$$

i.e., the algorithm has a large probability of $1 - \varepsilon$ to take an action that is optimal with respect to the latest estimates of the $Q$-values at the current state $x(t)$, and it has a small probability of $\varepsilon$ to take an arbitrary action in the action space $\mathcal{U}$.

During the learning process, violations of prescribed safety conditions may occur, due to, e.g., the application of random exploratory actions. A common strategy is to impose a penalty for such violations so that the control policy gradually learns to satisfy the safety conditions. However, such a strategy learns from safety violations and thus cannot avoid the occurrence of safety violations during learning. In addition,
even after a sufficient learning phase and no longer applying any further exploratory actions, safety violations may still occur, due to, e.g., errors in the estimated \( Q \)-values or the fact that maximizing the expectation of reward allows small (but non-zero) probability of safety violations and penalties. An integration of Q-learning and an AG according to Fig. 1 can avoid the application of any unsafe actions while maintaining the ability of Q-learning to evolve the control policy to improve performance. Such an integration is presented in Algorithm 2.

Algorithm 2 updates a \( Q \)-function estimate, \( \tilde{Q}(x, u) \), in a mini-batch manner, where \( T_{\text{max}} > 0 \) represents the maximum number of batch updates and \( t_{\text{batch}} > 0 \) represents the batch size. In Step 7, we consider a modified reward function \( \tilde{r}(x, u_1) \), which is defined as the original reward \( r(x, u) \) minus a penalty for the difference between the original action \( u_1 \) and the action after AG adjustment \( u \). This modified reward function achieves two goals: 1) If the original action \( u_1 \) is safe and the AG passes \( u_1 \) through (i.e., \( u = u_1 \)), then the modified reward \( \tilde{r}(x, u_1) \) is equal to the original reward \( r(x, u) \). 2) If the original action \( u_1 \) is unsafe and the AG adjusts \( u_1 \) to another action \( u \), then the algorithm is informed by the penalty and will learn to avoid choosing the unsafe action \( u_1 \) at the state \( x \).

**Algorithm 2 Safe Q-learning using AG**

1. Initialize \( Q \)-function estimate, \( \tilde{Q}(x, u) \); create empty buffer, \( D \); initialize state, \( x(0) \);
2. for \( T = 0, 1, \ldots, T_{\text{max}} - 1 \) do
3. \hspace{1em} for \( t = T t_{\text{batch}}, \ldots, (T + 1) t_{\text{batch}} - 1 \) do
4. \hspace{2em} Select an action \( u_1(t) \) according to (19);
5. \hspace{2em} Adjust \( u_1(t) \) to a safe action \( u(t) \) according to (7)-(9);
6. \hspace{2em} Apply \( u(t) \) to system and observe next state \( x(t + 1) \);
7. \hspace{2em} Evaluate a modified single-step reward according to
\[
\tilde{r}(x(t), u_1(t)) = r(x(t), u(t)) - M \text{dist}_{x(t)}(u_1(t), u(t))
\]
where \( M > 0 \) is a large penalty coefficient;
8. \hspace{2em} Calculate a new estimate of the \( Q \)-value of state-action pair \( (x(t), u_1(t)) \) according to
\[
Q(x(t), u_1(t)) = (1 - \alpha) \tilde{Q}(x(t), u_1(t)) + \alpha \tilde{r}(x(t), u_1(t)) + \gamma \max_{u \in U} \tilde{Q}(x(t + 1), u)
\]
where \( \alpha \in (0, 1] \) is a learning rate, \( \gamma \in (0, 1) \) is a discount factor;
9. \hspace{2em} Store \( Q(x(t), u_1(t)) \) in buffer \( D \);
10. end for
11. Update the \( Q \)-function estimate \( \tilde{Q}(x, u) \) according to data in buffer \( D \);
12. Empty the buffer \( D \);
13. end for

**Remark 1** (Use Case). In many applications, it is easier to establish a set bound for an uncertainty/disturbance, \( w(t) \in W \), while more difficult to know the statistics (or, probability distribution) of the uncertainty within the set bound, \( w(t) \sim P_w \), which may even be state and control action-dependent, \( P_w = P_w(x, u) \). On the one hand, to ensure safety, constraints such as (2) shall be enforced robustly (i.e., for all \( w(t) \in W \)). On the other hand, optimal control, in terms of achieving highest average/expected
performance, typically depends on the unknown statistics of $w(t)$. In such a scenario, an integrated safe RL algorithm will be very useful, where the AG ensures robust safety while RL evolves the control policy according to online data (which contain information of the underlying statistics of $w(t)$) to improve average performance. Algorithm 2 exemplifies such an integration with a control objective represented by the reward function $r(x, u)$. For applications where the control objective may change with time and is represented by a reference input $v(t) \in \mathcal{V}$, a reward function parameterized by $v \in \mathcal{V}$, $r_v(x, u) = r(x, u, v)$, can be used for RL. Correspondingly, the $Q$-values and optimal control policy will also depend on $v$, i.e., $Q^*_v(x, u) = Q^*(x, u, v)$ and $\pi^*_v(x) = \pi^*(x, v)$. Algorithm 2 can be extended to account for such a case. In addition, while there exist more advanced RL algorithms such as DDQN (Van Hasselt, Guez, & Silver, 2016), PPO (Schulman, Wolski, Dhariwal, Radford, & Klimov, 2017), they can be integrated with the AG to realize safe RL similarly as Algorithm 2.

4.2. Safe data-driven Koopman control

The Koopman operator theory provides a framework for converting a nonlinear control problem into a higher-dimensional linear control problem for which many linear control techniques such as LQR and linear MPC may be used (Korda & Mezić, 2018). Specifically, for a given system with dynamics $x(t+1) = f(x(t), u(t))$, the idea is to consider a set of functions, $g = \{g_1, g_2, \ldots\}$, $g_i : \mathcal{X} \to \mathbb{R}$, called observables, and identify a pair of matrices $(A, B)$ such that

$$g(x(t+1)) \approx Ag(x(t)) + Bu(t).$$

(20)

Then, by treating $z = g(x)$ as the lifted state, one obtains the following linear model:

$$z(t+1) = Az(t) + Bu(t)$$

(21)

to which linear control techniques can be applied so that the determined $u(t)$ can satisfactorily control the original system $x(t+1) = f(x(t), u(t))$. To accomplish this, the original state $x$ is typically included in the set of observables (Korda & Mezić, 2018).

While the computational advantages of this Koopman operator approach are appealing, its practical applications are still facing many major challenges yet to be addressed. One of such challenges relates to guaranteed safety in terms of strictly satisfying prescribed state and control constraints. Even if these constraints are enforced during the determination of control based on the Koopman linear model, undesirable constraint violations may still occur in actual application of the determined control to the original nonlinear system, due to errors between the nonlinear system and its Koopman linear model. Such errors mainly come from two sources (Tafazzol, Li, Kolmanovsky, & Filev, 2024): First, although a Koopman operator may be able to fully linearize nonlinear dynamics by lifting the dynamics into an infinite-dimensional space of observables, a finite-dimensional approximation of the operator is typically used in practical applications, due to computational reasons, resulting in approximation residuals. Second, a common strategy for calculating an approximate Koopman operator is data-driven, where data are sampled system trajectories (Korda & Mezić, 2018; Bruder et al., 2020). Although it is possible to monitor and minimize the model errors on the samples, the errors at unsampled locations are not known. An integration of the Koopman method and an AG according to Fig. 1 can address this safety challenge.
The AG algorithm (7)-(9) to adjust control actions can be considered as a function \( \pi_a \) that takes system state \( x(t) \) and action before adjustment \( u_1(t) \) as inputs and produces an adjusted action \( u(t) \) as output, i.e.,

\[
u(t) = \pi_a(x(t), u_1(t)).
\]

(22)

Then, with the AG in the loop, the dynamics can be written as

\[
x(t + 1) = f_a(x(t), u_1(t), w(t))
\]

\[= f(x(t), \pi_a(x(t), u_1(t)), w(t)).
\]

(23)

Note that even if the original system \( f \) is linear, the system with AG, \( f_a \), is typically nonlinear, due to the AG function \( \pi_a \) that enforces constraints and is typically nonlinear. In this case, one may apply the Koopman method and estimate a linear model for \( f_a \) in a higher-dimensional space of observables. As mentioned above, a typical approach to estimating a Koopman linear model is data-driven: Suppose a set of observables \( g \) has been chosen and a set of trajectory data \( \{x^k(t), u_1^k(t), x^k(t + 1)\}_{k=1}^{k_{\text{max}}} \) has been collected. Then, one can estimate an \((A, B)\) pair that best fits the data using a linear relation (21) in the least-squares sense by solving the following minimization problem:

\[
\min_{A,B} \|Z^+ - AZ - BU_1\|_F
\]

(24)

where \( Z^+ = [g(x^1(t + 1)), \ldots, g(x^{k_{\text{max}}}(t + 1))] \), \( Z = [g^1(x(t)), \ldots, g^{k_{\text{max}}}(x(t))] \), \( U_1 = [u_1^1(t), \ldots, u_1^{k_{\text{max}}}(t)] \), and \( \| \cdot \|_F \) denotes the Frobenius norm. An analytical solution to (24) is

\[
[A, B] = Z^+ \left[ \begin{array}{c} Z \\ U_1 \end{array} \right] = Z^+ \left[ \begin{array}{c} Z \\ U_1 \end{array} \right] \left[ \begin{array}{c} Z \\ U_1 \end{array} \right]^\top \left[ \begin{array}{c} Z \\ U_1 \end{array} \right]^\top \left[ \begin{array}{c} Z \\ U_1 \end{array} \right]^{-1}
\]

(25)

where \(( \cdot )^\top \) denotes the Moore-Penrose inverse. The estimation of \((A, B)\) can also be performed in an online recursive manner (Calderón, Schulz, Oehlschlägel, & Werner, 2021): Let \((A_{t-1}, B_{t-1})\) denote the previous estimate. When a new data point \((x(t - 1), u_1(t - 1), x(t))\) becomes available at time \( t \), the estimate can be updated according to

\[
[A_t, B_t] = [A_{t-1}, B_{t-1}] + \varepsilon(t) \gamma(t)
\]

(26)

where \( \varepsilon(t) = g(x(t)) - A_{t-1}g(x(t - 1)) - B_{t-1}u_1(t - 1) \) represents the prediction error, and \( \gamma(t) \) represents a correction vector, updated according to

\[
\gamma(t) = \frac{\left[ g(x(t - 1)) \right]^\top \Gamma(t) \left[ \begin{array}{c} g(x(t - 1)) \\ u_1(t - 1) \end{array} \right] + \lambda}{\left[ g(x(t - 1)) \right]^\top \Gamma(t) \left[ \begin{array}{c} g(x(t - 1)) \\ u_1(t - 1) \end{array} \right]}
\]

(27a)

\[
\Gamma(t + 1) = \frac{1}{\lambda} \Gamma(t) \left( I - \left[ g(x(t - 1)) \right]^\top \left[ \begin{array}{c} g(x(t - 1)) \\ u_1(t - 1) \end{array} \right] \right) \gamma(t)
\]

(27b)
where \( \lambda \in (0, 1] \) is a forgetting factor and \( \lambda = 1 \) corresponds to no forgetting. This recursive estimation procedure enables online learning/updating of the Koopman model using data collected as the system operates so that the model most accurately represents the dynamics corresponding to the present system parameter values and environmental (e.g., temperature, humidity, wind, etc.) conditions. An integration of online learning Koopman model, model-based determination of control, and an AG to enforce safety is presented in Algorithm 3.

**Algorithm 3** Safe data-driven Koopman control using AG

1. Select observables, \( g \); initialize Koopman model, \((A_0, B_0)\); initialize state, \( x(0) \), and calculate lifted state, \( z(0) = g(x(0)) \);
2. \textbf{for} \( t = 0, 1, 2, \ldots \) \textbf{do}
3. Determine a control action \( u_1(t) \) using the linear model \( z(t + k + 1) = A_t z(t + k) + B_t u_1(t + k) \), \( k = 0, 1, \ldots \) [e.g., according to (28)];
4. Adjust \( u_1(t) \) to a safe action \( u(t) \) using (7)-(9);
5. Apply \( u(t) \) to system and observe next state \( x(t + 1) \);
6. Calculate lifted state \( z(t + 1) = g(x(t + 1)) \);
7. Update Koopman model \((A_t, B_t)\) to \((A_{t+1}, B_{t+1})\) using (26)-(27);
8. \textbf{end for}

In Step 3, given the linear model \( z(t + k + 1) = A_t z(t + k) + B_t u_1(t + k) \), where \( k = 0, 1, \ldots \) correspond to predicted steps into the future, various strategies can be used to determine the control action \( u_1(t) \). Here, we consider a stabilization setting and a strategy based on MPC: At each time step \( t \), a sequence of actions, \( \{u_1(k|t)\}_{k=0}^{N-1} \), is determined according to:

\[
\begin{align*}
\min_{u_1(k|t)} \sum_{k=0}^{N-1} & \left( z(k|t)^\top Q z(k|t) + u_1(k|t)^\top R u_1(k|t) \right) + z(N|t)^\top Q_f z(N|t) \\
\text{s.t. } & z(k + 1|t) = A_t z(k|t) + B_t u_1(k|t) \\
& z(0|t) = g(x(t)) \quad (28)
\end{align*}
\]

where \( Q, R \) and \( Q_f \) are penalty matrices, and \( N \) is a selected prediction horizon; then, the first action, \( u_1(0|t) \), is applied, i.e., \( u_1(t) = u_1(0|t) \). This procedure is repeated at every step \( t \) in a receding horizon manner. Among other strategies, the MPC strategy (28) is chosen due to the following two reasons: 1) It does not have any constraints except for the dynamic equation and initial condition as equality constraints. Indeed, (28) is a discrete-time finite-horizon LQR problem – it admits a closed-form solution (Lewis, Vrabie, & Syrmos, 2012) and, thereby, solving for \( u_1(0|t) \) is computationally very easy. Note that the system with AG in the loop, (23), will never violate constraints (2). Therefore, if the learned Koopman linear model accurately reproduces the dynamics of (23), its response will never violate constraints [even though the constraints are not explicitly incorporated in the control determination process (28)].

Although the model may have certain errors, thanks to the AG supervision and adjustment of \( u_1(t) \) to \( u(t) \), the actual system response will never violate constraints. 2) We consider a finite-horizon LQR instead of an infinite-horizon LQR because for an infinite-horizon LQR solution to converge certain controllability condition of the pair \((A_t, B_t)\) is needed (Lewis et al., 2012), which may not always be satisfied during online data-driven update of \((A_t, B_t)\).
Remark 2 (Use Case). Potential use cases of the safe data-driven Koopman control method introduced in this section are similar to those of safe Q-learning stated in Remark 1, especially in applications where a set bound $W$ for uncertainty/disturbance $w(t)$ is known while the probability distribution of $w(t)$ in $W$, $P_w = P_w(x,u)$ [which also covers the case where $w$ is a deterministic function of $(x,u)$] is a priori unknown. Through online data-driven update of $(A_t,B_t)$, the statistical characteristics of $w(t)$ are contained in the learned Koopman model, and therefore control of higher performance can be determined with the learned model. Similar to Algorithm 2, the control determination process (28) can be extended from stabilization to a reference tracking setting. One major advantage of this safe data-driven Koopman control method over the safe Q-learning method is that, although both methods guarantee safety, the Koopman control method learns a (linear) model of the system based on which the control is determined, and this control determination process is more transparent, making it easier to build trust. However, in this Koopman control method, the eventual control performance is influenced, even significantly, by the choice of observables (Cibulka, Haniš, & Hromčík, 2019; Tafazzol et al., 2024). How to select (or, optimize the selection of) observables in Koopman-based control is still an open problem, which, although beyond the scope of this paper, will be investigated in future research.

5. Illustrative Example

In this section, we use a numerical example to illustrate the proposed AG framework for safe control and learning.

Consider the following discrete-time system:

$$ x(t+1) = Ax(t) + Bu(t) + Ew(t) $$

(29)

where $x(t) = [x_1(t), x_2(t)]^\top \in \mathbb{R}^2$ is a two-dimensional vector state, $u(t) \in \mathbb{R}$ is a scalar control action input, and $w(t) \in \mathbb{R}$ depends on the state as follows:

$$ w(t) = \sin(10x_1(t)). $$

(30)

Due to the expression (30) of $w(t)$, this system is nonlinear. The linearization about the origin yields a Jacobian $f_x(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which is different from the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in (29); while for $|x_1| \gg 0$, the term $w(t)$ changes between $-1$ and $1$ rapidly. A practical strategy for control of this system is to treat it as a linear system with a bounded disturbance $w(t) \in W = [-1,1]$. Following this strategy, for stabilization, one may design a linear controller as follows:

$$ u(t) = Kx(t) = [-0.2054, -0.7835]x(t) $$

(31)

where $K = [-0.2054, -0.7835]$ is the infinite-horizon LQR gain corresponding to $Q = \text{diag}(1,1)$ and $R = 10$. However, due to the nonlinearity $w(t)$ that takes values according to (30), such a linear controller can at most drive the state $x(t)$ to a
neighborhood of the origin, as shown by the blue curve in Fig. 2. Furthermore, we assume that the system operation must satisfy the following constraints:

\[-20 \leq x_1(t) \leq 20, \quad -4 \leq x_2(t) \leq 10, \quad -6 \leq u(t) \leq 6. \tag{32}\]

The linear controller (31) does not have the ability to handle these constraints. We use the generalized AG to handle constraints and improve control performance with safe learning.

5.1. Action governor design and safe control

The first step of designing a generalized AG for handling the constraints in (32) is to consider a nominal control policy $\pi_0$ for which there exists a nonempty safe and returnable set. We consider the following policy:

$$u_0(t) = \pi_0(x(t), v(t)) = K x(t) + L v(t) \tag{33}$$

where $K = [-0.2054, -0.7835]$ is the same gain matrix as in (31) such that the closed-loop matrix $\tilde{A} = A + BK$ is stable, and $L = -K [0, 1]^T = 0.2054$. Then, treating (29) as a linear system with a bounded disturbance $w(t) \in \mathcal{W} = [-1, 1]$ and following the approach in Section 3.1, we can compute the MOAS $\tilde{O}_\infty$ and set $\Pi_{\pi_0} = \tilde{O}_\infty$. The projection of the computed $\tilde{O}_\infty$ onto the state space, $\text{proj}_x(\tilde{O}_\infty)$, is shown in Fig. 2, where the black curve indicates the boundary. Then, we can design an AG that uses $\Pi_{\pi_0} = \tilde{O}_\infty$ and the online optimization problem (17) with $\text{dist}_{x(t)}(u_1(t), u) = |u_1(t) - u|$ to adjust control actions to enforce the constraints in (32).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{State trajectories under nominal control without AG, nominal control with AG, nominal control with CBF, and learned Koopman control with AG.}
\end{figure}

The trajectories of (29) starting with initial condition $x(0) = (14, 6)$ under nominal controller (31) without and with AG supervision are shown in Fig. 2 by the blue and red curves, respectively. It can be observed that without AG supervision, the trajectory
violates the constraints; while with AG supervision, the trajectory strictly satisfies the constraints. For comparison, we have also implemented a control barrier function (CBF) on this example. Because the system is discrete-time and has a relative degree of 2, we have adopted the approach of Xiong et al. (2022) to implement a discrete-time exponential CBF. The system trajectory under CBF guarding is shown in Fig. 2 by the magenta curve. Similar to the case with AG supervision, the trajectory under CBF guarding also strictly satisfies the constraints. However, we note that our generalized AG approach has guaranteed recursive feasibility when there are both state and control constraints (as shown in Proposition 3), while the CBF approach of Xiong et al. (2022) does not have such a recursive feasibility guarantee when there are constraints on the control input. This difference represents a significant advantage of our generalized AG approach over the CBF approach of Xiong et al. (2022) for handling state and control constraints simultaneously.

To illustrate Algorithm 1 for computing $\Pi_{\pi_0}$ for discrete systems, we also consider a discretized version of the closed-loop system. Specifically, we grid the range $[-25, 25] \times [-10, 15]$ for state $x$ with a resolution of $\Delta x_1 \times \Delta x_2 = 0.5 \times 0.5$, the range $[-25, 25]$ for reference input $v$ with a resolution of $\Delta v = 0.5$, and the range $[-1, 1]$ for disturbance input $w$ with a resolution of $\Delta w = 0.1$. The above ranges cover the safe operation ranges of $x$ and $u$ defined in (32) and the set bound for $w$, $W = [-1, 1]$. The discrete transition function $f$ maps each $(x, v, w)$ on the grids to $x^+$ on the grid that is closest to $\tilde{A}x + BLv + Ew$ measured by the Euclidean distance. Recall that Algorithm 1 requires an initial set $\Pi_{\pi_0}$ that satisfies the conditions in (18). For computing a valid $\Pi_{\pi_0}$, we consider the ellipsoidal set $\mathcal{E}(v) = \{x \in \mathbb{R}^2 : (x - x_v(v))^\top P^{-1}(x - x_v(v)) \leq 1\}$, where $x_v(v) = (I - \tilde{A})^{-1}BLv = \begin{bmatrix} v \\ 0 \end{bmatrix}$ is the steady state corresponding to constant $v$ and $w \equiv 0$, and $P$ is the solution to the discrete-time Lyapunov equation $\frac{1}{\alpha} \tilde{A}P\tilde{A}^\top - P + \frac{1}{1-\alpha} EE^\top = 0$, in which $\alpha = 0.75$ is a parameter that is chosen strictly between $\rho(\tilde{A})^2$, the square of the spectral radius of $\tilde{A}$, and 1. According to Theorem 3 of Polyak, Nazin, Topunov, and Nazin (2006), $\mathcal{E}(v)$ is positively invariant under the closed-loop dynamics $x^+ = \tilde{A}x + BLv + Ew$, constant $v$, and $w \in W = [-1, 1]$. Therefore, $\Pi_{\pi_0}$ can...
be constructed as the collection of all \((x, v)\) on the grids and in \(\bigcup_v (E(v), v)\) where the union is taken over all \(v \in V\) such that any \(x \in E(v)\) satisfies \((x, \pi_0(x, v), v) \in C\). With \(\Pi^0_{\pi_0}\), we can run Algorithm 1 to compute \(\Pi^0_{\pi_0}\). The projections of \(\Pi^0_{\pi_0}\) and computed \(\Pi^0_{\pi_0}\) onto the state space are shown in Fig. 3 by the blue and red points, respectively, where the boundary of projection of \(\bar{O}_\infty\) computed using the linear systems approach is also shown by the black dashed curve. The following observations can be made: 1) The safe set is significantly enlarged from the initial set \(\Pi^0_{\pi_0}\) (blue) to the final set \(\Pi^0_{\pi_0}\) (red) by Algorithm 1. 2) The final safe set computed using Algorithm 1 based on discrete approximation of the original continuous system almost overlaps with that computed using the linear systems approach. The differences are mainly due to discrete approximation errors (when next state is mapped to the nearest grid point). These two observations verify the effectiveness of Algorithm 1.

5.2. Safe online learning using action governor

Although with the AG supervision constraints are enforced, the system performance under the controller (31) is not satisfactory – as shown by the blue and red curves in Fig. 2, the state \(x(t) = (x_1(t), x_2(t))\) significantly oscillates around the origin, which is due to the nonlinearity \(w(t)\). In many practical applications, such nonlinearities as well as other possibly state-dependent disturbances are a priori unknown or hard to model. In such a case a viable option for improving control performance is through safe online learning. Here, we focus on illustrating the newly proposed safe data-driven Koopman control algorithm in Section 4.2.

Inspired by the approach proposed in Mamakoukas, Castano, Tan, and Murphey (2021) to selecting observables based on higher-order derivatives, for system (29) with nonlinearity (30), we consider the observables:

\[
  z(t) = g(x(t)) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \sin(10x_1(t)) \\ \sin(10x_1(t) + 10x_2(t)) \end{bmatrix}.
\]

(34)

Then, we implement Algorithm 3 with initial model:

\[
  A_0 = \begin{bmatrix} A \\ 0_{2 \times 2} \\ 0_{2 \times 2} \end{bmatrix}, \quad B_0 = \begin{bmatrix} B \\ 0_{2 \times 1} \end{bmatrix}
\]

(35)

so that the initial model is the linear model without accounting for the nonlinearity. For this example, we determine control \(u_1(t)\) according to infinite-horizon LQR with \(Q' = \text{diag}(1, 1, 0, 0)\) and \(R' = 10\). This choice aims to minimize the contribution of difference in cost functions to the difference in control performance – recall that the original control (31) is infinite-horizon LQR with \(Q = \text{diag}(1, 1)\) and \(R = 10\). We note that infinite-horizon LQR control can be obtained from the finite-horizon LQR formulation (28) by choosing the terminal cost matrix \(Q_f\) as the solution to the discrete-time algebraic Riccati equation. During learning, we reset the state \(x(t)\) uniformly at random within \(\text{proj}_x(\Pi_{\pi_0})\) every \(\Delta t = 20\) discrete-time steps to have trajectories sufficiently covering the safe operation range of \(x\). Similar to the average reward used in RL to monitor policy improvement, we consider the cost function \(c(t) = x(t)^\top Q x(t) + u(t)^\top R u(t) = \norm{x(t)}^2 + 10 |u(t)|^2\), which is the single-step cost of the LQR infinite-horizon cumulative cost, and calculate the average cost \(\bar{c}(t) = \frac{1}{t+1} \sum_{k=0}^t c(k)\).
to monitor control performance during learning. The average cost over learning is shown in Fig. 4. It can be observed that the average cost decreases and converges as learning proceeds, which indicates improvement of control performance. Due to the AG supervision, no constraint violation has occurred during the entire learning process. Therefore, a plot showing constraint violation during learning is omitted.

![Figure 4. Average cost during learning.](image)

The trajectories of system (29) starting with initial condition $x(0) = (14, 6)$ under the initial controller (31) and under the controls determined based on the learned Koopman model (both with AG supervision) are shown in Fig. 2 by the red and green curves, respectively. It can be observed that the learned Koopman control drives and maintains the state $x(t)$ within a significantly smaller neighborhood around the origin compared to the initial controller (31), illustrating the effectiveness of online learning for improving control performance.

6. Conclusions

This article introduced the generalized action governor, a supervisory scheme for augmenting a nominal closed-loop system with the capability of strictly handling safety constraints on system state and control variables. After presenting the generalized action governor algorithm and analyzing its properties for general systems, we introduced tailored, computationally-efficient design approaches for linear and discrete systems with bounded uncertainties. Then, we discussed the application of the generalized action governor to facilitate safe online learning, where we described two different safe learning algorithms based on the integration of 1) reinforcement learning and 2) data-driven Koopman operator-based control, respectively, with the generalized action governor. The developments were illustrated with a numerical example, which indicated that the generalized action governor provides a viable framework for realizing safe autonomy. Future work includes applying the introduced generalized action governor framework to addressing more practical and challenging safety-critical control engineering problems.
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