A numerical method based on the reproducing kernel Hilbert space method for the solution of fifth-order boundary-value problems

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Abstract: In this paper, we present a fast and accurate numerical scheme for the solution of fifth-order boundary-value problems. We apply the reproducing kernel Hilbert space method (RKHSM) for solving this problem. The analytical results of the equations have been obtained in terms of convergent series with easily computable components. We compare our results with spline methods, decomposition method, variational iteration method, Sinc-Galerkin method and homotopy perturbation methods. The comparison of the results with exact ones is made to confirm the validity and efficiency.

Keywords: Reproducing kernel method, Series solutions, fifth-order boundary-value problems, Reproducing kernel space.

1. Introduction

In this work we consider the numerical approximation for the fifth-order boundary-value problems of the form

\[ y^{(v)} = f(x)y + g(x), \quad x \in [a, b], \]  \hspace{1cm} (1.1)

with boundary conditions

\[ y(a) = A_0, \quad y'(a) = A_1, \quad y''(a) = A_2, \quad y(b) = B_0, \quad y'(b) = B_1, \]  \hspace{1cm} (1.2)

where the functions \( f(x) \) and \( g(x) \) are continuous on \([a, b]\) and \( A_0, A_1, A_2, B_0, B_1 \) are finite real constants. For more details about computational code of boundary value problems, the reader is referred to [1-3].
This type of boundary-value problems arise in the mathematical modelling of viscoelastic flows and other branches of mathematical, physical and engineering sciences [4,5] and references therein. Theorems which list the conditions for the existence and uniqueness of solutions of such problems are thoroughly discussed in a book by Agarwal [6]. Khan [7] investigated the fifth-order boundary-value problems by using finite difference methods. Wazwaz [8] applied Adomian decomposition method for solution of such type of boundary-value problems. The use of spline function in the context of fifth-order boundary-value problems was studied by Fyfe [9], who used the quintic polynomial spline functions to develop consistency relations connecting the values of solution with fifth-order derivatives at the respective nodes. Polynomial sextic spline functions were used [10] to develop the smooth approximations to the solution of problems (1.1) and (1.2). Caglar et al. [11] have used sixth-degree B-spline functions to develop first-order accurate method for the solution two-point special fifth-order boundary-value problems. Noor and Mohyud-Din [12,13] applied variational iteration and homotopy perturbation methods for solving the problems (1.1) and (1.2), respectively. Khan [14] have used the non-polynomial sextic spline functions for the solution fifth-order boundary-value problems. El-Gamel [15] employed the sinc-Galerkin method to solve the problems (1.1) and (1.2). Lamnii et al. [16] developed and analyzed two sextic spline collocation methods for the problem. Siddiqi et al. [17,18] used the non-polynomial sextic spline method for special fifth-order problems (1.1) and (1.2). Wang et al. [19] attempted to obtain upper and lower approximate solutions of such problems by applying the sixth-degree B-spline residual correction method.

In this paper, the RKHSM [20,21] will be used to investigate the fifth-order boundary-value problems. In recent years, a lot of attention has been devoted to the study of RKHSM to investigate various scientific models. The RKHSM which accurately computes the series solution is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that can be elegantly computed. The efficiency of the method was used by many authors to investigate several scientific applications. Geng and Cui [22] applied the RKHSM to handle the second-order boundary value problems. Yao and Cui [23] and Wang et al. [24] investigated a class of singular boundary value problems by this method and the obtained results were good. Zhou et al. [25] used the RKHSM effectively to solve second-order boundary value problems. In
the method was used to solve nonlinear infinite-delay-differential equations. Wang and Chao [27], Li and Cui [28], Zhou and Cui [29] independently employed the RKHSM to variable-coefficient partial differential equations. Geng and Cui [30], Du and Cui [31] investigated the approximate solution of the forced Duffing equation with integral boundary conditions by combining the homotopy perturbation method and the RKHSM. Lv and Cui [32] presented a new algorithm to solve linear fifth-order boundary value problems. In [33,34], authors developed a new existence proof of solutions for nonlinear boundary value problems. Cui and Du [35] obtained the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations by using the reproducing kernel Hilbert space method. Wu and Li [36] applied iterative reproducing kernel method to obtain the analytical approximate solution of a nonlinear oscillator with discontinuities. Recently, the method was applied the fractional partial differential equations and multi-point boundary value problems [34-37]. For more details about RKHSM and the modified forms and its effectiveness, see [20-43] and the references therein.

The paper is organized as follows. Section 2 is devoted to several reproducing kernel spaces and a linear operator is introduced. Solution representation in $W^5_2[a, b]$ has been presented in Section 3. It provides the main results, the exact and approximate solution of (1.1) and an iterative method are developed for the kind of problems in the reproducing kernel space. We have proved that the approximate solution converges to the exact solution uniformly. Some numerical experiments are illustrated in Section 4. There are some conclusions in the last section.

2. Preliminaries

2.1. Reproducing Kernel Spaces

In this section, we define some useful reproducing kernel spaces.

**Definition 2.1.** (Reproducing kernel). Let $E$ be a nonempty abstract set. A function $K : E \times E \rightarrow C$ is a reproducing kernel of the Hilbert space $H$ if and only if

$$
\begin{align*}
\forall t \in E, \ K(.,t) \in H, \\
\forall t \in E, \forall \varphi \in H, \ \langle \varphi(.), K(.,t) \rangle = \varphi(t).
\end{align*}
$$

The last condition is called “the reproducing property”: the value of the function $\varphi$ at the point $t$ is reproduced by the inner product of $\varphi$ with $K(.,t)$.
Definition 2.2.

\[ W^6_2[0,1] = \left\{ u(x) \mid u(x), u'(x), u''(x), u'''(x), u^{(4)}(x), u^{(5)}(x) \right\}, \]

where \( u(x) \) are absolutely continuous in \([0,1]\), \( u^{(6)}(x) \in L^2[0,1], x \in [0,1] \), \( u(0) = u(1) = u'(0) = u'(1) = 0 = u''(0) = 0 \).

The sixth derivative of \( u(x) \) exists almost everywhere since \( u^{(5)}(x) \) is absolutely continuous.

The inner product and the norm in \( W^6_2[0,1] \) are defined respectively by

\[
\langle u(x), g(x) \rangle_{W^6_2} = \sum_{i=0}^{5} u^{(i)}(0)g^{(i)}(0) + \int_{0}^{1} u^{(6)}(x)g^{(6)}(x)dx, \quad u(x), g(x) \in W^6_2[0,1],
\]

and

\[
\|u\|_{W^6_2} = \sqrt{\langle u, u \rangle_{W^6_2}}, \quad u \in W^6_2[0,1].
\]

The space \( W^6_2[0,1] \) is a reproducing kernel space, i.e., for each fixed \( y \in [0,1] \) and any \( u(x) \in W^6_2[0,1] \), there exists a function \( R_y(x) \) such that

\[
u(y) = \langle u(x), R_y(x) \rangle_{W^6_2}.
\]

Definition 2.3.

\[ W^1_2[0,1] = \left\{ u(x) \mid u(x) \text{ is absolutely continuous in } [0,1] \right\}, \]

where \( u'(x) \in L^2[0,1], x \in [0,1] \).

The inner product and the norm in \( W^1_2[0,1] \) are defined respectively by

\[
\langle u(x), g(x) \rangle_{W^1_2} = u(0)g(0) + \int_{0}^{1} u'(x)g'(x)dx, \quad u(x), g(x) \in W^1_2[0,1],
\]

and

\[
\|u\|_{W^1_2} = \sqrt{\langle u, u \rangle_{W^1_2}}, \quad u \in W^1_2[0,1].
\]

The space \( W^1_2[0,1] \) is a reproducing kernel space and its reproducing kernel function \( T_x(y) \) is given by
Theorem 2.1. The space $W^6_2[0, 1]$ is a reproducing kernel Hilbert space whose reproducing kernel function is given by,

$$R_y(x) = \begin{cases} 
\sum_{i=1}^{12} c_i(y) x^{i-1}, & x \leq y, \\
\sum_{i=1}^{12} d_i(y) x^{i-1}, & x > y,
\end{cases}$$

where

$$c_1(y) = 0,$$

$$c_2(y) = 0,$$

$$c_3(y) = 0,$$

$$c_4(y) = -\frac{158419}{1353663648} y^5 - \frac{158419}{8121981888} y^6 + \frac{99305}{14213468304} y^7,$$

$$c_5(y) = -\frac{157481}{341123239296} y^5 - \frac{335}{16243963776} y^8 + \frac{728021}{2707327296} y^4 - \frac{11725}{84603978} y^3 - \frac{196265}{170561619648} y^9 - \frac{2687}{42640404912} y^{11},$$

$$T_x(y) = \begin{cases} 
1 + x, & x \leq y, \\
1 + y, & x > y.
\end{cases}$$

(2.2)
\[ c_6(y) = \frac{4056701}{67683182400} y^5 - \frac{2011}{126905967} y^6 + \frac{158419}{284269366080} y^7 \]
\[ + \frac{11843}{304574320800} y^8 + \frac{2461}{284269366080} y^9 - \frac{158419}{1353663648} y^{10} \]
\[ + \frac{2461}{42301989} y^{11} - \frac{168263}{1705616196480} y^{12} - \frac{8999}{1705616196480} y^{13}, \]

\[ c_7(y) = \frac{4056701}{406099094400} y^5 - \frac{2011}{7614358020} y^6 + \frac{158419}{1705616196480} y^7 \]
\[ + \frac{11843}{1827445924800} y^8 + \frac{2461}{1705616196480} y^9 - \frac{158419}{8121981888} y^{10} \]
\[ + \frac{2461}{253811934} y^{11} - \frac{168263}{1023697178880} y^{12} - \frac{8999}{1023697178880} y^{13}, \]

\[ c_8(y) = \frac{158419}{284269366080} y^5 + \frac{158419}{1705616196480} y^6 - \frac{19861}{596965668768} y^7 \]
\[ - \frac{157481}{71635880252160} y^8 + \frac{67}{682246478592} y^9 - \frac{104003}{81219818880} y^{10} \]
\[ + \frac{335}{507623868} y^{11} + \frac{39253}{7163588025216} y^{12} + \frac{2687}{895485031520} y^{13}, \]

\[ c_9(y) = \frac{2461}{284269366080} y^5 + \frac{2461}{1705616196480} y^6 + \frac{67}{682246478592} y^7 \]
\[ - \frac{8321}{71635880252160} y^8 - \frac{1465}{2387862675072} y^9 - \frac{335}{16243963776} y^{10} \]
\[ + \frac{12221}{1137077464320} y^{11} + \frac{11251}{28654352100864} y^{12} + \frac{2003}{143271760504320} y^{13}, \]
\[ c_{10}(y) = -\frac{168263}{1705616196480} y^5 - \frac{168263}{10233697178880} y^6 + \frac{39253}{716358802516} y^7 \\
+ \frac{38153}{85963056302592} y^{10} + \frac{11251}{28654352100864} y^8 - \frac{196265}{170561619648} y^4 \\
+ \frac{56255}{21320202456} y^3 - \frac{6313}{5372691018912} y^9 - \frac{12751}{214907640756480} y^{11} - \frac{1}{725760} y^2, \]

\[ c_{11}(y) = \frac{11843}{304574320800} y^5 + \frac{11843}{1827445924800} y^6 - \frac{157481}{7163588025160} y^7 \\
- \frac{45611}{268634550945600} y^{10} - \frac{8321}{7163588025160} y^8 + \frac{157481}{341123239296} y^4 \\
- \frac{8321}{10660101228} y^3 + \frac{38153}{85963056302592} y^9 + \frac{49001}{214907640756480} y^{11} + \frac{1}{3628800} y^1, \]

\[ c_{12}(y) = -\frac{8999}{1705616196480} y^5 - \frac{8999}{10233697178880} y^6 + \frac{2687}{8954485031520} y^7 \\
+ \frac{49001}{214907640756480} y^{10} + \frac{2003}{143271760504320} y^8 - \frac{2687}{42640404912} y^4 \\
+ \frac{2003}{21320202456} y^3 - \frac{12751}{214907640756480} y^9 - \frac{725}{236398404832128} y^{11} - \frac{1}{39916800}, \]

\[ d_1(y) = -\frac{1}{39916800} y^{11}, \]

\[ d_2(y) = \frac{1}{3628800} y^{10}, \]

\[ d_3(y) = -\frac{1}{725760} y^9, \]
\begin{align*}
d_4(y) &= \frac{12221}{169207956} y^3 + \frac{12221}{1137077464320} y^8 + \frac{2461}{42301989} y^5 \\
&\quad + \frac{2461}{253811934} y^6 + \frac{335}{507623868} y^7 - \frac{8321}{10660101228} y^{10} \\
&\quad - \frac{11725}{84603978} y^4 + \frac{56255}{21320202456} y^9 + \frac{2003}{21320202456} y^{11},
\end{align*}

\begin{align*}
d_5(y) &= \frac{728021}{2707327296} y^4 - \frac{104003}{81219818880} y^7 - \frac{158419}{1353663648} y^5 \\
&\quad - \frac{158419}{8121981888} y^6 + \frac{157481}{3411123239296} y^{10} - \frac{335}{16243963776} y^8 \\
&\quad - \frac{11725}{84603978} y^3 - \frac{196265}{170561619648} y^9 - \frac{2687}{42640404912} y^{11},
\end{align*}

\begin{align*}
d_6(y) &= \frac{4056701}{67683182400} y^5 + \frac{4056701}{406099094400} y^6 + \frac{158419}{28426936608} y^7 \\
&\quad + \frac{11843}{304574320800} y^{10} + \frac{2461}{284269366080} y^8 - \frac{158419}{1353663648} y^4 \\
&\quad + \frac{2461}{42301989} y^3 - \frac{168263}{1705616196480} y^9 - \frac{8999}{1705616196480} y^{11},
\end{align*}

\begin{align*}
d_7(y) &= -\frac{2011}{6768318240} y^5 - \frac{2011}{7614358020} y^6 + \frac{158419}{1705616196480} y^7 \\
&\quad + \frac{11843}{1827445924800} y^{10} + \frac{2461}{1705616196480} y^8 - \frac{158419}{8121981888} y^4 \\
&\quad + \frac{2461}{253811934} y^3 - \frac{168263}{10233697178880} y^9 - \frac{8999}{10233697178880} y^{11},
\end{align*}
\[ d_8(y) = \frac{158419}{284269366080} y^5 + \frac{158419}{1705616196480} y^6 - \frac{19861}{596965668768} y^7 - \frac{157481}{71635880252160} y^{10} + \frac{67}{682246478592} y^8 + \frac{99305}{14213468304} y^4 + \frac{335}{507623868} y^3 + \frac{39253}{7163588025216} y^9 + \frac{2687}{8954485031520} y^{11}, \]

\[ d_9(y) = \frac{2461}{284269366080} y^5 + \frac{2461}{1705616196480} y^6 + \frac{67}{682246478592} y^7 - \frac{8321}{71635880252160} y^{10} - \frac{1465}{2387862675072} y^8 - \frac{335}{16243963776} y^4 - \frac{7325}{1776683538} y^3 + \frac{11251}{28654352100864} y^9 + \frac{2003}{1432717605432} y^{11}, \]

\[ d_{10}(y) = -\frac{168263}{1705616196480} y^5 - \frac{168263}{10233697178880} y^6 + \frac{39253}{7163588025216} y^7 + \frac{38153}{85963056302592} y^{10} + \frac{11251}{28654352100864} y^8 - \frac{196265}{170561619648} y^4 + \frac{56255}{21320202456} y^3 - \frac{6313}{5372691018912} y^9 - \frac{12751}{214907640756480} y^{11}, \]

\[ d_{11}(y) = \frac{11843}{304574320800} y^5 + \frac{11843}{1827445924800} y^6 - \frac{157481}{71635880252160} y^7 - \frac{45611}{268634550945600} y^{10} - \frac{8321}{71635880252160} y^8 + \frac{157481}{341123239296} y^4 - \frac{8321}{10660101228} y^3 + \frac{38153}{85963056302592} y^9 + \frac{49001}{2149076407564800} y^{11}, \]
\[ d_{12}(y) = -\frac{2687}{42640404912} y^4 + \frac{2003}{21320202456} y^3 - \frac{8999}{1705616196480} y^5 \]
\[-\frac{8999}{10233697178880} y^6 + \frac{2687}{8954485031520} y^7 + \frac{2003}{143271760504320} y^8 \]
\[-\frac{12751}{214907640756480} y^9 - \frac{725}{236398404832128} y^{11} + \frac{49001}{2149076407564800} y^{10}, \]

**Proof:**

\[ \langle u(x), R_y(x) \rangle_{W^6_2} = \sum_{i=0}^{5} u^{(i)}(0) R_y^{(i)}(0) + \int_0^1 u^{(6)}(x) R_y^{(6)}(x) dx, \tag{2.3} \]
\[ (u(x), R_y(x) \in W^6_2[0,1]), \]

Through several integrations by parts for (2.3) we have

\[ \langle u(x), R_y(x) \rangle_{W^6_2} = \sum_{i=0}^{5} u^{(i)}(0) \left[ R_y^{(i)}(0) - (-1)^{(5-i)} R_y^{(11-i)}(0) \right] \tag{2.4} \]
\[ + \sum_{i=0}^{5} (-1)^{(5-i)} u^{(i)}(1) R_y^{(11-i)}(1) + \int_0^1 u(x) R_y^{(12)}(x) dx. \]

Note that property of the reproducing kernel

\[ \langle u(x), R_y(x) \rangle_{W^6_2} = u(y), \]

If

\[ \begin{align*} 
R_y^{(5)}(0) - R_y^{(6)}(0) &= 0, \\
R_y^{(4)}(0) + R_y^{(7)}(0) &= 0, \\
R_y^{(11)}(0) - R_y^{(8)}(0) &= 0, \\
R_y^{(6)}(1) &= 0, \\
R_y^{(7)}(1) &= 0, \\
R_y^{(8)}(1) &= 0, \\
R_y^{(9)}(1) &= 0, \\
R_y^{(10)}(1) &= 0, \\
R_y^{(11)}(1) &= 0. \tag{2.5} \end{align*} \]

then (2.4) implies that,
\( R_{y}^{(12)}(x) = \delta(x - y), \)

When \( x \neq y, \)

\( R_{y}^{(12)}(x) = 0, \)

therefore

\[
R_{y}(x) = \begin{cases} 
\sum_{i=1}^{12} c_i(y)x^{i-1}, & x \leq y, \\
\sum_{i=1}^{12} d_i(y)x^{i-1}, & x > y.
\end{cases}
\]

Since

\( R_{y}^{(12)}(x) = \delta(x - y), \)

we have

\[
\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \tag{2.6}
\]

and

\[
\partial^{11} R_{y^+}(y) - \partial^{11} R_{y^-}(y) = 1. \tag{2.7}
\]

Since \( R_{y}(x) \in W^6_2[0, 1], \) it follows that

\[
R_{y}(0) = 0, \quad R_{y}(1) = 0, \quad R_{y}'(0) = 0, \quad R_{y}'(1) = 0, \quad R_{y}''(0) = 0, \quad R_{y}''(1) = 0. \tag{2.8}
\]

From (2.5)-(2.8), the unknown coefficients \( c_i(y) \) ve \( d_i(y) (i = 1, 2, \ldots, 12) \) can be obtained. Thus for \( x \leq y, \) \( R_{y}(x) \) is given by,
\[ R_y(x) = \frac{2461}{42301989}x^3y^5 + \frac{2461}{253811934}x^3y^6 + \frac{335}{507623868}x^3y^7 - \frac{8321}{10660101228}x^3y^{10} \]
\[- \frac{7325}{1776683538}x^3y^8 - \frac{11725}{84603978}x^3y^4 + \frac{12221}{169207956}x^3y^3 + \frac{56255}{21320202456}x^3y^9 \]
\[ + \frac{2003}{21320202456}x^3y^{11} - \frac{158419}{1353663648}x^4y^5 - \frac{158419}{8121981888}x^4y^6 + \frac{99305}{14213468304}x^4y^7 \]
\[ + \frac{157481}{341123239296}x^4y^{10} - \frac{335}{16243963776}x^4y^8 + \frac{728021}{2707327296}x^4y^4 - \frac{11725}{84603978}x^4y^3 \]
\[ - \frac{196265}{170561619648}x^4y^9 - \frac{2687}{42640404912}x^4y^{11} + \frac{4056701}{6763182400}x^5y^5 - \frac{2011}{126905967}x^5y^6 \]
\[ + \frac{158419}{284269366080}x^5y^7 + \frac{11843}{30457320800}x^5y^{10} + \frac{2461}{284269366080}x^5y^8 - \frac{158419}{135366364}x^5y^4 \]
\[ + \frac{2461}{42301989}x^5y^3 - \frac{168263}{1705616196480}x^5y^9 - \frac{8999}{1705616196480}x^5y^{11} + \frac{4056701}{40609909440}x^6y^5 \]
\[ - \frac{2011}{7614358020}x^6y^6 + \frac{158419}{1705616196480}x^6y^7 + \frac{11843}{182745924800}x^6y^{10} + \frac{2461}{170561619}x^6y^8 \]
\[ - \frac{158419}{8121981888}x^6y^4 + \frac{2461}{253811934}x^6y^3 - \frac{168263}{1023369717888}x^6y^9 - \frac{8999}{1023369717888}x^6y^{11} \]
\[ + \frac{158419}{284269366080}x^7y^5 + \frac{158419}{1705616196480}x^7y^6 - \frac{19861}{596956568768}x^7y^7 - \frac{157481}{716358802}x^7y^{10} \]
\[ + \frac{67}{682246478592}x^7y^8 - \frac{104003}{81219818880}x^7y^4 + \frac{335}{507623868}x^7y^3 + \frac{39253}{7163588025216}x^7y^9 \]
\[ + \frac{2687}{8954485031520}x^7y^{11} + \frac{2461}{284269366080}x^8y^5 + \frac{2461}{1705616196480}x^8y^6 + \frac{67}{682246478}x^8y^7 \]
3. Solution representation in $W_2^6[0, 1]$

In this section, the solution of equation (1.1) is given in the reproducing kernel space $W_2^6[0, 1]$.

On defining the linear operator $L : W_2^6[0, 1] \rightarrow W_2^1[0, 1]$ as

$$Lu = u^{(5)}(x) - f(x)u(x).$$

Model problem (1.1) changes the following problem:
\[
\begin{aligned}
Lu = K(x), \quad x \in [0, 1] \\
u(a) = 0, \quad u'(a) = 0, \quad u''(a) = 0, \quad u(b) = 0, \quad u'(b) = 0.
\end{aligned}
\] (3.1)

3.1. The Linear boundedness of operator \(L\).

**Lemma 3.1.** If \(u(x) \in W^6_2[a, b]\), then \(\|u^{(k)}(x)\|_{L^\infty} \leq M_k \|u(x)\|_{W^6_2}\), where \(M_k (k = 0, 1, \ldots, 5)\) are positive constants.

**Proof:** For any \(x \in [a, b]\) it holds that

\[
\|R_x(y)\|_{W^6_2} = \sqrt{\langle R_x(y), R_x(y) \rangle_{W^6_2}} = R_x(x),
\]

from the continuity of \(R_x(x)\), there exists a constant \(M_0 > 0\), such that \(\|R_x(y)\|_{W^6_2} \leq M_0\).

By (2.1) one gets

\[
|u(x)| = \left| \langle u(y), R_x(y) \rangle_{W^6_2} \right| \leq \|R_x(y)\|_{W^6_2} \|u(y)\|_{W^6_2} \leq M_0 \|u(y)\|_{W^6_2}.
\] (3.2)

For any \(x, y \in [a, b]\), there exists \(M_k (k = 1, 2, \ldots, 5)\), such that

\[
\left\| R^{(k)}_x(y) \right\|_{W^6_2} \leq M_k \quad (k = 1, 2, \ldots, 5),
\]

we have

\[
\left| u^{(k)}(x) \right| = \left| \langle u(y), R^{(k)}_x(y) \rangle_{W^6_2} \right| \leq \left\| R^{(k)}_x(y) \right\|_{W^6_2} \|u(y)\|_{W^6_2} \leq M_k \|u(y)\|_{W^6_2} \quad (k = 1, 2, \ldots, 5).
\] (3.3)

Combining (3.2) and (3.3), it follows that

\[
\left\| u^{(k)}(x) \right\|_{L^\infty} \leq M_k \|u(y)\|_{W^6_2} \quad (k = 1, 2, \ldots, 5).
\]

**Theorem 3.1.** Suppose \(f_i' \in L^2[a, b] \quad (i = 0, 1, 2, 3, 4)\), Then \(L : W^6_2[a, b] \rightarrow W^1_2[a, b]\) is a bounded linear operator.

**Proof:** (i) By the definition of the operator it is clear that \(L\) is a linear operator.
(ii) Due to the definition of $W^1_2[a,b]$, we have

\[ \|(Lu)(x)\|_{W^1_2}^2 = \langle (Lu)(x), (Lu)(x) \rangle_{W^1_2} \]

\[ = [(Lu)(a)]^2 + \int_a^b [(Lu)'(x)]^2 dx \]

\[ = \left[ \sum_{i=0}^{5} f_i(a)u^{(i)}(a) \right]^2 + \int_a^b \left[ \left( \sum_{i=0}^{5} f_i(x)u^{(i)}(x) \right)^2 \right] dx. \]

\[ \int_a^b [(Lu)'(x)]^2 dx = \int_a^b \left[ u^{(6)}(x) + \sum_{i=0}^{4} (f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x)) \right]^2 dx \]

\[ = \int_a^b \left[ u^{(6)}(x) \right]^2 dx + 2 \int_a^b \left[ u^{(6)}(x) \sum_{i=0}^{4} (f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x)) \right] dx \]

\[ + \int_a^b \left[ \sum_{i=0}^{4} (f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x)) \right]^2 dx, \]

where

\[ \int_a^b \left[ u^{(6)}(x) \right]^2 dx \leq \|u(x)\|_{W^6_2}^2, \]

and

\[ \int_a^b \left[ u^{(6)}(x) \sum_{i=0}^{4} (f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x)) \right] dx \]

\[ \leq \left\{ \int_a^b \left[ u^{(6)}(x) \right]^2 dx \right\}^2 \left\{ \int_a^b \left[ \sum_{i=0}^{4} (f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x)) \right]^2 dx \right\}^2. \]

By lemma (3.1) and $f_i'(x) \in L^2[a,b]$, we can obtain a constant $N > 0$, satisfying

\[ \int_a^b \left[ \sum_{i=0}^{4} (f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x)) \right]^2 dx \leq N(b-a) \|u(x)\|_{W^6_2}^2. \]

Furthermore one gets

\[ \int_a^b [(Lu)'(x)]^2 dx \leq \|u(x)\|_{W^6_2}^2 + 2\sqrt{N(b-a) \|u(x)\|_{W^6_2}^2} + N(b-a) \|u(x)\|_{W^6_2}^2, \]
let $G = \left(1 + 2\sqrt{N(b-a)} + N(b-a)\right) > 0$, then

$$\int_a^b [(Lu)'(x)]^2 \, dx \leq G \|u(x)\|^2_{W^6_2}.$$ 

Therefore $L$ is a bounded operator. So we obtain the result as required. \qed

**3.2. The normal orthogonal function system of $W^6_2[a, b]$**

We choose $\{x_i\}_{i=1}^\infty$ as any dense set in $[a, b]$ and let $\Psi_x(y) = L^*T_x(y)$, where $L^*$ is conjugate operator of $L$ and $T_x(y)$ is given by (2.2). Furthermore, for simplicity let $\Psi_i(x) = \Psi_{x_i}(x)$, namely,

$$\Psi_i(x) \overset{\text{def}}{=} \Psi_{x_i}(x) = L^*T_{x_i}(x).$$

Now several lemmas are given.

**Lemma 3.2.** $\{\Psi_i(x)\}_{i=1}^\infty$ is complete system of $W^6_2[a, b]$.

**Proof:** For $u(x) \in W^6_2[a, b]$, let $\langle u(x), \Psi_i(x) \rangle = 0 \ (i = 1, 2, ...)$, that is

$$\langle u(x), L^*T_{x_i}(x) \rangle = (Lu)(x_i) = 0.$$ 

Note that $\{x_i\}_{i=1}^\infty$ is the dense set in $[a, b]$, therefore $(Lu)(x) = 0$. It follows that $u(x) = 0$ from the existence of $L^{-1}$. \qed

**Lemma 3.3.** The following formula holds

$$\Psi_i(x) = (L\eta R_x(\eta))(x_i),$$

where the subscript $\eta$ of operator $L\eta$ indicates that the operator $L$ applies to function of $\eta$.

**Proof:**

\[
\Psi_i(x) = \langle \Psi_i(\xi), R_x(\xi) \rangle_{W^6_2[a, b]} \\
= \langle L^*T_{x_i}(\xi), R_x(\xi) \rangle_{W^6_2[a, b]} \\
= \langle (T_{x_i})(\xi), (L\eta R_x(\eta))(\xi) \rangle_{W^6_2[a, b]} \\
= (L\eta R_x(\eta))(x_i).
\]
This completes the proof. □

**Remark 3.1.** The orthonormal system \( \{ \Psi_i(x) \}_{i=1}^{\infty} \) of \( W_2^6[a, b] \) can be derived from Gram-Schmidt orthogonalization process of \( \{ \Psi_i(x) \}_{i=1}^{\infty} \),

\[
\Psi_i(x) = \sum_{k=1}^{i} \beta_{ik} \Psi_k(x), \quad (\beta_{ii} > 0, \ i = 1, 2, \ldots)
\] (3.7)

where \( \beta_{ik} \) are orthogonal coefficients.

In the following, we will give the representation of the exact solution of Eq.(1.1) in the reproducing kernel space \( W_2^6[a, b] \).

### 3.3. The structure of the solution and the main results

**Theorem 3.2.** If \( u(x) \) is the exact solution of Eq.(1.1), then

\[
u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} g(x_k) \Psi_i(x),
\]

where \( \{ x_i \}_{i=1}^{\infty} \) is a dense set in \([a, b]\).

**Proof:** From the (3.7) and uniqueness of solution of (1.1) (see [32]), we have

\[
u(x) = \sum_{i=1}^{\infty} \langle u(x), \Psi_i(x) \rangle_{W_2^6} \Psi_i(x)
\]

\[
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), L^* T_{x_k}(x) \rangle_{W_2^6} \Psi_i(x)
\]

\[
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Lu(x), T_{x_k}(x) \rangle_{W_2^6} \Psi_i(x)
\]

\[
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle g(x), T_{x_k}(x) \rangle_{W_2^6} \Psi_i(x)
\]

\[
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} g(x_k) \Psi_i(x). \quad \square
\]

Now the approximate solution \( u_n(x) \) can be obtained by truncating the \( n- \) term of the exact solution \( u(x) \),
\[ u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} g(x_k) \Psi_i(x). \]

**Theorem 3.3.** Assume \( u(x) \) is the solution of Eq.(1.1) and \( r_n(x) \) is the error between the approximate solution \( u_n(x) \) and the exact solution \( u(x) \). Then the error sequence \( r_n(x) \) is monotone decreasing in the sense of \( \| \cdot \|_{W^6_2} \) and \( \| r_n(x) \|_{W^6_2} \to 0 \) [23].

4. Numerical Results

In this section, four numerical examples are provided to show the accuracy of the present method. All computations are performed by Maple 13. The RKHSM does not require discretization of the variables, i.e., time and space, it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The accuracy of the RKHSM for the fifth-order boundary value problems is controllable and absolute errors are small with present choice of \( x \) (see Table 1-4). The numerical results we obtained justify the advantage of this methodology.

**Example 4.1.** ([8,11,12]). We first consider the linear boundary value problem

\[
\begin{cases}
y^{(5)}(x) = y - 15e^x - 10xe^x, & 0 < x < 1 \\
y(0) = 0, \; y'(0) = 1, \; y''(0) = 0, \; y(1) = 0, \; y'(1) = -e
\end{cases}
\]  

(4.1)

The exact solution of (4.1) is

\[ y(x) = x(1 - x)e^x. \]
If we homogenize the boundary conditions of (4.1), then the following (4.2) is obtained

\[
\begin{aligned}
\begin{aligned}
&u^{(5)}(x) - u(x) = 1 - 5e^x \left[ 2 - 3x + 3x^2\left(2 - \frac{5}{e}\right) + 4x^3\left(\frac{-3}{2} + \frac{4}{e}\right) \right] \\
&\quad -10e^x \left[ -3 + 6x\left(2 - \frac{5}{e}\right) + 12x^2\left(\frac{-3}{2} + \frac{4}{e}\right) \right] \\
&\quad -10e^x \left[ 6\left(2 - \frac{5}{e}\right) + 24x\left(\frac{-3}{2} + \frac{4}{e}\right) \right] - 5e^x \left[ 24\left(\frac{-3}{2} + \frac{4}{e}\right) \right] \\
&\quad -15e^x - 10xe^x, \\
&0 < x < 1
\end{aligned}
\end{aligned}
\]

\[u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \ u(1) = 0, \ u'(1) = 0\]

\[u(5)(x) - u(x) = 1 - 5e^x \left[ 2 - 3x + 3x^2\left(2 - \frac{5}{e}\right) + 4x^3\left(\frac{-3}{2} + \frac{4}{e}\right) \right] \]

\[\begin{aligned}
&\quad -10e^x \left[ -3 + 6x\left(2 - \frac{5}{e}\right) + 12x^2\left(\frac{-3}{2} + \frac{4}{e}\right) \right] \\
&\quad -10e^x \left[ 6\left(2 - \frac{5}{e}\right) + 24x\left(\frac{-3}{2} + \frac{4}{e}\right) \right] - 5e^x \left[ 24\left(\frac{-3}{2} + \frac{4}{e}\right) \right] \\
&\quad -15e^x - 10xe^x, \\
&0 < x < 1
\end{aligned}\)

\[u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \ u(1) = 0, \ u'(1) = 0\]

Example 4.2. ([8,12]). We now consider the nonlinear BVP

\[
\begin{aligned}
\begin{aligned}
&y^{(5)}(x) = e^{-x}y^2(x), \ 0 < x < 1 \\
&y(0) = 1, \ y'(0) = 1, \ y''(0) = 1, \ y(1) = e, \ y'(1) = e
\end{aligned}
\end{aligned}
\]

\[y(x) = e^x\]
If we homogenize the boundary conditions of (4.3), then the following (4.4) is obtained

$$\begin{align*}
&u^{(5)}(x) - 2\ e^{-x} \left[ 1 + e^x (x - \frac{x^2}{2} + x^3 (2 - \frac{5}{e}) + x^4 (\frac{-3}{2} + \frac{4}{e})) \right] u(x) \\
&= e^{-x} u^2(x) + e^{-x} \left[ 1 + e^x (x - \frac{x^2}{2} + x^3 (2 - \frac{5}{e}) + x^4 (\frac{-3}{2} + \frac{4}{e})) \right]^2 \\
&- e^x \left[ x - \frac{x^2}{2} + x^3 (2 - \frac{5}{e}) + x^4 (\frac{-3}{2} + \frac{4}{e}) \right] \\
&- 5e^x \left[ 1 - x + 3x^2 (2 - \frac{5}{e}) + 4x^3 (\frac{-3}{2} + \frac{4}{e}) \right] \\
&- 10e^x \left[ -1 + 6x (2 - \frac{5}{e}) + 12x^2 (\frac{-3}{2} + \frac{4}{e}) \right] \\
&- 10e^x \left[ 6(2 - \frac{5}{e}) + 24x (\frac{-3}{2} + \frac{4}{e}) \right] \\
&- 5e^x \left[ 24 (\frac{-3}{2} + \frac{4}{e}) \right], \quad 0 < x < 1
\end{align*}$$

$$u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \ u(1) = 0, \ u'(1) = 0$$

(4.4)

**Example 4.3.** ([15]). Consider the nonlinear BVP

$$\begin{align*}
y^{(5)}(x) &= -24e^{-y(x)} + \frac{48}{(1+x)}, \quad 0 < x < 1 \\
y(0) &= 0, \ y'(0) = 1, \ y''(0) = -1, \ y(1) = \ln 2, \ y'(1) = 0.5
\end{align*}$$

(4.5)

The exact solution of (4.5) is

$$y(x) = \ln(x + 1).$$

If we homogenize the boundary conditions of (4.5), then the following (4.6) is obtained

$$\begin{align*}
&u^{(5)}(x) = -24e^{-u(x)} + \frac{48}{(1+x)}, \\
&u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \ u(1) = 0, \ u'(1) = 0
\end{align*}$$

(4.6)
Example 4.4. ([15]). This is the nonlinear BVP

\[
\begin{align*}
\begin{cases}
y^{(5)}(x) + y^{(4)}(x) + e^{-2x}y^2(x) = 2e^x + 1 & 0 < x < 1 \\
y(0) = 0, \ y'(0) = 1, \ y''(0) = 1, \ y(1) = e, \ y'(1) = e
\end{cases}
\end{align*}
\] (4.7)

The exact solution of (4.7) is

\[y(x) = e^x.\]

If we homogenize the boundary conditions of (4.7), then the following (4.8) is obtained

\[
\begin{align*}
\begin{cases}
u^{(5)}(x) + u^{(4)}(x) = -e^{-2x}(u(x) + 1 + x + \frac{x^2}{2} + x^3(3e - 8) + x^4(\frac{11}{2} - 2e))^2 + 2e^x + 48e - 131 \\
u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \ u(1) = 0, \ u'(1) = 0
\end{cases}
\end{align*}
\] (4.8)

Remark 4.1. Lamnii et al. [16] solved the problem (5.1) by using sextic spline collocation method. He obtained the accurate approximate solutions of this problem for the small \(h\) values. Zhang [44] investigated approximate solution of the problem (5.1) by using variational iteration method. In addition, the same problem is solved by Noor and Mohyud-Din [12] previously and they got better results by using the variational iteration method.

Lv and Cui [32] studied only the linear fifth-order two-point boundary value problems by using reproducing kernel Hilbert space method. We use the RKHS for the same linear problem with different boundary conditions and also use different reproducing kernel function for computations. We also use the RKHS for the nonlinear problems.

Using our method we chose 36 points on \([0, 1]\). In Tables 1-4, we computed the absolute errors \(|u(x, t) - u_n(x, t)|\) at the points \(\{(x_i) : x_i = i, \ i = 0.0, 0.1, \ldots, 1.0\}\).
| $x$  | Exact Sol.      | RKHS | HPM [13] | B-Spline [11] | ADM [8] | Sinc [15] |
|------|----------------|------|----------|---------------|---------|-----------|
| 0.0  | 0.0000         | 0.000| 0.0000   | 0.0000        | 0.0000  | 0.0000    |
| 0.1  | 0.099465382    | 5.89 $\times 10^{-7}$ | 3 $\times 10^{-11}$ | 8.0 $\times 10^{-3}$ | 3 $\times 10^{-11}$ | 0.0000 |
| 0.2  | 0.195424441    | 1.73 $\times 10^{-8}$ | 2 $\times 10^{-10}$ | 1.2 $\times 10^{-3}$ | 2 $\times 10^{-10}$ | 0.1 $\times 10^{-5}$ |
| 0.3  | 0.283470349    | 6.02 $\times 10^{-7}$ | 4 $\times 10^{-10}$ | 5.0 $\times 10^{-3}$ | 4 $\times 10^{-10}$ | 0.3 $\times 10^{-5}$ |
| 0.4  | 0.358037927    | 7.42 $\times 10^{-7}$ | 8 $\times 10^{-10}$ | 3.0 $\times 10^{-3}$ | 8 $\times 10^{-10}$ | 0.3 $\times 10^{-5}$ |
| 0.5  | 0.412180317    | 3.32 $\times 10^{-7}$ | 1.2 $\times 10^{-9}$ | 8.0 $\times 10^{-3}$ | 1.2 $\times 10^{-9}$ | 0.0000 |
| 0.6  | 0.437308512    | 3.10 $\times 10^{-7}$ | 2 $\times 10^{-9}$ | 6.0 $\times 10^{-3}$ | 2 $\times 10^{-9}$ | 0.5 $\times 10^{-5}$ |
| 0.7  | 0.422888068    | 3.08 $\times 10^{-7}$ | 2.2 $\times 10^{-9}$ | 0.000         | 2.2 $\times 10^{-9}$ | 0.9 $\times 10^{-5}$ |
| 0.8  | 0.356086548    | 4.58 $\times 10^{-7}$ | 1.9 $\times 10^{-9}$ | 9.0 $\times 10^{-3}$ | 1.9 $\times 10^{-9}$ | 0.2 $\times 10^{-5}$ |
| 0.9  | 0.221364280    | 4.30 $\times 10^{-7}$ | 1.4 $\times 10^{-9}$ | 9.0 $\times 10^{-3}$ | 1.4 $\times 10^{-9}$ | 0.1 $\times 10^{-5}$ |
| 1.0  | 0.0000         | 2.36 $\times 10^{-13}$ | 0.0000   | 0.0000        | 0.0000  | 0.0000    |

**Table 1.** The absolute error of Example 5.1 for boundary conditions at $0.0 \leq x \leq 1.0$.  

| $x$  | Exact Solution | RKHS | HPM [13] | B-Spline[11] | ADM [8] | VIM [12] |
|------|----------------|------|----------|---------------|---------|-----------|
| 0.0  | 0.              | 0.000| 0.0000   | 0.0000        | 0.0000  | 0.0000    |
| 0.1  | 1.105170918     | 5.19 $\times 10^{-7}$ | 1 $\times 10^{-9}$ | 7.0 $\times 10^{-4}$ | 1 $\times 10^{-9}$ | 1 $\times 10^{-9}$ |
| 0.2  | 1.221402758     | 0.60 $\times 10^{-7}$ | 2 $\times 10^{-9}$ | 7.2 $\times 10^{-4}$ | 2 $\times 10^{-9}$ | 2 $\times 10^{-9}$ |
| 0.3  | 1.349858898     | 3.19 $\times 10^{-7}$ | 1 $\times 10^{-9}$ | 4.1 $\times 10^{-4}$ | 1 $\times 10^{-9}$ | 1 $\times 10^{-9}$ |
| 0.4  | 1.491824698     | 2.50 $\times 10^{-7}$ | 2 $\times 10^{-8}$ | 4.6 $\times 10^{-4}$ | 2 $\times 10^{-8}$ | 2 $\times 10^{-8}$ |
| 0.5  | 1.648721271     | 3.03 $\times 10^{-7}$ | 3.1 $\times 10^{-8}$ | 4.7 $\times 10^{-4}$ | 3.1 $\times 10^{-8}$ | 3.1 $\times 10^{-8}$ |
| 0.6  | 1.822118800     | 9.60 $\times 10^{-7}$ | 3.7 $\times 10^{-8}$ | 4.8 $\times 10^{-4}$ | 3.7 $\times 10^{-8}$ | 3.7 $\times 10^{-8}$ |
| 0.7  | 2.013752707     | 4.20 $\times 10^{-7}$ | 4.1 $\times 10^{-8}$ | 3.9 $\times 10^{-4}$ | 4.1 $\times 10^{-8}$ | 4.1 $\times 10^{-8}$ |
| 0.8  | 2.225540928     | 4.09 $\times 10^{-7}$ | 3.1 $\times 10^{-8}$ | 3.1 $\times 10^{-4}$ | 3.1 $\times 10^{-8}$ | 3.1 $\times 10^{-8}$ |
| 0.9  | 2.459603111     | 5.46 $\times 10^{-7}$ | 1.4 $\times 10^{-8}$ | 1.6 $\times 10^{-4}$ | 1.4 $\times 10^{-8}$ | 1.4 $\times 10^{-8}$ |
| 1.0  | 2.718281828     | 5.34 $\times 10^{-7}$ | 0.0000   | 0.0000        | 0.0000  | 0.0000    |

**Table 2.** The absolute error of Example 5.2 for boundary conditions at $0.0 \leq x \leq 1.0$.  

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| x     | Exact Solution | RKHS M | AE, 1.0E − 8 RKHS M | AE, 1.0E − 4 [15] Sinc-Galerkin |
|-------|---------------|--------|---------------------|---------------------------------|
| 0.0   | 0.0           | 0.0    | 0.0                 | 0.0                             |
| 0.0806| 0.0775164243  | 0.07751634304 | 0.003              | 0.0                             |
| 0.1648| 0.1525493985  | 0.1525493860 | 1.25               | 0.2                             |
| 0.2285| 0.2057939130  | 0.2057939053 | 0.77               | 0.2                             |
| 0.3999| 0.3364008055  | 0.3364007134 | 9.21               | 0.4                             |
| 0.5   | 0.4054651081  | 0.4054650667 | 4.14               | 0.1                             |
| 0.6923| 0.5260885504  | 0.5260885142 | 3.62               | 0.2                             |
| 0.7714| 0.5717701944  | 0.5717701752 | 1.92               | 0.3                             |
| 0.8836| 0.6331848394  | 0.6331848843 | 4.49               | 0.2                             |
| 0.9447| 0.6651077235  | 0.6651077287 | 0.52               | 0.5                             |
| 1.0   | 0.6931471806  | 0.6931471783 | 0.23               | 0.0                             |

**Table 3.** The absolute error (AE) of Example 5.3 for boundary conditions at $0.0 \leq x \leq 1.0.$
| $x$  | Exact Solution | RKHSM  | AE RKHSM | AE,1.0$E$ − 3 [15] Sinc-Galerkin |
|------|----------------|--------|----------|----------------------------------|
| 0.0  | 1.0            | 1.0    | 0.0      | 0.0                             |
| 0.0100 | 1.105170918    | 1.105170918 | 0.0 | 0.0                             |
| 0.1184 | 1.125694299    | 1.125694299 | 0.0 | 0.0                             |
| 0.1517 | 1.163811041    | 1.163811041 | 0.0 | 0.1                             |
| 0.2410 | 1.272521035    | 1.272521035 | 0.0 | 0.0                             |
| 0.3604 | 1.433902861    | 1.433902861 | 0.0 | 0.1                             |
| 0.4287 | 1.535260387    | 1.535260387 | 0.0 | 0.0                             |
| 0.5000 | 1.648721271    | 1.648721271 | 0.0 | 0.2                             |
| 0.6395 | 1.895532876    | 1.895532876 | 0.0 | 0.1                             |
| 0.8482 | 2.335439276    | 2.335439276 | 0.0 | 0.2                             |
| 0.9996 | 2.717194734    | 2.717194734 | $1 \times 10^{-9}$ | 0.2 |
| 1.0  | 2.718281828    | 2.718281828 | 0.0 | 0.0                             |

Table 4. The absolute error of Example 4 for boundary conditions at $0.0 \leq x \leq 1.0$.

Remark 4.2. The RKHSM tested on four problems, one linear and three nonlinear. A comparison with decomposition method by Wazwaz [8], sixth B-spline method by Caglar et al. [11], variational iteration and homotopy perturbation methods by Noor and Mohyiddin-Din [12,13] and sinc-galerkin method by Gamel [15] are made and it was seen that the present method yields good results (see Tables 1-4).

6. Conclusion

In this paper, we introduce an algorithm for solving the fifth-order problem with boundary conditions. For illustration purposes, chose four examples which were selected to show the computational accuracy. It may be concluded that, the RKHSM is very powerful and efficient in finding exact solution for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. The approximate solution obtained by the present method is uniformly convergent.
Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems [20-43] as well. However, if the problem becomes nonlinear, then the RKHSM does not require discretization or perturbation and it does not make closure approximation. Results of numerical examples show that the present method is an accurate and reliable analytical method for the fifth order problem with boundary conditions.

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