The Stieltjes constants, their relation to the $\eta_j$ coefficients, and representation of the Hurwitz zeta function

Mark W. Coffey

Received: February 18, 2009

Summary: The Stieltjes constants $\gamma_k(a)$ are the expansion coefficients in the Laurent series for the Hurwitz zeta function about its only pole at $s = 1$. We present the relation of $\gamma_k(1)$ to the $\eta_j$ coefficients that appear in the Laurent expansion of the logarithmic derivative of the Riemann zeta function about its pole at $s = 1$. We obtain novel integral representations of the Stieltjes constants and new decompositions such as $S_2(n) = S'_\gamma(n) + S_\Lambda(n)$ for the crucial oscillatory subsum of the Li criterion for the Riemann hypothesis. The sum $S'_\gamma(n)$ is $O(n)$ and we present various integral representations for it. We present novel series representations of $S_2(n)$. We additionally present a rapidly convergent expression for $\gamma_k = \gamma_k(1)$ and a variety of results pertinent to a parameterized representation of the Riemann and Hurwitz zeta functions.

1 Introduction and statement of results

In the Laurent expansion of the Riemann zeta function about $s = 1$,

$$\zeta(s) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s - 1)^n, \quad (1.1)$$

the Stieltjes constants $\gamma_k$ [9, 10, 22, 25, 27, 32] can be written in the form

$$\gamma_k = \lim_{N \to \infty} \left( \sum_{m=1}^{N} \frac{1}{m} \ln^k m - \frac{\ln^{k+1} N}{k+1} \right). \quad (1.2)$$

From the expansion around $s = 1$ of the logarithmic derivative of the zeta function,

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s - 1} - \sum_{p=0}^{\infty} \eta_p (s - 1)^p, \quad |s - 1| < 3, \quad (1.3)$$
we have
\[ \ln \zeta(s) = -\ln(s - 1) - \sum_{p=1}^{\infty} \frac{\eta_p - 1}{p} (s - 1)^p. \tag{1.4} \]

The constants \( \eta_j \) can be written as
\[ \eta_k = \frac{(-1)^k}{k!} \lim_{N \to \infty} \left( \sum_{m=1}^{N} \frac{1}{m} \Lambda(m) \ln^k m - \frac{\ln^{k+1} N}{k+1} \right), \tag{1.5} \]
where \( \Lambda \) is the von Mangoldt function \([15, 22, 24, 33, 30]\), defined by \( \Lambda(n) = \ln p \) if \( n = p^k \) for a prime number \( p \) and some integer \( k \geq 1 \), and \( \Lambda(n) = 0 \) otherwise. The radius of convergence of the expansion (1.3) is 3, as the first singularity encountered is the trivial zero of \( \zeta(s) \) at \( s = -2 \). The Dirichlet series corresponding to Eqs. (1.3) and (1.4) valid for \( \text{Re} \, s > 1 \) are
\[ \frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \quad \text{and} \quad \ln \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \ln n}. \tag{1.6} \]

Indeed, Eqs. (1.6) hold on \( \sigma = \text{Re} \, s = 1 \) if \( t = \text{Im} \, s \neq 0 \) \([33]\).

We note that there are various explicit expressions for the von Mangoldt function. These include
\[ \exp[\Lambda(n)] = \frac{\text{LCM}(1, \ldots, n)}{\text{LCM}(1, \ldots, n - 1)}, \tag{1.7} \]
where LCM is the least common multiple function, and Linnik’s identity \([28, \text{pp.} 21–22]\)
\[ \Lambda_1 = \frac{\Lambda(n)}{\ln n} = -\sum_{k=1}^{\ln n / \ln 2} \frac{(-1)^k}{k} \tau'_k(n). \tag{1.8} \]

In regard to Eq. (1.8), the strict \( \tau'_k \) and exact \( \tau_k \) divisor functions are related by
\[ \tau'_k(n) = |\{n_1, \ldots, n_k \geq 2; n_1 \cdots n_k = n\}| \tag{1.9a} \]
\[ = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \tau(\ell)(n). \tag{1.9b} \]

In particular, there is a finite sum on the right side of Eq. (1.8).

We recall that the function \( \ln \zeta(s) \) is intimately connected with the prime counting function \( \pi(x) \), the number of primes less than \( x \). We have
\[ \ln \zeta(s) = s \int_{2}^{\infty} \frac{\pi(x)}{x(x^s - 1)} \, dx. \tag{1.10} \]

Hence the behaviour of the function \( \pi(x) \) is related to the important coefficients \( \eta_j \). For further background on the classical zeta function we refer to standard texts \([15, 22, 24, 33, 30]\).
The Hurwitz zeta function, defined by $\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}$ for $\text{Re} \, s > 1$ and $\text{Re} \, a > 0$ extends to a meromorphic function in the entire complex $s$-plane. The generalization of the Laurent expansion (1.1) is

$$\zeta(s, a) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a)(s - 1)^n. \quad (1.11)$$

As shown by Eq. (1.1), by convention one takes $\gamma_k = \gamma_k(1)$.

Having set the notation above we may now state our main results.

**Proposition 1.1** For integers $k \geq 0$ we have

$$\frac{(-1)^k}{k!} \gamma_k - \eta_k = \frac{(-1)^k}{k!} \sum_{m=1}^{\infty} \frac{[1 - \Lambda(m)]}{m} \ln^k m, \quad (1.12)$$

where the sum on the right starts with $m = 2$ when $k > 0$. The case of $k = 0$ yields the identity

**Corollary 1.2**

$$\gamma = \frac{1}{2} \sum_{m=1}^{\infty} \frac{[1 - \Lambda(m)]}{m}, \quad (1.13)$$

where $\gamma$ is the Euler constant.

Put

$$S_2(n) \equiv - \sum_{m=1}^{n} \binom{n}{m} \eta_{m-1}. \quad (1.14)$$

Let $L_n^\alpha(x)$ be the Laguerre polynomial of degree $n$ (e.g., [4, 17]), and parameter $\alpha$ and $P_1(t) = B_1(t-[t]) = t-[t]-1/2$ the first periodized Bernoulli polynomial (e.g., [22, 33]). Then we have the series representation given in

**Proposition 1.3** For integers $n \geq 1$ we have (a)

$$S_2(n) = S_\gamma(n) + S_\Lambda(n), \quad (1.15)$$

where

$$S_\gamma(n) = \sum_{k=1}^{n} \frac{(-1)^k}{(k - 1)!} \binom{n}{k} \gamma_{k-1} = \int_1^{\infty} \frac{1}{t} L_{n-1}^1(\ln t) dP_1(t), \quad (1.16)$$

$$S_\Lambda(n) = \sum_{m=1}^{\infty} \frac{[1 - \Lambda(m)]}{m} L_{n-1}^1(\ln m)$$

$$\equiv n + S_{2\Lambda}(n) = n + \sum_{m=2}^{\infty} \frac{[1 - \Lambda(m)]}{m} L_{n-1}^1(\ln m), \quad (1.17)$$
\( S_{\gamma}(n) = O(n) \), (c) the average values

\[
\frac{1}{M} \sum_{n=1}^{M} S_{\gamma}(n) = \frac{1}{M} \int_{1}^{\infty} \frac{1}{t} L_{M-1}^{2}(\ln t) dP_1(t), \quad (1.18a)
\]

\[
\frac{1}{M} \sum_{n=1}^{M} S_{\Lambda}(n) = \frac{1}{M} \sum_{m=1}^{\infty} \frac{[1 - \Lambda(m)]}{m} L_{M-1}^{2}(\ln m), \quad (1.18b)
\]

and (d) for \( N \geq 1 \) a fixed integer

\[
S_{\gamma}(n) = -\sum_{\nu=1}^{N} \frac{L_{n-1}^{1}(\ln \nu)}{\nu} - L_{n}(\ln N) + 1 + \frac{1}{2N} L_{n-1}^{1}(\ln N) + O\left(\frac{1}{N^{2-\epsilon}}\right) L_{n-1}^{1}(1),
\]

where \( \epsilon > 0 \) is arbitrary.

We have found new integral representations of the Stieltjes constants, given in Proposition 1.4

\[ \text{Proposition 1.4} \] Let \( \text{Re } a > 0 \). Then we have (a)

\[
\gamma_k(a) = \frac{1}{2a} \ln^k a - \frac{\ln^{k+1} a}{k+1} + \frac{2}{a} \text{Re} \int_{0}^{\infty} \frac{(y/a - i) \ln^k(a - iy)}{(1 + y^2/a^2)(e^{2\pi y} - 1)} dy. \quad (1.19)
\]

(b) Let \( n \geq 1 \) be an integer. Then we have

\[
\gamma_k(a) = \sum_{m=0}^{n-1} \frac{\ln^k(m + a)}{m + a} + \frac{\ln^k(n + a)}{2(n + a)} - \frac{\ln^{k+1}(n + a)}{k + 1} \]

\[
+ \frac{2}{n + a} \text{Re} \int_{0}^{\infty} \frac{[y/(n + a) - i] \ln^k(n + a - iy)}{[1 + y^2/(n + a)^2](e^{2\pi y} - 1)} dy. \quad (1.20)
\]

(c)

\[
\gamma_k(a) = -\frac{\pi}{2} \frac{1}{k + 1} \text{Re} \int_{0}^{\infty} \frac{\ln^{k+1}(2a - 1 - it)}{\cosh^2(\pi t/2)} dt
\]

\[
+ \frac{\pi}{2} \frac{k+1}{2!} \sum_{j=1}^{k+1} \frac{\ln^2}{j!} \frac{(-1)^{j+1}}{(k - j + 1)!} \text{Re} \int_{0}^{\infty} \frac{\ln^{k-j+1}(2a - 1 - it)}{\cosh^2(\pi t/2)} dt. \quad (1.21)
\]

Let \( \psi(z) = \Gamma'(z)/\Gamma(z) \) be the digamma function, where \( \Gamma \) is the Gamma function \([1, 4, 17]\).

From Proposition 1.4 follows Corollaries 1.5 and 1.6. From the case \( m = 0 \) in Proposition 1.4 we have

\[ \text{Corollary 1.5} \] (a)

\[
\gamma_0(a) = -\psi(a) = \frac{1}{2a} - \ln a + \frac{2}{a} \text{Re} \int_{0}^{\infty} \frac{(y/a - i)}{(1 + y^2/a^2)(e^{2\pi y} - 1)} dy, \quad (1.22)
\]
(b) for integers \( n \geq 1 \),

\[
\gamma_0(a) = -\psi(a) = \sum_{m=0}^{n-1} \frac{1}{m+a} + \frac{1}{2(n+a)} - \ln(n+a)
+ \frac{2}{n+a} \operatorname{Re} \int_0^\infty \frac{[y/(n+a) - i]}{[1 + y^2/(n+a)^2](e^{2\pi y} - 1)}
\, dy, \tag{1.23}
\]

and (c)

\[
\gamma_0(a) = -\psi(a) = -\frac{\pi}{2} \operatorname{Re} \int_0^\infty \frac{\ln(2a - 1 - it)}{\cosh^2(\pi t/2)}
\, dt + \ln 2
= -\frac{\pi}{4} \int_0^\infty \frac{\ln(2a - 1)^2 + t^2)}{\cosh^2(\pi t/2)}
\, dt + \ln 2. \tag{1.24}
\]

Based upon the \( a = 1 \) case of Proposition 1.4 we obtain additional new integral representations of the sum \( S_\gamma \) defined in Eq. (1.16):

**Corollary 1.6**

\[
S_\gamma(n) = -\int_0^\infty \frac{1}{(1 + y^2)(e^{2\pi y} - 1)}
\left[(y - i)L_{n-1}^1[\ln(1 - iy)] + (y + i)L_{n-1}^1[\ln(1 + iy)]\right] \, dy, \tag{1.25}
\]

and

\[
S_\gamma(n) = \frac{\pi}{4} \int_0^\infty \left[L_n(2a - 1 - it) + L_n(2a - 1 + it) - 2\right]
\frac{dt}{\cosh^2(\pi t/2)}
- \frac{\pi}{4} \sum_{j=1}^n \frac{\ln^j 2}{j!} \left(\begin{array}{c} n \\ j \end{array}\right) \int_0^\infty \left\{1 F_1[j - n; j + 1; \ln(2a - 1 - it)]
+ 1 F_1[j - n; j + 1; \ln(2a - 1 + it)]\right\}
\frac{dt}{\cosh^2(\pi t/2)}, \tag{1.26}
\]

where \( 1 F_1 \) is the confluent hypergeometric function, with \( \left(\begin{array}{c} n \\ j \end{array}\right) 1 F_1[j - n; j + 1; x] = L_{n-j}^j(x) \).

Let \( d(n) \) be the number of divisors of \( n \). Then we have

**Proposition 1.7** (a) We have for integers \( m \geq 0 \)

\[
(-1)^m \sum_{n=1}^\infty \frac{\ln^n n}{n} [d(n) - \ln n - 2\gamma]
= (-1)^{m+1} \left(1 + \frac{2}{m+1}\right) \gamma_{m+1} - 2\gamma(-1)^m \gamma_m
+ \sum_{\ell=0}^m (-1)\ell \left(\begin{array}{c} m \\ \ell \end{array}\right) \gamma_{m-\ell}. \tag{1.27}
\]

In particular we have at \( m = 0 \)
Corollary 1.8

\[
\sum_{n=1}^{\infty} \frac{1}{n} [d(n) - \ln n - 2\gamma] = -3\gamma_1 - \gamma^2 = -3\gamma_1 - \frac{1}{4} \left( \sum_{m=1}^{\infty} \frac{[1 - \Lambda(m)]}{m} \right)^2,
\] (1.28)
giving

Corollary 1.9

\[
\sum_{m=1}^{\infty} \frac{d(m)}{m} - \lim_{n \to \infty} \left[ \frac{1}{2} \ln^2 n + 2\gamma \ln n \right] = \gamma^2 - 2\gamma_1 > 0.
\] (1.29)

(b) We have for integers \( m \geq 0 \)

\[
(-1)^m \sum_{n=1}^{\infty} \frac{\ln^m n}{n} [d(n) - \Lambda(n) \ln n - 2\gamma] = (m + 1)\eta_{m+1} + \frac{2(-1)^{m+1}}{m+1} \gamma_{m+1} - 2\gamma(-1)^m \gamma_m + \sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} \gamma_\ell \gamma_{m-\ell}.
\] (1.29)

In particular we have at \( m = 0 \)

Corollary 1.10

\[
\sum_{n=1}^{\infty} \frac{1}{n} [d(n) - \Lambda(n) \ln n - 2\gamma] = \eta_1 - 2\gamma_1 - \gamma^2 = 0.
\] (1.30)

We may obtain new series representations of \( S_\gamma(n) \) from part (a) and of \( S_2(n) \) from part (b). As an illustration we have

Corollary 1.11 For integers \( n \geq 1 \) we have

\[
S_2(n) = \gamma n + \sum_{m=2}^{\infty} \frac{L_{n-1}(\ln m)}{m \ln m} [d(m) - \Lambda(m) \ln m - 2\gamma] \\
- 2 \sum_{m=2}^{n} (-1)^m \binom{n}{m} \frac{\gamma_{m-1}}{(m-1)(m-1)!} \\
- 2\gamma \sum_{m=2}^{n} (-1)^m \binom{n}{m} \frac{\gamma_{m-2}}{(m-1)!} \\
+ \sum_{m=2}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{\ell=0}^{m-2} (-1)^\ell \binom{m-1}{\ell} \gamma_\ell \gamma_{m-\ell-2}.
\] (1.31)

Let \( \mu(n) \) be the Möbius function.
Proposition 1.12 Then we have (i)

\[ \eta_0 = -\gamma = \frac{1}{2} \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \ln^2 k, \]  

and for integers \( \ell \geq 1 \)

\[ (-1)^{\ell} \eta_{\ell} = \frac{1}{(\ell + 2)!} \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \ln^{\ell + 2} k + \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \sum_{p=1}^{\ell} \frac{\ln^p k}{p!} \frac{\gamma_{\ell-p+1}}{(\ell-p)!}. \]  

(ii) For integers \( j \geq 1 \) we have

\[ \eta_j = \frac{(-1)^{j+1}}{(j+1)!} \sum_{\ell=2}^{\infty} \frac{\mu(\ell)}{\ell} \ln^{j+2} \ell + \frac{(-1)^{j}}{j!} \gamma_j \]

\[ - \sum_{\ell=2}^{\infty} \frac{\mu(\ell)}{\ell} \sum_{n=0}^{j} \frac{(-1)^n}{n!} \gamma_n \frac{(-1)^j}{(j-n)!} \ln^{j-n+1} \ell. \]

From Proposition 1.12(i) follows

Corollary 1.13 We have

\[ S_2(n) = \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \ln k \left[ \frac{1}{n+1} L_n^1(\ln k) - 1 \right] \]

\[ + \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \sum_{\ell=1}^{n} (-1)^{\ell} \left( \frac{n}{\ell} \right) \sum_{p=0}^{\ell-1} \frac{\ln^p k}{p!} \frac{\gamma_{\ell-p}}{(\ell-p)!}. \]  

Proposition 1.14 Let \( B_j \) represent the Bernoulli numbers. For integers \( m \geq 1 \) and \( \Re \lambda > 0 \) we have

\[ \gamma_m = -\frac{m!}{1+\lambda} \sum_{\ell=1}^{m} \frac{B_{m-\ell+1}}{(m-\ell+1)!} \frac{\ln^{m-\ell}}{\ell!} \sum_{k=1}^{\infty} \frac{\lambda}{1+\lambda} \sum_{j=1}^{k} (-1)^{j} \binom{k}{j} \frac{1}{\lambda^j} \frac{\ln^{\ell}(j+1)}{(j+1)} \]

\[ - \frac{1}{(1+\lambda) \ln 2} \frac{1}{(m+1)} \sum_{k=1}^{\infty} \frac{\lambda}{1+\lambda} \sum_{j=1}^{k} (-1)^{j} \binom{k}{j} \frac{\ln^{m+1}(j+1)}{(j+1)} \]

\[ - \frac{B_{m+1}}{(m+1)} \ln^{m+1} 2. \]  

For \( m = 0 \) we have

\[ \gamma_0 = \gamma = \frac{1}{2} \ln 2 - \frac{1}{(1+\lambda) \ln 2} \sum_{k=1}^{\infty} \frac{\lambda}{1+\lambda} \sum_{j=1}^{k} (-1)^{j} \binom{k}{j} \frac{\ln(j+1)}{(j+1)}. \]  

(1.36)
Proposition 1.15 (Parameterized series representation of the Hurwitz zeta function)
For \( \Re s > 1, \Re a > 0, \) and \( \Re \lambda > 0 \) we have the representation
\[
\zeta(s, a) = \sum_{k=0}^{\infty} \frac{\lambda^k}{(1 + \lambda)^{k+1}} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{\lambda^j (a+j)^s}.
\] (1.37)

Proposition 1.16 Let the polylogarithm function \( \text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z \Phi(z, s, 1) \) where \( \Phi \) is the Lerch zeta function, \( s \in \mathbb{C} \) when \( |z| < 1 \) and \( \Re s > 1 \) when \( |z| = 1 \). Then for \( q > 1, \Re t > 0, \) and \( m > 0 \) an integer we have the integral
\[
\int_{0}^{1} \frac{u^{t-1}}{\ln u} (1-u)^{m-1} \text{Li}_q(1-u)du = \sum_{n=m}^{\infty} \frac{1}{(n-m+1)^q} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \ln(t+j),
\] (1.38)
giving

Corollary 1.17 With \( m = 1 \), we have the special case
\[
\int_{0}^{1} \frac{u^{t-1}}{\ln u} \text{Li}_q(1-u)du = \sum_{n=1}^{\infty} \frac{1}{n^q} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \ln(t+j).
\] (1.39)

Let for \( \Re x > 0 \) and \( \Re y > 0 \)
\[
B(z_1, z_2; x, y) \equiv \int_{z_1}^{z_2} t^{x-1} (1-t)^{y-1} dt
\] (1.40)
be the generalized incomplete Beta function and \( \text{Ei}(x) \equiv \int_{-\infty}^{x} e^t/t dt \) the exponential integral. Then we have

Proposition 1.18 (Context of the logarithmic integral)
\[
\int_{z_1}^{z_2} \frac{(u^{t-1} - u^{a-1})}{\ln u} (1-u)^{y-1} du
= \sum_{j=0}^{\infty} (-1)^j \binom{y-1}{j} \left\{ \text{Ei}[(a + j) \ln z_1] - \text{Ei}[(t + j) \ln z_1] - \text{Ei}[(a + 1) \ln z_1] + \text{Ei}[(t + 1) \ln z_1]
- \text{Ei}[(a + 1) \ln z_2] + \text{Ei}[(t + 1) \ln z_2] \right\},
\] (1.41)
giving

Corollary 1.19
\[
\int_{z_1}^{z_2} \frac{(u^{t-1} - u^{a-1})}{\ln u} du = \text{Ei}[a \ln z_1] - \text{Ei}[t \ln z_1] - \text{Ei}[a \ln z_2] + \text{Ei}[t \ln z_2]
- \text{Ei}[(a + 1) \ln z_1] + \text{Ei}[(t + 1) \ln z_1] + \text{Ei}[(a + 1) \ln z_2]
- \text{Ei}[(t + 1) \ln z_2].
\] (1.42)
Propositions 1.1 and 1.3 were first obtained by the author several years ago as complements to Refs. [8] and [11] concerning the Li criterion for the Riemann hypothesis. The proof of Proposition 1.3 given in the sequel is more direct than the original. The sort of alternating binomial sums that occur in Proposition 1.14 motivates a study of integrals such as appear in Propositions 1.16 and 1.18. After the proofs of these Propositions we present Discussion that contains additional examples and extensions.

2 Proofs of the Propositions

Proof of Proposition 1.1: In order to derive Eq. (1.12), we add Eqs. (1.1) and (1.3) and make use of Eq. (1.6), resulting in

\[ \zeta(s) + \frac{\zeta'(s)}{\zeta(s)} = \sum_{n=0}^{\infty} \left[ (-1)^n \frac{\gamma_n - \eta_n}{n!} \right] (s-1)^n = \sum_{k=1}^{\infty} \frac{[1 - \Lambda(k)]}{k^s}, \]  

wherein the pole at \( s = 1 \) has been removed. Repeated differentiation of Eq. (2.1) gives

Corollary 2.1

\[ \zeta^{(j)}(s) + \left[ \frac{\zeta'(s)}{\zeta(s)} \right]^{(j)} = \sum_{n=j}^{\infty} \left[ (-1)^n \frac{\gamma_n - \eta_n}{n!} \right] n(n-1)(n-2) \cdots (n-j+1)(s-1)^{n-j} \]

\[ = (-1)^j \sum_{k=1}^{\infty} \frac{[1 - \Lambda(k)]}{k^s} \ln^j k, \]  

where the sum on the right starts with \( k = 2 \) when \( j > 0 \). Taking \( s \to 1^+ \) in Eq. (2.2) gives Eq. (1.12).

Remarks 2.2

(i) Taking different values of \( s \) in Eq. (2.2) yields various connections between the \( \eta \)'s and the Stieltjes constants. For instance, with \( s = 2 \) in Eq. (2.2) we have

\[ \sum_{n=j}^{\infty} \left[ (-1)^n \frac{\gamma_n - \eta_n}{n!} \right] (-n)_j = \sum_{k=1}^{\infty} \frac{[1 - \Lambda(k)]}{k^2} \ln^j k, \]

where \( (a)_\ell = \Gamma(a + \ell)/\Gamma(a) \) is the Pochhammer symbol.

(ii) For Eq. (1.13) we use the values \( \gamma_0 = \gamma \) and \( \eta_0 = -\gamma \) (e.g., [8, 11]). This special case (1.13) for the Euler constant has unfortunately appeared in several places in the literature including [16] and [19, p. 109] with the sum starting at 2 instead of 1, missing a dominant contribution of 1/2.
(iii) Proposition 1.1 may also be proved by forming

\[
\ln \zeta(s) - \int_s^\infty [\zeta(y) - 1]dy = \sum_{n=2}^{\infty} \frac{[\Lambda(n) - 1]}{n^s \ln n}.
\]

(vi) The Stieltjes and \( \eta_j \) constants are strongly connected and one may easily write a recursion relation between the two sets of constants [8, Appendix]. For instance, \( \eta_1 = \gamma^2 + 2\gamma_1 \). Then from Eq. (1.12) we may write either

\[
\eta_1 = \frac{1}{3} \gamma^2 + \frac{2}{3} \sum_{m=1}^{\infty} \frac{[1 - \Lambda(m)]}{m} \ln m,
\]

or

\[
\gamma_1 = \frac{1}{3} \left[ \sum_{m=1}^{\infty} \frac{[1 - \Lambda(m)]}{m} \ln m - \gamma^2 \right],
\]

with the corresponding formula for \( \gamma \) given in Eq. (1.13). Similarly, we have corollary expressions for all of the \( \gamma_k \) and \( \eta_j \) constants.

**Proof of Proposition 1.3:** By applying Proposition 1.1, the definition (1.14) of \( S_2(n) \), and the power series definition of \( L_n^\alpha \), we obtain

\[
S_2(n) = \sum_{k=1}^{n} \frac{(-1)^k}{(k - 1)!} \binom{n}{k} \gamma_{k-1} + S_\Lambda(n).
\]

The second line of Eq. (1.17) follows since \( \Lambda(1) = 0 \) and \( L_{n-1}^1(0) = n \). We obtain alternative forms of the sum \( S_\gamma(n) \) of Eq. (1.16) by using the integral representation [22]

\[
\gamma_k = -\int_1^\infty \frac{1}{t^k} \ln t P_1(t) dt = \int_1^\infty \frac{\ln^{k-1} t}{t^2} (k - \ln t) P_1(t) dt - \delta_{k0}/2
\]

\[
= \frac{(-1)^{k-1}}{(k - 1)!} \int_1^\infty P_k(t) \left( \frac{d}{dt} \right)^k \frac{\ln^k t}{t} dt - \delta_{k0}/2,
\]

where \( \delta_{jk} \) is the Kronecker symbol. In Eq. (2.8), where we integrated by parts \( k - 1 \) times, \( P_k(t) = B_k(t - [t]) \) with \( B_k \) the \( k \)th degree Bernoulli polynomial, such that \( P'_{k+1}(t) = (k + 1) P_k(t) \). By then applying the definition (1.16) and the power series form of the Laguerre polynomials we have

\[
S_\gamma(n) = \int_1^\infty \frac{P_1(t)}{t^2} \left[ L_{n-2}^2(\ln t) + L_{n-1}^1(\ln t) \right] dt + n/2.
\]

This representation of \( S_\gamma \) is equivalent to integration by parts on the expression given on the right side of Eq. (1.16).
For part (b) we observe $P_1(t) = O(1)$. So for a constant $C > 0$ the integral term in Eq. (2.9) is majorized by
\[ C \int_1^\infty \frac{1}{t^2} \left[ L_{n-2}^2(\ln t) + L_{n-1}^1(\ln t) \right] dt = C \int_0^\infty e^{-u} \left[ L_{n-2}^2(u) + L_{n-1}^1(u) \right] du = Cn, \quad n \geq 2, \quad (2.10) \]
where we used the value of a Laplace transform \[17\]
\[ \int_0^\infty e^{-x} L_{n-\nu}^\nu(x) dx = \frac{1}{\Gamma(\nu)} \frac{(n-1)!}{(n-\nu)!}. \quad (2.11) \]
Then $S_y(n) = O(n)$. For part (c) we apply Eqs. (1.16) and (1.17) and interchange the order of operations.

For part (d), we make use of \[27, \text{Section 2}\]
\[ \gamma_k = \sum_{\nu=1}^{n} \ln^k \nu - \frac{\ln^{k+1} n}{k+1} - \frac{\ln^k n}{2} \frac{1}{n} + O \left( \frac{1}{n^{2-\epsilon}} \right), \quad (2.12) \]
with $\epsilon > 0$ arbitrary. This equation is essentially an asymptotic form of $\gamma_k$ for $k \gg 1$. We substitute this equation into the definition (1.16) of $S_y(n)$, apply the power series form of the Laguerre polynomials, and Proposition 1.3 is completed. \qed

**Proof of Proposition 1.4:** Part (a) is based upon the well known Hermite formula for $\zeta(s, a)$:
\[ \zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + 2 \int_0^\infty \frac{\sin(s \tan^{-1} y/a)}{(y^2 + a^2)^{s/2}} \frac{dy}{(e^{2\pi y} - 1)}. \quad (2.13) \]
We use the expression
\[ \tan^{-1} w = \frac{1}{2i} \ln \left( \frac{1 + iw}{1 - iw} \right), \quad (2.14) \]
so that $\exp[\pm is \tan^{-1}(y/a)] = [(1 + iy/a)/(1 - iy/a)]^{s/2}$. Then the integral term in Eq. (2.13) becomes
\[ 2 \int_0^\infty \frac{\sin(s \tan^{-1} y/a)}{(y^2 + a^2)^{s/2}} \frac{dy}{(e^{2\pi y} - 1)} \]
\[ = -i \int_0^\infty \frac{1}{(y^2 + a^2)^{s/2}} \left[ \frac{(1 + iy/a)^{s/2}}{1 - iy/a} - \frac{(1 - iy/a)^{s/2}}{1 + iy/a} \right] \frac{dy}{(e^{2\pi y} - 1)} \]
\[ = -i a^{-s} \int_0^\infty \frac{1}{(1 + y^2/a^2)^{s-1}} \left[ \frac{(1 + iy/a)^{s-1}}{1 - iy/a} - \frac{(1 - iy/a)^{s-1}}{1 + iy/a} \right] \frac{dy}{(e^{2\pi y} - 1)} \]
\[ = \int_0^\infty \frac{1}{(1 + y^2/a^2)} \left[ (y/a - i)e^{-(s-1)\ln(a-iy)} + (y/a + i)e^{-(s-1)\ln(a+iy)} \right] \frac{dy}{(e^{2\pi y} - 1)}. \quad (2.15) \]
For the other terms in Eq. (2.13) we have
\[
\frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} = \frac{e^{-(s-1) \ln a}}{2a} + \frac{e^{-(s-1) \ln a}}{s-1} = \frac{1}{s-1} + \frac{1}{2a} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \ln^j a (s-1)^j + \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(j+1)!} \ln^{j+1} a (s-1)^j. \tag{2.16}
\]

We then expand the right side of Eq. (2.15) in powers of \( s-1 \), combine the result with Eq. (2.16), and compare with the defining expansion (1.11) for \( \gamma_k(a) \). We find
\[
\gamma_k(a) = \frac{1}{2a} \ln^k a - \frac{\ln^{k+1} a}{k+1} + \frac{1}{a} \int_0^\infty \frac{(y/a - i) \ln^k (a - iy) + (y/a + i) \ln^k (a + iy)}{1 + y^2/a^2 (e^{2\pi y} - 1)} dy, \tag{2.17}
\]
that is the same as Eq. (1.19) and part (a) is proved.

For part (b) we apply Abel–Plana summation (e.g., [31, p. 90]) to write for all complex \( s \neq 1 \)
\[
\zeta(s, a) = \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} + \frac{(n+a)^{-s}}{2} + \frac{(n+a)^{1-s}}{s-1} + 2 \int_0^\infty \frac{\sin [s \tan^{-1} (y/(n+a))]}{[y^2 + (n+a)^2]^{s/2}} \frac{dy}{(e^{2\pi y} - 1)}. \tag{2.18}
\]
We then expand the terms of this equation in powers of \( s-1 \) in like manner to part (a).

Part (c) makes use of the integral representation for \( \Re a > 1/2 \) [23]
\[
\zeta(s, a) = \frac{\pi^{s-2}}{(s-1)} \int_0^\infty \left[ r^2 + (2a - 1)^2 \right]^{(1-s)/2} \frac{\cos [((s-1) \tan^{-1} [r/(2a - 1)])]}{\cosh^2(\pi r/2)} dr. \tag{2.19}
\]
We again apply Eq. (2.14) so that we are able to write
\[
\zeta(s, a)
= \frac{\pi}{4} \frac{1}{s-1} \sum_{j=0}^{\infty} \frac{\ln^j 2}{j!} (s-1)^j
\int_0^\infty \left[ \frac{1}{(2a - 1 - it)^{s-1}} + \frac{1}{(2a - 1 + it)^{s-1}} \right] \frac{dt}{\cosh^2(\pi t/2)}
= \frac{\pi}{4} \frac{1}{s-1}
\int_0^\infty \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \left[ \ln^\ell (2a - 1 - it) + \ln^\ell (2a - 1 + it) \right] (s-1)^\ell \frac{dt}{\cosh^2(\pi t/2)}
+ \frac{\pi}{4} \frac{1}{s-1} \sum_{j=1}^{\infty} \frac{\ln^2 2}{j!} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!}
\int_0^\infty \left[ \ln^j (2a - 1 - it) + \ln^j (2a - 1 + it) \right] (s-1)^{j+\ell} \frac{dt}{\cosh^2(\pi t/2)}. \tag{2.20}
\]
We then (i) separate out the \( \ell = 0 \) term two lines above and use the integral 
\[
\int_0^\infty \text{sech}^2 (\pi t/2) dt = 2/\pi,
\]
and (ii) reorder the double sum in the last line of Eq. (2.15), thereby obtaining
\[
\zeta(s, a) = \frac{1}{s-1} + \frac{\pi}{4} \sum_{\ell=0}^\infty \frac{(-1)^{\ell+1}}{(\ell+1)!} \int_0^\infty \left[ \ln^{\ell+1}(2a - 1 - it) + \ln^{\ell+1}(2a - 1 + it) \right] \frac{dt}{\cosh^2(\pi t/2)} (s-1)^\ell 
\]
\[+ \frac{\pi}{4} \sum_{m=0}^\infty (-1)^m \sum_{j=1}^{m+1} \frac{\ln^j 2}{j!} \frac{(-1)^{j+1}}{(m-j+1)!} \int_0^\infty \left[ \ln^{m-j+1}(2a - 1 - it) + \ln^{m-j+1}(2a - 1 + it) \right] \times \frac{dt}{\cosh^2(\pi t/2)} (s-1)^m.
\]
(2.21)

Comparing Eq. (2.21) with the Laurent expansion (1.11) gives part (b) and Proposition 1.4 is proved.

For Corollary 1.5 we put \( m = 0 \) in Eqs. (1.19), (1.20), and (1.21) and use the fact that \( \gamma_0(x) = -\psi(x) \) (e.g., [9]).

**Remarks 2.3** As it must, Eq. (2.18) satisfies \( \zeta(0, a) = 1/2 - a \), \( \zeta'(0, a) = \ln \Gamma(a) - (1/2) \ln(2\pi) \), and \( B_m(x) = -m\zeta(1-m, x) \) for positive integers \( m \).

Equation (1.22) recovers the result of differentiating Binet’s second expression for \( \ln \Gamma(z) \). Equation (1.24) subsumes the special case at \( a = 1/2 \) given in [17, p. 580].

For Corollary 1.6 we apply the definition (1.16) of \( S_\gamma(n) \) together with

**Lemma 2.4** We have
\[
\sum_{j=v}^{n} \frac{(-1)^{j-1}}{(j-v)!} \binom{n}{j} w^{j-v} = (-1)^{v-1} L_n^{v}(w),
\]
(2.22)
that follows from the power series form of the Laguerre polynomials. The second line of Eq. (1.26) follows first from reordering a double sum to obtain
\[
\sum_{k=1}^{n} (-1)^k \binom{n}{k} \sum_{j=1}^{k} \frac{\ln^j 2}{j!} \frac{(-1)^{j+1}}{(k-j)!} \ln^{k-j}(2a - 1 \mp it) 
\]
\[
= \sum_{j=1}^{n} \sum_{k=0}^{n-j} (-1)^{k+j} \binom{n}{k+j} \frac{\ln^{k}(2a - 1 \mp it)}{k!} \frac{\ln^j 2}{j!} (-1)^{j+1}.
\]
(2.23)
We then use various relations to express \( \binom{n}{k+j} \) in terms of Pochhammer symbols:

\[
\binom{n}{k+j} = \frac{(-1)^{k+j}}{(k+j)!} (-n)_{j+k} = \frac{(-1)^{k+j}}{(k+j)!} (-n)_{j-n} = \frac{(-1)^{k+j} (-1)^{j-n}}{(k+j)! (n-j)!} (j-n)_k,
\]

where \((k+j)! = j!(j+1)_k\). We then apply the power series definition of the confluent hypergeometric function, Eq. (1.26) follows and Corollary 1.6 is completed.

**Remarks 2.5** Similar to Proposition 1.4 we have obtained further integral representations of the Stieltjes constants and the sum \( S_\gamma(n) \) from related integral representations of the Riemann zeta function.

In particular, we have for \( s \in C/\{1\} \) [23]

\[
\zeta(s) = \frac{2^{s-1}}{s-1} - 2^s \int_0^\infty (1 + t^2)^{-s/2} \sin(s \tan^{-1} t) \frac{dt}{e^{\pi t} + 1},
\]

and

\[
\zeta(s) = \frac{2^{s-1}}{1 - 2^{1-s}} \int_0^\infty (1 + t^2)^{-s/2} \frac{\cos(s \tan^{-1} t)}{\cosh(\pi t/2)} dt.
\]

As a byproduct we have from Eq. (2.26)

**Corollary 2.6** For all complex \( s \neq 1 \) we have

\[
\zeta(s) = \frac{1}{2(1 - 2^{1-s})} \int_0^\infty \left\{ \frac{1}{(1/2 - iw)^s} + \frac{1}{(1/2 + iw)^s} \right\} \frac{dw}{\cosh(\pi w)}
\]

\[
= \frac{1}{2(1 - 2^{1-s})} \int_{-\infty}^\infty \frac{1}{(1/2 - iw)^s} \frac{dw}{\cosh(\pi w)}.
\]

Equation 2.27 follows from Eq. (2.26) with a simple change of variable and the use of relation (2.14).

Of a slightly different flavor, we have used the representation [14] valid for \( 0 < \text{Re } s < 2 \)

\[
\zeta(s) = \frac{1}{s-1} + \frac{\sin \pi s}{\pi(s-1)} \int_0^\infty \left[ \psi'(t) - \frac{1}{1+t} \right] t^{s-1} dt,
\]

to obtain

**Corollary 2.7** We have for integers \( m \geq 0 \)

\[
\gamma_m = (-1)^m m! \sum_{k=0}^{[m/2]} \frac{(-1)^{k+1}}{(2k+1)!} \frac{\pi^{2k}}{(m-2k)!} \int_0^\infty \ln^{m-2k} t \left[ \psi'(t) - \frac{1}{1+t} \right] dt.
\]

The proof of Corollary 2.7 makes use of the Taylor series

\[
\sin \pi s = \sum_{k=0}^\infty \frac{(-1)^{k+1}}{(2k+1)!} \pi^{2k+1} (s-1)^{2k+1},
\]

together with series manipulations. The case of \( m = 0 \) in Eq. (2.29) recovers \( \gamma_0 = \gamma \), as is readily seen by a limiting argument. Corollary 2.7 provides another form of the sum \( S_\gamma(n) \), that we omit.
We have found that Proposition 1.4(a) subsumes the $a = 1$ case derived in Ref. [2] by a contour integration. This reference evidences that Proposition 1.4 should be a practical method for computing $\gamma_k(a)$. Indeed, there is now an arbitrary precision Python implementation [29].

**Proof of Proposition 1.7:** For part (a) we form the combination of Dirichlet series

$$\zeta^2(s) + \zeta'(s) - 2\gamma\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} [d(n) - \ln n - 2\gamma],$$

(2.31)

wherein simple and double polar terms have been eliminated. By the use of Eq. (1.1) and series manipulations we find

$$\sum_{n=1}^{\infty} \frac{1}{n^s} [d(n) - \ln n - 2\gamma] = \sum_{j=0}^{\infty} \left[ \frac{2(-1)^{j+1}}{(j + 1)!} \gamma_{j+1} + \frac{(-1)^{j+1}}{j!} \gamma_{j+1} - 2\gamma \frac{(-1)^j}{j!} \gamma_j \right] + \sum_{\ell=0}^{j} \frac{(-1)^\ell}{\ell!(j - \ell)!} \gamma_{\ell} \gamma_{j-\ell}(s - 1)^j.$$

(2.32)

Taking the limit as $s \to 1^+$ in this equation gives Corollary 1.8. Corollary 1.9 then follows from the limit relations (1.2). Taking $m$ derivatives with respect to $s$ in Eq. (2.32) and putting $s \to 1^+$ yields

$$(-1)^m \sum_{n=1}^{\infty} \frac{\ln^m n}{n} [d(n) - \ln n - 2\gamma] = \left[ \frac{(-1)^{m+1}}{m!} \left( 1 + \frac{2}{m + 1} \right) \gamma_{m+1} - 2\gamma \frac{(-1)^m}{m!} \gamma_m \right] + \sum_{\ell=0}^{m} \frac{(-1)^\ell}{\ell!(m - \ell)!} \gamma_{\ell} \gamma_{m-\ell} m!,$$

(2.33)

and this gives Eq. (1.27).

For part (b) we form the combination of Dirichlet series

$$\zeta^2(s) - \left( \frac{\zeta'}{\zeta} \right)'(s) - 2\gamma\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} [d(n) - \Lambda(n) \ln n - 2\gamma].$$

(2.34)

We then use both Eqs. (1.1) and (1.3) and series manipulations to find

$$\sum_{n=1}^{\infty} \frac{1}{n^s} [d(n) - \Lambda(n) \ln n - 2\gamma] = \sum_{j=0}^{\infty} \left[ \frac{2(-1)^{j+1}}{(j + 1)!} \gamma_{j+1} + (j + 1) \eta_{j+1} - 2\gamma \frac{(-1)^j}{j!} \gamma_j \right] + \sum_{\ell=0}^{j} \frac{(-1)^\ell}{\ell!(j - \ell)!} \gamma_{\ell} \gamma_{j-\ell}(s - 1)^j, \quad |s - 1| < 3.$$

(2.35)
Taking the limit as \( s \to 1^+ \) in this equation gives Corollary 1.10. More generally, by taking \( m \) derivatives of relation (2.35) we have

\[
(-1)^m \sum_{n=1}^{\infty} \ln^n n \frac{[d(n) - \Lambda(n) \ln n - 2\gamma]}{n^s} \\
= \sum_{j=m}^{\infty} \left[ \frac{2(-1)^{j+1}}{(j+1)!} \gamma_{j+1} + (j+1)\eta_{j+1} - 2\gamma \frac{(-1)^j}{j!} \gamma_j + \sum_{\ell=0}^{j} \frac{(-1)^\ell}{\ell!(j-\ell)!} \gamma_{\ell} \gamma_{j-\ell} \right] \\
\times j(j-1)(j-2) \cdots (j-m+1)(s-1)^{j-m}, \quad |s-1| < 3, \quad (2.36)
\]

wherein term-by-term differentiation is justified within the stated radius of convergence. Taking \( s \to 1^+ \) in this equation gives Eq. (1.29).

For Corollary 1.11 we write \( \eta_{m-1} \) from Eq. (1.29) and apply the definition of \( S_2 \) of Eq. (1.14) in the form

\[
S_2(n) = \gamma n - \sum_{m=2}^{n} \binom{n}{m} \eta_{m-1}.
\]

We then apply Lemma 2.4 for the power series form of \( L_1^{1/n-1} \) to obtain Eq. (1.31).

**Proof of Proposition 1.12:** (i) We begin by writing

\[
\frac{\xi'(s)}{\xi(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \xi'(s), \quad \text{Re } s > 1, \quad (2.37)
\]

and forming \( \xi'(s) \) from Eq. (1.1). This gives

\[
\frac{\xi'(s)}{\xi(s)} = \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell!} \ln^\ell k(s-1)^\ell \\
\left[ -\frac{1}{(s-1)^2} + \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j!} j! \gamma_{j+1}(s-1)^j \right], \quad (2.38)
\]

where we have used \( \sum_{k=1}^{\infty} \mu(k)/k = 0 \). We then multiply the series, reorder a double summation, and make use of

\[
-\sum_{k=2}^{\infty} \frac{\mu(k)}{k} \ln k = \lim_{s \to 1^+} \frac{1}{ds} \frac{\xi(s)}{\xi(s)} = 1, \quad (2.39)
\]

wherein we appealed to a Tauberian theorem ([34, Ch. V]). After these operations we compare with the defining Laurent expansion (1.3) for the \( \eta_j \) coefficients, and Proposition 1.12(i) follows.

For part (ii) we instead use the identity

\[
-\frac{\xi'(s)}{\xi(s)} = \xi(s) \frac{d}{ds} \frac{1}{\xi(s)}. \quad \tag{2.40}
\]
We use the Laurent expansion (1.3) for the left side, and expansion (1.1) for \(\zeta(s)\) and the Dirichlet series for \(1/\zeta(s)\) on the right side. Expanding in powers of \(s-1\) gives

\[
\frac{1}{s-1} + \sum_{j=0}^{\infty} \eta_j(s-1)^j = -\left[ \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(s-1)^n \right] \\
\times \sum_{\ell=2}^{\infty} \frac{\mu(\ell)}{\ell} \ln \ell \left[ 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \ln^j (s-1)^j \right]. \tag{2.41}
\]

We then expand the right side, reordering the last series, apply relation (2.39) and (ii) follows.

Corollary 1.13 follows from part (i) by using the definition (1.14) of \(S_2(n)\) and applying Lemma 2.4. Similarly, another form of \(S_2(n)\) could be written based upon the expression for \(\eta_j\) given in part (ii).

**Proof of Proposition 1.14:** The key starting point of the proof is the Amore representation of the Riemann zeta function [3]

\[
\zeta(s) = \frac{1}{1 - 2^{1-s}} \frac{1}{(1+\lambda)} \sum_{k=0}^{\infty} \left( \frac{\lambda}{1+\lambda} \right)^k \sum_{j=0}^{k} \frac{(-1)^j}{j!} \frac{1}{\lambda^j (j+1)^s}, \quad s \in \mathbb{C}, s \neq 1. \tag{2.42}
\]

Beyond the condition \(\text{Re} s > 0\) stated in Ref. [3], the representation (2.42) holds for all complex \(s \neq 1\), as the summation continues to converge for \(\text{Re} s \leq 0\). (We further discuss Eq. (2.42) in the following section.) So Eq. (2.42) is globally convergent, as is its \(\lambda = 1\) special case embodied in the Hasse representation [18, 9]. Moreover, beyond the original condition \(\lambda > 0\) of Ref. [3], we may take \(\lambda\) complex with \(\text{Re} \lambda > 0\).

The proof now proceeds as in Proposition 6.1 of Ref. [10]. However, that description was terse and we now have the (arbitrary) complex parameter \(\lambda\). Therefore, we believe it is worthwhile to supply some more details. We first write again

\[
(j + 1)^{-s} = (j + 1)^{-1} \exp[-\ln(j+1)(s-1)] \\
= \frac{1}{j+1} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \ln^\ell (j+1)(s-1)^\ell. \tag{2.43}
\]

From the generating function of the Bernoulli numbers, we have

\[
\frac{te^t}{e^t - 1} = \frac{t}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n(1) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n B_n \frac{t^n}{n!}, \quad |t| < 2\pi, \tag{2.44}
\]

where \(B_n(x)\) is the \(n\)-th Bernoulli polynomial. We then put \(t = (s-1) \ln 2\) in Eq. (2.44) and obtain

\[
(1 - 2^{1-s})^{-1} = \frac{1}{\ln 2(s-1)} - \sum_{j=0}^{\infty} \frac{(-1)^j B_{j+1}}{(j+1)!} \ln^j 2(s-1)^j, \quad |s-1| < \frac{2\pi}{\ln 2}. \tag{2.45}
\]
We substitute Eqs. (2.43) and (2.45) into (2.42), writing the sum over \( \ell \) in Eq. (2.42) as the \( \ell = 0 \) term and the rest of the terms. We use

\[
\sum_{k=0}^{\infty} \left( \frac{\lambda}{1+\lambda} \right)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\lambda^j} \frac{1}{j+1} = (\lambda + 1) \ln 2, \tag{2.46}
\]

obtaining

\[
\zeta(s) = \frac{1}{1+\lambda} \left[ \frac{1}{\ln 2(s-1)} - \sum_{j=0}^{\infty} (-1)^j B_{j+1} \frac{1}{(j+1)!} \ln^j 2(s-1)^j \right] \\
\times \left[ (\lambda + 1) \ln 2 + \sum_{k=0}^{\infty} \left( \frac{\lambda}{1+\lambda} \right)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\lambda^j} \frac{1}{j+1} \\
\times \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell!} \ln^\ell (j+1)(s-1)^\ell \right]. \tag{2.47}
\]

We multiply the terms in Eq. (2.47), separate out the simple polar part, and compare with the defining expansion (1.1) for the Stieltjes constants. For the last product of series in Eq. (2.47) we use the reordering

\[
\sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} (\cdots)(s-1)^{n+\ell} = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} (\cdots)(s-1)^m = \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} (\cdots)(s-1)^m, \tag{2.48}
\]

and the Proposition follows. \( \square \)

In particular, at \( \lambda = 1/2 \) we obtain

**Corollary 2.8**

\[
\gamma_m = -\frac{2}{3} m! \sum_{\ell=1}^{m} \frac{B_{m-\ell+1}}{(m-\ell+1)!} \ln^{m-\ell} 2 \sum_{k=1}^{\infty} \left( \frac{1}{3} \right) \binom{k}{j} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{2j}{(j+1)} \ln^j (j+1) \\
- \frac{2}{3 \ln 2} \sum_{k=1}^{\infty} \left( \frac{1}{3} \right) \binom{k}{j} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \ln^{m+1} (j+1) \frac{2j}{(j+1)} \ln^m (k+1) \\
- \frac{B_{m+1}}{(m+1)!} \ln^{m+1} 2. \tag{2.49}
\]

**Proof of Proposition 1.15:** The Hurwitz zeta function has for \( \text{Re} s > 1 \) and \( \text{Re} a > 0 \) the integral representation

\[
\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-(a-1)x}}{e^x - 1} \, dx = \frac{1}{\Gamma(s)} \int_0^{1} \frac{x^{a-1} [\ln(1/x)]^{s-1}}{(1-x)} \, dx. \tag{2.50}
\]
Then we introduce a parameter $\lambda$ with $\text{Re} \, \lambda > 0$, make use of a geometric series expansion for $x \in [0, 1]$, and follow this with a binomial expansion:

$$
\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^1 \frac{x^{a-1} \ln(1/x)^{s-1}}{1 - \left(\frac{x + \lambda}{1 + \lambda}\right)} \, dx
$$

$$
= \frac{1}{\Gamma(s)} \int_0^1 \frac{x^{a-1} \ln(1/x)^{s-1}}{1 + \lambda} \sum_{k=0}^\infty \left(\frac{x + \lambda}{1 + \lambda}\right)^k \, dx
$$

$$
= \frac{1}{\Gamma(s)} \sum_{k=0}^\infty \frac{\lambda^k}{(1 + \lambda)^{k+1}} \sum_{j=0}^k \binom{k}{j} \frac{1}{\lambda^j} \int_0^1 x^{a+j-1} \ln(1/x)^{s-1} \, dx. \quad (2.51)
$$

Performing the integral in terms of the $\Gamma$ function gives the Proposition. □

**Proof of Proposition 1.16:** We first obtain an alternating binomial sum by integrating the Beta function. We have

$$
B(x, y) = \int_0^1 u^{x-1} (1-u)^{y-1} \, du = 2 \int_0^{\pi/2} \sin^{2x-1} \varphi \cos^{2y-1} \varphi \, d\varphi = B(y, x),
$$

$$
\text{min}[\text{Re} \, x, \text{Re} \, y] > 0,
$$

(2.52)

so that

$$
\int_a^t B(x, y) \, dx = \int_0^1 \frac{(u^{t-1} - u^{a-1})}{\ln u} (1-u)^{y-1} \, du
$$

$$
= \sum_{j=0}^\infty (-1)^j \binom{y-1}{j} \int_a^t \frac{dx}{x+j}
$$

$$
= \sum_{j=0}^\infty (-1)^j \binom{y-1}{j} [\ln(t+j) - \ln(a+j)], \quad (2.53a)
$$

where the form (2.53b) follows by binomial expansion in Eq. (2.52). Upon comparing Eq. (2.53a) and (2.53c) we have

$$
\int_0^1 \frac{u^{t-1}}{\ln u} (1-u)^{y-1} \, du = \sum_{j=0}^\infty (-1)^j \binom{y-1}{j} \ln(t+j) + C,
$$

(2.54)

where $C$ is a constant to be determined. A simple way to do this is to put $y = 2$ whereupon

$$
\int_0^1 u^{t-1}[(1-u)/(\ln u)] \, du = \ln t - \ln(1+t).
$$

Therefore, $C = 0$ and

$$
\int_0^1 \frac{u^{t-1}}{\ln u} (1-u)^{y-1} \, du = \sum_{j=0}^\infty (-1)^j \binom{y-1}{j} \ln(t+j), \quad (2.55)
$$
a result closely related to tabulated integrals when \( y \) is an integer \([17, \text{p. 546}]\). Now if \( y = n + 1, n \geq 1 \) an integer, the sum in Eq. (2.55) terminates:

\[
\int_0^1 \frac{u^{t-1}}{\ln u} (1 - u)^n \, du = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \ln(t + j).
\] (2.56)

We next multiply each side of this equation by \( 1/(n - m + 1)^q \) and sum on \( n \) from \( m \) to \( \infty \). We shift the summation index on the left side, apply the series definition of the polylogarithm function, and the Proposition follows.

\[\square\]

**Remark 2.9** The Proposition may be extended by analytically continuing to \( \text{Re} \, q \geq 1 \).

**Proof of Proposition 1.18:** By applying binomial expansion to Eq. (1.40) we have

\[
B(z_1, z_2; x, y) = \sum_{j=0}^{\infty} (-1)^j \left( \frac{y - 1}{j} \right) \left( \frac{z_2^{x+j} - z_1^{x+j}}{(x+j)} \right).
\] (2.57)

We then integrate this expression on \( x \) from \( a \) to \( t \) and compare with the integrated form of Eq. (1.40),

\[
\int_a^t B(z_1, z_2; x, y) \, dx = \int_{z_1}^{z_2} \frac{(u^{t-1} - u^{a-1})}{\ln u} (1 - u)^{y-1} \, du,
\] (2.58)

and the Proposition follows.

Corollary 1.19 is the special case of \( y = 1 \).

As a further special case we have

**Corollary 2.10**

\[
\int_a^1 B(2, z_2; x, 1) \, dx = \int_{2}^{z_2} \frac{(1 - u^{a-1})}{\ln u} \, du,
\] (2.59)

**showing the close connection with the logarithmic integral** \( \text{Li}(z) \equiv \int_2^z dt/\ln t \).

### 3 Discussion

Amplifying that the representation (2.42) holds for all complex \( s \neq 1 \) we easily verify that

\[
\zeta(0) = -\frac{1}{1 + \lambda} \sum_{k=0}^{\infty} \left( \frac{\lambda - 1}{\lambda + 1} \right)^k = -\frac{1}{2}.
\] (3.1)

Furthermore, we have

\[
\zeta'(s) = -\frac{1}{1 - 2^{1-s}} \left[ 2^{1-s} \ln 2 \zeta(s) + \frac{1}{1 + \lambda} \sum_{k=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^k \right.
\]

\[
\times \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\lambda^j} \ln(1 + j)^s \left( j + 1 \right)^s \right], \quad s \in \mathbb{C}, \ s \neq 1,
\] (3.2)
so that
\[ \zeta'(0) = -\ln 2 + \frac{1}{1 + \lambda} \sum_{k=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^k \frac{1}{\lambda^k} \ln(1 + j). \] (3.3)

This value is easily shown to be \(-\frac{1}{2}\ln 2\) at \(\lambda = 1\) and otherwise Eq. (3.3) shows that the summation term must evaluate to \(\frac{1}{2}\ln(2/\pi)\). In fact, we may demonstrate

**Lemma 3.1** For \(\text{Re} \lambda > 0, \text{Re} x > 0, \text{and} \text{Re} y > 0\) we have (a)
\[ \sum_{\ell=0}^{n} \left( -\frac{1}{\lambda} \right)^{\ell} \binom{n}{\ell} \ln \left( \frac{y + \ell}{x + \ell} \right) = \int_{0}^{1} \frac{(u^{y-1} - u^{x-1})}{\ln u} \frac{du}{(1 - u/\lambda)^n}, \quad n \geq 0, \] (3.4)
and (b)
\[ \sum_{n=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^n \sum_{\ell=0}^{n} \left( -\frac{1}{\lambda} \right)^{\ell} \binom{n}{\ell} \ln \left( \frac{y + \ell}{x + \ell} \right) \]
\[ = (\lambda + 1) \int_{0}^{1} \frac{(u^{y-1} - u^{x-1})}{\ln u} \frac{du}{(u + 1)} \] (3.5)
\[ = (\lambda + 1) \sum_{m=0}^{\infty} (-1)^m \ln \left( \frac{y + m}{x + m} \right) \] (3.6)
\[ = (\lambda + 1) \left[ \ln \frac{\Gamma \left( \frac{y+1}{2} \right)}{\Gamma(y/2)} - \ln \frac{\Gamma \left( \frac{x+1}{2} \right)}{\Gamma(x/2)} \right]. \] (3.7)

For the proof of part (a), we proceed as in Proposition 1.16, defining the integral
\[ I(q, z, a) \equiv \int_{0}^{1} u^{q-1} (1 - au)^{-1} du. \] (3.8)
We then evaluate
\[ \int_{x}^{y} I(q, z, a) dq = \int_{0}^{1} \frac{(u^{y-1} - u^{x-1})}{\ln u} (1 - au)^{-1} du \] (3.9)
also by means of binomial expansion of the integrand. Putting \(z - 1 = n \geq 0\) an integer then gives the first part of the Lemma. For part (b) we first use the result of part (a), interchanging summation and integration and obtaining Eq. (3.5). If we expand the integrand factor \(1/(1 + u)\) of Eq. (3.5) as a geometric series, valid for \(|u| < 1\), and then use a tabulated integral [17, p. 543] we find Eq. (2.57). Equation 3.7 may be found directly from a known integral [17, pp. 543 or 544] applied to Eq. (3.5), or by using the Hadamard product representation of the Gamma function in conjunction with Eq. (3.6).

For the latter we note (cf. [17, p. 936])
\[ \frac{\Gamma \left( \frac{y+1}{2} \right)}{\Gamma(y/2)} \frac{\Gamma(x/2)}{\Gamma \left( \frac{x+1}{2} \right)} = \prod_{k=0}^{\infty} \left( 1 - \frac{1}{(2k + y + 1)} \right) \left( 1 + \frac{1}{2k + x} \right). \] (3.10)
Therefore we obtain
\[
\ln \frac{\Gamma \left( \frac{y+1}{2} \right)}{\Gamma(y/2)} \frac{\Gamma(x/2)}{\Gamma \left( \frac{x+1}{2} \right)} = \sum_{k=0}^{\infty} \ln \left( \frac{2k+y}{2k+x} \right) \left( \frac{2k+1+y}{2k+1+x} \right) \\
= \sum_{k=0}^{\infty} \left[ \ln \left( \frac{2k+y}{2k+x} \right) - \ln \left( \frac{2k+1+y}{2k+1+x} \right) \right] \\
= \sum_{m=0}^{\infty} (-1)^m \ln \left( \frac{m+y}{m+x} \right),
\]
(3.11)
and the Lemma is again completed. Alternatively, we may directly relate the left side of (3.5) to the right side of (3.6) simply by reordering the double summation as \(\sum_{n=0}^{\infty} \sum_{\ell=0}^{n} = \sum_{\ell=0}^{\infty} \sum_{n=\ell}^{\infty}\).

Based upon a very special case of Lemma 3.1(a) we have

**Corollary 3.2** We have (a) for \(\text{Re } y > -1\)
\[
\int_{0}^{1} \frac{(u^y - 1)}{\ln u} \, du = \ln(y + 1),
\]
(3.12)
(b)
\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\lambda^j} \ln(j + 1) = \int_{0}^{1} \left[ \left( 1 - \frac{u}{\lambda} \right)^k - \left( 1 - \frac{1}{\lambda} \right)^k \right] \frac{du}{\ln u},
\]
(3.13)
and (c)
\[
\sum_{k=0}^{\infty} \left( \frac{\lambda}{\lambda + 1} \right)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\lambda^j} \ln(j + 1) = \frac{1}{2} (\lambda + 1) \ln \left( \frac{2}{\pi} \right).
\]
(3.14)

For part (a), we put \(n = 0, x = 1,\) and \(y \to y + 1\) in Eq. (3.4). For part (b), we use the integral representation of part (a) and evaluate the two binomial sums. For part (c) we find, using part (b),
\[
\sum_{k=0}^{\infty} \left( \frac{\lambda}{\lambda + 1} \right)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\lambda^j} \ln(j + 1) \\
= \sum_{k=0}^{\infty} \int_{0}^{1} \frac{1}{(\lambda + 1)^k} \left[ \lambda - u \right]^k - (\lambda - 1)^k \right] \frac{du}{\ln u} \\
= \frac{(\lambda + 1)}{2} \int_{0}^{1} \left( 1 - u \right) \frac{du}{1 + u} \ln u = \frac{1}{2} (\lambda + 1) \ln \left( \frac{2}{\pi} \right).
\]
(3.15)

Of many ways to evaluate the last integral of Eq. (3.15), one may use [17, p. 542]. The Corollary is demonstrated and Eq. (3.3) is affirmed.
Moreover, we must recover the values \( \zeta(-n) = (-1)^n B_{n+1}/(n + 1) \) for \( n \) an integer from Eq. (2.42). For \( n \) even this includes the trivial zeros of the zeta function, whereby \( B_{2m+1} = 0 \) for \( m \geq 1 \). We have from Eq. (2.42)

\[
\zeta(-n) = \frac{1}{(1 - 2^{n+1})} \frac{1}{(1 + \lambda)} \sum_{k=0}^{\infty} \left( \frac{\lambda}{1 + \lambda} \right)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\lambda^j} (j+1)^n. \tag{3.16}
\]

For \( \lambda = 1 \) this equation recovers the old formula of Worpitsky for the Bernoulli numbers \([7, 35]\).

As a byproduct of this work we obtain interesting infinite series (or products) for fundamental constants such as the Euler constant and \( \ln \pi \). The rapidity of convergence may make some of these suitable for computational applications. We omit many such binomial summations that may be obtained by methods very close to Propositions 1.16 and 1.18.

With respect to the right side of Eq. (1.12), integers \( m \) that are powers of 2 play a special role. It is only these contributions for which \( 1 - \Lambda(m) = 1 - \ln 2 > 0 \), while all other integral powers of primes give \( 1 - \Lambda(m) < 0 \).

Within the Li criterion for the Riemann hypothesis \([26]\), the sum \( S_2(n) \) is given by (e.g., \([11]\))

\[
S_2(n) = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \left( \frac{d}{ds} \right)^m [\ln(s - 1)\zeta(s)]_{s = 1}.
\tag{3.17}
\]

Therefore, by using the Hadamard product representation of the Riemann zeta function, it is easy to see that \( S_2(n) \) is connected with sums over its nontrivial zeros. If \( S_2(n) \) has linear or sublinear growth in \( n \), the Riemann hypothesis holds. According to our decomposition (1.15), the validity of the Riemann hypothesis is equivalent to the sum \( S_\Lambda(n) \) having linear or sublinear growth in \( n \). In fact, we conjecture (see below) that \( |S_\Lambda(n)| = O(n^{1/2 + \epsilon}) \) for \( \epsilon > 0 \) arbitrary. (This conjecture is stronger than the Riemann hypothesis itself.)

In regard to Proposition 1.7, we put

\[
\Delta_2(x) = \sum_{n \leq x} d(n) - x \ln x - (2\gamma - 1)x = O(x^{\alpha_2^{+\epsilon}}), \quad \epsilon > 0,
\tag{3.18}
\]

wherein \( \alpha_2 \) is the least such number for every positive \( \epsilon \). Dirichlet knew that \( \alpha_2 \leq 1/2 \) and the best result to date may be \( \alpha_2 \leq 131/416 \) \([21]\). In fact, Huxley showed that \( \Delta_2(x) = O(x^{23/73} \ln^{461/146} x) \) \([20]\) and improved this very recently to \( \Delta_2(x) = O(x^{131/416}) \) \([21]\). The smallest possible value of \( \alpha_2 \) is \( 1/4 \), and we prove that \( |S_\gamma(n) + n| = O(n^{1/4}) \) (see after Eq. (3.19) below). The method of Propositions 1.1 and 1.7 is easily extended to many other pole-free combinations of Dirichlet series.

Proposition 1.14, its special case Proposition 6.1 of Ref. [9], or other series representations of the Stieltjes constants may be used to obtain alternative summation representations of the \( O(n) \) sum \( S_\gamma(n) \).

Numerical investigations indicate that \( S_\gamma(n) \) is close to \( -n \) together with a small oscillatory component, while \( S_\Lambda(n) \) is close to \( n \) with a small oscillatory component.
Therefore, the crucial sum $S_2(n)$ appears to arise from substantial cancellation of $O(n)$, leaving a slowly growing, oscillatory contribution. A demonstration that $S_2(n)$ satisfies a one-sided subexponential bound would suffice to verify the Riemann hypothesis.

As a point of emphasis, the Riemann hypothesis will only fail if a Li–Keiper constant $\lambda_k$ becomes exponentially large in magnitude and negative. In particular, the Criterion (c) of Ref. [6] now carries over to the crucial subsum $S_{2\Lambda}(n)$. Therefore the Riemann hypothesis is invalid only if this sum becomes negative and exponentially large in magnitude for some $n$. We may spell this out in the following way.

**Condition 3.3** Suppose that there is a value of $1/2 \leq p < \infty$ such that $|S_{2\Lambda}(n)| = O(n^p)$.

Then upon Condition 3.3 the Riemann hypothesis will follow as a Corollary. It is compelling within the Li criterion approach that the optimal order of the sum $S_{2\Lambda}(n)$ is not necessarily required. As indicated, we suspect that the lowest possible order of this sum is close to $O(n^{1/2})$.

From Fejér’s formula for the asymptotic form of $L_n^\alpha(x)$ we have for $n \to \infty$

$$L_n^1(x) = \frac{1}{\sqrt{\pi}} e^{x^2/2} x^{-3/4} (n - 1)^{1/4} \cos(2\sqrt{(n - 1)x} - 3\pi/4) + O(n^{-1/4}), \quad x > 0.$$  \hspace{1cm} (3.19)

Therefore we now show that the oscillatory component of $S_n^\gamma(n) + n$ grows as $O(n^{1/4})$. Indeed, the last two terms on the right side of Proposition 2(d) contribute at $O(n^{1/4})$, with $\cos(2\sqrt{n - 1}y + \phi)$ factors. The last three terms on the right side additionally contribute at the next lowest order of $n^{-1/4}$. For the remaining term on the right side of Proposition 2(d) we have

$$- \sum_{\nu=1}^N \frac{L_n^1(\ln \nu)}{\nu} = -n + \sum_{\nu=2}^N \frac{L_n^1(\ln \nu)}{\nu} = -n + O(N^{1/2+\delta} n^{1/4}),$$  \hspace{1cm} (3.20)

with $\delta > 0$. We may therefore summarize as

**Corollary 3.4** As $n \to \infty$ we have $|S_n^\gamma(n) + n| = O(n^{1/4})$.

Besides the indications given in Ref. [11] that the Laguerre calculus is pervasive within the Li–Keiper formulation of the Riemann hypothesis, we have very recently systematically presented the structural origins of this framework [12]. The Li–Keiper constants arise as a sum over complex zeta zeros of a Laplace transform of the Laguerre polynomial $L_n^1(x)$. We have

$$L_n^1(\rho) \equiv \int_0^\infty e^{-\rho u} L_n^1(u)du = 1 - \left(1 - \frac{1}{\rho}\right)^n,$$  \hspace{1cm} (3.21)

that vanishes for $\rho_j = (1 - e^{2\pi ij})^{-1}$, with $j = 1, \ldots, n - 1$. These Laplace transform zeros have real part $1/2$:

$$\rho_j = \frac{1}{2} \frac{(1 - e^{-2\pi ij/n})}{[1 - \cos(2\pi j/n)]} = \frac{1}{2} \left[1 + i \cot\left(\frac{\pi j}{n}\right)\right].$$  \hspace{1cm} (3.22)
That these quantities lie on the critical line may be more than just a curiosity and may partly explain the distinguished role of the polynomial $L_{n-1}^1$.

Furthermore, we recall that [11, Appendix I]

$$L_n(s) = \int_0^1 x^{s-1} L_{n-1}^1(-\ln x) dx = \frac{n}{s} \frac{\, _2F_1\left(1-n, 1; 1; \frac{1}{s}\right)}{\, _2F_1\left(1-n, 1; 2; \frac{1}{s}\right)}.$$  \hspace{1cm} (3.23)

in terms of the Gauss hypergeometric function. We put $L_n(s) = s^n L_n(s)$ and have

**Corollary 3.5** We have the functional equation

$$L_n(s) = -\left(1 - \frac{1}{s}\right)^n L_n(1-s),$$  \hspace{1cm} (3.24)

and

$$L_n(s) = (-1)^{n+1} L_n(1-s),$$  \hspace{1cm} (3.25)

that follows immediately.

Equation (3.24) may be obtained directly or by applying the transformation formula [17]

$$\quad \frac{\, _2F_1(\alpha, \beta; \gamma; z)}{\left(1 - z\right)^{-\alpha}} = \frac{\, _2F_1(\alpha, \gamma - \beta; \gamma; z)}{\frac{z}{z - 1}}$$ \hspace{1cm} (3.26)

to the right side of Eq. (3.23).

Propositions 1.1, 1.3, 1.4, 1.7, 1.12 and other results that we have obtained help to expose more of the analytic structure of the Stieltjes and $\eta_j$ constants. We have obtained novel integral and other representations of the Stieltjes constants that enable new integral and series representations of the sums $S_\gamma(n)$ and $S_2(n)$. The growth behavior of $S_{2\lambda}(n)$ and $S_2(n)$ have direct implication for the validity of the Riemann hypothesis.

**Acknowledgements.** I thank R. Kreminski for access to high-precision values of $\gamma_k$.

**References**

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Washington, National Bureau of Standards (1964).

[2] O. R. Ainsworth and L. W. Howell, *An Integral Representation of the Generalized Euler–Mascheroni Constants*, NASA Technical Paper 2456 (1985).

[3] P. Amore, Convergence acceleration of series through a variational approach, arXiv: math-ph/0408036 v4 (2005), *J. Math. Anal. Appl.* 323, 63–77 (2006).

[4] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press (1999).
[5] R. P. Boas, Jr., Chapter 8: Convergence, divergence, and the computer, in: *Mathematical Plums*, R. Honsberger, ed., Math. Assoc. of America (1979).

[6] E. Bombieri and J. C. Lagarias, Complements to Li’s criterion for the Riemann hypothesis, *J. Number Theory* 77, 274–287 (1999).

[7] L. Carlitz, Remark on a formula for the Bernoulli numbers, *Proc. Amer. Math. Soc.* 4, 400–401 (1953).

[8] M. W. Coffey, Relations and positivity results for derivatives of the Riemann ξ function, *J. Comput. Appl. Math.* 166, 525–534 (2004).

[9] M. W. Coffey, New results on the Stieltjes constants: Asymptotic and exact evaluation, *J. Math. Anal. Appl.* 317, 603–612 (2006); arXiv:math-ph/0506061.

[10] M. W. Coffey, New summation relations for the Stieltjes constants, *Proc. Royal Soc. A* 462, 2563–2573 (2006).

[11] M. W. Coffey, Towards verification of the Riemann hypothesis, *Math. Phys., Analysis and Geometry* 8, 211–255 (2005).

[12] M. W. Coffey, The theta-Laguerre calculus formulation of the Li–Keiper constants, *J. Approx. Theory* 146, 267–275 (2007).

[13] M. W. Coffey, Conjecturing the optimal order of the components of the Li–Keiper constants, *AMS Proc. Contemp. Math.* 457, 135–159 (2008).

[14] N. G. de Bruijn, Integralen voor de ζ-functie van Riemann, *Mathematica (Zutphen)* B5, 170–180 (1937).

[15] H. M. Edwards, *Riemann’s Zeta Function*, Academic Press, New York (1974).

[16] X. Gourdon and P. Sebah, *Collection of formulae for the Euler’s constant γ*, http://numbers.computation.free.fr/Constants/Gamma/gammaFormulas.html, (2003).

[17] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York (1980).

[18] H. Hasse, Ein Summierungsverfahren für die Riemannsche Zeta-Reihe, *Math. Z.* 32, 458–464 (1930).

[19] J. Havil, *Gamma: Exploring Euler’s Constant*, Princeton University Press (2003).

[20] M. N. Huxley, Exponential sums and lattice points, II, *Proc. London Math. Soc.* 66, 279–301 (1993).

[21] M. N. Huxley, Exponential sums and lattice points, III, *Proc. London Math. Soc.* 87, 591–609 (2003).

[22] A. Ivić, *The Riemann Zeta-Function*, Wiley (1985).

[23] J. L. W. V. Jensen, *Sur la fonction ξ(s) de Riemann*, C. R. Acad. Sci. Paris 104, 1156–1159 (1887).
[24] A. A. Karatsuba and S. M. Voronin, *The Riemann Zeta-Function*, Walter de Gruyter, New York (1992).

[25] R. Kreminski, Newton–Cotes integration for approximating Stieltjes (generalized Euler) constants, *Math. Comp.* **72**, 1379–1397 (2003).

[26] X.-J. Li, The positivity of a sequence of numbers and the Riemann hypothesis, *J. Number Th.* **65**, 325–333 (1997).

[27] J. J. Y. Liang and J. Todd, The Stieltjes constants, *J. Res. Natl. Bur. Stand.* **768**, 161–178 (1972).

[28] Yu. V. Linnik, The dispersion method in binary additive problems, *Amer. Math. Soc.* (1963).

[29] http://mpmath.googlecode.com/svn/trunk/doc/build/functions/zeta.html

[30] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monats. Preuss. Akad. Wiss.* **671** (Nov. 1859).

[31] H. M. Srivastava and J. Choi, *Series Associated With the Zeta and Related Functions*, Kluwer (2001).

[32] T. J. Stieltjes, *Correspondance d’Hermite et de Stieltjes*, Volumes 1 and 2, Gauthier-Villars, Paris (1905).

[33] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford University Press, Oxford (1986).

[34] D. V. Widder, *The Laplace Transform*, Princeton University Press (1946).

[35] J. Worpitsky, Studien über die Bernoullischen und Eulerschen Zahlen, *J. Reine Angew. Math.* **94**, 203–232 (1883).

Mark W. Coffey
Department of Physics
Colorado School of Mines
Golden, CO 80401
USA
mcoffey@mines.edu