The Poincare lemma, antiexact forms, and fermionic quantum harmonic oscillator

Radosław Antoni Kycia$^{1,2,a}$

$^1$Masaryk University
Department of Mathematics and Statistics
Kotlářská 267/2, 611 37 Brno, The Czech Republic

$^2$Cracow University of Technology
Faculty of Physics, Mathematics and Computer Science
Warszawska 24, Kraków, 31-155, Poland

$^a$kycia.radoslaw@gmail.com

Abstract

Connection of Poincaré lemma and resulting Edelen homotopy operator, Bittner’s operator calculus and the fermionic quantum harmonic oscillator will be presented. Dual concept to homotopy operator will be derived in terms of extrusions. The considerations are presented in the local setup in star-shaped region natural for the validity of the Poincaré lemma.

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1 Introduction

The Poincaré lemma is one of the most important tools of local exterior calculus. There are various formulations and the most general one is the following well-known form (Corollary 4.1.1 of [4])

Corollary 1. (The Poincaré lemma) $H^*(\mathbb{R}^n) = H^*(\text{point}) = \begin{cases} \mathbb{R}, & (n = 0) \\ 0, & (n > 0) \end{cases}$

It can be formulated in the simpler form as (e.g. Theorem 11.49 of [15])

Theorem 1. (The Poincaré lemma) If $U$ is a star-shaped open subset of $\mathbb{R}^n$, then every closed covector field on $U$ is exact.
There is standard proof of the Lemma using Homotopy Invariance of de Rham complex (see Paragraph 29 of [19] or in [1]), which introduces some example of homotopy operators which locally inverts exterior derivative, however, the importance of this entity was pointed out explicitly by D.G.B. Edelen in [8] with applications. The homotopy operator of Edelen (to be defined below) distinguish a new (local) class of differential forms that split the exterior module into direct sum.

The homotopy operator, together with exterior differential make into, as it will be shown below, some example of operator calculus discovered by R. Bittner [2] to generalize differentiation and integration operations as it will be described below. This approach will occur to be fruitful in recovering the structure of creation and annihilation operators for fermionic particles used in Quantum Mechanics, e.g., Chapter 5 of [5]. There is, however a problem with the selection of Hilbert space on which the operators act. Therefore the Stokes theorem is used as a tool to construct dual operator to homotopy using extrusion. This formulation has strong connections with geometric integration theory formulated by J. Harrison and co-workers [11] and her version of Poincaré lemma [12]. The extrusion was found useful also in construction of discrete version of the Poincaré lemma [6, 7]. Another way that will lead to Hilbert space is to use densities.

The main aim of this paper is to present the view which unifies all these viewpoints and emphasize the interconnection between these different disciplines, and it can be treated as an ‘interpolation’ that is useful in most cases met in smooth differential geometric applications. All considerations will be local in a star-shaped region, and hence Poincaré lemma will be valid - closed and exact forms are locally the same.

The paper is organized as follows: In the next section, the formula for Edelen homotopy operator will be derived from the Homotopy Invariance of de Rham complex, and the definition of antieexact forms will be recalled. Then the next section will contain connection of these formulas with operator calculus and fermionic quantum harmonic oscillator. Finally, the dual version of the Poincaré lemma using extrusions will be formulated. The paper concludes in a discussion on possible applications. In the Appendix development of the theory of homotopy operator for complex manifold is given.

2 Homotopy operator

In this section, the connection between standard proof of the Poincaré lemma and the Edelen homotopy operator will be provided.

To begin with, introduce on the module of forms $\Omega(M \times \mathbb{R})$, where $M$ is some manifold (or open subset of a manifold) the operator

$$G\omega := \int_0^1 (\partial_t \omega) dt,$$

(1)
for $\omega \in \Omega(M \times \mathbb{R})$ and $\nu \in \mathfrak{X}(M \times \mathbb{R})$. Define now a homotopy $F : [0, 1] \times M \to M$ between $f$ and $g$, that is $F(0, .) = f(\cdot)$ and $F(1, \cdot) = g(\cdot)$. Using the homotopy we can define the operator for $\omega \in \Omega(M)$ by

$$\tilde{H}\omega = G \circ F^*(\omega). \quad (2)$$

This operator has important property, which can be introduced using Homotopy Invariance Formula (see Paragraph 29 of [19]), namely

**Theorem 2. (Homotopy Invariance Formula for de Rham complex)**

$$dG + Gd = i^*_1 - i^*_0, \quad (3)$$

where $i_t(x) = (t, x)$ for $t \in \mathbb{R}$ and $x \in M$.

Using this formula we have the well known formula

$$\tilde{H}d + d\tilde{H})\omega = GdF^*\omega + dGF^*\omega = i^*_1 F^*\omega - i^*_0 F^*\omega = g^*\omega - f^*\omega. \quad (4)$$

In order to derive Edelen homotopy operator $H$ one have to choose a special form of homotopy, namely the homotopy between identity ($g(x) = x$) and the constant map ($f(x) = x_0$) for some fixed point $x_0 \in M$. To provide correct definition of the homotopy we assume that the $M$ is star-shaped region that is the rays from $x$ to $x_0$ are within $M$. For such homotopy we define

**Definition 1. (Edelen homotopy operator)**

$$H\omega := \int_0^1 K_{\omega F(t, x)} t^{k-1} dt, \quad (5)$$

for $\omega \in \Omega(M)$, $K := (x - x_0)^i \partial_i$, $k = \deg(\omega)$, and $F(t, x) = x_0 + t(x - x_0)$ is a homotopy between the identity map $I : x \to x$ and $x \to x_0$ constant map. The form $\omega$ under the integral is evaluated at the point $F(t, x)$.

The form of the operator is a special case of $\tilde{H}$ for the homotopy $F(t, x) = x_0 + t(x - x_0)$ and its explicit derivation is simple application of pullback by $F$. This way of deriving of it is new and more general that presented in [8]. It has various properties described by Theorem 5-3.1 of [8], from which the most important in later use is $H^2 = 0$ and results from the double application of the inner product under the integral of (5).

The operator $H$ has its own Homotopy Invariance Formula

**Theorem 3. (Homotopy Invariance Formula for Edelen H operator)**

$$dH + Hd = I^* - s^*_{x_0}, \quad (6)$$

where $s_{x_0}(x) = x_0$ is the constant map and $I$ is the identity.
This formula was provided in Theorem 5-3.1 of [8] as a piecewise definition
\[
\begin{cases}
    Hd + dH = I, & \text{on } \Omega^k, k > 0, \\
    (Hdf)(x) = f(x) - f(x_0) & \text{for } f \in \Omega^0,
\end{cases}
\] (7)
which results from the fact that the pullback along the constant function \(s^*_x \omega = 0\) for \(\deg(\omega) > 0\), and from the fact that \(K . f = 0\).

One can also note that (6) is correct for any homotopy \(F\) between identity and the constant map, however, in such a case, the explicit formula (5) is not valid.

If a form \(\omega\) fulfills \(d\omega = 0\) then it is \emph{closed}, and in the star-shaped region \(M\) (which we will assume hereafter), by the Poincaré lemma, it is also \emph{exact}, which means that there is a form \(\alpha\) of degree \(\deg(\alpha) = \deg(\omega) - 1\) such that \(\omega = d\alpha\). The exact (and hence closed) forms form a subspace \(E(M)\) of \(\Omega(M)\). By Lemma 5-4.1 and 5-4.2 of [8], the operator \(dH\) is the projection operator from \(\Omega(M)\) to \(E(M)\), and \(d\) locally inverses \(H\) on \(E(M)\) by (6). In addition \(E^0(M)\) - the set of exact functions over \(M\) is empty.

Using \(H\), as in [8], we can also single out \emph{antiexact} forms that are image of the complementary projection operator \(Hd = I^* - dH - s^*_x\). This means that for antiexact form \(\omega\) there is an exact form \(\alpha = d\beta\) such that \(\omega = H\alpha\). They form a submodule \(A(M)\) of \(\Omega(M)\) and can be characterized by (Lemma 5-5.1 of [8]) \(K . \alpha = 0, \alpha_{x_0} = 0\) for \(\alpha \in A(M)\). In addition \(A^n(M)\) for \(n = \dim(M)\) is empty set.

These properties of exact and antiexact spaces provide finer than in [8] (compare with Chapter 5-6 therein), in the form of the diagram 1.

At each degree \(k \geq 0\) there is \(\Omega^k = E^k \oplus A^k\). The relations \(d^2 = 0 = H^2\) when traversing up or down are also visible. From the diagram it is easy visible that for fixed \(0 < k < n\) (assume that \(\dim(M) > 0\)) there is separate ’subdiagram’ depicted in Fig. 2 which will be starting point for fermionic harmonic oscillator construction below.

For \(k = 0\) the kernel \(Ker(d)\) is the field over which \(\Omega\) is the vector space, e.g., \(\mathbb{R}\). This field can be treated as a constant 0-forms and therefore, it is exact.

The general formula (6) is the starting point for considering operator algebra of \(H, d, I\) and \(s_{x_0}\) in terms of operator calculus of Bittner [2] which will be the subject of the next section.

3 Bittner’s operator calculus

The Bittner’s operator calculus [2] is a way to redefine derivative and integral in an abstract form using the properties
\[
\frac{d}{dx} \int_q^x f(x')dx' = f(x), \quad \int_q^x \frac{df(x')}{dx'} dx' = f(x) - f(q),
\] (8)
for \(f\) being e.g. \(C^1\) function.

It was generalized [2] as follows
Figure 1: Decomposition of $\Omega$ into exact and antiexact subspaces with respect to the degree.

$$
0 \xleftarrow{d} E_n \xrightarrow{H} 0
$$

$$
0 \xleftarrow{d} E_{n-1} \xrightarrow{H} A^{n-1} \xrightarrow{H} 0
$$

$$
0 \xleftarrow{d} E_{n-2} \xrightarrow{H} A^{n-2} \xrightarrow{H} 0
$$

$$
0 \xleftarrow{d} \ldots \xrightarrow{H} \ldots \xrightarrow{H} 0
$$

$$
0 \xleftarrow{d} E^1 \xrightarrow{H} A^1 \xrightarrow{H} 0
$$

$$
0 \xleftarrow{d} \mathbb{R} \xrightarrow{H} A^0
$$

Figure 2: Part of the decomposition from Fig.1 for $0 < k < n$.

$$
0 \xleftarrow{d} E^k \xrightarrow{H} A^{k-1} \xrightarrow{H} 0
$$
Figure 3: Part of decomposition of $\Omega^{k-1} = A^{k-1} \oplus E^{k-1}$ for fixed $0 < k < n$. Note that $\ker(d) = E^{k-1}$ and $\text{im}(d) = E^k$.

**Definition 2.** Consider two linear spaces $L^0$ and $L^1$ and define an abstract derivative as surjective mapping $S \in \text{Hom}(L^1, L^0)$. Element of $\ker(S)$ are called constants of the derivative $S$. Define also $T_q \in \text{Hom}(L^0, L^1)$ for some constant $q \in \ker(S)$ such that

$$ST_q = I, \quad T_qS = I - s_q,$$

(9)

where $s_q$ is projection operator on $\ker(S)$ associated with $q$. $T_q$ is called an abstract integral.

For instance in (8) $s_qf = f(q)$.

Let us consider the diagram from Fig. 2 for $k > 1$. In this case $Hd = I$ on $A^{k-1}$ and $dH = I$ on $E^k$ and therefore there is no projection on boundary data and $H$ and $d$ are inverses to each other. In conclusion $dH + Hd = I^*$ since $s^*_x_0$ term vanishes. This is also true for $k = n$.

For $k > 0$ this can be also seen as a mapping on Fig. 3. In this case at the level $k - 1$ the space $\Omega^{k-1} = A^{k-1} \oplus E^{k-1}$. This is decomposition according to the action of $d$ since $\ker(d) = E^{k-1}$ and $\text{im}(d) = E^k$. In this view the formula $Hd = I^*$ is the special case of the second formula of (9) where $s^*_x_0 = 0$.

For $k = 1$ the second viewpoint of Fig. 3 is proper since the pullback along the constant $s^*_x_0$ is essential, and therefore, the formula (6) leads to

$$dH = I, \quad Hd = I^* - s^*_x_0.$$

(10)

In this case the resemblance to (9) is even closer with $s_q = s^*_x_0$. In case of $A^0$ the ‘constants’ of $d$ are constants of derivative that are constants and pullback projects on them any function from $A^0$.

This observation allows us to formulate abstract differential equations on $\Omega$ using $d$ and $H$ as ‘derivate’ and ’integral’. However, these operations are nilpotent, which put additional constraints on such ‘differential calculus’ and suggest that they can be used to define fermionic harmonic oscillator. This track will be followed in the next section.

\footnote{Note that $I^* = I$.}
4 Fermionic harmonic oscillator

Fermionic quantum harmonic oscillator is defined by the Hamilton operator

$$\hat{H} = a^\dagger a - aa^\dagger,$$  \hspace{1cm} (11)

where creation $a^\dagger$ and annihilation $a$ operator fulfills the anticommutation rules

$$\{a, a\} = 0, \quad \{a^\dagger, a\} = 0, \quad \{a, a^\dagger\} = I.$$  \hspace{1cm} (12)

These operators act on Hilbert space, however, we restrict ourselves to vector space only, and then $a^\dagger$ is an operator which is not the adjoint of $a$ since there is no inner product which can be used to form such adjoint. The standard representation is

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. $$  \hspace{1cm} (13)

The algebra of $d$ and $H$ is the same as for $a$ and $a^\dagger$, namely,

$$dd = 0, \quad HH = 0, \quad Hd + dH = I^s - s_{x_0},$$  \hspace{1cm} (14)

where the term $s_{x_0}$ is zero when $\deg(\omega) = k > 0$. It is therefore natural, by analogy, to define fermionic Hamiltonian operator

$$\hat{H} := Hd - dH.$$  \hspace{1cm} (15)

The most important problem in application is to solve the eigenvalue problem for (15), namely,

$$\hat{H}\omega = \lambda\omega,$$  \hspace{1cm} (16)

where $\lambda \in \mathbb{R}$ and $\omega \in \Omega^k$. We have to consider three cases:

- $0 < k < n$: The equation (15) is of the form

$$2Hd\omega = (\lambda + 1)\omega, \hspace{1cm} (17)$$

and we are left in two cases:

  - $\lambda = -1$: for which $Hd\omega = 0$ that is $\omega \in \text{Ker}(Hd)$, which gives that $\omega \in \mathcal{E}^k$.
  
  - $\lambda \neq -1$: since $Hd\omega \in \mathcal{A}^k$ so $\omega \in \mathcal{A}^k$. Therefore $Hd\omega = \omega$ and the equation (15) is $2\omega = (\lambda + 1)\omega$, which gives $\lambda = 1$ only.

- $k = 0$: take $f \in \Omega^0$, then $Hf = 0$. If $f \in \mathcal{E}^0 = \text{ker}(S)$ is a constant function then the eigenvalue problem for (15) has the trivial solution $f = 0$. Therefore we assume that $f \in \mathcal{A}^0$. Then $Hdf = f - fx_0$ and the eigenvalue problem is

$$f - fx_0 = \lambda f \quad \iff \quad (1 - \lambda)f = fx_0.$$  \hspace{1cm} (18)

For $f \in \mathcal{A}^0$ we have $fx_0 = 0$ and there are two cases.
\( \lambda = 1 \): then \( f \) is an arbitrary element of \( A^0 \).

\( \lambda \neq 1 \): then \( f = 0 \).

- \( k = n \): let \( \mu \in E^n \), then \( d\mu = 0 \) and \( dH\mu = \mu \). The eigenvalue problem for (15) has the form

\[-dH\mu = \lambda \mu \iff (\lambda + 1)\mu = 0, \tag{19}\]

which gives two cases:

- \( \lambda = -1 \): then \( \mu \in E^n \) is arbitrary.

- \( \lambda \neq -1 \): then \( \mu = 0 \).

The above computation shows that the harmonic oscillator for \( 0 < k < n \) only picks exact (\( \lambda = -1 \)) or antiexact (\( \lambda = 1 \)) form and does not impose additional conditions. For \( k = 0 \) and \( k = n \) there is only antiexact or exact solution respectively. This shows that the tower of states from Fig. 1 has a deficiency at the top and bottom.

As in Quantum Mechanics [5] there is also a top-down method for base generation. We present two cases:

- Let \( \omega \in E^k \), \( k > 0 \). Then (since \( d\omega = 0 \)) locally \( \omega = d\mu \) for \( \mu \in A^{k-1} \). Then \( H\omega = -dHd\omega = -d\mu = -\omega \), where the property \( dHd = d \) of [8] was used. Therefore such \( \omega \) is an eigenvalue \( \lambda = -1 \) eigenvector.

- Likewise, let \( \omega \in A^k \), \( k < n \). Then \( H\omega = 0 \) and therefore \( \omega = H\mu \). Finally, \( H\omega = HdH\mu = H\mu = \omega \), where the property \( HdH = H \) of [8] was used. Therefore \( \omega \) is an eigenvector to the eigenvalue \( \lambda = 1 \).

These two cases completely describe the diagram from Fig. 2 and show how starting from one eigenvalue obtain the remaining one.

The missing part for the full analogy with Quantum Mechanics is a Hilbert space. In the next section, we present some consolation to this issue - the dual space and the analog of \( H \) operator.

## 5 Extrusion and duals

To find dual for differential, the natural object to pair with exact and antiexact differential forms is some (sub)manifold \( M \) or in our local case a regular star-shaped subset to provide that the antiexact forms and homotopy operator are properly defined, that is the pairing is

\[(M, \omega) := \int_M \omega. \tag{20}\]
Then we can use the Stokes theorem which in the simplest form states (see, e.g., [19, 15]) that for compactly supported \( n - 1 \) form \( \omega \) on the orientable manifold \( M \) there is

\[
\int_M d\omega = \int_{\partial M} i^\ast \omega, \tag{21}
\]

where in the right-hand part the inclusion \( i : \partial M \hookrightarrow M \) was used, and we usually omit such inclusion as it is the standard approach in literature. In this sense, the dual to \( d \) is the boundary operator \( \partial \), which is a well-known fact. The less known is the dual for the homotopy operator \( H \). This involves the construction of the analog of the prism operator [13] from algebraic topology for the case of smooth manifolds. This task is easy performed using extrusions, which will be our main focus now.

The term ‘extrusion’, originated from material science, in differential geometry, to our best knowledge, appeared in the context of discretization of differential equations in [3] as it is indicated in [7]. It was also used in geometric integration theory of J. Harrison to make dual to the operation of insertion of a vector into a differential form on the level of Dirac chains, see [11] for details. However, the concept presented here is closer to the ‘flowing chain’ of [11]. We use it in a ‘smooth’ case in an analogous way to define the dual to the ‘insertion operator’ and then also the dual to (5). The definition of extrusion is provided in Definition 10.1 of [7]:

**Definition 3.** (Extrusion) Given a manifold \( M \), and \( S \), a \( k \)-dimensional submanifold of \( M \), and a vector field \( X \in \mathfrak{X}(M) \), we call the manifold obtained by sweeping \( S \) along the flow of \( X \) for time \( t \) as the extrusion of \( S \) by \( X \) for time \( t \), and denote \( E^t_X(S) \). The manifold \( S \) carried by the flow for time \( t \) will be denoted \( \phi^t_X(S) \).

If a vector field is singular for some \( 0 < \tau < t \) or tangential to \( S \) then it can induce some pathological situations, e.g., \( E^t_X \) will be of the same dimension as \( S \) or will have singularities. Therefore in our approach, it is assumed that \( X \) is non-tangential and has no singularities in \([0; t]\) interval. The situation is presented in Fig. [1] part a).

The extrusion operation is an analog of the prism operator for simpimplexes [13]. The difference is that the base of the extrusion is a submanifold \( S \) and not a simplex.

The standard theorem for extrusion is the following (compare with Lemma 10.1 of [7] and with Theorem 8.2.3 of [11])

**Lemma 1.** For a submanifold \( S \subset M \) and a nonsingular vector field \( X \) on \( M \) that is not tangent to \( S \), then for \( \text{deg}(\omega) > 0 \) there is

\[
\int_S X \cdot \omega = \frac{d}{dt}\big|_{t=0} \int_{E^t_X(S)} \omega, \tag{22}
\]

or alternatively,

\[
\int_0^t \left( \int_{\phi^\tau_X(S)} X \cdot \omega \right) d\tau = \int_{E^t_X(S)} \omega. \tag{23}
\]
The general outline for the proof was remarked in [7] and for simplicial complex in [3]. We present it here in details for smooth case.

**Proof.** Since $X \notin TS$ and since $S$ is a submanifold, we can use the theorem on straightening the flow locally (see e.g. Theorem 9.22 of [12] or Theorem 5.0 in [14]) to get extrusion: Consider coordinate cover around $S$ of the form $(U, \{x^i\})$, where $S \subset U$. By a change of coordinates we can define $S$ to be $x^j = 0$ for $k + 1 < j$ and $j = 1$ where $k = \deg(\omega)$. Then the flow map $\phi_X(\tau, p(x^1, \ldots, x^{k+1}))$ is a diffeomorphism of a neighborhood of $p \in S$ into $(-\epsilon, \epsilon) \times S \subset U$, and defines coordinates in this neighborhood. The flow in the new coordinates is $\partial_{x^1}$ and $X = \phi_X(\partial_{x^1})$ - see Fig. 4, part b). Since the flow is nonsingular we can extend it for the whole interval $\tau \in [0; t]$, which gives 'a prism' that is $E^1_{\partial_{x^1}}$. In these coordinates the integral (23) on the left hand side is exactly the integral over extrusion, and hence on the right hand side.

The extrusion operator has obvious properties (compare with Propostion 8.2.4 of [11]),

- $E^t_X E^r_X = 0$,
- $\{E_X, E_Y \} = 0$ (anticommutator),
- $\int_{E^t_X(S)} \omega = \int_{E^r_X(S)} f \omega$.

We are ready for defining the dual of $\tilde{H}$ operator, that is,

**Definition 4.** The dual operator to $\tilde{H}$ acting on submanifold $S$ is $\tilde{H}^*_s(S) = F(E^r_X(S))$, where $F$ is a homotopy.

The derivation is straightforward by means of Lemma [1]

$$(S, H \omega) = \int_S \int_0^1 (\partial_{x^1} E^r_x \omega) d\tau = \int_{E^r_X(S)} F^*_r \omega = \int_{F(E^r_X(S))} \omega. \quad (24)$$
For Edelen homotopy operator $H$, the homotopy has the form $F(t, x) = x_0 + t(x - x_0)$ and this specifies explicit form of the operator. In this case one have to assume that $S$ is star-shaped. This proof is presented for Edelen operator in graphical form in Fig. 5. Note that the resulting 'cone' that is the image of $H_s$ is not a subspace of original manifold $M$, instead, it can be seen as a graph of the image of the extrusion in $[0, 1] \times M$ by the homotopy into $M$ or the submanifold $S$ covered by the image of the homotopy $F$.

The remaining question is, what are dual annihilators to exact and antiexact forms. In case of exact form, by Poincaré lemma, they are of the form $d\mu$ and therefore by Stokes theorem all boundaries (coexact) $\partial S$ annihilate exact forms, which is the well-known result. In case of antiexact forms, which are $H\omega$ locally, the annihilator is of the form of the 'cone' (coantiexact) $H^*_s(S)$ due to the Stokes theorem and the property $HH = 0$.

The above definition allows one to dualize $Hd + dH$ operator. Obviously dual to $I^*$ is the identity operator and therefore for $k = \dim(S) > 0$ one gets dual to (6) in the form

$$\partial H^*_s + H^*_s \partial = I. \tag{25}$$

The additional definition is required for $k = 0$ since the formulas from Lemma 1 does not work here. However, if the integration will be replaced for $k = 0$ by the evaluation at the point (0-chain), then we have

$$Hdf|_x = f|_x - f|_{x_0}, \tag{26}$$

for $f \in \Omega^0$. This shows that for $k = 0$ we have formula (since $dHf = 0$) dual to (6)

$$\partial H^*_s = I - s_{x_0}, \tag{27}$$

that acts on 0-chains, that is, points in $S$. This formula is presented in Fig. 6. The evaluation at a point (with attached $k$-bivector) is approach from geometric integration theory from [1] and in a smooth case it is needed only for $k = 0$.

The above duality can also be used to rewrite all basic operations in differential geometry, e.g. Lie derivative ($L_X = \{d, X\}$), since we have the dual representation for basic elements of superalgebra, namely for the derivation $d$ of degree 1 and
the antiderivation \( \partial \) of degree \(-1\), for details on derivations see e.g., [19] (problem 4.7) or [17].

In the next section, one way to form a Hilbert space is presented.

## 6 Hilbert space

As it was pointed out in [10] a Hilbert space can be formulated for half-densities, which for orientable manifolds can be considered as top dimension forms with coefficients that are square-integrable functions. This is link between presented above theory and a Hilbert space. We will focus on complex-valued differential forms for generality. The product of two half-densities \( \omega_1 = \rho_1 \text{Vol} \) and \( \omega_2 = \rho_2 \text{Vol} \) is defined as

\[
< \omega_1, \omega_2 > := \int_S \bar{\rho}_1 \rho_2 \text{Vol},
\]

where \( \bar{\rho}_1 \) is the complex conjugate of \( \rho_1 \), and where \( S \) is a compact \( n \)-dimensional subset of a manifold \( M \) of \( \text{dim}(M) = n \). This defines a pre-Hilbert space which can be completed to a Hilbert space [10], however we will not follow this standard path. Instead, we will focus on calculating adjoint operator \( H^\dagger \) to the Edelen homotopy operator \( H \), which fulfills \( < \omega_1, H \omega_2 > = < H^\dagger \omega_1, \omega_2 > \). Some parts of calculations are similar to those of computing different kind of \( H \) adjoint presented in [9].

We can define the inner product (28) not only for top \( n \)-degree forms but for any \( k \)-form by selecting a star-shaped \( S \) of dimension \( k \). Assuming for simplicity that the center of \( S \) is \( x_0 = 0 \) and the homotopy \( F(x,t) = tx \), we have\(^2\)

\[
< \omega_1, H \omega_2 > = \int_S K \text{Vol} t^k \int_0^1 dt \bar{\rho}_1(x) \rho_2(tx) t^{k-1},
\]

\(^2\)In this derivation we will be denoting integral as operators, that is, \( \int dx \, f(x) \) instead of \( \int f(x)dx \).
where \( K = x^\mu \partial_\mu \), and where \( Vol^k \) is a combination of \( k \)-base forms. Due to linearity we can assume that \( Vol^k \) is a simple \( k \)-form. Using spherical coordinates where \( Vol^k = r^{k-1} dr \, d\Omega \), and \( d\Omega \) is a \( k \)-dimensional sphere solid angle element, we have

\[
<\omega_1, H\omega_2 > = \int \hat{K}_\omega d\Omega \int_0^{R(\Omega)} dr \, r^{k-1} \int_0^1 dt \, \rho_1(r, \Omega) \rho_2(tr, \Omega) t^{k-1},
\]

where \( R(\Omega) \) is the radial coordinate of the point in the direction in the angle \( \Omega \), and \( \hat{K} = K/r \). Introducing the new variable \( u = tr \) we get

\[
<\omega_1, H\omega_2 > = \int \hat{K}_\omega d\Omega \int_0^1 dt \int_0^{R(\Omega)/r} du \, u^{k-1} \rho_1(u/t, \Omega) \rho_2(u, \Omega),
\]

Interchanging integration order of iterated integrals gives

\[
<\omega_1, H\omega_2 > = \int \hat{K}_\omega d\Omega \int_0^1 dt \int_{r/R(\Omega)}^{R(\Omega)/r} du \, u^{k-1} \rho_1(u/t, \Omega) \rho_2(u, \Omega).
\]

Finally, renaming \( u \) into \( r \) and introducing as in \([9]\)

\[
e(r/t, \Omega) = \begin{cases} 
1 & r/t \geq r/R(\Omega) \\
0 & r/t < r/R(\Omega)
\end{cases},
\]

we get

\[
<\omega_1, H\omega_2 > = \int_S K_\omega Vol^k \int_0^1 dt \, e(x/t) \rho_1(x/t) \rho_2(x) = < H^\dagger \omega_1, \omega_2 >,
\]

where

\[
e(x/t) = \begin{cases} 
1 & x/t \in \bar{S} \\
0 & r/t \notin \bar{S}.
\end{cases}
\]

We arrived at

\[
H^\dagger \omega_1 = \int_1^\infty e(tx) \rho_1(tx) dt \, K_\omega Vol^k.
\]

We can also compute \( d^\dagger \) for \( k < n \) since for \( k = n, d = 0 \). We have

\[
<\omega_1, d\omega_2 > = \int_S \frac{\partial \rho_2}{\partial x^\mu} dx^\mu \wedge Vol^k = \int_S -\frac{\partial \rho_1}{\partial x^\mu} \rho_2 dx^\mu \wedge Vol^k + \int_{\partial S} \overline{\rho_1} \rho_2 Vol^k.
\]

When we assume that \( \omega_1, \omega_2 \) are from the class of densities that vanish on the boundary \( \partial S \) then \( d^\dagger = -d \). In addition, when we consider \( id \) operator then \( (id)^\dagger = id \), that is, \( id \) is self-adjoint in the assumed class of forms. This resembles the momentum operator of Quantum Mechanics \([5]\), in fact, on the level of the coefficients of the densities it is exactly the momentum operator.

In the next section, some motivations for studying such structures will be provided.
7 Possible applications

The main motivation for this research was to find a suitable structure that can be used to describe the vacuum state in quantum field theory (QFT). There is a formulation of QFT [2] where the vacuum state is an integration measure that allows forming a Hilbert space with respect to this measure. A similar construction is present in quantization [2], where the boundary distributions on the position of particles in phase space are given by half-measures. The results presented above show that this part of geometry contains a lot of connections between differential geometry and operator calculus.

The operator approach to geometric objects as presented above, to our knowledge, is new and can stimulate the development of abstract calculus on differential forms and their duals.

We also hope that due to the local approach and computational nature, it will be useful in technical sciences, e.g., in material science.

8 Conclusions

In this paper, the local result related to the Poincaré lemma was used to derive analogies with operator calculus and Quantum Mechanics. The dual object to homotopy operator of Edelen was provided along with their properties. The approach assumes smoothness, and presented results have an intuitive picture. Therefore the paper serves as an interpolation between a more abstract approach to the subject in geometric integration theory, which for manifolds with ‘weird’ topological properties is inevitable.

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A Edelen homotopy operator for complex manifolds

This section contains an extension of the above theory for complex manifolds. We use the fact that the Edelen homotopy operator does not ‘feel’ the underlying field.

Complex manifold [16] is a smooth even dimensional manifold $M$ with holomorphic structure (of transition maps between coordinate patches). Such man-
ifold has complex structure $J$ which eigenspaces defining split of tangent space $T_pM = T_pM^+ \oplus T_pM^-$, where the $+$ space is spanned by holomorphic vector fields with the base $\{\partial_{z^\nu}\}^n_{\mu=1}$ and the space $-$ is spanned by anti-holomorphic vector fields with the base $\{\partial_{\bar{z}^\nu}\}^n_{\mu=1}$, where $2n = \text{dim}(M)$, and

$$\partial_{z^\mu} := \frac{1}{2} (\partial_{x^\mu} - i\partial_{y^\mu}), \quad \partial_{\bar{z}^\mu} := \frac{1}{2} (\partial_{x^\mu} + i\partial_{y^\mu}),$$

(38)

and where $\{z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n\}$ and $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$ are local complex and real coordinates related by the standard formula $z^\mu = x^\mu + iy^\mu$.

This induces similar structure on cotangent space, where the dual base has $n$ covector base $dz^\mu$ of bidegree $(1,0)$ and covector base $d\bar{z}^\mu$ of bidegree $(0,1)$. This constitutes the base of 1- forms $\Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$. Using exterior product higher bidegree spaces can be constructed.

The exterior derivative $d$ can be decomposed as $d = \partial + \bar{\partial}$, where Dolbeault operators are defined as

$$\partial : \Omega^{p,q}(M) \rightarrow \Omega^{p+1,q}(M), \quad \bar{\partial} := \frac{d}{\partial \bar{z}^\mu}, \quad \partial := \frac{d}{\partial z^\mu},$$

(39)

Since from $d^2 = 0$ it results that $\partial^2 = 0, \bar{\partial}^2 = 0$ and $\partial \bar{\partial} + \bar{\partial} \partial = 0$ therefore they define double complex on $\Omega^{p,q}(M)$.

Selecting a star-shaped region $S \subset M$ we can define, by analogy to (5), the homotopy operator where now $K := (x-x_0)^\mu \partial_{x^\mu} + (y-y_0)^\mu \partial_{y^\mu}$, and the homotopy is $F(t,z) = (1-t)(x-x_0)^\mu, (y-y_0)^\mu)$. It is however more instructive to reformulate $H$ in terms of $z^\mu$ and $\bar{z}^\mu$. In this case

$$K = K^+ + K^-,$$

(40)

where

$$K^+ = (z-z_0)^\mu \partial_{z^\mu}, \quad K^- = \bar{K}^-.$$

(41)

Then homotopy is $F(t,z) = z_0 + t(z-z_0)$ and similar for complex conjugate. This allows one to split $H$ into

$$H = H^+ + H^-,$$

(42)

where

$$H^\pm \omega = \int_{t_0}^{1} K^\pm \omega F(t,z) t^{k-1} dt.$$

(43)

These operators act as follows

$$H^+ : \Omega^{p,q}(M) \rightarrow \Omega^{p-1,q}(M), \quad H^- : \Omega^{p,q}(M) \rightarrow \Omega^{p,q-1}(M),$$

(44)

for $p, q > 1$ and is vanish when $p-1 < 0$ or $q-1 < 0$.

Then similarly to $H$ we have obvious (see [43] and double application of insertion operator to antisymmetric form) properties

$$H^+ H^+ = 0 = H^- H^-,$$

(45)
\[ H^+ H^- + H^- H^+ = 0, \] (46)

which results from \( HH = 0 \). Therefore \( H^\pm \) also define a double complex dual to the Dolbeault complex. The complexes are visualized in Fig. 7.

However, the formula (6) becomes more elaborate

\[ \text{Id} - s^*_{(\bar{z}_0, z_0)} = (\partial + \bar{\partial})(\partial + \bar{\partial})(H^+ + H^-) = (\partial \partial H^+) + (H^- \partial + \partial H^-) + (H^+ \partial + \partial H^+). \] (47)

The formula (47) in general cannot be simplified to corresponding formulas for the pairs \((\partial, H^+)\) and \((\bar{\partial}, H^-)\) as it is presented in the following example. Consider a differential \((1, 0)\) form \( \omega = \bar{z}dz \). Nonzero elements of (47) are

\[
\begin{align*}
\partial H^+ \omega &= (\bar{z}_0 + \frac{1}{2}(\bar{z} - \bar{z}_0))dz, \\
\bar{\partial} H^+ \omega &= \frac{1}{2}(z - z_0)d\bar{z}, \\
H^+ \partial \omega &= -\frac{1}{2}(z - z_0)d\bar{z}, \\
H^- \bar{\partial} \omega &= \frac{1}{2}(\bar{z} - \bar{z}_0)dz.
\end{align*}
\] (48)

Summing these terms up we get \((Hd + dH)\omega = \bar{z}dz = I(\bar{z}dz) - s^*_{(\bar{z}_0, z_0)}(\bar{z}dz)\) as required. Therefore all ingredients of (47) are in general case significant.

There are also two basic cases for which the pairs split into subcomplexes:

- \( \bar{\partial} \omega = 0 \) (holomorphic), \( \omega \in \Omega^{p,0}, p \in \mathbb{N} \) - with no \( d\bar{z} \) terms in local representation, that is \( \omega = \omega(z)_{\mu_1, \ldots, \mu_p}dz^{\mu_1} \land \ldots \land dz^{\mu_p} \). In this case \( H^- \omega = 0 \).
(anti-\(\bar{\partial}\)-exact), and \(\bar{\partial}H^+\omega = 0\). Then (47) has a simple form

\[
H^+\partial + \partial H^+ = I - s_{z_0}^*.
\]

(49)

This defines the subcomplex \((\Omega^{p,0}, \partial, H^+)\).

• \(\partial\omega = 0\) (antiholomorphic), \(\omega \in \Omega^{0,p}, p \in \mathbb{N}\) - with no \(dz\) terms in local representation, that is \(\omega = \omega(z)\mu_1 \cdots \mu_p d\bar{z}^{\mu_1} \cdots d\bar{z}^{\mu_p}\). In this case \(H^+\omega = 0\) (anti-\(\partial\)-exact), and \(\partial H^-\omega = 0\). Then (47) has a simple form

\[
H^-\bar{\partial} + \bar{\partial}H^- = I - s_{\bar{z}_0}^*.
\]

(50)

Likewise, this defines the subcomplex \((\Omega^{0,p}, \bar{\partial}, H^-)\).

Both of these subcomplexes lie on the boundary of Fig. 7. It is also interesting to check these properties for quaternionic manifolds, which deserves another paper.

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