Detecting systematic anomalies affecting systems when inputs are stationary time series

Ričardas Zitikis

School of Mathematical and Statistical Sciences
Western University, Ontario

Risk and Insurance Studies Centre
York University, Ontario

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You are most welcome to e-mail: rzitikis@uwo.ca
Life as usual: ups and downs...

...or is it?


- Inputs \((X_t)_{t \in \mathbb{Z}}\)

- Transfer function \(h\)

- Outputs \((Y_t)_{t \in \mathbb{Z}}\)

- Observable pairs \((X_1, Y_1), \ldots, (X_n, Y_n)\)

Think of \((X_t)_{t \in \mathbb{Z}}\) as a stationary and causal time series
\[ X_0, X_1, \ldots, X_n, X_{n+1}, \ldots \xrightarrow{(\delta_t)} h \xrightarrow{(\varepsilon_t)} Y_0, Y_1, \ldots, Y_n, Y_{n+1}, \ldots \]

- Input risks \((\delta_t)_{t \in \mathbb{Z}}\)
- Output risks \((\varepsilon_t)_{t \in \mathbb{Z}}\)

Think of risks as measurement errors → miscalculations
oversights

- \(Y_t = \begin{cases} 
  h(X_t + \delta_t) & \text{when only inputs are directly affected} \\
  h(X_t) + \varepsilon_t & \text{when only outputs are affected} \\
  h(X_t + \delta_t) + \varepsilon_t & \text{when inputs & outputs are affected}
\end{cases} \)
If $h$ were known, we could use, e.g.,

$$\frac{1}{n} \sum_{i=1}^{n} (h(X_i) - Y_i)^2 \begin{cases} = 0 & \text{when the system is risk free} \\ > 0 & \text{when the system is risk affected} \end{cases}$$

But we only know that $h \in \mathcal{H}$ (model uncertainty)
Our philosophy

We check (for risks) only those systems that were in reasonable order when newly installed

What does “be in reasonable order” mean?

Whose definition to use?

- Manufacturer’s definition? Maybe, but likely only indirectly
- Our definition? Yes, because it is aligned with our goals

...and it is on the next slide
**Definition.** The risk-free outputs $Y_t = h(X_t)$ are in reasonable order if

$$B_n^0 := \frac{1}{n^{1/2}} \sum_{i=2}^{n} |h(X_{i:n}) - h(X_{i-1:n})| = O_P(1)$$

where $X_{1:n} \leq \cdots \leq X_{n:n}$ are the ordered inputs $X_1, \ldots, X_n$.

**Example (to work out intuition).** If $h \in \text{Lipschitz}$, then the risk-free outputs are in reasonable order because

$$B_n^0 \leq \|h\|_{\text{Lip}} \frac{n^{1/2}}{n^{1/2}} \sum_{i=2}^{n} |X_{i:n} - X_{i-1:n}| = \|h\|_{\text{Lip}} \frac{X_{n:n} - X_{1:n}}{n^{1/2}} = \|h\|_{\text{Lip}} \frac{\text{Range}(X_1, X_2, \ldots, X_n)}{n^{1/2}} = O_P(1)$$
Suppose that a brand new system was in reasonable order

- If it is still risk free, then $l_n \to \mathbb{P} \text{ somewhere } \neq 0.5$
- If it gets risk affected, then $l_n \to \mathbb{P} 0.5$

What is this magical $l_n$?

The system-monitoring index (statistic)

$$l_n = \frac{\sum_{i=2}^{n} (Y_{i,n} - Y_{i-1,n})_+}{\sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}|}$$

where $Y_{1,n}, \ldots, Y_{n,n}$ are the concomitants of $X_1, \ldots, X_n$

Example |
---|---|
$(X_i, Y_i)$ | $(X_{i:n}, Y_{i:n})$
---|---|
$(5, 3)$ | $(1, 9)$
$(1, 9)$ | $(2, 6)$
$(4, 2)$ | $(4, 2)$
$(2, 6)$ | $(5, 3)$
A mathematical insight into the meaning of $I_n$

$$I_n = \frac{\sum_{i=2}^{n} (Y_{i,n} - Y_{i-1,n})^+}{\sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}|} = \frac{1}{2} \left( 1 + \frac{Y_{n,n} - Y_{1,n}}{\sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}|} \right)$$

because $x_+ = (|x| + x)/2$

$$= \frac{1}{2} \left( 1 + \frac{\text{Pseudo Range}(Y_1, Y_2, \ldots, Y_n)}{\text{Total Variation}(Y_1, Y_2, \ldots, Y_n)} \right)$$

if there are no risks and $n$ is large

$$\approx \frac{1}{2} \left( 1 + \frac{\int h'(x)dx}{\int |h'(x)|dx} \right)$$
Introduction

ARMA inputs and AVR transfer

How far beyond ARMA can we go?

Final notes
Example. Let risk free ($X_t$) be ARMA(1, 1) and follow

$$(X_t - 120) = 0.6(X_{t-1} - 120) + \eta_t + 0.4\eta_{t-1}$$

with iid Gaussian innovations $\eta_t \sim \mathcal{N}(0, \sigma^2_{\eta})$ where $\sigma^2_{\eta}$ is such that

$$X_t \sim \mathcal{N}(120, 9)$$
Automatic voltage regulators (AVR’s) often keep voltage between

\[ 120 \pm 6 \text{ volts} \quad (\pm 5\% \text{ of the nominal voltage}) \]

The transfer function

\[
h(x) = \begin{cases} 
  x_{\text{min}} & \text{when } x < x_{\text{min}} \\
  x & \text{when } x_{\text{min}} \leq x \leq x_{\text{max}} \\
  x_{\text{max}} & \text{when } x > x_{\text{max}}
\end{cases}
\]

with

\[ x_{\text{min}} = 114 \quad \& \quad x_{\text{max}} = 126 \]

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**Note.** Insurance layers often have similar transfer functions: deductible \( x_{\text{min}} \), policy limit \( x_{\text{max}} \), etc.
When the risk-free AVR is in reasonable order, we see

\[ I_n^0 = \frac{\sum_{i=2}^{n} (h(X_{i:n}) - h(X_{i-1:n})) + \sum_{i=2}^{n} |h(X_{i:n}) - h(X_{i-1:n})|}{\sum_{i=2}^{n} |h(X_{i:n}) - h(X_{i-1:n})|} \]

\[ = 1 \neq 0.5 \text{ (risk free)} \]

\[ B_n^0 = \frac{1}{n^{1/2}} \sum_{i=2}^{n} |h(X_{i:n}) - h(X_{i-1:n})| \]

\[ = O_p(1) \text{ (in reasonable order)} \]
Illustrative risk specifications: let \((\delta_t)\) and \((\varepsilon_t)\) be

- independent of \((X_t)\)
- independent of each other
- iid Lomax\((\alpha, 1)\) and thus have the means

\[
\mathbb{E}(\delta_t) = \mathbb{E}(\varepsilon_t) = \frac{1}{\alpha - 1}
\]

**Examples:**

\(\alpha = 1.2 \implies \mathbb{E}(\delta_t) = \mathbb{E}(\varepsilon_t) = 5\)

\(\alpha = 11 \implies \mathbb{E}(\delta_t) = \mathbb{E}(\varepsilon_t) = 0.1\)
\[ \mathbb{E}(\delta_t) = \mathbb{E}(\epsilon_t) = 5 \] (fast converging \( I_n \) and \( B_n \))

\[ I_n = \frac{\sum_{i=2}^{n} (Y_{i,n} - Y_{i-1,n}) + \sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}|}{\sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}|} \]

\[ \rightarrow \mathbb{P} \ 0.5 \] (risk affected)

\[ B_n = \frac{1}{n^{1/2}} \sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}| \]

\[ \rightarrow \mathbb{P} \ \infty \] (out of reasonable order)
\[ \mathbb{E}(\delta_t) = \mathbb{E}(\epsilon_t) = 0.1 \] (slowly converging \( I_n \) and \( B_n \))

\[ I_n = \frac{\sum_{i=2}^{n} (Y_{i,n} - Y_{i-1,n})}{\sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}|} \rightarrow \mathbb{P} 0.5 \text{ (risk affected)} \]

\[ B_n = \frac{1}{n^{1/2}} \sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}| \rightarrow \mathbb{P} \infty \text{ (out of reasonable order)} \]
|   | Introduction               | ARMA inputs and AVR transfer | How far beyond ARMA can we go? | Final notes       |
|---|----------------------------|-------------------------------|-------------------------------|-------------------|
| 1 | Introduction               |                               |                               |                   |
| 2 | ARMA inputs and AVR transfer|                               |                               |                   |
| 3 | How far beyond ARMA can we go? |                               |                               |                   |
| 4 | Final notes                |                               |                               |                   |
The answer depends on the validity of

\[ l_n = \frac{\sum_{i=2}^{n} (Y_{i,n} - Y_{i-1,n})}{\sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}|} \xrightarrow{\mathbb{P}} l_\infty \]

\[ \begin{cases} = 0.5 & \text{when risk affected} \\ \neq 0.5 & \text{when risk free} \end{cases} \]

assuming

\[ B_0^n = \frac{1}{n^{1/2}} \sum_{i=2}^{n} |h(X_i:n) - h(X_{i-1:n})| = O_{\mathbb{P}}(1) \]

i.e., when risk-free outputs are in reasonable order

**Definition.** The outputs \((Y_t)\) are **out of reasonable order** if

\[ B_n := \frac{1}{n^{1/2}} \sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}| \xrightarrow{\mathbb{P}} \infty \]

...and then necessarily \( l_n \xrightarrow{\mathbb{P}} 0.5 \)
When do $Y_t$’s become out of reasonable order?

Roughly speaking, this happens when $X_t$’s are stationary and at least one of the risks does not vanish from $I_n$, that is, when $\delta_t$’s and $\varepsilon_t$’s are

- noticeable (e.g., if $h(x) = c$, then $\delta_t$’s vanish from $I_n$)
- non-degenerate (e.g., if $\varepsilon_t = c$, then they vanish from $I_n$)
- not disguised as $X_t$’s (e.g., if $\delta_t = X_t$, $\varepsilon_t = X_t$, and $h(x) = x$, then $Y_t = 3X_t$ and the risk-identifying “3” vanishes from $I_n$)
In the risk-free scenario, when do we have

\[ I_n \xrightarrow{\mathbb{P}} I_\infty \neq 0.5 \]

Let the stationary inputs \((X_t)\) satisfy the Glivenko-Cantelli property and be temperately dependent (next two slides).

Let \(h\) be almost everywhere differentiable with vanishing derivative outside an interval \([a, b]\) (recall the AVR function).

Then in the risk-free scenario we have

\[
I_n \xrightarrow{\mathbb{P}} I_\infty := \frac{\int_a^b (h'(x))_+ \, dx}{\int_a^b |h'(x)| \, dx} = \frac{1}{2} \left( 1 + \frac{\int_a^b h'(x) \, dx}{\int_a^b |h'(x)| \, dx} \right) \\
\neq 0.5 \quad \text{unless} \quad h(b) = h(a)
\]
**Definition.** Inputs \((X_t)\) with the same marginal cdf’s \(F\) satisfy the Glivenko-Cantelli property if \(X_1, \ldots, X_n\) asymptotically identify \(F\), that is,

\[
\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \xrightarrow{P} 0 \quad \text{when} \quad n \to \infty
\]

where \(F_n\) is the empirical cdf based on \(X_1, \ldots, X_n\)

- If the inputs \((X_t)\) follow the stationary ARMA\((p, q)\) model driven by iid innovations \((\eta_n)\) with densities, then they satisfy the Glivenko-Cantelli property
**Definition.** Inputs \((X_t)\) with the same marginal cdf’s \(F\) are temperately dependent if

\[
P(X_{1:n} \geq x) \to 0 \quad \text{and} \quad P(X_{n:n} \leq x) \to 0
\]

for all \(x\) such that \(F(x) \in (0,1)\)

**Examples**

- If \((X_t)\) are iid, then they are temperately dependent.
- If \(X_t = X\) for all \(t \in \mathbb{Z}\), then they are not temperately dependent (they are super-strongly dependent).
- If \((X_t)\) are strictly stationary and \(\alpha\)-mixing, then they are temperately dependent.
1. Introduction

2. ARMA inputs and AVR transfer

3. How far beyond ARMA can we go?

4. Final notes
The system-monitoring index (statistic)

\[ I_n = \frac{\sum_{i=2}^{n} (Y_{i,n} - Y_{i-1,n}) + \sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}|}{\sum_{i=2}^{n} |Y_{i,n} - Y_{i-1,n}|} \rightarrow _{P} I_{\infty} \]

\( \begin{align*} 
= 0.5 & \quad \text{risk affected} \\
\neq 0.5 & \quad \text{risk free} 
\end{align*} \)

- is simple to implement
- works as intended in practically plausible situations
- jointly with another index, helps to determine \textit{when} the system gets affected by risks: at the input, output, or both stages

A “stopping” question

When to sound the alarm?

\( n = 50? \ 100? \ \ldots \)