SMOOTH EMBEDDINGS OF RATIONAL HOMOLOGY BALLS

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Abstract. The rational homology balls \( B_n \) appeared in Fintushel and Stern’s rational blow-down construction [FS] and were subsequently used in [Pa2, FS4] to construct exotic smooth manifolds with small Euler numbers. We show that a large class of smooth 4-manifolds have all of the \( B_n \)’s for odd \( n \geq 3 \) embedded in them. In particular, we give explicit examples, using Kirby calculus, of several families of smooth embeddings of the rational homology balls \( B_n \).

1. Introduction

In 1997, Fintushel and Stern [FS] defined the rational blow-down operation for smooth 4-manifolds, a generalization of the standard blow-down operation. For smooth 4-manifolds, the standard blow-down is performed by removing a neighborhood of a sphere with self-intersection \((-1)\) and replacing it with a standard 4-ball \( B^4 \). The rational blow-down involves replacing a negative definite plumbing 4-manifold with a rational homology ball. In order to define it, we first begin with a description of the negative definite plumbing 4-manifold \( C_n, n \geq 2 \), as seen in Figure 1, where each dot represents a sphere, \( S_i \), in the plumbing configuration. The integers above the dots are the self-intersection numbers of the plumbed spheres: \([S_1]^2 = -(n + 2)\) and \([S_i]^2 = -2\) for \( 2 \leq i \leq n - 1 \).

![Plumbing diagram of \( C_n, n \geq 2 \)]

The boundary of \( C_n \) is the lens space \( L(n^2, n - 1) \), thus \( \pi_1(\partial C_n) \cong H_1(\partial C_n; \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z} \). (Note, when we write the lens space \( L(p,q) \), we mean it is the 3-manifold obtained by performing \(-\frac{p}{q}\) surgery on the unknot.) This follows from the fact that \([-n - 2, -2, \ldots, -2]\), with \((n - 2)\) many \((-2)\)’s is the continued fraction expansion of \( \frac{n^2}{1-n} \).
Let $B_n$ be the 4-manifold as defined by the Kirby diagram in Figure 2 (for a more extensive description of $B_n$, see section 2). The manifold $B_n$ is a rational homology ball, i.e. \( H_*(B_n; \mathbb{Q}) \cong H_*(B^4; \mathbb{Q}) \). The boundary of $B_n$ is also the lens space $L(n^2, n - 1)$ [CH]. Moreover, any self-diffeomorphism of $\partial B_n$ extends to $B_n$ [FS]. Now, we can define the rational blow-down of a 4-manifold $X$ below in Definition 1.1. Fintushel and Stern [FS] also showed how to compute Seiberg-Witten and Donaldson invariants of $X$ from the respective invariants of $X$. In addition, they showed that certain smooth logarithmic transforms can be alternatively expressed as a series of blow-ups and rational blow-downs. The rational blow-down was particularly useful in constructing 4-manifolds with exotic smooth structures, with small Euler numbers.

**Definition 1.1.** ([FS], also see [GS]) Let $X$ be a smooth 4-manifold. Assume that $C_n$ embeds in $X$, so that $X = C_n \cup_{L(n^2, n-1)} X_0$. The 4-manifold $X_{(n)} = B_n \cup_{L(n^2, n-1)} X_0$ is by definition the rational blow-down of $X$ along the given copy of $C_n$.

One can define the (smooth) rational blow-up operation in a similar manner: if there exists a smoothly embedded $B_n$ in a 4-manifold $X$ then one can rationally blow-up $X$ by removing the $B_n$ and gluing in the $C_n$, along the lens space $L(n^2, n - 1)$. Consequently, one can ask: which 4-manifolds can be smoothly rationally blown up? Equivalently, which 4-manifolds contain a smoothly embedded rational homology ball $B_n$? We prove the following results regarding smooth embeddings of the rational homology balls $B_n$:

**Theorem 1.2.** Let $V_{-n-1}$ be a neighborhood of a sphere with self-intersection number $(-n - 1)$. There exists an embedding of the rational homology balls $B_n \hookrightarrow V_{-n-1}$, for all $n \geq 2$.

**Theorem 1.3.** Let $V_{-4}$ be a neighborhood of a sphere with self-intersection number $(-4)$. For all $n \geq 3$ odd, there exists an embedding of the rational homology balls $B_n \hookrightarrow V_{-4}$. For all $n \geq 2$ even, there exists an embedding of the rational homology balls $B_n \hookrightarrow B_2 \# \mathbb{C}P^2$.

Theorems 1.2 and 1.3 above show that there is little obstruction to smoothly embedding the rational homology balls $B_n$ into a smooth 4-manifold. In particular, Theorem 1.3 implies that if a smooth 4-manifold $X$ contains a
sphere with self-intersection \((-4)\), then one can smoothly embed the rational homology balls \(B_n\) into \(X\) for all odd \(n \geq 3\).

One of the implications of Theorem 1.3 is that for a given smooth 4-manifold \(X\), there does not exist an \(N\), such that for all \(n \geq N\) one cannot find a smooth embedding \(B_n \hookrightarrow X\). In the setting of this sort in algebraic geometry, for rational homology ball smoothings of certain surface singularities, such a bound on \(n\) does exist, in terms of \((c_1^2, \chi_h)\) invariants of an algebraic surface [KSB, Wa].

In section 2 we give a brief description of the rational homology balls \(B_n\). In section 3 we describe some previously known embeddings of \(B_n\) in order to illustrate the differences between them and those embeddings in Theorems 1.2 and 1.3. In sections 4 and 5 we prove Theorems 1.2 and 1.3 respectively. Finally, in section 6 we define the notion of “simple” embeddings of \(B_n\).

2. Description of the rational homology balls \(B_n\)

There are several ways to give a description of the rational homology balls \(B_n\). One of them is a Kirby calculus diagram seen in Figure 2. This represents the following handle decomposition: Start with a 0-handle, a standard 4-disk \(D^4\), attach to it a 1-handle \(D^1 \times D^3\). Call the resultant space \(X_1\), it is diffeomorphic to \(S^1 \times D^3\) and has boundary \(\partial X_1 = S^1 \times S^2\). Finally, we attach a 2-handle \(D^2 \times D^2\). The boundary of the core disk of the 2-handle gets attached to the closed curve, \(K\), in \(\partial X_1\) which wraps \(n\) times around the \(S^1 \times \ast\) in \(S^1 \times S^2\). We can also represent \(B_n\) by a slightly different Kirby diagram, which is more cumbersome to manipulate but is more visually informative, as seen in Figure 3, where the 1-handle is represented by a pair of balls. It is worthwhile to note that \(B_2\) can also be described as an unoriented disk bundle over \(\mathbb{R}P^2\), where \(K\) is the boundary of the Mobius band in \(\mathbb{R}P^2\).

![Another Kirby diagram of \(B_n\)](image)

3. Embeddings of \(B_n\) obtained from blow-ups of elliptical surfaces

In the existing literature, the straightforward examples of smooth 4-manifolds containing smoothly embedded rational homology balls \(B_n\), are
those manifolds obtained via a rational blow-down. Examples of such manifolds first appeared in Fintushel and Stern’s original paper [FS] on rational blow-downs: logarithmic transforms $E(m)_n$ of elliptic surfaces $E(m)$. In these manifolds, one starts with a fishtail fiber of $E(m)$, which has homological self-intersection 0, blows it up $(n - 2)$ times, and then one obtains a configuration of spheres $C_n$, which one rationally blows down (see Figure 4). Consequently, one obtains a manifold $E(m)_n$, having the same $(c_1^2, c_2)$ numbers but different Seiberg-Witten invariants as $E(m)$, which contains an embedded rational homology ball $B_n$.

![Figure 4. Fishtail fiber in $E(m)$ blown up $(n - 1)$ times](image)

Most other examples of smooth 4-manifolds that contain embedded rational homology balls $B_n$, are obtained in a similar manner, one blows up a smooth manifold several times, then finds a particular configuration of spheres $C_n$ which one rationally blows down. Often, one ends up with a manifold with lower betti number $b_2$ than the original manifold one started with. In fact, in a lot of these examples, since one can compute the betti numbers of the resultant manifold, by Freedman’s theorem [Fr, FQ] one can conclude they are homeomorphic to $k\mathbb{C}P^2 \# \ell \mathbb{C}P^2$, for some $k$ and $\ell$. However, after a computation of the Seiberg-Witten invariants, one can often show that the resultant manifolds are not diffeomorphic to $k\mathbb{C}P^2 \# \ell \mathbb{C}P^2$, and thus possess an exotic smooth structure, which is frequently the goal. In fact, one can sometimes find an infinite family of exotic 4-manifolds which are homeomorphic but not diffeomorphic to $k\mathbb{C}P^2 \# \ell \mathbb{C}P^2$. For example, using these techniques, exotic $\mathbb{C}P^2 \# 7 \mathbb{C}P^2$ manifolds were constructed in [Pa2]. Additionally, using a generalized rational blow-down [Pa1], exotic $\mathbb{C}P^2 \# 6 \mathbb{C}P^2$ manifolds were constructed in [SS].

4. Proof of Theorem 1.2

In this section we prove Theorem 1.3.

Proof. Proof of Theorem 1.2

We prove this theorem using Kirby calculus (see [GS] for detailed exposition). We start with the Kirby diagram for $V_{-n-1}$, Figure 7, blow it
up \((n-1)\) times, and obtain the configuration of spheres \(C_n\) with an additional sphere \(\Sigma_{-1}\) with \([\Sigma_{-1}]^2 = -1\), attached to the last sphere with self-intersection \((-2)\), \(S_{n-1}\), Figure 10. In Figures 11-15 we proceed to do the standard Kirby calculus manipulation where one changes the Kirby diagram of \(C_n\) from the one in Figure 5 to the one in Figure 6 (see [GS], p. 516), by first adding a cancelling 1/2 handle pair (Figure 11) and performing a series of handleslides. However, in our case the additional sphere \(\Sigma_{-1}\) is present, and intersects with the \(C_n\) configuration in a non-trivial way. As a result, when we perform the last handleslide to get \(C_n\) to look like Figure 6, the sphere \(\Sigma_{-1}\) intersects with \(C_n\) as seen in Figure 15.

Next, in Figure 16 we perform the rational blow-down, thus replacing \(C_n\) with \(B_n\). This is done by first swapping the one-handle and the 0-framed two-handle and then blowing down the \((n-1)\) spheres with self-intersection \((-1)\). Consequently, after rationally blowing down, we obtain \(B_n\) with an additional 0-framed two-handle. When we slide then \((n-1)\)-framed two-handle of \(B_n\) over that 0-framed two-handle, we obtain Figure 17. We proceed to slide the same handle over the 0-framed two-handle \((n-2)\) more
times, and obtain Figure 18. Finally, we remove the cancelling $1/2$ handle pair, and obtain a single $(-n - 1)$-framed two-handle, Figure 19, which is
the manifold $V_{n-1}$. Consequently, since to get from Figure 16 to Figure 19 we only performed handleslides, it follows that $B_n \hookrightarrow V_{n-1}$. \hfill \Box

**Corollary 4.1.** For $n \geq 2$, the rational blow-up of $B_n \hookrightarrow V_{n-1}$ is diffeomorphic to $V_{n-1} \# (n-1) \mathbb{CP}^2$.

Corollary 4.1 follows directly from the proof of Theorem 4.2. If we follow the Kirby moves backwards from Figure 19 to Figure 10, it follows that if we start with a $V_{n-1}$, and rationally blow up the $B_n \hookrightarrow V_{n-1}$, then we end up with $V_{n-1} \# (n-1) \mathbb{CP}^2$. 
5. Proof of Theorem 1.3

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Proof. Proof of Theorem 1.3

We prove Theorem 1.3 using similar Kirby calculus techniques as in the proof of Theorem 1.2. (Note, the case $n = 2$ is trivial and the case $n = 3$ is covered in Theorem 1.2, so here we can assume $n \geq 4$.) We start with the Kirby diagram for $V_{-4}$, Figure 20. We blow up $V_{-4}$ $(n-1)$ times, as seen in Figures 21 through 24 in such a manner that we end up with a plumbing tree of spheres as seen in Figure 24. This configuration of spheres is $C_n$ with an extra sphere $\Sigma'_{-1}$, with self-intersection $(-1)$, which intersects only with the first sphere with self-intersection $(-2)$, $S_2$, (compare with Figure 10).

As was done in the proof of the previous theorem, we proceed with a series of Kirby moves that will change the presentation of $C_n$ in Figure 24, from a linear plumbing of spheres, Figure 5, to the one in Figure 6. We start
by adding a cancelling $1/2$-handle pair in Figure 25. We proceed by sliding
the $(-n - 2)$-framed two-handle over the two-handle which was added in the
previous step (Figure 26). Following this, we perform $(n - 2)$ handleslides in
order to slide off the $(-2)$-framed two-handles in Figures 27-29. As a result,
the $(-1)$-framed two-handle corresponding to the sphere $\Sigma_{-1}^2$ intersects
once with each of the spheres corresponding to the $(n - 2)$ $(-1)$-framed
two-handles, as seen in Figure 29. Next, we slide the \((-n+1)\)-framed two-handle off of each of the \((n-1)\) \((-1)\)-framed two-handles, Figures 30 and 31. Consequently, in Figure 31 we obtain a presentation of \(C_n\) as in Figure 6, with the extra sphere \(\Sigma_{-1}\).

Next, we perform the rational blow-down procedure, by exchanging the one-handle and the 0-framed two-handle, and blowing down along the \((n-1)\) spheres with self-intersection \((-1)\), and obtain the Kirby diagram of \(B_n\) with an additional \((n-3)\)-framed two-handle, Figure 32. Next, we slide the \((n-1)\)-framed two-handle over the \((n-3)\)-framed two-handle and obtain the Kirby diagram in Figure 33 with the \((n-1)\)-framed two-handle becoming a 0-framed two-handle. At this point, the unknot corresponding to the \((n-3)\)-framed two-handle is linked with the unknot corresponding to the
If \( n \) is even, then after \( \frac{n-2}{2} \) such handleslides we obtain the diagram in Figure 35 (equivalent to the one in Figure 36), which is just \( B_2 \) blown up once, i.e. \( B_2 \# \mathbb{CP}^2 \). Consequently, if we start with \( B_2 \# \mathbb{CP}^2 \), and follow...
the Kirby moves backwards from Figure 36 to Figure 32, then we see that $B_n \hookrightarrow B_2 \# \mathbb{C}P^2$, for $n$ even.

If $n$ is odd, then if we start with the diagram in Figure 33 and slide off the $(n - 3)$-framed two-handle $\frac{n-3}{2}$ times, we obtain the diagram in Figure 37. Following this, we slide the 0-framed two handle (the one on the bottom of the diagram), over the other 0-framed two-handle and obtain the diagram in Figure 38. We then perform another handleslide, and slide off the $(-2)$-framed two-handle off of the 0-framed two-handle and get the diagram in Figure 39. Finally, we remove the cancelling 1/2-handle pair and are left with one $(-4)$-framed two-handle, Figure 40, which represents the manifold $V_{-4}$. Consequently, if we follow the Kirby moves backwards from Figure 40 to Figure 32 (skipping Figures 35 and 36, as these are for the case when $n$ is even), then we can conclude that $B_n \hookrightarrow V_{-4}$ for $n$ odd. □
The difference between the embeddings in Theorem 1.3 with \( n \) odd and even occurs because for \( n \) odd the rational homology balls \( B_n \) are spin and for \( n \) even they are not.

**Corollary 5.1.** For odd \( n \geq 3 \), the rational blow-up of \( B_n \hookrightarrow V_{-4} \) is diffeomorphic to \( V_{-4} \# (n - 1)\mathbb{C}P^2 \).

Similarly to the proof of Corollary 4.1, Corollary 5.1 follows directly from the proof of Theorem 1.3. From the proof of Theorem 1.3, we can represent \( V_{-4} \) with the Kirby diagram in Figure 32, where we can see the \( B_n \hookrightarrow \rightarrow V_{-4} \).

If we rationally blow up this \( B_n \), then we obtain the Kirby diagram in Figure 31, which by a sequence of Kirby moves gets us back to the diagram in Figure 24, which is precisely \( V_{-4} \# (n - 1)\mathbb{C}P^2 \).

6. **“Simple” embeddings**

The embeddings of the \( B_n \)'s in Theorems 1.2 and 1.3 are inherently different from the embeddings of \( B_n \hookrightarrow E(m)_n \), as discussed in the beginning of section 3. As seen from Corollaries 4.1 and 5.1, the embeddings of \( B_n \hookrightarrow V_{-n-1}, V_{-4} \) are such that if one rationally blows up those \( B_n \)'s and then performs the regular blow-down \((n - 1)\) times, then one gets back the manifolds \( V_{-n-1}, V_{-4} \) respectively. One could also do these two steps in reverse: if one starts with \( V_{-n-1}, V_{-4} \), blows them up \((n - 1)\) times and then rationally blows down the obtained \( C_n \) configuration, then one again obtains the manifolds \( V_{-n-1}, V_{-4} \) respectively. This is summarized in the following diagrams for the embeddings of \( B_n \hookrightarrow V_{-n-1} \) for \( n \geq 2 \) and for \( B_n \hookrightarrow V_{-4} \) for odd \( n \geq 3 \) respectively:

\[
\begin{align*}
V_{-n-1} & \xrightarrow{\text{RBU the } B_n} V_{-n-1} \# (n - 1)\mathbb{C}P^2 \\
& \quad \downarrow \text{BU (n-1) times} \\
V_{-n-1} \# (n - 1)\mathbb{C}P^2 & \xrightarrow{\text{RBD the } C_n} V_{-n-1} \\
V_{-4} & \xrightarrow{\text{RBU the } B_n} V_{-4} \# (n - 1)\mathbb{C}P^2 \\
& \quad \downarrow \text{BU (n-1) times} \\
V_{-4} \# (n - 1)\mathbb{C}P^2 & \xrightarrow{\text{RBD the } C_n} V_{-4}
\end{align*}
\]

This is not the case with the embeddings of \( B_n \hookrightarrow E(m)_n \), since the rational blow-ups of those \( B_n \)'s result in \( E(m)_n \# (n - 1)\mathbb{C}P^2 \) (and not \( E(m)_n \# (n - 1)\mathbb{C}P^2 \)) and so blowing down \((n - 1)\) times yields the manifold \( E(m) \) and not \( E(m)_n \), the manifold we started with.

As a result, one can call an embedding of \( B_n \hookrightarrow X \) “simple” if rationally blowing up and then blowing down \((n - 1)\) times yields back the same 4-manifold \( X \) (the top and right arrows of the diagram below). Equivalently,
an embedding $B_n \hookrightarrow X$ is “simple” if blowing up $(n - 1)$ times followed by rationally blowing down the $C_n$, yields back the same 4-manifold $X$ (the left and bottom arrows of the diagram below).

\[
\begin{array}{ccc}
X & \xrightarrow{\text{RBU the } B_n} & X \#(n - 1)\mathbb{C}P^2 \\
& \downarrow \text{BU (n-1) times} & \downarrow \text{BD (n-1) times} \\
X \#(n - 1)\mathbb{C}P^2 & \xrightarrow{\text{RBD the } C_n} & X
\end{array}
\]

It follows that the embeddings of of $B_n \hookrightarrow V_{-n-1}$ for $n \geq 2$ and for $B_n \hookrightarrow V_{-4}$ for odd $n \geq 3$ are “simple”, whereas the embedding $B_n \hookrightarrow E(m)_n$ is not “simple”. Therefore, one can ask the following question: Are there obstructions to embedding the $B_n$s in a “non-simple” way?

Nevertheless, Theorem 1.3 prevents one from finding an upper bound on $n$ for a smooth 4-manifold $X$ to contain an embedded $B_n$. However, one can ask whether such a bound exists for “non-simple” embeddings of $B_n \hookrightarrow X$.

The Kirby diagrams in the proofs of Theorems 1.2 and 1.3 strongly suggest that the key to determining whether an embedding of a rational homology ball $B_n$ is “simple” lies in analyzing how the extra sphere with self-intersection $(-1)$ intersects with the spheres of the $C_n$ configuration after one rationally blows up the $B_n$. For example, if one starts with $B_n \hookrightarrow V_{-n-1}$ for $n \geq 2$, and rationally blows it up, one obtains the Kirby diagram seen in Figure 10. In this case, the extra sphere with self-intersection $(-1)$ intersects with the last sphere of self-intersection $(-2)$ ($S_{n-1}$ in Figure 5) in the $C_n$ configuration. Likewise, if one starts with $B_n \hookrightarrow V_{-4}$ for odd $n \geq 3$, and rationally blows it up, one obtains the Kirby diagram seen in Figure 24. In this case, the extra sphere with self-intersection $(-1)$ intersects with the first sphere of self-intersection $(-2)$ ($S_2$ in Figure 5) in the $C_n$ configuration. In the “non-simple” embedding case of $B_n \hookrightarrow E(m)_n$, if one rationally blows up those rational homology balls, then the extra sphere of self-intersection $(-1)$ intersects with the first and last spheres of the $C_n$ configuration ($S_1$ and $S_{n-1}$, respectively, in Figure 6), as seen in Figure 4.

In those instances where an exotic smooth manifold $X_{(n)}$ is obtained after rationally blowing down $X$ (with $B_n$), then the embedding $B_n \hookrightarrow X_{(n)}$ must be “non-simple”. Moreover, the intersection patterns of certain spheres of self-intersection $(-1)$ with the $C_n$ configuration back up in $X$, may be directly related to the obtained exotic smooth structure of $X_{(n)}$. Consequently, understanding the precise way of how the rational homology balls embed in 4-manifolds, may give us a better understanding of exotic smooth structures of 4-manifolds which were obtained as a result of a rational blow-down.

**References**

[Ak] A. Akhmedov, *Construction of exotic smooth structures*, Topology and its Applications 154 (2007), no. 6, 1134-1140
A. Casson and J. Harer, Some homology lens spaces which bound rational balls, Pacific J. Math. 96 (1981), 23-36

R. Fintushel and R. Stern, Rational blowdowns of smooth 4-manifolds, J. Diff. Geom. 46 (1997) 181-235

R. Fintushel and R. Stern, Double node neighborhoods and families of simply connected 4-manifolds with $b^+ = 1$, J. Amer. Math. Soc. 19 (2006), 171-180.

M. Freedman, The topology of four-dimensional manifolds, J. Diff. Geom. 17 (1982), 357-453

R. Fintushel and R. Stern, Double node neighborhoods and families of simply connected 4-manifolds with $b^+ = 1$, J. Amer. Math. Soc. 19 (2006), 171-180.

M. Freedman and F. Quinn, Topology of 4-manifolds, Princeton Mathematical Series 39, Princeton University Press, 1990

R. Gompf and A. Stipsicz, An introduction to 4-manifolds and Kirby calculus, Graduate Studies in Mathematics 20, (American Mathematical Society, Providence, RI, 1999)

J. Harer, A. Kas and R. Kirby, Handlebody decompositions of complex surfaces, Memoirs AMS 62 (1986), no. 350

J. Kollár and N. I. Shepherd-Barron, Threefolds and deformations of surface singularities, Invent. Math. 91 (1988), no. 2, 299-338

J. Park. Seiberg-Witten invariants of generalised rational blow-downs. Bull. Austral. Math. Soc., 56 (1997), no. 3, 363-384

J. Park. Simply connected symplectic 4-manifolds with $b^+_1 = 1$ and $c_1^2 = 2$, Invent. Math. 159 (2005), 657-667

A. Stipsicz and Z. Szabó, An exotic smooth structure on $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$, Geom. Topol. 9 (2005), 813-832

J. Wahl, Miyaoka-Yau inequality for normal surfaces and local analogues, Contemporary Mathematics 162 (1994), 381-402