Gaugeon Formalism for Spin-3/2 Rarita-Schwinger Gauge Field

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Abstract

We provide a gauge covariant formalism of the canonically quantized theory of spin-3/2 Rarita-Schwinger gauge field. The theory admits a quantum gauge transformation by which we can shift the gauge fixing parameter. The quantum gauge transformation does not change the BRST charge. Thus, the physical Hilbert space is trivially independent of the gauge fixing parameter.

1 Introduction

In the standard formalism of canonically quantized gauge theories[1, 2] we do not consider the gauge transformation freely. There are no quantum gauge freedom since the quantum theory is defined only after the gauge fixing.

Yokoyama’s gaugeon formalism[3, 4, 5, 6, 8, 7] provides a wider framework in which we can consider the quantum gauge transformation among a family of Lorentz covariant linear gauges. In this formalism a set of extra fields, so called gaugeon fields, is introduced as the quantum gauge freedom. This theory was proposed for the quantum electrodynamics[3, 4, 5] and for the Yang-Mills theory.[6, 7] Owing to the quantum gauge freedom it becomes almost trivial to check the gauge parameter independence of the physical S-matrix.[8]

We should ensure that the gaugeon modes do not contribute to the physical processes. In fact, the gaugeon fields yield negative normed states.[3] To remove these unphysical modes Yokoyama imposed a Gupta-Bleuler type subsidiary condition,[3, 6, 7] which is not applicable if interaction exists for gaugeon fields. Yokoyama’s condition has been improved by introducing BRST symmetry for gaugeon fields.[9, 10, 11, 12] Unphysical gaugeon modes, as well as unphysical modes of the gauge field, are removed by the single Kugo-Ojima type condition.[2] Thus, the formalism is now applicable even in the background gravitational field. The BRST symmetry is also very helpful in the analysis of the gauge structure of the Fock space in the gaugeon formalism.[3, 13]

By now, we have the BRST symmetric gaugeon formalism of the electromagnetic gauge theory[10, 11] and of the Yang-Mills gauge theory.[9, 12] There are, however, other types
of gauge fields, such as the gravitational field, the gravitino (spin-3/2 gauge field), the anti-symmetric tensor gauge fields and the string theory. One might wonder whether the gaugeon formalism is applicable to these gauge fields. In the present paper, we formulate a BRST symmetric gaugeon formalism for the spin-3/2 Rarita-Schwinger gauge field. Although we treat it mainly in the free field case, we can straightforwardly incorporate the interaction with the Ricci flat background gravitational field.

The paper is organized as the following. In §2, we briefly review the theory of Hata and Kugo[14] as a standard formalism of the canonically quantized spin-3/2 gauge field. In §3, we propose a BRST symmetric gaugeon formalism for the spin-2/3 gauge field, where gauge fixing parameter can be shifted by a $q$-number gauge transformation. We see in §4 how the Fock space of the standard formalism is embedded in the wider Fock space of the present formalism. Section 5 is devoted to comments and discussion, including the remarks on other types of gaugeon formalism.

2 Standard formalism

The classical Lagrangian of the free gravitino field $\psi_\mu$ in $n (\geq 3)$ dimensional flat spacetime is given by

\[ L_{RS} = -\frac{i}{2} \bar{\psi_\mu} \gamma^{\mu\nu\lambda} \partial_\nu \psi_\lambda, \]

where $\gamma^{\mu\nu\lambda}$ is the matrix $\gamma^\mu \gamma^\nu \gamma^\lambda$ antisymmetrized with respect to $\mu$, $\nu$ and $\lambda$,

\[ \gamma^{\mu\nu\lambda} = \frac{1}{6} (\gamma^\mu \gamma^\nu \gamma^\lambda \pm 5 \text{ terms}). \]

The factor $1/2$ arises in (2.1) since we assumed the field $\psi_\mu$ to be a Majorana spinor-vector. The Lagrangian (2.1) is invariant up to total derivatives under the gauge transformation

\[ \delta \psi_\mu = \partial_\mu \Lambda, \]

where $\Lambda$ is an arbitrary spinor field.

To carry out the quantization, it is necessary to add a gauge fixing term and a corresponding Faddeev-Popov (FP) ghost term. As a standard formalism, we use the theory of Hata and Kugo.[14] Their quantum Lagrangian is given by

\[ L_{HK} = L_{RS} + B \bar{\phi}(\gamma \psi) - \frac{ia}{2} B \bar{\phi} B - i \partial_\mu \bar{c} \partial^\mu c, \]

where $\gamma \psi = \gamma^\mu \psi_\mu$, $\bar{\phi} = \gamma^\mu \partial_\mu$, $B$ is a spinor multiplier (subject to the Fermi statistics), $c$ and $c^*$ are the spinor FP ghosts (subject to the Bose statistics), and $a$ is a numerical gauge fixing parameter. Note that the FP ghost fields satisfy the second order differential
equation. Owing to this property, FP ghosts $c$ and $c^*$ together with the multiplier $B$ realize the correct ghost counting.\[15, 16\]

The field equations are given by

$$-i\gamma^{\mu\nu\lambda} \partial_\nu \psi_\lambda = \gamma^\mu \bar{\phi},$$
$$\partial_\mu (\gamma \psi) = i a \partial B,$$
$$\Box c = \Box c^* = 0,$$  \hspace{1cm} (2.5)

from which we also have

$$\Box B = 0,$$
$$\Box (\gamma \psi) = 0.$$  \hspace{1cm} (2.6)

The Lagrangian (2.4) leads to the following $n$-dimensional (anti)commutation relations:

$$\{ \psi_\mu(x), \bar{\psi}_\nu(y) \} = \left[ g_{\mu\nu} \partial + \frac{1}{n-2} \gamma_\mu \bar{\phi} \gamma_\nu - \frac{2}{n-2} (\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu) \right] D(x - y)$$
$$+ \left( \frac{4}{n-2} - a \right) \partial_\mu \partial_\nu \bar{\phi} E(x - y),$$

$$\{ B(x), \bar{\psi}_\nu(y) \} = i \partial_\nu D(x - y),$$
$$\{ B(x), B(y) \} = 0,$$

$$[ c(x), c^*(y) ] = -D(x - y),$$  \hspace{1cm} (2.7)

where the functions $D$ and $E$ are defined by

$$\Box D(x) = 0, \hspace{0.5cm} D(0, x) = 0, \hspace{0.5cm} \dot{D}(0, x) = -\delta^{(n-1)}(x),$$
$$\Box E(x) = D(x), \hspace{0.5cm} E(0, x) = \dot{E}(0, x) = 0.$$  \hspace{1cm} (2.8)

From the first equation of (2.7) we have

$$\{ \gamma \psi(x), \bar{\psi}(y) \gamma \} = -a \bar{\phi} D(x - y),$$  \hspace{1cm} (2.9)

There are two special gauges. One is the Landau gauge ($a = 0$), in which $\{ \gamma \psi, \bar{\psi} \gamma \} = 0$ so that the $\gamma \psi$ mode has vanishing norm. The other is the Feynman gauge ($a = 4/(n - 2)$), in which $\psi_\mu$ does not include dipole modes: the Feynman propagator $\langle T(\psi_\mu \bar{\psi}_\nu) \rangle$ does not have $1/p^4$ term.\[17\]

The Lagrangian (2.4) is invariant up to total derivative terms under the following BRST transformation:

$$\delta_B \psi_\mu = i \partial_\mu c,$$
$$\delta_B c^* = B,$$
$$\delta_B B = \delta_B c = 0.$$  \hspace{1cm} (2.10)

\[\text{In the Feynman gauge, it is convenient to use a field variable } \phi_\mu = \psi_\mu - \frac{i}{2} \gamma_\mu (\gamma \psi).\] When $a = 4/(n - 2)$, it satisfies the Dirac equation $\partial_\mu \phi_\mu = 0$, and the anticommutation relation becomes

$$\{ \phi_\mu(x), \bar{\phi}_\nu(y) \} = g_{\mu\nu} \bar{\phi} D(x - y).$$
The corresponding conserved BRST charge is given by

\[ Q_{B(HK)} = -i \int B \delta_0 c \, d^{n-1}x, \tag{2.11} \]

where \( \delta_0 = \partial_0 - \tilde{\partial}_0 \). Using this charge we can define the physical subspace \( \mathcal{V}_{phys}^{(HK)} \) as a space of the states which satisfy the physical subsidiary condition of Kugo-Ojima,

\[ Q_{B(HK)} \langle \text{phys} \rangle = 0. \tag{2.12} \]

There are many unphysical zero-normed states in the physical subspace \( \mathcal{V}_{phys}^{(HK)} \). In fact, \( \mathcal{V}_{phys}^{(HK)} \) has a zero-normed subspace

\[ \text{Im} \, Q_{B(HK)} = \left\{ |\Phi\rangle ; |\Phi\rangle = Q_{B(HK)}|\star\rangle \right\}. \]

Considering the quotient space of \( \mathcal{V}_{phys}^{(HK)} \) by this subspace, we can define the physical Hilbert space,

\[ \mathcal{H}_{phys}^{(HK)} = \mathcal{V}_{phys}^{(HK)}/\text{Im} \, Q_{B(HK)}, \tag{2.13} \]

which has positive definite metric.

### 3 Gaugeon formalism

We start from the Lagrangian

\[ \mathcal{L} = \mathcal{L}_{RS} + \bar{B} \phi (\gamma \psi) - \frac{i\epsilon}{2} (\bar{Y}_s + \alpha B) \phi (Y_s + \alpha B) - \partial_\mu \bar{Y}_s \partial^\mu Y - i \partial_\mu \bar{c} \partial^\mu c - i \partial_\mu \bar{K}_s \partial^\mu K, \tag{3.1} \]

where, in addition to the usual multiplier \( B \) and FP ghosts \( c \) and \( c^* \), we have introduced spinor gaugeon fields \( Y \) and \( Y_s \) (subject to the Fermi statistics) and corresponding spinor FP ghosts \( K \) and \( K_s \) (subject to the Bose statistics). In (3.1), \( \epsilon \) denotes a sign factor \( (\epsilon = \pm 1) \) and \( \alpha \) is a numerical gauge fixing parameter. As seen below, the standard gauge fixing parameter, which is denoted by \( a \) in the present paper, can be identified with

\[ a = \epsilon \alpha^2. \tag{3.2} \]

#### 3.1 Field equations and (anti)commutation relations

Field equations which follow from (3.1) are

\[ -i \gamma^{\mu \nu \lambda} \partial_\nu \psi_\lambda = \gamma^\mu \phi B, \]
\[ \phi (\gamma \psi) = i\epsilon \alpha \phi (Y_s + \alpha B), \]
\[ \Box Y = i\epsilon \phi (Y_s + \alpha B), \]
\[ \Box Y_s = 0 \]
\[ \Box c = \Box c_s = 0, \]
\[ \Box K = \Box K_s = 0. \tag{3.3} \]
From these equations we also have
\[ \Box B = 0, \]
\[ \Box (\gamma \psi) = 0, \]
\[ \phi \Box Y = 0. \] (3.4)

The canonical prescription of quantization leads to the following \( n \)-dimensional (anti)commutation relations: Among the usual fields \( (\psi_\mu, B, c, c^*) \), we have
\[ \{ \psi_\mu(x), \bar{\psi}_\nu(y) \} = \frac{g_{\mu\nu} \Box + \frac{1}{n-2} \gamma_\mu \gamma_\nu - \frac{2}{n-2} (\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu)}{D(x-y)} \]
\[ + \left( \frac{4}{n-2} - \varepsilon \alpha^2 \right) \partial_\mu \partial_\nu \phi E(x-y), \]
\[ \{ B(x), \bar{\psi}_\nu(y) \} = i \partial_\nu D(x-y), \]
\[ \{ B(x), B(y) \} = 0, \]
\[ [ c(x), c^*(y) ] = -D(x-y). \] (3.5)

Among the gaugeons and their FP ghosts \( (Y, Y^*, K, K^*) \), we have
\[ \{ Y^*_\nu(x), \bar{Y}^*_\nu(y) \} = 0, \]
\[ \{ Y_\nu(x), \bar{Y}^*_\nu(y) \} = -iD(x-y), \]
\[ \{ Y(x), \bar{Y}^*(y) \} = \varepsilon \phi E(x-y), \]
\[ [ K(x), \bar{K}^*(y) ] = -D(x-y). \] (3.6)

Anticommutators between the gaugeons and the usual fields are given by
\[ \{ Y_\nu(x), \bar{B}(y) \} = \{ Y^*_\nu(x), \bar{\psi}_\mu(y) \} = 0, \]
\[ \{ Y(x), \bar{B}(y) \} = 0, \]
\[ \{ Y(x), \bar{\psi}_\mu(y) \} = -\varepsilon \alpha \partial_\mu \phi E(x-y). \] (3.7)

The (anti)commutation relations (3.5) are exactly the same with (2.7) if we assume (3.2). In particular, \( \alpha = 0 \) corresponds to the Landau gauge, and \( \alpha = 2/\sqrt{n-2} \) (together with \( \varepsilon = +1 \)) leads to the Feynman gauge. We notice from (3.7) that in the Landau gauge \( (\alpha = 0) \) the gaugeon modes \( Y \) and \( Y^*_\nu \) completely decouple from the usual fields \( \psi_\mu \) and \( B \).

### 3.2 BRST symmetry

The Lagrangian (3.1) is invariant up to total derivatives under the following BRST transformation:
\[ \delta_B \psi_\mu = i \partial_\mu c, \]
\[ \delta_B c^* = B, \]
\[ \delta_B B = \delta_B c = 0, \]
\[ \delta_B Y = i K, \]
\[ \delta_B K^* = Y^*_\nu, \]
\[ \delta_B Y^*_\nu = \delta_B K = 0. \] (3.8)
which obviously satisfies the nilpotency, \( \delta B^2 = 0 \). The corresponding conserved BRST charge is given by

\[ Q_B = -i \int \left( \bar{B} \hat{\partial}_0 c + \bar{Y}_* \hat{\partial}_0 K \right) d^{n-1}x. \] (3.9)

By the help of this charge we can define the physical subspace \( V_{\text{phys}} \) as a space of the states satisfying

\[ Q_B |\text{phys}\rangle = 0. \] (3.10)

This subsidiary condition removes the gaugeon modes as well as the unphysical gravitino modes from the physical subspace; \( Y \) and \( Y_* \) together with \( K \) and \( K_* \) constitute the BRST quartet.[2]

### 3.3 \( q \)-number gauge transformation

The Lagrangian (3.1) admits a \( q \)-number gauge transformation. Under the field redefinition

\[
\begin{align*}
\hat{\psi}_\mu &= \psi_\mu + \tau \partial_\mu Y, \\
\hat{Y}_* &= Y_* - \tau B, \\
\hat{B} &= B, \quad \hat{Y} = Y, \\
\hat{c} &= c + \tau K, \\
\hat{K}_* &= K_* - \tau c_*, \\
\hat{c}_* &= c_*, \quad \hat{K} = K,
\end{align*}
\] (3.11)

with \( \tau \) being a numerical parameter, the Lagrangian (3.1) is form invariant (up to total derivative terms), that is, it satisfies

\[ \mathcal{L}(\varphi_A, \alpha) = \mathcal{L}(\hat{\varphi}_A, \hat{\alpha}) + \text{total derivatives}, \] (3.12)

where \( \varphi_A \) stands for any of the relevant fields and \( \hat{\alpha} \) is defined by

\[ \hat{\alpha} = \alpha + \tau. \] (3.13)

An immediate conclusion from the form invariance (3.12) is the following: All the field equations and all the (anti)commutation relations are gauge covariant under the \( q \)-number gauge transformation (3.11): \( \hat{\varphi}_A \) satisfies the field equations (3.3), (3.4) and the (anti)commutation relations (3.5) ~ (3.7) with \( \alpha \) replaced by \( \hat{\alpha} \).

It should be noted that the \( q \)-number gauge transformation (3.11) commutes with the BRST transformation (3.8). As a result, our BRST charge (3.9) is invariant under the \( q \)-number gauge transformation:

\[ \hat{Q}_B = Q_B. \] (3.14)

The physical subspace \( V_{\text{phys}} \) is, therefore, invariant under the \( q \)-number gauge transformation:

\[ \hat{V}_{\text{phys}} = V_{\text{phys}}. \] (3.15)

Similarly, our physical Hilbert space \( H_{\text{phys}} = V_{\text{phys}} / \text{Im} Q_B \) is also gauge invariant:

\[ \hat{H}_{\text{phys}} = H_{\text{phys}}. \] (3.16)
4 Gauge structure of the Fock space

As well as the BRST symmetry (3.3), the Lagrangian (3.1) has several other symmetries. In particular, we have the following BRST-like conserved charges:

\[
Q_{B(HK)} = -i \int \bar{c} \partial_0 B d^{n-1}x,
\]

\[
Q_{B(Y)} = -i \int \bar{K} \partial_0 Y_\ast d^{n-1}x,
\]

\[
Q'_{B(HK)} = -i \int \bar{K} \partial_0 B d^{n-1}x,
\]

\[
Q'_{B(Y)} = -i \int \bar{c} \partial_0 Y_\ast d^{n-1}x,
\]

(4.1)

all of which satisfy the nilpotency condition. Our BRST charge \( Q_B \) can be decomposed as

\[
Q_B = Q_{B(HK)} + Q_{B(Y)}.
\]

(4.2)

The charge \( Q_{B(HK)} \) generates the BRST transformation only for the usual fields \( \psi_\mu, B, c \) and \( c_\ast \), while \( Q_{B(Y)} \) applies only for \( Y, Y_\ast, K \) and \( K_\ast \). The charge \( Q'_{B(HK)} \) generates the BRST transformation for \( \psi_\mu \) and \( B \) but with \( K \) and \( K_\ast \) treated as their FP ghosts. Similarly, \( Q'_{B(Y)} \) generates the BRST transformation for \( Y \) and \( Y_\ast \) with \( c \) and \( c_\ast \) as their FP ghosts.

In the last section, we have taken (3.10) as a physical condition. Instead of it, however, we may choose the condition as

\[
Q_{B(HK)} |_{\text{phys}} = 0,
\]

\[
Q_{B(Y)} |_{\text{phys}} = 0.
\]

(4.3)

The unphysical modes of gravitino are removed by the first equation, while the gaugeon modes by the second. We express the space of states satisfying (4.3) by \( V^{(\alpha)}_{\text{phys}} \). As easily seen, this space is a subspace of \( V_{\text{phys}} \) defined in the last section:

\[
V^{(\alpha)}_{\text{phys}} \subset V_{\text{phys}}.
\]

(4.4)

We have attached the index \( \alpha \) to \( V^{(\alpha)}_{\text{phys}} \) to emphasize that its definition depends on the gauge fixing parameter \( \alpha \). In fact, the BRST charges \( Q_{B(HK)} \) and \( Q_{B(Y)} \) transform as

\[
\hat{Q}_{B(HK)} = Q_{B(HK)} + \tau Q'_{B(HK)},
\]

\[
\hat{Q}_{B(Y)} = Q_{B(Y)} - \tau Q'_{B(HK)},
\]

(4.5)

while their sum \( Q_B \) (and thus \( V^{(\alpha)}_{\text{phys}} \)) remains invariant.

Let us define a subspace \( V^{(\alpha)} \) of the total Fock space \( V \) by

\[
V^{(\alpha)} = \ker Q_{B(Y)} = \{ |\Phi\rangle \in V; Q_{B(Y)} |\Phi\rangle = 0 \} \subset V,
\]

(4.6)

which includes \( V^{(\alpha)}_{\text{phys}} \) as a subspace since by definition \( V^{(\alpha)}_{\text{phys}} \) can be expressed as

\[
V^{(\alpha)}_{\text{phys}} = \{ |\Phi\rangle \in V^{(\alpha)}; Q_{B(HK)} |\Phi\rangle = 0 \} \subset V^{(\alpha)}.
\]

(4.7)
The space $V^{(\alpha)}$ corresponds to the total Fock space of the standard formalism in $a = \varepsilon \alpha^2$ gauge. And thus, as seen from (4.7), $V^{(\alpha)}_{\text{phys}}$ corresponds to the physical subspace $V^{(\text{HK})}_{\text{phys}}$ of the standard formalism in $a = \varepsilon \alpha^2$ gauge. This can be understood from the facts that

1. The modes of gaugeons and their FP ghosts are removed from the space $V^{(\alpha)}$ by the condition $Q_B(Y)|_{\text{phys}} = 0$.

2. The usual fields $\psi, B, c$ and $c^*$ satisfy the (anti)commutation relations exactly the same with those of the standard formalism in $a = \varepsilon \alpha^2$ gauge.

One may understand the first fact by expressing the Lagrangian (3.1) as

$$\mathcal{L} = \mathcal{L}_{\text{HK}}(a = \varepsilon \alpha^2) - i \left\{ Q_B(Y), \bar{\partial}_{\mu} \bar{K}_s \partial^\mu Y + i \bar{K}_s \partial \left( \frac{1}{2} Y_s + \alpha B \right) \right\} + \text{total derivatives},$$

(4.8)

where $\mathcal{L}_{\text{HK}}(a = \varepsilon \alpha^2)$ denotes the Lagrangian of the $a = \varepsilon \alpha^2$ standard formalism.

We emphasize that the above arguments are also valid if we start from the $q$-number gauge transformed charges (4.5) rather than $Q_B(\text{HK})$ and $Q_B(Y)$. For example, we can define the subspaces $V^{(\alpha+\tau)}$ and $V^{(\alpha+\tau)}_{\text{phys}}$ by

$$V^{(\alpha+\tau)} = \ker \widehat{Q}_B(Y),$$

$$V^{(\alpha+\tau)}_{\text{phys}} = \ker \widehat{Q}_B(\text{HK}) \cap \ker \widehat{Q}_B(Y).$$

(4.9)

$V^{(\alpha+\tau)}$ can be identified with the Fock space of the standard formalism in $a = \varepsilon (\alpha + \tau)^2$ gauge, and $V^{(\alpha+\tau)}_{\text{phys}}$ corresponds to its physical subspace. Thus various Fock spaces of the standard formalism in different gauges are embedded in the single Fock space $V$ of our theory.\footnote{Strictly speaking, we have two theories corresponding to the value of $\varepsilon = \pm 1$. Consequently, we have two Fock spaces, to which we refer as $V_+$ and $V_-$ corresponding to the value of $\varepsilon$. Thus the statement becomes as follows: All of the Fock spaces of the standard formalism for all values of $a \geq 0 \ [a \leq 0]$ are embedded in the single Fock space $V_+ \ [V_-]$ of our theory.}

### 5 Comments and discussion

#### 5.1 Type II theory

We have seen in §3 that the gauge fixing parameter $\alpha$ can be shifted freely by the $q$-number gauge transformation. However, we cannot change the sign of the standard gauge parameter $a = \varepsilon \alpha^2$. The situation is analogous to the Type I gaugeon formalism for QED. There are two types of the gaugeon theory for QED.\footnote{We have two theories corresponding to the value of $\varepsilon = \pm 1$. Consequently, we have two Fock spaces, to which we refer as $V_+$ and $V_-$ corresponding to the value of $\varepsilon$. Thus the statement becomes as follows: All of the Fock spaces of the standard formalism for all values of $a \geq 0 \ [a \leq 0]$ are embedded in the single Fock space $V_+ \ [V_-]$ of our theory.} One of them is the Type I theory where the standard gauge parameter is expressed as $a = \varepsilon \alpha^2$, and the other is the Type II theory where the $a = \alpha$. In both types of the theory, $\alpha$ can be shifted as $\hat{\alpha} = \alpha + \tau$ by the $q$-number gauge transformation. Thus, in the Type II theory, we can shift the standard gauge parameter quite freely. We comment here that the Type II theory can be also formulated for the spin-3/2 gauge field.
Let us consider the following Lagrangian,

\[ \mathcal{L}_{II} = \mathcal{L}_{RS} + \bar{B} \bar{\phi} (\gamma \psi) - \frac{i}{2} \bar{B} \partial \phi B - \frac{i}{2} \bar{Y}_s \phi B - \partial \mu \bar{Y}_s \partial^\mu Y \\
- i \partial \mu \bar{c}_s \partial^\mu c - i \partial \mu \bar{K}_s \partial^\mu K. \] (5.1)

Under the q-number gauge transformation [3.11], this Lagrangian is also form invariant (up to total derivatives):

\[ \mathcal{L}_{II}(\varphi_A, \alpha) = \mathcal{L}_{II}(\hat{\varphi}_A, \hat{\alpha}) + \text{total derivatives}, \] (5.2)

with \( \hat{\alpha} \) defined by [3.13]. As easily seen, the Lagrangian (5.1) is also invariant (up to total derivatives) under all of the transformations corresponding to the BRST charges (4.1). Using the charge \( Q_{B(Y)} \) we can express the Lagrangian as

\[ \mathcal{L}_{II} = \mathcal{L}_{HK}(a = \alpha) - i \left\{ Q_{B(Y)} , \partial \mu \bar{K}_s \partial^\mu Y + \frac{i}{2} \bar{K}_s \phi B \right\} + \text{total derivatives}, \] (5.3)

which leads to the identification

\[ a = \alpha, \] (5.4)

nothing but the characteristic of a Type II theory. It should be noted that all the arguments given in §4 also apply to this Type II theory.

### 5.2 Extended Type I theory

If we put \( \alpha = 0 \) in the Type I Lagrangian (3.1), the gaugeon sector decouples from the rest. Then the remaining sector has the same form with the Lagrangian of the standard formalism in the Landau gauge. Thus, the equivalence of the theory to the standard formalism is manifest in the Landau gauge. This situation does not occur in the Type II theory. The gaugeon sector in (3.1) does not decouple for any value of \( \alpha \). In this sense, the Type I theory is preferable to the Type II theory. As seen above, however, we cannot change the sign of the standard gauge parameter \( a \) in the Type I theory, while in the Type II theory we can shift it quite freely.

For the QED case, an extended Type I theory is known,[13] where we can shift the standard gauge parameter quite freely. In this theory, two sets of gaugeons and their FP ghosts are introduced. In the following, we provide an extended type I gaugeon formalism for the spin-3/2 gauge field.

We start from the Lagrangian,

\[ \mathcal{L}'_I = \mathcal{L}_{RS} + \bar{B} \bar{\phi} (\gamma \psi) - i(\bar{Y}_1 + \alpha_1 \bar{B}) \phi (Y_2 + \alpha_2 B) - \partial \mu \bar{Y}_1 \partial^\mu Y \\
- \partial \mu \bar{Y}_2 \partial^\mu Y_2 - i \partial \mu \bar{c}_s \partial^\mu c - i \partial \mu \bar{K}_s \partial^\mu K, \] (5.5)

where we have introduced two sets of gaugeon fields \( Y_i, Y_{is} \) and their FP ghosts \( K_i, K_{is} \), and two gauge fixing parameters \( \alpha_i \) \( (i = 1, 2) \). If we put \( \alpha_1 = \alpha_2 = 0 \), then the gaugeon sector decouples form the rest and the remaining sector is equal to the Landau gauge Lagrangian of the standard formalism. In this sense, this is an extension of the Type I theory.
The Lagrangian (5.5) is invariant up to total derivatives under the following BRST transformation:

\[
\begin{align*}
\delta_B \psi_\mu &= i \partial_\mu \alpha, \\
\delta_B c &= B, \\
\delta_B B &= \delta_B c = 0, \\
\delta_B Y_1 &= i K_1, \\
\delta_B K_1 &= Y_{1s}, \\
\delta_B Y_{1s} &= \delta_B K_1 = 0, \quad (i = 1, 2) \\
\delta_B Y_2 &= i K_2, \\
\delta_B K_2 &= Y_{2s}, \\
\delta_B Y_{2s} &= \delta_B K_2 = 0, \quad (i = 1, 2)
\end{align*}
\]

which satisfies the nilpotency, \(\delta_B^2 = 0\). The corresponding BRST charge is a sum of three nilpotent BRST-like charges:

\[
Q_B = Q_B(HK) + Q_B(Y1) + Q_B(Y2), \quad (5.7)
\]

where \(Q_B(HK)\) is defined by (2.11) and \(Q_B(Y_i)\)'s are given by

\[
Q_B(Y_i) = -i \int \bar{K}_i \partial_0 Y_{1s} d^{n-1}x. \quad (i = 1, 2) \quad (5.8)
\]

As usual, the physical subspace is defined by this BRST charge: \(V_{\text{phys}} = \ker Q_B\).

We define the \(q\)-number gauge transformation by

\[
\begin{align*}
\hat{\psi}_\mu &= \psi_\mu + \tau_1 \partial_\mu Y_1 + \tau_2 \partial_\mu Y_2, \\
\hat{Y}_{1s} &= Y_{1s} - \tau_1 B, \\
\hat{B} &= B, \quad \hat{Y}_i = Y_i, \\
\hat{c} &= c + \tau_1 K_1 + \tau_2 K_2, \\
\hat{K}_{1s} &= K_{1s} - \tau_1 c_{1s}, \\
\hat{c}_{1s} &= c_{1s}, \quad \hat{K}_i = K_i \quad (i = 1, 2)
\end{align*}
\]

where \(\tau_i\) is the parameter of the transformation \((i = 1, 2)\). Under this transformation, the Lagrangian is form invariant (up to total derivatives):

\[
L'_1(\varphi_A, \alpha_1, \alpha_2) = L'_1(\hat{\varphi}_A, \hat{\alpha}_1, \hat{\alpha}_2) + \text{total derivatives}, \quad (5.10)
\]

where \(\varphi_A\) stands for any of the relevant fields and \(\hat{\alpha}_i\)'s are defined by

\[
\hat{\alpha}_i = \alpha_i + \tau_i, \quad (i = 1, 2) \quad (5.11)
\]

The BRST charge \(Q_B\) (5.7) is invariant under the \(q\)-number transformation (5.9). As a result, the physical subspace \(V_{\text{phys}}\) is gauge invariant.

To see the relation of the theory to the standard formalism, we may express the Lagrangian (5.5) as

\[
L'_1 = L_{HK}(a = \alpha_1 \alpha_2) - i \{Q_B(Y), \partial_\mu \bar{K}_{1s} \partial^\mu Y_1 + \partial_\mu \bar{K}_{2s} \partial^\mu Y_2 \\
+ i \bar{K}_{1s} \partial B + i \alpha_2 \bar{K}_{1s} \partial B + i \alpha_1 \bar{K}_{2s} \partial B \} + \text{total derivatives}, \quad (5.12)
\]
where $Q_{B(Y)}$ is a nilpotent BRST charge defined by $Q_{B(Y)} = Q_{B(Y1)} + Q_{B(Y2)}$. This leads us to

$$a = 2\alpha_1\alpha_2,$$

which can be shifted into an arbitrary value by the $q$-number gauge transformation (5.9).

We can show that both of the Fock spaces of Type I and Type II theory are embedded into the total Fock space of this theory. The arguments are quite parallel to the case of QED. For example, by the $q$-number gauge transformation we can always shift the parameters $\alpha_2$ into $\alpha_2 = 1/2$. With this value of the parameter, the Lagrangian (5.5) can be expressed as

$$L'_1 = L_{RS} + \bar{B}\partial(\gamma\psi) - \frac{i\alpha_1}{2}\bar{B}\partial B - \frac{i}{2}\bar{Y}_1\partial B - \partial_\mu\bar{Y}_1\partial^\mu Y_1$$

$$-i\partial_\mu\bar{c}\partial^\mu c - i\partial_\mu\bar{K}_{*1}\partial^\mu K_1$$

$$-i\{Q_{B(Y2)}, \partial_\mu\bar{K}_{*2}\partial^\mu Y_2 + i(\bar{Y}_1 + \alpha_1\bar{B})\partial Y_2\},$$

(5.14)

which is the same expression of the Type II Lagrangian (5.1) up to $Q_{B(Y2)}$-exact operators. Consequently, the subspace $V_{II} = \ker Q_{B(Y2)}$ can be identified with the Fock space of the Type II theory.

5.3 Gauge invariance

We have seen in §4 that the subspace $V^{(\alpha)} = \ker Q_{B(Y)} \subset V$ can be identified with the total Fock space $V^{(HK)}$ of the standard formalism in $a = \varepsilon\alpha^2$ gauge. This does not mean, however, that these spaces are isomorphic to each other. Instead, we can show the following isomorphism:

$$V^{(\alpha)} / \text{Im} Q_{B(Y)} \cong V^{(HK)}.$$  

(5.15)

Namely, by considering the quotient space we can ignore the the $Q_{B(Y)}$-exact states (states having the form $Q_{B(Y)}|*\rangle$), which have no corresponding states in $V^{(HK)}$. Eq.(5.15) is the precise statement that our theory includes the standard formalism as a sub-theory. As for the Hilbert spaces, it can be shown that

$$H^{(\alpha)}_{\text{phys}} \cong H^{(HK)}_{\text{phys}},$$

(5.16)

where $H^{(\alpha)}_{\text{phys}}$ is a physical Hilbert space defined by

$$H^{(\alpha)}_{\text{phys}} = V^{(\alpha)}_{\text{phys}} / N^{(\alpha)},$$

(5.17)

with $N^{(\alpha)}$ being a zero-normed subspace of $V^{(\alpha)}_{\text{phys}}$. Furthermore, we can also verify that our gauge invariant Hilbert space $H_{\text{phys}}$ is isomorphic to $H^{(\alpha)}_{\text{phys}}$. Therefore, we are lead to the gauge invariant result:

$$H^{(HK)}_{\text{phys}} \cong H^{(\alpha)}_{\text{phys}} \cong H_{\text{phys}},$$

(5.18)

The detailed arguments of (5.15) and (5.16) will be reported elsewhere. Similar discussion holds for the theories of Type II and extended Type I.
5.4 Background gravitational field

We have considered so far the theory in the flat space-time. It is straightforward to incorporate the interaction with the background gravity, if it satisfies Ricci flatness.

In the background gravitational field \( g_{\mu\nu} \), the classical Lagrangian (2.1) becomes

\[
\mathcal{L}_{RS} = -\frac{i}{2} \sqrt{g} \bar{\psi}_\mu \gamma^{\mu\nu\lambda} D_\nu \psi_\lambda,
\]

where \( g = |\det g_{\mu\nu}| \), \( D_\nu \) is the covariant derivative, and the Greek indices of \( \gamma^{\mu\nu\lambda} \) are now of the world coordinate thus having vielbein dependence. The Lagrangian (5.19) is invariant up to total derivatives under the gauge transformation

\[
\delta \psi_\mu = D_\mu \Lambda,
\]

if the background gravitational field satisfy the vacuum Einstein equation:

\[
R_{\mu\nu} = 0.
\]

The quantum Lagrangian (3.1) is now given by

\[
\mathcal{L} = \mathcal{L}_{RS} + \sqrt{g} \bar{B} \bar{\psi}(\gamma \psi) - \frac{i}{2} \sqrt{g}(\bar{Y}_* + \alpha \bar{B}) \bar{\psi}(Y_* + \alpha B) - \sqrt{g} \bar{Y}_* \bar{\psi} \bar{\psi} Y
\]

\[-i \sqrt{g} \bar{c}_* \bar{\psi} \bar{\psi} c - i \sqrt{g} \bar{K}_* \bar{\psi} \bar{\psi} K.
\]

This Lagrangian is invariant up to total derivatives under the BRST transformation (3.8) with an exception for \( \psi_\mu \), which transforms now as

\[
\delta_B \psi_\mu = i D_\mu c.
\]

The corresponding BRST charge is given by

\[
Q_B = \int \sqrt{g} J_B^0 d^{n-1}x,
\]

where \( J_B^\mu \) is the BRST current defined by

\[
J_B^\mu = \bar{B} \bar{D}^\mu c + \bar{Y}_* \bar{D}^\mu K.
\]

(Note that this current is actually conserved since the fields \( \varphi = B, Y_*, c \) and \( K \) satisfy the Klein-Gordon equation

\[
\Box \varphi = D^\mu D_\mu \varphi = 0.
\]

This is due to the Ricci flatness:

\[
\bar{\psi}^2 \varphi = \left( D^\mu D_\mu + \frac{1}{4} R \right) \varphi = D^\mu D_\mu \varphi,
\]

where \( R \) is the scalar curvature \( R = g^\mu\nu R_{\mu\nu} (= 0) \). Thus we can consistently define the physical subspace by \( \mathcal{V}_{\text{phys}} = \ker Q_B \).

The form invariance (3.12) also holds for the Lagrangian (5.22) under the \( q \)-number gauge transformation (3.11) with an exception for \( \psi_\mu \), which transforms now as

\[
\hat{\psi}_\mu = \psi_\mu + \tau D_\mu Y.
\]

All the arguments in §3 hold also in the present case. Especially, the BRST transformation commutes with the \( q \)-number gauge transformation, which leads to the gauge invariance of the physical subspace (and the physical Hilbert space).
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