LARGE DEVIATION EXPANSIONS FOR THE COEFFICIENTS OF RANDOM WALKS ON THE GENERAL LINEAR GROUP

HUI XIAO, ION GRAMA, AND QUANSHENG LIU

Abstract. Let $(g_n)_{n\geq 1}$ be a sequence of independent and identically distributed elements of the general linear group $GL(d, \mathbb{R})$. Consider the random walk $G_n := g_n \ldots g_1$. Under suitable conditions, we establish Bahadur-Rao-Petrov type large deviation expansion for the coefficients $\langle f, G_n v \rangle$, where $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$. In particular, our result implies the large deviation principle with an explicit rate function, thus improving significantly the large deviation bounds established earlier. Moreover, we establish Bahadur-Rao-Petrov type large deviation expansion for the coefficients $\langle f, G_n v \rangle$ under the changed measure. Toward this end we prove the Hölder regularity of the stationary measure corresponding to the Markov chain $G_n v / |G_n v|$ under the changed measure, which is of independent interest. In addition, we also prove local limit theorems with large deviations for the coefficients of $G_n$.

1. Introduction

1.1. Background and objectives. Let $d \geq 2$ be an integer. Assume that on the probability space $(\Omega, \mathcal{F}, P)$ we are given a sequence of real random $d \times d$ matrices $(g_n)_{n\geq 1}$ which are independent and identically distributed (i.i.d.) with common law $\mu$. A great deal of research has been devoted to studying the random matrix product $G_n := g_n \ldots g_1$. Many fundamental results related to $G_n$, such as the strong law of large numbers, the central limit theorem, the law of iterated logarithm and large deviations have been established by Furstenberg and Kesten [27], Kingman [48], Le Page [49], Guivarc’h and Raugi [42], Bougerol and Lacroix [8], Gol’dsheid and Margulis [29], Hennion [44], Furman [25], Guivarc’h and Le Page [41], Benoist and Quint [5, 6], to name only a few. These limit theorems turn out to be very useful in various areas, such as the spectral theory of random Schrödinger operators [8, 16], disordered systems and chaotic dynamics coming from
statistical physics [20], the multidimensional stochastic recursion [47, 41], the dynamics of group actions [9, 4], and the survival probabilities and conditioned limit theorems of branching processes in random environment [35, 50, 34].

Denote by \( \langle f, G_n v \rangle \) the coefficients of the matrix \( G_n \), where \( f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \), and \( \langle \cdot, \cdot \rangle \) is the corresponding dual bracket. There has been of growing interest in the study of the asymptotic behavior of \( \langle f, G_n v \rangle \), since the seminal work of Furstenberg and Kesten [27], where the following strong law of large numbers has been established for positive matrices:

\[
\lim_{n \to \infty} \frac{1}{n} \log |\langle f, G_n v \rangle| = \lambda, \quad \text{a.s.,}
\]

with \( \lambda \) a constant called the first Lyapunov exponent of the sequence \( (g_n)_{n \geq 1} \). In [27] the central limit theorem has also been proved, thus giving an affirmative answer to Bellman’s conjecture in [3]. In the case of invertible matrices, Guivarc’h and Raugi [42] have established the strong law of large numbers and the central limit theorem for the coefficients \( \langle f, G_n v \rangle \), where the proof turns out to be more involved than that in [27], and is based on the regularity of the stationary measure of the Markov chain \( G_n x := G_n v / |G_n v| \) with \( x = \mathbb{R} v \) a starting point on the projective space \( \mathbb{P}^{d-1} \). Recently, Benoist and Quint [6] have proved the following large deviation bound: for any \( q > \lambda \), there exists a constant \( c > 0 \) such that

\[
P\left( \log |\langle f, G_n v \rangle| > nq \right) \leq e^{-cn}.
\]  

(1.1)

But the precise decay rate on the large deviation probability in (1.1) is not known. The goal of this paper is to establish an exact large deviation asymptotic for the coefficients \( \langle f, G_n v \rangle \), called Bahadur-Rao-Petrov type large deviations following the groundwork by Bahadur-Rao [2] and Petrov [52] for sums of i.i.d. real-valued random variables. Our result will imply the large deviation principle with an explicit rate function, which improves (1.1). Moreover, we shall also establish Bahadur-Rao-Petrov type upper tail large deviation asymptotics for the couple \((G_n x, \log |\langle f, G_n v \rangle|)\) with target functions, which is of independent interest; in particular it implies a new result on the local limit theorem with large deviations for coefficients \( \langle f, G_n v \rangle \). Similar results for lower tail large deviations are also obtained, whose proof turns out to be more delicate.

1.2. Brief overview of the main results. Let \( I_\mu = \{ s \geq 0 : \mathbb{E}(\|g_1\|^s) < \infty \} \), where \( \|g\| \) is the operator norm of a matrix \( g \). For any \( s \in I_\mu \), define \( \kappa(s) = \lim_{n \to \infty} \left( \mathbb{E}(\|G_n\|^s) \right)^{1/s} \). Set \( \Lambda = \log \kappa \) and consider its Fenchel-Legendre transform \( \Lambda^* \), which satisfies \( \Lambda^*(q) = sq - \Lambda(s) > 0 \) for \( q = \Lambda'(s) \) and \( s \in I_\mu^0 \) (the interior of the interval \( I_\mu \)). In the sequel \( \langle \cdot, \cdot \rangle \) and \( |\cdot| \) denote respectively the dual bracket and the Euclidean norm. Denote by \( \mathbb{P}^{d-1} := \{ x = \mathbb{R} v : v \in \mathbb{R}^d \setminus \{0\} \} \),
\( \mathbb{R}^d \setminus \{0\} \) the projective space in \( \mathbb{R}^d \); the projective space \((\mathbb{P}^{d-1})^* \) in \( \mathbb{R}^d \) is defined similarly. For any \( x = \mathbb{R}v \in \mathbb{P}^{d-1} \) and \( y = \mathbb{R}f \in (\mathbb{P}^{d-1})^* \) we define \( \delta(y,x) = \vec{|f|/\|y\|} \). For any \( g \in \text{GL}(d, \mathbb{R}) \) and \( x = \mathbb{R}v \in \mathbb{P}^{d-1} \), let \( gx = \mathbb{R}gv \in \mathbb{P}^{d-1} \), and denote by \( gv \in \mathbb{R}^d \) the image of the automorphism \( v \mapsto gv \) on \( \mathbb{R}^d \).

Consider the transfer operator \( P_s \) defined by \( P_s \varphi(x) = \mathbb{E} [e^{s\sigma(g_1,x)} \varphi(g_1x)] \), \( x = \mathbb{R}v \in \mathbb{P}^{d-1} \), where \( \sigma(g,x) = \log \vec{|g|/|v|} \), and \( \varphi \) is a continuous function on \( \mathbb{P}^{d-1} \); the conjugate transfer operator \( P_s^* \) is defined similarly: see (2.4). The operators \( P_s \) and \( P_s^* \) have continuous strictly positive eigenfunctions \( r_s \) and \( r_s^* \) on \( \mathbb{P}^{d-1} \) which are unique up to a scaling constant, and unique probability eigenmeasures \( \nu_s \) and \( \nu_s^* \), satisfying \( P_s r_s = \kappa(s) r_s, \ P_s \nu_s = \kappa(s) \nu_s, \ P_s^* r_s^* = \kappa(s) r_s^* \) and \( P_s^* \nu_s^* = \kappa(s) \nu_s^* \). Denote \( \sigma_s := \sqrt{\Lambda'(s)} > 0 \). For details see Section 3.1.

Our first objective is to establish a Bahadur-Rao type large deviation asymptotic for the coefficients \( \langle f, G_n v \rangle \); we refer to Bahadur and Rao [2] for the case of i.i.d. real-valued random variables. More precisely, we prove that, for any \( s \in I^0, v \in \mathbb{R}^d \) and \( f \in (\mathbb{P}^{d-1})^* \) with \( |v| = |f| = 1 \), \( q = \Lambda'(s), \ x = \mathbb{R}v \) and \( y = \mathbb{R}f \), as \( n \to \infty \),

\[
\mathbb{P} \left( \log |\langle f, G_n v \rangle| \geq nq \right) = \frac{r_s(x) r_s^*(y) \exp \left( -n \Lambda^*(q) \right)}{\sigma_s \sqrt{2\pi n}} \left[ 1 + o(1) \right], \tag{1.2}
\]

where \( \sigma_s = \nu_s(r_s) = \nu_s^*(r_s^*) > 0 \). The asymptotic (1.2) clearly implies the large deviation principle for \( \langle f, G_n v \rangle \) with the rate function \( \Lambda^* \), which obviously improves the large deviation bound (1.1).

In fact, we shall extend (1.2) to the couple \((G_n x, \log |\langle f, G_n v \rangle|)\) with target functions. Precisely, for any \( s \in I^0, \) any Hölder continuous function \( \varphi \) on \( \mathbb{P}^{d-1} \) and any measurable function \( \psi \) on \( \mathbb{R} \) such that \( u \mapsto e^{-s_1 u} \psi(u) \) is directly Riemann integrable for some \( s_1 \in (0, s) \), we prove that as \( n \to \infty \),

\[
\mathbb{E} \left[ \varphi(G_n x) \psi(\log |\langle f, G_n v \rangle| - nq) \right], \tag{1.3}
\]

\[
= \frac{r_s(x) \exp \left( -n \Lambda^*(q) \right)}{\sigma_s \sqrt{2\pi n}} \left[ \int_{\mathbb{P}^{d-1}} \varphi(y,x) \delta(y,x) \nu_s(dx) \int_{\mathbb{R}} e^{-su} \psi(u) du + o(1) \right].
\]

Our second objective is to establish a Bahadur-Rao type result for the lower large deviation probabilities \( \mathbb{P}(\log |\langle f, G_n v \rangle| \leq nq) \), where \( q = \Lambda'(s) < \lambda \) with \( s < 0 \) sufficiently close to 0. Specifically, for \( s < 0 \) small enough, we prove that, as \( n \to \infty \),

\[
\mathbb{P}(\log |\langle f, G_n v \rangle| \leq nq) = \frac{r_s(x) r_s^*(y) \exp \left( -n \Lambda^*(q) \right)}{-s \sigma_s \sqrt{2\pi n}} \left[ 1 + o(1) \right], \tag{1.4}
\]

where \( r_s, r_s^*, \sigma_s, \Lambda^* \) and \( \sigma_s \) are defined in Section 3.1, which are strictly positive, similarly to the case \( s > 0 \). The asymptotic (1.4) is of course much sharper than the corresponding lower tail large deviation principle for
More generally, we extend the lower tail large deviation expansion (1.4) to the couple $(G_n x, \log |\langle f, G_n v \rangle|)$ with target functions, in the same line as (1.3).

For a brief description of the main ideas of the approach see Section 2.5.

The assertions (1.2), (1.3) and (1.4) stated above concern Bahadur-Rao type large deviation asymptotics. Actually we shall establish an extended version of these results with an additional vanishing perturbation on $q$, which in the literature is known as Bahadur-Rao-Petrov type large deviation expansion. Such type of extensions has important and interesting implications, for instance, to local limit theorems with large deviations for the coefficients $\langle f, G_n v \rangle$: see Theorem 2.5. Recently, Buraczewski, Collamore, Damek and Zienkiewicz [15] have established a law of large numbers, a central limit theorem and large deviation results for perpetuities using the Bahadur-Rao-Petrov large deviation asymptotic for sums of i.i.d. real valued random variables. With the help of our large deviation results for products of random matrices it is possible to extend these results to multivariate perpetuity sequences arising in financial mathematics. Another potential application of our results is in the study of multitype branching processes and branching random walks governed by products of random matrices; we refer to Mentemeier [51], Bui, Grama and Liu [11, 12] for details.

It is worth mentioning that using the approach developed in this paper, it is possible to establish new limit theorems for the Gromov product of random walks on hyperbolic groups; we refer to Gouëzel [30, 31] on this topic.

We also mention that our approach opens a way to study invariance principles for the coefficients $\langle f, G_n v \rangle$; recent progress in the study of invariance principles can be found in Cuny, Dedecker and Jan [18] and Cuny, Dedecker and Merlevède [19], where the vector norm $|G_n v|$ and the operator norm $\|G_n\|$ have been studied via the martingale approximation approach.

2. Main results

In this section we present our main results and the strategy of the proofs.

2.1. Notation and conditions. Denote by $c$, $C$ absolute constants whose values may change from line to line. By $c_\alpha$, $C_\alpha$ we mean constants depending only on the parameter $\alpha$. For any integrable function $\rho : \mathbb{R} \to \mathbb{C}$, denote its Fourier transform by $\hat{\rho}(t) = \int_{\mathbb{R}} e^{-ity} \rho(y) dy$, $t \in \mathbb{R}$. For a measure $\nu$ and a function $\varphi$ we write $\nu(\varphi) = \int \varphi dv$. Let $\mathbb{N} = \{1, 2, \ldots\}$. By convention $\log 0 = -\infty$.

The space $\mathbb{R}^d$ is equipped with the standard scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $|\cdot|$. For any integer $d \geq 2$, denote by $G := GL(d, \mathbb{R})$ the general linear group of invertible $d \times d$ matrices with coefficients in $\mathbb{R}$. The projective space $\mathbb{P}^{d-1}$ of $\mathbb{R}^d$ is the set of elements $x = \mathbb{R}v$, where $v \in \mathbb{R}^d \setminus \{0\}$. 
The projective space of $\mathbb{R}_d^*$ is denoted by $(\mathbb{P}^{d-1})^*$. We equip $\mathbb{P}^{d-1}$ with the angular distance $d$ (see [41]), i.e., for any $x, x' \in \mathbb{P}^{d-1}$ with $x \in \mathbb{R} v$ and $x' \in \mathbb{R} v'$, $d(x, y) = (1 - \langle v, v' \rangle)^{1/2}$.

Let $\mathcal{C}(\mathbb{P}^{d-1})$ be the space of complex-valued continuous functions on $\mathbb{P}^{d-1}$. We write $1$ for the identity function $1(x), x \in \mathbb{P}^{d-1}$. Throughout this paper, $\gamma > 0$ is a fixed sufficiently small constant. For any $\varphi \in \mathcal{C}(\mathbb{P}^{d-1})$, set

$$
\|\varphi\|_{\infty} := \sup_{x \in \mathbb{P}^{d-1}} |\varphi(x)| \quad \text{and} \quad \|\varphi\|_\gamma := \|\varphi\|_{\infty} + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\gamma},
$$

and consider the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathbb{P}^{d-1}) : \|\varphi\|_\gamma < +\infty\}$.

All over the paper $(g_n)_{n \geq 1}$ is a sequence of i.i.d. elements of the same probability law $\mu$ on $\mathbb{G}$. Denote by $\Gamma_\mu$ the smallest closed semigroup generated by the support of $\mu$. For any $g \in \mathbb{G}$, denote $\|g\| = \sup_{v \in \mathbb{R}^d \setminus \{0\}} \frac{|g\|_v}{|v|}$. Let

$$I_\mu = \{s \geq 0 : \mathbb{E}(\|g\|^s) < +\infty\},$$

and $I_\mu^o$ be its interior. In the sequel we always assume that there exists $s > 0$ such that $\mathbb{E}(\|g\|^s) < +\infty$, so that $I_\mu^o$ is non-empty open interval of $\mathbb{R}$.

For any $g \in \mathbb{G}$, set $\iota(g) = \inf_{v \in \mathbb{R}^d \setminus \{0\}} \frac{|g\|_v}{|v|}$, and it holds that $\iota(g) = \|g^{-1}\|^{-1}$. We will need the following exponential moment condition:

**A1.** There exist $s \in I_\mu^o$ and $\beta \in (0, 1)$ such that

$$
\int_\mathbb{G} \|g\|^{s+\beta} \iota(g)^{-\beta} \mu(dg) < +\infty.
$$

Moreover, we shall use the following two-sided moment condition. Denote $N(g) = \max\{\|g\|, \|g^{-1}\|\}$ for any $g \in \mathbb{G}$.

**A2.** There exists a constant $\eta > 0$ such that $\mathbb{E}(N(g_1)^\eta) < +\infty$.

A matrix $g \in \mathbb{G}$ is called proximal if it has an algebraic simple dominant eigenvalue, namely, $g$ has an eigenvalue $\lambda_g$ satisfying $|\lambda_g| > |\lambda'_g|$ for all other eigenvalues $\lambda'_g$ of $g$. It is easy to verify that $\lambda_g \in \mathbb{R}$. The eigenvector $v_g$ with unit norm $|v_g| = 1$, corresponding to the eigenvalue $\lambda_g$, is called the dominant eigenvector. We will need the following strong irreducibility and proximality conditions:

**A3.** (i) (Strong irreducibility) No finite union of proper subspaces of $\mathbb{R}^d$ is $\Gamma_\mu$-invariant.

(ii) (Proximality) $\Gamma_\mu$ contains at least one proximal matrix.

For any $g \in \mathbb{G}$ and $x = \mathbb{R} v \in \mathbb{P}^{d-1}$, let $g x = \mathbb{R} g v \in \mathbb{P}^{d-1}$ and

$$G_0 x := x, \quad G_n x := \mathbb{R} G_n v, \quad n \geq 1. \quad (2.1)$$
Then \((G_n x)_{n \geq 0}\) forms a Markov chain on the projective space \(\mathbb{P}^{d-1}\). Moreover, under condition \(A3\), \((G_n x)_{n \geq 0}\) has a unique stationary probability measure \(\nu\) on \(\mathbb{P}^{d-1}\) such that for any \(\varphi \in \mathcal{C}(\mathbb{P}^{d-1})\),

\[
\int_{\mathbb{P}^{d-1}} \int_{\mathbb{G}} \varphi(g x) \mu(dg) \nu(dx) = \int_{\mathbb{P}^{d-1}} \varphi(x) \nu(dx).
\] (2.2)

Furthermore, the support of \(\nu\) is given by

\[
\text{supp } \nu = \{ v_g \in \mathbb{P}^{d-1} : g \in \Gamma_\mu, \text{ } g \text{ is proximal} \}.
\] (2.3)

For any \(s \in (-s_0, 0) \cup I_\mu\) with small enough \(s_0 > 0\), define the transfer operator \(P_s\) and the conjugate transfer operator \(P_s^*\) as follows: for any \(\varphi \in \mathcal{C}(\mathbb{P}^{d-1})\),

\[
P_s \varphi(x) = \int_{\mathbb{G}} e^{s\sigma(g, x)} \varphi(g x) \mu(dg), \quad x \in \mathbb{P}^{d-1},
\] (2.4)

where \(\sigma(g, x) = \log \frac{|g v|}{|v|}\), and for any \(\varphi \in \mathcal{C}((\mathbb{P}^{d-1})^*)\),

\[
P_s^* \varphi(y) = \int_{\mathbb{G}} e^{s\sigma^*(g^*, y)} \varphi(g^* y) \mu(dg), \quad y \in (\mathbb{P}^{d-1})^*.
\] (2.5)

where \(g^*\) denotes the adjoint automorphism of the matrix \(g\). Under suitable conditions, the transfer operator \(P_s\) has a unique probability eigenmeasure \(\nu_s\) on \(\mathbb{P}^{d-1}\) corresponding to the eigenvalue \(\kappa(s)\): \(P_s \nu_s = \kappa(s) \nu_s\). Similarly, the conjugate transfer operator \(P_s^*\) has a unique probability eigenmeasure \(\nu_s^*\) corresponding to the eigenvalue \(\kappa(s)\): \(P_s^* \nu_s^* = \kappa(s) \nu_s^*\). For any \(x = R v \in \mathbb{P}^{d-1}\) and \(y = R f \in (\mathbb{P}^{d-1})^*\) with \(v \in \mathbb{R}^d \setminus \{0\}\) and \(f \in (\mathbb{R}^d)^* \setminus \{0\}\), denote \(\delta(y, x) = \frac{|f(x)|}{|f||v|}\) and set

\[
r_s(x) = \int_{(\mathbb{P}^{d-1})^*} \delta(y, x)^s \nu_s^*(dy), \quad r_s^*(y) = \int_{\mathbb{P}^{d-1}} \delta(y, x)^s \nu_s(dx).
\]

Then, \(r_s\) is the unique, up to a scaling constant, strictly positive eigenfunction of \(P_s\): \(P_s r_s = \kappa(s) r_s\); similarly \(r_s^*\) is the unique, up to a scaling constant, strictly positive eigenfunction of \(P_s^*\): \(P_s^* r_s^* = \kappa(s) r_s^*\). It is easy to see that

\[
\nu_s(r_s) = \nu_s^*(r_s^*) =: g_s.
\]

The stationary measure \(\pi_s\) is defined by \(\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{g_s}\), for any \(\varphi \in \mathcal{C}(\mathbb{P}^{d-1})\). We refer to Section 3.1 for details.

Define \(\Lambda = \log \kappa : (-s_0, 0) \cup I_\mu \to \mathbb{R}\), then the function \(\Lambda\) is convex and analytic. Condition \(A3\) implies that \(\sigma_s = \Lambda'(s)\) is strictly positive for any \(s \in (-s_0, 0) \cup I_\mu\). Denote by \(\Lambda^*\) the Fenchel-Legendre transform of \(\Lambda\), then it holds that \(\Lambda^*(q) = sq - \Lambda(s) > 0\) if \(q = \Lambda'(s)\) for \(s \in (-s_0, 0) \cup I_\mu\).
2.2. Precise large deviations for coefficients. The goal of this section is to state exact large deviation asymptotics for the coefficients \(|f, G_n v|\), where \(f \in (\mathbb{R}^d)^*\) and \(v \in \mathbb{R}^d\). To the best of our knowledge, the precise large deviations and even the large deviation principle for \(|f, G_n v|\) have not been studied by now in the literature. Our first result is a large deviation asymptotic of the Bahadur-Rao type (see [2]) for the upper tails of \(|f, G_n v|\).

Recall the notation \(x = \mathbb{R} v\) and \(y = \mathbb{R} f\) for any \(v \in \mathbb{R}^d \setminus \{0\}\) and \(f \in (\mathbb{R}^d)^* \setminus \{0\}\).

**Theorem 2.1.** Assume conditions \(A1\) and \(A3\). Let \(s \in I_\mu^0\) and \(q = \Lambda^*(s)\). Then, as \(n \to \infty\), uniformly in \(f \in (\mathbb{R}^d)^*\) and \(v \in \mathbb{R}^d\) with \(|f| = |v| = 1\),

\[
P \left( \log |\langle f, G_n v \rangle| \geq nq \right) = \frac{r_s(x)r_s^*(y) \exp \left(-n\Lambda^*(q)\right)}{\sigma_s \sqrt{2\pi n}} \left[1 + o(1)\right]. \tag{2.6}
\]

In particular, if we fix a basis \((e_i^s)_{1 \leq i \leq d}\) in \((\mathbb{R}^d)^*\) and a basis \((e_j)_{1 \leq j \leq d}\) in \(\mathbb{R}^d\), then taking \(f = e_i^s\) and \(v = e_j\) in (2.6), we get the Bahadur-Rao type large deviation asymptotic for the \((i, j)\)-th entry \(G_{n}^{i,j}\) of the matrix product \(G_n\). It is easy to verify that the large deviation asymptotic (2.6) implies a large deviation principle, as stated below: under the assumptions of Theorem 2.1, we have, uniformly in \(f \in (\mathbb{R}^d)^*\) and \(v \in \mathbb{R}^d\) with \(|f| = |v| = 1\),

\[
\lim_{n \to \infty} \frac{1}{n} \log P \left( \log |\langle f, G_n v \rangle| \geq nq \right) = -\Lambda^*(q). \tag{2.7}
\]

In its turn, the asymptotic (2.7) improves significantly the bound (1.1).

An important field of applications of large deviation asymptotics for the coefficients of type (2.6) is the study of asymptotic behaviors of multi-type branching processes in random environment. For results in the case of single-type branching processes we refer to [36, 37] and for the relation between the coefficients of products of random matrices and the multi-type branching processes we refer to [17].

Our next result is an improvement of Theorem 2.1 by allowing a vanishing perturbation \(l\) on \(q = \Lambda^*(s)\), in the spirit of the Petrov result [52], called the Bahadur-Rao-Petrov type large deviation. Large deviations with a perturbation \(l\) have been used for example in Buraczewski, Collamore, Damek and Zienkiewicz [15] for a recent application to the asymptotic of the ruin time in some models of financial mathematics. These results are also useful to deduce local limit theorems with large deviations, see subsection 2.3.

**Theorem 2.2.** Assume conditions \(A1\) and \(A3\). Let \(s \in I_\mu^0\) and \(q = \Lambda^*(s)\). Let \((l_n)_{n \geq 1}\) be any positive sequence satisfying \(\lim_{n \to \infty} l_n = 0\). Then, we have, as \(n \to \infty\), uniformly in \(|l| \leq l_n\), \(f \in (\mathbb{R}^d)^*\) and \(v \in \mathbb{R}^d\) with \(|f| = |v| = 1\),

\[
\lim_{n \to \infty} \frac{1}{n} \log P \left( \log |\langle f, G_n v \rangle| \geq nq \right) = -\Lambda^*(q). \tag{2.8}
\]
$|v| = 1,$

$$
P\left( \log |\langle f, G_n v \rangle| \geq n(q + l) \right) = \frac{r_s(x)r^*_s(y) \exp (-n\Lambda^*(q + l))}{\varrho_s} s\sigma_s \sqrt{2\pi n} \left[ 1 + o(1) \right].$$

More generally, for any measurable function $\psi$ on $\mathbb{R}$ such that $u \mapsto e^{-s' u} \psi(u)$ is directly Riemann integrable for some $s' \in (0, s)$, we have, as $n \to \infty$, uniformly in $|l| \leq l_n$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$, and $\varphi \in B_\gamma$,

$$
\mathbb{E} \left[ \varphi(G_n x) \psi \left( \log |\langle f, G_n v \rangle| - n(q + l) \right) \right] = \frac{r_s(x) \exp (-n\Lambda^*(q + l))}{\varrho_s} \frac{1}{\sigma_s \sqrt{2\pi n}} \left[ \int_{\mathbb{R}^d} \varphi(x) \delta(y, x)^* \nu_s(dx) \int_{\mathbb{R}} e^{-su} \psi(u) du + o(1) \right].
$$

A more general version of Theorem 2.2 is given in Theorem 5.7, where it is shown that the above large deviation asymptotics hold uniformly in $s \in K_\mu$ with any compact set $K_\mu \subset I_\mu^0$.

Consider the reversed random walk $M_n$ defined by $M_n = g_1 \ldots g_n$. Since the two probabilities $\mathbb{P}(\log |\langle f, G_n v \rangle| \geq n(q + l))$ and $\mathbb{P}(\log |\langle f, M_n v \rangle| \geq n(q + l))$ are equal (as $G_n$ and $M_n$ have the same law), for $M_n$ we have the same large deviation expansions as for $G_n$.

Now we are going to give exact asymptotics of the lower tail large deviation probabilities $\mathbb{P}(\log |\langle f, G_n v \rangle| \leq nq)$, where $q = \Lambda'(s) < \lambda = \Lambda'(0)$ for $s < 0$. These asymptotics cannot be deduced from Theorems 2.1 and 2.2; the proofs turn out to be more delicate and require to develop the corresponding spectral gap theory for the transfer operator $P_s$ and to establish the Hölder regularity for the stationary measure $\pi_s$ with $s < 0$.

**Theorem 2.3.** Assume conditions **A2** and **A3**. Then, there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, as $n \to \infty$, uniformly in $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$
\mathbb{P} \left( \log |\langle f, G_n v \rangle| \leq nq \right) = \frac{r_s(x)r^*_s(y) \exp (-n\Lambda^*(q))}{\varrho_s} - \frac{1}{\sigma_s \sqrt{2\pi n}} [1 + o(1)].
$$

In particular, fixing a basis $(e^*_i)_{1 \leq i \leq d}$ in $(\mathbb{R}^d)^*$ and a basis $(e_j)_{1 \leq j \leq d}$ in $\mathbb{R}^d$, with $f = e^*_i$ and $v = e_j$ in (2.9), we obtain the Bahadur-Rao type lower tail large deviation asymptotic for the entries $G^i_j$. From (2.9) we get a lower tail large deviation principle under the assumptions of Theorem 2.3: uniformly in $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \log |\langle f, G_n v \rangle| \leq nq \right) = -\Lambda^*(q).
$$

The result (2.10) sharpens the following lower tail large deviation bound established by Benoist and Quint [6, Theorem 14.21]: for $q < \lambda$, there exists
a constant $c > 0$ such that for all $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$
P\left( \log |\langle f, G_n v \rangle| < nq \right) \leq e^{-cn}.
$$

Now we give a Bahadur-Rao-Petrov version of the above theorem.

**Theorem 2.4.** Assume conditions **A2** and **A3**. Let $(l_n)_{n \geq 1}$ be any positive sequence satisfying $\lim_{n \to \infty} l_n = 0$. Then, there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, we have, as $n \to \infty$, uniformly in $|l| \leq l_n$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$
P\left( \log |\langle f, G_n v \rangle| \leq n(q + l) \right) = \frac{r_s(x) r_s^*(y) \exp \left( -n\Lambda^s(q + l) \right)}{\varrho_s} \frac{1}{-s\sigma_s \sqrt{2\pi n}} [1 + o(1)].
$$

More generally, for any $\varphi \in \mathcal{B}_\gamma$ and any measurable function $\psi$ on $\mathbb{R}$ such that $u \mapsto e^{-s^u \psi(u)}$ is directly Riemann integrable for some $s' \in (-s_0, s)$, we have, as $n \to \infty$, uniformly in $|l| \leq l_n$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$
\mathbb{E} \left[ \varphi(G_n x) \left( \log |\langle f, G_n v \rangle| - n(q + l) \right) \right] = \frac{r_s(x) \exp \left( -n\Lambda^s(q + l) \right)}{\varrho_s} \frac{1}{\sigma_s \sqrt{2\pi n}} \left[ \int_{\mathbb{R}^d} \varphi(x) \delta(y, x)^s \nu_s(dx) \int_\mathbb{R} e^{-s^u \psi(u)} du + o(1) \right].
$$

### 2.3. Local limit theorems with large deviations for coefficients.

In this subsection we formulate the precise local limit theorems with large deviations for the coefficients $\langle f, G_n v \rangle$. For sums of independent real-valued random variables, local limit theorems with large and moderate deviations can be found for instance in Gnedenko [28], Shepp [54], Stone [55], Borovkov and Borovkov [7], Breuillard [10], Varju [56]. For products of random matrices, such types of local limit theorems for the vector norm $|G_n v|$ have been recently established in [6, 57, 58]. Our following theorem extends the results in [57, 58] for the vector norm $|G_n v|$ to the case of the coefficients $\langle f, G_n v \rangle$.

**Theorem 2.5.** Let $(l_n)_{n \geq 1}$ be any positive sequence satisfying $\lim_{n \to \infty} l_n = 0$. Let $-\infty < a_1 < a_2 < \infty$ be real numbers.

1. Assume conditions **A1** and **A3**. Let $s \in I^*_{\mu}$ and $q = \Lambda'(s)$. Then, as $n \to \infty$, uniformly in $|l| \leq l_n$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$
P\left( \log |\langle f, G_n v \rangle| \in [a_1, a_2] + n(q + l) \right) = \left( e^{-sa_1} - e^{-sa_2} \right) \frac{r_s(x) r_s^*(y) \exp \left( -n\Lambda^s(q + l) \right)}{\varrho_s} \frac{1}{s\sigma_s \sqrt{2\pi n}} [1 + o(1)]. \quad (2.11)
$$

2. Assume conditions **A2** and **A3**. Then, there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, as $n \to \infty$, uniformly
in $|l| \leq l_n$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\mathbb{P}\left(\log |\langle f, G_n v \rangle| \in [a_1, a_2] + n(q + l)\right)$$

$$= (e^{-sa_2} - e^{-sa_1}) \frac{r_s(x)r_s^*(y) \exp(-n\Lambda^*(q + l))}{g_s} [1 + o(1)]. \quad (2.12)$$

Taking $\varphi = 1$ and $\psi = 1_{[a_1, a_2]}$ with real numbers $a_1 < a_2$, it is easy to see that Theorem 2.2 and Theorem 2.4 respectively recover the local limit theorem with large deviations (2.11) and (2.12).

2.4. Precise large deviations for coefficients under the changed measure. We now give Bahadur-Rao-Petrov type large deviations for the coefficients $\langle f, G_n v \rangle$ under the changed measure $\mathbb{Q}^x_s$, which are useful for example in the study of branching processes and branching random walks.

We first deal with the upper tail case. The following result is a more general version of Theorems 2.1 and 2.2. Denote $q_s = \Lambda(s)$ and $q_t = \Lambda(t)$ for any $s, t \in (-s_0, 0] \cup I^*_\mu$ with $s < t$.

**Theorem 2.6.** Assume conditions A1, A2 and A3. Let $s_\infty = \sup\{s : s \in I^*_\mu\}$. Then, there exists a constant $s_0 > 0$ such that for any fixed $s \in (-s_0, s_\infty)$ and any compact set $K_\mu \subset (s, s_\infty)$, we have, as $n \to \infty$, uniformly in $t \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\mathbb{Q}^x_s\left(\log |\langle f, G_n v \rangle| \geq nq_t\right) = \frac{r_t(x) \exp\{-n(\Lambda^*(q_t) - \Lambda^*(q_s) - s(q_t - q_s))\}}{r_s(x)} \frac{1}{(t - s)\sigma_t\sqrt{2\pi n}}$$

$$\times \int_{\mathbb{R}^d} \delta(y, x)^t \frac{r_s(x)}{r_t(x)} \pi_t(dx)[1 + o(1)]. \quad (2.13)$$

More generally, there exists a constant $s_0 > 0$ such that for any fixed $s \in (-s_0, s_\infty)$ and any compact set $K_\mu \subset (s, s_\infty)$, for any measurable function $\psi$ on $\mathbb{R}$ such that $u \mapsto e^{-s^\prime u}\psi(u)$ is directly Riemann integrable for any $s^\prime \in K_\mu^* := \{s^\prime \in \mathbb{R} : |s^\prime - s| < \epsilon, s \in K_\mu\}$ with $\epsilon > 0$ small enough, we have, as $n \to \infty$, uniformly in $t \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\mathbb{E}_{\mathbb{Q}^x_s}\left[\varphi(G_n x)\psi(\log |\langle f, G_n v \rangle| - nq_t)\right]$$

$$= \frac{r_t(x)}{r_s(x)} \frac{1}{\sigma_t\sqrt{2\pi n}}$$

$$\times \left[\int_{\mathbb{R}^d} \varphi(x)\delta(y, x)^t \frac{r_s(x)}{r_t(x)} \pi_t(dx) \int_\mathbb{R} e^{-(t-s)u}\psi(u)du + o(1)\right]. \quad (2.14)$$

We next consider the lower tail case. The following result is an extension of Theorems 2.3 and 2.4. Denote $q_s = \Lambda(s)$ and $q_t = \Lambda(t)$ for any $s, t \in (-s_0, 0] \cup I^*_\mu$ with $s > t$. 
Theorem 2.7. Assume conditions A1, A2 and A3. Then, there exists a constant $s_0 > 0$ such that for any fixed $s \in (-s_0, 0] \cup I_µ^o$ and any compact set $K_µ \subset (-s_0, 0)$, we have, as $n \to \infty$, uniformly in $t \in K_µ$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$Q_s^n \left( \log |\langle f, G_n v \rangle| \leq n q_t \right) = \int_{\mathbb{R}^d} \delta(y, x)^t \frac{r_s(x)}{r_t(x)} \pi_t(dx) \times r_s(x) \exp\{-n(\Lambda^*(q_t) - \Lambda^*(q_s)) - s(q_t - q_s)\} \frac{1}{(s - t)\sigma_t \sqrt{2\pi n}} [1 + o(1)].$$

More generally, there exists a constant $s_0 > 0$ such that for any fixed $s \in (-s_0, 0] \cup I_µ^o$ and any compact set $K_µ \subset (-s_0, 0)$, for any measurable function $\psi$ on $\mathbb{R}$ such that $u \mapsto e^{-s'u} \psi(u)$ is directly Riemann integrable for any $s' \in K_µ^e := \{s' \in \mathbb{R}: |s' - s| < \epsilon, s \in K_µ\}$ with $\epsilon > 0$ small enough, we have, as $n \to \infty$, uniformly in $t \in K_µ$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$E_{Q_s^n} \left[ \varphi(G_n x) \psi(\log |\langle f, G_n v \rangle| - n q_t) \right] = \frac{r_t(x)}{r_s(x)} \exp\{-n(\Lambda^*(q_t) - \Lambda^*(q_s)) - s(q_t - q_s)\} \times \frac{1}{\sigma_t \sqrt{2\pi n}} \left[ \int_{\mathbb{R}^d} \varphi(x) \delta(y, x)^t \frac{r_s(x)}{r_t(x)} \pi_t(dx) \int_{\mathbb{R}} e^{-(t-s)u} \psi(u) du + o(1) \right].$$

The proof of Theorem 2.7 relies essentially on the Hölder regularity of the stationary measure $\pi_s$, which will be presented in Section 3.

2.5. Proof strategy. The standard approach to obtain precise large deviations for i.i.d. real-valued random variables consists in performing a change of measure and proving an Edgeworth expansion under the changed measure (see e.g. [2, 52, 21]). Applying this strategy to the coefficients $\langle f, G_n v \rangle$ of products of random matrices turns out to be way more difficult. We have to overcome three main difficulties: state an Edgeworth expansion for the couple $(\langle G_n x, \log |\langle f, G_n v \rangle| \rangle$ with a target function $\varphi$ on the Markov chain $G_n x$ under the changed measure; give a precise control of the difference between $\log |\langle f, G_n v \rangle|$ and $\log |G_n v|$; establish the regularity of the eigenmeasure $\nu_s$.

For the first point, it turns out that the techniques which work for the quantity $\log |\langle f, G_n v \rangle|$ alone cannot be applied for the couple. Dealing with a couple $(\langle G_n x, \log |\langle f, G_n v \rangle| \rangle$ with a target function on $G_n x$ needs considering a new kind of smoothing inequality on a complex contour, instead of the usual Esseen one on the real line. We make use of the saddle point method to obtain precise asymptotics for the integrals of the corresponding Laplace transforms on the complex plane. For this method we refer to a recent work of the authors [58] where the Edgeworth expansion with a target function on $G_n x$ for the norm cocycle $\log |G_n v|$ has been established.
Secondly, from the previous work on limit theorems such as the strong law of large numbers, the central limit theorem and the law of iterated logarithm for the coefficients \( \langle f, G_n v \rangle \), see e.g. [42, 8, 44, 6], we know that the difference \(| \log |\langle f, G_n v \rangle| - \log |G_n v| | \) generally diverges to infinity as \( n \to \infty \). It is controlled by the corresponding norming factors in these limit theorems. However, such a control is not enough to obtain precise large deviation expansions for \( \langle f, G_n v \rangle \), nor even for a large deviation principle with explicit rate function. A precise account of the contribution of the error term is given by the following decomposition: for any \( x = \mathbb{R}v \) and \( y = \mathbb{R}f \) with \(|f| = |v| = 1\),

\[
\log |\langle f, G_n v \rangle| = \log |G_n v| + \log \delta(y, G_n x), \quad n \geq 1, \tag{2.16}
\]

where \( \delta(y, x) = \frac{|\langle f, v \rangle|}{|f|}\). The exact decomposition (2.16) allows us to deduce the precise large deviation asymptotic from the results for the couple \((G_n x, \log |G_n v|)\) with a target function on \(G_n x\) established in [57]. The idea is as follows: with \( Q_\varepsilon^s \) the changed measure defined in Section 3.1, we have

\[
\frac{e^{n\Lambda^*(q)}}{r_s(x)} \mathbb{P}(\log |\langle f, G_n v \rangle| \geq nq) = \mathbb{E}_{Q_\varepsilon^s} \left[ \frac{e^{-s(\log |G_n v| - nq)}}{r_s(G_n x)} 1_{\{\log |\langle f, G_n v \rangle| - nq \geq 0\}} \right]. \tag{2.17}
\]

We only sketch how to cope with the upper bound of the right-hand side of (2.17). Consider a partition \( I_k := (-\eta k, -\eta(k - 1)] \), \( k \geq 1 \), of the interval \((-\infty, 0]\), where \( \eta > 0 \) is a sufficiently small constant. Using (2.16) we get the upper bound

\[
1_{\{\log |\langle f, G_n v \rangle| - nq \geq 0\}} \leq \sum_{k=1}^{\infty} 1_{\{\log |G_n v| - nq - \eta(k - 1) \geq 0\}} 1_{\{\log \delta(y, G_n x) \in I_k\}},
\]

which we substitute into (2.17). Thus we are led to the estimation of the sum

\[
\sum_{k=1}^{\infty} e^{-\eta(k - 1)} \mathbb{E}_{Q_\varepsilon^s} \left[ \psi_s(\log |G_n v| - nq - \eta(k - 1)) \right] 1_{\{\log \delta(y, G_n x) \in I_k\}}, \tag{2.18}
\]

where \( \psi_s(u) = e^{-su}1_{\{u \geq 0\}}, u \in \mathbb{R} \). Let \( R_{s, it}(\varphi)(x) = \mathbb{E}_{Q_\varepsilon^s} e^{it(\sigma(g_1 x) - q)\varphi(g_1 x)} \) be the perturbed transfer operator defined for any Hölder continuous function \( \varphi \) on \( \mathbb{R}^{d-1} \), and \( R_{n, it} \) be its \( n \)-th iteration. Then, by the Fourier inversion formula, the sum in (2.18) is bounded from above by

\[
\frac{1}{2\pi} \sum_{k=1}^{\infty} e^{-\eta(k - 1)} \int_{\mathbb{R}} e^{-ity(k - 1)} R_{s, it}(r_s^{-1}\Phi_{s, k, \varepsilon_2})(x) \Psi_{s, \varepsilon_1}(t) dt, \tag{2.19}
\]

where we choose some appropriate smooth functions \( \Phi_{s, k, \varepsilon_2} \) and \( \Psi_{s, \varepsilon_1} \), for \( \varepsilon_1, \varepsilon_2 > 0 \), which dominate \( 1_{\{\log \delta(y, \cdot) \in I_k\}} \) and \( \psi_s \), respectively. Using spectral gap properties of \( R_{s, it} \), it has been established recently in [57] (see
Proposition 4.3) that, for any $k \geq 1$, the term under the sign of the infinite sum in (2.19), say $I_n(k)$, converges as $n \to \infty$ to a limit, say $I(k) = \frac{\sqrt{2\pi}}{s_\nu f_\nu(s_\eta)} e^{-s_\eta (k-1)} \nu_s(\Phi_{s,k,\varepsilon_2})$. The interchangeability of the limit as $n \to \infty$ and of the summation over $k$ in (2.19) is justified by specifying the rate in the convergence of $I_n(k)$ to $I(k)$, as argued in [57]. This implies that as $n \to \infty$ and $\varepsilon_1 \to 0$, (2.19) converges to $\sum_{k=1}^{\infty} I_k$. It remains to show that the last sum converges to $r^*_s(y)$, as $\eta \to 0$ and $\varepsilon_2 \to 0$. For this we have to make use of the zero-one law of the eigenmeasure $\nu_s$ established recently in [38]: for any $y \in (\mathbb{P}^{d-1})^*$ and any $t \in (-\infty, 0)$,

$$\nu_s \left( \{ x \in \mathbb{P}^{d-1} : \log \delta(y, x) = t \} \right) = 0 \text{ or } 1. \tag{2.20}$$

With $s = 0$ it was used in [38] to prove a local limit theorem for the coefficients $\langle f, G_nv \rangle$.

The proof of the lower large deviation asymptotic (1.4) can be carried out in a similar way as that of upper large deviation asymptotic (1.2). The novelty here consists in the use of the change of measure formula for $Q_x^s$ when $s < 0$ and of the spectral gap theory under the changed measure as stated in [58] for $s < 0$. In addition we need the Hölder regularity of the eigenmeasure $\nu_s$ for $s < 0$ sufficiently close to 0.

In some applications it is very useful to extend the large deviation results (1.2), (1.3) and (1.4) to the setting under the changed measure $Q_x^s$; see Theorems 2.6 and 2.7. To obtain these results, an important step is to establish the Hölder regularity of the eigenmeasure $\nu_s$ when $s > 0$; see Proposition 3.4. For this we adapt the arguments from [42] and [8] where (2.20) was established for $s = 0$. For $s > 0$ the arguments are much more delicate. One of the difficulties is that the sequence $(g_n)_{n \geq 1}$ becomes dependent under the changed measure $Q_x^s$. We need to extend the results in [8] to this case. Of crucial importance are the simplicity of the dominant Lyapunov exponent for $G_n$ under the changed measure recently established in [41] (see Lemma 7.6), and the key proximality property which states that $M_n m$ (here $M_n = g_1 \ldots g_n$) converges weakly to the Dirac measure $\delta_{Z_s}$, where $Z_s$ is a random variable whose law is the stationary measure $\pi_s$ of $G_n x$, for $s > 0$ (see Lemma 7.2), and $m$ is the unique rotation invariant measure on $\mathbb{P}^{d-1}$.

3. **Spectral gap properties and Hölder regularity of the stationary measure**

In this section we present some preliminaries on the spectral gap properties and state some new results on the regularity of the stationary measure $\pi_s$. The spectral gap and regularity properties will be used in the proofs of the main theorems. In particular, the regularity properties of the stationary measure $\pi_s$, will play an important role in the proof of Theorem 2.7.
As other applications of the regularity properties, we will obtain a law of large numbers and a central limit theorem for coefficients under the changed measure.

3.1. Spectral gap properties and a change of measure. Recall that the transfer operator $P_s$ and the conjugate transfer operator $P_s^*$ are defined by (2.4). Below $P_s \nu_s$ stands for the measure on $\mathbb{P}^{d-1}$ such that $P_s \nu_s(\varphi) = \nu_s(P_s \varphi)$, for any continuous functions $\varphi$ on $\mathbb{P}^{d-1}$, and $P_s^* \nu_s^*$ is defined similarly. The spectral gap properties of $P_s$ and $P_s^*$ are summarized in the following proposition which was proved in [41].

**Proposition 3.1.** Assume condition A3. Then, for any $s \in I_\mu^0$, the following assertions hold:

1. the spectral radii of the operators $P_s$ and $P_s^*$ are both equal to $\kappa(s)$ and there exist a unique, up to a scaling constant, strictly positive Hölder continuous function $r_s$ and a unique probability measure $\nu_s$ on $\mathbb{P}^{d-1}$ such that
   \[ P_s r_s = \kappa(s) r_s, \quad P_s \nu_s = \kappa(s) \nu_s; \]

2. there exist a unique strictly positive Hölder continuous function $r_s^*$ and a unique probability measure $\nu_s^*$ on $\mathbb{P}^{d-1}$ such that
   \[ P_s^* r_s^* = \kappa(s) r_s^*, \quad P_s^* \nu_s^* = \kappa(s) \nu_s^*; \]

   moreover, the function $\kappa : I_\mu^0 \mapsto \mathbb{R}$ is analytic.

The case of $s < 0$ is not covered by Proposition 3.1. We state below the corresponding result, which was proved in [38, 57].

**Proposition 3.2.** Assume conditions A2 and A3. Then there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$, the assertions (1) and (2) of Proposition 3.1 remain valid. Moreover, the function $\kappa : (-s_0, 0) \mapsto \mathbb{R}$ is analytic.

Now we give explicit formulae for the eigenfunctions $r_s$ and $r_s^*$.

**Lemma 3.3.** (1) Assume condition A3. Then, for $s \in I_\mu^0$, the eigenfunctions $r_s$ and $r_s^*$ are given as follows: for any $x \in \mathbb{P}^{d-1}$ and $y \in (\mathbb{P}^{d-1})^*$,

\[ r_s(x) = \int_{(\mathbb{P}^{d-1})^*} \delta(x,y)^s \nu_s^*(dy), \quad r_s^*(y) = \int_{\mathbb{P}^{d-1}} \delta(x,y)^s \nu_s(dx). \]  

(2) Assume conditions A2 and A3. Then there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$, the expressions in (3.1) remain valid.
The first assertion of Lemma 3.3 for \( s > 0 \) was proved in [41]. The proof of the second one for \( s < 0 \) is quite different from that in the case \( s > 0 \) and was proved in [38]. It is based on the Hölder regularity of the eigenmeasures \( \nu_s \) and \( \nu_s^* \), which is the subject of the next section.

By Propositions 3.1 and 3.2, the eigenvalue \( \kappa(s) \) and the eigenfunction \( r_s \) are both strictly positive. This allows to perform a change of measure, as shown below. Under the corresponding assumptions of Propositions 3.1 and 3.2, for any \( s \in (-s_0,0) \cup I_\mu \), the family of probability kernels \( q_s^n(x,g) = \frac{e^{s\sigma_g(q_s^n,\gamma)}}{r_n(x)}r_n(x) \), \( n \geq 1 \), satisfies the cocycle property: for any \( x \in \mathbb{P}^{d-1} \) and \( g_1, g_2 \in \Gamma_\mu \),

\[
q_s^n(x,g_1)q_m^s(g_1x,g_2) = q_{n+m}^s(x,g_2g_1). \tag{3.2}
\]

Thus the probability measures \( q_s^n(x,g_\ldots g_1)\mu(dg_1)\ldots\mu(dg_n) \) form a projective system on \( \mathbb{G}^{\mathbb{N}^*} \). By the Kolmogorov extension theorem, there exists a unique probability measure \( Q_s^\mathbb{N} \) on \( \mathbb{G}^{\mathbb{N}^*} \). The corresponding expectation is denoted by \( E_{Q_s^\mathbb{N}} \). Then the change of measure formula follows: for any measurable function \( h \) on \( (\mathbb{P}^{d-1} \times \mathbb{R})^n \),

\[
\frac{1}{\kappa^n(s)r_s^n(x)}E\left[r_s^n(G_nx)e^{s\sigma(G_n,\gamma)}h(G_1x,\sigma(G_1,x),\ldots,G_nx,\sigma(G_n,x))\right]
= E_{Q_s^\mathbb{N}}[h(G_1x,\sigma(G_1,x),\ldots,G_nx,\sigma(G_n,x))]. \tag{3.3}
\]

Under the changed measure \( Q_s^\mathbb{N} \), the process \( (G_nx)_{n \geq 0} \) defined by (2.1) still constitutes a Markov chain on \( \mathbb{P}^{d-1} \) with the transition operator given by

\[
Q_s\varphi(x) = \frac{1}{\kappa(s)r_s(x)}P_s(\varphi r_s)(x), \quad x \in \mathbb{P}^{d-1}. \tag{3.4}
\]

The Markov operator \( Q_s \) has a unique stationary probability measure \( \pi_s \) satisfying that there exists constants \( c, C > 0 \) such that for any \( \varphi \in \mathcal{B}_\gamma \),

\[
\|Q^n_s\varphi - \pi_s(\varphi)\|_\gamma \leq Ce^{-cn}\|\varphi\|_\gamma, \quad \text{where} \quad \pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}. \tag{3.5}
\]

For any \( s \in (-s_0,0) \cup I_\mu \) and \( t \in \mathbb{R} \), define a family of perturbed operators \( R_{s,t} \) as follows: for any \( \varphi \in \mathcal{B}_\gamma \),

\[
R_{s,t}\varphi(x) = E_{Q_s^\mathbb{N}}\left[e^{it\sigma(g_1,\gamma)}\varphi(g_1x)\right], \quad x \in \mathbb{P}^{d-1}. \tag{3.6}
\]

It follows from the cocycle property (3.2) that

\[
R^n_{s,t}\varphi(x) = E_{Q_s^\mathbb{N}}\left[e^{it\sigma(G_n,\gamma)}\varphi(G_nx)\right], \quad x \in \mathbb{P}^{d-1}.
\]

Under various restrictions on \( s \), it was shown in [14, 57, 58] that the operator \( R_{s,t} \) acts onto the Banach space \( \mathcal{B}_\gamma \) and has a spectral gap.
3.2. Hölder regularity of the stationary measure. In this section we present our results on the Hölder regularity of the stationary measure $\pi_s$ and of the eigenmeasure $\nu_s$. The regularity of $\pi_s$ and $\nu_s$ is central to establishing the precise large deviation asymptotics for the coefficients $\langle f, G_n v \rangle$ under the changed measure $Q^{x}_{s}$ and is also of independent interest. Below we denote $B(y, r) = \{ x \in \mathbb{P}^{d-1} : \delta(y, x) \leq r \}$ for $y \in (\mathbb{P}^{d-1})^*$ and $r \geq 0$.

**Proposition 3.4.** Assume conditions $A1$ and $A3$. Then, for any $s \in I_{\mu}^{o}$, there exists a constant $\alpha > 0$ such that
\[
\sup_{y \in (\mathbb{P}^{d-1})^*} \int_{\mathbb{P}^{d-1}} \frac{1}{\delta(y, x)\alpha} \pi_s(dx) < +\infty. \tag{3.7}
\]
In particular, for any $s \in I_{\mu}^{o}$, there exist constants $\alpha, C > 0$ such that for any $r \geq 0$,
\[
\sup_{y \in (\mathbb{P}^{d-1})^*} \pi_s(B(y, r)) \leq Cr^\alpha. \tag{3.8}
\]
Moreover, the assertions (3.7) and (3.8) remain valid when the stationary measure $\pi_s$ is replaced by the eigenmeasure $\nu_s$.

The proof of Proposition 3.4 is technically involved and is postponed to Section 7.

By (3.8) and the Frostman lemma, it follows that the Hausdorff dimension of the stationary measure $\pi_s$ is at least $\alpha$.

For $s = 0$ the Hölder regularity of the stationary measure $\nu$ ($\nu = \pi_0 = \nu_0$) is due to Guivarc’h [39]. We also refer to [8] for a detailed description of the method used in [39] and to [9, 6] for a different approach. Such regularity is of great importance in the study of products of random matrices. For example, it turns out to be crucial for establishing limit theorems for the coefficients $\langle f, G_n v \rangle$ and for the spectral radius $\rho(G_n)$ of $G_n$. However, similar result has not been established in the literature for the stationary measure $\pi_s$ when $s \in I_{\mu}^{o}$. The proof of the assertion (3.7) is based on the asymptotic properties of the components in the Cartan and Iwasawa decompositions of the reversed random matrix product $M_n = g_1 \ldots g_n$ and on the simplicity of the dominant Lyapunov exponent of $G_n$ under the changed measure $Q^{x}_{s}$: see Section 7.

When $s$ is non-positive and sufficiently close to 0, we also give the Hölder regularity of the stationary measure $\pi_s$.

**Proposition 3.5.** Assume conditions $A2$ and $A3$. Then, there exist constants $\alpha, s_0, C > 0$ such that the statements (3.7) and (3.8) hold for any $s \in (-s_0, 0]$.

Proposition 3.5 has been recently established in [38] using the Hölder regularity of the stationary measure $\nu$ and the analyticity of the eigenfunction $\kappa$. 
We will establish the following assertion, which is a stronger version of Proposition 3.4.

**Proposition 3.6.** Assume conditions **A1** and **A3**. Let \( s \in I_{\mu}^c \). Then, for any \( \varepsilon > 0 \), there exist constants \( c := c(s) > 0 \) and \( n_0 := n_0(s) \geq 1 \) such that for all \( n \geq k \geq n_0, x \in \mathbb{P}^{d-1} \) and \( y \in (\mathbb{P}^{d-1})^* \),

\[
Q_s^x \left( \delta(y, G_n x) \leq e^{-\varepsilon k} \right) \leq e^{-ck}.
\]

Similarly, the following result is a stronger version of Proposition 3.5.

**Proposition 3.7.** Assume conditions **A2** and **A3**. Let \( s \in (-s_0, s_0) \), where \( s_0 > 0 \) is small enough. Then, for any \( \varepsilon > 0 \), there exist constants \( c := c(s) > 0 \) and \( n_0 := n_0(s) \geq 1 \) such that for all \( n \geq k \geq n_0, x \in \mathbb{P}^{d-1} \) and \( y \in (\mathbb{P}^{d-1})^* \),

\[
Q_s^x \left( \delta(y, G_n x) \leq e^{-\varepsilon k} \right) \leq e^{-ck}.
\]

It turns out that Propositions 3.6 and 3.7 play an important role for establishing the Bahadur-Rao-Petrov type lower tail large deviations for the coefficients \( \langle f, G_n v \rangle \) under the changed measure \( Q_s^x \), see Theorem 2.7. Moreover, they are very useful to obtain the strong law of large numbers (SLLN) and the central limit theorem (CLT) for the coefficients \( \langle f, G_n v \rangle \) under the changed measure \( Q_s^x \), see the next section.

### 3.3. Applications to SLLN and CLT for the coefficients

In this section we formulate the SLLN and the CLT for the coefficients \( \langle f, G_n v \rangle \) under the changed measure \( Q_s^x \). These assertions are not used in the proofs of our large deviation results, but are of independent interest. They are deduced from the SLLN and the CLT for the norm cocycle \( \log |G_n v| \) using the Hölder regularity of stationary measure \( \pi_s \) stated in Propositions 3.4 and 3.5.

When \( s \in I_{\mu} \), the SLLN for \( \log |G_n v| \) was established in [41]: under conditions **A2** and **A3**, for any \( x = \mathbb{R}v \in \mathbb{P}^{d-1} \),

\[
\lim_{n \to \infty} \frac{1}{n} \log |G_n v| = \Lambda'(s), \quad Q_s^x \text{-a.s.}, \tag{3.9}
\]

where \( \Lambda'(s) = \frac{\kappa'(s)}{\sigma(s)} \) with the function \( \kappa \) defined in Proposition 3.1. The CLT for \( \log |G_n v| \) under the changed measure \( Q_s^x \) was proved in [14]: for any \( s \in I_{\mu} \) and \( t \in \mathbb{R} \), it holds uniformly in \( x = \mathbb{R}v \in \mathbb{P}^{d-1} \) with \( |v| = 1 \) that

\[
\lim_{n \to \infty} Q_s^x \left( \frac{\log |G_n v| - n\Lambda'(s)}{\sigma \sqrt{n}} \leq t \right) = \Phi(t), \tag{3.10}
\]

where \( \Phi \) is the standard normal distribution function on \( \mathbb{R} \).

When \( s \in (-s_0, 0) \) with small enough \( s_0 > 0 \), the SLLN and the CLT for \( \log |G_n v| \) under the measure \( Q_s^x \) have been recently established in [58].
We now give the SLLN and the CLT for the coefficients $\langle f, G_n v \rangle$ under the measure $Q^x_s$.

**Proposition 3.8.**

1. Assume conditions A1 and A3. Then, for any $s \in I_\mu$, uniformly in $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\lim_{n \to \infty} \frac{1}{n} \log |\langle f, G_n v \rangle| = \Lambda'(s), \quad Q^x_s\text{-a.s..}$$

Moreover, for any $s \in I_\mu$ and $t \in \mathbb{R}$, we have, uniformly in $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\lim_{n \to \infty} Q^x_s \left( \frac{\log |\langle f, G_n v \rangle| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq t \right) = \Phi(t).$$

2. Assume conditions A2 and A3. Then, there exists $s_0 > 0$ such that for any $s \in (-s_0, 0)$, the assertions (3.11) and (3.12) hold.

The proof of Proposition 3.8 relies on Propositions 3.4 and 3.5 and is postponed to Section 7.

### 4. Auxiliary results

In this section we state some preliminary results about the Taylor’s expansion of $\Lambda^*(q + l)$, a smoothing inequality, and some asymptotics of the perturbed operator $R_{s,it}$, which will be used to establish Bahadur-Rao-Petrov type large deviations.

The following lemma is proved in [57] and gives Taylor’s expansion of $\Lambda^*(q + l)$ with respect to the perturbation $l$. Recall that under conditions A1, A2 and A3, the moment generating function $\Lambda = \log \kappa$ is strictly convex and analytic on $(-s_0, 0) \cup I_\mu$; see e.g. [41, 14, 57]. Set $\gamma_{s,k} = \Lambda^{(k)}(s)$, $k \geq 1$. In particular, $\gamma_{s,2} = \Lambda''(s) = \sigma_s^2$. Under the changed measure $Q^x_s$, define the Cramér series $\zeta_s$ (see Petrov [53]) by

$$\zeta_s(t) = \frac{\gamma_{s,3}}{3!} + \frac{\gamma_{s,4}\gamma_{s,2} - 3\gamma_{s,3}^2}{24\gamma_{s,2}^3} \frac{t^2}{\gamma_{s,2}^2} + \frac{\gamma_{s,5}\gamma_{s,2} - 10\gamma_{s,4}\gamma_{s,3}\gamma_{s,2} + 15\gamma_{s,3}^2}{120\gamma_{s,2}^9/2} \gamma_{s,2}^2 \frac{t^2}{\gamma_{s,2}^2} + \ldots,$$

which converges for small enough $|t|$.

**Lemma 4.1.** Assume either conditions A1 and A3 when $s \in I_\mu^c$, or conditions A2 and A3 when $s \in (-s_0, 0)$ with small enough $s_0 > 0$. Let $q = \Lambda'(s)$. Then, there exists a constant $\eta > 0$ such that for any $|l| \leq \eta$,

$$\Lambda^*(q + l) = \Lambda^*(q) + sl + h_s(l),$$

where $h_s$ is linked to the Cramér series $\zeta_s$ by the identity

$$h_s(l) = \frac{l^2}{2\sigma_s^2} - \frac{l^3}{3\sigma_s^3} \zeta_s\left(\frac{l}{\sigma_s}\right).$$
In the sequel let us fix a non-negative density function $\rho$ on $\mathbb{R}$ with $\int_{\mathbb{R}} \rho(u) du = 1$, whose Fourier transform $\hat{\rho}$ is supported on $[-1, 1]$. Moreover, there exists a constant $C > 0$ such that $\rho(u) \leq \frac{C}{1 + u^2}$ for all $u \in \mathbb{R}$. For any $\varepsilon > 0$, define the scaled density function $\rho_{\varepsilon}$ by $\rho_{\varepsilon}(u) = \frac{1}{\varepsilon} \rho\left(\frac{u}{\varepsilon}\right)$, $u \in \mathbb{R}$, whose Fourier transform $\hat{\rho}_{\varepsilon}$ is supported on $[-\varepsilon^{-1}, \varepsilon^{-1}]$. For any non-negative integrable function $\psi$ on $\mathbb{R}$, we introduce two modified functions related to $\psi$ as follows: for any $u \in \mathbb{R}$, set $\mathcal{B}_\varepsilon(u) = \{ u' \in \mathbb{R} : |u' - u| \leq \varepsilon \}$ and

$$
\psi_{\varepsilon}^+(u) = \sup_{u' \in \mathcal{B}_\varepsilon(u)} \psi(u') \quad \text{and} \quad \psi_{\varepsilon}^-(u) = \inf_{u' \in \mathcal{B}_\varepsilon(u)} \psi(u').
$$

(4.1)

The following smoothing inequality gives two-sided bounds of $\psi$.

**Lemma 4.2.** Suppose that $\psi$ is a non-negative integrable function on $\mathbb{R}$ and that $\psi_{\varepsilon}^+$ and $\psi_{\varepsilon}^-$ are measurable for any $\varepsilon > 0$. Then, for $0 < \varepsilon < 1$, there exists a positive constant $C_\rho(\varepsilon)$ with $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for any $u \in \mathbb{R}$,

$$
\psi_{\varepsilon}^+(u) \rho_{\varepsilon^2}(u) - \int_{|w| \geq \varepsilon} \psi_{\varepsilon}^-(u - w) \rho_{\varepsilon^2}(w) dw \leq \psi(u) \leq (1 + C_\rho(\varepsilon)) \psi_{\varepsilon}^+ \rho_{\varepsilon^2}(u).
$$

The proof of the above lemma is similar to that of Lemma 5.2 in [33], and will not be detailed here.

The next proposition gives precise asymptotics of the perturbed operator $R_{s,it}$, which will be used to establish Bahadur-Rao-Petrov type large deviations for the coefficients $\langle f, G_n v \rangle$. Its proof is based on the spectral gap properties of the perturbed operator $R_{s,it}$.

**Proposition 4.3.** Suppose that $\psi : \mathbb{R} \to \mathbb{C}$ is bounded measurable function with compact support, and that $\psi$ is differentiable in a small neighborhood of $0$ in $\mathbb{R}$.

1. **Assume conditions A1 and A3.** Then, for any compact set $K_\mu \subset I^\mu_\gamma$, there exist constants $\delta = \delta(K) > 0$, $c = c(K) > 0$, $C = C(K) > 0$ such that for all $x \in \mathbb{P}^{d-1}$, $s \in K_\mu$, $|l| = O\left(\frac{1}{\sqrt{n}}\right)$, $\varphi \in \mathcal{B}_\gamma$ and $n \geq 1$,

$$
\left| \sigma_s \sqrt{n} e^{-\frac{sl^2}{2n}} \int_{\mathbb{R}} e^{-it\ln R_{s,it}^n(\varphi)}(x) \psi(t) dt - \sqrt{2\pi} \psi(0) \pi_s(\varphi) \right| \leq \frac{C}{\sqrt{n}} \|\varphi\|_\gamma + C \frac{\|\varphi\|_\gamma}{n} \sup_{|t| \leq \delta} \left( |\psi(t)| + |\psi'(t)| \right) + C e^{-cn} \|\varphi\|_\gamma \int_{\mathbb{R}} |\psi(t)| dt.
$$

(4.2)

2. **Assume conditions A2 and A3.** Then, there exist constants $s_0 > 0$, $\delta = \delta(s_0) > 0$, $c = c(s_0) > 0$, $C = C(s_0) > 0$ such that for any compact set $K_\mu \subset (-s_0, 0)$, the inequality (4.2) holds uniformly in $x \in \mathbb{P}^{d-1}$, $s \in K_\mu$, $|l| = O\left(\frac{1}{\sqrt{n}}\right)$, $\varphi \in \mathcal{B}_\gamma$ and $n \geq 1$.

The assertions (1) and (2) of Proposition 4.3 were respectively established in [57] and [58]. The perturbation $l$ as well as the explicit rate of convergence...
in Proposition 4.3 are important in the sequel. They play a crucial role to establish the Bahadur-Rao type large deviations for the coefficients \( \langle f, G_n v \rangle \) in Theorems 2.1, 2.2, 2.3 and 2.4.

5. Proof of upper tail large deviations for coefficients

The aim of this section is to establish Theorems 2.1 and 2.2. Since Theorems 2.1 is a direct consequence of Theorem 2.2, it suffices to establish Theorem 2.2. We also establish a large deviation result under the changed measure.

5.1. Zero-one laws for the stationary measure. We first present some zero-one laws for the stationary measure which will be used in the proof of Theorem 2.2.

Lemma 5.1. Assume condition A3. Then, for any \( s \in I_\mu^0 \) and any proper projective subspace \( Y \subseteq \mathbb{P}^{d-1} \), it holds that \( \pi_s(Y) = 0 \).

Lemma 5.2. Assume conditions A2 and A3. Then, there exists a constant \( s_0 > 0 \) such that for any \( s \in (-s_0, 0) \) and any proper projective subspace \( Y \subseteq \mathbb{P}^{d-1} \), it holds that \( \pi_s(Y) = 0 \).

Lemma 5.1 was established by Guivarc’h and Le Page [41] using the strategy of Furstenberg [26]. Lemma 5.2 was proved in [38] based on the Hölder regularity of the stationary measure \( \nu \). Note that the results in [41] and [38] are stated for the eigenmeasure \( \nu_s \), but they also hold for the stationary measure \( \pi_s \) since the measures \( \pi_s \) and \( \nu_s \) are equivalent.

We shall also need the following zero-one law of the stationary measure \( \pi_s \) recently established in [38].

Lemma 5.3. Assume condition A3. Then, for any \( s \in I_\mu^0 \) and any algebraic subset \( Y \) of \( \mathbb{P}^{d-1} \), it holds that either \( \pi_s(Y) = 0 \) or \( \pi_s(Y) = 1 \). In particular, for any \( y \in (\mathbb{P}^{d-1})^* \) and any \( t \in (-\infty, 0) \),

\[
\pi_s \left( \left\{ x \in \mathbb{P}^{d-1} : \log \delta(y, x) = t \right\} \right) = 0 \text{ or } 1. 
\] (5.1)

Lemma 5.4. Assume conditions A2 and A3. Then, there exists a constant \( s_0 > 0 \) such that for any \( s \in (-s_0, 0) \) and any algebraic subset \( Y \) of \( \mathbb{P}^{d-1} \), it holds that either \( \pi_s(Y) = 0 \) or \( \pi_s(Y) = 1 \). In particular, for any \( y \in (\mathbb{P}^{d-1})^* \) and any \( t \in (-\infty, 0) \),

\[
\pi_s \left( \left\{ x \in \mathbb{P}^{d-1} : \log \delta(y, x) = t \right\} \right) = 0 \text{ or } 1. 
\] (5.2)

The assertions (5.1) and (5.2) are sufficient for us to establish the Bahadur-Rao type large deviation asymptotics for the coefficient \( \langle f, G_n v \rangle \) (cf. Theorems 2.1 and 2.3). However, in order to obtain the Petrov type extensions (cf. Theorems 2.2 and 2.4), we need the following slightly stronger statements than (5.1) and (5.2).
Lemma 5.5. Assume condition A3. Then, for any \( y \in (\mathbb{P}^{d-1})^* \) and any \( t \in (−∞, 0) \), if
\[
\pi_s \left( \left\{ x \in \mathbb{P}^{d-1} : \log \delta(y, x) = t \right\} \right) = 0 \tag{5.3}
\]
holds for some \( s \in I^o_\mu \), then (5.3) holds for all \( s \in I^o_\mu \).

Lemma 5.6. Assume conditions A2 and A3. Then, for any \( y \in (\mathbb{P}^{d-1})^* \) and any \( t \in (−∞, 0) \), if
\[
\pi_s \left( \left\{ x \in \mathbb{P}^{d-1} : \log \delta(y, x) = t \right\} \right) = 0 \tag{5.4}
\]
holds for some \( s \in (−s_0, 0) \) with \( s_0 > 0 \) small enough, then (5.4) holds for all \( s \in (−s_0, 0) \).

Proof of Lemmas 5.5 and 5.6. We first prove Lemma 5.5. For any \( y \in (\mathbb{P}^{d-1})^* \) and any \( t \in (−∞, 0) \), denote \( Y_{y,t} = \{ x \in \mathbb{P}^{d-1} : \log \delta(y, x) = t \} \). Suppose that there exist \( s_1, s_2 \in I^o_\mu \) with \( s_1 \neq s_2 \) such that \( \pi_{s_1}(Y_{y,t}) = 0 \) and \( \pi_{s_2}(Y_{y,t}) \neq 0 \). Then by Lemma 5.3 we have \( \pi_{s_2}(Y_{y,t}) = 1 \). Since \( Y_{y,t} \) is a closed set in \( \mathbb{P}^{d-1} \), by the definition of the support of the measure we get that \( \text{supp} \pi_{s_2} \subset Y_{y,t} \). Since it is proved in [41] that \( \text{supp} \pi_{s_1} = \text{supp} \pi_{s_2} \) (both coincide with \( \text{supp} \nu \) defined by (2.3)), it follows that \( \text{supp} \pi_{s_1} \subset Y_{y,t} \) and hence \( \pi_{s_1}(Y_{y,t}) = 1 \). This contradicts to the assumption \( \pi_{s_1}(Y_{y,t}) = 0 \). Therefore, if (5.3) holds for some \( s \in I^o_\mu \), then it holds for all \( s \in I^o_\mu \).

The proof of Lemma 5.6 is similar by using the fact that \( \text{supp} \pi_s = \text{supp} \nu \) for any \( s \in (−s_0, 0) \), which is proved in [38].

5.2. Proof of Theorem 2.2. Now we are equipped to establish Theorem 2.2. This theorem is a direct consequence of the following more general result. Recall that \( s \) and \( q \) are related by \( q = \Lambda'(s) \).

Theorem 5.7. Assume conditions A1 and A3. Let \( K_\mu \subset I^o_\mu \) be any compact set in \( \mathbb{R} \). Then, we have, as \( n \to \infty \), uniformly in \( s \in K_\mu \), \( f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1 \),
\[
\mathbb{P} \left( \log |⟨f, G_n v⟩| \geq nq \right) = \frac{r_s(x)r_s^*(y)}{q_s} \exp \left( -n\Lambda^*(q) \right) \frac{1}{\sigma_s \sqrt{2\pi n}} \left[ 1 + o(1) \right]. \tag{5.5}
\]

More generally, for any measurable function \( \psi \) on \( \mathbb{R} \) such that \( u \mapsto e^{-s'\psi(u)} \) is directly Riemann integrable for any \( s' \in K'_\mu := \{ s' \in \mathbb{R} : |s' - s| < \epsilon, s \in K_\mu \} \) with \( \epsilon > 0 \) small enough, we have, as \( n \to \infty \), uniformly in \( s \in K_\mu \), \( f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1 \),
\[
\mathbb{E} \left[ \varphi(G_n v) \psi \left( \log |⟨f, G_n v⟩| - nq \right) \right] = \frac{r_s(x)}{q_s} \exp \left( -n\Lambda^*(q) \right) \frac{1}{\sigma_s \sqrt{2\pi n}} \left[ \int_{\mathbb{P}^{d-1}} \varphi(x) \delta(y, x)^s \nu_s(dx) \int_{\mathbb{R}} e^{-su} \psi(u) du + o(1) \right]. \tag{5.6}
\]
Recall that and \(\lfloor u \rfloor\) and using the change of measure formula (5.6), we denote the target functions \(\varphi\) and \(\psi\) are non-negative. For brevity, denote \(\psi_s(u) = e^{-su}\psi(u)\) for \(s \in I^*_\mu\), and
\[
\psi^+_{s,\varepsilon}(u) = \sup_{u' \in \mathbb{B}_s(u)} \psi_s(u'), \quad \psi^-_{s,\varepsilon}(u) = \inf_{u' \in \mathbb{B}_s(u)} \psi_s(u').
\]
Introduce the following condition: for any \(\varepsilon > 0\), the functions \(u \mapsto \psi^+_{s,\varepsilon}(u)\) and \(u \mapsto \psi^-_{s,\varepsilon}(u)\) are measurable and
\[
\lim_{\varepsilon \to 0^+} \int_R \psi^+_{s,\varepsilon}(u)du = \lim_{\varepsilon \to 0^+} \int_R \psi^-_{s,\varepsilon}(u)du = \int_R e^{-su}\psi(u)du < +\infty. \tag{5.7}
\]
To prove (5.6), we can assume additionally that the function \(\psi\) satisfies the condition (5.7). In fact, using the approximation techniques similar to that in [57], we can prove that if (5.6) holds under (5.7), then it also holds under the directly Riemann integrability condition introduced in the theorem. So in the following we assume (5.7).

Note that we have \(f \in (\mathbb{R}^d)^s\) and \(v \in \mathbb{R}^d\) with \(|f| = |v| = 1\), and \(y = \mathbb{R}f \in (\mathbb{P}^{d-1})^s\) and \(x = \mathbb{R}v \in \mathbb{P}^{d-1}\). Hence \(\log |\langle f, G_n v \rangle| = \log |G_n v| + \log \delta(y, G_n x)\), and that \(\log |\langle f, G_n v \rangle| = -\infty\) if and only if \(\log \delta(y, G_n x) = -\infty\). Taking into account that \(\psi(-\infty) = 0\), we can replace the logarithm of the coefficient \(\log |\langle f, G_n v \rangle|\) by the sum \(\log |G_n v| + \log \delta(y, G_n x)\) as follows:
\[
A := \sigma_s \sqrt{2\pi n} \frac{e^{n\Lambda^*(q)}}{r_s(x)} \mathbb{E}\left[\varphi(G_n x)\psi(\log |\langle f, G_n v \rangle| - nq)\right]
\]
\[
= \sigma_s \sqrt{2\pi n} \frac{e^{n\Lambda^*(q)}}{r_s(x)} \mathbb{E}\left[\varphi(G_n x)\psi(\log |G_n x| + \log \delta(y, G_n x) - nq)\right].
\]
For short, we denote for any \(y = \mathbb{R}f \in (\mathbb{P}^{d-1})^s\) and \(x = \mathbb{R}v \in \mathbb{P}^{d-1}\),
\[
T^y_n := \log |G_n v| - nq, \quad Y^{x,y}_n := \log \delta(y, G_n x).
\]
Recall that \(q = \Lambda'(s)\). Taking into account that \(e^{n\Lambda^*(q)} = e^{nsg_nK^-n(s)}\) and using the change of measure formula (3.3), we get
\[
A = \sigma_s \sqrt{2\pi n} \mathbb{E}_{q_s}(\varphi r_s^{-1}(G_n x)e^{-sT^n_y}\psi(T^n_y + Y^{x,y}_n)). \tag{5.8}
\]
For any fixed small constant \(0 < \eta < 1\), denote \(I_k := (-\eta k, -\eta(k - 1)]\), \(k \geq 1\). Let \(M_n := \lfloor C_1 \log n \rfloor\), where \(C_1 > 0\) is a sufficiently large constant and \(\lfloor a \rfloor\) denotes the integer part of \(a \in \mathbb{R}\). Then from (5.8) we have the following decomposition:
\[
A = A_1 + A_2, \tag{5.9}
\]
where
\[ A_1 := \sigma_s \sqrt{2\pi n} \mathbb{E}_{Q_n^+} \left[ (\varphi r_s^{-1})(G_n x)e^{-sT_n^w} \psi(T_n^w + Y_n^{x,y})1_{\{Y_n^{x,y} \leq -\eta M_n\}} \right], \]
\[ A_2 := \sigma_s \sqrt{2\pi n} \sum_{k=1}^{M_n} \mathbb{E}_{Q_n^+} \left[ (\varphi r_s^{-1})(G_n x)e^{-sT_n^w} \psi(T_n^w + Y_n^{x,y})1_{\{Y_n^{x,y} \in I_k\}} \right]. \]

We now give a bound for the first term \( A_1 \). Since the function \( u \mapsto e^{-s u} \psi(u) \) is directly Riemann integrable on \( \mathbb{R} \) for any \( s' \in K_\epsilon := \{ s' \in \mathbb{R} : |s' - s| < \epsilon, s \in K \} \) with \( \epsilon > 0 \) small enough, one can verify that the function \( u \mapsto e^{-s u} \psi(u) \) is bounded on \( \mathbb{R} \), uniformly in \( s \in K_\mu \), and hence there exists a constant \( C > 0 \) such that for all \( s \in K_\mu \),
\[ e^{-sT_n^w} \psi(T_n^w + Y_n^{x,y})1_{\{Y_n^{x,y} \leq -\eta M_n\}} \leq C e^{sY_n^{x,y}}1_{\{Y_n^{x,y} \leq -\eta M_n\}} \leq C e^{-s\eta M_n}. \]

Since the function \( \varphi r_s^{-1} \) is uniformly bounded on \( \mathbb{R}^{d-1} \), uniformly in \( s \in K_\mu \), we get the following upper bound for \( A_1 \): as \( n \to \infty \), uniformly in \( s \in K_\mu \), \( f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1 \),
\[ A_1 \leq C \sqrt{n} e^{-s\eta M_n} \leq C n^{-\frac{s\eta C - 1}{2}} \to 0. \] (5.10)

The remaining part of the proof is devoted to establishing upper and lower bounds for the second term \( A_2 \) defined by (5.9).

**Upper bound for \( A_2 \).** On the event \( \{Y_n^{x,y} \in I_k\} \), we have \( Y_n^{x,y} + \eta(k - 1) \in (0, \eta] \). With the notation \( \psi^+_\eta(u) = \sup_{u' \in B_\eta(u)} \psi(u') \), we get
\[ \psi(T_n^w - n l + Y_n^{x,y}) \leq \psi^+_\eta(T_n^w - n l - \eta(k - 1)). \]

It follows that
\[ A_2 \leq \sigma_s \sqrt{2\pi n} \sum_{k=1}^{M_n} \mathbb{E}_{Q_n^+} \left[ (\varphi r_s^{-1})(G_n x)e^{-sT_n^w} \psi^+_\eta(T_n^w - \eta(k - 1))1_{\{Y_n^{x,y} \in I_k\}} \right]. \]

We choose a small constant \( \epsilon > \eta \) and set
\[ \Psi_{s,\eta}(u) = e^{-s u} \psi^+_\eta(u), \quad \Psi^+_{s,\eta,\epsilon}(u) = \sup_{u' \in B_\epsilon(u)} \Psi_{s,\eta}(u'), \quad u \in \mathbb{R}. \] (5.11)

Since the function \( \Psi^+_{s,\eta,\epsilon} \) is non-negative and integrable on the real line, using Lemma 4.2, we get
\[ A_2 \leq (1 + C_P(\epsilon)) \sigma_s \sqrt{2\pi n} \sum_{k=1}^{\infty} \mathbb{1}_{\{k \leq M_n\}} e^{-s\eta(k-1)} \times \mathbb{E}_{Q_n^+} \left[ (\varphi r_s^{-1})(G_n x)1_{\{Y_n^{x,y} \in I_k\}} (\Psi^+_{s,\eta,\epsilon} \rho_{s,\epsilon})(T_n^w - \eta(k - 1)) \right], \] (5.12)
where \( C_P(\epsilon) > 0 \) is a constant converging to 0 as \( \epsilon \to 0 \). For fixed small constant \( \epsilon_1 > 0 \), introduce the density function \( \tilde{\rho}_{\epsilon_1} \) defined as follows:
\[ \tilde{\rho}_{\epsilon_1}(u) = \frac{1}{\epsilon_1}(1 - \frac{|u|}{\epsilon_1}) \text{ for } u \in [-\epsilon_1, \epsilon_1], \text{ and } \tilde{\rho}_{\epsilon_1}(u) = 0 \text{ otherwise}. \] For any
Let \( \hat{\chi}(u) := \langle u \in \mathcal{I}_k \rangle \) and \( \chi_{k,\varepsilon}^\pm (u) = \sup_{u' \in \mathcal{E}^\varepsilon} \chi_k(u') \), one can verify that the following smoothing inequality holds:

\[
\chi_k(u) \leq (\chi_{k,\varepsilon}^+ * \bar{\rho}_\varepsilon)(u) \leq \chi_{k,2\varepsilon}^+(u), \quad u \in \mathbb{R}.
\]

(5.13)

For short, we denote \( \tilde{\chi}(u) := (\chi_{k,\varepsilon}^+ * \bar{\rho}_\varepsilon)(u) \), \( u \in \mathbb{R} \), and

\[
\varphi^y_{s,k,\varepsilon}(x) = (\varphi_{rs}^{-1})(x) \tilde{\chi}(\log \delta(y,x)), \quad x \in \mathbb{P}^{d-1}.
\]

(5.14)

In view of (5.12), using the smoothing inequality (5.13) leads to

\[
A_2 \leq (1 + C_\rho(\varepsilon)) \sigma_s \sqrt{2\pi n} \sum_{k=1}^{\infty} \mathbb{I}_{\{k \leq M_n\}} e^{-s\eta(k-1)}
\]

\[
\times \mathbb{E}_{\tilde{Q}_x^\varepsilon} \left[ \varphi^y_{s,k,\varepsilon}(G_n(x))(\Psi_{s,\eta,\varepsilon}^+ \ast \rho_{\varepsilon^2})(T_n^\varepsilon - \eta(k-1)) \right]
\]

\[
=: A_2^+.
\]

(5.15)

Let \( \tilde{\Psi}_{s,\eta,\varepsilon}^+ \) be the Fourier transform of \( \Psi_{s,\eta,\varepsilon}^+ \). By the Fourier inversion formula,

\[
\Psi_{s,\eta,\varepsilon}^+ \ast \rho_{\varepsilon^2}(u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itu} \tilde{\Psi}_{s,\eta,\varepsilon}^+(t) \tilde{\rho}_{\varepsilon^2}(t) dt, \quad u \in \mathbb{R}.
\]

Substituting \( y = T_n^\varepsilon - nl - \eta(k-1) \), taking expectation with respect to \( \mathbb{E}_{\tilde{Q}_x^\varepsilon} \), and using Fubini’s theorem, we obtain

\[
\mathbb{E}_{\tilde{Q}_x^\varepsilon} \left[ \varphi^y_{s,k,\varepsilon}(G_n(x))(\Psi_{s,\eta,\varepsilon}^+ \ast \rho_{\varepsilon^2})(T_n^\varepsilon - \eta(k-1)) \right]
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\eta(k-1)} R^\eta_{s,\varepsilon^2}(\varphi^y_{s,k,\varepsilon})(x) \tilde{\Psi}_{s,\eta,\varepsilon}^+(t) \tilde{\rho}_{\varepsilon^2}(t) dt,
\]

(5.16)

where

\[
R^\eta_{s,\varepsilon^2}(\varphi^y_{s,k,\varepsilon})(x) = \mathbb{E}_{\tilde{Q}_x^\varepsilon} \left[ e^{iT_n^\varepsilon} \varphi^y_{s,k,\varepsilon}(G_n(x)) \right].
\]

Substituting (5.16) into (5.15), we get

\[
A_2^+ = (1 + C_\rho(\varepsilon)) \sigma_s \sqrt{\frac{n}{2\pi}} \sum_{k=1}^{\infty} \mathbb{I}_{\{k \leq M_n\}} e^{-s\eta(k-1)}
\]

\[
\times \int_{\mathbb{R}} e^{-it\eta(k-1)} R^\eta_{s,\varepsilon^2}(\varphi^y_{s,k,\varepsilon})(x) \tilde{\Psi}_{s,\eta,\varepsilon}^+(t) \tilde{\rho}_{\varepsilon^2}(t) dt.
\]

(5.17)

We shall use Proposition 4.3 to handle the integral in (5.17) for each fixed \( k \geq 1 \). Let us first check the conditions stated in Proposition 4.3. Since the function \( \tilde{\chi} \) is Hölder continuous on the real line, one can check that \( \varphi^y_{s,k,\varepsilon} \) defined by (5.14) is also Hölder continuous on the projective space \( \mathbb{P}^{d-1} \). Using the fact that the function \( u \mapsto e^{-s'\omega}(u) \) is directly Riemann integrable on \( \mathbb{R} \) for any \( s' \in K \) := \{ \{s' \in \mathbb{R} : |s' - s| < \epsilon, s \in K \} \) with \( \epsilon > 0 \) small enough, one can also verify that the function \( \tilde{\Psi}_{s,\eta,\varepsilon}^+ \tilde{\rho}_{\varepsilon^2} \) is compactly
supported in $\mathbb{R}$. Moreover, for any $s \in K_\mu$, the function $\hat{\Psi}_{s,\eta,\varepsilon}^+ \hat{\rho}_{\varepsilon^2}$ is differentiable in a small neighborhood of 0 on the real line. Hence, applying Proposition 4.3 with $\varphi = \varphi_{s,k,\varepsilon^1}$, $\psi = \hat{\Psi}_{s,\eta,\varepsilon}^+ \hat{\rho}_{\varepsilon^2}$ and with $l = l_{n,k} := \frac{n(k-1)}{n}$, we obtain that for sufficiently large $n$, uniformly in $1 \leq k \leq M_n$, $s \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$I_k^+ := \left| \sigma \sqrt{n} e^{n \eta s(l_{n,k})} \int e^{-i \eta s l_{n,k} R_{s,\eta,\varepsilon}^n} (\varphi_{s,k,\varepsilon^1}^y(x) \hat{\Psi}_{s,\eta,\varepsilon}^+ \hat{\rho}_{\varepsilon^2}(t) - B^+(k)) dt \right| \leq C \frac{\log n}{\sqrt{n}} \| \varphi_{s,k,\varepsilon^1}^y \|_\gamma,$$

where

$$B^+(k) := \sqrt{2\pi} \hat{\Psi}_{s,\eta,\varepsilon}^+ (0) \hat{\rho}_{\varepsilon^2}(0) \pi_s (\varphi_{s,k,\varepsilon^1}^y).$$

Taking into account that $1 \leq k \leq M_n = \lfloor C_1 \log n \rfloor$, by Lemma 4.1, we get that $|e^{n \eta s(l_{n,k})} - 1| \leq \frac{C}{\sqrt{n}} \log n$, uniformly in $1 \leq k \leq M_n$ and $s \in K_\mu$. Using (5.18) and the fact that $B^+(k)$ is dominated by $\| \varphi_{s,k,\varepsilon^1}^y \|_\gamma$, we can replace $e^{n \eta s(l_{n,k})}$ by 1, yielding that uniformly in $s \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\left| \sigma \sqrt{n} \int e^{-i \eta s l_{n,k} R_{s,\eta,\varepsilon}^n} (\varphi_{s,k,\varepsilon^1}^y(x) \hat{\Psi}_{s,\eta,\varepsilon}^+ \hat{\rho}_{\varepsilon^2}(t) dt - B^+(k)) \right| \leq I_k^+ e^{-n \eta s(l_{n,k})} + |e^{-n \eta s(l_{n,k})} - 1| B^+(k) \leq C \frac{\log n}{\sqrt{n}} \| \varphi_{s,k,\varepsilon^1}^y \|_\gamma.$$

By calculations, one can get that $\gamma$-Hölder norm $\| \varphi_{s,k,\varepsilon^1}^y \|_\gamma$ is bounded by $\frac{\log n}{\sqrt{n}}$. Taking sufficiently small $\gamma > 0$, we obtain that the series

$$\sum_{k=1}^{\infty} e^{-sn(k-1)} \| \varphi_{s,k,\varepsilon^1}^y \|_\gamma$$

is convergent, and moreover, its limit is 0 as $n \to \infty$. Consequently, we are allowed to interchange the limit as $n \to \infty$ and the infinite summation over $k$ in (5.17). Therefore, from (5.17), (5.18) and (5.19) we deduce that, uniformly in $s \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\limsup_{n \to \infty} A_2^+ \leq (1 + C \rho(\varepsilon)) \hat{\Psi}_{s,\eta,\varepsilon}^+ (0) \hat{\rho}_{\varepsilon^2}(0) \sum_{k=1}^{\infty} e^{-sn(k-1)} \pi_s (\varphi_{s,k,\varepsilon^1}^y).$$

In order to calculate the sum in (5.20), we shall make use of the zero-one law of the stationary measure $\pi_s$. Note that $\hat{\rho}_{\varepsilon^2}(0) = 1$. Using (5.13), we
have $\bar{\chi}_k \leq \chi^{+}_{k, 2\varepsilon_1}$. Therefore, we obtain

$$\limsup_{n \to \infty} A_2^+ \leq (1 + C_p(\varepsilon)) \hat{\Psi}^+_s, \eta, \varepsilon(0) \sum_{k=1}^{\infty} e^{-s\eta(k-1)} \pi_s (\hat{\varphi}^y_{s, k, \varepsilon_1}),$$  \hspace{1cm} (5.21)$$

where

$$\hat{\varphi}^y_{s, k, \varepsilon_1}(x) = (\varphi_{r_s^{-1}})(x) 1_{\{ \log \delta(y, \cdot) \in I_k \}}(x) + (\varphi_{r_s^{-1}})(x) 1_{\{ \log \delta(y, \cdot) \in I_{k, \varepsilon_1} \}}(x),$$  \hspace{1cm} (5.22)$$

and $I_{k, \varepsilon_1} = (-\eta k - 2\varepsilon_1, -\eta k] \cup (-\eta (k - 1), -\eta (k - 1) + 2\varepsilon_1]$. For the first term on the right hand-side of (5.22), we claim that uniformly in $s \in K_\mu$,

$$\lim_{\eta \to 0} \sum_{k=1}^{\infty} e^{-s\eta(k-1)} \pi_s \left( (\varphi_{r_s^{-1}}) 1_{\{ \log \delta(y, \cdot) \in I_k \}} \right) = \int_{\mathbb{P}^{d-1}} \delta(y, x)^s \varphi(x) r_s^{-1}(x) \pi_s(dx).$$  \hspace{1cm} (5.23)$$

Indeed, recalling that $I_k = (-\eta k, -\eta (k - 1)]$, we have

$$\sum_{k=1}^{\infty} e^{-s\eta(k-1)} \pi_s \left( (\varphi_{r_s^{-1}}) 1_{\{ \log \delta(y, \cdot) \in I_k \}} \right) \geq \sum_{k=1}^{\infty} \pi_s \left( (\varphi_{r_s^{-1}}) \delta(y, \cdot)^s 1_{\{ \log \delta(y, \cdot) \in I_k \}} \right) = \int_{\mathbb{P}^{d-1}} \delta(y, x)^s \varphi(x) r_s^{-1}(x) \pi_s(dx).$$

On the other hand, we have, as $\eta \to 0$, uniformly in $s \in K_\mu$,

$$\sum_{k=1}^{\infty} e^{-s\eta(k-1)} \pi_s \left( (\varphi_{r_s^{-1}}) 1_{\{ \log \delta(y, \cdot) \in I_k \}} \right) \leq e^s \sum_{k=1}^{\infty} \pi_s \left( (\varphi_{r_s^{-1}}) \delta(y, \cdot)^s 1_{\{ \log \delta(y, \cdot) \in I_k \}} \right) \to \int_{\mathbb{P}^{d-1}} \delta(y, x)^s \varphi(x) r_s^{-1}(x) \pi_s(dx).$$

Hence (5.23) holds.

To deal with the second term on the right-hand side of (5.22), we need to apply Lemma 5.1 and the zero-one law of the stationary measure $\pi_s$ stated in Lemma 5.3. Specifically, taking into account that the function $\varphi_{r_s^{-1}}$ is uniformly bounded on the projective space $\mathbb{P}^{d-1}$, using the Lebesgue dominated convergence theorem we get that there exists a constant $C_1 > 0$. 

such that for all $y \in (\mathbb{P}^{d-1})^*$,
\[
\lim_{\epsilon_1 \to 0} \sum_{k=1}^{\infty} e^{-s \eta (k-1) \pi_s} \left( \varphi r_s^{-1}(\mathbb{P}) \mathbf{1}_{\{ \log \delta(y,x) \in I_k, \epsilon_1 \}} \right) 
\leq C_1 \lim_{\epsilon_1 \to 0} \sum_{k=1}^{\infty} e^{-s \eta (k-1) \pi_s} \left( x \in \mathbb{P}^{d-1} : \log \delta(y,x) \in I_k, \epsilon_1 \right) 
= C_1 \sum_{k=1}^{\infty} e^{-s \eta (k-1) \pi_s} \left( x \in \mathbb{P}^{d-1} : \log \delta(y,x) = -\eta k \right) 
+ C_1 \sum_{k=1}^{\infty} e^{-s \eta (k-1) \pi_s} \left( x \in \mathbb{P}^{d-1} : \log \delta(y,x) = -\eta (k-1) \right) 
= 2C_1 \sum_{k=1}^{\infty} e^{-s \eta (k-1) \pi_s} \left( x \in \mathbb{P}^{d-1} : \log \delta(y,x) = -\eta k \right),
\]  
(5.24)
where the last equality holds due to Lemma 5.1. Now we are going to apply Lemma 5.5 to prove that there exists a constant $0 < \eta < 1$ such that
\[
\sum_{k=1}^{\infty} e^{-s \eta (k-1) \pi_s} \left( x \in \mathbb{P}^{d-1} : \log \delta(y,x) = -\eta k \right) = 0.
\]  
(5.25)
Indeed, by Lemma 5.5, we get that, for any $y \in (\mathbb{P}^{d-1})^*$ and any set $Y_{y,t} = \{ x \in \mathbb{P}^{d-1} : \log \delta(y,x) = t \}$ with $t \in (-\infty, 0)$, it holds that either $\pi_s(Y_{y,t}) = 0$ or $\pi_s(Y_{y,t}) = 1$ for all $s \in K_\mu$. If $\pi_s(Y_{y,t}) = 0$ for all $y \in (\mathbb{P}^{d-1})^*$ and $t \in (-\infty, 0)$, then clearly (5.25) holds. If $\pi_s(Y_{y_0,t_0}) = 1$ for some $y_0 \in (\mathbb{P}^{d-1})^*$ and $t_0 \in (-\infty, 0)$, then we can always choose $0 < \eta < 1$ in such a way that $-\eta k \neq t_0$ for all $k \geq 1$, so that we also obtain that (5.25) holds for all $s \in K_\mu$. Hence, in view of (5.21), combining (5.23) and (5.25) we obtain that uniformly in $s \in K_\mu$,
\[
\lim_{\eta \to 0} \lim_{\epsilon_1 \to 0} \sum_{k=1}^{\infty} e^{-s \eta (k-1) \pi_s} \left( \bar{z}^y_{s,k,\epsilon_1} \right) = \int_{\mathbb{P}^{d-1}} \delta(y,x)^{\psi} \varphi(x) r_s^{-1}(x) \pi_s(dx).
\]  
(5.26)
Since the target function $\psi$ satisfies the condition (5.7), from (5.11) we get
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} e^{-s u} \psi(u) du = 0.
\]  
(5.27)
Consequently, recalling that $A_2 \leq A_2^+$ and $C_\rho(\epsilon) \to 0$ as $\epsilon \to 0$, we obtain the desired upper bound for $A_2$: uniformly in $s \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,
\[
\lim_{\epsilon \to 0} \lim_{\eta \to 0} \lim_{\epsilon_1 \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^d} e^{-s u} \psi(u) du \int_{\mathbb{P}^{d-1}} \delta(y,x)^{\psi} \varphi(x) r_s^{-1}(x) \pi_s(dx).
\]  
(5.28)
Lower bound for $A_2$. We are going to establish the lower bound for $A_2$ given by (5.9). Recall that $Y_n^{x,y} = \log \delta(y, G_n x)$. On the event $\{Y_n^{x,y} \in I_k\}$ we have $Y_n^{x,y} + \eta (k-1) \in (0, \eta]$. With the notation $\psi_\eta^- (u) = \inf_{u' \in \mathbb{B}_\eta (u)} \psi (u)$, we get

$$\psi(T_n^u + Y_n^{x,y}) \geq \psi_\eta^- (T_n^u - \eta k).$$

In view of (5.9), using Fatou’s lemma, it follows that

$$\liminf_{n \to \infty} A_2 \geq \sum_{k=1}^\infty \liminf_{n \to \infty} \sigma_s \sqrt{2 \pi n} \mathbb{1}_{\{k \leq M_n\}}$$

$$\times \mathbb{E}_{\mathbb{Q}_n^\mathbb{2}} \left[ (\varphi r_s^{-1})(G_n x) e^{-s T_n^u} \psi^-_\eta (T_n^u - \eta k) \mathbb{1}_{\{Y_n^{x,y} \in I_k\}} \right].$$

We choose a small constant $\varepsilon > \eta$ and set

$$\Psi_{s,\eta,\varepsilon} (u) = e^{-s u } \psi^-_\eta (u), \quad \Psi_{s,\eta,\varepsilon}^- (u) = \inf_{u' \in \mathbb{B}_\varepsilon (u)} \Psi_{s,\eta,\varepsilon} (u'), \quad u \in \mathbb{R}. \quad (5.30)$$

Noting that the function $\Psi_{s,\eta,\varepsilon}^\prime$ is non-negative and integrable on the real line, by Lemma 4.2, from (5.29) we get the following lower bound:

$$\liminf_{n \to \infty} A_2 \geq \sum_{k=1}^\infty \liminf_{n \to \infty} A_3 - \sum_{k=1}^\infty \limsup_{n \to \infty} A_4, \quad (5.31)$$

where, with the notation $a_{n,k} = \sigma_s \sqrt{2 \pi n} e^{-s \eta k} \mathbb{1}_{\{k \leq M_n\}}$,

$$A_3 = a_{n,k} \mathbb{E}_{\mathbb{Q}_n^\mathbb{2}} \left[ (\varphi r_s^{-1})(G_n x) \mathbb{1}_{\{Y_n^{x,y} \in I_k\}} \left( \Psi_{s,\eta,\varepsilon}^- \ast \rho_{\varepsilon^2} \right)(T_n^u - \eta k) \right],$$

$$A_4 = a_{n,k} \int_{|u| \geq \varepsilon} \mathbb{E}_{\mathbb{Q}_n^\mathbb{2}} \left[ (\varphi r_s^{-1})(G_n x) \mathbb{1}_{\{Y_n^{x,y} \in I_k\}} \Psi_{s,\eta,\varepsilon}^- (T_n^u - \eta k - u) \rho_{\varepsilon^2} (u) \right] du.$$
Denote by $\hat{\psi}_{s,\eta,\varepsilon}$ the Fourier transform of $\Psi_{s,\eta,\varepsilon}^-$.
Applying the Fourier inversion formula to $\Psi_{s,\eta,\varepsilon}^- \ast \rho_{\varepsilon^2}$, and using Fubini’s theorem, we get

$$
\mathbb{E}_{\mathbb{Q}_x} \left[ \phi_{s,k,\varepsilon_1}^y (G_n x)(\Psi_{s,\eta,\varepsilon}^- \ast \rho_{\varepsilon^2})(T_n - \eta k) \right] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itn^k} R_{s,\varepsilon}(\phi_{s,k,\varepsilon_1}^y)(t) \hat{\psi}_{s,\eta,\varepsilon}^-(t) \hat{\rho}_{\varepsilon^2}(t) dt,
$$

(5.35)

where

$$
R_{s,\varepsilon}(\phi_{s,k,\varepsilon_1}^y)(x) = \mathbb{E}_{\mathbb{Q}_x} \left[ e^{itT_n^s \phi_{s,k,\varepsilon_1}^y} (G_n x) \right], \quad x \in \mathbb{P}^{d-1}.
$$

Substituting (5.35) into (5.34), we obtain

$$
A_n \geq \frac{a_{n,k}}{2\pi} \int_{\mathbb{R}} e^{-itn^k} R_{s,\varepsilon}(\phi_{s,k,\varepsilon_1}^y)(t) \hat{\psi}_{s,\eta,\varepsilon}^-(t) \hat{\rho}_{\varepsilon^2}(t) dt.
$$

(5.36)

We shall use Proposition 4.3 to give a precise asymptotic for the above integral. Let us first verify the conditions of Proposition 4.3. Since the function $\hat{\chi}_k$ is Hölder continuous for any fixed $k \geq 1$, one can check that $\phi_{s,k,\varepsilon_1}^y$ is Hölder continuous on the projective space $\mathbb{P}^{d-1}$. Since the function $u \mapsto e^{-s' u} \psi(u)$ is directly Riemann integrable on $\mathbb{R}$ for any $s' \in K_\varepsilon := \{ s' \in \mathbb{R} : |s' - s| < \varepsilon, s \in K \}$ with $\varepsilon > 0$ small enough, one can also verify that the function $\hat{\psi}_{s,\eta,\varepsilon}^-(t) \hat{\rho}_{\varepsilon^2}$ has compact support in $\mathbb{R}$, and that $\hat{\psi}_{s,\eta,\varepsilon}^- \hat{\rho}_{\varepsilon^2}$ is differentiable in a small neighborhood of 0 on the real line, for all $s \in K_\mu$.

Thus, using Proposition 4.3 with $\varphi = \phi_{s,k,\varepsilon_1}^y$, $\psi = \hat{\psi}_{s,\eta,\varepsilon}^- \hat{\rho}_{\varepsilon^2}$ and $l = l_{n,k} := \frac{\hat{\psi}_{s,\eta,\varepsilon}}{n}$, we obtain that for sufficiently large $n$, there exists a constant $C > 0$ such that for all $1 \leq k \leq M_n = [C_1 \log n]$, $s \in K_\mu$, $f \in (\mathbb{R}^d)^\ast$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$
I_k := |\sigma_n \sqrt{\pi e^{nh_s(l'_{n,k})}} \int e^{-itn^k} R_{s,\varepsilon}(\phi_{s,k,\varepsilon_1}^y)(t) \hat{\psi}_{s,\eta,\varepsilon}^-(t) \hat{\rho}_{\varepsilon^2}(t) dt - B^-(k)| \leq C \frac{\|\phi_{s,k,\varepsilon_1}^y\|_\gamma}{\sqrt{n}},
$$

(5.37)

where

$$
B^-(k) := \sqrt{2\pi} \hat{\psi}_{s,\eta,\varepsilon}^-(0) \hat{\rho}_{\varepsilon^2}(0) \pi_s (\phi_{s,k,\varepsilon_1}^y).
$$

Since $1 \leq k \leq M_n = [C_1 \log n]$, using Lemma 4.1 there exists a constant $C > 0$ such that for all $1 \leq k \leq M_n$, $s \in K_\mu$ and $n \geq 1$, it holds that $|e^{-nh_s(l'_{n,k})} - 1| \leq \frac{C_1 \log n}{\sqrt{n}}$. In a similar way as in the proof of (5.19), we can replace $e^{-nh_s(l'_{n,k})}$ by 1 to obtain that, uniformly in $s \in K_\mu$, $f \in (\mathbb{R}^d)^\ast$ and
$v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$\left| \sigma_s \sqrt{n} \int_{\mathbb{R}} e^{-intl^v_{n,k} R^n_{s,t}(\phi_{s,k,\varepsilon})(x)} \hat{\psi}_{s,n,\varepsilon}(t) \hat{\rho}_{\varepsilon z}(t) dt - B^{-}(k) \right| \leq \int_k e^{-nhs(t^v_{n,k})} + |e^{-nhs(t^v_{n,k})} - 1| B^{-}(k)

\leq C \frac{\log n}{\sqrt{n}} \|\phi_{s,k,\varepsilon}^y\|_\gamma + C \frac{\log n}{\sqrt{n}} \|\phi_{s,k,\varepsilon}^y\|_\gamma

\leq C \frac{\log n}{\sqrt{n}} \|\phi_{s,k,\varepsilon}^y\|_\gamma.

Since the $\gamma$-Hölder norm $\|\phi_{s,k,\varepsilon}^y\|_\gamma$ is bounded by $\frac{\rho_{\gamma k}}{(e^{\varepsilon 1 - 1})^\gamma}$, taking sufficiently small $\gamma > 0$, we obtain that the series $\frac{\log n}{\sqrt{n}} \sum_{k=1}^{\infty} e^{-\gamma k} \|\phi_{s,k,\varepsilon}^y\|_\gamma$ converges to 0 as $n \to \infty$. As a result, by virtue of (5.37), we obtain that, uniformly in $s \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\sum_{k=1}^{\infty} \liminf_{n \to \infty} A_3 \geq \hat{\Psi}_{s,n,\varepsilon}(0) \hat{\rho}_{\varepsilon z}(0) \sum_{k=1}^{\infty} e^{-\gamma k} \pi_s(\phi_{s,k,\varepsilon}^y).$$

Note that $\hat{\rho}_{\varepsilon z}(0) = 1$. Using (5.32), we have that $\tilde{\chi}_k \geq \chi_k \varepsilon_1$. Consequently, we obtain the lower bound for the first term on the right hand side of (5.31): uniformly in $s \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\sum_{k=1}^{\infty} \liminf_{n \to \infty} A_3 \geq \hat{\Psi}_{s,n,\varepsilon}(0) \sum_{k=1}^{\infty} e^{-\gamma k} \pi_s(\tilde{\phi}_{s,k,\varepsilon}^y). \quad (5.38)$$

where

$$\tilde{\phi}_{s,k,\varepsilon}^y(x) = (\varphi r_s^{-1})(x) 1_{\{\log \delta(y) \in I_k\}}(x) - (\varphi r_s^{-1})(x) 1_{\{\log \delta(y) \in \bar{I}_{k,\varepsilon}\}}(x), \quad (5.39)$$

and $\bar{I}_{k,\varepsilon} = (-\eta k, -\eta k + 2\varepsilon_1) \cup (-\eta (k - 1) + 2\varepsilon_1, -\eta (k - 1)]$. For the first term on the right hand-side of (5.39), since $I_k = (-\eta k, -\eta (k - 1)]$, in a similar way as in the proof of (5.23), it holds uniformly in $s \in K_\mu$ that

$$\lim_{\eta \to 0} \sum_{k=1}^{\infty} e^{-\gamma k} \pi_s((\varphi r_s^{-1})(x) 1_{\{\log \delta(y) \in I_k\}}) = \int_{\mathbb{R}^{d-1}} \delta(y,x)^s \varphi(x) r_s^{-1}(x) \pi_s(dx). \quad (5.40)$$

To handle the second term on the right-hand side of (5.39), we make use of Lemma 5.1 and the zero-one law of the stationary measure $\pi_s$ shown in Lemma 5.5. Specifically, similarly to the proof of (5.24), since the function $\varphi r_s^{-1}$ is bounded on $\mathbb{R}^{d-1}$, uniformly in $s \in K_\mu$, using the Lebesgue dominated convergence theorem we get that there exists a constant $C_1 > 0$ such
that for all \( y \in (\mathbb{R}^{d-1})^* \) and \( s \in K_\mu \),
\[
\lim_{\epsilon_1 \to 0} \sum_{k=1}^{\infty} e^{-snk} \pi_s \left( (\varphi^{-1}_s) \mathbb{1}_{\{\log \delta(y) \in \tilde{I}_k, \epsilon_1\}} \right) \leq C_1 \lim_{\epsilon_1 \to 0} \sum_{k=1}^{\infty} e^{-snk} \pi_s \left( x \in \mathbb{R}^{d-1} : \log \delta(y, x) \in \tilde{I}_k, \epsilon_1 \right)
\]
\[
= C_1 \sum_{k=1}^{\infty} e^{-snk} \pi_s \left( x \in \mathbb{R}^{d-1} : \log \delta(y, x) = -\eta k \right) + C_1 \sum_{k=1}^{\infty} e^{-snk} \pi_s \left( x \in \mathbb{R}^{d-1} : \log \delta(y, x) = -\eta(k - 1) \right)
\]
\[
= 2C_1 \sum_{k=1}^{\infty} e^{-snk} \pi_s \left( x \in \mathbb{R}^{d-1} : \log \delta(y, x) = -\eta k \right),
\]
(5.41)

where in the last equality we used Lemma 5.1. In the same way as in the proof of (5.25), applying Lemma 5.5 we can obtain that there exists a constant \( 0 < \eta < 1 \) such that for all \( s \in K_\mu \),
\[
\sum_{k=1}^{\infty} e^{-snk} \pi_s \left( x \in \mathbb{R}^{d-1} : \log \delta(y, x) = -\eta k \right) = 0.
\]
(5.42)

Since the target function \( \psi \) satisfies the condition (5.7), from (5.11) we get that uniformly in \( s \in K_\mu \),
\[
\lim_{\epsilon \to 0} \lim_{\eta \to 0} \tilde{\Psi}^-_{s, \eta, \epsilon}(0) = \int_{\mathbb{R}} e^{-su} \psi(u) du.
\]
(5.43)

Consequently, in view of (5.38), combining (5.40), (5.41), (5.42) and (5.43), we get the desired lower bound for \( A_3 \): uniformly in \( s \in K_\mu \),
\[
\lim_{\epsilon \to 0} \lim_{\eta \to 0} \sum_{k=1}^{\infty} \lim_{\eta \to \infty} \inf A_3 \geq \int_{\mathbb{R}^{d-1}} e^{-su} \psi(u) du \int_{\mathbb{R}^{d-1}} \delta(y, x)^s \varphi(x) r^{-1}_s(x) \pi_s(dx).
\]
(5.44)

Now we proceed to establish an upper bound for the term \( A_4 \) in (5.31). Note that \( \tilde{\Psi}^-_{s, \eta, \epsilon} \leq \Psi_{s, \eta} \), where \( \Psi_{s, \eta}(u) = e^{-sn} \psi^\pm_{s, \eta}(u) \), \( u \in \mathbb{R} \). Then it follows from Lemma 4.2 that \( \tilde{\Psi}^-_{s, \eta, \epsilon} \leq (1 + C_\rho(\epsilon)) \tilde{\Psi}^+_{s, \eta, \epsilon} \rho_{\epsilon z} \), where \( \tilde{\Psi}^+_{s, \eta, \epsilon}(u) = \sup_{u' \in B_{\epsilon}(u)} \Psi_{s, \eta}(u') \), \( u \in \mathbb{R} \). Moreover, using (5.13), we get \( \mathbb{1}_{\{y \in \tilde{I}_k\}} \leq \tilde{x}_k(u) = (x^+_{k, \epsilon_1} \ast \rho_{\epsilon z})(u) \). Consequently, similarly to the proof of (5.17), we can get the upper bound for \( A_4 \): uniformly in \( s \in K_\mu \),
\[
A_4 \leq (1 + C_\rho(\epsilon)) \frac{\alpha_{n,k}}{2\pi}
\]
\[
\times \int_{|u| \geq \epsilon} \int_{\mathbb{R}} e^{-i(tNK + u)} R^n_{s,t}(\varphi^y_{s,k,\epsilon_1})(x) \tilde{\Psi}^+_{s,\eta,\epsilon}(t) \rho_{\epsilon z}(t) dt \rho_{\epsilon z}(u) du.
\]
(5.45)
In order to handle the above integral, we first use the Lebesgue dominated convergence theorem to interchange the limit \( n \to \infty \) and the integral \( \int_{|u| \geq \varepsilon} \), and then we apply Proposition 4.3. An important issue is to find a dominating function, which can be done as follows. We split the integral \( \int_{|u| \geq \varepsilon} \) on the right hand side of (5.45) into two parts: \( \int_{\varepsilon \leq |u| \leq \sqrt{n}} \) and \( \int_{|u| > \sqrt{n}} \). For the first part, by elementary calculations it holds that \( e^{-nh_s(n^{\frac{k}{2}} + \frac{s^2}{n}}) \to 1 \), uniformly in \( s \in K_{\mu} \), \( 1 \leq k \leq M_n \) and \( |u| \leq \sqrt{n} \) as \( n \to \infty \). Hence, using Proposition 4.3, the function on the right hand side of (5.45) under the integral \( \int_{\varepsilon \leq |u| \leq \sqrt{n}} \) is dominated by \( C \rho_{\varepsilon^2} \), which is integrable on \( \mathbb{R} \). For the second part \( \int_{|u| > \sqrt{n}} \), since the density function \( \rho \) has polynomial decay, i.e. \( \rho_{\varepsilon^2}(u) \leq \frac{C}{1 + |u|^2} \), \( |u| > \sqrt{n} \), we get that \( \sqrt{n} \rho_{\varepsilon^2}(u) \leq \frac{C}{1 + |u|^2} \), which is clearly integrable on \( \mathbb{R} \). Therefore, we can pass the limit as \( n \to \infty \) under the integral \( \int_{|u| \geq \varepsilon} \) and then we use Proposition 4.3 to obtain the desired upper bound for \( A_4 \): uniformly in \( s \in K_{\mu} \),

\[
\sum_{k=1}^{\infty} \limsup_{n \to \infty} A_4 \leq (1 + C_{\rho}(\varepsilon)) \sum_{k=1}^{\infty} e^{-s\eta^k} \pi_s(\varphi_{s,k,\varepsilon}) \\
\quad \times \widehat{\Psi}_{s,\eta,\varepsilon}^+(0) \rho_{\varepsilon^2}(0) \int_{|u| \geq \varepsilon} \rho_{\varepsilon^2}(u) du.
\]

In the same way as in the proof of (5.26), by Lemmas 5.1 and 5.5, we can get that uniformly in \( s \in K_{\mu} \),

\[
\lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{k=1}^{\infty} e^{-s\eta^k} \pi_s(\varphi_{s,k,\varepsilon}) = \int_{\mathbb{R}^{d-1}} \delta(y,x)^{s} \varphi(x) r_s^{-1}(x) \pi_s(dx).
\]

Using (5.27) and noting that \( \widehat{\rho}_{\varepsilon^2}(0) = 1 \), it follows that uniformly in \( s \in K_{\mu} \),

\[
\lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{k=1}^{\infty} \limsup_{n \to \infty} A_4 \leq (1 + C_{\rho}(\varepsilon)) \int_{\mathbb{R}^{d-1}} \delta(y,x)^{s} \varphi(x) r_s^{-1}(x) \pi_s(dx) \\
\quad \times \int_{\mathbb{R}} e^{-su}\psi(u) du \int_{|u| \geq \varepsilon} \rho_{\varepsilon^2}(u) du.
\]

Since \( C_{\rho}(\varepsilon) \to 0 \) and \( \int_{|u| \geq \varepsilon} \rho_{\varepsilon^2}(u) du \to 0 \) as \( \varepsilon \to 0 \), this implies

\[
\lim_{\varepsilon \to 0} \lim_{\eta \to 0} \limsup_{n \to \infty} A_4 = 0.
\]

Combining this with (5.31) and (5.44), we get the desired lower bound for \( A_2 \): uniformly in \( s \in K_{\mu} \), \( f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1 \),

\[
\lim_{\varepsilon \to 0} \lim_{\eta \to 0} \limsup_{n \to \infty} A_2 \geq \int_{\mathbb{R}} e^{-su}\psi(u) du \int_{\mathbb{R}^{d-1}} \delta(y,x)^{s} \varphi(x) r_s^{-1}(x) \pi_s(dx).
\]
This, together with (5.9), (5.10) and (5.28), proves the desired asymptotic (5.6). This concludes the proof of Theorem 5.7 as well as Theorem 2.2. □

5.3. Proof of Theorem 2.6.

Proof of Theorem 2.6. It suffices to prove (2.14) since (2.13) follows from (2.14) by taking \( \varphi = 1 \) and \( \psi(u) = 1_{\{u \geq 0\}} \). Using the change of measure formula (3.3) twice, we get

\[
\mathbb{E}_{\mathbb{Q}_t^x} \left[ \varphi(G_n x) \psi \left( \log |\langle f, G_n v \rangle | - n q_t \right) \right] \\
= \frac{1}{\kappa^n(s) r_s(x)} \mathbb{E} \left[ (\varphi_{r_s}) (G_n x) e^{s \sigma(G_n x)} \psi \left( \log |\langle f, G_n v \rangle | - n q_t \right) \right] \\
= \frac{\kappa^n(t) r_t(x)}{\kappa^n(s) r_s(x)} \mathbb{E}_{\mathbb{Q}_t} \left[ (\varphi_{r_s r_t^{-1}}) (G_n x) e^{-(t-s) \sigma(G_n x)} \psi \left( \log |\langle f, G_n v \rangle | - n q_t \right) \right] \\
= \frac{\kappa^n(t) r_t(x)}{\kappa^n(s) r_s(x)} e^{-(t-s) n q_t} \times \\
\mathbb{E}_{\mathbb{Q}_t} \left[ (\varphi_{r_s r_t^{-1}}) (G_n x) e^{-(t-s) \sigma(G_n x) - n q_t} \psi \left( \log |\langle f, G_n v \rangle | - n q_t \right) \right], \quad (5.46)
\]

where \( \mathbb{Q}_t^x \) is the changed measure defined in the same way as \( \mathbb{Q}_s^x \) with \( s \) replaced by \( t \). Following the proof of Theorem 2.2, one can verify that, as \( n \to \infty \), uniformly in \( t \in K_\mu \), \( f \in (\mathbb{R}^d)^s \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1 \),

\[
\mathbb{E}_{\mathbb{Q}_t} \left[ (\varphi_{r_s r_t^{-1}}) (G_n x) e^{-(t-s) \sigma(G_n x) - n q_t} \psi \left( \log |\langle f, G_n v \rangle | - n q_t \right) \right] \\
= \frac{1}{\sigma_t \sqrt{2 \pi n}} \left[ \int_{\mathbb{R}^{d-1}} \varphi(x) \delta(y, x)^t \frac{r_s(x)}{r_t(x)} \pi_t(dx) \int_{\mathbb{R}} e^{-(t-s) u} \psi(u) du + o(1) \right].
\]

The result follows by taking into account that \( \Lambda^*(q_s) = sq_s - \Lambda(s) \), \( \Lambda^*(q_t) = tq_t - \Lambda(t) \), \( \Lambda(s) = \log \kappa(s) \) and \( \Lambda(t) = \log \kappa(t) \). □

6. Proof of lower tail large deviations for coefficients

The goal of this section is to establish Theorems 2.3 and 2.4 on Bahadur-Rao-Petrov type lower tail large deviations. In contrast to the proof of Theorems 2.1 and 2.2, it turns out that the proof of Theorems 2.3 and 2.4 is more delicate.

It suffices to prove Theorem 2.4 since Theorem 2.3 is a direct consequence of Theorem 2.4 by taking \( l = 0 \), \( \varphi = 1 \) and \( \psi(u) = 1_{\{u \geq 0\}} \), \( u \in \mathbb{R} \).

6.1. Proof of Theorem 2.4. We shall need the Hölder regularity of the stationary measure \( \pi_s \) (for sufficiently small \( s \)) recently established in [38].

Lemma 6.1. Assume conditions \( A2 \) and \( A3 \). Then, for any \( \varepsilon > 0 \), there exist constants \( s_0 > 0 \), \( k_0 \in \mathbb{N} \) and \( c, C > 0 \) such that for all \( s \in (-s_0, s_0) \),
\( n \geq k \geq k_0, \ y \in (\mathbb{P}^{d-1})^* \) and \( x \in \mathbb{P}^{d-1} \),
\[
\mathbb{Q}_x \left( \log \delta(y, G_n x) \leq -\varepsilon k \right) \leq C e^{-ck}.
\] (6.1)

Note that (6.1) is stronger than the following assertion of the Hölder regularity of the stationary measure \( \pi_s \):
there exist constants \( s_0, \alpha > 0 \) such that
\[
\sup_{s \in (-s_0, s_0)} \sup_{y \in (\mathbb{P}^{d-1})^*} \int_{\mathbb{P}^{d-1}} \frac{1}{\delta(y, x)^\alpha} \pi_s(dx) < +\infty.
\]

As an application of Lemma 6.1, we show the following result about the high-order negative moment of the \( \delta(y, G_n x) \) under the changed measure \( \mathbb{Q}_x \), which will play important role in the proof of Theorem 2.4.

**Lemma 6.2.** Assume conditions \( A_2 \) and \( A_3 \). Let \( p > 0 \) be any fixed constant. Then, there exists a constant \( s_0 > 0 \) such that
\[
\sup_{n \geq 1} \sup_{s \in (-s_0, s_0)} \sup_{y \in (\mathbb{P}^{d-1})^*} \sup_{x \in \mathbb{P}^{d-1}} \mathbb{E}_{\mathbb{Q}_x} \left( \frac{1}{\delta(y, G_n x)^p |s|} \right) < +\infty.
\]

**Proof.** By Lemma 6.1, for any \( \varepsilon > 0 \), there exist constants \( s_0 > 0, k_0 \in \mathbb{N} \) and \( c, C > 0 \) such that for all \( s \in (-s_0, s_0), n \geq k \geq k_0 \) and \( y \in (\mathbb{P}^{d-1})^*, x \in \mathbb{P}^{d-1} \),
\[
\mathbb{Q}_x \left( \delta(y, G_n x) \leq e^{-\varepsilon k} \right) \leq C e^{-ck}.
\] (6.2)

For any \( y \in (\mathbb{P}^{d-1})^* \) and \( k \geq k_0 \), we denote
\[
B_{n,k} = \left\{ x \in \mathbb{P}^{d-1} : e^{-\varepsilon (k+1)} \leq \delta(y, G_n x) \leq e^{-\varepsilon k} \right\}.
\]

By (6.2), it follows that there exist constants \( c, C > 0 \) such that for all \( s \in (-s_0, s_0) \),
\[
\mathbb{E}_{\mathbb{Q}_x} \left( \frac{1}{\delta(y, G_n x)^p |s|} \right) = \mathbb{E}_{\mathbb{Q}_x} \left( \frac{1}{\delta(y, G_n x)^p |s|} 1_{\{\delta(y, G_n x) > e^{-ck_0}\}} \right) + \sum_{k=k_0}^{\infty} \mathbb{E}_{\mathbb{Q}_x} \left( \frac{1}{\delta(y, G_n x)^p |s|} 1_{B_{n,k}} \right)
\]
\[
\leq e^{\varepsilon k |p|} + C \sum_{k=k_0}^{\infty} e^{\varepsilon (k+1)p |s|} e^{-ck},
\]

which is finite by taking \( s_0 > 0 \) small enough. This proves Lemma 6.2. \( \Box \)

Now we are in a position to establish Theorem 2.4. In the same spirit as in Theorem 5.7, we are able to prove the following result which is stronger than Theorem 2.4.
Theorem 6.3. Assume conditions A2 and A3. Then, there exists a constant $s_0 > 0$ such that for any compact set $K_\mu \subset (-s_0,0)$, we have, as $n \to \infty$, uniformly in $s \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\mathbb{P}\left(\log |\langle f, G_n v \rangle| \leq nq\right) = \frac{r_s(x) r_s(y) \exp\left(-n\Lambda^*(q)\right)}{\varrho_s} \left[1 + o(1)\right].$$

More generally, for any $\varphi \in \mathcal{B}_\gamma$ and any measurable function $\psi$ on $\mathbb{R}$ such that $u \mapsto e^{-s u} \psi(u)$ is directly Riemann integrable for all $s' \in K_\mu^* := \{s' \in \mathbb{R} : |s' - s| < \epsilon, s \in K_\mu\}$ with $\epsilon > 0$ small enough, we have, as $n \to \infty$, uniformly in $s \in K_\mu$, $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$,

$$\mathbb{E}\left[\varphi(G_n x) \psi(\log |\langle f, G_n v \rangle| - nq)\right] = \frac{r_s(x) \exp\left(-n\Lambda^*(q)\right)}{\varrho_s} \left[\int_{\mathbb{R}^d-1} \varphi(x) \mu_s(x) \nu_s(dx) \int_{\mathbb{R}} e^{-su} \psi(u) du + o(1)\right].$$

Proof of Theorems 2.4 and 6.3. We only need to prove Theorem 6.3 since Theorem 2.4 is a direct consequence of Theorem 6.3.

It suffices to prove the second assertion of Theorem 6.3, since the first one follows from the second by choosing $\varphi = 1$ and $\psi(u) = 1_{\{u \leq 0\}}$, $u \in \mathbb{R}$. As in the proof of Theorem 2.2, we assume that the target functions $\varphi$ and $\psi$ are non-negative, and that the function $\psi$ satisfies the condition (5.7).

Since $f \in (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ with $|f| = |v| = 1$, and $y = \mathbb{R} f$ and $x \in \mathbb{R} v$, we have $\log |\langle f, G_n v \rangle| = \log |G_n v| + \log \delta(y, G_n x)$. Hence we can replace the logarithm of the coefficient $\log |\langle f, G_n v \rangle|$ by the sum $\log |G_n v| + \log \delta(y, G_n x)$ as follows:

$$J := \sigma_s \sqrt{2\pi n} e^{n\Lambda^*(q)} r_s(x) \mathbb{E}\left[\varphi(G_n x) \psi(\log |\langle f, G_n v \rangle| - nq)\right] = \sigma_s \sqrt{2\pi n} e^{n\Lambda^*(q)} r_s(x) \mathbb{E}\left[\varphi(G_n x) \psi(\log |G_n x| + \log \delta(y, G_n x) - nq)\right].$$

As in the proof of Theorem 2.2, we denote for any $y \in \mathbb{R} f \in (\mathbb{P}^{d-1})^*$ and $x \in \mathbb{R} v \in \mathbb{P}^{d-1}$,

$$T_n^y := \log |G_n v| - nq, \quad Y_n^{x,y} := \log \delta(y, G_n x).$$

Taking into account that $q = \Lambda'(s)$ and $e^{n\Lambda^*(q)} = e^{nsq} \kappa^{-n}(s)$, and using the change of measure formula (3.3), we get

$$J = \sigma_s \sqrt{2\pi n} \mathbb{E}_{Q^x} \left[(\varphi r_s^{-1})(G_n x) e^{-sT_n^y} \psi(T_n^y + Y_n^{x,y})\right]. \quad (6.3)$$

For any fixed small constant $0 < \eta < 1$, denote $I_k := (-\eta k, -\eta(k - 1)]$, $k \geq 1$. Let $M_n := |C_1 \log n|$, where $C_1 > 0$ is a sufficiently large constant.
and \([a]\) denotes the integer part of \(a \in \mathbb{R}\). Then from (6.3) we have the following decomposition:

\[
J = J_1 + J_2,
\]

where

\[
J_1 := \sigma_s \sqrt{2\pi n} \mathbb{E}_{Q_s^x} \left[ (\phi r_s^{-1})(G_n x) e^{-sT_n^u} \psi(T_n^u + Y_n^{x,y}) 1_{\{Y_n^{x,y} \leq -\eta M_n\}} \right],
\]

\[
J_2 := \sigma_s \sqrt{2\pi n} \sum_{k=1}^{M_n} \mathbb{E}_{Q_s^x} \left[ (\phi r_s^{-1})(G_n x) e^{-sT_n^u} \psi(T_n^u + Y_n^{x,y}) 1_{\{Y_n^{x,y} \in I_k\}} \right].
\]

**Upper bound of** \(J_1\). Since the function \(u \mapsto e^{-s'T'\psi(u)}\) is directly Riemann integrable on \(\mathbb{R}\) for some \(s' \in (0, s)\), one can verify that the function \(u \mapsto e^{-s\psi(u)}\) is bounded on \(\mathbb{R}\) and hence there exists a constant \(C > 0\) such that for all \(s \in (-s_0, 0]\),

\[
e^{-sT_n^u} \psi(T_n^u + Y_n^{x,y}) \leq Ce^{sY_n^{x,y}}.
\]

Hence, by the Hölder inequality, Lemma 6.1 and Lemma 6.2, we obtain that as \(n \to \infty\), uniformly in \(s \in (-s_0, 0]\),

\[
J_1 \leq C \sqrt{n} \left \{ \mathbb{E}_{Q_s^x} \left( \frac{1}{\delta(y, G_n x)^{-2s}} \right) Q_s^x \left( \log \delta(y, G_n x) \leq -\eta [C_1 \log n] \right) \right \}^{1/2}
\]

\[
\leq C \sqrt{n} e^{-c_0 [C_1 \log n]} \to 0.
\]

**Upper bound of** \(J_2\). Following the proof of (5.17), one has

\[
J_2 \leq (1 + C_\rho(\varepsilon)) \sigma_s \sqrt{\frac{n}{2\pi}} \sum_{k=1}^{\infty} 1_{\{k \leq M_n\}} e^{-s\eta(k-1)}
\]

\[
\times \int_{\mathbb{R}} e^{-it\eta(k-1)} R_{n, it} (\varphi y_{s,k,\varepsilon_1})(x) \tilde{\Psi}_{s,\eta,\varepsilon}(t) \tilde{\rho}_{\varepsilon_2}(t) dt,
\]

where \(\varphi y_{s,k,\varepsilon_1}\) and \(\tilde{\Psi}_{s,\eta,\varepsilon}\) are respectively defined by (5.14) and (5.11). Since \(|s|\) and \(\gamma > 0\) are sufficiently small, by elementary calculations we get that we obtain that the series \(\frac{\log n}{\sqrt{n}} \sum_{k=1}^{M_n} e^{-s\eta(k-1)} \Vert \varphi y_{s,k,\varepsilon_1} \Vert_{\gamma}\) converges to 0 as \(n \to \infty\). Hence, we can apply Proposition 4.3 (2) instead of Proposition 4.3 (1), and follow the proof of (5.18), (5.19) and (5.20) to obtain that uniformly in \(s \in K_\mu\), \(f \in (\mathbb{R}^d)^*\) and \(v \in \mathbb{R}^d\) with \(|f| = |v| = 1\),

\[
\limsup_{n \to \infty} J_2 \leq (1 + C_\rho(\varepsilon)) \tilde{\Psi}_{s,\eta,\varepsilon}(0) \tilde{\rho}_{\varepsilon_2}(0) \sum_{k=1}^{\infty} e^{-s\eta(k-1)} \pi_s(\varphi y_{s,k,\varepsilon_1}).
\]

Then, we can proceed in a similar way as in the proof of (5.21), (5.23), (5.24) and (5.25). One of the main differences is that in (5.24) we need to use the Hölder regularity of the stationary measure \(\pi_s\) stated in Lemma 6.1 to justify the applicability of the Lebesgue dominated convergence theorem, when we interchange the limit \(\varepsilon_1 \to 0\) and the sum over \(k\) in (5.24).
Another difference is that in (5.25) it is necessary to use the zero-one law for the stationary measure \( \pi_s \) shown in Lemma 5.6 instead of Lemma 5.5. Consequently, one can obtain the desired upper bound of \( J_2 \) which is similar to (5.28): uniformly in \( s \in K_\mu, f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1, \)

\[
\lim_{\varepsilon \to 0} \lim_{\eta \to 0} \lim_{n \to \infty} \limsup_{n \to \infty} J_2 \leq \int e^{-su} \psi(u) du \int_{\mathbb{R}^d} \delta(y, x)^{s} \varphi(x) r_{s}^{-1}(x) \pi_s(dx).
\]

The lower bound of \( J_2 \) can be carried out in a similar way and hence we omit the details. □

By Theorem 2.2 and 2.4, we now give a proof of Theorem 2.5 on the local limit theorem with large deviations for coefficients \( \langle f, G_n v \rangle \).

**Proof of Theorem 2.5.** The asymptotic (2.11) follows from Theorem 2.2 by taking \( \varphi = 1 \) and \( \psi(u) = 1_{\{u \in [a_1, a_2]\}}(u), u \in \mathbb{R} \). In the same way, the asymptotic (2.12) is a direct consequence of Theorem 2.4. □

**7. Proof of the Hölder regularity of the stationary measure**

In this section we prove Proposition 3.4 on the Hölder regularity of the stationary measure \( \pi_s \) for any \( s \in I_\mu^0 \). This result is of independent interest and plays a crucial role for establishing the precise large deviation asymptotics for the coefficients \( \langle f, G_n v \rangle \) under the changed measure \( \mathbb{Q}_x^s \), see Theorem 2.2.

The study of the regularity of the stationary measure \( \nu \) defined by (2.2), attracted a great deal of attention, see e.g. [1, 4, 5, 6, 8, 9, 20, 22, 39, 42]. As far as we know, there are three different approaches to establish the regularity of \( \nu \). The first one is originally due to Guivarc’h [39], see also [8]. The approach in [39] consists in investigating the asymptotic behaviors of the components in the Cartan and Iwasawa decompositions of the random matrix product \( M_n = g_1 \cdots g_n \). The second one is developed in [9] for the study of the regularity of the stationary measure on the torus \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \), and has been applied to the setting of products of random matrices in [5, 6], where the large deviation bounds for the Iwasawa cocycle and for the Cartan projection play a crucial role. The third one, which is recently developed in [22] for the special linear group \( SL(2, \mathbb{C}) \) consisting of complex \( 2 \times 2 \) matrices with determinant one, is based on the theory of super-potentials introduced in [23]. All of the results mentioned above are concerned with the regularity of the stationary measure \( \nu \). However, the regularity of the eigenmeasure \( \nu_s \) or of the stationary measure \( \pi_s \) for \( s \) different from 0 was not known before in the literature.

In order to prove Proposition 3.4, we first extend some convergence results concerning the Cartan and Iwasawa decompositions of the matrix product
Similarly to (3.4), for any $s \in I_\mu$, we define the conjugate Markov operator $Q^*_s$ as follows: for any $\varphi \in \mathcal{C}(\mathbb{P}^{d-1})^*$,

$$Q^*_s \varphi(y) = \frac{1}{\kappa(s)r^*_s(y)} P^*_s(\varphi r^*_s)(y), \quad y \in (\mathbb{P}^{d-1})^*.$$ 

Then $Q^*_s$ has a unique stationary measure $\pi^*_s$ given by $\pi^*_s(\varphi) = \frac{\nu^*_s(\varphi r^*_s)}{\nu^*_s(r^*_s)}$ for any $\varphi \in \mathcal{C}(\mathbb{P}^{d-1})^*$.

### 7.1. Asymptotics for the Cartan decomposition

Recall that $G_n = g_n \cdots g_1$. We are going to investigate asymptotic behaviors of the components of the Cartan decomposition of the transposed matrix product

$$G^*_n = g^*_1 g^*_2 \cdots g^*_n, \quad n \geq 1,$$

where $g^*$ is the adjoint automorphism of the matrix $g$. Let $K = SO(d, \mathbb{R})$ be the orthogonal group, and $A^+$ be the set of diagonal matrices whose diagonal entries starting from the upper left corner are strictly positive and decreasing. With these notation, the well known Cartan decomposition states that $GL(d, \mathbb{R}) = KA^+K$. The Cartan decomposition of $G^*_n$ is written as $G^*_n = k_n a_n k'_n$, where $k_n, k'_n \in K$ and $a_n \in A^+$ with its diagonal elements (singular values) satisfying $a_{n,1}^2 \geq a_{n,2}^2 \geq \cdots \geq a_{n,d}^2 > 0$. Note that the diagonal matrix $a_n$ is uniquely determined, but the orthogonal matrices $k_n$ and $k'_n$ are not unique. We choose one such decomposition of $G^*_n$. Denote by $e^*_1, \ldots, e^*_d$ the dual basis of $(\mathbb{R}^d)^*$. The vector $k_n e^*_1 \in (\mathbb{P}^{d-1})^*$ is called the density point of $G^*_n$. It plays an important role in the study of products of random matrices: see [9, 6]. The following result shows that the density point converges almost surely to the random variable $Z^*_s$ of the law $\pi^*_s$ under the changed measure $Q_s := \int_{\mathbb{P}^{d-1}} Q^*_s \pi_s(dx)$. Note that by definition the measure $Q_s$ is shift-invariant and ergodic since $\pi_s$ is the unique stationary measure of the Markov operator $Q_s$. Recall that $\delta(y, x) = \frac{|(f,v)|}{|f||v|}$ for any $y = \mathbb{R} f \in (\mathbb{P}^{d-1})^*$ and $x = \mathbb{R} v \in \mathbb{P}^{d-1}$.

**Lemma 7.1.** Let $s \in I^\circ_\mu$. Under condition A3, with the above notation, we have

$$\lim_{n \to \infty} \frac{a_{n,1}^2}{a_{n,1}^2} = 0, \quad Q_s \text{-a.s.} \quad \text{and} \quad \lim_{n \to \infty} k_n e_1 = Z^*_s, \quad Q_s \text{-a.s.,}$$  

and for any $x = \mathbb{R} v \in \mathbb{P}^{d-1}$ with $v \in \mathbb{R}^d \setminus \{0\}$,

$$\lim_{n \to \infty} \frac{|G_nv|}{\|G_n||v|} = \delta(Z^*_s, x), \quad Q_s \text{-a.s.},$$
where the law of the random variable $Z_n^*$ (on $(\mathbb{P}^{d-1})^*$) is the stationary measure $\pi_n^*$. Moreover, the assertions (7.1) and (7.2) also hold true with the measure $Q_\mu$ replaced by $Q_\mu^x$, for any starting point $x \in \mathbb{P}^{d-1}$.

Before proceeding to proving Lemma 7.1, let us first recall the following two results which were established in [41]. In the sequel, let $m^*$ be the unique rotation invariant probability measure on the projective space $(\mathbb{P}^{d-1})^*$. For any matrix $g \in GL(d, \mathbb{R})$, denote by $g^*m^*$ the probability measure on $(\mathbb{P}^{d-1})^*$ such that for any measurable function $\varphi$ on $(\mathbb{P}^{d-1})^*$,

$$\int_{(\mathbb{P}^{d-1})^*} \varphi(y)(g^*m^*)(dy) = \int_{(\mathbb{P}^{d-1})^*} \varphi(g^*y)m^*(dy).$$

**Lemma 7.2.** Assume condition A3. Let $s \in I_\mu^0$. Then, the probability measure $G_n^*m^*$ converges weakly to the Dirac measure $\delta_{Z_n^*}$, $Q_s$-a.s., where the law of the random variable $Z_n^*$ under the measure $Q_\mu^x$ is given by $\pi^*_n$.

**Proof.** This result has been recently established in [41, Theorem 3.2]. □

The following result is proved in [41, Lemma 3.5].

**Lemma 7.3.** Assume condition A3. Let $s \in I_\mu^0$. Then, there exists a constant $c_s > 0$ such that for any $x \in \mathbb{P}^{d-1}$, it holds that $Q_\mu^x \leq c_s Q_s$.

The assertion of Lemma 7.3 implies that the measure $Q_\mu^x$ is absolutely continuous with respect to $Q_s$. Using Lemmas 7.2 and 7.3, we are now in a position to prove Lemma 7.1.

**Proof of Lemma 7.1.** By the Cartan decomposition of $G_n^*$, we have $G_n^* = k_n^*a_n^*k_n^*$, where $k_n^*, k_n' \in K$ and $a_n \in A^+$. By Lemma 7.2, the probability measure $G_n^*m^*$ converges weakly to the Dirac measure $\delta_{Z_n^*}$, $Q_s$-a.s.. Since $m^*$ is a rotation invariant measure on $(\mathbb{P}^{d-1})^*$, it follows that $(k_n^*a_n)m^*$ converges weakly to the random variable $Z_n^*$, $Q_s$-a.s.. Taking into account that $a_n$ is a diagonal random matrix with decreasing diagonal entries, we deduce that, as $n \to \infty$, we have $a_n^*m^* \to \delta_{e_1^*}$, $a_n^{2,2}/a_n^{1,1} \to 0$ and $k_ne_1^* \to Z_n^*$, $Q_s$-a.s.. This concludes the proof of the assertion (7.1). To show (7.2), using again the decomposition $G_n^* = k_n^*a_n^*k_n'$, it follows that for any $x = \mathbb{R}v \in \mathbb{P}^{d-1}$,

$$\frac{|G_nv|^2}{|v|^2} = \frac{(a_n k_n^*v, a_n k_n^*v)}{|v|^2} = \sum_{j=1}^d (a_n^{i,j})^2 \frac{|k_n^* v, e_j^*|^2}{|v|^2} = \sum_{j=1}^d (a_n^{i,j})^2 \delta(k_n e_j^*, x)^2.$$ 

This, together with the fact that $\|G_n\| = a_n^{1,1}$, implies (7.2). Taking into account Lemma 7.3, we see that the assertions (7.1) and (7.2) remain valid with the measure $Q_\mu$ replaced by $Q_\mu^x$. □
7.2. Asymptotics for the Iwasawa decomposition. In this subsection we study the asymptotics of the components in the Iwasawa decomposition of \( G_n^* \) under the changed measure \( Q^x_s \). Denote by \( L \) the group of lower triangular matrices with 1 in the diagonal elements, by \( A \) the group of diagonal matrices with strictly positive entries in the diagonal elements, and as before by \( K \) the group of orthogonal matrices. The Iwasawa decomposition states that \( GL(d, \mathbb{R}) = LAK \) and such decomposition is unique. Hence, for the product \( G_n^* \), there exist unique \( L(G_n^*) \in L, A(G_n^*) \in A \) and \( K(G_n^*) \in K \) such that \( G_n^* = L(G_n^*)A(G_n^*)K(G_n^*) \). The following result shows that \( L(G_n^*)e_1^* \) converges almost surely under the measures \( Q_s \) and \( Q^x_s \).

**Lemma 7.4.** Let \( s \in I^0_\mu \). Under condition A3, for any \( x \in \mathbb{R}^{d-1} \),

\[
\lim_{n \to \infty} L(G_n^*)e_1^* = \frac{Z_s^*}{\langle Z_s^*, e_1 \rangle}, \quad Q_s^x - \text{a.s. and } Q_s^x - \text{a.s.}
\]

where \( Z_s^* \) is a random variable given by Lemma 7.1.

**Proof.** In view of Lemma 7.3, it suffices to prove the assertion under the measure \( Q_s \). Using the Iwasawa decomposition \( G_n^* = L(G_n^*)A(G_n^*)K(G_n^*) \) and noticing that \( K(G_n^*) \) is an orthogonal matrix, it follows that

\[
\frac{G_n^* G_n e_1^*}{|G_n e_1^*|^2} = \frac{L(G_n^*) A(G_n^*)^2 L(G_n^*) e_1^*}{|A(G_n^*) L(G_n^*) e_1^*|^2} = L(G_n^*) e_1^*, \quad (7.3)
\]

where the second equality holds due to the fact that \( A(G_n^*)^2 L(G_n^*) e_1^* = |A(G_n^*) L(G_n^*) e_1^*|^2 e_1^* \). By the Cartan decomposition of \( G_n^* \) we have \( G_n^* = k_n a_n k'_n \), where \( k_n, k'_n \) are two orthogonal matrices. Hence, for any \( v \in \mathbb{R}^d \),

\[
\langle G_n^* G_n e_1^*, v \rangle = \langle (a_n)^2 k_n^* e_1^*, k_n v \rangle = \langle a_n^{1,1^2} \langle k_n^* e_1^*, e_1 \rangle \langle e_1^*, k_n v \rangle + O(a_n^{1,1} a_n^{2,2}) \rangle = \langle a_n^{1,1^2} \langle k_n^* e_1^*, e_1 \rangle \langle k_n e_1^*, v \rangle + O(a_n^{1,1} a_n^{2,2}) \rangle. \quad (7.4)
\]

Consequently, by (7.3) and (7.4) we obtain that \( Q_s - \text{a.s.}, \)

\[
\lim_{n \to \infty} \langle L(G_n^*) e_1^*, v \rangle = \lim_{n \to \infty} \frac{\langle G_n^* G_n e_1^*, v \rangle}{\langle G_n^* G_n e_1^*, e_1 \rangle} = \lim_{n \to \infty} \frac{\langle k_n e_1^*, v \rangle}{\langle k_n e_1^*, e_1 \rangle} = \langle Z_s^*, v \rangle = \langle \langle Z_s^*, e_1 \rangle, v \rangle \}
\]

where in the first equality we used (7.3), in the second one we used (7.4) and Lemma 7.1, and in the last one we applied again Lemma 7.1. Since \( v \in \mathbb{R}^d \) is arbitrary, the proof of Lemma 7.4 is complete. \( \square \)

For any \( 1 \leq k \leq d \), we briefly recall the notion of exterior algebra \( \wedge^k(\mathbb{R}^d) \) of the vector space \( \mathbb{R}^d \). The space \( \wedge^k(\mathbb{R}^d) \) is endowed with the dual bracket \( \langle \cdot, \cdot \rangle \) and the norm \( |\cdot| \); we use the same notation as in \( \mathbb{R}^d \) and the distinction
should be clear from the context. The scalar product in $\wedge^k(\mathbb{R}^d)$ satisfies the following property: for any $u_i, v_j \in \mathbb{R}^d$, $1 \leq i, j \leq d$,

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle u_i, v_j \rangle)_{1 \leq i, j \leq d},$$

where $\det((\langle u_i, v_j \rangle))_{1 \leq i, j \leq d}$ denotes the determinant of the associated matrix. It is well known that $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}, 1 \leq i_1 < i_2 < \cdots < i_k \leq d\}$ forms a basis of $\wedge^k(\mathbb{R}^d)$, $1 \leq k \leq d$, and that $v_1 \wedge \cdots \wedge v_k$ is nonzero if and only if $v_1, \ldots, v_k$ are linearly independent in $\mathbb{R}^d$. For any $g \in GL(d, \mathbb{R})$ and $1 \leq k \leq d$, the exterior product $\wedge^k g$ of the matrix $g$ is defined as follows: for any $v_1, \ldots, v_k \in \mathbb{R}^d$,

$$\wedge^k g(v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k.$$

Set $\|\wedge^k g\| = \sup\{|(\wedge^k g)v| : v \in \wedge^k(\mathbb{R}^d), |v| = 1\}$. Since $\wedge^k (g g') = (\wedge^k g)(\wedge^k g')$, it holds that $\|\wedge^k (g g')\| \leq \|\wedge^k g\|\|\wedge^k g'\|$ for any $g, g' \in GL(d, \mathbb{R})$. Besides, if we denote by $a_{11}, \ldots, a_{dd}$ the singular values of the matrix $g$, then $\|\wedge^k g\| = a_{11} \cdots a_{kk}$. In particular, we have $\|\wedge^k g\| \leq \|g\|^k$.

The following lemma was proved in [8]. For any $g \in GL(d, \mathbb{R})$, by the Iwasawa decomposition we have $g = L(g)A(g)K(g)$, where $L(g) \in L$, $A(g) \in A$ and $K(g) \in K$. In the sequel, we denote $N(g) = \max\{|g|, \|g^{-1}\|\}$.

**Lemma 7.5.** For any integers $n, m \geq 0$, we have

$$\left|L(G^*_n)^{n+m} e_1^n - L(G^*_n)^{n} e_1^n\right| \leq \sum_{j=n}^{n+m-1} \left|\frac{\wedge^2 G^*_j}{|G^*_j e_1|^2}\right| e^{2 \log N(g^*_j, 1)} \left|G^*_j e_1\right|^2,$$

where we use the convention that $L(G_0) = 0$ and $\|\wedge^2 G_0\| = 0$.

The following result shows the simplicity of the dominant Lyapunov exponent for $G_n$ under the changed measure $Q^x_n$.

**Lemma 7.6.** Assume condition A3. Let $s \in T_0^\mu$. Then, uniformly in $x = \mathbb{R}v \in \mathbb{P}^{d-1}$,

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}_{Q^x_n}(\sigma(G_n, x)) = \lambda_1(s),$$

and

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}_{Q^x_n}(\log \|\wedge^2 G_n\|) = \lambda_1(s) + \lambda_2(s),$$

where $\lambda_1(s) > \lambda_2(s)$ are called the first two Lyapunov exponents of $G_n$ under the measure $Q^x_n$.

The assertion (7.5) is proved in [41, Theorem 3.10]. The assertion (7.6) follows by combining Theorems 3.10 and 3.17 in [41]. The fact that $\lambda_1(s) > \lambda_2(s)$ will play an essential role in the proof of the Hölder regularity of the stationary measure $\pi_s$, see Proposition 3.4.
Using the simplicity of the Lyapunov exponent (see Lemma 7.6) we can complement the convergence result in Lemma 7.4 by giving the rate of convergence. This result is not used in the proofs, but is of independent interest.

Proposition 7.7. Assume condition A3. Let \( s \in I_\mu^o \). Then, there exist constants \( \alpha, C > 0 \) such that uniformly in \( x \in \mathbb{P}^{d-1} \) and \( n \geq 1 \),

\[
\mathbb{E}_{Q_s} \left| L(G_n^*) e_1^* - \frac{Z_s^*}{(Z_s^*, e_1^*)} \right|^\alpha \leq e^{-Cn}. \tag{7.7}
\]

Moreover, the assertion (7.7) remains valid when the measure \( Q_s^x \) is replaced by \( Q_s \).

The proof of Proposition 7.7 is postponed to subsection 7.3.

By Jensen’s inequality, the bound (7.7) implies that there exists a constant \( C > 0 \) such that uniformly in \( x \in \mathbb{P}^{d-1} \),

\[
\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}_{Q_s} \log \left| L(G_n^*) e_1^* - \frac{Z_s^*}{(Z_s^*, e_1^*)} \right| \leq -C. \tag{7.8}
\]

When \( s = 0 \), it was proved in [8] that \( C = \lambda_1(0) - \lambda_2(0) \). We conjecture that \( C = \lambda_1(s) - \lambda_2(s) \) also for \( s > 0 \), but the proof eluded us.

7.3. Proof of Propositions 3.4 and 7.7. With the results established in subsections 7.1 and 7.2, we are well equipped to prove Propositions 3.4 and 7.7.

Proof of Proposition 3.4. Since \( r_s \) is bounded away from infinity and 0 uniformly on \( \mathbb{P}^{d-1} \), it suffices to establish (3.7) and (3.8) for the stationary measure \( \pi_s \).

Define the function \( \rho : GL(d, \mathbb{R}) \times \mathbb{P}^{d-1} \to \mathbb{R} \) as follows: for \( g \in GL(d, \mathbb{R}) \) and \( x \in \mathbb{P}^{d-1} \),

\[
\rho(g, x) = \log \| \wedge^2 g \| - 2 \log |gx|.
\]

It is clear that

\[
\mathbb{E}_{Q_s} \rho(G_n, x) = \mathbb{E}_{Q_s} (\log \| \wedge^2 G_n \|) - 2 \mathbb{E}_{Q_s} (\log |G_n x|).
\]

By Lemma 7.6, we see that

\[
\lim_{n \to \infty} \frac{1}{n} \sup_{x \in \mathbb{P}^{d-1}} \mathbb{E}_{Q_s} \rho(G_n, x) < 0,
\]

which clearly implies that, for large enough \( n \),

\[
\sup_{x \in \mathbb{P}^{d-1}} \mathbb{E}_{Q_s} \rho(G_n, x) < 0. \tag{7.8}
\]

We claim that there exists a constant \( \alpha > 0 \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in \mathbb{P}^{d-1}} \mathbb{E}_{Q_s} \frac{\| \wedge^2 G_n \|^\alpha}{|G_n x|^{2\alpha}} < 0. \tag{7.9}
\]
To prove (7.9), we denote \( a_n = \log \left( \sup_{x \in \mathbb{P}^{d-1}} \mathbb{E}_{Q^s_d} \left( e^{\alpha \rho(G_n,x)} \right) \right) \), for sufficiently small constant \( \alpha > 0 \). Using the cocycle property (3.2) and the fact that \( \rho \) is subadditive, we get that for any \( n, m \geq 1 \),

\[
\mathbb{E}_{Q^s_d} \left( e^{\alpha \rho(G_{n+m},x)} \right) \\
\leq \mathbb{E} \left( q_m^*(x,G_m) e^{\alpha \rho(G_n,x)} \right) \mathbb{E} \left( q_n^*(x,g_{m+1} \ldots g_{m+n}) e^{\alpha \rho(g_{m+n+1},x)} \right) \\
= \mathbb{E}_{Q^s_d} \left( e^{\alpha \rho(G_m,x)} \right) \mathbb{E}_{Q^s_d} \left( e^{\alpha \rho(G_n,x)} \right).
\]

Taking supremum on both sides of the above inequality, we see that the sequence \( (a_n)_{n \geq 1} \) satisfies the subadditive property: \( a_{n+m} \leq a_m + a_n \). Hence we get \( a = \lim_{n \to \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n} \). To show that \( a < 0 \), it suffices to check that there exists some integer \( p \geq 1 \) such that

\[
\sup_{x \in \mathbb{P}^{d-1}} \mathbb{E}_{Q^s_d} \left( e^{\alpha \rho(G_p,x)} \right) < 1. \tag{7.10}
\]

We proceed to verify (7.10). Using the fact that \( \sup_x |\rho(g,x)| \leq 4 \log N(g) \) and the basic inequality \( e^y \leq 1 + y + \frac{y^2}{2} e^{|y|} \), \( y \in \mathbb{R} \), we obtain

\[
\mathbb{E}_{Q^s_d} \left( e^{\alpha \rho(G_p,x)} \right) \leq 1 + \alpha \mathbb{E}_{Q^s_d} \left( \rho(G_p,x) \right) + \frac{\alpha^2}{2} \mathbb{E}_{Q^s_d} \left( 16 \log^2 N(G_p) e^{4\alpha \log N(G_p)} \right). \tag{7.11}
\]

The second term on the right-hand side of (7.11) is strictly negative by using the bound (7.8) and taking large enough \( p \). The third term is finite due to the moment condition A2. Consequently, taking \( \alpha > 0 \) small enough, we obtain the inequality (7.10) and thus the desired assertion (7.9) follows.

Since the bound (7.9) holds uniformly in \( x \in \mathbb{P}^{d-1} \), taking into account that \( Q_s = \int_{\mathbb{P}^{d-1}} Q^x_s \pi_s (dx) \), it follows that there exist constants \( C > 0 \) and \( 0 < r < 1 \) such that

\[
\mathbb{E}_{Q_s} \| \wedge^{G_n}_{G_n,x} \|^{\alpha}_{2 \alpha} \leq C r^n. \tag{7.12}
\]

Using Lemma 7.4, Fatou’s lemma and the fact that \( |Z^*_s| = 1 \), we obtain that for sufficiently small constant \( \alpha > 0 \),

\[
\mathbb{E}_{Q_s} \frac{1}{|\langle Z^*_s, e^*_1 \rangle|^\alpha} \leq \liminf_{n \to \infty} \mathbb{E}_{Q_s} (|L(G_n^*) e^*_1|^\alpha). \tag{7.13}
\]

From Lemma 7.5 with \( n = 0 \), it follows that

\[
|L(G_n^*) e^*_1|^\alpha \leq \sum_{j=1}^{\infty} \frac{\| \wedge^2 G_j \|^{2 \alpha}}{|G_j e_1|^{2 \alpha}} e^{2 \alpha \log N(G_j^+)}. \]
Notice that $G_j$ and $g_j^* + 1$ are not independent under the measure $Q_s$. Using Fubini’s theorem, Hölder’s inequality and the bound (7.12), we get

$$
E_{Q_s} \left( |L(G_n^*)e_1|^\alpha \right) \leq \sum_{j=1}^{\infty} \left[ E_{Q_s} \left( \frac{\Lambda^2 G_j}{|G_je_1|} \right)^{2\alpha} \right]^{1/2} \left[ E_{Q_s} e^{4\alpha \log N(g_j^* + 1)} \right]^{1/2} \leq C E_{Q_s} (e^{4\alpha \log N(g_j^*)}) \sum_{j=1}^{\infty} r_j < +\infty.
$$

Combining this with (7.13) leads to $E_{Q_s} \left( |\langle Z^*_s, e_1 \rangle|^{\alpha} \right) < +\infty$. Note that for any $y \in (\mathbb{P}^{d-1})^*$, we can choose an orthogonal matrix $k$ such that $ke_1^* = y$. If we replace $g_j^*$ by $k^{-1}g_j^*k$, then it is easy to see that $G_n^*$ is replaced by $k^{-1}G_n^*k$.

Moreover, in view of Lemma 7.2, the random variable $Z_s^*$ is replaced by $k^{-1}Z_s^*$. Since the bound (7.12) holds uniformly in $x \in \mathbb{P}^{d-1}$, it follows that

$$
E_{Q_s} \left( \frac{1}{|\langle k^{-1}Z_s^*, e_1^* \rangle|^\alpha} \right) \leq C E_{Q_s} (e^{4\alpha \log N(k^{-1}g_j^*k)}) \sum_{j=1}^{\infty} r_j < +\infty.
$$

Observe that $N(k^{-1}g_j^*k) = N(g_j^*)$ and $\langle k^{-1}Z_s^*, e_1 \rangle = \langle Z_s^*, y \rangle$. Therefore, for any $s \in I^\circ$, there exists a constant $\alpha > 0$ such that

$$
\sup_{x \in \mathbb{P}^{d-1}} \int_{(\mathbb{P}^{d-1})^*} \frac{1}{\delta(y, x)^\alpha} \pi_s^\circ(dy) = \sup_{y \in (\mathbb{P}^{d-1})^*} \frac{1}{\delta(y, x)^\alpha} \pi_s^\circ(dy) < +\infty.
$$

This implies that there exists a constant $C > 0$ such that for any $0 < t < 1$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$
\pi_s^\circ \left( \left\{ y \in (\mathbb{P}^{d-1})^* : \delta(y, x) \leq t \right\} \right) \leq t^\alpha \int_{(\mathbb{P}^{d-1})^*} \frac{1}{\delta(y, x)^\alpha} \pi_s^\circ(dy) \leq Ct^\alpha.
$$

The proof of Proposition 3.4 is complete. \hfill \Box

**Proof of Proposition 7.7.** In view of Lemma 7.3, it suffices to prove the assertion of the proposition with $Q_s$ instead of $Q_s^\circ$, i.e. we show that there exist constants $\alpha, C > 0$ such that for all $n \geq 1$,

$$
E_{Q_s} \left( |L(G_n^*)e_1 - \frac{Z_s^*}{\langle Z_s^*, e_1 \rangle}|^\alpha \right) < e^{-Cn}, \quad (7.14)
$$
Using Lemma 7.5 and Hölder’s inequality, for sufficiently small constant \( \alpha > 0 \) and for any \( n, m \geq 1 \), we get
\[
\mathbb{E}_{Q_s} \left| L(G_{n+m}^*)e_1^* - L(G_n^*)e_1^* \right|^\alpha \\
\leq \sum_{j=n}^{n+m-1} \left[ \mathbb{E}_{Q_s} \frac{\| A^2 G_j \|^{2\alpha}}{|G_j e_1|^{4\alpha}} \right]^{1/2} \left[ \mathbb{E}_{Q_s} e^{4\alpha \log N(g_j^*)} \right]^{1/2} \\
\leq C \sum_{j=n}^{n+m-1} \left[ \mathbb{E}_{Q_s} \frac{\| A^2 G_j \|^{2\alpha}}{|G_j e_1|^{4\alpha}} \right]^{1/2},
\]
where the last inequality holds due to the moment condition \( A2 \). By the Fatou lemma, taking the limit as \( m \to \infty \), we see that
\[
\mathbb{E}_{Q_s} \left| L(G_n^*)e_1^* - \frac{Z_n^*}{\langle Z_n^*, e_1^* \rangle} \right|^\alpha \leq C \sum_{j=n}^{\infty} \left[ \mathbb{E}_{Q_s} \frac{\| A^2 G_j \|^{2\alpha}}{|G_j e_1|^{4\alpha}} \right]^{1/2} \leq Ce^{-Cn},
\]
where the last inequality holds due to the bound (7.12).

7.4 Proofs of Propositions 3.6, 3.7, 3.8 and Theorem 2.7. We first establish Propositions 3.6 and 3.7 based on Propositions 3.4 and 3.5, respectively, together with the fact that, under the changed measure \( Q_s^x \), the Markov chain \( (X_n^x)_{n \geq 0} \) converges exponentially fast to the stationary measure \( \pi_s \).

Proof of Propositions 3.6 and 3.7. For any \( 1 \leq k \leq n \) and \( \varepsilon > 0 \), denote \( \chi_k(u) := 1_{u \in (-\infty, -\varepsilon k]} \) and \( \chi_{k,\varepsilon_1}^+(u) = \sup_{u' \in \mathbb{R}} \chi_k(u') \) for \( \varepsilon_1 > 0 \). In the same way as in (5.13), we have the following smoothing inequality:
\[
\chi_k(u) \leq (\chi_{k,\varepsilon_1}^+ * \tilde{\rho}_{\varepsilon_1})(u) =: \tilde{\chi}_k(u), \quad u \in \mathbb{R}, \tag{7.15}
\]
where \( \tilde{\rho}_{\varepsilon_1} \) is the density function given in (5.13). For brevity, we denote
\[
\varphi^y_{k,\varepsilon_1}(x) = \tilde{\chi}_k(\log \delta(y, x)), \quad x \in \mathbb{R}^{d-1}. \tag{7.16}
\]
By (7.15) and (7.16), it follows that
\[
\mathbb{Q}_{s}^x \left( \delta(y, G_n x) \leq e^{-\varepsilon k} \right) \leq \mathbb{E}_{Q_s^x} \left[ \varphi^y_{k,\varepsilon_1}(G_n x) \right] \\
\leq \left| \mathbb{E}_{Q_s^x} \left[ \varphi^y_{k,\varepsilon_1}(G_n x) \right] - \pi_s(\varphi^y_{k,\varepsilon_1}) \right| + \pi_s(\varphi^y_{k,\varepsilon_1}).
\]
For the first term, note first that \( \| \varphi^y_{k,\varepsilon_1} \|_\gamma \leq \frac{e^{\varepsilon k}h}{(1-e^{-\varepsilon k})^\gamma} \). Using (3.5) and taking \( \gamma > 0 \) sufficiently small, we get that for any \( 1 \leq k \leq n \),
\[
\left| \mathbb{E}_{Q_s^x} \left[ \varphi^y_{k,\varepsilon_1}(G_n x) \right] - \pi_s(\varphi^y_{k,\varepsilon_1}) \right| \leq Ce^{-cn}\| \varphi^y_{k,\varepsilon_1} \|_\gamma \leq Ce^{-cn/2}. \tag{7.17}
\]
For the second term, using the fact that \( \chi_k(u) \leq \chi_{k,2\varepsilon_1}(u) = 1_{\{u \in (-\infty, -\varepsilon + 2\varepsilon_1]\}} \), and applying Propositions 3.4 and 3.5 (respectively for \( s \in I^0_\mu \) and \( s \in (-s_0, 0) \)), we obtain that there exist constants \( c, C > 0 \) such that
\[
\pi_s(\varphi^y_{k,\varepsilon_1}) \leq \pi_s(x \in \mathbb{R}^{d-1} : \delta(y, x) \in [0, e^{-ck + 2\varepsilon_1}]) \leq C e^{-ck}.
\] (7.18)

Putting together (7.17) and (7.18), we conclude the proof of Propositions 3.6 and 3.7.

Using Propositions 3.6 and 3.7, we are now in a position to establish Proposition 3.8 on the SLLN and the CLT for the coefficients \( \langle f, G_n v \rangle \) under the measure \( Q^x_s \).

\textbf{Proof of Proposition 3.8.} (1) We first prove (3.11). By Proposition 3.6 and Borel-Cantelli’s lemma, we get that for any \( \varepsilon > 0 \) and \( s \in I^0_\mu \), uniformly in \( f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1 \),
\[
\liminf_{n \to \infty} \frac{1}{n} \log \frac{|\langle f, G_n v \rangle|}{|G_n v|} \geq -\varepsilon, \quad Q^x_s\text{-a.s.}
\]
Since \( \varepsilon > 0 \) can be arbitrary small, this together with (3.9) implies the desired lower bound: uniformly in \( f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1 \),
\[
\liminf_{n \to \infty} \frac{1}{n} \log |\langle f, G_n v \rangle| \geq \Lambda'(s), \quad Q^x_s\text{-a.s.}
\]
The upper bound follows easily from (3.9) and the fact that \( \log |\langle f, G_n v \rangle| \leq \log |G_n v| \). Hence (3.11) holds.

We next prove (3.12). Using Proposition 3.6 with \( k = \sqrt{n} \), we get the following convergence in probability: for any \( \varepsilon > 0 \), uniformly in \( f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1 \),
\[
\lim_{n \to \infty} Q^x_s \left( \frac{\log |G_n v| - \log |\langle f, G_n v \rangle|}{\sigma_s \sqrt{n}} \right) \geq \varepsilon = 0.
\]
This yields (3.12) using (3.10) together with Slutsky’s lemma.

(2) The proof of part (2) can be carried out in an analogous way using Proposition 3.7, the SLLN and the CLT for the norm cocycle \( \log |G_n v| \) under the changed measure \( Q^x_s \) when \( s < 0 \) established in [58].

Using Propositions 3.6 and 3.7, we are able to prove Theorem 2.7.

\textbf{Proof of Theorem 2.7.} In a similar way as in the proof of (5.46), one can verify that for any \( s < t \) with \( s \in (-s_0, 0] \cup I^0_\mu \) and \( t \in K_\mu \subset (-s_0, s) \),
\[
\mathbb{E}_{Q^x_s} \left[ \varphi(G_n x) \psi(\log |\langle f, G_n v \rangle| - nq_t) \right] = \kappa^n(t) r_t(x) \kappa^n(s) r_s(x) e^{(s-t)nq_t} \times
\]
\[
\mathbb{E}_{Q^x_t} \left[ \langle \varphi r_t^{-1} \rangle(G_n x) e^{(s-t)|\log |G_n v|| - nq_t|} \psi(\log |\langle f, G_n v \rangle| - nq_t) \right].
\]
Recalling that \( \Lambda^*(q_0) = sq_0 - \Lambda(s) \), \( \Lambda^*(q_t) = tq_t - \Lambda(t) \), \( \Lambda(s) = \log \kappa(s) \) and \( \Lambda(t) = \log \kappa(t) \), we have

\[
\frac{\kappa^n(t)}{\kappa^n(s)} e^{(s-t)q_t} = \exp\{-n(\Lambda^*(q_t) - \Lambda^*(q_s) - s(q_t - q_s))\}.
\]

Hence, to prove Theorem 2.7, we are led to handle

\[
J := \sigma_t \sqrt{2\pi n} \mathbb{E}_{Q^s_t} \left[ (C_{r_t}^{-1})(G_n x) e^{(s-t)T^n_s} \psi(T^n_s + Y_{n,x,y}^n) 1_{\{Y_{n,x,y}^n \leq -\eta M_n\}} \right].
\]

For simplicity, denote

\[
T^n_s := \log |G_n v| - nq_t, \quad Y_{n,x,y}^n := \log \delta(y,G_n x).
\]

For any fixed small constant \( 0 < \eta < 1 \), set \( I_k := (-\eta k, -\eta (k-1)) \), \( k \geq 1 \). Take a sufficiently large constant \( C_1 > 0 \) and let \( M_n := \lfloor C_1 \log n \rfloor \). Then,

\[
J = J_1 + J_2,
\]

where

\[
J_1 := \sigma_t \sqrt{2\pi n} \mathbb{E}_{Q^s_t} \left[ (C_{r_t}^{-1})(G_n x) e^{(s-t)T^n_s} \psi(T^n_s + Y_{n,x,y}^n) 1_{\{Y_{n,x,y}^n \leq -\eta M_n\}} \right],
\]

\[
J_2 := \sigma_t \sqrt{2\pi n} \sum_{k=1}^{M_n} \mathbb{E}_{Q^s_t} \left[ (C_{r_t}^{-1})(G_n x) e^{(s-t)T^n_s} \psi(T^n_s + Y_{n,x,y}^n) 1_{\{Y_{n,x,y}^n \in I_k\}} \right].
\]

For \( J_1 \), since the function \( u \mapsto e^{-s'}u \psi(u) \) is directly Riemann integrable on \( \mathbb{R} \) for any \( s' \in K^c \), we see that the function \( u \mapsto e^{(s-t)u} \psi(u) \) is bounded on \( \mathbb{R} \) and so there exists a constant \( C > 0 \) such that for all \( s \in (-s_0,0] \cup I^c_\mu \),

\[
\left| e^{(s-t)T^n_s} \psi(T^n_s + Y_{n,x,y}^n) \right| \leq Ce^{(t-s)Y_{n,x,y}^n}.
\]

Hence, using Propositions 3.6 and 3.7, we get that as \( n \to \infty \),

\[
J_1 \leq C \sqrt{n} \mathbb{Q}^s_t \left( \log \delta(y,G_n x) \leq -\eta(C_1 \log n) \right) \leq C \sqrt{n} e^{-C_0 |C_1 \log n|} \to 0.
\]

For \( J_2 \), one can follow the proof of Theorem 2.4 to obtain that as \( n \to \infty \), uniformly in \( f \in (\mathbb{R}^d)^* \) and \( v \in \mathbb{R}^d \) with \( |f| = |v| = 1 \),

\[
J_2 = \int_{\mathbb{R}^{d-1}} \varphi(x) \delta(y,x)^t \frac{r_s(x)}{r_t(x)} \pi_t(dx) \int_{\mathbb{R}} e^{-(t-s)u} \psi(u)du + o(1).
\]

This ends the proof of Theorem 2.7. \( \square \)

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Current address, Xiao, H.: Université de Bretagne-Sud, LMBA UMR CNRS 6205, Vannes, France
Email address: hui.xiao@univ-ubs.fr

Current address, Grama, I.: Université de Bretagne-Sud, LMBA UMR CNRS 6205, Vannes, France
Email address: ion.grama@univ-ubs.fr

Current address, Liu, Q.: Université de Bretagne-Sud, LMBA UMR CNRS 6205, Vannes, France
Email address: quansheng.liu@univ-ubs.fr