REAL ZEROS OF RANDOM SUMS WITH I.I.D. COEFFICIENTS

BY

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Abstract. Let \( \{f_k\} \) be a sequence of entire functions that are real valued on the real-line. We study the expected number of real zeros of random sums of the form \( P_n(z) = \sum_{k=0}^{n} \eta_k f_k(z) \), where \( \{\eta_k\} \) are real valued i.i.d. random variables. We establish a formula for the density function \( \rho_n \) for that number. As a corollary, taking \( \{\eta_k\} \) to be i.i.d. standard Gaussian, appealing to Fourier inversion we recover the representation for the density function previously given by Vanderbei by means of a different proof. Placing the restrictions on the common characteristic function \( \phi \) of \( \{\eta_k\} \) that \( |\phi(s)| \leq (1 + as^2)^{-q} \), with \( a > 0 \) and \( q \geq 1 \), as well as that \( \phi \) is three times differentiable with both the second and third derivatives uniformly bounded, we achieve an upper bound on \( \rho_n \) with explicit constants that depend only on the restrictions on \( \phi \). As an application we consider the limiting value of \( \rho_n \) when the spanning functions \( f_k(z) \) are \( p_k(z) \), \( k = 0, 1, \ldots, n \), where \( \{p_k\} \) are the Bergman polynomials on the unit disk.

1. Introduction. The systematic study of the expected number of real zeros of polynomials
\[
P_n(z) = \eta_n z^n + \eta_{n-1} z^{n-1} + \cdots + \eta_1 z + \eta_0
\]
with random coefficients \( \{\eta_j\} \), called random algebraic polynomials (or Kac polynomials), dates back to the early 1930s. For early results in this area we refer the reader to the works [5], [18]–[22], as well as the books by Bharucha-Reid and Sambandham [3] and Farahmand [11].

Let \( E \) denote the mathematical expectation and \( N_n(S) \) the number of zeros of \( P_n \) in a set \( S \). In 1943, for a measurable \( \Omega \subset \mathbb{R} \), Kac [16] produced an integral equation for \( E[N_n(\Omega)] \) when the random variables \( \{\eta_j\} \) are i.i.d. standard Gaussian. Independently while studying random noise in 1945, Rice [30] derived a similar formula for \( E[N_n(\mathbb{R})] \) in the Gaussian setting. After Kac established the integral equation for \( E[N_n(\Omega)] \), he proved the asymptotic
\[
E[N_n(\mathbb{R})] = \frac{2 + o(1)}{\pi} \log n \quad \text{as } n \to \infty.
\]
The error term above was further sharpened by Hammersley [12], Edelman and Kostlan [9], and finally Wilkins [33].

Kac conjectured that a similar asymptotic should hold when the random variables are i.i.d. uniform on $[-1, 1]$. Realizing that the same proof would not go through, he was able in [17] to produce the asymptotic (1.1) in the uniform case. For other results concerning non-Gaussian random algebraic polynomials see [10], [13], [14], [15], [23], [24], [29], and [27].

Due to the work of Kac and Rice, formulas for the density function that give the expected number of real zeros of the random polynomial $P_n$ of the form

$$(1.2) \quad \rho_n(x) = \int_{\mathbb{R}} |\eta| D_n(0, \eta; x) \, d\eta,$$

where $D_n(\xi, \eta; x)$ is the joint density distribution of $P_n(x)$ and $P_n'(x)$, with

$$\mathbb{E}[N_n(\Omega)] = \int_{\Omega} \rho_n(x) \, dx,$$

are called Kac–Rice formulas. We note that $\rho_n$ is also referred to as the intensity function or the first correlation function.

When dealing with the expected number of real zeros of

$$P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z),$$

where $\{f_j\}$ are no longer monomials, and $\{\eta_j\}$ are non-Gaussian random variables, the Kac–Rice formula (1.2) still holds. However, the evaluation of this formula is very difficult. In fact little is known about a workable shape of the intensity function in this non-Gaussian setting.

Instead of altering the spanning functions by taking non-monomials, many authors have kept the monomial basis but introduced weights that can help determine the asymptotic of the intensity function. In this case the random sums take the shape

$$(1.3) \quad G_n(z) = \sum_{j=0}^{n} \eta_j c_j z^j,$$

where $\{\eta_j\}$ are i.i.d. random variables and $\{c_j\}$ are deterministic weights. For results concerning the weighted random polynomial $G_n$ we direct the reader to [9], [31], and [8].

Appealing to Fourier transforms of distribution functions, Bleher and Di [4] gave a universality result for the expected number of real zeros of $G_n$ defined in (1.3) when the coefficients $\{c_i\}$ are elliptical weights, that is, $c_i = \sqrt{\binom{n}{i}}$, and $\{\eta_i\}$ are i.i.d. random variables with mean zero and variance one. To achieve their result, Bleher and Di assume that the common
characteristic function for the i.i.d. random variables,

$$\phi(s) = \int \mathbb{R} r(t)e^{its} \, dt,$$

satisfies $|\phi(s)| \leq (1 + as^2)^{-q}$ with $a, q > 0$, and $\sup_{s \in \mathbb{R}}|\frac{d^j}{ds^j}\phi(s)| \leq C_j$ for $j = 2, 3$, where $C_2, C_3 > 0$ are constants. Under these assumptions, for $x \neq 0$ they show that

$$\lim_{n \to \infty} \rho_n(x) \frac{1}{\sqrt{n}} = \frac{1}{\pi(1 + x^2)},$$

where $\rho_n$ is the intensity function for the random sum $G_n$. Under further assumptions on the shape of $\phi(s)$ and that the derivatives up to the sixth order are bounded, they show that (1.4) also holds for $x = 0$. In light of the work by Edelmon and Kostlan [9], the result (1.4) matches up with the case when the random variables of the weighted random sum are i.i.d. standard Gaussian. The technique that Bleher and Di use also allows them to extend their result to higher order correlation functions and to non-Gaussian multivariate weighted random polynomials.

Applying the techniques of Bleher and Di [4], we achieve a workable representation for the density function of the expected number of zeros of a random sum spanned by entire functions that are real valued on the real line. To specify these results, let $\{\eta_k\}$ be real valued i.i.d. random variables such that

$$(1.5) \quad \mathbb{E}[\eta_k] = 0, \quad \mathbb{E}[\eta_k^2] = 1, \quad k = 0, 1, \ldots, n.$$ 

Consider the random linear combination

$$(1.6) \quad P_n(z) = \eta_n f_n(z) + \eta_{n-1} f_{n-1}(z) + \cdots + \eta_1 f_1(z) + \eta_0 f_0(z),$$

with $f_n, f_{n-1}, \ldots, f_0$ being entire functions that are real valued on the real line. Let $\rho_n(x)$ be the density function for the expected number of real zeros of $P_n(x)$. Define

$$(1.7) \quad K_n(x) := \sqrt{K_n^{(1,1)}(x,x)K_n(x,x) - K_n^{(0,1)}(x,x)^2},$$

where

$$(1.8) \quad K_n^{(1,1)}(x,x) = \sum_{j=0}^{n} f_j(x)^2, \quad K_n^{(0,1)}(x,x) = \sum_{j=0}^{n} f_j(x)f_j'(x),$$

$$(1.9) \quad K_n^{(1,1)}(x,x) = \sum_{j=0}^{n} f_j'(x)^2.$$
Theorem 1.1. The density function $\rho_n(x)$ of the real zero distribution of the random sum $P_n(x)$ given by (1.6) can be written as

\begin{equation}
\rho_n(x) = \mathcal{K}_n(x) \int_{\mathbb{R}} |\eta| \tilde{D}_n(0, \eta; x) \, d\eta,
\end{equation}

where $\tilde{D}_n(\xi, \eta; x)$ is the joint distribution density of the random variables

\begin{align*}
g_n(x) &= \sum_{k=0}^{n} \mu_k(x) \eta_k, & h_n(x) &= \sum_{k=0}^{n} \lambda_k(x) \eta_k,
\end{align*}

with

\begin{align*}
\mu_k(x) &= \frac{f_k(x)}{K_n(x, x)^{1/2}}, \\
\lambda_k(x) &= \frac{K_n(x, x) f'_k(x) - K_n^{(0,1)}(x, x) f_k(x)}{[K_n(x, x)(K_n^{(1,1)}(x, x) K_n(x, x) - K_n^{(0,1)}(x, x)^2)]^{1/2}}.
\end{align*}

Furthermore, $\{\mu_k\}$ and $\{\lambda_k\}$ satisfy

\begin{equation}
\sum_{k=0}^{n} \mu_k(x)^2 = \sum_{k=0}^{n} \lambda_k(x)^2 = 1 \quad \text{and} \quad \sum_{k=0}^{n} \lambda_k(x) \mu_k(x) = 0.
\end{equation}

Corollary 1.2. Let the common characteristic function of the i.i.d. random variables $\{\eta_k\}$ be $\phi(s) = \exp(-as^2)$, where $a \in (0, \infty)$ is a fixed number. Then

\begin{equation}
\rho_n(x) = \frac{1}{\pi} \mathcal{K}_n(x).
\end{equation}

In particular, when $a = 1/2$, that is, when the random variables $\{\eta_k\}$ are i.i.d. standard Gaussian, the above theorem recovers the result proven by Vanderbei [32, Theorem 1.2]. The proof of Corollary 1.2 differs from Vanderbei’s in that it uses the representation (1.9) and Fourier transforms instead of relying on the argument principle. Furthermore, our approach allows showing that (1.11) holds for i.i.d. scaled mean zero Gaussian random variables.

For the next result we will need some assumptions on the common characteristic function $\phi(s)$ of the i.i.d. random variables $\{\eta_k\}$. Assume that for fixed $a > 0$ and $q \geq 1$,

\begin{equation}
|\phi(s)| \leq \frac{1}{(1 + as^2)^q}, \quad s \in \mathbb{R},
\end{equation}

and $\phi(s)$ is three times differentiable with the property that there exist constants $C_2, C_3 > 0$ such that

\begin{equation}
\sup_{-\infty < s < \infty} \left| \frac{d^j \phi(s)}{ds^j} \right| \leq C_j, \quad j = 2, 3.
\end{equation}
Theorem 1.3. Suppose that the characteristic function for the collection \( \{ \eta_i \} \) of i.i.d. random variables satisfies conditions (1.12) and (1.13). Then the density function \( \rho_n(x) \) of the real zero distribution of the random sum \( P_n(x) \) satisfies

\[
\rho_n(x) \leq K_n(x) \frac{1}{aq} \left[ k_1 + C_3 k_2 + \frac{1}{\sqrt{aq}} C_2 (k_3 + C_2 k_4) \right].
\]

Here \( k_1, k_2, k_3, k_4 \) are constants that depend only on the conditions (1.12) and (1.13), with \( k_1 \approx 0.36, k_2 \approx 0.27, k_3 \approx 0.21, \) and \( k_4 \approx 1.18. \)

We note that the constants \( k_i \) in Theorem 1.3 are given explicitly in the proof.

Example 1.4. Consider a Laplace distribution with characteristic function

\[
\phi(s) = \frac{1}{1 + s^2/2}.
\]

Note that this function is no smaller than the Gaussian characteristic function \( \exp(-s^2/2) \). In this example we have \( C_2 = 1 \) and \( C_3 \approx 1.65 \), so that along with Theorem 1.3 we see that

\[
\rho_n(x) \leq K_n(x) \times 5.56174 \ldots = \frac{1}{\pi} K_n(x) \times 17.4727 \ldots.
\]

Thus in light of Corollary 1.11, \( \rho_n(x) \) is at most \( 17.4727 \ldots \) times larger than when the random variables are from the Gaussian distribution with characteristic function \( \exp(-as^2) \), with \( a > 0. \)

Asymptotics for \( \rho_n(x) \) when the \( \{ \eta_k \} \) are i.i.d. standard Gaussian has been well studied when the spanning functions are trigonometric functions [32], orthogonal polynomials on the real line ([8], [7], [2], [25], [26], [28], [35]), and orthogonal polynomials on the unit circle ([35], [1], [34]). As an application we consider the case \( P_n(x) = \sum_{k=0}^{n} \eta_k f_k(x) \) with \( f_k(x) = p_k(x) \), where \( p_k(z) = (k+1)\pi z^k \) are the Bergman polynomials on the unit disk, i.e. polynomials orthogonal with respect to area measure over the unit disk.

Theorem 1.5. Let \( f_k(x) = p_k(x) = (k+1)/\pi z^k \), \( k = 0, \ldots, n \), be the Bergman polynomials on the unit disk. Then the function \( K_n(x) \) defined at (1.7) satisfies

\[
\lim_{n \to \infty} K_n(x) = \begin{cases} 
\frac{\sqrt{2}}{1-x^2}, & |x| < 1, \\
\frac{1}{x^2-1}, & |x| > 1. 
\end{cases}
\]

Furthermore, the above convergence holds locally uniformly on the respective domains, and on the boundary we have

\[
K_n(\pm 1) = \frac{1}{3} \sqrt{n(n+3)/2}.
\]
The above theorem in connection with Theorem 1.2 allows one to find the limiting value of the intensity function for the random sum $P_n(z)$ when the random variables are scaled Gaussian, and also gives an upper bound for the limiting value of the intensity function in the non-Gaussian setting of Theorem 1.3.

2. The proofs

2.1. Proof of Theorem 1.1. Since the coefficients $\eta_0, \eta_1, \ldots, \eta_n$ satisfy (1.5), we have

\[ E[P_n(x)] = E[P_n'(x)] = 0, \]
\[ E[P_n(x)^2] = \sum_{k=0}^{n} f_k(x)^2 = K_n(x, x), \]
\[ E[P_n(x)P_n'(x)] = \sum_{k=0}^{n} f_k(x)f_k'(x) = K_n^{(0,1)}(x, x), \]
\[ E[P_n'(x)^2] = \sum_{k=0}^{n} f_k'(x)^2 = K_n^{(1,1)}(x, x). \]

Following the method of Bleher and Di [4], we now rescale $P_n(x)$ and $P_n'(x)$ as follows:

\[ g_n(x) := \frac{P_n(x)}{\sqrt{K_n(x, x)}} = \sum_{k=0}^{n} \mu_k(x)\eta_k, \]
\[ \tilde{g}_n(x) := \frac{P_n'(x)}{\sqrt{K_n^{(1,1)}(x, x)}} = \sum_{k=0}^{n} \nu_k(x)\eta_k, \]

where

\[ \mu_k(x) = \frac{f_k(x)}{\sqrt{K_n(x, x)}}, \quad \nu_k(x) = \frac{f_k'(x)}{\sqrt{K_n^{(1,1)}(x, x)}}, \quad k = 0, 1, \ldots, n. \]

Let $\tilde{D}_n(\xi, \eta; x)$ be the joint distribution density of $g_n(x)$ and $\tilde{g}_n(x)$. By a change of variables we have

\[ D_n(\xi, \eta; x) = \frac{1}{\sqrt{K_n(x, x)K_n^{(1,1)}(x, x)}} \tilde{D}_n\left(\frac{\xi}{\sqrt{K_n(x, x)}}, \frac{\eta}{\sqrt{K_n^{(1,1)}(x, x)}}; x\right), \]

so that the Kac–Rice equation (1.2) is now

\[ \rho_n(x) = \sqrt{\frac{K_n^{(1,1)}(x, x)}{K_n(x, x)}} \int_{\mathbb{R}} |\eta|\tilde{D}_n(0, \eta; x) d\eta. \]
We now change the joint distribution density

\[ K_n^{(0,1)}(x, x) \]

\[ K_n(x, x)K_n^{(1,1)}(x, x) \]

Observe that \((2.3)\) and \((2.1)\) give

\[ \sum_{k=0}^{n} \mu_k(x)^2 = \sum_{k=0}^{n} \nu_k(x)^2 = 1, \]

\[ (2.5) \quad \sum_{k=0}^{n} \mu_k(x)\nu_k(x) = \frac{K_n^{(0,1)}(x, x)}{\sqrt{K_n(x, x)K_n^{(1,1)}(x, x)}}. \]

We now change the joint distribution density \( \tilde{D}_n(\xi, \eta; x) \) for \( g_n(x) \) and \( \tilde{g}_n(x) \) to that of one for \( g_n(x) \) and \( h_n(x) \), where

\[ h_n(x) := \frac{\tilde{g}_n(x) - (\nu(x), \mu(x))g_n(x)}{\tau_n(x)} = \sum_{k=0}^{n} \lambda_k(x)\eta_k, \]

with

\[ \mu(x) = (\mu_0(x), \ldots, \mu_n(x)), \quad \nu(x) = (\nu_0(x), \ldots, \nu_n(x)), \]

\[ (\nu(x), \mu(x)) = \mathbb{E}[g_n(x)\tilde{g}_n(x)] = \sum_{k=0}^{n} \nu_k(x)\mu_k(x), \]

\[ \lambda(x) = (\lambda_0(x), \ldots, \lambda_n(x)) = \frac{\nu(x) - (\nu(x), \mu(x))\mu(x)}{\tau_n(x)}, \]

\[ \tau_n(x) = \|\nu(x) - (\nu(x), \mu(x))\mu(x)\| = \left(\sum_{k=0}^{n} [\nu_k(x) - (\nu(x), \mu(x))\mu_k(x)]^2\right)^{1/2}. \]

From the definitions of \( \mu(x), \nu(x), \) and \( \lambda(x) \), it follows that

\[ \sum_{k=0}^{n} \lambda_k(x)^2 = \sum_{k=0}^{n} \left( \frac{\nu_k(x) - (\nu(x), \mu(x))\mu_k(x)}{\tau_n(x)} \right)^2 = 1 \]

and

\[ \sum_{k=0}^{n} \lambda_k(x)\mu_k(x) = \sum_{k=0}^{n} \left( \frac{\nu_k(x) - (\nu(x), \mu(x))\mu_k(x)}{\tau_n(x)} \right)\mu_k(x) = 0. \]

By the above calculations, along with the first equation in \((2.5)\), we find that the vectors \( \mu(x) \) and \( \lambda(x) \) satisfy condition \((1.10)\) of Theorem 1.1.

Let us also write

\[ \tau_n(x) = (1 - 2(\nu(x), \mu(x))^2 + (\nu(x), \mu(x))^2)^{1/2} \]

\[ = \sqrt{1 - \frac{K_n^{(0,1)}(x, x)^2}{K_n(x, x)K_n^{(1,1)}(x, x)}} = \frac{K_n(x, x)K_n^{(1,1)}(x, x) - K_n^{(0,1)}(x, x)^2}{K_n(x, x)K_n^{(1,1)}(x, x)} \]
and

\begin{equation}
\lambda_k(x) = \frac{\nu_k(x) - (\nu(x), \mu(x))\mu_k(x)}{\tau_n(x)}
\end{equation}

\begin{equation}
= \frac{f_k'(x)}{K_n^{(1,1)}(x,x)^{1/2}} - \frac{h_k'(x)}{(K_n(x,x)K_n^{(1,1)}(x,x))^{1/2}} - \frac{K_n(0,1)(x,x)^{1/2}}{K_n(x,x)^{1/2}}
\end{equation}

\sqrt{\frac{K_n(x,x) - K_n(0,1)(x,x)^2}{K_n(x,x)^2}}

\begin{equation}
= \frac{K_n(x,x)f_k'(x) - K_n(0,1)(x,x)f_k(x)}{[K_n(x,x)(K_n^{(1,1)}(x,x)K_n(x,x) - K_n(0,1)(x,x)^2)]^{1/2}}.
\end{equation}

Let \( \hat{D}_n(\xi, \eta; x) \) be the joint distribution density of \( g_n(x) \) and \( h_n(x) \). By another change of variables we have

\[
\hat{D}_n(\xi, \eta; x) = \frac{1}{\tau_n(x)} \hat{D}_n\left(\xi, \eta - (\nu(x), \mu(x))\xi; x\right)
\]

so that now the transformation of the Kac–Rice formula (2.4) reduces to

\begin{equation}
\rho_n(x) = \sqrt{\frac{K_n^{(1,1)}(x,x)}{K_n(x,x)}} \tau_n(x) \int_\mathbb{R} |\eta| \hat{D}_n(0, \eta; x) d\eta.
\end{equation}

Furthermore, since

\[
\sqrt{\frac{K_n^{(1,1)}(x,x)}{K_n(x,x)}} \tau_n(x) = \sqrt{\frac{K_n^{(1,1)}(x,x)K_n(x,x) - K_n(0,1)(x,x)^2}{K_n(x,x)^2}} = K_n(x),
\]

we have completed the proof of the theorem.

2.2. Proof of Corollary 1.2. Assume that the common characteristic function for the i.i.d. random variables \{\eta_k\} is \( \phi(s) = \exp(-as^2) \), where \( a \in (0, \infty) \) is a fixed number. Note that in this case, the variance of the i.i.d. random variables \{\eta_k\} is \( 2a \). This assumption yields

\[
\mathbb{E}[P_n(x)^2] = 2aK_n(x, x), \quad \mathbb{E}[P_n(x)P'_n(x)] = 2aK_n(0,1)(x, x), \quad \mathbb{E}[P'_n(x)^2] = 2aK_n^{(1,1)}(x, x).
\]

Thus in the formula for \( K_n(x) \) in (1.7), this extra factor for the variance cancels out algebraically.

For \( \hat{D}_n(\xi, \eta; x) \) being the joint distribution function of \( g_n(x) \) and \( h_n(x) \) given by the first equation in (2.2) and (2.6) respectively, the characteristic function is

\[
\Phi_n(\alpha, \beta) = \int_\mathbb{R} \int_\mathbb{R} \hat{D}_n(\xi, \eta; x)e^{i\alpha \xi + i\beta \eta} d\xi d\eta.
\]

Let \( \omega_k = \alpha \mu_k(x) + \beta \lambda_k(x) \), \( k \in \{0, 1, \ldots, n\} \). By (1.10), \( \sum_{j=0}^{n} \omega_j^2 = \alpha^2 + \beta^2 = \gamma^2 \) with \( \gamma = (\alpha, \beta) \). Using the above assumption that \( \phi \) is a scaled mean
zero Gaussian, and the assumption that the random variables are i.i.d., we see that
\[ \Phi_n(\gamma) = \prod_{k=0}^{n} \phi(\omega_k) = \exp\left(-\sum_{k=0}^{n} a \omega_k^2\right) = \exp(-a|\gamma|^2). \]

From Theorem 1.1 to get the desired equality it suffices to compute
\[ \int_{\mathbb{R}} |\eta| \hat{D}_n(0, \eta; x) d\eta, \quad \text{where} \quad \hat{D}_n(0, \eta; x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp(-i\beta\eta) \Phi_n(\gamma) d\gamma. \]

To this end, observe that
\[ \hat{D}_n(0, \eta; x) = \frac{1}{4a\pi} \exp\left(-\eta^2 \frac{4a}{4a}\right). \]

Therefore
\[ \int_{\mathbb{R}} |\eta| \hat{D}_n(0, \eta; x) d\eta = \frac{1}{4a\pi} \int_{\mathbb{R}} |\eta| \exp\left(-\eta^2 \frac{4a}{4a}\right) d\eta = \frac{1}{\pi}, \]
which gives the desired result.

### 2.3. Proof of Theorem 1.3

As in the proof of Theorem 1.2, since the random variables \( \{\eta_k\} \) are independent and
\[ g_n(x) = \sum_{k=0}^{n} \mu_k(x) \eta_k \quad \text{and} \quad h_n(x) = \sum_{k=0}^{n} \lambda_k(x) \eta_k, \]
the joint distribution function \( \hat{D}_n(\xi, \eta; x) \) of \( g_n \) and \( h_n \) has characteristic function satisfying
\[ (2.13) \quad \Phi_n(\gamma) = \Phi_n(\alpha, \beta) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{D}_n(\xi, \eta; x)e^{i\alpha\xi + i\beta\eta} d\xi d\eta = \prod_{k=0}^{n} \phi(\omega_k), \]
where \( \gamma = (\alpha, \beta) \) and \( \omega_k = \mu_k(x)\alpha + \lambda_k(x)\beta, \ k = 0, \ldots, n. \)

We now present our version of [4, Lemma 4.1]. The main differences in the proof below are that we do not require \( \omega_k^2 = \mathcal{O}(n^{-1/2} |\gamma|^2) \), and we give a slightly different partition of the index set \( \{k : k = 0, 1, \ldots, n\} \) for the product (2.13).

**Lemma 2.1.** If \( \phi(s) \) satisfies (1.12), then
\[ (2.14) \quad |\Phi_n(\gamma)| \leq \frac{1}{(1 + a_0|\gamma|^2)^L} \]
with \( a_0 = a/T \) and \( L = Tq \), where \( 2 \leq T \leq n + 1 \); here \( L \) is the number of partitions of the index set \( \{0, 1, \ldots, n\} \) constructed in the proof.
Proof. Using (1.12) and (2.13) we have

\begin{equation}
|\Phi_n(\gamma)| \leq \prod_{k=0}^{n} \frac{1}{(1 + a\omega_k^2)^q}.
\end{equation}

Since (1.10) and the definition of \(\omega_k\) give \(\sum_{k=0}^{n} \omega_k^2 = \alpha^2 + \beta^2 = |\gamma|^2\), we partition \(\{0, 1, \ldots, n\}\) into \(T\) groups, with \(2 \leq T \leq n + 1\), which we call \(M_j\), to have on each group

\[\sum_{k \in M_j} \omega_k^2 \geq \frac{1}{T}|\gamma|^2.\]

The partition and estimate yield

\[1 + \frac{a}{T}|\gamma|^2 \leq 1 + a \sum_{k \in M_j} \omega_k^2 \leq \prod_{k \in M_j} (1 + a\omega_k^2).\]

Thus using the above bound and (2.15), we have

\begin{equation}
|\Phi_n(\gamma)| \leq \prod_{j=1}^{T} \prod_{k \in M_j} \frac{1}{(1 + a\omega_k^2)^q} \leq \prod_{j=1}^{T} \frac{1}{(1 + \frac{a}{T}|\gamma|^2)^q} = \frac{1}{(1 + \frac{a}{T}|\gamma|^2)^{Tq}},
\end{equation}

where \(a_0 = a/T\) and \(L = qT\). □

Our next lemmas are modifications of [4, Lemmas 4.2 and 4.3]. The modifications allow us to keep track of all constants. The first lemma is given in [4, (4.15)].

**Lemma 2.2 (Bleher and Di [4, Lemma 4.2]).** If \(\phi(s)\) satisfies (1.12) and (1.13), then

\begin{equation}
\left|\frac{\partial \Phi_n(\gamma)}{\partial \beta}\right| \leq \frac{C_2|\gamma|}{(1 + a_0|\gamma|^2)L}.
\end{equation}

**Lemma 2.3.** If \(\phi(s)\) satisfies (1.12) and (1.13), then

\begin{equation}
\left|\frac{\partial^3 \Phi_n(\gamma)}{\partial \beta^3}\right| \leq \frac{C_3}{(1 + a_0|\gamma|^2)L} + \frac{2C_2^2|\gamma|}{(1 + a_0|\gamma|^2)L}.
\end{equation}

Proof. Observe that

\[
\frac{\partial^3 \Phi_n(\gamma)}{\partial \beta^3} = \frac{\partial^3}{\partial \beta^3} \prod_{k=0}^{n} \phi(\omega_k)
\]

\[
= \sum_{k=0}^{n} \lambda_k^3 \phi'''(\omega_k) \prod_{l \neq k} \phi(\omega_l) + \sum_{k=0}^{n} \sum_{i \neq k} \lambda_i^2 \lambda_k \phi''(\omega_i) \phi'(\omega_k) \prod_{l \neq i, k} \phi(\omega_l)
\]

\[
+ \sum_{k=0}^{n} \sum_{i \neq k} \lambda_i \lambda_k^2 \phi'(\omega_i) \phi''(\omega_k) \prod_{l \neq i, k} \phi(\omega_l).
\]
As in the proof of Lemma 2.1 it follows that
\[ \left| \prod_{l \neq k} \phi(\omega_l) \right|, \left| \prod_{l \neq i,k} \phi(\omega_l) \right| \leq \frac{1}{(1 + a_0 |\gamma|^2)^L}. \]

Using the above and the estimate on \( \phi'''' \) in (1.13) we achieve
\[ \left| \sum_{k=0}^{n} \lambda_k^3 \phi''''(\omega_k) \prod_{l \neq k} \phi(\omega_l) \right| \leq \frac{C_3}{(1 + a_0 |\gamma|^2)^L} \sum_{k=0}^{n} |\lambda_k^3| \]
\[ \leq \frac{C_3}{(1 + a_0 |\gamma|^2)^L} \left( \sum_{k=0}^{n} \lambda_k^2 \right)^2 = \frac{C_3}{(1 + a_0 |\gamma|^2)^L}, \]
where we have appealed to Cauchy–Schwarz in the second inequality and to the fact that the \( \lambda_k \)'s are normalized in the last equality.

To complete the estimates, let us first note that since \( E[\eta_k] = 0 \) for \( k = 0, \ldots, n \), the characteristic function \( \phi(s) \) satisfies \( \phi'(0) = 0 \). Hence under the assumption of (1.13), for \( s \in \mathbb{R} \) we have \( |d\phi(s)/ds| \leq C_2 |s| \). With this in mind, estimating as previously gives
\[ \left| \sum_{k=0}^{n} \sum_{i \neq k} \lambda_i^2 \lambda_k \phi''''(\omega_i) \phi'(\omega_k) \prod_{l \neq i,k} \phi(\omega_l) \right| \]
\[ \leq \frac{C_2^2}{(1 + a_0 |\gamma|^2)^L} \sum_{k=0}^{n} \sum_{i \neq k} |\lambda_i^2 \lambda_k \omega_k| \leq \frac{C_2^2}{(1 + a_0 |\gamma|^2)^L} \sum_{k=0}^{n} |\lambda_k \omega_k| \sum_{i=0}^{n} \lambda_i^2 \]
\[ \leq \frac{C_2^2 |\gamma|}{(1 + a_0 |\gamma|^2)^L}. \]
The estimate for (2.18) is done similarly and has the same bound. Combining (2.19) and twice that of (2.20) gives (2.17).

**Lemma 2.4.** If \( \phi(s) \) satisfies (1.12) and (1.13), then
\[ |\mathcal{D}_n(0, \eta; x)| \leq \frac{K_1}{(1 + |\eta|)}, \quad K_1 = \frac{1}{2 \pi a q} + \frac{C_2}{(2a q)^{3/2}}, \]
\[ |\mathcal{D}_n(0, \eta; x)| \leq \frac{K_2}{(1 + |\eta|^3)}, \quad K_2 = \frac{1}{2 \pi a q} + \frac{C_2^2}{\sqrt{2}(a q)^{3/2}} + \frac{C_3}{2 \pi a q}. \]

**Proof.** Since
\[ \Phi_n(\gamma) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{D}_n(\xi, \eta; x) e^{i\alpha \xi + i\beta \eta} d\xi d\eta, \]
differentiating and using Fourier inversion gives
\[ \eta^k \mathcal{D}_n(0, \eta; x) = \frac{(-i)^k}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\beta \gamma} \frac{\partial^k \Phi_n(\gamma)}{\partial \beta^k} d\gamma. \]
Using the above with Lemmas 2.1 and 2.2 yields

\[ (1 + |\eta|)|\hat{D}_n(0, \eta; x)| \leq \frac{1}{(2\pi)^2} \int \Phi_n(\gamma) d\gamma + \frac{1}{(2\pi)^2} \int \left| \frac{\partial \Phi_n(\gamma)}{\partial \beta} \right| d\gamma \]

\[ \leq \frac{1}{(2\pi)^2} \int \frac{1}{(1 + a_0|\gamma|^2)L} d\gamma \]

\[ + \frac{C_2}{(2\pi)^2} \int \frac{|\gamma|}{(1 + a_0|\gamma|^2)L} d\gamma \]

\[ = \frac{1}{4\pi a_0(L - 1)} + \frac{C_2 \Gamma(L - 3/2)}{8\pi^{1/2}a_0^{3/2} \Gamma(L)}, \]

where \( \Gamma \) is the usual Gamma function.

Similarly with the aid of Lemma 2.3, we get

\[ (1 + |\eta|^3)|\hat{D}_n(0, \eta; x)| \]

\[ \leq \frac{1}{(2\pi)^2} \int \Phi_n(\gamma) d\gamma + \frac{1}{(2\pi)^2} \int \left| \frac{\partial^3 \Phi_n(\gamma)}{\partial \beta^3} \right| d\gamma \]

\[ \leq \frac{1}{(2\pi)^2} \int \frac{1}{(1 + a_0|\gamma|^2)L} d\gamma + \frac{2C_2^2}{(2\pi)^2} \int \frac{|\gamma|}{(1 + a_0|\gamma|^2)L} d\gamma \]

\[ + \frac{C_3}{(2\pi)^2} \int \frac{1}{(1 + a_0|\gamma|^2)L} d\gamma \]

\[ = \frac{1}{4\pi a_0(L - 1)} + \frac{2C_2 \Gamma(L - 3/2)}{4\pi^{1/2}a_0^{3/2} \Gamma(L)} + \frac{C_3}{4a_0\pi(L - 1)}. \]

To complete the estimates, note that

\[ \frac{1}{4\pi a_0(L - 1)} = \frac{T}{4\pi a(Tq - 1)} = \frac{Tq}{Tq - 1} \leq \frac{1}{2\pi aq}, \]

since the function \( x/(x - 1) \) is decreasing and \( Tq \geq 2 \) given that \( T \geq 2 \) along with the assumption that \( q \geq 1 \). This estimate also gives

\[ \frac{C_3}{4a_0\pi(L - 1)} \leq \frac{C_3}{2aq\pi}. \]

Also observe

\[ \frac{\Gamma(L - 3/2)}{\pi^{1/2}a_0^{3/2} \Gamma(L)} = \frac{T^{3/2}\Gamma(Tq - 3/2)}{\pi^{1/2}a^{3/2} \Gamma(Tq)} = \frac{(Tq)^{3/2}\Gamma(Tq - 3/2)}{\pi^{1/2}(aq)^{3/2} \Gamma(Tq)} \]

\[ \leq \frac{2^{3/2}}{(aq)^{3/2}}, \]
because the function $x^{3/2} \Gamma(x - 3/2)/\Gamma(x)$ is decreasing on $[2, \infty)$, and has value $2\sqrt{2\pi}$ at $x = 2$.

Combining (2.25)–(2.27) with (2.23) and (2.24) completes the estimates needed for (2.21) and (2.22).

We are now finally ready to give the proof of the main theorem.

**Proof of Theorem 1.3** From Theorem 1.1 we have

$$\rho_n(x) = K_n(x) \int_{\mathbb{R}} |\eta| \hat{D}_n(0, \eta; x) d\eta.$$  

Using Lemma 2.4 yields

$$\int_{\mathbb{R}} |\eta| \hat{D}_n(0, \eta; x) d\eta = \left( \int_{-1}^{1} + \int_{|\eta| \geq 1} \right) |\eta| \hat{D}_n(0, \eta; x) d\eta$$

$$\leq K_1 \int_{-1}^{1} \frac{|\eta|}{1 + |\eta|} d\eta + K_2 \int_{|\eta| \geq 1} \frac{|\eta|}{1 + |\eta|^3} d\eta$$

$$= 2K_1(1 - \log 2) + \frac{2}{9}K_2(\sqrt{3} \pi + \log 8)$$

$$= \frac{1}{aq}(k_1 + C_3k_2) + \frac{1}{(aq)^{3/2}}[C_2(k_3 + C_2k_4)],$$

where

$$k_1 = \frac{1}{\pi} \left( 1 - \log 2 + \frac{\pi\sqrt{3} + \log 8}{9} \right) = 0.36367 \ldots,$$

$$k_2 = \frac{1}{9\pi}(\pi\sqrt{3} + \log 8) = 0.265995 \ldots,$$

$$k_3 = \frac{1}{\sqrt{2}}(1 - \log 2) = 0.216978 \ldots,$$

$$k_4 = \frac{\sqrt{2}}{9}(\pi\sqrt{3} + \log 8) = 1.18179 \ldots,$$

and this completes the proof.

**2.4. Proof of Theorem 1.5** For the Bergman polynomials $p_k(z) = \sqrt{(k+1)/\pi} z^k$ observe that

(2.28)  

$$K_n(z, w) = \sum_{k=0}^{n} p_k(z)p_k(w) = \frac{1}{\pi} \sum_{k=0}^{n} (k + 1)(z\bar{w})^k$$

$$= \frac{1 - (z\bar{w})^{n+1} - (2 - z\bar{w})}{\pi(1 - z\bar{w})^2} - \frac{n(z\bar{w})^{n+1}}{\pi(1 - z\bar{w})}.$$
Thus for $x \in \mathbb{R} \setminus \{\pm 1\}$, as $n \to \infty$ locally uniformly on the respective domains we have

\begin{equation}
K_n(x, x) = \begin{cases} 
\frac{1}{\pi(1 - x^2)^2} + o(1), & |x| < 1, \\
\frac{n x^{2n+2}}{\pi(x^2 - 1)}(1 + o(1)), & |x| > 1.
\end{cases}
\end{equation}

Taking derivatives of (2.28) and then evaluating on the diagonal to form the other needed kernels $K_n^{(0,1)}(x, x)$ and $K_n^{(1,1)}(x, x)$, after algebraic simplification one sees that

\begin{equation}
K_n(x, x)K_n^{(1,1)}(x, x) - K_n^{(0,1)}(x, x)^2
= \begin{cases} 
\frac{2}{\pi^2(1 - x^2)^6} + o(1), & |x| < 1, \\
\frac{n^2 x^{4n+4}}{\pi^2(x^2 - 1)^4}(1 + o(1)), & |x| > 1.
\end{cases}
\end{equation}

Therefore, combining (2.29) and (2.30), locally uniformly for $x \in \mathbb{R} \setminus \{\pm 1\}$ we have

\[
\lim_{n \to \infty} K_n(x) = \lim_{n \to \infty} \sqrt{\frac{K_n(x, x)K_n^{(1,1)}(x, x) - K_n^{(0,1)}(x, x)^2}{K_n(x, x)^2}}
= \begin{cases} 
\sqrt{\frac{2}{1 - x^2}}, & |x| < 1, \\
\frac{1}{x^2 - 1}, & |x| > 1.
\end{cases}
\]

When $x = \pm 1$, by using summation formulas the kernels can be evaluated directly to give

$$K_n(x, x) = \frac{(n + 2)(n + 1)}{2},$$
$$K_n^{(0,1)}(x, x)^2 = \left(\frac{n(n + 1)(n + 2)}{3}\right)^2,$$
$$K_n^{(1,1)}(x, x) = \frac{n(n + 1)(n + 2)(3n + 1)}{12}.$$  

Hence

$$K_n(x) = \frac{1}{3} \sqrt{\frac{n(n + 3)}{2}} \quad \text{for } x = \pm 1.$$ 

**Acknowledgements.** The author would like to thank his Ph.D. advisor Igor Pritsker for helpful conversations concerning this project, and he acknowledges financial support from the Vaughn Foundation on behalf of Anthony Kable.
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