A least-squares finite element approach to model fluid-structure interaction problems

Solveigh Averweg1,*, Alexander Schwarz1, Carina Nisters1, and Jörg Schröder1

1 University Duisburg-Essen, Institute of Mechanics, Univeritätstraße 15, 45141 Essen, Germany

In this contribution an approach to model fluid-structure interaction (FSI) problems with monolithic coupling is presented. The fluid as well as the structural domain are discretized using the least-squares finite element method (LSFEM), whose application results in a minimization problem with symmetric positive definite systems also for non self-adjoint operators, see e.g. [2]. In this study, the second-order systems are reduced to first-order systems by introducing new variables, which leads to least-squares formulations for both domains based on the stresses and velocities as presented in e.g. [5] and [7]. A conforming discretization of the unknown fields in $H^1$ and $H(div)$ using Lagrange interpolation polynomials and vector-valued Raviart-Thomas interpolations functions, respectively, leads to the inherent fulfillment of the FSI coupling conditions. In more detail, a discretization in $H^1$ ensures continuity of the velocity field and a discretization in $H(div)$ results in continuity of the normal stress components at the interface.

© 2019 The Authors Proceedings in Applied Mathematics & Mechanics published by Wiley-VCH Verlag GmbH & Co. KGaA Weinheim

1 Introduction

The simulation of fluid-structure interaction problems is of high interest in a wide range of fields, such as biomechanics and aerodynamics. One of the main challenging tasks herein is the fulfillment of the coupling conditions at the interface of the fluid and the solid domain. In the following, we investigate a FSI approach based on least-squares (LS) formulations in terms of stresses and velocities for both domains, with inherently fulfilled coupling conditions due to an appropriate choice of interpolation functions for the unknowns. The governing equations are based on the incompressible Navier-Stokes equations in an Arbitrary-Lagrangian-Eulerian (ALE) framework for the fluid domain and on linear elastodynamics for the solid domain.

2 Theoretical Framework

The considered least-squares stress-velocity formulation for linear elastodynamics based on the balance of momentum and the constitutive relation for linear elasticity is given with the positive weighting factors $\omega_{11}$ and $\omega_{22}$ as

$$\mathcal{F}^S(\sigma, v) = \frac{1}{2} \left( \left\| \omega_{11} (\text{div} \sigma + f - \rho_s a(v)) \right\|_{L^2(\Omega_s)}^2 + \left\| \omega_{22} (\mathbb{Q}^{-1} : \sigma - \nabla^T u(v)) \right\|_{L^2(\Omega_s)}^2 \right).$$

(1)

with the Cauchy stresses $\sigma$, the accelerations $a(v)$, the displacements $u(v)$, the velocities $v$, the body force $f$, the solid density $\rho_s$ and the linear elasticity tensor $\mathbb{Q} = \lambda \mathbb{I} \otimes 1 + 2\mu \mathbb{I}$, with the Lamé constants $\lambda$ and $\mu$. More details on the constructed LS formulation can be found e.g. in [7].

For the description of the time-dependent flow of an incompressible fluid, a first-order system can be constructed based on the Navier-Stokes equations by introducing the Cauchy stresses $\sigma = 2\rho_f \nu_f \nabla^* v - p \mathbb{I}$. This results in three residual terms

$$R^F_1 := \rho_f a(v) - \text{div} \sigma + \rho_f \nabla v \cdot v,$$

$$R^F_2 := \text{dev} \sigma - 2\rho_f \nu_f \nabla^* v,$$

$$R^F_3 := \text{div} v,$$

(2)

with the pressure $p$, the fluid density $\rho_f$ and the kinematic viscosity $\nu_f$. For more details, see, e.g., [1], [5] and [6]. To consider the deformation of the fluid domain, an Arbitrary-Lagrangian-Eulerian framework is used by introducing a mesh velocity $\bar{v}$, which is determined with a linear elasticity approach and subtracted from the fluid velocity. Thus, the first residual term transforms to $R^F_1 := \rho_f a(v) - \text{div} \sigma + \rho_f \nabla v \cdot (v - \bar{v})$. The LS functional for fluid dynamics is then constructed with help of the quadratic $L^2$-norm of the given residual terms as $\mathcal{F}^F(\sigma, v) = \frac{1}{2} \left( \sum_{i=1}^3 \left\| \omega_{fi}(R^F_i) \right\|_{L^2(\Omega_f)}^2 \right)$, with the weighting factors $\omega_{fi}, i \in (1, 3)$. The solution of a coupled problem requires the fulfillment of the interface conditions, which is the equality of the tractions and the velocities on the boundary of the domains $\sigma_f \cdot n = \sigma_s \cdot n$ and $v_f = v_s$ on $\Gamma_i$, with $n$ denoting the outward normal vector.

2.1 Discretization in Space and Time

For an inherent fulfillment of the coupling conditions we choose conforming approximation functions for the velocities and stresses with

$$V_h = \{ v \in H^1(\Omega) : v|_{\Omega_t} \in P_k(\Omega_t)^2 \forall \Omega_t \}, \quad W_h = \{ \sigma \in H(div, \Omega) : \sigma|_{\Omega_s} \in RT_{m}(\Omega_s)^2 \forall \Omega_s \},$$

where $P_k(\Omega_t)^2$ and $RT_{m}(\Omega_s)^2$ denote Lagrangian polynomials and vector-valued Raviart-Thomas functions, respectively. For the interpolation orders herein we select $m = 2$ and $k = 3$, leading to the mixed finite element description $RT_k P_3$.

* Corresponding author: e-mail solveigh averweg@uni-due.de, phone +00 49 201 183 2683, fax +00 49 201 183 2680
For the time discretization the Houbolt method is applied for both domains. The approximations for the accelerations and displacements depending on the actual velocities and on known values for the displacements of previous steps are given by

\[
\begin{align*}
\mathbf{u}_{n+1} &= \frac{6}{11} \mathbf{v}_{n+1} \Delta t + \frac{18}{11} \mathbf{u}_n - \frac{9}{11} \mathbf{u}_{n-1} + \frac{2}{11} \mathbf{u}_{n-2}, \\
\mathbf{a}_{n+1} &= \frac{1}{\Delta t^2} \left[ \frac{12}{11} \mathbf{v}_{n+1} + \frac{19}{11} \mathbf{u}_n + \frac{26}{11} \mathbf{u}_{n-1} - \frac{7}{11} \mathbf{u}_{n-2} \right].
\end{align*}
\]

3 Numerical Example

For the validation of the presented formulations we investigate the flow over an elastic wall (FOW). The geometry, boundary conditions and total velocity field for the steady state at \( T = 60 \text{s} \) is illustrated in Fig. 1 and the corresponding model parameters are specified in Tab. 1.

![Fig. 1: Geometry, boundary conditions and velocity field](image)

The inflow velocity is increased up to a time of \( T = 20 \text{s} \) and hold, while the simulation is continued until the field is fully developed at approximately \( T = 40 \text{s} \). We investigate different mesh sizes with a number of elements from \( n_{\text{ele}} = 104 \) to \( n_{\text{ele}} = 2600 \) as well as time increments \( \Delta t \) and find convergent results. As an example, the displacement \( u_1 \) at \((3.2,2)\)m has been measured over time and is presented in Fig. 2.

![Fig. 2: Displacement \( u_1 \) at \((3.2,2)\)m over time in s for different element sizes and different time step sizes](image)

All calculations have been carried out using the AceGen and AceFEM software packages (version 6.815), see [3] and [4], embedded in Mathematica (version 11.1).

4 Conclusion

We present LS finite element formulations in terms of stresses and velocities for fluid dynamics and elastodynamics where the choice of Raviart-Thomas and Lagrangian interpolation functions leads to an inherent fulfillment of the FSI coupling conditions. The fluid formulation is given in an ALE framework to consider the deformation of the fluid domain. The flow over an elastic wall has been investigated in order to validate the formulations and check for the correct interface treatment.

Acknowledgements: The authors gratefully acknowledge the DFG (SCHW 1355/3-1 and SCHR570/31-1) for the financial support and Jože Korelc for the development and ongoing support when using AceGen and AceFEM.

References

[1] S. Averweg, A. Schwarz, C. Nisters, and J. Schröder. Implicit time discretization schemes for least-squares finite element formulations to model incompressible flows. Proc. Appl. Math. Mech., 18:e201800166 (2018).
[2] P.B. Bochev and M.D. Gunzburger. Least-Squares Finite Element Methods, 1st edn. Springer, New York (2009).
[3] J. Korelc. Automatic generation of finite-element code by simultaneous optimization of expressions. Theor. Comput. Sci., 187, 231-248 (1997).
[4] J. Korelc. Multi-language and Multi-environment Generation of Nonlinear Finite Element Codes. Eng. Comput., 18, 312-327 (2002).
[5] C. Nisters, and A. Schwarz. Efficient stress-velocity least-squares finite element formulations for the incompressible Navier-Stokes equations. Comput. Methods in Appl. Mech. Eng., 141, 333-359 (1997).
[6] C. Nisters, A. Schwarz, S. Averweg, J. Schröder. Remarks on a Fluid-Structure Interaction Scheme Based on the Least-Squares Finite Element Method at Small Strains. Adv. Mech. Materials and Structural Analysis, 261-279 (Springer, 2018).
[7] C. Nisters, A. Schwarz, K. Steeger, and J. Schröder. A stress-velocity least-squares mixed finite element formulation for incompressible elastodynamics. Proc. Appl. Math. Mech., 15, 217-218 (2015).
[8] A. Schwarz, J. Schröder, S. Serdas, S. Turek, A. Ouazzi, and M. Nickaen. Performance aspects of a mixed s-v LSFEM for the incompressible Navier-Stokes equations with improved mass conservation. Proc. Appl. Math. Mech. 13, 97-98 (2013)

© 2019 The Authors Proceedings in Applied Mathematics & Mechanics published by Wiley-VCH Verlag GmbH & Co. KGaA Weinheim www.gamm-proceedings.com