Knots and Numbers in $\phi^4$ Theory to 7 Loops and Beyond$^*$

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We evaluate all the primitive divergences contributing to the 7–loop $\beta$–function of $\phi^4$ theory, i.e. all 59 diagrams that are free of subdivergences and hence give scheme–independent contributions. Guided by the association of diagrams with knots, we obtain analytical results for 56 diagrams. The remaining three diagrams, associated with the knots $10_124$, $10_139$, and $10_152$, are evaluated numerically, to 10 sf. Only one satellite knot with 11 crossings is encountered and the transcendental number associated with it is found. Thus we achieve an analytical result for the 6–loop contributions, and a numerical result at 7 loops that is accurate to one part in $10^{11}$. The series of ‘zig–zag’ counterterms, \{(6\zeta_3, 20\zeta_5, 4418\zeta_7, 168\zeta_9, \ldots)\}, previously known for $n = 3, 4, 5, 6$ loops, is evaluated to 10 loops, corresponding to 17 crossings, revealing that the $n$–loop zig–zag term is $4C_{n-1} \sum_{p>0} \frac{(-1)^p p^{n-p}}{p^{2n-3}}$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ are the Catalan numbers, familiar in knot theory. The investigations reported here entailed intensive use of REDUCE, to generate $O(10^4)$ lines of code for multiple precision FORTRAN computations, enabled by Bailey’s MPFUN routines, running for $O(10^3)$ CPUhours on DecAlpha machines.

1. Introduction
At the AI–92 and AI–93 workshops, I reported\cite{1} on a wide variety of multi–loop results. Here, at AI–95, I shall confine myself to a single topic, namely the very recent results obtained by Dirk Kreimer and myself, at 7 loops, and beyond, by combining knot theory with state–of–the–art symbolic and numerical calculation. I refer you to the talks of my other collaborators – Pavel Baikov, Jochem Fleischer, John Gracey, Andrey Grozin, Oleg Tarasov – for reports of progress in other areas, and to the talk by Dirk Kreimer, for discussion of the knot theory used here.

The investigation concerns the primitive divergences contributing to the $\beta$–function of $\phi^4$ theory, i.e. the counterterms that arise from diagrams that are free of subdivergences and hence make scheme–independent contributions to the $\beta$–function. A famous series of such diagrams is the ‘zig–zag’ series, beginning with

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$Z(3) = 6\zeta_3$
$Z(4) = 20\zeta_5$
$Z(5) = \frac{441}{8}\zeta_7$
$Z(6) = 168\zeta_9$

To find the corresponding counterterms, one nullifies the 4 external momenta, and cuts the horizontal line, to obtain planar two–point diagrams, whose values (with unit external momentum, and loop–measure $d^4k/\pi^2$) yield the indicated counterterms. The first two terms are trivial to obtain: $Z(3) = W(3)$, $Z(4) = W(4)$, where $W(n) = nC_{n-1}\zeta_{2n-3}$ is the value of any diagram obtained by cutting the $n$–loop vacuum diagram consisting of a wheel with $n$ spokes, and $C_n = \frac{1}{2n+1}\binom{2n}{n}$ are the Catalan numbers. Thereafter, the rational numbers in the zig–zag series differ from those in the ladder series. The 5–loop result was obtained in and verified in. The zig–zag diagrams, at 3, 4, and 5 loops, account for 44%, 46%, and 47% of the terms in the MS $\beta$–function, first calculated in and corrected in. Moreover, they account for 100%, 100%, and 79% of the scheme–independent, primitive contributions. Hence there has long been an interest in obtaining the $n$–loop result, which we shall infer from numerical results, to $n = 10$ loops. We also compute all the primitive terms in the $\beta$–function, to $n = 7$ loops, and compare them with expectations based on knot theory.

2. Expectations from Knot Theory

The recent association of knots with transcendental numbers, in the counterterms of field theory, does not (yet, at least) allow us to obtain the value of a diagram directly from the associated knot (or knots). It does, however, provide a powerful guide to the transcendentals that are expected to occur, with rational coefficients, in the counterterm obtained from any diagram that is free of subdivergences. (The restriction to such primitive diagrams means that we do not need to specify a scheme.) As explained in Dirk Kreimer’s talk, one can turn any $\phi^3$ diagram into a link diagram, to which a skein relation is applied, yielding knots that are closures of positive braids. There are very few such knots. Ignoring factor knots, only 10 knots can be generated by diagrams with up to 7 loops. Of these, the 8 knots with up to 10 crossings are to be found in standard tables, where they are known as $3_1$, $5_1$, $7_1$, $8_{19}$, $9_1$, $10_{124}$, $10_{139}$, $10_{152}$, with the first number indicating the number of crossings. We denote the two positive–braid 11–crossing knots as $11_1$ and $11_{353}$, for reasons that will become apparent when we identify the numbers that they entail.

With the exceptions of the satellite knots $10_{139}$, $10_{152}$, and $11_{353}$, all positive knots with up to 11 crossings are torus knots. The torus knot $(p, q) = (q, p)$, with $p$ and $q$ having no common factor, is formed by a closed loop on a torus, winding round one axis $p$ times, while it winds round the other $q$ times, before rejoining itself. The $n$–loop ladder diagram yields the torus knot $(2n - 3, 2)$, with $2n - 3$ crossings, thereby giving us the entries $N_1 \sim \zeta_N$ in the knot–to–number dictionary, with
$N = 2n - 3$ crossings at the $n$–loop level. Considering $\phi^4$ theory to be obtained from a $\phi^2 \sigma$ coupling, with a non–propagating $\sigma \sim \phi^3$, one concludes\[^\text{12}\] that the $n$–loop zig–zag counterterm will be a rational multiple of $\zeta_{2n-3}$.

There are two more torus knots encountered at up to 7 loops: $8_{19} = (4, 3)$ and $10_{124} = (5, 3)$. (With $\{(N, 2) | N = 3, 5, 7, 9, 11\}$, they exhaust the mutually prime pairs $(p, q)$ such that the number of crossing, $pq - \max(p, q)$, does not exceed 11.) We shall identify the two double–sums associated with these two non–zeta knots.

Exact expressions for the numbers associated with $10_{139}$ and $10_{152}$ are not yet available; we shall be content with numerical evaluation of three diagrams whose skeinings yield combinations of the three positive 10–crossing knots, $10_{124}$, $10_{139}$ and $10_{152}$. We show that the triple–sum $F_{353} \equiv \sum_{l > m > n > 0} l^{-3}m^{-5}n^{-3}$ is associated with $11_{353}$, and obtain analytical results for all the diagrams that produce it.

Our methods involve high–precision numerical evaluations, yielding numbers for which we seek analytical fits in knot–theoretically prescribed search spaces. These search spaces correspond to rational combinations of the transcendentals associated with knots obtained by skeining\[^\text{13}\] the link diagrams that encode the topology of the intertwining of loop momenta in the counterterm diagrams.

### 3. Non–zeta Numbers and their Knots

With such a plethora of diagrams to consider, it is convenient to represent primitive counterterms by angular diagrams\[^\text{14}\] such as

$$G(n_1, n_2, n_3) \quad M(a_1, \ldots, n_4) \quad C(n_1, \ldots, n_5) \quad D(n_1, \ldots, n_5)$$

where, for example, the arguments of $G(n_1, n_2, n_3)$ specify the number of dots to be placed on the three lines, and all dots are connected to an origin (which is not shown). Then one uses the Gegenbauer expansion for the massless propagator with $L$ dots, which is known in closed form\[^\text{15}\] for arbitrary $L$. The radial integrations can be done analytically. (We used REDUCE\[^\text{16}\].) The angular integrations yield, in the case of the first topology, the triangle function $\Delta(l, m, n)$, which is 1, or 0, according as $(l + m + n + 1)/2$ is, or is not, an integer greater than all of the Gegenbauer expansion indices, $\{l, m, n\}$, introduced in expanding the three ‘dotted’ propagators. For this topology, we obtain a finite number of infinite triple–sums:

$$G(\alpha, \beta, \gamma) = \sum_{i, j, k} \frac{(2\alpha - i)}{\alpha} \frac{(2\beta - j)}{\beta} \frac{(2\gamma - k)}{\gamma} \frac{(i + j + k)!}{i!j!k!} \sum_{l, m, n} \frac{\Delta(l, m, n)}{l^{2\alpha - i}m^{2\beta - j}n^{2\gamma - k}}$$

$$\times \left[ \frac{2}{l + m + n - 1} \right]^{i+j+k+1} + \left( \frac{2}{l + m + n + 1} \right)^{i+j+k+1}.$$  

Manipulating the sums\[^\text{17}\] $F_{ab} \equiv \sum_{l > m > 0} l^{-a}m^{-b}$, $F_{abc} \equiv \sum_{l > m > n > 0} l^{-a}m^{-b}n^{-c}$, we
were able to obtain analytical results for all diagrams of type G, up to 7 loops, in terms of zetas and a single non-zeta level–11 transcendental

\[ K_{353} \equiv \frac{G(3,2,0) - 583\zeta_{11}}{40} = \frac{3}{5}(F_{353} - \zeta_3 F_{53}) + \frac{69}{40}\zeta_{11} - 583\zeta_3^2 = 0.2019352393 \]

associated with a unique, 11–crossing, positive–4–braid, satellite knot, which we consequently denote as 11_{353}. In knot theory, it is identified by the braid word \( \sigma_1^2\sigma_2^3\sigma_1\sigma_3\sigma_2^3\). A REDUCE program, running for 2 days on a DecAlpha, established its uniqueness, by computation of \( 3^{11} = 177147 \) Jones polynomial.

To find the new transcendentals entailed by the non–zeta torus knots \( 8_{19} = (4,3) \) and \( 10_{124} = (5,3) \), we studied the 4–fold sums generated by angular diagrams of type M, evaluating all such diagrams, up to 7 loops. Thanks to powerful methods of accelerated convergence, multiple–precision FORTRAN, and efficient integer–relation search routines, it was easy to find, at 15 sf, and to verify, up to 45 sf, the analytical form of the sole level–8 non–zeta transcendental

\[ K_{53} \equiv 7\zeta_5\zeta_3 - \frac{1}{39} M(2,1,1,0) = \frac{1}{5}(29\zeta_8 - 12F_{53}) = 5.733150251 \]

whose appearance we associate with the knot \( 8_{19} = (4,3) \). (The probability of an accidental fit is of order \( 10^{-30} \).) This is the only non–zeta number appearing in the expansion, to level 9, of the master two–loop diagram \( 18_{19} \), where it enters via

\[ U_{6,2} \equiv \sum_{n>m>0} \frac{(-1)^{n-m}}{n^m m^2} = \frac{277}{266} \zeta_8 - \frac{3}{10} K_{53} = -0.01476857328. \]

At level 10, we obtained the knot \( 10_{124} = (5,3) \) from skeining \( M(2,1,1,1) \), and found, at high precision, a fit to this diagram in the search space \( \{F_{73}, \zeta_{10}, \zeta_5^2, \zeta_3, \zeta_7, K_{53}, \zeta_9, K_{73}, \zeta_{11}, K_{353}\} \). So as to simplify the appearance of other results that involve \( 10_{124} \), we find it convenient to express the result as follows:

\[ K_{73} \equiv \frac{90\zeta_5^2 - 21\zeta_7\zeta_3 - \frac{1}{39} M(2,1,1,1)}{7} = \frac{94F_{73} + 982\zeta_5^2 - 793\zeta_{10}}{28} = 9.388141968. \]

We thus identify \( \{\zeta_3, \zeta_5, \zeta_7, K_{53}, \zeta_9, K_{73}, \zeta_{11}, K_{353}\} \) (and their products, corresponding to factor knots) as the only transcendentals that may appear in primitive diagrams whose skeinings do not involve the satellite knots \( 10_{139} \) and \( 10_{152} \). By methods too laborious to recount here, we succeeded in evaluating all primitive diagrams of types M, C and D, up to 7 loops, to 50 sf, and (to our great delight) found integer relations, of the expected form, i.e. with no trace of \( 10_{139} \) or \( 10_{152} \). (The probability of accidental fitting is typically \( 10^{-20} \), since the search routines need less than 30 sf, to produce results which hold to 50 sf.)

Our tables of new analytical results are too long to reproduce here. They have a feature that is notable: every planar diagram, and every diagram with 4–point (or lower) vertices (whether planar or non–planar), is a rational combination of transcendentals at the same level. Only for diagrams that are non–planar and also contain 5–point (or higher) vertices do we (sometimes) observe level–mixing. For
example, the planar diagrams \( G(3, 2, 0), G(4, 1, 0), C(3, 0, 0, 0, 1), C(1, 0, 2, 0, 1), D(3, 0, 0, 0, 1), D(1, 0, 2, 0, 1) \) are pure level–11, whilst \( G(2, 2, 1) \) and \( G(3, 1, 1) \) are non–planar diagrams, with 6–point vertices, and mix levels 6 through 10.

### 4. Primitive \( \phi^4 \) Counterterms, to 7 Loops and Beyond

Encouraged by the foregoing, we sought to evaluate all the primitive \( \phi^4 \) counterterms, up to 7 loops, and the zig–zag counterterms to 10 loops. This entails more complex angular diagrams, from which REDUCE generates summands, thousands of lines long, processed by MPFUN routines that evaluate sums over as many as 8 integers, with angular factors that may involve squares of 6–j symbols.

Thanks to about \( 10^3 \) CPU hours on a pair of DecAlphas, and a commensurate human effort, to develop methods of accelerated convergence, we succeeded in finding high–precision, knot–theory inspired, analytical fits to all but 3 of the 59 diagrams with up to 7 loops (generated automatically, by a purpose–built Wick–contracting program). At 3 loops, there is one diagram, \( Z(3) = W(3) \). At 4 loops, there is one diagram, \( Z(4) = W(4) \). At 5 loops, there are 3 diagrams, reduced by conformal transformation to 2 distinct values: \( Z(5) = \frac{63}{56} W(5) \) and \( M(1, 1, 1, 0) = [W(3)]^2 \). At 6 loops, there are 10 diagrams, reduced by conformal transformations to 5 distinct values: \( Z(6) = \frac{4}{7} W(6), W(3)W(4), M(2, 1, 1, 0), M(1, 1, 1, 1) = 12W(3)W(4) - 16M(2, 1, 1, 0), \) and \( D(2, 0, 0, 0, 1) = \frac{1063}{119} \zeta_9 + 8\zeta_3^3 \).

For the coefficients, \( \beta_n^{NS} \), of the primitive (i.e. no–subdivergence) terms in \( \beta(g) = \mu^2 \frac{dg}{d\mu^2} = \sum \beta_n (-g)^{n+1} \), with an interaction \( H_{int} = (4\pi)^2 g \phi^4 / 4! \), we obtain \( \beta_3^{NS} = 6\zeta_3, \beta_4^{NS} = 60\zeta_5, \beta_5^{NS} = \frac{1322}{3} \zeta_7 + 126\zeta_3^3 \), and the new 6–loop result

\[
\beta_6^{NS} = \frac{23056}{3} \zeta_9 + 384\zeta_3^3 + 4512\zeta_5 \zeta_3 - 336K_{53} = 12\,065.365\,126\,645.
\]

At 7 loops, there are 44 diagrams, reduced to 12 distinct values, by intensive use of conformal transformations, and re–interpretation of \( x \)–space variables as \( p \)–space variables, which transforms a planar diagram into one whose lines cross those of the original. The weights and values of the 12 numbers in \( \beta_7^{NS} = \sum_n w_n B_n \) are

\[
\begin{array}{cccc}
n & w_n & B_n & \end{array}
\]

\[
\begin{array}{cccc}
1 & \frac{8}{3} & 216\zeta_3^3 & \\
2 & 18 & 400\zeta_5^2 & \\
3 & 12 & \frac{3375}{64} \zeta_{11} & \\
4 & 108 & \frac{4805}{64} \zeta_{11} + 120\zeta_5 \zeta_3^2 + 85K_{353} & \\
5 & 81 & 420\zeta_7 \zeta_3 - 200\zeta_5^2 & \\
6 & 27 & 183032 420 030 498 901 717 011 912 3(1) & \\
7 & 108 & \frac{67925}{192} \zeta_{11} + 20\zeta_5 \zeta_3^2 - 15K_{353} & \\
8 & 108 & \frac{15001}{56} \zeta_{11} - 28\zeta_5 \zeta_3^2 - 48K_{353} & \\
9 & 18 & 216.919 375 55(6) & \\
10 & 30 & 450\zeta_7^2 - 189\zeta_7 \zeta_3 & \\
11 & 54 & \frac{1323}{137} \zeta_7 \zeta_3 & \\
12 & 12 & 200.357 566 43(2) & \\
\end{array}
\]

where \( B_{1,2,11} \) are factor knots; \( B_3 = Z(7) \) is the 7–loop zig–zag; \( B_{4,7,8} \) entail \( 11_{353} \).
$B_{5,10}$ are combinations of 10–crossing factor knots; $B_{6,9,12}$ entail $10_{124}$, $10_{139}$, $10_{152}$.

We regard the fit between knots and numbers, in 59 $\phi^4$ counterterms, to 7 loops, as strongly indicative of the kinship of knot theory and field theory. We lack only a single, 6–element, integer–relation, between $\{B_6, B_9, B_{12}\}$ and $\{K_{73}, \zeta_3, \zeta_2\}$, which the accuracy achieved for the fearsome 8–fold sums in $B_{9,12}$ is insufficient to reveal. We have found such relations for diagrams with 5– and 6–point vertices.

From $Z(7) = \frac{1073}{1792}W(7)$ and $Z(8) = \frac{1}{2}W(8)$, we infer the complete zig–zag series

$$Z(n) = 4C_{n-1} \sum_{p=1}^{\infty} \frac{(-1)^{pm-n}}{p^{2n-3}} = \left\{ \begin{array}{ll}
4C_{n-1} \zeta_{2n-3}, & \text{for even } n, \\
(4 - 4^{3-n})C_{n-1} \zeta_{2n-3}, & \text{for odd } n,
\end{array} \right.$$

which we have verified, to high precision, up to $n = 10$ loops. (An all–order proof is lacking, as yet.) This series gives a convergent contribution to the $\beta$–function, and hence is of decreasing importance at higher orders, where we observe the growth

$$\frac{\beta_5^{NS}}{\beta_4^{NS}} = 13.647456527; \quad \frac{\beta_6^{NS}}{\beta_5^{NS}} = 14.209833853; \quad \frac{\beta_7^{NS}}{\beta_6^{NS}} = 15.371460754$$

which we shall compare with asymptotic expectations in a more detailed paper.

I conclude by thanking David Bailey and Andrey Grozin for their generous help, and by noting, ruefully, my failure to confute Dirk Kreimer’s exciting new ideas.

References

1. D. J. Broadhurst, in New Computing Techniques in Physics Research II, ed. D. Perret–Gallix (World Scientific, Singapore, 1992) 579.
2. D. J. Broadhurst, in New Computing Techniques in Physics Research III, ed. K.–H. Becks, D. Perret–Gallix (World Scientific, Singapore, 1994) 511.
3. D. J. Broadhurst, Phys. Lett. B164 (1985) 356.
4. D. J. Broadhurst, Phys. Lett. B307 (1993) 132.
5. V. V. Belokurov, N. I. Ussyukina, J. Phys. A16 (1983) 2811.
6. N. I. Ussyukina, A. I. Davydychev, Phys. Lett. B305 (1993) 136; B298 (1993) 363.
7. D. I. Kazakov, Phys. Lett. B133 (1983) 406; TM $\Phi_58$ (1984) 343.
8. D. J. Broadhurst, Open University report OUT–4102–18 (1985).
9. N. I. Ussyukina, TM$\Phi$ 88 (1991) 14; Phys. Lett. B267 (1991) 382.
10. K. G. Chetyrkin, S. G. Gorishny, S. A. Larin, F. V. Tkachov, Phys. Lett. B132 (1983) 351; Moscow preprint INR P–0453 (1986).
11. H. Kleinert, J. Neu, V. Schulte–Frohlinde, K. G. Chetyrkin, S. A. Larin, Phys. Lett. B272 (1991) 39; B319 (1993) 545 (erratum).
12. D. Kreimer, UTAS–PHYS–94–25; UTAS–PHYS–95–10; UTAS–PHYS–95–11.
13. V. F. R. Jones, Annals of Math. 126 (1987) 335.
14. D. Rolfsen, Knots and Links (Publish or Perish, Berkeley, 1976).
15. K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, Nucl. Phys. B174 (1980) 345.
16. A. C. Hearn, REDUCE User’s Manual, Version 3.5, Rand publication CP78 (1993).
17. D. H. Bailey, NASA technical report RNR–90–022; RNR–91–025; RNR–94–013.
18. D. J. Broadhurst, Z. Phys. C32 (1986) 249.
19. D. T. Barfoot, D. J. Broadhurst, Z. Phys. C41 (1988) 81.
20. B. G. Nickel, J. Math. Phys. 19 (1978) 542.
21. L. N. Lipatov, JETP 72 (1977) 411.
22. D. I. Kazakov, D. V. Shirkov, O. V. Tarasov, TM$\Phi$ 38 (1979) 15.