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Checking Identifiability of Covariance Parameters in Linear Mixed Effects Models

Supplementary Material

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1. Histograms of $\hat{\sigma}$ and $\hat{\rho}$ (MLE)
Figure 1. Maximum likelihood estimated covariance parameters in model A. Vertical lines indicate the true parameter values: $\sigma = \sqrt{0.6} \approx 0.77$, and $\rho = 0.4$.

Figure 2. Maximum likelihood estimated covariance parameters in model B. Vertical lines indicate the true parameter values: $\sigma = \sqrt{0.6} \approx 0.77$, and $\rho = 0.5$.

2. Histograms of $\hat{\sigma}$ and $\hat{\rho}$ (REML)
Figure 3. REML estimated covariance parameters in model A. Vertical lines indicate the true parameter values: \( \sigma = \sqrt{0.6} \approx 0.77 \), and \( \rho = 0.4 \).

Figure 4. REML estimated covariance parameters in model B. Vertical lines indicate the true parameter values: \( \sigma = \sqrt{0.6} \approx 0.77 \), and \( \rho = 0.5 \).

3. Coverage plots of \( \rho \)
Figure 5. Coverage plot of the parameter $\rho$ in model A. True value is $\rho = 0.4$.

Figure 6. Coverage plot of the parameter $\rho$ in model B. True value is $\rho = 0.5$.

4. Proofs

In our study, a matrix $\Sigma$ with parameter $\theta$ is obtained through the map $\Sigma = \sum_{k=1}^{m} \theta_k E_{I_k}$ with the domain $\Theta = \{\theta : \Sigma \in S^m_{12}\}$. We first show that the parameter space $\Theta$ is open in $\mathbb{R}^m$. This result helps to establish the necessary conditions of model identifiability.
in the consequent results. The openness is immediate to visualize for some structured $\Sigma$, e.g. an MI-$\Sigma$. For other structures such as UN or Toeplitz, the verification is not straightforward. Although it is well known that $S^n_+$ is an open subset of $S^n$ [1, page 55], the set $S^n_+$ itself is too general for our discussion since each $\Sigma_j$ is parameterized by $\theta_j$ and $\theta = (\theta'_1 \cdots \theta'_J)'$ is the target of the study. In general, we have the following result. By the lemma, each parameter space $\Theta_j$ is open in $R^{m_j}$. Hence, the space $\Theta = \Theta_1 \times \cdots \times \Theta_J$ of $\theta$ is open in $R^{\sum_j m_j}$.

**Lemma 4.1** $\Theta = \{ \theta : \Sigma \in S^n_+ \}$ is an open set of $R^n$.

**Proof of Lemma 4.1:**
We consider the Euclidean norm $\| \cdot \|$ of $\theta$. Let $\theta$ be an arbitrary point of $\Theta$. We define a ball centered at $\theta$ with radius $r$ as $B(\theta, r) = \{ x : \| \theta - x \| < r \}$. We need to show that given $\theta$, there exists an $r$ such that $B(\theta, r) \subset \Theta$.

Since $\theta \in \Theta$, all the leading principal minors of $\Sigma$ are positive. For a given positive leading principal minor, $\theta$ lies in an open set (not necessarily $\Theta$). There is a radius such that a ball with this radius centered at $\theta$ is in this open set. Considering all the leading principal minors together, there exists a minimum radius enabling a ball to be in all these open sets. That is, every point of $\Theta$ is an interior point.

**Proof of Theorem 3.1:**
We prove that the model is not identifiable if and only if elements of $F$ are linearly dependent. Let $c_j$ be an arbitrary vector of length $m_j$ with elements $c_{jk}$, $k = 1, \ldots, m_j$. Let $c = (c'_1 \cdots c'_J)' \neq 0$. Suppose $\sum_{j=1}^J Z_j \left( \sum_{k=1}^{m_j} c_{jk} E_{jk} \right) Z'_j = 0$. Linear dependence of elements of $F$ is equivalent to $c \neq 0$. Define $\theta^* = \theta + c$. It is clear that both $\theta^*$ and $\theta$ produce the same $\Sigma_y$. Let $d$ be a constant and $\theta^* = \theta + dc$. By Lemma 4.1, $\theta^* \in \Theta$ for small enough $|d|$.

**Proof of Corollary 3.1:**
We have $\text{vec}\Sigma_y = V\theta$. Define $\theta^* = \theta + c$ where $c$ is arbitrary. Suppose $\text{vec}\Sigma_y = V\theta^*$. Then, $V$ of full column rank is equivalent to $c = 0$. For small enough $|d|$, $\theta^* = \theta + dc \in \Theta$ by Lemma 4.1.

**Proof of Theorem 3.2:**
Proof of the theorem is very similar to the proof of Corollary 3.1, and is omitted.

**References**

[1] A. Berman. *Cones, Matrices and Mathematical Programming*. Springer-Verlag Berlin, 1973.