Average Size of a Suffix Tree for Markov Sources

Philippe Jacquet\textsuperscript{1}, Wojciech Szpankowski\textsuperscript{2}\textsuperscript{†}

\textsuperscript{1}Bell Labs, Alcatel-Lucent, France.
\textsuperscript{2}Department of Computer Science, Purdue University, USA

We study a suffix tree built from a sequence generated by a Markovian source. Such sources are more realistic probabilistic models for text generation, data compression, molecular applications, and so forth. We prove that the average size of such a suffix tree is asymptotically equivalent to the average size of a trie built over \( n \) independent sequences from the same Markovian source. This equivalence is only known for memoryless sources. We then derive a formula for the size of a trie under Markovian model to complete the analysis for suffix trees. We accomplish our goal by applying some novel techniques of analytic combinatorics on words also known as analytic pattern matching.

Keywords: Suffix tree, Markov sources, digital trees, size, pattern matching, number of occurrences.

1 Introduction

Suffix trees are the most popular data structures on words. They find myriad of applications in computer science and telecommunications, most notably in algorithms on strings, data compressions (Lempel-Ziv’77 scheme), and codes. Despite this, little is still known about their typical behaviors for general probabilistic models (see [5, 1, 3]).

A suffix tree is a trie (a digital tree; see [9]) built from the suffixes of a single string. In Figure 1 we show the suffix tree constructed for the first four suffixes of the string \( X = 0101101110 \). More precisely, we actually build a suffix tree on the first \( n \) infinite suffixes of a string \( X \) as shown in Figure 1. We shall call it simply a suffix tree which we study in this paper. Such a tree consists of internal (branching) nodes and external node storing the suffixes. Our goal is to analyze the number of internal nodes called also the size of a suffix tree built from a sequence \( X \) generated by a Markov source. We accomplish it by employing powerful techniques of analytic combinatorics on words known also as analytic pattern matching [9].

In recent years there has been a resurgence of interest in algorithmic and combinatorial problems on words due to a number of novel applications in computer science, telecommunications, and most notably in molecular biology. A few possible applications are listed below. The reader is referred to our recent

\textsuperscript{†}W. Szpankowski is also with the Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology, Poland. His work was supported by NSF Center for Science of Information (CSoI) Grant CCF-0939370, and in addition by NSF Grants CCF-1524312, and NIH Grant 1U01CA198941-01, and the NCN grant, grant UMO-2013/09/B/ST6/02258.
book [9] for more details. In computer science and molecular biology many algorithms depend on a solution to the following problem: given a word $X$ and a set of arbitrary $b + 1$ suffixes $S_1, \ldots, S_{b+1}$ of $X$, what is the longest common prefix of these suffixes. In coding theory (e.g., prefix codes) one asks for the shortest prefix of a suffix $S_i$ which is not a prefix of any other suffixes $S_j, 1 \leq j \leq n$ of a given sequence $X$ (cf. [14]). In data compression schemes, the following problem is of prime interest: for a given “data base” sequence of length $n$, find the longest prefix of the $(n+1)$st suffix $S_{n+1}$ which is not a prefix of any other suffixes $S_i (1 \leq i \leq n)$ of the data base sequence. And last but not least, in molecular sequences comparison (e.g., finding homology between DNA sequences), one may search for the longest run of a given motif, a unique sequence, the longest alignment, and the number of common subwords [9]. These, and several other problems on words, can be efficiently solved and analyzed by a clever manipulation of a data structure known as a suffix tree. In literature other names have been also coined for this structure, and among these we mention here position trees, subword trees, directed acyclic graphs, etc.

The extension of suffix tree analysis to Markov sources is quite significant, especially when the suffix tree is used for natural languages. Indeed, Markov sources of finite memory approximate very well realistic texts. For example, the following quote is generated by a memoryless source with the letter statistic of the Declaration of Independence:

```
esdehTe,a; psseCed vcenseuirh vra f uetaiapgnuev n cosb mgffgfl itbahhr nijue n S uee,ru
s,k smodpztrnno.eeteespfgeetmtr i aur oiyr
```

which should be compared to the following quote generated by a Markov source of order 3 trained on the same text:

```
We hat Government of Governments long that their right of abuses are these rights, it, and or
themselves and are disposed according Men, der.
```
In this paper we analyze the average number of internal nodes (size) of a suffix tree built from \(n\) (infinite) suffixes of a string generated by a Markov source with positive transition probabilities. We first prove in Theorem 1 that the average size of a suffix tree under Markovian model is asymptotically equivalent to the size of a trie that is built from \(n\) independently generated strings, each string emitted by the corresponding Markovian source. To accomplish this, we study another quantity, namely the number of occurrences of a given pattern \(w\) in a string of length \(n\) generated by a Markovian source. We use its properties to establish our asymptotic equivalence between suffix trees and tries. Finally, we compare the average size of suffix trees to trie size under Markovian model (see Theorem 2), which – to the best of our knowledge – is only partially known [2].

In fact, there is extensive literature on tries [9] and very scarce one on suffix trees. An analysis of the depth in a Markovian trie has been presented earlier in [12]. A rigorous analysis of the depth of suffix tree was first presented in [5] for memoryless sources, and then extended in [3] to Markov sources. We should point out that depth grows like \(O(\log n)\) which makes the analysis manageable. In fact, height and fillup level for suffix tree – which are also of logarithmic growth – were analyzed in [15] (see also [1, 14]). But the average size grows like \(O(n)\) and is harder to study. For memoryless sources it was analyzed in [11] for tries and in [5] for suffix trees. We also know that some parameters of suffix trees (e.g., profile) cannot be inferred from tries, see [4]. Markov sources add additional level of complications in the analysis of suffix trees as well documented in [1]. In fact, the average size of tries under general dynamic sources was analyzed in [2], however, specifications to Markov sources requires extra care, especially for the so called rational Markov sources.

The proof of the convergence of the average size of the suffix tree to the average size of the trie borrows many fundamental elements of the depth analysis in [3], for example the term \(q_n(w)\) (see next section), but the extension of the depth analysis to the size analysis require the introduction of a new term \(d_n(w)\) which has non trivial properties. The analysis of average size of the trie in a Markovian model has been made by several author before but surprisingly we could not find a clear statement about the periodic case. This is the reason why we have to present a sketched proof here.

2 Main Results

We consider a stationary source generating a sequence of symbols drawn from a finite alphabet \(\mathcal{A}\).

We first derive a formula for the average size of a suffix tree in terms of the number of pattern occurrences. Let \(w\) be a word over \(\mathcal{A}\). We denote by \(O_n(w)\) the number of occurrences of word \(w\) in a sequence of length \(n\) generated by a Markov source with the transition matrix \(P\). We observe [5] that the average size \(s_n\) of a suffix tree built over a sequence of length \(n\) is

\[
s_n = \sum_{w \in \mathcal{A}^*} P(O_n(w) \geq 2).
\] (1)

In fact, (1) holds for any probabilistic source. We compare it to the average size \(t_n\) of trie built over \(n\) independent Markov sequences. If \(N_n(w)\) is the number of words which begin with \(w\) in a trie build with \(n\) words, we have

\[
t_n = \sum_{w \in \mathcal{A}^*} P(N_n(w) \geq 2).
\] (2)

Let \(P(w)\) be the probability of observing \(w\) in a Markov sequence, \(N_n(w)\) is a Bernoulli \((n, P(w))\) and
random variable $t_n$ can be written as

$$t_n = \sum_{w \in \mathcal{A}^*} 1 - (1 - P(w))^n - nP(w)(1 - P(w))^{n-1}. \quad (3)$$

We specifically consider a Markovian source. We assume that the source is stationary and ergodic. We will consider a Markovian process of order 1 with a positive transition matrix $P = [P(a|b)]_{a,b \in \mathcal{A}}$. Extensions to higher order Markov is possible since a Markovian source of order $r$ is simply a Markovian source of order 1 over the alphabet $\mathcal{A}^r$. Notice that contrary to previous analysis we don’t assume that $P(a|b) > 0$ for all $(a, b) \in \mathcal{A}^2$, since we allow that some transition may be forbidden and some other mandatory (while keeping the source ergodic).

Our main result of the paper is formulated next,

**Theorem 1** Consider a suffix tree built over $n$ suffixes of a sequence of length $n$ generated by a Markov source with a positive state transition matrix $P$. There exists $\varepsilon > 0$ such that

$$s_n - t_n = O(n^{1-\varepsilon}) \quad (4)$$

for large $n$.

In order to apply Theorem 1 one needs to estimate the average size of a trie under Markovian model. This seems to be unknown except for some general dynamic sources [2]. In fact, analysis of tries under Markovian sources is quite challenging (see [6]). But we can offer the following result for the average size of a trie under Markovian assumptions. A sketch of the proof is presented in Section 4.

**Theorem 2** Consider a trie built over $n$ independent sequences generated by a Markov source with positive transition probabilities. For $(a, b, c) \in \mathcal{A}^3$ define

$$\alpha_{abc} = \log \left[ \frac{P(a|b)P(c|a)}{P(c|b)} \right], \quad (5)$$

Then:

(aperiodic case) If not all $\{\alpha_{abc}\}$ are commensurable, then

$$t_n = \frac{n}{h} + o(n)$$

where $h = -\sum_{a,b} \pi_a P(b|a) \log P(b|a)$ is the entropy rate of the underlying Markov source with $\pi_a$, $a \in \mathcal{A}$, denoting the stationary probability.

(periodic case) If all $\{\alpha_{abc}\}$ are commensurable, then

$$t_n = \frac{n}{h}(1 + Q(n)) + O(n^{1-\varepsilon})$$

where $Q(n)$ is a periodic function and some $\varepsilon > 0$. 
Remark  We recall that a set of real numbers are commensurable (also known as “rationally related”) when their ratios are rational numbers. We observe that if for all \((a, b) \in \mathcal{A}^2\), the \(a_{abc}\) are commensurable for one \(c \in \mathcal{A}\), then \(a_{abc}\) are commensurable for all values of \(c\). Furthermore in the aperiodic case the \(o(n)\) term can have a growth rate arbitrary close to order \(n\), depending on source settings as shown in [7] in the memoryless case.

In the rest of this section, we present a road map of the proof of (4). For this we will make use of ordinary generating functions. Let \(w \in \mathcal{A}^k\) be a word of length \(k\). We also define \(N_0(z, w) = \sum_{n>0} P(O_n(w) = 0) z^n\) and \(N_1(z, w) = \sum_{n>0} P(O_n(w) = 1) z^n\) for \(z \in \mathbb{C}\). We know from [9] that

\[
N_0(z, w) = \frac{S_w(z)}{D_w(z)}, \quad N_1(z, w) = \frac{z^k P(w)}{D_w^2(z)},
\]

where \(S_w(z)\) is the autocorrelation polynomial of word \(w\) and \(D_w(z)\) is defined as follows

\[
D_w(z) = S_w(z)(1-z) + z^k P(w) (1 + F_w(z)(1-z)),
\]

The memoryless case considers \(F_w(z) = 0\). The addition of a non zero \(F_w(z)\) is a significant change from the analysis in the memoryless case. In fact it captures the correlations between characters in the sequence and leads to non trivial developments. Here \(F_w(z)\) for \(w \in \mathcal{A}^* - \{\varepsilon\}\) is a function that depends on the Markov parameters of the source. It also depends only on the first and last character of \(w\), say respectively \(a\) and \(b\) for \((a, b) \in \mathcal{A}^2\) as described below.

Let \(P\) be the transition matrix of the Markov source and \(\pi\) be its stationary vector with \(\pi_a\) its coefficient at symbol \(a \in \mathcal{A}\). The vector \(1\) is the vector with all coefficients equal to 1 and \(I\) is the identity matrix. Assuming that \(a \in \mathcal{A}\) (resp. \(b\)) is the first (resp. last) symbol of \(w\), we have [13, 9]

\[
F_w(z) = \frac{1}{\pi_a} \left[ (P - \pi \otimes 1)(I - z(P + \pi \otimes 1))^{-1} \right]_{b,a}
\]

where \([A]_{a,b}\) indicates the \((a, b)\) coefficient of the matrix \(A\), and \(\otimes\) represents the tensor product. An alternative way to express \(F_w(z)\) is

\[
F_w(z) = \frac{1}{\pi_a} \langle e_a(P - \pi \otimes 1)(I - z(P + \pi \otimes 1))^{-1} e_b \rangle
\]

where \(e_c\) for \(c \in \mathcal{A}\) is the vector with a 1 at the position corresponding to symbol \(c\) and all other coefficients are 0. Here \(\langle x, y \rangle\) represents the scalar product of \(x\) and \(y\).

Let us define two important quantities:

\[
d_n(w) = P(O_n(w) = 0) - (1 - P(w))^n, \\
g_n(w) = P(O_n(w) = 1) - n P(w)(1 - P(w))^{n-1},\]

and their corresponding generating functions

\[
\Delta_w(z) = \sum_{n>0} d_n(w) z^n, \\
Q_w(z) = \sum_{n>0} g_n(w) z^n.
\]
Observe that \( t_n - s_n = \sum_{w \in A^*} d_n(w) + q_n(w) \). Thus we need to estimate \( d_n(w) \) and \( q_n(w) \) for all \( w \in A^* \).

We denote \( B_k \) the set of words of length \( k \) that do not overlap with themselves over more than \( k/2 \) symbols (see [9, 5, 3] for more precise definition). To be precise \( w \in A^k - B_k \) if there exist \( j > k/2 \) and \( v \in A^j \) and \( (u_1, u_2) \in A^{k-j} \) such that \( w = u_1 v = vu_2 \). This set plays a fundamental role in the analysis and it is already proven in [3] that

\[
\sum_{w \in A^k - B_k} P(w) = O(\delta_1^k)
\]

where \( \delta_1 \) is the largest coefficient in the Markovian transition matrix \( P \). Since the authors of [3] only consider strictly positive matrix \( P \) we have \( \delta_1 < 1 \). Anyhow in the present paper we allow some coefficients to be equal to 1 or 0, as long the source is ergodic. Therefore \( \delta_1 \) may be equal to 1. To cope with this minor problem we define

\[
p = \exp \left( \limsup_{k, w \in A^k} \frac{\log P(w)}{k} \right)
\]

\[
q = \exp \left( \liminf_{k, w \in A^k, P(w) \neq 0} \frac{\log P(w)}{k} \right)
\]

These quantities exist and are smaller than 1 since \( A \) is a finite alphabet. From now we set \( \delta = \sqrt[p]{p} \) which replaces the parameter \( \delta_1 \) in the previous statements.

Now we are in the position to present two crucial lemmas, proved in the next section, from which Theorem 1 follows.

**Lemma 1** There exist \( \varepsilon < 1 \) such that \( \sum_{w \in A^*} q_n(w) = O(n^{\varepsilon}) \).

**Lemma 2** There exists a sequence \( R_n(w) \), for \( w \in A^* \) such for all \( 1 > \varepsilon > 0 \) we have

- (i) for \( w \in B_k \): \( d_n(w) = O((nP(w))^{\varepsilon}k\delta^k) + R_n(w) \);

- (ii) for \( w \in A^k - B_k \): \( d_n(w) = O((nP(w))^{\varepsilon}) + R_n(w) \),

where \( R_n(w) \) is such that \( \sum_{w \in A^*} R_n(w) = O(1) \).

**Remark:** The sequence \( d_n(w) \) is the main new element which makes the difference between the suffix tree depth analysis done in [3] and the suffix tree size analysis. The later was done in [9] for the memoryless case. The sequence \( R_n(w) \) reflects the impact of the Markovian source on the analysis in particular is a consequence of the introduction of a non zero function \( F_w(z) \).

**Proof of Theorem 1:** We already know via Lemma 1 that there exists \( \varepsilon < 1 \) such that \( \sum_{w \in A^*} q_n(w) = O(n^{\varepsilon}) \). Let now \( d_n^{(1)} = \sum_{k} \sum_{w \in B_k} (d_n(w) - R_n(w)) \) and since for all \( \varepsilon > 0 \) observe that

\[
d_n^{(1)} = \sum_{k} \sum_{w \in B_k} O(n^{\varepsilon}P^k(w)k\delta^k) = \sum_{k} O(n^{\varepsilon}k(p^\varepsilon \delta)^k),
\]
hence it converges for all $\varepsilon > 0$. Also let $d_n^{(2)} = \sum_k \sum_{w \in A^k - B_k} (d_n(w) - R_n(w))$. Observe that

$$d_n^{(2)} = \sum_k \sum_{w \in A^k - B_k} O(n^\varepsilon P^{\varepsilon-1}(w)P(w))$$

$$= \sum_k \sum_{w \in A^k - B_k} O(n^\varepsilon q^{(\varepsilon-1)k}P(w))$$

$$= \sum_k O(n^\varepsilon (\delta q^{\varepsilon-1})^k),$$

which converges for all $\varepsilon$ such that $\delta q^{\varepsilon-1} < 1$ (take $\varepsilon < 1$ close enough to 1) and is $O(n^\varepsilon)$. Finally $d_n^{(1)} + d_n^{(2)} + \sum_{w \in A^*} R_n(w)$ is also $O(n^\varepsilon)$ for $\varepsilon > 0$ since $\sum_{w \in A^*} R_n(w)$ is finitely bounded. This completes the proof of Theorem 1.

3 Proof of Lemmas

In this section we prove Lemma 1 and Lemma 2. In the proof of Lemma 1 we shall use some facts from [3], however, our proof follows the pattern matching approach developed in [9].

3.1 Proof of Lemma 1

The result is in fact already proven in [3]. Define

$$Q_w(z) = P(w) \left( \frac{z^k}{D_w^z(z)} - \frac{z}{(1 - (1 - P(w))z)^2} \right).$$

(9)

In [3] one defines $Q_n(1) = \frac{1}{n} \sum_{w \in A^*} q_n(w)$ and it is proven there that $Q_n(1) = O(n^{-\varepsilon})$ for some $\varepsilon > 0$.

3.2 Proof of Lemma 2

First we have the following simple lemma. The largest eigenvalue of $P$ is 1, let $\lambda_1, \lambda_2, \ldots$ be a sequence of other eigenvalues in the decreasing order of their modulus.

Lemma 3 Uniformly for all $w \in A^*$ we find $F_w(z) = O(\frac{1}{1 - |\lambda_1| z})$.

Proof: By the spectral representation of $P$ we know that $P = \pi \otimes 1 + \sum_{i > 0} \lambda_i u_i \otimes \zeta_i$, where $u_i$ (resp. $\zeta_i$) are the corresponding right (resp. left) eigenvectors. In fact we can introduce the matrices $D = \pi \otimes 1$ and $R = \sum_{i > 0} \lambda_i u_i \otimes \zeta_i$, whose spectral radius is $|\lambda_1|$ and satisfies the orthogonal property: $RD = DR = 0$.

We have $M(z) = P - \pi \otimes 1)(I - z(P + \pi \otimes 1))^{-1}$ we have $M(z) = R(1 - zR)^{-1}$. Since $R^k = O(|\lambda_1|^k R(I - zR)^{-1}$ is defined for all $z$ such that $|z| < \frac{1}{|\lambda_1|}$ and is $O(\frac{1}{1 - |\lambda_1| z})$, and so is $F_w(z) = [M(z)]_{a,b}$.

The next lemma is important.
Lemma 4 For \( z \) such that \( |\lambda_1 z| < 1 \) we have for all integers \( k \)

\[
\sum_{w \in \mathcal{A}^{k+1}} P(w)F_w(z) = O(\lambda_1^k).
\]  

(10)

Proof: The function \( F_w(z) \) depends only on the first and last symbol of \( w \). Considering a pair of symbols \( (a, b) \in \mathcal{A}^2 \) the sum of the probabilities of the words of length \( k+1 \) starting with \( a \) and ending with \( b \), \( \sum_{awb \in \mathcal{A}^{k+1}} P(w) \), equals \( \pi_a \langle e_b P^k e_a \rangle \). Easy algebra leads to

\[
\sum_{w \in \mathcal{A}^{k+1}} P(w)F_w(z) = \sum_{(a,b) \in \mathcal{A}^2} \langle e_a M(z)e_b \rangle \langle e_b P^k e_a \rangle.
\]

(11)

But since \( P^k = D + R^k \) and \( M(z)D = 0 \) and \( R^k = O(|\lambda_1|^k) \), we conclude the proof

\[ \square \]

We now follow a parallel approach to the approach developed in [3] and in [5, 9].

The generating function \( \Delta_w(z) = \sum_{n \geq 0} d_n(w)z^n \) becomes

\[
\Delta_w(z) = \frac{P(w)z}{1 - z} \left( \frac{1 + (1 - z)F_w(z)}{D_w(z)} - \frac{1}{1 - z + P(w)z} \right).
\]

(13)

We have

\[
d_n(w) = \frac{1}{2\pi i} \int_{|z| = \rho} \Delta_w(z) \frac{dz}{z^{n+1}},
\]

integrated on any loop encircling the origin in the definition domain of \( d_n(w) \). Extending the result in [5], the authors of [3] show that there exists \( \rho > 1 \) such that the function \( D_w(z) \) has a single root in the disk of radius \( \rho \). Let \( A_w \) be such a root. We have via the residue formula

\[
d_n(w) = \text{Res}(\Delta_w(z), A_w)A_w^{-n} - (1 - P(w))^n + d_n(w, \rho),
\]

(14)

where \( \text{Res}(f(z), A) \) denotes the residue of function \( f(z) \) on complex number \( A \) and

\[
d_n(w, \rho) = \frac{1}{2\pi i} \int_{|z| = \rho} \Delta_w(z) \frac{dz}{z^{n+1}}.
\]

(15)

We have

\[
\text{Res}(\Delta_w(z), A_w) = \frac{P(w)\left(1 + (1 - A_w)F_w(A_w)\right)}{(1 - A_w)C_w}
\]

(16)

where \( C_w = D'_w(A_w) \). But since \( D_w(A_w) = 0 \) we can write

\[
\text{Res}(\Delta_w(z), A_w) = -\frac{A_w^{-k}S_w(A_w)}{C_w}
\]

(17)
We now consider asymptotic expansion of $A_w$ and $C_w$ as it is described in [9], in Lemma 8.1.8 and Theorem 8.2.2. Although the expansions were presented for memoryless case, but for Markov source we simply replace $S_w(1)$ by $S_w(1) + P(w)F_w(1)$. We find
\begin{align}
A_w &= 1 + \frac{P(w)}{S_w(1)}
+ P(w)^2 \left( \frac{k - F_w(1)}{S_w(1)} - \frac{S_w'(1)}{S_w(1)} \right) + O(P(w)^3) \\
C_w &= -S_w(1) + P(w) \left( k - F_w(1) - 2 \frac{S_w'(1)}{S_w(1)} \right)
+ O(P(w)^2)
\end{align}

Notice that these expansions in the Markov model first appeared in [3].
From now follow the proof of Theorem 8.2.2 in [9]. We define the function
\[\delta_w(x) = \frac{A_w^{-k}S_w(A_w)}{C_w} A_w^{-x} - (1 - P(w))^x.\]

More precisely we define the function
\[\bar{\delta}_w(x) = \delta_w(x) - \delta_w(0)e^{-x}\]
which has a Mellin transform $\delta_w^*(s)\Gamma(s) = \int_0^\infty \bar{\delta}_w(x)x^{s-1}dx$ defined for all $\Re(s) \in (-1,0)$ with
\[\delta_w^*(s) = \frac{A_w^{-k}S_w(A_w)}{C_w} \left( (\log A_w)^{-s} - 1 \right) + 1 - \left( \log(1 - P(w)) \right)^{-s}.\]

When $w \in B_k$ with the expansion of $A_w$ and since $S_w(1) = 1 + O(\delta^k)$ and $S_w'(1) = O(k\delta^k)$, we find that similarly as shown in [9]
\[\delta_w^*(s) = O(|s|k\delta^k)P(w)^{1-s}.\]

Therefore, by the reverse Mellin transform, for all $1 > \varepsilon > 0$:
\[\bar{\delta}(n,w) = \frac{1}{2\pi i} \int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} \delta_w^*(s)\Gamma(s)n^{-s}ds = O(n^{1-\varepsilon}P(w)^{1-\varepsilon}k\delta^k)\]

When $w \in A^k - B_k$ we don’t have the $S_w(1) = 1 + O(\delta^k)$. But it is shown in [3] that there exists $\alpha > 0$ such that for all $w \in A^*$: $S_w(z) > \alpha$ for all $z$ such that $|z| \leq \rho$. Therefore we get
\[\bar{\delta}(n,w) = O(n^{1-\varepsilon}P(w)^{1-\varepsilon}).\]

We set
\[R_w(w) = d_w(0)e^{-n} + d_n(w,\rho).\]

We first investigate the quantity $d_w(0)$. We need to prove that $\sum_{w \in A^*} d_w(0)$ converges. For this, noticing that
\[S_w(A_w) = S_w(1) + \frac{P(w)}{S_w(1)} S_w'(1) + O(P(w)^2)\]
we obtain
\[ -\frac{A_w^{-k}S_w(A_w)}{C_w} = 1 - \frac{P(w)}{S_w(1)} \left( F_w(1) + \frac{S_w'(1)}{S_w(1)} \right) + O(P(w)^2). \] (24)

Thus
\[ d_w(0) = -\frac{P(w)}{S_w(1)} \left( F_w(1) + \frac{S_w'(1)}{S_w(1)} \right) + O(P(w)^2). \] (25)

Without the term \( F_w(1) \) we would have the same expression as in [9] whose sum over \( w \in A^* \) converges. Therefore we need to prove that the sum \( \sum_{w \in A^*} \frac{P(w)}{S_w(1)} F_w(1) \) converges. It is clear that the sum
\[ \sum_{w \in A^*} \frac{P(w)}{S_w(1)} F_w(1) \]
converges since
\[ \sum_{w \in A^*} P(w) = O(\delta^k) \]
and \( F_w(1) \) is uniformly bounded. Now we consider the other part
\[ \sum_{w \in A^*} \frac{P(w)}{S_w(1)} F_w(1). \]

We know that \( S_w(1) = 1 + O(\delta^k) \), therefore
\[ \sum_{w \in B_k} \frac{P(w)}{S_w(1)} F_w(1) = \sum_{w \in B_k} P(w) F_w(1) + O(\delta^k). \] (26)

But
\[ \sum_{w \in B_k} P(w) F_w(1) = \sum_{w \in A^*} P(w) F_w(1) + O(\delta^k), \]
and we know by Lemma 4 that \( \sum_{w \in A^*} P(w) F_w(1) = O(\lambda^k_1) \). Thus the sum \( \sum_{w \in B_k} \frac{P(w)}{S_w(1)} F_w(1) \) converges.

The second and last effort concentrates on the term \( d_n(w, \rho) \). We proceed as in the proof of Theorem 8.2.2 in [9]. We first have \( d_n(w, \rho) = O(P(w)\rho^{-n}) \) which is \( O(n^c P(w)^c) \) without any condition on \( w \).

The issue is now to work on \( w \in B_k \). In this case we have \( S_w(z) = 1 + O(\delta^k) \) and therefore
\[ d_n(w, \rho) = \frac{1}{2i\pi} \oint P(w) \left( \frac{1}{1 - z} \left( \frac{1}{D_w(z)} - \frac{1}{1 - z + zP(w)} \right) \right) \frac{dz}{z^{n+1}} \]
\[ + \frac{1}{2i\pi} \oint P(w) F_w(z) \frac{dz}{D_w(z) z^{n+1}}. \] (27)

We notice that the function
\[ \frac{P(w)}{1 - z} \left( \frac{1}{D_w(z)} - \frac{1}{1 - z + zP(w)} \right) \]
is \( O(P(w)\delta^k) + O(P(w)^2) \), therefore the first integral is \( O(P(w)\delta^k \rho^{-n}) \). The second function \( P(w) \frac{F_w(z)}{D_w(z)} \)
is equal to \( P(w) F_w(z) + O(P(w)\delta^k) \). We already know that \( \sum_{w \in B_k} P(w) F_w(z) = O(\lambda^k_1) \), thus the series converges and the lemma is proven.
4 Sketch of the Proof of Theorem 2

Let \( a \in \mathcal{A} \). We denote by \( t_{a,n} \) the average size of a trie over \( n \) independent Markovian sequences, all starting with the same symbol \( a \). Then for \( n \geq 2 \)

\[
t_n = 1 + \sum_{a \in \mathcal{A}} \sum_{k=0}^{n} \binom{n}{k} \pi_a^k (1 - \pi_a)^{n-k} t_{a,k},
\]

and similarly for \( b \in \mathcal{A} \)

\[
t_{n,b} = 1 + \sum_{a \in \mathcal{A}} \sum_{k=0}^{n} \binom{n}{k} P(a|b)^k (1 - P(a|b))^{n-k} t_{a,k},
\]

where we recall \( P(a|b) \) is the element of matrix \( P \). Let \( T(z) = \sum_n t_n \frac{z^n}{n!} e^{-z} \) and \( T_a(z) = \sum_n t_{a,n} \frac{z^n}{n!} e^{-z} \) be the familiar Poisson transforms. Using (28) and (29) we find

\[
T(z) = 1 - (1 + z) e^{-z} + \sum_{a \in \mathcal{A}} T_a(\pi_a z),
\]

\[
T_b(z) = 1 - (1 + z) e^{-z} + \sum_{a \in \mathcal{A}} T_a(P(a|b) z).
\]

Using dePoissonization arguments (see [8]) we shall obtain \( t_n = T(n) + O\left(\frac{1}{n} T(n)\right) \). Thus we need to study \( T(z) \) for large \( z \) in a cone around the real axis. For this we apply the Mellin transform that we describe next. In fact the convergence between the quantities \( t_n \) and \( T_n \) could also be derived by the application of the Rice method on the Mellin transform, since the latter as an explicit form.

Let now \( T(z) \) be the vector consisting of \( T_a(z) \) for every \( a \in \mathcal{A} \). It is not hard to see that its Mellin transform

\[
T^*(s) = \int_0^\infty T(z) z^{s-1} dz
\]

is defined for \(-1 > \Re(s) > -2\) (since \( T(z) = O(z^2) \) when \( z \to 0 \)), and

\[
T^*(s) = -(1 + s) \Gamma(s) \mathbf{1} + P(s) T^*(s)
\]

(32)

where \( P(s) \) is the matrix consisting of \( P(a|b)^{-s} \) if \( P(a|b) > 0 \) and 0 otherwise. This identity leads to

\[
T^*(s) = -(1 + s) \Gamma(s)(\mathbf{I} - P(s))^{-1} \mathbf{1}
\]

where \( \mathbf{I} \) is the identity matrix. Similarly the Mellin transform \( T^*(s) \) of \( T(z) \) satisfies

\[
T^*(s) = -(1 + s) \Gamma(s) + (\pi(s), T^*(s)).
\]

(33)

where \( \pi(s) \) is the vector composed of \( \pi_a^{-s} \).

The inverse Mellin transform of \( T^*(s) \) is defined as

\[
T(n) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} T^*(s)n^{-s} ds, \quad -1 > c > -2.
\]

(34)
In order to find asymptotic behavior of $T(z)$ as $z \to \infty$ we need to study the poles of $T^*(s)$ for $-2 < \Re(s)$. As discussed in [6, 9] this is equivalent to analyzing the poles of $T^*(s)$. Since $(1+s)\Gamma(s)$ has no pole on $-2 < \Re(s) < 0$ we must consider poles of $(I - P(s))^{-1}$. In other words (see [6, 9]) we need to find $s$ for which the eigenvalue of largest modulus $\lambda(s)$ of $P(s)$ is equal to 1. It is easy to see that $\lambda(-1) = 1$ since $P(-1) = P$. The residue at $s = -1$ of $n^{-s}(I - P(s))^{-1}1$ is equal to $\frac{2}{n}1$ where $h$ is the entropy rate of the Markovian source.

As explained in [6] in the periodic case there are multiple values of $s$ such that $\lambda(s) = 1$ and $\Re(s) = -1$. Since these poles are regularly spaced on the axis $\Re(s) = 0$, they contribute to the oscillating terms (function $Q$ in Theorem 2) in the asymptotic expansion of $t_n$. Furthermore, the location of zeros of $\lambda(s) = 1$ in the periodic case tells us that there exists $\varepsilon$ such that $(I - P(s))$ has no pole for $-1 < \Re(s) < -1 + \varepsilon$ leading to the error term $O(n^{1-\varepsilon})$.

In the aperiodic case there is only one pole on the line $\Re(s) = -1$, thus the oscillating term disappears. However, zeros of $\lambda(s) = 1$ can lie arbitrarily close to the line $\Re(s) = 1$, therefore the error term is just $o(n)$.

References

[1] P. Cénacl, B. Chauvin, F. Paccaut, and N. Pouyanne, Uncommon suffix tries, Random Structures & Algorithms, vol. 46, 117-141, 2015

[2] J. Clement, P. Flajolet, and B. Vallée, Dynamic Sources in Information Theory: A General Analysis of Trie Structures, Algorithmica, 29, 307-369, 2001.

[3] Fayolle, J., Ward, M. D. Analysis of the average depth in a suffix tree under a Markov model. In International Conference on Analysis of Algorithms DMTCS, proc. AD (Vol. 95, p. 104), 2005.

[4] J. Geithner and M. Ward, Variance of the Profile in Suffix Trees, submitted to AofA 2016.

[5] Jacquet, P., Szpankowski, W. Autocorrelation on words and its applications: analysis of suffix trees by string-ruler approach. J. Combinatorial Theory, Series A, 66(2), 237-269, 1994.

[6] P. Jacquet, W. Szpankowski, and J. Tang, Average Profile of the Lempel-Ziv Parsing Scheme for a Markovian Source, Algorithmica, 2002.

[7] Flajolet, P., Roux, M., & Valle, B., Digital trees and memoryless sources: from arithmetics to analysis. DMTCS Proceedings, (01), 2010.

[8] Jacquet, P., Szpankowski, W. (1998). Analytical depoissonization and its applications. Theoretical Computer Science, 201(1), 1-62.

[9] Jacquet, P., Szpankowski, W. Analytic Pattern Matching: From DNA to Twitter. Cambridge University Press, 2015.

[10] N. Merhav and W. Szpankowski, Average Redundancy of the Shannon Code for Markov Sources, IEEE Trans. Information Theory, 59, 7186-7193, 2013.

[11] M. Regnier, and P. Jacquet, New Results on the Size of Tries, IEEE Trans. Information Theory, 35, 203–205, 1989.
[12] Jacquet, P., Szpankowski, W. (1991). Analysis of digital tries with Markovian dependency. IEEE Transactions on Information Theory, 37(5), 1470-1475.

[13] M. Régnier, and W. Szpankowski, On Pattern Frequency Occurrences in a Markovian Sequence, Algorithmica, 22, 631–649, 1998.

[14] P. Shields, The Ergodic Theory of Discrete Sample Paths, American Mathematical Society, Providence, 1996.

[15] W. Szpankowski, A Generalized Suffix Tree and Its (Un)Expected Asymptotic Behaviors, SIAM J. Computing, 22, 1176–1198, 1993.