AN ELEMENTARY WAY TO INTRODUCE A PERRON-LIKE INTEGRAL

HANA BENDOVÁ AND JAN MALÝ

Abstract. We give an alternative definition of integral at the generality of the Perron integral and propose an exposition of the foundations of integral theory starting from this new definition. Both definition and proofs needed for the development are unexpectedly simple. We show how to adapt the definition to cover the multidimensional and Stieltjes case and prove that our integral is equivalent to the Henstock-Kurzweil-(Stieltjes) integral.

1. Introduction

The aim of this paper is to built self-contained foundations of the theory of non-absolutely convergent integral based on a new definition. Our definition is a slight modification of definitions used previously, but provides a possibility of a surprisingly comprehensible development of the theory.

We are focused on integrals which include the Lebesgue integral and integrate all derivatives. First such a construction was done by Denjoy [5] in 1912, shortly followed by Luzin [20]. The integral of Perron [23] from 1914 uses families of major and minor functions instead of a single antiderivative. A “weighted” analogue of the Perron integral is the Perron-Stieltjes integral introduced by Ward [30]. In 1957, Kurzweil [16] introduced a gauge generalized Riemann type integral, which is equivalent to the Perron integral. The same construction was found independently by Henstock [8], see also [9], [12]. The advantage of this construction is that it is based on Riemann sums which are commonly used to illustrate and motivate the concept of integral. There have been made serious attempts to build an elementary course of integration on basis of the Henstock-Kurzweil integral, e.g. [18]. A completely different idea of a curiously simple definition of integral in Perron-like generality is due to Tolstov [28].

We present a definition of integral which is also equivalent with Perron’s definition. An intermediate step between our integral and the Perron integral is the variational integral. This has been introduced by Henstock [7] and admits various formulations, see e.g. [10]. In [6], Definition 11.7, we may find a version which can be stated as follows:

A function $f : [a, b] \rightarrow \mathbb{R}$ is variational integrable if there exists a function $F : [a, b] \rightarrow \mathbb{R}$ with the following property: For each $\varepsilon > 0$ there exists an increasing function $\varphi_\varepsilon : [a, b] \rightarrow \mathbb{R}$ and a strictly positive function $\delta : [a, b] \rightarrow \mathbb{R}$ such that

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\[ \left[ \varphi_{\varepsilon} \right]_a^b < \varepsilon \] and for each \( x, y \in [a, b] \) we have
\[ |y - x| \leq \delta(x) \implies |F(y) - F(x) - f(x)(y - x)| \leq |\varphi_{\varepsilon}(y) - \varphi_{\varepsilon}(x)|. \]

The integral of \( f \) over \([a, b]\) is then \([F]_a^b\).

We simplify further the definition of variational integral: we use a single control function and replace the explicit description of \( \varepsilon-\delta \) dependence by an ordinary limit. This enables, among others, to use a language in which the definition looks almost like the ordinary definition of (anti)derivative and proofs of tools like integration by parts and change of variables are short and elegant.

The idea of a single control function appears in analysis also in other contexts: Cornea [3, 4] (see also [19]) uses a control function to modify Perron’s construction of solution of the Dirichlet problem in potential theory. Notice that there is an parallel with our construction; also in this case, the original Perron’s idea is based on upper and lower functions. Another relevant concept is that of delta-convex mappings. Originally, (scalar) delta-convex functions are differences of convex functions. Veselý and Zajíček [29] use control functions to generalize delta-convexity to the vector valued case.

Now, our definition is the following:

**Definition 1.** Let \( I = (a, b) \subset \mathbb{R} \) be an interval and \( f, F : I \to \mathbb{R} \) be functions. We say that \( f \) is an \( MC \)-derivative (monotonically controlled derivative) of \( F \) if there exists a strictly increasing function \( \varphi : I \to \mathbb{R} \) (the so-called control function to the pair \((F, f)\)) such that
\[ \lim_{y \to x} \frac{F(y) - F(x) - f(x)(y - x)}{\varphi(y) - \varphi(x)} = 0, \quad x \in I. \]
We also say that \( F \) is an \( MC \)-antiderivative (or an indefinite \( MC \)-integral) of \( f \).

Recall that the \( MC \)-integral coincides with the Perron integral for which the development of calculus is well known. Despite of this we hope that it is valuable to present an independent development of the theory. Indeed, we believe that the concept of \( MC \)-integral is comprehensible for students-beginners. Therefore we want to indicate how the theory of integral can be developed from scratch. However, we address this exposition to experienced mathematicians, so that our text is not exactly in the style of a course for beginners; this will be a task for a textbook project.

Any reasonable notion of indefinite integral must have the property that two indefinite integrals of the same function can differ only by an additive constant. This is mostly established by the observation that if \( F' \) is positive, the \( F \) is increasing. This property is valid also for \( MC \)-differentiation as shown in Section 2.

Then our exposition includes basic tools for integration like integration by parts and change of variables, this will be in Section 3. The results are well known in this generality, see e.g. [24], [17]. However, our definition leads to proofs which are very simple.

In Section 4 we prove the monotone convergence theorem. This opens a gateway for a development of the theory of integral in spirit of courses of Lebesgue integration.

The construction of the \( MC \)-integral can be easily adapted to more general situations. We show a simultaneous generalization to functions of several variables and to Stieltjes integration (in another terminology, integration with respect to Radon
measures), this is done in Section 5. The main result of this section is that, even in this generality, the $MC$-integral coincides with a corresponding integral defined by the Henstock-Kurzweil construction (and thus also with the Perron integral). This result is not so hard once we know that the Henstock-Kurzweil integral coincides with the variational integral \cite{9}; the crucial step of this equivalence is the Henstock lemma. We demonstrate the correspondence on a model case, but the idea indicates that practically each integral constructed via gage-fine tagged partitions has its $MC$-version and vice versa. For various such definition of multidimensional integrals, discussion of problems and further bibliography we refer e.g. to \cite{9}, \cite{12}, \cite{13}, \cite{14}, \cite{17}, \cite{21}, \cite{22}, \cite{23}, \cite{26}.

We will not develop the foundations of the multidimensional theory as in the first part of the paper, because here the advance in simplicity is already not so distinct. However, it is worth to mention that once started the development of integration theory with the $MC$-integral, it is possible to proceed to multidimensional integration and obtain the same results as in the theory of multidimensional Henstock-Kurzweil(-Stieltjes) integral. From the didactical point of view, it is perhaps recommendable to restrict soon the attention to the class of absolutely integrable functions, which are exactly the Lebesgue(-Stieltjes) integrable functions.

In this paper, positive means $\geq 0$ whereas $> 0$ is labelled as strictly positive. Similar convention applies to the terms increasing and strictly increasing.

2. $MC$-derivatives

In this section we prove some basic properties of $MC$-differentiation. First, we note that pairs $(F, f)$ such that $f$ is a $MC$-derivative of $F$ form a vector space. Also, it is evident that any ordinary derivative is an $MC$-derivative. However, there is one serious difference. Ordinary derivatives are unique. If we want to have a concept of derivative general enough to differentiate any indefinite Lebesgue integral, we necessarily lose uniqueness, namely, the derivative is pointwise determined only up to a set of measure zero. At this stage of exposition we do not need to speak on sets of measure zero, however, it may be useful to note that the exceptional sets are small. To illustrate this phenomenon, we assume that $f$ and $g$ are $MC$-derivatives of $F$, with control functions $\varphi$ and $\psi$, respectively. Then it is easy to observe that the monotone function $\eta = \varphi + \psi$ has infinite derivative at each point of the set $\{x : f(x) \neq g(x)\}$.

It is useful to notice that if we add an increasing function to a control function to $(F, f)$, we obtain also a control function to $(F, f)$.

If $\varphi$ is a control function to $(F, f)$, then any function of the form $\alpha \varphi + \beta$, where $\alpha, \beta$ are constants, $\alpha > 0$, is also a control function to $(F, f)$. Such a modification of a control function is called a rescaling.

**Proposition 1.** Let $F$ be an indefinite $MC$-integral of a function $f$ on an interval $I \subset \mathbb{R}$. Then $F$ is continuous.

**Proof.** Let $x \in I$. Since $\varphi$ is locally bounded, from (1) we obtain

$$\lim_{y \to x}(F(y) - F(x) - f(x)(y - x)) = 0.$$  

It follows that $F$ is continuous at $x$. \hfill $\square$

In the following theorem we prove that an indefinite $MC$-integral of a positive function is increasing.
Theorem 1. Let $F$ be an indefinite MC-integral of a function $f \geq 0$ on an open interval $I = (a_0, b_0) \subset \mathbb{R}$. Then $F$ is increasing.

Proof. Suppose that there exist $a, b \in I$ such that $a < b$ and $F(b) - F(a)$.

By rescaling we find a control function $\varphi$ with $(F + \varphi)(a) > (F + \varphi)(b)$.

We denote $G = F + \varphi$. We set $a_1 = a$, $b_1 = a$ and $c_1 = \frac{1}{2}(a_1 + b_1)$.

We choose $[a_2, b_2]$ among the intervals $[a_1, c_1]$, $[c_1, b_1]$ such that $G(a_2) > G(b_2)$.

We continue recursively and construct a nested sequence of closed intervals $[a_k, b_k]$ such that $b_k - a_k = 2^{-k}(b - a)$ and $G(a_k) > G(b_k)$.

There exists a point $x$ in the intersection of all intervals $[a_k, b_k]$. For each $k$ we can choose $x_k$ among the points $a_k$, $b_k$ such that $x_k \neq x$ and $G(x_k) - G(x)$ have the opposite sign to $x_k - x$.

Since

$$
\lim_{y \to x} \frac{F(y) - F(x) - f(x)(y - x)}{\varphi(y) - \varphi(x)} = 0,
$$

we obtain

$$
\lim_{k \to \infty} \frac{G(x_k) - G(x) - f(x)(x_k - x)}{\varphi(x_k) - \varphi(x)} = 1,
$$

however,

$$
\frac{G(x_k) - G(x)}{\varphi(x_k) - \varphi(x)} < 0 \quad \text{and} \quad \frac{f(x)(x_k - x)}{\varphi(x_k) - \varphi(x)} \geq 0, \quad k = 1, 2, \ldots
$$

This is a contradiction.

\[\square\]

Corollary 1. If $F$, $G$ are indefinite MC-integrals of a function $f$, then $F - G$ is constant.

Proof. The function $F - G$ is an indefinite MC-integral of the zero function and thus it is both increasing and decreasing.

\[\square\]

3. Calculus of MC-integral

Definition 2. If $f$ has an indefinite MC-integral $F$ on $(a, b)$ and $F$ has one-sided proper limits $F(a_+)$, $F(b_-)$ at the endpoints, then the (definite) MC-integral of $f$ over $(a, b)$ is defined as the increment of $F$:

$$
\int_a^b f(x) \, dx = [F]_a^b,
$$

where $[F]_a^b$ denotes $F(b) - F(a)$. By Corollary 1 this definition is correct, namely, it does not depend on the choice of the indefinite integral. It is obvious that integral is a linear functional. One could also define extended-real-valued integrals this way, but our convention will be that all integrals are real. Notice that in this text, all integral symbols refer to MC-integration unless specified otherwise.

Proposition 2. Suppose that $f$ is an MC-derivative of $F$ and $g$ is an MC-derivative of $G$ on $I = (a, b)$. Then $fG + Fg$ is an MC-derivative of $FG$. Hence the formula on integration by parts

$$
\int_a^b f(x)G(x) \, dx = [FG]_a^b - \int_a^b F(x)g(x) \, dx
$$

holds if the increment and the integral on the right are well defined.
Proof. Let \( \varphi \) control the pair \((F, f)\) and \( \psi \) control the pair \((G, g)\). We have
\[
F(y)G(y) - F(x)G(x) - (f(x)G(x) + F(x)g(x))(x-a)
\]
\[
= F(y) \left( G(y) - G(x) - g(x)(y-x) \right)
\]
\[
+ G(x) \left( F(y) - F(x) - f(x)(y-x) \right)
\]
\[
+ g(x)(y-x) \left( F(y) - F(x) - f(x)(y-x) \right)
\]
\[
+ f(x)g(x)(y-x)^2, \quad x, y \in I.
\]
By Proposition \(\blacksquare\) \(F\) is continuous. It is then easily seen from \(\mathfrak{2}\) that the pair \((FG, fG)\) is controlled by \(\eta(x) = \varphi(x) + \psi(x) + x\). Now, by the assumptions, there exists an indefinite \(MC\)-integral \(H\) of \(Fg\) with a well defined increment. It follows that \(FG - H\) is an indefinite \(MC\)-integral of \(fG\) with a well defined increment. \(\square\)

**Proposition 3.** Suppose that \(F\) is a strictly increasing function which maps open interval \((a, b)\) onto an open interval \((c, d)\). Let \(G : (c, d) \to \mathbb{R}\) be a function. Let \(f\) be an \(MC\)-derivative of \(F\) on \((a, b)\) and \(g\) be an \(MC\)-derivative of \(G\) on \((c, d)\). Then \(x \mapsto g(F(x))f(x)\) is an \(MC\)-derivative of \(G \circ F\) on \((a, b)\). Hence, the formula on change of variables
\[
\int_c^d g(y) dy = \int_a^b g(F(x))f(x) dx
\]
holds provided that the above assumptions are satisfied and at least one of the integrals converges.

**Proof.** Let \( \varphi \) control the pair \((F, f)\) and \( \psi \) control the pair \((G, g)\). We have
\[
G(F(x')) - G(F(x)) - g(F(x))(x' - x)
\]
\[
= G(F(x')) - G(F(x)) - g(F(x))(F(x') - F(x))
\]
\[
+ g(F(x))(F(x') - F(x)) - f(x)(x' - x)), \quad x, x' \in (a, b).
\]
Since
\[
\lim_{x' \to x} \frac{G(F(x')) - G(F(x)) - g(F(x))(F(x') - F(x))}{\psi(F(x')) - \psi(F(x))}
\]
\[
= \lim_{y' \to y} \frac{G(y') - G(y) - g(y)(y' - y)}{\psi(y') - \psi(y)} = 0, \quad x \in (a, b), \ y = F(x),
\]
we easily infer that \( \psi \circ F + \varphi \) controls the pair \((G \circ F, g \circ F)\). The statement concerning integration follows immediately. \(\square\)

**Lemma 1.** Let \(F, f : (a, b) \to \mathbb{R}\) be functions and \((a_k), (b_k)\) be sequences of real numbers. Suppose that \(a_k \searrow a\) and \(b_k \nearrow b\). If \(f\) is an \(MC\)-derivative of \(F\) on each \((a_k, b_k)\), then \(f\) is an \(MC\)-derivative on \(F\) on \((a, b)\). Moreover, the control function on \((a, b)\) can be chosen to be bounded.

**Proof.** We may assume that \((a_k)\) is strictly decreasing and \((b_k)\) is strictly decreasing. For each \(k\), let \( \varphi_k \) be a control function to \((F, f)\) on \((a_{k+1}, b_{k+1})\). Then
\( \varphi_k \) is bounded on \((a_k, b_k)\) and by a rescaling we may assume that \(0 < \varphi_k < 1\) on \((a_k, b_k)\). Set

\[
\psi_k(x) = \begin{cases} 
0, & x \leq a_k, \\
\varphi_k(x), & a_k < x < b_k, \\
1, & x \geq b_k.
\end{cases}
\]

Then the function

\[
\varphi = \sum_{k=1}^{\infty} 2^{-k} \psi_k
\]

is obviously a bounded control function to \((F, f)\) on \((a, b)\).

**Proposition 4.** Let \(a, b, c \in \mathbb{R}, a < b < c\). Let \(f : (a, c) \to \mathbb{R}\) be \(MC\)-integrable on \((a, b)\) and \((b, c)\). Then \(f\) is \(MC\)-integrable on \((a, c)\) and

\[
\int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx.
\]

**Proof.** Let \(F_1\) be an indefinite \(MC\)-integral of \(f\) on \((a, b)\) controlled by \(\varphi_1\) and \(F_2\) be an indefinite \(MC\)-integral of \(f\) on \((b, c)\) controlled by \(\varphi_2\). By Lemma 1 we can assume that \(\varphi_1, \varphi_2\) are bounded. Then the one-sided proper limits \(F_1(b_-), F_2(b_+)\) and \(\varphi_1(b_-), \varphi_2(b_+)\) exist. By adding suitable constants to \(F_1\) and rescaling \(\varphi_1, i = 1, 2\), we can arrange that \(F_1(b_-) = F_2(b_+) = 0\) and \(\varphi_1(b_-) < 0 < \varphi_2(b_+).\)

We set

\[
F(x) = \begin{cases} 
F_1(x), & x \in (a, b), \\
0, & x = b, \\
F_2(x), & x \in (b, c),
\end{cases}
\]

\[
\varphi(x) = \begin{cases} 
\varphi_1(x), & x \in (a, b), \\
0, & x = b, \\
\varphi_2(x), & x \in (b, c).
\end{cases}
\]

We claim that the function \(F\) is an indefinite \(MC\)-integral of \(f\) on \((a, c)\) controlled by \(\varphi\). Indeed, (11) holds obviously at each \(x \in (a, b) \cup (b, c)\). For \(x = b\) we use the jump of \(\varphi\) at \(b\) to observe that the limit in (11) reduces to

\[
\lim_{y \to x} (F(y) - F(x) - f(x)(y - x)).
\]

But the last limit clearly vanishes by the continuity of \(F\).

4. **Monotone convergence theorem**

In this section we establish the monotone convergence theorem for the \(MC\)-integral. This can be applied to show that the \(MC\)-integral includes the Lebesgue integral. Namely, constants are integrable over bounded intervals. Using Proposition 4 we obtain that all step (=piecewise constant) functions are integrable. We can define measurable sets as those sets \(M\), for which the characteristic function \(\chi_M\) has an indefinite \(MC\)-integral. It is well known that a system of sets which contains all intervals and is closed under monotone unions and intersections contains already all Borel sets, see e.g. [1], 1.3.9. Alternatively we can use Dynkin systems, see e.g. [2], Section 1.6. This is the step in which the monotone convergence theorem below is needed.

If we define the measure of a measurable \(M\) as the integral of the characteristic function of \(M\) (or as \(\infty\) if this integral diverges), we observe that the “measure” is complete (all subsets of null sets are measurable) and thus the class of all measurable
sets contains all Lebesgue measurable sets. This argumentation not only leads to a proof that the MC-integral includes the Lebesgue integral, but also bypasses some difficult steps in construction of the Lebesgue measure.

**Theorem 2** (Monotone convergence theorem). Let \( I = (a, b) \) be an open interval and \((f_k)_k\) be a sequence of MC-integrable functions on \( I \), \( f_n \not\rightarrow f \). If

\[
\lim_{k \to \infty} \int_a^b f_k(x) \, dx < +\infty,
\]

then \( f \) is MC-integrable over \( I \) and

\[
\int_a^b f(x) \, dx = \lim_{k \to \infty} \int_a^b f_k(x) \, dx.
\]

**Proof.** By subtracting \( f_1 \) we may achieve that \( f_k \geq 0 \), \( k = 1, 2, \ldots \). For each \( k \), let \( F_k \) be the indefinite MC-integral of \( f_k \) normalized by \( F_k(a) = 0 \). Then \((F_k)_k\) is an increasing sequence of increasing functions and we can define \( F = \lim_{k \to \infty} F_k \). From (3) we infer that \( F \) is bounded in \((a, b)\), also it is easy to observe that \( F(b) = \lim_{k \to \infty} F_k(b) \). Replacing, if necessary, \((F_k)_k\) by a subsequence, we may assume that

\[
F_k(b) > F(b) - 2^{-k}.
\]

Since \( f_j - f_k \geq 0 \) for \( j > k \), the function \( F_j - F_k \) is increasing by Theorem 1. Passing to the limit we obtain that each \( F - F_k \) is increasing. For each \( k \), let \( \varphi_k \) be a control function to \((F_k, f_k)\). By Lemma 1 we may assume that \( \varphi_k \) is bounded and thus it may be rescaled to satisfy \( 0 < \varphi_k < 1 \). We set

\[
\varphi(x) = \sum_{k=1}^{\infty} 2^{-k} \varphi_k(x) + \sum_{k=1}^{\infty} k(F(x) - F_k(x)) + x.
\]

From (4) we infer that \( \varphi \) is finite in \((a, b)\); obviously it is strictly increasing. We claim that \( \varphi \) controls \((F, f)\). We choose \( x \in I \) and \( \varepsilon > 0 \). We find an integer \( k > 0 \) such that \( x \frac{1}{k} < \varepsilon \) and \( f(x) - f_k(x) < \varepsilon \). Then we estimate

\[
F(y) - F(x) - f(x)(y - x) = \left( F_k(y) - F_k(x) - f_k(x)(y - x) \right)
+ \left( F(y) - F(x) - (F_k(y) - F_k(x)) \right)
+ (f_k(x) - f(x))(y - x)
\leq \left( F_k(y) - F_k(x) - f_k(x)(y - x) \right) + \left( \frac{1}{k} + \varepsilon \right) (\varphi(y) - \varphi(x)).
\]

Therefore

\[
\limsup_{y \to x} \frac{F(y) - F(x) - f(x)(y - x)}{\varphi(y) - \varphi(x)} \leq 2\varepsilon,
\]

which verifies the claim. Hence \( F \) is an indefinite MC-integral of \( f \) and

\[
\int_a^b f(x) \, dx = [F]_a^b = \lim_{k \to \infty} [F_k]_a^b = \lim_{k \to \infty} \int_a^b f_k(x) \, dx.
\]
5. A comparison with the Henstock-Kurzweil integral

In this section we show that our MC-integral coincides with the Henstock-Kurzweil integral. This will be done in the framework of multidimensional Stieltjes integration.

The multidimensional integration requires the language of interval functions. For an introduction to manipulation with interval functions, in particular to their differentiation, we refer to Saks [27]. By interval in \( \mathbb{R}^n \) we mean a Cartesian product of one-dimensional intervals. We denote by \( \mathcal{I} \) the collection of all nondegenerate bounded closed intervals in \( \mathbb{R}^n \). A finite set \( D \subset \mathcal{I} \) is called a partition of and interval \( I \in \mathcal{I} \) if the intervals from \( D \) are nonoverlapping (i.e. have disjoint interiors) and \( \bigcup_{Q \in D} Q = I \). A function \( F : \mathcal{I} \to \mathbb{R} \) is said to be

- additive, if for each interval \( I \in \mathcal{I} \) and each partition \( D \) of \( I \) we have
  \[
  \sum_{Q \in D} F(Q) = F(I),
  \]

- superadditive, if for each interval \( I \in \mathcal{I} \) and each partition \( D \) of \( I \) we have
  \[
  \sum_{Q \in D} F(Q) \leq F(I),
  \]

There are many possibilities how to modify the definition below, for example to require some “regularity” of intervals in the limiting process like in [21]. This will yield a variety of non-equivalent integrals. We illustrate our approach on the simplest model case. We consider only indefinite integrals. The definite integrals over intervals \( \not \in \mathcal{I} \) can be defined by an appropriate limit process.

**Definition 3.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function and \( F : \mathcal{I} \to \mathbb{R} \), \( G : \mathcal{I} \to \mathbb{R} \) be additive interval functions. We say that \( f \) is an MC-derivative of \( F \) with respect to \( G \), or that \( F \) is an indefinite MC-integral of \( f \) with respect to \( G \), if there exists a superadditive interval function \( \Phi > 0 \) (called a control function) such that for each \( x \in \mathbb{R}^n \) and for each sequence \( (Q_k)_k \) of intervals from \( \mathcal{I} \) such that \( x \in \bigcap_k Q_k \) and \( \text{diam} Q_k \to 0 \) we have

\[
\lim_{k \to \infty} \frac{F(Q_k) - f(x)G(Q_k)}{\Phi(Q_k)} = 0.
\]

If in the definition of MC-derivative (MC-integral) we require \( \Phi \) to be additive, we denote the result as AMC-derivative (AMC-integral).

Each superadditive interval function \( \Phi > 0 \) has the property that

\[
P, Q \in \mathcal{I}, \quad P \subset Q \implies \Phi(P) \leq \Phi(Q).
\]

Therefore the terminology “monotonically controlled” is again reasonable.

**Remark 1.** Additive interval functions in \( \mathbb{R} \) have the form

\[
G([a, b]) = G(b) - G(a),
\]

where \( G : \mathbb{R} \to \mathbb{R} \) is an “ordinary” function. For the one-dimensional Stieltjes differentiation and integration, (5) reduces to

\[
\lim_{y \to x} \frac{F(y) - F(x) - f(x)(G(y) - G(x))}{\varphi(y) - \varphi(x)} = 0.
\]
There is no need to use superadditive control functions in $\mathbb{R}$ because each superadditive function $\Phi > 0$ is easily majorized by an additive function, using the increasing function

$$\varphi(x) = \begin{cases} 
\Phi([0,x]), & x > 0, \\
0, & x = 0, \\
-\Phi([x,0]), & x < 0.
\end{cases}$$

In higher dimension, the relation between the $MC$ and $AMC$ definition is not so clear, see [10]. Since the $MC$-integral includes the $AMC$-integral and is more easy to handle, we prefer $MC$-integration. On the other hand, the notion of additive functions may seem to be more elementary and for the purpose of absolute integration the concept of $AMC$-integration is sufficient.

We recall the definition of $HK$-integral with respect to an additive interval function $G$ as it is defined e.g. in [17].

**Definition 4.** Let $I \in \mathcal{I}$ be an interval. A tagged partition of $I$ is defined as a couple $(D, \tau)$ where $D$ is a partition of $I$ and $\tau : D \to \mathbb{R}^n$ is a mapping such that

$$\tau(Q) \in Q, \quad Q \in D.$$  

The condition (6) is not always required in literature (it should be dropped for McShane integration), but it should be assumed for the purpose of $HK$-integration. We identify a tagged partition $(D, \tau)$ with the set $$\{(Q, x) : Q \in D, \quad x = \tau(Q)\}.$$ By a gage we mean a strictly positive function $\delta : \mathbb{R}^n \to \mathbb{R}$. Given a gage $\delta$, we say that a tagged partition $(D, \tau)$ of $I$ is $\delta$-fine if for each $(Q, x)$ we have $\text{diam } Q < \delta(x)$. Let $f : I \to \mathbb{R}$ be a function and $\alpha \in \mathbb{R}$. We say that $\alpha$ is a $HK$ (Henstock-Kurzweil version of Stieltjes) integral of $f$ over $I$ if for each $\varepsilon > 0$ there exists a gage $\delta$ such that for each $\delta$-fine partition $(D, \tau)$ of $I$ we have

$$\left| \sum_{(Q, x) \in (D, \tau)} f(x)G(Q) - \alpha \right| < \varepsilon.$$  

The $HK$-integral is unique if it exists. We define the indefinite $HK$-integral of $f : \mathbb{R}^n \to \mathbb{R}$ as the interval function which assign to each $Q \in \mathcal{I}$ the $HK$-integral of $f$ over $Q$ with respect to $G$. It is an additive interval function.

**Definition 5.** Let $\Psi : \mathcal{I} \times \mathbb{R}^n \to \mathbb{R}$ be a function and $\delta$ be a gage. The $\delta$-variation of $\Psi$ is defined as

$$V_\delta(P, \Psi) = \sup \left\{ \sum_{(Q, x) \in (D, \tau)} |\Psi(Q, x)| : (D, \tau) \text{ is a } \delta \text{-fine partition of } P \right\}, \quad P \in \mathcal{I}.$$  

If the interval function $V_\delta(\cdot, \Psi)$ is finite, then it is superadditive.

The following statement establishes the equivalence of the Henstock-Kurzweil integral and the so-called variational integral. The only if part is known as Henstock’s lemma. For the proof see e.g. [11], Theorem 44.6, [12].

**Proposition 5.** Let $F, G$ be additive interval functions on $\mathcal{I}$, $G \geq 0$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Let $\Psi : \mathcal{I} \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\Psi(Q, x) = f(x)G(Q) - F(Q), \quad Q \in \mathcal{I}, \quad x \in \mathbb{R}^n.$$
Then $\mathbf{F}$ is an indefinite HK integral of $f$ with respect to $\mathbf{G}$ if and only if
\[
\inf \left\{ V_\delta(I, \Psi) : \delta \text{ is a gage} \right\} = 0
\]
for each $I \in \mathcal{I}$.

Now, we are ready to compare our MC-definition of integral with the HK-integral.

**Theorem 3.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $\mathbf{F} : \mathcal{I} \to \mathbb{R}$ be an interval function. Then $\mathbf{F}$ is an indefinite MC-integral of $f$ with respect to $\mathbf{G}$ if and only if $\mathbf{F}$ is an indefinite HK-integral of $f$ with respect to $\mathbf{G}$.

**Proof.** Let $\Phi : \mathcal{I} \to \mathbb{R}$ be a control function to $(\mathbf{F}, f)$ with respect to $\mathbf{G}$. Choose $I \in \mathcal{I}$ and $\varepsilon > 0$. For each $x \in I$ there exists $\delta(x) > 0$ such that for all $Q \in \mathcal{I}$ containing $x$ with $\text{diam } Q < \delta$ we have
\[
\left| \mathbf{F}(Q) - f(x) \mathbf{G}(Q) \right| < \varepsilon \Phi(Q).
\]
We claim that $\delta$ is the desired gage. If $(D, \tau)$ is a $\delta$-fine partition of an interval $I \in \mathcal{I}$, then
\[
\left| \sum_{(Q,x) \in (D, \tau)} f(x) \mathbf{G}(Q) - \mathbf{F}(I) \right| = \left| \sum_{(Q,x) \in (D, \tau)} (f(x) \mathbf{G}(Q) - \mathbf{F}(Q)) \right|
\leq \sum_{(Q,x) \in (D, \tau)} \left| (f(x) \mathbf{G}(Q) - \mathbf{F}(Q)) \right|
\leq \sum_{(Q,x) \in (D, \tau)} \varepsilon \Phi(Q) < \varepsilon \Phi(I).
\]
It follows that $\mathbf{F}$ is an indefinite HK-integral of $f$ with respect to $\mathbf{G}$. Conversely, suppose that $\mathbf{F}$ is an indefinite HK-integral of $f$ with respect to $\mathbf{G}$ and denote
\[
\Psi(Q, x) = f(x) \mathbf{G}(Q) - \mathbf{F}(Q), \quad (Q, x) \in \mathcal{I} \times \mathbb{R}^n.
\]
We consider the intervals $I_k = [-k, k]^n$. Using the Henstock lemma (Proposition 5), for each integer $k > 0$ we find a gage $\delta_k$ on $I_k$ such that
\[
V_{\delta_k}(I_k, \Psi) \leq 2^{-k}.
\]
Set
\[
\Phi(Q) = |Q| + \sum_{k=1}^{\infty} k V_{\delta_k}(Q \cap I_k).
\]
Then $\Phi(Q)$ is a strictly positive finite superadditive function on $\mathcal{I}$. Given $x \in \mathbb{R}$ and $\varepsilon > 0$ we find an integer $k > 1$ such that $\frac{1}{k} < \varepsilon$ and $x \in I_{k-1}$. Let $Q \in \mathcal{I}$ be such that $x \in Q \subset I_k$ and $\text{diam } Q < \delta_k(x)$. Then
\[
|f(x) \mathbf{G}(Q) - \mathbf{F}(Q)| = |\Psi(x, Q)| \leq V_{\delta_k}(Q, \Psi) \leq \frac{1}{k} \Phi(Q) < \varepsilon \Phi(Q).
\]
This shows that $\mathbf{F}$ is an indefinite MC-integral of $f$ with respect to $\mathbf{G}$ controlled by $\Phi$. \qed

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Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic
E-mail address: haanja@gmail.com

Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic, and
Department of Mathematics, J. E. Purkyně University, České mládež 8, 400 96 Ústí nad Labem, Czech Republic
E-mail address: maly@karlin.mff.cuni.cz