Three-dimensional cellular automata as a model of a seismic fault

G Gálvez and A Muñoz
Department of Basic Sciences, UPBII Instituto Politécnico Nacional, Av. Acueducto S/N, Barrio la Laguna Ticomán, C. P. 07340, Ciudad de México, Mexico

E-mail: ggalvezcoyt@gmail, amunozdiosdado@gmail

Abstract. The Earth's crust is broken into a series of plates, whose borders are the seismic fault lines and it is where most of the earthquakes occur. This plating system can in principle be described by a set of nonlinear coupled equations describing the motion of the plates, its stresses, strains and other characteristics. Such a system of equations is very difficult to solve, and nonlinear parts leads to a chaotic behavior, which is not predictable. In 1989, Bak and Tang presented an earthquake model based on the sand pile cellular automata. The model though simple, provides similar results to those observed in actual earthquakes. In this work the cellular automata in three dimensions is proposed as a best model to approximate a seismic fault. It is noted that the three-dimensional model reproduces similar properties to those observed in real seismicity, especially, the Gutenberg-Richter law.

Introduction

According to the theory of plate tectonics earthquakes occur in the contact surfaces between the plates, which are called seismic faults, although some of them occur between the plates and they cannot be explained in an easy way. The rocks move and break along either side of the fault, movement along the seismic fault causes seismic waves. Seismographs recorded seismic waves through seismograms. The magnitude of an earthquake can be determined in principle from the amplitude of the seismic waves recorded on seismograms. The magnitude reflects the amount of energy released by the earthquake fault and is directly related to the caused damage.

Gutenberg and Richter developed a scale to measure the magnitude of an earthquake, which reflects the existence of invariant quantities of scale, also known as fractals.

A compilation that include at least the following details of an earthquake: date and exact time of the occurrence, geographic location, magnitude of earthquakes and the caused damage, constitutes what is called a seismic catalogue. The information is published by seismographic networks. Usually seismic catalogues are incomplete and only include earthquakes from a significant magnitude.

Although the theory of plate tectonics explains the seismic production, it says nothing about the possible magnitude of the earthquake and much less on its location and occurrence.

The proper description of a seismic fault by equations is very complicated to solve. Some seismic fault models more simple have been proposed, these simple models have some advantages, among them are relatively easy to implement, can generate seismic catalogs full synthetics, furthermore, despite its simplicity they properly reproduce some statistics properties of earthquakes, and one of these models is the two-dimensional sand pile model that is represented by a cellular automaton. This particular
model has been very successful in describing the properties observed actual seismicity. A natural step for description of a seismic fault is extending the sandpile model to three dimensions. In this paper, the 3-D sand pile model is used; again, it is obtained as a result synthetic seismic catalogs the analysis of these catalogs shows properties similar to those observed in the behavior of a actual seismic fault. Some parameters of 3-D sand pile model sand pile may suitably be selected for obtain a behavior more similar to that observed in actual seismicity. The main objective of this paper is to show that the generalization of automata sandpile two to three dimensions is adequate, Besides, synthetic catalogs generated by the model can then be used to investigate more properties of the seismic faults.

Gutenberg-Richter Law
Gutenberg and Richter [1] developed the scale of seismic measurement, using the data occurred in southern California USA, between 1934 and 1943, they noted that the number of earthquakes of local magnitude between 3.5 and 7.0 decreased exponentially as the magnitude increased, based on this observation they suggested an empirical relationship in 1944 for the logarithm of the frequency density vs the local magnitude, per unit time, $\log_{10} N(M)$, which may be expressed as:

$$\log_{10} N(M) = a - bM$$  \hspace{1cm} (1)

The parameters $a$ and $b$ vary according to factors such as sampling time and the regional level of seismicity of the region, $b$ has characteristic values for different regions of the Earth, with values in the range of $0.75 < b < 1.5$ [2]. Subsequent studies showed that the law is fulfilled for all regions of the world. The parameter $b$ describes the relative size distribution of the events. That is, a small slope implies that it has a greater number of large earthquakes, whereas if the slope is greater, there are fewer, so the parameter $b$ can be related to the risk or danger of the seismic region. In Figure 1 the plot of the logarithm of the frequency against the earthquake magnitude for real data from the area shown Guerrero Mexico. The graph shows that there are two distinct regions, can establish a Gutenberg-Richter law for $4 \leq M \leq 6.5$ and other in the region $7.5 \leq M$ [3].

Another important feature of the seismicity is the graph of the cumulative number of earthquakes against time for cutting magnitude $M_c$ in Figure 2, is presented the graph for number of earthquake vs time for $M_c \geq 4.3$ in the same of Guerrero México region [3]. The graph has a staircase shape, which a big scale follows a linear trend, as it is shown in the same figure 2, it is further noted that abrupt changes in the staircase correspond to earthquakes, some of appreciable magnitude.
Self-organized criticality

In the study of the dynamics of non-equilibrium systems (phase transitions) and heterogeneous systems has been made popular the concept of complexity, that is, systems with a large number of interacting parts, that change energy, matter or information with their environment are naturally complex systems, they self-organize its internal structure with new and surprising macroscopic properties, for example, has been observed the existence of fractal structures in space and time (which is called 1/f noise) [4-7].

The concept of self-organized criticality (SOC) was proposed by Bak, Tang and Wiesenfeld (BTW) in 1987 [8, 9] and provides a connection between nonlinear dynamics and the appearance of spatial auto similarity and 1/f noise in a natural and robust way.

Power laws

Mathematically a power law means that a number $N$ can be expressed by a power of another quantity $s$,

$$ N(s) = K s^{-\tau} $$

(2)

For example, $s$ can be a characteristic length and $N(s)$ the number of objects needed to cover a region with squares of length $s$, this fact would indicate the fractal dimension of the object, that is, fractals are characterized by power laws. If in the above expression is applied the logarithm is obtained that:

$$ \log N(s) = \log(K) - \tau \log s $$

(3)

That is, by plotting $\log N(s)$ vs. $\log(s)$ a straight line is obtained, its slope is $-\tau$. The scale invariance can be seen simply from the fact that the straight line looks the same at any scale.

Experimentally Utsu and Seki in 1954 found the following relationship:

$$ \log_{10} A = M - 4 $$

(4)

Where $A$ is the area of replicas of large earthquakes and $M$ is the magnitude of an earthquake. Using the relation 4 can show that the Gutenberg-Richter law is equivalent to a power law, that is:

$$ \log_{10} N(A) = \log K_i - b \log_{10} A $$

(5)

In this case $\log K_i = a - 4b$.

The explanation of the features observed in complex systems can be expressed mathematically as the explanation of the power laws and particularly the value of the exponents. In short, self-organized criticality is a hypothesis to cover a multitude of complex phenomena observed in nature. It is a theory of internal interactions of large systems. Specifically, it states that large interactive systems could self-organize in a critical state (governed by a power law). Once in this state small perturbations produce chain reactions that may affect any number of elements within the system [1].
Sand pile model
The model “sand pile” it was proposed by Per Bak [4], is normally used to represent or generate complexity and self-organization [1], and consist in adding randomly a grain of sand every time, inside a box, the first grains accumulates in separate places and then, some small mounds are formed. We continue the process; the slope of the mounds is growing. In the future, the slope locally reaches a critical value such that the sum of a grain results in an "avalanche". The avalanche filled empty box areas with sand. With the addition of even one grain of sand the box could be filled and when continuing adding grains of sand, they will leave the system. When the aggregate grains account equals the amount lost (on average) then, according to the theory, the pile of sand grains is self-organized in a critical condition. Of course the notion of nature as a heap of sand seems overly simple, but it is a good model to represent complex systems.

The two-dimensional sand pile is described below using cellular automata [1]. The space is divided by a homogeneous mesh network and each square is associated with a point (x, y), to each cell is assigned a number of grains of sand by the function \( Z(x, y) \). The addition of a grain of sand to the square of the grid is at random \( Z(x, y) \rightarrow Z(x, y) + 1 \), this process is repeated until the number of grains on a site reaches a critical value, it is normally considered \( Z_c = 4 \) grains. When the critical condition is achieved in one of the grid points the following rules apply:

\[
\begin{align*}
Z(x, y) & \rightarrow Z(x, y) - 4 \\
Z(x \pm 1, y) & \rightarrow Z(x \pm 1, y) + 1 \\
Z(x, y \pm 1) & \rightarrow Z(x, y \pm 1) + 1
\end{align*}
\]

(6)

The process is repeated each time on sites that reach the critical state until stability is reached. The number of sites \( s \) that were activated before reaching stability is the size of the avalanche and the number of steps \( t_d \) to reach the stable position is equal to the duration of the event, plus some particles may be lost by falling outside the borders.

The process is illustrated in Figure 3, in which it is shown an automaton of size 5x5, the sequence of figures should be from left to right from top to bottom. The first drawing shows an automaton in a critical state, then, in the second, it is added a particle in the center cell (the cell increases its value 3 to 4) and thus the site reaches an unstable condition and the process of redistribution of particles to neighboring sites begins, the first distribution is shown in the third figure, the process continues until it reaches a critically stable state, it is shown in the following drawings. The avalanche size in this case is \( s = 9 \) because a site was relaxed twice, and the duration of the event is \( t_d = 7 \).

The frequency distribution of magnitude leads to a power law, \( N(s) \approx s^{-\tau} \) and distribution of duration of event also results in a power law, \( N(t_d) \approx t_d^{-\sigma} \).

![Figure 3](image-url) Illustration of an avalanche in the sand pile model 5x5. A grain falling at the center of the grid, the site changes its value from 3 to 4. The cell becomes unstable starting an avalanche. The continuous distribution process until it reaches a stable condition. The last picture shows the eight sites exceeded the critical value. As can be seen the avalanche has a size \( s = 9 \) (a site relaxed twice) and a duration \( t_d = 7 \).
A natural step in the development of the models is to generalize the previous model to the three-dimensional (3D) sand pile, in this case we have to modify some parameters, for example, you have to modify the critical value that becomes unstable the cell. The three-dimensional cell is represented by a cube, each site in the cube has 6 direct neighbors and it is natural to choose now the critical value as 6 particles or grains. It is appropriate in the context of a seismic fault to change the idea of molehill or particle by the concept of stress to which it is subjected a cell and the logarithm of size of the avalanche as the earthquake magnitude, that is, \( M = \log(s) \). In this context, the duration of an earthquake is directly related to the \( t_d \) parameter. As in the two dimensional case \( s \) can be defined in three dimensions as the number of cells before reaching relaxed stability, this is the magnitude of the avalanche. Moreover, the number of steps to reach \( t_d \) stability is the duration of the event. Generalized rules for 3D sand pile are given below:

\[
\begin{align*}
Z(x,y,z) & \rightarrow Z(x,y,z)-6, \\
Z(x\pm1,y,z) & \rightarrow Z(x\pm1,y,z)+1 \\
Z(x,y\pm1,z) & \rightarrow Z(x,y\pm1,z)+1
\end{align*}
\]

Results

The cellular automaton 3D sand pile model was programmed complying with the generalized rules. The cellular automaton was test for many sizes, Figure 4 shows the graph of the number of events \( N(s) \) with the size \( s \) in log-log scale to a size 15x15x15 and for 30 million of events, as shown, the relationship is a typical power law (on the left), it should be noted that this power law is equivalent to the Gutenberg-Richter law using \( M = \log(s) \). The fit was made only by using the linear region of the graph to obtain a slope of the line \( b = 1.33 \), the value obtained is within the range reported for the parameter \( b \) of the Gutenberg-Richter law (0.75 <\( b \) <1.5).

The analysis of \( t_d \) (physically related to the duration of an earthquake), which also follows a power law is shown in Figure 5, done for the 3D sand pile model, size 15x15x15. The linear part for the time duration \( t_d \) leads to a slope \( b_t = 1.29 \), which as expected is slightly less in absolute value to the slope of the power law for the magnitude \( b = 1.33 \). It is expected that because the time duration \( t_d \) is less than the magnitude \( M \) (in the computer sense), the straight slope in the linear part of the log-log plot of time duration \( t_d \) is less in absolute value than that obtained in the graph of the magnitude \( M \). Tests with different sizes of the 3-D sand pile model were performed; the results are very similar among them. One question we want to know is, which is the relation of the straight line slope and el number \( Np \) of events?, to answer this question a model of 15x15x15 pile of sand was used, and \( Np \) took the set of values 10000, 100000, 1000000 and 10000000. Figure 6 shows the results; it can be seen the graphs for each of the values of \( Np \). For \( Np=10000 \), the data are widely dispersed, however, the slope is -1.26. Increasing the value of \( Np \), the data dispersion decreases and the slope improves.
With \( N_p = 10000000 \), a slope of -1.33 is obtained. It is further noted that the graph of Figure 1 where actual data are analyzed is similar to the graph of Figure 6. In real seismicity it was found that adding the magnitude of earthquakes with respect to time a graphic resembling a staircase is obtained. Using different sizes of sand pile model, we obtained graphics type staircases. It can be seen from Figure 5 that the slope of the line that can be associated to each staircase increases with model size.

\[ \log(N(t_d)) = -1.29 \log(t_d) + 0.45 \]

7. Conclusions
The model sand pile in two dimensions can expand smoothly to three dimensions suitably modifying the distribution rules. The consideration that the number of grains of sand or particles is directly related to stress of stone makes the 3D cellular automata a suitable model to describe a real seismic fault. The power laws for the magnitude have slopes that are consistent with the expected values for the parameter \( b \) of the Gutenberg-Richter law, also the duration of avalanches \( t_d \) complies with power law, which is related to fractal properties observed in complex systems. Several properties of seismic faults are obtained with the 3D sand pile model, as the staircase graphs for the cumulative magnitude. The 3D sand pile model produces synthetic seismic catalogs with properties very similar to the actual seismicity catalogs. Setting some parameters could help to adjust \( b \) the value of the Gutenberg-Richter law by more appropriate values. Synthetic seismic catalogs can be used to investigate properties of the actual seismicity [11-14].

Acknowledgments
We thank COFAA and EDI from the Instituto Politécnico Nacional for supporting this work.
References

[1] Gutenberg B and Richter C F 1944 Bull. Seismol. Soc. Am. 34 185
[2] Olami Z, Feder H J S and Christensen K 1992 Phys. Rev. Lett. 68 12441247
[3] Muñoz, A, Rudolf A H and Angulo F 2012 Rev. Mex. Fis. S 58 (1) 96–103
[4] Bak P 1996 How nature works (Springer-Verlag New York. USA)
[5] Mandelbrot B 1983 The fractal geometry of nature (Freeman, New York)
[6] Mandelbrot B and Van J W 1968 SIAM Rev 10 422
[7] Mandelbrot B and Wallis J R 1969 Water Resour. Res. 5 321
[8] Bak P, Tang C and Weisenfeld K 1987 Phys. Rev. Lett. 59, 381
[9] Bak P, Tang C and Weisenfeld K 1988 Phys. Rev. A 38 364
[10] Utsu T and Seki A 1954 Journal of Seismological Society of Japan v. 7 p. 233-240.
[11] Bizzarri A 201 Earth-Science Reviews 115 304–318
[12] Jiménez A 2013 Acta Geophysica, 61 (6) 1325-1350
[13] Asadi A 2014 Iranian Journal of Science & Technology 38 35-41